EMBEDDINGS OF HYPERBOLIC KAC-MOODY
ALGEBRAS INTO $E_{10}$

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ABSTRACT. We show that the rank 10 hyperbolic Kac-Moody algebra $E_{10}$ contains every simply laced hyperbolic Kac-Moody algebra as a Lie subalgebra. Our method is based on an extension of earlier work of Feingold and Nicolai.

1. Introduction

Since their discovery [9,15] by Kac and Moody, Kac-Moody algebras have been playing increasingly important roles in diverse subfields of mathematics and physics. The affine Kac-Moody algebras are by now as well understood as the finite dimensional simple Lie algebras classified by Cartan and Killing. Indefinite type Kac-Moody algebras however remain a notoriously intractable part of the theory. In spite of much work in this direction (see [7,11] and the references in [8]), obtaining detailed information about the structure of these Lie algebras seems out of reach at present. Most of the available results concern the subclass of hyperbolic Kac-Moody algebras. Such algebras only exist in ranks 2-10 and can be completely classified [16] (see also [5] for some missing diagrams). Among these, the algebra $E_{10}$ has been singled out for its relevance to string theory and has received much attention in recent times (e.g. [1–4,12–14]).

In [8], Feingold and Nicolai studied subalgebras of hyperbolic Kac-Moody algebras. They showed that the rank 3 hyperbolic Kac-Moody algebra $\mathcal{F} = HA_1^{(1)}$ contains every rank 2 hyperbolic Kac-Moody algebra with symmetric generalized Cartan matrix; in fact $\mathcal{F}$ was also shown to contain an infinite series of indefinite type Kac-Moody algebras. Analogously, it was shown that there are infinitely many inequivalent Kac-Moody algebras of indefinite type that occur as Lie subalgebras of $E_{10}$.

This motivates the main question of this letter: which Kac-Moody algebras of hyperbolic type occur as Lie subalgebras of $E_{10}$? To answer this question, we extend the method of Feingold-Nicolai and formulate some general principles for constructing Lie subalgebras. These principles are then used to prove our main result: Every simply laced hyperbolic Kac-Moody algebra occurs as a Lie subalgebra of $E_{10}$. This statement can be viewed as further evidence of the distinguished role played by $E_{10}$ in the family of

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simply laced hyperbolic Kac-Moody algebras. Though these subalgebras, being of hyperbolic type themselves, are poorly understood objects, their embeddings in $E_{10}$ may nevertheless shed more light on the structure of $E_{10}$.

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2. Simply laced hyperbolic diagrams

We start with a brief summary of the basic definitions and notation concerning Kac-Moody algebras. The reader is referred to [10] for full details. An $n \times n$ integer matrix $A = (a_{ij})$ is called a generalized Cartan matrix if it satisfies (i) $a_{ii} = 2 \forall i$, (ii) $a_{ij} \leq 0 \forall i \neq j$, and (iii) $a_{ij} = 0 \implies a_{ji} = 0$. Given a generalized Cartan matrix $A$, one defines the associated Kac-Moody algebra $g = g(A)$ (of rank $n$) to be the Lie algebra with $3n$ generators $\alpha_i, e_i, f_i \; (i = 1, \cdots, n)$ subject to the relations $[\alpha_i, \alpha_j] = 0, [\alpha_i, e_j] = a_{ij}e_j, [\alpha_i, f_j] = -a_{ij}f_j, [e_i, f_j] = \delta_{ij}\alpha_i$ for all $i, j$, and the Serre relations $(\text{ad} \; e_i)^{-a_{ij}+1}(e_j) = 0 = (\text{ad} \; f_i)^{-a_{ij}+1}(f_j)$ for $i \neq j$. If $A$ is nonsingular, the Cartan subalgebra $h$ is the span of the $\alpha_i$; if $A$ is singular, one must also include certain derivations in $h$. The simple roots $\alpha_j, \; j = 1 \cdots n$ are linearly independent elements in $h^*$ which satisfy the relation $\alpha_j(\alpha_i) = a_{ij} \forall i, j$; the integer lattice $Z(\alpha_1, \cdots, \alpha_n)$ spanned by them is the root lattice $Q$. One has the root space decomposition:

$$g = h \oplus (\oplus_{\alpha \in \Phi} g_\alpha)$$

where each $g_\alpha = \{x \in g : [h, x] = \alpha(h)x, \forall h \in h\}$ is finite dimensional and $\Phi = \{\alpha \in h^* : g_\alpha \neq 0\} \subset Q$ is the set of roots of $g$. With $Q^+ = \bigoplus_{\alpha \in \Phi} \mathbb{Z}_{>0}(\alpha)$, $Q^- = \bigoplus_{\alpha \in \Phi} \mathbb{Z}_{<0}(\alpha)$, and $Q = Q^+ \cup Q^-$, we also have $\Phi = \Phi^+ \cup \Phi^-$. The Weyl group $W \subset \text{GL}(h^*)$ of $g$ is generated by the simple reflections $r_i$ of $h^*$ defined by $r_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i \forall j$. The elements of $\Phi$ that are $W$-conjugate to a simple root are termed real roots; the remaining are called imaginary roots. Letting $\Phi_{re}$ and $\Phi_{im}$ denote the subsets of real and imaginary roots respectively, we have $\Phi = \Phi_{re} \cup \Phi_{im}$.

If the matrix $A$ is symmetric, $g$ is termed a simply laced Kac-Moody algebra. We will only consider simply laced algebras in the rest of this letter. In this case, the bilinear form $(,)$ on $h^*$ defined by $(\alpha_i, \alpha_j) = a_{ij} \forall i, j$ is symmetric and $W$-invariant (more generally, such a form exists when $A$ is symmetrizable, i.e., when there exists a diagonal matrix $D$ such that $DA$ is symmetric). As mentioned above, the real roots are $W$-conjugates of the simple roots; since $(\alpha_i, \alpha_i) = 2 \forall i$, the $W$-invariance of the form means $(\beta, \beta) = 2$ for all real roots $\beta$. For $\gamma \in \Phi_{im}$, it is true that $(\gamma, \gamma) \leq 0$ [10]. The Dynkin diagram associated to a symmetric generalized Cartan matrix $A$ is a graph on $n$ vertices, with vertices $i$ and $j$ joined by $|a_{ij}|$ edges for
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Table 1. “Regular” simply laced hyperbolic diagrams

| Diagram $HA_k^{(1)}$ | Diagram $HD_k^{(1)}$ |
|----------------------|----------------------|
| $2 \leq k \leq 7$   | $4 \leq k \leq 8$   |

Let $X$ be an indefinite type Dynkin diagram. If every proper, connected subdiagram of $X$ is of finite or affine type, then $X$ (or alternately, its associated Kac-Moody algebra) is said to be of hyperbolic type. It is clear that $X$ is of hyperbolic type if and only if the deletion of any one vertex of $X$ results in a diagram each of whose connected components is of finite or affine type. Hyperbolic Kac-Moody algebras only exist in ranks $\leq 10$. For each $a \geq 3$, $g([\begin{pmatrix} 2 & -a \\ -a & 2 \end{pmatrix}])$ is a simply laced, hyperbolic Kac-Moody algebra of rank 2. In ranks 3-10, there are exactly 23 simply laced hyperbolic Kac-Moody algebras [16]. Their Dynkin diagrams can be organized into 3 finite families (containing 15 diagrams, see Table [1] and 8 exceptions (termed irregular diagrams, Table [2]). We remark that if $X$ is a simply laced affine diagram, the notation $HX$ refers to its “hyperbolic extension”, obtained by adding one more vertex, connected only to the zeroth node of $X$. We use $E_{10}$ as alternative notation for $HE_8^{(1)}$; we will also let $E_{10}$ denote the Kac-Moody algebra whose Dynkin diagram is $E_{10}$.

The main result of this letter is the following:
Theorem 1. Given any simply laced hyperbolic Kac-Moody algebra \( \mathfrak{g} \), there is a Lie subalgebra of \( E_{10} \) that is isomorphic to \( \mathfrak{g} \).

To prove this theorem, we will use the following result [8, Theorem 3.1] (restated for our situation):

Theorem 2. (Feingold-Nicolai) Let \( A \) be an \( n \times n \) symmetric generalized Cartan matrix and \( \mathfrak{g} = \mathfrak{g}(A) \). Suppose \( k \leq n \) and \( \beta_1, \ldots, \beta_k \in \Phi^+_\text{re} \) such that \( \beta_i - \beta_j \notin \Phi \) \( \forall i \neq j \). Choose nonzero elements \( E_{\beta_i} \in \mathfrak{g}_{\beta_i} \) and \( F_{\beta_i} \in \mathfrak{g}_{-\beta_i} \) for all \( i \). The Lie subalgebra of \( \mathfrak{g} \) generated by \( \{E_{\beta_i}, F_{\beta_i} : i = 1, \ldots, k\} \) is a Kac-Moody algebra of rank \( k \), with generalized Cartan matrix \( C = [c_{ij}] = [(\beta_i, \beta_j)] \).

We also state the following lemma (implicit in [8]) that simplifies the task of verifying the hypothesis of theorem 2.

Lemma 1. With notation as in theorem 2, suppose \( \beta, \gamma \in \Phi^+_\text{re} \) satisfy \( (\beta, \gamma) \leq 0 \), then \( \beta - \gamma \notin \Phi \).
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Proof: Since $(\beta, \beta) = (\gamma, \gamma) = 2$, we have $(\beta - \gamma, \beta - \gamma) = (\beta, \beta) + (\gamma, \gamma) - 2(\beta, \gamma) \geq 4$. Since $\beta - \gamma$ has positive norm, it cannot be in $\Phi_{im}$; since its norm is not 2, it cannot be in $\Phi_{re}$ either.

Thus, in theorem 2 if $\beta_1, \ldots, \beta_k \in \Phi_{re}^+$ are such that $C := [(\beta_i, \beta_j)]$ is a generalized Cartan matrix (i.e., $(\beta_i, \beta_j) \leq 0 \forall i \neq j$), the subalgebra generated by the $E_{\beta_i}, F_{\beta_i}$ is automatically a Kac-Moody algebra with generalized Cartan matrix $C$.

**Definition 1.** In the notation of theorem 2 if $D$ denotes the Dynkin diagram of $g$ and $D'$ is the Dynkin diagram corresponding to the generalized Cartan matrix $C = [(\beta_i, \beta_j)]$, we will call $D'$ a root subdiagram of $D$. We will denote this by $D' \preceq D$.

As is clear from the definition, root subdiagrams need not be subdiagrams of $D$. The following statement is easily seen to imply theorem 1.

**Proposition 1.** Every simply laced hyperbolic Dynkin diagram occurs as a root subdiagram of $E_{10}$.

Sections 3 and 4 will be devoted to the proof of proposition 1.

### 3. Constructing root subdiagrams

The goal of this section is to formulate four general principles for constructing root subdiagrams of certain kinds of Dynkin diagrams. For the first two principles, we let $X$ denote a simply laced affine Dynkin diagram and $Y = HX$, the hyperbolic extension of $X$:

![Diagram](image)

Let the simple roots of $X$ be $\alpha_0, \alpha_1, \ldots, \alpha_r$ and let $\alpha_{-1}$ be the simple root corresponding to the extra vertex of $Y$. Let $\delta_X$ denote the null root of $X$; thus $\delta_X = \sum_{i=0}^{r} n_i \alpha_i$ ($n_i \in \mathbb{Z}_{\geq 0}$, $n_0 = 1$) and $(\delta_X, \alpha_i) = 0 \forall i = 0, \ldots, r$. Note that this means $(\delta_X, \alpha_{-1}) = -1$.

**Principle A:** Define $\beta_{-1}, \beta_0, \ldots, \beta_r$ as follows:

$$\beta_i = \alpha_i \quad (0 \leq i \leq r), \quad \beta_{-1} = r_{-1}(\delta_X + \alpha_0)$$

where $r_{-1} \in W$ is the simple reflection corresponding to $\alpha_{-1}$. Since $\delta_X + \alpha_0 = 2\alpha_0 + \sum_{i=1}^{r} n_i \alpha_i$ and the vertex $-1$ is only connected to vertex $0$, one observes that

$$\beta_{-1} = \delta_X + \alpha_0 + 2\alpha_{-1}$$

Since $\delta_X + \alpha_0$ is a real root of the affine Kac-Moody algebra $g(X)$, it is also a real root of $g(Y)$; thus its $W$-conjugate, viz $\beta_{-1}$, is in $\Phi_{re}^+$. Next, the bilinear form $(,) :$

$$(\beta_i, \beta_j) = (\alpha_i, \alpha_j) \quad (0 \leq i, j \leq r)$$
For $0 \leq j \leq r$, we also have :

\[
(\beta_{-1}, \beta_j) = (\delta_X + \alpha_0 + 2\alpha_{-1}, \alpha_j) = (\alpha_0 + 2\alpha_{-1}, \alpha_j)
\]

\[
= \begin{cases} 
0 & j \neq 0 \text{ and } j \text{ is not connected to vertex 0} \\
(\alpha_0, \alpha_j) & j \neq 0 \text{ and } j \text{ is connected to vertex 0} \\
0 & j = 0 
\end{cases}
\]

The $\beta_i$ thus satisfy the hypothesis of theorem 2. Next consider the root subdiagram of $Y$ formed by the $\beta_{-1}, \beta_0, \cdots, \beta_r$: we observe that the diagram formed by its subset $\{\beta_i\}_{i=0}^r$ is just $X$; we then add an extra vertex corresponding to $\beta_{-1}$. This vertex is not connected to vertex 0 anymore, but instead to all the neighbors of vertex 0 in $X$. For instance, applying principle A to $Y = E_{10}$, $X = E_8^{(1)}$, the $\beta_i$ generate the following root subdiagram (upto renumbering of vertices, this is $HD_8^{(1)}$ from Table 1):

![Diagram](image)

Thus

\[(3.1)\quad HD_8^{(1)} \preceq E_{10}\]

**Principle B:** Let $X, Y$, $\alpha_i$ ($-1 \leq i \leq r$), $\delta_X$ be as above. Choose a vertex $1 \leq p \leq r$ of $X$, i.e, a vertex of the finite type diagram underlying $X$. Define

\[
\beta_i = \alpha_i \quad (-1 \leq i \leq r, i \neq p), \quad \beta_p = \alpha_p + \delta_X
\]

Again, $\beta_i \in \Phi^+ \forall i = -1, \cdots, r$ and the bilinear form satisfies

\[
(\beta_i, \beta_j) = (\alpha_i, \alpha_j), \quad -1 \leq i, j \leq r; \quad i, j \neq p
\]

\[
(\beta_p, \beta_i) = (\alpha_p + \delta_X, \beta_i), \quad -1 \leq i \leq r; \quad i \neq p
\]

\[
= \begin{cases} 
(\alpha_p, \alpha_i) & 0 \leq i \leq r; \quad i \neq p \\
-1 & i = -1
\end{cases}
\]

The root subdiagram of $Y$ formed by the $\beta_{-1}, \beta_0, \cdots, \beta_r$ is thus the same as the original diagram $Y$ with one additional edge connecting the vertex $-1$ to the chosen node $p$ of $X$. For instance, figures 1 and 2 show the root subdiagrams $HA_8^{(1)}$ and $P_{10}$ obtained by applying Principle B to $Y = E_{10}$ and $p = 7, p = 8$.

Thus $HA_8^{(1)} \preceq E_{10}$ and $P_{10} \preceq E_{10}$; we remark that $HA_8^{(1)}$ and $P_{10}$ are indefinite type diagrams that are not hyperbolic.
Principle B': In fact, one can pick any subset $F \subset \{1, 2, \cdots, r\}$ and let
\[
\beta_i = \alpha_i \quad (-1 \leq i \leq r, i \notin F), \quad \beta_i = \alpha_i + \delta_X \quad (i \in F)
\]
A similar argument shows that the resulting root subdiagram is obtained by connecting vertex $-1$ to all the vertices in $F$.

Principle C: (shrinking) Suppose a Dynkin diagram $Z$ with $n$ vertices labeled $1 \cdots n$ has a subset $\{k, k+1, \cdots, k+p-1\}$ of $p$ vertices which forms a subdiagram isomorphic to the finite type diagram $A_p$:

Letting $\alpha_i, i = 1, \cdots, n$ denote the simple roots of $Z$, define
\[
\beta_j = \alpha_j, \quad \text{if } j < k \text{ or } j \geq k + p; \quad \bar{\beta} = \sum_{i=k}^{k+p-1} \alpha_i
\]
The root subdiagram formed by $\{\beta_j : j < k\} \cup \{\bar{\beta}\} \cup \{\beta_j : j \geq k+p\}$ has $p-1$ fewer vertices than $Z$; the vertex corresponding to $\bar{\beta}$ is now connected to the original neighbors of vertex $k$ as well to those of vertex $k+p-1$, while the rest of the diagram remains unchanged. We note that if vertices $k$ and $k+p-1$ have a common neighbor $s$, then the vertex $\bar{\beta}$ is connected to $s$ by 2 lines (with arrows pointing both toward $s$ and $\bar{\beta}$). An application of Principle C shows for instance that all members of the family $HA_k^{(1)} (1 \leq k \leq 7)$ occur as root subdiagrams of $HA_8^{(1)}$. Similarly $HD_k^{(1)} (4 \leq k \leq 8)$ occur as root subdiagrams of $HD_8^{(1)}$.

Principle D: (deletion) Given a Dynkin diagram $Z$ with $n$ vertices, deleting any subset of vertices (and all incident edges) clearly gives us a root subdiagram of $Z$.

4. Proof of main result

The goal of this section is to prove proposition 1 using these four principles.

Proof of proposition 1. In [8], it was shown that all the rank 2 simply laced hyperbolic Dynkin diagrams occur as root subdiagrams of $HA_1^{(1)} = F$. So, it is enough to show that the 23 diagrams in ranks 3-10 (Tables 1, 2) occur as root subdiagrams of $E_{10}$, since $HA_1^{(1)}$ occurs among these 23. Now, we had already observed that (i) $HD_8^{(1)} \preceq E_{10}$ (equation (3.1)) (ii) $HA_8^{(1)} \preceq E_{10}$ (figure 1) (iii) $P_{10} \preceq E_{10}$ (figure 2). As remarked earlier, principle C then
implies that $H_{A_k}^{(1)} (1 \leq k \leq 7)$ and $H_{D_k}^{(1)} (4 \leq k \leq 8)$ occur as root subdiagrams of $E_{10}$.

Next, we consider $H_{E_k}^{(1)} (k = 6, 7, 8)$. When $k = 8$, this is just $E_{10}$ itself. Consider $k = 7$; we observe that a diagram isomorphic to $H_{E_7}^{(1)}$ is obtained on deletion of vertex 0 from $H_{A_8}^{(1)}$ (figure 1). Thus $H_{E_7}^{(1)} \neq H_{A_8}^{(1)} \neq E_{10}$.

Similarly, observe that $H_{E_6}^{(1)} \neq P_{10}$, and is obtained by deletion of the two vertices of $P_{10}$ numbered 0 and 1 in figure 2. So, the hyperbolic diagrams in the three families $H_{A_k}^{(1)}, H_{D_k}^{(1)}, H_{E_k}^{(1)}$ are all root subdiagrams of $E_{10}$.

For the 8 irregular diagrams (Table 2):

(1) $X_6$: We start with $H_{D_4}^{(1)} \neq E_{10}$. Applying principle $A$ to $H_{D_4}^{(1)}$ clearly gives us $X_6$. Thus $X_6 \neq H_{D_4}^{(1)} \neq E_{10}$.

(2) Diagrams $Y_k$ ($k = 3, 4, 5$) are clearly obtained from $H_{A_k}^{(1)} - 2$ by application of principle $B$ (for suitable choice of $p$). For instance we have

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which realizes two interesting disconnected Dynkin diagrams of rank 10 as root subdiagrams of $E_{10}$.

**Notation:** If $X, Y$ are two connected Dynkin diagrams, we will let $X \oplus Y$ denote their disjoint union (i.e., the disconnected diagram with components isomorphic to $X$ and $Y$).

**Proposition 2.** The rank 10 diagrams $HE_7^{(1)} \oplus A_1$ and $HE_6^{(1)} \oplus A_2$ occur as root subdiagrams of $E_{10}$.

To prove this proposition, we will use the following characterization of real roots [10, Proposition 5.10].

**Theorem 3.** Let $\mathfrak{g}$ be a simply laced Kac-Moody algebra of finite, affine or hyperbolic type and let $Q$ be its root lattice. Then

$$\Phi_{re} = \{\beta \in Q : (\beta, \beta) = 2\}$$

**Proof of proposition 2.** Let $\alpha_i$ ($-1 \leq i \leq 8$) be the simple roots of $E_{10}$ and $\delta$ be the null root of $E_8^{(1)}$. We recall that $HE_7^{(1)} \leq HA_8^{(1)} \leq E_{10}$. Unravelling this chain of inclusions, we obtain the simple roots $\{\beta_i\}_{i=1}^{9}$ of $HE_7^{(1)}$ to be

$$\beta_i = \alpha_i \ (1 \leq i \leq 6), \quad \beta_7 = \delta + \alpha_7, \quad \beta_8 = \alpha_8, \quad \beta_9 = \alpha_{-1}$$

Note that we are now numbering the vertices of $HE_7^{(1)}$ with the labels $1 \cdots 9$ rather than $-1 \cdots 7$. If we can produce $\gamma \in \Phi^+_{re}$ such that

$$(\gamma, \beta_i) = 0 \text{ for all } 1 \leq i \leq 9$$

then the elements $\{\beta_i\}_{i=1}^{9} \cup \{\gamma\}$ will generate $HE_7^{(1)} \oplus A_1$. Let $\{\Lambda_i\}_{i=1}^{8}$ denote the fundamental weights of $E_{10}$ i.e. $(\Lambda_i, \alpha_j) = \delta_{ij}$ $\forall i,j$ (see [11] for a table of the $\Lambda_i$ as linear combinations of $\alpha_j$). Equation (5.1) can be easily seen to imply that $\gamma \in \text{span} (\Lambda_7 - 3\Lambda_0)$. Another easy computation gives $(\Lambda_7 - 3\Lambda_0, \Lambda_7 - 3\Lambda_0) = 2$ and $\Lambda_7 - 3\Lambda_0 \in \mathbb{Z}_{\geq 0}(\alpha_{-1}, \cdots, \alpha_8)$. Thus, using theorem 3, $\gamma = \Lambda_7 - 3\Lambda_0$ is the required root.

Next, we have $HE_6^{(1)} \leq P_{10} \leq E_{10}$. The simple roots $\{\beta_i\}_{i=1}^{8}$ of $HE_6^{(1)}$ are now:

$$\beta_1 = \alpha_{-1}, \quad \beta_i = \alpha_i \ (2 \leq i \leq 7), \quad \beta_8 = \alpha_8 + \delta$$

As above, we look for all $\gamma \in \Phi^+_{re}$ such that $(\gamma, \beta_i) = 0 \forall i = 1, \cdots, 8$. This condition implies $\gamma \in \text{span} (\Lambda_8 - 2\Lambda_1, \Lambda_1 - 2\Lambda_0)$. Letting $\gamma_1 = \Lambda_8 - 2\Lambda_1$ and $\gamma_2 = \Lambda_1 - 2\Lambda_0$, one computes $(\gamma_j, \gamma_j) = 2$ and $\gamma_j \in \mathbb{Z}_{\geq 0}(\alpha_{-1}, \cdots, \alpha_8)$ for $j = 1, 2$. Further $(\gamma_1, \gamma_2) = -1$. Thus $\{\beta_i\}_{i=1}^{8} \cup \{\gamma_1, \gamma_2\}$ generates $HE_6^{(1)} \oplus A_2$. This completes the proof. \qed

**Corollary 1.** $E_{10}$ has Lie subalgebras isomorphic to $\mathfrak{g}(HE_6^{(1)}) \oplus \mathfrak{sl}_3$ and $\mathfrak{g}(HE_7^{(1)}) \oplus \mathfrak{sl}_2$. 
References

[1] T. Damour, M. Henneaux, and H. Nicolai. \( E_{10} \) and a small tension expansion of M theory. *Phys. Rev. Lett.*, 89(22):221601, 4, 2002.

[2] Thibault Damour. Cosmological singularities, billiards and Lorentzian Kac-Moody algebras. In *Deserfest*, pages 55–76. World Sci. Publ., Hackensack, NJ, 2006.

[3] Thibault Damour and Marc Henneaux. \( E_{10}, BE_{10} \) and arithmetical chaos in superstring cosmology. *Phys. Rev. Lett.*, 86(21):4749–4752, 2001.

[4] Thibault Damour and Hermann Nicolai. Higher-order M-theory corrections and the Kac-Moody algebra \( E_{10} \). *Classical Quantum Gravity*, 22(14):2849–2879, 2005.

[5] Sophie de Buyl and Christiane Schomblond. Hyperbolic Kac Moody algebras and Einstein billiards. *J. Math. Phys.*, 45(12):4464–4492, 2004.

[6] M. Dyer. Reflection subgroups of Coxeter systems. *J. Algebra*, 135(1):57–73, 1990.

[7] Alex J. Feingold and Igor B. Frenkel. A hyperbolic Kac-Moody algebra and the theory of Siegel modular forms of genus 2. *Math. Ann.*, 263(1):87–144, 1983.

[8] Alex J. Feingold and Hermann Nicolai. Subalgebras of hyperbolic Kac-Moody algebras. In *Kac-Moody Lie algebras and related topics*, volume 343 of *Contemp. Math.*, pages 97–114. Amer. Math. Soc., Providence, RI, 2004.

[9] V. G. Kac. Simple irreducible graded Lie algebras of finite growth. *Izv. Akad. Nauk SSSR Ser. Mat.*, 32:1323–1367, 1968.

[10] V. G. Kac. *Infinite-dimensional Lie algebras*. Cambridge University Press, Cambridge, third edition, 1990.

[11] V. G. Kac, R. V. Moody, and M. Wakimoto. On \( E_{10} \). In *Differential geometrical methods in theoretical physics* (Como, 1987), volume 250 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 109–128. Kluwer Acad. Publ., Dordrecht, 1988.

[12] Axel Kleinschmidt and Hermann Nicolai. \( E_{10} \) and \( SO(9,9) \) invariant supergravity. *J. High Energy Phys.*, (7):041, 45 pp. (electronic), 2004.

[13] Axel Kleinschmidt and Hermann Nicolai. \( E_{10} \) cosmology. *J. High Energy Phys.*, (1):137, 13 pp. (electronic), 2006.

[14] Axel Kleinschmidt and Hermann Nicolai. Maximal supergravities and the \( E_{10} \) coset model. *Internat. J. Modern Phys. D*, 15(10):1619–1642, 2006.

[15] Robert V. Moody. A new class of Lie algebras. *J. Algebra*, 10:211–230, 1968.

[16] Cihan Saclioglu. Dynkin diagrams for hyperbolic Kac-Moody algebras. *J. Phys. A*, 22(18):3753–3769, 1989.

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