The Multi-Spectral Reordering (MSR) technique for the narrow band modeling of the radiative properties of nonuniform gaseous paths: the correlated/uncorrelated approximations

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Abstract. A general path averaging strategy is proposed to calculate the equivalent parameters to be used with the narrow band modeling of the radiative properties of gases in nonuniform media. Approximate formulas are in particular derived for the cases of correlated and uncorrelated gas spectra. These approximations are applied to the Narrow Band MSR-SLMB (Spectral-Line Moment-Based model together with the Multi-Spectral Reordering technique) model and assessed by comparison with LBL data. Both approximations are accurate with an absolute error less than 0.05 for transmission functions. This demonstrates the improvement that MSR modeling provides when compared to usual narrow band models, for which larger errors are observed.

1. Introduction
Radiative heat transfer in molecular gases is significant in many situations encountered in combustion or planetary atmosphere studies. Due to the excessive computational costs of the most reliable approximate model available, the so-called Line-By-Line (LBL) method, more efficient modeling techniques have to be used to get radiative properties estimates. But choosing the right approach among the existing ones can be a delicate problem especially when the gaseous medium is strongly nonuniform. Indeed, many gas radiative properties models frequently used in moderately nonuniform media, and more particularly those based on the assumption of scaled or correlated spectra, may be inappropriate for this task and their application can lead to incorrect outputs.

One of the trickiest problems consists in calculating IR emission spectra of flames after their transmission through cold gaseous atmospheres containing the same molecular species (mainly CO₂ and H₂O) as those produced by the flames. Such a situation is frequently encountered as it is representative of a large number of remote sensing problems, ranging from the tracking of missiles or aircrafts for military applications, to the sensing of exhaust gases from industrial plants or volcano for environmental purposes. It is also one of the most difficult modeling situations. Indeed, due to the so-called spectral correlation effects that may or may not exist between the radiative emission by the hot gases and its absorption by the atmosphere, the use of a scaled/correlated-\(k\) assumption alone may produce very large errors, both over narrow and wide bands. For this reason, several sophisticated techniques were proposed during the past decades to handle highly nonuniform situations. Among

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them we can cite the fictitious gas approach [1], the multi-group method [2] by Modest and co-workers or mapping techniques such as those described, for example, in West et al [3]. Those methods were introduced in the 90s-mid 2000s to improve radiative models for nonuniform calculations, a goal that they succeeded to achieve at the cost of higher computational requirements than with the standard models. More recently [4], we have proposed a new and efficient modeling, called the multi-spectrum reordering (MSR) technique, of radiative heat transfer in nonuniform media. Fundamentally, it shares many similarities with the mapping approaches, to the extent that both methods are based on the multiple reordering of gas spectra in distinct thermophysical conditions. One main difference is that the MSR technique is used together with the Spectral-Line Moment-Based (SLMB) modeling, which enables to get very accurate results without requiring many successive reorderings as with the usual mapping schemes. In fact, “standard” mapping techniques involve the reordering of spectra over each uniform sub-paths encountered along nonuniform paths until nearly constant absorption coefficients, viz. first order moments, are obtained over each of the MSR bands and for each uniform sub-path. This process can be computationally consuming for large nonuniformities. With the Narrow Band MSR-SLMB model, on the other hand, the use of higher order k-moments lowers the number of successive reorderings but it requires, as a set-off, path-averaging strategies. Two original ones are introduced in the present paper.

In particular, the aim of the present work is to investigate the impact of using a correlation or uncorrelation assumption of gas spectra in various thermophysical states when they are applied over multi-spectrally reordered spectral intervals. On one hand, the uncorrelation assumption is known (see Ref. [5] for example) to provide inaccurate results when applied over narrow and wide bands. In this case, correlated models, viz. the narrow band correlated-k as well as the usual full spectrum models such as ADF, SLW or FSK, were demonstrated to give better results. Nevertheless, the uncorrelation assumption has never been studied together with the NB MSR-SLMB modeling. This is also one purpose of the present work. On the other hand, the correlation assumption is more usual for this kind of calculations. An approach based on k-moments was proposed in Ref. [6].

In this paper, we derive the general formula for the calculation of the path-averaged k-moments in nonuniform gaseous media. Starting from this formula, two simpler ones, obtained under the correlated and uncorrelated assumptions, are given. The uncorrelation assumption provides a nonlinear formula: it is to our knowledge the first one of this kind proposed for radiative transfer applications. Then, the MSR modeling to which we apply the previous equations is described. The building of Narrow Band MSR-SLMB model databases for CO$_2$ and H$_2$O diluted in N$_2$ is also briefly discussed. Finally, results for several test cases are given to assess the nonuniform models against LBL reference calculations.

2. Nonuniform narrow band k-moment modeling

2.1. Preliminary comments about the correlation/uncorrelation assumptions

We consider a narrow spectral interval $\Delta \nu$ over which: A1/ the Planck function can be assumed constant, A2/ in a thermophysical condition chosen arbitrarily as a reference, the absorption coefficient is nearly constant; a nonuniform optical path and two locations along this path, $s'$ and $s''$, with distinct spectral absorption coefficients $\kappa_v(s') \neq \kappa_v(s'')$. For the sake of simplicity, the spectral absorption coefficient $\kappa_v[\phi(s')] = \kappa_v[x(s') P(s') T(s')]$ will be written $\kappa_v(s')$. Similar simplifications will be used for all local quantities along the path.

Following Ref. [7] (Eq. (19.133) p. 625), gas spectra are correlated (resp. scaled) if there exists a monotonously increasing (resp. linear) function $h$ such that $\kappa_v(s')$ can be written as $h[s', s'', \kappa_v(s')]$. In this case, the Pearson product-moment correlation coefficient:
\[
\Gamma(s',s^*) = \frac{1}{\sqrt{\sigma^2(s')\sigma^2(s^*)}} \left[ \frac{1}{\Delta \nu} \int_{s'}^{s^*} k(s') \kappa(s^*) \, d\nu - \bar{k}(s') \bar{k}(s^*) \right] \approx 1
\]  
(1)

where \(\bar{k}(s')\) and \(\sigma^2(s')\) are the mean and variance of the \(k\)-distribution at abscissa \(s'\). The dependency of \(\Gamma(s',s)\) on the spectral interval \(\Delta \nu\) is not explicitly written here to abridge the notations. The strict equality in Eq. (1) is achieved when gas spectra are linearly correlated, which is almost the case if assumption A2 is verified.

Contrariwise, if gas spectra are uncorrelated, \(\Gamma(s',s^*) = 0\).

In general, gas spectra are neither fully correlated nor fully uncorrelated over any spectral interval containing several lines. This is illustrated (very schematically, for pedagogical purposes) on Figure 1: high resolution spectra are shown for pure CO\(_2\) at three temperatures, \(500\) K (bottom), \(1,500\) K (center) and \(2,500\) K (top). From this figure, it is clear that over this narrow band, gas spectra cannot be rigorously assumed correlated or uncorrelated. Nevertheless, over smaller spectral intervals, these assumptions can be well thought-out as valid. Indeed, the correlation assumption seems acceptable over spectral intervals \(Z1\), which correspond to wavenumbers that are located near the center of spectral lines observed at each selected temperature. Contrariwise, uncorrelation seems more accurate over intervals \(Z2\), since some lines that appear in the two high temperature spectra (the so-called “hot lines”) are missing in the spectrum of the coldest one. The rigorous processing of this problem by the \(k\)-moment approach is presented in the next two sections.

![Figure 1](image_url)  
**Figure 1.** Illustration of the correlation and uncorrelation assumptions. Absorption coefficients are in \(\text{cm}^{-1}.\text{atm}^{-1}\).

2.2. The moment problem in nonuniform media
For an homogeneous isothermal gaseous column of length \( l \) between two abscissa 0 and \( s \) (see Fig.2), the transmission function averaged over \( \Delta \nu \) is given by [5,7]:

\[
\bar{\tau}^{\Delta \nu} (0,s) = \frac{1}{\Delta \nu} \int_{\Delta \nu} \exp \left[ - \kappa_0 (\phi) l \right] d\nu = \int_{0}^{\infty} f(k) \exp \left( - k P l \right) dk
\]

(2)

in which \( f(k) \) is the \( k \)-distribution, defined as:

\[
f(k) dk = \int_{\nu \in \Delta \nu / \kappa_0 (\phi) \in [k, k+dk]} \frac{d\nu}{\Delta \nu}
\]

(3)

\( f(k) \) represents the fraction of wavenumbers inside \( \Delta \nu \) such that the absorption coefficient of the gas in the thermophysical state \( \phi \) is between \( k \) and \( k+dk \). It can be obtained as the inverse Laplace transform of the transmission function but can also be reconstructed from the knowledge of the \( k \)-moments using the so-called \( k \)-moment method introduced in Ref. [9]. Some theoretical links between the inverse Laplace transform and the moment problem was studied for instance in Ref. [10].

In theory, applying the moment approach to recover the actual \( k \)-distribution requires to solve an infinite set of moment equations of the following form:

\[
\forall n \in \mathbb{Z}, \int_{0}^{\infty} f(k) k^n \ dk = \frac{1}{\Delta \nu} \int_{\Delta \nu} \left[ \kappa_0 (\phi) \right]^n d\nu = \mu_n (\phi)
\]

(4)

In practice, considering \( n = 1, 2 \) values is enough to provide accurate estimates of the transmission function (see for example Ref. [9]) in uniform media. \( \mu_n \) corresponds to the band averaged absorption coefficient \( \bar{\kappa} \).

For nonuniform paths, Eq. (2) becomes:

\[
\bar{\tau}^{\Delta \nu} (0,s) = \frac{1}{\Delta \nu} \int_{\Delta \nu} \exp \left[ - \int_{0}^{s} x(s') P(s') \kappa_0 (s') \ ds' \right] d\nu = \frac{1}{\Delta \nu} \int_{\Delta \nu} \exp \left[ - \kappa_{0,e} (0,s) u_0 (0,s) \right] d\nu
\]

(5)

where we have introduced the two path-averaged quantities (noted with the index “\( e \)”):

\[
\begin{align*}
    u_0 (0,s) &= \int_{0}^{s} x(s') P(s') \ ds' \\
    \kappa_{0,e} (0,s) u_0 (0,s) &= \int_{0}^{s} x(s') P(s') \kappa_0 (s') \ ds'
\end{align*}
\]

(6)

It is possible to write Eq. (5) in a \( k \)-distribution form. In this case, the distribution is associated with the path-averaged absorption coefficient \( \kappa_{0,e} (0,s) \):

\[
\bar{\tau}^{\Delta \nu} (0,s) = \int_{0}^{\infty} f_e (k) \exp \left[ - k \ u_0 (0,s) \right] dk
\]

(7)

where as a generalization of Eq. (3):

\[
f_e (k) dk = \int_{\nu \in \Delta \nu / \kappa_{0,e} (0,s) \in [k, k+dk]} \frac{d\nu}{\Delta \nu}
\]

(8)

Associated moment equations are:
\[ \int_{0}^{\infty} f_{s}(k) k^{n} \, dk = \frac{1}{\Delta V} \int_{\Delta V} \left[ \kappa_{s,c}(0,s) \right]^{n} \, dv = \mu_{s,c}(0,s) \quad (9) \]

Applying Eqs. (9) to infer \( f_{s} \) requires to define strategies to estimate the path-averaged \( k \)-moments, which is the aim of the next section, for \( n \) limited to the 1st and 2nd orders.

### 2.3. Nonuniform path-averaging strategies for \( k \)-moments

The processing of the first-order moment equation is obvious and provides:

\[ \overline{k}_{s,c}(0,s) u_{s,c}(0,s) = \int_{0}^{s} x(s') P(s') \overline{k}(s') \, ds' \]

This equation, together with the definition of \( u_{s,c}(0,s) \) given by Eq. (6), is common to all nonuniform narrow band models, including the usual Curtis-Godson and Lindquist-Simmons ones [5].

The case of the second-order moment equation is a little bit more complicated. Indeed, Eq. (9) provides:

\[ \mu_{s,c}(0,s) [u_{s,c}(0,s)]^{2} = \frac{1}{\Delta V} \int_{\Delta V} \int_{0}^{s} x(s') P(s') \kappa_{s,c}(s') \, ds' \int_{0}^{s} x(s'') P(s'') \kappa_{s,c}(s'') \, ds'' \, dv \]

If we separate the second path-integral into two components, \( T(s') = \{ s'' \in [0,s] / \forall v \in \Delta V, \kappa_{s,c}(s') = \kappa_{s,c}(s'') \} \) and \( U(s') = \{ s'' \in [0,s] / \exists v \in \Delta V, \kappa_{s,c}(s') \neq \kappa_{s,c}(s'') \} \), with \( T(s') \cup U(s') = [0,s] \), we obtain:

\[ \mu_{s,c}(0,s) [u_{s,c}(0,s)]^{2} = \int_{s'-0}^{s'} \int_{v \in \Delta V} x(s') P(s') \kappa_{s,c}(s') \kappa_{s,c}(s') \, ds' \, ds'' \]

(12)

Let’s introduce the variance of the \( k \)-distribution \( f_{s,c} \) defined as:

\[ \sigma_{s,c}^{2}(0,s) = \frac{1}{\Delta V} \int_{\Delta V} \left[ \kappa_{s,c}(0,s) - \overline{k}_{s,c}(0,s) \right]^{2} \, dv = \mu_{s,c}(0,s) - \left[ \overline{k}_{s,c}(0,s) \right]^{2} \]

(13)

Then, together with Eq. (1), Eq. (12) can be written as:

\[ \sigma_{s,c}^{2}(0,s) [u_{s,c}(0,s)]^{2} = \int_{s'-0}^{s'} \int_{v \in \Delta V} x(s') P(s') x(s'') P(s'') \Gamma(s',s'' = s') \sqrt{\sigma_{s,c}^{2}(s') \sigma_{s,c}^{2}(s'' = s')} \, ds' \, ds'' \]

\[ + \int_{s'-0}^{s'} \int_{v \in \Delta V} x(s') P(s') x(s'') P(s'') \Gamma(s',s'' \neq s') \sqrt{\sigma_{s,c}^{2}(s') \sigma_{s,c}^{2}(s'') \sigma_{s,c}^{2}(s'')} \, ds' \, ds'' \]

(14)

It can be put in the following compact form:

\[ \sigma_{s,c}^{2}(0,s) = \sigma_{s,c}^{2}(0,s) + \Omega(0,s) \left[ \sigma_{s,c}^{2}(0,s) - \sigma_{s,c}^{2}(0,s) \right] \]

(15)

in which we have introduced the path-averaged correlation coefficient \( \Omega(0,s) \) defined as:
\[ \Omega(0,s) = \frac{\omega(0,s,\Gamma)}{\omega(0,s,1)} \, , \, \omega(0,s,X) = \int_{s' = 0}^{s' = \Delta(s')} x(s') P(s') X(s',s'') \sqrt{\sigma^2(s') \sigma^2(s'')} ds'ds'' \]

where \( X \) is a function of abscissa \( s' \) and \( s'' \). Quantities \( \sigma_{e,C}^2(0,s) \) and \( \sigma_{e,U}^2(0,s) \) are defined hereafter.

If over the nonuniform path \([0,s]\), gas spectra are assumed uncorrelated (index “\( U \)”) then if \( s' \in U(s') \), \( \Gamma(s',s') = 0 \). From Eq. (16), \( \Omega(0,s) = 0 \) and:

\[ \sigma_{e,U}^2(0,s) = \sigma_{e,U}^2(0,s) = \left[ u_e(0,s) \right]^2 \int_{s' = 0}^{s' = \Delta(s')} x(s') P(s') \sqrt{\sigma^2/s' \sigma^2/s''} ds' ds'' \]

Contrariwise, if over [0,s], gas spectra are assumed correlated (index “\( C \)”) then for any \( s' \in U(s') \), \( \Gamma(s',s') = 1 \) and \( \Omega(0,s) = 1 \) that provides:

\[ \sigma_{e,C}^2(0,s) = \left[ u_e(0,s) \right]^2 \int_{s' = 0}^{s' = \Delta(s')} x(s') P(s') \sqrt{\sigma^2/s' \sigma^2/s''} ds' ds'' \]

Eq. (18) is exactly the same as the one obtained in one of our previous works, using similar assumptions [6]. Eqs. (17,18) can be rewritten in a discrete form (see Figure 2 for details) by noticing that condition \( \kappa_e(s_j) = \kappa_e(s_j') \) is equivalent to consider that \( T(s_j) = T(s_j') \) and \( \gamma_L(s_j) = \gamma_L(s_j') \) for Lorentz lines, assumed to have the same Lorentz half width at half maximum:

\[
\begin{aligned}
&\text{correlation: } u_e(0,s) \sqrt{\sigma_{e,C}^2(0,s)} = \sum_{i=0}^{N} x(s_i) P(s_i) \sqrt{\sigma^2(s_i'')} \delta s_i \\
&\text{uncorrelation: } \left[ u_e(0,s) \right]^2 \sigma_{e,U}^2(0,s) = \sum_{i=0}^{N} x(s_i) P(s_i) \sigma^2(s_i) \Pi(s_i') \delta s_i \]
&\text{with: } \Pi(s_i') = \sum_{i: \text{such that } \tau(s_j') = \tau(s_i'}, \\
&\text{and } \gamma_L(s_j') = \gamma_L(s_i') \}
\end{aligned}
\]

The correlation assumption provides a linear path-averaging equation in terms of parameters \( \delta s_i, i = 1,...,N \), like usual nonuniform approximations such as the Curtis-Godson one. The uncorrelation formulation yields a nonlinear relation.

Eq. (15) cannot be used in many cases because the path-averaged correlation coefficient defined by Eq. (16) is usually not known, but is required in a general frame. Thus, the simplest case corresponding to uncorrelated (\( \Omega(0,s) = 0 \)) or correlated (\( \Omega(0,s) = 1 \)) spectra barely represent the actual spectral correlations. Nevertheless, they are simple enough to be compared and assessed against LBL reference calculations for radiative transfer applications. This will be detailed after some elements about the multi-spectral modeling are described in section 3.
3. Application to the narrow band multi-spectral reordering technique

Path-averaging equations given in Section 2 can be used over any kind of spectral interval, as long as it is narrow enough to assume that the Planck function is constant over it. In this section, $\Delta \nu$ is defined using the multi-spectral reordering technique for the SLMB modeling of the radiative properties of the gas, whose principles can be found in Ref. [4].

The MSR technique can be formulated as follows. Let’s consider two (arbitrary) reference thermophysical conditions and define, for a set of real positive parameters $k_1$, $k_2$, the following two-dimensional distribution:

$$f(k_1, k_2)dk_1dk_2 = \int_{\Delta \nu(k_1, k_2)} \frac{d\nu}{\Delta \nu} = \frac{\Delta \nu(k_1, k_2)}{\Delta \nu}$$

(20)

where:

$$\Delta \nu(k_1, k_2) = \{\nu \in \Delta \nu / \kappa (\phi^{(\nu)}) \in [k_1, k_1 + dk_1] \text{ and } \kappa (\phi^{(\nu)}) \in [k_2, k_2 + dk_2]\}$$

(21)

Eq. (20) extends Eq. (2) to more than one “reference”. It should be noticed that usual narrow band $k$-distribution models do not explicitly make use of “reference” conditions. Indeed, these models assume that all spectra are correlated, so that any thermophysical state is implicitly a reference.

Using this two-dimensional $k$-distribution, we can rewrite the transmission function in Eq. (6) as:

$$\mathcal{P}^{\Delta \nu}(0, s) = \int_{k_1=0}^{k_1=0} \int_{k_2=0}^{k_2=0} f(k_1, k_2)\left\{\frac{1}{\Delta \nu(k_1, k_2)}\int_{\Delta \nu(k_1, k_2)} \exp[-\kappa_{\nu}(0, s)u_{\nu}(0, s)]d\nu\right\} dk_1dk_2$$

(22)

The term in brackets can be estimated by using the $k$-moment method which provides, assuming an inverse Gaussian $k$-distribution over $\Delta \nu(k_1, k_2)$ [11]:

$$\frac{1}{\Delta \nu(k_1, k_2)}\int_{\Delta \nu(k_1, k_2)} \exp[-\kappa_{\nu}(0, s)u_{\nu}(0, s)]d\nu \approx \exp\left\{-\frac{k_{\nu}(0, s)^2}{\sigma_{\nu}^2(0, s)}\left[\sqrt{1 + 2\frac{u_{\nu}(0, s)}{k_{\nu}(0, s)}} - 1\right]\right\}$$

(23)

$k_{\nu}(0, s)$ and $\sigma_{\nu}^2(0, s)$ are quantities spectrally averaged over $\Delta \nu(k_1, k_2)$. They can be obtained from the application of Eqs. (9,19), using the mean and variance of the absorption coefficient on each uniform sub-path along the nonuniform one.
NB MSR-SLMB databases were built to easily compute the mean and variance of the absorption coefficient using parameters (see Ref. [12] for details) estimated from LBL spectra for 25 temperatures between 300 K and 2,700 K by 100 K steps for CO$_2$ and H$_2$O. For H$_2$O, a second discretization as a function of H$_2$O molar fraction is made. Details about these calculations can be found in Ref. [4]. Their application for any thermophysical condition consists in using a simple linear interpolation formula [12]. The multi-spectral reordering strategy proposed in Ref. [4], with $T^*_{ref} = 1,500$ K as reference temperatures, was chosen. $k$-discretizations use 8 intervals of $k$ values which means that with the bi-spectral reordering, up to 64 spectral intervals (over each narrow band) can result from the reordering. Nevertheless, most of them are empty and thus are not involved in the calculations. A pure gas at atmospheric pressure was selected as the reference composition for CO$_2$ and 10 % H$_2$O in nitrogen was chosen for water vapor. Finally, the narrow spectral bands proposed in Ref. [13] were used, which corresponds to 96 bands for CO$_2$ and 367 for H$_2$O. This discretization covers the $[150 \text{ cm}^{-1} ; 9,300 \text{ cm}^{-1}]$ spectral range. NB transmission functions for mixtures of CO$_2$-H$_2$O-N$_2$ were calculated as the product of those of the single gases.

Using the NB MSR-SLMB model does not require excessive computational storage: CO$_2$ database occupies 4 Mo and H$_2$O 87 Mo. The calculation of around 80 path-integrals per narrow band is required. It is thus more demanding than the usual narrow band model (the Curtis-Godson approximation involves 3 path integrals only) but remains around 30 times less expensive than LBL calculations (2,500 path integrals per narrow band at $10^{-2} \text{ cm}^{-1}$ resolution).

4. Results

In order to assess the MSR modeling and the various nonuniform approximations described in this work, we chose two test cases representative of engineering applications. The NB MSR-SLMB model with the correlated and uncorrelated approximations were used for the calculations, as well as the usual NBKM/SNB model with the Curtis-Godson approximation [5,12] and the NB MSR-SLMB together with the following formula (see Ref. [8]):

$$u_e(0, s) \frac{\sigma_{\text{e,2OKM}}(0, s)}{k_e(0, s)} = \sum_{i=0}^{N} x(s_i) P(s_i) \frac{\sigma^2(s_i)}{k(s_i)} \delta s_i$$

(24)

The model based on Eq. (24) will be referred as 2OKM from now on. The 1st order MSR-SLMB model, which assumes a Dirac $k$-distribution function instead of the Inverse Gaussian one was also considered for the comparisons. In the case of the NBKM/SNB model, nonuniformities are processed by applying the Curtis-Godson approximation:

$$\frac{u_e(0, s) \left[ k_e(0, s) \right]^3}{\sigma_{\text{e,CG}}^2(0, s)} = \sum_{i=0}^{N} x(s_i) P(s_i) \left[ \frac{k(s_i)}{\sigma^2(s_i)} \right]^3 \delta s_i$$

(25)

As a first test we considered the temperature and molar fraction profiles corresponding to an oxy-fuel combustion scenario described in Ref. [14]. This profile was chosen because it corresponds to a high molar fraction of CO$_2$ (higher than 0.6) for which the optically thin approximations cannot be used a priori. It should be noticed that the influence of the variance is more sensitive at the optically thick than at the thin limit which justifies this choice. Results are depicted on Figure 3. In this case, all models provide similar errors that do not exceed 0.05 for the MSR models but can reach 0.27 with the usual NBKM/SNB approach. The correlated and 2OKM models are in this case the most accurate.

The second test case is taken from Ref. [15] (see Fig. 10 in this reference for a description of the specie and temperature profiles). It is representative of a plume signature situation, for which radiation emitted by the hot gases is measured after its transmission through a long path of cold air. As CO$_2$ and
H$_2$O are both in the exhaust gases and the air, but at different temperatures and concentrations, spectral correlation effects are very important in this kind of situation. Results are given in Figure 4. In this case, it is clear that the MSR modeling performs better than the standard NBKM/SNB model. The uncorrelated model provides here the best accuracy.

From these two test cases, it is difficult to conclude which model, correlated or uncorrelated, is the best one in a general frame. In fact, results indicate that some information about the correlation between spectra along the path, represented by parameter $\Omega(0,s)$, is likely to partially correct errors due to these assumptions. From a practical point of view, calculating $\Omega(0,s)$ from LBL data is possible, but would require to build specific databases, including the parameters to estimate $\Gamma(s',s)$. Such an approach would increase the computation costs drastically as it would also require the computation of both the correlated and uncorrelated solutions involved in Eq. (15). Nevertheless, for radiative sensing and spectrum analysis purposes, such a full modeling might be useful.

![Figure 3](image-url)  
**Figure 3.** Narrow band transmission functions for the nonuniform oxy-fuel temperature and specie concentration profiles (from Ref. [14]).
Figure 4. Narrow band intensities for an IR signature test case, taken from Ref. [15].

4. Conclusion
In this paper, a general nonuniform formula for the path-averaged narrow band model parameters has been proposed, as well as two simpler ones based on the correlation/uncorrelation assumptions. They have been applied to a NB MSR-SLMB model and assessed against LBL reference data for CO$_2$-N$_2$ or H$_2$O-N$_2$ mixtures. Several comparisons with a standard NBKM/SNB model and the Curtis-Godson approximation have also been discussed. From this study, it follows that all MSR models based on second order $k$-moments provide a similar accuracy, better than those resulting from standard narrow band models, at the cost of extra but reasonable calculation costs. Correlated/uncorrelated approaches combined with the MSR technique are acceptable for radiative transfer applications, whereas the full model might be required for fine radiative analysis. This statement will be refined in future works.

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