Uniqueness of the modified Schrödinger map in $H^{3/4+\varepsilon}(\mathbb{R}^2)$

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Abstract

We establish the local well-posedness of the modified Schrödinger map in $H^{3/4+\varepsilon}(\mathbb{R}^2)$.

1 Introduction

Let $M$ and $N$ be smooth Riemannian manifolds. Smooth harmonic maps are smooth maps $f : M \rightarrow N$ which locally minimize the energy

$$\frac{1}{2} \int_M |Df|^2 dvol$$

where $|Df|$ denotes the Hilbert-Schmidt norm of the derivative of $f$ and where $dvol$ is the Riemannian measure on $M$. They provide an interesting tool for constructing submanifolds given as the image of $f$. The problem is particularly rich if the dimension of $M$ is 2. See [7, 11] for the intricate relation between geometry and analysis in that context.

If $M$ is a Minkowski space instead then the Euler-Lagrange equations are the so called wave map equations, which are an intensively studied prototypical class of nonlinear wave equations. They occur in several models in physics. See Tataru [17] and Tao [16] for a recent results on wave maps.

Their Schrödinger version, the Schrödinger maps, has been studied only recently. In the simplest case one is given a map $s_0 \in \dot{H}^1(\mathbb{R}^2; \mathbb{R}^3)$ where $\dot{H}^1$ denotes the homogeneous space with $|s_0| = 1$ and one searches a map $s$ which satisfies

$$\partial_t s = s \times \Delta s,$$
This problem occurs as continuous limit of a cubic lattice of classical spins evolving in the magnetic field created by their closed neighbors (Sulem, Sulem and Bardos [15], Sulem and Sulem [14], and Chang, Shatah and Uhlenbeck [3]).

To begin with, we introduce the Schrödinger map from \( \mathbb{R} \times \mathbb{R}^2 \) to the unit sphere \( S^2 \). For the map \( s(t, \cdot) : \mathbb{R}^2 \to S^2 \), we always assume that the energy i.e. the Dirichlet integral of the map is finite. In that case the limit

\[
\lim_{R \to \infty} \frac{1}{\pi R^2} \int_{B_R(0)} s(t, x) \, dx
\]

exists. We may and do assume that this point is the north pole \( N \) after a rotation. Then, we identify the Riemann surface \(( \mathbb{C}, g d\bar{z} dz)\) with \( S^2 \setminus N \) by using stereographic projection, where \( g \) is determined by \( g(z, \bar{z}) = (1 + |z|^2)^{-2} \) through the relation

\[
\mathbb{C} \ni z \mapsto \left( \frac{2 \text{Re} z}{1 + |z|^2}, \frac{2 \text{Im} z}{1 + |z|^2}, \frac{1 - |z|^2}{1 + |z|^2} \right) \in S^2,
\]

and consider the map \( z : \mathbb{R} \times \mathbb{R}^2 \to (\mathbb{C}, g d\bar{z} dz) \).

For each \( t \in \mathbb{R} \), the energy of the map \( z(t) : \mathbb{R}^2 \to (\mathbb{C}, g d\bar{z} dz) \) is defined by

\[
E(z(t)) = \frac{1}{2} \int_{\mathbb{R}^2} \frac{|
abla z(t)|^2}{(1 + |z(t)|^2)^2} \, dx. \tag{1}
\]

The Euler-Lagrange equation of the energy functional above is given by

\[
\sum_{j=1}^{2} \left( \frac{\partial}{\partial x_j} - 2 \frac{\bar{z} \partial_{x_j} z}{1 + |z|^2} \right) \frac{\partial z}{\partial x_j} = 0.
\]

There is a simple geometric meaning of the left hand side: It is the most natural Laplacian and

\[
\nabla_j = \frac{\partial}{\partial x_j} - 2 \frac{\bar{z} \partial_{x_j} z}{1 + |z|^2}
\]

is the pull back covariant derivative by the map \( z \) from \( \mathbb{R} \times \mathbb{R}^2 \) to \( (\mathbb{C}, g d\bar{z} dz) \). Here, we notice that \( \nabla_0 \), the pull back covariant derivative along the direction \( t \), has the similar representation as above denoting \( j = 0 \) and \( x_0 = t \). Then, the Schrödinger map equation is given as the evolution equation of the form

\[
\frac{\partial z}{\partial t} = i \sum_{j=1}^{2} \left( \frac{\partial}{\partial x_j} - 2 \frac{\bar{z} \partial_{x_j} z}{1 + |z|^2} \right) \frac{\partial z}{\partial x_j}. \tag{2}
\]
Here, we notice that the solution of the equation (2) preserves the energy (1).

The equation (2) is regarded as a nonlinear Schrödinger equation with a derivative nonlinearity, to which the standard energy method cannot be applied to prove the local well-posedness. Although there are many studies on the local well-posedness of the initial value problem for such a class of the derivative nonlinear Schrödinger equations (e.g. Kenig-Ponce-Vega [10], Hayashi-Ozawa [6], Chihara [2]), they require many derivatives in $L^2$ of the initial data. The purpose of this paper is to consider the local well-posedness of the initial value problem for the Schrödinger map (2) for low regularity initial data, which have a derivative nonlinearity of a specific form.

Following the approach of Nahmod, Stefanov and Uhlenbeck [12] we apply the “gauge transformations”

$$u_j = e^{i\psi} \frac{\partial x_j z}{1 + |z|^2}, \quad j = 1, 2 \tag{3}$$

to derive the system of the nonlinear Schrödinger equations on $u_j$’s by choosing the “gauge” $\psi$ appropriately, which is called the modified Schrödinger map equation:

$$i\partial_t u_1 + \Delta u_1 = -2iA \cdot \nabla u_1 + A_0 u_1 + |A|^2 u_1 + 4i \text{Im}(u_1 \bar{u}_1) u_2,$$

$$i\partial_t u_2 + \Delta u_2 = -2iA \cdot \nabla u_2 + A_0 u_2 + |A|^2 u_2 + 4i \text{Im}(u_1 \bar{u}_2) u_1, \tag{4}$$

where $A = (A_1[u, u], A_2[u, u])$ and $A_0 = A_0[u, u]$ are defined by

$$A_j[u, v] = 2 G_j \ast \text{Im}(u_1 \bar{v}_2 + v_1 \bar{u}_2), \quad j = 1, 2, \tag{5}$$

$$G_1(x) = \frac{1}{2\pi} \frac{x_2}{|x|^2}, \quad G_2(x) = -\frac{1}{2\pi} \frac{x_1}{|x|^2}, \tag{6}$$

$$A_0[u, v] = 2 \sum_{j,k=1}^2 R_j R_k \Re(u_j \bar{v}_k + v_j \bar{u}_k) + 2 \Re(u_1 \bar{v}_1 + \bar{v}_2 u_2), \tag{7}$$

and $R_j$ is the Riesz transform defined by the Fourier multiplier $i\xi_j/|\xi|$. We summarize the derivation of the modified Schrödinger map equation in the appendix. It is useful to observe that $\nabla \cdot A = 0$ and hence $A \cdot \nabla f = \nabla \cdot (Af)$ for all $f \in C^1(\mathbb{R}^2)$, which is due to the choice of the “gauge” $\psi$ (see the appendix).

**Remark 1.** It is not completely obvious whether the Schrödinger map problem (2) and the modified Schrödinger map problem (4) are equivalent. This has been shown by Nahmod and Kenig in cases which cover the range of solutions considered in this paper.

For the modified Schrödinger maps (4), Nahmod, Stefanov and Uhlenbeck [13, 12] have proven existence of unique solutions for initial data in $H^{1+\varepsilon}(\mathbb{R}^2)$.
The existence part has been extended by the first author and independently by Nahmod and Kenig [2] to the construction of solutions for initial data in $H^{1/2+\varepsilon}(\mathbb{R}^2)$. In an even more general context on the original Schrödinger map (2), Ding and Wang [4, 5] have shown existence of solution if $u_0 \in H^2$ and uniqueness if $u_0 \in H^1$. We consider the initial value problem for the modified Schrödinger maps. Our main result is the following.

**Theorem 2.** Suppose that $s > 3/4$ and $u_0 \in H^s(\mathbb{R}^2)$. Then there exists a unique local in time solution $u$ to the modified Schrödinger map equation (4) which satisfies

$$u \in C([0, T]; H^s(\mathbb{R}^2)) \cap L^2(0, T; C^{s-1/2-\varepsilon}(\mathbb{R}^2))$$

for all $T > 0$ in the domain of existence with $\varepsilon > 0$ sufficiently small. More precisely, suppose that $u_0, v_0 \in H^s(\mathbb{R}^2)$ and that

$$u, v \in C([0, T]; H^s(\mathbb{R}^2)) \cap L^2(0, T; C^{s-1/2-\varepsilon}(\mathbb{R}^2)).$$

Then, with $c$ depending on $\|u\|_{L^\infty H^{1/2}}$ and $\|v_0\|_{L^\infty H^{1/2}},$

$$\|u(t) - v(t)\|_{H^{-1/2}} \leq \exp\{c(1 + \|u\|^2_{L^4 B^1_{4+\varepsilon, 2}} + \|v\|^2_{L^4 B^1_{4+\varepsilon, 2}})\} \|u(s) - v(s)\|_{H^{-1/2}}. \tag{8}$$

The Besov spaces $B^s_{p,q}$ will be discussed in the next section. There we will see that

$$L^\infty H^s(\mathbb{R}^2) \cap L^2 C^\sigma(\mathbb{R}^2) \hookrightarrow L^4 B^{1/2}_{4+\varepsilon, 2}(\mathbb{R}^2)$$

for small $s > 3/4$ and $\sigma > 1/4$.

The existence part is essentially due to the first author in [8]. In [8], the existence of at least one solution was shown for the data in $H^s$ with $s > 1/2$ (see Theorem 3 below). The energy estimate of the difference of two solutions in $L^2$ implies uniqueness only for the data in $H^1$, which is due to the loss of derivatives in the nonlinearity. The main ingredient of this paper is to bound the difference of the two solutions in the function space $H^{-1/2}$ to overcome this difficulty. Indeed, the estimate (8) enables us to prove the uniqueness of the solution for the data in $H^s$ with $s > 3/4$, which improves the previous result in $H^1$.

**Notation:** We denote the standard $L^2$ Sobolev space with $s$ derivatives in $L^2$ by $H^s$. The Besov spaces $B^s_{p,q}$ and the Hölder spaces $C^s$ are defined below. If $X$ is a Banach space we denote by $C X$ resp. $L^p X$ the spaces of continuous functions resp. weakly measurable $p$ integrable functions with values in $X$, equipped with the obvious norm. Constants may change from line to line. We use $\sim$ with its standard meaning.
2 Preliminary results

We shall use the Besov space $B_{p,q}^s$ with $s \in \mathbb{R}$, $2 \leq p, q \leq \infty$. Let $\phi \in C_0^{\infty}(\mathbb{R}^2)$, $0 \leq \phi \leq 1$, $\phi(\xi) = 1$ for $|\xi| \leq 1$, $\phi(\xi) = 0$ for $|\xi| \geq 5/4$, $\phi(2^{-j+1}\xi)$ if $j \geq 1$ and $\phi_0(\xi) = \phi(\xi)$. Note that $\text{supp} \phi_j \subset \{2^{-j-1} \leq |\xi| \leq 5 \cdot 2^j/4\}$. We define $S^j$ through the Fourier multiplier $\phi_j$ and

$$S_j = \sum_{l=0}^j S^l.$$ 

We define the spaces $B_{p,q}^s$ through the norm

$$\|f\|_{B_{p,q}^s} = \left(\sum_{j=0}^{\infty} 2^{sj} \|S^j f\|_{L^p}^q\right)^{1/q}$$

with the obvious modification if $q = \infty$. The Hölder space $C^s$ with $s > 0$ is defined through the norm

$$\|f\|_{C^s} = \|f\|_{C^{[s]}} + \sum_{|\alpha| = [s]} \sup_{x \neq y} \frac{|\partial^\alpha f(x) - \partial^\alpha f(y)|}{|x - y|^{[s]}},$$

where $s = [s] + \{s\}$ with $[s]$ integer and $0 \leq \{s\} < 1$. If $s \notin \mathbb{Z}_+$, it is known that $C^s = B_{\infty,\infty}^s$ with the equivalent quasi-norm.

Existence of somewhat regular solutions was obtained in [8, Theorem 1.1], which shows the existence of the solution to the modified Schrödinger map for the data in $H^s(\mathbb{R}^2)$ with $s > 1/2$.

**Theorem 3.** Let $u_0 \in H^s(\mathbb{R}^2)$ with $s > 1/2$. Then there exists $T > 0$ satisfying

$$\min\{1, C/(1 + \|u_0\|_{L^2})\} \leq T \leq 1,$$

and at least one solution $u \in L^\infty(0,T; H^s) \cap C([0,T]; H^s)$ to (4) such that

$$u \in L^p(0,T; B_{p,q}^s(\mathbb{R}^2)),$$

for all $2 \leq p, q \leq \infty$ and $s \in \mathbb{R}$ with

$$s - 1/p > \sigma \geq 0, \quad \text{and} \quad 1/p = 1/2 - 1/q.$$

More precisely, given initial data $u_0 \in H^s(\mathbb{R}^2)$ with $s > 1/2$, and choosing a small $\varepsilon > 0$, a solution $u \in C(L^2)$ is constructed in Theorem [8, Theorem 1.1] which satisfies

$$\|(1 - \Delta)^{(s-1/2-\varepsilon)/2} u\|_{L_{T,L}^{p,q}} \leq C \|u_0\|_{H^s(\mathbb{R}^2)},$$

$$\|u\|_{L_{T,L}^{\infty} H^s} \leq C(\|u_0\|_{H^s}) \|u_0\|_{H^s(\mathbb{R}^2)}.$$
where \( s - 1/2 - \varepsilon > 2/q > 0, 1/p = 1/2 - 1/q, \) and \( C(\|u_0\|_{H^s}) \) denotes the constant which depends on \( \|u_0\|_{H^s} \).

The estimate (11) is the a priori estimate for the solution of [8, Theorem 3.1], and (12) follows from the energy estimate [8, Proposition 2.5] combined with (11). We choose \( q \) large and apply a Sobolev embedding to see that \( q = \infty \) is allowed (11). In this process we have to change \( \varepsilon \) slightly. More precisely we even obtain (10) with \( q = \infty \).

We interpolate the norm in (11) with \( q = \infty \) and the energy inequality (12) to obtain the full inequality (10) by using the following lemma.

**Lemma 4.** Let \( 0 \leq \theta \leq 1 \). We suppose \( s_0, s_1 \in \mathbb{R}, 1 \leq q_0, q_1 \leq \infty, \) and \( s = (1 - \theta)s_0 + \theta s_1, 1/q = (1 - \theta)/q_0 + \theta/q_1. \) Then, we have

\[
\|f\|_{B_{q_2}^s} \leq \|f\|_{B_{q_0}^{s_0}}^{1-\theta}\|f\|_{B_{q_1}^{s_1}}^\theta.
\]

**Proof.** We estimate

\[
\sum_{j=0}^\infty (2^{sj} \|S^j f\|_{L^p})^2 \leq \sum_{j=0}^\infty (2^{s_0j} \|S^j f\|_{L^{q_0}})^{2(1-\theta)}(2^{s_1j} \|S^j f\|_{L^{q_1}})^{2\theta}
\]

\[
\leq \|f\|_{B_{q_0}^{s_0}}^{2(1-\theta)}\|f\|_{B_{q_1}^{s_1}}^{2\theta}.
\]

Now the inequality

\[
\|u(t)\|_{B_{q_2}^{s-1/p-\varepsilon}} \leq c\|u(t)\|_{B_{q_2}^s}^{1-\frac{2}{p}}\|u(t)\|_{B_{q_2}^s}^{\frac{2}{p}}\|u(t)\|_{B_{q_2}^s}^{-\frac{2}{p}}
\]

(13)

holds by using the relation \( 1/p = 1/2 - 1/q \). Hence, we obtain

\[
\|u\|_{L^p B_{q_2}^{s-1/p-\varepsilon}}^p \leq \int_0^T \|u(t)\|_{B_{q_2}^s}^{p-2} \|u(t)\|_{B_{q_2}^s}^2 dt \leq \|u\|_{L^\infty B_{q_2}^{s-1/p-\varepsilon}}^2 \|u\|_{L^2 B_{q_2}^{s-1/p-\varepsilon}}^2.
\]

(14)

Here, we notice that the \( H^s \)-valued strong continuity of the solution in time variable is also obtained once we prove the uniqueness of the solution in the class of solution in Theorem 3. In fact, it is shown in [8] that with a constant depending on \( \|u_0\|_{L^2} \),

\[
\left| \frac{d}{dt} \|u_j(t)\|_{H^s}^2 \right| \leq C\|(1 - \Delta)^{(s-1/2-\varepsilon)/2}u_j(t)\|_{L^2}^2 \|u_j(t)\|_{H^s}^2
\]

holds for smooth solution \( u_j \) converging to the solution \( u \) of Theorem 3 which is constructed regularizing the initial data by the mollifier. Thus in the limit

\[
\lim_{s \to 0} \|u(s)\|_{H^s} \leq \|u(0)\|_{H^s}.
\]
Since the solution is weakly continuous in time,
\[ \|u(0)\|_{H^s} \leq \liminf_{s \to 0} \|u(s)\|_{H^s}. \]
These facts imply the solution is strongly continuous at \( t = 0 \). Since the solution in this class is unique by our main result, Theorem 2, if \( s > 3/4 \), we are able to apply the argument above for all \( t \in [0, T] \). Thus, if \( s > 3/4 \) we obtain
\[ u \in C([0, T]; H^s). \]

3 Calculus in Sobolev spaces

In this section we establish several estimates of products in Sobolev spaces and the Besov spaces, which are used in the proof of Theorem 2.

Lemma 5. Let \( n = 2 \) and \( q > 4 \). Then the following inequalities are true:
\[ \|fg\|_{H^{1/2}} \lesssim \|f\|_{H^{1/2}} \|g\|_{B^{1/2}_{q, 2}}, \tag{16} \]
\[ \|fg\|_{B^{1/2}_{q, 2}} \lesssim \|f\|_{B^{1/2}_{q, 2}} \|g\|_{B^{1/2}_{q, 2}}. \tag{17} \]

Proof. For the proof of the inequalities (16) and (17) we use the paraproduct decomposition
\[ f \cdot g = \Pi_1(f, g) + \Pi_2(f, g) + \Pi_1(g, f), \tag{18} \]
where
\[ \Pi_1(f, g) = \sum_{k=2}^{\infty} S^k f \cdot S_{k-2} g, \]
\[ \Pi_2(f, g) = \sum_{k=0}^{\infty} S^k f \cdot \tilde{S}^k g, \]
and \( \tilde{S}^k = \sum_{l=(k-1)\vee 0}^{k+1} S^l \). Here, we denote \( a \vee b = \max(a, b) \) for \( a, b \geq 0 \). The important point in the decomposition above is that
\[ \text{supp } \mathcal{F}[S^k f \cdot S_{k-2} g] \subset \{ \xi; 2^{k-3} \leq |\xi| \leq 2^{k+1} \}, \]
\[ \text{supp } \mathcal{F}[S^k f \cdot \tilde{S}^k g] \subset \{ \xi; |\xi| \leq 2^{k+2} \}. \]
hold. For the estimate of the $B^s_{p,2}$-norm of each term on the right hand side of (15) for $s \geq 0$, $1 \leq p \leq \infty$, the support conditions above are used as follows:

$$\|\Pi_1 (f, g)\|_{B^s_{p,2}}^2 = \sum_{j=0}^{\infty} 2^{2sj} \|S^j \Pi_1 (f, g)\|_{L^p}^2$$

$$= \sum_{j=0}^{\infty} 2^{2sj} \left\| S^j \left( \sum_{k=(j-1)\vee 0}^{j+3} S^k f \cdot S_{k-2} g \right) \right\|_{L^p}^2$$

$$\lesssim \sum_{k=2}^{\infty} 2^{sk} \|S^k f \cdot S_{k-2} g\|_{L^p}^2,$$

and similarly

$$\|\Pi_2 (f, g)\|_{B^s_{p,2}} = \left\{ \sum_{j=0}^{\infty} 2^{2sj} \left\| S^j \left( \sum_{k=(j-2)\vee 0}^{\infty} S^k f \cdot \tilde{S}_{k} g \right) \right\|_{L^p}^2 \right\}^{1/2}$$

$$\lesssim \left\{ \sum_{j=0}^{\infty} \left( \sum_{k=(j-2)\vee 0}^{\infty} 2^{sj} \|S^k f \cdot \tilde{S}_{k} g\|_{L^p} \right)^2 \right\}^{1/2}$$

$$\lesssim \sum_{k=0}^{\infty} 2^{sk} \|S^k f \cdot \tilde{S}_{k} g\|_{L^p}.$$

Then, it is easy to see the estimate

$$\|fg\|_{B^1_{q,2}} \lesssim \|f\|_{B^1_{q,2}} \|g\|_{B^0_{\infty,1}} + \|f\|_{B^0_{\infty,1}} \|g\|_{B^1_{q,2}},$$

(19)

holds, because

$$\|\Pi_1 (f, g)\|_{B^1_{q,2}} \lesssim \|g\|_{L^\infty} \|f\|_{B^1_{q,2}} \lesssim \|g\|_{B^0_{\infty,1}} \|f\|_{B^1_{q,2}};$$

$$\|\Pi_2 (f, g)\|_{B^1_{q,2}} \lesssim \|g\|_{B^0_{\infty,1}} \|f\|_{B^1_{q,2}}.$$

Inequality (17) is an immediate consequences of the inequality above since $B^1_{q,2} \hookrightarrow B^0_{\infty,1}$ for $q > 4$.

Now we turn to the proof of the inequality (16). Since $H^s = B^s_{2,2}$ with the equivalent norms, we have

$$\|\Pi_1 (f, g)\|_{H^{1/2}} \lesssim \left\{ \sum_{k=2}^{\infty} 2^k \|S^k f\|_{L^2}^2 \|S_{k-2} g\|_{L^\infty}^2 \right\}^{1/2}$$

$$\lesssim \|g\|_{L^\infty} \left\{ \sum_{k=2}^{\infty} 2^k \|S^k f\|_{L^2}^2 \right\}^{1/2}$$

$$\lesssim \|g\|_{B^{1/2}_{q,2}} \|f\|_{H^{1/2}};$$

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since $B^{1/2}_{q,2} \hookrightarrow L^\infty$ for $q > 4$. For the estimate of the other terms, we set $1/p + 1/q = 1/2$ with $q > 4$, and we use the Sobolev embedding

$$\|S^k f\|_{L^p} \lesssim 2^{2k/q} \|S^k f\|_{L^2}$$

(20) to obtain

$$\|\Pi_2(f, g)\|_{H^{1/2}} \lesssim \sum_{k=0}^{\infty} 2^{k/2} \|S^k f\|_{L^p} \|\tilde{S}^k g\|_{L^q}$$

$$\lesssim \sum_{k=0}^{\infty} 2^{k/2} \|S^k f\|_{L^2} 2^{2k/q} \|\tilde{S}^k g\|_{L^q}$$

$$\lesssim \|f\|_{H^{1/2}} \|g\|_{B^{2/q}_{q,2}}.$$  

Note that $B^{1/2}_{q,2} \hookrightarrow B^{2/q}_{q,2}$ holds for $q > 4$. Finally, we use the inequality (20) again to obtain

$$\|\Pi_1(g, f)\|_{H^{1/2}} \lesssim \left\{ \sum_{k=2}^{\infty} 2^k \|S_{k-2} f\|_{L^p} \|S^k g\|_{L^q} \right\}^{1/2}$$

$$\lesssim \left\{ \sum_{k=2}^{\infty} 2^k \left( \sum_{l=0}^{k-2} \|S^l f\|_{L^p} \right)^2 \|S^k g\|_{L^q} \right\}^{1/2}$$

$$\lesssim \left\{ \sum_{k=2}^{\infty} 2^k \left( \sum_{l=0}^{k-2} 2^{2l/q} \|S^l f\|_{L^2} \right)^2 \|S^k g\|_{L^q} \right\}^{1/2}$$

$$\lesssim \left\{ \sum_{l=0}^{\infty} \left( \sum_{k=l+2}^{\infty} 2^k \|S^k g\|_{L^q}^2 \right)^{1/2} 2^{2l/q} \|S^l f\|_{L^2} \right\}^{1/2}$$

$$= \|g\|_{B^{1/2}_{q,2}} \sum_{l=0}^{\infty} 2^{-(1/2-2/q)l} 2^{l/2} \|S^l f\|_{L^2}$$

$$\lesssim \|g\|_{B^{1/2}_{q,2}} \|f\|_{H^{1/2}}$$

for $q > 4$. This completes the proof of the inequality (16). 

Lemma 6. The following inequalities are true

$$\|fg\|_{H^{-1/2}} \lesssim \|f\|_{H^{-1/2}} \|g\|_{B^{1/2}_{q,2}}$$

(21)

and

$$\|f \nabla g\|_{H^{-1/2}} \lesssim \|f\|_{H^{1/2}} \|g\|_{B^{1/2}_{q,2}}.$$  

(22)
Proof. The first inequality follows by duality (consider $f \to fg$ with fixed $g$) from the previous lemma. For the second inequality we use $f \nabla g = \nabla (fg) - (\nabla f)g$ and estimate

$$\|f \nabla g\|_{H^{-1/2}} \leq \|\nabla (fg)\|_{H^{-1/2}} + \|(\nabla f)g\|_{H^{-1/2}}$$

$$\lesssim \|fg\|_{H^{1/2}} + \|\nabla f\|_{B_{1/2}^{1/2}} \|g\|_{B_{1/2}^{1/2}}$$

$$\lesssim \|f\|_{H^{1/2}} \|g\|_{B_{1/2}^{1/2}}.$$

In what follows, we collect several estimates on the vector field $A[f, g] = (A_{1}[f, g], A_{2}[f, g])$ defined by (5), (6) by using the estimates above. It is easy to see that each $A_{j}$ is real, and $\nabla \cdot A = 0$ holds.

Lemma 7. The vector field $A$ defined by (5), (6) satisfies

$$\|\nabla A[f, g]\|_{L^{\infty}} \lesssim \|f\|_{B_{4/2}^{1/2}} \|g\|_{B_{4/2}^{1/2}} + \|f\|_{L^{2}} \|g\|_{L^{2}}, \quad \text{(23)}$$

$$\|A[f, g]\|_{B_{4/2}^{1/2}} \lesssim \|f\|_{B_{4/2}^{1/2}} \|g\|_{B_{4/2}^{1/2}} + \|f\|_{L^{2}} \|g\|_{L^{2}}, \quad \text{(24)}$$

for $q > 4$.

Proof. The inequality (23) is derived as follows.

$$\|\nabla A[f, g]\|_{L^{\infty}} \lesssim \|fg\|_{C^{\infty}} + \|fg\|_{L^{1}}$$

$$\lesssim \|f\|_{C^{\infty}} \|g\|_{C^{\infty}} + \|f\|_{L^{2}} \|g\|_{L^{2}}$$

$$\lesssim \|f\|_{B_{4/2}^{1/2}} \|g\|_{B_{4/2}^{1/2}} + \|f\|_{L^{2}} \|g\|_{L^{2}}, \quad \text{(25)}$$

where we used elliptic regularity and the embedding $B_{4/2}^{1/2}(\mathbb{R}^{2}) \subset C^{\infty}(\mathbb{R}^{2})$.

To derive the inequality (24) we decompose $A[f, g]$ into the high frequency part and the low frequency part,

$$A[f, g] = S_{0}A[f, g] + (I - S_{0})A[f, g],$$

where $S_{0}$ is the operator defined in Section 2, which is the Fourier multiplier $\phi$. Then, the high frequency part is easily estimated as

$$\|(I - S_{0})A[f, g]\|_{B_{4/2}^{1/2}} \lesssim \|fg\|_{B_{4/2}^{1/2}}$$

$$\lesssim \|fg\|_{B_{4/2}^{1/2}}$$

$$\lesssim \|f\|_{B_{4/2}^{1/2}} \|g\|_{B_{4/2}^{1/2}}.$$

Here the first inequality follows from the elliptic regularity (it is not hard to see that there no difficulty from the low frequency part), and the third from (16).
To estimate the low frequency part, we first observe that
\[ S_0 A[f, g] = F^{-1}[\phi] \ast A[f, g] \]
\[ \sim F^{-1}[\phi] \ast K \ast (fg), \]
where \( F^{-1} \) denotes the inverse Fourier transform and \( K \) is the smooth homogeneous function of degree \(-1\). Then, it is easy to see \( \Phi \equiv F^{-1}[\phi] \ast K \in L^r(\mathbb{R}^2) \) for \( 2 < r < \infty \) by using the Hardy-Littlewood-Sobolev inequality. Thus, we obtain by Young’s inequality
\[
\| S_0 A[f, g] \|_{B^{1/2}_{q,2}} \lesssim \| S_0 A[f, g] \|_{L^q}
\]
\[ = \| \Phi \ast (fg) \|_{L^q} \]
\[ \lesssim \| \Phi \|_{L^q} \| f \|_{L^2} \| g \|_{L^2}, \]
where we also use the embedding \( B_{q,q}^{1/2+\varepsilon} \hookrightarrow B_{q,2}^{1/2} \), the second inequality follows from the fact that we are concerned with the low frequency part, and the third from the fact \( B_{q,q}^0 = L^q \) with the equivalent norm. This completes the proof of the inequality (24).

**Lemma 8.** Let \( n = 2 \) and \( q > 4 \). Then the following inequality holds.
\[
\| A[f, g] \nabla h \|_{H^{-1/2}} \lesssim (\| g \|_{H^{1/2}} \| h \|_{H^{1/2}} + \| g \|_{B_{q,2}^{1/2}} \| h \|_{B_{q,2}^{1/2}}) \| f \|_{H^{-1/2}}, \quad (26)
\]
where \( A[f, g] \) is the vector field defined by (5), (6).

**Proof.** We use that \( A \) is divergence free and hence
\[
\| A[f, g] \nabla h \|_{H^{-1/2}} = \| \nabla (A[f, g] h) \|_{H^{-1/2}} \lesssim \| A[f, g] h \|_{H^{1/2}}.
\]
As in the proof of the previous lemma, we decompose \( A[f, g] \) into the high frequency part and the low frequency part,
\[
A[f, g] = S_0 A[f, g] + (I - S_0) A[f, g].
\]
Then, the high frequency part is easily estimated as follows.
\[
\| (I - S_0) A[f, g] \cdot h \|_{H^{1/2}} \lesssim \| h \|_{B_{q,2}^{1/2}} \| (I - S_0) A[f, g] \|_{H^{1/2}}
\]
\[ \lesssim \| h \|_{B_{q,2}^{1/2}} \| fg \|_{H^{-1/2}} \]
\[ \lesssim \| h \|_{B_{q,2}^{1/2}} \| g \|_{B_{q,2}^{1/2}} \| f \|_{H^{-1/2}}. \quad (27)
\]
Here the first inequality follows from (16), the second from elliptic regularity, and the third inequality from (21).
For the estimate on the low frequency part, we observe that a direct calculation shows that
\[ \|uv\|_{L^2} \leq \|u\|_{L^\infty}\|v\|_{L^2} \]
and
\[ \|\nabla(uv)\|_{L^2} \leq (\|u\|_{L^\infty} + \|\nabla u\|_{L^\infty})\|v\|_{H^1}, \]
hence by interpolation
\[ \|S_0A[f, g] \cdot h\|_{H^{1/2}} \leq 2\|S_0A[f, g]\|_{W^{1,\infty}}\|h\|_{H^{1/2}}. \]
To complete the proof we have to bound \( S_0A[f, g] \) and its gradient. By translation invariance it suffices to do this at the origin. The argument for \( S_0A[f, g] \) and for its gradient is the same. Now using the notation in the proof of the previous lemma, we observe that
\[ S_0A[f, g](0) \sim \{ \Phi \ast (fg) \}(0) \]
\[ = \int_{\mathbb{R}^2} \Phi(y)f(y)g(y)dy. \]
Thus,
\[ |S_0A[f, g](0)| \lesssim \left| \int_{\mathbb{R}^2} \Phi(y)f(y)g(y)dy \right| \]
\[ \lesssim \|\Phi\|_{H^{1/2}}\|f\|_{H^{-1/2}} \]
\[ \lesssim \|\Phi\|_{B^{1/2}_{q,2}}\|g\|_{H^{1/2}}\|f\|_{H^{-1/2}}. \]
Finally, we notice that
\[ \|\Phi\|_{B^{1/2}_{q,2}} \lesssim \|\Phi\|_{B^{1/2+\epsilon}_{q,2}} \lesssim \|\Phi\|_{B^{0}_{q,2}} \sim \|\Phi\|_{L^q} < \infty, \]
since \( \Phi \) is supported in the low frequency part in the Fourier space, and \( \Phi \in L^r(\mathbb{R}^2) \) for \( 2 < r < \infty \). This completes the proof.

4 The energy estimates

In this section we derive elementary \( L^2 \) inequalities for Schrödinger equations with a drift term. We first consider
\[ i\partial_t u + \Delta u + iv \cdot \nabla u = F. \]

Lemma 9. Suppose that \( v \) is real valued and
\[ \int_0^T \|\nabla v(\tau)\|_{L^\infty}d\tau < \infty, \]
where \( \| \nabla v \|_{L^\infty} = \| \nabla v \|_{H^S} \) with \( | \cdot |_{H^S} \) the Hilbert-Schmidt norm, more precisely the Euclidean length of the vector in the case here. Then the following inequality holds for \( 0 \leq s \leq 1 \)

\[
\| u(t) \|_{H^s(\mathbb{R})} \leq e^{4 \int_0^t \| \nabla v \|_{L^\infty} d\tau} \left( \| u(0) \|_{H^s(\mathbb{R})} + \int_0^t \| F(\tau) \|_{H^s} d\tau \right).
\]

**Proof.** Let \( u(t) = \tilde{S}(t, \tau) \) be the solution at time \( t \) to

\[
i \partial_t u + \Delta u + iv \cdot \nabla u = 0, \quad (t, x) \in (0, T) \times \mathbb{R}^2
\]

with initial data \( u(\tau) = f \). It suffices to show that \( \tilde{S}(t, \tau) \) is bounded on \( H^s \), \( 0 \leq s \leq 1 \). Interpolation reduces the claim to \( s = 0 \) and \( s = 1 \). If \( s = 0 \) then

\[
\frac{d}{dt} \| u(t) \|^2_{L^2} = \int v \cdot \nabla |u|^2 dx \\
\leq \| \nabla v(t) \|_{L^\infty} \| u(t) \|^2_{L^2}
\]

and the assertion follows by application of Gronwall’s Lemma. If \( s = 1 \) then

\[
\frac{d}{dt} \| \nabla u(t) \|^2_{L^2} = \int v \cdot (\nabla u \Delta \pi + \nabla \pi \Delta u) dx \\
\leq 4 \| \nabla v(t) \|_{L^\infty} \| \nabla u(t) \|^2_{L^2}
\]

and the assertion follows as above. \( \Box \)

We also consider the dual problem:

\[
i \partial_t u + \Delta u + i\nabla \cdot (vu) = F.
\]

**Lemma 10.** Suppose that \( v \) is real valued and

\[
\int_0^T \| \nabla v(\tau) \|_{L^\infty} d\tau < \infty.
\]

Then, if \( -1 \leq s \leq 0 \)

\[
\| u(t) \|_{H^s(\mathbb{R})} \leq e^{4 \int_0^t \| \nabla v \|_{L^\infty} d\tau} \left( \| u(0) \|_{H^s(\mathbb{R})} + \int_0^t \| F(\tau) \|_{H^s} d\tau \right).
\]

**Proof.** It suffices to study the case \( F \equiv 0 \), since the general case follows by variation of constants as above. Let \( S(\tau, t)g \) be the solution to

\[
i \partial_t u + \Delta u + i\nabla \cdot (vu) = 0, \quad (\tau, x) \in (0, T) \times \mathbb{R}^2
\]

evaluated at time \( \tau \) with initial data \( u(t) = g \). We have to show that

\[
\| S(\tau, t)g \|_{H^s} \leq e^{4 \int_0^t \| \nabla v \|_{L^\infty} d\tau} \| g \|_{H^s}.
\]
Let $\tilde{S}(\tau, t)f$ be the solution to

$$i\partial_t u + \Delta u + iv \cdot \nabla u = 0, \quad (t, x) \in (0, T) \times \mathbb{R}^2,$$

at time $t$ with initial data $u(\tau) = f$. Then $\tilde{S}(t, \tau)$ is the adjoint operator of $S(\tau, t)$ since

$$\frac{d}{dt} \langle S(t', t)f, \tilde{S}(t', \tau)g \rangle = \langle i\Delta S(t', t)f, \tilde{S}(t', \tau)g \rangle + \langle S(t', t)f, i\Delta \tilde{S}(t', \tau)g \rangle - \langle \nabla \cdot (vS(t', t)f), \tilde{S}(t', \tau)g \rangle - \langle S(t', t)f, v \cdot \nabla \tilde{S}(t', \tau)g \rangle = 0,$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $L^2$. Now Lemma 3 can be applied to $\tilde{S}$ and by duality we obtain (28).

5 The difference of two solutions

In this section we give the proof of the estimate (8). This estimate combined with Theorem 3 completes the proof of Theorem 2.

Let $u_0, v_0 \in H^s(\mathbb{R}^2)$ with $s > 3/4$. According to Theorem 3 there exist solutions in

$$L^\infty(0, T; H^s(\mathbb{R}^2)) \cap L^4(0, T; B_{q, 2}^{1/2})$$

for some $0 < T \leq 1, q > 4$. Any solution satisfying the assumptions of Theorem 2 lies in that space. Let $u$ and $v$ be two such solutions and let $w$ be their difference. It satisfies

$$i\partial_tw_1 + \Delta w_1 + 2i\nabla \cdot (A[u, u]w_1) = -2iA[u + v, w] \cdot \nabla v_1 + (A_0[u, u] + |A[u, u]|^2)w_1 + (A_0[u + v, w] + |A[u, u]|^2 - |A[v, v]|^2)v_1 + 4i\{ \text{Im} (u_2\overline{w}_1)w_2 + \text{Im} ((u_2 + v_2)\overline{w}_1 + w_2(\overline{w}_1 + \overline{v}_1))v_2 \},$$

$$i\partial_tw_2 + \Delta w_2 + 2i\nabla \cdot (A[u, u]w_2) = -2iA[u + v, w] \cdot \nabla v_2 + (A_0[u, u] + |A[u, u]|^2)w_2 + (A_0[u + v, w] + |A[u, u]|^2 - |A[v, v]|^2)v_2 + 4i\{ \text{Im} (u_1\overline{w}_2)w_1 + \text{Im} ((u_1 + v_1)\overline{w}_1 + w_1(\overline{w}_2 + \overline{v}_2))v_1 \}. \quad (29)$$

Here, we notice that $A[u, v]$ and $A_0[u, v]$ are bilinear and symmetric in $u, v$.

We want to estimate $\|w(t)\|_{H^{-1/2}}$. Using Lemma 11 for the equations on $w$ above, we shall show for $t > s$ with $c$ depending on $\|u\|_{L^4B_{q, 2}^{1/2}}, \|v\|_{L^4B_{q, 2}^{1/2}}$.
\[ \| u \|_{L^\infty H^{1/2}}, \text{ and } \| v \|_{L^\infty H^{1/2}}, \]

\[ \| w(t) \|_{H^{-1/2}(\mathbb{R}^2)} \leq c \| w(s) \|_{H^{-1/2}(\mathbb{R}^2)} + c \int_s^t (1 + \| u(\tau) \|^2_{B^{1/2}_{q,2}} + \| v(\tau) \|^2_{B^{1/2}_{q,2}})^2 \| w(\tau) \|_{H^{-1/2}} d\tau \]

(30)

and hence, by Gronwall’s inequality

\[ \| w(t) \|_{H^{-1/2}} \leq c \| w(0) \|_{H^{-1/2}} \exp \left\{ (1 + \| u \|^2_{L^4([0,T], B^{1/2}_{q,2})} + \| v \|^2_{L^4([0,T], B^{1/2}_{q,2})})^2 \right\}, \]

(31)

which implies uniqueness of the solutions.

To establish (30) we use Lemma 10. We first observe that the assumption of Lemma 10

\[ \int_0^T \| \nabla A[u, u](\tau) \|_\infty d\tau \lesssim \| u \|^2_{L^2_T L^{3/2}_{q,2}} + \| u \|^2_{L^\infty_T L^2} < \infty \]

is verified by (23).

Then it suffices to estimate each term on the right hand sides of the equations (29).

The following terms are easily estimated.

\[ \| \text{Im} (f \overline{g}) h \|_{H^{-1/2}} \lesssim \| \text{Im} (f \overline{g}) \|_{B^{1/2}_{q,2}} \| h \|_{H^{-1/2}} \]

\[ \lesssim \| f \|_{B^{1/2}_{q,2}} \| g \|_{B^{1/2}_{q,2}} \| h \|_{H^{-1/2}}, \]

(32)

where the first inequality is a consequence of (21), and the second is a consequence of (17). Similarly

\[ \| \text{Im} (f \overline{g}) h \|_{H^{-1/2}} \lesssim \| \text{Im} (f \overline{g}) \|_{H^{-1/2}} \| h \|_{B^{1/2}_{q,2}} \]

\[ \lesssim \| f \|_{B^{1/2}_{q,2}} \| h \|_{B^{1/2}_{q,2}} \| g \|_{H^{-1/2}}, \]

(33)

and using the boundedness of the Riesz transforms

\[ \| R_j R_k \text{Re} (f \overline{g}) h \|_{H^{-1/2}} \lesssim \| R_j R_k \text{Re} (f \overline{g}) \|_{B^{1/2}_{q,2}} \| h \|_{H^{-1/2}} \]

\[ \lesssim \| f \|_{B^{1/2}_{q,2}} \| g \|_{B^{1/2}_{q,2}} \| h \|_{H^{-1/2}} \]

(34)

and

\[ \| R_j R_k \text{Re} (f \overline{g}) h \|_{H^{-1/2}} \lesssim \| R_j R_k \text{Im} (f \overline{g}) \|_{H^{-1/2}} \| h \|_{B^{1/2}_{q,2}} \]

\[ \lesssim \| f \|_{B^{1/2}_{q,2}} \| h \|_{B^{1/2}_{q,2}} \| g \|_{H^{-1/2}}, \]

(35)
It remains to control the terms containing $A$. In particular the estimate of the term containing derivatives of $v$ is crucial. This term is controlled by Lemma\textsuperscript{8}

$$
\|A[u, w] \nabla v\|_{H^{-1/2}} \lesssim (\|u\|_{H^{1/2}} \|v\|_{H^{1/2}} + \|u\|_{B^{1/2}_{q, 2}} \|v\|_{B^{1/2}_{q, 2}}) \|w\|_{H^{-1/2}}.
$$

Finally, we observe that

$$
|A[u, u]|^2 - |A[v, v]|^2 = (A[u, u] + A[v, v]) A[u + v, w]
$$

and the following two estimates complete the proof. The first one follows from (21) and (24),

$$
\|A[u, u] A[f, g] h\|_{H^{-1/2}}
\lesssim \|A[u, u]\|_{B^{1/2}_{q, 2}} \|A[f, g]\|_{H^{-1/2}}
\lesssim \|A[u, u]\|_{B^{1/2}_{q, 2}} \|A[f, g]\|_{B^{1/2}_{q, 2}} \|h\|_{H^{-1/2}}
\lesssim (\|u\|_{B^{1/2}_{q, 2}}^2 + \|u\|_{L^2}^2) (\|f\|_{B^{1/2}_{q, 2}} \|g\|_{B^{1/2}_{q, 2}} + \|f\|_{L^2} \|g\|_{L^2}) \|h\|_{H^{-1/2}}.
$$

The second one follows from (21), (24), and Lemma\textsuperscript{8},

$$
\|A[u, u] A[f, g] h\|_{H^{-1/2}}
\lesssim \|A[u, u]\|_{B^{1/2}_{q, 2}} \|A[f, g]\|_{H^{-1/2}}
\lesssim \|A[u, u]\|_{B^{1/2}_{q, 2}} \|A[f, g]\|_{H^{1/2}}
\lesssim (\|u\|_{B^{1/2}_{q, 2}}^2 + \|u\|_{L^2}^2) (\|f\|_{B^{1/2}_{q, 2}} \|h\|_{B^{1/2}_{q, 2}} + \|f\|_{H^{1/2}} \|h\|_{H^{1/2}}) \|g\|_{H^{-1/2}}.
$$

This completes the proof of (30).

A Appendix

In this appendix, we briefly describe the derivation of the modified Schrödinger map [14], which is due to Nahmod, Stefanov and Uhlenbeck [12]. Recall that the Schrödinger map $z$ from $\mathbb{R} \times \mathbb{R}^2$ to $(\mathbb{C}, g dz d\bar{z}) \simeq S^2$ is given by

$$
\frac{\partial z}{\partial t} = i \sum_{j=1}^2 \left( \frac{\partial}{\partial x_j} - 2 \frac{\bar{z} \partial x_j z}{1 + |z|^2} \right) \frac{\partial z}{\partial x_j}.
$$

(39)

It is not hard to check that

$$
\left( \frac{\partial}{\partial x_j} - 2 \frac{\bar{z} \partial x_j z}{1 + |z|^2} \right) \frac{\partial z}{\partial x_k} = \left( \frac{\partial}{\partial x_k} - 2 \frac{\bar{z} \partial x_k z}{1 + |z|^2} \right) \frac{\partial z}{\partial x_j}.
$$

(40)
and
\[ \left[ \frac{\partial}{\partial x_j} - \frac{2}{1 + |z|^2} \overline{z} \frac{\partial}{\partial x_k} - \frac{2}{1 + |z|^2} \right] = -4i \text{Im} (\overline{b}_j b_k) \] (41)
with
\[ b_j = \frac{\partial_x z}{1 + |z|^2} \]
hold for \( j, k = 0, 1, 2 \). Then we set
\[ u_j = e^{i\psi} \frac{\partial_x z}{1 + |z|^2}, \]
and
\[ D_j = (1 + |z|^2)^{-1} e^{-i\psi} \circ \nabla_j \circ (1 + |z|^2)^{i\psi} = \partial_x z + iA_j \]
for \( j = 0, 1, 2 \), where \( x_0 = t \), where the real-valued function \( \psi \) is determined later. Note that
\[ A_j = -\partial_x \psi - i \frac{z \partial_x \overline{z} - \overline{z} \partial_x z}{1 + |z|^2} = -\partial_x \psi + 2 \text{Im} (\overline{b}_j z) \]
is real-valued. By using the notation above, the equations (39), (40), and (41) are rewritten as
\[ u_0 = i \sum_{j=1}^{2} D_j u_j \] (42)
\[ D_j u_k = D_k u_j, \] (43)
\[ [D_j, D_k] = i \left( \frac{\partial A_k}{\partial x_j} - \frac{\partial A_j}{\partial x_k} \right) = -4i \text{Im} (u_j \overline{u}_k) \] (44)
for \( j, k = 0, 1, 2 \). Here, we notice that the equations (42), (43), and (44) are invariant for arbitrary choice of \( \psi \). Then, the system of the nonlinear Schrödinger equations on \( u_j \)'s is derived as follows. For \( j = 1, 2 \) we first notice that
\[ D_j u_0 = D_0 u_j = \partial_t u_j + iA_0 u_j \]
holds by (43). On the other hand, by using (42), (43), and (44) we have
\[ D_j u_0 = i \sum_{k=1}^{2} D_j D_k u_k \]
\[ = i \sum_{k=1}^{2} (D_k D_j u_k + [D_j, D_k] u_k) \]
\[ = i \sum_{k=1}^{2} D_k^2 u_j + 4 \sum_{k=1}^{2} \text{Im} (u_j \overline{u}_k) u_k. \]
Thus, for $j = 1, 2$ we obtain in a somewhat brief notation

\[ \partial_t u_j = i(\nabla + iA)^2 u_j - iA_0 u_j + 4 \sum_{k=1}^{2} \text{Im}(u_j \bar{u}_k) u_k, \] (45)

where we denote $A = (A_1, A_2)$.

Now we determine the gauge $\psi$. For each $t$ we define $\psi$ – up to constants – by

\[ \Delta \psi = -i \sum_{j=1}^{2} \partial_{x_j} z \partial_{x_j} \bar{z} \frac{1}{1 + |z|^2} = 2 \sum_{j=1}^{2} \partial_{x_j} \text{Im}(\bar{b}_j z) \]

so that

\[ \nabla \cdot A = 0, \quad A_j \to 0 \text{ as } x \to \infty. \] (46)

This condition and (44) enable us to determine $A$ and $A_0$ in terms of $u_1, u_2$. In fact,

\[ -\Delta A_1 = \partial_{x_2} (\partial_{x_1} A_2 - \partial_{x_2} A_1) = -4 \partial_{x_2} \text{Im}(\bar{u}_1 u_2) \] (47)

holds by using the first equality of (46) and (44). Thus,

\[ A_1 = 4 G_1 * \text{Im}(\bar{u}_1 u_2), \] (48)

where $2\pi G_1(x) = x_2/|x|^2$. Similarly, we obtain

\[ A_2 = 4 G_2 * \text{Im}(\bar{u}_1 u_2) \] (49)

with $2\pi G_2(x) = -x_1/|x|^2$. We use (44) and (46) again with $j = 0$ to determine $A_0$,

\[ -\Delta A_0 = -\sum_{k=1}^{2} \partial_{x_k}^2 A_0 \]

\[ = \sum_{k=1}^{2} \partial_{x_k} (-\partial_t A_k + 4 \text{Im}(\bar{u}_k u_0)) \]

\[ = 4 \sum_{k=1}^{2} \partial_{x_k} \text{Im}(\bar{u}_k u_0). \] (50)

Then we apply (42) to obtain

\[ \text{Im}(\bar{u}_k u_0) = \text{Im}(\bar{u}_k i \sum_{j=1}^{2} D_j u_j) \]

\[ = \sum_{j=1}^{2} \text{Re}(\bar{u}_k D_j u_j) \]

\[ = \sum_{j=1}^{2} \{\partial_{x_j} \text{Re}(\bar{u}_k u_j) - \text{Re}(\overline{(D_j u_k)} u_j)\}, \]
where we used the relation \( D_j f \cdot \overline{g} = \partial_{x_j} (f \cdot \overline{g}) - f \cdot \overline{D_j g} \). Since this relation also implies
\[
\text{Re} \left( (D_j u_k) u_j \right) = \text{Re} \left( (D_k u_j) u_j \right) = \partial_{x_k} |u_j|^2 / 2,
\]
we obtain
\[
-\Delta A_0 = 4 \sum_{j,k=1}^2 \partial_{x_j} \partial_{x_k} \text{Re} \left( u_j \overline{u_k} \right) - 2|u|^2,
\]
where we denote \( u = (u_1, u_2) \) and \(|u|^2 = |u_1|^2 + |u_2|^2\). Therefore,
\[
A_0 = 4 \sum_{j,k=1}^2 R_j R_k \text{Re} \left( u_j \overline{u_k} \right) + 2|u|^2, \tag{51}
\]
where \( R_j \) denotes the Riesz transforms. Therefore, we derive the system with (48), (49), and (51), which is the modified Schrödinger map \( (4) \).

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