FAMILIES OF QUASI-BI-HAMILTONIAN SYSTEMS AND SEPARABILITY

YUNBO ZENG\(^*\) and WEN-XIU MA\(^†\)

*Department of Mathematical Sciences, Tsinghua University
Beijing 100084, China
† Department of Mathematics, City University of Hong Kong
Kowloon, Hong Kong, China

Abstract. It is shown how to construct an infinite number of families of quasi-bi-Hamiltonian (QBH) systems by means of the constrained flows of soliton equations. Three explicit QBH structures are presented for the first three families of the constrained flows. The Nijenhuis coordinates defined by the Nijenhuis tensor for the corresponding families of QBH systems are proved to be exactly the same as the separated variables introduced by mean of the Lax matrices for the constrained flows.

Keywords: quasi-bi-Hamiltonian systems, constrained flows of soliton equations, Nijenhuis coordinates, separated variables, Lax matrices, separability.

PACS codes: 02.90.+p, 03.40.-t

\(^*\)E-mail:yzeng@tsinghua.edu.cn
\(^†\)E-mail:mawx@cityu.edu.hk
I. Introduction.

As is known, some integrable systems possess bi-Hamiltonian structure. We recall some known results. Let $M$ be a differential manifold, $TM$ and $T^*M$ its tangent and cotangent bundle, and $\theta_0$ and $\theta_1 : T^*M \to TM$ two compatible Poisson tensors on $M$ [1]. A vector field $X$ is said to be bi-Hamiltonian (BH) with respect to $\theta_0$ and $\theta_1$, if two smooth functions, $H, F \in C^\infty(M)$, exist such that

$$X = \theta_0 dH = \theta_1 dF$$  \hspace{1cm} (1.1)

where $dF$ denotes the differential of $F$ (gradient $\nabla F$ for finite system and variation $\delta F$ for field system). If $\theta_0$ is invertible, the tensor $\Phi = \theta_1 \theta_0^{-1}$ is a Nijenhuis tensor or hereditary operator. The operator $\Phi$ maps a given BH vector field into another BH vector field. Hence having a Nijenhuis tensor, one can construct a hierarchy of Hamiltonian symmetries, and a related hierarchy of integrals of motion for the underlying system. The BH structure (1.1) ensures that the resulting integrals of motion are pairwise in involution with respect to both Poisson brackets. Thus the BH structure of a given system is important for its integrability.

Unfortunately, for a majority of the BH finite-dimensional systems, none of the $\theta_0$ and $\theta_1$ is invertible. In fact, all the known BH finite-dimensional systems arising from the constrained flows or stationary flows of soliton equations usually exist in an extended phase space and both $\theta_0$ and $\theta_1$ are degenerated (see, for example, [2-8]). In their natural phase space these systems may satisfy a weaker condition than the BH one. The notion of a quasi-bi-Hamiltonian (QBH) system was introduced [9,10]. According to [10], for $\dim M = 2n$, a vector field, $X$, is said to be a QBH vector field with respect to Poisson tensors, $\theta_0$ and $\theta_1$, if there exist three smooth functions $H, F, \rho$, such that

$$X = \theta_0 \nabla H = \frac{1}{\rho} \theta_1 \nabla F$$  \hspace{1cm} (1.2)

where two Poisson tensors $\theta_0$ and $\theta_1$ are compatible and nondegenerated (invertible). The function $\rho$ is called an integrating factor. On a $2n$-dimensional symplectic manifold $M$, let $(q = (q_1, ..., q_n), p = (p_1, ..., p_n))$ be a set of canonical coordinates and $\theta_0$ the canonical Poisson matrix $\theta_0 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ ($I$ denoting the $n \times n$ identity matrix). As $\theta_0$ and $\theta_1$ are compatible and invertible, the Nijenhuis tensor $\Phi = \theta_1 \theta_0^{-1}$ is maximal, i.e. it has $n$ distinct eigenvalues $\mu = (\mu_1, ..., \mu_n)$. As is known [11], in a neighborhood of a regular point, where the eigenvalues $\mu$ are distinct, one can construct a canonical transformation $(q, p) \mapsto (\mu, \nu)$ ($(\mu, \nu)$ referred to as the Nijenhuis coordinates) such that $\theta_1$ and $\Phi$ take the Darboux form

$$\theta_1 = \begin{pmatrix} 0 & \Lambda_1 \\ -\Lambda_1 & 0 \end{pmatrix}, \quad \Phi = \begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_1 \end{pmatrix}, \quad \Lambda_1 = \text{diag}(\mu_1, ..., \mu_n).$$  \hspace{1cm} (1.3)
A QBH vector field is said to be Pfaffian [10] if, in the Nijenhuis coordinates, an integrating factor $\rho$ in equation (1.2) is the product of the eigenvalues of $\Phi$, i.e.

$$
\rho = \prod_{i=1}^{n} \mu_i.
$$

(1.4)

In the Pfaffian case, the general solutions, $H$ and $F$, of equation (1.2) are obtained and the Hamilton-Jacobi equation for $H$ is shown to be separable by verifying the Levi-Civita conditions [12]. Some relationship between BH and QBH structure is discussed in [13]. Several QBH systems are presented [9,10,12-14]. It is in general quite difficult to directly construct a BH or QBH structure for a given integrable Hamiltonian vector field. In recent years much work has been devoted to the constrained flows of soliton equations (see, for example, [2-8,15-24]). One of the aims of this paper is to show how to construct an infinite number of families of QBH systems from the constrained flows of soliton equations. We have presented some families of the constrained flows in order to study the dynamical $r$-matrices in [24]. We now describe the explicit QBH structures for these families of the constrained flows.

The Lax representation for the constrained flows of soliton equations can always be deduced from the adjoint representation of the Lax pair for soliton equations [16,17]. There is an effective way for the separation of variables for some finite-dimensional integrable Hamiltonian systems with some kind of Lax matrices [25,26]. The separated variables for some constrained flows can be introduced and the Jacobi inversion problems for the constrained flows can be established by means of the Lax representation [27,28]. We are interested in the relationship between the two methods for the separability mentioned above. Another main aim of this paper is to prove that the Nijenhuis coordinates for the underlying families of QBH systems are usually the same as the separated variables introduced by the Lax matrices.

The paper is organized as follows.

In section 2 we present a new QBH system. We directly construct the second compatible Poisson tensor by using a map relating this system to its modified version, and prove that the Nijenhuis coordinates for this system is equivalent to the separated variables defined by Lax matrix. We make some comparison of the two methods for separability.

In section 3 and section 4, by using the constrained flows associated with the polynomial second order spectral problems and the higher-order symmetry constraints, we propose a way to construct an infinite number of families of QBH systems. The explicit QBH structures of the first two families of constrained flows are given. The equivalence of the Nijenhuis coordinates and the separated variables is proved. In section 5 we point out that the two compatible Poisson tensors $\theta_0, \theta_1$ and the integrating factor $\rho$ given by the QBH structure (2.28) and (2.29a) are just that for the third family of OBH systems. Also some conclusions and a conjecture are given.

II. New QBH system.
In this section we present a new QBH system. By using a map relating this system to its modified version, the second compatible Poisson tensor is obtained from the image of the Poisson tensor for the modified version under the map. We use this system to illustrate how to prove the equivalence of the Nijenhuis coordinates and the separated variables introduced by the Lax matrix.

A. New finite-dimensional integrable Hamiltonian system.

For Jaulent–Miodek (JM) spectral problem [29]

\[
\psi_x = U(u, \lambda)\psi, \quad U(u, \lambda) = \begin{pmatrix} 0 & 1 \\ \lambda^2 - u_1\lambda - u_0 & 0 \end{pmatrix}, \quad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad u = \begin{pmatrix} u_1 \\ u_0 \end{pmatrix},
\]

its adjoint representation is defined by [30]

\[
V_x = [U, V] \equiv UV - VU,
\]

where \( V \) is taken as

\[
V = \sum_{i=0}^{\infty} V_i \lambda^{-i}, \quad V_i = \begin{pmatrix} a_i & b_i \\ c_i & -a_i \end{pmatrix}.
\]

Then equation (2.2) and (2.3) yields

\[
a_0 = a_1 = a_2 = b_0 = b_1 = 0, \quad b_2 = 1, \quad b_3 = \frac{1}{2}u_1, \quad a_3 = -\frac{1}{4}u_{1,x}, \quad c_0 = 1, \quad c_1 = -\frac{1}{2}u_1, \ldots,
\]

and in general

\[
\begin{pmatrix} b_{k+2} \\ b_{k+1} \end{pmatrix} = L \begin{pmatrix} b_{k+1} \\ b_k \end{pmatrix}, \quad k = 1, 2, \ldots
\]

\[
a_k = -\frac{1}{2}b_{k,x}, \quad c_k = a_{k,x} - u_0 b_k - u_1 b_{k+1} + b_{k+2}, \quad k = 1, 2, \ldots,
\]

where

\[
L = \begin{pmatrix} u_1 - \frac{1}{2} D^{-1} u_{1,x} & \frac{1}{4} D^2 + u_0 - \frac{1}{2} D^{-1} u_{0,x} \\ 1 & 0 \end{pmatrix}, \quad D = \frac{\partial}{\partial x}, \quad DD^{-1} = D^{-1} D = 1.
\]

The Jaulent-Miodek hierarchy associated with (2.1) can be written as an infinite-dimensional Hamiltonian system

\[
u_{t_n} = \begin{pmatrix} u_1 \\ u_0 \end{pmatrix}_{t_n} = J \begin{pmatrix} b_{n+2} \\ b_{n+1} \end{pmatrix} = J \frac{\delta H_n}{\delta u}, \quad n = 1, 2, \ldots
\]
where the Hamiltonian $H_n$ and the Hamiltonian operator $J$ are given by

$$J = \begin{pmatrix} 0 & 2D \\ 2D & -u_1x - 2u_1D \end{pmatrix}, \quad H_n = \frac{1}{n}(2b_{n+3} - u_1b_{n+2}).$$

Under zero boundary condition we have

$$\frac{\delta \lambda}{\delta u} = \lambda \psi_1^2, \quad L \frac{\delta \lambda}{\delta u} = \lambda \frac{\delta \lambda}{\delta u}.$$ (2.6)

The constrained flow of (2.5) consists of the equations obtained from the spectral problem (2.1) for $N$ distinct $\lambda_j$ and the restriction of the variational derivatives for conserved quantities $H_l$ (for any fixed $l$) and $\lambda_j$ [15-17]:

$$\Psi_{1,x} = \Psi_2, \quad \Psi_{2,x} = \Lambda^2 \Psi_1 - u_1 \Lambda \Psi_1 - u_0 \Psi_1,$$ (2.7a)

$$\frac{\delta H_l}{\delta u} - \frac{1}{2} \sum_{j=1}^{N} \frac{\delta \lambda_j}{\delta u} = \left(\frac{b_{l+2}}{b_{l+1}}\right) - \frac{1}{2} \left(\langle \Lambda \Psi_1, \Psi_1 \rangle \right) = 0,$$ (2.7b)

which has been recognized as a symmetry constraint [18-20]. Hereafter we denote the inner product in $\mathbb{R}^N$ by $\langle \cdot, \cdot \rangle$ and 

$$\Psi_i = (\psi_{i1}, \cdots, \psi_{iN})^T, \quad i = 1, 2, \Lambda = diag(\lambda_1, \cdots, \lambda_N).$$

For $l = 4$, we have

$$H_4 = \frac{7}{128} u_1^5 + \frac{5}{16} u_1^3 u_0 - \frac{5}{32} u_1^2 u_1 + \frac{3}{8} u_0^2 u_1 - \frac{1}{8} u_1x u_0x.$$ (2.8)

By introducing the Jacobi-Ostrogradsky coordinates

$$q_1 = u_1, \quad q_2 = u_0,$$

$$p_1 = \frac{\delta H_4}{\delta u_1x} = -\frac{5}{16} u_1 u_1x - \frac{1}{8} u_0x, \quad p_2 = \frac{\delta H_4}{\delta u_0x} = -\frac{1}{8} u_1x,$$ (2.9)

the equations (2.7) for $l = 4$ are transformed into a finite-dimensional Hamiltonian system (FDHS)

$$\Psi_{1,x} = \frac{\partial F_1}{\partial \Psi_2} = \Psi_2, \quad q_{1,x} = \frac{\partial F_1}{\partial p_1} = -8p_2, \quad q_{2,x} = \frac{\partial F_1}{\partial p_2} = -8p_1 + 20q_1 p_2,$$ (2.10a)

$$\Psi_{2,x} = -\frac{\partial F_1}{\partial \Psi_1} = \Lambda^2 \Psi_1 - q_1 \Lambda \Psi_1 - q_2 \Psi_1,$$ (2.10b)

$$p_{1,x} = -\frac{\partial F_1}{\partial q_1} = \frac{35}{128} q_1^4 + \frac{15}{16} q_1^2 q_2 - 10 p_2^2 + \frac{3}{8} q_2^2 - \frac{1}{2} \langle \Lambda \Psi_1, \Psi_1 \rangle,$$ (2.10c)
or equivalently
\[ P_x = \theta_0 \nabla F_1, \]
where
\[ P = (\Psi_1^T, q_1, q_2, \Psi_2^T, p_1, p_2)^T, \quad \theta_0 = \left( \begin{array}{cc} 0 & I_{(N+2) \times (N+2)} \\ -I_{(N+2) \times (N+2)} & 0 \end{array} \right), \]
\[ F_1 = \frac{1}{2} \langle \Psi_2, \Psi_2 \rangle - \frac{1}{2} \langle \Lambda^2 \Psi_1, \Psi_1 \rangle + \frac{1}{2} q_1 < \Lambda \Psi_1, \Psi_1 > + \frac{1}{2} q_2 < \Psi_1, \Psi_1 > - 8 p_1 p_2 + 10 q_1 p_2 - \frac{5}{16} q_1^3 q_2^2 - \frac{3}{8} q_1 q_2^2 - \frac{7}{128} q_1. \]

The Lax representation for FDHS (2.10) can be deduced from the adjoint representation (2.2) by using the method in [16,17] which is sketched as follows. Due to (2.4a), (2.6) and (2.7b), we may define
\[ \tilde{b}_m = \frac{1}{2} < \Lambda^{m-5} \Psi_1, \Psi_1 >, \quad m = 5, 6, \ldots, \]
which together with (2.4b) and (2.10) yields
\[ \tilde{a}_m = -\frac{1}{2} < \Lambda^{m-5} \Psi_1, \Psi_2 >, \quad \tilde{c}_m = -\frac{1}{2} < \Lambda^{m-5} \Psi_2, \Psi_2 >, \quad m = 5, 6, \ldots. \]

Set
\[ \tilde{a}_m = a_m, \quad \tilde{b}_m = b_m, \quad \tilde{c}_m = c_m, \quad m = 0, 1, 2, 3, 4. \]

Then the construction of \( \tilde{a}_m, \tilde{b}_m, \tilde{c}_m \) ensures that under (2.10)
\[ \tilde{V} = \sum_{i=0}^{\infty} \tilde{V}_i \lambda^{-i}, \quad \tilde{V}_i = \left( \begin{array}{c} \tilde{a}_i \\ \tilde{b}_i \\ \tilde{c}_i \\ -\tilde{a}_i \end{array} \right), \]
also satisfies (2.2). Notice that
\[ \sum_{m=5}^{\infty} \tilde{a}_m \lambda^{-m+4} = -\frac{1}{2} \sum_{m=0}^{\infty} \sum_{j=1}^{N} \left( \frac{\lambda}{\lambda} \right)^m \psi_{1j} \psi_{2j} = -\frac{1}{2} \sum_{j=1}^{N} \frac{\psi_{1j} \psi_{2j}}{\lambda - \lambda_j}, \]
set
\[ Q \equiv \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & -A(\lambda) \end{pmatrix} = \lambda^4 \tilde{V}, \quad (2.12a) \]
we have

\[ A(\lambda) = 2p_2\lambda + 2p_1 - 2q_1p_2 - \frac{1}{2} \sum_{j=1}^{N} \frac{\psi_{1j}\psi_{2j}}{\lambda - \lambda_j}, \tag{2.12b} \]

\[ B(\lambda) = \lambda^2 + \frac{1}{2}q_1\lambda + \frac{3}{8}q_1^2 + \frac{1}{2}q_2 + \frac{1}{2} \sum_{j=1}^{N} \frac{\psi_{1j}^2}{\lambda - \lambda_j}, \tag{2.12c} \]

\[ C(\lambda) = \lambda^4 - \frac{1}{2}q_1\lambda^3 - (\frac{1}{2}q_2 + \frac{1}{8}q_1^2)\lambda^2 + (\frac{1}{4}q_1^2 + \frac{1}{2}q_1q_2 - \frac{1}{2}) < \Psi_1, \Psi_1 > \lambda \]

\[ + \frac{1}{4}q_2 - \frac{5}{64}q_1^4 - 4p_2^2 - \frac{1}{2} < \Lambda\Psi_1, \Psi_1 > + \frac{1}{2}q_1 < \Psi_1, \Psi_1 > - \frac{1}{2} \sum_{j=1}^{N} \frac{\psi_{1j}^2}{\lambda - \lambda_j}. \tag{2.12d} \]

Since \( \tilde{V} \) under (2.10) satisfies (2.2), then \( Q \) under (2.10) satisfies (2.2), too, namely

\[ Q_x = [U, Q], \tag{2.13} \]

which presents the Lax representation for (2.10). This can also be verified by a direct calculation. The equation (2.13) implies that \( \frac{1}{2}TrQ^2(\lambda) = A^2(\lambda) + B(\lambda)C(\lambda) \) is the generating function of the integrals of motion for (2.10). We have

\[ A^2(\lambda) + B(\lambda)C(\lambda) = \lambda^6 - F_1\lambda + F_2 + \sum_{i=1}^{N} \frac{F^{(i)}}{\lambda - \lambda_i}, \tag{2.14} \]

\[ F_2 = -\frac{1}{2} < \Lambda\Psi_2, \Psi_2 > + \frac{1}{2} < \Lambda^3\Psi_1, \Psi_1 > - \frac{1}{4}q_1 < \Lambda^2\Psi_1, \Psi_1 > \]

\[ + \left( \frac{1}{8}q_1^2 + \frac{1}{4}q_1q_2 - \frac{1}{4} < \Psi_1, \Psi_1 > \right) < \Psi_1, \Psi_1 > - \frac{1}{4}q_1 < \Psi_2, \Psi_2 > \]

\[ + \left( \frac{3}{8}q_1^2 + \frac{1}{2}q_2 \right) (-4p_2^2 - \frac{5}{64}q_1^4 + \frac{1}{4}q_2^2 - \frac{1}{2} < \Lambda\Psi_1, \Psi_1 > + \frac{1}{2}q_1 < \Psi_1, \Psi_1 > ) \]

\[ - \left( \frac{1}{4}q_2 + \frac{1}{16}q_1^2 \right) < \Lambda\Psi_1, \Psi_1 > - 2p_2 < \Psi_1, \Psi_2 > + 4(p_1 - q_1p_2)^2, \tag{2.15} \]

\[ F^{(i)} = (-2p_2\lambda_i + 2q_1p_2 - 2p_1)\psi_{1i}\psi_{2i} - \frac{1}{4}(\lambda_i^2 + \frac{1}{2}q_1\lambda_i + \frac{3}{8}q_1^2 + \frac{1}{2}q_2)\psi_{1i}^2 \]

\[ + \frac{1}{2} \left( \lambda_i^4 - \frac{1}{2}q_1\lambda_i^3 - (\frac{1}{2}q_1^2 + \frac{1}{2}q_2)\lambda_i^2 + (\frac{1}{4}q_1^2 + \frac{1}{2}q_1q_2 - \frac{1}{2}) < \Psi_1, \Psi_1 > \right) \lambda_i + \frac{1}{4}q_2^2 - 4p_2^2 \]

\[ - \frac{5}{64}q_1^4 - \frac{1}{2} < \Lambda\Psi_1, \Psi_1 > + \frac{1}{2}q_1 < \Psi_1, \Psi_1 > \lambda_i + \frac{1}{4} \sum_{k=1}^{N} \frac{(\psi_{1k}\psi_{2k} - \psi_{1k}\psi_{2i})^2}{\lambda_k - \lambda_i}, \quad i = 1, \ldots, N, \tag{2.16} \]
where \( F^{(i)}, i = 1, \ldots, N, F_1, F_2 \) are \( N + 2 \) independent integrals of motion for (2.10). By means of the \( r \)-matrix, it can be shown that the equation (2.10) is a finite-dimensional integrable Hamiltonian system (FDIHS).

In order to find the QBH structure for (2.10), we need to use the modified system of (2.10). Let us consider the modified Jaulent–Miodek (MJM) spectral problem [31]

\[
\phi_x = U(v, \lambda)\phi, \quad U(v, \lambda) = \begin{pmatrix} v_0 & \lambda \\ \lambda - v_1 & -v_0 \end{pmatrix}, \quad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \quad v = \begin{pmatrix} v_0 \\ v_1 \end{pmatrix}. \tag{2.17}
\]

The equations (2.2) and (2.3) yield

\[
\begin{align*}
a_0 &= 0, & b_0 &= 1, & b_1 &= \frac{1}{2} v_1, & a_1 &= v_0, & c_0 &= 1, & c_1 &= -\frac{1}{2} v_1, \\
(2a_{k+1} - b_{k+1}) &= L \begin{pmatrix} 2a_k \\ -b_k \end{pmatrix}, & k &= 1, 2, \ldots, \\
L &= \begin{pmatrix} \frac{1}{4} D + \frac{1}{2} D^{-1} v_0 D & -2v_0 + D \\ \frac{1}{2} v_1 + \frac{1}{2} D^{-1} v_1 D \end{pmatrix}.
\end{align*} \tag{2.18}
\]

The MJM hierarchy associated with (2.17) can also be written as an infinite-dimensional Hamiltonian system

\[
v_t = \begin{pmatrix} v_0 \\ v_1 \end{pmatrix}, \quad \delta H_n = J \frac{\delta H_n}{\delta v}, \quad n = 1, 2, \ldots, \tag{2.19}
\]

where the Hamiltonian \( H_n \) and the Hamiltonian operator \( J \) are given by

\[
J = \begin{pmatrix} \frac{1}{4} D & 0 \\ 0 & -2D \end{pmatrix}, \quad H_n = -\frac{1}{n} [a_{n,xx} - v_1 b_n + 2b_{n+1}].
\]

Also we have

\[
\frac{\delta \lambda}{\delta v} = \begin{pmatrix} 2\phi_1 \phi_2 \\ \phi_1^2 \end{pmatrix}. \tag{2.20}
\]

In the similar way as for (2.7), the constrained flow of (2.19) is defined by

\[
\Phi_{1,x} = v_0 \Phi_1 + \Lambda \Phi_2, \quad \Phi_{2,x} = (\Lambda - v_1)\Phi_1 - v_0 \Phi_2, \tag{2.21a}
\]

\[
\frac{\delta H_l}{\delta v} + \frac{1}{2} \left( 2 \langle \Phi_1, \Phi_2 \rangle \right) = 0, \tag{2.21b}
\]

where \( \Phi_i = (\phi_{i1}, \ldots, \phi_{iN})^T, i = 1, 2. \)

For \( l = 3, \)

\[
H_3 = -\left( \frac{1}{4} v_{0x}^2 - \frac{1}{16} v_{1x}^2 + \frac{1}{4} v_0^4 + \frac{5}{64} v_1^4 - \frac{3}{8} v_{0x} v_1^2 - \frac{3}{8} v_0^2 v_1^2 \right).
\]
By introducing the Jacobi-Ostrogradsky coordinates
\[ \tilde{q}_1 = v_1, \quad \tilde{q}_2 = v_0, \quad (2.22a) \]
\[ \tilde{p}_1 = -\frac{\delta H_3}{\delta v_{1x}} = -\frac{1}{8}v_{1x}, \quad \tilde{p}_2 = -\frac{\delta H_3}{\delta v_{0x}} = \frac{1}{2}v_{0x} - \frac{3}{8}v_1^2, \quad (2.22b) \]
the equations (2.21) for \( l = 3 \) are transformed into a FDHS

\[ \Phi_{1,x} = \frac{\partial \tilde{F}_1}{\partial \Phi_2} = \tilde{q}_2\Phi_1 + \Lambda\Phi_2, \quad \tilde{q}_{1x} = \frac{\partial \tilde{F}_1}{\partial \tilde{p}_1} = -8\tilde{p}_1, \quad \tilde{q}_{2x} = \frac{\partial \tilde{F}_1}{\partial \tilde{p}_2} = 2\tilde{p}_2 + \frac{3}{4}\tilde{q}_1^2, \quad (2.23a) \]

\[ \Phi_{2,x} = -\frac{\partial \tilde{F}_1}{\partial \Phi_1} = \Lambda\Phi_1 - \tilde{q}_1\Phi_1 - \tilde{q}_2\Phi_2, \quad (2.23b) \]

\[ \tilde{p}_{1x} = -\frac{\partial \tilde{F}_1}{\partial \tilde{q}_1} = \frac{3}{2}\tilde{q}_1\tilde{p}_2 - \frac{3}{4}\tilde{q}_1\tilde{q}_2 - \frac{1}{4}\tilde{q}_1^3 - \frac{1}{2}<\Phi_1,\Phi_1>, \quad (2.23c) \]

\[ \tilde{p}_{2x} = -\frac{\partial \tilde{F}_1}{\partial \tilde{q}_2} = \tilde{q}_2^3 - \frac{3}{4}\tilde{q}_1^2\tilde{q}_2^2 <\Phi_1,\Phi_2>, \quad (2.23d) \]

or

\[ \tilde{P}_x = \theta_0 \nabla \tilde{F}_1, \]

where

\[ \tilde{P} = (\Phi_1^T, \tilde{q}_1, \tilde{q}_2, \Phi_2^T, \tilde{p}_1, \tilde{p}_2)^T, \]

\[ \tilde{F}_1 = -4\tilde{p}_1^2 + \tilde{p}_2^2 + \frac{3}{4}\tilde{q}_1\tilde{p}_2 + \frac{3}{8}\tilde{q}_1^2\tilde{q}_2^2 + \frac{1}{16}\tilde{q}_1^4 - \frac{1}{4}\tilde{q}_2^4 \]

\[ + \tilde{q}_2 <\Phi_1,\Phi_2> + \frac{1}{2} <\Lambda\Phi_2,\Phi_2 > - \frac{1}{2} <\Lambda\Phi_1,\Phi_1 > + \frac{1}{2}\tilde{q}_1 <\Phi_1,\Phi_1 >. \]

**B. The QBH structure for the FDIHS (2.10).**

We now establish a map relating FDIHS (2.10) to (2.23), then use the map to construct the second compatible Poisson tensor for the FDIHS (2.10).

It is known [31] that a gauge transformation between the JM and MJM spectral problem is as follows

\[ \psi_1 = \phi_1, \quad \psi_2 = \lambda\phi_2 + v_0\phi_1, \quad u_1 = v_1, \quad u_0 = -v_{0x} - v_0^2, \quad (2.24) \]

which, together with (2.9) and (2.22), gives rise to the map relating (2.10) to (2.23), i.e. \( P = M(\tilde{P}) \):

\[ \Psi_1 = \Phi_1, \quad \Psi_2 = \Lambda\Phi_2 + \tilde{q}_2\Phi_1, \quad q_1 = \tilde{q}_1, \]

\[ q_2 = -2\tilde{p}_2 - \frac{3}{4}\tilde{q}_1^2 - \tilde{q}_2^2, \quad p_1 = \tilde{q}_1\tilde{p}_1 + \frac{1}{4}\tilde{q}_2^2 + \frac{1}{2}\tilde{q}_2\tilde{p}_2 - \frac{1}{4} <\Phi_1,\Phi_2>, \quad p_2 = \tilde{p}_1. \quad (2.25) \]
The map $M$ given by (2.25) transforms all equations in (2.10) except for (2.10c) into the corresponding equations in (2.23) except for (2.23c). In fact, the equation (2.10c) with an additive constant term $c = -\frac{1}{2}F_1$ is transformed into (2.23c) under the map (2.25). However, the second Poisson tensor constructed later by using the map (2.25) is valid for an arbitrary $c$, therefore we can take $c = 0$. The Jacobi $M'$ of the map $M$ take the form

$$M'(\vec{P}) = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & -\frac{3}{2}\tilde{q}_1 & -2\tilde{q}_2 & 0 & 0 & -2 \\
-\frac{1}{2}\Phi^T I & 0 & \Phi_1 & \Lambda & 0 & 0 \\
0 & 0 & \rho_1 & \frac{3}{4}q_2 + \frac{1}{2}\tilde{q}_2 & -\frac{1}{4}\Phi_1 I & \tilde{q}_1 + \frac{1}{2}\tilde{q}_2
\end{pmatrix}.$$  \hspace{5cm} (2.26)

According to the standard procedure [32], the image of the Poisson tensor of the FDIHS (2.23) under the map $M$ gives rise to the second compatible Poisson tensor for the FDIHS (2.10). That is

$$\theta_1 = M'\theta_0 M'^T \mid_{P=M(\vec{P})} = \begin{pmatrix}
0_{(N+2) \times (N+2)} & A_1 \\
-\Lambda I & B_1
\end{pmatrix},$$  \hspace{5cm} (2.27a)

$$A_1 = \begin{pmatrix}
\Lambda & -\frac{1}{4}\Psi_1 & 0_{N \times 1} \\
0_{1 \times N} & q_1 & 1 \\
2\Psi^T_1 & -\frac{1}{2}q_2 - \frac{15}{8}q_1^2 & -\frac{3}{2}q_1
\end{pmatrix}, \quad B_1 = \begin{pmatrix}
0_{N \times N} & \frac{1}{4}\Psi_2 & 0_{N \times 1} \\
-\frac{1}{4}\Psi^T_1 & 0 & p_2 \\
0_{1 \times N} & -p_2 & 0
\end{pmatrix}. \hspace{5cm} (2.27b)$$

Furthermore, by a straightforward calculation, we can show the following proposition.

**Proposition 1.** The system (2.10) possesses the QBH representation

$$P_x = \theta_0 \triangledown F_1 = \frac{1}{\rho} \theta_1 \triangledown E_1$$  \hspace{5cm} (2.28)

where

$$\rho = B(\lambda)|_{\lambda=0} = \frac{3}{8}q_1^2 + \frac{1}{2}q_2 - \frac{1}{2} < \Lambda^{-1}\Psi_1, \Psi_1 >,$$  \hspace{5cm} (2.29a)

$$E_1 = [A^2(\lambda) + B(\lambda)C(\lambda)]|_{\lambda=0} = F_2 - \sum_{i=1}^{N} \lambda_i^{-1}F^{(i)}$$

$$= \left(\frac{3}{16}q_1^2 + \frac{1}{4}q_2\right)(< \Lambda^{-1}\Psi_2, \Psi_2 > - < \Lambda\Psi_1, \Psi_1 >)$$

$$+ (2p_2^2 + \frac{5}{128}q_1^4 - \frac{1}{8}q_2^2 + \frac{1}{4} < \Lambda\Psi_1, \Psi_1 > - \frac{1}{4}q_1 < \Psi_1, \Psi_1 >) < \Lambda^{-1}\Psi_1, \Psi_1 >$$

$$+ \left(\frac{3}{8}q_1^2 + \frac{1}{2}q_2\right)\left(\frac{1}{4}q_2^2 - 4p_2^2 - \frac{5}{64}q_1^4\right) + 4(p_1 - q_1p_2)^2$$

$$+ \frac{1}{4} < \Lambda^{-1}\Psi_1, \Psi_2 >^2 - < \Lambda^{-1}\Psi_1, \Psi_1 > < \Lambda^{-1}\Psi_2, \Psi_2 >].$$  \hspace{5cm} (2.29b)
C. The Nijenhuis coordinates.

We now prove that the Nijenhuis coordinates for QBH system (2.28) are the same as the separated variables defined by means of the Lax matrix (2.12b). As $\theta_0$ and $\theta_1$ are compatible and invertible, the matrix $\theta_1\theta_0^{-1}$ is maximal, it has $N+2$ distinct eigenvalues $\mu = (\mu_1, \ldots, \mu_{N+2})$. The explicit form of the canonical transformation from $P$ to the Nijenhuis coordinates $(\mu, \nu)$ is given in what follows. The eigenvalues $\mu_1, \ldots, \mu_{N+2}$ are defined by the roots of the equation

$$f(\lambda) = |\lambda I - A_1| = 0,$$

which, since $A_1$ depends only on $(\Psi_1, q_1, q_2)$, gives rise to

$$\mu_j = f_j(\Psi_1, q_1, q_2), \quad j = 1, \ldots, N + 2,$$

$$\psi_{1j} = g_j(\mu), \quad j = 1, \ldots, N, \quad q_1 = g_{N+1}(\mu), \quad q_2 = g_{N+2}(\mu).$$

Then we introduce the generating function by

$$S = \sum_{j=1}^{N} \psi_{2j} g_j(\mu) + p_1 g_{N+1}(\mu) + p_2 g_{N+2}(\mu),$$

such that

$$\psi_{1j} = \frac{\partial S}{\partial \psi_{2j}}, \quad j = 1, \ldots, N, \quad q_1 = \frac{\partial S}{\partial p_1}, \quad q_2 = \frac{\partial S}{\partial p_2},$$

$$\nu_j = \frac{\partial S}{\partial \mu_j} = \sum_{j=1}^{N} \psi_{2j} \frac{\partial g_j}{\partial \mu_j} + p_1 \frac{\partial g_{N+1}}{\partial \mu_j} + p_2 \frac{\partial g_{N+2}}{\partial \mu_j}, \quad j = 1, \ldots, N + 2.$$

The equations (2.33b) reconstruct (2.31) or (2.32), the equations (2.33c) give the expression for $\nu_j$. The system (2.10) written in terms of $(\mu, \nu)$ can be shown to be separable.

On the other hand, the separated variables $(\bar{\mu}, \bar{\nu})$ for (2.10) can be constructed by means of the Lax matrix in the following way [27,28]. The coordinates $\bar{\mu}_1, \ldots, \bar{\mu}_{N+2}$ are introduced by the zeros of $B(\lambda)$:

$$B(\lambda) = \lambda^2 + \frac{1}{2} q_1 \lambda + \frac{3}{8} q_1^2 + \frac{1}{2} q_2 + \frac{1}{2} \sum_{j=1}^{N} \frac{\psi_{1j}^2}{\lambda - \lambda_j} = \frac{R(\lambda)}{K(\lambda)},$$

with

$$R(\lambda) = \prod_{k=1}^{N+2} (\lambda - \bar{\mu}_k) = \sum_{k=0}^{N+2} \beta_k \lambda^{N+2-k}, \quad K(\lambda) = \prod_{k=1}^{N} (\lambda - \lambda_k) = \sum_{k=0}^{N} \alpha_k \lambda^{N-k},$$
\[ \alpha_0 = 1, \quad \alpha_1 = -\sum_{j=1}^{N} \lambda_j, \quad \ldots, \quad \alpha_N = (-1)^N \prod_{j=1}^{N} \lambda_j, \]

\[ \beta_0 = 1, \quad \beta_1 = -\sum_{j=1}^{N+2} \bar{\mu}_j, \quad \ldots, \quad \beta_{N+2} = (-1)^N \prod_{j=1}^{N+2} \bar{\mu}_j, \]

and the canonically conjugate coordinates \( \bar{\nu}_1, \ldots, \bar{\nu}_{N+2} \) are defined by

\[ \bar{\nu}_k = -A(\bar{\mu}_k) = -2p_2 \bar{\mu}_k - 2p_1 + 2q_1p_2 + \frac{1}{2} \sum_{j=1}^{N} \frac{\psi_{1j} \psi_{2j}}{\bar{\mu}_k - \lambda_j}, \quad k = 1, \ldots, N + 2. \] (2.36)

The FDIHS (2.10) in terms of the coordinates \( (\bar{\mu}, \bar{\nu}) \) will be shown to be separable later.

We have the following proposition.

**Proposition 2.** The Nijenhuis coordinates \( (\mu, \nu) \) defined by (2.30) and (2.33) are exactly the same as the separated variables \( (\bar{\mu}, \bar{\nu}) \) defined by (2.34) and (2.36). The QBH vector field (2.28) is Pfaffian in the Nijenhuis coordinates.

Proof. We first show that

\[ f(\lambda) = B(\lambda)K(\lambda) = R(\lambda). \] (2.37)

We denote \( f(\lambda) \) by \( f_N(\lambda; \lambda_1, \ldots, \lambda_N) \) in order to prove (2.37) by induction. Obviously, (2.37) holds for \( N = 1 \). Then we have by induction

\[
\begin{align*}
 f_N(\lambda; \lambda_1, \ldots, \lambda_N) &= \left| \begin{array}{cccccc}
 \lambda - \lambda_1 & 0 & \ldots & 0 & \frac{1}{4} \psi_{11} & 0 \\
 0 & \lambda - \lambda_2 & \ldots & 0 & \frac{1}{4} \psi_{12} & 0 \\
 \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
 0 & 0 & \ldots & \lambda - \lambda_N & \frac{1}{4} \psi_{1N} & 0 \\
 0 & 0 & \ldots & 0 & \lambda - q_1 & -1 \\
 -2\psi_{11} & -2\psi_{12} & \ldots & -2\psi_{1N} & \frac{1}{2} q_2 + \frac{15}{8} q_1^2 & \lambda + \frac{3}{2} q_1
\end{array} \right| \\
&= (\lambda - \lambda_1) f_{N-1}(\lambda; \lambda_2, \ldots, \lambda_N) + \frac{1}{2} \psi_{11}^2 \prod_{k=2}^{N} (\lambda - \lambda_k) \\
&\quad + \frac{1}{2} q_1 \lambda + \frac{3}{8} q_1^2 + \frac{1}{2} q_2 + \frac{1}{2} \sum_{j=2}^{N} \frac{\psi_{1j}^2}{\lambda - \lambda_j} K(\lambda) + \frac{1}{2} \frac{\psi_{11}^2}{\lambda - \lambda_1} K(\lambda) = B(\lambda)K(\lambda). 
\end{align*}
\] (2.38)

The equation (2.38) implies that \( \lambda_1 \), similarly \( \lambda_k, k = 2, \ldots, N, \) is not the zero of \( f(\lambda) \). Thus (2.37) indicates that \( f(\lambda) \) and \( B(\lambda) \) have the same zeros, i.e. \( \mu_k = \bar{\mu}_k \).

It follows from (2.34) that

\[ \psi_{1j}^2 = \frac{2 R(\lambda_j)}{K'(\lambda_j)}, \quad q_1 = 2(\beta_1 - \alpha_1), \] (2.39a)
\[ \frac{1}{2} q_2 + \frac{3}{8} q_1^2 = \frac{1}{2} < \Lambda^{-1} \Psi_1, \Psi_1 > + \frac{\beta_{N+2}}{\alpha_N}, \]  
(2.39b)

where the prime denotes differentiation with respect to \( \lambda \). The equations (2.39a) and (2.39b) yield

\[ q_2 = 2 \sum_{j=1}^{N} \frac{R(\lambda_j)}{\lambda_j K'(\lambda_j)} - 3(\beta_1 - \alpha_1)^2 + 2 \frac{\beta_{N+2}}{\alpha_N}. \]  
(2.39c)

According to (2.33a), one gets

\[ S = \sum_{j=1}^{N} \psi_{2j} \sqrt{\frac{2R(\lambda_j)}{K'(\lambda_j)}} + 2 p_1 (\beta_1 - \alpha_1) + p_2 \sum_{j=1}^{N} \frac{2R(\lambda_j)}{\lambda_j K'(\lambda_j)} - 3(\beta_1 - \alpha_1)^2 + 2 \frac{\beta_{N+2}}{\alpha_N}. \]

Notice that

\[ \frac{\partial}{\partial \mu_k} \sum_{j=1}^{N} \psi_{2j} \sqrt{\frac{2R(\lambda_j)}{K'(\lambda_j)}} = \sum_{j=1}^{N} \frac{\psi_{2j} R(\lambda_j)}{\sqrt{2R(\lambda_j)K'(\lambda_j)(\mu_k - \lambda_j)}} = \frac{1}{2} \sum_{j=1}^{N} \frac{\psi_{1j} \psi_{2j}}{\mu_k - \lambda_j}, \]

\[ \frac{\partial}{\partial \mu_k} \sum_{j=1}^{N} \frac{2R(\lambda_j)}{\lambda_j K'(\lambda_j)} = \sum_{j=1}^{N} \frac{2R(\lambda_j)}{\lambda_j K'(\lambda_j)(\mu_k - \lambda_j)} = \frac{1}{\mu_k} \sum_{j=1}^{N} \frac{2R(\lambda_j)}{\lambda_j K'(\lambda_j)(\mu_k - \lambda_j)}, \]

\[ \frac{\partial \beta_{N+2}}{\partial \mu_k} = \frac{\beta_{N+2}}{\mu_k}, \quad \frac{\partial (\beta_1 - \alpha_1)^2}{\partial \mu_k} = -q_1, \quad \frac{\partial (\beta_1 - \alpha_1)}{\partial \mu_k} = -1, \]

and using \( B(\mu_k) = 0 \), one finds from (2.33c) that \( \nu_k, \mu_k \) satisfy (2.36). Finally, it follows from (2.39b) that

\[ \rho = B(\lambda)|_{\lambda=0} = \frac{1}{2} q_2 + \frac{3}{8} q_1^2 - \frac{1}{2} < \Lambda^{-1} \Psi_1, \Psi_1 > + \frac{\beta_{N+2}}{\alpha_N} = \frac{(-1)^N}{\alpha_N} \prod_{j=1}^{N+2} \mu_j. \]  
(2.40)

This completes the proof.

D. Comparison of the two methods for separability.

For the FDIHS with QBH structure, the separated variables, i.e. the Nijenhuis coordinates, can be introduced by the Nijenhuis tensor. Then the separability of the Hamilton-Jacobi equation for the system can be shown by varifying the Levi-Civita conditions. For the FDIHS with some kind of Lax representation, the separated variables can be found and the separability of the Hamilton Jacobi equation for the system can
be shown by means of the Lax representation. So far there is not an effective way to define separated variables for the FDIHSs with some kind of Lax matrices, such as the $3 \times 3$ Lax matrices \cite{22} or the Lax matrices admitting dynamical $r$-matrix. However, if the separated variables can be introduced by the Lax matrix, one can further establish the Jacobi inversion problem for the system by means of the Lax representation. By using the standard Jacobi inversion technique, the solution to the system can be found.

We now use the Lax representation (2.12) to construct the Jacobi inversion problem for (2.10). Set

$$A^2(\lambda) + B(\lambda)C(\lambda) = \frac{W(\lambda)}{K(\lambda)}, \quad W(\lambda) = \sum_{i=0}^{N+6} P_i\lambda^i, \quad (2.41)$$

then $P_i$ are also the integrals of motion for (2.10). By substituting (2.13) and using (2.14), (2.41) leads to

$$P_{N+6} = 1, \quad P_{N+6-i} = \alpha_i, \quad i = 1, 2, 3, 4,$$

$$F_1 = -P_{N+1} + \alpha_5, \quad F_2 = P_N - \alpha_1 P_{N+1} + \alpha_1 \alpha_5 - \alpha_6, .... \quad (2.42)$$

The equations (2.34), (2.36) and (2.41) give rise to

$$\nu_k = \sqrt{\frac{W(\mu_k)}{K(\mu_k)}}, \quad k = 1, ..., N + 2, \quad (2.43)$$

which indicates that the Hamilton-Jacobi equation is separable. Replacing $\nu_k$ by $\frac{\partial S_k}{\partial \mu_k}$ and interpreting the $P_i$ as integration constants, one gets the generating function $S$ of the canonical transformation from (2.43)

$$S(\mu_1, ..., \mu_{N+2}; P_0, ..., P_{N+1}) = \sum_{k=1}^{N+2} \int_{\mu_k}^{\mu_{k+1}} \sqrt{\frac{W(\lambda)}{K(\lambda)}} d\lambda. \quad (2.44)$$

The linearizing coordinates are then

$$Q_i = \frac{\partial S}{\partial P_i} = \frac{1}{2} \sum_{k=1}^{N+2} \int_{\mu_k}^{\mu_{k+1}} \frac{\lambda^i}{\sqrt{W(\lambda)K(\lambda)}} d\lambda, \quad i = 0, 1, ..., N + 1. \quad (2.45)$$

The linear flow induced by (2.10) is then given by (using (2.42))

$$Q_i = \gamma_i + x\frac{\partial F_1}{\partial P_i} = \gamma_i - x\delta_{i,N+1}, \quad i = 0, 1, ..., N + 1, \quad (2.46)$$

where $\gamma_i$ are arbitrary constants. Combining the equation (2.45) with the equation (2.46) leads to the Jacobi inversion problem for the FDIHS (2.10)

$$\frac{1}{2} \sum_{k=1}^{N+2} \int_{\mu_k}^{\mu_{k+1}} \frac{\lambda^i}{\sqrt{W(\lambda)K(\lambda)}} d\lambda = \gamma_i - x\delta_{i,N+1}, \quad i = 0, 1, ..., N + 1. \quad (2.47)$$

Since $\psi_{1j}, q_1, q_2$ defined by (2.39) are the symmetric functions of $\mu_k, k = 1, ..., N + 2$ by using the standard Jacobi inversion technique \cite{33}, they can be solved in terms of Riemann theta functions from (2.47). After having $\psi_{1j}, q_1, q_2$, the $\psi_{2j}, p_1, p_2$ can be found by using (2.10a). In this way the solution to (2.10) is obtained.
III. The first family of QBH systems.

In the following sections, by using the method described in the previous section, we will present QBH representation for some families of FDIHSs given in [24], and prove the equivalence of two sets of separated variables.

A. The first family of FDIHSs.

We first recall the constrained flows of the hierarchy of nonlinear evolution equations (NLEE) associated with the following polynomial second order spectral problem [31]

\[ \psi_x = U(u, \lambda)\psi, \quad U(u, \lambda) = \begin{pmatrix} 0 & 1 \\ - \sum_{i=0}^{m} u_i \lambda^i & 0 \end{pmatrix}, \quad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad (3.1) \]

where \( u_m = -1, \quad u = (u_{m-1}, ..., u_0)^T \). The adjoint representation (2.2) of (3.1) yields

\[ a_0 = \ldots = a_m = b_0 = \ldots = b_{m-1} = 0, \quad b_m = 1, \quad b_{m+1} = \frac{1}{2} u_{m-1}, \]

\[ a_{m+1} = -\frac{1}{4} u_{m-1,x}, \quad c_0 = 1, \quad c_1 = -\frac{1}{2} u_{m-1}, \ldots, \]

and in general

\[ \begin{pmatrix} b_{k+m} \\ \vdots \\ b_{k+1} \end{pmatrix} = L \begin{pmatrix} b_{k+m-1} \\ \vdots \\ b_k \end{pmatrix}, \quad (3.2a) \]

\[ a_k = -\frac{1}{2} b_{k,x}, \quad c_k = -\frac{1}{2} b_{k,xx} - \sum_{i=0}^{m} u_i b_{k+i}, \quad k = 1, 2, \ldots, \quad (3.2b) \]

where

\[ L = \begin{pmatrix} L_{m-1} & L_{m-2} & \ldots & L_1 & L_0 \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{pmatrix} \]

\[ L_0 = \frac{1}{4} D^2 + u_0 - \frac{1}{2} D^{-1} u_{0,x}, \quad L_i = u_i - \frac{1}{2} D^{-1} u_{i,x}, \quad i = 1, \ldots, m-1. \]

The hierarchy of NLEEs associated with (3.1) can be written as an infinite-dimensional Hamiltonian system

\[ u_{tn} = \begin{pmatrix} u_{m-1} \\ \vdots \\ u_0 \end{pmatrix}, \quad J \begin{pmatrix} b_{n+m} \\ \vdots \\ b_{n+1} \end{pmatrix} = J \frac{\delta H_n}{\delta u}, \quad n = 1, 2, \ldots, \quad (3.3) \]
where the Hamiltonian $H_n$ and the Hamiltonian operator $J$ are defined by

$$J = \begin{pmatrix}
0 & 0 & \ldots & 0 & 2D \\
0 & 0 & \ldots & 2D & J_{m-1} \\
0 & 0 & \ldots & J_{m-1} & J_{m-2} \\
& & \ddots & \ddots & \vdots \\
2D & J_{m-1} & \ldots & J_1 & J_0
\end{pmatrix},$$

$$J_i = -u_{i,x} - 2u_i D, \quad i = 0, 1, \ldots, m - 1, \quad H_n = \frac{2}{m-2n-2} \sum_{i=1}^{m} i u_i b_{n+i+1}.$$  

Under zero boundary condition we have

$$\frac{\delta \lambda}{\delta u} = (\lambda^{m-1} \psi_1^2, \lambda^{m-2} \psi_1^2, \ldots, \psi_1^2)^T, \quad L \frac{\delta \lambda}{\delta u} = \lambda \frac{\delta \lambda}{\delta u}. \quad (3.4)$$

Similarly, the constrained flows of the NLEEs (3.3) are defined by [24]

$$\Psi_{1,x} = \Psi_2, \quad \Psi_{2,x} = \Lambda^m \Psi_1 - \sum_{i=0}^{m-1} u_i \Lambda^i \Psi_1, \quad (3.5a)$$

$$\frac{\delta H_l}{\delta u} - \frac{1}{2} \sum_{j=1}^{N} \frac{\delta \lambda_j}{\delta u} = \begin{pmatrix}
0 \\
\vdots \\
0
\end{pmatrix} - \frac{1}{2} \begin{pmatrix}
< \Lambda^{m-1} \Psi_1, \Psi_1 > \\
< \Lambda^{m-2} \Psi_1, \Psi_1 > \\
< \Lambda^{m-1} \Psi_1, \Psi_1 >
\end{pmatrix} = 0. \quad (3.5b)$$

For $l = m$, (3.5b) leads to

$$u_{m-k} = \sum_{j=1}^{k} (-1)^{j-1} \frac{j+1}{2^j} \sum_{l_1 + \ldots + l_j = k-j} < \Lambda^{l_1} \Psi_1, \Psi_1 > \ldots < \Lambda^{l_j} \Psi_1, \Psi_1 >,$$

$$k = 1, \ldots, m, \quad (3.6)$$

where $l_1 \geq 0, \ldots, l_j \geq 0$. By substituting (3.6) into (3.5a), the first constrained flow of (3.3) can be written as a canonical FDHS

$$\Psi_{1,x} = \frac{\partial F_0}{\partial \Psi_2}, \quad \Psi_{2,x} = -\frac{\partial F_0}{\partial \Psi_1}, \quad (3.7)$$

or

$$P_x = \theta_0 \nabla F_0,$$

where

$$P = (\Psi_1^T, \Psi_2^T)^T, \quad \theta_0 = \begin{pmatrix}
0 & I_{N \times N} \\
-I_{N \times N} & 0
\end{pmatrix},$$
F_{0} = \frac{1}{2} \langle \Psi_2, \Psi_2 \rangle > + \sum_{j=0}^{m} \left( -\frac{1}{2} \right)^{j+1} \sum_{l_1+\ldots+l_{j+1}=m-j}^{} \langle \Lambda^{l_1} \Psi_1, \Psi_1 \rangle \ldots \langle \Lambda^{l_{j+1}} \Psi_1, \Psi_1 \rangle >.

The entries of the Lax matrix for (3.7) are given by [24]

\begin{align}
A(\lambda) &= -\frac{1}{2} \sum_{j=1}^{N} \frac{\psi_{1j} \psi_{2j}}{\lambda - \lambda_j}, & B(\lambda) &= 1 + \frac{1}{2} \sum_{j=1}^{N} \frac{\psi_{1j}^2}{\lambda - \lambda_j}, (3.8a) \\
C(\lambda) &= \lambda^m + \sum_{k=1}^{m} \lambda^{m-k} \sum_{j=1}^{k} (-\frac{1}{2})^j \sum_{l_1+\ldots+l_{j}=k-j}^{} \langle \Lambda^{l_1} \Psi_1, \Psi_1 \rangle \ldots \langle \Lambda^{l_j} \Psi_1, \Psi_1 \rangle > - \frac{1}{2} \sum_{j=1}^{N} \frac{\psi_{2j}^2}{\lambda - \lambda_j}. (3.8b)
\end{align}

We have

\begin{equation}
A(\lambda)^2 + B(\lambda)C(\lambda) = \lambda^m + \sum_{i=1}^{N} \frac{F^{(i)}}{\lambda - \lambda_i}, (3.9)
\end{equation}

\begin{equation}
F^{(i)} = \frac{1}{2} \left[ \lambda_i^m + \sum_{k=1}^{m} \lambda_i^{m-k} \sum_{j=1}^{k} (-\frac{1}{2})^j \sum_{l_1+\ldots+l_{j}=k-j}^{} \langle \Lambda^{l_1} \Psi_1, \Psi_1 \rangle \ldots \langle \Lambda^{l_j} \Psi_1, \Psi_1 \rangle > \psi_{1i}^2 \\
- \frac{1}{2} \psi_{2i}^2 + \frac{1}{4} \sum_{k \neq i} \frac{(\psi_{1k} \psi_{2k} - \psi_{1k} \psi_{2k})^2}{\lambda_k - \lambda_i}, \quad i = 1, \ldots, N,
\end{equation}

where \( F^{(i)}, i = 1, \ldots, N, \) are independent integrals of motion for (3.7) and \( F_0 = \sum_{i=0}^{N} F^{(i)}. \)

It can be shown that the system (3.7) is integrable in the Liouville’s sense. The systems with \( m = 1, 2, \ldots \) give rise to a family of FDIHSs which include the well-known Garnier system as the first member (\( m=1 \)). This family of FDIHSs was first given in [34].

In order to find the QBH structure for (3.7), we need to consider the following modified polynomial second order spectral problem [31]

\begin{equation}
\phi_x = U(v, \lambda) \phi, \quad U(v, \lambda) = \begin{pmatrix} \sum_{i=1}^{m} v_i \lambda^{i-1} & \lambda \\ -v_0 \end{pmatrix}, \quad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, (3.10)
\end{equation}

where \( v_m = -1, v = (v_0, \ldots, v_{m-1})^T. \) The equations (2.2) and (2.3) yield

\begin{align}
a_0 = \ldots = a_{m-2} = b_0 = \ldots = b_{m-3} = 0, & \quad b_{m-2} = 1, & \quad b_{m-1} = \frac{1}{2} v_{m-1}, \\
a_{m-1} = v_0, \quad c_0 = 1, & \quad c_1 = -\frac{1}{2} v_{m-1}, \ldots,
\end{align}
and in general
\[
\begin{pmatrix}
2a_{k+1} \\
-b_{k+1} \\
\vdots \\
-b_{k+m-1}
\end{pmatrix}
= L
\begin{pmatrix}
2a_k \\
b_k \\
\vdots \\
b_{k+m-2}
\end{pmatrix},
\]
(3.11a)

\[
c_{k+1} = a_{k,x} - \sum_{i=1}^{m} v_i b_{k+i-1}, \quad k = 1, 2, \ldots,
\]
(3.11b)

where
\[
L = \begin{pmatrix}
0 & -2v_0 + D & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 1 \\
L_0 & L_1 & L_2 & \cdots & L_{m-2} & L_{m-1}
\end{pmatrix},
\]

\[
L_0 = \frac{1}{4}D + \frac{1}{2}D^{-1}v_0 D, \quad L_i = \frac{1}{2}v_i + \frac{1}{2}D^{-1}v_i D, \quad i = 1, \ldots, m-1.
\]

The hierarchy of NLEEs associated with (3.10) is
\[
v_{tn} = \begin{pmatrix}
v_0 \\
\vdots \\
v_{m-1}
\end{pmatrix}_t = J
\begin{pmatrix}
2a_n \\
b_n \\
\vdots \\
b_{n+m-2}
\end{pmatrix} = J \frac{\delta H_n}{\delta u}, \quad n = 1, 2, \ldots,
\]
(3.12)

where the Hamiltonian \(H_n\) and the Hamiltonian operator \(J\) are given by
\[
J = \begin{pmatrix}
\frac{1}{2}D & 0 & 0 & \cdots & 0 & 0 \\
0 & J_2 & J_3 & \cdots & J_{m-1} & -2D \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & J_{m-1} & -2D & \cdots & 0 & 0 \\
0 & -2D & 0 & \cdots & 0 & 0
\end{pmatrix},
\]

\[
J_i = v_{i,x} + 2v_i D, \quad i = 0, 1, \ldots, m-1, \quad H_n = \frac{2}{m-2n-2}[a_{n,x} - \sum_{i=1}^{m} iv_i b_{n+i-1}].
\]

Also we have
\[
\frac{\delta \lambda}{\delta u} = (2\phi_1 \phi_2, \phi_1^2, \lambda \phi_1^2, \ldots, \lambda^{m-2} \phi_1^2)^T.
\]
(3.13)

The constrained flows of (3.12) are defined by
\[
\Phi_{1,x} = v_0 \Phi_1 + \Lambda \Phi_2, \quad \Phi_{2,x} = (\Lambda^{m-1} - \sum_{i=1}^{m-1} v_i \Lambda^{i-1})\Phi_1 - v_0 \Phi_2,
\]
(3.14a)
\[ \frac{\delta H_l}{\delta v} + \frac{1}{2} \sum_{j=1}^{N} \delta \lambda_j \frac{\delta v}{\delta v} = \begin{pmatrix} \frac{2a_l}{b_l} \\ \vdots \\ -\frac{b_{l+m-2}}{} \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 2 < \Phi_1, \Phi_2 > \\ < \Phi_1, \Phi_1 > \\ \vdots \\ < \Lambda^{m-2} \Phi_1, \Phi_1 > \end{pmatrix} = 0. \] (3.14b)

For \( l = m - 1 \), (3.14b) leads to

\[ v_0 = -\frac{1}{2} < \Phi_1, \Phi_2 >, \] (3.15a)

\[ v_{m-k} = \sum_{j=1}^{k} (-1)^{j-1} \frac{j+1}{2j} \sum_{l_1+\ldots+l_j=k-j} < \Lambda^{l_1} \Phi_1, \Phi_1 > \ldots < \Lambda^{l_j} \Phi_1, \Phi_1 >, \] (3.15b)

By substituting (3.15) into (3.14a), the first constrained flow of NLEE (3.12) can be written as a canonical FDHS

\[ \Phi_{1,x} = \frac{\partial \tilde{F}_0}{\partial \Phi_2}, \quad \Phi_{2,x} = -\frac{\partial \tilde{F}_0}{\partial \Phi_1}, \] (3.16a)

or

\[ \tilde{P}_x = \theta_0 \nabla \tilde{F}_0, \]

where

\[ \tilde{P} = \begin{pmatrix} \Phi_1^T \\ \Phi_2^T \end{pmatrix}^T, \quad \theta_0 = \begin{pmatrix} 0 & I_{N \times N} \\ -I_{N \times N} & 0 \end{pmatrix}, \]

\[ \tilde{F}_0 = \frac{1}{2} < \Lambda \Phi_2, \Phi_2 > -\frac{1}{4} < \Phi_1, \Phi_2 >^2 \]

\[ + \sum_{j=1}^{m} (-\frac{1}{2})^j \sum_{l_1+\ldots+l_j=m-j} < \Lambda^{l_1} \Phi_1, \Phi_1 > \ldots < \Lambda^{l_j} \Phi_1, \Phi_1 >. \] (3.16b)

**B. The QBH structure for the family of FDIHD (3.7).**

It is known [31] that the gauge transformation between the spectral problems (3.1) and (3.10) is given by

\[ \psi_1 = \phi_1, \quad \psi_2 = \lambda \phi_2 + v_0 \phi_1, \]

\[ u_i = v_i, \quad i = 1, \ldots, m-1, \quad u_0 = -v_{0x} - v_0^2, \] (3.17)

which, together with (3.6) and (3.15), gives rise to the map relating (3.7) to (3.16), i.e. \( P = M(\tilde{P}) \):

\[ \Psi_1 = \Phi_1, \quad \Psi_2 = \Lambda \Phi_2 - \frac{1}{2} < \Phi_1, \Phi_2 > \Phi_1. \] (3.18)
In fact the map \( M \) transforms the first equation and the second equation with an additive term \(-c\Psi_1(c = \tilde{F}_0)\), in (3.7) into the corresponding equations in (3.16). Since the \( \theta_1 \) constructed in the following is valid for an arbitrary \( c \), so we can take \( c = 0 \). The Jacobi \( M' \) of the map \( M \) takes the form

\[
M'(\tilde{P}) = \begin{pmatrix}
I_{N \times N} & 0_{N \times N} \\
\frac{1}{2} \Phi_1 \Phi_2^T & \Lambda - \frac{1}{2} \Phi_1 \Phi_1^T
\end{pmatrix}.
\]

(3.19)

Then the second compatible Poisson tensor for the vector field (3.7) is

\[
\theta_1 = M'\theta_0 M'^T |_{P = \tilde{P}} = \begin{pmatrix}
0_{N \times N} & A_1 \\
-A_1^T & B_1
\end{pmatrix},
\]

(3.20)

\[
A_1 = \Lambda - \frac{1}{2} \Psi_1 \Psi_1^T, \quad B_1 = \frac{1}{2} \Psi_2 \Psi_1^T - \frac{1}{2} \Psi_1 \Psi_2^T.
\]

By a straightforward calculation, we have the following proposition.

**Proposition 3.** The system (3.7) possesses the QBH representation

\[
P_x = \theta_0 \nabla F_0 = \frac{1}{\rho} \theta_1 \nabla E_1
\]

(3.21a)

where

\[
\rho = B(\lambda)|_{\lambda=0} = 1 - \frac{1}{2} < \Lambda^{-1} \Psi_1, \Psi_1 >, \quad (3.21b)
\]

\[
E_1 = [A^2(\lambda) + B(\lambda)C(\lambda)]|_{\lambda=0} = - \sum_{i=1}^{N} \lambda_i^{-1} F(i)
\]

\[
= \frac{1}{2} < \Lambda^{-1} \Psi_2, \Psi_2 > + \frac{1}{4} [ < \Lambda^{-1} \Psi_1, \Psi_2 >^2 - < \Lambda^{-1} \Psi_1, \Psi_1 > < \Lambda^{-1} \Psi_2, \Psi_2 >]
\]

\[
+ \sum_{j=0}^{m} (-\frac{1}{2})^{j+1} \sum_{l_1 + \ldots + l_j+1 = m-j} < \Lambda^{l_1} \Psi_1, \Psi_1 > \ldots < \Lambda^{l_{j+1}} \Psi_1, \Psi_1 >.
\]

(3.21c)

**C. The Nijenhuis coordinates.**

In the same way as for (2.30)-(2.33), the eigenvalues of the Nijenhuis tensor \( \mu_1, ..., \mu_N \) are defined by the roots of the equation

\[
f(\lambda) = | \lambda I - A_1 | = 0,
\]

(3.22a)

which gives

\[
\psi_{1j} = g_j(\mu) \quad j = 1, ..., N.
\]
Then one defines
\[ \nu_j = \frac{\partial S}{\partial \mu_j} = \sum_{j=1}^{N} \psi_{2j} \frac{\partial g_j}{\partial \mu_j}, \quad j = 1, \ldots, N. \] (3.22b)

On the other hand, the generalized elliptic coordinates \((\bar{\mu}, \bar{\nu})\) are defined by means of the Lax matrix in the following way [24]. The coordinates \(\bar{\mu}_1, \ldots, \bar{\mu}_N\) are introduced by the zeros of \(B(\lambda)\):
\[ B(\lambda) = 1 + \frac{1}{2} \sum_{j=1}^{N} \frac{\psi_{1j}^2}{\lambda - \lambda_j} = \frac{R(\lambda)}{K(\lambda)}, \] (3.23a)
where \(K(\lambda)\) is defined by (2.35) and
\[ R(\lambda) = \prod_{k=1}^{N} (\lambda - \bar{\mu}_k) = \sum_{k=0}^{N} \beta_k \lambda^{N-k}, \] (3.23b)
and the canonically conjugate coordinates \(\bar{\nu}_1, \ldots, \bar{\nu}_N\) are defined by
\[ \bar{\nu}_k = -A(\bar{\mu}_k) = \frac{1}{2} \sum_{j=1}^{N} \psi_{1j} \psi_{2j} \frac{1}{\bar{\mu}_k - \lambda_j}, \quad k = 1, \ldots, N. \] (3.23c)

We have the following proposition.

**Proposition 4.** The Nijenhuis coordinates \((\mu, \nu)\) defined by (3.22) are exactly the same as the generalized elliptic coordinates \((\bar{\mu}, \bar{\nu})\) defined by (3.23). The QBH vector field (3.21) is Pfaffian in the Nijenhuis coordinates.

Proof. Similarly, we have by induction

\[
\begin{align*}
f_N(\lambda; \lambda_1, \ldots, \lambda_N) &= |\lambda I - A_1| \\
&= \begin{vmatrix}
\lambda - \lambda_1 + \frac{1}{2} \psi_{11}^2 & \frac{1}{2} \psi_{11} \psi_{12} & \cdots & \frac{1}{2} \psi_{11} \psi_{1N} \\
\frac{1}{2} \psi_{12} \psi_{11} & \lambda - \lambda_2 + \frac{1}{2} \psi_{12}^2 & \cdots & \frac{1}{2} \psi_{12} \psi_{1N} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{1}{2} \psi_{1N} \psi_{11} & \frac{1}{2} \psi_{1N} \psi_{12} & \cdots & \lambda - \lambda_N + \frac{1}{2} \psi_{1N}^2 \\
\end{vmatrix} \\
&= (\lambda - \lambda_1) f_{N-1}(\lambda; \lambda_2, \ldots, \lambda_N) + \frac{1}{2} \psi_{11}^2 \prod_{k=2}^{N} (\lambda - \lambda_k) = B(\lambda) K(\lambda),
\end{align*}
\] (3.24)
which shows that $\mu_k = \bar{\mu}_k$. It follows from (3.23a) that
\[
\psi_{1j}^2 = 2 \frac{R(\lambda_j)}{K'(\lambda_j)}, \quad j = 1, ..., N.
\] (3.25)
Thus we have
\[
\nu_k = \sum_{j=1}^{N} \psi_{2j} \frac{\partial}{\partial \mu_k} \sqrt{2R(\lambda_j) K'(\lambda_j)}(\mu_k - \lambda_j) = \frac{1}{2} \sum_{j=1}^{N} \psi_{1j} \psi_{2j},
\] (3.26)
which implies that $\nu_k = \bar{\nu}_k$, since $\mu_k = \bar{\mu}_k$. Finally, it is found from (3.23a) that
\[
\rho = B(\lambda)|_{\lambda=0} = 1 - \frac{1}{2} < \Lambda^{-1} \Psi_1, \Psi_1 > = \frac{\beta_N}{\alpha_N}.
\]
This completes the proof.

IV. The second family of QBH systems.

For $l = m + 1$, it is found from (3.5b) [24] that
\[
u_{m-k} = (-\frac{1}{2})^k u_{m-1}^{k-1} + \sum_{i=0}^{k-2} u_{m-1}^i \sum_{j=1}^{\frac{k-1}{2}} E_{i,j} \sum_{l_1+...+l_j = k-i-2} < \Lambda^{l_1} \Psi_1, \Psi_1 > ... < \Lambda^{l_j} \Psi_1, \Psi_1 >, \quad k = 2, ..., m,
\]
\[
L_0 u_{m-1} = < \Lambda^{m-1} \Psi_1, \Psi_1 > - \sum_{i=1}^{m-1} L_i < \Lambda^{i-1} \Psi_1, \Psi_1 >,
\]
(4.1a)
where
\[
E_{i,j} = -(i+j+1) \beta_{i,j}, \quad \beta_{i,j} = (-\frac{1}{2})^{i+j} \frac{(i+j)!}{i!j!}.
\]

Denote
\[
q = u_{m-1}, \quad p = -\frac{1}{8} u_{m-1,x}.
\]
By substituting (4.1a), (3.5a) and (4.1b) become a canonical FDHS
\[
P_x = \theta_0 \nabla F_1,
\]
(4.2a)
where
\[
P = (\Psi_1^T, q, \Psi_2^T, p)^T, \quad \theta_0 = \begin{pmatrix} 0 & I_{(N+1)\times(N+1)} \\ -I_{(N+1)\times(N+1)} & 0 \end{pmatrix},
\]
\[ F_1 = \frac{1}{2} < \Psi_2, \Psi_2 > + \left(-\frac{1}{2}q\right)^{m+2} - 4p^2 \]

\[ + \sum_{i=0}^{m} q^i \sum_{j=1}^{\left[\frac{m+2}{i}\right]} \beta_{i,j} \sum_{l_1 + \ldots + l_j = m+2-i-2j} < \Lambda^{i_1} \Psi_1, \Psi_1 > \ldots < \Lambda^{i_j} \Psi_1, \Psi_1 >. \tag{4.2b} \]

The entries of the Lax matrix \( Q \) for (4.2) are of the form \[ A(\lambda) = 2p - \frac{1}{2} \sum_{j=1}^{N} \frac{\psi_{1j} \psi_{2j}}{\lambda - \lambda_j}, \quad B(\lambda) = \lambda + \frac{1}{2} q + \frac{1}{2} \sum_{j=1}^{N} \frac{\psi_{1j}^2}{\lambda - \lambda_j}, \tag{4.3a} \]

\[ C(\lambda) = \sum_{k=0}^{m+1} \lambda^{m+1-k} \tilde{c}_k - \frac{1}{2} \sum_{j=1}^{N} \frac{\psi_{2j}^2}{\lambda - \lambda_j}, \tag{4.3b} \]

where

\[ \tilde{c}_k = \left(-\frac{1}{2}q\right)^k + \sum_{i=0}^{k-2} q^i \sum_{j=1}^{\left[\frac{k+1}{i}\right]} \beta_{i,j} \sum_{l_1 + \ldots + l_j = k-i-2j} < \Lambda^{i_1} \Psi_1, \Psi_1 > \ldots < \Lambda^{i_j} \Psi_1, \Psi_1 >, \quad k = 1, \ldots, m+1, \tag{4.3c} \]

\[ \tilde{c}_{m+2+k} = -\frac{1}{2} < \Lambda^k \Psi_2, \Psi_2 >, \quad k = 0, 1, \ldots \tag{4.3d} \]

Similarly, the equality

\[ A^2(\lambda) + B(\lambda)C(\lambda) = \lambda^{m+2} - F_1 + \sum_{i=1}^{N} \frac{F^{(i)}}{\lambda - \lambda_i}, \tag{4.4} \]

\[ F^{(i)} = -2p\psi_{1i} \psi_{2i} - \frac{1}{2} (\lambda_i + \frac{1}{2} q) \psi_{2i}^2 + \frac{1}{2} \sum_{k=0}^{m+1} \tilde{c}_k \lambda_i^{m+1-k} \psi_{1i}^2 \]

\[ + \frac{1}{4} \sum_{k \neq i} \frac{(\psi_{1i} \psi_{2k} - \psi_{1k} \psi_{2i})^2}{\lambda_k - \lambda_i}, \quad i = 1, \ldots, N, \]

determines \( N + 1 \) independent integrals of motion \( F_0, F^{(i)}, i = 1, \ldots, N, \) for the FDHS (4.2). The systems (4.2) for \( m = 1, 2, \ldots \) give the second family of FDIHSs. By taking \( m = 1 \) (4.2) gives rises to the multidimensional Henon-Heiles system. The system (4.2) was also studied by a recurrence relation in [35], however no explicit expressions like (4.2b) and (4.3) were given in that paper.

In the exactly the same way as we did in the previous section, we can obtain another FDHS from (3.14) for \( l = m \), find the map relating this FDHS to the FDHS (4.2) and
finally, by using this map, obtain the second compatible Poisson tensor for the vector field for (4.2)

$$\theta_1 = \begin{pmatrix} 0_{(N+1)\times(N+1)} & A_1 \\ -A_1^T & B_1 \end{pmatrix},$$

$$A_1 = \begin{pmatrix} \Lambda & -\frac{1}{4}\Psi_1 \\ 2\Psi_1^T & -\frac{1}{2}q \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0_{N\times N} & \frac{1}{4}\Psi_2 \\ -\frac{1}{4}\Psi_2^T & 0 \end{pmatrix}. \tag{4.5}$$

By a straightforward calculation, we can show the following proposition.

**Proposition 5.** The system (4.2) possesses the QBH representation

$$P_x = \theta_0 \nabla F_1 = \frac{1}{\rho} \theta_1 \nabla E_1 \tag{4.6}$$

where

$$\rho = B(\lambda)|_{\lambda=0} = \frac{1}{2}q - \frac{1}{2} < \Lambda^{-1}\Psi_1, \Psi_1 >,$$ \tag{4.7a}

$$E_1 = [A^2(\lambda) + B(\lambda)C(\lambda)]|_{\lambda=0} = -F_1 - \sum_{i=1}^{N} \lambda_i^{-1} F^{(i)}$$

$$= 2p < \Lambda^{-1}\Psi_1, \Psi_2 > + 4q < \Lambda^{-1}\Psi_2, \Psi_2 > + 4p^2 - \left(\frac{1}{2}q\right)^{m+2}$$

$$- \sum_{i=0}^{m} q^i \sum_{j=1}^{[m+2-i]} \beta_{i,j} \sum_{l_1+...+l_j=m+2-i-2j} < \Lambda^{l_1}\Psi_1, \Psi_1 > ... < \Lambda^{l_j}\Psi_1, \Psi_1 >$$

$$+ \frac{1}{4} \left[ < \Lambda^{-1}\Psi_1, \Psi_2 >^2 - < \Lambda^{-1}\Psi_1, \Psi_1 > < \Lambda^{-1}\Psi_2, \Psi_2 > \right] - \frac{1}{2} \sum_{i=0}^{m+1} q^i \sum_{j=0}^{[m+1-i]} \beta_{i,j}$$

$$\times \sum_{l_1+...+l_{j+1}=m+1-i-2j} < \Lambda^{l_1}\Psi_1, \Psi_1 > ... < \Lambda^{l_j}\Psi_1, \Psi_1 > < \Lambda^{l_{j+1}-1}\Psi_1, \Psi_1 >. \tag{4.7b}$$

In the same way, \(\mu_1, ..., \mu_{N+1}\) in the Nijenhuis coordinates are defined by the roots of the equation

$$f(\lambda) = |\lambda I - A_1| = 0,$$ \tag{4.8a}

which gives

$$\psi_{1j} = g_j(\mu) \quad j = 1, ..., N, \quad q = g_{N+1}(\mu).$$

Then one defines

$$\nu_j = \frac{\partial S}{\partial \mu_j} = \sum_{j=1}^{N} \psi_{2j} \frac{\partial g_j}{\partial \mu_j} + p \frac{\partial g_{N+1}}{\partial \mu_j}, \quad j = 1, ..., N+1. \tag{4.8b}$$
On the other hand, the generalized parabolic coordinates \((\bar{\mu}, \bar{\nu})\) are defined by means of the Lax matrix in the following way \([24]\). The coordinates \(\bar{\mu}_1, ..., \bar{\mu}_{N+1}\) are introduced by the zeros of \(B(\lambda)\):

\[
B(\lambda) = \lambda + \frac{1}{2} q + \frac{1}{2} \sum_{j=1}^{N} \frac{\psi_{1j}^2}{\lambda - \lambda_j} = \frac{R(\lambda)}{K(\lambda)},
\]  

(4.9a)

where \(K(\lambda)\) is defined by \((2.35)\) and \(R(\lambda)\) by

\[
R(\lambda) = \prod_{k=1}^{N+1} (\lambda - \bar{\mu}_k) = \sum_{k=0}^{N+1} \beta_k \lambda^{N+1-k},
\]

\[
\beta_0 = 1, \quad \beta_1 = -\sum_{j=1}^{N} \bar{\mu}_j, ..., \quad \beta_{N+1} = (-1)^{N+1} \prod_{j=1}^{N} \bar{\mu}_j,
\]

and the canonically conjugate coordinates \(\bar{\nu}_1, ..., \bar{\nu}_{N+1}\) are defined by

\[
\bar{\nu}_k = -A(\bar{\mu}_k) = -2p + \frac{1}{2} \sum_{j=1}^{N} \frac{\psi_{1j} \psi_{2j}}{\bar{\mu}_k - \lambda_j}, \quad k = 1, ..., N + 1.
\]  

(4.9b)

We have the following proposition.

**Proposition 4.** The Nijenhuis coordinates \((\mu, \nu)\) defined by \((4.8)\) are exactly the same as the generalized parabolic coordinates \((\bar{\mu}, \bar{\nu})\) defined by \((4.9)\). The QBH vector field \((4.6)\) is Pfaffian in the Nijenhuis coordinates.

Proof. In a similar way, we can show by induction that

\[
f(\lambda) = B(\lambda)K(\lambda).
\]  

(4.10)

It follows from \((4.9a)\) that

\[
\psi_{1j}^2 = 2 \frac{R(\lambda_j)}{K'(\lambda_j)}, \quad j = 1, ..., N,
\]

\[
q = \langle \Lambda^{-1} \Psi_1, \Psi_1 \rangle + 2 \frac{\beta_{N+1}}{\alpha_N} = \sum_{j=1}^{N} \frac{2R(\lambda_j)}{\lambda_j K'(\lambda_j)} + \frac{2 \beta_{N+1}}{\alpha_N}.
\]  

(4.11)

Then it is similar to find that

\[
\nu_k = -A(\mu_k),
\]

\[
\rho = B(\lambda)|_{\lambda=0} = \frac{1}{2} q - \frac{1}{2} < \Lambda^{-1} \Psi_1, \Psi_1 >= \frac{\beta_{N+1}}{\alpha_N}.
\]

This completes the proof.
V. Concluding remarks.

In the exactly same way as we did in the previous two sections, we can construct the third family of QBH systems from the constrained flows (3.5) for \( l = m + 2, m = 1, 2, \ldots \). The QBH system (2.28) is just the second member \((m = 2)\) in the third family of QBH systems, and \( \theta_1 \) and \( \rho \) given by (2.27) and (2.29a) are the second compatible Poisson tensor and the integrating factor for the third family of QBH systems.

In general, the constrained flow (3.5) for \( l = m + k \) can be transformed into a FDIHS by introducing the Jacobi-Ostrogradsky coordinates. Under the map relating this FDIHS to that obtained from the modified constrained flow (3.14) for \( l = m + k - 1 \), the image of the Poisson tensor \( \theta_0 \) for the latter gives rise to the second compatible Poisson tensor \( \theta_1 \) for the former. In this way, for each \( k \) we can obtain a family of QBH systems with \( m = 1, 2, \ldots \). The results obtained in the previous sections suggest the following conjecture: each family of QBH systems \((l = m + k, m = 1, 2, \ldots)\) shares the same \( \theta_1 \) and \( \rho \) for the QBH structure

\[
\theta_0 \nabla F_1 = \frac{1}{\rho} \theta_1 \nabla E_1,
\]

and, in general, by means of the Lax matrix \( Q = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & -A(\lambda) \end{pmatrix} \) and the expression

\[
A^2(\lambda) + B(\lambda)C(\lambda) = \sum_{i=0}^{m+2k} F_1 \lambda^i + \sum_{i=1}^{N} \frac{F^{(i)}}{\lambda - \lambda_i},
\]

we have

\[
\rho = B(\lambda)|_{\lambda=0}, \quad E_1 = [A^2(\lambda) + B(\lambda)C(\lambda)]|_{\lambda=0} = F_0 - \sum_{i=1}^{N} F^{(i)} \lambda_i^{-1}.
\]

For \( k = 1, 2, \ldots \), we find an infinite number of families of QBH systems. Furthermore we can show in a similar way that the Nijenhuis coordinates introduced by the Nijenhuis tensor are exactly the same as the separated variables defined by means of the Lax matrix for the QBH system in the family, and each QBH vector field is Pfaffian in the Nijenhuis coordinates.

Acknowledgment.

This work was supported by the Chinese Basic Research Project “Nonlinear Science”, the City University of Hong Kong and the Research Grants Council of Hong Kong. One of the authors (Y.B.Zeng) wishes to express his gratitude to Department of Mathematics of the City University of Hong Kong for warm hospitality.
References.
1. F. Magri, J. Math. Phys. 19, 1156 (1978).
2. M. Antonowicz, A. P. Fordy and S. Rauch-Wojciechowski, Phys. Lett. A 124, 143 (1987).
3. M. Antonowicz and S. Rauch-Wojciechowski, J. Phys. A 24, 5043 (1991).
4. M. Antonowicz and S. Rauch-Wojciechowski, Phys. Lett. A 163, 167 (1992).
5. M. Antonowicz and S. Rauch-Wojciechowski, J. Math. Phys. 33, 2115 (1992).
6. Yunbo Zeng, J. Phys. A 24, L11 (1993).
7. Yunbo Zeng, J. Math. Phys. 34, 4742 (1993).
8. M. Blaszak, J. Phys. A 26, 5985 (1993).
9. R. Caboz, V. Ravoson and L. Gavrilo, J. Phys. A 24, L523 (1991).
10. R. Brouzet, R. Caboz, J. Rabenivo and V. Ravoson, J. Phys. A 29, 2069 (1996).
11. F. Magri and T. Marsico, Electromagnetism and Geometrical structures, ed G. Ferrarese, Bologna: Pitagora, 1996, p207.
12. C. Morosi and G. Tondo, J. Phys. A 30, 2799 (1997).
13. M. Blaszak, J. Math. Phys. 39, 3213 (1998).
14. J. Rabenivo, J. Phys. A 34, 7113 (1998).
15. Yunbo Zeng, Phys. Lett. A 160, 541 (1991).
16. Yunbo Zeng and Yishen Li, J. Phys. A 26, L273 (1993).
17. Yunbo Zeng, Physica D73, 171 (1994).
18. Yunbo Zeng, J. Phys. A 24, L1065 (1991).
19. W.X. Ma and W. Strampp, Phys. Lett. A 185, 277 (1994).
20. W.X. Ma, J. Phys. Soc. Jpn. 64, 1085 (1995).
21. W.X. Ma, B. Fuchssteiner and W. Oevel, Physica A 233, 331 (1996).
22. W.X. Ma, Q. Ding, W. G. Zhang and B. Q. Lu, IL Nuovo Cimento B 111, 1135 (1996).
23. O. Ragnisco and S. Rauch-Wojciechowski, Inverse Problems 8, 245 (1992).
24. Yunbo Zeng and Runlian Lin, Families of dynamical r-matrices and Jacobi inversion problem for nonlinear evolution equations, J. Math. Phys., 39, 5964 (1998).
25. E. K. Sklyanin, Prog. Theor. Phys. Suppl. 118, 35 (1995).
26. J. Harnad and P. Winternitz, Commun. Math. Phys. 172, 263 (1995).
27. Yunbo Zeng, J. Phys. A 30, 3719 (1997).
28. Yunbo Zeng, J. Phys. Society of Japan 66, 2277 (1997).
29. M. Jaulent and K. Miodek, Lett. Math. Phys. 1, 243 (1976).
30. A. C. Newell, Solitons in Mathematics and Physics (SIAM, Philadelphia, 1985).
31. M. Antonowicz and A. P. Fordy, Commun. Math. Phys. 124, 465 (1989).
32. B.A. Kupershmidt and J. Wilson, Invent. Math. 62, 403 (1981).
33. A.D. Dubrovin, Russian Math. Survey 36, 11 (1981).
34. Yunbo Zeng and Yishen Li, J. Math. Phys. 31, 2835 (1990).
35. J. C. Eilbeck, V. Z. Enol’skii, V. B. Kuznetsov and A. V. Tsiganov, J. Phys. A 27, 567 (1994).