Modeling of Tilting-Pad Journal Bearings with Transfer Functions

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Tilting-pad journal bearings are widely used to promote stability in modern rotating machinery. However, the dynamics associated with pad motion alters this stabilizing capacity depending on the operating speed of the machine and the bearing geometric parameters, particularly the bearing preload. In modeling the dynamics of the entire rotor-bearing system, the rotor is augmented with a model of the bearings. This model may explicitly include the pad degrees of freedom or may implicitly include them by using dynamic matrix reduction methods. The dynamic reduction models may be represented as a set of polynomials in the eigenvalues of the system used to determine stability. All tilting-pad bearings can then be represented by a fixed size matrix with polynomial elements interacting with the rotor. This paper presents a procedure to calculate the coefficients of polynomials for implicit bearing models. The order of the polynomials changes to reflect the number of pads in the bearings. This results in a very compact and computationally efficient method for fully including the dynamics of tilting-pad bearings or other multiple degrees of freedom components that interact with rotors. The fixed size of the dynamic reduction matrices permits the method to be easily incorporated into rotor dynamic stability codes. A recursive algorithm is developed and presented for calculating the coefficients of the polynomials. The method is applied to stability calculations for a model of a typical industrial compressor.

Keywords: Rotating machine dynamics, Tilting-pad bearings, Dynamic reduction, Stability calculations, Transfer functions

INTRODUCTION

Barrett et al. (1988) and Brockett and Barrett (1993) showed the importance of including all dynamic coefficients of the tilting-pad bearing for stability analysis. In general a tilting-pad bearing, like that shown in Fig. 1, with \( n \) pads has \( 2(5n + 4) \) coefficients. The variable number of coefficients could present a problem in some analysis programs since the number of equations of motion completely...
representing the bearings will change with the number of pads. Brockett and Barrett (1993) showed that by using dynamic condensation equivalent stiffness and damping coefficients can be obtained which are functions of the complex frequency s. Thus the number of equations representing a tilting-pad bearing remains fixed although the bearing coefficients are variable throughout an analysis. The dynamic condensation method used by Brockett and Barrett (1993) follows that developed by Leung (1978, 1989) for structural analysis. Conversely, if a single input – single output transfer function is known as a ratio of polynomials with known coefficients, an equivalent mass–stiffness–damping matrix model can be obtained (Maslen and Bielk, 1992).

The work presented here shows another approach. Instead of reducing the pad degrees of freedom for each complex frequency s, complex dynamic coefficients are calculated as a function of the complex variable s. The complex dynamic coefficients are represented by a ratio of polynomials or transfer functions. The advantage of this approach is that the resultant dynamic coefficient matrix is always $2 \times 2$. A different number of pads will only change the order of the polynomials but the general size of the matrix remains the same. This approach facilitates the inclusion of tilting-pad bearings in analysis tools because it requires few modifications from the standard eight coefficient hydrodynamic bearings. If the analysis tools already include magnetic bearing control transfer function models, this method can be used directly and no modifications are required (Brockett and Barrett, 1995). The transfer function representation of tilting-pad bearings also facilitates understanding the behavior of the bearing as a function of the vibration frequency.

**PROCEDURE**

Tilting-pad bearing models can be expressed by

$$
\begin{bmatrix}
M & 0 \\
0 & I_p
\end{bmatrix}
\begin{bmatrix}
\ddot{u} \\
\dot{\delta}
\end{bmatrix} +
\begin{bmatrix}
C_{au} & C_{al} \\
C_{bu} & C_{bl}
\end{bmatrix}
\begin{bmatrix}
\dot{u} \\
\dot{\delta}
\end{bmatrix} +
\begin{bmatrix}
K_{au} & K_{al} \\
K_{bu} & K_{bl}
\end{bmatrix}
\begin{bmatrix}
u \\
\delta
\end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}.
$$

(1)

Dynamically reducing the pad degrees of freedom from Eq. (1) we get

$$
s^2[M]\{u\} + \left\{s[C_{au} + K_{au} - [sC_{al} + K_{al}]
\times [s^2I_p + sC_{bl} + K_{bl}]^{-1}
\times [sC_{bu} + K_{bu}]\right\}\{u\} = \{f\}.
$$

(2)

**FIGURE 1** Schematic of a tilting-pad journal bearing.
or

\[ s^2[M]\{u\} + \{K_{eq}(s)\}\{u\} = \{f\}, \]  

where

\[ K_{eq}(s) = \left\{ \begin{array}{l}
[sC_{xx} + K_{xx}] - [sC_{xy} + K_{xy}]
\times [s^2I_p + sC_{\delta\delta} + K_{\delta\delta}]^{-1} [sC_{\delta\delta} + K_{\delta\delta}]
\end{array} \right\}. \]

For an \( n \) pad bearing

\[ K_{eq}(s) = \begin{bmatrix}
 sC_{xx} + K_{xx} & sC_{xy} + K_{xy} \\
 sC_{yx} + K_{yx} & sC_{yy} + K_{yy}
\end{bmatrix} - \sum_{i=1}^{n} \frac{(sC_{i\delta} + K_{i\delta})(sC_{\delta\delta} + K_{\delta\delta})}{s^2I_p + sC_{\delta\delta} + K_{\delta\delta}}, \]

The \( l, m \) equivalent complex bearing coefficients can be written as

\[ K_{eq,lm}(s) = sC_{lm} + K_{lm} - \sum_{i=1}^{n} \frac{(sC_{i\delta} + K_{i\delta})(sC_{\delta\delta} + K_{\delta\delta})}{s^2I_p + sC_{\delta\delta} + K_{\delta\delta}}. \]

Using partial-fraction expansion the summation term can be written as

\[ \sum_{i=1}^{n} \frac{(sC_{i\delta} + K_{i\delta})(sC_{\delta\delta} + K_{\delta\delta})}{s^2I_p + sC_{\delta\delta} + K_{\delta\delta}} = \sum_{i=1}^{n} \frac{(sC_{i\delta} + K_{i\delta})(sC_{\delta\delta} + K_{\delta\delta})}{s^2I_p + sC_{\delta\delta} + K_{\delta\delta}} \]

Then Eq. (7) becomes

\[ \sum_{i=1}^{n} \frac{(sC_{i\delta} + K_{i\delta})(sC_{\delta\delta} + K_{\delta\delta})}{s^2I_p + sC_{\delta\delta} + K_{\delta\delta}} = \sum_{i=1}^{2n} sA_{di} + B_{di}, \]

or

\[ \sum_{i=1}^{n} \frac{(sC_{i\delta} + K_{i\delta})(sC_{\delta\delta} + K_{\delta\delta})}{s^2I_p + sC_{\delta\delta} + K_{\delta\delta}} = f_{2n}s^{2n} + f_{2n-1}s^{2n-1} + \ldots + f_1s + f_0. \]

After some manipulation it can be shown that

\[ g_{2n} = 1, \]

\[ g_{2n-1} = \sum_{i=1}^{2n} D_{di}, \]

\[ g_{2n-2} = \sum_{i=1}^{2n-1} D_{di} \sum_{j=1}^{2n} D_{dj}, \]

\[ g_{2n-3} = \sum_{i=1}^{2n-2} D_{di} \sum_{j=1}^{2n-1} D_{dj} \sum_{k=1}^{2n} D_{dk}, \]
\( g_{2n-4} = \sum_{i=1}^{2n-3} Dd_i \sum_{j=i+1}^{2n-2} Dd_j \sum_{k=j+1}^{2n-1} Dd_k \sum_{l=k+1}^{2n} Dd_l \)

\[
\vdots
\]

\( g_1 = \sum_{i=1}^{2n} Dd_i \sum_{j=i+1}^{3} Dd_j \sum_{k=j+1}^{4} Dd_k \cdots \times \sum_{r_{2n-1}=r_{2n-2}+1}^{2n} Dd_{2n-1} \)

\( g_0 = \sum_{i=1}^{2n} Dd_i \sum_{j=i+1}^{2} Dd_j \sum_{k=j+1}^{2n} Dd_k \cdots \sum_{r_{2n-1}=r_{2n-2}+1}^{2n} Dd_{2n-1} \)

\[
= \prod_{i=1}^{2n} Dd_i,
\]

(12)

Therefore

\( K_{eq,m}(s) = \frac{f_{2n+1}^2 s^{2n+1} + f_{2n}^2 s^{2n} + \cdots + f_1^2 s + f_0^2}{g_{2n+1}^2 s^{2n} + g_{2n-1}^2 s^{2n-1} + \cdots + g_1 s + g_0} \)  

(14)

where

\[
f_0' = K_{lm}g_0 - f_0,
\]

\[
f_1' = C_{lm}g_0 + K_{lm}g_1 - f_1,
\]

\[
\vdots
\]

\[
f_{2n-1}' = C_{lm}g_{2n-2} + K_{lm}g_{2n-1} - f_{2n-1},
\]

\[
f_{2n}' = C_{lm}g_{2n-1} + K_{lm}g_{2n} - f_{2n},
\]

\[
f_{2n+1}' = C_{lm}g_{2n}.
\]

Numerically, the complex stiffness coefficients can be expressed as equivalent real stiffness and real damping coefficients for any oscillatory complex frequency, \( s \), from the relationships:

\( k_{eq,m}(s) = \text{Re}(K_{eq,m}(s)) - \frac{\text{Re}(s)}{\text{Im}(s)} \text{Im}(K_{eq,m}(s)) \)

(16)

\( c_{eq,m}(s) = \frac{1}{\text{Im}(s)} \text{Im}(K_{eq,m}(s)) \).

(17)

**NUMERICAL EXAMPLE**

The rotor used for this example, shown in Fig. 2, is an eight stage centrifugal compressor used for natural gas re-injection at an offshore drilling site, running at 5626 rpm. The rotor is approximately 2.8 m long and 954 kg with the mass center near the mid-span. Table I lists the model for the rotor. It is
composed of 35 mass stations with identical tilting-pad bearings acting at nodes 4 and 31. The complete set of tilting-pad bearing coefficients is shown in Table II. The stiffness and damping coefficients for these bearings can be calculated by using several methods (Shapiro and Colsher, 1977; Branagan, 1988). The geometry of the bearings in this analysis are given in Table III. Aerodynamic cross-coupling was assumed to act at station 16 with a value of $2.539 \times 10^6 \text{N/m}$. A stability analysis is performed for this rotor using synchronously reduced coefficients for the bearings assuming that $s = i\omega$ in Eq. (5), where $\omega$ is the shaft spin frequency; a method often employed in rotor stability calculations. The analysis is repeated using the transfer function representation of the bearings calculated using the algorithm presented in this paper. The synchronously reduced bearing coefficients are shown in Table IV. Table V shows the coefficients of the four transfer functions obtained from the coefficients of Table II using the method described in this paper (Eq. (12) and (15)).

### DISCUSSION OF RESULTS

The eigenvalues obtained using the frequency dependent transfer function representation proposed in this paper are shown in Table VI. The
### TABLE II  Tilting-pad bearing coefficients

| Pad coefficients | Number | Kx (N/m) | Ky (N/m) | Kxy (N/m) | Kyx (N/m) | Cxx (N s/m) | Cyx (N s/m) | Cyx (N s/m) | Cyy (N s/m) |
|------------------|--------|----------|----------|-----------|-----------|-------------|-------------|-------------|-------------|
| Pad number       | 1      | 6.08E+07 | 7.33E+07 | 1.83E+08  | 2.19E+08  | 2.65E+05    | -3.10E+04  | -3.07E+04  | 6.32E+05    |
| Pad number       | 2      | 6.08E+07 | 7.33E+07 | 1.83E+08  | 2.19E+08  | 2.65E+05    | -3.10E+04  | -3.07E+04  | 6.32E+05    |
| Pad number       | 3      | 6.08E+07 | 7.33E+07 | 1.83E+08  | 2.19E+08  | 2.65E+05    | -3.10E+04  | -3.07E+04  | 6.32E+05    |
| Pad number       | 4      | 6.08E+07 | 7.33E+07 | 1.83E+08  | 2.19E+08  | 2.65E+05    | -3.10E+04  | -3.07E+04  | 6.32E+05    |
| Pad number       | 5      | 6.08E+07 | 7.33E+07 | 1.83E+08  | 2.19E+08  | 2.65E+05    | -3.10E+04  | -3.07E+04  | 6.32E+05    |

### TABLE III  Bearing characteristics

| Number of pads | Bearing load | m (preload) | Load direction | Radius | Radial clearance | Length | Rotor speed | Offset factor | Pad arc length | Pad moment |
|----------------|--------------|-------------|----------------|--------|-----------------|--------|-------------|---------------|----------------|------------|
| 5              | 4680 N       | 0.0         | On pad         | 63.5 mm| 0.1016 mm       | 63.5 mm| 5626 rpm    | 0.5           | 60°            | 1.13 x 10^-4 kg/m² |

### TABLE IV  Synchronously reduced coefficients

| Number | Bearing load | m (preload) | Load direction | Radius | Radial clearance | Length | Rotor speed | Offset factor | Pad arc length | Pad moment |
|--------|--------------|-------------|----------------|--------|-----------------|--------|-------------|---------------|----------------|------------|
| 5      | 4680 N       | 0.0         | On pad         | 63.5 mm| 0.1016 mm       | 63.5 mm| 5626 rpm    | 0.5           | 60°            | 1.13 x 10^-4 kg/m² |

When using the synchronously reduced bearing coefficients it is not possible to locate some of the eigenvalues in Table VI. The reason for this is that some of the dynamics of the system are neglected when using the synchronously reduced bearing coefficients. This changes the dynamic system being analyzed and different results are expected. In particular, eigenvalues 9–12 in Table VI are directly related to the natural frequencies of the bearing pads, while eigenvalues 3–6 in Table VI are the result of the dependency of the bearing coefficients on the eigenvalues. Figure 3 shows the effect of the reduction frequency on the equivalent coefficients. The coefficients are plotted as a ratio of the synchronously reduced coefficients. It is shown that the reduction frequency has a great effect on
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TABLE V Transfer functions representing the tilting-pad bearings

| Power of s | Num$_{yx}$ | Den$_{xx}$ | Num$_{xy}$ | Den$_{yx}$ | Num$_{yy}$ | Den$_{yy}$ |
|------------|------------|------------|------------|------------|------------|------------|
| 11         | 2.650E+05  | -3.096E+04 | -3.072E+04 | 6.324E+05  |            |            |
| 10         | 2.504E+11  | 1.000E+00  | -6.291E+09 | 1.000E+00  | 5.839E+11  | 1.000E+00  |
| 9          | 5.734E+16  | 1.000E+06  | -2.940E+14 | 1.000E+06  | 1.322E+17  | 1.000E+06  |
| 8          | 4.010E+21  | 2.349E+11  | 2.349E+11  | 9.322E+21  | 2.349E+11  |            |
| 7          | 2.271E+25  | 1.674E+16  | 1.674E+16  | 5.282E+23  | 1.674E+16  |            |
| 6          | 6.774E+28  | 9.598E+19  | 9.598E+19  | 9.598E+19  | 9.598E+19  |            |
| 5          | 1.233E+32  | 2.866E+23  | 2.866E+23  | 2.866E+23  | 2.866E+23  |            |
| 4          | 1.460E+35  | 5.185E+26  | 5.185E+26  | 5.185E+26  | 5.185E+26  |            |
| 3          | 1.106E+38  | 6.001E+29  | 6.001E+29  | 6.001E+29  | 6.001E+29  |            |
| 2          | 5.066E+40  | 4.280E+32  | 4.280E+32  | 4.280E+32  | 4.280E+32  |            |
| 1          | 1.377E+43  | 1.645E+35  | 1.645E+35  | 1.645E+35  | 1.645E+35  |            |
| 0          | 1.888E+45  | 2.542E+37  | 2.542E+37  | 5.853E+45  | 2.542E+37  |            |

TABLE VI Eigenvalues using the transfer function representation of the tilting-pad bearings

| Eigenvalue no. | Damping exp. (s$^{-1}$) | Frequency (rpm) |
|----------------|--------------------------|-----------------|
| 1              | -11.74                   | 2008            |
| 2              | -2.76                    | 2101            |
| 3              | -124.8                   | 3234            |
| 4              | -156.6                   | 3707            |
| 5              | -393.5                   | 4889            |
| 6              | -338.8                   | 5005            |
| 7              | -13.55                   | 7994            |
| 8              | -17.50                   | 8135            |
| 9              | -981.8                   | 11690           |
| 10             | -979.5                   | 11700           |
| 11             | -977.7                   | 11710           |
| 12             | -979.1                   | 11710           |
| 13             | -275.1                   | 13550           |
| 14             | -132.0                   | 15420           |
| 15             | -198.8                   | 16430           |

TABLE VII Eigenvalues using the synchronously reduced coefficients

| Eigenvalue no. | Damping exp. (s$^{-1}$) | Frequency (rpm) |
|----------------|--------------------------|-----------------|
| 1              | -44.31                   | 1996            |
| 2              | -1.23                    | 2122            |
| 3              | -14.44                   | 7977            |
| 4              | -20.67                   | 8147            |
| 5              | -583.7                   | 11430           |
| 6              | -104.7                   | 15640           |
| 7              | -190.3                   | 15850           |

The value of the equivalent coefficients. Therefore the full dynamics of the bearing should be used for stability calculations.

CONCLUSIONS

It is possible to model a tilting-pad bearing with transfer functions using the algorithm presented here. The stability analysis shows that synchronously reduced coefficients can predict eigenvalues significantly different than those predicted using a complete model of the tilting-pad bearings. It is shown that using the synchronously reduced bearing coefficients tends to over-predict the stability of the system while some eigenvalues are eliminated completely from the model. A case is presented where the synchronously reduced coefficients predicted the rotor to be stable while a complete model of the tilting-pad bearing predicted instability. The evidence shown in this work supports the requirement of using all dynamics of the bearing when performing stability analysis. The transfer function representation is a simple way to include all dynamics of the bearing in existing computer codes.

The algorithm presented in this paper is just one of many ways to calculate the coefficients of the transfer functions representing a tilting-pad bearing. This one was chosen partly because it is possible to write each coefficient independently.
FIGURE 3 (a) Effect of the reduction frequency on the equivalent bearing stiffness coefficients. (b) Effect of the reduction frequency on the equivalent bearing damping coefficients.
NOMENCLATURE

\( a_i, b_i, c_i \)  
Coefficients for the partial-fraction expansion

\( d_i, e_i, f_i \)  
Coefficients for the partial-fraction expansion

\( A_i, B_i, C_i, D_i, E_i \)  
Coefficients for the partial-fraction expansion

\( Ad, Bd, Dd \)  
Change of variables to obtain the transfer function coefficients

[\( c_{eq}(s) \)]  
Equivalent bearing damping coefficient matrix as a function of the complex frequency \( s \), \( Ns/m \)

\( C_{ij} \)  
The \( i,j \) equivalent damping coefficients, reduced at a frequency different than synchronous (used in Fig. 3), \( Ns/m \)

\( C_{sij} \)  
The \( i,j \) synchronously reduced damping coefficients (used in Fig. 3), \( Ns/m \)

[\( C_{mn} \)]  
Damping coefficient matrix for the shaft degrees of freedom, \( Ns/m \)

[\( C_{uo} \), \( C_{bu} \)]  
Cross-coupled damping coefficient matrices between the shaft and pad degrees of freedom, \( Ns \)

[\( C_{ss} \)]  
Damping coefficient matrix for the pad degrees of freedom, \( Ns \)

\( f_i, f_t, g_t \)  
Coefficients in the transfer function

[\( I_p \)]  
Pad inertia matrix, \( kg m^2 \)

[\( k_{eq}(s) \)]  
Equivalent bearing stiffness coefficient matrix as a function of the complex frequency \( s \), \( N/m \)

[\( K_{eq}(s) \)]  
Equivalent complex bearing coefficient matrix as a function of the complex frequency \( s \), \( N/m \)

\( K_{eq,i}(s) \)  
The \( l,m \) equivalent bearing coefficients, where \( l \) or \( m \) could be \( x \) or \( y \)

\( K_{ij} \)  
The \( i,j \) equivalent stiffness coefficients, reduced at a frequency different than synchronous (used in Fig. 3), \( N/m \)

\( K_{sij} \)  
The \( i,j \) synchronously reduced stiffness coefficients (used in Fig. 3, \( N/m \)

\( [K_{mn}] \)  
Stiffness coefficient matrix for the shaft degrees of freedom, \( N/m \)

\( [K_{uo}], [K_{bu}] \)  
Cross-coupled stiffness coefficient matrices between the shaft and pad degrees of freedom, \( N \)

\( [K_{ss}] \)  
Stiffness coefficient matrix for the pad degrees of freedom, \( N-m \)

\( m \)  
Pad preload, dim.

\( n \)  
Number of pads

\( s \)  
Complex frequency, \( 1/s \)

\( \{u\} \)  
Displacement vector for the shaft degrees of freedom \( \equiv \{x,y\}^T \), \( m \)

\( \{x,y\} \)  
Displacement coordinates of the shaft in the horizontal and vertical directions, \( m \)

\( \alpha \)  
Pad offset factor, dim.

\( \delta_1, \delta_2, \ldots, \delta_n \)  
Rotation coordinates of pad

\( 1, 2, \ldots, n \), dim.

\( \psi \)  
Pad arc length, dim.

\( \psi_p \)  
Pivot arc, dim.

\( \omega \)  
Reduction frequency for the bearing coefficients (used in Fig. 3), \( rad/s \)

\( \omega_r \)  
Spin speed of the rotor, \( rad/s \)

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