DEFORMATIONS OF HYPERBOLIC CONE-STRUCTURES: STUDY OF THE COLLAPSING CASE

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Dedicated to professor Paulo Sabini (1972 - 2007)

Abstract. This work is devoted to the study of deformations of hyperbolic cone structures under the assumption that the lengths of the singularity remain uniformly bounded over the deformation. Given a sequence $(M_i, p_i)$ of pointed hyperbolic cone-manifolds with topological type $(M, \Sigma)$, where $M$ is a closed, orientable and irreducible 3-manifold and $\Sigma$ an embedded link in $M$. If the sequence $M_i$ collapses and assuming that the lengths of the singularity remain uniformly bounded, we prove that $M$ is either a Seifert fibered or a Sol manifold. We apply this result to a question stated by Thurston and to the study of convergent sequences of holonomies.

1. Introduction

Fixed a closed, orientable and irreducible 3-manifold $M$, this text focus deformations of hyperbolic cone structures on $M$ which are singular along a fixed embedded link $\Sigma = \Sigma_1 \sqcup \ldots \sqcup \Sigma_l$ in $M$. A hyperbolic cone structure with topological type $(M, \Sigma)$ is a complete intrinsic metric on $M$ (see section 2 for the definition) such that every non-singular point (i.e. every point in $M - \Sigma$) has a neighborhood isometric to an open set of $H^3$, the hyperbolic space of dimension 3, and that every singular point (i.e. every point in $\Sigma$) has a neighborhood isometric to an open neighborhood of a singular point of $H^3(\alpha)$, the space obtained by identifying the sides of a dihedral of angle $\alpha \in (0, 2\pi]$ in $H^3$ by a rotation about the axis of the dihedral. The angles $\alpha$ are called cone angles and they may vary from one connected component of $\Sigma$ to the other. By convention, the complete structure on $M - \Sigma$ is considered as a hyperbolic cone structure with topological type $(M, \Sigma)$ and cone angles equal to zero.

Unlike hyperbolic structures, which are rigid after Mostow, the hyperbolic cone structures can be deformed (cf. [HK2]). The difficulty to understand these deformations lies in the possibility of degenerating the structure. In other words, the Hausdorff-Gromov limit of the deformation (see section 2 for the definition) is only an Alexandrov space which may have dimension strictly smaller than 3, although its curvature remains bounded from below by $-1$ (cf. [Koj]). In fact, the works of Kojima, Hodgson-Kerckhoff and Fuji (see [Koj], [HK] and [Fuj]) show that the degeneration of the hyperbolic cone structures occurs if and only if the singular link of these structures intersects itself over the deformation.

When $M$ is a hyperbolic manifold and $\Sigma$ is a finite union of simple closed geodesics of $M$, it is know that $M - \Sigma$ admits a complete hyperbolic structure (see [Koj]). We are interested in studying the following question that was raised by W.Thuston in 80’s:

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 Question 1.1. Let $M$ be a closed and orientable hyperbolic manifold and suppose the existence of a simple closed geodesic $\Sigma$ in $M$. Can the hyperbolic structure of $M$ be deformed to the complete hyperbolic structure on $M - \Sigma$ through a path $M_\alpha$ of hyperbolic cone structures with topological type $(M, \Sigma)$ and parametrized by the cone angles $\alpha \in [0, 2\pi]$?

When the deformation proposed by Thurston exists, it is a consequence of Thurston’s hyperbolic Dehn surgery Theorem that the length of $\Sigma$ must converge to zero. In particular, we have that the length of the singular link remains uniformly bounded over the deformation. Motivated by this fact, we are interested in studying the deformations of hyperbolic cone structures with this additional hypothesis on the length of the singular link. This assumption is automatically verified when the holonomy representations of the hyperbolic cone structures are convergent. This happens (cf. [CS]), for example, when $\Sigma$ is a small link in $M$. Note that this additional hypothesis avoids the undesirable case where the singularity becomes dense in the limiting Alexandrov space.

In this paper, $M_i$ will always denote a sequence of hyperbolic cone-manifolds with topological type $(M, \Sigma)$. Given points $p_i \in M_i$, suppose that the sequence $(M_i, p_i)$ converges (in the Hausdorff-Gromov sense) to a pointed Alexandrov space $(Z, z_0)$. Our goal is to understand the metric properties of the Alexandrov space $Z$ and exploit them to obtain further topological information on the manifold $M$. To do this, we need the important notion of collapse for a sequence and a continuous deformation of hyperbolic cone-manifolds.

Definition 1.2. We say that a sequence $M_i$ of hyperbolic cone-manifolds with topological type $(M, \Sigma)$ collapses if, for every sequence of points $p_i \in M - \Sigma$, the sequence $r_{iM_i - \Sigma}^M(p_i)$ consisting of their Riemannian injectivity radii in $M_i - \Sigma$ converges to zero. Otherwise, we say that the sequence $M_i$ does not collapse.

When a convergent sequence of hyperbolic cone-manifolds does not collapse, the limit Alexandrov space must have dimension 3. In this case, several geometric techniques are known to study the topological type of $M$. On the collapsing case, however, most of the geometric information can be lost. This happens because the dimension of the limit Alexandrov space may be strictly smaller than 3 (see [3.4]). Therefore, finding conditions that eliminate the possibility of collapse is very useful.

Denote by $L_{M_i}(\Sigma_j)$ the length of the connected component $\Sigma_j$ of $\Sigma$ in the hyperbolic cone-manifold $M_i$. Using this notation, the principal result of this paper is the following one:

Theorem 1.3. Let $M$ be a closed, orientable and irreducible 3-manifold and let $\Sigma = \Sigma_1 \sqcup \ldots \sqcup \Sigma_l$ be an embedded link in $M$. Suppose the existence of a sequence $M_i$ of hyperbolic cone-manifolds with topological type $(M, \Sigma)$ and having cone angles $\alpha_{ij} \in (0, 2\pi]$. If
\[
\sup \{ L_{M_i}(\Sigma_j) : i \in \mathbb{N} \text{ and } j \in \{1, \ldots, l\} \} < \infty
\]
and the sequence $M_i$ collapses, then $M$ is Seifert fibered or a Sol manifold.

As a consequence of the theorem (1.3), we obtain the following result related to the Thurston’s question (1.1).

Corollary 1.4. Let $M$ be a closed and orientable hyperbolic 3-manifold and suppose the existence of a finite union of simple closed geodesics $\Sigma$ in $M$. Let $M_\alpha$ be a (angle decreasing) deformation of this structure along a continuous path of hyperbolic cone-structures with topological type $(M, \Sigma)$ and having the same cone angle $\alpha \in (\theta, 2\pi] \subset [0, 2\pi]$ for all components of $\Sigma$. If
\[
\sup \{ L_{M_\alpha}(\Sigma_j) : \alpha \in (\theta, 2\pi] \text{ and } j \in \{1, \ldots, l\} \} < \infty,
\]
then every convergent (in the Hausdorff-Gromov sense) sequence $M_{\alpha_i}$, with $\alpha_i$ converging to $\theta$, does not collapses.

If the deformation in the statement of the corollary is supposed to be maximal, we remind that $\theta = 0$ if and only if $\theta \leq \pi$. This is a consequence of Kojima’s work in [Koj].

2. Metric Geometry

Given a metric space $Z$, the metric on $Z$ will always be denoted by $d_Z (\cdot,\cdot)$. The open ball of radius $r > 0$ about a subset $A$ of $Z$ is going to be denoted by

$$B_Z (A, r) = \bigcup_{a \in A} \{ z \in Z : d_Z (z, a) < r \}.$$ 

A metric space $Z$ is called a length space (and its metric is called intrinsic) when the distance between every pair of points in $Z$ is given by the infimum of the lengths of all rectifiable curves connecting them. When a minimizing geodesic between every pair of points exists, we say that $Z$ is complete.

For all $k \in \mathbb{R}$, denote $M^2_k$ the complete and simply connected two dimensional Riemannian manifold of constant sectional curvature equal to $k$. Given a triple of points $(x, y, z)$ of $Z$, a comparison triangle for the triple is nothing but a geodesic triangle $\Delta_k (\overline{x}, \overline{y}, \overline{z})$ in $M^2_k$ with vertices $\overline{x}$, $\overline{y}$ and $\overline{z}$ such that

$$d_{M^2_k} (\overline{x}, \overline{y}) = d_Z (x, y), \ d_{M^2_k} (\overline{y}, \overline{z}) = d_Z (y, z) \text{ and } d_{M^2_k} (\overline{z}, \overline{x}) = d_Z (z, x).$$

Note that a comparison triangle always exists when $k \leq 0$. The $k$-angle of the triple $(x, y, z)$ is, by definition, the angle $\angle_k (x; y, z)$ of a comparison triangle $\Delta_k (\overline{x}, \overline{y}, \overline{z})$ at the vertex $\overline{x}$ (assuming the triangle exists).

**Definition 2.1.** A finite dimensional (in the Hausdorff sense) length space $Z$ is called an Alexandrov space of curvature not smaller than $k \in \mathbb{R}$ if every point has a neighborhood $U$ such that, for all points $x, y, z \in U$, the angles $\angle_k (x; y, z)$, $\angle_k (y; x, z)$ and $\angle_k (z; x, y)$ are well defined and satisfy

$$\angle_k (x; y, z) + \angle_k (y; x, z) + \angle_k (z; x, y) \leq 2\pi.$$

We point out that every hyperbolic cone-manifold is an Alexandrov space of curvature not smaller than $-1$.

Suppose from now on that $Z$ is a $n$ dimensional Alexandrov space of curvature not smaller than $k \in \mathbb{R}$. Consider $z \in Z$ and $\lambda \in (0, \pi)$. The point $z$ is said to be $\lambda$-strained if there exists a set $\{(a_i, b_i) \in Z \times Z : i \in \{1, \ldots, n\}\}$, called a $\lambda$-strainer at $z$, such that $\angle_k (x; a_i, b_i) > \pi - \lambda$ and

$$\max \left\{ \left| \angle_k (x; a_i, a_j) - \frac{\pi}{2} \right|, \left| \angle_k (x; b_i, b_j) - \frac{\pi}{2} \right|, \left| \angle_k (x; a_i, b_j) - \frac{\pi}{2} \right| \right\} < \lambda$$

for all $i \neq j \in \{1, \ldots, n\}$. The set $R_\lambda (Z)$ of $\lambda$-strained points of $Z$ is called the set of $\lambda$-regular points of $Z$. It is a remarkable fact that $R_\lambda (Z)$ is an open and dense subset of $Z$.

Recall now, the notion of (pointed) Hausdorff-Gromov convergence (see [BBI]):

**Definition 2.2.** Let $(Z_i, z_i)$ be a sequence of pointed metric spaces. We say that the sequence $(Z_i, z_i)$ converges in the (pointed) Hausdorff-Gromov sense to a pointed metric space $(Z, z_0)$, if the following holds: For every $r > \varepsilon > 0$, there exist $i_0 \in \mathbb{N}$ and a sequence of (may be non continuous) maps $f_i : B_{Z_i} (z_i, r) \to Z (i > i_0)$ such that

1. $f_i (z_i) = z_0$,
2. $\sup \{ d_Z (f_i (z_1), f_i (z_2)) - d_Z (z_1, z_2) : z_1, z_2 \in Z \} < \varepsilon$, 
3. $\lim_{i \to \infty} d_Z (f_i (z_i), z_0) = 0$. 

This is a consequence of Kojima’s work in [Koj].
iii. \( B_Z(z_0, r - \varepsilon) \subset B_Z(f_i(B_{Z_i}(z_i, r)), \varepsilon) \),
iv. \( f_i(B_{Z_i}(z_i, r)) \subset B_Z(z_0, r + \varepsilon) \).

Its a fundamental fact that the class of Alexandrov spaces of curvature not smaller than \( k \in \mathbb{R} \) is pre-compact with respect to the notion of convergence in the Hausdorff-Gromov sense. In particular, every pointed sequence of hyperbolic cone-manifolds with constant topological type has a subsequence converging (in the Hausdorff-Gromov sense) to a pointed Alexandrov space which may have dimension strictly smaller than 3, although its curvature remains bounded from below by \(-1\).

The Fibration Theorem is an useful tool for the study of sequences of Alexandrov spaces. The first version of this theorem, due to Fukaya (see \( \text{Fuk} \)), concerns the Riemannian manifolds with pinched curvature. Later this theorem was extended by Takao Yamaguchi to Alexandrov Spaces (see \( \text{Yam} \), \( \text{SY} \) and \( \text{Bel} \)). For the convenience of the reader, we present here the precise statement of the Yamaguchi theorem that is going to be needed in this work (see \( \text{Bar} \)):

**Theorem 2.3** (Yamaguchi). Let \((M_i, p_i)\) be a sequence of pointed complete Riemannian manifolds of dimension 3 with sectional curvature bounded from below by \(-1\). Suppose that the sequence \((M_i, p_i)\) converges in the Hausdorff-Gromov sense to a pointed complete length space \((Z, z)\) of dimension 2 (resp. dimension 1), then there exists a constant \( \lambda > 0 \) satisfying the following condition: For every compact domain \( Y \) of \( Z \) contained in \( R_\lambda(Z) \) and for sufficiently large \( i_0 = i_0(Y) \in \mathbb{N} \), there exist

- a sequence \( \tau_i > 0 \) (\( i > i_o \)) converging to zero,
- a sequence \( N_i \) (\( i > i_o \)) of compact 3-submanifolds of \( M_i \) (perhaps with boundary)
- a sequence of \( \tau_i \)-approximations \( p : N_i \rightarrow Y \) which induces a structure of locally trivial fibre bundle on \( N_i \). Furthermore, the fibers of this fibration are circles (resp. spheres, tori).

### 3. Sequences of Hyperbolic cone-manifolds

Recall that \( M \) denotes a closed, orientable and irreducible differential manifold of dimension 3 and that \( \Sigma = \Sigma_1 \sqcup \ldots \Sigma_l \) denotes an embedded link in \( M \). A sequence of hyperbolic cone-manifolds with topological type \((M, \Sigma)\) will always be denoted by \( M_i \).

Given a sequence \( M_i \) as above, fix indices \( i \in \mathbb{N} \) and \( j \in \{1, \ldots, l\} \). For sufficiently small radius \( R > 0 \), the metric neighborhood

\[
B_{M_i}(\Sigma_j, R) = \{ x \in M_i : d_{M_i}(x, \Sigma_j) < R \}
\]

of \( \Sigma \) is a solid torus embedded in \( M_i \). The supremum of the radius \( R > 0 \) satisfying the above property will be called normal injectivity radius of \( \Sigma_j \) in \( M_i \) and it is going to be denoted by \( R_i(\Sigma_j) \). Analogously we can define \( R_i(\Sigma) \), the normal injectivity radius of \( \Sigma \). It is a remarkable fact (see \( \text{Fu} \) and \( \text{HK} \)) that the existence of a uniform lower bound for \( R_i(\Sigma) \) ensures the existence of a sequence of points \( p_{ik} \in M_i \) such that the sequence \((M_{ik}, p_{ik})\) converges in the Hausdorff-Gromov sense to a pointed hyperbolic cone-manifold \((M_\infty, p_\infty)\) with topological type \((M, \Sigma)\). Moreover, \( M_\infty \) must be compact provides that cone angles of \( M_{ik} \) are uniformly bounded from below.

The purpose of this section is to prove the Theorem 1.3. This section is divided into two parts. The first part contains some preliminary results whereas the remaining one presents the proof of the theorem.

Let us point out that, throughout the rest of the paper, the term "component" is going to stand for "connected component"
3.1. Preliminary results. Let us begin with three elementary lemmas which will be important for the proof of Theorem 1.3.

**Lemma 3.1.** Suppose that \( M - \Sigma \) is hyperbolic and let \( T \) be a two dimensional torus embedded in \( M - \Sigma \). Then \( T \) separates \( M \). Moreover, one and only one of the following statements holds:

i. \( T \) is parallel to a component of \( \Sigma \) (hence it bounds a solid torus in \( M \)),

ii. \( T \) is not parallel to a component of \( \Sigma \) and it bounds a solid torus in \( M - \Sigma \),

iii. \( T \) is not parallel to a component of \( \Sigma \) and it is contained in a ball \( B \) of \( M - \Sigma \). Furthermore, \( T \) bounds a region in \( B \) which is homeomorphic to the exterior of a knot in \( S^3 \).

**Proof.** Suppose that \( T \) is not parallel to a connected component of \( \Sigma \). Since \( M - \Sigma \) is hyperbolic, \( T \) is compressible in \( M - \Sigma \) and hence it splits \( M \) into two components, say \( U \) and \( V \). Denote by \( \Sigma_U \) and by \( \Sigma_V \) the subsets of \( \Sigma \) contained respectively in \( U \) and in \( V \). Without loss of generality suppose \( \Sigma_U \neq \emptyset \).

Let \( D \) be a compression disk for \( T \). Since \( M - \Sigma \) is irreducible and \( \Sigma_U \neq \emptyset \), the sphere obtained by the compression of \( T \) along \( D \) bounds a ball \( B \subset M - \Sigma \) in the opposite side of \( \Sigma_U \). In particular, this implies that \( \Sigma_U = \emptyset \), since \( \Sigma_U \) and \( \Sigma_V \) lie in different sides of \( S \), by construction. Thus \( V \) must satisfy one of the assertions (ii) or (iii) according to whether \( D \subset V \) or \( D \subset U \). \( \square \)

**Lemma 3.2.** Given a closed ball \( B \subset M - \Sigma \) and an embedded two dimensional torus \( T \subset B \), let \( U \) and \( W \) be the compact three manifolds (with toral boundary) obtained by surgery of \( M \) along \( T \) (suppose \( W \subset B \)). If \( W \) is homeomorphic to the exterior of a knot in \( S^3 \), then we can replace \( W \) by a solid torus \( V \) without changing the topological type of \( M \).

**Proof.** Let \( B' \) be a closed ball and \( h : \partial B \rightarrow \partial B' \) a diffeomorphism such that \( B \cup_h B' \), the manifold obtained by gluing \( B \) and \( B' \) through \( h \), is diffeomorphic to \( S^3 \). Note that the manifold \( V' = (U \cap B) \cup_h B' \) is a closed solid torus. Let \( V \) be a closed solid torus and \( g : \partial V \rightarrow \partial V' \) a diffeomorphism such that \( V \cup_g V' \), the manifold obtained by gluing \( V \) and \( V' \) through \( g \), is diffeomorphic to \( S^3 \). Thus the compact three manifold \( B = (V' \cup_g V) - \text{int} (B') = (U \cap B) \cup_g V \) is diffeomorphic to a closed ball.

It is a standard result of low dimensional topology that the manifold

\[
(M - \text{int} (B)) \cup_{Id} B = (M - W) \cup_g V
\]

obtained by using \( Id : \partial B \rightarrow \partial B \) as gluing map, is diffeomorphic to \( M \). \( \square \)

**Lemma 3.3.** Consider \( \varepsilon \in (0,1) \) and let \( \mathcal{M} \) be a hyperbolic cone-manifold with topological type \((M, \Sigma)\) together with a metric neighborhood \( V \) of \( \Sigma \) whose components are homeomorphic to solid tori. Denote by \( g \) the hyperbolic metric on \( M - \Sigma \) and let \( h \) be a Riemannian metric on \( M \) such that:

i. \( h \) coincides with \( g \) on \( M - V \),

ii. \( h \) is a \( \varepsilon \)-perturbation of \( g \) on \( M - \Sigma \), i.e.

\[
\left\| v \right\|_g - \left\| v \right\|_h < \varepsilon,
\]

for every \( x \in M - \Sigma \) and \( v \in K_x = \left\{ u \in T_x (M - \Sigma) : \left\| u \right\|_g \leq 2 \right\} \).

Then the Hausdorff-Gromov distance between \( \mathcal{M} \) and the Riemannian manifold \( \mathcal{N} = (M, h) \) is smaller than \( 4\varepsilon D \), where \( D = 1 + \text{diam}_\mathcal{M} (M) \).
Proof. To prove the lemma, it suffices to check that the identity map $Id : \mathcal{M} \rightarrow \mathcal{N}$ is a $2\varepsilon D$-isometry. Fix $x, y \in M$. Since $\Sigma$ has codimension 2, there exists a differentiable curve $\alpha : [0, L] \rightarrow \mathcal{M}$ (parametrized by arc length in $\mathcal{M}$) such that $\alpha((0, L)) \subset \mathcal{M} - \Sigma$ and $L < d_\mathcal{M}(x, y) + \frac{\varepsilon}{2}$. Then

$$d_\mathcal{N}(x, y) - d_\mathcal{M}(x, y) \leq L_\mathcal{N}(\alpha) - L_\mathcal{M}(\alpha) + \frac{\varepsilon}{2}$$

$$\leq \int_0^L \|\alpha'(t)\|_g - \|\alpha'(t)\|_h \, dt + \frac{\varepsilon}{2}$$

$$< \varepsilon d_\mathcal{M}(x, y) + \frac{\varepsilon^2}{2} + \frac{\varepsilon}{2} \leq \varepsilon D$$

and, in particular, $\text{diam}_\mathcal{N}(M) \leq 2D$.

Similarly, there exists a differentiable curve $\beta : [0, L'] \rightarrow \mathcal{N}$ (parametrized by arc length in $\mathcal{N}$) such that $\beta((0, L')) \subset \mathcal{N} - \Sigma$ and $L' < d_\mathcal{N}(x, y) + \frac{\varepsilon}{2}$. Inequality (3.1) implies that

$$\left\| \beta'(t) \right\|_g - \left\| \beta'(t) \right\|_h < \varepsilon,$$

for all $t \in [0, L']$. Analogously to the preceding case it then follows that $d_\mathcal{M}(x, y) - d_\mathcal{N}(x, y) \leq 2\varepsilon D$.

3.2. Proof of the Theorem (1.3). The purpose of this section is to study a collapsing sequence $M_i$. According to [Fuj] theorem 1, the sequence $M_i$ cannot collapse when $\lim_{i \rightarrow \infty} R_i(\Sigma_j) = \infty$, for all components $\Sigma_j$ of $\Sigma$. Given $p \in \Sigma_1$, we can assume without loss of generality that $\sup \{R_i(\Sigma_1) : i \in \mathbb{N}\} < \infty$ and that the sequence $(M_i, p)$ converges in the Hausdorff-Gromov sense to a pointed Alexandrov space $(Z, z_0)$. We are interested in the case where the length of the singularity remains uniformly bounded, i.e. where

$$\sup \{L_{M_i}(\Sigma_j) : i \in \mathbb{N}, j \in \{1, \ldots, l\}\} < \infty.$$

Again an Ascoli-type argument implies (passing to a subsequence if necessary) that each component $\Sigma_j$ of $\Sigma$ satisfies one, and only one, of the following statements:

1. $\sup \{d_{M_i}(p, \Sigma_j) : i \in \mathbb{N}\} < \infty$ and $\Sigma_j$ converges in the Hausdorff-Gromov sense to a closed curve $\Sigma_j^Z \subset Z$;
2. $\lim_{i \in \mathbb{N}} d_{M_i}(p, \Sigma_j) = \infty$.

This dichotomy allows us to write $\Sigma = \Sigma_0 \cup \Sigma_\infty$, where $\Sigma_0$ contains the components $\Sigma_j$ of $\Sigma$ which satisfies item (1) (in particular $\Sigma_1 \subset \Sigma_0$) and $\Sigma_\infty$ those that satisfies the item (2). Now consider the compact set

$$\Sigma Z = \bigcup_{\Sigma_j \subset \Sigma_0} \Sigma_j^Z \subset Z.$$

The following proposition provides information on the dimension of the limit of a sequence of hyperbolic cone-manifolds which collapses.

Lemma 3.4. Given a sequence $p_i \in M$, suppose that the sequence $(M_i, p_i)$ converges in the Hausdorff-Gromov sense to a pointed Alexandrov space $(Z, z_0)$. If the sequence $M_i$ collapses and verifies

$$\sup \{L_{M_i}(\Sigma_j) : i \in \mathbb{N}, j \in \{1, \ldots, l\}\} < \infty,$$

then the dimension of $Z$ is strictly smaller than 3.
Proof. We can assume that there exists $z_\infty \in Z - \Sigma Z$, since the equality $\Sigma Z = Z$ can only happen when the dimension of $Z$ is strictly smaller than 3. Set $\delta = d_{Z} (z_\infty, \Sigma Z) > 0$ and take a sequence $q_i \in M - \Sigma$ such that

$$\lim_{i \to \infty} q_i = z_\infty \quad \text{and} \quad B_{M_i} \left( q_i, \frac{\delta}{3} \right) \cap B_{M_i} \left( \Sigma, \frac{\delta}{3} \right) = \emptyset.$$  

By definition, the sequence $(M_i, q_i)$ converges in the Hausdorff-Gromov sense to $(Z, z_\infty)$. Since the sequence $M_i$ collapses, we have $\lim_{i \to \infty} r_{M_i}^{M_j} (q_i) = 0$.

According to [HK] Lemma 3.8] we can change the metrics of $M_i$ on $B_{M_i} \left( \Sigma, \frac{\delta}{3} \right) - \Sigma$ to obtain a sequence of complete Riemannian manifolds $N_i$ homeomorphic to $M - \Sigma$ which have finite volume and pinched sectional curvature. Without loss of generality, we can assume that the sequence $(N_i, q_i)$ converges in the Hausdorff-Gromov sense to a pointed Alexandrov space $(Y, y_\infty)$. Note that, by construction, the balls $B_{N_i} \left( q_i, \frac{\delta}{3} \right)$ and $B_{M_i} \left( q_i, \frac{\delta}{3} \right)$ are isometric. Then the balls $B_Y \left( y_\infty, \frac{\delta}{3} \right)$ and $B_Z \left( z_\infty, \frac{\delta}{3} \right)$ are also isometric and we have $\lim_{i \to \infty} r_{N_i}^{N_j} (q_i) = 0$.

According to [Gro] Corollary 8.39, $B_Z \left( z_\infty, \frac{\delta}{3} \right)$ has Hausdorff dimension strictly smaller than 3. Given that the dimension of an Alexandrov space is a well-defined integer, it follows that $Z$ also has dimension strictly smaller than 3. \qed

The preceding lemma says that the only possibilities for the dimension of $Z$ are 2, 1 or 0. The proof of Theorem 1.3 will accordingly be split in three cases.

3.2.1. Case where $Z$ has dimension 2. It is a classical result of metric geometry that all two dimensiona1 Alexandrov spaces are topological manifolds of dimension 2, possibly with boundary. We can describe the boundary of $Z$ as

$$\partial Z = \bigsqcup_{k \in \Gamma} \partial Z_k$$

where $\partial Z_k$ are the connected components of $\partial Z$. Note that each component $\partial Z_k$ is homeomorphic to a circle or to a straight line.

Theorem 3.5 (collapsing - dimension 2). Suppose the existence of a point $p \in \Sigma_1$ such that the sequence $(M_i, p)$ converges (in the Hausdorff-Gromov sense) to a two dimensional pointed Alexandrov space $(Z, z_0)$ and also that $\sup \{ R_i (\Sigma_1) : i \in \mathbb{N} \} < \infty$. Assuming that

$$\sup \{ L_{M_i} (\Sigma_j) : i \in \mathbb{N}, j \in \{ 1, \ldots, \ell \} \} < \infty,$$

it follows that:

i. $M$ is Seifert fibered,

ii. if $\partial Z \neq \emptyset$, then $\partial Z$ has only one connected component and $M$ is a lens space,

iii. if $Z$ is not compact, then $\lim_{i \in \mathbb{N}} R_i (\Sigma_j) = \infty$, for any component $\Sigma_j$ of $\Sigma_\infty$.

Proof. According to [Sal], the metric of hyperbolic cone-manifolds $M_i$ in some small neighborhoods of $\Sigma$ can be deformed to yield a sequence of complete Riemannian manifolds $N_i$ (homeomorphic to $M$) with sectional curvature not smaller than $-1$. Furthermore thanks to Lemma (3.3), we can suppose that the sequence $(N_i, p)$ converges (in the Hausdorff-Gromov sense) to $(Z, z_0)$.

If $Z$ is compact, the statement follows directly from the results of Shioya and Yamaguchi in [SY]. To be more precise, when $Z$ does not have boundary, [SY] Theorem 0.2] applied to the sequence $N_i$ gives that $M$ is Seifert fibered. When $Z$ has boundary, [SY] Corollary 0.4] applied to the sequence $N_i$ implies (M is irreducible) that $Z$ is a closed disk with at most one cone in its
interior. Furthermore, [SY] Theorem 0.3 implies that $M$ is a lens space (in particular it is also Seifert fibered).

Suppose from now on that $Z$ is not compact. Since the normal injectivity radius of components of $\Sigma_0$ are uniformly bounded, there exists $R > 0$ such that

$$B_{M_i} (\Sigma_j, R (\Sigma_j)) \subset B_{M_i} (p, R/2)$$

for all $i \in \mathbb{N}$ and for every component $\Sigma_j \subset \Sigma_0$. Let $K$ be a compact and connected two dimensional submanifold of $Z$ such that $B_Z (z_0, R) \subset K$ (therefore $\Sigma_Z \subset K$), $\partial K$ is a disjoint union of circles and $Z - K$ is a disjoint union of components of infinite diameter.

Set $\Lambda = \{ k \in \Gamma : \partial Z_k \cap K \neq \emptyset \}$. If $\partial Z \neq \emptyset$, we assume also that:

- $\Lambda \neq \emptyset$, that is $K \cap \partial Z \neq \emptyset$,
- $\partial Z_k \cap K = \partial Z_k$, for all $k \in \Lambda$ such that the component $\partial Z_k$ is compact,
- $\partial Z_k \cap K$ is connected, for all $k \in \Lambda$.

For each $k \in \Lambda$, denote by $\partial K_k$ the (unique by construction) connected component of $\partial K$ such that $\partial Z_k \cap \partial K_k \neq \emptyset$. The boundary of $K$ is given by

$$\partial K = \bigcup_{k \in \Lambda} \partial K_k \cup \bigcup_{m \in \Gamma - \Lambda} \partial K_m ,$$

where $\partial K_m$ are the components of $\partial K$ which does not intersect the boundary of $Z$.

Let $\lambda$ be the constant given by the Fibration Theorem [2.3]. Since $K$ is compact, the sets of $\lambda$-cone-points

$$C = \{ z \in K - \partial Z ; \mathcal{L} (T_z Z) \leq 2\pi - \lambda \}$$

$$C_k = \{ z \in \partial Z_k \cap K ; \mathcal{L} (T_z Z) \leq \pi - \lambda \} \subset \partial K_k \quad (k \in \Lambda)$$

are finite (cf. [SY] Theorem 2.1]). Modulo increasing $K$, we can assume that $C$ is contained in its interior. Let $s_1 \gg s_2 > 0$ be such that

- $B_K [z, s_1] \subset \text{int} (K)$ and $B_K [z, s_1]$ is homeomorphic to $D^2$, for all $z \in C$,
- $B_K [z, s_1] \cap B_K [z', s_1] = \emptyset$, for all $z, z' \in C \cup \bigcup_{k \in \Lambda} C_k$,
- For all $z \in C \cup \bigcup_{k \in \Lambda} C_k$ and for all $k \in \Lambda$, we have

$$B_K (\partial K_k, s_2) \cap B_K (z, s_1) \neq \emptyset \quad \text{if and only if} \quad z \in C_k.$$

Consider the compact subset $Y$ of $Z$ defined by

$$Y = K - \bigcup_{z \in C} B_K (z, s_1) - \bigcup_{k \in \Lambda} U_k \subset R_\lambda (Z) ,$$

where $U_k = B_K (\partial Z_k, s_2) \cup \bigcup_{z \in C_k} B_K (z, s_1)$, for every $k \in \Lambda$. The boundary of $Y$ is given by

$$\partial Y = \bigcup_{k \in \Lambda} \partial Y_k \cup \bigcup_{z \in C} \partial Y_z \cup \bigcup_{m \in \Gamma - \Lambda} \partial K_m ,$$

where $\partial Y_k$ and $\partial Y_z$ are the components of $\partial Y$ which intersect respectively the regions $U_k$ and $B_K (z, s_1)$.

Without loss of generality, the Fibration Theorem [2.3] gives us a sequence $\mathcal{N}_i$ of compact and connected three dimensional submanifolds of $M$, a sequence $\tau_i > 0$ that converges to zero and a sequence of $\tau_i$-approximations $p_i : \mathcal{N}_i \to Y$ which induces a structure of locally trivial circle bundle on $\mathcal{N}_i$. Since the manifold $M$ is orientable, it follows that the connected components of $\partial \mathcal{N}_i = p_i^{-1} (\partial Y)$ are two dimensional tori.
Let $C$ be a component of $\partial Y$ and $T_i$ the component of $\partial N_i$ associated with it $(T_i = p_i^{-1}(C))$. By construction, we can choose base points $q_i \in T_i$ and $q \in C$ such that the pointed components $(B_i, q_i)$ of $N_i - \text{int}(N_i)$ associated with the torus $T_i$ $(\partial B_i = T_i)$, converges in the Hausdorff-Gromov sense to the pointed component $(B, q)$ of $Z - \text{int}(Y)$ associated to $C$ $(\partial B = C)$.

Figure 1.

The above paragraph shows that the bijection between the boundary components of $Z - Y$ and those of $N_i - N_i$ induces a bijection between the components of $Z - Y$ and those of $N_i - N_i$. Moreover, the diameter of the components of $N_i - N_i$ associated with a component $B$ of $Z - Y$ goes to infinity if and only if $B$ is noncompact.

Without loss of generality, we can suppose that $\Sigma_\infty \subset N_i - N_i$, for all $i \in \mathbb{N}$. It follows from lemma 3.1 that the singular components of $N_i - N_i$ are solid tori whose souls are components of $\Sigma_\infty$. Moreover, as the boundary components of $N_i$ remain a finite distance from the base point, the normal injectivity radii of the components of $\Sigma_\infty$ became infinite with $i$. Note that this implies that the singular components of $N_i - N_i$ must be associated with noncompact components of $Z - Y$.

**Lemma 3.6.** There exists $i_0 \in \mathbb{N}$ such that, for every $i > i_0$, the components of $N_i - N_i$ are homeomorphic to either a solid torus or to the exterior of a knot in $S^3$ which, in addition, is contained in an embedded ball in $M - \Sigma$.

**Proof of Lemma 3.6:** A boundary torus of $N_i - N_i$ has 3 possible natures according to the decomposition (3.2) of $\partial Y$. We shall consider separately each of these possibilities.

1st type: The torus $T_i^z = p_i^{-1}(\partial Y_z) \subset \partial N_i$, $z \in C$.

Consider a point $z \in C$. By rescaling the metric with respect to the length of the fibers and using the stability theorem of Perelman (cf. [Kap]), Shiota and Yamaguchi show in [SY, theorem 0.2] that the region $B_i^z$ of $N_i - N_i$ bounded by the torus $T_i^z$ is a solid torus, for sufficiently large $i$. Furthermore, the solid torus $B_i^z$ is glued on $N_i$ without "killing" the fiber.
2nd type: The torus $T^k_i = p_i^{-1}(\partial Y_k) \subset \partial N_i$, $k \in \Lambda$.

Given an index $k \in \Lambda$, we can decompose the component $\partial Y_k$ in two simple arcs $\alpha$ and $\beta$ with the same ends, say $P$ and $Q$, and such that $\alpha = U_k \cap \partial Y_k$. Note that $\alpha = \partial Y_k$ and $\beta$ is reduced to a single point (in particular $P = Q$) when $\partial Z_k$ is compact.

Let $B^k_i$ be the region of $N_i - N_i$ bounded by the torus $T^k_i$. Using the same techniques, [SY, Theorem 0.3] shows that we can choose, for sufficiently large $i$, a region $B^\alpha_i \subset B^k_i$ which is homeomorphic to $\alpha \times D^2$. Moreover:

- the sequence formed by the regions $B^\alpha_i$ converges in the Hausdorff-Gromov sense to the region $U_k$,
- the circle $\{z\} \times \partial D^2$ is a fiber of the Seifert fibration on the boundary of $N_i$, for every $z \in \alpha$.

Then, for sufficiently large $i$, the regions $B^k_i$ are homeomorphic to solid tori provided that $\partial Z_k$ is compact. Suppose now that $\partial Z_k$ is not compact.

Let $D^P_i$ and $D^Q_i$ be the disks of $B^\alpha_i$ which are glued respectively on the fibers $p^{-1}(P)$ and $p^{-1}(Q)$. The fibers above $\beta$ together with the disks $D^P_i$ and $D^Q_i$ yield an embedded sphere in $N_i - \Sigma$. This sphere splits $N_i$ into two regions: the region $B^\beta_i = B^k_i - \text{int}(B^\alpha_i)$ and the region $N_i - \text{int}(B^\beta_i)$ which contain the singular set $\Sigma_0 \neq \emptyset$. Since the manifold $N_i - \Sigma$ is irreducible, the region $B^\beta_i$ must be homeomorphic to a ball and therefore, for sufficiently large $i$, the region $B^k_i = B^\alpha_i \cup B^\beta_i$ is homeomorphic to a solid torus.

3rd type: The torus $T^m_i = p^{-1}(\partial K_m) \subset \partial N_i$, $m \in \Gamma - \Lambda$.

Given an index $m \in \Gamma - \Lambda$, let $B^m_i$ be the region of $N_i - N_i$ bounded by the torus $T^m_i$. Since $B^m_\infty(z_0, R) \subset K$, the torus $T^m_i$ is contained in $M - \Sigma$ and cannot be parallel to a component of $\Sigma_0$. According to Lemma (3.1), the region $B^m_i$ is homeomorphic either to a solid torus (possibly having a component of $\Sigma_\infty$ as its soul) or to the exterior of a knot in $S^3$ which is contained in an embedded ball in $M - \Sigma$.  

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{torus_diagram.pdf}
\caption{Tori of second type}
\end{figure}
Fix from now on a index $i > i_0$. According to Lemma (3.2), the preceding claim shows that all components of $N_i - N_i^t$ can be supposed to be homeomorphic to solid tori. Let $W_i$ be the three manifold obtained by the union of $N_i$ with all torus components of $N_i - N_i^t$ which are glued to the boundary of $N_i$ without "killing" the fibers of the Seifert fibration. For example, $B^i_z \subset W_i$ and $B^i_k \subset N_i - W_i$, for all $z \in C$ and $k \in \Lambda$.

The Seifert fibration on $N_i$ (with base $Y$) naturally extends to a Seifert fibration on all off $W_i$. Moreover, the underlying space of its basis is the topological surface $Y$ obtained by gluing disks on the components of $\partial Y$ associated with the components of $W_i - N_i$. If $W_i = N_i$ (so that $\partial Y = \emptyset$), the manifold $M$ is Seifert fibered and $\partial Z = \emptyset$. Thus let us assume from now on that $W_i \neq N_i$ (so that $\partial Y \neq \emptyset$ and $\partial W_i \neq \emptyset$).

Suppose the existence of an essential arc $\gamma$ properly immersed in $Y$. Then the fibration above $\gamma$ provides an essential annulus embedded in $W_i$. Since all the components of $N_i - W_i$ are solid torus that are glued to $\partial W_i$ so as to "kill" the fibers of the Seifert fibration, this essential annulus becomes an essential sphere in $N_i$. Therefore $Y$ has non-negative Euler characteristic, it has exactly one boundary component and it contains at most one interior cone point. In particular, it follows that $Z$ has at most one boundary component and at most one interior cone point. In fact, if $Z$ has two or more interior cone points (boundary components), then the compact $K$ can be chosen so as to contains more than one interior cone point (boundary components) and this would contradict to the non existence of a second interior cone point (boundary components) in $Y$.

Because $Y$ is compact, the discussion above implies that $Y$ is homeomorphic to a disk $D^2$. So $W_i$ is homeomorphic to a solid torus and thus that $M$ is a lens space.

3.2.2. Case where $Z$ has dimension 1. It is also a standard result of metric geometry that all one dimensional Alexandrov spaces are topological manifolds of dimension 1. Precisely, they are homeomorphic to on of the following models: $S^1$, $[0,1]$, $[0,\infty)$ or $(-\infty, \infty)$.

**Theorem 3.7** (collapsing - dimension 1). Suppose the existence of a point $p \in \Sigma_1$ such that the sequence $(M_i, p)$ converges (in the Hausdorff-Gromov sense) to a one dimensional pointed Alexandrov space $(Z, z_0)$ and also that $\sup \{L(M, \Sigma_j) : i \in \mathbb{N} \} < \infty$. Assuming that

$$\sup \{ L(M, \Sigma_j) : i \in \mathbb{N}, j \in \{1, \ldots, l\} \} < \infty,$$

it follows that:

i. $Z$ is homeomorphic to $S^1$ and $M$ is a euclidean, Nil or Sol manifold,

ii. $Z$ is homeomorphic to $[0,1]$ and $M$ is a lens space or a euclidean, Nil, Sol or prism manifold,

iii. $Z$ is homeomorphic to $[0,\infty)$ or $(-\infty, \infty)$, $M$ is a lens space or prism manifold and

$$\lim_{i \in \mathbb{N}} R_i(\Sigma_j) = \infty,$$

for every component $\Sigma_j$ of $\Sigma_{\infty}$.

**Proof.** If $Z$ is compact (i.e. homeomorphic to $S^1$ or $[0,1]$), then the assertions above follow directly from the results of Shioya and Yamaguchi in [SY]. To be more precise, when $Z$ is homeomorphic to $S^1$, the irreducibility of $M$ implies that it is an euclidean, Nil or Sol manifold (see remark before Theorem 0.5 and table 1 of [SY]). When $Z$ is homeomorphic to $[0,1]$, the irreducibility of $M$ implies that it is a lens space or a euclidean, Nil, Sol or prism manifold (cf. [SY] table 1).

Suppose from now on that $Z$ is not compact, i.e. that it is homeomorphic to $[0,\infty)$ or $(-\infty, \infty)$. According to [Sal], the metric of hyperbolic cone-manifolds $M_i$ on some small neighborhoods of $\Sigma$ can be deformed to yield a sequence of complete Riemannian manifolds $N_i$ (homeomorphic to $M$)
with sectional curvature not smaller than \(-1\). Furthermore thanks to Lemma (3.3), we can suppose that the sequence \(\{N_i, p\}\) converges (in the Hausdorff-Gromov sense) to \((Z, z_0)\).

Since \(\sup \{R_i(\Sigma_j) : i \in \mathbb{N}, \Sigma_j \in \Sigma_0\} < \infty\), there exists \(R > 0\) such that

\[
B_{M_i}(\Sigma_j, R) \subset B_{M_i}(p, R/2)
\]

for all \(i \in \mathbb{N}\) and for every component \(\Sigma_j\) of \(\Sigma_0\). Consider a compact \(K\) of \(Z\) which is homeomorphic to \([0, 1]\) and contains \(B_Z(z_0, R) \cup \partial Z\).

Let \(\lambda\) be the constant given by the Fibration Theorem (2.3) and set \(Y = K - U\), where \(U\) is empty or a small open neighborhood of \(\partial Z\) according to whether or not the boundary of \(Z\) is empty.

Without loss of generality, the Fibration Theorem (2.3) gives us a sequence \(N_i\) of compact and connected three dimensional submanifolds of \(M\), a sequence \(\tau_i > 0\) that converges to zero and a sequence of \(\tau_i\)-approximations \(p_i : N_i \rightarrow Y\) which induces a structure of locally trivial bundle on \(N_i\) whose fibers are either two dimensional spheres or tori.

**Lemma 3.8.** The fibers of \(p_i : N_i \rightarrow Y\) are tori.

**Proof of Lemma (3.8):** Since the fibers are either spheres or tori, it suffices to rule out the first possibility. Looking for a contradiction, let us suppose that the fibers are spheres. Given \(z_1 \in Z - B\) and a positive constant \(\delta < \text{diam}_Z(K)\), consider a sequence of points \(q_i \in N_i - \Sigma\) converging (in the Hausdorff-Gromov sense) to \(z_1\) and such that \(d_{HG}(q_i, z_1) < \delta\) for all \(i \in \mathbb{N}\).

**Claim:** There exists \(i_0 \in \mathbb{N}\) such that, for every \(i > i_0\), we can find a homotopically nontrivial loop \((M - \Sigma) \gamma_i\) based on \(q_i\) whose length does not exceed \(\delta\).

**Proof of Claim:** Consider the loops based on \(q_i\) which are constituted by two minimizing geodesic segments with same ends and having equal lengths bounded by \(\frac{\delta}{2}\). Note that these loops are always homotopically nontrivial.

To say that a point \(q_i\) is not a base point for a loop as above is equivalent to say that its injectivity radius not less than \(\frac{\delta}{2}\). This is a contradiction with the collapsing assumption and therefore the claim follows. \(\diamondsuit\)

Now fix \(i > i_0\). Consider a boundary sphere of \(N_i\) associates with an unbounded component of \(Z - K\). By construction, this sphere separates \(M\) in two components: one of these components contains the singular link \(\Sigma_0 \neq \emptyset\) (recall that \(\Sigma_1 \subset \Sigma_0\) by hypothesis) and the other contains a homotopically nontrivial loop as in the claim. Since \(M - \Sigma\) is irreducible, this cannot happen and therefore the fibers must be solid tori. This establishes the lemma. \(\diamondsuit\)

As in the two dimensional case, the component(s) of \(N_i - N_i\) associated with unbounded components of \(Z - Y\) are solid tori. When \(Z\) is homeomorphic to \([0, \infty)\), Shioya and Yamaguchi showed (see [SY] Theorem 0.5 and Table 1) that the component of \(N_i - N_i\) associated with the bounded component of \(Z - Y\) is homeomorphic to either a solid torus or to \(M \tilde{\otimes} S^1\). It follows that being irreducible \(M\) is according homeomorphic to a lens space or a prism manifold. \(\square\)

### 3.2.3. Case where \(Z\) has dimension 0.

This case was treated in [SY]. For the convenience of the reader, we state here their result:

**Theorem 3.9** (collapsing - dimension 0). Suppose the existence of a point \(p \in \Sigma_1\) such that the sequence \((M_i, p)\) converges (in the Hausdorff-Gromov sense) to a zero dimensional pointed Alexandrov space \((Z, z_0)\) and also that \(\sup \{R_i(\Sigma_1) : i \in \mathbb{N}\} < \infty\). Then \(M\) is an euclidean, spherical or Nil manifold.
4. Applications

4.1. Proof of Corollary (1.4).

Proof. Let $M_\alpha$ be a convergent sequence as in the statement of the corollary. Since $M$ is a hyperbolic manifold, it is not Seifert fibered or a Sol manifold. Therefore, in the presence of the hypothesis (1.2), is a consequence of the Theorem (1.3) that the sequence $M_\alpha$, cannot collapses. □

4.2. Volume of Representations. Recall that $M$ denotes a closed, orientable (not necessarily irreducible) differential manifold of dimension 3 and that $\Sigma = \Sigma_1 \sqcup \ldots \sqcup \Sigma_k$ denotes an embedded link in $M$. Let us recall the definition of the volume of representations and some of its properties. For the interested reader, we refer to [Fra] and [Dun] for details.

Let $U = U_1 \sqcup \ldots \sqcup U_k$ be an open neighborhood of $\Sigma$, where each $U_i$ is a neighborhood of $\Sigma_i$ homeomorphic to a solid torus. Consider product structures $(T^2 \times [0, \infty))_i$ on the open sets $U_i = U_1 - \Sigma$ called cuspidal ends of $M - \Sigma$. We will denote the universal cover of $M - \Sigma$ by $\pi : \widetilde{M - \Sigma} \rightarrow M - \Sigma$. The connected components of $\pi^{-1}(U_i)$ are denoted by $V_{ij} \approx (\pi^{-1}(T^2) \times [0, \infty))_{ij}$.

Definition 4.1. Given a representation $\rho : \pi_1(M - \Sigma) \rightarrow PSL_2(\mathbb{C})$, a positive piecewise differentiable map $d : \Sigma \rightarrow \mathbb{H}^3$ is called a pseudo-developing map for $\rho$ if it verify:

i. (equivariance) $d \circ \gamma = \rho(\gamma) \circ d$, for all $\gamma \in \pi_1(M - \Sigma)$

ii. There is $r_0 \in [0, \infty)$ such that, for every $(p, r_0)_{ij} \in V_{ij}$, the curve 

$$ t \in [r_0, \infty) \mapsto d\left((p, t)_{ij}\right) \in \mathbb{H}^3 $$

is a geodesic of $\mathbb{H}^3$ parameterized by arc length. In addition, there are points $\zeta_{ij} \in \partial \mathbb{H}^3$ (depending only on the indexes $i$ and $j$), such that

$$ \lim_{t \rightarrow \infty} d\left((p, t)_{ij}\right) = \zeta_{ij} \in \partial \mathbb{H}^3. $$

Given a pseudo-developing map for $\rho$, let $\omega_d$ be the unique differential form of degree 3 on $M - \Sigma$ verifying

$$ \pi^* \omega_d = d^* V_{\mathbb{H}^3}, $$

where $V_{\mathbb{H}^3}$ denotes the volume form of $\mathbb{H}^3$. The volume of $\rho$ is defined by

$$ Vol_{M - \Sigma}(\rho) = \int_{M - \Sigma} \omega_d = \int_R d^* V_{\mathbb{H}^3} < \infty, $$

where $R \subset \widetilde{M - \Sigma}$ is a fundamental region for the action of $\pi_1(M - \Sigma)$ on $\widetilde{M - \Sigma}$.

Given a representation $\rho : \pi_1(M - \Sigma) \rightarrow PSL_2(\mathbb{C})$, note that a pseudo-developing map for $\rho$ always exists. Moreover, the volume of $\rho$ does not depend on the chosen pseudo-developing map. The following result presents some properties of the volume function

$$ Vol_{M - \Sigma} : R(M - \Sigma) \longrightarrow \mathbb{R}^+, $$

where $R(M - \Sigma)$ denotes the set of all representations of $\pi_1(M - \Sigma)$ in $PSL_2(\mathbb{C})$ with its canonical structure of algebraic variety.

Proposition 4.2. The function $Vol_{M - \Sigma}$ is continuous and verifies the following properties:

i. $Vol_{M - \Sigma}(\rho) = 0$, for any reducible representation $\rho \in R(M - \Sigma)$. Recall that a representation $\rho \in R(M - \Sigma)$ is called reducible if the group $\rho(\pi_1(M - \Sigma)) \subset PSL_2(\mathbb{C})$ fixes a point in $\partial \mathbb{H}^3$. 


ii. The function $Vol_{M-\Sigma}$ factorizes to the quotient $R(M-\Sigma)/\text{PSL}_2(\mathbb{C})$. That is,

$$Vol_{M-\Sigma}(\rho) = Vol_{M-\Sigma}(A \circ i)$$

for every representation $\rho \in R(M-\Sigma)$ and every element $A \in \text{PSL}_2(\mathbb{C})$.

iii. Consider a closed, orientable, differentiable manifold $N$ of dimension 3. Let $\Omega$ be an embedded link in $N$. If $f : (N, \Omega) \to (M, \Sigma)$ is a diffeomorphism, then

$$Vol_{M-\Sigma}(\rho) = Vol_{N-\Omega}(\rho \circ f_*),$$

for every representation $\rho \in R(M-\Sigma)$.

The following propositions recall two essential properties of the volume function. The first concerns the operation of Dehn filling. The second one establishes the relationship between the Riemannian volume of a hyperbolic cone-manifold and the volume of its holonomy.

**Proposition 4.3.** Given a compact, orientable, differentiable 3-manifold $W$ with tori boundary, let $M$ be a closed 3-manifold obtained from $W$ by Dehn filling on its tori boundary components. For every representation $\rho \in R(M)$ we have that

$$Vol_M(\rho) = Vol_{M-\Sigma}(\rho \circ i_*),$$

where $\Sigma$ denotes the union of souls cores of the tori added to $W$ and

$$i_* : \pi_1(M-\Sigma) \to \pi_1(M)$$

is the canonical map induced by the inclusion $i : M-\Sigma \hookrightarrow M$.

**Proposition 4.4.** Let $M$ be a hyperbolic cone-manifold with topological type $(M, \Sigma)$. If $\rho_M \in R(M-\Sigma)$ is a holonomy representation for $M$, then

$$Vol(M-\Sigma) = Vol_{M-\Sigma}(\rho_M),$$

where $Vol(M-\Sigma)$ is the Riemannian volume of $M-\Sigma$.

Suppose from now on that $M$ is irreducible. Given a sequence $M_i$ of hyperbolic cone-manifolds with topological type $(M, \Sigma)$, let $\rho_i$ be an associated sequence of holonomy representations. Suppose also that the sequence $\rho_i$ converges to a representation $\rho_\infty \in R(M-\Sigma)$.

As a consequence of the above propositions, we have two applications of the Theorem (1.3) concerning the volume of holonomy representations of a hyperbolic cone-manifold.

**Corollary 4.5.** If $Vol_{M-\Sigma}(\rho_\infty) = 0$ (in particular, if $\rho_\infty$ is reducible), then $M$ is Seifert fibered or a Sol manifold.

**Proof.** According to the proposition (4.4), we have

$$Vol(M_i) = Vol_{M-\Sigma}(\rho_i),$$

for every $i \in \mathbb{N}$. Since the function $Vol_{M-\Sigma}$ is continuous we also have

$$\lim_{i \to \infty} Vol(M_i) = \lim_{i \to \infty} Vol_{M-\Sigma}(\rho_i) = Vol_{M-\Sigma}(\rho_\infty) = 0.$$

As a consequence, the sequence $M_i$ necessarily collapses.

Recall that the convergence hypothesis implies the condition (1.1). The result is now a direct consequence of the Theorem (1.3). \qed
Corollary 4.6. Denote by $\alpha_{ij}$ the cone angle of the component $\Sigma_j$ in $M_i$. If $M$ has zero simplicial volume and
\[
\lim_{i \to \infty} \alpha_{ij} = 2\pi \quad j \in \{1, \ldots, k\},
\]
then $M$ is Seifert fibered or a Sol manifold.

Proof. Since the cone angles converge to $2\pi$, we have
\[
\rho_{\infty}([\mu]_{\pi_1(M - \Sigma)}) = 1_{PSL_2(\mathbb{C})},
\]
for all meridian $\mu$ of $\Sigma$. Then the representation $\rho_{\infty} \in R(M - \Sigma)$ factorizes to a representation $\psi_{\infty} \in R(M)$. More specifically
\[
\rho_{\infty} = \psi_{\infty} \circ i_*,
\]
where $i_*$ is the map induced by the inclusion $i : M - \Sigma \hookrightarrow M$. Since the simplicial volume of $M$ is zero, we have (see [Fra])
\[
Vol_M(\psi_{\infty}) = 0
\]
and then (Proposition 4.3)
\[
Vol_{M - \Sigma}(\rho_{\infty}) = Vol_{M - \Sigma}(\psi_{\infty} \circ i_*) = Vol_M(\psi_{\infty}) = 0.
\]
The assertion follows from the preceding corollary. \qed

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