WEAK MEAN EQUICONTINUITY FOR A COUNTABLE DISCRETE AMENABLE GROUP ACTION

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Abstract. The weak mean equicontinuous properties for a countable discrete amenable group $G$ acting continuously on a compact metrizable space $X$ are studied. It is shown that the weak mean equicontinuity of $(X \times X, G)$ is equivalent to the mean equicontinuity of $(X, G)$. Moreover, when $(X, G)$ has full measure center or $G$ is abelian, it is shown that $(X, G)$ is weak mean equicontinuous if and only if all points in $X$ are uniquely ergodic points and the map $x \mapsto \mu_x^G$ is continuous, where $\mu_x^G$ is the unique ergodic measure on $\{\text{Orb}(x), G\}$.

1. Introduction

Throughout this paper, $G$ is a countable infinite discrete amenable group. By a $G$-system we mean a pair $(X, G)$, where $X$ is a compact metric space with metric $d$ and $G$ acts as a group of continuous maps from $X$ to itself. In particular, when $G = \mathbb{Z}$ the $\mathbb{Z}$-system $(X, \mathbb{Z})$ can be considered inducing by a homeomorphism $T$ from $X$ to itself. In this case we say that $(X, T)$ is a topological dynamical system (t.d.s. for short).

In the theory of topological dynamical systems, several notions of continuity have been studied. People firstly focused on equicontinuous systems, because they have simple dynamical behaviors [1, 7]. In recent years, mean equicontinuity has received keen interest. We refer to [5, 6, 8, 10, 12] for further study on mean equicontinuity and related subjects. But in the measure theoretic point of view, we are only interested in the cumulative effect of points in orbits, in which the order makes no sense. To ignore the order, Zheng and Zheng [18] introduced the notion weak mean equicontinuity. They show that a t.d.s. is weak mean equicontinuous if and only if the time average of any continuous function converges to a continuous function.

The aim of this paper is to study the weak mean equicontinuity for $G$-actions. In order to make clear statement of our results, we introduce the following notations.

A countable infinite discrete group $G$ is amenable [3] if there exists a sequence of finite nonempty subsets $\{F_n\}_{n=1}^{\infty}$ of $G$, which is called a left Følner sequence, such that $\lim_{n\to+\infty} \frac{|gF_n \Delta F_n|}{|F_n|} = 0$ for every $g \in G$, where $|\cdot|$ denotes the cardinality of a
set. Every abelian group is amenable. For example, if $G = \mathbb{Z}^d$, then $\{F_n\}_{n=1}^\infty$ is a left Følner sequence of $G$, where $F_n = [-n, n]^d \cap \mathbb{Z}^d$ for all $n \in \mathbb{N}^+$.

Denote by $M(X, G)$ the set of all $G$-invariant Borel probability measures. It is well known that for a countable infinite amenable group $G$, $M(X, G)$ is not empty. In particular, we say $(X, G)$ is uniquely ergodic if $M(X, G)$ is singleton.

Given a $G$-system $(X, G)$, a finite nonempty subset $F$ of $G$ and $x \in X$, define

$$\mu_{x,F} = \frac{1}{|F|} \sum_{g \in F} \delta_{gx}$$

where $\delta_z$ is the Dirac measure which has full measure on point $z$. For a left Følner sequence $F = \{F_n\}_{n=1}^\infty$ of $G$ and $x, y \in X$, define

$$W_F(x, y) = \limsup_{n \to +\infty} W(\mu_{x,F_n}, \mu_{y,F_n}),$$

where $W(\cdot, \cdot)$ is the Wasserstein distance (see Section 2 for definition) on Borel probability measure space. Since $W(\cdot, \cdot)$ satisfies triangle inequality, we have that $W_F(x, y)$ also satisfies triangle inequality. Hence it is a pseudometric on $X$.

**Definition 1.1.** Let $(X, G)$ be a $G$-system and $F$ be a left Følner sequence of $G$. We say $(X, G)$ is weak mean equicontinuous with respect to $F$ if for any $\varepsilon > 0$, there is $\delta > 0$ such that $W_F(x, y) < \varepsilon$ whenever $x, y \in X$ with $d(x, y) < \delta$. Specially, we call $(X, G)$ is weak mean equicontinuous if it is weak mean equicontinuous with respect to all left Følner sequences of $G$.

**Remark 1.2.** As in [18], we can define $W_F(\cdot, \cdot)$ by ignoring the order. In Appendix, we will show that the two definitions are the same.

Let $F = \{F_n\}_{n=1}^\infty$ be a left Følner sequence of $G$. Then $(X, G)$ is weak mean equicontinuous with respect to $F$ if and only if $W_F(x, y)$ is continuous on $X \times X$. Recall that $(X, G)$ is mean equicontinuous with respect to $F$ if for any $\varepsilon > 0$, there is $\delta > 0$ such that

$$\limsup_{n \to +\infty} \frac{1}{|F_n|} \sum_{g \in F_n} d(gx, gy) < \varepsilon$$

whenever $x, y \in X$ and $d(x, y) < \delta$. Hence, the relation of mean equicontinuity and weak mean equicontinuity can be stated as follows.

**Theorem 1.3.** Let $(X, G)$ be a $G$-system and $F$ be a left Følner sequence of $G$. Then the following two statements are equivalent:

1. $(X, G)$ is mean equicontinuous with respect to $F$;
2. $(X \times X, G)$ is weak mean equicontinuous with respect to $F$.

Here $(X \times X, G)$ is the product system of $(X, G)$ such that $g(x, y) = (gx, gy)$ holds for any $g \in G$ and $(x, y) \in X \times X$. The metric $\tilde{d}$ on $X \times X$ is defined by

$$\tilde{d}((x, y), (x', y')) = d(x, x') + d(y, y')$$

for $x, x', y, y' \in X$. 


For a $G$-system $(X, G)$, the measure center is $\cup_{\mu \in M(X,G)} \text{supp} \mu$, which is the minimal compact subset of $X$ with full measure for any invariant measure $\mu \in M(X, G)$. We say $x \in X$ is a uniquely ergodic point if the $G$-system $(\text{Orb}(x), G)$ is uniquely ergodic, where $\text{Orb}(x) = \{gx : g \in G\}$ is the orbit of $x$. If $x \in X$ is uniquely ergodic point, we note the unique ergodic measure of $(\text{Orb}(x), G)$ by $\mu^G_x$. Recently, Frantzikinakis and Host [4] prove that in a zero entropy system, all uniquely ergodic points satisfy the logarithmic Sarnak conjecture. Thus it is interesting to consider the case when all the points in a $G$-system are uniquely ergodic points. In this paper, we prove that in a weak mean equicontinuous system, all the points in the measure center are uniquely ergodic points.

We also focus on uniform convergence, a classic topic of dynamical systems for it meets with more favor in mathematical physics. In 1952, Oxtoby [14] shows that the Birkhoff average of any continuous function converges uniformly in uniquely ergodic systems. In 1982, Johnson and Moser [9] prove that for a t.d.s. $(X, T)$, if a continuous function $f \in C(X)$ is orthogonal to all invariant measures (i.e. $\int f d\mu = 0$ holds for any $\mu \in M(X, T)$), then the Birkhoff average converges to 0 uniformly. Recently, Zhang and Zhou [17] also study related topics. In this paper, we show that for weak mean equicontinuous systems, Birkhoff averages of continuous functions are uniformly convergent on measure center.

**Theorem 1.4.** Let $(X, G)$ be a $G$-system. If the measure center of $(X, G)$ is $X$, then the following statements are equivalent.

1. $(X, G)$ is weak mean equicontinuous;
2. $(X, G)$ is weak mean equicontinuous with respect to a Følner sequence of $G$;
3. All points in $X$ are uniquely ergodic points and the map $x \to \mu^G_x$ is continuous;
4. For any $f \in C(X)$, there exists $f^* \in C(X)$ such that for any left Følner sequence $F = \{F_n\}$ of $G$,
\[ \frac{1}{|F_n|} \sum_{g \in F_n} f(gx) \text{ converges to } f^*(x) \text{ uniformly.} \]

By Theorem 1.4, we have the following.

**Corollary 1.5.** Let $(X, G)$ be a minimal $G$-system. Then the following are equivalent:

1. $(X, G)$ is weak mean equicontinuous.
2. All points in $X$ are uniquely ergodic points and the map $x \to \mu^G_x$ is continuous;
3. $(X, G)$ is uniquely ergodic.

**Question 1.6.** Is the condition “If the measure center of $(X, G)$ is $X$” necessary in Theorem 1.4?

We call a sequence of finite nonempty subsets $\{L_n\}_{n=1}^\infty$ of $G$ is a right Følner sequence if $\lim_{n\to\infty} \frac{|L_n \Delta L_n g|}{|L_n|} = 0$ for every $g \in G$. It is easy to check that $\{L_n\}_{n=1}^\infty$
is a right Følner sequence if and only if \( \{L_n^{-1}\}_{n=1}^{\infty} \) is a left Følner sequence, where \( L_n^{-1} = \{g^{-1} : g \in L_n\} \) for all \( n \in \mathbb{N}^+ \).

**Theorem 1.7.** Let \((X, G)\) be a \(G\)-system and \(\mathcal{L}\) be a left and right Følner sequence of \(G\). Then the following statements are equivalent.

1. \((X, G)\) is weak mean equicontinuous;
2. \((X, G)\) is weak mean equicontinuous with respect to \(\mathcal{L}\);
3. All points in \(X\) are uniquely ergodic points and the map \(x \mapsto \mu^G_x\) is continuous;
4. For any \(f \in C(X)\), there exists \(f^* \in C(X)\) such that for any left Følner sequence \(\mathcal{F} = \{F_n\}\) of \(G\),
   \[
   \frac{1}{|F_n|} \sum_{g \in F_n} f(gx) \text{ converges to } f^*(x) \text{ uniformly.}
   \]

**Remark 1.8.**

1. If \(G\) is a finite extension of a countable discrete abelian group, then every left Følner sequence of \(G\) is a right Følner sequence (see for example [13], Section 1.1). Thus, the group which satisfies the condition of Theorem 1.7 is not need to be abelian group.
2. For some amenable group \(G\), there exist left Følner sequences of \(G\) which are not right Følner sequences (see for example [13], Section 1.1).

This paper is organized as follows. In Section 2 we introduce some basic notions and results needed in the paper. In Section 3 we prove Theorem 1.3. In Section 4 we prove Theorem 1.4 and Theorem 1.7. In Appendix A we show that our definition is the same as in [18].

## 2. Preliminaries

In this section we recall some notions and results of \(G\)-systems which are needed in our paper. Note that \(\mathbb{N}^+\) denotes the set of all positive integers in this paper.

### 2.1. Measures.

Suppose that \((X, G)\) is a \(G\)-system. Denote by \(\mathcal{B}(X)\) the Borel \(\sigma\)-algebra of \(X\) and \(M(X)\) the set of all Borel probability measures on \(X\). For \(\mu \in M(X)\), denote by \(\text{supp} \mu\) the support of \(\mu\), i.e. the smallest closed subset \(W \subset X\) such that \(\mu(W) = 1\). In the weak*-topology, \(M(X)\) is a nonempty compact convex space.

#### 2.1.1. Metrics on measure space.

Given \(\mu, \nu \in M(X)\), a transport plan from \(\mu\) to \(\nu\) is a probability measure \(\pi\) on the product space \(X \times X\) such that

\[
(p_1)_* \pi = \mu \quad \text{and} \quad (p_2)_* \pi = \nu,
\]

where \(p_1, p_2 : X \times X \to X\) are the canonical projections, and \((p_i)_* \pi = \pi \circ p_i^{-1}\) for \(i = 1, 2\). Let \(\Pi(\mu, \nu)\) denote the set of all transport plans from \(\mu\) to \(\nu\). Then the
Wasserstein distance between $\mu$ and $\nu$ is defined as:

$$W(\mu, \nu) = \inf_{\pi \in \Pi(\mu, \nu)} \int d(x, y)d\pi(x, y).$$

The integral in the above formula is called the cost of the transport plan $\pi$, and the infimum is always attained. The Wasserstein distance is a metric on $M(X)$ and the induced topology is just the weak*-topology on $M(X)$ (see for example [15], Theorem 7.3 and Theorem 7.12).

Let $\{f_n\}_{n=1}^\infty$ be a countable dense subset of continuous functions space $C(X)$. For any $\mu, \nu \in M(X)$, define

$$\rho(\mu, \nu) = \sum_{i=1}^\infty \frac{\left| \int f_id\mu - \int f_id\nu \right|}{2^i(||f_i|| + 1)},$$

where $||f_i|| = \max\{|f_i(x)| : x \in X\}$ for all $i \in \mathbb{N}^+$. Then $\rho$ is a metric on $M(X)$ and the induced topology is also the weak*-topology on $M(X)$ (see for example [16], Theorem 6.4). Thus, $W(\cdot, \cdot)$ and $\rho(\cdot, \cdot)$ are compatible.

2.1.2. Invariant measures. We say $\mu \in M(X)$ is $G$-invariant if $\mu(g^{-1}B) = \mu(B)$ holds for all $B \in B(X)$ and $g \in G$. Denote by $M(X, G)$ the collection of all $G$-invariant Borel probability measures. In the weak*-topology, $M(X, G)$ is a nonempty compact convex space.

We say $B \in B(X)$ is $G$-invariant if $g^{-1}B = B$ for any $g \in G$. And $\mu \in M(X, G)$ is ergodic if for any $G$-invariant Borel set $B \in B(X)$, $\mu(B) = 0$ or $\mu(B) = 1$ holds. Denote by $M^e(X, G)$ the collection of all ergodic measures on $(X, G)$. It is well known that $M^e(X, G)$ is the collection of all extreme points of $M(X, G)$. Thus, $M^e(X, G)$ is nonempty. Using Choquet representation theorem, for each $\mu \in M(X, G)$ there is a unique measure $\tau$ on the Borel subsets of the compact space $M(X, G)$ such that $\tau(M^e(X, G)) = 1$ and $\mu = \int_{M^e(X, G)} md\tau(m)$, which is called the ergodic decomposition of $\mu$.

We say $(X, G)$ is uniquely ergodic if $M^e(X, G)$ is singleton. Since $M^e(X, G)$ is the set of extreme points of $M(X, G)$, then $(X, G)$ is uniquely ergodic if and only if $M(X, G)$ is singleton.

Similar to Birkhoff pointwise ergodic, Lindenstrauss established the pointwise ergodic theorem on $G$-systems [11]. A left Følner sequence $\mathcal{F} = \{F_n\}_{n=1}^\infty$ of $G$ is tempered if there exists some $C > 0$ such that

$$| \bigcup_{k<n} F_k^{-1}F_n | \leq C|F_n|$$

for all $n \in \mathbb{N}^+$. And for every left Følner sequence, there is a subsequence which is tempered.

**Lemma 2.1** (Pointwise Ergodic Theorem). *Let $(X, G)$ be a $G$-system and $\mu$ be a $G$-invariant Borel probability measure. Suppose $\mathcal{F} = \{F_n\}_{n=1}^\infty$ is a tempered Følner sequence of $G$. Then for any $f \in L^1(X, \mu)$, there is a $G$-invariant $f^* \in L^1(X, \mu)$ such that $f^*(x) = \frac{1}{|F_n|} \int_{F_n} f(y)dy$ for all $x \in F_n$. Furthermore, this limit is independent of the choice of $n$.***
such that
\[ \lim_{n \to +\infty} \frac{1}{|F_n|} \sum_{g \in F_n} f(gx) = f^*(x) \quad a.e. \]

In particular, if \( \mu \) is ergodic, one has
\[ \lim_{n \to +\infty} \frac{1}{|F_n|} \sum_{g \in F_n} f(gx) = \int f(x) d\mu(x) \quad a.e. \]

Given a left Følner sequence \( \mathcal{F} = \{F_n\}_{n=1}^\infty \) of \( G \) and \( x \in X \), we have \( \{\mu_{x,F_n}\}_{n=1}^\infty \subset M(X) \). Denote by \( M_F^x \) the collection of all limit points of \( \{\mu_{x,F_n}\}_{n=1}^\infty \). Since \( M(X) \) is compact, we have \( M_F^x \neq \emptyset \). Moreover, \( M_F^x \subset M(X,G) \). If \( M_F^x = \{\mu\} \), say \( x \) is a generic point of \( \mu \) with respect to \( \mathcal{F} \). And the pointwise ergodic theorem shows that if \( \mu \in M^e(X,T) \) and \( H \) is a tempered Følner sequence of \( G \), then the set of all generic points of \( \mu \) with respect to \( H \) has full measure.

3. Proof of Theorem 1.3

In this section, we will prove that \( (X, G) \) is weak mean equicontinuous with respect to a left Følner sequence \( \mathcal{F} \) of \( G \) if and only if \( (X \times X, G) \) is mean equicontinuous with respect to \( \mathcal{F} \).

Proof of Theorem 1.3. Let \( (X, G) \) be a \( G \)-system and \( \mathcal{F} = \{F_n\}_{n \in \mathbb{N}} \) be a left Følner sequence of \( G \). Firstly we assume that \( (X, G) \) is mean equicontinuous with respect to \( \mathcal{F} \). Then for given \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that
\[ \limsup_{n \to +\infty} \frac{1}{|F_n|} \sum_{g \in F_n} \tilde{d}(gx, gy) < \varepsilon, \]
whenever \( x, y \in X \) with \( d(x, y) < \delta \). Hence for \( (x_1, y_1), (x_2, y_2) \in X \times X \) satisfy \( d(x_1, x_2) < \delta \) and \( d(y_1, y_2) < \delta \), since \( \mu((x_1, y_1), (x_2, y_2)), F_n \in \Pi(\mu(x_1,y_1), F_n, \mu(x_2,y_2), F_n) \) for any \( n \in \mathbb{N}^+ \), one has
\[ W_\mathcal{F}((x_1, y_1), (x_2, y_2)) = \limsup_{n \to +\infty} W(\mu(x_1,y_1), F_n, \mu(x_1,y_1), F_n) \]
\[ \leq \limsup_{n \to +\infty} \int \tilde{d}((x, y), (x', y')) d\mu((x_1, y_1), (x_2, y_2)), F_n \]
\[ = \limsup_{n \to +\infty} \frac{1}{|F_n|} \sum_{g \in F_n} \tilde{d}(g(x_1, y_1), g(x_2, y_2)) \]
\[ = \limsup_{n \to +\infty} \frac{1}{|F_n|} \sum_{g \in F_n} (d(gx_1, gx_2) + d(gy_1, gy_2)) \]
\[ < 2\varepsilon. \]
This implies that \( (X \times X, G) \) is weak mean equicontinuous with respect to \( \mathcal{F} \).

Now assume that \( (X \times X, G) \) is weak mean equicontinuous with respect to \( \mathcal{F} \). Then for given \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that if \( (x_1, y_1), (x_2, y_2) \in X \times X \) satisfy
\[ d(x_1, x_2) < \delta \text{ and } d(y_1, y_2) < \delta, \text{ then } W_F((x_1, y_1), (x_2, y_2)) < \varepsilon. \text{ Hence for } x, y \in X \text{ with } d(x, y) < \delta, \text{ one has} \]

\[ \epsilon > W_F((x, y), (y, y)) = \limsup_{n \to \infty} W(\mu_{(x,y),F_n}, \mu_{(y,y),F_n}) \]

\[ \geq \limsup_{n \to \infty} \int_{X \times X} d(x', y') d\mu_{(x,y),F_n}(x', y') \]

\[ = \limsup_{n \to +\infty} \frac{1}{|F_n|} \sum_{g \in F_n} d(gx, gy). \]

Therefore, \((X, G)\) is mean equicontinuous with respect to \(F\). This ends the proof of Theorem 1.3. \(\Box\)

4. Proof of Theorem 1.4 and Theorem 1.7

In this section, we study uniquely ergodic points and uniform convergence. Firstly we show some properties of uniquely ergodic points, which are useful to prove Theorem 1.4 and Theorem 1.7.

**Lemma 4.1.** Let \((X, G)\) be a \(G\)-system and \(x \in X\) be a uniquely ergodic point. Then for any left Følner sequence \(F\) of \(G\), we have \(\mu_{x,F}\) exists and \(\mu_{x,F} = \mu_G^x\).

**Proof.** Since \(x\) is a uniquely ergodic point, then \(M(\text{Orb}(x), G) = \{\mu_G^x\}\). Thus, \(M_x^F = \{\mu_G^x\}\), which implies \(\mu_{x,F}\) exists and \(\mu_{x,F} = \mu_G^x\). \(\Box\)

**Lemma 4.2.** Let \((X, G)\) be a \(G\)-system and \(F\) be a left Følner sequence of \(G\). Then the following two statements are equivalent:

1. \((X, G)\) is uniquely ergodic;
2. \(W_F(x, y) = 0\) for all \(x, y \in X\).

**Proof.** (1) \(\Rightarrow\) (2) Let \(\mu\) be a uniquely ergodic measure on \((X, G)\). Given \(x, y \in X\), by Lemma 4.1 we have \(\mu_{x,F} = \mu_{y,F} = \mu\). Thus,

\[ W_F(x, y) = W(\mu_{x,F}, \mu_{y,F}) = W(\mu, \mu) = 0. \]

(2) \(\Rightarrow\) (1) Let \(H\) be a tempered subsequence of \(F\). Assume that \((X, G)\) is not uniquely ergodic, then there exist distinct ergodic measures \(\mu\) and \(\nu\) in \(M(X, G)\). By pointwise ergodic theorem, there is \(x \in X\) (resp. \(y \in X\)) which is generic point of \(\mu\) (resp. \(\nu\)) with respect to \(H\). Then

\[ W_F(x, y) \geq W_H(x, y) = W(\mu, \nu) > 0. \]

This is impossible. Therefore, \((X, G)\) is uniquely ergodic. \(\Box\)

**Lemma 4.3.** Let \((X, G)\) be a weak mean equicontinuous \(G\)-system with respect to a left Følner sequence \(F\) of \(G\). For \(x \in X\), the following two statements are equivalent:

1. \(x\) is a uniquely ergodic point;
2. \(W_F(x, gx) = 0\) for all \(g \in G\).
Proof. (1) ⇒ (2) is immediately from Lemma 4.2.

(2) ⇒ (1) Since \( W_F(x, gx) = 0 \) for all \( g \in G \) and \( (X, G) \) is weak mean equicontinuous with respect to \( F \), one has \( W_F(x, y) = 0 \) for all \( y \in \text{Orb}(x) \). Then for all \( y_1, y_2 \in \text{Orb}(x) \), one has

\[
W_F(y_1, y_2) \leq W_F(x, y_1) + W_F(x, y_2) = 0.
\]

Hence, \( W_F(y_1, y_2) = 0 \) for all \( y_1, y_2 \in \text{Orb}(x) \). By Lemma 4.2, \( (\text{Orb}(x), G) \) is uniquely ergodic. Hence, \( x \) is a uniquely ergodic point. \( \square \)

Lemma 4.4. Let \( (X, G) \) be a weak mean equicontinuous \( G \)-system with respect to a left Følner sequence \( F \) of \( G \). If \( \mu \in M^e(X, G) \), then \( (\text{supp} \mu, G) \) is uniquely ergodic. Thus all the points in \( \text{supp} \mu \) are uniquely ergodic points.

Proof. Assume that \( (\text{supp} \mu, G) \) is not uniquely ergodic. Then there exists \( \nu \in M^e(X, G) \) with \( \mu \neq \nu \) such that \( \text{supp} \nu \subset \text{supp} \mu \).

Let \( H \) be a tempered subsequence of \( F \). Then there exist \( x_m, y \) such that \( \mu_{x_m, H} = \mu \) for any \( m \in \mathbb{N}^+ \), \( \mu_{y, H} = \nu \) and \( \lim_{m \to \infty} x_m = y \).

Thus,

\[
W_F(x_m, y) \geq W_H(x_m, y) = W(\mu, \nu) > 0 \text{ for any } m \in \mathbb{N}^+.
\]

This contradicts with the condition that \( (X, G) \) is weak mean equicontinuous with respect to \( F \). Hence, \( (\text{supp} \mu, G) \) is uniquely ergodic. \( \square \)

Now we prove Theorem 1.4.

Proof of Theorem 1.4. We will show that (1) ⇒ (2) ⇒ (3) ⇒ (4) ⇒ (1).

(1) ⇒ (2) is obvious.

(2) ⇒ (3) Assume that \( (X, G) \) is weak mean equicontinuous with respect to a left Følner sequence \( F \). Lemma 4.2 shows that all points in \( \bigcup_{\mu \in M^e(X, G)} \text{supp} \mu \) are uniquely ergodic points. And by Lemma 4.3 one has \( W_F(x, gx) = 0 \) for every \( g \in G \) and \( x \in \bigcup_{\mu \in M^e(X, G)} \text{supp} \mu \). Then by the weak equicontinuity, one has \( W_F(x, gx) = 0 \) for all \( g \in G \) and \( x \in \bigcup_{\mu \in M^e(X, G)} \text{supp} \mu \). Thus, all points in \( \bigcup_{\mu \in M^e(X, G)} \text{supp} \mu \) are uniquely ergodic points by Lemma 4.3.

For \( x \in X \), one has \( x \in \bigcup_{\mu \in M^e(X, G)} \text{supp} \mu = \bigcup_{\mu \in M^e(X, G)} \text{supp} \mu \), which shows that \( x \) is a uniquely ergodic point. Thus by Lemma 4.1 one deduces that \( \mu_{x, F} \) exists and \( \mu_{x, F} = \mu_x^G \). The continuity of the map \( x \to \mu_x^G \) is immediately from the assumption that \( (X, G) \) is weak mean equicontinuous.

(3) ⇒ (4) Given a left Følner sequence \( F = \{F_n\}_{n=1}^\infty \) of \( G \), one has that

\[
\lim_{n \to \infty} \mu_{x, F_n} = \mu_x^G \text{ for any } x \in X.
\]
To our aim, it is enough to show that the convergence is uniformly. Assume that the convergence is not uniformly. Then there exists $\varepsilon > 0$, $x_i \in X$ and $n_i \to \infty$ such that

$$\rho(\mu_{x_i}, F_{n_i}, \mu_{x_i}^G) \geq \varepsilon$$

for all $i \in \mathbb{N}^+$. Passing by a subsequence, we can assume that

$$\lim_{i \to \infty} x_i = x^*$$

and

$$\lim_{i \to \infty} \mu_{x_i}, F_{n_i} = \mu.$$

It is clear that $\mu \in M(X, G)$ and

$$\rho(\mu, \mu_{x^*}^G) = \rho(\lim_{i \to \infty} \mu_{x_i}, F_{n_i}, \lim_{i \to \infty} \mu_{x_i}^G) = \lim_{i \to \infty} \rho(\mu_{x_i}, F_{n_i}, \mu_{x_i}^G) \geq \varepsilon.$$

By the ergodic decomposition theorem, there exists $\tilde{\mu} \in M^c(X, G)$ such that

$$\rho(\tilde{\mu}, \mu_{x^*}^G) \geq \varepsilon$$

and $\text{supp} \tilde{\mu} \subseteq \text{supp} \mu$. Fix $y \in \text{supp} \tilde{\mu}$, then $\tilde{\mu} = \mu_y^G$. And there exists $y_i \in \overline{Orb}(x_i), i \in \mathbb{N}^+$ such that

$$\lim_{i \to \infty} y_i = y. \quad (4.1)$$

For $i \in \mathbb{N}^+$, since $\overline{Orb}(x_i)$ is uniquely ergodic, we have $\mu_{x_i}^G = \mu_{y_i}^G$. Thus,

$$\lim_{i \to \infty} \sup \rho(\mu_{y_i}^G, \mu_{y}^G) = \lim_{i \to \infty} \rho(\mu_{x_i}^G, \mu_{y}^G) = \rho(\mu_{x^*}, \tilde{\mu}) \geq \varepsilon.$$

Combining this with (4.1), the map $z \to \mu_z^G$ is not continuous which is contradictory to (3). Thus, $(3) \Rightarrow (4)$ is valid.

$(4) \Rightarrow (1)$ Given a left Folner sequence $\mathcal{F} = \{F_n\}_{n=1}^\infty$ of $G$, then by (4) for any $x \in X$, $\mu_{x, \mathcal{F}}$ exists. Now we assume that $(X, G)$ is not weak mean equicontinuous with respect to $\mathcal{F}$. Then there are $\varepsilon > 0$ and $y_m, y \in X$ with $y_m \to y$ such that

$$W_{\mathcal{F}}(y_m, y) > \varepsilon.$$

This implies that $W(\mu_{y_m, \mathcal{F}}, \mu_{y, \mathcal{F}}) > \varepsilon$ for any $m \in \mathbb{N}^+$. Since $W(\cdot, \cdot)$ and $\rho(\cdot, \cdot)$ are compatible, there is $\delta > 0$ such that $\rho(\mu_{y_m, \mathcal{F}}, \mu_{y, \mathcal{F}}) > \delta$ for any $m \in \mathbb{N}^+$. Passing by a subsequence, there exist $\delta' \in (0, \infty)$ and $f \in C(X)$ such that

$$| \int_X f d\mu_{y, \mathcal{F}} - \int_X f d\mu_{y_m, \mathcal{F}} | > \delta' \text{ for all } m \in \mathbb{N}^+.$$

Then

$$| f^*(y) - f^*(y_m) | > \delta' \text{ for all } m \in \mathbb{N}^+.$$ 

Since $\lim_{m \to \infty} y_m = y$, one has $f^*$ is not continuous. This contradicts with the assumption. Hence, $(X, G)$ is weak mean equicontinuous.

To prove Theorem 1.7, we need only to show $(2) \Rightarrow (3)$, for the proof of $(3) \Rightarrow (4) \Rightarrow (1) \Rightarrow (2)$ is as same as the proof of Theorem 1.4.
Proof of Theorem 1.7 (2) ⇒ (3)

Fix $x \in X$ and assume that $\mathcal{L} = \{L_n\}_{n=1}^{\infty}$. Then for $g \in G$, by Theorem A.1 one has

$$W_{\mathcal{L}}(x, gx) = \limsup_{n \to \infty} W(\mu_{x,L_{n}}, \mu_{gx,L_{n}}) = \limsup_{n \to \infty} \inf_{h \in \text{Aut}(L_{n})} \frac{1}{|L_{n}|} \sum_{l \in L_{n}} d(lx, l^{h}gx) = 0,$$

where $\text{Aut}(F)$ is the permutation group of $F$, and the last equality comes from that $\mathcal{L}$ is right Følner sequence. Then by Lemma 4.3 one has that $x$ is a uniquely ergodic point. With the same reason in Theorem 1.4, one has that map $x \to \mu_{x}^{G}$ is continuous. □

**Appendix A.**

In this section, we will prove our definition of weak mean equicontinuity is the same as in [18]. To our aim, we need some definitions and lemmas which are stated as follow.

Let $(X, G)$ be a $G$-system and $F = \{F_{n}\}_{n=1}^{\infty}$ be a left Følner sequence of $G$. For any $x, y \in X$, define

$$W_{F}^{\text{Aut}}(x, y) = \limsup_{n \to \infty} \inf_{h \in \text{Aut}(F_{n})} \frac{1}{|F_{n}|} \sum_{g \in F_{n}} d(gx, g^{h}y),$$

where $\text{Aut}(F)$ is the permutation group of $F$ and $g^{h} = h(g)$.

Our main result in this section is the following.

**Theorem A.1.** Let $(X, G)$ be a $G$-system and $F = \{F_{n}\}_{n=1}^{\infty}$ be a left Følner sequence of $G$. Then $W_{F}^{\text{Aut}}(x, y) = W_{\mathcal{L}}(x, y)$ for every $x, y \in X$.

**Definition A.2.** An $n \times n$ matrix $A = (a_{ij}) : i, j = 1, 2, \cdots, n$ is called doubly stochastic provided it is non-negative and the sum of entries in every row and every column is 1. And denote by $\text{DM}(n)$ the set of all $n \times n$ doubly stochastic matrices.

Given $A \in \text{DM}(|F_{n}|)$. For $z_{1}, z_{2} \in X$, put

$$\pi_{A}(z_{1}, z_{2}) = \frac{1}{|F_{n}|} \sum_{g_{1}x = z_{1}, g_{2}y = z_{2}} A(g_{1}, g_{2}).$$

Then $\pi_{A} \in \Pi(\mu_{x,F_{n}}, \mu_{y,F_{n}})$. Define $\Psi : \text{DM}(|F_{n}|) \longrightarrow \Pi(\mu_{x,F_{n}}, \mu_{y,F_{n}})$ such that $\Psi(A) = \pi_{A}$. It is easy to check that $\Psi$ is surjective.

**Definition A.3.** Let $\sigma \in S_{n}$, where $S_{n}$ is the $n$-order permutation group. The permutation matrix $A^{\sigma}$ is the $n \times n$ matrix $A^{\sigma} = (a^{\sigma}_{ij}) : i, j = 1, 2, \cdots, n$, defined as follows:

$$a^{\sigma}_{ij} = \begin{cases} 1 & \text{if } \sigma(j) = i, \\ 0 & \text{otherwise.} \end{cases}$$

And denote by $\text{PM}(n)$ the set of all $n \times n$ permutation matrices.
For $A \in PM(|F_n|)$, there is $h \in Aut(F_n)$ such that $\Psi(A) = \frac{1}{|F_n|} \sum_{g \in F_n} \delta_{(g_x,g^h y)}$. On the other hand, for $h \in Aut(F_n)$, let $\pi_h = \frac{1}{|F_n|} \sum_{g \in F_n} \delta_{(g_x,g^h y)}$ and

$$A_h(g_1, g_2) = \begin{cases} 1 & \text{if } g_2 = g_1^h \\ 0 & \text{otherwise.} \end{cases}$$

Then $A_h \in PM(|F_n|)$ and $\Psi(A_h) = \pi_h$. Thus,

$$\Psi(PM(|F_n|)) = \{ \frac{1}{|F_n|} \sum_{g \in F_n} \delta_{(g_x,g^h y)} : h \in Aut(F_n) \}.$$

The relation between $DM(n)$ and $PM(n)$ was shown by Birkhoff and Von Neumann (see for example [2], Chapter 2, Theorem 5.2).

**Lemma A.4** (Birkhoff-Von Neumann Theorem). The extreme points of $DM(n)$ are exactly the $n \times n$ permutation matrices $PM(n)$.

Now we prove Theorem [A.1].

**Proof of Theorem [A.1].** Since $\Psi : DM(|F_n|) \rightarrow \Pi(\mu_x,F_n,\mu_y,F_n)$ is surjective, one has

$$W(\mu_x,F_n,\mu_y,F_n) = \inf_{A \in DM(|F_n|)} \sum_{g_1, g_2 \in F_n} \frac{A(g_1, g_2) d(g_1 x, g_2 y)}{|F_n|}.$$

Then by Birkhoff-Von Neumann Theorem, we deduce that

$$W(\mu_x,F_n,\mu_y,F_n) = \inf_{A \in PM(|F_n|)} \sum_{g_1, g_2 \in F_n} \frac{A(g_1, g_2) d(g_1 x, g_2 y)}{|F_n|} = \inf_{h \in Aut(F_n)} \frac{1}{|F_n|} \sum_{g \in F_n} d(g x, g^h y),$$

the last equality comes from $\Psi(PM(|F_n|)) = \{ \frac{1}{|F_n|} \sum_{g \in F_n} \delta_{(g_x,g^h y)} : h \in Aut(F_n) \}$. This ends the proof of Theorem [A.1].

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