Abstract. In this paper we characterize the stabilization for some thermo-elastic type system with Cattaneo law and we prove that the exponential or polynomial stability of this system implies a polynomial stability of the corresponding thermoelastic system with the Fourier law. The proof of the main results uses, respectively, the methodology introduced in Ammari-Tucsnak [3], where the exponential stability for the closed loop problem is reduced to an observability estimate for the corresponding uncontrolled system, and a characterization of the polynomial stability for a \(C_0\)-semigroup, in a Hilbert space, by a polynomial estimation of the resolvent of its generator obtained by Borichev-Tomilov [5]. An illustrating example are given.

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1 Introduction and main results

Let \(H_i\) be a Hilbert space equipped with the norm \(\| \cdot \|_{H_i}, i = 1, 2\), and let \(A_1 : \mathcal{D}(A_1) \subset H_1 \to H_1\) and \(A : \mathcal{D}(A) \subset H_2 \to H_2\) are positive self-adjoint operators. We introduce the scale of Hilbert spaces \(H_{\alpha}, \alpha \in \mathbb{R},\) as follows: for every \(\alpha \geq 0, H_{\alpha} = \mathcal{D}(A_1^\alpha),\) with the norm \(\|z\|_{1,\alpha} = \|A_1^\alpha z\|_{H_1}\) and \(H_{2,\alpha} = \mathcal{D}(A^\alpha),\) with the norm \(\|z\|_{2,\alpha} = \|A^\alpha z\|_{H_2}\.\) The space \(H_{i,-\alpha}\) is defined by duality with respect to the pivot space \(H_i\) as follows: \(H_{i,-\alpha} = H_{i,\alpha}^*,\) for \(\alpha > 0, i = 1, 2\). The operators \(A_1\) and \(A\) can be extended (or restricted) to each \(H_{i,\alpha}\), such that it becomes a bounded operator

\[
A_1 : H_{1,\alpha} \to H_{1,\alpha-1}, A : H_{2,\alpha} \to H_{2,\alpha-1}, \quad \forall \alpha \in \mathbb{R}. \tag{1.1}
\]

We assume that the operator \(A\) can be written as \(A = A_2A_2^*\), where \(A_2 \in \mathcal{L}(H_1, H_{2,-\frac{1}{2}})\), which can be extended (or restricted) to \(H_{1,\alpha}\), such that it becomes an operator of \(\mathcal{L}(H_{1,\alpha}, H_{2,\alpha-\frac{1}{2}}), \alpha \in \mathbb{R},\) and \(A_2^* \in \mathcal{L}(H_2, H_{1,-\frac{1}{2}})\), which can be extended (or restricted) to \(H_{2,\alpha}\), such that it becomes an operator of \(\mathcal{L}(H_{2,\alpha}, H_{1,\alpha-\frac{1}{2}}), \alpha \in \mathbb{R}.\) Let \(C \in \mathcal{L}(H_2, H_{1,-\frac{1}{2}})\)
and $C^* \in \mathcal{L}(H_1^{1,\frac{1}{2}}, H_2)$, which can be extended or restricted to $H_{2,\alpha}, H_{1,\alpha}$, such that it belongs to $\mathcal{L}(H_{2,\alpha}, H_{1,\alpha-\frac{1}{2}}, H_{1,\alpha-\frac{1}{2}}, H_{2,\alpha-\frac{1}{2}})$, $\alpha \in \mathbb{R}$, respectively. We denote by $H_1^\tau$ the space $H_1$ equipped with the inner product $<u,v>_{H_1^\tau} = \tau <u,v>_{H_1}$, $u,v \in H_1$.

The aim of this paper is to show first that the exponential and polynomial decay of the energy $E$ satisfies the following equality

$$E(t) = \frac{1}{2} \| (w_1, \dot{w}_1, w_2, w_3) \|^2_{\mathcal{H}_t}, \quad t \geq 0,$$

satisfies the following equality

$$E(0) - E(t) = \int_0^t \| w_3(s) \|^2_{H_1} \, ds, \quad t \geq 0. \quad (1.6)$$

The aim of this paper is to show first that the exponential and polynomial decay of the energy $E(t)$ is reduced to an observability inequality for a corresponding conservative adjoint system, as in [1, 2, 3, 6].

For $\tau = 0$, the thermo-elastic problem with Cattaneo law (1.2)-(1.5) is just the following classical thermo-elastic system (with Fourier law)

$$\ddot{w}_1(t) + A_1 w_1(t) + C w_2(t) = 0, \quad (1.7)$$

$$\ddot{w}_2(t) + A_2 w_2(t) - C^* \dot{w}_1(t) = 0, \quad (1.8)$$

$$w_1(0) = w_1^0, \quad \dot{w}_1(0) = w_1^1, \quad w_2(0) = w_2^0, \quad (1.9)$$

whose the energy

$$E_0(t) = \frac{1}{2} \| (w_1, \dot{w}_1, w_2) \|^2_{\mathcal{H}_0}, \quad t \geq 0,$$

where $\mathcal{H}_0 := H_{1,\frac{1}{2}} \times H_1 \times H_2$, satisfies the energy equality

$$E_0(0) - E_0(t) = \int_0^t \| A_2^* w_2(s) \|^2_{H_2} \, ds, \quad t \geq 0. \quad (1.10)$$

The second main result in this paper is to show that the exponential and polynomial decay of the energy $E$ of the abstract thermo-elastic system with Cattaneo law provides a polynomial decay of the energy $E_0$ of the classical thermo-elastic system (1.7)-(1.9).
This is done by a spectral technic using a recent characterization of polynomial stability of \( C_0 \)-semigroups in Hilbert spaces due Borichev-Tomilov [5].

Consider now the conservative adjoint problem

\[
\begin{align*}
\ddot{\phi}_1(t) + A_1 \phi_1(t) + C \phi_2(t) &= 0, \\
\dot{\phi}_2(t) + A_2 \phi_3(t) - C^* \dot{\phi}_1(t) &= 0 \\
\tau \ddot{\phi}_3(t) - A_2^* \ddot{\phi}_2(t) &= 0 \\
\phi_1(0) &= \phi_1^0, \quad \dot{\phi}_1(0) = \phi_1^1, \\
\phi_2(0) &= \phi_2^0, \quad \phi_3(0) = \phi_3^0,
\end{align*}
\]

and the unbounded linear operators

\[
A_d : \mathcal{D}(A_d) \subset \mathcal{H} \to \mathcal{H},
A_c : \mathcal{D}(A_c) \subset \mathcal{H} \to \mathcal{H},
\]

where

\[
\mathcal{D}(A_d) = \mathcal{D}(A_c) = H_{1,1} \times H_{1,\frac{1}{2}} \times H_{2,\frac{1}{2}} \times H_{1,\frac{1}{2}},
\]

and

\[
\mathcal{D}(A) = H_{1,1} \times H_{1,\frac{1}{2}} \times H_{2,1}.
\]

We transform the system (1.2)-(1.5) into a first-order system of evolution equation type. For this, let

\[
W := (w_1, \dot{w}_1, w_2, w_3), \quad W(0) = W^0 := (w_1^0, w_1^1, w_2^0, w_3^0).
\]

Then, \( W \) satisfies

\[
\dot{W}(t) = A_d W(t), \quad t \geq 0, \quad W(0) = W^0.
\]

For the polynomial energy decay of the classical thermo-elastic system, we assume also the following assumption:

**Assumption H.** \( i \mathbb{R} \subset \rho(A) \), where \( A \) is the operator defined by (1.17) and \( \rho(A) \) is the resolvent set of \( A \).

The main result of this paper is the following theorem.

**Theorem 1.1.** 1. The system described by (1.2)-(1.5) is exponentially stable in \( \mathcal{H} \) if and only if there exists \( T, C > 0 \) such that

\[
\int_0^T ||\phi_3(t)||_{H_1}^2 \, dt \lesssim ||(\phi_0^0, \phi_1^1, \phi_2^0, \phi_3^0)||_{H}^2
\]

\[
\forall \ (\phi_1^0, \phi_1^1, \phi_2^0, \phi_3^0) \in \mathcal{H}.
\]
2. If the system described by (1.2)-(1.5) is exponentially stable in \( \mathcal{H} \) then \((w_1, \tilde{w}_1, w_2)\) solution of (1.7)-(1.9) is polynomially stable for all initial data in \( H_{1,1} \times H_{1,2} \times H_{2,1}, \) i.e., there exists a constant \( C > 0 \) such that for all \((w^0_1, w^1_1, w^2_2) \in \mathcal{D}(A)\) we have

\[
\|(w_1(t), \tilde{w}_1(t), w_2(t))\|_{\mathcal{H}_0} \leq \frac{C}{\sqrt{t}} \|(w^0_1, w^1_1, w^2_2)\|_{\mathcal{D}(A)}, \quad \forall t > 0. \tag{1.19}
\]

3. If there exist \( \alpha, T, C > 0 \) such that

\[
\int_0^T \|\phi_3(t)\|^2_{\mathcal{H}_1} \, dt \leq \|\langle \phi^0_1, \phi^1_1, \phi^0_2, \phi^0_3 \rangle\|^2_{\mathcal{H}_{-\alpha}} \tag{1.20}
\]

for all \((\phi^0_1, \phi^1_1, \phi^0_2, \phi^0_3) \in \mathcal{H}_{-\alpha} = H_{1,-\frac{1}{2}} \times H_{1,-\frac{3}{2}} \times H_{2,-\frac{1}{2}} \times H_{1,-\frac{1}{2}}\) then, there exists a constant \( C > 0 \) such that for all \((w^0_1, w^1_1, w^0_2, w^0_3) \in \mathcal{D}(A_d)\) we have

\[
E(t) \leq \frac{C}{t^{\alpha+1}} \|(w^0_1, w^1_1, w^0_2, w^0_3)\|^2_{\mathcal{D}(A_d)}, \quad \forall t > 0. \tag{1.21}
\]

4. If the solution of the system described by (1.2)-(1.5) satisfies (1.21) then the solution of (1.7)-(1.9) satisfies

\[
E_0(t) \leq \frac{C}{t^{\frac{1}{\alpha+1}}} \|(w^0_1, w^1_1, w^2_2)\|^2_{\mathcal{D}(A)}, \quad \forall t > 0 \tag{1.22}
\]

for some constant \( C > 0 \) and all \((w^0_1, w^1_1, w^2_2) \in \mathcal{D}(A)\).

As a direct consequence we have the following corollary.

**Corollary 1.2.** 1. If the system (1.2)-(1.5) satisfies (1.18) for all initial data in \( \mathcal{D}(A_d) \) then the system (1.7)-(1.9) satisfies (1.19) for all initial data in \( \mathcal{D}(A) \).

2. If the system (1.2)-(1.5) satisfies (1.20) for all initial data in \( \mathcal{D}(A_d) \) then the system (1.2)-(1.5) satisfies (1.21) for all initial data in \( \mathcal{D}(A) \).

The paper is organized as follows. In Section 2, we show the well-posedness of the evolution system (1.2)-(1.5), by showing that the operator \((A_d, \mathcal{D}(A_d))\) generates a contraction \(C_0\)-semigroup in the space \( \mathcal{H} \). In the third section we give some results in the regularity for some infinite dimensional systems needed of the proof of the main result. Section 4 contains the proof of the main results. Some applications are given in Section 5.

## 2 Well-posedness

Let \( \mathcal{H} := H_{1,\frac{1}{2}} \times H_1 \times H_2 \times H^*_1 \) the Hilbert space endowed with the inner product

\[
\left\langle \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} \right\rangle_{\mathcal{H}} = \left\langle A_{\frac{1}{2}}^{\frac{1}{2}} u_1, A_{\frac{1}{2}}^{\frac{1}{2}} v_1 \right\rangle_{H_1} + \left\langle u_2, v_2 \right\rangle_{H_1} + \left\langle u_3, v_3 \right\rangle_{H_2} + \tau \left\langle u_4, v_4 \right\rangle_{H_1}.
\]

We have the following fundamental result.
Theorem 2.1. The operator \( A_d \), respectively \( A \), generates a strongly continuous contraction semigroup \( (T(t))_{t \geq 0} \) on \( \mathcal{H} \), respectively on \( \mathcal{H}_0 \).

Proof. Take \( \begin{pmatrix} u_1 \\ u_2 \\ v \\ w \end{pmatrix} \in D(A_d) \). We have

\[
\langle A_d \begin{pmatrix} u_1 \\ u_2 \\ v \\ w \end{pmatrix}, \begin{pmatrix} u_1 \\ u_2 \\ v \\ w \end{pmatrix} \rangle_{\mathcal{H}} = \langle \begin{pmatrix} w_2 \\ -A_1 u_1 - C v \\ C^* u_2 - A_2 w \\ \frac{1}{\tau} A_2^* v - \frac{1}{\tau} w \end{pmatrix}, \begin{pmatrix} u_1 \\ u_2 \\ v \\ w \end{pmatrix} \rangle_{\mathcal{H}} = -\|w\|_{\mathcal{H}_1}^2.
\]

Thus \( A_d \) is dissipative. The density of \( D(A_d) \) is obvious.

Next, we are going to show that \( A_d \) is closed and

\[
D(A_d^*) = D(A_d), \quad A_d^* = \begin{pmatrix} 0 & I & 0 & 0 \\ -A_1 & 0 & -C^* & 0 \\ 0 & C & 0 & -A_2^* \\ 0 & 0 & \frac{1}{\tau} A_2 & -\frac{1}{\tau} I \end{pmatrix}.
\] (2.1)

Let \( (W_n) \subset D(A_d), W_n \rightarrow W \in \mathcal{H}, A_d W_n \rightarrow Z \in \mathcal{H} \) as \( n \rightarrow \infty \). Then

\[
\langle A_d W_n, \Phi \rangle_{\mathcal{H}} \rightarrow \langle Z, \Phi \rangle_{\mathcal{H}}.
\]

Choosing successively \( \Phi = (\Phi^1, 0, 0, 0), \Phi^1 \in H_{1,1}, \Phi = (0, 0, 0, \Phi^3), \Phi^3 \in H_{2,1}, \Phi = (0, 0, 0, \Phi^4), \Phi^4 \in H_{1,2}, \Phi = (0, 0, 0, 0), \Phi^2 \in H_{1,2} \), we obtain

\[
W^2 \in H_{1,\frac{1}{2}}, W^4 \in H_{1,\frac{1}{2}}, C^* W^2 - A_2 W^4 = Z^3, \quad W^3 \in H_{2,\frac{1}{2}}, A_2^* W^3 - W^4 = \tau Z^3, \quad W^1 \in H_{1,1}, -A_1 W^1 - CW^3 = Z^2,
\]

which yields that \( W \in D(A_d) \) and \( A_d W = Z \).

\[
V \in D(A_d^*) \iff \exists Z \in \mathcal{H} \forall \Phi \in D(A_d); \langle A_d \Phi, Z \rangle_{\mathcal{H}} = \langle \Phi, Z \rangle_{\mathcal{H}}.
\]

Choosing \( \Phi \) appropriately as in above, the conclusion (2.1) follows. Finally, the Hille-Yosida theorem leads to the claim.

By the same way we can prove that \( A \) generates a \( C_0 \)-semigroup of contractions on \( \mathcal{H}_0 \).

\( \square \)
3 Regularity of some coupled systems

We consider the initial and boundary value problems

\[
\ddot{\phi}_1(t) + A_1\phi_1(t) + C\phi_2(t) = 0, \quad \dot{\psi}(t) + A_2\phi_3 - C^*\dot{\phi}_1(t) = 0, \quad \tau \phi_3(t) - A_2^*\phi_2(t) = 0 \quad (3.1)
\]

\[
\phi_1(0) = w_1^0, \quad \dot{\phi}_1(0) = w_1^1, \quad \phi_2(0) = w_2^0, \quad \phi_3(0) = w_3^0, \quad (3.2)
\]

and

\[
\ddot{\psi}(t) + A_1\phi(t) + C\psi(t) = 0, \quad \dot{\psi}(t) + A_2w(t) - C^*\dot{\phi}(t) = 0, \quad \tau \dot{\psi}(t) - A_2^*\psi(t) = g(t) \quad (3.3)
\]

\[
\phi(0) = 0, \quad \dot{\phi}(0) = 0, \quad \psi(0) = 0, \quad w(0) = 0. \quad (3.4)
\]

We have the following proposition.

**Proposition 3.1.** Let \( g \in L^2(0,T;H_2) \). Then the system (3.3)-(3.4) admits a unique solution

\[
\left(\phi, \dot{\phi}, \psi, w\right) \in C(0,T;H_{1,\frac{1}{2}} \times H_1 \times H_2 \times H_1). \quad (3.5)
\]

Moreover \( w \in L^2(0,T;H_1) \) and there exists a constant \( C > 0 \) such that

\[
\|w\|_{L^2(0,T;H_2)} \leq C \|g\|_{L^2(0,T;H_1)}, \quad \forall g \in L^2(0,T;H_1). \quad (3.6)
\]

For proving Proposition 3.1, we should study the conservative system (without dissipation) associated to problem (1.2)-(1.5). We have the following result.

**Lemma 3.2.** For all \( (w_1^0,w_1^1,w_2^0,w_3^0) \in H_{1,\frac{1}{2}} \times H_1 \times H_2 \times H_1 \) the system (3.1)-(3.2) admits a unique solution \( \left(\phi_1, \dot{\phi}_1, \phi_2, \phi_3\right) \in C(0,T;H_{1,\frac{1}{2}} \times H_1 \times H_2 \times H_1) \). Then \( \phi_3 \in L^2(0,T;H_1) \) and there exists a constant \( C > 0 \) such that

\[
\|\phi_3\|_{L^2(0,T;H_1)} \leq C \left\|\left(w_1^0,w_1^1,w_2^0,w_3^0\right)\right\|_{H_{1,\frac{1}{2}} \times H_1 \times H_2 \times H_1}, \quad (3.7)
\]

\[
\forall (w_1^0,w_1^1,w_2^0,w_3^0) \in H_{1,\frac{1}{2}} \times H_1 \times H_2 \times H_1.
\]

**Proof.** By the classical semigroup theory, see [11], we prove that for all \( (w_1^0,w_1^1,w_2^0,w_3^0) \in H_{1,\frac{1}{2}} \times H_1 \times H_2 \times H_1 \) the system (3.1)-(3.2) admits a unique solution \( \left(\phi_1, \dot{\phi}_1, \phi_2, \phi_3\right) \in C(0,T;H_{1,\frac{1}{2}} \times H_1 \times H_2 \times H_1) \). We obtain that \( \phi_3 \in L^2(0,T;H_1) \) and that (3.7) holds. \( \square \)

Now we can give the proof of Proposition 3.1.

**Proof.** of Proposition 3.1.

Let the operator

\[
A_c : D(A_c) = H_{1,1} \times H_{1,\frac{1}{2}} \times H_{2,\frac{1}{2}} \times H_{2,\frac{1}{2}} \subset \mathcal{H} \to \mathcal{H},
\]
defined by
\[
\mathcal{A}_c \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} u_2 \\ -A_1 u_1 - C u_3 \\ C^* u_2 \\ \frac{1}{T} A_2^* u_2 \end{pmatrix}, \quad \forall (u_1, u_2, u_3, u_4) \in \mathcal{D}(A).
\]
\(\mathcal{A}_c\) is a skew-adjoint operator and generates a group of isometries \((S(t))_{t \in \mathbb{R}}\) on \(\mathcal{H}\). Moreover we define the operator
\[
\mathcal{B} : H_2 \to \mathcal{H}, \quad \mathcal{B} k = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{\sqrt{T}} k \end{pmatrix}, \quad \forall k \in H_1.
\]
(3.8)
The problem (3.3)-(3.4) can be rewritten as a Cauchy problem on \(\mathcal{H}\) under the form
\[
\begin{pmatrix} \dot{\phi} \\ \phi \\ \dot{\psi} \\ \psi \\ \dot{w} \\ w \end{pmatrix} (t) = \mathcal{A}_c \begin{pmatrix} \phi \\ \phi \\ \phi \\ \phi \end{pmatrix} (t) - \mathcal{B} g(t), \quad t > 0,
\]
(3.9)
\[
\phi(0) = 0, \quad \dot{\phi}(0) = 0, \quad \psi(0) = 0, \quad w(0) = 0.
\]
(3.10)
We can see that the operator \(\mathcal{B}^* : \mathcal{H} \to H_1\) is given by
\[
\mathcal{B}^* \begin{pmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \end{pmatrix} = \frac{1}{\sqrt{T}} v_2, \quad \forall (u_1, u_2, v_1, v_2) \in \mathcal{H},
\]
which implies that
\[
\mathcal{B}^* S(t) \begin{pmatrix} u_1^0 \\ u_2^0 \\ v_1^0 \\ v_2^0 \end{pmatrix} = \mathcal{B}^* \begin{pmatrix} \phi_1(t) \\ \phi_2(t) \\ \phi_3(t) \end{pmatrix} = \frac{1}{\sqrt{T}} \phi_3(t), \quad \forall (u_1^0, u_1^1, u_2^0, w_3^0) \in \mathcal{D}(\mathcal{A}_c),
\]
(3.11)
with \((\phi_1, \phi_2, \phi_3)\) is the solution of (3.1)-(3.2). According to semigroup theory, see [11], we have that (3.3)-(3.4) admits a unique solution
\[
\begin{pmatrix} \phi, \dot{\phi}, \psi, \dot{w} \end{pmatrix} (t) = \int_0^t S(t - s) B g(s) \, ds \in C(0, T; \mathcal{H})
\]
which satisfies the regularity (3.6).
4 Proof of the main result

Let \((w_1, \dot{w}_1, w_2, w_3) \in C(0, T; H_{1, \frac{1}{2}} \times H_1 \times H_2 \times H_1)\) be the solution of (1.2)-(1.5) for a given initial data \((w_1^0, \dot{w}_1^0, w_2^0, w_3^0)\). Then \((w_1, \dot{w}_1, w_2, w_3)\) can be written as

\[
(w_1, \dot{w}_1, w_2, w_3) = (\phi_1, \dot{\phi}_1, \phi_2, \phi_3) + (\phi, \dot{\phi}, \psi, w),
\]

(4.1)

where \((\phi_1, \phi_2, \phi_3)\) satisfies (3.1)-(3.2) and \((\phi, \dot{\phi}, \psi, w)\) satisfies (3.3)-(3.4) with \(g = -w_3\).

The main ingredient of the proof of Theorem 1.1 is the following result.

**Lemma 4.1.** Let \((w_1^0, \dot{w}_1^0, w_2^0, w_3^0) \in H_{1, \frac{1}{2}} \times H_1 \times H_2 \times H_1\). Then the solution \((w_1, \dot{w}_1, w_2, w_3)\) of (1.2)-(1.5) and the solution \((\phi_1, \phi_2, \phi_3)\) of (3.1)-(3.2) satisfy

\[
C_1 \int_0^T ||\phi_3(t)||^2_{H_1} dt \leq \int_0^T ||w_3(t)||^2_{H_1} dt \leq 4 \int_0^T ||\phi_3(t)||^2_{H_1} dt,
\]

(4.2)

where \(C_1 > 0\) is a constant independent of \((w_1^0, \dot{w}_1^0, w_2^0, w_3^0)\).

**Proof.** We prove (4.2) for \((w_1, \dot{w}_1, w_2, w_3)\) satisfying (1.2)-(1.5) and \((\phi_1, \phi_2, \phi_3)\) solution of (3.1)-(3.2). We know that \(w_3 \in L^2(0, T; H_1)\) and that (1.6) holds true. Relation (1.1) implies that

\[
\int_0^T ||\phi_3(t)||^2_{H_1} dt \leq 2 \left\{ \int_0^T ||w_3(t)||^2_{H_1} dt + \int_0^T ||w(t)||^2_{H_1} dt \right\}.
\]

By applying now Proposition 3.1 with \(g = -w_3 \in L^2(0, T; H_1)\) we obtain that

\[
\int_0^T ||w(t)||^2_{H_1} dt \leq C \int_0^T ||w_3(t)||^2_{H_1} dt.
\]

(4.3)

Then the first inequality of (4.2) holds true.

On the other hand, according to relation (1.1) we have that

\[
\phi_3 \in L^2(0, T; H_1),
\]

and

\[
\ddot{\phi}(t) + A_1 \phi(t) + C \psi(t) = 0, \ \dot{\psi}(t) + A_2 w(t) - C^* \dot{\phi}(t) = 0, \ \dot{w}(t) - A_2^* \psi(t) + w(t) = -\phi_3(t).
\]

(4.4)

We still denote by \(\phi_3\) the extension by 0, \(t \in \mathbb{R} \setminus [0, T]\). We still also denote by \((\phi(t), \dot{\phi}(t), \psi(t), w(t))\) the functions \((1_{[0,T]} \phi(t), 1_{[0,T]} \dot{\phi}(t), 1_{[0,T]} \psi(t), 1_{[0,T]} w(t))\). It is clear that these functions satisfy the equation on the line \(\mathbb{R}\)

\[
\begin{cases}
\ddot{\phi}(t) + A_1 \phi(t) + C \psi(t) = 0, \ \dot{\psi}(t) + A_2 w(t) - C^* \dot{\phi}(t) = 0, \\
\dot{w}(t) - A_2^* \psi(t) + w(t) = -\phi_3(t), \ t \in \mathbb{R}, \ \phi(0) = 0, \ \dot{\phi}(0) = 0, \ \psi(0) = 0, \ w(0) = 0.
\end{cases}
\]

(4.5)

Taking the Laplace transform we obtain

\[
\lambda^2 \hat{\phi}(\lambda) + A_1 \hat{\phi}(\lambda) + C \hat{\psi}(\lambda) = 0, \ \lambda \hat{\psi}(\lambda) + A_2 \hat{w}(\lambda) - \lambda C^* \hat{\phi}(\lambda) = 0,
\]

\[
\lambda^2 \hat{\phi}(\lambda) + A_1 \hat{\phi}(\lambda) + C \hat{\psi}(\lambda) = 0, \ \lambda \hat{\psi}(\lambda) + A_2 \hat{w}(\lambda) - \lambda C^* \hat{\phi}(\lambda) = 0,
\]
\[ \lambda \tau \hat{w}(\lambda) - A_2^* \hat{\psi}(\lambda) + \hat{w}(\lambda) = -\hat{\phi}_3(\lambda), \quad \forall \lambda = \gamma + i\eta, \gamma > 0. \]

The equality above holds in \( H_{1, \frac{1}{2}}, H_{2, -\frac{1}{2}}, H_{2, -\frac{1}{2}}, \) respectively. By applying \( \lambda \hat{\phi} \in H_{1, \frac{1}{2}}, \hat{\psi} \in H_2, \hat{w} \in H_1 \) respectively to first, second and to the third equation on the equalities above, we get by taking the real part,

\[ \gamma \lambda ||\hat{\phi}(\lambda)||_{H_1}^2 + \gamma ||A_1^* \hat{\phi}(\lambda)||_{H_1}^2 + \gamma ||\hat{\psi}(\lambda)||_{H_2}^2 + (\gamma \tau + 1) ||\hat{w}(\lambda)||_{H_1}^2 = -\Re < \hat{\phi}_3(\lambda), \hat{w}(\lambda) >_{H_1}. \]

We get,

\[ \int_{\mathbb{R}} ||\hat{w}(\lambda)||_{H_1}^2 d\eta \leq \frac{1}{2} \int_{\mathbb{R}} ||\hat{\phi}_3(\lambda)||_{H_1}^2 d\eta + \frac{1}{2} \int_{\mathbb{R}} ||\hat{w}(\lambda)||_{H_1}^2 d\eta. \]

Parseval identity implies

\[ ||w||_{L^2(0,T;H_1)}^2 \leq ||\phi_3||_{L^2(0,T;H_1)}^2, \quad (4.6) \]

and with relation \((4.1)\), we have

\[ ||w_3||_{L^2(0,T;H_1)}^2 \leq 4 ||\phi_3||_{L^2(0,T;H_1)}^2, \quad (4.7) \]

This achieves the proof.

We can now prove Theorem 1.1.

**Proof of the first assertion.** All finite energy solutions of \((1.2)-(1.5)\) satisfy the estimate

\[ E(t) \leq M e^{-\omega t} E(0), \quad \forall t \geq 0, \quad (4.8) \]

where \( M, \omega > 0 \) are constants independent of \( (w_0^0, w_1^0, w_2^0, w_3^0) \), if and only if there exist a time \( T > 0 \) and a constant \( C > 0 \) (depending on \( T \)) such that

\[ E(0) - E(T) \geq CE(0), \quad \forall (w_0^0, w_1^0, w_2^0, w_3^0) \in H_{1, \frac{1}{2}} \times H_1 \times H_2 \times H_1. \]

By \((1.6)\) relation above is equivalent to the inequality

\[ \int_0^T ||w_3(s)||_{H_1}^2 ds \geq CE(0), \quad \forall (w_0^0, w_1^0, w_2^0, w_3^0) \in H_{1, \frac{1}{2}} \times H_1 \times H_2 \times H_1. \]

From Lemma 4.1 it follows that the system \((1.2)-(1.5)\) is exponentially stable if and only if

\[ \int_0^T ||\phi_3(s)||_{H_1}^2 ds \geq CE(0), \quad \forall (w_0^0, w_1^0, w_2^0, w_3^0) \in H_{1, \frac{1}{2}} \times H_1 \times H_2 \times H_1 \]

holds true. It follows that \((1.2)-(1.5)\) is exponentially stable if and only if \((1.18)\) holds true. This ends up the proof of the first assertion of Theorem 1.1.

**Proof of the third assertion.**
We have that for all \((\phi_1^0, \phi_1^1, \phi_2^0, \phi_3^1) \in \mathcal{H}\)

\[
\int_0^T ||\phi_3(t)||_{\mathcal{H}_1}^2 \, dt \geq C ||(\phi_1^0, \phi_1^1, \phi_2^0, \phi_3^1)||_{\mathcal{H}_{-\alpha}}^2.
\] (4.9)

Then, by Lemma 4.1 combined with (4.9) and (1.10) imply the existence of a constant \(K > 0\) such that

\[
|| (w_1(T), w_1'(T), w_2(T), w_3(T)) ||_{\mathcal{H}}^2 \leq \frac{|| (w_1^0, w_1^1, w_2^0, w_3^0) ||_{\mathcal{H}}^2 - K \frac{|| (w_1^0, w_1^1, w_2^0, w_3^0) ||_{\mathcal{H}}^{2+2\alpha}}{|| (w_1^0, w_1^1, w_2^0, w_3^0) ||_{\mathcal{H}}^{2\alpha}},
\]

\[
\forall (w_1^0, w_1^1, w_2^0, w_3^0) \in \mathcal{D}(A_d).
\] (4.10)

Estimate (4.10) remains valid in successive intervals \([kT, (k+1)T]\) and since \(A_d\) generates a semigroup of contractions in \(\mathcal{D}(A_d)\) and the graph norm on \(\mathcal{D}(A_d)\) is equivalent to \(||.||_{\mathcal{H}_1}\). We obtain the existence of a constant \(C > 0\) such that for all \(k \geq 0\) we have

\[
|| (w_1((k+1)T), w_1'((k+1)T), w_2((k+1)T), w_3((k+1)T)) ||_{\mathcal{H}}^2 \leq
\]

\[
- C \frac{|| (w_1((k+1)T), w_1'((k+1)T), w_2((k+1)T), w_3((k+1)T)) ||_{\mathcal{H}}^{2+2\alpha}}{|| (w_1^0, w_1^1, w_2^0, w_3^0) ||_{\mathcal{D}(A_d)}^{2\alpha}},
\]

\[
\forall (w_1^0, w_1^1, w_2^0, w_3^0) \in \mathcal{D}(A_d).
\] (4.11)

If we adopt the notation

\[
\mathcal{H}_k = \frac{|| (w_1(kT), w_1'(kT), w_2(kT), w_3(kT)) ||_{\mathcal{H}}^2}{|| (w_1^0, w_1^1, w_2^0, w_3^0) ||_{\mathcal{D}(A_d)}^{2\alpha}}.
\] (4.12)

relation (4.11) gives

\[
\mathcal{H}_{k+1} \leq \mathcal{H}_k - C\mathcal{H}_k^{1+\alpha}, \quad \forall k \geq 0.
\] (4.13)

By applying the following lemma.

Lemma 4.2. [4] Lemma 5.2] Let \((\mathcal{E}_k)\) be a sequence of positive real numbers satisfying

\[
\mathcal{E}_{k+1} \leq \mathcal{E}_k - C\mathcal{E}_k^{2+\delta}, \quad \forall k \geq 0,
\] (4.14)

where \(C > 0\) and \(\delta > -1\) are constants. Then there exists a positive constant \(M\) such that

\[
\mathcal{E}_k \leq \frac{M}{(k+1)^{-\delta}}, \quad \forall k \geq 0.
\] (4.15)

and using relation (4.13) we obtain the existence of a constant \(M > 0\) such that

\[
|| (w_1(kT), w_1'(kT), w_2(kT), w_3(kT)) ||_{\mathcal{H}}^2 \leq \frac{M|| (w_1^0, w_1^1, w_2^0, w_3^0) ||_{\mathcal{D}(A_d)}^{2\alpha}}{(k+1)^{\frac{\delta}{2}}}, \quad \forall k \geq 0,
\]

which obviously implies (4.21).
Proof of the second assertion.
The second assertion of Theorem 1.1 is equivalent to the following

\[ \rho(A_d) \supset \{ i\beta \mid \beta \in \mathbb{R} \} \equiv i\mathbb{R}, \quad (4.16) \]

and

\[ \limsup_{|\beta| \to \infty} \| (i\beta - A_d)^{-1} \| < \infty \quad (4.17) \]

implies that by a result of Borichev-Tomilov [5] that \( A \) satisfies the following two conditions:

\[ \rho(A) \supset \{ i\beta \mid \beta \in \mathbb{R} \} \equiv i\mathbb{R}, \quad (4.18) \]

and

\[ \limsup_{|\beta| \to \infty} \frac{1}{\beta^2} \| (i\beta - A)^{-1} \| < \infty, \quad (4.19) \]

where \( \rho(A) \), respectively \( \rho(A_d) \), denotes the resolvent set of the operator \( A \), respectively of \( A_d \).

By assumption H the conditions (4.18), (4.16) are satisfied. Now for proving the above implication, suppose that the condition (4.19) is false. By the Banach-Steinhaus Theorem, there exist a sequence of real numbers \( \beta_n \to \infty \) and a sequence of vectors

\[ Z_n = \begin{pmatrix} u_n \\ \varphi_n \\ \theta_n \end{pmatrix} \in D(A) \]

with \( \| Z_n \|_{H_0} = 1 \) such that

\[ \| \beta_n^2 (i\beta_n I - A) Z_n \|_{H_0} \to 0 \quad \text{as} \quad n \to \infty, \quad (4.20) \]

i.e.,

\[ \beta_n^2 (i\beta_n u_n - \varphi_n) \to 0 \quad \text{in} \quad H_{1,\frac{1}{2}}, \quad (4.21) \]

\[ \beta_n^2 (i\beta_n \varphi_n + A_1 u_n + C\theta_n) \to 0 \quad \text{in} \quad H_1, \quad (4.22) \]

\[ \beta_n^2 (i\beta_n \theta_n + A\theta_n - C^* \varphi_n) \to 0 \quad \text{in} \quad H_2. \quad (4.23) \]

We notice that we have

\[ \| \beta_n^2 (i\beta_n I - A) Z_n \|_{H_0} \geq |\mathbb{R} (\langle \beta_n^2 (i\beta_n I - A) Z_n, Z_n \rangle_{H_0}) |. \quad (4.24) \]

Then, by (4.20)

\[ \beta_n A_2^* \theta_n \to 0, \quad A_2^* \theta_n \to 0. \]

Let \( q_n = A_2^* \theta_n \),

\[ i\beta_n q_n + \frac{1}{\tau} q_n - A_2^* \theta_n \to 0, \quad (4.25) \]

which implies that

\[ i\beta_n u_n - \varphi_n \to 0 \quad \text{in} \quad H_{1,\frac{1}{2}}, \quad (4.26) \]

\[ i\beta_n \varphi_n + A_1 u_n + C\theta_n \to 0 \quad \text{in} \quad H_1, \quad (4.27) \]

\[ i\beta_n \theta_n + A_2 q_n - C^* \varphi_n \to 0 \quad \text{in} \quad H_2. \quad (4.28) \]
\[
i\beta_n q_n + \frac{1}{\tau} q_n - A_2^* \theta_n \to 0 \quad \text{in} \ H_1.
\]

(4.29)

i.e. \( \tilde{Z}_n = \begin{pmatrix} u_n \\ \varphi_n \\ \theta_n \\ q_n \end{pmatrix} \in D(A_d) \) with \( \| \tilde{Z}_n \|_H \) bounded such that

\[
\| (i\beta_n I - A_d) \tilde{Z}_n \|_H \to 0 \quad \text{as} \ n \to \infty,
\]

(4.30)

which implies that \((4.17)\) is false and ends the proof of the second assertion of Theorem 1.1.

**Proof of the fourth assertion of Theorem 1.1.**

By the same way as above, we can prove the fourth assertion of Theorem 1.1, i.e.,

the fourth assertion of Theorem 1.1 is equivalent to following: For \( \alpha > 0 \),

\[
\rho(A_d) \supset \{ i\beta \mid \beta \in \mathbb{R} \} \equiv i\mathbb{R},
\]

(4.31)

and

\[
\limsup_{|\beta| \to \infty} \frac{1}{\beta^{2\alpha}} \| (i\beta - A_d)^{-1} \| < \infty,
\]

(4.32)

implies that by a result of Borichev-Tomilov \([5, \text{Theorem 2.4}]\) that \( A \) satisfies the following two conditions:

\[
\rho(A) \supset \{ i\beta \mid \beta \in \mathbb{R} \} \equiv i\mathbb{R},
\]

(4.33)

and

\[
\limsup_{|\beta| \to \infty} \frac{1}{\beta^{2\alpha+2}} \| (i\beta - A)^{-1} \| < \infty.
\]

(4.34)

5 Applications to stabilization for a thermo-elastic system

5.1 First example

We consider the following initial and boundary problem

\[
\begin{aligned}
\ddot{u}_1 - \partial_x^2 u_1 + \partial_x u_2 &= 0, \quad (0, +\infty) \times (0, 1), \\
\ddot{u}_2 - \partial_x^2 u_3 + \partial_x \dot{u}_1 &= 0, \quad (0, +\infty) \times (0, 1), \\
\tau \dot{u}_3 - \partial_x u_2 + u_3 &= 0, \quad (0, +\infty) \times (0, 1), \\
u_1(t, 0) &= u_1(t, 1) = 0, \quad (0, +\infty), \\
u_3(t, 0) &= u_3(t, 1) = 0, \quad (0, +\infty), \\
u_1(0, x) &= u_1^0(x), \quad \dot{u}_1(0, x) = u_1^1, \quad u_2(0, x) = u_2^0, \quad u_3(0, x) = u_3^0, \quad x \in (0, 1),
\end{aligned}
\]

(5.1)

where \( 0 < \tau \) and satisfies \( \sqrt{\frac{\tau}{1 + \tau}} \notin \mathbb{Q} \). In this case, we have:

\[
H_1 = H_2 = L^2(0, 1), \quad H_{1, \frac{1}{2}} = H^1_0(0, 1),
\]

and

\[
A_1 = -\frac{d^2}{dx^2}, \quad D(A_1) = H^2(0, 1) \cap H^1_0(0, 1), \quad A_2 = -\frac{d}{dx}, \quad D(A_2) = H^1(0, 1),
\]

12
\[ A_2^* = \frac{d}{dx}, \mathcal{D}(A_2^*) = H_0^1(0, 1), \quad C = \frac{d}{dx} : H^1(0, 1) \to L^2(0, 1), \]

\[ C^* = -\frac{d}{dx} : H_0^1(0, 1) \to L^2(0, 1). \]  

(5.2)

Then, \( \mathcal{A}_d \) is given by

\[ \mathcal{A}_d : \mathcal{D}(\mathcal{A}_d) \to H_0^1(0, 1) \times L^2(0, 1) \times L^2(0, 1) \times L^2(0, 1), \]

\[ \mathcal{A}_d = \begin{pmatrix} 0 & I & 0 & 0 \\ \frac{d^2}{dx^2} & 0 & -\frac{d}{dx} & 0 \\ 0 & \frac{d}{dx} & 0 & -\frac{1}{\tau} I \\ 0 & -\frac{d}{dx} & 0 & \frac{d}{dx} \end{pmatrix}, \]

where

\[ \mathcal{D}(\mathcal{A}_d) = [H^2(0, 1) \cap H_0^1(0, 1)] \times H_0^1(0, 1) \times H^1(0, 1) \times H_0^1(0, 1). \]

Stability results for (5.1), for \( \tau = 0 \), are then a consequence of Theorem 1.1.

In this case the problem (1.11)-(1.14) becomes

\[ \ddot{\phi}_1 - \partial_x^2 \phi_1 + \partial_x \phi_2 = 0, \quad (0, 1) \times (0, +\infty), \]  

(5.3)

\[ \dot{\phi}_2 - \partial_x \phi_3 + \partial_x \dot{\phi}_1 = 0, \quad (0, 1) \times (0, +\infty), \]  

(5.4)

\[ \tau \dot{\phi}_3 - \partial_x \phi_2 = 0, \quad (0, 1) \times (0, +\infty), \]  

(5.5)

\[ \phi_i(0, t) = \phi_i(1, t) = 0, \quad (0, +\infty), \quad i = 1, 3, \]  

(5.6)

\[ \phi_i(x, 0) = u_i^0(x), \quad \dot{\phi}_1(x, 0) = u_1^1(x), \quad (0, 1), \quad i = 1, 2, 3. \]  

(5.7)

The observability inequality concerning the solutions of (5.3)-(5.7) is given in the proposition below.

**Proposition 5.1.** Let \( T > 2 \) be fixed. Then the following assertions hold true.

*The solution \((\phi_1, \phi_2, \phi_3)\) of (5.3)-(5.7) satisfies*

\[ \int_0^T \int_0^1 |\phi_3(x, t)|^2 \, dx \, dt \geq C \left\| \left( u_1^0, u_1^1, u_2^0, u_3^0 \right) \right\|_{H_0^1(0, 1) \times L^2(0, 1) \times L^2(0, 1) \times L^2(0, 1)}, \]

\[ \forall (u_1^0, u_1^1, u_2^0, u_3^0) \in \mathcal{H}, \]  

(5.8)

where \( C > 0 \) is a constant and

\[ \mathcal{H} = \left\{ (u_1, u_2, u_3, u_4) \in H_0^1(0, 1) \times L^2(0, 1) \times L^2(0, 1) \times L^2(0, 1), \int_0^1 u_3(x) \, dx = 0 \right\} = \langle (0, 0, 1, 0)^t \rangle^+. \]
Proof. If we put
\[
\begin{pmatrix}
u^0_1 \\
u^1_1 \\
u^0_2 \\
u^0_3 \\
\end{pmatrix} \in \mathcal{H}, \ \text{i.e.}, \ \begin{pmatrix}
u^0_1 \\
u^1_1 \\
u^0_2 \\
u^0_3 \\
\end{pmatrix}(x) = \sum_{n \in \mathbb{Z}^*} a_n \varphi_n(x)
\]
where \((a_n)_{n \in \mathbb{Z}^*} \in l^2\), and

\[
\varphi_n(x) = \begin{pmatrix}
\sin(n\pi x) \\
\lambda_n \sin(n\pi x) \\
-n\pi \left(\frac{1}{\lambda_n} \left(\lambda_n - \frac{n\pi}{\lambda_n}\right) + 1\right) \cos(n\pi x) \\
\frac{1}{\tau} \left(\lambda_n - \frac{n\pi}{\lambda_n}\right) \sin(n\pi x)
\end{pmatrix}, \ n \in \mathbb{Z}^*,
\]
with

\[
(\lambda_n)_{n \in \mathbb{Z}^*} = \left\{i n\pi \sqrt{\frac{1 + \tau}{\tau}}, \ n \in \mathbb{Z}^* \right\} \cup \left\{i n\pi, \ n \in \mathbb{Z}^* \right\}.
\]

Then, we clearly have

\[
\phi_3(x,t) = \sum_{n \in \mathbb{Z}^*} a_n \frac{1}{\tau} \left(\lambda_n - \frac{n\pi}{\lambda_n}\right) e^{\lambda_n t} \sin(n\pi x).
\] (5.9)

From Ingham’s inequality (see Ingham [9]) we obtain, for all \(T > 2\), the existence of a constant \(C_T > 0\) such that the solution \((\phi_1, \phi_2, \phi_3)\) of (5.3)-(5.7) satisfies

\[
\int_0^T \int_0^1 |\phi_3(x,t)|^2 \, dx \, dt \geq C_T \sum_{n \in \mathbb{Z}^*} |\lambda_n a_n|^2,
\] (5.10)

which is exactly (5.8).

Now, as an immediate consequence of Theorem 1.1 we have the following stability result for \((u_1, \dot{u}_1, u_2)\) solution of (5.1) with \(\tau = 0\).

**Proposition 5.2.** There exists a constant \(C > 0\) such that for all \((u^0_1, u^1_1, u^0_2) \in \mathcal{D}(\hat{A}) = \mathcal{D}(A) \cap \dot{\mathcal{H}},\)

\[
\|(u_1, \dot{u}_1, u_2)\|_{\dot{\mathcal{H}}} \leq C \sqrt{t} \|(u^0_1, u^1_1, u^0_2)\|_{\mathcal{D}(A)}, \ \forall t > 0.
\]

**Remark 5.3.** We can obtain the same result, as above, by application of an exponential stability result obtained by Racke for (5.1) in [13, Theorem 2.1] and Theorem 7.1.

### 5.2 Second example

Let \(\Omega\) be a bounded smooth domain of \(\mathbb{R}^2\). We consider the following initial and boundary problem:

\[
\begin{align*}
\ddot{u} - \mu \Delta u - (\lambda + \mu) \nabla \text{div} u + \nabla \theta &= 0, \ (0, +\infty) \times \Omega, \\
\dot{\theta} + \text{div} q + \text{div} \dot{u} &= 0, \ (0, +\infty) \times (0, 1), \\
\tau \dot{q} + \nabla \theta + q &= 0, \ (0, +\infty) \times (0, 1),
\end{align*}
\]
\[
\begin{align*}
u &= 0, \theta = 0, \ \Omega \times (0, +\infty),
\end{align*}
\]
\[
\begin{align*}
u(0, x) = u^0(x), \ \dot{u}(0, x) = u^1(x), \ \theta(0, x) = \theta^0, \ q(0, x) = q^0(x), \ x \in \Omega,
\end{align*}
\]
The parameters $\tau, \mu, \lambda$ are positive constants which satisfy $\lambda + 2\mu > 0$.

In this case, we have:

$$H_1 = (L^2(\Omega))^2, H_2 = L^2(\Omega), \quad H_{1/2} = (H^1_0(\Omega))^2,$$

and

$$A_1 = -\mu \Delta - (\mu + \lambda)\nabla \text{div}, \quad \mathcal{D}(A_1) = (H^2(\Omega) \cap H^1_0(\Omega))^2, \quad A_2 = \text{div}, \quad \mathcal{D}(A_2) = H^1(\Omega),$$

$$A^*_2 = -\nabla, \quad \mathcal{D}(A^*_2) = (H^1_0(\Omega))^2, \quad C = \nabla : H^1(\Omega) \to (L^2(\Omega))^2,$$

$$C^* = -\text{div} : (H^1_0(\Omega))^2 \to L^2(\Omega).$$

Then, $A_d$ is given by

$$A_d : \mathcal{D}(A_d) \to (H^1_0(\Omega) \times L^2(\Omega))^2 \times L^2(\Omega) \times (L^2(\Omega))^2,$$

where

$$\mathcal{D}(A_d) = \left[H^2(\Omega) \cap H^1_0(\Omega)\right]^2 \times (H^1_0(\Omega))^2 \times H^1_0(\Omega) \times (H^1(\Omega))^2.$$

Stability result for (5.11), with $\tau = 0$, are then an immediate consequence of Theorem 1.1 and of [12, Theorem 3.1]. We have the following result

**Proposition 5.4.** Let $\Omega$ be a a radially symmetric and let the initial data $(u^0, u^1, \theta^0, \varrho^0)$ be radially symmetric. Then, there exists a constant $C > 0$ such that for all $(u^0, u^1, \theta^0) \in [H^2(\Omega) \cap H^1_0(\Omega)] \times H^1_0(\Omega) \times [H^2(\Omega) \cap H^1(\Omega)]$,

$$\|(u, \dot{u}, \theta)\|_H \leq C \sqrt{t} \|(u^0, u^1, \theta^0)\|_{[H^2(\Omega) \cap H^1_0(\Omega)]^2 \times H^1_0(\Omega) \times [H^2(\Omega) \cap H^1_0(\Omega)]}, \quad \forall t > 0.$$

**Remark 5.5.** We remark that we obtain the same stability result as Lebeau-Zuazua in [10].

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*see [8] page 327 for definitions.*
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