SPECTRUM OF THE KOHN LAPLACIAN ON THE ROSSI SPHERE

TAWFIK ABBAS, MADELYNE M. BROWN, RAVIKUMAR RAMASAMI,
AND YUNUS E. ZEYTUNCU

Abstract. We study the spectrum of the Kohn Laplacian \( \Box_b \) on the Rossi example \((S^3, L_t)\). In particular we show that 0 is in the essential spectrum of \( \Box_b \), which yields another proof of the global non-embeddability of the Rossi example.

1. Introduction

When is an abstract CR-manifold globally CR-embeddable into \( \mathbb{C}^N \)? Rossi showed that the CR-manifold \((S^3, L_t)\) is not CR-embeddable [Ros65], where \( S^3 \) is the 3-sphere in \( \mathbb{C}^2 \),

\[
L_t = \frac{1}{z_1} \frac{\partial}{\partial z_2} - \frac{1}{z_2} \frac{\partial}{\partial z_1} + \overline{t} \left( \frac{1}{z_1} \frac{\partial}{\partial z_2} - \frac{1}{z_2} \frac{\partial}{\partial z_1} \right),
\]

and \(|t| < 1\). In the case of strictly pseudoconvex CR-manifolds Boutet de Monvel proved that if the real dimension of the manifold is at least 5, then it can always be globally CR-embedded into \( \mathbb{C}^N \) for some \( N \) [BdM75]. Later Burns approached this problem in the \( \overline{\partial} \) context and showed that if the tangential operator \( \overline{\partial}_{b,t} \) has closed range and the Szegö projection is bounded, then the CR-manifold is CR-embeddable into \( \mathbb{C}^N \) [Bur79]. Later in 1986, Kohn showed that CR-embeddability is equivalent to showing that the tangential Cauchy-Riemann operator \( \overline{\partial}_{b,t} \) has closed range [Koh85]. We refer to [CS01, Chapter 12] for a full account of these results and also to [Bog91] for general theory of CR-manifolds.

In the setting of the Rossi example, as an application of the closed graph theorem, \( \overline{\partial}_{b,t} \) has closed range if and only if the Kohn Laplacian

\[
\Box_b = -L_t \frac{1 + |t|^2}{(1 - |t|^2)^2} \overline{L}_t
\]

has closed range, see [BE90, 0.5]. Furthermore, the closed range property is equivalent to the positivity of the essential spectrum of \( \Box_b \), see [Fu05] for similar discussion. In this note we tackle the problem of embeddability, from the perspective of spectral analysis. In particular, we show that 0 is in the essential spectrum of \( \Box_b \), so the Rossi sphere is not globally CR-embeddable in \( \mathbb{C}^N \). This provides a different approach to the results in [Bur79, Koh85].

We start our analysis with the spectrum of \( \Box_b \). We utilize spherical harmonics to construct finite dimensional subspaces of \( L^2(S^3) \) such that \( \Box_b \) has tridiagonal matrix representations on these subspaces. We then use these matrices to compute eigenvalues of \( \Box_b \). We also present numerical results obtained by Mathematica that motivate most of our theoretical
results. We then present an upper bound for small eigenvalues and we exploit this bound to find a sequence of eigenvalues that converge to 0.

In addition to particular results in this note, our approach can be adopted to study possible other perturbations of the standard CR-structure on the 3-sphere, such as in [BE90]. Furthermore, our approach also leads some information on the growth rate of the eigenvalues and possible connections to finite-type (order of contact with complex varieties) results similar to the ones in [Fu08]. We plan to address these issues in future papers.

2. Analysis of $\Box_b$ on $\mathcal{H}_{p,q}(S^3)$

2.1. Spherical Harmonics. We start with a quick overview of spherical harmonics, we refer to [ABR01] for a detailed discussion. We will state the relevant theorems on $C^2$ and $S^3 \subseteq C^2$. A polynomial in $C^2$ looks like

$$p(z, \overline{z}) = \sum_{\alpha, \beta} c_{\alpha, \beta} z^\alpha \overline{z}^\beta$$

where each $c_{\alpha, \beta} \in C$, and $\alpha$, $\beta$ are multi-indices. That is, $\alpha = (\alpha_1, \alpha_2)$, $z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2}$, and $|\alpha| = \alpha_1 + \alpha_2$.

We denote the space of all homogeneous polynomials on $C^2$ of degree $m$ by $P_m(C^2)$, and we let $H_m(C^2)$ denote the subspace of $P_m(C^2)$ that consists of all harmonic homogeneous polynomials on $C^2$ of degree $m$. We use $P_m(S^3)$ and $H_m(S^3)$ to denote the restriction of $P_m(C^2)$ and $H_m(C^2)$ onto $S^3$. We denote the space of complex homogenous polynomials on $C^2$ of bidegree $p, q$ by $P_{p,q}(C^2)$, and those polynomials that are homogeneous and harmonic by $H_{p,q}(C^2)$. As before, we denote $P_{p,q}(S^3)$ and $H_{p,q}(S^3)$ as the polynomials of the previous spaces, but restricted to $S^3$. We recall that on $C^2$, the Laplacian is defined as $\Delta = 4(\frac{\partial^2}{\partial z_1 \partial \overline{z}_1} + \frac{\partial^2}{\partial z_2 \partial \overline{z}_2})$. As an example, the polynomial $z_1 \overline{z}_2 - 2z_2 \overline{z}_1 \in P_{1,1}(C^2)$, and $z_1 \overline{z}_2^2 \in H_{1,2}(C^2)$. We take our first step by stating the following theorem.

**Theorem 2.1.** [ABR01] Theorem 5.1] If $p$ is a polynomial on $C^2$ of degree $m$, then

$$P[p|_{S^3}] = (1 - |z|^2)q + p$$

for some polynomial $q$ of degree at most $m - 2$.

This theorem highlights how the Poisson integral of an $m$ degree polynomial on $S^3$ can be represented by a polynomial decomposition. As the Poisson integral yields a harmonic polynomial, the polynomial decomposition will be harmonic.

Similarly, we have the following decomposition for the space of homogeneous polynomials into a space of harmonic polynomials and a space of homogeneous polynomials with a factor of $|z|^2$.

**Theorem 2.2.** [ABR01] Theorem 5.5] If $m \geq 2$, then

$$P_m(C^2) = H_m(C^2) \oplus |z|^2 P_{m-2}(C^2),$$

and

$$P_{p,q}(C^2) = H_{p,q}(C^2) \oplus |z|^2 P_{p-1,q-1}(C^2).$$

By applying the previous statement multiple times to the homogeneous part of a polynomial decomposition, we arrive at the following theorem.
Theorem 2.3. [ABR01] Theorem 5.7] Every \( p \in \mathcal{P}_m(\mathbb{C}^2) \) can be uniquely written in the form
\[
p = p_m + |z|^2 p_{m-2} + \ldots + |z|^{2k} p_{m-2k}
\]
where \( k = \left\lfloor \frac{m}{2} \right\rfloor \) and each \( p_i \in \mathcal{H}_m(\mathbb{C}^2) \), where \( [x] \) means the nearest integer to \( x \).

This yields to the following decomposition of the space of square integrable functions on \( S^3 \).

Theorem 2.4. [ABR01] Theorem 5.12] \( L^2(S^3) = \bigoplus_{m=0}^{\infty} \mathcal{H}_m(S^3) \).

The previous theorem is essential to the spectral analysis of \( \Box^b \) on \( L^2(S^3) \) since it decomposes the infinite dimensional space \( L^2(S^3) \) into finite dimensional pieces, which is necessary for obtaining the matrix representation of \( \Box^b \). In order to get such a matrix representation, we need a method for obtaining a basis for \( \mathcal{H}_m(S^3) \). Theorem 2.6 presents a method to do so for \( \mathcal{H}_m(\mathbb{C}^2) \) and Theorem 2.8 presents a method for \( \mathcal{H}_{p,q}(\mathbb{C}^2) \). The dimension of the matrix representation on a particular \( \mathcal{H}_m(S^3) \) is the dimension of the subspace \( \mathcal{H}_m(S^3) \), which is given below and analogously given for \( \mathcal{H}_{p,q}(\mathbb{C}^2) \).

Theorem 2.5. [ABR01] Proposition 5.8] If \( m \geq 2 \), then
\[
\dim \mathcal{H}_m(\mathbb{C}^2) = \binom{n + m - 1}{n - 1} - \binom{n + m - 3}{n - 1},
\]
\[
\dim \mathcal{P}_{p,q}(\mathbb{C}^2) = (p + 1)(q + 1),
\]
and
\[
\dim \mathcal{H}_{p,q}(\mathbb{C}^2) = p + q + 1
\]
\[
\dim \mathcal{H}_k(\mathbb{C}^2) = (k + 1)^2.
\]

Now we present a method to obtain explicit bases of spaces of spherical harmonics. These bases play an essential role in explicit calculations in the next section. Here, \( K \) denotes the Kelvin trasform,
\[
Kg(z) = |z|^{-2} g \left( \frac{z}{|z|^2} \right).
\]

Theorem 2.6. [ABR01] Theorem 5.25] If \( n > 2 \) then the set
\[
\{ K[D^\alpha |z|^{-2}] : |\alpha| = m \text{ and } \alpha_1 \leq 1 \}
\]
is a vector space basis of \( \mathcal{H}_m(\mathbb{C}^2) \), and the set
\[
\{ D^\alpha |z|^{-2} : |\alpha| = m \text{ and } \alpha_1 \leq 1 \}
\]
is a vector space basis of \( \mathcal{H}_m(S^3) \).

It follows from the previous definition that the homogenous polynomials of degree \( k \) can be written as the sum of polynomials of bidegree \( p, q \) such that \( p + q = k \).

Theorem 2.7. \( \mathcal{P}_k(\mathbb{C}^2) = \bigoplus_{p+q=k} \mathcal{P}_{p,q}(\mathbb{C}^2) \).

Analogous to the version in Theorem 2.6, we use the following method to construct an orthogonal basis for \( \mathcal{H}_{p,q}(\mathbb{C}^2) \) and \( \mathcal{H}_{p,q}(S^3) \).
Theorem 2.8. The set
\[ \left\{ K[D^\alpha D^\beta |z|^{-2}] \mid |\alpha| = p, |\beta| = q, \alpha_1 = 0 \text{ or } \beta_1 = 0 \right\} \]
is a basis for \( \mathcal{H}_{p,q}(\mathbb{C}^2) \), and the set
\[ \left\{ D^\alpha D^\beta |z|^{-2} \mid |\alpha| = p, |\beta| = q, \alpha_1 = 0 \text{ or } \beta_1 = 0 \right\} \]
is an orthogonal basis for \( \mathcal{H}_{p,q}(S^3) \).

2.2. \( \Box_b \) on \( \mathcal{H}_{p,q}(S^3) \). Before we study the operator \( \Box_b \), we first need some background on a simpler operator we call \( \Box_b \). It arises from the CR-manifold \( (S^3, \mathcal{L}) \), and is defined as
\[ \Box_b = -\mathcal{L}\overline{\mathcal{L}}. \]

We note that this CR-structure is induced from \( \mathbb{C}^2 \) and this manifold is naturally embedded. By the machinery above we can compute the eigenvalues of \( \Box_b \).

Theorem 2.9. Suppose \( f \in \mathcal{H}_{p,q}(S^3) \). Then
\[ \Box_b f = (pq + q) f. \]

Proof. Expanding the definition, we get
\[ \Box_b = \left( \bar{z}_2 \frac{\partial}{\partial z_1} - \bar{z}_1 \frac{\partial}{\partial z_2} \right) \left( z_2 \frac{\partial}{\partial \bar{z}_1} - z_1 \frac{\partial}{\partial \bar{z}_2} \right) \]
\[ = \bar{z}_2 \frac{\partial}{\partial z_1} \left( z_2 \frac{\partial}{\partial \bar{z}_1} - z_1 \frac{\partial}{\partial \bar{z}_2} \right) + \bar{z}_1 \frac{\partial}{\partial z_2} \left( \frac{\partial}{\partial \bar{z}_1} - z_1 \frac{\partial}{\partial \bar{z}_2} \right) \]
\[ = -z_2 \bar{z}_2 \frac{\partial^2}{\partial z_1 \partial \bar{z}_2} + \bar{z}_2 \frac{\partial}{\partial \bar{z}_1} + z_1 \bar{z}_2 \frac{\partial}{\partial \bar{z}_2} \]
\[ - z_1 \bar{z}_1 \frac{\partial^2}{\partial \bar{z}_1 \partial \bar{z}_2} + \bar{z}_1 \frac{\partial}{\partial \bar{z}_1} + z_2 \bar{z}_1 \frac{\partial}{\partial \bar{z}_2} \]
Now, let \( f \in \mathcal{H}_{p,q}(S^3) \). Since \( f \) is harmonic, we know that \( \frac{\partial^2}{\partial z_1 \partial \bar{z}_1} = -\frac{\partial^2}{\partial \bar{z}_2 \partial z_2} \). Substituting, we get
\[ = z_2 \bar{z}_2 \frac{\partial^2}{\partial z_2 \partial \bar{z}_2} + \bar{z}_2 \frac{\partial}{\partial \bar{z}_2} + z_1 \bar{z}_2 \frac{\partial}{\partial \bar{z}_1} \frac{\partial^2}{\partial z_1 \partial \bar{z}_2} \]
\[ + z_1 \bar{z}_1 \frac{\partial^2}{\partial \bar{z}_1 \partial \bar{z}_1} + \bar{z}_1 \frac{\partial}{\partial \bar{z}_1} + z_2 \bar{z}_1 \frac{\partial}{\partial \bar{z}_2} \frac{\partial^2}{\partial z_2 \partial \bar{z}_1} \]
Since \( f \) is a polynomial and \( \Box_b \) is linear, it suffices to show that if \( f = z^{\alpha_1} \bar{z}^{\beta_1} = z_1^{\alpha_1} z_2^{\alpha_2} \bar{z}_1^{\beta_1} \bar{z}_2^{\beta_2} \), where \( \alpha_1 + \alpha_2 = p \) and \( \beta_1 + \beta_2 = q \), then the claim holds. Using the expansion we got, each derivative expression simply becomes a multiple of \( f \). So we get
\[ \Box_b f = (\alpha_2 \beta_2 + \beta_2 + \alpha_1 \beta_1 + \alpha_1 \beta_1 + \beta_1 + \alpha_1 \beta_1) f \]
\[ = ((\alpha_1 + \alpha_2) (\beta_1 + \beta_2) + (\beta_1 + \beta_2)) f \]
\[ = (pq + q) f \]
and we are done. \( \square \)
In a similar manner, we can show that \(-\mathcal{L}f = (pq + p)f\). For this case, we actually have that \(\text{spec}(\Box_b) = \{pq + q \mid p, q \in \mathbb{N}\}\), so \(0 \notin \text{essspec}(\Box_b)\) since it is not an accumulation point of the set above.

3. Experimental Results in Mathematica

Using the symbolic computation environment provided by Mathematica, we were able to write a program to streamline our calculations\(^1\). We implemented the algorithm provided in Theorem 2.8 to construct the vector space basis of \(H_k(S^3)\) for a specified \(k\). As an example, our code produced the following basis of \(H_3(S^3)\):

\[
\{-6z_2^3, -6z_1z_2^2, -6z_1^2z_2, -6z_1^3, 4z_1z_2^2z_2 - 2z_2^3z_2^2, 2z_1z_1^2z_2 - 4z_2z_1z_2^2, -6z_2z_1z_2^2, -6z_1z_2^2, 4z_1z_2z_1 - 2z_2z_2^2, -6z_2^2z_1z_2^2, -6z_2^2z_2, -6z_2^3, -6z_1z_2^2, -6z_1^2z_2, -6z_1^3\}.
\]

Now, with the basis for \(H_k(S^3)\), the matrix representation of \(\Box_b^k\) on \(H_k(S^3)\) can be computed for each \(k\). In particular, we used this program to construct the matrix representations for \(1 \leq k \leq 12\). For a specific \(k\), the code applies \(\Box_b^k\) to each basis element of \(H_k(S^3)\) obtained by the results in the previous sections. Then, using the inner product defined by,

\[
\langle f, g \rangle = \int_{S^3} f \bar{g} \, d\sigma,
\]

where \(\sigma\) is the standard surface area measure, the software computes \(\langle \Box_b^k f_i, f_j \rangle\), where \(f_i, f_j\) are basis vectors for \(H_k(S^3)\). With these results, Mathematica yields the matrix representation for the imputed value of \(k\). For example, the program produced the matrix representation for \(k = 3\) seen in Figure 3.1. Since each entry had a common normalization factor,

\[
h = \frac{1 + |t|^2}{(1 - |t|^2)^2},
\]

this constant has been factored out. With Mathematica’s Eigenvalue function, the eigenvalues were then calculated for these matrix representations. Our numerical results suggest that

\[\begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -6t & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 0 & -6t & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 + 3|t|^2 & 0 & 0 & 0 & 0 & 0 & -6t & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 + 3|t|^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 + 3|t|^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 + 3|t|^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2t & 0 & 0 & 0 & 0 & 0 & 3 + 4|t|^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2t & 0 & 0 & 0 & 0 & 0 & 3 + 4|t|^2 & 0 & 0 & 0 & 0 \\ 0 & 2t & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 + 4|t|^2 & 0 & 0 & 0 \\ 0 & -2t & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 + 4|t|^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6t & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3|t|^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -6t & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3|t|^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -6t & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3|t|^2 \end{pmatrix}
\]

**Figure 3.1.** Matrix Representation of \(\Box_b^3\) on \(H_3(S^3)\)

\(^1\)Our code for this and the other symbolic computations described below is available on our website at [https://sites.google.com/a/umich.edu/zeytuncu/home/publ](https://sites.google.com/a/umich.edu/zeytuncu/home/publ)
the smallest non-zero eigenvalue of $\Box_b$ on $H_{2k-1}(S^3)$ decreases as $k$ increases. Conversely, the smallest non-zero eigenvalue of $\Box_b$ on $H_{2k}(S^3)$ increases with $k$. The smallest eigenvalue of $H_{2k-1}(S^3)$ is plotted for $1 \leq k \leq 5$ and $0 < |t| < 1$ in Figure 3.2. It is apparent that $\lambda_{\min,1} \leq \lambda_{\min,3} \leq \lambda_{\min,5} \leq \lambda_{\min,7} \leq \lambda_{\min,9}$ where $\lambda_{\min,k}$ denotes the smallest non-zero eigenvalue of $\Box_b$ on $H_k(S^3)$. These initial numerical results suggest that $\lim_{k \to \infty} \lambda_{\min,2k-1} = 0$ for $0 < |t| < 1$, which agrees with our final result.

![Figure 3.2. Smallest Non-Zero Eigenvalues for $k = 1, 3, 5, 7, 9$.](image)

4. INVARIANT SUBSPACES OF $H_{2k-1}(S^3)$ UNDER $\Box_b$

In this section we fix $k \geq 1$ and work on $H_{2k-1}(S^3)$. As we have seen, $\Box_b$ can be expanded in the following way:

$$\Box_b = -(\mathcal{L} + \overline{t}\mathcal{L}) \frac{1 + |t|^2}{(1 - |t|^2)^2}(\mathcal{L} + t\mathcal{L})$$

$$= -h(\mathcal{L}\overline{t}\mathcal{L} + |t|^2\mathcal{L}^2 + t\mathcal{L}^2 + \overline{t}\mathcal{L}^2)$$

(1)

This is because of the linearity of $\mathcal{L}$ and $\overline{t}\mathcal{L}$. Now, we need the following property.

**Lemma 4.1.** If $\langle f_i, f_j \rangle = 0$ for $i \neq j$ and $f_i, f_j \in H_{0,2k-1}(S^3)$, then $\langle \mathcal{L}^\sigma f_i, \overline{t}\mathcal{L}^\sigma f_j \rangle = 0$ for $0 \leq \sigma \leq 2k-1$.

**Proof.** Choose $f_i$ and $f_j$, $i \neq j$ from an orthogonal basis for $H_{0,2k-1}(S^3)$. We show that for $0 \leq \sigma \leq 2k-1$, $\mathcal{L}^\sigma f_i$ and $\overline{t}\mathcal{L}^\sigma f_j$ are orthogonal. To do this we use induction on $\sigma$. Suppose $\langle \mathcal{L}^\sigma f_i, \overline{t}\mathcal{L}^\sigma f_j \rangle = 0$, and we show that $\langle \mathcal{L}^\sigma f_i, \overline{t}\mathcal{L}^\sigma f_j \rangle = 0$. Note that,

$$\langle \mathcal{L}^\sigma f_i, \overline{t}\mathcal{L}^\sigma f_j \rangle = \langle \overline{t}\mathcal{L}^\sigma f_i, \mathcal{L}^\sigma f_j \rangle$$

$$= \langle \overline{t}\mathcal{L}^\sigma f_i, (\mathcal{L}\overline{t}\mathcal{L})^\sigma f_j \rangle$$

$$= \langle \overline{t}\mathcal{L}^\sigma f_i, -\Box_b^\overline{t}\mathcal{L}^\sigma f_j \rangle.$$. 
However since $\mathcal{L}^{\sigma-1} f_j \in \mathcal{H}_{\sigma-1,2k-1-\sigma-1}(S^3)$, we know that $\square_b \mathcal{L}^{\sigma-1} f_j = (\sigma)(2k - \sigma - 2)\mathcal{L}^{\sigma-1} f_j$. Therefore,
\[
\langle \mathcal{L}^{\sigma-1} f_i, -\square_b \mathcal{L}^{\sigma-1} f_j \rangle = \langle \mathcal{L}^{\sigma-1} f_i, -(\sigma)(2k - \sigma - 2)\mathcal{L}^{\sigma-1} f_j \rangle \\
= -(\sigma)(2k - \sigma - 2)\langle \mathcal{L}^{\sigma-1} f_i, \mathcal{L}^{\sigma-1} f_j \rangle \\
= 0,
\]
by our induction hypothesis as desired. \qed

With this, we first note that if $\{f_0, \ldots, f_{2k-1}\}$ is an orthogonal basis for $\mathcal{H}_{0,2k-1}(S^3)$, then $\{\mathcal{L}^\sigma f_0, \ldots, \mathcal{L}^\sigma f_{2k-1}\}$ is an orthogonal basis for $\mathcal{H}_{\sigma,2k-1-\sigma}(S^3)$. Now, we define the following subspaces of $\mathcal{H}_{2k-1}(S^3)$.

**Definition 4.2.** Suppose $\{f_0, \ldots, f_{2k-1}\}$ is the orthogonal basis for $\mathcal{H}_{0,2k-1}(S^3)$. Then we define
\[
V_i = \text{span}\{f_i, \mathcal{L}^j f_i, \ldots, \mathcal{L}^{2j-2} f_i, \ldots, \mathcal{L}^{2k-2} f_i\}, \\
W_i = \text{span}\{\mathcal{L} f_i, \mathcal{L}^3 f_i, \ldots, \mathcal{L}^{2j-1} f_i, \ldots, \mathcal{L}^{2k-1} f_i\}.
\]

Denote the basis elements of $V_i$ by $v_{i,1}, \ldots, v_{i,k}$ and for $W_i$ by $w_{i,1}, \ldots, w_{i,k}$. We first note that since each bidegree space $\mathcal{H}_{p,q}(S^3) \subseteq \mathcal{H}_{2k-1}(S^3)$ has $2k$ elements, we have $2k$ $V_i$ spaces and $2k$ $W_i$ spaces. We now note the following fact.

**Theorem 4.3.** $\bigoplus_{i=0}^{2k-1} V_i \oplus W_i = \mathcal{H}_{2k-1}(S^3)$.

**Proof.** We first note that by Theorem 2.7,
\[
\mathcal{H}_{2k-1}(S^3) = \bigoplus_{i=0}^{2k-1} \mathcal{H}_{i,2k-1-i}(S^3)
\]
but by Lemma 4.1 we see that this is really just
\[
= \bigoplus_{i=0}^{2k-1} \mathcal{L}^i f_0 \oplus \cdots \oplus \mathcal{L}^i f_{2k-1}.
\]
Manipulating this, we have
\[
= \bigoplus_{i=0}^{2k-1} f_i \oplus \mathcal{L} f_i \cdots \oplus \mathcal{L}^{2k-1} f_i \\
= \bigoplus_{i=0}^{2k-1} f_i \oplus \mathcal{L}^2 f_i \oplus \cdots \oplus \mathcal{L}^{2k-2} f_i \oplus \mathcal{L} f_i \oplus \mathcal{L}^3 f_i \oplus \cdots \oplus \mathcal{L}^{2k-1} f_i \\
= \bigoplus_{i=0}^{2k-1} V_i \oplus W_i,
\]
which is our goal. \qed

The advantage of constructing these spaces in the first place is due to the following fact.

**Theorem 4.4.** $\square_b^t$ is invariant on $V_i$ and $W_i$. 

\( \Box_b = -h(\mathcal{L}\mathcal{L} + |t|^2\mathcal{L}\mathcal{L} + t\mathcal{L}^2 + i\mathcal{L}^2) \)

Since the fraction in front is a constant, we can ignore it and only consider the expression in the parentheses. Let \( f \in \mathcal{H}_{0,2k-1}(\mathbb{S}^3) \), and define \( v_\sigma = \mathcal{T}^{-\sigma} f \) to be a basis element of either \( V_i \) or \( W_i \), since they have the same form. We first note that \( v_\sigma \in \mathcal{H}_{\sigma,2k-1-\sigma}(\mathbb{S}^3) \). Then by our expansion we have that

\[
\Box_b v_\sigma = -h(\mathcal{L}\mathcal{L} v_\sigma + |t|^2\mathcal{L}\mathcal{L} v_\sigma + t\mathcal{L}^2 v_\sigma + i\mathcal{L}^2 v_\sigma)
\]

We already know \( \mathcal{L}\mathcal{L} v_\sigma \) and \( \mathcal{L}\mathcal{L} v_\sigma \) will simply be a multiple of \( v_\sigma \), so we consider \( \mathcal{L}^2 v_\sigma \) and \( \mathcal{L}^2 v_\sigma \).

\[
\mathcal{L}^2 v_\sigma = \mathcal{L}^2 \mathcal{T}^{-\sigma} f \\
= \mathcal{L} \left[ \mathcal{L}\mathcal{L} \left[ \mathcal{T}^{-\sigma-1} f \right] \right] \\
= -(\sigma)(2k - \sigma)\mathcal{L}\mathcal{L} \left[ \mathcal{T}^{-\sigma-2} f \right] \\
= (\sigma)(\sigma - 1)(2k + 1 - \sigma)(2k - \sigma)\mathcal{L}^{-\sigma-2} f \\
= (\sigma)(\sigma - 1)(2k + 1 - \sigma)(2k - \sigma)v_{\sigma-2}
\]

\[
\mathcal{L}^2 v_\sigma = \mathcal{L}^2 \left[ \mathcal{T}^{-\sigma} f \right] \\
= \mathcal{T}^{\sigma+2} f \\
= v_{\sigma+2}
\]

so we get multiples of \( v_{\sigma-2} \) and \( v_{\sigma+2} \). Relating this back to \( V_i \) and \( W_i \), we see that if \( \sigma = 2j - 2 \), then \( \mathcal{L}^2 v_{i,j} \) is a multiple of \( v_{i,j-1} \), and \( \mathcal{L}^2 v_{i,j} \) is a multiple of \( v_{i,j+1} \). If \( \sigma = 2j - 1 \), we get a similar result for \( w_{i,j} \). So we indeed have that \( \Box_b ^i \) is invariant on \( V_i \) and \( W_i \), and we are done.

In light of this fact, we can consider \( \Box_b ^i \) not on the whole space \( L^2(\mathbb{S}^3) \) or \( \mathcal{H}_{2k-1}(\mathbb{S}^3) \), but rather on these \( V_i \) and \( W_i \) spaces. In fact, we actually have a representation of \( \Box_b ^i \) on these spaces.

**Theorem 4.5.** The matrix representation of \( \Box_b ^i \), \( m(\Box_b ^i) \), on \( V_i \) and \( W_i \) is tridiagonal, where \( m(\Box_b ^i) \) on \( V_i \) is

\[
m(\Box_b ^i) = h \begin{pmatrix}
  d_1 & u_1 & & \\
  -\bar{t} & d_2 & u_2 & \\
  & -\bar{t} & d_3 & \ddots \\
  & & \ddots & \ddots & u_{k-1} \\
  & & & -\bar{t} & d_k \\
  & & & & 8
\end{pmatrix}
\]
Proof. Using equations (2a) and (2b), along with Theorem 2.9, we can entirely describe the action of each piece of $\square_b$ on a basis element $v_{i,j}$ or $w_{i,j}$:

$$m(\square_b^j) = h\begin{pmatrix} d_1 & u_1 & \cdots & \cdots & u_{k-1} \\ -t & d_2 & \cdots & \cdots & -t \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -t & \cdots & \cdots & \cdots & -t \\ d_k & \cdots & \cdots & \cdots & d_1 \end{pmatrix}$$

where $u_j = -t \cdot (2j)(2j-1)(2k - 2j)(2k - 1 - 2j)$ and $d_j = (2j - 1)(2k + 1 - 2j) + |t|^2 \cdot (2j - 2)(2k + 2 - 2j)$. For $W_i$, we get something similar:

$$m(\square_b^j) = h\begin{pmatrix} d_1 & u_1 & \cdots & \cdots & u_{k-1} \\ -t & d_2 & \cdots & \cdots & -t \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -t & \cdots & \cdots & \cdots & -t \\ d_k & \cdots & \cdots & \cdots & d_1 \end{pmatrix}$$

where $u_j = -t \cdot (2j + 1)(2j)(2k - 2j)(2k - 1 - 2j)$ and $d_j = (2j)(2k - 2j) + |t|^2 \cdot (2j - 1)(2k + 1 - 2j)$.

We note that the above definitions don’t depend on $i$; in other words, each of these matrices are the same on $V_i$ and $W_i$, regardless of the choice of $i$.

By looking at it this way, we notice the tridiagonal structure. So with these observations, we can state that

$$\square_b^j v_{i,j} = h\left( -t \cdot (2j - 2)(2j - 3)(2k + 3 - 2j)(2k + 2 - 2j) v_{i,j-1} \\ + (2j - 1)(2k + 1 - 2j) + |t|^2 \cdot (2j - 2)(2k + 2 - 2j) \right) v_{i,j} \\ - \bar{t} \cdot v_{i,j+1} \right)$$

$$\square_b^j w_{i,j} = h\left( -t \cdot (2j - 1)(2j - 2)(2k + 2 - 2j)(2k + 1 - 2j) w_{i,j-1} \\ + (2j)(2k - 2j) + |t|^2 \cdot (2j - 1)(2k - 1 - 2j) \right) w_{i,j} \\ - \bar{t} \cdot w_{i,j+1} \right)$$

Now that we have this formula, we can find $m(\square_b^j)$ on $V_i$ and $W_i$ by computing their effect on the basis vectors $v_{i,j}$ and $w_{i,j}$: when we do this for $V_i$, we get

$$d_j = (2j - 1)(2k + 1 - 2j) + |t|^2 \cdot (2j - 2)(2k + 2 - 2j)$$

$$u_{j-1} = -t \cdot (2j - 2)(2j - 3)(2k + 3 - 2j)(2k + 2 - 2j)$$

$$\implies u_j = -t \cdot (2j)(2j - 1)(2k - 2j)(2k - 1 - 2j)$$
and for $W_i$, we get
\[ d_j = (2j)(2k - 2j) + |t|^2 \cdot (2j - 1)(2k - 1 - 2j) \]
\[ u_{j-1} = -t \cdot (2j - 1)(2j - 2)(2k + 2 - 2j)(2k + 1 - 2j) \]
\[ \implies u_j = -t \cdot (2j + 1)(2j - 2j)(2k - 1 - 2j). \]

Finally, by factoring out $h$ and simply substituting each portion in we obtain the matrix representations above. \[ \square \]

An immediate consequence of this is that each $V_i$ subspace contributes the same set of eigenvalues to the spectrum of $\square_{b_i}^t$, and similarly for each $W_i$. Furthermore, we note that the matrices are rank $k$. Since the choice of $i$ does not change $m(\square_{b_i}^t)$ on these spaces, we will fix an arbitrary $i$ and call the spaces $V$ and $W$ instead.

5. Bottom of the Spectrum of $\square_{b_i}^t$

Now that we have a matrix representation for $\square_{b_i}^t$ on these $V$ and $W$ spaces inside $\mathcal{H}_{2k-1}(S^3)$, we can begin to analyze their eigenvalues as $k$ varies. First, we go over some facts about tridiagonal matrices.

**Theorem 5.1.** Suppose $A$ is a tridiagonal matrix,

\[
A = \begin{pmatrix}
  d_1 & u_1 & & \\
  l_1 & d_2 & u_2 & \\
    & l_2 & d_3 & \ddots \\
    &   & \ddots & \ddots & u_{k-1} \\
    &   & & l_{k-1} & d_k
\end{pmatrix}
\]

and the products $u_id_i > 0$ for $1 \leq i \leq k$, then $A$ is similar to a symmetric tridiagonal matrix.

**Proof.** One can verify that if

\[
S = \begin{pmatrix}
  1 & \sqrt{\frac{u_1}{l_1}} & & \\
    & \sqrt{\frac{u_1u_2}{l_1l_2}} & \ddots & \\
    & & \ddots & \ddots \\
    & & & \sqrt{\frac{u_1\ldots u_{k-1}}{l_1\ldots l_{k-1}}}
\end{pmatrix}
\]

then $A = SBS^{-1}$, where

\[
B = \begin{pmatrix}
  d_1 & \sqrt{u_1l_1} & & \\
  \sqrt{u_1l_1} & d_2 & \sqrt{u_2l_2} & \\
    & \sqrt{u_2l_2} & d_3 & \ddots \\
    & & \ddots & \ddots & \sqrt{u_{k-1}l_{k-1}} \\
    & & & \sqrt{u_{k-1}l_{k-1}} & d_k
\end{pmatrix}
\]
Therefore, $A$ is similar to a symmetric tridiagonal matrix. \hfill \square

Another special property of tridiagonal matrices is the continuant.

**Definition 5.2.** Let $A$ be a tridiagonal matrix, like the above. Then we define the continuant of $A$ to be a recursive sequence: $f_1 = d_1$, and $f_i = d_{i-1}f_{i-1} - u_{i-2}l_{i-2}f_{i-2}$, where $f_0 = 1$.

The reason we define this is because $\det(A) = f_k$. In addition, if we denote $A_i$ to mean the square submatrix of $A$ formed by the first $i$ rows and columns, then $\det(A_i) = f_i$.

With this background, we will now start analyzing $\mathbb{C}^k$ on $W$.

To get bounds on the eigenvalues, we will invoke the Cauchy interlacing theorem, see [Hwa04].

**Theorem 5.3.** Suppose $A$ is an $n \times n$ Hermitian matrix of rank $n$, and $B$ is an $n - 1 \times n - 1$ matrix minor of $A$. If the eigenvalues of $A$ are $\lambda_1 \leq \cdots \leq \lambda_n$ and the eigenvalues of $B$ are $\nu_1 \leq \cdots \leq \nu_{n-1}$, then the eigenvalues of $A$ and $B$ interlace:

$$0 < \lambda_1 \leq \nu_1 \leq \lambda_2 \leq \nu_2 \leq \cdots \leq \lambda_{n-1} \leq \nu_{n-1} \leq \lambda_n$$

Now, we can get an intermediate bound on the smallest eigenvalue.

**Theorem 5.4.** Suppose $A$ is a Hermitian matrix of rank $n$, and $\lambda_1 \leq \cdots \leq \lambda_n$ are its eigenvalues. Then

$$\lambda_1 \leq \frac{\det(A)}{\det(A_{k-1})}$$

where $A_{k-1}$ is $A$ without the last row and column.

**Proof.** Since $A_{k-1}$ is a $k-1 \times k-1$ matrix minor of $A$, we can apply the Cauchy interlacing theorem. If the eigenvalues of $A_{k-1}$ are $\nu_1 \leq \cdots \leq \nu_{k-1}$, then

$$\lambda_1 \leq \nu_1 \leq \lambda_2 \leq \nu_2 \leq \cdots \leq \lambda_{n-1} \leq \nu_{n-1} \leq \lambda_n$$

Now, we claim that

$$\lambda_1 \det(A_{k-1}) \leq \det(A)$$

To see why this is true, first observe that the determinant of a matrix is simply the product of all its eigenvalues. In particular,

$$\lambda_1 \det(A_{k-1}) = \lambda_1 \nu_1 \cdots \nu_{k-1}$$

But we can simply apply the Cauchy interlacing theorem: since $\nu_1 \leq \lambda_2$, $\nu_2 \leq \lambda_3$, and so on, we get

$$\lambda_1 \nu_1 \cdots \nu_{k-1} \leq \lambda_1 \lambda_2 \cdots \lambda_k$$

$$= \det(A)$$

so the claim is proven. Now, dividing both sides by $\det(A_{k-1})$,

$$\lambda_1 \leq \frac{\det(A)}{\det(A_{k-1})}$$

as desired. \hfill \square
Since \( m(\square^T) \) on \( W \) satisfies the conditions of Theorem 5.1, we find it is similar to this Hermitian tridiagonal matrix:

\[
A = \begin{pmatrix}
    a_1 + b_1|t|^2 & c_1|t| & & \\
    c_1|t| & a_2 + b_2|t|^2 & c_2|t| & \\
    & c_2|t| & a_3 + b_3|t|^2 & \cdots \\
    & & \ddots & \ddots & c_{k-1}|t| \\
    & & & c_{k-1}|t| & a_k + b_k|t|^2
\end{pmatrix}
\]

where \( a_i = (2i)(2k-2i), b_i = (2i-1)(2k+1-2i), \) and \( c_i = \sqrt{(2i+1)(2i)(2k-2i)(2k-1-2i)}. \) Note that we are ignoring the constant \( h \) for now, which we will add back later. If we can find \( \det(A_i) \), then by Theorem 5.4 we can get a closed form for the bound on the smallest eigenvalue. With the following lemma, this is possible:

**Lemma 5.5.** \( a_i b_{i+1} = c_i^2 \)

**Proof.** We can simply work through the formulas to figure this out: \( a_i = (2i)(2k-2i), b_{i+1} = (2i+1)(2k+1-2i), \) and \( c_i^2 = (2i+1)(2i)(2k-2i)(2k-1-2i). \) The products clearly match up. \( \square \)

**Theorem 5.6.** The determinant of \( A_i \) is

\[
\det(A_i) = a_1 a_2 \cdots a_i a_i \\
+ b_1 a_2 \cdots a_{i-1} a_i |t|^2 \\
+ \cdots \\
+ b_1 b_2 \cdots b_{i-1} a_i |t|^{2i-2} \\
+ b_1 b_2 \cdots b_{i-1} b_i |t|^{2i}
\]

In each row, we replace a particular \( a_j \) with \( b_j \), and multiply by \(|t|^2\). Note that if \( i = k \), then \( a_k = 0 \) and all terms but the last term are 0.

**Proof.** We will prove this using strong induction on \( i \). The base case is \( i = 1 \), where \( \det(A_1) = a_1 + b_1|t|^2 \), which does indeed match up with our formula. Now, assume the formula works for \( A_{i-1} \) and \( A_i \). We need to show that the formula works for \( A_{i+1} \). Using the formula for the continuant, we get

\[
\det(A_{i+1}) = (a_{i+1} + b_{i+1}|t|^2) \det(A_i) - c_i^2 |t|^2 \det(A_{i-1})
\]

Now, use Lemma 5.5:

\[
= (a_{i+1} + b_{i+1}|t|^2) \det(A_i) - a_{i+1} b_{i+1} |t|^2 \det(A_{i-1})
\]
Now, we use our induction hypothesis:
\[
\begin{align*}
&= (a_{i+1} + b_{i+1}|t|^2)(a_1a_2 \cdots a_i + b_1a_2 \cdots a_i|t|^2 + \cdots + b_1b_2 \cdots b_i|t|^{2i}) \\
&- a_ib_{i+1}|t|^2(a_1a_2 \cdots a_{i-1} + b_1a_2 \cdots a_{i-1}|t|^2 + \cdots + b_1b_2 \cdots b_{i-1}|t|^{2i-2}) \\
&= a_1a_2 \cdots a_{i+1} + b_1a_2 \cdots a_{i+1}|t|^2 + \cdots + b_1b_2 \cdots b_{i+1}|t|^{2i} \\
&+ a_1a_2 \cdots a_ib_{i+1}|t|^2 + b_1a_2 \cdots a_ib_{i+1}|t|^2 + \cdots + b_1b_2 \cdots b_{i-1}a_ib_{i+1}|t|^{2i-2} + b_1b_2 \cdots b_{i+1}|t|^{2i-2} \\
&- a_1a_2 \cdots a_ib_{i+1}|t|^2 - b_1a_2 \cdots a_ib_{i+1}|t|^2 + \cdots - b_1b_2 \cdots b_{i-1}a_ib_{i+1}|t|^{2i-2} \\
&= a_1a_2 \cdots a_{i+1} + b_1a_2 \cdots a_{i+1}|t|^2 + \cdots + b_1b_2 \cdots b_{i+1}|t|^{2i} + b_1b_2 \cdots b_{i+1}|t|^{2i-2}
\end{align*}
\]
which is the formula for $A_{i+1}$, and we are done. 

With this knowledge, we are finally able to prove our theorem.

**Theorem 5.7.** $0 \in \text{essspec}(\Box^k_B)$

**Proof.** By Theorem \ref{thm:essential_spectrum}, we have that on $W$ in $H_{2k-1}(S^3)$, $m(\Box^k_B)$ is similar to
\[
A = h\begin{pmatrix}
(a_1 + b_1|t|^2) & c_1|t| \\
c_1|t| & (a_2 + b_2|t|^2) & c_2|t| \\
c_2|t| & a_3 + b_3|t|^2 & \ddots \\
\vdots & \ddots & \ddots & c_{k-1}|t| \\
c_{k-1}|t| & a_k + b_k|t|^2
\end{pmatrix}
\]
where $a_j = (2j)(2k-2j)$, $b_j = (2j-1)(2k+1-2j)$, and $c_j = \sqrt{(2j + 1)(2j)(2k - 2j)(2k - 1 - 2j)}$.

Now, by Theorem \ref{thm:essential_spectrum} above, we know that
\[
\lambda_{\text{min}} \leq \frac{\det(A)}{\det(A_{k-1})}.
\]
Recall that $A_{k-1}$ denotes the submatrix formed by deleting the last row and column of the $k \times k$ matrix $A$. To show that $0 \in \text{essspec}(\Box^k_B)$, we want to show that $\det(A)/\det(A_{k-1}) \to 0$ as $k \to \infty$. For this purpose we find an upper bound for $\det(A)/\det(A_{k-1})$ and show that this converges to 0. Notice that Theorem \ref{thm:essential_spectrum} implies that,
\[
\frac{\det(A)}{\det(A_{k-1})} = h \frac{b_1b_2 \cdots b_{k-1}b_k|t|^{2k}}{a_1a_2 \cdots a_{k-1} + b_1a_2 \cdots a_{k-1}|t|^2 + b_1b_2 \cdots a_{k-1}|t|^4 + \cdots + b_1b_2 \cdots b_{k-1}|t|^{2k-2}}
\leq h \frac{b_1b_2 \cdots b_{k-1}b_k|t|^{2k}}{a_1a_2 \cdots a_{k-1}}.
\]
(3)
since, $a_j, b_j, |t| > 0$. Now using the formulas for $a_j$ and $b_j$, notice that (3) can be written as
\[
h(2k - 1)|t|^{2k} \prod_{j=1}^{k-1} \frac{(2j + 1)(2k - 2j - 1)}{(2j)(2k - 2j)}.
\]
However, we know that for all $k$ and $1 \leq j \leq k - 1$,
\[
\frac{(2k - 2j - 1)}{(2k - 2j)} < 1,
\]

and so,
\[ h(2k-1)|t|^{2k} \prod_{j=1}^{k-1} \frac{(2j+1)(2k-2j-1)}{(2j)(2k-2j)} \leq h(2k-1)|t|^{2k} \prod_{j=1}^{k-1} \frac{(2j+1)}{(2j)} = h(2k-1)|t|^{2k} \prod_{j=1}^{k-1} 1 + \frac{1}{2j}. \]
Furthermore, we have
\[ h(2k-1)|t|^{2k} \prod_{j=1}^{k-1} 1 + \frac{1}{2j} \leq h(2k-1)|t|^{2k} \exp \left( \sum_{j=1}^{k-1} \frac{1}{2j} \right). \]
Note that
\[ \sum_{j=1}^{k-1} \frac{1}{2j} \leq \frac{1}{2} \ln k \]
so our expression becomes
\[ \frac{\det(A)}{\det(A_{k-1})} \leq h(2k-1)|t|^{2k} \exp \left( \frac{1}{2} \ln k \right) = h(2k-1)\sqrt{k}|t|^{2k} \]
and our problem reduces to showing that \( \lim_{k \to \infty} h(2k-1)\sqrt{k}|t|^{2k} = 0 \). We note that \( h \) is a constant and \( |t| < 1 \); therefore, by L'Hospital’s rule the last expression indeed goes to 0.
Finally, we have,
\[ 0 \leq \lim_{k \to \infty} \lambda_{\text{min}} \leq \lim_{k \to \infty} \frac{\det(A)}{\det(A_{k-1})} \leq \lim_{k \to \infty} h(2k-1)\sqrt{k}|t|^{2k} = 0, \]
and so \( \lambda_{\text{min}} \to 0 \). Hence \( 0 \in \text{essspec}(\square^b) \).

We note that by the discussion in the introduction, this means that the CR-manifold \((\mathcal{L}_t, S^3)\) is not embeddable into any \( \mathbb{C}^N \).

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(Tawfik Abbas) MICHIGAN STATE UNIVERSITY, DEPARTMENT OF MATHEMATICS, EAST LANSING, MI 48824, USA
E-mail address: abbastaw@msu.edu

(Madelyne M. Brown) BUCKNELL UNIVERSITY, DEPARTMENT OF MATHEMATICS, LEWISBURG, PA 17837, USA
E-mail address: mmb021@bucknell.edu

(Ravikumar Ramasami) UNIVERSITY OF MICHIGAN–DEARBORN, DEPARTMENT OF MATHEMATICS & STATISTICS, DEARBORN, MI 48128, USA
E-mail address: rramasam@umich.edu

(Yunus E. Zeytuncu) UNIVERSITY OF MICHIGAN–DEARBORN, DEPARTMENT OF MATHEMATICS & STATISTICS, DEARBORN, MI 48128, USA
E-mail address: zeytuncu@umich.edu

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