PROPERTY (T) AND ACTIONS ON INFINITE MEASURE SPACES

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ABSTRACT. The aim of the article is to provide a characterization of Kazhdan’s property (T) for locally compact, second countable pairs of groups $H \subset G$ in terms of actions on infinite, $\sigma$-finite measure spaces. It is inspired by the recent characterization of the Haagerup property by similar actions due to T. Delabie, A. Zumbrunnen and the author.

1. Introduction

Throughout this article, $G$ denotes a locally compact, second countable group (lcsc group for short); we assume furthermore that it is non-compact.

In [6], the author and his co-authors T. Delabie and A. Zumbrunnen present a characterization of the Haagerup property in terms of actions on infinite, $\sigma$-finite measure spaces having an invariant mean and whose associated permutation representations are $C_0$.

Citing A. Valette in Chapter 7 of [4], “According to the philosophy that, to any characterization of property (T) there is a parallel characterization of the Haagerup property”, the aim of the present note is to propose a sort of reciprocal to that statement, namely, to propose a new characterization of property (T) inspired by the above mentioned one of the Haagerup property.

The objects under study here are what we call dynamical systems: given a lcsc group $G$, such a dynamical system is a quadruple $(\Omega, \mathcal{B}, \mu, G)$ where $(\Omega, \mathcal{B}, \mu)$ is a measure space on which $G$ acts by $\mu$-preserving automorphisms. For brevity, if $\mu$ is infinite, we denote by $B_f$ the subset of elements $B \in \mathcal{B}$ such that $0 \leq \mu(B) < \infty$.

Throughout the article, if $(\Omega, \mathcal{B}, \mu, G)$ is a dynamical system as above, we denote by $\pi_\Omega : G \to U(L^2(\Omega, \mathcal{B}, \mu))$ the associated permutation representation defined by

$$ (\pi_\Omega(g)\xi)(\omega) := \xi(g^{-1}\omega) $$

for $g \in G$ and $\omega \in \Omega$.

Let $(\pi, \mathcal{H})$ be a unitary representation of the lcsc group $G$. Then recall from Definition 1.1.1 of [2] that $\pi$ almost has invariant vectors if, for every compact set $\varnothing \neq Q \subset G$ and for every $\varepsilon > 0$, there exists a unit vector $\xi \in \mathcal{H}$ such that

$$ \sup_{g \in Q} |\pi(g)\xi - \xi| < \varepsilon. $$

Recall also from Definition 1.1 of [9] that, if $H$ is a closed subgroup of a lcsc group $G$, then the pair $H \subset G$ has Property (T) if, for every unitary representation...
π of G which almost has invariant vectors, there exists a vector ξ \neq 0 such that 
π(h)ξ = ξ for every h ∈ H. In particular, G has property (T) if the pair G ⊂ G has property (T).

Before stating our main result, we need two definitions. The first one is a weakening of C0-dynamical systems from [6]: recall from Definition 1.2 of [6] that a dynamical system \((Ω, B, μ, G)\) is C0 if, for all \(A, B ∈ B_f\), one has \(\lim_{g → ∞} μ(gA ∩ B) = 0\).

**Definition 1.1.** Let \((Ω, B, μ, G)\) be a dynamical system as above. We say that it is weakly C0 if, for all \(A, B ∈ B_f\) and for any \(ε > 0\), there exists \(g ∈ G\) such that \(μ(gA ∩ B) < ε\).

Next, the following type of sequence \((B_m)_{m ≥ 1} ⊂ B_f\) plays a crucial role in the proof of the main result of [6] (proof of Proposition 2.8):

**Definition 1.2.** Let \((Ω, B, μ, G)\) be a dynamical system. A sequence \((B_m)_{m ≥ 1} ⊂ B_f\) is said to be almost invariant if \(μ(B_m) = 1\) for every \(m ≥ 1\) and if, for every compact set \(K ⊂ G\), one has

\[\lim_{m → ∞} \sup_{g ∈ K} μ(gB_m ∩ B_m) = 1.\]

**Remark 1.3.** Let \((Ω, B, μ, G)\) be a weakly C0 dynamical system.

1. If \(A ∈ B_f\) is G-invariant then \(μ(A) = 0\). In particular, a non-trivial weakly C0 dynamical system has an infinite measure.
2. As will be explained in more details in Section 2, the unitary representation \(π_Ω\) is weakly mixing in the sense of [3], which means, by Theorem 1.9 of [3], that \(π_Ω\) has no non-trivial finite-dimensional subrepresentation.

**Remark 1.4.** Let \((Ω, B, μ, G)\) be a dynamical system which contains an almost invariant sequence of sets \((B_m)_{m ≥ 1} ⊂ B_f\). Then the representation \((π_Ω, L^2(Ω, B, μ))\) almost has invariant vectors namely, \(ξ_m := χ_{B_m} ∈ L^2(Ω, B, μ)\) is a unit vector for every \(m ≥ 1\) and, for every compact set \(Q ⊂ G\) and every \(ε > 0\), one has

\[\sup_{g ∈ Q} \|π_Ω(g)ξ_m − ξ_m\| < ε\]

for every large enough \(m\).

Indeed, one has for all \(g\) and \(m\)

\[\|π_Ω(g)ξ_m − ξ_m\|^2 = 2(1 − \text{Re}\langle π_Ω(g)ξ_m|ξ_m⟩) = 2(1 − \int_Ω χ_{gB_m}(ω)χ_{B_m}(ω)dμ(ω)) = 2(1 − μ(gB_m ∩ B_m))\]

which converges to 0 uniformly on \(Q\) as \(m → ∞\).

Here is our main result.

**Theorem 1.5.** Let G be a lcsc group and let \(H ⊂ G\) be a closed subgroup of G.

1. If the pair \(H ⊂ G\) has Property (T) and if \((Ω, B, μ, G)\) is a dynamical system such that \((Ω, B, μ, H)\) is weakly C0, then \((Ω, B, μ, G)\) admits no almost invariant sequence.
2. If the pair \(H ⊂ G\) does not have Property (T), then there exists a σ-finite, dynamical system \((Ω, B, μ, G)\) which has an almost invariant sequence and whose restriction \((Ω, B, μ, H)\) is weakly C0.
The proof of Theorem 1.5 will be given in Section 3, and it rests partly on weakly mixing representations and weakly mixing actions of groups on probability measure-preserving spaces, which is the subject of the next section.

2. Weakly mixing representations, weakly mixing actions

As mentioned in Section 1, we will see that the notion of weakly $C_0$ dynamical systems is closely related to weakly mixing representations and weakly mixing actions.

Our references for the latter properties are the article [3] of V. Bergelson and J. Rosenblatt on the one hand, and Chapters 1 and 2 of the monograph [7] by E. Glasner on the other hand.

**Notation** Let $G$ be a lcsc group and let $(S, B_S, \nu)$ be a probability space on which $G$ acts by Borel automorphisms and preserves $\nu$. Then the subset

$$L_0^2(S, \nu) := \left\{ \xi \in L^2(S, \nu) : \int_S \xi d\nu = 0 \right\}$$

is a $G$-invariant closed subspace of $L^2(S, \nu)$, and we denote by $\pi_{S,0}$ the restriction of $\pi_S$ to $L_0^2(S, \nu)$.

**Definition 2.1.** Let $G$ be a lcsc group.

1. A unitary representation $(\pi, \mathcal{H})$ of $G$ is *weakly mixing* if it contains no non-trivial finite-dimensional subrepresentation.

2. Let $(S, B_S, \nu, G)$ be a dynamical system where $\nu$ is a $G$-invariant probability measure. We say that the action of $G$ on $S$ is *weakly mixing* if the representation $\pi_{S,0}$ is a weakly mixing representation.

**Remark 2.2.** The original definition of weakly mixing representations involve the space of weakly almost periodic functions $WAP(G)$ on $G$ and the unique invariant mean on it (cf. [3], Definition 1.1, and [7], Definition 3.2), and the characterization in terms of finite-dimensional subrepresentations is for instance Theorem 1.9 of [3]. We have chosen not to introduce $WAP(G)$ because we feel that it is useless in the present context.

We gather some characterisations of weakly mixing representations that will be used in the next section.

**Proposition 2.3.** Let $G$ be a lcsc group, let $(\pi, \mathcal{H})$ be a unitary representation of $G$ and let $\mathcal{T}$ be a total subset of $\mathcal{H}$. The following conditions are equivalent:

1. $(\pi, \mathcal{H})$ is a weakly mixing representation of $G$;
2. for every $\varepsilon > 0$ and for all $\xi_1, \ldots, \xi_m \in \mathcal{H}$, there exists $g \in G$ such that $|\langle \pi(g)\xi_j | \xi_j \rangle| < \varepsilon$ for all $j = 1, \ldots, m$;
3. for every $\varepsilon > 0$ and for all $\xi_1, \ldots, \xi_m \in \mathcal{T}$, there exists $g \in G$ such that $|\langle \pi(g)\xi_j | \xi_k \rangle| < \varepsilon$ for all $j, k = 1, \ldots, m$.

**Proof.** Equivalence between (a) and (b) follows from Corollary 1.6 and from Theorem 1.9 of [3], and equivalence between (b) and (c) is a consequence of density of the span of $\mathcal{T}$ and of polar decomposition of scalar products in Hilbert spaces;
for all $\xi, \eta \in \mathcal{H}$.  

**Corollary 2.4.** Let $G$ be a lcsc group and let $(\Omega, \mathcal{B}, \mu, G)$ be a dynamical system. Then the following conditions are equivalent:

(a) $(\Omega, \mathcal{B}, \mu, G)$ is weakly $C_0$;

(b) the permutation representation $\pi_{\Omega}$ is weakly mixing.

**Proof.** Assume that condition (a) holds. This means that for all $A, B \in \mathcal{B}_f$, and for every $\varepsilon > 0$, there exists $g \in G$ such that $|\langle \pi_{\Omega}(g)\chi_A |\chi_B \rangle| < \varepsilon$. If $A_1, \ldots, A_m \in \mathcal{B}_f$ and if $\varepsilon > 0$, then considering $A = B = \bigcup_{j=1}^m A_j$ which belongs to $\mathcal{B}_f$, there exists $g \in G$ such that $|\langle \pi_{\Omega}(g)\chi_{A_j} |\chi_{A_k} \rangle| < \varepsilon$ for all $j, k = 1, \ldots, m$. As linear combinations of characteristic functions $\chi_A$ with $A \in \mathcal{B}_f$ are dense in $L^2(\Omega, \mathcal{B}, \mu)$, this proves that condition (b) holds.

Conversely, if (b) holds, then (a) is the special case $m = 2$ with $\xi_1 = \chi_A$ and $\xi_2 = \chi_B$ in condition (c) of Proposition 2.3.  

**Lemma 2.5.** Let $(S, \mathcal{B}_S, \nu)$ be a probability space on which $G$ acts by $\nu$-preserving Borel automorphisms, and assume that the action is weakly mixing. Then, for every $\varepsilon > 0$ and for all $A_1, \ldots, A_m \in \mathcal{B}_S$, there exists $g \in G$ such that

$$|\nu(gA_j \cap A_k) - \nu(A_j)\nu(A_k)| < \varepsilon$$

for all $j, k = 1, \ldots, m$.

**Proof.** For $j = 1, \ldots, m$, set $\xi_j := \chi_{A_j} - \nu(A_j)$. Then $\xi_j \in L^2_0(S, \nu)$ for every $j$, and, for every $\varepsilon > 0$, by Proposition 2.3, there exists $g \in G$ such that $|\langle \pi_{\nu}(g)\xi_j |\xi_k \rangle| < \varepsilon$ for all $j, k = 1, \ldots, m$. But we have

$$\langle \pi_{\nu}(g)\xi_j |\xi_k \rangle = \langle \chi_{gA_j} - \nu(A_j) |\chi_{A_k} - \nu(A_k) \rangle$$

$$= \nu(gA_j \cap A_k) - 2\nu(A_j)\nu(A_k) + \nu(A_j)\nu(A_k)$$

$$= \nu(gA_j \cap A_k) - \nu(A_j)\nu(A_k)$$

for all $j, k = 1, \ldots, m$.  

3. **Proof of Theorem 1.5**

Let $H \subset G$ be a pair of lcsc groups with Property (T) as in statement (1) of Theorem 1.5 and let $(\Omega, \mathcal{B}, \mu, G)$ be a dynamical system whose restriction to $(\Omega, \mathcal{B}, \mu, H)$ is weakly $C_0$. Corollary 2.4 implies that the permutation representation $\pi_{\Omega}$ of $H$ is weakly mixing, hence that it does not have any non-trivial finite-dimensional subrepresentation. If there existed an almost invariant sequence of sets $(B_m) \subset \mathcal{B}_f$ as in Definition 1.2, then the unitary representation $\pi_{\Omega}$ of $G$ would almost have invariant vectors, hence there would exist a non-zero vector $\xi$ such that $\pi_{\Omega}(h)\xi = \xi$ for every $h \in \mathcal{H}$ by Property (T), but this contradicts the weakly mixing property of $\pi_{\Omega}$ restricted to $H$. Hence $(\Omega, \mathcal{B}, \mu, G)$ has no almost invariant sequence.

The rest of the present section is devoted to the proof of statement (2) of Theorem 1.5. Thus we assume henceforth that the pair $H \subset G$ does not have Property (T). Let us choose an increasing sequence of compact subsets $(K_n)_{n \geq 1}$ of $G$ with the
following properties: \( e \in \hat{K}_1, K_n \subset \hat{K}_{n+1} \) for every \( n \geq 1 \) and \( G = \bigcup_{n \geq 1} K_n \).
Hence, \( K_1 \) is a compact neighbourhood of \( e \), and for every compact set \( K \subset G \), there exists \( m \) such that \( K \subset K_m \).

**Lemma 3.1.** With the above hypotheses, there exists a conditionally negative definite function \( \psi : G \to \mathbb{R}_+ \) whose restriction to \( H \) is unbounded.

**Proof.** By Theorem 1.2 of [9], there exists a sequence \( (\psi_n)_{n \geq 1} \) of real-valued, positive definite and normalized functions on \( G \) which converges to 1 uniformly on compacts sets, but such that

\[
\sup_{h \in H} |\psi_n(h) - 1| \to 0.
\]

Extracting subsequences if necessary, we assume that there exists \( c > 0 \) and a sequence \( (h_n)_{n \geq 1} \subset H \) such that \( 1 - \psi_n(h_n) \geq c \) and

\[
\sup_{g \in K_n} |\psi_n(g) - 1| \leq \frac{1}{n^2}
\]

for all \( n \). Then set

\[
\psi = \sum_{n \geq 1} \sqrt{n}(1 - \psi_n).
\]

It defines a conditionally negative definite function on \( G \) which satisfies

\[
\psi(h_k) = \sum_{n+k} \sqrt{n}(1 - \psi_n(h_k)) + \sqrt{k}(1 - \psi_k(h_k)) \geq \sqrt{k}c.
\]

Hence \( \psi \) is unbounded on \( H \). \( \square \)

An adaptation of Proposition 2.2.3 of [4], of [5] and of Theorem A.1 of [8] shows that there exists a measure-preserving \( G \)-action on a standard probability space \((S, B_S, \nu)\) with the following properties:

(a) the restriction to \( H \) of the action is weakly mixing (this is where the existence of \( \psi \) in Lemma 3.1 is needed; see details below);

(b) there exists a non-trivial asymptotically invariant sequence of Borel subsets of \( S \), namely, there exists a sequence \( (A_n)_{n \geq 1} \subset B_S \) such that \( \nu(A_n) = 1/2 \) for every \( n \) and such that, for every compact set \( K \subset G \),

\[
\lim_{n \to \infty} \sup_{g \in K} \nu(gA_n \triangle A_n) = 0
\]

where \( A \triangle B = A \setminus B \cup B \setminus A \) for all sets \( A, B \).

Furthermore, the proof of Lemma 1.3 of [1] shows that we can (and will) assume that \( S \) is a compact metric space on which \( G \) acts continuously, and that \( \nu \) has support \( S \).

We think that it is helpful to describe the construction of the probability \( G \)-space \((S, B_S, \nu)\) with some details. We follow faithfully the proof of Theorem 2.2.2 of [4].

By Lemma 3.1, let \( \psi : G \to \mathbb{R}_+ \) be a conditionally negative definite function which is unbounded on \( H \). For \( n \geq 1 \), we set \( \varphi_n = \exp(-\psi/n) \), and we denote by \((\pi_n, \mathcal{H}_n, \xi_n)\) the associated Gel’fand-Naimark-Segal triple. Since \( \varphi_n \) is real-valued, there exists a real Hilbert subspace \( \mathcal{H}'_n \) of \( \mathcal{H}_n \), containing \( \xi_n \), such that

\[
\mathcal{H}_n = \mathcal{H}'_n \oplus i\mathcal{H}'_n \quad \text{and} \quad \pi_n(g)\mathcal{H}'_n = \mathcal{H}'_n.
\]
for all $n$ and $g$. We set $\mathcal{H} = \bigoplus_{n \geq 1} \mathcal{H}_n$, $\mathcal{H}' = \bigoplus_{n \geq 1} \mathcal{H}'_n$ so that $\mathcal{H} = \mathcal{H}' \oplus i\mathcal{H}'$, and $\pi = \bigoplus_{n \geq 1} \pi_n$. Finally, we identify $\xi_n$ with the corresponding vector

$$0 \oplus \ldots \oplus \xi_n \oplus \ldots \in \mathcal{H}'$$

and we observe that $\xi_n \perp \xi_m$ when $n \neq m$.

Next, set $\mathcal{H}^k = \bigoplus_{k \geq 0} \mathcal{H}^{xk}$, where $\mathcal{H}^{x0} = \mathbb{C}$, and for $k > 0$, $\mathcal{H}^{xk}$ is the $k$-th symmetric tensor product of $\mathcal{H}$, that is, the closed subspace of the Hilbert tensor product space $\mathcal{H}^\otimes k$ generated by the vectors of the form

$$\sum_{s \in S_k} \eta_{s(1)} \otimes \ldots \otimes \eta_{s(k)}$$

where $S_k$ denotes the usual permutation group. Then the representation $\pi$ extends in a natural way to a representation $\pi^\sigma$ of $G$ on $\mathcal{H}^\sigma$ which leaves the subspace $\mathcal{H}^\sigma_0 = \mathcal{H}^\sigma \ominus \mathcal{H}^{x0}$ invariant. Finally, we denote by $\pi^\sigma_0$ the restriction of $\pi^\sigma$ to $\mathcal{H}^\sigma_0$.

**Lemma 3.2.** The restriction of $\pi^\sigma_0$ to $H$ is weakly mixing.

**Proof.** Let $(h_\ell)_{\ell \geq 1} \subset H$ be a sequence such that $\psi(h_\ell) \to \infty$ as $\ell \to \infty$. Then, as in the proof of Lemma 2.1 of [4], for every finite set $F \subset G$, one has

$$\max_{g,g' \in F} \psi(gh_\ell g') \to \infty$$

as $\ell \to \infty$. As the set of vectors $\{\pi_n(g)\xi_n : g \in G, n \geq 1\}$ is total in $\mathcal{H}$, it suffices, by Proposition 2.3, to prove that, for all $g_1, \ldots, g_m \in G$, for every $n \geq 1$ and for every $\varepsilon > 0$ there exists $\ell \geq 1$ such that $|\langle \pi_n(g_1h_\ell g_k)\xi_n|\xi_n\rangle| < \varepsilon$ for all $j,k$. But

$$|\langle \pi_n(g_1h_\ell g_k)\xi_n|\xi_n\rangle| = \exp(-\psi(g_1h_\ell g_k)/n) \to 0$$

as $\ell \to \infty$. \qed

In order to define $(S, \nu)$, we choose a countable orthonormal basis $\mathcal{B}$ of $\mathcal{H}'$ which contains $\{\xi_n : n \geq 1\}$, we set

$$(S, \nu) := \prod_{b \in \mathcal{B}} \left( \mathbb{R}, \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{x^2}{2} \right) dx \right)$$

and we define the random variable $X_b : S \to \mathbb{R}$ by $X_b((\omega_\nu)_b \in \mathcal{B}) = \omega_b$ for every $b$. Then the map

$$\xi = \sum_{b \in \mathcal{B}} \xi_b b \mapsto \sum_{b \in \mathcal{B}} \xi_b X_b$$

from $\mathcal{H}'$ to $L^2(S, \nu)$ extends to a surjective isometry $u : \mathcal{H}_\nu \to L^2(S, \nu)$ which sends $\mathcal{H}^{x0}$ onto the space of constant functions on $S$ and such that

$$u \left( \sum_{s \in \mathcal{S}_n} b_{s(1)} \otimes \ldots \otimes b_{s(n)} \right) = n! X_{b_1} \ldots X_{b_n}$$

for all $b_1, \ldots, b_n \in \mathcal{B}$. Moreover, there is a $\nu$-preserving action of $G$ on $(S, \nu)$ such that $u^* \pi_S(g)u = \pi^\sigma(g)$ and $u^* \pi_{S,0}(g)u = \pi^\sigma_0(g)$ for all $g \in G$. In particular, Lemma 3.2 implies that the action of $H$ on $(S, \nu)$ is weakly mixing. Finally, the non-trivial asymptotically invariant sequence $(A_n) \subset \mathcal{B}_S$ of condition (b) above is obtained by setting $A_n = \{\omega \in S : X_{\xi_n}(\omega) \geq 0\}$. See pages 23 and 24 of [4] for further details.

Let us describe now our construction of the dynamical system $(\Omega, \mathcal{B}, \mu, G)$ which is taken from [6].
Let \((A_n)_{n \geq 1} \subset B_S\) be the above non-trivial asymptotically invariant sequence. Then the following inequalities
\[
|\nu(gA_n \cap A_n) - 1/2| = |\nu(gA_n \cap A_n) - \nu(A_n)| = \int_S \chi_{A_n}(\chi_{gA_n} - \chi_{A_n})d\nu \leq \int_S |\chi_{gA_n} - \chi_{A_n}|d\nu = \nu(gA_n \triangle A_n)
\]
show that, for all positive integers \(m\) and \(k\), there exists an integer \(n(k,m)\) such that
\[
\sup_{g \in K_m} |\nu(gA_{n(k,m)} \cap A_{n(k,m)}) - 1/2| \leq 1/2 \left(1 - e^{-\frac{1}{m2^k}}\right).
\]
Then set
\[
B_m := \prod_{k \geq 1} A_{n(k,m)} \subset X := \prod_{k \geq 1} S
\]
for every \(m > 0\).

We equip the set \(X\) with a \(\sigma\)-algebra \(\mathcal{C}\) containing \((B_m)_{m \geq 1}\) and with a measure \(\mu : \mathcal{C} \to [0, \infty]\) which have the following properties (see [6], Propositions 2.8, 3.4 and 3.5):

(i) The \(\sigma\)-algebra \(\mathcal{C}\) is generated by the collection (denoted by \(\mathcal{F}_{c,0}\) in Definition 3.1 of [6]) of sets \(C = \prod_n C_n\) such that \(\prod_n 2\nu(C_n) := \lim_{N \to \infty} \prod_{n=1}^N 2\nu(C_n)\) exists in \([0, \infty)\) and such that \(\prod_n 2\nu(C_n) = 0\) or
\[
(3.1) \quad \lim_{N \to \infty} \prod_{n=N}^\infty 2\nu(gC_n \cap C_n) = 1
\]
uniformly for \(g \in K_1\).

(ii) The measure \(\mu\) on \(\mathcal{C}\) satisfies
\[
\mu(C) = \prod_n 2\nu(C_n)
\]
for every set \(C = \prod_n C_n \in \mathcal{F}_{c,0}\).

(iii) The diagonal action of \(G\) on \(X\) is \(\mathcal{C}\)-measurable.

(iv) For every \(A \in \mathcal{C}\) such that \(\mu(A) < \infty\), one has
\[
\lim_{g \to e} \mu(gA \triangle A) = 0.
\]

Remark 3.3. (1) The measure \(\mu\) on \(X\) is not necessarily \(\sigma\)-finite, as is proved in Proposition 2.6 of [6]. Thus, equality (3.1) in (i) and property (iv) are continuity properties that are needed to restrict \(\mu\) to a \(G\)-invariant subset \(\Omega \subset \mathcal{C}\) on which it is \(\sigma\)-finite: see Section 3 of [6].

(2) We observe that the sequence \((B_m)_{m \geq 1}\) is contained in \(\mathcal{C}\): indeed, as
\[
e^{-\frac{1}{m2^k}} \leq 2\nu(gA_{n(k,m)} \cap A_{n(k,m)}) \leq 1.
\]
for every \( g \in K_m \) and for every \( k \geq 1 \), we get that
\[
1 \geq \lim_{N \to \infty} \prod_{k=N}^{\infty} 2\nu(gA_{n(k,m)} \cap A_{n(k,m)}) \\
\geq \lim_{N \to \infty} \prod_{k=N}^{\infty} e^{-\frac{1}{2\nu(A)}} \\
= e^{-\frac{1}{2}} \lim_{N \to \infty} \sum_{k=N}^{\infty} \frac{1}{2\nu(A)} = 1.
\]

In particular, it converges uniformly for \( g \in K_1 \).

The following proposition is inspired by Proposition 2.8 of [6].

**Proposition 3.4.** The dynamical system \((X,\mathcal{C},\mu,H)\) is weakly \(C_0\), namely, for all \( A, B \in \mathcal{C} \) such that \( \mu(A), \mu(B) < \infty \) and for every \( \epsilon > 0 \), there exists \( h \in H \) such that \( \mu(hA \cap B) < \epsilon \).

**Proof.** Assume first that \( A, B \in \mathcal{F}_{c,0} \) have positive measures, and write \( A = \prod_n A_n \) and \( B = \prod_n B_n \), so that
\[
\mu(A) = \prod_{n \geq 1} 2\nu(A_n) \quad \text{and} \quad \mu(B) = \prod_{n \geq 1} 2\nu(B_n).
\]

Let \( \epsilon > 0 \) be fixed and take \( \epsilon' > 0 \) small enough in order that
\[
\delta := \frac{1}{2} + \frac{\epsilon'}{2 - \epsilon'} < 1.
\]

Since \( 0 < \mu(A), \mu(B) < \infty \), there exists \( N \) large enough such that
\[
\frac{1}{2} - \epsilon' < \nu(A_n), \nu(B_n) < \frac{1}{2} + \epsilon' \quad \forall n \geq N.
\]

Since \( \delta < 1 \), there exists \( m \) large enough such that \( \delta^{m+1} < \epsilon / \mu(A) \). The action of \( H \) on \((S,\nu)\) being weakly mixing, by Lemma 2.5, there exists \( h \in H \) such that
\[
|\nu(hA_n \cap B_n) - \nu(A_n)\nu(B_n)| \leq \epsilon'
\]
for all \( n \in \{N, \ldots, N + m\} \). Then we have
\[
\mu(hA \cap B) \leq \prod_{n=1}^{N-1} 2\nu(A_n) \cdot \prod_{n=N}^{N+m} 2(\nu(A_n)\nu(B_n) + \epsilon') \cdot \prod_{n \geq N+m+1} 2\nu(A_n)
\]
\[
= \mu(A) \cdot \prod_{n=N}^{N+m} \left( \frac{2\nu(A_n)\nu(B_n) + 2\epsilon'}{2\nu(A_n)} \right)
\]
\[
= \mu(A) \left( \prod_{n=N}^{N+m} \left( \frac{1}{2} + \frac{\epsilon'}{2 - \epsilon'} \right) \right)
\]
\[
< \mu(A) \left( \frac{1}{2} + \epsilon' + \frac{\epsilon'}{2 - \epsilon'} \right)
\]
\[
= \mu(A)\delta^{m+1} < \epsilon.
\]

Hence the claim holds for all sets \( A, B \in \mathcal{F}_{c,0} \).

The same claim holds for \( A, B \) which belong to the semiring \( \mathcal{F}_c \) generated by \( \mathcal{F}_{c,0} \) because, by Proposition 3.4 of [6], for all \( A, B \in \mathcal{F}_c \), there exist \( C, D \in \mathcal{F}_{c,0} \) such that \( A \subset C \) and \( B \subset D \).
Moreover, it also holds for all elements of the ring $\mathcal{R}(F_c)$ generated by $F_c$. Finally, if $A, B \in \mathcal{C}$ are such that $0 < \mu(A), \mu(B) < \infty$, if $\varepsilon > 0$ is given, there exist two sequences $(C_k)_{k \geq 1}, (D_\ell)_{\ell \geq 1} \subset \mathcal{R}(F_c)$ such that

$$A \subset \bigcup_{k \geq 1} C_k \quad \text{and} \quad B \subset \bigcup_{\ell \geq 1} D_\ell$$

and

$$\mu(A) \leq \sum_k \mu(C_k) < \mu(A) + \varepsilon \quad \text{and} \quad \mu(B) \leq \sum_\ell \mu(D_\ell) < \mu(B) + \varepsilon.$$ 

Choose first $N$ large enough so that $\sum_{\ell > N} \mu(D_\ell) < \varepsilon/3$. Then, as

$$gA \cap B \subset \left( \bigcup_{\ell = 1}^N gA \cap D_\ell \right) \cup \left( \bigcup_{\ell > N} D_\ell \right),$$

we get

$$\mu(gA \cap B) \leq \sum_{\ell = 1}^N \mu(gA \cap D_\ell) + \varepsilon/3$$

for every $g \in G$.

Choose next $M$ large enough so that $\sum_{k > M} \mu(C_k) < \varepsilon/3N$. Then, as

$$gA \cap D_\ell \subset \left( \bigcup_{k = 1}^M gC_k \cap D_\ell \right) \cup \left( \bigcup_{k > M} gC_k \right)$$

for every $1 \leq \ell \leq N$, and since $\mu$ is $G$-invariant, we get

$$\mu(gA \cap B) \leq \sum_{\ell = 1}^N \sum_{k = 1}^M \mu(gC_k \cap D_\ell) + 2\varepsilon/3$$

for every $g \in G$. By the previous part of the proof applied to $C := \bigcup_{k = 1}^M C_k$ and $D := \bigcup_{\ell = 1}^N D_\ell$, there exists $h \in H$ such that

$$\mu(hA \cap B) < \varepsilon.$$ 

The proof of Theorem 1.5 will be complete if we prove that there is a $G$-invariant set $\Omega \in \mathcal{C}$ on which $\mu$ is $\sigma$-finite.

As in Proposition 3.8 of [6], we fix a countable dense subset $D \ni e$ of $G$, we set $Y = \bigcup_{m \geq 1} B_m$ and

$$\Omega = \bigcup_{h \in D} hY$$

on which $\mu$ is obviously $\sigma$-finite.

The continuity condition (iv) implies that $\Omega$ is $G$-invariant (see the proof of Proposition 3.8 of [6]), thus the proof of Theorem 1.5 is complete.
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