The Jacobian Conjecture as a problem in combinatorics

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Abstract

The Jacobian Conjecture has been reduced to the symmetric homogeneous case. In this paper we give an inversion formula for the symmetric case and relate it to a combinatoric structure called the Grossman-Larson Algebra. We use these tools to prove the symmetric Jacobian Conjecture for the case $F = X - H$ with $H$ homogeneous and $JH^3 = 0$. Other special results are also derived. We pose a combinatorial statement which would give a complete proof the Jacobian Conjecture.

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Dedication. This paper is submitted in honor of Professor Masayoshi Miyanishi, who has profoundly impacted the field of Affine Algebraic Geometry, giving
inspiration to many mathematicians working in this and related areas, including
this author.

1 The Jacobian Conjecture

1.1 The General Assertion

The Jacobian Conjecture is:

Conjecture 1.1 (JC). For any integer \( n \geq 1 \) and polynomials \( F_1, \ldots, F_n \in \mathbb{C}[X_1, \ldots, X_n] \), the polynomial map \( F = (F_1, \ldots, F_n) : \mathbb{C}^n \to \mathbb{C}^n \) is an automorphism if the determinant \( |JF| \) of the Jacobian matrix \( JF = (D_iF_j) \) is a nonzero constant.

Here and throughout this paper we write \( D_i \) for \( \partial/\partial X_i \). We will continue to write \( JF \) for the Jacobian matrix of a polynomial map \( F \), and the determinant of this matrix will be denoted by \( |JF| \).

For technical reasons, it will be convenient to henceforth consider polynomial maps (and later power series maps) with coefficients in an arbitrary commutative \( \mathbb{Q} \)-algebra \( K \). Proving the Jacobian Conjecture is equivalent to proving the conjecture as stated in 1.1 with \( \mathbb{C} \) replaced by \( K \) (and \( \mathbb{C}^n \) by \( \text{Spec} \ K[X_1, \ldots, X_n] \)).

1.2 The Homogeneous Symmetric Reduction

This paper is based on the following result, which puts together two well-known reductions:

Theorem 1.2 (Symmetric Reduction). The Jacobian Conjecture is true if it holds for all polynomial maps \( F \) having the form \( F = X - H \) with \( H \) homogeneous of degree \( d \geq 2 \) and \( JH \) is a symmetric matrix. In fact, it suffices to prove the case \( d = 3 \).

The reduction to the homogeneous cubic case was proved in [1]; the reduction to the symmetric situation is due to de Bondt and van den Essen [2].

Definition 1.3. A polynomial map \( F = X - H \) of the form prescribed in Theorem 1.2, with \( d \geq 2 \) arbitrary, will be said to be of symmetric homogeneous type.

The condition \( JH \) is symmetric is equivalent to the existence of a homogeneous polynomial \( P \in K[X_1, \ldots, X_n] \) with \( H = \nabla P \). \( P \) is called the potential function for \( H \). Thus the symmetric case occurs precisely when the Jacobian matrix of \( H \) is the Hessian matrix of \( P \):

\[
JH = \text{Hess} \ P = (D_iD_j \ P)
\]

If \( H \) is homogeneous of degree \( d \), \( P \) can, of course, be taken to be homogeneous of degree \( d + 1 \).
2 Formulas for the Formal Inverse

The formulas for the formal inverse given in this section provide means for the Jacobian Conjecture to be addressed as a problem in combinatorics. See [8] for a full discussion of this approach. These formulas are valid for systems of power series $F = (F_1, \ldots, F_n)$ where, for $i = 1, \ldots, n$, $F_i \in K[[X_1, \ldots, X_n]]$ has the form $X_i + \text{higher degree terms}$. We call such a map a formal map of special type. Such a map has a unique formal inverse, that is, a formal map $F^{-1} = G = (G_1, \ldots, G_n)$ of special type having the property that $F \circ G = G \circ F = (X_1, \ldots, X_n)$.

2.1 The Tree Formula of Bass-Connell-Wright

Let $\mathbb{T}_r$ be the set of isomorphism classes of finite rooted trees. For $G = F^{-1}$, the Tree Formula of Bass-Connell-Wright (best reference for this is [7]) states:

**Theorem 2.1 (BCW Tree Formula).** Let $F = X - H$ be a formal map, and let $G = (G_1, \ldots, G_n)$ be the formal inverse. Writing $G = X + N$, with $N = (N_1, \ldots, N_n)$, we have

$$N_i = \sum_{T \in \mathbb{T}_r} \frac{1}{|\text{Aut} T|} \, \mathcal{P}_{T,H,i}$$

where

$$\mathcal{P}_{T,H,i} = \sum_{\ell : V(T) \to \{1, \ldots, n\}, \ell(v) = i} \prod_{v \in V(T)} D_{\ell'(v)} H_{\ell(v)}.$$  

In this expression $v^+$ is the set $\{w_1, \ldots, w_t\}$ of children of $v$ and $D_{\ell'(v)} = D_{\ell'(w_1)} \cdots D_{\ell'(w_t)}$.

In the case where $F = X - H$ with $H$ homogeneous of degree $d \geq 2$, the polynomial $\mathcal{P}_{T,H,i}$ is homogeneous of degree $m(d-1) + 1$ where $m = |V(T)|$, the number of vertices in the tree $T$. Hence letting $\mathbb{T}_{r,m}$ be the set of trees in $\mathbb{T}_r$ having $m$ vertices, and letting $\mathcal{P}_{T,H} = (\mathcal{P}_{T,H,1}, \ldots, \mathcal{P}_{T,H,n})$, we have:

**Theorem 2.2 (Bass-Connell-Wright Homogeneous Tree Formula).** Let $F = X - H$ be a polynomial map with $H$ homogeneous of degree $d$. Then the formal inverse has the form $G = X + N$ where

$$N = N^{(1)} + N^{(2)} + N^{(3)} + \cdots$$

with $N^{(m)}$ homogeneous of degree $m(d-1) + 1$ and given by the formula

$$N^{(m)} = \sum_{T \in \mathbb{T}_{r,m}} \frac{1}{|\text{Aut} T|} \, \mathcal{P}_{T,H}.$$  \hspace{1cm} (2.1)
2.2 The Tree Formula for the Symmetric Case

The formula of Bass-Connell-Wright takes on a simpler form in the symmetric case. We now let $T$ be the set of isomorphism classes of finite free trees (i.e., having no designated root).

**Theorem 2.3 (Symmetric Tree Formula).** Let $F = X - \triangledown P$ be a symmetric formal map, and let $G = (G_1, \ldots, G_n)$ be its inverse. Then $G = X + \triangledown Q$ with

$$Q = \sum_{T \in T} \frac{1}{\lvert \text{Aut} T \rvert} Q_{T,P}$$

where

$$Q_{T,P} = \sum_{\ell : E(T) \to \{1, \ldots, n\}} \prod_{v \in V(T)} D_{\text{adj}(v)} P,$$

Here $\text{adj}(v)$ is the set $\{e_1, \ldots, e_s\}$ of edges adjacent to $v$ and $D_{\text{adj}(v)} = D_{\ell(e_1)} \cdots D_{\ell(e_s)}$.

A somewhat similar formula appears without proof in [5].

**Proof.** In the case where $H = \triangledown P$, the expression $H_{i(v)}$ becomes $D_{i(v)} P$, hence

$$\mathcal{P}_{T,H} = \prod_{v \in V(T)} D_{k(v)} D_{i(v)} P = \prod_{v \in V(T)} D_{k(v) + e_i(v)} P.$$

Given $i \in \{1, \ldots, n\}$, $T \in T$, $\ell : E(T) \to \{1, \ldots, n\}$, and $w \in V(T)$, we create a rooted tree $T_w$ by declaring $w$ to be the root, and create a labeling $\ell_w : V(T) \to \{1, \ldots, n\}$ by giving $w$ the label $i$ and moving the label of each edge $e \in E(V)$ to the vertex $v$ adjacent to $e$ which is farthest from $w$. Let $k_w(v)$ be the child type of $v$ in $T_v$ resulting from this labeling.

We claim that $Q$ as defined in the theorem is the potential function for

$$N = \sum_{T \in T_{\text{rt}}} \frac{1}{\lvert \text{Aut} T \rvert} \mathcal{P}_{T,H},$$

that is to say, $D_i Q = N_i$ for $i = 1, \ldots, n$. To see this, note that:

$$D_i Q = D_i \sum_{T \in T} \frac{1}{\lvert \text{Aut} T \rvert} Q_{T,P}$$

$$= D_i \left( \sum_{T \in T} \frac{1}{\lvert \text{Aut} T \rvert} \sum_{\ell : E(T) \to \{1, \ldots, n\}} \prod_{v \in V(T)} D_{\text{adj}(v)} P \right)$$

$$= \sum_{T \in T} \frac{1}{\lvert \text{Aut} T \rvert} \sum_{\ell : E(T) \to \{1, \ldots, n\}} \sum_{w \in V(T)} \prod_{v \in V(T)} D_{\text{adj}(v) + \delta_{w,v,e_i}} P$$
Denoting by $\bar{S}$, for $S \in T_{\text{rt}}$, the unrooted tree determined by $S$, ignoring the root, we have

$$= \sum_{S \in T_{\text{rt}}} \sum_{w \in V(\bar{S}) \atop \bar{S}_w \cong_{T_{\text{rt}}} S} \frac{1}{|\text{Aut } \bar{S}|} \mathcal{P}_{S, \nabla P, i} \mathcal{P}_{T_w, \nabla P, i}$$

$$= \sum_{S \in T_{\text{rt}}} \frac{1}{|\text{Aut } \bar{S}|} \mathcal{P}_{S, \nabla P, i}$$

$$= \sum_{S \in T_{\text{rt}}} \left| \frac{\{ w \in V(\bar{S}) \mid \bar{S}_w \cong_{T_{\text{rt}}} S \}}{|\text{Aut } \bar{S}|} \right| \mathcal{P}_{S, \nabla P, i}$$

$\text{Aut } \bar{S}$ acts on $V(\bar{S}) = V(S)$, the orbit of the root $r$ of $S$ being the set $\{ w \in V(\bar{S}) \mid \bar{S}_w \cong_{T_{\text{rt}}} S \}$. The stabilizer of $r$ in $\text{Aut } \bar{S}$ is $\text{Aut}_{T_{\text{rt}}} S$, so

$$\left| \{ w \in V(\bar{S}) \mid \bar{S}_w \cong_{T_{\text{rt}}} S \} \right| = \frac{|\text{Aut } \bar{S}|}{|\text{Aut}_{T_{\text{rt}}} S|}$$

and we get

$$= \sum_{S \in T_{\text{rt}}} \frac{1}{|\text{Aut}_{T_{\text{rt}}} S|} \mathcal{P}_{S, \nabla P, i}$$

$$= N_i,$$

which, since $\nabla P = H$, completes the proof. \qed

For the symmetric homogeneous case Theorem 2.3 gives the following. Here we let $T_m$ be the set of isomorphism classes of free (i.e., non-rooted) trees having $m$ vertices.

**Theorem 2.4 (Symmetric Homogeneous Tree Formula).** Suppose $F$ has the form $F = X - \nabla P$ with $P$ homogeneous of degree $d + 1$. Let $G$ be the formal
inverse of $F$. Then $G = X + \nabla Q$ with

$$Q = Q^{(1)} + Q^{(2)} + Q^{(3)} + \ldots$$

and

$$Q^{(m)} = \sum_{T \in \mathcal{T}_m} \frac{1}{|\text{Aut } T|} Q_T.$$

$Q^{(m)}$ is homogeneous of degree $m(d-1) + 2$.

It is clear that in this situation $N^{(m)} = \nabla Q^{(m)}$, where $N^{(m)}$ is as in Theorem 2.1.

2.3 Zhao’s Formulas and the Gap Theorem

The formula below of Zhao, proved in [9], has an important consequence for this discussion, namely the Gap Theorem (Theorem 2.6).

**Theorem 2.5 (Zhao’s Formula for the Symmetric Case).** As in Theorem 2.4, let $Q^{(m)}$, $m \geq 1$, be the homogeneous summands of the potential function for $N = G - X$, where $G$ is formal inverse of a degree $d$ polynomial map $F = X - \nabla P$ of symmetric homogeneous type. Then $Q^{(1)} = P$ and, for $m \geq 2$,

$$Q^{(m)} = \frac{1}{2(m-1)} \sum_{k+\ell=m, k, \ell \geq 1} \left( \nabla Q^{(k)} \cdot \nabla Q^{(\ell)} \right).$$

(Here $(\nabla Q^{(k)} \cdot \nabla Q^{(\ell)})$ denotes the usual dot product of vectors.)

Again it should be noted that this theorem holds in the nonhomogeneous case as well, giving nonhomogeneous, formally converging summands for the potential function for $N$.

The following theorem gives explicit finitude to showing that the polynomial inverse of a polynomial map of symmetric homogeneous type is a polynomial.

**Theorem 2.6 (Gap Theorem for the Symmetric Case).** Given the situation of Theorem 2.4, then $F$ is invertible, i.e., $G$ is a polynomial map, if we have

$$Q^{(M+1)} = Q^{(M+2)} = \ldots = Q^{(2M)} = 0$$

for some positive integer $M$.

**Proof.** This is immediate from formula 2.3 in Theorem 2.5.

3 Consequences

3.1 Trees with naked chains

In the case where $F = X - H$ with $H$ homogeneous of degree $d \geq 2$, then the invertibility of $JF$ is equivalent to $JH$ being nilpotent, in which case we must have $(JH)^n = 0$ (where $n$ is the number of variables). This motivates the following theorem:
Theorem 3.1 (Chain Vanishing Theorem). Suppose $P \in K[[X_1, \ldots, X_n]]$ with $(\text{Hess } P)^r = 0$ for some $r \geq 1$, and suppose $T$ is a tree which contains a “naked $r$-chain,” that is, a geodesic

$$(e_0) \bullet e_1 \bullet e_2 \bullet \cdots \bullet e_{r-1} \bullet (e_r)$$

meaning the vertices $v_2, \ldots, v_{r-1}$ have degree 2 and the two vertices $v_1, v_r$ have degree 1 or 2. Assume either (a) $P$ is a homogeneous polynomial of degree $\geq 2$, or (b) both $v_1$ and $v_r$ have degree 2. Then $Q_{T,P} = 0$.

Proof. First we assume (b) holds, i.e., $e_0$ and $e_r$ are actually there. Write $E(T)$ as the disjoint union $\{e_1, \ldots, e_{r-1}\} \cup E'$ and $V(T)$ as the disjoint union $\{v_1, \ldots, v_r\} \cup V'$. By definition (see Theorem 2.3) we have

$$Q_{T,P} = \sum_{\ell:E(T) \to \{1, \ldots, n\}} \prod_{v \in V(T)} D_{\text{adj}(v)} P$$

$$= \sum_{\ell':E' \to \{1, \ldots, n\}} \prod_{e \in E'} D_{\text{adj}(e)} P \sum_{\ell:e_{1},...,e_{r-1}\to\{1,\ldots,n\}} \prod_{v \in \{v_1, \ldots, v_r\}} D_{\text{adj}(v)} P$$

$$= \sum_{\ell':E' \to \{1, \ldots, n\}} \prod_{v \in V'} D_{\text{adj}(v)} P \sum_{i_1, \ldots, i_{r-1}} (D_{\ell'(e_0)} i_1 P) (D_{i_1 i_2} P) \cdots (D_{i_{r-2} i_{r-1}} P) (D_{i_{r-1} \ell'(e_r)} P)$$

$$= \sum_{i_1, \ldots, i_{r-1}} (D_{i_1} P) (D_{i_1 i_2} P) \cdots (D_{i_{r-2} i_{r-1}} P) (D_{i_{r-1} \ell'(e_r)} P) .$$

Since the $(ij)^{th}$ entry in the matrix Hess $P$ is $D_{ij} P$, the final summation above gives the $(\ell'(e_0) \ell'(e_r))^{th}$ entry in $(\text{Hess } P)^r$, which is zero by hypothesis. Therefore $Q_{T,P} = 0$.

Now assume (a) holds. We proceed as before and all the equalities above are valid except the last one, which assumes the existence of $e_0$ and $e_r$. If, say, $e_r$ is present but $e_0$ is not, then the final summation reads:

$$= \sum_{i_1, \ldots, i_{r-1}} (D_{i_1} P) (D_{i_1 i_2} P) \cdots (D_{i_{r-2} i_{r-1}} P) (D_{i_{r-1} \ell'(e_r)} P) .$$

Since $D_{ij} P$ is homogeneous of degree $d - 1$, Euler’s formula says $D_{ij} P = \frac{1}{d-1} \sum_{i_0=1}^{n} D_{i_0 i_j} P$. Thus the above sum is

$$= \frac{1}{d-1} \sum_{i_0, i_1, \ldots, i_{r-1}} (D_{i_0 i_1} P) (D_{i_1 i_2} P) \cdots (D_{i_{r-2} i_{r-1}} P) (D_{i_{r-1} \ell'(e_r)} P) ,$$

which vanishes, since $(\text{Hess } P)^r = 0$. Finally, of both $e_0$ and $e_r$ are absent, then

$$= \sum_{i_1, \ldots, i_{r-1}} (D_{i_1} P) (D_{i_1 i_2} P) \cdots (D_{i_{r-2} i_{r-1}} P) (D_{i_{r-1}} P) ,$$

which completes the proof.
and the proof is completed by applying Euler’s formula to both end factors $D_1 P$ and $D_r P$.

\[ \square \]

### 3.2 The Symmetric $JH^3 = 0$ Case

The following new result for the symmetric situation, announced in [8], will use the Symmetric Homogeneous Tree Formula and the Chain Vanishing Theorem (Theorems [2.4] and [3.1]).

**Theorem 3.2 (Symmetric Cube Zero Case).** If $F = X - H$ is a polynomial map with symmetric Jacobian matrix of homogeneous type with $(JH)^3 = 0$, then $F$ is invertible with

\[ F^{-1} = X + N^{(1)} + N^{(2)}. \]

In particular, the degree of $F^{-1}$ is $\leq 2d - 1$, where $d = \deg H$ (independent of $n$).

**Remark 3.3.** What is remarkable about the above statement is that it is independent of $n$, the number of variables. Moreover the form of $F^{-1}$ is independent of the degree $d$ of $H$. (The known bound for the degree of the inverse of an invertible polynomial map of degree $d$ is $d^n - 1$ (Gabber’s Theorem). See [1].)

**Proof.** By Gap Theorem (Theorem 2.6) it suffices to show that $Q^{(3)} = Q^{(4)} = 0$, where $Q^{(m)}$ is as defined in the Symmetric Homogeneous Tree Formula (Theorem [2.4]). But this is immediate from the following proposition.

**Proposition 3.4.** If $P \in K[X_1, \ldots, X_n]$ is homogeneous of degree $\geq 2$ with $(\text{Hess} P)^3 = 0$ and if $T$ is a tree with 3 or 4 vertices, then $Q_{T,P} = 0$.

**Proof.** The only tree with three vertices is the 3-chain $T_1 = \bullet \bullet \bullet$, and in this case $Q_{T_1,P} = 0$ by the Chain Vanishing theorem [8.1]. There are two trees with four vertices, namely

\[ T_1 = \bullet \bullet \bullet \bullet \quad \text{and} \quad T_2 = \bullet \bullet \bullet \bullet \]

We have $Q_{T_1,P} = 0$ by the Chain Vanishing Theorem. To get the vanishing of $Q_{T_2,P}$ we apply the operator $\sum_{i=1}^n (D_i P) D_i$ to $Q_{T_2,P} (= 0)$, where $T$, as above, is the 3-chain. We get:

\[
0 = \sum_{i=1}^n (D_i P)(D_i Q_{T,P})
= \sum_{i=1}^n (D_i P) \left( D_i \sum_{j,k} (D_j P)(D_{jk} P)(D_{k} P) \right)
\]
which becomes, using the product rule:

\[
= \sum_{i,j,k} (D_i P)(D_{ij} P)(D_{jk} P)(D_k P) \quad (3.2)
\]

\[
+ \sum_{i,j,k} (D_i P)(D_{ij} P)(D_{ijk} P)(D_k P) \quad (3.3)
\]

\[
+ \sum_{i,j,k} (D_i P)(D_{jk} P)(D_{ki} P)(D_{k} P). \quad (3.4)
\]

Note that (3.2) and (3.4) are each equal to \( Q_{T_1, P} \), and the (3.3) is \( Q_{T_2, P} \). Thus we have

\[2Q_{T_1, P} + Q_{T_2, P} = 0.\]

Since \( Q_{T_1, P} = 0 \), we must have \( Q_{T_2, P} = 0 \) as well.

### 3.3 The Grossman-Larson Algebra

The proof of Proposition 3.4 entails operations that hearken to a ring defined by Grossman and Larson in [3], which we will now define as a \( \mathbb{Q} \)-algebra.

Let \( \mathcal{H}_{GL} \), or simply \( \mathcal{H} \), be the vector space over \( \mathbb{Q} \) spanned by \( \mathcal{T}_{rt} \), the set of all rooted trees. To explain multiplication in \( \mathcal{H} \) it will be necessary to introduce some concepts and notations.

First, let \( S \) be a rooted tree, \( T \) a (possibly non-rooted) tree, and let \( v \in V(T) \). We denote by \( v \in V(T) \) the tree which joins \( T \) to \( S \) by introducing a new edge \( e \) which connects \( \text{rt}_S \) to \( v \). If \( T \) is a rooted tree, then \( S \rightarrow_v T \) is rooted by \( \text{rt}_T \).

Similarly if \( S_1, \ldots, S_r \) are rooted trees and \( v_1, \ldots, v_r \in V(T) \), we can form the tree

\[
(S_1, \ldots, S_r) \rightarrow_{(v_1, \ldots, v_r)} T,
\]

which attaches \( S_i \) to \( T \) at \( v_i \), for \( i = 1, \ldots, r \). Again, if \( T \) is rooted, we take \( \text{rt}_T \) to be the root of the newly formed tree.

Secondly, if \( S \) is a rooted tree, let \( \text{DelRoot}(S) \) denote the forest (meaning a set with multiplicity of rooted trees) of branches of \( \text{rt}_S \). This means we delete the root of \( S \) and its adjacent edges; the children of \( \text{rt}_S \) become the roots of the trees in \( \text{DelRoot}(S) \).

Now we define the multiplication in \( \mathcal{H} \). For rooted trees \( S \) and \( T \), we write \( \text{DelRoot}(S) = \{S_1, \ldots, S_r\} \) (incorporating multiplicity) and we define the product \( S \cdot T \) by

\[
S \cdot T = \sum_{(v_1, \ldots, v_r) \in V(T)^r} [(S_1, \ldots, S_r) \rightarrow_{(v_1, \ldots, v_r)} T] \quad (3.5)
\]

This multiplication is extended to \( \mathcal{H} \) by distributivity. One quickly checks that the singleton serves as a left and right multiplicative identity element. In [3] it is shown that the multiplication is associative (a fact which is not hard to verify), and that \( \mathcal{H} \) has the additional structure of a Hopf algebra, a property which will not be used here.

An important thing to note is that \( \mathcal{H} \) is a graded ring by the grading \( \mathcal{H} = \bigoplus_{i=0}^{\infty} \mathcal{H}_i \), where \( \mathcal{H}_i \) spanned by trees having \( i \) non-root vertices, i.e., by \( T_{rt_{i+1}} \).
Now we let $M$ be the $\mathbb{Q}$-vector space spanned by the set $T$ of all non-rooted trees. We observed that $S \triangleleft v T$ forms a non-rooted tree when $S$ is rooted and $T$ is non-rooted; one can use $3.5$ to endow $M$ with the structure of an $H$-module, which we will call the tree module. In fact, $M$ is a graded $H$-module $M = \bigoplus_{i=1}^{\infty} M_i$ taking $M_i$ to be the vector space spanned by $T_i$.

**Definition 3.5 (Free tree quotient modules).** For a positive integer $r$, let $\mathcal{C}(r)$ denote the sub-$H$-module of $M$ generated by all trees containing a naked $r$-chain (see Theorem 3.1 for the definition). Let $\mathcal{V}(r)$ denote the sub-vector space (over $\mathbb{Q}$) generated by all trees which have at least one vertex of degree $\geq r + 1$. It is easily seen that $\mathcal{V}(r)$ is also a sub-$H$-module. For positive integers $r, e$, let $N(r, e) = C(r) + V(e)$. These are graded submodules of $M$. Finally, let $\overline{M}(r, e) = M/N(r, e)$ and let $\overline{M}(r, \infty) = M/C(r)$. The $H$-modules $\overline{M}(r, e)$ ($e$ possibly being $\infty$) will be called the tree quotient modules.

Given $\gamma \in M$ we will often denote by $\gamma$ its image in $\overline{M}(r, e)$, where $r$ and $e$ are understood in the context of the discussion.

**3.4 Relationship to the Ring of Differential Operators**

We write $\mathcal{D}[X] = \mathcal{D}[X_1, \ldots, X_n]$ for the ring of differential operators on $K[X] = K[X_1, \ldots, X_n]$. A polynomial $P \in K[X]$ gives rise to a ring homomorphism

$$\varphi_P : \mathcal{H} \to \mathcal{D}[X]$$

which we will be defined as follows: For a rooted tree $S$, we let $e_1, \ldots, e_r$ be the edges adjacent to $rt_S$ and define the differential operator $\partial_{S, P} \in \mathcal{D}[X]$ by

$$\partial_{S, P} = \sum_{E(S) \to \{1, \ldots, n\}} \left( \prod_{v \in V(S) - \{rt_S\}} D_{\text{adj}(e)} P \right) D_{t(e_1), t(e_2), \ldots, t(e_r)}.$$

Note the similarity with the definition of the polynomial $Q_{T, P}$ (Theorem 2.3) for a free tree $T$; the difference is that here we omit $rt_S$ from the product and leave “open” the derivatives corresponding to edges adjacent to $rt_S$.

Taking $\varphi_P(S) = \partial_{S, P}$ defines $\varphi_P$ on $\mathcal{H}$ as a $\mathbb{Q}$-linear map; in fact, it is straightforward to show that $\varphi_P$ is a ring homomorphism.

Now we define a map

$$\rho_P : M \to K[X]$$

by sending an unrooted tree $T$ to $\varphi_T$. Again, it is straightforward to verify that this map is compatible with the structures of $M$ as an $H$-module and $K[X]$ as a $\mathcal{D}[X]$-module, that is, the diagram

$$\mathcal{H} \times M \to M \quad \quad \quad \quad \downarrow \quad \quad \quad \quad \downarrow$$

$$\mathcal{D}[X] \times K[X] \to K[X]$$

commutes, where the horizontal arrows are induced by the module structures and the vertical arrows are $\varphi_P \times \rho_P$ and $\rho_P$. Now we observe:
Proposition 3.6. For $P \in K[X_1, \ldots, X_n]$ and positive integers $r, e$ we have:

1. If $P$ is homogeneous with $\text{Hess}(P)^r = 0$, then $\rho_P(\mathcal{C}(r)) = 0$.

2. If $\deg P \leq e$, then $\rho_P(\mathcal{V}(e)) = 0$.

Thus if $P$ is homogeneous of degree $\leq e$ with $\text{Hess}(P)^r = 0$ then $\rho_P$ induces a homomorphism $\mathcal{P}_P(r, e) : \mathcal{M}(r, e) \to K[X]$ such that

$$\mathcal{H} \times \mathcal{M}(r, e) \to \mathcal{M}(r, e)$$

$$\mathcal{D}[X] \times K[X] \to K[X]$$

(3.9)

commutes, where the horizontal arrows are induced by the module structures and the vertical arrows are $\varphi_P \times \mathcal{P}_P(r, e)$ and $\mathcal{P}_P(r, e)$. The last statement also holds for $e = \infty$.

Proof. Statement 1 follows from the Chain Vanishing Theorem 3.1. Statement 2 follows from the definition of $\mathcal{Q}_T,P(r) \leq 2.2$ with the observation that $D_{\text{adj}(\nu)}P = 0$ if $\nu \in V(T)$ has degree $\geq e + 1$.

Definition 3.7. For $m \geq 1$ let $\nu_m \in \mathcal{M}$ be defined by

$$\nu_m = \sum_{T \in \mathcal{T}_m} \frac{1}{|\text{Aut} T|} T.$$ 

Note that $\nu_m$ is homogeneous of degree $m$, i.e., $\nu_m \in \mathcal{M}_m$, and that, for $P \in K[X]$ homogeneous, $\rho_P(\nu_m) = Q^{(m)}$, where $Q^{(m)}$ is as defined in Theorem 2.3. It follows that

$$\mathcal{P}_P(\nu_m) = Q^{(m)}$$

(3.10)

where $\mathcal{P}_m$ is the image of $\nu_m$ in $\overline{\mathcal{M}}(r, e)$, whenever $\mathcal{P}_P$ makes sense by virtue of Proposition 3.6.

3.5 The Symmetric $JH^3 = 0$ Case Revisited

We will now observe that the proof of Theorem 2.2 boils down to a statement about the $\mathcal{H}$-module $\mathcal{M}(3, \infty)$. The theorem followed from the fact that $Q^{(3)} = Q^{(4)} = 0$ when $P$ is homogeneous and $\text{Hess}(P)^3 = 0$. Since $\mathcal{P}_P$ is defined for $r = 3, e = \infty$, in this situation (by Proposition 3.6), this would follow from $\mathcal{P}_3 = 0$ in $\overline{\mathcal{M}}(3, \infty)$, by 3.10. But in fact we have, more strongly:

Proposition 3.8. In the graded module $\overline{\mathcal{M}}(3, \infty) = \bigoplus_{i=1}^{\infty} \mathcal{M}(3, \infty)_i$ the homogeneous summands $\mathcal{M}(3, \infty)_3$ and $\mathcal{M}(3, \infty)_4$ are both zero.

Proof. Let $T, T_1$, and $T_2$ be as defined in the proof of Proposition 3.1 We will write $\bar{S}$ for the image in $\overline{\mathcal{M}}(3, \infty)$ of a tree $S$. Since $\mathcal{M}_3 = \mathcal{Q} \cdot T$ and $T$ is the chain of length 3, $\bar{T} = 0$ and hence $\overline{\mathcal{M}}(3, \infty)_3 = 0$. We have $\mathcal{M}_4 = \mathcal{Q} \cdot T_1 \oplus \mathcal{Q} \cdot T_2$.
and since $T_1 = 0$ (since $T_1$ is the chain of length 4), $\overline{\mathcal{M}}(3, \infty)$ is generated over $\mathbb{Q}$ by $T_2$. Now note that, letting $S$ be the rooted chain of length 2, i.e.,

$$S = \begin{array}{c}
\includegraphics[width=0.1\textwidth]{tree.png}
\end{array}$$

then the $\mathcal{H}$ action on $\mathcal{M}$ gives $S \cdot T = 2T_1 + T_2$, from which it follows that $T_2 \in \mathcal{C}(3)$. Therefore $\overline{T}_2 = 0$ and so $\overline{\mathcal{M}}(3, \infty)_4 = 0$, completing the proof. \(\square\)

### 3.6 The Quadratic Symmetric $JH^4 = 0$ Case

Computations in the $\mathcal{H}$-modules $\overline{\mathcal{M}}(r, e)$ allow us to obtain certain specific results when $JH$ is nilpotent of higher order, for certain specific degrees. For example:

**Theorem 3.9.** Let $F = X - H$ be a polynomial map having symmetric Jacobian matrix, with $H$ quadratic homogeneous and $(JH)^4 = 0$. Then $F$ is invertible with

$$F^{-1} = X + N^{(1)} + N^{(2)} + N^{(3)} + N^{(4)}.$$  

In particular, the degree of $F^{-1}$ is $\leq 5$.

**Remark 3.10.** Of course the Jacobian Conjecture is known to be true for quadratic maps. This was proved by S. Wang; a simple proof due to S. Oda can be found in [1]. However, Theorem 3.9 yields more strongly the uniform degree bound of 5 for $F^{-1}$, when $F$ is as in the theorem, independent of the number of variables. Again recall that the general known degree bound here is $2n - 1$ (see remark 3.3).

**Proof.** The proof will entail an explicit computation in the $\mathcal{H}$-module $\overline{\mathcal{M}}(4, 3)$. It follows from the Gap Theorem (Theorem 2.6) that it suffices to prove $Q^{(m)} = 0$ for $5 \leq m \leq 8$, where $Q^{(m)}$ is defined as in Theorem 2.4.

We have $H = \nabla P$ where $P \in K[X]$ is homogeneous cubic. According to Proposition 3.11, the map $\overline{\mathcal{M}}(4, 3)_m = 0$ for $5 \leq m \leq 8$.

**Proposition 3.11.** In the graded module $\overline{\mathcal{M}}(4, 3) = \bigoplus_{i=1}^{\infty} \overline{\mathcal{M}}(4, 3)_i$ we have $\overline{\mathcal{M}}(4, 3)_m = 0$ for $5 \leq m \leq 8$.

**Proof.** Let $S, S', S'', S''', S'''''$ and $S''''''$ be the rooted trees appearing in Figure [1], the bottom vertex being the root. These will be viewed as elements of the Grossman-Larson algebra $\mathcal{H}$.

Figures 2 through 5 give complete lists of free (i.e., unrooted) trees with $m$ vertices, for $5 \leq m \leq 8$. Viewing these free trees as elements of $\overline{\mathcal{M}}$, our goal is

\footnote{In these lists the trees are ordered by their maximal rooted planar representative. Rooted planar trees are ordered by considering first the number of vertices, then, if those are the same the lexicographical ordering of the root branches, considered left to right. Inductively, this}
We first consider the three trees with 5 vertices, identified in Figure 2. Obviously $A_1 \in \mathcal{C}(4)$ and $A_3 \in \mathcal{V}(3)$. Furthermore, letting $A$ be the chain with four vertices (hence $A \in \mathcal{C}(4)$), we have $S \cdot A = 2A_1 + 2A_2$, which shows $A_2 \in \mathcal{C}(4)$.

Therefore $M(4, 3)_5 = 0$.

Figure 3 lists and labels the six trees with 6 vertices. In $\mathcal{M}_6$, note that $B_1 \in \mathcal{C}(4)$ and that $B_4, B_6 \in \mathcal{V}(3)$. Furthermore we have $S' \cdot A = 2B_1 + 2B_3$ which gives $B_3 \in \mathcal{C}(4)$. The equation $S \cdot A_1 = 2B_1 + 2B_2 + B_3$ shows $B_2 \in \mathcal{C}(4)$. Finally, we note that $S'' \cdot A = 2B_1 + 6B_2 + 4B_3 + 2B_4 + 2B_5$, which shows that $B_5 \in \mathcal{C}(4)$ as well. Hence $M(4, 3)_6 = 0$.

gives a total ordering of rooted planar trees. Letting $T_n$ be the number of rooted trees having $n$ vertices, let $T(x) = \sum_{p=1}^{\infty} T_p x^p$ be the generating function for rooted trees. Then $T_n$ can be calculated using the following formula, due to G. Pólya:

$$T(x) = x \exp \left\{ \frac{\sum_{k=1}^{\infty} T(x^k)}{k} \right\}$$

Then the number $t_n$ of free trees with $n$ vertices is determined by the formula of R. Otter:

$$t(x) = T(x) - \frac{1}{2} \left\{ (T(x))^2 - T(x^2) \right\}$$

where $t(x) = \sum_{p=1}^{\infty} t_p x^p$. The first few values of $t_n$ have been found to be:

$t_1 = 1, t_2 = 1, t_3 = 1, t_4 = 2, t_5 = 3, t_6 = 6, t_7 = 11, t_8 = 23, t_9 = 47, t_{10} = 106, t_{11} = 235$

This confirms that the lists in Figures 2 through 5 are complete. See [4] as a reference for the facts in this footnote.
$M_7$ is generated over $\mathbb{Q}$ by the eleven trees $C_1, \ldots, C_{11}$ listed in Figure 4. Note that $C_1$ and $C_2$ lie in $C(4)$ and that $C_4, C_7, C_8, C_{10}, C_{11} \in V(3)$, which leaves $C_3, C_5, C_6,$ and $C_9$. We have

\begin{align*}
S \cdot B_1 &= 2C_1 + 2C_2 + 2C_3 \implies C_3 \in C(4) \\
S' \cdot A_1 &= 2C_1 + 2C_3 + C_6 \implies C_6 \in C(4) \\
S'' \cdot A &= 2C_2 + 2C_5 \implies C_5 \in C(4) \\
S \cdot B_2 &= C_2 + 2C_3 + C_4 + C_5 + C_9 \implies C_9 \in N(4, 3)
\end{align*}

This establishes that $M(4, 3)_{7} = 0$.

Lastly we tackle $M_8$, which is generated over $\mathbb{Q}$ by the unrooted trees
$D_1, \ldots, D_{23}$ given in Figure 5. Apparently $D_1, D_2, D_3, D_4 \in \mathcal{C}(4)$ and $D_4, D_8, D_9, D_{12}, D_{14}, D_{15}, D_{16}, D_{17}, D_{19}, D_{21}, D_{22}, D_{23} \in \mathcal{V}(3)$, leaving us to deal with $D_5, D_6, D_7, D_{10}, D_{11}, D_{13}, D_{18}, D_{20}$. Toward that end we observe

\begin{align*}
S \cdot C_1 &= 2D_1 + 2D_2 + 2D_3 + D_5 \implies D_5 \in \mathcal{C}(4) \\
S' \cdot B_1 &= 2D_1 + 2D_3 + 2D_7 \implies D_7 \in \mathcal{C}(4) \\
S''' \cdot A &= 2D_2 + 2D_{10} \implies D_{10} \in \mathcal{C}(4) \\
S''' \cdot B_2 &= D_2 + 2D_5 + D_8 + D_{10} + D_{13} \implies D_{13} \in \mathcal{N}(4,3) \\
S'' \cdot A_1 &= 2D_2 + 2D_6 + D_{13} \implies D_6 \in \mathcal{N}(4,3) \\
S \cdot C_2 &= D_2 + 2D_3 + D_4 + D_6 + D_{10} + D_{18} \implies D_{18} \in \mathcal{N}(4,3) \\
S'' \cdot A_2 &= 2D_{10} + D_{14} + D_{18} + D_{20} \implies D_{20} \in \mathcal{N}(4,3)
\end{align*}

showing that $M(4,3)_8 = 0$ and completing the proof.

\[ \square \]

### 3.7 Questions About the Tree Quotient Modules

The proofs in sections 3.5 and 3.6 raise interesting questions about the tree quotient modules $M(r,e)$. For example, we established in Propositions 3.8 and 3.11 that $M(3,\infty)_3 = M(3,\infty)_4 = 0$. In fact, the author can prove a far stronger statement which shows that the tree quotient module $M(3,\infty)$ is quite small:

**Theorem 3.12.** $M(3,\infty)_m = 0$ for $m \geq 3$, i.e.,

\[
M(3,\infty) = M(3,\infty)_1 \oplus M(3,\infty)_2,
\]

each of these two summands having vector space dimension 1 over $\mathbb{Q}$.

The proof will not be given here as it seems to have no implications for the Jacobian Conjecture.

We also established that $M(4,3)_m = 0$ for $5 \leq m \leq 8$. One can use the same methods to prove the vanishing of $M(4,3)_m$ for some larger values of $m$. So we ask:

**Question 3.13.** Is $M(4,3)_m = 0$ for $m \geq 5$?

Of course, an affirmative answer would (seemingly) not resolve any additional cases of the Symmetric Jacobian Conjecture. However, an affirmative answer to following question certainly would:

**Question 3.14.** Let $r$ be a positive integer. Does there exist a positive integer $M_r$ such that $M(r,4)_m = 0$ when $M_r + 1 \leq m \leq 2M_r$?

Or one could ask the weaker question:
Question 3.15. Let $r$ be a positive integer. Does there exist a positive integer $M_r$ such that $\tau_m = 0$ in $\mathfrak{M}(r,4)_m$ (see Definition 3.7) when $M_r + 1 \leq m \leq 2M_r$?

Or one could ask the stronger question:
**Question 3.16.** Let \( r \) be a positive integer. Does \( \mathbb{M}(r,4) \) have finite rank as a \( \mathbb{Q} \)-vector space?

which is equivalent to asking if \( \mathbb{M}(r,4)_m = 0 \) for \( m >> 0 \). It is obvious that the proof of Theorem 3.9 can be mimicked to show that:

**Theorem 3.17.** Let \( r \) be a positive integer. Assume Question 3.15 (or 3.14, or 3.16) has an affirmative answer for \( r \), and let \( F = X - H \) be a polynomial map of symmetric homogeneous type with \( H \) cubic and \( (JH)^r = 0 \). Then \( F \) is invertible with

\[
F^{-1} = X + N^{(1)} + N^{(2)} + N^{(3)} + \cdots + N^{(M_k)}.
\]

In particular, the degree of \( F^{-1} \) is \( \leq 2M_k + 1 \).

Whence, in light of Theorem 1.2:

**Theorem 3.18.** If Question 3.15 has an affirmative answer for all positive integers \( r >> 0 \), then the Jacobian Conjecture is true.

### 3.8 The Cubic Symmetric \( JH^4 = 0 \) Case

Questions 3.14 and 3.15 can be resolved by computer algorithm for any fixed \( r \), subject to time/space limitations. A computer program has been written and run by Li-Yang Tan which appears to resolve the cubic symmetric \( JH^4 = 0 \) case of the Jacobian Conjecture [6]. The result is intriguing. The program shows \( \mathbb{M}(4,4)_m = 0 \) for \( m = 8, 9, 10, 11, 12, 14 \). Curiously, \( \mathbb{M}(4,4)_{13} \neq 0 \) but rather has rank one. However the vector \( \mathbf{v}_m \) (see Definition 3.7) is zero in \( \mathbb{M}(4,4)_{13} \). Thus we have \( \mathbf{v}_m = 0 \) for \( m = 8, \ldots, 14 \), so the \( JH^4 = 0 \) case is solved, by Theorem 3.17. We state the theorem thus proved by computer:

**Theorem 3.19.** Let \( F = X - H \) be a polynomial map having symmetric Jacobian matrix, with \( H \) cubic homogeneous and \( (JH)^4 = 0 \). Then \( F \) is invertible with

\[
F^{-1} = X + N^{(1)} + N^{(2)} + N^{(3)} + N^{(4)} + N^{(5)} + N^{(6)} + N^{(7)}.
\]

In particular, the degree of \( F^{-1} \) is \( \leq 15 \).

### 3.9 Ideal Membership Theorems

In [8] the author formulated certain ideal membership questions, some of which can be answered in light of the results of this paper. Theorems 3.20 and 3.22 below, which were announced in [8], are strengthenings of Theorems 3.2 and 3.9, respectively. In these theorems, the ring \( R_{n,d}^{\text{sym}} \) is the \( \mathbb{Q} \)-algebra generated by the formal coefficients (indeterminates) \( c^q \) of the formal homogeneous polynomial

\[
P = \sum_{\| q \| = d+1} c^q X^q \text{ of degree } d.\]

Here \( q = (q_1, \ldots, q^n) \in \mathbb{N}^n, \| q \| = q_1 + \cdots + q_n \), and \( X^q = X^{q_1} \cdots X^{q_n} \). The reader is referred to [8] for further explanation of the notation.
Theorem 3.20. Let $F = X - H$ be the formal degree $d \geq 2$ polynomial map of symmetric homogeneous type in dimension $n$. In other words $H = \nabla P$ where $P$ is as above. Let $\mathfrak{I}$ be the ideal in $\mathfrak{R}_{n,[d]}^{\text{sym}}$ generated by the coefficients of $(JH)^3$. Then all coefficients $d^q$ of $Q^{(m)}$ for $m \geq 3$ (hence all $d^q$ with $|q| = m(d-1) + 2$ with $m \geq 3$) are in $\mathfrak{I}$.

Proof. This is immediate from Theorem 3.2, taking $K = \mathfrak{R}_{n,[d]}^{\text{sym}}/\mathfrak{I}$. (This is an advantage to allowing $K$ to be any $\mathbb{Q}$-algebra.)

In similar fashion, the following theorems results from Theorems 3.9 and 3.19:

Theorem 3.21. Let $F = X - H$ be the formal degree 2 polynomial map of symmetric homogeneous type in dimension $n$, and let $\mathfrak{I}$ be the ideal in $\mathfrak{R}_{n,[2]}^{\text{sym}}$ generated by the coefficients of $(JH)^4$. Then all coefficients $d^q$ of $Q^{(m)}$ for $m \geq 5$ (i.e. $|q| = 2m + 2$ for $m \geq 5$) are in $\mathfrak{I}$.

Theorem 3.22. Let $F = X - H$ be the formal degree 3 polynomial map of symmetric homogeneous type in dimension $n$, and let $\mathfrak{I}$ be the ideal in $\mathfrak{R}_{n,[3]}^{\text{sym}}$ generated by the coefficients of $(JH)^4$. Then all coefficients $d^q$ of $Q^{(m)}$ for $m \geq 8$ (i.e. $|q| = m + 2$ for $m \geq 8$) are in $\mathfrak{I}$.

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