THE BISHOP-PHELPS-BOLLOBÁS PROPERTY FOR OPERATORS BETWEEN SPACES OF CONTINUOUS FUNCTIONS

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Abstract. We show that the space of bounded linear operators between spaces of continuous functions on compact Hausdorff topological spaces has the Bishop-Phelps-Bollobás property. A similar result is also proved for the class of compact operators from the space of continuous functions vanishing at infinity on a locally compact and Hausdorff topological space into a uniformly convex space, and for the class of compact operators from a Banach space into a predual of an $L_1$-space.

1. Introduction

E. Bishop and R. Phelps proved in 1961 [7] that every (continuous linear) functional $x^*$ on a Banach space $X$ can be approximated by a norm attaining functional $y^*$. This result is called the Bishop-Phelps Theorem. Shortly thereafter, B. Bollobás [8] showed that this approximation can be done in such a way that, moreover, the point at which $x^*$ almost attains its norm is close in norm to a point at which $y^*$ attains its norm. This is a quantitative version of the Bishop-Phelps Theorem, known as the Bishop-Phelps-Bollobás Theorem.

For a real or complex Banach space $X$, we denote by $S_X$, $B_X$ and $X^*$ the unit sphere, the closed unit ball and the dual space of $X$, respectively.

Theorem 1.1 (Bishop-Phelps-Bollobás Theorem, [8, Theorem 1]). Let $X$ be a Banach space and $0 < \varepsilon < 1/2$. Given $x \in B_X$ and $x^* \in S_{X^*}$ with $|1 - x^*(x)| < \varepsilon^2/2$, there are elements $y \in S_X$ and $y^* \in S_{X^*}$ such that $y^*(y) = 1$, $\|y - x\| < \varepsilon + \varepsilon^2$ and $\|y^* - x^*\| < \varepsilon$.

We refer the reader to the recent paper [10] for a more accurate version of the above theorem.

In 2008, M.D. Acosta, R.M. Aron, D. García and M. Maestre introduced the so-called Bishop-Phelps-Bollobás property for operators [1, Definition 1.1]. For our purposes, it will be useful to recall an appropriate version of this property for classes of operators defined in [2, Definition 1.3]. Given two Banach spaces $X$ and $Y$, $\mathcal{L}(X,Y)$ denotes the space of all (bounded and linear) operators from $X$ into $Y$. The subspace of $\mathcal{L}(X,Y)$ of finite-rank operators $\mathcal{F}(X,Y)$; $\mathcal{K}(X,Y)$ will denote the subspace of all compact operators.

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Definition 1.2. Let $X$ and $Y$ be Banach spaces and $\mathcal{M}$ a linear subspace of $\mathcal{L}(X,Y)$. We say that $\mathcal{M}$ satisfies the Bishop-Phelps-Bollobás property if given $\varepsilon > 0$, there is $\eta(\varepsilon) > 0$ such that whenever $T \in \mathcal{M}$ and $x_0 \in S_X$ satisfy that $\|Tx_0\| > 1 - \eta(\varepsilon)$, then there exist a point $u_0 \in S_X$ and an operator $S \in \mathcal{M}$ satisfying the following conditions:

$$\|Su_0\| = 1, \quad \|u_0 - x_0\| < \varepsilon, \quad \text{and} \quad \|S - T\| < \varepsilon.$$ 

In case that $\mathcal{M} = \mathcal{L}(X,Y)$ satisfies the previous property it is said that the pair $(X,Y)$ has the Bishop-Phelps-Bollobás property for operators (shortly BPBp for operators).

Observe that the BPBp of a pair $(X,Y)$ means that one is able to approximate any pair of an operator and a point at which the operator almost attains its norm by a new pair of a norm-attaining operator and a point at which this new operator attains its norm. In particular, if a pair $(X,Y)$ has the BPBp, the set of norm-attaining operators is dense in $\mathcal{L}(X,Y)$. The reverse result is far from being true: there are Banach spaces $Y$ such that the pair $(\ell^2_1, Y)$ does not have the BPBp (see [1]).

In [1] the authors provided the first version of the Bishop-Phelps-Bollobás Theorem for operators. Amongst them, a sufficient condition on a Banach space $X$, the pair $(X,Y)$ has the BPBp for operators, which is satisfied, for instance, by $Y = c_0$ or $Y = \ell^\infty$. A characterization of the Banach spaces $Y$ such that the pair $(\ell^1_1, Y)$ has the BPBp for operators is also given. There are also positive results for operators from $L_1(\mu)$ into $L_\infty(\nu)$ [5,12], for operators from $L_1(\mu)$ into $L_1(\nu)$ [12], for certain ideals of operators from $L_1(\mu)$ into another Banach space [11,12], for operators from an Asplund space into $C_0(L)$ or into a uniform algebra [3,9], and for operators from a uniformly convex space into an arbitrary Banach space [3,10]. For some more recent results, see also [6]. Let us also point out that the set of norm attaining operators from $L_1[0,1]$ into $C[0,1]$ is not dense in $\mathcal{L}(L_1[0,1], C[0,1])$ [19].

Our aim in this paper is to provide classes of Banach spaces satisfying a version of the Bishop-Phelps-Bollobás Theorem for operators. The first result, which is the content of section 2 states that given arbitrary compact Hausdorff topological spaces $K$ and $S$, the pair $(C(K), C(S))$ satisfies the BPBp for operators in the real case. This result extends the one by J. Johnson and J. Wolfe [13] that the set of norm attaining operators from $C(K)$ into $C(S)$ is dense in $\mathcal{L}(C(K), C(S))$. In section 3 we prove that the space $K(C_0(L), Y)$ satisfies the Bishop-Phelps-Bollobás property whenever $L$ is a locally compact Hausdorff topological space and $Y$ is uniformly convex in both the real and the complex case. Let us remark that it was also proved in [14] that the set of norm-attaining weakly compact operators from $C(K)$ into $Y$ is dense in the space of all weakly compact operators. But, as commented above, there are Banach spaces $Y$ such that the pair $(\ell^2_1, Y)$ does not satisfy the BPBp for operators (in the real case, $\ell^2_\infty \equiv \ell^1_2$), so some assumption on $Y$ is needed to get the Bishop-Phelps-Bollobás property. Finally, we devote section 4 to show that the space $K(X,Y)$ has the Bishop-Phelps-Bollobás property when $X$ is an arbitrary Banach space and $Y$ is a predual of an $L_1$-space in both the real and the complex case. This extends the result of [14] that the set of norm-attaining finite-rank operators from an arbitrary Banach space into a predual of an $L_1$-space is dense in the space of compact operators. In particular, for $Y = C_0(L)$ for some locally compact Hausdorff topological space $L$, the result is a consequence of the already cited paper [4].

2. Operators between spaces of continuous functions

Throughout this section, $K$ and $S$ are compact Hausdorff topological spaces. Here $C(K)$ is the space of real valued continuous functions on $K$. $M(K)$ denotes the space of regular Borel finite
Lemma 2.1 (Theorem 1, p. 490). Let $X$ be a Banach space and let $S$ be a compact Hausdorff topological space. Given an operator $A : X \to C(S)$, define $\mu : S \to X^*$ by $\mu(s) = A^*(\delta_s)$ for every $s \in S$. Then the relationship $$(Ax)(s) = \mu(s)(x), \quad \forall x \in X, s \in S$$ defines an isometric isomorphism between $\mathcal{L}(X, C(S))$ and the space of $w^*$-continuous functions from $S$ to $X^*$, endowed with the supremum norm, i.e. $\|\mu\| = \sup\{\|\mu(s)\| : s \in S\}$. Compact operators correspond to norm continuous functions.

Lemma 2.2 (Lemma 2.2). Let $\mu : S \to M(K)$ be $w^*$-continuous. Let $\varepsilon > 0$, $s_0 \in S$ and an open subset $V$ of $K$ be given. Then there exists an open neighborhood $U$ of $s_0$ such that if $s \in U$, then $\|\mu(s)(V)\| \geq \|\mu(s_0)(V)\| - \varepsilon$.

The next result is a version of [14, Lemma 2.3] in which the main difference is that we start with an operator and a function in the unit sphere of $C(K)$ where the operator almost attains its norm and construct a new operator and a new function, both close to the previous elements and satisfying additional restrictions. Condition iii) is the new ingredient that will be useful to our purpose.

Lemma 2.3. Let $\mu : S \to M(K)$ be a $w^*$-continuous function satisfying $\|\mu\| = 1$ and $0 < \delta < 1$. Suppose that $s_0 \in S$ and $f_0 \in S_{C(K)}$ satisfy $\int_K f_0 \, d\mu(s_0) > 1 - \frac{\delta^2}{12}$. Then there exist a $w^*$-continuous mapping $\mu' : S \to M(K)$, an open set $U$ in $S$, an open set $V$ of $K$ and $h_0 \in C(K)$ satisfying the following conditions:

i) $\|\mu'(s)(V)\| = 0$ for every $s \in U$.
ii) $\int_K h_0 \, d\mu'(s) \geq \|\mu'\| - \delta$ for every $s \in U$.
iii) $\|h_0 - f_0\| < \delta$.
iv) $\|h_0\| = 1$ and $|h_0(t)| = 1$ for every $t \in K \setminus V$.
v) $\|\mu' - \mu\| < \delta$.

Proof. Let us write $\mu_0 := \mu(s_0)$. By the Hahn decomposition theorem, there is a partition of $K$ into two measurable sets $K^+$ and $K^-$ such that $K^+$ is a positive set for $\mu_0$ and $K^-$ is negative for $\mu_0$. For every $0 < x < 1$, consider two open subsets of $K$ given by

$$O^+_x := \{t \in K : f_0(t) > x\}, \quad O^-_x := \{t \in K : f_0(t) < -x\},$$

and consider the set

$$D_x := (K^+ \cap O^+_x) \cup (K^- \cap O^-_x).$$

Write $\alpha = \frac{\delta^2}{12}$. By the assumption, we have

$$1 - \alpha < \int_K f_0 \, d\mu_0 \leq |\mu_0|(D_x) + x|\mu_0|(K \setminus D_x) \leq |\mu_0|(D_x) + x(1 - |\mu_0|(D_x)) = (1 - x)|\mu_0|(D_x) + x.$$

Hence,

$$(1) \quad |\mu_0|(D_x) > 1 - \frac{\alpha}{1 - x}.$$ 

Next, consider the open subset $W_x$ of $K$ given by

$$W_x := O^+_x \cup O^-_x = \{t \in K : \mu_0(t) > x\},$$
and observe that, since $D_x \subset W_x$, we have
\begin{equation}
(2) \quad |\mu_0(W_x)| \geq |\mu_0(D_x)| \geq 1 - \frac{\alpha}{1 - x}.
\end{equation}
Write $c := 1 - \frac{\delta}{4}$ and choose real numbers $a$ and $b$ with $1 - \delta < a < b < c < 1$. As the open subset $W_a$ contains $O_b^+ \cup O_b^-$, there is $u \in C(K)$ such that $0 \leq u \leq 1$, $u \equiv 1$ on $O_b^+ \cup O_b^-$ and $\text{supp } u \subset W_a$. Since the support of $u$ is contained in $W_a$ (where $f_0$ is separated from 0), the function $h_0$ defined on $K$ by
\[ h_0(t) = \frac{f_0(t)}{|f_0(t)|} u(t) + (1 - u(t)) f_0(t) \quad \text{if } f_0(t) \neq 0, \quad h_0(t) = 0 \quad \text{otherwise}, \]
is continuous and, actually, $h_0 \in B_{C(K)}$. We claim that $\|h_0 - f_0\| < \delta$, which guarantees condition iii). Indeed, if $t \in K \setminus W_a$, then $u(s) = 0$ and so $h_0(t) = f_0(t)$; if otherwise $t \in W_a$, we have that
\[ |h_0(t) - f_0(t)| = u(t) \left| \frac{f_0(t)}{|f_0(t)|} - f_0(t) \right| \leq 1 - |f_0(t)| < 1 - a < \delta, \]
proving the claim. On the other hand, we know that $u \equiv 1$ in $O_b^+ \cup O_b^-$, so $|h_0| = 1$ on $O_b^+ \cup O_b^-$. Therefore, if we write $V := K \setminus \left( O_b^+ \cup O_b^- \right)$, which is an open subset of $K$, the second part of condition iv) is satisfied.

Next, as the open subsets $V$ and $W_c$ satisfy $V \cap W_c = \emptyset$, there is a function $f \in C(K)$ such that $0 \leq f \leq 1$, $f \equiv 1$ on $V$ and $\text{supp } f \subset K \setminus W_c$. Since $D_c \subset W_c$, we have that
\[ \int_{W_c} h_0 \, d\mu_0 = \int_{D_c} h_0 \, d\mu_0 + \int_{W_c \setminus D_c} h_0 \, d\mu_0 = |\mu_0|(D_c) + \int_{W_c \setminus D_c} h_0 \, d\mu_0 \geq |\mu_0|(D_c) - |\mu_0|(W_c \setminus D_c) \geq |\mu_0|(D_c) - |\mu_0|(K \setminus D_c) \geq 2|\mu_0|(D_c) - |\mu_0|(K). \]
Therefore, by using (1), we obtain that
\begin{equation}
(3) \quad \int_{W_c} h_0 \, d\mu_0 > 1 - 2 \frac{\alpha}{1 - c} = 1 - \frac{2}{3} \delta.
\end{equation}
As a consequence,
\[ \int_K h_0(1 - f) \, d\mu_0 = \int_{W_c} h_0(1 - f) \, d\mu_0 + \int_{K \setminus W_c} h_0(1 - f) \, d\mu_0 \geq \int_{W_c} h_0(1 - f) \, d\mu_0 - |\mu_0|(K \setminus W_c) \]
\[ > 1 - 2 \frac{\alpha}{1 - c} - (1 - |\mu_0|(W_c)) \quad \text{(by (3))} \]
\[ > 1 - 3 \frac{\alpha}{1 - c} = 1 - \delta \quad \text{(by (2))} \]
Now, in view of the $w^*$-continuity of $\mu$, the previous inequality, condition (2) and Lemma 2.2, we get that there exists an open neighborhood $U_0$ of $s_0$ such that
\begin{equation}
(4) \quad \int_K h_0(1 - f) d\mu(s) > 1 - \delta \quad \text{and} \quad |\mu(s)|(W_c) > 1 - \delta, \quad \forall s \in U_0.
\end{equation}
We can also choose an open subset $U$ of $S$ such that $s_0 \in U$ and $\overline{U} \subset U_0$, and a function $g \in C(S)$ such that $0 \leq g \leq 1$, $g(U) = \{1\}$ and $\text{supp} \ g \subset U_0$. Define $\mu' : S \rightarrow M(K)$ by

$$
\mu'(s) = (1 - g(s) f) \mu(s), \quad (s \in S),
$$

that is, $\mu'(s)$ is the unique Borel measure on $K$ satisfying

$$
\int_K \varphi \, d\mu'(s) = \int_K (1 - g(s) f) \varphi \, d\mu(s) \quad \forall \varphi \in C(K).
$$

It is clear that $\mu'$ is $w^*$-continuous. If $s \in U$, $g(s) = 1$ and $f(V) = \{1\}$, so condition i) is satisfied. Since $0 \leq f, g \leq 1$, then $\|\mu'\| \leq \|\mu\| = 1$ and hence, in view of (2), for every $s \in U$ we have that

$$
\int_K h_0 \, d\mu'(s) = \int_K (1 - g(s) f) h_0 \, d\mu(s) > 1 - \delta \geq \|\mu'\| - \delta,
$$

so condition ii) is also satisfied.

We only have to check condition v), that is, $\|\mu' - \mu\| < \delta$. Indeed, if $s \in S \setminus U_0$, then $g(s) = 0$ and so $\mu'(s) = \mu(s)$; if, otherwise, $s \in U_0$, by (4) and the fact that $f(V_c) = \{0\}$, we obtain that

$$
\left| \int_K \varphi (d(\mu(s) - \mu'(s)) \right| = \left| \int_K \varphi g(s) f \, d\mu(s) \right| \leq \|\mu(s)\|(K \setminus W_c) \leq \|\mu\| - \|\mu(s)\|(W_c) < \delta
$$

for every $\varphi \in B_{C(K)}$, proving the claim.

Finally, since $\|\mu\| = 1$, condition v) implies that $\mu' \neq 0$ and, in view of i), we deduce that $K \neq V$, so $K \setminus V$ is not empty and $\|h_0\| = 1$ since $|h_0(t)| = 1$ for every $t \in K \setminus V$.

The last ingredient that we will use is the next iteration result due to Johnson and Wolfe.

**Lemma 2.4** ([14, Lemma 2.4]). Let $\mu : S \rightarrow M(K)$ be $w^*$-continuous and $\delta > 0$. Suppose there is an open set $U \subset S$, an open set $V \subset K$, $s_0 \in U$ and $h_0 \in C(K)$ with $\|h_0\| = 1$ such that

1. if $s \in U$, then $\|\mu(s)\|(V) = 0$,
2. $\int_K h_0 \, d\mu(s_0) \geq \|\mu\| - \delta$,
3. $|h_0(t)| = 1$ for $t \in K \setminus V$.

Then, for any $\frac{\delta}{3} < r < 1$ there exist a $w^*$-continuous function $\mu' : S \rightarrow M(K)$ and a point $s_1 \in U$ such that

1. if $s \in U$, then $\|\mu'(s)\|(V) = 0$,
2. $\int_K h_0 \, d\mu'(s_1) \geq \|\mu'\| - r\delta$,
3. $\|\mu' - \mu\| \leq r\delta$.

The next result improves [13, Theorem 1].

**Theorem 2.5.** Let $K$ and $S$ be compact Hausdorff topological spaces. Then the pair $(C(K), C(S))$ has the Bishop-Phelps-Bollobás property for operators in the real case. Moreover, the function $\eta$ satisfying Definition 1.2 does not depend on the spaces $K$ and $S$ (in fact one can take $\eta(\varepsilon) = \frac{\varepsilon^2}{12s^2}$).

**Proof.** Let us fix $\frac{\delta}{3} < r < 1$. Given $0 < \varepsilon < 2$ let us choose $0 < \delta < \varepsilon \frac{1 - \frac{r^2}{12}}{1 - \frac{\delta^2}{12}}$. Assume that $T_0 \in S_L(C(K), C(S))$ and $f_0 \in S(C(K))$ satisfy that $\|T_0(f_0)\| > 1 - \frac{\delta^2}{12}$. Then, there is an element $s_1 \in S$ such that $\|T_0(f_0)(s_1)\| > 1 - \frac{\delta^2}{12}$. By using $-f_0$ instead of $f_0$, if necessary, we may assume that $T_0(f_0)(s_1) > 1 - \frac{\delta^2}{12}$. Therefore, we can apply Lemma 2.3 to the $w^*$-continuous function $\mu_0 : S \rightarrow M(K)$ associated with the operator $T_0$ (i.e. $\mu_0(s) = T_0^*(\delta_s)$ for every $s \in S$).
to get that there exist a function $h_0 \in S_{C(K)}$, an open set $U$ in $S$, an open set $V$ of $K$ and a $w^*$-continuous function $\mu_1 : S \to M(K)$ satisfying the following conditions:

i) $|\mu_1(s)|(V) = 0$ for every $s \in U$.
ii) $\int_K h_0 \, d\mu_1(s) \geq \|\mu_1\| - \delta$ for every $s \in U$.
iii) $\|h_0 - f_0\| < \delta$.
iv) $\|h_0\| = 1$ and $|h_0(t)| = 1 \quad \forall t \in K \setminus V$.
v) $\|\mu_1 - \mu_0\| < \delta$.

Now, by using Lemma 2.4, we inductively construct a sequence $\{\mu_n\}$ of $w^*$-continuous functions from $S$ into $M(K)$ and a sequence $\{s_n\}$ in $U$ satisfying

i) $\|\mu_{n+1} - \mu_n\| \leq r^n \delta$.
ii) $\|\mu_n\| \leq \int_K h_0 \, d\mu_n(s_n) + r^n \delta$
iii) $\|\mu_n(s)(V) = 0$ for every $s \in U$ and $n \in \mathbb{N}$.

If for every $n \in \mathbb{N}$, we write $T_n \in \mathcal{L}(C(K), C(S))$ to denote the bounded linear operator associated with the function $\mu_n$, we may rewrite i) and ii) as

$$\|T_{n+1} - T_n\| \leq r^n \delta \quad \text{and} \quad \|T_n\| \leq \|T_n(h_0)\| + r^n \delta.$$

Since $0 < r < 1$, the previous condition implies that $\{T_n\}$ is a Cauchy sequence, so it converges to an operator $T \in \mathcal{L}(C(K), C(S))$ satisfying

$$\|T - T_0\| \leq \sum_{k=0}^{\infty} \|T_k - T_k\| \leq \sum_{k=0}^{\infty} r^k \delta = \delta \frac{1}{1-r} < \varepsilon.$$ 

By taking limit in the right-hand side of (5), we also have that

$$\|T\| \leq \|T(h_0)\|$$

and, since $h_0 \in S_{C(K)}$, $T$ attains its norm at $h_0$.

Finally, we have that

$$|1 - \|T\|| \leq \|T_0\| - \|T\| \leq \|T_0 - T\| \frac{\varepsilon}{2} < 1,$$

so $T \neq 0$, $\frac{T}{\|T\|}$ also attains its norm at $h_0$ and

$$\left\| \frac{T}{\|T\|} - T_0 \right\| \leq \left\| \frac{T}{\|T\|} - T \right\| + \|T - T_0\| = \left| 1 - \|T\| \right| + \left| T - T_0 \right| < \varepsilon.$$

As we already knew that $\|h_0 - f_0\| < \delta < \varepsilon$, this shows that the pair $(C(K), C(S))$ satisfies the Bishop-Phelps-Bollobás Theorem for operators with $\eta = \frac{\delta^2}{12}$.

3. COMPACT OPERATORS FROM A SPACE OF CONTINUOUS FUNCTIONS INTO A UNIFORMLY CONVEX SPACE

Our purpose now is to prove the Bishop-Phelps-Bollobás property for compact operators. The following result due to Kim will play an essential role:

**Lemma 3.1 (Kim, Theorem 2.5).** Let $Y$ be a uniformly convex space. For every $0 < \varepsilon < 1$, there is $0 < \gamma(\varepsilon) < 1$ with the following property:
given $n \in \mathbb{N}$, $T \in S_{\mathcal{L}(\ell_\infty, Y)}$ and $x_0 \in S_{\ell_\infty}$ such that $\|Tx_0\| > 1 - \gamma(\varepsilon)$, there exist $S \in S_{\mathcal{L}(\ell_\infty, Y)}$ and $x_1 \in S_{\ell_\infty}$ satisfying

$$\|Sx_1\| = 1, \quad \|S - T\| < \varepsilon \quad \text{and} \quad \|x_1 - x_0\| < \varepsilon.$$
It is easy to show that the function $\gamma$ in the previous result satisfies $\lim_{t \to 0^+} \gamma(t) = 0$.

Let $L$ be a locally compact Hausdorff topological space. As usual, $C_0(L)$ will be the space either of real or complex continuous functions on $L$ with limit zero at infinity. We recall that $C_0(L)^*$ can be identified with the space $M(L)$ of regular Borel measures on $L$ by the Riesz representation theorem.

For every $f \in C_0(L)$ and every (non-empty) set $S \subset L$, we define $\text{Osc}(f, S)$ by

$$\text{Osc}(f, S) = \sup_{x,y \in S} |f(x) - f(y)|$$

The next result generalizes Lemma 3.1 and Proposition 3.2 of [14] to $C_0(L)$. Its proof is actually based on the proof of these results.

**Proposition 3.2.** Let $L$ be a locally compact Hausdorff topological space and let $Y$ be a Banach space. For every $\varepsilon > 0$, $T \in \mathcal{K}(C_0(L), Y)$ and $f_0 \in C_0(L)$, there exist a positive regular Borel measure $\mu$, a non-negative integer $m$, pairwise disjoint compact subsets $K_j$ of $L$ and $\varphi_j \in C_0(L)$ for $1 \leq j \leq m$, satisfying the following conditions:

1. $\text{Osc}(f_0, K_j) < \varepsilon$.
2. $0 \leq \varphi_j \leq 1$ and $\varphi_j \equiv 1$ on $K_j$.
3. $\text{supp} \varphi_i \cap \text{supp} \varphi_j = \emptyset$ for $i \neq j$.
4. The operator $P : C_0(L) \to C_0(L)$ given by

$$P(f) := \sum_{j=1}^{m} \frac{1}{\mu(K_j)} \left( \int_{K_j} f \, d\mu \right) \varphi_j, \quad \forall f \in C_0(L),$$

is a norm-one projection from $C_0(L)$ onto the linear span of $\{\varphi_1, \ldots, \varphi_m\}$ that also satisfies $\|T - TP\| < \varepsilon$.

**Proof.** Since $T$ is a compact operator, the adjoint operator $T^*$ is a compact operator from $Y^*$ into $C_0(L)^*$, so we may take a finite $\frac{\varepsilon}{2}$-net $\{\mu_1, \ldots, \mu_t\}$ of $T^*(B_{Y^*}) \subset C_0(L)^* \equiv M(L)$. We define the (finite regular) measure $\mu$ by $\mu = \sum_{i=1}^{t} |\mu_i|$. For each $1 \leq i \leq t$, we have that $\mu_i \ll \mu$, hence the Radon-Nikodým theorem allows us to find a function $g_i \in L_1(\mu)$ such that $\mu_i = g_i \mu$. Since the set of simple functions is dense in $L_1(\mu)$, we may choose a set of simple functions $\{s_i : i = 1, \ldots, t\}$ such that $\|g_i - s_i\|_1 < \frac{\varepsilon}{2m}$ for every $1 \leq i \leq t$. Next, consider a finite family $(A_j)_{j=1}^{m_0}$ of pairwise disjoint measurable sets such that for every $1 \leq i \leq t$, there is a family $(\alpha_{ij})_{j=1}^{m_0}$ of scalars such that $s_i = \sum_{j=1}^{m_0} \alpha_{ij} \chi_{A_j}$. Let $M$ be a positive real number satisfying

$$M \geq \max \{|\alpha_{ij}| : 1 \leq i \leq t, 1 \leq j \leq m_0\}.$$

Since $\mu$ is regular, for each $1 \leq j \leq m_0$, we find a compact set $C_j \subset A_j$ such that $\mu(A_j \setminus C_j) < \frac{\varepsilon}{2m_0M}$. As $f_0$ is continuous and each $C_j$ is compact, we may divide each $C_j$ into a family of Borel sets $(B^p_j)_{p=1}^{n_j}$ such that

$$\text{Osc}(f_0, B^p_j) < \varepsilon \quad \forall 1 \leq j \leq m_0, 1 \leq p \leq n_j.$$

Applying the regularity again, for each $j$ and $p$, there is a compact set $K^p_j \subset B^p_j$ such that $\mu(B^p_j \setminus K^p_j) < \frac{\varepsilon}{2m_0n_jM}$. Finally, choose suitable $m \in \mathbb{N}$, a rearrangement $(K_j)_{j=1}^{m}$ of the family $\{K^p_j : 1 \leq j \leq m_0, 1 \leq p \leq n_j, \mu(K^p_j) > 0\}$ and scalars $(\beta_j)$ for $j \leq m$ and $i \leq t$ such that

$$\sum_{j=1}^{m} \beta_j \chi_{K_j} = \sum_{j=1}^{m} \alpha_{ij} \left( \sum_{p=1}^{n_j} \chi_{K^p_j} \right).$$
Using Urysohn lemma, we may choose a family $(\varphi_j)_{j=1}^m$ in $C_0(L)$ satisfying that $0 \leq \varphi_j \leq 1$, $\varphi_j \equiv 1$ on $K_j$ for each $j \leq m$ and $\text{supp} \varphi_i \cap \text{supp} \varphi_j = \emptyset$ for every $i \neq j$.

To finish the proof, we only have to check (4). Indeed, for $i = 1, \ldots, t$, we write $\nu_i = \sum_{j=1}^m \beta_j^i \chi_{K_j} \mu \in M(L) = C_0(L)^*$. By defining the operator $P$ as in condition (4), it is easy to check that $P$ is a norm one projection onto the linear span on $\{\varphi_1, \ldots, \varphi_m\}$ and $P^* \nu_i = \nu_i$ for each $1 \leq i \leq t$. Therefore, for every $1 \leq i \leq t$ we have that

$$\|\mu_i - P^* \nu_i\| = \|g_i \mu - \nu_i\| \leq \|g_i \mu - s_i \mu\| + \|s_i \mu - \nu_i\|$$

$$\leq \|g_i - s_i\| + \left\|s_i \mu - \sum_{j=1}^{m_0} \alpha_j^i \chi_{C_j} \mu\right\| + \left\|\sum_{j=1}^{m_0} \alpha_j^i \chi_{C_j} \mu - \sum_{j=1}^m \beta_j^i \chi_{K_j} \mu\right\|$$

$$< \frac{\varepsilon}{12} + \sum_{j=1}^{m_0} \left|\alpha_j^i\right| \mu(A_j \setminus C_j) + \sum_{j=1}^{m_0} \left|\alpha_j^i\right| \sum_{p=1}^{n_j} \lambda(B_j \setminus K_j)$$

$$< \frac{\varepsilon}{12} + \sum_{j=1}^{m_0} \left|\alpha_j^i\right| \sum_{p=1}^{n_j} \lambda(B_j \setminus K_j)$$

Since $\{\mu_1, \ldots, \mu_t\}$ is a $\frac{\varepsilon}{t}$-net of $T^*(B_{Y^*})$, the above inequality shows that $\{\nu_1, \ldots, \nu_t\}$ is a $\frac{\varepsilon}{t}$-net of $T^*(B_{Y^*})$. Now, given $y^* \in B_{Y^*}$, we can choose $i \leq t$ satisfying $\|\nu_i - T^* y^*\| < \frac{\varepsilon}{2}$ and observe that

$$\|T^* y^* - P^* T^* y^*\| \leq \|T^* y^* - \nu_i\| + \|\nu_i - P^* T^* y^*\|$$

$$= \|T^* y^* - \nu_i\| + \|P^* \nu_i - P^* T^* y^*\| \leq 2 \|T^* y^* - \nu_i\| < \varepsilon.$$ 

Hence, we have $\|T - TP\| = \|T^* - P^* T^*\| < \varepsilon$, as desired.

The following result shows that $\mathcal{K}(C_0(L), Y)$ satisfies the Bishop-Phelps-Bollobás property for every locally compact Hausdorff topological space $L$ and every uniformly convex space $Y$, and that the function $\eta(\varepsilon)$ involved in the definition of the property does not depend on $L$.

**Theorem 3.3.** Let $Y$ be a uniformly convex Banach space. For every $0 < \varepsilon < 1$ there is $0 < \eta(\varepsilon) < 1$ such that for any locally compact Hausdorff topological space $L$, if $T \in \mathcal{S}_{\mathcal{K}(C_0(L), Y)}$ and $f_0 \in SC_{C_0(L)}$ satisfy $\|T f_0\| > 1 - \eta(\varepsilon)$, there exist $S \in \mathcal{S}_{\mathcal{K}(C_0(L), Y)}$ and $g_0 \in SC_{C_0(L)}$ such that

$$\|S g_0\| = 1, \quad \|S - T\| < \varepsilon \quad \text{and} \quad \|g_0 - f_0\| < \varepsilon.$$

**Proof.** Given $0 < \varepsilon < 1$, we choose $0 < \delta < \frac{\varepsilon}{4}$ such that $0 < \gamma(\delta) < \frac{\varepsilon}{4}$, where $\gamma(\delta)$ satisfies the statement of Lemma 3.2. We also consider $\alpha$ such that $0 < \alpha < \min\{\delta, \frac{\gamma(\delta)}{2}\}$ and $\eta(\varepsilon) := \alpha > 0$.

Fix $0 < \varepsilon < 1$, $T \in \mathcal{S}_{\mathcal{K}(C_0(L), Y)}$ and $f_0 \in SC_{C_0(L)}$ with $\|T f_0\| > 1 - \eta(\varepsilon) = 1 - \alpha$. Applying Proposition 3.2 we get a positive regular Borel measure $\mu$ on $L$, a non-negative integer $m$, pairwise disjoint compact subsets $K_j$ of $L$ and $\varphi_j \in C_0(L)$ $(1 \leq j \leq m)$ such that

1. $\text{Osc}(f_0, K_j) < \alpha$,
2. For every $1 \leq j \leq m$, $0 \leq \varphi_j \leq 1$ and $\varphi_j \equiv 1$ on $K_j$,
3. $\text{supp} \varphi_i \cap \text{supp} \varphi_j = \emptyset$ for $i \neq j$,
4. $\|T - TP\| < \alpha$,
where $P \in \mathcal{L}(C_0(L))$ is given by

$$P(f) := \sum_{j=1}^{m} \frac{1}{\mu(K_j)} \left( \int_{K_j} f \, d\mu \right) \varphi_j \quad (f \in C_0(L)),$$

and it is a norm-one projection onto the linear span of $\{\varphi_1, \ldots, \varphi_m\}$.

Now, if $t \in K_j$ for some $j = 1, \ldots, m$, we obtain that

$$\|P(f_0)(t) - f_0(t)\| = \left| \frac{1}{\mu(K_j)} \int_{K_j} (f_0(s) - f_0(t)) \, d\mu(s) \right| \leq \frac{1}{\mu(K_j)} \int_{K_j} |f_0(s) - f_0(t)| \, d\mu(s) \leq \alpha.$$

Hence

$$\max \left\{ \|P(f_0) - f_0\| : t \in \bigcup_{j=1}^{m} K_j \right\} \leq \alpha. \tag{6}$$

We also have that

$$\|TP(f_0)\| \geq \|T(f_0)\| - \|T - TP\| > 1 - 2\alpha > 1 - \gamma(\delta), \tag{7}$$

and this implies that

$$1 - 2\alpha \leq \|TP\| \leq 1 \quad \text{and} \quad 1 - 2\alpha \leq \|P(f_0)\|.$$
Next, we estimate the distance between $S$ and $T$ as follows
\[
\|S - T\| = \|V \Phi P - T\| \leq \|V \Phi P - TP\| + \|TP - T\|
\leq \|V \Phi P - U \Phi P\| + \|U \Phi P - TP\| + \alpha
\leq \|V - U\| + \|U \Phi P - T \Phi^{-1} \Phi P\| + \alpha
\leq \delta + \|U - T \Phi^{-1}\| + \alpha \quad \text{(by (8))}
\leq 2\delta + \|U - U_1\| \leq 2\delta + \gamma(\delta) < \varepsilon \quad \text{(by (9)).}
\]
On the other hand,
\[
\max \{ |f_1 - f_0|(t)| : t \in \bigcup_{j=1}^m K_j \} = \max_{1 \leq j \leq m} \max_{t \in K_j} \max |x^1(j) - f_0(t)|
\leq \max_{1 \leq j \leq m} \left\{ |x^1(j) - x^0(j)| + \max_{t \in K_j} |x^0(j) - f_0(t)| \right\}
\leq \|x^1 - x^0\| + \max_{1 \leq j \leq m} \max_{t \in K_j} |P(f_0) - f_0(t)|
\leq \delta + \gamma(\delta) + \alpha < 2\delta + \gamma(\delta) \quad \text{(by (10) and (11)).}
\]
Hence, there exists an open set $O \subseteq L$ such that
\[
(11) \quad \bigcup_{j=1}^m K_j \subset O, \quad |f_1 - f_0|(t)| < 3\delta + \gamma(\delta) \quad (t \in O).
\]
By Urysohn Lemma again, there is $\psi \in C_0(L)$ such that $0 \leq \psi \leq 1$, $\psi \equiv 1$ on $\bigcup_{j=1}^m K_j$ and supp $\psi \subset O$. We write $g_0 := \psi f_1 + (1 - \psi) f_0 \in B_{C_0(L)}$ and we claim that $S$ attains its norm at $g_0$ and that $\|f - g_0\| < \varepsilon$, which finishes the proof. Indeed, on one hand, it is clear that the restriction of $g_0$ to $\bigcup_{j=1}^m K_j$ coincides with $f_1$. It follows that $P(g_0) = P(f_1)$ and so $S(g_0) = S(f_1)$ and $S$ attains its norm at $g_0$. On the other hand, for $t \in L \setminus O$ we have $g(t) = f_0(t)$. If, otherwise, $t \in O$, condition (11) gives that
\[
|g_0(t) - f_0(t)| = |\psi(t)(f_1(t) - f_0(t))| < 3\delta + \gamma(\delta) < \varepsilon. \quad \square
\]

4. Compact operators into a predual of an $L_1(\mu)$-space

Our goal is to show that the space of compact operators from an arbitrary Banach space into an isometric predual of an $L_1$-space has the Bishop-Phelps-Bollobás property in both the real and the complex case.

We need a preliminary result which follows easily from the Bishop-Phelps-Bollobás theorem. It is also a very particular case of [II, Theorem 2.2].

**Lemma 4.1.** For every $0 < \varepsilon < 1$, there is $0 < \eta'(\varepsilon) < 1$ such that for every positive integer $n$ and every Banach space $X$, the pair $(X, \ell^\infty_n)$ has the BPBp for operators with this function $\eta'(\varepsilon)$. More concretely, given an operator $U \in S_{L_1(X, \ell^\infty_n)}$ and an element $x_0 \in S_X$ such that $\|U(x_0)\| > 1 - \eta'(\varepsilon)$, there exist $V \in S_{L_1(X, \ell^\infty_n)}$ and $z_0 \in S_X$ satisfying
\[
\|V z_0\| = 1, \quad \|z_0 - x_0\| < \varepsilon \quad \text{and} \quad \|V - U\| < \varepsilon.
\]

**Theorem 4.2.** For every $0 < \varepsilon < 1$ there is $\eta(\varepsilon) > 0$ such that if $X$ is any Banach space, $Y$ is a predual of an $L_1$-space, $T \in S_{K(X,Y)}$ and $x_0 \in S_X$ satisfy $\|Tx_0\| > 1 - \eta(\varepsilon)$, then there exist $S \in S_{F(X,Y)}$ and $z_0 \in S_Y$ with
\[
\|S z_0\| = 1, \quad \|z_0 - x_0\| < \varepsilon \quad \text{and} \quad \|S - T\| < \varepsilon.
\]
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Proof. For any $0 < \varepsilon < 1$ we take $\eta(\varepsilon) = \min\{\frac{\varepsilon}{4}, \eta'(\varepsilon/2)\}$, where $\eta'$ is the function provided by the previous lemma.

Fix $0 < \varepsilon < 1$, $T \in S_{K(X,Y)}$ and $x_0 \in S_X$ satisfying $\|Tx_0\| > 1 - \eta(\varepsilon)$. Let us choose a positive number $\delta$ with $\delta < \frac{1}{4}\min\{\frac{\varepsilon}{4}, \|T(x_0)\| - 1 + \eta'(\frac{\varepsilon}{2})\}$ and let $\{y_1, \ldots, y_n\}$ be a $\delta$-net of $T(B_X)$. In view of [17, Theorem 3.1] and [18, Theorem 1.3], there is a subspace $E \subset Y$ isometric to $\ell^m_{\infty}$ for some natural number $m$ and such that $\text{dist}(y_i, E) < \delta$ for every $i \leq n$. Let $P: Y \to Y$ be a norm one projection onto $E$. We will check that $\|PT - T\| < 4\delta$. In order to show that we fix any element $x \in B_X$ and so $\|Tx - y_i\| < \delta$ for some $i \leq n$. Let $e \in E$ be any element satisfying $\|e - y_i\| < \delta$. Then we have

$$
\|T(x) - PT(x)\| \leq \|T(x) - y_i\| + \|y_i - e\| + \|e - PT(x)\| \\
\leq 2\delta + \|P(e) - PT(x)\| + \|e - T(x)\| \\
\leq 2\delta + \|e - y_i\| + \|y_i - T(x)\| < 4\delta.
$$

So $\|PT\| > \|T\| - 4\delta = 1 - 4\delta > 0$. As a consequence we also obtain that

$$
\|PT(x_0)\| > \|T(x_0)\| - 4\delta > 1 - \eta'(\frac{\varepsilon}{2}).
$$

Hence the operator $R = \frac{PT}{\|PT\|}$ satisfies $\|R(x_0)\| > 1 - \eta'(\frac{\varepsilon}{2})$. Since $E$ is isometric to $\ell^m_{\infty}$, by Lemma 1.1 there exist an operator $S \in L(X, E) \subset L(X, Y)$ with $\|S\| = 1$ and $z_0 \in S_X$ satisfying that

$$
\|S - R\| < \frac{\varepsilon}{2}, \quad \|z_0 - x_0\| < \frac{\varepsilon}{2}, \quad \text{and} \quad \|Sz_0\| = 1.
$$

Finally, we have that

$$
\|S - T\| \leq \|S - R\| + \|R - PT\| + \|PT - T\| \\
\leq \|S - R\| + \|R - PT\| + 4\delta \\
\leq \frac{\varepsilon}{2} + 4\delta < \varepsilon.
$$

References

[1] M.D. Acosta, R.M. Aron, D. García and M. Maestre, The Bishop-Phelps-Bollobás theorem for operators, J. Funct. Anal. 254 (11) (2008), 2780–2799.

[2] M.D. Acosta, J. Becerra-Guerrero, D. García, S.K. Kim and M. Maestre, Bishop-Phelps-Bollobás property for certain spaces of operators, preprint 2013.

[3] M.D. Acosta, J. Becerra-Guerrero, D. García and M. Maestre, The Bishop-Phelps-Bollobás Theorem for bilinear forms, Trans. Amer. Math. Soc. (to appear). Available at http://dx.doi.org/10.1090/S0002-9947-2013-05881-3.

[4] R.M. Aron, B. Cascales and O. Kozuhshkina, The Bishop-Phelps-Bollobás Theorem and Asplund operators, Proc. Amer. Math. Soc. 139 (10) (2011), 3553–3560.

[5] R.M. Aron, Y.S. Choi, D. García and M. Maestre, The Bishop-Phelps-Bollobás theorem for $L(L_1(\mu), L_\infty[0, 1])$, Adv. Math. 228 (1) (2011), 617–628.

[6] R.M. Aron, Y.S. Choi, S.K. Kim, H.J. Lee and M. Martín, The Bishop-Phelps-Bollobás version of Lindenstrauss properties A and B, preprint 2013. Available at http://arxiv.org/abs/1305.6420.

[7] E. Bishop and R.R. Phelps, A proof that every Banach space is subreflexive, Bull. Amer. Math. Soc. (N.S.) 67 (1961), 97–98.

[8] B. Bollobás, An extension to the theorem of Bishop and Phelps, Bull. London Math. Soc. 2 (1970), 181–182.

[9] B. Cascales, A.J. Guirao, and V. Kadets, A Bishop-Phelps-Bollobás type theorem for uniform algebras, Adv. Math. 240 (2013), 370–382.

[10] M. Chica, V. Kadets, M. Martín, S. Moreno-Pulido and F. Rambla-Barreno, Bishop-Phelps-Bollobás moduli of a Banach space, preprint 2013. Available at http://arxiv.org/abs/1304.0376.

[11] Y.S. Choi and S.K. Kim, The Bishop-Phelps-Bollobás theorem for operators from $L_1(\mu)$ to Banach spaces with the Radon-Nikodým property, J. Funct. Anal. 261 (6) (2011), 1446–1456.

[12] Y.S. Choi, S.K. Kim, H.J. Lee and M. Martín, The Bishop-Phelps-Bollobás theorem for operators on $L_1(\mu)$, preprint 2013. Available at http://arxiv.org/abs/1305.6078.
[13] N. Dunford and J.T. Schwartz, *Linear operators*, Volume I, Interscience, New York, 1958.
[14] J. Johnson and J. Wolfe, *Norm attaining operators*, Studia Math. 65 (1) (1979), 7–19.
[15] S.K. Kim, *The Bishop-Phelps-Bollobás Theorem for operators from $c_0$ to uniformly convex spaces*, Israel J. Math. (to appear). Available at http://dx.doi.org/10.1007/s11856-012-0186-x.
[16] S.K. Kim and H.J. Lee, *Uniform convexity and Bishop-Phelps-Bollobás property*, Canadian J. Math. (to appear). Available at http://dx.doi.org/10.4153/CJM-2013-009-2.
[17] A.J. Lazar and J. Lindenstrauss, *Banach spaces whose duals are $L_1$ spaces and their representing matrices*, Acta Math. 126 (1971), 165–193.
[18] N.J. Nielsen and G.H. Olsen, *Complex preduals of $L_1$ and subspaces of $\ell_\infty^n(C)$*, Math. Scand. 40 (2) (1977), 271–287.
[19] W. Schachermayer, *Norm attaining operators on some classical Banach spaces*, Pacific J. Math. 105 (2) (1983), 427–438.