More about the Laughlin droplet

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Abstract

We compare the exact diagonalization ground state wave function (calculated without any restriction) of a two-dimensional droplet in a perpendicular magnetic field with the Laughlin ansatz, as the number of electrons increases. The fully spin-polarized case for filling factor 1/3 without lateral confining potential or Zeeman effect is considered. We observe that the overlap decreases as the number of electrons increases.

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Most of the knowledge and understanding of the fractional quantum Hall effect states is due to the existence of Laughlin’s wave function, a simple and intuitive analytical expression that matches very well with the ground states (gs) of the system characterized by the filling factor $\nu = 1/q$, where $q$ is an odd integer. This has been tested by numerical calculations of finite systems for different geometries as well as experimental results related with measurable quantities which can be derived from its properties, as is the case of the fractional charge of some of the excited states. We analyze the overlap between this ansatz and the gs wave function that comes out from numerical exact diagonalization of the Hamiltonian for finite systems. We also consider other properties of the gs.

Laughlin’s wave function for $\nu = 1/q$ is given by:

$$\Psi_L = \prod_{i<j} (z_i - z_j)^q e^{-\sum |z_i|^2/4} \tag{1}$$

where $z = x + iy$ (in units of $l_B = \sqrt{\hbar eB / c}$, $B$ being the magnetic field) and where the symmetric gauge has been assumed. The physical meaning is easily deduced from its form: the minimization of the energy in the partially filled lowest Landau level is obtained when each electron sees zeros of order $q$ at the positions of the other electrons in such a way that there are non free zeros which would increase the energy. It can be verified that if the polynomial part is expanded, one obtains a combination of Slater determinants built up from Fock Darwin single particle wave functions of the type:

$$\phi_{sp} \sim z^m e^{-|z|^2/4} \tag{2}$$

(solutions of the non-interacting system) where $m$ is the single particle angular momentum. The expansion is a homogeneous polynomial of degree $M$ (the total angular momentum) and contains single particle angular momenta up to a maximum given by,

$$m_L = q(N - 1) \tag{3}$$

where $N$ is the number of electrons. In order to analyze the evolution of the wave function given by Eq.(1) with $N$, we consider the cases $N = 2, 3$ and 4 keeping the condition given by $\nu = 1/3$. We compare Eq.(1) with the wave function that comes out from the exact diagonalization (we only considered the fully spin polarized case). In the exact calculation no Zeeman or kinetic energy contributions are considered, electrons are confined within a disk due to the restriction on the total angular momentum.
For $N = 2$ the dimension of the subspace characterized by the well defined quantum numbers $M = 3$, the angular momentum needed to obtain $\nu = 1/3$ for $N = 2$ ($M = qN(N - 1)/2$) and total spin along the z-direction $S_z = 1$ is equal to two, that is to say, the exact gs is a combination of two Slater determinants in which the largest possible single particle angular momentum is given by,

$$m_{ex} = M - \frac{(N - 1)(N - 2)}{2}$$

(4)
or $m_{ex} = 3$ in this case, in agreement with the value obtained from Eq.(3). Furthermore, the linear combination of Slaters is the same in both cases (in Eq.(1) and in the diagonalization) and consequently the overlap is equal to one. That is to say, for $N = 2$ the Laughlin wave function is exact. For $N = 3$ ($M = 9$) the situation is not the same: the number of Slaters involved in the g.s. is not the same, the maximum $m$ is also different and as a consequence the overlap is lower than one. This tendency is increased for $N = 4$. A summary of the results is given in Table 1 and Figs.1 and 2 below. $N_L$ and $N_{ex}$ are the total number of Slater determinants included in the Laughlin and exact ground states respectively, i.e., $N_{ex}$ is the dimension of the Hilbert space characterized by $M$ and $S_z$. $P_L$ and $P_{ex}$ are the normalized weights of the most important Slater within the expansion of the g.s. respectively. This Slater determinant contains, for all the values of $N$ considered, a packet of successive single particle angular momenta separated from the center of the dot in such a way that they produce a compact ring. Remarkably this compact structure made up of successive single particle angular momenta wavefunctions ($m = 2, 3$ and $4$ for $N = 3$) differs from the structure intuitively suggested by that of electrons surrounded by magnetic quantum fluxes. Namely, the zeros at the electronic positions seem to emphasize the short range character of the interaction rather than the separation between electrons. Fig.1 shows the overlap $S = |\langle \Psi_L | \Psi_{ex} \rangle|^2$ for $N = 2, 3$ and 4 and Fig.2 shows the values of $P_L$ and $P_{ex}$ for different $N$. The tendency seems to indicate that the overlap will decrease as $N$ increases and so, as the function given by Eq.(1) is a especially good approximation for a low number of electrons, it must be taken with some care for large $N$ in a finite system.

It must be emphasized that the tendency of the overlap to worsen as the number of electrons increases would be a trivial result if the gs were a single Slater determinant and also if we used different single particle wavefunctions: the exact wavefunctions for $\Psi_{ex}$ and approximate wavefunctions for $\Psi_L$. In this case, the overlap would scale as $(1 - \epsilon)^N$, $\epsilon$ being
a small number for good trial functions. However this is not our case: the gs is a linear combination of several Slater determinants and in addition, we use the same Slaters to build up the linear combinations in each case, that is to say, the overlap depends on the coefficients, namely, on the electron-electron interaction and is not directly related to \( N \).

As a consequence, for example, Laughlin’s ansatz is the exact result for filling factor \( \nu = 1 \) independently of \( N \), since only one determinant is involved.

Special attention must be paid to the increasing difference between \( N_L \) and \( N_{ex} \) due to the fact that the ratio between the weights of the Slater determinants that are lacking in \( \Psi_L \) related to the weights of the Slaters which are included is not negligible. As an example, for \( N = 4 \) one of the Slaters lacking in the \( \Psi_L \) has a normalized weight of 0.002 compared with that of the most important one which is 0.331. Furthermore, \( \Psi_L \) loses the small and the large values of \( m \) and so the center as well as the edge of the droplet are poorly reproduced. As an example for \( N = 5 \), there are 27 Slaters that contain \( m = 0 \) and only 16 of them are included in the expansion of \( \Psi_L \). However, in spite of the fact that the overlap of the gs wavefunction and Laughlin’s ansatz worsens as \( N \) increases, it is not the case for some expected values, especially for the energy. If \( H \) has only the Coulomb contribution (as is appropriate for a fully spin polarized system in the lowest Landau level regime), and if we define the discrepancy \( D \) as \( [(E_{ex} - E_L)/E_{ex}] \times 100 \) where \( E_{ex} = \langle \Psi_{ex} | H | \Psi_{ex} \rangle \) and \( E_L = \langle \Psi_L | H | \Psi_L \rangle \) the results obtained are:

For \( N = 2 \), \( D = 0 \), \( D = 10.5 \) for \( N = 3 \) and \( D = 1.3 \) for \( N = 4 \), i.e. improving the result as \( N \) increases.

There are several previous studies on the Laughlin wave function for finite systems which devoted special attention to the study of the edge states: Mitra and MacDonald have analyzed the angular momentum distribution function for a droplet and found that the occupations are peaked at the edge (for \( N = 15, 20 \) and \( 25 \)) and that it has a rapid decline. We believe that it can be a consequence of the reduced base implied in \( \Psi_L \) as it loses the small and large values of the single particle angular momenta. Tsiper and Goldman compare the density of a droplet obtained from Eq.(1) and from exact diagonalization for \( N = 5 \) to 12 and \( \nu = 1/3 \). They obtain important differences at the centre of the droplet and a nearly exact coincidence at the edge. The difference in the electron-electron interaction implied in the Laughlin ansatz (short range interaction) and in the exact calculation (Coulomb interaction).
interaction) is invoked in order to explain the difference and the formation of striped states. We believe that their result could be a consequence of the procedure used to obtain the results: within the exact calculation they truncate the base of the Hilbert space, i.e., the $m'_{ex}$ considered is obtained by increasing Hilbert space until overlap $S$ converges to at least three significant digits. However this procedure forces a precise coincidence with $\Psi_L$ at the edge, giving no information about the total weight of the rest of the members of the base of the Hilbert space, which is not necessarily negligible, as we mentioned previously. As an example, for $N = 12$ they consider $m'_{ex} = 35$ ($m_L = 33$) while $m_{ex} = 143$ is the exact single-particle maximum angular momentum involved. At the center, their numerical calculation contains all the Slaters without restriction and they obtain strip-like oscillations on the radial electron densities which has been proved to be responsible for the observed unexpected behavior of the current-voltage power law \cite{10, 11}.

Finally we conclude that for finite systems the overlap between the Laughlin wave function and the exact results is worse as $N$ increases. This result is similar to that obtained previously by Haldane \cite{2} for spherical geometry that mimics a two dimensional homogeneous system. His conclusion was that for $N = 3$ the Laughlin type function is the exact solution (even for Coulomb interaction) but it is not for $N \geq 4$. In a recent paper by Yannouleas and Landman \cite{12} a systematic study of a system of 6 electrons in a range of filling factors from $\nu = 1/5$ to $\nu = 1/9$ is reported. They conclude that the analytical model of collectively rotating electron molecules (REM) \cite{10} provides better representation of the system. Other references \cite{2, 4} have tested some results obtained with Eq.(1) for some particular values of $N$, however our aim is to study the tendency as $N$ increases. The contribution of our report refers to the edge as well as the central properties of finite systems which can be appreciably different from those properties obtained by the use of the Laughlin droplet for large $N$ and as a consequence, the differences can be significant at the thermodynamic limit. However, the evolution of the overlap of the wave functions does not necessarily characterize the evolution of the expected values of some operators as was previously pointed out for the energy operator.

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FIG. 1: Overlap between the Laughlin wave function and the exact diagonalization result as a function of $N$. 

![Graph showing the overlap between the Laughlin wave function and the exact diagonalization result as a function of N. The graph displays a decreasing trend from 1 to 0.97 as N increases from 2 to 4.]
FIG. 2: Evolution of the normalized weight of the most important Slater determinant within the wave function expansion as $N$ increases.
Table 1: Fully spin polarized droplet for $S_z = N/2$ and $\nu = 1/3$. $M$ is the total angular momentum, $m_L$ and $m_{ex}$ are the maximum single particle angular momentum for each $M$. $N_L$ and $N_{ex}$ are the number of Slater determinants involved in the expansion of the ground state. $P_L$ and $P_{ex}$ are the normalized weights of the most important Slater within the ground state; for $N = 2$ it contains: $m_1 = 1$ and $m_2 = 2$ (let’s say (1,2)), $N = 3$: (2,3,4), $N = 4$: (3,4,5,6), $N = 5$: (4,5,6,7,8). $S$ is the overlap (see text).

| $N$ | 2  | 3  | 4  | 5  |
|-----|----|----|----|----|
| $M$ | 3  | 9  | 18 | 30 |
| $m_L$ | 3  | 6  | 9  | 12 |
| $m_{ex}$ | 3  | 8  | 15 | 24 |
| $N_L$ | 2  | 5  | 16 | 50 |
| $N_{ex}$ | 2  | 7  | 18 | 192 |
| $P_L$ | 0.75 | 0.48 | 0.25 | 0.05 |
| $P_{ex}$ | 0.75 | 0.54 | 0.33 |  |
| $S$ | 1 | 0.991 | 0.979 |  |