Construction of holomorphic local conformal framed nets

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May 5, 2014

Abstract

We construct holomorphic local conformal framed nets extended from a tensor power of the Virasoro net with $c = 1/2$ with a pair of binary codes $(C, D)$ satisfying the conditions given by Lam and Yamada for holomorphic framed vertex operator algebras. Our result is an operator algebraic counterpart of theirs, but our proof is entirely different. We apply the $\alpha$-induction in order to identify the representation theory of “code local conformal net” and this gives rise to the existence of the desired local conformal net.

∗Supported in part by Global OE Program “The research and training center for new development in mathematics”, the Mitsubishi Foundation Research Grants and the Grants-in-Aid for Scientific Research, JSPS.
†Supported by MEXT scholarship
Introduction

The concept of states and observables are fundamental in quantum physics. Here, the observables are the measurements of physical quantities and states are the physical quantities of interest. In algebraic quantum field theory, states are vectors in a Hilbert space and observables are self-adjoint operators defined on it. As the observation is done in a certain spacetime domain, the observables are defined on each specific spacetime domain. These ideas are translated into mathematical structures using the language of operator algebras [1,13]. When we have conformal symmetry on the 1+1 dimensional Minkowski spacetime, we have full conformal field theory. In the language of operator algebra, we have a family of von Neumann algebras \( \{ \mathcal{A}(O) \}_O \) where the regions \( O \) are double cones which are rectangles with the sides parallel to the lines \( x = \pm t \). The family \( \{ \mathcal{A}(O) \}_O \) satisfies isotony, locality, conformal covariance, existence of vacuum vector and positivity of energy where the locality is defined by spacelike separation. We call the family \( \{ \mathcal{A}(O) \}_O \) a 2-dimensional local conformal net. We can restrict \( \{ \mathcal{A}(O) \}_O \) to the lines \( x = \pm t \) and replace the spacelike separation by non-intersection which is simpler. (See [15] for more details.) Through this process, we have two chiral conformal field theories.

Local conformal nets in chiral conformal field theory arise from the restriction of the rectangles \( O \) to the line \( x = \pm t \). They are of the form \( \{ \mathcal{A}(I) \}_I \) where each \( \mathcal{A}(I) \) is a von Neumann algebra abd \( I \) is an open interval in the “spacetime” which is now the circle \( S^1 \). We have another mathematical approach to the same physical theory which is the language of vertex operator algebra. Vertex operators arise from the Fourier series expansion of operator-valued distributions on \( S^1 \). Conceptually, it should be possible to choose certain vertex operators and test functions with the supports in the interval \( I \), and form a family of mathematical objects equivalent to local conformal nets. Unlike local conformal nets where the state spaces are Hilbert spaces, there is no assumption on the existence of an inner product and the completeness on the state spaces in the theory of vertex operator algebra. This motivates an open problem whether these two mathematical objects correspond to each other or not, i.e., if an example of one of the objects is given, the counterpart in the form of the other object should be found. There are several pieces of evidence which emphasize that the answer to this open problem is positive, at least under some extra nice conditions. Both mathematical objects have examples associated to affine Lie algebras and
the Virasoro algebra. Given a positive definite lattice $L$, one can construct a lattice vertex operator algebra and their twisted orbifolds as in [9]. The construction of local conformal nets for a positive definite lattice $L$ is given in [25]. An operator algebraic counterpart of the twisted orbifolds of lattice vertex operator algebras are also constructed by extending lattice conformal nets in [8].

For a given central charge $c < 1$, Virasoro conformal nets $\text{Vir}_c$ and simple Virasoro vertex operator algebras $L(c, 0)$ are essentially the same objects in many aspects. They have the same representation theory and their irreducible representations obey the same fusion rules. Our main interest is in the simple current extensions of their tensor products which are holomorphic (defined to have only one irreducible representation which is the trivial one). By definition, framed vertex operator algebras contain $L(1/2, 0)^{\otimes n}$ as a subalgebra with the same conformal element [7, 20, 23]. In [20], Lam and Yamauchi extended the tensor products of the simple Virasoro vertex operator algebra with central charge $1/2$ to holomorphic framed vertex operator algebras using a pair of even binary codes $(C, D)$ where $C$ is the dual code of $D$ and $D$ satisfies the following conditions:

1. The length of $D$ is $16n$ where $n$ is a positive integer;

2. The Hamming weights of all elements in $D$ are divisible by 8;

3. The all-one word is in $D$.

This kind of extension exists for $L(1/2, 0)^{\otimes 16n}$. Here, the word $(c_1, c_2, \ldots, c_{16n})$ in $C$ correspond to the module $\bigotimes_{i=1}^{16n} L(1/2, c_i/2)$ and the word $(d_1, d_2, \ldots, d_{16n})$ in $D$ correspond to the module of $\bigoplus_{c \in C} \bigotimes_{i=1}^{16n} L(1/2, c_i/2)$ whose decompositions into $L(1/2, 0)^{\otimes 16n}$-modules have the conformal weights $1/16$ for the entries $d_i = 1$.

In this manuscript, we give an operator algebraic counterpart of a holomorphic framed vertex operator algebra which is a holomorphic local conformal framed net in the sense of [16]. A local conformal framed net $\mathcal{A}$ is defined as an irreducible extension of $\text{Vir}_{1/2}^{\otimes n}$. The structure of local conformal framed nets as the simple current extension $\text{Vir}_{1/2}^{\otimes n} \rtimes \mathbb{Z}_k \rtimes \mathbb{Z}_l$ for some integers $k$ and $l$ is shown in [16, Theorem 4.3], and the corresponding result for vertex operator algebras has been obtained in [20]. We construct holomorphic local conformal framed nets which are extended from tensor products of Virasoro conformal nets with central charge $1/2$ by using a pair of even binary codes
(C, D) satisfying the above conditions for D with C = D⊥. To show the existence of \( \text{Vir}^{\otimes 16n} \rtimes C \rtimes D \), we need to know the representation theory of \( \text{Vir}^{\otimes 16n} \rtimes C \), which is our counterpart of the code vertex operator algebra and decide whether an appropriate action of D on \( \text{Vir}^{\otimes n} \rtimes C \) exists or not. (See [23] and references therein for code vertex operator algebras.)

A representation of a local conformal net corresponds to a unitary equivalence class of endomorphisms called a sector. For chiral conformal field theory, a sector \( \rho \) consists of endomorphisms \( \rho_I \) for each \( A_I \) such that each \( \rho_I \) leaves \( A_I' \) invariant where \( I' \) is the set of the interior points of the complement of \( I \). The local conformal net \( \text{Vir}^{\otimes 16n} \rtimes C \) is completely rational in the sense of [17, Definition 8]. It has the \( \mu \)-index which is the square sum of the statistical dimensions of all irreducible sectors equal to \( 2^{32n} \). The \( \mu \)-index of \( \text{Vir}^{\otimes 16n} \rtimes C \) is \( 2^{32n} / |C|^2 \) [17]. This provides some conditions on the statistical dimensions of all irreducible sectors of \( \text{Vir}^{\otimes 16n} \rtimes C \).

The simple current extension \( \text{Vir}^{\otimes 16n} \rtimes C \) is an irreducible extension of \( \text{Vir}^{\otimes 16n} \). As defined by Longo and Rehren in [21], the \( \alpha^\pm \)-induction \( \rho \mapsto \alpha^\rho \) transforms a sector of \( \text{Vir}^{\otimes 16n} \) into a sector of \( \text{Vir}^{\otimes 16n} \rtimes C \). (See also [3, 26] for general properties of the \( \alpha \)-induction.) Note that the \( \alpha^\pm \)-induction depends on the code \( C \) as it is defined using a canonical endomorphism \( \gamma \) from \( \text{Vir}^{\otimes 16n} \rtimes C \) into \( \text{Vir}^{\otimes 16n} \) for each spacetime domain. Let \( \theta \) be the restriction on \( \text{Vir}^{\otimes 16n} \) of the canonical endomorphism \( \gamma \). The endomorphism \( \theta \) decomposes into a direct sum of the irreducible sectors of \( \text{Vir}^{\otimes 16n} \) corresponding to \( C \) with multiplicity 1. Since the system of irreducible sectors of \( \text{Vir}^{\otimes 16n} \rtimes C \) is non-degenerate in the sense of [24], we have that the irreducible sectors of \( \text{Vir}^{\otimes 16n} \rtimes C \) are the intersection between the subsectors of the \( \alpha^\pm \)-induction of the irreducible sectors of \( \text{Vir}^{\otimes 16n} \). We can first consider the \( \alpha^\pm \)-induction of the irreducible sectors \( \rho \) such that \( \alpha^\rho_+ = \alpha^\rho_- \) which holds if and only if \( \varepsilon^\pm(\rho, \theta) = 1 \) where \( \varepsilon^\pm \) is the statistics operator which is unitary [3]. Using [3, part III: Lemma 3.8], we study the \( S \)-matrix of \( \text{Vir}^{\otimes 16n} \) to investigate whether \( \varepsilon^\pm(\rho, \sigma)\varepsilon^\pm(\rho, \sigma) = 1 \) for each irreducible subsector \( \sigma \) of \( \theta \). Some of the irreducible sectors may give the \( \alpha^\pm \)-induction identical to that of some other irreducible sectors. Some of them split into a direct sum of irreducible sectors of \( \text{Vir}^{\otimes 16n} \rtimes C \) as the \( \alpha^\pm \)-induction does not preserve the irreducibility. At this point, we apply the \( \mu \)-index of \( \text{Vir}^{\otimes 16n} \rtimes C \) to identify the family of irreducible sectors of \( \text{Vir}^{\otimes 16n} \rtimes C \). The strategy was previously used in the
identification of the representation theory of \( \text{Vir}_{1/2}^{\otimes 2} \times \{(0, 0), (1, 1)\} \) as given in [16, Proposition 2.2].

For the case of \( \text{Vir}_{1/2}^{\otimes 16n} \times C \rtimes D \) with \( C = D \perp \), the problem is divided into three cases according to the cardinality of the triply even binary code \( D \). When the cardinality of \( D \) is either 2 or 4, the problem is similar to the case of \( \text{Vir}_{1/2}^{\otimes 2} \times \{(0, 0), (1, 1)\} \) and we can obtain the answer by applying the same strategy directly. For \( D \) with higher cardinality, we have difficulties to deal with the contribution to the \( \mu \)-index of the \( \alpha^\pm \)-induction since it may only give irreducible sectors with the statistical dimensions more than 1. When this happens, the irreducible subsectors of such \( \alpha^\pm \)-induction are not automorphisms. Thus, we prove by mathematical induction by constructing a decreasing sequence of triply even codes \( D = D_1 \supset D_2 \supset \cdots \supset D_{p-1} \supset D_p = \{(0)_{16n}, (1)_{16n}\} \) which gives an increasing series of even codes \( C = C_1 \subset C_2 \subset \cdots \subset C_{p-1} \subset C_p \). Here, we choose \( D_i \)'s such that \( |D_i|/|D_{i+1}| \) is 2 and \( D_{p-1} \) is generated by \( \beta \) and \( (1, 1, \ldots, 1) \) for some \( \beta \) different from the identity and the all-one word in \( D \). We draw a conclusion that a sector associated to the codeword \( \beta \) of \( \text{Vir}_{1/2}^{\otimes 16n} \) gives irreducible sectors of \( \text{Vir}_{1/2}^{\otimes 16n} \times C \) with statistical dimensions 1 since the \( \alpha^\pm \)-induction from \( \text{Vir}_{1/2}^{\otimes 16n} \) to \( \text{Vir}_{1/2}^{\otimes 16n} \rtimes C_r \) and the double \( \alpha^\pm \)-induction from \( \text{Vir}_{1/2}^{\otimes 16n} \times C \) to and to \( \text{Vir}_{1/2}^{\otimes 16n} \rtimes C_r \) are the same for \( r = 2, 3, \ldots, p \).

1 Preliminaries

1.1 Local conformal nets

Denote the collection of all non-dense open intervals in \( S^1 \) by \( \mathcal{I} \). For each \( I \) in \( \mathcal{I} \), assign a von Neumann algebra \( \mathcal{A}(I) \) on a Hilbert space \( \mathcal{H} \).

**Definition 1.1.** The family \( \{\mathcal{A}(I)\}_{I \in \mathcal{I}} \) is called a local net, simply denoted by \( \mathcal{A} \), if it satisfies the following conditions for all \( I, I_1 \) and \( I_2 \) in \( \mathcal{I} \).

(a) (Isotony) The condition \( I_1 \subset I_2 \) implies \( \mathcal{A}(I_1) \subset \mathcal{A}(I_2) \).

(b) (Locality) The condition \( I_1 \cap I_2 = \emptyset \) implies \([\mathcal{A}(I_1), \mathcal{A}(I_2)] = \{0\}\), where the Lie bracket denotes the commutator.

(c) (Möbius covariance) There exists a strongly continuous unitary representation \( U \) of \( \text{PSL}(2, \mathbb{R}) \) on \( \mathcal{H} \) such that \( U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(gI) \), for all \( g \in \text{PSL}(2, \mathbb{R}) \), \( I \in \mathcal{I} \).
(d) (Positivity of the energy) Denote the rotational subgroup of $\text{PSL}(2, \mathbb{R})$ by $R(\cdot)$. If $e^{-i\theta L_0} = U(R(\theta))$, then $L_0$ is positive.

(e) (Existence of the vacuum) There exists a $U$-invariant unit vector $\Omega \in \mathcal{H}$ called a vacuum vector.

(f) (Irreducibility) The von Neumann algebra generated by all $\mathcal{I}$ is equal to $\mathcal{B}(\mathcal{H})$, i.e., $\bigvee_{I \in \mathcal{I}} \mathcal{A}(I) = \mathcal{B}(\mathcal{H})$. This condition is equivalent to the uniqueness of $\Omega$ up to phase factor and $\mathcal{A}(I)$ being type $\text{III}_1$ factor unless it is $\mathbb{C}$.

There are some important consequences from this definition.

**Proposition 1.2.** Let $\mathcal{A}$ be a local net.

1. (Reeh-Schlieder theorem) $\Omega$ is cyclic and separating for all $\mathcal{A}(I)$ where $I$ is in $\mathcal{I}$.

2. (Haag’s Duality) $\mathcal{A}(I') = \mathcal{A}(I)'^{\prime}$ where $I'$ is the set of interior points of $S^1 \setminus I$ and $\mathcal{A}(I)'^{\prime}$ is the commutant of $\mathcal{A}(I)$ with respect to $\mathcal{B}(\mathcal{H})$.

**Definition 1.3.** The local net $\mathcal{A}$ is called a local conformal net if (c) (Möbius covariance) is extended to the following condition.

(c’) (Conformal covariance) $U$ extends to a projective unitary representation of Diff$(S^1)$, the group of smooth orientation preserving diffeomorphisms and

$$U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(gI), \text{ for all } g \in \text{Diff}(S^1), I \in \mathcal{I};$$

$$U(g)xU(g)^* = x, \text{ for all } g \in \text{Diff}(I'), x \in \mathcal{A}(I).$$

Here, Diff$(I)$ is the subset of Diff$(S^1)$ that leaves every element in $I'$ invariant.

**Remark 1.4.** We call the six conditions in the definition of local conformal nets the axioms of chiral conformal field theories. Local conformal nets with the locality defined by spacelike separation on 1 + 1 Minkowski space can be restricted to two chiral conformal field theories on the lines $x = \pm t$ as shown in [15].
Definition 1.5. A DHR (Doplicher-Haag-Roberts) representation $\pi$ of a local conformal net $\mathcal{A}$ on a Hilbert space $\mathcal{K}$ is a map

$$\mathcal{I} \ni I \mapsto \pi_I \subset \mathcal{B}(\mathcal{K})$$

where $\pi_I$ is a normal representation of $\mathcal{A}(I)$.

A DHR representation is Möbius covariant with positive energy if there exists a projective unitary representation $U_\pi$ of $\text{PSL}(2, \mathbb{R})$ with positive energy such that for any $I$ in $\mathcal{I}$, $x$ in $\mathcal{A}(I)$ and $g$ in $\text{PSL}(2, \mathbb{R})$

$$U_\pi(g)\pi_I(x)U_\pi(g)^* = \pi_{gI}(U(g)xU(g)^*)$$

A DHR representation is conformal covariant with positive energy if there exists a projective unitary representation $U_\pi$ of $\text{Diff}^{(\infty)}(S^1)$, the universal cover of $\text{Diff}(S^1)$, with positive energy such that for any $I$ in $\mathcal{I}$, $x$ in $\mathcal{A}(I)$ and $g$ in $\text{Diff}^{(\infty)}(S^1)$

$$U_\pi(g)\pi_I(x)U_\pi(g)^* = \pi_{\dot{g}I}(U(\dot{g})xU(\dot{g})^*)$$

where $\dot{g}$ is the image of $g$ in $\text{Diff}(S^1)$.

By Doplicher-Haag-Roberts sector theory, $\pi$ corresponds bijectively to an equivalent class of endomorphisms $[\rho_I]_{I \in \mathcal{I}}$ on $\mathcal{B}(\mathcal{H})$ where $\rho_I$ is a localized endomorphism of $\mathcal{A}(I)$, i.e., leaving $\mathcal{A}(I')$ invariant(cf. [1]). The class $[\rho_I]_{I \in \mathcal{I}}$ is often simply denoted by $\rho$. We call such a class a DHR sector or simply a sector. For each $I$ in $\mathcal{I}$, $\rho_I(\mathcal{A}(I))$ is a subfactor by the construction of $\rho$. We define the statistical dimension or the dimension $d_\rho$ of $\rho$ as

$$d_\rho = ([\mathcal{A}_I : \rho_I(\mathcal{A}(I))])^{\frac{1}{2}}.$$

1.2 Complete rationality

Let $\mathcal{A}$ be a local conformal net on a fixed Hilbert space $\mathcal{H}$.

Definition 1.6. [17, 22] The local conformal net $\mathcal{A}$ is completely rational if it satisfies the following properties:

1. (Split property) $\mathcal{A}(I_1) \lor \mathcal{A}(I_2) \cong \mathcal{A}(I_1) \otimes \mathcal{A}(I_2)$ where $\overline{T_1}$ and $\overline{T_2}$ are disjoint.

2. (Finite $\mu$-index) Split $S^1$ into fours intervals $I_1, I_2, I_3$ and $I_4$ anti-clockwise. The $\mu$-index of $\mathcal{A}$, $\mu_A = \left(\left(\mathcal{A}(I_2) \lor \mathcal{A}(I_4)\right)' : \mathcal{A}(I_1) \lor \mathcal{A}(I_3)\right]$, is finite.
The local conformal net \( \mathcal{A} \) is holomorphic if it is completely rational and its \( \mu \)-index is equal to one.

**Theorem 1.7.** [17, 22] Let \( \mathcal{A} \) be a local conformal net with split property. If \( \mathcal{A} \) has finitely many irreducible DHR sectors with positive energy up to isomorphism and the statistical dimension of each irreducible DHR sector is finite, then \( \mathcal{A} \) is completely rational and \( \mu \mathcal{A} = \sum_{i=1}^{k} d_{\rho_{i}} \) where \( i \) runs over the irreducible DHR sectors of \( \mathcal{A} \).

### 1.3 The Virasoro net \( \text{Vir}_{1/2} \)

The Virasoro algebra is the Lie algebra generated by \( \{L_n\}_{n \in \mathbb{Z}} \) and \( c \) such that

\[
[L_n, L_m] = (m - n)L_{m+n} + \frac{m^3 - m}{12}\delta_{m+n,0}c
\]

and \( [L_n, c] = 0 \) for every \( n \) in \( \mathbb{Z} \). By [10] and [11], either the central charge \( c \geq 1 \) or \( c = 1 - 6m(m+1) \), \( m = 2, 3, 4, \ldots \) in an irreducible unitary representation. The lowest eigenvalue \( h \) of \( L_0 \) is called the conformal weight on the unitary representations. By [11], for a given \( c \), the possible values of \( h \) are

\[
h = \frac{(m+1)p - mq}{4m(m+1)} = \frac{1}{4m(m+1)}, \text{ where } p \in \{1, 2, \ldots, m-1\}, q \in \{1, 2, \ldots, m\}.
\]

For \( c = 1/2 \), there are three unitary representations with conformal weights 0, 1/16 and 1/2, respectively. Let \( U \) be the unitary representation with conformal weight 0. Define the Virasoro net \( \text{Vir}_{1/2} \) by (cf. [14, 27]):

\[
\text{Vir}_{1/2}(I) = U(\text{Diff}(I))''.
\]

By [14], \( \text{Vir}_{1/2} \) is completely rational. There are three inequivalent irreducible DHR sectors arising from the unitary representations of the Virasoro algebra with conformal weights 0, 1/16, 1/2, respectively. They have statistical dimensions \( 1, \sqrt{2}, 1 \), respectively, and hence the \( \mu \)-index of \( \text{Vir}_{1/2} \) is 4. Denote these sectors by \( \lambda_0, \lambda_{1/16} \) and \( \lambda_{1/2} \), respectively. They obey the following fusion rules where \( \lambda(0) \) is the identity sector.

\[
\lambda_{1/2} \circ \lambda_{1/2} = \lambda_0, \quad \lambda_{1/2} \circ \lambda_{1/16} = \lambda_{1/16}, \\
\lambda_{1/16} \circ \lambda_{1/16} = \lambda_0 \oplus \lambda_{1/2}.
\]
Denote the conformal weight by $h$. By spin-statistics theorem, we have the conformal spin $\omega = e^{2\pi ih}$ [12]. The $S$-matrix of Vir$_{1/2}$ is given as follows. (See [6], for example.)

$$S = \begin{pmatrix}
\frac{1}{2} & \frac{\sqrt{2}}{2} & \frac{1}{2} \\
\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\
\frac{1}{2} & -\frac{\sqrt{2}}{2} & \frac{1}{2}
\end{pmatrix}$$
in the order of $\lambda_0$, $\lambda_{1/16}$ and $\lambda_{1/2}$.

### 1.4 Binary codes

Here, we define some notations on binary codes that we will use later (cf., [2]). A binary code $C$ of length $n$ is a subgroup of $\mathbb{Z}_2^n$. A member $c$ in $C$ is called a word or a codeword in $C$. The dimension of $C$, denoted by $\dim(C)$, is an integer $k$ such that $C$ is isomorphic to $\mathbb{Z}_2^k$. The support of $c = (x_1, x_2, \ldots, x_n)$ is the set $\text{supp}(c) = \{i = 1, 2, \ldots, n \mid c_i = 1\}$. The Hamming weight or simply the weight of $c$ is the cardinality of $\text{supp}(c)$. The code $C$ is called even, doubly even and triply even if the Hamming weights of the words in $C$ are divisible by 2, 4 and 8, respectively. We denote the all-zero word and all-one word of length $n$ by $(0)_n$ and $(1)_n$, respectively. Let $c_1 = (x_1, x_2, \ldots, x_n)$ and $c_2 = (y_1, y_2, \ldots, y_n)$ be in $C$. Define the inner product as

$$c_1 \cdot c_2 = \sum_{i=1}^{n} x_i y_i$$

where the multiplication is the multiplication of real numbers and the addition is the addition in $\mathbb{Z}_2$. The dual $C^\perp$ of $C$ is defined as

$$C^\perp = \{ c' \in \mathbb{Z}_2^n \mid c \cdot c' = 0, \forall c \in C \}$$

and $\dim(C) + \dim(C^\perp) = n$. Let $D$ be a binary code of length $m$ and $d = (z_1, z_2, \ldots, z_m)$ be a word in $D$. The direct sums between words and codes are defined as follows:

$$c \oplus d = (x_1, x_2, \ldots, x_n, d_1, d_2, \ldots, d_m);$$

$$C \oplus D = \{ c \oplus d \mid c \in C \text{ and } d \in D \}.$$
2 Framed nets

Here we recall some basics of framed nets in \[16\]. \(\operatorname{Vir}_{1/2} \otimes n\) has \(3^n\) inequivalent irreducible sectors. We label the DHR sectors of \(\operatorname{Vir}_{1/2} \otimes n\) as 0, 1/16, 1/2 using the conformal weights as usual. Then each

\[
\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \{0, 1/16, 1/2\}^n
\]

represents an irreducible DHR sector of \(\operatorname{Vir}_{1/2} \otimes n\) and any irreducible DHR sector of \(\operatorname{Vir}_{1/2} \otimes n\) is of this form. The fusion rules follow componentwise. The conformal weight of \(\lambda\) is \(\sum_{i=1}^n \lambda_i\). The statistical dimension of \(\lambda\) is \(2^k\) where \(k\) is the number of entries \(\lambda_i = 1/16\). Hence, the \(\mu\)-index is \(4^n\). We can extend the net \(\operatorname{Vir}_{1/2} \otimes n\) by the following lemma.

**Lemma 2.1.** [16] Let \(A\) be a local conformal net and \(\{\lambda_i\}_i\) be a finite system of irreducible sectors of \(A\) with statistical dimension 1 and conformal spin 1. Then, the crossed product of \(A\) by the finite abelian group \(G\) given by \(\{\lambda_i\}_i\) produces a local extension of the net \(A\). We call the extended net a simple current extension of \(A\) denoted by \(A \rtimes G\).

In the criteria of the preceding lemma, we can form a simple current extension of \(\operatorname{Vir}_{1/2} \otimes n\) by using the group \(\lambda(C)\) consisting of

\[
\lambda(c) = (c_1/2, c_2/2, \ldots, c_n/2)
\]

where \(c = (c_1, c_2, \ldots, c_n)\) belongs to an even binary code \(C\) of length \(n\). We call \(C\) a 1/2-code as in [20]. On the other hand, \(\lambda\) with some \(\lambda_i = 1/16\) is not an automorphism so a class of such DHR sectors does not give any simple current extension immediately.

**Definition 2.2.** [16] A local conformal net \(A\) is called a framed net if it is an irreducible extension of \(\operatorname{Vir}_{1/2} \otimes n\) for some positive integer \(n\).

The next theorem shows the relationship between framed nets and simple current extensions of \(\operatorname{Vir}_{1/2} \otimes n\).

**Theorem 2.3.** [16] Let \(A\) be a framed net extended from \(\operatorname{Vir}_{1/2} \otimes n\). There exist integers \(k, l\) and actions of \(\mathbb{Z}_2^k, \mathbb{Z}_2^l\) such that \(A\) is isomorphic to a simple current extension of a simple current extension of \(\operatorname{Vir}_{1/2} \otimes n\) as follows.

\[
A \cong (\operatorname{Vir}_{1/2} \rtimes \mathbb{Z}_2^k) \rtimes \mathbb{Z}_2^l.
\]
We introduce the proof of the next theorem since its method will be useful later.

**Theorem 2.4.** [10] $\text{Vir}_{1/2}^{\otimes 2} \rtimes C$ where $C = \{(0,0), (1,1)\}$ has four inequivalent irreducible sectors with conformal weights $0, 1/8, 1/2$ and $1/8$, respectively. The fusion rules are given by $\mathbb{Z}_4$.

**Proof.** $\text{Vir}_{1/2}^{\otimes 2}$ has nine irreducible sectors. The $S$-matrix is given by

$$S = \begin{pmatrix}
\frac{1}{16} & \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} & \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & \frac{1}{16} \\
\frac{1}{16} & 0 & \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & \frac{1}{4} & 0 & -\frac{1}{2} & 0 & -\frac{\sqrt{2}}{4} \\
\frac{1}{16} & \frac{\sqrt{2}}{4} & 0 & \frac{\sqrt{2}}{4} & \frac{1}{4} & 0 & 0 & -\frac{1}{2} & -\frac{\sqrt{2}}{4} \\
\frac{1}{16} & \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} & -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & \frac{1}{4} \\
\frac{1}{16} & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} & \frac{1}{2} & -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} \\
\frac{1}{16} & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{16} & -\frac{1}{2} & 0 & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & 0 & 0 & \frac{1}{2} & -\frac{\sqrt{2}}{4} \\
\frac{1}{16} & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} & -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} \\
\frac{1}{16} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2}
\end{pmatrix}$$

in the order of id, $(0,1/16)$, $(1/16,0)$, $(1,1/2)$, $(1/2,0)$, $(1/16,1/16)$, $(1/16,1/2)$, $(1/2,1/16)$ and $(1/2,1/2)$.

In this case, $\theta = \text{id} \oplus (1/2,1/2)$. Applying Proposition 3.23 of part I and Proposition 5.1 of part III of [13] and Theorem 5.10 of [14], $\alpha^\pm$-induction of id, $(0,1/2)$, $(1,1/2)$, $(1/2,1/2)$ and $(1/16,1/16)$ gives irreducible sectors of $\text{Vir}_{1/2}^{\otimes 2} \rtimes C$. By $\langle \alpha_{\rho_1}^\pm, \alpha_{\rho_2}^\pm \rangle_{\text{Vir}_{1/2}^{\otimes 2} \rtimes C} = \langle \theta \circ \rho_1, \rho_2 \rangle_{\text{Vir}_{1/2}^{\otimes 2}}$, the $\alpha^\pm$-induction of $(0,1/2)$ is the same as that of $(1/2,0)$. Likewise, the $\alpha^\pm$-induction of id is the same as that of $(1/2,1/2)$. The $\alpha^\pm$-induction of $(1/16,1/16)$ has the statistical dimension 2. By [17] Proposition 24, the $\mu$-index of $\text{Vir}_{1/2}^{\otimes 2} \rtimes C$ is 4. So $\text{Vir}_{1/2}^{\otimes 2} \rtimes C$ has four inequivalent irreducible sectors of statistical dimension 1. Since the sectors have statistical dimension 1, there are two possibilities for the fusion rules which are $\mathbb{Z}_2^2$ and $\mathbb{Z}_4$. The latter case occurs since by [12] the conformal spins are preserved and $\mathbb{Z}_2^2$ violates section 3 of [21].

The preceding theorem shows that the $\alpha^\pm$-induction of $\lambda$ with some $\lambda_i = 1/16$ may give irreducible sectors of statistical dimension 1. It may be possible to find a binary code $D$ that acts on these irreducible sectors such that the simple current extension $\text{Vir}_{1/2}^{\otimes n} \rtimes C \rtimes D$ occurs. We call $D$ a...
1/16-code, its element $d \in D$ a 1/16-word or a $\tau$-word and the pair $(C, D)$ structure codes as in [20]. We have the following theorem that resembles framed vertex operator algebras.

**Proposition 2.5.** Suppose that the simple current extension $\text{Vir}_{1/2} \otimes C \otimes D$ is well-defined. The following statements hold.

1. $C$ is an even binary code and $D$ is a triply even binary code, i.e., the Hamming weights of the elements in $D$ are divisible by 8.
2. $C \subset D^\perp$.

**Proof.**
1. This follows from the definition of simple current extension since only DHR sectors with triply even 1/16-words can have the conformal spins 1.

2. Suppose that $C$ is not a subset of $D^\perp$. Then there exist $c$ in $C$ and $d$ in $D$ such that the cardinality of $\text{supp}(c) \cap \text{supp}(d)$ is odd. The $S$-matrix element associating to $c$ and $d$ is negative as the $S$-matrix of $\text{Vir}_{1/2} \otimes n$ is the $n$th tensor power of the $S$-matrix of $\text{Vir}_{1/2}$. So the $\alpha^{\pm}$-induction of $\lambda$ such that the 1/16-word is $d$ does not give irreducible sectors of $\text{Vir}_{1/2} \otimes C$.

3 Holomorphic framed nets

There some more restrictions on the codes $C$ and $D$ if the framed net $\text{Vir}_{1/2} \otimes C \otimes D$ is holomorphic. Let $C$ be a binary code of length 16 satisfying the condition in [20] where $D = C^\perp$ as follows.

1. The length of $D$ is $16n$ where $n$ is a positive integer.
2. The code $D$ is triply even.
3. The word $(1)_{16n}$ is in $D$.

Our aim is to construct a holomorphic local conformal net with the central charge $c = 8n$ and structure codes $(C, D)$ in the form of $\text{Vir}_{1/2} \otimes 16n \otimes C \otimes D$. Let $C$ be isomorphic to $\mathbb{Z}_2^k$ as an abstract group.
Recall that the $S$-matrix of $\text{Vir}_{1/2}$ is given as
\[
\begin{pmatrix}
\frac{1}{2} & \frac{\sqrt{2}}{2} & \frac{1}{2} \\
\frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\
\frac{1}{2} & -\frac{\sqrt{2}}{2} & \frac{1}{2}
\end{pmatrix},
\]
where the orders of rows and columns are given by conformal weights as 0, 1/16, 1/2, respectively.

Note that since $D$ contains $(1)_{16n}$, all the codewords in $C$ are even, so $\lambda(C) = \{\lambda(c) \mid c \in C\}$ naturally gives an action of $C$ on $\text{Vir}_{1/2} \otimes 16n$, since the conformal spin of the DHR sector $\lambda_{1/2}$ is $-1$. We then have a crossed product net $\text{Vir}_{1/2} \otimes 16n \rtimes C$ through this action by [16] Lemma 2.1. This is an operator algebraic counterpart of the code vertex operator algebra.

The next step is to find an appropriate action of $D$ on $\text{Vir}_{1/2} \otimes 16n \rtimes C$ and to prove that $\text{Vir}_{1/2} \otimes 16n \rtimes C \rtimes D$ has a structure code $(C, D)$. (It is trivial that this net is holomorphic since its $\mu$-index is equal to $4^{16n}/|C||D|^2 = 1$.)

For this purpose, we need to know the representation theory of $\text{Vir}_{1/2} \otimes 16n \rtimes C$. Note that the dual canonical endomorphism $\theta$ for the inclusion
\[\text{Vir}_{1/2} \otimes 16n \subset \text{Vir}_{1/2} \otimes 16n \rtimes C\]
is given as $\bigoplus_{c \in C} \lambda(c)$, so for a general $\lambda \in \{0, 1/16, 1/2\}^{16n}$, we have $\alpha^+(\lambda) = \alpha^-(\lambda)$ if and only if $\varepsilon^+(\lambda, \theta) = \varepsilon^-(\lambda, \theta)$ if and only if $\tau(\lambda) \in C^\perp = D$, where $\tau(\lambda)$ is the $\tau$-word. (This follows from the form of the $S$-matrix above as the $S$-matrix of $\text{Vir}_{1/2} \otimes 16n$ is the 16nth tensor power of the $S$-matrix of $\text{Vir}_{1/2}$.)

From now on, we consider only $\lambda$ in $D$. Then by [4] Theorem 5.10, $\alpha^+(\lambda)$ gives a (possibly reducible) DHR sector of $\text{Vir}_{1/2} \otimes 16n \rtimes C$. Since we always have $\alpha^+_\lambda = \alpha^-_\lambda$, we simply write $\alpha_\lambda$. We are going to prove that all the irreducible DHR sectors of this local conformal net arises from irreducible decomposition of $\alpha_\lambda$ of $\lambda \in D$.

First note $\langle \alpha_\lambda, \alpha_\mu \rangle = \langle \lambda \mu, \theta \rangle$, and this implies that $\langle \alpha_\lambda, \alpha_\mu \rangle \neq 0$ if and only if $\lambda \mu \in \lambda(C)$.

Fix $\beta \in D$ with weight $8j$ and consider a DHR sector $\lambda$ of $\text{Vir}_{1/2} \otimes 16n$ with $\tau(\lambda) = \beta$. The number of such $\lambda$’s is $2^{16n-8j}$. As in [20], set
\[C_\beta = \{c \in C \mid \text{supp}(c) \subset \beta\}.
\]
Note that we have $|C_\beta| \geq 2^{4j}$ by [20] Remark 6. The number of distinct $\alpha_\lambda$ is now $2^{16n-8j}|C_\beta|/|C|$. We also have $d_\lambda = d_{\alpha_\lambda} = 2^{4j}$. Note that
\[\langle \alpha_\lambda, \alpha_\lambda \rangle = \sum_{c \in C} \langle \lambda^2, c \rangle = |C_\beta| \geq 2^{4j}.
\]
Suppose that we have an irreducible decomposition \( \alpha_\lambda = \bigoplus_i m_i \sigma_i \), where \( m_i \) is the multiplicity of the irreducible DHR sector \( \sigma_i \). We then have \( \langle \alpha_\lambda, \alpha_\lambda \rangle = \sum_i m_i^2 \). Set \( d_{\sigma_i} = d_i \) and consider the possible lowest value of \( \sum_i d_i^2 \). Since we have \( \sum_i m_i d_i = 2^{4j} \), the possible lowest value of \( \sum_i d_i^2 \) is \( 2^{8j}/|C_\beta| \) by the Cauchy-Schwarz inequality and this happens when we have \( |C_\beta|/2^{4j} = m_i/d_i \) for all \( i \). If we have this lowest value for all \( \lambda \), the total contribution to the \( \mu \)-index of \( \text{Vir}_{1/2} \otimes^{16n} \times C \) is equal to \( 2^{16k}/|C| = |D| \). Now the number of \( \beta \) is \( |D| \), so all the contribution to the \( \mu \)-index of \( \text{Vir}_{1/2} \otimes^{16n} \times C \) arising in this way is at least \( |D|^2 \), which is the right \( \mu \)-index of \( \text{Vir}_{1/2} \otimes^{16n} \times C \). This shows that we already have all the irreducible DHR sectors of \( \text{Vir}_{1/2} \otimes^{16n} \times C \), and we have \( |C_\beta|/2^{4j} = m_i/d_i \) for all \( \beta, \lambda \) and \( i \). Note that this equality implies \( d_i \) is rational, but it is also an algebraic integer, so each \( d_i \) has to be an integer.

**Lemma 3.1.** Fix \( \lambda \) and \( \beta \) as above, and consider the irreducible decomposition \( \alpha_\lambda = \bigoplus_i m_i \sigma_i \). Then all \( m_i \)'s are equal.

**Proof.** By the fusion rules and \( \langle \alpha_\lambda, \alpha_\mu \rangle = \langle \lambda \mu, \theta \rangle \), we have

\[
\alpha_\lambda \circ \alpha_\lambda = \alpha_{\lambda^2} = \bigoplus_m |C_\beta| \alpha_{\mu_m},
\]

where the number of \( m \)'s is \( 2^{8j}/|C_\beta| \), \( |C_\beta| \) is the multiplicity, all \( \mu_m \) are mutually inequivalent, and each \( \mu_m \) is in \( \{0, 1/2\}^{16n} \), hence has a dimension 1. Now take \( \sigma_i, \sigma_l \) appearing in the irreducible decomposition of \( \alpha_\lambda \). Then \( \sigma_i \cdot \sigma_l \) is a direct sum of DHR sectors with dimension 1 of the form \( \alpha_{\mu} \), \( \mu \in \{0, 1/2\}^{16n} \) possibly with some multiplicity. Choose one such \( \mu \). Then the Frobenius reciprocity implies \( \langle \sigma_i, \sigma_l \alpha_\mu \rangle \geq 1 \), but the dimension of \( \alpha_\mu \) is 1, so we have \( \sigma_i = \sigma_l \alpha_\mu \) (cf. [4]). Since \( \text{supp}(\mu) \) is contained in \( \beta = \{h \mid \lambda_h = 1/16\} \), we have \( \alpha_{\lambda \mu} = \alpha_\lambda \), which implies the equality of the multiplicities, \( m_i = m_l \). That is, for fixed \( \lambda \), all the \( m_i \)'s in the irreducible decomposition \( \alpha_\lambda = \bigoplus_i m_i \sigma_i \) are equal, and we simply write \( \alpha_\lambda = m \bigoplus_i \sigma_i \). This also gives \( d_{\sigma_i} = m2^{4j}/|C_\beta| \).

Here we prove the following general lemma on a modular tensor category.

**Lemma 3.2.** Fix a modular tensor category and suppose that the dimensions of the irreducible objects are all 1 and the conformal spins of the irreducible objects are all \( \pm 1 \). Then all the nontrivial elements in this modular tensor category has order 2.
Proof. For irreducible objects $\lambda, \mu, \nu$ in this tensor category, we have

$$Y_{\lambda\mu} = \frac{\omega_\lambda \omega_\mu}{\omega_{\lambda\mu}} = \pm 1$$

since $N_{\lambda\mu} \neq 0$ only if $\lambda\mu = \nu$. Then $S_{\lambda\mu} = w^{-1/2}Y_{\lambda\mu}$, and the Verlinde formula gives

$$N_{\mu\mu}^0 = \sum_{\lambda} \frac{S_{\mu\lambda}S_{\mu\lambda}^*}{S_{0\lambda}} = 1,$$

so this means $\mu\mu = 0$ for all non-trivial irreducible object $\mu$. \hfill \Box

We want to show that an automorphism group $\Delta$ isomorphic to $D$ on $\text{Vir}_{1/2} \otimes 16n \rtimes C$ exists which implies the existence of the holomorphic framed net $\text{Vir}_{1/2} \otimes 16n \rtimes C \rtimes D$.

**Proposition 3.3.** In the above setting, there exists a set $\Delta$ of irreducible DHR sectors of $\text{Vir}_{1/2} \otimes 16n \rtimes C$ satisfying the following.

1. Each element in $\Delta$ has dimension 1.
2. Each element in $\Delta$ has conformal spin 1.
3. The set $\Delta$ is closed under the sector multiplication and conjugation.
4. For each $\beta \in D$, we have exactly one DHR sector in $\Delta$ which arises from the irreducible decomposition of any $\alpha_{\lambda}$ where the $\tau$-word of $\lambda$ is $\beta$, and through this, we have a group isomorphism of $\Delta$ and $D$.

**Proof.** We divide the problem into three cases according to the dimension of $D$.

**Case 1:** $\dim(D) = 1$

The code $D$ has only two words which are $(0)_{16n}$ and $(1)_{16n}$ so $D = \langle(1)_{16n}\rangle$. By looking at the $S$-matrix and following the method given in the proof of theorem 2.4 the $\alpha$-induction of the following sectors on $\text{Vir}_{1/2}^{\otimes 16n}$ give irreducible DHR sectors on $\text{Vir}_{1/2}^{\otimes 16n} \rtimes \langle(1)_{16n}\rangle^\perp$: id, $(0)_{16n-1} \oplus (1/2)$ and $(1/16)_{16n}$ with statistical dimensions 1, 1 and $2^{8n}$, respectively. $\langle(1)_{16n}\rangle^\perp$ has $2^{16n-1}$ words so the $\mu$-index of $\mathcal{A} \cong \text{Vir}_{1/2}^{\otimes 16n} \rtimes \langle(1)_{16n}\rangle^\perp$ is 4.

There are the following possibilities.
1. The $\alpha$-induction of $(1/16)_{16n}$ splits into irreducible DHR sectors of statistical dimension $2^k$, for some positive integer $k$.

2. The $\alpha$-induction of $(1/16)_{16n}$ splits into a number different from two of inequivalent irreducible DHR sectors of statistical dimension 1.

3. The $\alpha$-induction of $(1/16)_{16n}$ splits into two inequivalent irreducible DHR sectors of statistical dimension 1 with the multiplicity $2^{8n-1}$.

Case 3. occurs by the value of the $\mu$-index. Denote the irreducible DHR subsectors of the $\alpha$-induction of $(1/16)_{16n}$ by $\sigma_1$ and $\sigma_2$. The conformal spins of $\sigma_1$ and $\sigma_2$ are 1. By lemma 3.2, all the four irreducible DHR sectors have order 2. We have $\Delta = \{id, \sigma_i\}$ where $i$ can be either 1 or 2.

**Case 2:** $\dim(D) = 2$

There are four words in $D$. Suppose that $d$ is a word in $D$ different from $(0)_{16n}$ and $(1)_{16n}$. Then, the other word in $D$ is $(1)_{16n} + d$. The Hamming weight of $d$ is $8i$, $i = 1, 2, \ldots, 2n - 1$, if and only if the Hamming weight of $(1)_{16n} + d$ is $16n - 8i$. Rewrite $D$ as

$$D = \langle (1)_{8i} \oplus (0)_{16n-8i}, (0)_{8i} \oplus (1)_{16n-8i} \rangle$$

Then, $D^\perp = \langle (1)_{8i} \rangle^\perp \oplus \langle (1)_{16n-8i} \rangle^\perp$. By the definition of crossed product von Neumann algebras,

$$\text{Vir}_{1/2}^{\otimes 8i} \rtimes (\langle (1)_{8i} \rangle^\perp \oplus \langle (1)_{16n-8i} \rangle^\perp)$$

$$\cong (\text{Vir}_{1/2}^{\otimes 8i} \rtimes \langle (1)_{8i} \rangle^\perp) \otimes (\text{Vir}_{1/2}^{\otimes (16n-8i)} \rtimes \langle (1)_{16n-8i} \rangle^\perp).$$

Since $i$ is arbitrary, it is sufficient to consider only $\text{Vir}_{1/2}^{\otimes 8i} \rtimes \langle (1)_{8i} \rangle^\perp$. Using the same argument as the case $\dim(D) = 1$, $\text{Vir}_{1/2}^{\otimes 8i} \rtimes \langle (1)_{8i} \rangle^\perp$ totally has four irreducible sectors of quantum dimension 1 where id and $(0)_{8i-1} \oplus (1/2)_1$ induce one sector each and the $\alpha$-induction of $(1/16)_{8i}$ splits into two inequivalent irreducible sectors of statistical dimension 1. Denote the splitted irreducible subsectors by $\beta_1$ and $\beta_2$. By lemma 3.2, the four irreducible sectors obey the fusion rules given by $\mathbb{Z}_2^2$. $\beta_1$ and $\beta_2$ satisfy the following equation:

$$\beta_1 \alpha_{(0)_{8i-1} \oplus (1/2)_1} = \beta_2.$$
For $\text{Vir}_{1/2}^{\otimes (16n-8i)} \rtimes \langle (1)_{16n-8i} \rangle^\perp$, denote the splitted irreducible subsectors of the $\alpha$-induction of $(1/16)_{16n-8i}$ by $\beta_1'$ and $\beta_2'$. $\beta_1'$ and $\beta_2'$ satisfy the following equation:

$$\beta_1' \alpha_0(0)_{16n-8i-1} \oplus (1/2) = \beta_2' .$$

By the representation theory of $\text{Vir}_{1/2}^{\otimes 8i} \rtimes \langle (1)_{8i} \rangle^\perp$ and $\text{Vir}_{1/2}^{\otimes (16n-8i)} \rtimes \langle (1)_{16n-8i} \rangle^\perp$ above, we have 4 irreducible DHR sectors with dimension 1 and the $\tau$-word $(0)_{16n-8i}$, with conformal spins $1, -1, -1, 1$. We also have 4 irreducible DHR sectors with dimension 1 with the $\tau$-word $(1)_{8i} \oplus (0)_{16n-8i}$, and their conformal spins are $1, -1, -1, 1$ by the same argument. We also get the same conclusion for the $\tau$-word $(0)_{8i} \oplus (1)_{16n-8i}$. For $\lambda$'s with their $\tau$-words $(1)_{16n}$, we have only one $\alpha_\lambda$ and its contribution to the $\mu$-index is 4. We note that all have conformal spin 1. If we multiply an irreducible DHR sector arising from $\alpha_\lambda$ with $\lambda$'s $\tau$-word $(1)_{8i} \oplus (0)_{16n-8i}$ and another such a sector arising from $\alpha_\lambda$ with $\lambda$'s $\tau$-word $(0)_{8i} \oplus (1)_{16n-8i}$, we obtain an irreducible DHR sector of dimension 1 arising from $\alpha_\lambda$ with $\lambda$'s $\tau$-word $(1)_{16n}$. There are 4 irreducible DHR sectors with the $\tau$-word $(1)_{16n}$. We thus see that all the statistical dimensions of all the irreducible DHR sectors are 1, and we can apply Lemma 3.2. Then we first choose the identity sector id. We next choose $\sigma_1$ with dimension 1 and conformal spin 1 from an irreducible decomposition of $\alpha_\lambda$ with $\lambda$'s $\tau$-word $(1)_{8i} \oplus (0)_{16n-8i}$. This is possible for both even and odd $i$. We also choose $\sigma_2$ with dimension 1 and conformal spin 1 from an irreducible decomposition of $\alpha_\lambda$ with $\lambda$'s $\tau$-word $(0)_{8i} \oplus (1)_{16n-8i}$. Then we set $\Delta = \{ \text{id}, \sigma_1, \sigma_2, \sigma_1 \sigma_2 \}$.

**Case 3:** $\dim(D) \geq 3$

We now proceed by induction. Suppose we have $D$ and we have proved the proposition for the case where the order is $D$ is smaller.

Choose $\beta \in D$ with $\beta \neq (0)_{16n}$ and $\beta \neq (1)_{16n}$. Choose an irreducible DHR sector $\sigma$ appearing in the decomposition of some $\alpha_\lambda$ with $\lambda$'s $\tau$-word $\beta$. Suppose the dimension $d_\sigma$ is larger than 1, and we will derive a contradiction. As in the proof of Lemma 3.1, we know that $\sigma \sigma$ decomposes into a direct sum of irreducible DHR sectors of dimension 1. The Frobenius reciprocity shows that all of these have multiplicity 1. This means that the endomorphism $\sigma$ gives a crossed product subfactor with an abelian group of order $d_\sigma^2$. Let $G$ be this abelian group. This is a subgroup of $\mathbb{Z}_{16n}^\perp / C$. 

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First assume all the irreducible DHR sectors in \( G \) have conformal spin 1. We then construct a decreasing sequence
\[
D = D_1 \supset D_2 \supset \cdots \supset D_{p-1} = \langle \beta, (1)_{16n} \rangle \supset D_p = \langle (1)_{16n} \rangle
\]
and an increasing sequence
\[
C = C_1 \subset C_2 \subset \cdots \subset C_{p-1} \subset C_p,
\]
where \([D_r : D_{r-1}] = 2\) and \(D_r = C_r\) and some \(C_r\) has the property \(C_r/C = G\). This is easy by choosing \(D_p, D_{p-1}, D_{p-2}, \ldots\) in this order. Note that all \(D_r\) are triply even codes containing \((1)_{16n}\) so that we can apply the above general argument to \(D_r\). Consider the \(\alpha\)-induction of \(\sigma\) from \(\text{Vir}_{1/2} \otimes^{16n} \rtimes C\) to \(\text{Vir}_{1/2} \otimes^{16n} \rtimes C_r\). By the \(\alpha\sigma\)-reciprocity in [3] and that fact that \(G\) is an abelian group, we know that this \(\alpha\)-induction produces an irreducible DHR sector of dimension equal to \(d_{\sigma}\), which is bigger than 1. This \(\alpha\)-induction is equal to the \(\alpha\)-induction of \(\lambda\) from \(\text{Vir}_{1/2} \otimes^{16n} \rtimes C_{r}\) by [21], so this contradicts the induction hypothesis.

Next consider the case some of the irreducible DHR sectors in \( G \) have conformal spin \(-1\). Note that the conformal spins are multiplicative on \( G \) because they arise from \(\alpha\)-inductions of DHR sectors in \(\{0, 1/2\}^{16n}\). Then the extension \(\text{Vir}_{1/2} \otimes^{16n} \rtimes C \rtimes G\) is rewritten as \(\text{Vir}_{1/2} \otimes^{16n} \rtimes C \rtimes \tilde{G} \rtimes \mathbb{Z}_2\), where \(\tilde{G}\) is a maximal subgroup giving a local extension of \(\mathbb{Z}_{16n}/C\) and the last crossed product by \(\mathbb{Z}_2\) is a Fermionic extension (the locality is replaced by graded locality, cf. [5]). In this case the order of \(G\) is \(d_{\sigma}' \geq 4\), so the group \(\tilde{G}\) is nontrivial. We again construct a decreasing sequence
\[
D = D_1 \supset D_2 \supset \cdots \supset D_{p-1} \supset D_p
\]
and an increasing sequence
\[
C = C_1 \subset C_2 \subset \cdots \subset C_{p-1} \subset C_p
\]
in a similar way to the above case, where now some \(C_r\) has the property \(C_r/C = \tilde{G}\). Now again consider the \(\alpha\)-induction of \(\sigma\) from \(\text{Vir}_{1/2} \otimes^{16n} \rtimes C\) to \(\text{Vir}_{1/2} \otimes^{16n} \rtimes C_r \rtimes \mathbb{Z}_2\). We again know that this \(\alpha\)-induction produces an irreducible DHR sector of dimension equal to \(d_{\sigma}\), which is bigger than 1. This \(\alpha\)-induction is equal to the \(\alpha\)-induction of \(\lambda\) from \(\text{Vir}_{1/2} \otimes^{16n} \rtimes C_r \rtimes \mathbb{Z}_2\) to \(\text{Vir}_{1/2} \otimes^{16n} \rtimes C_{r} \rtimes \mathbb{Z}_2\). Every irreducible DHR sector of \(\text{Vir}_{1/2} \otimes^{16n} \rtimes C_r\) has dimension 1.
by the induction hypothesis, so its $\alpha$-induction to $\text{Vir}_{1/2} \otimes 16n \rtimes C \rtimes \mathbb{Z}_2$ again has index 1 and this is a contradiction.

We have thus proved that all the irreducible DHR sectors of $\text{Vir}_{1/2} \otimes 16n \rtimes C$ have dimension 1. Then we choose the generators \{\beta_s\} of $D$, and for each $\beta_s$, we choose an irreducible DHR sector $\sigma_s$ with conformal spin 1 appearing in the irreducible decomposition of $\alpha_\lambda$ with $\lambda$'s $\tau$-word $\beta_s$. Then the group generated by the irreducible DHR sectors $\beta_s$ gives the desired set $\Delta$. 

Using the preceding proposition, we immediately obtain the following main theorem.

**Theorem 3.4.** Suppose the codes $C, D$ are as in [20]. Then we have a holomorphic framed local conformal net with structure codes $(C, D)$.

**Proof.** We simply make a two-step crossed product $\text{Vir}_{1/2} \otimes 16n \rtimes C \rtimes D$, where the action of $D$ is given by $\Delta$ in Proposition 3.3. Construct the simple current extension of $\text{Vir}_{1/2} \otimes 16n \rtimes C$ using $\Delta$. It is easy to see that the $\mu$-index is 1 and the structure codes are $(C, D)$. 

**Remark 3.5.** As mentioned earlier, Lam and Yamauchi proved the existence of holomorphic framed vertex operator algebras associated to binary codes in [20]. The classification of the maximal triply even binary codes given by Betsumiya and Muenasa in [2] leads to the classification of holomorphic framed vertex operator algebras extended from $L(1/2,0) \otimes 16n$, $n = 1, 2, 3$, by Lam and Shimakura in [19]. Concrete examples of such vertex operator algebras are constructed and identified. We have the corresponding local conformal nets, and the vacuum characters are clearly the same in the both approaches. At $c = 24$, the vacuum character is uniquely determined and equal to the $j$-function except for the constant term, so if the constant terms are different in the setting of [19], the corresponding local conformal nets are also different, but Lam and Shimakura have examples where the constant terms of the vacuum characters, which are the dimensions of the weight 1 spaces, are the same, but have different Lie algebra structures. They are different as vertex operator algebras, and we expect that the corresponding local conformal nets are also different, but we do not have a proof so far.

**Remark 3.6.** Since our argument relies on only structures of the tensor categories, it is possible to use $SU(2)_2$ instead of $\text{Vir}_{1/2}$. (Note that the $S$-matrices of these two are the same, and one of the three conformal weights differ by 1/8, but we use only triply even codes, so this difference does not
matter.) However, as shown in [18, Theorem A.2], the vertex operator algebra corresponding to $SU(2)_2$ itself is an extension of $L(1/2,0) \otimes L(1/2,0) \otimes L(1/2,0)$, so this does not give new examples.

**Acknowledgements.** Part of this work of Y. K. has been done at Università di Roma “Tor Vergata”. He gratefully acknowledges the hospitality there. He also thanks R. Longo and H. Yamauchi for helpful comments. The second-named author thanks the first-named author for the supervision.

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