A generalization of carries process and a relation to riffle shuffles

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Abstract

As a continuation to our previous work [6], we consider a generalization of carries process. Our results are: (i) right eigenvectors of the transition probability matrix, (ii) correlation of carries between different steps, and (iii) a relation to generalized riffle shuffles.

1 Introduction

1.1 Background and definition

Carries process is the Markov chain of carries in adding array of numbers. It was Holte [4] who first studied the carries process, and he found many beautiful properties, e.g., the eigenvalues of the transition probability matrix consist of negative powers of the base \( b \), the eigenvectors are independent of \( b \), and Eulerian numbers appear in the stationary distribution. Diaconis-Fulman [1, 2, 3] found connections to different subjects, e.g., the carries process has the same distribution of the descent of the riffle shuffle, and the left eigenvectors of the transition probability coincides with the Foulkes character table of \( S_n \).

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In [6], we considered the generalization of the carries process in the sense that (i) we take various digit sets, and (ii) we also consider negative base, and studied the same properties as was done by [4, 2]. There we obtained (i) the left eigenvectors of the transition probabilities, (ii) the limit theorem which yields the distribution of the sum of i.i.d. uniformly distributed random variables on $[0, 1]$. This paper is the continuation of [6]; here we study (i) the right eigenvectors of the transition probabilities, which yields the correlation of carries of different steps, and (ii) a relation to generalized riffle shuffles on the colored permutation group. [7] is a review article of our results obtained so far.

In what follows, we first recall the definitions and some results in [6] (subsection 1.2), and then state our results of this paper (subsection 1.3). To simplify the statements, we shall discuss the positive/negative base simultaneously. Let $\pm b \in \mathbb{Z} (b \geq 2)$ be the base and let $D_d := \{d, d+1, \cdots, d+b-1\}$ be the digit set such that $1-b \leq d \leq 0$ to have $0 \in D_d$. Then any $x \in \mathbb{N}$ has the unique representation

$$x = a_N (+b)^N + a_{N-1} (+b)^{N-1} + \cdots + a_0, \quad a_k \in D_d,$$

$$x = a'_N (-b)^N + a'_{N-1} (-b)^{N-1} + \cdots + a'_0, \quad a'_k \in D_d.$$

In adding $n$ numbers under this representation, let $C_{k-1}^\pm$ be the carry from the $(k-1)$-th digit which belongs to a set $\mathcal{C}(\pm b, n)$ to be specified later in Proposition [41]. In the $k$-th digit, we take $X_{1,k}, \cdots, X_{n,k}$ uniformly at random from $D_d$ and then the carry $C_k^\pm \in \mathcal{C}(\pm b, n)$ to the $(k+1)$-th digit is determined by the following equation.

$$C_{k-1}^\pm + X_{1,k} + \cdots + X_{n,k} = C_k^\pm (+b) + r, \quad r \in D_d.$$

The process $\{C_k^\pm\}_{k=0}^\infty$ is Markovian with state space $\mathcal{C}(\pm b, n)$, which we call the $n$-carries process over $(\pm b, D_d)$. Holte’s carries process corresponds to the case where the base is positive and $d = 0$. 

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1.2 Our previous results

In this subsection we recall some results in [6] which is necessary to state the results in this paper. Let

\[ l = l(\pm b, d) := \begin{cases} \frac{d}{b} & ((+b)\text{-case}) \\ \frac{b+d}{b+1} & ((-b)\text{-case}) \end{cases} \]

Then the carry set \( C(\pm b, n) \) is explicitly given by Proposition 1.1 below.

**Proposition 1.1**

1. The carry set \( C(\pm b, n) \) in the \( n \)-carries process over \( (\pm b, D_d) \) is given by

\[ C(\pm b, n) = \{ s, s+1, \cdots, t \} \]

where

\[ s := \lfloor (n-1)l \rfloor = -\lceil (n-1)(-l) \rceil, \quad t := \lceil (n-1)(l+1) \rceil. \]

2. The number of elements of \( C(\pm b, n) \) is

\[ \#C(\pm b, n) = \begin{cases} n & ((n-1)l \in \mathbb{Z}) \\ n+1 & ((n-1)l \notin \mathbb{Z}) \end{cases} \]

\( \#S \) is the number of elements of a finite set \( S \). We remark that if \((n-1)l \notin \mathbb{Z}\), \( \#C(\pm b, n) \) is larger than that of the Holte’s carries process.

We introduce the following notation which is an important parameter to describe our results.

\[ p = p(\pm b, d, n) := \frac{1}{1 - \langle (n-1)l \rangle} = \begin{cases} 1 & ((n-1)l \in \mathbb{Z}) \\ (n-1)(-l)^{-1} & ((n-1)l \notin \mathbb{Z}) \end{cases} \]

where \( \langle x \rangle := x - \lfloor x \rfloor \) is the fractional part of \( x \). \( \#C(b, n) = n \) if and only if \( p = 1 \), including the case of Holte’s carries process. For simplicity, we shall modify the suffix of the transition probability such that it ranges from 0 to \( \#C(\pm b, n) - 1 \).

\[ \tilde{P}_{i,j}^\pm := P_{i+s,j+s}^\pm = P \left( C_1^\pm = j + s \mid C_0^\pm = i + s \right) \]

where \( i, j = 0, 1, \cdots, \#C(\pm b, n) - 1 \), and \( s := \min C(\pm b, n) = \lfloor (n-1)l \rfloor \). Then \( \tilde{P}^\pm \) is computed as follows.
Proposition 1.2

\[ \tilde{P}_{ij}^\pm = \frac{1}{b^n} \sum_{r \geq 0} (-1)^r \binom{n + 1}{r} \binom{n + A(i,j) - br}{n} 1(A(i,j) - br \geq 0) \]

\[ A(i,j) := \begin{cases} (j + \frac{1}{p}) b - (i + \frac{1}{p}) & ((+b) - case) \\ (-j + 1 - \frac{1}{p}) b - (i + \frac{1}{p}) + nb & ((-b) - case) \end{cases} \]

\[ i,j = 0,1,\ldots, \#C(\pm b,n) - 1. \]

where \( 1(E) \) is the indicator function of the event \( E \), that is, \( 1(E) = 1 \) if \( E \) is true and \( 1(E) = 0 \) otherwise.

When \( p = 1 \) (resp. \( p = 2 \)), \( \tilde{P}^\pm \) coincides with that of Holte’s carries process (resp. with that of the type B process discussed in [2]). The matrix \( \tilde{P}^\pm = \{\tilde{P}_{ij}^\pm\} \) is determined by the triple \((\pm b,n,p)\). It is clearly not true in general that for given \((\pm b,n,p)\) we can find \( d \) with \(-(b-1) \leq d \leq 0\) such that \( \tilde{P}^\pm \) is the transition probability of the \( n \)-carries process over \((\pm b,D_d)\). However, there are some cases where \( \tilde{P} \) is a stochastic matrix, even if it does not correspond to carries processes.

Proposition 1.3 (1) if \( p \geq 1 \) and \( \frac{(\pm b)-1}{p} \in \mathbb{Z} \), then \( \tilde{P}^\pm \) is a stochastic matrix,

(2) if \( p < 1 \) and \( \frac{(\pm b)-1}{p} \in \mathbb{Z} \), then \( \tilde{P}^\pm \) is not a stochastic matrix.

Proposition 1.3 is proved in Appendix. We shall henceforth assume that the condition (1) of Proposition 1.3 is always satisfied, so that \( \tilde{P}^\pm \) defines a Markov chain \( \{\kappa_{i}^\pm\}_{i=0}^{\infty} \) on \( \{0,1,\ldots, \#C(\pm b,n) - 1\} \); we call it \((\pm b,n,p)\)-carries process. The left eigenvectors of \( \tilde{P} \) are given by Theorem 1.4 below.

Theorem 1.4 The eigenvalues and the left eigenvectors of \( \tilde{P}^\pm \) are given by

\[ \tilde{P}^\pm = L_p^{-1} D_b^\pm L_p \]

where \( D_b^\pm := \text{diag} \left( 1, \left( \pm \frac{1}{b} \right), \ldots, \left( \pm \frac{1}{b} \right)^{\#C(\pm b,n)-1} \right) \)

\[ L_p := \{v_{ij}^{(p)}(n)\}_{0 \leq i,j \leq \#C(\pm b,n)-1} \]

\[ v_{ij}^{(p)}(n) := \sum_{r=0}^{j} (-1)^r \binom{n + 1}{r} \binom{p(j - r) + 1}{n - i}. \]
Remark 1.1 When $p = 2$, $\{v_{0,k}^{(2)}(n)\}_{k=0}^n$ is called the (array of) MacMahon numbers: $v_{0,k}^{(2)}(n)$ is equal to the number of elements of the hyperoctahedral group (signed permutation group) whose type B descent is equal to $k$. More generally, if $p \in \mathbb{N}$, $v_{0,k}^{(p)}(n)$ is equal to the number of elements in the colored permutation group $G_{p,n} \cong \mathbb{Z}_p \wr S_n$ whose descent is equal to $k$ [8].

Remark 1.2 Miller [5] studied the Foulkes character in the general complex reflection groups. According to his results, if $p \in \mathbb{N}$, the array of left eigenvectors $\{v_{i,j}^{(p)}(n)\}_{i,j}$ coincides with the Foulkes character table of $\mathbb{Z}_p \wr S_n$.

Remark 1.3 Let $p^*$ be the dual exponent of $p > 1: \frac{1}{p} + \frac{1}{p^*} = 1$. Then the row eigenvectors of $L_{p^*}$ are the “reverse” of those of $L_p$ in the following sense.

$$v_{i,j}^{(p^*)}(n) = (-1)^i \left(\frac{p^*}{p}\right)^{n-i} v_{i,n-j}^{(p)}(n).$$

1.3 Results in this paper

Here we state the results obtained in this paper.

1.3.1 Right eigenvectors

We first derive the right eigenvectors of $\tilde{P}^\pm$.

Theorem 1.5 Let

$$R_p := L_p^{-1} = \{u_{i,j}^{(p)}(n)\}_{i,j}$$

be the matrix composed of the right eigenvectors of $\tilde{P}^\pm$. Then its elements are given by

$$u_{i,j}^{(p)}(n) = \sum_{k=0}^n \sum_{l=0}^k \frac{s(k,l)(-1)^{n-j-l}}{k! \ p^l} \left(\frac{l}{n-j}\right) \left(\frac{n-i}{n-k}\right).$$

$s(n,k)$ is the Stirling number of the first kind:

$$s(n,k) := (-1)^{n-k} \{\sigma \in S_n \mid \sigma \text{ has } k \text{ cycles} \}.$$ 

The formula in Theorem 1.5 appears in [5] in a different but related context (the inverse of the Foulkes character table), and it is a matter of computation to show that $R_p$ is the inverse of $L_p : R_p = L_p^{-1}$. The counterpart of the duality of $L_p$ (Remark 1.3) is
Remark 1.4

\[ u^{(p^*)}_{ij}(n) = (-1)^j \left( \frac{p}{p^*} \right)^{n-j} u^{(p)}_{n-i,j}(n). \]

In other words, the column vectors of \( R_p \) and \( R_{p^*} \) are reverse of each other, up to constants.

1.3.2 Correlation of carries

As is done in [3], we can make use of Theorem 1.5 to compute the expectations and correlations between carries at different steps. As we stated before, \( \{\kappa^{\pm}_r\}_{r=0}^\infty \) is the \((\pm b, n, p)\)-carries process.

**Theorem 1.6** Let \( E[\cdot | \kappa^{\pm}_0 = i] \) be the expectation value conditioned \( \kappa^{\pm}_0 = i \). Then we have

1. \( E[\kappa^{\pm}_r | \kappa^{\pm}_0 = i] = \frac{1}{(\pm b)^r} \left( i \right) \left( \frac{1}{p} n + \frac{1}{2} \right) - \frac{1}{p} + \frac{n+1}{2} \)

2. \( \text{Var} (\kappa^{\pm}_r | \kappa^{\pm}_0 = i) = \frac{n+1}{12} \left( 1 - \frac{1}{(\pm b)^{2r}} \right) \)

3. \( \text{Cov} (\kappa^{\pm}_s, \kappa^{\pm}_{s+r} | \kappa^{\pm}_0 = i) = \frac{1}{(\pm b)^r} \frac{n+1}{12} \left( 1 - \frac{1}{(\pm b)^{2s}} \right) \).

**Theorem 1.7** Let \( E_\pi[\cdot] \) be the expectation value conditioned that \( \kappa^{\pm}_0 \) obeys the stationary distribution. Then we have

1. \( E_\pi[\kappa^{\pm}_0] = \frac{n+1}{2} - \frac{1}{p} \)

2. \( \text{Cov}_\pi(\kappa^{\pm}_r, \kappa^{\pm}_0) = \frac{1}{(\pm b)^r} \cdot \frac{n+1}{12} \).

We note that the variances and covariances do not depend on \( p \). For \((-b)\)-case, \( \kappa^- \) and \( \kappa^-_0 \) are negatively correlated when \( r \) is odd.

1.3.3 Relation to the riffle shuffle

\((+b)\)-case  We study the relation between the carries process and the (generalized) riffle shuffles, which is a generalization to any \( p \in \mathbb{N} \) of the work by
Diaconis-Fulman [2]. First of all, we set a ordering on the set \( \Sigma := [n] \times \mathbb{Z}_p \) as follows \( ([n] := \{1, 2, \cdots, n\}) \).

\[
(1, 0) < (2, 0) < \cdots < (n, 0) < \\
(1, p - 1) < (2, p - 1) < \cdots < (n, p - 1) < \\
(1, p - 2) < (2, p - 2) < \cdots < (n, p - 2) < \\
\cdots \\
(1, 1) < \cdots < (n, 1).
\]

For \( q \in \mathbb{Z}_p \) let \( T_q : \Sigma \to \Sigma \) be the shift given by \( T_q(i, r) := (i, r + q) \), \((i, r) \in [n] \times \mathbb{Z}_p \). The group \( G_{p,n} \simeq \mathbb{Z}_p \wr S_n \) of colored permutations is the set of bijections \( \sigma : \Sigma \to \Sigma \) s.t. \( \sigma(i, 1) = T_1(\sigma(i, 0)) \). Writing \( \sigma(i, 0) =: (\sigma(i), r_i), \ i = 1, 2, \cdots, n, \ \sigma \in G_{p,n} \) is characterized by \(((\sigma(1), r_1), (\sigma(2), r_2), \cdots, (\sigma(n), r_n)) \). We say that \( \sigma \in G_{p,n} \) have a descent at \( i \) if and only if \( (\sigma(i), r_i) > (\sigma(i + 1), r_{i+1}) \) (if \( i = 1, 2, \cdots, n - 1 \) or \( r_n \neq 0 \). We denote by \( d(\sigma) \) the number of descents of \( \sigma \). Let \( b = pc + 1, c \in \mathbb{N} \).

The \((+b, n, p)\)-shuffle is defined as follows : (i) Divide \( n \) “cards with \( p \) colors” \((1, r_1), \cdots, (n, r_n)\), \( r_j \in \mathbb{Z}_p \), into \( b \)-piles, and (ii) Riffle them together as the usual riffle shuffle, except that for the \((jp+r)\)-th pile \((j = 0, \cdots, c, \ r = 0, \cdots (p - 1), \) counted from the 0-th), apply \( T_r \)-shift to them. Riffle shuffle(resp. type B shuffle) is the special case where \( p = 1 \) (resp. \( p = 2 \)).

Let \( \{\sigma_r\}_{r=0}^{\infty} \) be the sequence of \((+b, n, p)\)-shuffles starting at \( \sigma_0 = id \), which is a Markov chain on \( G_{p,n} \). Then we have the following theorem, which is a straightforward generalization of the results by Diaconis-Fulman[2].

**Theorem 1.8** Let \( b = pc + 1, c \in \mathbb{N}, \ p \geq 2, \ p \in \mathbb{N} \), and let \( \{\kappa_r^+\}_{r=0}^{\infty} \) be the \((+b, n, p)\)-carries process with \( \kappa_0^+ = 0 \). Then for any \( N \in \mathbb{N} \) and for any \( j \in \{0, 1, \cdots, n\} \)

\[
P \left( \kappa_N^+ = j \mid \kappa_0^+ = 0 \right) = P \left( d(\sigma_N) = j \mid \sigma_0 = id. \right)
\]

**(-b)-case** We consider a “shuffle” corresponding to the carries process for negative base, for \( p = 1, 2 \). For given \( \sigma = (\sigma(1), \cdots, \sigma(n)) \in S_n \), let \( R_1 \sigma \in S_n \)

\[
(R_1 \sigma)(k) := n + 1 - \sigma(k), \quad k = 1, 2, \cdots, n
\]

be its “reverse”. Let

\[
S_1^- := R_1 \circ (b\text{-shuffle})
\]
be the operation of carrying out the reverse after a $b$-shuffle. $b$-shuffle means the $(+b, n, 1)$-shuffle. Let $\{\tilde{\sigma}_r\}_{r=0}^\infty (\tilde{\sigma}_r := (S_1^{-r})^\sigma_0, \ r = 1, 2, \cdots)$ be a Markov chain on $S_n$ starting at $\sigma_0 = id$. Then the $(-b, n, 1)$-carries process $\{\kappa_r^-\}_r$ starting at $\kappa_0^- = 0$ has the same distribution to that of the descent $\{d(\sigma_r)\}_{r=0}^\infty$ of $\{\sigma_r\}_{r=0}^\infty$.

**Theorem 1.9** Let $\{\kappa_r^-\}_r$ be the $(b, n, 1)$-carries process with $\kappa_0^- = 0$. Then for any $j_1, \cdots, j_N \in \{0, 1, \cdots, n - 1\}$, we have

$$P \left( \kappa_1^- = j_1, \ k_2^- = j_2, \cdots, \ k_N^- = j_N \mid \kappa_0^- = 0 \right) = P \left( d(\tilde{\sigma}_1) = j_1, \ d(\tilde{\sigma}_2) = j_2, \cdots, d(\tilde{\sigma}_N) = j_N \mid \sigma_0 = id \right).$$

Similar result also holds for $p = 2$ case. Let $p = 2, b: \text{odd}$. Let

$$(R_2\sigma)(i, 0) := (n + 1 - \sigma(i), r_{i+1}), \quad S_2^- := R_2 \circ ((+b, n, 2)-shuffle)$$

where we set $(\sigma(i), r_i) := \sigma(i, 0), \ i = 1, \cdots, n$. Let $\{\tilde{\sigma}_r\}_{r=0}^\infty (\tilde{\sigma}_r := (S_2^{-r})^\sigma_0, \ r = 1, 2, \cdots)$ be a Markov chain on $G_{2,n}$ starting at $\sigma_0 = id$. Then we have the following relation between the $(b, n, 2)$-carries process and the riffle shuffles.

**Theorem 1.10** Let $\{\kappa_r^-\}_r$ be the $(b, n, 2)$-carries process with $\kappa_0^- = 0$. Then for $j \in \{0, 1, \cdots, n\}$, we have

$$P \left( \kappa_N^- = j \mid \kappa_0^- = 0 \right) = P \left( d(\tilde{\sigma}_N) = j \mid \sigma_0 = id \right).$$

For the proof of Theorems 1.9, 1.10, we construct the connection between $(+b, n, p)$-carries process and $(-b, n, p)$-carries process, and use the result in [1, 2] to relate the $(+b, n, p)$-carries process and the riffle shuffle.

So far we discussed properties of the $(\pm b, n, p)$-process and the relation to the riffle shuffle. There are still some unsolved problems:

1. Theorem 1.8 does not imply that the $(+b, n, p)$-carries process $(p \in \mathbb{N})$ and the descent of $(+b, n, p)$-shuffles have the same distribution. Since it is true for $p = 1$ [1], we expect that it is also true for $p \geq 2$.

2. If $p \in \mathbb{N}$, the $(+b, n, p)$-carries process is related to the descent statistics (Remark 1.1), the Foulkes character (Remark 1.2), and the $(+b, n, p)$-shuffles on $\mathbb{Z}_p \wr S_n$ (Theorem 1.8). It is desirable to have the corresponding results (if any) for $p \notin \mathbb{N}$.

3. It is desirable to construct a $(-b, n, p)$-shuffle for $p \neq 1, 2$.

In the following sections, we prove theorems stated above.
2 Right Eigenvectors and Their Duality

2.1 Right eigenvectors

Proof of Theorem 1.5
It suffices to show \( R_p = L_p^{-1} \). In what follows we omit \( n \)-dependence and compute

\[
\sum_{m=0}^{n} u^{(p)}_{im} v^{(p)}_{mj} = \sum_{k=i}^{n} \frac{1}{k!} \left( \begin{array}{c} n - i \\ n - k \end{array} \right) \sum_{r=0}^{j} (-1)^r \left( \begin{array}{c} n + 1 \\ r \end{array} \right) \sum_{l=0}^{k} \frac{s(k, l)}{p^l} \\
\times \sum_{m=n-l}^{n} (-1)^{n-m-l} \left( \begin{array}{c} l \\ n - m \end{array} \right) \{p(j - r) + 1\}^{n-m}.
\]

Here we changed the order of summation noting that \( n - m \leq l \leq k \) implies \( n - l \leq m \leq n \). By the binomial theorem, we have

\[
\sum_{m=n-l}^{n} (-1)^{n-m-l} \left( \begin{array}{c} l \\ n - m \end{array} \right) \{p(j - r) + 1\}^{n-m} = \{p(j - r)\}^l.
\]

It follows that

\[
\sum_{m=0}^{n} u^{(p)}_{im} v^{(p)}_{mj} = \sum_{k=i}^{n} \frac{1}{k!} \left( \begin{array}{c} n - i \\ n - k \end{array} \right) \sum_{r=0}^{j} (-1)^r \left( \begin{array}{c} n + 1 \\ r \end{array} \right) \sum_{l=0}^{k} s(k, l)(j - r)^l.
\]

It is well known that \( s(n, k) \) satisfies the following equation

\[
x(x - 1)(x - 2) \cdots (x - n + 1) = n! \left( \begin{array}{c} x \\ n \end{array} \right) = \sum_{k \geq 0} s(n, k)x^k
\]

which implies

\[
\sum_{l=0}^{k} s(k, l)(j - r)^l = k! \left( \begin{array}{c} j - r \\ k \end{array} \right).
\]

Therefore

\[
\sum_{m=0}^{n} u^{(p)}_{im} v^{(p)}_{mj} = \sum_{r=0}^{j} (-1)^r \left( \begin{array}{c} n + 1 \\ r \end{array} \right) \sum_{k=i}^{n} \left( \begin{array}{c} n - i \\ n - k \end{array} \right) \left( \begin{array}{c} j - r \\ k \end{array} \right)
\]

\[
= \sum_{r=0}^{j} (-1)^r \left( \begin{array}{c} n + 1 \\ r \end{array} \right) \left( \begin{array}{c} n - i + j - r \\ n \end{array} \right).
\]
Here we used the formula \( \binom{a + b}{n} = \sum_i \binom{a}{i} \binom{b}{n-i} \). Since RHS is equal to the coefficient of \( x^j \) in \((1 - x)^{n+1} \times x^i(1-x)^{-(n+1)}\), we have
\[
\sum_{m=0}^{n} u_{im}^{(p)} v_{m,j}^{(p)} = \delta_{ij}.
\]

\[\square\]

### 2.2 Some examples

In this subsection we give some examples of the duality of the right eigenvectors (Remark [1.4]). To simplify the notation, let
\[
\tilde{R}_p := \{ \tilde{u}_{ij}^{(p)}(n) \}_{ij}, \quad \tilde{u}_{ij}^{(p)}(n) := n!p^n u_{ij}^{(p)}(n).
\]

Then the statement in Remark [1.4] is equivalent to
\[
\tilde{u}_{ij}^{(p^+)}(n) = \left(-\frac{p^+}{p}\right)^j \tilde{u}_{n-i,j}^{(p)}(n).
\]

In the examples below, we take \( n = 3 \).

(1)
\[
\tilde{R}_2 = \begin{pmatrix}
1 & 9 & 23 & 15 \\
1 & 3 & -1 & -3 \\
1 & -3 & 1 & 3 \\
1 & -9 & 23 & -15
\end{pmatrix}.
\]

Each column vectors of \( \tilde{R}_2 \) is symmetric up to sign.

(2)
\[
\tilde{R}_3 = \begin{pmatrix}
1 & 15 & 66 & 80 \\
1 & 6 & 3 & -10 \\
1 & -3 & -6 & 8 \\
1 & -12 & 39 & -28
\end{pmatrix}, \quad \tilde{R}_{3/2} = \begin{pmatrix}
1 & 6 & \frac{39}{4} & \frac{7}{2} \\
1 & \frac{2}{3} & -\frac{3}{2} & -1 \\
1 & -3 & \frac{3}{2} & \frac{5}{4} \\
1 & -\frac{15}{2} & \frac{33}{2} & -10
\end{pmatrix}.
\]

If we multiply each column vectors of \( \tilde{R}_{3/2} \) by \( 1, (-2), (-2)^2, (-2)^3 \), and then turn them over, we obtain \( \tilde{R}_3 \).
3 Expectation and Variance of Carries

In this section we prove Theorems 1.6, 1.7.

Proof of Theorem 1.6

We discuss \((+b)\)-case only. \((-b)\)-case can be proved similarly.

(1) By [5], Corollary 9.1,

\[ u_{ij}^{(p)} = \text{coefficient of } x^{n-j} \text{ in } \left( n + \frac{x-1}{p} - i \right). \]

Hence

\( \tilde{u}_1 = \{ \tilde{u}_{i,1} \}_{i=0}^n, \quad \tilde{u}_1(i) := i + \frac{1}{p} - \frac{n+1}{2} \)

is an eigenvector of \( \tilde{P}^+ \) with eigenvalue \( b^{-1} \). It then follows that

\[
E[\kappa_r \mid \kappa_0 = i] = \sum_j jP^{(r)}(i, j)
\]

\[
= \sum_j P^{(r)}(i, j) \left( \tilde{u}_1(j) - \frac{1}{p} + \frac{n+1}{2} \right)
\]

\[
= \frac{1}{br} \tilde{u}_1(i) - \frac{1}{p} + \frac{n+1}{2}
\]

\[
= \frac{1}{br} \left( i + \frac{1}{p} - \frac{n+1}{2} \right) - \frac{1}{p} + \frac{n+1}{2}.
\]

(2) At first we compute \( E[\kappa_r^2 \mid \kappa_0 = i] \). In order to do that, we need to have an eigenvector \( u_2 = \{ u_{i,2} \}_{i=0}^n \) with eigenvalue \( b^{-2} \). After some computation, we have

\[ \tilde{u}_2(i) = \left( i + \frac{1}{p} - \frac{n+1}{2} \right)^2 - \frac{n+1}{12}. \]

Computing similarly as we did in (1), we obtain

\[ E \left[ \left( \kappa_r + \frac{1}{p} - \frac{n+1}{12} \right)^2 \mid \kappa_0 = i \right] \]

\[ \quad = \frac{1}{b^{2r}} \left( \left( i + \frac{1}{p} - \frac{n+1}{2} \right)^2 - \frac{n+1}{12} \right) + \frac{n+1}{12}. \quad (3.1) \]
On the other hand
\[
E \left[ \left( \kappa_r + \frac{1}{p} - \frac{n+1}{12} \right)^2 \mid \kappa_0 = i \right] = E[\kappa_r^2 \mid \kappa_0 = i] \\
+ 2 \left( \frac{1}{p} - \frac{n+1}{12} \right) E[\kappa_r \mid \kappa_0 = i] + \left( \frac{1}{p} - \frac{n+1}{12} \right)^2. \tag{3.2}
\]

By (3.1), (3.2) we have
\[
E\left[\kappa_r^2 \mid \kappa_0 = i\right] = \frac{1}{b^2r} \left( i + \frac{1}{p} - \frac{n+1}{2} \right)^2 + \frac{n+1}{12} \left( 1 - \frac{1}{b^2r} \right) - 2 \left( \frac{1}{p} - \frac{n+1}{2} \right) E[\kappa_r \mid \kappa_0 = i] - \left( \frac{1}{p} - \frac{n+1}{2} \right)^2. \tag{3.3}
\]

By squaring the result in (1), we have
\[
E[\kappa_r \mid \kappa_0 = i]^2 = \frac{1}{b^2r} \left( i + \frac{1}{p} - \frac{n+1}{2} \right)^2 \\
+ 2 E[\kappa_r \mid \kappa_0 = i] \left( -\frac{1}{p} + \frac{n+1}{2} \right) - \left( -\frac{1}{p} + \frac{n+1}{2} \right)^2. \tag{3.4}
\]

Taking difference between (3.3) and (3.4), we obtain the conclusion.

(3) We compute
\[
E[\kappa_s \kappa_s + r \mid \kappa_0 = i] \\
= \sum_j E[\kappa_s + r \mid \kappa_s = j] \cdot j \cdot P(\kappa_s = j \mid \kappa_0 = i) \\
= \sum_j E[\kappa_r \mid \kappa_0 = i] \cdot j \cdot P(\kappa_s = j \mid \kappa_0 = i) \\
= \sum_j \left\{ \frac{j}{b^r} + \left( \frac{n+1}{2} - \frac{1}{p} \right) \left( 1 - \frac{1}{b^r} \right) \right\} P(\kappa_s = j \mid \kappa_0 = i) \\
= \frac{1}{b^r} E[\kappa_s^2 \mid \kappa_0 = i] + \left( \frac{n+1}{2} - \frac{1}{p} \right) \left( 1 - \frac{1}{b^r} \right) E[\kappa_s \mid \kappa_0 = i] \\
= \frac{1}{b^r} \frac{n+1}{12} \left( 1 - \frac{1}{b^2s} \right) + \frac{1}{b^r} E[\kappa_s \mid \kappa_0 = i]^2 \\
+ \left( \frac{n+1}{2} - \frac{1}{p} \right) \left( 1 - \frac{1}{b^r} \right) E[\kappa_s \mid \kappa_0 = i].
\]
In the last equality, we used the result in (2). We further compute, using the result in (1).

\[
E[\kappa_{s+r}|\kappa_0 = i] = \frac{1}{b^r} \cdot \left(1 - \frac{1}{b^{2s}}\right)
\]

\[
+ \frac{1}{b^r} \left\{ \frac{1}{b^s} \left( i + \frac{1}{p} - \frac{n+1}{2} \right) + \left( \frac{n+1}{2} - \frac{1}{p} \right) \right\} \cdot E[\kappa_s | \kappa_0 = i]
\]

\[
+ \left( \frac{n+1}{2} - \frac{1}{p} \right) \cdot \left(1 - \frac{1}{b^r}\right) \cdot E[\kappa_s | \kappa_0 = i]
\]

which leads to the conclusion. \(\square\)

**Proof of Theorem 1.7**

By the Markov chain limit theorem, \(E[\kappa_r | \kappa_0 = i] \to_{r \to \infty} E[\pi | \kappa_0]\). Hence we can prove (1) (resp. (2)) by taking \(r \to \infty\) (resp. \(s \to \infty\)) in Theorem 1.6(1) (resp. in Theorem 1.6(3)). \(\square\)

### 4 Relation to Riffle Shuffles

#### 4.1 \((+b)\)-case

The proof of Theorem 1.8 can be done by the same methods as that in [2], Theorems 2.3, 4.4. However we give them for the sake of completeness.

**Lemma 4.1** Let \(b = pc + 1\) and let \(\sigma \in G_{p,n}\). The probability of obtaining \(\sigma\) after \((+b,n,p)\)-shuffles \(r\) times is equal to

\[
P (\sigma_r = \sigma) = b^{-rn} \left( n + \frac{(pc+1)^r - 1}{p} - d(\sigma^{-1}) \right).
\]

**Proof.** **Step 1** We suppose \(r = 1\). A \((+b,n,p)\)-shuffle generating \(\sigma \in G_{p,n}\) is obtained in putting \((c - d(\sigma^{-1}))\)-slits in the array \((\sigma^{-1}(1,0), \cdots, \sigma^{-1}(n,0))\) of \(n\)-elements, the numbers of which is equal to

\[
\left( \frac{n + c - d(\sigma^{-1})}{n} \right)
\]
yielding the conclusion for \( r = 1 \).

We give an example of \( p = 3, c = 2, \) and \( n = 7 \). The numbers below “GSR” is the GSR representation of the \((7, 7, 3)\)-shuffle generating \( \sigma \). We denote \((\sigma(i), r_i)\) by \( \sigma(i)_{r_i} \) for simplicity. Since \( d(\sigma^{-1}) = 2 \), there are no other ones generating \( \sigma \).

| GSR | \( \sigma \) | \( \sigma^{-1} \) |
|-----|-------------|---------------|
| 5   | 6_2         | 5             |
| 4   | 5_1 3_2     |               |
| 1   | 2_1 4_1     |               |
| 2   | 3_2 7       |               |
| 0   | 1 2_2       |               |
| 6   | 7 1_1       |               |
| 3   | 4 6         |               |

Step 2 It is now sufficient to show that a composition of \((pa + 1, n, p)\)-shuffle and \((pb + 1, n, p)\)-shuffle is equivalent to a \(((pa + 1)(pb + 1), n, p)\)-shuffle. It can easily be done by using the star map introduced in [1], Lemma 3.5. Let \( \pi \) be a map sending the GSR representation \( A \) of a \((b, n, p)\)-shuffle to the corresponding element in \( G_{p,n} \). We shall explain the star map by using the following example, where we take \( n = 6, p = 3, \) and carry out a composition of \((3 \cdot 1 + 1, 6, 3)\)-shuffle \( A_1 \) and \((3 \cdot 2 + 1, 6, 3)\)-shuffle \( A_2 \).

| \( A_2 \) | \( A_1 \) | \( (A_2A_1)^* \) | \( B \) | \( \pi(A_2) \) | \( \pi(A_1) \) | \( \pi(A_2)\pi(A_1) \) |
|-----------|-----------|-----------------|------|-------------|-------------|-----------------|
| 5 1       | 0 1       | 1               |      | 5_2         | 2_1         | 1_1             |
| 0 3       | 3 3       | 15              |      | 1_0         | 6_0         | 3_0             |
| 3 2       | 4 2       | 18              |      | 2_0         | 4_2         | 4_0             |
| 4 0       | 5 0       | 20              |      | 4_1         | 1_0         | 5_2             |
| 6 1       | 3 1       | 13              |      | 6_0         | 3_1         | 2_1             |
| 3 2       | 6 2       | 26              |      | 3_0         | 5_2         | 6_2             |

To construct \((A_2A_1)^*\), the column on the right of that is equal to \( A_1 \). The column on the left is obtained by rearranging the numbers in \( A_2 \) along the permutation generated by \( \pi[A_1] \). Then we consider the lexicographic order on \( \{0, \cdots, 6\} \times \{0, \cdots, 3\} \), and \( \pi[(A_2A_1)^*] \) is defined to arrange numbers under that order. Since the star map satisfies

\[
\pi[A_2] \circ \pi[A_1] = \pi[(A_2A_1)^*],
\]
we have that \((A_2A_1)^*\) is equivalent to the \((28, 6, 3)\)-shuffle \(B\). The table on the right shows the corresponding elements in \(G_{3,6}\). □

The following lemma is a (trivial) extension of Diaconis-Fulman [2], Proposition 4.1.

**Lemma 4.2** Let \(\sigma \in G_{p,n}\) with \(d(\sigma) = d\) and let

\[
c_{ij}^d := \# \{(\tau, \mu) \in G_{p,n} \times G_{p,n} \mid d(\tau) = i, \quad d(\mu) = j, \quad \tau\mu = \sigma\}.
\]

Then this number is independent of the choice of \(\sigma\) s.t. \(d(\sigma) = d\) and satisfies the following equation.

\[
\sum_{i,j \geq 0} \frac{c_{ij}^d s^i t^j}{(1 - s)^{n+1}(1 - t)^{n+1}} = \sum_{a,b \geq 0} \binom{n + pab + a + b - d}{n} s^a t^b.
\]

**Proof.** Since a composition of \((pa + 1, n, p)\)-shuffle and \((pb + 1, n, p)\)-shuffle is equivalent to \(((pa + 1)(pb + 1), n, p)\)-shuffle, we have

\[
\sum_{\mu} \binom{n + a - d(\mu)}{n} \mu^{-1} \sum_{\tau} \binom{n + b - d(\tau)}{n} \tau^{-1} = \sum_{\sigma} \binom{n + pab + a + b - d(\sigma)}{n} \sigma^{-1}.
\]

We multiply both sides by \(s^a \cdot t^b\), take summation in \(a, b \geq 0\), and then take the coefficient of \(\sigma\). Noting that \(\tau\mu = \sigma\), we have

\[
\sum_{\tau\mu = \sigma} \sum_{a} \binom{n + a - d(\mu)}{n} s^a \sum_{b} \binom{n + b - d(\tau)}{n} t^b = \sum_{\tau\mu = \sigma} \frac{c_{ij}^d}{(1 - s)^{n+1}} \cdot \frac{t^d(\tau)}{(1 - t)^{n+1}} = \sum_{i,j} c_{ij}^d \frac{s^i}{(1 - s)^{n+1}} \cdot \frac{t^j}{(1 - t)^{n+1}}.
\]

□

**Proof of Theorem 1.8**

By Lemma 4.1, we compute

\[
P(d(\sigma_r) = j) = \sum_{i \geq 0} \sum_{d(\sigma^{-1}) = i, d(\sigma) = j} P(\sigma_r = \sigma)
\]

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\[ \sum_{i \geq 0} c^{ij}_0 \left( n + \frac{(pc+1)^r-1}{p} - i \right) \cdot b^{-rn}. \]

Putting \( d = 0 \) in Lemma 4.2 and taking the coefficient of \( s^m \), \( m := \frac{(pc+1)^r-1}{p} \) on both sides, we have

\[ \sum_{i,k \geq 0} c^{ij}_0 \left( n + \frac{(pc+1)^r-1}{p} - i \right) \frac{t^k}{(1-t)^{n+1}} = \sum_d \left( n + \frac{(pc+1)^r-1}{p} (pd+1)+d \right) t^d. \]

We multiply \( (1-t)^{n+1} \) both sides and take the coefficient of \( t^j \) on both sides.

\[ P(d(\sigma_r) = j) \]

\[ = \frac{1}{(pc+1)^r n} \left\{ (1-t)^{n+1} \sum_d \left( n + \frac{(pc+1)^r-1}{p} (pd+1)+d \right) t^d \right\} \]

\[ = \frac{1}{(pc+1)^r n} (-1)^r \begin{pmatrix} n + 1 \\ r \end{pmatrix} \left( n + \frac{(pc+1)^r}{p} (j-r) + \frac{(pc+1)^r-1}{p} \right) \]

\[ = P \left( \kappa_r^+ = j \mid \kappa_0^+ = 0 \right) \]

In the last equality, we used the fact that, by Theorem 1.4, \( \kappa_r^+(b) \) and \( \kappa_1^{(b')} \) have the same distribution, where we denoted \((+b, n, p)-process by \( \{ \kappa_{b'} \}_{r=0}^{\infty} \) \]

**4.2 \((-b)-case**

We prove Theorem 1.9 only, for the proof of Theorem 1.10 is similar.

**Proof of Theorem 1.9**

Let \( i' = r(i) \), \( X' \) be the “reverse” of \( i, X \), respectively:

\[ i' = r(i) := n - 1 - i, \quad i \in \mathcal{C} := \{0, 1, \ldots, n-1\} \]

\[ X' := b - 1 - X, \quad X \in \mathcal{D} := \{0, 1, \ldots, b-1\} \]

By Proposition 5.1(1) and by the symmetry of \( \bar{P} \) for \( p = 1 \) [4] :

\[ \bar{P}^+(i, j) = \bar{P}^+(n - 1 - i, n - 1 - j) = \bar{P}^+(r(i), r(j)), \]

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we have
\[
P \left( \kappa_1^- = j_1, \kappa_2^- = j_2, \kappa_3^- = j_3, \cdots, \kappa_N^- = j_N \mid \kappa_0^- = 0 \right)
= P \left( \kappa_1^+ = r(j_1), \kappa_2^+ = j_2, \kappa_3^+ = r(j_3), \cdots, \kappa_N^+ = r^N(j_N) \mid \kappa_0^+ = 0 \right).
\]

Let \(\{\sigma_r\}_{r=0}^{\infty}\) be the usual riffle shuffle \(((b, n, 1)\text{-shuffle})\) starting at \(\sigma_0 = \text{id}\).

Then by the result in [1], Theorem 3.1,
\[
P \left( \kappa_1^+ = r(j_1), \kappa_2^+ = j_2, \cdots, \kappa_N^+ = r^N(j_N) \mid \kappa_0^+ = 0 \right)
= P \left( d(\sigma_1) = r(j_1), d(\sigma_2) = j_2, \cdots, d(\sigma_N) = r^N(j_N) \mid \sigma_0 = \text{id} \right). \quad (4.1)
\]

Let \(\sigma^R \in S_n\) be the “reverse” of \(\sigma \in S_n\) defined by
\[
(\sigma^R)(j) := n+1-j, \quad j = 1, 2, \cdots, n
\]
and consider a Markov chain \(\{\sigma^R_r\}_{r=0}^{\infty}\). Noting
\[
d(\sigma^R) = n - 1 - d(\sigma) = r(d(\sigma)),
\]
the descent of \(\sigma_r\) and that of \(\sigma^R_r\) evolve as the following figure where we denoted \(j' = r(j)\).
\[
d(\sigma_0) = 0 \rightarrow d(\sigma_1) = j'_1 \rightarrow d(\sigma_2) = j_2 \rightarrow d(\sigma_3) = j'_3
\]
\[
d(\sigma_0^R) = 0 \rightarrow d(\sigma_1^R) = j_1 \rightarrow d(\sigma_2^R) = j'_2 \rightarrow d(\sigma_3^R) = j_3
\]

Since
\[
\tilde{\sigma}_1 = \sigma_1^R, \quad \tilde{\sigma}_2 = \sigma_2, \quad \tilde{\sigma}_3 = \sigma_3^R, \quad \cdots
\]
we have
\[
P \left( d(\sigma_1) = r(j_1), d(\sigma_2) = j_2, \cdots, d(\sigma_N) = r^N(j_N) \mid \sigma_0 = \text{id} \right)
= P \left( d(\tilde{\sigma}_1) = j_1, d(\tilde{\sigma}_2) = j_2, \cdots, d(\tilde{\sigma}_N) = j_N \mid \sigma_0 = \text{id} \right). \quad (4.2)
\]

By (4.1), (4.2) we complete the proof of Theorem 1.9.
5 Appendix

5.1 A condition to be a stochastic matrix

We prove Proposition 1.3 which gives a sufficient condition that \( \tilde{P} \) is a stochastic matrix.

Proof of Proposition 1.3

(1) We prove \((+b)\)-case only, since the proof for \((-b)\)-case follows similarly. We may assume \( p \neq 1 \). Suppose \( k := \frac{b-1}{p} \in \mathbb{N} \). Then \( A_{ij} := A(i, j) = jb - i + k \in \mathbb{N} \). Since \( Q_{ij} := \tilde{P}_{ij}b^n \) is equal to the coefficient of \( x^{A_{ij}} \) in \((1 + x + \cdots + x^{b-1})^{n+1} \), we have \( Q_{ij} \geq 0 \). Moreover \( Q_{ij} \) is equal to the number of elements \((X_1, X_2, \ldots, X_n, Y) \in D^{n+1} \), \( D := \{0, 1, \ldots, b-1\} \) such that \( X_1 + \cdots + X_n + Y = A_{ij} \). Fix \( i \in C := \{0, 1, \ldots, n\} \) and let

\[ \mathcal{A}_i := \{ A_{ij} \mid j \in C \}, \quad S_n := X_1 + \cdots + X_n. \]

To prove \( \sum_j Q_{ij} = b^n \), it suffices to show the following fact\(^1\): for any \((X_1, \ldots, X_n) \in D^n \) we can find \( Y \in D \) uniquely such that

\[ S_n + Y \in \mathcal{A}_i. \]

We note that \( \min S_n = 0, \max S_n = n(b-1), \) and \( \min \mathcal{A}_i = -i + k, \max \mathcal{A}_i = nb - i + k. \) Since \( p \geq 1, \) we have \( 1 \leq k \leq b-1 \) so that

\[ -b - i + k + 1 \leq 0, \quad n(b-1) + 1 \leq nb - i + k. \]

Therefore for any \( S_n \in \{0, 1, \ldots, n(b-1)\} \) we can find \( l = -1, 0, \ldots, n-1 \) uniquely such that

\[ lb - i + k + 1 \leq S_n \leq (l+1)b - i + k. \]

Then \( Y := (l+1)b - i + k - S_n \) satisfies \( Y \in D \) and

\[ X_1 + \cdots + X_n + Y = (l+1)b - i + k \in \mathcal{A}_i. \]

Since \( \mathcal{A}_i \) consists of \((n+1)\)-points at intervals of \( b \), \( Y \) is unique.

\(^1\)It is in fact equivalent.
(2) Suppose $\frac{b-1}{p} \in \mathbb{N}$ but $p < 1$. Then $\frac{b-1}{p} \geq b$. Hence if $i = 0$,

$$\min A_0 = A_{0,0} = \frac{b-1}{p} \geq b$$

However, if $X_1 = X_2 = \cdots = X_n = 0$, there is no $Y \in \mathcal{D}$ with

$$X_1 + \cdots + X_n + Y \in A_0.$$

We show some examples. Let $b = 5$, $n = 3$. Then it must be $(-d) = 0, 1, 3, 4$ in order to have $0 \in \mathcal{D}_d$. Since we have

$$\frac{1}{p} = \left\langle \frac{(n-1)(-d)}{b-1} \right\rangle = \left\langle \frac{2(-d)}{4} \right\rangle = 1, \quad \frac{1}{2}, \quad (-d) = 0, 1, 3, 4,$$

we should have $p = 1, 2$ if the $(5, 3, p)$-process corresponds to a $n$-carries process over $(b, \mathcal{D}_d)$. However, the condition in Proposition $1.3$ $p \geq 1$, $\frac{b-1}{p} = \frac{4}{p} \in \mathbb{N}$ is satisfied if and only if $p = 4, 2, \frac{4}{3}, 1$. In fact, if we take $p = 4,

$$\tilde{P} = \begin{pmatrix}
\frac{4}{125} & \frac{68}{125} & \frac{52}{125} & \frac{1}{125} \\
\frac{1}{125} & \frac{52}{125} & \frac{68}{125} & \frac{4}{125} \\
0 & \frac{7}{25} & \frac{16}{25} & \frac{2}{25} \\
0 & \frac{4}{25} & \frac{17}{25} & \frac{4}{25}
\end{pmatrix}$$

is a stochastic matrix. On the other hand, $p = 4/5$ satisfies $\frac{b-1}{p} \in \mathbb{N}$ but $p < 1$. In fact, in this case

$$\tilde{P} = \begin{pmatrix}
\frac{52}{125} & \frac{68}{125} & \frac{4}{125} & 0 \\
\frac{7}{25} & \frac{16}{25} & \frac{2}{25} & 0 \\
\frac{4}{25} & \frac{17}{25} & \frac{4}{25} & 0 \\
\frac{2}{25} & \frac{16}{25} & \frac{7}{25} & 0
\end{pmatrix}$$

is not a stochastic matrix, for the sum of the elements of 0-th row vector is less than 1.

### 5.2 Symmetry of $\tilde{P}$

We show the symmetry properties of $\tilde{P}$ which are used for the proof of Theorems 1.9, 1.10. First of all, for $p = 1, 2$, the transition probability
of \((+b, n, p)\)-carries process and that of \((-b, n, p)\)-carries process satisfy the following simple relation. We note that the triple \((-b, n, 1)\) always satisfies the condition in Proposition 1.3.

**Proposition 5.1** (1) If \(p = 1\), \(A^-(i, j) = A^+(i, n - 1 - j)\). Hence
\[
P^-(i, j) = P^+(i, n - 1 - j), \quad i, j \in \{0, 1, \cdots, n - 1\}.
\]
(2) If \(p = 2\), \(b\):odd so that \(\frac{(\pm b) - 1}{p} \in \mathbb{N}\), then \(A^-(i, j) = A^+(i, n - j)\) and hence
\[
P^-(i, j) = P^+(i, n - j), \quad i, j \in \{0, 1, \cdots, n\}.
\]

The second proposition refers to the “cross-symmetry”.

**Proposition 5.2** Let \(p > 1\), with \(p^* > 1\) being its dual exponent. We then have
\[
\tilde{P}^+_p(i, j) = \tilde{P}^+_{p^*}(n - i, n - j).
\]

\((+b, n, 1)\)-carries process satisfies the same relation where \(n\) in above formula is replaced by \((n - 1)\).

**Proof.** Let \(A_p(i, j) := \left(j + \frac{1}{p}\right)b - \left(i + \frac{1}{p}\right)\). As we have seen in the proof of Proposition 1.3, \(Q_{ij} := b^n P_{ij}\) is equal to the number of \((X_1, \cdots, X_n, Y) \in D^{n+1}\) such that
\[
X_1 + X_2 + \cdots + X_n + Y = A_p(i, j) \quad (5.1)
\]
Let \(X'_j := b - 1 - X_j, Y' := b - 1 - Y\) be the “reverse” of \(X_j, Y\), respectively. Using
\[
(n + 1)(b - 1) - A_p(i, j) = A_{p^*}(n - i, n - j),
\]
equation \((5.1)\) is equivalent to
\[
X'_1 + X'_2 + \cdots + X'_n + Y' = A_{p^*}(n - i, n - j).
\]
yielding
\[
\tilde{P}^+_p(i, j) = \tilde{P}^+_{p^*}(n - i, n - j).
\]

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