Interactive, Intelligent Tutoring for Auxiliary Constructions in Geometry Proofs

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ABSTRACT
Geometry theorem proving forms a major and challenging component in the K-12 mathematics curriculum. A particular difficult task is to add auxiliary constructions (i.e., additional lines or points) to aid proof discovery. Although there exist many intelligent tutoring systems proposed for geometry proofs, few teach students how to find auxiliary constructions. And the few exceptions are all limited by their underlying reasoning processes for supporting auxiliary constructions. This paper tackles these weaknesses of prior systems by introducing a new interactive geometry tutor, the Advanced Geometry Proof Tutor (AGPT). AGPT leverages a recent automated geometry prover to provide combined benefits that any geometry theorem prover or intelligent tutoring system alone cannot accomplish. In particular, AGPT not only can automatically process images of geometry problems directly, but also can interactively train and guide students toward discovering auxiliary constructions. We have evaluated AGPT via a pilot study with 78 high school students. The study results show that, on training students how to find auxiliary constructions, there is no significant perceived difference between AGPT and human tutors, and AGPT is significantly more effective than the state-of-the-art geometry solver that produces human-readable proofs.

INTRODUCTION
Geometry is an important, mandatory subject in the secondary school curriculum. Problem solving is an interesting part of learning geometry and offer unique challenges to students, both visually and mathematically [9, 17]. They also make geometry one of the most difficult subjects for students. The standard format of a geometry proof problem consists of a geometry figure, a set of constraints and a goal to be proven. Students are asked to write a step-by-step deduction using Euclidean axioms. Figure 1 depicts a sample problem, which is used throughout the paper to illustrate our approach. When a problem requires auxiliary constructions, i.e., drawing additional geometrical elements on the original problem figure, as part of the proof, its difficulty significantly increases since determining what and where to draw can be very challenging. No robust and efficient method for geometry theorem proving with constructions was known [15].

Consequently, easy-to-use and effective intelligent tutoring systems can help students learn the concepts and practice their problem solving skills. Unfortunately, the majority of existing geometry proof tutors [1, 11, 12] do not support teaching and training students how to add auxiliary constructions. Although the importance of auxiliary constructions motivates several efforts in the field of artificial intelligence [15, 18, 20], it is fair to question their suitability and effectiveness from the students’ learning perspective. Specifically, even if students have access to a solution/proof generated by such systems, and may understand how the auxiliary constructions help solve the given problem, can they find similar constructions when facing different problems? The answer is likely no without the students understanding the approach the systems adopt to add auxiliary constructions.

In this paper, we propose an interactive geometry tutor, the Advanced Geometry Proof Tutor (AGPT), that targets auxiliary constructions in geometry proof problems. Our key insight is that, in order for intelligent tutoring systems to be effective, they need to utilize the inner workings of powerful solvers (and systems in general) to train and help users. Specifically, AGPT leverages a recent automated geometry theorem prover [20], iGeoTutor, as the core back-end engine to offer the combined functionalities that neither a geometry theorem prover nor an intelligent tutoring system alone can accomplish. Traditional intelligent tutoring systems do not emphasize problem-solving capabilities and can handle only a limited spectrum of problems, thus constraining their tutoring support. In contrast, AGPT can take any user-supplied problem and leverage the power of the underlying geometry theorem prover. Compared to automated geometry theorem provers that focus on automated construction and solution discovery, AGPT refrains from directly presenting solutions to students. Rather, it interactively guides students toward discovering a solution on their own. AGPT borrows the classical pedagogical philosophy from an intelligent tutoring system’s stand point of view — forcing and guiding students to think on their own, but providing hints and tips whenever necessary.

We have conducted a pilot study with 78 high school students. In our study, we compare AGPT with both human tutors and...
Given the output from the Hough transform, we first remove distances between the line and the origin, and another parameter the angle of the vector from the origin to this closest point. Care is needed when removing the similar lines. According to our experimentation with the Hough transform, the detected lines are often shorter than the original lines. Thus, instead of removing one or the other when two lines are recognized as identical, we form a new line by picking the two furthest endpoints among the total four points extracted from the two original lines. In other words, we generate a longest line with the same polar coordinates by stretching the original lines. Finally, we keep the generated line and safely remove any duplicates.

2. In terms of the polar coordinates, the data now represent the same set of lines as those in the original geometry figure. The focus of the second step is to deal with possible discrepancies regarding the position of each endpoint. We adopt a simple, direct approach by defining the neighborhood of each endpoint and merging those that appear in the same neighborhood by calculating the mean.

3. The intersection point computation step is to recover relationships among points and lines that are not automatically inferred by the Hough transform. We consider two types of intersections: (1) two lines where one endpoint of one line lies on the other line, and (2) two lines crossing each other (i.e. in the shape of an “X”). In the first case, the goal is to decide where the target point lies. This can be accomplished simply by computing the distance between the target point and a candidate line. However, it is worth noting that due to variations on the position of the detected point and line by the Hough transform, the point is frequently found off the line, such as falling short or exceeding or tending toward the “X” shape of the intersection. Thus, we introduce a threshold to take those minor differences into account. In the second case, intersection points can be computed by combining the linear functions derived from the existing endpoints and incorporating relationships specifying that the intersection falls on both lines.

### Constraint Synthesis

Apart from recognizing the geometry figure in the user-uploaded image, the detection system also automatically extracts relevant constraints for users to select to further increase the user-friendliness of AGPT.

The geometrical elements for which the tutoring system attempts to infer constraints are segments, angles and special polygons. A segment is measured from both the quantitative and positional perspectives to produce four kinds of predicates, including perpendicularity, parallelism, equality and addition of length expressed as $A + B = C$. As for angles, our system only checks the occurrence of angle bisectors to allow a clean, concise constraint representation. The angle equality constraints can often be derived from existing line constraints. The special polygons that we consider include parallelograms, rectangles, diamonds, squares and equilateral triangles. To avoid possible duplicate constraints, as soon as a special polygon is discovered, our system automatically removes the constraints that can be derived from it.
Problem Specification Representation

Given the geometry figure that has been detected and refined along with its associated constraints on the target elements, our system can present the problem specification using SVG elements to the user. The lines displayed on the canvas are all attached with action listeners to allow the user to interactively supplement missing constraints, if any. Alternatively, the user may designate a line (or angle) with its endpoints’ identifiers. The user interface depicted in Figure 3 shows the result of the three-step image processing on a user uploaded picture.

Tutoring System

After the user has specified the problem through the detection system, the tutoring system will take control and start handling all interactions with the user. We first briefly review iGeoTutor’s template matching-based proof technique, upon which our tutoring methodology is built. Then, we present the major decisions that we have made in designing the tutoring interactions. Next, we describe the entire interactive tutoring process coordinated by the tutorial planner. Finally, we end with a live AGPT’s demo session.

Background: Template Matching-Based Proof Technique

iGeoTutor is introduced recently as an automated geometry theorem prover capable of producing human-readable proofs. It is shown to significantly outperform the previous state-of-the-art system GRAMY [15]. The main technique behind iGeoTutor is its template matching-based approach to tackle the key challenge in geometry theorem proving, i.e. auxiliary construction. The idea is to guide the search for additional constructions by completing a set of standard shapes, which are called templates. Indeed, the procedure starts by finding suitable candidate shapes among the templates; Figure 4 depicts the six templates that iGeoTutor currently employs. A shape is a good candidate if the given problem figure can closely match it in terms of the given constraints. Once a match is identified, iGeoTutor can supply the missing constructions to complete the problem figure. These steps are repeated until a proof is found or a search limit is reached.

Algorithms 1 and 2 summarize iGeoTutor’s high-level proof procedure. Function Prove takes as input a problem figure and infers knowledge that describes the problem. If a proof is found without auxiliary constructions, the function terminates with the proof. Otherwise, the recursive routine ProofByTemplate is invoked. It first ranks the templates according to how closely they match the given problem figure. Then, it iterates over the ranked list of templates, and for each, it adds auxiliary constructions to complete the current problem figure. With the new knowledge in the current problem figure, it attempts to find a proof again. This recursive process repeats until the system either finds a proof or reaches a certain search depth. Evaluation results suggest that a search depth of three is often sufficient in practice.

Design of Interactive Tutoring

This section presents in detail our design of the tutoring interactions, including the high-level tutoring strategy, user interface, feedback mechanism, and the degree to which the tutoring system manages student interactions.

High-Level Tutoring Strategy

Given the presented evidence [20], iGeoTutor is clearly highly effective in finding auxiliary constructions in geometry theorem proving. However, iGeoTutor’s underlying technique for solving problems is opaque to the students. Thus, we design tutoring interactions

| Procedure | Prove (GeometryProblem gp) |
|------------|---------------------------|
| begin      |                           |
| knowledge  |← KnowledgeExtraction(gp)  |
| if knowledge.reasoning() then |
| return knowledge.getProof()  |
| else       |                           |
| return ProofByTemplate(0, knowledge) |

| Function | ProofByTemplate (int depth, Knowledge knowledge) |
|----------|--------------------------------------------------|
| begin    |                                                  |
| if depth < maxDepth then |
| matchedTemplates ← TemplateSearchProcedure(knowledge) |
| foreach template in matchedTemplates do |
| constructions ← SynthesisConstruction(template) |
| foreach construction in constructions do |
| knowledge.addConstruction (construction) |
| if knowledge.reasoning() then |
| return knowledge.getProof()  |
| else                           |
| return ProofByTemplate(depth + 1, knowledge) |

Figure 3: Extracted geometry figure and constraints.
to expose iGeoTutor’s template matching-based approach to students, and engage them in the problem-solving process so that they may practice and eventually master the skill.

**User Interface** The interactive tutoring provided by AGPT follows an iterative process. During each iteration where iGeoTutor automatically finds a template to instantiate, AGPT pauses and asks the student to manually match a template. Because the student does not have access to the underlying template figures, AGPT tailors the exercise of template matching as a two-phase task. During the first phase, AGPT presents a multiple choice question by (1) displaying the exact template figures to be recovered (annotated as the yellow bounding box in Figure 5) and (2) describing each choice via a help message on how the template figure can be recovered (annotated as the blue bounding box). It also reminds the student the known conditions and conclusion (annotated by the red bounding box). Figure 5 depicts the layout of the entire interface.

After the student selects a template, she will be presented with the second part of the task to specify the constructions to recover her selected template. In particular, the student needs to make use of the existing conditions and conclusion (for backward reasoning if necessary), and obtain the template shape from the original problem figure by drawing additional lines (e.g., connecting existing points or introducing points to make new segments). To train students the concept of template matching, AGPT allows them to draw any segments, anywhere on the canvas. AGPT also offers support to help students conveniently convey their intended constructions: (1) allowing students to draw segments on the problem figure via mouse movements, (2) automatically adjusting the drawn segments to align with existing nearby points on the figure, and (3) labeling new intersection points whenever lines intersect.

The correctness of user input is determined by checking whether the student’s drawing along with some existing segments satisfy the geometrical constraints associated with the selected template figure. It is worth mentioning that when AGPT verifies the user construction, it can tolerate certain

![Figure 4: The six distilled template figures.](image)

**Algorithm 3: Interactive tutoring process**

```plaintext
function InitiateInteraction (GeometryProblem gp)
begin
  proof ← Prove(gp)
  templateTrace ← GetTemplateTrace(proof)
  knowledge ← KnowledgeExtraction(gp)
  return TemplateChoicesGeneration (knowledge, templateTrace)
end

function TemplateChoicesGeneration (Knowledge knowledge, Templates templateTrace)
begin
  matchedTemplates ← TemplateSearchProcedure(knowledge)
  return SortTemplateChoices(matchedTemplates, templateTrace)
end

function BackTrackKnowledge (int turningPoint, Templates templateTrace)
begin
  knowledge ← RestoreKnowledgeBase(turningPoint)
  return TemplateChoicesGeneration (knowledge, Templates templateTrace)
end

function ReasoningWithChoice (Knowledge knowledge, Templates templateTrace, int depth, int turningPoint, Template choice, Construction model)
begin
  if ¬isCorrectConstruction(choice, model) then
    turningPoint ← depth
    SaveKnowledgeBase(knowledge, depth)
    knowledge.addConstruction(model)
    if knowledge.reasoning() then
      DisplayCompleteProof()
    else if + + depth == maxDepth then
      BackTrackKnowledge (turningPoint, templateTrace)
    else
      choices ← TemplateChoicesGeneration (knowledge, templateTrace)
      if choices = NULL then
        BackTrackKnowledge (turningPoint, templateTrace)
      else
        return choices
    end
  end
end
```
degree of inaccuracy in line and point positioning to increase AGPT’s ease-of-use.

Given the construction that the student has specified, if it is not correct, AGPT will initiate a new iteration of the two-phase task. Otherwise, AGPT will end the tutoring session.

**Feedback Mechanism** AGPT provides feedback exclusively in the second phase of the question when students provide constructions. During the first several failed attempts, AGPT provides minimal feedback (e.g., “Try again”). Afterward, if the student makes further failed attempts, AGPT highlights the relevant matched segments on the current problem figure. If the student still cannot discover a correct construction, AGPT will reveal a solution and perform the corresponding step.

**System Control vs. Student Free Exploration** A good balance is to let students explore the solution space on their own, and whenever they need help, AGPT uses the closest solution to guide students based on their own effort. This approach requires AGPT to query iGeoTutor for all possible solutions either before the tutoring session begins or right after each time when students make an attempt. Both would be computationally expensive and lead to a poor user experience. To alleviate the computational burden, AGPT requests iGeoTutor to compute only one solution and use it to guide students. Note that AGPT does not force students to follow this solution to maximize their learning experience. However, AGPT does take special measures to help students stay on the solution path: (1) it always includes the solution template among the template choices in the first-phase’s multiple choice question, and (2) it produces hints solely based on the solution construction after a correct template has been selected and the user seeks feedback from AGPT.

**Tutorial Planner**

In this section, we describe how the tutorial planner coordinates different functional modules within the tutoring system to produce interactions with the student.

**Overview** The tutorial planner begins by invoking `Prove` in Algorithm 1 to find a proof to the given problem. If a proof is found, it collects the full construction trace (including template searching and matching) and uses the trace to direct the subsequent interactions with the student.

Note that the interactions will not be interrupted unless the student (1) reaches the depth bound without finding a construction, or (2) has selected a template that leads to a fruitless path (for instance, there does not exist any matching templates in the next step). In either case, the tutorial planner will backtrack to the last step where the student has made a wrong choice or an invalid construction, and resume from there.

**Core Algorithms** Algorithm 3 describes the main routines that the tutorial planner employs to produce the interactive tutoring process. The function `InitializeInteraction` starts the interaction process by invoking `Prove` to find the proof’s construction trace. It then invokes the function `TemplateChoicesGeneration` to enumerate the template choices before presenting them to the student. Once the student has selected a template, the next step is handled by `ReasoningWithStudentChoice`: according to the student’s selection, either another round of template choice generation is initiated when a solution has not yet been found, or a proof is synthesized when the student succeeds in finding a correct auxiliary construction. Note that

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1The simplest solution is defined by the number of auxiliary lines to be added and templates to be matched according to Algorithm 2.
ReasoningWithStudentChoice also performs backtracking when needed as discussed earlier.

Generating Template Choices As shown above, the function TemplateChoicesGeneration delegates the task of finding matched templates to TemplateSearchProcedure. After all candidate templates have been returned from TemplateSearchProcedure, up to four templates will be used and randomly ordered. As mentioned earlier, when the student is still on the right search path toward a solution, TemplateChoicesGeneration needs to include the solution template from the construction trace.

An Example Tutoring Session Given the example problem in Figure 1, the student can consult AGPT by first capturing the problem figure using the camera on their mobile devices, and then uploading the picture to our system. Next, the student needs to input the constraints and goal for the given problem figure through the UI displayed in Figure 3, after which the student will be presented with the initial round of template selection shown in Figure 5.

As explained earlier, the template selection is presented as a multiple choice question, placed together with the help message and template shape. The student, upon selecting a template, will be led to a canvas shown in Figure 6, where she can practice by drawing freely any auxiliary lines to match the selected template. In order to differentiate the student’s constructions from existing figure elements, her drawing is displayed as black dashed lines. As shown in Figure 6, the problem figure is automatically centered on the canvas to best accommodate the student’s drawing that falls outside the problem figure.

Found a proof, now guide you to discover it on your own

![Diagram of a triangle with dashed lines](image)

Figure 6: A live AGPT’s tutoring session: user construction.

AGPT, upon taking the student’s constructions into account will either initiate a new round of template selection based on the student’s added constructions or backtrack to a previous round if a solution is yet to be found. Otherwise, our system will bring the session to its end, where the student is offered the opportunity to view the entire proof. Our submitted supplemental materials contain a demo video to give a more detailed and concrete illustration. It is worth to mention that proof writing on a complete geometry problem figure is not the focus of our work considering that many of the existing geometry proof tutors are shown effective both clinically and empirically. Consequently, we allow students to access the proof whenever they have added the correct auxiliary constructions.

EVALUATION This section presents the pilot study that we conducted to evaluate the utility of AGPT in helping students learn auxiliary constructions in geometry proof problems. In particular, we compare AGPT against human tutors and iGeoTutor, a recent non-interactive geometry theorem prover [20].

Participants We have contacted the geometry teacher in a neighboring local high school, who helped us recruit all ninth grade students (78 in total) for this study. Each participant is given a booklet that contains all the geometry theorems that they should find helpful during the study.

Procedure First, all participants took a pre-test. Then, we asked the geometry teacher to divide the students into three groups (26 students each). The first group interacted with a human tutor (the same geometry teacher with more than five years experience in teaching geometry) regarding the questions on the pre-test (A Group). The second group used iGeoTutor to explore the solutions to the pre-test problems (B Group). The third group interacted with AGPT (C Group). We allocated up to 15 minutes for students to practice and familiarize with the UIs of iGeoTutor and AGPT (both running on Samsung Chromebook Plus provided by the local high school), during which students are allowed to seek help from the study assistants. However, we restrict the questions to be only UI-related. Students could only work individually during the tutoring sessions with iGeoTutor or AGPT. Finally, all students took a post-test. Both the pre-test and post-test were paper based and contained three geometry proof problems shown in Table 1. Each participant was given 30 minutes to complete the pre-test and post-test, and one hour for interacting with the human tutor (the geometry teacher), iGeoTutor, or AGPT (excluding the 15 minutes training on the UIs).

Student Interactions Student in Group A went through a standard class, during which the geometry teacher explained the pre-test problems and solutions, and answered their questions. For Group B students, there was little interaction to be recorded. Students first drew the problem figure and entered the geometrical constraints, and then waited for a solution, which they spent the majority of their time studying. We logged the detailed interaction history for each student using AGPT and report them in Table 2. For all Group C students, none managed to find a solution to any problem after seven attempts. In each attempt, facing the two-phase task, a student can have the following responses:

- **W**: The student does not choose the same template as the solution template at the same iteration. For example, assume

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2Our geometry tutor accepts both printed and handwritten figures.

3Ninth grade is the typical grade level for meeting the Geometry standards in the United States.
### Geometry Proof Problem

| Problem  | Auxiliary Construction | Template Matching |
|----------|------------------------|-------------------|
| **PRE1.** | ![Diagram](image1)  
**Goal:** $DF = EF$  
**Given:** $AE \perp CD$,  
$AE \perp BE$, $BF = CF$  

(I). Extend BE and DF to meet at G to match the template Opposite Triangle with shape $BGFDC$.  

OR  

(II). Extend EF to intersect DC at H to match the template Opposite Triangle with shape $BEFHC$. |
| **PRE2.** | ![Diagram](image2)  
**Goal:** $\angle ABD = \angle CBD$  
**Given:** $AC \perp CB$,  
$AC = CB$, $AE \perp BE$  
$BD = 2AE$  

(I). Extend AE and BC to meet at F to match the template Isosceles Triangle (1) with shape $BAEF$. |
| **PRE3.** | ![Diagram](image3)  
**Goal:** $AB = AC + CD$  
**Given:** $\angle C = 2\angle B$,  
$\angle BAD = \angle CAD$  

(I). Introduce a point F on AB to make $AF = AC$ and connect DF to match the template Congruent Triangle with shape $AFDACD$.  

OR  

(II). Extend AC to E such that $AE = AB$ and connect DE to match the template Congruent Triangle with shape $ABDAED$. |
| **POST1.** | ![Diagram](image4)  
**Goal:** $\angle BAE = \angle DAE$,  
$\angle ABE = \angle CBE$  
**Given:** $AB = AD + BC$,  
$DE = EC$, $AD \parallel BC$  

(I). Extend AD and BE to meet at G to match the template Isosceles Triangle (1) with shape $ABEG$.  

OR  

(II). Extend AE and BC to meet at F to match the template Isosceles Triangle (1) with shape $BAEF$. |
| **POST2.** | ![Diagram](image5)  
**Goal:** $KH \parallel BC$  
**Given:** $\angle ACQ = \angle BCQ$,  
$\angle ABP = \angle CBP$, $AK \perp CQ$,  
$AH \perp BP$  

(I). Extend AK to intersect BC at D to match the template Isosceles Triangle with shape $CAKD$. Extend AH to intersect BC at E to match the template Isosceles Triangle (1) with shape $BAHE$. |
| **POST3.** | ![Diagram](image6)  
**Goal:** $DC \perp AC$  
**Given:** $\angle BAD = \angle CAD$,  
$AB = 2AC$, $AD = BD$  

(I). Introduce a point F on AB to make $AF = AC$ and connect DF to match the template Congruent Triangle with shape $AFDACD$.  

OR  

(II). Extend AC to E such that $AE = AB$ and connect DE to match the template Congruent Triangle with shape $ABDAED$. |

Table 1: All the problems used in our evaluation. **PRE1-PRE3** are the problems on Pre-Test, similarly **POST1-POST3** are the problems on Post-Test. For problems on the Pre-Test, (I) is AGPT’s adopted solution while (II) is discovered by students. The solutions to the Post-Test problems are all discovered by students.
that a solution template trace is only an instance of Isosceles Triangle (1). If the student selects the Opposite Triangle template in her first attempt or Isosceles Triangle (1) in her second attempt, in either case the student’s response will be considered W.

- **M:** The student selects the solution template at the right step and obtains a solution construction through AGPT’s feedback mechanism.

- **C:** The student selects the solution template at the right step and supplies a solution construction on his own.

- **J:** The student selects the solution template at the right step and supplies a non-solution construction on his own.

It is worth to mention that a non-solution template/construction does not necessarily indicate a wrong solution. In fact, students managed to find correct alternatives to PRE1 and PRE3, marked as (II) in the last column in Table 2. There are two other symbols expressed in parenthesis B, meaning that AGPT backtracked the student to where she was diverted away from the solution path. D means that the student pressed the back button to return to the previous stage. From Table 2, we highlight below a few important observations:

- There are twenty-one students who solved at least one problem, among whom fifteen found solutions through AGPT’s feedback mechanism, indicating that the feedback was generally helpful. Five of the fifteen students did not give a single correct construction on their own, and relied completely on AGPT’s feedback. In addition, eight students worked out different solutions from AGPT’s, likely indicating that these students engaged in thoughtful problem-solving.

- There were 452 attempts (392 ended up in W) and 153 redirections (103 backtracks with 50 self-control) made in total by group C students. The data suggests that it takes each student on average almost six attempts and two redirections to solve one problem. This provides strong evidence that students have experienced through a difficult process. More importantly, AGPT’s support of encouraging students to explore their own ideas has benefited them according to their performance on the post-test.

- Some students displayed noticeable frustrations throughout the course of the study. They likely ran out of patience after seven attempts and gave up. How to incorporate richer support without compromising students’ learning would be interesting future work.

## Results

Table 3 compiles each student’s performance on the Pre-Test and Post-Test. In particular, C represents the number of correct constructions a student found, S represents the number of correct solutions a student was able to work out, and P indicates the number of correct constructions a student found, S represents the number of correct solutions a student was able to work out, and P introduces a new measure of recording students’ performance which we will explain in detail next.

### Analysis

All students did better on the post-test than the pre-test according to Table 3. We have run a paired t test to compare each participant’s performance (in terms of number of solved problems) across the two test sets within each group. The results show that all three groups of students’ post-test performance is significantly better than that of the pre-test ([t(25) = 3.2609, p = 0.0032], [t(25) = 2.5733, p = 0.0164], and [t(25) = 3.5634, p = 0.0015] for Group A, B and C respectively).

Using the improvement score, defined as the difference between the number of correctly solved problems on the post test over the pre-test, we have run an ANOVA test across the three groups of students. The results do not reveal any significant difference among the three interaction approaches ([F(2, 75) = 2.13, P = 0.13]). Based on an inspection of the post-test student solutions, we conjecture that the results were due to: (1) the number of problems on both tests is too few such that the (in)correctness of each problem would lead to
significant differences in the scores, and (2) students still have difficulties in solving proof problems even on problems that do not require auxiliary constructions. To mitigate these two factors, we awarded partial credits for students’ incomplete proofs (w.r.t. to the P column in Table 3). Partial credits were only given to solutions with correctly identified auxiliary constructions. The rationale is that student’s failure in completely solving a problem with the correct auxiliary constructions is due to the student’s lack of knowledge and skills in proof construction rather than auxiliary construction. The criterion used to grade partial solutions is based on comparing students’ answers with the reference solutions. In particular, we calculate the ratio of the number of correct proof steps students did manage to work out and the total number of steps.

Taking partial credits into account, we recomputed the pre-test and post-test scores for each student registered under the P column in Table 3. Then, we ran an ANOVA test on the improvement scores again. The results do reveal significant difference among the three interaction approaches ($F(2, 75) = 3.23, P = 0.045$). Post hoc Tukey test indicated no significant difference between Group A group and Group B, nor between Group A group and Group C, whereas the difference between Group B and Group C is significant ($P < 0.05$). To conclude based on the two ANOVA tests we ran, AGPT is shown to be significantly more effective than iGeoTutor in helping students learn auxiliary constructions, even though there is no significant difference detected between AGPT and the human tutor.

Discussion
The results from our relatively large scale study were consistent with what we had hypothesized. Below we summarize reasons for AGPT’s effectiveness. First, iGeoTutor’s powerful underlying proof strategy and technique facilitated AGPT to be successful. More importantly, iGeoTutor alone is insufficient, as evidenced by Group B students in the study who showed no improvement when they approached new, but similar problems. Instead, Group C students benefited significantly more from AGPT, our interactive tutoring system built upon iGeoTutor, and designed to train students its underlying techniques via template matching and interactive construction of auxiliary elements. Finally, AGPT’s tutoring style of leading students to go through a challenging, engaging, and thoughtful process might have frustrated some of them occasionally, but left a positive impact overall in helping them learn.

RELATED WORK
This section surveys closely related work on automated geometry theorem provers and intelligent geometry proof tutors.

Automated Geometry Theorem Proving
A series of automated geometry proof systems based on decision methods have dominated the field for decades. Notable
work includes Wu-Ritt’s characteristic set method [3–5], the Gröbner method [2,13], the resultant method [10], the elimination method [19], and the parallel numerical method [21]. The general approach behind these provers is to use sophisticated algebraic theories to determine the validity of algebraic formulations of the geometry problems. While they are powerful in handling a large number of nontrivial theorems in practice, they have significant drawbacks for use in education. The most important is that the generated proofs are unnatural and incomprehensible as these systems do not approach geometry problems like how secondary school students are taught to.

Recognizing the weaknesses of the aforementioned systems, Chou has done seminal work on the automated generation of human-understandable proofs for geometry proof problems. Their work is the pioneer in attempting to bridge the gap between automated geometry theorem proving to intelligent geometry proof tutoring. The proposed area method [6] has been the most powerful geometry theorem proving algorithm in this domain. However, the work has only limited success in educational settings because the generated proofs follow specialized, nonstandard area axioms rather than the standard Euclidean axioms from traditional geometry textbooks.

GRAMY [15] designed by Matsuda et al. is the first proof system capable of generating human-readable proofs using Euclidean axioms that students learn in schools. However, as the recent work on iGeoTutor [20] shows that GRAMY’s approach is quite inexpressive, and also ineffective and can cause combinatorial explosion when multiple construction steps are required.

Intelligent Geometry Proof Tutor
The application of intelligent tutoring systems in the domain of high school geometry proof problems dates back decade ago. Among those geometry proof tutors [1,11,12], Geometry Tutor [1] is one of the earliest that are based on formal geometry rules to solve problems as well as to teach students. A major weakness of Geometry Tutor (as well as its companions) is that they cannot handle problems that require auxiliary constructions; the rules designed in the knowledge are based on the assumption that a given geometry figure is sufficient for constructing a proof. Cobo et al. propose AgentGeom [7] to offer students the cognitive and metacognitive support to help them develop problem solving and mathematical reasoning skills. A major advantage of AgentGeom over Geometry Tutor is its ability to capture the students’ thought process by adopting discursive and graphic style of interactions. Additionally, it can feed students hints that match their state of mind if necessary. Although AgentGeom’s corpus does contain problems that require auxiliary constructions, its tutoring activities only follow hard-coded proofs, and more importantly do not teach students a general, systematic approach for finding auxiliary constructions. Perhaps the closest to AGPT is Matsuda et al. [14,16]’s systems, which focus on teaching students how to find auxiliary constructions in geometry proof problems. However, both systems’ tutoring capabilities are based on GRAMY’s auxiliary construction procedure and thus suffers from poor efficiency and expressivity as mentioned earlier, especially when dealing with challenging problems. In addition, neither demonstrates student learning gains from the interactive tutoring support. In contrast, AGPT builds upon the state-of-the-art technology for auxiliary constructions, and our pilot study shows that AGPT helps students acquire the skills to find auxiliary constructions as effectively as human tutors, and significantly more effectively than the geometry theorem prover alone.

CONCLUSION
In this paper, we have presented AGPT, a geometry tutoring system that leverages its underlying geometry theorem prover to interactively train and guide students on discovering auxiliary constructions on their own. AGPT’s effectiveness is substantiated by our pilot study with 78 high school students. We envision that AGPT’s high-level conceptual approach of disclosing and imparting the inner working of powerful solvers to students can be fruitfully adapted to other important educational topics, such as algebra, calculus, physics, etc..

Our immediate future work is to deploy AGPT to the participating high schools in our pilot study and continue refining our system based on student and teacher feedback.

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