Minimal $P$-symmetric periodic solutions of nonlinear Hamiltonian systems

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Abstract

In this paper some existence results for the minimal $P$-symmetric periodic solutions are proved for first order autonomous Hamiltonian systems when the Hamiltonian function is superquadratic, asymptotically linear and subquadratic. These are done by using critical points theory, Galerkin approximation procedure, Maslov $P$-index theory and its iteration inequalities.

Keywords: Maslov $P$-index, iteration inequality, minimal $P$-symmetric periodic solutions, Hamiltonian systems

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1 Introduction and main results

We study the $P$-boundary problem of first order autonomous Hamiltonian systems:

\[
\begin{aligned}
\dot{x} &= JH'(x), \quad \forall x \in \mathbb{R}^{2n}, \\
x(\tau) &= Px(0),
\end{aligned}
\]

where $\tau > 0$, $P \in Sp(2n)$, $H \in C^2(\mathbb{R}^{2n}, \mathbb{R})$ and $H(Px) = H(x)$, $\forall x \in \mathbb{R}^{2n}$. $H'(x)$ denote its gradient, $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$ is the standard symplectic matrix, $I_n$ is the identity matrix on $\mathbb{R}^n$ and $n$ is the positive integer.

A solution $(\tau, x)$ of the problem (1.1) is called $P$-solution of the Hamiltonian systems. It is a kind of generalized periodic solution of Hamiltonian systems. The problem (1.1) has relation with the the closed geodesics on Riemannian manifold (cf.\cite{12}) and symmetric periodic solution or the quasi-periodic solution problem (cf.\cite{13}). In addition, C. Liu in \cite{14} transformed some periodic boundary problem for asymptotically linear delay differential systems and some asymptotically linear delay Hamiltonian systems to $P$-boundary problems of Hamiltonian systems as above, we

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also refer \[3, 9, 15, 16\] and references therein for the background of \(P\)-boundary problems in \(N\)-body problems.

Suppose \(P\) satisfies \(P^k = I\), here \(k\) is assumed to be the smallest positive integer such that \(P^k = I\) (this condition for \(P\) is called \((P)_k\) condition in the sequel), so the \(P\)-solution \((\tau, x)\) can be extended as a \(k\tau\)-periodic solution \((k\tau, x^k)\). We say that a \(T\)-periodic solution \((T, x)\) of a Hamiltonian system in (1.1) is \(P\)-symmetric if \(x(T) = P x(0)\). \(T\) is the \(P\)-symmetric period of \(x\).

We define \(T\) be the minimal \(P\)-symmetric period of \(x\) if \(T = \min\{\lambda > 0 \mid x(t + \lambda) = P x(t), \forall t \in \mathbb{R}\}\). Note that \(T\) might not be the minimal period of \(x\) although it is the minimal \(P\)-symmetric period of \(x\).

In recent years, Maslov P-index theory was developed to study the existence and multiplicity of \(P\)-solutions (cf.\[7, 8, 19, 20\]), specially, the corresponding iteration theory was built to estimate the minimality of the period of \(P\)-solution (i.e., the minimal \(P\)-symmetric period) (cf.\[21, 23\]) and look for geometrically distinct \(P\)-solutions (i.e., subharmonic \(P\)-solutions) (cf.\[24\]). It is meaningful to study the minimal \(P\)-symmetric periodic solutions of (1.1). So far there are very few papers about it.

In the following, we always suppose \(P \in Sp(2n)\) satisfies the \((P)_k\) condition.

In this paper, combining the Galerkin approximation procedure (cf.\[22, 23, 24\]) with the method with C. Liu and me (cf.\[23\]), we study the minimal \(P\)-symmetric periodic solutions of (1.1) when the Hamiltonian function \(H\) is superquadratic, asymptotically linear and sub-quadratic respectively.

For \(\tau > 0\), we define

\[
S_\tau(H) = \{ x \in C^1([0, \tau], \mathbb{R}^{2n}) : x \neq constant, x \text{ is a } P\text{-solution of (1.1)} \}.
\]

We now state the main results as follows.

**Theorem 1.1.** Suppose \(P \in Sp(2n)\) satisfies the \((P)_k\) condition, and \(H\) satisfies the following conditions:

\(\text{(H0)}\) \(H \in C^2(\mathbb{R}^{2n}, \mathbb{R})\) with \(H(Px) = H(x), \forall x \in \mathbb{R}^{2n}\);

\(\text{(H1)}\) \(H(x) = \frac{1}{2}(h_0x, x) + o(|x|^2)\) as \(|x| \to 0\);

\(\text{(H2)}\) \(H(x) - \frac{1}{2}(h_0x, x) \geq 0, \forall x \in \mathbb{R}^{2n}\),

where \(h_0\) is semi-positive definite symmetric matrix with \(P^T h_0 P = h_0\);

\(\text{(H3)}\) There exist constants \(\mu > 2\) and \(R_0 > 0\) such that

\[
0 < \mu H(x) \leq H'(x) \cdot x, \quad \forall |x| \geq R_0;
\]

\(\text{(HX1)}\) \(H''(x(t)) \geq 0\) for every \(x \in S_\tau(H)\) and \(t \in \mathbb{R}\);

\(\text{(HX2)}\) \(\int_0^\tau H''(x(t)) dt > 0\) for every \(x \in S_\tau(H)\);

\(\text{(HX3)}\) \(i_P(h_0) + \nu_P(h_0) \leq \dim \ker_{\mathbb{R}}(P - I)\), where \((i_P(h_0), \nu_P(h_0))\) denote the Maslov P-index of \(h_0\).
Then (1.1) possesses a \(P\)-solution \(x\) with the minimal \(P\)-symmetric period \(k\tau\) or \(\frac{k\tau}{k+1}\).

**Remark 1.2.** Specially, if \(h_0 = 0\), then \(i_P(h_0) = 0\), \(\nu_P(h_0) = \dim \ker_{\mathbb{R}}(P - I), \forall P \in Sp(2n)\).

At the moment, (HX3) holds automatically. Our result generalize the corresponding one in [21].

For the asymptotically linear Hamiltonian systems, we consider the case that the asymptotical matrix may be degenerate and the get the following two theorems:

**Theorem 1.3.** Suppose \(P \in Sp(2n)\) satisfies the \((P)_k\) condition, and \(H\) satisfies \((H0),(H1), (H2), (HX1), (HX2)\) and the following conditions:

\((H4)\) There exists constant \(a_1, a_2\) and some \(s \in (1, \infty)\) such that

\[|H''(x)| \leq a_1|x|^s + a_2;\]

\((H5)\) There exists semi-positive definite symmetric matrix \(h_\infty\) with \(P^T h_\infty P = h_\infty\) such that

\[H'(x) = h_\infty x + o(|x|) \text{ as } |x| \to \infty;\]

\((H6)\) \(h_\infty - h_0\) is positive definite, \(h_\infty h_0 = h_0 h_\infty\), where \(h_0 \in \Sigma_4(\mathbb{R}^{2n})\) is the matrix given in \((H1)\) and \((H2)\);

\((HX4)\) \(i_P(h_\infty) > i_P(h_0) + \nu_P(h_0), i_P(h_\infty) + \nu_P(h_0) \leq \dim \ker_{\mathbb{R}}(P - I), \text{ where } (i_P(h_\infty), \nu_P(h_\infty))\) denote the Maslov P-index of \(h_\infty\).

Then (1.1) possesses a \(P\)-solution \(x\) with the minimal \(P\)-symmetric period \(k\tau\) or \(\frac{k\tau}{k+1}\) provided one of the following cases occurs:

1. \(\nu_P(h_\infty) = 0;\)
2. \(\nu_P(h_\infty) > 0\) and \(G_\infty(x) = H(x) - \frac{1}{2}(h_\infty x, x)\) satisfies

\[|G'_\infty(x)| \leq M \text{ for } x \in \mathbb{R}^{2n}, \ G_\infty(x) \to +\infty \text{ as } |x| \to \infty.\] (1.2)

**Theorem 1.4.** Suppose \(P \in Sp(2n)\) satisfies the \((P)_k\) condition, and \(H\) satisfies \((H0),(H1), (H2),(H4), (H5), (HX1), (HX2)\) and the following conditions:

\((H7)\) \(\{x \in \mathbb{R}^{2n} : H'(x) = 0\} = \{0\};\)

\((HX5)\) \(i_P(h_\infty) + \nu_P(h_\infty) \leq \dim \ker_{\mathbb{R}}(P - I) + 1, \ i_P(h_\infty) + \nu_P(h_\infty) \notin [i_P(h_0), i_P(h_0) + \nu_P(h_0)].\)

Then (1.1) possesses a \(P\)-solution \(x\) with the minimal \(P\)-symmetric period \(k\tau\) or \(\frac{k\tau}{k+1}\) provided one of the following cases occurs:

1. \(\nu_P(h_\infty) = 0;\)
2. \(\nu_P(h_\infty) > 0\) and \(G_\infty(x) = H(x) - \frac{1}{2}(h_\infty x, x)\) satisfies

\[|G'_\infty(x)| \leq M \text{ for } x \in \mathbb{R}^{2n}, \ G_\infty(x) \to +\infty \text{ as } |x| \to \infty.\] (1.3)
Remark 1.5. In Theorem 1.4 we do not need the condition (H6).

The following theorem studies the minimal \( P \)-symmetric periodic solutions of subquadratic Hamiltonian systems with \( P \)-boundary

\[
\begin{cases}
\dot{x} = \lambda JH'(x), \quad \forall x \in \mathbb{R}^{2n}, \quad \lambda \in \mathbb{R}, \\
x(\tau) = Px(0).
\end{cases}
\] (1.4)

This is motivated by [2, 11].

**Theorem 1.6.** Suppose \( P \in Sp(2n) \) satisfies the \((P)_k\) condition, and \( H \) satisfies (H0) and (H8)

\((H8)\) \(|H'(x)| \leq M \) for \( x \in \mathbb{R}^{2n} \), and \( H(x) \to +\infty \) as \(|x| \to \infty\);

\((H9)\) \( H(0) = 0 \) and \( H'(x), H(x) > 0 \) for \( x \neq 0 \).

Suppose \( \tau > 0 \), (HX1) and (HX2) hold. There exists \( \lambda_\tau > 0 \) such that for any \( \lambda \geq \lambda_\tau \), (1.4) possesses a \( P \)-solution \( x \) with the minimal \( P \)-symmetric period \( k\tau \) or \( k\tau + 1 \).

In order to get the information about the Maslov \( P \)-index of the \( P \)-solution, we need the relation between the Maslov \( P \)-index and Morse index. This has been done in Section 2 by using the Galerkin approximation procedure and the Maslov P-index theory. The main idea comes from [11] and [21].

## 2 Maslov P-index and Morse index

Maslov P-index was first studied in [7] and [19] independently for any symplectic matrix \( P \) with different treatment, it was generalized by C. Liu and the author in [22, 23]. And then C. Liu used relative index theory to develop Maslov P-index in [21] which is consistent with the definition in [22, 23]. In fact, when the symplectic matrix \( P = diag\{-I_{n-k}, I_k, -I_{n-k}, I_k\}, \) \( 0 \leq k \leq n \), the \((P, \omega)\)-index theory and its iteration theory were studied in [8] and then be successfully used to study the multiplicity of closed characteristics on partially symmetric convex compact hypersurfaces in \( \mathbb{R}^{2n} \). Here we use the notions and results in [21, 22, 23].

For \( \tau > 0 \), \( P \in Sp(2n) \), \( \mathcal{L}_s(\mathbb{R}^{2n}) \) denotes all symmetric real \( 2n \times 2n \) matrices. For \( B(t) \in C(\mathbb{R}, \mathcal{L}_s(\mathbb{R}^{2n})) \) and satisfies \( P^T B(t + \tau) P = B(t) \). If \( \gamma \) is the fundamental solution of the linear Hamiltonian systems

\[
\dot{y}(t) = JB(t)y, \quad y \in \mathbb{R}^{2n}.
\] (2.1)

Then the Maslov \( P \)-index pair of \( \gamma \) is defined as a pair of integers

\[(i_P, \nu_P) \equiv (i_P(\gamma), \nu_P(\gamma)) \in \mathbb{Z} \times \{0, 1, \cdots, 2n\},\]

where \( i_P \) is the index part and

\[\nu_P = \dim \ker(\gamma(\tau) - P)\]
Theorem 2.1. Suppose \( B(t) \in C(\mathbb{R}, \mathcal{L}_s(\mathbb{R}^{2n})) \) and satisfies \( P^T B(t+\tau)P = B(t) \) with the Maslov P-index \((i_P(B), \nu_P(B))\), for any constant \( 0 < d < 1/4\|A-B\|^2 \), there exists an \( m_0 > 0 \) such that for \( m \geq m_0 \), there holds

\[
\begin{align*}
    m_d^\pm(P_m(A-B)) &= m + \dim \ker \mathbb{R}(P-I) - i_P(B) - \nu_P(B), \\
    m_d^\pm(P_m(A-B)) &= m + i_P(B), \\
    m_d^0(P_m(A-B)) &= \nu_P(B),
\end{align*}
\]

where \( B \) be the operator defined by \((2.2)\) corresponding to \( B(t) \).
Proof. Let \( x(t) = \gamma_P(t)\xi(t) \in W_P, \xi \in W^{1/2,2}(S_T, \mathbb{R}^{2n}) \), \( \gamma_P(t) \) is defined in \([22, 24]\) is a symplectic path which satisfies \( \gamma_P(0) = I \) and \( \gamma_P(\tau) = P \). Then we have

\[
\langle Ax, x \rangle = \int_0^T (-J \dot{x}(t), x(t)) dt
\]

\[
= \int_0^T \left[ (-J \dot{\xi}(t), \xi(t)) - (\gamma_P(t)^T J \dot{\gamma}_P(t) \xi(t), \xi(t)) \right] dt
\]

\[
= \int_0^T \left[ (-J \dot{\xi}(t), \xi(t)) - (B_{\gamma_P}(t) \xi(t), \xi(t)) \right] dt,
\]

\[
\langle (A - B)x, x \rangle = \int_0^T \left[ (-J \dot{x}(t), x(t)) - (B(t)x(t), x(t)) \right] dt
\]

\[
= \int_0^T \left[ (-J \dot{\xi}(t), \xi(t)) - (\gamma_P(t)^T J \dot{\gamma}_P(t) \xi(t), \xi(t)) - (\gamma_P(t)^T B(t) \gamma_P(t) \xi(t), \xi(t)) \right] dt
\]

\[
= \int_0^T \left[ (-J \dot{\xi}(t), \xi(t)) - (\tilde{B}_{\gamma_P}(t) \xi(t), \xi(t)) \right] dt,
\]

where \( \tilde{B}_{\gamma_P}(t) = \gamma_P(t)^T J \dot{\gamma}_P(t), \tilde{B}_{\gamma_P}(t) = \gamma_P(t)^T J \dot{\gamma}_P(t) + \gamma_P(t)^T B(t) \gamma_P(t) \). By the definitions of \( \gamma_P(t) \) and \( B(t) \), \( \tilde{B}_{\gamma_P}(t) \) and \( \tilde{B}_{\gamma_P}(t) \) are both symmetric matrix functions and \( \tilde{B}_{\gamma_P}(0) = \tilde{B}_{\gamma_P}(\tau), \tilde{B}_{\gamma_P}(0) = \tilde{B}_{\gamma_P}(\tau) \). The operators \( A \) and \( A - B \) defined in \( W_P \) correspond to the operators \( -J \frac{d}{dt} \tilde{B}_{\gamma_P} \) and \( -J \frac{d}{dt} \tilde{B}_{\gamma_P} \) defined in \( W^{1/2,2}(S_T, \mathbb{R}^{2n}) \). Suppose \( \gamma \) is the fundamental solution of \( \dot{z}(t) = JB(t)z(t) \).

Consider the following linear Hamiltonian systems

\[ \dot{z}(t) = J\tilde{B}_{\gamma_P}(t)z(t), \quad z(t) \in \mathbb{R}^{2n}. \tag{2.4} \]

Suppose \( \tilde{\gamma}(t) \) is the fundamental solution of \( \tilde{z}(t) = J\tilde{B}_{\gamma_P}(t)z(t) \). By direct computation, we obtain

\[ \tilde{\gamma}(t) = \gamma_P(t)^{-1}\gamma(t) = \gamma_2. \]

And similarly, \( \gamma_P(t)^{-1} \) is the fundamental solution of \( \dot{z}(t) = J\tilde{B}_{\gamma_P}(t)z(t) \). By Theorem 7.1 in \([25]\), there exists an \( m^* > 0 \) such that for \( m \geq m^* \) such that

\[
m_d^+(P_m(A - B)P_m) = m + i(\tilde{B}_{\gamma_P}) - i(\tilde{B}_{\gamma_P}) + \nu(\tilde{B}_{\gamma_P}) - \nu(\tilde{B}_{\gamma_P}),
\]

\[
m_d^+(P_m(A - B)P_m) = m - i(\tilde{B}_{\gamma_P}) + i(\tilde{B}_{\gamma_P}),
\]

\[
m_d^0(P_m(A - B)P_m) = \nu(\tilde{B}_{\gamma_P})
\]

where \( \tilde{B}_{\gamma_P} \) and \( \tilde{B}_{\gamma_P} \) be the compact operator defined by \([22]\) corresponding to \( \tilde{B}_{\gamma_P}(t) \) and \( \tilde{B}_{\gamma_P}(t) \). \( i(\tilde{B}_{\gamma_P}), \nu(\tilde{B}_{\gamma_P}) \) \( \) and \( (i(\tilde{B}_{\gamma_P}), \nu(\tilde{B}_{\gamma_P})) \) is the Maslov-type index of \( \tilde{B}_{\gamma_P}(t) \) and \( \tilde{B}_{\gamma_P}(t) \) in \([25]\). Now by Theorem 3.3 in \([22]\), we have

\[ i(\tilde{B}_{\gamma_P}) = i_P(0) - i(\gamma_P) - n = -i(\gamma_P) - n, \quad i(\tilde{B}_{\gamma_P}) = i_P(B) - i(\gamma_P) - n. \tag{2.6} \]

Note that

\[ \nu(\tilde{B}_{\gamma_P}) = \nu(\gamma_P(t)^{-1}) = \dim \ker_{\mathbb{R}}(P - I), \quad \nu(\tilde{B}_{\gamma_P}) = \nu(\tilde{\gamma}) = \nu_P(\gamma_2) = \nu_P(\gamma) = \nu_P(B). \tag{2.7} \]

Finally we get \([23]\) by \([2.4, 2.6]\).
The following theorem was proved in [21] by relative index theory and iteration theory of Maslov P-index.

**Theorem 2.2.** Suppose $H \in C^2(\mathbb{R}^{2n}, \mathbb{R})$ and $P \in Sp(2n)$ satisfies the $(P)_k$ condition. For $\tau > 0$, let $x_0$ be a $P$-solution of (1.1). If the Maslov $P$-index of $x_0$ satisfies
\[ i_P(x_0) \leq \dim \ker (P - I) + 1, \]
and further satisfies $(HX1)$ and $(HX2)$. Then the minimal $P$-symmetric period of $x_0$ is $k\tau$ or $\frac{k\tau}{k+1}$.

In order to estimate the Maslov $P$-index of a critical point of the functional we considered, we need the following result which was proved in [12, 17, 27].

**Theorem 2.3.** Let $E$ be a real Hilbert space with orthogonal decomposition $E = X \oplus Y$, where $\dim X < +\infty$. Suppose $f \in C^2(E, \mathbb{R})$ satisfies the $(P.S)$ condition and the following conditions:

1. There exist $\rho$ and $\alpha > 0$ such that
   \[ f(w) \geq \alpha, \quad \forall w \in \partial B_\rho(0) \cap Y. \]
2. There exist $e \in \partial B_1(0) \cap Y$ and $R > \rho$ such that
   \[ f(w) < \alpha, \quad \forall w \in \partial Q, \]
   where $Q = (B_R(0) \cap X) \oplus \{re \mid 0 \leq r \leq R\}$.

Then

1. $f$ possesses a critical value $c \geq \alpha$, which is given by
   \[ c = \inf_{h \in \Lambda} \max_{w \in Q} f(h(w)), \]
   where $\Lambda = \{h \in C(\overline{Q}, E) \mid h = \text{id} \text{ on } \partial Q\}$.
2. If $f''(w)$ is Fredholm for $w \in \mathcal{K}_c(f) \equiv \{w \in E : f'(w) = 0, f(w) = c\}$, then there exists an element $w_0 \in \mathcal{K}_c(f)$ such that the negative Morse index $m^-(w_0)$ and nullity $m^0(w_0)$ of $f$ at $w_0$ satisfies
   \[ m^-(w_0) \leq \dim X + 1 \leq m^-(w_0) + m^0(w_0). \]  

**Definition 2.4.** [12] Let $E$ be a $C^2$-Riemannian manifold, $D$ is a closed subset of $E$. A family $\mathcal{F}(\alpha)$ is said to be a homological family of dimension $q$ with boundary $D$ if for some nontrival class $\alpha \in H_q(E, D)$ the family $\mathcal{F}(\alpha)$ is defined by
\[ \mathcal{F}(\alpha) = \{G \subset E : \alpha \text{ is in the image of } i_* : H_q(G, D) \to H_q(E, D)\}, \]
where $i_*$ is the homomorphism induced by the immersion $i : G \to E$. 


Theorem 2.5. [12] As in the definition 2.4, for given $E$, $D$ and $\alpha$, let $\mathcal{F}(\alpha)$ be a homological family of dimension $q$ with boundary $D$. Suppose that $f \in C^2(E, \mathbb{R})$ satisfies (P.S) condition. Define
\[
c \equiv c(f, \mathcal{F}(\alpha)) = \inf_{G \in \mathcal{F}(\alpha)} \sup_{w \in G} f(w).
\] (2.9)

Suppose that $\sup_{w \in D} f(w) < c$ and $f'$ is Fredholm on
\[
K_c = \{ x \in E : f'(x) = 0, f(x) = c \}. \tag{2.10}
\]
Then there exists $x \in K_c$ such that the Morse indices $m^-(x)$ and $m^0(x)$ of the functional $f$ at $x$ satisfy
\[
q - m^0(x) \leq m^-(x) \leq q.
\]

3 Superquadratic Hamiltonian systems

In this section, we study the minimal $P$-symmetric periodic solution of superquadratic Hamiltonian systems with $P$-boundary conditions. In order to prove Theorem 1.1, we need the following arguments.

For $z \in W_P$, we define
\[
f(z) = \frac{1}{2} \int_0^{k\tau} (-J \dot{z}(t), z(t)) dt - \int_0^{k\tau} H(z) dt = k \left( \frac{1}{2} \langle Az, z \rangle - \int_0^\tau H(z) dt \right). \tag{3.1}
\]
It is well known that $f \in C^2(W_P, \mathbb{R})$ whenever
\[
H \in C^2(\mathbb{R}^{2n}, \mathbb{R}) \text{ and } |H''(x)| \leq a_1 |x|^s + a_2; \tag{3.2}
\]
for some $s \in (1, \infty)$ and all $x \in \mathbb{R}^{2n}$. Looking for solutions of (1.1) is equivalent to looking for critical points of $f$.

Proof of Theorem 1.1. We carry out the proof in several steps.

Step 1. Since the growth condition (3.2) has not been assumed for $H$, we need to truncate the function $H$ at infinite. We follow the method in Rabinowitz’s pioneering work [26].

Let $K > 0$ and $\chi \in C^\infty(\mathbb{R}, \mathbb{R})$ such that $\chi(y) \equiv 1$ if $y \leq K$, $\chi(y) \equiv 0$ if $y \geq K + 1$, and $\chi'(y) < 0$ if $y \in (K, K + 1)$, where $K$ is free for now. Set
\[
H_K(z) = \chi(|z|)H(z) + (1 - \chi(|z|))R_K |z|^4, \tag{3.3}
\]
where the constant $R_K$ satisfies
\[
R_K \geq \max_{K \leq |z| \leq K + 1} \frac{H(z)}{|z|^4}.
\]
Then $H_K \in C^2(\mathbb{R}^{2n}, \mathbb{R})$, and there is $K_0 > 0$ such that for $K \geq K_0$, $H_K$ satisfies (H1), (H2) and (3.2) with $s = 2$. Moreover a straightforward computation shows (H3) hold with $\nu = \min\{\mu, 4\}$. Integrating this inequality then yields

$$H_K(z) \geq a_1|z|^\nu - a_2$$

(3.4)

for all $z \in \mathbb{R}^{2n}$, where $a_1$, $a_2 > 0$ are independent of $K$.

Let $G_K(z) = H_K(z) - \frac{1}{2}(h_0z, z)$, then by (3.3) it is easy to show that

$$G_K(z) \geq a_3|z|^\nu - a_4$$

(3.5)

for all $z \in \mathbb{R}^{2n}$, where $a_3$, $a_4 > 0$ are independent of $K$.

Finally, we set

$$f_K(z) = \frac{1}{2} \int_0^{k_T} (-J\dot{z}(t), z(t))dt - \int_0^{k_T} H_K(z)dt = \frac{k}{2}(Az, z) - \int_0^{k_T} H_K(z)dt, \quad \forall z \in W_P,$$

(3.6)

then $f_K \in C^2(W_P, \mathbb{R})$.

**Step 2.** For $m > 0$, let $f_{K,m} = f_K|_{W_P^m}$. We will show that $f_{K,m}$ satisfies the hypotheses of Theorem 2.3.

By (H1) and (3.3), for any $\epsilon > 0$, there is a $M = M(\epsilon, K) > 0$ such that

$$G_K(z) \leq \epsilon|z|^2 + M|z|^4, \quad \forall x \in \mathbb{R}^{2n}.$$ (3.7)

Let $B_0$ be the operator defined by (2.2) corresponding to $h_0$, and let

$$X_m = M^-(P_m(A - B_0)P_m) \oplus M^0(P_m(A - B_0)P_m), \quad Y_m = M^+(P_m(A - B_0)P_m).$$

For $z \in Y_m$, by (3.7) and the fact that $P_nB_0 = B_0P_n$ for $n \geq 0$, we have

$$f_{K,m}(z) = \frac{k}{2}\langle(A - B_0)z, z\rangle - \int_0^{k_T} G_K(z)dt$$

$$\geq \frac{k}{2}\|(A - B_0)^2\|^{-1}\|z\|^2 - (\epsilon\alpha_2 + M\alpha_4\|z\|^2)\|z\|^2.$$ (3.8)

So there are constant $\rho = \rho(K) > 0$ and $\alpha = \alpha(K) > 0$, which are sufficiently small and independent of $m$, such that

$$f_{K,m}(z) \geq \alpha, \quad \forall z \in \partial B_\rho(0) \cap Y_m.$$ (3.9)

Let $e \in \partial B_1(0) \cap Y_m$ and set

$$Q_m = \{re \mid 0 \leq r \leq r_1\} \oplus (B_{r_1}(0) \cap X_m),$$

where $r_1$ is free for the moment. Let $z = z^- + z^0 \in B_{r_1}(0) \cap X_m$, then

$$f_{K,m}(z + re) = \frac{k}{2}\langle(A - B_0)z^-, z^-\rangle + \frac{k}{2}r^2\langle(A - B_0)e, e\rangle - \int_0^{k_T} G_K(z + re)dt$$

$$\leq \frac{k}{2}r^2\|A - B_0\| - \frac{k}{2}\|(A - B_0)^2\|^{-1}\|z^-\|^2 - \int_0^{k_T} G_K(z + re)dt.$$ (3.10)
If $r = 0$, there holds
\[ f_{K,m}(z + re) \leq -\frac{k}{2} \|(A - B_0)^2\|^{-1}\|z\|^2. \] (3.10)

If $r = r_1$ or $\|z\| = r_1$, by (3.5), there holds
\[ \int_0^{k\tau} G_K(z + re) dt \geq \int_0^{\tau} G_K(z + re) dt \geq a_3 \int_0^{k\tau} \|z + re\|^{\nu} dt - k\tau a_4 \geq a_5(\|z\|^{\nu} + r^{\nu}) - a_6 \] (3.11)

Combining (3.9) with (3.11) yields
\[ f_{K,m}(z + re) \leq a_7 r^2 - a_8 \|z\|^2 - a_5(\|z\|^{\nu} + r^{\nu}) + a_6. \]

So we can choose $r_1$ large enough which is independent of $K$ and $m$ such that
\[ f_{K,m}(z + re) \leq 0, \quad \forall z \in \partial Q_m. \] (3.12)

Now using the same argument as ([23], Theorem 4.2), we have $f_{K,m}$ has a critical value $c_{K,m} \geq \alpha$, which is given by
\[ c_{K,m} = \inf_{g \in \Lambda_m} \max_{w \in Q_m} f_{K,m}(g(w)), \] (3.13)
where $\Lambda_m = \{ g \in C(Q_m, W^0_P) \mid g = id \text{ on } \partial Q_m \}$. Moreover, there is a critical point $x_{K,m}$ of $f_{K,m}$ which satisfies
\[ m^{-}(x_{K,m}) \leq \dim X_m + 1. \] (3.14)

**Step 3.** Since $id \in \Lambda_m$, by (3.9) and (H2) we have
\[ c_{K,m} \leq \sup_{w \in Q_m} f_{K,m}(w) \leq \frac{k}{2} r^2_1 \|A - B_0\|. \] (3.15)

Then in the sense of subsequence we have
\[ c_{K,m} \to c_K, \quad \alpha \leq c_K \leq \frac{k}{2} r^2_1 \|A - B_0\|. \] (3.16)

Using the same argument as (4.40)-(4.43) in [23], we have that $f_K$ satisfies the (P.S)* condition on $W_P$, i.e., any sequence $\{z_m\} \subset W_P$ satisfying $z_m \in W^m_P$, $f_{K,m}(z_m)$ is bounded and $f_{K,m}(z_m) \to 0$ possesses a convergent subsequence in $W_P$. Hence in the sense of the subsequence we have
\[ x_{K,m} \to x_K, \quad f_K(x_K) = c_K, \quad f'_K(x_K) = 0. \] (3.17)

By the standard argument as in [23], $x_K$ is a classical nonconstant $P$-solution of
\[ \begin{cases} \dot{x} = JH'_K(x), \quad \forall x \in \mathbb{R}^{2n}, \\ x(\tau) = P_x(0). \end{cases} \] (3.18)
Indeed, if $x_K(t)$ is a constant solution of (3.18), by (H2), then

$$f_K(x_K) = \frac{k}{2}(Ax_K, x_K) - \int_0^{kt} \frac{1}{2}(h_0 x_K, x_K) dt - \int_0^{kt} [H_K(x_K) - \frac{1}{2}(h_0 x_K, x_K)] dt \leq 0.$$  (3.19)

This contradicts to $f_K(x_K) = c_K \geq \alpha > 0$.

And there is a $K_0 > 0$ such that for all $K \geq K_0$, $\|x_K\|_{L^\infty} < K$. Then $H'_K(x_K) = H'(x_K)$ and $x_K$ is a non-constant $P$-solution of (3.18). We denote it simply by $x := x_K$.

**Step 4.** Let $B(t) = H''_K(x(t))$ and $B$ be the operator defined by (2.2) corresponding to $B(t)$. By direct computation, we get

$$\langle f''_K(z), w \rangle - k((A-B)w, w) = \int_0^{kt} [H''_K(x(t))w, w] - (H''_K(z(t))w, w)] dt, \ \forall w \in W_P.$$  Then by the continuous of $H''_K$,

$$\|f''_K(z) - k(A-B)\| \to 0 \quad \text{as} \quad \|z - x\| \to 0.$$  (3.20)

Let $d = \frac{1}{4}\|\| (A-B)^2 \|^2$ and $r_0 > 0$ such that

$$\|f''_K(z) - k(A-B)\| < \frac{1}{4}d, \ \forall z \in V_{r_0} = \{z \in W_P : \|z - x\| \leq r_0\}.$$  Hence for $m$ large enough, there holds

$$\|f''_{k,m}(z) - kP_m(A-B)P_m\| < \frac{1}{2}d, \ \forall z \in V_{r_0} \cap W_P^m.$$  (3.21)

For $z \in V_{r_0} \cap W_P^m$, $\forall w \in M^-_d (P_m(A-B)P_m) \setminus \{0\}$, from (3.21) we have

$$\langle f''_{k,m}(z), w \rangle \leq k(P_m(A-B)P_m w, w) + \|f''_{k,m}(z) - kP_m(A-B)P_m\| \cdot \|w\|^2$$

$$\leq -d \|w\|^2 + \frac{1}{2}d \|w\|^2 = -\frac{1}{2}d \|w\|^2 < \theta.$$  Then

$$\dim M^-(f''_{k,m}(z)) \geq \dim M^d_-(P_m(A-B)P_m), \ \forall z \in V_{r_0} \cap W_P^m.$$  (3.22)

Similiary to the proof of (3.22), for large $m$, there holds

$$\dim M^+(f''_{k,m}(z)) \geq \dim M^d_+(P_m(A-B)P_m), \ \forall z \in V_{r_0} \cap W_P^m.$$  (3.23)

By (3.14), (3.17), (3.22) and Theorem 2.1 for large $m$ we have

$$m + \nu_P(h_0) + 1 \geq \dim X_m + \dim (x_{k,m}) \geq \dim M^d_-(P_m(A-B)P_m) = m + \nu_P(x).$$

Then by (HX3), we have

$$\nu_P(x) \leq \nu_P(h_0) + 1 \leq \dim \ker_{R}(P-I) + 1.$$  (3.24)

Finally, by (3.24), (HX1), (HX2) and Theorem 2.2, the proof is completed. 

\[ \square \]
4 Asymptotically linear Hamiltonian systems

Proof of Theorem 1.4. Let $W_P$, $A$, $P_m$ be as in Section 2, and let $f$ be defined by (3.1). Then (H4) implies that $f \in C^2(W_P, \mathbb{R})$. Let $B_0$ and $B_\infty$ be the operator defined by (2.2) corresponding to $h_0$ and $h_\infty$ respectively.

For $m > 0$, let $f_m = f|_{W_P^m}$. We carry out the proof in several steps.

Step 1. By (H1), it is easy to prove that

$$f(z) = \frac{k}{2}((A - B_0)z, z) + o(\|z\|^2) \quad \text{as} \quad z \to 0. \quad (4.1)$$

Let

$$X_m = M^-(P_m(A - B_0)P_m) \oplus M^0(P_m(A - B_0)P_m), \quad Y_m = M^+(P_m(A - B_0)P_m).$$

For $z \in Y_m$, by (4.1) and the fact that $P_nB_0 = B_0P_n$ for $n \geq 0$, there exists $\rho > 0$ small enough that

$$f_{K,m}(z) = \frac{k}{2}((A - B_0)z, z) + o(\|z\|^2)
\geq \frac{k}{2}\|(A - B_0)^2\|^{-1}\|z\|^2 + o(\|z\|^2)
\geq \alpha = \frac{k}{4}\|(A - B_0)^2\|^{-1}\|\rho\|^2 > 0, \quad \forall \ z \in \partial B_\rho(0) \cap Y_m. \quad (4.2)$$

Step 2. Since $P_nB_\infty = B_\inftyP_n$ for $n \geq 0$, it is easy to show that there exists $m_0 > 0$ such that

$$\ker(A - B_0) \subset W_P^m.$$ 

On the other hand, there is $m_1 > 0$ such that for $m \geq m_1$

$$\dim \ker(P_m(A - B_\infty)P_m) \leq \dim \ker(A - B_\infty). \quad (4.3)$$

Then there exists $m_1 \geq m_0$ such that for $m \geq m_1$

$$\ker P_m(A - B_\infty)P_m = \ker(A - B_\infty). \quad (4.4)$$

This implies that

$$\text{Im} P_m(A - B_\infty)P_m \subset \text{Im}(A - B_\infty).$$

Then for any $z \in \text{Im} P_m(A - B_\infty)P_m$, we have

$$\|P_m(A - B_\infty)P_m\| = \|(A - B_\infty)\| \geq \|(A - B_\infty)^2\|^{-1}\|z\|.$$ 

Then for any $0 < d \leq \frac{1}{4}\|(A - B_\infty)^2\|^{-1}$,

$$M_d^*(P_m(A - B_\infty)P_m) = M^*(P_m(A - B_\infty)P_m), \quad \text{where} \quad * = +, -, 0. \quad (4.5)$$
By Theorem 2.1 there exist $m_2 \geq m_1$ such that for $m \geq m_2$,
\[
\dim M^-(P_m(A-B_\infty)P_m) = m + i_P(h_\infty).
\] (4.6)

Similarly, there exists $m_3 > 0$ such that for $m \geq m_3$,
\[
\dim M^-(P_m(A-B_0)P_m) = m + i_P(h_0),
\]
\[
\dim M^0(P_m(A-B_0)P_m) = \nu_P(h_0),
\] (4.7)

Let $m_4 = \max\{m_2, m_3\}$. For $m \geq m_4$, by (4.6), (4.7) and (H4) we have
\[
\dim M^-(P_m(A-B_\infty)P_m) > \dim X_m.
\]

It implies that there exists
\[
y \in M^-(P_m(A-B_\infty)P_m) \cap Y_m, \quad \|y\| = 1.
\] (4.8)

By (4.8), we have $(A-B_\infty)y \in Y_m, (A-B_0)y \in Y_m$. For any $z = z_- + z_0 \in X_m$,
\[
\langle (B_\infty - B_0) y, z \rangle = -\langle (A-B_\infty)y, z \rangle + \langle (A-B_0)y, z \rangle = 0.
\] (4.9)

By (H6) we have that $B_\infty - B_0$ is positive definite and
\[
\langle (B_\infty - B_0)z_-, z_- \rangle \geq \lambda_0\|z_0\|^2, \quad \text{where } \lambda_0 > 0,
\] (4.10)

\[
[(A-B_\infty) - (A-B_0)]^2 = (B_0-B_\infty)^2 = (B_\infty - B_0)^2 = [(A-B_0) - (A-B_\infty)]^2.
\] (4.11)

(4.11) implies that
\[
(A-B_\infty)(A-B_0) = (A-B_0)(A-B_\infty).
\] (4.12)

Hence
\[
0 = \langle (A-B_\infty)z_-, (A-B_0)z_0 \rangle = \langle (A-B_0)(A-B_\infty)z_-, z_0 \rangle
\]
\[
= \langle (A-B_\infty)(A-B_0)z_-, z_0 \rangle = \langle (A-B_0)z_-, (A-B_\infty)z_0 \rangle,
\] (4.13)

it implies that $\langle z_-, (A-B_\infty)z_0 \rangle = 0$. Hence
\[
\langle (B_\infty - B_0)z_-, z_0 \rangle = \langle z_-, (B_\infty - B_0)z_0 \rangle
\]
\[
= -\langle z_-, (A-B_\infty)z_0 \rangle + \langle z_-, (A-B_0)z_0 \rangle = 0.
\] (4.14)

Set
\[
Q_m = \{z = ry + z_- + z_0 \in W^m_P : z_- + z_0 \in X_m, \|z_- + z_0\| \leq r_1, 0 \leq r \leq r_1\},
\] (4.15)

$r_1 > 0$ will be determined later. For $z = ry + z_- + z_0 \in Q_m$, by (4.9), (4.10), (4.14) and (H5), we have
\[
f_m(z) = \frac{k}{2}\langle (A-B_\infty)z, z \rangle - \int_0^{k\tau} G_\infty(z)dt
\]
\[
= \frac{k}{2}\langle (A-B_0)z_-, z_- \rangle + \frac{kr^2}{2}\langle (A-B_\infty)y, y \rangle
\]
\[
- \frac{k}{2}\langle (B_\infty - B_0)z_- + z_0, z_0 \rangle - \frac{kr^2}{2}\langle (B_\infty - B_0)z_0, z_0 \rangle + o(\|z\|^2)
\]
\[
\leq -\frac{k}{2}\|A-B_0\|^2\|z_\|^{-1}\|z_-\|^2 - \frac{kr^2}{2}\|A-B_\infty\|^2\|z_-\|^2 - \frac{k\lambda_0}{2}\|z_0\|^2 + o(\|z\|^2)
\]
\[
\leq -\frac{k}{2}\min\{\|A-B_0\|^2, r^2\|A-B_\infty\|^2\}z_\|^2 + o(\|z\|^2).
\]
Then taking $r_1 > 0$ to be large enough we have
\[ f_m(z) \leq 0, \quad \forall \ z \in \partial Q_m. \quad (4.16) \]

**Step 2.** Using the same arguments as the proof of Lemma 2.1 in [18] and Lemma 7.1 in [28], we have that $f_m$ satisfies (P.S) condition and $f$ satisfies (P.S)$^*$ condition either (H5) with $\nu_P(h_\infty) = 0$ or the condition (2) in Theorem 1.3. By (4.12), (4.16) and Theorem 2.3, $f_m$ has a critical value $c_m \geq \alpha$, which is given by
\[ c_m = \inf_{g \in \Lambda_m} \max_{w \in Q_m} f_m(g(w)), \quad (4.17) \]
where $\Lambda_m = \{ g \in C(Q_m, W_P^m) | g = id \ on \ \partial Q_m \}$. Moreover, there is a critical point $x_m$ of $f_m$ which satisfies
\[ m^-(x_m) \leq \dim X_m + 1. \quad (4.18) \]
Since $id \in \Lambda_m$, by (4.17) and (H2) we have
\[ c_m \leq \sup_{w \in Q_m} f_m(w) \leq \beta = kr_1^2 \| A - B_0 \|. \quad (4.19) \]
Then in the sense of subsequence we have
\[ c_m \to c, \quad 0 < \alpha \leq c \leq \beta. \quad (4.20) \]
Since $f$ satisfies the (P.S)$^*$ condition on $W_P$, hence in the sense of the subsequence we have
\[ x_m \to x, \quad f(x) = c, \quad f'(x) = 0. \quad (4.21) \]

Now using the same arguments as (3.19)-(3.24), by (1.18)-(1.24) and (HX4), we have that $x$ is a non-constant $P$-solution of (1.1) with its Maslov $P$-index $i_P(x)$ satisfying
\[ i_P(x) \leq i_P(h_0) + \nu_P(h_0) + 1 \leq \dim \ker_{\mathbb{R}}(P - I) + 1. \quad (4.22) \]
The proof is completed by (4.22), (HX1), (HX2) and Theorem 2.2. 
\[ \square \]

**Proof of Theorem 1.4. Step 1.** Let $W_P, A, P_m$ be as in Section 2, and let $f$ be defined by (3.1). Then (H4) implies that $f \in C^2(W_P, \mathbb{R})$. Let $B_\infty$ be the operator defined by (2.2) corresponding to $h_\infty$.

For $m > 0$, let $f_m = f|_{W_P^m}$. Using the same arguments as the proof of Lemma 2.1 in [18] and Lemma 7.1 in [28], we have that $f_m$ satisfies (P.S) condition and $f$ satisfies (P.S)$^*$ condition either (H5) with $\nu_P(h_\infty) = 0$ or the condition (2) in Theorem 1.3. Let
\[ X_m = M^{-}(P_m(A - B_\infty)P_m) \oplus M^{0}(P_m(A - B_\infty)P_m), \quad Y_m = M^{+}(P_m(A - B_\infty)P_m). \]
For $z \in Y_m$, by (1.3) we have
\[
 f_m(z) = \frac{k}{2} \langle (A - B_\infty)z, z \rangle - \int_0^{k^\tau} G_\infty(z) dt \\
\geq \frac{k}{2} \| (A - B_0)^2 \|^{-1} \| z \|^2 - M_1 \| z \|^2 \\
\geq \alpha = -\frac{k}{2} \| (A - B_0)^2 \|^{-1} M_1.
\]

(4.23)

For $z = z_+ + z_0 \in X_m$, by (1.3) we have
\[
 f_m(z) = \frac{k}{2} \langle (A - B_\infty)z_+, z_+ \rangle - \int_0^{k^\tau} G_\infty(z) dt \\
\leq -\frac{k}{2} \| (A - B_0)^2 \|^{-1} \| z_\| \|^2 + M_1 \| z_\| - \int_0^{k^\tau} G_\infty(z_0) dt.
\]

Since $B_\infty P_n = P_n B_\infty$, there exist $m_1 > 0$ such that for $m \geq m_1$,
\[
 \ker P_m (A - B_\infty) P_m = \ker (A - B_\infty).
\]

Then by (1.3),
\[
 \int_0^{k^\tau} G_\infty(z_0) dt \to +\infty, \quad \text{as } \| z_0 \| \to \infty.
\]

(4.25)

By (4.24) and (4.25), there exist $r_1 > 0$ and $\beta < \alpha$ such that
\[
 f_m(z) \leq \beta, \quad \forall \ z \in \partial Q_m,
\]

(4.26)

where $Q_m = \{ z \in X_m : \| z \| \leq r_1 \}$. The constants $\alpha$, $\beta$ and $r_1$ in the above are independent of $m$.

**Step 2.** Let $S = Y_m$, then $\partial Q_m$ and $S$ homologically link (cf.[4]). Let $D = \partial Q_m$ and $\delta = [Q_m] \in H_l(W^p_{\infty}, D)$, where $l = \dim X_m$. Then $\delta$ is nontrivial and $\mathcal{F}(\delta)$ defined by Definite 2.4 is a homological family of dimension $l$ with boundary $D$. It is well known that $f'_m$ is Fredholm on $\mathcal{K}_{c_m}$ defined by (2.9) and (2.10). By (4.23) and (4.26), we obtain
\[
 \sup_{w \in D} f_m(w) \leq \beta < \alpha \leq c_m = c(f_m, \mathcal{F}(\delta)).
\]

Then by Theorem 2.5, there exists $x_m \in \mathcal{K}_{c_m}$ such that the Morse indices $m^-(x_m)$ and $m^0(x_m)$ of $f_m$ at $x_m$ satisfies
\[
 \dim X_m - m^0(x_m) \leq m^-(x_m) \leq \dim X_m.
\]

(4.27)

Since $Q_m \in \mathcal{F}(\delta)$, by (4.24) we have
\[
 c_m \leq \sup_{w \in Q_m} f_m(w) \leq \frac{k}{2} r_1^2 \| A - B_\infty \| + M_1 r_1 = M_2.
\]

Hence in the sense of subsequence we have
\[
 c_m \to c, \quad \alpha \leq c \leq M_2.
\]
Since $f$ satisfies $(P.S)^*$ condition, in the sense of subsequence,
\[ x_m \to x_0, \quad f(x_0) = c, \quad f'(x_0) = 0. \quad (4.28) \]

Using the standard arguments we have $x_0$ is a classical $P$-solution of $(1.1)$. Now using the same arguments as $(3.20)-(3.22)$, there exists $r_2 > 0$ such that
\[ \dim M^\pm(f_m''(z)) \geq \dim M^d_m(A - B)m, \quad \forall z \in \{ z \in W^m_P : \| z - x_0 \| \leq r_2 \}, \quad (4.29) \]

where $B$ be the operator defined by $(2.2)$ corresponding to $B(t) = H''(x_0(t))$.

By $(4.5)$, $(4.27)-(4.29)$ and Theorem 2.1, there exists $m_2 \geq m_1$ such that for $m \geq m_2$,
\[
\begin{align*}
& m + i_P(h_\infty) + \nu_P(h_\infty) = \dim X_m \geq m^-(x_m) \\
& \quad \geq \dim M^d_m(A - B)m = m + i_P(x_0) \\
& m + i_P(h_\infty) + \nu_P(h_\infty) = \dim X_m \leq m^-(x_m) + m^0(x_m) \\
& \quad \leq \dim(M^d_m(A - B)m) \oplus M^0_m(A - B)m = m + i_P(x_0) + \nu_P(x_0).
\end{align*}
\]

Thus there holds
\[ i_P(h_\infty) + \nu_P(h_\infty) - \nu_P(x_0) \leq i_P(x_0) \leq i_P(h_\infty) + \nu_P(h_\infty). \quad (4.30) \]

Combining $(4.30)$ with (HX5) yields that $x_0 \neq 0$, or by (H2) we have
\[ B(t) = H''(x_0(t)) = h_0, \quad \text{and} \quad i_P(x_0) = i_P(h_0), \quad \nu_P(x_0) = \nu_P(h_0). \quad (4.31) \]

So $(4.30)$ contradicts to (HX5). Further, we have that $x_0$ is non-constant by (H7).

Now our conclusion follows from $(4.30)$, (HX5), (HX1), (HX2) and Theorem 2.2 The proof is complete.

\[ \square \]

5 Subquadratic Hamiltonian systems

Proof of Theorem 1.6. Let $W_P, A, P_m$ and $W^m_P$ be defined as in Section 2, let
\[ g(z) = \lambda \int_0^{k\tau} H(z)dt - \frac{k}{2} (Az, z), \quad \forall z \in W_P. \quad (5.1) \]

Set $g_m = g|_{W^m_P}$ for $m > 0$, it is easy to prove that $g_m$ satisfies $(P.S)$ condition and $g$ satisfies $(P.S)^*$ condition under the condition (H8)(cf.[2]). Let
\[ X_m = P_m(M^+(A)), \quad Y_m = P_m(M^-(A) \oplus M^0(A)). \]
For \( z \in X_m \), by (H8), (H9) and (5.1),
\[
g(z) \leq \lambda M_1 \|z\| - \frac{k}{2} \|A^\sharp\|^{-1} \|z\|^2.
\]

So there exists \( r_\lambda > 1 \) and \( Q_m = \{ z \in W_P^m : \|z\| \leq r_\lambda \} \) such that
\[
g(z) \leq 0, \quad \forall z \in \partial Q_m.
\]

(5.2)

Let \( v \in Q_m \) with \( \|z\| = 1 \) and \( S_m = v + Y_m \). For \( z = v + z_+ + z_0 \in S_m \),
\[
g(z) = \lambda \int_0^{k\tau} H(z) dt - \frac{k}{2} \langle Az, z^- \rangle - \frac{k}{2} \langle Av, v \rangle
\]
\[
\leq \lambda \int_0^{k\tau} H(z) dt + \frac{k}{2} \|A^\sharp\|^{-1} \|z^-\|^2 - \frac{k}{2} \|A\|.
\]

(5.3)

Following [2], three cases are needed to be considered.

Case 1 \( \|z_+\|^2 > a_0 = 3 \|A^\sharp\||\|A\| \). Then by (H10) and (5.3),
\[
g(z) \geq \frac{k}{2} (\|A^\sharp\|^{-1} \|z^-\|^2 - \|A\|) \geq \|A\|.
\]

Case 2 \( \|z_+\|^2 \leq a_0 = 3 \|A^\sharp\||\|A\| \) and \( \|z_0\| > a_1 \). Then by (H9) and (5.3),
\[
g(z) \geq \lambda k\tau H(z_0) - \lambda M_1 \|z_+ + v\| + \frac{k}{2} \|A^\sharp\|^{-1} \|z^-\|^2 - \frac{k}{2} \|A\| \geq 1
\]
if \( \lambda \geq 1 \) and \( a_1 \) is large enough.

Case 3 \( \|z_+\|^2 \leq a_0 \) and \( \|z_0\| \leq a_1 \). Let \( S = v + M^-(A) \oplus M^0(A) \) and \( \Omega = \{ z \in S : \|z_+\|^2 \leq a_0, \|z_0\| \leq a_1 \} \), then \( \Omega \) is convex and weakly compact. Since \( \int_0^{k\tau} H(z) dt \) is weakly continuous, it achieves its infimum \( \sigma \) on \( \Omega \) at \( \hat{z} = v + \hat{z}_- + \hat{z}_0 \). By (H10) and the fact \( \hat{z} \neq 0 \), we have \( \sigma > 0 \). Therefore,
\[
g(z) \geq \lambda \sigma + \frac{k}{2} \|A^\sharp\|^{-1} \|z^-\|^2 - \frac{k}{2} \|A\| \geq 1
\]
if \( \lambda \geq \sigma^{-1} \left( \frac{k}{2} \|A\| + 1 \right) \) and \( z \in \Omega \). Hence
\[
g(z) \geq 1, \quad \forall z \in \Omega_m = \{ z \in S_m : \|z_+\|^2 \leq a_0, \|z_0\| \leq a_1 \}.
\]

Combining the three cases, we have the constants
\[
\lambda_r = \sigma^{-1} \left( \frac{k}{2} \|A\| + 1 \right) + 1, \quad \alpha = \min \{ \|A\|, 1 \} > 0
\]
such that for \( \lambda \geq \lambda_r \), we have
\[
g(z) \geq \alpha, \quad \forall z \in S_m.
\]

(5.4)
Since $\partial Q_m$ and $S_m$ homologically link, by Theorem II.1.2 and Definition II.1.2 in [4], $\partial Q_m$ and $S$ homologically link. By (5.2) and (5.4), using the same argument as step 2 in the proof of Theorem 1.4 there is a classical $P$-solution $x_0$ of (1.4) such that

$$i_P(x_0) \leq \dim \ker_2(P - I) \quad (5.5)$$

$$g(x_0) = c \geq \alpha > 0, \quad g'(x_0) = 0. \quad (5.6)$$

By (H9) and (5.6), $x_0$ is non-constant. By (5.5), (HX1), (HX2) and Theorem 2.2 we complete the proof.

References

[1] H. Amann and E. Zehnder, Nontrivial solutions for a class of nonresonance problems and applications to nonlinear differential equations, Ann. Sc. Super. Pisa, Cl. Sci. Serie IV, VII(4) (1980), 539–603.

[2] V. Benci and P. Rabinowitz, Critical point theorems for indefinite functionals, Inv. Math. 52 (1979), 241–273.

[3] A. Chenciner and R. Montgomery, A remarkable periodic solution of the three body problem in the case of equal masses, Ann. of Math. 152 (2000), no.3, 881–901.

[4] K. C. Chang, Infinite dimensional Morse theory and multiple solution problems, in Progress in Nonlinear Differential Equations and Their Applications, Vol 6 (1993).

[5] K. C. Chang, J. Liu and M. Liu, Nontrivial periodic solutions for strong resonance Hamiltonian systems, Ann. Inst. H. Poincare Anal. Non. lineaire 14(1) (1997), 103–117.

[6] Y. Dong, Maslov type index theory for linear Hamiltonian systems with Bolza boundary value conditions and multiple solutions for nonlinear Hamiltonian systems, Pacific J. Math. 221: 2 (2005), 253–280.

[7] Y. Dong, P-index theory for linear Hamiltonian systems and multiple solutions for nonlinear Hamiltonian systems, Nonlinearity 19: 6 (2006), 1275–1294.

[8] Y. Dong and Y. Long, Closed characteristics on partially symmetric compact convex hypersurfaces in $\mathbb{R}^{2n}$, J. Diff. Equa. 196 (2004), 226-248.

[9] D. Ferrario and S. Terracini, On the existence of collisionless equivariant minimizers for the classical $n$-body problem, Invent. Math. 155: 2 (2004), 305–362.

[10] G. Fei and Q. Qiu, Periodic solutions of asymptotically linear Hamiltonian systems, Chin. Ann. Math., 18B(3) (1997), 359–372.

[11] G. Fei and Q. Qiu, Minimal period solutions of nonlinear Hamiltonian systems, Nonlinear Analysis, Theory, Method Applications 27(7) (1996), 821–839.
[12] N. Ghoussoub, *Location, multiplicity and Morse indices of min-max critical points*, J. Reine Angew Math. **417** (1991), 27–76.

[13] X. Hu and S. Sun, *Morse index and the stability of closed geodesics*, Sci. China Math. **53**(5) (2010), 1207–1212.

[14] X. Hu and S. Sun, *Index and stability of symmetric periodic orbits in Hamiltonian systems with application to figure-eight orbit*, Comm. Math. Phys. **290** (2009), 737–777.

[15] X. Hu and S. Sun, *Stability of relative equilibria and Morse index of central configurations*, C. R. Acad. Sci. Paris **347** (2009), 1309–1312.

[16] X. Hu and P. Wang, *Conditional Fredholm determinant for the S-periodic orbits in Hamiltonian systems*, Journal of Functional Analysis **261** (2011), 3247–3278.

[17] A. Lazer and S. Solomini, *Nontrivial solution of operator equations and Morse indices of critical points of min-max type*, Nonlinear Anal. **12** (1988), 761–775.

[18] S. Li and J. Liu, *Morse theory and asymptotically linear Hamiltonian systems*, J. Diff. Equa. **78** (1989), 53–73.

[19] C. Liu, *Maslov P-index theory for a symplectic path with applications*, Chin. Ann. Math. **4** (2006), 441–458.

[20] C. Liu, *Periodic solutions of asymptotically linear delay differential systems via Hamiltonian systems*, J. Differential Equations **252** (2012), 5712–5734.

[21] C. Liu, *Relative index theories and applications*, 2015.

[22] C. Liu and S. Tang, *Maslov (P,ω)-index theory for symplectic paths*, Advanced Nonlinear Studies **15** (2015), 963–990.

[23] C. Liu and S. Tang, *Iteration inequalities of the Maslov P-index theory with applications*, Nonlinear Analysis **127** (2015), 215–234.

[24] C. Liu and S. Tang, *Subharmonic P-solutions of first order Hamiltonian systems*, submitted.

[25] Y. Long, *Index theory for symplectic paths with application*, Progress in Mathematics, Vol. 207, Birkhäuser Verlag, 2002.

[26] P. H. Rabinowitz, *Periodic solutions of Hamiltonian systems*, Comm. Pure Appl. Math. **31** (1978), 157–184.

[27] S. Solimini, *Morse index estimates in min-max theorems*, Manuscripta Math. **63** (1989), 421–453.

[28] A. Szulkin, *Cohomology and Morse theory for strongly indefinite functionals*, Math. Z. **209** (1992), 375–418.