Naimark complements in the building of simplices

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Abstract. We define frames for a finite dimensional Hilbert space $\mathbb{H}$ as the complete systems in $\mathbb{H}$. An equiangular tight frame (ETF) is an equal norm tight frame with the same sharp angles between the vectors. A regular simplex is a special type of ETF in which the number of vectors is one more than the dimension of the space they span. A detailed and independent from other sources presentation of recent results by M. Fickus, J. Jasper, E. J. King and D. G. Mixon is given, in which a lower bound for the spark of the system of equal norm vectors is obtained using the restricted isometry property. The existence of the regular $s$-simplices for an arbitrary positive integer $s$ is proved using Naimark complement. A review of recent results towards resolving the known Paulsen problem is given.

1. Introduction

Let $n$ and $d$ be positive integers with $n \geq d$, and let $\mathbb{F}$ be either $\mathbb{R}$ or $\mathbb{C}$. A finite frame is a spanning set for a $d$-dimensional Hilbert space $\mathbb{H}_d$ over $\mathbb{F}$ that generalizes the notion of a basis by relaxing the need for linear independence. In other words, a family of vectors $\{ \varphi_j \}_{j=1}^n$ is a frame for a real or complex $\mathbb{H}_d$ if there are constants $0 < A \leq B < \infty$ such that for all $x \in \mathbb{H}$,

$$A\|x\|^2 \leq \sum_{j=1}^n |\langle x, \varphi_j \rangle|^2 \leq B\|x\|^2.$$

In a finite-dimensional space the concept of a frame is equivalent to the completeness of the system, i.e. span$\{ \varphi_j \}_{j=1}^n = \mathbb{H}_d$.

The synthesis operator of a finite sequence of vectors $\{ \varphi_j \}_{j=1}^n$ in $\mathbb{H}$ is $\Phi : \mathbb{F}^n \rightarrow \mathbb{H}_d$,

$$\Phi x := \sum_{j=1}^n x(j)\varphi_j,$$

where $x(j)$ denotes the $j$th entry of $x \in \mathbb{F}^n$. Its adjoint is the corresponding analysis operator $\Phi^* : \mathbb{H}_d \rightarrow \mathbb{F}^n$ which satisfies $(\Phi^* y)(j) = \langle \varphi_j, y \rangle$ for all $j = 1, \ldots, n$. Here and throughout, inner products are taken to be conjugate-linear in its first argument and linear in its second, i.e.

$$\langle x, y \rangle = \sum_j x(j)\overline{y(j)}.$$

Applying the analysis operator to the synthesis operator yields the Gram matrix $\Phi^* \Phi : \mathbb{F}^n \rightarrow \mathbb{F}^n$, an $n \times n$ matrix whose $(j, j')$th entry is $(\Phi^\ast \Phi)(j, j') = \langle \varphi_j, \varphi_{j'} \rangle$. Taking the reverse composition gives the frame operator $\Phi \Phi^* : \mathbb{H} \rightarrow \mathbb{H}$, $\Phi \Phi^* y = \sum_{j=1}^n \langle \varphi_j, y \rangle \varphi_j$. It is well known that sequences $\{ \varphi_j \}_{j=1}^n$ in $\mathbb{H}$ and $\{ \tilde{\varphi}_j \}_{j=1}^n$ in $\hat{\mathbb{H}}$ have the same Gram matrix if and only if there exists a unitary operator $U : \mathbb{H} \rightarrow \hat{\mathbb{H}}$ such that $U \varphi_j = \tilde{\varphi}_j$ for all $j = 1, \ldots, n$. 

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For the single vector \( \{ \varphi_j \} \) we define the synthesis and analysis operators for each \( j = 1, \ldots, n \) following [1]:

\[
\varphi_j : \mathbb{F} \to \mathbb{H}, \quad \varphi_j x = x \varphi_j;
\]

\[
\varphi^*_j : \mathbb{H} \to \mathbb{F}, \quad \varphi^*_j y = \langle \varphi_j, y \rangle.
\]

Using this notation we can write the frame operator as

\[
\Phi \Phi^* = \sum_{j=1}^{n} \varphi_j \varphi^*_j.
\]

As such the \( d \times n \) matrix representation of the synthesis operator \( \Phi \) has the frame elements \( \{ \varphi_j \}_{j=1}^{n} \) as columns.

In particular, if the frame bounds are equal, the frame operator has the form \( \Phi \Phi^* = aI \) with \( a > 0 \), and so signal reconstruction is rather painless: \( x = \frac{1}{a} \Phi \Phi^* x, \ x \in \mathbb{H}_d. \) In this case the frame is called tight or \( a \)-tight.

Oftentimes, it is additionally desirable for the frame elements to have equal or unit norms, in these cases the frames are equal norm or unit norm respectively. The frame \( \{ \varphi_j \}_{j=1}^{n} \) is called equal norm frame if there exists \( c > 0 \) such that \( \| \varphi_j \|^2 = c, \ j = 1, \ldots, n. \)

If the frame is \( a \)-tight and equal norm simultaneously, we obtain the following relations between \( d = \text{dim}(\mathbb{H}) \), \( c \) and \( a \):

\[
da = \text{Tr}(aI) = \text{Tr}(\Phi \Phi^*) = \text{Tr}(\Phi^* \Phi) = \sum_{j=1}^{n} \| \varphi_j \|^2 = nc.
\]

In the case \( a = 1 \) (Parseval or normalized tight frames) we have \( d = nc \), so such equal norm frames always have norms less than 1.

The first achievement in the construction of the equal norm tight frame with \( n \) vectors in \( \mathbb{R}^d \) with arbitrary \( n \geq d \) is by A. I. Maltsev [2].

Nowadays it is clear that equal norm tight frames exist for any pair \( (d, n) \) with \( n \geq d \) as in the real and also in the complex spaces. The systematic constructions of unit norm tight frames for \( \mathbb{R}^d \) are based on two interconnected methods, such as Spectral Tetris and Sparsity [3, 4].

The next step in the restriction of the reasonable family of frames is the equiangular frames which are equal norm by definition.

The equal norm frame \( \{ \varphi_j \}_{j=1}^{n} \) is called an equiangular frame if there exists \( w \geq 0 \) such that \( |\langle \varphi_j, \varphi_{j'} \rangle|^2 = w \) for all \( j \neq j' \).

Such frames are optimal for the various inequalities. For example, let’s see at the (mutual) coherence of any sequence of equal norm vectors \( \{ \varphi_j \}_{j=1}^{n} \) with \( \| \varphi_j \|^2 = c, \ j = 1, \ldots, n \) in \( d \)-dimensional Hilbert space \( \mathbb{H}_d \) over \( \mathbb{F} \):

\[
\mu := \max_{j \neq j'} \frac{|\langle \varphi_j, \varphi_{j'} \rangle|}{c}.
\]

In the real case, each vector \( \varphi_j \) spans a line and \( \mu \) is the cosine of the smallest angle between any pair of these lines.

Welch [5] gives a lower bound on \( \mu \) in \( \mathbb{H}_d \):

\[
\mu \geq \left[ \frac{n - d}{d(n - 1)} \right]^{1/2}.
\]
The equality in this estimate draws attention [1, 6]. These papers were the starting point for the present work. We tried to detail, clarify some of the results and make the presentation independent.

**Lemma 1.**

For any vectors \( \{ \varphi_j \}_{j=1}^n \) in \( \mathbb{F}^m \) let \( \Phi \) be the matrix whose \( j \)-th column is \( \varphi_j \) for all \( j \). For any \( a > 0 \) the following are equivalent:
1) \( \{ \varphi_j \}_{j=1}^n \) forms an \( a \)-tight frame for its span;
2) \( \Phi \Phi^* \Phi = a \Phi \);
3) \( (\Phi \Phi^*)^2 = a \Phi \Phi^* \);
4) \( (\Phi^* \Phi)^2 = a \Phi \Phi^* \).

Also, if \( \{ \varphi_j \}_{j=1}^n \subset \mathbb{H}_d \subset \mathbb{F}^m \), then it forms an \( a \)-tight frame for \( \mathbb{H}_d \) if and only if \( \Phi \Phi^* = a \mathbf{Pr}_{\mathbb{H}_d} \) where \( \mathbf{Pr}_{\mathbb{H}_d} \) denotes the \( m \times m \) matrix of the orthogonal projection on the subspace \( \mathbb{H}_d \).

As such, \( \{ \varphi_j \}_{j=1}^n \) forms an ETF for its span if and only if one of the assertions 2)— 4) holds and \( \{ \varphi_j \}_{j=1}^n \) is equiangular. In this case, with \( c = \| \varphi_j \| \), the dimension \( d \) is connected with \( a, n, c \) and equiangularity constant \( w \):

\[
a = \frac{cn}{d}, \quad w = c^2 \left[ \frac{n - d}{d(n-1)} \right].
\]

So, equal norm vectors \( \{ \varphi_j \}_{j=1}^n \) in a subspace \( \mathbb{H}_d \subset \mathbb{F}^m \), form an ETF for \( \mathbb{H}_d \) if and only if they achieve equality in Welch inequality.

**Proof.** Fix \( \{ \varphi_j \}_{j=1}^n \) in \( \mathbb{F}^m \) and \( a > 0 \), and let \( \mathbb{H}_d \) be any \( d \)-dimensional subspace of \( \mathbb{F}^m \) that contains \( \{ \varphi_j \}_{j=1}^n \).

1) \( \Leftrightarrow \) 2)

The synthesis operator

\[
y \in \mathbb{F}^n \to \Phi y \in \mathbb{F}^m
\]

is surjective if and only if its codomain \( \{ \Phi y : y \in \mathbb{F}^n \} = \text{span}\{ \varphi_j \}_{j=1}^n \).

If we want \( \{ \varphi_j \}_{j=1}^n \) to form an \( a \)-tight frame for \( \mathbb{H}_d \), it leads to the equality

\[
\Phi \Phi^* = a \mathbf{I}_{\mathbb{H}_d} \quad \text{or} \quad \Phi \Phi^* x = a x, \quad x \in \mathbb{H}_d.
\]

In this case

\[
\mathbb{H}_d = \text{span}\{ \varphi_j \}_{j=1}^n = \{ \Phi y : y \in \mathbb{F}^n \}
\]

and we have 2):

\[
\Phi \Phi^* \Phi y = a \Phi y, \quad y \in \mathbb{F}^n.
\]

2) \( \Leftrightarrow \) 3)

On the other hand, writing \( \mathbb{H}_d = \{ \mathbf{Pr}_{\mathbb{H}_d} x : x \in \mathbb{F}^m \} \), we see that \( \{ \varphi_j \}_{j=1}^n \) forms an \( a \)-tight frame for \( \mathbb{H}_d \) if and only if

\[
\Phi \Phi^* \mathbf{Pr}_{\mathbb{H}_d} x = a \mathbf{Pr}_{\mathbb{H}_d} x, \quad x \in \mathbb{F}^m.
\]

Let’s see at

\[
\mathbf{Pr}_{\mathbb{H}_d} \Phi y = \mathbf{Pr}_{\mathbb{H}_d} \left( \sum_{j=1}^n y(j) \varphi_j \right) = \sum_{j=1}^n y(j) \varphi_j = \Phi y, \quad y \in \mathbb{F}^n,
\]

that is \( \mathbf{Pr}_{\mathbb{H}_d} \Phi = \Phi \).

So we have \( \Phi^* \mathbf{Pr}_{\mathbb{H}_d} = (\mathbf{Pr}_{\mathbb{H}_d} \Phi)^* = \Phi^* \), and \( \Phi \Phi^* = a \mathbf{Pr}_{\mathbb{H}_d} \),

\[
\Phi \Phi^* \Phi \Phi^* = a^2 (\mathbf{Pr}_{\mathbb{H}_d})^2 = a^2 \mathbf{Pr}_{\mathbb{H}_d} = a a \mathbf{Pr}_{\mathbb{H}_d} = a \Phi \Phi^*.
\]
Multiplying 2) by $\Phi^*$ on the right or left gives 3) or 4) respectively.

If either 3) or 4) hold we’ll use the singular value decomposition $\Phi = UDV^*$, where $D$ is the diagonal matrix with 0 and $\sqrt{a}$ on the diagonal. From this we obtain $DD^*D = aD$ and then 2).

Now let’s assume that $\|\phi_j\|_2^2 = c$ for all $j$, and let’s see at the Frobenius norm of $\Phi\Phi^* - \frac{cn}{d}Pr_{H_d}$. This quantity is the measure of tightness of $\{\phi_j\}_{j=1}^n$[6]. Remind that

$$\|A\|_{Fro} = \text{Tr}(A^*A) = \text{Tr}(AA^*) = \sum_{i,j=1}^n |a_{i,j}|^2,$$

and $\text{Tr}(\Phi^*\Phi) = \text{Tr}(\Phi\Phi^*) = nc$.

So we have

$$0 \leq \text{Tr} \left( \left( \Phi\Phi^* - \frac{cn}{d}Pr_{H_d} \right)^2 \right) =$$

$$= \text{Tr} \left( \Phi\Phi^*\Phi\Phi^* - 2\frac{cn}{d}\text{Tr}(\Phi\Phi^*Pr_{H_d}) + \frac{(cn)^2}{d} \right) =$$

$$= \sum_{i,j=1}^n |\langle \phi_i, \phi_j \rangle|^2 - 2\left(\frac{cn}{d}\right)^2 + \left(\frac{cn}{d}\right)^2 \leq$$

$$= c^2 \sum_{i \neq j} \left(\frac{|\langle \phi_i, \phi_j \rangle|}{\|\phi_i\|\|\phi_j\|}\right)^2 + nc^2 - \left(\frac{cn}{d}\right)^2 \leq$$

$$\leq c^2 n(n-1) \left(\max_{i \neq j} \frac{|\langle \phi_i, \phi_j \rangle|}{\|\phi_i\|\|\phi_j\|}\right)^2 - \frac{c^2 n(n-d)}{d}.$$  \hspace{1cm} (2)

Solving for the coherence here give the Welch bound:

$$\left(\max_{i \neq j} \frac{|\langle \phi_i, \phi_j \rangle|}{\|\phi_i\|\|\phi_j\|}\right)^2 \geq \frac{n-d}{d(n-1)}.$$  

If $\{\phi_j\}_{j=1}^n$ is an ETF for $\mathbb{H}_d$ with $|\langle \phi_i, \phi_j \rangle|^2 = w$ for all $i \neq j$, the two above inequalities become equalities and

$$w = c^2 \frac{n-d}{d(n-1)}.$$  

Conversely, if Welch inequality holds with equality, the final equality in equation (2) is 0 implying both inequalities are equalities and so $\{\phi_j\}_{j=1}^n$ is tight and equiangular.

2. The RIP - property and the lower estimate for the spark

**Definition 1.** The set of vectors $\{\phi_j\}_{j=1}^n \subset \mathbb{H}$ is said to have the restricted isometry property (RIP) for a given integer $k$ and $\delta \in [0,1)$ if

$$(1-\delta)\|x\|^2 \leq \|\Phi x\|^2 \leq (1+\delta)\|x\|^2$$  \hspace{1cm} (3)

for all $k$-sparse vectors $x \in \mathbb{F}_n$ that is for all $x \in \mathbb{F}_n$ with at most $k$ nonzero entries.

**Lemma 2.** [7] The set $\{\phi_j\}_{j=1}^n \in (k, \delta)$-RIP if and only if

$$\|\Phi_K\Phi_K - I\|_2 \leq \delta.$$
for all \( k \)-element subsets \( K \) of \([n] = \{1, \ldots, n\}\), where \( \Phi_K \) denote the synthesis operator of \( \{\phi_j\}_{j \in K} \).

Proof. We start with noting that equation (3) is equivalent to

\[
\|\Phi_K x\|_2^2 - \|x\|_2^2 \leq \delta \|x\|_2^2
\]

for all \( K \subseteq [n] \), \( x \in \mathbb{F}^n \), \( x \) is \( k \)-sparse.

Then we observe that

\[
\|\Phi_K x\|_2^2 - \|x\|_2^2 = \langle \Phi_K x, \Phi_K x \rangle - \langle x, x \rangle = \langle (\Phi_K^* \Phi_K - I) x, x \rangle.
\]

For the Hermitian matrix \( (\Phi_K^* \Phi_K - I) \) we have the Rayleigh-Ritz equality [8]:

\[
\|\Phi_K^* \Phi_K - I\|_2 = \max_{x - k\text{-sparse}, x \neq 0} \frac{\langle (\Phi_K^* \Phi_K - I) x, x \rangle}{\|x\|_2^2},
\]

and equation (3) is equivalent to

\[
\max_{|K| \leq k} \|\Phi_K^* \Phi_K - I\|_2 \leq \delta.
\]

\[ \square \]

**Lemma 3.** If \( \{\phi_j\}_{j=1}^n \) is \((k, \delta)\)-RIP for some \( k \geq 2 \) and \( \delta \in (0, 1) \), then \( \text{spark} \{\phi_j\}_{j=1}^n \geq k + 1 \).

Proof. Note that since \( \delta < 1 \), from lemma 2 we obtain that each Gram matrix \( \Phi_K^* \Phi_K \) is invertible. It seems that any \( k \) columns \( \{\phi_j\}_{j \in K} \) are linearly independent, and so

\[
\text{spark} \{\phi_j\}_{j=1}^n \geq k + 1.
\]

\[ \square \]

**Lemma 4.** For equal norm \((k, \delta)\)-RIP vectors \( \{\phi_j\}_{j=1}^n \) with \( k \geq 2 \) the inequality \( \delta \geq \mu \), where \( \mu \) is the coherence of \( \{\phi_j\}_{j=1}^n \), is valid.

Proof. Let’s assume that \( \|\phi_j\| = 1 \), \( j = 1, \ldots, n \). From \((k, \delta)\)-RIP we obtain

\[
\delta \geq \max_{|K| = 2} \|\Phi_K^* \Phi_K - I\|_2 = \max_{j \neq j'} \left\| \begin{bmatrix} 0 & \langle \phi_j, \phi_{j'} \rangle \\ \langle \phi_{j'}, \phi_j \rangle & 0 \end{bmatrix} \right\|_2 = \max_{j \neq j'} |\langle \phi_j, \phi_{j'} \rangle| = \mu.
\]

\[ \square \]

**Theorem 1.** Let \( \{\phi_j\}_{j=1}^n \) is a set of unit norm vectors and \( \mu \left( \{\phi_j\}_{j=1}^n \right) \) denotes the coherence of this set. If \( k < 1/\mu + 1 \), then \( \{\phi_j\}_{j=1}^n \) is \((k, \delta)\)-RIP with \( \delta \leq (k - 1)\mu \).

Proof. According to lemma 2 we are to obtain the upper bound to

\[
\max_{|K| = k} \|\Phi_K^* \Phi_K - I\|_2,
\]
which is equal to the largest eigenvalue of the matrix $\Phi^*_K \Phi_K - I$. It's important to note that this matrix has zeros on the main diagonal.

As stated in Gershgorin theorem [8] all eigenvalues of the matrix $(\Phi_K^* \Phi_K - I)$ lies in the circle with the center at 0 and radius

$$\max_{j \in K} \sum_{j' \in K, j' \neq j} |\langle \varphi_{j'}, \varphi_j \rangle| \leq (k - 1) \mu.$$ 

So we obtain the upper bound on the RIP-bound $\delta \leq (k - 1) \mu$.

**Corollary 1.** We have the inequality

$$\text{spark}\{\varphi_j\}_{j=1}^n \geq \frac{1}{\mu} + 1$$

for any equal norm set of vectors $\{\varphi_j\}_{j=1}^n$.

**Proof.** If $k < \frac{1}{\mu} + 1$, then $\{\varphi_j\}_{j=1}^n$ is $(k, \delta)$-RIP for some $\delta \in [0, 1)$ (cf. th. 1). So $\text{spark}\{\varphi_j\}_{j=1}^n \geq k + 1$ (cf. lemma 3). Passing to supremum above $k$, we obtain 

$$\text{spark}\{\varphi_j\}_{j=1}^n \geq \frac{1}{\mu} + 1.$$ 



3. Naimark complements in the building of simplices

**Definition 2.** Let $s$ be a positive integer. The sequence $\{\varphi_j\}_{j=1}^{s+1} \subset \mathbb{H}$ is a regular $s$-simplex, if it is an ETF for the span $\langle \{\varphi_j\}_{j=1}^{s+1} \rangle$ and $\dim \text{span} \langle \{\varphi_j\}_{j=1}^{s+1} \rangle = s$.

**Lemma 5.** If $\{\varphi_j\}_{j=1}^{s+1}$ is a regular $s$-simplex, then $\mu \left( \{\varphi_j\}_{j=1}^{s+1} \right) = \frac{1}{s}$.

**Proof.** According to lemma 1 the vectors $\{\varphi_j\}_{j=1}^{s+1}$ form an ETF for the span $\langle \{\varphi_j\}_{j=1}^{s+1} \rangle$ with $\dim \text{span} \langle \{\varphi_j\}_{j=1}^{s+1} \rangle = s$ if and only if

$$\mu \left( \{\varphi_j\}_{j=1}^{s+1} \right) = \left[ \frac{s + 1 - s}{s(s + 1 - 1)} \right]^{1/2} = \frac{1}{s}.$$ 



**Corollary 2.** If $\{\varphi_j\}_{j=1}^n$ is an ETF for $\mathbb{H}_d$ and there is a regular $s$-simplex among $\{\varphi_j\}_{j=1}^n$ with $s \leq n$, then

$$\mu \left( \{\varphi_j\}_{j=1}^n \right) = \left[ \frac{n - d}{d(n - 1)} \right]^{1/2} = \frac{1}{s},$$

**Proof.** As $\{\varphi_j\}_{j=1}^n$ is an ETF for $\mathbb{H}_d$, the "angle" $|\langle \varphi_j, \varphi_{j'} \rangle|$ is the same for all pairs $j \neq j'$. So we have

$$\mu \left( \{\varphi_j\}_{j=1}^n \right) = \left[ \frac{n - d}{d(n - 1)} \right]^{1/2} = \mu \left( \{\varphi_{j_k}\}_{k=1}^{s+1} \right) = \frac{1}{s},$$
where the vectors $\varphi_{j_1}, \ldots, \varphi_{j_{d+1}}$ form a regular $s$-simplex contained in $\{\varphi_j\}_{j=1}^n$.

\[\varphi_l = \frac{1}{a} \sum_{j=1}^n \langle \varphi_l, \varphi_j \rangle \varphi_j, \ l = 1, \ldots, n.\]

For $k \neq l$ it follows that

\[\langle \varphi_k, \varphi_l \rangle = \left( \varphi_k, \frac{1}{a} \sum_{j=1}^n \langle \varphi_l, \varphi_j \rangle \varphi_j \right) = \frac{1}{a} \sum_{j=1}^n \langle \varphi_j, \varphi_k \rangle \langle \varphi_k, \varphi_j \rangle, \text{ or } \Phi^* \Phi = \frac{1}{a} (\Phi^* \Phi)^2.\]

For $P := \frac{1}{a} \Phi^* \Phi$ we deduce that $P = P^2$.

Thus, given that $P$ is a Hermitian matrix we get that $P$ is an orthogonal projection matrix on a $d$-dimensional subspace. As any orthogonal projection matrix it has $d$ eigenvalues 1 and $n - d$ eigenvalues 0, trace$(P) = \text{rank}(P) = d$. From this we obtain that $\Phi^* \Phi = aP$ has $d$ eigenvalues $a$ and $n - d$ eigenvalues 0,

\[\text{trace}(\Phi^* \Phi) = ad = \sum_{j=1}^n \|\varphi_j\|^2, \ \text{rank}(\Phi^* \Phi) = d.\]

\[\text{Lemma 6.} \ \text{Let } \{\varphi_j\}_{j=1}^n \text{ be an } a \text{-tight frame for its span } \left(\{\varphi_j\}_{j=1}^n\right) \text{ and dim span } \left(\{\varphi_j\}_{j=1}^n\right) = d < n. \text{ Then the } n \times n \text{ matrix } \frac{1}{a} \Phi^* \Phi \text{ is an orthogonal projection matrix on a } d \text{-dimensional subspace of } \mathbb{F}^n. \text{ Gram matrix } \Phi^* \Phi \text{ has eigenvalue } a \text{ with multiplicity } d \text{ and eigenvalue 0 with multiplicity } n - d.\]

\[\text{Proof.} \ \text{It’s directly verified that the matrix } Q := I - \frac{1}{a} \Phi^* \Phi \text{ satisfies } Q = Q^* = Q^2 \text{ and its eigenvalues are 1 and 0 with multiplicities } n - d \text{ and } d \text{ respectively. It means that the columns of } Q, \psi_j := Q e_j, \ j = 1, \ldots, n, \text{ where } \{e_j\}_{j=1}^n \text{ is the orthonormal basis in } \mathbb{F}^n, \text{ defines the span } \left(\{\psi_j\}_{j=1}^n\right) \text{ and dim span } \left(\{\psi_j\}_{j=1}^n\right) = n - d.\]

If $f \in \text{span} \left(\{\psi_j\}_{j=1}^n\right)$, then $Q f = f$,

\[f = Q \left( \sum_{j=1}^n (Q f, e_j) e_j \right) = \sum_{j=1}^n (f, Q e_j) Q e_j = \sum_{j=1}^n (f, \tilde{\psi}_j) \tilde{\psi}_j,\]

\[\text{Theorem 2.} \ \text{Any } a \text{-tight frame for } d \text{-dimensional span } \left(\{\varphi_j\}_{j=1}^n\right) \text{ has Naimark complement } \{\psi_j\}_{j=1}^n \text{ such that:}\]

1. $\{\psi_j\}_{j=1}^n$ is an $a$-tight frame for the $(n - d)$-dimensional span $\left(\{\psi_j\}_{j=1}^n\right)$.
2. Gram matrix $\Psi^* \Psi = a I - \Phi^* \Phi$.
3. $\Phi \Psi^* = 0$.

\[\text{Proof.} \ \text{It’s directly verified that the matrix } Q := I - \frac{1}{a} \Phi^* \Phi \text{ satisfies } Q = Q^* = Q^2 \text{ and its eigenvalues are 1 and 0 with multiplicities } n - d \text{ and } d \text{ respectively. It means that the columns of } Q, \psi_j := Q e_j, \ j = 1, \ldots, n, \text{ where } \{e_j\}_{j=1}^n \text{ is the orthonormal basis in } \mathbb{F}^n, \text{ defines the span } \left(\{\psi_j\}_{j=1}^n\right) \text{ and dim span } \left(\{\psi_j\}_{j=1}^n\right) = n - d.\]

If $f \in \text{span} \left(\{\psi_j\}_{j=1}^n\right)$, then $Q f = f$,
i. e. the set \( \{ \tilde{\psi}_j \}_{j=1}^n \) forms Parseval (or normalized tight) frame for the span \( \left( \{ \tilde{\psi}_j \}_{j=1}^n \right) \).

The vectors \( \tilde{\psi}_j = \sqrt{a} \tilde{\psi}_j \), \( j = 1, \ldots, n \) form an \( \alpha \)-tight frame for the span \( \left( \{ \psi_j \}_{j=1}^n \right) = \text{span} \left( \{ \tilde{\psi}_j \}_{j=1}^n \right) \) and \( \Psi^* \Phi = aI - \Phi^* \Phi \).

(3) The lemma 1 gives that
\[
(\Psi \Phi^*)^* (\Psi \Phi^*) = \Phi \Psi^* \Phi \Phi^* = \Phi (aI - \Phi^* \Phi) \Phi^* = a \Phi \Phi^* - (\Phi \Phi^*)^2 = 0.
\]

The Frobenius norm of the matrix
\[
\| \Psi \Phi^* \|_F^2 = \text{trace} ((\Psi \Phi^*)^* (\Psi \Phi^*)) = \text{trace} 0 = 0.
\]
Consequently, \( \Psi \Phi^* = 0 \), and its conjugate \( \Phi \Psi^* = 0 \).

\[ \square \]

**Corollary 3.** If \( \{ \varphi_j \}_{j=1}^n \) is an ETF for the span \( \left( \{ \varphi_j \}_{j=1}^n \right) \), then each one of its Naimark complements \( \{ \psi_j \}_{j=1}^n \) is an ETF for the span \( \left( \{ \psi_j \}_{j=1}^n \right) \).

Proof. In the notation of the theorem 2 we have that (cf. lemma 1)
\[
(\Psi \Phi^*)^2 = (aI - \Phi^* \Phi)^2 = a^2 I - 2a \Phi^* \Phi + \Phi^* \Phi \Phi^* \Phi =
\]
\[
= a^2 I - 2a \Phi^* \Phi + a \Phi^* \Phi = a \Phi (aI - \Phi^* \Phi) = a \Phi \Psi^* \Phi,
\]
i. e. (again lemma 1) \( \{ \psi_j \}_{j=1}^n \) is an \( \alpha \)-tight frame for the span \( \left( \{ \psi_j \}_{j=1}^n \right) \).

If \( \{ \varphi_j \}_{j=1}^n \) is an ETF with \( \| \varphi_j \|^2 = c, j = 1, \ldots, n \), then \( \| \psi_j \|^2 = a - \| \varphi_j \|^2 = a - c, j = 1, \ldots, n \). Besides we have that \( \langle \psi_j, \varphi_{j'} \rangle = - \langle \varphi_j, \varphi_{j'} \rangle \) for all \( j \neq j' \).

\[ \square \]

**Corollary 4.** A regular \( s \)-simplex exists for any positive integer \( s \).

Proof. Let \( s \) be a positive integer. Any sequence \( \{ c_j \}_{j=1}^{s+1} \) of unimodular scalars \( (|c_j|^2 := u, j = 1, \ldots, s + 1) \) is a \( (s + 1) \alpha \)-tight equiangular frame for \( \mathbb{F}^s \). Really if \( \alpha \in \mathbb{R} \), then
\[
\alpha = \frac{1}{u(s+1)} \sum_{j=1}^{s+1} c_j \alpha c_j, \quad (s+1)u\alpha^2 = \sum_{j=1}^{s+1} (c_j \alpha)^2;
\]
if \( \alpha \in \mathbb{C} \), then
\[
\alpha = \frac{1}{u(s+1)} \sum_{j=1}^{s+1} \overline{c_j} \alpha c_j, \quad u(s+1)|\alpha|^2 = \sum_{j=1}^{s+1} |\overline{c_j} \alpha|^2.
\]
Besides, we have that \( \Phi \Phi^* = u(s+1), |c_j c_{j'}| = u \) for all \( j \neq j' \), and
\[
\Phi^* \Phi = \begin{pmatrix}
\frac{u}{c_1} & \frac{c_1}{c_2} & \ldots & \frac{c_1}{c_{s+1}} \\
\frac{c_2}{c_1} & u & \ldots & \frac{c_2}{c_{s+1}} \\
\ldots & \ldots & \ldots & \ldots \\
\frac{c_{s+1}}{c_1} & \frac{c_{s+1}}{c_2} & \ldots & u
\end{pmatrix}.
\]

8
According to the theorem 2 the Gram matrix for the Naimark complement to the set \( \{ c_j \}_{j=1}^{s+1} \) is the matrix
\[
\Psi^* \Psi = (s + 1)uI - \Phi^* \Phi = \begin{pmatrix}
    su & -c_1^* c_2 & \ldots & -c_1^* c_{s+1} \\
    -c_2^* c_1 & su & \ldots & -c_2^* c_{s+1} \\
    \ldots & \ldots & \ldots & \ldots \\
    -c_{s+1}^* c_1 & -c_{s+1}^* c_2 & \ldots & su \\
\end{pmatrix}.
\]

Any Gram matrix defines the set of unitary equivalent families of vectors. Looking at the matrix \( \Psi^* \Psi \) it is clearly seen that any such family of vectors forms \((s+1)u\)-tight equiangular \(\sqrt{su}\)-norm frame for \(s\)-dimensional span \(\{ \psi_j \}_{j=1}^{s+1} \).

Conversely, let \( \{ \varphi_j \}_{j=1}^{s+1} \) be a regular \(s\)-simplex with \(\|\varphi_j\|^2 = c, j = 1, \ldots, s + 1\). Then \(a = \frac{c(s+1)}{s}\) and \(\langle \varphi_j , \varphi_j' \rangle = \frac{c}{s}\). So, the Gram matrix for its Naimark complement is
\[
\Psi^* \Psi = \frac{c(s+1)}{s}I - \Phi^* \Phi = \begin{pmatrix}
    \frac{c}{s} & -\langle \varphi_1, \varphi_2 \rangle & \ldots & -\langle \varphi_1, \varphi_{s+1} \rangle \\
    -\langle \varphi_2, \varphi_1 \rangle & \frac{c}{s} & \ldots & -\langle \varphi_2, \varphi_{s+1} \rangle \\
    \ldots & \ldots & \ldots & \ldots \\
    -\langle \varphi_{s+1}, \varphi_1 \rangle & -\langle \varphi_{s+1}, \varphi_2 \rangle & \ldots & \frac{c}{s} \\
\end{pmatrix},
\]

moreover, all entries of the matrix \( \Psi^* \Psi \) are unimodular. Thus Naimark complement to the \( \{ \varphi_j \}_{j=1}^{s+1} \) is unitary equivalent to the set of unimodular scalars.

\( \square \)

4. The Paulsen problem

Remind, that a frame \( \{ \varphi_j \}_{j=1}^{n} \) is called \(a\)-tight if
\[
\sum_{j=1}^{n} |\langle \varphi_j, x \rangle|^2 = a \|x\|^2, \quad x \in \mathbb{H}_d \quad \text{or} \quad \Phi \Phi^* = aI.
\]

In the case \(a = 1\) it’s called Parseval frame or normalized tight frame.

The frame \( \{ \varphi_j \}_{j=1}^{n} \) is called equiangular tight frame (ETF) if it is an equiangular and tight simultaneously.

If \( \{ \varphi_j \}_{j=1}^{n} \) is Parseval frame with analysis operator \( \Phi^* \), then \( \Phi^* \) is an isometry, since
\[
\| \Phi^* x \|_2^2 = \sum_{j=1}^{n} |\langle \varphi_j, x \rangle|^2 = \|x\|^2, \quad x \in \mathbb{H}_d.
\]

Conversely, if an \( n \times d \) matrix is an isometry, then it is the analysis operator of the Parseval frame.

Parseval frames (and only such frames) satisfy the reconstruction identity
\[
x = \sum_{j=1}^{n} \langle x, \varphi_j \rangle \varphi_j,
\]
or \( x = \Phi \Phi^* x, \Phi \Phi^* = I_{\mathbb{H}_d} \).
The following theorem was the first in the Frame theory, and maybe one of the most important.

**Theorem 3.** [9] If \( \Phi = \{ \phi_j \}_{j=1}^n \) is Parseval frame for \( \mathbb{H}_d \), then there exists an \( n \)-dimensional Hilbert space \( \mathbb{H}_n \), and an orthonormal basis \( \{ b_j \}_{j=1}^n \subset \mathbb{H}_n \) such that \( \mathbb{H}_d \) is a linear subspace of \( \mathbb{H}_n \) and \( \phi_j = \text{Pr}_{\mathbb{H}_d} b_j \) for all \( j \), where \( \text{Pr}_{\mathbb{H}_d} \) denotes the orthogonal projection of \( \mathbb{H}_n \) onto \( \mathbb{H}_d \).

The converse statement is also true.

**Theorem 4.** If \( \{ b_j \}_{j=1}^n \) is an orthonormal basis for \( \mathbb{H}_n \), and \( \mathbb{H}_d \subset \mathbb{H}_n \) is any \( d \)-dimensional linear subspace, then \( \Phi = \{ \text{Pr}_{\mathbb{H}_d} b_j \}_{j=1}^n \) is Parseval frame for \( \mathbb{H}_d \), where \( \text{Pr}_{\mathbb{H}_d} \) denotes the orthogonal projection of \( \mathbb{H}_n \) onto \( \mathbb{H}_d \).

Proof. Denote by \( \phi_j = \text{Pr}_{\mathbb{H}_d} b_j \), \( j = 1, \ldots, n \). We have for \( x \in \mathbb{H}_d \)

\[
\|x\|^2 = \|\text{Pr}_{\mathbb{H}_d} x\|^2 = \sum_{j=1}^n |\langle \text{Pr}_{\mathbb{H}_d} x, b_j \rangle|^2 = \sum_{j=1}^n |\langle x, \text{Pr}_{\mathbb{H}_d} b_j \rangle|^2 = \sum_{j=1}^n |\langle x, \phi_j \rangle|^2,
\]

i.e. \( \Phi \) is Parseval frame.

Theorems 3 and 4 are generalized in [10] to frames of a general form, which are projections of Riesz bases. Projections of orthogonal systems (generally speaking, incomplete) are considered in detail in [11].

Parseval frame \( \{ \text{Pr}_{\mathbb{H}_d} b_j \}_{j=1}^n \) from the theorems 3, 4 is not equal-norm frame. In fact, the first equal-norm Parseval frame was built by A.I.Maltsev in [2] (of course without such terms).

Almost all known equal-norm Parseval frame designs are based on a discrete Fourier transform matrix. The elements of this matrix are formed by complex numbers, however, the correct choice of columns and rows of the matrix and simple arithmetic operations with them lead to matrices of synthesis operators for real frames with given properties.

**Theorem 5.** Equal norm Parseval Frame \( \Phi = \{ \phi_j \}_{j=1}^n \) exists in \( \mathbb{R}_d \) for any \( n \geq d \).

The explicit construction of such a frame is given in the article [12].

Any frame has a Parseval frame as a natural satellite. Indeed, if \( \{ e_j \}_{j=1}^n \) is an orthonormal basis for \( \mathbb{H}_n \), then

\[
Sx := \Phi \Phi^* x = \sum_{j=1}^n \langle x, \phi_j \rangle \phi_j.
\]

Hence,

\[
\langle Sx, x \rangle = \sum_{j=1}^n |\langle x, \phi_j \rangle|^2.
\]

Moreover, \( \{ \phi_j \}_{j=1}^n \) is a frame with bounds \( A, B > 0 \) if and only if \( AI_d \leq S \leq BI_d \).

Also we have that

\[
x = SS^{-1} x = \sum_{j=1}^n \langle S^{-1} x, \phi_j \rangle \phi_j = \sum_{j=1}^n \langle x, S^{-1} \phi_j \rangle \phi_j = \sum_{j=1}^n \langle x, S^{-1/2} \phi_j \rangle S^{-1/2} \phi_j,
\]

i.e \( \{ S^{-1/2} \phi_j \}_{j=1}^n \) is a Parseval frame.
Now we define the distance between frames. The $\ell^2$-distance between two frames $\Phi = \{ \varphi_j \}_{j=1}^n$ and $\Phi' = \{ \varphi'_j \}_{j=1}^n$ in $\mathbb{H}_d$ is defined by

$$\text{dist}(\Phi, \Phi') := \left( \sum_{j=1}^n \| \varphi_j - \varphi'_j \|^2 \right)^{1/2}.$$ 

**Theorem 6.** If $\Phi = \{ \varphi_j \}_{j=1}^n$ is a frame for $\mathbb{H}_d$ with the frame operator $S$, then $\{ S^{-1/2} \varphi_j \}_{j=1}^n$ minimizes the $\ell^2$-distance between $\Phi$ and all possible choices of Parseval frames.

Theorem 6 was proved in [13], cf. also [12].

The frame $\{ S^{-1/2} \varphi_j \}_{j=1}^n$ is not an equal-norm frame.

If $\Phi = \{ \varphi_j \}_{j=1}^n$ is an equal norm Parseval frame for $\mathbb{R}_d$, then $S = I_d$ and $\| \varphi_j \|^2 = \frac{d}{n}$, $j = 1, \ldots, n$.

We say that $\Phi$ is an $\epsilon$-nearly equal norm Parseval frame if

$$(1 - \epsilon)I_d \leq S \leq (1 + \epsilon)I_d,$$

and

$$(1 - \epsilon) \frac{d}{n} \leq \| \varphi_j \|^2 \leq (1 + \epsilon) \frac{d}{n}, \quad j = 1, \ldots, n.$$

Let $\text{ENPF}$ be the set of all equal norm Parseval frames. The **Paulsen problem** is one of the most known and attractive problems in frame theory.

**Paulsen problem.** For every $\epsilon$-nearly equal norm Parseval frame $\Phi$, is

$$\inf_{\Psi \in \text{ENPF}} \text{dist}^2(\Phi, \Psi)$$

bounded by a fixed polynomial in $\epsilon$ and $d$?

Early results gave bounds on the squared distance that were polynomial in $\epsilon$, $d$ and $n$ [16, 4]. The first bound that was polynomial in $\epsilon$ and $d$ was obtained in [14]. They proved that squared distance is at most $O(\epsilon d^{13/2})$.

A much simpler way to a better bound was found in [15].

**Theorem 7.** For any $\epsilon$-nearly equal norm Parseval frame $\Phi$ there is $\Psi \in \text{ENPF}$ such that

$$\text{dist}^2(\Phi, \Psi) \leq 20\epsilon d^2.$$ 

Cahill and Casazza [16] gave a family of examples of $\epsilon$-nearly equal norm Parseval frames where the

$$\text{dist}^2(\Phi, \Psi) \geq c \epsilon d.$$

It is an interesting open question to close the gap.

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