Network Monitoring under Strategic Disruptions

Mathieu Dahan
Center for Computational Engineering, Massachusetts Institute of Technology, Cambridge, MA 02139, mdahan@mit.edu

Lina Sela
Department of Civil, Architectural and Environmental Engineering, University of Texas, Austin, TX 78712, linasela@utexas.edu

Saurabh Amin
Department of Civil and Environmental Engineering, and Institute for Data, Systems, and Society, Massachusetts Institute of Technology, Cambridge, MA 02139, amins@mit.edu

This article considers a resource allocation problem for monitoring infrastructure networks facing strategic disruptions. The network operator is interested in determining the minimum number of sensors and a sensing strategy, to ensure a desired detection performance against simultaneous failures induced by a resource-constrained attacker. To address this problem, we formulate a mathematical program with constraints involving the mixed strategy Nash equilibria of an operator-attacker game. The set of player strategies in this game are determined by the network structure and players’ resources, and grow combinatorially with the network size. Thus, well-known algorithms for computing equilibria in strategic games cannot be used to evaluate the constraints in our problem. We present a solution approach based on two combinatorial optimization problems, formulated as minimum set cover and maximum set packing problems. By using a combination of game-theoretic and combinatorial arguments, we show that the resulting solution has guarantees on the detection performance and admits a small optimality gap in practical settings. Importantly, this approach is scalable to large-scale networks. We also identify a sufficient condition on the network structure for this solution to be optimal. Finally, we demonstrate the scalability and optimality guarantee of our approach using a set of benchmark water networks.

Subject classifications: Games/group decisions: noncooperative; Military: search/surveillance.

1. Introduction

This article studies a monitoring problem on networks in the face of simultaneous disruptions caused by a strategic adversary. Our setup is motivated by applications in infrastructure management where the operator is interested in strategically allocating available sensing resources to detect adversarial disruptions. Recent incidents on infrastructure systems (Brown et al. 2006, Gleick 2006, Russon 2016) highlight the need to improve the operator’s situation awareness to security attacks. In this article, we investigate (randomized) sensing strategies for network monitoring under strategic failure events. We introduce a sensing model that can be used to define an operator-attacker strategic game played on the network. Solving this game is challenging because the sets of actions of both players grow combinatorially with the size of the network. Hence, commonly known
algorithms (Nisan et al. 2007, Porter et al. 2008) cannot be used to compute equilibrium player strategies in a scalable manner. Instead, we construct an approximate equilibrium strategy of this large-scale game based on the properties of minimum set cover and maximum set packing problems. This construction enables a scalable solution approach for our network monitoring problem.

Specifically, we focus on the following question: *How many sensors are required and how to strategically place them in the network to detect adversarial attacks?* In Section 2, we formulate this network monitoring problem as a Mathematical Program with Equilibrium Constraints (Luo et al. 1996). In this problem, the network operator wants to minimize the number of sensors such that the expected rate of detection of adversarial failures (i.e., detection performance) is above a pre-specified threshold in any equilibrium of the induced operator-attacker game. This game models the strategic interaction between the operator who chooses where to allocate her sensors, and the attacker who chooses to target and disrupt one or more network components. Each player is resource-constrained in that the operator (resp. the attacker) cannot place more sensors (resp. cannot target more network components) than her number of resources. In this game, the operator’s objective is to maximize the number of failure events that are detected by her sensing strategy, while the attacker’s objective is to maximize the number of failure events that remain undetected. The problem also entails computing one equilibrium of this operator-attacker game. Our formulation is related to previous work in network sensing problems (Ostfeld and Salomons 2004, Krause et al. 2008b, Berry et al. 2006), security games (Mavronicolas et al. 2008, Alpern et al. 2011, Baykal-Gursoy et al. 2014), search games (Von Neumann 1953, Garnaev et al. 1997), and interdiction problems (Washburn and Wood 1995, Cormican et al. 1998, Smith and Lim 2008).

In the operator-attacker game, both players can randomize over their respective sets of actions. We adopt (approximate) Nash equilibrium in mixed strategies as the solution concept to capture the inherent uncertainty in the players’ strategies. Our analysis shows that this is a natural choice for our network monitoring problem because, for a range of parameters, the operator benefits from choosing a randomized sensing strategy when facing the attacker; similarly, the attacker prefers randomized attacks over fixed ones. Randomized sensing has become practically feasible in the context of infrastructure defense, thanks to the recent technological advances that allow for flexible on/off schedules and simultaneous collection of measurements at desired network locations. For example, Pita et al. (2008) used randomized strategies to assist the police at the Los Angeles International Airport; in particular, for the scheduling of checkpoints and for generating patrolling schedules for canine units. Another motivating example is the sensor network deployments in urban water systems such as PipeNet@Boston (Stoianov et al. 2007), WaterWise@Singapore (Allen et al. 2011), and SWND@Bristol (Wright et al. 2015). Such deployments enable the implementation of (randomized) sensing strategies to detect pipe failures (leak and burst events) and/or the presence
of contaminants (Wu et al. 2014). Importantly, these disruptions can be introduced by malicious entities who can target multiple network components (Kroll 2006), thus necessitating the strategic allocation of sensing resources by the operator.

In this article, we adopt a sensing model following Ostfeld and Salomons (2004) and Perelman et al. (2016) in which the detection of a failure event by a sensor is binary and only depends on the location of the sensor and the network component that is targeted. This sensing model is, in practice, constructed based on the sensors’ technological features (e.g., sensing range), type of failures (e.g., pipe bursts versus contamination events), and network topology. We use this model to define the players’ payoff functions in the simultaneous operator-attacker game.

1.1. Related Work

Sensor Placement. The network sensing problem in the case of reliability failures (or faults) has been well studied. Typically, it can be modeled as a problem of optimal allocation of limited sensing resources to optimize a performance metric, such as observability of the network (Chakrabarti et al. 2009), uncertainty about failure events (Krause et al. 2008b), or proportion of affected population (Berry et al. 2006). Some of these formulations can be solved using mixed-integer programming methods, while others exhibit properties of submodular optimization leading to the application of greedy algorithms with approximation guarantees. The main feature of this optimization-based approach is fixed sensing, i.e., continuous operation of sensors placed at fixed locations in the network. However, in many adversarial situations, all network failures cannot be fully detected due to the strategic nature of the attack and due to the operator’s resource constraints. Thus, a fixed sensing strategy can face a significant loss of detection performance in such situations.

Robust Sensor Placement. One approach to address the above-mentioned issue for certain performance metrics is to consider the problem of robust sensor placement, which aims to maximize the worst-case detection performance against a set of possible failure scenarios. Krause et al. (2008a) studied a robust network sensing problem where these possible failure scenarios are modeled as normalized monotone submodular functions. The authors propose an efficient approximation algorithm to solve this problem. They also provide applications in outbreak detection, and robustness against sensor failures. Orlin et al. (2016), motivated by the latter application, provided a constant factor approximation algorithm to find a sensor placement that is robust against a subset of sensors being faulty. Similarly to these works, we define the set of actions of the resource-constrained operator (and also of the attacker) using cardinality constraints, and model the players’ payoffs using a monotone submodular sensing function. However, our setting is different in that we allow randomized sensing as opposed to restricting the operator’s strategy to fixed sensor placements. Indeed, in our model, if the operator chooses a fixed sensor placement and leaves some parts of
the network unmonitored due to her resource constraints, the strategic attacker will target these unmonitored parts to avoid detection. Thus, randomized sensing is a key feature of our analysis.

**Security Games.** Related to our setting is the work by Mavronicolas et al. (2008), who consider a security game on a computer network in which the nodes are vulnerable to attacks and the defender can allocate a defense mechanism on an edge. The defender’s goal is to maximize the number of attack detections by randomizing the placement of her defense mechanism. Simultaneously, the attackers randomize their attacks to avoid being detected. Our model encompasses this network security game, and extends it by considering a more general sensing model and by allowing the defender to place several defenses to simultaneously monitor different parts of the network.

**Search Games.** Our setting also generalizes two well-known search games. The first one is the Hide-and-Seek game defined in Ch. 3.2 of Karlin and Peres (2016) and first introduced by Von Neumann (1953). In this zero-sum game, a robber hides in one of a set of “safe-houses” located at roads intersections, and a cop simultaneously chooses to travel along one road to find the robber. Our setup also generalizes the Infiltration Game with a Cable, as defined in Ch. 2.1 of Garnaev (2000). In this problem, an infiltrator wants to cross a (discretized) channel and a guard uses an electric cable to detect the infiltrator. Both these problems can be modeled with our game by restricting our sensing model and by giving each player only one unit of resources.

### 1.2. Our Results

We solve the network monitoring problem under strategic attack by first evaluating the expected detection rate of failures in any Nash equilibrium of the operator-attacker game in terms of both players’ resources; and then constructing an approximate Nash equilibrium which provides guarantees in terms of the detection performance.

Our game is strategically equivalent to a zero-sum game; hence, equilibrium strategies can be computed via linear programming. However, the size of both players’ sets of actions makes the computation of these linear programs unscalable for real networks. Lipton et al. (2003) showed that for bimatrix games, there exists an $\epsilon-$Nash equilibrium with a support that is logarithmic in the number, $n$, of pure strategies; in particular, they provide a $n^{O(\ln n/\epsilon^2)}$ time algorithm for computing it. For the problem sizes considered in our paper, the number of available pure strategies for each player can easily reach $10^{60}$, which makes the algorithm in Lipton et al. (2003) practically inapplicable for our setting. Instead, we derive an approximate Nash equilibrium based on two combinatorial optimization problems (introduced in Section 3) that we formulate as a minimum set cover (MSC) problem and maximum set packing (MSP) problem (Vazirani 2001). An advantage of our approach is that the number of variables and constraints these problems require is much smaller, given by the number of sensor locations and network components, respectively. Although
both MSC and MSP problems are known to be NP-hard, we find that modern integer programming solvers can solve them even for large-scale networks. This leads to a scalable solution of our network monitoring problem with approximation guarantees.

We mainly focus on the setting in which the size of the network is large enough so it does not restrict the attacker’s capability to spread her attacks across the network. In our model, this setting is represented when the size of the MSPs is larger than the number of resources of the attacker. Furthermore, solving the game is trivial when the operator can simultaneously allocate enough sensors to monitor all the network components. Thus, we study the equilibria of the game when the operator’s resources do not permit complete monitoring of the network. We emphasize, however, that our approach to solving the problem also extends to the settings when the attacker has more resources than the size of MSPs (see Section EC.6).

In Section 4.1, we present the structural properties that are satisfied by all mixed strategy Nash equilibria of our game. To prove these properties, we exploit the features of our sensing function (e.g., monotonicity and submodularity), and combine them with game-theoretic and combinatorial arguments. We show (Prop. 6) that in equilibrium, both players must randomize their actions (i.e., no equilibrium exists in pure strategies). Furthermore, both players use all the available resources in equilibrium (Prop. 3). More interestingly, we show that the expected detection rate of failures has a unique value in all equilibria. We provide lower and upper bounds on this expected detection rate in equilibrium in terms of both players’ available resources, and the optimal values of our MSC and MSP problems (Thm. 1). Using solutions of the MSC and MSP problems, we construct a mixed strategy profile that not only is an ε-Nash equilibrium, but also provides each player a payoff that is ε-close to the payoff they would get in any Nash equilibrium (Thm. 2).

In Section 4.2, we specialize our results to the case where the duality gap between our two combinatorial optimization problems is zero, i.e., when the sizes of the MSCs and MSPs are the same. We show that, for this case, MSCs and MSPs can be directly used to construct a Nash equilibrium of the game (Prop. 8). We also deduce analytical expressions of the players’ payoffs and the expected detection rate of failures in any equilibrium (Cor. 1).

These results enable us to compute (in Section 5) an exact solution to the network monitoring problem for the case when the MSCs and MSPs are of same size, i.e., we provide an analytical expression of the optimum number of sensors, and an equilibrium strategy profile (Prop. 10). For the general case (i.e., when the size of MSCs is larger than or equal to the size of MSPs), we derive an approximate solution (Prop. 11) with guarantees on the detection performance and the corresponding optimality gap. Importantly, this solution approach is also valid when only approximate solutions of the MSC and MSP problems (obtained from standard greedy or heuristic algorithms) are available instead of the exact ones.
Finally, in Section 6, we demonstrate our results by solving the network monitoring problem for the case of adversarial disruptions in large-scale water networks. In particular, we illustrate the computational advantages and performance guarantees of our approach. We conclude and briefly discuss the extension of our work in Section 7. The complete proofs are provided in the electronic companion of this article.

2. Problem Formulation

In this section, we first define a generic sensing model and introduce a strategic game to model the interaction between the operator and an attacker. Next, we pose our network monitoring problem under strategic attack as a mathematical program with equilibrium constraints, and discuss the computational challenge in solving it.

2.1. Sensing Model

For a given network, let $\mathcal{V}$ be the set of locations where a sensor can be installed, and let $\mathcal{E}$ be the set of components that the operator is interested in monitoring using available sensors. Without loss of generality, we consider the situation where the operator wants to detect edge failures using sensors placed on the nodes of the network. Thus, the network can be represented as an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is the set of nodes and $\mathcal{E}$ is the set of edges of the network.

Following Ostfeld and Salomons (2004) and Perelman et al. (2016), we assume that the edges that are monitorable by a sensor only depend on its location (i.e., the node it is placed at). Additionally, we limit our attention to binary sensing, i.e., for a sensor placed at a given node, its monitoring capability for any edge is either 1 (monitored) or 0 (not monitored). Therefore, for each node $i \in \mathcal{V}$, the set of edges monitored by a sensor placed at node $i$ can be represented as a monitoring set $\mathcal{C}_i \in 2^\mathcal{E}$. Equivalently, we say that the failure of an edge $e \in \mathcal{E}$ is detected by a sensor placed at node $i$ if and only if $e \in \mathcal{C}_i$. Typically, the sets $\mathcal{C}_i$, $i \in \mathcal{V}$, can be obtained based on the outcome of a threshold-based rule that determines the sensors’ capability to monitor different failure events from each location (Allen et al. 2011, Romano et al. 2012, Perelman et al. 2016). For our purpose, the sensing model, represented by the collection of sets $\{\mathcal{C}_i, \ i \in \mathcal{V}\}$, is assumed to be known.

The notion of monitoring set $\mathcal{C}_i$ for a single sensor at node $i$ can be extended to the set of edges that can be monitored collectively by a set of sensors. We denote a sensor placement by a set $S \in 2^\mathcal{V}$ of nodes that receive sensors. For a given sensor placement $S$, let $\mathcal{C}_S$ denote the set of edges that are monitored by at least one sensor in $S$:

$$\mathcal{C}_S := \bigcup_{i \in S} \mathcal{C}_i.$$
To count the number of edges in any given subset of edges of $G$ that can be monitored using an arbitrary sensor placement, we define a sensing function $F : 2^V \times 2^E \rightarrow \mathbb{N}$. For a given sensor placement $S \in 2^V$ and a subset of edges $T \in 2^E$, the value of $F(S, T)$ is the number of edges of $T$ that are monitored by at least one sensor placed in $S$. Mathematically, $F$ is defined as follows:

$$
\forall (S, T) \in 2^V \times 2^E, \quad F(S, T) := |C_S \cap T| = \sum_{e \in T} 1_{\{e \in C_S\}}.
$$

(1)

With a slight abuse of notation, we denote $F(i, e) := F(\{i\}, \{e\})$, $\forall (i, e) \in V \times E$. Given a sensor placement $S \in 2^V$ and an edge $e \in E$, $F(S, e) = 1_{\{e \in C_S\}}$ determines whether or not edge $e$ is monitored (and hence, its failure is detected) by at least one sensor placed in $S$. In fact, the sensing function $F$ has features that are important in our analysis:

**Lemma 1.** The sensing function $F$ satisfies the following properties:

(i) For any subset of edges $T \in 2^E$, $F(\cdot, T)$ is submodular and monotone:

$$
\forall T \in 2^E, \forall (S, S') \in (2^V)^2, \begin{cases} F(S \cup S', T) + F(S \cap S', T) & \leq F(S, T) + F(S', T), \\ S \subseteq S' & \Rightarrow F(S, T) \leq F(S', T). \end{cases}
$$

(ii) For any sensor placement $S \in 2^V$, $F(S, \cdot)$ is finitely additive:

$$
\forall S \in 2^V, \forall (T, T') \in (2^E)^2 \mid T \cap T' = \emptyset, \quad F(S, T \cup T') = F(S, T) + F(S, T').
$$

The submodularity of $F(\cdot, T)$ results from the fact that adding a sensor to a smaller sensor placement increases the number of monitored edges in $T$ at least as much as when adding that sensor to a bigger sensor placement. The monotonicity is implied by the fact that adding a sensor to a sensor placement does not decrease the number of edges of $T$ that are monitored. Finally, the additivity of $F(S, \cdot)$ is a direct consequence of (1).

### 2.2. Operator-Attacker Game

For the network $G$ and the sensing model $\{C_i, \ i \in V\}$, we consider a two-player strategic game $\Gamma := (\{1, 2\}, (A_1, A_2), (u_1, u_2))$. In this game, Player 1 ($P1$) is the operator who is interested in detecting failure events on edges of the network $G$ by choosing a sensor placement on the nodes of $G$. The detection capability of $P1$ is limited in that she has a fixed amount of sensing resources that she cannot exceed. For simplicity, we assume that the cost of allocating one unit of resources (i.e., sensing a node) is identical for all nodes; without loss of generality, we let this cost to be 1. Let $b_1 \in \mathbb{N}$ be the maximum number of sensors she can simultaneously place on the nodes of $G$. Thus, $P1$’s set of pure actions can be modeled as the set of sensor placements of size at most $b_1$:

$$
A_1 = \{S \in 2^V \mid |S| \leq b_1\}.
$$

(2)
Player 2 (P2) is an attacker who simultaneously chooses to induce failure events on a subset of edges $T \in 2^E$ of the network. For simplicity, we assume that targeting an edge results in its failure, and incurs P2 a fixed cost that is identical across all edges. Again, without loss of generality, let the cost of targeting an edge be 1. The attack capability of P2 is limited in that she cannot target a subset of edges of size larger than $b_2 \in \mathbb{N}$. We call an attack a subset of edges of size at most $b_2$ that P2 chooses. Thus, P2’s set of pure actions can be modeled as follows:

$$A_2 = \{ T \in 2^E \mid |T| \leq b_2 \}. \quad (3)$$

The payoff of P1 is defined as the number of detected failure events:

$$u_1(S,T) = F(S,T),$$

and the payoff of P2 is defined as the number of undetected failure events:

$$u_2(S,T) = |T| - F(S,T),$$

where $|T|$ is the total number of edges that are targeted by P2, and $F(S,T)$ is the number of detected events given by $(1)$. In this model, the sensing (resp. attack) resources of P1 (resp. P2), and hence their pure actions, are constrained by the parameter $b_1$ (resp. $b_2$).

We allow both players to randomize over their set of pure actions (i.e., sensor placements for P1 and attacks for P2), and assume that the players’ strategies are independent randomizations. Let $\Delta(A_1) := \{ \sigma^1 \in [0,1]^{A_1} \mid \sum_{S \in A_1} \sigma^1(S) = 1 \}$ and $\Delta(A_2) := \{ \sigma^2 \in [0,1]^{A_2} \mid \sum_{T \in A_2} \sigma^2(T) = 1 \}$ denote the mixed extensions of P1 and P2 pure strategies, respectively. Here, $\sigma^1$ (resp. $\sigma^2$) represents a mixed sensing strategy (resp. mixed attack strategy) of P1 (resp. P2). For notational simplicity, we define $\sigma^1_S := \sigma^1(S)$ and $\sigma^2_T := \sigma^2(T)$. The respective expected payoffs for a strategy profile $\sigma = (\sigma^1, \sigma^2) \in \Delta(A_1) \times \Delta(A_2)$ can be expressed as:

$$U_1(\sigma^1, \sigma^2) = E_\sigma [F(S,T)], \quad (4)$$

$$U_2(\sigma^1, \sigma^2) = E_\sigma [|T| - F(S,T)]. \quad (5)$$

Thus, the mixed extension of the game $\Gamma$ is $\langle \{1, 2\}, (\Delta(A_1), \Delta(A_2)), (U_1, U_2) \rangle$. We will also use the notations $U_i(S, \sigma^2) = U_i(1(S), \sigma^2)$ and $U_i(\sigma^1, T) = U_i(\sigma^1, 1(T))$ for $i \in \{1, 2\}$. The support of $\sigma^1 \in \Delta(A_1)$ (resp. $\sigma^2 \in \Delta(A_2)$) is defined as $\text{supp}(\sigma^1) = \{ S \in A_1 \mid \sigma^1_S > 0 \}$ (resp. $\text{supp}(\sigma^2) = \{ T \in A_2 \mid \sigma^2_T > 0 \}$), i.e., the support of a mixed strategy is the set of pure actions that are assigned strictly positive probabilities.

Note that a mixed sensing (resp. attack) strategy entails a decision of P1 (resp. P2) to introduce randomness in her behavior, i.e., the manner in which nodes (resp. edges) of the network are sensed.
(resp. attacked). Consider, for example, the problem of sensing an urban water network to detect adversary-induced events on pipes. In this problem, a mixed strategy of $P_1$ can be viewed as “random scheduling” of a subset of fixed sensors that are installed on some nodes of the network, with the constraint that only up to $b_1$ of these sensors can be used simultaneously. A mixed strategy of $P_2$, constrained by the parameter $b_2$, can be similarly interpreted.

We now introduce the equilibrium concept of the game $\Gamma$. A strategy profile $(\sigma^1, \sigma^2) \in \Delta(A_1) \times \Delta(A_2)$ is a mixed strategy Nash Equilibrium (NE) if:

\begin{align*}
\forall \sigma^1 \in \Delta(A_1), & \quad U_1(\sigma^1, \sigma^2) \geq U_1(\sigma^1, \sigma^2^*), \\
\forall \sigma^2 \in \Delta(A_2), & \quad U_2(\sigma^1, \sigma^2) \geq U_2(\sigma^1^*, \sigma^2^*).
\end{align*}

(6) (7)

Equivalently, in a NE $(\sigma^1^*, \sigma^2^*)$, $\sigma^1^*$ (resp. $\sigma^2^*$) is a best response to $\sigma^2^*$ (resp. $\sigma^1^*$). Given the parameters $b_1$ and $b_2$, we denote $\Sigma(b_1, b_2)$ the set of NE of the game $\Gamma$.

Given $\epsilon \geq 0$, a strategy profile $(\sigma^1', \sigma^2') \in \Delta(A_1) \times \Delta(A_2)$ is an $\epsilon$-NE if:

\begin{align*}
\forall \sigma^1 \in \Delta(A_1), & \quad U_1(\sigma^1', \sigma^2') \geq U_1(\sigma^1, \sigma^2') - \epsilon, \\
\forall \sigma^2 \in \Delta(A_2), & \quad U_2(\sigma^1', \sigma^2') \geq U_2(\sigma^1, \sigma^2) - \epsilon.
\end{align*}

(8) (9)

Given the parameters $b_1$ and $b_2$, we denote $\Sigma_{\epsilon}(b_1, b_2)$ the set of $\epsilon$-NE of the game $\Gamma$. When there is no confusion, we will refer to $\Sigma(\epsilon)$ (resp. $\Sigma_{\epsilon}(b_1, b_2)$) simply as $\Sigma$ (resp. $\Sigma_{\epsilon}$).

From a game-theoretic viewpoint, a NE in mixed strategies describes the behavior of $P_1$ and $P_2$ who play the game $\Gamma$ repeatedly against each other and ignore any strategic relationship in-between plays. In such a repeated play, each player cannot guess the actions of her opponent in any particular round of play (Fudenberg and Levine 1998). Thus, NE in mixed strategies can be viewed as a stochastic steady state of the repeated interaction between players. By extension, we view $\epsilon$-NE as strategy profiles in which the players are almost at a stochastic steady state, i.e., no player has more than an incentive of $\epsilon$ to change her mixed strategy.

Henceforth, we assume without loss of generality that every edge in the network can be monitored from at least one node. Indeed, if $k$ edges cannot be monitored from any node in the network, then $P_2$’s incentive will be to always target those edges, and then allocate her remaining resources (if any) to target other edges. In this case, we can remove those $k$ edges from the network, and solve the resulting game where $P_2$ has now $b_2 - k$ resources that she can allocate on the resulting subnetwork (whose edges are now monitored from at least one node each); we will just need to increase her payoff by $k$ to account for the $k$ edges already targeted and never detected by $P_1$.

Finally, for any mixed strategy profile, we define the following performance metric:
Definition 1 (Expected Detection Rate). Given a strategy profile $\sigma \in \Delta(A_1) \times \Delta(A_2)$, the expected detection rate, $r(\sigma)$, is the expectation (under $\sigma$) of the ratio between the number of failure events that are detected and the total number of failure events:

$$r(\sigma) := \mathbb{E}_\sigma \left[ \frac{F(S,T)}{|T|} \right].$$

(10)

Now, we present the network monitoring problem. In particular, we use the expected detection rate in equilibrium of the game $\Gamma$ as the metric of monitoring performance resulting from the strategic interaction between the operator and the attacker.

2.3. Network Monitoring Problem

Consider the problem in which $\textbf{P}_1$ is concerned with monitoring the network $\mathcal{G}$, and is given the sensing model $\{C_i, i \in \mathcal{V}\}$. She faces a strategic adversary $\textbf{P}_2$ whose attack capability is constrained by $b_2$. In this problem, $\textbf{P}_1$ aims to minimize the number of sensing resources, $b_1$, and strategically allocate them to ensure that the expected detection rate in any equilibrium of the induced game $\Gamma$ does not fall below a pre-specified threshold. Formally, the network monitoring problem can be posed as follows:

$$(\mathcal{P}) : \begin{array}{ll}
\text{minimize} & b_1 \\
\text{subject to} & r(\sigma^*) \geq \alpha, \quad \forall \sigma^* \in \Sigma(b_1, b_2) \\
& \sigma^* \in \Sigma(b_1, b_2) \\
& b_1 \in \mathbb{N},
\end{array}$$

(11)

where $\alpha \in [0,1]$ denotes the target detection performance that $\textbf{P}_1$ must achieve (for e.g., to meet regulatory requirements on limiting network losses due to failures). Constraints (11) ensure that the expected detection rate in any equilibrium of the game induced by the chosen $b_1$ and the number of attack resources $b_2$ is at least $\alpha$. Constraint (12) simply requires the computation of one NE of the game $\Gamma$, which exists due to Nash’s Theorem (Nash 1950). The optimal value of $(\mathcal{P})$, denoted $b_1^\dagger$, is the minimal number of sensors for which the equilibrium constraints (11)-(12) are satisfied. Thus, $(b_1^\dagger, \sigma^\dagger)$ with $\sigma^\dagger \in \Sigma(b_1^\dagger, b_2)$ is an optimal solution of $(\mathcal{P})$, where the sensing strategy in $\sigma^\dagger$ specifies an allocation of the $b_1^\dagger$ sensors in equilibrium such that the target detection performance $\alpha$ is met. Since the constraints (11)-(12) are defined in terms of the equilibria of the game $\Gamma$, $(\mathcal{P})$ is a Mathematical Program with Equilibrium Constraints.

To solve $(\mathcal{P})$ in a brute force manner, one would need to compute every NE of the game $\Gamma$ for each $b_1 \in \mathbb{N}$, and check for each of them whether or not the expected detection rate is at least $\alpha$. Clearly, such an exhaustive approach is not computationally scalable. To handle the set of equilibrium constraints (11)-(12), we note that $\Gamma$ is strategically equivalent to a zero-sum game.
Lemma 2. Γ is strategically equivalent to the game \( \tilde{\Gamma} := \langle \{1, 2\}, (A_1, A_2), (-u_2, u_2) \rangle \). Therefore, the NE of Γ can be obtained by solving the following two linear programming problems:

\[
(LP_1) \quad \max_{\sigma_1' \in \Delta(A_1)} \min_{T \in A_2} -U_2(\sigma_1', T) \quad \quad \quad (LP_2) \quad \max_{\sigma_2' \in \Delta(A_2)} \min_{S \in A_1} U_2(S, \sigma_2')
\]

where \( U_2 \) is given by (5).

From the interchangeability of NE in zero-sum games, if \((\sigma_1^*, \sigma_2^*) \in \Sigma \) and \((\sigma_1^\dagger, \sigma_2^{*}) \in \Sigma \), then \((\sigma_1^*, \sigma_2^{*}) \in \Sigma \) and \((\sigma_1^\dagger, \sigma_2^{*}) \in \Sigma \); furthermore, \( \Sigma \) is a convex set.3

Lemma 2 implies that linear programming techniques can be used to compute NE of Γ. However, the computation of (LP_1) and (LP_2) quickly becomes unscalable as the size of the network increases. In particular, due to the size of the players’ sets of actions (\( |A_1| = \sum_{k=0}^{b_1} \binom{|V|}{k} \)) and \( |A_2| = \sum_{l=0}^{b_2} \binom{|E|}{l} \)), the number of variables and constraints in both LPs can be huge. For example, for a network consisting of 200 nodes and edges, and \( b_1 = b_2 = 10 \), computing the equilibria of game Γ will entail solving LPs containing \( 2.37 \cdot 10^{16} \) variables and constraints.

Instead, we develop a scalable approach to solve the network monitoring problem (P) which involves: (i) deriving properties satisfied by all equilibrium sensing and attack strategies, (ii) using these properties to compute an estimate \( b_1' \) of the optimal value \( b_1^\dagger \), and (iii) constructing an approximate NE of the resulting game Γ with parameters \( b_1' \) and \( b_2 \). Our approach is computationally scalable for large-scale networks, and provides an approximate solution of \( (P) \). In particular, our estimate of the number of required sensors \( b_1' \) is “close” to the optimal value \( b_1^\dagger \), and the expected detection rate of our approximate NE is also close to the expected detection rate in equilibrium. Furthermore, we identify a condition on the network and the sensing model under which our approach gives an optimal solution of \( (P) \). Our computational study in Section 6 shows that this condition is practically relevant.

3. Preliminaries

We now illustrate our solution approach using a simple example. To do this, we first need to introduce two combinatorial optimization problems.

3.1. Set Cover and Set Packing Problems

We say that a set of nodes \( S \in 2^V \) is a set cover if and only if every edge of \( G \) is monitored by at least one sensor placed in \( S \), i.e., \( F(S, e) = 1, \forall e \in E \). A set of nodes \( S \in 2^V \) is a minimal set cover if \( S \) is a set cover that is minimum with respect to inclusion, i.e., if any node of \( S \) is removed, the
resulting set is not a set cover anymore. A set of nodes \( S \in 2^V \) is a minimum set cover (MSC) if and only if it is an optimal solution of the following problem:

\[
(I_{MSC}): \quad \text{minimize } |S| \\
\text{subject to } F(S,e) = 1, \ \forall e \in E \ \\
S \in 2^V.
\]

(13)

Solving \((I_{MSC})\) amounts to determining the minimum number of sensors and their placement to monitor all the edges of the network. We denote the set (resp. the size) of MSCs by \( S \) (resp. \( n^* \)). Under our assumption that each edge can be monitored from at least one node in the network (Section 2.2), \((I_{MSC})\) becomes feasible and \( n^* \) exists. Note also that an MSC is a minimal set cover of minimum cardinality.

We say that a set of edges \( T \in 2^E \) is a set packing if and only if a sensor placed at any node \( i \) can monitor at most one edge in \( T \), i.e., \( F(i,T) \leq 1, \ \forall i \in V \). A set of edges \( T \in 2^E \) is a maximum set packing (MSP) if and only if it is an optimal solution of the following problem:

\[
(I_{MSP}): \quad \text{maximize } |T| \\
\text{subject to } F(i,T) \leq 1, \ \forall i \in V \\
T \in 2^E.
\]

(14)

Thus, solving \((I_{MSP})\) amounts to finding the maximum number of “independent” edges, i.e., a set of edges of maximum size such that each edge requires a different sensor to be monitored. We denote the set (resp. the size) of MSPs by \( M \) (resp. \( m^* \)).

Both \((I_{MSC})\) and \((I_{MSP})\) are known to be NP-hard (Vazirani 2001), and are related to each other in a particular manner. Specifically, their integer programming formulations (see (EC.16)-(EC.17)) have linear programming relaxations that are dual of each other. This provides the following relation between the sizes of MSCs and MSPs:

**Lemma 3.** The size of MSPs is no greater than the size of MSCs, i.e., \( m^* \leq n^* \).

This result follows from the fact that to monitor all the edges of an MSP, the network operator needs \( m^* \) sensors. From Lemma 3, we obtain that if \( S \in 2^V \) is a set cover and \( T \in 2^E \) is a set packing of same size, then \( S \) is an MSC and \( T \) is an MSP.

### 3.2. A Simple Example

Consider the network shown in Fig. 1. Assume a sensing model in which a sensor placed at any node can monitor edges at one-hop vertically, and at two hops horizontally. Thus, \( C_{i_1} = \{e_1,e_2,e_3\} \), \( C_{i_2} = \{e_1,e_2,e_4\} \), \( C_{i_3} = \{e_1,e_2,e_5\} \), \( C_{i_4} = \{e_1,e_6,e_7\} \), \( C_{i_5} = \{e_5,e_6,e_8\} \), \( C_{i_6} = \{e_3,e_9,e_{10}\} \), \( C_{i_7} = \{e_7,e_9,e_{10}\} \), \( C_{i_8} = \{e_8,e_9,e_{10}\} \).
Consider the problem (P) with the target expected detection rate \( \alpha = 0.75 \) and the number of attack resources \( b_2 = 2 \). For this small-sized problem, we can solve \((LP_1)\) and \((LP_2)\) to compute the NE of \( \Gamma \), and obtain the expected detection rate in any NE for each \( b_1 \in \mathbb{N} \); see Table 1. Notice that for a given \( b_1 \), the expected detection rate is the same in any NE.

### Table 1  
Expected detection rate in equilibrium for every \( b_1 \in \mathbb{N} \).

| \( b_1 \) | 0 | 1 | 2 | 3 | \( \geq 4 \) |
|---|---|---|---|---|---|
| \( r(\sigma^*) \) | 0 | \( \frac{2}{7} \) | \( \frac{1}{7} \) | \( \frac{4}{7} \) | 1 |

From Table 1, we conclude that the optimal value of (P) is \( b_1^* = 3 \), i.e., with 3 sensors, \( P1 \) is capable of detecting \( \frac{6}{7} \geq \alpha \) of the failure events in equilibrium. However, as mentioned in Section 2.3, this method does not extend to larger networks, as it requires solving large \((LP_1)\) and \((LP_2)\) for each \( b_1 \in \mathbb{N} \) and checking if the constraint on the expected detection rate in any equilibrium \((11)\) is met.

Now consider another approach that is based on our results in Sections 4 and 5: First, by solving \((I_{MSC})\), we obtain an MSC given by \( S^{min} = \{i_3, i_4, i_6, i_8\} \); thus \( n^* = 4 \). Then, we show that the expected detection rate in any equilibrium is lower bounded by \( \frac{2}{n} \). Thus, with \( b_1' := \lceil \alpha n^* \rceil = 3 \) sensors, the expected detection rate in any equilibrium is at least \( \alpha \).

Secondly, by solving \((I_{MSP})\), we obtain an MSP given by \( T^{max} = \{e_3, e_4, e_8\} \); thus \( m^* = 3 \). Then, we show that the expected detection rate in any equilibrium is upper bounded by \( \frac{2}{m^*} \), which implies that if \( b_1 < \lceil \alpha m^* \rceil \), the equilibrium constraints \((11)\) are not satisfied. This enables us to deduce that an optimality gap associated with \( b_1' \) is given by \( \lceil \alpha n^* \rceil - \lceil \alpha m^* \rceil \). In fact, for this example, we obtain that the optimality gap is \( 3 - 2.25 = 0 \). Therefore, by solving \((I_{MSC})\) and \((I_{MSP})\), we can estimate the number of sensing resources to ensure that the equilibrium constraints in (P) are satisfied; furthermore, we can verify that this number is optimal for this example, i.e., \( b_1' = b_1^* = 3 \).

Thirdly, given \( b_1' = 3 \) and \( b_2 = 2 \), we can use \( S^{min} \) and \( T^{max} \) to construct an approximate NE. Consider the strategy profile \( \sigma = (\sigma^1, \sigma^2) \), where \( \sigma^1_{\{i_3,i_4,i_6\}} = \sigma^1_{\{i_6,i_3,i_4\}} = \sigma^1_{\{i_6,i_8,i_3\}} = \sigma^1_{\{i_4,i_6,i_8\}} = \frac{1}{4} \),
and \( \sigma^2_{\{e_2,e_4\}} = \sigma^2_{\{e_2,e_4\}} = \sigma^2_{\{e_4,e_5\}} = \frac{1}{3} \); \( \sigma \) is illustrated in Fig. 2. We show that the above-constructed strategy profile \( \sigma \) is an \( \epsilon \)-NE, and provides each player a payoff that is \( \epsilon \)-close to their equilibrium payoff, with \( \epsilon = b_1' b_2 \left( \frac{1}{\max\{b_1',m^*\}} - \frac{1}{n} \right) = \frac{1}{7} \). Indeed, from (LP_1) and (LP_2), we can deduce that P1 and P2’s equilibrium payoffs in the game \( \Gamma \) are \( \frac{12}{7} \) and \( \frac{7}{7} \) respectively, while our strategy profile \( \sigma \) provides them with the respective payoffs \( \frac{3}{2} \) and \( \frac{1}{2} \). One can easily check that \( \sigma \) gives each player a payoff that is \( \frac{1}{11} \) (\( \leq \epsilon \)) close to their equilibrium payoff.

Finally, we give an upper bound on the loss in detection performance by choosing our sensing strategy \( \sigma^1 \) (see Fig. 2 (top)), instead of choosing an equilibrium strategy for P1. It turns out that, with 3 sensors, \( \sigma^1 \) detects at least \( \frac{2}{3} \) of the failures regardless of P2’s strategy, while in equilibrium \( \frac{5}{7} \) of the failures are detected (Table 1). Thus, the loss in detection performance is 12.5\%. This exact calculation is possible only because, for this example, we can solve (LP_1) and (LP_2) and compute the value of the expected detection rate in equilibrium. In contrast, we provide an upper bound on the loss in detection performance without solving (LP_1) and (LP_2), but instead by computing \( n^* \) and \( m^* \) from (EC.16) and (EC.17). This upper bound is given by \( 1 - \frac{\max\{b_1',m^*\}}{n^*} \) = 25\%. In the next two sections, we formalize these results.

### 4. Equilibrium Characterization

To evaluate the constraints (11)-(12) in the network monitoring problem \( (P) \), we need to study the equilibrium characteristics of the game \( \Gamma \) for given parameters \( b_1 \) and \( b_2 \). In this section, we
focus on the equilibrium characterization for the main case of interest when $P_2$’s number of attack resources is less than the size of MSPs, i.e., $b_2 < m^*$. Intuitively, for a given $b_2$, $P_2$’s incentive would be to allocate her resources to target edges that are “spread” across the network such that the number of detections by $P_1$ is minimized. A natural metric of the spread that $P_2$ can achieve is given by the notion of MSP (see (14)). Thus, the assumption $b_2 < m^*$ enables us to focus on the situations in which the network is large enough and $P_2$ exhausts her ability to spread attacks across the network, thereby making it most challenging for $P_1$ to detect the resulting disruptions. Indeed, as we briefly discuss in Section EC.6, the sensing strategy for the case $b_2 < m^*$ also ensures the required detection performance when $b_2 \geq m^*$; i.e., when the use of additional attack resources by $P_2$ in equilibrium enables $P_1$ to more easily detect (at least some) disruptions.

Additionally, note that solving $\Gamma$ is trivial when $b_1 \geq n^*$. Indeed, in this case, $P_1$ has enough resources to operate sensors placed on an MSC and monitor all the edges of the network with a single sensor placement. Thus, in this case, one can show that the expected detection rate in any NE is equal to 1, which implies that $P_1$ can achieve any target detection performance $\alpha \in [0, 1]$ with $n^*$ sensors. This implies that the optimal number of sensors in $(P)$, $b_1^\dagger$, is at most $n^*$.

The above discussion suggests that it is interesting and relevant to study the game $\Gamma$ when $b_1 < n^*$ and $b_2 < m^*$. We consider these limits on the players’ resources $b_1$ and $b_2$ and characterize the NE of the game $\Gamma$. Then, we show how optimal solutions of $(I_{MSC})$ and $(I_{MSP})$ can be used to construct an approximate NE of $\Gamma$, along with guarantees in terms of detection performance.

4.1. General Case: $m^* \leq n^*$

Before proceeding further, we define several quantities that are useful for interpreting the equilibria of the game. For a given $P_1$’s sensing strategy, the following quantities enable us to represent the nodes that are sensed, along with the probabilities of being sensed.

**Definition 2 (Sensing Probability and Node Basis).**

- For a strategy $\sigma^1 \in \Delta(A_1)$ of $P_1$, the *sensing probability* of node $i \in V$, denoted $\rho_{\sigma^1}(i)$, is the probability with which $i$ is sensed, i.e.:
  \[
  \forall \sigma^1 \in \Delta(A_1), \forall i \in V, \quad \rho_{\sigma^1}(i) := \sum_{\{S \in A_1 | i \in S\}} \frac{\sigma_1^S}{\sum_{S \in A_1} \sigma_1^S} \mathbb{1}_{\{i \in S\}}. \tag{15}
  \]

- The *node basis* of a strategy $\sigma^1 \in \Delta(A_1)$, denoted $V_{\sigma^1}$, is the set of nodes with non-zero sensing probability, i.e.:
  \[
  \forall \sigma^1 \in \Delta(A_1), \quad V_{\sigma^1} := \{i \in V \mid \rho_{\sigma^1}(i) > 0\}. \tag{16}
  \]

We can define analogous quantities for a given attack strategy of $P_2$. 


DEFINITION 3 (ATTACK PROBABILITY AND EDGE BASIS).

- Given a strategy \( \sigma^2 \in \Delta(A_2) \), the attack probability of edge \( e \in E \), denoted \( \rho_{\sigma^2}(e) \), is the probability with which \( e \) is targeted by \( \sigma^2 \), i.e.:

\[
\forall \sigma^2 \in \Delta(A_2), \forall e \in E, \quad \rho_{\sigma^2}(e) := \sum_{\{T \in A_2 \mid e \in T\}} \sigma^2_T = \sum_{T \in A_2} \sigma^2_T \mathbb{1}(e \in T).
\]

(17)

- The edge basis of a strategy \( \sigma^2 \in \Delta(A_2) \), denoted \( E_{\sigma^2} \), is the set of edges with non-zero attack probability, i.e.:

\[
\forall \sigma^2 \in \Delta(A_2), \quad E_{\sigma^2} := \{ e \in E \mid \rho_{\sigma^2}(e) > 0 \}.
\]

Now, consider a set of nodes \( S \) of size at least \( b_1 \) and a set of edges \( T \) of size at least \( b_2 \). Lemma 4 below shows (by construction) the existence of a mixed strategy profile with the following properties: (a) \( \textbf{P1} \) randomizes the placement of her sensors over subsets of \( S \) of size \( b_1 \) such that each node of \( S \) is sensed with an identical probability; (b) \( \textbf{P2} \) randomizes the attack of \( b_2 \) edges in \( T \) such that each edge is targeted with an identical probability.

LEMMA 4. Consider a set of nodes \( S \in 2^V \) of size \( n \geq b_1 \), and a set of edges \( T \in 2^E \) of size \( m \geq b_2 \). Then, there exists a strategy profile, denoted \( (\sigma^1(S,b_1),\sigma^2(T,b_2)) \in \Delta(A_1) \times \Delta(A_2) \), whose node basis and edge basis are \( S \) and \( T \) respectively, and such that:

\[
\forall i \in S, \quad \rho_{\sigma^1(S,b_1)}(i) = \frac{b_1}{n},
\]

(18)

\[
\forall e \in T, \quad \rho_{\sigma^2(T,b_2)}(e) = \frac{b_2}{m}.
\]

(19)

Here, \( \rho_{\sigma^1(S,b_1)} \) and \( \rho_{\sigma^2(T,b_2)} \) denote the node sensing and edge attack probabilities corresponding to the strategy profile \( (\sigma^1(S,b_1),\sigma^2(T,b_2)) \).

For details on the construction of \( (\sigma^1(S,b_1),\sigma^2(T,b_2)) \), we refer to Lemma EC.3. The main idea to construct the sensing strategy \( \sigma^1(S,b_1) \) is to “cycle” over size-\( b_1 \) subsets of \( S \), such that every node of \( S \) is sensed with an identical probability given by (18). Similarly, the attack strategy \( \sigma^2(T,b_2) \) can be constructed by cycling over size-\( b_2 \) subsets of \( T \) to ensure that (19) holds. For the example network in Section 3.2, such a construction is illustrated in Fig. 2 for \( S = \{i_3,i_4,i_6,i_8\} \)

\[
T = \{e_3,e_4,e_8\}, \quad b_1 = 3 \text{ and } b_2 = 2.
\]

One can easily check that each node in the set \( S \) is sensed with probability \( \frac{b_1}{|S|} = \frac{3}{4} \), and that each edge in the set \( T \) is targeted with probability \( \frac{b_2}{|T|} = \frac{2}{3} \).

Recall that if \( \textbf{P1} \) had the resources to operate at least \( n^* \) sensors (i.e., \( b_1 \geq n^* \)), a dominant strategy would be to choose a sensor placement of size \( n^* \) on an MSC. Thus, a natural question is to investigate whether MSCs can be used to construct \( \textbf{P1} \)'s sensing strategies when she has strictly less than \( n^* \) sensors (i.e., \( b_1 < n^* \)). In particular, in equilibrium of a game in which \( \textbf{P1} \) is restricted
to place her sensors on subsets of a set cover and all other aspects are identical to the game \( \Gamma \), can we provide a sensing strategy for \( P_1 \)?

Recall that in any equilibrium of \( \Gamma \), \( P_1 \)'s strategy is an optimal solution of \((LP_1)\). Therefore, for a given subset of nodes \( S' \), restricting \( P_1 \)'s actions to subsets of \( S' \) can be done by adding the constraint \( V_{\sigma_1} \subseteq S' \) to \((LP_1)\). Thus, the problem of finding \( P_1 \)'s strategies in equilibrium of the game when restricted to play over \( S' \) is given by the following linear program:

\[
\begin{align*}
(LP_{S'}) \quad & \max_{\{\sigma_1 \in \Delta(A_1) \mid V_{\sigma_1} \subseteq S'\}} \min_{T \in A_2} -U_2(\sigma_1, T).
\end{align*}
\]

Note that the optimal value of \((LP_{S'})\) is a lower bound on \( P_1 \)'s equilibrium payoff in the strategically equivalent game \( \tilde{\Gamma} \), which is given by the optimal value of \((LP_1)\). In fact, we can use the construction in Lemma 4 to solve \((LP_{S'})\) when \( S' \) is a minimal set cover (defined in Section 3.1):

**Proposition 1.** Given a minimal set cover \( S' \in 2^V \), the optimal value of \((LP_{S'})\) is \( b_2 \left( \frac{n}{|S'|} - 1 \right) \), and an optimal solution is given by \( \sigma_1(S', b_1) \), as introduced in Lemma 4.

From this proposition, we know that if \( P_1 \) were restricted to randomize over subsets of a minimal set cover \( S' \), then there exists an equilibrium in which \( P_1 \)'s sensing strategy is \( \sigma_1(S', b_1) \). Furthermore, for any minimal set cover \( S' \), \( b_2 \left( \frac{n}{|S'|} - 1 \right) \) is a lower bound on \( P_1 \)'s equilibrium payoff in the strategically equivalent zero-sum game \( \tilde{\Gamma} \). This lower bound can be maximized by decreasing the size of the minimal set cover. Since a minimal set cover of minimum cardinality is an MSC, the best lower bound on the optimal value of \((LP_1)\) that we can obtain from Prop. 1 is \( b_2 \left( \frac{n^*}{n^* - 1} - 1 \right) \), and this lower bound is achieved when \( P_1 \) is restricted to randomize the placement of her sensors over subsets of an MSC. Note that if we instead consider a set of size \( n^* - 1 \), then this set is not a set cover and we cannot apply Prop. 1 anymore.

Next, we derive a similar lower bound on \( P_2 \)'s equilibrium payoff (notice that \( P_2 \)'s payoff is the same in \( \Gamma \) and \( \tilde{\Gamma} \)), by guessing a good subset of edges to target. Intuitively, if \( P_2 \) decides to target edges that are close to each other, then it is likely that \( P_1 \) will be able to detect most (if not all) of these failures with few sensors. However, if \( P_2 \) targets edges that are spread, then it will be difficult for \( P_1 \) to detect many of these failure events due to her resource constraints.

A good candidate for subset of edges that can be targeted by \( P_2 \) is a set packing. Recall that in our context, a set packing is a subset of edges such that a sensor placed at any node of the network can monitor at most one of such edges (cf. Section 3.1). Thus, by targeting a set packing, \( P_2 \) can ensure that at most one failure event can be detected with a single sensor placed by \( P_1 \). To study \( P_2 \)'s strategies in equilibrium of a game similar to \( \Gamma \) but where \( P_2 \) is restricted to randomize her
attacks over subsets of a set packing, we can restrict \( (LP_2) \)'s feasible set. Formally, given a subset of edges \( T' \), consider the following linear program:

\[
(LP_{T'}) \quad \max_{\sigma^2 \in \Delta(A_2)} \min_{S \in A_1} \sum U_2(S, \sigma)^2.
\]

Since \( (LP_{T'}) \) is a restriction of \( (LP_2) \), the optimal value of \( (LP_{T'}) \) is a lower bound on \( P2 \)'s equilibrium payoff. Again, we can use the construction in Lemma 4 to optimally solve \( (LP_{T'}) \) when \( T' \) is a set packing of size at least \( b_2 \):

**Proposition 2.** Given a set packing \( T' \in 2^E \) of size at least \( b_2 \), the optimal value of \( (LP_{T'}) \) is

\[
\max \left\{ 0, b_2 \left( 1 - \frac{b_1}{m^*} \right) \right\},
\]

and an optimal solution is given by \( \sigma^2(T', b_2) \), as introduced in Lemma 4.

From this proposition, we can derive analogous conclusions to Prop. 1. First, we conclude that if \( P2 \) is restricted to randomize over a set packing of size no less than her amount of resources \( b_2 \), then there exists an equilibrium where \( P2 \)'s attack strategy is \( \sigma^2(T', b_2) \). We also deduce that for any set packing \( T' \),

\[
\max \left\{ 0, b_2 \left( 1 - \frac{b_1}{m^*} \right) \right\}
\]

is a lower bound on \( P2 \)'s equilibrium payoff. This lower bound can be maximized by increasing the size of the set packing. Thus, the best lower bound on the optimal value of \( (LP_2) \) that we can obtain from Prop. 2 is \( \max \left\{ 0, b_2 \left( 1 - \frac{b_1}{m^*} \right) \right\} \), i.e., when \( P2 \) is restricted to randomize over subsets of an MSP (see (14)). Again, if we consider a set of edges of size \( m^* + 1 \), then this set is not a set packing and we cannot apply Prop. 2 anymore.

In Prop. 1 (resp. Prop. 2), we saw that if the strategies of \( P1 \) (resp. \( P2 \)) were restricted to randomizations over the subsets of a minimal set cover \( S' \) (resp. a set packing \( T' \) of size no less than \( b_2 \)), then there exists an equilibrium in the corresponding restricted game where \( P1 \)'s strategy is \( \sigma^1(S', b_1) \) (resp. \( P2 \)'s strategy is \( \sigma^2(T', b_2) \)), which randomizes over actions that use all available resources; see (EC.21) (resp. (EC.22)). In fact, we can extend this result and show that in any NE of the game \( \Gamma \) with \( b_1 < n^* \) and \( b_2 < m^* \), \( P1 \)'s strategy is to randomize over sensor placements of size exactly \( b_1 \), and \( P2 \)'s strategy is to randomize her attacks over sets of \( b_2 \) edges. The proof of this result relies on a best response argument and involves using Prop. 1, the features of the sensing function \( F \) (Lemma 1), and the properties of \( (I_{MSC}) \) and \( (I_{MSP}) \), including weak duality between them (Lemma 3); see Section EC.3.

**Proposition 3.** In equilibrium, both players randomize over actions that use all their resources:

\[
\forall (\sigma^{1*}, \sigma^{2*}) \in \Sigma, \forall S \in \text{supp}(\sigma^{1*}), \ |S| = b_1, \tag{20}
\]

\[
\forall (\sigma^{1*}, \sigma^{2*}) \in \Sigma, \forall T \in \text{supp}(\sigma^{2*}), \ |T| = b_2. \tag{21}
\]

Prop. 3 has both structural and computational advantages. Structurally, it shows an interesting property of sensing and attack strategies in any equilibrium of \( \Gamma \). \( P1 \) (resp. \( P2 \)) would prefer to
use all her resources in order to detect more of P2’s failure events (resp. to have more failures that are not detected by P1). Note, however, that this result does not hold when $b_1 \geq n^*$ or $b_2 > m^*$; see Section EC.6.

Computationally, Prop. 3 enables us to restrict the set of candidate strategy profiles that may be equilibria to the ones that satisfy (20)-(21), thus reducing the number of variables of (LP$_1$) and (LP$_2$). Moreover, we can also reduce the number of constraints in (LP$_1$) and (LP$_2$) since pure actions that do not use full resources are weakly dominated. The resulting reduced linear programs are presented in the next proposition:

**Proposition 4.** The NE of $\Gamma$ can be obtained by solving the following two linear programs:

\[
\begin{align*}
(LP_1) \quad & \max_{\sigma^1 \in \Delta(\mathcal{A}_1)} \min_{T \in \mathcal{A}_2} -U_2(\sigma^1, T) \\
(LP_2) \quad & \max_{\sigma^2 \in \Delta(\mathcal{A}_2)} \min_{S \in \mathcal{A}_1} U_2(S, \sigma^2)
\end{align*}
\]

where $\mathcal{A}_1 := \{S \in 2^V \mid |S| = b_1\}$ and $\mathcal{A}_2 := \{T \in 2^E \mid |T| = b_2\}$.

Thanks to Prop. 4, we managed to reduce the number of variables and constraints of (LP$_1$) (and the number of constraints and variables of (LP$_2$)) from $\sum_{k=0}^{b_1} \binom{|V|}{k}$ and $\sum_{l=0}^{b_2} \binom{|E|}{l}$ to $(|V|_{b_1})$ and $(|E|_{b_2})$ respectively. Although this result can improve the computation of NE for small-sized networks or when the number of resources of both players is large, we obtain only a slight improvement for medium and large networks. For example, if we consider again a network where $|V| = |E| = 200$ and $b_1 = b_2 = 10$, then the number of variables and constraints only drop from $2.37 \cdot 10^{16}$ for (LP$_1$) and (LP$_2$) to $2.25 \cdot 10^{16}$ for (LP$_1$) and (LP$_2$). Therefore, using (LP$_1$) and (LP$_2$) to compute the NE of $\Gamma$ is still not scalable. Hence, we continue our analysis to further characterize the set of NE.

Recall from Def. 2 that the node basis of a sensing strategy is the set of nodes that receive a sensor with positive probability. The next result gives a property of the node basis in equilibrium.

**Proposition 5.** In any NE $(\sigma^{1*}, \sigma^{2*}) \in \Sigma$, the node basis $\mathcal{V}_{\sigma^{1*}}$ is a set cover.

Equivalently, in any NE, P1’s sensing strategy monitors every edge with positive probability (no edges are completely unmonitored). Although this result is intuitive, the proof is quite involved and requires checking if a strategy whose node basis is not a set cover does not satisfy the equilibrium conditions. Showing this entails using all the previously shown results; see Section EC.3.

Prop. 5 is a practically relevant result. Consider, for example, the setting where the pure strategies of P1 consist of placing and operating a set of sensors on a subset of locations (nodes) of the network that need to be initially prepared to receive sensors. The node basis of an equilibrium strategy would give the locations that need to be prepared, and the probability distribution $\sigma^1$ can be interpreted as the random scheduling of $b_1$ sensors on subsets of these locations. To minimize the number of locations that need to be prepared, we need to find an equilibrium where the node
basis of the sensing strategy is of minimum size. From Prop. 5, we can deduce that this number is at least \( n^* \), i.e., the size of an MSC.

Another implication of Prop. 5 is that there is no pure NE. Indeed, since the node basis in equilibrium is a set cover and since each sensor placement is of size at most \( b_1 < n^* \), \( P_1 \) must necessarily choose a randomized sensing strategy. In fact, we have the following (stronger) result:

**Proposition 6.** In equilibrium, both players must necessarily randomize their actions.

In other words, there is no equilibrium where at least one player chooses a pure strategy. The intuition is that, due to their resource constraints, both players need to randomize their actions in equilibrium to sense and attack across the network. This property does not hold when \( b_1 \geq n^* \) or \( b_2 \geq m^* \); see Section EC.6 for a counterexample.

We use the above-mentioned equilibrium properties to derive bounds on the players’ payoffs in any equilibrium of the game \( \Gamma \) in terms of both players’ resources, and the sizes of MSCs and MSPs. This leads to bounds on the expected detection rate in equilibrium, which is crucial for addressing the equilibrium constraints (11) in (\( P \)).

**Theorem 1.** For a given network \( \mathcal{G} \), a sensing model \( \{ \mathcal{C}_i, i \in \mathcal{V} \} \), and the players’ resources \( b_1 < n^* \) and \( b_2 < m^* \), the game \( \Gamma \) has the following properties:

(i) The equilibrium payoffs of both players are constant and bounded as follows:

\[
\forall (\sigma^{1*}, \sigma^{2*}) \in \Sigma, \quad \begin{cases} 
\frac{b_1 b_2}{n^*} \leq U_1(\sigma^{1*}, \sigma^{2*}) \leq \min \left\{ \frac{b_1 b_2}{m^*}, b_2 \right\}, \\
\max \left\{ 0, b_2 \left( 1 - \frac{b_1}{m^*} \right) \right\} \leq U_2(\sigma^{1*}, \sigma^{2*}) \leq b_2 \left( 1 - \frac{b_1}{n^*} \right).
\end{cases}
\]

(ii) In any equilibrium, the expected detection rate is constant and bounded as follows:

\[
\forall \sigma^* \in \Sigma, \quad \frac{b_1}{n^*} \leq \bar{r}(\sigma^*) \leq \min \left\{ \frac{b_1}{m^*}, 1 \right\}.
\]

Firstly, we note that the lower and upper bounds on \( P_1 \)’s equilibrium payoff are nondecreasing with respect to \( b_1 \) and \( b_2 \). The intuition is that the more sensing resources \( P_1 \) has, the more failure events she will be able to detect. Also, the more attack resources \( P_2 \) has, the more edges she will target (due to Prop. 3), resulting in more detections. From Prop. 5, we know that each edge is monitored with a positive probability in equilibrium, which implies that each additional edge failure will be detected with positive probability and will increase \( P_1 \)’s payoff. Secondly, these bounds are also nonincreasing with respect to \( n^* \) and \( m^* \). Indeed, as the network size becomes larger, both \( n^* \) and \( m^* \) increase because each monitoring set covers a smaller fraction of the network. Thus, it will be more difficult for \( P_1 \) to detect failure events (with the same amount of resources), reducing her payoff. Similar conclusions can be drawn regarding the bounds on \( P_2 \)’s equilibrium payoff.
Finally, the bounds on the expected detection rate in equilibrium are nondecreasing with $P1$’s sensing resources (because she can detect more failures), and are nonincreasing with respect to $n^*$ and $m^*$ (because $P2$ can spread her attacks further apart). Importantly, note that these bounds do not depend on the attack resources $b_2$.

Next, we use Thm. 1 and the results on equilibrium properties to construct an approximate NE (as defined in (8)-(9)) of the game $\Gamma$ based on an MSC and an MSP.

**Theorem 2.** Consider a network $\mathcal{G}$, a sensing model $\{C_i, i \in V\}$, and the players’ resources $b_1 < n^*$ and $b_2 < m^*$. Then, for any MSC $S^{min} \in \mathcal{S}$ and any MSP $T^{max} \in \mathcal{M}$, the strategy profile $(\sigma^1(S^{min}, b_1), \sigma^2(T^{max}, b_2))$, as introduced in Lemma 4, satisfies the following properties:

1. $(\sigma^1(S^{min}, b_1), \sigma^2(T^{max}, b_2)) \in \Sigma_1(b_1, b_2)$,
2. $|U_i(\sigma^1(S^{min}, b_1), \sigma^2(T^{max}, b_2)) - U_i(\sigma^{1*}, \sigma^{2*})| \leq \epsilon, \forall (\sigma^{1*}, \sigma^{2*}) \in \Sigma(b_1, b_2), \forall i \in \{1, 2\}$,
3. $r(\sigma^1(S^{min}, b_1), \sigma^2(T^{max}, b_2)) \geq \frac{b_1}{n^*}$,

where $\epsilon = b_1 b_2 \left(\frac{1}{\max\{b_1, m^*\}} - \frac{1}{n^*}\right)$.

From (i) in Thm. 2, we see that by computing an MSC and an MSP, we can construct an $\epsilon$–NE where $\epsilon$ depends on the sizes of the players’ resources and on the sizes of MSCs and MSPs. Furthermore, from (ii), we conclude that this $\epsilon$–NE has the additional property that it provides both players payoffs that are $\epsilon$–close to their respective equilibrium payoffs. We emphasize that this is not a generic property of $\epsilon$–NE, even for zero-sum games. Essentially, this property is a consequence of Props. 1-3. Finally, from (iii), we see that the expected detection rate of $(\sigma^1(S^{min}, b_1), \sigma^2(T^{max}, b_2))$ is guaranteed to be at least $\frac{b_1}{n^*}$. In fact, we can obtain a stronger performance guarantee for the sensing strategy $\sigma^1(S^{min}, b_1)$, as shown in our next result:

**Proposition 7.** Given an MSC $S^{min} \in \mathcal{S}$, the expected detection rate by allocating $b_1$ sensing resources according to $\sigma^1(S^{min}, b_1)$ provides the following detection guarantee:

$$\min_{\sigma^2 \in \Delta(\mathcal{A}_2)} r(\sigma^1(S^{min}, b_1), \sigma^2) = \frac{b_1}{n^*} \geq \frac{\max\{b_1, m^*\}}{n^*} r(\sigma^{1*}, \sigma^{2*}), \forall (\sigma^{1*}, \sigma^{2*}) \in \Sigma(b_1, b_2).$$

Thus, by using $\sigma^1(S^{min}, b_1)$ as sensing strategy, we can guarantee that the expected detection rate is at least $\frac{b_1}{n^*}$, regardless of the attack strategy chosen by $P2$. This result is interesting from the viewpoint of robustness of sensors placements, that is, if the operator wants to ensure that the expected detection rate is at least $\frac{b_1}{n^*}$ even in the worst-case scenario, then she can guarantee this performance by choosing $\sigma^1(S^{min}, b_1)$ as her sensing strategy. In fact, the worst detection performance that $P1$ achieves by choosing $\sigma^1(S^{min}, b_1)$ is close to the expected detection rate in equilibrium. In particular, it is no less than the fraction $\frac{\max\{b_1, m^*\}}{n^*}$ of the expected equilibrium detection rate.
Finally, note that when \( n^* \) and \( m^* \) become close to each other (or equivalently, as the duality gap between \( (I_{MSC}) \) and \( (I_{MSP}) \) decreases), the gap between the upper and lower bounds in Thm. 1 also becomes narrow, and \((\sigma^1(S^{\text{min}}, b_1), \sigma^2(T^{\text{max}}, b_2))\) in Thm. 2 becomes closer to a NE. Next, we further specialize these results to the case when \( n^* = m^* \).

### 4.2. Special Case: \( n^* = m^* \)

Recall that in general, we have \( n^* \leq m^* \). Now, we refine the results given in Thms. 1 and 2, and derive additional results on the structure of the NE of the game \( \Gamma \), when \( b_1 < n^* \), \( b_2 < m^* \), and \( n^* = m^* \).

First, by examining Thm. 1, we notice that for \( n^* = m^* \), the upper and lower bounds that we obtained for both players’ payoffs and the expected detection rate in equilibrium become equal. This observation is documented in the following corollary.

**Corollary 1 (of Theorem 1).** Consider a network \( \mathcal{G} \), a sensing model \( \{C_i, i \in \mathcal{V}\} \), and the players’ resources \( b_1 < n^* \) and \( b_2 < m^* \). If \( n^* = m^* \), then the game \( \Gamma \) has the following properties:

(i) The equilibrium payoffs of both players are constant and are given by:

\[
\forall (\sigma^1, \sigma^2) \in \Sigma, \quad \begin{cases} 
U_1(\sigma^1, \sigma^2) = b_1 b_2 n^*, \\
U_2(\sigma^1, \sigma^2) = b_2 \left(1 - \frac{b_1}{n^*}\right).
\end{cases}
\] (22)

(ii) In any equilibrium, the expected detection rate is constant and is given by:

\[
\forall \sigma^* \in \Sigma, \quad r(\sigma^*) = \frac{b_1}{n^*}.
\] (23)

Secondly, we can refine the results in Thm. 2. Indeed, it is easy to see that \( \epsilon \) given in Thm. 2 is equal to 0 when \( n^* = m^* \), which implies that the strategy profile defined in Lemma 4 is a NE in this case. In fact, we can use this result to provide a sufficient and necessary condition for a strategy profile constructed over an MSC and MSP to be a NE.

**Proposition 8.** If \( n^* = m^* \), \( b_1 < n^* \), and \( b_2 < m^* \), then for any MSC \( S^{\text{min}} \in \mathcal{S} \) and any MSP \( T^{\text{max}} \in \mathcal{M} \), a strategy profile \((\sigma^1, \sigma^2) \in \Delta(\mathcal{A}_1) \times \Delta(\mathcal{A}_2)\) whose node basis is \( S^{\text{min}} \) and whose edge basis is \( T^{\text{max}} \) is a NE if and only if:

\[
\forall i \in S^{\text{min}}, \quad \rho_{\sigma^1}(i) = \frac{b_1}{n^*}, \quad \text{and} \quad \forall e \in T^{\text{max}}, \quad \rho_{\sigma^2}(e) = \frac{b_2}{n^*}.
\] (24)

In other words, a strategy profile in which \( P1 \) and \( P2 \) randomize over subsets of an MSC and an MSP is a NE if and only if each node of the MSC is sensed with an identical probability, and each edge of the MSP is targeted with an identical probability, given by (24). Thus, from this proposition,
we obtained a complete characterization of the NE that are based on an MSC and an MSP. Lemma EC.3 provides an explicit construction of such a strategy profile \((σ^1(S^{\text{min}}, b_1), σ^2(T^{\text{max}}, b_2))\).

Additionally, we can derive another interesting result from Prop 8. Recall that there are settings where it is of interest to minimize the number of network locations that need to be prepared to receive a sensor (represented in our model by the node basis defined in (16)). From Prop. 5, we deduced that at least \(n^*\) locations are needed, since the node basis in equilibrium must be a set cover. Thanks to Prop. 8, we showed that, when \(n^* = m^*\), there exists a sensing strategy in equilibrium whose node basis is an MSC. Therefore, in this case, we can conclude that the minimum number of locations that need to be prepared to receive a sensor in equilibrium is exactly \(n^*\); these locations form an MSC, \(S^{\text{min}}\), and can be sensed according to the strategy \(σ^1(S^{\text{min}}, b_1)\).

We end our analysis of the case \(n^* = m^*\) with a further characterization of the support of the NE of \(Γ\). In particular, using the MSCs and the MSPs, we derive necessary conditions that are satisfied by every NE.

**Proposition 9.** If \(b_1 < n^*, b_2 < m^*,\) and \(n^* = m^*\), then the set of NE has the following properties:

(i) In any NE, a node receives a sensor with positive probability only if it monitors exactly one edge of any MSP:

\[
\forall (σ^1, σ^2) \in Σ, \forall i \in \mathcal{V}_i^*, \forall T^{\text{max}} \in \mathcal{M}, \ F(i, T^{\text{max}}) = 1.
\]

Furthermore, in any NE, a sensor placement is chosen with positive probability only if each of its sensors monitors a different edge of any MSP:

\[
\forall (σ^1, σ^2) \in Σ, \forall S \in \text{supp}(σ^1), \forall T^{\text{max}} \in \mathcal{M}, \forall e \in T^{\text{max}}, \ F(S, e) = \sum_{i \in S} F(i, e).
\]

(ii) In any NE, an edge is targeted with positive probability only if it is monitored by a unique node of any MSC:

\[
\forall (σ^1, σ^2) \in Σ, \forall e \in \mathcal{E}_{σ^1}, \forall S^{\text{min}} \in \mathcal{S}, \exists ! i \in S^{\text{min}} \ | \ F(i, e) = 1.
\]

In other words, if there exists at least one MSP, \(T^{\text{max}}\), such that a node \(i \in \mathcal{V}\) does not monitor any edge in \(T^{\text{max}}\), then \(i\) will never receive a sensor in equilibrium. Besides, Prop. 9 tells us that if there exists an edge \(e \in \mathcal{E}\) that is part of an MSP and such that at least two sensors in a sensor placement \(S\) monitor \(e\), then \(S\) will never be chosen in equilibrium; since \(P2\) targets edges that are spread across the network, \(P1\) must allocate her sensing resources to avoid redundant detections.

Similarly, if there exists at least one MSC, \(S^{\text{min}}\), such that an edge \(e\) is monitored by at least two nodes in \(S^{\text{min}}\), then \(e\) will never be targeted in equilibrium.
5. Network Monitoring Problem

In the previous section, we studied the game $\Gamma$ for any set of parameters $b_1$ and $b_2$ with the assumption that $b_2 < m^*$. We derived properties satisfied by all equilibrium sensing and attack strategies, and it enabled us to give a characterization of the expected detection rate in any equilibrium (Thm. 1 and Cor. 1). Furthermore, we showed how optimal solutions of ($I_{MSC}$) and ($I_{MSP}$) can be used to construct an (approximate) NE of $\Gamma$ (Thm. 2 and Prop. 8). Next, we use these results to derive an (approximate) solution of the network monitoring problem ($P$).

5.1. (Approximate) Solution

Let us first consider the special case when the network and the sensing model satisfy $n^* = m^*$. From (23), we can easily conclude that the minimum number of sensors that are needed for the expected detection rate to be at least $\alpha$ in equilibrium is $b_1^* = \lceil \alpha n^* \rceil$. Besides, for an MSC $S^{\min}$ and an MSP $T^{\max}$, we know from Prop. 8 that $(\sigma^1(S^{\min}, b_1^*), \sigma^2(T^{\max}, b_2))$ is a NE of the game $\Gamma$ induced by $b_1^*$ and $b_2$. Therefore, when $b_2 < m^*$ and $n^* = m^*$, an optimal solution of the network monitoring problem ($P$) can be obtained by solving the MSC and MSP problems. The results are summarized in the following proposition:

**Proposition 10.** Consider a network $G$ and sensing model $\{C_i, i \in V\}$ that satisfy $n^* = m^*$, a target detection rate $\alpha \in [0, 1]$, and $P_2$’s resources $b_2 < m^*$. Then, for any MSC $S^{\min} \in S$ and any MSP $T^{\max} \in M$, an optimal solution of ($P$) is given by $\lceil \alpha n^* \rceil$, $(\sigma^1(S^{\min}, \lceil \alpha n^* \rceil), \sigma^2(T^{\max}, b_2))$.

Interestingly, note that the optimal number of sensors $\lceil \alpha n^* \rceil$ does not depend on $P_2$’s resources $b_2$. This property can be intuitively explained as follows: Given $P_1$’s strategy, $P_2$’s best response is to target the edges that are monitored with the smallest probability. Therefore, $P_1$ can maximize the probability of monitoring these edges by randomizing over an MSC $S^{\min}$ according to $\sigma^1(S^{\min}, b_1)$. The monitoring probability is then identical across the least monitored edges (there are at least $m^*$ of them), and is equal to $\frac{b_1}{n^*}$, which is a consequence of (18). This explains why the expected detection rate in equilibrium (and the optimal number of sensors $b_1^*$) do not depend on $b_2$. Note however that this argument holds because the network is large in comparison to $P_2$’s resources, i.e., $b_2 < m^*$, and counterexamples can be found when $b_2 > m^*$; see Section EC.6.

We now consider the network monitoring problem ($P$) in the general case $m^* \leq n^*$, and address the following question: Can we extend the argument behind the optimality of the solution $\lceil \alpha n^* \rceil$, $(\sigma^1(S^{\min}, \lceil \alpha n^* \rceil), \sigma^2(T^{\max}, b_2))$ for the case when $n^* = m^*$ to obtain an approximate solution of ($P$) in the general case? To address this question, we admit $\epsilon$–NE as the equilibrium concept (instead of NE) and consider a relaxed version of ($P$) in which the constraint (12) is replaced by $\sigma^1 \in \Sigma_1(b_1, b_2)$, for some $\epsilon \geq 0$. Our next result shows that this relaxed problem, denoted ($P_\epsilon$), is approximately solvable using our approach.
PROPOSITION 11. Consider a network $G$, a sensing model $\{C_i, i \in V\}$, a target detection performance $\alpha \in [0,1]$, and $P_2$’s resources $b_2 < m^*$. Then, for any MSC $S^{min} \in S$ and any MSP $T^{max} \in \mathcal{M}$, $b'_1 := \lceil \alpha^* \rceil$ and $(\sigma^1(S^{min}, b'_1), \sigma^2(T^{max}, b_2))$ is an approximate solution of $(P_1)$, where $\epsilon = b'_1 b_2 \left( \frac{1}{\max(b'_1, m^*)} - \frac{1}{n^*} \right)$, with an optimality gap given by $[\alpha^*] - \lfloor \alpha^* \rfloor$.

This result follows from Thms. 1 and 2. Indeed, with $[\alpha^*]$ sensors, the lower bound on the expected detection rate in equilibrium of the induced game (given in Thm. 1) ensures that the equilibrium constraints (11) are satisfied. Furthermore, the upper bound on the equilibrium expected detection rate in Thm. 1 implies that $P_1$ needs at least $\lfloor \alpha m^* \rfloor$ to satisfy (11). Consequently, the optimal value of $(P)$ satisfies $\lfloor \alpha m^* \rfloor \leq b'_1 \leq \lceil \alpha^* \rceil$, which gives the optimality gap in Prop. 11. Furthermore, from Thm. 2, we know that our strategy profile constructed over an MSC and an MSP (according to Lemma EC.3) is an $\epsilon$–NE. Thus, we obtain an approximate solution of the relaxed problem (P_2), using solutions of the MSC and MSP problems.

Furthermore, our solution approach has three main advantages which we discuss next. Firstly, it significantly reduces the size of the optimization problems that are involved in computing a solution. Indeed, recall that the number of variables and constraints of $(LP_1)$ (and the number of constraints and variables of $(LP_2)$) is equal to $\sum_{k=0}^{b_1} \binom{|V|}{k}$ and $\sum_{i=0}^{b_2} \binom{|E|}{i}$ respectively. In comparison, the number of variables and constraints of $(I_{MSC})$ (and the number of constraints and variables of $(I_{MSP})$), is only $|V|$ and $|E|$ respectively; see (EC.16) and (EC.17). For example, if we consider again a network where $|V| = |E| = 200$ and $b_1 = b_2 = 10$, then the number of variables and constraints drop from $2.37 \cdot 10^{16}$ for $(LP_1)$ and $(LP_2)$ to only 200 for $(I_{MSC})$ and $(I_{MSP})$. The downside is that our approach involves solving two integer programs. Still, as we show in Section 6, modern mixed-integer optimization solvers can be used to optimally solve these integer programs.

Secondly, our solution approach requires solving $(I_{MSC})$ and $(I_{MSP})$ only once, regardless of both players’ resources and the target expected detection rate. In comparison, recall from Section 2.3 that to find an optimal solution of $(P)$ only using $(LP_1)$ and $(LP_2)$, we would need to solve them for increasing values of $b_1$. Furthermore, any change in $b_2$ would result in solving $(LP_1)$ and $(LP_2)$ all over again.

Finally, in our approach, $P_1$ does not need to know the exact amount of attack resources to determine the number of sensing resources she needs and how to allocate them (captured by $b^*_1$ and the sensing strategy in $\sigma^1$). Indeed, our results show that with $[\alpha^*]$ sensors that are allocated according to $\sigma^1(S^{min}, [\alpha^*])$, $P_1$ is guaranteed to meet the target detection rate $\alpha$. Furthermore, as long as $b_2 < m^*$, the performance guarantees and the optimality gap associated with our solution can directly be computed from $n^*$ and $m^*$. Instead, if $P_1$ were using $(LP_1)$, she would need to know $b_2$. This makes our solution approach especially relevant in settings where the operator does not have an exact knowledge of the attacker’s resources.
5.2. Extension to Set Covers and Set Packings

Indeed, the above-mentioned results rely on the computation of an MSC and an MSP. Although, $(I_{MSC})$ and $(I_{MSP})$ are known to be NP-hard problems (Vazirani 2001), we will see in Section 6 that modern mixed-integer optimization solvers can be used to optimally solve them. Now, even if one is unable to solve $(I_{MSC})$ and $(I_{MSP})$ to optimality, we can still extend the results we derived based only on the computation of a set cover and a set packing. Indeed, using a heuristic or greedy algorithm (Chvatal 1979, Afif et al. 1995), one can compute a set cover $S'$. Then, Prop. 11 can be extended to conclude that with $\lceil \alpha |S'| \rceil$ sensors, the expected detection rate in any equilibrium of the induced game is at least $\alpha$. We can construct a robust sensing strategy $\sigma_1(S', \lceil \alpha |S'| \rceil)$, which ensures that the expected detection rate is at least $\alpha$, regardless of $P_2$’s strategy. Of course, in this case, $P_1$ ends up using more sensing resources than if she had been able to compute an MSC.

Similarly, a set packing $T'$ can be computed using a heuristic (Halldorsson et al. 2000, Hifi 1997). Again, by suitably extending Prop. 11, we obtain an optimality gap associated with $\lceil \alpha |S'| \rceil$ of $\lceil \alpha |S'| \rceil - \lceil \alpha |T'| \rceil$. Finally, if $T'$ is of size at least $b_2$, we can also conclude that $(\sigma_1(S', \lceil \alpha |S'| \rceil), \sigma_2(T', b_2))$ is an $\epsilon'$-NE, where $\epsilon' = \lceil \alpha |S'| \rceil b_2 \left( \frac{1}{\max\{\lceil \alpha |S'| \rceil, |T'| \}} - \frac{1}{|S'|} \right)$ and provides each player a payoff that is $\epsilon'$-close to their equilibrium payoff. Note that when the set cover is smaller, our solution is better. Also, when the set packing is bigger, the optimality gap is tighter, and $\epsilon'$ is smaller.

6. Computational Results

We now apply our results to the problem of sensing in large-scale water networks in the face of simultaneous adversary-induced pipe disruptions. Table 2 lists the characteristics (i.e., total pipe length, and number of nodes and edges) of each of the 13 benchmark networks considered in our study; see University of Exeter (2014), Jolly et al. (2014) for the complete data on these networks.  

To demonstrate our approach, we first compute the monitoring sets $C_i, i \in \mathcal{V}$ (defined in Section 2.1), which depend on the type of failure events (i.e., pipe bursts) and the type of sensors. Then, we solve $(I_{MSC})$ and $(I_{MSP})$, as introduced in Section 3.1, and use their solutions to derive an approximate solution of $(P)$ according to Prop. 11. In fact, for 6 out of the 13 networks, the equality $n^* = m^*$ is satisfied; thus we have an optimal solution of $(P)$ for these networks (Prop. 10).

Let us first discuss the computation of the monitoring sets $C_i, i \in \mathcal{V}$. Each pipe disruption (burst event) causes a disturbance in the local flow conditions, which moves through the network as a pressure wave (signal). The propagating signal dissipates with the distance from the source of the event due to friction losses and attenuation at network elements (e.g., valves). Hence, a burst event can be detected by monitoring the signal at a sensor location provided that the event lies within its detection range, i.e., the signal does not dissipate significantly before it reaches the sensor.
(Romano et al. 2012). For well-calibrated networks, the detection range (and hence $C_i$, $i \in V$) can be determined by simulating the individual events on a commercially available transient simulation software, e.g., HAMMER (Bentley 2015) or Pipe2016: Surge (Wood and Lingireddy 2016).

For the sake of simplicity, we adopt the following alternative to conducting transient simulations (Deshpande et al. 2013, Perelman et al. 2016): We compute $C_i$, $i \in V$ based on a practical observation that the typical detection range of modern pressure sensors is (at least) 1 km (Visenti 2007). We determine whether an event is detected by a sensor or not by checking if the length of the shortest path from the origin of the event to the sensed node is smaller (or greater) than 1 km. Specifically, we compute the shortest path between each pair of nodes using Floyd-Warshall algorithm (Floyd 1962) or a more sophisticated algorithm (e.g. Han (2004)). Then, we mark the edges that are within 1 km from each node, thus forming the monitoring sets $C_i$, $i \in V$.

Secondly, for each network, we formulate $(\mathcal{I}_{MSC})$ and $(\mathcal{I}_{MSP})$ as integer programs (see (EC.16)-(EC.17)). Although these problems are NP-hard, we find that mixed-integer programming solvers such as Gurobi (Gurobi Optimization 2014) are able to solve them to optimality for all the water networks we considered. The optimal values of both problems (i.e., $n^*$ and $m^*$), along with the running times, are summarized in Table 2. In the following discussion, we assume that an MSC $S^{min}$ and an MSP $T^{max}$ are computed, noting that our solution approach also applies to the case where just a set cover and a set packing are available; see Section 5.2.

Thirdly, given a target detection performance $\alpha \in [0, 1]$ and for the case when $P2$’s resources satisfy $b_2 < m^*$, we know (from Prop. 11) that $b_1' = \lceil \alpha n^* \rceil$ and $(\sigma^1(S^{min}, b_1'), \sigma^2(T^{max}, b_2))$ is an approximate solution of $(P)$. That is, with $b_1'$ sensors, the expected detection rate in any equilibrium of the game induced by the resources $b_1'$ and $b_2$ is at least $\alpha$, and this value is no worse than $\lceil \alpha n^* \rceil - \lceil \alpha m^* \rceil$ with respect to the optimal number of sensors; these results are summarized in Table 2. Furthermore, $(\sigma^1(S^{min}, b_1'), \sigma^2(T^{max}, b_2))$ is an $\epsilon$–NE and provides each player a payoff that is $\epsilon$–close to their respective equilibrium payoffs in the induced game.

Finally, we compute an upper bound on the loss in detection performance in choosing $\sigma^1(S^{min}, b_1')$ with respect to the performance in equilibrium of the game induced by $b_1'$ and $b_2$. From Prop. 7, the relative difference between the expected detection rate in equilibrium and if $P1$ chooses $\sigma^1(S^{min}, b_1')$ is upper bounded by $1 - \frac{\max(b_1', m^*)}{n^*}$. We compute this upper bound for each network and summarize the results in the last column of Table 2.

As mentioned earlier, the sizes of MSCs and MSPs are equal for 6 out of the 13 networks. This means that for these 6 networks, we can optimally solve $(P)$. In particular, we can conclude from Prop. 10 that given a target detection rate $\alpha$ and $P2$’s resources $b_2 < n^*$, the optimum number of sensors that are needed is given by $\lceil \alpha n^* \rceil$, and a NE is given by $(\sigma^1(S^{min}, \lceil \alpha n^* \rceil), \sigma^2(T^{max}, b_2))$. 
Table 2  Network data and computational results, $\alpha = 0.75$.

| Network | Total length | No. of pipes | No. of nodes | Running time (s) | $m^*$ | $n^*$ | Optimality gap | Relative loss of performance |
|---------|--------------|--------------|--------------|------------------|-------|-------|----------------|-----------------------------|
| bwsn1   | 37.56        | 168          | 126          | 0.05             | 7     | 7     | 0%             | 0%                          |
| ky3     | 91.29        | 366          | 269          | 0.01             | 15    | 15    | 0%             | 0%                          |
| ky5     | 96.58        | 496          | 420          | 0.02             | 18    | 19    | 1 (7.14%)      | 5.3%                        |
| ky7     | 137.05       | 603          | 481          | 0.09             | 28    | 28    | 0%             | 0%                          |
| ky6     | 123.20       | 644          | 543          | 0.08             | 24    | 24    | 0%             | 0%                          |
| ky1     | 166.60       | 907          | 791          | 0.03             | 31    | 31    | 0%             | 0%                          |
| ky13    | 153.30       | 940          | 778          | 0.06             | 28    | 30    | 2 (9.52%)      | 6.7%                        |
| ky2     | 152.25       | 1124         | 811          | 0.39             | 18    | 19    | 1 (7.14%)      | 5.3%                        |
| ky4     | 260.24       | 1156         | 959          | 0.03             | 62    | 64    | 1 (2.13%)      | 3.1%                        |
| ky8     | 247.34       | 1614         | 1325         | 0.14             | 45    | 45    | 0%             | 0%                          |
| dover   | 779.86       | 16000        | 14965        | 4.34             | 119   | 121   | 1 (1.11%)      | 1.7%                        |
| bswn2   | 1,844.04     | 14822        | 12523        | 0.77             | 352   | 361   | 7 (2.65%)      | 2.5%                        |
| mnsr    | 476.67       | 25484        | 24681        | 58.89            | 50    | 52    | 1 (2.63%)      | 3.8%                        |

For the remaining networks, note that the relative difference between $n^*$ and $m^*$ is still small. This implies that our estimate of the optimal value of $(P), b'_1 = \lceil an^* \rceil$, is close to the optimal value $b'_1$. Indeed, for these networks, when $\alpha = 0.75$, our optimality gap implies that our solution is not off by more than 1 or 2 sensors (except for BSWN2 for which the relative optimality gap is 2.65%).

If we look at the loss in detection performance by choosing $\sigma^1(S_{\text{min},b'_1})$ in comparison to the performance in equilibrium, we can see that this loss is small (2.7% in average over all networks). Indeed, even for the networks for which we do not have $n^* = m^*$, we obtain a loss in detection performance between 1.7% and 6.7%.

Finally, note that the time to solve the integer programming formulations of $(I_{\text{MSC}})$ and $(I_{\text{MSP}})$ is fairly small. For networks with less than 1500 nodes and edges, Gurobi solved them in less than half a second, which directly enables us to construct an (approximate) NE. Recall that even for such medium-sized networks, we cannot use $(LP_1)$ and $(LP_2)$ to compute equilibrium strategies. Besides, for larger networks, we can obtain $n^*$ and $m^*$ in about a minute. Therefore, the approach that we propose is scalable to large-scale networks, thanks to modern optimization solvers.

7. Conclusion

In this article, we studied the strategic monitoring of infrastructure networks aided by sensing technology to detect adversarial disruptions. Specifically, we considered the problem of finding the minimum number of sensors, and their optimal allocation (placement) strategy to ensure a desirable detection performance against a strategic attacker who targets multiple network components. We formulated this network monitoring problem as a mathematical program with equilibrium constraints. To address these constraints, we studied the properties of the equilibrium strategies...
of the underlying (large-scale) operator-attacker game, and developed a computationally scalable approach to evaluate the detection performance in equilibrium. In particular, our approach relies on solving two combinatorial problems that can be formulated as minimum set cover and maximum set packing problems. We used game-theoretic arguments, and combined them with combinatorial properties of the sensing model to derive structural insights on the sensing and attack strategies in equilibrium. This enabled us to provide a solution for the network monitoring problem, along with approximation guarantees. Interestingly, the operator can implement our solution approach and obtain the performance guarantees without an exact knowledge of the attacker’s resources, as long as the network is large enough, which is captured by the condition $b_2 < m^*$. Finally, we showed that our solution yields small (or zero) optimality gaps for a batch of large-scale water networks.

This work can be extended by considering weights in the payoff functions of the operator-attacker game. One possibility is to add weights to network components to reflect their criticality. Such an extension will enable us to model the operator’s preference when monitoring pipes with different diameters in the case of burst events in water or gas networks, or regions with different distributions of population in the case of water contamination events. Another possibility is to generalize the binary sensing model by adding a weight to each sensed node to represent the probability of detecting a failure occurring within its monitoring set. This would enable us to account for the imperfect detection of disruption events by sensors when designing network sensing strategies. Finally, the above-mentioned extensions can be combined to consider a weight for each pair of sensor location and network component. Such an extension can model the effect of the distance between a sensor and a failure event on the probability of detection, and enable us to account for the time between the injection of a contaminant and the time it is detected. This extended model can be used in various practical situations, including the one in which the operator (resp. attacker) is interested in minimizing (resp. maximizing) the detection time of a strategic attack.

**Endnotes**

1. Our framework can be applied to a much more general class of monitoring problems. We only require $V$ and $E$ to be discrete sets.

2. In Section 6, we consider a typical sensing model for pipe bursts monitoring where the sensors can detect bursts occurring within a certain geographical range. In this context, for a sensor at location $i$, the monitoring set $C_i$ consists of the edges within the sensor’s range.

3. Note that, from Lemma 2, we can view all our results in Section 4 from a robust optimization perspective. Indeed, our strategic game is equivalent to the problem where the operator is interested in minimizing the maximum number of failure events that remain undetected (see (LP1)). Thus,
an equilibrium sensing strategy in $\Gamma$ can be viewed as a minimaximizing solution of this equivalent robust optimization problem.

Adding the constraint $V_{\sigma_i} \subseteq S'$ is equivalent to setting $\sigma_1^S = 0$ for every sensor placement $S$ that contains a node outside of the set $S'$.

There exist examples where $n^* = m^*$ is satisfied. For instance, in the case where a sensor at node $i$ can only monitor the edges that are adjacent to it, MSCs and MSPs respectively become minimum vertex covers and maximum matchings. Thus, the equality $n^* = m^*$ is satisfied for this sensing model in any bipartite graph due to König’s Theorem (see König (1931)).

Interested readers may contact the authors for more information regarding the networks and the code for the computational results.

Acknowledgments
This work was supported by NSF grant CNS 1239054, NSF CAREER award CNS 1453126, and MIT Schoetlter Fellowship. We are grateful to Asuman E. Ozdaglar, David Simchi-Levi, Demosthenis Teneketzis, and Georgia Perakis for suggestions and encouragements.

References
Afif, Mohamed, Mhand Hifi, Vangelis Th. Paschos, Vassilis Zissimopoulos. 1995. A new efficient heuristic for the minimum set covering problem. The Journal of the Operational Research Society 46(10) 1260–1268.

Allen, M., A. Preis, M. Iqbal, S. Stitangarajan, H. N. Lim, L. Girod, A. J. Whittle. 2011. Real time in-network monitoring to improve operational efficiently. J. Am. Water Works Assoc. 103(7) 63–75.

Alpern, Steve, Alec Morton, Katerina Papadaki. 2011. Patrolling games. Operations Research 59(5) 1246–1257.

Baykal-Gursoy, Melike, Zhe Duan, H. Vincent Poor, Andrey Garnaev. 2014. Infrastructure security games. European Journal of Operational Research 239(2) 469 – 478.

Bentley, Hammer. 2015. Bentley, water hammer and transient analysis software, version v8i. http://www.bentley.com/en-US/Products/HAMMER/, Accessed: 2015-04-14.

Berry, J., W. Hart, C. Phillips, J. Uber, J. Watson. 2006. Sensor placement in municipal water networks with temporal integer programming models. Journal of Water Resources Planning and Management 132(4) 218–224.

Brown, Gerald, Matthew Carlyle, Javier Salmeron, Kevin Wood. 2006. Defending critical infrastructure. Interfaces 36(6) 530–544.

Chakrabarti, S., E. Kyriakides, D.G. Eliades. 2009. Placement of synchronized measurements for power system observability. Power Delivery, IEEE Transactions on 24(1) 12–19.

Chvatal, V. 1979. A greedy heuristic for the set-covering problem. Mathematics of Operations Research 4(3) 233–235.
Cormican, Kelly J., David P. Morton, R. Kevin Wood. 1998. Stochastic network interdiction. *Operations Research* **46**(2) 184–197.

Deshpande, Ajay, Sanjay E Sarma, Kamal Youcef-Toumi, Samir Mekid. 2013. Optimal coverage of an infrastructure network using sensors with distance-decaying sensing quality. *Automatica* **49**(11) 3351–3358.

Floyd, Robert W. 1962. Algorithm 97: Shortest path. *Commun. ACM* **5**(6) 345–.

Fudenberg, D., D.K. Levine. 1998. *The Theory of Learning in Games*. EBSCO eBook Collection, MIT Press.

Garnaev, A. 2000. *Search Games and Other Applications of Game Theory*. Lecture Notes in Economics and Mathematical Systems, Springer Berlin Heidelberg.

Garnaev, A., G. Garnaeva, P. Goutal. 1997. On the infiltration game. *International Journal of Game Theory* **26**(2) 215–221.

Gleick, Peter. 2006. Water and terrorism. *Water Policy* **8** 481–503.

Gurobi Optimization, Inc. 2014. Gurobi optimizer reference manual. [http://www.gurobi.com](http://www.gurobi.com), Accessed: 2015-04-14.

Halldrsson, Magns M., Jan Kratochvl, Jan Arne Telle. 2000. Independent sets with domination constraints. *Discrete Applied Mathematics* **99**(13) 39 – 54.

Han, Yijie. 2004. Improved algorithm for all pairs shortest paths. *Inf. Process. Lett.* **91**(5) 245–250.

Hifi, M. 1997. A genetic algorithm-based heuristic for solving the weighted maximum independent set and some equivalent problems. *The Journal of the Operational Research Society* **48**(6) 612–622.

Jolly, M. D., A. D. Lothes, S. Bryson, L. Ormsbee. 2014. Research database of water distribution system models. *Journal of Water Resources Planning and Management* **140**(4) 410–416.

Karlin, A.R., Y. Peres. 2016. *Game Theory, Alive*. American Mathematical Society.

König, D. 1931. Gráfok és mátrixok. *Matematikai és Fizikai Lapok* **38** 116–119.

Krause, Andreas, Brendan McMahan, Carlos Guestrin, Anupam Gupta. 2008a. Robust submodular observation selection. *Journal of Machine Learning Research (JMLR)* **9** 2761–2801.

Krause, Andreas, Ajit Singh, Carlos Guestrin. 2008b. Near-optimal sensor placements in gaussian processes: Theory, efficient algorithms and empirical studies. *J. Mach. Learn. Res.* **9** 235–284.

Kroll, D.J. 2006. *Securing Our Water Supply: Protecting a Vulnerable Resource*. PennWell Corporation.

Lipton, Richard J., Evangelos Markakis, Aranyak Mehta. 2003. Playing large games using simple strategies. *Proceedings of the 4th ACM Conference on Electronic Commerce*. EC ’03, ACM, New York, NY, 36–41.

Luo, Z.Q., J.S. Pang, D. Ralph. 1996. *Mathematical Programs with Equilibrium Constraints*. Cambridge University Press.
Mavronicolas, Marios, Vicky Papadopoulou, Anna Philippou, Paul Spirakis. 2008. A network game with attackers and a defender. *Algorithmica* 51(3) 315–341.

Nash, John F. 1950. Equilibrium points in n-Person games. *Proceedings of the National Academy of Sciences of the United States of America* 36(1) 48–49.

Nisan, Noam, Tim Roughgarden, Eva Tardos, Vijay V. Vazirani. 2007. *Algorithmic Game Theory*. Cambridge University Press, New York, NY, USA.

Orlin, James B., Andreas S. Schulz, Rajan Udwani. 2016. *Robust Monotone Submodular Function Maximization*. Springer International Publishing, Cham, 312–324.

Ostfeld, Avi, Elad Salomons. 2004. Optimal layout of early warning detection stations for water distribution systems security. *Journal of Water Resources Planning & Management* 130(5) 377 – 385.

Perelman, Lina Sela, Waseem Abbas, Xenofon Koutsoukos, Saurabh Amin. 2016. Sensor placement for fault location identification in water networks: A minimum test cover approach. *Automatica* 72 166 – 176.

Pita, James, Manish Jain, Janusz Marecki, Fernando Ordoñez, Christopher Portway, Milind Tambe, Craig Western, Praveen Paruchuri, Sarit Kraus. 2008. Deployed armor protection: The application of a game theoretic model for security at the los angeles international airport. *Proceedings of the 7th International Joint Conference on Autonomous Agents and Multiagent Systems: Industrial Track*. AAMAS ’08, International Foundation for Autonomous Agents and Multiagent Systems, Richland, SC, 125–132.

Porter, Ryan, Eugene Nudelman, Yoav Shoham. 2008. Simple search methods for finding a nash equilibrium. *Games and Economic Behavior* 63(2) 642 – 662. Second World Congress of the Game Theory Society.

Romano, M., Z. Kapelan, D. A. Savic. 2012. Automated detection of pipe bursts and other events in water distribution systems. *Journal of Water Resources Planning and Management* 140(4) 457–467.

Russon, Mary-Ann. 2016. Hackers hijacking water treatment plant controls shows how easily civilians could be poisoned. *International Business Times*.

Smith, J Cole, Churlzu Lim. 2008. Algorithms for network interdiction and fortification games. *Pareto Optimality, Game Theory And Equilibria* 609–644.

Stoianov, Ivan, Lama Nachman, Sam Madden, Timur Tokmouline. 2007. PIPENET a wireless sensor network for pipeline monitoring. *Proceedings of the 6th International Conference on Information Processing in Sensor Networks*. IPSN ’07, ACM, New York, NY, USA, 264–273.

University of Exeter. 2014. Centre for Water Systems. http://emps.exeter.ac.uk/engineering/research/cws/resources/benchmarks/design-resilience-pareto-fronts/data-files/, Accessed: 2014-10-24.

Vazirani, Vijay V. 2001. *Approximation Algorithms*. Springer-Verlag New York, Inc., New York, NY, USA.

Visenti. 2007. http://www.visenti.com/. Accessed: 2015-04-14.

Von Neumann, John. 1953. A certain zero-sum two-person game equivalent to the optimal assignment problem. *Contributions to the Theory of Games* 2 5–12.
Washburn, Alan, Kevin Wood. 1995. Two-person zero-sum games for network interdiction. *Operations Research* 43(2) 243–251.

Wood, Don J., Srini Lingireddy. 2016. Pipe2016: Surge, kypipe. [http://kypipe.com/surge/](http://kypipe.com/surge/).

Wright, Robert, Edo Abraham, Panos Parpas, Ivan Stoianov. 2015. Control of water distribution networks with dynamic dma topology using strictly feasible sequential convex programming. *Water Resources Research* 51(12) 9925–9941.

Wu, D., D. Chatzigeorgiou, K. Youcef-Toumi, S. Mekid, R. Ben-Mansour. 2014. Channel-aware relay node placement in wireless sensor networks for pipeline inspection. *IEEE Transactions on Wireless Communications* 13(7) 3510–3523.
Proofs of Statements

**EC.1. Preliminary Results**

**Lemma EC.1.** The monitoring sets satisfy the following properties:

\[ \forall (S, S') \in (2^V)^2, \quad C_{S\cup S'} = C_S \cup C_{S'}, \quad (\text{EC.1}) \]

\[ C_{S\cap S'} \subseteq C_S \cap C_{S'}, \quad (\text{EC.2}) \]

\[ S \subseteq S' \implies C_S \subseteq C_{S'}. \quad (\text{EC.3}) \]

**Proof of Lemma EC.1.**

i) Trivially, we have:

\[ \forall (S, S') \in (2^V)^2, \quad C_{S\cup S'} = \bigcup_{i \in S \cup S'} C_i = \left( \bigcup_{i \in S} C_i \right) \cup \left( \bigcup_{i \in S'} C_i \right) = C_S \cup C_{S'}. \]

ii) Now, consider two sensor placements \( S, S' \in 2^V \), and consider an edge \( e \in E \). If \( e \in C_{S\cap S'} = \bigcup_{i \in S \cap S'} C_i \), then \( \exists i \in S \cap S' \mid e \in C_i \). Since \( i \in S \cap S' \), we deduce that \( C_i \subseteq C_S \) and \( C_i \subseteq C_{S'} \).

Therefore, \( e \in C_S \cap C_{S'} \). Since this is true for any \( e \in C_{S\cap S'} \), we showed that \( C_{S\cap S'} \subseteq C_S \cap C_{S'} \).

iii) Consider \((S, S') \in (2^V)^2 \mid S \subseteq S'\). Then:

\[ C_S = \bigcup_{i \in S} C_i \subseteq \bigcup_{i \in S'} C_i = C_{S'}. \]

\[ \square \]

Note that the reversed inclusion \( C_{S\cap S'} \supseteq C_S \cap C_{S'} \) does not hold in general (which is the main reason why \( F(\cdot, T) \) is submodular and not modular). Next, we use the properties of the monitoring sets to derive the properties of the sensing function \( F \).

**Proof of Lemma 1.**

(i) Let us show the submodularity and the monotonicity of \( F \) with respect to the first variable.

Consider a subset of edges \( T \in 2^E \). Then, for any pair of sensor placements \((S, S') \in (2^V)^2\), we have:

\[
F(S \cup S', T) + F(S \cap S', T) \overset{(1)}{=} |C_{S\cup S'} \cap T| + |C_{S\cap S'} \cap T| \\
= |(C_S \cup C_{S'}) \cap T| + |C_{S\cap S'} \cap T| \\
= |(C_S \cap T) \cup (C_{S'} \cap T)| + |C_{S\cap S'} \cap T| \\
= |C_S \cap T| + |C_{S'} \cap T| - |C_S \cap C_{S'} \cap T| + |C_{S\cap S'} \cap T| \\
\overset{(\text{EC.2})}{\leq} |C_S \cap T| + |C_{S'} \cap T| \\
\overset{(1)}{=} F(S, T) + F(S', T). \quad (\text{EC.4})
\]
Furthermore, if \( S \subseteq S' \), then:

\[
F(S, T) \overset{(1)}{=} |C_S \cap T| \overset{(EC.3)}{\leq} |C_{S'} \cap T| \overset{(1)}{=} F(S', T).
\] (EC.5)

(ii) Now, let us show the additivity of \( F \) with respect to the second variable. Consider a sensor placement \( S \in 2^V \). Then:

\[
\forall (T, T') \in (2^E)^2 \mid T \cap T' = \emptyset , \quad F(S, T \cup T') \overset{(1)}{=} |C_S \cap (T \cup T')| = |(C_S \cap T) \cup (C_S \cap T')| = |C_S \cap T| + |C_S \cap T'| - |C_S \cap T \cap T'| \overset{(1)}{=} F(S, T) + F(S, T').
\] (EC.6)

We can now derive additional properties of the sensing function that will be used in our analysis.

**Corollary EC.1.** The sensing function defined in (1) satisfies the following properties:

\[
\forall (S, T) \in 2^V \times 2^E, \quad F(S, T) \leq \sum_{i \in S} F(i, T),
\] (EC.7)

\[
\forall (S, T) \in 2^V \times 2^E, \quad F(S, T) = \sum_{e \in T} F(S, e).
\] (EC.8)

**Proof of Corollary EC.1.**

i) Consider \( T \in 2^E \). First, note that \( F(\cdot, T) \) is normalized, i.e., \( F(\emptyset, T) = |\cup_{i \in \emptyset} C_i \cap T| = 0 \). Besides, since \( F(\cdot, T) \) is submodular (Lemma 1), we have:

\[
\forall (S, S) \in (2^V)^2 \mid S \cap S = \emptyset , \quad F(S \cup S, T) = F(S \cup S, T) + F(S \cap S, T) \overset{(EC.4)}{\leq} F(S, T) + F(S', T).
\]

Therefore, by induction, we obtain:

\[
\forall S \in 2^V , \quad F(S, T) = F(\cup_{i \in S} \{i\}, T) \leq \sum_{i \in S} F(i, T).
\]

ii) Consider \( S \in 2^V \). Since \( F(S, \cdot) \) is additive (Lemma 1), we obtain by induction:

\[
\forall T \in 2^E , \quad F(S, T) = F(S, \cup_{e \in T} \{e\}) \overset{(EC.6)}{=} \sum_{e \in T} F(S, e).
\]

**Lemma EC.2.** Given a sensing strategy \( \sigma^1 \in \Delta(A_1) \), let \( \{i_1, \ldots, i_n\} \in 2^V \) denote a set that contains its node basis \( \mathcal{V}_{\sigma^1} \). After reordering the indices such that \( \rho_{\sigma^1}(i_1) \leq \cdots \leq \rho_{\sigma^1}(i_n) \), we have the following inequality:

\[
\forall b \in [1, n], \quad \sum_{k=1}^{b} \rho_{\sigma^1}(i_k) \leq \frac{b}{n} \mathbb{E}_{\sigma^1}[|S|].
\] (EC.9)
Similarly, given an attack strategy $\sigma^2 \in \Delta(A_2)$, let $\{e_1, \ldots, e_m\} \in 2^E$ denote a set that contains its edge basis $E_{\sigma^2}$. After reordering the indices such that $\rho_{\sigma^2}(e_1) \geq \cdots \geq \rho_{\sigma^2}(e_m)$, we have the following inequality:

$$\forall b \in [1, m], \sum_{i=1}^{b} \rho_{\sigma^2}(e_i) \geq \frac{b}{m} \mathbb{E}_{\sigma^2}[|T|]. \quad (EC.10)$$

**Proof of Lemma EC.2.**

i) Consider a sensing strategy $\sigma^1 \in \Delta(A_1)$ and a set of nodes $\{i_1, \ldots, i_n\} \in 2^V$ such that $\mathcal{V}_{\sigma^1} \subseteq \{i_1, \ldots, i_n\}$. Let us reorder the indices such that $\rho_{\sigma^1}(i_1) \leq \cdots \leq \rho_{\sigma^1}(i_n)$. Then, we want to show that $\forall b \in [1, n], \sum_{k=1}^{b} \rho_{\sigma^1}(i_k) \leq \frac{b}{n} \mathbb{E}_{\sigma^1}[|S|]$. Instead, let us assume the contrary, that is, $\exists b \in [1, n] \mid \sum_{k=1}^{b} \rho_{\sigma^1}(i_k) > \frac{b}{n} \mathbb{E}_{\sigma^1}[|S|]$.

First, we can deduce the following inequality:

$$\rho_{\sigma^1}(i_b) = \frac{1}{b} \sum_{k=1}^{b} \rho_{\sigma^1}(i_k) \geq \frac{1}{b} \sum_{k=1}^{b} \rho_{\sigma^1}(i_k) > \frac{b \mathbb{E}_{\sigma^1}[|S|]}{bn} = \frac{\mathbb{E}_{\sigma^1}[|S|]}{n}. \quad (EC.11)$$

Besides, since $\mathcal{V}_{\sigma^1} \subseteq \{i_1, \ldots, i_n\}$, then $\forall S \in \text{supp}(\sigma^1), S \subseteq \{i_1, \ldots, i_n\}$ and we have the following equality:

$$\forall S \in \text{supp}(\sigma^1), |S| = \sum_{k=1}^{n} 1_{\{i_k \in S\}}. \quad (EC.12)$$

Then, we obtain the following contradiction:

$$\mathbb{E}_{\sigma^1}[|S|] = \sum_{S \in A_1} \sigma^1_S |S| \quad (EC.12) \sum_{S \in A_1} \sigma^1_S \sum_{k=1}^{n} 1_{\{i_k \in S\}} = \sum_{k=1}^{n} \sum_{S \in A_1} \sigma^1_S 1_{\{i_k \in S\}}$$

$$\begin{align*}
&\overset{(15)}{=} \sum_{k=1}^{n} \rho_{\sigma^1}(i_k) = \sum_{k=1}^{b} \rho_{\sigma^1}(i_k) + \sum_{k=1}^{n} \rho_{\sigma^1}(i_k) \\
&> \frac{b \mathbb{E}_{\sigma^1}[|S|]}{n} + \sum_{k=1}^{n} \rho_{\sigma^1}(i_k) = \frac{b \mathbb{E}_{\sigma^1}[|S|]}{n} + (n - b) \rho_{\sigma^1}(i_b) \\
&> \frac{b \mathbb{E}_{\sigma^1}[|S|]}{n} + (n - b) \mathbb{E}_{\sigma^1}[|S|] \\
&= \mathbb{E}_{\sigma^1}[|S|].
\end{align*}$$

Therefore, $\forall b \in [1, n], \sum_{k=1}^{b} \rho_{\sigma^1}(i_k) \leq \frac{b}{n} \mathbb{E}_{\sigma^1}[|S|]$.

ii) Now, let us show that $\forall b \in [1, m], \sum_{i=1}^{b} \rho_{\sigma^2}(e_i) \geq \frac{b}{m} \mathbb{E}_{\sigma^2}[|T|]$. Instead, assume that $\exists b \in [1, m] \mid \sum_{i=1}^{b} \rho_{\sigma^2}(e_i) < \frac{b}{m} \mathbb{E}_{\sigma^2}[|T|]$. Similarly, we deduce the following inequality:

$$\rho_{\sigma^2}(e_b) = \frac{1}{b} \sum_{i=1}^{b} \rho_{\sigma^2}(e_i) \leq \frac{1}{b} \sum_{i=1}^{b} \rho_{\sigma^2}(e_i) < \frac{b \mathbb{E}_{\sigma^2}[|T|]}{bm} = \frac{\mathbb{E}_{\sigma^2}[|T|]}{m}. \quad (EC.13)$$
Moreover, since $\mathcal{E}_{\sigma^2} \subseteq \{e_1, \ldots, e_m\}$, we have:

$$\forall T \in \text{supp}(\sigma^2), \ |T| = \sum_{l=1}^{m} \mathbf{1}_{\{e_l \in T\}}.$$  \hfill (EC.14)

Then, we obtain the following contradiction:

$$\mathbb{E}_{\sigma^2}[|T|] = \sum_{T \in \mathcal{A}_2} \sigma^2_T |T| \overset{\text{(EC.14)}}{=} \sum_{T \in \mathcal{A}_2} \sigma^2_T \sum_{l=1}^{m} \mathbf{1}_{\{e_l \in T\}} = \sum_{l=1}^{m} \sum_{T \in \mathcal{A}_2} \sigma^2_T \mathbf{1}_{\{e_l \in T\}}$$

$$\overset{(17)}{=} \sum_{i=1}^{m} \rho_{\sigma^2}(e_i) = \sum_{i=1}^{b} \rho_{\sigma^2}(e_i) + \sum_{b+1}^{m} \rho_{\sigma^2}(e_i)$$

$$< \frac{b\mathbb{E}_{\sigma^2}[|T|]}{m} + \sum_{b+1}^{m} \rho_{\sigma^2}(e_i) = \frac{b\mathbb{E}_{\sigma^2}[|T|]}{m} + (m-b)\rho_{\sigma^2}(e_b)$$

$$\overset{\text{(EC.13)}}{<} \frac{b\mathbb{E}_{\sigma^2}[|T|]}{m} + (m-b)\frac{\mathbb{E}_{\sigma^2}[|T|]}{m} = \mathbb{E}_{\sigma^2}[|T|].$$

Therefore, $\forall b \in [1, m]$, $\sum_{i=1}^{b} \rho_{\sigma^2}(e_i) \geq \frac{b}{m} \mathbb{E}_{\sigma^2}[|T|]$. \hfill \Box

**Proof of Lemma 2.** Adding a term to $\textbf{P1}$’s payoff that only depends on $\textbf{P2}$’s action does not change the NE of the game. Thus, the following transformation preserves the set of NE:

$$\forall (S, T) \in \mathcal{A}_1 \times \mathcal{A}_2, \ u_1(S, T) = F(S, T) = F(S, T) = -u_2(S, T). \hfill (EC.15)$$

So $\Gamma$ and $\bar{\Gamma}$ are strategically equivalent, and have the same set of NE $\Sigma(b_1, b_2)$. \hfill \Box

**Proof of Lemma 3.** Consider an MSP $T^{\text{max}} = \{e_1, \ldots, e_m^{*}\} \in \mathcal{M}$ and an MSC $S^{\text{min}} = \{i_1, \ldots, i_n^{*}\} \in \mathcal{S}$. Then, we have the desired inequality:

$$m^{*} = \sum_{l=1}^{m^{*}} 1 \overset{(13)}{=} \sum_{l=1}^{m^{*}} F(S^{\text{min}}, e_l) \overset{\text{(EC.7)}}{\leq} \sum_{l=1}^{m^{*}} \sum_{k=1}^{n^{*}} F(i_k, e_l) = \sum_{k=1}^{n^{*}} \sum_{l=1}^{m^{*}} F(i_k, e_l) \overset{\text{(EC.8)}}{=} \sum_{k=1}^{n^{*}} F(i_k, T^{\text{max}}) \overset{(14)}{\leq} \sum_{k=1}^{n^{*}} 1 = n^{*}. \hfill \Box$$

**EC.2. Integer Programming Formulations**

To concisely represent the sensing model $\{C_i, i \in \mathcal{V}\}$ defined in Section 2.1, we define the influence matrix $\mathbf{F} = (F(i, e))_{(i, e) \in \mathcal{V} \times \mathcal{E}}$. $\mathbf{F}$ is a $|\mathcal{V}| \times |\mathcal{E}|$ binary matrix whose rows (resp. columns) are indexed by the nodes (resp. edges) of $\mathcal{G}$. Thus, a given row of $\mathbf{F}$ indicates the edges of $\mathcal{G}$ that a sensor
placed at the corresponding node monitors. Similarly, a given column of $F$ indicates the nodes of $G$ from where a sensor is capable of monitoring the corresponding edge.

Then, solving (I_MSC) is equivalent to selecting a subset of rows of the influence matrix $F$ of minimum cardinality such that each column of $F$ is covered, i.e., it has at least one 1 entry in the selected subset of rows. This problem can be formulated as the following integer program:

$$\begin{align*}
\text{minimize} & \quad 1^T_{|V|} x \\
\text{subject to} & \quad F^T x \geq 1_{|E|} \\
& \quad x \in \{0,1\}^{|V|},
\end{align*}$$  \tag{EC.16}

where $1_{|V|}$ (resp. $1_{|E|}$) represents the vector of length $|V|$ (resp. $|E|$) filled with ones.

Similarly, solving (I_MSP) is equivalent to selecting a subset of columns of $F$ of maximum cardinality such that each row of $F$ has at most one 1 entry in the selected subset of columns:

$$\begin{align*}
\text{maximize} & \quad 1^T_{|E|} y \\
\text{subject to} & \quad F y \leq 1_{|V|} \\
& \quad y \in \{0,1\}^{|E|}.
\end{align*}$$  \tag{EC.17}

From (EC.16) and (EC.17), it is easy to see that these integer programming formulations of (I_MSC) and (I_MSP) have linear programming relaxations that are dual of each other.

**Example EC.1.** Consider the example network in Section 3.2. Then, the corresponding influence matrix is given by:

$$
F = \begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1
\end{pmatrix}
$$

As mentioned in Section 3.2, a solution of (I_MSC) is given by $\{i_3,i_4,i_6,i_8\}$ and a solution of (I_MSP) is given by $\{e_3,e_4,e_8\}$. They are illustrated in Fig. EC.1.

**EC.3. Proofs of Section 4.1**

Before moving forward with the proofs, we introduce a few notations: Given two integers $(q,p) \in \mathbb{N}^* \times \mathbb{N}$, let $\tau_q(p) \in \mathbb{N}$ be the remainder of the Euclidean division of $p$ by $q$, and let $\mu(q,p) \in \mathbb{N}^*$ be the smallest positive integer such that there exists an integer, denoted $\nu(q,p) \in \mathbb{N}^*$, that satisfies:

$$\mu(q,p) \times p = \nu(q,p) \times q. \tag{EC.18}$$
Consider a set of nodes $S$ and complete with the first $b_1$ nodes in $S$ and define the first pure action (sensor placement) $S^1$. Then, we take the second $b_1$ nodes of $S$, and define the second sensor placement $S^2$, and so on, until there are $k < b_1$ nodes of $S$ remaining. If $k = 0$, i.e., there is no node remaining (meaning that the size of $S$ is divisible by $b_1$), then we stop. Otherwise, we take the remaining $k$ nodes, and complete with the first $b_1 - k$ nodes of $S$ to create another sensor placement. We repeat the procedure until $k = 0$. Once all these sensor placements are defined, $\sigma^1(S, b_1)$ is obtained by uniformly randomizing over them. A similar construction is obtained for $\sigma^2(T, b_2)$. The following lemma formally constructs this strategy profile.

**Lemma EC.3.** Consider a set of nodes $S = \{i_1, \ldots, i_n\} \in 2^V$ of size $n \geq b_1$, and a set of edges $T = \{e_1, \ldots, e_m\} \in 2^E$ of size $m \geq b_2$. We define the following pure actions:

\[
\forall k \in [1, \mu(n, b_1)], \quad S^k = \{i_{r_1((k-1)b_1+1)}, \ldots, i_{r_2(kb_1)}\}, \quad \text{EC.19}
\]

\[
\forall l \in [1, \mu(m, b_2)], \quad T^l = \{e_{r_3((l-1)b_2+1)}, \ldots, e_{r_4(mb_2)}\}, \quad \text{EC.20}
\]

and a strategy profile $(\sigma^1(S, b_1), \sigma^2(T, b_2)) \in \Delta(A_1) \times \Delta(A_2)$ supported over $\{S^1, \ldots, S^\mu(n, b_1)\}$ and $\{T^1, \ldots, T^\mu(m, b_2)\}$, where:

\[
\forall k \in [1, \mu(n, b_1)], \quad \sigma^1(S, b_1)_{S^k} := \frac{1}{\mu(n, b_1)}, \quad \text{EC.21}
\]

\[
\forall l \in [1, \mu(m, b_2)], \quad \sigma^2(T, b_2)_{T^l} := \frac{1}{\mu(m, b_2)}. \quad \text{EC.22}
\]

Then, the strategy profile $(\sigma^1(S, b_1), \sigma^2(T, b_2))$ has the following properties:
(i) Each node in \( S \) (resp. each edge in \( T \)) is sensed (resp. targeted) with an identical probability given by:

\[
\forall i \in S, \, \rho_{\sigma^1(S,b_1)}(i) = \frac{b_1}{n}, \quad \text{(EC.23)}
\]

\[
\forall e \in T, \, \rho_{\sigma^2(T,b_2)}(e) = \frac{b_2}{m}. \quad \text{(EC.24)}
\]

(ii) Each node in \( S \) (resp. each edge in \( T \)) belongs to \( \nu(n,b_1) \) actions (resp. \( \nu(m,b_2) \) actions) in the support of \( \sigma^1(S,b_1) \) (resp. \( \sigma^2(T,b_2) \)):

\[
\forall i \in S, \, |\{k \in [1,\mu(n,b_1)] \mid i \in S^k\}| = \nu(n,b_1), \quad \text{(EC.25)}
\]

\[
\forall e \in T, \, |\{l \in [1,\mu(m,b_2)] \mid e \in T^l\}| = \nu(m,b_2). \quad \text{(EC.26)}
\]

(iii) The following inequality is satisfied:

\[
\forall e \in \mathcal{E}, \, F(S,e) \leq \frac{1}{\nu(n,b_1)} \sum_{k=1}^{\mu(n,b_1)} F(S^k,e). \quad \text{(EC.27)}
\]

**Proof of Lemma EC.3.** We show the result for a set of nodes \( S \in 2^V \) of size \( n \geq b_1 \).

First we note, by construction, that each node \( i, l \in [1,n] \) belongs to the same number of sensor placements \( S^k, \, k \in [1,\mu(n,b_1)] \). Thus, we have:

\[
\mu(n,b_1)b_1 = \sum_{k=1}^{\mu(n,b_1)} b_1 = \sum_{k=1}^{\mu(n,b_1)} |S^k| = \sum_{k=1}^{\mu(n,b_1)} \sum_{i \in V \setminus S^k} \sum_{k=1}^{\mu(n,b_1)} \sum_{i \in S^k} 1_{\{i \in S^k\}}
\]

\[
= \sum_{i \in V} |\{k \in [1,\mu(n,b_1)] \mid i \in S^k\}|
\]

\[
= \sum_{i \in S} |\{k \in [1,\mu(n,b_1)] \mid i \in S^k\}|
\]

\[
= n \times |\{k \in [1,\mu(n,b_1)] \mid i \in S^k\}|, \quad \forall i \in S,
\]

which enables us to show (EC.25):

\[
\forall i \in S, \, |\{k \in [1,\mu(n,b_1)] \mid i \in S^k\}| = \frac{\mu(n,b_1)b_1}{n} = \frac{\mu(n,b_1)}{\nu(n,b_1)}. \quad \text{(EC.28)}
\]

In other words, every node in \( S \) is sensed by \( \nu(n,b_1) \) pure actions in the support of \( \sigma^1(S,b_1) \).

Thus, we can show (EC.23):

\[
\forall i \in S, \, \rho_{\sigma^1(S,b_1)}(i) = \sum_{S' \in A_2} \sigma^1(S,b_1)|S'\|1_{\{i \in S'\}} = \sum_{k=1}^{\mu(n,b_1)} \sigma^1(S,b_1)_{S^k} 1_{\{i \in S^k\}}
\]

\[
= \frac{1}{\mu(n,b_1)} \sum_{k=1}^{\mu(n,b_1)} 1_{\{i \in S^k\}} = \frac{1}{\mu(n,b_1)} |\{k \in [1,\mu(n,b_1)] \mid i \in S^k\}|
\]
An analogous proof can be applied to $T \in 2^E$ of size $m \geq b_2$ to show (EC.24) and (EC.26).

Finally, let us show (EC.27). Consider $e \in E$. If $F(S,e) = 1$, then $\exists i_0 \in S \mid F(i_0,e) = 1$. Since there are $\nu(n,b_1)$ sensor placements in $\{S^k, \ k \in [1,\mu(n,b_1)]\}$ that contain $i_0$, then $\frac{1}{\nu(n,b_1)} \sum_{k=1}^{\mu(n,b_1)} F(S^k,e) \geq 1 = F(S,e)$. If $F(S,e) = 0$, then $\forall i \in S, \ F(i,e) = 0$ and $\frac{1}{\nu(n,b_1)} \sum_{k=1}^{\mu(n,b_1)} F(S^k,e) = 0$.

Note that the inequality (EC.27) can also be derived from a property of the sensing function. Since $\forall T \in 2^E, F(\cdot,T)$ is submodular, monotone and nonnegative (Lemma 1), then it is fractionally subadditive. (EC.27) is then a direct application of this property combined with (EC.25).

We illustrate this lemma with an example:

**Example EC.2.** Consider a set of six nodes $S = \{i_1,\ldots,i_6\}$ and suppose that $P_1$ has four sensors ($b_1 = 4$). Let us construct $\sigma^1(S,b_1)$ in this case: First, we define the pure actions $S^k$, which are illustrated in Figure EC.2. In this case, we have $S^1 = \{i_1,i_2,i_3,i_4\}$, $S^2 = \{i_5,i_6,i_1,i_2\}$, and $S^3 = \{i_3,i_4,i_5,i_6\}$. There are 3 pure actions $S^k$ because $\mu(6,4) = 3$.

![Figure EC.2](image-url)  
**Figure EC.2** Support of $\sigma^1(S,b_1)$ when $S$ is composed of six nodes and $b_1 = 4$. Each action in the support of $\sigma^1(S,b_1)$ is in green.

Now, $\sigma^1(S,b_1)$ is obtained by uniformly randomizing over them (in this case, each pure action is assigned with probability $\frac{1}{3}$). It is easy to check that each node in $S$ belongs to $\nu(6,4) = 2$ actions in the support of $\sigma^1(S,b_1)$. This implies that each node is sensed with probability $\frac{\nu(6,4)}{\mu(6,4)} = \frac{2}{3} = \frac{b_1}{|S|}$.

**Proof of Proposition 1.** Consider a minimal set cover $S' = \{i_1,\ldots,i_n\} \in 2^V$ of size $n$. Then, necessarily, $S'$ is such that:

$$\forall k \in [1,n], \exists e_k \in E \mid F(i_k,e_k) = 1 \text{ and } F(i_j,e_k) = 0, \ \forall j \neq k. \quad (EC.29)$$
Indeed, if \( \exists k \in [1, n] \mid \forall e \in E : F(i_k, e) = 0 \) or \( \exists j \neq k \mid F(i_j, e) = 1 \), then any edge that is monitored by \( i_k \) is also monitored by another node of \( S' \). Therefore, removing \( i_k \) from \( S' \) will still produce a set cover, which contradicts \( S' \) being a minimal set cover.

First, let us show that \( \max_{\sigma^1 \in \Delta(A_1)} \min_{T \in A_2} -U_2(\sigma^1, T) \geq b_2 \left( \frac{b_1}{n} - 1 \right) \). Consider \( \sigma^1(S', b_1) \in \Delta(A_1) \) defined in (EC.21). Recall that \( \sigma^1(S', b_1) \) is such that \( V_{\sigma^1(S', b_1)} = S' \) and \( \forall k \in [1, n], \rho_{\sigma^1(S', b_1)}(i_k) = \frac{b_1}{n} \). Also, recall that we are in the case when \( b_1 < n^* \), implying that \( b_1 < n^* \leq n \).

Since \( S' \) is a set cover, then \( \forall e \in E, \exists k_e \in [1, n] \mid F(i_{k_e}, e) = 1 \). Then, by partitioning the set of actions \( A_1 \) into the set of sensor placements that contain \( i_{k_e} \) and the set of sensor placements that do not, we obtain:

\[
\forall e \in E, \quad \sum_{S \in A_1} \sigma^1(S', b_1)_S F(S, e) = \sum_{\{S \in A_1 \mid i_{k_e} \notin S\}} \sigma^1(S', b_1)_S F(S, e) + \sum_{\{S \in A_1 \mid i_{k_e} \in S\}} \sigma^1(S', b_1)_S F(S, e) \\
\geq \sum_{\{S \in A_1 \mid i_{k_e} \in S\}} \sigma^1(S', b_1)_S F(S, e) \\
\geq \sum_{\{S \in A_1 \mid i_{k_e} \in S\}} \sigma^1(S', b_1)_S F(i_{k_e}, e) \\
= \sum_{\{S \in A_1 \mid i_{k_e} \in S\}} \sigma^1(S', b_1)_S \\
= \frac{b_1}{n}. \tag{EC.30}
\]

Thus, we obtain:

\[
\forall T \in A_2, \quad -U_2(\sigma^1(S', b_1), T) = \sum_{S \in A_1} \sigma^1(S', b_1)_S F(S, T) - |T| \geq \sum_{e \in T} \sum_{S \in A_1} \sigma^1(S', b_1)_S F(S, e) - |T| \\
\geq \frac{b_1}{n} - |T| \\
= \left( \frac{b_1}{n} - 1 \right) |T| \\
\geq b_2 \left( \frac{b_1}{n} - 1 \right). \tag{EC.31}
\]

Therefore:

\[
\max_{\{\sigma^1 \in \Delta(A_1) \mid V_{\sigma^1} \subseteq S'\}} \min_{T \in A_2} -U_2(\sigma^1, T) \geq \min_{T \in A_2} -U_2(\sigma^1(S', b_1), T) \geq b_2 \left( \frac{b_1}{n} - 1 \right). \tag{EC.32}
\]

Note that the only property of \( \sigma^1(S', b_1) \) that was used to show (EC.32) is that its node basis is \( S' \) and that \( \forall i \in S' \), \( \rho_{\sigma^1(S', b_1)}(i) = \frac{b_1}{n} \).
Now, let us show the reverse inequality. Consider any $\sigma^1 \in \Delta(A_1) \mid V_{\sigma^1} \subseteq S' = \{i_1, \ldots, i_n\}$. Let us reorder the indices such that $\rho_{\sigma^1}(i_1) \leq \cdots \leq \rho_{\sigma^1}(i_n)$. Then, thanks to Lemma EC.2, we have $\sum_{k=1}^{b_2} \rho_{\sigma^1}(i_k) \leq \frac{b_2}{n} \mathbb{E}_{\sigma^1}[|S|]$.

Consider $T' = \{e_1, \ldots, e_{b_2}\}$ (where the $e_k$’s are defined in (EC.29)). Again, by partitioning $A_1$ into the set of sensor placements that contain $i_k$ and the set of sensor placements that do not, we obtain:

$$\forall k \in [1, b_2], \quad \sum_{S \in A_1} \sigma^1_S F(S, e_k) = \sum_{S \in A_1 \mid i_k \in S} \sigma^1_S F(S, e_k) + \sum_{S \in A_1 \mid i_k \notin S} \sigma^1_S F(S, e_k)$$

$$\leq \sum_{k=1}^{b_2} \rho_{\sigma^1}(i_k) - b_2 \quad \text{(EC.33)}$$

where we combined the fact that the node basis of $\sigma^1$ is a subset of $S'$ and that $i_k$ is the only node from $S'$ that monitors edge $e_k$ (by construction). This implies that:

$$-U_2(\sigma^1, T') = \sum_{S \in A_1} \sigma^1_S F(S, T') - |T'| \leq |T'| \quad \text{(EC.8)}$$

$$\leq \sum_{k=1}^{b_2} \rho_{\sigma^1}(i_k) - b_2 \quad \text{(EC.33)}$$

$$\leq \frac{b_2}{n} \mathbb{E}_{\sigma^1}[|S|] - b_2 \quad \text{(2)}$$

Thus, $\min_{T \in A_2} -U_2(\sigma^1, T) \leq b_2 \left( \frac{b_1}{n} - 1 \right)$. This upper bound is valid for any $\sigma^1 \in \Delta(A_1)$ such that $V_{\sigma^1} \subseteq S'$, and does not depend on $\sigma^1$. Therefore:

$$\max_{\{\sigma^1 \in \Delta(A_1) \mid V_{\sigma^1} \subseteq S'\}} \min_{T \in A_2} -U_2(\sigma^1, T) \leq b_2 \left( \frac{b_1}{n} - 1 \right). \quad \text{(EC.34)}$$

By combining (EC.32) and (EC.34), we can conclude that:

$$\max_{\{\sigma^1 \in \Delta(A_1) \mid V_{\sigma^1} \subseteq S'\}} \min_{T \in A_2} -U_2(\sigma^1, T) = \min_{T \in A_2} -U_2(\sigma^1(S', b_1), T) = b_2 \left( \frac{b_1}{n} - 1 \right).$$

The optimal value of $\text{(LP}_{S'})$ is $b_2 \left( \frac{b_1}{n} - 1 \right)$ and an optimal solution is given by $\sigma^1(S', b_1)$.

**Proof of Proposition 2.** Consider a set packing $T' = \{e_1, \ldots, e_m\} \in 2^E$ of size $m \geq b_2$. We separately consider the cases when $b_1 \geq m$ and $b_1 < m$.

**Case 1:** If $b_1 \geq m$, then $P_1$ can monitor all the edges of $T'$ with a single sensor placement $S'$. Therefore, no matter which strategy supported over $T'$ $P_2$ chooses, her attacks will always be detected by $S'$. Thus, the optimal value of $\text{(LP}_{T'})$ in this case is equal to $0 = \max\{0, b_2(1 - \frac{b_1}{m})\}$.

**Case 2:** Consider $b_1 < m$. In this case, note that $\max\{0, b_2(1 - \frac{b_1}{m})\} = b_2(1 - \frac{b_1}{m})$. 

– First, let us show that \( \max_{(\sigma^2 \in \Delta(A_2) | \mathcal{E}_2 \subseteq T')} \min_{S \in A_1} U_2(S, \sigma^2) \geq b_2 \left( 1 - \frac{b_1}{m} \right) \). Consider \( \sigma^2(T', b_2) \in \Delta(A_2) \) defined in (EC.22). Recall that \( \sigma^2(T', b_2) \) is such that its edge basis is \( \mathcal{E}_{\sigma^2(T', b_2)} = T' \) and \( \forall \ell \in [1, m] \), \( \rho_{\sigma^2(T', b_2)}(e_\ell) = \frac{b_2}{m} \).

Since \( T' \) is a set packing, then \( \forall i \in V \), \( F(i, T') \leq 1 \). This implies that:

\[
\forall S \in A_1, \quad U_2(S, \sigma^2(T', b_2)) \overset{\text{(5)}}{=} \sum_{T \in A_2} \sigma^2(T', b_2)_T (|T| - F(S, T)) \\
\overset{\text{(EC.7)}}{=} \sum_{T \in A_2} \sum_{T \subseteq A_2} \sigma^2(T', b_2)_T (|T| - \sum_{i \in S} F(i, T)) \\
= \sum_{T \in A_2} \sigma^2(T', b_2)_T \sum_{e \in T} (1 - \sum_{i \in S} F(i, e)) \\
= \sum_{T \in A_2} \sigma^2(T', b_2)_T \sum_{e \in T} (1 - \sum_{i \in S} F(i, e)) \mathbb{1}_{\{e \in T\}} \\
= \sum_{e \in T} (1 - \sum_{i \in S} F(i, e)) \sum_{T \in A_2} \sigma^2(T', b_2)_T \mathbb{1}_{\{e \in T\}} \\
\overset{\text{(17)}}{=} \sum_{e \in T} \sum_{i \in S} F(i, e) \rho_{\sigma^2(T', b_2)}(e) \\
\overset{\text{(EC.24)}}{=} \sum_{i=1}^{m} (1 - \sum_{i \in S} F(i, e_{\ell})) \frac{b_2}{m} \\
\overset{\text{(EC.8)}}{=} \sum_{i \in S} F(i, T') \\
\overset{\text{(2)}}{=} b_2 - \frac{b_2}{m} \sum_{i \in S} 1 \\
\geq b_2 - \frac{b_2}{m} \sum_{i \in S} 1 \\
\geq b_2 \left( 1 - \frac{b_1}{m} \right). 
\]

Therefore:

\[
\max_{(\sigma^2 \in \Delta(A_2) | \mathcal{E}_2 \subseteq T')} \min_{S \in A_1} U_2(S, \sigma^2) \geq \min_{S \in A_1} U_2(S, \sigma^2(T', b_2)) \geq b_2 \left( 1 - \frac{b_1}{m} \right). \tag{EC.35} 
\]

Similarly, note that the only property of \( \sigma^2(T', b_2) \) that was used to show (EC.35) is that its edge basis is \( T' \) and that \( \forall e \in T' \), \( \rho_{\sigma^2(T', b_2)}(e) = \frac{b_2}{m} \).

– Now, let us show the reverse inequality. Consider any \( \sigma^2 \in \Delta(A_2) | \mathcal{E}_2 \subseteq T' = \{ e_1, \ldots, e_m \} \). Let us reorder the indices such that \( \rho_{\sigma^2}(e_1) \geq \cdots \geq \rho_{\sigma^2}(e_m) \). Then, thanks to Lemma EC.2, we have \( \sum_{i=1}^{b_1} \rho_{\sigma^2}(e_i) \geq \frac{b_2}{m} E_{\sigma^2}[|T|] \).

In Section 2.2, we assumed that each edge can be monitored from at least one node. Therefore \( \forall \ell \in [1, m], \exists i_\ell \in V | F(i_\ell, e_\ell) = 1 \) (note that the \( i_\ell \)'s are not necessarily distinct). Now, consider the sensor placement \( S' = \{ i_1, \ldots, i_{b_1} \} \). \( S' \) monitors \( \{ e_1, \ldots, e_{b_1} \} \), which enables us to show:

\[
U_2(S', \sigma^2) \overset{\text{(5)}}{=} \sum_{T \in A_2} \sigma^2_T (|T| - F(S', T)) 
\]
\[ (EC.8) \quad \mathbb{E}_{\sigma^2}[|T|] - \sum_{T \in A_2} \sigma_T^2 \sum_{e \in T} F(S', e) \]
\[ = \mathbb{E}_{\sigma^2}[|T|] - \sum_{e \in \mathcal{E}} F(S', e) \sum_{T \in A_2} \sigma_T^2 \mathbbm{1}_{\{e \in T\}} \]
\[ = \mathbb{E}_{\sigma^2}[|T|] - \mathbb{E}_{\sigma^2}[|T|] - \sum_{e \in \mathcal{E}} F(S', e) \rho_{\sigma^2}(e) \]
\[ = \mathbb{E}_{\sigma^2}[|T|] - \sum_{e \in \mathcal{E}} F(S', e) \rho_{\sigma^2}(e) \]
\[ (EC.10) \quad \leq \mathbb{E}_{\sigma^2}[|T|] - \frac{b_1 \mathbb{E}_{\sigma^2}[|T|]}{m} \]
\[ = \left(1 - \frac{b_1}{m}\right) \mathbb{E}_{\sigma^2}[|T|] \]
\[ \leq \left(1 - \frac{b_1}{m}\right) b_2. \]

Thus, \( \min_{S \in A_1} U_2(S, \sigma^2) \leq b_2 \left(1 - \frac{b_1}{m}\right). \) This upper bound is valid for any \( \sigma^2 \in \Delta(A_2) \) such that \( \mathcal{E}_{\sigma^2} \subseteq T' \), and does not depend on \( \sigma^2 \). Thus, we deduce the following inequality:

\[
\max_{\{\sigma^2 \in \Delta(A_2) \mid \mathcal{E}_{\sigma^2} \subseteq T'\}} \min_{S \in A_1} U_2(S, \sigma^2) \leq b_2 \left(1 - \frac{b_1}{m}\right). \tag{EC.36}
\]

By combining (EC.35) and (EC.36), we can conclude in this case that:

\[
\max_{\{\sigma^2 \in \Delta(A_2) \mid \mathcal{E}_{\sigma^2} \subseteq T'\}} \min_{S \in A_1} U_2(S, \sigma^2) = \min_{S \in A_1} U_2(S, \sigma^2(T', b_2)) = b_2 \left(1 - \frac{b_1}{m}\right).
\]

The optimal value of (LP_{T'}) in this case \( b_1 < m \) is \( b_2 \left(1 - \frac{b_1}{m}\right) = \max\{0, b_2 \left(1 - \frac{b_1}{m}\right)\} \), and an optimal solution is given by \( \sigma^2(T', b_2) \).

\( \square \)

**Proof of Proposition 3.**

i) First, let us show by contradiction that \( \textbf{P1} \) uses all her resources in equilibrium. Suppose that \( \exists (\sigma^1, \sigma^2) \in \Sigma, \exists S_0 \in \text{supp}(\sigma^1^*) \mid |S_0| < b_1. \)

- The first step is to show that \( \textbf{P2}'s \) strategy \( \sigma^2 \) necessarily targets with positive probability at least one edge that is not monitored by \( S_0 \). On the contrary, assume that \( \forall e \in \mathcal{E} : \rho_{\sigma^2}(e) > 0 \implies F(S_0, e) = 1. \) Then, \( \textbf{P1} \) can detect all the attacks of \( \sigma^2 \) with the sensor.
placement $S^0$. Thus, $S^0$ is a best response for $P_1$ to $\sigma^2^*$, and $P_2$'s payoff in equilibrium is 0. Since $P_2$'s payoff is identical for any NE (direct consequence of Lemma 2), then $P_2$'s payoff for the initial NE we considered, $(\sigma^1^*, \sigma^2^*)$, is also 0. However, since $b_1 < n^*$, we know that $\exists e' \in \mathcal{E} \mid F(S^0, e') = 0$, i.e., $e'$ is not monitored by $S^0$. Since $S^0 \in \text{supp}(\sigma^1^*)$, then $e'$ is not monitored with positive probability. Therefore, if $P_2$ targets $e'$, she will get a positive payoff:

$$U_2(\sigma^1^*, e') \overset{(5)}{=} |\{e'\}| - \sum_{S \in A_1} \sigma^1_{S}^* F(S, e') = 1 - \sigma^1_{S^0}^* F(S^0, e') - \sum_{S \in A_1 \setminus \{S^0\}} \sigma^1_{S}^* F(S, e') \geq 1 - \sum_{S \in A_1 \setminus \{S^0\}} \sigma^1_{S}^*$$

$$= \sum_{S \in A_1} \sigma^1_{S}^* - \sum_{S \in A_1 \setminus \{S^0\}} \sigma^1_{S}^*$$

$$= \sigma^1_{S^0}^* > 0$$

$$> U_2(\sigma^1^*, \sigma^2^*),$$

which contradicts the equilibrium condition (7) for $(\sigma^1^*, \sigma^2^*)$. Therefore, $\exists e_0 \in \mathcal{E} \mid \rho_{\sigma^2^*}(e_0) > 0$ and $F(S^0, e_0) = 0$.

Now, we can show that $P_1$ can increase her payoff by placing one more sensor. Let us denote $i_0 \in \mathcal{V} \setminus S^0$ that satisfies $F(i_0, e_0) = 1$. Then, by considering the sensor placement $S' = S^0 \cup \{i_0\} \in A_1$, we obtain:

$$U_1(S', \sigma^2^*) \overset{(4)}{=} \sum_{T \in A_2} \sigma^2_{T} F(S', T) \overset{(EC.8)}{=} \sum_{T \in A_2} \sigma^2_{T} \sum_{e \in T} F(S', e) = \sum_{T \in A_2} \sigma^2_{T} \sum_{e \in \mathcal{E}} F(S', e) \mathbf{1}_{\{e \in T\}}$$

$$= \sum_{e \in \mathcal{E}} F(S', e) \sum_{T \in A_2} \sigma^2_{T} \mathbf{1}_{\{e \in T\}} \overset{(17)}{=} \sum_{e \in \mathcal{E}} F(S', e) \rho_{\sigma^2^*}(e)$$

$$= \rho_{\sigma^2^*}(e_0) + \sum_{e \in \mathcal{E} \setminus \{e_0\}} F(S', e) \rho_{\sigma^2^*}(e) \overset{(EC.5)}{\geq} \rho_{\sigma^2^*}(e_0) + \sum_{e \in \mathcal{E} \setminus \{e_0\}} F(S^0, e) \rho_{\sigma^2^*}(e) = \rho_{\sigma^2^*}(e_0) + \sum_{e \in \mathcal{E}} F(S^0, e) \rho_{\sigma^2^*}(e) \overset{(4)}{=} \rho_{\sigma^2^*}(e_0) + U_1(S^0, \sigma^2^*)$$

$$= \rho_{\sigma^2^*}(e_0) + U_1(S^0, \sigma^2^*) > U_1(S^0, \sigma^2^*),$$

which violates the equilibrium condition (6) for $(\sigma^1^*, \sigma^2^*)$. Therefore, $\forall S \in \text{supp}(\sigma^1^*), |S| = b_1$.

ii) Now, let us show that $P_2$ uses all her resources in equilibrium. By contradiction, suppose that $\exists (\sigma^1^*, \sigma^2^*) \in \Sigma$, $\exists T^0 \in \text{supp}(\sigma^2^*) \mid |T^0| < b_2$. 

The first step is to show that there exists an edge \( e' \) not in \( T^0 \) that is not monitored by every sensor placement in the support of \( \sigma^{1*} \); this would imply that \( \textsf{P2} \) can improve her payoff by using her remaining attack resources to target that edge. Let us assume the contrary, i.e., that all the edges not in \( T^0 \) are monitored by every sensor placement in the support of \( \sigma^{1*} \):

\[
\forall S \in \text{supp}(\sigma^{1*}), \forall e \notin T^0, \ F(S, e) = 1.
\]

First, let us denote \( T^1 \subseteq T^0 \) the subset of edges of \( T^0 \) that are unmonitored by at least one sensor placement \( S \in \text{supp}(\sigma^{1*}) \), i.e.:

\[
T^1 = \bigcup_{S \in \text{supp}(\sigma^{1*})} (\mathcal{E} \setminus \mathcal{C}_S) = \mathcal{E} \setminus \bigcap_{S \in \text{supp}(\sigma^{1*})} \mathcal{C}_S \subseteq T^0.
\]

For all \( S \in \text{supp}(\sigma^{1*}) \), let us denote \( k_S \) the number of edges of \( T^1 \) that are not monitored by \( S \). Since every edge outside of \( T^1 \) is monitored by every sensor placement in the support of \( \sigma^{1*} \), then targeting any of these edges will provide \( \textsf{P2} \) a payoff equal to 0. Therefore, \( \textsf{P2} \)'s best response to \( \sigma^{1*} \) is any attack \( T \in \mathcal{A}_2 \) such that \( T^1 \subseteq T \) (note that \( |T^1| \leq |T^0| \leq b_2 \)), and \( \textsf{P2} \)'s equilibrium payoff is equal to \( k^* := \mathbb{E}_{\sigma^{1*}} [k_S] \). This is shown in the following steps:

\[
\forall T \in \mathcal{A}_2 \ | \ T^1 \subseteq T, \ U_2(\sigma^{1*}, T) \overset{(5)}{=} |T| - \sum_{S \in \mathcal{A}_1} \sigma^{1*}_S F(S, T) \\
= |T| - \sum_{e \in T^1} \sum_{S \in \mathcal{A}_1} \sigma^{1*}_S F(S, e) - \sum_{e \in T \setminus T^1} \sum_{S \in \mathcal{A}_1} \sigma^{1*}_S F(S, e) \\
= |T| - \sum_{S \in \mathcal{A}_1} \sigma^{1*}_S \sum_{e \in T^1} F(S, e) - \sum_{e \in T \setminus T^1} \sum_{S \in \mathcal{A}_1} \sigma^{1*}_S \\
= |T| - \sum_{S \in \mathcal{A}_1} \sigma^{1*}_S F(S, T^1) - |T \setminus T^1| \\
= |T^1| - \sum_{S \in \mathcal{A}_1} \sigma^{1*}_S F(S, T^1) \\
= |T^1| - \sum_{S \in \mathcal{A}_1} \sigma^{1*}_S (|T^1| - k_S) \\
= |T^1| \left( 1 - \sum_{S \in \mathcal{A}_1} \sigma^{1*}_S \right) + \sum_{S \in \mathcal{A}_1} \sigma^{1*}_S k_S \\
= \mathbb{E}_{\sigma^{1*}} [k_S].
\]

This implies that \( \textsf{P1} \)'s equilibrium payoff in the strategically equivalent zero-sum game \( \tilde{\Gamma} \) is equal to \(-k^*\). Thanks to Prop. 1, we know that \(-k^* \geq b_2 (\frac{b_2}{m^*} - 1) \). Since we are in the case when \( b_2 < m^* \) and we have \( m^* \leq n^* \) (see Lemma 3), then we obtain:

\[
k^* \leq \frac{b_2}{n^*} (n^* - b_1) < \frac{m^*}{n^*} (n^* - b_1) \leq n^* - b_1.
\]
Consider \( S \in \text{supp}(\sigma^*) \). We know that \( S \) leaves \( k_S \) edges in \( \mathcal{G} \) unmonitored, that we denote \( e_1, \ldots, e_{k_S} \). For \( l \in [1,k_S] \), let \( i_l \) be a node from where a sensor can monitor edge \( e_l \). Then, \( S \cup \{i_1, \ldots, i_{k_S}\} \) can monitor every edge in \( \mathcal{G} \) and is a set cover (of size at most \( b_1 + k_S \)). Since \( n^* \) is the smallest size of the set covers, then necessarily, we have \( b_1 + k_S \geq n^* \). Therefore, \( \forall S \in \text{supp}(\sigma^*) \), \( k_S \geq n^* - b_1 \). This last result implies that \( k^* = \mathbb{E}_{\sigma^*} [k_S] \geq n^* - b_1 \), which contradicts (EC.37).

Thus, there exists an edge \( e' \) not in \( T_0 \) that is not monitored by at least one sensor placement, denoted \( S' \), in the support of \( \sigma^* \).

– Now, we can show that \( P_2 \) can increase her payoff by targeting edge \( e' \) and the edges in \( T_0 \). Let \( T' = T_0 \cup \{e'\} \in \mathcal{A}_2 \) (since \( |T_0| < b_2 \)). Then, we get:

\[
U_2(\sigma^*, T') = |T'| - \sum_{S \in \mathcal{A}_1} \sigma^*_S F(S, T') = |T'| - \sum_{S \in \mathcal{A}_1} \sigma^*_S \sum_{e \in T'} F(S, e)
\]

\[
= \sum_{e \in T'} \left( 1 - \sum_{S \in \mathcal{A}_1} \sigma^*_S F(S, e) \right)
\]

\[
= 1 - \sum_{S \in \mathcal{A}_1} \sigma^*_S F(S, e') + \sum_{e \in T' \setminus \{e'\}} \left( 1 - \sum_{S \in \mathcal{A}_1} \sigma^*_S F(S, e) \right)
\]

\[
= \sum_{S \in \mathcal{A}_1} \sigma^*_S (1 - F(S, e')) + \sum_{e \in T' \setminus \{e'\}} \left( 1 - \sum_{S \in \mathcal{A}_1} \sigma^*_S F(S, e) \right)
\]

\[
= \sigma^*_{S'} (1 - F(S', e')) + \sum_{S \in \mathcal{A}_1 \setminus \{S'\}} \sigma^*_S (1 - F(S, e')) + \sum_{e \in T' \setminus \{e'\}} \left( 1 - \sum_{S \in \mathcal{A}_1} \sigma^*_S F(S, e) \right)
\]

\[
\geq \sigma^*_{S'} + \sum_{e \in T' \setminus \{e'\}} \left( 1 - \sum_{S \in \mathcal{A}_1} \sigma^*_S F(S, e) \right)
\]

\[
> \sum_{e \in T' \setminus \{e'\}} \left( 1 - \sum_{S \in \mathcal{A}_1} \sigma^*_S F(S, e) \right)
\]

\[
= \sum_{e \in T^0} \left( 1 - \sum_{S \in \mathcal{A}_1} \sigma^*_S F(S, e) \right)
\]

\[
= |T^0| - \sum_{S \in \mathcal{A}_1} \sigma^*_S F(S, T^0)
\]

\[
= U_2(\sigma^*, T^0),
\]

which violates the equilibrium condition (7) for \((\sigma^1*, \sigma^2*)\). Therefore, \( \forall T \in \text{supp}(\sigma^2*) \), \( |T| = b_2 \).
Proof of Proposition 4.
i) First, from Prop. 3, we can easily see that the set of optimal solutions of (LP₁) is a subset of \(\Delta(\overline{A₁})\). Indeed, if there exists an optimal solution \(\sigma^{1^*}\) of (LP₁) that is not in \(\Delta(\overline{A₁})\), then \(\exists S \in \text{supp}(\sigma^{1^*}) \ | |S| < b₁\), which is a contradiction since optimal solutions of (LP₁) are the sensing strategies in equilibrium, and should satisfy (20) in Prop. 3. Therefore, the sensing strategies in equilibrium are the optimal solutions of the following linear program:

\[
\text{(LP₁)} \quad \max_{\sigma^1 \in \Delta(\overline{A₁})} \min_{T \in A₂} -U_2(\sigma^1, T).
\]

Now, consider a sensing strategy \(\sigma^1 \in \Delta(\overline{A₁})\). Let \(T^0 \in A₂\) be an attack that satisfies:

\[
T^0 \in \arg \min_{T \in A₂} -U_2(\sigma^1, T).
\]

Then, consider \(T' \in \overline{A₂} \mid T^0 \subseteq T'\). We can deduce that:

\[
\min_{T \in A₂} -U_2(\sigma^1, T) = -U_2(\sigma^1, T^0) \overset{(5)}{=} \sum_{S \in A₁} \sigma^1_S F(S, T^0) - |T^0| \overset{(EC.8)}{=} \sum_{S \in A₁} \left( \sum_{e \in T^0} \sigma^1_S F(S, e) - 1 \right) = \sum_{e \in T'} \left( \sum_{S \in A₁} \sigma^1_S F(S, e) - 1 \right) - \sum_{e \in T' \setminus T^0} \left( \sum_{S \in A₁} \sigma^1_S F(S, e) - 1 \right) \leq 0
\]

\[
\geq \sum_{e \in T'} \left( \sum_{S \in A₁} \sigma^1_S F(S, e) - 1 \right) \overset{(5)}{=} -U_2(\sigma^1, T') \geq \min_{T \in \overline{A₂}} -U_2(\sigma^1, T). \quad \text{(EC.38)}
\]

Furthermore, since \(\overline{A₂} \subset A₂\), then we trivially have:

\[
\min_{T \in A₂} -U_2(\sigma^1, T) \leq \min_{T \in \overline{A₂}} -U_2(\sigma^1, T). \quad \text{(EC.39)}
\]

By combining (EC.38) and (EC.39), we deduce that:

\[
\min_{T \in A₂} -U_2(\sigma^1, T) = \min_{T \in \overline{A₂}} -U_2(\sigma^1, T).
\]

Since this is true for any sensing strategy \(\sigma^1 \in \Delta(\overline{A₁})\), we can conclude that:

\[
\max_{\sigma^1 \in \Delta(\overline{A₁})} \min_{T \in A₂} -U_2(\sigma^1, T) = \max_{\sigma^1 \in \Delta(\overline{A₁})} \min_{T \in \overline{A₂}} -U_2(\sigma^1, T),
\]

and the sensing strategies in equilibrium are the optimal solutions of (LP₁).
ii) Again, from (21) in Prop. 3, we can deduce that the set of optimal solutions of (LP$_2$) is a subset of $\Delta(\mathcal{A}_2)$, and the attack strategies in equilibrium are the optimal solutions of the following linear program:

\[
(\tilde{\text{LP}}_2) \quad \max_{\sigma^2 \in \Delta(\mathcal{A}_2)} \min_{S \in \mathcal{A}_1} U_2(S, \sigma^2).
\]

Now, consider an attack strategy $\sigma^2 \in \Delta(\mathcal{A}_2)$, and let $S^0 \in \mathcal{A}_1$ be a sensor placement that satisfies:

\[
S^0 \in \arg \min_{S \in \mathcal{A}_1} U_2(S, \sigma^2).
\]

Then, consider $S' \in \mathcal{A}_1$ such that $S^0 \subseteq S'$. We can deduce that:

\[
\min_{S \in \mathcal{A}_1} U_2(S, \sigma^2) = U_2(S^0, \sigma^2) \overset{(5)}{=} \sum_{T \in \mathcal{A}_2} \sigma^2_T (|T| - F(S^0, T)) \geq \sum_{T \in \mathcal{A}_2} \sigma^2_T (|T| - F(S', T)) \overset{(5)}{=} U_2(S', \sigma^2) \geq \min_{S \in \mathcal{A}_1} U_2(S, \sigma^2).
\]

Since $\mathcal{A}_2 \subset \mathcal{A}_1$, then the reverse inequality is also true, which implies that:

\[
\min_{S \in \mathcal{A}_2} U_2(S, \sigma^2) = \min_{S \in \mathcal{A}_2} U_2(S, \sigma^2). \quad \text{(EC.40)}
\]

Since (EC.40) holds for any attack strategy $\sigma^2 \in \Delta(\mathcal{A}_2)$, this implies that:

\[
\max_{\sigma^2 \in \Delta(\mathcal{A}_2)} \min_{S \in \mathcal{A}_2} U_2(S, \sigma^2) = \max_{\sigma^2 \in \Delta(\mathcal{A}_2)} \min_{S \in \mathcal{A}_2} U_2(S, \sigma^2),
\]

and the attack strategies in equilibrium are the optimal solutions of (LP$_2$).

\[
\Box
\]

**Proof of Proposition 5.** We show the result by contradiction, that is, suppose that $\exists (\sigma^{1*}, \sigma^{2*}) \in \Sigma$ such that $\mathcal{V}_{\sigma^{1*}}$ is not a set cover. This implies that $\exists e \in \mathcal{E} \mid \forall S \in \text{supp}(\sigma^{1*}), F(S, e) = 0$.

For simplicity, we introduce the following notation:

\[
\forall e \in \mathcal{E}, \eta_{\sigma^{1*}}(e) := \mathbb{E}_{\sigma^{1*}}[F(S, e)], \quad \text{(EC.41)}
\]

which is the probability with which edge $e$ is monitored by $\sigma^{1*}$. Now, let us sort the edges in nondecreasing order of monitoring:

\[
\eta_{\sigma^{1*}}(e_1) \leq \eta_{\sigma^{1*}}(e_2) \leq \cdots \leq \eta_{\sigma^{1*}}(e_{|\mathcal{E}|}).
\]
Let us denote \( T^0 = \{ e_1, \ldots, e_k \} \in 2^E \) the edges that are not monitored by any sensor placement \( S \in \text{supp}(\sigma^1) \):

\[
\forall i \in [1, k], \forall S \in \text{supp}(\sigma^1), \ F(S, e_i) = 0.
\]

This implies that \( \eta_{\sigma^1}(e_1) = \cdots = \eta_{\sigma^1}(e_k) = 0 \). Then, since \( \textbf{P2} \) maximizes her payoff when targeting edges that are monitored with the smallest probability, it is easy to see that \( T' = \{ e_1, \ldots, e_{b_2} \} \) is a best response for \( \textbf{P2} \) (recall that \( \textbf{P2} \) uses all her resources; see Prop. 3).

This implies that \( \textbf{P2}'s \) equilibrium payoff is:

\[
U_2(\sigma^1, T') = |T'| - \sum_{S \in A_1} \sigma^1_S F(S, T') = b_2 - \sum_{S \in A_1} \sigma^1_S \sum_{e \in T'} F(S, e)
\]

\[= b_2 - \sum_{\eta_{\sigma^1'}(e)} \eta_{\sigma^1'}(e_i). \quad \text{(EC.41)} \]

Thus, \( \textbf{P1}'s \) equilibrium payoff in the game \( \tilde{\Gamma} \) is \(-b_2 + \sum_{i=k+1}^{b_2} \eta_{\sigma^1'}(e_i)\).

Now, to show the contradiction, we construct another strategy \( \tilde{\sigma}^1 \) that will provide a better payoff than \( \sigma^1 \) to \( \textbf{P1} \).

- **Case 1:** \( k \geq b_2 \), then \( \textbf{P1}'s \) equilibrium payoff in \( \tilde{\Gamma} \) is \(-b_2 \) (it corresponds to zero detections). However, she has an incentive to switch her strategy, and by randomizing over the nodes that can monitor the \( k \) edges \( \{ e_1, \ldots, e_k \} \), she will increase her payoff; which is a contradiction.

- **Case 2:** \( k < b_2 \), then \( \textbf{P2} \) will randomize over attacks that contain \( \{ e_1, \ldots, e_k \} \) (i.e. \( \rho_{\sigma^2}(e_1) = \cdots = \rho_{\sigma^2}(e_k) = 1 \)).

  - **Case 2.1:** \( k \geq b_1 \). Then \( \textbf{P1}'s \) equilibrium payoff in \( \tilde{\Gamma} \) is at least equal to \( b_1 - b_2 \) (since she can monitor \( b_1 \) edges in \( \{ e_1, \ldots, e_k \} \) that are always targeted). This implies that \( \textbf{P2}'s \) equilibrium payoff is at most \( b_2 - b_1 \). However, thanks to Prop. 2, we know that \( \textbf{P2}'s \) equilibrium payoff is larger than or equal to \( b_2 - \frac{b_1 b_2}{m*} > b_2 - b_1 \) (since \( b_2 < m* \)). Therefore there is a contradiction.

  - **Case 2.2:** \( k < b_1 \). Then, the idea is to construct another strategy that allocates \( k \) sensors to monitor edges \( \{ e_1, \ldots, e_k \} \) (that were previously unmonitored), and that randomizes the placement of the remaining \( b_1 - k \) sensors over the node basis of \( \sigma^1 \).

For now, assume that \( \textbf{P1} \) has \( b_1 - k \) sensors. For any sensor placement \( S \in \text{supp}(\sigma^1) \) (viewed as a set of nodes), let us consider \( \sigma^1(S, b_1 - k) \) defined in (EC.21) (in this case, we randomize the placement of \( b_1 - k \) sensors over the set \( S \) of size \( b_1 \)). Given \( S \in \text{supp}(\sigma^1) \), recall that \( \text{supp}(\sigma^1(S, b_1 - k)) = \{ S^1, \ldots, S^{\mu(b_1, b_1 - k)} \} \), and that \( \forall l \in [1, \mu(b_1, b_1 - k)] \), \( \sigma^1(S, b_1 - k)_{s_l} = \frac{1}{\mu(b_1, b_1 - k)} \).
Now, let us construct the following sensing strategy:

$$\sigma^1' = \sum_{S \in \text{supp}(\sigma^1^*)} \sigma^1_1^* \sigma^1(S, b_1 - k). \quad (\text{EC.43})$$

Then, \(\text{supp}(\sigma^1') = \bigcup_{S \in \text{supp}(\sigma^1^*)} \{S^1, \ldots, S^1_{\mu(b_1,b_1-k)}\}\). Notice that it is indeed a probability distribution, and then that \(\sigma^1^*\) is a probability distribution.

Thus, \(\sigma^1'\) is a probability distribution that randomizes over sensor placements of size \(b_1 - k\). Then, \(\forall S \in \text{supp}(\sigma^1')\), we augment \(S\) by placing \(k\) additional sensors to monitor edges \(\{e_1, \ldots, e_k\}\) that were previously unmonitored and that are always targeted in equilibrium by \(\text{P2}\). We denote \(\{i_1, \ldots, i_k\}\) the placement of such additional sensors, and we denote \(\hat{S} = S \cup \{i_1, \ldots, i_k\}\), \(\forall S \in \text{supp}(\sigma^1')\) the augmented sensor placement. Then, we consider the probability distribution \(\hat{\sigma}^1\) with support equal to \(\bigcup_{S \in \text{supp}(\sigma^1^*)} \{\hat{S}^1, \ldots, \hat{S}^1_{\mu(b_1,b_1-k)}\}\) and such that:

$$\forall S \in \text{supp}(\sigma^1'), \ \sigma^1'_S = \hat{\sigma}^1_S, \quad (\text{EC.44})$$

i.e., \(\hat{\sigma}^1\) is the same probability distribution as \(\sigma^1'\) except that it randomizes over the augmented sensor placements present in the support of \(\sigma^1'\).

Then, we can derive the following calculations which combine the previous construction of \(\hat{\sigma}^1\) with a property of the sensing function derived in (EC.27), and which will lead to a contradiction:

$$\forall T \in \mathcal{A_2}, \{e_1, \ldots, e_k\} \subset T \text{ and } |T| = b_2, \ -U_2(\hat{\sigma}^1, T) \overset{(5)}{=} -b_2 + \sum_{S \in \mathcal{A}_1} \hat{\sigma}^1_S F(S, T)$$

$$\overset{(\text{EC.8})}{=} -b_2 + \sum_{l=1}^{k} \sum_{S \in \mathcal{A}_1} \hat{\sigma}^1_S F(S, e_l) + \sum_{e \in T \setminus \{e_1, \ldots, e_k\}} \sum_{S \in \mathcal{A}_1} \hat{\sigma}^1_S F(S, e)$$

$$= -b_2 + k + \sum_{e \in T \setminus \{e_1, \ldots, e_k\}} \sum_{S \in \text{supp}(\sigma^1^*)} \sum_{l=1}^{\mu(b_2,b_2-k)} \hat{\sigma}^1_S F(\tilde{S}^l, e)$$

$$\overset{(\text{EC.44})}{=} -b_2 + k + \sum_{e \in T \setminus \{e_1, \ldots, e_k\}} \sum_{S \in \text{supp}(\sigma^1^*)} \sum_{l=1}^{\mu(b_1,b_1-k)} \sigma^1'_S F(\tilde{S}^l, e)$$
\[ \text{(EC.5)} \quad \geq -b_2 + k + \sum_{e \in T \setminus \{e_1, \ldots, e_k\}} \sum_{S \in \text{supp}(\sigma^1)} \sum_{l=1}^{\mu(b_1, b_1 - k)} \sigma^1_{Sl} F(S^l, e) \]

\[ \text{(EC.21), (EC.43)} \quad = -b_2 + k + \sum_{e \in T \setminus \{e_1, \ldots, e_k\}} \sum_{S \in \text{supp}(\sigma^1)} \sum_{l=1}^{\mu(b_1, b_1 - k)} \frac{1}{\mu(b_1, b_1 - k)} \sigma^1_{Sl} F(S^l, e) \]

\[ = -b_2 + k + \sum_{e \in T \setminus \{e_1, \ldots, e_k\}} \sum_{S \in \text{supp}(\sigma^1)} \frac{1}{\mu(b_1, b_1 - k)} \sigma^1_{Sl} F(S^l, e) \]

\[ \geq -b_2 + k + \frac{b_1 - k}{b_1} \sum_{i=k+1}^{b_2} \eta_{\sigma^1}(e_i) \]

\[ \geq -b_2 + \frac{b_2}{b_1} \sum_{i=k+1}^{b_2} \eta_{\sigma^1}(e_i) + k \left( 1 - \frac{1}{b_1} \sum_{i=k+1}^{b_2} \eta_{\sigma^1}(e_i) \right). \quad \text{(EC.45)} \]

In inequality (EC.45), we used the fact that the size of \( T \setminus \{e_1, \ldots, e_k\} \) is \( b_2 - k \), and no matter what these \( b_2 - k \) edges are, the sum of the \( \eta_{\sigma^1}(e) \) is always larger than or equal to \( \sum_{i=k+1}^{b_2} \eta_{\sigma^1}(e_i) \) (since we sorted the edges in nondecreasing order of monitoring).

Thanks to Prop. 2, we know that the equilibrium payoff of P2 is larger than or equal to \( b_2 - \frac{b_1 b_2}{m^*} \). Thus:

\[ U_2(\sigma^1^*, \sigma^2^*) \stackrel{\text{(EC.42)}}{=} b_2 - \sum_{i=k+1}^{b_2} \eta_{\sigma^1^*}(e_i) \geq b_2 - \frac{b_1 b_2}{m^*} \iff \sum_{i=k+1}^{b_2} \eta_{\sigma^1^*}(e_i) \leq \frac{b_1 b_2}{m^*} < b_1, \]

since \( b_2 < m^* \). Therefore, by combining the previous inequality with (EC.46), we get:

\[ \forall T \in A_2 \mid \{e_1, \ldots, e_k\} \subset T \text{ and } |T| = b_2, \quad -U_2(\bar{\sigma}^1, T) > -b_2 + \sum_{i=k+1}^{b_2} \eta_{\sigma^1^*}(e_i) \stackrel{\text{(EC.42)}}{=} -U_2(\sigma^1^*, \sigma^2^*). \]

Since we know that each attack in the support of an equilibrium strategy uses all the resources (Prop. 3), and since it must contain \( \{e_1, \ldots, e_k\} \) (beginning of Case 2), then:

\[ -U_2(\bar{\sigma}^1, \sigma^2^*) = E_{\sigma^2^*}[-U_2(\bar{\sigma}^1, T)] > -U_2(\sigma^1^*, \sigma^2^*), \]

which violates the equilibrium condition (6) for \((\sigma^1^*, \sigma^2^*)\) in the strategic equivalent game \( \bar{\Gamma} \).

Therefore, \( \forall (\sigma^1^*, \sigma^2^*) \in \Sigma, \mathcal{V}_{\sigma^1^*} \) is a set cover.

\[ \square \]

**Proof of Proposition 6.** From Prop. 5 and the fact that \( b_1 < n^* \), we know that P1 must randomize her sensor placements so the node basis is a set cover in equilibrium.
Now, assume that there exists a NE \((\sigma^{1*}, T) \in \Sigma\) such that \(P2\) chooses a pure strategy \(T\) (of size \(b_2\) from Prop. 3). There are two cases to consider.

- If \(b_1 \geq b_2\), then \(P1\) can detect all the failure events of \(T\) by placing \(b_2\) sensors at the nodes that can monitor the edges of \(T\), and \(P2\)'s equilibrium payoff would be 0. Again, since \(P2\)'s payoff is identical for any NE, then \(U_2(\sigma^{1*}, T) = 0\). However, we showed in the proof of Prop. 3 that there exists an edge outside of \(T\) that is not monitored with positive probability by \(\sigma^{1*}\). Therefore, \(P2\) can increase her payoff by targeting that edge, thus leading to a contradiction.

- If \(b_1 < b_2\), then \(P1\) can detect at least \(b_1\) failure events occurring on \(T\) by placing sensors on \(b_1\) nodes that can collectively monitor \(b_1\) edges of \(T\). The resulting payoff for \(P1\) in the game \(\Gamma\) is at least \(b_1 - b_2\). However, thanks to Prop. 2, we know that \(P2\)'s equilibrium payoff is at least \(\max\{0, b_2(1 - \frac{b_1}{m^*})\} \geq b_2(1 - \frac{b_1}{m^*}) > b_2 - b_1\) (since \(b_2 < m^*\)). Therefore, \(P1\)'s equilibrium payoff in \(\Gamma\) is strictly upper bounded by \(b_1 - b_2\), thus leading to a contradiction.

In both cases, we showed that \(P2\) needs to randomize. Therefore, in equilibrium, both players must randomize.

\(\square\)

Proof of Theorem 1.

(i) Since \(P2\)'s expected payoff in the game \(\Gamma\) is also her expected payoff in \(\Gamma\), we know that \(P2\)'s equilibrium payoff is constant.

From Prop. 1, we know that:
\[
\forall (\sigma^{1*}, \sigma^{2*}) \in \Sigma, \; -U_2(\sigma^{1*}, \sigma^{2*}) \geq b_2 \left(\frac{b_1}{m^*} - 1\right) \tag{EC.47}
\]

Now, from Prop. 2, we have:
\[
\forall (\sigma^{1*}, \sigma^{2*}) \in \Sigma, \; U_2(\sigma^{1*}, \sigma^{2*}) \geq \max\left\{0, b_2 \left(1 - \frac{b_1}{m^*}\right)\right\} \tag{EC.48}
\]

By combining (EC.47) and (EC.48), we obtain:
\[
\forall (\sigma^{1*}, \sigma^{2*}) \in \Sigma, \; \max\left\{0, b_2 \left(1 - \frac{b_1}{m^*}\right)\right\} \leq U_2(\sigma^{1*}, \sigma^{2*}) \leq b_2 \left(1 - \frac{b_1}{m^*}\right) \tag{EC.49}
\]

Since from (EC.15), \(\forall (\sigma^{1*}, \sigma^{2*}) \in \Delta(A_1) \times \Delta(A_2), \; U_1(\sigma^{1*}, \sigma^{2*}) = -U_2(\sigma^{1*}, \sigma^{2*}) + \mathbb{E}_{\sigma^*}[|T|]\), and thanks to Prop. 3, we can easily deduce that \(P1\)'s payoff in equilibrium is also constant, and can be bounded as follows:
\[
\forall (\sigma^{1*}, \sigma^{2*}) \in \Sigma, \; \frac{b_1 b_2}{m^*} \leq U_1(\sigma^{1*}, \sigma^{2*}) \leq \min\left\{\frac{b_1 b_2}{m^*}, b_2\right\} \tag{EC.50}
\]

(ii) Again, thanks to Prop. 3, we have:
\[
\forall \sigma^* \in \Sigma, \; r(\sigma^*) \overset{(10)}{=} \mathbb{E}_{\sigma^*} \left[\frac{F(S,T)}{|T|}\right] = \frac{1}{b_2} \mathbb{E}_{\sigma^*}[F(S,T)] = \frac{1}{b_2} U_1(\sigma^{1*}, \sigma^{2*}). \tag{EC.51}
\]
Therefore, from (EC.50) and (EC.51), we obtain that the expected detection rate in equilibrium is constant and bounded as follows:

\[ \forall \sigma^* \in \Sigma, \quad \frac{b_1}{n^*} \leq r(\sigma^*) \leq \min \left\{ \frac{b_1}{m^*}, 1 \right\}. \]

\[ \square \]

Proof of Theorem 2

(i) Let \( S^{min} \in S \) be an MSC and \( T^{max} \in M \) be an MSP. Then, let us show that \( (\sigma^1(S^{min}, b_1), \sigma^2(T^{max}, b_2)) \) is an \( \epsilon \)-NE where \( \epsilon = b_1 b_2 \left( \frac{1}{\max(b_1, m^*)} - \frac{1}{n^*} \right) \).

First, note that:

\[
\max \left\{ 0, b_2 \left( 1 - \frac{b_1}{m^*} \right) \right\} + b_2 \left( \frac{b_1}{n^*} - 1 \right) = \max \left\{ b_2 \left( \frac{b_1}{n^*} - 1 \right), b_1 b_2 \left( \frac{1}{n^*} - \frac{1}{m^*} \right) \right\} \\
= b_1 b_2 \max \left\{ \left( \frac{1}{n^*} - \frac{1}{b_1} \right), \left( \frac{1}{n^*} - \frac{1}{m^*} \right) \right\} \\
= b_1 b_2 \left( \frac{1}{n^*} - \min \left\{ \frac{1}{b_1}, \frac{1}{m^*} \right\} \right) \\
= b_1 b_2 \left( \frac{1}{n^*} - \frac{1}{\max\{b_1, m^*\}} \right) \\
= -\epsilon. \tag{EC.52}
\]

Then, we have:

\[
\forall \sigma^1 \in \Delta(A_1), \quad -U_2(\sigma^1(S^{min}, b_1), \sigma^2(T^{max}, b_2)) \geq \min_{\sigma^2 \in \Delta(A_2)} \left( -U_2(\sigma^1(S^{min}, b_1), \sigma^2) \right)
\]

\[ \overset{\text{Prop. 1}}{=} b_2 \left( \frac{b_1}{n^*} - 1 \right) \tag{EC.53} \]

\[ = - \max \left\{ 0, b_2 \left( 1 - \frac{b_1}{m^*} \right) \right\} + \max \left\{ 0, b_2 \left( 1 - \frac{b_1}{m^*} \right) \right\} + b_2 \left( \frac{1}{n^*} - 1 \right) \]

\[ \overset{(EC.52)}{=} - \max \left\{ 0, b_2 \left( 1 - \frac{b_1}{m^*} \right) \right\} - \epsilon \]

\[ \overset{\text{Prop. 2}}{=} - \min_{\sigma^2 \in \Delta(A_1)} U_2(\sigma^1, \sigma^2(T^{max}, b_2)) - \epsilon \\
\geq -U_2(\sigma^1, \sigma^2(T^{max}, b_2)) - \epsilon. \tag{EC.54}
\]

Therefore, \( \forall \sigma^1 \in \Delta(A_1) \):

\[
U_1(\sigma^1(S^{min}, b_1), \sigma^2(T^{max}, b_2)) \overset{(EC.15)}{=} -U_2(\sigma^1(S^{min}, b_1), \sigma^2(T^{max}, b_2)) + E_{\sigma^2(T^{max}, b_2)}[T] \]

\[ \overset{(EC.54)}{=} -U_2(\sigma^1, \sigma^2(T^{max}, b_2)) - \epsilon + E_{\sigma^2(T^{max}, b_2)}[T] \]

\[ \overset{(EC.15)}{=} U_1(\sigma^1, \sigma^2(T^{max}, b_2)) - \epsilon. \tag{EC.55}
\]
Similarly, we have:

\[ \forall \sigma^2 \in \Delta(A_2), \ U_2(\sigma^1(S^{\min}, b_1), \sigma^2(T^{\max}, b_2)) \geq \min_{\sigma^1 \in \Delta(A_1)} U_2(\sigma^1, \sigma^2(T^{\max}, b_2)) \]

\[ \overset{\text{Prop. 2}}{=} \max \left\{ 0, \left(1 - \frac{b_1}{n^*}\right) b_2 \right\} \]

\[ = -b_2 \left(\frac{b_1}{n^*} - 1\right) + b_2 \left(\frac{b_1}{n^*} - 1\right) + \max \left\{ 0, \left(1 - \frac{b_1}{m^*}\right) b_2 \right\} \]

\[ \overset{\text{EC.52}}{=} -b_2 \left(\frac{b_1}{n^*} - 1\right) - \epsilon \]

\[ \overset{\text{Prop. 1}}{=} 1 - \min_{\sigma^2 \in \Delta(A_2)} U_2(\sigma^1(S^{\min}, b_1), \sigma^2) - \epsilon \]

\[ \geq U_2(\sigma^1(S^{\min}, b_1), \sigma^2) - \epsilon. \] 

(\text{EC.57})

Therefore, (EC.55) and (EC.57) imply that \((\sigma^1(S^{\min}, b_1), \sigma^2(T^{\max}, b_2)) \in \Sigma_e.\)

(ii) From (EC.56) and (EC.53), we have:

\[ \max \left\{ 0, \left(1 - \frac{b_1}{n^*}\right) b_2 \right\} \leq U_2(\sigma^1(S^{\min}, b_1), \sigma^2(T^{\max}, b_2)) \leq b_2 \left(1 - \frac{b_1}{n^*}\right). \]

By combining it with (EC.49), we obtain for any NE \((\sigma^{1*}, \sigma^{2*}) \in \Sigma:\)

\[ |U_2(\sigma^1(S^{\min}, b_1), \sigma^2(T^{\max}, b_2)) - U_2(\sigma^{1*}, \sigma^{2*})| \leq b_2 \left(1 - \frac{b_1}{n^*}\right) - \max \left\{ 0, \left(1 - \frac{b_1}{m^*}\right) b_2 \right\} \]

\[ \overset{\text{EC.52}}{=} \epsilon. \] 

(\text{EC.58})

Furthermore, we know that \(\forall (\sigma^1, \sigma^2) \in \Delta(A_1) \times \Delta(A_2), \ U_1(\sigma^1, \sigma^2) = -U_2(\sigma^1, \sigma^2) + \mathbb{E}_{\sigma^2}[|T|] \)

(from (EC.15)), that \(\forall (\sigma^{1*}, \sigma^{2*}) \in \Sigma, \mathbb{E}_{\sigma^{2*}}[|T|] = b_2 \) (from (21)) and that \(\mathbb{E}_{\sigma^2(T^{\max}, b_2)}[|T|] = b_2 \)

(from (EC.19)). Then, for any NE \((\sigma^{1*}, \sigma^{2*}) \in \Sigma,\) we have:

\[ |U_1(\sigma^1(S^{\min}, b_1), \sigma^2(T^{\max}, b_2)) - U_1(\sigma^{1*}, \sigma^{2*})| = |U_2(\sigma^1(S^{\min}, b_1), \sigma^2(T^{\max}, b_2)) - U_2(\sigma^{1*}, \sigma^{2*})| \]

\[ \overset{\text{EC.58}}{\leq} \epsilon. \] 

(iii) Let us show a stronger result:

\[ \forall T \in A_2, \ U_1(\sigma^1(S^{\min}, b_1), T) = \frac{U_2(\sigma^1(S^{\min}, b_1), T)}{|T|} \]

\[ \overset{\text{EC.15}}{=} -U_2(\sigma^1(S^{\min}, b_1), T) + |T| \]

\[ \overset{\text{EC.31}}{\geq} \left(\frac{b_1}{n^*} - 1\right) \frac{|T|}{|T|} + 1 \]

\[ = \frac{b_1}{n^*}. \] 

(\text{EC.59})

Thus, by linearity of the expectation, we obtain:

\[ \forall \sigma^2 \in \Delta(A_2), \ r(\sigma^1(S^{\min}, b_1), \sigma^2) \overset{\text{(10)}}{=} \mathbb{E}_{\sigma^1(S^{\min}, b_1), \sigma^2}[F(S, T)] \]

\[ \overset{\text{Prop. 4}}{=} \mathbb{E}_{\sigma^2}[U_1(\sigma^1(S^{\min}, b_1), T)] \]

\[ \overset{\text{EC.59}}{\geq} \frac{b_1}{n^*}, \] 

(\text{EC.60})
Consider an MSC $S^{\text{min}} \in \mathcal{S}$. We already showed in (EC.60) that $\min_{\sigma^2 \in \Delta(A_2)} r(\sigma^1(S^{\text{min}}, b_1), \sigma^2) \geq \frac{b_1}{n^*}$. Now, consider $\iota' \in S^{\text{min}}$. Recall from (EC.29) that $\exists e' \in \mathcal{E} \mid F(i', e') = 1$ and $F(i, e') = 0$, $\forall i \in S^{\text{min}} \setminus \{i'\}$ (since an MSC is a minimal set cover). Then:

$$r(\sigma^1(S^{\text{min}}, b_1), e') = \mathbb{E}_{\sigma^1(S^{\text{min}}, b_1)}[F(S, e')] = \sum_{S \in A_1} \sigma^1(S^{\text{min}}, b_1)_S F(S, e')$$

$$= \sum_{\{S \in A_1 \mid \iota' \in S\}} \sigma^1(S^{\text{min}}, b_1)_S F(S, e') + \sum_{\{S \in A_1 \mid \iota' \notin S\}} \sigma^1(S^{\text{min}}, b_1)_S F(S, e')$$

$$= \sum_{\{S \in A_1 \mid \iota' \notin S\}} \sigma^1(S^{\text{min}}, b_1)_S = \rho_{\sigma^1(S^{\text{min}}, b_1)}(\iota')$$

$$\equiv \frac{b_1}{n^*}.$$

Therefore, $\min_{\sigma^2 \in \Delta(A_2)} r(\sigma^1(S^{\text{min}}, b_1), \sigma^2) = \frac{b_1}{n^*}$.

Since, from Thm. 1, we have $\forall (\sigma^1, \sigma^2) \in \Sigma$, $r(\sigma^1, \sigma^2) \leq \min \{\frac{b_1}{m^*}, 1\} = \frac{b_1}{\max\{b_1, m^*\}}$, then we deduce that:

$$\frac{\max\{b_1, m^*\}}{n^*} r(\sigma^1, \sigma^2) \leq \frac{b_1}{n^*} = \min_{\sigma^2 \in \Delta(A_2)} r(\sigma^1(S^{\text{min}}, b_1), \sigma^2).$$

**EC.4. Proofs of Section 4.2**

In this section, we consider the case when $b_1 < n^*$, $b_2 < m^*$, and $n^* = m^*$.

**Proof of Corollary 1.** By rewriting Thm. 1 when $n^* = m^*$, we obtain $\forall (\sigma^1, \sigma^2) \in \Sigma$:

$$\frac{b_1 b_2}{n^*} \leq U_1(\sigma^1, \sigma^2) \leq \min \left\{ \frac{b_1 b_2}{m^*}, b_2 \right\} = \min \left\{ \frac{b_1 b_2}{n^*}, b_2 \right\} = \frac{b_1 b_2}{n^*},$$

$$b_2 \left(1 - \frac{b_1}{n^*}\right) = \max \left\{ 0, b_2 \left(1 - \frac{b_1}{n^*}\right) \right\} = \max \left\{ 0, b_2 \left(1 - \frac{b_1}{m^*}\right) \right\} \leq U_2(\sigma^1, \sigma^2) \leq b_2 \left(1 - \frac{b_1}{n^*}\right),$$

since $b_1 < n^*$.

Similarly, for the expected detection rate in equilibrium, we obtain:

$$\forall \sigma^* \in \Sigma, \frac{b_1}{n^*} \leq r(\sigma^*) \leq \min \left\{ \frac{b_1}{m^*}, 1 \right\} = \min \left\{ \frac{b_1}{n^*}, 1 \right\} = \frac{b_1}{n^*}.$$

**Lemma EC.4.** For any MSC $S^{\text{min}} \in \mathcal{S}$ and any MSP $T^{\text{max}} \in \mathcal{M}$, each edge in $T^{\text{max}}$ is monitored by only one node in $S^{\text{min}}$, and each node in $S^{\text{min}}$ monitors only one edge in $T^{\text{max}}$. 

\[ \Box \]
Proof of Lemma EC.4. Consider an MSC $S^{min} \in S$ and an MSP $T^{max} \in M$. Since $T^{max}$ is an MSP, each node in $S^{min}$ monitors at most one edge in $T^{max}$. Now, assume that at least one node in $S^{min}$ does not monitor any edge in $T^{max}$. Since $S^{min}$ is an MSC, then the $n^*$ edges in $T^{max}$ are monitored by at most $n^* - 1$ nodes. From Dirichlet’s principle, there exists a node in $S^{min}$ that monitors at least two edges in $T^{max}$, which is a contradiction. Therefore, each node in $S^{min}$ monitors exactly one edge in $T^{max}$.

Thus, we can define a mapping $\psi : S^{min} \rightarrow T^{max}$ such that $\forall i \in S^{min}$, $\psi(i)$ is the edge in $T^{max}$ that is monitored by node $i$. Now, since $S^{min}$ is an MSC, for every edge $e \in T^{max}$, $\exists i \in S^{min}$ such that $e$ is monitored by $i$. Therefore, $\psi$ is surjective which is equivalent to $\psi$ being injective since its domain and codomain have the same number of elements. Therefore, each edge in $T^{max}$ is monitored by only one node in $S^{min}$.

This lemma tells us that no matter which MSC and which MSP we consider, there is a one-to-one correspondence between a node in the MSC and the edge in the MSP in monitors.

Thanks to Lemma EC.4, and without loss of generality, given an MSC $\{i_1, \ldots, i_{n^*}\} \in S$ and an MSP $\{e_1, \ldots, e_{n^*}\} \in M$, we can rearrange the indices such that $\forall k \in \llbracket 1, n^* \rrbracket$, $i_k$ monitors $e_k$.

Proof of Proposition 8. For simplicity, given an optimization problem ($Q$), we denote $OPT(Q)$ its objective value. Let $S^{min} \in S$ be an MSC and $T^{max} \in M$ be an MSP. Recall that from Props. 1 and 2, we have $OPT(LP_{S^{min}}) \leq OPT(LP_1) = -OPT(LP_2) \leq -OPT(LP_{T^{max}}).$ Since $n^* = m^*$ (and $b_1 < n^*$), then we also have:

$$OPT(LP_{S^{min}}) = b_2 \left(\frac{b_1}{n^*} - 1\right) = -b_2 \left(1 - \frac{b_1}{m^*}\right) = -\max \left\{0, b_2 \left(1 - \frac{b_1}{m^*}\right)\right\} = -OPT(LP_{T^{max}}).$$

Therefore, $OPT(LP_{S^{min}}) = OPT(LP_1)$ and $OPT(LP_{T^{max}}) = OPT(LP_2)$. Since $(LP_{S^{min}})$ (resp. $(LP_{T^{max}})$) is a restriction of $(LP_1)$ (resp. $(LP_2)$), then any sensing strategy $\sigma^{1^*}$ (resp. attack strategy $\sigma^{2^*}$) that is optimal for $(LP_{S^{min}})$ (resp. $(LP_{T^{max}})$) is optimal for $(LP_1)$ (resp. $(LP_2)$). Thus, any strategy profile $(\sigma^{1^*}, \sigma^{2^*})$ such that $\sigma^{1^*}$ and $\sigma^{2^*}$ are optimal solutions of $(LP_{S^{min}})$ and $(LP_{T^{max}})$ respectively is a NE.

To show that a sensing strategy $\sigma^{1^*}$ (whose node basis is $S^{min}$) is an optimal solution of $(LP_{S^{min}})$, it is sufficient to show that $\min_{\sigma^2 \in \Delta(A_2)} -U_2(\sigma^{1^*}, \sigma^2) \geq b_2 \left(\frac{b_1}{n^*} - 1\right) = OPT(LP_{S^{min}}).$ By applying (EC.32) for $S^{min}$ (which is a minimal set cover), we know that $\min_{\sigma^2 \in \Delta(A_2)} -U_2(\sigma^{1}(S^{min}, b_1), \sigma^2) \geq b_2 \left(\frac{b_1}{n^*} - 1\right).$ However, recall that to show this inequality, the only property from $\sigma^{1}(S^{min}, b_1)$ that we used was that its node basis is $S^{min}$ and that $\forall i \in S^{min}, \rho_{\sigma^{1}(S^{min}, b_1)}(i) = \frac{b_1}{n^*}$. Therefore, any sensing strategy that satisfies the same conditions also satisfies the same inequality and is an optimal solution of $(LP_{S^{min}})$. 
Similarly, we can easily deduce from (EC.35) that any attack strategy $\sigma^2$ whose edge basis is $T^{max}$ and which satisfies $\forall e \in T^{max}, \rho_{\sigma^2}(e) = \frac{b_2}{n}$ also satisfies the inequality $\min_{\sigma, \epsilon \in \Delta(A1)} U_2(\sigma^1, \sigma^2) \geq b_2 (1 - \frac{b_2}{n})$ and is an optimal solution of (LP$_{T^{max}}$).

Thus, any strategy profile $(\sigma^1, \sigma^2)$ whose node basis is $S^{min}$, whose edge basis is $T^{max}$, and that satisfies $\rho_{\sigma^1}(i) = \frac{b_2}{n}, \forall i \in S^{min}$, and $\rho_{\sigma^2}(e) = \frac{b_2}{n}, \forall e \in T^{max}$, is a NE (it is a sufficient condition).

Now, let us show (by contradiction) that this is also a necessary condition.

i) Consider a NE $(\sigma^1, \sigma^2) \in \Sigma$ whose node basis is an MSC $S^{min} = \{i_1, \ldots, i_n\} \in S$, and assume that the sensing probability is not identical among the nodes in $S^{min}$. Without loss of generality (by reordering the indices), assume that $\rho_{\sigma^1}(i_1) < \rho_{\sigma^1}(i_n)$.

Consider an MSP $T^{max} = \{e_1, \ldots, e_{n^*}\} \in M$. Recall that from Lemma EC.4, we can reorder the indices so that $\forall k \in [1, n^*]$, edge $e_k$ is only monitored by node $i_k$ in $S^{min}$. Since $b_2 < n^*$, then we can construct an attack strategy $\sigma^2$ whose edge basis is $T^{max}$, which targets each edge with an identical probability $\frac{b_2}{n}$, with the particularity that it assigns positive probabilities on the attacks $T^1 = \{e_1\} \cup \{e_2, \ldots, e_{b_2}\}$ and $T^{1'} = \{e_{n^*}\} \cup \{e_2, \ldots, e_{b_2}\}$. This can be achieved by taking a convex combination of two strategies built using Lemma EC.3 with a different ordering of the edges of $T^{max}$ (we know thanks to Lemma 2 that $\Sigma$ is a convex set).

Formally, consider $\sigma^2(T^{max}, b_2)$ defined in Lemma EC.3. Then, from (EC.20), we know that an action in its support is given by $T^1 = \{e_1, \ldots, e_{b_2}\}$. Now, let us reorder the indices of the edges in $T^{max}$ so that $T^{max} = \{e_{n^*}, e_2, e_3, \ldots, e_{n^*-1}, e_1\}$ (we swapped $e_1$ and $e_{n^*}$). Then, we can consider another strategy $\sigma^2(T^{max}, b_2)'$ defined in Lemma EC.3 with this new ordering and whose support contains the attack $T^{1'} = \{e_{n^*}, e_2, e_3, \ldots, e_{b_2}\}$. The edge basis of both $\sigma^2(T^{max}, b_2)$ and $\sigma^2(T^{max}, b_2)'$ is $T^{max}$. Furthermore, they both target each edge of $T^{max}$ with probability $\frac{b_2}{n}$. Therefore, $\sigma^2(T^{max}, b_2)$ and $\sigma^2(T^{max}, b_2)'$ both satisfy the sufficient conditions and are optimal solutions of (LP$_{T^{max}}$). This implies that $(\sigma^1, \sigma^2(T^{max}, b_2))$ and $(\sigma^1, \sigma^2(T^{max}, b_2)')$ are NE. Then, we consider the attack strategy $\sigma^2 := \frac{1}{2}\sigma^2(T^{max}, b_2) + \frac{1}{2}\sigma^2(T^{max}, b_2)'$. By convexity of the set of NE, we deduce that $(\sigma^1, \sigma^2)$ is also a NE. Besides, the support of $\sigma^2$ contains $T^1 = \{e_1\} \cup \{e_2, \ldots, e_{b_2}\}$ and $T^{1'} = \{e_{n^*}\} \cup \{e_2, \ldots, e_{b_2}\}$ (which are different since $b_2 < m^* = n^*$). Therefore, $T^1$ and $T^{1'}$ should give the same payoff to $P2$.

However, we have the following contradiction:

$$U_2(\sigma^1, T^1) \equiv |T^1| - \sum_{S \in A_1} \sigma^1_S F(S, T^1)$$

$$= \sum_{e \in T^1} \left(1 - \sum_{S \in A_1} \sigma^1_S F(S, e)\right)$$

$$= \left(1 - \sum_{S \in A_1} \sigma^1_S F(S, e_1)\right) + \sum_{e \in T^1 \setminus \{e_1\}} \left(1 - \sum_{S \in A_1} \sigma^1_S F(S, e)\right)$$

$$= \left(1 - \sum_{S \in A_1} \sigma^1_S F(S, e_1)\right) + \sum_{e \in T^1 \setminus \{e_1\}} \left(1 - \sum_{S \in A_1} \sigma^1_S F(S, e)\right)$$
\[(1 - \sum_{S \in A_1 | i \in S} \sigma_s^{1*}) + \sum_{e \in T^1 \setminus \{e_1\}} \left(1 - \sum_{S \in A_1} \sigma_s^{1*} F(S, e)\right) \quad \text{(EC.61)}\]

\[\quad \overset{(15)}{=} (1 - \rho_{\sigma^{1*}}(i_1)) + \sum_{e \in T^1 \setminus \{e_1^*\}} \left(1 - \sum_{S \in A_1} \sigma_s^{1*} F(S, e)\right) \quad \text{(EC.62)}\]

\[> (1 - \rho_{\sigma^{1*}}(i_{n^*})) + \sum_{e \in T^1 \setminus \{e_{n^*}\}} \left(1 - \sum_{S \in A_1} \sigma_s^{1*} F(S, e)\right)\]

\[\overset{(15)}{=} \left(1 - \sum_{S \in A_1} \sigma_s^{1*} F(S, e_{n^*})\right) + \sum_{e \in T^1 \setminus \{e_{n^*}\}} \left(1 - \sum_{S \in A_1} \sigma_s^{1*} F(S, e)\right) \quad \text{(EC.63)}\]

\[\overset{(5)}{=} U_2(\sigma^{1*}, T^1').\]

Note that in (EC.61) (resp. (EC.63)), we used the fact that the node basis of \(\sigma^{1*}\) is \(S^{min}\) and that \(e_1\) (resp. \(e_{n^*}\)) is only monitored by \(i_1\) (resp. \(i_{n^*}\)) in \(S^{min}\). In (EC.62) we used the fact that \(T^1 \setminus \{e_1\} = \{e_2, \ldots, e_{b_2}\} = T' \setminus \{e_{n^*}\} \).

Thus, the sensing probability is necessarily identical among the nodes of \(S^{min}\): \(\exists \gamma \in \mathbb{R} \mid \forall i \in S^{min}, \rho_{\sigma^{1*}}(i) = \sum_{\{S \in A_1 | i \in S\}} \sigma_s^{1*} = \gamma.\) By summing over the nodes of \(S^{min}\), and by combining Prop. 3 with the fact that the node basis of \(\sigma^{1*}\) is \(S^{min}\), we obtain:

\[n^* \gamma = \sum_{i \in S^{min}} \sum_{\{S \in A_1 | i \in S\}} \sigma_s^{1*} = \sum_{i \in S^{min}} \sum_{S \in A_1} \sum_{S \in S^{min}} \sigma_s^{1*} f_{\{i \in S\}} = \sum_{S \in A_1} \sigma_s^{1*} = b_1 \sum_{S \in A_1} \sigma_s^{1*} = b_1.\]

Therefore, \(\gamma = \frac{b_1}{n^*}\), meaning that in any NE \((\sigma^{1*}, \sigma^{2*})\) whose node basis is an MSC, the sensing probability must be equal to \(\frac{b_1}{n^*}\) for all the nodes of the MSC, which proves the necessary condition on the sensing strategy.

ii) Similarly, consider a NE \((\sigma^{1*}, \sigma^{2*}) \in \Sigma\) whose edge basis is an MSP \(T^{max} = \{e_1, \ldots, e_{n^*}\} \in \mathcal{M}\), and assume that the attack probability is not identical among the edges of \(T^{max}\). Again, without loss of generality, we assume that \(\rho_{\sigma^{2*}}(e_1) < \rho_{\sigma^{2*}}(e_{n^*})\).

Consider now an MSC \(S^{min} = \{i_1, \ldots, i_{n^*}\} \in \mathcal{S}\). Again, we reorder the indices such that \(\forall l \in [1, n^*]\), node \(i_l\) only monitors edge \(e_l\) in \(T^{max}\). An analogous construction can be done to obtain a sensing strategy \(\sigma'\) whose node basis is \(S^{min}\), which senses each node in \(S^{min}\) with probability \(\frac{b_1}{n^*}\), and which assigns positive probabilities on the sensor placements \(S' := \{i_1\} \cup \{i_2, \ldots, i_{b_1}\}\) and \(S':= \{i_{n^*}\} \cup \{i_2, \ldots, i_{b_1}\}\). From the sufficient conditions, we know that \(\sigma'\) is an optimal solution of \((LP_{S^{min}})\) and that \((\sigma', \sigma^{2*})\) is a NE. Therefore \(S'\) and \(S''\) should provide \(\textbf{P1}\) with the same payoff. However, we have the following contradiction:

\[U_1(S', \sigma^{2*}) \overset{(4)}{=} \sum_{T \in A_2} \sigma_T^{2*} F(S', T)\]
with the fact that the edge basis of $\sigma^2$ is $T_{\max}$. In (EC.65), we used the fact that $S^1$ only monitors edges $\{e_1\} \cup \{e_2, \ldots, e_{b_1}\}$ of $T_{\max}$. In (EC.66), we used the fact that $S^1 \setminus \{i_1\} = S^1' \setminus \{i_1'\}$.

Thus, the attack probability is identical among the edges of $T_{\max}$: \( \exists \gamma' \in \mathbb{R} \mid \forall e \in T_{\max}, \rho_{\sigma^2}(e) = \sum_{T \in A_2} \sigma^2_T = \gamma' \). By summing over the edges of the MSP and by combining Prop. 3 with the fact the edge basis of $\sigma^2$ is $T_{\max}$, we obtain:

\[
\begin{align*}
\sum_{e \in T_{\max}} \gamma' &= \sum_{e \in T_{\max}} \sum_{T \in A_2} \sigma^2_T = \sum_{e \in T_{\max}} \sum_{T \in A_2} \sigma^2_T \mathbb{1}_{\{e \in T\}} = \sum_{T \in A_2} \sum_{e \in T_{\max}} \mathbb{1}_{\{e \in T\}} \\
&= b_2 \sum_{T \in A_2} \sigma^2_T.
\end{align*}
\]

Thus, $\gamma' = \frac{b_2}{n^*}$, meaning that in any NE $(\sigma^1, \sigma^2)$ whose edge basis is an MSP, the attack probability is equal to $\frac{b_2}{n^*}$ for all the edges of the MSP, which proves the necessary condition on the attack strategy.

\[\square\]

**Proof of Proposition 9.** Let $\sigma^* = (\sigma^1, \sigma^2) \in \Sigma$. Consider an MSC $S_{\min} \in \mathcal{S}$, an MSP $T_{\max} \in \mathcal{M}$, and $(\sigma^1(S_{\min}, b_1), \sigma^2(T_{\max}, b_2))$ constructed in Lemma EC.3.
(i) Thanks to Lemma 2, we know that the NE of \( \Gamma \) are interchangeable. Prop. 8 implies that 
\((\sigma^1(S_{\text{min}}, b_1), \sigma^2(T_{\text{max}}, b_2)) \) defined in (EC.21)-(EC.22) is a NE. Therefore, \((\sigma^1, \sigma^2(T_{\text{max}}, b_2)) \) is also a NE, and we obtain:

\[
\frac{b_1 b_2}{n^*} = \sum_{S \in A_1} \sum_{T \in A_2} \sigma^1_S \sigma^2(T_{\text{max}}, b_2) T F(S, T)
\]

\[
= \sum_{S \in A_1} \sum_{l=1} \sigma^1_S \frac{1}{\mu(n^*, b_2)} \sum_{T \in T^l} F(S, T)
\]

\[
= \sum_{S \in A_1} \sigma^1_S \frac{1}{\mu(n^*, b_2)} \sum_{e \in E} F(S, e) \sum_{T \in T^l} \sum_{l=1} 1_{\{e \in T\}}
\]

\[
= \sum_{S \in A_1} \sigma^1_S \frac{1}{\mu(n^*, b_2)} \sum_{e \in T_{\text{max}}} F(S, e) \sum_{l=1} 1_{\{e \in T\}}
\]

\[
= \sum_{S \in A_1} \sigma^1_S \frac{1}{\mu(n^*, b_2)} \sum_{e \in T_{\text{max}}} F(S, e) \nu(n^*, b_2)
\]

\[
= \frac{b_2}{n^*} \sum_{S \in A_1} \sigma^1_S = \frac{b_2}{n^*} \mathbb{E}_{\sigma^*}[F(S, T_{\text{max}})]
\]

Therefore, we obtain:

\[
\forall \sigma^* \in \Sigma, \forall T_{\text{max}} \in \mathcal{M}, \mathbb{E}_{\sigma^*}[F(S, T_{\text{max}})] = b_1.
\]

(EC.67)

We can deduce further results from this equation. Consider \( \sigma^* \in \Sigma, T_{\text{max}} \in \mathcal{M} \), then:

\[
b_1 = \mathbb{E}_{\sigma^*}[F(S, T_{\text{max}})] \leq \mathbb{E}_{\sigma^*} \left[ \sum_{e \in T_{\text{max}}} F(S, e) \right] \leq \mathbb{E}_{\sigma^*} \left[ \sum_{i \in S} \sum_{e \in T_{\text{max}}} F(i, e) \right]
\]

\[
= \mathbb{E}_{\sigma^*} \left[ \sum_{i \in S} \sum_{e \in T_{\text{max}}} F(i, e) \right] \leq \mathbb{E}_{\sigma^*} \left[ \sum_{i \in S} F(i, T_{\text{max}}) \right]
\]

\[
\leq \mathbb{E}_{\sigma^*} \left[ \sum_{i \in S} 1 \right] = \mathbb{E}_{\sigma^*}[|S|] \Rightarrow b_1.
\]

Therefore, all the previous inequalities become equalities. The first one implies that:

\[
\forall S \in \text{supp}(\sigma^*), \forall e \in T_{\text{max}}, F(S, e) = \sum_{i \in S} F(i, e).
\]
The second induced equality implies that:

\[ \forall i \in \mathcal{V}_{\sigma^1}, \ \forall T^{\max} \in \mathcal{M}, \ F(i, T^{\max}) = 1. \]

(ii) Similarly, by interchangeability \((\sigma^1(S^{\min}, b_1), \sigma^{2^*}) \in \Sigma\). Then:

\[
\frac{b_1 b_2}{n^*} = U_1(\sigma^1(S^{\min}, b_1), \sigma^{2^*}) = \sum_{S \in A_1} \sum_{T \in A_2} \sigma^1(S^{\min}, b_1) T^{\sigma^{2^*}} F(S, T)
\]

\[
= \sum_{k=1} \sum_{T \in A_2} \sigma^1(S^{\min}, b_1) T^{\sigma^{2^*}} F(S^k, T)
\]

\[
= \frac{1}{\mu(n^*, b_1)} \sum_{k=1} \sum_{T \in A_2} \sigma_t^{2^*} \sum_{e \in T} F(S^k, e)
\]

\[
= \frac{1}{\mu(n^*, b_1)} \sum_{e \in A_2} \sigma_t^{2^*} \sum_{k=1} \sum_{e \in T} F(S^k, e).
\]

Since \(S^{\min} \in \mathcal{S}\), then \(\forall e \in \mathcal{E}, \ \exists i_e \in S^{\min}, F(i_e, e) = 1\). Therefore:

\[ \forall e \in \mathcal{E}, \ \sum_{k=1} \sum_{k \in [1, \mu(n^*, b_1)] \mid i_e \in S^k} 1 + \sum_{k \in [1, \mu(n^*, b_1)] \mid i_e \notin S^k} F(S^k, e) \geq |\{k \in [1, \mu(n^*, b_1)] \mid i_e \in S^k\}|
\]

\[ (\Sigma \Rightarrow) \nu(n^*, b_1).
\]

Thus, by plugging it in the previous equation, we obtain:

\[
\frac{b_1 b_2}{n^*} \geq \frac{1}{\mu(n^*, b_1)} \sum_{T \in A_2} \sigma_t^{2^*} \sum_{e \in T} \nu(n^*, b_1) = \frac{1}{\mu(n^*, b_1)} \sum_{T \in A_2} \sigma_t^{2^*} |T| = \frac{1}{n^*} \sum_{T \in A_2} \sigma_t^{2^*} |T| = \frac{b_1}{n^*} b_2.
\]

Therefore, we obtain the following equality:

\[ \forall T \in \text{supp}(\sigma^{2^*}), \ \forall e \in T, \ \forall k \in [1, \mu(n^*, b_1)] \mid i_e \notin S^k, F(S^k, e) = 0. \]  \( (\Sigma \Rightarrow) \)

Now, consider \(T \in \text{supp}(\sigma^{2^*}), e \in T\), and let \(i_e \in S^{\min}\) be a node that satisfies \(F(i_e, e) = 1\). Note that \((\Sigma \Rightarrow)\) can be derived starting from any initial ordering of the node indices in \(S^{\min}\). Consider \(i' \in S^{\min} \mid i' \neq i_e\). Since \(b_1 < n^*\), then we can find an ordering of the node indices in \(S^{\min}\) such that there is a sensor placement \(S^k\) in the support of \(\sigma^1(S^{\min}, b_1)\) that satisfies \(i' \in S^k\) and \(i_e \notin S^k\). From \((\Sigma \Rightarrow)\), we deduce that \(0 \leq F(i', e) \leq F(S^k, e) = 0\).

Therefore, for any \(i' \in S^{\min}\) such that \(i' \neq i_e\), \(i'\) does not monitor \(e\), which implies that \(e\) is only monitored by \(i_e\) in \(S^{\min}\). Thus:

\[ \forall e \in \mathcal{E}_{\sigma^{2^*}}, \ \forall S^{\min} \in \mathcal{S}, \ \exists! i \in S^{\min} \mid F(i, e) = 1. \]
EC.5. Solution of the Network Monitoring Problem

Proof of Proposition 10  When \( n^* = m^* \), we know from Cor. 1 that \( \forall \sigma^* \in \Sigma, r(\sigma^*) = \frac{b_1}{n^*} \). Therefore, \((P)\) can be rewritten as follows:

\[
\begin{align*}
\text{minimize} & \quad b_1 \\
\text{subject to} & \quad \frac{b_1}{n^*} \geq \alpha, \\
& \quad \sigma^\dagger \in \Sigma(b_1, b_2), \\
& \quad b_1 \in \mathbb{N}.
\end{align*}
\]

Then, the optimal value of \((P)\) in that case is \( b_1^\dagger = \lceil \alpha n^* \rceil \).

Now, consider an MSC \( S^{\min} \in \mathcal{S} \) and an MSP \( T^{\max} \in \mathcal{M} \). From Lemma 4 and Prop. 8, we know that \( (\sigma^1(S^{\min}, b_1^\dagger), \sigma^2(T^{\max}, b_2)) \in \Sigma(b_1^\dagger, b_2) \), i.e., is a NE of the game induced by \( b_1^\dagger \) and \( b_2 \). Therefore, \( \lceil \alpha n^* \rceil, (\sigma^1(S^{\min}, b_1^\dagger), \sigma^2(T^{\max}, b_2)) \) is an optimal solution of \((P)\).

Proof of Proposition 11  Recall that in the general case \( m^* < n^* \), we admit a relaxation of \((P)\) and consider instead the following mathematical program with equilibrium constraints:

\[
(P_\varepsilon) : \quad \begin{align*}
\text{minimize} & \quad b_1 \\
\text{subject to} & \quad r(\sigma^*) \geq \alpha, \\
& \quad \forall \sigma^* \in \Sigma(b_1, b_2) \\
& \quad \sigma^\dagger \in \Sigma_\varepsilon(b_1, b_2) \\
& \quad b_1 \in \mathbb{N},
\end{align*}
\]  

( EC.69 )

for some \( \varepsilon \geq 0 \).

In this case, we know from the lower bound in Thm. 1 that \( \forall b_1 < n^*, \forall \sigma^* \in \Sigma(b_1, b_2), \frac{b_1}{n^*} \leq r(\sigma^*) \). Therefore, constraint ( EC.69 ) is satisfied if \( b_1 \geq b_1^\dagger := \lceil \alpha n^* \rceil \). Now, consider an MSC \( S^{\min} \in \mathcal{S} \) and an MSP \( T^{\max} \in \mathcal{M} \). We know from Thm. 2 that \((\sigma^1(S^{\min}, b_1^\dagger), \sigma^2(T^{\max}, b_2)) \in \Sigma_\varepsilon(b_1^\dagger, b_2), \) where \( \epsilon = b_1^\dagger b_2 \left( \frac{1}{\max(b_1^\dagger, m^*)} - \frac{1}{n^*} \right) \). Therefore, \( b_1^\dagger, (\sigma^1(S^{\min}, b_1^\dagger), \sigma^2(T^{\max}, b_2)) \) is a feasible solution of \((P_\varepsilon)\) (with the same \( \epsilon \)), and the corresponding objective value is \( b_1^\dagger \).

Finally, from the upper bound in Thm. 1, we know that \( \forall b_1 < n^*, \forall \sigma^* \in \Sigma(b_1, b_2), r(\sigma^*) \leq \min \left\{ \frac{b_1}{m^*}, 1 \right\} \). Therefore, constraint ( EC.69 ) cannot be satisfied if \( b_1 < \lceil \alpha m^* \rceil \). This implies that an optimality gap associated with \( b_1^\dagger, (\sigma^1(S^{\min}, b_1^\dagger), \sigma^2(T^{\max}, b_2)) \) is given by \( \lceil \alpha n^* \rceil - \lceil \alpha m^* \rceil \).

\( \square \)

EC.6. Case when \( b_2 \geq m^* \)

As argued in Section 4, the network monitoring problem \((P)\) when \( b_2 \geq m^* \) is of limited practical interest. However, for the sake of completeness, we now briefly discuss this case. First, recall from Section 4 that the optimal value of \((P)\) is no more than \( n^* \), since \( \textbf{P1} \) can achieve any target
detection rate when she has at least $n^*$ resources; thus, we will continue to restrict our attention to the game $\Gamma$ when $b_1 < n^*$.

To evaluate the equilibrium constraints (11)-(12), recall that we derived in Section 4 equilibrium properties of the game $\Gamma$ that hold for the case when $b_1 < n^*$ and $b_2 < m^*$. Note that all these properties, except Props. 5 and 6, also hold when $b_1 < n^*$ and $b_2 = m^*$. This implies that Props. 10 and 11, i.e., our (approximate) solution for the network monitoring problem ($P$), are still valid when $b_2 = m^*$.

Unfortunately, most of these properties are not satisfied by the NE of $\Gamma$ when $b_1 < n^*$ and $b_2 > m^*$. For Prop. 1, we can only conclude that given any minimal set cover $S'$, the optimal value of $(LP_{S'})$ is at least $b_2 \left( \frac{b_1}{|S'|} - 1 \right)$. Prop. 2 cannot be applied anymore since there is no set packing of size at least $b_2$. We can find a counterexample (see example EC.3) that violates (21) in Prop. 3; this implies that Prop. 4 does not hold either. Props. 5 and 6 do not hold, even when $b_2 = m^*$: in example EC.3, we find a pure NE whose node basis is not a set cover. Thm. 1 is not valid anymore: we can find NE with different payoffs for $P_1$, and different expected detection rates. Furthermore, the bounds on the players’ equilibrium payoffs, and the upper bound on the equilibrium expected detection rates, do not hold anymore. Thm. 2 does not hold either: we cannot construct a “good” $\varepsilon$–NE from an MSC and an MSP. In Prop. 7, the worst expected detection rate when choosing $\sigma^1(S^{min}, b_1)$ may be arbitrarily far from an equilibrium expected detection rate. Cor. 1, and Props. 8 and 9 do not hold anymore.

Next, we present an illustrative example to discuss the equilibrium properties of the game $\Gamma$ when $b_1 < n^*$ and $b_2 > m^*$.

**Example EC.3.** Consider the network represented in Fig. EC.3.

---

**Figure EC.3** Network which violates some of the equilibrium properties derived in Section 4.
This network is composed of a subnetwork $G_1 = (V_1, E_1)$ and $n$ extra nodes $i_{n+1}, \ldots, i_{2n}$. In $G_1$, there are $n$ nodes $i_1, i_2, \ldots, i_n$ that are connected to $i_{n+1}, \ldots, i_{2n}$ according to Fig. EC.3. Note that $i_{n+1}, \ldots, i_{2n} \notin V_1$ and $e_1, \ldots, e_n \notin E_1$.

Assume that the sensing model is such that a sensor placed at any node of $V_1$ except $i_1, i_2, \ldots, i_n$ cannot monitor any of the edges $e_1, \ldots, e_n$. Then, we assume that a sensor placed at $i_k$ (for $k \in [1, n]$) can monitor $e_k$ and all the edges of $G_1$. Finally, we assume that a sensor placed at $i_{n+k}$ (for $k \in [1, n]$) can monitor $e_k$. More formally, we have:

\[
\forall i \in V_1 \setminus \{i_1, \ldots, i_n\}, \quad C_i \subseteq E_1,
\]

\[
\forall k \in [1, n], \quad C_{i_k} = E_1 \cup \{e_k\},
\]

\[
\forall k \in [1, n], \quad C_{i_{n+k}} = \{e_k\}.
\]

Note that in this example, the only MSC is $S^{\min} = \{i_1, \ldots, i_n\}$, and the only MSP is $T^{\max} = \{e_1, \ldots, e_n\}$; so $n^* = m^* = n$. Now, consider any $b_1 < n$ and any $b_2 \in [n, |E_1| + n]$. Then, one can check that $\forall T \in A_2 \mid T^{\max} \subseteq T, (\{i_1, \ldots, i_{b_1}\}, T) \in \Sigma$, i.e., is a pure NE. This argument shows that Props. 5 and 6 do not hold anymore when $b_2 \geq m^*$ (we found a pure NE, and its node basis is not a set cover).

Now, if we take for example $b_2 = |E_1| + n$, we just showed that we could construct a pure NE where $P2$ can use $n, n + 1, \ldots$, or $n + |E_1| = b_2$ resources. This means that Prop. 3 does not hold anymore (we have equilibria where $P2$ does not use all her resources). This also implies that $P1$’s payoff and the expected detection rate are not constant in equilibrium anymore: We found equilibria where $P1$’s payoff is equal to $b_1, b_1 + 1, \ldots, b_1 + |E_1|$, which violates Cor. 1. This corresponds to equilibrium detection rates equal to $\frac{b_1}{n^*}, \frac{b_1+1}{n^*}, \ldots, \frac{b_1+|E_1|}{n^*}$. Since $\frac{b_1+|E_1|}{n^*} \rightarrow 1$, then the upper bound on the expected detection rates given by Thm. 1 is violated. Furthermore, the bound derived in Prop. 7 is not valid anymore. By choosing $\sigma^1(S^{\min}, b_1)$, the expected detection rate may be arbitrarily far from an equilibrium expected detection rate: we can only trivially bound the difference with $1 - \frac{b_1}{n^*}$.

Still, some results remain valid when $b_1 < n^*$ and $b_2 > m^*$: In Prop. 3, (20) still holds. In Thm. 1, the lower bound on the equilibrium expected detection rates is still valid. In Prop. 7, when choosing $\sigma^1(S^{\min}, b_1)$, the expected detection rate is still guaranteed to be at least $\frac{b_1}{n^*}$, regardless of $P2$’s strategy.

From these remaining results, we can show that $b_1^* = \lceil \alpha n^* \rceil$ is still a sufficient condition for the expected detection rates in equilibrium to be at least $\alpha$, and provides an upper bound on the optimal value $b_1^*$ of (P). Furthermore, given an MSC, $S^{\min}$, if $P1$ allocates these $b_1^*$ sensors according to the sensing strategy $\sigma^1(S^{\min}, b_1^*)$, she will still be guaranteed to detect a fraction $\alpha$ of the failures in expectation, regardless of which strategy $P2$ chooses.