Factorization of symmetric polynomials

Vadim B. Kuznetsov and Evgeny K. Sklyanin

Abstract. We construct linear operators factorizing the three bases of symmetric polynomials: monomial symmetric functions \( m_\lambda(x) \), elementary symmetric polynomials \( E_\lambda(x) \), and Schur functions \( s_\lambda(x) \), into products of univariate polynomials.

1. Introduction

In our recent paper \([1]\) written together with V.V. Mangazeev we used the lore of the quantum integrability to provide a new insight into the theory of symmetric polynomials. The main result of \([1]\) is a construction of a separation of variables for the Jack polynomials. We have explicitly described an integral operator \( S_n \) depending on the parameter \( \alpha \) which maps any Jack polynomial \( P_\lambda^{(\alpha)}(x) \) of \( n \) variables \((x_1, \ldots, x_n) \equiv x \) labelled by a partition \( \lambda \) of length \( n \) into a product \( \prod_{j=1}^n q_\lambda(z_j) \) of univariate polynomials \( q_\lambda(z) \). Some other integral operators closely related to \( S_n \) were also constructed and studied.

In case of the generic Jack polynomials, the proofs presented in \([1]\) are quite involved, and the simple underlying philosophy of the separation of variables is somewhat overshadowed by the abundant technicalities. The purpose of the present paper is to provide a gentler pedagogical introduction into the subject of \([1]\). We revise the results of \([1]\) for the three degenerate cases of the Jack polynomials corresponding to the three particular values of the parameter \( \alpha \). These are the three well-known bases of symmetric polynomials: monomial symmetric functions \( m_\lambda(x) \), elementary symmetric polynomials \( E_\lambda(x) \), and Schur functions \( s_\lambda(x) \) corresponding respectively to the values \( \infty \), 0, and 1 of the parameter \( \alpha \). For these cases the constructions of \([1]\) simplify drastically, and we provide an independent elementary proof for each of the three cases.

Let \( \mathbb{C}[x] \) be the ring of polynomials in \( x_1, \ldots, x_n \), and \( \mathbb{C}[x]^{S_n} \) be the corresponding subring of symmetric polynomials. The three bases \( m_\lambda(x) \), \( E_\lambda(x) \), and \( s_\lambda(x) \) in the ring \( \mathbb{C}[x]^{S_n} \) we are going to work with are all labelled by partitions \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{N}^n, \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0 \) of length \( n \). The weight \( |\lambda| \) of a
partition \( \lambda \) being defined as
\[
(1.1) \quad |\lambda| = \sum_{i=1}^{n} \lambda_i,
\]
the dominance partial ordering \( \preceq \) for two partitions \( \mu \) and \( \lambda \) is defined as follows:
\[
(1.2) \quad \mu \preceq \lambda \iff \{|\mu| = |\lambda|; \sum_{j=1}^{k} \mu_j \leq \sum_{j=1}^{k} \lambda_j, \quad k = 1, \ldots, n-1\}.
\]

We shall often use the notation \( \lambda_{i,j} \equiv \lambda_i - \lambda_j \), in particular, \( \lambda_{i,i+1} = \lambda_i - \lambda_{i+1} \) assuming \( \lambda_{n+1} \equiv 0 \), so that \( \lambda_{n,n+1} = \lambda_n \). For the rest of this section \( P_\lambda(x) \) will denote a member of any of the three polynomial families in question.

In our analysis of the three cases we shall follow the unified plan. First, we describe a family of \( n \) commuting differential operators \( \{H_j\}_{j=1}^{n} \) whose joint eigenfunctions are the polynomials \( P_\lambda(x) \):
\[
(1.3) \quad [H_j, H_k] = 0, \quad H_j P_\lambda = h_j(\lambda) P_\lambda, \quad j = 1, \ldots, n,
\]
with \( h_j(\lambda) \) being the corresponding eigenvalues. Thus, the above spectral problem puts the polynomials \( P_\lambda(x) \) into the framework of the theory of quantum integrable systems.

It is convenient for our purposes to replace the standard normalization of the polynomials \( P_\lambda(x) \) by the condition of the unit value at the special point \( 1 \equiv (1, \ldots, 1) \). For any polynomial \( P_\lambda(x) \) thus define the polynomial \( \tilde{P}_\lambda(x) \equiv P_\lambda(x)/P_\lambda(1), \bar{P}_\lambda(1) = 1. \)

We introduce then the polynomial \( q_\lambda(z) \in \mathbb{C}[z] \) by fixing all the arguments of \( \bar{P}_\lambda(x) \) except one:
\[
(1.4) \quad q_\lambda(z) = \bar{P}_\lambda(z, 1, \ldots, 1), \quad q_\lambda(1) = 1,
\]
and the operator \( Q_z : \mathbb{C}[x]^{S_n} \rightarrow \mathbb{C}[x]^{S_n} \otimes \mathbb{C}[z] \) by its eigenvectors and eigenvalues:
\[
(1.5) \quad Q_z \bar{P}_\lambda = q_\lambda(z) \bar{P}_\lambda.
\]

The crucial point is that the operator \( Q_z \) (which we call \( Q \)-operator) admits a simple description in terms independent of the basis \( \bar{P}_\lambda \). Note that, by definition, the operators \( Q_z \) commute:
\[
(1.6) \quad [Q_{z_1}, Q_{z_2}] = 0, \quad \forall z_1, z_2 \in \mathbb{C}.
\]

We show that the polynomials \( q_\lambda(z) \) satisfy an ordinary differential equation in \( z \) with the coefficients depending linearly on all eigenvalues \( h_j(\lambda) \) of the operators \( H_j \). The aforementioned differential equation in \( z \) bears the name separation equation. In other words, the eigenvalue \( q_\lambda(z) \) of the \( Q \)-operator solves the multiparameter spectral problem arising in the process of separation of variables. Therefore, these univariate polynomials become separated polynomials in the unified approach to the method of separation of variables which is explained below.

The next step is to take the composition of \( n \) copies of the \( Q \)-operator \( Q_{z_1} \ldots Q_{z_n} \) and the linear functional
\[
(1.7) \quad \rho_0 : \mathbb{C}[x]^{S_n} \rightarrow \mathbb{C} : f(x) \mapsto f(1).
\]

As a consequence of the commutativity of \( Q_z \) the resulting operator
\[
(1.8) \quad S_n = \rho_0 Q_{z_1} \ldots Q_{z_n}
\]
acts from $\mathbb{C}[x]^S_n$ into $\mathbb{C}[z]^S_n$ and factorizes the polynomials $\bar{P}_\lambda(x)$

$$(1.9) \quad S_n : \bar{P}_\lambda(x) \mapsto \prod_{j=1}^n q_\lambda(z_j)$$

into products of the univariate polynomials $q_\lambda(z)$. In this way, the original spectral problem $(1.15)$ gets mapped into a *multiparameter spectral problem* for the factorized eigenfunctions $\prod_{j=1}^n q_\lambda(z_j)$.

The expression $(1.8)$ can be simplified due to the existence of additional relations between $\rho_0$ and $Q_z$. Define $\rho_k$, $k = 0, \ldots, n - 1$ as

$$(1.10) \quad \rho_k : \mathbb{C}[x_{k+1}, \ldots, x_n]^{S_{n-k}} \mapsto \mathbb{C} : f(x_{k+1}, \ldots, x_n) \mapsto f(1, \ldots, 1)$$

and let $\rho_n = 1$. We prove then that there exist operators

$$(1.11) \quad A_k : \mathbb{C}[x_1, \ldots, x_k]^{S_k} \mapsto \mathbb{C}[x_1, \ldots, x_{k-1}]^{S_k} \otimes \mathbb{C}[z_k], \quad k = 1, \ldots, n$$

such that

$$(1.12) \quad \rho_{k-1}Q_z = A_k\rho_k.$$ 

(see [1], Proposition 6.1).

Using the commutation relations $(1.12)$ we can transform the original expression $\rho_0Q_{z_1} \cdots Q_{z_n}$ for $S_n$ into $A_1\rho_1Q_{z_2} \cdots Q_{z_n}$, then into $A_1A_2\rho_2Q_{z_3} \cdots Q_{z_n}$ etc., and finally, using $\rho_n \equiv 1$, into

$$(1.13) \quad S_n = A_1A_2 \cdots A_n,$$

see [1], Theorem 6.1. The factorization $(1.13)$ of the operator $S_n$ exhibits a triangularity property in the sense that each operator $A_k$ acts on $k$ variables only.

Applying the relations $(1.12)$ to the polynomial $\bar{P}_\lambda(x)$ we get a set of remarkable equalities

$$(1.14) \quad A_k : \bar{P}_\lambda(x_1, \ldots, x_k, 1, \ldots, 1) \mapsto \bar{P}_\lambda(x_1, \ldots, x_{k-1}, 1, \ldots, 1)q_\lambda(z_k).$$

Note that the equalities $(1.14)$ cannot be used to define the operators $A_k$ directly because the restricted polynomials $\bar{P}_\lambda(x_1, \ldots, x_k, 1, \ldots, 1)$ are not linearly independent in the space $\mathbb{C}[x_1, \ldots, x_k]^{S_k}$ and, consequently, $(1.14)$ contain a lot of nontrivial identities.

Another application of the operator $Q_z$ is to produce the operators raising the number of variables in the polynomials $\bar{P}_\lambda$. As shown in [1] in case of generic Jack polynomials the operator $Q_z$ taken at $z = 0$ can be decomposed uniquely

$$(1.15) \quad Q_0 = Q'_0 \mathcal{P}$$

into the projector

$$(1.16) \quad \mathcal{P} : \mathbb{C}[x_1, \ldots, x_n]^S_n \mapsto \mathbb{C}[x_1, \ldots, x_{n-1}]^{S_{n-1}} : f(x_1, \ldots, x_n) \mapsto f(x_1, \ldots, x_{n-1}, 0)$$

and an operator

$$(1.17) \quad Q'_0 : \mathbb{C}[x_1, \ldots, x_{n-1}]^{S_{n-1}} \mapsto \mathbb{C}[x_1, \ldots, x_n]^S_n.$$ 

The operator $Q'_0$ can be used then to add an extra variable $x_n$ to the polynomials $\bar{P}_{\lambda_1 \ldots \lambda_{n-1}}$

$$(1.18) \quad Q'_0 : \bar{P}_{\lambda_1 \ldots \lambda_{n-1}}(x_1, \ldots, x_{n-1}) \mapsto \bar{P}_{\lambda_1 \ldots \lambda_{n-1}0}(x_1, \ldots, x_n).$$
In the next three sections the program sketched above is fulfilled for each of the three symmetric polynomial bases: \( m_\lambda(x) \), \( E_\lambda(x) \), and \( s_\lambda(x) \). In each case our presentation follows the same routine:

(A) Basis \( P_\lambda \) and the normalized polynomials \( \bar{P}_\lambda \).
(B) Commuting differential operators \( \{H_j\}_{j=1}^n \) with the eigenfunctions \( \bar{P}_\lambda(x) \).
(C) \( Q \)-operator \( Q_z \) and its eigenvalues \( q_\lambda(z) \). Differential equation for \( q_\lambda(z) \).
(D) Separating operator \( S_n \) and its factorization into the chain of \( A \)-operators.
(E) Lifting operator \( Q_0' \).

2. Factorizing the basis \( m_\lambda(x) \)

2A. Basis \( m_\lambda(x) \). Let \( x^a \equiv x_1^{a_1} \cdots x_n^{a_n} \) for any \( a = (a_1, \ldots, a_n) \). The monomial symmetric functions \( m_\lambda(x) \) are defined as the sum over all distinct permutations \( \nu = (\nu_1, \ldots, \nu_n) \) of the partition \( \lambda = (\lambda_1, \ldots, \lambda_n) \)

\[
m_\lambda(x) = \sum x_1^{\nu_1} \cdots x_n^{\nu_n} = x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n} + \text{permuted terms}
\]

and are known to form a basis in \( \mathbb{C}[x]^{S_n} \). We normalize them introducing \( \bar{m}_\lambda(x) \equiv m_\lambda(x)/m_\lambda(1) \), see Section 1. Note that \( \bar{m}_\lambda(x) \) can be represented as a sum

\[
\bar{m}_\lambda(x) = \frac{1}{n!} \sum_{\sigma \in S_n} x^{\sigma(\lambda)}, \quad \bar{m}_\lambda(1) = 1,
\]

over all permutations \( \sigma \in S_n \).

The particular functions \( m_\lambda \) for

\[
\lambda = 1^{r} 0^{n-r}, \quad r = 0, \ldots, n
\]

are known as elementary symmetric functions. Equivalently, \( e_r(x) \) is the sum of all products of \( r \) distinct variables \( x_i \), so that \( e_0(x) = 1 \) and

\[
e_r(x) = m_{(1^r 0^{n-r})}(x) = \sum_{1 \leq i_1 < \ldots < i_r \leq n} x_{i_1} x_{i_2} \cdots x_{i_r}.
\]

Their generating function is

\[
w_n(t) = \prod_{i=1}^{n} (1 + tx_i) = \sum_{j=0}^{n} e_j(x) t^j.
\]

2B. Operators \( H_j \). The simple monomials \( x^a \) form a basis in \( \mathbb{C}[x] \) and are already factorized. They are joint eigenfunctions of \( n \) commuting differential operators

\[
D_j = x_j \frac{\partial}{\partial x_j}, \quad D_j x^a = a_j x^a, \quad j = 1, \ldots, n.
\]

The symmetric monomial functions \( \bar{m}_\lambda \), however, are not factorized, and the corresponding differential operators \( H_j \) are obtained as elementary symmetric functions in \( D_j \):

\[
H_j = e_j(D_1, \ldots, D_n), \quad H_j \bar{m}_\lambda = e_j(\lambda) \bar{m}_\lambda,
\]

\[
[H_j, H_k] = 0, \quad \forall j, k = 1, \ldots, n.
\]
2C. \textit{Q}-operator.\ The \textit{Q}-operator $Q_z$ on $\mathbb{C}[x]^{S_n}$ is defined in terms of its eigenvectors $\bar{m}_\lambda$ and eigenvalues $q_\lambda(z)$

\begin{equation}
Q_z \bar{m}_\lambda = q_\lambda(z) \bar{m}_\lambda, \quad q_\lambda(z) = \bar{m}_\lambda(z, 1, \ldots, 1) = \frac{1}{n} \sum_{j=1}^{n} z^{\lambda_j}.
\end{equation}

**Proposition 2.1.** The operator $Q_z$ admits an alternative description independent of the basis $\bar{m}_\lambda$:

\begin{align}
(2.8a) & \quad Q_z = \frac{1}{n} \sum_{j=1}^{n} z^{D_j}, \\
(2.8b) & \quad (Q_z f)(x) = \frac{1}{n} \sum_{j=1}^{n} f(x_1, \ldots, x_{j-1}, z x_j, x_{j+1}, \ldots, x_n).
\end{align}

**Proof.** The expression (2.8) is well defined on any polynomial $f$, not necessarily symmetric. In particular, for the monomials $x^\lambda$ we have

\[ Q_z x^\lambda = \frac{1}{n} \sum_{j=1}^{n} z^{D_j} x^\lambda = \left( \frac{1}{n} \sum_{j=1}^{n} z^{\lambda_j} \right) x^\lambda = q_\lambda(z) x^\lambda. \]

Averaging the monomials $x^\lambda$ over the permutation group with (2.2) we recover (2.7).

2D. Separation of variables.

**Proposition 2.2.** For the separating operator $S_n$ defined by the formula (1.8) and satisfying (1.9) for $\bar{P}_\lambda \equiv \bar{m}_\lambda$ there exist uniquely defined operators $A_k$ satisfying the relations (1.12).

**Proof.** Applying the both sides of (1.12) to a polynomial $f \in \mathbb{C}[x]^{S_n}$ and using the formula (2.8b) for $Q_z f$ and the definition (1.10) of $\rho_k$ we obtain:

\begin{equation}
[\rho_{k-1}Q_z f](x_1, \ldots, x_n) = \frac{1}{n} f(z x_1, x_2, \ldots, x_{k-1}, 1, \ldots, 1) \\
+ \ldots \\
+ \frac{1}{n} f(x_1, \ldots, x_{k-2}, z x_{k-1}, 1, \ldots, 1) \\
+ \frac{1}{n} f(x_1, \ldots, x_{k-1}, z, 1, \ldots, 1) \\
+ \ldots \\
+ \frac{1}{n} f(x_1, \ldots, x_{k-1}, 1, \ldots, 1, z).
\end{equation}

The last $(n - k + 1)$ terms are all equal due to the symmetry of $f$. The right-hand side of (2.9) has obviously the form $A_k \rho_k$ where the operator $A_k$ is described in terms of the projector operators $P_{jk}$, $j < k$, defined as

\begin{equation}
(P_{jk} f)(x_1, \ldots, x_n) = f(\ldots, x_j x_k, \ldots, 1, \ldots).
\end{equation}
It is convenient to identify the variable $z_k$ in the target space with $x_k$. The formula for $A_k$ in terms of $P_{jk}$ is then

$$A_k = \frac{1}{n} \left( n - k + 1 + \sum_{j=1}^{k-1} P_{jk} \right), \quad A_1 \equiv 1.$$  

Note that the formula (2.11) allows to extend the operator $A_k$ to all nonsymmetric polynomials from $\mathbb{C}[x]$. It is not obvious at all that product of $A_k$ maps symmetric polynomials into symmetric. Nevertheless, the above analysis shows that it is true!

Using the relations

$$P_{jk} P_{lk} = P_{jk}, \quad \forall j, l = 1, \ldots, k - 1,$$

for the projectors $P_{jk}$ it is easy to invert $A_k$:

$$A^{-1}_k = \frac{1}{n - k + 1} \left( n - \sum_{j=1}^{k-1} P_{jk} \right), \quad k = 1, \ldots, n.$$

The above inversion formula for $A_k$ seems to be a peculiarity of the monomial functions case. Its analogs for the generic Jack polynomials are unknown.

**2E. Lifting operator.** Letting $z = 0$ in (2.8b) and using the symmetry of a polynomial $f \in \mathbb{C}[x]^{S_n}$ we can shuffle the arguments $x$ to obtain

$$Q_0 : f(x) \mapsto \frac{1}{n} \sum_{j=1}^{n} f(x)\big|_{x_j=0} = \frac{1}{n} \sum_{j=1}^{n} f(x_1, \ldots, \hat{x}_j, \ldots, x_n),$$

the hat over $x_j$ marking the omitted argument. Using the formula (2.14) for $Q_0$ and (1.15) we get the expression for the action of the operator $Q'_0$ on a polynomial $f \in \mathbb{C}[x_1, \ldots, x_{n-1}]^{S_{n-1}}$:

$$Q'_0 : f(x_1, \ldots, x_{n-1}) \mapsto \frac{1}{n} \sum_{j=1}^{n} f(x_1, \ldots, \hat{x}_j, \ldots, x_n).$$

**Proposition 2.3.** The operator $Q'_0$ defined by (2.15) acts on the basis $\bar{m}$ as follows:

$$Q'_0 : \bar{m}_{\lambda_1 \ldots \lambda_{n-1}}(x_1, \ldots, x_{n-1}) \mapsto \bar{m}_{\lambda_1 \ldots \lambda_{n-1}}0(x_1, \ldots, x_n).$$

**Proof.** The formula (2.15) defines $Q'_0$ for nonsymmetric polynomials from $\mathbb{C}[x_1, \ldots, x_{n-1}]$ as well. On the monomials we have

$$Q'_0 : x_1^{\lambda_1} \ldots x_{n-1}^{\lambda_{n-1}} \mapsto \frac{1}{n} \sum_{j=1}^{n} x_1^{\lambda_1} \ldots x_j^{\lambda_j-1} x_{j+1}^{\lambda_{j+1}} \ldots x_{n-1}^{\lambda_{n-1}}.$$  

After having symmetrized the left-hand side over $S_{n-1}$ we obtain in the right-hand side the average over $S_n$, hence (2.16).
3. Factorizing the basis $E_\lambda(x)$

3A. Basis $E_\lambda(x)$. For each partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ define the polynomials $E_\lambda(x)$ as

$$ E_\lambda(x) = e_1^{\lambda_1-\lambda_2}(x)e_2^{\lambda_2-\lambda_3}(x)\cdots e_n^{\lambda_n}(x) $$

where $e_j$, $j = 1, \ldots, n$, are elementary symmetric functions defined in (2.3)–(2.4).

The polynomials $E_\lambda$ form a basis in $\mathbb{C}[x]^{S_n}$, see [2]. Noticing that $e_j(1) = \binom{n}{j}$ we introduce the normalized polynomials

$$ \bar{e}_j(x) \equiv \frac{e_j(x)}{e_j(1)}, \quad \bar{E}_\lambda(x) \equiv \prod_{j=1}^{n} \bar{e}_j(x)^{\lambda_j+1}, $$

such that $\bar{e}_j(1) = 1$ and $\bar{E}_\lambda(1) = 1$.

3B. Operators $H_j$. Note that the polynomials $E_\lambda(x)$ are already given in a factorized form by the definition (3.1). The corresponding separation of variables is obtained by taking $\epsilon_j \equiv \bar{e}_j(x)$ as independent variables and using the isomorphism $\mathbb{C}[x]^{S_n} \simeq \mathbb{C}[\epsilon]$. The substitution $e_j(x) = \epsilon_j$ produces the separating linear operator

$$ \mathcal{S}_n^{(0)} : \mathbb{C}[x]^{S_n} \rightarrow \mathbb{C}[\epsilon] : E_\lambda(x) \mapsto \prod_{j=1}^{n} \epsilon_j^{\lambda_j+1}. $$

The monomials $\mathcal{S}_n^{(0)} E_\lambda$ are obviously the eigenfunctions of the commuting differential operators $D_j$ in variables $\epsilon$:

$$ D_j = \epsilon_j \frac{\partial}{\partial \epsilon_j}, \quad D_j \mathcal{S}_n^{(0)} E_\lambda = \lambda_{j,j+1}^{(0)} \mathcal{S}_n^{(0)} E_\lambda, \quad j = 1, \ldots, n. $$

It is an easy exercise to express the operators $D_j$ in terms of the original variables $x$. Indeed, using the generating function (2.4) for $e_j$ and taking the logarithmic differential of the both sides one obtains

$$ \sum_{i=1}^{n} \frac{t}{1+tx_i} \, dx_i = \sum_{j=1}^{n} \frac{t^j}{\prod_{m=1}^{n} (1+tx_m)} \, d\epsilon_j. $$

Comparing the residues at the poles one gets

$$ dx_i = \sum_{j=1}^{n} \frac{(-x_j)^{n-j}}{\prod_{m \neq i} (x_m - x_i)} \, d\epsilon_j, \quad i = 1, \ldots, n, $$

then

$$ \frac{\partial}{\partial \epsilon_j} = \sum_{i=1}^{n} \frac{(-x_i)^{n-j}}{\prod_{m \neq i} (x_m - x_i)} \frac{\partial}{\partial x_i}, \quad j = 1, \ldots, n, $$

and, finally,

$$ H_j \equiv (\mathcal{S}_n^{(0)})^{-1} D_j \mathcal{S}_n^{(0)} = e_j(x) \sum_{i=1}^{n} \frac{(-x_i)^{n-j}}{\prod_{m \neq i} (x_m - x_i)} \frac{\partial}{\partial x_i}, \quad j = 1, \ldots, n. $$

As a consequence, the operators $H_j$ mutually commute and have the functions $E_\lambda(x)$ as their eigenfunctions:

$$ [H_j, H_k] = 0, \quad H_j E_\lambda(x) = \lambda_{j,j+1} E_\lambda(x), \quad j = 1, \ldots, n. $$
The factorizing operator $S_n$ obtained from the constructions of $\Pi$ in the limit $\alpha \to 0$ and described below is, however, different from $S_n^{(0)}$ described above, illustrating thus the point that a separation of variables is not unique. Both operators $S_n$ and $S_n^{(0)}$ correspond to the same set of commuting differential operators $H_j$ but the separated functions and the separated equations are different.

3C. $Q$-operator. Define the operator $Q_z$ by its eigenvectors $\bar{E}_\lambda(x)$ and eigenvalues

\begin{equation}
Q_z: \bar{E}_\lambda \mapsto q_\lambda(z)\bar{E}_\lambda,
\end{equation}

\begin{equation}
q_\lambda(z) \equiv \bar{E}_\lambda(z,1,\ldots,1) = \prod_{j=1}^{n} \bar{e}_j(z,1,\ldots,1)^{\lambda_j,j+1} = \prod_{j=1}^{n} \left(1 + \frac{z - 1}{n}j\right)^{\lambda_j,j+1}.
\end{equation}

The polynomial $q_\lambda(z)$ is the unique polynomial solution of the differential equation

\begin{equation}
\frac{dq_\lambda(z)}{dz} = \sum_{j=1}^{n} \frac{\lambda_j,j+1}{z + \frac{n-j}{n}} q_\lambda(z)
\end{equation}
satisfying the normalization condition $q_\lambda(1) = 1$.

The factorized structure of the polynomials $E_\lambda(x)$ and $q_\lambda(z)$ implies that the action of the operator $Q_z$ on polynomials from $\mathbb{C}[x]$ is determined uniquely by its action on the elementary symmetric functions $\epsilon_j$:

\begin{equation}
Q_z : \epsilon_j \equiv \epsilon_j(x) \mapsto \left(1 + \frac{z - 1}{n}j\right) \epsilon_j.
\end{equation}

The formula (3.13) can be written in terms of the generating function (2.4) as follows:

\begin{equation}
Q_z : w_n(t) \mapsto \left(1 + \frac{z - 1}{n}t\frac{d}{dt}\right) w_n(t).
\end{equation}

Using the isomorphism $\mathbb{C}[x]^{S_n} \simeq \mathbb{C}[\epsilon]$ we observe that $Q_z$ acts on a polynomial $f \in \mathbb{C}[\epsilon]$ by substitution

\begin{equation}
Q_z : f(\epsilon_1,\ldots,\epsilon_n) \mapsto f(Q_z(\epsilon_1),\ldots,Q_z(\epsilon_n)).
\end{equation}

In fact, all operators we are going to construct in this section: $S_n$, $A_k$, $Q_0'$, share with $Q_z$ the same property of acting on $\mathbb{C}[\epsilon]$ by substitutions.

3D. Separation of variables. The separating operator $S_n$ is defined by the formula (1.8). By virtue of (1.9), it acts on the basis $\bar{E}_\lambda(x)$ as follows:

\begin{equation}
S_n : \bar{E}_\lambda \mapsto \prod_{j=1}^{n} q_\lambda(z_j).
\end{equation}

As explained above, the formula (3.10) implies that the operator $S_n$ acts on $\mathbb{C}[\epsilon]$ by substitutions:

\begin{equation}
S_n : \epsilon_j \mapsto \binom{n}{j} \prod_{i=1}^{n} \left(1 + \frac{z_i - 1}{n}j\right).
\end{equation}
(note the normalization factor \(n_j\) = \(e_j(1)\) in the right-hand side).

**Theorem 3.1.** For the separating operator \(S_n\) defined above there exist uniquely defined operators \(A_k\) satisfying the relations (1.12) and thus (1.13).

**Proof.** Applying the hypothetic operator equality (1.12) to the basis \(E_\lambda(x)\) we get the system of relations (1.14) for \(P_\lambda = E_\lambda\). The existence and uniqueness of the operators \(A_k\) will be established when we prove that the overdetermined system (1.14) has a unique solution. Because of the factorized nature of the polynomials \(E_\lambda(x)\) and \(q_\lambda(x)\) it is sufficient to consider the relations (1.14) for the elementary symmetric functions \(e_j(x)\) only. The resulting equations determining the action of \(A_k\) on \(\rho_k e_j\) are:

\[
A_k \rho_k e_j = \left(1 + \frac{z_k - 1}{n} j\right) \rho_{k-1} e_j, \quad j = 1, \ldots, n. \tag{3.18}
\]

The system (3.18) is apparently overdetermined because it counts \(n\) equations for only \(k\) independent quantities contained in \(\rho_k e_j\) (note that \(\rho_k e_j\) depend only on the variables \(x_1, \ldots, x_k\)). To prove that the equations (3.18) are nevertheless consistent let us sum them with the factor \(t^j\) and use the correspondence (3.13)–(3.14) to get

\[
A_k \rho_k w_n(t) = \left(1 + \frac{z_k - 1}{n} t \frac{d}{dt}\right) \rho_{k-1} w_n(t). \tag{3.19}
\]

From the definition (1.10) it follows that

\[
\rho_k w_n(t) = (1 + t)^{n-k} \prod_{i=1}^{k} (1 + tx_i) = (1 + t)^{n-k} w_k(t). \tag{3.20}
\]

Substituting \(\rho_k w_n(t)\) and \(\rho_{k-1} w_n(t)\) from (3.20) to (3.19) and dividing by \((1 + t)^{n-k}\) we get

\[
A_k w_k(t) = (1 + t)^{k-n} \left(1 + \frac{z_k - 1}{n} t \frac{d}{dt}\right) (1 + t)^{n-k+1} w_{k-1}(t). \tag{3.21}
\]

Simplifying the resulting expression and expanding it in \(t\) we finally get

\[
A_k : e_j^{(k)} \mapsto \left(1 + \frac{z_k - 1}{n} j\right) e_j^{(k-1)} + \epsilon^{(k-1)}_{j-1} \left(1 + \frac{z_k - 1}{n} (n - k + j)\right), \quad j = 1, \ldots, k - 1, \tag{3.22a}
\]

\[
A_k : e_k^{(k)} \mapsto z_k e_{k-1}^{(k-1)}, \tag{3.22b}
\]

where \(\epsilon^{(k)}_j\) is the \(j\)th elementary symmetric function in variables \(x_1, \ldots, x_k\).

Remarkably, because of the cancellation of the factor \((1 + t)^{n-k}\), the resulting system contains only \(k\) equations determining thus uniquely the action of \(A_k\) on \(e_j^{(k)}\).

**3E. Lifting operator.**

**Proposition 3.1.** The operator \(Q_z\) taken at \(z = 0\) admits a unique factorization (1.14) into the projector \(P\) (1.10) and the operator \(Q_0\) (1.17):

\[
Q_0 : E_{\lambda_1 \ldots \lambda_{n-1}}(x_1, \ldots, x_{n-1}) \mapsto E_{\lambda_1 \ldots \lambda_{n-1} 0}(x_1, \ldots, x_n). \tag{3.23}
\]
Proof. As usual, it is sufficient to consider the action of the studied operators on the elementary symmetric functions \( e_j \) only. Setting \( z = 0 \) in (3.13) we get

\[ Q_0 : e_j^{(n)} \mapsto \frac{n-j}{n} e_j^{(n)}. \]

From the definition (1.16) of the projector \( P \) it follows that

\[ P : e_j^{(n)} \mapsto e^{(n)}_{j-1}, \quad j = 1, \ldots, n-1, \]

\[ P : e_n^{(n)} \mapsto 0. \]

The unique factorization \( Q_0 = Q'_0 P \) is then obvious, with

\[ Q'_0 : e_j^{(n)} \mapsto \frac{n-j}{n} e_j^{(n)}, \quad j = 1, \ldots, n-1. \]

For the normalized polynomials (3.2) we get

\[ Q'_0 : \bar{e}_j^{(n)} \mapsto \bar{e}_j^{(n)}, \quad j = 1, \ldots, n-1, \]

hence (3.23).

4. Factorizing the basis \( s_\lambda(x) \)

4A. Basis \( s_\lambda(x) \). Let \( \delta = (n-1, n-2, \ldots, 0) \) and

\[ \mu = \lambda + \delta = (\lambda_1 + n - 1, \ldots, \lambda_{n-1} + 1, \lambda_n) \]

for any partition \( \lambda \). The Schur function \( s_\lambda(x) \) is defined then as the ratio of two antisymmetric polynomials

\[ s_\lambda(x) = \frac{\det\{x_\mu^i \}}{\det\{x_\delta^i \}} = \frac{a_\mu(x)}{a_\delta(x)}, \]

where

\[ a_\mu(x) = \det\{x_\mu^i \} = \begin{vmatrix} x_1^{\mu_1} & x_1^{\mu_2} & \cdots & x_1^{\mu_n} \\ x_2^{\mu_1} & x_2^{\mu_2} & \cdots & x_2^{\mu_n} \\ \vdots & \vdots & \ddots & \vdots \\ x_n^{\mu_1} & x_n^{\mu_2} & \cdots & x_n^{\mu_n} \end{vmatrix}, \]

and \( a_\delta(x) \) is the Vandermonde determinant \( \Delta_n(x) \)

\[ a_\delta(x) = \det\{x_\delta^i \} = \Delta_n(x) = \prod_{i<j} (x_i - x_j) \]

(note that the sign of \( \Delta_n(x) \) differs from \( \prod \) by the factor \((-1)^{n(n-1)/2}\)). In the formulas like (4.2) it is always assumed that in the expression under the det sign the index \( i \) is the number of the matrix row, and \( j \) of the column.

Lemma 4.1. The Schur function evaluated at \( x_k = \ldots = x_n = 1 \) is given by the formula

\[ s_\lambda(x_1, \ldots, x_{k-1}, 1, \ldots, 1) = \frac{a^{(k)}_\mu(x_1, \ldots, x_{k-1})}{a^{(k)}_\delta(x_1, \ldots, x_{k-1})}, \quad k = 1, \ldots, n, \]
H commuting differential operators

\[ a^{(k)}_{\mu}(x_1, \ldots, x_{k-1}) = \left| \begin{array}{cccc}
  x_1^{\mu_1} & x_1^{\mu_2} & \cdots & x_1^{\mu_n} \\
  \vdots & \vdots & & \vdots \\
  x_{k-1}^{\mu_1} & x_{k-1}^{\mu_2} & \cdots & x_{k-1}^{\mu_n} \\
  \mu_1 & \mu_2 & \cdots & \mu_n \\
  1 & 1 & \cdots & 1
\end{array} \right|.
\]

(4.5c) \[ a^{(k)}_{\delta}(x_1, \ldots, x_{k-1}) = \left( \prod_{i=1}^{n-k} i! \right) \left( \prod_{j=1}^{k-1} (x_j - 1)^{n-k+1} \right) \left( \prod_{i<j} (x_i - x_j) \right). \]

\textbf{Proof} is given by induction in \((n - k)\). For \(k = n\) the statement is obvious. Then, set \(x_{k-1} = 1 + \varepsilon\) in (4.5b) and expand \((1 + \varepsilon)^\mu\) in the \(k^{th}\) row in powers of \(\varepsilon\):

\[(1 + \varepsilon)^\mu = 1 + \left( \frac{\mu}{1} \right) \varepsilon + \cdots + \left( \frac{\mu}{n - k + 1} \right) \varepsilon^{n-k+1} + \cdots \]

The terms containing \(\mu\) in powers \(\leq (n - k)\) are cancelled from the determinant by subtracting from \(k^{th}\) row a linear combination of the lower rows. As a result, we get \(a^{(k)}_{\mu} \sim \varepsilon^{n-k+1} a^{k-1}_{\mu}/(n - k + 1)!\). The factor \(\varepsilon^{n-k+1}/(n - k + 1)!\) is cancelled then by the same factor in the expansion of the expression (4.5c) for \(a^{(k)}_{\delta}\).

In particular, for \(k = 1\) we get from (4.5) the formula for the normalization factor

\[ s_\lambda(1) = \prod_{i<j} \frac{\mu_i - \mu_j}{j - i} = \frac{\Delta_\lambda(\mu)}{\Delta_\lambda(\delta)}, \quad \Delta_\lambda(\delta) = 1!2!3! \cdots (n - 1)! \]

Using (4.6) we define the normalized Schur functions as \(s_\lambda(x) \equiv s_\lambda(x)/s_\lambda(1)\), so that \(s_\lambda(1) = 1\).

\textbf{4B. Operators} \(H_j\). Let \(\mathbb{C}[x]^{A_n}\) be the space of \textit{antisymmetric} polynomials. The multiplication by \(\Delta_\lambda(x)\) provides an isomorphism of linear spaces and the corresponding mapping of the bases

\[ \Delta_\lambda(x) : \mathbb{C}[x]^{A_n} \rightarrow \mathbb{C}[x]^{A_n} : s_\lambda(x) \mapsto a_\mu(x), \]

see (4.2) and (4.4). The polynomials \(a_\mu(x)\) given by (4.3) are linear combinations of monomials \(x^{\sigma(\mu)}\) and therefore are eigenfunctions of the following commuting differential operators in \(\mathbb{C}[x]^{A_n}\):

\[ H_j = e_j(D_1, \ldots, D_n), \quad \tilde{H}_j a_\mu = e_j(\mu) a_\mu. \]

\[ [\tilde{H}_j, \tilde{H}_k] = 0, \quad \forall j, k = 1, \ldots, n, \]

cf. (2.6). Conjugating the operators \(\tilde{H}_j\), with the factor \(\Delta_\lambda(x)\) we get the family of commuting differential operators \(H_j\) in \(\mathbb{C}[x]^{S_n}\) diagonalized by the basis \(s_\lambda(x)\):

\[ H_j = \Delta^{-1}_\lambda(x) e_j(D_1, \ldots, D_n) \Delta_\lambda(x), \quad H_j s_\lambda = e_j(\mu) s_\lambda, \]

\[ [H_j, H_k] = 0, \quad \forall j, k = 1, \ldots, n, \]
4C. \(Q\)-operator.

**Proposition 4.1.** The polynomial \(q_{\lambda}(z) \equiv \bar{s}_{\lambda}(z,1,\ldots,1)\) is given by the expression

\[
q_{\lambda}(z) = \frac{(n-1)!}{(z-1)^{n-1}} \phi_{\lambda}(z),
\]

where

\[
\phi_{\lambda}(z) = \sum_{j=1}^{n} c_j z^{\mu_j}, \quad c_j = \prod_{k \neq j} (\mu_j - \mu_k)^{-1}.
\]

**Proof.** Using the formulae (4.5a) and (4.6) we get

\[
q_{\lambda}(z) = \bar{s}_{\lambda}(z,1,\ldots,1) = s_{\lambda}(z,1,\ldots,1) = \Delta_n(\delta) a^{(2)}_{\mu}(z) \Delta_n(\mu).
\]

Substituting \(a^{(2)}_{\mu}(z) = (z-1)^{n-1}(n-2)!\ldots2!1!\) from (4.5c) and \(\Delta_n(\delta)\) from (4.6), and cancelling the factorials we transform (4.12) into

\[
q_{\lambda}(z) = \frac{(n-1)!}{(z-1)^{n-1}} \phi_{\lambda}(z), \quad \phi_{\lambda}(z) = \frac{a^{(2)}_{\mu}(z)}{\Delta_n(\mu)}.
\]

Noticing that, by (4.5b),

\[
a^{(2)}_{\mu}(z) = \begin{vmatrix} z^{\mu_1} & z^{\mu_2} & \cdots & z^{\mu_n} \\ \mu_1^{n-2} & \mu_2^{n-2} & \cdots & \mu_n^{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{vmatrix}
\]

we expand the above determinant along the first row and, cancelling the arising Vandermonde determinants, obtain (4.11).

The polynomial \(\phi_{\lambda}(z)\) defined by (4.11) is a linear combination of the monomials \(z^{\mu_j}, j = 1,\ldots,n\), and, therefore, satisfies the differential equation

\[
\sum_{j=1}^{n} \mu_j c_j = 0, \quad k = 0,\ldots,n-2,
\]

which, taken together with the normalization condition \(q_{\lambda}(1) = 1\) expressed as

\[
\sum_{j=1}^{n} \mu_j^{n-1} c_j = 1,
\]

follow from the identities

\[
\sum_{j=1}^{n} \frac{\mu_j^l}{\prod_{k \neq j}(\mu_j - \mu_k)} = \delta_{l,n-1}, \quad l = 0,\ldots,n-1,
\]
which are obtained by setting $t = 0$ in the partial fraction decomposition

$$
\prod_{k=1}^{n} (t - \mu_k)^{\delta_{k,n-1}} = \delta_{t,n-1} + \sum_{j=1}^{n} \frac{1}{t - \mu_j} \prod_{k \neq j} (\mu_j - \mu_k).
$$

In fact, the above properties can be used to characterize $\phi_\lambda(z)$.

**Proposition 4.2.** The solution $\phi_\lambda(z)$ to (4.12) is uniquely characterized by the conditions $[\partial^k_z \phi]_{z=1} = 0$, $k = 0, \ldots, n - 2$, which ensure that the polynomial $\phi_\lambda(z)$ is divisible by $(z - 1)^{n-1}$, and by $[\partial^{n-1}_z \phi]_{z=1} = 1$, which is equivalent to the normalization condition $q_\lambda(1) = 1$.

**Proof.** The general solution to (4.14) is $\phi = \sum_{j=1}^{n} c_j z^{\mu_j}$. The divisibility by $(z - 1)^{n-1}$ condition is equivalent then to the equations (4.16), and the normalization condition to the equation (4.17). Solving the system of linear equations (4.16)–(4.17) for $c_j$ via Cramer’s formula we recover the unique solution (4.11).

Note that the differential equation (4.15) can be written also as

$$
[z\partial_z]^n + \sum_{k=1}^{n} (-1)^k h_k(\lambda) (z\partial_z)^{n-k}] \phi = 0
$$

where $h_k(\lambda) = e_k(\mu)$ are the eigenvalues of the operators $H_k$ (4.9). The corresponding differential equation for $q_\lambda(z)$ is

$$
Z^n + \sum_{k=1}^{n} (-1)^k h_k(\lambda) Z^{n-k}] q_\lambda(z) = 0, \quad Z = z (\partial_z + \frac{n-1}{z-1}).
$$

Note that $q_\lambda(z)$ is the only, up to a coefficient, polynomial solution to (4.18).

As usual, we define the operator $Q_z$ in $\mathbb{C}[x]^{S_n}$ through its eigenvectors $\tilde{s}_\lambda(x)$ and eigenvalues $q_\lambda(z)$.

**Theorem 4.1.** Let $z > 1$ and $y_1 < y_2 < \ldots < y_n$. Then for any $f \in \mathbb{C}[x]^{S_n}$ the value $[Q_z f](y)$ is given by the integral

$$
(Q_z f)(y) = (n-1)! \frac{(z - 1)^{n-1}}{\Delta_n(y)} \int_{\Omega_x} dx \delta(x_1 \ldots x_n - zy_1 \ldots y_n) \Delta_n(x) f(x)
$$

where the integration domain $\Omega_x$ is defined by the inequalities

$$
0 < y_1 < x_1 < y_2 < \ldots < x_{n-1} < y_n < x_n.
$$

The above formula is obtained by setting $g = 1$ in the formula (4.2) in [1]. Here we present an elementary and independent proof.

**Proof.** From (4.20) it follows that $x_j > y_j$ for $j = 1, \ldots, n$, hence $x_1 \ldots x_n > y_1 \ldots y_n$. The condition $z > 1$ ensures then that the support of the delta function in (4.19) has a non-empty intersection with the domain $\Omega_x$.

Since the Schur functions $s_\lambda(x)$ form a basis in $\mathbb{C}[x]^{S_n}$ it is sufficient to verify (4.19) for $f = s_\lambda$:

$$
(n-1)! \frac{(z - 1)^{n-1}}{\Delta_n(y)} \int_{\Omega_x} dx \delta(x_1 \ldots x_n - zy_1 \ldots y_n) \Delta_n(x) s_\lambda(x) = q_\lambda(z) s_\lambda(y).
$$

Using the correspondence (4.7) of the symmetric and antisymmetric polynomials as well as the formula (4.11) we reduce the task to verifying the equality
\[ (4.22) \quad \int_{\Omega_\varepsilon} dx \delta(x_1 \cdots x_n - zy_1 \cdots y_n) a_\mu(x) = a_\mu(y) \phi_\lambda(z). \]

Expanding the determinantal expression (4.23) for \( a_\mu(x) \) along the last row we get
\[ (4.23) \quad a_\mu(x) = \sum_{k=1}^{n} (-1)^{k+n} x^\mu_k \det M^{(k)} \]
where the matrix \( M^{(k)} \) is
\[ (4.24) \quad [M^{(k)}]_{ij} = x^\mu_i, \quad i = 1, \ldots, n - 1; \quad j = 1, \ldots, k, \ldots, n \]
Integrating (4.23) in the variable \( x_n \) with the delta-function factor \( \delta(x_1 \cdots x_n - zy_1 \cdots y_n) \) replaces \( x^\mu_n \) with the factor
\[ \frac{(zy_1 \cdots y_n)^\mu_n}{(x_1 \cdots x_{n-1})^{\mu_k+1}}, \]

hence
\[ (4.25) \quad \int dx_n \delta(\ldots) a_\mu(x) = \sum_{k=1}^{n} (-1)^{k+n} (zy_1 \cdots y_n)^\mu_k x^\mu_n \det \{ x^\mu_j - \mu_k \} \]
where the matrix indices \( i,j \) run like in (4.24). Further integration is performed independently for each row of the determinant:
\[ (4.26) \quad \int_{y_i}^{y_{i+1}} dx_i x_i^{\mu_k-1} = \frac{y_i^{\mu_k} - y_{i+1}^{\mu_k}}{\mu_j - \mu_k} \]
(the difference \( \mu_{jk} \equiv \mu_j - \mu_k \) is never 0, so there are no logarithms). According to (4.14) the product of the factors \( (\mu_j - \mu_k)^{-1} \) produces the coefficient \( (-1)^{n-1} c_k \), and the left-hand side of (4.22) is transformed into
\[ (4.27) \quad \sum_{k=1}^{n} (-1)^{k-1}(zy_1 \cdots y_n)^\mu_k c_k \det \{ y_i^{\mu_k} - y_{i+1}^{\mu_k} \}. \]
The determinant of order \( (n-1) \) in (4.27) is transformed into a determinant of order \( n \) in the following way. Let \( t_i = y_i^{\mu_k} \). Then
\[ (4.28) \quad \begin{vmatrix} t_1^1 & t_1^2 & \cdots & t_1^{k-1} & t_1^{k+1} - t_1^1 & \cdots & t_1^{k+1} - t_1^{k-1} \\ t_2^1 & t_2^2 & \cdots & t_2^{k-1} & t_2^{k+1} - t_2^1 & \cdots & t_2^{k+1} - t_2^{k-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ t_n^1 & t_n^2 & \cdots & t_n^{k-1} & t_n^{k+1} - t_n^1 & \cdots & t_n^{k+1} - t_n^{k-1} \end{vmatrix} = (-1)^{k-1} \begin{vmatrix} t_1^1 & t_1^2 & \cdots & t_1^{k-1} & 1 & t_1^{k+1} & \cdots & t_1^n \\ t_2^1 & t_2^2 & \cdots & t_2^{k-1} & 1 & t_2^{k+1} & \cdots & t_2^n \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ t_n^1 & t_n^2 & \cdots & t_n^{k-1} & 1 & t_n^{k+1} & \cdots & t_n^n \end{vmatrix} \]
To prove the matrix identity (4.28) take its right-hand side and replace the \( i^{th} \) row, for \( i \) from 2 to \( n \), with the difference of the \( i^{th} \) and \( (i-1)^{th} \) row, so that \( k^{th} \) column becomes all zeroes except the element \( (1k) \). Expanding the determinant in the \( k^{th} \) column produces the left-hand side of (4.28) which coincides with the determinant in (4.27).
Using the factors \((zy_1\ldots y_n)^{\mu_k}\) in \((4.27)\) we transform the right-hand side of \((4.28)\) to the familiar form \((4.3)\) for \(a_\mu(y)\) and finally reproduce the right-hand side of \((4.22)\).

**4D. Separation of variables.**

**Theorem 4.2.** For the separating operator \(S_n\) defined by the formula \((1.8)\) and satisfying \((1.9)\) for \(P_\lambda \equiv s_\lambda\) there exist uniquely defined operators \(A_k\) satisfying the relations \((4.19)\). The value of \([A_k f](\tilde{y})\) is given by the integral

\[
(4.29) \quad [A_k f](\tilde{y}) = \frac{(-1)^{k-1}(n-1)!}{(n-k)!(z_k - 1)^{n-1}\Delta_k-1(\tilde{y})} \prod_{j=1}^{k-1}(\tilde{y}_j - 1)^{n-k+1} \times \int_{\tilde{\Omega}_x} d\tilde{x} \delta(\tilde{x}_1\ldots \tilde{x}_k - z_k \tilde{y}_1\ldots \tilde{y}_{k-1}) \Delta_k(\tilde{x}) \prod_{j=1}^{k}(\tilde{x}_j - 1)^{n-k} f(\tilde{x})
\]

where \(\tilde{x} = (\tilde{x}_1,\ldots, \tilde{x}_k)\), \(\tilde{y} = (\tilde{y}_1,\ldots, \tilde{y}_{k-1})\), and the integration domain is

\[
\tilde{\Omega}_x : \quad 1 < \tilde{x}_1 < \tilde{y}_1 < \ldots < \tilde{y}_{k-1} < \tilde{x}_k.
\]

**Proof.** The formula \((4.29)\) can be obtained directly from the formula \((6.18)\) from [1] by setting \(g = 1\). The proof given below is basically a simplified proof of the Proposition 6.1 from [1]. The mismatch of signs with respect to [1] is due to a changed definition of \(\Delta_n\).

Let us evaluate \(\rho_{k-1} Q_{z_k}\) in the left-hand side of \((4.12)\) using the definition \((4.10)\) of \(\rho_k\) and the integral formula \((4.19)\) for \(Q_{z_k}\). The operator \(\rho_{k-1}\) sets \((n-k+1)\) of the variables \(y_j\) in \((4.19)\) to the unit values. Since \([Q_{z_k} f](y)\) is a symmetric polynomial it does not matter which of \(y_j\) we choose to fix. Let us set

\[
y_j = 1 + \varepsilon v_j, \quad j = 1,\ldots, n - k + 1,
\]

\[
y_{n-k+j+1} = \tilde{y}_j, \quad j = 1,\ldots, k - 1,
\]

\[
x_j = 1 + \varepsilon u_j, \quad j = 1,\ldots, n - k,
\]

\[
x_{n-k+j} = \tilde{x}_j, \quad j = 1,\ldots, k
\]

and take the limit \(\varepsilon \to 0\). Due to the inequalities \((4.20)\) the variables \(x_1,\ldots, x_{n-k}\) are pinched between \(y_j\)’s and, therefore, forced to tend to 1 as well, which accounts for the factor \(\rho_{k+1}\) in the right-hand side \(A_k \rho_k\) of \((4.12)\). To calculate the kernel of the integral operator \(A_k\) we observe that in the limit \(\varepsilon \to 0\)

\[
\Delta_n(x) \sim \varepsilon^{\frac{(n-k)(n-k-1)}{2}} \Delta_{n-k}(u) \Delta_k(\tilde{x}) \prod_{j=1}^{k}(1 - \tilde{x}_j)^{n-k},
\]

\[
\Delta_n(y) \sim \varepsilon^{\frac{(n-k+1)(n-k)}{2}} \Delta_{n-k+1}(v) \Delta_{k-1}(\tilde{y}) \prod_{j=1}^{k-1}(1 - \tilde{y}_j)^{n-k+1}.
\]

Since \(dx_1\ldots dx_{n-k} \sim \varepsilon^{n-k} du_1\ldots du_{n-k}\), the factors \(\varepsilon\) cancel completely with those from \(\Delta_n(x)\) and \(\Delta_n(y)\). The integration in \(u_j\) produces the constant factor

\[
(4.30) \quad \int_{v_1}^{v_2} du_1 \cdots \int_{v_{n-k}}^{v_{n-k+1}} du_{n-k} \Delta_{n-k}(u) = \frac{(-1)^{n-k}}{(n-k)!} \Delta_{n-k+1}(v).
\]
The formula (4.30) is proved by integrating independently in \( u_i \) the \( i \)-th row of the determinant representing \( \Delta_{n-k}(u) \) and using an identity similar to (4.28). Collecting then all the factors and coefficients we finally get (4.30).

A peculiar feature of the Schur functions \((\alpha = 1)\) case is a beautiful inversion formula for the operator \( S_n \).

**Theorem 4.3.** The inverse of \( S_n \) is the differential operator on \( \mathbb{C}[x]^{S_n} \) given by the formula

\[
S_n^{-1} = \frac{(-1)^{\frac{n(n-1)}{2}}}{(n-1)!} \frac{\Delta_n(\delta)}{\Delta_n(x)} \circ K_n \circ \prod_{k=1}^{n} (x_k - 1)^{n-1},
\]

where \( K_n \) is the differential operator

\[
K_n = \det \{ D_i^\delta \} = \prod_{i<j} (D_i - D_j), \quad D_i \equiv x_i \frac{\partial}{\partial x_i}.
\]

Since \( S_n^{-1} \) is a differential operator it is convenient to identify \( x \) and \( z \) variables and assume that \( S_n^{-1} \) acts in \( \mathbb{C}[x]^{S_n} \). The formula for \( S_n^{-1} \) was mentioned in [3] without a proof (and in a slightly different notation). Here we present a detailed derivation.

**Proof.** It is sufficient to verify that

\[
S_n^{-1} : \prod_{j=1}^{n} q_\lambda(x_j) \mapsto s_\lambda(x).
\]

Using the formulae (4.2) for \( s_\lambda(x) \), (4.4) for \( \Delta_n(x) \), (4.6) for \( s_\lambda(1) \), (4.10) for \( q_\lambda(z) \), and (4.11) for \( \phi_\lambda(z) \) we reduce the task to proving the equality

\[
K_n : \prod_{i=1}^{n} \phi_\lambda(x_i) \mapsto (-1)^{\frac{n(n-1)}{2}} \frac{a_\mu(x)}{\Delta_n(\mu)}.
\]

Using the determinantal representation (4.32) for the operator \( K_n \) we represent \( K_n \prod \phi_\lambda(x_i) \) as the determinant of the matrix \( M \):

\[
M_i^j = D_i^\delta \phi_\lambda(x_i) = D_i^\delta \sum_{k=1}^{n} c_k x_i^{\mu_k} = \sum_{k=1}^{n} x_i^{\mu_k} \cdot c_k^\mu \delta_j.
\]

The resulting expression is recognized as a product of three matrices, therefore

\[
K_n \prod_{i=1}^{n} \phi_\lambda(x_i) = \det[M_i^j] = \det[x_i^{\mu_k}] \cdot \det[c_i^\mu] \cdot \det[\mu_j^\delta] = a_\mu(x) \cdot (c_1 \cdots c_n) \cdot \Delta_n(\mu).
\]

To obtain (4.34) and thus conclude the proof it remains to substitute into the above formula the values of \( c_n \) from (4.11).

The differential operator \( K_n \) appeared also in [4] though in a different context.

**4E. Lifting operator.**

**Proposition 4.3.** The operator \( Q_z \) taken at \( z = 0 \) admits a unique factorization (1.17) into the projector \( P \) (1.16) and the operator \( Q_0 \) (1.17):

\[
Q_0 : \bar{s}_{\lambda_1 \ldots \lambda_{n-1}}(x_1, \ldots, x_{n-1}) \mapsto \bar{s}_{\lambda_1 \ldots \lambda_{n-1}}(x_1, \ldots, x_n).
\]
For $y_1 < y_2 < \cdots < y_n$ the value $[Q_n^0 f](y)$ is given by the integral

\[
[Q_n^0 f](y) = (-1)^{n-1} \frac{(n-1)!}{\Delta_n(y)} \int_{\Omega^*_n} d\mathbf{x}' \Delta_{n-1}(\mathbf{x}')f(\mathbf{x}')
\]

where $\mathbf{x}' = (x_1, \ldots, x_{n-1})$ and

\[
\Omega^*_n : 0 < y_1 < y_2 < \cdots < x_{n-1} < y_n.
\]

**Proof.** We cannot set $z = 0$ directly in the integral (4.19) because of the restriction $z > 1$. Instead, we shall use the definition of $Q_z$ in terms of its eigenvectors $s_\lambda(\mathbf{x})$ and eigenvalues $q_\lambda(z)$.

In what follows we have to be careful to distinguish the partitions of length $n$ and $n-1$. Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $\lambda' = (\lambda_1, \ldots, \lambda_{n-1})$. Respectively, $\mu = (\mu_1, \ldots, \mu_n)$ and $\mu' = (\mu_1, \ldots, \mu_{n-1})$, where $\mu_i = \lambda + n - i$, $\mu'_i = \lambda_i + n - 1 - i$. Note that $\mu_i = \mu'_i + 1$, for $i = 1, \ldots, n-1$.

From (4.10) and (4.11) it follows that $q_\lambda(0)$ is nonzero only for $\mu_n \equiv \lambda_n = 0$.

In this case

\[
s_{\lambda_1,\ldots,\lambda_{n-1},0}(x_1, \ldots, x_{n-1}) = s_{\lambda_1,\ldots,\lambda_{n-1}}(x_1, \ldots, x_{n-1}).
\]

Using (4.40) we get, respectively, for the normalized Schur functions

\[
s_{\lambda_1,\ldots,\lambda_{n-1},0}(x_1, \ldots, x_{n-1}) = \frac{(n-1)!}{\prod_{i=1}^{n-1} \mu_i}.
\]

Similarly, from (4.2) and (4.3) we conclude that $P s_\lambda$ does not vanish only for $\lambda_n = 0$, and

\[
s_{\lambda_1,\ldots,\lambda_{n-1},0}(x_1, \ldots, x_{n-1}) = s_{\lambda_1,\ldots,\lambda_{n-1}}(x_1, \ldots, x_{n-1}).
\]

From the last formula together with (4.40) the factorization (1.15) and the formula (4.37) follow immediately.

The integral formula (4.38) can be obtained, in principle, by setting $g = 1$ in the formula (7.10) from (1) and taking into account the different definition of $\Delta_n$. We provide, however, an independent proof. It is sufficient to verify (4.38) on the basis $f = s_\lambda(\mathbf{x}')$. Using (4.2) and (4.4), and cancelling the arising coefficients we reduce the problem to verifying the identity

\[
\int_{\Omega^*_n} d\mathbf{x}' \, a_\mu(\mathbf{x}') = (-1)^{n-1} \frac{(n-1)!}{\prod_{i=1}^{n-1} \mu_i} a_\mu(\mathbf{y}).
\]

To prove the last identity we integrate independently in $x_i$ from $y_i$ to $y_{i+1}$ each row of the determinant representing $a_\mu(\mathbf{x}')$ and then use $\mu_i = \mu'_i + 1$ and a variant of the determinantal identity (2.28).

\[\blacksquare\]

5. Concluding remarks

We have considered three important standard bases in the linear space of symmetric polynomials in $n$ variables. These bases have been related to the special cases of the Jack polynomials which, in their turn, solve the famous quantum integrable system (Calogero-Sutherland model). The main objective was to demonstrate the main features of the (quantum) separation of variables, which is designed to produce explicit factorization and representation for the multivariate special functions through the application of suitable (integral) operators.
The method of quantum separation of variable has its counterpart in the classical Hamiltonian mechanics which is applicable to a wide class of Liouville integrable systems. Usually, it is the classical system that gets separated in the first place, followed by the problem of quantization. For the three bases considered in this paper such approach is valid only for the $E_\lambda$ one because the Jack’s parameter $\alpha$ is proportional to the Plank constant. The two other bases, $m_\lambda$ and $s_\lambda$, do not have classical analogs, so that the separation of variables that we have found for them in this paper is purely quantum.

In this paper we considered polynomials symmetric under permutation group $S_n$, or Weyl group of the root system $A_{n-1}$. It is a challenging problem to find the factorizing operators for the bases of polynomials symmetric with respect to Weyl groups corresponding to other root systems, such as, for example, $B_n$, $C_n$, or $D_n$.

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