ON CLIFFORD THEORY WITH GALOIS ACTION

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Abstract. Let $\hat{G}$ be a finite group, $N$ a normal subgroup of $\hat{G}$ and $\vartheta \in \text{Irr } N$. Let $F$ be a subfield of the complex numbers and assume that the Galois orbit of $\vartheta$ over $F$ is invariant in $\hat{G}$. We show that there is another triple $(\hat{G}_1, N_1, \vartheta_1)$ of the same form, such that the character theories of $\hat{G}$ over $\vartheta$ and of $\hat{G}_1$ over $\vartheta_1$ are essentially “the same” over the field $F$ and such that the following holds: $\hat{G}_1$ has a cyclic normal subgroup $C$ contained in $N_1$, such that $\vartheta_1 = \lambda^{N_1}$ for some linear character $\lambda$ of $C$, and such that $N_1/C$ is isomorphic to the (abelian) Galois group of the field extension $F(\lambda)/F(\vartheta_1)$. More precisely, having “the same” character theory means that both triples yield the same element of the Brauer-Clifford group $\text{BrCliff}(\hat{G}, F(\vartheta))$ defined by A. Turull.

1. Introduction

1.1. Motivation. Clifford theory is concerned with the characters of a finite group lying over one fixed character of a normal subgroup. So let $\hat{G}$ be a finite group and $N \trianglelefteq \hat{G}$ a normal subgroup. Let $\vartheta \in \text{Irr } N$, where $\text{Irr } N$ denotes the set of irreducible complex valued characters of the group $N$, as usual. We write $\text{Irr}(\hat{G} | \vartheta)$ for the set of irreducible characters of $\hat{G}$ which lie above $\vartheta$ in the sense that their restriction to $N$ has $\vartheta$ as constituent.

In studying $\text{Irr}(\hat{G} | \vartheta)$, it is usually no loss of generality to assume that $\vartheta$ is invariant in $\hat{G}$, using the well known Clifford correspondence [9, Theorem 6.11]. In this situation, $(\hat{G}, N, \vartheta)$ is often called a character triple. Then a well known theorem tells us that there is an “isomorphic” character triple $(\hat{G}_1, N_1, \vartheta_1)$ such that $N_1 \subseteq \mathbb{Z}(\hat{G}_1)$ [9, Theorem 11.28]. Questions about $\text{Irr}(\hat{G} | \vartheta)$ can often be reduced to questions about $\text{Irr}(\hat{G}_1 | \vartheta_1)$, which are usually easier to handle. This result is extremely useful, for example in reducing questions about characters of finite groups to questions about characters of finite simple or quasisimple groups.

Some of these questions involve Galois automorphisms or even Schur indices [13, 21] (over some fixed field $F \subseteq \mathbb{C}$, say). Unfortunately, both of the above reductions are not well behaved with respect to Galois action on characters and other rationality questions (like Schur indices of the involved characters). The first reduction (Clifford correspondence) can be replaced by a reduction to the case where $\vartheta$ is semi-invariant.
over the given field $\mathbb{F}$ [15, Theorem 1]. (This means that the Galois orbit of $\vartheta$ is invariant in the group $\hat{G}$.)

Now assuming that the character triple $(\hat{G}, N, \vartheta)$ is such that $\vartheta$ is semi-invariant in $\hat{G}$ over some field $\mathbb{F}$, usually we can not find a character triple $(\hat{G}_1, N_1, \vartheta_1)$ with $N_1 \subseteq \mathbb{Z}(\hat{G}_1)$, and such that these character triples are “isomorphic over the field $\mathbb{F}$”. We will give an exact definition of “isomorphic over $\mathbb{F}$” below, using machinery developed by Alexandre Turull [23, Definition 7.7]. For the moment, it suffices to say that a correct definition should imply that $\hat{G}/N \cong \hat{G}_1/N_1$ and that there is a bijection between $\bigcup \iota \text{Irr}(\hat{G} \mid \vartheta^\alpha)$ and $\bigcup \iota \text{Irr}(\hat{G}_1 \mid \vartheta_1^\alpha)$ (unions over a Galois group) commuting with field automorphisms over $\mathbb{F}$ and preserving Schur indices over $\mathbb{F}$. Now if, for example, $\mathbb{Q}(\vartheta) = \mathbb{Q}(\sqrt{5})$ (say), then it is clear that we can not have $\mathbb{Q}(\vartheta) = \mathbb{Q}(\vartheta_1)$ with $\vartheta_1$ linear. The main result of this paper, as described in the abstract, provides a possible substitute: At least we can find an “isomorphic” character triple $(\hat{G}_1, N_1, \vartheta_1)$, where $N_1$ is cyclic by abelian and $\vartheta_1$ is induced from a cyclic normal subgroup of $\hat{G}_1$. This result is probably the best one can hope for, if one wants to take into account Galois action and Schur indices.

1.2. Notation. To state the main result precisely, we need some notation. Instead of character triples, we find it more convenient to use Clifford pairs as introduced in [12]. We recall the definition. Let $\hat{G}$ and $G$ be finite groups and let $\kappa: \hat{G} \to G$ be a surjective group homomorphism with kernel $\text{Ker} \kappa = N$. Thus

$$1 \longrightarrow N \longrightarrow \hat{G} \xrightarrow{\kappa} G \longrightarrow 1$$

is an exact sequence, and $\hat{G}/N \cong G$ via $\kappa$. We say that $(\vartheta, \kappa)$ is a Clifford pair over $G$. (Note that $\hat{G}$, $G$ and $N$ are determined by $\kappa$ as the domain, the image and the kernel of $\kappa$, respectively.) We usually want to compare different Clifford pairs over the same group $G$, but with different groups $\hat{G}$ and $N$.

Let $\mathbb{F} \subseteq \mathbb{C}$ be a field. (For simplicity of notation, we work with subfields of $\mathbb{C}$, the complex numbers, but of course one can replace $\mathbb{C}$ by any algebraically closed field of characteristic 0 and assume that all characters take values in this field.) Then $\vartheta \in \text{Irr} N$ is called semi-invariant in $\hat{G}$ over $\mathbb{F}$ (where $N \subseteq \hat{G}$), if for every $g \in \hat{G}$, there is a field automorphism $\alpha = \alpha_g \in \text{Gal}(\mathbb{F}(\vartheta)/\mathbb{F})$ such that $\vartheta^{\alpha g} = \vartheta$. In this situation, the map $g \mapsto \alpha_g$ actually defines an action of $G$ on the field $\mathbb{F}(\vartheta)$ [8, Lemma 2.1].

To handle Clifford theory over small fields, Turull [23, 24] has introduced the Brauer-Clifford group. For the moment, it is enough to know that the Brauer-Clifford group is a certain set $\text{BrCliff}(G, E)$ for any group $G$ and any field $E$ on which $G$ acts. Given a Clifford pair $(\vartheta, \kappa)$ and a field $\mathbb{F}$ such that $\vartheta$ is semi-invariant over $\mathbb{F}$, the group $G$ acts on $\mathbb{F}(\vartheta)$ and the Brauer-Clifford group $\text{BrCliff}(G, \mathbb{F}(\vartheta))$ is defined. Turull [23, Definition 7.7] shows how to associate a certain element $[\vartheta, \kappa, \mathbb{F}] \in \text{BrCliff}(G, \mathbb{F}(\vartheta))$ with $(\vartheta, \kappa)$ and $\mathbb{F}$. Moreover, if $(\vartheta, \kappa)$ and $(\vartheta_1, \kappa_1)$ are two pairs over $G$ such that $\vartheta$ and $\vartheta_1$ are semi-invariant over $\mathbb{F}$ and induce the
same action of $G$ on $\mathbb{F}(\vartheta) = \mathbb{F}(\vartheta_1)$, and if $[\vartheta, \kappa, \mathbb{F}] = [\vartheta_1, \kappa_1, \mathbb{F}]$, then the character theories of $\hat{G}$ over $\vartheta$ and of $\hat{G}_1$ over $\vartheta_1$ are essentially “the same”, including rationality properties over the field $\mathbb{F}$. (See [23, Theorem 7.12] for the exact statement.) This justifies it to view such Clifford pairs as “isomorphic over $\mathbb{F}$”.

1.3. Main result. The following is the main result of this paper. For simplicity, we state it for subfields of the complex numbers $\mathbb{C}$, but in fact $\mathbb{C}$ can stand for any algebraically closed field of characteristic 0, if all characters are assumed to take values in that fixed field $\mathbb{C}$. (As usual, $\text{Lin}\mathbb{C}$ denotes the set of linear characters of a group $\mathbb{C}$.)

**Theorem A.** Let $\mathbb{F} \subseteq \mathbb{C}$ be a field, let

$$1 \longrightarrow N \longrightarrow \hat{G} \xrightarrow{\kappa} G \longrightarrow 1$$

be an exact sequence of finite groups and let $\vartheta \in \text{Irr}\; N$ be semi-invariant in $\hat{G}$ over $\mathbb{F}$. Then there is another exact sequence of finite groups

$$1 \longrightarrow N_1 \longrightarrow \hat{G}_1 \xrightarrow{\kappa_1} G \longrightarrow 1$$

and $\vartheta_1 \in \text{Irr}\; N_1$, such that $\mathbb{F}(\vartheta) = \mathbb{F}(\vartheta_1)$ as $G$-fields, such that

$$[\vartheta, \kappa, \mathbb{F}] = [\vartheta_1, \kappa_1, \mathbb{F}] \quad \text{in} \quad \text{BrCliff}(G, \mathbb{F}(\vartheta)),$$

and such that the following hold:

(a) $\hat{G}_1$ has a cyclic normal subgroup $C \trianglelefteq \hat{G}_1$ with $C \leq N_1$,
(b) there is a faithful $\lambda \in \text{Lin}\mathbb{C}$ with $\vartheta_1 = \lambda^{N_1}$,
(c) $\lambda$ is semi-invariant in $\hat{G}_1$ over $\mathbb{F}$ and
(d) $N_1/C \cong \text{Gal}(\mathbb{F}(\lambda)/\mathbb{F}(\vartheta))$.

As mentioned before, Turull’s result [23, Theorem 7.12] yields that in the situation of Theorem A, there are correspondences of characters with good compatibility properties. We mention a few properties here, and refer the reader to Turull’s paper for a more complete list:

**Corollary B.** In the situation of Theorem A, for each subgroup $H \leq G$ there is a bijection between

$$\mathbb{Z}[\text{Irr}(\kappa^{-1}(H) \mid \vartheta)] \quad \text{and} \quad \mathbb{Z}[\text{Irr}(\kappa_1^{-1}(H) \mid \vartheta_1)].$$

The bijections can be chosen such that their union commutes with restriction and induction of characters, with field automorphisms over the field $\mathbb{F}$, and with multiplications of characters of $H$, and such that it preserves the inner product of class functions and fields of values and Schur indices (even elements in the Brauer group) over $\mathbb{F}$.

We can say something more about the group $\hat{G}_1$ in the main theorem.
**Proposition C.** In the situation of Theorem A and for \( \hat{U}_1 = (\hat{G}_1)_{\vartheta_1} \) and \( V = (\hat{G}_1)_\lambda \) (the inertia groups of \( \vartheta_1 \) and \( \lambda \) in \( \hat{G}_1 \)), we have \( \hat{U}_1 = VN_1 \) and \( N_1 \cap V = C \) (cf. Figure 1), and

\[
\left[ (\vartheta_1, (\kappa_1)_{\hat{G}_1}, \mathbb{F}(\lambda)) \right] = \left[ (\lambda, (\kappa_1)_V, \mathbb{F}(\lambda)) \right].
\]

For every subgroup \( X \) with \( N_1 \leq X \leq \hat{U}_1 \), induction yields a bijection

\[
\text{Irr}(X \cap V \mid \lambda) \ni \psi \mapsto \psi^X \in \text{Irr}(X \mid \vartheta)
\]

commuting with field automorphisms over \( \mathbb{F}(\lambda) \) and preserving Schur indices over \( \mathbb{F}(\lambda) \).

**Figure 1.** Subgroups of \( \hat{G}_1 \) in Theorem A

Observe that since \( \lambda \) is faithful, we actually have \( V = \mathbf{C}_{\hat{G}_1}(C) \) and \( C \subseteq \mathbb{Z}(V) \). So over the bigger field \( \mathbb{F}(\lambda) \) and in the smaller group \( \hat{U}_1 \), we can replace the Clifford pair \((\vartheta_1, (\kappa_1)_{\hat{G}_1})\) by the even simpler pair \((\lambda, (\kappa_1)_V)\). This is, of course, just the classical result mentioned before, which is usually proved using the theory of projective representations and covering groups.

We will prove Proposition C at the end of Section 3 as a special case of other, auxiliary results, which are also needed for the proof of Theorem A. Everything else in this paper is devoted to the proof of the main result, Theorem A, which is proved through a series of reductions.

1.4. **Relation to earlier results.** A number of people have studied Clifford theory over small fields, including Dade [2, 3, 4], Isaacs [8], Schmid [16, 17] and Riese [14], cf. [15]. While we use some ideas of these authors, most important for our paper is the theory of the Brauer-Clifford group as developed by Turull [23, 22, 24], which supersedes in some sense his earlier theory of Clifford classes [19]. In particular, it is essential for our proof that the Brauer-Clifford group and certain subsets of it are indeed groups, a fact which, it seems to me, has not been important in the applications of the Brauer-Clifford group [20, 27] so far.

A paper of Dade [2] contains, between the lines, a result similar to our main theorem (but in a more narrow situation): Dade studies the situation where (in
our notation) \( \vartheta \) is invariant in \( \hat{G} \) and has values in \( F \). Then a cohomology class \([\vartheta] \in H^2(G, F^*)\) is defined. This class determines part of the character theory of \( \hat{G} \) over \( \vartheta \), but not completely, since it does not take into account the Schur index of \( \vartheta \) itself. After proving some properties of this cohomology class, Dade shows that all cohomology classes with these properties occur, by constructing examples. When examining this construction, one will find that the examples have almost the same properties as the group \( \hat{G}_1 \) in Theorem A. One can deduce that all such cohomology classes come from character triples as in Theorem A.

Theorem A contains the classical result that every simple direct summand of the group algebra \( \mathbb{E}N \) of a finite group \( N \) over a field \( \mathbb{E} \) of characteristic zero is equivalent to a cyclotomic algebra [28]. (This is the case \( G = 1 \) of Theorem A.) Our proof of Theorem A uses ideas from the proof of this result, as presented by Yamada [28].

2. Review of the Brauer-Clifford group

Let us briefly recall the relevant definitions. Details can be found in the papers of Turull [23, 24], see also [6]. Let \( G \) be a group. A \( G \)-ring is a ring \( Z \) (with 1) together with an action of \( G \) on \( Z \) via (unital) ring automorphisms. We use exponential notation \( z \mapsto z^g \) to denote such an action. Let \( Z \) be a commutative \( G \)-ring. A \( G \)-algebra over \( Z \) is a \( G \) ring \( A \) together with a homomorphism of \( G \)-rings \( \varepsilon : Z \to Z(A) \). This means that \( A \) is an algebra over \( Z \) and that the algebra unit \( \varepsilon : Z \to A \) has the property \( \varepsilon(z^g) = \varepsilon(z)^g \). We usually suppress mention of the algebra unit \( \varepsilon \) and simply write \( za \) for \( \varepsilon(z)a \).

We only need \( G \)-algebras over fields in this paper. Let \( Z \) be a field on which \( G \) acts. A \( G \)-algebra over \( Z \) is called central simple, if it is central simple as algebra over \( Z \). The Brauer-Clifford group \( \text{BrCliff}(G, Z) \) is the set of equivalence classes of central simple \( G \)-algebras over \( Z \) under a certain equivalence relation. To define this equivalence relation, we need the skew group ring \( ZG \) of \( G \) over \( Z \) (also called the crossed product), which is the set of formal sums \( \sum_{g \in G} gc_g \), \((c_g \in Z)\) with multiplication defined by

\[
\left( \sum_{g \in G} gc_g \right) \left( \sum_{h \in G} hd_h \right) = \sum_{g,h \in G} ghc_{gh}^hd_h.
\]

If \( V \) is a right \( ZG \)-module, then \( \text{End}_Z V \) is a \( G \)-algebra over \( Z \), called a trivial \( G \)-algebra. Two \( G \)-algebras \( S \) and \( T \) over \( Z \) are called equivalent, if there are \( ZG \)-modules \( V \) and \( W \) such that

\[
S \otimes_Z \text{End}_Z V \cong T \otimes_Z \text{End}_Z W
\]

as \( G \)-algebras over \( Z \). The Brauer-Clifford group \( \text{BrCliff}(G, Z) \) is the set of equivalence classes of central simple \( G \)-algebras over \( Z \), with multiplication induced by tensoring over \( Z \).
The Brauer-Clifford group is an abelian torsion group [23, Theorem 3.10, 6, Theorem 5].

Next let \((\vartheta, \kappa): \hat{G} \to G\) be a Clifford pair and \(F \subseteq \mathbb{C}\) a field. Assume that \(\vartheta\) is semi-invariant in \(\hat{G}\) over \(F\). (The last assumption is not necessary in Turull’s theory, but we only need this case, and the notation can be simplified somewhat in this case.) So for every \(g \in \hat{G}\), there is a field automorphism \(\alpha_g \in \text{Gal}(F(\vartheta)/F)\) such that \(\vartheta^{\alpha_g} = \vartheta\). The map \(g \mapsto \alpha_g\) defines an action of \(G\) on the field \(F(\vartheta)\) [8, Lemma 2.1] and thus the Brauer-Clifford group \(\text{BrCliff}(G, F(\vartheta))\) is defined. Turull showed [23, Definition 7.7] how to associate an element

\[
[\vartheta, \kappa, F] \in \text{BrCliff}(G, F(\vartheta))
\]
to \((\vartheta, \kappa, F)\). We recall the construction. Let \(e = e_{\vartheta(\kappa, F)} \in \mathbb{Z}(F\mathcal{N})\) be the central primitive idempotent of \(F\mathcal{N}\) corresponding to \(\vartheta\). Note that \(e\) is invariant in \(\hat{G}\) since \(\vartheta\) is semi-invariant. An \(F\hat{G}\)-module \(V\) is called \(\vartheta\)-quasihomogeneous if \(Ve = V\). Let \(V\) be a nonzero quasihomogeneous \(F\hat{G}\)-module. Then \(S = \text{End}_{F\mathcal{N}} V\) is a \(G\)-algebra. For \(z \in \mathbb{Z}(F(N\mathcal{e}))\), the map \(v \mapsto vz\) is in \(S\), and this defines a canonical isomorphism \(\mathbb{Z}(F(N\mathcal{e})) \cong \mathbb{Z}(S)\) of \(G\)-algebras. The central character \(\omega_\vartheta\) belonging to \(\vartheta\) defines an isomorphism \(\mathbb{Z}(F(N\mathcal{e})) \cong F(\vartheta)\), which commutes with the action of \(G\) (namely, we have \(\omega_\vartheta(z^g) = \omega_\vartheta(z)^{\alpha_g}\)). Thus we can view \(S\) as a central simple \(G\)-algebra over \(F(\vartheta)\). The equivalence class of \(S\) does not depend on the choice of the quasihomogeneous module \(V\) [23, Theorem 7.6]. Following Turull, we write \([\vartheta, \kappa, F]\) for this equivalence class. (Actually, Turull defines \([\vartheta, \kappa, F]\) as an element of \(\text{BrCliff}(G, \mathbb{Z})\), where \(\mathbb{Z} = \mathbb{Z}(F(N\mathcal{e}))\).)

Now suppose that \((\vartheta, \kappa)\) and \((\vartheta_1, \kappa_1)\) are two semi-invariant Clifford pairs over \(G\) such that \(F(\vartheta) = F(\vartheta_1)\). Assume that both Clifford pairs induce the same action of \(G\) on \(F(\vartheta)\), that is, for each \(g \in G\) there is \(\alpha_g \in \text{Gal}(F(\vartheta)/F)\) such that \(\vartheta^{\alpha_g} = \vartheta\) and \(\vartheta_1^{\alpha_g} = \vartheta_1\) for all \(g \in G\). If

\[
[\vartheta, \kappa, F] = [\vartheta_1, \kappa_1, F],
\]
then the character theory of \(\hat{G}\) over the Galois conjugates of \(\vartheta\) and the character theory of \(\hat{G}_1\) over the Galois conjugates of \(\vartheta_1\) are essentially “the same” [23, Theorem 7.12], as we mentioned in the introduction.

3. Subextensions

Let

\[
1 \longrightarrow K \longrightarrow \hat{G} \overset{\kappa}{\longrightarrow} G \longrightarrow 1
\]

be an exact sequence and \(\vartheta \in \text{Irr} K\). (Thus \((\vartheta, \kappa)\) is a Clifford pair.) Suppose that \(\hat{H}\) is a supplement of \(K\) in \(\hat{G}\), so that \(\hat{G} = \hat{H}K\). Set \(L = \hat{H} \cap K\). Then \(\hat{H}/L \cong \hat{G}/K \cong G\). Suppose that \(\varphi \in \text{Irr} L\). We want to compare the elements \([\vartheta, \kappa, F]\) and \([\varphi, \kappa_{\hat{H}}, F]\). We do this under additional assumptions, which we collect here for convenient reference:
3.1. Hypothesis. Let

\[
\begin{array}{ccc}
1 & \longrightarrow & L \\
& \downarrow & \downarrow \\
1 & \longrightarrow & K
\end{array} \quad \begin{array}{ccc}
\longrightarrow & \longrightarrow & \longrightarrow \\
& \kappa & \hat{\kappa} \\
\longrightarrow & \longrightarrow & \longrightarrow \\
& G & \longrightarrow \end{array} \quad \begin{array}{ccc}
& \longrightarrow & \longrightarrow \\
& G & \longrightarrow \\
& \longrightarrow & \longrightarrow \\
& 1 & \longrightarrow
\end{array}
\]

be a commutative diagram of finite groups with exact rows, let \( \vartheta \in \text{Irr} K \) and \( \varphi \in \text{Irr} L \) be irreducible characters and let \( F \) be a field of characteristic zero such that the following conditions hold:

- (a) \( n = [\vartheta_L, \varphi] > 0 \).
- (b) \( F(\varphi) = F(\vartheta) =: E \).
- (c) For every \( h \in \hat{H} \) there is \( \gamma = \gamma_h \in \text{Gal}(E/F) \) such that \( \vartheta^{h\gamma} = \vartheta \) and \( \varphi^{h\gamma} = \varphi \).

To reduce visual clutter, we write

\[ e = e_{(\vartheta, F)} = \sum_{\gamma \in \text{Gal}(E/F)} e_{\vartheta}^{\gamma} \text{ and } f = e_{(\varphi, F)} = \sum_{\gamma \in \text{Gal}(E/F)} e_{\varphi}^{\gamma} \]

for the corresponding central primitive idempotents in \( FK \) and \( FL \), respectively.

3.2. Lemma ([10, Lemma 6.3]).

\[ i := \sum_{\gamma \in \text{Gal}(E/F)} e_{\vartheta}^{\gamma} e_{\varphi}^{\gamma} \]

is a \( \hat{H} \)-stable nonzero idempotent in \( FKe \), and we have \( ei = i = ie \) and \( fi = i = if \).

3.3. Lemma ([10, Lemma 6.4]).

\[ Z(iFKi) \cong Z(FKe) \cong F(\vartheta) \cong F(\varphi) \cong Z(FLf) \]

as \( G \)-rings.

3.4. Lemma ([10, Lemma 6.5]). Set \( Z = Z(iFKi) \) and let \( S = (iFKi)_L \). Then \( S \) is a central simple \( G \)-algebra over \( Z \cong E = F(\vartheta) \) with dimension \( n^2 \) over \( Z \), and

\[ C_{iFKi}(S) = F Li \cong FLf. \]

Thus the equivalence class of \( S \) defines an element in \( \text{BrCliff}(G, E) \). The main result of this section is:

3.5. Theorem. Assume Hypothesis 3.1. Then

\[ [\vartheta, \kappa, F] \cdot [S] = [\varphi, \kappa_{\hat{H}}^{-1}, F] \text{ (in BrCliff}(G, E)). \]

This result is of course related to the results in [10], but we didn’t use the language of the Brauer-Clifford group in our previous paper. Thus we give a translation here. Theorem 3.5 is also related to results in [25, 26].

The precise result depends on the following conventions: All modules over \( \mathbb{F}G \) and other group algebras are right modules, and endomorphism rings also operate
from the right. Thus for any \( F \)-algebra \( A \) operating on a module \( V \) we get a homomorphism \( A \hookrightarrow \text{End}_F(V) \), both operating from the right. Then \( \text{End}_A(V) \) is simply the centralizer of the image of \( A \) in \( \text{End}_F(V) \). With other conventions, one may have to replace \([S] \) by \([S]^{-1}\) in the formula of Theorem 3.5.

**Proof of Theorem 3.5.** Let \( V \) be an \( \mathcal{F}\hat{G} \)-module with \( Ve = V \). Then the \( G \)-algebra \( \text{End}_{\mathcal{F}K}(V) \) is in \([\vartheta, \kappa, F]\). Moreover, set \( W = Vi \). Then \( W \) is an \( \mathcal{F}\hat{H} \)-module with \( Wf = Vif = Vi = W \), and thus \( \text{End}_{\mathcal{F}L}(W) \) is a \( G \)-algebra in \([\varphi, \kappa_iH, F]\).

We may also consider \( W = Vi \) as a module over \( i\mathcal{F}Ki \). Consider the algebra \( A = \text{End}_{i\mathcal{F}K}(Vi) \). Since \( \hat{H} \) acts on \( Vi \) and elements of \( L \) commute with elements of \( A \), we see that \( G \cong \hat{H}/L \) acts on \( A \). We claim that

\[
A \cong \text{End}_{\mathcal{F}K}(V)
\]

as \( G \)-algebras over \( Z \). To see this, consider the map \( \text{End}_{\mathcal{F}K}(V) \to A \) sending \( \varphi \in \text{End}_{\mathcal{F}K}(V) \) to \( \varphi_i \in A \) defined by \( (vi)(\varphi_i) = vi\varphi_i \). It is easily verified that this map is an homomorphism of algebras. To see that it commutes with the action of \( G \cong \hat{H}/L \), observe that for \( h \in \hat{H} \), we have

\[
v\varphi^h i = vh^{-1}\varphi hi = vh^{-1}\varphi ih = v(\varphi i)^h.
\]

We get the inverse of \( \varphi \mapsto \varphi_i \) as follows: Since \( \mathcal{F}Ke \) is a simple ring, we have \( \mathcal{F}Ke = \mathcal{F}Ki\mathcal{F}K \). Thus there are elements \( a_\nu, b_\nu \in \mathcal{F}K \) such that \( e = \sum_\nu a_\nu ib_\nu \). Then the map sending \( \psi \in A = \text{End}_{i\mathcal{F}K}(Vi) \) to \( \hat{\psi} \in \text{End}_{\mathcal{F}K}(V) \) defined by \( v\hat{\psi} = \sum_\nu va_\nu ib_\nu \) is the inverse of the above map. (In fact, it is well known that \( \text{End}_{R}(V) \cong \text{End}_{iR}(Vi) \) for idempotents \( i \) in a ring \( R \) with \( R = RiR \).) The claim is proved.

It follows from the claim that \( A = \text{End}_{i\mathcal{F}K}(Vi) \in [\vartheta, \kappa, F] \). To finish the proof it suffices to show that \( A \otimes_{\mathcal{E}} S \cong B := \text{End}_{\mathcal{F}L}(W) \).

Each of \( A, B, i\mathcal{F}Ki \) and \( S \) acts faithfully on \( W = Vi \) by (right) multiplication, and thus we may identify these algebras with subalgebras of \( T := \text{End}_{\mathcal{E}}(W) \). Clearly, we have

\[
A, S \subseteq B,
\]

and \( A \) and \( S \) centralize each other. Because \( A = C_T(i\mathcal{F}Ki) \) by definition, we have \( C_T(A) = i\mathcal{F}Ki \) by the double centralizer property [5, Theorem 3.15]. It follows that

\[
C_B(A) = C_T(A) \cap B = i\mathcal{F}Ki \cap C_T(i\mathcal{F}Li) = S
\]

by the definitions of \( B \) and \( S \). Since \( A, B \) and \( S \) are central simple algebras over \( \mathcal{E} \), it follows that \( B \cong A \otimes_{\mathcal{E}} S \) [5, Corollary 3.16].

**3.6. Corollary.** Assume Hypothesis 3.1 with \( n = [\vartheta_L, \varphi] = 1 \). Then \([\vartheta, \kappa, F] = [\varphi, \kappa_iH, F]\).

**Proof.** Clear since then \( S \cong \mathcal{E} \) yields the trivial element of \( \text{BrCliff}(G, \mathcal{E}) \).
The assumptions of the corollary hold in particular when \( \vartheta_L = \varphi \in \text{Irr} \, L \) and \( F(\vartheta) = F(\varphi) \). To see another situation where Hypothesis 3.1 holds with \( n = 1 \), we need a simple lemma, which is a minor extension of a result of Riese and Schmid [15, Theorem 1]. (The bijection in the following lemma also preserves Schur indices over \( F \), but we will not need this fact.)

3.7. Lemma. Let \( A \trianglelefteq \hat{G} \) be a normal subgroup and \( \tau \in \text{Irr} \, A \). Let \( F \) be a field and \( \Gamma := \text{Gal}(F(\tau)/F) \). Define

\[
\hat{H} = \{ g \in \hat{G} \mid \exists \alpha \in \Gamma: \tau^{g\alpha} = \tau \}.
\]

Then induction defines a bijection between

\[
\bigcup_{\alpha \in \Gamma} \text{Irr}(\hat{H} \mid \tau^\alpha) \quad \text{and} \quad \bigcup_{\alpha \in \Gamma} \text{Irr}(\hat{G} \mid \tau^\alpha)
\]

commuting with field automorphisms over \( F \). In particular, \( F(\psi \hat{G}) = F(\psi) \) for \( \psi \in \text{Irr}(\hat{H} \mid \tau) \).

Proof. Write \( \hat{G}_\tau \) to denote the inertia group of \( \tau \). Clifford correspondence [9, Theorem 6.11] yields that induction defines bijections from \( \text{Irr}(\hat{G}_\tau \mid \tau) \) onto \( \text{Irr}(\hat{H} \mid \tau) \), and from \( \text{Irr}(\hat{G}_\tau \mid \tau) \) onto \( \text{Irr}(\hat{G} \mid \tau) \). Thus induction defines a bijection from \( \text{Irr}(\hat{H} \mid \tau) \) onto \( \text{Irr}(\hat{G} \mid \tau) \). The same statement holds with \( \tau \) replaced by \( \tau^\alpha \). (Of course, we have \( \hat{G}_{\tau^\alpha} = \hat{G}_\tau \).

It follows that induction yields a surjective map

\[
\bigcup_{\alpha \in \Gamma} \text{Irr}(\hat{H} \mid \tau^\alpha) \to \bigcup_{\alpha \in \Gamma} \text{Irr}(\hat{G} \mid \tau^\alpha)
\]

which is injective when restricted to some \( \text{Irr}(\hat{H} \mid \tau^\alpha) \). It remains to show that there occurs no collapsing between two different sets \( \text{Irr}(\hat{H} \mid \tau^{\alpha_1}) \) and \( \text{Irr}(\hat{H} \mid \tau^{\alpha_2}) \). So assume that \( \psi \hat{G} = \xi \hat{G} = \chi \) for \( \psi \in \text{Irr}(\hat{H} \mid \tau^{\alpha_1}) \) and \( \xi \in \text{Irr}(\hat{H} \mid \tau^{\alpha_2}) \). Then \( \tau^{\alpha_1} \) and \( \tau^{\alpha_2} \) are both irreducible constituents of \( \chi_A \). By Clifford’s theorem there is \( g \in \hat{G} \) with \( \tau^{g\alpha_1} = \tau^{\alpha_2} \) or \( \tau^{g\alpha_1\alpha_2^{-1}} = \tau \). By the very definition of \( \hat{H} \) we see that \( g \in \hat{H} \). But then \( \psi = \psi^g \in \text{Irr}(\hat{H} \mid \tau^{\alpha_2}) \). Since induction is bijective when restricted to \( \text{Irr}(\hat{H} \mid \tau^{\alpha_2}) \), it follows that \( \psi = \xi \) as wanted.

Obviously, induction commutes with field automorphisms: We have \( (\psi^{\alpha}) \hat{G} = (\psi \hat{G})^\alpha \). We always have \( F(\psi \hat{G}) \subseteq F(\psi) \). If \( \alpha \in \text{Gal}(F(\psi)/F(\psi \hat{G})) \), then \( \psi \hat{G} = (\psi \hat{G})^\alpha = (\psi^\alpha) \hat{G} \) and thus \( \psi = \psi^\alpha \) by bijectivity. It follows \( \alpha = 1 \) and thus \( F(\psi) = F(\psi \hat{G}) \) as claimed. The proof is finished.

3.8. Corollary. Let

\[
1 \longrightarrow K \longrightarrow \hat{G} \longrightarrow G \longrightarrow 1
\]

be an exact sequence of finite groups and \( \vartheta \in \text{Irr} \, K \) be semi-invariant over \( F \) in \( \hat{G} \). Assume there is \( A \trianglelefteq \hat{G} \) with \( A \trianglelefteq K \) and let \( \tau \) be an irreducible constituent of \( \vartheta_A \).
Set
\[ \tilde{H} = \{ g \in \tilde{G} \mid \exists \alpha \in \text{Gal}(F(\tau)/F): \tau^{g \alpha} = \tau \}. \]
Then \( \tilde{G} = \tilde{H} K \), and for \( L = \tilde{H} \cap K \) there is a unique \( \varphi \in \text{Irr}(L \mid \tau) \) with \( \vartheta = \varphi^K \), and we have \( F(\vartheta) = F(\varphi) \) and \( [\vartheta, \kappa, F] = [\varphi, \kappa, \tilde{H}, F] \) for this \( \varphi \).

**Proof.** We first show that \( \tilde{G} = \tilde{H} K \). Let \( g \in \tilde{G} \). Since \( \vartheta \) is semi-invariant in \( \tilde{G} \), there is an \( \alpha \in \text{Gal}(F(\vartheta)/F) \) such that \( \vartheta^{g \alpha} = \vartheta \). We may extend \( \alpha \) to an automorphism \( \beta \) of \( F(\vartheta, \tau) \). Then \( \tau^{g \beta} \) is a constituent of \( \vartheta_A \). By Clifford’s theorem, we have \( \tau^{g \beta k} = \tau \) for some \( k \in K \). Thus \( g k \in \tilde{H} \), which shows \( \tilde{G} = \tilde{H} K \).

Lemma 3.7 yields the existence of a \( \varphi \) with \( \vartheta = \varphi^K \).

We verify that Hypothesis 3.1 holds. We have \( n = [\vartheta_L, \varphi] = [\vartheta, \varphi^K] = 1 \). Lemma 3.7 yields that \( F(\varphi) = F(\vartheta) \). Finally, let \( h \in \tilde{H} \). Then there is \( \alpha \in \text{Gal}(F(\vartheta)/F) \) with \( \vartheta^{h \alpha} = \vartheta \). Then \( (\varphi^{h \alpha})^K = \vartheta = \varphi^K \). Since \( \varphi^{h \alpha} \in \text{Irr}(L \mid \tau^{h \alpha}) \) and \( \tau^{h \alpha} = \tau^\beta \) for some field automorphism \( \beta \), it follows from Lemma 3.7 that \( \varphi^{h \alpha} = \varphi \).

We have now shown that Hypothesis 3.1 holds with \( n = 1 \). Corollary 3.6 yields the result. \( \square \)

3.9. **Remark.** It follows directly from Lemma 3.7 that for each subgroup \( U \) with \( K \leq U \leq \tilde{G} \) there is a correspondence between \( \text{Irr}(U \mid \tau) \) and \( \text{Irr}(U \cap \tilde{H} \mid \tau) \). It is also elementary to prove that these correspondences commute with restriction and induction of characters. In fact, every property of the correspondence that follows from the equality \( [\vartheta, \kappa, F] = [\varphi, \kappa, \tilde{H}, F] \) and the results of Turull [23] can be proved elementarily, without using the Brauer-Clifford group. But we will need Corollary 3.8 in inductive arguments to come later, and there we can not do without the language of the Brauer-Clifford group.

Proposition C is a special case of Corollary 3.8, applied over a bigger field:

**Proof of Proposition C.** Assume the situation of Theorem A. Recall that \( C, N_1 \leq \tilde{G}_1 \) with \( C \leq N_1 \), and that \( \vartheta_1 = \lambda^{N_1} \) with \( \lambda \in \text{Irr} C \). By definition in Proposition C, \( \tilde{U}_1 = (\tilde{G}_1)_{\vartheta_1} \) is the inertia group of \( \vartheta_1 \) in \( \tilde{G}_1 \), and \( V = (\tilde{G}_1)_{\lambda} \) is the inertia group of \( \lambda \) in \( \tilde{G}_1 \). Clearly, \( \lambda^{N_1} = \vartheta_1 \in \text{Irr} N_1 \) implies that \( V \cap N_1 = C \).

We apply Corollary 3.8 to the exact sequence
\[ 1 \longrightarrow N_1 \longrightarrow \tilde{U}_1 \longrightarrow U \longrightarrow 1 \]
with \( C \) instead of \( A, \lambda \) instead of \( \tau \) and the field \( F(\lambda) \) instead of \( F \). Note that \( \tilde{H} = V \).

We get that \( \tilde{U}_1 = V N_1 \) and \( [\vartheta_1, (\kappa_1)_{\tilde{U}_1}, F(\lambda)] = [\lambda, (\kappa_1)_V, F(\lambda)] \). By Lemma 3.7, induction induces a bijection \( \text{Irr}(X \cap V \mid \lambda) \rightarrow \text{Irr}(X \mid \vartheta_1) \). \( \square \)

4. **A Subgroup of the Schur-Clifford Group**

Let \( E \) be a field and \( G \) a group which acts on \( E \) and fixes the subfield \( F \). Recall that in [12] we defined the **Schur-Clifford group** \( S^C(G, E) \) to be the subset of \( \text{BrCliff}(G, E) \) of all \( [\vartheta, \kappa, F] \) such that \( (\vartheta, \kappa) \) is a Clifford pair over \( G \) that induces
the given action of $G$ on $F(\vartheta) = E$. We also showed that $SC(F)(G, E)$ is a subgroup of the Brauer-Clifford group, if $E$ is contained in a cyclotomic extension of $F$. Further, in that case we have

$$SC(F)(G, E) = SC(E)(G, E) =: SC(G, E).$$

Fix a finite group $G$ and a field $F$. We consider two subclasses of Clifford pairs $(\vartheta, \kappa)$ where, as usual, the homomorphism $\kappa$ is part of an exact sequence

$$1 \rightarrow N \rightarrow \hat{G} \xrightarrow{\kappa} G \rightarrow 1$$

and $\vartheta \in \text{Irr} N$. The main result of this section is that both subclasses yield the same subgroup of the Schur-Clifford group.

### 4.1. Definition

Let $M$ be the class of all Clifford pairs $(\vartheta, \kappa)$ such that

(a) there is $A \trianglelefteq \hat{G}$ with $A \subseteq N$,
(b) there is a linear character $\lambda \in \text{Lin} A$ such that $\vartheta = \lambda N$.

Let $C$ be the subclass of $M$ containing the pairs in $M$ such that

(c) the $A$ above is cyclic and $\lambda$ is faithful,
(d) $\lambda$ is semi-invariant in $\hat{G}$ over $F$, and
(e) $N/A \cong \text{Gal}(F(\lambda)/F(\vartheta))$.

Recall that for a class $X$ of Clifford pairs, we defined

$$SC_X(G, E) = \{ [\vartheta, \kappa, F] \in \text{BrCliff}(G, E) \mid (\vartheta, \kappa) \in X \}.$$

In this terminology, Theorem A says that $SC(G, E) = SC_C(G, E)$.

There is some redundancy in these conditions:

### 4.2. Lemma

If (a), (b) and (d) of Definition 4.1 hold for some Clifford pair $(\vartheta, \kappa)$, so does (e). (Thus Condition (d) in Theorem A follows from the other conditions in Theorem A.)

**Proof.** Since $\lambda$ is semi-invariant in $\hat{G}$ over $F$, there is for every $n \in N$ an $\alpha_n \in \text{Gal}(F(\lambda)/F)$ such that $\lambda^{\alpha_n} = \lambda$. Since $(\lambda^n)^N = \vartheta^n = \vartheta$, we must have $\alpha_n \in \text{Gal}(F(\lambda)/F(\vartheta))$. This defines an homomorphism from $N/A$ into $\text{Gal}(F(\lambda)/F(\vartheta))$ with kernel $N/\lambda/A$ [8, Lemma 2.1]. Since $\lambda^N \in \text{Irr} N$, we must have $N/\lambda = A$.

Now let $\alpha \in \text{Gal}(F(\lambda)/F(\vartheta))$. Then $\lambda^\alpha$ and $\lambda$ are constituents of $\vartheta = \vartheta^\alpha$ and thus conjugate in $N$, so that $\lambda^{\alpha n} = \lambda$ for some $n \in N$. It follows that $\alpha = \alpha_n \in \text{Gal}(F(\lambda)/F(\vartheta))$. Thus $n \mapsto \alpha_n \in \text{Gal}(F(\lambda)/F(\vartheta))$ is surjective. Thus Condition (e) holds.

The following result follows directly from the definitions.

### 4.3. Lemma

Let $(\vartheta, \kappa: \hat{G} \rightarrow G)$ be a Clifford pair and $K \subseteq \text{Ker} \vartheta$ a normal subgroup of $\hat{G}$. Then $[\vartheta, \kappa, F] = [\vartheta, \kappa, F]$, where $\pi: \hat{G}/K \rightarrow G$ is the map induced by $\kappa$ and $\vartheta \in \text{Irr}(N/K)$ is the character $\vartheta$ viewed as character of $N/K$ (where $N = \text{Ker} \kappa$).
Note that when Conditions (a), (b) and (d) above hold, then $K = \text{Ker} \lambda$ is normal in $\hat{G}$, and also $K = \text{Ker} \vartheta$. Thus we can factor out $K$ and get a Clifford pair such that (c) is true, too.

4.4. Proposition. Let $E$ be a field extension of $F$ on which $G$ acts as $F$-algebra. Then

$$SC_M(G, E) = SC_C(G, E).$$

Proof. By definition, we have $SC_C(G, E) \subseteq SC_M(G, E)$. To show the converse inclusion, we begin with a Clifford pair $(\vartheta, \kappa) \in M$ that induces the given action on $E = F(\vartheta)$. Here $\kappa: \hat{G} \to G$ and $\vartheta \in \text{Irr} \ N$ with $N = \text{ker} \kappa$ as usual. Let $A \subseteq N$ be a normal subgroup of $\hat{G}$ and $\lambda \in \text{Lin} \ A$ a linear character with $\vartheta = \lambda^N$.

By assumption, the character $\vartheta$ is semi-invariant in $\hat{G}$. Set

$$\hat{G}_0 = \{ g \in \hat{G} \mid \text{there is } \alpha \in \text{Gal}(F(\lambda)/F) \text{ such that } \lambda^{g\alpha} = \lambda \},$$

$$N_0 = N \cap \hat{G}_0, \quad \kappa_0 = \kappa|_{\hat{G}_0}, \text{ and } \vartheta_0 = \lambda^{N_0}.$$

Note that $\vartheta_0^N = \lambda^N = \vartheta$. By Corollary 3.8, we have that $\hat{G} = \hat{G}_0 N$, that $F(\vartheta) = F(\vartheta_0)$, and that $[\vartheta, \kappa, F] = [\vartheta_0, \kappa_0, F]$. Condition (d) (and thus (e)) holds for the Clifford pair $(\vartheta_0, \kappa_0)$. By Lemma 4.3 and the remark following it, we can factor out the kernel of $\lambda$ and we get a Clifford pair in $C$ that yields the same element of the Brauer-Clifford group as $(\vartheta, \kappa)$. \qed

4.5. Corollary. $SC_C(G, E)$ is a subgroup of $SC(G, E)$ and $\text{BrCliff}(G, E)$.

Proof. It suffices to show that $SC_M(G, E)$ is a subgroup. By Theorem 5.7 from [12], we have to show the following:

(a) $SC_M(G, E) \neq \emptyset$.
(b) When $(\vartheta, \kappa) \in M$, then $(\vartheta, \kappa) \in M$ (here, $\vartheta$ denotes the complex conjugate of $\vartheta$).
(c) When $(\vartheta_1, \kappa_1)$ and $(\vartheta_2, \kappa_2) \in M$, then $(\vartheta_1 \times \vartheta_2, \kappa_1 \times \kappa_2) \in M$.

That $SC_M(G, E) \neq \emptyset$ follows from Corollary 5.2 in [12], and (b) is clear. For (c), assume $A_i \subseteq N_i$ is the abelian normal subgroup of $\hat{G}_i$, where

$$1 \longrightarrow N_i \longrightarrow \hat{G}_i \overset{\kappa_i}{\longrightarrow} G \longrightarrow 1$$

is the exact sequence belonging to $\kappa_i$, and let $\lambda_i \in \text{Lin} \ A_i$ with $\vartheta_i = \lambda_i^{N_i}$ for $i = 1, 2$.

Recall that the Clifford pair $(\vartheta_1 \times \vartheta_2, \kappa_1 \times \kappa_2)$ is defined as follows: Let

$$\hat{G}_1 \times_G \hat{G}_2 = \{ (g_1, g_2) \in \hat{G}_1 \times \hat{G}_2 \mid g_1 \kappa_1 = g_2 \kappa_2 \}$$

be the pullback and $\kappa_1 \times \kappa_2: \hat{G}_1 \times_G \hat{G}_2 \to G$ the canonical homomorphism mapping $(g_1, g_2)$ to $g_1 \kappa_1 = g_2 \kappa_2$. Then $\text{Ker}(\kappa_1 \times \kappa_2) = N_1 \times N_2$, and $\vartheta_1 \times \vartheta_2 \in \text{Irr} (N_1 \times N_2)$. It follows that $A_1 \times A_2 \subseteq \hat{G}_1 \times_G \hat{G}_2$ is an abelian normal subgroup and $\vartheta_1 \times \vartheta_2 = \lambda_1^{N_1} \times \lambda_2^{N_2} = (\lambda_1 \times \lambda_2)^{N_1 \times N_2}$. This shows (c). \qed
5. Reduction to prime power groups

5.1. Lemma. Let \( a \in \mathcal{SC}(G, \mathbb{E}) \). Then \( a \in \mathcal{SC}_C(G, \mathbb{E}) \) if and only if the \( p \)-parts \( a_p \) of \( a \) are in \( \mathcal{SC}_C(G, \mathbb{E}) \) for all primes \( p \).

Proof. Recall that \( \text{BrCliff}(G, \mathbb{E}) \) is torsion. Thus \( a = \prod_p a_p \) is the product of its \( p \)-parts \( a_p \), and \( a_p \in \langle a \rangle \). The result follows since \( \mathcal{SC}(G, \mathbb{E}) \) and \( \mathcal{SC}_C(G, \mathbb{E}) \) are subgroups (Corollary 4.5).

For any subgroup \( H \leq G \), there is a group homomorphism

\[ \text{Res}^G_H : \text{BrCliff}(G, \mathbb{E}) \to \text{BrCliff}(H, \mathbb{E}) \]

which is induced by viewing a \( G \)-algebra as an \( H \)-algebra. This restriction homomorphism sends \( \left[ \psi, \kappa, \mathbb{F} \right] \) to \( \left[ \psi, \pi, \mathbb{F} \right] \), where \( \pi \) is the restriction of \( \kappa \) to the preimage \( \kappa^{-1}(H) \) of \( H \) [12, Proposition 7.1].

In the proof of the next result, we also need the corestriction map

\[ \text{Cores}^G_H : \text{BrCliff}(H, \mathbb{E}) \to \text{BrCliff}(G, \mathbb{E}) \]

defined in [11].

5.2. Lemma. Let \( a \in \mathcal{SC}(G, \mathbb{E}) \) have \( p \)-power order for the prime \( p \) and let \( P \leq G \) be a Sylow \( p \)-subgroup of \( G \). Then \( a \in \mathcal{SC}_C(G, \mathbb{E}) \) if and only if \( \text{Res}^G_P(a) \in \mathcal{SC}_C(P, \mathbb{E}) \).

Proof. The “only” if part is clear and does not depend on \( a \in \mathcal{SC}(G, \mathbb{E}) \) having \( p \)-power order.

Now assume \( \text{Res}^G_P(a) \in \mathcal{SC}_C(P, \mathbb{E}) \). Apply the corestriction map

\[ \text{Cores}^G_P : \text{BrCliff}(P, \mathbb{E}) \to \text{BrCliff}(G, \mathbb{E}) \]

It follows from [12, Corollary 9.2] that \( \text{Cores}^G_P \) maps \( \mathcal{SC}_M(P, \mathbb{E}) \) into \( \mathcal{SC}_M(G, \mathbb{E}) \). Since \( \mathcal{SC}_M(G, \mathbb{E}) = \mathcal{SC}_C(G, \mathbb{E}) \) by Proposition 4.4, we have \( \text{Cores}^G_P(\text{Res}^G_P(a)) \in \mathcal{SC}_C(G, \mathbb{E}) \). By [11, Theorem 6.3], we get \( \text{Cores}^G_P(\text{Res}^G_P(a)) = \langle a^{[G:P]} \rangle \). Since \( a \) has \( p \)-power order, it follows that \( a \in \langle a^{[G:P]} \rangle \). Thus \( a \in \mathcal{SC}_C(G, \mathbb{E}) \) as claimed.

By the last two results, to show that an arbitrary element \( a \in \mathcal{SC}(G, \mathbb{E}) \) is in fact contained in \( \mathcal{SC}_C(G, \mathbb{E}) \), we can assume that \( a \) has \( p \)-power order and that \( G \) is a \( p \)-group. Note that it was essential in the above proofs that \( \mathcal{SC}(G, \mathbb{E}) \) and \( \mathcal{SC}_C(G, \mathbb{E}) \) are groups.

5.3. Remark. By similar arguments, one can show: An element \( a \in \text{BrCliff}(G, \mathbb{E}) \) is in \( \mathcal{SC}_C(G, \mathbb{E}) \) if and only if \( \text{Res}^G_P(a) \in \mathcal{SC}_C(P, \mathbb{E}) \) for all Sylow subgroups \( P \) of \( G \). Note, however, that even when \( P \) is a \( p \)-group, the exponent of \( \text{BrCliff}(P, \mathbb{E}) \) or \( \mathcal{SC}(P, \mathbb{E}) \) is in general not a \( p \)-power. For example, \( \text{BrCliff}(1, \mathbb{E}) = \text{Br}(\mathbb{E}) \), and \( \mathcal{SC}(1, \mathbb{E}) \) is in general not the trivial group.
6. Reduction to a larger field

In this section, we show that when $G$ is a $p$-group and $[\vartheta, \kappa, F]$ has $p$-power-order, then we can replace the fields $E$ and $F$ by certain larger fields to prove the main theorem (Theorem A).

First, we assume the following situation: Let $E$ be a field on which a group $G$ acts, and let $F$ be a field contained in the fixed field $E^G$. Suppose that there are two other fields $K \geq L \geq F$ such that $K = EL$ and $E \cap L = F$. Assume that $L/F$ is Galois and let $\Delta = \text{Gal}(L/F) \cong \text{Gal}(K/E)$. The situation is summarized in the following picture:

\[
\begin{array}{c}
&L \\
E & \Delta & K \\
&F
\end{array}
\]

In this situation, $G \times \Delta$ acts on $K \cong E \otimes_F L$. If $S$ is a central simple $G$-algebra over $E$, then $S \otimes_E L \cong S \otimes_K K$ is a central simple $(G \times \Delta)$-algebra over $K$. This defines a group homomorphism $\text{BrCliff}(G, E) \to \text{BrCliff}(G \times \Delta, K)$. In fact, this group homomorphism is the composition of

\[
\text{BrCliff}(G, E) \xrightarrow{\text{Inf}} \text{BrCliff}(G \times \Delta, E) \xrightarrow{} \text{BrCliff}(G \times \Delta, K),
\]

where the first map is the inflation map induced by the epimorphism $G \times \Delta \to G$, and the second map is induced by scalar extension from $E$ to $K$. Our first goal is to show that the above group homomorphism is actually an isomorphism:

6.1. Proposition. The maps $S \mapsto S \otimes_K K$ and $T \mapsto T^\Delta := C_T(\Delta)$ define inverse maps between the isomorphism classes of central simple $G$-algebras over $E$ and central simple $(G \times \Delta)$-algebras over $K$. Moreover, they induce mutually inverse isomorphisms

\[
\text{BrCliff}(G, E) \cong \text{BrCliff}(G \times \Delta, K).
\]

It is clear that $S \cong (S \otimes_K K)^\Delta$, and it is a result of Hochschild [7, Lemma 1.2] that $T \cong T^\Delta \otimes_K K$. But to see that the inverse map sending $T$ to $T^\Delta$ respects equivalence classes, we need some more general arguments, and it will be more convenient for us to reprove Hochschild’s result.

Let $K\Delta$ denote the skew group ring with respect to the action of $\Delta$ on $K$ (see Section 2). The ring $K\Delta$ acts on $K$ from the right by

\[
x \circ \sum_{\sigma \in \Delta} x^\sigma c_\sigma = \sum_{\sigma} x^\sigma c_\sigma.
\]
This makes $\mathbb{K}$ into a right $\mathbb{K}\Delta$-module. So if $V$ is a vector space over $\mathbb{E}$, then $V \otimes_{\mathbb{E}} \mathbb{K}$ is a right $\mathbb{K}\Delta$-module. For a right $\mathbb{K}\Delta$-module $W$, we still write $W^\Delta = \{ w \in W \mid w\sigma = w \text{ for all } \sigma \in \Delta \}$.

The next lemma basically follows from the fact that $\mathbb{K}\Delta \cong \mathbb{M}_{|\Delta|}(\mathbb{E})$ which is well known from Galois cohomology.

6.2. Lemma. For every $\mathbb{E}$-vector space $V$ and every right $\mathbb{K}\Delta$-module $W$, we have

$$V \cong (V \otimes_{\mathbb{E}} \mathbb{K})^\Delta \quad \text{and} \quad W \cong W^{\Delta} \otimes_{\mathbb{E}} \mathbb{K}$$

naturally, and $\text{End}_{\mathbb{K}\Delta}(W) \cong \text{End}_{\mathbb{E}}(W^\Delta)$. (In fact, we have a category equivalence between the category of vector spaces over $\mathbb{E}$ and the category of modules over $\mathbb{K}\Delta$.)

Proof. It is clear that $(V \otimes_{\mathbb{E}} \mathbb{K})^\Delta \cong V$ naturally for any $\mathbb{E}$-vector space $V$.

Conversely, let $W$ be a $\mathbb{K}\Delta$-module. We have to show that $w \otimes k \mapsto wk$ is an isomorphism $W^\Delta \otimes_{\mathbb{E}} \mathbb{K} \cong W$.

First, we observe the following identity in $\mathbb{K}\Delta$: For any $a \in \mathbb{K}$, we have

$$\left( \sum_{\sigma \in \Delta} \sigma \right) a \left( \sum_{\tau \in \Delta} \tau \right) = \sum_{\sigma, \tau \in \Delta} \sigma \tau a^r = \left( \sum_{\sigma \in \Delta} \sigma \right) \text{Tr}_{\mathbb{E}}^\Delta(a).$$

Let $b_1, \ldots, b_n$ be a basis of $\mathbb{K}$ over $\mathbb{E}$ and let $a_1, \ldots, a_n$ be the dual basis with respect to the form $(x, y) \mapsto \text{Tr}_{\mathbb{E}}^\Delta(xy)$, so that $\text{Tr}_{\mathbb{E}}^\Delta(b_i a_j) = \delta_{ij}$. Then for all $k \in \mathbb{K}$ we have

$$k = \sum_{i=1}^n \text{Tr}_{\mathbb{E}}^\Delta(k a_i) b_i = \sum_{i=1}^n \text{Tr}_{\mathbb{E}}^\Delta(k b_i) a_i.$$

Set

$$E_{ij} := a_i \left( \sum_{\sigma \in \Delta} \sigma \right) b_j \in \mathbb{K}\Delta.$$

It follows that

$$E_{ij} E_{rs} = a_i \left( \sum_{\sigma \in \Delta} \sigma \right) \text{Tr}_{\mathbb{E}}^\Delta(b_j a_r) b_s = \delta_{jr} E_{is},$$

where the first equality follows from the identity in the last paragraph. In particular, the $E_{ij}$'s are linearly independent over $\mathbb{E}$ and so, by counting dimensions, form a basis of $\mathbb{K}\Delta$. Since $(\sum_i E_{ii}) E_{rs} = E_{rs}$, it follows that $\sum_i E_{ii} = 1$. We have now proved that the $E_{ij}$'s form a complete set of matrix units. Using this, it is routine to verify that

$$W \ni w \mapsto \sum_{i=1}^n w a_i \left( \sum_{\sigma \in \Delta} \sigma \right) \otimes b_i \in W^\Delta \otimes_{\mathbb{E}} \mathbb{K}$$

is the inverse of the natural map $W^\Delta \otimes_{\mathbb{E}} \mathbb{K} \to W$ sending $w_0 \otimes k$ to $wk$.

The isomorphism $\text{End}_{\mathbb{K}\Delta}(W) \cong \text{End}_{\mathbb{E}}(W^\Delta)$ follows from $W \cong W^\Delta \otimes_{\mathbb{E}} \mathbb{K}$.

6.3. Lemma. Let $W_1$ and $W_2$ be two $\mathbb{K}\Delta$-modules. Then $(W_1 \otimes_{\mathbb{K}} W_2)^\Delta \cong W_1^\Delta \otimes_{\mathbb{E}} W_2^\Delta$.

Proof. We have a natural injection of $W_1^\Delta \otimes_{\mathbb{E}} W_2^\Delta$ into $(W_1 \otimes_{\mathbb{K}} W_2)^\Delta$. It follows from Lemma 6.2 that both spaces have dimension $(\dim_{\mathbb{K}} W_1)(\dim_{\mathbb{K}} W_2)$ over $\mathbb{E}$. Thus the injection is an isomorphism.
Proof of Proposition 6.1. A \((G \times \Delta)\)-algebra over \(\mathbb{K}\) can be viewed as a \(\mathbb{K}\Delta\)-module. It is easy to check that the isomorphisms of Lemma 6.2 are isomorphisms of \((G \times \Delta)\)-algebras.

If \(S_1\) and \(S_2\) are equivalent, then \(S_1 \otimes_{\mathbb{E}} \mathbb{K}\) and \(S_2 \otimes_{\mathbb{E}} \mathbb{K}\) are equivalent, as explained at the beginning of this section.

Now assume that \([T_1^\Delta] = [T_2^\Delta]\) in \(\text{BrCliff}(G \times \Delta, \mathbb{K})\). To finish the proof, we need to show that \(T_1\Delta\) and \(T_2\Delta\) are equivalent. By assumption, there are \(\mathbb{K}[G \times \Delta]\)-modules \(P_1\) and \(P_2\) such that

\[T_1 \otimes_{\mathbb{K}} \text{End}_{\mathbb{K}}(P_1) \cong T_2 \otimes_{\mathbb{K}} \text{End}_{\mathbb{K}}(P_2).\]

Taking centralizers of \(\Delta\) and using Lemma 6.3, it follows that \(T_1\Delta\) and \(T_2\Delta\) are equivalent. By assumption, there are \(\mathbb{K}[G \times \Delta]\)-modules \(P_1\) and \(P_2\) such that

\[T_1 \otimes_{\mathbb{K}} \text{End}_{\mathbb{K}}(P_1) \cong T_2 \otimes_{\mathbb{K}} \text{End}_{\mathbb{K}}(P_2).\]

(Note that \((\text{End}_{\mathbb{K}}(P_i)) = \text{End}_{\mathbb{K}}(P_{\Delta})\) from Lemma 6.2, it follows that \(\text{End}_{\mathbb{E}}(P_{\Delta})\) is a trivial \(G\)-algebra over \(\mathbb{E}\) (for \(i = 1, 2\)), and \(T_1\Delta\) and \(T_2\Delta\) are equivalent. \(\square\)

We write

\[C(\Delta): \text{BrCliff}(G \times \Delta, \mathbb{K}) \to \text{BrCliff}(G, \mathbb{E}), \quad C(\Delta)[T] = [T^\Delta].\]

6.4. Lemma. We have

\[C(\Delta)(\mathcal{S}(G \times \Delta, \mathbb{K})) \subseteq \mathcal{S}(G, \mathbb{E}) \quad \text{and} \quad C(\Delta)(\mathcal{S}_M(G \times \Delta, \mathbb{K})) \subseteq \mathcal{S}_M(G, \mathbb{E}).\]

Proof. Let

\[1 \to N \to \hat{G} \to G \times \Delta \to 1\]

be an exact sequence and \(V\) an \(F\hat{G}\)-module with \(S = \text{End}_{\hat{G}N}(V)\), and such that the class of \(S\) is in \(\mathcal{S}(G \times \Delta, \mathbb{K})\). Then for \(M = \kappa^{-1}(\Delta)\) we have an exact sequence

\[1 \to M \to \hat{G} \to G \times \Delta \to 1,\]

where \(\pi\) is \(\hat{G} \to G \times \Delta \to G\). The equality \(\text{End}_{F\hat{G}}(V) = \text{S}^\Delta\) proves the first assertion.

Let \(\vartheta \in \text{Irr} N\) be an irreducible constituent of the character of \(V_N\). Then \(F(\vartheta) = \mathbb{K} \cong \text{Z}(S)\). Since \(\Delta\) acts faithfully on \(\mathbb{K}\), it follows that \(\vartheta^m \neq \vartheta\) for all \(m \in M \setminus N\). Thus \(\vartheta^M\) is irreducible, and it follows that \([S^\Delta] = [\vartheta^M, \pi, F]\). This shows, in particular, the second assertion. \(\square\)

6.5. Lemma. Let \([S] \in \text{BrCliff}(G, \mathbb{E})\). If \([S \otimes_{\mathbb{F}} \mathbb{L}] \in \mathcal{S}_M(G, \mathbb{K})\), then \([S]^{|\Delta|} \in \mathcal{S}_M(G, \mathbb{E})\).
Proof. Note that we can view $S \otimes_F L$ as a $(G \times \Delta)$-algebra. Applying the co restricition map with $G \leq G \times \Delta$, we get [11, Theorem 6.3]
\[
\text{Core}_{G}^{G \times \Delta} \text{Res}_{G}^{G \times \Delta}[S \otimes_F L] = [S \otimes_F L]^{|\Delta|}.
\]
Since we assume $\text{Res}_{G}^{G \times \Delta}[S \otimes_F L] \in SC_{M}(G, K)$, it follows [12, Corollary 9.2] that $[S \otimes_F L]^{|\Delta|} \in SC_{M}(G \times \Delta, K)$.

Applying $C(\Delta)$ yields, by Lemma 6.4, that $[S]^{|\Delta|} \in SC_{M}(G, E)$. \hfill \qed

Recall that we want to prove that an arbitrary element $a \in SC(G, E)$ is in fact an element of $SC_{C}(G, E)$, and that we have already reduced to the case where $a$ has $p$-power order and $G$ is a $p$-group. The results of this section yield a further reduction:

6.6. Corollary. Let $G$ be a $p$-group and let
\[
a = [\vartheta, \kappa : \hat{G} \to G, F] \in SC(G, E)
\]
have $p$-power order. (So $\vartheta$ is semi-invariant over $F$ in $\hat{G}$ and $F(\vartheta) = E$.) Let $\varepsilon$ be a primitive $|\hat{G}|$-th root of unity. Then there is a unique field $L$ such that $[L : F]$ is a $p'$-number and $|F(\varepsilon) : L|$ is a $p$-power. If $[\vartheta, \kappa, L] \in SC_{C}(G, L(\vartheta))$, then $a = [\vartheta, \kappa, F] \in SC_{C}(G, E)$.

Proof. The field $L$ is uniquely determined as the fixed field of a Sylow $p$-subgroup of the abelian Galois group $\text{Gal}(F(\varepsilon)/F)$.

By assumption, $E = F(\vartheta)$. By a result from my previous paper [12, Lemma 6.1], we know that $[\vartheta, \kappa, F] = [\vartheta, \kappa, E^G]$. Therefore, we may assume without loss of generality that $E^G = F$. Then $|E : F|$ is a power of $p$, since $G$ is a $p$-group. Thus $E \cap L = F$. With $K = EL \cong E \otimes_F L$, the assumptions from the beginning of the section hold.

Now $L(\vartheta) = LF(\vartheta) = LE = K$ and $[\vartheta, \kappa, L] = [\vartheta, \kappa, F] \otimes_F L$. If $[\vartheta, \kappa, L] \in SC_{M}(G, K)$ then $[\vartheta, \kappa, F]^{[L : F]} \in SC_{M}(G, E)$ by Lemma 6.5. Since $a = [\vartheta, \kappa, F]$ has $p$-power order, it follows that $a \in (a^{[L : F]}) \subseteq SC_{M}(G, E) = SC_{C}(G, E)$. This proves the corollary. \hfill \qed

7. Reduction to elementary groups

Let $L$ be a field of characteristic 0. Recall that a group $H$ is called $L$-elementary for the prime $p$ or $L$-$p$-elementary, if the following two conditions hold:

(a) $H = PC$ is the semidirect product of a normal cyclic $p'$-group $C$ and a $p$-group $P$.

(b) The linear characters of $C$ are semi-invariant over $L$ in $H$.

The second condition is sometimes expressed differently. We explain the connection. Let $\lambda \in \text{Lin} C$ be a faithful character of $C$. Then $\lambda$ is semi-invariant in $H = PC$ if for every $y \in P$ there is $\sigma \in \text{Gal}(L(\lambda)/L)$ such that $\lambda^{y\sigma} = \lambda$. Now $L(\lambda) = L(\zeta)$, where $\zeta$ is a primitive $|C|$-th root of unity. Then for $\sigma \in \text{Gal}(L(\zeta)/L)$, there is
a unique \( k = k(\sigma) \in (\mathbb{Z}/|C|\mathbb{Z})^* \) with \( \zeta^\sigma = \zeta^{k(\sigma)} \). The second condition above is equivalent to: For every \( y \in P \) there is \( \sigma \in \text{Gal}(\bar{L}(\zeta)/L) \) such that \( c^y = c^{k(\sigma)} \) for all \( c \in C \).

We need the following part of the generalized induction theorem:

7.1. **Proposition** (Brauer, Berman, Witt [18, § 12.6]). Let \( \hat{G} \) be a finite group with \( |\hat{G}| = b p^k \), where the prime \( p \) does not divide \( b \), and let \( L \) be a field. Then we can write

\[
b_1\hat{G} = \sum_{X,\tau} a_{X,\tau}\tau_{\hat{G}}
\]

with integers \( a_{X,\tau} \), where \((X, \tau)\) runs through the set of pairs such that \( X \) is an \( L\)-\( p \)-elementary subgroup of \( \hat{G} \) and \( \tau \) is the character of an \( LX \)-module.

The next result is similar to results of Dade [1, cf. 9, Theorem 8.24] and Schmid [17, Lemma 5.1]. See also [28, Proposition 3.3].

7.2. **Theorem.** Let \( \hat{G} \) be a finite group with normal subgroup \( N \trianglelefteq \hat{G} \) such that \( \hat{G}/N \) is a \( p \)-group for some prime \( p \). Let \( \vartheta \in \text{Irr}(N) \) be \( L \)-semi-invariant, where \( L \) is some field of characteristic 0 such that \( |L(\zeta) : L| \) is a \( p \)-power, where \( \zeta \) is an \( \exp(\hat{G}) \)-th root of unity. Then there exists an \( L\)-\( p \)-elementary group \( X \leq \hat{G} \) and \( \varphi \in \text{Irr}(N \cap X) \) such that the following hold:

(a) \( \hat{G} = XN \).
(b) \([\vartheta, \varphi^N] \neq 0 \mod p \).
(c) \( L(\vartheta) = L(\varphi) \).
(d) For each \( x \in X \) there is \( \alpha = \alpha_x \in \text{Gal}(L(\vartheta)/L) \) such that \( \vartheta^{x\alpha} = \vartheta \) and \( \varphi^{x\alpha} = \varphi \).

**Proof.** Write

\[
b_1\hat{G} = \sum_{X,\tau} a_{X,\tau}\tau_{\hat{G}},
\]

as in Proposition 7.1. Set \( \Gamma = \text{Gal}(L(\vartheta)/L) \) and

\[
\psi = \text{Tr}_L^{L(\vartheta)}(\vartheta) = \sum_{\sigma \in \Gamma} \vartheta^\sigma.
\]
Then
\[ b|\Gamma| = [\psi, \psi b \hat{G}]_N \]
\[ = \sum_{X, \tau} a_{X, \tau} [\psi \overline{\psi}, (\tau \hat{G})_N] \]
\[ = \sum_{X, \tau} a_{X, \tau} [\psi \overline{\psi}, \sum_{t \in [\hat{G} : XN]} (\tau'^t \chi_N)_N] \quad \text{(Mackey)} \]
\[ = \sum_{X, \tau} a_{X, \tau} \hat{G} : XN [\psi \overline{\psi}, (\tau \chi_N)_N] \quad \text{(}\psi \overline{\psi}\text{ is invariant in } \hat{G}) \]
\[ = \sum_{X, \tau} a_{X, \tau} \hat{G} : XN [\psi, \psi (\tau \chi_N)_N] \]
\[ = [\Gamma] \sum_{X, \tau} a_{X, \tau} \hat{G} : XN [\vartheta, \psi (\tau \chi_N)_N], \]

since \( \psi (\tau \chi_N)_N \) is a character with values in \( \mathbb{L} \). Since \( b \not\equiv 0 \mod p \), it follows that there is an \( \mathbb{L}_p \)-elementary subgroup \( X \) and a character \( \tau \) of an \( \mathbb{L}X \)-module, such that
\[ a_{X, \tau} \hat{G} : XN [\vartheta, \psi (\tau \chi_N)_N] \not\equiv 0 \mod p. \]

Fix such an \( X \) and \( \tau \) and set \( Y = X \cap N \). First we note that since \( \hat{G} / N \) is a \( p \)-group and \( |\hat{G} : XN| \not\equiv 0 \mod p \), it follows that \( \hat{G} = XN \).

Let \( \hat{\Gamma} = \text{Gal}(\mathbb{L}(\zeta) / \mathbb{L}) \) and set \( P = (X/Y) \times \hat{\Gamma} \). This group acts on the characters of \( N \) and of \( Y \). The character \( \tau_Y \) is invariant under the action of \( P \), since it is the restriction from a character of \( X \) which has values in \( \mathbb{L} \). Since \( \vartheta \) is semi-invariant in \( \hat{G} \), it follows that every \( P \)-conjugate of \( \vartheta \) is of the form \( \vartheta^\sigma \) with \( \sigma \in \Gamma = \text{Gal}(\mathbb{L}(\vartheta) / \mathbb{L}) \) and thus \( \psi \) is \( P \)-invariant, as is \( \psi_Y \), of course. Thus \( \psi_Y \tau_Y \) is invariant under \( P \).

For each \( \varphi \in \text{Irr } Y \), let \( S(\varphi) \) be the sum of the characters in the \( P \)-orbit of \( \varphi \). Since \( \psi_Y \tau_Y \) is invariant, we may write
\[ \psi_Y \tau_Y = \sum_{\varphi} c_{\varphi} S(\varphi), \quad (c_{\varphi} = [\psi_Y \tau_Y, \varphi]_Y) \]

where the sum runs over a set of representatives of the \( P \)-orbits of \( \text{Irr } Y \). It follows from
\[ 0 \not\equiv [\vartheta, \psi_Y \tau_Y] = [\vartheta_Y, \psi_Y \tau_Y] = \sum_{\varphi} c_{\varphi} [\vartheta_Y, S(\varphi)] \]

that there is \( \varphi \in \text{Irr } Y \) such that \( c_{\varphi} [\vartheta_Y, S(\varphi)] \not\equiv 0 \mod p \). After replacing \( \varphi \) by another character in its orbit (if necessary), we may assume that \( [\vartheta_Y, \varphi] \not\equiv 0 \mod p \).

For the rest of the proof, we fix a \( \varphi \in \text{Irr } Y \) such that
\[ [\psi_Y \tau_Y, \varphi][\vartheta_Y, S(\varphi)][\vartheta_Y, \varphi] \not\equiv 0 \mod p, \]

and we show that this \( \varphi \) has the desired properties. Of course we already have that
\[ [\vartheta, \varphi_N] = [\vartheta_Y, \varphi] \not\equiv 0 \mod p. \]

It remains to show (c) and (d).

The group \( P = (X/Y) \times \hat{\Gamma} \) (with \( \hat{\Gamma} = \text{Gal}(\mathbb{L}(\zeta) / \mathbb{L}) \) as above) is a \( p \)-group by our assumption. Write \( P_\varphi \) and \( P_\vartheta \) for the stabilizers of \( \varphi \) and \( \vartheta \) in \( P \). We claim
that \( P_\varphi = P_\vartheta \). First we show \( P_\vartheta \subseteq P_\varphi \). Since \( \vartheta \) is semi-invariant in \( X \), we have \( P_\vartheta = P_\varphi \). Thus \( P_\vartheta \) is normal in \( P \) and \( P/P_\vartheta \cong X/X_\vartheta \cong \Gamma \) is abelian. Thus every \( P_\vartheta \)-orbit contained in the \( P \)-orbit of \( \varphi \) has the same length \( |P_\vartheta : P_\vartheta \cap P_\varphi| \). Since \( [\vartheta_Y, \varphi^x] = [\vartheta_Y, \varphi^x] \) for \( y \in P_\vartheta \), it follows that \( |P_\vartheta : P_\vartheta \cap P_\varphi| \) divides
\[
[\vartheta_Y, S(\varphi)] = \sum_{x \in [P : P_\varphi]} [\vartheta_Y, \varphi^x] \neq 0 \mod p.
\]
Thus \( |P_\vartheta : P_\vartheta \cap P_\varphi| = 1 \) and \( P_\vartheta \subseteq P_\varphi \).

As we mentioned before, \( \psi \) is the \( P \)-orbit sum of \( \vartheta \) and \( \tau_Y \) is \( P \)-invariant. It follows that
\[
[\psi_Y \tau_Y, \varphi] = \sum_{x \in [P : P_\varphi]} [(\vartheta^x)_Y \tau_Y, \varphi] \neq 0 \mod p
\]
is divisible by the \( p \)-power \( |P_\varphi : P_\vartheta| \). Therefore, \( P_\varphi = P_\vartheta \) as claimed.

From \( P_\varphi \cap \hat{\Gamma} = P_\varphi \cap \Gamma \) it follows that a field automorphism over \( \mathbb{L} \) fixes \( \varphi \) if and only if it fixes \( \vartheta \). Thus \( \mathbb{L}(\vartheta) = \mathbb{L}(\varphi) \). Since \( \vartheta \) is semi-invariant in \( \hat{G} \), there is, for every \( x \in X \), an \( \alpha_x \in \Gamma \) such that \( \vartheta^{x \alpha_x} = \vartheta \). If \( \beta \in \hat{\Gamma} \) is any extension of \( \alpha_x \) to \( \mathbb{L}(\zeta) \), then \((xY, \beta) \in P_\vartheta = P_\varphi \). It follows that also \( \vartheta^{x \alpha_x} = \varphi \). The proof is finished. \( \square \)

7.3. Corollary. In the situation of Theorem 7.2, let \( \kappa: \hat{G} \to G \) be an epimorphism with kernel \( N \). Then \([\vartheta_\kappa, \kappa, \mathbb{L}] = [\varphi, \kappa_\mathbb{L}, \mathbb{L}]\).

\textbf{Proof.} It follows from Theorem 7.2 that Hypothesis 3.1 holds for
\[
\begin{array}{cccccc}
1 & \longrightarrow & Y & \longrightarrow & X & \longrightarrow \mathbb{K}_\mathbb{L} \longrightarrow \mathbb{G} & \longrightarrow 1 \\
& \downarrow & & \downarrow & & \downarrow & \\
1 & \longrightarrow & N & \longrightarrow & \mathbb{G} & \longrightarrow \mathbb{G} & \longrightarrow 1
\end{array}
\]
over the field \( \mathbb{L} \). Thus Theorem 3.5 applies and yields that
\[
[\varphi, \kappa_\mathbb{L}, \mathbb{L}] = [S] \cdot [\vartheta_\kappa, \kappa, \mathbb{L}],
\]
where \([S]\) is the equivalence class of the algebra \( S = (i \mathbb{L} N i)^Y \), with
\[
i = \sum_{\alpha \in \text{Gal}(\mathbb{L}(\vartheta) / \mathbb{L})} (e_{\vartheta e_{\varphi}})^{\alpha}.
\]
It remains to show that \([S] = 1\), in other words, \( S \) is a trivial \( G \)-algebra over \( \mathbb{L}(\vartheta) \).

Pick \( G \)-algebras \( A \in [\vartheta, \kappa, \mathbb{L}] \) and \( B \in [\varphi, \kappa_\mathbb{L}, \mathbb{L}] \) with \( A \otimes_{\mathbb{L}(\vartheta)} S \cong B \), as in the proof of Theorem 3.5. By Lemma 3.4, we know that \( S \) is central simple of dimension \( n^2 \) over \( \mathbb{L}(\vartheta) \), where \( n = [\vartheta_Y, \varphi] \neq 0 \mod p \). Since \( \mathbb{L}(\zeta) \) is a splitting field of all groups involved, it follows that \( A \otimes_{\mathbb{L}(\zeta)} S \) is a direct product of matrix rings over \( \mathbb{L}(\zeta) \), and \( A \otimes_{\mathbb{L}(\vartheta)} \mathbb{L}(\zeta) \) is a matrix ring over \( \mathbb{L}(\zeta) \). The same holds for \( B \), and thus \( S \otimes_{\mathbb{L}(\vartheta)} \mathbb{L}(\zeta) \cong \mathbb{M}_n(\mathbb{L}(\zeta)) \). Since \( [\mathbb{L}(\zeta) : \mathbb{L}(\vartheta)] \) is a power of \( p \) and \( n \) is prime to \( p \), it follows that \( S \cong \mathbb{M}_n(\mathbb{L}(\vartheta)) \).

The action of \( G \) on \( \mathbb{L}(\vartheta) \) extends naturally to an action of \( G \) on \( \mathbb{M}_n(\mathbb{L}(\vartheta)) \). Via some fixed isomorphism \( S \cong \mathbb{M}_n(\mathbb{L}(\vartheta)) \), we get an action of \( G \) on \( S \). We write
\[ \varepsilon : G \to \text{Aut } S \] for this action, that is we write \( s^{\varepsilon(g)} \) to denote the effect of this action. This action is different from the \( G \)-algebra action on \( S = (iL\mathcal{N}i)^Y \) we already have, but both actions agree on \( \mathbb{Z}(S) \cong \mathbb{L}(\vartheta) \).

It follows that for \( g \in G \), the map \( S \ni s \mapsto s^{\varepsilon(g^{-1})g} \) is an \( \mathbb{L}(\vartheta) \)-algebra automorphism of \( S \). Thus there is \( \sigma(g) \in S^* \) such that \( s^{\varepsilon(g^{-1})g} = \sigma(g)^{-1} s \sigma(g) \) for all \( s \in S \), or \( s^\vartheta = s^{\varepsilon(g)} \sigma(g) \). By comparing left and right hand side of
\[
\varepsilon(xy) \sigma(xy) = \sigma(xy) \varepsilon(x) \varepsilon(y) \sigma(y) = \varepsilon(xy) \varepsilon(x) \varepsilon(y) \sigma(y),
\]
we see that
\[
\sigma(x)^{\varepsilon(y)} \sigma(y) = \alpha(x, y) \sigma(xy) \text{ for some } \alpha(x, y) \in \mathbb{L}(\vartheta)^*.
\]
Thus \( \sigma : G \to S \cong M_n(\mathbb{L}(\vartheta)) \) is a twisted projective representation and \( \alpha \in Z^2(G, \mathbb{L}(\vartheta)^*) \).

Taking determinants in \( \sigma(x)^{\varepsilon(y)} \sigma(y) = \alpha(x, y) \sigma(xy) \) yields that \( \alpha^n \) is a coboundary. We know also that \( \alpha^{[G]} \) is a coboundary. Since \( |G| \) is a power of \( p \) and \( n \) is prime to \( p \), it follows that \( \alpha \) itself is a coboundary. So after multiplying \( \sigma \) with a coboundary, we may assume that \( \sigma(x)^{\varepsilon(y)} \sigma(y) = \sigma(xy) \) for all \( x, y \in G \).

Now we can show that \( S \) is a trivial \( G \)-algebra. Let \( V = \mathbb{L}(\vartheta)^n \) and view \( V \) as a right \( S \)-module via our fixed isomorphism \( S \cong M_n(\mathbb{L}(\vartheta)) \). We have to define a right \( G \)-module structure on \( V \) such that \( S \cong \text{End}_{\mathbb{L}(\vartheta)}(V) \) as \( G \)-algebras over \( \mathbb{L}(\vartheta) \).

The action of \( G \) on \( \mathbb{L}(\vartheta) \) defines an action of \( G \) on \( V \) which we denote by \( v \mapsto v^\vartheta \). It has the property
\[
(v s)^\vartheta = v^\vartheta s^{\varepsilon(g)} \text{ for } s \in S.
\]
Define \( v \circ g := v^\vartheta \sigma(g) \). Then
\[
(v \circ x) \circ y = (v^x \sigma(x))^y \sigma(y) = v^{xy} \sigma(x)^{\varepsilon(y)} \sigma(y) = v^{xy} \sigma(xy) = v \circ (xy)
\]
for \( v \in V \) and \( x, y \in G \), and \( (v \lambda) \circ g = (v \circ g) \lambda^\vartheta \) for \( v \in V \), \( \lambda \in \mathbb{L}(\vartheta) \) and \( g \in G \). Thus \( V \) is a right module over the crossed product \( \mathbb{L}(\vartheta) \backslash G \). The computation
\[
((v \circ g^{-1}) s) \circ g = (v^{g^{-1}} \sigma(g^{-1}) s)^\vartheta \sigma(g) = v \sigma(g^{-1})^{\varepsilon(g)} s^{\varepsilon(g)} \sigma(g)
\]
\[
= v \sigma(g)^{-1} s^{\varepsilon(g)} \sigma(g)
\]
\[
= v s^\vartheta
\]
shows that \( S \cong \text{End}_{\mathbb{L}(\vartheta)}(V) \) as \( G \)-algebras. This finishes the proof that \( [S] = 1 \) in \( \text{BrCliff}(G, \mathbb{L}(\vartheta)) \). \( \square \)

8. Proof of the Theorem for Elementary Groups

Let \( \mathbb{F} \) be a field of characteristic 0. We need the following easy lemma which states that semi-invariance is in some sense transitive:

8.1. Lemma. Let \( M, N \) be normal subgroups of \( \widehat{G} \) with \( M \leq N \). Suppose that \( \tau \in \text{Irr } M \) is \( \mathbb{F} \)-semi-invariant in \( N \) and that \( \vartheta \in \text{Irr } (N \mid \tau) \) is \( \mathbb{F} \)-semi-invariant in \( \widehat{G} \). Then \( \tau \) is \( \mathbb{F} \)-semi-invariant in \( \widehat{G} \).
Proof. Let \( g \in \hat{G} \). There is \( \alpha \in \text{Gal}(\mathbb{F}(\vartheta)/\mathbb{F}) \) such that \( \vartheta^{g\alpha} = \vartheta \). We may extend \( \alpha \) to \( \mathbb{F}(\vartheta, \tau) \). For simplicity, we denote this element of \( \text{Gal}(\mathbb{F}(\vartheta)/\mathbb{F}) \) by \( \alpha \), too.

Then \( \tau^{g\alpha} \) is a constituent of \( \vartheta^{g\alpha} = \vartheta \) and so there is \( n \in N \) such that \( \tau^{g\alpha} = \tau^n \).

Since \( \tau \) is semi-invariant in \( N \), there is \( \beta \in \text{Gal}(\mathbb{F}(\tau)/\mathbb{F}) \) such that \( \tau^{n\beta} = \tau \). Thus \( \tau^{g\alpha\beta} = \tau \). Since \( g \in \hat{G} \) was arbitrary, \( \tau \) is semi-invariant in \( \hat{G} \) over \( \mathbb{F} \).

Let \( \hat{G} \) be an \( \mathbb{F} \)-elementary group with respect to some prime \( p \). Then \( \hat{G} = PC \) where \( P \) is a Sylow \( p \)-subgroup and \( C \) is a normal cyclic subgroup of \( p' \)-order. In the next result, we will only need that \( \hat{G} \) has a normal abelian subgroup \( C \) such that \( \hat{G}/C \) is nilpotent. In this case, every irreducible character of \( \hat{G} \) is induced from a linear character of some subgroup containing the normal subgroup \( C \) [9, Theorem 6.22]. (This is even true if \( \hat{G}/C \) is supersolvable, but we will also need the nilpotence of \( \hat{G}/C \).)

8.2. Theorem. Let

\[
1 \longrightarrow N \longrightarrow \hat{G} \longrightarrow G \longrightarrow 1
\]

be an exact sequence of groups. Assume that there is a normal abelian subgroup \( C \leq N \) such that \( \hat{G}/C \) is nilpotent. Suppose that \( \vartheta \in \text{Irr} \ N \) is \( \mathbb{F} \)-semi-invariant in \( \hat{G} \). Then there are subgroups \( A \leq M \leq H \) of \( \hat{G} \) and a linear character \( \lambda \in \text{Lin} \ A \) such that

(a) \( A \triangleleft H \), \( \hat{G} = HN \) and \( C \leq A \leq M = H \cap N \),

(b) \( \vartheta = \lambda^N \),

(c) \( \mathbb{F}(\vartheta) = \mathbb{F}(\varphi) \), where \( \varphi = \lambda^M \),

(d) \( [\vartheta, \kappa, \mathbb{F}] = [\varphi, \kappa|_H, \mathbb{F}] \).

Our proof of this result follows closely the proof of Proposition 3.6 in Yamada’s book [28]. (Proposition 3.6 in Yamada’s book is the case \( N = \hat{G} \) (that is, \( G = 1 \)) of Theorem 8.2,(a)–(c).)

Proof of Theorem 8.2. Consider the set of all subgroups \( T \leq N \) such that

(1) \( C \leq T \leq \hat{G} \),

(2) there is \( \tau \in \text{Irr} \ T \) such that \( \vartheta = \tau^N \),

(3) for all \( n \in N \), there is \( \alpha \in \text{Gal}(\mathbb{F}(\tau)/\mathbb{F}) \) such that \( \tau^{n\alpha} = \tau \).
Let $T$ be minimal among the subgroups having Properties (1)–(3). (Note that $T = N$ is such a subgroup.) If the corresponding $\tau$ is linear, we set $A = T$, $\lambda = \tau$, $H = \hat{G}$ and $M = N$. Then $\varphi = \lambda^M = \vartheta$ and (a)–(d) of the proposition are trivially true. So in the case where $\tau(1) = 1$, the proof is finished.

Assume $\tau(1) > 1$. Our first goal is to find a proper subgroup $T_0 < T$ which is normal in $\hat{G}$ and such that $\tau$ is induced from a character $\tau_0 \in \text{Irr} T_0$. The non-linear character $\tau$ is induced from a linear character of some subgroup $B$ such that $C \leq B < T$. Let $S$ be a maximal subgroup of $T$ containing $B$, and thus $C$. Then $\tau$ is induced from some character of $S$. Since $S \triangleleft T$ (because $T/C$ is nilpotent), we see that $\tau_{T \setminus S} = 0$. Set

$$ U = \bigcap_{g \in \hat{G}} S^g. $$

Then $C \leq U \leq \hat{G}$. We claim that $\tau$ vanishes on $T \setminus U$. Since $\tau$ vanishes outside $S$, it follows that $\tau^g$ vanishes outside $S^g$ for $g \in \hat{G}$. But $\tau^g$ and $\tau$ are Galois conjugate (by Lemma 8.1), and thus $\tau$ vanishes outside $S^g$. Since $g \in \hat{G}$ was arbitrary, it follows that $\tau$ vanishes outside $U$ as claimed.

Since $U, T \leq \hat{G}$ and $\hat{G}/U$ is nilpotent, there exists $T_0 \leq \hat{G}$ such that $U \leq T_0 < T$ and $[T : T_0]$ is a prime $p$. Since $\tau$ vanishes outside $U$, it vanishes outside $T_0$. Since $|T : T_0| = p$, we see that $\tau = \tau_0^T$ for some $\tau_0 \in \text{Irr} T_0$. We have found a $T_0$ as wanted.

In the following, set

$$ \hat{G}_0 = \{ g \in \hat{G} \mid \exists \alpha \in \text{Gal}(\mathbb{F}(\tau_0)/\mathbb{F}): \tau_0^g = \tau_0 \}, $$

$$ N_0 = \hat{G}_0 \cap N \text{ and } \vartheta_0 = \tau_0^{N_0}. $$

Then $\vartheta_0^N = \tau_0^N = (\tau_0^T)^N = \tau^N = \vartheta$ and $\vartheta_0 \in \text{Irr}(N_0 \mid \tau_0)$ is the unique element with $\vartheta_0^N = \vartheta$ (Lemma 3.7). Corollary 3.8, applied to the extension

$$ 1 \longrightarrow N \longrightarrow \hat{G} \xrightarrow{\kappa} G \longrightarrow 1 $$

and the normal subgroup $T_0 \subseteq N$, yields the following hold:

(\begin{enumerate}
\item $\hat{G} = \hat{G}_0 N$,
\item $\vartheta = \vartheta_0^N$,
\item $\mathbb{F}(\vartheta) = \mathbb{F}(\vartheta_0)$,
\item $[\vartheta, \kappa, \mathbb{F}] = [\vartheta_0, \kappa_{\hat{G}_0}, \mathbb{F}].$
\end{enumerate})

(We can also show that $\hat{G} = \hat{G}_0 T$ by applying Corollary 3.8 to the extension $T \to \hat{G} \to \hat{G}/T$, but actually we will not need this for the proof.)

Note that (1) and (2) hold for $T_0$ and $\tau_0$. It follows from the minimality of $T$ that (3) does not hold for $T_0$, which means that $N_0 < N$. Thus we may apply induction (on $|N/C|$) to conclude that there are subgroups $A \leq M \leq H \leq \hat{G}_0$ and $\lambda \in \text{Lin} A$ such that Properties (a)–(d) hold with $\hat{G}$, $N$, $\vartheta$ replaced by $\hat{G}_0$, $N_0$, $\vartheta_0$.

It follows that $\hat{G} = \hat{G}_0 N = H N$ and $M = H \cap N_0 = H \cap \hat{G}_0 \cap N = H \cap N$ (see Figure 2), that $\vartheta = \vartheta_0^N = \lambda^N$, and that $\mathbb{F}(\vartheta) = \mathbb{F}(\vartheta_0) = \mathbb{F}(\varphi)$. Finally, we have that

$$ [\vartheta, \kappa, \mathbb{F}] = [\vartheta_0, \kappa_{\hat{G}_0}, \mathbb{F}] = [\varphi, \kappa_H, \mathbb{F}]. \square $$
8.3. **Corollary.** In the situation of Theorem 8.2, $[[\vartheta, \kappa, \mathbb{F}]] \in \mathcal{SC}_C(G, \mathbb{F}(\vartheta))$.

*Proof. By Proposition 4.4, it suffices to show that $[[\vartheta, \kappa, \mathbb{F}]] = \mathcal{M}(G, \mathbb{F}(\vartheta))$. By Theorem 8.2, we have $[[\vartheta, \kappa, \mathbb{F}]] = [[[\varphi, \kappa_H, \mathbb{F}]]]$, and it is obvious that the conditions in Theorem 8.2 yield that $[[\varphi, \kappa_H, \mathbb{F}]]$ is in $\mathcal{SC}_M(G, \mathbb{F}(\vartheta))$. (See Definition 4.1.) □

8.4. **Remark.** In fact, the construction from the proof of Theorem 8.2 itself yields an $M$ that also has the following properties:

(e) $M = \{n \in N \mid \exists \alpha \in \text{Gal}(\mathbb{F}(\lambda)/\mathbb{F}) : \lambda^{n\alpha} = \lambda\}$,

(f) $M/A \cong \text{Gal}(\mathbb{F}(\lambda)/\mathbb{F}(\vartheta))$.

*Proof. By Lemma 4.2, (e) together with $\lambda^M \in \text{Irr} M$ implies (f).

To see (e), let $N_0$ be as in the proof above and write $\tilde{M} = \{n \in N \mid \exists \alpha \in \text{Gal}(\mathbb{F}(\lambda)/\mathbb{F}(\vartheta)) : \lambda^{n\alpha} = \lambda\}$. By induction we may assume that $\tilde{M} \cap N_0 = M$. Since $\lambda^N = \vartheta \in \text{Irr} N$ and $\lambda^{N_0} = \vartheta_0 \in \text{Irr} N_0$, we know from (f) respective Lemma 4.2 that $\tilde{M}/A \cong \text{Gal}(\mathbb{F}(\lambda)/\mathbb{F}(\vartheta))$ and $M/A \cong \text{Gal}(\mathbb{F}(\lambda)/\mathbb{F}(\vartheta_0))$. But since $\mathbb{F}(\vartheta) = \mathbb{F}(\vartheta_0)$, it follows that $M = \tilde{M}$. The proof is finished. □

9. **Proof of the main theorem**

For the convenience of the reader, we summarize here how Theorem A from the introduction follows from the various results proved so far. We restate Theorem A, using the notation introduced in Section 4.

9.1. **Theorem.** Let $\mathbb{E}$ be a field on which the finite group $G$ acts. Then $\mathcal{SC}(G, \mathbb{E}) = \mathcal{SC}_C(G, \mathbb{E})$.

*Proof. Let $a = [[\vartheta, \kappa, \mathbb{F}]] \in \mathcal{SC}(G, \mathbb{E})$. We have to show that $a \in \mathcal{SC}_C(G, \mathbb{E})$. By Lemmas 5.1 and 5.2, we may assume that $a$ has $p$-power order and that $G$ is a $p$-group. By Corollary 6.6, it suffices to show that $[[\vartheta, \kappa, \mathbb{L}]] \in \mathcal{SC}_C(G, \mathbb{L}(\vartheta))$, where $\mathbb{L}$ is such that a splitting field of $\hat{G}$ and all its subgroups has $p$-power degree over $\mathbb{F}$.
L. Then Corollary 7.3 yields that we may assume that \( \hat{G} \) is \( L \)-elementary for the prime \( p \). In this case, Corollary 8.3 yields that \( [\hat{\vartheta}, \kappa, L] \in SC_c(G, L(\hat{\vartheta})) \).

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