Isolated singularities of solutions to the Yamabe equation in dimension 6

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Abstract

We study the asymptotic behavior of local solutions to the Yamabe equation near an isolated singularity, when the metric is not conformally flat. We prove that, in dimension 6, any solution is asymptotically close to a Fowler solution, which is an extension of the same result for lower dimensions by F.C. Marques in 2008.

1 Introduction

Let $g$ be a smooth Riemannian metric on the unit ball $B_1 \subset \mathbb{R}^n$, $n \geq 3$. We study positive solutions of the Yamabe equation in the punctured ball

$$-L_g u = n(n-2)u^{\frac{n+2}{n-2}} \quad \text{in } B_1 \setminus \{0\},$$

where $L_g = \Delta_g - \frac{(n-2)}{4(n-1)}R_g$ is the conformal Laplacian, $\Delta_g$ is the Laplace-Beltrami operator and $R_g$ is the scalar curvature of $g$. We will always assume that solutions are smooth away from the singular point.

When $g$ is conformally flat and 0 is a non-removable singularity, Caffarelli-Gidas-Spruck [2] proved that

$$u(x) = u_0(x)(1 + o(1)) \quad \text{as } x \to 0,$$

where $u_0$ is a Fowler solution. Here Fowler solutions are referred to the singular positive solutions of

$$-\Delta u_0 = n(n-2)u_0^{\frac{n+2}{n-2}} \quad \text{in } \mathbb{R}^n \setminus \{0\},$$

which were proved to be radially symmetric and classified in the same paper [2]. A different proof and refinement of this result were given by Korevaar-Mazzeo-Pacard-Schoen [14], in particular, they improved the $o(1)$ remainder term to a $O(|x|^\alpha)$ for some $\alpha > 0$. Namely,

$$u(x) = u_0(x)(1 + O(|x|^\alpha)).$$

When $g$ is not conformally flat and $3 \leq n \leq 5$, Marques [21] established the same asymptotic behavior. In this paper, we show that this still holds in dimension 6, which appears to be the borderline of the current method.

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Theorem 1.1. Suppose that \( n = 6 \) and \( u \in C^2(B_1 \setminus \{0\}) \) is a positive solution of (1). If 0 is not a removable singularity of \( u \), then

\[
u(x) = u_0(1 + O(|x|^\alpha)) \quad \text{as } x \to 0,
\]

where \( u_0 \) is a Fowler solution and \( \alpha > 0 \).

Once the convergence to a Fowler solution is established, the arguments of [14] and [21] can be used to improve the approximation by deformed Flower solutions. See Han-Li-Li [10] for expansions up to arbitrary orders when the metric is conformally flat. Existence of solutions of (1) is related to the study of local solutions of the singular Yamabe problem, which has been studied by Schoen [25], Mazzeo-Smale [24], Mazzeo-Pollack-Uhlenbeck [23], Mazzeo-Pacard [22] and etc, after the resolution of the Yamabe problem by Yamabe [30], Trudinger [29], Aubin [1] and Schoen [26].

The difficulty to establish asymptotical symmetry of solutions near isolated singularities is that (1) has no symmetry when \( g \) is not conformally flat. Similar difficulty also happens to the prescribing scalar curvature equation

\[-\Delta u = n(n-2)K u^{\frac{n+2}{n-2}} \quad \text{in } B_1 \setminus \{0\}, \quad u > 0,
\]

where \( \Delta \) is the Laplace operator and \( K > 0 \) is a \( C^1 \) function in \( B_1 \). Under the flatness condition

\[c_1|x|^{l-1} \leq |\nabla K(x)| \leq c_1|x|^{l-1},
\]

where \( c_1, c_2 > 0 \) and \( l \geq \frac{n-2}{2} \) are constants, Chen-Lin [5, 6] established (2) for non-removable singularities. On the other hand, they constructed a singular solution which does not satisfy (2) when \( l < \frac{n-2}{2} \). As for (1), the metric in normal coordinates centered at any point \( \bar{x} \in B_1 \) has the flatness

\[g_{ij} = \delta_{ij} + O(|x|^\tau)
\]

with \( \tau = 2 \). Marques’ proof would be possible to give (3) in all dimensions, if \( \tau > \frac{n-2}{2} \) in a neighborhood of 0. Obviously, this condition holds automatically in dimensions in 3, 4, 5 but does not in dimension 6.

By Theorem 8 of [21], (3) follows from

\[
\frac{1}{C}d_g(x, 0)^{-\frac{n-2}{2}} \leq u(x) \leq C d_g(x, 0)^{-\frac{n-2}{2}}
\]

for some \( C \geq 1 \) independent of \( x \), where \( d_g \) is the distance function with respect to \( g \). The proof of both the upper bound and lower bound in (6) for dimension 6 requires delicate analysis to handle difficulties related to borderline cases. To establish the upper bound we use the moving spheres method which requires a careful construction of a test function. In order to overcome certain difficulties we build our argument on properties of the conformal normal coordinates and apply the maximum principle only on selected domains. See [20, 19, 18, 17, 31] about the moving spheres method. To prove the lower bound in dimension 6, we deform the metric conformally to one with negative scalar curvature and take advantage of certain monotonicity properties of the solutions. As a result we can improve a differential inequality of [21] that plays a crucial role in the proof of the lower bound. The dimension 6 case shows some similarity to the borderline \( l = \frac{n-2}{2} \) of [5, 6], but the proof in this article is more involved.
To end this section, we would like to mention some related papers about the isolated singularities problem for the Yamabe equation; see [3, 8, 7, 12, 13, 15, 16, 27, 28, 31] and references therein.

This paper is organized as follows. In section 2, we establish the upper bound. In section 3, we establish the criteria of singularity removability in terms of Pohozaev integral and prove the main theorem.

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2 The upper bound

We will use $B^g_\rho(x)$ to denote the geodesic ball with respect to $g$ centered at $x$ with radius $\rho > 0$, the superscript $g$ in $B^g_\rho(x)$ will be dropped when there is no ambiguity.

Theorem 2.1. Suppose $n = 6$ and $u \in C^2(B_1 \setminus \{0\})$ is a solution of (1). Then

$$\limsup_{x\to 0} d_g(x,0)^{\frac{n-2}{2}} u(x) < \infty.$$ (7)

Proof. Without loss of generality, we assume $B_1$ is the normal coordinates chart of $g$ centered at 0. If (7) were invalid, there exists a sequence $x_k \to 0$ such that

$$d_g(0, x_k)^{\frac{n-2}{2}} u(x_k) \to \infty \text{ as } k \to \infty.$$ (8)

We shall divide the remaining proof into four steps.

Step 1. Blow-up analysis.

Claim: The sequence $x_k$ in (8) can be selected to be local maximum points of $u$.

The proof of this fact is standard and we briefly describe it for readers’ convenience. Set

$$f_k(y) = u(y)(d_k - d_g(y, x_k))^{\frac{n-2}{2}} \text{ for } d_g(y, x_k) \leq d_k,$$ where $d_k = d_g(x_k, 0)/2$. Clearly, $f_k(x_k) \to \infty$ and $f_k = 0$ on $\partial B_d(x_k)$. Let $f_k(\hat{x}_k)$ be a maximum of $f_k$ on $B_{d_k}(x_k)$ and set

$$\alpha_k = \frac{1}{2}(d_k - d_g(\hat{x}_k, x_k)).$$

By the definition of $\hat{x}_k$ we have

$$u(\hat{x}_k)(2\alpha_k)^{\frac{n-2}{2}} \geq u(x_k)d_k^{\frac{n-2}{2}} \to \infty$$ (9)

and for $y \in B_{\alpha_k}(\hat{x}_k)$,

$$u(y) \leq u(\hat{x}_k)(\frac{2\alpha_k}{d_k - d_g(y, x_k)})^{\frac{n-2}{2}} \leq u(\hat{x}_k)2^{\frac{n-2}{2}},$$ (10)

where we have used $d_k - d_g(y, x_k) \geq d_k - d_g(\hat{x}_k, x_k) - d_g(\hat{x}_k, y) \geq \alpha_k$ in the last inequality.
As a consequence of (9), (10), the sequence of functions \( \hat{v}_k \) defined by
\[
\hat{v}_k(y) = u(\hat{x}_k)^{-1}u(\exp_{\hat{x}_k} u(\hat{x}_k) - \frac{2}{n-2} y)
\]
has a subsequence (still denoted as \( v_k \)) that converges in \( C^2_{\text{loc}}(\mathbb{R}^n) \) to \( U \) of
\[
\Delta U + n(n-2)U^{\frac{n+2}{n-2}} = 0 \quad \text{in} \quad \mathbb{R}^n.
\]
(11)

By the classification theorem of Caffarelli-Gidas-Spruck [2],
\[
U(y) = \left( \frac{\lambda}{1 + \lambda^2 |y - y_0|^2} \right)^{\frac{n-2}{2}}
\]
for some \( \lambda \geq 1 \) and \( y_0 \in \mathbb{R}^n \). Since \( y_0 \) is the maximal point of \( U \) and \( \nabla^2 U(y_0) \) is negative definite, there exist \( y_k \to y_0 \) where \( y_k \) is a local maximum of \( v_k \). Thus from the beginning we can assume \( x_k \) to be the pre-image of \( y_k \). Thus \( x_k \) is a local maximum point of \( u \) which also satisfies
\[
u(x_k) \xrightarrow[k \to \infty]{} \infty \quad \text{as} \quad k \to \infty.
\]
(12)

We shall use the conformal normal coordinates centered at \( x_k \). Namely, we can find a smooth positive function \( \kappa_k \) to deform the metric conformally: \( \bar{g} := \kappa_k^{-\frac{4}{n-2}}g \). In the normal coordinates of \( \bar{g} \) centered at \( x_k \) there holds
\[
det \bar{g}(x) = 1 \quad \text{for} \quad |x| < \delta,
\]
where \( \delta > 0 \) is independent of \( k \); see Cao [4] and Günther [9]. (Without causing much confusion, we did not label \( k \) to \( \bar{g} \).) It is easy to check that
\[
\kappa_k(0) = 1 \quad \text{and} \quad \nabla \kappa_k(0) = 0.
\]
(13)

Let \( u_k = \kappa_k u \), then \( u_k \) satisfies the following equation based on the conformal invariant property of \( L_{\bar{g}} \):
\[
-L_{\bar{g}} u_k(x) = n(n-2)u_k(x)^{\frac{n+2}{n-2}} \quad \text{in} \quad B_\delta \setminus \{z_k\},
\]
where \( z_k \) is the singular point in the new coordinate and \( \delta \) is a positive small number. For scaling we set \( M_k = u_k(x_k) \) and
\[
v_k(y) = M_k^{-\frac{2}{n-2}} u_k(\exp_{x_k} (M_k^{-\frac{2}{n-2}} y)).
\]
Then the conformal invariant property further carries us to the equation for \( v_k \):
\[
-L_{g_k} v_k(y) = n(n-2)v_k(y)^{\frac{n+2}{n-2}} \quad \text{in} \quad B_{\delta M_k^{-\frac{2}{n-2}}} \setminus \{S_k\},
\]
(14)
where \( (g_k)_{ij} = \bar{g}_{ij}(M_k^{-\frac{2}{n-2}} y) \) and \( S_k = M_k^{-\frac{2}{n-2}} z_k \). By the discussion about the location of \( x_k \) we have
\[
|S_k| \to \infty
\]
and
\[
v_k(y) \to U(y) \quad \text{in} \quad C^2_{\text{loc}}(\mathbb{R}^n)
\]
as \( k \to \infty \), where \( U \geq 0 \) satisfies (11). Since \( x_k \) is a local maximum of \( u \) and (13) holds, we have

\[
U(0) = 1 \quad \text{and} \quad \nabla U(0) = 0.
\]

By the classification theorem of Caffarelli-Gidas-Spruck [2],

\[
U(y) = \left( \frac{1}{1 + |y|^2} \right)^{\frac{n-2}{2}}.
\]

Before further investigation we mention two lower bounds of \( v_k \) which will be used in two different occasions. The first one is

\[
v_k(y) \geq \Lambda M_k^{-1}, \quad |y| \leq \delta M_k^{\frac{n-2}{2}},
\]

which follows from the maximum principle and the definition of \( v_k \). Indeed, choose \( \delta > 0 \) small so that the first eigenvalue of \(-L_g\) is positive in \( B_\delta \) (with respect to Dirichlet boundary condition). Let \( u \geq \Lambda_1 > 0 \) on \( \partial B_\delta \) and \( \phi \) be the solution of \( L_g \phi = 0 \) in \( B_\delta \) with \( \phi = \Lambda_1 \) on \( \partial B_\delta \). Then we see that \( u \geq \phi \) by the maximum principle and \( \phi > \Lambda > 0 \) for some \( \Lambda \in (0, \Lambda_1) \) by standard Harnack inequality. Thus (15) holds. The second lower bound is stated in the following proposition:

**Proposition 2.2.** There exists \( C > 0 \) independent of \( k \) such that

\[
v_k(y) \geq \frac{1}{C} (1 + |y|)^{2-n} \quad \text{for } y \in B_{\delta M_k^{\frac{n-2}{2}}}(x_k).
\]

**Proof.** Fix \( \tau > 0 \) so that \(-L_g\) is coercive in \( H_0^1(B_\tau) \). We assume \( \delta < \tau/2 \). Let \( G_k \) be the solution of

\[
-L_g G_k(x) = \delta x \quad \text{in } B_\tau, \quad G_k(x) = 0 \quad \text{on } \partial B_\tau,
\]

where \( \delta x \) is the Dirac measure centered at \( x_k \). Then \( G_k \) satisfies

\[
\frac{1}{A} |y|^{2-n} \leq G_k(\exp_{x_k} y) \leq A |y|^{2-n} \quad \text{for } y \in B_\delta \setminus \{0\},
\]

\[
\lim_{y \to 0} G(\exp_{x_k} y) |y|^{n-2} = \frac{1}{(n-2)\omega_n},
\]

where \( \omega_n \) denotes the volume of the standard \((n-1)\)-sphere and \( A > 0 \) is independent of \( k \). Since \( v_k(y) \to U(y) \) as \( k \to \infty \) for \( |y| = 1 \), there exists \( C > 0 \) independent of \( k \) such that

\[
u \geq \frac{1}{C} G_k \quad \text{on } \partial B_{M_k^{\frac{n-2}{2}}}(x_k).
\]

By the comparison principle, we have \( u \geq \frac{1}{C} G_k \) in \( B_\tau \setminus B_{M_k^{\frac{n-2}{2}}}(x_k) \). Hence,

\[
v_k(y) \geq \frac{1}{C} M_k^{-1} G_k(\exp_{x_k} M_k^{-\frac{n-2}{2}} y)
\geq \frac{1}{AC} M_k^{-1} M_k |y|^{2-n} = \frac{1}{AC} |y|^{2-n} \quad \text{for } |y| \geq 1.
\]

When \( |y| \leq 1 \), we used \( v_k \to U \) again to have the lower bound. Therefore, the proposition is proved. \( \square \)
Recall that in the conformal normal coordinates,
\[ \bar{g}_{ij}(x) = \delta_{ij} + O(|x|^2), \quad \det \bar{g} = 1, \quad R_{\bar{g}}(x) = O(|x|^2). \]
It follows that
\[ \Delta \bar{g} = \partial_i(\bar{g}^{ij}\partial_j) = \Delta + b_j \partial_j + d_{ij} \partial_{ij}, \]
where
\[ b_j(x) = \partial_i \bar{g}^{ij}(x) = O(|x|), \]
and
\[ d_{ij}(x) = \bar{g}^{ij}(x) - \delta_{ij} = O(|x|^2). \]
Thus
\[ -L_{g_k} = -\Delta_{g_k} + c(n) R_{g_k} = -\Delta - \tilde{b}_j \partial_j - \tilde{d}_{ij} \partial_{ij} + \tilde{c}, \]
where \( c(n) = \frac{(n-2)}{4(n-1)} \).
\begin{align}
\tilde{b}_j(y) &= M_k^{-\frac{2}{n-2}} b_j(M_k^{-\frac{2}{n-2}} y) = O(M_k^{-\frac{1}{n-2}} |y|), \\
\tilde{d}_{ij}(y) &= d_{ij}(M_k^{-\frac{2}{n-2}} y) = O(M_k^{-\frac{4}{n-2}} |y|^2), \\
\tilde{c}(y) &= c(n) R_g(M_k^{-\frac{2}{n-2}} y) M_k^{-\frac{1}{2}} = O(M_k^{-\frac{8}{n-2}} |y|^2). 
\end{align}

Note that the subscripts \( k \) are dropped for convenience. The equation of \( v_k \) becomes
\[ (\Delta + \tilde{b}_j \partial_j + \tilde{d}_{ij} \partial_{ij} - \tilde{c})v_k + n(n-2)v_k^{\frac{n+2}{n-2}} = 0 \quad \text{in} \quad B_{\delta M_k^{-\frac{2}{n-2}}} \setminus \{S_k\}. \]

**Step 2. Setting up the moving spheres framework.**

For \( \lambda > 0 \) and any function \( v \), define
\[ v^\lambda(y) := \left( \frac{\lambda}{|y|} \right)^{n-2} v(y^\lambda), \quad y^\lambda := \frac{\lambda^2 y}{|y|^2} \]
as the Kelvin transformation of \( v \) with respect to \( \partial B_\lambda \). To carry out the method of moving spheres we restrict our discussion on \( \Sigma_\lambda \setminus \{S_k\} \), where \( \Sigma_\lambda \) is defined as
\[ \Sigma_\lambda := B_{\delta M_k^{-\frac{2}{n-2}}} \setminus B_\lambda = \{ y \mid \lambda < |y| < \delta M_k^{-\frac{2}{n-2}} \}. \]

Let
\[ w_\lambda(y) := v_k(y) - v_k^\lambda(y), \quad y \in \Sigma_\lambda \setminus \{S_k\}. \]

A straightforward computation yields
\[ \Delta w_\lambda + \tilde{b}_j \partial_j w_\lambda + \tilde{d}_{ij} \partial_{ij} w_\lambda - \tilde{c} w_\lambda + n(n+2)\xi^{\frac{4}{n-2}} w_\lambda = E_\lambda \quad \text{in} \quad \Sigma_\lambda \setminus \{S_k\}, \]
where $\xi > 0$ is given by

$$n(n+2)\frac{\sqrt{n^2+2}}{v_k-v_k^\lambda}, \quad v_k \neq v_k^\lambda,$$

and

$$n(n+2)\frac{4}{v_k}, \quad v_k = v_k^\lambda,$$

and

$$E_\lambda(y) = \left( c(y)v_k^\lambda(y) - \left( \frac{\lambda}{|y|} \right)^n c(y)v_k(y^\lambda) \right) - (\bar{b}_j \partial_j v_k^\lambda + \bar{d}_{ij} \partial_{ij} v_k^\lambda)$$

$$+ \left( \frac{\lambda}{|y|} \right)^{n+2} \left( \bar{b}_j (y^\lambda) \partial_j v_k(y^\lambda) + \bar{d}_{ij} (y^\lambda) \partial_{ij} v_k(y^\lambda) \right).$$

(22)

Here we note that we shall always require $\lambda \in [1/2, 2]$. Since $|S_k| \to \infty$ as $k \to \infty$, $v_k$ is smooth in $B_\lambda$ and $v_k^\lambda$ is smooth in $\Sigma_\lambda$.

The following estimate of $E_\lambda$ is crucial to the construction of auxiliary functions in the sequel. Since it is related to the smallness of $v_k - U$ in $B_2$, we use the a notation to represent this quantity:

$$\sigma_k := \|v_k - U\|_{C^2(B_2)}, \quad \sigma_k \to 0 \quad \text{as} \quad k \to \infty.$$ 

(23)

**Proposition 2.3.** Let $E_\lambda$ be defined in (22), then for $\lambda \in [1/2, 2]$ and $y \in \Sigma_\lambda$, we have

$$|E_\lambda| \leq C_0 M_k^{-\frac{\lambda}{n-2}} |y|^{1-n} + C_0 \sigma_k M_k^{-\frac{4}{n-2}} |y|^{-n},$$

(24)

where $C_0 > 0$ is some constant independent of $y$ and $k$.

Proposition 2.3 is an easy corollary of Proposition 2.1 of [17]. We include a proof here for readers’ convenience.

**Proof of Proposition 2.3.** First we estimate the second term of $E_\lambda$:

$$I := (\bar{b}_j \partial_j v_k^\lambda + \bar{d}_{ij} \partial_{ij} v_k^\lambda).$$

Since in the conformal normal coordinates

$$0 = (\Delta_{g_k} - \Delta) w = (\bar{b}_j \partial_j + \bar{d}_{ij} \partial_{ij}) w$$

(25)

for any smooth radial function $w(y)$, we have

$$I = (\bar{b}_j \partial_j + \bar{d}_{ij} \partial_{ij}) [(v_k - U)^\lambda].$$

By a direct computation,

$$\partial_j \left\{ \frac{\lambda}{|y|}^{n-2} (v_k - U)(y^\lambda) \right\} = \partial_j \left\{ \frac{\lambda}{|y|}^{n-2} \right\} (v_k - U)(y^\lambda) + \frac{\lambda}{|y|}^{n-2} \partial_j \left\{ (v_k - U)(y^\lambda) \right\},$$

$$\partial_j \left\{ \frac{\lambda}{|y|}^{n-2} (v_k - U)(y^\lambda) \right\} = \partial_j \left\{ \frac{\lambda}{|y|}^{n-2} \right\} (v_k - U)(y^\lambda) + \frac{\lambda}{|y|}^{n-2} \partial_j \left\{ (v_k - U)(y^\lambda) \right\},$$

(26)
\[
\partial_{ij} \left\{ \left( \frac{\lambda}{|y|} \right)^{n-2} (v_k - U)(y^\lambda) \right\} \\
= \partial_{ij} \left\{ \left( \frac{\lambda}{|y|} \right)^{n-2} (v_k - U)(y^\lambda) \right\} + \partial_i \left\{ \left( \frac{\lambda}{|y|} \right)^{n-2} \right\} \partial_j \left\{ (v_k - U)(y^\lambda) \right\} \\
+ \partial_j \left\{ \left( \frac{\lambda}{|y|} \right)^{n-2} \right\} \partial_i \left\{ (v_k - U)(y^\lambda) \right\} + \frac{\lambda^2}{|y|^{n-2}} \partial_{ij} \left\{ (v_k - U)(y^\lambda) \right\}.
\]

Since \( \bar{d}_{ij} \equiv \tilde{d}_{ji} \), using (25) with \( w = \left( \frac{\lambda}{|y|} \right)^{n-2} \) we have

\[
I = \left( \frac{\lambda}{|y|} \right)^{n-2} b_j \partial_j \left\{ (v_k - U)(y^\lambda) \right\} + 2 \bar{d}_{ij} \partial_i \left\{ \left( \frac{\lambda}{|y|} \right)^{n-2} \right\} \partial_j \left\{ (v_k - U)(y^\lambda) \right\} \\
+ \left( \frac{\lambda}{|y|} \right)^{n-2} \bar{d}_{ij} \partial_j \left\{ (v_k - U)(y^\lambda) \right\}.
\]

To evaluate terms in \( I \), we observe that for \( z \in B_2 \),

\[
(v_k - U)(z) = O(\sigma_k) |z|^2,
\]

\[
|\nabla_z (v_k - U)(z)| = O(\sigma_k),
\]

\[
|\nabla^2_z (v_k - U)(z)| = O(\sigma_k),
\]

where we have used \((v_k - U)(0) = |\nabla (v_k - U)(0)| = 0 \). It follows from (18) and (26) that

\[
I = O(1) \sigma_k M_k^{-\frac{4}{n-2}} \left( |y|^{2-n} |y^\lambda| |\nabla y^\lambda| + |y^2| |y^{1-n} |y^\lambda| |\nabla y^\lambda| \\
+ |y|^{2-n} |y| |\nabla^2 y^\lambda| + |\nabla y^\lambda|^2 \right)
\]

\[
= O(1) \sigma_k M_k^{-\frac{4}{n-2}} |y|^{-n}.
\]

Similarly,

\[
\left( \frac{\lambda}{|y|} \right)^{n+2} \left( \hat{b}_j (y^\lambda) \partial_j v_k(y^\lambda) + \bar{d}_{ij} (y^\lambda) v_k(y^\lambda) \right) = O(1) \sigma_k M_k^{-\frac{4}{n-2}} |y|^{-n}
\]

and

\[
|\hat{c}(y)||v_k^\lambda(y) - U^\lambda(y)| + \left( \frac{\lambda}{|y|} \right)^{n+2} |\hat{c}(y^\lambda)||v_k(y^\lambda) - U(y^\lambda)| = O(1) \sigma_k M_k^{-\frac{4}{n-2}} |y|^{-n}.
\]

Finally, the estimate on \( \hat{c} \) gives

\[
\hat{c}(y)U^\lambda(y) - \left( \frac{\lambda}{|y|} \right)^{n+2} \hat{c}(y^\lambda)U(y^\lambda) = O(M_k^{-\frac{8}{n-2}} |y|^{4-n}).
\]

Therefore, Proposition 2.3 is established. 

\(\square\)

**Step 3. Constructing an auxiliary function.**
If \( n = 6 \), (24) reads
\[
|E_\lambda(y)| \leq C_0 \sigma_k M_k^{-1} |y|^{-6} + C_0 M_k^{-2} |y|^{-2} \quad \text{for } y \in \Sigma_\lambda.
\] (28)

Consider the linear ordinary differential equation
\[
h''(r) + \frac{5}{r} h'(r) = -2C_0 \sigma_k M_k^{-1} r^{-6} - 2C_0 M_k^{-2} r^{-2} \quad \text{for } \lambda < r < \delta M_k^{\frac{1}{2}},
\] (29)

with the initial data
\[
h_\lambda(\lambda) = h'_\lambda(\lambda) = 0.
\] (30)

It is easy to find out that
\[
h_\lambda(r) = C_0 \sigma_k M_k^{-1} (r^{-4} \ln \frac{r}{\lambda} + \frac{r^{-4}}{4} - \frac{\lambda^{-4}}{4}) - C_0 M_k^{-2} (\ln \frac{r}{\lambda} + \frac{1}{4} (\lambda^{-4} - 1))
\] (31)
is the unique solution. An immediate observation is that \( h_\lambda(r) \leq 0 \) because \( h_\lambda \) is super-harmonic, (30) forces \( h_\lambda \) to be negative for \( r > \lambda \).

Setting \( h_\lambda(y) = h_\lambda(|y|) = h_\lambda(r), \) we have
\[
\Delta g_k h_\lambda = \Delta h_\lambda = h'''(r) + \frac{5}{r} h''(r) = -2C_0 \sigma_k M_k^{-2} |y|^{-2} - 2C_0 M_k^{-1} r^{-6}.
\]

By (28),
\[
\Delta g_k h_\lambda + E_\lambda < -C_0 \sigma_k M_k^{-1} |y|^{-6} - C_0 M_k^{-2} |y|^{-2} \quad \text{in } \Sigma_\lambda.
\] (32)

Next, we verify that
\[
(\Delta g_k - \bar{c} + 48\xi) h_\lambda(y) + E_\lambda(y) < 0 \quad \text{for } y \in \Omega_\lambda,
\] (33)

where
\[
\Omega_\lambda := \left\{ y \in \Sigma_\lambda \setminus \{S_k\} | v_k(y) < 2v_k^\lambda(y) + 2|h_\lambda(y)| \right\}.
\]

Indeed, since \( |\bar{c}| \leq AM_k^{-2} |y|^{2} \) for some \( A > 0 \), by the lower bound of \( v_k \) in (16),
\[
48\xi - \bar{c} \geq C|y|^{-4} - AM_k^{-2} |y|^{2} > 0, \quad \text{if } |y| < \delta_1 M_k^{1/3}
\] (34)

for some \( \delta_1 > 0 \). On the other hand, for \( |y| \in [\delta_1 M_k^{1/3}, \delta M_k^{\frac{3}{4}}] \) and large \( k \),
\[
2v_k^\lambda(y) + 2|h_\lambda(y)| \leq C|y|^{-4} + C\sigma_k M_k^{-1} + C M_k^{-2} \ln |y| < \Lambda M_k^{-1} \leq v_k(y),
\] (35)

where we have used (15) in the last inequality. Hence,
\[
\Omega_\lambda \subset B_{\delta_1 M_k^{1/3}}.
\] (36)

Since \( h_\lambda \leq 0 \), (33) follows immediately from (32), (34) and (36).

**Step 4. Completing the proof of the upper bound of \( u \).**

The benchmark of the moving sphere method is the following inequality that can be verified by direct computation:
\[
U(y) - U^\lambda(y) > (\leq, <) 0 \quad \text{for } |y| > \lambda, \quad \text{if } \lambda < (\leq, >) 1.
\] (37)
First we show that
\[ w_{\lambda_0} + h_{\lambda_0} > 0 \quad \text{in} \quad \Sigma_{\lambda_0} \setminus \{S_k\}, \quad \text{for} \quad \lambda_0 \in \left[ \frac{1}{2}, \frac{3}{5} \right]. \tag{38} \]

The proof of (38) starts from (37): For \( \lambda_0 \in \left[ \frac{1}{7}, \frac{3}{5} \right] \), there is a universal constant \( \epsilon_0 > 0 \) such that
\[ U(y) - U^{\lambda_0}(y) > \epsilon_0(|y| - \lambda_0)|y|^{-5} \quad \text{for} \quad |y| > \lambda_0. \]

By the convergence of \( v_k \) to \( U \) in \( C_{\text{loc}}^2(\mathbb{R}^n) \), for any fixed \( R >> 1 \),
\[ v_k(y) - v_k^{\lambda_0}(y) > \frac{\epsilon_0}{2}(|y| - \lambda_0)|y|^{-5}, \quad \text{if} \quad \lambda_0 < |y| < R \tag{39} \]
and \( k \) is sufficiently large. In particular for \( |y| = R \),
\[ v_k(y) \geq (1 - \frac{\epsilon_0}{2})|y|^{-4} \quad \text{and} \quad v_k^{\lambda_0}(y) \leq (1 - 3\epsilon_0)|y|^{-4} \quad \text{for} \quad |y| = R. \tag{40} \]

Thus the gap between \( v_k \) and \( v_k^{\lambda_0} \) is enough to engulf \( h_{\lambda_0} \). By the explicit expression of \( h_{\lambda} \) and (39), we see that for large \( k \)
\[ w_{\lambda_0}(y) + h_{\lambda_0}(y) > 0 \quad \text{for} \quad \lambda_0 < |y| < R. \]

To prove (38) for \( R < |y| < \delta M_k^{1/2} \), we first determine an upper bound for \( \overline{c} \) and construct a test function \( \phi \) over this region. By (18), we can find \( A > 0 \) to have
\[ |\overline{c}| \leq AM_k^{-2}|y|^2. \tag{41} \]

Then we set
\[ \phi(y) = (1 - \epsilon_0)|y|^{-4} + \frac{\Lambda}{2M_k} + AM_k^{-2}|y|^2, \quad R < |y| < \delta M_k^{1/2}, \]
where \( \Lambda \) is the constant in (15). It is easy to check that
\[ L_{g_k}\phi = \Delta_{g_k}\phi - \overline{c}\phi = \Delta\phi - \overline{c}\phi \geq \frac{7A}{M_k^2} - AM_k^{-2}|y|^{-2} - A\Lambda|y|^2M_k^{-3}. \]

By choosing \( \delta > 0 \) small enough (independent of \( k \) when \( k \) is large), we have
\[ L_{g_k}\phi > 0, \quad R < |y| < \delta M_k^{1/2}. \]

Then the standard maximum principle gives \( v_k \geq \phi \) on this annulus because \( L_{g_k}(v_k - \phi) < 0 \) and (see (40) and (15))
\[ v_k \geq \phi \quad \text{on} \quad \partial B_{\delta M_k^{1/2}} \cup \partial B_R. \]

Since
\[ v_k^{\lambda_0}(y) \leq (1 - 2\epsilon_0)|y|^{-4} \quad \text{for} \quad |y| > R \tag{42} \]
for large \( k \) and \( R \), we have
\[ v_k^{\lambda_0}(y) - h_{\lambda_0} \leq \phi(y) \quad \text{for} \quad |y| > R. \]
Hence, we conclude that (38) holds because
\[ v_k(y) - v_k^{\lambda_0}(y) + h_{\lambda_0}(y) > 0 \quad \text{for} \quad R < |y| < \delta M_k^{1/2}. \]

The critical position in the moving sphere method is defined by
\[ \bar{\lambda} := \sup\{ \lambda \in [1/2, 2] | v_k(y) > v_k^{\mu}(y) - h_{\mu}(y), \quad \forall \ y \in \Sigma_{\mu} \setminus \{S_k\} \text{ and } 1/2 < \mu < \lambda \}. \]

By (38), \( \bar{\lambda} \) is well-defined. In order to reach to the final contradiction we claim that \( \bar{\lambda} = 2 \).

If \( \bar{\lambda} < 2 \), by (35) we still have \( v_k > v_k^{\bar{\lambda}} - h_{\bar{\lambda}} \) on \( \partial B_{\delta M_k^{1/2}} \). By the maximum principle, \( v_k - v_k^{\bar{\lambda}} + h_{\bar{\lambda}} \) is strictly positive in \( \Sigma_{\bar{\lambda}} \) and \( \frac{\partial}{\partial r}(v_k - v_k^{\bar{\lambda}} + h_{\bar{\lambda}}) > 0 \) on \( \partial B_{\bar{\lambda}} \). By a standard argument in moving spheres method, we can move spheres a little further than \( \bar{\lambda} \). This contradicts the definition of \( \bar{\lambda} \). Therefore the claim is proved.

Sending \( k \) to \( \infty \) in the inequality
\[ v_k(y) > v_k^{\bar{\lambda}}(y) - h_{\bar{\lambda}}(y) \quad \text{for} \quad \bar{\lambda} < |y| < \delta M_k^{1/2}, \]
we have
\[ U(y) \geq U^{\bar{\lambda}}(y) \quad \text{for} \quad \bar{\lambda} < |y|, \]
which is a clear violation of (37) because \( \bar{\lambda} = 2 \). This contradiction concludes the proof of Theorem 2.1.

\[ \square \]

**Corollary 2.4.** Under the same assumptions in Theorem 2.1, we have
\[ \max_{r/2 \leq |x| \leq 2r} u \leq C_1 \min_{r/2 \leq |x| \leq 2r} u \]
for every \( 0 < r < 1/4 \), where \( C_1 \) is independent of \( r \). Moreover, for \( 0 < |x| < 1/4 \),
\[ |\nabla u(x)| \leq C_1 |x|^{-1} u(x), \]
\[ |\nabla^2 u(x)| \leq C_1 |x|^{-2} u(x). \]

**Proof.** The corollary follows from Theorem 2.1 by using the standard local estimates for the rescaled function \( v(y) = r^2 u(ry) \). We omit the details.

\[ \square \]

### 3 Lower bound and removability

In this section, we shall show that either 0 is a removable singularity or \( u(x)|x|^\frac{n-2}{2} \geq c \) for some \( c > 0 \) when \( n = 6 \). This is based on a delicate analysis using the Pohozaev identity.

We shall make a conformal change of the metric around the origin. Suppose that \( \{y_1, \ldots, y_n\} \) is a conformal normal coordinates system centered at 0. Using the polar coordinates, we have
\[ g = dr^2 + r^2 h(r, \theta), \]
where $h$ is a metric on $\mathbb{S}^{n-1}$ and $\det h = 1$, $r = |y|$ and $\theta = \frac{y}{|y|}$. Let 

$$f(r) = (1 - r^2)^{-\frac{n-2}{2}},$$

which is a solution of

$$\Delta f = n(n-2)f^{\frac{n+2}{n-2}}.$$  \hfill (43)

Let

$$\tilde{g} = f^{\frac{4}{n-2}}g$$

be a conformal metric of $g$, then the conformal covariance property of $L_g$ gives

$$c(n)R_{\tilde{g}} = -L_{\tilde{g}}(1) = -f^{-\frac{n+2}{n-2}}L_g f$$

$$= -f^{-\frac{n+2}{n-2}}(\Delta f + c(n)R_g f) = -n(n-2) + O(|y|^2),$$ \hfill (44)

where $|R_g| \leq Cr^2$ in the conformal normal coordinates was used. We shall use geodesic normal polar coordinates of $\tilde{g}$, in which

$$\tilde{g} = f^{\frac{4}{n-2}} dr^2 + f^{\frac{4}{n-2}} r^2 h(r, \theta) = d\rho^2 + \rho^2 \tilde{h}(\rho, \theta),$$ \hfill (45)

where $\rho = \frac{1}{2} \ln \frac{1+r}{1-r}$ and

$$\sqrt{\det h} = \sqrt{\det \tilde{g}} = f^{\frac{2n}{n-2}} = (1-r^2)^{-\frac{n}{2}} =: \zeta(\rho).$$

Then the Laplace-Beltrami operator can be written as

$$\Delta \tilde{g} = \partial_\rho^2 + \frac{1}{\rho^{n-1}\zeta} \partial_\rho (\rho^{n-1} \zeta) \partial_\rho + \frac{1}{\rho^2} \Delta \tilde{h}$$

$$= \partial_\rho^2 + \frac{n-1}{\rho} \partial_\rho + \partial_\rho \ln \zeta \partial_\rho + \frac{1}{\rho^2} \Delta \tilde{h}. \hfill (46)$$

Suppose $u$ is a positive solution of

$$-L_{\tilde{g}} u = n(n-2)u^{\frac{n+2}{n-2}} \quad \text{in } B_1 \setminus \{0\},$$ \hfill (47)

$\{x_1, \ldots, x_n\}$ is a normal coordinates system of $\tilde{g}$ centered at 0, we let

$$P(r, u) := \int_{\partial B_r} \left( \frac{n-2}{2} \frac{\partial u}{\partial r} - \frac{1}{2} r |\nabla u|^2 + r \left| \frac{\partial u}{\partial r} \right|^2 + \frac{(n-2)^2}{2} r u^{\frac{2n}{n-2}} \right) dS_r$$

be the Pohozaev integral, where $dS_r$ is the standard area measure on $\partial B_r$. The Pohozaev identity asserts that, for any $0 < s \leq r < 1$,

$$P(r, u) - P(s, u) = - \int_{s \leq |x| \leq r} \left( x^k \partial_k u + \frac{n-2}{2} u \right) (L_{\tilde{g}} u - \Delta u) dx. \hfill (48)$$

By Corollary 2.4, we have

$$\left| (x^k \partial_k u + \frac{n-2}{2} u)(L_{\tilde{g}} u - \Delta u) \right| \leq C|x|^{2-n},$$

which implies that the following limit can be defined:

$$P(u) := \lim_{r \to 0} P(r, u).$$
**Theorem 3.1.** Assume $n = 6$ and $u > 0$ is a solution of (47). Then $P(u) \leq 0$ and the equality holds if and only if 0 is an removable singularity of $u$.

When $n = 3, 4, 5$, Theorem 3.1 was proved by Marques [21] by an argument similar to that of Chen-Lin [6] for the prescribing scalar curvature.

**Proof.** If 0 is removable, it is easy to check that $P(u) = 0$. Suppose $P(u) \geq 0$. We will show that $P(u) = 0$ and 0 is removable. Thus the theorem follows.

**Claim 1.**

$$\lim_{x \to 0} u(x)|x|^{\frac{n-2}{2}} = 0. \quad (49)$$

With the establishment of the upper bound of $u$, the proof of (49) is standard (See page 359 of [21]). Roughly speaking, if $u(x)|x|^{\frac{n-2}{2}} \geq c$, then for any $r_i \to 0$, $v_i(y) = r_i^{\frac{n+2}{2}} u(r_iy)$ converges along a subsequence to $v$ of

$$\Delta v + n(n-2)v^{\frac{n+2}{2}} = 0 \quad \text{in } \mathbb{R}^n \setminus \{0\},$$

which has a non-removable singularity at the origin. By [2], $v$ is a Fowler solution and $P(v) < 0$. Then we obtain a contradiction from

$$0 > P(v) = P(1, v) = \lim_{i \to \infty} P(1, v_i) = \lim_{i \to \infty} P(r_i, u) = P(u) \geq 0.$$

**Claim 2.**

$$\lim_{x \to 0} u(x)|x|^{\frac{n-2}{2}} = 0. \quad (50)$$

In the geodesic normal polar coordinates system, using (47), (46) and Corollary 2.4 we have

$$\bar{u}_{\rho \rho} + \frac{n-1}{\rho} \bar{u}_\rho = \int_{\partial B_\rho} \left( -\partial_\rho \ln \zeta \partial_\rho u - \frac{1}{\rho^2} \Delta_{\tilde{h}} u + c(n) R_{\tilde{g}} u - u^{\frac{n+2}{2}} \right),$$

$$= -\partial_\rho \ln \zeta \bar{u}_\rho + \int_{\partial B_\rho} \left( c(n) R_{\tilde{g}} u - u^{\frac{n+2}{2}} \right) \leq -\partial_\rho \ln \zeta \bar{u}_\rho - (n(n-2) + O(\rho^2))\bar{u} - c_2 \bar{u}^{\frac{n+2}{2}}, \quad (51)$$

where $\bar{u}$ is the average of $u$ with the standard metric and $c_2 > 0$, and we have used that $\det \tilde{h}$ depends only on $\rho$ and

$$\int_{\partial B_\rho} \Delta_{\tilde{h}} u \rho^{n-1} d\text{vol}_{g_{2n-1}} = \frac{\rho^{n-1}}{\sqrt{\det \tilde{h}}} \int_{\partial B_\rho} \Delta_{\tilde{h}} u d\text{vol}_{\tilde{h}} = 0.$$

Let $t = -\ln \rho$ and $\bar{u}(\rho) = e^{\frac{n-2}{2} t} w(t)$. By a direct computation,

$$\bar{u}_\rho = -e^{\frac{n-2}{2} t} \left( \frac{n-2}{2} w + w_t \right),$$

$$\bar{u}_{\rho \rho} = e^{\frac{n+2}{2} t} \left( \frac{n(n-2)}{4} w + (n-1) w_t + w_{tt} \right).$$

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Therefore, we have
\[ \ddot{u}_{\rho \rho} + \frac{n - 1}{\rho} \dot{u}_\rho = e^{ \frac{n + 2}{\rho} t} \left( w_{tt} - \left( \frac{n - 2}{2} \right)^2 w \right), \]
and
\[ w_{tt} - \left( \frac{n - 2}{2} \right)^2 w \leq e^{-t} [\partial_{\rho} \ln \zeta \left( \frac{n - 2}{2} w + w_i \right)] - (n(n - 2) + O(e^{-2t})) e^{-2t} w - c_2 w^{\frac{n+2}{n-2}}. \]
Note that
\[ r = \frac{e^{2\rho} - 1}{e^{2\rho} + 1}, \quad \frac{dr}{d\rho} = \frac{4e^{2\rho}}{(e^{2\rho} + 1)^2}, \quad \frac{d^2 r}{d\rho^2} \bigg|_{\rho = 0} = 0, \]
and thus
\[ r = \rho + O(\rho^3). \]
It follows that
\[ \partial_{\rho} \ln \zeta = \frac{2nr}{1 - r^2 (e^{2\rho} + 1)^2} = 2n(\rho + O(\rho^3))(1 + O(\rho^2))(1 + O(\rho^2)) = 2n\rho + O(\rho^3) = 2ne^{-t} + O(e^{-3t}). \]
Hence,
\[ e^{-t} [\partial_{\rho} \ln \zeta \left( \frac{n - 2}{2} w + w_i \right)] - [n(n - 2) + O(e^{-2t})] e^{-2t} w \]
\[ = e^{-2t}(2n + O(e^{-2t})) w_{t} + O(e^{-4t}) w. \] (52)
Thus the upper bound of \( w_{tt} - \left( \frac{n - 2}{2} \right)^2 w \) can be determined as
\[ w_{tt} - \left( \frac{n - 2}{2} \right)^2 w \leq e^{-2t}(2n + O(e^{-2t})) w_{t} + O(e^{-4t}) w - c_2 w^{\frac{n+2}{n-2}}. \] (53)

By Corollary 2.4, we have \(|w_{t}(t)| \leq C w(t)\). Using (52) and first two lines of (51), we obtain a lower bound of \( w_{tt} - \left( \frac{n - 2}{2} \right)^2 w \):
\[ w_{tt} - \left( \frac{n - 2}{2} \right)^2 w \geq -c_1 w^{\frac{n+2}{n-2}} - c_3 e^{-2t} w. \] (54)
If Claim 2 were not true, by Claim 1 and Corollary 2.4, we can choose \( \varepsilon_0 > 0 \) sufficiently small so that there exist sequences \( \bar{t}_i \leq t_i \leq t_i^* \) with \( \lim_{i \to \infty} \bar{t}_i = +\infty \), such that \( w(\bar{t}_i) = w(t_i^*) = \varepsilon_0 \), \( w_{t}(t_i) = 0 \), and \( \lim_{i \to \infty} w(t_i) = 0 \). Also the smallness of \( w(t) \) implies
\[ \frac{1}{C} w \leq w_{tt} \leq C w \quad \text{for} \quad \bar{t}_i \leq t \leq t_i^*. \] (55)
Hence, for \( \bar{t}_i \leq t \leq t_i \), we have \( w_{t} \leq 0 \)
\[ w_{t}(t) \leq - \frac{1}{C} \int_{\bar{t}_i}^{t_i} w \, ds. \]
It follows that for \( \bar{t}_i \leq t \leq t_i - 1 \)
\[ w_{t}(t) \leq - \frac{1}{C} \int_{t}^{t+1} w \, ds \leq - \frac{1}{C} w(t + 1) \leq - \frac{1}{C} w(t), \]
where we used Harnack inequality in Corollary 2.4. By (53), we obtain, for large $i$
\[w_{tt} - \left(\frac{n-2}{2}\right)^2 w \leq -c_2 w^{\frac{n+2}{n-2}}\text{ for } \bar{t}_i \leq t \leq t_i - 1.\]

In conclusion,
\[-c_1 w^{\frac{n+2}{n-2}} - c_3 e^{-2t} w \leq w_{tt} - \left(\frac{n-2}{2}\right)^2 w \leq c_3 e^{-4t_i} w - c_2 w^{\frac{n+2}{n-2}}\text{ for } \bar{t}_i \leq t \leq t_i.\]

and
\[-c_1 w^{\frac{n+2}{n-2}} - c_3 e^{-2t} w \leq w_{tt} - \frac{n-2}{2} w \leq -c_2 w^{\frac{n+2}{n-2}} + c_3 e^{-2t} w \text{ for } t_i \leq t \leq t_i^*.\]

Now we use (57) and (58) to derive pointwise estimates of $w(t)$. This part is similar to the proof of (27) and (28) in [21], the main improvement is the first inequality of (60), where $e^{-4t_i}$ replaces $e^{-2t_i}$ of (28) in [21].

**Lemma 3.2.** The following two estimates hold:
\[(\frac{2}{n-2} - ce^{-2t_i}) \ln \frac{w(t)}{w(t_i)} \leq t - t_i \leq (\frac{2}{n-2} + ce^{-2t_i}) \ln \frac{w(t)}{w(t_i)} + c\]
for $t_i \leq t \leq t_i^*$, and
\[(\frac{2}{n-2} - ce^{-4t_i}) \ln \frac{w(t)}{w(t_i)} \leq t_i - t \leq (\frac{2}{n-2} + ce^{-2t_i}) \ln \frac{w(t)}{w(t_i)} + c\]
for $\bar{t}_i \leq t \leq t_i$.

**Proof of Lemma 3.2.** We only prove the first inequality in (60), since the other three were proved in [21]. By the second inequality of (57) we have
\[w_{tt} - \left(\frac{n-2}{2}\right)^2 w \leq 0, \text{ for } \bar{t}_i < t < t_i.\]

Multiplying $w'(t)$ (which is non-positive) on both sides we have
\[\frac{d}{dt} \left( w_i^2 - \left(\frac{n-2}{2}\right)^2 + c_3 e^{-4t_i} w_i^2 \right) \geq 0.\]

It follows that
\[w_i(t)^2 - \left(\frac{n-2}{2}\right)^2 + c_3 e^{-4t_i} w(t)^2 \leq -\left(\frac{n-2}{2}\right)^2 + c_3 e^{-4t_i} w(t_i)^2 \text{ for } \bar{t}_i < t < t_i.\]

Hence,
\[\frac{dt}{dw} = \frac{1}{w(t)} \leq -\left(\frac{n-2}{2}\right)^2 + c_3 e^{-4t_i} \frac{1}{\sqrt{w(t)^2 - w(t_i)^2}}.\]

Integrating the above inequality, we have
\[t_i - t = -\int_{w(t_i)}^{w(t)} \frac{dt}{dw} dw \geq \left(\frac{2}{n-2} - ce^{-4t_i}\right) \int_{w(t_i)}^{w(t)} \frac{1}{\sqrt{w^2 - w(t_i)^2}} dw \geq \left(\frac{2}{n-2} - ce^{-4t_i}\right) \ln \frac{w(t)}{w(t_i)}.\]
where we have used the estimate
\[ \int_{1}^{a} \frac{1}{\sqrt{s^2 - 1}} \, ds = \int_{0}^{\ln a} \frac{e^\xi}{\sqrt{e^{2\xi} - 1}} \, d\xi \geq \int_{0}^{\ln a} 1 \, d\xi = \ln a \quad \text{for } a > 1. \]

Therefore, Lemma 3.2 is proved.

At \(|x| = \rho_i = e^{-t_i}\), we have
\[ u(x) = \bar{u}(r_i)(1 + o(1)), \quad |\nabla u(x)| = -\bar{u}'(r_i)(1 + o(1)). \quad (61) \]

Indeed, let \(h_i(y) = \frac{u(\rho_i y)}{u(\rho_i e_1)}\), where \(e_1 = (1, 0, \ldots, 0)\). We have
\[ L_{g_i} h_i + n(n-2)\left(\frac{n-2}{\rho_i^2} u_i(\rho_i e_1)\right)^{\frac{4}{n-2}} h_i^{\frac{n+2}{n-2}} = 0 \quad \text{in } B_{1}/\rho_i \setminus \{0\}, \]
where \((g_i)_{kl} = g_{kl}(\rho_i y)\). By Corollary 2.4, \(h_i\) is locally uniformly bounded in \(\mathbb{R}^n \setminus \{0\}\). By the choice of \(\rho_i, \rho_i^\frac{n-2}{2} u_i(\rho_i e_1) \to 0\) as \(i \to \infty\). Hence, \(h_i \to h\) in \(C^2_{\text{loc}}(\mathbb{R}^n \setminus \{0\})\) for some \(h\) satisfying
\[-\Delta h = 0 \quad \text{in } \mathbb{R}^n \setminus \{0\}, \quad h \geq 0 \]
and \(h(e_1) = 1\) and \(\partial_\rho(h(y)\rho^\frac{n-2}{2}) = 0\). By the Bôcher theorem, \(h(y) = a|y|^{2-n} + b\) with \(a = b = \frac{1}{2}\).
Hence, (61) follows.
By (61), we have
\[ P(\rho_i, u) = |S^{n-1}| \left( \frac{1}{2} w'(t_i)^2 - \frac{1}{2} \left(\frac{n-2}{2}\right)^2 w^2(t_i) + \frac{(n-2)^2}{2} w^{\frac{2n}{n-2}}(t_i) \right) (1 + o(1)). \]
Hence for sufficiently large \(i\)
\[ w^2(t_i) \leq c_n |P(\rho_i, u)|. \quad (62) \]
By the choice of \(t_i\), we have
\[ P(u) = \lim_{i \to \infty} P(\rho_i, u) = 0. \quad (63) \]
It follows the Pohozaev identity (48) and (63) that
\[ |P(\rho_i, u)| \leq \int_{B_{\rho_i} \setminus B_{\rho_i^*}} |\mathcal{A}(u)| \, dx + \int_{B_{\rho_i^*}} |\mathcal{A}(u)| \, dx =: I_1 + I_2, \]
where \(\rho_i^* = e^{-t_i}\),
\[ \mathcal{A}(u) = \left( x^k \partial_k u + \frac{n-2}{2} u \right) (L_{g_i} u - \Delta u). \]
By Corollary 2.4, we have
\[ |\mathcal{A}(u)| \leq C |x|^{2-n}. \]
Hence,
\[ I_2 \leq C(\rho_i^*)^2 = C e^{-2t_i}. \]
By the first inequality in (59), we have

\[ w(t) \leq w(t_i) \exp \left( \frac{n-2}{2} + ce^{-2t_i} \right) (t - t_i), \]

which implies

\[ u(x) \leq cw(t_i) \exp \left( -\left( \frac{n-2}{2} + ce^{-2t_i} \right) t_i \right) |x|^{2 - n - ce^{-2t_i}} \text{ for } \rho_i^* \leq |x| \leq \rho_i. \]

By Corollary 2.4, we have

\[ |A(u)| \leq Cu^2. \]

Hence,

\[ I_1 \leq Cw(t_i)^2e^{-(n-2)t_i} \int_{\rho_i^* \leq |x| \leq \rho_i} |x|^{4 - 2n - 2ce^{-2t_i}} \, dx. \]

By (59) and (60), we see that

\[ t_i^* - t_i \leq \left( \frac{2}{n-2} + ce^{-2t_i} \right) \ln \frac{\varepsilon_0}{w(t_i)} + c, \]

\[ t_i - \bar{t}_i \geq \left( \frac{2}{n-2} - ce^{-4t_i} \right) \ln \frac{\varepsilon_0}{w(t_i)}. \]

Hence,

\[ \frac{t_i^* - t_i}{t_i - t_i} \leq 1 + ce^{-2t_i} + C(\ln \frac{\varepsilon_0}{w(t_i)})^{-1}. \quad (64) \]

Using the second inequality of (60), we have \((t_i - \bar{t}_i)(\ln \frac{\varepsilon_0}{w(t_i)})^{-1} \leq C\). Thus (64) implies

\[ t_i^* \leq 2t_i - \bar{t}_i + C. \quad (65) \]

Using (65) we can estimate \(I_1\) more precisely:

\[ I_1 \leq Cw(t_i)^2e^{-(n-2)t_i}((\rho_i^*)^{4-n} - \rho_i^{4-n}) \]

\[ = Cw(t_i)^2e^{-(n-2)t_i}(e^{(n-4)t_i} - e^{(n-4)\bar{t}_i}) \]

\[ \leq Cw(t_i)^2(Ce^{(n-6)t_i - (n-4)\bar{t}_i} - e^{-2t_i}) \leq Cw(t_i)^2e^{-2\bar{t}_i}, \]

where in the final step we used \(n = 6\). Combing the estimates of \(I_1\) and \(I_2\), we have, for \(n = 6\),

\[ |P(\rho_i, u)| \leq Cw(t_i)^2e^{-2\bar{t}_i} + Ce^{-2t_i^*}. \quad (66) \]

Using (62) and (66), we can combine terms to obtain

\[ w(t_i)^2 \leq Ce^{-2t_i^*} \]

for \(i\) large. From the first inequality of (60) and the first inequality of (59), we have, for \(n = 6\),

\[ t_i - \bar{t}_i \geq \left( \frac{1}{2} - ce^{-4t_i} \right) \ln \frac{\varepsilon_0}{w(t_i)}. \]
and

\[ t^*_i - t_i \geq \left( \frac{1}{2} - ce^{-2t_i} \right) \ln \frac{\varepsilon_0}{w(t_i)}. \]

Adding them up and using (67) and (65), we have

\[ t^*_i - \bar{t}_i \geq -(1 - ce^{-2t_i}) \ln w(t_i) - C \geq (1 - ce^{-2t_i})t^*_i - C \geq t^*_i - C, \]

which implies

\[ \bar{t}_i \leq C. \]

This contradicts to \( \bar{t}_i \to \infty \). Therefore, Claim 2 is proved.

Based on Claim 2 we clearly have \( w'(t) < 0 \) for \( t > T_1 \). Equation (54) now implies

\[ w_{tt} - (4 - \delta)w \geq 0 \quad \text{for} \quad t \geq T_1, \]

where \( \delta > 0 \) is some small constant. Thus for \( t \geq T_1 \), \( w_t^2 - (4 - \delta)w^2 \) is non-increasing, and the integration of this quantity leads to

\[ w(t) \leq w(T_1) \exp(- (4 - \delta)(t - T_1)), \quad t > T_1, \]

whose equivalent form is

\[ u(x) \leq C(\delta)|x|^{-\delta}. \]

Then standard elliptic estimate immediately implies that \( u \) has a removable singularity at the origin. Therefore, we complete the proof of Theorem 3.1.

**Corollary 3.3.** Assume \( n = 6 \) and \( u > 0 \) is a solution of (47). If \( 0 \) is not removable, then

\[ u(x) \geq \frac{1}{C}|x|^{-2}, \]

where \( C > 1 \) is independent of \( x \).

**Proof.** If it were false, then \( \lim \inf_{x \to 0} |x|^2 u(x) = 0 \). As the proof of (63), we have \( P(u) = 0 \) and thus \( 0 \) is removable. We obtain a contradiction. The corollary is proved.

**Proof of Theorem 1.1.** Suppose that \( 0 \) is not a removable singularity of \( u \). After conformal changes, using Theorem 2.1 and Corollary 3.3 we have

\[ \frac{1}{C}|x|^{-2} \leq u(x) \leq C|x|^{-2}. \]

Theorem 1.1 follows immediately from Theorem 8 of [21].
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