EFFECTIVE DIFFUSION ON RIEMANNIAN FIBER BUNDLES

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Abstract. The purpose of this paper is to provide equations to model the evolution of effective diffusion over a Riemannian fiber bundle (under the hypothesis of infinite diffusion rate along compact fibers). These equations are obtained by projecting the diffusion equation onto the base manifold of the fiber bundle. The projection (or dimensional reduction) is achieved by integrating the diffusion equation along fibers of the bundle. This work generalizes and puts into a general framework previous work on effective diffusion over channels and the interfaces between curved surfaces.

1. Introduction

Understanding spatially constrained diffusion is of fundamental importance in various sciences, such as biology, chemistry and nano-technology. However, solving the diffusion equation in arbitrarily constrained geometries is a very difficult task. One way to tackle it consists in reducing the degrees of freedom of the problem by considering only the main direction(s) of transport. For example, the study of diffusion on thin channels can be carried out by reducing several spatial degrees of freedom to a single one by means of a projection method. More concretely, consider a diffusive process in a channel modeled by a density function \( P \) that obeys the diffusion equation

\[
\frac{\partial P}{\partial t}(x, t) = D_0 \Delta P(x, t),
\]

subject to the restriction that there is no density flow along the channel’s walls. We can construct an effective density function by letting

\[
\rho(u, t) = \lim_{h \to 0} \frac{\text{Total concentration of } P \text{ in } R(h)}{h},
\]

where the region \( R(h) \) is the section of the channel within two transversal cross sections that are an arc distance of \( h \) apart over a base curve that “follows” the channel’s geometry (see Figure 1), and the variable \( u \) is the arc-length parameter on this curve. It turns out that this effective density function \( \rho \) obeys in an approximate manner an equation of the form (known as a generalized Fick-Jacobs equation)

\[
\frac{\partial \rho}{\partial t}(u, t) = \frac{\partial}{\partial u} \left( \sigma(u) D(u) \frac{\partial}{\partial u} \left( \frac{\rho(u, t)}{\sigma(u)} \right) \right),
\]

Date: 7 October 2014.
Partially supported by CONACyT grant 135106.
Figure 1.1. Region between two transversal cross section of a channel.

where $\sigma$ is given by

\[
\sigma(u) = \lim_{h \to 0} \frac{\text{Area}(R(h))}{h}.
\]

The function $\mathcal{D}$ is known as the effective diffusion coefficient and it encapsulates the effect of the channel’s geometry on the diffusion process along the base curve. Much work has been done (see [1, 8, 3, 11, 6, 5, 7, 9, 10]) to find explicit formulas for $\mathcal{D}$ in terms of geometrical quantities associated to the channel, so that the Fick-Jacobs equation models the evolution of $\rho$ as closely as possible. We can distinguish two cases.

1. **Infinite transverse diffusion rate.** In this case it is assumed that the density function $P$ stabilizes instantly in the transversal directions of the channel. In mathematical terms this means that $P$ is constant along these transversal directions. This assumption results in an effective diffusion coefficient that depends on the curvature function of the curve and 0-th order geometrical quantities of the cross sections, such as width or area.

2. **Finite transverse diffusion rate.** In this case the finite time of transversal stabilization of $P$ is taken into account. This is characterized mathematically in that the resulting formulas for $\mathcal{D}$ involve the curvature function of the base curve, and tangential and curvature information of the channel’s wall(s).

The selection of the base curve is very important in the dimension-reduction technique describe above. This is demonstrated by the fact (see [10, 9]) that if for channel’s of constant width the base curve is chosen properly then the formulas for $\mathcal{D}$ coincide for the finite and infinite transversal diffusion rate cases.
Motivated by the above discussion, we know describe the main purpose of the paper. We develop a very general theory, in the infinite transversal diffusion rate case, for projecting the diffusion equation in a space of dimension $n$ to a base space of dimension $m$ with $m < n$. We do this in the context of the theory of fibre bundles. Such objects have a total space $E$, a base space $M$ and a projection map $\pi : E \to M$. In the case discussed above, the total space $E$ is the channel, $M$ is the base curve, and $\pi$ sends points on the transversal cross sections (fibers) to their base point in the curve. The process of passing from the density $P$ to the reduced density $\rho$ is a particular case of a very well known construction: that of integrating a differentiable form along the fibers of the bundle. Using these tools we are able to give global and coordinate-free proofs of all our results. In this general setting, the effective diffusion $D$ becomes an endomorphism of the tangent bundle of $M$, i.e for every $x$ in $M$ we have that $D(x)$ is a linear map in the tangent space $T_x M$ of $M$. We will compute the effective diffusion using local frames instead of local coordinates, which results in a simpler and more geometric way of doing calculations. The paper is organized as follows

- In section 2 we show how to reduce the continuity equation in fibre bundle $E$ to a reduced continuity equation in its base space $M$. We prove that if Fick’s law holds on $E$ (for a constant diffusion coefficient $D_0$) and if the we have infinite diffusion rate in fiber direction, then the reduced continuity equation becomes a diffusion equation in $M$ (see Proposition 3). This last equation involves an effective diffusion coefficient $D$, which is a bundle endomorphism of the tangent space of $M$.

- In section 3 we compute the effective diffusion for a channel of constant width over an arbitrary curve on the plane. We have obtained this result previously in [10], and our re-derivation of the formula serves as a test case of our general theory.

- In section 4 we compute the effective diffusion endomorphism $D$ corresponding to the interface of two equidistant surfaces in 3-dimensional space. We do this by showing that the principal directions of the base surface are eigenvectors of $D$, and the computing the eigenvalues of $D$ in these directions.

- In section 5 we compute the effective diffusion on the surface (not the interior) of a circular channel over an arbitrary curve in $\mathbb{R}^3$. The main point of the calculation is to illustrate how our techniques still apply to fibre bundles whose fibers are manifolds without boundary (in this case, circles).

- In the Appendix we make a brief review of the geometrical concepts needed for the construction of the effective diffusion endomorphism.

2. THE EFFECTIVE DIFFUSION EQUATION

Let $E$ be fiber bundle over an $m$-dimensional manifold $M$, having compact fibers (with or without boundary) of dimension $k$, and projection map $\pi : E \to M$. We will assume that $M$ and $E$ are orientable, and with Riemannian metrics $\langle , \rangle_E$ and $\langle , \rangle_M$.  

The continuity equation. The continuity equation on the fibre bundle $E$ is given by

$$\frac{\partial P}{\partial t} + \text{div}(J) = 0,$$

where the density function $P$ is a time dependent function on $E$, and the density flow $J$ is a time dependent vector field in $E$. The divergence of $J$ is given by

$$\text{div}(J) = (\ast (d(\ast J^\flat)))^\sharp,$$

where $d$ is the differential operator acting on differentiable forms in $E$, $\ast$ is the Hodge star operator, and the $\flat$-operator converts vector fields to 1-forms (see the Appendix).

Integrating the density function along the fibers. Since we are assuming that the fibers of $E$ are compact, by integrating along the fiber we can define the operator $\pi_*$ sending $l$-forms in $E$ to $(l - k)$-forms in $M$. For an $l$-form $\omega$ on $E$ we define

$$\pi_*(\omega)_x(X_1, \ldots, X_{l-k}) = \int_{\pi^{-1}(x)} \beta_\omega$$

where

$$\beta_\omega(Y_1, \ldots, Y_k) = \omega(\tilde{X}_1, \ldots, \tilde{X}_{l-k}, Y_1, \ldots, Y_k)$$

and $\tilde{X}_i$ is a lift of $X_i$, i.e $D\pi(\tilde{X}_i) = X_i$. Observe that $\tilde{X}_i$ is defined on the whole fiber $\pi^{-1}(x)$. If the fibers of $E$ are manifolds without boundary then $\pi_*$ commutes with $d$ (see [2, pg. 62]), i.e for any differential form $\omega$ we have that

$$d(\pi_* \omega) = \pi_* (d\omega)$$

If the fibers of $E$ are manifolds with boundary then a sufficient condition for (2.3) to hold is that $\omega$ vanishes on all the vectors perpendicular to the boundary $\partial E$ of $E$. We will always assume that this condition holds for $J^\flat$, which means that there is no density flow across $\partial E$.

By applying $\ast$ to (2.1) and using formula (2.2) we get

$$\frac{\partial}{\partial t}(\ast P) + d(\ast J^\flat) = 0.$$ 

If we apply $\pi_*$ and then $\ast$ to (2.4) (and use the fact that $d$ commutes with $\pi_*$) we obtain

$$\frac{\partial \rho}{\partial t} + \ast(d(\pi_*(\ast J^\flat))) = 0,$$

where

$$\rho = \ast(\pi_*(\ast P)) = \pi_*(P \mu_E) / \mu_M,$$

and $\mu_E$ and $\mu_M$ be the metric volume forms in $E$ and $M$.

Remark. Formula (2.6) is a generalization of formula (1.1).
Integrating the density flow along the fibers. We would like to write equation \ref{eq:2.5} as a continuity equation in $M$. To do this we need to find a time dependent vector field $j$ in $M$ such that

$$\text{div}(j) = *(d(\pi_*(\ast J)))$$

which is equivalent to

$$*(d(*j)) = *(d(\pi_*(\ast J)))$$

This last equation is satisfied if we let

$$j = (-1)^{m-1}(\pi_*(\ast J))^\sharp,$$

where the $\sharp$-operator converts 1-forms to vector fields (see Appendix).

The effective continuity equation. We will refer to the time dependent function $\rho$ given by formula \ref{eq:2.6} as the effective density function, and to the time dependent vector field $j$ given by \ref{eq:2.7} as the effective density flow. We proved above that these objects satisfy the equation

$$\frac{\partial \rho}{\partial t} + \text{div}(j) = 0,$$

which we will refer to as the effective continuity equation.

The diffusion equation. Fick’s law establishes that

$$J(x, t) = -D(x)\nabla P(x, t),$$

where for $x$ in $M$ we have that $D(x)$ is a linear operator from $T_xE$ to $T_xE$, i.e and endomorphism of $TE_x$. The simplest choice of $D$ is to let it be scalar multiplication by a constant $D_0$. By using more general $D$’s we can model the inhomogeneity or anisotropy of $E$. Assuming Fick’s law, the continuity equation in $E$ becomes the diffusion equation

$$\frac{\partial P}{\partial t}(x, t) = \text{div}(D(x)P(x, t)),$$

In our work we will always assume that the endomorphism $D$ on $TE$ is multiplication by a positive scalar $D_0$, in which case the diffusion equation becomes

$$\frac{\partial P}{\partial t}(x, t) = D_0\Delta P(x, t),$$

where the laplacian operator $\Delta$ applied to $P$ is

$$\Delta P = \text{div}(\nabla P).$$

Problem. If Fick’s law holds in $E$ for $D = D_0$, does the effective density $\rho$ obeys a diffusion equation in $M$?

There is an important case when we can answer the above question positively (see Proposition \ref{prop:3} in the next paragraph).
An effective diffusion equation for infinite fiber diffusion rate. If the fibers of $E$ are “small enough” compared with the “size” of $M$, then it is to be expected that the density $P$ will stabilize faster along the fibers than along $M$. If we assume that this stabilization occurs infinitely fast, then $P$ must be constant along the fibers of $E$. We borrow the nomenclature from the physics literature, and refer to this situation by saying that there is an infinite fiber diffusion rate. Under this assumption, can write

$$P = \pi^* Q = Q \circ \pi$$

for a time dependent function $Q$ in $M$. Using formula 2.6 we then have that

$$\rho = \frac{\pi_*(\pi^*(Q)\mu_E)}{\mu_M} = \frac{Q\pi_*(\mu_E)}{\mu_M},$$

which allows us to obtain $Q$ in terms of $\rho$ as

(2.10) $Q = \frac{\rho}{\sigma}$,

where

(2.11) $\sigma = \frac{\pi_*(\mu_E)}{\mu_M}$.

Observe that if $R$ is a region in $M$ then

(2.12) $\int_R \sigma \mu_M = \int_{\pi^{-1}(R)} \mu_E = \text{vol}_E(\pi^{-1}(R))$.

Remark. Formula 2.11 is a generalization of that given by 1.3.

If Fick’s law holds in $E$ for $D = D_0$, then we have

$$J^b = -D_0 \pi^* dQ = -D_0 d\pi^* (dQ),$$

where $\pi^*$ is the pull-back of forms under $\pi$. Hence, we can write the effective density flow 2.7 as

(2.13) $j = -(-1)^{m-1} D_0 \ast (\pi_* (\ast d(\pi \ast Q)))^\sharp$

(2.14) $= -(-1)^{m-1} D_0 \ast (\pi_* (\ast (dQ)))^\sharp$

If we define $D$ by

(2.15) $D = (-1)^{m-1} \left( \frac{D_0}{\sigma} \right) (\sharp \circ \ast \circ \pi_* \circ \ast \circ \pi^* \circ b)$,

where $\circ$ stands for composition of operators, then by combining formulas 2.10 and 2.13 we obtain

(2.16) $j = -\sigma D(\nabla Q) = -\sigma D(\nabla \left( \frac{\rho}{\sigma} \right))$.

For a given point $x$ in $M$, the following sequence illustrate the spaces $D$ transverses to go from $T_x M$ to $T_x M$

$$T_x M \xrightarrow{\pi} \Omega^1_x M \xrightarrow{\pi} \Gamma(\Omega^1_x E) \xrightarrow{\pi} \Gamma(\Omega^{n-1}_x E) \text{ and } \Omega^{n-1}_x M \xrightarrow{\pi} \Omega^1_x M \xrightarrow{\pi} T_x M.$$
Proposition 1. For the case of infinite fiber diffusion rate, the effective density function $\rho$ satisfies the equation

$$\frac{\partial \rho}{\partial t}(x,t) = \text{div} \left( \sigma(x) D(x) \left( \nabla \left( \frac{\rho(x,t)}{\sigma(x)} \right) \right) \right),$$

where $D : TX \to TX$ is the vector bundle morphism defined in formula 2.15.

Proof. This follows from the effective continuity equation 2.8 and formula 2.16. Observe that $D$ is a bundle morphism since for any given $x$ in $M$ the map $D(x) : T_x M \to T_x M$ is a composition of linear maps. □

Definition 2. We will refer to the endomorphism $D$ of $TM$ given by formula 2.15 as the effective diffusion endomorphism.

Remark. Equation 2.17 is a generalization of 1.2.

Proposition 3. For the case of infinite fiber diffusion rate, there exists a metric in $M$ such that the effective density function $\rho$ satisfies the diffusion equation

$$\frac{\partial \rho}{\partial t}(x,t) = \text{div}(D(x) (\nabla \rho(x,t))),$$

where $D$ is the effective diffusion endomorphism. Furthermore, this choice of metric is such that for any region $R$ in $M$ we have that $\text{vol}_M(R) = \text{vol}_E(\pi^{-1}(R))$.

Proof. Let $<,>_E$ and $<,>_M$ be any metrics in $E$ and $M$. Define a new metric $<,>'_M$ in $M$ by the formula

$$<X,Y>'_M = \sigma^{2/m} <X,Y>_M,$$

for $\sigma$ defined in 2.11. Then, since

$$\sigma^{2/m} = \left( \frac{\pi_* (\mu_E)}{\mu_M} \right)^{2/m},$$

we have that

$$\sigma'_M = \frac{\pi_* (\mu_E)}{\mu'_M} = \frac{\pi_* (\mu_E)}{(\sigma^{2/m})^{m/2} \mu_M} = 1.$$

Equation 2.18 then follows directly from Proposition 1 and the last part of the Proposition follows from equation 2.12. □

3. Constant width channels on the plane

In this section we compute the effective diffusion for a channel of constant width $w > 0$ over a curve $C$ on the plane. Such a channel can be represented as the set

$$E = \{ x + vN(x) | x \in C \text{ and } -w/2 \leq v \leq w/2 \},$$

where $N$ is a unit normal field to $C$. For reasonable curves (e.g compact) and small $w$, the space $E$ is a fibre bundle with projection map $\pi : E \to M$ given by $\pi(p) = x$ where

$$p = x + vN(x).$$
Let $T$ be a unit tangent field to $C$ that makes the frame $T, N$ positively oriented. To perform integration over the fibers of $E$ we need to compute the lift of $T$ to $E$. If we define

$$T(p) = (1 - \kappa(x)v)T(x)$$

$$N(p) = N(x)$$

then $T$ is such a lift of $T$ (see Figure 3.1), and $D\pi(N) = 0$. Let $T^*$ be the dual field to $T$, and $T^*, N^*$ be the dual frame to $T, N$. The matrix of $<,>_E$ in the frame $T, N$ is given by

$$g = \begin{pmatrix} (1 - \kappa v)^2 & 0 \\ 0 & 1 \end{pmatrix}.$$  

**Computing $\sigma$.** The volume element in $E$ is

$$\mu_E = (1 - \kappa v)T^* \wedge N^*,$$

and hence

$$\pi_*(\mu_E) = \left( \int_{-w/2}^{w/2} (1 - \kappa v) dv \right) T^* = wT^*.$$  

Since the volume form in $C$ is $T^*$, we conclude that

$$\sigma = \frac{wT^*}{T^*} = w.$$  

**Computing $D$.** Observe that

$$\pi^*(T^*) = T^*,$$

and by formula 7.1 in the Appendix, we have that

$$*(T^*) = g^{11} \det(g)^{1/2}N^* = (1 - \kappa v)^{-1}N^*.$$
Hence
\[
\pi_*(\pi^*(T^*))) = \int_{-w/2}^{w/2} (1 - \kappa v)^{-1} dv
\]
\[
= \frac{1}{\kappa} \log \left( \frac{1 + \kappa w/2}{1 - \kappa w/2} \right)
\]
Using formula 2.15 we obtain
\[
D = \frac{D_0}{\kappa w} \log \left( \frac{1 + \kappa w/2}{1 - \kappa w/2} \right)
\]
(3.1)
\[
= \left( \frac{2D_0}{\kappa w} \right) \arctanh(\kappa w/2).
\]
This is a formula we have previously obtained in [10] by different methods.

4. The interface between to equidistant surfaces in 3-d space
Let \( S \) be a orientable surface in \( \mathbb{R}^3 \) and \( N \) a unit normal field to this surface. For small \( w > 0 \) and a “reasonable surface” the space
\[
E = \{ x + vN(x) | x \in S \text{ and } -w/2 \leq v \leq w/2 \},
\]
is a fibre bundle over \( S \) with projection map \( \pi : E \to M \) given by \( \pi(p) = x \), where \( p = x + vN(x) \).
The surface \( S \) has principal directions fields \( T_1 \) and \( T_2 \), with corresponding principal curvatures \( \kappa_1 \) and \( \kappa_2 \). If we define
\[
T_1(p) = (1 - \kappa_1(x)v)T_1,
\]
\[
T_2(p) = (1 - \kappa_2(x)v)T_2,
\]
\[
N(p) = N(x),
\]
then \( T_1 \) and \( T_2 \) are lifts of \( T_1 \) and \( T_2 \), and \( D\pi(N) = 0 \). The metric in \( <,>_E \) is represented in the \( T_1, T_2, N \) frame by the matrix
\[
g = \begin{pmatrix}
(1 - \kappa_1v)^2 & 0 & 0 \\
0 & (1 - \kappa_2v)^2 & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]
We will let \( T_1^*, T_2^*, N^* \) be the dual frame to \( T_1, T_2, N \).

Computing \( \sigma \). The volume form in \( E \) is
\[
\mu_E = (1 - \kappa_1 v)(1 - \kappa_2 v)T_1^* \wedge T_2^* \wedge N^*,
\]
and hence
\[
\pi_*(\mu_E) = \left( \int_{-w/2}^{w/2} (1 - \kappa_1 v)(1 - \kappa_2 v)dv \right) T_1^* \wedge T_2^*,
\]
\[
= w(1 + \kappa_1 \kappa_2 w^2/12)T_1^* \wedge T_2^*.
\]
Since the volume form in \( M \) is \( T_1^* \wedge T_2^* \), we conclude from formula 2.11 that
\[
\sigma = w(1 + \kappa_1 \kappa_2 w^2/12).
\]
Computing $\mathcal{D}$. Using formula 7.1 in the Appendix we obtain that

\[
\ast(T_1^*) = g^{11} \det(g)^{1/2} T_2^* \wedge N^* = \left(\frac{1 - \kappa_2 v}{1 - \kappa_1 v}\right) T_2^* \wedge N^*,
\]

\[
\ast(T_2^*) = -g^{22} \det(g)^{1/2} T_1^* \wedge N^* = -\left(\frac{1 - \kappa_1 v}{1 - \kappa_2 v}\right) T_1^* \wedge N^*.
\]

From these formulas and the identities $\ast T_1^* = T_2^*, \ast T_2^* = -T_1^*$, we obtain

\[
\ast(\pi_\ast(\ast(T_1^*))) = -\left(\int_{-w/2}^{w/2} \frac{1 - \kappa_2 v}{1 - \kappa_1 v} dv\right) T_1^*,
\]

\[
\ast(\pi_\ast(\ast(T_2^*))) = -\left(\int_{-w/2}^{w/2} \frac{1 - \kappa_1 v}{1 - \kappa_2 v} dv\right) T_2^*.
\]

By evaluating the above integrals, using formulas \(\pi_\ast(T_1^*) = T_1^*, \pi_\ast(T_2^*) = T_2^*\), and the fact that the matrix of the metric $<,>_M$ is the identity, formula 2.15 yields

\[
\mathcal{D}(T_1) = D_1 T_1 \quad \text{and} \quad \mathcal{D}(T_2) = D_2 T_2.
\]

where

\[
D_1 = \frac{D_0 \left(w \kappa_1 \kappa_2 + 2(\kappa_2 - \kappa_1) \arctanh(\kappa_1 w/2)\right)}{\kappa_1^2 w(1 + \kappa_1 \kappa_2 w^2/12)},
\]

\[
D_2 = \frac{D_0 \left(w \kappa_1 \kappa_2 - 2(\kappa_2 - \kappa_1) \arctanh(\kappa_2 w/2)\right)}{\kappa_2^2 w(1 + \kappa_1 \kappa_2 w^2/12)}.
\]

We have just proven the following result.

**Proposition 4.** Let $S$ be a surface in $\mathbb{R}^3$ with principal direction fields $T_1$ and $T_2$, and corresponding principal curvatures $\kappa_1$ and $\kappa_2$. For the bundle $E$ given by 4.1, we have that the fields $T_1$ and $T_2$ are eigenvectors of $\mathcal{D}$ with corresponding eigenvalues $D_1$ and $D_2$ given 4.2 and 4.3.

Remark. At an umbilical point $x$ of $S$ (i.e where $\kappa_1(x) = \kappa_2(x)$) the above proposition is still valid for any pair of orthonormal vectors $T_1, T_2$ in $T_x S$.

It is important to observe that if we want to write the effective diffusion equation 2.18 in coordinates, we need to express the principal direction fields $T_1$ and $T_2$ in terms of the corresponding coordinate fields. The reason for this is that the divergence operator that enters into the effective diffusion equation needs coordinates for its computation.

**Remark.** We can express eigenvalues of $\mathcal{D}$ in terms of the gaussian and mean curvatures

\[
K = \kappa_1 \kappa_2 \quad \text{and} \quad H = \frac{1}{2}(\kappa_1 + \kappa_2)
\]

by using the identities

\[
\kappa_1 = H + \sqrt{H^2 - K} \quad \text{and} \quad \kappa_2 = H - \sqrt{H^2 - K}.
\]

We now discuss some applications of Proposition 4 to specific families of surfaces.
Figure 4.1. Eigenvalues $D_1$ and $D_2$, of the effective diffusion $D$ for $D_0 = 1$, on a torus with inner radius $r = 1$, outer radius $R = 2$ and width $w = 1/4$.

**Spheres.** In this case we have that the principal curvatures $\kappa_1$ and $\kappa_2$ satisfy

$$\kappa_1 = \kappa_2 = 1/r,$$

where $r$ is the radius of the sphere. From formulas 4.2 and 4.3 we obtain that the eigenvalues of $D$ are

$$D_1 = D_2 = \frac{D_0 12r^2}{12r^2 + w^2}. \tag{4.4}$$

**Recovering the one dimensional case.** Let $C$ be any curve on the plane with curvature function $\kappa$, and let $S = C \times \mathbb{R} \subset \mathbb{R}^3$. The principal fields of $S$ are the unit tangent $T$ to the curve and $(0, 0, 1)$, with corresponding eigenvalues $\kappa$ and 0. Using formulas 4.2 and 4.3 we obtain

$$D_1 = \left( \frac{2D_0}{\kappa w} \right) \arctanh(\kappa w/2),$$

$$D_2 = D_0.$$

Hence, the eigenvalue $D_1$ of $D$ coincides with the case of curves in the plane discussed in section 3 (see Formula 3.1).

**The torus.** For a torus with inner radius $r$ and outer radius $R$ we have that

$$\kappa_1 = -1/r \quad \text{and} \quad \kappa_2 = -\frac{\cos(\theta)}{R + r \cos(\theta)},$$

where $\theta$ is the variable parametrizing the parallels of the torus. In Figure 4.1 we show the graphs of $D_1$ and $D_2$ obtained by using the above values of $\kappa_1$ and $\kappa_2$ for specific values of $r, R$ and $w$. 
5. Effective diffusion in the surface of a tube

Let $C$ be a curve in three-dimensional space, and let $T, N, B$ be the corresponding Serret-Frenet frame. We will let $E$ be the set of points of the form (for a constant radius $r$)

$$p = x + r \cos(\theta)N + r \sin(\theta)B \quad \text{where} \quad x \in C.$$ 

We want to construct a lift of $T$ to $E$. To do this, consider $x = x(s), T = T(s), N = N(s)$ and $B = B(s)$ as functions of the arc length parameter $s$ of $C$. From the formula

$$\pi(p(s)) = x(s)$$

we obtain

$$D\pi(T) = T \quad \text{and} \quad D\pi(H) = 0,$$

where

$$T = \frac{dp}{du} \quad \text{and} \quad N = \frac{dp}{d\theta}.$$ 

By using the Frenet-Serret formulas we obtain

$$T = (1 - \kappa r \cos(\theta))T + \tau r (-\sin(\theta)N + \cos(\theta)B),$$

$$N = -r \sin(\theta)N + r \cos(\theta)B,$$

where $\kappa$ and $\tau$ are the curvature and torsion of $C$. The metric matrix in the $T, N$ frame is

$$g = \begin{pmatrix} (1 - \kappa r \cos(\theta))^2 + \tau^2 r^2 & \tau r^2 \\ \tau r^2 & r^2 \end{pmatrix}$$

and its inverse

$$g^{-1} = \frac{1}{(1 - \kappa r \cos(\theta))^2} \begin{pmatrix} 1 & -\tau \\ -\tau & \frac{-\tau}{(1 - \kappa r \cos(\theta))^2} \end{pmatrix}.$$ 

Computing $\sigma$. The volume form in $E$ is

$$\mu_E = r(1 - \kappa r \cos(\theta))T^* \wedge H^*,$$

so that

$$\pi_*(\mu_E) = \left( \int_0^{2\pi} r(1 - r\kappa \cos(\theta))d\theta \right) T^* = 2\pi r T^*$$

and

$$\sigma = \frac{2\pi r T^*}{T^*} = 2\pi r.$$ 

Computing $D$. We have that

$$\pi^*(T^*) = T^*,$$

and

$$*T^* = g^{11} \det(g) = r(1 - \kappa r \cos(\theta))^{-1} H^*,$$

so that

$$\pi_*(*T^*) = \int_0^{2\pi} \frac{r}{1 - \kappa r \cos(\theta)} d\theta = \frac{2\pi r}{1 + r\kappa} \left( \sqrt{\frac{2}{1 - r\kappa}} - 1 \right) T^*.$$ 

We conclude that

$$D = \frac{D_0}{(1 + r\kappa)} \left( \sqrt{\frac{2}{1 - r\kappa}} - 1 \right).$$
6. Conclusions

We have shown that the diffusion equation on the total space of a fibre bundle can be converted into a diffusion equation on its base space, under the hypothesis of infinite diffusion rate along the fibers. We provided a general formula for the effective diffusion endomorphism of the reduced diffusion equation, that we later applied to obtain explicit formulas for diverse fiber bundles. Of particular interest was the computation of the effective diffusion endomorphism associated to the the interface of two equidistant surfaces in 3-dimensional space, in terms of the principal curvatures of the base surface.

7. Appendix

The sharp and flat operators. We can see a 1-form $\alpha$ as a vector field $\alpha^\flat$ defined by

$$<\alpha^\flat, X> = \alpha(X),$$

where $<,>$ is the metric of the space under consideration. Similarly, a vector field $X$ can be seen as a 1-form $X^\flat$ defined by

$$X^\flat(Y) = <X, Y>.$$

For a local frame $X_1, \ldots, X_n$ the metric can be expressed as a symmetric matrix $g$ with coefficients

$$g_{ij} = <X_i, X_j>,$$

and we will write

$$g^{ij} = (g^{-1})_{ij}.$$

Let $X^1, \ldots, X^n$ be the 1-forms forming the dual frame to $X_1, \ldots, X_n$, so that $X^i(X_j) = \delta_j^i$. For a vector field

$$X = \sum_{i=1}^n a^i X_i,$$

we have that

$$X^\flat = \sum_{i=1}^n a_i X^i$$

where $a_i = \sum_{j=1}^n g_{ij} a^j$.

For a 1-form

$$\alpha = \sum_{i=1}^n \alpha_i X^i,$$

we have that

$$\alpha^\flat = \sum_{i=1}^n \alpha^i X_i$$

where $\alpha^i = \sum_{j=1}^n g^{ij} \omega_j$. 

Hodge star operator. The Hodge star operator $\ast$ maps $l$-forms to $(n-l)$ forms, where $n$ is the dimension of the space under consideration, and it is defined so that for $l$-forms $\omega$ and $\eta$ we have that

$$\omega \wedge (\ast \eta) = \langle \omega, \eta \rangle \mu,$$

where $\mu$ is the volume form of the metric. For monomials $\omega = \alpha_1 \wedge \ldots \wedge \alpha_l$ and $\eta = \beta_1 \wedge \ldots \beta_l$, where the $\alpha_i$'s and the $\beta_j$'s are 1-forms, we have that

$$\langle \omega, \eta \rangle = \det(\langle \alpha_i, \beta_j \rangle) = \langle \alpha_i^\sharp, \beta_j^\sharp \rangle.$$

The $\ast$-operator satisfies the duality relation, where for an $l$-form $\omega$ we have that

$$\ast \ast \omega = (-1)^{l(n-l)} \omega.$$

If for a local frame of vector fields $X_1, \ldots, X_n$ we have metric matrix $g_{ij} = \langle X_i, X_j \rangle$, then the coefficients $g^{ij}$ of $g^{-1}$ are

$$g^{ij} = \langle X_i, X_j \rangle.$$

The metric volume form is given by

$$\mu = \det(g)^{1/2} X^1 \wedge \ldots \wedge X^n.$$

Using the above formulas we obtain

$$\ast(X^1 \wedge \ldots \wedge X^n) = \det(g)^{-1/2},$$

and

$$\ast X^i = g^{ii} \det(g)^{1/2} t_i(X^1 \wedge \ldots \wedge X^n),$$

for

$$t_i(X^1 \wedge \ldots \wedge X^n) = (-1)^{i+1} X^1 \wedge \ldots \wedge X^{i-1} \wedge \hat{X}^i \wedge X^{i+1} \wedge \ldots \wedge X^n,$$

where notation $\hat{X}^i$ indicates that that term has been removed as a factor in the above wedge product.

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