Comment on “Exposed-Key Weakness of $\alpha \eta$”
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Abstract

We show that the insecurity claim of the $\alpha \eta$ cryptosystem made by C. Ahn and K. Birnbaum in Phys. Lett. A 370 (2007) 131-135 under heterodyne attack is based on invalid extrapolations of Shannon’s random cipher analysis and on an invalid statistical independence assumption. We show, both for standard ciphers and $\alpha \eta$, that expressions of the kind given by Ahn and Birnbaum can at best be interpreted as security lower bounds.

Key words: Quantum cryptography, Data Encryption, Random Cipher
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In [1], Ahn and Birnbaum claim to establish, by an approximate analysis, the information-theoretic insecurity of the $\alpha \eta$ encryption system [2,4] even for ciphertext-only attacks in which Eve makes heterodyne measurements followed by classical processing. While information-theoretic security in the asymptotic limit against such attacks has been claimed by us to be unlikely in [3,5], the main purpose of this comment is to show that the arguments of [1] do not establish insecurity of either the asymptotic or finite cases. We prove the asymptotic insecurity of $\alpha \eta$ ciphertext-only attacks conjectured by us in [3,5], and comment on its lack of practical significance. We also give some new lower bounds on the average number of spurious keys of $\alpha \eta$ and other random ciphers.

In Section 1, we describe the claim of [1] in the light of known results and conjectures to explain that, despite its quantitative appearance, they have not given a precise claim that can in principle be falsified. We also summarize our position regarding Shannon’s random cipher and the claims of [1].

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Section 2, we review the concepts of ‘unicity distance’ and average number
of spurious keys $N_k$ of a cipher and the available results on them in the
standard cryptography literature. In Section 3, we extend these results to
random ciphers like $\alpha\eta$. In Section 4, we critique the analysis of Ahn and
Birnbaum in detail. We also show that their approximate expressions can be
replaced by rigorous lower bounds (rather than approximate equalities) of
similar form in the light of our results of Section 3 and that these bounds
cannot be used to argue insecurity of any cipher. We also show that, for $\alpha\eta$, a
true unicity point is never reached for finite $n$ under known-plaintext attacks,
making it more information-theoretically secure than standard ciphers at least
for those attacks, contrary to the claim of [1]. Some concluding remarks are
given in Section 5.

1 Background and the Claim of [1]

Some specific security analyses and claims on $\alpha\eta$ have been given in [2,3,4,5]. In
particular, we have expressed [3,5] our belief that $\alpha\eta$ in its original form is not
information-theoretically secure under ciphertext-only and known-plaintext
attack for large enough $n$. Let $H(K|Y_n)$ be Eve’s key uncertainty given the
$n$-length ciphertext $Y_n$. In other words, we claimed without a proof that, even
for ciphertext-only attacks, we would have

$$\lim_{n \to \infty} H(K|Y_n) = 0.$$

We sketch a proof of this result for ciphertext-only attacks here. Since statisti-
cal and known-plaintext attacks give Eve more information, (1) should be
expected to hold for these attacks as well. An LFSR gives a periodic output
of period $2^{|K|} - 1$ bits. In consequence, observation of the heterodyne attack
output over each such running key period provides Eve with successive ob-
servations of the same key corrupted by independent noise coming from the
totally random data. At worst, Eve can make her optimum estimate of the
key in each period and take a majority vote at the end. Intuitively, her prob-
ability of success goes to unity as the number of periods goes to infinity in
the same way as the average of many measurements of some quantity with
independent noise in each measurement tends to the true value as the number
of measurements goes to infinity.

Note however that eq. (1) has no practical implication on the security of $\alpha\eta$
in real use since it is merely an asymptotic statement; see the discussion in ref.
[2]. In particular, the PRNG embedded into $\alpha\eta$ is not to be used longer than its
period $2^L$ as in the case of standard ciphers, where $L$ is the register length, thus
rendering the above insecurity argument inapplicable. For this reason, and the
that the attack just mentioned is hugely inefficient, we have not gone into a detailed proof. Ahn and Birnbaum in [1] implicitly make the same claim (1) along with a purported proof based on analogy with Shannon’s random cipher [6]. We stress that the argument just given is completely independent of that of Ahn & Birnbaum, who provide no evidence for it beyond the analogy with Shannon’s random cipher, regarding which we will outline our position below.

Against this background regarding (1), the only new claim with any quantitative justification in [1] is their approximation

\[ H_E(K) \approx L - QU \quad \text{for} \quad Q \ll n_0 = L/U, \quad (2) \]

where \( H_E(K) \) is Eve’s equivocation on the key, \( L \) the seed key length, \( Q \) the length of the data sequence, and \( U \) their upper bound on Eve’s information per data bit. By analogy with the Shannon random cipher, the authors then claim “Eve can determine \( K \) with high probability when ..”

\[ Q(U + 1) \gg L + H_E(K) \quad (3) \]

and thus “the \( \alpha \eta \) protocol is worse than the simple additive stream cipher”.

We find the extrapolation from (2) to (3) completely unwarranted and that (3) itself has no more clear meaning than (1). Firstly, since there is no commonly agreed meaning of the symbols “\( \approx \)” and “\( \ll \)”, these statements are not well-defined. As they stand, they cannot be falsified, the possibility of the latter being the hallmark of a meaningful scientific statement. More significantly for our purpose, there is no reason why (2) is a good approximation in any sense while there is reason to think that it is not, as we will show in detail in this Letter.

Ahn and Birnbaum’s argument supporting (2), which is a heuristic one and not a proof, is again based on a Shannon random cipher analogy which they suggest would be applicable if the PRNG used satisfies a certain pairwise independence condition between any two running key values.

On the one hand, we discuss in detail in Section 4 why their pairwise independence condition is unlikely to lead to an approximate satisfaction of (2) in whatever sense and degree they mean, which they have not specified.

On the other hand, we argue that an appeal by analogy to Shannon’s random cipher ensemble cannot establish insecurity of any concrete cipher. Since Shannon’s argument uses a large ensemble of ciphers, the average behavior of this ensemble cannot be expected to resemble that of a given concrete cipher. In consequence, we would not consider any security or insecurity claim that is essentially based on a Shannon random cipher analogy to be reliable. While
we have not examined the issue in detail, we believe that the agreement to this model observed by Shannon for his ‘unicity distance’ for some concrete ciphers for encrypting English is likely the result of the very special statistics of English (or any other natural language) that make any cipher encrypting English text quite weak.

However, we attempted to extract the possible meaning and identify the possible validity of the claim [2] above. It turns out that such a possible rendering of [2], similar in form but not in content, has been given before [7,8] for nonrandom ciphers without the necessity of appealing to Shannon’s random cipher assumptions. In Section 3, we extend the results of Hellman [7] and Beauchemin and Brassard [8] (HBB) to random ciphers (in the sense of Section 2 (see also [3]), not that of Shannon) like $\alpha \eta$ and analyze it along this rigorous rendering, different though it is from Ahn and Birnbaum’s claim.

2 Average number of Spurious Keys $\overline{N}_k$ and ‘Unicity distance’

The general form of a random cipher consists of an encryption map $E_k(\cdot)$ applied by the sender Alice to a plaintext $n$-sequence $X^n = X_1 \ldots X_n$ of symbols each picked from an alphabet $\mathcal{X}$ resulting in a ciphertext $n$-sequence $Y^n = Y_1 \ldots Y_n$:

$$Y^n = E_k(X^n, R^n).$$

with the ciphertext symbols belonging to a possibly different alphabet $\mathcal{Y}$. Note that the encryption map is indexed by the secret key selected randomly from a possible set of values $\mathcal{K}$ and known only to Alice and the receiver Bob and that the ciphertext is not determined by the key and plaintext alone but rather requires an additional random variable $R^n$ generated by Alice for its complete specification. The key length $|K|$ is typically of the order of a few 100 bits for standard ciphers like the Advanced Encryption Standard (AES). The ciphertext may be openly read by the eavesdropper Eve before reaching Bob, who applies a corresponding decryption map $D_k(\cdot)$ to recover the plaintext:

$$X^n = D_k(Y^n).$$

Observe that the decryption map $D_k$ must function without Bob knowing $R^n$. Further details on random ciphers may be found in [3] – we note here that a random cipher usually uses a larger ciphertext alphabet so that $\mathcal{Y} \neq \mathcal{X}$ - the former may even be continuous as it is for $\alpha \eta$ under heterodyne attack.
Fixing a particular attack on a given cryptosystem, random or otherwise, means that the eavesdropper Eve is assumed to know the joint probability distribution $\Pr[X^nY^nK]$ of the plaintext, ciphertext, and key, and is in possession of the corresponding ciphertext random variable $Y^n$. In the case of $\alpha\eta$, where information is coded into quantum states, one must additionally specify a quantum measurement whose result becomes the ciphertext $Y^n$. In the case of $\alpha\eta$ under heterodyne attack, $X = \{0, 1\}$ and $Y$ is $\mathbb{R}^2$ or $\mathbb{C}$ since the heterodyne measurement gives two real numbers. Actually, only the argument of the complex number result is useful to Eve and thus $Y$ may be taken to be the circle $S^1$. In this Letter, we will consider only information-theoretic security (IT security) and allow unlimited computational power to Eve.

In the cryptography literature, beginning with Shannon [6], the ‘unicity distance’ has been proposed as a measure of IT security of a cipher. The concept may precisely be defined as the smallest length of plaintext for which only one key value can lead to the observed ciphertext, thus marking the point where the system is totally broken. Unfortunately, for most data statistics, there is never a point where the key becomes fixed with probability one and the choice of a particular unicity point involves an implicit choice of a probability that is viewed as ‘small enough’ and must, in our opinion, be specified in any insecurity claims. In [6], Shannon estimated the unicity distance of an ensemble of ciphers satisfying certain ideal conditions that are in general not satisfied for a given cipher. Even for Shannon’s random cipher (Throughout this Letter, we will use ‘Shannon’s random cipher’ to denote the ensemble of ciphers defined in [6] and ‘random cipher’ to denote any cipher of the form of Eq. (4). The reader should keep in mind that they are completely different concepts), there is no point where the key is fixed with probability one. However, the probability that the key is erroneously determined by Eve at a designated ‘unicity point’ can be calculated for this case, as has been done by Hellman in [7] (see Theorem 1 and Corollary 1 therein). This probability calculation appears extremely difficult to do for any concrete cipher, random or otherwise.

In view of the generic non-existence of a true unicity point for a cipher, we prefer to work with a closely related concept defined by Hellman [7] for this reason and used also by Beauchemin and Brassard [8]. This is the average number of spurious keys $N_k$ seen by the attacker that we define below following [8].

Under a given attack, for each ciphertext $y$, we define the set $K_y$ as:

$$K_y = \{k \in \mathcal{K} \mid \Pr[D_k(y)] > 0\}. \quad (6)$$

Thus $K_y$ is the set of keys that could give rise to the observed ciphertext $y$. Since only one of these keys is the actual one used, the number of spurious
keys $N_k(y)$ is

$$N_k(y) = |K_y| - 1.$$  

(7)

The average number of spurious keys $\overline{N}_k$ is defined to be the expectation of $N_k(y)$ over $Y$:

$$\overline{N}_k = \sum_y \Pr[y] N_k(y)$$

(8)

Since each $N_k(y)$ is non-negative, if $\overline{N}_k = 0$, $N_k(y) = 0$ for all $y$ and the cipher is broken with probability one.

It is significant to note that a unicity distance $n_0(p)$, which gives the shortest data length $n_0$ from which Eve could determine the key with probability $p$, is a useful operational measure of security that one may try to determine numerically or bound analytically for various types of attacks. Some special $p$ cases have been obtained previously for known-plaintext quantum joint attacks [9] that yield the fundamental security limit. It is also meaningful to evaluate $n_0(p)$ under heterodyne or other attacks. Indeed, this is being pursued by different groups in Europe, Japan, and the US on $\alpha\eta$ and similar cryptosystems.

We stress here that we do not consider $\overline{N}_k$ by itself to be an operationally meaningful IT security measure, although it may well provide bounds on such a measure. Among its drawbacks are the fact that the cardinality alone of each set $K_y$ defined above gives no feel for the numerical probabilities of its elements. In addition, the operational meaning of averaging over $y$ may be questioned. As an example of an operational security measure closely related to the unicity distance $n_0(p)$, we suggest the following ‘$\Pi-$’ function defined, as a function of the data length $n$ for a given attack on a given cipher as:

$$\Pi(n) := \max_{y^n} \max_{k \in K_{y^n}} \Pr[k|y^n].$$

(9)

Thus, $\Pi(n)$ is Eve’s probability on the most likely key maximized over all possible ciphertext observations of length $n$. As such, for a chosen $\epsilon$, if it can be shown that $\Pi \leq \epsilon$ for the data length of operation, the user can be guaranteed that the system is broken with a probability not greater than $\epsilon$ no matter what observation Eve gets. In this Letter, we do not study the $\Pi$-function as a security measure since the results on $\overline{N}_k$, both those available and those proven here, are closer in spirit and content to the claims of [1] and are sufficient to point out the inadequacies in their arguments. $\overline{N}_k$ can be
estimated exactly for the Shannon random cipher and equals (see [7]):

\[ N_k = (2^{H(K)} - 1)2^{-nD} = 2^{H(K) - nD}, \]  

where \( D \) is the per symbol data redundancy in bits, i.e.,

\[ D := \log_2(|X|) - \frac{H(X^n)}{n}. \]  

Note that \( N_k \) never becomes exactly zero, so the cipher is never broken with probability one. However, Shannon took the point where \( N_k = 1 \) to be the ‘unicity distance’ \( n_0 \), so that \( n_0 = H(K)/D \) using the approximation in Eq. (10).

For the case of an arbitrary endomorphic nonrandom cipher, i.e., one for which \( \mathcal{X} = \mathcal{Y} \), the following result due to Hellman and Beuchemin and Brassard holds (see [8]):

**Theorem 1** (HBB result) For any nonrandom cipher with \( \mathcal{X} = \mathcal{Y} \),

\[ N_k \geq 2^{H(K) - nD} - 1, \]  

Note that, in contrast to Eq. (10), the RHS of Eq. (12) can reach zero. However, since Theorem 1 gives just a lower bound on \( N_k \), the vanishing of its RHS does not establish insecurity in any conceivable definition. The approximate equality of the right-hand sides of (10) and (12) led Hellman [7] to state that Shannon “random ciphers are essentially the worst possible” in the sense of having the lowest possible \( N_k \).

Under some restricted assumptions that we do not get into here, Hellman goes further and gives upper bounds on the probability that \( N_k \leq m \) for any integer \( m \). These can obviously be translated into lower bounds on the probability that \( N_k > m \). We do not give the expressions here, because the important point in our context is that, to judge the insecurity level of a cipher, we would rather be interested in upper bounds on the probability \( \Pr[N_k > m] \) which are not available in the analyses [7] and [8] or elsewhere.

In sum, the available results on \( N_k \) for nonrandom ciphers are only lower bounds. As such, they cannot in principle be used to establish insecurity of a system, but may conceivably be used in conjunction with a meaningful security measure to ensure a certain security level.
3 Lower Bound on $\overline{N}_k$ for Random Ciphers

The HBB result quoted above can be extended to include random ciphers with arbitrary ciphertext alphabet $\mathcal{Y}$, including continuous alphabet ciphers such as $\alpha\eta$. We prove the extended lower bound in this section.

We need the following lemma that is easily established from standard properties of entropy and mutual information:

**Lemma:** For any cipher with plaintext sequence $X^n = X_1 \ldots X_n$, ciphertext sequence $Y^n = Y_1 \ldots Y_n$, and key $K$, with arbitrary plaintext alphabet $\mathcal{X}$ and arbitrary ciphertext alphabet $\mathcal{Y}$, random or non-random,

$$H(K|Y^n) = H(X^n) + H(K) - I(X^nK;Y^n).$$  \hspace{1cm} (13)

**Theorem 2:** For any cipher with plaintext sequence $X^n = X_1 \ldots X_n$, ciphertext sequence $Y^n = Y_1 \ldots Y_n$, and key $K$, with arbitrary plaintext alphabet $\mathcal{X}$ and arbitrary ciphertext alphabet $\mathcal{Y}$, random or non-random,

$$\overline{N}_k \geq 2^{H(K)+n(\log_2 |\mathcal{X}| - D)} - I(X^nK;Y^n) - 1.$$ \hspace{1cm} (14)

$D$ is defined as before by Eq. (11). Theorem 1 can be recovered from (14) by observing that $I(X^nK;Y^n) \leq n \log_2 |\mathcal{Y}| = n \log_2 |\mathcal{X}|$ when $\mathcal{X} = \mathcal{Y}$.

**Proof:** We proceed as in [8]. We have

$$H(K|Y^n) = \sum_y \Pr[y]H(K|y) \leq \sum_y \Pr[y] \log_2 (N_k(y) + 1) \hspace{1cm} (15)$$

$$\leq \log_2 (\sum_y \Pr[y](N_k(y) + 1)) = \log_2 (\overline{N}_k + 1). \hspace{1cm} (16)$$

The inequality (15) follows from the definition eq. (7) of $N_k(y)$ and (16) from the concavity of the log function [10]. The result follows on substituting for $H(K|Y^n)$ using Lemma 1 and exponentiating both sides. ■

The necessary and sufficient conditions for the inequality of Theorem 2 to be satisfied with equality are: The keys in the set $K_y$ must be equiprobable for every $y$ and $|K_y|$ must be the same for all ciphertexts $y$. Intuitively, these constraints would not be satisfied for an arbitrary cipher, so the lower bound cannot be expected to be tight without a detailed analysis on the given cipher.

We have thus extended the HBB result to random ciphers and observed that it is still just a lower bound on the average number of spurious keys and cannot therefore provide a basis for an insecurity claim.
4 Application to $\alpha\eta$ and the analysis of Ahn & Birnbaum

We assume the description of the $\alpha\eta$ cryptosystem to be familiar to the reader from [1] – we use essentially the same notations here. Further details on the system may be found in [2,3,4,5].

In order to estimate the lower bound in Theorem 2, one needs to estimate $I(X^nK;Y^n)$ for the cipher being studied. For $\alpha\eta$, it is useful to define a signal random variable $S^n = S_1 \ldots S_n$ as

$$S^n = f^{(n)}(X^n, K),$$  \hspace{1cm} (17)

where $f^{(n)}$ is simply the function of the data $n$-sequence and the key that outputs the corresponding $n$-sequence of signal angles on the coherent state circle. $f^{(n)}$ depends on the particular PRNG (denoted ENC hereafter to conform with usage in earlier papers) used, but its explicit form does not concern us here. Each $S_i$ is an $M$-ary random variable. Now the ciphertext $Y^n = Y_1 \ldots Y_n$ is the $n$-sequence of continuous-variable heterodyne measurements made by Eve, and may be represented as

$$Y^n = S^n + R^n,$$  \hspace{1cm} (18)

where $R^n = R_1 \ldots R_n$, and the $\{R_i\}$ are independent identically distributed random variables having an approximately Gaussian distribution with zero mean and standard deviation $\sigma = \frac{M}{\sqrt{N}}$. $N$ being the mean photon number of each transmitted coherent state. They represent the heterodyne measurement noise of each symbol $i$. For this two-step model of generation of the ciphertext, note that, for each $i$, $(X_iK) \rightarrow S_i \rightarrow Y_i$ is a Markov chain, and hence so is $Y_i \rightarrow S_i \rightarrow (X_iK)$ and consequently, $Y^n \rightarrow S^n \rightarrow (X^nK)$. Therefore, by the data processing inequality [10], we have for all $n$,

$$I(X^nK;Y^n) \leq I(S^n;Y^n).$$  \hspace{1cm} (19)

Let us denote the running key sequence emitted by an arbitrary ENC seeded with a seed key of length $|K|$ by $K' = K'_1 \ldots K'_n \ldots$, where each $K'_i$ is of length $\log_2(M/2)$ bits – the length needed to choose a basis on the coherent state circle. It is clear that the $\{K'_i\}, 1 \leq i \leq n$ cannot be statistically independent beyond a certain $n$ if each segment of the running key has a uniform marginal distribution (as is the case for a pseudo-random number generator), since the seed key entropy is limited to $|K|$ and the running key is a deterministic function of the seed key. This fact shows that, for an arbitrary ENC, there exists a running key length $n_{dep}$ measured in running-key symbols, beyond
which \( \{K'_i\}, 1 \leq i \leq n \) are statistically dependent, and that

\[
n_{\text{dep}} \leq |K|/\log_2(M/2)
\]

(20)

for an arbitrary ENC. \( n_{\text{dep}} \) is referred to as the ‘dependency distance’. When a linear feedback shift register (LFSR) is used as an ENC, knowing any \( |K| \) consecutive bits of the output running key fixes the seed key and vice versa. Therefore, for an LFSR, \( n_{\text{dep}} = |K|/\log_2(M/2) \equiv n_{\text{dep}}(LFSR) \). Note also that if the \( \{K'_i\}, 1 \leq i \leq n \) are statistically dependent, so are the signal random variables \( \{S_i\}, 1 \leq i \leq n \).

4.1 Ciphertext-only heterodyne attack

Consider first the case of ciphertext-only heterodyne attack on \( \alpha \eta \), for which \( D = 0 \). Also the plaintext alphabet size \( |\mathcal{X}| = 2 \) for \( \alpha \eta \). Ahn and Birnbaum calculate in [1], a quantity \( U \), that is, in our notation:

\[
U = I(S_i; Y_i) \quad \forall \ i.
\]

(21)

This definition makes sense for the LFSR case (it needs a proof in the general case) because the \( \{S_i\} \) do indeed have the same (in fact, uniform) marginal distributions for each \( i \) when the plaintext is uniformly random. It is also true that \( I(S^n; Y^n) = nU \) for all \( n \leq n_{\text{dep}} \) because, for such data lengths, the \( i \)-th signal symbol in the \( n \)-sequence is statistically independent of the rest as mentioned above. However, this information estimate that is linear in \( n \) is not valid beyond the dependency distance because the running key has correlations beyond \( n_{\text{dep}} \). The argument in [1] that the “pseudo-random number generator redistributes Eve’s prior probabilities back to the flat distribution for each new symbol” merely makes \( U \) of Eq. (21) well-defined but does not justify the above estimate. It is the joint probability distribution of the \( \{S_i\} \) that goes into the calculation of \( I(S^n; Y^n) \) and not the marginal per symbol probability distribution. In fact, it follows from Theorem 4.2.1 of [10] that

\[
I(S^n; Y^n) < nU \quad \forall n > n_{\text{dep}},
\]

(22)

and the inequality is strict because the \( \{S_i\} \) are not statistically independent. Even if \( I(X^nK; Y^n) \) is taken to be equal to \( I(S^n; Y^n) \) (see (19)), the claim in [1] that the former quantity increases linearly up to \( n_0 = |K|/U \) cannot be true. Note that \( U \approx \frac{1}{2} \log_2 N + 1.6 \ll \log_2 M \) in the regime \( \sigma = M/(2\sqrt{N}) \gg 1 \) assumed in the calculation of [1] and thus \( n_0 \gg n_{\text{dep}}(LFSR) \), and thus we are already well into the region where (22) is a strict inequality. This argument is
unchanged for a general ENC by virtue of the inequality (20) – the running-key dependency sets in not later than it does for the LFSR case.

The key argument which to them would make Shannon’s random cipher analysis in the form of Eq. (2) (or equivalently, Eq. (22) in the form of an equality) applicable is that for two different running key segments \( k_s \) and \( k_q \) at the output of the PRNG “.. values of \( K \) which have similar values of \( k_q \) will have uncorrelated values of \( k_s \) for \( s \neq q \).” From the Theorem just cited, (22) is an equality if and only if the \( \{S_i\} \) from 1 to \( n \) are jointly statistically independent, a condition that cannot be satisfied for \( n > n_{\text{dep}} \). Their condition quoted above seems to be the strictly weaker one that the key segments need only be pairwise statistically independent. In view of the well-known difference between pairwise and complete statistical independence of a sequence of random variables, we feel justified in demanding a rigorous proof of how Eq. (22) may be “approximately” true under their weaker assumption. We have assumed that by the word ‘uncorrelated’ in the quotation above, Ahn and Birnbaum mean ‘statistically independent’ although this is not clear from [1]. Whatever their meaning of the term, we urge them to prove how and to what degree it leads to an ‘approximate’ satisfaction of the ‘only if’ condition of Gallager’s theorem that renders (22) an equality.

Therefore, the only conclusion on \( I(X^n K; Y^n) \) derivable from the analysis of [1] is that

\[
I(X^n K; Y^n) \leq nU \quad \forall \ n. \tag{23}
\]

Using this in conjunction with Theorem 2 yields the following lower bound on \( N_k \):

\[
N_k \geq 2^{H(K)+n(1-U)} - 1. \tag{24}
\]

If we choose to find the data length \( n_{\text{unicity}}' \) at which the lower bound reads \( N_k \geq 0 \), we find

\[
n_{\text{unicity}}' = H(K)/(U - 1), \tag{25}
\]

which is claimed in [1] to be the ‘unicity distance’ of \( \alpha \eta \), beyond which “Eve’s entropy on the key will transition from linear decline to asymptotic decay by analogy to the unicity distance of a classical deterministic cipher...” It is also claimed that “Eve may have enough information to determine the key with high probability when \( n \gg n_{\text{unicity}}' \).”

There are several things amiss with such claims. The fact that the linear decline of Eve’s entropy on the key has already ended at \( n_{\text{dep}} \) has been noted.
In addition, the analogy with Shannon’s random cipher does not exist. As stressed in Sections 2 and 3, for concrete ciphers, the only available results are lower bounds on $N_k$ against which the analysis of [1] is no exception. As a matter of principle, a lower bound on $N_k$ cannot prove insecurity of a cipher. If Ahn and Birnbaum wish to claim that $N_k$ is indeed close to zero at $n$ ‘unicity’, they must show both the reasons why the bound of Theorem 2 is tight for $\alpha \eta$ and also why $I(X^nK; Y^n) \approx nU$ is a good approximation for $\alpha \eta$ beyond $n = n_{dep}$. Also, if $N_k$ is not claimed to be exactly zero (so the key is not determined with probability one – it is shown in Section 4.2 below that $N_k$ for $\alpha \eta$ is never exactly zero for any finite data length $n$ under known-plaintext heterodyne attack and consequently also for the weaker ciphertext-only attack) – Ahn and Birnbaum need to estimate the probability with which Eve obtains the key correctly. As per the discussion of Sections 2 and 3, this probability can be determined for Shannon’s random cipher but has never been done for any standard cipher, let alone $\alpha \eta$. This fact does not make all previous work in cryptography meaningless because the bulk of it is concerned with complexity-based security under specific attacks and not information-theoretic security which is under consideration here. Without such a calculation, a statement like “Eve may have enough information to determine the key with high probability when $n \gg n$ ‘unicity’.” is unfalsifiable – it does not satisfy the requirement of being a scientific claim over and above [1] without quantifying both how high the probability is and how much greater than $n$ ‘unicity’ $n$ needs to be.

4.2 Statistical and Known-Plaintext Attacks

For general statistical attacks, i.e., those for which $H(X^n) < n$, Ahn and Birnbaum claim that a simple additive stream cipher (ASC) is broken “with high probability” when

$$n - H(X^n) \gg |K|,$$

and, by comparison, $\alpha \eta$ is broken “with high probability” when

$$n(U + 1) - H(X^n) \gg |K|.$$

These assertions are again justified by analogy to Shannon’s random cipher analysis, and are interpreted as implying that $\alpha \eta$ is broken at smaller data lengths than the ASC because of the added factor of $(U + 1)$ in equation (27).

As before, since the terms “high probability” and “$\gg$” have no precise meaning, these claims are unfalsifiable until they are quantified. As with standard ciphers under many statistical attacks, by choosing $n$ large enough, we can
drive the probability of finding the key as close to 1 as desired. This is the content of eq. (1), but the equations above ostensibly claim more than that. We contend that what they claim is not well-defined without quantitative meaning given to “high probability” and “$\gg$”.

Rigorous bounds, different from (26) and (27) although similar in form, can be obtained from an application of our Theorem 2. For the ASC, we have trivially that $I(X^n K; Y^n) \leq H(Y^n) \leq n$. Substituting this into the RHS of Theorem 2 gives the lower bound

$$N_k \geq 2^{H(K) - nD - 1},$$

(28)

which is just the HBB result. As we did for ciphertext-only attacks, setting the lower bound to zero gives the condition (compare (26))

$$n - H(X^n) \geq |K|$$

(29)

that must be satisfied if $N_k = 0$. As such, this is simply a necessary condition for $N_k = 0$ and does not imply the latter.

For $\alpha\eta$, using Eq. (22) in Theorem 2 and rewriting $D$ in terms of $H(X^n)$ gives the lower bound

$$N_k \geq 2^{H(K) + H(X^n) - nU - 1}.$$ 

(30)

Setting the RHS to zero, gives the necessary condition (compare (27))

$$nU - H(X^n) \geq |K|$$

(31)

for $N_k = 0$. It is not a sufficient condition for the latter, which, as we show below, is never true except at $n = \infty$ even for known-plaintext attacks. As is the case for all applications of Theorem 2, there is no proof that $N_k$ approximately equals the RHS of Eq. (30) which would be needed to make insecurity claims on its basis. Again, it is essential to provide estimates of the probability that the key is found correctly by Eve to prove insecurity.

Indeed, there is no evidence that (25) is valid as an approximate estimate of ‘unicity distance’. The numerical result quoted in [1] for the simulation of [11] yields a ‘unicity distance’ too small by a factor $\sim 300$, which shows $U \sim 1$ when (25) is used instead of $U \sim 300$. While such comparison has little meaning when the attack success probability is not specified, it surely is unreasonable to claim, as in [1], that such a large discrepancy exists because of the suboptimal processing used in [11].
Intuitively, the measurement noise in $\alpha \eta$ would make it more secure than an additive stream cipher instead of worse as claimed in [1] at least for the case of known-plaintext attacks where $H(X^n) = 0$. In this case, an ASC is broken with probability 1 at the nondegeneracy distance $n_d$ defined in [2], which is just $n_d = |K|$ for an LFSR. On the other hand, it is clearly not possible to pin down the seed key at this $n$ with probability 1 in the case of $\alpha \eta$. As a matter of fact, the true unicity point of $\alpha \eta$ using any ENC, i.e., the point where the key is determined with probability one, is infinite under even known-plaintext attacks. To see this, note that, irrespective of what ENC is, in the more exact continuous Gaussian-noise model of the noise $R_i$ used in [1] (as opposed to the wedge approximation used in [3]), there is always a non-zero probability, however small, that a $Y^n$ that is close to any given $n$-sequence of signal points on the coherent state circle may arise from any data sequence $X^n$ and any running key and thus seed key $K$. Furthermore, a large fraction (in terms of probability) of such events for Eve occur without giving rise to any detection error for Bob. In particular, the close approximation to $R_i$ consisting of a continuous probability distribution cut off at 90° on each side of the signal point $S_i$ would give zero error for Bob and infinite unicity distance because every allowed basis $n$-sequence is still possible given the ciphertext, albeit some are highly unlikely. The above argument shows that the true unicity point is not reached for any finite $n$. Together with the fact that $\lim_{n \to \infty} N_k = 0$ proven in Section 1 we have that the unicity distance is infinite. This fact that $N_k \neq 0$ for any specified finite distance underscores the necessity of providing probability estimates to any claims that the system is broken at that distance. These estimates are not provided in [1] and seem thus far difficult to obtain, although some progress is currently being made by various research groups.

5 Conclusion

In conclusion, we have shown, both by arguing the non-existence of an analogy to Shannon’s random cipher and by a direct analysis of their final claim (2), that the arguments of Ahn and Birnbaum do not establish the insecurity of $\alpha \eta$. Rigorously true results similar in form to their expressions are derived as corollaries of the lower bound on $N_k$ (Theorem 2). It is noted that these results, being lower bounds, cannot in principle establish insecurity of any system. We also noted the lack of any estimates by Ahn and Birnbaum of the probability that $\alpha \eta$ is broken at the claimed distance (which they also did not clearly delimit) to be a serious loophole insofar as it makes their claim of insecurity unfalsifiable.

There are other points mentioned in [1] that we disagree with but cannot get into in any detail here. One concerns the comparison of $\alpha \eta$ with DSR to an ASC, and another about the existence of a proven secure concrete BB84
cryptosystem. While the work in [1] does not throw light on the true security level of $\alpha\eta$, further efforts in this direction are possible and welcome.

6 Added Comment

In their response [12] to our Comment, Ahn and Birnbaum abandon the Shannon random cipher analogy argument of their original paper [1] and repeat their other qualitative argument that PRNG outputs "will mimic those of a true random number generator". If that is the case, there is no need for all the work on cryptographic encryption. Given the previously known condition (1) of our Comment, the problem here is quantitative. For example, for a fixed data bit length $n$ equal to the seedkey length under known-plaintext attack, a conventional cipher is broken with probability 1. If the bare $\alpha\eta$ is broken with probability $\sim 10^{-4}$ irrespective of complexity, it is already a significant improvement. Since we have given intuitive as well as rigorous arguments on why these authors’ main claim, our (2), is merely a lower bound that can yield no insecurity conclusion, its validity can only be established by rigorous quantitative reasoning the authors have not provided.

They also give a simulation example for seedkey size = 13 with other fixed system parameters. It is not spelled out exactly how the reported simulation was carried out. In particular, we cannot assess whether or not the method of updating the eavesdropper’s probabilities in the simulation uses any of the assumptions the simulation purports to validate. We do not comment on it not only for this reason but also because their original paper used only theoretical arguments to support their conclusion. These arguments remain the same, and our theoretical refutation still stands. The importance of addressing our theoretical refutation lies in the fact that a single numerical example cannot validate a general quantitative conclusion. It would indeed be interesting if a complete numerical study can be carried out for realistic key sizes to show the dependence of the results on the system parameters. However, such study appears exceedingly difficult due to the complexity involved.

The authors do not dispute the unfalsifiability of their claim given by our (2) in the absence of meaning given to "$\approx$" and "$\ll$". Regarding their claim given by (3) in our Letter, insofar as it is meant to say something over and above our (1), the alleged counter-example in their Reply does not make (3) falsifiable because the example does not satisfy (1). The issue could be easily resolved if these authors just define their symbols and give the success probability estimate, which they still have not done. There is no analogy between their claim and the examples they give, their (7) with $\tan x \approx x$ and their (8). The elementary point to be made here is that the approximation error in those cases can be readily estimated rigorously on demand. In contrast, their main
result given by our (2) is merely a lower bound on $H_E(K)$ – the crux of the matter is that the gap between $H_E(K)$ and the right-hand side is unknown.

We cannot go into here the other side issues raised by these authors in their Reply. We may just mention that the rigorous examination of unicity distance for given success probability under a given attack is possible and being pursued by us and other groups.

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