Generalized conformal derivations of Lie conformal algebras

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Abstract: Let $R$ be a Lie conformal algebra. The purpose of this paper is to investigate the conformal derivation algebra $CDer(R)$, the conformal quasiderivation algebra $QDer(R)$ and the generalized conformal derivation algebra $GDer(R)$. The generalized conformal derivation algebra is a natural generalization of the conformal derivation algebra. Obviously, we have the following tower $CDer(R) \subseteq QDer(R) \subseteq GDer(R) \subseteq gc(R)$, where $gc(R)$ is the general Lie conformal algebra. Furthermore, we mainly research the connection of these generalized conformal derivations. Finally, the conformal $(\alpha, \beta, \gamma)$-derivations of Lie conformal algebras are studied. Moreover, we obtain some connections between several specific generalized conformal derivations and the conformal $(\alpha, \beta, \gamma)$-derivations. In addition, all conformal $(\alpha, \beta, \gamma)$-derivations of finite simple Lie conformal algebras are characterized.

Key words: Lie conformal algebras, conformal derivations, generalized conformal derivations, conformal quasiderivations, conformal $(\alpha, \beta, \gamma)$-derivations

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1 Introduction

Throughout this paper, we denote by $\mathbb{C}$ the set of complex numbers and $\mathbb{C}^*$ the set of non-zero complex numbers. We also assume that $\lambda, \mu$ and $\nu$ are indeterminate variables. For convenience, $\mathcal{A}$ will denote the ring $\mathbb{C}[\partial]$ of polynomials in the indeterminate $\partial$. Moreover, if $V$ is a vector space, the space of polynomials of $\lambda$ with coefficients in $V$ is denoted by $V[\lambda]$.

It is well known that derivations and generalized derivations are very important subjects in the research of both Lie algebras and their generalizations. All kinds of generalized derivations were studied in [2, 10, 12, 13, 18]. In recent years, many authors have investigated generalized derivations of nonassociative algebras, Lie algebras, Lie superalgebras and Lie color algebras in [3, 7, 11, 16]. In [11], G. Leger and E. Lucks investigated the generalized derivations of Lie algebras and their subalgebras. Moreover, they obtained many nice properties of the generalized derivation algebras and their subalgebras. Later on, many authors got many useful results similar to Lie algebras, such as Lie superalgebras, Lie color algebras.

Lie conformal algebra as a natural generalization of Lie algebra, was initially introduced by V. G. Kac as a formal language describing the singular part of the operator product expansion in conformal field theory. It is very useful for studying infinite dimensional Lie algebras satisfying the local property. Since then, the structure theories and representation theories of some Lie conformal algebras have been extensively undertaken in [1, 4, 6, 8]. In the last few years, the conformal

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derivations of some concrete Lie conformal algebras have been studied by many authors in [5,15]. We believe that the generalized conformal derivations of Lie conformal algebras would help to promote the development of structure theories of Lie conformal algebras. This is our motivation to present this paper.

In the present work, we would like to study the generalized conformal derivations of Lie conformal algebras. We mainly discuss the conformal derivation algebra $CDer(R)$, the center conformal derivation algebra $ZDer(R)$, the conformal quasiderivation algebra $QDer(R)$ and the generalized conformal derivation algebra $GDer(R)$ of a Lie conformal algebra $R$. It may be useful for investigating the structures of Lie conformal algebras. This is just the beginning of the study on generalized conformal derivations of Lie conformal algebras. Furthermore, we get many nice properties for Lie conformal algebras.

In [14,17], $(\alpha, \beta, \gamma)$-derivations of Lie algebras and Lie superalgebras were studied. Moreover, they also obtained some properties of $(\alpha, \beta, \gamma)$-derivations of Lie (super)algebras. In particular, the spaces of $(\alpha, \beta, \gamma)$-derivations for simple Lie (super)algebras were presented. In this article, we will investigate the conformal $(\alpha, \beta, \gamma)$-derivations of Lie conformal algebras. And, the connection between generalized conformal derivations and the conformal $(\alpha, \beta, \gamma)$-derivations is studied. Moreover, conformal $(\alpha, \beta, \gamma)$-derivations of all finite simple Lie conformal algebras are characterized.

Let us briefly describe the structure of the article. In Section 2, we review the basic notations which are used in this paper. In Section 3, similar to the definitions of the generalized derivations of Lie algebras, some definitions of the generalized conformal derivations are presented first. Then, we study the connections of these generalized conformal derivations. Moreover, the generalized conformal derivations of Virasoro Lie conformal algebra are determined. In Section 4, we introduce the concept of conformal $(\alpha, \beta, \gamma)$-derivations and obtain some connections between the generalized conformal derivations and the conformal $(\alpha, \beta, \gamma)$-derivations. Finally, we classify the conformal $(\alpha, \beta, \gamma)$-derivations and characterize the conformal $(\alpha, \beta, \gamma)$-derivations of all finite simple Lie conformal algebras.

2 Preliminaries

In this section, for the reader’s convenience, we shall summarize some basic facts about Lie conformal algebras used in this paper, see [4,8,9].

**Definition 2.1.** A Lie conformal algebra $R$ is an $\mathcal{A}$-module endowed with a $\mathbb{C}$-bilinear map

$$R \otimes R \rightarrow R[\lambda], \quad a \otimes b \rightarrow [a_\lambda b],$$
satisfying the following three axioms \((a, b, c \in R)\):

- **Conformal sesquilinearity**: \(\partial a \lambda b = -\lambda[a_\lambda b], \quad [a_\lambda \partial b] = (\partial + \lambda)[a_\lambda b];\)

- **Skew symmetry**: \(a_\lambda b = -[b_\lambda - \partial a];\)

- **Jacobi identity**: \(a_\lambda [b_\lambda c] = [[a_\lambda b]_\lambda + c] + [b_\lambda [a_\lambda c]].\)

The notions of a homomorphism, ideal and subalgebra of a Lie conformal algebra are defined as usual. A Lie conformal algebra is called *finite* if it is finitely generated as an \(\mathcal{A}\)-module. The *rank* of a Lie conformal algebra \(R\) is its rank as an \(\mathcal{A}\)-module. A Lie conformal algebra \(R\) is *simple* if its only ideals are trivial and it is not abelian.

Suppose \(A\) and \(B\) are subspaces of a Lie conformal algebra \(R\). We define \([A, B]\) as the \(\mathbb{C}\)-linear span of all \(\lambda\)-coefficients in the products \([a_\lambda b]\), where \(a \in A, b \in B\). Moreover, if \(B\) is an \(\mathcal{A}\)-submodule of \(R\), by conformal sesquilinearity, \([A, B]\) is also an \(\mathcal{A}\)-submodule of \(R\).

There are two important examples of Lie conformal algebras.

**Example 2.2.** The Virasoro Lie conformal algebra \(\mathit{Vir}\) is the simplest nontrivial example of Lie conformal algebras. It is defined by

\[
\mathit{Vir} = \mathbb{C}[\partial] L, \quad [L_\lambda L] = (\partial + 2\lambda)L.
\]

**Example 2.3.** Let \(\mathfrak{g}\) be a Lie algebra. The current Lie conformal algebra associated to \(\mathfrak{g}\) is defined by:

\[
\text{Cur}\mathfrak{g} = \mathbb{C}[\partial] \otimes_{\mathbb{C}} \mathfrak{g}, \quad [a_\lambda b] = [a, b], \quad \text{for all } a, b \in \mathfrak{g}.
\]

It is shown in [4] that any finite simple Lie conformal algebra is either isomorphic to \(\mathit{Vir}\) or \(\text{Cur}\mathfrak{g}\) where \(\mathfrak{g}\) is a finite-dimensional simple Lie algebra.

**Definition 2.4.** Suppose \(R\) and \(S\) are two \(\mathcal{A}\)-modules. A *conformal linear map* from \(R\) to \(S\) is a \(\mathbb{C}\)-linear map \(\phi_\lambda : R \rightarrow S[\lambda]\) such that

\[
\phi_\lambda(\partial a) = (\partial + \lambda)\phi_\lambda(a), \quad \text{for all } a \in R.
\]

**Remark 2.5.** Obviously, by Definition 2.4 \(\phi_\lambda\) does not depend on the choice of the indeterminate variable \(\lambda\). Therefore, for convenience, in the sequel, we also write this conformal linear map as \(\phi_\mu\) for some other indeterminate variable \(\mu\).

Set \(\phi_\lambda = \sum_{i=0}^{n} \lambda^i\phi_i\) where \(\phi_i : R \rightarrow S\) is a \(\mathbb{C}\)-linear map for any \(i \in \{0, \ldots, n\}\). Denote \(\ker(\phi) = \bigcap_{i=0}^{n} \ker(\phi_i)\) and \(\text{Im}(\phi) = \bigcup_{i=0}^{n} \text{Im}(\phi_i)\). Obviously, \(a \in \ker(\phi)\) if and only if \(\phi_\lambda(a) = 0\) and \(b \in \text{Im}(\phi)\) if and only if it appears as a coefficient in a polynomial \(\phi_\lambda(a)\) for some \(a \in R\).

Denote by \(\text{Chom}(R, S)\) the space of conformal linear maps between \(\mathcal{A}\)-modules \(R\) and \(S\). For simplicity, we write \(\text{Cend}(R)\) for \(\text{Chom}(R, R)\). It can be made into an \(\mathcal{A}\)-module via \((\partial \phi)_\lambda(a) = -\lambda \phi_\lambda(a)\) for all \(a \in R\).
Definition 2.6. The $\lambda$-bracket on $\text{Cend}(R)$ given by

$$[\phi \lambda \varphi]_\mu a = \phi_\lambda (\phi_{\mu-\lambda} a) - \phi_{\mu-\lambda} (\phi_\lambda a), \text{ for any } a \in R,$$

defines a Lie conformal algebra structure on $\text{Cend}(R)$. This is called the general Lie conformal algebra on $R$ and we denote it by $gc(R)$.

Definition 2.7. Let $R$ be a Lie conformal algebra. A conformal linear map $D_\lambda : R \rightarrow R[\lambda]$ is called a conformal derivation if

$$D_\lambda ([a \mu b]) = [(D_\lambda a)_{\lambda+\mu} b] + [a_\mu (D_\lambda b)], \text{ for all } a, b \in R.$$

It is easy to see that for any $a \in R$, the map $(\text{ad}_a)_\lambda$, defined by $(\text{ad}_a)_\lambda b = [a_\lambda b]$ for any $b \in R$, is a conformal derivation of $R$. All conformal derivations of this kind are called inner conformal derivations. Denote by $CDer(R)$ and $CInn(R)$ the vector spaces of all conformal derivations and inner conformal derivations of $R$, respectively.

Obviously, $CInn(R)$ and $CDer(R)$ are conformal subalgebras of $gc(R)$ and we have the following tower $CInn(R) \subseteq CDer(R) \subseteq gc(R)$.

Definition 2.8. The center of Lie conformal algebra $R$ is

$$Z(R) = \{a \in R \mid [a_\lambda b] = 0, \text{ for all } b \in R\}.$$

3 Generalized conformal derivation algebras

Firstly, we will give some definitions and notations of the generalized conformal derivations of Lie conformal algebras.

An element $D_\lambda \in gc(R)$ is said to be a conformal quasiderivation of a Lie conformal algebra $R$ if there exists an element $D'_\lambda \in gc(R)$ such that

$$[(D_\lambda a)_{\lambda+\mu} b] + [a_\mu (D_\lambda b)] = D'_\lambda ([a_\mu b]), \text{ for all } a, b \in R. \quad (3.1)$$

Furthermore, if there exist two elements $D'_\lambda, D''_\lambda \in gc(R)$ such that

$$[(D_\lambda a)_{\lambda+\mu} b] + [a_\mu (D'_\lambda b)] = D''_\lambda ([a_\mu b]), \text{ for all } a, b \in R, \quad (3.2)$$

we call $D_\lambda$ a generalized conformal derivation.

Denote by $QDer(R)$ and $GDer(R)$ the sets of conformal quasiderivations and generalized conformal derivations of $R$. By setting $D'_\lambda = D_\lambda$ in $(3.2)$, we can get $QDer(R) \subseteq GDer(R)$.

An element $D_\lambda \in gc(R)$ is called a conformal centroid of $R$, if it satisfies the following equality

$$D_\lambda ([a_\mu b]) = [(D_\lambda a)_{\lambda+\mu} b] = [a_\mu (D_\lambda b)], \text{ for all } a, b \in R. \quad (3.3)$$
If $D_\lambda \in gc(R)$ satisfies the following condition:

$$[(D_\lambda a)_{\lambda + \mu} b] = [a_\mu (D_\lambda b)], \quad \text{for all } a, b \in R,$$

(3.4)

$D_\lambda$ is called a conformal quasicentroid of $R$. From now on, for convenience, we denote by $C(R)$ and $QC(R)$ the sets of conformal centroids and quasicentroids. It is easy to see that $C(R) \subseteq QC(R)$.

Denote by $ZDer(R)$ the set of elements $D_\lambda \in gc(R)$ satisfying

$$[(D_\lambda a)_{\mu} b] = D_\lambda ([a_\mu b]) = 0, \quad \text{for all } a, b \in R.$$  

(3.5)

It is easy to verify that $ZDer(R)$ is a Lie conformal ideal of $GDer(R)$ (see Proposition 3.4). We also have $D_\lambda a \in Z(R)[\lambda]$. If $Z(R) = 0$, then $ZDer(R) = 0$.

For the centerless Lie conformal algebra $R$, we have the following tower:

$$R \cong CI\!nn(R) \subseteq CDer(R) \subseteq QDer(R) \subseteq GDer(R) \subseteq gc(R).$$

(3.6)

Moreover, when $R$ is abelian, it is easy to see $ZDer(R) = C(R) = QC(R) = CDer(R) = QDer(R) = GDer(R) = gc(R)$.

It is known that $CDer(Vir) = CI\!nn(Vir)$ (see [4]). Then, we present a characterization of $C(Vir), QC(Vir), CDer(Vir), QDer(Vir)$ and $GDer(Vir)$.

**Proposition 3.1.** For the Virasoro Lie conformal algebra, we can get

1. $GDer(Vir) = QDer(Vir) = \{D_\lambda = a_0(\lambda) + a_1(\lambda)\partial \mid a_0(\lambda), a_1(\lambda) \in \mathbb{C}[\lambda]\}$.
2. $C(Vir) = QC(Vir) = 0$.

**Proof.** Let $D_\lambda \in GDer(Vir)$. Then, there exist two elements $D'_\lambda, D''_\lambda \in gc(Vir)$ such that

$$[(D_\lambda L)_{\lambda + \mu} L] + [L_\mu (D'_\lambda L)] = D''_\lambda ([L_\mu L]).$$

(3.7)

Assume that $D_\lambda (L) = \sum_{i=0}^{m} a_i(\lambda) \partial^i L$, $D'_\lambda (L) = \sum_{i=0}^{n} b_i(\lambda) \partial^i L$ and $D''_\lambda (L) = \sum_{i=0}^{s} c_i(\lambda) \partial^i L$, where $a_i(\lambda), b_i(\lambda), c_i(\lambda) \in \mathbb{C}[\lambda]$ and $a_m(\lambda) \neq 0$, $b_n(\lambda) \neq 0$, $c_s(\lambda) \neq 0$. Taking them into (3.7) and comparing the coefficients of $L$, we can get

$$(\partial + 2\lambda + 2\mu) \sum_{i=0}^{m} a_i(\lambda)(-\lambda - \mu)^i + (\partial + 2\mu) \sum_{i=0}^{n} b_i(\lambda)(\mu + \partial)^i = (\partial + \lambda + 2\mu) \sum_{i=0}^{s} c_i(\lambda) \partial^i.$$  

(3.8)

Obviously, by comparing the highest degrees of $\partial$ in both two sides of (3.8), $s$ must be equal to $n$.

If $n > 1$, letting $\mu = 0$ in (3.8) and comparing the coefficients of $\partial^{n+1}$ and $\partial^n$, we obtain

$$b_n(\lambda) = c_n(\lambda), \quad b_{n-1}(\lambda) = c_{n-1}(\lambda) + c_n(\lambda)\lambda.$$  

(3.9)

Then, according to (3.9) and by comparing the coefficients of $\partial^n$ in (3.8), we get $n\mu b_n(\lambda) = 0$. This shows that $b_n(\lambda) = 0$, a contradiction. Then, by comparing the coefficients of $\mu$, we
obtain \( m \leq 1 \). Therefore, \( D_{\lambda}(L) = (a_0(\lambda) + a_1(\lambda) \partial)L, D_{\lambda}'(L) = (b_0(\lambda) + b_1(\lambda) \partial)L \) and \( D_{\lambda}''(L) = (c_0(\lambda) + c_1(\lambda) \partial)L \). Plugging them into (3.8), we can get \( b_1(\lambda) = c_1(\lambda) = a_1(\lambda), b_0(\lambda) = a_0(\lambda) \) and \( c_0(\lambda) = 2(a_0(\lambda) - a_1(\lambda)). \)

Thus, we obtain

\[
D_{\lambda}(L) = D_{\lambda}'(L) = (a_0(\lambda) + a_1(\lambda) \partial)L, \quad D_{\lambda}''(L) = (2(a_0(\lambda) - a_1(\lambda)) + a_1(\lambda) \partial)L. \tag{3.10}
\]

Therefore, by the definitions of conformal quasiderivation and generalized conformal derivation, we get \( GDer(Vir) = QDer(Vir) = \{ D_{\lambda} = a_0(\lambda) + a_1(\lambda) \partial \mid a_0(\lambda), a_1(\lambda) \in \mathbb{C}[\lambda] \}. \)

Similarly, it is easy to see \( C(Vir) = QC(Vir) = 0. \)

**Remark 3.2.** By Proposition 3.1, we can see that \( CDer(Vir) \neq GDer(Vir) = QDer(Vir). \)

Next, we begin to study the connections of these generalized conformal derivations.

**Proposition 3.3.** \( GDer(R), QDer(R) \) and \( C(R) \) are Lie conformal subalgebras of \( gc(R) \).

**Proof.** Here, we only prove that for any \( D_{1,\lambda}, D_{2,\lambda} \in GDer(R), [D_{1,\lambda} D_{2,\lambda}]_{\mu} \in GDer(R)[\lambda]. \) The other two cases can be proved similarly.

According to the definition of \( gc(R) \), we get

\[
[D_{1,\lambda} D_{2,\lambda}]_{\mu} = D_{1,\lambda} D_{2,\mu-\lambda} - D_{2,\mu-\lambda} D_{1,\lambda}.
\]

Since \( D_{1,\lambda} \) and \( D_{2,\lambda} \in GDer(R) \), there exist four elements \( D_{1,\lambda}', D_{1,\lambda}'', D_{2,\lambda}', D_{2,\lambda}'' \in gc(R) \) such that

\[
[(D_{1,\lambda} a)_{\lambda + \mu} b] + [a_{\mu}(D_{1,\lambda} b)] = D_{1,\lambda}''([a_{\mu} b]),
\]

\[
[(D_{2,\lambda} a)_{\lambda + \mu} b] + [a_{\mu}(D_{2,\lambda} b)] = D_{2,\lambda}''([a_{\mu} b]), \quad \text{for all } a, b \in R.
\]

Then, we obtain

\[
[(D_{1,\lambda}(D_{2,\mu-\lambda} a))_{\mu+v} b] = D_{1,\lambda}''([(D_{2,\mu-\lambda} a)_{\nu+\mu-\lambda} b]) - [(D_{2,\mu-\lambda} a)_{\nu+\mu-\lambda} (D_{1,\lambda} b)]
\]

\[
= D_{1,\lambda}''(D_{2,\mu-\lambda}([a_v b]) - [a_v(D_{2,\mu-\lambda} b)])
\]

\[
- D_{2,\mu-\lambda}([a_v(D_{1,\lambda} b)]) + [a_v(D_{2,\mu-\lambda}(D_{1,\lambda} b))].
\]

Similarly, we have

\[
[(D_{2,\mu-\lambda}(D_{1,\lambda} a))_{\mu+v} b] = D_{2,\mu-\lambda}''(D_{1,\lambda}''([a_v b]) - [a_v (D_{1,\lambda} b)])
\]

\[
- D_{1,\lambda}''([a_v(D_{2,\mu-\lambda} b)]) + [a_v(D_{1,\lambda}(D_{2,\mu-\lambda} b))].
\]

Thus, it follows at once that

\[
[(D_{1,\lambda} D_{2,\mu} a)_{\mu+v} b] + [a_{\nu}([D_{1,\lambda} D_{2,\lambda}']_{\mu} b)] = [D_{1,\lambda} D_{2,\lambda}'' a]_{\mu} ([a_v b]).
\]

Therefore, \( [D_{1,\lambda} D_{2,\mu}] \in GDer(R)[\lambda], \) i.e. \( GDer(R) \) is a Lie conformal subalgebra of \( gc(R). \) \( \square \)
Proposition 3.4. $\mathcal{Z}Der(R)$ is a Lie conformal ideal of $\mathcal{G}Der(R)$.

Proof. Firstly, it is readily seen that $\mathcal{Z}Der(R) \subseteq \mathcal{G}Der(R)$.

Next, we only need to prove that $[D_{1\lambda} D_{2\mu}] \in \mathcal{Z}Der(R)[\lambda]$ for any $D_{1\lambda} \in \mathcal{Z}Der(R)$, $D_{2\lambda} \in \mathcal{G}Der(R)$. Since $D_{2\lambda} \in \mathcal{G}Der(R)$, there exist two elements $D'_{2\lambda}$, $D''_{2\lambda} \in gc(R)$ such that

\[
[(D_{2\lambda} a) \lambda + \mu b] + [a \mu (D'_{2\lambda} b)] = D''_{2\lambda} ([a b]), \text{ for all } a, b \in R.
\]

Then, we obtain

\[
[[D_{1\lambda} D_{2\mu}] a b] = [(D_{1\lambda} (D_{2\mu} a) \lambda b)] - [(D_{2\mu} a (D_{1\lambda} b)) \lambda b]
\]

\[= 0 - (D_{2\mu-\lambda} ([D_{1\lambda} a] \nu + \lambda b] - [(D_{1\lambda} a) \nu + \lambda \mu (D'_{2\mu-\lambda} b)])
\]

\[= 0 - (D_{2\mu-\lambda} (0) - 0)
\]

\[= 0.
\]

And, it is easy to see that $[D_{1\lambda} D_{2\mu}] ([a b]) = 0$ for any $a, b \in R$.

Hence, we have completed the proof. \qed

Remark 3.5. Let $R$ be a Lie conformal algebra which is free and of finite rank as an $\mathcal{A}$-module. It is known that $gc(R)$ is a simple Lie conformal algebra. Therefore, if $\mathcal{G}Der(R) = gc(R)$, by Proposition 3.4 in this case, $\mathcal{Z}Der(R)$ is equal to either 0 or $gc(R)$.

Proposition 3.6. If $Z(R) = 0$, then $C(R)$ is a commutative Lie conformal subalgebra of $\mathcal{G}Der(R)$.

Proof. Obviously, we have $C(R) \subseteq \mathcal{G}Der(R)$ by their definitions.

Let $D_{1\lambda}, D_{2\lambda} \in C(R)$. By some computations, we deduce that

\[
[(D_{1\lambda} (D_{2\mu} a)) \nu b] = D_{1\lambda} ([[(D_{2\mu} \lambda a) \nu b]]
\]

\[= D_{1\lambda} ([a \nu (D_{2\mu} b)])
\]

\[= [(D_{1\lambda} a) \nu + \lambda \mu (D_{2\mu} b)]
\]

\[= [(D_{2\mu} \lambda (D_{1\lambda} a)) \nu b].
\]

It follows that

\[
[([D_{1\lambda} D_{2\mu}] a b) \nu b] = 0.
\]

Due to the arbitrary of $b$ and $Z(R) = 0$, we obtain that

\[
[D_{1\lambda} D_{2\mu}] a = 0.
\]

This means $[D_{1\lambda} D_{2\mu}] = 0$.

Now we get the expected result. \qed
Lemma 3.7. The following properties hold:

(1) \( C(R) + CDer(R) \subseteq QDer(R) \).

(2) \( C(R) \subseteq QDer(R) \cap QC(R) \).

(3) \( GDer(R) = QDer(R) + QC(R) \).

Proof. (1) For any \( D_\lambda \in C(R) \), it follows that

\[
[(D_\lambda a)_{\lambda+\mu} b] + [a_\mu (D_\lambda b)] = 2D_\lambda ([a_\mu b]).
\]

Thus, we can choose \( D'_\lambda = 2D_\lambda \) such that \( D_\lambda \in QDer(R) \).

Due to the definition of \( CDer(R) \) and \( QDer(R) \), we have seen that \( CDer(R) \subseteq QDer(R) \). Therefore, (1) holds.

(2) It is obvious that \( C(R) \subseteq QC(R) \). Furthermore, according to (1), then (2) holds.

(3) Assume \( D_\lambda \in GDer(R) \). Then, there exist two elements \( D'_\lambda, D''_\lambda \in gc(R) \) such that

\[
[(D_\lambda a)_{\lambda+\mu} b] + [a_\mu (D'_\lambda b)] = D''_\lambda ([a_\mu b]), \quad \text{for all } a, b \in R. \tag{3.11}
\]

Set \( D_1 \lambda = \frac{D_\lambda + D'_\lambda}{2} \) and \( D_2 \lambda = \frac{D_\lambda - D'_\lambda}{2} \), we only need to show that \( D_1 \lambda \in QDer(R) \) and \( D_2 \lambda \in QC(R) \).

By skew symmetry and (3.11), for any \( a, b \in R \), we get

\[
[b_{-\lambda-\mu-\partial}(D_\lambda a)] + [(D'_\lambda b)_{-\mu-\partial}a] = D''_\lambda ([b_{-\mu-\partial}a]). \tag{3.12}
\]

By the conformal sesquilinearity and replacing \( -\lambda - \mu - \partial \) by \( \mu' \) in (3.12), we obtain

\[
[b_{\mu'}(D_\lambda a)] + [(D'_\lambda b)_{\lambda+\mu'}a] = D''_\lambda ([b_{\mu'}a]). \tag{3.13}
\]

Then, by changing the place of \( a, b \) and replacing \( \mu' \) by \( \mu \) in (3.13), we have

\[
[a_\mu (D_\lambda b)] + [(D'_\lambda a)_{\lambda+\mu}b] = D''_\lambda ([a_\mu b]). \tag{3.14}
\]

For one thing, by (3.11) and (3.14), we have

\[
[(D_1 \lambda a)_{\lambda+\mu} b] + [a_\mu (D_1 \lambda b)] = D''_\lambda ([a_\mu b]), \quad \text{for all } a, b \in R. \tag{3.15}
\]

Hence, \( D_1 \lambda \in QDer(R) \).

For another, by (3.11) and (3.14), we also obtain that for any \( a, b \in R \),

\[
[(D_2 \lambda a)_{\lambda+\mu} b] = [a_\mu (D_2 \lambda b)]. \tag{3.16}
\]

Therefore, \( D_2 \lambda \in QC(R) \). \( \square \)
Lemma 3.8. We have the following conclusions:

(1) \([CDer(R) \lambda C(R)] \mu \subseteq C(R)[\lambda]\).

(2) \([QDer(R) \lambda QC(R)] \mu \subseteq QC(R)[\lambda]\).

(3) \([QC(R) \lambda QC(R)] \mu \subseteq QDer[R]\).

Proof. (1) Let \(D_1 \lambda \in CDer(R), D_2 \lambda \in C(R)\). Then we get

\[
[D_1 \lambda (D_2 \mu - \lambda a)]_{\mu + v b} = D_1 \lambda ([D_2 \lambda - \lambda a]_{\mu + v - \lambda} (D_1 \lambda b)]
\]

\[
= D_1 \lambda (D_2 \mu - \lambda ([a_v b])) - D_2 \mu - \lambda ([a_v (D_1 \lambda b)]).
\]

Similarly, we can obtain

\[
[D_1 \lambda D_2 \mu a]_{\mu + v b} = [a_v ([D_1 \lambda D_2 \mu b]).
\]

For (2), using the similar method of proving (1), we can conclude that (2) holds.

For (3), suppose that \(D_1 \lambda, D_2 \lambda \in QC(R)\), we can obtain that

\[
[D_1 \lambda (D_2 \mu - \lambda a)]_{\mu + v b} = [a_v ([D_1 \lambda D_2 \mu b]).
\]

Hence, we get

\[
([D_1 \lambda D_2 \mu a]_{\mu + v b}] + [a_v ([D_1 \lambda D_2 \mu b])] = 0([a_v b]).
\]

This completes the proof. \(\square\)

Proposition 3.9. \(QC(R) + [QC(R), QC(R)]\) is a Lie conformal ideal of \(GDer(R)\).

Proof. In view of Lemma 3.7 (3) and Lemma 3.8 (2), (3), it suffices to prove the statement. \(\square\)
Proposition 3.10. If $Z(R) = 0$, then $QC(R)$ is a Lie conformal algebra if and only if $[QC(R) \lambda QC(R)]_\mu = 0$. In particular, if $Z(R) = 0$, then $C(R)$ centralizes $QC(R)$.

Proof. $\Leftarrow$ It is nothing to prove.

$\Rightarrow$ Assume that $D_1 \lambda, D_2 \lambda \in QC(R)$, then we have $[D_1 \lambda D_2]_\mu \in QC(R)[\lambda]$. Furthermore, we obtain that

$$[[D_1 \lambda D_2]_\mu a_v b] = [a_v ([D_1 \lambda D_2]_\mu b)].$$

According to Lemma 3.8 (3), we obtain

$$[[D_1 \lambda D_2]_\mu a_v b] + [a_v ([D_1 \lambda D_2]_\mu b)] = 0([a_v b]).$$

Finally, it shows that

$$[[D_1 \lambda D_2]_\mu a_v b] = 0.$$

According to $Z(R) = 0$, this forces $[D_1 \lambda D_2]_\mu = 0$.

This proves the proposition. $\square$

Proposition 3.11. If $D_\lambda \in C(R)$, then $ker(D)$ and $Im(D)$ are ideals of $R$.

Proof. Assume that $D_\lambda \in C(R), b \in ker(D)$. By the definition of $ker(D)$, we know $D_\lambda(b) = 0$. Then, for any $a \in R$, by the definition of $C(R)$, we have

$$D_\lambda([a_\mu b]) = [(D_\lambda a)_\lambda + \mu b] = [a_\mu (D_\lambda b)] = 0.$$

Therefore, $[a_\mu b] \in ker(D)[\mu]$.

Suppose that $c \in Im(D)$. Then, we assume that $c$ appears as a coefficient in $D_\lambda d$ for some $d \in R$. Therefore, we get

$$[(D_\lambda d)_\mu b] = D_\lambda([d_\mu - \lambda b]).$$

Therefore, $[c_\mu b] \in Im(D)[\mu]$.

Hence, this proves $ker(D)$ and $Im(D)$ are ideals of $R$. $\square$

Let $R$ be a centerless Lie conformal algebra. Assume that $R = S \oplus T$, where $S, T$ are ideals of $R$. If $D_\lambda \in GDer(R)$, it is easy to see that $D_\lambda(S) \subseteq S[\lambda], D_\lambda(T) \subseteq T[\lambda]$. Furthermore, we get the following results.

Proposition 3.12. (1) $GDer(R) = GDer(S) \oplus GDer(T)$.

(2) $QDer(R) = QDer(S) \oplus QDer(T)$.

(3) $C(R) = C(S) \oplus C(T)$.

(4) $QC(R) = QC(S) \oplus QC(T)$. 

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Similar to the investigation of quasiderivations of Lie algebras, we shall deduce that the conformal quasiderivations of some Lie conformal algebras can be embedded as conformal derivations in some larger Lie conformal algebras.

Let \( R \) be a Lie conformal algebra and \( t \) an indeterminate. We define \( \tilde{R} \), canonically associated to \( R \), as \( \tilde{R} = R[t \mathbb{C}[t] / (t^3)] = Rt + Rt^2 \). \( \tilde{R} \) can be endowed with a natural \( \mathcal{A} \)-module structure as follows: \( \partial (a \otimes t^i) = (\partial a) \otimes t^i \) for any \( a \in R \) and \( i = 1, 2 \). It is readily seen that \( \tilde{R} \) is a Lie conformal algebra with the \( \lambda \)-bracket:

\[
[a \otimes t^i, b \otimes t^j] = [a, b] \otimes t^{i+j},
\]

where \( a, b \in R, i, j \in \mathbb{Z} \). For simplicity, we write \( at \) and \( at^2 \) in place of \( a \otimes t \) and \( a \otimes t^2 \) respectively.

In the following of this section, we present an assumption: \( R \) has a decomposition of \( \mathcal{A} \)-modules, i.e.

\[
R = [R, R] \oplus U, \quad \text{where } U \text{ is an } \mathcal{A} \text{-submodule of } R. \quad (3.17)
\]

Then \( \tilde{R} = Rt + Rt^2 = Rt + [R, R]t^2 + Ut^2 \). Now we define a map \( f : QDer(R) \longrightarrow gc(\tilde{R}) \) satisfying \( f(D_\lambda)(at + br^2 + ut^2) = D_\lambda (a)t + D'_\lambda (b)t^2 \), where \( a \in R, b \in [R, R], u \in U \), and \( D'_\lambda \) is an element in \( gc(R) \) associated with \( D_\lambda \) in (3.1). Obviously, by the definition, \( f(D_\lambda) \) is a conformal linear map.

**Theorem 3.13.** Let \( R \) be the Lie conformal algebra satisfying (3.17). Then, we have the following conclusions:

1. \( f \) is injective.
2. \( f(D_\lambda) \) does not depend on the choice of \( D'_\lambda \).
3. \( f(QDer(R)) \subseteq CDer(\tilde{R}) \).

**Proof.** (1) If \( f(D_{1 \lambda}) = f(D_{2 \lambda}) \), then we get \( f(D_{1 \lambda})(at + br^2 + ut^2) = f(D_{2 \lambda})(at + br^2 + ut^2) \).

This shows that \( D_{1 \lambda} (a)t + D'_{1 \lambda} (b)t^2 = D_{2 \lambda} (a)t + D'_{2 \lambda} (b)t^2 \). Therefore, we conclude that \( D_{1 \lambda} (a) = D_{2 \lambda} (a) \) for any \( a \in R \). Hence \( D_{1 \lambda} = D_{2 \lambda} \), and \( f \) is injective.

(2) Assume that there exists \( D''_{\lambda} \in gc(R) \) such that \( f(D_\lambda)(at + br^2 + ut^2) = D_\lambda (a)t + D''_{\lambda} (b)t^2 \).

Furthermore, for any \( c, d \in R \), we have

\[
[(D_\lambda c)\lambda + \mu d] + [c \mu (D_\lambda d)] = D'_{\lambda}([c \mu d]), \quad (3.18)
\]

\[
[(D_\lambda c)\lambda + \mu d] + [c \mu (D_\lambda d)] = D''_{\lambda}([c \mu d]). \quad (3.19)
\]

It follows that \( D'_{\lambda}([c \mu d]) = D''_{\lambda}([c \mu d]) \) for any \( c, d \in R \). Thus, for any \( b \in [R, R] \), we get \( D'_{\lambda} (b) = D''_{\lambda} (b) \). Hence

\[
f(D_\lambda)(at + br^2 + ut^2) = D_\lambda (a)t + D'_{\lambda} (b)t^2 = D_\lambda (a)t + D''_{\lambda} (b)t^2,
\]

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which shows that \( f(D_\lambda) \) does not depend on the choice of \( D'_\lambda \).

(3) According to the definition of \( \tilde{R} \), we obtain \([at^i \mu bt^j] = 0 \) for \( i + j \geq 3 \). Therefore, we only need to prove the following equality \( f(D_\lambda)([at \mu bt]) = [(f(D_\lambda)at)_{\lambda+\mu} bt] + [at_\mu (f(D_\lambda)bt)] \).

By some computations, one has

\[
\begin{align*}
f(D_\lambda)([at \mu bt]) &= f(D_\lambda)([a \mu b]t^2) \\
&= D'_\lambda([a \mu b])t^2 \\
&= ([D_\lambda a)_{\lambda+\mu} b] + [a_\mu (D_\lambda b)]t^2 \\
&= [(f(D_\lambda)at)_{\lambda+\mu} bt] + [at_\mu (f(D_\lambda)bt)].
\end{align*}
\]

Finally, we have completed the proof. \( \square \)

**Remark 3.14.** Obviously, not all finite Lie conformal algebras satisfy the assumption. For example, \( R = C[\partial]L \oplus C[\partial]W \) is a Lie conformal algebra of rank 2 with the \( \lambda \)-bracket as follows:

\[
[L_\lambda L] = (\partial + 2\lambda)L, \quad [L_\lambda W] = \partial W, \quad [W_\lambda W] = 0. \tag{3.20}
\]

Then, \([R, R] = C[\partial]L \oplus C[\partial]\partial W\). Therefore, there does not exist any \( \mathfrak{g} \)-submodule \( U \) such that \( R = [R, R] \oplus U \).

Of course, there are also many Lie conformal algebras satisfying the assumption. All finite simple, semisimple Lie conformal algebras and \( \text{Cur}\mathfrak{g} \) where \( \mathfrak{g} \) is a finite-dimensional Lie algebra are all such Lie conformal algebras. In addition, there are some solvable Lie conformal algebras satisfying the assumption. For example, \( R = C[\partial]a \oplus C[\partial]b \) is a Lie conformal algebra of rank 2 with the \( \lambda \)-bracket as follows:

\[
[a_\lambda a] = 0, \quad [a_\lambda b] = d(\lambda)b, \quad [b_\lambda b] = 0, \tag{3.21}
\]

where \( d(\lambda) \in C[\lambda] \). Then, if \( d(\lambda) = 0 \), \([R, R] = 0\), i.e. \( U = R \). If \( d(\lambda) \neq 0 \), \([R, R] = C[\partial]b \). Then, \( U = C[\partial]a \).

**Theorem 3.15.** Let \( R \) be the Lie conformal algebra satisfying (3.17). If \( Z(R) = 0 \), and \( \tilde{R} \), \( f \) are defined as above, then \( \text{CDer}(\tilde{R}) = f(Q\text{Der}(R)) \oplus Z\text{Der}(\tilde{R}) \).

**Proof.** From \( Z(R) = 0 \), it follows at once that \( Z(\tilde{R}) = Rt^2 \). Suppose \( \tilde{D}_\lambda \in \text{CDer}(\tilde{R}) \), then we obtain \( \tilde{D}_\lambda(Z(\tilde{R})) \subseteq Z(\tilde{R})[\lambda] \). Thus, \( \tilde{D}_\lambda(Ut^2) \subseteq \tilde{D}_\lambda(Z(\tilde{R})) \subseteq Z(\tilde{R})[\lambda] = Rt^2[\lambda] \).

Now define a conformal linear map: \( g_\lambda : Rt + [R, R]t^2 + Ut^2 \rightarrow Rt^2[\lambda] \) by

\[
g_\lambda(x) = \begin{cases} 
\tilde{D}_\lambda(x) \cap \tilde{R}t^2[\lambda] & \text{if } x \in Rt, \\
\tilde{D}_\lambda(x) & \text{if } x \in Ut^2, \\
0 & \text{if } x \in [R, R]t^2. 
\end{cases} \tag{3.22}
\]
Obviously, we know that $g_{\lambda}([\tilde{R}, \tilde{R}]) \subseteq g_{\lambda}([R, R]t^2) = 0$. Furthermore, we have $[g_{\lambda}(\tilde{R})\mu\tilde{R}] \subseteq [Rt^2[\lambda] \mu (Rt + Rt^2)] = 0$. Hence, we get $g_{\lambda} \in ZDer(\tilde{R})$.

By the definition of $g_{\lambda}$, we have $(\bar{D}_\lambda - g_{\lambda})(Rt) = \bar{D}_\lambda(Rt) \cap Rt[\lambda] \subseteq Rt[\lambda]$. We also have $(\bar{D}_\lambda - g_{\lambda})(Rt^2) = 0$ and $(\bar{D}_\lambda - g_{\lambda})([R, R]t^2) = \bar{D}_\lambda([\tilde{R}, \tilde{R}] [\lambda] \subseteq [R, R][t^2][\lambda]$.

Therefore, there exist $D_{\lambda}, D'_{\lambda} \in gc(R)$ for all $a \in R, b \in [R, R]$ such that $(\bar{D}_\lambda - g_{\lambda})(at) = D_{\lambda}(a)t$, $(\bar{D}_\lambda - g_{\lambda})(bt^2) = D'_{\lambda}(b)t^2$.

Since $\bar{D}_\lambda - g_{\lambda} \in CDer(\tilde{R})$ and by the definition of $CDer(\tilde{R})$, we have $[(\bar{D}_\lambda - g_{\lambda})(a_1t) \mu a_2t] + [a_1t \mu (\bar{D}_\lambda - g_{\lambda})(a_2t)] = (\bar{D}_\lambda - g_{\lambda})[a_1t \mu a_2t]$. Thus, $[D_{\lambda}(a_1) \mu a_2] + [a_1 \mu D_{\lambda}(a_2)] = D'_{\lambda}([a_1 \mu a_2])$. Therefore, $D_{\lambda} \in QDer(R)$.

Furthermore, we deduce $\bar{D}_\lambda - g_{\lambda} = f(D_{\lambda}) \subseteq f(QDer(R))$, then $CDer(\tilde{R}) = f(QDer(R)) + ZDer(\tilde{R})$.

For any $\bar{D}_\lambda \in f(QDer(R)) \cap ZDer(\tilde{R})$, there exists an element $\bar{D}_\lambda \in QDer(R)$ such that $\bar{D}_\lambda = f(D_{\lambda})$. Then, $\bar{D}_\lambda(at + bt + ut^2) = f(D_{\lambda})(at + bt + ut^2) = D_{\lambda}(a)t + D'_{\lambda}(b)t^2$, for $a \in R, b \in [R, R]$. Since $\bar{D}_\lambda \in ZDer(\tilde{R})$, then $\bar{D}_\lambda(at + bt + ut^2) \in Z(\tilde{R})[\lambda] = Rt^2[\lambda]$. Therefore, $\bar{D}_\lambda(a) = 0$ for any $a \in R$. Finally, we conclude $\bar{D}_\lambda = 0$.

Thus, we have completed the proof. \qed

4 Conformal $(\alpha, \beta, \gamma)$-derivations

Throughout this section, we are concerned with the conformal $(\alpha, \beta, \gamma)$-derivations of Lie conformal algebras. The conformal $(\alpha, \beta, \gamma)$-derivations are natural generalizations of conformal derivations.

Let $R$ be an arbitrary Lie conformal algebra. We call a conformal linear map $D_{\lambda} \in CendR$ a conformal $(\alpha, \beta, \gamma)$-derivation of $R$ if there exist $\alpha, \beta, \gamma \in \mathbb{C}$ such that for any $a, b \in R$, the following relation is satisfied:

$$\alpha D_{\lambda}([a \mu b]) = ([\beta D_{\lambda} a]_{\lambda + \mu} b) + [a \mu (\gamma D_{\lambda} b)].$$ (4.1)

For any given $\alpha, \beta, \gamma \in \mathbb{C}$, we denote the set of all conformal $(\alpha, \beta, \gamma)$-derivations by $CDer_{(\alpha, \beta, \gamma)}R$, i.e.

$$CDer_{(\alpha, \beta, \gamma)}R = \{D_{\lambda} \in CendR \mid \alpha D_{\lambda}([a \mu b]) = ([\beta D_{\lambda} a]_{\lambda + \mu} b) + [a \mu (\gamma D_{\lambda} b)], \forall a, b \in R\}.$$ 

Obviously, we have the following conclusions.

**Proposition 4.1.** (1) $CDer_{(1,1,1)}R = CDer(R)$.

(2) $CDer_{(0,1,-1)}R = QC(R)$.

(3) $CDer_{(1,0,0)}R \cap CDer_{(0,1,0)}R = ZDer(R)$. 

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(4) $CDer(\alpha, \beta, \gamma)R = CDer(\triangle \alpha, \triangle \beta, \triangle \gamma)R$, for any $\triangle \in \mathbb{C}^*$.

Furthermore, we have the following proposition. The proposition will be used later on.

**Proposition 4.2.** $CDer(\alpha, \beta, \gamma)R = CDer(0, \beta - \gamma, -\beta)R \cap CDer(2\alpha, \beta + \gamma, \beta + \gamma)R$, where $\alpha, \beta, \gamma \in \mathbb{C}$.

**Proof.** For any $D_\lambda \in CDer(\alpha, \beta, \gamma)R$ and $a, b \in R$, we get the following equality:

$$\alpha D_\lambda ([a \mu b]) = [(\beta D_\lambda a)_{\lambda + \mu} b] + [a \mu (\gamma D_\lambda b)]. \quad (4.2)$$

With a similar discussion as that in the proof of Lemma 3.7 (3), we can also get

$$\alpha D_\lambda ([a \mu b]) = [a \mu (\beta D_\lambda b)] + [(\gamma D_\lambda a)_{\lambda + \mu} b]. \quad (4.3)$$

Then, by (4.2) and (4.3), we get

$$0 = (\beta - \gamma)\left([(D_\lambda a)_{\lambda + \mu} b] - [a \mu (D_\lambda b)]\right), \quad (4.4)$$

and

$$2\alpha D_\lambda ([a \mu b]) = (\beta + \gamma)\left([(D_\lambda a)_{\lambda + \mu} b] + [a \mu (D_\lambda b)]\right). \quad (4.5)$$

Therefore, $CDer(\alpha, \beta, \gamma)R \subseteq CDer(0, \beta - \gamma, -\beta)R \cap CDer(2\alpha, \beta + \gamma, \beta + \gamma)R$. Similarly, from (4.4) and (4.5), we can also conclude that (4.2) holds.

Now, this proposition is proved. \qed

In the following, we would like to find the connection of these three parameters. Furthermore, we formulate that the three original parameters are depended on only one form from the following theorem.

Using the similar method in [14], we have the following theorem.

**Theorem 4.3.** For any $\alpha, \beta, \gamma \in \mathbb{C}$, there exists $\delta \in \mathbb{C}$ such that the subspace $CDer(\alpha, \beta, \gamma)R \subseteq CendR$ has only the following four cases: (1) $CDer(\delta, 0, 0)R$, (2) $CDer(\delta, 1, -1)R$, (3) $CDer(\delta, 1, 0)R$, (4) $CDer(\delta, 1, 1)R$.

From above discussions, we get the main result of this section.

**Theorem 4.4.** For any $\alpha, \beta, \gamma \in \mathbb{C}$, then $CDer(\alpha, \beta, \gamma)R$ is equal to one of the following subspaces of $CendR$:

(i) $CDer(0, 0, 0)R = Cend R$.
(ii) $CDer(1, 0, 0)R = \{D_\lambda \in Cend R \mid D_\lambda([R, R]) = 0\}$.
(iii) $CDer(0, 1, -1)R = QC(R)$.
(iv) $CDer(\delta, 1, -1)R = CDer(0, 1, -1)R \cap CDer(1, 0, 0)R$.
(v) $CDer(\delta, 1, 1)R$, $\delta \in \mathbb{C}$.  

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(vi) $\text{CDer}_{(\delta,1,0)}R = \text{CDer}_{(0, 1, -1)}R \cap \text{CDer}_{(2\delta, 1, 1)}R$.

Finally, we give a characterization of all conformal $(\alpha, \beta, \gamma)$-derivations of finite simple Lie conformal algebras.

**Theorem 4.5.** (1) For the Virasoro Lie conformal algebra and any $\alpha, \beta, \gamma \in \mathbb{C}$, then $\text{CDer}_{(\alpha, \beta, \gamma)}\text{Vir}$ is equal to one of the following subspaces of $\text{Cend Vir}$:

(i) $\text{CDer}_{(0, 0, 0)}\text{Vir} = \text{Cend Vir}$,

(ii) $\text{CDer}_{(1, 0, 0)}\text{Vir} = \text{CDer}_{(0,1, -1)}\text{Vir} = \text{CDer}_{(\delta, 1, -1)}\text{Vir} = \text{CDer}_{(\delta, 1, 0)}\text{Vir} = 0$,

(iii) $\text{CDer}_{(\delta, 1, 1)}\text{Vir} = 0$, if $\delta \neq 1$ and $\delta \neq 2$; $\text{CDer}_{(1,1,1)}\text{Vir} = \text{ClInn(Vir)}$ and $\text{CDer}_{(2,1,1)}\text{Vir} = \{D_{\lambda} \mid D_{\lambda}(L) = a(\lambda)L, \text{where } a(\lambda) \in \mathbb{C}[\lambda]\}$.

(2) For the current Lie conformal algebra $A = \text{Curg}$ when $g$ is a finite-dimensional simple Lie algebra and any $\alpha, \beta, \gamma \in \mathbb{C}$, then $\text{CDer}_{(\alpha, \beta, \gamma)}A$ is equal to one of the following subspaces of $\text{Cend A}$:

(i) $\text{CDer}_{(0, 0, 0)}A = \text{Cend A}$,

(ii) $\text{CDer}_{(1, 0, 0)}A = \text{CDer}_{(\delta, 1, -1)}A = 0$ if $\delta \neq 0$,

(iii) $\text{CDer}_{(0,1, -1)}A = \{D_{\lambda} \mid D_{\lambda}(a) = f(\lambda)a, \text{where } a \in g \text{ and } f(\lambda) \in \mathbb{C}[\lambda]\}$.

(iv) $\text{CDer}_{(\delta, 1, 1)}A = 0$ if $\delta \neq 1$ and $\delta \neq 2$; $\text{CDer}_{(1,1,1)}A = \{D_{\lambda} \mid D_{\lambda}(a) = P(\partial)(\partial + \lambda)a + d_{\lambda}(a), \text{where } a \in g, P(\partial) \in \mathbb{C}[\partial] \text{ and } d_{\lambda} \in \text{ClInn(A)},\}$ and $\text{CDer}_{(2,1,1)}A = \{D_{\lambda} \mid D_{\lambda}(a) = f(\lambda)a, \text{where } a \in g, f(\lambda) \in \mathbb{C}[\lambda]\}$.

(v) $\text{CDer}_{(\delta, 1, 0)}A = 0$ if $\delta \neq 1$; $\text{CDer}_{(1,1,0)}A = \{D_{\lambda} \mid D_{\lambda}(a) = f(\lambda)a, \text{where } a \in g, f(\lambda) \in \mathbb{C}[\lambda]\}$.

**Proof.** (1) It can be directly obtained from the proof of Proposition 3.1.

(2) Let $D_{\lambda} \in \text{CDer}_{(\alpha, \beta, \gamma)}A$. Since $\text{Curg}$ is a free $\mathfrak{g}$-module of finite rank, we can set $D_{\lambda}(a) = \sum_{i=0}^{n} \partial^{i}d_{\lambda}^{i}(a)$ for every $a \in g$, where $d_{\lambda}^{i}$ are $\mathbb{C}$-linear maps from $g$ to $g[\lambda]$. According to (4.1), we can get

$$\alpha \sum_{i=0}^{n} \partial^{i}d_{\lambda}^{i}([a, b]) = \beta \sum_{i=0}^{n} (-\lambda - \mu)^{i}[d_{\lambda}^{i}a, b] + \gamma \sum_{i=0}^{n} (\partial + \mu)^{i}[a, d_{\lambda}^{i}(b)]. \tag{4.6}$$

By Theorem 4.4 and due to the simplicity of $A$, it is easy to see that (i) and (ii) hold.

Next, let us consider the case when $\alpha = 0$, $\beta = 1$, $\gamma = -1$. By (4.6), we get $n = 0$. Therefore, for any $D_{\lambda} \in \text{CDer}_{(0,1,-1)}A$, we set $D_{\lambda}(a) = d_{\lambda}(a)$ for any $a \in g$, where $d_{\lambda}$ is a $\mathbb{C}$-linear map from $g$ to $g[\lambda]$. Thus, by (4.6), we get

$$[d_{\lambda}a, b] = [a, d_{\lambda}b]. \tag{4.7}$$

Set $d_{\lambda}(a) = \sum_{i=0}^{m} \lambda^{i}d_{i}(a)$ for every $a \in g$, where $d_{i}$ are $\mathbb{C}$-linear maps from $g$ to $g$. Taking it into (4.7), we obtain

$$[d_{i}(a), b] = [a, d_{i}(b)], \text{ for all } i \in \{0, \cdots, m\}. \tag{4.8}$$
It is known that \( d_i = \alpha_i \text{Id}_g \), where \( \alpha_i \in \mathbb{C} \) (see Theorem 13 in [17]). Therefore, \( D_\lambda(a) = f(\lambda)a \) for any \( f(\lambda) \in \mathbb{C}[\lambda] \). Thus, (iii) holds.

Then, we consider the case when \( \alpha = \delta, \beta = \gamma = 1 \). Switching the place of \( a \) and \( b \) in (4.6), we get

\[
-\delta \sum_{i=0}^{n} \partial^i d^i_\lambda ([a,b]) = \sum_{i=0}^{n} (-\lambda - \mu)^i [d^i_\lambda b, a] + \sum_{i=0}^{n} (\partial + \mu)^i [b, d^i_\lambda(a)].
\]  

(4.9)

Setting \( \mu = 0 \) in (4.6) and (4.9) and adding them up, we obtain

\[
\sum_{i=0}^{n} \partial^i ([a, d^i_\lambda (b)] + [b, d^i_\lambda (a)]) = \sum_{i=0}^{n} (-\lambda)^i ([a, d^i_\lambda (b)] + [b, d^i_\lambda (a)]).
\]  

(4.10)

Therefore, by (4.10), we get \([a, d^i_\lambda (b)] + [b, d^i_\lambda (a)] = 0 \) for any \( i > 1 \) and \( a, b \in g \). By Lemma 6.3 in [4], \( d^i_\lambda (a) = f_i(\lambda)a \), where \( f_i(\lambda) \in \mathbb{C}[\lambda] \) for \( i > 1 \). Thus, \( D_\lambda \) is of the following form:

\[
D_\lambda(a) = d^0_\lambda(a) + P(\lambda, \partial)\partial a,
\]  

(4.11)

where \( P(\lambda, \partial) \in \mathbb{C}[\lambda, \partial] \). Setting \( P(\lambda, \partial) = \sum_{i=0}^{m} p_i(\lambda) \partial^i \) and by (4.6), we have

\[
\delta(d^0_\lambda(a, b) + \sum_{i=0}^{m} p_i(\lambda) \partial^{i+1}([a,b]) = [d^0_\lambda(a, b) + \sum_{i=0}^{m} (-\lambda - \mu)^{i+1} p_i(\lambda)[a,b] + [a, d^0_\lambda(b)] + \sum_{i=0}^{m} (\partial + \mu)^{i+1} p_i(\lambda)[a,b].
\]  

(4.12)

Obviously, if \( \delta = 0 \), by (4.12), \( D_\lambda = 0 \). In the case of \( \delta \neq 0 \), if \( m > 1 \), comparing the coefficients of \( \partial^m \), we get \( (m+1)\mu p_m(\lambda) + (1-\delta) p_{m-1}(\lambda) = 0 \) which gives \( p_m(\lambda) = 0 \). Therefore, we can set \( D_\lambda(a) = d^0_\lambda(a) + p(\lambda)\partial a \). Taking it into (4.12), we get

\[
\delta(d^0_\lambda(a,b) + \delta p(\lambda)\partial(a,b) = [d^0_\lambda(a,b)] + [a, d^0_\lambda(b)] - \lambda p(\lambda)[a,b] + \partial p(\lambda)[a,b].
\]  

(4.13)

When \( \delta = 1 \), \( CDer_{(1,1,1)}A = CDer(A) = \{ D_\lambda \mid D_\lambda(a) = P(\partial)(\partial + \lambda)a + d_\lambda(a), \text{ where } a \in g, P(\partial) \in \mathbb{C}[\partial] \text{ and } d_\lambda \in CImm(A) \} \) (see Lemma 6.2 in [4]). When \( \delta \neq 0 \) and \( \delta \neq 1 \), by (4.13), we have \( p(\lambda) = 0 \). Thus, in this case, \( D_\lambda(a) = d^0_\lambda(a) \) and \( d^0_\lambda(a) \) satisfies

\[
\delta(d^0_\lambda(a,b)) = [d^0_\lambda(a,b)] + [a, d^0_\lambda(b)].
\]  

(4.14)

It is known from Theorem 13 in [17] that any \((\delta, 1, 1)\)-derivation of the Lie algebra \( g \) is zero when \( \delta \neq 1 \) and \( \delta \neq 2 \), and any \((2, 1, 1)\)-derivation of \( g \) is equal to \( \alpha \text{Id}_g \) for some complex number \( \alpha \). Therefore, by (4.14), we conclude that when \( \delta \neq 1 \) and \( \delta \neq 2 \), \( D_\lambda(a) = 0 \) and when \( \delta = 2 \), \( D_\lambda(a) = f(\lambda)a \) for some \( f(\lambda) \in \mathbb{C}[\lambda] \). Thus, (iv) holds.

Finally, according to Theorem 4.4, \( CDer_{(\delta,1,0)}A = CDer_{(0,1,-1)}A \cap CDer_{(2\delta,1,1)}A \). Then, (v) can be easily obtained from (iii) and (iv).
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