Topological Obstructions To Maximal Slices

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A necessary condition for a globally hyperbolic spacetime $\mathbb{R} \times \Sigma$ to admit a maximal slice is that the Cauchy slice $\Sigma$ admit a metric with nonnegative scalar curvature, $R \geq 0$. In this paper, the two cases considered are the closed spatial manifold and the asymptotically flat spatial manifold. Although most results here will apply in four or more spacetime dimensions, this work will mainly consider 4-dimensional spacetimes. For $\Sigma$ closed or asymptotically flat, all topologies are allowed by the field equations. Since all $\Sigma$ occur as Cauchy slices of solutions to the Einstein equations and most $\Sigma$ do not admit metrics with $R \geq 0$, it follows that most globally hyperbolic spacetimes never admit a maximal slice, i.e. a slice with zero mean extrinsic curvature. In particular, asymptotically flat globally hyperbolic spacetimes which admit maximal slices are the exception rather than the rule. The reason for this is due to topological obstructions to constructing such slices. In the asymptotically flat case, this will be shown by smooth compactification of the manifold in order to use the results for spatially closed manifolds.

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I. INTRODUCTION

Finding necessary and sufficient conditions for the existence of constant mean curvature (CMC) hypersurfaces in physically reasonable spacetimes has been an outstanding problem in classical relativity. Such slices are useful in studying the behavior of singularities, numerical relativity, and for calculating conserved quantities \cite{1,2}. A particular class of such hypersurfaces are maximal slices, namely, slices with zero extrinsic curvature. Although, in spatially closed spacetimes, maximal slices are often scarce unless the spacetime is static, it was a common belief until later work that not only do all asymptotically flat globally hyperbolic spacetimes have a maximal slice but, in fact, have an entire foliation by maximal slices. In particular, spacetimes sufficiently near to Minkowski spacetime admit maximal slices. Additionally, spacetimes admitting time functions obeying certain boundary conditions admit maximal slices; the existence theorems are due to R. Bartnik \cite{3}. There are additional results for slices outside black hole regions \cite{5}. On the other hand, D. Brill, gave examples of some spatially closed spacetimes and an asymptotically flat spacetime that admit no maximal slices \cite{5,6}. The asymptotically flat example of Brill was not smooth but only piecewise smooth and also was for only a limited set of topologies. These restrictions were removed in \cite{7}. Hence smooth, asymptotically flat spacetimes generically exhibit a topological obstruction to maximal slices.

Historically, a less well known problem was whether every 3-manifold admits physically reasonable initial data for a spacetime. The solution of this problem gives insight into not only into the mathematical structure of the classical field equations but also has implications in the quantum theory. In quantum gravity, it is believed that the topology of spacetime is highly nontrivial on the microscopic scale. Since one expects that, at a minimum, classically allowed topologies occur in the quantum theory, if every 3-manifold admits physically reasonable initial data then it is sensible to discuss all spatial topologies when studying quantum gravity.

For the first time ever it was shown in \cite{7} that all 3-manifolds do in fact have physically reasonable initial data. Therefore, there are no obstructions to the allowed topologies from the Einstein equations. Moreover, this result combined with the work on the classification of 3-manifolds which admit metrics with nonnegative scalar curvature implies that the generic situation is that globally hyperbolic spacetimes do not admit maximal slices. In other words, a spacetime which admits a maximal slice is the exception rather than the rule. The purpose of the present paper is to provide the details not given in the earlier paper \cite{7}.

In \cite{8}, the case of physically reasonable matter was treated both in the case of cosmological models and asymptotically flat spacetimes. The vacuum case for cosmological models was also treated. The asymptotically flat vacuum case was treated in n-dimensions in \cite{9}. For a review of the initial data problem see \cite{8}.

Physically, all topologies are allowed as spatial topologies of the asymptotically flat case and cosmological case. However, it turns out that topological censorship implies that structures with nontrivial first homotopy are censored in the asymptotically flat and asymptotically locally anti-de Sitter case; the structures collapses to form black holes or, more specifically are hidden behind black hole horizons \cite{10,11,12,13}. In fact the original form of topological censorship, first proven in \cite{14}, was motivated by the solutions given in \cite{7}.

In the present paper, it will be shown that every nonnegative function on a closed 3-manifold is the energy density of an initial data set with $J^a = 0$. In particular, every closed 3-manifold has vacuum initial data. Furthermore, there
exists initial data on every closed 3-manifold such that it is a CMC hypersurface for some constant. Next, it will be shown that every asymptotically flat $\Sigma$ has initial data. Moreover, every asymptotically flat $\Sigma$ has initial data with a CMC hypersurface. Finally, the initial data sets constructed here can be evolved into globally hyperbolic spacetimes of the form $\mathbb{R} \times \Sigma$. Therefore, it makes sense to consider all manifolds $\Sigma$ as a possible topology of an initial data set for the Einstein equations.

A necessary condition for a globally hyperbolic spacetime to admit a maximal slice is that $\Sigma$ admit a metric with $R \geq 0$. For closed 3-manifolds, M. Gromov and H. Lawson have shown that such manifolds comprise a small fraction of all closed 3-manifolds [14]. Therefore, only a small fraction of spatially closed, globally hyperbolic spacetimes can admit maximal slices. In order to classify the asymptotically flat 3-manifolds $\Sigma$ which admit metrics with $R \geq 0$, a compactification theorem is proven using the Green’s function of the operator $-8D^2 + R$. The compactification theorem reduces the classification of asymptotically flat 3-manifolds to that of closed ones with $R > 0$. Again, it follows that most globally hyperbolic spacetimes do not have maximal slices. On the other hand, there are no topological obstructions to finding CMC hypersurfaces because initial data with $p$ equal to a constant can be constructed regardless of the topology of $\Sigma$.

The mathematical techniques presented here in this paper provide a set of tools useful to solving other related problems of interest in gravitational physics. For example, the theorems proven here apply in higher dimensions and were used to prove existence of certain vacuum solutions in gravity with compact extra dimensions [15]. The authors of [15] acknowledge this as a private communication with the author of this paper.

II. INITIAL DATA SETS

A Cauchy surface is a spacelike hypersurface such that every non-spacelike curve intersects this surface exactly once. A partial Cauchy surface is a surface that satisfies the weaker condition that each non-spacelike curve intersects the surface at most once.

A spacetime is globally hyperbolic if and only if it admits a Cauchy slice. Alternately, a spacetime $M$, possibly with boundary, is globally hyperbolic if it is strongly causal and the sets $J^+(p, M) \cap J^-(q, M)$ are compact for all $p, q \in M$.

This definition is a generalization of that of a globally hyperbolic spacetime without boundary and is satisfied by asymptotically locally anti-deSitter (ALADS) spacetimes. Also, note that the Penrose compactification of an asymptotically flat (AF) globally hyperbolic spacetime (which is itself globally hyperbolic by the usual definition) is globally hyperbolic in this general sense.

The domain of outer communications (DOC) is the portion of a spacetime $M$ which is exterior to event horizons. Precisely $D = I^-(I_0^+) \cap I^+(I_0^-)$ for a connected component $I_0$ for an AF spacetime and $D = I^-(I_0) \cap I^+(I_0)$ for an ALADS spacetime. Intuitively, the DOC is the subset of $M$ that is in causal contact with $I$. Note that $D$ is the interior of an $(n+1)$-dimensional spacetime-with-boundary $D' = D \cup I$ and that $D'$ is itself a globally hyperbolic spacetime with boundary.

An event horizon is the boundary of the DOC. More specifically, a future event horizon is the boundary of the causal past of a connected component of the boundary at infinity, $I_0 \times J^-(I_0, M')$, a past event horizon is the boundary of the causal future of $I_0$, $J^+(I_0, M')$ and the event horizon is the union of future and past event horizons.

An initial data set for the Cauchy problem in general relativity consists of a 3-manifold $\Sigma$, riemannian metric $g_{ab}$, symmetric tensor $p_{ab}$ (which will be the extrinsic curvature of $\Sigma$ in the evolved globally hyperbolic spacetime $\mathbb{R} \times \Sigma$), energy density $\rho$, and momentum density $J^a$ which satisfies the Hamiltonian and momentum constraints

$$R - p_{ab}p^{ab} + p^2 = 16\pi\rho,$$

and

$$D_b(p^{ab} - pg^{ab}) = 8\pi J^a.$$

Here $R$ is the scalar curvature of the metric $g_{ab}$, $D_b$ is the covariant derivative defined by $g_{ab}$, and $p \equiv p_a^a$. Initial data is called physically reasonable if it is smooth (i.e. $C^\infty$), $\Sigma$ is geodesically complete with respect to $g_{ab}$, and

1. The timelike future (causal future) of a set $S$ relative to $U$, $I^+(S, U)$ ($J^+(S, U)$), is the set of all points that can be reached from $S$ by a future directed timelike curve (causal curve) in $U$. The interchange of the past with future in the previous definition yields $I^-(S, U)$ ($J^-(S, U)$).

2. In fact, it is that used in the proof of topological censorship in ALADS spacetimes [12].
the sources obey the local energy condition \( \rho \geq \sqrt{(J^a J^a)} \). From here on, initial data will always refer to physically reasonable data. When the energy and momentum densities correspond to classical nondissipative matter sources or vacuum, \( \rho = J^a = 0 \), the coupled Einstein-matter equations can be used to evolve the initial data into a globally hyperbolic spacetime \([4, 10]\). Moreover, \( \Sigma \) is a spacelike hypersurface in the evolved globally hyperbolic spacetime, and the constraints are the orthogonal and parallel projections of the 4-vector arising from contracting the field equations with the normal to \( \Sigma \). In the evolved globally hyperbolic spacetime, the local energy condition is the dominant energy condition, i.e. the stress-energy tensor satisfies \( T_{\alpha \beta} W^\alpha W^\beta \geq 0 \), and \( T_{\alpha \beta} W^\beta T^\alpha_{\ \ \gamma} W^\gamma \leq 0 \) for all \( W^\alpha \) on \( \mathbb{R} \times \Sigma \) with \( W_\alpha W^\alpha < 0 \). Usually, \( \Sigma \) is required to satisfy the boundary condition that it be a closed or asymptotically flat 3-manifold when describing a cosmological model or an isolated system, respectively. A 3-manifold is closed if it is compact and has no boundary. An asymptotically flat 3-manifold \( \Sigma \) is: a 3-manifold such that for some compact \( C \subset \Sigma \), \( \Sigma - C \) consists of a finite number of disconnected components each of which is diffeomorphic to \( \mathbb{R}^3 \) minus a ball \( B \). (Note that, the definition of asymptotically flat manifold used refers only to differentiable manifolds with no further structure, i.e. no metric, connection, or other geometric structure.) Initial data on asymptotically flat 3-manifolds is usually required to satisfy certain fall off conditions in the asymptotic regions. The most standard conditions are the following: Initial data on \( \Sigma \) is asymptotically flat initial data, if the metric \( g_{ab} \) and the extrinsic curvature \( p_{ab} \) and \( \delta_{ab} \) satisfy \( g_{ab} - \delta_{ab} = O(\frac{1}{r}) \), \( \partial_r g_{ab} = O(\frac{1}{r}) \), \( \partial_a \partial_r g_{ab} = O(\frac{1}{r^2}) \), \( \tilde{p}_{ab} = O(\frac{1}{r}) \), and \( \partial_r \tilde{p}_{ab} = O(\frac{1}{r^2}) \) where \( \tilde{g}_{ab} \) and \( \tilde{p}_{ab} \) are the pullbacks of \( g_{ab} \) and \( p_{ab} \) from \( \Sigma - C \) onto \( \mathbb{R}^3 - B \). Another type of initial data on an asymptotically flat 3-manifold is asymptotically null initial data. This is used when using \( p = \text{constant} \neq 0 \) on \( \Sigma \) and it will be discussed later. We will now adopt the convention that initial data on asymptotically flat 3-manifolds is asymptotically flat initial data unless stated otherwise.

A CMC hypersurface in a globally hyperbolic spacetime \( \mathbb{R} \times \Sigma \) is a spacelike hypersurface diffeomorphic to \( \Sigma \) with \( p = \text{constant} \). A maximal slice is a CMC hypersurface for which \( p = 0 \). Since the Hamiltonian constraint must be satisfied on the maximal slice, it follows that \( R \geq 0 \) on \( \Sigma \). Therefore, a necessary condition for a globally hyperbolic spacetime \( \mathbb{R} \times \Sigma \) to admit a maximal slice is that \( \Sigma \) admit a metric with nonnegative scalar curvature.

Geometrically, a maximal slice is an extremum of the area functional

\[
A(\Sigma) = \int_{\Sigma} d\sigma_g
\]

with respect to timelike pushes into the globally hyperbolic spacetime \( \mathbb{R} \times \Sigma \). In order to see this take the directional derivative

\[
\delta_n A(\Sigma) = \delta_n \int_{\Sigma} d\sigma_g = \int_{\Sigma} \delta_n d\sigma_g = \frac{1}{2} \int_{\Sigma} g^{ab} \delta_n g_{ab} d\sigma_g
\]

where \( n \) is the unit normal to \( \Sigma \).

Now, the extrinsic curvature in the evolved spacetime is given by the Lie derivative

\[
p_{ab} = -\frac{1}{2N} L_{Nn} g_{ab} = -\frac{1}{2N} \delta_n g_{ab}
\]

where \( N \) is the lapse function. Finally, using this expression for the extrinsic curvature, one obtains

\[
\delta_n A(\Sigma) = \frac{1}{2} \int_{\Sigma} g^{ab} \delta_n g_{ab} d\sigma_g = -\int_{\Sigma} N g^{ab} p_{ab} d\sigma_g
\]

Hence, \( \delta_n A(\Sigma) = 0 \) implies \( p = 0 \). Therefore, the maximal slice is extremal or maximal. CMC slices can be obtained from a similar variational principle by using a Lagrange multiplier.

The goal is of this section is to construct initial data on a given manifold \( \Sigma \). This is accomplished by showing that every nonnegative function on a closed 3-manifold is the energy density of an initial data set with \( J^a = 0 \). In particular, every closed 3-manifold has vacuum initial data. Furthermore, there exists initial data on every closed 3-manifold so that it is a CMC hypersurface for some constant. Next, it will be shown that every asymptotically flat \( \Sigma \) has initial data. Moreover, every asymptotically flat \( \Sigma \) has initial data with \( p \) equal to a nonzero constant.

The first case considered will be if \( \Sigma \) is a closed 3-manifold. In order to construct the initial data, we need the following theorem due to J. Kazdan and F. Warner \([17]\).

**Theorem 1.** Given any closed manifold \( M^n \) with \( n \geq 3 \) and a smooth function which is negative somewhere on \( M^n \), there exists a riemannian metric with the prescribed function as its scalar curvature.
We will now use this theorem to mimic the initial data for a Robertson-Walker spacetime. Recall that for Robertson-Walker spacetimes the extrinsic curvature $p_{ab}$ is proportional to the metric.

**Theorem 2.** Given any closed 3-manifold $\Sigma$ and smooth function $\rho$ there exists initial data on $\Sigma$ with energy density $\rho$ and $J^a = 0$. Furthermore, $p$ is equal to a constant. In the case that $\rho = 0$, given any nonzero constant $C$ there exists vacuum initial data with $p = C$.

**Proof.** Since $\Sigma$ is compact and $\rho$ is smooth, $\rho$ attains a maximum on $\Sigma$, call it $\rho_0$. Now, define the smooth function $f$ by $f \equiv 16\pi \rho - 6A_0^2$ where $6A_0^2 \equiv 16\pi \rho_0 + \epsilon$ and $\epsilon$ is any positive number. The function $f$ is always negative on $\Sigma$; and therefore, theorem 1 applies to $f$. Let $g_{ab}$ be a metric on $\Sigma$ with scalar curvature $f$. Now, define $p_{ab} = A g_{ab}$. Clearly, $g_{ab}, p_{ab}, \rho$, and $J^a = 0$ form an initial data set on $\Sigma$. Taking the trace of $p_{ab}$ yields $p = 3A_0 = \text{constant}$ which completes the first part of the proof. If $\rho = 0$, then by rescaling the metric by a constant, the scalar curvature can be given any negative value $R = -\frac{2}{3}C^2$ where $C$ is an arbitrary constant. Let $g_{ab}$ be the metric with $R = -\frac{2}{3}C^2$, then $g_{ab}$ and $p_{ab} = \frac{C}{3}g_{ab}$ satisfy the constraints and $p = C$.

Having just proven the existence theorem, it is useful to construct an explicit concrete example. Begin with the 3-manifold $\mathbb{R}^3$ with the metric given by $g = \frac{dr^2}{1+k^2r^2} + r^2d\Omega^2$ where $r$ is the radial coordinate and $d\Omega^2$ is the standard metric on the unit 2-sphere. The scalar curvature of the metric $g$ is $R = -6k^2$. This space is just hyperbolic 3-space. By identifying points of this space via an appropriate group of discrete isometries, it is possible to obtain a closed manifold. Furthermore, locally the metric of the resulting closed manifold is the same as hyperbolic 3-space, because the identifications are done via isometries. A particular example of such a manifold is the hyperbolic dodecahedron space. It is obtained from identifying opposite faces of a solid dodecahedron after a counter-clockwise rotation of $3\pi/5$ radians. One choice of initial data on these spaces is $g_{ab}, p_{ab} = A g_{ab}, \rho = 3/8\pi(A^2 - k^2)$, and $J^a = 0$. Moreover, if $\rho$ is taken to be the energy density of dust, i.e. $T_{\alpha\beta} = \rho u_{\alpha}u_{\beta}$ where $u_{\alpha}u^\alpha = -1$, then the time evolution of the initial data is a Robertson-Walker spacetime of negative spatial curvature. If $k^2 = A^2$, then the resulting globally hyperbolic spacetime is spherically symmetric and vacuum. Thus, it is Minkowski spacetime with points identified via a subgroup of the Lorentz group.

Since the initial data constructed in theorem 2 is a generalization of Robertson-Walker initial data, a natural choice of a stress-energy tensor is that of a dust source. With this in mind, we have the following theorem.

**Theorem 3.** For every closed 3-manifold $\Sigma$, there is a spacetime $\mathbb{R} \times \Sigma$ which is physically reasonable, and there is at least a single CMC hypersurface. Further, there is a vacuum globally hyperbolic spacetime of that form.

**Proof.** If $\rho$ is defined as in the proof of theorem 2 and is taken to be the energy density of dust, i.e. $T_{\alpha\beta} = \rho u_{\alpha}u_{\beta}$ where $u_{\alpha}u^\alpha = -1$, then the initial data can be evolved into a globally hyperbolic spacetime. This is proven by showing that the coupled Einstein-matter equations form a strictly hyperbolic Leray system, and then applying the existence theorems 2, 16. If $\rho$ is identically zero, then the evolution exists and gives a globally hyperbolic vacuum spacetime for the same reasons as in the dust filled case. Furthermore, in both cases the initial data has $p = \text{constant}$, so the initial hypersurface is a CMC hypersurface in the evolved globally hyperbolic spacetime.

The above theorem took $\rho$ to be the energy density of dust. However, one has the same result for other sources such as the energy-momentum tensor for a minimally coupled scalar field $\phi$. It is given by

$$T_{\alpha\beta} = \nabla_\alpha \phi \nabla_\beta \phi - 1/2 g_{\alpha\beta} \nabla_\gamma \phi \nabla^\gamma \phi.$$  

Assuming an initially time-independent scalar field, this gives the energy-momentum current vector $J_\alpha = T_{0\alpha} = -1/2g_{0\alpha}(\nabla \phi)^2$ ($\alpha = 1, 2, 3$). One is free to choose a gauge such that $g_{0\alpha} = 0$. With this choice, our assumption that $J^\alpha = 0$ is satisfied. For the energy density $T_{00}$, we find

$$T_{00} = -g_{00} \frac{1}{2}(\nabla \phi)^2 = \rho.$$  

Additionally, a cosmological constant can be also used as a source; pick $\rho = \text{constant}$ and $J^\alpha = 0$.

The technique just used to construct initial data on closed 3-manifolds cannot be applied to asymptotically flat spacetimes for two reasons: If the extrinsic curvature is proportional to the metric, then it will not approach zero at infinity and if other asymptotic boundary conditions are applied in order to make $p_{ab} = A g_{ab}$ hold at infinity, then an evolution theorem for such initial data sets would have to be proven. Furthermore, one would have to show that an asymptotically flat slice also existed in the evolved spacetime. Both of these statements are rather difficult to prove and may not even hold in general. Thus for asymptotically flat 3-manifolds a new procedure is needed for constructing...
initial data. Since by definition every asymptotically flat 3-manifold arises from the removal of a finite number of points from a closed 3-manifold, one would like to find out how to remove points in such a way that the metric and other fields have the correct asymptotic behavior. Motivated by the matching techniques used in describing the gravitational collapse of dust, we will remove a finite number of balls from a closed 3-manifold with initial data on it, and then smoothly glue in a spacelike hypersurface of the Schwarzschild spacetime producing the desired asymptotic behavior.

The technique just described will now be used to prove a lemma which guarantees the existence of initial data for manifolds admitting a special type of metric.

**Lemma 1.** Let $S$ be a closed 3-manifold with initial data. Suppose that in the neighborhood of some point $i_0$ of $S$ the initial data satisfies the following conditions: the metric $g_{ab}$ is spherically symmetric with scalar curvature $R = -6k^2$, $p_{ab} = A g_{ab}$, $J^a = 0$, and $\rho = 3/8\pi (A^2 - k^2)$. Then $S - \{i_0\}$ has asymptotically flat initial data.

**Proof.** The spherical symmetry in a neighborhood of $i_0$ means that in spherical coordinates centered at $i_0$ the metric takes the form $g = \xi dr^2 + r^2 d\Omega^2$ for $r < r_2$ ($r_2$ fixed) where $\xi$ only depends on $r$. Since the metric is geodesically complete and $R = -6k^2$, it follows that $\xi = (1 + k^2 r^2)^{-1}$. Now, the goal is to smoothly match the above spherically symmetric initial data to the initial data for the Schwarzschild spacetime by using a small amount of dust. However, before doing this the constraints for general spherically symmetric initial data will be written.

A general spherically symmetric metric and extrinsic curvature can be written as $g = \xi dr^2 + r^2 d\Omega^2$ and $p = a dr^2 + \beta r^2 d\Omega^2$ where $\xi$, $\alpha$, and $\beta$ only depend on $r$. The constraints can be written as

$$\rho = \frac{1}{16\pi} \left[ \frac{2\xi^{-2} \xi'}{r} + \frac{2(1 - \xi^{-1})}{r^2} + 4\alpha \beta^{-1} + 2\beta^2 \right]$$

and

$$J^r = \frac{1}{4\pi} \left[ -\beta' \xi^{-1} + \frac{\xi^{-1}}{r} (\alpha \xi^{-1} - \beta) \right].$$

Now, assuming that $J^r = 0$, the constraints reduce to

$$\rho = \frac{1}{8\pi r^2} \frac{d}{dr} \left[ \beta^2 r^3 - \xi^{-1} r + r \right]$$

and $\alpha = \xi(\beta' r + \beta)$. Integrating the Hamiltonian constraint from $r$ to $r_1$ yields $\beta^2 r^3 - \xi^{-1} r + r + \int_r^{r_1} 8\pi \rho r^2 dr \equiv 2C$. Therefore,

$$\beta^2 = \xi^{-1} - \left( 1 - \frac{2M(r)}{r} \right)$$

where $M(r) \equiv C - \int_r^{r_1} 4\pi \rho r^2 dr$.

Now, the match to the initial data for Schwarzschild initial data will be performed. Choose any numbers $r_0$ and $r_1$ such that $r_2 > r_1 > r_0 > (A^2 - k^2)^{1/3}$. Let $\eta$ be a smooth monotonically decreasing function which is zero for $r_2 > r > r_1$ and one for $r \leq r_0$. Now, choose $\rho$ to be equal to $\frac{1}{8\pi} (A^2 - k^2) (1 - \eta)$. At $r = r_1$, the quantities $\xi$, $\alpha$, $\beta$, and $\rho$ should correspond to the initial data on $S$, and in order for this to hold $C$ must be equal to $\frac{(A^2 - k^2) r_1^3}{2}$. On the other hand the metric and other expressions should become the Schwarzschild initial data for $r \leq r_0$ which implies $\beta = 0$ and $\alpha = 0$. This means $\xi^{-1} = 1 - \frac{2M(r)}{r}$ for $r \leq r_0$. Since the metric is required to equal the metric with $R = -6k^2$ for $r \geq r_1$, $\xi^{-1}$ must equal $1 + k^2 r^2$ for $r \geq r_1$. In order to smoothly interpolate between the two metrics, a natural choice of the metric is

$$\xi = \left( 1 - \frac{2M(r)}{r} \right)^{-1} \eta + (1 - \eta) (1 + k^2 r^2)^{-1}$$

for $r_0 \leq r \leq r_1$. Both $\xi$ and the induced $\beta^2$ are smooth and positive. Further, they go smoothly to the appropriate functions at the points $r_0$ and $r_1$. Finally, choosing $\alpha = \xi(\beta' r + \beta)$ the constraints are satisfied by construction, and the initial data is smooth. Moreover, the initial data is equal to the Schwarzschild initial data for $r \leq r_0$ and the initial data on $S$ for $r \geq r_1$. Therefore, in order to obtain asymptotically flat initial data, smoothly extend the Schwarzschild initial data across the throat at $r = 2M(r_0)$. \(\square\)
Theorem 5. Every asymptotically flat 3-manifold has initial data.

Proof. By definition, every asymptotically flat 3-manifold \( \Sigma \) has a compact subset \( C \) such that \( \Sigma - C \) has a finite number of disconnected components each of which is diffeomorphic to \( \mathbb{R}^3 \) minus a ball \( B \). One can easily compactify \( \mathbb{R}^3 - B \) in a smooth way to obtain \( S^3 - B \). Using this compactification for each asymptotic region of \( \Sigma \), it follows that \( \Sigma \) is diffeomorphic to a closed 3-manifold \( \Sigma\) minus a finite number of points. (Remember that when we refer to manifolds, they only have a differentiable structure and no other structure.) This means that every asymptotically flat 3-manifold \( \Sigma \) arises by removing points from a closed 3-manifold \( \tilde{\Sigma} \).

Now, let \( \tilde{\Sigma} \) be a closed 3-manifold and \( \{x_1, x_2, \ldots, x_n\} \) be any set of a finite number of points in \( \tilde{\Sigma} \). Pick any metric \( g_{ab} \) on \( \tilde{\Sigma} \). Next, pick a neighborhood \( N_j \) about each point \( x_j \) such that \( N_j \) is diffeomorphic to a ball in \( \mathbb{R}^3 \) with radius \( r_3 \) via a diffeomorphism \( \psi_j: N_j \to B \). Choose a smooth monotonic decreasing function \( \eta \) which is equal to zero for \( r_3 > r > r_2 \) and one for \( r_2 > r_1 > r \). Let \( \tilde{g}_{ab} \) be the metric defined on \( B \) by

\[
\tilde{g}_{ab} = (1 - \eta)\psi_j^{-1} g_{ab\mid N_j} + \eta h_{ab}
\]

where \( \psi_j^{-1} g_{ab\mid N_j} \) is the pullback of \( g_{ab\mid N_j} \) onto \( B \) and \( h_{ab} \) is the spherically symmetric metric with constant scalar curvature \(-6k^2\) on \( \mathbb{R}^3 \). Finally, define a new metric \( \tilde{g}_{ab} \) on \( \tilde{\Sigma} \) to be equal to \( g_{ab} \) on \( \tilde{\Sigma} - \bigcup_j N_j \) and \( \psi_j \tilde{g}_{ab} \) on each \( N_j \). Then \( \tilde{g}_{ab} \) is spherically symmetric and has constant scalar curvature in neighborhoods of each of the points \( x_j \).

Next, initial data will be constructed. Let \( \mathcal{R} \) be the scalar curvature of \( \tilde{g}_{ab} \). The scalar curvature \( \mathcal{R} \) is smooth and \( \tilde{\Sigma} \) is compact, this means \( \mathcal{R} \) attains a minimum denote it by \( \mathcal{R}_0 \). Now, define a smooth function \( \rho = \frac{\mathcal{R} + 6A^2}{16\pi} \) where \( 6A^2 \equiv |\mathcal{R}_0| + \epsilon \), \( \epsilon > 0 \). Since \( \rho \geq 0 \), the choice \( J^a = 0 \) means the local energy condition is satisfied. Choose \( \tilde{g}_{ab} \), \( p_{ab} = A\tilde{g}_{ab} \), \( J^a = 0 \), and \( \rho = \frac{\mathcal{R} + 6A^2}{16\pi} \) as initial data on \( \tilde{\Sigma} \).

Finally, apply lemma [1] at each point in \( \{x_1, x_2, \ldots, x_n\} \). This can be done because the proof of lemma [1] only uses the local structure of the metric and other fields. Therefore, \( \tilde{\Sigma} - \{x_1, x_2, \ldots, x_n\} \) has asymptotically flat initial data.

The initial data constructed in the above theorem has \( p = \) constant in a compact region but vanishing in the asymptotic regions. However, it is sometimes useful to use initial data with \( p \) constant everywhere, for example when studying the behavior of singularities [1]. Although, it will be shown that initial data with \( p \) everywhere zero does not always exist for all manifolds, the following theorem is an existence theorem for initial data with \( p \) equal to a nonzero constant on an arbitrary asymptotically flat manifold. For initial data of this form, different asymptotic boundary conditions are imposed on the metric and other fields in order that the Hamiltonian constraint hold at infinity. Initial data with \( p \) equal to a constant is a particular example of asymptotically null initial data. The reason for the terminology is that the hypersurface reaches null infinity in the evolved spacetime. Usually, one requires the metric and extrinsic curvature to approach the metric and extrinsic curvature of a CMC hypersurface of the Schwarzschild spacetime. However, at the present time there are no standardized conditions on the rate at which the metric and extrinsic curvature approach the Schwarzschild asymptotically null initial data, so we will prove existence of asymptotically null initial data with arbitrary topology and \( p = \) constant under the strongest conditions, namely, that outside a compact set the initial data in each of the asymptotic regions is equal to the initial data of a CMC hypersurface in the Schwarzschild spacetime.

Theorem 5. Every asymptotically flat 3-manifold has asymptotically null initial data with \( p = \) constant.

Proof. Let \( \tilde{\Sigma} \) be a closed 3-manifold and \( \{x_1, x_2, \ldots, x_n\} \) a finite set of points in \( \tilde{\Sigma} \). Now, let \( \tilde{g}_{ab} \) be the metric defined in the proof of theorem [1]. Pick the initial data on \( \tilde{\Sigma} \) to be the initial data from the proof of theorem [4], i.e. \( \tilde{g}_{ab} \), \( p_{ab} = A\tilde{g}_{ab} \), \( J^a = 0 \), and \( \rho = \frac{\mathcal{R} + 6A^2}{16\pi} \). Recall the initial data is spherically symmetric in neighborhoods of each of the points \( \{x_1, x_2, \ldots, x_n\} \). The goal is to match this initial data to the Schwarzschild initial data for a CMC hypersurface. Before doing this, the constraints for spherically symmetric initial data with \( p \) equal to a constant will be derived.
Given a spherically symmetric metric and extrinsic curvature, they can be written as \( g = \xi dr^2 + r^2d\Omega^2 \) and \( p = \alpha dr^2 + \beta r^2d\Omega^2 \), where the coefficients only depend on \( r \). The constraints can be written as

\[
\rho = \frac{1}{16\pi} \left[ \frac{2\xi^2 - 2\xi'}{r} + \frac{2(1 - \xi^{-1})}{r^2} + 4\alpha\beta\xi^{-1} + 2\beta' \right]
\]

and

\[
J^r = \frac{1}{4\pi} \left[ -\beta'\xi^{-1} + \frac{\xi^{-1}}{r} (\alpha\xi^{-1} - \beta) \right].
\]

Now, assuming that the extrinsic curvature is proportional to the metric, i.e. \( p_{ab} = Ag_{ab} \), it follows that \( J^r = 0 \) and the Hamiltonian constraint becomes

\[
\rho = \frac{1}{16\pi} \left[ \frac{2\xi^2 - 2\xi'}{r} + \frac{2(1 - \xi^{-1})}{r^2} + 6A^2 \right] = \frac{1}{8\pi r^2} \left[ 1 - \frac{d}{dr}(r\xi^{-1}) \right] + \frac{3}{8\pi} A^2.
\]

In our case, pick one of the points \( x_j \) for which the the initial data on \( \tilde{\Sigma} \) is spherically symmetric, then \( \rho = \frac{3}{8\pi} (A^2 - k^2) \) in a neighborhood of \( x_j \). As in lemma 4 let \( r_2 \) be a fixed number for which the metric and extrinsic curvature can be expressed as \( g = \xi dr^2 + r^2d\Omega^2 \) and \( p = \alpha dr^2 + \beta r^2d\Omega^2 \) for all \( r < r_2 \). Choose any numbers \( r_0 \) and \( r_1 \) such that \( r_2 > r_1 > r_0 > (A^2 - k^2)r_1^3 \). For the above initial data \( \rho = \frac{3}{8\pi} (A^2 - k^2) \) for \( r < r_2 \). Let \( \gamma \) be a smooth monotonically increasing function which is equal to one for \( r > r_1 \) and zero for \( r < r_0 \). Now, define \( \bar{\rho} \equiv \rho \gamma \). Substituting \( \bar{\rho} \) into the Hamiltonian constraint and integrating from \( r \) to \( r_1 \) yields \( \xi^{-1} = 1 - \frac{2M(r)}{r} + A^2r^2 \) where

\[
M(r) \equiv \frac{(A^2 - k^2)r^3}{2} - \int_0^r 4\pi \bar{\rho}r^2dr.
\]

By definition \( \xi \) is smooth, and it is easily shown to be positive. For \( r > r_1 \), \( \xi^{-1} = 1 + k^2r^2 \). Now, if \( r < r_0 \), then \( \xi^{-1} = 1 - \frac{2M(r_0)}{r} + A^2r^2 \) which is the metric for a CMC hypersurface in the Schwarzschild spacetime. Finally, extend \( \xi \) across the throat by using Kruskal coordinates. This gives an asymptotic region as in the proof of lemma 4. Since the above arguments were all only local, they can be applied at each of the points \( \{x_1, x_2, \ldots, x_n\} \). Therefore, there is initial data on \( \tilde{\Sigma} - \{x_1, x_2, \ldots, x_n\} \) with \( p = 3A \). \( \square \)

Since the initial data constructed in theorems 4 and 3 has momentum density equal to zero, \( J^a = 0 \), one choice for the matter source is dust, and just as in the case of the closed manifolds the initial data can be evolved into a globally hyperbolic spacetime. Each asymptotic region of one these evolved spacetimes is just the asymptotic region of the Schwarzschild spacetime, while the interior region is a piece of one of the spatially closed globally hyperbolic spacetimes constructed in theorem 3. Using dust as the source of \( \rho \) yields the following theorem.

**Theorem 6.** For every asymptotically flat 3-manifold \( \Sigma \), there is a globally hyperbolic spacetime \( \mathbb{R} \times \Sigma \) which is physically reasonable. Furthermore, one can always find a globally hyperbolic spacetime \( \mathbb{R} \times \Sigma \) with a CMC hypersurface.

**Proof.** Choose as the matter source \( T_{\alpha\beta} = \rho u_{\alpha}u_{\beta} \) where \( u_{\alpha}u^{\alpha} = -1 \). Then invoking the same existence theorems as in theorem 3 and using the initial data of theorem 4 yields the first part of the theorem. To obtain the second result just repeat the procedure using the initial data from theorem 4. \( \square \)

The fact that all closed and asymptotically flat 3-manifolds are allowed by the classical field equations as the spatial topologies of globally hyperbolic spacetimes will be used in the next section to prove the nonexistence of maximal slices in general. The nonexistence of maximal slices for these globally hyperbolic spacetimes only depends on the topology of the 3-manifold and not on the matter sources. The results of this section also show that CMC hypersurfaces for some \( p \neq 0 \) exist for all topologies. Hence, there are no topological obstructions to general CMC hypersurfaces.

**III. TOPOLOGICAL OBSTRUCTIONS**

Suppose that a globally hyperbolic spacetime \( \mathbb{R} \times \Sigma \) has a maximal slice and obeys the dominant energy condition, then \( p = 0 \) and \( \rho \geq 0 \) on the slice. Furthermore, the constraints must also hold on the slice, in particular \( R = 16\pi + p_{ab}q^{ab} - \rho^2 \). Combining these facts together implies that \( \Sigma \) has a metric with \( R \geq 0 \). Therefore, a necessary condition for the existence of a maximal slice is that \( \Sigma \) admit a metric with \( R \geq 0 \). Using a natural method of counting 3-manifolds, the set of all closed 3-manifolds which admit a metric with \( R \geq 0 \) comprise a small fraction of all closed 3-manifolds. Since theorem 3 proves that all closed 3-manifolds occur as hypersurfaces of globally hyperbolic spacetimes, it follows that only a small fraction of spatially closed globally hyperbolic spacetimes admit maximal slices. For asymptotically flat globally hyperbolic spacetimes, a maximal slice implies that \( R \geq 0 \) on the asymptotically flat 3-manifold. It can be shown that an asymptotically flat 3-manifold with \( R \geq 0 \) has a smooth compactification with...
R > 0; and therefore, only a small fraction of asymptotically flat globally hyperbolic spacetimes have maximal slices. The results for closed 3-manifolds will now be discussed, and the compactification theorem will be proven at the end of this section.

Given a closed manifold there can be topological obstructions to placing a metric on it with \( R \geq 0 \). The most well known example of such an obstruction is the Euler characteristic in the case of 2-manifolds. The idea is assume some 2-manifold has a metric with \( R \geq 0 \), then integrate the scalar curvature over \( M \). Next, applying the Gauss-Bonnet theorem \( \chi(M) = \frac{1}{8\pi} \int_M R dA \), it follows that \( \chi(M) \geq 0 \). There are only four closed 2-manifolds with nonnegative Euler characteristic, namely, the sphere, projective plane, torus, and Klein bottle. Since there are a countable number of distinct closed 2-manifolds and only four of them which admit metrics with \( R \geq 0 \), this proves not only are there obstructions to admitting a metric with \( R \geq 0 \) but also that manifolds with \( R \geq 0 \) are rare.

An example of a topological obstruction which is more closely related to the obstruction for admitting a metric with \( R \geq 0 \) on manifolds of dimension greater than two is the first Betti number. More precisely, if a closed manifold has a metric with positive Ricci curvature, then the first Betti number vanishes. First, define \( \Omega^p(M^n) \) to be the vector space of all p-forms on \( M^n \). We have the following elliptic complex

\[
0 \rightarrow \Omega^0(M^n) \xrightarrow{d} \Omega^1(M^n) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^p(M^n) \xrightarrow{d} \Omega^{p+1}(M^n) \rightarrow \cdots \xrightarrow{d} \Omega^{n-2}(M^n) \xrightarrow{d} \Omega^{n-1}(M^n) \xrightarrow{d} \Omega^n(M^n) \rightarrow 0
\]

where \( d \) is the exterior derivative. The \( p^{th} \) cohomology \( H^p(M^n) \) is defined by

\[
Ker(\Omega^p(M^n) \xrightarrow{d} \Omega^{p+1}(M^n))/Im(\Omega^{p-1}(M^n) \xrightarrow{d} \Omega^p(M^n))
\]

in other words, it is the space of closed p-forms modulo exact p-forms. For closed n-manifolds de Rham’s theorem says that \( H^p(M^n) \) is isomorphic to the real singular cohomology, i.e. the usual cohomology which is calculated using only the topology of \( M^n \). Picking a metric on \( M^n \) allows the choice of a unique representative of each cohomology class. In order to choose the unique representative, first define an inner product of forms

\[
(\alpha, \beta) = \int_{M^n} \alpha_{ab...c} \beta^{ab...c} dV.
\]

The adjoint of \( d \) with respect to this inner product denoted by \( \delta \) can be written as \( \delta \alpha = -\nabla^a \alpha_{ab...c} \). Now, define the energy of a form to be

\[
E(\alpha) = \int_{M^n} \alpha_{ab...c} \alpha^{ab...c} dV.
\]

Given a cohomology class \( [\alpha_0] \in H^p(M^n) \), then the class can be represented by \( \alpha = \alpha_0 + d\beta \) where \( \beta \) ranges over all \((p-1)\)-forms. The goal is given a fixed \( \alpha_0 \) find a \( \beta \) such that \( \alpha \) has least energy. Taking the variation, the Euler-Lagrange equations imply that \( \delta \alpha = 0 \). Since \( \alpha \) was closed to begin with, it follows that \( \delta \alpha = 0 \) and \( d\alpha = 0 \). Observe that for 2-forms the above equations are just the vacuum Maxwell equations for a positive definite metric. Let the Laplacian of a p-form \( \alpha \) be defined by \( \Delta \alpha = (d d + d \delta) \alpha \). The conditions that \( \delta \alpha = 0 \) and \( d\alpha = 0 \) are equivalent to \( \Delta \alpha = 0 \), because if \( \alpha \) is harmonic then

\[
0 = (\Delta \alpha, \alpha) = (\delta \alpha, \delta \alpha) + (d\alpha, d\alpha).
\]

The above arguments imply that the kernel of \( \Delta \) or the space of harmonic p-forms is isomorphic to the \( p^{th} \)-cohomology vector space \( H^p(M^n) \); this result is known as Hodge’s theorem.

Finally, Hodge theory is applied to the problem of finding obstructions to admitting metrics with positive Ricci curvature, \( R_{ab} \xi^a \xi^b > 0 \) if \( \xi \) is not zero everywhere. One can verify that

\[
\Delta \xi_a = -\nabla^2 \xi_a + R_{ab} \xi^b
\]

by using the facts that \( d\omega = \nabla_{[a} \omega_{bc...d]} \) and \( \delta \alpha = -\nabla^a \alpha_{ab...c} \). Suppose that \( \xi \in ker \Delta \), then

\[
0 = \int_{M^n} \xi^a \Delta \xi_a dV = -\int_{M^n} \xi^a \nabla^2 \xi_a dV + \int_{M^n} R_{ab} \xi^a \xi^b dV
\]

but

\[
-\int_{M^n} \xi^a \nabla^2 \xi_a dV + \int_{M^n} R_{ab} \xi^a \xi^b dV = \int_{M^n} \nabla^b \xi^a \nabla_b \xi_a dV + \int_{M^n} R_{ab} \xi^a \xi^b dV > 0
\]
assuming positive Ricci curvature. This means \( \xi = 0 \). Therefore, positive Ricci curvature implies that the kernel of \( \Delta \) is trivial and that \( H^2(M^n) \) is zero. Since the first Betti number \( b_1(M^n) \) is the dimension of \( H^1(M^n) \), this result is equivalent to \( b_1(M^n) = 0 \). For the n-torus \( b_1(T^n) = n \), therefore the n-torus admits no metric with positive Ricci curvature. Another example of a manifold which admits no metric with positive Ricci curvature is any closed manifold of the form \( N \times T^n \) where \( N \) is an arbitrary manifold. This is true because \( b_1(N \times T^n) = b_1(N) + b_1(T^n) = b_1(N) + n > 0 \). With the above examples in mind, the case of positive scalar curvature will now be considered.

The key to finding an obstruction to positive Ricci curvature was finding a generalized Laplacian which split into the usual Laplacian and a linear term involving the curvature. The object is to find an operator which satisfies these conditions with the linear term being the scalar curvature. Given a manifold with spinors on it, i.e. a spin manifold, there is such an operator, namely, the Dirac operator, denoted by \( \hat{D} \). The \textit{Weitzenböck formula} for the Dirac operator is \( \hat{D}^2 = -\nabla^2 \psi + \frac{1}{4} R \psi \). The operator \( \hat{D}^2 \) is the Laplacian operator for spinors, and the kernel of \( \hat{D}^2 \) is the space of harmonic spinors. If \( R > 0 \), then the \textit{Weitzenböck} formula implies that \( \ker \hat{D}^2 = 0 \). However, a difficulty with the Dirac operator is that there is no analogue of Hodge’s theorem. In fact, the space of harmonic spinors in general depends on the metric. To avoid this problem, one might consider the index of the Dirac operator or an operator constructed from it because the Atiyah-Singer index theorem guarantees that the index is a topological invariant. However, in the case of interest, namely 3-manifolds, all such invariants vanish, so no information on which 3-manifolds admit metric with \( R > 0 \) can be obtained directly. In order to overcome this difficulty, replace the closed 3-manifold \( \Sigma \) with the 4-manifold \( \Sigma \times S^1 \). Observe that although such a manifold admits no metric with positive Ricci curvature, the scalar curvature is much weaker; and in fact, if \( \Sigma \) admits a metric with \( R > 0 \) then \( \Sigma \times S^1 \) admits a metric with \( R > 0 \). To prove this, let the metric on \( \Sigma \times S^1 \) be given by \( ds^2 = g_{ij} dx^i dx^j + d\theta^2 \) where \( g \) is the metric on \( \Sigma \) which has \( R > 0 \), then this product metric has positive scalar curvature. Now, look at the Dirac operator on closed 4-manifolds with \( R > 0 \). First, on any even dimensional manifold the bundle of spinors \( S \) can be decomposed into the sum of \( S_+ \oplus S_- \), namely, the spinors of \( \frac{1}{4} \)-helicity and \( -\frac{1}{4} \)-helicity. Now, define the operator \( \hat{D}^+ \) to be the restriction of \( \hat{D} \) to \( S_+ \). It is easily proven that \( \hat{D}^+ \) maps \( S_+ \) into \( S_- \) and that the adjoint of \( \hat{D}^+ \) is the restriction of \( \hat{D} \) to \( S_- \) which is denoted by \( \hat{D}^- \). The index of a differential operator is the dimension of the kernel minus the dimension of the cokernel of the operator. When the operator has an adjoint the cokernel of the operator is the kernel of the operator’s adjoint. Thus, the index of \( \hat{D}^+ \) is \( \text{ind} \hat{D}^+ = \dim \ker \hat{D}^+ - \dim \ker \hat{D}^- \). Since the 4-manifold being considered has \( R > 0 \), then \( \ker \hat{D}^- = 0 \) but \( \hat{D} \) is self-adjoint so \( \ker \hat{D} = \ker \hat{D}^2 = 0 \). Furthermore, \( S = S_+ \oplus S_- \) implies that \( \ker \hat{D} = \ker \hat{D}^+ \oplus \ker \hat{D}^- \) which means \( R > 0 \) implies \( \ker \hat{D}^+ = 0 \) and \( \ker \hat{D}^- = 0 \). Hence, \( R > 0 \) implies \( \text{ind} \hat{D}^+ = 0 \). Identifying the topological invariant associated with \( \text{ind} \hat{D}^+ = 0 \) is a rather involved procedure, so the details will not be given here. The invariant is the \( \tilde{A} \)-genus and in four dimensions it is proportional to the signature. The \textit{signature} of a closed 4n-manifold \( M^{4n} \) is the signature of the bilinear form \( (\alpha, * \beta) \) where \( \alpha, \beta \in H^{2n}(M^{4n}), * \beta \) is the dual form of \( \beta \), and \( (, ) \) is the inner product of forms defined above. The signature of closed manifolds of other dimensions is taken to be zero. Now, with the invariant in hand let us return to the manifold \( \Sigma \times S^1 \); the signature \( \tau \) of \( \Sigma \times S^1 \) is \( \tau(\Sigma \times S^1) = \tau(\Sigma) \tau(S^1) \). Since \( \tau(\Sigma) = \tau(S^1) = 0 \), it follows that \( \tau(\Sigma \times S^1) = 0 \) which implies the \( \tilde{A} \)-genus vanishes. The problem of the \( \tilde{A} \)-genus vanishing is caused by the fact that the signature obeys the product rule. In order to obtain a nontrivial invariant, M. Gromov and H. B. Lawson define a family of generalized Dirac operators on general bundles of spinors over \( \Sigma \times S^1 \). These generalized Dirac operators still satisfy generalized Weitzenböck formulae which imply; if \( R > 0 \), then the generalized \( \tilde{A} \)-genera all must vanish. The difference between the classical \( \tilde{A} \)-genus and the generalized \( \tilde{A} \)-genera is that the generalized ones do not satisfy the product rule. Using the generalized \( \tilde{A} \)-genera M. Gromov and H. B. Lawson were able to prove the following classification theorem [14].

**Theorem 7.** A closed orientable 3-manifold \( M^3 \) (if it is nonorientable, then take its double cover) which has a \( K(\pi, 1) \) as a prime factor\(^3\) in its prime decomposition\(^4\) admits no metric with \( R > 0 \). In fact, any metric on \( M^3 \) with \( R \geq 0 \) must be flat.

The work of W. Thurston [19] on hyperbolic 3-manifolds implies most 3-manifolds are in fact \( K(\pi, 1) \)'s in the following way: First, a knot is the continuous embedding of a circle in a 3-manifold and a link is a finite number of disjoint knots. Next, one preforms Dehn surgery along a link in a manifold by removing tubular neighborhoods

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\(^3\) A \( K(\pi, 1) \) is a closed 3-manifold with a contractible universal covering space. For example, the 3-torus is a \( K(\pi, 1) \). See reference [18] for more details.

\(^4\) The connected sum of two 3-manifolds \( M_1 \) and \( M_2 \) is the 3-manifold \( M_1 \# M_2 \) obtained from removing a ball from each, and then gluing the resulting manifolds together along their boundaries. A closed 3-manifold is prime if \( M = M_1 \# M_2 \) implies that \( M_1 \) or \( M_2 \) is a 3-sphere. Every 3-manifold has a unique prime decomposition into the connected sum of a finite number of prime factors. Also see reference [18].
of each knot, and then gluing back the removed solid tori differently. More precisely, one identifies the boundary of each hole left with the boundary of another solid torus via a homeomorphism of the boundary different from the one defined by the inclusion of the removed torus in the manifold. W. B. R. Lickorish proved that every closed orientable 3-manifold can be obtained from Dehn surgery on the 3-sphere. More recently, W. Thurston has proven that every closed orientable 3-manifold is obtained from the 3-sphere $S^3$ by Dehn surgery along links $L$ such that the 3-sphere minus $L$ is a hyperbolic 3-manifold, call such links hyperbolic. Furthermore, all but a finite number of 3-manifolds obtained obtained from $S^3$ by Dehn surgeries along a given hyperbolic link $L$ are closed hyperbolic 3-manifolds. Hence, in this way of counting 3-manifolds most closed 3-manifolds are hyperbolic. Since all hyperbolic manifolds are covered by $\mathbb{R}^3$, it follows that they are all $K(\pi, 1)$’s. Therefore, most 3-manifolds are $K(\pi, 1)$’s.

The above result combined with theorem 7 implies most 3-manifolds never admit a metric with $R > 0$. Furthermore, there are only ten flat closed 3-manifolds which means most closed 3-manifolds do not admit metric with $R \geq 0$. Combining these ideas with the existence theorems for globally hyperbolic spacetimes with arbitrary spatial topology yields the following result.

**Theorem 8.** Most spatially closed globally hyperbolic spacetimes $\mathbb{R} \times \Sigma$ never admit a maximal slice.

**Proof.** Existence of a maximal slice implies $\Sigma$ admits a metric with $R \geq 0$, and theorem 3 implies all closed 3-manifolds occur as hypersurfaces. Therefore, theorem 7 implies most globally hyperbolic spacetimes $\mathbb{R} \times \Sigma$ admit no maximal slice.

Finally, the classification of asymptotically flat 3-manifolds which admit metrics with $R \geq 0$ will be discussed. This classification theorem is proven by showing that every asymptotically flat 3-manifold with a metric having $R \geq 0$ has a smooth compactification with $R > 0$. The classification is completed by applying theorem 7 to the compactifications. Before proving the compactification theorem, several technical propositions are required which are now presented.

The first technical proposition needed is the maximum principle for second order elliptic operators. One important feature of this version of the maximum principle is there are no restrictions on the sign of $c$.

**Theorem 9.** Let $L$ be a elliptic differential operator defined by

$$Lu = \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} + cu$$

on an open subset $U \subseteq \mathbb{R}^n$. The coefficients $a$, $b$, and $c$ are locally bounded; $a$ is symmetric; and in the neighborhood of any point of $U$ there are two positive constants $m$ and $M$ such that

$$m \sum_{i=1}^{n} \xi_i^2 \leq \sum_{i,j=1}^{n} a_{ij} \xi_i \xi_j \leq M \sum_{i=1}^{n} \xi_i^2$$

for all $\xi \in \mathbb{R}^n$. If $u \in C^2(U)$ with $Lu \geq 0$ and $u \leq 0$, then $u(x_0) = 0$ for some $x_0 \in U$ implies $u \equiv 0$ in $U$.

The next proposition is an existence lemma for a family of “bump functions” with a prescribed type of asymptotic behavior. The existence of such a one parameter family of functions is an essential ingredient in the proof of the compactification.

**Lemma 2.** Given $c > 0$, there is a family of smooth decreasing functions $\alpha_\rho$ for $\rho > c$ such that $\alpha_\rho = 1$ for $r \leq \rho$, $\alpha_\rho = 0$ for $r \geq 2\rho$, $|\alpha'_\rho| \leq \frac{A}{\rho}$, and $|\alpha''_\rho| \leq \frac{A}{\rho^2}$ where $A$ is a constant independent of $\rho$ and $\alpha'_\rho \equiv \frac{d}{dr}\alpha_\rho(r)$.

**Proof.** Let $\gamma_\rho$ be any smooth function which is equal to one for $\rho + \epsilon \leq r \leq \frac{3}{2}\rho - \epsilon$ where $\epsilon = \frac{1}{6}c$, and zero otherwise. Likewise, $\gamma_\rho$ is one for $\frac{5}{4}\rho + \epsilon \leq r \leq 2\rho - \epsilon$ and zero otherwise. Now, define smooth increasing and decreasing functions by the following expressions

$$\alpha_\rho^- (r) \equiv \frac{\int_{\rho}^{r} \gamma_\rho^- dt}{\int_{\frac{3}{2}\rho}^{\frac{5}{4}\rho} \gamma_\rho^- dt}$$

and

$$\alpha_\rho^+ (r) \equiv \frac{\int_{\rho}^{r} \gamma_\rho^+ dt}{\int_{\frac{3}{2}\rho}^{\frac{5}{4}\rho} \gamma_\rho^+ dt}.$$
Observe that $|\alpha'_L| \leq \left( \int_{\rho}^{\frac{5}{4} \rho} t^{-\epsilon} dt \right)^{-1} \frac{1}{r-1}$ and $|\alpha'_R| \leq \left( \int_{\rho}^{2 \rho} t^{-\epsilon} dt \right)^{-1} \frac{1}{r-1}$. Furthermore,

$$\int_{\rho}^{\frac{5}{4} \rho} \frac{\gamma}{t} dt \geq \int_{\rho}^{\frac{5}{4} \rho} \frac{1}{t} dt = \log \frac{\frac{5}{4} \rho - \epsilon}{\rho + \epsilon} \geq \log \frac{4}{3}$$

and

$$\int_{\frac{5}{4} \rho}^{2 \rho} \frac{\gamma}{t} dt \geq \int_{\frac{5}{4} \rho}^{2 \rho} \frac{1}{t} dt = \log \frac{2 \rho - \epsilon}{\frac{5}{4} \rho + \epsilon} \geq \log \frac{6}{5} .$$

Hence the derivatives of $\alpha_L$ and $\alpha_R$ having the following behavior $|\alpha'_L| \leq \left( \log \frac{4}{3} \right)^{-1} \frac{1}{r-1}$ and $|\alpha'_R| \leq \left( \log \frac{4}{3} \right)^{-1} \frac{1}{r-1}$. Next, define another smooth function by the following expression

$$\gamma_{\rho}(r) \equiv \begin{cases} \alpha_{L}(r), & \text{if } \rho \leq r \leq \frac{4}{5} \rho; \\ 1, & \text{if } \frac{4}{5} \rho \leq r \leq \frac{5}{4} \rho; \\ \alpha_{R}(r), & \text{if } \frac{5}{4} \rho \leq r \leq 2 \rho; \\ 0, & \text{otherwise.} \end{cases}$$

It follows that $|\gamma'_\rho| \leq \left( \log \frac{4}{3} \right)^{-1} \frac{1}{r-1}$. Finally, let

$$\alpha_{\rho}(r) \equiv \int_{\rho}^{\frac{5}{4} \rho} \frac{\gamma}{t} dt .$$

By definition $\alpha_{\rho}$ is smooth, $\alpha_{\rho} = 1$ for $r \leq \rho$, and $\alpha_{\rho} = 0$ for $r \geq 2 \rho$. Observe that

$$\int_{\rho}^{2 \rho} \frac{\gamma_{\rho}}{t} dt \geq \int_{\frac{5}{4} \rho}^{2 \rho} \frac{1}{t} dt = \log \frac{5}{4} ,$$

this implies $|\alpha'_\rho| \leq \left( \log \frac{5}{4} \right)^{-1} \frac{1}{r-1}$. Furthermore,

$$|\alpha'_\rho| \leq \left( \log \frac{5}{4} \right)^{-1} \left[ \frac{|\gamma'_\rho|}{r} + \frac{|\gamma|}{r^2} \right] \leq \left( \log \frac{5}{4} \right)^{-1} \left( \log \frac{4}{3} \right)^{-1} \left[ \left( \log \frac{4}{3} \right)^{-1} + 1 \right] r^{-2} .$$

Therefore, $|\alpha'_\rho| \leq \frac{A}{r}$ and $|\alpha'_\rho| \leq \frac{A}{r^2}$ where $A = \left( \log \frac{5}{4} \right)^{-1} \left( \left( \log \frac{4}{3} \right)^{-1} + 1 \right)$.

The next lemma is an existence theorem for the ground state of the operator $-D^2 + V$. This lemma is used in constructing a metric with positive scalar curvature on the compactified manifold.

**Lemma 3.** If $M$ is a closed $n$-manifold and $L = -D^2 + V$ where $V$ is a smooth function on $M$, then there is a smooth function $\psi_0 > 0$ and real number $\lambda_0$ such that $L\psi_0 = \lambda_0 \psi_0$.

**Proof.** Let $H_1(M)$ be the Sobolev space of all $L^2(M)$ functions whose generalized derivative is also in $L^2(M)$. $H_1(M)$ is actually a Hilbert space with inner product given by $(\phi, \psi) = \int_M (D_a \phi D^a \psi + \phi \psi) dV$. Now, define a functional in the following way:

$$I(\psi) = \frac{\int_M (D_a \psi D^a \psi + V \psi^2) dV}{\int_M \psi^2 dV}$$

---

5 The Sobolev spaces $H_k(M^n)$ for any nonnegative integer $k$ can be defined as the completion of smooth functions in the norm

$$\|f\|_{(k)} = \sqrt{\sum_{i=0}^{k} \|D^i f\|^2}$$

where $\| \cdot \|_2$ is the $L^2$ norm, $D^{(0)} f \equiv f$, and $D^{(i)} f \equiv (D_{a_1} a_2 \ldots a_i f D^{a_1 a_2 \ldots a_i} f)^{1/2}$.

6 The *generalized derivative* with respect to $x^i$ of a locally integrable function on an open subset $U \subseteq \mathbb{R}^n$ is the locally integrable function $\partial_i f$ such that

$$\int_U \phi \partial_i f \, d^n x = - \int_U f \frac{\partial \phi}{\partial x^i} \, d^n x$$

for all $\phi \in C^\infty(U)$. 
for \( \psi \) in \( H_1(M) \) and not identically zero. Let \( \mathcal{S} = \{ b \in \mathbb{R} \mid b \leq I(\psi) \text{ for all } \psi \in H_1(M) \text{ and } \psi \neq 0 \} \). Since the gradient term is nonnegative and \( V \) is bounded, \( I \) is bounded from below and consequently \( \mathcal{S} \) is nonempty. Further, \( \mathcal{S} \) is bounded above. Since \( \mathcal{S} \) is a nonempty set of real numbers which is bounded above it has a least upper bound, call it \( \lambda_0 \). By definition, it follows that

\[
\lambda_0 = \inf_{\psi \in H_1(M) \setminus \{0\}} I(\psi).
\]

Furthermore, it follows that there is a sequence \( \{\psi_k\} \) in \( H_1(M) \) such that \( I(\psi_k) \to \lambda_0 \) as \( k \to \infty \) and \( \int \psi_k^2 \, dV = 1 \). Clearly, \( \{\psi_k\} \) is a bounded subset of \( H_1(M) \). The embedding \( H_1(M) \subset L^2(M) \) is compact, i.e. every bounded set is mapped to a compact one. Therefore, there is a subsequence \( \{\psi_{k_i}\} \) of \( \{\psi_k\} \) such that \( \psi_{k_i} \to \psi_0 \) in \( L^2(M) \). Moreover, \( I(\psi_{k_i}) \to \lambda_0 \) as \( k_i \to \infty \) because all subsequences of a convergent sequence converge to the same limit. Next, it will be shown that \( \psi_0 \in H_1(M) \) and \( I(\psi_0) = \lambda_0 \).

Since \( \{\psi_{k_i}\} \) is a bounded subset of \( H_1(M) \), \( \{\psi_{k_i}\} \) is compact in the weak topology. This is because bounded subsets of Hilbert spaces are weakly compact. Hence there is a subsequence \( \{\psi_{k_i}\} \) of \( \{\psi_{k_i}\} \) which converges weakly in \( H_1(M) \) to \( \phi_0 \). Now, the embedding \( H_1(M) \subset L^2(M) \) is continuous so it must also be continuous in the weak topology. Therefore, \( \{\psi_{k_i}\} \) converges weakly in \( L^2(M) \) to \( \phi_0 \). However, \( \{\psi_{k_i}\} \) converges in \( L^2(M) \) to \( \phi_0 \) by the previous arguments. Hence \( \psi_{k_i} \to \psi_0 \) weakly in \( L^2(M) \). This means \( \psi_0 = \phi_0 \) in \( L^2(M) \) because \( \{\psi_{k_i}\} \) is a subsequence of the weakly convergent sequence \( \{\psi_{k_i}\} \). Therefore, \( \psi_0 \) is in \( H_1(M) \). From now on, let the sequence \( \{\psi_{k_i}\} \) be denoted by \( \{\psi_m\} \). Weak convergence of \( \{\psi_m\} \) means that for all linear functionals, \( A, A(\psi_m) \to A(\psi_0) \) as \( m \to \infty \). In particular, weak convergence and Schwarz’s inequality imply \( (\psi_0, \psi_0) \leq \lim_{m \to \infty} (\psi_m, \psi_0) \leq \lim_{m \to \infty} ||\psi_m|| ||\psi_0|| \), where \( || \cdot || \) and \( (, ) \) are the norm and inner product on \( H_1 \). Hence, \( \psi_0(\psi_0) \leq \lim_{m \to \infty} (\psi_m, \psi_m) \). Further, the \( L^2(M) \) convergence of \( \{\psi_m\} \) and the fact that \( V \) is bounded imply \( \int (V-1)\psi_m^2 \, dV \to \int (V-1)\psi_0^2 \, dV \). This is proven by using Hölder’s inequality\(^7\) and the triangle inequality, namely,

\[
\left| \int (V-1)\psi_m^2 \, dV - \int (V-1)\psi_0^2 \, dV \right| \leq K \int |\psi_m^2 - \psi_0^2| \, dV \leq K||\psi_m - \psi_0||_2^2
\]

where \( |V-1| \leq K \) and \( || \cdot ||_2 \) is the \( L^2(M) \) norm. Choosing \( V = 2 \), this argument also implies that \( ||\psi_0||_2 = 1 \). Using the above limits, it follows that

\[
\lambda_0 = \lim_{m \to \infty} I(\psi_m) = \lim_{m \to \infty} \frac{\psi_m, \psi_m + \int (V-1)\psi_m^2 \, dV}{||\psi_m||_2^2} \geq \frac{(\psi_0, \psi_0) + \int (V-1)\psi_0^2 \, dV}{||\psi_0||_2^2} = I(\psi_0) \geq \lambda_0.
\]

Therefore, \( \psi_0 \) is in \( H_1(M) \) and an extremum of \( I \).

The smoothness of \( \psi_0 \) will now be demonstrated. Since the functional \( I \) attains an extremum, the derivative at \( \psi_0 \) must vanish in all directions, i.e.

\[
D_{\psi}I(\psi_0) = \int_M (D_{\psi}\psi_0 D_{\psi} \phi + V\psi_0 \phi - \lambda_0 \psi_0 \phi) \, dV = 0.
\]

In particular, it vanishes for smooth \( \phi \) which means

\[
-D^2\psi_0 + V\psi_0 = \lambda_0 \psi_0
\]

in the sense of distributions or in other words \( \psi_0 \) is a weak solution. Now, rewrite the equation as \( D^2\psi_0 = (\lambda_0 - V)\psi_0 \) which means \( D^2\psi_0 \) is in \( L^2(M) \) because \( \psi_0 \) is in \( H_1(M) \) and \( V \) is smooth. Further, \( |D_{\psi}\psi_0| \) is in \( L^2(M) \) and \( V \) is

\(^7\) Hölder’s inequality is: Given \( f \in L^p \) and \( g \in L^q \), then

\[
\int |fg| \, dV \leq \left( \int |f|^p \, dV \right)^{\frac{1}{p}} \left( \int |g|^q \, dV \right)^{\frac{1}{q}}
\]

for \( \frac{1}{p} + \frac{1}{q} = 1 \) with \( 1 \leq p \leq \infty \).
smooth so it follows that $|D_a D^2 \psi_0|$ is in $L^2(M)$. By using induction, one can prove that $D^{2k} \psi_0$ and $|D_a D^{2k} \psi_0|$ are in $L^2(M)$. Now, one can prove that the following identity

$$D_a D_b T_{c \ldots e} - D_b D_a T_{c \ldots e} = R_{abc} f T_{d \ldots e} + R_{abd} f T_{c \ldots e} + \cdots + R_{abc} f T_{c \ldots f},$$

holds. Using this identity and the facts about the derivatives of $\psi_0$, it follows that

$$\int D_a D_b \psi_0 D^a D^b \psi_0 dV = \int (D^2 \psi_0)^2 dV - \int R_{abc} D^a \psi_0 D^b \psi_0 dV$$

and the righthand side is finite. Therefore, $\psi_0 \in H_2(M)$. Repeating this procedure for $D_a D_b D_c \psi_0$, it follows that

$$\int D_a D_b D_c \psi_0 D^a D^b D^c \psi_0 dV = \int D_c D^2 \psi_0 D^a D^2 \psi_0 dV - \int R_{abc} D^a \psi_0 D^b D^c \psi_0 dV - 2 \int R_{abc} D^a \psi_0 D^b D^c \psi_0 dV$$

and again the righthand side is finite. Therefore, $\psi_0 \in H_3(M)$. By using this bootstrap technique and induction, one can prove that any weak solution to the differential equation is in $H_k(M)$ for all $k$. However, the Sobolev embedding theorem for compact n-manifolds says that

$$H_k(M) \subset C^r(M)$$

for $k > \frac{n}{2} + r$. Therefore, all weak solutions are smooth.

Finally, let $\psi_0$ be any solution, then the above argument implies it is smooth. Since $\psi_0$ is smooth, it follows that $|\psi_0|$ is also in $H_1(M)$. Furthermore, $I(\psi_0) = I(|\psi_0|)$ so $|\psi_0|$ is also a solution. Further, the regularity of weak solutions implies $|\psi_0|$ must be smooth. Therefore, $\psi_0$ cannot change sign which means $\psi_0$ can be chosen to be nonnegative. In order to prove that $\psi_0 > 0$, let $u = -\psi_0$ and then apply the maximum principle theorem.

The next lemma is used in producing asymptotically flat 3-manifolds from closed ones. Although, one usually proves this using operator techniques on Hilbert spaces, it will proven via a variational principle below.

**Lemma 4.** If $\lambda_0 > 0$ for the operator $L = -D^2 + V$, then $L$ is an isomorphism of $C^\infty(M)$ to itself.

**Proof.** First, $\lambda_0 > 0$ and its variational characterization imply

$$0 < \lambda_0 ||\phi||^2 \leq \int_M (D_a \phi D^a \phi + V \phi^2) dV$$

for all $\phi \in C^\infty(M)$ not identically zero. This means $L$ is one to one. Now, define an inner product on $C^\infty(M)$ by

$$(\phi, \psi)_L \equiv \int_M (D_a \phi D^a \psi + V \phi \psi) dV$$

Furthermore, there is a constant $K > 0$ such that $K^{-1} ||\phi|| \leq ||\phi||_L \leq K ||\phi||$ where $||\phi||_L$ is the standard norm on $H_1(M)$. Therefore, the completion of $C^\infty(M)$ in the norm $||\phi||_L$ is not only a Hilbert space but it is $H_1(M)$.

Next, pick a smooth function $f$ and define the functional $F(\psi) \equiv \int f \psi dV$ for all $\psi \in H_1(M)$. Observe that

$$|F(\psi)| \leq \int_M |f \psi| dV \leq ||f||_2 ||\psi||_2 \leq C ||\psi||_L$$

where $C$ is a constant. Therefore, $F$ is a bounded linear functional on $H_1(M)$. The Riesz representation theorem implies there is a unique $\phi \in H_1(M)$ such that $(\psi, \phi) = F(\psi)$ for all $\psi \in H_1(M)$. Therefore,

$$\int_M (D_a \phi D^a \phi + V \phi \phi - f \psi) dV = 0$$

for all $\psi$, in other words $\phi$ is a weak solution of $L \phi = f$. Since $f$ and $V$ are smooth, the regularity argument of lemma implies $\phi$ is smooth. \hfill \Box

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8 Riesz Representation Theorem: Given any bounded linear functional $F$ on a Hilbert space $\mathcal{H}$ there is a uniquely determined vector $f$ in $\mathcal{H}$ such that $F(x) = (x, f)$ for all $x \in \mathcal{H}$. Furthermore, $||F|| = ||f||$. Note: $\mathcal{H}$ need not be separable.
The following Sobolev inequality is needed to control the norm of the scalar curvature as the metric is deformed using a family of bump functions. This inequality is well known and proven in several different papers \[21\]. However, the proof will be given here for the sake of completeness.

**Lemma 5.** If \( \Sigma \) is an asymptotically flat 3-manifold with metric \( g_{ab} \) satisfying \( g_{ab} = \delta_{ab} + h_{ab} \) outside a compact set \( C \) where \( |h_{ab}| \leq \frac{A}{r^4} \), \( |\partial_c h_{ab}| \leq \frac{A}{r^7} \), and \( |\partial_d \partial_c h_{ab}| \leq \frac{A}{r^8} \); then

\[
\left( \int_{\Sigma} |f|^6 \, dV \right)^{\frac{1}{3}} \leq K \left( \int_{\Sigma} D_a f D^a f \, dV \right)^{\frac{1}{3}}
\]

for all \( f \in C^\infty_0 (\Sigma) \). Furthermore, if two metrics agree on \( C' \supset C \) and are bounded by the same constant \( A \), then \( K \) is the same for both of them.

**Proof.** The asymptotic behavior of the metric implies that there is a constant \( \lambda \) depending on \( A \) such that \( \lambda^{-1} \delta_{ab} \xi^a \xi^b \leq g_{ab} \xi^a \xi^b \leq \lambda \delta_{ab} \xi^a \xi^b \) on \( \Sigma - C \). Now, the above Sobolev inequality does hold on \( \mathbb{R}^3 \) minus a ball with the standard metric. This follows from the same inequality on \( \mathbb{R}^3 \). The asymptotic bounds on the metric and the Sobolev inequality for the flat metric on \( \mathbb{R}^3 - B \) imply

\[
\left( \int_{\Sigma - C} |f|^6 \, dV \right)^{\frac{1}{3}} \leq \lambda \left( \int_{\Sigma - C} |f|^6 \, d^3 x \right)^{\frac{1}{3}} \leq \mathcal{K} \lambda \left( \int_{\Sigma - C} \delta^{ab} D_a f D_b f \, dV \right)^{\frac{1}{3}} \leq \mathcal{K} \lambda \left( \int_{\Sigma - C} D_a f D^a f \, dV \right)^{\frac{1}{3}}
\]

where \( \mathcal{K} \) is the Sobolev constant for \( \mathbb{R}^3 - B \) times the number of asymptotic regions. This establishes the Sobolev inequality on the asymptotic regions. Now, suppose the inequality fails on \( \Sigma \); then there is a function \( f_n \in C^\infty_0 (\Sigma) \) for positive integer \( n \) such that \( \int_{\Sigma} |f|^6 \, dV = 1 \) and \( \int_{\Sigma} D_a f_n D^a f_n \, dV < \frac{1}{n} \). Now, apply the Sobolev inequality to the sequence \( f_n \) restricted to \( \Sigma - C \) to obtain \( \left( \int_{\Sigma - C} |f_n|^6 \, dV \right)^{\frac{1}{3}} < \frac{\mathcal{K} \lambda \pi}{n} \). Taking the limit as \( n \to \infty \) yields \( f_n \to 0 \) in the \( L^6 \) norm on \( \Sigma - C \).

One other inequality is needed to complete the proof, namely, the following: Given any compact 3-manifold (with or without boundary) \( N \) and any smooth function on it which satisfies \( \int_N f \, dV = 0 \), then there is a constant \( K \) independent of \( f \) such that

\[
\left( \int_{N} |f|^6 \, dV \right)^{\frac{1}{3}} \leq K \left( \int_{N} D_a f D^a f \, dV \right)^{\frac{1}{3}}.
\]

One can prove this inequality by choosing a covering of \( N \) by charts and a partition of unity. Next, apply the Sobolev inequality to each chart and sum the resulting inequalities using the partition of unity; for details see \[22\]. This yields the inequality

\[
\left( \int_{N} |f|^6 \, dV \right)^{\frac{1}{3}} \leq \tilde{K} \left[ \left( \int_{N} D_a f D^a f \, dV \right)^{\frac{1}{3}} + \left( \int_{N} f^2 \, dV \right)^{\frac{1}{3}} \right] .
\]

The additional term is due to the gradients of the functions in the partition of unity. In order to express the last integral in terms of the \( L^2 \) norm of the gradient of \( f \), the first nontrivial eigenvalue of the Laplacian on \( N \) is used. More precisely, let

\[
\lambda_1 = \inf_{\phi} \frac{\int_N D_a \phi D^a \phi \, dV}{\int_N \phi^2 \, dV},
\]

subject to the constraint \( \int_N \phi \, dV = 0 \). Since \( \lambda_1 \) is a minimum of the functional on the righthand side, it follows that

\[
\int_N f^2 \, dV \leq \frac{1}{\lambda_1} \int_N D_a f D^a f \, dV,
\]

for all smooth \( f \) which satisfy \( \int_N f \, dV = 0 \). Combining this inequality with the previous one yields the desired inequality.
Now, let $\beta_n = \int_{C_1} f_n \, dV$ where $C \subset C_1 \subset C'$ and $C_1$ is compact, then

$$\left( \int_{C_1} |f_n - \beta_n|^6 \, dV \right)^{\frac{1}{6}} \leq \tilde{K} \left( \int_{C_1} D_x f_n D^3 f_n \, dV \right)^{\frac{1}{6}} < \frac{\tilde{K}}{n^2},$$

because of the inequality in the previous paragraph. Taking the limit as $n \to \infty$ yields $f_n \to \beta_n$ in the $L^6$ norm on $C_1$ which means $f_n \to \beta_n$ in the $L^6$ norm on $(\Sigma - C) \cap C_1$. However, $f_n \to 0$ on $\Sigma - C$ from our arguments on the asymptotic regions, which implies $\beta_n \to 0$. Therefore, $f_n \to 0$ in the $L^6$ norm on $\Sigma$, a contradiction to $\int_{\Sigma} |f_n|^6 \, dV = 1$. Therefore, the Sobolev inequality must hold for all $f \in C^\infty_\text{c} (\Sigma)$.

Finally, if two metrics agree on $C'$ and are asymptotically bounded by the same constants, then the Sobolev constant is the same for both of them. This is true because the metrics agree on a compact set and must have the same Sobolev constant on that region. Next, in the asymptotic regions both metrics have the same bounding constants. Moreover, the argument which established the inequality on the asymptotic regions implies both metrics have the same constant in the asymptotic regions. Therefore, it follows that $K$ can be chosen to be the same for both metrics. \[\square\]

The final proposition is an existence theorem for conformal metrics with $R = 0$. This result was proven by several authors\,[21, 23], however, the proof is given here for the sake of completeness and the technique used here is simpler. The norm $|| \ ||_2$ is the norm on $L^2$.

**Lemma 6.** If $\Sigma$ is an asymptotically flat 3-manifold and $L = -D^2 + V$, then $L\phi = f$ has a unique solution which is smooth and $O(\frac{1}{r})$ whenever $V$ and $f$ are smooth, $O(\frac{1}{r^3})$, and $||V-|| \leq \frac{1}{r^3}$ where $K$ is a Sobolev constant (from lemma 5) and $V_-$ is the absolute value of the negative part of $V$.

**Proof.** Let

$$||\phi||_L^2 \equiv \int (D_a \phi D^a \phi + V \phi^2) \, dV$$

for all $\phi \in C^\infty_\text{c} (\Sigma)$. Using Hölder’s inequality and the fact that $V = V_+ - V_-$ where $V_+$ and $V_-$ are absolute values of the nonnegative and negative parts of $V$, respectively, it follows that

$$\int_{\Sigma} D_a \phi D^a \phi \, dV - ||V_-||_2 \phi^2 ||_3 \leq \int_{\Sigma} (D_a \phi D^a \phi - V_- \phi^2) \, dV \leq \int_{\Sigma} (D_a \phi D^a \phi + V \phi^2) \, dV$$

where $|| \ ||_2$ and $|| \ ||_3$ are the $L^2$ and $L^3$ norms respectively. Next, observe that observe that $||\phi||_2^2 = ||\phi^2||_p$ for $L^2^p$ and $L^p$ norms, respectively so

$$\int_{\Sigma} D_a \phi D^a \phi \, dV - ||V_-||_2 \phi^2 ||_6 = \int_{\Sigma} D_a \phi D^a \phi \, dV - ||V_-||_2 \phi^2 ||_3 \leq \int_{\Sigma} (D_a \phi D^a \phi + V \phi^2) \, dV$$

Lemma 5 implies that

$$0 < \frac{1}{K^2} - ||V_-||_2 \phi^2 \phi ||_6 \leq ||\phi||^2_2 .$$

Therefore, $|| \ ||_L$ is a norm and its completion, $H_L$, is contained in $L^6(\Sigma)$. Furthermore, $H_L$ is a Hilbert space because $||\phi||^2_2 = (\phi, \phi)_L$ where

$$(\phi, \psi)_L \equiv \int_{\Sigma} (D_a \phi D^a \psi + V \phi \psi) \, dV .$$

Next, let $F(\psi) \equiv \int f \psi \, dV$; then Hölder’s inequality implies

$$|F(\psi)| \leq \int |f \phi| \, dV \leq ||f||_6 ||\phi||_6 \leq C||\psi||_L$$

where $C$ is a constant. Hence $F$ is a bounded linear functional on $H_L$. The Riesz representation theorem implies there exists a unique $\phi \in H_L$ such that $(\psi, \phi)_L = F(\psi)$ for all $\psi \in H_L$. Therefore, $L\phi = f$ weakly.
Finally, \( \phi \in L^6 \) which means \( \phi \) is locally in \( L^6 \). Since \( L^6_{\text{loc}}(\Sigma) \subset L^2_{\text{loc}}(\Sigma) \), it follows that \( \phi \in L^2_{\text{loc}}(\Sigma) \). This means the regularity argument of lemma 3 applies locally. Hence, \( \phi \) is locally smooth on \( \Sigma \) and therefore smooth everywhere. Furthermore, using the fall off conditions on \( V \) and \( f \), one can show \( \phi \) has \( \frac{1}{r} \) fall off.

The compactification theorem will now be proven using a gluing procedure. The technique involves deforming the metric to a flat metric in the asymptotic regions, and regulating the norm of the scalar curvature so that the operator \( L = -8D^2 + R \) remains positive. Then, the asymptotic regions are compactified by smoothly gluing in a ball onto each of the regions. Next, positivity of the operator \( L \) is used to construct a Green’s function of the operator \( L \) on the compactified manifold. Finally, the properties of the Green’s function are used to find a smooth metric on the compactification with \( R > 0 \).

**Theorem 10.** Every asymptotically flat 3-manifold \( \Sigma \) which admits a metric with nonnegative scalar curvature has a smooth compactification \( \bar{\Sigma} \) which admits a metric with positive scalar curvature.

**Proof.** Let \( g_{ab} \) be a metric on \( \Sigma \) with \( R \geq 0 \). Now, define a one parameter family of metrics \( \bar{g}_{ab}(t) = \alpha_t g_{ab} + (1 - \alpha_t) \delta_{ab} \) where \( \alpha_t \) is a smooth function on \( \Sigma \) which is equal to one in the interior region of \( \Sigma \); and on the asymptotic regions has the behavior \( \alpha = 1 \) for \( r \leq t \), \( \alpha = 0 \) for \( r \geq 2t \), \(|\alpha'| \leq \frac{C}{r^2} \), and \(|\alpha''| \leq \frac{C}{r^3} \) where \( C \) is a constant independent of \( t \). The existence of \( \alpha \) follows from lemma 2. Using the fact that \( g_{ab} \) is asymptotically flat and the behavior of \( \alpha \), it follows that \( \bar{g}_{ab}(t) \) is a one parameter family of asymptotically flat metrics. The goal is to find a value of \( t \) for which the corresponding metric is conformal to one which has \( R = 0 \) and then use the conformal factor as a Green’s function on the compactified manifold. In general, one can never have \( R \geq 0 \) for any metric in the family unless \( \Sigma = \mathbb{R}^3 \) and \( g_{ab} \) is flat.

In order to find the conformal factor, the procedure is to pick \( t \) large enough, and therefore make \( R_- \) small enough, so that lemma 6 applies with \( V = \frac{1}{2} R \) and \( f = -V \). In order to prove the bound of lemma 6 recall that the family of metrics has the form \( \bar{g}_{ab}(t) = \delta_{ab} + \alpha_t h_{ab} \) in the asymptotic regions, where \( h_{ab} = O(\frac{1}{t}) \). Further, there is a constant \( A \) such that \(|\alpha_t h_{ab}| \leq \frac{A}{r^3} \), \(|\partial_r (\alpha_t h_{ab})| \leq \frac{A}{r^2} \), and \(|\partial_r \partial_r (\alpha_t h_{ab})| \leq \frac{A}{r} \) for all large \( t \). These results combined with the additional observation that \( R_- = 0 \) unless \( t \leq r < 2t \) imply that there are two constants \( A \) and \( B \) both independent of \( t \) such that

\[
\left( \int_{\Sigma} |R_-|^\frac{3}{2} \, dv \right)^{\frac{2}{3}} \leq \left( \int_{\Sigma} |R|^\frac{3}{2} \, dv \right)^{\frac{2}{3}} \leq A \left( \int_{t}^{2t} \left( \frac{1}{r^3} \right. \right. \left. \left. r^2 dr \right) \right)^{\frac{2}{3}} \leq B \frac{1}{t}.
\]

Now, fix a \( t \) large enough so that \( \frac{B^2 A}{t} < 8 \); denote this metric by \( \bar{g}_{ab} \). For the metric \( \bar{g}_{ab} \), lemma 6 can be applied and it follows that there is a smooth \( \psi = O(\frac{1}{t}) \). Let \( \bar{G} \equiv 1 + \psi \), then \( \bar{G} \) satisfies

\[
-8D^2 \bar{G} + \bar{R} \bar{G} = 0
\]

on \( \Sigma \) with respect to \( \bar{g}_{ab} \). Next, it will be shown that \( \bar{G} \) is positive.

Define a family of operators \( L_\lambda = -8D^2 + \lambda \bar{R} \) for \( 0 \leq \lambda \leq 1 \). Lemma 4 still applies and it follows that a family of solutions \( \bar{G}_\lambda = 1 + \psi_\lambda \) exists. Let \( m_\lambda \) denote the minimum of \( \bar{G}_\lambda \); then \( m_\lambda \) is a continuous function of \( \lambda \). Suppose that for some \( \lambda \), \( m_\lambda \leq 0 \); then continuity implies there is some smaller value of \( \lambda \) for which \( m_\lambda = 0 \). The maximum principle then implies \( \bar{G}_\lambda = 0 \) for all \( x \in \Sigma \). This contradicts the fact that \( \bar{G}_\lambda \rightarrow 1 \) as \( r \rightarrow \infty \). Therefore, \( \bar{G}_\lambda > 0 \) for \( 0 \leq \lambda \leq 1 \). In particular, it holds for \( \lambda = 1 \).

The manifold \( \Sigma \) will now be compactified. Recall in the asymptotic regions \( \bar{g}_{ab} = \delta_{ab} + \alpha_t h_{ab} \) where \( \alpha \) is as above, in particular \( \alpha = 0 \) for \( r > 2t \). Now, define a new metric \( \tilde{g}_{ab} \equiv \phi^2 \bar{g}_{ab} \) where \( \phi \equiv \gamma + \frac{(1-\gamma)}{r} \) and \( \gamma \) is a smooth decreasing function which is one for \( r < 3t \) and zero for \( r > 4t \). Given \( r > 4t \) the metric becomes \( \tilde{g}_{ab} = \frac{1}{r^3} \delta_{ab} \). Because of the form of this metric, the point at infinity \( i_k \) on each asymptotic region can be smoothly added. Hence \( \Sigma = \Sigma \cup \{ i_1, i_2, \ldots, i_n \} \) is smooth and \( \tilde{g}_{ab} \) is a smooth metric on \( \Sigma \).

Next, a positive Green’s function is constructed for the operator \( L = -8D^2 + \tilde{R} \) with respect to the metric \( \tilde{g}_{ab} \). Let \( \bar{G} > 0 \) be the conformal factor found above for the metric \( \bar{g}_{ab} \) on \( \Sigma \). The function \( \bar{G} \) satisfies the equation

\[
-8D^2 \bar{G} + \bar{R} \bar{G} = 0
\]

\[ A \text{ function is locally in } L^p(\Sigma) \text{ if it is in } L^p(K) \text{ for all compact sets, } K, \text{ contained in } \Sigma. \text{ The space of all such functions is denoted by } L^p_{\text{loc}}(\Sigma) \text{ and the topology is given by requiring sequences to converge in } L^p(K) \text{ for all } K. \]
on $\Sigma$ with respect to the metric $\tilde{g}_{ab}$. It follows that $\tilde{G} \equiv \phi^{-1}\vec{G}$ satisfies the equation $-\Delta \tilde{G} + \vec{R} \tilde{G} = 0$, on $\Sigma - \{i_1, i_2, \ldots, i_n\}$ with respect to the metric $\tilde{g}_{ab}$. Now, the goal is to prove that $\tilde{G}$ is the Green’s function of the operator $L = -8\tilde{D}^2 + \tilde{R}$ on $\Sigma$ with respect to the metric $\tilde{g}_{ab}$. For $r > 2\tilde{t}$, it follows that

$$\tilde{G}_k = 1 + \frac{M_k}{2\tilde{t}} + \sum_{l=1}^{\infty} \sum_{m=-l}^{l} \frac{A_{lm}(k)Y_{lm}(\theta, \phi)}{r^{l+1}}$$

where $k$ denotes which asymptotic region, the $Y_{lm}$’s are spherical harmonics, $M_k$ is the mass measured at $i_k$, and the $A_{lm}(k)$’s are higher moments measured at $i_k$. Thus for $r > 4\tilde{t}$,

$$\tilde{G}_k = \frac{1}{\tilde{r}} + \frac{M_k}{2} + \sum_{l=1}^{\infty} \sum_{m=-l}^{l} A_{lm}(k)Y_{lm}(\theta, \phi) \tilde{r}^l$$

where $\tilde{r} \equiv \frac{1}{\tilde{r}}$. Since the point at infinity $i_k$ for each asymptotic region has radial coordinate $\tilde{r} = 0$, it follows that $L\tilde{G}_k = 8\delta_{ik}$ on the compactification of each asymptotic region. It is important to note that the delta function at the point $i_k$ is denoted by $\delta_{ik}$ and it is linear functional on smooth test functions of the compactified manifold. It has the well known feature that $(\delta_{ik}, \phi) = c\phi(i_k)$, for any test function $\phi$ where $c$ is a positive constant determined by the choice of normalization of the delta function.

Hence, $LG = 8\sum_{k=1}^{n} \delta_{ik}$ on $\Sigma$. Furthermore, $\tilde{G} > 0$ because both $\phi$ and $G$ are positive. Therefore, $\tilde{G}$ is a positive Green’s function for $L$ on $\Sigma$.

Finally, the above Green’s function is used to construct a metric with positive scalar curvature on $\Sigma$. Let $\psi_0$ denote the ground state and $\lambda_0$ the corresponding eigenvalue of the operator $L$. By lemma 3, a smooth positive ground state $\psi_0$ always exits. Now, using the fact that $\tilde{G}$ is the Green’s function, it follows that

$$8 \sum_{k=1}^{n} \psi_0(i_k) = \sum_{k=1}^{n} \frac{1}{c} (\delta_{ik}, \psi_0) = \frac{1}{c} (L\tilde{G}, \psi_0) = \frac{1}{c} (\tilde{G}, L\psi_0) = \frac{\lambda_0}{c} (\tilde{G}, \psi_0) .$$

This combined with the positivity of both $\psi_0$ and $\tilde{G}$ imply $\lambda_0 > 0$. Finally, define a new smooth metric on $\Sigma$ by $\hat{g}_{ab} = \psi_0^4 \tilde{g}_{ab}$. The scalar curvature of $g_{ab}$ is

$$\hat{R} = \psi_0^{-5}(-8\tilde{D}a\tilde{D}a\psi_0 + \tilde{R}\psi_0) = \lambda_0\psi_0^{-4} > 0 .$$

Therefore, $\tilde{\Sigma}$ has a metric with positive scalar curvature.

Conversely, given any closed 3-manifold with positive scalar curvature one can remove points to obtain an asymptotically flat 3-manifold with metric having $R = 0$. The converse of the compactification was first suggested by R. Geroch [24]. Combining the above compactification theorem with Geroch’s result yields the following corollary:

**Corollary 1.** An asymptotically flat 3-manifold $\Sigma$ has a metric with $R = 0$ if and only if it has a compactification $\tilde{\Sigma}$ which has a metric with $R > 0$.

**Proof.** The compactification is an immediate consequence of theorem [10] Given a closed 3-manifold $\tilde{\Sigma}$ with metric having $\tilde{R} > 0$. Let $L$ be the operator $-8\tilde{D}^2 + \tilde{R}$. Take any finite set of points $\{i_1, i_2, \ldots, i_n\}$ in $\tilde{\Sigma}$ and solve the equation $L\phi = \delta$ where $\delta = 8\sum_{k=1}^{n} \delta_{ik}$. Since $\tilde{R} > 0$, $\lambda_0 > 0$ and lemma 4 implies $L$ is an isomorphism on $C^\infty(\tilde{\Sigma})$. Let $\{f_n\}$ be a sequence of smooth functions which converge to $\delta$ in the sense of distributions. Since $L$ is an isomorphism on smooth functions, there is a sequence of smooth functions $\{\phi_n\}$ such that $L\phi_n = \delta$. It follows that

$$\langle \phi_n - \phi_m, L\psi \rangle = \langle L(\phi_n - \phi_m), \psi \rangle = \langle f_n - f_m, \psi \rangle$$

for all smooth $\psi$. Furthermore, the right-hand side goes to zero as $m, n \to \infty$ because $\{f_n\}$ converges. This implies $\langle \phi_n - \phi_m, \hat{\psi} \rangle \to 0$ for all smooth functions $\hat{\psi}$ because $L$ is an isomorphism. Hence $\{\phi_n\}$ is a Cauchy sequence in the space of distributions. Moreover, completeness of the space of distributions implies $\phi_n \to \phi$. Therefore, $L\phi = \delta$.

Since $L$ has smooth coefficients and $L\phi = 0$ on $\Sigma - \{i_1, i_2, \ldots, i_n\}$, a standard bootstrap argument implies $\phi$ must also be smooth on $\Sigma - \{i_1, i_2, \ldots, i_n\}$. Using normal coordinates about each of the points $i_k$, one can show that $\phi$ has the correct asymptotic behavior on $\Sigma - \{i_1, i_2, \ldots, i_n\}$. Finally, an argument using the maximum principle implies that $\phi > 0$. \[\square\]
Now, the compactification theorem and the classification theorem are combined to prove that most asymptotically flat globally hyperbolic spacetimes do not admit maximal slices. This result is generic in that it only depends on the spatial topology of the spacetime.

**Theorem 11.** Most asymptotically flat globally hyperbolic spacetimes do not admit a maximal slice.

**Proof.** The existence of a maximal slice implies \( \Sigma \) admits a metric with \( R \geq 0 \). The compactification theorem implies the closed manifold \( \tilde{\Sigma} \) admits a metric with positive scalar curvature. This combined with theorem 7 implies most \( \Sigma \) never admit a metric with \( R \geq 0 \). Theorem 6 implies all \( \Sigma \) are allowed as the spatial topology of spacetimes. Therefore, most asymptotically flat globally hyperbolic spacetimes never admit a maximal slice.

A more explicit description of the spatial topology of globally hyperbolic spacetimes that do admit a maximal slice can be given by decomposing the orientable closed 3-manifolds in terms of three basic types of prime orientable factors. They are closed 3-manifolds with finite fundamental group, the \( K(\pi, 1) \)’s, and the handle \( S^2 \times S^1 \). By theorem 7, the only prime factors which can possibly admit a metric with \( R > 0 \) are the ones with finite fundamental group and the handle. The handle does have a metric with positive scalar curvature, one such metric is the product metric. The prime 3-manifolds with finite fundamental group are the spherical spaces. Furthermore, the only orientable prime factors admitting metrics with \( R > 0 \) are these spherical spaces and the handle. If \( M^3 \) admits a metric with \( R \geq 0 \), then theorem 7 implies \( M^3 \) either has a metric with positive scalar curvature or it is flat but there are only six orientable closed flat spaces. Thus, the form of any closed orientable 3-manifold with \( R \geq 0 \) is completely known if these conjectures are assumed. This means that, given any spatially closed globally hyperbolic spacetime which admits a maximal slice the spatial topology is known, more precisely it is the connected sum of spherical spaces and handles, or it is one of six flat spaces. Of course, if the globally hyperbolic spacetime is spatially nonorientable, then the double cover is of the above form. In the asymptotically flat case, these results imply that the spatial topology is the connected sum of spherical spaces and handles minus a finite number of points.

**IV. CONCLUSIONS**

Although the generic situation is that asymptotically flat globally hyperbolic spacetimes do not admit maximal slices, the question still remains what are both necessary and sufficient conditions for the maximal slices to exist. It is important to realize that just because there are no topological obstructions to maximal slices does not mean that maximal slices exist. This is best illustrated with the following example: Let the spatial topology be that of a 3-torus \( T^3 \). If one takes \( \mathbb{R} \times T^3 \) with the metric \( ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \), i.e. Minkowski space with points identified, then the \( t = \) constant slices yield a foliation by maximal slices. On the other hand, if one takes the same topology but with a different metric, namely, \( ds^2 = -dt^2 + C t^2 (dx^2 + dy^2 + dz^2) \), where \( C > 0 \) is a constant, i.e. a dust filled Robertson-Walker spacetime with points identified, then this spacetime has no maximal slice because \( T^3 \) expands as \( t \) increases. This is an example of a spatially closed spacetime but with a little bit of work it is conceivable that such examples can be constructed in the asymptotically flat case also.

 Likewise, it has been shown that CMC hypersurfaces which are not maximal can exist regardless of the spatial topology but this does not assure that they always exist. This only means that there are no topological obstructions to finding such hypersurfaces. In fact, examples of spatially closed vacuum spacetimes which do not have any CMC hypersurface given by 25, 26.

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