Modular forms and K3 surfaces

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Abstract

For every known Hecke eigenform of weight 3 with rational eigenvalues we exhibit a K3 surface over \( \mathbb{Q} \) associated to the form. This answers a question asked independently by Mazur and van Straten. The proof builds on a classification of CM forms by the second author.

Keywords: Singular K3 surface, modular form, complex multiplication

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1 Introduction

The question of modularity for algebraic varieties over \( \mathbb{Q} \) has been studied in great detail in recent years. Historically, it began with an observation by Eichler who noticed the modularity of an elliptic curve over \( \mathbb{Q} \) of conductor 11. Shimura then proved that every Hecke eigenform of weight 2 is associated to an abelian variety over \( \mathbb{Q} \). In the case of rational eigenvalues, the corresponding variety is an elliptic curve.

Conversely, the Taniyama–Shimura–Weil conjecture stated that every elliptic curve over \( \mathbb{Q} \) is modular. The celebrated proof of this conjecture by Wiles et al. [35, 33, 4] not only implies Fermat’s Last Theorem, but also catalyzed many further developments in this area, notably the recent great progress on Serre’s conjecture.

By now, we know modularity for several classes of varieties. For instance, Dieulefait and Manoharmayum [7] prove modularity of rigid Calabi–Yau threefolds over \( \mathbb{Q} \) under a mild condition. There is also work by Livné [10] on modularity for two-dimensional motives with complex multiplication (CM) that we use here, citing it as Theorem 3.

On the other hand, the problem of geometric realizations is harder for Hecke eigenforms of weight greater than two. Deligne [6] gives a geometric construction of \( \ell \)-adic Galois representations for Hecke eigenforms. However, the varieties involved vary greatly with the level. In this sense, his construction is not as uniform as one might wish (cf. Remark 5).

This paper gives the first case of higher weight where we can realize all known Hecke eigenforms with rational eigenvalues in a single class of varieties:

**Theorem 1**

Assume the extended Riemann Hypothesis (ERH) for odd real Dirichlet characters. Then every Hecke eigenform of weight 3 with rational eigenvalues is associated to a K3 surface over \( \mathbb{Q} \).
This result answers a question asked independently by Mazur and van Straten. It builds on the classification of CM forms with rational coefficients by the second author which we recall in section 3. That section also explains the dependence of Theorem 1 on the ERH. Section 2 recalls the notion of singular K3 surfaces and Livné’s modularity result. We review the relevant known examples and obstructions in sections 4 and 5. Our main technique for proving Theorem 1 is constructing one-dimensional families of K3 surfaces and searching for singular specializations over \( \mathbb{Q} \). This is explained in section 6 and exhibited in detail for one particular family in section 7. The paper concludes with the remaining surfaces needed to prove Theorem 1.

2 Singular K3 surfaces

A K3 surface is a smooth, projective, simply connected surface \( X \) with trivial canonical bundle \( \omega_X = O_X \). The most prominent examples are smooth quartics in \( \mathbb{P}^3 \) and Kummer surfaces. Later we will work with elliptic K3 surfaces.

Throughout this paper, modularity will refer to classical modular forms (cf. sect. 3). This classical kind of modularity is a very special property of a variety; a general K3 surface over \( \mathbb{Q} \) cannot be modular for several reasons (cf. the discussion before Theorem 3), though the Langlands Program predicts a correspondence with some automorphic forms.

K3 surfaces and their moduli have been studied in great detail. We will come back to these questions in section 6. The only complex K3 surfaces that can be classically modular are those that have no moduli at all. In terms of the Picard number \( \rho(X) = \text{rk} \text{NS}(X) \), the condition that \( X \) have no moduli is that

\[
\rho(X) = 20,
\]

the maximum in characteristic zero. K3 surfaces with Picard number 20 are often referred to as singular K3 surfaces. The terminology is reflected in the Shioda–Inose structure (cf. sect. 4) which relates any singular K3 surface to a product of two isogenous elliptic curves with complex multiplication (CM), thus with singular moduli.

Our results will often be stated in terms of the discriminant \( d = d(X) \) of a singular K3 surface \( X \), i.e. the discriminant of the intersection form on the Néron–Severi lattice, which is the Néron–Severi group endowed with the cup-product pairing:

\[
d = d(X) = \text{disc(NS}(X)).
\]

Example 2

The Fermat quartic in \( \mathbb{P}^3 \),

\[
S = \{(x_0 : x_1 : x_2 : x_3) \in \mathbb{P}^3 \mid x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0\}
\]
is a singular K3 surface. There are 48 obvious lines on \( S \), all defined over the 8th cyclotomic field \( \mathbb{Q}(e^{2\pi i/8}) \). Pjateckiĭ-Šapiro and Šafarevič [18] proved that the lines generate \( \text{NS}(S) \), with \( \text{disc}(\text{NS}(S)) = d = -64 \). However, the proof depended on a claim by Demjanenko whose argument turned out to be incorrect. The proof was later independently completed by Cassels [5] and Inose [13].

For any smooth projective surface \( X \) over \( \mathbb{C} \), we define the transcendental lattice \( T(X) \) as the following sublattice of \( H^2(X, \mathbb{Z}) \):

\[
T(X) = \text{NS}(X)^\perp \subset H^2(X, \mathbb{Z}).
\]
When $X$ is a K3 surface, $H^2(X, \mathbb{Z})$ has rank 22 and signature $(3, 19)$. Hence $T(X)$ has rank $22 - \rho(X)$ and signature $(2, 20 - \rho(X))$.

If $X$ is a singular K3 surface, then $T(X)$ has rank 2, and is even, positive-definite and equipped with an orientation $\mathbb{Z}$. Using the intersection form, we will identify the transcendental lattice with a $2 \times 2$ matrix

$$T(X) \leftrightarrow \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}$$

with integral coefficients and discriminant $d = b^2 - 4ac < 0$. Applying an $\text{SL}_2(\mathbb{Z})$ change of basis, we will always use a reduced representative with $-a < b \leq a \leq c$. We will also use the shorthand $[2a, b, 2c]$ for $T(X)$.

In consequence, if the singular K3 surface $X$ is defined over some number field $L$, then $T(X)$ gives rise to a two-dimensional Galois representation $\rho$ over $L$. Over some extension of the ground field, Shioda and Inose associated a Hecke character to this Galois representation $\mathbb{Z}$. Livné then proved modularity over $\mathbb{Q}$ as an application of a more general result concerning motives:

**Theorem 3 (Livné [16])**

*Every singular K3 surface over $\mathbb{Q}$ is modular: there is a Hecke eigenform with system of eigenvalues $\{c_p\}$, such that*

$$\text{trace } \rho(\text{Frob}_p) = c_p \text{ for almost all } p.$$  

The corresponding Hecke eigenform $f$ has weight 3 and CM by the imaginary quadratic field $K = \mathbb{Q}(\sqrt{d})$, where $d = \text{disc}(\text{NS}(X)) < 0$ is the discriminant of $X$. The Hecke eigenvalues of $f$ are all rational integers.

For instance, the Hecke eigenform corresponding to the Fermat quartic has CM by $\mathbb{Q}(\sqrt{-1})$. For the model $S$ in Example 2, the newform has level 16, as given in [22 Table 1].

### 3 CM newforms

Modular forms for congruence subgroups come with weight, level and nebentypus character that appear in the transformation law. For fixed invariants, the modular forms constitute a module over the Hecke algebra. On cusp forms, the Hecke operators can be diagonalized simultaneously. The set of eigenvalues as in Theorem 3 determines a unique primitive normalized eigenform, a so-called newform.

By the transformation law, a modular form $f$ always has a Fourier expansion

$$f = f(\tau) = \sum_{n \geq 0} a_n q^n, \quad q = e^{2\pi i \tau}.$$  

Here a newform is characterized by the property that the Mellin transform

$$L(f, s) = \sum_{n \geq 0} a_n n^{-s}$$

has an Euler product. In particular, the Fourier coefficients $a_p$ are multiplicative and equal the Hecke eigenvalues $c_p$. Throughout this paper, we will use the terms Fourier coefficients and Hecke eigenvalues for a newform interchangeably.
We are interested in newforms of weight 3 with rational eigenvalues. Generally, odd weight is special in the sense that the nebentypus character is necessarily nontrivial. In fact, by a result of Ribet [19], a newform $f$ of odd weight with real Hecke eigenvalues has CM by its nebentypus character. In particular, $f$ comes from a Hecke character.

CM newforms with rational coefficients have been classified by the second author in [22]. The analysis of the associated Hecke characters revealed the following structure:

**Theorem 4 (Sch"{u}tt [22])**

For fixed weight $k + 1$, there is a bijective correspondence

$$
\begin{align*}
\text{CM newforms of weight } k + 1 \\
\text{with rational eigenvalues} \\
\text{up to twisting}
\end{align*}
\xrightarrow{1:1}
\begin{align*}
\text{Imaginary-quadratic fields } K \\
\text{with group } Cl(K) \subseteq (\mathbb{Z}/k\mathbb{Z})^g \\
\text{for some } g \in \mathbb{N}
\end{align*}
$$

Unless $K = \mathbb{Q}(\sqrt{-1})$ or $\mathbb{Q}(\sqrt{-3})$, twisting refers to modifying the Fourier coefficients by a quadratic Dirichlet character $\chi$ (since otherwise the rationality of Fourier coefficients is not preserved):

$$
\tag{1} f \otimes \chi = \sum_{n \geq 1} a_n \chi(n) q^n.
$$

Any twist of a newform is again a newform, although the level and the nebentypus character will differ in general. If $K = \mathbb{Q}(\sqrt{-1})$ or $\mathbb{Q}(\sqrt{-3})$, then we can also twist the associated Hecke character by a biquadratic resp. cubic character. For geometric equivalents of twisting, see [3] and the following explanations.

For fixed weight, we will refer to the Hecke eigenforms from Theorem 4 as newforms of class number $h(K)$. In this terminology, it should be understood that the newforms have rational Hecke eigenvalues.

Conjecturally, the number of imaginary quadratic fields on the right hand side in Theorem 4 is always finite. The case of $k = 1$ was proven by Heilbronn and made explicit by Heilbronn and Linfoot [13]. The exponents $k = 2, 3$ are due to Weinberger [33] (the latter also proven by Boyd and Kisilevsky [3]). Recently, Heath-Brown proved finiteness for $k = 5, 2^a$, or $3 \cdot 2^a$ [12].

Most of these results are not effective. However, for $k = 2$ (and $k = 1$), our knowledge goes much further. There are 65 imaginary quadratic fields whose class groups are at most two-torsion. Their class numbers go up to 16 for $K = \mathbb{Q}(\sqrt{-1365})$. Many of these fields were already known to Euler through his search for idoneal numbers. We list them by discriminants $d_K$ and class numbers $h(d_K)$:

| $h(d_K)$ | $d_K$ |
|----------|-------|
| 1        | $-3, -4, -7, -8, -11, -19, -43, -67, -163$ |
| 2        | $-15, -20, -24, -35, -40, -51, -52, -88, -91, -115, -123, -148, -187, -232, -267, -403, -427$ |
| 4        | $-84, -120, -132, -168, -195, -228, -280, -312, -340, -372, -408, -435, -483, -520, -532, -555, -595, -627, -708, -715, -760, -795, -1012, -1435$ |
| 8        | $-420, -660, -840, -1092, -1155, -1320, -1380, -1428, -1540, -1848, -1995, -3003, -3315$ |
| 16       | $-5460$ |
In [34], Weinberger proved that there is at most one further imaginary quadratic field with class group exponent 2. Assuming the absence of Siegel–Landau zeros for the L-series of odd real Dirichlet characters, the known list is complete. This condition would be a consequence of the extended Riemann hypothesis (ERH) for odd real Dirichlet characters. Thus the (stronger) assumption in Theorem 1.

This paper solves the geometric realization problem in weight 3. By Theorem 1 our primary task is to find a singular K3 surface over \( \mathbb{Q} \) for each of the 65 newforms (up to twisting) corresponding to imaginary quadratic fields with class group exponent 2. As we will work with elliptic surfaces, we need not worry about twisting (cf. the discussion around (1) and (3)).

**Remark 5 (Elliptic modular surfaces)**

There is a canonical way to realize newforms of weight 3, via the elliptic modular surface (or universal elliptic curve) for the corresponding congruence subgroup \( \Gamma_1(N) \) [28]. However, this correspondence is not uniform in several respects. First, the complex surfaces involved vary greatly with \( N \). Even if one considers a newform and a twist as in (1), the corresponding surfaces will in general not be \( \mathbb{Q} \)-isomorphic. Moreover, the surfaces will in general have more than one associated newform (i.e. \( p_g = h^{2,0} > 1 \)).

There are two classes of surfaces which might come to mind first when thinking of modular forms: abelian surfaces and K3 surfaces — both generalizations of elliptic curves to dimension two. They will be discussed in the next section.

### 4 Singular abelian surfaces and Kummer surfaces

Shioda and Inose derived a canonical way to produce a singular K3 surface of given isomorphism class (i.e. of given transcendental lattice) [30]. The main object in their construction is the Kummer surface \( \text{Km}(A) \) of the abelian surface \( A \) with the given transcendental lattice. Hence we shall briefly discuss singular abelian surfaces over \( \mathbb{Q} \).

A complex abelian surface \( A \) has \( H^2(A, \mathbb{Z}) \) of rank 6. The surface is called singular if \( \rho(A) = 4 \), that is, if its transcendental lattice \( T(A) \) has rank 2. By a result of Shioda and Mitani [31], every singular abelian surface is isomorphic to a product of two isogenous elliptic curves \( E, E' \) with CM. These can be given explicitly in terms of the transcendental lattice \( T(A) \).

If \( E \) and \( E' \) are defined over \( \mathbb{Q} \), we deduce that the product \( A = E \times E' \) realizes a newform of weight 3 with CM by the corresponding imaginary quadratic field of class number 1.

For singular abelian surfaces over \( \mathbb{Q} \), there is one further possible construction: If \( K \) has class number 2, consider an elliptic curve \( E \) with CM by the ring of integers \( \mathcal{O}_K \) in \( K \). Then \( E \) is defined over a quadratic extension of \( \mathbb{Q} \). Let \( A \) be the Weil restriction of \( E \) to \( \mathbb{Q} \). By [31], \( A \) is a singular abelian surface with transcendental lattice corresponding to the non-principal class in \( Cl(K) \). Since \( E \) is associated to some Hecke character, \( A \) geometrically realizes a newform of weight 3 with CM by \( K \).

It follows that we can realize all newforms of weight 3 and class number 1 or 2 in singular abelian surfaces over \( \mathbb{Q} \). We cannot pursue this approach any further because there is an obstruction coming from the cohomological structure of singular abelian surfaces:
Lemma 6

Let $A$ be a singular abelian surface of discriminant $d$. Assume that $A$ is defined over a number field $L$. Let $H(d)$ denote the ring class field. Then

$$[L.H(d) : L] | 4.$$ 

Proof: Consider the compatible system of $\lambda$-adic Galois representations on $H^1(A)$. By [31], $A$ is isomorphic over some extension of $L$ to the product of two isogenous CM elliptic curves. The eigenvalues of Frobenius on $H^1(A)$ therefore lie in the ring class field $H(d)$. By class field theory, the extension $(H(d).L)/L$ has degree dividing the dimension of the Galois representation on $H^1(A)$, which is 4. 

By the lemma, we can only define singular abelian surfaces of class number 1 and 2 over $\mathbb{Q}$. In order to realize the newforms of weight 3 and greater class number geometrically, we therefore turn to singular K3 surfaces. This has two advantages: On the one hand, the above constructions for class number 1 and 2 carry over to the corresponding Kummer surfaces (cf. Lemma 7). On the other, the descent obstructions are much milder for singular K3 surfaces. This will be studied in the next section.

We now turn to Kummer surfaces. Here we consider the quotient of an abelian surface $A$ by its involution $-1$. The minimal resolution of the resulting 16 singularities of type $A_1$ is a K3 surface. One multiplies the intersection form on $T(A)$ by 2 to obtain $T(Km(A)) = T(A)[2]$. For example, the Fermat surface (Example 2) is $\mathbb{Q}$-isomorphic to the Kummer surface of the product of elliptic curves with CM by $\mathbb{Z}[\sqrt{-1}]$ and $\mathbb{Z}[2\sqrt{-1}]$.

Lemma 7

Any newform of weight 3 and class number 1 or 2 is associated to a Kummer surface over $\mathbb{Q}$.

Proof: The Kummer quotient is defined over the ground field of the abelian surface (cf. [2]). For each imaginary quadratic field $K$ of class number 1 or 2, we have found a singular abelian surface over $\mathbb{Q}$. Therefore we obtain a Kummer surface over $\mathbb{Q}$ for some newform with CM by $K$. We now address the issue of twisting.

We will use elliptic fibrations. Let $E, E'$ be elliptic curves. Denote the $j$-invariants by $j, j'$. Inose exhibited an explicit elliptic fibration on $Km(E \times E')$ in [15]. By [22, proof of Prop. 4.1], this elliptic fibration can be defined over $L = \mathbb{Q}(j + j', j \cdot j')$:

$$Km(E \times E') : \quad y^2 = x^3 + B(t)x + C(t), \quad B, C \in L[t].$$ (2)

In the present situation, the abelian surface $A$ that we start with is isomorphic to the product of two isogenous CM elliptic curves of class number 1 or 2. Then $L = \mathbb{Q}$ because $j, j'$ are equal for class number 1 and quadratic conjugate for class number 2. Let $f$ denote the associated newform. Then the $\mathbb{Q}(\sqrt{7})$-isomorphic fibration

$$Km(A) : \quad dy^2 = x^3 + B(t)x + C(t), \quad d \in \mathbb{Q}^*$$ (3)

realizes the twist of $f$ by the quadratic Dirichlet character $\left(\frac{d}{4}\right)$ as in [11]. This leaves cubic and biquadratic twists in case of $K = \mathbb{Q}(\sqrt{-3}), \mathbb{Q}(\sqrt{-1})$. Here we use the fact that for the elliptic curves with $j$-invariant, $j = 0$ resp. 1728, the Kummer surface admits an automorphism of order 6 resp. 4. In fact, in the first case, Inose’s elliptic fibration [2] is isotrivial ($B \equiv 0$). Hence every fiber admits an automorphism of order 6, so we can apply cubic twists fiberwise. In the second case we could also argue directly with the Fermat quartic (Example 2), since it is Kummer. 

\qed
By Lemma 7, we are left to find singular K3 surfaces over \( \mathbb{Q} \) for all newforms of weight 3 with class number 4, 8 or 16. Up to twisting, there are 38 such forms.

## 5 Obstructions for singular K3 surfaces

Before continuing our search for singular K3 surfaces over \( \mathbb{Q} \), we discuss obstructions to the field of definition. These are much milder than for singular abelian surfaces (cf. Lemma 6), notably because \( H^1(X) \) is trivial for a K3 surface \( X \). Moreover, not every singular K3 surface is Kummer although the relation with Kummer surfaces is very close.

Shioda and Inose showed in [30] how to produce a singular K3 surface with given transcendental lattice. Every singular K3 surface \( X \) admits a Nikulin involution such that the quotient is Kummer. The resulting picture is often referred to as Shioda–Inose structure. We sketch it in the following figure. Here \( A \) and \( X \) are chosen with the same transcendental lattice \( T(X) = T(A) \), and \( T(\text{Km}(A)) = T(A)[2] \).

\[
\begin{array}{c}
A \\
\downarrow
\end{array}
\xrightarrow{T(X)}
\begin{array}{c}
X \\
\downarrow
\end{array}
\xrightarrow{T(\text{Km}(A))}
\text{Km}(A)
\]

The Shioda–Inose construction is exhibited over some finite extension of the field of definition of \( A \): the elliptic fibration [2] on the Kummer surface is a base change from \( X \). Hence the question is when we can descend \( X \) to \( \mathbb{Q} \). We now discuss the known obstructions.

The first obstruction comes from lattice theory. It involves the genus of a lattice and was first studied by Shimada [26] in the case of fundamental discriminant. The second author then proved the general case in [24].

**Theorem 8 (Schütt, Shimada)**

Let \( X \) be a singular K3 surface over some number field. Let \( d \) denote the discriminant of \( X \) and \( K = \mathbb{Q}(\sqrt{d}) \). Then

\[
\{T(X^\sigma) \mid \sigma \in \text{Aut}(\mathbb{C}/K)\} = \text{genus of } T(X).
\]

If \( X \) is defined over \( \mathbb{Q} \), the genus of \( T(X) \) consists of a single class. This implies that \( \text{Cl}(K) \) is at most two-torsion. Independently this follows from Theorems 3 and 4.

The second obstruction is related to class field theory. It essentially says that even if a singular K3 surface descends to some number field \( L \), it still carries the structure of the ring class field \( H(d) \) through the Galois action on the Néron–Severi group. This property was first noted by the first author in [8]. The second author gave an alternative proof in [24].

**Theorem 9 (Elkies)**

Let \( X \) be a singular K3 surface of discriminant \( d \). Let \( H(d) \) be the ring class field for \( d \). Let \( L \) be a number field such that \( \text{NS}(X) \) is generated by divisors over \( L \). Then \( H(d) \subseteq L(\sqrt{d}) \).

In consequence, as \( h(d) \) increases, singular K3 surfaces become more and more complicated. In particular the number of singular K3 surfaces over \( \mathbb{Q} \) up to \( \mathbb{C} \)-isomorphism
is finite. This result is originally due to Šafarevič [20]. Hence, in our attempt to find singular K3 surfaces for all newforms of weight three with rational Fourier coefficients, we are searching within a finite set.

We now resume our search for singular K3 surfaces over \( \mathbb{Q} \) for the newforms of weight 3 and class numbers 4, 8 and 16. Here we mention another class of singular K3 surfaces which has been investigated before: extremal elliptic K3 surfaces. These are singular K3 surfaces admitting an elliptic fibration with finite group of sections. Up to the torsion sections, such surfaces are determined by the configuration of singular fibers. Extremal elliptic K3 fibrations were classified by Shimada and Zhang in [27]. They are finite in number.

Many explicit defining equations have been obtained by the second author in [21] and by Beukers and Montanus in [2]. In addition to previous newforms of class number 1 and 2, they realize ten discriminants of class number 4 and 8. The next table lists the discriminants and one possible configuration of singular fibers and Mordell–Weil group for each newform. In each case the surface is semistable, that is, all reducible fibers are of type \( I_n \) for some \( n \geq 1 \); so we simplify the notation by listing only the indices \( n \).

For completeness, we also give the transcendental lattice \( T_X \) in the shorthand notation \([2a, b, 2c]\) for the intersection form.

| discriminant | configuration | MW        | \( T_X \) |
|--------------|---------------|-----------|-----------|
| -84          | \([1, 2, 2, 3, 14]\) | \( \mathbb{Z}/2\mathbb{Z} \) | \([2, 0, 42]\) |
| -120         | \([1, 2, 2, 5, 12]\) | \( \mathbb{Z}/2\mathbb{Z} \) | \([6, 0, 20]\) |
| -4 \cdot 132 | \([1, 1, 3, 4, 11]\) | (0)       | \([24, 12, 28]\) |
| -168         | \([1, 1, 3, 4, 14]\) | (0)       | \([4, 0, 42]\) |
| -195         | \([1, 1, 3, 5, 13]\) | (0)       | \([6, 3, 34]\) |
| -280         | \([1, 1, 4, 7, 10]\) | (0)       | \([2, 0, 140]\) |
| -312         | \([1, 1, 3, 4, 13]\) | (0)       | \([6, 0, 52]\) |
| -4 \cdot 420 | \([1, 3, 4, 5, 7]\) | (0)       | \([24, 12, 76]\) |
| -660         | \([1, 2, 3, 5, 11]\) | (0)       | \([2, 0, 330]\) |
| -840         | \([1, 1, 4, 5, 6, 7]\) | (0)       | \([12, 0, 70]\) |

By Theorem 9, the Galois group of the ring class field \( H(d) \) acts nontrivially on the Néron–Severi group \( \text{NS}(X) \), i.e. on the reducible fibers. A nontrivial action is possible when a fiber has type \( I_n \) for \( n \geq 3 \), allowing an involution of the components that preserves the identity component and incidence relation, or when several fibers of the same type lie over points on the base curve \( \mathbb{P}^1 \) that are Galois conjugates. Later in this paper we again need a term for points on \( \mathbb{P}^1 \) lying under reducible fibers of an elliptic surface; we call them “cusps”, consistent with the special case of the universal elliptic curve over the modular curve \( X(N) \) or \( X_1(N) \).

In terms of the associated newforms, the discriminants in the above table exhaust the extremal elliptic K3 surfaces over \( \mathbb{Q} \). Note that since we consider elliptic surfaces, we can again twist as in [3]. In the next section we will explain our main techniques to exhibit singular K3 surfaces for the remaining 28 imaginary quadratic fields.

6 The main techniques

We search for singular K3 surfaces over \( \mathbb{Q} \) with particular discriminants. The main idea is to take advantage of the moduli theory of complex K3 surfaces. Any one-dimensional family of K3 surfaces of Picard number \( \rho \geq 19 \) has infinitely many specializations with
\( \rho = 20 \). We will only search for the specializations over \( \mathbb{Q} \) (which are finite in number by Šafarevič’s result \([20]\)). Because of Theorem \( \mathcal{3} \) these surfaces and therefore the families involved will be very special. Hence one of the key steps will be to construct suitable families. This will be achieved in the next section. Here we explain how we find the specializations of a given family.

Given a one-dimensional family of K3 surfaces \( X_\lambda \) over \( \mathbb{Q} \) satisfying \( \rho(X_\lambda) \geq 19 \) for all \( \lambda \), there is an easily checked necessary condition that must be satisfied by any member \( X \) with \( \rho(X) = 20 \). The condition is based on the Lefschetz fixed point formula at a good prime \( p \), formulated in terms of \( \ell \)-adic étale cohomology \( H^i_\ell(X, \mathbb{Q}_\ell) \) for some prime \( \ell \neq p \). Here we work with the base change \( \overline{X} \) of the reduction of \( X \mod p \) to an algebraic closure of \( \mathbb{F}_p \). For simplicity, we will just write \( H^i(X) \) for \( H^i_\ell(X, \mathbb{Q}_\ell) \) in the following.

The cohomology groups \( H^i(X) \) are equipped with an induced action of the geometric Frobenius morphism \( \text{Frob}_p \). The set \( X(\mathbb{F}_p) \) of \( \mathbb{F}_p \)-rational points on \( X \) is exactly the fixed set of \( \text{Frob}_p \). For a K3 surface \( X/\mathbb{Q} \) with good reduction at \( p \), the Lefschetz fixed point formula simplifies to

\[
\#X(\overline{\mathbb{F}}_p) = 1 + \text{trace} \, \text{Frob}_p^*(H^2(X)) + p^2. \tag{4}
\]

Because \( \rho(X) \geq 19 \), we can predict 19 of the 22 eigenvalues of \( \text{Frob}_p^* \) on \( H^2(X) \). Since the absolute Galois group operates through a permutation on the algebraic cycles, all these eigenvalues have the form \( \zeta \cdot p \) for some roots of unity \( \zeta \). By the Weil conjectures, one further eigenvalue has the form \( \pm p \), and the remaining two eigenvalues are algebraic integers \( \alpha_p, \beta_p \) of absolute value \( p \). In particular, the unordered pair \((\alpha_p, \beta_p)\) is determined by \((4)\) and the sign of the other eigenvalue \( \pm p \). In general, \( \beta_p = \bar{\alpha}_p \); the only exception is the supersingular case where \( \alpha_p + \beta_p = 0 \).

If the specialization \( X \) at some \( \lambda_0 \in \mathbb{Q} \) is a singular K3 surface, then it is modular by Theorem \( \mathcal{3} \). Hence, for the right choice of sign in the other eigenvalue,

\[
\alpha_p + \beta_p = a_p \tag{5}
\]

where \( a_p \) is the Fourier coefficient of the corresponding newform \( f \) of weight 3. In particular, both \( \alpha_p \) and \( \beta_p \) lie in the imaginary quadratic extension \( K \) associated to \( f \). By Theorem \( \mathcal{4} \) \( K \) has class group exponent 2. Moreover, \( K \) remains fixed when \( p \) varies. This gives a criterion to either rule out \( \lambda_0 \) or collect evidence for \( \rho(X) = 20 \).

As it stands, our condition for \( \rho = 20 \) is necessary but not sufficient. To search for the CM-specializations, we will use the condition in a different, almost opposite approach: we search for good parameters mod \( p \) and try to lift \( \lambda_0 \) to \( \mathbb{Q} \). We use the following algorithm:

**Algorithm 10**

Let \( X_\lambda \) be a family of K3 surfaces over \( \mathbb{Q} \) with Picard number \( \rho \geq 19 \). Then the following algorithm returns candidate parameters \( \lambda_0 \in \mathbb{Q} \) such that the specialization \( X \) at \( \lambda_0 \) might have \( \rho(X) = 20 \):

(i) Fix one of the 65 weight 3 newforms \( f \) up to twisting (that is, one of the imaginary quadratic fields of class group exponent two).

(ii) For each of several primes \( p = p_i \) \((i = 1, \ldots, n)\), use \((4)\) to compute \( \alpha_{p_i}, \beta_{p_i} \) for every \( \lambda \in \mathbb{F}_{p_i} \) and find all \( \lambda \) such that one choice of sign for the other eigenvalue leads to \( \alpha_{p_i}, \beta_{p_i} \) matching \( f \) as in \((4)\). Even though \( f \) varies, we need only compute \( \#X_\lambda(\mathbb{F}_{p_i}) \) once for each pair \((p, \lambda)\).
(iii) For a collection of parameters $\lambda_1 \mod p_1, \ldots, \lambda_n \mod p_n$ matching the newform $f$, compute a lift $\lambda_0 \in \mathbb{Q}$ of small height using the Chinese Remainder Theorem and the Euclidean algorithm.

For each newform $f$, the algorithm returns a number of guesses for specializations in $\mathbb{Q}$, if any. Often, there will be one $\lambda_0$ among them which looks particularly likely (small height, small primes involved etc.). At this point, we can continue to collect numerical evidence by running the above test for further primes $p$. In the end, however, we want to prove that $\rho(X_{\lambda_0}) = 20$. Therefore we have to find explicitly an additional divisor on the specialization $X$ at $\lambda_0$. This is where we turn to elliptic surfaces. Until this point the procedure works for any one-dimensional family. For instance, we computed candidates for the singular specializations for the Dwork pencil

$$x_0^4 + x_1^4 + x_2^4 + x_3^4 = \lambda x_0 x_1 x_2 x_3$$

of deformations of the Fermat quartic from Example 2 (cf. [11] for a detailed account). But the only way we know to systematically search for extra divisors on a K3 family $X_\lambda$ uses models of $X_\lambda$ as elliptic surfaces.

The advantage of working with elliptic surfaces is the following. By the formula of Shioda and Tate, their Néron–Severi groups are always generated by horizontal and vertical divisors, i.e. by fiber components and sections. Hence for the Picard number to increase in a family, either the singular fibers degenerate further (which happens for only finitely many $\lambda_0 \in \mathbb{C}$), or there is an additional section $P$. Then the discriminant $d$ of the specialization can be computed purely in terms of the intersection behavior of $P$ with the singular fibers and the other sections. This is made explicit through the theory of Mordell–Weil lattices.

In our setting, $d$ is predicted up to a square factor by the newform $f$ and its CM-field $K$ by Theorem [3] and the explanation following it. This provides us with additional information about the conjectural section $P$, information that is often crucial to the feasibility of a direct computation. The next section illustrates this by the detailed analysis of a particular family of K3 surfaces.

7 Specializations of a one-dimensional family

We want to search in one-dimensional families of K3 surfaces with Picard number $\rho \geq 19$ for singular specializations over $\mathbb{Q}$. In fact, it is a nontrivial task to find such families with interesting specializations in the first place. This difficulty is due to the nontrivial Galois action that the Néron–Severi group must admit by Theorem [3]. In this section, we discuss one particular family in detail.

We start with the following two-dimensional family in extended Weierstrass form with parameters $\lambda \in \mathbb{P}^1, \mu \neq 0$:

$$X_{\lambda, \mu} : y^2 = x^3 + (t - \lambda)Ax^2 + t^2(t - 1)(t - \lambda)^2Bx + t^4(t - 1)^2(t - \lambda)^3C,$$

$$A = \frac{1}{24}\left(\frac{1}{9}(2\mu + 9)^3t^3 - (22\mu - 9)(2\mu - 27)t^2 - 27(14\mu - 9)t - 81\right),$$
$$B = \mu\left(\frac{1}{9}(2\mu + 9)^3t^2 - 2(10\mu - 9)(2\mu - 9)t - 27(2\mu - 3)\right),$$
$$C = \frac{2}{3}\mu^2((2\mu + 9)^3t - 81(2\mu - 3)^2).$$
This elliptic surface has discriminant
\[
\Delta = 36\mu^4 r^5 (t - 1)^3 (t - \lambda)^6
\]
\[\((2\mu + 9)^4 t^3 - 9(32\mu + 27)(2\mu + 9)^2 t^2 + 81(308\mu^2 + 243 - 864\mu)t + 729(4\mu - 9)\).
\]
It has the following singular fibers:

| fiber | \(I_5\) | \(I_3\) | \(I_7\) | \(I_5^*\) | \(I_1, I_3, I_1\) |
|-------|--------|--------|--------|--------|-----------------|

In general, \(X_{\lambda, \mu}\) has Néron–Severi lattice
\[NS(X_{\lambda, \mu}) = U + A_2 + A_4 + A_6 + D_4.\]

Here \(U\) denotes the hyperbolic plane (generated by the 0-section \(O\) and a general fiber \(F\)), and \(A_i, D_i\) are the root lattices corresponding to the reducible singular fibers. In particular, we deduce that
\[\rho(X_{\lambda, \mu}) \geq 18.\]

We briefly explain how we found the above family. As in [9], we work with an extended Weierstrass form. Here we can translate \(x\) so that all singular fibers have their singularities at \(x = y = 0\). In the present situation, this gives rise to the family
\[y^2 = x^3 + (t - \lambda) a_2(t) x^2 + t^2 (t - 1) (t - \lambda)^2 a_1(t) x + t^4 (t - 1)^2 (t - \lambda)^3 a_0(t).\]

In general, the \(a_i(t)\) are polynomials of degree \(\deg(a_i) \leq i + 1\). Hence we have ten parameters to choose including \(\lambda\), relative to one normalization by rescaling \(x, y\). The above extended Weierstrass form guarantees that the fiber types are at least \(I_4, I_2, I_0^*, I_2\) at \(t = 0, 1, \lambda, \infty\) respectively. We can easily choose the coefficients of \(a_2(t)\) to promote the fiber at \(\infty\) to type \(I_6\). Then we solve a system of three nonlinear equations in the five coefficients of \(a_0(t), a_1(t)\) to derive the family \(X_{\lambda, \mu}\). This can be achieved by appropriate combinations of the equations and a suitable choice of normalization.

The family \(X_{\lambda, \mu}\) can easily be specialized to a family \(X_{\lambda}\) with \(\rho(X_{\lambda}) \geq 19\) by degenerating the singular fibers, i.e. merging fibers. There are two independent ways to do so. On the one hand, we can match \(\lambda\) with one of the cusps. This results in four families. Each of them has several interesting specializations. The case where we merge \(I_0^*\) with a \(I_1\)-fiber will be taken up in the next section (Example [17]). On the other hand, we can merge one of the \(I_1\) fibers with another singular fiber using \(\mu\). We now discuss one particular case.

We merge two fibers of type \(I_1\) by setting \(\mu = \frac{443}{10}\). We obtain the one-dimensional family \(X_{\lambda}\) with the following singular fibers:

| fiber | \(I_5\) | \(I_3\) | \(I_2\) | \(I_1\) | \(I_0^*\) |
|-------|--------|--------|--------|--------|-----------------|

The general member \(X_{\lambda}\) has Néron–Severi lattice
\[NS(X_{\lambda}) = U + A_1 + A_2 + A_4 + A_6 + D_4,\]

so \(\rho(X_{\lambda}) \geq 19\). The Galois action on the Néron–Severi group is encoded in the fields where the singular fibers with at least three components split (depending on \(\lambda\)):

| fiber | \(I_5\) | \(I_3\) | \(I_7\) | \(I_0^*\) |
|-------|--------|--------|--------|-----------------|
| splitting field | \(\mathbb{Q}(\sqrt{6\lambda})\) | \(\mathbb{Q}(\sqrt{10(1 - \lambda)})\) | \(\mathbb{Q}(\sqrt{15})\) | \(\mathbb{Q}(f(\lambda))\) |
Here \( \mathbb{Q}(f(\lambda)) \) is the splitting field of the cubic polynomial

\[
f(\lambda) = x^3 + \left( \frac{4096}{15} \lambda^3 - 115 \lambda - 146 \lambda^2 - \frac{25}{24} \right) x^2 + 6 \lambda^2 (\lambda - 1) (8192 \lambda^2 - 7150 \lambda - 475) x + 1080 \lambda^3 (\lambda - 1)^2 (2048 \lambda - 1805).
\]

There are five obvious specializations with \( \rho = 20 \) where we match \( \lambda \) (i.e. the \( I^*_0 \) fiber) with another cusp. This way, we obtain explicit equations for extremal elliptic K3 surfaces over \( \mathbb{Q} \) that were not derived in [2] or [21]. These fibrations provide alternative realizations of weight 3 newforms for the following discriminants previously realized by semistable extremal elliptic K3 surfaces (cf. Section 4):

| \( \lambda \) | degeneration in \( \text{NS}(X_\lambda) \) | disc \( \text{NS}(X_\lambda) \) | \( T(X_\lambda) \) |
|---|---|---|---|
| \(-\frac{5}{1024}\) | \( D_4 \sim D_5 \) | \(-840\) | \([20, 0, 42]\) |
| \(\frac{35}{32}\) | \( A_1 + D_4 \sim D_6 \) | \(-420\) | \([6, 0, 70]\) |
| \(1\) | \( A_2 + D_4 \sim D_7 \) | \(-280\) | \([4, 0, 70]\) |
| \(0\) | \( A_4 + D_4 \sim D_9 \) | \(-168\) | \([4, 0, 42]\) |
| \(\infty\) | \( A_6 + D_4 \sim D_{11} \) | \(-120\) | \([6, 0, 20]\) |

Every other specialization with \( \rho = 20 \) requires the existence of an additional section.

We ran Algorithm 10 through the first 30 primes \( p \) to find candidate parameters \( \lambda_0 \in \mathbb{Q} \).

In step (ii), we restricted to those \( p \) which split in the field \( K \) corresponding to a given newform \( f \). In almost every case, this implied that there was exactly one candidate parameter \( \lambda \mod p \) (if there were any at all). Some lifts to \( \mathbb{Q} \) are given in Table 1.

We shall now explain how the structure of \( X_\lambda \) as an elliptic surface helps us verify that the K3 surface at such a lift \( \lambda_0 \) is singular.

### Elliptic surfaces

Once we know the group of sections of an elliptic surface, we easily compute the discriminant of its Néron–Severi lattice using the theory of Mordell–Weil lattices [29]. In the case at hand, the specialization \( X \) at \( \lambda_0 \) has a conjectural section \( P \neq O \), generating the Mordell–Weil group (for the general situation cf. sect. 8). By [29] Thm. 8.6 the discriminant is given by the product of the discriminants of the root lattices and the height of \( P \):

\[
\text{disc NS}(X) = -2 \cdot 3 \cdot 5 \cdot 7 \cdot 4 \cdot \hat{h}(P) = -840 \hat{h}(P).
\]  

(6)

The height of \( P \) is determined by the intersection number \( (P, O) \) of \( P \) and \( O \) in \( \text{NS}(X) \) and some correction terms \( \text{corr}_v(P) \) according to the fiber components which \( P \) meets:

\[
\hat{h}(P) = 4 + 2 (P, O) - \sum_v \text{corr}_v(P).
\]  

(7)

To describe the correction term at a \( I_n \) fiber, we number the components cyclically \( \Theta_0, \ldots, \Theta_{n-1} \) such that \( \Theta_0 \) is the identity component (i.e. the component meeting \( O \)). In this notation,

\[
\text{corr}_v(P) = \begin{cases} 
0 & \text{if } P \text{ meets the identity component of the fiber at } v, \\
1 & \text{if } P \text{ meets a non-identity component at a } I^*_0 \text{ fiber,} \\
\frac{i(n-1)}{n} & \text{if } P \text{ meets the component } \Theta_i \text{ of a } I_n \text{ fiber.}
\end{cases}
\]
Assume that we have a candidate parameter $\lambda_0$ for a specialization $X$ to realize some newform $f$. Let $K$ denote the imaginary quadratic field corresponding to $f$. Then we have to arrange for a section $P$ such that $\text{NS}(X)$ has discriminant $d < 0$ so that $\mathbb{Q}(\sqrt{d}) = K$. Let us look at one example in detail.

Example 11

Let $f$ be a newform for the field $K = \mathbb{Q}(\sqrt{-1540})$ of discriminant $d = -1540$. Then Algorithm 10 suggests the candidate parameter $\lambda_0 = 7^{21}/29$. We want to find a section $P$ such that $\text{disc NS}(X) = d$.

Consider the formula (6) for the discriminant of the elliptic surface with section $P$. Our target discriminant $d = -1540$ requires that we eliminate a factor 6 from (6) while preserving 5 and 7. Hence $P$ must meet the $I_5$ and $I_7$ fibers on their identity components, but meet the $I_2$ and $I_3$ fibers on non-identity components. Now we assume that $(P.O) = 0$ and that $P$ intersects a non-identity component of the $I_0^*$ fiber. Then (7) reads

$$\hat{h}(P) = 4 - \frac{1}{2} - \frac{2}{3} - 1 = \frac{11}{6}. $$

Hence (4) gives

$$\text{disc NS}(X) = -840 \cdot \frac{11}{6} = -1540$$

as required. The assumption $(P.O) = 0$ implies that $P$ has coordinates $(u, v)$ for polynomials $u, v$ of degree 4 resp. 6 in the coordinate $t$ of the base curve. Note that inserting $P$ into the Weierstrass equation of $X$ allows us to express the coefficients of $v$ in terms of those of $u$. For the fibers to be met as prescribed, we must have

$$(t - 1) \left( t - \frac{35}{32} \right) (t - \lambda_0)^2 | v, \quad (t - 1)(t - \lambda_0) | u.$$ 

Moreover, the fibers at $\lambda_0$ and $\frac{35}{32}$ give two more linear relations in the coefficients of $u$. Since $u$ has degree four, there is only one degree of freedom left. Thus the problem is easily solved using a computer algebra system. The solution $P$ and the resulting transcendental lattice can be found in Table 1.

Often we are not lucky enough to arrive at a system of equations that we can solve directly as in Example 11. If we cannot find a direct solution, we apply the following algorithm, essentially a $p$-adic Newton iteration in several variables.

Algorithm 12

Given a system $f_1 = \ldots = f_n = 0$ of algebraically independent polynomial equations over $\mathbb{Q}$ in $n$ variables $z_1, \ldots, z_n$. The following procedure tests for a solution over $\mathbb{Q}$:

1. With an exhaustive search, find a solution $(\tilde{z}_1, \ldots, \tilde{z}_n)$ modulo some prime $p$.
2. Using difference quotients, double the $p$-adic accuracy of the solution a few times.
3. Compute a lift in $\mathbb{Q}$ with the Euclidean algorithm.
4. Test whether the lift solves the system of equations over $\mathbb{Q}$. While it does not, return to step (2) to double the precision once more and try (3) and (4) again.

For step (2) to converge, we need some regularity assumptions for the polynomials $f_i$. For instance, it will converge if the coefficients of the $f_i$ are $p$-adic integers and the Jacobian determinant $|\partial(f_1, \ldots, f_n)/\partial(z_1, \ldots, z_n)|$ does not vanish at $(\tilde{z}_1, \ldots, \tilde{z}_n)$.
We next outline some further implementation issues specific to our setting.

Algorithm 12 does not require that the section itself be defined over \( \mathbb{Q} \), only its \( x \)-coordinate. Even if the \( y \)-coordinate involves a square root of a rational number as a factor, we can always arrange to solve a system of equations over \( \mathbb{Q} \).

A more delicate point about Algorithm 12 is the choice of the prime \( p \). Here we distinguish whether or not \( p \) splits in the fixed imaginary quadratic field \( K \).

For a singular K3 surface \( X \) over a number field \( L \), one can predict the geometric Picard number of the reductions. Namely, let \( p \) denote a prime of \( L \) above \( p \). If \( X \) has good reduction mod \( p \), write \( X_p \) for the reduced K3 surface. Then

\[
\rho(X_p) = \begin{cases} 
20, & \text{if } p \text{ splits in } K, \\
22, & \text{if } p \text{ is inert or ramified in } K.
\end{cases}
\]  

(8)

This follows from the Shioda–Inose structure (cf. sect. 5) since the above cases decide exactly whether the elliptic curves \( E, E' \) (and the abelian surface \( E \times E' \)) are supersingular.

In more detail, we apply the Tate conjecture \cite{32} to a (conjectural) singular K3 surface \( X \) over \( \mathbb{Q} \). Here we consider elliptic K3 surfaces with section, so the Tate conjecture holds true by \cite{1}. Note that the assumption implies that the primes ramifying in \( K \) are always bad, since they divide the level of the associated Hecke eigenform by Theorem 3.

At the inert primes \( p \) in \( K \), the Fourier coefficient \( a_p \) of the associated newform is zero. The resulting eigenvalues of Frobenius are \( p \) and \(-p\). Hence the Tate conjecture predicts that the reduction \( X_p \) has an additional algebraic cycle over \( \mathbb{F}_p \) and one more over \( \mathbb{F}_{p^2} \). On an elliptic surface, these extra cycles would either change the configuration of reducible fibers or appear as extra Mordell–Weil sections. In the former case we might miss the reduced surface entirely. In the latter case, reduction mod \( p \) would increase the Mordell–Weil rank compared to the rank over \( \mathbb{Q} \). Depending on the conditions we impose — on \((P,O)\) and the fiber components met — step (1) of Algorithm 12 might then return more than one section. However, only one of these sections would lift to \( \mathbb{Q} \), so we would have to make the right choice.

Therefore, we always run Algorithm 12 at a prime \( p \) that splits in \( K \). Under the assumption that the K3 surface is singular, the Picard number in (8) guarantees that the Mordell–Weil rank is constant at 20 upon reduction.

**Remark 13**

We will apply Algorithm 12 after guessing the parameter \( \lambda_0 \) with the help of Algorithm 10. Hence there is one further equation \( f_0 = 0 \) in \( z_1, \ldots, z_n \). After step 4, we thus also have to verify that \( f_0 \) vanishes at the lift.

The same ideas can also be applied without a candidate parameter \( \lambda_0 \) at hand. In Example 11, this would have sufficed as well: in the end, we would have to solve two equations in two variables. However, if we have to apply Algorithm 12 to find the section explicitly, it is computationally very convenient for step 4 to have one parameter fewer.

The following table collects the specializations of \( X_{\lambda} \) where we verified \( \rho = 20 \). The additional section \( P \) can be recovered from its \( x \)-coordinate \( u(t) \) by taking \( x = u(t) \) in the Weierstrass equation and choosing a root for \( v(t) \). We also list the height \( \hat{h}(P) \) of \( P \) and the discriminant of NS(\( X \)). The discriminant will be justified below.
The computation of the transcendental lattices $T(X)$ will be explained in the next subsection.

| $\lambda$ | disc NS(X) | $P \cdot u(t)$ | $h(P)$ | $T(X)$ |
|-----------|------------|----------------|--------|--------|
| $-\frac{1}{2}$ | $-225.4$ | $-\frac{\sqrt{2}}{5} \cdot t^3 (2t+35) (t-1) (2t+1)$ | $\frac{11}{2}$ | $(30.0,30)$ |
| $\frac{3}{2}$ | $-4.43$ | $-\frac{36}{7} \cdot t^2 (t-1)$ | $\frac{11}{2}$ | [4.2,44] |
| $\frac{3}{2}$ | $-4.67$ | $-\frac{36}{7} \cdot t (t-1)(128t-105)$ | $\frac{11}{2}$ | [4.2,84] |
| $\frac{1}{2}$ | $-88$ | $-\frac{36}{7} \cdot t (t-1)(32t-5)$ | $\frac{11}{2}$ | [2.0,44] |
| $\frac{3}{2}$ | $-4.163$ | $-\frac{36}{7} \cdot t^2 (t-1)$ | $\frac{11}{2}$ | [4.2,164] |
| $\frac{3}{2}$ | $-228$ | $-\frac{36}{7} \cdot t (2t-5) (8t-7)$ | $\frac{11}{2}$ | [2.0,114] |
| $\frac{3}{2}$ | $-312$ | $-\frac{36}{7} \cdot t^2 (32t-45)$ | $\frac{11}{2}$ | [6.0,52] |
| $\frac{3}{2}$ | $-340$ | $-\frac{36}{7} \cdot (t-1)(2t+1)(24t+35)$ | $\frac{11}{2}$ | [20.10,22] |
| $\frac{3}{2}$ | $-372$ | $-\frac{36}{7} \cdot t^2 (512t-605)$ | $\frac{11}{2}$ | [6.0,62] |
| $\frac{3}{2}$ | $-408$ | $\frac{36}{7} \cdot t (t-1)(9t-10)(-96t+81)$ | $\frac{11}{2}$ | [6.0,68] |
| $\frac{3}{2}$ | $-4.435$ | $\frac{3}{7} \cdot (-7776t^3+4816t^2+1660t+175)$ | $\frac{11}{2}$ | [20.10,92] |
| $\frac{5}{2}$ | $-4.483$ | $\frac{3}{7} \cdot (-5242880t^3+9202816t-3988061)$ | $\frac{11}{2}$ | [4.2,484] |
| $\frac{5}{2}$ | $-520$ | $-\frac{36}{7} \cdot t (t-1)(32t+5)(49t+5)$ | $\frac{11}{2}$ | [20.0,26] |
| $\frac{5}{2}$ | $-532$ | $-\frac{36}{7} \cdot t^2 (608t-245)$ | $\frac{11}{2}$ | [4.2,134] |
| $\frac{5}{2}$ | $-4.555$ | $\frac{36}{7} \cdot (-31648t^3+86320t^2-78300t+23625)$ | $\frac{11}{2}$ | [4.2,556] |
| $\frac{5}{2}$ | $-4.595$ | $\frac{3}{7} \cdot (-1048576t^3-716160t^2-45225t-875)$ | $\frac{11}{2}$ | [20.10,124] |
| $\frac{5}{2}$ | $-660$ | $\frac{3}{7} \cdot (32t+1)(-3072t^2+2592t+105)$ | $\frac{11}{2}$ | [20.10,38] |
| $\frac{5}{2}$ | $-708$ | $\frac{36}{7} \cdot t (98t-605)(-53176t+46585)$ | $\frac{11}{2}$ | [6.0,118] |
| $\frac{5}{2}$ | $-760$ | $\frac{36}{7} \cdot t (t-1)(1568t+19)(-75t-1)$ | $\frac{11}{2}$ | [20.0,38] |
| $\frac{5}{2}$ | $-4.795$ | $\frac{3}{7} \cdot U(t)(15360t+7)^2$ | $\frac{11}{2}$ | [20.10,164] |
| $\frac{5}{2}$ | $-1.092$ | $\frac{36}{7} \cdot t^2 (512t-875)(-4096t+2765)$ | $\frac{11}{2}$ | [6.0,182] |
| $\frac{5}{2}$ | $-1.320$ | $-\frac{36}{7} \cdot (1024t-495)(-2097152t^2+5225472t-1607445)$ | $\frac{11}{2}$ | [4.0,330] |
| $\frac{5}{2}$ | $-1.380$ | $\frac{36}{7} \cdot (96t-121)(-4427776t^2+9923936t-563785)$ | $\frac{11}{2}$ | [6.0,210] |
| $\frac{5}{2}$ | $-1.428$ | $\frac{36}{7} \cdot t (32t-375)(32t^2-99880t+87500)$ | $\frac{11}{2}$ | [6.0,238] |
| $\frac{5}{2}$ | $-1.540$ | $\frac{36}{7} \cdot (512t-539)(-1)(512000t^2-1097257t+588245)$ | $\frac{11}{2}$ | [22.0,70] |
| $\frac{5}{2}$ | $-1.848$ | $\frac{36}{7} \cdot t (9216t+35)(7558272t^2+631582t+1323)$ | $\frac{11}{2}$ | [42.0,44] |

Table 1: Singular specializations over $\mathbb{Q}$ in the family $X_{\lambda}$

For the largest discriminant $-4 \cdot 795$, the section $P$ is not integral. Here $U(t)$, the numerator of $u(t)$, is given by

$$U(t) = 229323571200000 \cdot t^5 - 191371714560000 \cdot t^4 - 9553942361376 \cdot t^3 - 151103350160 \cdot t^2 - 953437100 \cdot t - 2100875.$$
The table is not complete. For each of the four discriminants
\[ d = -4 \cdot 1435, \ -4 \cdot 1155, \ -4 \cdot 1995, \ -5460, \]
we find using Algorithm 10 the lift
\[ \lambda_0 = -\frac{5 \cdot 7 \cdot 11^2}{2^3 \cdot 41}, \ -\frac{7}{2^3}, \ -\frac{7 \cdot 19}{2^7 \cdot 5}, \ -\frac{3 \cdot 5 \cdot 7^2}{2 \cdot 19^2} \]
respectively for which we expect \( X_{\lambda_0} \) to be singular with Néron–Severi discriminant \( d \); but the additional section is too complicated for us to compute easily, even using Algorithm 12. In the next section we realize each of the remaining nine discriminants, including these four, in a singular K3 surface not in this family \( X_{\lambda} \).

**Lemma 14**
For all K3 surfaces in the table, \( \text{NS}(X) \) is generated by fiber components and the sections \( O \) and \( P \). In particular, the discriminant is as claimed.

*Proof:* The fiber components together with the sections \( O \) and \( P \) generate a lattice \( N \) of rank 20. We must show that \( N = \text{NS}(X) \). The discriminant \( d \) of \( N \) is given by (6) and (7). If the index of \( N \) in \( \text{NS}(X) \) were greater than 1, then \( \text{NS}(X) \) would have discriminant \( d \) divided by the square of the index.

For any elliptic K3 surface \( X_{\lambda} \) in our family, the Mordell–Weil group is torsion-free. Using the formula (7), we see that \( 2h(P) \) is a 2-adic integer for any section \( P \), because \( (P,O) \) is an integer and each correction term \( \text{corr}_c(P) \) is in \( \frac{1}{2}\mathbb{Z}_2 \). By (6), we derive the general relation
\[ 4 \mid \text{disc}(\text{NS}(X_{\lambda})). \] (9)
In each case in the table, the quotient \( d/\text{disc}(\text{NS}(X)) \) can therefore only be a square if it equals 1. The claim follows.

**Transcendental lattices**

The transcendental lattices for the singular K3 surfaces above were computed using lattice theory as developed by Nikulin [17]. Here we sketch the argument.

Given an even integral lattice \( L \), we denote its dual by \( L^\vee \). In [17], Nikulin introduced a quadratic form on the quotient \( L^\vee/L \) which he called the discriminant form:
\[ q_L: \quad L^\vee/L \to \mathbb{Q} \mod 2\mathbb{Z} \]
\[ x \mapsto x^2 + 2\mathbb{Z} \]
For each singular K3 surface \( X \) in the table, we know the Néron–Severi lattice \( \text{NS}(X) \) by Lemma 14. Hence we can compute its discriminant form.

**Theorem 15** ([Nikulin [17], Prop. 1.6.1])
Let \( N \) be an even integral unimodular lattice. Let \( L \) be a primitive non-degenerate sublattice and \( M = L^\perp \). Then
\[ q_L = -q_M. \]
Since \( \text{NS}(X) \) always embeds primitively into \( H^2(X,\mathbb{Z}) \), the theorem provides the discriminant form of the transcendental lattice \( T(X) \).
8 The remaining discriminants

Theorem 16 (Nikulin [17, Cor. 1.9.4])
The genus of an even integral lattice is determined by its signature and discriminant form.

We now use the fact that each singular K3 surface $X$ in the table is defined over $\mathbb{Q}$. By Theorem 16, the genus of the transcendental lattice $T(X)$ consists of a single class. Hence the discriminant form of the Néron–Severi lattice $\text{NS}(X)$ determines $T(X)$ uniquely via the above two theorems. Thus the computation is completed by verifying that for the given transcendental lattices $q_{T(X)} = -q_{\text{NS}(X)}$.

8 The remaining discriminants

So far we have matched all but nine newforms of weight 3 from Section 3 with singular K3 surfaces over $\mathbb{Q}$. For some of the remaining discriminants, we found candidate surfaces in other one-dimensional families of K3 surfaces with $\rho \geq 19$, see Examples 17–20. For the other forms, we used slightly different techniques to derive elliptic K3 surfaces designed for those particular forms. These will be sketched in Examples 21–24. The transcendental lattices are computed using the discriminant form as before.

Example 17 (Discriminants $-1155, -1995$)

In the family $X_{\lambda, \mu}$ we choose $\lambda$ to merge fibers of type $I^*_0$ and $I^*_1$. A general member $X$ of the resulting family has

$$\text{NS}(X) = U + A_2 + A_4 + A_6 + D_5,$$

so $\rho(X) \geq 19$. For a specialization to be defined over $\mathbb{Q}$, we furthermore need a rational cusp at a $I^*_1$ fiber, i.e. the cubic factor of $\Delta$ encoding the $I^*_1$ fibers in terms of $\mu$ must have a rational zero. This can be achieved by the following substitution:

$$\mu = 9 \frac{(\nu + 1)^3}{5\nu^3 + 15\nu^2 - 5\nu + 1}.$$

Then the rational cusp gives

$$\lambda = \frac{-(\nu - 3)(5\nu^3 + 15\nu^2 - 5\nu + 1)}{(7\nu^2 + 1)^2}.$$

Algorithm 10 suggests several singular specializations over $\mathbb{Q}$. Here we verify two of them. The corresponding newforms seem to occur in the family $X_{\lambda}$ from the previous section as well, but there the conjectural sections have double height because of the relation (3).

Discriminant $-1155$: Let $\nu = -3/5$. Then there is a section $P$ of height $\hat{h}(P) = 11/4$. Its $x$-coordinate is given by

$$u(t) = -\frac{2 \cdot 27}{5^8 \cdot 7 \cdot 11^3 \cdot 13} (242 t - 585) (46060586 t^3 + 422472710 t^2 + 32588325 t + 8292375).$$

This singular K3 surface has discriminant $d = -1155$ and transcendental lattice $T(X) = [6, 3, 194]$.

Discriminant $-1995$: Let $\nu = 9/35$. Then there is a section $P$ of height $\hat{h}(P) = 19/4$. Its $x$-coordinate is given by

$$u(t) = -\frac{27 \cdot 11^3}{2^5 \cdot 5^{10} \cdot 7^7 \cdot 53} \frac{(784 t - 795) U(t)}{(8757 t - 9010)^2}.$$
where
\[ U(t) = 519278509294553530368 t^5 - 2767640394056706623700 t^4 \\
+ 590818374571257772625 t^3 - 6312492415348218806875 t^2 \\
+ 3374618170228790821875 t - 721947602876973103125. \]

This singular K3 surface has discriminant \( d = -1995 \) and transcendental lattice \( T(X) = [46, 11, 46] \).

For the remaining discriminants, we constructed single families of elliptic K3 surfaces. Here we give only the specializations in extended Weierstrass form
\[ X : \quad y^2 = x^3 + A x^2 + B x + C. \] (10)

**Example 18 (Discriminant \(-627\))**

We consider an elliptic K3 surface with singular fibers \( I_3, I_6, I_{11} \) at \( \frac{1}{7}, 0, \infty \):
\[
A = \frac{25}{24} t^4 + \frac{293}{6} t^3 - \frac{23645}{16} t^2 + \frac{1705}{12} t - \frac{1331}{384}, \\
B = -(7t - 1) t^2 (200 t^3 + 9276 t^2 - 92442 t + 4477), \\
C = 96 (7t - 1)^2 t^4 (100 t^2 + 4588 t - 15059). 
\]

It has a section \( P \) of height \( \hat{h}(P) = 19/6 \) and \( x \)-coordinate
\[
u(t) = -\frac{3}{2 \cdot 72 \cdot 115} t (52734375 t^3 + 538828125 t^2 - 2025538427 t + 1004475087). \]

The transcendental lattice is \( T(X) = [22, 11, 34] \) with discriminant \( d = -627 \).

**Example 19 (Discriminant \(-715\))**

We consider an elliptic K3 surface with singular fibers \( I_4, I_5, I_{11} \) at \( 1, 0, \infty \):
\[
A = -\frac{19487171}{3808800} t^4 + \frac{3674891}{13800} t^3 - \frac{247797}{80} t^2 + \frac{37743}{8} t + \frac{23805}{32}, \\
B = -\frac{2}{5} t (t - 1) (161051 t^3 - 5251521 t^2 + 16877745 t + 8212725), \\
C = -152352 t^2 (t - 1)^2 (1331 t^2 - 17526 t - 23805). 
\]

It has a section \( P \) of height \( \hat{h}(P) = 13/4 \) and \( x \)-coordinate
\[
u(t) = -\frac{15}{11776} (t - 1) (44289025 t^3 - 35970275 t^2 + 11995075 t - 1058529). \]

The transcendental lattice is \( T(X) = [22, 11, 38] \) with discriminant \( d = -715 \).

**Example 20 (Discriminant \(-1435\))**

We consider an elliptic K3 surface with singular fibers \( I_5, I_7, I_8 \) at \( 1, 0, \infty \):
\[
A = -\frac{16807}{332928} t^4 - \frac{490}{2601} t^3 + \frac{333135767}{13872} t^2 - \frac{275656745}{5202} t + \frac{603551125}{20808}, \\
B = -t^2 (t - 1) (t + 50) (2401 t^2 - 114044 t + 114244), \\
C = -83232 t^4 (t - 1)^2 (343 t^2 + 436 t - 3380). 
\]
8 The remaining discriminants

It has a section $P$ of height $\hat{h}(P) = 41/8$ and $x$-coordinate

\[ u(t) = \frac{51^2}{2 \cdot 5^4 \cdot 7} \frac{U(t)}{(20003760 \, t + 208409617)^2}, \]

where

\[ U(t) = 94791757788196875 \, t^5 - 13440531435036024375 \, t^4 \\
+ 311827388703362736750 \, t^3 - 2250368299914898266350 \, t^2 \\
+ 5701998864279209056695 \, t - 4700672234454567466251. \]

The transcendental lattice is $T(X) = [38, 3, 38]$ with discriminant $d = -1435$.

The four remaining discriminants require more work for two reasons. For three of them $(-1012, -3003, -3315)$, the discriminant has two large prime factors. Hence representing one of them by a singular fiber of an elliptic fibration would be too restrictive for the other singular fibers in a one-dimensional family. Instead, we use two-dimensional families to find singular K3 surfaces over $\mathbb{Q}$ with these discriminants. This approach will be sketched in the next subsection.

The final discriminant $d = -5460$ illustrates the constraints that we are facing: The Hilbert class field of $\mathbb{Q}(\sqrt{d})$ has Galois group $(\mathbb{Z}/2\mathbb{Z})^4$. Hence Theorem 9 implies that a family of K3 surfaces with a specialization of discriminant $d$ over $\mathbb{Q}$ has to be fairly complicated.

In fact, Algorithm 10 suggests that both one-dimensional families in Section 7 and Example 17 admit a specialization with discriminant $d$ over $\mathbb{Q}$. However, the conjectural section would have too big height for computations with Algorithm 12.

In other words, we must find the right balance between two competing constraints: a family of K3 surfaces that is not too complicated to parametrize, but admits the required Galois action; and a specialization with the given discriminant that is not too complicated to allow an explicit verification by Algorithm 12.

These complexity problems can be circumvented as follows. We let go the first step of guessing the candidate parameter for a singular specialization on a family of K3 surfaces over $\mathbb{Q}$ by Algorithm 10. Instead we apply Algorithm 12 directly to a family of elliptic K3 surfaces over $\mathbb{F}_p$. To start the algorithm, we need only determine all family members over $\mathbb{F}_p$ — i.e. a finite set. By point counting, we can already filter some surfaces with the test from Section 6. After finding an appropriate section on one of these surfaces, we increase the $p$-adic accuracy and finally compute a lift for both the surface and the section. Then we verify as before that the lifted surface has the prescribed configuration of singular fibers and that the section lifts to this surface.

**Example 21 (Discriminant $-5460$)**

Using the above approach for $p = 37$, we found an elliptic K3 surface $X$ with the following singular fibers:

| cusp  | $\infty$ | 0 | 1 | $7^{3 \cdot 11}_{23}$ | $\alpha, \alpha^\sigma$ | $-\frac{1}{125}$ |
|-------|----------|---|---|---------------------|-------------------------|-----------------|
| fiber | $I_7$    | $I_5$ | $I_4$ | $I_3$ | $I_2$ | $I_1$ |

Here $\alpha, \alpha^\sigma$ are the roots of the polynomial $7625 \, t^2 - 1367158 \, t - 57967$. The surface is
given in extended Weierstrass form \textsuperscript{[10]} with coefficients

\[
A = 3125 t^4 - 784700 t^3 - 40778898 t^2 - 18971036 t - 218491,
\]
\[
B = -2^{30} \cdot 3^3 t^2 (t - 1) (625 t^3 - 151380 t^2 - 1599171 t + 62426),
\]
\[
C = 2^{38} \cdot 3^{10} t^4 (t - 1)^2 (125 t^2 - 29164 t - 17836).
\]

It has a section $P$ of height $\hat{h}(P) = 13/4$ and $x$-coordinate

\[
u(t) = -\frac{2^{19}}{3^3 \cdot 74} (t - 1) (45273407 t^3 - 3678666 t^2 + 168432 t - 5324).\]

See \textsuperscript{[10]} for the details of the computation. The transcendental lattice is $T(X) = [42,0,130]$ with discriminant $-5460$.

Two-dimensional families

For the remaining three discriminants, we work with two-dimensional families of K3 surfaces with $\rho \geq 18$ because of the two large prime factors of the discriminant. We determine a specialization with two independent sections $P,Q$. The discriminant is then computed in terms of the height pairing on the Mordell–Weil lattice \textsuperscript{[29]}.

On an elliptic K3 surface $X$, the height pairing of two sections $P,Q$ is by \textsuperscript{[29]} Thm. 8.6

\[
\langle P,Q \rangle = 2 + (P,O) + (Q,O) - (P,Q) - \sum_v \text{corr}_v(P,Q).
\]

The height pairing involves intersection numbers in $\text{NS}(X)$ and correction terms according to the non-identity fiber components met by the sections. Since we consider only semistable elliptic fibrations at this time, we require only the correction term at a $I_n$ fiber (or root lattice $A_{n-1}$). Using the cyclical numbering of components such that $O$ meets $\Theta_0$, we have

\[
\text{corr}_v(P,Q) = \frac{i (n - j)}{n}, \text{ if } P \text{ meets } \Theta_i \text{ and } Q \text{ meets } \Theta_j \text{ of a } I_n \text{ fiber.}
\]

If necessary, we interchange $P$ and $Q$ or renumber the components so that $i < j$. In case $P = Q$, the height pairing specializes to $\hat{h}(P)$ in \textsuperscript{[7]}.

The height pairing endows the Mordell–Weil group modulo torsion with the structure of a positive-definite lattice, the so-called \textit{Mordell–Weil lattice} $\text{MWL}(X)$. Note that the discriminant of the Mordell–Weil lattice need not be integral if there are reducible fibers.

One way to derive the height pairing is via the orthogonal projection in $\text{NS}(X) \otimes \mathbb{Q}$ with respect to zero section and fiber components. Hence the discriminant of an elliptic surface $X$ satisfies

\[
(-1)^{\text{rk}(\text{MW}(X))} |\text{MW}(X)_{\text{tor}}|^2 \text{disc}(\text{NS}(X)) = \text{disc}(\text{MWL}(X)) \cdot \prod_v \text{disc}(F_v). \tag{11}
\]

Here the product runs over all reducible fibers $F_v$ and involves the discriminants of the corresponding root lattices. We are now ready to exhibit the singular K3 surfaces for the remaining three discriminants. Due to the configuration of singular fibers, the Mordell–Weil groups are always torsion-free.
Example 22 (Discriminant $-1012$)

We consider an elliptic K3 surface $X$ with singular fibers $I_3, I_5, I_{11}$ at $t = 3861/28124, 0, \text{ and } \infty$:

\[
\begin{align*}
A &= 66 \left(351384 t^4 - 372196 t^3 + 113098 t^2 - 13539 t + 594\right), \\
B &= \frac{2^7 \cdot 11}{3} t (28124 t - 3861) (47916 t^3 - 44484 t^2 + 9479 t - 594), \\
C &= 2^{11} \cdot 11 t^2 (28124 t - 3861)^2 (242 t^2 - 193 t + 22).
\end{align*}
\]

The elliptic surface $X$ has sections $P, P^\sigma$ that are conjugate over $\mathbb{Q}(\sqrt{23})$. We found them by an extension of Algorithm 12 using lattice reduction instead of the Euclidean algorithm to recognize algebraic numbers of degree greater than 1. The sections have height $\hat{h}(P) = \hat{h}(P^\sigma) = 38/15$ and pairing $\langle P, P^\sigma \rangle = 8/15$. Let

\[
U(t) = (16427508 + 3425424 \sqrt{23}) t^2 + (2159201 + 450164 \sqrt{23}) t + 293522 + 61224 \sqrt{23}.
\]

The $x$-coordinates of the sections $P$ and $P^\sigma$ are

\[
u(t) = 3 \frac{(6 - \sqrt{23})^2 t (28124 t - 3861) U(t)}{(392 + 95 \sqrt{23})^2}
\]

and the Galois conjugate of $u$. By (11), the elliptic surface $X$ has discriminant

\[
\text{disc}(\text{NS}(X)) = -3 \cdot 5 \cdot 11 \cdot \left| \frac{\langle P, P \rangle}{\langle P, P^\sigma \rangle} \cdot \frac{\langle P^\sigma, P^\sigma \rangle}{\langle P^\sigma, P^\sigma \rangle} \right| = -1012.
\]

We compute the transcendental lattice $T(X) = [22, 0, 46]$.

For the final two discriminants, we combined two-dimensional families with the mod $p$ approach for Example 21.

Example 23 (Discriminant $-3003$)

We consider an elliptic K3 surface in extended Weierstrass form (10):

\[
\begin{align*}
A &= 2197 t^4 - 48516 t^3 + 393636 t^2 - 1208764 t + 411600, \\
B &= 216 t (3 t - 25) (10 t - 67) (13351 t^3 - 200168 t^2 + 834514 t - 411600), \\
C &= 108^2 t^2 (3 t - 25)^2 (10 t - 67)^2 (81133 t^2 - 641164 t + 411600).
\end{align*}
\]

This surface has the following reducible singular fibers:

| fiber | $I_2$ | $I_3$ | $I_3$ | $I_6$ | $I_7$ |
|-------|------|------|------|------|------|
| cusp  | $\frac{45}{8}$ | $\frac{45}{7}$ | $\frac{47}{8}$ | 0    | $\infty$ |

There are orthogonal sections $P, Q$ of heights $\hat{h}(P) = 11/6$ and $\hat{h}(Q) = 13/6$. Their $x$-coordinates are

\[
\begin{align*}
P &: u(t) = 27 t (3 t/25 - 1) (31671 (t/25)^2 - 675 t - 6700), \\
Q &: u(t) = -\frac{1}{27} t (5984 t^3 - 65032 t^2 + 16442 t - 949725).
\end{align*}
\]

By (11), $X$ has discriminant $-3003$. The transcendental lattice is $T(X) = [6, 3, 502]$. 
Example 24 (Discriminant $-3315$)
We consider an elliptic K3 surface $X$ in extended Weierstrass form:

\[
A = 1105 (15268125 t^4 - 131340300 t^3 + 35302566 t^2 - 4215388 t + 574821), \\
B = -3200 \cdot 11^3 \cdot 13 \cdot 17^2 t (119 t - 26) (741 t - 289) \\
\quad \times (1795625 t^3 - 1311895 t^2 + 317027 t - 33813), \\
C = 80^3 \cdot 11^6 \cdot 13^2 \cdot 17^2 t^2 (119 t - 26)^2 (741 t - 289)^2 (27625 t^2 - 16594 t + 2601).
\]

This surface has the following reducible singular fibers:

| Fiber | $I_3$ | $I_4$ | $I_5$ | $I_8$ |
|-------|-------|-------|-------|-------|
| Cusp  | 28/117 | 280/141 | 0 | $\infty$ |

There are orthogonal sections $P, Q$ of heights $\hat{h}(P) = 17/8$ and $\hat{h}(Q) = 13/4$. Their $x$-coordinates are

\[
P : \qquad u(t) = 44^3 (88179 t^2 - 58882 t + 2415744/221), \\
Q : \qquad u(t) = -\frac{5}{208} (741 t - 289) (1960038171 t^3 + 503840883 t^2 - 379180711 t + 5454297).
\]

By (11), $X$ has discriminant $-3315$. The transcendental lattice is $T(X) = [2,1,1658]$.

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8 The remaining discriminants

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