HIGH ORDER RANDOM WALKS: BEYOND SPECTRAL GAP

TALI KAUFMAN*, IZHAR OPPENHEIM†

Received August 7, 2017
Revised January 10, 2019
Online First March 4, 2020

We study high order random walks in high dimensional expanders; namely, in complexes which are local spectral expanders. Recent works have studied the spectrum of high order walks and deduced fast mixing. However, the spectral gap of high order walks is inherently small, due to natural obstructions (called coboundaries) that do not happen for walks on expander graphs.

In this work we go beyond spectral gap, and relate the shrinkage of a $k$-cochain by the walk operator, to its structure under the assumption of local spectral expansion. A simplicial complex is called a one-sided local spectral expander, if its links have large spectral gaps and a two-sided local spectral expander if its links have large two-sided spectral gaps.

We show two Decomposition Theorems (one per one-sided/two-sided local spectral assumption): For every $k$-cochain $\phi$ defined on an $n$-dimensional local spectral expander, there exists a decomposition of $\phi$ into “orthogonal” parts that are, roughly speaking, the “projections” on the $j$-dimensional cochains for $0 \leq j \leq k$. The random walk shrinks each of these parts by a factor of $\frac{k+1-j}{k+2}$ plus an error term that depends on the spectral expansion.

When assuming one-sided local spectral gap, our Decomposition Theorem yields an optimal mixing for the high order random walk operator. Namely, negative eigenvalues of the links do not matter! This improves over [5] that assumed two-sided spectral gap in the links to get optimal mixing. This improvement is crucial in a recent breakthrough [1] proving a conjecture of Mihail and Vazirani. Additionally, we get an optimal mixing for high order random walks on Ramanujan complexes (whose links are one-sided local spectral expanders).

When assuming two-sided local spectral gap, our Decomposition Theorem allows us to describe the whole spectrum of the random walk operator (up to an error term that is determined by the spectral gap) and give an explicit orthogonal decomposition of the

* Research supported by ERC and BSF.
† Research supported by ISF.

Mathematics Subject Classification (2010): 05E45, 05C81
spaces of cochains that approximates the decomposition to eigenspaces of the random walk operator.

1. Introduction

In this paper we study high order random walks on simplicial complexes whose links are spectral expanders. High order random walks are strongly related to PCP agreement tests; direct product testing and direct sum testing [5]. This relation influenced, in part, the study of high order random walks.

The focus of previous works [10,5] was on bounding the second largest eigenvalue (in absolute value) of the high order walk operator in complexes whose links are good spectral expanders. Namely, previous works have shown that in complexes with links that are good spectral expanders, every $k$-cochain that is orthogonal to the constant functions is shrinked by the $k$-order random walk operator $M_k^+$, i.e., by the walk that walks from a $k$-face to a $k$-face through a $(k+1)$ face. The shrinkage rate is immediately determined by the second largest eigenvalue of the walk operator. However, due to natural obstructions (such as coboundaries) this second largest eigenvalue cannot be very small.

However, it could well be the case that $k$-cochains with some specific structures are shrinked much better than the bound obtained by the bound on the second largest eigenvalue. This is similar in spirit to the small set expansion question in, say, the noisy hypercube [2]. Indeed, the noisy hypercube is not a good expander so we can not say that general sets expand well. However, methods beyond the spectral gap enabled showing that small sets of the noisy hypercube expand very well.

The focus of this work is to relate the structure of a $k$-cochain $\phi$ to the amount of its shrinkage by the random walk operator $M_k^+$, in complexes that are local spectral expanders. We provide two decomposition theorems that relates the amount of shrinkage of a $k$-cochain to its "projections" on the spaces of cochains of lower dimensions. Exact formulations of this statement are given below.

1.1. On simplicial complexes and localization

A pure $n$-dimensional simplicial complex $X$ is a simplicial complex in which every simplex is contained in an $n$-dimensional simplex. In other words, it is an $(n+1)$-hypergraph with a closure property: for every hyperedge in the hypergraph, all of its subsets are also hyperedges in the hypergraph. The
sets with \((i+1)\)-elements are denoted \(X(i), 0 \leq i \leq n\). The one-skeleton of the complex \(X\) is its underlying graph obtained by \(X(0) \cup X(1)\). A set \(\tau \in X(i)\) is called a face. The link of \(\tau\) denoted \(X_\tau\) is the complex obtained by taking all faces in \(X\) that contain \(\tau\) and removing \(\tau\) from them. Thus, if \(\tau\) is of dimension \(i\) (i.e. \(\tau \in X(i)\)), then \(X_\tau\) is of dimension \(n-i-1\).

For \(-1 \leq k \leq n-1\), we denote by \(C^k(X, \mathbb{R})\) the set of all functions \(\phi: X(k) \to \mathbb{R}\). Abusing the terminology, we will call the space \(C^k(X, \mathbb{R})\) the space of (non-oriented) \(k\)-cochains. Below we define an inner-product on \(C^k(X, \mathbb{R})\) and denote by \(\|\cdot\|\) the norm induced by this inner-product (see exact details in Section 2).

For every \(-1 \leq i \leq n-2\), the one skeleton of \(X_\tau\) is a graph. The second largest eigenvalue of the random walk on \(X_\tau\) (with suitable weights - see below) is denoted \(\mu_\tau\); the smallest eigenvalue of the random walk on \(X_\tau\) (with suitable weights) is denoted \(\nu_\tau\).

**Definition 1.1 (One sided local spectral expander).** A pure \(n\)-dimensional complex \(X\) is called a one-sided \(\lambda\)-local-spectral expander if for every \(-1 \leq i \leq n-2\), and for every \(\tau \in X(i)\), \(\mu_\tau \leq \lambda\).

**Definition 1.2 (Two sided local spectral expander).** A pure \(n\)-dimensional complex \(X\) is a two-sided \(\lambda\)-local-spectral expander if for every \(-1 \leq i \leq n-2\), and for every \(\tau \in X(i)\), \(-\lambda \leq \nu_\tau, \mu_\tau \leq \lambda\).

1.2. On high order random walks and related results

We study the random walk operator, \(M_k^+\), which is the transition matrix of a random walk on the \(k\)-faces of \(X\) defined as follows: given a \(k\)-dimensional face of \(X\), \(\tau \in X(k)\), choose randomly (according to a weight function) a \((k+1)\)-face \(\sigma \in X(k+1)\) such that \(\tau \subset \sigma\) and then choose uniformly a \(k\)-face \(\tau' \in X(k)\) such that \(\tau' \subset \sigma\) (for an exact definition see Section 3). We normalize the operator so that the largest eigenvalue is 1. Such walks were first introduced and studied in [10] where it was shown that in one-sided \(\lambda\)-local-spectral expanders a \(k\)-cochain \(\phi\) orthogonal to the constant functions satisfies the following:

\[
\|M_k^+\phi\| \leq \left(1 - \frac{\varepsilon(k, \lambda)}{2(k+2)^2}\right) ||\phi||,
\]

where \(\varepsilon(k, \lambda) \leq 1\).

In the recent work of [5] it was shown that for two-sided \(\lambda\)-local-spectral expanders, a \(k\)-cochain \(\phi\) orthogonal to the constant functions satisfies the
following:
\[ \|M_k^+ \phi\| \leq \left( \left(1 - \frac{1}{k+2}\right) + O((k+1)\lambda) \right) \|\phi\|. \]

This bound can be shown to be optimal in the sense that it is essentially equivalent (for small enough \( \lambda \)) to the bound obtained in the complete complex. The work of [5] have used this optimal bound in order to obtain a complete de-randomization of the direct product testing. The method that they introduced is the method of decreasing differences. The seemingly mild improvement that [5] achieves over [10] is crucial for their application. Note that [5] requires two-sided \( \lambda \)-local-spectral expanders. As the Ramanujan complexes are only one-sided \( \lambda \)-local-spectral expanders, the result of [5] does not apply to the Ramanujan complexes themselves but only to other complexes that could be built from them.

1.3. Decomposition theorems for high order random walks and their implications

In this paper we show two decomposition theorems: the first for one-sided \( \lambda \)-local-spectral expanders and the second for two-sided \( \lambda \)-local-spectral expanders. Both theorems follow the same philosophy: a \( k \)-cochain \( \phi \) orthogonal to the constants can be decomposed into “orthogonal” parts \( \phi^0, \ldots, \phi^k \), where, roughly speaking, each such part is the “projection” on the \( j \)-dimensional cochains for \( 0 \leq j \leq k \) and applying \( M_k^+ \) on each such \( \phi^j \) shrinks it by a factor of \( \frac{k+1-j}{k+2} + O(\lambda) \). The different assumptions on the local spectral gaps (one-sided or two-sided) lead to different implementations of this philosophy.

Theorem 1.3 (Decomposition Theorem for one-sided local spectral expanders, informal, for formal see Theorem 5.2, Corollary 5.3). Given a pure \( n \)-dimensional one-sided \( \lambda \)-local-spectral expander \( X \). For a \( k \)-cochain \( \phi \) \( (k \leq n-1) \) orthogonal to the constant functions there exist \( j \)-cochains \( \phi^j \) for every \( 0 \leq j \leq k \) such that \( \|\phi\|^2 = \sum_{j=0}^k \|\phi^j\|^2 \) and

\[ \langle M_k^+ \phi, \phi \rangle \leq \sum_{j=0}^k \left( \frac{k+1-j}{k+2} + f(k,j)\lambda \right) \|\phi^j\|^2, \]

where \( f(k,j) \) are explicit positive constants (independent of \( X \)).
As a corollary of the decomposition theorem we derive optimal bounds on the second largest eigenvalue (in absolute value) of $M_k^+$ for $X$ which is a one-sided $\lambda$-local-spectral expander. This result is stronger than [5] that applies only for two-sided $\lambda$-local-spectral expander.

**Theorem 1.4 (Bounding the second eigenvalue of the $k$-walk Theorem, informal, for formal see Theorem 5.4).** Given a pure, $n$-dimensional, one-sided $\lambda$-local-spectral expander $X$. For a $k$-cochain $\phi$ ($k \leq n - 1$) orthogonal to the constant functions:

$$\|M_k^+ \phi\| \leq \left( \left(1 - \frac{1}{k+2}\right) + \frac{k+1}{2}\lambda \right) \|\phi\|.$$

*On the usefulness of Theorem 1.4 in the recent breakthrough [1].* Recently, Theorem 1.4 was used by Anari, Liu, Gharan and Vinzant [1] in their proof of a famous conjecture of Mihail and Vazirani. We note that in the cases considered in their proof, local two-sided spectral expansion does not hold and therefore it was important that the bound on mixing rate of the $k$-random walk can be derived only from the local one-sided spectral expansion.

The Decomposition Theorem in the case of one-sided local spectral expanders has two problems: first, the “orthogonal” parts here are at different dimensions, i.e., we do not really have an orthogonal decomposition of the spaces of $k$-cochains into orthogonal subspaces. Second, while the $j$-cochains $\phi^j$ are described explicitly, their definition involves taking a pre-image of the square root of $M_k^+$ and thus their actual computation is hard to implement. These two problems are resolved when assuming two-sided local spectral gap. In that cases, under the assumption that the local spectral gap is sufficiently small, we show that the spectrum of $M_k^+$ is contained in small neighborhoods around $\frac{k+1-j}{k+2}$ for $j = 0, \ldots, k$ and that the eigenspaces for these eigenvalues can be approximated by an explicit orthogonal decomposition of the space of $k$-cochains.

In order to state this result precisely, we will need some additional definitions and notations: denote by $C^k(X)$ the spaces of $k$-cochains, i.e., maps of the form $\phi: X(k) \to \mathbb{R}$ and denote by $C^k_0(X)$ be the subspace of $k$-cochains orthogonal to the constant functions (the exact inner-product is defined in Section 2 below). For $0 \leq j < k \leq n$, let $d_{j,k}: C^j(X) \to C^k(X)$ to be the operator induced by the incidence matrix of $j$-simplices in $k$-simplices, i.e., for
every $\sigma \in C^k(X)$ and every $\phi \in C^j(X)$, $d_j k \phi(\sigma) = \sum_{\tau \in X(j), \tau \subset \sigma} \phi(\tau)$. Define

$$U^j_k = \begin{cases} 
    d_0 k (C^0_0(X)) & j = 0, \\
    d_j k (C^j_0(X)) \cap (d_{j-1} k (C^{j-1}_0(X)))^\perp & j = 1, \ldots, k - 1, \\
    (d_{k-1} k (C^{k-1}_0(X)))^\perp & j = k.
  \end{cases}$$

We show that for every $0 \leq k \leq n - 1$, $U^0_k \oplus \ldots \oplus U^k_k$ is an orthogonal decomposition of $C^k_0(X)$ (this is a general result that does not rely on any spectral assumptions) and that if $X$ is a two-sided $\lambda$-local spectral expander with $\lambda$ small enough, the spaces $U^j_k$ approximate the eigenspaces of $M^+_k$:

**Theorem 1.5 (Decomposition Theorem for two-sided local spectral expanders, informal, for formal see Corollary 5.8, Theorem 5.9, Theorem 5.10, Corollary 5.11).** Let $X$ be a two-sided $\lambda$-local spectral expander. If $\lambda$ is sufficiently small there are explicit constants $f(k), g(k), h(k), b(k)$ such that the following holds:

1. For every $0 \leq k \leq n - 1$, every $0 \leq j \leq k$ and every $\phi \in C^k_0(X)$, the projection of $\phi$ on $U^j_k$ is an almost $\frac{k+1-j}{k+2}$ eigenvector, i.e.,

$$\left\| M^+_k (P_{U^j_k} \phi) - \left( \frac{k + 1 - j}{k + 2} \right) (P_{U^j_k} \phi) \right\| \leq f(k) \lambda \| P_{U^j_k} \phi \|,$$

where $P_{U^j_k} \phi$ is the orthogonal projection on $U^j_k$.

2. For every $0 \leq k \leq n - 1$, the non-trivial spectrum of $M^+_k$ is concentrated in small neighborhood of $\frac{k+1-j}{k+2}$, $j = 0, \ldots, k$, explicitly

$$\text{Spec}(M^+_k) \subseteq \{1\} \cup \bigcup_{j=0}^{k} \left[ \frac{k + 1 - j}{k + 2} - g(k) \lambda, \frac{k + 1 - j}{k + 2} + g(k) \lambda \right].$$

3. For every $0 \leq k \leq n$, the shrinkage of the random walk $M^+_k$ after any number of steps can be determined based on the size of the projections on the $U^j_k$’s: for every $i \in \mathbb{N}$ and every $\phi \in C^k_0(X)$,

$$\| (M^+_k)^i \phi \| \leq \sqrt{\sum_{j=0}^{k} \left( \frac{k + 1 - j}{k + 2} + h(k) \lambda \right)^{2i} (1 + b(k) \lambda)^2 \| P_{U^j_k} \phi \|^2},$$

and

$$\| (M^+_k)^i \phi \| \geq \sqrt{\sum_{j=0}^{k} \left( \frac{k + 1 - j}{k + 2} - h(k) \lambda \right)^{2i} (1 - b(k) \lambda)^2 \| P_{U^j_k} \phi \|^2}.$$
On the relation of Theorem 1.5 to the work of [4]. In an earlier version of this article, we proved only the Decomposition Theorem for one-sided local spectral expanders. In a subsequent work, Dikstein, Dinur, Filmus and Harsha [4] proved a decomposition theorem for two-sided local spectral expander that is similar to the first assertion Theorem 1.5. Theorem 1.5 is based on the ideas of [4], but, in our opinion, it gives a more complete treatment to the same problem. For comparison, in [4], the subspaces in the decomposition are almost-orthogonal (and not orthogonal) and it does not include the results regarding the spectrum and the shrinkage of $M^+_k$ that are stated in the second and third assertions of Theorem 1.5.

1.4. On small set expansion phenomenon, the Grassmann complex and our work

As we have explained above, we study the amount of shrinkage of a $k$-cochain by the random walk operator $M^+_k$. Our motivation is to go beyond spectral gap and to relate the shrinkage of a $k$-cochain by the operator, to its structure. Similar questions are asked in the study of small set expansion in the noisy hypercube [2]. Recently, it was shown that studying the structure of non expanding $k$-cochains of the Grassmann complex is strongly related to the “2-to-1 games Conjecture” [6,7], which is a weaker form of the famous Unique Game Conjecture. Our work here, is of the same flavor. However, instead of working with a specific complex (e.g., the Grassmann complex) we work with simplicial complexes, whose links are good spectral expanders. We characterize non expanding $k$-cochains as those whose mass is concentrated on the lower levels of the decomposition that we construct.

1.5. On different definitions of local spectral expanders

In [12], the second author gave a different (not strictly equivalent) definition for the notion of $\lambda$-local spectral expansion. The definition in [12] goes as follows: a simplicial complex $X$ is called a one sided $\lambda$-local spectral expansion in [12] if all its links of dimension $>0$ are connected and if for every $\tau \in X(n-2)$, $\mu_\tau \leq \lambda$. Also, a simplicial complex $X$ is called a two sided $(\lambda, \kappa)$-local spectral expansion in [12] if all its links of dimension $>0$ are connected and if for every $\tau \in X(n-2)$, $\mu_\tau \leq \lambda$ and $\nu_\tau \geq \kappa$.

Although these definitions are not strictly equivalent, a main result in [12] (see also Corollary 4.7 below) shows that they are equivalent up to changing $\lambda$, i.e., for any $0 < \lambda$ there is $0 < \lambda' = \frac{\lambda}{1+(n-1)\lambda}$ such that for every pure $n$-dimensional simplicial complex $X$, 
• $X$ is a one-sided $\lambda$-local spectral expander by the definition of this paper if and only if $X$ is a one-sided $\lambda'$-local spectral expander by the definition of [12].

• $X$ is a two-sided $\lambda$-local spectral expander by the definition of this paper if and only if $X$ is a two-sided $(\lambda',-\lambda')$-local spectral expander by the definition of [12].

This equivalence up to changing $\lambda$ is useful, because in examples it is sometimes easier to bound just the spectrum of the 1-dimensional links and not to have to analyse the spectrum of all the links.

1.6. Organisation

The paper is organized as follows. Section 2 contains the basic definitions regarding weighted complexes and the inner product and norm induced by the weights. Section 3 contains the definitions of the upper and lower random walks, and what we call the “signless differential” and show how these definitions are connected. Section 4 contains a connection in the spirit of the so called “Garland method” between the norm of the signless differential and the norm of the upper random walk operator in the links. Section 5 contains our main results regarding decomposition theorems for the upper random walk and their corollaries.

2. Weighted simplicial complexes

Let $X$ be a pure $n$-dimensional finite simplicial complex. For $-1 \leq k \leq n$, define $X(k)$ to be the set of all $k$-simplices in $X$ ($X(-1) = \{\emptyset\}$). A weight function $m$ on $X$ is a function:

$$ m: \bigcup_{-1 \leq k \leq n} X(k) \to \mathbb{R}^+, $$

such that for every $-1 \leq k \leq n-1$ and for every $\tau \in X(k)$ we have that

$$ m(\tau) = \sum_{\sigma \in X(k+1), \tau \subseteq \sigma} m(\sigma). $$

By its definition, it is clear that $m$ is determined by the values it takes on $X(n)$. A simplicial complex with a weight function will be called a weighted simplicial complex. The most basic (and most important) example of a weight function is the homogeneous weight $m$ that is the weight function defined by $m(\sigma) = 1$ for every $\sigma \in X(n)$. The following facts already appear in [12] and therefore the proofs are omitted.
Proposition 2.1 ([12, Proposition 2.7]). For every $-1 \leq k \leq n$ and every $\tau \in X(k)$ we have that
\[
\frac{1}{(n-k)!} m(\tau) = \sum_{\sigma \in X(n), \tau \subseteq \sigma} m(\sigma),
\]
where $\tau \subseteq \sigma$ means that $\tau$ is a face of $\sigma$.

In particular, the homogeneous weight $m$ on $X$ can be written explicitly as $\forall -1 \leq k \leq n$, $\forall \tau \in X(k)$,
\[
\frac{1}{(n-k)!} m(\tau) = |\{\sigma \in X(n): \tau \subseteq \sigma\}|.
\]

Corollary 2.2 ([12, Corollary 2.8]). For every $-1 \leq k < l \leq n$ and every $\tau \in X(k)$ we have
\[
\frac{1}{(l-k)!} m(\tau) = \sum_{\sigma \in X(l), \tau \subset \sigma} m(\sigma).
\]

Throughout this article, $X$ is a pure $n$-dimensional finite weighted simplicial complex with a weight function $m$.

Recall that in the introduction we defined the space of $k$-cochains, $C^k(X, \mathbb{R})$, as the set of all functions $\phi: X(k) \rightarrow \mathbb{R}$. On $C^k(X, \mathbb{R})$ define the following inner-product:
\[
\forall \phi, \psi \in C^k(X, \mathbb{R}), \langle \phi, \psi \rangle = \sum_{\sigma \in X(k)} m(\sigma) \phi(\sigma) \psi(\sigma).
\]

Denote by $\| . \|$ the norm induced from this inner-product.

3. Higher dimensional random walks and the signless differential

3.1. Upper and lower random walks

For $X$ as above, we define the following random walks on simplices of $X$:

Definition 3.1. For $0 \leq k \leq n-1$, the upper random walk on $k$-simplices is defined by the transition probability matrix $M^+_k : X(k) \times X(k) \rightarrow \mathbb{R}$:
\[
M^+_k(\tau, \tau') = \begin{cases} 
\frac{1}{k+2} m(\tau \cup \tau') & \text{if } \tau = \tau', \\
\frac{m(\tau \cup \tau')}{(k+2)m(\tau)} & \text{if } \tau \cup \tau' \in X(k+1), \\
0 & \text{otherwise}.
\end{cases}
\]

Definition 3.2. For $0 \leq k \leq n$, the lower random walk on $k$-simplices is defined by the transition probability matrix $M_k^- : X(k) \times X(k) \to \mathbb{R}$:

$$M_k^-(\tau, \tau') = \begin{cases} 
\frac{m(\tau)}{(k+1)m(\tau')} & \tau = \tau', \\
\frac{m(\tau \cap \tau')}{(k+1)m(\tau \cap \tau')} & \tau \cap \tau' \in X(k-1), \\
0 & \text{otherwise.}
\end{cases}$$

We leave it to the reader to check that those are in fact transition probability matrices, i.e., that for every $\tau$, $\sum_{\tau'} M_k^-(\tau, \tau') = 1$. We note that both random walks defined above are lazy in the sense that $M_k^\pm(\tau, \tau) \neq 0$. In the case of the upper random walk, one can easily define a non-lazy random walk as follows:

Definition 3.3. For $0 \leq k \leq n - 1$, the non-lazy upper random walk on $k$-simplices is defined by the transition probability matrix $(M')_k^+ : X(k) \times X(k) \to \mathbb{R}$:

$$(M')_k^+ = \frac{k + 2}{k + 1} \left( M_k^+ - \frac{1}{k + 2} I \right) = \frac{k + 2}{k + 1} M_k^+ - \frac{1}{k + 1} I.$$ 

It is standard to view $M_k^\pm, (M')_k^+$ as averaging operators on $C^k(X, \mathbb{R})$ and we will not make the distinction between the transition probability matrix and the averaging operator it induces.

It is worth noting that $M_0^-$ and $(M')_0^+$ are familiar operators/matrices: $M_0^-$ is a projection on the space of the constant functions (on vertices) with respect to the inner-product defined above, and $(M')_0^+$ is the weighted (normalized) adjacency matrix of the 1-skeleton of $X$.

### 3.2. The signless differential

We recall that when considering oriented simplicial complexes, the upper and lower Laplacians are obtained for the differential (i.e., coboundary) operator using discrete Hodge theory (see, for instance, [12, Section 2.2]). In our setting, disregarding orientation yields a similar result: below we define a signless differential and show that the upper and lower random walk operators are obtained from this signless differential in exactly the same way that the upper and lower Laplacians are obtained for the usual differential in the oriented setting - see exact formulation in Corollary 3.7 below.
Definition 3.4. For \(-1 \leq k \leq n-1\), the signless \(k\)-differential is an operator 
\(d_k : C^k(X, \mathbb{R}) \rightarrow C^{k+1}(X, \mathbb{R})\) defined as:
\[ \forall \phi \in C^k(X, \mathbb{R}), \forall \sigma \in X(k+1), d_k \phi(\sigma) = \sum_{\tau \subset \sigma, \tau \in X(k)} \phi(\tau). \]

Define \((d_k)^* : C^{k+1}(X, \mathbb{R}) \rightarrow C^k(X, \mathbb{R})\) to be the adjoint operator to \(d_k\), i.e., the operator such that for every \(\phi \in C^k(X, \mathbb{R})\), \(\psi \in C^{k+1}(X, \mathbb{R})\), \(\langle d_k \phi, \psi \rangle = \langle \phi, (d_k)^* \psi \rangle\).

Remark 3.5. We note that the signless differential is not a differential in the usual sense, since \(d_k^{k+1} \neq 0\). The name signless differential stems from the fact that this is the operator we will use in lieu of the differential in our setting (note that since our non-oriented cochains are defined without using orientation of simplices, we cannot use the usual differential).

Below, we will usually omit the index of signless differential and its adjoint and just denote \(d,d^*\) where \(k\) will be implicit.

Lemma 3.6. For \(-1 \leq k \leq n-1\), \(d^* : C^{k+1}(X, \mathbb{R}) \rightarrow C^k(X, \mathbb{R})\) is the operator
\[ \forall \psi \in C^{k+1}(X, \mathbb{R}), \forall \tau \in X(k), d^* \psi(\tau) = \sum_{\sigma \in X(k+1), \tau \subset \sigma} m(\sigma) m(\tau)^{-1} \psi(\sigma). \]

Proof. The proof is very similar to the proof of [12, Proposition 2.11], in which an analogous fact is proven in the oriented setting. We leave the adaption to the non-oriented setting to the reader.

Corollary 3.7. For \(0 \leq k \leq n-1\) and \(\phi \in C^k(X, \mathbb{R})\), \(d^*d\phi = (k+2)M^+\phi\) and \(dd^*\phi = (k+1)M^-\phi\).

Proof. Let \(\phi \in C^k(X, \mathbb{R})\) and \(\tau \in X(k)\), then
\[ d^*d\phi(\tau) = \sum_{\sigma \in X(k+1), \tau \subset \sigma} m(\sigma) m(\tau)^{-1} d\phi(\sigma) \]
\[ = \sum_{\sigma \in X(k+1), \tau \subset \sigma} m(\sigma) m(\tau)^{-1} \sum_{\tau' \in X(k), \tau' \subset \sigma} \phi(\tau') \]
\[ = \sum_{\sigma \in X(k+1), \tau \subset \sigma} m(\sigma) m(\tau)^{-1} \sum_{\tau' \in X(k), \tau' \subset \sigma, \tau' \neq \tau} \phi(\tau') \]
\[ + \sum_{\sigma \in X(k+1), \tau \subset \sigma} m(\sigma) m(\tau)^{-1} \phi(\tau). \]
Note that

\[
\sum_{\sigma \in X(k+1), \tau \subset \sigma} \frac{m(\sigma)}{m(\tau)} \phi(\tau) = \frac{m(\tau)}{m(\tau)} \phi(\tau) = \phi(\tau).
\]

Also note that

\[
\sum_{\sigma \in X(k+1), \tau \subset \sigma} \frac{m(\sigma)}{m(\tau)} \sum_{\tau' \in X(k), \tau' \subset \sigma, \tau' \neq \tau} \phi(\tau')
\]

\[
= \sum_{\sigma \in X(k+1), \tau \subset \sigma} \sum_{\tau' \in X(k), \tau' \subset \sigma, \tau' \neq \tau} \frac{m(\tau' \cup \tau)}{m(\tau)} \phi(\tau')
\]

\[
= \sum_{\tau' \in X(k), \tau \cup \tau' \in X(k+1)} \frac{m(\tau \cup \tau')}{m(\tau)} \phi(\tau').
\]

Therefore

\[
d^* d\phi(\tau) = \phi(\tau) + \sum_{\tau' \in X(k), \tau \cup \tau' \in X(k+1)} \frac{m(\tau \cup \tau')}{m(\tau)} \phi(\tau') = (k + 2)M^+ \phi(\tau).
\]

Similarly,

\[
d d^* \phi(\tau) = \sum_{\eta \in X(k-1), \eta \subset \tau} d^* \phi(\eta) = \sum_{\eta \in X(k-1), \eta \subset \tau} \sum_{\tau' \in X(k), \eta \subset \tau', \eta \neq \tau} \frac{m(\tau')}{m(\eta)} \phi(\tau')
\]

\[
= \sum_{\eta \in X(k-1), \eta \subset \tau} \sum_{\tau' \in X(k), \tau' \neq \eta, \tau' \subset \eta} \frac{m(\tau')}{m(\eta)} \phi(\tau')
\]

\[
+ \sum_{\eta \in X(k-1), \eta \subset \tau} \frac{m(\tau)}{m(\eta)} \phi(\tau)
\]

\[
= \sum_{\eta \in X(k-1), \eta \subset \tau} \sum_{\tau' \in X(k), \tau' \neq \eta, \tau' \subset \eta} \frac{m(\tau')}{m(\tau \cap \tau')} \phi(\tau')
\]

\[
+ \sum_{\eta \in X(k-1), \eta \subset \tau} \frac{m(\tau)}{m(\eta)} \phi(\tau)
\]

\[
= \sum_{\tau' \in X(k), \tau' \cap \tau' \in X(k-1)} \frac{m(\tau')}{m(\tau \cap \tau')} \phi(\tau') + \sum_{\eta \in X(k-1), \eta \subset \tau} \frac{m(\tau)}{m(\eta)} \phi(\tau)
\]

\[
= (k + 1)M^- \phi(\tau).
\]
4. Links and localization

Let $X$ be a pure $n$-dimensional finite simplicial complex with a weight function $m$. Recall that for $-1 \leq k \leq n-1$, $\tau \in X(k)$, the link of $\tau$, denoted $X_\tau$, is a pure $(n-k-1)$-simplicial complex defined as:

$$\eta \in X_\tau(l) \Leftrightarrow \eta \in X(l) \text{ and } \tau \cup \eta \in X(k+l+1).$$

On $X_\tau$ we define the weight function $m_\tau$ induced by $m$ as

$$m_\tau(\eta) = m(\tau \cup \eta).$$

Using this weight function, the inner-product and the norm on $C^l(X_\tau, \mathbb{R})$ are defined as above. The operators $M^\pm_{\tau,l}, (M^+)^\tau_{l,l}$ and $d_\tau, d^*_\tau$ are also defined on $C^l(X_\tau, \mathbb{R})$ as above.

Given a cochain $\phi \in C^l(X, \mathbb{R})$ and a simplex $\tau \in X(k)$ with $-1 \leq k < l$, we define the localization of $\phi$ on $X_\tau$, denoted $\phi_\tau$ as a cochain $\phi_\tau \in C^{l-k-1}(X_\tau, \mathbb{R})$ defined as

$$\phi_\tau(\eta) = \phi(\tau \cup \eta).$$

We note that $\tau$ and $\eta$ are, by definition, disjoint and $\tau \cup \eta = \tau \dot{\cup} \eta$. However, to avoid cumbersome notation, we will not use the disjoint union symbol.

The key observation (which was initially due to Garland [8], but is now considered standard - see [3], [9]) is that the inner-products of $\phi, d^*\phi,$ and $d\phi$ can be calculated via their localizations:

**Proposition 4.1.** Let $-1 \leq k < l \leq n$ and let $\phi, \psi \in C^l(X, \mathbb{R})$, then

1. \[
\frac{l+1}{k+1} \langle \phi, \psi \rangle = \sum_{\tau \in X(k)} \langle \phi_\tau, \psi_\tau \rangle
\]
   and in particular for $\psi = \psi$, \[
\left( \frac{l+1}{k+1} \right) \| \phi \|^2 = \sum_{\tau \in X(k)} \| \phi_\tau \|^2;
\]

2. \[
\frac{l}{k+1} \langle d^*\phi, d^*\psi \rangle = \sum_{\tau \in X(k)} \langle d^*_\tau \phi_\tau, d^*_\tau \psi_\tau \rangle
\]
   and in particular for $\psi = \psi$, \[
\left( \frac{l}{k+1} \right) \| d^*\phi \|^2 = \sum_{\tau \in X(k)} \| d^*_\tau \phi_\tau \|^2.
\]

3. If $l < n$, then \[
\langle d\phi, d\psi \rangle = \sum_{\tau \in X(l-1)} \left( \langle d_\tau \phi_\tau, d_\tau \psi_\tau \rangle - \frac{l}{l+1} \langle \phi_\tau, \psi_\tau \rangle \right)
\]
   and in particular for $\psi = \psi$, \[
\| d\phi \|^2 = \sum_{\tau \in X(l-1)} \left( \| d_\tau \phi_\tau \|^2 - \frac{l}{l+1} \| \phi_\tau \|^2 \right).
\]
Proof. The facts stated in this Proposition were already proven in the oriented setting in [12] (see [12, Lemma 3.4, Lemma 3.5]). The proofs in the non-oriented setting are very similar and are detailed in the appendix. ■

As a result of Proposition 4.1 we deduce the following:

**Proposition 4.2.** Let $0 \leq k \leq n-1$ and let $\phi, \psi \in C^k(X, \mathbb{R})$, then

$$\langle d\phi, d\psi \rangle = \langle d^*\phi, d^*\psi \rangle + \langle \phi, \psi \rangle + \sum_{\tau \in X(k-1)} \langle (M')_{\tau,0}^{\tau,0}(I - M_{\tau,0}^-)\phi_{\tau}, \psi_{\tau} \rangle$$

and in particular for $\phi = \psi$,

$$\|d\phi\|^2 = \|d^*\phi\|^2 + \|\phi\|^2 + \sum_{\tau \in X(k-1)} \langle (M')_{\tau,0}^{\tau,0}(I - M_{\tau,0}^-)\phi_{\tau}, \phi_{\tau} \rangle.$$

**Proof.** Let $\phi, \psi \in C^k(X, \mathbb{R})$. Note that for every $\tau \in X(k-1)$, $M_{\tau,0}^-$ is the orthogonal projection on the space of constant functions in $C^0(X_{\tau}, \mathbb{R})$ and therefore $(M')_{\tau,0}^{\tau,0}M_{\tau,0}^- = M_{\tau,0}^-.$

By Corollary 3.7, for every $\tau \in X(k-1)$,

$$\langle d\tau\phi_{\tau}, d\tau\psi_{\tau} \rangle = \langle 2M_{\tau,0}^+\phi_{\tau}, \psi_{\tau} \rangle = \langle ((M')_{\tau,0}^{\tau,0} + I)\phi_{\tau}, \psi_{\tau} \rangle$$

$$= \langle (M')_{\tau,0}^{\tau,0}\phi_{\tau}, \psi_{\tau} \rangle + \langle \phi_{\tau}, \psi_{\tau} \rangle$$

$$= \langle (M')_{\tau,0}^{\tau,0}M_{\tau,0}^+\phi_{\tau}, \psi_{\tau} \rangle + \langle (M')_{\tau,0}^{\tau,0}(I - M_{\tau,0}^-)\phi_{\tau}, \psi_{\tau} \rangle + \langle \phi_{\tau}, \psi_{\tau} \rangle$$

$$= \langle M_{\tau,0}^-\phi_{\tau}, \psi_{\tau} \rangle + \langle (M')_{\tau,0}^{\tau,0}(I - M_{\tau,0}^-)\phi_{\tau}, \psi_{\tau} \rangle + \langle \phi_{\tau}, \psi_{\tau} \rangle$$

$$= \langle d^*\phi_{\tau}, d^*\psi_{\tau} \rangle + \langle (M')_{\tau,0}^{\tau,0}(I - M_{\tau,0}^-)\phi_{\tau}, \psi_{\tau} \rangle + \langle \phi_{\tau}, \psi_{\tau} \rangle.$$

Combining this equality with Proposition 4.1 yields

$$\langle d\phi, d\psi \rangle = \sum_{\tau \in X(k-1)} \left( \langle d\tau\phi_{\tau}, d\tau\psi_{\tau} \rangle - \frac{k}{k+1} \langle \phi_{\tau}, \psi_{\tau} \rangle \right)$$

$$= \sum_{\tau \in X(k-1)} \left( \langle d^*\phi_{\tau}, d^*\psi_{\tau} \rangle + \langle (M')_{\tau,0}^{\tau,0}(I - M_{\tau,0}^-)\phi_{\tau}, \psi_{\tau} \rangle + \langle \phi_{\tau}, \psi_{\tau} \rangle - \frac{k}{k+1} \langle \phi_{\tau}, \psi_{\tau} \rangle \right)$$

$$= \langle d^*\phi, d^*\psi \rangle + \langle \phi, \psi \rangle + \sum_{\tau \in X(k-1)} \langle (M')_{\tau,0}^{\tau,0}(I - M_{\tau,0}^-)\phi_{\tau}, \psi_{\tau} \rangle,$$

as needed. ■
In light of the above corollary, we will want to bound the expression

\[ \sum_{\tau \in X(k-1)} \langle (M')^+_{\tau,0} (I - M^-_{\tau,0}) \phi_{\tau}, \psi_{\tau} \rangle, \]

using spectral information about \( X \). To make this precise, we will recall/define the following. For \( 0 \leq k \leq n - 1 \) and \( \tau \in X(k-1) \), recall that by Corollary 3.7, \( (M')^+_{\tau,0} = d^*_\tau d_\tau - I \) and therefore the eigenvalues of \( (M')^+_{\tau,0} \) are real. Denote \( \mu_{\tau} \) to be the second largest eigenvalue of \( (M')^+_{\tau,0} \) and \( \nu_{\tau} \) to be the smallest eigenvalue of \( (M')^+_{\tau,0} \). Note that if 1-skeleton of \( X_\tau \) is connected, then for every eigenfunction \( \varphi \in C^0(X_\tau, \mathbb{R}) \), if \( \varphi \perp \text{Im} M^-_{\tau,0} \), then \( (M')^+_{\tau,0} \varphi = \mu \varphi \) with \( \nu_{\tau} \leq \mu \leq \mu_{\tau} < 1 \). Denote

\[ \mu_k = \max_{\tau \in X(k-1)} \mu_{\tau}, \nu_k = \min_{\tau \in X(k-1)} \nu_{\tau}. \]

**Lemma 4.3.** For every \( 0 \leq k \leq n - 1 \) and every \( \phi, \psi \in C^k(X, \mathbb{R}) \) we have that

\[ \sum_{\tau \in X(k-1)} \langle (M')^+_{\tau,0} (I - M^-_{\tau,0}) \phi_{\tau}, \phi_{\tau} \rangle \leq (k + 1) \mu_k \| \phi \|^2, \]

and

\[ \sum_{\tau \in X(k-1)} \left| \langle (M')^+_{\tau,0} (I - M^-_{\tau,0}) \phi_{\tau}, \psi_{\tau} \rangle \right| \leq (k + 1) \max\{\mu_k, -\nu_k\} \| \phi \| \| \psi \|. \]

**Proof.** Let \( \phi, \psi \) be as above. Recall that for every \( \tau \in X(k-1) \), \( \phi_{\tau} \) decomposes orthogonally as

\[ \phi_{\tau} = (I - M^-_{\tau,0}) \phi_{\tau} + M^-_{\tau,0} \phi_{\tau}, \]

Therefore

\[ \langle (M')^+_{\tau,0} (I - M^-_{\tau,0}) \phi_{\tau}, \phi_{\tau} \rangle \]

\[ = \langle (M')^+_{\tau,0} (I - M^-_{\tau,0}) \phi_{\tau}, (I - M^-_{\tau,0}) \phi_{\tau} \rangle + \langle (M')^+_{\tau,0} (I - M^-_{\tau,0}) \phi_{\tau}, M^-_{\tau,0} \phi_{\tau} \rangle. \]
As explained above, \((M')_\tau^+ (I - M_{\tau,0}^-) \phi_\tau \in \text{Im}(I - M_{\tau,0}^-)\) and therefore
\[
\langle (M')_\tau^+ (I - M_{\tau,0}^-) \phi_\tau, M_{\tau,0}^- \phi_\tau \rangle = 0.
\]
This yields that
\[
\langle (M')_\tau^+ (I - M_{\tau,0}^-) \phi_\tau, \phi_\tau \rangle = \langle (M')_\tau^+ (I - M_{\tau,0}^-) \phi_\tau, (I - M_{\tau,0}^-) \phi_\tau \rangle.
\]
Note that by the definition of \(\mu_k\)
\[
\langle (M')_\tau^+ (I - M_{\tau,0}^-) \phi_\tau, (I - M_{\tau,0}^-) \phi_\tau \rangle \leq \mu_k \| (I - M_{\tau,0}^-) \phi_\tau \|^2 \leq \mu_k \| \phi_\tau \|^2.
\]
Summing over all \(\tau \in X(k - 1)\) and applying Proposition 4.1 yields the needed results, i.e.,
\[
\sum_{\tau \in X(k-1)} \langle (M')_\tau^+ (I - M_{\tau,0}^-) \phi_\tau, \phi_\tau \rangle
= \sum_{\tau \in X(k-1)} \langle (M')_\tau^+ (I - M_{\tau,0}^-) \phi_\tau, (I - M_{\tau,0}^-) \phi_\tau \rangle
\leq \sum_{\tau \in X(k-1)} \mu_k \| \phi_\tau \|^2 = (k + 1) \mu_k \| \phi \|^2.
\]
The second inequality follows from Cauchy-Schwarz:
\[
\sum_{\tau \in X(k-1)} \left| \langle (M')_\tau^+ (I - M_{\tau,0}^-) \phi_\tau, \psi_\tau \rangle \right|
\leq \sum_{\tau \in X(k-1)} \| (M')_\tau^+ (I - M_{\tau,0}^-) \phi_\tau \| \| \psi_\tau \|
\leq \sum_{\tau \in X(k-1)} \max\{\mu_k, -\nu_k\} \| \phi_\tau \| \| \psi_\tau \|
\leq \max\{\mu_k, -\nu_k\} \left( \sum_{\tau \in X(k-1)} \| \phi_\tau \|^2 \right)^{\frac{1}{2}} \left( \sum_{\tau \in X(k-1)} \| \psi_\tau \|^2 \right)^{\frac{1}{2}}
= (k + 1) \max\{\mu_k, -\nu_k\} \| \phi \| \| \psi \|.
\]

We recall the following definition from the introduction:

**Definition 4.4 (Local spectral expander).** An \(n\)-dimensional complex \(X\) is a one-sided \(\lambda\)-local-spectral expander if for every \(0 \leq k \leq n - 1\), \(\mu_k \leq \lambda\). An \(n\)-dimensional complex \(X\) is a two-sided \(\lambda\)-local-spectral expander if for every \(0 \leq k \leq n - 1\), \(-\lambda \leq \mu_k \leq \lambda\).
Next, we recall the following result appearing in [12, Lemma 5.1] (see also [11, Proposition 3.7]):

**Lemma 4.5.** Let $X$ be a weighted pure $n$-dimensional simplicial complex, such that all the links of $X$ of dimension $\geq 1$ (including $X$ itself) are connected, then for every $0 \leq k \leq n-2$,

$$\mu_k \leq \frac{\mu_{k+1}}{1 - \mu_{k+1}},$$
$$\nu_k \geq \frac{\nu_{k+1}}{1 - \nu_{k+1}}.$$

A simple induction leads to the following:

**Corollary 4.6.** Let $X$ be a weighted pure $n$-dimensional simplicial complex, such that all the links of $X$ of dimension $\geq 1$ (including $X$ itself) are connected, then for every $0 \leq k \leq n-2$,

$$\mu_k \leq \frac{\mu_{n-1}}{1 - (n - 1 - k)\mu_{n-1}},$$
$$\nu_k \geq \frac{\nu_{n-1}}{1 - (n - 1 - k)\nu_{n-1}}.$$

A corollary of the above corollary is the following:

**Corollary 4.7.** Let $X$ be a weighted pure $n$-dimensional simplicial complex, such that all the links of $X$ of dimension $\geq 1$ (including $X$ itself) are connected, and $0 < \lambda \leq 1$ be some constant. If $\mu_{n-1} \leq \frac{\lambda}{1+(n-1)\lambda}$, then $X$ is a one-sided $\lambda$-spectral expander. Moreover, if $\mu_{n-1} \leq \frac{\lambda}{1+(n-1)\lambda}$ and $\frac{-\lambda}{1+(n-1)\lambda} \leq \nu_{n-1}$, then $X$ is a two-sided $\lambda$-spectral expander.

**Proof.** By the above corollary, if $\mu_{n-1} \leq \frac{\lambda}{1+(n-1)\lambda}$, then for every $0 \leq k \leq n-2$ we have that

$$\mu_k \leq \frac{\mu_{n-1}}{1 - (n - 1 - k)\mu_{n-1}} \leq \frac{\lambda}{1+(n-1)\lambda} \leq \frac{\lambda}{1 - (n - 1)\frac{\lambda}{1+(n-1)\lambda}} \leq \lambda,$$

and therefore $X$ is a one-sided $\lambda$-spectral expander. The proof of the second assertion is similar. ■
Remark 4.8. The reader should note that in [12, Lemma 5.1] the results of Lemma 4.5 are phrased in the language of spectral gaps of the Laplacians on the links $\Delta^+_{\tau,0}$ and not as the spectral gaps of $(M')^+_{\tau,0}$. However, the translation of the spectral gaps is easy once one recalls that $\Delta^+_{\tau,0} = I - (M')^+_{\tau,0}$.

5. Decomposition theorems for upper random walks

Roughly speaking, we will show below that given $0 \leq k \leq n-1$, the space of $k$-cochains orthogonal to the constants can be decomposed into “orthogonal” parts coming from the degrees $0 \leq j \leq k$ of the simplicial complex and this decomposition allows us to bound the shrinkage of $M^+_k$. We will prove this type of results under two sets of assumptions: first, we will prove a decomposition theorem under the assumption of one-sided local spectral gap. Under this assumption the decomposition is not really an orthogonal decomposition, but it already gives a bound for the maximal non-trivial eigenvalue of $M^+_k$ and some insight for what type of cochains $M^+_k$ shrinks better than the bound given by this eigenvalue. Second, we will assume to prove a decomposition theorem under the assumption of two-sided local spectral gap. Under this more restrictive assumption, we are able to give a rather comprehensive description of the spectral theory of $M^+_k$. Namely, we show that the spectrum of $M^+_k$ is concentrate in small interval centered at $\frac{j}{k+2}, j = 1, \ldots, k+1$ and that we give an explicit orthogonal decomposition that approximates the decomposition into eigenspaces.

5.1. The space of cochains orthogonal to the constants

For every $0 \leq k \leq n-1$, we denote $C^k_0(X, \mathbb{R})$ to be

$$C^k_0(X, \mathbb{R}) = \left\{ \phi \in C^k(X, \mathbb{R}) : \sum_{\sigma \in X(k)} m(\sigma)\phi(\sigma) = 0 \right\}.$$ 

Let $1_k$ be the constant 1 function in $C^k(X, \mathbb{R})$, then by definition for every $\phi \in C^k_0(X, \mathbb{R})$, we have that

$$\langle \phi, 1_k \rangle = \sum_{\sigma \in X(k)} m(\sigma)\phi(\sigma) = 0,$$

and one can see that $C^k(X, \mathbb{R})$ has the orthogonal decomposition $C^k(X, \mathbb{R}) = \text{span}\{1_k\} \oplus C^k_0(X, \mathbb{R})$. It is easy to see that $M^\pm_k 1_k = 1_k$ and since, by Corollary
Lemma 5.1. For $0 \leq k \leq n-1$, $\ker((d_{k-1})^*) \subseteq C^k_0(X,\mathbb{R})$ and for every $\psi \in C^{k-1}(X,\mathbb{R})$, $\psi \in C^{k-1}_0(X,\mathbb{R})$ if and only if $d_{k-1}\psi \in C^k_0(X,\mathbb{R})$.

Proof. We note that by definition $d_{k-1}1_{k-1} = (k+1)1_k$, and, by Lemma 3.6, $(d_{k-1})^*1_k = 1_{k-1}$. Therefore for every $\phi \in \ker((d_{k-1})^*)$, we have that

$$0 = \langle (d_{k-1})^*\phi, 1_{k-1} \rangle = \langle \phi, (d_{k-1})1_{k-1} \rangle = (k+1)\langle \phi, 1_k \rangle \Rightarrow \phi \in C^k_0(X,\mathbb{R}).$$

Second, let $\psi \in C^{k-1}(X,\mathbb{R})$. If $\psi \in C^{k-1}_0(X,\mathbb{R})$, then $M^+_{k-1}\psi \in C^{k-1}_0(X,\mathbb{R})$ and therefore by Corollary 3.7

$$0 = \langle M^+_{k-1}\psi, 1_{k-1} \rangle = \langle d_{k-1}\psi, \frac{1}{k+1}d_{k-1}1_{k-1} \rangle = \langle d_{k-1}\psi, 1_k \rangle,$$

i.e., $d_{k-1}\psi \in C^k_0(X,\mathbb{R})$.

Conversely, assume that $d_{k-1}\psi \in C^k_0(X,\mathbb{R})$, then

$$0 = \langle d_{k-1}\psi, 1_k \rangle = \langle \psi, (d_{k-1})^*1_k \rangle = \langle \psi, 1_{k-1} \rangle \Rightarrow \psi \in C^{k-1}_0(X,\mathbb{R}).$$

5.2. Decomposition Theorem for one-sided local spectral expanders

Theorem 5.2 (Decomposition Theorem). For every $0 \leq k \leq n-1$ and every $\phi \in C^k_0(X,\mathbb{R})$, there are $\phi^0 \in C^0_0(X,\mathbb{R}), \phi^k, (\phi^{k-1})' \in C^{k-1}_0(X,\mathbb{R}), \ldots, \phi^0, (\phi^0)' \in C^0_0(X,\mathbb{R})$ such that if we denote $(\phi^k)' = \phi$, then the following holds:

1. For every $0 \leq j \leq k$,

$$\| (\phi^j)' \|^2 = \| \phi^j \|^2 + \| \phi^{j-1} \|^2 + \ldots + \| \phi^0 \|^2.$$

2. $\|d_k\phi\|^2 = \sum_{j=0}^k (k+1-j)\| \phi^j \|^2 + \sum_{j=0}^k \sum_{\tau \in X(j-1)} \langle (M'_\tau)^+ (I-M^-_{\tau,0}) (\phi^j)'_\tau, (\phi^j)'_\tau \rangle.$

Proof. We will prove the theorem by induction on $k$. For $k = 0$ and $\phi \in C^0_0(X,\mathbb{R})$, we take $\phi^0 = \phi$ and check that the theorem holds for this choice.

1. This condition holds trivially.

2. We note that $\phi \in C^0_0(X,\mathbb{R})$ implies that $d^*\phi = 0$ and therefore this condition follows from Proposition 4.2.
Assume next that \( k > 0 \) and that the theorem holds for \( k - 1 \). For \( \phi \in C^k_0(X, \mathbb{R}) \), we first decompose \( \phi \) as \( \phi = \phi^k + \phi' \), where \( \phi^k \in \ker((d_{k-1})^*) \) and \( \phi' \in (\ker(d_{k-1}))^\perp \). This is an orthogonal decomposition and therefore

\[
\|\phi\|^2 = \|\phi^k\|^2 + \|\phi'\|^2.
\]

Also, by Proposition 4.2,

(1) \[
\|d_k\phi\|^2 = \|\phi\|^2 + \|d_{k-1}^\ast \phi'\|^2 + \sum_{\tau \in X^{(k-1)}} \langle (M')_{\tau,0}^+ (I - M_{\tau,0}^-) \phi_{\tau}, \phi_{\tau} \rangle.
\]

We note that \( (\ker(d_{k-1})^\perp)^\ast = \text{Im}(d_{k-1}) \) and therefore, by using Lemma 5.1, there is \( \psi \in C^{k-1}_0(X, \mathbb{R}) \) such that \( d_{k-1}\psi = \phi' \). This yields that there is \( \psi \in C^{k-1}_0(X, \mathbb{R}) \), such that \( \|d_{k-1}\psi\|^2 = \|\phi'\|^2 \) and

\[
\|d_k\phi\|^2 = \|\phi\|^2 + \|d_{k-1}^\ast d_{k-1}\psi\|^2 + \sum_{\tau \in X^{(k-1)}} \langle (M')_{\tau,0}^+ (I - M_{\tau,0}^-) \phi_{\tau}, \phi_{\tau} \rangle.
\]

We recall that since \( (d_{k-1})^\ast d_{k-1} \) is a self-adjoint operator, with non-negative eigenvalues, \( \sqrt{(d_{k-1})^\ast d_{k-1}} \) is the self-adjoint operator, with non-negative eigenvalues defined as follows: for every eigenfunction \( \varphi \) of \( (d_{k-1})^\ast d_{k-1} \) with an eigenvalue \( \mu \), \( \varphi \) is an eigenfunction of \( \sqrt{(d_{k-1})^\ast d_{k-1}} \) with the eigenvalue \( \sqrt{\mu} \).

We will take \( (\phi^{k-1})' = \sqrt{(d_{k-1})^\ast d_{k-1}}\psi \) and check that the theorem holds for this choice.

First, we note that, using Corollary 3.7, \( (d_{k-1})^\ast d_{k-1}(C^{k-1}_0(X, \mathbb{R})) \subseteq C^{k-1}_0(X, \mathbb{R}) \), and therefore \( \sqrt{(d_{k-1})^\ast d_{k-1}(C^{k-1}_0(X, \mathbb{R}))} \subseteq C^{k-1}_0(X, \mathbb{R}) \), which implies that \( (\phi^{k-1})' = \sqrt{(d_{k-1})^\ast d_{k-1}}\psi \in C^{k-1}_0(X, \mathbb{R}) \).

Second, we note that

\[
\|d_{k-1}\psi\|^2 = \langle (d_{k-1})^\ast d_{k-1}\psi, \psi \rangle
\]

\[
= \langle \sqrt{(d_{k-1})^\ast d_{k-1}}\psi, \sqrt{(d_{k-1})^\ast d_{k-1}}\psi \rangle
\]

\[
= \|\sqrt{(d_{k-1})^\ast d_{k-1}}\psi\|^2.
\]

Therefore \( \sqrt{(d_{k-1})^\ast d_{k-1}}\psi \in C^{k-1}_0(X, \mathbb{R}) \) and \( \|\sqrt{(d_{k-1})^\ast d_{k-1}}\psi\| = \|\phi'\| \).

This yields that

\[
\|\phi\|^2 = \|\phi^k\|^2 + \|\phi^{k-1}\|^2 + \ldots + \|\phi^0\|^2,
\]

and by the induction assumption

(2) \[
\|\phi\|^2 = \|\phi^k\|^2 + \|\phi^{k-1}\|^2 + \ldots + \|\phi^0\|^2.
\]

Last, we note that

\[
\|(d_{k-1})^\ast \phi'\|^2 = \|(d_{k-1})^\ast d_{k-1}\psi\|^2.
\]
By the induction assumption,

\[ \langle (d_{k-1})^* d_{k-1} \psi, (d_{k-1})^* d_{k-1} \psi \rangle \]

\[ = \langle (d_{k-1})^* d_{k-1} \sqrt{(d_{k-1})^* d_{k-1} \psi}, \sqrt{(d_{k-1})^* d_{k-1} \psi} \rangle \]

\[ = \| d_{k-1} \sqrt{(d_{k-1})^* d_{k-1} \psi} \|^2 \]

\[ = \| d_{k-1} (\phi^{k-1})' \|^2. \]

Combining this with (1), we get that

\[ \| d\phi \|^2 = \| \phi \|^2 + \| d_{k-1} (\phi^{k-1})' \|^2 + \sum_{\tau \in X(k-1)} \langle (M')^+_{\tau,0} (I - M^-_{\tau,0}) \phi_{\tau}, \phi_{\tau} \rangle. \]

By the induction assumption,

\[ \| d_{k-1} (\phi^{k-1})' \|^2 \]

\[ = \sum_{j=0}^{k-1} (k - j) \| \phi^j \|^2 + \sum_{j=0}^{k-1} \sum_{\tau \in X(j-1)} \langle (M')^+_{\tau,0} (I - M^-_{\tau,0}) (\phi^j)_{\tau}', (\phi^j)_{\tau}' \rangle. \]

Therefore

\[ \| d\phi \|^2 = \| \phi \|^2 + \| d_{k-1} (\phi^{k-1})' \|^2 + \sum_{\tau \in X(k-1)} \langle (M')^+_{\tau,0} (I - M^-_{\tau,0}) \phi_{\tau}, \phi_{\tau} \rangle \]

\[ = \| \phi \|^2 + \sum_{j=0}^{k-1} (k - j) \| \phi^j \|^2 + \sum_{j=0}^{k-1} \sum_{\tau \in X(j-1)} \langle (M')^+_{\tau,0} (I - M^-_{\tau,0}) (\phi^j)_{\tau}', (\phi^j)_{\tau}' \rangle \]

\[ + \sum_{\tau \in X(k-1)} \langle (M')^+_{\tau,0} (I - M^-_{\tau,0}) \phi_{\tau}, \phi_{\tau} \rangle \]

\[ = \| \phi \|^2 + \sum_{j=0}^{k-1} (k - j) \| \phi^j \|^2 + \sum_{j=0}^{k} \sum_{\tau \in X(j-1)} \langle (M')^+_{\tau,0} (I - M^-_{\tau,0}) (\phi^j)_{\tau}', (\phi^j)_{\tau}' \rangle \]

\[ = \sum_{j=0}^{k} (k + 1 - j) \| \phi^j \|^2 + \sum_{j=0}^{k} \sum_{\tau \in X(j-1)} \langle (M')^+_{\tau,0} (I - M^-_{\tau,0}) (\phi^j)_{\tau}', (\phi^j)_{\tau}' \rangle, \]

where the last equality is due to (2).

**Corollary 5.3.** Let \( X \) be a pure \( n \)-dimensional weighted simplicial complex such that all the links of \( X \) of dimension \( \geq 1 \) are connected (including \( X \) itself) and let \( 0 \leq k \leq n-1 \). Then for every \( \phi \in C^k_0(X, \mathbb{R}) \), there are
φ^k ∈ C^k_0(X, ℜ), φ^{k-1} ∈ C^{k-1}_0(X, ℜ), ..., φ^0 ∈ C^0_0(X, ℜ), such that

∥φ∥^2 = ∥φ^k∥^2 + ... + ∥φ^0∥^2,

and

∥dφ∥^2 ≤ \sum_{j=0}^{k} \left( k + 1 - j + \sum_{i=j}^{k} (i + 1) \mu_i \right) ∥φ^j∥^2.

In particular, if X is a one-sided λ-local-spectral expander, then

∥dφ∥^2 ≤ \sum_{j=0}^{k} \left( k + 1 - j + \frac{(k + j + 2)(k + 1 - j)}{2} \lambda \right) ∥φ^j∥^2.

**Proof.** Let φ ∈ C^k_0(X, ℜ) and

φ^k ∈ C^k_0(X, ℜ), φ^{k-1}, (φ^{k-1})' ∈ C^{k-1}_0(X, ℜ), ..., φ^0, (φ^0)' ∈ C^0_0(X, ℜ)

as in the Decomposition Theorem. Then

∥φ∥^2 = ∥φ^k∥^2 + ... + ∥φ^0∥^2,

and we will prove that

∥dφ∥^2 ≤ \sum_{j=0}^{k} (k + 1 - j + \sum_{i=j}^{k} (i + 1) \mu_i) ∥φ^j∥^2,

(the proof of the second inequality is similar and therefore it is left to the reader).

Note that for every 0 ≤ j ≤ k, we have by Lemma 4.3 that

\sum_{τ ∈ X(j-1)} ((M')^+_τ(I - M^-_{τ,0})(φ^j)'_τ, (φ^j)'_τ) ≤ (j + 1)μ_j ∥(φ^j)'∥^2.

Therefore

\sum_{j=0}^{k} \sum_{τ ∈ X(j-1)} ((M')^+_τ(I - M^-_{τ,0})(φ^j)'_τ, (φ^j)'_τ)

\leq \sum_{j=0}^{k} (j + 1)μ_j \sum_{i=j}^{k} ∥φ^i∥^2

= \sum_{i=0}^{k} ∥φ^i∥^2 \sum_{j=i}^{k} (j + 1)μ_j.
Replacing the roles of $i$ and $j$ in the above inequality and combining it with the equation if the Decomposition Theorem for $\|d\phi\|^2$ yields the needed inequality.

A consequence of this corollary is the following mixing results for $\lambda$-local spectral expanders:

**Theorem 5.4 (Mixing of the random walks).** Let $X$ be a weighted pure $n$-dimensional simplicial complex and let $0 \leq \lambda \leq 1$ be some constant. If $X$ is a one-sided $\lambda$-local spectral expander, then for every $0 \leq k \leq n-1$

$$\forall \phi \in C^k_0(X, \mathbb{R}), \|M_k^+ \phi\| \leq \left(\frac{k + 1}{k + 2} + \frac{k + 1}{2} \lambda\right) \|\phi\|.$$  

**Proof.** Let $0 \leq k \leq n-1$ and $\phi \in C^k_0(X, \mathbb{R})$. Assume that $X$ is a one-sided $\lambda$-local spectral expander, then by Corollary 5.3 we get

$$\|d\phi\|^2 \leq \sum_{i=0}^{k} \left(k + 1 - i + \frac{(k + i + 2)(k + 1 - i)}{2}\right) \|\phi^i\|^2$$

$$\leq \sum_{i=0}^{k} \left(k + 1 + \frac{(k + 2)(k + 1)}{2}\right) \|\phi^i\|^2$$

$$= \left(k + 1 + \frac{(k + 2)(k + 1)}{2}\right) \|\phi\|^2.$$  

Recall that by Corollary 3.7 $(d_k)^* d_k = (k + 2)M_k^+$ and therefore

$$\langle M_k^+ \phi, \phi \rangle = \frac{1}{k + 2} \|d\phi\|^2 \leq \left(\frac{k + 1}{k + 2} + \frac{k + 1}{2} \lambda\right) \|\phi\|^2.$$  

$M_k^+$ is a positive operator that maps $C^k_0(X, \mathbb{R})$ into itself and by the above inequality, any eigenvector of $M_k^+$ in $C^k_0(X, \mathbb{R})$ has an eigenvalue $\leq \frac{k + 1}{k + 2} + (k + 1)\lambda$.  

5.3. Decomposition Theorem for two-sided local spectral expanders

Let $X$ be a weighted simplicial complex of dimension $n$ and $0 \leq k \leq n-1$. Define the following subspaces of $C^k_0(X)$:

1. For $k=0$, define $V_0^0 = C^k_0(X)$.  

2. For $1 \leq k \leq n-1$ and $j = 0, \ldots, k-1$, define $V^j_k = d_{k-1} \ldots d_j(C^j_0(X))$ and also define $V^k_k = (d_{k-1}(C^k_0(X)))^\perp = (V^{k-1}_k)^\perp = \ker(d^*_{k-1})$.

By Lemma 5.1 $V^0_k, \ldots, V^k_k$ are subspaces of $C^k_0(X)$. Denote

$$U^j_k = \begin{cases} V^0_k & j = 0, \\
V^j_k \cap (V^{j-1}_k)^\perp & j = 1, \ldots, k-1, \\
V^k_k & j = k.
\end{cases}$$

We note that for every $2 \leq k$ and every $0 \leq j < j + 1 \leq k-1$, $V^j_k \subseteq V^j_{k+1}$ and therefore the following is an orthogonal decomposition:

$$C^k_0(X) = U^k_k \oplus U^{k-1}_k \oplus \ldots \oplus U^0_k.$$

Conceptually, recall that in the introduction we defined the incidence operator of the $j$-simplices in $k$-simplices as follows: for $0 \leq j < k \leq n$, define $d_{j \nearrow k} : C^j(X) \to C^k(X)$ to be the operator

$$\forall \sigma \in X(k), d_{j \nearrow k} \phi(\sigma) = \sum_{\tau \in X(j), \tau \subseteq \sigma} \phi(\tau).$$

We note that by definition $d_{k-1 \nearrow k} = d_{k-1}$ and we prove below that for every $j < k-1$, $d_{j \nearrow k} = \frac{1}{(k-j)!}d_{k-1} \ldots d_j$.

**Proposition 5.5.** For $0 \leq j < k \leq n$ and $0 \leq j \leq k-1$, $d_{j \nearrow k} = \frac{1}{(k-j)!}d_{k-1} \ldots d_j$

**Proof.** For $j = k-1$, $d_{k-1 \nearrow k} = d_{k-1}$ by definition. We proceed by downward induction. Assume that $d_{j+1 \nearrow k} = \frac{1}{(k-(j+1))!}d_{k-1} \ldots d_{j+1}$, then for every $\phi \in C^j(X)$ and every $\sigma \in X(k)$,

$$(d_{k-1} \ldots d_j \phi)(\sigma) = (d_{k-1} \ldots d_{j+1}(d_j \phi))(\sigma)$$

$$= (k-1-j)! \sum_{\eta \in X(j+1), \eta \subseteq \sigma} (d_j \phi)(\eta)$$

$$= (k-1-j)! \sum_{\eta \in X(j+1), \eta \subseteq \sigma} \sum_{\tau \in X(j), \tau \subseteq \eta} \phi(\tau)$$

$$= (k-1-j)! \sum_{\tau \in X(j), \tau \subseteq \sigma} \phi(\tau) \left( \sum_{\eta \in X(j+1), \eta \subseteq \sigma, \tau \subseteq \eta} 1 \right)$$

$$= (k-1-j)! (k-j) \sum_{\tau \in X(j), \tau \subseteq \sigma} \phi(\tau) = (k-j)! d_{j \nearrow k} \phi(\sigma),$$

as needed. \qed
As a conclusion of the above Proposition, \( V_j^k = d_j \wedge_k (C_0^j(X)) \), i.e., this definition of the spaces \( U_j^k \) coincides with the one given in the introduction.

**Theorem 5.6.** Let \( X \) be a two-sided \( \lambda \)-local spectral expander, and let \( \{ \varepsilon_k : 0 \leq k \leq n-1 \} \) be constants defined as

\[
\varepsilon_k = \begin{cases} 
\lambda & k = 0, \\
2k(1 + 2k\sqrt{k})\varepsilon_{k-1} + (k + 1)\lambda & 0 < k.
\end{cases}
\]

If \( \varepsilon_k \leq \frac{1}{2(1+2(k+1)\sqrt{k+1})} \) for all \( 0 \leq k \leq n-2 \), then for every \( 0 \leq k \leq n-1 \), every \( 0 \leq j \leq k \) and every \( \phi \in C_0^k(X) \),

\[
\|d^*_{k-1}d_kP_{U_j^k}\phi - (k + 1 - j)P_{U_j^k}\phi\| \leq \varepsilon_k\|P_{U_j^k}\phi\|.
\]

**Remark 5.7.** Note that \( \lim_{\lambda \to 0} \varepsilon_k = 0 \) and therefore the conditions of Theorem 5.6 holds if \( \lambda \) is small enough with respect to \( n \).

**Proof.** The proof is by induction on \( k \).

For \( k = 0 \), by definition \( \phi \in C_0^0(X) \) is equal to \( P_{U_0^0}\phi \). By our assumption, \( \phi \) is orthogonal to the constant functions on \( X(0) \) and \( d^*d\phi = 2M_0^+\phi \), where \( M_0^+ \) is the lazy random walk on the vertices. Since \( X \) is a two-sided \( \lambda \)-local spectral expander, it follows that the spectrum of \( 2M_0^+ \) on \( C_0^k(X) \) is contained in the interval \([1 - \lambda, 1 + \lambda]\). Therefore

\[
\|d^*d\phi - \phi\| \leq \lambda\|\phi\|,
\]

i.e., for \( k = j = 0 \),

\[
\|d^*dP_{U_0^0}\phi - (0 + 1 - 0)P_{U_0^0}\phi\| \leq \varepsilon_0\|P_{U_0^0}\phi\| \text{ as needed.}
\]

Let \( 1 \leq k \leq n-1 \) and assume that for every \( \psi \in C_{k-1}^k(X) \) and every \( j = 0, \ldots, k-1 \),

\[
\|d^*_{k-1}d_kP_{U_j^k}\psi - (k - j)P_{U_j^k}\psi\| \leq \varepsilon_{k-1}\|P_{U_j^k}\psi\|.
\]

Fix \( \phi \in C_0^k(X) \) and \( 0 \leq j \leq k \). Denote \( \varphi = P_{U_k^j}\phi \). With this notation, we need to prove that for every \( \varphi' \in C_0^k(X) \)

\[
\|\langle d^*\psi, \varphi' \rangle - (k + 1 - j)\langle \varphi, \varphi' \rangle\| \leq \varepsilon_k\|\varphi\||\varphi'\|.
\]

By Proposition 4.2, this is equivalent to proving

\[
\left| \langle d^*\varphi, d^*\varphi' \rangle - (k - j)\langle \varphi, \varphi' \rangle + \sum_{\tau \in X(k-1)} \langle (M')_\tau^+(I - M^-_\tau,0)\varphi, \varphi'_\tau \rangle \right| \leq \varepsilon_k\|\varphi\||\varphi'\|.
\]
By Lemma 4.3,
\[ \sum_{\tau \in X(k-1)} \left| \langle (M' \tau_0^+(I - M_{\tau,0}^-) \varphi_\tau, \varphi'_\tau) \rangle \right| \leq (k + 1)\lambda \| \varphi \| \| \varphi' \| , \]
and we are left with bounding
\[ \left| \langle d^*_k \varphi, d^*_k \varphi' \rangle - (k - j) \langle \varphi, \varphi' \rangle \right| . \]
If \( j = k \), then \( \varphi \in \ker(d^*_k) \) and \( k - j = 0 \) and therefore
\[ \left| \langle d^*_k \varphi, d^*_k \varphi' \rangle - (k - j) \langle \varphi, \varphi' \rangle \right| = 0. \]
Therefore, when \( j = k \),
\[ \left| \langle d^* \varphi, d^* \varphi' \rangle - (k - j) \langle \varphi, \varphi' \rangle + \sum_{\tau \in X(k-1)} \langle (M' \tau_0^+(I - M_{\tau,0}^-) \varphi_\tau, \varphi'_\tau) \rangle \right| \]
\[ \leq (k + 1)\lambda \| \varphi \| \| \varphi' \| , \]
as needed.

We will complete the proof by assuming that \( j < k \) and showing
\[ \left| \langle d^*_k \varphi, d^*_k \varphi' \rangle - (k - j) \langle \varphi, \varphi' \rangle \right| \leq 2k(1 + 2k\sqrt{k})\varepsilon_{k-1}\| \varphi \| \| \varphi' \|. \]
We note that \( j < k \) implies that \( \varphi \in V^j_k \subseteq \text{Im}(d^*_k) \) and in particular \( \varphi \perp V^k_k \).
Therefore
\[ \langle \varphi, \varphi' \rangle = \langle \varphi, P_{V^k_k} \varphi' + P_{V^k_k}^\perp \varphi' \rangle = \langle \varphi, P_{V^k_k} \varphi' \rangle. \]
We also note that \( d^*_k P_{V^k_k} = 0 \) and therefore
\[ \langle d^*_k \varphi, d^*_k \varphi' \rangle = \langle d^*_k \varphi, d^*_k P_{V^k_k} \varphi' \rangle. \]
As a consequence, when bounding
\[ \left| \langle d^*_k \varphi, d^*_k \varphi' \rangle - (k - j) \langle \varphi, \varphi' \rangle \right| \]
we can assume without loss of generality that \( \varphi' \in V^{k-1}_k \). By this assumption, there are \( \psi, \psi' \in C^k(X) \) such that \( d_{k-1} \psi = \varphi, d_{k-1} \psi' = \varphi' \), i.e.,
\[ \left| \langle d^*_k \varphi, d^*_k \varphi' \rangle - (k - j) \langle \varphi, \varphi' \rangle \right| \]
\[ = \left| \langle d^*_k d_{k-1} \psi, d^*_k d_{k-1} \psi' \rangle - (k - j)\langle d_{k-1} \psi, d_{k-1} \psi' \rangle \right| . \]
We note that (as in the proof of Theorem 5.2)
\[ \| \varphi \|^2 = \langle d_{k-1} \psi, d_{k-1} \psi \rangle \]
\begin{align*}
= \left\langle \sqrt{d_{k-1}^*}d_{k-1}^\ast \psi, \sqrt{d_{k-1}^*}d_{k-1}^\ast \psi \right\rangle \\
= \left\| \sqrt{d_{k-1}^*}d_{k-1}^\ast \psi \right\|^2,
\end{align*}

and similarly, \( \|\varphi\|^2 = \|\sqrt{d_{k-1}^*}d_{k-1}^\ast \varphi\|^2 \). As a consequence of the above equalities, we need to show that

\begin{equation}
\left| \langle d_{k-1}^\ast d_{k-1}^\ast \psi, d_{k-1}^\ast d_{k-1}^\ast \psi \rangle \rangle - (k-j)\langle d_{k-1}^\ast \psi, d_{k-1}^\ast \psi \rangle \right| \\
\leq 2k(1 + 2k\sqrt{k})\varepsilon_{k-1} \left\| \sqrt{d_{k-1}^*}d_{k-1}^\ast \psi \right\| \left\| \sqrt{d_{k-1}^*}d_{k-1}^\ast \psi \right\|.
\end{equation}

By \( \varphi \in U^j_k \), \( \varphi \in \text{Im}(d_{k-1} \ldots d_j) \cap \ker(d_{j-1}^* \ldots d_{k-1}^*) \) for \( j > 0 \) and \( \varphi \in \text{Im}(d_{k-1} \ldots d_0) \) for \( j = 0 \). We note that by the definition of \( \psi \), this implies that \( \psi \in \text{Im}(d_{k-2} \ldots d_j) \), i.e., \( \psi = \sum_{i=0}^j P_{U^i_{k-1}}^j \psi \). We also note that for \( j > 0 \), \( d_{j-1}^* \ldots d_{k-1}^* d_{k-1}^\ast \psi = 0 \), i.e., \( d_{k-1}^* d_{k-1}^\ast \psi \in \ker(d_{j-1}^* \ldots d_{k-1}^*) = (V_{k-1}^j)^\perp \). We will use these two facts to show that the projection of \( \psi \) on the subspace \( \bigcup_{i=0}^{j-1} U_{k-1}^i = V_{k-1}^j \) is small. If \( j = 0 \), this holds vacuously. Assume that \( j > 0 \). By the definition of \( U^i_{k-1} \),

\( \forall i < j, P_{V_{k-1}^j}^i P_{U_{k-1}^i}^j = P_{U_{k-1}^j}^i, P_{V_{k-1}^j}^i P_{U_{k-1}^j}^i = 0. \)

Also, \( d_{k-1}^* d_{k-1}^\ast \psi \in (V_{k-1}^j)^\perp \) implies that \( P_{V_{k-1}^j}^i d_{k-1}^* d_{k-1}^\ast \psi = 0 \). Using these equalities and the induction assumption yields

\begin{align*}
\left\| P_{V_{k-1}^j}^i \psi \right\|^2 &\leq \left\| P_{V_{k-1}^j}^i \left( \sum_{i=0}^j (k-i)P_{U_{k-1}^i} \right) \psi \right\|^2 \\
&= \left\| P_{V_{k-1}^j}^i \left( \sum_{i=0}^j (k-i)P_{U_{k-1}^i} \psi \right) - P_{V_{k-1}^j}^i d_{k-1}^* d_{k-1}^\ast \psi \right\|^2 \\
&= \left\| P_{V_{k-1}^j}^i \left( \sum_{i=0}^j (k-i)P_{U_{k-1}^i} \psi - d_{k-1}^* d_{k-1}^\ast P_{U_{k-1}^i} \psi \right) \right\|^2 \\
&\leq \left\| P_{V_{k-1}^j}^i \left( \sum_{i=0}^j (k-i)P_{U_{k-1}^i} \psi - d_{k-1}^* d_{k-1}^\ast P_{U_{k-1}^i} \psi \right) \right\|^2 \\
&\leq \sum_{i=0}^j \left\| (k-i)P_{U_{k-1}^i} \psi - d_{k-1}^* d_{k-1}^\ast P_{U_{k-1}^i} \psi \right\|^2.
\end{align*}
\[ \leq k \varepsilon^2 \sum_{i=0}^{j} \left\| P_{U_{k-1}^i} \psi \right\|^2 \]

\[ = k \varepsilon^2 \left\| \psi \right\|^2. \]

This shows that \( \left\| P_{V_{k-1}^j} \psi \right\| \leq \varepsilon_{k-1} \sqrt{k} \left\| \psi \right\|. \)

Recall that \( \psi \in V_{k-1}^j \) and therefore

\[ \left\| d_{k-1}^j d_{k-1}^j \psi - (k - j) \psi \right\| \]

\[ \leq \left\| d_{k-1}^j d_{k-1}^j P_{U_{k-1}^j} \psi - (k - j) P_{U_{k-1}^j} \psi \right\| \]

\[ + \left\| d_{k-1}^j d_{k-1}^j P_{V_{k-1}^j} \psi - (k - j) P_{V_{k-1}^j} \psi \right\| \]

\[ \leq \varepsilon_{k-1} \left\| P_{U_{k-1}^j} \psi \right\| + \left\| d_{k-1}^j d_{k-1}^j \left\| P_{V_{k-1}^j} \psi \right\| + (k - j) \left\| P_{V_{k-1}^j} \psi \right\| \]

\[ \leq \varepsilon_{k-1} (1 + k \sqrt{k} + (k - j) \sqrt{k}) \left\| \psi \right\| \leq \varepsilon_{k-1} (1 + 2k \sqrt{k}) \left\| \psi \right\|. \]

This implies

\[ \left\| \left\langle d_{k-1}^j d_{k-1}^j \psi, d_{k-1}^j d_{k-1}^j \psi' \right\rangle - (k - j) \left\langle d_{k-1}^j d_{k-1}^j \psi, d_{k-1}^j d_{k-1}^j \psi' \right\rangle \right\| \]

\[ \leq \varepsilon_{k-1} (1 + 2k \sqrt{k}) \left\| \psi \right\| \left\| \sqrt{d_{k-1}^j d_{k-1}^j} \right\| \left\| \sqrt{d_{k-1}^j d_{k-1}^j} \psi \right\| \]

\[ \leq \varepsilon_{k-1} (1 + 2k \sqrt{k}) \sqrt{k} \left\| \psi \right\| \left\| \sqrt{d_{k-1}^j d_{k-1}^j} \psi \right\|. \]

In order to complete the proof of (4), we need to show that

\[ \left\| \psi \right\| \leq 2 \sqrt{k} \left\| \sqrt{d_{k-1}^j d_{k-1}^j} \psi \right\|. \]

Recall that by our assumptions \( \varepsilon_{k-1} \leq \frac{1}{2(1+2k \sqrt{k})} \) and therefore, using inequality (5),

\[ 2 \sqrt{k} \left\| \sqrt{d_{k-1}^j d_{k-1}^j} \psi \right\| \geq 2 \left\| \sqrt{d_{k-1}^j d_{k-1}^j} \right\| \left\| \sqrt{d_{k-1}^j d_{k-1}^j} \psi \right\| \]

\[ \geq 2 \left\| d_{k-1}^j d_{k-1}^j \psi \right\| \]

\[ \geq 2 \left( (k - j) \left\| \psi \right\| - \left\| d_{k-1}^j d_{k-1}^j \psi - (k - j) \psi \right\| \right) \]

\[ \geq 2(k - j) - \varepsilon_{k-1} 2(1 + 2k \sqrt{k}) \left\| \psi \right\| \geq \left\| \psi \right\|, \]

as needed.
Corollary 5.8. Let $X$ be a two-sided $\lambda$-local spectral expander, and let $\{\varepsilon_k: 0 \leq k \leq n - 1\}$ be constants defined in Theorem 5.6. If
\[
\varepsilon_k \leq \frac{1}{2(1+2(k+1)\sqrt{k+1})}
\]
for all $0 \leq k \leq n - 2$, then for every $0 \leq k \leq n - 1$, every $0 \leq j \leq k$ and every $\phi \in C_k^0(X)$,
\[
\left\| M_k^+ P_{U^j_k} \phi - \left( \frac{k + 1 - j}{k + 2} \right) P_{U^j_k} \phi \right\| \leq \frac{\varepsilon_k}{k + 2} \| P_{U^j_k} \phi \|.
\]

Proof. By Corollary 3.7, $\frac{1}{k+2} d^*_k d_k = M_k^+$ and the inequality stated above follows.

The above Corollary allows us to determine the spectrum of $M_k^+$ given that the constants $\varepsilon_k, k = 0, \ldots, n - 1$ are small enough:

Theorem 5.9. Let $X$ be a two-sided $\lambda$-local spectral expander, and let $\{\varepsilon_k: 0 \leq k \leq n - 1\}$ be constants defined in Theorem 5.6. If
\[
\varepsilon_k \leq \frac{1}{2(1+2(k+1)\sqrt{k+1})}
\]
for all $0 \leq k \leq n - 2$ and $\varepsilon_{n-1} < \frac{1}{2\sqrt{n}}$, then
\[
\text{Spec}(M_k^+) \subseteq \{1\} \cup \bigcup_{j=0}^{k} \left[ \frac{k + 1 - j}{k + 2} - \frac{\sqrt{k+1}}{k + 2} \varepsilon_k, \frac{k + 1 - j}{k + 2} + \frac{\sqrt{k+1}}{k + 2} \varepsilon_k \right].
\]

Moreover, for $\phi \in C_k^0(X)$ such that $M_k^+ \phi = \mu \phi$, if
\[
\mu \in \left[ \frac{k + 1 - j}{k + 2} - \frac{\sqrt{k+1}}{k + 2} \varepsilon_k, \frac{k + 1 - j}{k + 2} + \frac{\sqrt{k+1}}{k + 2} \varepsilon_k \right],
\]
then
\[
\| \phi - P_{U^j_k} \phi \| \leq \frac{\sqrt{k+1} \varepsilon_k}{1 - \sqrt{k+1} \varepsilon_k} \| \phi \|.
\]

Proof. Let $\phi \in C_k^0(X)$ be an eigenvector of $M_k^+$ with eigenvalue $\mu$. Then by Corollary 5.8 and by the fact that $U^0_k \oplus \cdots \oplus U^k_k$ is an orthogonal decomposition
of $C^k_0(X)$, it holds that

\[
\sum_{j=0}^{k} \left| \mu - \frac{k + 1 - j}{k + 2} \right|^2 \left\| P_{U_k^j} \phi \right\|^2 \\
= \left\| \sum_{j=0}^{k} \left( \mu - \frac{k + 1 - j}{k + 2} \right) P_{U_k^j} \phi \right\|^2 \\
= \left\| \mu \phi - \sum_{j=0}^{k} \frac{k + 1 - j}{k + 2} P_{U_k^j} \phi \right\|^2 \\
= \left\| M_k^+ \phi - \sum_{j=0}^{k} \frac{k + 1 - j}{k + 2} P_{U_k^j} \phi \right\|^2 \\
= \left\| \sum_{j=0}^{k} M_k^+ P_{U_k^j} \phi - \frac{k + 1 - j}{k + 2} P_{U_k^j} \phi \right\|^2 \\
\leq (k + 1) \sum_{j=0}^{k} \left\| M_k^+ P_{U_k^j} \phi - \frac{k + 1 - j}{k + 2} P_{U_k^j} \phi \right\|^2 \\
\leq (k + 1) \sum_{j=0}^{k} \frac{\varepsilon_k^2}{(k + 2)^2} \left\| P_{U_k^j} \phi \right\|^2 \\
= (k + 1) \frac{\varepsilon_k^2}{(k + 2)^2} \left\| \phi \right\|^2.
\]

Therefore

\[
\sum_{j=0}^{k} \frac{\left\| P_{U_k^j} \phi \right\|^2}{\left\| \phi \right\|^2} \left| \mu - \frac{k + 1 - j}{k + 2} \right|^2 \leq (k + 1) \frac{\varepsilon_k^2}{(k + 2)^2}.
\]

Noting that

\[
\sum_{j=0}^{k} \frac{\left\| P_{U_k^j} \phi \right\|^2}{\left\| \phi \right\|^2} = 1,
\]

it follows that there is $j_0$ such that $\left| \mu - \frac{k + 1 - j_0}{k + 2} \right| \leq \sqrt{k + 1} \frac{\varepsilon_k}{k + 2}$. We also note that

\[
(k + 1) \frac{\varepsilon_k^2}{(k + 2)^2} \left\| \phi \right\|^2
\]
\[
\geq \sum_{j=0, j \neq j_0}^k \|P_{U_j^k} \phi\|^2 \left| \mu - \frac{k + 1 - j}{k + 2} \right|^2 \\
\geq \sum_{j=0, j \neq j_0}^k \|P_{U_j^k} \phi\|^2 \left( \left| \frac{k + 1 - j}{k + 2} - \frac{k + 1 - j_0}{k + 2} \right| - \left| \mu - \frac{k + 1 - j_0}{k + 2} \right| \right)^2 \\
\geq \sum_{j=0, j \neq j_0}^k \|P_{U_j^k} \phi\|^2 \left( \frac{1}{k + 2} - \sqrt{\frac{k + 1}{k + 2}} \varepsilon_k \right)^2 \\
= \sum_{j=0, j \neq j_0}^k \|P_{U_j^k} \phi\|^2 \left( 1 - \sqrt{\frac{k + 1}{k + 2}} \varepsilon_k \right)^2 \\
= \left( \frac{1 - \sqrt{k + 1} \varepsilon_k}{k + 2} \right)^2 \| (I - P_{U_j^k}) \phi \|^2.
\]

Therefore
\[
\| \phi - P_{U_j^0} \phi \| \leq \sqrt{k + 1} \varepsilon_k \| \phi \|. 
\]

Relying on the above Theorem, we denote \( W_j^k \) to be the subspaces of \( C_0^k(X) \) spanned by eigenvectors of \( M_k^+ \) with eigenvalues in the interval \([k + 1 - j \sqrt{k + 1} \varepsilon_k, \frac{k + 1 - j}{k + 2} + \sqrt{k + 1} \varepsilon_k]\), i.e.,
\[
W_j^k = \text{Span} \left\{ \varphi: M_k^+ \varphi = \mu \varphi, \mu \in \left[ \frac{k + 1 - j}{k + 2} - \sqrt{\frac{k + 1}{k + 2}} \varepsilon_k, \frac{k + 1 - j}{k + 2} + \sqrt{\frac{k + 1}{k + 2}} \varepsilon_k \right] \right\}.
\]

We note that if the constants \( \varepsilon_k \) are small enough, then these subspaces intersect trivially and \( W_0^k \oplus \ldots \oplus W_k^k \) is an orthogonal decomposition of \( C_0^k(X) \). Next, we show that for every \( j \), and every \( \phi \in C_0^k(X) \), the norm of projection of \( \phi \) on \( W_j^k \) can be approximate by the projection of \( \phi \) on \( U_j^k \).

**Theorem 5.10.** Let \( X \) be a two-sided \( \lambda \)-local spectral expander, and let \( \{ \varepsilon_k: 0 \leq k \leq n - 1 \} \) be the constants defined in Theorem 5.6. For every \( 0 \leq k \leq n - 1 \) and \( 0 \leq j_0 \leq k \), if \( \varphi \in U_j^k \), then
\[
\| \varphi - P_{W_j^k} \varphi \| \leq \sqrt{k + 2} \varepsilon_k \| \varphi \|.
\]

Moreover, for every \( \phi \in C_0^k(X) \),
\[
(1 - (k + 1) \sqrt{k + 2} \varepsilon_k) \| P_{U_j^k} \phi \| \leq \| P_{W_j^k} \phi \| \leq (1 + (k + 1) \sqrt{k + 2} \varepsilon_k) \| P_{U_j^k} \phi \|.
\]
Proof. Note that by our assumptions, $W_k^0, \ldots, W_k^k$ have trivial intersection and since $M_k^+$ is self-adjoint, these are orthogonal spaces, i.e., $W_k^0 \oplus \ldots \oplus W_k^k$ is an orthogonal decomposition of $C_k^0(X)$. Fix some $0 \leq j_0 \leq k$ and assume that $\varphi \in U_{j_0}^k$. Then as in (6) above,

$$\frac{1}{(k+2)^2} \left\| \varphi - P_{W_k^{j_0}} \varphi \right\|^2 \leq \sum_{j=0}^{k} \left| \frac{k+1-j_0}{k+2} - \frac{k+1-j}{k+2} \right|^2 \left\| P_{W_k^j} \varphi \right\|^2$$

$$= \left\| \frac{k+1-j_0}{k+2} \varphi - \sum_{j=0}^{k} \frac{k+1-j}{k+2} P_{W_k^j} \varphi \right\|^2 = \left\| \frac{k+1-j_0}{k+2} \varphi - M_k^+ \varphi + \sum_{j=0}^{k} M_k^+ P_{W_k^j} \varphi - \frac{k+1-j}{k+2} P_{W_k^j} \varphi \right\|^2$$

$$\leq (k+2) \left( \left\| \frac{k+1-j_0}{k+2} \varphi - M_k^+ \varphi \right\|^2 + \sum_{j=0}^{k} \left\| M_k^+ P_{W_k^j} \varphi - \frac{k+1-j}{k+2} P_{W_k^j} \varphi \right\|^2 \right) \leq (k+2) \left( \frac{\varepsilon_k^2}{(k+2)^2} \left\| \varphi \right\|^2 + \sum_{j=0}^{k} \frac{k+1}{(k+2)^2} \varepsilon_k^2 \left\| P_{W_k^j} \varphi \right\|^2 \right) = \frac{\varepsilon_k^2}{k+2} \left\| \varphi \right\|^2$$

and therefore

$$\left\| \varphi - P_{W_k^{j_0}} \varphi \right\| \leq \sqrt{k+2}\varepsilon_k \left\| \varphi \right\|,$$

as needed.

Next, let $\phi \in C_k^0(X)$. The inequality proven above shows that for every $j$,

$$\left\| P_{W_k^j} U_k^j - P_{U_k^j} \right\| = \left\| (I - P_{W_k^j}) U_k^j \right\| \leq \sqrt{k+2}\varepsilon_k.$$

Note that $(I - P_{W_k^j})$ is a projection on $(W_k^j)^{\perp}$ and therefore for every $j' \neq j$,

$$\left\| P_{W_k^{j'}} U_k^{j'} \right\| \leq \left\| (I - P_{W_k^j}) U_k^{j'} \right\| \leq \sqrt{k+2}\varepsilon_k.$$
It follows that
\[
\left\| P_{W_k^j} \phi \right\| = \left\| P_{W_k^j} \sum_{j'=0}^{k} P_{U_k^{j'}} \phi \right\|
\leq \left\| P_{W_k^j} \sum_{j'=0, j' \neq j}^{k} P_{U_k^{j'}} \phi \right\| + \left\| P_{W_k^j} P_{U_k^j} \phi - P_{U_k^j} \phi \right\| + \left\| P_{U_k^j} \phi \right\|
\leq (1 + (k+1)\sqrt{k+2\varepsilon_k}) \left\| P_{U_k^j} \phi \right\|.
\]

Thus,
\[
\left\| P_{W_k^j} \phi \right\| \leq (1 + (k+1)\sqrt{k+2\varepsilon_k}) \left\| P_{U_k^j} \phi \right\|,
\]
as needed. The proof of the inequality
\[
\left\| P_{W_k^j} \phi \right\| \geq (1 - (k+1)\sqrt{k+2\varepsilon_k}) \left\| P_{U_k^j} \phi \right\|
\]
is similar and left for the reader.

The above Theorem allows us to determine the rate of decay for the iterated random walk for a cochain \( \phi \) based on the size of its projection of the spaces \( U_k^0, \ldots, U_k^k \):

**Corollary 5.11.** Let \( X \) be a two-sided \( \lambda \)-local spectral expander, and let \( \{ \varepsilon_k : 0 \leq k \leq n-1 \} \) be constants defined in Theorem 5.6. If \( \varepsilon_k \leq \frac{1}{2(1+2(k+1)\sqrt{k+1})} \) for all \( 0 \leq k \leq n-2 \) and \( \varepsilon_{n-1} < \frac{1}{2\sqrt{n}} \), then for every \( 0 \leq k \leq n \), every \( i \in \mathbb{N} \) and every \( \phi \in C_0^k(X) \),

\[
\left\| (M_k^+)^i \phi \right\| \leq \sqrt{\sum_{j=0}^{k} \left( \frac{k+1-j}{k+2} + \frac{\sqrt{k+1}}{k+2} \varepsilon_k \right)^{2i} (1 + (k+1)\sqrt{k+2\varepsilon_k})^2 \left\| P_{U_k^j} \phi \right\|^{2}},
\]
and

\[
\left\| (M_k^+)^i \phi \right\| \geq \sqrt{\sum_{j=0}^{k} \left( \frac{k+1-j}{k+2} - \frac{\sqrt{k+1}}{k+2} \varepsilon_k \right)^{2i} (1 - (k+1)\sqrt{k+2\varepsilon_k})^2 \left\| P_{U_k^j} \phi \right\|^{2}}.
\]
Proof. The proofs of the two inequalities are similar and we will prove only the first one and leave the second one to the reader. Let $W^j_k$ be as above. We note that for every $\phi \in C^k_0(X)$, the orthogonal projections of $\phi$ on the subspaces $W^j_k$, $j = 0, \ldots, k$ yield an orthogonal decomposition of $\phi$. Also, by definition, $M^+_k P^j_k = P^j_k M^+_k P^j_k$ and $\|M^+_k P^j_k\| \leq \frac{k+1-j}{k+2} + \frac{\sqrt{k+1}}{k+2} \epsilon_k$. Combining these facts yields

$$\| (M^+_k)^i \phi \|^2 = \sum_{j=0}^k \left\| P^j_k (M^+_k)^i \phi \right\|^2$$

$$= \sum_{j=0}^k \left\| (P^j_k M^+_k)^i P^j_k \phi \right\|^2$$

$$\leq \sum_{j=0}^k \left( \frac{k+1-j}{k+2} + \frac{\sqrt{k+1}}{k+2} \epsilon_k \right)^2 \left\| P^j_k \phi \right\|^2.$$ 

By Theorem 5.10,

$$\left\| P^j_k \phi \right\| \leq (1 + (k+1) \sqrt{k+2} \epsilon_k) \left\| P^j_k U_k \phi \right\|,$$

and it follows that

$$\| (M^+_k)^i \phi \| \leq \sqrt{\sum_{j=0}^k \left( \frac{k+1-j}{k+2} + \frac{\sqrt{k+1}}{k+2} \epsilon_k \right)^2 (1 + (k+1) \sqrt{k+2} \epsilon_k)^2 \left\| P^j_k U_k \phi \right\|^2}. \quad \blacksquare$$

References

[1] N. Anari, K. Liu, Sh. O. Gharan and C. Vinzant: Log-concave polynomials II: High-dimensional walks and an FPRAS for counting bases of a matroid, arXiv:1811.01816, 2018.

[2] B. Barak, P. Gopalan, J. Håstad, R. Meka, P. Raghavendra and D. Steurer: Making the long code shorter, SIAM J. Comput. 44 (2015), 1287–1324.

[3] W. Ballmann and J. Świątkowski: On $L^2$-cohomology and property (T) for automorphism groups of polyhedral cell complexes, Geom. Funct. Anal. 7 (1997), 615–645.

[4] Y. Dikstein, I. Dinur, Y. Filmus and P. Harsha: Boolean functions on high-dimensional expanders, arXiv:1804.08155, 2018.

[5] I. Dinur and T. Kaufman: High dimensional expanders imply agreement expanders, Electronic Colloquium on Computational Complexity (ECCC), 2017.

[6] I. Dinur, S. Khot, G. Kindler, D. Minzer and M. Safra: Towards a proof of the 2-to-1 games conjecture?, Electronic Colloquium on Computational Complexity (ECCC) 23 (2016), 198.
A. Proof of Proposition 4.1

Proof. Let $\phi, \psi \in C^l(X, \mathbb{R})$, then

$$
\sum_{\tau \in X(k)} \langle \phi_{\tau}, \psi_{\tau} \rangle = \sum_{\tau \in X(k)} \sum_{\eta \in X^{(l-k-1)}_{\tau}} m(\tau \cup \eta) \phi_{\tau \cup \eta}(\eta) \psi_{\tau \cup \eta}(\eta)
= \sum_{\tau \in X(k)} \sum_{\eta \in X^{(l-k-1)}_{\tau}} m(\tau \cup \eta) \phi_{\tau \cup \eta}(\eta) \psi_{\tau \cup \eta}(\eta)
= \sum_{\tau \in X(k)} \sum_{\sigma \in X(l), \tau \subset \sigma} m(\tau) \phi_{\tau}(\sigma) \psi_{\tau}(\sigma)
= \sum_{\tau \in X(k)} \sum_{\sigma \in X(l), \tau \subset \sigma} m(\tau) \phi_{\tau}(\sigma) \psi_{\tau}(\sigma)
= (l+1) \sum_{\sigma \in X(l)} m(\sigma) \phi_{\tau}(\sigma) \psi_{\tau}(\sigma)
= \left(\frac{l+1}{k+1}\right) \|\phi\|^2.
$$

In order to prove the second equality, we notice that for every $\tau \in X(k)$ and every $\eta \in X^{(l-k-2)}_{\tau}$, we have that

$$(d^* \phi)_{\tau}(\eta) = d^* \phi_{\tau \cup \eta} = \sum_{\sigma \in X(l), \tau \cup \eta \subset \sigma} \frac{m(\sigma)}{m(\tau \cup \eta)} \phi_{\tau}(\sigma)
= \sum_{\sigma \setminus \tau \in X(l-k-1), \eta \subset \sigma \setminus \tau} \frac{m_{\tau}(\sigma \setminus \tau)}{m_{\tau}(\eta)} \phi_{\tau}(\sigma \setminus \tau)
= d^*_{\tau} \phi_{\tau}(\eta).$$
Therefore, \((d^*\phi)_\tau = d^*_\tau \phi_\tau\) and similarly, \((d^*\psi)_\tau = d^*_\tau \psi_\tau\). By the equality proven above

\[
\binom{l}{k+1} \langle d^*\phi, d^*\psi \rangle = \sum_{\tau \in X(k)} \langle (d^*\phi)_\tau, (d^*\psi)_\tau \rangle = \sum_{\tau \in X(k)} \langle d^*_\tau \phi_\tau, d^*_\tau \psi_\tau \rangle.
\]

Assume now that \(l < n\), then for every \(\sigma \in X(l+1)\), the following holds:

\[
(d\phi(\sigma))(d\psi(\sigma))
\]

\[
= \left( \sum_{\eta \in X(l), \eta \subset \sigma} \phi(\eta) \right) \left( \sum_{\eta \in X(l), \eta \subset \sigma} \psi(\eta) \right)
\]

\[
= \sum_{\eta \in X(l), \eta \subset \sigma} \phi(\eta)\psi(\eta) + \sum_{\eta, \eta' \in X(l), \eta \neq \eta', \eta, \eta' \subset \sigma} (\phi(\eta)\psi(\eta')) + (\phi(\eta')\psi(\eta))
\]

\[
= \sum_{\eta, \eta' \in X(l), \eta \neq \eta', \eta, \eta' \subset \sigma} (\phi(\eta) + \phi(\eta'))(\psi(\eta) + \psi(\eta')) - l \sum_{\eta \in X(l), \eta \subset \sigma} \phi(\eta)\psi(\eta).
\]

Therefore

\[
\langle d\phi, d\psi \rangle = \sum_{\sigma \in X(l+1)} m(\sigma)(d\phi(\sigma))(d\psi(\sigma))
\]

\[
= \sum_{\sigma \in X(l+1)} m(\sigma) \sum_{\tau \in X(l+1), \tau \subset \sigma} (d^*_\tau \phi_\tau(\sigma \setminus \tau))(d^*_\tau \psi_\tau(\sigma \setminus \tau))
\]

\[
- l \sum_{\sigma \in X(l+1)} m(\sigma) \sum_{\eta \in X(l), \eta \subset \sigma} \phi(\eta)\psi(\eta).
\]

We note that

\[
\sum_{\sigma \in X(l+1)} m(\sigma) \sum_{\tau \in X(l+1), \tau \subset \sigma} (d^*_\tau \phi_\tau(\sigma \setminus \tau))(d^*_\tau \psi_\tau(\sigma \setminus \tau))
\]

\[
= \sum_{\tau \in X(l+1)} \sum_{\sigma \in X(l+1), \tau \subset \sigma} m(\sigma)(d^*_\tau \phi_\tau(\sigma \setminus \tau))(d^*_\tau \psi_\tau(\sigma \setminus \tau))
\]

\[
= \sum_{\tau \in X(l+1), \gamma \in X(1)\gamma} m(\gamma)(d^*_\tau \phi_\tau(\gamma))(d^*_\tau \psi_\tau(\gamma))
\]

\[
= \sum_{\tau \in X(l+1)} \langle d^*_\tau \phi_\tau, d^*_\tau \psi_\tau \rangle.
\]
and also

\[ l \sum_{\sigma \in X(l+1)} m(\sigma) \sum_{\eta \in X(l), \eta \subset \sigma} \phi(\eta)\psi(\eta) \]

\[ = l \sum_{\eta \in X(l)} \phi(\eta)\psi(\eta) \sum_{\sigma \in X(l+1), \eta \subset \sigma} m(\sigma) \]

\[ = l \sum_{\eta \in X(l)} m(\eta)\phi(\eta)\psi(\eta) \]

\[ = l \langle \phi, \psi \rangle = \sum_{\tau \in X(l-1)} \frac{1}{l+1} \langle \phi_\tau, \psi_\tau \rangle, \]

where the last equality is due to the equality

\[ (l + 1)\langle \phi, \psi \rangle = \sum_{\tau \in X(l-1)} \langle \phi_\tau, \psi_\tau \rangle, \]

proven above.

Tali Kaufman  
Department of Computer Science  
Bar-Ilan University  
kaufmant@mit.edu

Izhar Oppenheim  
Department of Mathematics  
Ben-Gurion University of the Negev  
Be’er Sheva 84105, Israel  
izharo@bgu.ac.il