Discrete dynamics and non-Markovianity

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Abstract
We study discrete quantum dynamics where a single evolution step consists of unitary system transformation followed by decoherence via coupling to an environment. Often, non-Markovian memory effects are attributed to structured environments, whereas, here, we take a more general approach within a discrete setting. In addition of controlling the structure of the environment, we are interested in how local unitaries on the open system allow the appearance and control of memory effects. Our first simple qubit model where local unitary is followed by dephasing illustrates how memory effects arise, despite having no structure in the environment the system is coupled with. We, then, elaborate on this observation by constructing a model for an open quantum walk where the unitary coin and transfer operation is augmented with the dephasing of the coin. The results demonstrate that in the limit of strong dephasing within each evolution step, the combined coin-position open system always displays memory effects, and their quantities are independent of the structure of the environment. Our construction makes possible an experimentally realizable open quantum walk with photons exhibiting non-Markovian features.

Keywords: open quantum systems, non-Markovian processes, quantum walk

(Some figures may appear in colour only in the online journal)

1. Introduction
In recent years, there has been a growing interest on defining and quantifying quantum memory effects in open system dynamics [1–11]. These are based on a number of different approaches, ranging, e.g., from the concepts of information flow [2] to nondivisibility [3, 8] and mutual information [5]. Based on these theoretical developments, both theoretical [12–14] and experimental realizations for detecting and controlling non-Markovianity has become feasible [15, 16], and a number of proposals to exploit memory effects, e.g., for quantum information tasks, has recently been proposed [7, 17, 18]. The previous studies on non-Markovian quantum dynamics mostly focused on continuous coupling between the open system and its environment. However, there also exist other possibilities to study memory effects, for example, discrete dynamics that we consider here. In this case, the open system couples, in a stepwise manner in time, to its environment. One can also envisage the possibility of having unitary transformations changing the state of the open systems between nonunitary evolution steps caused by the environment. We are interested in what is the state of the open system after each step that consists of consecutive unitary and nonunitary parts. Thereby, the theoretical challenge is to construct a discrete dynamical map for the stepwise evolution of the system of interest, and this map should contain information about the local unitary within the system of interest and the properties of the environment it is interacting with.

Quantum walks provide a promising base to combine the study of discrete dynamics and memory effects. Generally, in quantum walks, the system can evolve discretely or continuously [19–21], and a study on the relation between the two cases can be found in [22]. Quantum walks have proven to be important in such diverse fields as quantum information processing [23], complex networks [24, 25], and the physics of topological phases [26, 27]. Moreover, the dynamics of quantum walks show a very broad range of different dynamical behaviors from ballistic spreading to localization [28–36]. By introducing noise to the quantum walk, the effects of the transition from unitary to nonunitary dynamics and the
classical limit can be studied [19, 32, 34, 37–50]. Recently, quantum walks have been implemented experimentally, for example, using trapped ions [51, 52], atoms in optical lattices [53], linear optics [27, 41], optical fibers [32, 54], and waveguides [29, 55, 56]. Optical implementations of quantum walks have recently been reported to realize a walk with a time dependent coin [57] and a single qubit positive operator valued measurement [58].

In this article, we study non-Markovian discrete dynamics of a discrete quantum walk in a line. For this purpose, we introduce a new type of open quantum walk where the dynamics is characterized by a nondivisible discrete dynamical map. It is worth pointing out that most of the work studying environmental effects in quantum walk dynamics is done in terms of models that are described by a dynamical semigroup. Exceptions concerning more general dynamics are presented in [59, 60]. See a more detailed discussion on [60] at the end of section 5.2.3.

We are interested in how to induce and to control memory effects for quantum walks. The article is structured in the following way. We first introduce the concept of a discrete dynamical map and formulate the suitable quantifiers for memory effects. To understand better the origin of memory effects in the considered models, we then, review the standard dephasing model for a qubit with non-Markovian dynamics. We, then, study discrete open quantum dynamics with a unitary control operation and show how the addition of the control transforms the dynamics of the open system inherently non-Markovian. To elaborate on the insight obtained from the simple qubit model, we construct a discrete quantum walk where we identify the coin operator as the local control operation and proceed with memory effects in a 1D discrete open quantum walk. Throughout this paper, the interaction between the open system and its environment is of a pure dephasing type, and in the quantum walk model, the coin is coupled to an environment. The formulation follows the path and emphasizes the experimental realization of the models with photons.

2. Discrete dynamical map

Quantum dynamics is generated by quantum dynamical maps, a family of completely positive and trace preserving (CPT) maps $\Phi_n, n \geq 0$ such that $\Phi_0 = I$. If the state of the quantum system, described by a density operator or matrix, is initially $\rho_0$, then, $\rho_n = \Phi_n(\rho_0)$ defines the evolution of the state. Discrete dynamics for a quantum system emerge if the parameter $n$ indexing the family of CPT maps $\Phi_n$ takes discrete values only, e.g., $n \in \mathbb{N}$. We assume that $0 \in \mathbb{N}$.

CPT maps suitable for studying discrete dynamics can be constructed by using Stinespring’s dilation theorem [61]. It states that, for every CPT map, it is possible to assign a unitary evolution in an enlarged Hilbert space. The total Hilbert space is, thus, $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_E$, and the map is generated as $\rho_n = \text{tr}_E(U^\dagger \rho_0 \otimes \chi(U)^n)$. Here, $U$ is a unitary operator acting on the total Hilbert space, and $\chi$ is the fixed initial state on the auxiliary Hilbert space. This construction guarantees that $\Phi_n$ is CPT for each $n$. The physical interpretation for this construction in the context of open quantum systems is that $\mathcal{H}_E$ is a Hilbert space for the external environment to which the system is coupled unitarily.

Quantum dynamical maps can be classified in the following way by their divisibility properties. If the dynamical map $\Phi_n$ satisfies the following decomposition law:

$$\Phi_{n+1}(\tau) = \Phi_n \circ \Phi_0$$

for all $n \in \mathbb{N}$, then, the dynamical map is Markovian and forms a discrete dynamical semigroup. The dynamical map is called CP divisible if

$$\Phi_{n+m} = \Phi_m \circ \Phi_n$$

for $m, n \in \mathbb{N}$ and where $\Phi_{n+m}$ is completely positive and TP. If the dynamical map can be composed as

$$\Phi_{n+m} = V_{n,m} \circ \Phi_m,$$

where $V_{n,m}$ is positive and TP, then, the dynamical map is called $P$ divisible [62].

It is clear that the divisibility property of the dynamical map goes beyond the concept of the semigroup. However, the various definitions of quantum non-Markovianity are still under active discussion [11]. In the next section, we will introduce the measure used in this work to quantify the non-Markovianity of the discrete dynamical map.

3. Measure for non-Markovianity

Following [2], we use a measure based on distinguishability of quantum states, quantified by the T distance

$$d(\rho_1, \rho_2) = \frac{1}{2} \text{Tr} |\rho_1 - \rho_2|.$$  

T distance is contractive under positive and TP maps [63], hence, also under completely positive and TP maps. Thus, the evolution of T distance for the fixed initial pair under a P-divisible map is contractive, which means that $D_{\rho_0,\rho_1}(m) \geq D_{\rho_0,\rho_1}(n)$ for all $m < n$ where $D_{\rho_0,\rho_1}(m) = d(\Phi_m(\rho_0), \rho_1)$. If we find that the T distance increases, $D_{\rho_0,\rho_1}(m) > D_{\rho_0,\rho_1}(n)$ for some $n > m$, then, we know that the dynamical map is not P divisible, and we say that the dynamics is non-Markovian.

To construct a measure for discrete dynamics, we define the increment of the T distance evolution,

$$\Delta_{1,2}(n) = D_{\rho_0,\rho_1}(n) - D_{\rho_0,\rho_1}(n - 1), \quad n \geq 1,$$

and the measure for non-Markovianity is defined in terms of the increment as

$$\hat{N}(\Phi)(n) = \max_{\rho_0,\rho_1} \sum_{n \in S, m \in \mathbb{N}, |\Delta_{1,2}(n)| > 0} \Delta_{1,2}(n).$$

It can be shown that it is sufficient to make the maximization over orthogonal pairs of states [64], and it has a local representation [15]. The measure has a physical interpretation in terms of information flow between the system and the environment. When $\Delta_{1,2}(n) < 0$, information flows away from the system to the environment, and the ability to distinguish the two states decreases. When $\Delta_{1,2}(n) > 0$, there
is a backflow of information from the environment to the system, which improves the distinguishability. In general, a lower bound for non-Markovianity is quite straightforward to obtain using only a small number of initial states. It is also worth noting that there exists evolutions that are not \( P \) divisible, but the \( T \) distance between evolving states might still decrease at all points in time. In this work, we do not present the results for full optimization over all the initial states but, instead, focus on specific pairs of initial states that allow witnessing the non-Markovianity of the dynamics.

We would also like to mention that the properties between the time discrete dynamical maps and their time continuous limits in terms of divisibility might be interesting. However, this study is out of the scope of this work since we restrict ourselves to the time discrete dynamics.

4. Discrete qubit dynamics

In this section, we define a discrete in time dephasing model, and we study its properties in terms of non-Markovianity. However, first we introduce a continuous in time dephasing model and the associated master equation that generates the dynamical maps. We also discuss the incompatibility of performing the local control on the level of master equation with respect to adding the control to the full system environment model.

4.1. Time continuous dephasing

The genuine quantum effect on open quantum system dynamics is the loss of quantum coherences without energy exchange between the system and the environment. This effect is called pure dephasing. Our motivation to study this effect on qubit dynamics is due to the possibility of experimental implementation using optical elements [16]. Coupling between the system and the environment is given by

\[
U_\text{deph} = \int d\omega \sum_{\nu=L,R} e^{i\nu \omega \Delta \mu} \langle \nu \rangle \otimes |\omega\rangle \langle \omega |,
\]

where \( |\nu\rangle \) labels the qubit degree of freedom and \( |\omega\rangle \) corresponds to the environmental degree of freedom. With an optical implementation in mind, this type of unitary dynamics describes the interaction of a photon with a birefringent medium, e.g., quartz [16]. Different basis states \( |\nu\rangle \) correspond to the different polarization states, \( n_\nu \) is the polarization dependent index of refraction, \( \omega \) is the frequency of the photon, and \( \Delta \mu \) corresponds to the thickness of the quartz plate.

For a fixed product initial state \( \rho = \rho(0) \otimes \chi \), this coupling generates pure dephasing dynamics, given by a dynamical map \( \Phi_{\text{deph}}^{\text{PD}}(\rho(0)) = \text{tr}_E(U_{\text{deph}} \rho(0) U_{\text{deph}}^\dagger) \), expressed in the \( |\nu\rangle \) basis as

\[
\Phi_{\text{deph}}^{\text{PD}}: \begin{cases} 
|\nu\rangle \langle \nu | & \mapsto |\nu\rangle \langle \nu |, \\
|L\rangle \langle R | & \mapsto \kappa(\Delta \mu) |L\rangle \langle R |, \\
|R\rangle \langle L | & \mapsto \kappa^*(\Delta \mu) |R\rangle \langle L |.
\end{cases}
\]

The map is determined by the function \( \kappa: \mathbb{R}_+ \mapsto \mathbb{C}_+, \kappa(\Delta \mu) = \int d\omega |\Delta \kappa\omega| \langle \chi(\omega) \rangle^2 \), where \( \Delta \kappa = n_\kappa - n_\kappa \). Throughout this work, we take the environment initial population distribution \( |\chi(\omega)|^2 \) (spectrum) to be constructed of two Gaussians with widths \( \sigma_0 \), amplitudes \( \frac{A}{1+A^2} \) and \( \frac{1}{1+A^2} \), where \( A \in [0, 1] \), central frequencies of the peaks \( \mu_1, \mu_2 \), and peak separation \( \Delta \omega = \mu_2 - \mu_1 \) so that

\[
|\chi(\omega)|^2 = \frac{1}{1 + A} \left( 1 + \frac{1}{2\pi\sigma^2} \right) \exp(-\omega^2/(2\sigma^2)) + A \exp(-\omega^2/(2\sigma^2))
\]

(7)

This choice is experimentally motivated [16], and, with parameters \( A \) and \( \sigma \), it is possible to control the structure of the environment.

It can be shown that the dynamics over a period \( \Delta \mu \) can be Markovian or non-Markovian depending on the properties of the population distribution \( |\chi(\omega)|^2 \) of the initial state of the environment. Dynamics as a function of \( \Delta \mu \) is plotted in figure 1. \( T \) distance dynamics for the optimal pair of initial states is given by \( D_{\text{op}}{}_{\rho(\Delta \mu)}(\Delta \mu) = |\kappa(\Delta \mu)| \). This type of dynamics is analyzed in [16]. An alternative approach for decoherence suppression in open system dynamics is presented in [65].

To further motivate our work, we note that the dynamics inside the quartz plate is given by the following time local master equation \( \dot{\rho} = -i\left[ -\frac{1}{2} \frac{\Delta n_\mu}{\sigma_0} \rho, \rho \right] - \frac{1}{2} \gamma(t) (\sigma_0 \rho \sigma_0 - \rho) \).

For a single Gaussian environment spectrum (i.e., \( A = 0 \)), the dephasing rate is given by \( \gamma(t) = \Delta n_\mu^2 t \) and where \( \sigma_0^2 \) is the variance and \( \mu \) is the mean value of the Gaussian environment spectrum. Since \( t \geq 0 \) and \( \sigma_0^2 > 0 \), the decay rate is positive. When \( \gamma(t) \geq 0 \), this will define a divisible map [66] that will produce Markovian dynamics for the open system according to our choice of measure. If we would take a phenomenological approach, we could add an arbitrary time dependent Hamiltonian \( H(t) \) to the master equation, but it does not necessarily have full microscopic justification.

This is in stark contrast with our results in the following sections where the local control is added to the full system environment dynamics and, indeed, induces non-Markovian dynamics. Lastly, we would like to point out that, for simplicity of the argument, here, we used a time continuous model for dephasing. A time discrete description would be obtained if the added Hamiltonian \( H(t) \) would be nonzero only on disjoint time intervals and act on such a time scale that all other dynamics could be neglected during a particular time interval, and the dynamics would still be Markovian.

4.2. Discrete dephasing with local control

We consider now discrete dynamics where, at each step in the dilatation space, we have an action of a local control unitary operation [67] followed by dephasing. The single step unitary in the total space is, then,

\[
V = U_{\text{deph}} \cdot (C_\eta \otimes I_E).
\]

(8)
In this work, we consider only the following local control unitaries $C_{\eta}$:

$$C_{\eta} = \sqrt{\eta} \langle L \mid L \rangle + \sqrt{1 - \eta} (\langle L \mid R \rangle + \langle R \mid L \rangle).$$

(9)

These correspond to biased beam splitter transformations and $\eta = \frac{1}{4}$ being the Hadamard transformation also known as the balanced beam splitter transformation. The local unitary operator transforms the polarization basis into a new basis that is not generally simultaneously diagonalizable with the decoherence basis. This proves to be crucial for the non-Markovianity of the quantum dynamics as we will show. The reduced dynamics generated by $V^n$ is given by $\Phi^n_\mu$, which is defined as

$$\rho(n) = \Phi^n_\mu \rho = \text{tr}_C \{ V^n (\rho \otimes \chi) (V^n)^{\dagger} \}. \quad (10)$$

The state of the qubit can be expressed as $\rho = \frac{1}{2} (I + \vec{r} \cdot \vec{\sigma})$, where $\vec{r} = (r_1, r_2, r_3)^T$ is the Bloch vector and $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)^T$ where $\sigma_i$ are the usual Pauli matrices. For arbitrary values of $\eta$, the dynamics is most conveniently given in terms of the Bloch vector. The dynamical map can be written as

$$\vec{r}(m) = \int d\omega \, |\chi(\omega)|^2 \, M(\omega)^m \, \vec{r}. \quad (11)$$

The matrix $M(\omega)$ is given by

$$M(\omega) = \begin{pmatrix}
-\cos \Delta n \omega & -\sin \Delta n \omega & \alpha \cos \Delta n \omega \\
\beta \sin \Delta n \omega & -\cos \Delta n \omega & -\alpha \sin \Delta n \omega \\
\alpha & 0 & \beta
\end{pmatrix},$$

(12)

where $\alpha = 2\sqrt{(1 - \eta)\eta}$, $\beta = 2\eta - 1$, and $\Delta n = n_L - n_R$. Matrix $M(\omega)$ is periodic, e.g., $M(\omega + \Omega) = M(\omega)$, where $\Omega = \frac{2\pi}{\delta \omega/(2\pi)}$. The numerical integration of $M(\omega)^m$ must be done carefully since integrands are highly oscillatory. However, reliable numerical results can be obtained. In the strong dephasing limit, $\tilde{\Omega} \ll \sigma$, and, for some particular values of $\eta$, it is possible to obtain analytical results.

Figures 2, 3 present the results for the values of $\eta = 0, 0.5, 1.0$ in the case of weak and intermediate dephasing strengths. It is worth noting that, for all of the cases, we have $A = 0$, i.e., the environmental spectrum is flat corresponding to Markovian dynamics without local control.

For $\eta = 1$, the local control is $C_1 = \sigma_z$, and we have

$$\langle \nu, \omega | (U_0 (\sigma_z \otimes I)_C)^m | L \rangle \omega = \delta_{\nu, \omega} e^{i m \omega \delta \tau}, \quad (13)$$

$$\langle \nu, \omega | (U_0 (\sigma_z \otimes I)_C)^m | R \rangle \omega = (-1)^m \delta_{\nu, \omega} e^{i m \omega \delta \tau}. \quad (14)$$

This leads to the following dynamical map:

$$\Phi_m^Q : \begin{cases}
|\nu\rangle \langle \nu | \rightarrow |\nu\rangle \langle \nu |, \quad &\forall \nu, \\
|L\rangle \langle R | \rightarrow \kappa^m (m \delta \tau) (-1)^m |L\rangle \langle R |,
\end{cases} \quad (15)$$

Compared to the uncontrolled case, the sign of the coherences is flipped. This does not affect the memory effects since the dephasing process moves the states toward the z-axis on the xy plane of the Bloch sphere, and the sign change in the coherences keeps the distance of the state from the z-axis constant. The dynamics is displayed in panels (c), (d) of figures 2, 3.

For $\eta = 0$, the local control is $\sigma_z$, and we have

$$\langle \nu, \omega | (U_0 (\sigma_z \otimes I)_C)^m | \nu', \omega | \rightarrow \delta_{\nu, \nu'} e^{i m \omega \delta \tau}, \quad (16)$$

$$\langle \nu, \omega | (U_0 (\sigma_z \otimes I)_C)^m | \nu', \omega | \rightarrow (1 - \delta_{\nu, \nu'}) e^{i m \omega \delta \tau}. \quad (17)$$
Figure 2. A weak dephasing interaction. Panels (a) $\eta = 0$, (b) $\eta = 0.5$, (c) $\eta = 1.0$, and we have plotted the time evolution of nonzero Bloch vector components. Panel (d) plots the measure for non-Markovianity for the fixed pair of initial states for the used three values of $\eta$. Initial states in all figures are $\vec{r}_1 = \frac{1}{\sqrt{2}} (1, 0, 1)^T$ and $\vec{r}_2 = -\vec{r}_1$ expressed in terms of the Bloch vector. The parameters of the environment are $A = 0$ and $\delta \omega = 9\sigma$. The parameters for the interaction are $\Delta n = 0.009$ and $\delta t = 0.014 \frac{2\pi}{\delta \omega \Delta n}$. This gives $\frac{\omega}{\sigma} \approx 643 \gg 1$.

Figure 3. The intermediate dephasing interaction. Panels (a) $\eta = 0$, (b) $\eta = 0.5$, (c) $\eta = 1.0$, and we have plotted the time evolution of the nonzero Bloch vector components. Panel (d) plots the measure for non-Markovianity for the fixed pair of initial states for the used three values of $\eta$. The initial states in all figures are $\vec{r}_1 = \frac{1}{\sqrt{2}} (1, 0, 1)^T$ and $\vec{r}_2 = -\vec{r}_1$ expressed in terms of a Bloch vector. The parameters of the environment are $A = 0$ and $\delta \omega = 9\sigma$. The parameters for the interaction are $\Delta n = 0.009$ and $\delta t = 0.014 \frac{2\pi}{\delta \omega \Delta n}$. This gives $\frac{\omega}{\sigma} = 4.5 \sim 1$. 
This gives the dynamical map, \( \Phi_{2m}^\nu : |\nu\rangle \langle \nu'| \mapsto |\nu\rangle \langle \nu'|, \)
\begin{equation}
\Phi_{2m+1}^\nu : \begin{cases}
|\nu\rangle \langle \nu'| \mapsto |\nu\rangle \langle \nu'|, & \forall \nu \neq \nu', \\
|L\rangle \langle R| \mapsto \kappa (\delta t)^m |R\rangle \langle L|,
|R\rangle \langle L| \mapsto \kappa (\delta t) |L\rangle \langle R|.
\end{cases}
\end{equation}
In this case, the information flow between the system and the environment is maximal in the sense that the local control is able to completely eliminate the effect of the environment after an even number of steps, see panels (a), (d) in figures 2, 3.

4.3. Strong dephasing limit

Let us assume that we have a ‘flat’ population distribution for the initial environmental state. What we mean by this is that \( |\chi (\omega)|^2 \) stays almost constant for \( \omega \in [\omega' - \bar{\Omega}/2, \omega' + \bar{\Omega}/2] \), where \( \omega' \in [0, \infty) \).

Now, this means that the effects of the environment in this limit are generic in the sense that the global structure of \( |\chi (\omega)|^2 \) does not play a role. This type of environment is usually called Markovian. It is intuitively clear that, in this type of situation, it is enough to integrate only over a single period of length \( \bar{\Omega} \) in equation (11) to a very good approximation. In terms of equations, this means
\begin{equation}
r' (m) = \frac{1}{\bar{\Omega}} \int_0^{\bar{\Omega}} d\omega M (\omega) m r.
\end{equation}
The physical idea behind this approximation is that there is no distinguished frequency of the environment because of the flatness of the spectrum. Then, all the frequencies in the interval of length \( \bar{\Omega} \) ‘mix’ the state of the reduced system with equal weights, and, hence, it is sufficient to consider only one of these intervals.

In general, we have to validate this approximation numerically, but for the special case where \( \eta = \frac{1}{2} \) and \( A = 0 \), we can obtain analytical expressions for the dynamical map in the strong dephasing limit, which we will do next. It turns out that we need the condition \( \sigma \gg \bar{\Omega} \) for the analytical calculation, but the numerical data show that, already, for intermediate dephasing, \( \sigma \sim \bar{\Omega} \), the strong dephasing approximation (20) works quite well, see figure 4. Dynamics in the intermediate and strong dephasing regimes for \( \eta = \frac{1}{2} \) are plotted in panel (b) of figures 2, 3.

We give now a more detailed proof of equation (20) for \( \eta = \frac{1}{2} \) and \( A = 0 \). Let \( \eta = \frac{1}{2} \) and \( |\chi (\omega)|^2 = \frac{1}{2} \text{sech}^2 \frac{\omega - \omega'}{2} \).

We assume that \( \bar{\Omega} \ll \sigma \), which means that the spectral distribution varies on a much larger scale than the integral kernel \( M (\omega) \), which takes the following form:
\begin{equation}
M (\omega) = \begin{pmatrix}
0 & -\cos (\Delta n t \omega) \\
-\sin (\Delta n t \omega) & 0 \\
0 & 0
\end{pmatrix}.
\end{equation}
We decompose the interval \( \mathbb{R}_+ = [0, \infty) \) as \( \bigcup_{k \in \mathbb{N} \setminus \{0\}} [\bar{\Omega} k, \bar{\Omega} k + \bar{\Omega}] \), where \( \bar{\Omega} = [\bar{\Omega} (k - 1), \bar{\Omega}, k) \). Since \( \int_{\bar{\Omega} k} d\omega |\chi (\omega)|^2 \) is continuous on closed interval \( \bar{\Omega} k \) and differentiable on \( \bar{\Omega} k \), the mean value theorem states that we can always find \( \omega_k \in \bar{\Omega} k \) such that
\begin{equation}
|\chi (\omega_k)|^2 \bar{\Omega} = \int_{\bar{\Omega} k} d\omega |\chi (\omega)|^2.
\end{equation}
As an approximation, we choose the midpoint of each interval \( \omega_k = \bar{\Omega} k + \frac{\bar{\Omega}}{2} \), then,
\begin{equation}
\sum_{k \in \mathbb{N} \setminus \{0\}} |\chi (\omega_k)|^2 \bar{\Omega} = \bar{\Omega} \left( \frac{1}{2} - \mu / \bar{\Omega}, e^{-2 \pi^2 \sigma^2 / \bar{\Omega}^2} \right).
\end{equation}
where \( \vartheta_{\sigma} (\mu, q) \) is the Jacobi \( \vartheta \) function.

Condition \( \bar{\Omega} \ll \sigma \) allows for performing the following approximation:
\begin{equation}
\sum_{k} \int_{\bar{\Omega} k} d\omega |\chi (\omega)|^2 M (\omega)^m \\
\approx \sum_{k} |\chi (\omega_k)|^2 \int_{\bar{\Omega} k} d\omega M (\omega)^m = \sum_{k} |\chi (\omega_k)|^2 |\tilde{M} (m)|
\end{equation}
where \( \tilde{M} (m) = \int_{\bar{\Omega} k} d\omega M (\omega)^m \) and we used equation (23) in the last step. When we take the strong dephasing limit \( \sigma \to \infty \) and use the property \( \lim_{\sigma \to 0} \vartheta_{\sigma} (\mu, q) = 1 \) of the Jacobi \( \vartheta \) function, we obtain equation (20) [68]. Next, we will construct the dynamical map in this special case.

4.4. Dynamical map in the strong dephasing limit for \( A = 0 \) and \( \eta = \frac{1}{2} \)

In the strong dephasing limit for the single Gaussian spectral distribution and \( \eta = \frac{1}{2} \), we obtain the following analytical
form for the dynamical map $\Lambda_m = \frac{1}{\Omega} \int_0^{\Omega} M(\omega)^m$:

$$
\Lambda_0 = I, \quad \Lambda_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \Lambda_2 = \frac{1}{2} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \Lambda_3 = \frac{1}{2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},
$$

$$
\Lambda_{2m} = \begin{pmatrix} a_{m-2} & 0 & a_{m-1} \\ 0 & b_{m-1} & 0 \\ a_{m-2} & 0 & a_{m-1} \end{pmatrix}, \quad m \geq 2,
$$

$$
\Lambda_{2m-1} = \begin{pmatrix} a_{m-2} & 0 & a_{m-1} \\ 0 & b_{m-1} & 0 \\ a_{m-3} & 0 & a_{m-2} \end{pmatrix}, \quad m \geq 3,
$$

where

$$
a_k = \sum_{i=0}^{k} (2i + 1) C(i) (-8)^i, \quad b_k = \sum_{i=0}^{k} C(i) (-8)^i, \quad a_k = b_k = 0, \quad k < 0, \quad C(k) = \frac{1}{k+1} \begin{pmatrix} 2k \end{pmatrix}, \quad k \in \mathbb{N}.
$$

$C(k)$ is called a Catalan number. $a_k$ and $b_k$ have the following limiting behavior:

$$
\lim_{k \to \infty} b_k = (\sqrt{2} - 1), \quad \lim_{k \to \infty} a_k = 1 - \frac{1}{\sqrt{2}}.
$$

Thus, the dynamical map takes the following form in the limit of an infinite number of steps:

$$
\Lambda_{m \to \infty} = \begin{pmatrix} 1 - \frac{1}{\sqrt{2}} & 0 & 1 - \frac{1}{\sqrt{2}} \\ 0 & \sqrt{2} - 1 & 0 \\ 1 - \frac{1}{\sqrt{2}} & 0 & 1 - \frac{1}{\sqrt{2}} \end{pmatrix}.
$$

4.5. Discussion

The results above show that, in a discrete dephasing model for a qubit, the addition of local unitary can induce non-Markovian dynamics even for a flat Markovian spectral structure. A local unitary operation can be seen as a periodic control that dynamically decouples the open system from the environment giving rise to a partial revival of the populations and coherences in the open system dynamics. Periodicity of this control operation can be seen from the discreteness of the dynamics, i.e., we ‘watch’ the system only at the integer multiples of the control period. It is also worth noting that the local unitary changes the open system state, while the earlier created correlations between the system and the environment still persist. The local change in the system state allows, then, the existing system-environment correlations to be converted back to the increased distinguishability of the system states and backflow of information.

Another effect of the local control, in the case that it does not commute with the dephasing basis, e.g., when $C_0 = \sigma_x$, is that it transforms the open system dynamics from pure dephasing to dissipative. For the case when local control is $C_l = \sigma_y$, which commutes with the dephasing operator $U_t$, the dynamics is a pure dephasing type, and the action of the local control does not have an effect on the non-Markovianity of the dynamics, i.e., non-Markovian dynamics can emerge from the spectral structure only. We also show that the non-Markovianity induced by the local control is generic in the sense that the structure of the environmental spectrum does not play a role in the strong dephasing regime. In the special case of Hadamard control $\eta = 0.5$ and $A = 0$, we were able to derive an analytical expression for the dynamical map.

In the following section, we study a more complicated situation with a one dimensional discrete quantum walk. Our findings will be better understood with the help of the physical intuition gained from the present section.

5. Open quantum walk

5.1. Quantum walk

Quantum walks are either continuous or discrete time unitary protocols that evolve as a quantum state on a Hilbert space that is constructed from the underlying graph where the walk takes place. In this section, we will mostly adapt to the notation of [69].

In this work, we limit the discussion to a discrete quantum walk on a line. The Hilbert space for the walk is $\mathcal{H}_W = \mathcal{H}_C \otimes \mathcal{H}_P = C^2 \otimes \ell^2(\mathbb{Z})$. It consists of the coin and the position spaces. The unitary operator $W$ evolving the state of the walker over a single step is the following:

$$
W = (|L\rangle \langle L| \otimes I + |R\rangle \langle R| \otimes S^\dagger)(C_0 \otimes I_P),
$$

where $S = \sum_{x} |x-1\rangle \langle x|$ and $C_0 = C_2$, see equation (9). We choose to focus only on Hadamard walks, meaning that the coin unitary is the Hadamard matrix. The unitary operator $W$ can be diagonalized if we move into a quasimomentum picture by the Fourier transform $|k\rangle = \sum_{x} e^{i k x} |x\rangle$, $k \in [-\pi, \pi]$. $k$ is called the quasimomentum since it is periodic. Note that the Fourier- or quasimomentum basis is not normalizable but, nevertheless, very useful when used carefully. The inverse transformation is defined as $|\gamma\rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} dk e^{-ik}|k\rangle$.

In this work, we always initialize the position of the walker to the origin. In the quasimomentum picture, the unitary operator $W$ acts on an arbitrary state initialized from origin $|\phi_0\rangle = |\phi\rangle \otimes |0\rangle$ as

$$
|\phi_m\rangle = W^m|\phi_0\rangle = \int_{-\pi}^{\pi} \frac{dk}{2\pi} (M_k)^m|\phi\rangle \otimes |k\rangle,
$$

where $M_k$ is the following $2 \times 2$ matrix:

$$
M_k = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-ik} & e^{-ik} \\ e^{ik} & e^{ik} \end{pmatrix}.
$$

Eigenvalues of $M_k$ are $e^{-\imath \nu_k}$, where $\nu_k$ is defined by

$$
\sin k = \sqrt{2} \sin \nu_k.
$$
Using the quasimomentum representation for solving the dynamics and, then, transforming back to the position representation, we obtain the following expression for a general initial state starting from the origin $|\psi_0\rangle = (c_L|L\rangle + c_R|R\rangle) \otimes |0\rangle$ and evolving $m$ steps,

$$|\psi_m\rangle = W^m|\psi_0\rangle = \sum_{x=-m}^{m} \left[ (c_L A^m_L(x) + c_R A^m_R(x))|L\rangle + (c_L B^m_L(x) + c_R B^m_R(x))|R\rangle \right] \otimes |x\rangle.$$

(34)

Analytical expressions for the coefficient functions $A^m_L(x)$, $A^m_R(x)$, $B^m_L(x)$, and $B^m_R(x)$, are obtained most easily from the quasimomentum picture. They are

$$A^m_L(x) = \frac{1 + (-1)^{m+x}}{2} \alpha^m(x) + \beta^m(x),$$

$$A^m_R(x) = \frac{1 + (-1)^{m+x}}{2} \beta^m(x) - \gamma^m(x),$$

$$B^m_L(x) = \frac{1 + (-1)^{m+x}}{2} \beta^m(x) + \gamma^m(x),$$

$$B^m_R(x) = \frac{1 + (-1)^{m+x}}{2} \alpha^m(x) + \beta^m(x),$$

(35)

(36)

where

$$\alpha^m(x) = \int_{-\pi}^{\pi} \frac{dk}{2\pi} e^{ikx},$$

$$\beta^m(x) = \int_{-\pi}^{\pi} \frac{dk}{2\pi} \cos \frac{k}{\sqrt{1 + \cos k}} e^{ikx},$$

$$\gamma^m(x) = \int_{-\pi}^{\pi} \frac{dk}{2\pi} \sin \frac{k}{\sqrt{1 + \cos k}} e^{ikx}.$$

(37)

(38)

(39)

Using the method of stationary phase, asymptotic expressions for equations (35), (36) may be obtained.

After an $m$ number of steps, the walker has nonzero probability to be found in positions $-m, -m + 2, \ldots, m - 2, m$. From this follows that, after an even number of steps, the walker has nonzero probability only at even vertices and similarly for the odd number of steps. This property is sometimes called *modularity*.

5.2. Open quantum walk

We extend the Hilbert space to take into account the effect of an external environment. The extended Hilbert space is $\mathcal{H} = \mathcal{H}_C \otimes \mathcal{H}_P \otimes \mathcal{H}_E \equiv \mathcal{H}_G \otimes \mathcal{H}_E$, and now our system consists of the coin and position degrees of freedom.

We couple the coin degrees of freedom to the external environment, and the coupling unitary is given by the trivial extension of equation (5),

$$U_0 = \int d\omega \sum_{\sigma = L, R} e^{i\omega \sigma \cdot \sigma} \sigma \otimes \mathbb{I}_P \otimes |\omega\rangle \langle \omega|.$$

(40)

For simplicity, we consider only the homogeneous case, meaning that the coupling operator acts identically on each vertex of the graph. The walk operator is extended to act on the enlarged Hilbert space trivially $W \equiv W \otimes \mathbb{I}_E$.

The dynamical map for the open quantum walk is constructed as usual,

$$\Phi^W_m(\rho_0) = \text{tr}_C \{ (U_0 W)^m \rho_0 \otimes \chi ((U_0 W)^m \rho_0) \},$$

(41)

where $\chi$ is, again, the initial state for the environment. Remarkably, an analytical expression for the dynamical map can be found.

5.2.1. Quantum dynamical map. The matrix elements of the dynamical map are $[\Phi^W_m]_{\sigma', \sigma, x'; \tau', y'} = \langle \sigma', x' | \Phi^W_m(\sigma, x) \langle \tau, y | xK_{\sigma'} \cdots xK_{\sigma} | 0 \rangle | x' \rangle | y' \rangle$. The matrix elements can be written explicitly by using the definition of the quantum walk,

$$[\Phi^W_m]_{\sigma', \sigma, x'; \tau', y'} = \int d\omega \ |\chi(\omega)\rangle \langle \chi(\omega)|$$

$$\times \sum_{k_0, \ldots, k_n = (L, R)} e^{i\omega \sum_{n=1}^{N} (n_k - n_{k-1})}$$

$$\times \langle e^{i\chi_{k_0, \ldots, k_n}} \langle x_{k_0, \ldots, k_n} | \langle x_{k_0} \cdots x_{k_n} | 0 \rangle | K'_n \cdots K'_1 | y' \rangle \rangle,$$

(42)

where $K_L = S$, $K_R = S'$, and $e^{i\chi_{k_0, \ldots, k_n}} = (\sigma'(C_{H_\sigma})^n)$.

Each non-zero inner product $\langle x_{k_0} \cdots x_{k_n} | 0 \rangle | K'_n \cdots K'_1 | y' \rangle$ in equation (42) corresponds to a path $0 \to x'$ in an $n$-level binary tree. Each path $0 \to x$ has $N_L(x) = \frac{n + x}{2}$ and $N_R(x) = \frac{n - x}{2}$ left and right turns. This allows writing the exponential part of equation (42) as

$$e^{i\omega \sum_{n=1}^{N} (n_k - n_{k-1})} = e^{i\omega \Delta n (y' - x')},$$

(43)

where $\Delta n = n_L - n_R$ since left and right turns each contribute to the sum in total $N_L(x)$ and $N_R(x)$ times the terms $n_L$ and $n_R$, respectively. Then, by suppressing the explicit matrix products, we can write the matrix element as

$$[\Phi^W_m]_{\sigma', \sigma, x'; \tau', y'} = \int d\omega \ |\chi(\omega)\rangle \langle \chi(\omega)|$$

$$\times \langle \sigma', x' | W^n | \sigma, 0 \rangle \langle \tau, 0 | (W^n)^* | \tau', y' \rangle.$$

(44)

This shows that the coupling of the coin degrees of freedom to the external environment has an effect only on the coherences between different sites. It does not have any effect on the position distribution or on the propagation speed.

Also, if we would trace over position degrees of freedom in equation (41), we would obtain a valid dynamical map for the coin degrees of freedom provided that the initial state of the walker does not contain correlations between the coin and the position degrees of freedom.

5.2.2. Strong dephasing limit. In the strong dephasing limit, the state of the quantum system is block diagonal since the coherences between any two different sites in the lattice are destroyed. This means that the state of the quantum walker...
can be written as
\[ \rho_m = \bigoplus_{x \in \{-m, m\}} \tilde{\rho}^m(x), \]
where \( \tilde{\rho}^m(x) = \sum_{\sigma, \sigma'}^{\sigma(L), \sigma(R)} \tilde{\rho}^{m}_{\sigma, \sigma'}(x) |\sigma, x\rangle \langle \sigma', x| \). Eigenvalues of a self-adjoint \( 2 \times 2 \) matrix \( \begin{pmatrix} a & b \\ b & c \end{pmatrix} \) are \( \lambda_{\pm} = \frac{1}{2} (a + b \pm \sqrt{(a - c)^2 + 4 |b|^2}) \).

The pure initial state \( |\psi_0\rangle = c_L |L, 0\rangle + c_R |R, 0\rangle \) evolves in \( m \) steps in the strong dephasing limit into \( \Phi^W_m (|\psi_0\rangle \langle \psi_0|) \). The expression for block \( x \) is
\[ \langle x | \Phi^W_m (|\psi_0\rangle \langle \psi_0|) |x\rangle = \langle x | W^m |\psi_0\rangle \langle \psi_0| (W^*)^m |x\rangle \equiv \rho_{\psi_0} (x). \]

Characterization of non-Markovianity now becomes more feasible since we need to diagonalize at the \( m \)th step \( 2m + 1 \) \( 2 \times 2 \) self-adjoint matrices, instead of the \( m^2 \) valued matrix. An analytical expression by using equations (34), (35), (36) could be obtained, but in this work, we do not analyze it further.

5.2.3. Non-Markovianity of the open quantum walk. For this model, we first study the coin state dependence of non-Markovianity. We do this by fixing both initial states to be pure and to be localized at site 0. We parametrize the coin degree of freedom in terms of the angles \( \theta \in [0, \pi] \) and \( \phi \in [0, 2\pi] \) of the Bloch vector. We have fixed the parameters of the environment to be \( \delta \omega = 9\sigma \), \( \Delta \omega = 0.009 \), and we chose three different values for \( A \). Also, we fixed the rest of the parameters by choosing \( \delta t \Delta \omega \Delta \eta = 2.5 \), where \( \Omega = \frac{\omega}{\Omega} \). The results can be seen in figure 5. We see that a good choice for the initial state pair is \( |L\rangle, |R\rangle \) that witnesses non-Markovianity of the dynamics for all values of \( A \), which characterizes the structure of the environment.

Next, we studied the behavior of the non-Markovianity measure for fixed initial state pair \( |L\rangle, |R\rangle \) when varying \( \delta t \Delta \omega \Delta \eta \). The results have been plotted in figure 6. We can see that the measure for non-Markovianity is periodic when we increase \( \delta t \), which gives the duration of the dephasing step. Remarkably, for specific values of \( \delta t \), the structure of the environment is irrelevant, i.e., for a specific duration of dephasing, the quantity of memory effects remain the same irrespective of the structure of the environment. The periodicity of the measure is related to the separation of the peaks of the environmental spectrum in the sense that \( N_{A=0} = N_{A=\omega} \) when \( \delta t = \frac{2\pi}{\Delta \omega \Delta \eta} \). It is also very interesting to note that, for the strong dephasing limit, the value of non-Markovianity approaches a constant value that is, again, independent of the structure of the environment that resembles the simple discrete qubit dynamics case. However, before the strong dephasing limit when \( \delta t \) has small or intermediate values, non-Markovianity has two sources: the local unitary and the environmental structure. For \( A = 0 \), the non-Markovianity originates from the coin flipping only (local unitary), while for other presented values of \( A \), the structure of the environment gives an additional contribution to the quantity of the memory effects.

There exist previous works on non-Markovian quantum walks, e.g., [59, 60]. However, we would like to point out that the interpretation of the results presented in [60] is extremely challenging for two reasons. First, the authors consider the dephasing type interaction for the coin, but they calculate all the relevant time dependent functions from a model that describes amplitude damping. Second, in the paper, they use a time local master equation, which is the generator of a dynamical map that gives the time evolution of the system part of the total initial state up to time \( \tau \) from time \( t_0 \) when the system and the environment are prepared into a product state. The authors, nevertheless, use the same map to evolve the state of the two level system from some arbitrary time \( t_1 \) to \( t_2 > t_1 \), which can be done only if the map satisfies the semigroup property, i.e., it is Markovian. This is in contrast
with the claimed non-Markovianity of the dephasing dynamics.

6. Summary

We have analyzed, in detail, two different models for discrete open quantum system dynamics: a simple qubit dynamics with dephasing and a more elaborate full open quantum walk model. The results show that the addition of a local unitary operation can dramatically change the nature of the open system dynamics and, in particular, the appearance of memory effects. We have discussed how memory effects generated within our models can become generic in the sense that the structure of the environment spectrum does not play a role—the amount of memory effects is independent of the form of the spectra—and non-Markovianity is solely induced by a local unitary operation in these cases. The considered models and observed phenomena give rise to approximation schemes that simplify the description of the dynamics considerably that may prove useful for other contexts too. We have also discussed, in detail, for the first time, a model for an open quantum walk where memory effects can be introduced and can be controlled in an experimentally relevant way. The presented open quantum walk is experimentally realizable using linear optics.

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