Some double-angle formulas related to a generalized lemniscate function

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Dedicated to Professor Tetsutaro Shibata on the occasion of his 60th birthday

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Abstract
In this paper, we will establish some double-angle formulas related to the inverse function of \( \int_0^x \frac{dt}{\sqrt{1 - t^6}} \). This function appears in Ramanujan’s Notebooks and is regarded as a generalized version of the lemniscate function.

Keywords Generalized trigonometric functions · Double-angle formulas · Lemniscate function · Jacobian elliptic functions · \( p \)-Laplacian

Mathematics Subject Classification 33E05 · 34L40

1 Introduction

Let \( 1 < p, q < \infty \) and

\[
F_{p,q}(x) := \int_0^x \frac{dt}{(1 - t^q)^{1/p}}, \quad x \in [0, 1].
\]

We will denote by \( \sin_{p,q} \) the inverse function of \( F_{p,q} \), i.e.,

\[
\sin_{p,q} x := F_{p,q}^{-1}(x).
\]
Clearly, \( \sin_{p,q} x \) is an increasing function mapping \([0, \pi_{p,q}/2]\) to \([0, 1]\), where

\[
\pi_{p,q} := 2F_{p,q}(1) = 2 \int_0^1 \frac{dr}{(1-r^q)^{1/p}}.
\]

We extend \( \sin_{p,q} x \) to \((\pi_{p,q}/2, \pi_{p,q})\) by \( \sin_{p,q} (\pi_{p,q} - x) \) and to the whole real line \( \mathbb{R} \) as the odd \( 2\pi_{p,q} \)-periodic continuation of the function. Since \( \sin_{p,q} x \in C^1(\mathbb{R}) \), we also define \( \cos_{p,q} x \) by \( \cos_{p,q} x := (d/dx)(\sin_{p,q} x) \). Then, it follows that

\[
|\cos_{p,q} x|^p + |\sin_{p,q} x|^q = 1.
\]

In case \((p, q) = (2, 2)\), it is obvious that \( \sin_{p,q} x, \cos_{p,q} x \) and \( \pi_{p,q} \) are reduced to the ordinary \( \sin x, \cos x \) and \( \pi \), respectively. This is a reason why these functions and the constant are called generalized trigonometric functions (with parameter \((p, q)\)) and the generalized \( \pi \), respectively.

The generalized trigonometric functions are well studied in the context of nonlinear differential equations (see [4,6,7] and the references given there). Suppose that \( u \) is a solution of the initial value problem of the \( p \)-Laplacian

\[
-(|u'|^{p-2}u')' = \frac{(p-1)q}{p}|u'|^{q-2}u, \quad u(0) = 0, \quad u'(0) = 1,
\]

which is reduced to the equation \(-u'' = u\) of simple harmonic motion for \( u = \sin x \) in case \((p, q) = (2, 2)\). Then,

\[
\frac{d}{dx}(|u'|^p + |u|^q) = \left( \frac{p}{p-1}(|u'|^{p-2}u')' + q|u|^{q-2}u \right) u' = 0.
\]

Therefore, \(|u'|^p + |u|^q = 1\). It is possible to show that \( u \) coincides with \( \sin_{p,q} \) defined as above. The generalized trigonometric functions are often applied to the eigenvalue problem of the \( p \)-Laplacian.

Now, we are interested in finding double-angle formulas for generalized trigonometric functions. It is possible to discuss addition formulas for these functions: for instance \( \sin_{2,6} \) has the addition formula (3) with (2) below (see also [5] for \( \sin_{4/3,4} \)), but for simplicity we will not develop this point here.

We have known the double-angle formulas of \( \sin_{2,q}, \sin_{q^*,q}, \) and \( \sin_{q^*,2} \) for \( q = 2, 3, 4 \) except for \( \sin_{3/2,2} \), where \( q^* := q/(q-1) \) (Table 1). For details for each formula, we refer the reader to [8] (after having proved the formula for \( \sin_{2,3} \) in the co-authored paper [8], the author noticed that the formula has already been obtained as \( \varphi(2s) \) by Cox and Shurman [3, p. 697]). It is worth pointing out that Lemma 3.1 (resp. Lemma 3.2) below connects the parameter \((2, q)\) to \((q^*, q)\) (resp. \((q^*, 2)\)) and yields the possibility to obtain the other formula from one formula. Indeed, in this way, the formulas of \( \sin_{4/3,4} \) and \( \sin_{4,3/2} \) follow from that of \( \sin_{2,3} \) ([10, Subsect. 3.1] and [8, Theorem 1.1], respectively), and the formula of \( \sin_{2,3} \) follows from that of \( \sin_{3/2,3} \) ([8, Theorem 1.2]). Nevertheless, the parameter \((3/2, 2)\) is still open because of the difficulty of the inverse problem corresponding to (10).
Table 1 The parameters for which the double-angle formulas have been obtained

| q | \((q^*, 2)\) | \((2, q)\) | \((q^*, q)\) |
|---|---|---|---|
| 2 | \((2, 2)\) by Abu al-Wafa’ | \((2, 2)\) by Abu al-Wafa’ | \((2, 2)\) by Abu al-Wafa’ |
| 3 | \((3/2, 2)\) open | \((2, 3)\) by Cox–Shurman | \((3/2, 3)\) by Dixon |
| 4 | \((4/3, 2)\) by Sato–Takeuchi | \((2, 4)\) by Fagnano | \((4/3, 4)\) by Edmunds et al. |
| 6 | \((6/5, 2)\) Theorem 1.2 | \((2, 6)\) by Shinohara | \((6/5, 6)\) Theorem 1.1 |

In this paper, we wish to investigate the double-angle formula of the function \(\sin_{2,6}x\), whose inverse function is defined as

\[
\sin^{-1}_{2,6}x = \int_{0}^{x} \frac{dr}{\sqrt{1 - t^6}}.
\]

The function \(\sin_{2,6}x\) appears as the inverse of \(”H(v)”\) in Ramanujan’s Notebooks [1, p. 246] and is regarded as a generalized version of the lemniscate function \(\sin_{2,4}x\). For the function, Shinohara [9] gives the novel double-angle formula

\[
\sin_{2,6}(2x) = \frac{2 \sin_{2,6}x \cos_{2,6}x}{\sqrt{1 + 8 \sin^6_{2,6}x}}, \quad x \in [0, \pi_{2,6}/2]. \quad (1)
\]

In fact, he found (1) in “trial and error calculations” (according to private communication), but instead we will give a proof of (1) in Sect. 2. Moreover, as mentioned above, we can show the following counterparts of (1) for \(\sin_{6/5,6}\) and \(\sin_{6/5,2}\), respectively.

**Theorem 1.1** Let \((p, q) = (6/5, 6)\). Then, for \(x \in [0, \pi_{6/5,6}/4]\),

\[
\sin_{6/5,6}(2x) = 2^{1/6} \sin_{6/5,6}x \cos^{1/5}_{6/5,6}x \left(3 + \sqrt{1 + 32 \sin^6_{6/5,6}x \cos^{6/5}_{6/5,6}x}\right)^{1/2} \left(1 + 32 \sin^6_{6/5,6}x \cos^{6/5}_{6/5,6}x\right)^{1/6}.
\]

**Theorem 1.2** Let \((p, q) = (6/5, 2)\). Then, for \(x \in [0, \pi_{6/5,2}/2]\),

\[
\sin_{6/5,2}(2x) = \sqrt{1 - \left(9 - 8 \sin^2_{6/5,2}x - 4 \sin^2_{6/5,2}x \cos^{2/5}_{6/5,2}x\right) \left(9 - 8 \sin^2_{6/5,2}x + 8 \sin^2_{6/5,2}x \cos^{2/5}_{6/5,2}x\right)^3}.
\]
\section{Proof of (1)}

The change of variable $s = t^2$ leads to the representation

$$\sin_{2,6}^{-1} x = \frac{1}{2} \int_0^{x^2} \frac{ds}{\sqrt{s(1-s^3)}}, \quad 0 \leq x \leq 1.$$ 

The furthermore change of variable ([2, 576.00 in p. 256])

$$\cn u = \frac{1 - (\sqrt{3} + 1)s}{1 + (\sqrt{3} - 1)s}, \quad k^2 = \frac{2 - \sqrt{3}}{4}$$

gives

$$\sin_{2,6}^{-1} x = \frac{1}{2} \int_0^{\cn^{-1} \phi(x)} \frac{(2\sqrt{3} \sn u \dn u)}{((\sqrt{3} + 1) + (\sqrt{3} - 1) \cn u)^2} \, du$$

$$= \frac{1}{2} \cdot 3^{1/4} \int_0^{\cn^{-1} \phi(x)} \, du$$

$$= \frac{1}{2} \cdot 3^{1/4} \cn^{-1} \phi(x),$$

where $\sn u = \sn(u, k)$, $\cn u = \cn(u, k)$, and $\dn u = \dn(u, k)$ are the Jacobian elliptic functions (see e.g., [11, Chap. XXII] for more details), and

$$\phi(x) = \frac{1 - (\sqrt{3} + 1)x^2}{1 + (\sqrt{3} - 1)x^2}. \quad (2)$$

Thus,

$$\sin_{2,6} u = \phi^{-1}(\cn(2 \cdot 3^{1/4} u)), \quad 0 \leq u \leq \pi_{2,6}/2 = K/(3^{1/4}),$$

where $K = K(k)$ is the complete elliptic integral of the first kind and

$$\phi^{-1}(x) = \frac{1 - x}{\sqrt{(\sqrt{3} + 1) + (\sqrt{3} - 1)x}}.$$

Now, we use the addition formula of \cn. For $u, \quad v, \quad u \pm v \in [0, K/(3^{1/4})]$,

$$\sin_{2,6}(u \pm v) = \phi^{-1}(\cn(\tilde{u} \pm \tilde{v})) = \phi^{-1} \left( \frac{\cn \tilde{u} \cn \tilde{v} \mp \sn \tilde{u} \sn \tilde{v} \dn \tilde{u} \dn \tilde{v}}{1 - k^2 \sn^2 \tilde{u} \sn^2 \tilde{v}} \right)$$
where $\tilde{u} := 2 \cdot 3^{1/4} u$ and $\tilde{v} := 2 \cdot 3^{1/4} v$. Recall that $\text{sn}^2 x + \text{cn}^2 x = 1$ and $k^2 \text{sn}^2 x + \text{dn}^2 x = 1$; then the last equality gives

$$\sin^2 \left( \frac{2}{3} u \pm v \right) = \phi^{-1} \left( \frac{\phi(U)\phi(V) \mp \sqrt{(1 - \phi(U)^2)(1 - \phi(V)^2)}(1 - k^2(1 - \phi(U)^2))}{1 - k^2(1 - \phi(U)^2)(1 - \phi(V)^2)} \right), \quad (3)$$

where $U := \sin_{2,6} u$ and $V := \sin_{2,6} v$.

With $u = v$ and the observation that

$$1 - \phi(U)^2 = \frac{4\sqrt{3}U^2(1 - U^2)}{(1 + (\sqrt{3} - 1)U^2)^2};$$

$$1 - k^2(1 - \phi(U)^2) = \frac{1 + U^2 + U^4}{(1 + (\sqrt{3} - 1)U^2)^2},$$

this implies that

$$\sin_{2,6} (2u) = \phi^{-1} \left( \frac{\phi(U)^2 - (1 - \phi(U)^2)(1 - k^2(1 - \phi(U)^2))}{1 - k^2(1 - \phi(U)^2)^2} \right)$$

$$= \phi^{-1} \left( \frac{1 - 4(\sqrt{3} + 1)U^2 + 8U^6 + 4(\sqrt{3} + 1)U^8}{1 + 4(\sqrt{3} - 1)U^2 + 8U^6 - 4(\sqrt{3} - 1)U^8} \right),$$

Routine simplification now results in the formula

$$\sin_{2,6} (2u) = \frac{2U \sqrt{1 - U^6}}{\sqrt{1 + 8U^6}} = \frac{2 \sin_{2,6} u \cos_{2,6} u}{\sqrt{1 + 8 \sin_{2,6}^2 u}},$$

and the proof is complete.

### 3 Proofs of theorems

To prove Theorem 1.1, we use the following multiple-angle formulas.

**Lemma 3.1** ([10]) Let $1 < q < \infty$ and $q^* := q/(q - 1)$. If $x \in [0, \pi_{2,q}((2^2/q)) = [0, \pi_{q^*,q}/2]$, then

$$\sin_{2,q} \left( \frac{2^2/q \cdot x}{2} \right) = 2^{2/q} \sin_{q^*,q} x \cos_{q^*,q}^{-1} x, \quad (4)$$

$$\cos_{2,q} \left( \frac{2^2/q \cdot x}{2} \right) = \cos_{q^*,q} x - \sin_{q^*,q} x$$

$$= 1 - 2 \sin_{q^*,q} x = 2 \cos_{q^*,q}^2 x - 1. \quad (5)$$
Proof of Theorem 1.1 Let $x \in [0, \pi_{6,5,6}/4]$. Applying (4) of Lemma 3.1 in case $q = 6$ with $x$ replaced by $2x \in [0, \pi_{6,5,6}/2]$, we get

$$\sin_{2,6}(2 \cdot 2^{1/3}x) = 2^{1/3} \sin_{6/5,6}(2x)(1 - \sin_{6/5,6}^6(2x))^{1/6}. \quad (6)$$

First, we consider the case

$$0 \leq x < \frac{\pi_{6,5,6}}{8}.\]$$

Then, since $0 \leq 2 \sin_{6/5,6}^6(2x) < 1$ by [10, Lemma 2.1], Eq. (6) gives

$$2 \sin_{6/5,6}^6(2x) = 1 - \sqrt{1 - \sin_{2,6}^6(2 \cdot 2^{1/3}x)}.$$ 

Set $S = S(x) := \sin_{2,6}(2^{1/3}x)$. Using the double-angle formula (1) for $\sin_{2,6} x$, we have

$$2 \sin_{6/5,6}^6(2x) = 1 - \sqrt{1 - \left(\frac{2S\sqrt{1 - S^6}}{\sqrt{1 + 8S^6}}\right)^6}$$

$$= 1 - \frac{\sqrt{1 - 40S^6 + 384S^{12} + 320S^{18} + 64S^{24}}}{(1 + 8S^6)^{3/2}}$$

$$= 1 - \frac{|1 - 20S^6 - 8S^{12}|}{(1 + 8S^6)^{3/2}}. \quad (7)$$

Since $0 \leq S^6 < \sin_{2,6}^6(\pi_{2,6}/4) = (3\sqrt{3} - 5)/4$, evaluated by (1), we see that $1 - 20S^6 - 8S^{12} > 0$. Thus,

$$2 \sin_{6/5,6}^6(2x) = 1 - \frac{1 - 20S^6 - 8S^{12}}{(1 + 8S^6)^{3/2}}$$

$$= \frac{(\sqrt{1 + 8S^6} - 1)(\sqrt{1 + 8S^6} + 3)^3}{8(1 + 8S^6)^{3/2}}$$

$$= \frac{S^6(3 + \sqrt{1 + 8S^6})^3}{(1 + 8S^6)^{3/2}(1 + \sqrt{1 + 8S^6})}.$$ (7)

Therefore, by (4),

$$\sin_{6/5,6}(2x) = \frac{2^{1/6} \sin_{6/5,6} x \cos_{6/5,6}^{1/5} x \left(3 + \sqrt{1 + 32 \sin_{6/5,6}^6 x \cos_{6/5,6}^{6/5} x}\right)^{1/2}}{\left(1 + 32 \sin_{6/5,6}^6 x \cos_{6/5,6}^{6/5} x\right)^{1/4} \left(1 + \sqrt{1 + 32 \sin_{6/5,6}^6 x \cos_{6/5,6}^{6/5} x}\right)^{1/6}}.$$
In the remaining case

\[ \frac{\pi/6}{8} \leq x \leq \frac{\pi/6}{4}, \]

it follows easily that \( 1 \leq 2 \sin^6_{6/5,6} (2x) < 2 \) and \( 1 - 2056 - 8512 \leq 0 \), hence we obtain (7) again. The proof is complete. \( \square \)

To show Theorem 1.2, the following lemma is useful.

**Lemma 3.2** ([5,6]) Let \( 1 < p, q < \infty \). For \( x \in [0, 2] \),

\[ q \pi_{p,q} = p^* \pi_{q^*,p^*}, \]

\[ \sin_{p,q} \left( \frac{p \cdot q \cdot x}{2} \right) = \cos_{q^*,p^*} \left( \frac{q \cdot p \cdot x}{2} (1 - x) \right). \]

**Proof of Theorem 1.2** Let \( x \in [0, \pi/6, 2/2] \). Then, since \( 4x/\pi/6, 2 \in [0, 2] \), it follows from Lemma 3.2 that

\[ \sin_{6/5,2} (2x) = \cos_{2,6} \left( \frac{\pi \cdot 2}{6} \left( 1 - \frac{4x}{\pi/6, 2} \right) \right) = \cos_{2,6} \left( \frac{\pi \cdot 2}{6} - \frac{2x}{3} \right). \]

Thus,

\[ \sin_{6/5,2} 2x = \sqrt{1 - \sin_{2,6}^2 \left( \frac{\pi \cdot 2}{6} - \frac{2x}{3} \right)}. \quad (8) \]

The function \( \sin_{2,6} \) has the addition formula (3). Letting \( u = \pi \cdot 2/6 \) and \( v = 2x/3 \), we have

\[ \sin_{2,6} \left( \frac{\pi \cdot 2}{6} - \frac{2x}{3} \right) = \phi^{-1}(\phi(V)) = \sqrt{\frac{1 - V^2}{1 + 2V^2}}, \quad (9) \]

where \( V := \sin_{2,6} (2x/3) \). Applying (9) to the right-hand side of (8), we obtain

\[ \sin_{6/5,2} 2x = \sqrt{1 - \left( \frac{1 - \sin_{2,6}^2 (2x/3) \cdot \sin_{2,6}^2 (2x/3)}{1 + 2 \sin_{2,6}^2 (2x/3)} \right)^3}. \]

Let \( f(x) := \sin_{6/5,2} x \) and \( g(x) := \sin_{2,6} (2x/3) \). Then

\[ f(2x) = \sqrt{1 - \left( \frac{1 - g(x)^2}{1 + 2g(x)^2} \right)^3}. \quad (10) \]
Therefore, it is easy to see that
\[ g(x) = \sqrt[1/3]{\frac{1 - (1 - f(2x)^2)^{1/3}}{1 + 2(1 - f(2x)^2)^{1/3}}}. \tag{11} \]

On the other hand, by (1) with \( x \) replaced with \( x/2 \), we see that \( g(x) \) satisfies
\[ g(x) = \frac{2g(x/2)\sqrt{1 - g(x/2)^6}}{\sqrt{1 + 8g(x/2)^6}}. \]

Applying (11) with \( x \) replaced with \( x/2 \) to the right-hand side, we obtain
\[ g(x) = \frac{2f(x)(1 - f(x)^2)^{1/6}}{\sqrt{9 - 8f(x)^2}}. \tag{12} \]

Substituting (12) into (10), we can express \( f(2x) \) in terms of \( f(x) \), i.e.,
\[ f(2x) = \sqrt[3]{1 - \left(\frac{9 - 8f(x)^2}{9 - 8f(x)^2 + 8f(x)^2(1 - f(x)^2)^{1/3}}\right)^3}. \]

Since \( 1 - f(x)^2 = \cos^6 \frac{x}{2} \), the proof is complete. \( \square \)

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