MINIMALITY OF THE HOROCYCLE FLOW ON LAMINATIONS BY HYPERBOLIC SURFACES WITH NON-TRIVIAL TOPOLOGY

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Abstract. We consider a minimal compact lamination by hyperbolic surfaces.
We prove that if no leaf is simply connected, then the horocycle flow on its
unitary tangent bundle is minimal.

1. Introduction. The geodesic and horocycle flows on hyperbolic surfaces are two
classical examples of flows in homogeneous spaces. Their dynamical and ergodic
properties were studied in the 1930s by E. Hopf and G.A. Hedlund.

A compact hyperbolic surface $S$ is the quotient of the Poincaré upper-half plane
$\mathbb{H}$ by a cocompact torsion-free Fuchsian group $\Gamma$. Its unit tangent bundle $T^1S$, that
can be seen as the quotient of $PSL(2, \mathbb{R})$ by $\Gamma$, is the phase space of these flows. The
geodesic flow is uniformly hyperbolic on $T^1S$, and its stable manifolds are precisely
the horocyclic orbits. Alternatively, these flows can be seen as coming from the
diagonal and upper-triangular unipotent one-parameter subgroups of $PSL(2, \mathbb{R})$
acting on $\Gamma \backslash PSL(2, \mathbb{R})$. In this context, Hedlund proved in 1936 that the horocycle
flow is minimal — that is, all its orbits are dense [12].

M. Ratner completed in the 1990s the ergodic-theoretical and topological de-
scriptions of the dynamics of unipotent groups on homogeneous spaces, with a con-
clusive theorem in a subject that had seen important contributions by S.G. Dani,
H. Furstenberg, G.A. Margulis, J. Smillie, among others.
Hedlund’s theorem has a converse (see for example [7]): If the horocycle flow on a hyperbolic surface $S$ is minimal, then $S$ must be compact. In fact the dynamics of the horocycle flow is also well understood for geometrically finite surfaces in general (i.e. those whose fundamental group is finitely generated): in the restriction to the non wandering set of the flow, the orbits are dense or periodic (see for example [8]).

The topological behavior of horocycles poses many questions in the case where the surface is geometrically infinite (i.e. its fundamental group is not finitely generated). M. Kulikov has constructed in [16] a geometrically infinite surface without minimal sets for the horocycle flow and S. Matsumoto has announced other examples in [18] and revisited them in [19]. The dynamics of the recurrent horocycle orbits under different assumptions has been extensively studied by Y. Coudène, F. Ledrappier, F. Maucourant and B. Schapira, among others.

From the point of view of ergodic theory, in the infinite-volume setting, A.N. Starkov conjectured that this flow is ergodic (with respect to Liouville measure) if and only if the action of $\Gamma$ on the boundary $\partial H$ is ergodic (with respect to Lebesgue measure). By duality, this is equivalent to the ergodicity of the joint action of the geodesic and horocycle flows on $T^1S$, which is just the action of the affine group $B$. This result was proved in a special case by M. Babillot and F. Ledrappier and independently by M. Pollicott, and in general by V. Kaimanovich. For these and other related results see for example [15] and [23]. See also [20] for a complete survey in the more general setting of pinched Hadamard manifolds.

In the present paper we consider a compact lamination $(M, F)$ by hyperbolic surfaces. Its unit tangent bundle is a lamination obtained by taking the unit tangent bundles of the leaves of $F$, and it is defined by a continuous $\text{PSL}(2, \mathbb{R})$-action. We study the horocycle flow—that is, the action of the upper triangular unipotent subgroup $U$ of $\text{PSL}(2, \mathbb{R})$— on this space when the lamination is minimal. The actions of the diagonal group $D$ and the affine group $B$ play an important role in this study. The idea of studying the geodesic flow for compact laminations is not new. Its dynamics and its relation with the dynamics of the lamination $F$ have been studied in several contexts, see for example [3], [14].

As the lamination is minimal, one might expect that the compactness of the ambient space forces the minimality of the horocycle flow, like for compact hyperbolic surfaces. Nevertheless, this is not the case, since there are examples where not even the $B$-action is minimal (see [1] and [17]). The last two authors have posed the following question some years ago: Is the minimality of the $B$-action equivalent to the minimality of the horocycle flow?

The main result of this paper gives a positive answer to the question above for a large family of laminations by hyperbolic surfaces:

**Theorem 1.1.** Let $(M, F)$ be a minimal compact lamination by hyperbolic surfaces. If $F$ has no simply connected leaves, then the horocycle flow on the unitary tangent bundle $T^1F$ is minimal.

Minimality of the foliated horocycle flow has already been proved in an algebraic setting. More precisely, in Remarks on the dynamics of the horocycle flow for homogeneous foliations by hyperbolic surfaces ([1]), the authors prove the minimality of the horocyclic flow for the homogeneous case using algebraic methods. On the other hand, in the work Horocycle flows for laminations by hyperbolic Riemann surfaces and Hedlund’s theorem ([17]), the authors consider a general compact minimal lamination by hyperbolic surfaces and they mostly study, with topological tools, the
action of the affine group generated by the joint action of the horocycle and geodesic flows. Its main contribution when it comes to the subject of the horocycle flow is to prove that under certain conditions it is topologically transitive, and to present examples where different types of dynamical behaviour appear, which in turn leads to the conjecture which is addressed in the present paper.

A foliation by hyperbolic surfaces that satisfies the hypotheses of Theorem 1.1 is the Hirsch foliation, briefly described below. A more detailed description of it can be found in [5, Page 371]. It was this foliation that motivated many of the ideas present in this paper. Furthermore, we believe that in this example, because of the simplicity of its construction, many of the arguments we use become particularly transparent.

**Example 1** (The Hirsch foliation). Let $P$ be the closed unit disk minus two open disks of radius $1/4$ centered at $-1/2$ and $1/2$ in the complex plane; namely, a pair of pants. The set $P$ is then invariant under the involution $\sigma : Z \in P \mapsto -Z \in P$. The suspension $S$ of $\sigma$ is a non-trivial fiber bundle over $S^1$, whose holonomy interchanges the two interior boundary components of the pair of pants. Therefore, it can be obtained from a solid torus removing from its interior a thinner solid torus wrapped two times around the generator of its fundamental group. Let $p : S \rightarrow S^1$ be the fibration. The boundary of $S$ consists of two tori $T_1$ and $T_2$. The restriction of $p$ to the external torus $T_1$ is itself a fibration $p_1$ over $S^1$ with fiber $S^1$, while the restriction of $p$ to the internal torus $T_2$ is obtained from the fibration $p_2$ over $S^1$ with fiber $S^1$ by composing with the covering map sending $z = e^{2\pi i \theta} \in S^1$ to $z^2 = e^{2\pi i \frac{\theta}{2}} \in S^1$ (see Figure 1(a)).

Let $M$ be the 3-manifold obtained from $S$ by glueing smoothly $T_1$ and $T_2$ via a diffeomorphism $h$ that sends $p_1^{-1}(z)$ to $p_2^{-1}(z)$ for every $z$ in $S^1$. For example,
where we identify each point \((Z, z) \in \partial P \times S^1\) with its image in \(T_1\). The fibration in \(S\) projects onto a foliation \(F\) having two types of leaves: there are countably many leaves of genus one and a Cantor set of ends, which correspond to periodic points of the doubling map \(f(\theta) = 2\theta \pmod{1}\), and all other leaves have no genus and a Cantor set of ends (they are so-called **Cantor trees**). All leaves are dense and those of genus one are exactly the ones with non-trivial holonomy (see Figure 1(b)-1(c)).

The Hirsch foliation admits many structures of foliation by **hyperbolic surfaces**, as explained in Section 2. The moduli space of such structures has recently been described in [2].

The non-homogeneous Lie foliations constructed by G. Hector, Matsumoto and G. Meigniez in [11] are other examples of minimal foliations satisfying the hypotheses of Theorem 1.1 for which the horocycle flow is always minimal.

This paper has two main parts. The first one, contained in Sections 3 and 4, prove a statement that is apparently weaker than Theorem 1.1 (see Theorem 3.1 in Section 3). Basically, this result states that for laminations which are similar to the Hirsch foliation the horocycle flow is minimal. The second one, contained in Section 5, which is purely topological and is unrelated to the horocycle flow, says that, roughly speaking, laminations without simply connected leaves are similar to the Hirsch foliation. More precisely, its main result is the following:

**Theorem 1.2.** Let \((M, F)\) be a minimal compact lamination by hyperbolic surfaces. The following conditions are equivalent:

(i) No leaf is simply connected.
(ii) There is a leaf that has an essential loop with trivial holonomy.
(iii) There is a leaf which is geometrically infinite.
(iv) All leaves are coarsely tame.

The concept of **coarsely tame** hyperbolic surfaces will be defined in Section 2. Coarsely tame surfaces are a special kind of geometrically infinite hyperbolic surfaces.

Together, Theorems 1.2 and 3.1 prove Theorem 1.1. Furthermore, Theorem 1.2 tells us precisely for which laminations the topological dynamics of the horocycle flow is still unclear. In fact, rephrasing Theorem 1.2 we get:

**Corollary 1.** Let \((M, F)\) be a minimal compact lamination by hyperbolic surfaces. The following conditions are equivalent:

(i) There is a simply connected leaf.
(ii) There is a leaf that is geometrically finite.
(iii) All leaves are geometrically finite.
(iv) All essential loops have non-trivial holonomy.

For some of these laminations, though, the horocycle flow is known not to be minimal, but neither is the corresponding \(B\)-action. Thus, in the geometrically finite case, the question whether the minimality of the \(B\)-action implies the minimality of the horocycle flow remains open.

Theorem 1.1 is valid for any compact lamination by surfaces of variable negative curvature. Such a example can be obtained, for example, by multiplying the uniformized hyperbolic metric by any positive function which is close to 1 in the \(C^2\) topology. The geometric arguments used in Section 4 can be adapted to any surface...
of pinched negative curvature whereas the uniformization theorem of [4] and [24] can be used to prove that if no leaf is simply connected, then all the leaves satisfy the hypothesis of the Key Lemma.

2. Preliminaries and notation.

**Hyperbolic surfaces.** A hyperbolic surface $S$ is the quotient of the hyperbolic plane $\mathbb{H}$ under the left action of a torsion-free discrete subgroup $\Gamma$ of the group $\text{PSL}(2, \mathbb{R})$ of orientation preserving isometries of $\mathbb{H}$. This group acts freely and transitively on the unit tangent bundle $T^1 \mathbb{H}$ of $\mathbb{H}$, which means that we can make the identification $T^1 S = \Gamma \backslash \text{PSL}(2, \mathbb{R})$.

When $\Gamma$ is of finite type we say that $S$ is geometrically finite, else it is geometrically infinite.

**Definition 2.1.** We say that $S$ is coarsely tame if it is noncompact and there exists a constant $b > 0$ such that any geodesic ray in $S$ either stays in a compact region or intersects infinitely many closed geodesics whose lengths are bounded from above by $b$.

**Remark 1.** (i) A coarsely tame surface is geometrically infinite and its fundamental group is a purely hyperbolic Fuchsian group of the first kind— that is, its limit set is the whole boundary at infinity of the hyperbolic plane. In particular, it has infinite area.

(ii) If furthermore it has bounded geometry as defined in [6, Section 3], there exists a lower bound $a > 0$ of the injectivity radius of $S$, hence any geodesic ray in $S$ either stays in a compact region or intersects infinitely many closed geodesics whose lengths are bounded between $a$ and $b$.

For instance, the leaves of the Hirsch foliation are coarsely tame surfaces of bounded geometry. Also, if $\Gamma$ is a cocompact Fuchsian group and $N$ is a proper normal subgroup of infinite index, then $N \backslash \mathbb{H}$ is coarsely tame of bounded geometry (see [22]).

Let $D$ and $U$ be the diagonal and unipotent subgroups of $\text{PSL}(2, \mathbb{R})$

$$D = \left\{ \left( \begin{array}{cc} e^{\frac{t}{2}} & 0 \\ 0 & e^{-\frac{t}{2}} \end{array} \right) : t \in \mathbb{R} \right\} \quad \text{and} \quad U = \left\{ \left( \begin{array}{cc} 1 & s \\ 0 & 1 \end{array} \right) : s \in \mathbb{R} \right\}.$$

Their right actions define the geodesic flow $g_t$ and the horocycle flow $h_s$ on $T^1 S$, respectively. Therefore, the joint action of $g_t$ and $h_s$ is the action of the affine group

$$B = \left\{ \left( \begin{array}{cc} a & b \\ 0 & a^{-1} \end{array} \right) : a > 0, \ b \in \mathbb{R} \right\}.$$

If $I \subset \mathbb{R}$, the subset $\{g_t(u) : t \in I\}$ of the orbit of a point $u \in T^1 S$ under the geodesic flow is $g_I(u)$, and a similar notation will be used for subsets of horocycle orbits.

The projection $\pi : T^1 S \to S$ is the canonical projection that assigns to each vector in $T^1 S$ its base point.

**Laminations.** A compact lamination by surfaces $(M, \mathcal{F})$ (or simply $\mathcal{F}$) consists of a compact metrizable space $M$ together with a family $\{(U_\alpha, \varphi_\alpha)\}$ such that

1. $\{U_\alpha\}$ is an open covering of $M$,
2. $\varphi_\alpha : U_\alpha \to D \times T$ is a homeomorphism, where $D$ is a disk in $\mathbb{R}^2$ and $T$ is a separable locally compact metrizable space, and
(3) for \((x,t) ∈ ϕ_β(\cup_{α} U)\), \(\varphi_α ∘ \varphi_β^{-1}(x,t) = (λ_{αβ}(x), τ_{αβ}(t))\), where \(λ_{αβ}\) is smooth and depends continuously on \(t\) in the \(C^∞\) topology.

We will always work in the leafwise smooth setting unless otherwise stated, although \(C^3\) leafwise regularity would be enough for all our purposes.

Each \(U_α\) is called a foliated chart, a set of the form \(\varphi_α^{-1}(x) × T\) being its transversal. The sets of the form \(\varphi_α^{-1}(D × \{t\})\), called plaques, glue together to form maximal connected surfaces called leaves.

A foliation is said to be minimal if all its leaves are dense.

The tangent bundle of the foliation \(F\) is the \(R^2\)-bundle over \(M\) which can be trivialized on each foliated chart \((U_α ∩ U_β) × R^2 \rightarrow (U_α ∩ U_β) × R^2 (p,v) \mapsto (p,dλ_{αβ}(p)(v))\).

It is itself a (noncompact) foliation, whose leaves are the tangent bundles of the leaves of \(F\).

**Laminations by hyperbolic surfaces.** In each foliated chart we can endow each plaque with a Riemannian metric, in a continuous way. Glueing these local metrics with partitions of unity gives a Riemannian metric on each leaf, which varies continuously in the \(C^∞\) topology as we move from leaf to leaf. In particular, leaves of a compact foliation always have bounded geometry.

When \((M,F)\) is endowed with such a metric, we define the unit tangent bundle \(T^1F\) of \(F\) as the subset of the tangent bundle containing vectors of unit length. It is a circle bundle over \(M\), and it is a foliation whose leaves are the unit tangent bundles of the leaves of \(F\).

Furthermore, we will assume that there is a Riemannian metric for which all leaves are hyperbolic surfaces, that is, they have constant curvature \(-1\). In fact, any given Riemannian metric endows each leaf with a conformal structure, or equivalently, with a Riemann surface structure. If all leaves are uniformized by the disk, then the uniformization is continuous and leaves become hyperbolic surfaces. See [4] and [24]. The existence of these hyperbolic metrics turns out to be a purely topological condition. It is equivalent to the universal cover of every leaf having positive volume entropy. This is independent of the metric because, \(M\) being compact, the restriction to leaves of all Riemannian metrics are quasi-isometric.

Once each leaf has a hyperbolic structure, there is a right continuous \(PSL(2,R)\)-action on \(T^1F\) whose orbits are the unit tangent bundles to the leaves. We will also denote by \(g_t\) and \(h_s\) the geodesic and horocycle flows on \(T^1F\) defined by the action of the one-parameter subgroups \(D\) and \(U\). The natural right \(B\)-action on \(T^1F\) combines these two flows.

3. The horocycle flow on a minimal foliation by coarsely tame leaves. In next sections, we will prove the following (apparently weaker) version of the main theorem which was stated in the Introduction:

**Theorem 3.1.** Let \((M,F)\) be a minimal compact foliation by hyperbolic coarsely tame surfaces. Then the horocycle flow on the unitary tangent bundle \(T^1F\) is minimal.

The best known example of an object in the hypotheses of this theorem is probably the Hirsch foliation (described in the introduction). Studying the dynamics of the horocycle flow in its unit tangent bundle will involve the understanding of the
dynamics of the horocycle flow on a single leaf, and also considerations about the interplay between the dynamics of the lamination and that of the horocycle flow on individual leaves.

**Proposition 1.** Let \((M, F)\) be a minimal compact lamination such that
(i) the affine action \(T^1 F \cdot B\) is minimal,
(ii) the horocycle flow on \(T^1 F\) is transitive, i.e. \(\exists u \in T^1 F\) such that \(h_{R^u}(u) = T^1 F\).

Let \(M \neq \emptyset\) be a minimal set for the horocycle flow in \(T^1 F\), i.e. a non-empty closed \(U\)-invariant subset of \(T^1 F\) which is minimal under inclusion. If there is a real number \(t_0 > 0\) such that \(g_{t_0}(M) = M\), then \(M = T^1 F\).

**Proof.** If \(M \neq T^1 F\), then
\[
C = \{ t \in \mathbb{R} : g_t(M) \cap M \neq \emptyset \} = \{ t \in \mathbb{R} : g_t(M) = M \}
\]
is a discrete subgroup of \(\mathbb{R}\), so it is either trivial or cyclic.

By hypothesis, \(C\) is nontrivial and hence the \(B\)-invariant set
\[
g_t(M) = \bigcup_{t \in \mathbb{R}} g_t(M) = \bigcup_{t \in [0,t_0]} g_t(M)
\]
is closed. Then
\[
g_t(M) = T^1 F
\]
by \(B\)-minimality.

Let \(u \in T^1 F\) be a point with a dense horocycle orbit. There is \(t \in [0,t_0]\) such that \(g_t(u) \in M\). Since \(g_{t}(h_{R}^{-1}(u)) = h_{R}^{-1}(g_{t}(u))\) and \(M\) is \(h_{R}\)-invariant, \(g_{t}(h_{R}^{-1}(u)) \subset M\) and hence \(T^1 F = M\). \(\square\)

Therefore, to prove Theorem 3.1, we have to see that it verifies the hypotheses of Proposition 1.

The first condition, the minimality of the affine action, follows from the minimality of \(F\) and the fact that the \(B\)-action restricted to the unitary tangent bundle of each (coarsely tame) leaf \(L = \Gamma \setminus \mathbb{H}\) is dual to the action of its fundamental group on \(PSL(2, \mathbb{R})/B = \partial \mathbb{H}\), and hence both actions are minimal.

As for the second condition, the transitivity of the horocycle flow, since \(F\) is minimal, it is enough to prove that the horocycle flow is transitive when restricted to some leaf \(L\). This is true of all surfaces having a fundamental group of the first kind, see for example [8].

**Proof of Theorem 3.1.** The condition on the minimal set \(M\) is the object of the next section. More precisely, in the next section we will see that if \(S\) is a coarsely tame surface of bounded geometry and \(C \subset T^1 S\) is closed and invariant under the horocycle flow, then either \(C = T^1 S\) or there exist a \(t_0 > 0\) such that \(g_{t_0}(C) \cap C \neq \emptyset\).

Taking \(M\) as in Proposition 1 and \(C\) to be its intersection with the unit tangent bundle to a leaf, we obtain Theorem 3.1. \(\square\)

4. The horocycle flow on coarsely tame surfaces of bounded geometry.
Let \(S\) be a coarsely tame hyperbolic surface of bounded geometry.

Recall that the dense horocyclic trajectories on \(T^1 S\) are characterized as follows (see [8, Theorem 3.1]). Let \(\hat{u} \in T^1 \mathbb{H}\), and denote \(\hat{u}(+\infty) \in \partial \mathbb{H}\) the extremity of the geodesic ray defined by \(\hat{u}\). The following conditions are equivalent:

(1) Any horodisk centered at \(\hat{u}(+\infty)\) contains points of the orbit \(\Gamma \zeta\) for some \(\zeta \in \mathbb{H}\) (i.e. \(\hat{u}(+\infty)\) is horocyclic).
\( \gamma_n^+ \) and \( \gamma_n^- \) be the attracting and repelling fixed points of \( \gamma_n \) on the boundary at infinity of \( \mathbb{H} \) (see Figure 2). Taking the upper-half-plane model of the hyperbolic plane, we can assume that the geodesic ray directed by \( \tilde{u} \) is the vertical half-line \( r = \{ ie^t : t \geq 0 \} \), and that the axes \( \tilde{\alpha}_n \) are half-circles orthogonal to the real line. By hypothesis, they intersect \( r \). Furthermore, we will assume without loss of generality that when we orient \( \tilde{\alpha}_n \) going from \( \gamma_n^- \) to \( \gamma_n^+ \), the angle between \( r \) and \( \tilde{\alpha}_n \) is smaller or equal than \( \pi/2 \) for all \( n \). Let \( r(t_n) \) be the point \( ie^{t_n} \) where \( r \) intersects \( \tilde{\alpha}_n \).

We will also denote by \( g \) and \( h \), respectively, the geodesic and horocycle flows in \( T^1\mathbb{H} \).

Proving that there exists \( t_0 > 0 \) such that \( g_{t_0}(u) \in \overline{h_b(u)} \) amounts to finding a sequence \( \{ \gamma'_n \}_{n \geq 1} \) in \( \Gamma \) and a sequence of times \( s_n \) such that

\[
\lim_{n \to +\infty} \gamma'_n h_{s_n}(\tilde{u}) = g_{t_0}(\tilde{u}).
\]

Let \( H \) be the horizontal line passing through \( i \), that is, the projection to \( \mathbb{H} \) of the horocycle orbit of the point \( \tilde{u} \). The above condition simply states that \( \gamma'_n(H) \) approaches the horizontal line passing through \( ie^{t_0} \) (see Figure 3). Therefore, it suffices to prove that there exist a sequence \( \{ \gamma'_n \}_{n \geq 1} \) in \( \Gamma \) and a constant \( t_0 > 0 \) such that

(i) \( \gamma'_n \to \infty \) and

(2) The horocyclic orbit of the projected point in \( T^1S \) is dense.

Using this characterization, we see that \( u \in T^1S \) has a dense horocyclic orbit whenever \( g_{R^+}(u) \) accumulates in \( T^1S \).

**Key Lemma.** Let \( S \) be a hyperbolic surface and \( u \in T^1S \). If there is a sequence \( \{ \alpha_n \}_{n \geq 1} \) of closed geodesics in \( S \) of lengths \( \ell(\alpha_n) \in [a,b] \), with \( b \geq a > 0 \), and a sequence of times \( \{ t_n \}_{n \geq 1} \) such that \( t_n \to +\infty \) and \( \pi(g_{t_n}(u)) \in \alpha_n \), then there exists a time \( t_0 > 0 \) such that \( g_{t_0}(u) \in \overline{h_b(u)} \).

**Proof.** The universal cover of \( S \) is the hyperbolic plane \( \mathbb{H} \), and its unit tangent bundle is \( T^1\mathbb{H} \). If \( \Gamma = \pi_1(S) \), we write \( S = \Gamma \setminus \mathbb{H} \). We lift \( u \) to \( \tilde{u} \in T^1\mathbb{H} \), that defines a geodesic ray \( r \). Since each geodesic \( \alpha_n \) is closed on \( S \), it lifts to a geodesic \( \tilde{\alpha}_n \) on \( \mathbb{H} \) which is the axis of a hyperbolic element \( \gamma_n \in \Gamma \). Let \( \gamma_n^+ \) and \( \gamma_n^- \) be the attracting and repelling fixed points of \( \gamma_n \) on the boundary at infinity of \( \mathbb{H} \) (see Figure 2).

Taking the upper-half-plane model of the hyperbolic plane, we can assume that the geodesic ray directed by \( \tilde{u} \) is the vertical half-line \( r = \{ ie^t : t \geq 0 \} \), and that the axes \( \tilde{\alpha}_n \) are half-circles orthogonal to the real line. By hypothesis, they intersect \( r \). Furthermore, we will assume without loss of generality that when we orient \( \tilde{\alpha}_n \) going from \( \gamma_n^- \) to \( \gamma_n^+ \), the angle between \( r \) and \( \tilde{\alpha}_n \) is smaller or equal than \( \pi/2 \) for all \( n \). Let \( r(t_n) \) be the point \( ie^{t_n} \) where \( r \) intersects \( \tilde{\alpha}_n \).

We will also denote by \( g \) and \( h \), respectively, the geodesic and horocycle flows in \( T^1\mathbb{H} \).

Proving that there exists \( t_0 > 0 \) such that \( g_{t_0}(u) \in \overline{h_b(u)} \) amounts to finding a sequence \( \{ \gamma'_n \}_{n \geq 1} \) in \( \Gamma \) and a sequence of times \( s_n \) such that

\[
\lim_{n \to +\infty} \gamma'_n h_{s_n}(\tilde{u}) = g_{t_0}(\tilde{u}).
\]
We will show that \( \gamma \), Therefore \( \pi \).

This is inconsistent with our assumption that this angle is bounded above by \( r \).

Therefore \( \alpha \) approaches \( \xi \).

\( B_{\gamma_n}(i, \gamma_n'i) \rightarrow t_0 \).

where \( B \) is the Busemann function given by \( B_{\xi}(x,y) = \lim_{z \rightarrow \xi} [d(y,z) - d(x,z)] \).

Notice that \( B(x,y) \leq 0 \) if and only if \( y \) belongs to the closed horoball centered at \( \xi \) passing through \( x \).

We will prove (i) and (ii) for \( \gamma_n \), a convenient subsequence of iterates of \( \gamma_n \).

We divide the proof in several steps.

**Step 1.** We will show that \( \gamma_n \rightarrow \infty \). Since \( r(t_n) \rightarrow \infty \) and \( r(t_n) \rightarrow \infty \) and it belongs to the axis \( \alpha_n \) of \( \gamma_n \), one of the endpoints \( \gamma_n^+ \) or \( \gamma_n^- \) of \( \alpha_n \) goes to infinity. If there is a subsequence for which \( \gamma_n^+ \rightarrow \xi \neq \infty \), then \( \gamma_n^- \rightarrow \infty \) and the angle at the point \( r(t_n) \) between the ray \( r \) and the oriented geodesic \( \alpha_n \) would approach \( \pi \) as \( n \) grew. This is inconsistent with our assumption that this angle is bounded above by \( \frac{\pi}{2} \).

Therefore \( \gamma_n^+ \rightarrow \infty \).

Looking at the dynamics of \( \gamma_n \in PSL(2, \mathbb{R}) \) acting on \( \mathbb{H} \cup \partial \mathbb{H} \), we see that \( \gamma_n \rightarrow \infty \) does not belong to the interval with endpoints \( \gamma_n^+ \) and \( \gamma_n^- \), and therefore \( \gamma_n \rightarrow \infty \) as stated.

**Step 2.** We will show that

\[
B_{\gamma_n}(i, \gamma_n i) = B_{\infty}(\gamma_n^{-1}i, i) \leq b. 
\]

We have that

\[
B_{\infty}(\gamma_n^{-1}i, i) = B_{\infty}(\gamma_n^{-1}i, \gamma_n^{-1}r(t_n)) + B_{\infty}(\gamma_n^{-1}r(t_n), r(t_n)) + B_{\infty}(r(t_n), i).
\]

Computing these three terms gives

1. \( B_{\infty}(r(t_n), i) = -t_n \),
2. \( B_{\infty}(\gamma_n^{-1}r(t_n), r(t_n)) \leq d(\gamma_n^{-1}r(t_n), r(t_n)) \leq b \), since both \( \gamma_n^{-1}r(t_n) \) and \( r(t_n) \) belong to the closed geodesic \( \alpha_n \) the length of which is bounded by \( b \), and
3. \( B_{\infty}(\gamma_n^{-1}i, \gamma_n^{-1}r(t_n)) \leq d(\gamma_n^{-1}i, \gamma_n^{-1}r(t_n)) = d(i, r(t_n)) = t_n \).

Therefore

\[
B_{\gamma_n}(i, \gamma_n i) \leq -t_n + b + t_n = b.
\]

Notice that, using the same proof, we obtain that for \( k > 0 \), \( B_{\gamma_n}(i, \gamma_n i) \leq kb \).

**Step 3.** We would like to show that there exists \( C > 0 \) such that

\[
B_{\gamma_n}(i, \gamma_n i) \geq C.
\]
In fact we will see that there is a positive $k$ such that for every $n$

$$B_{\gamma_n^k}(i, \gamma_n^k i) \geq C.$$ 

As before, we consider the decomposition

$$B_{\gamma_n} = B_{\gamma_n}^1 + B_{\gamma_n}^2 + B_{\gamma_n}^3,$$

and we will compute each of these three terms.

For the first term we consider the geodesic ray $c_n$ going from $r(t_n)$ to $\gamma_n \infty$. We have that

$$B_{\gamma_n}(i, r(t_n)) = \lim_{t \to +\infty} d(i, c_n(t)) - d(r(t_n), c_n(t)).$$

Take the geodesic path from $\gamma_n^t r(t_n)$ to $r(t_n)$ and the one from $r(t_n)$ to $i$. They form an angle greater or equal to $\pi/2$ (see Figure 4). Therefore, we have

$$d(\gamma_n r(t_n), r(t_n)) \geq d(\gamma_n r(t_n), r(t_n)) + (t - t_n) - \log(2).$$

Now we will deal with the second term.

$$B_{\gamma_n}(\gamma_n^{-1} r(t_n), r(t_n)) = \lim_{t \to \infty} d(r(t), \gamma_n^{-1} r(t_n)) - d(r(t), r(t_n)).$$

Remark that $d(r(t), r(t_n)) = t - t_n$.

The angle formed by the geodesic path from $\gamma_n^{-1} r(t_n)$ to $r(t_n)$ and the one from $r(t_n)$ to $r(t)$ is greater or equal to $\pi/2$ (see Figure 4). Therefore, we have

$$d(\gamma_n^{-1} r(t_n), r(t)) \geq d(\gamma_n^{-1} r(t_n), r(t_n)) + (t - t_n) - \log(2).$$
We also know that \( d(\gamma_n^{-1} r(t_n), r(t_n)) = \text{length}(\alpha_n) \geq a \), so we have that
\[
d(\gamma_n^{-1} r(t_n), r(t)) - (t - t_n) \geq a - \log(2).
\]
As before, the third term is equal to \( t_n \).
Putting everything together, we have that
\[
B_{\gamma_n \infty}(i, \gamma_n i) = B_{\infty}(\gamma_n^{-1} i, i) \geq t_n - \log(2) + a - \log(2) - t_n = a - 2 \log(2).
\]
To obtain the desired conclusion, it would be enough to verify that
\[
a - 2 \log(2) > 0.
\]
This might not hold, but if we replace \( \gamma_n \) with \( \gamma_n^k \) for a sufficiently large \( k \), we will get
\[
B_{\gamma_n^k \infty}(i, \gamma_n^k i) \geq ka - 2 \log(2) > 0.
\]
An appropriate subsequence of the \( \gamma_n^k \) satisfies conditions (i) and (ii). \( \square \)

Notice that the Key Lemma also holds if \( S \) is a Riemannian surface whose sectional curvature is less or equal than \(-1\).
Using Remark 1 and Lemma 4, we get the following corollary:

**Corollary 2.** Let \( S \) be a coarsely tame hyperbolic surface of bounded geometry. Then, for any tangent vector \( u \in T^1 S \), either \( \tilde{h}_u(u) = T^1 S \) or there is a real number \( t_0 > 0 \) such that \( g_{\tilde{h}_u}(u) \in \tilde{h}_u(u) \).

Notice that Corollary 2 holds in the more general context where \( S \) admits a pair of pants decomposition such that any geodesic ray either stays in a compact region or intersects infinitely many closed geodesics whose length belongs to some interval \([a, b]\), with \( b \geq a > 0 \), depending on the ray. Surfaces for which there exists \( b \) are called weakly tame in \([22]\).

Using this corollary, we retrieve a result due to S.Matsumoto (see \([18],[19]\)) saying that the horocycle flow on \( T^1 S \) admits no minimal sets.

To see this, let \( S = \Gamma \setminus \mathbb{H} \) be a coarsely tame hyperbolic surface of bounded geometry, and suppose that \( C \subset T^1 S \) is a minimal set. Since \( S \) is non-compact, \( C \neq T^1 S \) (see \([8]\)). Using Corollary 2 we get \( t_0 \) such that \( g_{\tilde{h}_u}(C) = C \). On the other hand, if \( u \in C \) and \( \tilde{u} \) is a lift of \( u \) to \( T^1 \mathbb{H} \), the point at infinity \( \tilde{u}(+\infty) \) is non-horocyclic. For any \( k \in \mathbb{Z} \), let \( \zeta_k \) be the base point of \( \tilde{g}_{kt_0}^{\zeta_k}(\tilde{u}) \), where \( \tilde{g}_\mathbb{R} \) denotes the geodesic flow. Notice that since \( g_{\tilde{h}_u}(u) \in C \), there exist \( (t_n)_{n \geq 0} \) in \( \mathbb{R} \) and \( (\gamma_n)_{n \geq 0} \) in \( \Gamma \) such that \( \gamma_n \tilde{h}_u(\tilde{u}) \) converges to \( \tilde{g}_{kt_0}(u) \). This implies that \( B_{\gamma_n(\tilde{u}(+\infty))}(\zeta_0, \gamma_n \zeta_0) = B_{\tilde{u}(+\infty)}(\gamma_n^{-1} \zeta_0, \zeta_k) \) converges to \( B_{\tilde{u}(+\infty)}(\zeta_0, \zeta_k) = kt_0 \). Hence \( \Gamma \zeta_0 \) meets a horodisk centered at \( \tilde{u}(+\infty) \) but this is not the case, since \( \tilde{u}(+\infty) \) is not horocyclic.

5. **Topology of leaves.** The purpose of this section is to show that there are in fact many foliations which are similar to the Hirsch foliation. More precisely, we will prove Theorem 1.2, which has been stated in the Introduction.

The proof of this theorem will make use of two preliminary results. The first one is about simply connected leaves and the second one is the construction of a certain type of uniformization of \((M, F)\). We also will make use of the following remark, which is in itself an interesting fact about compact laminations by hyperbolic surfaces:
Remark 2. A compact lamination by hyperbolic surfaces has the property that no leaf has a cusp. This is because the injectivity radius of all leaves is bounded below by a positive constant. A more dynamical way to see it is the following: If there were a leaf with a cusp, the horocycle flow on the unit tangent bundle would have a closed orbit $h$ (a closed horocycle) and the geodesic flow $g_t$ would have the property that the diameter of $g_t(h)$ converges to 0. Taking a subsequence the set $g_{t_n}(h)$ would converge to a fixed point of the geodesic flow, which is a contradiction.

Proposition 2. Let $(M, \mathcal{F})$ be minimal compact lamination by hyperbolic surfaces. Then $\mathcal{F}$ has no simply connected leaves if and only if there is a leaf that contains an essential loop with trivial holonomy.

Proof. Assume that a compact lamination $(M, \mathcal{F})$, which is not necessarily minimal, has no simply connected leaves. A theorem due to D.B.A. Epstein, K.C. Millet and D. Tischler [9] and independently to G. Hector [10] states that the leaves with trivial holonomy form a residual set. Since any leaf in this residual set is not simply connected, it must contain an essential loop which has obviously trivial holonomy. Reciprocally, assume that there is a leaf $L$ that contains an essential loop $c_0 : S^1 \to L$ without holonomy. By the standard length-shortening arguments $c$ is freely homotopic to a closed geodesic in $L$, and we can assume that it is in fact a closed geodesic parametrized by arc length by a periodic function $c_0 : \mathbb{R} \to L$ of period $s_0$. Since it has trivial holonomy and the metric varies continuously from leaf to leaf in the $C^\infty$ topology (although $C^3$ is enough), Reeb’s stability theorem implies that for all $\varepsilon > 0$, there exist a small transversal $T$ through the point $t_0 = c_0(0)$ and a smooth map $\hat{c} : \mathbb{R} \times T \to M$ such that:

1. $\hat{c}(s, t_0) = c_0(s)$ for all $s \in \mathbb{R}$.
2. $\hat{c}(s + s_0, t) = \hat{c}(s, t)$ for all $s \in \mathbb{R}$, $t \in T$. Let $c_t(s) = \hat{c}(s, t)$.
3. For all $t$ in $T$ the curve $c_t : \mathbb{R} \to M$ has image in a leaf $L_t$ of the lamination passing through $t \in T$.
4. For all $t$ in $T - \{t_0\}$ the curve $c_t : \mathbb{R} \to M$ is closed and there is a geodesic $\alpha_t$ in $L_t$ such that $d(\alpha_t(s), c_t(s)) < \varepsilon$ for $s \in [0, s_0]$.

Let us join the points $\alpha_t(s_0)$ and $\alpha_t(0)$ by the geodesic segment $\beta_t$ in $L_t$ that realizes the distance between them. If $\varepsilon$ is sufficiently small, the concatenation of $\alpha_t|_{[0, s_0]}$ and $\beta_t$ is freely homotopic to $c_t$. Since $L_t$ has negative curvature, this means that $c_t$ is essential in $L_t$. Finally, as $\mathcal{F}$ is minimal, any leaf $L'$ meets the image of $\hat{c}$ and hence there exists $t \in T$ such that $L' = L_t$ is not simply connected.

By construction, the continuous map $\hat{c} : \mathbb{R} \times T \to M$ factors over a continuous extension $c : S^1 \times T \to M$ of the closed geodesic $c_0$ so that the loop $c_t = c|_{S^1 \times \{t\}}$ is essential in the leaf $L_t$ for each $t \in T$. In fact, using again Reeb’s stability theorem as above, the inclusion of a $\varepsilon$-tubular neighborhood $A$ of $c_0$ into $M$ extends to a continuous map $c : A \times T \to M$ such that $c(A \times \{t\})$ is a $\varepsilon$-tubular neighborhood of the essential loop $c_t$ into the leaf $L_t$ for all $t \in T$. Note the persistence of the closed geodesic in the free homotopic class of $c_t$ when the hyperbolic metric varies in the transverse direction. In fact, by the normal hyperbolicity of the geodesic flow in the sense of M.W. Hirsch, C.C. Pugh and M. Shub [13], the closed geodesic varies continuously in the $C^\infty$ topology, and we can finally assume that every essential loop $c_t$ is in fact a simple closed geodesic.

Definition 5.1. Let $(M, \mathcal{F})$ be a compact minimal lamination by hyperbolic surfaces. Then a smooth map $\Phi : D \times T \to M$ is a uniform systole cylinder cover if:
(1) $D$ is diffeomorphic to a cylinder $S^1 \times (0,1)$.

(2) $\Phi$ is a local homeomorphism (which is not necessarily a covering map) and $\Phi(D \times T) = M$.

(3) For each $t \in T$ the map $\Phi_t : D \to M$, $x \mapsto \Phi(x,t)$, has image in the leaf $L_t$ of the lamination passing through $\Phi(x_0,t)$, where $x_0$ is a fixed point of $D$, and $\Phi_t : D \to \Phi_t(D) \subset L_t$ is a local diffeomorphism.

(4) Let $g^*$ be the continuous leafwise Riemannian metric on $D \times T$ which is the pull-back metric, under $\Phi$, of the continuous metric $g$ on $M$ which renders all leaves of $\mathcal{F}$ hyperbolic. Under these two metrics restricted to the leaves, $\Phi_t$ is a local isometry between $D \times \{t\}$ and its image in $L_t$.

(5) For each $t \in T$ the surface $D \times \{t\}$ with the metric induced by $g^*$ has a unique closed geodesic $c_t$, the systole of the cylinder, and it is a tubular neighborhood of this geodesic whose fiber has uniformly bounded length.

**Theorem 5.2.** Let $(M,\mathcal{F})$ be a compact minimal lamination by hyperbolic surfaces. If there is an essential loop without holonomy, then $\mathcal{F}$ admits a uniform systole cylinder cover.

**Proof.** According to the proof of Proposition 2 and the subsequent discussion, we have a continuous map $c : S^1 \times T \to M$, where each $c_t$ is a simple closed geodesic on a leaf $L_t$.

Let $\mathbb{D}$ be the Poincaré disk with the Poincaré metric and $F : \mathbb{D} \times T \to M$ be the global uniformization map (see [4] and [24]), i.e. the continuous map that satisfies the following conditions:

(1) The restriction $F_t$ of $F$ to $\mathbb{D} \times \{t\}$ is a uniformization of $L_t$, i.e. it is a covering map which is a local isometry between the Poincaré disc and the hyperbolic leaf $L_t$.

(2) $F(0,t) = c_t(1)$ and $\frac{d}{dz}(F(z,t))|_{z=0} = \dot{c}_t(1)$.

Let $\tilde{c}_t$ be the lift of $c_t$ which passes through $0 \in \mathbb{D}$, and $\gamma_t \in PSL(2,\mathbb{R})$ be the Möbius transformation of $\mathbb{D}$ that has axis $\tilde{c}_t$ and such that $c_t$ is the quotient of $\tilde{c}_t$ by the cyclic group generated by $\gamma_t$. Remark $\gamma_t$ varies continuously with $t$.

We define the skew-cylinder $N = \mathbb{D} \times T/\sim$ where $(z,t) \sim (\gamma_t z,t)$, with the induced Riemannian metric along the $z$-direction.

Then the uniformization map $F$ can be factorized through $N$.

Then the uniformization map $F$ can be factorized through $N$

$$
\begin{array}{ccc}
\mathbb{D} \times T & \xrightarrow{\pi} & N = \mathbb{D} \times T/\sim \\
\downarrow{F} & & \downarrow{\varphi} \\
M & \xrightarrow{\psi} & \\
\end{array}
$$

verifying

(1) There exists a diffeomorphism $f : S^1 \times \mathbb{R} \times T \to N$ which sends $S^1 \times \{0\} \times \{t\}$ onto the systole $c_t$.

(2) The map $\varphi$ is a local homeomorphism; in particular it is an open map.

(3) Restricted to $\mathbb{D} \times \{t\}$ and $\pi(\mathbb{D} \times \{t\})$ respectively, both $\pi$ and $\varphi$ are locally isometric covering maps.

For every $k \in \mathbb{N}$, take $N_k = f(S^1 \times (-k,k) \times T)$. Its image under $\varphi$ is open in $M$, and $\{\varphi(N_k) : k \in \mathbb{N}\}$ is a cover of $M$. Since $M$ is compact, there exists a $k_0$
We are now in a position to prove Theorem 1.2.

Proof of Theorem 1.2. The equivalence (i) \iff (ii) has been proved in Proposition 2, and now we will prove (ii) \implies (iv) \implies (iii) \implies (ii).

(i) \implies (iii) Since \(\mathcal{F}\) is minimal, it is clear that all its leaves are noncompact. We wish to prove that there are positive constants \(a\) and \(b\) such that if \(L\) is a leaf of \(\mathcal{F}\) and \(r\) is any geodesic ray going to infinity in \(L\), then \(r\) must intersect infinitely many closed geodesics with lengths between \(a\) and \(b\). This is equivalent to proving that any geodesic ray going to infinity in \(L\) intersects one geodesic with length between \(a\) and \(b\).

We start by constructing a uniform systole cylinder cover and later we proceed in three steps. Indeed, under the hypotheses that \(\mathcal{F}\) is minimal and that it has a leaf containing an essential loop without holonomy, Theorem 5.2 gives us a uniform systole cylinder cover \(\Phi : D \times T = \mathbb{S}^1 \times (-k_0, k_0) \times T \to N_{k_0} \to M\).

\(\square\)

Step 1. In the first step, each leaf \(L\) is covered by the countable family of surfaces

\[
\mathcal{U}_L = \{\Phi(D_t) : L = L_t \}\.
\]

Each element \(S \in \mathcal{U}_L\) is an open hyperbolic surface of finite genus, which is relatively compact in \(L\). Since \(\Phi|_{D_t}\) arises from the uniformization covering \(F_t : \mathbb{D} \to L_t\), its boundary is a union of finitely many piecewise smooth simple closed curves contained in the images of boundary components of \(D_t\). We will actually refer to each of these by the term ‘boundary component’ of \(S\) even if in general the
connected components of the boundary are non-simple curves composed of several of these 'boundary components'. Notice that even if some boundary components of $S$ can be inessential in $L$, there are always essential boundary components of $S$ in $L$.

**Step 2.** Let $r$ be a geodesic ray that goes to infinity on $L$. It must intersect infinitely many of the surfaces $S \in \mathcal{U}_L$. In this second step, we will see that for infinitely many of them, there is at least one boundary component of $S$ whose algebraic intersection number with $r$ is not zero. To define the intersection number of a ray with a closed curve, we must remove from $L$ the starting point of the ray and consider the intersection number in this new noncompact surface. We proceed in two stages:

**Step 2.1.** First, we fix a surface $S \in \mathcal{U}_L$ containing the starting point of $r$ and we assume that the intersection number of $r$ with every boundary component of $S$ is zero. Let $C$ be the first boundary component of $S$ that is intersected by $r$ with the orientation induced by the orientation of $S$. Consider a parametrization $r : [0, +\infty) \to L$ of the ray $r$. Since $r$ intersects $C$, it does so a finite number of times $\{t_1, \ldots, t_{2n}\}$, having intersection number +1 for $n$ times (including $t_1$) and intersection number −1 for other $n$ times.

There are two consecutive parameter $t_i$ and $t_{i+1}$ where the intersection numbers have opposite sign. Then we substitute $r$ for the concatenation of $r|_{[0,t_i]}$, $\beta_i$ and $r|_{[t_{i+1}, +\infty)}$, where $\beta_i$ is a segment of $C$ that joins $r(t_i)$ and $r(t_{i+1})$. We have thus reduced the number of intersection points of the ray $r$ with the boundary component $C$. Arguing inductively, we obtain a new curve that does not cut $C$ but that still goes to infinity. Until it meets another boundary component of $S$, the new curve remains in the closure of $S$. Since we can do the same for the next boundary component of $S$ that is intersected by $r$, we can inductively construct a curve that goes to the same point at infinity as $r$ remaining in the closure of $S$, which is absurd.

**Step 2.2.** In general, $L$ is covered by an increasing sequence of relatively compact open hyperbolic surfaces which are made up of finitely many surfaces in $\mathcal{U}_L$. Repeating the same argument used for $S$ for each of those surfaces, we deduce that the algebraic intersection of $r$ with at least one boundary component is not zero for infinitely many surfaces in $\mathcal{U}_L$.

**Step 3.** In the final step, we consider a surface $S \in \mathcal{U}_L$ such that:

1. its boundary is far from the starting point of $r$ in the sense that no boundary component of $S$ meets an open ball $B$ centered at this point,
2. the intersection number of $r$ with a boundary component $C$ of $S$ is not zero.

Then the surface which is obtained by removing the starting point admits a complete Riemannian metric having the same closed geodesics of length ≤ $b$ than the original one outside of the neighborhood $B$ of the new end. Then $C$ is freely homotopic to a closed geodesic $\alpha$, and the ray $r$ must intersect it since their intersection number is still nonzero. Furthermore, since the length of $C$ is between $a$ and $b$ and $\alpha$ cannot be longer than $C$, we have proved that $r$ intersects a closed geodesic of length between $a$ and $b$, as desired.

This shows that $L$ is coarsely tame with bounded geometry.

(iii) ⇒ (ii) Any surface which is coarsely tame with bounded geometry is geometrically infinite, so in fact all leaves are geometrically infinite.

(ii) ⇒ (i) Let $L$ be a leaf which is geometrically infinite and take $x \in L$. Its fundamental group $\pi_1(L, x)$ is not finitely generated, and its holonomy group $\text{Hol}(L, x)$
is a finitely generated quotient of $\pi_1(L,x)$. The non-trivial kernel of the homomorphism $\pi_1(L,x) \to \text{Hol}(L,x)$ is precisely the set of homotopy classes of loops based at $x$ without holonomy.

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