A noise-controlled dynamic bifurcation

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Abstract

We consider a slow passage through a point of loss of stability. If the passage is sufficiently slow, the dynamics are controlled by additive random disturbances, even if they are extremely small. We derive expressions for the ‘exit value’ distribution when the parameter is explicitly a function of time and the dynamics are controlled by additive Gaussian noise. We derive a new expression for the small correction introduced if the noise is coloured (exponentially correlated). There is good agreement with results obtained from simulation of sample paths of the appropriate stochastic differential equations. Multiplicative noise does not produce noise-controlled dynamics in this fashion.

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1. Introduction

Bifurcations are normally observed by slowly changing a parameter (‘sweeping’) and waiting for a qualitative change in behaviour. The standard theory of bifurcations, however, does not have explicit time-dependence of parameters [1]. Here we study the effect of noise on the loss of stability in the simple equation

\[ \dot{y} = gy. \]  

(1)

This is a linearised dynamic bifurcation: a bifurcation in which the parameter \( g \) is explicitly a function of time.

Normally it is said that loss of stability in (1) is signalled by exponential growth of \(|y|\) for \( g > 0 \). However if we continuously increase \( g \) from \( g < 0 \) and wait for \(|y|\) to exceed some fixed value, we find that this occurs well after \( g = 0 \) [2]. In this paper we set

\[ g = \mu t \]  

(2)

and let \( y \) take (fixed) initial value \( y_0 \) at some \( g = g_0 < 0 \). Then the solution of (1),

\[ y = y_0 e^{\frac{\mu}{2} (g^2 - g_0^2)}, \]  

(3)

reveals that \(|y| < |y_0|\) until \( g = -g_0 \), regardless of the magnitude of \( g_0 \) and how small the sweep rate \( \mu \) is. However if \( \mu \) is sufficiently small the dynamics enter a noise-controlled régime, in which the value of \( g \) when \(|y|\) reaches \(|y_0|\) is a random variable controlled by the size of additive random disturbances [3,4,5]. This random variable we call the exit value; it is more akin to a mean first passage time than to the usual definition of a bifurcation point [6].

This work is chiefly concerned with the probability distribution of the exit value in the “noise-controlled régime” [7] when (1) is perturbed by additive white and coloured noise. Because (1) is a non-autonomous linear equation, it is possible to calculate an exact solution for the probability distribution of \( y \) as a function of time and thus to calculate the probability that \(|y| > |y_0|\), which we will denote by \( \Pr(|y| > |y_0|) \). The probability distribution of the exit value is the derivative with respect to \( g \) of \( \Pr(|y| > |y_0|) \). Additive noise reduces the average exit value by defining a level below which \( \langle y^2 \rangle \) does not fall.

The noise-controlled régime is in force if

\[ \mu |\ln \epsilon| < \frac{1}{2} g_0^2 + \mathcal{O}(\mu) \]  

(4)
where $\epsilon$ is the (r.m.s.) noise level. This is the most likely régime for small $\mu$. Concentrating on this régime makes possible a simpler analytical treatment than previously reported [8, 6], including explicit derivation of the probability distribution of exit values. This is presented in section 2. We have chosen $|y| = |y_0|$ as the fixed exit level; this makes transparent the difference between the noise-controlled régime and the deterministic limit, but we emphasise that, as long as the noise-controlled régime remains in force, the dynamics are independent of the initial conditions.

Our analytical results are in excellent agreement with results obtained from simulation of sample paths. These simulations we performed using an algorithm with weak second order convergence (the errors in the moments of the exit value distribution are proportional to the second power of the timestep [9]). Our results are also consistent with the reported results of analogue and digital simulations [10, 6] performed in different parameter ranges.

If the noise is coloured, the probability distribution of $y$ is narrowed and the mean exit value is increased [8, 5]. The correction is, however, small except for very large values of the correlation time $\tau$ [11]. In section 3 we derive a new formula for this correction which we successfully compare with numerical results in the (physically realistic) régime $\mu \tau^2 < \mathcal{O}(1)$. In section 4 we show that multiplicative noise has a qualitatively different (and in general much less dramatic) effect on the dynamics. In section 5 we discuss the situation where $y$ has spatial degrees of freedom and the passage through the point of stability loss causes a growth of spatial order.

Noise-controlled dynamics are useful in understanding the behaviour of systems of ordinary differential equations used as low-order models in fluid mechanics [12, 13] and laser physics [14, 15]. In these cases the time-dependence of the parameter corresponding to $g$ is in general more complicated; our system serves as a linearised model which, being a good approximation near $g = 0$, captures the essential dynamics.

2. Additive white noise

To study the effect of additive white noise we replace the ordinary differential equation (ODE)

$$dy_t = g y_t dt + \epsilon dW_t$$

where $\epsilon$ is constant with $0 \leq \epsilon \ll 1$ and $W_t$ is the Wiener process [16]. The solution of this SDE with $g = \mu t$ is

$$y_t = e^{\frac{1}{2} \mu (t^2 - t_0^2)} \left( y_0 + \epsilon \int_{t_0}^{t} e^{-\frac{1}{2} \mu (t^2 - s^2)} dW_s \right)$$

(6)
where \( t_0 = g_0/\mu \). For each \( t \), \( y \) is a Gaussian random variable with mean equal to the solution (3) of the corresponding deterministic equation (1) and variance a function of time given by

\[
\sigma^2 = \langle y_t^2 \rangle - \langle y_t \rangle^2 = \epsilon^2 e^{\frac{1}{2}g^2} \int_{t_0}^{t} e^{-\mu s^2} ds.
\]  

(7)

If \(-g_0 > O(\sqrt{\mu})\) then, for \( g > O(\sqrt{\mu})\),

\[
\sigma^2 = \epsilon^2 e^{\frac{1}{2}g^2} \sqrt{\frac{\pi}{\mu}}.
\]  

(8)

The noise-controlled régime is in force when \( \sigma^2 \gg \langle y_t \rangle^2 \) for \( g > O(\sqrt{\mu}) \), so that all memory of the initial conditions \((g_0, y_0)\) is lost and noise controls the exit value distribution. For this we need

\[
\mu |\ln \epsilon| < \frac{1}{2} g_0^2 + \mu \left( \frac{1}{4} \ln \frac{\pi}{\mu} - \ln y_0 \right).
\]  

(9)

Since the right hand side is \( O(1) \) in general, (9) holds even for extremely small values of \( \epsilon \) if \( \mu \) is moderately small \((\mu < 0.1)\) [7].

In what follows, we assume that the noise level, \( \epsilon \), is \( \ll |y_0| \) but large enough that (9) holds. For additive noise this means that, for \( g > O(\sqrt{\mu}) \), we can set \( \langle y_t \rangle = 0 \). The probability distribution of exit values is then given by

\[
\frac{d}{dg} \Pr(|y| > |y_0|) = \frac{d}{dg} \sqrt{\frac{2}{\pi \sigma}} \int_{y_0}^{\infty} e^{-\frac{x^2}{2\sigma^2}} dx
\]

\[
= \sqrt{\frac{2}{\pi \mu \sigma^2}} \frac{d\sigma}{dt} \left( - \int_{y_0}^{\infty} e^{-\frac{x^2}{2\sigma^2}} dx + \int_{y_0}^{\infty} \frac{x^2}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}} dx \right)
\]

\[
= \sqrt{\frac{2}{\pi \mu y_0}} e^{\frac{-y_0^2}{2\sigma^2}}.
\]  

(10)

using integration by parts.

Let \( \hat{g} \) be the value of \( g \) such that \( \sigma^2(\hat{g}) = y_0^2 \) and let \( v = -(g - \hat{g})(\hat{g}/\mu) \). Then \( \sigma \simeq e^v \) and the probability that the exit value of \( g \) lies between \( g \) and \( g + dg \) for additive white noise is \( R_A(g)dg \) where, to lowest order in \( \mu \),

\[
R_A(g) \simeq \sqrt{\frac{2}{\pi}} e^{\frac{\mu}{2} e^{2v}}.
\]  

(11)
Distribution of exit values in the noise-controlled régime with white noise. Sample paths of (5) were simulated numerically numerous times, with $y_0 = 1$ and $g_0 = -1$. The next value of $g$ at which $y = 1$ was recorded each time and the resulting distributions of exit values are plotted along with the calculated distribution (11). The most probable value, given by (12), is indicated by a vertical line. (Without noise, the exit value is always 1; if $\mu|\ln \epsilon| > 0.5 + (\mu/4) \ln(\pi/\mu)$ the noise-controlled régime is not in force and the exit value distribution is narrow, Gaussian and centred on $g = 1$.)

This distribution is compared with numerical results in Figure 1; it is the probability distribution of $\frac{1}{2} \ln n^2$ where $n$ is a Gaussian random variable with unit variance. The most probable value of $v$ is 0, so the most probable exit value is $\hat{g}$ where

$$\hat{g} = \sqrt{2\mu|\ln \epsilon| - \frac{\mu}{2} \ln \frac{\pi}{\mu} + 2\mu \ln y_0}. \quad (12)$$

We can obtain exact expressions for the moments of the exit value distribution in the noise-controlled régime by noting that $v = \frac{1}{2} (v' + \ln 2)$ where $v' = \ln \frac{1}{2} n^2$. 

Figure 1.
The characteristic function, \( \phi(s) \), of \( \nu' \) is known [17]:

\[
\phi(s) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-e^{su}u^2} e^{isu} \, du = \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{1}{2} + is\right). \tag{13}
\]

The mean exit value, \( \langle g_e \rangle \) is

\[
\langle g_e \rangle = \hat{g} + \frac{\mu}{2\hat{g}} (\gamma + \ln 2) \tag{14}
\]

where \( \gamma = 0.57721 \ldots \) (Euler’s constant) and and the variance of the distribution of exit values is

\[
\langle g_e^2 \rangle - \langle g_e \rangle^2 = \frac{\pi^2 \mu^2}{8 \hat{g}^2}. \tag{15}
\]

It is instructive to consider how these results arise in terms of trajectories (sample paths) of (5). Firstly note that, once \( g > O(\sqrt{\mu}) \), most trajectories have \( |y| \gg \epsilon \), and so are no longer affected by the noise. Additive noise acts near \( g = 0 \), effectively providing a random initial condition for the subsequent evolution [18, 5]. The most probable exit value is determined by the time, \( O(\sqrt{\ln(y_0/\epsilon)/\mu}) \), taken for the variance \( \langle y^2 \rangle \) to rise from a level proportional to \( \epsilon^2 \) at \( g = 0 \) to \( y_0^2 \) at \( g = \hat{g} \). The probability distribution of exit values has width \( O(\mu) \) because most members of the ensemble of trajectories cross \( |y| = |y_0| \) within a time \( O(1) \) of each other, corresponding to an \( O(\mu) \) spread in \( g \). Now consider the exponential rise of neighbouring trajectories; the difference in their exit values is proportional to the log of the ratio of their starting points. The Gaussian distribution in \( y \) near \( g = 0 \) is thus transformed into a distribution of exit values whose characteristic shape is that of \( \frac{1}{2} \ln n^2 \), where \( n \) is a Gaussian random variable.

3. Additive coloured noise

Noise is often added to an ordinary differential equation as a way of approximating the influence of neglected small effects while retaining simplicity of description. Often “coloured noise”, i.e. noise with non-zero correlation time, is more appropriate than white noise. If \( y \) is subject to coloured noise, we replace (5) by

\[
dy_t = \mu ty_t \, dt + \epsilon \eta_t \, dt. \tag{16}
\]

For \( \eta_t \) we use the most commonly studied coloured noise process [19], the Ornstein-Uhlenbeck process satisfying the SDE

\[
d\eta_t = \frac{\eta_t}{\tau} \, dt + dW_t \tag{17}
\]
which has
\[ \langle \eta_t \rangle = 0 \quad \text{and} \quad \langle \eta_s \eta_t \rangle = \frac{1}{2\tau} e^{-|s-t|/\tau} \quad \forall s, t. \]  

The solution of (16) with \( y = 1 \) at \( t = t_0 \) is
\[ y_t = e^{\frac{1}{2} \mu(t^2-t_0^2)} + \epsilon \int_{t_0}^{t} e^{-\mu(t^2-s^2)} \eta(s) ds. \]  

We restrict ourselves to the noise-controlled régime where, for \( t > \mathcal{O}(\frac{1}{\sqrt{\mu}}) \),
\[ \langle y_t^2 \rangle \simeq e^{2\mu t^2} \frac{1}{\tau} \int_{t_0}^{t} \int_{t_0}^{s} e^{-\frac{1}{2}(s^2+u^2)-(s-u)/\tau} du ds \]
\[ = e^{2\mu t^2} \frac{1}{\tau} \int_{t_0}^{t} e^{-\frac{1}{2} s^2 - s/\tau + \frac{1}{2\mu\tau^2} \sqrt{\frac{2}{\mu}} \int_{\alpha_0}^{\alpha} e^{-v^2} dv} ds \]
where \( \alpha = \sqrt{\mu/2}(s - (\mu\tau)^{-1}) \) and \( \alpha_0 = \sqrt{\mu/2}(t_0 - (\mu\tau)^{-1}) \). Most of the support of the integrand is near \( s = 0 \). We therefore use the following approximation for the error function [20],
\[ \int_{\alpha_0}^{\alpha} e^{-v^2} dv \simeq \frac{e^{-\alpha^2}}{-\alpha + \sqrt{\alpha^2 + \frac{1}{\pi}}} \]
and expand the denominator in (21), about \( s = 0 \), in a form convenient for the integration over \( s \):
\[ \sqrt{\frac{2}{\mu} \frac{1}{\tau}} - \alpha + \sqrt{\alpha^2 + \frac{1}{\pi}} \simeq \frac{2}{1 + \kappa} e^{\frac{\mu}{\kappa} s} \]
where \( \kappa = \sqrt{1 + \frac{8}{\pi} \mu\tau^2} \). Thus for \( t-\tau \) and \( |t_0|-\tau > \mathcal{O}(1/\sqrt{\mu}) \), \( \langle y_t^2 \rangle \) is approximately given by
\[ \langle y_t^2 \rangle \simeq \sqrt{\frac{\pi}{\mu}} e^{2\mu t^2} \frac{e^{\mu^2/4\kappa^2}}{1 + \kappa} \frac{2}{1 + \kappa}. \]  

This is smaller than the white noise value (8) by a factor which is a function of \( \mu\tau^2 \). The solution of (16) also differs from that of (5) in multi-time correlations \( \langle y_t y_s \rangle \) but this is not important for the purposes of calculating the distribution of exit values of \( g \). We make the approximation that noise with correlation time \( \tau \) is equivalent to white noise at a slightly smaller level \( \epsilon' \) where
\[ \epsilon' \simeq \epsilon \sqrt{\frac{2}{1 + \kappa} e^{\mu^2/8\kappa^2}}. \]
The mean exit value in the presence of exponentially correlated noise with correlation time $\tau$ is therefore given by:

$$\langle g_e \rangle = \sqrt{2\mu|\ln \epsilon| - \mu f(\mu \tau^2)} + \frac{\mu}{2\hat{g}}(\gamma + \ln 2) + O(\mu^2)$$  \hspace{1cm} (25)

where

$$f(\mu \tau^2) = \ln\left(\frac{2}{1 + \kappa}\right) - \frac{\mu \tau^2}{4\kappa^2} - \frac{1}{2} \ln\left(\frac{\pi}{\mu}\right),$$  \hspace{1cm} (26)

$\gamma = 0.57721 \ldots$ (Euler’s constant) and $\hat{g}$ is given by (12). This formula is compared with numerical results in Figure 2. The most probable exit value and variance of the exit value distribution can be calculated similarly.

4. Multiplicative white noise

If $y$ is subject to noise which is proportional to $y$ then we replace (1) by the SDE

$$dy = gyt \, dt + \epsilon y_t \, dW_t.$$  \hspace{1cm} (27)

(The most natural way for multiplicative noise to arise is if $g$ itself is subject to fluctuations.) The solution of (27) with $y = y_0$ at $t - t_0$ is [16]

$$y = y_0 e^{\frac{1}{2} \mu (t^2 - t_0^2) - \frac{1}{2} \epsilon^2 (t - t_0)} e^{\epsilon (W_t - W_{t_0})}$$  \hspace{1cm} (28)

where $t_0 = g_0 / \mu$. The median value of $y$ is $y_0 \exp(\frac{1}{2\mu} (g^2 - g_0^2 - \epsilon^2 (g - g_0)))$; the factor $\exp(\epsilon (W_t - W_{t_0}))$ is always positive with mean $\exp(\frac{1}{2} \epsilon^2 (t - t_0))$. As before, we first calculate the probability that $y > y_0$ as a function of $g$.

$$Pr(|y| > |y_0|) = Pr(e^{\epsilon (W_t - W_{t_0})} > \frac{1}{m}) = \frac{1}{\sqrt{2\pi \sigma_m^2}} \int_{-\ln m}^{\infty} e^{-\frac{x^2}{2\sigma_m^2}} \, dx$$  \hspace{1cm} (29)

where $m = \exp(\frac{1}{2} \mu (t^2 - t_0^2) - \frac{1}{2} \epsilon^2 (t - t_0))$ and $\sigma_m^2 = \epsilon^2 (t - t_0)$. When $\epsilon \ll 1$ the possibility that one trajectory crosses $|y| = |y_0|$ more than once can be ignored; the exit value distribution is then Gaussian, centred on $-g_0 + \epsilon^2$, with variance $(2\mu/|g_0|)\epsilon^2$.

Using our exit value definition, we find that multiplicative noise increases the median delay even beyond the deterministic value. Other definitions, based for example on the trajectory of $\langle y_t^2 \rangle$ [4, 8], may lead to the opposite conclusion because $\langle y_t \rangle$ and $\sqrt{\langle y_t^2 \rangle}$ are greater than the median value of $y$. Regardless of definition, the effect is small if $\epsilon$ is small. For multiplicative noise, there is no analogue of the noise-controlled régime; we need $O(1)$ multiplicative noise to produce $O(1)$
Figure 2. Distribution of exit values in the noise-controlled régime with coloured noise. In (a) and (b) the distribution obtained by simulation of sample paths solution of (16) is shown along with the calculated distribution (11) (smooth curve) calculated with $\epsilon$ replaced by $\epsilon'$, according to (24), to account for the effect of non-zero correlation time in the noise. Graph (c) shows the calculated mean exit value as a function of $\tau$ (25) with $\mu = 0.01$ and $\epsilon = 10^{-10}$ (solid line) and some numerical results (crosses).

effects. We have demonstrated this when the noise is simply proportional to $y$; it will remain true if the noise is any $\mathcal{O}(1)$ function of $y$ because the noise also becomes tiny as $y$ does, and therefore does not dominate at any stage. In the presence of both additive and multiplicative noise, an exact solution can still be found for the probability distribution of $y$ as a function of time; to derive the exit value statistics, it is useful to note that additive noise acts primarily near $g = 0$ and multiplicative noise later [18, 11].
5. Conclusion

When the sweep through the point of stability loss is slow, the dynamics are controlled by very small random additive noise and memory of initial conditions is lost. For small values of the sweep rate ($\mu < 0.1$), noise produces $\mathcal{O}(1)$ effects even if its magnitude is extremely small. In this noise-controlled régime the most probable value of $g$ when $y$ reaches an $\mathcal{O}(1)$ predetermined level is $\hat{g} = \sqrt{2\mu \ln \epsilon} + \mathcal{O}(\mu)$ and the probability distribution of exit values has standard deviation proportional to $\mu$. As long as the noise is additive and Gaussian, its correlation time affects the exit value distribution only slightly. Multiplicative noise does not produce noise-controlled dynamics in this way.

A physical picture of this noise-controlled system is the dynamics of a strongly damped object allowed to slide in a slowly-changing potential and continuously subject to random influences. The variable $y$ is the distance from the extremum of the potential. For $t < 0$, the extremum is the centre of a well; for $t > 0$ the extremum is the crest of a hill. Once $t > 0$, the object lingers near the crest for a time controlled by the noise level before sliding away.

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