Inertia property determination by spectrum analysis of damped oscillation detection signal

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Abstract
Oscillation method is the most effective approach to determine the inertia properties of rigid bodies. This paper proposes a new signal processing method to determine the moment of inertia (MoI) accurately in the presence of high frictional damping when using a torsion pendulum setup. A mathematical model of damped oscillation is established, and the analytic form of Fourier transformation of the damped torsional oscillation is derived. A formula for calculating the MoI in relation to the dominant frequency and damping ratio is also derived. An algorithm regarding periodic extension and normalization of torsional oscillation signal is proposed to calculate the dominant frequency and damping ratio of torsional oscillation in the frequency domain. The feasibility and accuracy of this algorithm are verified by both numerical simulations and measurement experiments. The experimental results show that the proposed method has good measurement repeatability, and the relative error of measurement is within 0.80%.

1  |  INTRODUCTION

Moment of inertia (MoI) is a physical parameter that represents the inertia of an object in rotational motion. It usually can be regarded as a measure of motion performance of an object. Together with mass, centre of mass, product of inertia etc., it constitutes mass characteristics (properties) [1,2]. All machines with rotational behaviour, such as aircraft, satellites, automobiles and robots, need to be tested for inertia properties to verify their rotational performance [3–10]. Precision measurement of inertia properties is of significance for air vehicles, which need precise attitude and orbit control, as well as missile, high supersonic aircraft, interceptor and satellite [11,12]. The inertia properties provide design input for motion analysis and attitude control system of an air vehicle and help verify the rationality of its payload structure layout [13–18]. In addition, it plays an important role in the development of power generation equipment [19–21].

The oscillation method [22–26], including torsion pendulum method, compound pendulum method and multi-filar pendulum method, is the most widely used, reliable and accurate method for determining the inertia properties of large objects. It calculates the MoI of an object being measured by making use of the fact that the MoI about the torsion axis is proportional to the square of the oscillation period. Although a high-precision torsion pendulum measuring system is usually equipped with an advanced air bearing for the torsion rod, which significantly reduces the frictional damping of the bearing [27], the influence of air resistance caused by the complex shape of the measured object and internal damping of the torsion rod still cannot be ignored [28]. Furthermore, the use of air bearing greatly increases the complexity of the measurement system and the economic cost of measurement. Nowadays, most of the research works on inertia measurement technology focus on the measurement method of inertia properties for large size objects and the integrated measurement method of mass characteristics. For example, Teng et al. [29] developed a measurement system based on the principles of three-point measurement and constant torque (torsion pendulum method). This measurement system can determine all mass characteristics of an aircraft. Brancati et al. [30] described a piece of equipment and a procedure for determination of large objects’ inertia parameters, and the procedure is based on a complete motion equation of rigid bodies, in which a least-squares optimization is used to identify the inertia tensor. Olmedo et al. [31] developed an instrumented torsion platform used to estimate the...
inertia tensor of rigid bodies with complex geometries, claiming that their method is capable of estimating the moment and product of inertia and the centre of mass location of a rigid body. Tondji and Botez [32] proposed a semi-empirical estimation and experimental method for determining inertial properties of unmanned aerial vehicles. They used Raymer and DAT-COM statistical-empirical methods to estimate all mass characteristics of an unmanned aerial vehicle. Previati et al. [33] developed a measurement system based on multi-filar pendulum to determine the mass properties of a large-sized and heavy rigid body and analysed the effect of deformation of the testing structure on the measurement of large or heavy bodies by numerical and experimental approaches. However, research works focused on demodulation of oscillation signals and effect of frictional damping have been scarce in the literature. When using an oscillation method to measure the inertia properties of an object, the oscillation motion will be strongly affected by the frictional damping caused by bearing friction, air resistance, and molecular friction of the torsion rod. Meanwhile, the frictional damping will change the frequency of oscillation, resulting in errors in the calculation of inertia properties. In particular, in some low-cost measurement systems, the role of damping cannot be ignored. In order to realise measurement with low cost and high precision, it is of great significance to study the measurement technology of inertia properties when high damping exists during oscillation test.

The conventional processing algorithm of oscillation method is designed to calculate the MoI according to the temporal-domain characteristics of oscillation signals [34] and extract the temporal-domain characteristics of such signals, namely, its zero and extremum. These characteristic points are then used to estimate period of oscillation and to calculate inertia properties. However, an oscillation detection signal usually contains white noise and other environmental noise, and consequently, the extracted temporal-domain characteristic points always have a certain error, which limits the improvement of inertia measurement accuracy. Moreover, when the damping is very high, the amplitude of oscillation detection signals attenuates very quickly. As a result, the number of characteristic points that can be extracted is often very few, which makes the calculation of inertia properties unreliable. As mentioned above, frictional damping can change the oscillation frequency and the spectrum distribution characteristics of an oscillation detection signal. The spectrum of a damped oscillation detection signal contains important information related to the measured inertia. At present, there is no literature on the spectrum distribution characteristics of damped oscillation detection signals, nor is there any report on the measurement of inertia properties using the information of frequency domain.

Different from previous studies, this paper focuses on the frequency-domain distribution characteristics of damped oscillation signals, and an algorithm for calculating measured inertia in the frequency domain is proposed. This paper is organised as follows. The basic principles of oscillation method are first introduced along with the conventional demodulation method for oscillation signals, followed by description of a frequency-domain processing algorithm. Results of numerical simulations

are also presented to verify the effectiveness of the algorithm in this part. In the last part, the effectiveness and feasibility of the proposed method is further verified through testing and measurement of specimen with known inertia properties.

2 | MEASUREMENT PRINCIPLE OF MOMENT OF INERTIA

2.1 | Mathematical model of torsional oscillation

The oscillation method usually uses a torsion pendulum to measure the inertia properties of an object. The structure of the torsion pendulum measuring setup is shown in Figure 1. The torsion rod is the core part of the measuring setup and is used to stimulate torsional oscillation.

After being rotated by an angle in the horizontal plane perpendicular to the torsion rod, the object under test begins to make reciprocating torsional oscillation around the central axis of the torsion rod under the restoring torque of the torsion rod. As described in Equation (1), the restoring moment \( M \) generated by the torsion rod is proportional to the rotation angle \( \theta \) according to Hooke’s law

\[
M = -k\theta
\]

where \( M \) is the restoring moment, \( k \) is torsional stiffness constant and \( \theta \) represents the angle of rotation. Generally, the torsional oscillation will be damped, and the damping effect is mainly generated by the air friction resistance that depends on the shape of the object under test, the friction resistance of the bearing and the internal molecular friction of the torsion rod. The friction resistance of the torsion rod bearing is the main source of damping, which can be neglected when an air bearing
is used. If the measuring system adopts a low-cost mechanical bearing, the resulting damping will make a significant influence on the measurement of inertia properties. In order to realise low-cost measurement, an accurate mathematical model of torsional oscillation under the condition of high frictional damping must be established.

Such frictional damping can be mainly processed as a linear damping equivalent, which is proportional to the angular velocity of torsional oscillation. There is also some nonlinear damping in the torsional vibration system, but compared with linear damping, it is quite low, and its influence on the measurement of inertia is so small that it can be ignored. Defining \( \varepsilon \) as the damping proportion coefficient, and \( \omega \) as the angular velocity, the damping torque can be expressed as \( M_z = -k\omega - \varepsilon\omega \).

Then, the combined torque \( M_z \) of the system can be expressed as

\[
M_z = -k\theta - \varepsilon \omega.
\]  

(2)

The definition of the law of rotation can be described by

\[
M_z = I\beta.
\]  

(3)

where \( I \) is the MoI of the moving body and \( \beta \) is the angular acceleration.

The kinetic differential equation of torsional oscillation can be established according to the law of rotation, as shown as

\[
I \ddot{\theta} + \varepsilon \omega \dot{\theta} + k\theta = 0.
\]  

(4)

For the convenience of expression, \( \zeta \) is defined as the damping ratio of torsional oscillation: \( \zeta = \frac{\varepsilon}{2\sqrt{I}} \). And \( \omega_n \) is defined as the undamped natural frequency of torsional oscillation system: \( \omega_n = \sqrt{\frac{k}{I}} \). Then, Equation (4) can be rewritten as

\[
I \ddot{\theta} + 2\zeta \omega_n \dot{\theta} + \omega_n^2 \theta = 0.
\]  

(5)

Obviously, Equation (5) is a typical second-order differential equation with constant coefficients. To obtain the analytic solution of differential equation (5), the roots of its characteristic equation must be derived. The characteristic equation of differential equation (5) has two conjugate complex roots, as shown in

\[
\begin{align*}
\lambda_1 &= -\zeta \omega_n + j\omega_n \sqrt{1 - \zeta^2} \\
\lambda_2 &= -\zeta \omega_n - j\omega_n \sqrt{1 - \zeta^2}.
\end{align*}
\]  

(6)

Therefore, the differential equation (5) has a general solution in the form described in

\[
\theta(t) = c_1 \cos \left( \sqrt{1 - \zeta^2} \omega_n t \right) + c_2 \sin \left( \sqrt{1 - \zeta^2} \omega_n t \right)
\]  

(7)

where \( c_1 \) and \( c_2 \) are arbitrary constant coefficients. Assuming that the initial pendulum angle is \( \theta_0 \), the first initial condition can be achieved: \( \theta(0) = \theta_0 \). According to the law of rotation, the other initial condition can be achieved as well: \( \frac{\dot{\theta}(t)}{\omega_n} = \omega_n \theta(t) \).

Then, it can be derived that \( C_1 = \theta_0 \) and \( C_2 = 0 \). Therefore, the equation has the following solution:

\[
\theta(t) = \theta_0 \exp \left( -\zeta \omega_n t \right) \cos \left( \sqrt{1 - \zeta^2} \omega_n t \right)
\]  

(8)

where \( \theta_0 \) is the initial excitation angle. For torsion pendulum measurement systems, the torsional angle is usually very small (within 0.1 rad), and the linear displacement of torsional oscillation at the radius of \( R \) from the centre axis has the following relationship with the angular displacement:

\[
I \dot{\theta} = \theta \dot{R}.
\]  

(9)

### 2.2 Determination of MoI in frequency domain

In engineering practice, the linear displacement of torsional oscillation can be easily detected by grating sensor, position sensitive detector (PSD) or any other displacement sensors. The torsion pendulum method determines the MoI about the torsion rod axis by measuring the torsional oscillation period. It can be seen that the period measurement error or the frequency measurement error of torsional oscillation has a great influence on the measurement results.

According to Equation (4), the existence of frictional damping has a significant impact on the frequency of torsional oscillation. In order to analyse the influence of frictional damping on the amplitude–frequency distribution of torsional oscillation, a Fourier transform of torsional oscillation is needed. The constant coefficients \( \theta_0 \) and \( R \) are ignored for convenience (i.e. the initial amplitude of damped torsional oscillation is normalised), and the Fourier transform of the torsional displacement \( \theta(t) \) is described as

\[
\mathcal{F} \{ \theta(t) \} = \frac{1}{2} \left[ \frac{1}{\zeta \omega_n + \left( \omega - \sqrt{1 - \zeta^2} \omega_n \right)} + \frac{1}{\zeta \omega_n + \left( \omega + \sqrt{1 - \zeta^2} \omega_n \right)} \right].
\]  

(10)

From Equation (10), it can be concluded that the damped torsional oscillation is no longer a harmonic signal, but an amplitude- and frequency-modulated signal with a certain frequency bandwidth. Its dominant frequency changes to \( \sqrt{1 - \zeta^2} \omega_n \), and torsional oscillation period changes to \( \sqrt{1 - \zeta^2} \omega_n \), with \( T_o \) denoting the period of natural torsional oscillation. If this torsional oscillation period is used to calculate the MoI, the measurement error will be very large. Defining \( \omega_p \) as the dominant frequency of torsional oscillation, the calculation formula of the measured MoI is

\[
I = \frac{(1 - \zeta^2) k}{\omega_p^2}.
\]  

(11)
Thus, it can be seen that accurate identification of the damping ratio $\zeta$ and damped torsional oscillation frequency $\omega_p$ will be the prerequisite for precision measurement of MoI. $A(\omega)$ is defined as the amplitude–frequency distribution function of torsional oscillation, and considering unilateral spectrum, the peak value of Fourier amplitude spectrum of torsional oscillation is

$$A(\omega_p) = \frac{1}{\Delta \omega}.$$  \hspace{1cm} (12)

Obviously, the Fourier spectrum peak of damped torsional oscillation is a function of the damping ratio $\zeta$ and the undamped frequency $\omega_u$. The torsional oscillation damping ratio $\zeta$ can be calculated through the dominant frequency $\omega_u$ and its Fourier amplitude spectrum $A(\omega_u)$, i.e. the measured inertia properties can be determined in Fourier frequency domain.

### 2.3 Conventional method for determination of MoI

The conventional method calculates the MoI using the characteristic points, which are extracted from the temporal-domain waveform of torsional oscillation detection signals. $t_n$ is defined as the $n$th maxima (or minima) of the oscillation signal, and the corresponding angular displacement is denoted as $\Theta_n$. Then, the dominant frequency $\omega_u$ and damping ratio $\zeta$ can be calculated as follows:

$$\begin{align*}
\omega_u & = \frac{2\pi n}{t_n}, \\
\zeta & = -\frac{\ln \left( \frac{\Theta_n}{\Theta_0} \right)}{2\pi n}.
\end{align*}$$ \hspace{1cm} (13)

The dominant frequency $\omega_u$ can be calculated by the zeros of the torsional oscillation signal as well. In this case Equation (11) can be applied to calculate the measured MoI.

It can be seen that measurement accuracy of the conventional method is easily affected by signal noise. As described by Equation (10), the damped torsional oscillation angular displacement curve is an asymptotic single-frequency one. When using empirical mode decomposition (EMD) algorithm [35] to denote the detection signal, the damped torsional oscillation component can be regarded as a typical intrinsic mode functions (IMF) component. Therefore, the EMD algorithm is commonly used to pre-process the oscillation detection signal [28]. However, the EMD algorithm often fails to achieve ideal denoising, as do some improved mode decomposition algorithms. As a result, the residual noise may still have great influence on the calculation of torsional oscillation period and damping ratio in temporal domain. The processing flow of the conventional method is shown in Figure 2.

It should be noted that when the damping ratio is large, the amplitude of the torsional oscillation signal will attenuate rapidly, and few temporal-domain characteristic points can be extracted, which further limits the improvement of measurement accuracy of inertia properties.

### 3 Demodulation Algorithm Based on Fourier Spectrum Analysis

#### 3.1 Determination of inertia in frequency domain

In engineering practice, the Fast Fourier transform (FFT) algorithm is often used to realise the Fourier transform of a digital signal to obtain its amplitude–frequency distribution and phase–frequency distribution. In the process of FFT algorithm, the amplitude of the integral kernel of Fourier transform is normalised, which makes the spectrum amplitude of the discrete Fourier transform different from that of continuous Fourier transform. When the FFT algorithm is used to process a torsional oscillation signal, the amplitude of the dominant component of torsional oscillation can be calculated as

$$A(\omega_p) = \frac{1}{N\Delta t} \sum_{n=0}^{N-1} \exp \left( -\zeta \omega_u T_s \right) \cong \frac{1 - \exp \left( -\zeta \omega_u T_s \right)}{\zeta \omega_u T_s}$$ \hspace{1cm} (14)

where $N$ is the torsional oscillation signal sampling number, $\Delta t$ is the sampling period, and $T_s$ is the observation time, with $N = T_o/\Delta t$. Thus, when using the FFT algorithm, the peak value of Fourier amplitude spectrum of a torsional oscillation signal is a function of damping ratio $\zeta$, undamped oscillation frequency $\omega_u$ and signal observation time $T_o$. At a certain value of the observing time $T_o$, the spectrum peak $A(\omega_p)$ is a function of torsional oscillation damping ratio $\zeta$, undamped frequency $\omega_u$, and damped oscillation frequency $\omega_p$. Thus, the Fourier spectrum analysis of the torsional oscillation signal can be used to simultaneously achieve the estimates of the torsional damping ratio $\zeta$ and the damped torsional oscillation frequency $\omega_p$.

The proposed algorithm to determine the MoI in the frequency domain is shown in Figure 3. The noise in the torsional oscillation signal output of the sensor is eliminated by mode decomposition or a smoothing filter. After pre-processing, the torsional oscillation signal is divided into two channels to transform to the Fourier spectrums. Under general load conditions, torsional oscillation has a relatively long period. When the torsional oscillation has high damping, the amplitude of the torsional angle will quickly attenuate to 0, and the acquisition time of torsional oscillation signal is relatively short. Thus, when the FFT algorithm is used to obtain the amplitude–frequency distribution of a torsional oscillation signal, the frequency resolution is very high. Hence, the extracted torsional oscillation frequency contains a large data truncation error or rounding error, preventing it from being directly used in the calculation.
3.2 | Numerical simulations

The main simulation parameters are set as follows: the signal sampling frequency is 3 kHz, the torsional oscillation signal observation time is 5 s, the initial amplitude is 1.5 rad, the damping ratio of the torsional oscillation is 0.06 and the undamped natural frequency is $2\pi$ rad/s. Therefore, the dominant frequency of the simulation signal is 6.2719 rad/s, and the peak value of the Fourier amplitude spectrum should be 0.4500 rad.

This periodic extension method can keep the continuity of the slope at the connection point and optimise the spectrum distribution of the extended signal. The extended signal $S_{pe}$ can be described as

$$S_{pe} = C(t) \cos \left( \sqrt{1-\xi^2} \omega_n t \right) \quad (15)$$

where $C(t)$ represents the periodic amplitude modulation function. It has a modulation period of $T_m = \frac{2(\pi-\xi^2)}{\omega_n \sqrt{1-\xi^2}}$, and the corresponding angular frequency is $\omega_m = \sqrt{\frac{\xi^2}{(\pi-1)^2}} \omega_n$. Amplitude modulation function $C(t)$ can be described as Equation (16), where $m$ is a non-negative integer, its maximum value being equal to the number of extension connections and $n$ is the number of zeros.

According to the Fourier transform, any periodic signal can be transformed into the sum of a finite number of harmonic components with different initial phases and amplitudes. When the simple harmonic signal $\cos \left( \sqrt{1-\xi^2} \omega_n t \right)$ is multiplied by a low-frequency periodic signal $C(t)$, in temporal domain, the amplitude of harmonic signal will be modulated by this low-frequency signal, and in the frequency domain, multiple side-lobe components will appear with the same frequency interval of $\sqrt{\frac{\xi^2}{(\pi-1)^2}} \omega_n$ [i.e. the frequency of periodic signal $C(t)$]. Furthermore, these sidelobe components are symmetrically distributed.
FIGURE 5  Normalised simulation signal in temporal domain and in Fourier frequency domain

FIGURE 6  Extended simulation signal in temporal domain and in Fourier frequency domain

with the centre of dominant frequency \( \sqrt{1 - \zeta^2} \omega \).

\[
C(t) = \begin{cases}
\exp \left\{ \zeta \omega \left[ t - \frac{(4m+2n+1)\pi}{2\omega \sqrt{1 - \zeta^2}} \right] \right\}, & t \in \left[ \frac{2(m+1)(n+1)\pi}{\omega \sqrt{1 - \zeta^2}}, \frac{2(m+1)(n+1)\pi}{\omega \sqrt{1 - \zeta^2}} \right] \\
-\zeta \omega \left[ t - \frac{(4m+2n+1)\pi}{2\omega \sqrt{1 - \zeta^2}} \right] \right\}, & t \in \left[ \frac{2(m+1)(n+1)\pi}{\omega \sqrt{1 - \zeta^2}}, \frac{2(m+1)(n+1)\pi}{\omega \sqrt{1 - \zeta^2}} \right]
\end{cases}
\tag{16}
\]

In the simulation, the final extended signal is made up of 1000 base signal connections (FFT frequency resolution is optimised from 1.2566 rad/s to 6.9615 × 10^{-4} rad/s). Figure 5 shows the temporal-domain distribution and the Fourier spectrum distribution of the normalised torsional oscillation signal, and the Fourier spectrum peak of the normalised simulation signal is 0.4505 rad, which is almost exactly consistent with the set value. Based on the calculated Fourier spectrum maxima, the final calculated damping ratio is 0.0601, and the undamped natural oscillation frequency is 6.2829 rad/s, both of which are almost exactly consistent with the simulation settings. The errors are mainly caused by the truncation error or rounding error of the computer during calculation. The frequency difference between the two closer sidelobe frequency components and the dominant component is ±0.6968 rad/s, which is also completely consistent with the previous analysis. The simulation results show the effectiveness of the proposed algorithm.

In order to further verify the accuracy of the proposed algorithm in calculating the damping ratio \( \zeta \) and the oscillation dominant frequency \( \omega_p \), the simulation signals with different damping ratios and different dominant frequencies are processed. As shown in Figure 7(a), when the dominant frequency of oscillation signal is 2 rad/s, the maximum calculation error of the damping ratio is only 0.0004, which produces almost no error for the calculation of inertia properties. Figure 7(b) shows the calculation results of different oscillation dominant frequencies when the damping ratio is 0.06. The maximum absolute calculation error of oscillation dominant frequency is within 0.0004 rad/s. It is mainly caused by truncation error or rounding error, and it has little effect on the measurement of inertia properties.

Sensor noise, electromagnetic interference in signal transmission and other factors tend to generate noise of a certain power in the oscillation detection signal, especially in some low-cost measurement systems where the role of noise could be more significant. At present, there is no perfect denoising method for amplitude modulation/frequency modulation signal to eliminate the noise in all frequency bands, and the residual
TABLE 1 Calculation results of different simulation signals

| SNR | Conventional method | Proposed algorithm |
|-----|---------------------|--------------------|
|     | Damping ratio $\zeta$ | Damping ratio $\zeta$ | Dominant frequency $\omega_p$ (rad/s) | Dominant frequency $\omega_p$ (rad/s) |
| 40  | 0.0601 | 6.2735 | 0.0599 | 6.2716 |
| 35  | 0.0602 | 6.2827 | 0.0597 | 6.2711 |
| 30  | 0.0602 | 6.2628 | 0.0598 | 6.2708 |
| 25  | 0.0602 | 6.2769 | 0.0599 | 6.2693 |
| 20  | 0.0602 | 6.2811 | 0.0604 | 6.2744 |
| 15  | 0.0585 | 6.2869 | 0.0606 | 6.2646 |
| 10  | 0.0598 | 6.2561 | 0.0587 | 6.2660 |
| 5   | 0.0544 | 6.1842 | 0.0632 | 6.2739 |

low-frequency noise has a certain influence on the measurement of inertia. In order to study the effect of signal noise on the performance of the frequency-domain processing algorithm, the simulation signals with different signal-to-noise ratios (SNRs) are processed by both the proposed algorithm and the conventional method. The aforementioned simulation signal with a frequency of 6.2719 rad/s and a damping ratio of 0.06 is superposed with Gaussian white noise of different powers, and the variational mode decomposition algorithm [36] is used to pre-process the oscillation detection signal. Then, the proposed algorithm and the conventional method are applied to process the simulation signal. The calculation results of the dominant frequency and the damping ratio obtained by two methods are shown in Table 1, in which the dominant frequency results of the conventional method are calculated from the average value of several torsional oscillation periods. It can be seen from the table that under different SNR conditions, the two methods have the same accuracy in calculation of the damping ratio, while the method described in this work is obviously superior to the conventional method in calculation of the oscillation frequency. Numerical simulation shows that the calculation error of the MoI derived using the proposed demodulation algorithm is within 0.01 kg·m², i.e., the proposed algorithm will not lose measurement accuracy in the process of signal demodulation.

Assuming that the torsional stiffness constant $k$ equals 300 kg·m²/s², the measured MoI is calculated according to the damping ratio and dominant frequency in Table 1, and the absolute value of the measurement error of the two methods is shown in Figure 8. It can be seen from the figure that the measurement accuracy of the method described in this work is obviously superior to that of the conventional method.

4 | MEASUREMENT EXPERIMENTS

4.1 | Measuring setup

Based on laboratory conditions, the method proposed in this work is experimentally verified by using a torsion pendulum measurement system. The schematic diagram of the system composition is shown in Figure 9(a), and the real experimental system is shown in Figure 9(b). The experimental system is mainly composed of a torsion pendulum table, a grating sensor data processing card and a computer, among which the torsion pendulum table is equipped with a damping mechanical module and a grating sensor. The torsion rod of the pendulum is equipped with an air bearing. In order to simulate frictional damping, the damping mechanical module is used to apply a pressure to the torsional plate to obtain frictional damping. The grating sensor is used to record the torsional oscillation in real
time, and the data processing card is used to sample and demodulate the grating signals, thus realizing the direction discrimination and subdivision of oscillation displacement. The maximum sampling rate is 3 kHz.

As shown in Figure 9(b), a cross beam is fixed with the loading platform of the torsion pendulum table, and some circular adaptors are designed on the cross beam to mount the tested loads. Before the experiments, the torsional stiffness constant $k$ of the torsion rod is calibrated by using several sets of standard weights with known MoI, and the final calibration result is $k = 203.8102 \text{ kg·m}^2/\text{s}^2$. The MoI of the auxiliary parts of the measuring table should also be measured. The measured torsional oscillation frequency is 2.5561 rad/s, the damping ratio is 0.0003 and the MoI of the auxiliary parts is 30.9504 kg·m².

### 4.2 Repetitive measurement experiments

In order to verify the effectiveness of the proposed algorithm, two cylindrical parts of uniform density are machined with main parameters given as follows: their masses are 19.8365 and 19.8257 kg, respectively, their diameters are 198.68 and 198.57 mm, respectively, and their moments of inertia about their rotating axis are 0.0982 and 0.0980 kg·m², respectively. One measured cylindrical part of 19.8365 kg is placed on the adaptor, which is 499.776 mm away from the centre of the torsion pendulum, and the other measured cylindrical part is placed on the adaptor, which is 499.830 mm away from the centre of the torsion pendulum. Then, according to the parallel axis theorem [37], the MoI of the measured loads should be 41.0543 kg·m². This value can be regarded as the conventional true value of the measured MoI.

The MoI of the loads is measured, and the damping mechanical module is used to generate frictional damping to the torsion pendulum system during the measurement. As shown in Figure 10, the actual torsional oscillation signal is deeply amplitude modulated. The SNR of the torsional oscillation detection signal is about 17.5 dB.

The actual torsional oscillation signal is processed by the algorithm shown in Figure 3, and then, the dominant frequency and damping ratio can be calculated in the frequency domain. Figure 11(a) shows the normalised torsional oscillation signal, and Figure 11(b) shows its Fourier spectrum distribution, with the amplitude spectrum peak at 0.3344 rad. Figure 11(c) shows the periodic extended torsional oscillation signal (part intervals).

In the actual processing, the number of base signal connections is 900, and the frequency resolution of FFT is within 0.0001 rad/s. Figure 11(d) shows the Fourier spectrum distribution of the extended signal, and the dominant frequency is 2.2183 rad/s.

According to the amplitude spectrum peak of 0.3344 rad and the dominant frequency of 2.2183 rad/s, the damping ratio $\zeta$ can be calculated as 0.0560, and then, the MoI can be calculated as 41.2873 kg·m². The calculated result is in good agreement with the set value in the experiment, with a deviation of +0.23 kg·m², which verifies the accuracy of the method described in this study.

To test the repeatability of the measurements, the current loads are measured 10 times under the same conditions. The torsional oscillation signals obtained from 10 measurements are processed by the proposed algorithm, and the damping ratio $\zeta$ and dominant frequency $\omega_p$ are determined, based on which the moments of inertia of loads are calculated. The measurement results of 10 experiments are shown in Figure 12. The measurement standard deviation of MoI is about 0.14 kg·m², indicating that the method described in this study has high measurement repeatability. In fact, the fluctuation of measurement results is mainly caused by the experimental states of 10 measurement experiments, especially the uniformity of friction damping. The proposed algorithm has no repeatable calculation error for the same torsional oscillation signal.

### 4.3 Measurement experiments for different loads

In order to further verify the accuracy of the proposed algorithm, while maintaining stable friction of the damping mechanical module, MoI measurement experiments are carried out for eight different loads. Each group of loads is measured five times. The damping ratio and the dominant frequency of torsional oscillation signal are calculated by using the proposed method before the moments of inertia of different loads are calculated. The measurement results are shown in Table 2. In this table, the true value of the MoI is calculated by the inertia parameters of loads and the distances between the loads and the centre of the torsion pendulum.

According to the measurement results, the relative measurement error of the MoI is within 0.80%. Under the same experimental conditions, as the measured MoI increases, the damping ratio increases slightly. One possible reason can be that the larger measured loads result in the greater internal damping of the torsion rod (the large measured loads increase the heating of the torsion rod). These measurement experiments demonstrate the accuracy of the proposed method.

### 4.4 Comparison experiments

In order to verify the better performance of the proposed algorithm, several measurement experiments are carried out for different loads. The proposed method and the convenient method
are both used to determine the moments of inertia of different loads. Figure 13 gives the result of comparison.

Figure 13(a) shows the measurement errors by using the proposed method, and Figure 13(b) shows the measurement errors by using the convenient method. As can be seen from the figures, the repeatability and accuracy of the convenient method are both lower than those of the proposed method. The experimental results verify that the proposed method can better reduce the influence of noise in torsional oscillation signals.

When there is DC bias and low frequency drift in the oscillation detection signal, the time-domain feature points extracted by the conventional method usually contain a large error. The proposed method uses the spectrum distribution of detection signal to calculate the measurand. The DC bias and low-frequency drift can be distinguished from the dominant frequency component in the frequency domain of the detection signal. Therefore, the proposed method presents better measurement accuracy than the conventional method. It should be pointed out that the algorithm proposed in this work is more time-consuming than the conventional method due to the adoption of periodic extension. However, compared with the time of the whole measurement process, the running time of the proposed demodulation algorithm can be completely ignored.

5 | DISCUSSION

It can be seen from the above experimental results that the polarity of the measurement error remains unchanged, which may be caused by three reasons.

1. The calibration error of torsional stiffness constant $k$ leads to the presence of a large systematic error in the measurement results.

| No. | Damping ratio $\zeta$ | Dominant frequency $\omega_p$ (rad/s) | True value of MoI (kg m$^2$) | Measured MoI (kg m$^2$) |
|-----|----------------------|--------------------------------------|-----------------------------|------------------------|
| 1   | 0.0553               | 2.4529                               | 33.6242                     | 33.7704                |
| 2   | 0.0554               | 2.3608                               | 36.3033                     | 36.4563                |
| 3   | 0.0561               | 2.2183                               | 41.0543                     | 41.2873                |
| 4   | 0.0572               | 1.9874                               | 51.1789                     | 51.4318                |
| 5   | 0.0564               | 1.9441                               | 53.4604                     | 53.7533                |
| 6   | 0.0567               | 1.6895                               | 70.7683                     | 71.1723                |
| 7   | 0.0581               | 1.6331                               | 75.6164                     | 76.1609                |
| 8   | 0.0609               | 1.4732                               | 93.0436                     | 93.5597                |
2. In this study, frictional damping is equivalent to linear damping. In fact, some damping effects are nonlinear.
3. The grating sensor may output a low-frequency drift, which is difficult to be filtered.

In order to eliminate these error factors, more accurate calibration of torsional stiffness constant and more damping effect should be considered, as should a better denoising method.

Compared to the conventional temporal-domain method, the frequency-domain analysis method proposed in this work proves more effective against the signal noise. Generally, the sensor output signal is always accompanied by noise caused by environmental or other factors. Such noise has a great influence on the temporal-domain method. The extraction of zeros and extrema of torsional oscillation signal produces large errors, which will significantly reduce the calculation accuracy of torsional oscillation period and damping ratio. The frequency-domain analysis method described in this work is based on spectrum analysis of torsional oscillation signal, which can partly overcome the shortcomings of the conventional method. Although the extraction of zeros is also employed during the construction of the extension base signal, due to which the extraction error will cause certain spectrum leakage of the extended signal in the frequency domain and generate sidelobe spectral lines, such extraction will not change the dominant frequency. The influence of residual noise on the method described in this work is obviously weaker than that on the conventional method.

6 | CONCLUSIONS

This paper presents a frequency-domain demodulation algorithm for measuring MoI using oscillation method. The proposed method can accurately extract the dominant frequency and the damping ratio of an oscillation signal in the frequency domain, thus providing better performance. A new signal demodulation method is provided for measuring the MoI. This paper mainly reaches the following conclusions.

A kinetic differential equation of damped torsional oscillation is created in this work, and its analytic solution is derived. The analytic form of Fourier transform of damped torsional oscillation is also derived, which indicates that the damping ratio and the dominant frequency of damped torsional oscillation can be determined by Fourier spectrum analysis. The FFT algorithm is used to obtain the amplitude–frequency distribution of a torsional oscillation signal, and the spectrum peak is a function of the damping ratio and the frequency of undamped natural oscillation. Hence, it is feasible to determine the damping ratio and inertia properties by Fourier spectrum analysis.

1. An algorithm based on Fourier spectrum analysis is proposed to determine the damping ratio and the dominant frequency of damped torsional oscillation, as well as the measured MoI. Numerical simulations and measurement experiments demonstrate the effectiveness of this algorithm. Compared with the conventional temporal-domain method, the proposed frequency-domain method provides better accuracy.
2. Because DC bias and low frequency drift can be distinguished with the dominant frequency component from the oscillation detection signal in the frequency domain, the proposed method is more effective in eliminating the effects of sensor bias and low-frequency noise compared with conventional methods.
3. The proposed signal processing algorithm achieves good accuracy even when high friction damping exists during torsional oscillation. This means during precision measurement of inertia properties, expensive air bearing and its auxiliary parts are no longer needed to reduce the frictional damping. The relative measurement error of MoI is within 0.80% with the proposed method. In sum, the cost of measuring inertia properties can be significantly reduced.

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