DIFFUSIVE MAGNETOHYDRODYNAMIC INSTABILITIES BEYOND THE CHANDRASEKHAR THEOREM

GÜNTER RÜDIGER,1,2 MANFRED SCHULTZ1,2, FRANK STEFANI,2 AND MICHAEL MOND3

1 Leibniz-Institut für Astrophysik Potsdam, An der Sternwarte 16, D-14482 Potsdam, Germany; gunter@iap.de, mcschultz@iap.de
2 Helmholtz-Zentrum Dresden-Rossendorf, POB 510119, D-01314 Dresden, Germany; f.stefani@hzdr.de
3 Department of Mechanical Engineering, Ben-Gurion University, Beer-Sheva, Israel; mond@bgu.ac.il

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ABSTRACT

We consider the stability of axially unbounded cylindrical flows that contain a toroidal magnetic background field with the same radial profile as their azimuthal velocity. For ideal fluids, Chandrasekhar had shown the stability of this configuration if the Alfvén velocity of the field equals the velocity of the background flow, i.e., if the magnetic Mach number \( M_m = 1 \). We demonstrate that magnetized Taylor–Couette flows with such profiles become unstable against non-axisymmetric perturbations if at least one of the diffusivities is finite. We also find that for small magnetic Prandtl numbers \( \text{Pr} \) the lines of marginal instability scale with the Reynolds number and the Hartmann number. In the limit \( \text{Pr} \to 0 \) the lines of marginal instability completely lie below the line for \( M_m = 1 \) and for \( \text{Pr} \to \infty \) they completely lie above this line. For any finite value of \( \text{Pr} \), however, the lines of marginal instability cross the line \( M_m = 1 \), which separates slow from fast rotation. The minimum values of the field strength and the rotation rate that are needed for the instability (slightly) grow if the rotation law becomes flat. In this case, the electric current of the background field becomes so strong that the current-driven Taylor instability (which also exists without rotation) appears in the bifurcation map at low Hartmann numbers.

Key words: galaxies: magnetic fields – instabilities – magnetohydrodynamics (MHD) – stars: magnetars – stars: magnetic field

1. MOTIVATION

Plane and parallel hydrodynamic shear flows are only unstable in the inviscid theory against infinitesimal perturbations if their span-wise velocity profile has an inflexion point (Rayleigh 1880). There is no such inflexion for a plane Poiseuille flow with the profile \( 1 - y^2 \) between the walls at \( y = \pm 1 \) so that they are stable for vanishing viscosity. Plane Poiseuille flows with finite viscosity, however, are unstable against infinitesimal disturbances if their Reynolds number \( UL/\nu \) exceeds the (high) value 5772 (Drazin & Reid 1981). Such flows are destabilized by the finite viscosity. If the linear instability is considered to be a structure-forming process then only the viscosity gives rise to the structure in this case. This result opposes the expectation that any diffusivity should act against the formation of local and also global maxima and minima. Note that dissipation-induced instabilities are also well-known in a number of finite-dimensional mechanical systems, as comprehensively surveyed by Krechetnikov & Marsden (2007).

Magnetohydrodynamics (MHD) provides yet another wide range of phenomena in which equilibria that are stable under zero magnetic diffusivity may no longer be stable when finite electrical conductivity is considered (Furth et al. 1963; Coppi et al. 1966). The resulting resistive instabilities such as the tearing modes drive many observed phenomena in astrophysics, in space as well as in laboratory plasma (Connor et al. 2009; Landi & Bettarini 2012). Examples include solar flares and coronal mass ejections that are related to the tearing mode-driven magnetic reconnection, which is also known to limit the operational regimes in Tokamaks.

A similarly central role as Rayleigh’s inflexion point theorem in hydrodynamics is played in MHD by a theorem of Chandrasekhar (1956), who stated that the solution

\[
U = U_\Lambda
\]  

of the MHD equations is linearly stable for ideal and incompressible flows. Here, \( U \) is the flow aligned with the magnetic field \( B = \sqrt{4\pi \rho} U_\Lambda \), with \( U_\Lambda \) denoting the Alfvén velocity of the field. The fluid mass density is \( \rho \) and the vacuum permeability is \( \mu_0 \). Tataronis & Mond (1987) studied the stability of the plasma for the more general setting

\[
U = M_m U_\Lambda
\]  

with the magnetic Mach number \( M_m \) as a constant, and discussed the destabilizing effects of \( M_m \approx 1 \). In the present paper we shall present the destabilizing effects of finite diffusivities (viscosity and/or magnetic diffusivity) for a special realization of Equations (1) and (2), i.e., for Taylor–Couette flows of electrically conducting fluids between rotating concentric cylinders. We shall show that the Chandrasekhar theorem no longer holds if at least one of the two diffusivities has a finite value. For such fluids, it is no problem to find unstable solutions even for \( M_m = 1 \). This work is also concerned with results by a short-wave approximation for deriving an analytical expression for marginal instabilities in the inductionless limit (Kirillov & Stefani 2013; Kirillov et al. 2014).

As for the astrophysical relevance of our study, we note that the interaction of rotating flows with (dominantly) toroidal magnetic fields is a ubiquitous phenomenon in a multitude of cosmic objects, ranging from planetary cores via (neutron) stars to accretion disks and galaxies. The magnetic Mach number almost always exceeds unity, with typical values of 10 for galactic (Elstner et al. 2014), about 30 for the solar tachocline (when assuming a field of 1 kG), and values between 0.1 and 1 for magnetars (e.g., 1E1547.0+5028 with a rotation period of 2 s and a dipolar field of \( 2 \times 10^{14} \) G) and 1000 for neutron stars. Typical field strengths and rotation velocities of White Dwarfs are 1 MG and \( 2 \text{ km s}^{-1} \). Combined with the characteristic
density of $10^6$ g cm$^{-3}$, the magnetic Mach number of these compact objects turns out to be about 600. Even for the largest observed fields of 100 MG the numerical value of Mm is larger than one.

A similar variety can be found for the dependency of the angular velocity $\Omega = U_0/R$ on the radius $R$, which is often quantified by $q = \partial \log \Omega / \partial \log R$, and for the corresponding radial dependence of the azimuthal magnetic field $B_\phi$. While the rotation laws of thin accretion disks, as well as the rotation within the cores of low-mass red giant stars (Ceillier et al. 2012), are characterized by $q \approx -1.5$ ("Kepler rotation"), galaxies have a mean value of $q = -1$ which, however, vanishes in the inner rigidly rotating parts of the galaxy. A particularly rich behavior is observed in the solar tachocline, with $q \approx -10$ toward the poles but positive values at the equator (Parfrey & Menou 2007), so the latitudinal shear should play the dominant role for the instability theory of the tachocline (Watson 1981; Arlt et al. 2007; Kitchatinov & Rüdiger 2008).

Much less is known about the detailed radial dependence of the azimuthal field $B_\phi$. The hypothetical case $B_\phi \propto R$ corresponds to an insulated central electric current, while the $B_\phi \propto R$ results from a constant axial current. The latter configuration would be prone to the current-driven, kink-type Taylor instability (Pitts & Tayler 1985), while the former may be subject to the shear-driven azimuthal magnetorotational instability “AMRI” (Rüdiger et al. 2014b). Both types of instabilities have indeed been discussed to be at work, separately or together, in various regions of the solar tachocline (Kagan & Wheeler 2014).

It is, however, not the intention of the present paper to discuss in detail the peculiarities of specific cosmic bodies. By choosing Chandrasekhar’s solution as a theoretical reference point, and discussing various radial profiles of $\Omega$ and $B_\phi$, as well as various values of Mm, we rather hope to contribute to a better systematics of shear and current-driven instabilities, with particular focus on the role of the magnetic Prandtl number.

Cosmic objects possess various magnetic Prandtl numbers (see Brandenburg & Subramanian 2005, Table 1) reaching from $Pr \approx 0.05$ for the solar tachocline (Gough 2007) and $Pr \approx 1$ for low-mass red giants (Rüdiger et al. 2014b), to $Pr \gg 1$ for neutron stars. If the interstellar turbulence is imagined as being driven by supernova explosions and stellar winds then $Pr \approx 1$ can also be assumed for galaxies (based on the turbulent values of viscosity and resistivity). With the molecular values of the interstellar medium its magnetic Prandtl number would be much higher. We shall see how the combination of magnetic Mach number and magnetic Prandtl number influences the stability/instability of the considered magnetized fluids.

2. BASIC EQUATIONS

Consider the interaction of the differential rotation in an axially unbounded Taylor–Couette container and a toroidal magnetic field between the inner and the outer cylinder that is maintained by axial electric currents outside and/or inside the inner cylinder. The fluid possesses the microscopic viscosity $\nu$ and the magnetic resistivity $\eta = 1/\mu_0 \sigma$ ($\sigma$ the electric conductivity). The general solution of the stationary and axisymmetric equations is

$$U_0 = R \Omega = a_\Omega R + \frac{b_\Omega}{R}, \quad B_\phi = a_B R + \frac{b_B}{R},$$  \hspace{2cm} (3)

where $a_\Omega$, $b_\Omega$, $a_B$, and $b_B$ are constants that fulfill the condition (1) if $a_\Omega = a_B/\sqrt{\mu_0 \sigma}$ and $b_\Omega = b_B/\sqrt{\mu_0 \sigma}$. The most popular realization of the condition (1) is the rotating pinch in which an axial and uniform-in-radius electric current is subject to rigid-body rotation where both the azimuthal flow and the azimuthal field linearly depend on the radius $R$ (Roberts 1956; Acheson 1978; Pitts & Tayler 1985; Rüdiger & Schulz 2010).

Another very special example of the stability problem appears for $a_\Omega = a_B = 0$, describing the interaction of the rotation law $\Omega \propto 1/R^2$ (the Rayleigh limit) with the field $B_\phi \propto 1/R$, which is current-free between the cylinders. These profiles fulfill the condition (2), but even for Mm = 1 they become unstable against non-axisymmetric perturbations with the azimuthal wave number $m = \pm 1$ if one of the two diffusivities does not vanish. Because of its current-free character we have called it the AMRI (Hollerbach et al. 2010; Rüdiger et al. 2013), which recently has been observed in a laboratory experiment with the liquid alloy GaInSn as the conducting fluid (Seilmayer et al. 2014).

The magnetized Taylor–Couette flow is governed by the ratios

$$r_m = \frac{R_{in}}{R_{out}}, \quad \mu_\Omega = \frac{\Omega_{out}}{\Omega_{in}}, \quad \mu_B = \frac{B_{out}}{B_{in}}. \hspace{2cm} (4)$$

$R_{in}$ and $R_{out}$ are the radii of the inner and the outer cylinder, $\Omega_{in}$ and $\Omega_{out}$ their rotation rates, and $B_{in}$ and $B_{out}$ are the azimuthal magnetic fields at the inner and outer cylinders. Conditions (1) and (2) then require $\mu_\Omega = \mu_B = 1$. For our standard model, which works with $r_m = 0.5$, one finds $\mu_\Omega = 0.25$ and $\mu_B = 0.35$, modeling the Rayleigh limit of uniform angular momentum and a quasi-Keplerian rotation law with cylinders rotating as $R^{-3/2}$ (like planets).

The governing dimensionless and linearized equations for the evolution of the flow perturbations $u$ and the field perturbations $b$ are

$$\text{Re} \left( \frac{\partial u}{\partial t} + (U \cdot \nabla)u + (u \cdot \nabla)U \right) = -\nabla p + \Delta u + \nabla^2 \vec{H} \cdot (\nabla \times b \times B + \nabla \times \nabla \times b) \hspace{2cm} (5)$$

and

$$\text{PmRe} \left( \frac{\partial b}{\partial t} - \nabla \times (U \times b) \right) = \nabla \times \nabla \times \nabla \times b. \hspace{2cm} (6)$$

Here the unit of the fluctuating and mean magnetic fields is a characteristic scale $B_0$ of the background field. The mean flow $U$ is normalized with a characteristic flow amplitude $U_0$, the flow perturbations with $\eta/L$ (with $L$ as a characteristic distance), and the time with $L/U_0$. The equation system is numerically solved with div $u = \text{div} b = 0$ for no-slip boundary conditions and for insulating or perfect-conducting cylinders unbound in axial direction. The boundary conditions are applied at both $R_{in}$ and $R_{out}$.

The dimensionless free parameters in (5) and (6) are the Reynolds number (Re) and the Hartmann number (Ha), defined
For very small \( H \) approximately at \( W = 2 \) for \( H \) the plane \( P_m \) turns \( m \) in \( R_m \) plane in Figure 2. A1 numbers. They are plotted in the \( R_m - \) Ha-plane (left panel) and in the \( R_m - \) Ha plane (right panel). Note the crossing points of the line \( M_m = 1 \) (dotted) with the curve of marginal instability. The results are obtained for perfect conducting boundaries.

\[
\text{Re} = \frac{\Omega_m D^2}{\nu}, \quad \text{Ha} = \frac{B_m D}{\sqrt{\mu_0 \rho \nu}},
\]

where \( D = R_{\text{out}} - R_m \) is the unit of length which here is always \( D = R_{\text{out}}/2 \). With the magnetic Prandtl number \( P_m = \nu/\eta \) the magnetic Reynolds number of the rotation is \( R_m = P_m \text{Re} \). The Lundquist number of the magnetic field is \( S = \sqrt{P_m \text{Ha}} \). As will be shown below, the modified magnetic Reynolds number

\[
\bar{R}_m = \sqrt{\text{Re} \bar{R}_m}
\]

will also play an important role as a counterpart of the Hartmann number. The reason is that \( \bar{R}_m/\text{Ha} \) defines the magnetic Mach number \( M_m \) in (2), which does not depend on the diffusivities. The well-established linear code by Rüdiger & Schultz (2010) solves the above equation system for \( m = \pm 1 \) after a Fourier transformation in the azimuthal direction together with the appropriate boundary conditions. In the present paper we mainly, but not always, apply vacuum boundary conditions to the magnetic fields. It is important to stress that the calculations showed our basic findings as independent of the choice of the boundary conditions. All the minima of the instability curves in the \( \text{Ha} - \) Re-plane and the characteristic eigenvalues \( \text{Ha}_{\text{Tap}} \) at the \( \text{Ha} \) axis for resting cylinders exist for both sorts of boundary conditions. For the most part, the numerical values of the characteristic Reynolds and Hartmann numbers for conducting cylinders slightly exceed those for insulating ones.

3. THE RAYLEIGH LIMIT

Figure 1 shows the lines of marginal stability of the non-axisymmetric \( m = \pm 1 \) mode for the rotation law with \( U_0 \propto B_0 \propto 1/R \) (i.e., \( \mu_\theta = 0.25, \mu_\phi = 0.5 \)). In hydrodynamics, \( \mu_\theta = 0.25 \) corresponds to the so-called Rayleigh limit of radially constant angular momentum that distinguishes between hydrodynamically unstable \( (\mu_\theta < 0.25) \) and stable \( (\mu_\theta > 0.25) \) flow profiles.

For a given supercritical Hartmann number the instability mode always exists between a minimum Reynolds number and a maximum Reynolds number. The lower branch of the instability cone defines the critical rotation rate, which is necessary to provide the needed energy for the pattern maintenance. The upper branch limits the instability domain by suppressing the non-axisymmetric instability using shear that is too strong.

For very small \( P_m \) the curves converge and form a common minimum Reynolds number at a certain critical Hartmann number (Figure 1, left panel). While the value of the minimum Reynolds number decreases monotonically for growing magnetic Prandtl number, the lowest critical Hartmann number is reached for \( P_m \) of order unity. For very small \( P_m \) the minimum of the instability cone scales with \( \text{Re} \), here with a value of about \( \text{Re} \approx 800 \), while the typical Hartmann number is approximately 10 times less.

However, the modified magnetic Reynolds number \( \bar{R}_m \) turns out to be most appropriate for illustrating the destabilization of Chandrasekhar’s solution. The dotted line in Figure 1 (right panel) is defined by \( \bar{R}_m = \text{Ha} \), representing all points fulfilling (1). Remarkably, all of the bifurcation lines for finite \( P_m \) cross the (dotted) Chandrasekhar line at some distinct points. At these crossing points the flow becomes unstable despite the stability criterion (1). Since the instability at these crossing points arises only from the action of the diffusivities, the instability should be called a diffusive one. Away from these crossing points the character of the instability is less obvious.

The numerical values of the crossing points—where the diffusive character of the instability is evident—are plotted in three different representations as functions of \( P_m \) in Figure 2. The solid (dashed) lines correspond to models with perfect-conducting (insulating) cylinders. Except for \( P_m \approx 1 \), both cases lead to very similar results. In the \( R_m - P_m \) plane (left panel) one finds minimal \( \bar{R}_m \) approximately at \( P_m \sim 10^{-2} \) and \( P_m \sim 10^2 \). For \( P_m = 1 \) the curves have a local maximum that reflects the phenomenon that the main part of the cones for \( P_m > 1 \) lies above the dotted line in Figure 1 (right), while it lies below the dotted line for \( P_m < 1 \). For \( P_m \rightarrow 0 \) the complete cone formed by the line of marginal instability lies above \( M_m = 1 \), while its upper branch is approaching \( M_m = 1 \). For \( P_m \rightarrow \infty \) this cone is completely located above \( M_m = 1 \) but in this case its lower branch approaches \( M_m = 1 \). Note also that the Hartmann number for instability takes its smallest values for \( P_m \approx 1 \). For a small enough \( P_m \) the minimum Hartmann numbers for instability no longer depend on the magnetic Prandtl number.
In the limit $Pm \to 0$ the Hartmann numbers and—what is here the same—the modified magnetic Reynolds numbers $Rm$ of the crossing points run with $Pm^{-1/2}$, which means that the magnetic Reynolds number $Rm$ of the rotation (and the Lundquist number $S$ of the magnetic field) remain finite. The middle panel of Figure 2 demonstrates that the solutions for $Pm \to 0$ possess values of $Rm \simeq 0.8$ for perfect-conducting cylinders and $Rm \simeq 2$ for insulating cylinders. Translated to the $Rm$–$Ha$-plane in Figure 1 (right), in the limit $Pm \to 0$ the instability line does not cross the Chandrasekhar line $Mm = 1$ for finite values of $Rm$. For finite but small $Pm$ (i.e., $Pm \lesssim 0.01$) it always crosses this line for $Rm = O(1)$, which means that the form of the instability line follows the quadratic rule $Re \simeq Ha^2$, which for $Mm = 1$ indeed yields $Rm \simeq 1$. For infinitely small $Pm$ the rotation rate of the crossing point remains finite as it runs as $\Omega \simeq \eta/D^2$.

Similar results, but now scaling with the Reynolds number $Re$, are obtained in the limit $Pm \to \infty$ (Figure 2, right). The influence of the boundary conditions is here much weaker and it can be seen in both cases that $\Omega \simeq 2.9\eta/D^2$ for finite but large $Pm$.

To summarize this section, we have shown that the special Chandrasekhar solution with a rotation law of the Rayleigh limit and a current-free magnetic field becomes unstable under the influence of viscosity and/or resistivity. The limits $Pm \to 0$ and $Pm \to \infty$ demonstrate that only one non-vanishing diffusivity is needed for the destabilization. For infinitely small $Pm$ the line of marginal instability lies completely below the line $Mm = 1$, i.e., for very small magnetic Prandtl number the instability only exists for slow rotation, $Mm < 1$. For infinitely large $Pm$ the lines of marginal instability always lie above the Chandrasekhar line $Mm = 1$ so that the instability would only exist for rapid rotation, i.e., $Mm > 1$.

4. QUASI-KEPLERIAN ROTATION

The quasi-Keplerian rotation profile in our Taylor–Couette setup approximately corresponds to $\mu_\Omega = 0.35$. With this value fixed, and selecting $Pm = 0.001$, Figure 3 shows the bifurcation maps of the instability of the $m = \pm 1$ modes for various radial profiles of the azimuthal magnetic field. Toroidal fields whose radial profiles differ from the above case with $B_\phi \propto R^{-1}$ can only be maintained by an additional axial current within the fluid.\footnote{In experiments these profiles could be adjusted by changing the ratio of the electric current through the liquid to the current along the axis.} For resting cylinders ($Re = 0$), an immediate consequence is the appearance of the current-driven, kink-type Taylor instability at a certain critical Hartmann number.

The curve with $\mu_B = 0.5$ again corresponds to a purely central current, whereas $\mu_B = 0.7$ fulfills the condition (2) for the considered quasi-Keplerian flow $\mu_\Omega = 0.35$. Though not completely shown in the figure, all curves, except the current-free one with $\mu_B = 0.5$, meet the horizontal axis at finite values $Ha_{Tay}$ that do not depend on the value of $Pm$. The calculations lead to $Ha_{Tay} = 2565$ for $\mu_B = 0.7$ (not shown) and $Ha_{Tay} = 760$ for $\mu_B = 0.75$ (shown). These critical Hartmann numbers are significantly reduced if the instability is allowed to tap into the additional energy source of differential rotation. We find $Ha_{min} = 250$ for $\mu_B = 0.7$. As the comparison of the curves for $\mu_B = 0.5$ and $\mu_B = 0.7$ in Figure 3 shows, the minimal Reynolds number is also reduced by the inclusion of electric currents through the fluid.

The instability cones in the Re–Ha-plane for the quasi-Keplerian radial profiles $\mu_\Omega = \mu_B/2 = 0.35$ are given in Figure 4 (left) for various magnetic Prandtl numbers. Evidently, the minimum critical Hartmann and Reynolds numbers exceed the corresponding values for the Rayleigh limit profiles (see Figure 1, left) by almost one order of magnitude. For small magnetic Prandtl numbers the instability cones do not depend on $Pm$, so they thus again scale with the Reynolds number $Re$ and the Hartmann number $Ha$. This is in great contrast to the combination of quasi-Keplerian rotation ($\mu_\Omega = 0.35$) with the current-free magnetic field ($\mu_B = 0.5$) shown in Figure 3, which is known to scale for small $Pm$ with $Rm$ rather than with $Re$, and with the Lundquist number $S$.\footnote{In experiments these profiles could be adjusted by changing the ratio of the electric current through the liquid to the current along the axis.}
Figure 4. Same as in Figure 1 but for the quasi-Keplerian rotation law with $\mu_{\Omega} = \frac{\mu_a}{2} = 0.35$. The curves are plotted in the Re–Ha plane (left) as well as in the $R_m$–Ha plane (right). For a small enough $Pm$ the minimum Hartmann numbers and the minimum Reynolds numbers for instability no longer depend on the magnetic Prandtl number. In the limit $Pm \to 0$ the curve completely lies below the Chandrasekhar line $Mm = 1$ (dotted). The results are obtained for insulating boundaries.

Figure 5. Same as in Figure 2 but for the quasi-Keplerian rotation law with $\mu_{\Omega} = \frac{\mu_a}{2} = 0.35$. The results are obtained for perfect conducting boundaries (solid) and insulating boundaries (dashed).

rather than Ha (Rüdiger et al. 2013, 2014). Obviously, the additional energy source connected with the axial electric current in the fluid basically changes the scaling rules for small $Pm$.

The same instability numbers but in the $R_m$–Ha-plane are given in Figure 4 (right). Again, for small $Pm$ the minimum critical Hartmann number and modified magnetic Reynolds number for the Kepler profiles exceed the corresponding values for the Rayleigh limit profiles. Except for these differences the basic features of the instability maps of Figure 4 are similar to those of Figure 1. For decreasing $Pm$ the cones migrate downward but continue to cross the Chandrasekhar line (1) in only one point. Each of these crossing points represents a marginal unstable solution for a configuration that would fulfill the stability condition (1) for ideal fluids.

As in Figure 2, in Figure 5 the actual coordinates of the crossing points are plotted in dependence on the magnetic Prandtl number for both sets of boundary conditions. In contrast to the situation at the Rayleigh line for $Pm \to 0$, neither $R_m$ nor $Rm$ show constant limit values. One finds instead $Rm \propto Pm^{1/3}$, hence for small $Pm$

$$\frac{\Omega m D^2}{\sqrt{\nu \eta^2}} \approx 100.$$  

(9)

There is a characteristic difference, therefore, of the behavior of the crossing points for $Pm \to 0$. While the magnetic Reynolds number for the flow at the Rayleigh limit remains finite, it slowly vanishes for quasi-Keplerian rotation. The behavior of the critical Reynolds number, however, is the same: in all cases it becomes infinitely large for $Pm \to 0$ so that again in this case the line of marginal instability completely lies below $Mm = 1$ (the dotted line in Figure 4).

Comparing Figures 2 and 5, one finds that for $Pm$ of order $10^{-3}$ or $10^{-6}$ (the values for fluid metals like sodium or gallium) the magnetic Reynolds number and the Hartmann number of the crossing points in both cases are almost equal. This remains approximately true for higher magnetic Prandtl numbers up to $Pm \lesssim 0.01$.

5. BEYOND THE KEPLER LIMIT

In this section flow and field profiles that are more flat than the Keplerian one are considered, again focusing on $\mu_R = 2\mu_{\Omega}$.

In Figure 6 we show that for the rotation laws with $\mu_{\Omega} = 0.35$–0.5 for $Pm \to 0$ the instability cones in the $Rm$–Ha-plane also do not change their location for various but small magnetic Prandtl numbers $Pm$. For all of the considered models with $\mu_R = 2\mu_{\Omega}$ the instability therefore scales with the Hartmann and Reynolds number. One can even conclude from the linear system (5), (6) that all of their solutions that exist in the quasi-stationary approximation for $Pm = 0$ (which is not identical to $\nu = 0$) basically scale with Ha and Re.

Typically, the instability domains in Figure 6 form cones that are open for large values of Re and Ha. Until $\mu_{\Omega} = 0.37$ the instability exhibits a minimum and two branches with positive
slope, i.e., the rotation can be too slow or too fast and the magnetic field can be too weak or too strong for the instability.

Since all models with \( \mu_B > 0.5 \) contain an axial electric current within the fluid that becomes Tayler-unstable at sufficiently slow rotation (Tayler 1973), the instability line for these modes always meets the abscissa (i.e., \( \text{Re} = 0 \)) at some critical Hartmann number \( H_{\text{Tay}} \). The latter value becomes smaller for increasing electric current. As Figure 6 shows, for \( \mu_{1G} = 0.38 \) the \( H_{\text{Tay}} \) already becomes so small that the former typical minimum of the critical curve disappears. The slope of the lower branch of the instability cone becomes negative so that the requirement of a minimum critical rotation rate no longer exists. It is obvious that the stronger electric current in the fluid changes the character of the instability from shear-driven to current-driven. Figure 6 also demonstrates that the curves for \( Pm = 10^{-5} \) and \( Pm = 10^{-6} \) are almost identical, hence the scaling law of the solution with \( \text{Re} \) and \( H \) for small \( Pm \) also remains valid in this current-dominated case. The lines of marginal instability of the modes \( m = \pm 1 \) that we computed for all models with \( U_0 \propto B_0 \) scale with \( \text{Re} \) and \( H \) for \( Pm \rightarrow 0 \).

Galaxies are the only cosmic objects whose internal flows and fields can simultaneously be observed. Their magnetic fields of several tens of \( \mu_G \), with densities of order \( 10^{-24} \text{ g cm}^{-3} \), correspond to an Alfvén-velocity of about \( 35 \text{ km s}^{-1} \), which is rather uniform with radial scales of \( 1-10 \text{ kpc} \). The almost uniform linear velocity of the rotation (here \( \mu_{1G} = 0.5 \)) lies between \( 100 \) and \( 200 \text{ km s}^{-1} \) hence also galaxies are super Alfvén flows. If the radial profile of \( B_0 \) in the outer parts of the disk is assumed to be almost uniform (i.e., \( \mu_B = 1 \)) they fulfill condition (2), with \( \text{Mm} \approx 5-10 \). If the interstellar turbulence can be imagined as externally driven then \( Pm \approx 1 \) and \( \eta \approx 3 \times 10^{26} \text{ cm}^2 \text{ s}^{-1} \) (Gressel et al. 2008). The Reynolds number thus is of order \( 10^3 \) and the Hartmann number is only slightly larger than 100. For \( \mu_B = 1 \) it is \( H_{\text{Tay}} = 109 \) for vacuum boundary conditions. After the right plot of Figure 6 for \( Pm = 1 \) and \( \mu_B = 1 \) one finds \( H_a < 100 \) as the stability condition for \( \text{Re} = 1000 \) (dotted line), so the majority of the galaxies could marginally be unstable. As the magnetic Mach numbers in the halos are reported as close to unity the existence of the magnetic instability along the Chandrasekhar line (1) is suggested there. Our model, however, is not able to describe real galaxies with vertical variations of the background fields.

### 6. Azimuthal Pattern Drift and Growth Rates

The change in the character of the instability between shear-driven and current-driven is also reflected in the drift rates of the \( m = \pm 1 \) modes. For a Fourier mode with the definition \( \exp(i(\omega t + m\phi)) \) the drift rate (as the real part of the complex Fourier frequency \( \omega \))

\[
\omega_d = -m \frac{\partial \phi}{\partial t}
\]

has the opposite sign as the azimuthal migration of the pattern. While (at small \( Pm \)) the instability patterns for the Rayleigh rotation (\( \mu_{1G} = 0.25 \)) and the Kepler rotation (\( \mu_{1G} = 0.35 \)) migrate in the direction of the global rotation, it nearly rests in the laboratory system for the sub-Keplerian rotation laws with \( \mu_{1G} > 0.35 \) (see Figure 7). Both extrema are known: the pattern of the AMRI at the Rayleigh line tends to rotate with the outer cylinder while the Tayler instability without rotation basically rests in the laboratory system.

While this paper focuses on the marginal lines for the non-axisymmetric modes with \( |m| = 1 \) and their scaling for small magnetic Prandtl numbers, the necessary next step for applications to particular astrophysical problems is a detailed study of the growth rates, including those of the modes with larger \( m \). What is known is the typical growth rate of the AMRI being proportional to \( \Omega \) (although with a smaller pre-factor than...
for standard MRI), and the two scalings for the Tayler instability \( \sim \Omega_\Lambda \) in the high-conductivity limit and \( \sim \Omega_\Lambda^2/\Omega \) in the low-conductivity limit. The corresponding scaling on the Chandrasekhar line as well as the stability characteristics of the modes with \( |m| > 1 \) are yet to be explored.

7. CONCLUSIONS

The numerical study of azimuthal fields interacting with Taylor–Couette flows of conducting fluids has also shown that the configuration fulfilling Equation (1) can become unstable against non-axisymmetric perturbations in fluids with finite resistivity and/or viscosity. This diffusive or dissipation-induced instability is a perfect illustration of Montgomery’s (1993) verdict “…that for fluid equations of the Navier–Stokes type the ideal limit with zero dissipation coefficients has essentially nothing to do with the case of small but finite dissipation coefficient.” In particular, at the Rayleigh limit with the rotation law \( \Omega \propto R^{-2} \) the existence of at least one of the two diffusivities enables the instability to occur. For \( Pm \to 0 \) we find the condition

\[
\frac{\Omega_m D^2}{\eta} \approx 0.8, \quad (11)
\]

(for conducting cylinders) where \( \Omega_m \) is the maximally possible rotation rate, which is very low for stellar parameters. Equation (11) also means that \( S = 0.8 \), which is extremely low for stars. A condition similar to (11) holds for \( Pm \to \infty \), with \( \eta \) replaced by \( \nu \), which, however, now defines a minimal rotation rate. The Reynolds number of neutron stars that belong to the AMRI at the Rayleigh limit but it is now clear that configurations with (2) belong to the same class (see Figure 6).

The curves always possess a minimum Hartmann number, and for \( \mu_\Omega \lesssim 0.37 \), they also possess a local minimum of the Reynolds number. For \( \mu_\Omega \gtrsim 0.38 \) this minimum disappears so that the lower branch acquires a negative slope everywhere until it meets the abcissa where \( Re = 0 \). The current-driven Tayler instability appears for all \( \mu_B > 0.5 \) and sufficiently strong field amplitude. The critical Hartmann number \( H_{\text{Tay}} \) (for resting cylinders) is decreasing for increasing \( \mu_B \). For \( \mu_B = 1 \) it is \( H_{\text{Tay}} \approx 109 \) (150) for vacuum (conducting) boundary conditions. The \( H_{\text{Tay}} \) never depends on the magnetic Prandtl number of the fluid. Rigid rotation suppresses the Tayler instability while differential rotation with \( \mu_\Omega > 1 \) acts supportingly. Hence, for slow rotation the instability lines above the Ha axis always turn to the left. Again, for \( Pm \ll 1 \) these positions do not depend on \( Pm \).

One can also recognize the transition from the AMRI to the Tayler instability for increasing \( \mu_\Omega \) by means of the drift frequency of the magnetic non-axisymmetric pattern that develops from corotation with the outer cylinder (for AMRI) to resting in the laboratory system (for Tayler instability). The corresponding experiments have meanwhile been realized in the laboratory using liquid metals (Seilmayer et al. 2012, 2014). Indeed, after Figure 7 the instability pattern for rotation laws that are flatter than the Keplerian one increasingly loses its azimuthal migration in the laboratory system.

For the range \( 0.25 \leq \mu_\Omega \leq 0.5 \) we have proven that for background fields and background flows with the same radial profile (i) for \( Pm \ll 1 \) the bifurcation lines for \( m = \pm 1 \) in the Ha–Re-plane do not depend on \( Pm \) and (ii) for infinitely small \( Pm \) the magnetic Mach number along the line of marginal instability does not exceed unity. The same should also be true for a rigidly rotating pinch with \( \mu_\Omega = 1 \) and \( \mu_B = 2 \), which also belongs to the class of models fulfilling condition (2). These magnetic field profiles—and all with \( \mu_B = 1/\mu_r \)—are due to a homogeneous electric current that axially flows through the fluid conductor. If the flow and field cannot be expressed in form of (2) then the instability is governed by the Lundquist number \( S \) of the field and the magnetic Reynolds number \( Rm \) of the rotation, as is the case for those AMRI solutions whose rotation laws are flatter than in the Rayleigh limit.

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