NONVANISHING OF THE DIFFERENTIAL OF HOLOMORPHIC MAPPINGS AT BOUNDARY POINTS

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§0 Introduction

Let $M$ and $M'$ be two smooth hypersurfaces in $\mathbb{C}^n$. A smooth mapping $h : M \to M'$ is a CR mapping if its components are annihilated by the induced Cauchy-Riemann operator on $M$. Let $p_0 \in M$ and suppose that near $p_0$, $h$ is the restriction of a holomorphic mapping $H$ defined on one side of $M$ near $p_0$ and smooth up to $M$. We shall say that $h$ satisfies the Hopf lemma property at $p_0$ if the component of $H$ normal to $M_2$ has a nonzero derivative at $p_0$ in the normal direction to $M_1$. The hypersurface $M$ is said to be minimal at $p_0 \in M$ if there is no germ of a complex hypersurface contained in $M$ through $p_0$. Recall the theorem of Trépreau [T] that if $M$ is minimal at $p_0$, then every CR function defined on $M$ near $p_0$ extends holomorphically to at least one side of $M$ in $\mathbb{C}^n$ near $p_0$. A stronger condition on a hypersurface $M$ at a point $p_0$ is that of essential finiteness (as defined in [BJT], [BR3], [DA2]). We will recall this definition in §1. We note here that if $M$ is of D’Angelo finite type at $p_0$ [DA1], then $M$ is essentially finite at $p_0$ (and hence minimal at $p_0$).

In this paper we prove a general result of the “Hopf lemma” type for CR mappings, with nonidentically vanishing Jacobians, between real hypersurfaces in $\mathbb{C}^n$. Applications of this result to finiteness and holomorphic extendibility of such mappings are also given. The novelty here is that we make no assumption on the non-flatness of the mapping or its Jacobian, nor do we assume that the hypersurfaces are pseudoconvex or minimally convex.

Theorem 1. Let $M$ be a smooth, connected, orientable hypersurface in $\mathbb{C}^n$ which is essentially finite at all points. Let $h : M \to M'$ be a smooth CR mapping from $M$ to another smooth hypersurface $M' \subset \mathbb{C}^n$, with $\text{Jac } h \neq 0$. Let $p_0 \in M$, and suppose that $h^{-1}(h(p_0))$ is a compact subset of $M$. Then $h$ satisfies the Hopf lemma property at $p_0$.

If the hypersurfaces are pseudoconvex, the result above follows from the classical Hopf lemma for harmonic functions, as proved in Fornaess [F] (see also [BBR]). Other results of the Hopf lemma type for CR mappings were previously obtained in [BR3] and [BR5].

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As shown in [BBR], [BR1], and [DF], the Hopf lemma property can be used to prove holomorphic extension of CR mappings between real analytic hypersurfaces. From Theorem 1 we obtain the following corollary.

**Corollary 1.** Let $M, M', h$ be as in Theorem 1, and assume in addition that $M$ and $M'$ are real analytic. Then $h$ extends holomorphically to a neighborhood of $p_0$ in $\mathbb{C}^n$.

In the global case, i.e. when $M$ and $M'$ are compact boundaries, the compactness of $h^{-1}(h(p_0))$ is automatically satisfied, yielding the following result.

**Theorem 2.** Let $\Omega$ and $\Omega'$ be bounded domains in $\mathbb{C}^n$ with smooth boundaries, such that $\partial \Omega$ is essentially finite at all points. Suppose $H : \Omega \to \Omega'$ is a proper holomorphic mapping, smooth up to $\partial \Omega$. Then $H$ satisfies the Hopf lemma property at every point $p \in \partial \Omega$. Furthermore, $H$ is finite-to-one on $\Omega$.

When $\Omega$ and $\Omega'$ in Theorem 2 are real analytic, we obtain a new proof of the following result of the second author and Pan [HP], extending earlier results in [BR1], [DF], [BR3].

**Corollary 2.** If $\Omega$ and $\Omega'$ are bounded domains in $\mathbb{C}^n$ with real analytic boundaries, and $H : \Omega \to \Omega'$ is a proper holomorphic mapping, smooth up to $\partial \Omega$, then $H$ extends holomorphically to a neighborhood of $\Omega$ in $\mathbb{C}^n$.

It should be noted that Theorem 2 and Corollary 2 may be proved more directly (see Remark 2.2).

Another application of Theorem 1 is a propagation result of the Hopf lemma property (Theorem 3), as well as real analyticity (Corollary 3), analogous to the classical Hartog’s theorem for extension of holomorphic functions.

**Theorem 3.** Let $M$ be a smooth, orientable, connected hypersurface in $\mathbb{C}^n$ which is essentially finite at all points, and let $h : M \to M'$ be a smooth CR mapping from $M$ to another smooth hypersurface $M' \subset \mathbb{C}^n$, with $\text{Jac } h \not\equiv 0$. Suppose that $U_1$ and $U$ are relatively compact open subsets of $M$, with $\overline{U_1} \subset U$. Then if the Hopf lemma property holds at every point in $U \setminus U_1$, it also holds everywhere in $U$.

**Corollary 3.** Let $M$ be a real analytic, orientable, connected hypersurface in $\mathbb{C}^n$ which is essentially finite at all points, and let $h : M \to M'$ be a smooth CR mapping from $M$ to another real analytic hypersurface $M' \subset \mathbb{C}^n$, with $\text{Jac } h \not\equiv 0$. Suppose that $U_1$ and $U$ are relatively compact open subsets of $M$, with $\overline{U_1} \subset U$. If $h$ is real analytic in $U \setminus U_1$, then $h$ is real analytic everywhere in $U$ and hence extends holomorphically to an open neighborhood of $U$ in $\mathbb{C}^n$.

**Remark 0.1.** In Theorem 1 and Corollary 1, the condition $\text{Jac } h \not\equiv 0$ may be replaced by the stronger condition that $M'$ does not contain any nontrivial complex variety through $p_0$. (See e.g. [BR4].) A similar statement holds for Theorem 3 and Corollary 3.

Some of the results of the present paper, including Corollary 1, were announced earlier by the second author. Also, a recent preprint of Y. Pan [P] contains a special case of Corollary 1 above and other related results.

§1 *Preliminaries*

Let $M$ be a smooth real hypersurface in $\mathbb{C}^n$. For $p \in M$, we denote by $T_pM$ the real tangent space of $M$ at $p$ and by $\mathbb{C}T_pM$ its complexification. We denote
by $V_p M$ the complex subspace of $\mathbb{C} T_p M$ consisting of all antiholomorphic vectors
tangent to $M$ at $p$, and by $T^c_p M = \text{Re } V_p M$ the complex tangent space of $M$ at $p$
considered as a real subspace of $T_p M$. Note that if $h$ is a smooth CR map from $M$
to a hypersurface $M'$, then $h$ satisfies the Hopf lemma property mentioned in §0 if and only if
$$dh(T_{p_0} M) \not\subset T^c_{h(p_0)} M'.$$
Note that for this form of the definition, it is not necessary to assume that $h$ extends
holomorphically to one side of $M$.

If $\rho(z, \overline{z})$ is a defining function for $M$ near $p_0 = 0$, with $\rho(0) = 0$ and $d\rho(0) \neq 0$,
we consider the formal Taylor series of $\rho$ in $z$ and $\overline{z}$ at $0$ and write $R(z, \zeta)$ for
its complexification, i.e. $R(z, \zeta) \equiv \sum a_{\alpha, \beta} z^\alpha \zeta^\beta$, where $\alpha! \beta! a_{\alpha, \beta} = \rho_{z_\alpha \overline{z}_\beta}(0)$. Let
$X_1, \ldots, X_n$, be the vector fields with formal power series coefficients given by
$$X_j = R_{\zeta_j}(0, \zeta) \frac{\partial}{\partial \zeta_j} - R_{\zeta_j}(0, \zeta) \frac{\partial}{\partial \zeta_n}, \quad j = 1, \ldots, n - 1,$$
where we have assumed $\rho_{z_n}(0) \neq 0$. For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_{n-1})$ we define
c$_\alpha(z)$ in the ring of convergent power series in $n$ complex variables, by
$$(1.1) \quad c_\alpha(z) = X^\alpha R(z, \zeta)|_{\zeta = 0},$$
where $X^\alpha = X_1^{\alpha_1} \cdots X_{n-1}^{\alpha_{n-1}}$. We say that $M$ is essentially finite at $0$ if the ideal
$(c_\alpha(z))$ generated by the $c_\alpha(z)$ in the ring $\mathbb{C}[[z]]$ of formal power series is of finite
codimension. It should be noted that this definition is independent of the choice of
coordinates and defining function $\rho$; it is given in a slightly different form in [BR3].
Note also that if $M$ is essentially finite at $p_0$, then $M$ is minimal at $p_0$, and that if $M$
is of D’Angelo finite type, then it is essentially finite.

Recall that an analytic disc in $\mathbb{C}^n$ is a continuous mapping $A : \overline{\Delta} \to \mathbb{C}^n$ which
is holomorphic in $\Delta$, where $\Delta$ is the open unit disc in the plane. We say that $A$
is attached to $M$ if $A(\partial \Delta) \subset M$. Let $M$ be a smooth hypersurface minimal at $p_0$. As
in [BR5], we say that $M$ is minimally convex at $p_0$ if $M$ is minimal at $p_0$, and there is
a neighborhood $U$ of $p_0$ in $M$ and a side of the hyperplane $T_{p_0} M$ in $\mathbb{C}^n$ such
that the real derivatives $\frac{\partial}{\partial \zeta} [A(\zeta)]|_{\zeta = 1}$ lie on that side or in $T_{p_0} M$, for all sufficiently
smooth analytic discs $A$ attached to $U$ with $A(1) = p_0$. Here $\zeta = \xi + i\eta$, with
$\xi \in \Delta$.

For the convenience of the reader we begin by stating a number of known results,
Theorems A, B, C and D below, which will be important for the proofs of Theorems
1. Theorem A is a consequence of a result of Tumanov [Tu], as observed in [BR5].

**Theorem A [Tu, BR5]**. Let $M$ be a smooth, real hypersurface in $\mathbb{C}^n$, and assume
that $M$ is minimal at $p_0$. Then one of the following two conditions holds.

1. $M$ is minimally convex at $p_0$.
2. Every CR function defined in a neighborhood of $p_0$ in $M$ extends holomorphically
to a full neighborhood of $p_0$ in $\mathbb{C}^n$.

**Theorem B [BR5]**. If $h$ is a smooth CR mapping from a smooth hypersurface
$M$ to another smooth hypersurface $M'$, with $M$ minimal at $p_0$, $\text{Jac } h \neq 0$, and $M'$
minimally convex at $p_0 = h(p_0)$, then the Hopf lemma property holds at $p_0$.

We need also to recall a result which follows from Theorem 4 in [BR3].
Theorem C [BR3]. Let $H$ be a holomorphic map defined in a neighborhood of a smooth hypersurface $M$ essentially finite at $p_0$, $H(M) \subset M'$, with $M'$ another smooth hypersurface of $\mathbb{C}^n$, and Jac $H \neq 0$. Then $H$ satisfies the Hopf lemma property at $p_0$ and $H$ is finite-to-one in a neighborhood of $p_0$.

We shall also need a stronger version of this result, and we indicate here its proof.

Theorem D. Let $H$ be a holomorphic map defined on one side of a smooth hypersurface $M$ essentially finite at $p_0$, with $H$ smooth up to $M$. Suppose $H(M) \subset M'$, with $M'$ another smooth hypersurface of $\mathbb{C}^n$. Then Jac $H$ is not flat at $p_0$ if and only if $H$ satisfies the Hopf lemma property at $p_0$. In addition, if either of these equivalent conditions is satisfied, then any smooth extension of $H$ to a sufficiently small neighborhood of $p_0$ in $\mathbb{C}^n$ is finite-to-one.

Proof of Theorem D. If Jac $H$ is not flat at $p_0$, we conclude that if $G$ is a formal transversal component of $H$ (as defined in [BR3]), then $G \neq 0$. Hence, by Theorem 4 of [BR3], it follows that $H$ satisfies the Hopf lemma property and is of finite multiplicity. Conversely, if $H$ satisfies the Hopf Lemma property at $p_0$, by Theorem 4 of [BR3] it follows again that $H$ is of finite multiplicity and also that $M'$ is essentially finite at $H(p_0)$. By Theorem 3 of [BR3], we conclude also that $H$ is not totally degenerate at $p_0$, in the sense of [BR3], and hence, using again the Hopf lemma property, Jac $H$ is not flat at $p_0$.

We may assume $p_0 = H(p_0) = 0$. Since $H$ is holomorphic on one side of $M$ and smooth up to the boundary, its Taylor series at 0 defines a formal (not necessarily convergent) holomorphic map $H = (\sum a^1_\alpha z^\alpha, \ldots, \sum a^n_\alpha z^\alpha)$. The equivalent conditions above imply that $H$ is finite as a formal map. That is, the ideal generated by the $\sum a^j_\alpha z^\alpha$, $j = 1, \ldots, n$, is of finite codimension in the ring $\mathbb{C}[[z]]$ of formal power series in $z$. Since the Taylor series of $H$ coincides with that of any smooth extension of $H$ to $\mathbb{C}^n = \mathbb{R}^{2n}$, we conclude e.g. by [GG], [EL], that this extension is finite-to-one near 0.

§2 Inverse image of a nonminimally convex point.

In this section we shall state and prove a new result, Theorem 4 below, from which Theorem 1 will follow.

Theorem 4. Let $h: M \to M'$ be a CR map, where $M$, $M'$, $h$, $p_0 \in M$ satisfy all the conditions of Theorem 1. In addition, suppose that $M'$ is not minimally convex at $p_0' = h(p_0)$. Then all CR functions on $M$ extend holomorphically to a full neighborhood of $p_0$ in $\mathbb{C}^n$. In particular, $h$ extends holomorphically to a neighborhood of $p_0$ and satisfies the Hopf lemma property at $p_0$.

Before proving Theorem 4, we note that Theorem 1 is a consequence of Theorem 4 and Theorem B above. Indeed, if $M'$ is minimally convex at $h(p_0)$, then since any essentially finite hypersurface is minimal at all points, Theorem 1 follows from Theorem B. On the other hand, if $h(p_0)$ is not minimally convex, Theorem 1 is an immediate consequence of Theorem 4.

In the rest of this section, we shall prove Theorem 4. We may assume that $p_0 = p_0' = 0$, and we let $Z_M = h^{-1}(0)$. Note that $Z_M$ is a compact subset of $M$ by the assumptions of the theorem. Hence without loss of generality, we shall assume that $M$ is bounded.
The following lemma shows that we can reduce the proof of the theorem to the case where \( h \) extends holomorphically to one side of \( M \).

**Lemma 2.1.** Under the assumptions of Theorem 1, there exists an open neighborhood \( U \) of \( 0 \) in \( C^n \) such that \( Z_M \cap U \) is compact in \( M \cap U \) and \( h \) extends holomorphically to \( U^+ \), one side of \( M \) in \( U \).

**Proof.** Since \( M \) is essentially finite and hence minimal at all points, it follows that \( h \) extends holomorphically to at least one side of \( M \) at each point. Since \( M \) is orientable, it is given by a global smooth defining function \( \rho \) with nonvanishing gradient on \( M \). We may assume that \( h \) extends to the plus side of \( M \), (i.e. where \( \rho(z) > 0 \)) near \( 0 \). Let \( M_1 \) be the largest connected open subset of \( M \) containing \( 0 \) such that \( h \) extends holomorphically to the plus side of \( M \) near every point of \( M_1 \).

If \( M_1 = M \), then the Lemma is an immediate consequence of the assumptions of Theorem 1. Assume therefore that \( M_1 \) is a proper subset of \( M \) and let \( \partial M_1 \) be its boundary in \( M \). For \( \delta > 0 \), let \( M_1^\delta = \{ p \in M_1 : \text{dist}(p, \partial M_1) > \delta \} \). Since at every point of \( M \), \( h \) extends holomorphically to at least one side of \( M \), it follows from the definition of \( M_1 \) that there is an open neighborhood \( U \) of \( \partial M_1 \) in \( M \) such that \( h \) extends holomorphically to both sides of \( M \) at every point in \( U \cap M_1 \). Applying Theorem C, we conclude that \( Z_M \cap U \cap M_1 \) is a discrete set.

Let \( \partial M \) be the boundary of \( M \) in \( C^n \) and choose \( a > 0 \) such that \( a < \text{dist}(Z_M, \partial M) \) (which is possible by the assumption of the theorem). Denote by \( M^a = \{ p \in M : \text{dist}(p, \partial M) > a \} \). Note that \( \partial M_1 \cap \overline{M^a} \) is compact in \( M \). Therefore, there exists \( \delta_0 > 0 \) such that for all \( \delta, \delta_0 < \delta < \delta_0 \), we have

\[
\partial M_1^\delta \cap Z_M = \overline{M^a} \cap \partial M_1^\delta \cap Z_M \subset U \cap M_1.
\]

By compactness and the discreteness mentioned above, we conclude that \( \partial M_1^\delta \cap Z_M \) is a finite set. Since these sets are all disjoint for different \( \delta \)'s, we conclude that there exists \( \delta_1 \), with \( 0 < \delta_1 < \delta_0 \), such that \( \partial M_1^{\delta_1} \cap Z_M = \emptyset \). Now the lemma follows by taking \( U \) to be a sufficiently small open neighborhood of \( M_1^{\delta_1} \) in \( C^n \). \( \square \)

By Lemma 2.1, after shrinking \( M \) if necessary, we may now assume that there is a connected open set \( \mathcal{O} \) in \( C^n \) such that:

(i) \( \mathcal{O} \cup M \) is a manifold with boundary of class \( C^\infty \).
(ii) \( h \) extends holomorphically to \( \mathcal{O} \); if \( H \) denotes the holomorphic extension of \( h \), then \( H \in C^\infty(\overline{\mathcal{O}}) \).
(iii) \( h(0) = 0 \).
(iv) \( Z_M = h^{-1}(0) \) is a compact subset of \( M \).

We write

\[
Z = H^{-1}(0) \cap \mathcal{O}.
\]

We shall show that we can take \( H \) to be a proper mapping of an open domain to its image. The following lemma is crucial in this construction.

**Lemma 2.2.** Let \( V \) be a connected open neighborhood of \( Z_M \) in \( M \), with \( \overline{V} \) a compact subset of \( M \). For \( \delta > 0 \), let

\[
\mathcal{O}^\delta = \{ z \in \mathcal{O} : \text{dist}(z, \overline{V}) < \delta \},
\]


\begin{itemize}
\item[(i)] \( \mathcal{O} \cup M \) is a manifold with boundary of class \( C^\infty \).
\item[(ii)] \( h \) extends holomorphically to \( \mathcal{O} \); if \( H \) denotes the holomorphic extension of \( h \), then \( H \in C^\infty(\overline{\mathcal{O}}) \).
\item[(iii)] \( h(0) = 0 \).
\item[(iv)] \( Z_M = h^{-1}(0) \) is a compact subset of \( M \).
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\[
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\]
and \( \partial O^\delta = S^\delta_1 \cup S^\delta_2 \), with \( S^\delta_1 = \partial O^\delta \cap M \) and \( S^\delta_2 = \partial O^\delta \setminus S^\delta_1 \). Then for any \( \delta_0 > 0 \) there exists \( \delta, 0 < \delta < \delta_0 \), such that \( S^\delta_2 \cap Z = \emptyset \).

**Proof.** Note that by assumption, \( \overline{S^\delta_2} \cap Z_M = \emptyset \). Hence there exists \( \epsilon > 0 \) such that for any \( \delta \) sufficiently small,

\[
Z \cap \{ z \in S^\delta_2 : \text{dist}(z, \overline{S^\delta_2} \cap M) < \epsilon \} = \emptyset.
\]

Let \( Z' = \{ z \in Z : z \) is not an isolated point of \( Z \} \). If there exists \( \delta > 0 \) such that \( Z' \cap O^\delta = \emptyset \), then \( Z \cap O^\delta \) is countable, and the conclusion of the lemma follows since the sets \( S^\delta_2 \), as \( \delta \) varies, are disjoint.

To complete the proof of the lemma, we shall assume

\[
Z' \cap O^\delta \neq \emptyset,
\]

for all \( \delta \) sufficiently small, and reach a contradiction. It is clear that under condition (2.5) we have

\[
\overline{Z} \cap M \neq \emptyset.
\]

We claim that \( \text{Jac} \ H \) vanishes to infinite order at every point \( p \in \overline{Z} \cap M \). Indeed if \( \text{Jac} \ H \) does not vanish to infinite order at a point \( p \in Z_M \), then by Theorem D, \( p \) is an isolated point of \( H^{-1}(0) \) in \( O \). Since this cannot be the case for \( p \in \overline{Z} \cap M \subset Z_M \), the claim follows.

Now let \( T^\delta_2 = \{ z \in S^\delta_2 : \text{dist}(z, \overline{S^\delta_2} \cap M) \geq \epsilon \} \), where \( \epsilon \) satisfies (2.4). Using (2.4), we note that

\[
\overline{Z} \cap \partial O^\delta \subset T^\delta_2 \cup (Z_M \cap \overline{Z'}).
\]

Since \( T^\delta_2 \) is compactly contained in \( O \) for sufficiently small \( \delta \), and \( \text{Jac} \ H \) is holomorphic in \( O \), there exists \( C > 0 \) such that for all multi-indices \( \alpha \)

\[
\sup_{z \in T^\delta_2} \text{sup} |D^\alpha \text{Jac} \ H(z)| \leq C^{|\alpha|+1} \alpha!.
\]

By the maximum principle on complex varieties (see e.g. [N1]) we have,

\[
\sup_{z \in \overline{Z} \cap \partial O^\delta} \text{sup} |D^\alpha \text{Jac} \ H(z)| = \sup_{z \in \overline{Z} \cap \partial O^\delta} |D^\alpha \text{Jac} \ H(z)|.
\]

However, as proved in the claim above, \( \text{Jac} \ H \) vanishes to infinite order on \( \overline{Z} \cap M \). Hence, in view of (2.7) and (2.8)

\[
\sup_{z \in \overline{Z} \cap \partial O^\delta} |D^\alpha \text{Jac} \ H(z)| = \sup_{z \in T^\delta_2} |D^\alpha \text{Jac} \ H(z)| \leq C^{|\alpha|+1} \alpha!.
\]

This proves that the radius of convergence of \( \text{Jac} \ H(z), z \in Z' \), is greater than a positive constant which is independent of the distance to \( M \). Hence \( \text{Jac} \ H \) extends holomorphically to a full neighborhood in \( \mathbb{C}^n \) of each point of \( \overline{Z} \cap M \). Since \( \text{Jac} \ H \) vanishes to infinite order there, it follows that \( \text{Jac} \ H \equiv 0 \), contrary to assumption. We conclude that (2.6), and hence (2.5), cannot hold, which completes the proof of Lemma 2.2. \( \Box \)

In reducing the proof of Theorem 4 to the global case of a proper mapping we shall use the following.
Proposition 2.10. Let $M$ be a connected hypersurface of class $C^0$ with $M \subset \mathcal{O}$, where $\mathcal{O}$ is an open bounded domain in $\mathbb{C}^n$, and let $H$ be a holomorphic mapping in $\mathcal{O}$, continuous up to the boundary, $\text{Jac } H \neq 0$, with $H(M)$ contained in another hypersurface $M'$ of class $C^0$. Suppose $0 \in M$, $H(0) = 0$, and

\begin{equation}
H^{-1}(0) \cap \partial \mathcal{O} \subset M.
\end{equation}

Then there is a subdomain $\mathcal{O}_1 \subset \mathcal{O}$ satisfying

(i) $0 \in \partial \mathcal{O}_1$, and there exists a sequence $\{z_j\} \subset \mathcal{O}_1$ such that $z_j \to 0$ and $H(z_j)$ stays strictly on one side of $M'$;

(ii) there exists $U$, a neighborhood of $0$ in $M'$, with $\overline{H(\mathcal{O}_1)} \supset U$;

(iii) $H : \mathcal{O}_1 \to H(\mathcal{O}_1)$ is a proper map.

Proof. We begin with the following lemma, which describes a well-known construction, see e.g. [BC].

Lemma 2.12. Let $\mathcal{O} \subset \mathbb{C}^n$ be an open bounded domain, and suppose $H : \mathcal{O} \to \mathbb{C}^n$ is a holomorphic mapping, continuous up to $\partial \mathcal{O}$. Let

\[ D = \{ z \in \mathcal{O} : H(z) \notin H(\partial \mathcal{O}) \}. \]

If $D \neq \emptyset$, then $H : D \to H(D)$ is finite-to-one and hence open. Furthermore, if $D_1$ is any connected component of $H(D)$, and $D'_1$ a connected component of $H^{-1}(D'_1)$, then $H : D_1 \to D'_1$ is a proper map.

Proof. Since this result is in the “folklore”, we shall be brief. We assume $D \neq \emptyset$. If $H$ is not finite-to-one, there exists $w \in H(D)$ for which $H^{-1}(w)$ has an accumulation point $z_0$ in $\overline{D}$ (and hence $H(z_0) = w$). By the definition of $D$,

\begin{equation}
H(D) \cap H(\partial \mathcal{O}) = \emptyset, \quad \text{and } H(\partial D) \subset H(\partial \mathcal{O}).
\end{equation}

Hence $z_0 \notin \partial D$. On the other hand, if $z_0 \in D$, then there is a nontrivial variety contained in $H^{-1}(w)$, which would necessarily intersect $\partial D$. Since this is also impossible, by the definition of $D$, $H$ is finite-to-one and hence open (see e.g. [R]).

To show that $H : D_1 \to D'_1$ is proper, suppose $z_j \to z_0$, $z_j \in D_1$, $z_0 \in \partial D_1$. Then by continuity $H(z_j) \to H(z_0) = w_0 \in \overline{D'_1}$. We claim that $w_0 \in \partial D'_1$. Indeed, if $w_0$ is an interior point of $D'_1$, let $V'$ be an neighborhood of $w_0$ in $D'_1$. Consider $H$ as a map from $\mathcal{O}$ to $\mathbb{C}^n$. Then a component of $H^{-1}(V')$ is contained in $D_1$, by the definition of $D$. Then $z_0$ would be an interior point of $D_1$, contrary to assumption. This proves Lemma 2.12. □

We may now complete the proof of Proposition 2.10. Let $\rho'$ be a defining function for $M'$ near 0. Without loss of generality, we may assume that there exist $z_j \in \mathcal{O}, j = 1, 2, \ldots$, with

\begin{equation}
\lim_{j} z_j = 0 \quad \text{and } \rho'(H(z_j)) > 0.
\end{equation}

Indeed, we first select $z_j \in \mathcal{O}$ with $\text{Jac } H(z_j) \neq 0$. Since $H$ is open near such a $z_j$, by slightly moving $z_j$ if necessary, we may assume $H(z_j) \notin M'$. Replacing $\rho'$ by $-\rho'$ and selecting a subsequence if necessary, we reach the desired conclusion (2.14).
Let $D$ be as in Lemma 2.12. By hypothesis (2.11) and the continuity of $H$ it follows that $H(\partial\mathcal{O}, M)$ is a compact set which does not contain $0$. Hence, by taking $z_j$ sufficiently close to $0$, we may assume that the points $z_j$ chosen in (2.14) are in $D$. We shall show that there exists $\epsilon > 0$, arbitrarily small, such that

$$\{ w \in \mathbb{C}^n : |w| < \epsilon \text{ and } \rho'(w) > 0 \} \equiv W_\epsilon \subset H(D). \tag{2.15}$$

Suppose that (2.15) is proved. Let $D'_1$ be the connected component of $H(D)$ containing the connected open set $W_\epsilon$. We claim that there is a connected component $D_1$ of $H^{-1}(D'_1)$ such that $0 \in \partial D_1$. Indeed, by Lemma 2.12, the restriction of $H$ to $D$ is finite-to-one, and the restriction to any connected component of $H^{-1}(D'_1)$ is proper, and hence onto $D'_1$. Therefore, $H^{-1}(D'_1)$ consists of finitely many connected components $D_k$. Choose one of these components, say $D_1$, which contains infinitely many of the $z_j$. Then $0 \in \partial D_1$. Since, by Lemma 2.12, the restriction of $H$ to $D_1$ is proper onto $H(D_1) = D'_1$, Proposition 2.10 will follow by taking $\mathcal{O}_1 = D_1$.

It remains to prove (2.15). Choose $\epsilon$ such that

$$0 < \epsilon < \text{dist}(0, H(\partial\mathcal{O}, M)), \tag{2.16}$$

and such that the open set $W_\epsilon$ defined in (2.15) is connected. Let $j_0$ be such that $H(z_{j_0}) \in W_\epsilon$. Let $w \in W_\epsilon$ be arbitrary, and $\gamma(t), 0 \leq t \leq 1$, be a continuous curve connecting $H(z_{j_0})$ and $w$ and contained in $W_\epsilon$. Assume by contradiction that $w \notin H(D)$. Since $H(D)$ is open, there exists $t', 0 < t' \leq 1$, such that $\gamma(t') \in H(D)$ for $0 \leq t < t'$, but $\gamma(t') \notin H(D)$. Now choose a sequence $t_k < t'$, with $t_k \to t'$, and $p_k \in D$ with $H(p_k) = \gamma(t_k)$ and $p_k \to p' \in \overline{D}$. Since $H(p') = \gamma(t') \notin H(D)$, it follows from the definition of $D$ that $p' \in \partial D$. Recall that $H(\partial D) \subset H(\partial\mathcal{O})$. Hence $H(p') \in H(\partial\mathcal{O})$. In view of (2.16) and the fact that $H$ maps $M$ into $M'$, we must have $H(p') \in M'$. We reach a contradiction, since $H(p') = \gamma(t') \in W_\epsilon$. The proof of Proposition 2.10 is now complete. □

Let $\Omega$ be a bounded domain in $\mathbb{C}^n$ and $p_0 \in \mathbb{C}^n$. Recall that $p_0$ is in the holomorphic hull of $\Omega$ if there is a compact subset $K \subset \Omega$ such that $p_0 \in \overline{K}$, where

$$\hat{K} = \{ z \in \Omega : |f(z)| \leq \sup_{w \in K} |f(w)| \text{ for all } f \in H(\Omega) \}.$$ 

Here $H(\Omega)$ denotes the space of all holomorphic functions in $\Omega$. We observe that when $p_0$ is a boundary point of $\Omega$, then $p_0$ is in the holomorphic hull of $\Omega$ if and only if any function in $H(\Omega)$ extends holomorphically to some larger domain which contains $p_0$ as an interior point.

**Proposition 2.17.** Let $\Omega$ and $\Omega'$ be two bounded domains in $\mathbb{C}^n$ and $H$ a proper holomorphic mapping from $\Omega$ to $\Omega'$. Suppose that $p_0$ and $p'_0$ are boundary points of $\Omega$ and $\Omega'$ respectively, and that there is a sequence $\{ z_j \}_{j=1}^\infty \subset \Omega$ converging to $p_0$ such that $\lim_j H(z_j) = p'_0$. Suppose that any function in $H(\Omega')$ is bounded on the sequence $\{ H(z_j) \}_{j=1}^\infty$. Then $p_0$ is in the holomorphic hull of $\Omega$.

**Remark 2.18.** Note that the hypothesis of the proposition is satisfied if $p'_0$ is in the envelope of holomorphy of $\Omega'$. 
Proof. By using a standard result (see e.g. [N2] Chapter 7, Lemma 2), it suffices to prove the following claim:

Each function in $\mathcal{H}(\Omega)$ is bounded on $\{z_j\}$. More precisely, for any $f \in \mathcal{H}(\Omega)$, there exists a constant $C_f > 0$ such that $|f(z_j)| \leq C_f$ for all $j$.

To prove the claim, we note that $H$ is finite-to-one on $\Omega$ since it is proper. Hence there exists $m$ such that each $w \in \Omega'$ has $m$ pre-images, $g_k(w), k = 1, \ldots, m$, counted with multiplicity (see e.g. [R]). Now let $f \in \mathcal{H}(\Omega)$ and denote by $\sigma_1(w), \ldots, \sigma_m(w)$ the elementary symmetric functions of $g_k(w), k = 1, \ldots, m$. By well known results (see e.g. [R]) the $\sigma_k(w)$ are holomorphic in $\Omega'$ and hence, by hypothesis, uniformly bounded on the sequence $\{H(z_j)\}$. If we let $w_j = H(z_j)$, we observe that $f(z_j)$ is one of the roots of the polynomial $X^m - \sigma_1(w_j)X^{m-1} + \ldots + (-1)^m\sigma_m(w_j)$. Since the coefficients of this polynomial are bounded, independently of $j$, it follows that the $f(z_j)$ are bounded, independently of $j$. This proves the claim and hence Proposition 2.17. □

Proof of Theorem 4. First, we prove that under the assumptions of Theorem 4, $M'$ is minimal at $p'_0 = h(p_0)$. Indeed, suppose not. Then there is a complex hypersurface $\Sigma$ contained in $M'$ through $p'_0$. Hence, this hypersurface must contain all small analytic disks $A'$ attached to $M'$ with $A'(1) = p'_0$. On the other hand since $M$ is minimal at $p_0$, the boundaries of small analytic discs $A$ attached to $M$ with $A(1) = p_0$ cover a full neighborhood of $p_0$ in $M$ [Tu]. Since we can take $A' = h \circ A$, this contradicts the assumption that Jac $h \neq 0$. (See also [E] for related results.)

By Lemma 2.1, we may assume that $h$ admits a holomorphic extension $H$ to one side of $M$, and that conditions (i)–(iv) preceding Lemma 2.2 are satisfied, so that we may apply Lemma 2.2. If $\delta$ satisfies the conclusion of Lemma 2.2, then $H$ satisfies the hypotheses of Proposition 2.10 with $\mathcal{O} = \mathcal{O}^{\delta}$. We then obtain from Proposition 2.10 a subdomain $\mathcal{O}_1$ of $\mathcal{O}$ such that the restriction of $H$ to $\mathcal{O}_1$ is a proper mapping from $\mathcal{O}_1$ to $\mathcal{O}_1'$, continuous up to the boundary, with $p_0 \in \partial \mathcal{O}_1$ and $p'_0 = H(p_0) \in \partial \mathcal{O}_1'$. Moreover, there exists a sequence $\{z_j\} \subset \mathcal{O}_1$ such that $w_j = H(z_j) \to p'_0$, with $\{w_j\}$ strictly on one side of $M'$.

Since $M'$ is minimal, but not minimally convex at $p'_0$, by assumption, it follows from Theorem A that any CR function defined near $p'_0 \in M'$ extends holomorphically to a full neighborhood of $p'_0$ in $\mathbb{C}^n$. Now, by using the Baire Category Theorem (see e.g. [BR2, Theorems 7 and 8] for a more general result) we conclude that there is a connected neighborhood $\mathcal{U}'$ of $p'_0$ in $\mathbb{C}^n$ with $\mathcal{U}' \cap \mathcal{O}_1' \neq \emptyset$ such that every function in $\mathcal{H}(\mathcal{O}_1')$ extends holomorphically to $\mathcal{U}'$. In particular, we see that any such function is uniformly bounded on $\{w_j\}$. Using Proposition 2.17 we conclude that $p_0$ is in the holomorphic hull of $\mathcal{O}_1$, which lies on the side of $M$ to which every CR function near $p_0$ extends. It follows immediately that every CR function near $p_0$ on $M$ extends holomorphically to a full neighborhood of $p_0$ in $\mathbb{C}^n$. The Hopf lemma property then follows from Theorem C above. The proof of Theorem 4 (and hence that of Theorem 1) is now complete. □

§3 Consequences of Theorem 1 and remarks

In this section we prove the other results stated in the introduction and make some remarks.

We first note that Corollary 1 follows easily from Theorem 1 and the following holomorphic extendibility result, which is a consequence of Theorem 1 of [BR1]:

Theorem E [BR1]. Let \( h : M \to M' \) be a smooth CR map, with \( M \) and \( M' \) real analytic hypersurfaces in \( \mathbb{C}^n \). Assume that \( M \) is essentially finite at \( p_0 \) and that \( h \) satisfies the Hopf Lemma property at \( p_0 \). Then \( h \) extends holomorphically to a full neighborhood of \( p_0 \) in \( \mathbb{C}^n \).

Proof of Theorem 2. In order to apply Theorem 1, we note first that since \( H \) is proper, \( \text{Jac} \ H \not\equiv 0 \) in \( \Omega \). Hence its boundary value on \( \partial \Omega \) does not vanish identically.

Note also that for any \( p_0 \in \partial \Omega \), \( H^{-1}(H(p_0)) \) is closed in \( \partial \Omega \) and hence compact. We may now conclude by Theorem 1 that the Hopf lemma property holds at each point in \( \partial \Omega \).

To prove that \( H \) is finite-to-one in \( \Omega \), we observe first that \( H \) is finite-to-one in \( \Omega \), since it is proper (see e.g. [R]). Since the Hopf lemma property holds at \( p_0 \), we may apply the last part of Theorem D to conclude that for any \( p_0 \in \partial \Omega \), \( H \) is finite-to-one in a neighborhood of \( p_0 \) in \( \Omega \). The desired result then follows by compactness of \( \Omega \).

Remark 2.1. It also follows from Theorem 4 in [BR3] that under the hypotheses of Theorem 2, \( \partial \Omega' \) is also essentially finite at all points.

Proof of Corollary 2. By a result of Diederich and Fornaess [DF], any compact real analytic boundary in \( \mathbb{C}^n \) does not contain a nontrivial complex variety and hence is essentially finite. We may then apply Theorem 2 to conclude that the Hopf lemma property is satisfied at every point of \( M \). The conclusion of Corollary 2 then follows from Theorem E.

Remark 2.2. In fact, Corollary 2 may be proved much more directly by using Proposition 2.17 together with Theorems A, B, C, and E.

Proof of Theorem 3. It suffices to show that if \( p_0 \in U_1 \), then \( h \) satisfies the Hopf lemma property at \( p_0 \). By taking the connected components of \( U \) and \( U_1 \) containing \( p_0 \), we may assume, without loss of generality, that \( U_1 \) and \( U \) are connected. Let \( E = \{ p \in U : h(p) = h(p_0) \} \). Since \( M \) is essentially finite, \( h \) extends holomorphically to one side of \( M \) near any point. Therefore, since by assumption the Hopf lemma property holds in \( U \setminus U_1 \), it follows from Theorem D that \( E \cap (U \setminus U_1) \) is a discrete set.

For \( \delta > 0 \), sufficiently small, let

\[ U^\delta = \{ p \in U : \text{dist}(p, \partial U) > \delta \}, \]

and let \( \partial U^\delta \) be its boundary. By the discreteness established above and the compactness of \( \partial U^\delta \), we conclude that for sufficiently small \( \delta \), \( \partial U^\delta \cap E \) is finite. Hence there exists \( \delta_1 > 0 \) for which the set \( \partial U^{\delta_1} \cap E \) is empty. It is now easy to check that the hypotheses of Theorem 1 are satisfied for \( h \) and \( p_0 \) by taking \( M = U^{\delta_1} \). This proves Theorem 3.

Proof of Corollary 3. Since \( h \) is real analytic at all points of \( U \setminus U_1 \) and \( M \) is real analytic, \( h \) extends holomorphically to a full neighborhood in \( \mathbb{C}^n \) of each such point. By Theorem C, \( h \) then satisfies the Hopf lemma property in all of \( U \setminus U_1 \) and hence in all of \( U \) by Theorem 3. Applying Corollary 1, we then have that \( h \) extends holomorphically to a full neighborhood of \( U \) in \( \mathbb{C}^n \).

□
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