SOME COMPLEXITY RESULTS IN THE THEORY OF NORMAL NUMBERS

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Abstract. Let $N(b)$ be the set of real numbers which are normal to base $b$. A well-known result of H. Ki and T. Linton [17] is that $N(b)$ is $\Pi^0_3$-complete. We show that the set $N^\perp(b)$ of reals which preserve $N(b)$ under addition is also $\Pi^0_3$-complete. We use the characterization of $N^\perp(b)$ given by G. Rauzy in terms of an entropy-like quantity called the noise. It follows from our results that no further characterization theorems could result in a still better bound on the complexity of $N^\perp(b)$. We compute the exact descriptive complexity of other naturally occurring sets associated with noise. One of these is complete at the $\Pi^0_4$ level. Finally, we get upper and lower bounds on the Hausdorff dimension of the level sets associated with the noise.

1. Introduction

Let $b \geq 2$ be a positive integer. Every real number $x$ has a base $b$ expansion $x = [x] + \sum_{n=0}^{\infty} \frac{c_n}{b^n}$, and this expansion is unique if we adopt the convention that a tail of the coefficients $c_n$ cannot be equal to $b-1$. Recall $x$ is $b$-normal if for every $B = (i_0, \ldots, i_{\ell-1}) \in b^{<\omega}$ we have that $\lim_{N \to \infty} \frac{1}{N} |I(x, B, N)| = \frac{1}{b^\ell}$, where $I(x, B, N) = \{ i < N : (c_i, c_{i+1}, \ldots, c_{i+\ell-1}) = B \}$. For a real number $r$, define real functions $\pi_r$ and $\sigma_r$ by $\pi_r(x) = rx$ and $\sigma_r(x) = r + x$. We let $N(b)$ denote the set of reals $x$ which are normal to base $b$. We let

$$N^\perp(b) = \{ y : \forall x \in N(b) \ (x + y) \in N(b) \}.$$

1.1. Normality preserving functions. Let $f$ be a function from $\mathbb{R}$ to $\mathbb{R}$. We say that $f$ preserves $b$-normality if $f(N(b)) \subseteq N(b)$. We can make a similar definition for preserving normality with respect to the regular continued fraction expansion, $\beta$-expansions, Cantor series expansions, the Li"{o}roth series expansion, etc.

Several authors have studied $b$-normality preserving functions. They naturally arise in H. Furstenberg’s work on disjointness in ergodic theory [12]. V. N. Agafonov [1], T. Kamae [13], T. Kamae and B. Weiss [15], and W. Merkle and J. Reimann [20] studied $b$-normality preserving selection rules. The situation for continued fractions is very different. Let $[a_1, a_2, a_3, \ldots]$ be normal with respect to the continued fraction expansion. B. Heersink and J. Vandehey [13] recently proved that for any integers $m \geq 2, k \geq 1$, the continued fraction $[a_k, a_{m+k}, a_{2m+k}, a_{3m+k}, \ldots]$ is never normal with respect to the continued fraction expansion.

In 1949 D. D. Wall proved in his Ph.D. thesis [24] that for non-zero rational $r$ the function $\pi_r$ is $b$-normality preserving for all $b$ and that the function $\sigma_r$ is $b$-normality preserving for all $b$ whenever $r$ is rational. These results were also independently proven by K. T. Chang in 1976 [9]. D. D. Wall’s method relies on the well known characterization that a real number $x$ is normal in base $b$ if and only if the sequence $(b^n x)$ is uniformly distributed mod 1 [18].
D. Doty, J. H. Lutz, and S. Nandakumar took a substantially different approach from D. D. Wall and strengthened his result. They proved in [11] that for every real number \( x \) and every non-zero rational number \( r \) the \( b \)-ary expansions of \( x, \pi, (x) \), and \( \sigma_r (x) \) all have the same finite-state dimension and the same finite-state strong dimension. It follows that \( \pi_r \) and \( \sigma_r \) preserve \( b \)-normality. It should be noted that their proof uses different methods from those used by D. D. Wall and is unlikely to be proven using similar machinery.

C. Aistleitner generalized D. D. Wall’s result on \( \sigma_r \) in [4]. Suppose that \( q \) is a rational number and that the digits of the \( b \)-ary expansion of \( z \) are non-zero on a set of indices of density zero. He proved that the function \( \sigma_qz \) is \( b \)-normality preserving. G. Rauzy obtained a complete characterization of \( \mathcal{N}^k (b) \) in [21]. M. Bernay used this characterization to prove that \( \Sigma_\alpha \) has zero Hausdorff dimension [7]. One of the main results of this paper, stated in Corollary [3], is to obtain an exact determination of the descriptive set theoretic complexity of \( \mathcal{N}^k (b) \). A significance of this is explained at the end of [13] below.

M. Mendès France asked in [19] if the function \( \pi_r \) preserves simple normality with respect to the regular continued fraction for every non-zero rational \( r \). This was recently settled by J. Vandehey [22] who showed that \( \frac{a}{b} + \frac{c}{d} \) is normal with respect to the continued fraction when \( x \) is normal with respect to the continued fraction expansion and integers \( a, b, c, d \) satisfy \( ad - bc \neq 0 \). Work was also done on the normality preserving properties of the functions \( \pi_r \) and \( \sigma_r \) for the Cantor series expansions by the first and third author in [2] and additionally with J. Vandehey in [3]. However, these functions are not well understood in this context.

1.2. Descriptive Complexity. In any topological space \( X \), the collection of Borel sets \( B(X) \) is the smallest \( \sigma \)-algebra containing the open sets. They are stratified into levels, the Borel hierarchy, by defining \( \Sigma^1_0 = \) the open sets, \( \Pi^1_0 = \neg \Sigma^1_0 = \{ X - A \mid A \in \Sigma^0_0 \} \) = the closed sets, and for \( \alpha < \omega_1 \) we let \( \Sigma^\alpha_0 \), \( \Pi^\alpha_0 \) be the collection of countable unions \( A = \bigcup_n A_n \) where each \( A_n \in \Pi^\alpha_\alpha_n \) for some \( \alpha_n < \alpha \). We also let \( \Pi^0_\alpha = \neg \Sigma^\alpha_0 \). Alternatively, \( A \in \Pi^\alpha_0 \) if \( A = \bigcap_n A_n \) where each \( A_n \in \Sigma^\alpha_0 \) where each \( \alpha_n < \alpha \). We also set \( \Delta^\alpha_0 = \Sigma^\alpha_0 \cap \Pi^\alpha_0 \), in particular \( \Delta^1_0 \) is the collection of clopen sets. For any topological space, \( B(X) = \bigcup_{\alpha < \omega_1} \Sigma^\alpha_0 = \bigcup_{\alpha < \omega_1} \Pi^\alpha_0 \). All of the collections \( \Delta^\alpha_0, \Sigma^\alpha_0, \Pi^\alpha_0 \) are pointclasses, that is, they are closed under inverse images of continuous functions. A basic fact (see [16]) is that for any uncountable Polish space \( X \), there is no collapse in the levels of the Borel hierarchy, that is, all the various pointclasses \( \Delta^\alpha_0, \Sigma^\alpha_0, \Pi^\alpha_0 \), for \( \alpha < \omega_1 \), are all distinct. Thus, these levels of the Borel hierarchy can be used to calibrate the descriptive complexity of a set. We say a set \( A \subseteq X \) is \( \Sigma^\alpha_0 \) (resp. \( \Pi^\alpha_0 \)) hard if \( A \notin \Pi^\alpha_0 \) (resp. \( A \notin \Sigma^\alpha_0 \)). This says \( A \) is “no simpler” than a \( \Sigma^\alpha_0 \) set. We say \( A \) is \( \Sigma^\alpha_0 \)-complete if \( A \in \Sigma^\alpha_0 \), that is, \( A \in \Sigma^\alpha_0 \) and \( A \notin \Pi^\alpha_0 \). This says \( A \) is exactly at the complexity level \( \Sigma^\alpha_0 \). Likewise, \( A \) is \( \Pi^\alpha_0 \)-complete if \( A \in \Pi^\alpha_0 - \Sigma^\alpha_0 \).

A fundamental result of Suslin (see [13]) says that in any Polish space \( B(X) = \Delta^1_1 = \Sigma^1_1 \cap \Pi^1_1 \), where \( \Pi^1_1 = \neg \Sigma^1_1 \), and \( \Sigma^1_1 \) is the pointclass of continuous images of Borel sets. Equivalently, \( A \in \Sigma^1_1 \) iff \( A \) can be written as \( x \in a \iff \exists y \ (x, y) \in B \) where \( B \subseteq X \times Y \) is Borel (for some Polish space \( Y \)). Similarly, \( A \in \Pi^1_1 \) iff it is of the form \( x \in A \iff \forall y \ (x, y) \in B \) for a Borel \( B \). The \( \Sigma^1_1 \) sets are also called the \textit{analytic sets}, and \( \Pi^1_1 \) the \textit{co-analytic sets}. We also have \( \Sigma^1_1 \neq \Pi^1_1 \) for any uncountable Polish space.
H. Ki and T. Linton [17] proved that the set \( \mathcal{N}(b) \) is \( \Pi^0_\alpha(\mathbb{R}) \)-complete. Further work was done by V. Becher, P. A. Heiber, and T. A. Slaman [5] who settled a conjecture of A. S. Kechris by showing that the set of absolutely normal numbers is \( \Pi^0_\beta(\mathbb{R}) \)-complete. Furthermore, V. Becher and T. A. Slaman [6] proved that the set of numbers normal in at least one base is \( \Sigma^0_\gamma(\mathbb{R}) \)-complete.

K. Beros considered sets involving normal numbers in the difference hierarchy in [8]. Let \( N_k(b) \) be the set of numbers normal of order \( k \) in base \( b \). He proved that for \( b \geq 2 \) and \( s > r \geq 1 \), the set \( N_r(b) \setminus N_s(b) \) is \( D^2_\alpha(\Pi^0_\beta(\mathbb{R})) \)-complete (see [16] for a definition of the difference hierarchy). Additionally, the set \( \bigcup_k N_{2k+1}(2) \setminus N_{2k+2}(2) \) is shown to be \( D_\omega(\Pi^0_\gamma(\mathbb{R})) \)-complete.

1.3. Results of this paper. We are interested in determining the exact descriptive set theoretic complexity of \( N_\perp(b) \) and some related sets. The definition of \( N_\perp(b) \) shows that \( N_\perp(b) \) is \( \Pi^1_\alpha(\mathbb{R}) \), since it involves a universal quantification. It is not immediately clear if \( N_\perp(b) \) is a Borel set, but this in fact follows from a result of Rauzy. Specifically, Rauzy [21] characterized \( N_\perp(b) \) in terms of an entropy-like condition called the noise. We recall this condition and associated notation from [21]. For any positive integer length \( \ell \), let \( E_\ell \) denote the set of functions from \( b^\ell \) to \( b \). We call an \( E \in E_\ell \) a block function of width \( \ell \). As in [21] we set, for \( x \in \mathbb{R} \),

\[
\beta(x, N) = \inf_{E \in E_\ell} \frac{1}{N} \sum_{n < N} \inf \{ 1, |c_n - E(c_{n+1}, \ldots, c_{n+\ell})| \},
\]

where \( c_0, c_1, \ldots \) is the (fractional part) of the base \( b \) expansion of \( x \).

We also let for \( E \in E \)

\[
\beta_E(x, N) = \frac{1}{N} \sum_{n < N} \inf \{ 1, |c_n - E(c_{n+1}, \ldots, c_{n+\ell})| \}.
\]

We then define the lower and upper noises \( \beta^-(x), \beta^+(x) \) of \( x \) by:

\[
\beta^-(x) = \lim_{\ell \to \infty} \beta^{-\ell}_E(x),
\]

where

\[
\beta^{-\ell}_E(x) = \lim_{N \to \infty} \inf \beta_E(x, N).
\]

The upper entropy \( \beta^+(x) \) is defined similarly using

\[
\beta^+(x) = \lim_{\ell \to \infty} \beta^{+\ell}_E(x)
\]

where

\[
\beta^{+\ell}_E(x) = \lim_{N \to \infty} \sup \beta_E(x, N).
\]

For a fixed \( E \in E \) we also let

\[
\beta^{-E}_E(x) = \lim_{N \to \infty} \inf \beta_E(x, N),
\]

and similarly for \( \beta^{+E}_E(x) \).

Rauzy showed that \( x \in \mathcal{N}(b) \) iff it has the maximal possible noise in that \( \beta^-(x) = \frac{b-1}{b} \). Furthermore, \( x \in \mathcal{N}^\perp(b) \) iff it has minimal possible noise in that \( \beta^+(x) = 0 \).

It is therefore natural to ask for any \( s \in [0, \frac{b-1}{b}] \), what are the complexities of the lower and upper noise sets associated to \( s \). Specifically, we introduce the following four sets.
Definition 1. Let \( s \in [0, \frac{b-1}{b}] \). Let
\[
A_1(s) = \{ x : \beta^-(x) \leq s \}, \quad A_2(s) = \{ x : \beta^-(x) \geq s \}
\]
(1)
\[
A_3(s) = \{ x : \beta^+(x) \leq s \}, \quad A_4(s) = \{ x : \beta^+(x) \geq s \}
\]
Finally, we let
\[
L(s) = A_1(s) \cap A_2(s) = \{ x : \beta^-(x) = s \}
\]
\[
U(s) = A_3(s) \cap A_4(s) = \{ x : \beta^+(x) = s \}.
\]

Thus, \( \mathcal{N}(b) = L(\frac{b-1}{b}) \), and \( \mathcal{N}^\perp(b) = U(0) \). The Ki-Linton result shows that \( \mathcal{N}(b) \), and thus \( L(\frac{b-1}{b}) \) is \( \Pi^0_3 \)-complete for any base \( b \). Recall also the Becher-Slaman result which shows that the set of reals which are normal to some base \( b \) forms a \( \Sigma^0_3 \)-complete set.

We have the following complexity results.

Theorem 2. For any \( s \in [0, \frac{b-1}{b}] \), the set \( A_1(s) \) is \( \Pi^0_1 \)-complete and the set \( A_3(s) \) is \( \Pi^0_3 \)-complete. For any \( s \in (0, \frac{b-1}{b}] \), the set \( A_2(s) \) is \( \Pi^0_2 \)-complete, and the set \( A_4(s) \) is \( \Pi^0_4 \)-complete. For \( s \in (0, \frac{b-1}{b}) \), the set \( L(s) \) is \( \Pi^0_1 \)-complete, and the set \( U(s) \) is \( \Pi^0_3 \)-complete.

As a corollary we obtain the Ki-Linton result as well as the determination of the exact complexity of \( \mathcal{N}^\perp(b) \).

Corollary 3. The sets \( \mathcal{N}(b) \) and \( \mathcal{N}^\perp(b) \) are both \( \Pi^0_3 \)-complete.

Proof. We have \( x \in \mathcal{N}(b) \) iff \( x \in A_2(\frac{b-1}{b}) \), and \( x \in \mathcal{N}^\perp(b) \) iff \( x \in A_3(0) \), so the result follows immediately from Theorem 2.

Remark 4. In defining the noise, it is sometimes convenient to use the minor variation
\[
\beta_E(x, N) = \frac{1}{N} \sum_{\ell \leq n < N} \inf \{ 1, |c_n - \varphi(e_{n-\ell}, \ldots, e_{n-1})| \},
\]
that is, the block function predicts the next digit rather than the previous digit. In this case we must start the sum at \( \ell + 1 \) rather than 1, but this does not affect any of the limits used in defining \( \beta^-(x) \) or \( \beta^+(x) \).

Remark 5. In proving Theorem 2 we will work with \( x \) in the sequence space \( X = b^\omega \cap G \) where \( G \) is the set of \( x \in b^\omega \) which do not end in a tail of \( b-1 \)'s. This is a Polish space as \( b^\omega \) is a compact Polish space (with the usual product of discrete topologies on \( \{0, 1, \ldots, b-1\} \) and \( G \) is a \( G_\delta \) (that is, \( \Pi^0_3 \)) subset of \( b^\omega \). This is permissible as the map \( f : X \to \mathbb{R} \) given by \( f(e_0, c_1, \ldots) = \sum c_n b^n \) is continuous. So, for example, given that \( B_3(s) \) is \( \Pi^0_3 \)-complete, where \( B_3(s) \subseteq X \) is defined as \( A_3(s) \), except we consider directly \( x \in b^\omega \), then it follows that \( A_3(s) \) is \( \Pi^0_3 \)-complete. For if \( A_3(s) \) were in \( \Sigma^0_3 \), then so would be \( B_3(s) \) since \( x \in B_3(s) \iff f(x) \in A_3(s) \), that is, \( B_3(s) = f^{-1}(A_3(s)) \).

We remark on the significance of complexity classifications such Theorem 2. Aside from the intrinsic interest to descriptive set theory, results of this form can be viewed as ruling out the existence of further theorems which would reduce the complexity of the sets. For example, Rauzy’s theorem reduces the complexity of \( \mathcal{N}^\perp(b) \) from \( \Pi^0_3 \) to \( \Pi^0_3 \). The fact that \( A_3(0) \) is \( \Pi^0_3 \)-complete tells us that there cannot be other theorems which result in a yet simpler characterization of \( \mathcal{N}^\perp(b) \).
Lastly, we wish to approximate the Hausdorff dimension of the sets $A(s), U(s),$ and $L(s)$. Put $H(s) = -s \log_2 s - (1 - s) \log_2(1 - s)$.

**Theorem 6.** For $s \in \left[0, \frac{b - 1}{b}\right]$ we have

$$
\dim_H(A_1(s)) = 1 \\
\dim_H(A_2(s)) = 1 \\
\frac{1}{\log b} H(s) + \frac{\log(b - 1)}{\log b}s \leq \dim_H(A_3(s)) \leq \frac{1}{\log b} H(s) + s \\
\dim_H(A_4(s)) = 1.
$$

Furthermore

$$
\frac{1}{\log b} H(s) + \frac{\log(b - 1)}{\log b}s \leq \dim_H(U(s)) \leq \frac{1}{\log b} H(s) + s \\
\dim_H(L(s)) = 1.
$$

2. A Property of Noise

Before proving the main theorem in §3 we show a property of the noise which shows that one must be careful when attempting to construct reals with a desired lower-bound for the noise. If we have a set $A \subseteq \omega$ with density $d$, then for almost all $x \in b^\omega$, if $x_A$ is the result of copying $x$ to the set $A$ and taking value $0$ off of $A$, then we easily have that $\beta(x_A) = d(\frac{b - 1}{b})$. The next lemma shows that it might be possible to lower the noise by taking non-zero values off of $A$.

**Theorem 7.** There is a periodic set $A \subseteq \omega$, say with period $p$ and density $\frac{4}{p}$, such that if $u \in b^\omega$ satisfies $u \upharpoonright (\omega - A) = 0$ and $\beta^-(u) = d(\frac{b - 1}{b})$, then there is a $v \in b^\omega$ with $v \upharpoonright A = u \upharpoonright A$ and $\beta^+(v) < d(\frac{b - 1}{b})$.

**Proof.** Fix a positive integer $k$, and let $\ell > 10kb^k$. Let $A$ be the set with period $\ell + k$ with $A \cap [0, \ell + k - 1] = \{\ell, \ell + 1, \ldots, \ell + k - 1\}$. Suppose $u \in b^\omega$ with $u \upharpoonright (\omega - A) = 0$ and $\beta^-(u) = d(\frac{b - 1}{b})$, where $d = \frac{4}{p}$. Let $s \in b^\omega$ such that $s \upharpoonright b^s \in b^\omega$, where $p = \lfloor k \log_2(b) \rfloor + 1$, be the binary representation of the integer represented by the base $b$ expansion of $s$, where we put the least significant digits first. Let $c(s) \in 2^n$ be the result of translating the digits of $b(s)$ according to $0 \mapsto 11001, 1 \mapsto 11101$. Note that $c(s)$ will never have more than $4$ consecutive 1's.

Let $w = 2(5p + k + 8)$. We will describe a particular block function $E: b^w \to b$ and $v \in b^\omega$ as above. We describe $v \upharpoonright B_n - B'_n$, which only depends on $u \upharpoonright B'_n$. We say a sequence $t \in 2^\omega$ is canonical if it is a subsequence of a sequence of the form $t_0 \cdots t_r$, where $t_i = a_i \cdot b_i - 0 \cdot 1 \cdots 11111$ where $a_i = c(b_i)$ and $b_i$ is the $k$-digit base $b$ expansion of $i$ (with least significant digit first). We assume here that $r \leq b^k$. Thus, a canonical sequence is a way of “counting” from $0$ to $r$.

The block function $E$ operates as follows. If $s \in b^w$ is the constant 0 sequence, then $E(s) = 0$. If the first 1 (reading from the left) in $s$ occurs at position $p$ to the right of the midpoint of $s$, then $E(s)$ is the next digit of a canonical sequence
starting at the position of this 1, that is, \( E(s) = t(w - p) \) for a canonical sequence \( t \) of length \( > w \). If \( p < \frac{|s|}{2} \), then \( E \) checks to see if there is a sequence of 5 consecutive 1’s in \( s \). If not, \( E(s) = 0 \). If so, let \( q \) be the position which is the start of the first such sequence. Note that \( w \) is large enough so that if \( q < \frac{|s|}{2} \), then reading to right from \( q \) there are enough positions in \( s \) to see a complete “cycle” of a canonical sequence (that is, some \( t_i \)). Likewise if \( q > \frac{|s|}{2} \) there is enough room to see a cycle to the left of \( q \). Then \( E \) checks to see if the positions to the right (left if \( q > \frac{|s|}{2} \)) of \( q \) give a cycle of a canonical sequence. If so, we check to see if \( s \) is the subsequence of the corresponding canonical sequence. If so, \( E(s) \) is the next digit of this canonical sequence, and if not we set \( E(s) = 0 \). This completes the definition of \( E \).

We now define \( v \upharpoonright B_n - B'_n \). This consists of a canonical sequence \( t \) starting at the unique position \( q \in B_n - B'_n \) so that the \( k \) digits in \( B'_n \) correspond to the substring \( b_i \) of some \( t_i \) in \( t \). This completes the definition of \( v \).

We claim that for each \( q \in B_n \), if we let \( u_q = u \upharpoonright [q - w, q - 1] \), then
\[
|\{ q \in B_n : E(u_q) \neq u(q) \}| \leq 2.
\]
To see this, first note that \( E \) will predict a 0 at the start of the block \( B_n \) (corresponding to the 0 after a \( b_i \)), which is the correct value, and then predict a 1 (corresponding to the start of a 11111 sequence) at the next position, which is incorrect; note that \( B_n \) is large enough so that the first 1 in it is far to the right of the start of the block. Given the two initial 0’s in \( B_n \), \( E \) will continue to predict 0 until \( E \) reaches the point \( q \in B_n \) which is the first 1 (as \( E \) cannot find the 5 consecutive 1’s it needs to consider outputting a non-zero value). At position \( q \), \( E \) will also predict a 0, which is incorrect. After position \( q \), \( E \) will make correct predictions through the end of \( B'_n \), as \( u \upharpoonright [q, q'] \) is canonical, where \( q' = (n + 1)(\ell + k) - 1 \) is the last position of \( B'_n \).

Thus, \( \beta^+(v) \leq \frac{2}{\ell+1} \), while \( \beta^-(u) = d\left(\frac{b-1}{b}\right) = \frac{k}{1+k} \frac{b-1}{b} \). So, for \( k \frac{b-1}{b} > 2 \), we may choose \( \ell \) large enough so that the above construction of \( v \) works, and we then have \( \beta^+(v) \leq \beta^-(u) \). \( \square \)

### 3. Proof of Theorem 2

The upper bounds for the complexities of the sets of Theorem 2 all follow by straightforward computations from the definitions of these sets. For example, consider \( A_1(s) \). We have
\[
x \in A_1(s) \leftrightarrow \forall \epsilon \in \mathbb{Q}^+ \, \exists m \, \forall \ell \geq m \, \beta^{-}_{\ell}(x) \leq \epsilon
\]
\[
\leftrightarrow \forall \epsilon \in \mathbb{Q}^+ \, \exists m \, \forall \ell \geq m \, \forall k \, \exists N \geq k \, (\beta_{k}(x, N) \leq \epsilon)
\]
Since for fixed \( \epsilon, \ell, N \) the set \( \{ x : \beta_{\ell}(x, N) \leq \epsilon \} \) is clopen, this shows that \( A_1(s) \in \Pi^0_1 \).

The following lemma and its proof will be used several times in the proofs for the lower bounds on the complexities in Theorem 2.

**Lemma 8.** Let \( A \subseteq \omega \) be an infinite set with upper density \( d \), and let \( y \in b^{\omega} \). Then for almost all \( x \in b^{\omega} \), if \( x' \in b^{\omega} \) is defined by
\[
x'(n) = \begin{cases} x(k) & \text{if } n \text{ is the } k\text{th element of } A \\ y(k) & \text{if } n \text{ is the } k\text{th element of } \omega - A \end{cases}
\]
then \( \beta^+(x') \geq d\left(\frac{b-1}{b}\right) \).
Proof: Fix $\epsilon_n = \frac{1}{n^2}$, and fix a sufficiently fast growing sequence $n_0 < n_1 < \cdots$ (we will specify how fast the sequence needs to grow below). We also choose the $n_k$ such that $d(A, n_k) \geq d - \epsilon_k$, where $d(A, n) = \frac{|A \cap n|}{n}$ is the density of $A$ among $\{0, \ldots, n-1\}$. Consider the block of integers $B_k = [n_{k-1}, n_k)$. Let $m_k = |A \cap B_k|$, so $m_k \geq (d - \epsilon_k)n_k - n_{k-1}$. Let $l_k = b^{m_k}$ be the number of block functions of width $k$. Consider one of these block functions $E: b^k \to b$. Consider the function $\tau: b^{m_k} \to b^{m_k}$ defined as follows. If $s \in b^{m_k}$, let $s'$ be the result of copying $s$ to $B_k \cap A$ and copying $y$ to $B_k - A$ (that is, for $k \in B_k - A$, $s'(k) = y(t)$ where $k$ is the $t$th element of $\omega - A$). If $p_i$ is the $i$th element of $B_k \cap A$, let $\tau(s)(i) = E(s' | [p_i - k, p_i - 1]) - s'(p_i) \mod b$. So, $\tau(s) \in b^{m_k}$. Clearly $\tau$ is a bijection from $b^{m_k}$ to $b^{m_k}$. So, the number of $s \in b^{m_k}$ for which there are exactly a many $i < m_k$ such that $E(s' | [p_i - k, p_i - 1]) \neq s'(p_i)$ is equal to the number of $s \in b^{m_k}$ such that $s$ has exactly $a$ non-zero digits. So, the number $e(m_k)$ of $s \in b^{m_k}$ such that

\[ |\{i < m_k: E(s' | [p_i - k, p_i - 1]) \neq s'(p_i)\}| \geq m_k \left(\frac{b-1}{b} - \epsilon_k\right) \]

is at least as big as the number of $s \in b^{m_k}$ such that $|\{i < m_k: s(i) \neq 0\}| \geq m_k \left(\frac{b-1}{b} - \epsilon_k\right)$. From the law of large numbers we have that $\lim_{m_k \to \infty} \frac{e(m_k)}{m_k} = 1$. So, for large enough $m_k$ we have that for all $t \in b^{[0,n_k)}$

\[ \frac{1}{b^{m_k}} \left| \left\{ s \in b^{m_k} : \left| \left\{i < m_k: E(t^{s'} | [p_i - k, p_i - 1]) \neq t^{s'}(p_i)\right\} \right| \geq m_k \left(\frac{b-1}{b} - \epsilon_k\right) \right\} \right| \geq 1 - \frac{\epsilon_k}{l_k} = 1 - \epsilon_k. \]

It follows that for fixed $n_k - 1$ that for all sufficiently large $n_k > n_{k-1}$ we have that for all $t \in b^{[0,n_k-1]}$ that

\[ \frac{1}{b^{m_k}} \left| \left\{ s \in b^{m_k} : \forall E \in E_k \left| \left\{i < n_k: E(t^{s'} | [i-k, i-1]) \neq t^{s'}(i)\right\} \right| \geq m_k \left(\frac{b-1}{b} - \epsilon_k\right) \right\} \right| \geq 1 - \frac{\epsilon_k}{l_k} = 1 - \epsilon_k. \]

Since

\[ m_k \left(\frac{b-1}{b} - \epsilon_k\right) \geq \left[(d - \epsilon_k)n_k - n_{k-1}\right] \left(\frac{b-1}{b} - \epsilon_k\right) \]

\[ = dn_k \left(\frac{b-1}{b}\right)
[1 - \epsilon_k - \epsilon_k \frac{b-1}{b} - \frac{n_{k-1}}{dn_k}] + n_k \epsilon_k^2 + n_{k-1} \epsilon_k \]

\[ \geq \frac{d(b-1)n_k}{b}(1 - 3b \epsilon_k) \]

(assuming that $\frac{n_k - 1}{n_k} < b \epsilon_k$), it follows that for all large enough $n_k$ and all $t \in b^{[0,n_k-1]}$ we have that

\[ \frac{1}{b^{m_k}} \left| \left\{ s \in b^{m_k} : \forall E \in E_k \left| \left\{i < n_k: E(t^{s'} | [i-k, i-1]) \neq t^{s'}(i)\right\} \right| \geq \frac{d(b-1)n_k}{b}(1 - 3b \epsilon_k) \right\} \right| \geq 1 - \epsilon_k. \]
We assume now that the $n_k$ are sufficiently fast growing to satisfy these inequalities. Since $\sum_{k} \epsilon_k < \infty$, it follows from Borel-Cantelli that for $\mu$ almost all $x \in b^\omega$ that for any $E \in \mathcal{E}$, there are cofinitely many $k$ such that

$$|\{i < n_k : E(x' \upharpoonright [i - k, i - 1]) \neq x'(i)\}| \geq \frac{d(b - 1)n_k}{b} (1 - 3b\epsilon_k)$$

and thus for $\mu$ almost all $x \in b^\omega$ and all $E \in \mathcal{E}$ we have that

$$\limsup_k \frac{1}{k} |\{i < k : E(x' \upharpoonright [i - k, i - 1]) \neq x'(i)\}| \geq \frac{d(b - 1)}{b},$$

and so $\beta^+(x) \geq \frac{(b-1)}{b}$.

$\square$

The next lemma suffices to show that $N^\omega(b)$ is $\Pi^0_3$-complete. We give an alternative, somewhat simpler, proof of the lemma after the current proof. However, the first proof more resembles the proofs of the other parts of Theorem 2 to follow.

**Lemma 9.** For any $s \in [0, \frac{b-1}{b})$, $A_3(s)$ is $\Pi^0_3$-hard.

**Proof.** For $x \in 2^\omega$, we view $x$ as coding the sequence $x_0, x_1, \ldots$ in $2^\omega$ where $x_i(j) = x(i(j))$. We let $P = \{x : \forall i \exists j \omega \geq j_0 x_i(j) = 0\}$. It is well-known that $P$ is $\Pi^0_3$-complete.

We define a partition of $\omega$ into disjoint arithmetical sequences as follows. Let $I_0 = \{n : n \equiv 0 \mod 2\}$, be the set of even integers, and in general let $I_n = \{n : n \equiv 2^n - 1 \mod 2^{n+1}\}$. The $\{I_n\}$ are pairwise disjoint arithmetic progressions, and $\omega = \bigcup_n I_n$. Note that $\omega - \bigcup_{k \leq m} I_k = \{n : n \equiv 2^{m+1} - 1 \mod 2^{m+1}\}$. Each $I_n$ clearly has density $\frac{1}{2^n}$.

Fix a set $J \subseteq \omega$ such that $s = \frac{b-1}{b} (1 - \sum_{i \in J} \frac{1}{2^i}) = s$. Let $d_i = \frac{1}{2^i}$ be the density of $I_i$. Let $B = \bigcup_{i \in J} I_k$ and let $d_B = 1 - \sum_{i \in J} \frac{1}{2^i}$ be the density of $B$.

We will take two fast growing sequences $\{a_i\}_{i \in \omega}$ and $\{b_j\}_{j \in \omega}$ of natural numbers (the $a_i$ will grow faster than the $b_j$). We will then set $n_{i,j} = a_i b_j$. Also, let $B_{i,j} = [n_{i,j-1}, n_{i,j})$ and $m_{i,j} = |B_{i,j} \cap I_i|$. Note that $|m_{i,j} - \frac{1}{2^i} (n_{i,j} - n_{i,j-1})| \leq 1$.

We first define the $b_j$.

Assume $b_0 < b_1 < \cdots < b_{j-1}$ have been defined. Let $b_j > b_{j-1}$ be large enough so that

1. For each $i \leq j$ with $i \in J$, the density of $B \cup I_i$ in $[n_{j-1}, n_j)$ is at least $(d_B + d_i)(1 - \frac{1}{8j})$.
2. $\frac{n_{j-1} - n_{j-1}}{n_{j-1}} > \frac{1 - \frac{1}{8j}}{1 - \frac{1}{4j}}$.
3. For every $i \leq j$ and any $m \geq (d_B + d_i)(n_j - n_{j-1})(1 - \frac{1}{8j})$, if $A \subseteq [n_{j-1}, n_j)$ has size at least $m$, and $E \in \mathcal{E}_j$, then we have that $\frac{p_m}{b^m} \geq 1 - \frac{1}{2^j}$, where $p_m$ is the number of $s \in b^{[n_{j-1}, n_j)}$ such that $s$ is $0$ off of $A$ and we have that
   $$|\{k \in [n_{j-1}, n_j) : E(s(k - j), \ldots, s(k - 1)) \neq s(k)\}|$$
   $$\geq (d_B + d_i)(n_j - n_{j-1}) \frac{b - 1}{b} \left(1 - \frac{1}{4j}\right).$$

Properties (1) and (2) are easily satisfied, and property (3) can be satisfied as in the proof of Lemma 8. Also, the properties continue to hold if we replace $n_{j-1}$ and $n_j$ with $a_{n_{j-1}}$ and $a_{n_j}$ for any positive integer $a$. 
We next define the sequence \(a_0 < a_1 < \cdots\), which will be sufficiently fast growing with respect to the \(\{b_j\}\). Namely, such that for any \(i\) and any \(j > 1\), \(|i': a_i \leq a_ib_j| \leq j\). We could, for example, take \(a_i = \prod_{i' \leq 1} b_{i'}\). Set \(n_{i,j} = a_ib_j\) and let \(B_{i,j} = [n_{i,j-1}, n_{i,j}]\). We define now the map \(\psi: 2^\omega \to b^\omega\) which will be our reduction from \(P\) to \(A_3(s)\). We first define a particular real \(u \in 2^\omega\). To do this, consider integers \(i < j\).

Claim 10. \(|\{(i', j'): i' \geq i \land j' > 0 \land B_{i', j'} \cap B_{i,j} \neq \emptyset\}| \leq j^2\).

Proof. For fixed \(i' \geq i\), the number of \(j'\) such that \(B_{i', j'} \cap B_{i,j} \neq \emptyset\) has size at most \(j\), since \(B_{i,j} = [n_{i,j-1}, n_{i,j}] = [a_ib_{j-1}, a_i, b_j]\), \(B_{i', j+1} = [a_{i'}b_{j-1}, a_{i'}, b_{j+1}]\), and \(a_{i'} \geq a_i\). Also, the number of \(i' \geq i\) such that \(B_{i', j'} \cap B_{i,j} \neq \emptyset\) for some \(j' > 0\) is at most \(j\). This is because \(n_{i', j' - 1} = a_{i'}b_{j'-1}\), and if \(n_{i', j' - 1} \leq n_{i,j}\) then we must have \(a_{i'} \leq \frac{a_{i}b_{j-1}}{a_{i-1}} \leq a_i b_j\). From the choice of the \(a_i\), this set has size at most \(j\). \(\square\)

Fix for the moment any \(j > i\), and consider the block \(B_{i,j}\). Let \(W_{i,j} \subseteq \omega^2\) be the set of \((i', j')\) with \(i' \geq i, j' > 0\), and with \(B_{i', j'} \cap B_{i,j} \neq \emptyset\). By Claim 10 \(|W_{i,j}| \leq j^2\). Consider the set \(P_{i,j}\) of “patterns,” by which we mean elements of \(2^{W_{i,j}}\). For each pattern \(p \in P_{i,j}\), and real \(u \in b^\omega\), let \(s(i, j, p, u) \in b^{B_{i,j}}\) be the sequence defined by:

\[
s(i, j, p, u)(k) = \begin{cases} u(k) & \text{if } k \in B \\ u(k) & \text{if } \exists i', j' \in W_{i,j} (P(i', j') = 1 \land k \in (I_{i'} \cap B_{i', j'})) \\ 0 & \text{otherwise} \end{cases}
\]

For each fixed pattern \(p \in P_{i,j}\), from properties \(2, 3\) of the \(b_j\) it follows that the \(\mu\) measure of the set of \(u \in b^\omega\) such that for all \(t \in b^{[0, n_{i,j} - 1]}\), if \(s' = t^c s(i, j, p, u)\) then for all \(E \in \mathcal{E}_j\) we have

\[
|\{k \in B_{i,j}: E(s'(k - j), \ldots, s'(k - 1)) \neq s'(k)\}| \geq (d_B + d_j)(n_j - n_{i,j-1}) \frac{b - 1}{b} \left(1 - \frac{1}{4j}\right)
\]

is at least \(1 - \frac{1}{j^2 2^j}\). Since the number of patterns in \(P_{i,j}\) is at most \(2j^2\), we have that the \(\mu\) measure of the set of \(u \in b^\omega\) such that for all patterns \(p \in P_{i,j}\) and all \(E \in \mathcal{E}_j\) equation (2) holds is at least \(1 - \frac{1}{j^2}\). By Borel-Cantelli it follows that for fixed \(i\) that for \(\mu\) almost all \(u \in b^\omega\) that there are cofinitely many \(j\) such that \((s_{i,j})\) for all \(E \in \mathcal{E}_j\), and all \(t \in b^{[0, n_{i,j} - 1]}\), and all patterns \(p \in W_{i,j}\), if \(s' = t^c s(i, j, p, u)\) as above, then equation (2) holds.

By countable additivity of \(\mu\), for \(\mu\) almost all \(u \in b^\omega\) the previous statement holds for all \(i\). Fix \(u \in b^\omega\) in this measure one set.

We now define the continuous map \(\psi: 2^\omega \to b^\omega\) which will be the reduction from \(P\) to \(A_3(s)\). Let \(x \in 2^\omega\) code the \(x_i\), where \(x_i(j) = x(i(j))\). We define \(\psi(x)\) by:

\[
\psi(x)(k) = \begin{cases} u(k) & \text{if } k \in B \\ u(k) & \text{if } \exists i, j (j > 0 \land k \in (B_{i,j} \cap I_i) \land x_{i-1}(j) = 1) \text{ where } i \text{ is the } \overline{i-1}\text{-th element of } J \\ 0 & \text{otherwise} \end{cases}
\]

The map \(\psi\) is clearly continuous. We show that \(x \in P\) iff \(\beta^+(\psi(x)) \leq s\). Suppose first that \(x \in P\), so \(\forall i \exists j_0 \forall j \geq j_0 \ x_i(j) = 0\). Then for all \(i \in J\) we have that for large
enough $k \in I_i$ that $\psi(x)(k) = 0$. The $E \in \mathcal{E}$ (of width 1) which constantly outputs 0 will then correctly compute $\psi(x)(k)$ for all sufficiently large $k \in I_i$ whenever $i \in J$.

For $i \notin J$, we will have $\lim_{n \to \infty} \frac{1}{n} \mathbb{I}\{k < n : (k \in I_i) \wedge E(x \upharpoonright [k-1, k-1]) \neq x(k)\} = \frac{1+b^{-1}}{b}$. From this it follows that $\lim_{n \to \infty} \frac{1}{n} \mathbb{I}\{k < n : E(x \upharpoonright [k-1, k-1]) \neq x(k)\} = \frac{1+b^{-1}}{b} \sum_{i \notin J} \frac{1}{b} = s$. So, $\beta^*(\psi(x)) = s$ and so $\psi(x) \notin A_3(s)$.

Suppose next that $x \notin P$. Let $i_0$ be least such that there are infinitely many $j$ such that $x_{i_0}(j) = 1$. Let $i_0^+$ be the $i_0$th element of $J$. If $i < i_0^+$ is in $J$, then for large enough $k \in I_i$ we have $\psi(x)(k) = 0$. If $i < i_0^+$ is not in $J$, then for large enough $k \in I_i$ we have $\psi(x)(k) = u(k)$. Also, for infinitely many $j$ we have that $\psi(x)(k) = u(k)$ for all $k \in I_{i_0^+} \cap B_{i_0^+}$. Fix $E \in \mathcal{E}$. Say $E \in \mathcal{E}_{j_0}$. From the definition of $u$, there are cofinitely many $j$ such that $\mathfrak{C}_{i_0^+ \cdot j}$ holds. Intersecting a cofinite and an infinite set gives a $j \geq \max\{i_0^+, j_0\}$ such that $x_{i_0}(j) = 1$, $\mathfrak{C}_{i_0^+ \cdot j}$ holds, and for all $i < i_0$ we have that $x_j(k) = 0$ for all $k \geq j$. Thus, for all $i < i_0^+$ which are in $J$ we have that $u(k) = 0$ for all $k \geq j$ with $k \in I_i$. Let $p_0 \in P_{i_0^+ \cdot j}$ be the pattern such that $p_0(i', j') = 1$ iff $i' \notin J$ or $[i, j] \wedge x_{i'}(j') = 1$ where $i'$ is the $i'$th element of $J$. From $\mathfrak{C}_{i_0^+ \cdot j}$ applied to $t = \psi(x) \upharpoonright n_{i_0^+, j-1}$ and the pattern $p = p_0$, and noting that from the definition of $\psi(x)$ that $\psi(x) \upharpoonright B_{i_0^+} = \psi(x) \upharpoonright [n_{i_0^+, j-1}, n_{i_0^+}] = s(i_0^+, j, p_0, u)$, we have that

$$ |\{k \in B_{i_0^+} : E(\psi(x)(k-j_0), \ldots, \psi(x)(k-1)) \neq \psi(x)(k)\}| \geq (d_B + d_I)(n_{i_0^+, j} - n_{i_0^+, j-1}) \frac{b-1}{b} \left( 1 - \frac{1}{4j} \right) $$

(3)

$$ = \left( s + d_i \frac{b-1}{b} \right) (n_{i_0^+, j} - n_{i_0^+, j-1}) \left( 1 - \frac{1}{4j} \right), $$

and thus by property (2) of the $b_j$, the density of $k \in [0, n_{i_0^+, j})$ such that

$$ E(\psi(x)(k-j_0), \ldots, \psi(x)(k-1)) \neq \psi(x)(k) $$

is at least $\left( s + d_i \frac{b-1}{b} \right) \left( 1 - \frac{1}{4j} \right)$. Since this holds for infinitely many $j$, we have that $\beta^+(\psi(x)) \geq s + d_i \frac{b-1}{b} > s$. Thus, $\psi(x) \notin A_3(s)$.

We now present the alternate simpler proof of Lemma 9.

**Proof.** Let $0 \leq s < \frac{b-1}{b}$. For each $i \in \mathbb{N}$ pick a probability vector $(p_{i,0}, \ldots, p_{i,u-1})$ such that $1 - \max p_n, d = 1 - p_{i,0} = s + \frac{1-s-1/b}{n}$ and construct real numbers $u_n$ such that for any $k$

$$ \lim_{n \to \infty} \frac{I(u_n, b_1 \ldots b_k, v)}{\mathbb{E}} = \prod_{j=1}^k p_n, b_j. $$

Then $\beta^*(u_i) = s + \frac{1-s-1/b}{n}$ for every $\ell$ and the constant 0 block function $E_0$ is the minimizer of $\inf_{E \in \mathcal{E}} \limsup_{n} \beta_E(x, N)$. Let $(a_j)$ be a sufficiently quickly growing sequence of integers with $a_1 = 1$ such that for $1 \leq \ell \leq j$, $1 \leq i \leq j$, and for all $n \geq \frac{a_j}{1-1/j}$ we have

$$ |\beta_{E_0}(u_i, n) - \beta_i(u_i)| = |\beta_{\ell}(u_i, n) - \beta_i(u_i)| < \frac{1}{j} $$


and
\[ \frac{a_{j+1}}{a_j} > j^2. \]

Put \( B_j = [a_j, a_{j+1}) \).

We view \( x \in 2^\omega \) as coding the sequence \( x_0, x_1, \ldots \) in \( 2^\omega \) where \( x_i(j) = x(i, j) \).

Define \( m(j) = \min\{j, \min\{i : x_i(j) = 1\}\} \) and define \( \psi(x) \) by
\[
\psi(x)_{B_j}(k) = u_{m(j)}(k - a_j).
\]

The map \( \psi \) is continuous. Suppose \( x \in P \). Then \( \liminf_j m(j) = \infty \). Fix \( \ell \) and note
\[
\inf_{E \in \mathcal{E}_\ell} \limsup_N \frac{1}{N} \{ k \leq N : \psi(x)(k) \neq E(\psi(x)(k+1), \ldots, \psi(x)(k+\ell)) \} \]
\[
= \limsup_N \frac{1}{N} \{ k \leq N : \psi(x)(k) \neq 0 \} \]
\[
= \limsup_N \sup_j \sum_{k=1}^{j-1} \frac{1}{N} \{|v \leq |B_k| : u_{m(k)}(v) \neq 0\} + \frac{1}{N} \{0 \leq v \leq N - a_j : u_{m(j)}(v) \neq 0\}.
\]

Now for \( N \geq a_j \) and \( k \leq j - 2 \)
\[
\frac{1}{N} \{1 \leq v \leq |B_k| : u_{m(k)}(v) \neq 0\} = |B_k| N \beta_\ell(u_{m(k)}, |B_k|) \leq \frac{a_{k+1} - a_k}{a_j} \leq \frac{a_{j-1}}{a_j} < \frac{1}{(j-1)^2}.
\]

and
\[
\frac{1}{N} \{1 \leq v \leq |B_j - 1| : u_{m(j-1)}(v) \neq 0\} \leq \frac{a_j - a_{j-1}}{N} \left( \beta_\ell(u_{m(j-1)}) + \frac{1}{j-1} \right) \leq \frac{a_j}{N} \left( \beta_\ell(u_{m(j-1)}) + \frac{1}{j-1} \right).
\]

For \( a_j \leq N \leq \frac{a_j}{j-1} \leq a_{j+1} \) we have
\[
\frac{1}{N} \{0 \leq v \leq N - a_j : u_{m(j)}(v) \neq 0\} \leq \frac{N - a_j}{N} \leq \frac{a_{j+1} - a_j}{a_j} = \frac{1}{j-1}.
\]

On the other hand for \( \frac{a_j}{j-1} \leq N \leq a_{j+1} \)
\[
\frac{1}{N} \{0 \leq v \leq N - a_j : u_{m(j)}(v) \neq 0\} = \frac{N - a_j}{N} \beta_\ell(u_{m(j)}, N - a_j) \leq \frac{N - a_j}{N} \left( \beta_\ell(u_{m(j)}) + \frac{1}{j} \right).
\]

Thus
\[
\lim_{\ell} \beta^{+}_\ell(\psi(x)) \leq \lim_{\ell} \sup_j \sum_{t} \left( t \beta_\ell(u_{m(j-1)}(v)) + (1 - t) \beta_\ell(u_{m(j)}(v)) : t \in [0, 1] \right) + \frac{j - 1}{(j-1)^2} + \frac{2}{j-1} + \frac{1}{j} = \lim_{\ell} \left( s + \sup_j \max \left\{ \frac{1 - s - 1/b}{m(j-1)}, \frac{1 - s - 1/b}{m(j)} \right\} \right) = s.
\]
Now suppose \( x \notin P \). Then \( \liminf_n m(n) = M < \infty \). Fix \( \ell \) and note
\[
\inf_{E \in \mathcal{E}_\ell} \limsup_N \frac{1}{N} \left| \left\{ k \leq N : \psi(x)(k) \neq E(\psi(x)(k + 1), \ldots, \psi(x)(k + \ell)) \right\} \right|
\]
\[
= \limsup_j \sup_{N \in B_j} \frac{1}{N} \sum_{k=1}^{j-1} |B_k| \beta(\ceil{u_m(k)}, |B_k|) + \frac{N - a_j}{N} \beta(\ceil{u_m(j)}, N - a_j)
\]
\[
\geq \limsup_j \frac{a_j + 1 - a_j}{a_j + 1} \beta(\ceil{u_m(j)}, a_j + 1 - a_j)
\]
which implies
\[
\lim \beta_\ell^+(\psi(x)) \geq s + \frac{1 - s - 1/b}{M} > s.
\]

We next show the lower bound for \( A_2(s) \). The proof is similar to that for \( A_3(s) \).

**Lemma 11.** For any \( s \in (0, \frac{b-1}{b}] \) the set \( A_2(s) \) is \( \Pi^0_3 \)-complete.

**Proof.** We reduce the set \( Q = \{ x \in 2^\omega : \forall i \exists j_0 \forall j \geq j_0 \ x_i(j) = 1 \} \) to \( A_2(s) \). Let \( J \subseteq \omega \) be such that \( \frac{b-1}{b} \sum j \in J d_j = s \), where we recall \( d_j = \frac{1}{2^j} \) is the density of \( I_i \).

Let the sequences \( \{a_i\}, \{b_j\} \) be as in Lemma 9 and as in that lemma let \( n_{i,j} = a_i b_j \), \( B_{i,j} = [n_{i,j-1}, n_{i,j}] \). We use also the notion of pattern defined as follows. For \( i < j \) let
\[
W_{i,j} = \{(i', j') : j' > 0 \land B_{i',j'} \cap B_{i,j} \neq \emptyset \}
\]
and let \( P_{i,j} \) be the set of patterns \( p \) from \( W_{i,j} \) to \{0, 1\} such that if \( i' < i \) then \( p(i', j_1) = p(i', j_2) \) for all \( j_1, j_2 \). Although \( W_{i,j} \) now has size greater than \( j^2 \), this restriction on \( p \) implies that \( |P_{i,j}| \leq 2^{j^2+i} \leq 2^{j^2+j} \).

Arguing as in Lemma 9 for each \( i \in \omega \) there is a \( \mu \) measure one set \( A_i \) of \( u \in b^\omega \) such that for co-finitely many \( j \), all \( E \in \mathcal{E}_j \), all \( t \in b^{[0,n_{j-1}]} \), and all patterns \( p \in P_{i,j} \), if we define \( s(i,j,p,u) \in b^{B_{i,j}} \) by
\[
s(i,j,p,u)(k) = \begin{cases} u(k) & \text{if } \exists i', j' \in W_{i,j} (P(i',j') = 1 \land k \in (I_{i'} \cap B_{i',j'})) \\ 0 & \text{otherwise} \end{cases}
\]
then if \( s' = t^\ast s(i,j,p,u) \) we have
\[
|\{k \in B_{i,j} : E(s'(k - j), \ldots, s'(k-1)) \neq s'(k)\}| \geq
\]
\[
\left( \sum_{i' \leq i} d_{i'} \left(n_j - n_{j-1}\right) \frac{b-1}{b} \left(1 - \frac{1}{4^j}\right) \right).
\]

We fix \( u \in b^\omega \) in all of the measure one sets \( A_i \). We define \( \psi : 2^\omega \to b^\omega \) by:
\[
\psi(x)(k) = \begin{cases} u(k) & \text{if } \exists i, j \ (i \in J \land j > 0 \land k \in (B_{i,j} \cap I_i) \land x_i(j) = 1) \\ 0 & \text{otherwise} \end{cases}
\]

We show \( \psi \) is a reduction from \( Q \) to \( A_2(s) \).
If \( x \notin Q \), then there is an \( i_0 \) such that for infinitely many \( j \) we have \( x_{i_0}(j) = 0 \). Let \( i_0^+ \) be the \( i_0 \)th element of \( J \). We may assume that \( u \) is such that for every \( i \) that \( \lim_{n \to \infty} \frac{1}{n} |\{ k \in I_i : u(k) = 0 \}| = \frac{d_i}{b} \). It follows that for \( i \neq i_0^+ \) and any \( \epsilon > 0 \) that for all sufficiently large \( j \) that
\[
\frac{1}{|B_{i_0^+, j}|} \left| \{ k \in B_{i_0^+, j} \cap I_i : u(k) \neq 0 \} \right| \leq (d_i + \epsilon) \frac{b - 1}{b}.
\]
It follows that for any \( i_1 > i_0^+ \) and \( \epsilon > 0 \) that for all sufficiently large \( j \) with \( x_{i_0}(j) = 0 \) that
\[
\frac{1}{|B_{i_0^+, j}|} \left| \{ k \in B_{i_0^+, j} : \psi(x)(k) \neq 0 \} \right| \leq \left( \sum_{i \leq i_1} d_i + \epsilon \right) \frac{b - 1}{b} + \sum_{i > i_1} d_i.
\]
It follows by considering the trivial \( E \in \mathcal{E} \) which always outputs 0 that
\[
\beta^-(\psi(x)) \leq \sum_{i \neq i_0^+} d_i \frac{b - 1}{b} < \sum_{i \in J} \frac{b - 1}{b} = s.
\]
Thus, \( \psi(x) \notin A_2(s) \).

Suppose next that \( x \in Q \). Let \( i \in J \). For all sufficiently large \( j \) we have that \( \psi(x)(k) = u(k) \) for all \( k \in B_{i_0^+, j} \cap I_i \), where \( i' \leq i \) and \( i' \in J \). For such \( j \), consider the pattern \( p \in \mathcal{P}_{i_0^+, j} \) such that \( p(i', j') = 1 \) if \( i' < i \), and for \( i' \geq i \) with \( i' \in J \) and \( j' > 0 \) we have \( p(i', j') = x_{i'} \cdot (j') \), where \( i' \) is the \( i' \)th element of \( J \). From the properties of \( u \) we have that for all \( E \in \mathcal{E}_j \) that
\[
\frac{1}{|B_{i_0^+, j}|} \left| \{ k \in B_{i_0^+, j} : E(\psi(x)(k - j), \ldots, \psi(x)(k - 1)) \neq \psi(x)(k) \} \right| \geq \left( \sum_{i' \leq i} d_{i'} \right) \left( \frac{b - 1}{b} \right) \left( 1 - \frac{1}{4j} \right).
\]
This gives that \( \beta^-(\psi(x)) \geq (\sum_{i \in J} d_i) \left( \frac{b - 1}{b} \right) = s \), and so \( \psi(x) \in A_2(s) \). \( \square \)

We next show that the set \( A_4(s) \) is \( \Pi^0_2 \)-complete for any \( s \in (0, \frac{b-1}{b}] \).

**Lemma 12.** For any \( s \in (0, \frac{b-1}{b}] \), the set \( A_4(s) \) is \( \Pi^0_2 \)-complete.

**Proof.** Let \( Q \subseteq 2^\omega \) be the set \( Q = \{ x : \forall i \exists j \geq i \ x(j) = 1 \} \). Let \( A \subseteq \omega \) be a set of density \( d = s \frac{b}{b-1} \in (0,1] \). We define the integers \( b_0 < b_1 < \cdots \) inductively, and set \( B_j = [b_{j-1}, b_j] \). Given \( b_0, \ldots, b_{j-1} \), we let \( b_j > b_{j-1} \) be large enough so that for each \( t \in 2^{b_{j-1}} \), we have that for all \( E \in \mathcal{E}_j \) that
\[
\left| \left\{ u \in 2^{B_j} : \frac{1}{|B_j|} \left| \{ k \in B_j : E(t \wedge u(k - j), \ldots, t \wedge u(k - 1)) \neq t \wedge u(k) \} \right| \geq \frac{b - 1}{b} \left( d - \frac{1}{j} \right) \right\} \right| \geq \left( 1 - \frac{1}{2j^2} \right) 2^{|B_j|}.
\]
This is possible by law of large numbers and Lemma 8. By Borel-Cantelli, for \( \mu \) almost all \( x \in 2^\omega \) we have that
\[
\exists j_0 \forall j \geq j_0 \forall t \in 2^B_j \forall E \in \mathcal{E}_j \frac{1}{|B_j|} \left| \left\{ k \in B_j: E(x_s(k-j), \ldots, x_t(k-1)) \neq x_t(k) \right\} \right| \geq s - \frac{1}{j},
\]
where \( x_t \) is the result of replacing \( x \upharpoonright [0, b_j-1) \) with \( t \). Fix \( u \in 2^\omega \) in this measure one set.

We define \( \psi: 2^\omega \to 2^\omega \) by:
\[
\psi(x)(k) = \begin{cases} u(k) & \text{if } \exists j \ (k \in B_j \land x(j) = 1) \\ 0 & \text{otherwise} \end{cases}
\]
If \( x \in Q \), then there are infinitely many \( j \) such that \( x(j) = 1 \) and thus \( \psi(x) \upharpoonright B_j = u \upharpoonright B_j \). From the definition of \( u \), there is a tail of these \( j \) for which
\[
\forall E \in \mathcal{E}_j \frac{1}{|B_j|} \left| \left\{ k \in B_j: E(\psi(x)(k-j), \ldots, \psi(x)(k-1)) \neq \psi(x)(k) \right\} \right| \geq s - \frac{1}{j}.
\]
Thus, for any \( E \in \mathcal{E} \) we have that
\[
\text{lim sup} \frac{1}{|B_j|} \left| \left\{ k \in B_j: E(\psi(x)(k-j), \ldots, \psi(x)(k-1)) \neq \psi(x)(k) \right\} \right| \geq s.
\]
We may assume that \( \sum_{x \in \mathcal{E}, j \in \mathbb{N}} \frac{|B_j|}{B_j} \to 0 \) with \( j \), and it follows that \( \beta^+(\psi(x)) \geq s \), so \( \psi(x) \in A_1(s) \).

If \( x \notin Q \), then \( \psi(x)(k) \) is 0 for all large enough \( k \). This gives that \( \beta^+(\psi(x)) = 0 < s \), and so \( \psi(x) \notin A_4(s) \).

We next show the lower-bound for \( A_1(s) \).

**Lemma 13.** For \( s \in [0, \frac{b_1}{b_1}) \), the set \( A_1(s) \) is \( \mathbf{\Pi}^0_4 \)-hard.

**Proof.** Fix \( s \in [0, \frac{b_1}{b_1}) \). Let \( R = \{ x \in (2^\omega)^3 \colon \forall i \exists j_0 \geq j_0 \exists k \ x(i, j, k) = 0 \} \). \( R \) is a \( \mathbf{\Pi}^0_4 \)-complete set, and so it suffices to reduce \( R \) to \( A_1(s) \). Let \( J \subseteq \omega \) be such that
\[
s = \frac{b_1}{b_1} \left( 1 - \sum_{i \in J} \frac{b_i}{b_i} \right).
\]
Let the \( I_i \) partition \( \omega \) as before, so \( d(I_i) = \frac{b_i}{b_i} \).

We will define a fast growing sequence \( b_0 < b_1 < \cdots \), and we will also let \( b_n = [b_n, b_{n+1}) \). Each \( n \) codes a triple \( n = (i_n, j_n, t_n) \), where \( i_n, j_n, t_n \leq n \). We will also define a certain sufficiently fast growing function \( g: \omega \to \omega \) (this will be the map \( j \mapsto p(j, t_n) \) of Claim 14). Also as in the proof of Lemma 9 we will fix a particular \( u \in 2^\omega \) from a certain \( \mu \) measure one set which will guide the construction of the reduction map \( \psi \). The construction will be similar to that of Lemma 9 the main difference being that at some points of the construction instead of copying 0s to parts of the block \( B_n \) we will copy a portion of \( u \) repeated with period \( g(j_n) \).

**Claim 14.** Let \( \omega = A \cup B \cup C \), a disjoint union, and assume \( A, B, C \) have period \( p \) (that is, \( \chi_A, \chi_B, \chi_C \) have period \( p \)). Let \( A, B, C \) have densities \( d_A, d_B, d_C \) respectively. Then for \( \mu \) almost all \( u \in 2^\omega \) (in fact, if \( u \) is normal in base \( b \) we have the following. There is an \( E \in \mathcal{E} \) such that for any \( \bar{u} \) with \( \bar{u} \upharpoonright A = u \upharpoonright A \) and \( \bar{u} \upharpoonright B \) of period \( p \), we have that \( \beta_E^+(\bar{u}) \leq d_A \frac{b_n}{b_n} + d_C \).

**Proof.** Let \( u \) be normal in base \( b \). Let \( \epsilon > 0 \). Consider sequences \( s \in b^{\mu p} \) for some integer \( n \). For large enough \( n \), the probability that a sequence \( s' \in b^{\mu n p} \) will have the property that \( A' = \{ i \in A \cap \mu p: s'(i) = s'(i-p) \} \) has size at least \( \frac{d_A \mu n}{b} (1 - \frac{\epsilon}{3}) \) is at least \( 1 - \frac{\epsilon}{3} \). This follows by the argument of Lemma 9. We call such an \( s' \)
good. Since $u$ is normal in base $b^{dA p n}$, it follows that for large enough $N$ that the number of $k \leq N$ such that $u \upharpoonright (A \cap [k p n, (k + 1)p n))$ is good is at least $N (1 - \frac{\epsilon}{2})$. For such $N$ we have that
\[
|\{i \in A \cap dA p n N : u(i) = u(i - p)\}| \geq N \left( 1 - \frac{\epsilon}{2} \right) \left( \frac{dA p n}{b} \right) \left( 1 - \frac{\epsilon}{3} \right)
\geq \left( \frac{dA p n}{b} \right) (1 - \epsilon).
\]

Let $E \in \mathcal{E}_{p+1}$ be the block function such that $E(s) = s(0)$, so $E$ is simply guessing that $x(n)$ will be $x(n - p)$. It follows that if $\bar{u} \in b^\omega$ is such that $\bar{u} \upharpoonright A$ is normal, then for large enough $N$ we have that
\[
|\{i \leq p n N : \bar{u}(i) = E(\bar{u}(i - p), \ldots, \bar{u}(i - 1))\}| \geq \left( \frac{dA p n N}{b} \right) (1 - \epsilon) + dB p n N
= p n N \left( \frac{dA}{b} (1 - \epsilon) + dB \right).
\]

Thus, $\beta_E^+ (\bar{u}) \leq 1 - \left( \frac{dA}{b} (1 - \epsilon) + dB \right) = dA \frac{b - 1}{b} + dC + \epsilon \frac{2A}{b}$. Since $\epsilon$ was arbitrary, $\beta_E^+ (\bar{u}) \leq dA \frac{b - 1}{b} + dC$.

For $u \in b^\omega$ and $p, q \in \omega$, we define $\tilde{u}(k, p)$ by $\tilde{u}(k, p)(m) = u(m)$ for $m < k$, and for $m \geq k$ we set $\tilde{u}(k, p)(m) = u(m - k \mod p)$. Thus, after the first $k$ digits of $u$, we repeat the digits of $u$ periodically with period $p$.

**Claim 15.** Let $j_0 \in \omega$, $\epsilon, \epsilon' > 0$. Then there is a $p = p(j_0, \epsilon) \in \omega$ which is a power of 2 and an $\eta = \eta(p, \epsilon')$ such that if $A, B, C \subseteq \omega$ are disjoint and of period $p$ with densities $d_A, d_B, d_C$ respectively and with $\omega = A \cup B \cup C$, then for any $n = p \ell \geq \eta$, we have that $\frac{|H|}{|b^{C \cap [0, n]}|} \geq 1 - \epsilon'$, where $H$ is the set of $u \in b^n$ such that for any $u_c \in b^{C \cap [0, n]}$ if $u' \in b^n$ is defined by
\[
\begin{aligned}
u'(i) = \begin{cases} u(i) & \text{if } i \in A \\ \bar{u}(0, p) & \text{if } i \in B \\ u_c(i) & \text{if } i \in C \end{cases}
\end{aligned}
\]
then for any $E \in \mathcal{E}_{j_0}$ we have
\[
\forall k \in [\eta, n] \left[ \frac{1}{k} \left| \{i \leq k : E(u'(i - j_0), \ldots, u'(i - 1)) \neq u'(i)\} \right| \right.
\geq (dA + dB) \frac{b - 1}{b} (1 - \epsilon) - j_0 dC \left. \right]
\]
Furthermore, this holds for all $p' \geq p$.

**Proof.** Fix $j_0, E \in \mathcal{E}_{j_0}$, $\epsilon > 0$. We show that for large enough $p \in \omega$, and $\omega = A \cup B \cup C$, a disjoint union with $A, B, C$ having period $p$, and $u_c \in b^C$, that the probability a $u \in b^{(A \cup B) \cap [0, n]}$ satisfies (*) is as close to 1 as desired.

Given $u \in b^{(A \cup B) \cap [0, n]}$, let $u'' \in b^{[0, n]}$ be defined as $u'$ above except we put $u''(i) = 0$ for $i \in C \cap [0, n]$. The argument of Lemma 5 shows that as $p$ grows, with probability approaching 1 in choosing $u \upharpoonright ((A \cup B) \cap [0, p])$ we have that
\[
\frac{1}{p} \left| \{i \leq p : E(u''(i - j_0), \ldots, u''(i - 1)) \neq u''(i)\} \right| \geq (d_A + d_B) \frac{b - 1}{b} \left( 1 - \frac{\epsilon}{2} \right).
\]
From the finite version of Fubini it follows that for any \( \delta > 0 \) that for large enough \( p \) that with probability at least \( 1 - \delta \) in choosing \( u_b = u \uparrow (B \cap [0,p]) \) that for probability at least \( 1 - \delta \) in choosing \( u_a = u \uparrow (A \cap [0,p]) \) we have that if \( u'' \) is defined from \( u_a \) and \( u_b \), then (**) holds.

Choose \( u_b \) in this set of measure at least \( 1 - \delta \). Let \( G \) be the set of \( u_a = u \uparrow (A \cap [0,p]) \) such that if \( u'' = u_a \cup u_b \cup u_c \) (with \( u_c = 0 \)) then \( u \) satisfies (**). So, \( G \) has measure at least \( 1 - \delta \). From the law of large numbers we have that the probability that a \( u \in b^\omega \) has the property that for all \( k \geq 1 \)

\[
|\{k' \leq k: u \uparrow (A \cap [k'p, (k' + 1)p)) \in G\}| \geq k \left( 1 - \frac{\epsilon}{2} \right)
\]

is at least \( 1 - h(\delta) \), where \( h(\delta) \to 0 \) as \( \delta \to 0 \). We choose \( \delta < \frac{\epsilon}{2} \) small enough so that \( h(\delta) < \frac{\epsilon}{2} \). It follows that for all \( n = \ell p \geq p \) that with probability at least \( 1 - \delta \), \( u_b \in b^{B \cap (0,p)} \) has the property that with probability at least \( 1 - h(\delta) \), \( u_A = u \uparrow (A \cap [0,n]) \) has the property that if \( u'' \in b^{[0,n)} \) is formed from \( u_A, u_b, u_c = 0 \) as above, then for all \( k' \leq \ell \)

\[
|\{i \leq k'p: E(u''(i - j_0), \ldots, u''(i - 1)) \neq u''(i)\}| \\
&\geq k'p \left( 1 - \frac{\epsilon}{2} \right) (d_A + d_B) \left( \frac{b - 1}{b} \right) \left( 1 - \frac{\epsilon}{2} \right) \\
&\geq k'p (d_A + d_B) \left( \frac{b - 1}{b} \right) (1 - \epsilon).
\] (5)

Since \( \delta, h(\delta) < \frac{\epsilon}{2} \) it follows that for any \( n = \ell p \geq p \) that with probability at least \( 1 - \epsilon \), \( u \in [0,n) \) has the property that if \( u'' \) is obtained from \( u \) and \( u_c \) as above, then for all \( k' \leq \ell \) we have that Equation (5) holds. If we set \( \eta(p, \epsilon) = \frac{\epsilon}{2} \), then we get the inequality of Equation (5) replacing \( i \leq k'p \) with \( i \leq k \) for any \( k \in [\eta, n] \).

Consider then \( u' \in b^{[0,n)} \), which is defined as \( u'' \) except we use the given \( u_c \) instead of the 0 function. We have that \( E(u''(i - j_0), \ldots, u''(i - 1)) \) and \( E(u'(i - j_0), \ldots, u'(i - 1)) \) can only disagree if \( C \cap [i - j_0, i) \neq \emptyset \). Thus, there can be at most \( d_{c \cap j_0} \) many \( i \in [0,n) \) where such a disagreement occurs. The inequality of the claim then follows.

\[\square\]

**Claim 16.** For every \( m, \epsilon, \epsilon' \) there is an \( \eta = \eta(m, \epsilon, \epsilon') \) such that for any \( \eta_0, j_0, m_0 \leq m \), and any \( p \geq 1 \), for all large enough \( n \) we have that if \( A = \bigcup_{i \leq m_0} I_i \bigcup_{i > m_0} I_i \),

\[
B = \bigcup_{i \leq \eta_0} I_i, \text{ and } C = \omega - (A \cup B), \text{ then } \frac{|H|}{m} \geq (1 - \epsilon'), \text{ where } H \text{ is the set of } u \in b^n
\]
such that if

\[
u'(i) = \begin{cases} u(i) & \text{if } i \in A \cup C \\ u(0,p) & \text{if } i \in B \end{cases}
\]

then for any \( E \in \mathcal{E}_{j_0} \) we have

\[
\forall k \in [\eta, n] \left[ \frac{1}{k} \sum_{i \leq k} \{ i \leq k: E(u'(i - j_0), \ldots, u'(i - 1)) \neq u'(i) \} \right] \\
\geq d_A \left( \frac{b - 1}{b} \right) (1 - \epsilon).
\]
Proof. It is enough to fix $i_0$, $j_0$, $m_0$, $\epsilon$, $\epsilon'$, and $p$ and we show that for large for large enough $n$ the stated property holds. Note that there are $\leq b^p$ many choice for $u_B = u \upharpoonright (B \cap [0,n))$. It is enough to fix a choice for $u_B$ are show that for large enough $n$ the property holds. This, however, follows immediately from the argument of Lemma 8.

From Claim 15 we have the following. Let $b_0 < b_1 < \cdots < b_n < \cdots$ be a sufficiently fast growing sequence of powers of 2 (exactly how fast the sequence needs to grow will be specified below).

**Claim 17.** For almost all $u \in b^x$ we have the following. For any $j_0$ and $E \in \mathcal{E}_{j_0}$, for all large enough $n$ we have that if $[b_n, b_{n+1}) = A \cup B$, a disjoint union of sets of period $\leq 2^n$, where $A = [b_n, b_{n+1}) \cap \bigcup_{i \leq j_0} I_i$ and $B = [b_n, b_{n+1}) \cap \bigcup_{i > j_0} I_i$ for some $i_0 \leq n$, and if $j \leq n$ and $p = p(j, \frac{1}{2})$, $\eta \geq \eta(p, \frac{1}{n^2})$ as in Claim 15, then if $u'$ is defined as in Claim 16 then we have:

(1) (large $j$ case) If $j \geq j_0$ then

$$\forall k \in [b_n + \eta, b_{n+1}) \left\lceil \frac{1}{k - b_n} \left| \{i \in [b_n, k) : E(u'(i - j), \ldots, u'(i - 1)) \neq u'(i)\} \right| \geq (d_A + d_B) \left(\frac{b - 1}{b}\right) \left(1 - \frac{j}{n}\right) \frac{b}{2^n}. \right.$$

(2) (general case)

$$\forall k \in [b_n + \eta, b_{n+1}) \left\lceil \frac{1}{k - b_n} \left| \{i \in [b_n, k) : E(u'(i - j), \ldots, u'(i - 1)) \neq u'(i)\} \right| \geq d_A \left(\frac{b - 1}{b}\right) \left(1 - \frac{1}{n}\right). \right.$$  

where $d_A$, $d_B$ are the densities of $A$, $B$ in $[b_n, b_{n+1})$.

Proof. It is enough to fix $j_0$, $E \in \mathcal{E}_{j_0}$, and show that almost all $u$ have the desired property for these values. By Borel-Cantelli it is enough to show that the probability that $u \upharpoonright [b_n, b_{n+1})$ satisfies the conclusion of the claim it at least $1 - \frac{1}{2^n}$. Fix $j_0$, and we show, assuming $b_{n+1}$ is sufficiently large compared to $b_n$, that the interval $[b_n, b_{n+1})$ has this property. There are at most $n$ many partitions $[b_n, b_{n+1}) = A \cup B$ of the type stated in the claim (since the choice of $A, B$ is determined by $i_0 \leq n$), so it enough to fix $A, B$ and show that with probability at least $1 - \frac{1}{2^n}$ in choosing $u \upharpoonright [b_n, b_{n+1})$ the statement of the claim holds. Note that the $j$ in the claim satisfy $j \leq n$, so there is a bound $\eta_n$ depending only on $n$, such that if $j \leq n$, $p = p(j, \frac{1}{2})$, then $\eta(p, \frac{1}{n^2}) \leq \eta_n$. We will assume that $b_n \geq \eta_n$ for all $n$, and in particular we will choose $b_{n+1}$ so that $\frac{\eta_n + b_n}{b_{n+1} - b_n} < \frac{1}{n}$. This is possible as $\eta_n+1$ depends only on $n + 1$ and not the value of $b_{n+1}$. Similarly we may fix $j \leq n$ and show that with probability at least $1 - \frac{1}{2^n}$ in choosing $u \upharpoonright [b_n, b_{n+1})$ we satisfy the claim. In case (1), that is, $j \geq j_0$, the conclusion follows immediately from Claim 15 assuming that $b_{n+1} > b_n + \eta_n$. In applying Claim 15 we use $C = \bigcup_{j > n} I_j$. In case (2), the conclusion follows immediately from Claim 15 assuming again that $b_{n+1}$ is sufficiently large compared to $b_n$ (the interval $[0, n)$ of Claim 16 becomes $[b_n, b_{n+1})$ here).
We now fix \( u \in 2^n \) in the measure one set described in Claim \( \text{(17)} \). We also fix the fast growing sequence \( b_0 < b_1 < \cdots < b_n < b_{n+1} < \cdots \). As we said in Claim \( \text{(17)} \) we take \( b_{n+1} > b_n + \eta_n \), where \( \eta_n \) is the maximum of \( \eta(p, b_n) \), where \( p = p(n, b_n) \), from Claim \( \text{(15)} \) and \( \eta(n, b_n) \) from Claim \( \text{(16)} \).

Claim \( \text{(17)} \) then gives the following property of \( u \) and the \( b_n \).

\[ \{\exists i \}\text{ : } E(i) = \{x(i), \ldots, x(i-1) \neq u(i)\} \]

then for \( b_n + \eta_n \leq k \leq b_{n+1} \) we have:

1. If \( j \geq j_0 \) then \( d \geq (d_A + d_B) \left( \frac{1}{n} \right) \left( 1 - \frac{1}{n} \right) - \frac{1}{n^2} \).
2. If \( j < j_0 \) then \( d \geq d_A \left( \frac{1}{n} \right) \left( 1 - \frac{1}{n} \right) \).

We now return to the proof of Lemma \( \text{(13)} \). Given \( x \in 2^{\omega^2} \), we define a function \( h(x): \omega^3 \to \omega^3 \) as follows. We let \( h(x)(i,j,0) \) be \((1, j + 1, 0)\) if \( x(i', j, 0) = 0 \) for all \( i' < i \). Otherwise, set \( h(x)(i,j,0) = (0, j,1) \).

In general, we define

\[
h(x)(i,j,t) = \begin{cases} 
    (1, (h(x)(i,j,t - 1)))_1 + 1, 0) & \text{if } \forall i' \leq i \exists j' \leq t \ x(i', (h(x)(i,j,t - 1)))_1 = 0 \\
    (0, (h(x)(i,j,t - 1)))_2 + 1) & \text{otherwise}
\end{cases}
\]

The function \( h \) does the following. The input \( i \) sets the “width” of the search, that is, it will search over the \( (x)_j \) for \( i' < i \). The input \( j \) sets the initial start of the search in that the search will begin at the \( x(i', j,0) \). The search checks to see if all of these are equal to 0. If so, it will output \( h(x)(i,j,0) = 1 \), denoting a successful search, and then replace \( j \) with \( j + 1 \) and begin a new search at \( x(i,j+1,0) \). The output \( h(x)(i,j,0)_1 \) records the new value \( j + 1 \), and the output \( h(x)(i,j,0)_2 \) records the new “height” of the search, will in this case be set back to 0. If not all the values \( x(i', j, 0) \) are 0, then the \( j \) value remains the same and we increment the height of the search. The means we will search the values \( x(i', j, k') \) where \( i' \leq i \), \( j \) is the current value of \( (h(x)(i,j,t))_1 \), and \( k' \leq (h(x)(i,j,t))_2 \), which is the current height. For a given \( j \), we keep incrementing the height \( k \) and see if \( \forall i' \leq i \exists k' \leq k \ x(i', j, k') = 0 \). If so, the search is successful, and we then increment \( j \) to \( j + 1 \) and reset the height \( k \) to 0. Otherwise, we continue to increment the height \( k \) and continue the search.

The search is attempting to verify, step by step, that

\[ \forall i' \leq i \forall j' \geq j \exists k' x(i', j', k') = 0. \]

If this condition holds for some \( i, j \), then \( h(x)(i,j,t) \) will tend to \( \infty \) with \( t \). If this condition fails for \( i, j \), then for \( j' \) the least integer \( \geq j \) such that \( \forall i' \leq i \exists k \ x(i', j', k) = 0 \) we have that \( h(x)(i,j,t) \) will be equal to \( j' \) for all large enough \( t \) (the search will “get stuck” at \( j' \)).
Recall that \( \{b_n\} \) is a sufficiently fast growing sequence, so that \( u \) and the \( \{b_n\} \) satisfy Claim \([17]\) As before, we let \( B_n = [b_n, b_{n+1}) \). We view \( n \) as coding a triple of integers which we write as \((i_n, j_n, t_n)\). We define the map \( \psi \colon 2^{\omega^2} \to 2^{\omega^2} \) as follows. Let \( x \in 2^{\omega^2} \). Recall the \( I_i \) are the pairwise disjoint arithmetical sequences of Lemma \([9]\) so \( I_i \) has density \( \frac{1}{2^{i+1}} \). Also, \( J \subseteq \omega \) is such that \( \sum_{n \in J} (1 - \sum_{j \in \omega} \frac{1}{2^{j+1}}) = s \). For \( i \notin J \), we will just copy \( \bar{u} \) to \( \psi \).

For \( m \in B_n \cap I_i \), where \( i \in J \), we define \( \psi(x)(m) \) through the following cases (the definition of \( \bar{u}(k,p) \) is given right before Claim \([15]\)).

1. If \( i \notin J \), we set \( \psi(x)(m) = u(m) \).
2. If \( i \in J \) and \( m \in B_n \), we set \( \psi(x)(m) = u(m) \) if \( i_n < i \).
3. If \( i \in J \), \( m \in B_n \), and \( i_n \geq i \), then if \( (h(x)(i_n, j_n, t_n))_0 = 0 \) (unsuccessful search at step \( t_n \) for \( (i_n, j_n) \)) we set \( \psi(x)(m) = u(m) \).
4. If \( i \in J \), \( m \in B_n \), \( i_n \geq i \), and \( (h(x)(i_n, j_n, t_n))_0 = 1 \) (successful search at step \( t_n \)), we set \( \psi(x)(m) = \bar{u}(b_n, p(j_n, \frac{1}{j_n}))(m) \), where \( p(j_n, \frac{1}{j_n}) \) is defined in Claim \([15]\).

We show that \( \psi \) is a reduction from the \( \Pi^0_1 \) set \( R \) to the set

\[ A_1(s) = \{ z \in b^{\omega^2} \colon \beta^-(z) \leq s \}. \]

First assume that \( x \in R \). Fix \( \epsilon > 0 \). Let \( i_0 \) be large enough so that

\[ \frac{b - 1}{b} \sum_{i \in J, i \leq i_0} d_i + \sum_{i > i_0} d_i < s + \frac{\epsilon}{2}, \]

where \( d_i = \frac{1}{2^{i+1}} \) is the density of \( I_i \). Let \( j_0 \) be large enough so that \( \forall i \leq i_0 \forall j \geq j_0 \exists k \ x(i,j,k) = 0 \). We can do this as \( x \in R \). Let \( A \subseteq \omega \) be given by

\[ n \in A \iff \exists n(i = i_0 \land j_n = j_0) \land \exists (h(x)(i_n, j_n, t_n))_0 = 1 \land \forall i \leq i_0 \forall j \geq j_0 \exists k \ x(i,j,k) = 0. \]

that is, the \((i_n, j_n)\) search at step \( t_n \) is successful. From the definition of \( j_0 \) and the properties of \( h(x) \) we have that \( A \) is infinite (that is, there are infinitely many \( t \) such that \( (h(x)(i_0, j_0, t))_0 = 1 \)). For any \( n \in A \), \( i \leq i_0 \) in \( J \), and \( m \in B_n \cap I_i \), we have that \( \psi(x)(m) = \bar{u}(b_n, p(j_n, \frac{1}{j_n})) \). It follows from Claim \([14]\) and the fact that the \( b_n \) grow sufficiently fast, that there is an \( E \in \mathcal{E} \) (with say \( E \in \mathcal{E}_r \)) such that for all large enough \( n \in A \) we have that

\[ \frac{1}{|B_n|} | \{ m \in B_n \colon (E(\psi(x)(m - r), \ldots, \psi(x)(m - 1)) \neq \psi(x)(m)) \} | \leq \frac{b - 1}{b} (1 - \sum_{i \leq i_0} d_i) + \frac{\epsilon}{2} < s + \epsilon. \]

Since \( \epsilon > 0 \) was arbitrary, we have that \( \beta^-(\psi(x)) \leq s \), that is, \( \psi(x) \in A_1 \).

Assume next that \( x \notin R \). Let \( i_0 \) be least so that \( \forall j \exists j' \geq j \forall k \ x(i, j', k) = 1 \). So, for any \( j \), \( (h(x)(i_0, j, t)) \) has a limiting value \( j' \geq j \) as \( t \) goes to infinity (i.e., the width \( i_0 \) search starting at \( j \) will always get stuck). Note that if \( i_1 > i_0 \), then \( (h(x)(i_1, j, t))_1 \) will reach its limiting value at or before when \( (h(x)(i_0, j, t))_1 \) does, that is, the \( h(x)(i_1, j, t) \) search will get stuck at or before when \( h(x)(i_0, j, t) \) gets stuck. So, for all \( j \) we have that for all sufficiently large \( n \) with \( i_n \geq i_0 \), and \( j_n = j \), that \( (h(x)(i_n, j_n, t_n))_0 = 0 \) (unsuccessful search at step \( t_n \)).
Consider $\psi(x) \upharpoonright I_i$ where $i \geq i_0$. So, for each $j$ we have that for all large enough $n$ with $j_n = j$ that either $i_n < i$, in which case $\psi(x) \upharpoonright (B_n \cap I_i) = u \upharpoonright (B_n \cap I_i)$, or else $i_n \geq i$, in which case (since $(h(i_n,j_i))_0 = 0$ we also have that $\psi(x) \upharpoonright (B_n \cap I_i) = u \upharpoonright (B_n \cap I_i)$. Thus, for any $j$ we have that for large enough $n$ that $\psi(x) \upharpoonright (B_n \cap I_i)$ is either of the form $u \upharpoonright (B_n \cap I_i)$ or else of the form $\tilde{u}(b_n,p)$ where $p > j$. That is, the periods $p$ used in truncating $u$ in the block $B_n$ go to infinity with $n$.

It follows from (†) (where the $i_0$ there is the current $i_n$) that for any $\epsilon > 0$ and any $E \in \mathcal{E}_{j_0}$ that for all large enough $n$ that in either case, $i_n < i_0$ or $i_n > i_0$ (in which case $j_n \geq j_0$ for large enough $n$), we have that for all $k$ with $b_n + \eta_n \leq k \leq b_{n+1}$ that if

$$d = \frac{1}{k - b_n} \{|m \in [b_n,k) : E(\psi(x)(m - j), \ldots, \psi(x)(m - 1)) \neq \psi(x)(m)|\}$$

then

$$d \geq \left(\frac{b - 1}{b}\right) \min \left\{ \left(1 - \frac{1}{j} \right), \left(1 - \frac{1}{n} \right) \left(\sum_{i \in J \cap \mathcal{I}_{j_0}} \frac{1}{2^{i+1}} + \sum_{i \in J \cap \mathcal{I}_{j_0}} \frac{1}{2^{i+1}}\right) \right\}$$

for any $j$ and all large enough $n$. For any $\epsilon > 0$ we have for all large enough $n$ that

$$d \geq \left(\frac{b - 1}{b}\right) \left(\sum_{i \in J \cap \mathcal{I}_{j_0}} \frac{1}{2^{i+1}} + \sum_{i \in J \cap \mathcal{I}_{j_0}} \frac{1}{2^{i+1}}\right) - \epsilon \geq (s - \epsilon) + \left(\frac{b - 1}{b}\right) \sum_{i \in J \cap \mathcal{I}_{j_0}} \frac{1}{2^{i+1}}.$$

We may assume the $b_n$ grow sufficiently fast so that $\frac{\eta_n}{b_{n-1}} \to 0$ and $\frac{\sum_{n \leq m} b_m}{b_n} \to 0$ as $n \to \infty$ (note here that $\eta_n$ is defined independently of $b_n$). Then for large enough $n$ we have the above inequality for $d$ holds using

$$d' = \frac{1}{k} \{|m \in [0,k) : E(\psi(x)(m - j), \ldots, \psi(x)(m - 1)) \neq \psi(x)(m)|\}$$

for all large enough $k$. Thus, $\beta^-(\psi(x)) \geq s + \left(\frac{b - 1}{b}\right)(\sum_{i \in J \cap \mathcal{I}_{j_0}} \frac{1}{2^{i+1}})$, and so $\psi(x) \notin A_1(s)$.

\[\square\]

4. Hausdorff dimension of real numbers with noise $s$

For each $s \in [0, \frac{b - 1}{b}]$, we know that $\mathcal{N}(b) \subset A_2(s)$ and $\mathcal{N}(b) \subset A_4(s)$ which implies $\dim_H(A_2(s)) = \dim_H(A_4(s)) = 1$. Since $A_1(s)$ and $A_3(s)$ do not contain $\mathcal{N}(b)$ we must introduce the following machinery to compute the Hausdorff dimension of these sets.

For $x \in b^\omega$ let $M(x)$ be the set of weak-* limit points of the sequence of measures $\mu_{x_n} = \frac{1}{n} \sum_{k=1}^{n} \delta_{T^k x}$ where $T(E_1E_2E_3 \cdots) = E_2E_3 \cdots$ is the shift on $b^\omega$. This is a closed convex subset of the shift-invariant probability measures on $b^\omega$ which we denote by $\mathcal{M}(b^\omega)$. We say a point $x \in b^\omega$ is generic for a measure $\mu$ if $M(x) = \{x \mu\}$. For a closed convex subset $M \subseteq \mathcal{M}(b^\omega)$ define $\mathcal{G}(M)$ to be the set of $x \in b^\omega$ such
that \( M(x) = M \). Note \( \mathcal{G}(\{\mu\}) \) is the set of generic points for \( \mu \). Recall the measure theoretic entropy of a shift-invariant measure \( \mu \) on \( b^\omega \) is

\[
h(\mu) = \lim_{N \to \infty} \frac{1}{N} \sum_{B \in b^N} -\mu[B] \log \mu[B].
\]

Colebroot proved the following result in [10].

**Theorem 18.** The Hausdorff dimension of \( \mathcal{G}(M) \) is \( \sup \mu \in M \frac{h(\mu)}{\log b} \).

In an analogous way to real numbers we can define the noise of a measure \( \mu \) as

\[
\beta(\mu) = 1 - \lim_{\ell \to \infty} \sum_{B \in b^\ell} \mu[B] \max_{0 \leq d \leq b-1} \frac{\mu[d^{-1}B]}{\mu[B]}.
\]

It is clear that the upper and lower noise of each element in \( \mathcal{G}(M) \) is \( \overline{\beta}(M) := \sup_{\mu \in M} \beta(\mu) \) and \( \underline{\beta}(M) := \inf_{\mu \in M} \beta(\mu) \) respectively. In general

\[
\overline{\beta}(x) = \overline{\beta}(M(x))
\]

\[
\underline{\beta}(x) = \underline{\beta}(M(x)).
\]

Thus

\[
A_1(s) = \bigcup_{M \subseteq A_3(s)} \mathcal{G}(M) \quad A_2(s) = \bigcup_{M \subseteq A_2(s)} \mathcal{G}(M)
\]

\[
A_3(s) = \bigcup_{M \subseteq A_3(s)} \mathcal{G}(M) \quad A_4(s) = \bigcup_{M \subseteq A_4(s)} \mathcal{G}(M).
\]

Furthermore

\[
U(s) = \bigcup_{M \subseteq U(s)} \mathcal{G}(M) \quad L(s) = \bigcup_{M \subseteq L(s)} \mathcal{G}(M).
\]

Now consider \( \lambda \) the uniform measure on \( b^\omega \) and \( \delta_0 \) the Dirac point mass at \((0,0,\cdots)\). Let \( M \) be the convex hull of \( \{\delta_0, \lambda\} \). Then \( M \) is a closed convex subset of \( \mathcal{M}(b^\omega) \) and we have \( \beta^-(M) = \beta(\delta_0) = 0 \). Thus \( \mathcal{G}(M) \subseteq L(s) \subseteq A_1(s) \) and \( \dim_H(L(s)) \geq \dim_H(\mathcal{G}(M)) = \frac{h(\lambda)}{\log b} = 1 \) which implies \( \dim_H(L(s)) = \dim_H(A_1(s)) = 1 \).

We also have the following lower bound on \( \dim_H A_3(s) \)

\[
\dim_H A_3(s) \geq \frac{1}{\log b} \sup_{\mu: \beta(\mu) \leq s} h(\mu).
\]

Note the same lower bound holds for \( \dim_H U(s) \) since

\[
\dim_H U(s) \geq \frac{1}{\log b} \sup_{\mu: \beta(\mu) = s} h(\mu) \geq \frac{1}{\log b} \sup_{\mu: \beta(\mu) \leq s} h(\mu).
\]

This second inequality follows since if \( \beta(\mu) < s \) we can find \( t \in [0,1] \) such that \( \beta(t\mu+(1-t)\lambda) = s \) where \( \lambda \) is the uniform measure on \( b^\omega \) and \( h(t\mu+(1-t)\lambda) \geq h(\mu) \) since \( \lambda \) is the measure of maximal entropy. We have that \( \beta(\mu) = 0 \) is equivalent to
\( h(\mu) = 0 \) and \( \beta(\mu) = \frac{b-1}{b} \) is equivalent to \( h(\mu) = \log b \) but we do not have a general expression for

\[
\sup_{\mu: \beta(\mu) \leq s} h(\mu).
\]

One approach to finding this supremum is to restrict our attention to measures which are \( k \)-step Markov, that is measures of the form

\[
\mu[b_1, b_2, \ldots, b_\ell] = \rho(b_1, \cdots, b_k) \prod_{i=1}^{\ell-k} P_{b_i b_{i+k-1} b_{i+1} \cdots b_{i+k}}
\]

where \( \rho \) is a probability distribution on \( b^k \) (which we view as a \( 1 \times b^k \) matrix) and \( P \) is a \( b^k \times b^k \) matrix of non-negative real numbers such that

\[
\sum_{B' \in b^k} P_{B, B'} = 1 \text{ for all } B \in b^k
\]

\[
\rho P = \rho
\]

\[
P(B, B') > 0 \Rightarrow B_i = B_{i-1}' \text{ for } 2 \leq i \leq k.
\]

A \( k \)-step Markov chain \( \mu \) with stationary distribution \( \rho \) and transition matrix \( P \) has entropy

\[
h(\mu) =: h_k(\rho, P) = \sum_{B \in b^k} \rho(B) \sum_{B' \in b^k} -P_{B, B'} \log P_{B, B'}
\]

and noise

\[
\beta(\mu) =: \beta_k(\rho, P) = 1 - \sum_{B \in b^k} \mu[B] \max_{0 \leq d \leq b-1} \frac{\mu[d^B]}{\mu[B]}
\]

\[
= 1 - \sum_{B \in b^k} \mu[B] P_{db_b \cdots b_{b-1} b_1 b_2 \cdots b_k}.
\]

Thus \( \sup_{(\rho, P): \beta_k(\rho, P) \leq s} h_k(\rho, P) \leq \sup_{\mu: \beta(\mu) \leq s} h(\mu) \). If we can compute this supremum over all stochastic matrices \( P \) with steady state \( \rho \) we have improved lower bounds on \( \dim(H(U(s))) \). Problems of this type are unfortunately quite difficult in general. This problem is tractable for small \( k \) however, and we now consider an easy special case.

**Lemma 19.** For \( s \in [0, \frac{b-1}{b}] \)

\[ \dim_h(A_3(s)) \geq \frac{1}{\log b} H(s) + s \frac{\log(b-1)}{\log b} \]

where \( H(s) = -s \log s - (1-s) \log(1-s) \).

**Proof.** For \( k = 1 \) and \( P \) with identical rows (so we now write \( P_j \) instead of \( P_{i, j} \)) we have the associated Markov measure \( \mu \) is Bernoulli which implies

\[
h(\mu) = - \sum_{d=0}^{b-1} P_d \log P_d
\]

\[
\beta(\mu) = 1 - \max_{0 \leq d \leq b-1} P_d.
Now if $\beta(\mu) \leq s$ then $\max_{0 \leq d \leq \frac{b}{b-1}} P_d \geq 1-s$ and we can without of loss of generality take this maximum to be at $d = 0$. Therefore

$$\dim_H A_3(s) \geq \sup_{P: P_0 \geq 1-s} -\sum_{d=0}^{b-1} P_d \log P_d.$$ 

Clearly this supremum is attained when $P_0 = 1-s$ since $1-s \geq \frac{1}{2}$. Thus, we must maximize $-\sum_{d=0}^{b-1} P_d \log P_d$ subject to the constraints $P_0 = 1-s$, $0 \leq P_d \leq 1-s$ and $\sum_{d=0}^{b-1} P_d = 1$, which is equivalent to maximizing $-\sum_{d=1}^{b-1} P_d \log P_d$ subject to the constraints $0 \leq P_d \leq 1-s$ and $\sum_{d=1}^{b-1} P_d = s$. The maximizer of $-\sum_{d=1}^{b-1} P_d \log P_d$ subject to the constraint $\sum_{d=1}^{b-1} P_d = s$ occurs when $P_d = \frac{s}{b-1}$. Since $s \leq \frac{b-1}{b}$ we have $\frac{s}{b-1} \leq 1-s$, so for each $1 \leq d \leq b-1$ we have $0 \leq P_d \leq 1-s$. Thus we have

$$\dim_H(A_3(s)) \geq \frac{1}{\log b} \left( -(1-s) \log(1-s) - \sum_{d=1}^{b-1} \frac{s}{b-1} \log \frac{s}{b-1} \right)$$

$$= \frac{1}{\log b} \left( -(1-s) \log(1-s) - s \log \frac{s}{b-1} \right)$$

$$= \frac{1}{\log b} H(s) + s \frac{\log(b-1)}{\log b}.$$

In order to obtain an upper bound on $\dim_H(A_3(s))$ we use the argument of M. Bernay [7] which showed $\dim_H N^\perp(b) = 0$. To do this define the sets

$$A(N_0, \ell, s, \epsilon) = \bigcap_{N \geq N_0} \bigcup_{E \in E_\ell} \left\{ \omega \in [0,1) : \sum_{n<N} \inf_{1 \leq n \leq N} |\omega_n - E(\omega_{n+1}, \cdots, \omega_{n+\ell})| \leq N(s + \epsilon) \right\}$$

and note

$$A_3(s) = \bigcap_{\epsilon > 0} \bigcup_{\ell=1}^{\infty} \bigcup_{N_0=1}^{\infty} A(N_0, \ell, s, \epsilon).$$

M. Bernay proved the following lemma.

**Lemma 20.** For $s \in [0, \frac{b-1}{b}]$

$$\dim_H A(N_0, \ell, s, \epsilon) \leq \frac{1}{\log b} H(s + \epsilon) + s + \epsilon.$$

This implies the following upper bound on $\dim_H(A_3(s))$.

**Lemma 21.** For $s \in [0, \frac{b-1}{b}]$

$$\dim_H A_3(s) \leq \frac{1}{\log b} H(s) + s.$$

Thus, we have proven Theorem 6.
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