Tensionless Strings, WZW Models at Critical Level
and
Massless Higher Spin Fields

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ABSTRACT

We discuss the notion of tensionless limit in quantum bosonic string theory, especially in flat Minkowski space, noncompact group manifolds (e.g., $SL(2, R)$) and coset manifolds (e.g., $AdS_d$). We show that in curved space typically there exists a critical value of the tension which is related to the critical value of the level of the corresponding affine algebra. We argue that at the critical level the string theory becomes tensionless and that there exists a huge new symmetry of the theory. We discuss the appearance of the higher spin massless states at the critical level.
1 Introduction

The aim of this paper is to initiate the discussion of the tensionless limit (i.e., $\alpha' \to \infty$ or $T = (2\pi\alpha') \to 0$) in the quantized string theory. The tensionless limit of string theory should give us an idea about a short-distance properties of the theory. Naively in this limit all particles will have vanishing mass and therefore new symmetries should appear. This has previously been shown to be the case for the classical tensionless string [1], and its quantized version in [2], [3], [4] in a flat background.

However a priori it is an open question if this limit gives rise to a consistent theory. In this letter we would like to argue that there is such limit for some target spaces and that the theory would have new symmetries associated with higher spin massless particles.

Unlike the previous studies of the limit cited above, we would like to consider the tensionless limit directly in the quantum string theory. Since the nature of the limit is highly quantum, in the path integral small tension corresponds to large $\hbar$, this is a natural thing to do. We shall be interested in the general case when the target space is a curved manifold. Our main examples will be group and coset manifolds.

An important incentive for the present study comes from recent discussions of the AdS/CFT correspondence at vanishing Yang-Mills coupling constant [6]. On the string side this correspondence requires taking the tensionless limit.

The paper is organized as follows. In Section 2 we consider the tensionless limit in flat Minkowski space. We discuss the limit at the level of Hilbert space and Virasoro constraints. We show that in the tensionless limit the Virasoro constrains give rise to Fronsdal’s conditions for free massless high spin fields. However it is highly probable that higher spin massless interactions cannot be constructed in flat space and hence that the tensionless limit is inconsistent in flat space as an interacting theory. In Section 3 we turn to the discussion of string theory on noncompact group manifolds. In this case the level $k$ of the corresponding affine algebra can be identified with a dimensionless analog of the string tension ($k = 2\pi T R^2$ where $R$ is the size of a group manifold). For the theory to be unitary the level is typically restricted to $-h^V < k < \infty$ where $h^V$ is the dual Coxeter number. We argue that the tensionless limit corresponds to taking the level to a critical value (i.e., $k = -h^V$) where the number of zero-norm states increases dramatically and thus indicate the appearance of new huge gauge symmetry. Finally in Section 4 we discuss the tensionless limit for coset manifolds, in particular, we consider $AdS_d$ space and we show how the free massless high spin fields may arise in the limit. A summary of our results and comments regarding the future directions of investigation are collected in Section 5.
In this letter we consider the bosonic string and ignore the questions of consistency of the theory. However we believe that similar results will hold for suprestrings.

2 Tensionless strings in flat space

In this section we consider the tensionless limit for the bosonic string in flat Minkowski space. Despite the fact that the subject has been around for 15 years, we think that some points have been overlooked. Besides the flat space example serves as a good starting point for a discussion of string theory on curved manifolds.

Let start from the standard bosonic string action in conformal gauge living in flat space

\[ S = \frac{1}{4\pi\alpha'} \int d^2\sigma \eta_{\mu\nu} \partial_\alpha X^\mu \partial_\alpha X^\nu, \]  

where \( \eta_{\mu\nu} = \text{diag}(-1,1,...,1) \). The parameter in front of the action is the string tension \( T = (2\pi\alpha')^{-1} \). In this section we discuss some aspect of string theory when the tension \( T \) is taken to zero, i.e. tensionless strings [1],[5]. This can be done in different ways and it has been discussed extensively in the literature. For example, one can take the limit at the level of classical action (2.1) [7],[8] and then quantize it [2] [3], [4]. Another approach is to consider the limit at the level of scattering amplitudes [9]-[14].

However here we discuss the tensionless limit in the free quantum theory, at the level of the Hilbert space. For the present discussion it is enough to work within the old covariant quantization program (for review, see [15]). For the sake of simplicity we consider the open string. However the whole discussion can be straightforwardly generalized to closed strings. The field \( X^\mu \) is expanded in modes which obey the commutation relations

\[ [a^\mu_n, a^\nu_m] = \eta^{\mu\nu} \delta_{n+m}, \quad [q^\mu, p_\nu] = i \delta^\mu_\nu \]  

where \( (a^\mu_n)^\dagger = a^\mu_{-n} \). The Fock space is built by the actions of \( a^\mu_n \) with \( n > 0 \) on the vacuum \( |0, k\rangle \) such that \( p_\mu |0, k\rangle = k_\mu |0, k\rangle \). Physical states are those that satisfy the Virasoro constraints

\[ (L_0 - 1)|\text{phys}\rangle = 0, \quad L_n|\text{phys}\rangle = 0, \quad n > 0 \]  

where the Hamiltonian \( L_0 \) is

\[ L_0 = \frac{1}{2} \alpha' p_\mu \eta^{\mu\nu} p_\nu + \sum_{n \neq 0} n(a^\mu_n)^\dagger \eta_{\mu\nu} a^\nu_n \]  

and the \( L_n \)'s are

\[ L_n = \sqrt{\alpha'} p_\mu a^\mu_n + \sum_{m=1}^\infty \sqrt{m(m+n)} a^\mu_{n+m} \eta_{\mu\nu} (a^\nu_m)^\dagger + \frac{1}{2} \sum_{m=1}^{n-1} \sqrt{m(n-m)} a^\mu_m \eta_{\mu\nu} a^\nu_{n-m} \]
It is important that our expressions are properly normalized. The momentum $p$ requires $\sqrt{\alpha'}$ and the creation and annihilation operators are taken dimensionless. Indeed Gross and Mende have used the same normalization in their study of high-energy string scattering [9], [11], [10].

Next we take the tensionless limit (i.e., $\alpha' \to \infty$) at the level of Virasoro constraints. The tensionless limit will result in modifications of the constraints (2.3), namely

\[
(p_\mu \eta^{\mu\nu} p_\nu) |\text{phys}\rangle = 0, \quad (p_\mu a^\mu_n) |\text{phys}\rangle = 0, \quad n > 0. \tag{2.6}
\]

These constraints, $l_0 = p_\mu \eta^{\mu\nu} p_\nu$ and $l_n = p_\mu a^\mu_n$ for $n > 0$ together with $l_{-n} = l^\dagger_n$, do not generate the Virasoro algebra, instead the corresponding algebra is

\[
[l_n, l_m] = \delta_{n+m} l_0, \quad [l_n, l_0] = 0, \quad n \neq 0, \quad m \neq 0 \tag{2.7}
\]

which is the Heisenberg algebra with $l_0$ being the central element. Now we can analyze the string spectrum using the new conditions on the physical spectrum. Following the standard prescription we construct the Fock space using the creation operators, the negative norm states are supposed to be projected out by the new conditions (2.6) and the physical states will be organized according to massless representations of the Poincaré group. Indeed for some of the states these mass-shell and transversality conditions (2.6) give us the Fronsdal’s massless free higher spin fields (in the specific on-shell gauge)\(^3\). To illustrate this point we consider as an example the sector build from $a^\mu_{-1}$. The Poincaré irreducible representation of spin $s$ corresponds to

\[
|\phi\rangle = \epsilon_{\mu_1...\mu_s}(k) a^\mu_{-1}...a^\mu_{s-1} |0, k\rangle, \tag{2.8}
\]

where $\epsilon_{\mu_1...\mu_s}(k)$ is a symmetric and traceless field (i.e., $\eta^{\mu_1\mu_2} \epsilon_{\mu_1\mu_2...\mu_s} = 0$) and therefore the representations of the corresponding flat space little group $O(d - 2)$ are characterized by Young tableaux with one row. The conditions (2.6) ensure that we are working with free massless higher spin fields,

\[
k^{\mu} \eta_{\mu\nu} k^{\nu} = 0, \quad k^{\mu_1} \epsilon_{\mu_1...\mu_s}(k) = 0. \tag{2.9}
\]

The second condition in (2.9) should be interpreted in same way as it done in QED when by imposing condition $\partial_\mu A^\mu = 0$ on the Fock space one kills unwanted states. The gauge transformations amount to a shift of the state by a null state (a physical state which is orthogonal to all physical states and therefore of zero norm):

\[
|\phi\rangle \longrightarrow |\phi\rangle + k_{\mu_1} \gamma_{\mu_2...\mu_s} a^\mu_{-1}...a^\mu_{s-1} |0, k\rangle. \tag{2.10}
\]

\(^3\)Previously the realization of the higher spin symmetries in free string field theory has been discussed in [16].
Here \( k^{\mu_2 \gamma_{\mu_2 ... \mu_s}} = 0 \) and \( \gamma \) is completely symmetric tensor by construction. Obviously in (2.10) the shifted state has zero norm on shell, where \( k^2 = 0 \). Alternatively we may rewrite the transformation (2.10) as follows

\[
|\phi \rangle \longrightarrow |\phi \rangle + l_{-1} \gamma_{\mu_2 ... \mu_s} a_{a_{-1} ... a_{-1}} |0, k \rangle.
\] (2.11)

In a similar fashion the other states in the Fock space may be analyzed. The important new property is that the number of null states is huge. For example, all states of the form

\[
l_{n_1} l_{n_2} ... l_{n_p} |0, k \rangle, \quad n_i > 0, \quad i = 1, ..., p
\] (2.12)

are null states. Thus we witness the appearance of a new large symmetry which corresponds to the gauge symmetries of massless higher spin free fields. A similar conclusions regarding the appearance of new symmetries have been made by Gross [10] in studying high-energy string scattering.

In this naive tensionless limit we see that there are massless free high spin fields in the spectrum. However we know that there is no consistent interacting theory for these fields in flat Minkowski space.\(^4\) Therefore we conclude that the present tensionless limit does not produce a consistent (non-free) theory. We should keep in mind, however, that in drawing the conclusion that no interacting theory exists, use is generally made of the Coleman-Mandula theorem [19], which in turn is proven under the assumption of a finite number of different particles [10].

### 3 Tensionless strings on group manifolds

From this section onwards, we discuss the notion of a tensionless limit in the setting of curved space. We try to repeat the idea from the previous section in that we first construct the tensionfull quantum theory and then only at the quantum level take the tensionless limit.

To begin with, we consider the tensionless limit of strings on group manifolds. The main example we have in mind is \( SL(2, R) \), however most of the discussion goes through for other noncompact groups.

Let us consider the sigma model (i.e., the gauge fixed string action) over a group manifold \( G \) with a Lie algebra \( g \)

\[
S = \frac{1}{4\pi \alpha'} \int d^2 \sigma \ (g_{\mu \nu} + B_{\mu \nu}) \partial_+ X^\mu \partial_- X^\nu,
\] (3.13)

\(^4\)Interacting higher spin theories typically require a non-zero cosmological constant, and have been extensively studied, starting in [17]. For recent progress, see [18] and references therein.
where \( X^\mu \) is coordinate on the target space (i.e., dimensionful field). With \( R \) a dimensionful parameter which characterizes the size of the manifold, we can rescale \( \phi^\mu = RX^\mu \) such that \( \phi \) is dimensionless. Using this dimensionless field \( \phi^\mu \) we rewrite the action in terms of group elements \( g \)

\[
S = \frac{k}{4\pi} \int d^2 \sigma \text{Tr}(g^{-1}\partial_\alpha gg^{-1}\partial^\alpha g) + \frac{k}{12\pi} \int d^3 \sigma \epsilon^{\alpha\beta\gamma} \text{Tr}(g^{-1}\partial_\alpha gg^{-1}\partial_\beta gg^{-1}\partial_\gamma g)
\]  

(3.14)

where \( k = R^2/\alpha' \) is the level. Thus for a group manifold \( k \) is a dimensionless analog of the string tension [20] and therefore, classically the tensionless limit amounts to taking the level \( k \) to 0. However, we believe that this is not in general an allowed limit at the quantum level.

For compact groups the level \( k \) is quantized and should be positive. Thus we cannot take it continuously to zero, and the smallest possible positive level in the theory does not have any special properties. Therefore we conclude that there are no meaningful tensionless limits for compact group manifold. This should not come as a surprise since massless particles on a compact manifold are problematic.

However if the group is noncompact then typically the level \( k \) is not quantized. In this case only the positivity of the central charge restricts the allowed values of \( k \). To understand these restrictions, let us spell out the steps in the Sugawara construction. For the WZW model the affine symmetry is given by the Kac-Moody algebra

\[
[J^A_n, J^B_m] = i f^{AB}_C J^C_{n+m} + k \eta^{AB} n \delta_{n+m}
\]

(3.15)

where \( f^{AB}_C \) are the structure constants of \( g \). We define the Sugawara operators as follows

\[
l_n = \frac{1}{2} \sum_{m=-\infty}^{+\infty} : J^A_m \eta^{AB} J^B_{n-m} :
\]

(3.16)

where dots denote normal ordering. They satisfy the following commutation relations

\[
[J^A_n, J^A_m] = -(k + h^V) m J^A_{n+m},
\]

(3.17)

\[
[l_n, J^A_m] = (k + h^V) \left( (n-m) l_{n+m} + \frac{k \dim g}{12} (n^3 - n) \delta_{n+m} \right),
\]

(3.18)

where \( h^V \) is the dual Coxeter number. If the \((k + h^V) \neq 0\) we define the Virasoro operators \( L_n \) by normalizing \( l_n \) as follows \( L_n \equiv (k + h^V)^{-1} l_n \). The \( L_n \)s obeys the standard Virasoro algebra

\[
[L_n, L_m] = (n-m) L_{n+m} + \frac{k \dim g}{12(k + h^V)} (n^3 - n) \delta_{n+m}.
\]

(3.19)

As a result, the world-sheet (properly normalized) Hamiltonian has the form

\[
L_0 = \frac{1}{2(k + h^V)} \sum_{m=-\infty}^{+\infty} : J^A_m \eta^{AB} J^B_{-m} :.
\]

(3.20)
Thus we conclude that the quantum tension is \((k + h^V)\) rather than \(k\). Note that unitarity (i.e., the positivity of the central charge) puts a bound on the level, \(k\): \(-h^V < k < \infty\) and that we normalize our objects such that \(k\) is positive.

For example, for \(SL(2, R)\) the unitarity bound is: \(2 < k < \infty\). Therefore \(k\) cannot be taken to zero in the quantum theory. However, we suggest that the limit \(k \to 2\) represents the tensionless limits in the quantum theory. The central charge of the model is

\[
    c = \frac{3k}{k - 2}
\]

and thus to embed the model into the critical bosonic string theory the following bound should be satisfied: \(c \leq 26\) (i.e., \(k \geq 13/2\)). Therefore one may think that the limit \(k \to 2\) cannot be done within the critical string theory. However it may happen that in the neighborhood of the point \(k = -h^V\) the theory should be redefined and the Virasoro algebra is not relevant anymore. Having this in mind we will ignore this problem in what follows.

When the level \(k\) is equal to \(-h^V\) (i.e., for \(SL(2, R)\) when \(k = 2\)) it is called the critical level. In mathematics WZW models at critical level have attracted a lot of attention and the representation theory of corresponding affine algebra has been considered in [24]. However in the present context we are interested in the possibility to interpret a noncompact WZW model at critical level as an (unconventional) string theory, possibly with a big new symmetry.

Let us state some relevant properties of WZW models at critical level. At the critical level we cannot introduce \(L_n\)s which obey the Virasoro algebra. However, there are still Sugawara operators \(l_n\) which commute with each other and with \(J_n^A\)

\[
    [l_n, J_m^A] = 0, \quad [l_n, l_m] = 0.
\]

Thus, at the critical level there is a large number of new null states, e.g., all states of the form

\[
    l_{n_1} l_{n_2} \ldots l_{n_p} |0, \alpha\rangle, \quad n_i > 0, \quad i = 1, \ldots, p
\]

are null states. In (3.23) \(|0, \alpha\rangle\) is a state with the property that \(J_{-n}^A |0, \alpha\rangle = 0\) for \(n > 0\) and \(\alpha\) is a label for a finite dimensional representation of \(g\), \(J_0^A |0, \alpha\rangle = \alpha J_0^A |0, \alpha\rangle\). Compared to the noncritical level, the number of zero-norm states increases dramatically when \(k = -h^V\) thus indicating the appearance of the gauge symmetry of the space-time theory we are seeking.

Another important question is what would happen with the Virasoro constraints in this limit (i.e., \(k \to -h^V\)). Naively the constraints will collapse to the following ones

\[
    l_n |\text{phys}\rangle = 0, \quad n \geq 0.
\]

Using the properties (3.22) the conditions (3.24) becomes just the single condition

\[
    C_2 |\text{phys}\rangle = 0,
\]

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where $C_2$ is a quadratic Casimir. However, this reasoning may be too naive. Let us look at the subset consisting of the following states

$$|\epsilon\rangle = \epsilon_{A_1 A_2 \ldots A_n} J_{-1}^{A_1} J_{-1}^{A_2} \ldots J_{-1}^{A_n} |0, \alpha\rangle$$

(3.26)

where $\epsilon_{A_1 A_2 \ldots A_n}$ is completely symmetric tensor. Away from critical level the Virasoro conditions imply that

$$C_2(\alpha) = (k + h^V)(n - 1), \quad \epsilon_{A_1 A_2 \ldots A_n} \alpha^{A_1} = 0,$$

(3.27)

where $C_2(\alpha)$ is a quadratic Casimir for the representation $\alpha$. When the level $k$ goes to the critical value $-h^V$ the states (3.26) become massless (i.e., $C_2(\alpha) = 0$), but the transversality condition remains true. Thus at the critical level we reproduce the analogue of the Fronsdal’s conditions in the fixed gauge. However at the critical level the trasversality condition does not arise from $l_n$, which it does in the flat case.

We now formalize the tensionless limit may somewhat. In flat space we scaled the zero and non-zero modes differently with respect to $\alpha'$. Thus the flat space tensionless limit can be formulated as follows: We introduce a parameter $R$ and rescale the parameters of the theory as $\eta_{\mu\nu} = R^{-1}\tilde{\eta}_{\mu\nu}$, $a^\mu_n = \sqrt{R}\tilde{a}^\mu_n$ and $p^\mu = \tilde{p}^\mu$ (i.e., we do not scale the contravariant zero mode). The limit $R \to \infty$ gives rise to the tensionless limit and it does not change the underlying Heisenberg algebra. However the Virasoro algebra ($L_0 = Rl_0$ and $L_n = \sqrt{R}l_n$, $n \neq 0$) gets contracted to the algebra (2.7).

Let us now turn to the affine algebra (3.15) and try to apply the same logic\(^5\). We have to rescale the zero and the nonzero modes as well as the metric $\eta_{AB}$ in some way (and, as a result, we also have to scale the structure constants since $f_{CD}^{AB} f_{BD}^{AC} \sim \eta^{AB}$). There is a scaling which would preserve the affine algebra: $\eta_{AB} = R^{-1}\tilde{\eta}_{AB}$, $f_{CD}^{AB} = \sqrt{R}\tilde{f}_{CD}^{AB}$, $J_n^A = \sqrt{R}\tilde{J}_n^A$ and $J_0^A = \sqrt{R}\tilde{J}_0^A$. However, this scaling does nothing with the Virasoro generators and it does not lead to anything new.

Next we can try to mimic the flat space case by scaling the zero and non-zero modes differently. In particular we can keep fixed the contravariant zero mode $J_{0A} = \tilde{J}_{0A}$ ($J_0^A = R\tilde{J}_0^A$) and scale the rest as before. We then obtain the following conditions on the physical states in the limit $R \to \infty$

$$ (\tilde{J}_0^A \tilde{\eta}_{AB} \tilde{J}_n^B)|\text{phys}\rangle = 0, \quad n \geq 0.$$

(3.28)

However the algebra (3.15) does not have a well defined limit in this case. We may continue and study other scalings of the affine algebra with well-defined limit. Typically the limit will lead to the contraction of the affine algebra and thus it will change the model drastically.

\(^5\)We thank Ergin Sezgin for a valuable discussion of this issue.
Although we were able to reproduce the Fronsdal’s conditions at the critical level we could not derive the “Virasoro” conditions responsible for the transversality condition (3.27). Indeed it is tempting to treat \((k + h^V)^{-1}\) as we treated \(\alpha'\) in the flat case. However for the group manifold we cannot perform the limit as in the flat space example considered in previous section. More physical intuition is needed to find the right prescription for the limit. In the next section we consider a more physical example, where we know what to expect from the tensionless string spectrum.

4 Tensionless strings in \(AdS_d\)

In this section we consider string theory over coset manifolds based on noncompact groups. The logic to a large extent follows that in the previous section. First we discuss \(AdS_d\) as a specific example.

Following the work of Fradkin-Linetetsky, [21], [22] we represent \(AdS\)-space as a coset symmetric space of the form

\[
AdS_d = \frac{SO(d-1,2)}{SO(d-1,1)},
\]

where \(SO(d-1,2)\) is the anti-de Sitter group in \(d\) dimensions and \(SO(d-1,1)\) is its Lorentz subgroup. The underlying CFT may be thought of as a \(SO(d-1,2)\) WZW model with gauged subgroup \(SO(d-1,1)\). We ignore the questions of consistency of this theory and the fact that there are other proposals for the string theory in \(AdS_d\), [23]. Our intension is to give a rough idea of how things may work, and this does not rely on the particularities of the coset constructions.

The affine Kac-Moody algebra \(SO(d-1,2)\) of the \(AdS_d\) coset model is of the form

\[
[M_n^{\mu\nu}, M_m^{\rho\sigma}] = i(\eta^{\mu[\sigma|} M_{n+m}^{\nu]\rho]} + \eta^{\nu[\rho|} M_{n+m}^{\mu]\sigma]} - kn(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho})\delta_{n+m}
\]

\[
[M_n^{\rho\mu}, P_m^\rho] = i(\eta^{\rho\rho} P_{n+m}^\mu - \eta^{\mu\rho} P_{n+m}^\nu)
\]

\[
[P_n^\mu, P_m^\nu] = iM_n^{\mu\nu} + kn\eta^{\mu\nu}\delta_{n+m}
\]

where \(k = T\Lambda^{-1}\) and \((-\Lambda)\) is the cosmological constant in \(AdS_d\).

The Virasoro generators are constructed according to standard Goddard-Kent-Olive construction

\[
L_n = \frac{1}{2} \sum_{m=-\infty}^{\infty} \left( \frac{1}{k - (d-1)} : P_m^{\mu\rho} \eta_{\mu\nu} P_n^{\nu\rho} : + \left( \frac{1}{k - (d-1)} - \frac{1}{k - (d-2)} \right) : M_m^{\mu\nu} \eta_{\mu\rho} \eta_{\nu\sigma} M_n^{\rho\sigma} : \right)
\]
where we used the fact that for $SO(d - 1, 2)$ and $SO(d - 1, 1)$ the dual Coxeter numbers are $(d - 1)$ and $(d - 2)$ respectively. Thus the level is bounded by the values: $(d - 1) < k < \infty$. The limit $k \to \infty$ corresponds to the flat space limit (with the affine currents appropriately rescaled). The tensionless limit would correspond to $k \to (d - 1)$, and as before we get a dramatic increase in the number of zero-norm states (they will be constructed out of Sugawara tensors for $SO(d - 1, 2)$ with positive $n$).

For generic noncritical $k > (d - 1)$ the theory has $SO(d - 1, 1)$ global symmetry since

$$[L_0, M_{0 \mu}^\nu] = 0$$

(4.34)

where $L_0$ is the worldsheet Hamiltonian. Thus all states of the theory are organized in the representations of $SO(d - 1, 1)$. If, in analogy with the flat limit, we define the tensionless limits as the massless limit of the theory (i.e., the limit when all states become massless) then we should have an enlargement of the symmetry to the AdS-group, $SO(d - 1, 2)$. However this never happens at noncritical values of $k$. Thus if the tensionless limit exists it must correspond to the theory at the critical level $k = (d - 1)$ where

$$[l_0, M_{0 \mu}^\nu] = 0, \quad [l_0, P_{0 \mu}^\nu] = 0.$$ 

(4.35)

$l_0$ is the worldsheet Hamiltonian at the critical level, related to the noncritical as follows $(k - (d - 1))L_0 = l_0$. From this simple argument we conclude that if the tensionless limit exists then it should be at the critical level limit. At the critical level we will thus necessary have higher spin massless states, unless the theory becomes trivial. To investigate if the tensionless theory is trivial or nontrivial one should study the behavior of the string spectrum in the vicinity of the critical level. This problem seems to be hard. Indeed nothing is known about the Regge trajectories in this coset model. We hope to come back to this question in the future.

There is another important point to be addressed. The tensionless limit of string theory and the appearance of massless higher spin states in the string theory are not equivalent notions. The tensionless limit (if it exists) implies the existence of massless higher spin states. However the presence of higher spin states does mean that the theory is in tensionless phase. The possibility that massive higher spin states coexist with massless cannot be excluded. For example, in the present model of $AdS_d$ we know that there are no higher spin states in the semiclassical regime (i.e., when $k$ is big), but we have very little knowledge of what happens when $k$ moves towards the critical value. In particular, we do not know if massless higher spin states appear at some value of $k$. However if they arise at $k \neq (d - 1)$ they will mix with the massive states. The completely massless spectrum will appear only at $k = (d - 1)$.

We finally note, that in units where $\alpha' = 1$, the critical level may be interpreted as a
critical radius of $AdS_d$. According to our discussion the symmetry at this radius is greatly enhanced.

5 Summary and discussion

In this short note we have discussed the tensionless limit in the quantum string theory at the level of the Fock space and the Virasoro constraints on physical states. We found that in flat space the truncated Virasoro constraints correctly reproduce the Fronsdal’s conditions for free higher spin massless fields. We then applied the same type of procedure in curved manifolds, in particular in $AdS_d$ space.

In curved space we found that there is a critical value of string tension related to the critical value of the level of WZW model. The theory has very special properties at the critical tension where the number of null states drastically increase, indicating the appearance of a very large gauge symmetry in the underlying space-time theory.

At present it is not clear if there is a similar critical tension for superstrings.\(^6\) Presumably this depends on the properties of the background under consideration.

Another interesting question is the $AdS/CFT$ interpretation of the critical tension (assuming it exists for $AdS_5 \times S^5$). The expression $g_{YM}^2 N = (R \sqrt{T})^4$ relates the Yang-Mills coupling to the radius of $AdS_5$ (and $S^5$). The existence of critical value for $g_{YM}^2 N$ seems unlikely. Therefore if there is a critical tension then, in analogy to $k \rightarrow k + hV$, it should lead to a modification of the expression according to $g_{YM}^2 N = (R \sqrt{T} + (R \sqrt{T})_{\text{crit}})^4$. Previously the existence of critical string tension has been argued and its relation to higher spin theories discussed in [26]. Further supportive argument in favour of critical tension has been recently considered in [27].

The obvious future directions of this investigation are to extend the discussion to superstrings as well as to make more rigorous some of the qualitative arguments presented here.

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\(^6\)Superstrings do seem to have an important relation to higher spin theories, however, see, e.g., [26]
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