A Generalized discrete Riesz transforms

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Abstract

In this paper, we introduce a discrete Riesz transforms associated with the non-symmetric trigonometric Heckman-Opdam polynomials of type $A_1$. We prove that they can be extended to a bounded operators on $\ell^p(\mathbb{Z})$, $1 < p < \infty$.

1 Introduction

Fourier series constitute a fundamental tool in the study of periodic functions. It is from this concept that a branch of mathematics known as harmonic analysis was developed. There are many generalizations of Fourier series that have proved to be useful and are all special cases of decompositions over an orthonormal basis of an inner product space. In this work, we consider the Fourier series generated by the non-symmetric trigonometric Heckman-Opdam polynomials of type $A_1$ which contains as special case ($k=0$) the classical Fourier series. Generally, the Non-symmetric Heckman-Opdam polynomials are family of orthogonal polynomials associated with a root system, introduced by Opdam [6] as eigenfunctions of Cherednik operators. In particular, for root system of type $A_1$ these polynomials are reduced to the non-symmetric ultraspherical or Gegenbauer polynomials.

The aim of this paper is to establish the $\ell^p$-boundedness of the discrete Riesz transforms associated with the non-symmetric Heckman-Opdam polynomials. We prove that they are a Calderón-Zygmund operators on the discrete homogeneous space $\mathbb{Z}$. The key point for this consideration is a discrete version of Hörmander type conditions given in [4], and closely connect with a continuous version given in [3]. We recall that discrete Riesz transforms for ultraspherical polynomials are recently studied in [4] and in [1] for Jacobi polynomials.

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2 Non-symmetric Heckmann-Opdam polynomials of type $A_1$

2.1 Definitions and properties

For the theory of Heckmann-Opdam polynomials associated to general root systems, we refer to [7].

Let $k \geq 0$, the Cherednick operator of type $A_1$ is defined by

$$T^k(f)(x) = f'(x) + 2k \frac{f(x) - f(-x)}{1 - e^{-2x}} - kf(x), \quad f \in C^1(\mathbb{R}). \quad (2.1)$$

We introduce the weight function

$$\delta_k(x) = |2\sin x|^{2k}$$

and the inner product on $L^2([0, 2\pi), \delta_k(x)dx)$

$$(f, g)_k = \frac{1}{2\pi} \int_0^{2\pi} f(x)g(x)\delta(x)dx,$$

with its associate norm $\|\cdot\|_k$.

Defining the partial ordering $\triangleright$ on $\mathbb{Z}$ as follows

$$j \triangleright n \iff \begin{cases} |j| < |n| \text{ and } |n| - |j| \in 2\mathbb{Z}^+, \\ \text{or,} \\ |j| = |n| \text{ and } n < j. \end{cases}$$

The non-symmetric Heckman-Opdam polynomials $E^k_n$, $n \in \mathbb{Z}$ are defined by the following conditions:

$$(a) \quad E^k_n(x) = e^{nx} + \sum_{j\triangleright n} c_{n,j} e^{jx} \quad (2.2)$$

$$(b) \quad (E^k_n(ix), e^{ijx})_k = 0, \quad \text{for any } j \triangleright n. \quad (2.3)$$

The polynomials $E^k_n$ diagonalize simultaneously the Cherednik operators and

$$T^k(E^k_n) = \tilde{n} E^k_n, \quad n \in \mathbb{Z}, \quad (2.4)$$

where $\tilde{n} = n + k$ if $n \geq 0$ and $\tilde{n} = n - k$ if $n < 0$. Clearly,

$$E^k_0(x) = 1, \quad \text{and} \quad E^k_1(x) = e^x. \quad (2.5)$$

A consequence the trigonometric polynomials $\{E^k_n(ix)\}_{n \in \mathbb{Z}}$ is an orthogonal basis of $L^2([0, 2\pi), \delta_k(x)dx)$.

The Heckman-Opdam- Jacobi polynomials $P^k_n$, $n \in \mathbb{Z}_+$ is given by

$$P^k_n(x) = E^k_n(x) + E^k_n(-x).$$
and they satisfy
\[(T^k)^2 P_n^k = (n+k)^2 P_n^k.\]
In particular if we denote \(L_k\) the differential operator
\[L_k(f) = f'' + 2k \frac{1+e^{-2x}}{1-e^{-2x}} f'(x) + k^2 f(x)\]
then \(P_n^k\) is the eigenfunction of \(L_k\), namely that
\[L_k(P_n^k) = (n+k)^2 P_n^k.\]
The Heckman opdam–Jacobi polynomials is expressed via the hypergeometric function \(2F_1\) by
\[P_n^k(x) = P_n^k(0) \(_2F_1(n+2k,-n,k+1/2\sinh^2(x/2))\)
where for all \(n \in \mathbb{Z}^+\)
\[P_n^k(0) = 2E_n^k(0) = \frac{\Gamma(k)\Gamma(n+2k)}{\Gamma(2k)\Gamma(n+k)}.\]
Noting also that
\[E_{-n}^k(0) = \frac{\Gamma(k)\Gamma(n+2k+1)}{2\Gamma(2k)\Gamma(n+k+1)}, \quad n > 0.\]
We have following relationship
\[E_n^k(x) = E_n^k(0) \left\{ p_n^k(x) + \frac{\tilde{n}+2k}{2k+1} \sinh x P_{n-1}^k(x) \right\}, \quad n \in \mathbb{Z}.\]

It is also worth mentioning that the non-symmetric Heckman-Opdam polynomials \(E_n^k\) are closely related to non-symmetric Jack polynomials see [8]. If \(\lambda = (\lambda_1, \lambda_2) \in \mathbb{Z} \times \mathbb{Z}\) and \(N_n^k\) the correspondent non-symmetric Jack polynomials then
\[E_{\lambda_2-\lambda_1}^k = N_{\lambda}^k(e^{-x}, e^x).\]
An important result due to Sahi [8] states that the coefficients \(c_{n,j}\) in (2.2) are all nonnegative.

The upcoming propositions are inspired by the work of [8].

**Proposition 2.1.** For all \(n \in \mathbb{Z}\), we have
\[E_{n+1}^k(x) = e^x E_{n}^k(-x).\]  

**Proof.** In view of (2.5) this identity is true for \(n = -1, 0\). Let \(n \in \mathbb{Z}, n \neq -1, 0\). Put \(H_n^k(x) = e^x E_{-n}^k(-x)\), it is enough to check that \(T^k(H_n) = (n+1)H_n\). We have
\[T^k(H_n)(x) = e^x E_{-n}^k(-x) - e^x (E_{-n}^k)'(-x) + 2k \frac{e^x E_{-n}^k(-x) - e^{-x} E_{-n}^k(x)}{1-e^{-2x}} - ke^x E_{-n}^k(-x)\]
\[= -e^x \left( (E_{-n}^k)'(-x) + 2k \frac{E_{-n}^k(-x) - E_{-n}^k(x)}{1-e^{2x}} - kE_{-n}^k(-x) \right) + e^x E_{-n}^k(-x)\]
\[= (1 - (-n))e^x E_{-n}^k(-x) = (n+1)e^x E_n^k(-x) = (n+1)H_n(x).\]
Then (2.6) follows by comparing the highest coefficient. \(\square\)
Proposition 2.2. For \( n \in \mathbb{Z}^+ \), \( n \neq 0 \), we have
\[
E_{-n}^k(x) = E_n^k(-x) + \frac{k}{n + k} E_n^k(x). \tag{2.7}
\]

Proof. Observe that for \( n \geq 1 \) we have
\[
T^k(E_n(-x)) = -2k E_n(x) - (n + k) E_n(-x).
\]
Then
\[
T^k \left( E_n^k(-x) + \frac{k}{n + k} E_n^k(x) \right) = -2k E_n(x) - (n + k) E_n^k(-x)
\]
\[
= -k E_n(x) - (n + k) E_n^k(-x)
\]
\[
= -(n + k) \left( E_n^k(-x) + \frac{k}{n + k} E_n^k(x) \right).
\]

We conclude the (2.7) by comparing the highest coefficient. \( \square \)

From (2.7) one can deduce the following identity.

Corollary 2.3. For \( n \in \mathbb{Z}^+ \), we have
\[
\left( 1 - \frac{k^2}{(n + k)^2} \right) E_n^k(x) = E_{-n}^k(-x) - \frac{k}{n + k} E_{-n}^k(x). \tag{2.8}
\]

Proposition 2.4. For all \( n \in \mathbb{Z}^+ \), we have
\[
\|E_{n+1}^k\|_k^2 = \|E_{-n}^k\|_k^2 = n! \frac{\Gamma(n + 2k + 1)}{\Gamma(n + k + 1)^2}.
\]

Proof. In view of (2.5), we write
\[
E_n^k(-x) = E_{-n}^k(x) - \frac{k}{n + k} E_{-n}^k(x), \quad n \geq 1.
\]

Since \( E_n^k \) and \( E_{-n}^k \) are orthogonal, then
\[
\|E_n^k\|_k^2 = \|E_{-n}^k\|_k^2 + \frac{k^2}{(n + k)^2} \|E_n^k\|_k^2
\]
and
\[
\|E_{n+1}^k\|_k^2 = \|E_{-n}^k\|_k^2 = \frac{n(n + 2k)}{(n + k)^2} \|E_n^k\|_k^2.
\]

In addition it not hard to see that
\[
\|E_1^k\|_k^2 = \|E_0^k\|_k^2 = \frac{\Gamma(2k + 1)}{\Gamma(k + 1)^2}.
\]

Therefore the desired formula follows. \( \square \)

In what follows we set
\[
E_n^k = \frac{E_n^k}{\|E_n^k\|_k}.
\]
From (2.8) we get
\[ \sqrt{\frac{n(n+2k)}{n+k}} \mathcal{E}^k_n(x) = e^{-x} \mathcal{E}^k_{n+1} - \frac{k}{n+k} \mathcal{E}^k_{-n} (x), \quad n \geq 0, \]
and so,
\[ e^{-x} \mathcal{E}^k_{n+1} = \sqrt{\frac{n(n+2k)}{n+k}} \mathcal{E}^k_n(x) + \frac{k}{n+k} \mathcal{E}^k_{-n} (x), \quad n \geq 0. \] (2.9)
However from (2.6) and (2.7) we have that
\[ \mathcal{E}^k_{-n} (x) = e^{-x} \mathcal{E}^k_{-n+1}(x) + \frac{k}{n+k} \mathcal{E}^k_{n} (x), \quad n \geq 1, \]
which implies that
\[ \sqrt{\frac{n(n+2k)}{n+k}} \mathcal{E}^k_{-n}(x) = e^{-x} \mathcal{E}^k_{-n+1} + \frac{k}{n+k} \mathcal{E}^k_{n} (x), \quad n \geq 1 \]
and then
\[ e^{-x} \mathcal{E}^k_{n+1} = \sqrt{-\frac{n(-n+2k)}{-n+k}} \mathcal{E}^k_n(x) - \frac{k}{-n+k} \mathcal{E}^k_{-n} (x), \quad n \leq -1. \] (2.10)
Now putting
\[ \alpha_n = \sqrt{|n|(|n|+2k)} \quad \text{and} \quad \beta_n = \varepsilon(n) \frac{k}{|n|+k}, \quad n \in \mathbb{Z}, \] (2.11)
with
\[ \varepsilon(n) = \begin{cases} 
1, & n \geq 0 \\
-1, & n < 0
\end{cases} \]
We then state the following.

**Proposition 2.5.** For all \( n \in \mathbb{Z} \), we have
\[ e^{-x} \mathcal{E}^k_{n+1} = \alpha_n \mathcal{E}^k_n + \beta_n \mathcal{E}^k_{-n} = \alpha_n \mathcal{E}^k_n - \beta_n \mathcal{E}^k_{-n}. \] (2.12)

Next we define on \( \mathbb{C}^\mathbb{N} \) the operators \( \Lambda_k \) and \( \Lambda_k^* \) by:
\[ \Lambda_k(f)(n) = \alpha_n f(n+1) + \beta_n f(-n+1) - f(n), \]
\[ \Lambda_k^*(f)(n) = \alpha_{n-1} f(n-1) + \beta_{n-1} f(-n+1) - f(n), \]
and the generalized discrete Laplace operator by
\[ \Delta_k = -\Lambda_k^* \Lambda_k. \]
It follows that
\[ \Delta_k(f)(n) = \alpha_n f(n+1) + \alpha_{n-1} f(n-1) - 2f(n) - (\beta_n - \beta_{n-1})f(-n+1), \quad f \in \ell^2(\mathbb{Z}). \]
In the case \( k = 0 \), we have
\[ \Lambda_0 f(n) = f(n+1) - f(n), \quad \Lambda_0^* f(n) = f(n-1) - f(n) \]
and
\[ \Delta_k(f)(n) = f(n + 1) - 2f(n) + f(n - 1). \]

Let us introduce the operator
\[ \tilde{T}_k(u)(x) = u'(x) + 2ki \frac{u(x) - u(-x)}{1 - e^{-2ix}}; \quad u \in C^1(\mathbb{R}). \]

In view of (2.1) and (2.4) we have

**Proposition 2.6.** For all \( n \in \mathbb{Z}^+ \) we have
\[ \tilde{T}_k(\mathcal{E}_n^k(ix))(x) = i(n + k)\mathcal{E}_n^k(ix). \]

### 2.2 Generalized discrete Fourier transform

For \( f \in \ell^2(\mathbb{Z}) \) we define the generalized discrete Fourier transform of \( f \) by
\[ \mathcal{F}_k(f)(x) = \sum_{n \in \mathbb{Z}} f(n)\mathcal{E}_n^k(ix), \quad n \in \mathbb{Z}. \]

As \( \{\mathcal{E}_n^k(ix), n \in \mathbb{Z}\} \) is an orthonormal basis of \( L^2([0, 2\pi], \delta_k(x)dx) \), then we can state,

**Theorem 2.7.** \( \mathcal{F}_k \) is an isometric isomorphism from \( \ell^2(\mathbb{Z}) \) onto \( L^2([0, 2\pi], \delta_k(x)dx) \). Precisely we have
\[ \|\mathcal{F}_k(f)\|_k^2 = \sum_{n \in \mathbb{Z}} |f(n)|^2 \]
and for \( n \in \mathbb{Z} \),
\[ f(n) = \frac{1}{2\pi} \int_0^{2\pi} \mathcal{F}_k(f)(x)\mathcal{E}_n^k(ix)\delta_k(x)dx. \]

It’s not hard to see that
\[ \mathcal{F}_k(\Lambda_k(f))(x) = (e^{-ix} - 1)\mathcal{F}_k(\Lambda_k(f))(x), \quad \mathcal{F}_k(\Lambda_k^*(f))(x) = (e^{ix} - 1)\mathcal{F}_k(\Lambda_k(f))(x) \]
and
\[ \mathcal{F}_k(\Delta_k(f))(x) = -4\sin^2(x/2)\mathcal{F}_k(f)(x). \]

Clearly \(-\Lambda_k\) is positive self-adjoint operator on \( \ell^2(\mathbb{Z}) \) and generates a \( C_0 \)-semigroup \( e^{t\Lambda_k} \), \( t \geq 0 \) given by
\[ \mathcal{F}_k(e^{t\Lambda_k}(f))(x) = e^{-4t \sin^2(x/2)}\mathcal{F}_k(f)(x) \]
It can be written as
\[ e^{t\Lambda_k}(f)(n) = \sum_{m \in \mathbb{Z}} H_t(n, m)f(m), \quad n \in \mathbb{Z}, \]
with
\[ H_t(n, m) = \frac{1}{2\pi} \int_0^{2\pi} e^{-4t \sin^2(x/2)}\mathcal{E}_m^k(ix)\mathcal{E}_n^k(-ix)\delta_k(x)dx, \]
which will be called the generalized discrete heat kernel.
We now introduce the generalized discrete Riesz Transforms given as a multiplier operators by

\[ F_k(R(f))(x) = -ie^{-ix/2}F_k(f)(x), \quad F_k(R^*(f))(x) = ie^{ix/2}F_k(f)(x), \quad x \in [0, 2\pi]. \]

They can be written as

\[ R = \Lambda_k(-\Delta_k)^{-1/2}, \quad \text{and} \quad R^* = \Lambda_k^*(-\Delta_k)^{-1/2}. \] (2.13)

It follows easily that \( R \) and \( R^* \) are a bounded operators on \( \ell^2(\mathbb{Z}) \) and they are a kernel operators, since we have

\[ R(f)(n) = -i \int_0^{2\pi} e^{-ix/2}F_k(f)(x)\mathcal{E}_n^k(-ix)\delta_k(x)dx = \frac{1}{2\pi i} \sum_{n \in \mathbb{Z}} f(m) \int_0^{2\pi} e^{-ix/2}\mathcal{E}_m^k(ix)\mathcal{E}_n^k(-ix)\delta_k(x)dx. \]

\[ = \sum_{m \in \mathbb{Z}} R(n, m)f(m), \]

where

\[ R(n, m) = \frac{1}{2\pi i} \int_0^{2\pi} e^{-ix/2}\mathcal{E}_m^k(ix)\mathcal{E}_n^k(-ix)\delta_k(x)dx. \]

\[ = \frac{1}{2\pi i} \int_{-\pi}^{\pi} h(x)\mathcal{E}_m^k(ix)\mathcal{E}_n^k(-ix)\delta_k(x)dx. \]

with \( h(x) = \text{sign}(x)e^{-ix/2} \). Similarly

\[ R^*(f)(n) = \sum_{m \in \mathbb{Z}} R^*(n, m)f(m) \]

where

\[ R^*(n, m) = -\frac{1}{2\pi i} \int_0^{2\pi} e^{ix/2}\mathcal{E}_m^k(ix)\mathcal{E}_n^k(-ix)\delta_k(x)dx. \]

3 \( \ell^p \)-boundedness of the generalized Riesz transform

The main tool to study the \( \ell^p \)-boundedness of a discrete integral operator is a discrete version of an \( \ell^p \)- theorem adopted in Dunkl theory by the first author [2, 3]. The ideas for this material come from [4, 1].

3.1 \( \ell^p \)-Theorem for a discrete integral operators

Let us begin by remember the following continuous version
Theorem 3.1. Let $K$ be a measurable function on $\mathbb{R}^2 \setminus \{(x, y); |x| \neq |y|\}$ and $S$ be a bounded operator on $L^2(\mathbb{R}, dx)$ such that

$$T(f)(x) = \int_{\mathbb{R}} K(x, y)f(y)dy,$$

for all a.e. $x \in \mathbb{R}$, such that $x, -x \notin \text{supp}(f)$. If $K$ satisfies the following Hörmander type condition

$$\sup_{y, y' \in \mathbb{R}} \int_{|x|-|y|>|y'-y|} \left( |K(x, y) - K(x, y')| + |K(y, x) - K(y', x)| \right) dx < \infty$$

then $T$ can be extended to a bounded operator from $L^p(\mathbb{R}, dx)$ onto itself for $1 < p < \infty$.

The proof of this theorem only requires some minor modifications of the known classical ones and can be easily adopted to discrete analogues version by considering the metric space $Z$ with the counting measure which is invariant by $-Id$. We state the following,

Theorem 3.2. Let $K$ be a measurable function on $\mathbb{Z} \times \mathbb{Z} \setminus \{(n, m); |m| \neq |n|\}$ and $T$ be a bounded operator on $\ell^2(\mathbb{Z})$ such that for a compact support function $f \in \ell^1(\mathbb{Z})$,

$$T(f)(n) = \sum_{m \in \mathbb{Z}} K(n, m)f(m),$$

for all $n \in \mathbb{Z}$, such that $f(n) = f(-n) = 0$. If $K$ satisfies the following Hörmander type condition

$$\sup_{m, \ell \in \mathbb{Z}} \sum_{||n|-|m||>||m|\ell|} \left( |K(n, m) - K(n, \ell)| + |K(m, n) - K(\ell, n)| \right) < \infty. \quad (3.1)$$

then $T$ can be extended to a bounded operator from $\ell^p(\mathbb{Z})$ onto itself, for all $1 < p < \infty$.

Our strategy in applying this theorem inspired by [4], through the following result,

Theorem 3.3. Suppose that $T$ is a linear and bounded operator on $\ell^2(\mathbb{Z})$ and such that there exists a function $K : \mathbb{Z} \times \mathbb{Z} \setminus \{(m, n), |m| \neq |n|\}$ such that for every $f = (f(n))_{n \in \mathbb{Z}}$

$$T(f)(n) = \sum_{m \in \mathbb{Z}} K(n, m)f(m) \quad (3.2)$$

for $n \in \mathbb{Z}$ such that $f(n) = f(-n) = 0$. In addition we assume that $K$ satisfies the following Hörmander conditions

(i) $|K(n, m)| \leq \frac{C}{|n-m|}$,

(ii) $|K(n, m) - K(n, \ell)| + |K(m, n) - K(\ell, n)| \leq C \frac{|n-\ell|}{(|m|-|n|)^2}$,

for all $m, n, \ell$ such that

$$||m| - |n|| > 2|n - \ell| \quad \text{and} \quad \frac{|n|}{2} \leq |m|, |\ell| \leq \frac{3|n|}{2}.$$  

Then $T$ can be extended to bounded operator from $\ell^p(\mathbb{Z})$ onto $\ell^p(\mathbb{Z})$, for all $1 < p < \infty$. 

Proof. We apply the same arguments that used in [4].

For \( n \in \mathbb{Z} \), we let
\[
W_n = \{ m \in \mathbb{Z}; \ |n|/2 \leq |m| \leq 3|n|/2 \}.
\]

Define the operators
\[
T_{\text{glob}}(f)(n) = T(\chi_{\mathbb{Z}\setminus W_n} f)(n), \quad n \in \mathbb{Z}
\]
and
\[
T_{\text{loc}}(f) = T(f) - T_{\text{glob}}(f),
\]
for compactly supported \( f = (f(n))_{n \in \mathbb{Z}} \). We first check the boundedness of \( T_{\text{glob}} \).

According to (3.2), one can write
\[
T_{\text{glob}}(f)(n) = \sum_{m \in \mathbb{Z}\setminus W_n} K(m, n) f(m) = \sum_{|m| \leq |n|/2} K(m, n) f(m) + \sum_{|m| \geq 3|n|/2} K(m, n) f(m).
\]

By using (i) we have that
\[
|T_{\text{glob}}(f)(n)| \leq C \left\{ \frac{1}{n} \sum_{|m| \leq |n|} |f(m)| + \sum_{|m| \geq |n|/2} \frac{|f(m)|}{|m|} \right\}
\]
\[
\leq C \left\{ H_0(\tilde{f})(|n|) + H_1(\tilde{f})(|n|) \right\}
\]
where \( \tilde{f}(n) = |f(n)| + |f(-n)| \), \( H_0 \) and \( H_1 \) are the discrete Hardy operators given on \( \mathbb{C}^\mathbb{Z}^+ \) by
\[
H_0(a)(n) = \frac{1}{n} \sum_{m=0}^n a(n), \quad H_1(a)(n) = \sum_{m \geq n} \frac{a(n)}{n}.
\]
Since it’s known that these operators are bounded on \( \ell^p(\mathbb{N}) \) for \( 1 < p < \infty \), then \( T_{\text{glob}} \) can be extended to bounded operator on \( \ell^p(\mathbb{Z}) \).

Next we focus on \( T_{\text{loc}} \). Define the kernel \( \tilde{K} \) by
\[
\tilde{K}(n, m) = \chi_{W_n}(m) K(n, m); \quad n, m \in \mathbb{Z}; \ |n| \neq |m|.
\]
We can write
\[
T_{\text{loc}}(f)(n) = \sum_{m \in \mathbb{Z}} \tilde{K}(n, m) f(m)
\]
for compactly supported function \( f \) with \( f(n) = f(-n) = 0 \). We will prove that \( \tilde{K} \) satisfies the Hörmander type condition (3.1).

Let \( m, \ell \in \mathbb{Z} \) with \( |m| < |\ell| \). We have
\[
\sum_{|n| - |m| \geq 2|m| - \ell} |\tilde{K}(n, m) - \tilde{K}(n, \ell)| = \sum_{|n| - |m| \geq 2|m| - \ell} |\tilde{K}(n, m) - \tilde{K}(n, \ell)| + \sum_{|n| - |m| \geq 2|m| - \ell} |\tilde{K}(n, m) - \tilde{K}(n, \ell)|.
\]
Making use of the assertion \((i)\)
\[
\sum_{\substack{|n| - |m| \geq 2|m-\ell| \\mid n < 2\ell/3}} |\tilde{K}(n, m) - \tilde{K}(n, \ell)| = \sum_{\substack{|n| - |m| \geq 2|m-\ell| \\mid n < 2\ell/3}} |\tilde{K}(n, m)|
\]
\[
\leq C \sum_{\substack{|n| - |m| \geq 2|m-\ell| \\mid 2|m|/3 \leq |n| \leq 2\ell/3}} \frac{1}{|n| - |m|}
\]
\[
\leq \frac{C}{|\ell| - |m|} \sum_{\substack{j \geq 2|m|/3}} \leq C.
\]
and by \((ii)\)
\[
\sum_{\substack{|n| - |m| \geq 2|m-\ell| \\mid n \geq 2\ell/3}} |\tilde{K}(n, m) - \tilde{K}(n, \ell)| \leq C |m - \ell| \sum_{\substack{|n| - |m| \geq 2|m-\ell|}} \frac{1}{(|n| - |m|)^2}
\]
\[
\leq C |m - \ell| \sum_{j \geq 2|m|} \frac{1}{j^2} \leq C.
\]
Hence, we conclude that
\[
\sum_{|n| - |m| \geq 2|m-\ell|} |\tilde{K}(n, m) - \tilde{K}(n, \ell)| \leq C
\]
Let us now prove that
\[
\sum_{|n| - |m| \geq 2|m-\ell|} |\tilde{K}(m, n) - \tilde{K}(\ell, n)| \leq C \tag{3.3}
\]
Noting first that if \(|n| - |m| \geq 2|m-\ell|\) then
\[
|n| - |\ell| \geq \frac{|n| - |m|}{2}. \tag{3.4}
\]
Begin with the case \(3|m| < |\ell|\). Then \(W_m \cap W_\ell = \emptyset\) and by using \((i)\) and \((3.4)\) we have that
\[
\sum_{|n| - |m| \geq 2|m-\ell|} |\tilde{K}(m, n) - \tilde{K}(\ell, n)|
\]
\[
= \sum_{\substack{|n| - |m| \geq 2|m-\ell| \\mid n \in W_m}} |\tilde{K}(m, n) - \tilde{K}(\ell, n)| + \sum_{\substack{|n| - |m| \geq 2|m-\ell| \\mid n \in W_\ell}} |\tilde{K}(m, n) - \tilde{K}(\ell, n)|
\]
\[
\leq C \left( \sum_{\substack{|n| - |m| \geq 2|m-\ell| \\mid n \in W_m}} \frac{1}{|n| - |m|} + \sum_{\substack{|n| - |m| \geq 2|m-\ell| \\mid n \in W_\ell}} \frac{1}{|m| - |m|} \right)
\]
\[
\leq \frac{C}{|\ell|} \left( \sum_{n \in W_m} + \sum_{n \in W_\ell} \right) \leq C \left( \frac{|m| + |\ell|}{|\ell|} \right) \leq C.
\]
Therefore, when $3|m| < |\ell|$,  
\[ \sum_{||n|−|m||\geq 2|m−\ell|} |\tilde{K}(m, n) − \tilde{K}(\ell, n)| \leq C. \quad (3.5) \]

Next, we assume that $|\ell| \leq 3|m|$. If further $|\ell| < 9|m|/4$, then we have $2|\ell|/3 < 3|m|/2$ and we write  
\[ \sum_{||n|−|m||\geq 2|m−\ell|} |\tilde{K}(m, n) − \tilde{K}(\ell, n)| \]
\[ = \sum_{||n|−|m||\geq 2|m−\ell|} |\tilde{K}(m, n) − \tilde{K}(\ell, n)| + \sum_{|n|/2\leq |n|<3|\ell|/3} |\tilde{K}(m, n) − \tilde{K}(\ell, n)| \]
\[ + \sum_{3|m|/2\leq |n|\leq 3|\ell|/2} |\tilde{K}(m, n) − \tilde{K}(\ell, n)|. \]

Since  
\[ \{ n \in \mathbb{Z}; \ 2|\ell|/3 \leq |n| \leq 3|m|/2 \} \subset \{ n \in \mathbb{Z}; \ |n|/2 \leq |\ell|, \ |m| < 3|n|/2 \} \]
so, using (ii) we get  
\[ \sum_{||n|−|m||\geq 2|m−\ell|} |\tilde{K}(m, n) − \tilde{K}(\ell, n)| \leq C, \quad \sum_{|n|−|m||\geq 2|m−\ell|} \frac{|m−\ell|}{|n|−|m|}^2 \leq C \]

According to (i) and (3.1) we obtain that  
\[ \sum_{|n|−|m||\geq 2|m−\ell|} |\tilde{K}(m, n) − \tilde{K}(\ell, n)| \leq C \sum_{|n|−|m||\geq 2|m−\ell|} \frac{1}{|n|−|m|} \]
\[ \leq C \sum_{3|m|/2\leq |n|\leq 3|\ell|/2} \frac{1}{|m−\ell|} \leq C. \quad (3.6) \]

and  
\[ \sum_{||n|−|m||\geq 2|m−\ell|} |\tilde{K}(m, n) − \tilde{K}(\ell, n)| \]
\[ \leq C \sum_{|n|−|m||\geq 2|m−\ell|} \frac{1}{|n|−|m|} \]
\[ \leq C \sum_{|n|−|m||\geq 2|m−\ell|} \frac{1}{|n|−|m|} + C \sum_{2|m|/3\leq |n|\leq 2|\ell|/3} \frac{1}{|m−\ell|} \leq C. \quad (3.7) \]
Now suppose that \( \ell > 9|m|/4 \). In this case we have \( 3|m|/2 < 2|\ell|/3 \) and

\[
\sum_{|n|-|m| \geq 2|m-\ell|} |\tilde{K}(m, n) - \tilde{K}(\ell, n)|
\leq \sum_{|n|-|m| \geq 2|m-\ell|} |\tilde{K}(m, n) - \tilde{K}(\ell, n)| + \sum_{|n|-|m| \geq 2|m-\ell|} |\tilde{K}(m, n) - \tilde{K}(\ell, n)| \leq C,
\]

which is an immediate consequence of (3.7) and (3.6). This achieves the proof of Theorem 3.3.

\[\square\]

### 3.2 Boundedness of the Riesz Transforms

We come now to the \( \ell^p \)-boundedness of the Riesz transform, we shall prove that the kernel \( R(n, m) \) satisfies the condition of Theorem 3.3.

**Lemma 3.4.** For all \( m, n \in \mathbb{Z} \), we have

\[
(m - n - 1/2)R(n, m) = \frac{k}{\pi} \int_{-\pi}^{\pi} e^{-ix} \mathcal{E}_m(ix) \mathcal{E}_n^k(ix) \cos(x/2) \delta_{k-1/2}(x) dx.
\]

**Proof.** From the proposition 2.6, we have

\[
i(m + k)R(n, m) = \frac{1}{2\pi i} \int_{-\pi}^{\pi} h(x) \tilde{T}^k \left( \mathcal{E}_m(ix) \right) \mathcal{E}_n^k(-ix) \delta_k(x) dx.
\]

Integrating by parts yields

\[
\int_{-\pi}^{\pi} h(x) \left( \mathcal{E}_m^k(ix) \right)' \mathcal{E}_n^k(-ix) \delta_k(x) dx =
\]

\[
\frac{i}{2} \int_{-\pi}^{\pi} h(x) \mathcal{E}_m^k(ix) \mathcal{E}_n^k(-ix) \delta_k(x) dx - \int_{-\pi}^{\pi} h(x) \mathcal{E}_m^k(ix) \left( \mathcal{E}_n^k(-ix) \right)' \delta_k(x) dx
\]

\[-2k \int_{-\pi}^{\pi} h(x) \mathcal{E}_m^k(ix) \mathcal{E}_n^k(-ix) \cot(x) \delta_k(x) dx.
\]

On the other hand, one can make the following

\[
2ki \int_{-\pi}^{\pi} h(x) \frac{\mathcal{E}_m^k(ix) - \mathcal{E}_m^k(-ix)}{1 - e^{-2ix}} \mathcal{E}_n^k(-ix) \delta_k(x) dx
\]

\[
= 2ki \int_{-\pi}^{\pi} \mathcal{E}_m^k(ix) \left\{ \frac{h(x) \mathcal{E}_n^k(-ix)}{1 - e^{-2ix}} - \frac{h(-x) \mathcal{E}_n^k(ix)}{1 - e^{2ix}} \right\} \delta_k(x) dx
\]

\[
= 2k \int_{-\pi}^{\pi} h(x) \mathcal{E}_m^k(ix) \mathcal{E}_n^k(-ix) \cot(x) \delta_k(x) dx
\]

\[
+ 2ki \int_{-\pi}^{\pi} h(x) \mathcal{E}_m^k(ix) \frac{\mathcal{E}_n^k(-ix) - \mathcal{E}_n^k(ix)}{1 - e^{2ix}} \delta_k(x) dx
\]

\[
- 2k \int_{-\pi}^{\pi} e^{-ix} \mathcal{E}_m^k(ix) \mathcal{E}_n^k(ix) \cos(x/2) \delta_{k-1/2}(x) dx.
\]
Therefore we obtain
\[
\int_{-\pi}^{\pi} h(x) \mathcal{T}^k\left(\mathcal{E}_m^k(ix)\right) \mathcal{E}_n^k(-ix) \delta_k(x) \, dx = -\int_{-\pi}^{\pi} h(x) \mathcal{E}_m^k(ix) \mathcal{T}^k\left(\mathcal{E}_n(ix)\right) \Lambda_k(x) \, dx - 2k \int_{-\pi}^{\pi} e^{-ix} \mathcal{E}_m^k(ix) \mathcal{E}_n^k(ix) \cos(x/2) \delta_{k-1/2}(x) \, dx + \frac{i}{2} \int_{-\pi}^{\pi} h(x) \mathcal{E}_m^k(ix) \mathcal{E}_n^k(-ix) \delta_k(x) \, dx
\]
and, since
\[
\int_{-\pi}^{\pi} h(x) \mathcal{E}_m^k(ix) \mathcal{T}^k\left(\mathcal{E}_n(ix)\right) \Lambda_k(x) \, dx = -i(\tilde{m} + k) \mathcal{R}(m, n)
\]
then the lemma follows. \(\square\)

As an immediately consequence of Lemma 3.4 we have the following estimate.

**Corollary 3.5.** For all \(m, n \in \mathbb{Z}\), such that \(n \neq m\) we have
\[
|\mathcal{R}(n, m)| \leq \frac{C}{|n - m|}, \tag{3.8}
\]
for some constant \(C\).

**Lemma 3.6.** The Riesz kernel \(\mathcal{R}\) satisfies
\[
|\mathcal{R}(n, m + 1) - \mathcal{R}(n, m)| \leq \frac{C}{|m| - |n|^2}, \quad \frac{|n|}{2} \leq |m| \leq \frac{3|n|}{2}.
\]

**Proof.** Let us write
\[
\int_{-\pi}^{\pi} e^{-ix} \mathcal{E}_m^k(ix) \mathcal{E}_n^k(ix) \cos(x/2) \delta_{k-1/2}(x) \, dx
\]
\[
\int_{-\pi}^{\pi} \left( e^{-ix} - 1 \right) \cos(x/2) \mathcal{E}_m^k(ix) \mathcal{E}_n^k(ix) \delta_{k-1/2}(x) \, dx + \int_{-\pi}^{\pi} \mathcal{E}_m^k(ix) \mathcal{E}_n^k(ix) \cos(x/2) \delta_{k-1/2}(x) \, dx
\]
\[
= -i \int_{-\pi}^{\pi} h(x) \mathcal{E}_m^k(ix) \mathcal{E}_n^k(ix) \delta_k(x) \, dx + \int_{-\pi}^{\pi} \mathcal{E}_m^k(ix) \mathcal{E}_n^k(ix) \cos(x/2) \delta_{k-1/2}(x) \, dx
\]
From Lemma 3.4 and the formula (2.12),
\[
(\tilde{m} + 1 - \tilde{n} - 1/2) \mathcal{R}(n, m + 1) = \frac{k}{\pi} \int_{-\pi}^{\pi} e^{-ix} \mathcal{E}_{m+1}^k(ix) \mathcal{E}_n^k(ix) \cos(x/2) \delta_{k-1/2}(x) \, dx
\]
\[
= \frac{k\alpha_m}{\pi} \int_{-\pi}^{\pi} \mathcal{E}_m^k(ix) \mathcal{E}_n^k(ix) \cos(x/2) \delta_{k-1/2}(x) \, dx + \frac{k\beta_m}{\pi} \int_{-\pi}^{\pi} \mathcal{E}_{m}^k(ix) \mathcal{E}_n^k(ix) \cos(x/2) \delta_{k-1/2}(x) \, dx.
\]
We then get
\[
(\tilde{m} + 1 - \tilde{n} - 1/2) \mathcal{R}(m + 1, n) - (\tilde{m} - \tilde{n} - 1/2) \mathcal{R}(m, n)
\]
\[
= \frac{k(\alpha_m - 1)}{\pi} \int_{-\pi}^{\pi} \mathcal{E}_m^k(ix) \mathcal{E}_n^k(ix) \cos(x/2) \delta_{k-1/2}(x) \, dx
\]
\[
+ \frac{k\beta_m}{\pi} \int_{-\pi}^{\pi} \mathcal{E}_{m}^k(ix) \mathcal{E}_n^k(ix) \cos(x/2) \delta_{k-1/2}(x) \, dx + \frac{k}{\pi} \int_{-\pi}^{\pi} h(x) \mathcal{E}_m^k(ix) \mathcal{E}_n^k(ix) \delta_k(x) \, dx.
\]
It follows that
\[
(m + 1 - \tilde{n} - 1/2) \left( R(n, m + 1) - R(n, m) \right) = (\tilde{m} - \tilde{m} + 1) R(m, n) + k(\alpha - 1) \int_{-\pi}^{\pi} E_m^k(ix) E_m^k(ix) \cos(x/2) \delta_{k-1/2}(x) dx + \int_{-\pi}^{\pi} h(x) E_m^k(ix) E_m^k(ix) \delta_k(x) dx.
\]

From (3.8) we have
\[
|\tilde{m} - \tilde{m} + 1| R(n, m) | \leq C |m - n|
\]
and in view of (2.11)
\[
|\alpha - 1| = \frac{k^2}{|m| + k} \left( |m| + |m| + k \right) \leq \frac{C}{|m|} \leq \frac{C}{||n| - |m||}
\]
and
\[
|\beta_m| \leq \frac{1}{|m|} \leq \frac{C}{||n| - |m||},
\]
since for |n|/2 \leq |m| \leq 3|n|/2, we have that ||n| - |m|| \leq |n|/2 \leq |m|. Moreover, by noting that
\[
\int_{-\pi}^{\pi} h(x) E_m^k(ix) E_m^k(ix) \delta_k(x) dx = R(n, -m + 1)
\]
so, by Lemma 3.4 one can obtain
\[
|R(n, -m + 1)| \leq \frac{C}{|n + m|} \leq \frac{C}{||n| - |m||}.
\]
Finally as
\[
\frac{1}{|m + 1 - \tilde{n} - 1/2|} \leq \frac{C}{|m - n|} \leq \frac{C}{||n| - |m||}
\]
the lemma is concluded.

**Theorem 3.7.** The kernel $K$ satisfies
\[
|K(n, \ell) - K(n, m)| \leq C \frac{||\ell - m||}{||m| - |n||^2}
\]
For all $n, m \in \mathbb{Z}$ such that $|n|/2 \leq |m|, |\ell| \leq 3|n|/2$ and $||n| - |m|| \geq 2|m - \ell|$.  

**Proof.** Assume that $|m| < |\ell|$. Let us first making the following observation. Considering the conditions: $||n| - |m|| \geq 2|m - \ell|$ and $|n|/2 \leq |m|, |\ell| \leq 3|n|/2$ the numbers $m$ and $\ell$ must have the same sign. Then for all integer $h$ between $m$ and $\ell$
\[
||n| - |h|| \geq ||n| - |\ell|| \geq \frac{|n| - |m||}{2}.
\]
Consequently,
\[
|R(n, \ell) - R(n, m)| \leq \sum_{|m| \leq |h| \leq |\ell| - 1} |K(n, h + 1) - K(n, h)|
\]
\[
\leq C \sum_{|m| \leq |h| \leq |\ell| - 1} \frac{1}{|n| - |h|^2}
\]
\[
\leq C \frac{|m - \ell|}{|n| - |m|^2},
\]
which is the desired estimate.

Now we can apply the same arguments to get the estimate,
\[
|R(m, n) - R(\ell, n)| \leq C \frac{|m - \ell|}{|n| - |m|^2}.
\]

As the Riesz transforms are bounded on $\ell^2(\mathbb{Z})$ we can then invoke the theorem 3.2 and finally we state the following.

**Theorem 3.8.** The generalized discrete Riesz transforms $R$ and $R^*$ are bounded operators on $\ell^p(\mathbb{Z})$ for all $1 < p < \infty$.

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