Brane-world black holes with post-Newtonian parameter: astrophysical signatures

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Abstract. The existence of the many unanswered questions in fundamental physics, in particular, in astrophysics allows for a great variety of theories to remain viable candidates for becoming the correct theory at energies not accessible in current experiments. One special class of these type of theories is the class of extra-dimensional brane-world models. Besides answering many fundamental problems, for instance, the hierarchy problem, they may produce testable predictions. In this work, we find and investigate brane-world induced black string horizon corrections, when the black hole solution has a post-Newtonian parameter. For suitable choices of such a parameter, the Hawking radiation on the brane is precluded, and the Hawking radiation in the bulk causes the black hole to slightly recoil into the bulk, which modifies the black hole apparent horizon. It has an impact on quasars luminosity and, therefore, might be detected and measured.

1. Introduction
Dark Matter and Dark Energy hypotheses are the basis of the answers to many open questions in contemporary fundamental physics. There exist a large number of theories that are aimed at the explanation of physics at scales not accessible in current terrestrial experiments as well as in astrophysics. The brane-world scenarios are a class of successful theories which, among other things, provide a solution of the hierarchy problem. The underlying idea is that, unlike matter fields, which live on a four-dimensional brane, the gravity lives in a five-dimensional space and, therefore, leaks into the bulk which greatly reduces its strength as measured on the brane.

There exists a high level of degeneracy among the candidate theories and observational tests are needed in order to make progress towards the correct theory. In this work, we propose an observational test to distinguish the brane-world scenarios from other classes of theories of gravity.

The black hole horizon is sensitive to the details of the extra-dimensional theories. This might impact astrophysical models of quasars, which invoke accretion of matter onto a black hole. The masses of black hole and the accretion onto them are detectable through the radiation caused by the accretion process, which also depends on the distribution of the mass in the galaxy.
hosting the black hole. Here, we present the Casadio-Fabbri-Mazzacurati black hole solutions, that possess a post-Newtonian parameter, and study how these solutions can be extended to the bulk, thus generalizing the concept of black string. For suitable values of the post-Newtonian parameter, the Hawking radiation on the brane is suppressed, but not the Hawking radiation in the bulk. It causes the black hole to slightly recoil into the bulk, which makes the apparent horizon on the brane to be modified.

In Section 2, we use the Gauss-Codazzi formalism to project the 5D space-time quantities on the 4D brane, taking into account the brane tension and junction conditions. The Taylor expansion of metric along the Gaussian coordinate perpendicular to the brane is carried out, providing the bulk metric (and also the black string warped horizon) near the brane. In the Section 3, we present a complete model and illustrate main results in figures. Section 4 is an explanation of how brane-world induced effects might be calculated and how they change quasars luminosity. Our results and conclusions are summarized in Sections 5 and 6.

2. Projection of 5D model on the 4D brane: the Gauss-Codazzi formalism and junction conditions

We are working in 5D space-time where \( y \) is a Gaussian coordinate orthogonal to the brane so the brane is defined by \( y = 0 \). The 5D metric is \( \hat{g}_{AB} dx^A dx^B = g_{\mu\nu}(x^\alpha, y) dx^\mu dx^\nu + dy^2 \), where \( g_{\mu\nu} \) is the metric on the brane. \( n^A \) is an orthonormal vector field normal to the brane so that the metrics on the brane and in the bulk are related as follows:

\[
g_{\mu\nu} = \hat{g}_{\mu\nu} - n_{\mu}n_{\nu}. \tag{1}\]

The Einstein’s field equations in 5D read

\[
G^{(5)}_{AB} = -\Lambda_5 g^{(5)}_{AB} + k_5^2 T^{(5)}_{AB}. \tag{2}\]

The sub-and superscripts ‘(5)’ indicate that the quantity is 5-dimensional.

As the phenomenology is accessible only on the brane, we need to project this equation onto the brane. Besides the projection, we need to take into account junction conditions

\[
g_{\mu\nu}^+ - g_{\mu\nu}^- = 0, \tag{3}\]

\[
K_{\mu\nu}^- - K_{\mu\nu}^+ = -k_5^2(T^{brane}_{\mu\nu} - \frac{1}{3}T^{brane}), \tag{4}\]

where \( T^{brane} \) is the energy-momentum tensor on the brane which is defined as

\[
T^{brane}_{\mu\nu} = T_{\mu\nu} - \lambda g_{\mu\nu}, \tag{5}\]

where \( \lambda \) is the brane tension, and \( K_{\mu\nu} \) is the extrinsic curvature. As a result, one obtains the known junction condition for the Lie derivative of the metric

\[
K_{\mu\nu} = -\frac{1}{2} k_5^2 (T_{\mu\nu} + \frac{1}{3}(\lambda - T) g_{\mu\nu}). \tag{6}\]

For more details on the derivation of the above equation see Ref. [1]. Using the junction conditions after the projection, one obtains the field equations on the brane (see again Ref. [1])

\[
G_{\mu\nu} = -\Lambda g_{\mu\nu} + k^2 T_{\mu\nu} + \frac{k^2}{\lambda} S_{\mu\nu} - \epsilon_{\mu\nu} + \frac{4}{\lambda} F_{\mu\nu}. \tag{7}\]

Here \( \epsilon_{\mu\nu} \) is a Weyl tensor, which provides the wave equations for Kaluza-Klein modes if we apply the covariant derivative to the field equation and take into account Bianchi identity and energy
conservation on the brane. Like the Weyl tensor, $S_{\mu \nu}$ is also a high energy term and contributes to the wave equation. $S_{\mu \nu}$ is defined as

$$S_{\mu \nu} = \frac{1}{12} T T_{\mu \nu} - \frac{1}{4} T_{\mu \alpha} T_{\nu}^{\alpha} + \frac{1}{24} [3 T_{\alpha \beta} T^{\alpha \beta} - T^2].$$

(8)

The last term in (7) vanishes if we take into account that there is no exchange between bulk and brane. Indeed, if one applies the Bianchi identity and energy conservation on the brane, one obtains $F_{\mu \nu} = 0$ with $F_{\mu \nu}$ defined as

$$F_{\mu \nu} = T_{AB}^\mu \eta^A_B - \frac{1}{4} T(5).$$

(9)

Interestingly, after the projection, new terms appear proportional to the square of the energy-momentum tensor. Junction condition and brane tension relate bulk with the brane, and so the brane holds information about the bulk. Therefore, we can expect that brane-world effects and corrections can appear in high energy processes due to $S$. For more details on Eq. (7) able see Ref. [1]. In the next section we will provide techniques of our calculation which is based on Taylor expansion.

3. Taylor expansion of metric

The field equation can not be exactly solved, so we have to apply different approaches to find some approximate solution. One successful and simple technique is an expansion in Taylor series. Thus, by using Lie derivatives and the definition of the extrinsic curvature $K_{\mu \nu}$, the equations above and the junction condition provide the bulk metric near the brane in terms of brane metric exclusively. This is achieved carrying out a Taylor expansion of the metric along the extra dimension $y$ in Gaussian coordinates.

This approach is discussed in several works (see Refs. [1, 2, 3]). We present just the final result from Ref. [3]. The bulk metric obtained from the metric on the brane, up to the fourth order along the Gaussian coordinate $y$, reads

$$g_{\mu \nu}(x^\alpha, y) = g_{\mu \nu} - \kappa_5^2 \left[ T_{\mu \nu} + \frac{1}{3} (\lambda - T) g_{\mu \nu} \right] |y|$$

$$+ \left[ \frac{1}{2} \kappa_5^3 T_{\mu \nu} T_{\alpha \nu} + \frac{2}{3} (\lambda - T) T_{\mu \nu} - 2 \mathcal{E}_{\mu \nu} + \frac{1}{3} \left( \frac{1}{6} \kappa_5^3 (\lambda - T)^2 - \Lambda_5 \right) g_{\mu \nu} \right] \frac{|y|^2}{2!}$$

$$+ \left[ 2 K_{\mu \beta} K_{\alpha}^{\beta} K_{\nu}^{\alpha} - \mathcal{E}_{(\mu |(\alpha} K_{\nu)_{\alpha}} - \nabla^\alpha \mathcal{E}_{\mu \nu} + \frac{1}{6} \Lambda_5 g_{\mu \nu} K + K_{\alpha \beta} R_{\mu \alpha \nu \beta} - K \mathcal{E}_{\mu \nu}ight.$$  

$$+ 3 K_{(\mu \nu |}^\alpha + K_{\mu \alpha} K_{\nu \beta} K^{\alpha \beta} - 2 K^2 K_{\mu \nu} \right] \frac{|y|^3}{3!}$$

$$+ \left[ \frac{\Lambda_5}{6} (R - \frac{\Lambda_5}{3} + K^2) g_{\mu \nu} + \left( \frac{K^2}{3} - \Lambda_5 \right) K_{\mu \alpha} K_{\nu}^{\alpha} + (R - \Lambda_5 + 2 K^2) \mathcal{E}_{\mu \nu} \right.$$  

$$+ \left( K_{\gamma}^{\mu} K_{\beta}^{\nu} + \mathcal{E}_{\alpha \beta}^\gamma + K K_{\alpha \beta} \right) R_{\mu \alpha \nu \beta} + K^2 K_{\mu \nu} - \frac{1}{6} \Lambda_5 R_{\mu \nu} + 2 K_{\mu \beta} K_{\nu}^{\rho} K_{\alpha}^{\sigma} \right] \frac{|y|^4}{4!} + \cdots$$

(10)

where $C_{\mu \sigma} = C_{\mu \sigma}$, and $C_{\mu \alpha \sigma}$, for any tensor $C$ that has rank 2, and $g_{\mu \nu}(x^\alpha, 0) \equiv g_{\mu \nu}$ is the metric on the brane. Given the 5D Weyl tensor $C_{\mu \nu \alpha \beta}$ one commonly defines the electric $\mathcal{E}_{\mu \nu} = C_{\mu \nu \alpha \beta} n^\alpha n^\beta$ and the magnetic $\mathcal{B}_{\mu \nu \alpha} = g_1^\beta g_\nu^\sigma C_{\mu \alpha \beta \gamma} n^\gamma$ Weyl tensor components.
We need a metric for vacuum on the brane which reads:
\[
g_{\mu\nu}(x, y) = g_{\mu\nu} - \frac{1}{3} \kappa^2 \lambda g_{\mu\nu} |y| + \left[ \frac{1}{6} \left( \frac{1}{6} \kappa^2 \lambda^2 - \Lambda_5 \right) g_{\mu\nu} - \mathcal{E}_{\mu\nu} \right] y^2 \\
- \frac{1}{6} \left( \left( \frac{193}{36} \lambda^3 \kappa^6 g_{\mu\nu} + \frac{5}{3} \Lambda_5 \kappa^2 \lambda \right) g_{\mu\nu} + \kappa^2 R_{\mu\nu} \right) \frac{|y|^3}{3!} + \\
+ \frac{1}{6} \Lambda_5 \left( \left( R - \frac{1}{3} \Lambda_5 - \frac{1}{18} \lambda^2 \kappa^4 \right) + \frac{7}{324} \lambda^4 \kappa_5 \right) g_{\mu\nu} + \left( R - \frac{1}{3} \Lambda_5 - \frac{19}{36} \lambda^2 \kappa_5 \right) \mathcal{E}_{\mu\nu} \\
+ \frac{1}{6} \left( \frac{37}{36} \lambda^2 \kappa_3 - \Lambda_5 \right) R_{\mu\nu} + \mathcal{E}^{\alpha\beta} R_{\mu\nu\alpha\beta} \right] y^4 + \cdots \tag{11}
\]

Therefore, in order to proceed, \( g_{\mu\nu} \) must be chosen. There are solutions that can be obtained on the brane, which differ from Schwarzschild one. These are the so called Casadio-Fabbri-Mazzacurati (CFM) brane-world solutions [4, 5]. In general, a static spherically symmetric metric can be written as follows
\[
g_{\mu\nu}dx^\mu dx^\nu = -N(r)dt^2 + A(r)dr^2 + r^2 d\Omega^2. \tag{12}
\]

CFM brane-world solutions have an additional post-Newtonian parameter \( \beta \). Although for solar system scales such parameter satisfies the relation \( |\beta - 1| \sim 10^{-4} \) [6], for extremal black holes this is not the case, as we shall see below. By a certain choice of \( \beta \) the black hole Hawking radiation can be suppressed on the brane so that we can analyze pure brane-world effect. Indeed for some values of \( \beta \) the Hawking radiation, although precluded on the brane, is not zero in the bulk, which makes the black hole to recoil into the bulk. Even more, for these solutions, Hawking radiation is altered and the evaporation of mini-black holes is slowed, so they can be detected at the LHC.

The CFM I brane-world solution reads [4]
\[
N(r) = 1 - \frac{2GM}{c^2 r} \quad\text{and}\quad A(r) = \frac{1 - \frac{3GM}{2c^2 r}}{(1 - \frac{2GM}{2c^2 r}(4\beta - 1))}. \tag{13}
\]

Hawking radiation for this CFM I brane-world solution is suppressed if \( \beta = 5/4 \).

The CFM II brane-world solution is given by [5]
\[
N(r) = 1 - \frac{2GM}{c^2 r} + \frac{2G^2 M^2}{c^4 r^2} (\beta - 1) \quad\text{and}\quad A(r) = \frac{1 - \frac{3GM}{2c^2 r}}{(1 - \frac{2GM}{2c^2 r}(4\beta - 1))}. \tag{14}
\]

Now we need to choose \( \beta = 3/2 \) since for this value of \( \beta \), Hawking radiation is suppressed and gives an opportunity to demonstrate pure brane-world effects. In the following Section, we show an imprint of brane-world solutions on the black-hole accretion rate, i.e. on the quasar luminosity, which is the physical quantity that is observed.

4. Accretion and brane-world effects

Accretion depends on the black hole mass and horizon and its rate is proportional to the derivative of mass so that for the luminosity \( L \) can be written as
\[
L(\ell) = \eta(\ell) M c^2, \tag{15}
\]

where \( \ell \) is the bulk curvature radius parameter. And accretion efficiency \( \eta \) is commonly given by \( \eta = \frac{GM}{6c^2 R} \), where \( R \) denotes the black hole horizon. Thus, for both the CFM I and the CFM
II brane-world solutions, the Taylor expansion of Eq. (11) provides the black string horizon that equals the black hole horizon when it slightly recoils from the brane into the bulk, which is denoted by \( R(y) = \sqrt{g_{\theta\theta}(x^a, y)} \) and is given by Eq. (11). Then, we compare the luminosity with the Schwarzschild case \( R_S \). Indeed, \( R(y) \) is the black hole horizon when the black hole recoils from the brane. Therefore, the result for quasar luminosity correction can be given by the following form

\[
\Delta L = \frac{GM}{6c^2} \left( R(y)^{-1} - R_S^{-1} \right) \dot{M} c^2 = \frac{1}{12} \left( \frac{R_S}{\sqrt{g_{\theta\theta}(x^a, y)}} - 1 \right) \dot{M} c^2. \tag{16}
\]

All the analysis depends upon the previous expressions and we are ready to provide our main results in figures in the next Section.

5. Results

As a result, we illustrate how the luminosity varies as the black hole recoils into the extra dimension \( y \). The most important and distinguishable feature of quasar luminosity in brane-world scenarios is its dependence on the black hole mass. In Fig. 1, we show plots of normalized correction of luminosity versus the size of extra dimension \( y \), for different masses. We see that for masses lower than \( M = 10^{6} M_\odot \), the relative correction is a decreasing function of the extra dimension. Here \( M_\odot \) denotes the mass of the sun. For the second model CFM II for all cases, the normalized correction of luminosity is always a decreasing function for all black hole masses, see Fig. 2. The correction is normalized by solar luminosity \( L_\odot \sim 3.91 \times 10^{33} \text{erg s}^{-1} \). For typical supermassive black hole of mass \( M \approx 10^9 M_\odot \), the luminosity correction can be improved by two orders if we keep in the expansion up to the fourth order terms (instead of restricting to second order). Therefore, now the correction is \( \Delta L \sim 2.1 \times 10^{-3} L_\odot \). In previous works it was found that \( \Delta L \sim 1.0 \times 10^{-5} L_\odot \). Now we can investigate the dependence of the luminosity on the black

![Figure 1. Plots of the normalized correction to luminosity \( \Delta L/\dot{M}c^2 \) in a CFM type I model as a function of the extra dimension \( y \) for different values of masses \( M/M_\odot = 10^7 \) (dashed black line) \( 10^6 \) (continuous black line), \( 10^5 \) (dash-dotted line), \( 10^4 \) (light-gray thick line), \( 10^3 \) (gray dashed line), \( 10^2 \) (dark-gray line).

![Figure 2. The same as in Fig. 1 but for the CFM II model. The labeling of curves is as follows: \( M/M_\odot = 10^7 \) - dashed black line, \( 10^6 \) - continuous black line, \( 10^5 \) - dash-dotted line, \( 10^4 \) - dark dotted line, \( 10^3 \) - light-gray thick line, for \( 10^2 \) - gray dashed line.](image-url)
the bulk. For all values for the mass in the range analysed, the relative luminosity is essentially the same. On the another hand, the CFM II black hole has a completely distinct behavior, for the same range of the black hole mass. As it can be seen at Fig. 4, as the mass varies in the range illustrated, the normalized correction of the quasar luminosity is as abrupt as the mass increases. Indeed, it is in full compliance to Eq.(14) describing the CFM II metric, that differs from the CFM I metric (13) solely by the post-Newtonian correction for the Schwarzschild term \( N(r) \) in (13). As such term contains the a term \( M^2 \), the profile in Fig. 4 is expected.

6. Conclusion
In this work we carried out an analysis of possible signatures of extra-dimensional theories. Our analysis shows that there could be variations in quasar luminosity in different braneworld scenarios. These variations provide a promising observational test of these intriguing theories. The corrections to the quasar luminosity could be three-orders of magnitude and may be detectable in future observations. While these corrections emerge at all the wavelengths of radiation, they should be still detectable for quasars with their narrow range of radiation wavelengths. In particular, the corrections to the luminosity of a quasar (which is modeled as a black hole combined with the canonical model of accretion) show a new feature: the luminosity is a function of black hole’s mass, see Figs. 1 and 2. We find that even if the size of the extra-dimension is very small, the correction is detectable for the both models (CFM I and CFM II) considered in this study. The work on other models, which feature non-constant accretion rate and time dependent mass of black hole is in progress.

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References
[1] Maartens R and Koyama K 2010 Living Rev. Relativity 13 5
[2] da Rocha R and da Silva J H 2012 Phys. Rev. D 85 046009
[3] da Rocha R, Piloyan A, Kuerten A M and Coimbra-Araujo C H 2013 Class. Quantum Grav. 30 045014
[4] Casadio R, Fabbri A and Mazzacurati L 2002 Phys. Rev. D 65 084040
[5] Casadio R and Mazzacurati L 2003 Mod. Phys. Lett. A 18 651–660
[6] Williams J G, Turyshiev S G and Boggs D H 2004 Phys. Rev. Lett. 93 261101