The statistical nature of the second order corrections to the thermal SZE

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Abstract

This paper shows that the accepted expressions for the second order corrections in the parameter $z$ to the thermal Sunyaev-Zel’dovich effect can be accurately reproduced by a simple convolution integral approach. This representation allows to separate the second order SZE corrections into two type of components. One associated to a single line broadening, directly related to the even derivative terms present in the distortion intensity curve, while the other is related to a frequency shift, which is in turn related to the first derivative term.

The thermal Sunyaev-Zel’dovich effect (SZE)\textsuperscript{11}\textsuperscript{12} arises from the frequency shift of CMB photons that are scattered by a hot electron gas. It has been detected in observations of some rich, X-ray luminous clusters. The SZE can be applied to the determination of cosmological parameters\textsuperscript{13} \textsuperscript{14}.

The distorted spectra arising from Compton scattering can be expressed as a power series containing terms which are even powers of the frequency $\nu$, except for a first order term in $\nu$ that is associated to the photon diffusion approximation\textsuperscript{15} \textsuperscript{16} \textsuperscript{17} \textsuperscript{18}. The parameter $z = \frac{kT}{mc^2}$, used as a discriminant to indicate when the relativistic corrections are important is usually kept up to second order effects.

In this paper we show that the scattering law approach to the SZE\textsuperscript{19} is able to reproduce the distortion curve contained in Refs.\textsuperscript{20} \textsuperscript{21} \textsuperscript{22} \textsuperscript{23} \textsuperscript{24} providing new insight of the physics inherent in distortion spectrum. The basic idea is simple. The distorted spectrum can be obtained by performing a convolution integral between the CMB blackbody curve and a kernel corresponding to a single line profile, slightly
broadened by the interaction between photons and the thermal electrons present in the gas. For the case of distortion curves linear in \( z \) and in the optical depth \( \tau \), it has already been shown that the line broadening corresponding to a Maxwellian reproduces the Kompaneets equation 11.

To obtain the widely accepted spectra from a scattering law, in the relativistic case, we have chosen to start from a slight modification of a kernel recently proposed and perform explicitly the convolution integral. Having obtained the distortion expression we shall compare it with the distortion curve derived from references 3 4 5 6 7 8 10. A brief discussion is included at the end of this note.

The scattering law approach to CMB distortions is based upon the fact that the distorted radiation spectrum which originates from photon scattering in dilute medium is given by:

\[
I(\nu) = \int_0^\infty I_o(\bar{\nu})G_s(\bar{\nu}, \nu)d\bar{\nu}
\] (1)

Here, \( I(\nu) \) is the scattered radiation off the plasma, \( I_o(\nu) = \frac{2h\nu^3}{c^2}(\exp\left(\frac{h\nu}{kT_R}\right) - 1)^{-1} \) the undistorted spectrum, where \( T_R \) is the CMB temperature, and \( G_s(\bar{\nu}, \nu) \) the scattering law.

For the thermal non-relativistic effect we have successfully reproduced Kompaneets equation with a kernel \( G_s(\bar{\nu}, \nu) \) which is given by 12

\[
G_s(\bar{\nu}, \nu) = (1 - \tau)\delta(\bar{\nu} - \nu) + \tau G(\bar{\nu}, \nu) \] (2)

where

\[
G(\bar{\nu}, \nu) = \frac{1}{\sqrt{\pi W(\nu)}}e^{-\left(\frac{\bar{\nu} - \nu}{W(\nu)}\right)^2}. \] (3)

Here, \( m_e \) and \( T_e \) are the mass and temperature of an electron and the electron gas, respectively, while \( W^2(\nu) = 4kT_e/m_e c^2 \nu^2 = 4\zeta \nu^2 \) is, obviously, the square of the width of the spectral line at frequency \( \nu \).

We now introduce a modified kernel given by:

\[
G_s(\bar{\nu}, \nu) = (1 - 2\tau)\delta(\bar{\nu} - \nu) + \tau\sqrt{\pi W(\nu)}e^{-\left(\frac{\bar{\nu} - \nu}{W(\nu)}\right)^2} + \tau\delta(\bar{\nu} - (1 - 2\zeta)\nu) \] (4)

The proposed kernel, Eq. 4, simply implies that the single line profile is a purely Gaussian function peaked at \( \bar{\nu} = \nu \) 14 and that the central limit theorem governs the scattering of photons by electrons 14. Also, a systematic effect, associated to a 2\( \zeta \nu \) frequency shift is present consistently with the diffusion approximation. Both effects are small and are assumed additive. The Gaussian term in the corresponding integral becomes even in the variable

\[
\alpha = \frac{\bar{\nu} - \nu}{W(\nu)}
\] (5)

leading to the expression, up to second order in \( z \):

\[
I(\nu) = I_o(\nu) - 2\tau I_o(\nu) + \frac{\tau}{\sqrt{\pi W(\nu)}} \int_{-\infty}^\infty I_o(\nu + \Delta\nu(\alpha))e^{-\alpha^2}d\alpha + \tau I_o(\nu - 2\zeta\nu) \] (6)

where \( \Delta\nu(\alpha) = W(\nu)\alpha \).

The expression \( I_o(\nu + \Delta\nu(\alpha)) \) can be trivially expanded up to arbitrary order in \( z \), but for the purposes of this note we will only consider the second order corrections.
\[ \frac{\Delta I}{\tau} \propto \frac{(h c)^2}{2 \tau (k_b T)^3} \]

\[ \frac{\Delta I}{\tau} = -2z\nu \frac{\partial I_0}{\partial \nu} + (z + \frac{1}{2}z^2) \nu^2 \frac{\partial^2 I_0}{\partial \nu^2} + \frac{1}{2}z^2 \nu^4 \frac{\partial^4 I_0}{\partial \nu^4} \]  

Figure 1: CMB distortion for the case of Eq. (7) (dotted line) compared with the relativistic curve, Eq. (8) (solid line), \(kT_e = 10\) KeV. The long dashed line has been added as a reference and corresponds to the Kompaneets approximation. The frequency is given in Hz and \(\frac{\Delta I}{\tau}\) is given in units of \(\frac{(h c)^2}{2 \tau (k_b T)^3}\).

The resulting distortion curve reads:

\[ \frac{\Delta I}{\tau} = -2z\nu \frac{\partial I_0}{\partial \nu} + (z + \frac{1}{2}z^2) \nu^2 \frac{\partial^2 I_0}{\partial \nu^2} + \frac{1}{2}z^2 \nu^4 \frac{\partial^4 I_0}{\partial \nu^4} \]  

In obtaining Eq. (7), the last term of Eq. (6) was also expanded up to second order in \(z\). A careful examination of the result corresponding to the relativistic thermal SZE derived by Sazonov and Sunyaev for a static cluster \((V = 0)\) leads to the widely accepted expression [5]:

\[ \frac{\Delta I_{SS}}{\tau} = (-2z + \frac{17}{5}z^2) \nu \frac{\partial I_0}{\partial \nu} + (z - \frac{17}{10}z^2) \nu^2 \frac{\partial^2 I_0}{\partial \nu^2} + \frac{7}{100}z^2 \nu^4 \frac{\partial^4 I_0}{\partial \nu^4} \]  

Fig. (1) shows a direct comparison between the distortion curve that arises from Eq. (7) and the one arising form the convolution kernel (4), for a temperature of \(kT_e = 10\) KeV. The convolution kernel (4) reproduces the main features of the relativistic thermal SZE. Shimon and Rephaeli [15] have remarked that all the current independent approaches to the relativistic thermal SZE are compatible with Eq. (8). In our Eq. (7), the \(z^2\nu^4\) term arises from the purely \textit{non-relativistic} line broadening \((W(\nu))^4\) yielding a \(\frac{7}{100}\) factor instead of the \(\frac{7}{100}\) factor included in Eq. (8).

We wish to emphasize on the fact that the high frequency behavior of the distorted spectrum has been obtained using a superposition of the distortion effects associated to the Gaussian Kernel and the \(-2z\nu\) frequency shift established in Ref. [9]. It is also interesting to notice that the suppression of the \(\frac{7}{100}z^2\) and \(\frac{7}{100}z^2\) terms in Eq. (8) would
not alter in a significant way the spectrum up to temperatures of $kT_e = 10\,\text{keV}$. This means that the relativistic corrections to the thermal SZE may be interpreted as small second order modifications to the shift and width parameters included in Eq. (4).

The inclusion of the shift in the delta function in the convolution kernel preserves the even parity in the higher order derivative terms of the distortion curve up to second order in $z$. A possible alternative that would yield both even and odd higher order derivative terms has been discussed elsewhere [16].

Similar techniques as the one applied here to CMB physics have been discussed in other contexts [17], and are specially attractive to derive a first analysis of the nature of the distortions that arise from the interaction of photons and dilute systems.

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