CONICAL PLURISUBHARMONIC MEASURE AND NEW CROSS THEOREMS

VIỆT-ANH NGUYỄN

Abstract. We study the boundary behavior of the conical plurisubharmonic measure. As an application, we establish some new extension theorems for separately holomorphic mappings defined on cross sets.

1. Introduction and statement of the main results

Let $D$ be an open subset of $\mathbb{C}^n$ and $A \subset \partial D$. We suppose in addition that $D$ is locally $C^2$ smooth on $A$ (i.e. for any $\zeta \in A$, there exist an open neighborhood $U = U_\zeta$ of $\zeta$ in $\mathbb{C}^n$ and a real function $\rho = \rho_\zeta \in C^2(U)$ such that $D \cap U = \{z \in U : \rho(z) < 0\}$ and $d\rho(\zeta) \neq 0$). For $\zeta \in A$ and $1 < \alpha < \infty$, we consider the conical approach region

$A_\alpha(\zeta) := \{z \in D : |z - \zeta| < \alpha \cdot \text{dist}(z, T_\zeta)\},$

where $\text{dist}(z, T_\zeta)$ denotes the Euclidean distance from the point $z$ to the tangent hyperplane $T_\zeta$ of $\partial D$ at $\zeta$.

For any function $u$ defined on $D$, let

$\hat{u}(z) := \begin{cases} u(z), & z \in D, \\ \sup_{\alpha > 1} \limsup_{w \in A_\alpha(z), w \to z} u(w), & z \in \partial D. \end{cases}$

Next, consider the function $h_{A,D} := \sup_{u \in \mathcal{F}} u$, where

$\mathcal{F} := \{u \in PSH(D) : u \leq 1 \text{ on } D, \hat{u} \leq 0 \text{ on } A\}.$

Here $PSH(D)$ denotes the set of all functions plurisubharmonic on $D$. Then the conical plurisubharmonic measure of $A$ relative to $D$ is given by

$\omega(z, A, D) := h_{A,D}^*(z), \quad z \in D,$

where $u^*$ denotes the upper semicontinuous regularization of a function $u$.

A manifold $M \subset \mathbb{C}^n$ of class $C^2$ is said to be generic if, for every point $z \in M$, the complex linear hull of the tangent space $T_zM$ (to $M$ at $z$) coincides with the whole space $\mathbb{C}^n$.

The main purpose of this work is to investigate the boundary behavior of the conical plurisubharmonic measure in a special but important case, and thereafter to apply this study to the theory of separately holomorphic mappings. Now we are in the position to state the main result.

2000 Mathematics Subject Classification. Primary 32U, 32U05, 32D10, 32D15.

Key words and phrases. Cross theorem, holomorphic extension, plurisubharmonic measure.
Main Theorem. Let $M \subset \mathbb{C}^n$ be a generic manifold of class $C^2$ and $D$ a domain in $\mathbb{C}^n$ such that $M \subset \partial D$ and $D$ is locally $C^2$ smooth on $M$. Let $A \subset M$ be a measurable subset of positive measure. Then for all density points $z$ relative to $A$, $\hat{\omega}(z,A,D) = 0$.

This theorem describes the stable character of the the conical plurisubharmonic measure $\omega(\cdot, A, D)$ along the conical approach regions at all density points relative to $A$. It sharpens the previous results of A. Sadullaev (see [6]) and B. Coupet (see [1]) where the estimate $\omega(\cdot, A, D) < 1$ on $D$ was obtained. Our proof relies on the use of families of analytic discs attached to $M$ and on some fine estimates of plurisubharmonic functions.

This paper is organized as follows.

We begin Section 2 by collecting some results of the method of attaching analytic discs to a generic manifold. Next, we develop necessary estimates for the conical plurisubharmonic measure and then prove the Main Theorem. Section 3 concludes the article with various applications of the Main Theorem in the theory of separately holomorphic mappings.

Acknowledgment. The paper was written while the author was visiting the Université Pierre et Marie Curie, the Université Paris-Sud XI and the Korea Institute for Advanced Study (KIAS). He wishes to express his gratitude to these organizations.

2. Proof

A smooth generic manifold $M \subset \mathbb{C}^n$ is said to be totally real if $\dim_{\mathbb{R}} M = n$. In the first step we will focus on the case of totally real manifolds since their geometry is well-understood.

Now let $M \subset \mathbb{C}^n$ be a totally manifold of class $C^2$. We may assume without loss of generality that $0 \in M$ and $T_0M = \mathbb{R}^n$ (it suffices to perform an affine change of coordinates). $M$ is then defined in a neighborhood of $0 \in \mathbb{C}^n$ by the equation $z = x + ih(x)$, where $h$ is a function of class $C^2$ defined in a neighborhood $U$ of $0 \in \mathbb{R}^n$ with values in $\mathbb{R}^n$ satisfying $h(0) = 0$ and $dh(0) = 0$.

Let $\Delta$ be the open unit disc in $\mathbb{C}$ and $T := \partial \Delta$. A holomorphic disc is, by definition, a function $f$ in $C(\Delta) \cap O(\Delta)$. A holomorphic disc $f$ is said to be attached to $M$ on an arc $\gamma \subset T$ if $f(\gamma) \subset M$. Let $\Gamma$ be an open cone in $\mathbb{R}^n$ of vertex $0 \in \mathbb{R}^n$ and $U$ an open neighborhood of $0 \in \mathbb{C}^n$. A wedge $W$ of edge $M$ with the cone $\Gamma$ in $U$ is the set of the form

$$W = W_{\Gamma,U} := \{z \in U : \text{Im } z - h(\text{Re } z) \in \Gamma\}.$$

In what follows we fix two arbitrary open cones $\Gamma' \subset \Gamma''$ in $\mathbb{R}^n$ such that $\Gamma' \cap S^{n-1}$ is relatively compact in $\Gamma'' \cap S^{n-1}$, where $S^{n-1}$ is the unit sphere in $\mathbb{R}^n$. We also fix a neighborhood $U'$ of $0 \in \mathbb{R}^n$ such that $h$ is a function of class $C^2$ defined in a neighborhood $U'$ of $0 \in \mathbb{R}^n$ with values in $\mathbb{R}^n$ and $W_{\Gamma',U'} \subset D$. Fix a smooth function $\varphi$ defined on $\Delta$ harmonic on $\Delta$ such that $\varphi = 0$ on $\{e^{i\theta} \in T : |\theta| \leq \frac{\pi}{2}\}$
and \( \varphi < 0 \) on the remaining part of \( \mathbb{T} \) and \( \frac{\partial \varphi(s, \varphi' \theta)}{\partial s} |_{s=1, \theta=0} = 1 \). Here \([0, 1] \times [-\pi, \pi] \ni (s, \theta) \mapsto se^{i\theta} \in \Delta \).

For a (real) manifold \( M \) of \( \mathcal{C}^2 \) of dimension \( m \) let \( \text{mes}_M \) denote the \( m \)-dimensional Lebesgue measure on \( M \). When there is no fear of confusion we often write \( \text{mes} \) instead of \( \text{mes}_M \).

We recall the following result of B. Coupet in [1].

**Theorem 2.1.** Under the above hypothesis and notation there exist an open neighborhood \( U \subset U' \) of \( 0 \in \mathbb{R}^n \), an open neighborhood \( U \) of \( 0 \) in \( \mathbb{R}^{n-1} \), and open neighborhood \( V \) of \( 0 \) in \( \mathbb{R}^n \), a function \( G \) of class \( \mathcal{C}^1 \) defined on \( \Delta \times U \times (V \cap W_{r'_M}) \) with values in \( W_{r''_M} \), \( G \) being holomorphic with respect to the variable \( w \in \Delta \) and an open arc \( \gamma \subset \mathbb{T} \) with \( 1 \in \gamma \) and a nonzero real number \( \beta \) such that the following conditions are satisfied:

(i) For \((\zeta, \tau) \in U \times V\), the holomorphic disc \( G(\cdot, \zeta, \tau) \) is attached to \( M \) on \( \gamma \).

Moreover, \( G(\cdot, 0, 0) = 0 \) on \( \Delta \).

(ii) Given \( \zeta \in V \), the mapping \( G_{\tau} : (se^{i\theta}, \zeta) \mapsto G(se^{i\theta}, \zeta, \tau) \) defines, for each fixed \( \frac{1}{2} \leq s \leq 1 \), a diffeomorphism of \( \{se^{i\theta} : e^{i\theta} \in \gamma\} \times U \) into \( \mathbb{C}^n \). In particular, \( G_{\tau} \) maps \( \{e^{i\theta} : e^{i\theta} \in \gamma\} \times U \) onto an open neighborhood of \( 0 \) in \( M \). Moreover,

\[
\frac{\partial \text{Re} G_{\tau}(se^{i\theta}, \zeta)}{\partial (\theta, \zeta)}(\theta, \zeta) = \beta \frac{\partial \varphi}{\partial s}(se^{i\theta})|\tau| \cdot \text{Id}_\theta + \text{Id}_\zeta + o(|\tau|), \quad e^{i\theta} \in \gamma, \; \zeta \in U \text{ and } |\zeta| \leq 4|\tau|,
\]

where the left hand side is the Jacobian of \( \text{Re} G_{\tau} \) with respect to the variables \((\theta, \zeta)\).

(iii) There exists a conformal map \( \psi \) which maps \( \Delta \) onto a Jordan domain \( E \subset \Delta \) with smooth boundary such that \( \gamma \subset \partial E \) and \( \psi(1) = 1 \) and that

\[
\frac{\partial \text{Im} G(se^{i\theta}, \zeta, \tau)}{\partial \tau} = \varphi(\psi(se^{i\theta})).\text{Id}_\tau + o(\tau), \quad e^{i\theta} \in \gamma, \; 1 \leq |\zeta| \leq 4|\tau|.
\]

**Proof.** It follows implicitly from the construction of the function \( G \) and its properties given in Théorème 2 in [1].

Now we arrive at the

**Proof of the Main Theorem when \( M \) is totally real.** Suppose without loss of generality that \( 0 \) is a point of density of \( A \) in \( M \). The idea is to use families of holomorphic discs attached to \( M \) which parametrize a open neighborhood of \( 0 \) in \( M \). These families are supplied by Theorem 2.1.

For \( r > 0 \) let \( U_r := \{ \zeta \in U : |\zeta| < r \} \). Using the hypothesis that \( 0 \) is a point of density relative to \( A \) and applying Part (ii) of Theorem 2.1 we have

\[
(2.1) \quad \lim_{r \to 0} \inf_{r \in \mathbb{R}^+ : |r|=1} \frac{\text{mes} \left(A \cap G_{r'\tau}(\gamma \times U_r)\right)}{\text{mes} \left(G_{r'\tau}(\gamma \times U_r)\right)} = 1.
\]

We need the following elementary lemma.

**Lemma 2.2.** Let \( \gamma \) be an open arc in \( \mathbb{T} \) with \( e^{i\theta} \in \gamma \) for all \( \theta \in (-2\eta, 2\eta) \), where \( 0 < \eta < \frac{\pi}{2} \) is a fixed number. Then there exists \( 1 < C < \infty \) with the following...
property. If \( u \) is a subharmonic function defined in \( \Delta \) with \( u \leq 1 \) on \( \Delta \) and \( \dot{u} \leq 0 \) on \( B \), where \( B \) is a measurable subset of \( \gamma \) with \( \frac{\text{mes}(B)}{\text{mes}(\gamma)} \geq 1 - \frac{1}{N} \) for some \( N > 1 \), then
\[
\sup_{\rho=1-\frac{\eta}{N}, \mid \zeta \mid < \eta} u(\rho e^{i\zeta}) \leq \frac{C}{\sqrt{N}}.
\]

Proof. Observe that for \( \rho \in [0, 1] \) and \( \zeta \in [-\pi, \pi] \),
\[
u(\rho e^{i\zeta}) \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(1 - \rho^2)}{|e^{i\theta} - \rho e^{i\zeta}|^2} d\theta,
\]
where
\[
1_B(\theta) := \begin{cases} 0, & e^{i\theta} \in B, \\ 1, & e^{i\theta} \notin B. \end{cases}
\]

Therefore, an easy estimate shows that for \( \rho = 1 - \frac{1}{N} \) and \( |\zeta| < \eta \),
\[
u(\rho e^{i\zeta}) \leq \frac{1}{2\pi} \int_{-\pi}^{\frac{\eta}{2}} \frac{(1 - \rho^2)}{|1 - \rho|^2} d\theta + \frac{1}{2\pi} \int_{|\theta| > 2\eta} \frac{(1 - \rho^2)}{|e^{i\theta} - \rho e^{i\zeta}|^2} d\theta \leq \frac{C}{\sqrt{N}}.
\]

For \( 0 < \epsilon < 1 \) let \( \delta_\epsilon := \{(1 - \epsilon)e^{i\theta} : \theta \in (-\eta, \eta)\} \). Applying Lemma to the holomorphic discs \( G(\cdot, \zeta, \tau) \) attached to \( M \) on \( \gamma \) which are supplied by Theorem 2.1 we deduce from (2.1) that for every \( 0 < \epsilon < 1 \),
\[
\lim_{r \to 0} \inf_{0 \neq u \in \mathcal{F}, \tau \in \mathbb{R}^n : |r| = 1} \frac{\text{mes}(\{z \in G(\delta_\epsilon, U_r, r\tau) : u(z) \leq \epsilon\})}{\text{mes}(\{z \in G(\delta_\epsilon, U_r, r\tau)\})} = 1.
\]

For \( 1 < \alpha < \infty \) let \( \Gamma_\alpha \) denote the open cone
\[
\left\{ y = (y_1, \ldots, y_n) \in \mathbb{R}^n : \sqrt{y_1^2 + \cdots + y_n^2} < \alpha y_n \right\}.
\]

Suppose that \( \Gamma' := \Gamma_l \) for some \( 1 < l < \infty \). For \( 0 < r < r' < \infty \) let \( \Gamma'(r, r') := \{y \in \Gamma' : r \leq |y_n| \leq r'\} \). Define, for \( 0 < \epsilon < 1 \) the \( \epsilon\)-approximate function \( G_\epsilon \) of the function \( G|_{\delta_\epsilon \times U \times V} \) as follows
\[
G_\epsilon((1-\epsilon)e^{i\theta}, \zeta, \tau) := \left( \beta \frac{\partial \varphi}{\partial s} ((1-\epsilon)e^{i\theta})|\tau|, \zeta, \varphi((1-\epsilon)e^{i\theta}) \cdot \tau \right), \quad \theta \in (-\eta, \eta), \ (\zeta, \tau) \in U \times V.
\]

By Part (i) of Theorem 2.1 \( G(\cdot, 0, 0) = 0 \) on \( \overline{\Delta} \). This, combined with Part (ii) and (iii) of Theorem 2.1 and the explicit formula for \( G_\epsilon \), implies that when \( \tau \) is taken in a fixed compact set, the set \( \{z \in G(\delta_\epsilon, U_r, r\tau)\} \) is, in some sense, equal to the set \( \{z \in G_\epsilon((1-\epsilon)e^{i\theta}), \zeta, \tau) \in U \times V \} \) as \( r \to 0 \).

We now study the shape of the latter set. Using the property of \( \varphi \) and \( \psi \) we see that there exist finite positive constants \( c_1, c_2, c_3, c_4 \) such that for \( \epsilon > 0 \) small enough and for \( \theta \in (-\eta, \eta) \),
\[
c_1 < \frac{\partial \varphi}{\partial s} ((1-\epsilon)e^{i\theta}) < c_2.
\]
and

\[-c_4 \epsilon < \varphi((1 - \epsilon)e^{i\theta}) < -c_3 \epsilon.\]

By shrinking \(\eta\) (if necessary) we may assume without loss of generality that

\[1 < \frac{c_2}{c_1}, \frac{c_4}{c_3} < 2.\]

For \(0 < r, \epsilon < 1\) define the cubics

\[K'_{r,\epsilon} := [-c_1 \eta \frac{r}{4}, c_1 \eta \frac{r}{4}] \times [-r, r]^{n-1} \times \Gamma'\left(\epsilon c_3 \frac{r}{2}, 4 \epsilon c_3 r\right),\]

\[K''_{r,\epsilon} := [-4c_2 \eta r, 4c_2 \eta r] \times [-r, r]^{n-1} \times \Gamma'\left(\epsilon c_3 \frac{r}{4}, 4 \epsilon c_4 r\right).\]

The explicit formula of \(G^s\) and the above discussion together imply that

\[K'_{r,\epsilon} \subset \left\{ z \in G^s(\delta, U_r, -\Gamma'\left(\frac{r}{4}, 4r\right)) : u(z) \leq \epsilon \right\} \subset K''_{r,\epsilon}\]

as \(r \to 0\). This, combined with estimate (2.2), implies that for every \(0 < \epsilon < 1\),

\[
\liminf_{r \to 0} \frac{\mes\left(\{z \in K'_{r,\epsilon} : u(z) \leq \epsilon\}\right)}{\mes(K'_{r,\epsilon})} = 1.
\]

Now we need the final lemmas.

**Lemma 2.3.** For every \(0 < \epsilon < 1\) and \(0 < a, b, c, d < \infty\), there exists a finite constant \(C > 1\) that depend only on \(\epsilon\) and the quotients \(\frac{b}{a}, \frac{c}{a}, \frac{d}{a}\), with the following property. Consider the domains

\[H' := \{z \in \mathbb{C}^n : x_1 \in (-2d, 2d), x_2, \ldots, x_n \in (-2c, 2c), y_1, \ldots, y_{n-1} \in (-2b, 2b), y_n \in \left(\frac{a}{2}, 4a\right)\},\]

\[H := \{z \in \mathbb{C}^n : x_1 \in (-d, d), x_2, \ldots, x_n \in (-c, c), y_1, \ldots, y_{n-1} \in (-b, b), y_n \in (a, 2a)\}.
\]

Then for every \(u \in \mathcal{PSH}(H')\) and every \(0 < \epsilon < 1\) such that \(u \leq 1\) on \(H'\) and that

\[
\frac{\mes\left(\{z \in H' : u(z) \leq \frac{\epsilon}{2}\}\right)}{\mes(H')} > 1 - \frac{1}{C},
\]

we have \(u < \epsilon\) on \(H\).

**Proof.** Observe that there exists the maximum number \(0 < r < \infty\) such that the ball \(B(z, r)\) centered at \(z\) with radius \(r\) in \(\mathbb{C}^n\) is contained in \(H'\) for all \(z \in \overline{H}\). By the sub-mean property of plurisubharmonic functions we have

\[
u(z) \leq \frac{1}{\mes(B(z, r))} \int_{B(z, r)} u(w)dw, \quad z \in H.
\]

Setting \(C' := \frac{\mes(H')}{\mes(B(z, r))}\), we see that \(C'\) depends only on the quotients \(\frac{b}{a}, \frac{c}{a}, \frac{d}{a}\). We may choose \(1 < C < \infty\) large enough so that \((1 - \frac{C'}{C}) \frac{\epsilon}{2} + \frac{C'}{C} < \epsilon\). With this choice the desired conclusion follows immediately from (2.4). \(\Box\)
Lemma 2.4. For every \( \alpha > 1 \) and \( 0 < \epsilon < 1, 0 < \lambda, \mu < \infty \), there exists a finite constant \( C > 1 \) with the following property. For all \( a > 0 \) consider the domains

\[
K' := \left\{ z \in \mathbb{C}^n : x_1 \in (-2\lambda a, 2\lambda a), x_2, \ldots, x_n \in (-2\mu a, 2\mu a), y \in \Gamma_{8\sqrt{n}a}, y_n \in \left(\frac{a}{2}, 4a\right) \right\},
\]
\[
K := \left\{ z \in \mathbb{C}^n : x_1 \in (-\lambda a, \lambda a), x_2, \ldots, x_n \in (-\mu a, \mu a), y \in \Gamma_a, y_n \in (a, 2a) \right\}.
\]

Then for every \( u \in \mathcal{PSH}(K') \) such that \( u \leq 1 \) on \( K' \) and that

\[
\frac{\text{mes}\left(\{z \in K' : u(z) \leq \frac{\epsilon}{2}\}\right)}{\text{mes}(K')} > 1 - \frac{1}{C},
\]

we have \( u < \epsilon \) on \( K \).

Proof. Applying Lemma 2.3 to the case where \( b := 2\alpha a \) and observing that \( H' \subset K' \), \( K \subset H \), the desired conclusion follows. \( \square \)

Now we are in the position to complete the proof of the Main Theorem. Fix arbitrary \( 0 < \epsilon_0 < 1 \) and \( 1 < \alpha < \infty \). We only need to show that there exists a sufficiently small open neighborhood \( U \) of \( 0 \in \mathbb{C}^n \) such that \( u \leq \epsilon_0 \) on the wedge \( W_{\Gamma, \mu} \) for all \( u \in \mathcal{F} \), where \( \Gamma := \Gamma_a \). We choose \( \Gamma' := \Gamma_{8\sqrt{n}a} \), and let \( \Gamma'' \) be an arbitrary open cone in \( \mathbb{R}^n \) such that such that \( \Gamma'' \cap \mathbb{S}^{n-1} \) is relatively compact in \( \mathbb{R}^n \). Consequently, we may apply Theorem 2.1.

Let \( C_0 \) be the constant obtained from Lemma 2.4 applied to \( K' := K'_{r, \frac{\epsilon}{2}} \). Observe here that \( C_0 \) does not depend on \( r \). Using estimate (2.3) with \( \epsilon := \frac{\epsilon_0}{2} \), we may choose \( r_0 > 0 \) small enough such that for all \( 0 < r < r_0 \),

\[
\frac{\text{mes}\left(\{z \in K'_{r, \epsilon} : u(z) \leq \epsilon\}\right)}{\text{mes}(K'_{r, \epsilon})} > 1 - \frac{1}{C_0}, \quad \forall u \in \mathcal{F}.
\]

Since the hypothesis of Lemma 2.4 is now satisfied, we conclude that \( u(0, y) \leq \epsilon_0 \) for all \( y \in \Gamma \) with \( |y| < r_0 \). \( \square \)

The general case where \( M \subset \mathbb{C}^n \) is merely a generic manifold is, in principle, not difficult. We only need to parametrize \( M \) by a family of totally real manifolds \( M_\eta \), where the parameter \( \eta \) belongs to an open neighborhood of \( 0 \in \mathbb{R}^\dim(M) - n \). We leave the details to the interested reader.

3. Applications

Let \( X \) be an arbitrary complex manifold and \( D \subset X \) an open subset. We say that a set \( A \subset \partial D \) is locally contained in a generic manifold if there exist an (at most countable) index set \( J \neq \emptyset \), a family of open subsets \( (U_j)_{j \in J} \) of \( X \) and a family of generic manifolds \( (M_j)_{j \in J} \) such that \( A \cap U_j \subset M_j, j \in J \), and that \( A \subset \bigcup_{j \in J} U_j \). The dimensions of \( M_j \) may vary according to \( j \in J \). Given a set \( A \subset \partial D \) which is locally contained in a generic manifold, we say that \( A \) is of positive size if under the above notation \( \sum_{j \in J} \text{mes}_A(M_j) > 0 \). A point \( a \in A \) is said to be a density point relative to \( A \) if it is a density point relative to \( A \cap U_j \) on \( M_j \) for some \( j \in J \). Denote by \( A' \) the set of density points relative to \( A \).
Suppose now that $A \subset \partial D$ is of positive size. In what follows we keep the terminology and the notation introduced in Section 2 of [3]. We equip $D$ with the system of conical approach regions supported on $A$. The Main Theorem says that $A$ is locally pluriregular at all density points relative to $A$. Observe that $\text{mes}_{M_j} ((A \setminus \hat{A}) \cap U_j) = 0$ for $j \in J$. Therefore, it is not difficult to show that $A'$ is locally pluriregular. Choose an increasing sequence $(A_n)_{n=1}^{\infty}$ of subsets of $A$ such that $A_n \cap U_j$ is closed and $\text{mes}_{M_j} ((A \setminus \bigcup_{n=1}^{\infty} A_n) \cap U_j) = 0$ for $j \in J$. Observe that $A'_n$ is locally pluriregular, $A'_n \cap U_j \subset A$ for $j \in J$ and that $\hat{A} := \bigcup_{n=1}^{\infty} A'_n$ is locally pluriregular. Consequently, it follows from Definition 2.3 in [3] that

$$\tilde{\omega}(z, A, D) \leq \omega(z, \hat{A}, D), \quad z \in D.$$ 

Since $\text{mes}_{M_j} ((A' \setminus \hat{A}) \cap U_j) = 0$ for $j \in J$, we deduce from the Main Theorem that

$$\omega(z, \hat{A}, D) \leq \omega(z, A', D), \quad z \in D.$$ 

In summary, we have shown that

$$\tilde{\omega}(z, A, D) \leq \omega(z, A', D), \quad z \in D.$$ 

This estimate, combined with Theorem A in [3], implies the following results which have been stated in Theorem 10.4 and 10.5 in [3].

**Theorem 3.1.** Let $X, Y$ be two complex manifolds, let $D \subset X$, $G \subset Y$ be two domains, and let $A$ (resp. $B$) be a subset of $\partial D$ (resp. $\partial G$). $D$ (resp. $G$) is equipped with a system of conical approach regions $(A_\alpha(\zeta))_{\zeta \in \bar{D}, \alpha \in I_\zeta}$ (resp. $(A_\beta(\eta))_{\eta \in \bar{G}, \beta \in I_\eta}$) supported on $A$ (resp. on $B$). Suppose in addition that $A$ and $B$ are of positive size. Let $Z$ be a complex analytic space possessing the Hartogs extension property. Define

$$W' := X(A', B'; D, G),$$
$$\hat{W'} := \left\{ (z, w) \in D \times G : \omega(z, A', D) + \omega(w, B', G) < 1 \right\},$$

where $A'$ (resp. $B'$) is the set of density points relative to $A$ (resp. $B$).

Then, for every mapping $f : W \to Z$ which satisfies the following conditions:

- $f \in C_s(W, Z) \cap O_s(W^o, Z)$;
- $f$ is locally bounded;
- $f|_{A \times B}$ is continuous,

there exists a unique mapping $\hat{f} \in O(\hat{W'}, Z)$ which admits $A$-limit $f(\zeta, \eta)$ at every point $(\zeta, \eta) \in W \cap \hat{W'}$.

If, moreover, $Z = \mathbb{C}$ and $|f|_W < \infty$, then

$$|\hat{f}(z, w)| \leq |f|_{A \times B}^{1-\omega(z, A', D) - \omega(w, B', G)} |f|_W^{\omega(z, A', D) + \omega(w, B', G)}, \quad (z, w) \in \hat{W'}.$$

The second application is a very general mixed cross theorem.
Theorem 3.2. Let $X$, $Y$ be two complex manifolds, let $D \subset X$, $G \subset Y$ be two domains, let $A$ be a subset of $\partial D$, and let $B$ be a subset of $G$. $D$ is equipped with the system of conical approach regions $(A_\alpha(\zeta))_{\zeta \in \overline{D}, \alpha \in I_\zeta}$ supported on $A$ and $G$ is equipped with the canonical system of approach regions $(A_\beta(\eta))_{\eta \in \overline{G}, \beta \in I_\eta}$. Suppose in addition that $A$ is of positive size. Let $Z$ be a complex analytic space possessing the Hartogs extension property. Define

$$W' := \mathbb{X}(A', B^*; D, G),$$

$$\hat{W}' := \left\{ (z, w) \in D \times G : \omega(z, A', D) + \omega(w, B^*, G) < 1 \right\},$$

where $A'$ is the set of density points relative to $A$, and $B^*$ denotes the set of all points in $B \cap G$ at which $B$ is locally pluriregular.

Then, for every mapping $f : W \rightarrow Z$ which satisfies the following conditions:

- $f \in C_s(W, Z) \cap O_s(W^0, Z)$;
- $f$ is locally bounded along $A \times G$,

there exists a unique mapping $\hat{f} \in \mathcal{O}(\hat{W}', Z)$ which admits $A$-limit $f(\zeta, \eta)$ at every point $(\zeta, \eta) \in W \cap W'$.

If, moreover, $Z = \mathbb{C}$ and $|f|_W < \infty$, then

$$|\hat{f}(z, w)| \leq |f|_W^{1 - \omega(z, A', D) - \omega(w, B^*, G)} |f|^{\omega(z, A', D) + \omega(w, B^*, G)}_W, \quad (z, w) \in \hat{W}' .$$

The Main Theorem also implies Corollary 2 and 3 which were stated without proof in [4]. These corollaries generalize Theorem 3.1 and 3.2 to the case where some pluripolar or thin singularities are allowed (see [2] or [5] for more details on this issue).

References

[1] B. Coupet, Construction de disques analytiques et régularité de fonctions holomorphes au bord, Math. Z. 209 (1992), no. 2, 179–204.
[2] M. Jarnicki and P. Pflug, A general cross theorem with singularities, Analysis (Munich), 27 (2007), no. 2-3, 181–212.
[3] V.-A. Nguyễn, A unified approach to the theory of separately holomorphic mappings, Ann. Scuola Norm. Sup. Pisa Cl. Sci., (2008), serie V, Vol. VII(2), 181–240.
[4] V.-A. Nguyễn, Recent developments in the theory of separately holomorphic mappings, arXiv:math.CV.0901.1991.
[5] V.-A. Nguyễn and P. Pflug, Cross theorems with singularities, arXiv:math.CV.0901.3030.
[6] A. Sadullaev, A boundary uniqueness theorem in $\mathbb{C}^n$ (Russian), Mat. Sb. (N.S.) 101(143) (1976), no. 4, 568–583. English translation: Math. USSR-Sb. 30 (1976), no. 4, 501–514 (1978).

Viêt-Anh Nguyễn, VIETNAMESE ACADEMY OF SCIENCE AND TECHNOLOGY, INSTITUTE OF MATHEMATICS, DEPARTMENT OF ANALYSIS, 18 HOANG QUOC VIET ROAD, CAU GIAY DISTRICT, 10307 HANOI, VIETNAM

E-mail address: nvanh@math.ac.vn

Current address: SCHOOL OF MATHEMATICS, KOREA INSTITUTE FOR ADVANCED STUDY, 207-43CHEONGRYANGNI-2DONG, DONGDAEMUN-GU, SEOUL 130-722, KOREA

E-mail address: vietanh@kias.re.kr