Kähler manifolds and the curvature operator of the second kind

Xiaolong Li

Abstract
This article aims to investigate the curvature operator of the second kind on Kähler manifolds. The first result states that an \( m \)-dimensional Kähler manifold with \( \frac{3}{2}(m^2 - 1) \)-nonnegative (respectively, \( \frac{3}{2}(m^2 - 1) \)-nonpositive) curvature operator of the second kind must have constant nonnegative (respectively, nonpositive) holomorphic sectional curvature. The second result asserts that a closed \( m \)-dimensional Kähler manifold with \( \left( \frac{3m^3 - m + 2}{2m} \right) \)-positive curvature operator of the second kind has positive orthogonal bisectional curvature, thus being biholomorphic to \( \mathbb{CP}^m \). We also prove that \( \left( \frac{3m^3 + 2m^2 - 3m - 2}{2m} \right) \)-positive curvature operator of the second kind implies positive orthogonal Ricci curvature. Our approach is pointwise and algebraic.

Keywords Curvature operator of the second kind · Orthogonal bisectional curvature · Holomorphic sectional curvature · Rigidity theorems

Mathematics Subject Classification 53C55 · 53C21

1 Introduction

The Riemann curvature tensor on a Riemannian manifold \((M^n, g)\) induces a self-adjoint operator \( \bar{\mathcal{R}} : S^2(T_p M) \to S^2(T_p M) \) via

\[
\bar{\mathcal{R}}(\varphi)_{ij} = \sum_{k,l=1}^{n} R_{iklj} \varphi_{kl},
\]

where \( S^2(T_p M) \) is the space of symmetric two-tensors on the tangent space \( T_p M \). The curvature operator of the second kind, denoted by \( \bar{\mathcal{R}} \) throughout this article, refers to the...
symmetric bilinear form
\[ \hat{R} : S^2_0(T_p M) \times S^2_0(T_p M) \to \mathbb{R} \]
obtained by restricting \( \tilde{R} \) to \( S^2_0(T_p M) \), the space of traceless symmetric two-tensors. See [9] or [24] for a detailed discussion. This terminology is due to Nishikawa [40], who conjectured in 1986 that a closed Riemannian manifold with positive (respectively, nonnegative) curvature operator of the second kind is diffeomorphic to a spherical space form (respectively, Riemannian locally symmetric space).

Nishikawa’s conjecture had remained open for more than three decades before its positive part was resolved by Cao, Gursky, and Tran [9] recently and its nonnegative part was settled by the author [24] shortly after. The key observation in [9] is that two-positive curvature operator of the second kind implies the strictly PIC1 condition (i.e., \( M \times \mathbb{R} \) has positive isotropic curvature). This is sufficient since earlier work of Brendle [6] has shown that a solution to the normalized Ricci flow starting from a strictly PIC1 metric on a closed manifold exists for all time and converges to a metric of constant positive sectional curvature. Soon after that, the author [24] proved that strictly PIC1 is implied by three-positivity of \( \hat{R} \), thus getting an immediate improvement to the result in [9]. Furthermore, the author was able to resolve the nonnegative case of Nishikawa’s conjecture under three-nonnegativity of \( \hat{R} \). More recently, the conclusion has been strengthened by Nienhaus, Petersen, and Wink [38], who ruled out irreducible compact symmetric spaces by proving that an \( n \)-dimensional closed Riemannian manifold with \( \frac{n+2}{2} \)-nonnegative \( \hat{R} \) is either flat or a rational homology sphere. Combining these works together, we have

**Theorem 1.1** Let \((M^n, g)\) be a closed Riemannian manifold of dimension \( n \geq 3 \).

(1) If \( M \) has three-positive curvature operator of the second kind, then \( M \) is diffeomorphic to a spherical space form.

(2) If \( M \) has three-nonnegative curvature operator of the second kind, then \( M \) is either flat or diffeomorphic to a spherical space form.

To prove sharper results, the author [26] introduced the notion of \( \alpha \)-positive curvature operator of the second kind for \( \alpha \in [1, \dim(S^2_0(T_p M)) \) where \( N = \dim(S^2_0(T_p M)) \). Hereafter, \( \lfloor x \rfloor \) denotes the floor function defined by

\[ \lfloor x \rfloor := \max\{k \in \mathbb{Z} : k \leq x\} \]

**Definition** A Riemannian manifold \((M^n, g)\) is said to have \( \alpha \)-positive (respectively, \( \alpha \)-nonnegative) curvature operator of the second kind if for any \( p \in M \) and any orthonormal basis \( \{\varphi_i\}_{i=1}^N \) of \( S^2_0(T_p M) \), it holds that

\[ \sum_{i=1}^{\lfloor \alpha \rfloor} \hat{R}(\varphi_i, \varphi_i) + (\alpha - \lfloor \alpha \rfloor) \hat{R}(\varphi_{\lfloor \alpha \rfloor + 1}, \varphi_{\lfloor \alpha \rfloor + 1}) > 0 \quad (\text{respectively, } \geq 0) \tag{1.1} \]

Similarly, \((M^n, g)\) is said to have \( \alpha \)-negative (respectively, \( \alpha \)-nonpositive) curvature operator of the second kind if the reversed inequality holds.

In [26], the author proved that \((n + \frac{n-2}{n})\)-positive (respectively, \((n + \frac{n-2}{n})\)-nonnegative) curvature operator of the second kind implies positive (respectively, nonnegative) Ricci curvature in all dimensions. Combined with Hamilton’s work [16, 17], this immediately leads to an improvement of Theorem 1.1 in dimension three: a closed three-manifold with \( 3_1 \)-positive \( \hat{R} \) is diffeomorphic to a spherical space form and a closed three-manifold with \( 3_1 \)-nonnegative
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**Theorem 1.2** Let \((M^m, g, J)\) be a Kähler manifold of complex dimension \(m \geq 2\).

1. If \(M\) has \(\alpha\)-nonnegative (respectively, \(\alpha\)-nonpositive) curvature operator of the second kind for some \(\alpha < \frac{3}{2}(m^2 - 1)\), then \(M\) is flat.
2. If \(M\) has \(\frac{3}{2}(m^2 - 1)\)-nonnegative (respectively, \(\frac{3}{2}(m^2 - 1)\)-nonpositive) curvature operator of the second kind, then \(M\) has constant nonnegative (respectively, nonpositive) holomorphic sectional curvature.

Previously, the author [24, Theorem 1.9] proved that Kähler manifolds with four-nonnegative \(\hat{R}\) are flat, and Nienhaus, Petersen, Wink, and Wylie [39] proved the following result.

**Theorem 1.3** Let \((M^m, g, J)\) be a Kähler manifold of complex dimension \(m \geq 2\). Set

\[
A = \begin{cases} 
3m \frac{m+1}{m+2}, & \text{if } m \text{ is even,} \\
3m \frac{(m+1)(m^2-1)}{(m+2)(m^2+1)}, & \text{if } m \text{ is odd.}
\end{cases}
\]

If the curvature operator of the second kind of \(M\) is \(\alpha\)-nonnegative or \(\alpha\)-nonpositive for some \(\alpha < A\), then \(M\) is flat.

Theorem 1.2 improves Theorem 1.3. Moreover, the number \(\frac{3}{2}(m^2 - 1)\) is sharp as \((\mathbb{C}P^m, g_{FS})\) has \(\frac{3}{2}(m^2 - 1)\)-nonnegative \(\hat{R}\) and \((\mathbb{C}P^m, g_{\text{stand}})\), the complex hyperbolic space with constant negative holomorphic sectional curvature, has \(\frac{3}{2}(m^2 - 1)\)-nonpositive \(\hat{R}\). Theorem 1.2 has two immediate corollaries.

**Corollary 1.4** For \(\alpha \leq \frac{3}{2}(m^2 - 1)\), there do not exist \(m\)-dimensional Kähler manifolds with \(\alpha\)-positive or \(\alpha\)-negative curvature operator of the second kind.
Corollary 1.5 Let \((M^m, g, J)\) be a complete non-flat Kähler manifold of complex dimension \(m\). If \(M\) has \(\frac{3}{2}(m^2 - 1)\)-nonnegative (respectively, \(\frac{3}{2}(m^2 - 1)\)-nonpositive) curvature operator of the second kind, then \(M\) is isometric to \((\mathbb{CP}^m, g_{FS})\) (respectively, a quotient of \((\mathbb{CH}^m, g_{stand})\)), up to scaling.

In Kähler geometry, there are several positivity conditions on curvatures that characterize \(\mathbb{CP}^m\) among closed Kähler manifolds up to biholomorphism. For instance, a closed Kähler manifold with positive bisectional curvature is biholomorphic to \(\mathbb{CP}^m\). This was known as the Frankel conjecture [13] and it was proved independently by Mori [32] and Siu and Yau [43]. A weaker condition, called positive orthogonal bisectional curvature, also characterizes \(\mathbb{CP}^m\). This is due to Chen [11] and Gu and Zhang [19] (see also Wilking [45] for an alternative proof using Ricci flow). Therefore, it is natural to seek a positivity condition on \(R\) that characterizes \(\mathbb{CP}^m\). Regarding this question, we prove that

Theorem 1.6 A closed Kähler manifold of complex dimension \(m \geq 2\) with \(\alpha_m\)-positive curvature operator of the second kind, where
\[
\alpha_m := \frac{3m^3 - m + 2}{2m},
\]
is biholomorphic to \(\mathbb{CP}^m\).

Note that when \(m = 2\), \(\alpha_2 = 6\) is the best constant for Theorem 1.6 to hold, as \(\mathbb{CP}^1 \times \mathbb{CP}^1\) has \((6 + \epsilon)\)-positive \(\tilde{R}\) for \(\epsilon > 0\). For Kähler surfaces, the author proved in [25] that a closed Kähler surface with six-positive \(\tilde{R}\) is biholomorphic to \(\mathbb{CP}^2\), and a closed nonflat Kähler surface with six-nonnegative \(\tilde{R}\) is either biholomorphic to \(\mathbb{CP}^2\) or isometric to \(\mathbb{CP}^1 \times \mathbb{CP}^1\), up to scaling. The two different proofs in [25] only work for complex dimension two.

The number \(\alpha_m\), however, does not seem to be optimal for \(m \geq 3\). An ambitious question is

Question 1.7 What is the largest number \(C_m\) so that a closed \(m\)-dimensional Kähler manifold with \(C_m\)-positive curvature operator of the second kind is biholomorphic to \(\mathbb{CP}^m\)?

Theorem 1.6 implies \(\alpha_m \leq C_m\). We point out that
\[
C_m \leq \beta_m := \frac{3m^3 + 2m^2 - 3m - 2}{2m},
\]
as the product manifold \(\mathbb{CP}^{m-1} \times \mathbb{CP}^1\) has \(\alpha\)-positive \(\tilde{R}\) for any \(\alpha > \beta_m\) (see [27]). This determines the leading term of \(C_m\) to be \(\frac{3}{2}m^2\). In particular, we have
\[
\frac{40}{3} \leq C_3 \leq \frac{44}{3}.
\]
It remains an interesting question to determine \(C_m\) for \(m \geq 3\).

In the next result, we show that \(\beta_m\)-positivity of \(\tilde{R}\) implies positive orthogonal Ricci curvature.

Theorem 1.8 Let \((M^m, g, J)\) be a Kähler manifold of complex dimension \(m \geq 2\). If \(M\) has \(\beta_m\)-positive curvature operator of the second kind with \(\beta_m\) defined in (1.3), then \(M\) has positive orthogonal Ricci curvature, namely
\[
\text{Ric}(X, X) - R(X, JX, X, JX)/|X|^2 > 0
\]
for any \(0 \neq X \in T_pM\) and any \(p \in M\). If \(M\) is further assumed to be closed, then \(M\) has \(h^{p,0} = 0\) for any \(1 \leq p \leq m\), and in particular, \(M\) is simply-connected and projective.
The orthogonal Ricci curvature $\text{Ric}^\perp$ is defined as
\[
\text{Ric}^\perp(X, X) = \text{Ric}(X, X) - R(X, JX, X, JX)/|X|^2
\]
for $0 \neq X \in T_p M$. This notion of curvature was introduced by Ni and Zheng [36] in the study of Laplace comparison theorems on Kähler manifolds. We refer the reader to [37], [35], and [33] for a more detailed account of it. The constant $\beta_m$ in Theorem 1.8 is optimal, as $\mathbb{CP}^{n-1} \times \mathbb{CP}^1$ has $\beta_m$-nonnegative $\tilde{R}$ and it has nonnegative (but not positive) orthogonal Ricci curvature.

In addition, we prove a result similar to Theorem 1.8, which states that

**Theorem 1.9** Let $(M^m, g, J)$ be a Kähler manifold of complex dimension $m \geq 2$. Suppose $M$ has $\gamma_m$-positive curvature operator of the second kind, where
\[
\gamma_m := \frac{3m^2 + 2m - 1}{2}.
\]
Then for any $p \in M$ and $0 \neq X \in T_p M$, it holds that
\[
2\text{Ric}(X, X) - R(X, JX, X, JX)/|X|^2 > 0.
\]
If $M$ is further assumed to be closed, then $M$ has $h^{p,0} = 0$ for any $1 \leq p \leq m$, and in particular, $M$ is simply-connected and projective.

Chu, Lee, and Tam [12] introduced a family of curvature conditions for Kähler manifolds called mixed curvature. They are defined as
\[
C_{a, b}(X) := a\text{Ric}(X, \bar{X}) + bR(X, \bar{X}, X, \bar{X})/|X|^2
\]
for $a, b \in \mathbb{R}$. Theorem 1.9 establishes a connection between $\tilde{R}$ and the mixed curvature condition $C_{2, -1}$.

Finally, let’s discuss our strategies to prove the above-mentioned results. We will work pointwise and establish relationships between the curvature operator of the second kind and other frequently used curvature notions in Kähler geometry, such as holomorphic sectional curvature, orthogonal bisectional curvature, and orthogonal Ricci curvature. Theorems 1.2, 1.6, 1.8, and 1.9 follow immediately from parts (1), (2), (3), and (4) of the following theorem, respectively.

**Theorem 1.10** Let $(V, g, J)$ be a complex Euclidean vector space with complex dimension $m \geq 2$ and $R$ be a Kähler algebraic curvature operator on $V$ (see Definition 2.3). Then the following statements hold:

1. If $R$ has $\frac{3}{2}(m^2 - 1)$-nonnegative (respectively, $\frac{3}{2}(m^2 - 1)$-nonpositive) curvature operator of the second kind, then $R$ has constant nonnegative (respectively, nonpositive) holomorphic sectional curvature.
2. If $R$ has $\alpha_m$-nonnegative (respectively, $\alpha_m$-positive, $\alpha_m$-nonpositive, $\alpha_m$-negative) curvature operator of the second kind with $\alpha_m$ defined in (1.2), then $R$ has nonnegative (respectively, positive, nonpositive, negative) orthogonal bisectional curvature and nonnegative (respectively, positive, nonpositive, negative) holomorphic sectional curvature.
3. If $R$ has $\beta_m$-nonnegative (respectively, $\beta_m$-positive, $\beta_m$-nonpositive, $\beta_m$-negative) curvature operator of the second kind with $\beta_m$ defined in (1.3), then $R$ has nonnegative (respectively, positive, nonpositive, negative) orthogonal Ricci curvature.
(4) If $R$ has $\gamma_m$-nonnegative (respectively, $\gamma_m$-positive, $\gamma_m$-nonpositive, $\gamma_m$-negative) curvature operator of the second kind with $\gamma_m$ defined in (1.4), then the expression

$$2 \text{Ric}(X, X) - R(X, JX, X, JX)/|X|^2$$

is nonnegative (respectively, positive, nonpositive, negative) for any $0 \neq X \in V$.

The strategy to prove a statement in Theorem 1.10 is to choose a model space and apply $\hat{R}$ to the eigenvectors of the curvature operator of the second kind on this model space. A good model space leads to a sharp result. The model spaces we use for parts (1), (3), and (4) of Theorem 1.10 are $\mathbb{C}P^m$, $\mathbb{C}P^{m-1} \times \mathbb{C}P^1$, and $\mathbb{C}P^{m-1} \times \mathbb{C}$, respectively. For part (2) of Theorem 1.10, we use $\mathbb{C}P^m$ as the model space, but the result does not seem to be sharp for $m \geq 3$. Finally, it’s worth mentioning that this strategy has been used successfully by the author in several works [24–27] with $\mathbb{C}P^2$, $\mathbb{C}P^1 \times \mathbb{C}P^1$, $S^{n-1} \times S^1$, $S^k \times S^{n-k}$ and $\mathbb{C}P^k \times \mathbb{C}P^{m-k}$ as model spaces.

We emphasize that our approach is pointwise; therefore, many of our results are of pointwise nature and the completeness of the metric is not needed. Another feature is that our proofs are purely algebraic and work equally well for nonpositivity conditions on $\hat{R}$.

This article is organized as follows. Section 2 consists of three subsections. We fix some notation and conventions in Sect. 2.1 and give an introduction to the curvature operator of the second kind in Sect. 2.2. In Sect. 2.3, we review some basics about Kähler algebraic curvature operators. In Sect. 3, we collect some identities that will be frequently used in this paper. In Sect. 4, we construct an orthonormal basis of the space of traceless symmetric two-tensors on a complex Euclidean vector space and calculate the diagonal elements of the matrix representing $\hat{R}$ with respect to this basis. The proofs of Theorems 1.2, 1.6, 1.8, and 1.9 are given in Sects. 5, 6, 7, and 8, respectively.

2 Preliminaries

2.1 Notation and conventions

In the following, $(V, g)$ is a real Euclidean vector space of dimension $n \geq 2$ and $\{e_i\}_{i=1}^n$ is an orthonormal basis of $V$. We always identify $V$ with its dual space $V^*$ via the metric.

- $S^2(V)$ and $\Lambda^2(V)$ denote the space of symmetric two-tensors on $V$ and two-forms on $V$, respectively.
- $S^2_0(V)$ denotes the space of traceless symmetric two-tensors on $V$. Note that $S^2(V)$ splits into $O(V)$-irreducible subspaces as

$$S^2(V) = S^2_0(V) \oplus \mathbb{R}g.$$

- $S^2(\Lambda^2 V)$, the space of symmetric two-tensors on $\Lambda^2(V)$, has the orthogonal decomposition

$$S^2(\Lambda^2 V) = S^2_B(\Lambda^2 V) \oplus \Lambda^4 V,$$

where $S^2_B(\Lambda^2 V)$ consists of all tensors $R \in S^2(\Lambda^2(V))$ that also satisfy the first Bianchi identity. The space $S^2_B(\Lambda^2(V))$ is called the space of algebraic curvature operators (or tensors) on $V$.
- The tensor product is defined via

$$(e_i \otimes e_j)(e_k, e_l) = \delta_{ik}\delta_{jl}.$$
• \( \circ \) denotes the symmetric product defined by
\[
 u \circ v = u \otimes v + v \otimes u.
\]

• \( \wedge \) denotes the wedge product defined by
\[
 u \wedge v = u \otimes v - v \otimes u.
\]

• The inner product on \( S^2(V) \) is given by
\[
 \langle A, B \rangle = \text{tr}(A^T B).
\]

If \( \{ e_i \}_{i=1}^n \) is an orthonormal basis of \( V \), then \( \{ \frac{1}{\sqrt{2}} e_i \circ e_j \}_{1 \leq i < j \leq n} \cup \{ \frac{1}{2} e_i \circ e_i \}_{1 \leq i \leq n} \) is an orthonormal basis of \( S^2(V) \).

• The inner product on \( \Lambda^2(V) \) is given by
\[
 \langle A, B \rangle = \frac{1}{2} \text{tr}(A^T B).
\]

If \( \{ e_i \}_{i=1}^n \) is an orthonormal basis of \( V \), then \( \{ e_i \wedge e_j \}_{1 \leq i < j \leq n} \) is an orthonormal basis of \( \Lambda^2(V) \).

2.2 The curvature operator of the second kind

Given \( R \in S^2_B(\Lambda^2(V)) \), the induced self-adjoint operator \( \hat{R} : \Lambda^2(V) \to \Lambda^2(V) \) given by
\[
 \hat{R}(\omega)_{ij} = \frac{1}{2} \sum_{k,l=1}^n R_{ijkl} \omega_{kl},
\]
is called the curvature operator (or the curvature operator of the first kind by Nishikawa [40]). The most famous result concerning \( \hat{R} \) is perhaps the differentiable sphere theorem stating that a closed Riemannian manifold with two-positive curvature operator is diffeomorphic to a spherical space form. This is due to Hamilton [16] in dimension three, Hamilton [17] and Chen [10] in dimension four, and Böhm and Wilking [3] in all higher dimensions. Rigidity results for closed manifolds with two-nonnegative curvature operator are obtained by Hamilton [17] in dimension three, Hamilton [17] and Chen [10] in dimension four, and Ni and Wu [34] in all higher dimensions. For other important results regarding \( \hat{R} \), see for example [29], [14], [44], [42] and the references therein.

By the symmetries of \( R \in S^2_B(\Lambda^2(V)) \) (not including the first Bianchi identity), \( R \) also induces a self-adjoint operator \( \overline{R} : S^2(V) \to S^2(V) \) via
\[
 \overline{R}(\varphi)_{ij} = \sum_{k,l=1}^n R_{ijkl} \varphi_{kl}.
\]

However, the nonnegativity of this operator is too strong in the sense that \( \overline{R} : S^2(V) \to S^2(V) \) is nonnegative if and only if \( R = 0 \). Therefore, one usually considers the restriction of \( \overline{R} \) to the space of traceless symmetric two-tensors, i.e., the induced symmetric bilinear form \( \overline{R} : S^0_0(V) \times S^2_0(V) \to \mathbb{R} \) given by
\[
 \overline{R}(\varphi, \psi) = \sum_{i,j,k,l=1}^n R_{ijkl} \varphi_{il} \psi_{jk}.
\]
Following Nishikawa’s terminology [40], we call the symmetric bilinear form \( \hat{R} \) the curvature operator of the second kind.

The action of the Riemann curvature tensor on symmetric two-tensors indeed has a long history. It appeared for Kähler manifolds in the study of the deformation of complex analytic structures by Calabi and Vesentini [8]. They introduced the self-adjoint operator \( \xi_{\alpha\beta} \to R^\rho_{\alpha\beta} \sigma \xi_{\rho\sigma} \) from \( S^2(T_p^1M) \) to itself, and computed the eigenvalues of this operator on Hermitian symmetric spaces of classical type, with the exceptional ones handled shortly after by Borel [4]. In the Riemannian setting, the operator \( \hat{R} \) arises naturally in the context of deformations of Einstein structure in Berger and Ebin [1] (see also [22, 23] and [2]). In addition, it appears in the Bochner–Weitzenböck formulas for symmetric two-tensors (see for example [31]), for differential forms in [41], and for Riemannian curvature tensors in [20]. In another direction, curvature pinching estimates for \( \hat{R} \) were studied by Bourguignon and Karcher [5], and they calculated eigenvalues of \( \hat{R} \) on the complex projective space with the Fubini-Study metric and the quaternionic projective space with its canonical metric. Nevertheless, the operators \( \hat{R} \) and \( \hat{R} \hat{\hat{R}} \) are significantly less investigated than \( \hat{R} \).

Let \( N = \dim(S^2_0(V)) = \frac{(n-1)(n+2)}{2} \) and \( \{\varphi_i\}_{i=1}^N \) be an orthonormal basis of \( S^2_0(V) \). The \( N \times N \) matrix \( \hat{R}(\varphi_i, \varphi_j) \) is called the matrix representation of \( \hat{R} \) with respect to the orthonormal basis \( \{\varphi_i\}_{i=1}^N \). The eigenvalues of \( \hat{R} \) refer to the eigenvalues of any of its matrix representations. Note that the eigenvalues of \( \hat{R} \) are independent of the choices of the orthonormal bases.

For a positive integer \( 1 \leq k \leq N \), we say \( R \in S^2_0(\Lambda^2(V)) \) has \( k \)-nonnegative curvature operator of the second kind if the sum of the smallest \( k \) eigenvalues of \( \hat{R} \) is nonnegative. This was extended to all \( k \in [1, N] \) in [26] as follows.

**Definition 2.1** Let \( N = \frac{(n-1)(n+2)}{2} \) and \( \alpha \in [1, N] \).

1. We say \( R \in S^2_0(\Lambda^2(V)) \) has \( \alpha \)-nonnegative curvature operator of the second kind (\( \hat{R} \) is \( \alpha \)-nonnegative for short) if for any orthonormal basis \( \{\varphi_i\}_{i=1}^N \) of \( S^2_0(V) \), it holds that

\[
\sum_{i=1}^{\lfloor \alpha \rfloor} \hat{R}(\varphi_i, \varphi_i) + (\alpha - \lfloor \alpha \rfloor)\hat{R}(\varphi_{\lfloor \alpha \rfloor + 1}, \varphi_{\lfloor \alpha \rfloor + 1}) \geq 0.
\]

If the inequality is strict, then \( R \) is said to have \( \alpha \)-positive curvature operator of the second kind (\( \hat{R} \) is \( \alpha \)-positive for short).

2. We say \( R \in S^2_0(\Lambda^2(V)) \) has \( \alpha \)-nonpositive (respectively, \( \alpha \)-negative) curvature operator of the second kind if \( -R \) has \( \alpha \)-nonnegative (respectively, \( \alpha \)-positive) curvature operator of the second kind.

Note that when \( \alpha = k \) is an integer, this agrees with the usual definition. We always omit \( \alpha \) when \( \alpha = 1 \). Clearly, \( \alpha \)-nonnegativity of \( \hat{R} \) implies \( \beta \)-nonnegativity of \( \hat{R} \) if \( \alpha \leq \beta \). The same holds for positivity, negativity, and nonpositivity.

**Definition 2.2** A Riemannian manifold \((M^n, g)\) is said to have \( \alpha \)-nonnegative (respectively, \( \alpha \)-positive, \( \alpha \)-nonpositive, \( \alpha \)-negative) curvature operator of the second kind if \( R_p \in S^2_0(\Lambda^2 T_p M) \) has \( \alpha \)-nonnegative (respectively, \( \alpha \)-positive, \( \alpha \)-nonpositive, \( \alpha \)-negative) curvature operator of the second kind for each \( p \in M \).

The generalized definition is motivated by geometric examples. For instance, \( S^{n-1} \times S^1 \) has \( \alpha \)-nonnegative curvature operator of the second kind for any \( \alpha \geq (n + \frac{n-2}{n}) \), but not for any \( \alpha < (n + \frac{n-2}{n}) \). Another example is \((\mathbb{C}P^m, g_{FS})\), whose curvature operator of the second kind is \( \alpha \)-nonnegative for any \( \alpha \geq \frac{3}{2}(m^2 - 1) \), but not for any \( \alpha < \frac{3}{2}(m^2 - 1) \).
2.3 Kähler algebraic curvature operators

Throughout this subsection, let \((V, g, J)\) be a complex Euclidean vector space of complex dimension \(m \geq 1\). In other words, \((V, g)\) is a real Euclidean vector space of real dimension \(2m\) and \(J : V \to V\) is an endomorphism of \(V\) satisfying the following two properties:

1. \(J^2 = -\text{id on } V\),
2. \(g(X, Y) = g(JX, JY)\) for all \(X, Y \in V\).

\(J\) is called a complex structure on \(V\).

**Definition 2.3** \(R \in S_2^2(\Lambda^2 V)\) is called a Kähler algebraic curvature operator if it satisfies

\[
R(X, Y, Z, W) = R(X, JY, JZ, JW),
\]

for all \(X, Y, Z, W \in V\).

Note that a Kähler algebraic curvature operator \(R\) satisfies

\[
R(X, JX, Y, JY) = R(X, Y, X, JY) + R(X, JY, X, JY),
\]

for all \(X, Y, Z, W \in V\). (2.1) follows from the symmetries of \(R\) and (2.2) is a consequence of the first Bianchi identity and (2.1). The expression on the left-hand side of (2.2) is called bisectional curvature (see [15]), holomorphic sectional curvature if \(X = Y\) (see for instance [46]), and orthogonal bisectional curvature if \(g(X, Y) = g(X, JY) = 0\) (see for example [28]). We will use (2.1) and (2.2) frequently in the rest of this paper.

In view of (2.1) and (2.2), the Ricci tensor of a Kähler algebraic curvature operator \(R\) is given by

\[
\text{Ric}(X, Y) = \sum_{i=1}^{m} R(X, JY, e_i, Je_i),
\]

and the scalar curvature of \(R\), denoted by \(S\), is given by

\[
S = 2 \sum_{i, j=1}^{m} R(e_i, Je_i, e_j, Je_j),
\]

where \(\{e_1, \ldots, e_m, Je_1, \ldots, Je_m\}\) is an orthonormal basis of \(V\).

Next, we recall some definitions.

**Definition 2.4** A Kähler algebraic curvature operator \(R\) is said to have

1. nonnegative holomorphic sectional curvature if for any \(X \in V\),

\[
R(X, JX, X, JX) \geq 0.
\]

2. nonnegative orthogonal bisectional curvature if for any \(X, Y \in V\) with \(g(X, Y) = g(X, JY) = 0\),

\[
R(X, JX, Y, JY) \geq 0.
\]
nonnegative orthogonal Ricci curvature if for any $0 \neq X \in V$,
\[
\text{Ric}^\perp(X, X) := \text{Ric}(X, X) - R(X, JX, X, JX)/|X|^2 \geq 0
\]
Analogously, one defines the positivity, negativity, and non-positivity of holomorphic sectional curvature, orthogonal bisectional curvature, and orthogonal Ricci curvature. Finally, a Kähler manifold $(M^m, g, J)$ is said to satisfy a curvature condition if $R_p \in S^2_B(\Lambda^2 T_p M)$ satisfies the curvature condition at every $p \in M$.

3 Identities

In this section, we collect some identities that will be frequently used in subsequent sections. Many of them have been used explicitly or implicitly in earlier works such as [41], [9], [24–26], and [38].

Lemma 3.1 Let $(V, g)$ be a real Euclidean vector space of dimension $n \geq 2$ and $\{e_i\}_{i=1}^n$ be an orthonormal basis of $V$. Then we have
\[
\langle e_i \odot e_j, e_k \odot e_l \rangle = 2(\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}),
\]
and
\[
\check{R}(e_i \odot e_j, e_k \odot e_l) = 2(R_{iklj} + R_{ilkj}),
\]
for all $1 \leq i, j, k, l \leq n$.

Proof Using $(e_i \otimes e_j)(e_k, e_l) = \delta_{ik} \delta_{jl}$, we compute that
\[
\langle e_i \odot e_j, e_k \odot e_l \rangle = \sum_{p, q=1}^n (e_i \odot e_j)(e_p, e_q) \cdot (e_k \odot e_l)(e_p, e_q)
= \sum_{p, q=1}^n (\delta_{ip} \delta_{jq} + \delta_{iq} \delta_{jp})(\delta_{kp} \delta_{ql} + \delta_{kq} \delta_{lp})
= 2(\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}).
\]
This proves (3.1). To prove (3.2), we calculate that
\[
\check{R}(e_i \odot e_j, e_k \odot e_l) = \sum_{p, q, r, s=1}^n R_{prsq} \cdot (e_i \odot e_j)(e_p, e_q) \cdot (e_k \odot e_l)(e_r, e_s)
= \sum_{p, q, r, s=1}^n R_{prsq}(\delta_{ip} \delta_{jq} + \delta_{iq} \delta_{jp})(\delta_{kr} \delta_{ls} + \delta_{ks} \delta_{lr})
= R_{iklj} + R_{ilkj} + R_{jkl} + R_{jkl} + R_{jkl}
= 2(R_{iklj} + R_{ilkj}).
\]

Lemma 3.2 Let $\{e_i, e_j, e_k, e_l\}$ be an orthonormal four-frame in a Euclidean vector space $(V, g)$ of dimension $n \geq 4$. Define the following traceless symmetric two-tensors:
\[
h^\pm_1 = \frac{1}{2} (e_i \odot e_j \pm e_k \odot e_l),
\]
Let \( \{e_i, e_j\} \) be an orthonormal two-frame in a Euclidean vector space \((V, g)\) of dimension \(n \geq 2\). Define the following traceless symmetric two-tensors:

\[
h_3 = \frac{1}{2\sqrt{2}} (e_i \otimes e_i - e_j \otimes e_j),
\]
\[
h_4 = \frac{1}{\sqrt{2}} e_i \otimes e_j.
\]

Then we have \( \|h_3\| = \|h_4\| = 1 \) and

\[
\hat{R}(h_3, h_3) = \hat{R}(h_4, h_4) = R_{ijij}.
\]

**Lemma 3.3** Let \( \{e_i, e_j\} \) be an orthonormal two-frame in a Euclidean vector space \((V, g)\) of dimension \(n \geq 2\). Define the following traceless symmetric two-tensors:

\[
h_3 = \frac{1}{2\sqrt{2}} (e_i \otimes e_i - e_j \otimes e_j),
\]
\[
h_4 = \frac{1}{\sqrt{2}} e_i \otimes e_j.
\]

Then we have \( \|h_3\| = \|h_4\| = 1 \) and

\[
\hat{R}(h_3, h_3) = \hat{R}(h_4, h_4) = R_{ijij}.
\]

**Proof** This is a straightforward computation using (3.1) and (3.2). \( \square \)

**Lemma 3.4** Let \( \{e_1, \ldots, e_m, Je_1, \ldots, Je_m\} \) be an orthonormal basis of a complex Euclidean vector space \((V, g, J)\) of complex dimension \(m \geq 1\). Then for any \(1 \leq i, j \leq m\), we have

\[
\hat{R}(e_i \otimes e_i + Je_i \otimes Je_i, e_j \otimes e_j + Je_j \otimes Je_j) = -8R(e_i, Je_i, e_j, Je_j).
\]

**Proof** This follows from a routine computation using (3.2), (2.1), and (2.2) as follows:

\[
\hat{R}(e_i \otimes e_i + Je_i \otimes Je_i, e_j \otimes e_j + Je_j \otimes Je_j)
= \hat{R}(e_i \otimes e_i, e_j \otimes e_j) + \hat{R}(e_i \otimes e_i, Je_j \otimes Je_j)
\]
\[ + \hat{R}(J e_i \odot J e_i, e_j \odot e_j) + \hat{R}(J e_i \odot J e_i, e_j \odot J e_j) \]
\[ = -4R(e_i, e_j, e_i, e_j) - 4R(e_i, e_j, e_i, e_j) \]
\[ - 4R(J e_i, e_j, J e_i, e_j) - 4R(J e_i, J e_i, J e_i, J e_i) \]
\[ = -8R(e_i, e_j, e_i, e_j) - 8R(e_i, J e_i, e_i, J e_i) \]
\[ = -8R(e_i, J e_i, e_j, J e_j). \]

The author observed in [24, Proposition 4.1] (see also [38, Proposition 1.2]) that the trace of \( \hat{R} \) is equal to \( \frac{n + 2}{2n} S \), where \( S \) denotes the scalar curvature. That is to say, if \( \{ \varphi_i \}_{i=1}^N \) is an orthonormal basis of \( S^2_0(V) \), then
\[ \sum_{i=1}^N \hat{R}(\varphi_i, \varphi_i) = \frac{n + 2}{2n} S. \quad (3.4) \]

This implies that

**Lemma 3.5** \( R \in S^2_0(\Lambda^2 V) \) has \( \frac{(n-1)(n+2)}{2} \)-nonnegative (respectively, \( \frac{(n-1)(n+2)}{2} \)-nonpositive) curvature operator of the second kind if and only if \( R \) has nonnegative (respectively, nonpositive) scalar curvature \( S \).

**Lemma 3.6** Suppose that \( R \in S^2_0(\Lambda^2 V) \) has \( \alpha \)-nonnegative (respectively, \( \alpha \)-nonpositive) curvature operator of the second kind for some \( \alpha < \frac{(n-1)(n+2)}{2} \). If \( S = 0 \), then \( R = 0 \).

### 4 An orthonormal basis for \( S^2_0(V) \)

Below we construct an orthonormal basis for \( S^2_0(V) \) on a complex Euclidean vector space \( (V, g, J) \).

**Lemma 4.1** Let \( \{ e_1, \ldots, e_m, J e_1, \ldots, J e_m \} \) be an orthonormal basis of a complex Euclidean vector space \( (V, g, J) \). Let
\[ E^+ = \text{span}\{u \odot v - J u \odot J v : u, v \in V\}. \]

Define
\[ \varphi_{ij}^+ = \frac{1}{2} (e_i \odot e_j - J e_i \odot J e_j), \quad \text{for } 1 \leq i < j \leq m, \]
\[ \psi_{ij}^+ = \frac{1}{2} (e_i \odot J e_j + J e_i \odot e_j), \quad \text{for } 1 \leq i < j \leq m, \]
\[ \theta_i = \frac{1}{2\sqrt{2}} (e_i \odot e_i - J e_i \odot J e_i), \quad \text{for } 1 \leq i \leq m, \]
\[ \theta_{m+i} = \frac{1}{\sqrt{2}} e_i \odot J e_i, \quad \text{for } 1 \leq i \leq m. \]

Then
\[ E^+ = \{ \varphi_{ij}^+ \}_{1 \leq i < j \leq m} \cup \{ \psi_{ij}^+ \}_{1 \leq i < j \leq m} \cup \{ \theta_i \}_{i=1}^{2m} \quad (4.1) \]
forms an orthonormal basis of \( E^+ \). In particular, \( \dim(E^+) = m(m + 1) \).
Proof Clearly, $\mathcal{E}^+ \subset E^+$. Using (3.1), one verifies that $\mathcal{E}^+$ is an orthonormal subset of $E^+$. The statement $\mathcal{E}^+$ spans $E^+$ follows from the following observation. If

$$u = \sum_{i=1}^{m} x_i e_i + \sum_{i=1}^{m} y_i Je_i,$$

$$v = \sum_{i=1}^{m} z_i e_i + \sum_{i=1}^{m} w_i Je_i,$$

then

$$u \odot v - J u \odot J v = \sum_{i,j=1}^{m} (x_i z_j - y_i w_j)(e_i \odot e_j - Je_i \odot Je_j)$$

$$+ \sum_{i,j=1}^{m} (x_i w_j + y_i z_j)(e_i \odot Je_j + Je_i \odot e_j)$$

$$= 4 \sum_{1 \leq i < j \leq m} (x_i z_j - y_i w_j)\phi_{ij}^+ + 4 \sum_{1 \leq i < j \leq m} (x_i w_j + y_i z_j)\psi_{ij}^+$$

$$+ 2\sqrt{2} \sum_{i=1}^{m} (x_i z_i - y_i w_i)\theta_i + 2\sqrt{2} \sum_{i=1}^{m} (x_i w_i + y_i z_i)\theta_{m,i+i}. $$

Thus, $\mathcal{E}^+$ forms an orthonormal basis of $E^+$ and $\dim(E^+) = m(m+1).$ \hfill $\Box$

Lemma 4.2 Let $\{e_1, \ldots, e_m, Je_1, \ldots, Je_m\}$ be an orthonormal basis of a complex Euclidean vector space $(V, g, J)$. Let $E^- = (E^+)^\perp$ be the orthogonal complement of $E^+$, where $E^+$ is the subspace of $S^2_0(V)$ defined in Lemma 4.1. Define

$$\phi_{ij}^- = \frac{1}{2} (e_i \odot e_j + Je_i \odot Je_j), \text{ for } 1 \leq i < j \leq m,$$

$$\psi_{ij}^- = \frac{1}{2} (e_i \odot Je_j - Je_i \odot e_j), \text{ for } 1 \leq i < j \leq m,$$

and

$$\eta_k = \frac{k}{\sqrt{8k(k+1)}} (e_{k+1} \odot e_{k+1} + Je_{k+1} \odot Je_{k+1})$$

$$- \frac{1}{\sqrt{8k(k+1)}} \sum_{i=1}^{k} (e_i \odot e_i + Je_i \odot Je_i),$$

for $1 \leq k \leq m - 1$. Then

$$\mathcal{E}^- = \{\phi_{ij}^-\}_{1 \leq i < j \leq m} \cup \{\psi_{ij}^-\}_{1 \leq i < j \leq m} \cup \{\eta_k\}_{k=1}^{m-1} \quad (4.2)$$

forms an orthonormal basis of $E^-$. In particular, $\dim(E^-) = m^2 - 1$.

Proof Since $S^2_0(V) = E^+ \oplus E^-$, we have that

$$\dim(E^-) = \dim(S^2_0(V)) - \dim(E^+)$$

$$= (2m-1)(m+1) - m(m+1)$$

$$= m^2 - 1.$$
As the number of traceless symmetric two-tensors in $\mathcal{E}^-$ is equal to $\dim(E^-)$, it suffices to verify that $\mathcal{E}^+ \cup \mathcal{E}^-$ is an orthonormal basis of $S_0^2(V)$, which is a straightforward computation using (3.1).

We remark that on $(\mathbb{C}^{\mathbb{P}^m}, g_{FS})$, $E^+$ is the eigenspace associated with the eigenvalue 4, and $E^-$ is the eigenspace associated with the eigenvalue $-2$. The subspace $E^-$ is spanned by traceless symmetric two-tensors of the form $u \otimes v + Ju \otimes Jv$ with $g(u, v) = 0$ and $u \otimes u + Ju \otimes Jv - v \otimes v - Jv \otimes Jv$ with $g(u, u) = g(v, v)$. See [5, page 84].

The next step is to calculate the matrix representation of $\hat{R}$ with respect to the orthonormal basis $\mathcal{E}^+ \cup \mathcal{E}^-$ for $S_0^2(V)$. We only need the diagonal elements of this matrix.

**Lemma 4.3** For the basis $\mathcal{E}^+$ of $E^+ \subset S_0^2(V)$ defined in Lemma 4.1, we have

$$\hat{R}(\varphi_{ij}^+, \varphi_{ij}^+) = \hat{R}(\psi_{ij}^+, \psi_{ij}^+) = 2R(e_i, J e_i, e_j, J e_j)$$

for $1 \leq i < j \leq m$, and

$$\hat{R}(\theta_i, \theta_i) = \hat{R}(\theta_{m+i}, \theta_{m+i}) = R(e_i, J e_i, e_i, J e_i),$$

for $1 \leq i \leq m$. Moreover,

$$\sum_{1 \leq i < j \leq m} \left( \hat{R}(\varphi_{ij}^+, \varphi_{ij}^+) + \hat{R}(\psi_{ij}^+, \psi_{ij}^+) \right) + \sum_{i=1}^{2m} \hat{R}(\theta_i, \theta_i) = S.$$  (4.5)

**Proof** Applying Lemma 3.2 to the orthonormal four-frame $\{e_i, e_j, J e_i, J e_j\}$ yields

$$\hat{R}(\varphi_{ij}^+, \varphi_{ij}^+) = \frac{1}{2} \left( R(e_i, e_j, e_i, e_j) + R(J e_i, J e_j, e_i, J e_j) \right)$$

$$- R(e_i, e_j, J e_i, J e_j) - R(e_i, J e_j, e_i, J e_j)$$

$$= R(e_i, e_j, e_i, e_j) + R(e_i, J e_j, e_i, J e_j) + R(e_i, J e_i, e_i, J e_j)$$

$$= 2R(e_i, J e_i, e_i, J e_j),$$

where we have used (2.1) and (2.2). Similarly, we get

$$\hat{R}(\psi_{ij}^+, \psi_{ij}^+) = \frac{1}{2} \left( R(e_i, J e_j, e_i, J e_j) + R(J e_i, e_j, e_i, J e_j) \right)$$

$$+ R(e_i, J e_i, e_j, J e_j)$$

$$= R(e_i, J e_j, e_i, J e_j) + R(e_i, J e_i, e_i, J e_j) + R(e_i, e_j, e_i, e_j)$$

$$= 2R(e_i, J e_i, e_j, J e_j).$$

Now, (4.3) is proved.

For $1 \leq i \leq m$, we apply Lemma 3.3 to the orthonormal two-frame $\{e_i, J e_i\}$ and get

$$\hat{R}(\theta_i, \theta_i) = \hat{R}(\theta_{m+i}, \theta_{m+i}) = R(e_i, J e_i, e_i, J e_i).$$

This proves (4.4).

Finally, (4.5) follows from (4.3), (4.4), and (2.4).

**Lemma 4.4** For the basis $\mathcal{E}^-$ of $E^- \subset S_0^2(V)$ defined in Lemma 4.1, we have

$$\sum_{1 \leq i < j \leq m} \left( \hat{R}(\varphi_{ij}^-, \varphi_{ij}^-) + \hat{R}(\psi_{ij}^-, \psi_{ij}^-) \right) + \sum_{k=1}^{m-1} \hat{R}(\eta_k, \eta_k) = -\frac{m-1}{2m} S.$$  (4.6)

**Proof** Since $\mathcal{E}^+ \cup \mathcal{E}^-$ is an orthonormal basis for $S_0^2(V)$, (4.6) follows immediately from and (4.5), (3.4) and (2.4).
5 Flatness

We prove Theorem 1.2 in this section. The key ingredient is

**Proposition 5.1** Let $R$ be a Kähler algebraic curvature operator on a complex Euclidean vector space $(V, g, J)$ of complex dimension $m \geq 2$.

1. If $R$ has $\frac{3}{2}(m^2 - 1)$-nonnegative (respectively, $\frac{3}{2}(m^2 - 1)$-nonpositive) curvature operator of the second kind, then $R$ has constant nonnegative (respectively, nonpositive) holomorphic sectional curvature.

2. If $R$ has $\alpha$-nonnegative (respectively, $\alpha$-nonpositive) curvature operator of the second kind for some $\alpha < \frac{3}{2}(m^2 - 1)$, then $R$ is flat.

We need an elementary lemma.

**Lemma 5.1** Let $N$ be a positive integer and $A$ be a collection of $N$ real numbers. Denote by $a_i$ the $i$-th smallest number in $A$ for $1 \leq i \leq N$. Define a function $f(A, x)$ by

$$f(A, x) = \sum_{i=1}^{\lfloor x \rfloor} a_i + (x - \lfloor x \rfloor)a_{\lfloor x \rfloor + 1},$$

for $x \in [1, N]$. Then we have

$$f(A, x) \leq x\bar{a}, \quad (5.1)$$

where $\bar{a} := \frac{1}{N} \sum_{i=1}^{N} a_i$ is the average of all numbers in $A$. Moreover, the equality holds for some $x \in [1, N)$ if and only if $a_i = \bar{a}$ for all $1 \leq i \leq N$.

**Proof** We first show that Lemma 5.1 holds when $x = k$ is an integer. This is obvious if $k = 1$ or $k = N$. If $2 \leq k \leq N - 1$, we have

$$N(f(A, k) - k\bar{a}) = N\sum_{i=1}^{k} a_i - k\sum_{i=1}^{N} a_i = (N - k)\sum_{i=1}^{k} a_i - k\sum_{i=k+1}^{N} a_i.$$

Note that $\sum_{i=1}^{k} a_i \leq ka_{k+1}$ with equality if and only if $a_1 = \cdots = a_{k+1}$ and $\sum_{i=k+1}^{N} a_i \geq (N - k)a_{k+1}$ with equality if and only if $a_{k+1} = \cdots = a_N$. So, we get

$$N(f(A, k) - k\bar{a}) \leq (N - k)ka_{k+1} - k(N - k)a_{k+1} = 0.$$

Moreover, the equality holds if and only if $a_i = \bar{a}$ for all $1 \leq i \leq N$.

Next, for $x \in (1, N)$, we have

$$f(A, x) = \sum_{i=1}^{\lfloor x \rfloor} a_i + (x - \lfloor x \rfloor)a_{\lfloor x \rfloor + 1}$$

$$= (x - \lfloor x \rfloor)\sum_{i=1}^{\lfloor x \rfloor + 1} a_i + (1 - x + \lfloor x \rfloor)\sum_{i=1}^{\lfloor x \rfloor} a_i$$

$$= (x - \lfloor x \rfloor)f(A, \lfloor x \rfloor + 1) + (1 - x + \lfloor x \rfloor)f(A, \lfloor x \rfloor).$$
\[ \leq (x - \lfloor x \rfloor)(\lfloor x \rfloor + 1)\bar{a} + (1 - x + \lfloor x \rfloor)\lfloor x \rfloor \bar{a} \]

Here we have used \( f(A, \lfloor x \rfloor + 1) \leq (\lfloor x \rfloor + 1)\bar{a} \) and \( f(A, \lfloor x \rfloor) \leq \lfloor x \rfloor \bar{a} \) in getting the inequality step.

If the equality holds in (5.1) for \( x \in [1, N] \), then we must have \( f(A, \lfloor x \rfloor + 1) = (\lfloor x \rfloor + 1)\bar{a} \) and \( f(A, \lfloor x \rfloor) = \lfloor x \rfloor \bar{a} \), which implies \( a_i = \bar{a} \) for all \( 1 \leq i \leq N \). This completes the proof. \( \square \)

We are now ready to prove Proposition 5.1.

**Proof of Proposition 5.1** (1) Let \( \{e_1, \ldots, e_m, J e_1, \ldots, J e_m\} \) be an orthonormal basis of \( V \) and let \( \mathcal{E}^+ \cup \mathcal{E}^- \) be the orthonormal basis of \( S_0^2(V) \) constructed in Lemmas 4.1 and 4.1.

Let \( A \) be the collection of the values \( \hat{R}(\theta_i, \theta_i) \) for \( 1 \leq i \leq 2m, \hat{R}(\varphi_{ij}^+, \varphi_{ij}^-) \) and \( \hat{R}(\psi_{ij}^+, \psi_{ij}^-) \) for \( 1 \leq i < j \leq m \). By (4.5), the sum of all values in \( A \) is equal to \( S \), and \( \bar{a} \), the average of all values in \( A \), is given by

\[ \bar{a} = \frac{S}{m(m+1)}. \]

By Lemma 5.1, we have

\[ f \left( A, \frac{1}{2}(m^2 - 1) \right) \leq \frac{1}{2}(m^2 - 1)a = \frac{m-1}{2m}S, \tag{5.2} \]

where \( f(A, x) \) is the function defined in Lemma 5.1.

If \( R \) has \( \frac{3}{2}m(m-1) \)-nonnegative curvature operator of the second kind, then

\[ 0 \leq \sum_{1 \leq i < j \leq m} \left( \hat{R}(\varphi_{ij}^-, \varphi_{ij}^-) + \hat{R}(\psi_{ij}^+, \psi_{ij}^-) \right) + \sum_{k=1}^{m-1} \hat{R}(\eta_k, \eta_k) + f \left( A, \frac{1}{2}(m^2 - 1) \right). \]

Substituting (4.6) into the above inequality yields

\[ \frac{m-1}{2m}S \leq f \left( A, \frac{1}{2}(m^2 - 1) \right). \tag{5.3} \]

Therefore, (5.2) must hold as equality and we conclude from Lemma 5.1 that all values in \( A \) are equal to \( \frac{S}{m(m+1)} \). In particular,

\[ R(e_i, Je_i, e_i, Je_i) = \frac{S}{m(m+1)} \]

for all \( 1 \leq i \leq m \). Since the orthonormal basis \( \{e_1, \ldots, e_m, Je_1, \ldots, Je_m\} \) is arbitrary, we have

\[ R(X, JX, X, JX) = \frac{S}{m(m+1)} \]

for any unit vector \( X \in V \). Hence, \( R \) has constant holomorphic sectional curvature. Finally, we notice that \( S \geq 0 \) by Lemma 3.5.

If \( \hat{R} \) is \( \frac{3}{2}m(m-1) \) nonpositive, we apply the proved result to \(-R\) and get that \( R \) has constant nonpositive holomorphic sectional curvature.

(2) \( R \) has constant holomorphic sectional curvature by part (1). Thus, \( R = c R_{\mathbb{C}P^m} \) for some \( c \in \mathbb{R} \), where \( R_{\mathbb{C}P^m} \) denotes the Riemann curvature tensor of \((\mathbb{C}P^m, g_{FS})\). The desired
conclusion follows from the fact that \( cR_{\mathbb{C}P^m} \) has \( \alpha \)-nonnegative or \( \alpha \)-nonpositive curvature operator of the second kind for some \( \alpha < \frac{3}{2}(m^2 - 1) \) if and only if \( c = 0 \) (see [5]).

Alternatively, one can argue as follows. If \( \hat{R} \) is \( \left( \frac{3}{2}(m^2 - 1) - \epsilon \right) \)-nonnegative for some \( 1 > \epsilon > 0 \), then we would get in (5.3) that

\[
0 \leq -\frac{m-1}{2m} S + f \left( A, \frac{1}{2}(m^2 - 1) - \epsilon \right) \leq -\epsilon S \frac{m}{m(m+1)}.
\]

So, we have \( S = 0 \). By Lemma 3.6, \( R = 0 \). Similarly, \( \hat{R} \) is \( \alpha \)-nonpositive for some \( \alpha < \frac{3}{2}(m^2 - 1) \) if and only if \( R = 0 \).

We now prove Theorem 1.2 and its corollaries.

**Proof of Theorem 1.2** Theorem 1.2 follows immediately from Proposition 5.1 and Schur’s lemma for Kähler manifolds (see for instance [21, Theorem 7.5]).

**Proof of Corollary 1.4** If \( \hat{R} \) is \( \alpha \)-positive or \( \alpha \)-negative for some \( \alpha \leq \frac{3}{2}(m^2 - 1) \), then \( R \) must have constant holomorphic sectional curvature by Proposition 5.1. Thus, \( R = cR_{\mathbb{C}P^m} \) for some \( c \in \mathbb{R} \). This contradicts the fact that \( cR_{\mathbb{C}P^m} \) does not have \( \alpha \)-positive or \( \alpha \)-nonnegative curvature operator of the second kind for any \( \alpha \leq \frac{3}{2}(m^2 - 1) \).

**Proof of Corollary 1.5** By Theorem 1.2, \( M \) has constant holomorphic sectional curvature. The conclusion follows from the the classification of complete simply-connected Kähler manifolds with constant holomorphic sectional curvature.

### 6 Orthogonal bisectional curvature

Throughout this section, \( \alpha_m \) is the number defined in (1.2), i.e.,

\[
\alpha_m = \frac{3m^3 - m + 2}{2m}.
\]

We restate part (2) of Theorem 1.10 as two propositions.

**Proposition 6.1** Let \( R \) be a Kähler algebraic curvature operator on a complex Euclidean vector space \( (V, g, J) \) of complex dimension \( m \geq 2 \). If \( R \) has \( \alpha_m \)-nonnegative (respectively, \( \alpha_m \)-positive, \( \alpha_m \)-nonpositive, \( \alpha_m \)-negative) curvature operator of the second kind, then \( R \) has nonnegative (respectively, positive, nonpositive, negative) orthogonal bisectional curvature.

**Proposition 6.2** Let \( R \) be a Kähler algebraic curvature operator on a complex Euclidean vector space \( (V, g, J) \) of complex dimension \( m \geq 2 \). If \( R \) has \( \alpha_m \)-nonnegative (respectively, \( \alpha_m \)-positive, \( \alpha_m \)-nonpositive, \( \alpha_m \)-negative) curvature operator of the second kind, then \( R \) has nonnegative (respectively, positive, nonpositive, negative) holomorphic sectional curvature.

**Proof of Proposition 6.1** Let \( \{e_1, \ldots, e_m, Je_1, \ldots, Je_m\} \) be an orthonormal basis of \( V \) and let \( E^+ \cup E^- \) be the orthonormal basis of \( S^2_0(V) \) constructed in Lemma 4.1 and Lemma 4.1.

Let \( A \) be the collection of the values \( \hat{R}(\theta_i, \theta_i) \) for \( 1 \leq i \leq m \). By Lemma 4.3, \( A \) consists of one copy of \( R(e_i, Je_i, e_i, Je_i) \) for each \( 1 \leq i \leq m \), and its average \( \bar{a} \) is given by

\[
\bar{a} = \frac{1}{m} \sum_{i=1}^{m} R(e_i, Je_i, e_i, Je_i).
\]
Let $B$ be the collection of the values $\hat{R}(\varphi_{ij}^+, \varphi_{ij}^-)$ and $\hat{R}(\psi_{ij}^+, \psi_{ij}^-)$ for $1 \leq i < j \leq m$ with $(i, j) \neq (1, 2)$. According to Lemma 4.3, $B$ is made of two copies of $2R(e_i, Je_i, e_j, Je_j)$ for each $1 \leq i < j \leq m$ with $(i, j) \neq (1, 2)$. The average of all values in $B$, denoted by $\bar{b}$, is given by

$$\bar{b} = \frac{4}{(m - 2)(m + 1)} \left( \sum_{1 \leq i < j \leq m} R(e_i, Je_i, e_j, Je_j) - R(e_1, Je_1, e_2, Je_2) \right).$$

Next, let $f(A, x)$ and $f(B, x)$ be defined as in Lemma 5.1. By Lemma 5.1, we have

$$f(A, m - 1) \leq (m - 1)\bar{a} = \frac{m - 1}{m} \sum_{i=1}^m R(e_i, Je_i, e_1, Je_1), \quad (6.1)$$

and

$$f(B, \frac{(m - 2)(m^2 - 1)}{2m}) \leq \frac{(m - 2)(m^2 - 1)}{2m} \bar{b} = \frac{2m - 1}{m} \left( \sum_{1 \leq i < j \leq m} R(e_i, Je_i, e_j, Je_j) - R(e_1, Je_1, e_2, Je_2) \right). \quad (6.2)$$

Note that

$$\alpha_m = (m^2 - 1) + 2 + (m - 1) + \frac{(m - 2)(m^2 - 1)}{2m}.$$ 

If $R$ has $\alpha_m$-nonnegative curvature operator of the second kind, then we have

$$0 \leq \sum_{1 \leq i < j \leq m} \left( \hat{R}(\varphi_{ij}^+, \varphi_{ij}^-) + \hat{R}(\psi_{ij}^+, \psi_{ij}^-) \right) + \sum_{k=1}^{m-1} \hat{R}(\eta_k, \eta_k)$$

$$+ \hat{R}(\varphi_{12}^+, \varphi_{12}^-) + \hat{R}(\psi_{12}^+, \psi_{12}^-)$$

$$+ f(A, m - 1) + f(B, \frac{(m - 2)(m^2 - 1)}{2m}).$$

Substituting (4.6), (4.3), (6.1), and (6.2) into the above inequality yields

$$0 \leq -\frac{m - 1}{2m} S + 4R(e_1, Je_1, e_2, Je_2) + \frac{m - 1}{m} \sum_{i=1}^m R(e_i, Je_i, e_1, Je_1)$$

$$+ \frac{2(m - 1)}{m} \left( \sum_{1 \leq i < j \leq m} R(e_i, Je_i, e_j, Je_j) - R(e_1, Je_1, e_2, Je_2) \right)$$

$$= \frac{2(m + 1)}{m} R(e_1, Je_1, e_2, Je_2),$$

where we have used (2.4). The arbitrariness of the orthonormal frame allows us to conclude that $R(e_1, Je_1, e_2, Je_2) \geq 0$ for any orthonormal two-frame $\{e_1, e_2\}$. Hence, $R$ has nonnegative orthogonal bisectional curvature.

If $R$ has $\alpha_m$-positive curvature operator of the second kind, then the last two inequalities in the above argument become strict and one concludes that $R$ has positive orthogonal bisectional curvature. Applying the results to $-R$ then yields the statements for $\alpha_m$-negativity and $\alpha_m$-nonpositivity. \qed
Proof of Proposition 6.2} The idea is the same as in the proof of Proposition 6.1. The main difference is that we need to separate the terms involving $R(e_1, J e_1, e_1, J e_1)$. As before, let \{e_1, \ldots, e_m, J e_1, \ldots, J e_m\} be an orthonormal basis of $V$ and let $\mathcal{E}^+ \cup \mathcal{E}^-$ be the orthonormal basis of $S_0^2(V)$ constructed in Lemmas 4.1 and 4.1.

Let $A$ be the collection of the values $\hat{R}(\theta_i, \theta_i)$ for $2 \leq i \leq m$ and $m + 2 \leq i \leq 2m$, and the values $\hat{R}(\varphi_{ij}^+ \varphi_{ij}^+)$ and $\hat{R}(\psi_{ij}^+ \psi_{ij}^+)$ for $1 \leq i < j \leq m$. By Lemma 4.3, $A$ consists of two copies of $R(e_i, J e_i, e_j, J e_j)$ for each $2 \leq i \leq m$ and two copies of $2R(e_i, J e_i, e_j, J e_j)$ for each $1 \leq i < j \leq m$. The average of all values in $A$, denoted by $\tilde{a}$, is given by

$$\tilde{a} = \frac{1}{(m-1)(m+2)} \left( 2 \sum_{i=2}^{m} R(e_i, J e_i, e_i, J e_i) + 4 \sum_{1 \leq i < j \leq m} R(e_i, J e_i, e_j, J e_j) \right)$$

and let $\alpha_m = (m^2-1) + 2 + \frac{(m-1)^2(m+2)}{2m}$.

Since $\hat{R}$ is $\alpha_m$-nonnegative, we obtain

$$0 \leq \sum_{1 \leq i < j \leq m} (\hat{R}(\varphi_{ij}^+, \varphi_{ij}^+) + \hat{R}(\psi_{ij}^+, \psi_{ij}^+)) + \sum_{k=1}^{m-1} \hat{R}(\eta_k, \eta_k)
+ \hat{R}(\theta_1, \theta_1) + \hat{R}(\theta_{m+1}, \theta_{m+1})
+ f \left( A, \frac{(m-1)^2(m+2)}{2m} \right).$$

Substituting (4.6), (4.4), and (6.3) into the above inequality yields

$$0 \leq -\frac{m-1}{2m} S + 2R(e_1, J e_1, e_1, J e_1)
+ \frac{m-1}{2m} (S - 2R(e_1, J e_1, e_1, J e_1))
= \frac{m+1}{m} R(e_1, J e_1, e_1, J e_1).$$

Since the orthonormal frame \{e_1, \ldots, e_m, J e_1, \ldots, J e_m\} is arbitrary, we conclude that $R$ has nonnegative holomorphic sectional curvature. Other statements can be proved similarly. \qed

Proof of Theorem 1.6} By Proposition 6.1, $M$ has positive orthogonal bisectional curvature. $M$ is biholomorphic to $\mathbb{CP}^m$ by [11] and [19] (or [45]). \qed

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7 Orthogonal Ricci curvature

We prove part (3) of Theorem 1.10 in this section. For convenience, we restate it in an equivalent way with the only difference being shifting the dimension from $m$ to $m + 1$.

**Proposition 7.1** Let $R$ be a Kähler algebraic curvature operator on a complex Euclidean vector space $(V, g, J)$ of complex dimension $(m + 1) \geq 2$. Let

$$\tilde{\beta}_m := \beta_{m+1} = \frac{m(m + 2)(3m + 5)}{2(m + 1)}.$$ 

If $R$ has $\tilde{\beta}_m$-nonnegative (respectively, $\tilde{\beta}_m$-positive, $\tilde{\beta}_m$-nonpositive, $\tilde{\beta}_m$-negative) curvature operator of the second kind, then $R$ has nonnegative (respectively, positive, nonpositive, negative) orthogonal Ricci curvature.

**Proof** The key idea is to use $\mathbb{CP}^m \times \mathbb{CP}^1$ as a model space. The eigenvalues and their associated eigenvectors of the curvature operator of the second kind on $\mathbb{CP}^m \times \mathbb{CP}^1$ are determined in [27].

We construct an orthonormal basis of $S^2_0(V)$. Let $\{e_0, \ldots, e_m, Je_0, \ldots, Je_m\}$ be an orthonormal basis of $V$. Let

$$V_0 = \text{span}\{e_0, Je_0\},$$
$$V_1 = \text{span}\{e_1, \ldots, e_m, Je_1, \ldots, Je_m\}.$$ 

Then $V = V_0 \oplus V_1$.

Notice that $h \in S^2_0(V_i)$ can be viewed as an element in $S^2_0(V)$ via

$$h(X, Y) = h(\pi_i(X), \pi_i(Y))$$
where $\pi_i : V \to V_i$ is the projection map for $i = 1, 2$. Thus, $S^2_0(V_i)$ can be identified with a subspace of $S^2_0(V)$ for $i = 1, 2$.

Let $\mathcal{E}^+ \cup \mathcal{E}^-$ be the orthonormal basis of $S^2_0(V_1)$ constructed in Lemma 4.1 and Lemma 4.1. An orthonormal basis of $S^2_0(V_0)$ is given by

$$\tau_1 = \frac{1}{2\sqrt{2}} (e_0 \otimes e_0 - Je_0 \otimes Je_0),$$
$$\tau_2 = \frac{1}{\sqrt{2}} (e_0 \otimes Je_0)$$

Next, we define the following traceless symmetric two-tensors on $V$:

$$h_i = \frac{1}{\sqrt{2}} e_0 \otimes e_i \quad \text{for } 1 \leq i \leq m,$$
$$h_{m+i} = \frac{1}{\sqrt{2}} e_0 \otimes Je_i \quad \text{for } 1 \leq i \leq m,$$
$$h_{2m+i} = \frac{1}{\sqrt{2}} Je_0 \otimes e_i \quad \text{for } 1 \leq i \leq m,$$
$$h_{3m+i} = \frac{1}{\sqrt{2}} Je_0 \otimes Je_i \quad \text{for } 1 \leq i \leq m.$$ 

We also define

$$\zeta = \frac{1}{\sqrt{8m(m + 1)}} \left( m(e_0 \otimes e_0 + Je_0 \otimes Je_0) - \sum_{i=1}^{m} (e_i \otimes e_i + Je_i \otimes Je_i) \right).$$
forms an orthonormal basis of $S^2_0(V)$. This corresponds to the orthogonal decomposition

$$S^2_0(V) = S^2_0(V_1) \oplus S^2_0(V_0) \oplus \text{span} \{ u \circ v : u \in V_0, v \in V_1 \} \oplus \mathbb{R}\xi.$$

The next step is to calculate the diagonal elements of the matrix representation of $\hat{R}$ with respect to this basis. By Lemma 3.3, we have

$$\hat{R}(\tau_1, \tau_1) = \hat{R}(\tau_2, \tau_2) = R(e_0, J e_0, e_0, J e_0). \quad (7.1)$$

Using Lemma 3.3 again, we obtain

$$\sum_{i=1}^{4m} \hat{R}(h_i, h_i) = \sum_{i=1}^{m} R(e_0, e_i, e_i, e_i) + \sum_{i=1}^{m} R(e_0, J e_i, e_0, J e_i)$$

$$+ \sum_{i=1}^{m} R(J e_0, e_i, J e_0, e_i) + \sum_{i=1}^{m} R(J e_0, J e_i, J e_0, J e_i)$$

$$= 2 \sum_{i=1}^{m} (R(e_0, e_i, e_0, e_i) + R(e_0, J e_i, e_0, J e_i))$$

$$= 2 \sum_{i=1}^{m} R(e_0, J e_0, e_i, J e_i)$$

$$= 2 \text{Ric}^\perp(e_0, e_0).$$

We calculate using (3.3) that

$$\hat{R}(\xi, \xi) = \frac{m^2}{8m(m+1)} \hat{R}(e_0 \circ e_0 + J e_0 \circ J e_0, e_0 \circ e_0 + J e_0 \circ J e_0)$$

$$- \frac{2m}{8m(m+1)} \sum_{i=1}^{m} \hat{R}(e_0 \circ e_0 + J e_0 \circ J e_0, e_i \circ e_i + J e_i \circ J e_i)$$

$$+ \frac{1}{8m(m+1)} \sum_{i,j=1}^{m} \hat{R}(e_i \circ e_j + J e_i \circ J e_j, e_j \circ e_j + J e_j \circ J e_j)$$

$$= - \frac{m}{m+1} R(e_0, J e_0, e_0, J e_0) + \frac{2}{m+1} \text{Ric}^\perp(e_0, e_0)$$

$$- \frac{1}{m(m+1)} \sum_{i,j=1}^{m} R(e_i, J e_i, e_j, J e_j).$$

Let $A$ be the collection of the values $\hat{R}(\tau_i, \tau_i)$ for $i = 1, 2$, $\hat{R}(\theta_i, \theta_i)$ for $1 \leq i \leq 2m$, $\hat{R}(\varphi^+_{ij}, \varphi^+_{ij})$ and $\hat{R}(\psi^+_{ij}, \psi^+_{ij})$ for $1 \leq i < j \leq m$. By Lemma 4.3 and (7.1), we know that $A$ contains two copies of $R(e_i, J e_i, e_j, J e_j)$ for each $0 \leq i < m$ and two copies of $2R(e_i, J e_i, e_j, J e_j)$ for each $1 \leq i < j \leq m$. Therefore, $\bar{a}$, the average of all values in $A$, is given by

$$\bar{a} = \frac{2}{m^2 + m + 2} \left( \sum_{i,j=1}^{m} R(e_i, J e_i, e_j, J e_j) + R(e_0, J e_0, e_0, J e_0) \right).$$
Since $R$ has $\hat{\beta}_m$-nonnegative curvature operator of the second kind with
\[
\hat{\beta}_m = (m^2 - 1) + 4m + 1 + \frac{m(m^2 + m + 2)}{2(m + 1)},
\]
we obtain that
\[
0 \leq \sum_{1 \leq i < j \leq m} \left( \hat{R}(\psi_{ij}^-, \psi_{ij}^-) + \hat{R}(\psi_{ij}^-, \psi_{ij}^-) \right) + \sum_{k=1}^{m-1} \hat{R} (\eta_k, \eta_k) + \sum_{i=1}^{4m} \hat{R}(h_i, h_i) + \hat{R}(\zeta, \zeta) + f \left( A, \frac{m(m^2 + m + 2)}{2(m + 1)} \right), \tag{7.2}
\]
where $f(A, x)$ is the function defined in Lemma 5.1. By Lemma 5.1, we have
\[
f \left( A, \frac{m(m^2 + m + 2)}{2(m + 1)} \right) \leq \frac{m(m^2 + m + 2)}{2(m + 1)} \tilde{a}
\]
\[
= \frac{m}{m + 1} \left( \sum_{i, j=1}^{m} R(e_i, Je_i, e_j, Je_j) - R(e_0, Je_0, e_0, Je_0) \right). \tag{7.3}
\]
By (4.6) and (2.4), we have
\[
\sum_{1 \leq i < j \leq m} \left( \hat{R}(\psi_{ij}^-, \psi_{ij}^-) + \hat{R}(\psi_{ij}^-, \psi_{ij}^-) \right) + \sum_{k=1}^{m-1} \hat{R} (\eta_k, \eta_k) + \sum_{i=1}^{4m} \hat{R}(h_i, h_i)
\]
\[
= - \frac{m - 1}{m} \sum_{i, j=1}^{m} R(e_i, Je_i, e_j, Je_j). \tag{7.4}
\]
Substituting (7.4), the expressions for $\sum_{i=1}^{4m} \hat{R}(h_i, h_i)$ and $\hat{R}(\zeta, \zeta)$, and (7.3) into (7.2) yields
\[
0 \leq - \frac{m - 1}{m} \sum_{i, j=1}^{m} R(e_i, Je_i, e_j, Je_j) + 2 \text{Ric}^\perp(e_0, e_0)
\]
\[
- \frac{m}{m + 1} R(e_0, Je_0, e_0, Je_0) + \frac{2}{m + 1} \text{Ric}^\perp(e_0, e_0)
\]
\[
- \frac{1}{m(m + 1)} \sum_{i, j=1}^{m} R(e_i, Je_i, e_j, Je_j)
\]
\[
+ \frac{m}{m + 1} \left( \sum_{i, j=1}^{m} R(e_i, Je_i, e_j, Je_j) + R(e_0, Je_0, e_0, Je_0) \right)
\]
\[
= \frac{2(m + 2)}{m + 1} \text{Ric}^\perp(e_0, e_0).
\]
Since the orthonormal frame $\{e_0, \ldots, e_m, Je_0, \ldots, Je_m\}$ is arbitrary, we conclude that $\text{Ric}^\perp \geq 0$. Other statements can be proved similarly. $\square$

**Proof of Theorem 1.8** By Proposition 6.1, $M$ has positive orthogonal Ricci curvature. If $M$ is in addition closed, one can use \cite[Theorem 2.2]{33} to conclude that $h^{p,0} = 0$ for all $1 \leq p \leq m$. In particular, $M$ is simply-connected and projective. $\square$
By the classification of closed three-dimensional Kähler manifolds with positive orthogonal Ricci curvature in [35], we get the following corollary of Theorem 1.8.

**Corollary 7.1** A closed three-dimensional Kähler manifold with \( 4/3 \)-positive curvature operator of the second kind is either biholomorphic to \( \mathbb{C}P^3 \) or biholomorphic to \( \mathbb{Q}^3 \), the smooth quadratic hypersurface in \( \mathbb{C}P^4 \).

### 8 Mixed curvature

Part (4) of Theorem 1.10 follows immediately from the following proposition.

**Proposition 8.1** Let \((V, g, J)\) be a complex Euclidean vector space with complex dimension \( (m + 1) \geq 2 \). Let \( R \) be a Kähler algebraic curvature operator on \( V \). Let \( \tilde{\gamma}_m = \gamma_{m+1} = \frac{3m^2 + 8m + 4}{2} \).

If \( R \) has \( \tilde{\gamma}_m \)-nonnegative (respectively, \( \tilde{\gamma}_m \)-positive, \( \tilde{\gamma}_m \)-nonpositive, \( \tilde{\gamma}_m \)-negative) curvature operator of the second kind, then the expression

\[
2 \text{Ric}(X, X) - R(X, JX, X, JX)/|X|^2
\]

is nonnegative (respectively, positive, nonpositive, negative) for any nonzero vector \( X \) in \( V \).

**Proof of Proposition 8.1** The proof is similar to that of Proposition 7.1, but with the different model space \( \mathbb{C}P^m \times \mathbb{C} \). The eigenvalues and their associated eigenvectors of the curvature operator of the second kind on \( \mathbb{C}P^m \times \mathbb{C} \) are determined in [27].

Let \( \{e_0, \ldots, e_m, Je_0, \ldots, Je_m\} \) be an orthonormal basis of \( V \). Then \( V = V_0 \oplus V_1 \), where \( V_0 = \text{span}[e_0, Je_0] \) and \( V_1 = \text{span}[e_1, \ldots, e_m, Je_1, \ldots, Je_m] \). Let

\[
\mathcal{E}^+ \cup \mathcal{E}^- \cup \{\tau_i\}_{i=1}^{2m} \cup \{h_i\}_{i=1}^{4m} \cup \{\xi\}
\]

be the orthonormal basis of \( S^2_0(V) \) constructed in the proof of Proposition 7.1.

Let \( A \) be the collection of the values \( \tilde{R}(\theta_i, \theta_j) \) for \( 1 \leq i \leq 2m, \tilde{R}(\varphi_{ij}^+, \varphi_{ij}^+) \) and \( \tilde{R}(\psi_{ij}^+, \psi_{ij}^+) \) for \( 1 \leq i < j \leq m \). By Lemma 4.3, we know that \( A \) contains two copies of \( \tilde{R}(e_i, Je_i, e_j, Je_j) \) for each \( 1 \leq i < j \leq m \).

Therefore, \( \tilde{a} \), the average of all values in \( A \), is given by

\[
\tilde{a} = \frac{2}{m(m + 1)} \left( \sum_{i,j=1}^{m} R(e_i, Je_i, e_j, Je_j) \right).
\]

Let \( f(A, x) \) be the function defined in Lemma 5.1. Then we have

\[
f \left( A, \frac{1}{2} m^2 \right) \leq \frac{1}{2} m^2 \tilde{a} = \frac{m}{m + 1} \sum_{i,j=1}^{m} R(e_i, Je_i, e_j, Je_j). \tag{8.1}
\]

Note that

\[
\tilde{\gamma}_m = (m^2 - 1) + 4m + 3 + \frac{1}{2} m^2.
\]
Since $R$ has \( \tilde{\gamma}_m \)-nonnegative curvature operator of the second kind, we obtain that

\[
0 \leq \sum_{1 \leq i < j \leq m} \left( \tilde{R}(\psi_{ij}, \psi_{ij}) + \tilde{R}(\psi_{ij}, \psi_{ij}) \right) + \sum_{k=1}^{m-1} \tilde{R}(\eta_k, \eta_k)
\]

\[
+ \sum_{i=1}^{4m} \tilde{R}(h_i, h_i) + \tilde{R}(\zeta, \zeta) + \tilde{R}(\tau_1, \tau_1) + \tilde{R}(\tau_2, \tau_2)
\]

\[
+ f \left( A, \frac{1}{2} m^2 \right).
\]

Substituting (7.4), (8.1), (7.1), and the expressions for \( \sum_{i=1}^{4m} \tilde{R}(h_i, h_i) \) and \( \tilde{R}(\zeta, \zeta) \) into the above inequality produces

\[
0 \leq -\frac{m-1}{m} \sum_{i,j=1}^{m} R(e_i, Je_i, e_j, Je_j) + 2 \text{Ric}^+(e_0, e_0)
\]

\[
- \frac{m}{m+1} R(e_0, Je_0, e_0, Je_0) + \frac{2}{m+1} \text{Ric}^+(e_0, e_0)
\]

\[
- \frac{1}{m(m+1)} \sum_{i,j=1}^{m} R(e_i, Je_i, e_j, Je_j) + 2R(e_0, Je_0, e_0, Je_0)
\]

\[
+ \frac{m}{m+1} \sum_{i,j=1}^{m} R(e_i, Je_i, e_j, Je_j)
\]

\[
= \frac{m+2}{m+1} (2 \text{Ric}(e_0, e_0) - R(e_0, Je_0, e_0, Je_0)).
\]

Finally, the arbitrariness of the orthonormal frame \( \{e_0, \ldots, e_m, Je_0, \ldots, Je_m\} \) allows us to conclude that

\[
2 \text{Ric}(X, X) - R(X, JX, X, JX)/|X|^2 \geq 0
\]

for any nonzero vector \( X \in V \). Other statements can be proved similarly.

\[\square\]

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