RECURRENCE SPECTRUM IN SMOOTH DYNAMICAL SYSTEMS.

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Abstract. We prove that for conformal expanding maps the return time does have constant multifractal spectrum. This is the counterpart of the result by Feng and Wu in the symbolic setting.

1. Introduction

Let $f : M \to M$ be a map on a smooth compact manifold $M$ with metric $d$. Let $\Lambda \subset M$ be a compact $f$-invariant set. For $x \in M$ we can define the return times in $r$-neighborhoods by

$$\tau_r(x) = \inf \{ n > 0 : d(f^n x, x) < r \}.$$  

We put $\tau_r(x) = \infty$ if the orbit of $x$ does not come back close to $x$.

When $M$ is an interval, $f$ is a piecewise monotonic transformation with some regularity (piecewise $C^{1+\alpha}$ suffices) and $\mu$ is an $f$-invariant ergodic measure with nonzero entropy, Saussol, Troubetzkoy and Vaienti [11], building on results by Ornstein and Weiss [10] and Hofbauer and Raith [5, 6], proved that

$$\lim_{r \to 0} \frac{\log \tau_r(x)}{-\log r} = \dim_H \mu$$

for $\mu$-almost every $x$.

From now on we assume that $\Lambda \subset M$ is a conformal repeller of a $C^{1+\alpha}$ map $f : M \to M$. That is to say that $\Lambda$ is a compact forward invariant set and that $f$ is $C^{1+\alpha}$ on $M$, expanding and conformal on $\Lambda$. Moreover we suppose that there exists an open set $V \subset M$ such that $\Lambda = \bigcap_{n \geq 0} f^{-n} V$.

From the multifractal analysis of conformal expanding maps it is well known that for any real $s \in [0, \dim_H \Lambda]$ there exists an ergodic equilibrium state $\mu_s$ (in fact there are many such measures for non extremal values of $s$). Therefore Equation (1) together with basic properties of dimension of measures imply that

$$\dim_H \left\{ x \in \Lambda : \lim_{r \to 0} \frac{\log \tau_r(x)}{-\log r} \right\} \geq \dim_H \mu_s = s.$$
The reader familiar with multifractal analysis will believe that in fact the
dimension of this set is larger than this, and at this point may expect a nice
non-trivial (strictly) concave analytic curve. In fact, this is really not the
case, as our first theorem says

**Theorem 1.** Let \( \Lambda \subset \mathcal{M} \) be a conformal repeller of the \( C^{1+\alpha} \) map \( f : \mathcal{M} \to \mathcal{M} \). For any \( \alpha \leq \beta \in [0, \infty) \) we have

\[
\dim_H \left\{ x \in \Lambda : \lim_{r \to 0} \frac{\log \tau_{r}(x)}{-\log r} = \alpha \text{ and } \lim_{r \to 0} \frac{\log \tau_{r}(x)}{-\log r} = \beta \right\} = \dim_H \Lambda.
\]

Theorem 1 implies in particular that for all values of \( s \in [0, \infty) \) the di-
mension in the left hand side of Equation (2) equals \( \dim_H \Lambda \).

The corresponding result of Theorem 1 in the symbolic case was estab-
lished by Feng and Wu \[3\]. Although our method relies on some essential
ideas present in the above mentioned paper, we emphasize that our proof is
not a direct translation of the symbolic result. \(^1\)

In Section 2 we prove the theorem in the situation where the system \((f, \Lambda)\)
is conjugated to a subshift of finite type. It happens that our problem can
be reduced to this case and this is proven in Section 3.

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2. System conjugated to a subshift of finite type

In this section we give a proof of Theorem 1 in a special case.

**Theorem 2.** Let \( \Lambda \subset \mathcal{M} \) be such that \((f, \Lambda)\) is conjugated to a subshift of
finite type. Then the result of Theorem 1 holds.

The proof of this result is decomposed in several parts. First we give a
construction similar to the one by Feng and Wu of points with prescribed
recurrence in a symbolic space \[3\].

\(^1\)After the completion of this work we learn that L. Olsen has obtained — independ-
dently and with different techniques — a similar result for conformal Iterated Function
Systems satisfying some strong separation condition, which is equivalent to a conjugacy
to a subshift of finite type, the case considered in Theorem 2.
2.1. Producing points with given symbolic recurrence sequence. 
For a given sequence of integer \((\ell_n)\) such that
(a) \(\exists n_0, \ell_{n_0 + 1} \geq \ell_n + 2n\) for any \(n \geq n_0\) and
(b) \(\lim_{n \to \infty} \ell_n / n^2 = \infty\),
we define a function \(g\) nearly as in \([3]\) such that points in the image of \(g\) have \((\ell_n)\) as recurrence sequence. Let \(A \subset B\) be two alphabets (i.e. finite or countable sets) with \(A \not= B\) and \(#B \geq 3\). Choose a marker \(m \in B \setminus A\).

**Definition 3.** Let \(c \not= \bar{c} \in B \setminus \{m\}\). Define \(g : A^N \to B^N\) by \(g(w) = \lim w^{(k)}\), where the sequence \(w^{(k)}\) is constructed recursively as follows.

We put \(w^{(0)} = w^{(1)} = \cdots = w^{(n_0 - 1)} = mw_1w_2\cdots\). For \(k \geq n_0\) we construct \(w^{(k)}\) from \(w^{(k-1)}\) by inserting the block \(w_1^{(k-1)} \cdots w_k^{(k-1)} y_k\) at the position \(\ell_k + 1\), i.e.

\[
w^{(k)} = w_1^{(k-1)} w_2^{(k-1)} \cdots w_k^{(k-1)} w_1^{(k-1)} \cdots w_k^{(k-1)} y_k w_{\ell_k+1}^{(k-1)} w_{\ell_k+2}^{(k-1)} \cdots,
\]

where \(y_k = c\) if \(w_{k+1}^{(k-1)} \not= c\) or \(y_k = \bar{c}\) if \(w_{k+1}^{(k-1)} = c\), so that \(y_k \not= m, w_{k+1}^{(k-1)}\).

Note that all the \(w^{(k)}\) share the same \(\ell_{n+1}\) first letters whenever \(k \geq n\), hence the limit \(g(w)\) is well defined.

Let us define the \(k\)-repetition time of \(w\) by

\[R_k(w) = \inf\{n > 0 : w_{n+1}w_{n+2} \cdots w_{n+k} = w_1w_2 \cdots w_k\}\]

The main interest of the function \(g\) resides in the following

**Lemma 4 (3).** If \(w \in A^N\) and \(k \geq n_0\) then \(R_k(g(w)) = \ell_k\).

**Proof of Lemma 4.** We do it by induction. Let \(w^* = g(w)\). Since there is no \(m\) in \(w\) it is obvious that \(R_{n_0}(w^*) = \ell_{n_0}\). Assume that for some \(k \geq n_0\) we have proven that \(R_k(w^*) = \ell_k\). By construction this implies that \(\ell_k < R_{k+1}(w^*) \leq \ell_k + k\) or \(R_{k+1}(w^*) = \ell_k + 1\) (note that there can be no \(m\) between the positions \(\ell_k + k + 1\) and \(\ell_k + 1\) in \(w^*)\). Suppose that we are in the first case and let \(p = R_{k+1}(w^*)\). The number of \(m\) in the block \(w_{p+1}^* \cdots w_{p+k}^* = w_1^* \cdots w_k^*\) being equal to the number of \(m\) inside the block \(w_{\ell_k+1}^* \cdots w_{\ell_k+k}^*\) we get a contradiction if \(p > \ell_k\) (because \(w_{\ell_k+1}^* = m\) is missed and there is no \(m\) in \(w_{\ell_k+k+1}^* \cdots w_{p+k}^*\)).

Although one can imagine similar constructions for subshifts of finite type, we decided to work with fullshifts to make the core argument more transparent. However, we then have to work a bit to apply it to our systems, which are presumably only coded by subshifts of finite type. This is the aim of next section.

2.2. Building the source of large dimension. Suppose that \(\dim_H \Lambda > 0\) otherwise there is nothing to prove in Theorem 2. Let \((\Sigma, T)\) be a subshift of finite type conjugated to \((f, \Lambda)\) and denote by \(\pi : \Sigma \to \Lambda\) the conjugating homeomorphism. For our purpose there is no loss of generality to suppose that \((\Sigma, T)\) is topologically mixing. Thus at least one letter, say \(a\), has two successors \(b\) and \(c\). Let then \(B\) and \(C\) be the shortest paths (and smallest in the lexicographic order) starting in \(b\) and \(c\), respectively, and ending in \(a\). That is \(aB\) and \(aC\) are the smallest words of \(\Sigma\) with prefix \(ab\) and \(ac\), respectively, and suffix \(a\). It is clear that there is a one-to-one correspondence
between the set of words and the set of cylinders. Without confusion, we also use a word to denote its corresponding cylinder. Let \( A = aB \) and denote by \( t(\omega) \) the return time of \( \omega \in A \) into \( A \),

\[
t(\omega) = \inf\{t > 0 : T^t \omega \in A\}.
\]

Note that \( t \) is unbounded on \( A \) since there is a disjoint union contained in \( A \). More precisely,

\[
A \supset aBB \cup aBCB \cup aCCB \cup \cdots
\]

To simplify the exposition we will consider the new alphabet consisting in \( |A| - c \)-cylinders, where \( |A| \) denotes the length of the cylinder \( A \), together with its associated transition matrix. Let then \( Z_n \) be the partition of \( \Sigma \) by \( n \)-cylinders (the new ones), and denote by \( F_n \) the finite \( \sigma \)-algebra generated by \( Z_n \).

Let \( \mu \) be the equilibrium state of the potential \( \varphi = -\dim_H(\Lambda) \log |Df| \circ \pi \). It is well known that \( \mu \) is the measure of maximal dimension and the Bowen pressure formula gives \( P(\varphi|\Sigma) = 0 \). In addition the measure \( \mu \) is \( \psi \)-mixing (See e.g. [7] for properties of equilibrium states), which means that

\[
\psi(n) \overset{\text{def}}{=} \sup_{m} \sup_{U \in F_m, V \in B} \left| \frac{\mu(U \cap T^{-m}V)}{\mu(U)\mu(V)} - 1 \right| \xrightarrow{n \to \infty} 0.
\]

Hence there exists \( m_0 \) such that

\[
\mu(A \cap T^{-m_0}(\Sigma \setminus A) \cap T^{-2m_0}(\Sigma \setminus A) \cap \cdots \cap T^{-mm_0}(\Sigma \setminus A)) \leq \delta^m,
\]

where \( \delta \overset{\text{def}}{=} \mu(\Sigma \setminus A)(1 + \psi(m_0)) < 1 \). Set \( H_n = \{ \omega \in A : t(\omega) \geq n \} \). The preceding estimate yields

\[
\mu(H_n) \leq \delta^{n/m_0}.
\]

Moreover we have \( H_n = A \setminus \{ x \in A : t(x) \leq n - 1 \} \in F_n \). From Proposition [7] (see Section [8] below), setting \( \Sigma_n = (\Sigma \setminus H_n)^\infty \), we get that

\[ (3) \quad \lim_{n \to \infty} P(\varphi|\Sigma_n) = P(\varphi|\Sigma) = 0. \]

Let \( \nu_n \) be an ergodic equilibrium state of \( \varphi \) on \( \Sigma_n \). If \( n \) is sufficiently large then \( \nu_n(A) > 0 \) (otherwise \( \nu_n \) would be supported on the forward invariant set \( (\Sigma \setminus A)^\infty \), on which the pressure of \( \varphi \) is strictly less than \( P(\varphi|\Sigma) \)).

Let \( \hat{\Sigma} \subset A \) denotes the set of points returning infinitely many times in \( A \). Let us define the induced system \( (\hat{\Sigma}, \hat{T}, \hat{\nu}_n) \) by

\[
\hat{T}(x) = T^{t(x)}(x), \quad \hat{\nu}_n = \frac{1}{\nu_n(A)}\nu_n|_A.
\]

Since \( \hat{T} \) is the induced transformation on a cylinder of a subshift of finite type, \( (\hat{\Sigma}, \hat{T}) \) is a full shift with the countable alphabet

\[
\hat{Z} = \bigcup_{m=1}^{\infty} \{ Z \cap \hat{\Sigma} : Z \in \mathcal{Z}_{m+1}, Z \subset A, t|_Z = m \}.
\]

Since the return times in \( A \) are bounded (by \( n - 1 \)) on \( \Sigma \setminus H_n \), one has

\[
\hat{\Sigma}_n \overset{\text{def}}{=} A \cap \Sigma_n \subset \hat{\Sigma}.
\]
Observe that \((\hat{\Sigma}, \hat{T})\) is still a fullshift over the finite alphabet
\[
\hat{Z}^n = \bigcup_{m=1}^{n-1} \{Z \cap \hat{\Sigma}_n : Z \in \mathcal{Z}_{m+1}, Z \subset A, t|_Z = m\}.
\]

Set \(\psi = \log |Df| \circ \pi, \ S_k \psi(\omega) = \sum_{j=0}^{k-1} \psi(T^j \omega)\) and \(\hat{S}_k t(\omega) = \sum_{j=0}^{n-1} t(T^j \omega)\). Let us define the source of large dimension by
\[
G_n = \Big\{ \omega \in \hat{\Sigma}_n : (i) \lim_{k \to \infty} \frac{1}{k} S_k \psi(\omega) \to \nu(\psi), \ (ii) \lim_{k \to \infty} \frac{1}{k} \hat{S}_k t(\omega) \to \hat{\nu}(t) \Big\}.
\]
By Birkhoff’s Ergodic Theorem we have \(\hat{\nu}(G_n) = 1\) which provides the following lower bound
\[
\dim_H \pi \nu_n \geq \dim_H \pi \hat{\nu}_n = \dim_H \pi \nu_n|_A = \dim_H \pi \nu_n.
\]
The last equality follows by ergodicity of the equilibrium state \(\nu_n\). Moreover,
\[
\dim_H \pi \nu_n = \frac{h_{\nu_n}}{\int \psi d\nu_n} = \dim_H \Lambda + \frac{P(\varphi|\Sigma_n)}{\int \psi d\nu_n}.
\]
Observe that this together with (3) and (4) imply that
\[
\lim_{n \to \infty} \dim_H \pi G_n = \dim_H \Lambda.
\]

2.3. The image of the source has good symbolic recurrence. With the alphabets \(A = \mathcal{Z}^n\) and \(B = \hat{Z}\) we can use Definition 3 to produce a function \(g: \hat{\Sigma}_n \to \hat{\Sigma}\). For the marker \(m \in B \setminus A\) we choose a \(m \in \hat{Z}\) such that \(t|_m = n\).

Given an integer \(k\), let \(p\) be such that \(\ell_p \leq k < \ell_{p+1}\) and set \(\epsilon_k = (p + 2)^2 / \ell_p\). The property (b) of the sequence \(\ell_p\) implies that \(\epsilon_k \to 0\) as \(k \to \infty\).

Observe that \(t\) is \(\hat{Z}\) measurable, hence for any \(\omega \in G_n\) we have
\[
|\hat{S}_k t(g\omega) - \hat{S}_k t(\omega)| \leq k \epsilon_k n,
\]
since the symbolic sequence of \(g(\omega)\) and \(\omega\) differs only because of the block of length \(i\) inserted at position \(\ell_i\), which makes a difference of at most \(\sum_{i=n_0+1}^{p+1} i \leq k \epsilon_k\) letters. Hence for any \(\omega \in G_n\) we have
\[
\lim_{k \to \infty} \frac{1}{k} \hat{S}_k t(g\omega) = \hat{\nu}(t).
\]

Notice that a \(k\)-cylinder of \((\hat{\Sigma}, \hat{T})\) containing \(\omega\) is indeed \(\hat{S}_k t(\omega)\) cylinder of \((\Sigma, T)\). Hence Lemma 4 gives that for \(\omega \in G_n\) we have (the notation \(\hat{R}_k\) stands for the \(k\)-repetition time computed with \(\hat{T}\))
\[
\hat{R}_k t(g\omega) = \hat{R}_k (g\omega) = \ell_k.
\]
Passing to the limit using Equation (7) and the fact that the sequence \((R_h)_h\) is monotone gives us that the image \(gG_n\) has good symbolic recurrence. Namely, for any \(\omega \in G_n\),
\[
\lim_{h \to \infty} \left( \lim_{h \to \infty} \frac{\log R_h (g\omega)}{h} \right) = \lim_{k \to \infty} \left( \frac{1}{\hat{\nu}(t)} \right) \frac{\log \ell_k}{k}.
\]
2.4. The image has large dimension. The fact that the dimension of the image \( \pi g G_n \) is large will follow from (5) if we are able to prove that the inverse function of \( \gamma \) has some smoothness.

Let \( \omega \in G_n \). Denotes the \( h \)-cylinder of \( (\Sigma, T, Z) \) containing \( g(\omega) \) by \( Z(\omega, h) \). If \( k \) is such that
\[
\hat{S}_k t(g\omega) \leq h < \hat{S}_{k+1} t(g\omega)
\]
then
\[
Z(g\omega, h) \subset \hat{Z}(g\omega, k),
\]
where \( \hat{Z}(z, k) \) denotes the \( k \)-cylinder of \( (\hat{\Sigma}, \hat{T}, \hat{Z}) \) containing a point \( z \). By construction of \( g \) we have
\[
g^{-1}\hat{Z}(g\omega, k) \subset \hat{Z}(\omega, k(1 - \epsilon_k)).
\]
Because of (6) we have
\[
\hat{Z}(\omega, k(1 - \epsilon_k)) \subset Z(\omega, h(1 - \delta_h))
\]
for some sequence \( \delta_h \to 0 \) as \( h \to \infty \) (independent of \( \omega \)). Putting together this chain of inclusions gives us that
\[
g^{-1}Z(\omega, h) \subset Z(\omega, h(1 - \delta_h)).
\]
This estimate on cylinders may be exploited for balls using the conjugacy assumption.

**Proposition 5.** Let \( (f, \Lambda) \) be a \( C^{1+\alpha} \) conformal expanding map conjugated to a subshift. Then there exists a constant \( \kappa \in (0, 1) \) such that for any integer \( n \) and any \( x \in \Lambda \)
\[
B(x, \kappa|D_x f^n|^{-1}) \cap \Lambda \subset \pi Z(\pi^{-1}x, n) \subset B(x, \kappa^{-1}|D_x f^n|^{-1}) \cap \Lambda.
\]

**Proof.** We give only a sketch. Assume that diam \( \Lambda = 1 \) and \( |D_x f| > 1 \) for any \( x \in \Lambda \). Since \( f \) is conjugated to a subshift the minimal distance between two image \( \pi Z \) and \( \pi Z' \) of different elements \( Z, Z' \in Z \) is bounded from below by some constant \( \delta > 0 \). Moreover, by distortion there exists a constant \( D \) such that for any \( n \) and \( x, y \) in the same \( n \)-cylinder we have \( |D_x f^n| \leq D|D_y f^n| \). Let \( \kappa = \delta/D \).

Suppose that \( d(x, y) < \kappa|D_x f^n|^{-1} \). This implies that \( d(x, y) < \delta \), thus \( Z(\pi^{-1}x, 1) = Z(\pi^{-1}y, 1) \). If \( Z(\pi^{-1}x, k - 1) = Z(\pi^{-1}y, k - 1) \) for some \( k < n \), then we get \( d(f^k x, f^k y) \leq D|D_x f^k|d(x, y) < \delta \). Hence \( Z(\pi^{-1}x, k) = Z(\pi^{-1}y, k) \). By induction we get the first inclusion in Equation (10). The remaining inclusion is easier and we omit its proof. \( \square \)

**Lemma 6.** The function \( \gamma^{-1} \) is \( \alpha \)-Hölder continuous on \( g(G_n) \) for any \( \alpha \in (0, 1) \).

**Proof.** By arguments similar to those in the proof of Equation (5), but using the continuity of \( \psi \), we get that
\[
\lim_{n \to \infty} \sup_{\omega \in G_n} \left| \frac{1}{n} S_n \psi(\omega) - \frac{1}{n} S_n \psi(g\omega) \right| = 0.
\]
Let \( x, y \in \pi G_n \). Let \( n \) be such that
\[
\kappa \exp(-S_{n+1} \psi(g\pi^{-1} x)) \leq d(\gamma x, \gamma y) < \kappa \exp(-S_n \psi(g\pi^{-1} x)).
\]
It follows from Proposition 3 that \( g \pi^{-1} y \in \mathcal{E}(g \pi^{-1} x, n) \). Thus \( \pi^{-1} y \in \mathcal{E}(\pi^{-1} x, n(1 - \delta_n)) \) by (9). Using Proposition 5 again we find that
\[
d(x, y) < \kappa^{-1} \exp(-S_{n(1 - \delta_n)} \psi(\pi^{-1} x)).
\]

For any \( \alpha \in (0, 1) \), because of (11), the facts that \( \delta_n \to 0 \), \( \psi \) is strictly positive and bounded, we can find an \( n(\alpha) \) such that for all \( n \geq n(\alpha) \) and all \( x \) we have
\[
\exp(-S_{n(1 - \delta_n)} \psi(\pi^{-1} x)) \leq \kappa^2 \exp(-\alpha S_{n+1} \psi(g \pi^{-1} x))
\]
This shows that whenever \( d(\gamma x, \gamma y) < \kappa \inf \exp S_{n(\alpha)} \psi \) we have
\[
d(x, y) < d(\gamma x, \gamma y)^\alpha,
\]
proving the lemma. \( \square \)

Since \( \pi G_n = \gamma(\pi g G_n) \) Lemma 3 ensures that the image of the source has large dimension:
\[
\text{dim}_H \pi g(G_n) \geq \text{dim}_H \pi G_n.
\]

2.5. The image has good recurrence rates. By Proposition 5 we have
\[
\tau_\kappa \exp(S_k \psi(\pi^{-1} x))(x) \geq R_k(\pi^{-1} x) \geq \tau_{\kappa^{-1}} \exp(S_{k+1} \psi(\pi^{-1} x))(x),
\]
for any \( x \in \Lambda \). Let \( \lambda_n = \int \psi d\nu_n \). By Equation (11), \( g \) does not change the Birkhoff average of \( \psi \) thus for any \( x \in \pi g(G_n) \), by the property (ii) in the definition of \( G_n \) we have
\[
\lim_{k \to \infty} \frac{1}{k} S_k \psi(\pi^{-1} x) = \lambda.
\]

Then we choose the sequence \( (\ell_k) \) as in (3) such that
\[
\lim_{k \to \infty} \frac{\log \ell_k}{k} = \alpha \lambda_n \hat{\nu}_n(t) \quad \text{and} \quad \lim_{k \to \infty} \frac{\log \ell_k}{k} = \beta \lambda_n \hat{\nu}_n(t).
\]

Using the bounds in (13) and (14) we get that for any \( x \in \pi g(G_n) \)
\[
\lim_{r \to 0} \frac{\log \tau_r(\pi^{-1} x)}{-\log r} = \lim_{k \to \infty} \frac{\log R_k(\pi^{-1} x)}{\lambda_n k} = \lim_{k \to \infty} \frac{1}{\lambda_n \hat{\nu}_n(t)} \frac{\log \ell_k}{k} = \alpha,
\]
where the second equality follows from (3). The same arguments for the \( \lim \) gives that \( x \in E(\alpha, \beta) \), where
\[
E(\alpha, \beta) = \left\{ x \in \Lambda : \lim_{r \to 0} \frac{\log \tau_r(\pi^{-1} x)}{-\log r} = \alpha \quad \text{and} \quad \lim_{r \to 0} \frac{\log \tau_r(\pi^{-1} x)}{-\log r} = \beta \right\}.
\]

Since \( \pi g(G_n) \subset E(\alpha, \beta) \), Theorem 2 follows from (5) and (12).

3. Reduction to a system conjugated to a subshift

We want to ensure that the original system contains subsystems with arbitrary large dimension and which are conjugated to subshifts of finite type. By Bowen’s formula this is the same as finding such subsystems with pressure of the potential \( -\dim_H(\Lambda) \log |Df| \) arbitrarily close to 0. Fernandez, Ugalde and Urias prove that for maps of the interval with a Markov partition by intervals this is always possible [4]. Their proof relies on combinatorial arguments. We provide here a proof based on the spectral approach, close in
Let \( Z \) \( \sigma \) i.e. \( (16) \) var

Lemma 8. We postpone the proofs of Lemmas 8 and 9 below to Section 4.

have 1

3.1. Pressure for systems with small hole. Let \((X, T)\) be a subshift of finite type. Given an open set \( H \subset X \), the hole, we set \( Y = X \setminus H \) and denote by \( Y^\infty = \cap_{m \geq 0} T^{-m}Y \) the compact forward invariant set of points never falling into the hole \( H \). Our next result says that the pressure of \( Y^\infty \) tends to the pressure of the original system as the hole gets smaller. Notice that this is not a small perturbation for the potential, so that classical continuity of the pressure do not apply.

**Proposition 7.** Let \((X, T)\) be a topologically mixing subshift of finite type. Assume that \( \varphi : X \to \mathbb{R} \) is a Hölder continuous potential. Let \( \mu \) be the equilibrium measure of the potential \( \varphi \). If \( H_n \) is a decreasing sequence of \( \mathcal{F}_n \) measurable sets such that for some \( c, \varepsilon > 0 \)

\[
\mu(H_n) \leq cc^{-\varepsilon n}
\]

then

\[
\lim_{n \to \infty} P(\varphi|(X \setminus H_n)^\infty) = P(\varphi|X).
\]

**Proof.** The proof uses the characterization of the pressure by the logarithm of the spectral radius of the Perron-Frobenius operator associated to the system. Then we show that one can apply a perturbation result by Keller and Liverani \( 8 \) designed for these type of operators. Taking if necessary the new potential \( \varphi - P(\varphi|X) + \psi \circ T - \psi \) we can assume that \( P(\varphi|X) = 0 \) and \( L1 = 1 \), where \( L \) is the Perron-Frobenius operator associated to \((X, T, \varphi)\), i.e.

\[
L \psi(x) = \sum_{y : Ty = x} e^{\varphi(y)} f(y).
\]

Let \( Z_n \) be the partition of \( X \) by \( n \)-cylinders, and denote by \( \mathcal{F}_n \) the finite \( \sigma \)-algebra generated by \( Z_n \). By the Gibbs property of \( \mu \) there exists \( c_0 \) such that for any \( x \in Z \in Z_k \) we have \( 1/c_0 \leq \mu(Z)e^{-\varphi_k(x)} < c_0 \). We furthermore suppose that for any cylinders \( A, Z \) such that \( AZ \overset{\text{def}}{=} A \cap T^{-k}Z \neq \emptyset \) we have \( 1/c_0 \leq \mu(A)\mu(Z)/\mu(AZ) \leq c_0 \).

Given a function \( f : X \to \mathbb{R} \) and a set \( A \subset X \) let \( \text{osc}(f, A) = \sup_A f - \inf_A f \). Since \( \varphi \) is Hölder there exists \( c_1 \) and \( \alpha < 1 \) such that we have

\[
(16) \quad \text{var}_n(\varphi) \overset{\text{def}}{=} \sup_{Z \in Z_n} \text{osc}(\varphi, Z) \leq c_1 \alpha^n, \quad \text{for any } n \geq 1.
\]

Fix \( \theta \in (1, \min(\alpha^{-1}, e^{\varepsilon/2})) \). Define

\[
\text{osc}(f) = \sum_{n \geq 1} \theta^n \sum_{Z \in Z_n} \text{osc}(f, Z)\mu(Z).
\]

Let \( |f|_2 = (\int |f|^2 d\mu)^{1/2} \) and consider the Banach space

\[
B = \{ f : X \to \mathbb{R} : \| f \| \overset{\text{def}}{=} \text{osc}(f) + |f|_2 < \infty \}.
\]

We postpone the proofs of Lemmas \( 8 \) and \( 9 \) below to Section 4.

**Lemma 8.** There exists constants \( c_3 \) and \( c_4 \) such that for any \( k \geq 1 \)

\[
\text{osc}(T^k f) \leq c_3 \theta^{-k} \text{osc}(f) + c_4 |f|_2.
\]
Let \( H_n^k = H_n \cup T^{-1}H_n \cup \cdots T^{-k+1}H_n \). For simplicity we denote by \( A \) instead of \( 1_A \) the indicator function of a set \( A \).

**Lemma 9.** There exists a constant \( c_5 \) (depending on \( k \)) such that for any \( N \in \mathbb{N} \) and \( f \in B \),

\[
\text{osc}(f H^k_F) \leq \text{osc}(f) + c_5 |f|_2.
\]

For a set \( Y \subset X \) we let \( L_Y \) be the perturbed operator, \( L_Y f = L(Y f) \).

Observe that \( L^k_X f = L^k_X(Y_k f) \), where \( Y_k = Y \cap T^{-1}Y \cap \cdots T^{-k+1}Y \). Hence \( L^k_{X} f = L^k f - L^k (H^k_N f) \).

**Lemma 10.** There exists \( k \) and \( c_6 \) such that for any \( N \in \mathbb{N} \) and \( f \in B \) we have

\[
\text{osc}(L^k f) \leq \frac{1}{2} \text{osc}(f) + c_6 |f|_2.
\]

**Proof of Lemma 10.** Choose \( k \) such that \( c_3 \theta^{-k} \leq \frac{1}{2} \). We apply Lemma 8 and Lemma 9 successively

\[
\text{osc}(L^k f) \leq \text{osc}(L^k (H^k_N f)) + \text{osc}(L^k f) \\
\leq \frac{1}{4} \text{osc}(f) + c_4 |f|_2 + \frac{1}{4} \text{osc}(H^k_N f) + c_4 |H^k_N f|_2 \\
\leq \frac{1}{2} \text{osc}(f) + (2c_4 + c_5) |f|_2.
\]

\[ \square \]

The following lemma express that the hole is a perturbation of the original system as far as operators from \( B \) to \( L^2(\mu) \) are concerned.

**Lemma 11.** For any \( N \in \mathbb{N} \) and \( f \in B \) with \( \|f\| = 1 \) we have

\[
\|(L - L_{X \setminus H_N}) f \|_2 \leq c_0 \mu(H_N)^{\frac{1}{3}}.
\]

**Proof of Lemma 11.** Using Hölder inequality then the invariance of \( L \) we get

\[
\int_X |L(H_N f)|^2 d\mu \leq \sup_X |L| f \int_{H_N} |f| d\mu \leq \mu(H_N)^{\frac{1}{2}} |f|_2 \sup_X L |f|.
\]

Additionally, using again Hölder inequality, the supremum is bounded by

\[
\sup_X L |f| \leq \sum_{Z \in Z_1} c_0 \mu(Z) \sup_Z |f| \leq \frac{c_0}{\theta} \text{osc}(f) + c_0 |f|_2.
\]

\[ \square \]

It comes from the Folkolre Theorem that \( 1 = e^{P(\phi|X)} \) is a simple eigenvalue of \( L \) acting on \( B \) and that the rest of the spectrum is contained in a disc of radius \( \rho < 1 \). By Lemmas 8, 10 and 11 we can apply Keller and Liverani’s theorem ([8], Corollary 1) with the sequence of operators \( L_{X \setminus H_n} \) acting on \( B \). This gives that this spectral figure will be conserved: for \( n \) sufficiently large, the spectrum of \( L_{X \setminus H_n} \) on \( B \) outside the disc of radius \( \rho \) consists exactly in one simple eigenvalue \( \lambda_n \), with eigenvector \( h_n \), such that \( \lambda_n \to 1 \) and \( h_n \to h \) (presumably \( \lambda_n \) and \( h_n \) are positive, but we won’t use this fact).

Furthermore, the restriction of the eigenvector \( h_n \) in \((X \setminus H_n)^\infty \) is again an eigenvector of the Perron Frobenius operator \( L((X \setminus H_n)^\infty) \) of the system \(((X \setminus H_n)^\infty, T)\). Consequently the spectral radius of \( L((X \setminus H_n)^\infty) \) on \( B \) is
larger or equal to $|\lambda_n|$. Thus we get $P(\varphi|X) \geq P(\varphi|(X \setminus H_n)^\infty) \geq \log |\lambda_n|$, which proves the result.

3.2. Removing the boundary of the Markov partition. The following proposition is of independent interest.

**Proposition 12.** Let $\Lambda \subset \mathcal{M}$ be a conformal repeller of the $C^{1+\alpha}$ map $f: \mathcal{M} \to \mathcal{M}$. There exists subsets $\Lambda_n \subset \Lambda$ such that
(i) $(f, \Lambda_n)$ is conjugated to a subshift of finite type
(ii) $\lim_{n \to \infty} \dim_H \Lambda_n = \dim_H \Lambda$.

**Proof.** Let $\xi$ be a generating Markov partition for the system $(f, \Lambda)$ and $(X, T)$ the corresponding subshift of finite type semi-conjugated via $\pi: X \to \Lambda$ to $(f, \Lambda)$. We write $\varphi = -\dim_H(\Lambda) \log |Df|$.

Let $K = \pi^{-1}\partial\xi$ be the premiage of the boundary of the partition. The Markov property implies that $TK \subset K$. The compact invariant set $K$ is strictly contained in $X$ hence $P(\varphi|K) < 0$ and the $\mathcal{F}_n$ measurable sets $H_n = \mathcal{Z}_n(K)$ have a measure going exponentially to zero (see for example the appendix of [2] for a proof of these facts). By Proposition 7 for any $\epsilon > 0$ there exists $n(\epsilon)$ such that for any $n \geq n(\epsilon)$

$$P(\varphi|(X \setminus H_n)^\infty) > \epsilon \sup \varphi,$$

whence the subset $\Lambda_n \overset{\text{def}}{=} \pi(X \setminus H_n)^\infty \subset \Lambda$ has dimension

$$\dim_H(\Lambda_n) > (1 - \epsilon) \dim_H(\Lambda).$$

Moreover, $(f, \Lambda_n)$ is conjugated to a subshift of finite type. \hfill \Box

We are now able to give a proof of our main theorem.

**Proof of Theorem 1.** By (i) in Proposition 12 we can apply Theorem 2 to $(f, \Lambda_n)$. This provides us with a set $E_n(\alpha, \beta) \subset \Lambda_n$ (See (15)) such that $\dim E_n(\alpha, \beta) = \dim \Lambda_n$. By monotony of Hausdorff dimension we get,

$$\dim_H \Lambda \geq \dim_H E(\alpha, \beta) \geq \dim_H E_n(\alpha, \beta).$$

The result follows from (ii) in Proposition 12 taking the limit as $n \to \infty$. \hfill \Box

4. Proofs of the technical lemmas

**Proof of Lemma 8.** Let $\varphi_k = \varphi + \varphi \circ T + \cdots \varphi \circ T^{k-1}$. Note that using Equation (16) with $c_2 = c_1/(1 - \alpha)$, for any integer $n, k$, we have $\text{var}_{n+k}(\varphi_k) \leq c_2 \alpha^n$. For a cylinder $A \in \mathcal{Z}_k$ and a point $x$ we denote by
Ax the unique element of \( A \cap T^{-k}x \), if any. If \( Z \in \mathcal{Z}_n \) with \( n \geq 1 \) then

\[
\text{osc}(L^k f, Z) = \sup_{x,x' \in Z} L^k f(x) - L^k f(x')
\]

\[
= \sum_{x,x' \in Z} \sum_{A \in \mathcal{A}_k, AZ \neq \emptyset} e^{\varphi_k(Ax)} f(Ax) - e^{\varphi_k(Ax')} f(Ax')
\]

\[
\leq \sum_{x,x' \in Z} \sum_{A} e^{\varphi_k(Ax)} (f(Ax) - f(Ax')) + e^{\varphi_k(Ax)} (e^{\varphi_n + (\varphi_k) - 1}) |f|(Ax')
\]

\[
\leq \sum_{A} c_0 \mu(A) \text{osc}(f, AZ) + (e^{\varphi_n + (\varphi_k) - 1}) c_0 \mu(A) \sup_{A} |f|
\]

\[
\leq c_0 (1 + e^{c^2}) \sum_{A} \mu(A) \text{osc}(f, AZ) + c_0 e^{c^2} c_2 \alpha^n \sum_{A} \mu(A) \mu(AZ)^{-1/2} \left( \int_{AZ} |f|^2 d\mu \right)^{1/2}
\]

Summing up over \( Z \in \mathcal{Z}_n \) and then over \( n \) yields to

\[
\text{osc}(L^k f) \leq \sum_{n \geq 1} \theta^n \sum_{Z \in \mathcal{Z}_{n+k}} \left[ c_0^2 (1 + e^{c^2}) \mu(Z) \text{osc}(f, Z) + c_0^2 e^{c^2} c_2 \alpha^n \mu(Z)^{1/2} \left( \int_{Z} |f|^2 d\mu \right)^{1/2} \right]
\]

\[
\leq c_3 \theta^{-k} \text{osc}(f) + c_4 |f|_2.
\]

For the last inequality we have set \( c_3 = c_0^2 (1 + e^{c^2}) \) and \( c_4 = c_0^2 e^{c^2} c_2 / (1 - \alpha \theta) \) and use the Schwarz inequality.

\[\square\]

**Proof of Lemma** Since \( H_N^k \) is \( \mathcal{F}_{N+k} \) measurable we have

\[
\text{osc}(fH_N^k) = \sum_{n \geq 1} \theta^n \sum_{Z \in \mathcal{Z}_n} \text{osc}(fH_N^k, Z) \mu(Z)
\]

\[
\leq \sum_{n=1}^{N+k-1} \theta^n \sum_{Z \in \mathcal{Z}_n, Z \cap H_N^k \neq \emptyset} \mu(Z) \sup_{Z} |f| + \sum_{n \geq N+k} \theta^n \sum_{Z \in \mathcal{Z}_n} \text{osc}(f, Z) \mu(Z)
\]

\[
\leq \text{osc}(f) + \sum_{n=1}^{N+k-1} \theta^n \sum_{Z \in \mathcal{Z}_n, Z \cap H_N^k \neq \emptyset} \mu(Z)^{1/2} \left( \int_{Z} |f|^2 d\mu \right)^{1/2}
\]

\[
\leq \text{osc}(f) + \sum_{n=1}^{N+k-1} \theta^n \mu(\mathcal{Z}_n(H_N^k))^{1/2} |f|_2.
\]

For the last inequality we used Schwarz inequality and \( \mathcal{Z}_n(H_N^k) \) denotes the union of elements of \( \mathcal{Z}_n \) intersecting \( H_N^k \). For any \( n \leq N + k - 1 \) we have

\[
\mathcal{Z}_n(H_N^k) \subset \bigcup_{j=0}^{k-1} T^{-j} \mathcal{Z}_{n-j}(H_N) \subset \bigcup_{j=0}^{k-1} T^{-j} H_{n-k}.
\]

The result follows from the invariance of the measure by taking

\[
c_5 = \frac{\sqrt{c_0} e^{c^2}}{1 - \theta e^{-\frac{1}{2}}} \geq \sup_{N} \sum_{n=1}^{N+k-1} \theta^n (k c e^{-\epsilon(n-k)})^{1/2}.
\]

\[\square\]
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