Spatio-temporal Heterogeneity and Hyperuniformity in 2D Conserved Lattice Gas

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Hyperuniformity is a description of hidden correlations in point distributions revealed by an anomalous scaling in fluctuations of local density at various coarse-graining length scales. We demonstrate the existence of additional correlations hidden in the higher moments of the distribution of coarse-grained density, in the absorbing phase of a lattice gas model where fluctuations are anomalously suppressed up to a hyperuniform length scale that diverges at the critical point of a non-equilibrium phase transition. The density distribution at the hyperuniform length scale increasingly deviates from a Gaussian on approach to the critical point, revealing growing spatial heterogeneity that can be attributed to an excess in under-density sub-regions which we show are also the first regions that enter the absorbing phase during the preceding dynamics. Our results suggest that hidden order beyond hyperuniformity may generically be present in complex disordered systems.

The behaviour of long-wavelength density fluctuations and its anomalous suppression - hyperuniformity [1, 2], has been a subject of recent interest in the study of disordered systems for it provides an avenue to probe long-range order in problems that do not possess translational or bond-orientational symmetry [3–8, 13]. At the same time, hyperuniformity is also emergent in a diverse variety of naturally occurring or model systems, that range from granular or colloidal materials [4, 5, 9–12] to soft biological tissues [14–16]. This has lead to speculations on its universality [3, 6] and the need for greater understanding of its causal role in the organization and structure of complex systems.

For a given configuration of points \( \vec{r}_i \) in \( d \)-dimensional space with global number density \( \rho \), the local density \( \rho_R \equiv \frac{1}{V} \sum_{\vec{r}_i \in \Omega} \delta(\vec{r}_i - \vec{r}_i) \) defined over a subspace region \( \Omega \) of some finite length scale \( R \) is a coarse-grained variable characterized by a discrete probability distribution \( P(\rho_R) \). In the scenario where \( \vec{r}_i \) is generated randomly by an underlying Poisson process, \( \rho_R \) of disconnected regions in real space are uncorrelated such that \( P(\rho_R) \) is constrained by the central limit theorem (CLT) and its variance scales as \( \sigma^2(\rho) \sim R^{-d} \) in the limit of \( R \to \infty \). Hyperuniformity is the characterization of density fluctuations \( \sigma^2(\rho) \sim R^{-d} \) that are anomalously suppressed \( (a > d) \) even in the thermodynamic limit due to the presence of peculiar correlations in physical density fields. For systems that are not (ideally) hyperuniform, \( \sigma^2(\rho) \) is instead suppressed up to a finite length scale \( \xi_{R^\sigma} \) [5, 8, 15], which we refer hereafter as either the hyperuniform or fluctuation suppressed length scales.

Hyperuniformity analyses, therefore, focus on characterizing pairwise correlation through \( \sigma^2(R) \) and its Fourier equivalent - the structure factor \( S(k) \) [1, 2], with little regard for the \( R \) dependence of \( P(\rho_R) \). However, this perspective of the density field as viewed only from \( \sigma^2(\rho) \) may not capture essential features of \( P(\rho_R) \), missing out on additional pairwise or higher-ordered correlations found only in the higher moments of the distribution. These correlations are especially relevant at intermediate length scales where crucial information of the phase behaviour on the approach to a critical point may often be present [6, 10, 17, 18].

In this Letter, we show that additional non-trivial correlations are indeed hidden in \( P(\rho_R) \) for the 2D Conserved Lattice Gas (CLG) model [19–21], which is also known to exhibit a hyperuniformity crossover at the critical point of an active-absorbing state phase transition [6]. These correlations manifest as non-monotonic convergence of the higher moments to their respective Gaussian values [22]. In particular, the kurtosis has a maximum at a length proportionate to \( \xi_{R^\sigma} \) and the magnitude of this maximum diverges on approaching the critical point. The coarse-grained density field, therefore, becomes increasingly heterogeneous on approaching the critical point, signalling the existence of additional correlations over what is revealed by the suppression of density fluctuations.

We further demonstrate that the physical origin of these additional correlations is in the dynamical process leading up to the absorbing state, by showing that the local density \( \rho_R \) of a region of linear size \( R \) correlates with the time the region needs to reach the absorbing state, for relevant \( R \) values. The existence of additional correlations that are not captured by fluctuations in density fields uncovers the presence of a new form of hidden order beyond hyperuniformity that may prove to be generically present in complex disordered systems, such as with dynamical heterogeneities [24] that are generically present in systems exhibiting glassy dynamics. The behaviour of the higher moments at differing coarse-grain length scales may also serve as a probe towards understanding and characterizing non-equilibrium phase transitions in classical and active systems.

In the CLG model, particles are placed initially at random on a \( L \times L \) square lattice with mean density \( \rho \equiv N/L^2 \). A particle is deemed active if one or more of its immediate neighbouring sites is occupied and active particles move in each time step, randomly to an adjacent unoccupied site such that the global density \( \rho \) of the system is conserved throughout its dynamics [19–21]. Active particles are dynamically updated in parallel [6] where they simultaneously evolve at a given time step in favour of sequential dynamics in our simulations. Numerical results presented in the entirety of this work are derived from 50 unique realizations for each given \( \rho \) with systems of size \( L = 1024 \).
Below a critical density $\rho_c \sim 0.2391$ [23], the system evolves towards an absorbing state where there are no active particles at long time scales. Conversely, a finite fraction of particles remains active as part of a non-equilibrium steady state for $\rho > \rho_c$. This active-absorbing state transition belongs to the universality class of directed percolation [20, 21] and has recently been found to exhibit a hyperuniformity crossover. In the absorbing phase, $\sigma^2(R)$ is suppressed up to a length scale $\xi_{\sigma^2}$ that grows with $\rho$ and diverges at $\rho_c$, such that the system is believed to be hyperuniform at $\rho_c$ [6].

More specifically, at fluctuation suppressed length scales ($R < \xi_{\sigma^2}$) in the absorbing phase, $\sigma^2(R) \sim R^{-\lambda}$ where $\lambda = 2.45$ is found to be universal across the broader class of random organization models [6]. This universal behavior of $\sigma^2(R)$ in the vicinity of $\rho_c$ characterized by hyperuniformity can be seen in the departure from plateau regions at small $R$ for $\sigma^2 R^4$ in Fig.1(a). Here we note that the hyperuniformity exponent ($\gamma_{\sigma^2} \sim 0.67$) found to best scale collapse $\sigma^2(R)$ as seen in Fig.1(b) is comparable to what is reported in [6], and the universal exponent $\lambda$ is similarly recovered [25]. Corresponding information on the pair correlation and structure factor can be found in Fig.S6[29]. We also utilize an alternative method to numerically extract this exponent from its imprint on the higher moments of $P(\rho_R)$ which we discuss later.

Next, we investigate the $R$ dependence of the skewness $\gamma \equiv \langle((\rho_R - \mu)/\sigma)^3\rangle$ and kurtosis $\kappa \equiv \langle((\rho_R - \mu)/\sigma)^4\rangle$ in the absorbing phase. Fig.1(c,e) reveals that these higher moments are non-monotonic functions of $R$, a non-trivial behaviour that does not occur for randomly seeded configurations, represented by the black dashed lines. In particular, $\gamma$ and $\kappa$ have Gaussian-like values both at short length scales and, as expected from CLT, at large ones. This behaviour is also readily seen in $P(\rho_R)$ as shown in Fig.2, where Gaussian resemblance is present not only at longer length scales but also at much shorter length scales.

To obtain precise numerical estimates for the various length scales characterizing the skewness and the kurtosis, we perform polynomial fits (see Fig.S2 and Fig.S3 [29]) to the profiles of $\gamma$ and $\kappa$ found in Fig.1(b,c). The distances at which these polynomials equal zero identify the length scales $\xi_{\gamma}$ and $\xi_{\kappa}$ at which $\gamma$ and $\kappa$ first deviate from the Gaussian values. Indeed, we observe in Fig.1(d,e) that scaling $R$ by these lengths scale collapses $\gamma$ and $\kappa$ for different $\Delta \rho$ up to their deviations from the Gaussian behaviour. Consistent with what is seen in Fig.1(a,c,e) & Fig.2, these length scales grow with $\rho$ on its approach to $\rho_c$, as we illustrate in Fig. 3(a,c). Power-law fits
of these length scales suggest that they diverge at the critical point with exponents $v_\gamma = -0.59 \pm 0.04$ and $v_\kappa = -0.26 \pm 0.03$. Conversely, the lengths $\xi_\gamma^\star$ and $\xi_\kappa^\star$ at which $\gamma$ and $\kappa$ acquire their extreme values scale with the hyperuniformity exponent, as in Fig. 1(a,c). We further observe in Fig. 1(b,d) that the extreme value $\gamma^\star$ of the skewness grows but appears to saturate at a finite value on approaching the transition, while the maximum $\kappa^\star$ of the kurtosis diverges at the critical point.

This presents collectively, several alternative numerical approaches that can be used to estimate the fluctuation suppressed length scales and hyperuniformity exponent from $\gamma$ and $\kappa$ of $P(\rho R)$. If this imprint of the fluctuations on the higher moments are sufficiently prevalent or universal in fluctuation suppressed systems, this approach may outperform $\sigma_2^\gamma(R)$ based methods in precision, since it is not sensitive to the behavior and properties of the functional form that mediates the continuous transition in scaling behavior of $\sigma_2^\gamma(R)$ near the hyperuniformity crossover.

We remark that these critical exponents $v_\gamma$ and $v_\kappa$ controlling the length-scales of departure from the Gaussian behaviour are significantly different from, and smaller than the hyperuniformity exponent $v_{\gamma^\star}$ which controls the length above which Gaussian resemblance slowly recovers. Hence, if the power-law divergences of $\xi_\gamma^\star$, $\xi_\kappa$ and of $\kappa^\star$ hold in $\Delta \rho \to 0$ limit, then non-Gaussian behaviour becomes increasingly more relevant on approaching the critical point. This increasing deviation from the Gaussian behavior is indeed apparent in the distributions of Fig. 2 when considered at $R$ values close to $\xi_\kappa^\star$, e.g. $R \approx 25$ for $\rho = 0.235$, and $R \approx 150$ for $\rho = 0.2385$.

These observations establish an intriguing analogy between the behaviour of the coarse-grained density distribution $P(\rho R)$ on increasing $R$, and that of the displacement of the particles of supercooled liquids $P(\Delta R_f)$ on increasing the observation time $t$. As the fluctuations of $\rho R$ are suppressed up to a length scale $\xi_\gamma$: that diverges at the active-adsorbing phase transition, the CLT been gradually recovered at larger lengths; Similarly, the fluctuations of $\Delta R_f$ are suppressed up to the relaxation time $\tau$ that ideally diverges at the glass transition, the CLT been gradually recovered at longer timescales. The analogy extends to the behaviour of the kurtosis, or equivalently of the non-Gaussian parameter (excess kurtosis) $\kappa \approx 3$ most frequently studied in glassy systems; In the CLG model, $\kappa$ peaks at lengths that scale as $\xi_\kappa^\star$, and the peak height diverges at the glass transition; In supercooled liquids, $\kappa$ peaks at a time scaling as $\tau$, and its height diverges at the transition [26]. We remark, however, that the distribution is transiently skewed in the CLG model, while in liquids $\gamma = 0$ at all times.

In supercooled liquids, the above features of the displacement probability distribution reflect the existence of spatial heterogeneities on the relaxation timescale, particles with unusually large or small displacements being arranged in clusters [24]. The investigation of the distribution of the coarse-grained density, on a length scale $R$ close to that where $\kappa$ peaks, similarly reveal the presence of spatial heterogeneities in the CLG model, as Fig. 4b illustrates.

To understand this, and probe the physical origins of these under-density regions and their possible dynamical connections, we first introduce $t^\star(\vec{r}_f)$ - the time when a particle on site $\vec{r}_f$ was last active. From this, we then calculate a coarse-grained local time scale (analogous to $\rho R$ for the density) by averaging the dimensionless freezing time $t_f \equiv t^\star/\tau$ across patches of various length scales $R$, where $\tau$ is the relaxation time scale of the particular simulation (i.e. time for the entire lattice to reach the absorbing state).

In Fig. 4(a) we show the characteristic values of $t_f$ for local regions with given density $\rho R$ at various length scales $R$ (see Fig. 5 in [29] for scatter plots of $t_f - \rho R$). We find that $t_f$ peaks at $\rho R \sim \rho$ indicating that sub-regions with the slowest relaxation correspond to regions of small deviation from the mean density. More importantly, the under-density regions have freezing timescales several orders of magnitude smaller than the maximal dimensionless freezing time at $\rho R \sim \rho$, indicating that these under-density regions stem from rapid relaxation of fast sub-regions in real space where all particles freeze out early and are thus effectively passive for significant periods of the dynamics.

This correspondence between structural and dynamical properties that emerge at intermediate length scales can also be seen by comparing the values of $t_f$ and $\rho R$ across real space regions as shown in Fig. 4(b,c) for a given $R$ that is close to
the non-monotonic peaks of $\gamma$ and $\kappa$ in Fig. 4(a, c, e). Intensity plots of $\rho_R$ in real space with increasing $R$ which show the progressive emergence of heterogeneities in $\rho$, and the growth of these under-density regions that are strongly correlated to the freezing time scales, can be found in Fig. S4 [29]. Collectively, these results clarify that the spatial heterogeneities in the density originate from heterogeneities in the freezing timescale.

In this work, we establish the existence of hidden correlations beyond what is captured by the suppression of density fluctuations in the CLG model near the active-absorbing state phase transition where it exhibits hyperuniformity. These correlations are revealed by the dependence of the higher centralized scaled moments of density distributions on the coarse-grained length scale, and are measures not traditionally considered in hyperuniformity studies, which limit their attention to the scaling behavior of the second moment. Specifically, the higher moments reveal that the distribution becomes skewed and leptokurtic up to $R_c$ only, the higher moments reveal that the distribution becomes skewed and leptokurtic up to $R_c$ only, the higher moments reveal that the distribution becomes skewed and leptokurtic up to $R_c$, and of the higher moments (Gaussian/non-Gaussian) of the displacement probability distribution. The CLG is a Hyperuniform/non-Gaussian system, while maximally random jammed sphere packings [30] are (effectively) Hyperuniform and Gaussian systems. We have also evidence indicating that the random organization model [31] is Hyperuniform and non-Gaussian, while the Voronoi model for cell tissue [15] and the Quantizer problem [3] are (effectively) Hyperuniform and Gaussian. Accordingly, non-Gaussian behavior appears to occur in system exhibiting an absorbing transition and Gaussian behavior in jammed solids, but further work in this direction is certainly needed. Future investigations may also consider the possibility of artificially tuning the Gaussian behavior through local particle displacements [32].

More generally, these analyses based on the higher moments may provide for additional tools in probing the possible causal role of hyperuniformity in the self-organization of disordered systems by further characterizing the approach to criticality of non-equilibrium phase transitions.

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FIG. 4. (a) Average non-dimensional freezing time scale $t^* / \tau$ of particles within patches of lengthscale $R$ with local density $\rho_R$ for $\rho = 0.23$. A comparison of (b) $\log_{10}(t^* / \tau)$ and (c) $\rho_R$ for $R = 150$ of an absorbing state configuration ($\rho = 0.23$) indicates that regions of under(over)-density are strongly correlated to regions of shorter(longer) freezing timescales. Note that the white region in (b) represent vacant lattice sites of the absorbing state. An excess in under density regions at intermediate length scales are the physical origins of the sign and non-monotonic behavior observed in $\gamma$ and $\kappa$.

random organization models is an exciting avenue that demands further investigation. These explorations may, in turn, inform on its universality or conversely provide means for further taxonomy and classification of fluctuation suppressed or hyperuniform disordered systems. Hence, we envisage a parallel with the taxonomy recently introduced for diffusive systems [27, 28], where four main classes are identified based on the behaviour of the second moment (Fickian/non-Fickian) and of the higher moments (Gaussian/non-Gaussian) of the displacement probability distribution. The CLG is a Hyperuniform/non-Gaussian system, while maximally random jammed sphere packings [30] are (effectively) Hyperuniform and Gaussian systems. We have also evidence indicating that the random organization model [31] is Hyperuniform and non-Gaussian, while the Voronoi model for cell tissue [15] and the Quantizer problem [3] are (effectively) Hyperuniform and Gaussian. Accordingly, non-Gaussian behavior appears to occur in system exhibiting an absorbing transition and Gaussian behavior in jammed solids, but further work in this direction is certainly needed. Future investigations may also consider the possibility of artificially tuning the Gaussian behavior through local particle displacements [32].

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