NILPOTENT GROUPS AND COLIMITS

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Abstract. The colimit of the subgroups of a group $G$ which are nilpotent of class less than $q$ appears as the fundamental group of a subspace $B(q, G)$ of the classifying space $BG$ introduced in [1]. In this paper we study these groups from an algebraic point of view, and apply our results to deduce homotopy theoretical properties of the space $B(q, G)$. More precisely, we give a group theoretic condition which implies that the space $B(2, G)$ is not an Eilenberg–MacLane space of type $K(\pi, 1)$. First examples of such groups are given in [6].

1. Introduction

Let $G$ be a discrete group. In [1] a natural filtration

$$B(2, G) \subset \cdots \subset B(q, G) \subset \cdots \subset BG$$

of the classifying space $BG$ is introduced using the descending central series of free groups. The authors raised a question [1, page 15]: Is the space $B(q, G)$ Eilenberg–Maclane space of type $K(\pi, 1)$. First examples of groups where this property fails are given in [6]. In this paper we study the fundamental group of $B(q, G)$ and its relation to $G$ from an algebraic point of view. We give a group theoretic condition which implies that the space $B(q, G)$ fails to be a $K(\pi, 1)$ space, thus extending the list of such examples.

The fundamental group of $B(q, G)$ is isomorphic to the colimit $N_q(G)$ of the nilpotent subgroups $N \subset G$ of class less than $q$. It turns out that some of the basic homotopy theoretic properties of these spaces are determined by the groups $N_q(G)$. Corresponding to the fundamental groups of the spaces $B(q, G)$ in the filtration of $BG$ there is a sequence of group homomorphisms

$$N_2(G) \to \cdots \to N_q(G) \to \cdots \to G.$$ 

By the universal property of colimits for each $N \subset G$ of class less than $q$ there are maps $\eta_N : N \to N_q(G)$. We describe basic properties of $N_q(G)$, and state the following conjecture.

Conjecture 1.1. A group $G$ is nilpotent of class less than $q$ if and only if the natural map $\epsilon : N_q(G) \to G$ is an isomorphism.

We give a proof for the case of $q = 2$. The group $N_2(G)$ is the colimit of abelian subgroups of $G$, and is the main subject of study of this paper. For certain groups we describe it by studying the commutativity relations in $G$. These relations, we call symplectic sequences, are given by a set of non-identity elements $\{g_i\}_{i=1}^{2r}$ satisfying the commutation rules:

$$[g_i, g_{i+r}] = [g_j, g_{j+r}] \text{ for all } 1 \leq i, j \leq r,$$

$$[g_i, g_j] = 1 \text{ for any other pair.}$$
Let $D_2(G)$ denote the kernel of the homomorphism $G \times G \to G/[G,G]$ defined by the multiplication $(x,y) \mapsto xy[G,G]$. Here is the main result of the paper.

**Theorem 1.2.** Let $\{g_i\}_{i=1}^{2r}$ be a symplectic sequence in $G$ for some $r \geq 2$, and $S$ denote the subgroup generated by $\{g_i\}$. Then the natural map $N_2(S) \to D_2(S)$ is an homomorphism, moreover $N_2(S) \to N_2(G)$ is an inclusion whose image is the subgroup generated by $\{\eta(g_i)\}$.

When $G$ is finite this theorem gives an explicit group theoretic condition which implies that the space $B(2, G)$ is not an Eilenberg–Maclane space of type $K(\pi, 1)$.

**Theorem 1.3.** Suppose that $G$ is a finite group which has a non–trivial symplectic sequence $\{g_i\}_{i=1}^{2r}$ for some $r \geq 2$. Then $B(2, G)$ is not a $K(\pi, 1)$ space.

Extraspecial 2–groups are the first examples studied in [6]. Using symplectic sequences we extend the list of such groups to extraspecial $p$–groups, symmetric groups, and general linear groups.

The organization of the paper is as follows. In Section 2 we describe the basic properties of the groups $N_2(G)$, and also define the spaces $B(q, G)$. We introduce the group $D_2(G)$, and study its group theoretical properties in Section 3. Symplectic sequences are defined, and Theorem 1.2 is proved in Section 4. The special case of finite groups is examined in Section 5. In particular, important implications of Theorem 1.2 for finite groups are discussed. Applications of these results to the spaces $B(q, G)$, in particular Theorem 1.3 and related examples are given in Section 6.

## 2. Preliminaries

The descending central series of a group $G$ is the normal series

$$1 \subset \cdots \subset \Gamma^{q+1}(G) \subset \Gamma^q(G) \subset \cdots \subset \Gamma^2(G) \subset \Gamma^1(G) = G$$

defined inductively $\Gamma^1(G) = G$, $\Gamma^{q+1}(G) = [\Gamma^q(G), G]$. A group is nilpotent of class less than $q$ if $\Gamma^q(G) = 1$. The collection $N(q, G)$ of subgroups of class less than $q$ is a partially ordered set under inclusion. A natural way to study a group using the nilpotency information is to consider the natural map

$$N_q(G) = \colim_{N(q, G)} N \xrightarrow{\epsilon} G,$$

where the colimit is in the category of groups. The construction $G \mapsto N_q(G)$ is natural and defines an endofunctor $N_q$ of the category of groups $\text{Grp}$.

For each group $N$ in the collection $N(q, G)$ the commutativity of the diagram

$$\begin{array}{ccc}
N_q(G) & \xrightarrow{\epsilon} & G \\
\downarrow{\eta_N} & & \downarrow{\eta_N} \\
N & \xrightarrow{\eta_N} & G
\end{array}$$

implies that the natural maps $\eta_N$ are inclusions. In particular $\epsilon$ is surjective since all the cyclic subgroups of $G$ are contained in $N(q, G)$. We denote the image of $g$ under $\eta(g)$ simply
by $\eta(g)$ or $(g)$. The group $N_q(G)$ can be constructed as a quotient of the free group $F(G)$ generated on the set $\{(g)\mid g \in G\}$. Let $R_q$ denote the normal closure of the subgroup of $F(G)$ generated by the products $(g)(h)(gh)^{-1}$ of pairs of elements in $G$ such that the subgroup $\langle g, h \rangle \subset G$ generated by the pair is of class less than $q$. The natural map $(g) \mapsto \eta(g)$ establishes an isomorphism between $F(G)/R_q$ and $N_q(G)$. Its inverse is induced by the maps $N \to F(G)/R_q$. Therefore we obtain a presentation of the colimit

$$N_q(G) = \langle (g), g \in G \mid (g)(h) = (gh) \text{ if } \Gamma^q((g, h)) = 1 \rangle.$$  

(1)

When $q = 2$ the relations remember the multiplication between the commuting elements, as $q$ gets larger more relations are added which correspond to higher nilpotence information. In particular we have a sequence of surjections

$$N_2(G) \twoheadrightarrow N_3(G) \twoheadrightarrow \cdots \twoheadrightarrow N_q(G) \twoheadrightarrow N_{q+1}(G) \twoheadrightarrow \cdots \twoheadrightarrow G.$$  

Alternatively these groups can be described by their universal property: There is a natural bijection

$$\text{Grp}(\text{colim} N, H) \cong \lim_{\text{Grp}}(N, H).$$

This means that a homomorphism $N_q(G) \to H$ is a set map $\phi : G \to H$ which restricts to a group homomorphism on the members of $\mathcal{N}(q, G)$, call such a map a nil$_q$–map. Equivalently it is a set map satisfying $f(g_0)f(g_1) = f(g_0g_1)$ whenever $\Gamma^q((g_0, g_1)) = 1$. Therefore enlarging the morphisms of the category of groups by allowing such maps is equivalent to studying the image of the functor $N_q$. Define a category $\text{Grp}_q$ whose objects are groups and morphisms are nil$_q$–maps. Note that $N_q$ extends to this category and is the left adjoint of the inclusion functor

$$N_q : \text{Grp}_q \subseteq \text{Grp} : \iota_q$$

with the unit $\eta : G \to N_q(G)$ defined by $\eta(g) = \eta_q(g)$ and the counit $\epsilon : N_q(G) \to G$. Any nil$_q$–map $\phi : G \to H$ induces a group homomorphism $\epsilon_N(\phi) : N_q(G) \to H$. Note also that any nil$_r$–map is a nil$_q$–map for $r \geq q$. There is a sequence of inclusions of categories

$$\text{Grp} \subset \cdots \subset \text{Grp}_{q+1} \subset \text{Grp}_q \subset \cdots \subset \text{Grp}_3 \subset \text{Grp}_2.$$  

Note that if a group $G$ is nilpotent of class less than $q$ then every nil$_q$–map $G \to H$ is by definition a group homomorphism. Conversely, we want to characterize groups $G$ for which every nil$_q$–map $G \to H$ is a group homomorphism. Equivalently, we state the following characterization of nilpotent groups:

**Conjecture 2.1.** A group $G$ is nilpotent of class less than $q$ if and only if the natural map $\epsilon : N_q(G) \to G$ is an isomorphism.

We prove the case $q = 2$ which is special since there is a distinguished class of nil$_2$–maps $\omega_n : N_2(G) \to N_2(G)$ for each integer $n \in \mathbb{Z}$ defined by raising a generator to the $n$–th power $(g) \mapsto (g^n)$. The map $\omega_{-1}$ is an automorphism of $N_2(G)$ induced by the inversion map $i : G \to G$ which sends an element to its inverse.

**Proposition 2.2.** A group $G$ is abelian if and only if the natural map $\epsilon : N_2(G) \to G$ is an isomorphism.
Proof. The inversion map \( i \) induces a group homomorphism \( \omega_{-1} \) on \( N_2(G) \) hence by the commutative diagram of nil\(_2\)-maps

\[
\begin{array}{ccc}
G & \xrightarrow{i} & G \\
\downarrow{\eta} & & \uparrow{\epsilon} \\
N_2(G) & \xrightarrow{\omega_{-1}} & N_2(G)
\end{array}
\]

it induces a group homomorphism on \( G \).

The general case follows if one can prove that the natural map

\[
\text{colim}_{N(q,G)} (\Gamma^{q-1}(N) \times N) \to \text{colim}_{N(q,G)} \Gamma^{q-1}(N) \times \text{colim}_{N(q,G)} N
\]

is injective. In this case the inverse can be used to show that the multiplication map \( \Gamma^{q-1}(G) \times G \to G \) is a group homomorphism.

**Classifying spaces.** There is a general construction associated to a cosimplicial group

\[
Q : \Delta \to \text{Grp}, \ n \mapsto Q_n
\]

and a group \( H \) which gives a simplicial set \( B(Q,H) \) whose set of \( n \)-simplices is the set of group homomorphisms \( \text{Grp}(Q_n,H) \). The classifying space \( BG \) of a group \( G \) can be described in this way. The assignment

\[
n = (0 \to 1 \to \cdots \to n) \mapsto F_n = \langle e_1, e_2, \cdots, e_n \rangle
\]

defines a cosimplicial group \( F : \Delta \to \text{Grp} \). Then the associated simplicial set \( B(F,G) \) is the classifying space \( BG \) of \( G \). A natural transformation \( \lambda : F \to F' \) gives a map of simplicial sets \( B(F',G) \to B(F,G) \). For each \( q \geq 2 \) the natural transformation \( \lambda_n : F_n \to F_n/\Gamma^q(F_n) \) induces a map \( B(q,G) \to BG \) from the simplicial set associated to the cosimplicial group \( n \mapsto F_n/\Gamma^q(F_n) \). The set of \( n \)-simplices of \( B(q,G) \) is given by \( \text{Grp}(F_n/\Gamma^q(F_n),G) \), that is \( n \)-tuples of elements of \( G \) which generate a nilpotent group of class less than \( q \). Therefore the map \( B(q,G) \to BG \) is an inclusion. This construction gives a functor \( B(q,-) : \text{Grp} \to S \) which extends to \( \text{Grp}_q \). This follows from the fact that any nil\(_q\)-map from a nilpotent group of class less than \( q \) is actually a group homomorphism. Therefore a nil\(_q\)-map \( \phi : G \to H \) induces a set map \( \text{Grp}(F_n/\Gamma^q(F_n),G) \to \text{Grp}(F_n/\Gamma^q(F_n),H) \) compatible with the simplicial structure.

The fundamental group of \( B(q,G) \) is studied in [1]. They show that there is a natural isomorphism \( \pi_1(B(q,G)) \cong N_q(G) \). As the constructions behave well respect to direct products of groups same is true for the fundamental group.

**Proposition 2.3.** There is a natural isomorphism \( B(F',G_1 \times G_2) \to B(F',G_1) \times B(F',G_2) \) induced by the projections \( \pi_i : G_1 \times G_2 \to G_i \). In particular, the natural map \( N_q(G_1 \times G_2) \to N_q(G_1) \times N_q(G_2) \) is an isomorphism.

Proof. At the simplicial level there are natural maps

\[
\text{Grp}(F'(n),G_1 \times G_2) \to \text{Grp}(F'(n),G_1) \times \text{Grp}(F'(n),G_2)
\]
induced by the projections. These maps are bijections, and compatible with the simplicial structure. Specializing to $B(q, -)$, and looking at the fundamental group gives the desired result.

The nil$_q$–map $\eta : G \to N_q(G)$ and the projection $F_n \to F_n/\Gamma^q(F_n)$ induce inclusions between the sets

$$\text{Grp}(F_n/\Gamma^q(F_n), G) \xrightarrow{\eta} \text{Grp}(F_n/\Gamma^q(F_n), N_q(G)) \to \text{Grp}(F_n, N_q(G)),$$

where the first one splits by the map induced by $\epsilon : N_q(G) \to G$. These maps are compatible with the simplicial structure.

**Proposition 2.4.** There are inclusions of simplicial sets

$$B(q, G) \xrightarrow{\eta} B(q, N_q(G)) \to BN_q(G),$$

where the first one splits by the map induced by $\epsilon : N_q(G) \to G$. In particular, the natural homomorphism $N_q\eta : N_q(G) \to N_qN_q(G)$ splits by $N_q\epsilon$.

The composition $B(q, G) \to BN_q(G)$ induces an isomorphism on the fundamental groups, see also [1], [6].

### 3. Twisted products

A nil$_q$–map $\phi : G \to H$ induces a commutative diagram

$$
\begin{array}{ccc}
N_q(G) & \xrightarrow{N_q(\phi)} & N_q(H) \\
\downarrow{\epsilon_G} & & \downarrow{\epsilon_H} \\
G & \xrightarrow{\phi} & H
\end{array}
$$

of nil$_q$–maps. Let $\bar{H}$ denote the quotient of $H$ by the image of the kernel of $\epsilon_G$ under the composition $\epsilon_H N_q(\phi)$. Define $D_q(G, \phi)$ to be the pull–back of the diagram

$$
\begin{array}{ccc}
D_q(G, \phi) & \to & H \\
\downarrow & & \downarrow \\
G & \to & \bar{H}
\end{array}
$$

Note that $\phi$ induces a group homomorphism $G \to \bar{H}$. If two nil$_q$–maps differ by a group homomorphism $\theta$ then there is an induced homomorphism $D_q(G, \phi) \to D_q(G, \theta \phi)$. Therefore one can form the limit over the category whose objects are nil$_q$–maps $G \to H$ and morphisms are group homomorphisms making the triangles commute:
By the universal property of colimits this limit is isomorphic to $N_q(G)$ since $\epsilon$ is an initial object of this category. We will see certain types of groups where the inversion map $\omega_{-1}$ is initial among nil$_2$–maps defined on $G$ hence giving an isomorphism $N_2(G) \cong D_2(G, \omega_{-1})$. The latter group is denoted by $D_2(G)$ for simplicity of notation.

In this section we will study basic properties of the group $D_2(G)$. It is defined by the pull–back

$$
\begin{array}{ccc}
D_2(G) & \rightarrow & G \\
\downarrow & & \downarrow \\
G & \rightarrow & G
\end{array}
$$

where $\bar{G}$ is the quotient of $G$ by the image of $\eta(g)\eta(h)\eta(gh)^{-1}$ under $\epsilon N_q(\omega_{-1})$. This image is generated by $ghg^{-1}h^{-1}$, hence it is the commutator group. Therefore $D_2(G)$ can be identified with the kernel of the multiplication map

$$
G \times G \rightarrow G/[G, G], \quad (x, y) \mapsto xy[G, G].
$$

There is a commutative diagram

$$
\begin{array}{ccc}
N_2(G) & \stackrel{\bar{\epsilon}}{\longrightarrow} & D_2(G) \\
\downarrow \epsilon & & \downarrow \pi_1 \\
D_2(G) & \rightarrow & G
\end{array}
$$

where $\bar{\epsilon}(\eta(g)) = (g, g^{-1})$, and $\pi_1$ is induced by the projection onto the first coordinate. The map $\pi_1$ is clearly surjective, and its kernel is the subgroup $1 \times [G, G]$, that is $D_2(G)$ is an extension of $G$ by its commutator subgroup.

**Proposition 3.1.** The map $\bar{\epsilon}$ is surjective, and $\epsilon$ is the composition $\pi_1 \bar{\epsilon}$.

**Proof.** The surjectivity of $\epsilon$ follows from the equation

$$
((gh)^{-1}, gh)(g, g^{-1})(h, h^{-1}) = (1, [g, h]),
$$

which implies that $D_2(G)$ is generated by the pairs of the form $(g, g^{-1})$. \qed

Let $\bar{\eta} : G \rightarrow D_2(G)$ denote the nil$_2$–map $g \mapsto (g, g^{-1})$. There is a commutative diagram

$$
\begin{array}{ccc}
N_2(G) & \stackrel{\omega_{-1}}{\longrightarrow} & N_2(G) \\
\downarrow \bar{\epsilon} & & \downarrow \epsilon \\
D_2(G) & \stackrel{\tau}{\rightarrow} & D_2(G) \\
\downarrow \bar{\eta} & & \downarrow \bar{\eta} \\
G & \stackrel{i}{\rightarrow} & G
\end{array}
$$

where $\tau$ is the swap map $(x, y) \mapsto (y, x)$. That is the inversion map extends to a group automorphism, namely $\tau$, on $D_2(G)$. Alternatively the group $D_2(G)$ can be thought of as the smallest group which extends the inversion map $i$ to a group automorphism. In the
sense that if $\theta : T \to G$ is a surjective group homomorphism such that $i$ extends to a group automorphism of $T$ then there exists a unique surjective homomorphism $T \to D_2(G)$ such that $\theta$ factors as $T \to D_2(G) \to G$.

Both constructions $N_2(-)$ and $D_2(-)$ behave well with respect to central quotients. Let $Z$ be a subgroup contained in the center of $G$ then the natural map

$$N_2(G) = \colim N \to \colim N Z$$

is an isomorphism. We record the following for future reference.

**Proposition 3.2.** If $Z$ is a subgroup of $G$ contained in the center then there is a commutative diagram

$$\begin{array}{ccc}
Z & \to & N_2(G) \\
\downarrow & & \downarrow \varepsilon_G \\
Z & \to & D_2(G)
\end{array}$$

$$\begin{array}{ccc}
& & N_2(G/Z) \\
& & \downarrow \varepsilon_{G/Z} \\
& & N_2(G/Z)
\end{array}$$

$$\begin{array}{ccc}
& & D_2(G/Z) \\
& & \downarrow \pi_1 \\
& & D_2(G/Z)
\end{array}$$

$$\begin{array}{ccc}
& & G \\
& & \downarrow \pi_1 \\
& & G/Z
\end{array}$$

We can generalize $D_2(G)$ for values of $q$ greater than 2. Define a filtration of the commutator $[G, G]$ obtained from the image of the kernel of the natural map $N_2(G) \to N_q(G)$ under $\varepsilon$. This image can be identified with $d_q(G) = \langle [x, y] \mid \Gamma_q(x, y) = 1 \rangle$ and gives a normal series

$$1 = d_2(G) \subset d_3(G) \subset \cdots \subset d_q(G) \subset d_{q+1}(G) \subset \cdots \subset [G, G].$$

Denoting by $D_q(G)$ the quotient of $D_2(G)$ by the subgroup $1 \times d_q(G)$, we obtain a commutative diagram

$$\begin{array}{ccc}
N_q(G) & \xrightarrow{\varepsilon_q} & D_q(G) \\
\downarrow & & \downarrow \pi_1 \\
N_{q+1}(G) & \xrightarrow{\varepsilon_{q+1}} & D_{q+1}(G)
\end{array}$$

$$\begin{array}{ccc}
& & G \\
& & \downarrow \pi_1 \\
& & G
\end{array}$$

where all the maps are surjective, and $\varepsilon_q : N_q(G) \to G$ is the composition $\pi_1 \varepsilon_q$.

**Structure of $D_2(G)$ and twisted products.** The structure of the group $D_q(G, \phi)$ defined for any nil$_q$–map $\phi : G \to H$ can be described as follows. Let $H_\phi$ denote the image of the kernel of $\epsilon$ under $\phi$. One can define a group $G \times_\phi H_\phi$ which as a set is the direct product $G \times H_\phi$ and the multiplication is given by $(g, \alpha)(h, \beta) = (gh, \phi(gh)^{-1} \phi(g)\alpha\phi(h)\beta)$. In particular the group which correspond to the nil$_2$–map $i$ which sends an element to its inverse is the set $G \times [G, G]$ with the multiplication rule given by

$$(x, \alpha)(y, \beta) = (xy, [x, y] y\alpha\beta)$$

where $y\alpha$ stands for $y\alpha y^{-1}$. Inverse of an element $(x, \alpha)$ is given by $(x^{-1}, x^{-1}\alpha^{-1})$. To distinguish this group from the direct product we write $G \times [G, G]$. There is an isomorphism

$$G \times [G, G] \to D_2(G), \quad (g, \alpha) \mapsto (g, g^{-1}\alpha).$$
One can check that the swap map $\tau$ is represented by the automorphism $(x, \alpha) \mapsto (x^{-1} \alpha, x^{-1} \alpha)$.

The definition of $D_2$ is functorial, and preserves inclusions. Suppose that $N \lhd G \to H$ is an exact sequence of groups. The functor $D_2$ preserves exactness if and only if the restriction of the maps to the commutator subgroups is also exact. The necessity of this condition follows from the $3 \times 3$ lemma ([1, 1.3.2]) applied to the diagram

$$
\begin{array}{ccc}
[N, N] & \longrightarrow & [G, G] \\
\downarrow & & \downarrow \\
D_2(N) & \longrightarrow & D_2(G)
\end{array}
\begin{array}{ccc}
& \longrightarrow & [H, H] \\
& \downarrow & \downarrow \\
& D_2(H) & \\
N & \longrightarrow & G
\end{array}
\phi \to H
$$

and sufficiency is clear from the description of $D_2(G)$ as the twisted product: $(\phi(g), \phi(\alpha)) = (1, 1)$ implies $g \in N$ and $\alpha \in [N, N]$ by the exactness of the last and the first rows.

Next we describe the descending central subgroups $\Gamma^q(D_2(G))$. Next result is an exercise in commutator identities. We use the convention $[x, y] = xyx^{-1}y^{-1}$.

**Lemma 3.3.** Suppose that $\Gamma^{q+1}(G) = 1$ then

$$
[g_1^{-1}, [g_2^{-1}, \cdots [g_{q-1}^{-1}, g_q^{-1}] \cdots ]] = [g_1, [g_2, \cdots [g_{q-1}, g_q] \cdots ]^{(-1)^q}
$$

for all $g_i \in G$, $1 \leq i \leq q$.

**Proof.** This follows by induction on $q$. The equation clearly holds for any group of class less than 2. Assume it holds for any group of class less than $q$. Take $G$ of class less than $q + 1$ and consider the quotient map $G \to G/\Gamma^q(G)$. Since the quotient is of class less than $q$ we can write

$$
[g_2^{-1}, \cdots [g_{q-1}^{-1}, g_q^{-1}] \cdots ]] = [g_2, \cdots [g_{q-1}, g_q] \cdots ]^{(-1)^{q-1} \alpha}
$$

for some $\alpha$ in $\Gamma^q(G)$, which is central in $G$. Taking the commutator of $g_1^{-1}$ with both sides and applying the commutator identity $[x^{-1}, y] = x^{-1}y^{-1}[x, y]$ we obtain the desired equation. \hfill \square

**Proposition 3.4.** The following is a pull–back diagram

$$
\begin{array}{ccc}
\Gamma^q(D_2(G)) & \longrightarrow & \Gamma^q(G) \\
\downarrow & & \downarrow \pi \\
\Gamma^q(G) & \omega(-1)^q \pi & \Gamma^q(G)/\Gamma^{q+1}(G)
\end{array}
$$

where $\pi$ is the natural projection and $\omega(-1)^q$ is the inversion map if $q$ is odd and the identity otherwise. In particular $\Gamma^q(D_2(G))$ is of the form $\Delta_q(G) = \{(g, g^{(-1)^q}) | g \in \Gamma^q(G)\}$ if and only if $\Gamma^{q+1}(G) = 1$.

**Proof.** We prove the last statement first. Recall that $D_2(G)$ is generated by $(g, g^{-1})$ in $G \times G$. Assume that $\Gamma^q(G) \subset Z(G)$ then an iterated commutator of $q$ many elements
Now the first statement can be obtained by applying the last part to the quotient \( q \) direction. For the converse assume that \( \Gamma^q(D_2(G)) \) is the one described by the pull–back. \( \square \)

In this section we study the map \( \bar{\theta} : n \rightarrow D_2(G) \cong G \times [G, G] \), and give a group theoretic condition which implies that this map is an isomorphism. First we describe the kernel of \( \bar{\theta} \).

Recall that the map \( \bar{\theta} : N_2(G) \rightarrow G \times [G, G] \) is given by \( (g) \mapsto (g, 1) \). An arbitrary element \( \prod_{i=0}^{n} (g_i, 1) \) in \( N_2(G) \) maps to the product \( \prod_{i=0}^{n} (g_i, 1) \). This product can be written as

\[
\prod_{i=0}^{n} (g_i, 1) = \left( \prod_{i=0}^{n} g_i, \left[ \prod_{i=1}^{n} g_i, \prod_{i=2}^{n} g_i \right], \ldots, \left[ \prod_{i=i}^{n} g_i, \prod_{i=2}^{n} g_i \right] \right)
\]

by using the multiplication rule of the twisted product. One can expand the commutators in the second coordinate using the identity \( [x, yz] = [x, y]^q [x, z] \). As a result it becomes a product of conjugates of commutators. To describe this expression we introduce some notation. Let \( \bar{n} \) denote the ordered set \( \{1 < 2 < \cdots < n\} \), and consider the set \( \mathcal{P}_{2,n} \) of order preserving inclusions \( \theta : \bar{2} \rightarrow \bar{n} \). This set can be totally ordered by the lexicographic order: \( \theta_1 < \theta_2 \) if \( \theta_1(1) < \theta_2(1) \) or \( \theta_1(1) = \theta_2(1) \) and \( \theta_1(2) < \theta_2(2) \). Define a function on the \( n \)-fold direct product \( G^n \) for each such map

\[
\mathcal{P}_{2,n} \times G^n \rightarrow [G, G], \quad \theta(g_1, g_2, \cdots, g_n) = \theta^q[g_{\theta(1)}, g_{\theta(2)}]
\]

where \( \bar{\theta} \) is the ordered product \( \prod_{\theta(1) < \theta(2)} g_i \). In terms of this notation the image of an element under \( \bar{\theta} \) can be written as

\[
\bar{\theta}(\prod_{i=0}^{n} g_i) = \left( \prod_{i=0}^{n} g_i, \left[ \prod_{i=1}^{n} g_i, \prod_{i=2}^{n} g_i \right], \ldots, \left[ \prod_{i=n}^{n} g_i, \prod_{i=2}^{n} g_i \right] \right)
\]

where the last product is ordered. Therefore the kernel of \( \bar{\theta} \) can be described as follows.
Lemma 4.1. If an element $\prod_{i=0}^{n}(g_i)$ of $N_2(G)$ is contained in the kernel of $\bar{\epsilon}$ then the following equations hold in $G$:

1. $\prod_{i=0}^{n}g_i = 1$
2. $\Theta(g_1, g_2, \cdots, g_n) = \prod_{\theta \in T_2, n} \theta(g_1, g_2, \cdots, g_n) = 1$.

In particular, a nil-$2$–map $\phi : G \to H$ factors through $D_2(G)$ if and only if $\Theta(g_1, g_2, \cdots, g_n) = 1$ implies $\Theta(\phi(g_1), \phi(g_2), \cdots, \phi(g_n)) = 1$ for all $n$ and elements $g_1, \cdots, g_n$ in $G$. Since $\Theta$ consists of a product of commutators next result is an immediate consequence.

Corollary 4.2. Assume that $G$ satisfies the following: for every nil-$2$–map $\phi : G \to H$, and product of commutators $\prod_{i}[x_i, y_i] = 1$ the product of the commutators of the images $\prod_{i}[\phi(x_i), \phi(y_i)] = 1$. Then the natural map $\bar{\epsilon} : N_2(G) \to D_2(G)$ is an isomorphism.

Next definition is essential in producing examples of groups for which the map $\bar{\epsilon}$ is an isomorphism. In section [G] we will see that it has homotopy theoretic implications on the space $B(2, G) \subset BG$.

Definition 4.3. We call a sequence of elements $\{g_i\}_{i=1}^{2r}$ in $G$ a symplectic sequence if the following conditions are satisfied

1. $[g_i, g_{i+r}] = [g_j, g_{j+r}]$ for all $1 \leq i, j \leq r$,
2. $[g_i, g_j] = 1$ for any other pair.

The sequence is called non–trivial if the commutators $[g_i, g_{i+r}] \neq 1$, and otherwise trivial.

Note that the commutator $[g_i, g_{i+r}]$ commutes with all the other $g_j$ for all $j$. The key fact about symplectic sequences is that they are preserved under nil-$2$–maps. This will be a consequence of the following observation.

Lemma 4.4. Let $\{g_i\}_{i=1}^{2r}$ be a symplectic sequence in $G$, and $\phi : G \to H$ a nil-$2$–map. If $r \geq 2$ then for any positive integer $a, b, c, d$ the equation

$$[[\phi(g_i^a g_{i+r}^b), \phi(g_j^c g_{j+r}^d)] = [\phi(g_j), \phi(g_j)]^{ad-bc}, \ 1 \leq i, j \leq r,$$

holds in $H$.

Proof. For simplicity of notation set $t_i = g_i^a g_{i+r}^b, t_{i+r} = g_i^c g_{i+r}^d$, and $n = ad - bc$. For all $i \neq j$ the following holds in the group $H$

$$[\phi(t_i), \phi(t_{i+r})] = \phi(t_i)\phi(g_j^n g_j^{-a})\phi(t_{i+r})^{-1}\phi(g_j g_{j+r}^{-1} g_{j+r}^{-1})\phi(t_{i+r})^{-1}$$

$$= \phi(g_j^n)\phi(t_i)\phi(g_j^{-a} g_{j+r}^{-1})\phi(t_{i+r})^{-1}\phi(g_j^{-a} g_{j+r}^{-1})$$

$$= \phi(g_j^n)\phi(t_i)\phi(g_j^{-a} g_{j+r}^{-1})\phi(g_j^{-a} g_{j+r}^{-1})^{-1}$$

$$= [\phi(g_j), \phi(g_{j+r})]^{ad-bc}.$$

where we make use of the identity $\phi(g)^{-1} = \phi(g^{-1})$. The key step is the observation that $[g_j^{-a} t_{i+r}, t_i^{-1} g_{j+r}] = [g_j^{-a} g_{j+r} t_{i+r}, t_i^{-1}] = [g_j^n, g_{j+r}^{-1} t_{i+r}, t_i^{-1}] = 1$.

Next result is a particular case where $(a, b) = (1, 0)$ and $(c, d) = (0, 1)$.
Corollary 4.5. Suppose that \( \{g_i\}_{i=1}^{2r}, r \geq 2, \) is a symplectic sequence in \( G, \) and \( \phi : G \to H \)

is a nil_2-map. Then \( \{\phi(g_i)e^{2r}\}_{i=1}^{2r} \) is a symplectic sequence in \( H. \)

Let \( S \subset G \) denote the subgroup generated by \( \{g_i\}_{i=1}^{2r} \) where \( r \geq 2. \) If the sequence is

trivial then \( S \) is abelian, otherwise its commutator subgroup is a cyclic group generated by \( c_S = [g_i, g_{i+r}]. \) Let \( \phi : S \to H \) be a nil_2-map. Denote the subgroup of \( H \) generated by

the symplectic sequence \( \{\phi(g_i)e\} \) by \( T. \) Assume that the symplectic sequences \( \{g_i\} \) and \( \{\phi(g_i)e\} \)

are non-trivial. The commutator subgroup of \( T \) is generated by \( c_T = [\phi(g_i), \phi(g_{i+r}e)]. \)

Lemma 4.6. The order of \( c_T \) divides the order of \( c_S. \)

Proof. Since \( \phi \) is a nil_2-map it preserves commuting elements. Therefore if \( g_i \) commutes

with \( g_{i+r}e \) then their images \( \phi(g_i)e \) and \( \phi(g_{i+r}e) \) also commute with each other. Hence \( c_S = 1 \)

implies that \( c_T = 1. \)

Now Corollary 4.2 implies the following.

Corollary 4.7. Let \( \{g_i\}_{i=1}^{2r} \) be a symplectic sequence in \( G \) for some \( r \geq 2, \) and \( S \) denote

the subgroup generated by \( \{g_i\}. \) Then the natural map \( N_2(S) \to D_2(S) \) is an isomorphism,

moreover \( N_2(S) \to N_2(G) \) is an inclusion whose image is the subgroup generated by \( \{\eta(g_i)\}. \)

Proof. We will apply the condition in Corollary 4.2. Take a nil_q-map \( \phi : S \to H, \)

and assume that the symplectic sequence \( \{\phi(g_i)\} \) is non-trivial, since otherwise the condition is trivially satisfied. Let \( T \) denote the subgroup generated by \( \{\phi(g_i)\} \) in \( H. \) Consider an

arbitrary product \( \prod_k [x_k, y_k] = 1 \) in \( S \) and the corresponding product \( \prod_k [\phi(x_k), \phi(y_k)] \) in \( T. \)

Now by the commutation relations of \( S \) and \( T \) arbitrary products of commutators simplifies to

products consisting of commutators of the form \( [g_i^a g_{i+r}^b, g_i^c g_{i+r}^d] \) and \( [\phi(g_i^a g_{i+r}^b), \phi(g_{i+r}^c g_i^d)]. \)

The first product gives a certain power \( c_S^N = 1 \) and by Lemma 4.4 the latter one becomes \( c_T^N \)

which is also trivial by Lemma 4.6.

The second statement follows from the diagram

\[
\begin{array}{ccc}
N_2(S) & \longrightarrow & N_2(G) \\
\downarrow \cong & & \downarrow \\
D_2(S) & \longrightarrow & D_2(G)
\end{array}
\]

since the functor \( D_2 \) preserves injective group homomorphisms.

Central extensions of abelian groups. Let \( A \) and \( F \) be finitely generated abelian groups.

The natural map \( H^2(A; F) \to \text{Hom}(A \wedge A, F) \) defined by \( [\rho] \mapsto b = \rho - \rho' \) is surjective [4],

where \( \rho'(x, y) = \rho(y, x). \) A map \( A \wedge A \to F, \) by definition, corresponds to an alternating bilinear form \( b : A \times A \to F, \)

that is \( b(g, -) \) and \( b(-, g) \) are both homomorphism for all \( g \in A, \)

and \( b(g, g) = 1. \) This bilinear form coincides with the commutator map \( [-, -] : E_\rho \times E_\rho \to F \)

of the central extension

\[
F \triangleleft E_\rho \to A
\]

associated to the representative \( \rho. \)
Let $F$ be a field, or the ring of integers $\mathbb{Z}$. Take $A$ to be a vector space $V$ over $F$ (or a torsion free abelian group), and denote by $H(b)$ the group defined by

$$(v, \alpha)(w, \beta) = (v + w, b(v, w) + \alpha + \beta)$$
on the pairs $(v, \alpha), (w, \beta) \in V \times F$. Observe that the map $f \mapsto (f, 1)$ injects $F$ as a central subgroup in $E_\rho \times [E_\rho, E_\rho]$ whose quotient is isomorphic to $H(b)$. Identifying the twisted product with $D_2(E_\rho)$ we obtain a central extension $F \triangleleft D_2(E_\rho) \to H(b)$.

Given a subspace $W \subset V$ the set of vectors $v$ satisfying $b(v, w) = 0$ for all $w \in W$ is denoted by $W^\perp$. A subspace $I \subset V$ is called isotropic if $I \subset I^\perp$. The set of isotropic subspaces is partially ordered under inclusion. Writing $b = b' \perp 0|_V$ where $b'$ is non–degenerate induces an isomorphism of groups $H(b) \cong H(b') \times V^\perp$. The non–degenerate part $b'$ is a sum of symplectic planes $\mathbb{H} \perp \cdots \perp \mathbb{H}$ where each plane has a basis $\{e_i, e_{i+r}\}$ such that $b(e_i, e_{i+r}) = 1$, see [5]. Any lift $\{\tilde{e}_i\}$ of this basis to the group $E_\rho$ gives a symplectic sequence.

**Theorem 4.8.** If the rank $2r$ of the non–degenerate part $b'$ is at least 4 then $H(b)$ is isomorphic to the colimit, in the category of groups, of the isotropic subspaces of $V$ regarded as abelian groups.

**Proof.** We can assume $b$ is non–degenerate. Apply Corollary 4.7 to $E_\rho$ by taking the lifts $\{\tilde{e}_i\}_{i=1}^{2r}$ as our symplectic sequence. By Proposition 3.2 there is a commutative diagram

$$
\begin{array}{ccc}
F & \overset{N_2(E_\rho)}{\longrightarrow} & \underset{\cong}{\text{colim } I} \\
\| & \downarrow{\cong} & \downarrow{\cong} \\
F & \overset{D_2(E_\rho)}{\longrightarrow} & H(b).
\end{array}
$$

\[\square\]

**Corollary 4.9.** If $r \geq 2$ then the natural map $N_2(E_\rho) \to D_2(E_\rho)$ is an isomorphism.

5. **Finite Groups**

Throughout this section $G$ denotes a finite group. The collection $\mathcal{N}(q, G)$ can be reduced to a smaller collection without affecting the colimit.

**Proposition 5.1.** The group $N_q(G)$ is isomorphic to the colimit of the groups in the sub–collection $N_2 \subset \mathcal{N}(q, G)$ consisting of $p$–groups of rank at most 2 and direct products of a $p$–group of rank 1 with a $q$–group of rank 1 for distinct primes $p$ and $q$.

**Proof.** The proof uses the coset poset of Brown [3]; for the $q = 2$ case see [6, Theorem 6.1], which generalizes to nilpotent groups without difficulty. A proof of this statement is given in [7, Proposition 2.2.1]. \[\square\]

For a prime $p$ define a collection $\mathcal{N}(q, G)_p$ consisting of $p$–subgroups of nilpotency class less than $q$, and set $N_q(G)_p$ to be the colimit of the groups in this collection. There is a natural
map \( N_q(G)_p \rightarrow N_q(G) \) induced by the inclusion \( \mathcal{N}(q, G)_p \subset \mathcal{N}(q, G) \) for each prime \( p \) which divides the order of the group \( G \). Furthermore it follows from Proposition 5.1 that the kernel of the map

\[
\prod_{p|\lvert G\rvert} N_q(G)_p \rightarrow N_q(G)
\]

is contained in the kernel of the natural map \( \prod N_q(G)_p \rightarrow \prod N_q(G)_p \). Hence there exist a map \( N_q(G) \rightarrow \prod N_q(G)_p \) such that the following diagram commutes

\[
\begin{array}{ccc}
\prod N_q(G)_p & \longrightarrow & N_q(G) \\
\uparrow & & \downarrow \\
N_q(G)_p & \longleftarrow & \prod N_q(G)_p.
\end{array}
\] (2)

In particular, the natural map \( N_q(G)_p \rightarrow N_q(G) \) is an inclusion and it splits. In the case of \( q = 2 \) the splitting \( \mathcal{2} \) can be described by using the maps \( \omega_n : N_2(G) \rightarrow N_2(G) \) defined by \( (g) \mapsto (g^n) \). The composition \( \omega_p \omega_{p'} \) where \( |G| = p^r p' \) such that \( (p, p') = 1 \) and \( \bar{p} \equiv 1/p' \mod p \) splits the natural inclusion \( N_2(G)_p \rightarrow N_2(G) \). As a consequence of the splitting we obtain the following.

**Proposition 5.2.** If \( N_2(G) \) is finite then \( G \) is nilpotent, i.e. isomorphic to the product of its Sylow \( p \)-subgroups.

**Proof.** Assume that \( G \) has composite order and \( N_2(G) \) is finite. Let \( p \) and \( q \) be distinct primes dividing the order of \( G \). Take \( g \) and \( h \) in \( G \) of order a power of \( p \) and \( q \) respectively such that \( [g, h] \neq 1 \). The product \((g)(h)\) has infinite order in the free product \( \prod_{p|\lvert G\rvert} N_2(G)_p \) where \( (g) \) and \( (h) \) denotes the corresponding generators in \( N_2(G)_p \) and \( N_2(G)_q \). Assume that its order in \( N_2(G) \) is \( n \). From \( \mathcal{2} \) and the consideration of the natural map \( \alpha : N_2(G)_p \prod N_2(G)_q \rightarrow N_2(G)_p \prod N_2(G)_q \) we conclude that \( p \) and \( q \) divides \( n \). Now the product \(( (g)(h) )^n \) can be written as a product of commutators

\[
(g)(h)\cdots(g)(h) = [(g)(h)](g)(h)^2\cdots[(g)^{n-1}, (h)^{n-1}][(h)^{n-1}, (g)^n](g)^n(g)(h)^n
\]

which is assumed to be in the kernel of \( N_2(G)_p \prod N_2(G)_q \rightarrow N_2(G) \). This kernel is contained in the kernel of \( \alpha \) which is a free group with a free basis consisting of the commutators \([x, y]\) of elements in \( N_2(G)_p \) and \( N_2(G)_q \), see Serre [8 Section 1.3]. Therefore at least one of the successive commutators must be inverses of each other which would result in \( g = 1 \) or \( h = 1 \), a contradiction. \[\square\]

Symplectic sequences in finite groups produce torsion elements in the kernel of \( \epsilon : N_2(G) \rightarrow G \). This will have a consequence in determining the homotopy type of the space \( B(2, G) \) in \([\mathcal{6}]\).

**Proposition 5.3.** Suppose that \( G \) is a finite group which has a non–trivial symplectic sequence \( \{g_i\}_{i=1}^{2r} \) for some \( r \geq 2 \). Then the kernel of the natural map \( \epsilon : N_2(G) \rightarrow G \) has a torsion element of order \( p \) for each prime dividing the order of \( c = [g_i, g_{i+r}] \).
Proof. By Corollary 4.5 the set \( \{\eta(g_i)\} \) is a symplectic sequence in \( N_2(G) \), in particular, \( d = [\eta(g_i), \eta(g_{i+r})] \) is central in the subgroup generated by \( \{\eta(g_i)\} \). Note also that since \( c \) commutes with all \( g_j \) we have \( [\eta(c), \eta(g_j)] = 1 \) for all \( 1 \leq j \leq 2r \). Therefore the element \( t = [\eta(g_i), \eta(g_{i+r})]\eta([g_i, g_{i+r}])^{-1} \) in \( \ker \epsilon \) is torsion: Let \( n \) denote the order of \( c \) then \( t^n = [\eta(g_i), \eta(g_{i+r})]^n\eta([g_i, g_{i+r}]^n)^{-1} = d^n c^n = 1 \) since the order of \( d \) divides \( n \) by Lemma 4.6. In fact this shows that the order of \( t \) is equal to \( n \). \( \square \)

6. Examples and applications

In this section we list some of the consequences of the group theoretic results obtained in the previous sections. Recall that in §2 the functor

\[ B(q, -) : \text{Grp}_q \to S \]

is defined, and we have an isomorphism \( \pi_1(B(q, G)) \cong N_q(G) \). Let \( E(q, G) \to B(q, G) \) denote the principal \( G \)-bundle obtained by the pull–back of the universal bundle \( EG \to BG \):

\[
\begin{array}{ccc}
E(q, G) & \longrightarrow & EG \\
\| & & \| \\
B(q, G) & \longrightarrow & BG.
\end{array}
\]

The characterization in Proposition 2.2 of abelian groups can be translated into the following statement.

**Proposition 6.1.** A group \( G \) is abelian if and only if \( E(2, G) \) is contractible.

**Proof.** If \( G \) is abelian then by the description of \( B(2, G) \) as a colimit of the classifying spaces \( BA \) of abelian subgroups \( A \) of \( G \), see [1, Theorem 4.3], there is a homotopy equivalence \( B(2, G) \simeq BG \). Therefore the pull–back \( E(2, G) \) is homotopy equivalent to the contractible space \( EG \). Conversely, if the space \( E(2, G) \) is contractible then the kernel of \( N_2(G) \to G \) which is isomorphic to the fundamental group of \( E(2, G) \) is trivial. By Proposition 2.2 \( G \) is abelian. \( \square \)

More generally, one can ask when \( B(q, G) \) has the homotopy type of a \( K(\pi, 1) \) space, a question raised in [1]. Recall the natural map \( B(q, G) \to BN_q(G) \) we defined in §2.

**Proposition 6.2.** Suppose that \( G \) is a finite group. If the natural map \( B(q, G) \to BN_q(G) \) is a homotopy equivalence then the kernel of the natural map \( N_q(G) \to G \) is torsion free.

**Proof.** The fiber of the map \( B(q, G) \to BG \) is homotopy equivalent to a finite dimensional complex, see [1]. If a finite dimensional complex is a \( K(\pi, 1) \) space then its fundamental group is torsion free. This follows from the fact that the cohomological dimension of \( \pi \) is less than or equal to its geometric dimension, and \( \pi \) is torsion free if its cohomological dimension is finite [4, Chapter VIII]. \( \square \)
In this context Proposition 5.3 implies the following.

**Theorem 6.3.** Suppose that $G$ is a finite group which has a non-trivial symplectic sequence \( \{g_i\}_{i=1}^r \) for some $r \geq 2$. Then $B(2, G)$ is not a $K(\pi, 1)$ space.

**Proof.** By Proposition 5.3 the kernel of the natural map $N_2(G) \to G$ has torsion, hence Proposition 6.2 implies that $B(2, G)$ cannot be a $K(\pi, 1)$. \( \square \)

Next we consider some classes of groups which satisfy the conclusions of Proposition 5.3 and Theorem 6.3. We will need the fact that Sylow $p$–subgroups of the symmetric group $\Sigma_n$ on $n$ letters are described by the $p$–adic expansion of $n$ [2, §VI.1]. For example $\text{Syl}_p(\Sigma_{p^k})$ is the $k$–fold wreath product $\wr k \mathbb{Z}/p$.

(1) Extraspecial $p$–groups are central in the ideas developed in this paper, such as the definition of a symplectic sequence. An extraspecial $p$–group is a central extension $Z(P) = \mathbb{Z}/p \triangleleft E_{p,r}^\pm \to (\mathbb{Z}/p)^{2r}$.

There are two types up to isomorphism $+, -$. Assume $r \geq 2$, then by Corollary 4.9 there is an isomorphism $N_2(E_p) \cong D_2(E_p)$. Therefore $E_p$ satisfies the conclusion of Theorem 6.3. The group $D_2(E_p)$ is a central extension $Z(D_2(E_p)) = \mathbb{Z}/p \triangleleft D_2(E_p) \to H(b)$ where one can show that

$$H(b) \cong \begin{cases} (\mathbb{Z}/2)^{2r+1} & \text{if } p = 2 \\ E_{p,r}^+ & \text{if } p \text{ is odd}. \end{cases}$$

Here $E_{p,r}^+$ denotes the extraspecial $p$–group of exponent $p$ where $p$ is an odd prime. Proposition 3.4 implies that $[D_2(E_p), D_2(E_p)]$ is isomorphic to $\mathbb{Z}/p$. One can show that there are central extensions

- $\mathbb{Z}/p \triangleleft N_2(E_p^+) \to (\mathbb{Z}/p)^{2r+1}$
- $\mathbb{Z}/p \triangleleft N_2(E_p^-) \to (\mathbb{Z}/p)^{2r-1} \times \mathbb{Z}/p^2$
- $\mathbb{Z}/2 \triangleleft N_2(E_2^+) \to (\mathbb{Z}/2)^{2r+1}$

where $p$ is an odd prime.

(2) The general linear group $GL_n(\mathbb{F}_q)$, $n \geq 4$, over the finite field $\mathbb{F}_q$ of characteristic $p$ has a symplectic sequence given by the elementary matrices

$$\{g_1 = E_{12}, g_2 = E_{13}, g_3 = E_{2n}, g_4 = E_{3n}\}$$

where $E_{ij}$ has 1’s on the diagonal and in the $(i, j)$-slot. It follows easily from the commutation relations of elementary matrices that this sequence satisfies the required commutation relations of a symplectic sequence. The commutator $E_{1n} = [g_1, g_3]$ has order $q$, hence by 5.3 the kernel of $N_2(GL_n(\mathbb{F}_q)) \to GL_n(\mathbb{F}_q)$ has a torsion element of order $p$.

(3) Next we consider the $n$–fold wreath product $W_p(n) = \mathbb{Z}/p \wr \cdots \wr \mathbb{Z}/p$ for $n \geq 3$.

Regarding linear transformations of an $n$ dimensional vector space $V$ over $\mathbb{F}_p$, as the permutations of a set of cardinality $p^n$ gives an inclusion $\iota : GL_{p^n}(p) \to \Sigma_{p^n}$.
Consider the case of $n = 4$, and the images of the elements in $\{g_i\}_{i=1}^4$ under this embedding. The elements in $V$ fixed by all $g_i$ for $1 \leq i \leq 4$ is given by the sub-space $\{(x, 0, 0, 0) | x \in \mathbb{F}_p\}$ whose cardinality is $p$. Hence the images land inside the symmetric group $\Sigma_{p^4 - p} \subset \Sigma_{p^4}$ under the inclusion $\iota$. The $p$–Sylow subgroup $S = \prod_{p}^{p-1}(W_p(3) \times W_p(2) \times W_p(1))$ of $\Sigma_{p^4-p}$ contains the image of the symplectic sequence. Therefore there is a $p$ torsion in the kernel of $\epsilon_S : N_2(S) \rightarrow S$. Note that the isomorphism $N_2(G \times H) \cong N_2(G) \times N_2(H)$ for the product of any given two groups $G$ and $H$ induces an isomorphism $\ker \epsilon_{G \times H} \cong \ker \epsilon_G \times \ker \epsilon_H$. Since $S$ is a product of wreath products, the kernel $\ker \epsilon_{W_p(3)}$ which corresponds to the largest wreath product contains $p$ torsion. This torsion element comes from the symplectic sequence $\{\pi(g_i)\}_{i=1}^4$ where $\pi : S \rightarrow W_p(3)$ is the natural projection.

(4) The previous example implies that $\Sigma_n$ with $n = \sum_{i=1}^{m} \alpha_i p^i$ where $0 \leq \alpha_i \leq p-1$ contains $p$ torsion in the kernel of $\epsilon : N_2(\Sigma_n) \rightarrow \Sigma_n$ when $m \geq 3$. This is a consequence of the inclusion $\varpi^m \mathbb{Z}/p \subset \Sigma_n$. One can conclude similarly for the alternating group $A_n$. For an odd prime $p$ both $\Sigma_n$ and $A_n$ have the same Sylow $p$–subgroups, and the case $p = 2$ can be settled by the isomorphism $\text{GL}_4(\mathbb{F}_2) \cong A_8$.

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