A fast and stable test to check if a weakly diagonally dominant matrix is an M-matrix

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Abstract

We present a stable test for determining if a substochastic matrix is convergent. By establishing a duality between weakly chained diagonally dominant (w.c.d.d.) L-matrices and convergent substochastic matrices, we show that this test can be trivially extended to determine whether a weakly diagonally dominant (w.d.d.) matrix is an M-matrix. The test’s runtime is linear in the order of the input matrix if it is sparse and quadratic if it is dense. Depending only on the structure of the matrix, the test is unconditionally stable. This is a partial strengthening of the cubic test in [J. M. Peña., A stable test to check if a matrix is a nonsingular M-matrix, Math. Comp., 247, 1385–1392, 2004]. As a by-product of our analysis, we prove that a w.d.d. M-matrix is a w.c.d.d. L-matrix, a fact whose converse has been known since at least 1964. We point out that this strengthens some recent results on M-matrices in the literature.

1 Introduction

The substochastic matrices\(^1\) are real matrices with nonnegative entries and whose row-sums are at most one. We establish two results relating to this family:

(i) To each substochastic matrix \(B\) we associate a possibly infinite index of connectivity \(\hat{\text{conn}}B\) and show that for each nonnegative integer \(k\), \(B^k\) is a \(\|\cdot\|_\infty\)-contraction if and only if \(k > \hat{\text{conn}}B\).

(ii) We show that the index of connectivity of a sparse (resp. dense) square substochastic matrix is computable in time linear (resp. quadratic) in the order of the input matrix.

It follows immediately from (i) that a square substochastic matrix is convergent if and only if its index of connectivity is finite.

Letting \(\hat{J}(B)\) denote the set of rows of \(B\) whose row-sums are strictly less than one, it turns out that \(\hat{\text{conn}}B\) depends only on the adjacency digraph of \(B\) when interpreted as a bipartite graph whose disjoint vertex sets are \(\hat{J}(B)\) and its complement. Perhaps surprisingly,

\(^1\)Definitions of italicized items are deferred to the sequel.
\( \text{conn} B \) does not depend in any other way on the values of the entries of \( B \), and hence the computation mentioned in \((ii)\) requires no stability considerations whatsoever!

The results above are intimately related to weakly diagonally dominant \((\text{w.d.d.})\) \( M \)-matrices, which arise naturally from discretizations of differential operators and appear in the Bellman equation for optimal decision making on a controlled Markov chain \([BMZ09]\). As such, this class of matrices has attracted a significant amount of attention from the scientific computing and numerical analysis communities.

Weakly chained diagonally dominant \((\text{w.c.d.d.})\) matrices were studied in a wonderful work by P. N. Shivakumar and K. H. Chew \([SC74]\) in which they were proven to be nonsingular (see also \([AF16]\) for a short proof). Various authors have recently studied the family of w.c.d.d. \( M \)-matrices, obtaining bounds on the infinity norm of their inverses (i.e., \( \| A^{-1} \|_{\infty} \)) \([Shi+96; \ CH07; \ Li08; \ Wan09; \ HZ10]\). While a w.c.d.d. matrix is w.d.d. by definition, the converse is not necessarily true in general (e.g., \( \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \) is w.d.d. but not w.c.d.d.).

It has long been known (possibly as early as 1964; see \([BH64]\)) that a w.c.d.d. \( L \)-matrix is a w.d.d. \( M \)-matrix. We obtain a proof of the converse as a by-product of our analysis. In fact, we are able to go further, establishing that a w.d.d. \( L \)-matrix is either singular or an \( M \)-matrix (a non-w.d.d. \( L \)-matrix can be nonsingular and fail to be an \( M \)-matrix; e.g., \( \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix} \)). We summarize these facts by writing

\[
A \text{ is a w.d.d. } M\text{-matrix } \iff A \text{ is a nonsingular w.d.d. } L\text{-matrix } \\
\iff A \text{ is a w.c.d.d. } L\text{-matrix.} \tag{E}
\]

\( \text{(E)} \) immediately strengthens the results pertaining to norms of inverses listed in the previous paragraph, ensuring that they also apply to w.d.d. \( M \)-matrices.

\( \text{(E)} \) is also useful in that w.c.d.d. matrices give a graph-theoretic characterization of w.d.d. \( M \)-matrices. This characterization is often easier to use than the usual characterizations of \( M \)-matrices involving, say, inverse-positivity or positive principal minors \([Ple77]\).

By establishing a duality between w.c.d.d. \( L \)-matrices and convergent substochastic matrices, we use point \((ii)\) to obtain a test to determine whether a w.d.d. matrix is an \( M \)-matrix. Previous work in this regard is the test in \([Pn04]\) to determine if an arbitrary matrix (not necessarily w.d.d.) is an \( M \)-matrix, which has a cost asymptotically equivalent to Gaussian elimination (i.e., cubic in the order of the input matrix).

We list a few other interesting recent results concerning w.c.d.d. matrices and \( M \)-matrices here: \([TH10; \ LLZ13; \ YZL13; \ XLL14; \ WS15; \ Li+16; \ LL16; \ ZS16]\).

Section 2 introduces and establishes results on substochastic matrices, \( M \)-matrices, and w.c.d.d. matrices. Section 3 gives the procedure to compute the index of connectivity.

2 Matrix families

Substochastic matrices

**Definition 2.1.** A substochastic matrix is a real matrix \( B := (b_{ij}) \) with nonnegative entries (i.e., \( b_{ij} \geq 0 \)) and row-sums at most one (i.e., \( \sum_j b_{ij} \leq 1 \)). A stochastic (a.k.a. Markov) matrix is a substochastic matrix whose row-sums are exactly one.
Figure 2.1: An example of an $n \times n$ substochastic matrix and its graph

Note that in our definition above, we do not require $B$ to be square.

**Definition 2.2.** Let $A := (a_{ij})$ be an $m \times n$ complex matrix.

(i) The digraph of $A$, denoted graph $A$, is defined as follows:

(a) If $A$ is square, graph $A$ is a tuple $(V, E)$ consisting of the vertex set $V := \{1, \ldots, n\}$ and edge set $E \subset V \times V$ satisfying $(i, j) \in E$ if and only if $a_{ij} \neq 0$.

(b) If $A$ is not square, graph $A := \text{graph } A'$ where $A'$ is the smallest square matrix obtained by appending rows or columns of zeros to $A$.

(ii) A walk in graph $A \equiv (V, E)$ is a nonempty finite sequence of edges $(i_1, i_2), (i_2, i_3), \ldots, (i_{m-1}, i_m)$ in $E$. The set of all walks in graph $A$ is denoted $\text{walks } A$.

(iii) Let $p \in \text{walks } A$. The length of $p$, denoted $|p|$, is the total number of edges in $p$. head $p$ (resp. last $p$) is the first (resp. last) vertex in $p$.

To simplify matters, we hereafter denote edges by $i \rightarrow j$ instead of $(i, j)$ and walks by $i_1 \rightarrow i_2 \rightarrow \cdots \rightarrow i_m$ instead of $(i_1, i_2), (i_2, i_3), \ldots, (i_{m-1}, i_m)$. We use the terms “row” and “vertex” interchangeably.

Let $B := (b_{ij})$ be an $m \times n$ substochastic matrix. We define the sets

$$\hat{J}(B) := \left\{ i : \sum_j b_{ij} < 1 \right\}$$

and $\hat{P}_i(B) := \left\{ p \in \text{walks } B : \text{head } p = i \text{ and last } p \in \hat{J}(B) \right\}$.

It is understood that when we write $i \notin \hat{J}(B)$, we mean $i \in \hat{J}(B)^c := \{1, \ldots, m\} \setminus \hat{J}(B)$. We define the index of connectivity associated with $B$ by

$$\widehat{\text{conn}} B := \max \left( 0, \sup_{i \notin \hat{J}(B)} \left\{ \inf_{p \in \hat{P}_i(B)} |p| \right\} \right) \quad \text{(C)}$$

subject to the conventions $\inf \emptyset = \infty$ and $\sup \emptyset = -\infty$. We will see shortly that the matrix $B$ is convergent if and only if $\widehat{\text{conn}} B$ is finite.

**Example 2.3.** The $n \times n$ matrix $B$ in Figure 2.1 on page 3 satisfies $\hat{J}(B) = \{1\}$ and

$$\min_{p \in \hat{P}_1(B)} |p| = i - 1 \text{ for } i \notin \hat{J}(B).$$

It follows that $\widehat{\text{conn}} B = n - 1$. 

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An immediate consequence of the definition of the index of connectivity is below.

**Lemma 2.4.** If $B$ is an $m \times n$ substochastic matrix, $\text{com}B$ is either infinite or strictly less than $\min\{m, n\} + 1_{\{m>n\}}$.

Note that if $m = n$ (i.e., $B$ is square) in the above, $\min\{m, n\} + 1_{\{m>n\}} = n$.

**Proof.** Suppose $m \leq n$. Consider a walk $p$ without self-loops (i.e., edges of the form $i \rightarrow i$) in graph $B$ satisfying $|p| \geq m$. Since graph $B$ has exactly $m$ vertices, $p$ must have one or more cycles. By removing all such cycles, we obtain a new walk whose head and last vertices are unchanged, but whose length is strictly less than $m$.

The case of $m > n$ is handled similarly. ■

We are now ready to present our main result related to substochastic matrices. In the statement below, it is understood that if $B$ is a square matrix, $B^0 = I$.

**Theorem 2.5.** Let $B$ be a square substochastic matrix. If $\alpha := \text{com}B$ is finite,

\[ 1 = \|B^0\|_\infty = \cdots = \|B^\alpha\|_\infty > \|B^{\alpha+1}\|_\infty \geq \|B^{\alpha+2}\|_\infty \geq \cdots \]

Otherwise,

\[ 1 = \|B^1\|_\infty = \|B^2\|_\infty = \cdots \]

Before giving a proof, it is useful to record a few consequences of the above.

**Corollary 2.6.** Let $B$ be a square substochastic matrix. Then, its eigenvalues are no larger than one in magnitude. Moreover, the following statements are equivalent:

(i) $\text{com}B$ is finite.

(ii) $B$ is convergent.

(iii) $I - B$ is nonsingular.

**Remark 2.7.** Since a square stochastic (a.k.a. Markov) matrix is simply a special type of nonconvergent substochastic matrix, the above implies a few familiar results on the spectrum of a stochastic matrix (recall that for any matrix $M$, $I - M$ is singular if and only if $\lambda = 1$ is an eigenvalue of $M$).

**Proof.** The claim that $B$ admits no eigenvalues larger than one in magnitude is a direct consequence of the fact that $\|B\|_\infty \leq 1$.

$(i) \implies (ii)$ follows immediately from Theorem 2.5, while $(ii) \implies (iii)$ is true for any matrix. We prove below, by contrapositive, the claim $(iii) \implies (i)$.

Suppose $\text{com}B$ is infinite. Let $R$ be the set of rows $i \notin \hat{J}(B)$ for which $\hat{P}_i(B)$ is empty. Due to our assumptions, there is at least one such row and hence $R$ is nonempty. Without loss of generality, we may assume $R = \{1, \ldots, r\}$ for some $1 \leq r \leq n$ where $n$ is the order of $B$ (otherwise, simultaneously reorder the rows and columns of $B$). Let $e \in \mathbb{R}^r$ be the
column vector whose entries are all one. If \( r = n \), each row-sum of \( B \) is one (i.e., \( Be = e \)). Otherwise, \( B \) has the block structure

\[
B = \begin{pmatrix} B_1 & 0 \\ B_2 & B_3 \end{pmatrix}
\]

where \( B_1 \in \mathbb{R}^{r \times r} \).

The partition above ensures that for each row \( i \notin R \), \( i \in \hat{J}(B) \) or \( \hat{P}_i(B) \) is nonempty. Therefore, \( \text{coim} B_3 \) is finite, and hence the linear system \((I - B_3)x = B_2e\) has a unique solution \( x \). Moreover, since the row-sums of \( B_1 \) are one, \( B_1e = e \). It follows that

\[
(I - B) \begin{pmatrix} e \\ x \end{pmatrix} = \begin{pmatrix} e \\ x \end{pmatrix} - \begin{pmatrix} B_1e \\ B_2e + B_3x \end{pmatrix} = \begin{pmatrix} e \\ x \end{pmatrix} - \begin{pmatrix} e \\ x \end{pmatrix} = 0.
\]

\[\blacksquare\]

Corollary 2.8. An irreducible substochastic matrix (i.e., a substochastic matrix \( B \) whose digraph is strongly connected and with \( J(B) \) nonempty) is convergent.

Returning to our goal of proving Theorem 2.5, we first establish some lemmata related to substochastic matrices. The first lemma is a consequence of definitions and requires no proof.

**Lemma 2.9.** Let \( B \) be an \( m \times n \) matrix with nonnegative entries. Then, \( \|B\|_\infty < 1 \) if and only if \( \hat{J}(B) = \{1, \ldots, m\} \).

**Lemma 2.10.** Let \( B := (b_{ij}) \) and \( C := (c_{ij}) \) be compatible (i.e., the product \( BC \) is well-defined) substochastic matrices. Then,

(i) \( BC \) is a substochastic matrix.

(ii) If \( i \in \hat{J}(B) \), then \( i \in \hat{J}(BC) \).

(iii) If \( i \notin \hat{J}(B) \), then \( i \in \hat{J}(BC) \) if and only if there is an \( h \in \hat{J}(C) \) such that \( i \to h \) is an edge in graph \( B \).

(iv) \( i \to j \) is an edge in graph(\( BC \)) if and only if there exist edges \( i \to h \) and \( h \to j \) in graph \( B \) and graph \( C \), respectively.

**Proof.**

(i) \( BC \) has nonnegative entries and \( BCe \leq \|BC\|_\infty \leq \|B\|_\infty \|C\|_\infty \) \leq 1.

(ii) Note first that \( \sum_j [BC]_{ij} = \sum_j \sum_k b_{ik} c_{kj} = \sum_k b_{ik} \sum_j c_{kj} \leq \sum_k b_{ik} \) if \( i \in \hat{J}(B) \), \( \sum_k b_{ik} < 1 \) and the desired result follows.

(iii) Suppose \( i \notin \hat{J}(B) \). If there is an \( h \in \hat{J}(C) \) such that \( i \to h \) is an edge in graph \( B \), then \( \sum_j c_{hj} < 1 \) and \( \sum_j [BC]_{ij} = b_{ih} \sum_j c_{hj} + \sum_{k \neq h} b_{ik} \sum_j c_{kj} \leq \sum_k b_{ik} \leq 1 \). Otherwise, \( \sum_j c_{kj} = 1 \) for all \( k \) with \( b_{ik} \neq 0 \) and hence \( \sum_j [BC]_{ij} = \sum_k b_{ik} \sum_j c_{kj} = \sum_k b_{ik} = 1 \).

(iv) Suppose \( i \to h \) and \( h \to j \) are edges in graph \( B \) and graph \( C \), respectively. Then, \( [BC]_{ij} = \sum_k b_{ik} c_{kj} \geq b_{ih} c_{hj} > 0 \). Otherwise, for each \( k \), at least one of \( b_{ik} \) or \( c_{kj} \) is zero and hence \( [BC]_{ij} = 0 \). \[\blacksquare\]
Lemma 2.11. Let \( B \) be a square substochastic matrix, \( i \notin \hat{J}(B) \), and \( k \) be a positive integer. Then, \( i \in \hat{J}(B^k) \) if and only if there exists a walk \( p \) in \( \hat{P}_i(B) \) such that \( k > |p| \).

Proof. To simplify notation, let \( i_1 := i \).

Suppose there exists a walk \( i_1 \rightarrow i_2 \rightarrow \cdots \rightarrow i_m \) in \( \hat{P}_i(B) \). We claim that \( i_1 \rightarrow i_m \) is an edge in \( B^{m-1} \). If this is the case, Lemma 2.10 (ii) and (iii) guarantee that \( i_1 \in \hat{J}(B^{m-1}) = \hat{J}(B^m) \). If \( k \geq m \), \( i_1 \in \hat{J}(B^k) \) by Lemma 2.10 (ii).

We now return to our unproven claim. If \( m = 2 \), the claim is trivial. Suppose \( m > 2 \). Since \( i_1 \rightarrow i_2 \) and \( i_2 \rightarrow i_3 \) are edges in graph \( B \), an application of Lemma 2.10 (iv) implies that \( i_1 \rightarrow i_3 \) is an edge in graph \( B^2 \). Repeating this procedure, we arrive at the desired result.

As for the converse, one can apply the argument above “backwards” to construct a walk of length less than \( k \) in \( \hat{P}_i(B) \).

We are now ready to prove Theorem 2.5.

Proof of Theorem 2.5. Since \( \|B^{k+1}\|_\infty \leq \|B^k\|_\infty \|B\|_\infty \leq \|B^k\|_\infty \), the inequalities \( 1 \geq \|B^1\|_\infty \geq \|B^2\|_\infty \geq \cdots \) follow trivially.

The remaining inequalities in the theorem statement follow by applying Lemma 2.11 to each row not in \( \hat{J}(B) \) and invoking Lemma 2.9.

M-matrices

Definition 2.12. A monotone matrix is an \( n \times n \) real matrix \( A \) satisfying the following property: for each \( x \in \mathbb{R}^n \), if \( Ax \) consists only of nonnegative entries, so too must \( x \).

Proposition 2.13 ([Col64]). Let \( A \) be a real square matrix. \( A \) is monotone if and only if it is nonsingular and its inverse consists only of nonnegative entries.

Definition 2.14. A Z-matrix is a real matrix whose off-diagonal entries are nonpositive.

Definition 2.15. An L-matrix is a Z-matrix whose diagonal entries are positive.

Definition 2.16. An M-matrix is a monotone Z-matrix.

Proposition 2.17. An M-matrix is an L-matrix.

We find it useful to define the following enlargement of the family of L-matrices:

Definition 2.18. An \( L_0 \)-matrix is a Z-matrix whose diagonal entries are nonnegative (compare with Definition 2.15).

Weakly chained diagonally dominant (w.c.d.d.) matrices

Before we can define w.c.d.d. matrices, we require some preliminary definitions.

Definition 2.19. Let \( A := (a_{ij}) \) be a complex matrix.

(i) The \( i \)-th row of \( A \) is w.d.d. (resp. s.d.d.) if \( |a_{ii}| \geq \sum_{j \neq i} |a_{ij}| \) (resp. >).

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Let $A := (a_{ij})$ be an $m \times n$ complex w.d.d. matrix. We define the sets

$$J(A) := \{i: |a_{ii}| > \sum_{j \neq i} |a_{ij}|\}$$

and $P_i(A) := \{p \in \text{walks } A: \text{head } p = i \text{ and } \text{last } p \in J(A)\}$. We will see shortly that the sets $J(\cdot)$ and $P_i(\cdot)$ are related to $\hat{J}(\cdot)$ and $\hat{P}_i(\cdot)$.

We are now ready to introduce w.c.d.d. matrices:

**Definition 2.20.** A square complex matrix $A$ is w.c.d.d. if both points below are satisfied:

1. $A$ is w.d.d.
2. For each $i \notin J(A)$, $P_i(A)$ is nonempty.

The index of connectivity associated with a square w.d.d. $L_0$-matrix $A$ is

$$\text{conn } A := \max \left(0, \sup_{i \notin J(A)} \left\{ \inf_{p \in P_i(A)} |p| \right\} \right)$$

(compare this with $\widehat{\text{conn}}$ defined in (C)). The lemma below is a trivial consequence of the definitions above and as such requires no proof.

**Lemma 2.21.** If $A$ is a square w.d.d. $L_0$-matrix, $A$ is w.c.d.d. if and only if $\text{conn } A$ is finite.

Note that if $A := (a_{ij})$ is a square w.d.d. $L_0$-matrix with $a_{kk} = 0$ for some row $k$ (i.e., $A$ is not an $L$-matrix), $\text{conn } A$ is trivially infinite.

We are now able to establish a duality between w.d.d. $L$-matrices (or more accurately, $L_0$-matrices) and substochastic matrices that, as we will see, connects the nonsingularity of the former to the convergence of the latter.

**Lemma 2.22.** Let $A := (a_{ij})$ be an $n \times n$ w.d.d. $L_0$-matrix and $D := (d_{ij})$ be an $n \times n$ real diagonal matrix whose diagonal entries are positive and satisfy, for each $i$ such that $a_{ii} \neq 0$, $d_{ii} \leq 1/a_{ii}$ so that $B := I - DA$ is substochastic. Then,

$$\text{conn } A = \widehat{\text{conn}} B.$$  \hspace{1cm} (D)
Conversely, let $B$ be an $n \times n$ substochastic matrix and $D$ be an $n \times n$ diagonal matrix whose diagonal entries are positive so that $A := D(I - B)$ is a w.d.d. L$_0$-matrix. Then, (D) holds.

Proof. Let $A$ and $B := I - DA$ be given as above. That $B$ is substochastic is trivial. Letting graph $A \equiv (V, E)$ and graph $B \equiv (V', E')$, note that $V = V'$ and

$$E \setminus \{(i, i)\} \subset E' \subset E.$$ 

More concisely, graph $B$ is simply graph $A$ with zero or more self-loops (i.e., edges of the form $i \to i$) removed. Moreover, it is easily verified that $J(A) = \hat{J}(B)$. As a result of these facts, (D) follows immediately.

The converse is handled similarly. ■

Example 2.23. Let $A := (a_{ij})$ be a square w.d.d. L-matrix of order $n$ and

$$B_A := I - \text{diag}(a_{11}, \ldots, a_{nn})^{-1}A$$

denote the point Jacobi matrix associated with $A$ (cf. [Var00, Chapter 3]). By the previous results, $A$ is w.c.d.d. if and only if $\text{conn} A = \text{conn} B_A$ is finite.

The substochastic matrix in Figure 2.1 on page 3 is the point Jacobi matrix associated with the w.d.d. L-matrix in Figure 2.2 on page 7.

We now restate and prove characterization (E) from the introduction.

Theorem 2.24. The following are equivalent:

(i) $A$ is a w.d.d. M-matrix.

(ii) $A$ is a nonsingular w.d.d. L-matrix.

(iii) $A$ is a w.c.d.d. L-matrix.

Proof.

(i) $\implies$ (ii): An M-matrix is a nonsingular L-matrix by Proposition 2.13 and Proposition 2.17.

(ii) $\implies$ (iii): Let $A := (a_{ij})$ be a nonsingular w.d.d. L-matrix of order $n$. Then, the associated point Jacobi matrix $B_A$ is substochastic and $I - B_A$ is nonsingular since

$$I - B_A = \text{diag}(a_{11}, \ldots, a_{nn})^{-1}A.$$ 

By Corollary 2.6 and Lemma 2.22, $\text{conn} A = \text{conn} B_A$ is finite. By Lemma 2.21, $A$ is w.c.d.d.

(iii) $\implies$ (i): Let $A$ be a w.c.d.d. L-matrix of order $n$. Then, by Lemma 2.21 and Lemma 2.22, the associated point Jacobi matrix $B_A$ is substochastic with $\text{conn} B_A = \text{conn} A$ finite. By Corollary 2.6, $B_A$ is convergent and hence the Neumann series $I + B_A + B_A^2 + \cdots$ for the inverse of $I - B_A$ converges to a matrix whose entries are nonnegative. Therefore, $A$ is monotone by Proposition 2.13.

Corollary 2.25. An irreducible w.d.d. L-matrix (i.e., a w.d.d. L-matrix $A$ whose digraph is strongly connected and with $J(A)$ nonempty) is an M-matrix.
3 Computing the index of connectivity

In this section, we present a procedure to compute the index of connectivity \( \hat{\text{conn}} B \) of a substochastic matrix \( B \).

By the results of the previous section, this procedure can also be used to determine if an arbitrary w.d.d. matrix \( A \) is an M-matrix as follows. If \( A \) is not a square L-matrix, it is trivially not an M-matrix (Proposition 2.17). Otherwise, we can check the finitude of the index of connectivity of its associated point Jacobi matrix \( B_A \) to determine whether or not \( A \) is an M-matrix (recall Example 2.23 and Theorem 2.24; see also Remark 3.4).

Definition 3.1. Let \( G \equiv (V, E) \) be a graph, \( W \subset V \), \( w \) denote a new vertex (i.e., \( w \notin V \)), and \( f \) be a function which maps every vertex in \( V \setminus W \) to itself and every vertex in \( W \) to \( w \) (i.e., \( f|_{V \setminus W} = \text{id}_{V \setminus W} \) and \( f|_{W} (\cdot) = w \)). The vertex contraction of \( G \) with respect to \( W \) is a new graph \( G' \equiv (V', E') \) where \( V' := (V \setminus W) \cup \{w\} \) and \( E' := \{(f(i), f(j)) : (i, j) \in E\} \).

Remark 3.2. Vertex contraction (also known as vertex identification) is a generalization of the well-known notion of edge contraction from graph theory.

An overview of the procedure for computing the index of connectivity for an arbitrary substochastic matrix \( B \) is given below:

1. Obtain the vertex contraction of graph \( B \) with respect to \( \hat{J}(B) \). Label the new vertex in the contraction 0 and the new vertex set \( V' \). Note that \( V' = \hat{J}(B)^\complement \cup \{0\} \) (recall that the superscript \( \complement \) denotes complement).

2. Reverse all arcs in the resulting graph.

3. In the resulting graph, find the shortest distances \( d(i) \) from the new vertex 0 to all vertices \( i \in V' \) by a breadth-first search (BFS) starting at 0. It is understood that \( d(0) = 0 \) and that if \( i \) is unvisited in the BFS, \( d(i) = \infty \).

4. Return \( \max_{i \in V'} d(i) \).

That this procedure terminates is trivial (BFS is performed on a graph with finitely many vertices). As for the correctness of the procedure, it is easy to verify that

\[
d(i) = \inf_{p \in P_i(B)} |p| \text{ for } i \notin \hat{J}(B)
\]

so that \( \hat{\text{conn}} B = \max(0, \sup_{i \notin \hat{J}(B)} d(i)) = \max_{i \in V'} d(i) \).

Remark 3.3. Since BFS does not revisit vertices, the correctness of the procedure is unaffected if graph \( B \) is preprocessed to remove self-loops (i.e., edges of the form \( i \rightarrow i \)) and edges of the form \( i \rightarrow j \) with \( i \in \hat{J}(B) \).

Algorithm 1 gives precise pseudocode for steps (1) to (4). Without loss of generality, it is assumed that the input matrix is square (the rectangular case is obtained by a few trivial additions to the code).

It is obvious that if the input to Algorithm 1 is a dense matrix of order \( n \), \( \Theta(n^2) \) operations are required. Suppose instead that we restrict our inputs to matrices that are sparse in the
(a) Graph $B$ (vertices in $\hat{J}(B)$ are highlighted)

(b) The resulting graph

Figure 3.1: Steps (1) and (2) applied to an example

Algorithm 1 Computing the index of connectivity $\hat{\text{conn}} B$ for a square substochastic matrix $B := (b_{ij})_{1 \leq i, j \leq n}$ of order $n$

1: // Find all rows in $\hat{J}(B)$
2: $s \leftarrow 0$
3: $S[1, \ldots, n] \leftarrow \text{new} \text{ array of bools} \quad 21: \text{ if } s < n \text{ then} \quad 39: \text{ conn } \leftarrow \infty$
4: for all rows $i$ do
5: $t \leftarrow 0$
6: for all cols $j$ satisfying $b_{ij} \neq 0$ do
7: $t \leftarrow t + b_{ij}$
8: end for
9: if $t < 1$ then
10: $s++$
11: $S[i] \leftarrow \text{true}$
12: else
13: $S[i] \leftarrow \text{false}$
14: end if
15: end for
16: // Find neighbours of each vertex (ignoring extraneous edges as per Remark 3.3)
17: $N[0, \ldots, n] \leftarrow \text{new} \text{ array of lists}$
18: for all rows $i$ satisfying $S[i] = \text{false}$ do
19: for all cols $j \neq i$ satisfying $b_{ij} \neq 0$ do
20: $N[S[j] ? 0 : j].add(i)$
21: end for
22: end for
23: end for
24: // Perform BFS starting at 0
25: $Q \leftarrow \text{new} \text{ queue}$
26: $\text{conn } \leftarrow 0$
27: $Q.push((0, 0))$
28: while $Q$ is nonempty do
29: $(j, d) \leftarrow Q.pop()$
30: $\text{conn } \leftarrow \text{max}(\text{conn}, d++)$
31: for all $i$ in $N[j]$ satisfying $S[i] = \text{false}$ do
32: $s++$
33: $S[i] \leftarrow \text{true}$
34: $Q.push((i, d))$
35: end for
36: end while
37: end while
sense that they have at most \( c \) nonzero entries per row, where \( c \) is a fixed constant independent of the order \( n \). If the matrices are stored in an appropriate format (e.g., sparse row format, Ellpack-Itpack, etc. [Saa03]), the loops on lines 6 and 20 require only a constant number of iterations for each fixed \( i \). In this case, \( \Theta(n) \) operations are required.

**Remark 3.4.** Given a square w.d.d. L-matrix \( A : = (a_{ij}) \), computing conn \( A \) directly without storing its point Jacobi matrix \( B_A \) is trivial: one need only modify Algorithm 1 by replacing all instances of “\( b_{ij} \)” with “\( a_{ij} \)” (lines 6, 7, and 20) and “\( t < 1 \)” with “\( t > 0 \)” (line 9).

In the presence of inexact arithmetic, the comparison on line 9 of Algorithm 1 can fail to produce the desired results. However, this occurs only if the input substochastic matrix \( B \) is “nearly nonconvergent” (i.e., \( \rho(B) \approx 1 \)). This is best described by the next example.

**Example 3.5.** Consider the family of matrices \( \{B_\epsilon\}_{\epsilon > 0} \) given by

\[
B_\epsilon := \begin{pmatrix} 0 & 1/(1 + \epsilon) \\ 1 & 0 \end{pmatrix}.
\]

While each matrix \( B_\epsilon \) is convergent, the spectral radius \( \rho(B_\epsilon) \) tends to one as \( \epsilon \to 0 \). In inexact arithmetic \( 1/(1 + \epsilon) \approx 1 \) may cause the comparison on line 9 of Algorithm 1 to fail for the first row \( (i = 1) \).

Equivalently, if we are testing to see if a w.d.d. L-matrix is an M-matrix, an issue only occurs if the input w.d.d. L-matrix is “nearly singular” (i.e., \( \kappa(A) \gg 1 \)):

**Example 3.6.** Consider the family of matrices \( \{A_\epsilon\}_{\epsilon > 0} \) given by

\[
A_\epsilon := \begin{pmatrix} 1 + \epsilon & -1 \\ -1 & 1 \end{pmatrix}.
\]

While each matrix \( A_\epsilon \) is an M-matrix, the condition number \( \kappa(A_\epsilon) \) tends to infinity as \( \epsilon \to 0 \).

Note that \( B_\epsilon = I - \text{diag}(1/(1 + \epsilon), 1)A \) of the previous example is nothing other than the point Jacobi matrix associated with \( A_\epsilon \).

### A Generalizing Theorem 2.5

This appendix generalizes Theorem 2.5. To present the generalization, we first extend our notion of walks:

**Definition A.1.** Let \((A_n)_{n \geq 1}\) be a sequence of square complex matrices of uniform order.

(i) A walk in \((A_n)\) is a nonempty finite sequence of edges \((i_1, i_2), (i_2, i_3), \ldots, (i_{m-1}, i_m)\) such that each \((i_k, i_{k+1})\) is an edge in graph \(A_k\). The set of all walks in \((A_n)\) is denoted \(\text{walks}(A_1, A_2, \ldots)\).

(ii) For \(p \in \text{walks}(A_1, A_2, \ldots)\), head \(p\), last \(p\), and \(|p|\) are defined in the obvious way.
Note, in particular, that if we fix a square complex matrix $A$, we are returned to the original definition of a walk given in Section 2 if we take $A_n := A$ for all $n$.

It is also useful to generalize the sets $\hat{P}_i(\cdot)$ of Section 2. In particular, given a sequence $(B_n)_{n \geq 1}$ of compatible substochastic matrices (i.e., the product $B_k B_{k+1}$ is well-defined for each $k$), let

$$\hat{P}_i(B_1, B_2, \ldots) := \left\{ p \in \text{walks}(B_1, B_2, \ldots) : \text{head } p = i \text{ and last } p \in \hat{J}(B_{|p|+1}) \right\}.$$ 

We are now ready to give the generalization.

**Theorem A.2.** Let $(B_n)_{n \geq 1}$ be a sequence of compatible substochastic matrices, $(C_n)_{n \geq 0}$ be defined by $C_0 := I$ and $C_n := B_1 B_2 \cdots B_n$ whenever $n$ is a positive integer, and

$$\hat{\text{conn}}(B_1, B_2, \ldots) := \max \left\{ 0, \sup_{i \notin \hat{J}(B_1)} \left\{ \inf_{p \in \hat{P}_i(B_1, B_2, \ldots)} |p| \right\} \right\}.$$ 

If $\alpha := \hat{\text{conn}}(B_1, B_2, \ldots)$ is finite,

$$1 = \|C_0\|_\infty = \cdots = \|C_\alpha\|_\infty > \|C_{\alpha+1}\|_\infty \geq \|C_{\alpha+2}\|_\infty \geq \cdots$$

Otherwise,

$$1 = \|C_1\|_\infty = \|C_2\|_\infty = \cdots$$

The proof of the above is nearly identical to that of Theorem 2.5, requiring only a simple generalization of Lemma 2.11. However, in this general case, the finitude of the index of connectivity is no longer an indicator of convergence:

**Example A.3.** Let $(B_n)_{n \geq 1}$ be a sequence of compatible substochastic matrices satisfying $\|B_n\|_\infty = 1 - 1/2^n$ and $(C_n)_{n \geq 0}$ be defined as above. Clearly, each matrix $B_n$ is convergent, but $\|C_n\|_\infty = \prod_{k=1}^n (1 - 1/2^k) \not\to 0$ as $n \to \infty$.

Moreover, even if each $B_n$ is itself convergent, it is still possible that the index of connectivity is infinite:

**Example A.4.** Let $(B_n)_{n \geq 1}$ be given by

$$B_n := \frac{1}{2} \begin{pmatrix} 0 & 1 + (-1)^n \\ 1 - (-1)^n & 0 \end{pmatrix}.$$ 

Defining $(C_n)_{n \geq 0}$ as above, we find that

$$C_n := \frac{1}{2} \begin{pmatrix} 0 & 0 \\ 1 - (-1)^n & 1 + (-1)^n \end{pmatrix} \text{ for } n \geq 1.$$ 

That is, $\|C_n\|_\infty = 1$ independent of $n$.

It is not hard to find interesting cases in which $\hat{\text{conn}}(B_1, B_2, \ldots)$ is finite:
Example A.5. Let \((B_n)_{n \geq 1}\) be a sequence of square substochastic matrices of order \(n\) satisfying the following properties:

(i) \(B_1\) is convergent.

(ii) \(\hat{J}(B_1) = \hat{J}(B_n)\) and graph \(B_1 = \text{graph } B_n\) for all \(n\).

Then, \(\hat{\text{conn}}(B_1, B_2, \ldots) < n\).

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