LOCAL NEWFORMS FOR GENERIC REPRESENTATIONS OF UNRAMIFIED ODD UNITARY GROUPS AND FUNDAMENTAL LEMMA

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Abstract. In this paper, we establish the theory of local newforms for irreducible tempered generic representations of unramified odd unitary groups over a non-archimedean local field. For the proof, we prove an analogue of the fundamental lemma for our compact groups.

1. Introduction

For a connection between classical modular forms and automorphic representations, it is a significant problem in representation theory of $p$-adic groups to establish the theory of local newforms. It was initiated by Casselman [6] for $GL_2$, and generalized by Jacquet–Piatetski-Shapiro–Shalika [18] to the case for $GL_N$ (see also [17]). For other groups, there are some progresses:

- Roberts–Schmidt [39, 40] established the theory of local newforms for $PGSp_4$ and for $\tilde{SL}_2$, which is the double cover of $SL_2$.
- Okazaki [36] generalized results in [39] to $GSp_4$.
- Unramified $U(1,1)$ and $U(2,1)$ were treated by Lansky–Raghuram [25] and Miyauchi [28, 29, 30, 31], respectively.
- Tsai [42] proved Gross’ conjecture [12] on the local newforms for $SO_{2n+1}$ for generic supercuspidal representations.
- Recently, the first and third authors together with Kondo [2] extended results in [18] to all irreducible representations of $GL_N$.

In this paper, we treat unramified unitary groups $U_{2n+1}$ of odd variables. Let us describe our results more precisely. Let $E/F$ be an unramified quadratic extension of non-archimedean local fields of characteristic 0 and of residue characteristic $p > 2$. Denote by $\mathfrak{o}_E$ and $\mathfrak{p}_E$ the ring of integers of $E$ and its maximal ideal, respectively. We consider a quasi-split unitary group

$$H = U_{2n+1} = \{ g \in GL_{2n+1}(E) \mid gJ_{2n+1}^Tg = J_{2n+1} \}$$

with

$$J_N = \begin{pmatrix}
1 & & & \\
& -1 & & \\
& & \ddots & \\
& & & (-1)^{N-1}
\end{pmatrix}.$$
Set $K_{0,H} = GL_{2n+1}(\mathbb{O}_E) \cap H$. For a positive integer $m$, we define

$$K_{m,H} = 1 \left( \begin{array}{ccc} n & 1 & n \\ \mathbb{O}_E & \mathbb{O}_E & p_E^{-m} \\ p_E^n & 1 + p_E^n & \mathbb{O}_E \end{array} \right) \cap H.$$ 

The following is the first main result.

**Theorem 1.1.** Let $\pi$ be an irreducible tempered representation of $H$.

1. If $\pi$ is not generic, then $\pi^{K_{m,H}} = 0$ for all $m \geq 0$.
2. Suppose that $\pi$ is generic. We denote by $c(\phi)$ the conductor of the $L$-parameter $\phi$ of $\pi$ (see Section 4.2). Then

$$\dim(\pi^{K_{m,H}}) = \begin{cases} 0 & \text{if } m < c(\phi), \\ 1 & \text{if } m = c(\phi). \end{cases}$$

Moreover, if $m > c(\phi)$, then $\pi^{K_{m,H}} \neq 0$. We call a nonzero element in $\pi^{K_{c(\phi),H}}$ a local newform of $\pi$.

Theorem 1.1 (1) is an application of the local Gan–Gross–Prasad (GGP) conjecture ([9, Conjecture 17.3]) established by Beuzart-Plessis [3, 4, 5]. More precisely, if an irreducible tempered representation $\pi$ of $H$ satisfies $\pi^{K_{m,H}} \neq 0$ for some $m \geq 0$, then the argument of a lemma of Gan–Savin ([10, Lemma 12.5]) implies that there exists an irreducible unramified tempered representation $\pi'$ of $H' \cong U_{2n}$, which we regard as a subgroup of $H$, such that $\text{Hom}_{H'}(\pi \otimes \pi', \mathbb{C}) \neq 0$. In this situation, the local GGP conjecture tells us that $\pi$ is generic. For a detail, see Theorem 4.4 below.

In all of the previous works listed above, the proofs of analogues of Theorem 1.1 (2) used Rankin–Selberg integrals. However, we do not consider them in this paper. Instead of Rankin–Selberg integrals, we prove an analogue of the fundamental lemma for our compact groups.

The fundamental lemma is a core in the theory of endoscopy. Let $\tilde{G} = G \rtimes \theta$ be a twisted space with an involution $\theta$ on a reductive group $G$, and let $H$ be an endoscopic group of $\tilde{G}$. Suppose that both $G$ and $H$ are unramified. In this setting, the fundamental lemma asserts that the characteristic function $1_{K_H}$ is a transfer of $1_{\tilde{K}}$, where $\mathbb{K} \subset G$ and $\mathbb{K}_H \subset H$ are hyperspecial maximal compact subgroups, and we set $\mathbb{K} = \mathbb{K} \rtimes \theta \subset \tilde{G}$. For the notion of transfer, see Definition 2.4 below. The fundamental lemma was established by Ngô [33] after Waldspurger’s consecutive works [44, 45]. For some special cases, e.g., in Laumon– Ngô [26] and Kottwitz [22], the fundamental lemma was earlier proven.

The transfer conjecture, which is a consequence of the fundamental lemma ([43, 45]), says that for any $f \in C^\infty_c(\tilde{G})$, there exists $f^H \in C^\infty_c(H)$ such that $f^H$ is a transfer of $f$. However, its proof is not constructive. It is of independent interest to find explicit pairs $(f, f^H)$ such that $f^H$ is a transfer of $f$. Beyond the characteristic functions of hyperspecial maximal compact subgroups, a few cases are known. For example:
• Clozel [7] and Labesse [24] (resp. Haines [14]) established the base change fundamental lemma for spherical Hecke algebras (resp. for the central elements in parahoric Hecke algebras).

• Hales [15] and Lemaire–Mœglin–Waldspurger [27] extended the (usual) fundamental lemma to all elements in the spherical Hecke algebras.

• Kazhdan–Varshavsky [19, Theorem 2.2.6] proved a formula for the transfer of Deligne–Lusztig functions when $\theta = \text{id}$. It is an answer to a conjecture of Kottwitz. Also, it implies that the fundamental lemma for Iwahori subgroups ([19, Corollary 2.3.13]).

• The fundamental lemma for principal congruence subgroups was shown by Ferrari [8, Théorème 3.2.3] when $\theta = \text{id}$ (resp. by Ganapathy–Varma [11, Lemma 8.2.2] when $G = \text{GL}_N$ and $\theta \neq \text{id}$).

• The second author also showed the fundamental lemma for the Moy–Prasad filtrations of unitary groups ([35, Theorem 1.4]).

We remark that in these papers, some assumption on the residue characteristic $p$ might be imposed.

Now we state the second main result. Let $\mathbb{K}_0 = \text{GL}_{2n+1}(\mathfrak{o}_E) \subset G = \text{GL}_{2n+1}(E)$. For a positive integer $m > 0$, we define $\mathbb{K}_m$ by the compact subgroup of $G$ consisting of matrices $k$ of the form

$$
\begin{pmatrix}
1 & n & m \\
n & \mathfrak{o}_E & \mathfrak{o}_E \\
n & \mathfrak{o}_E & \mathfrak{o}_E \\
1 & \mathfrak{p}_E^m & 1 + \mathfrak{p}_E^m \\
1 & \mathfrak{p}_E^m & \mathfrak{p}_E^m \\
1 & \mathfrak{p}_E^m & \mathfrak{o}_E \\
1 & \mathfrak{p}_E^m & \mathfrak{o}_E
\end{pmatrix}
$$

with $\det(k) \in \mathfrak{o}_E^*$. Hence $\mathbb{K}_{m,H} = \mathbb{K}_m \cap H$. Set $\tilde{\mathbb{K}}_m = \mathbb{K}_m \times \theta \subset \tilde{G} = G \rtimes \theta$, where $\theta$ is an involution on $G$ such that $H$ is the centralizer of $\theta$ in $G$.

**Theorem 1.2** (Theorem 2.5). For $m > 0$, the function $\text{vol}(\mathbb{K}_{m,H}; dh)^{-1}\mathbf{1}_{\mathbb{K}_{m,H}} \in C_c^\infty(H)$ is a transfer of $\text{vol}(\mathbb{K}_m; dg)^{-1}\mathbf{1}_{\tilde{\mathbb{K}}_m} \in C_c^\infty(\tilde{G})$.

By this theorem, one can transfer a result in [18] on the local newforms for $G = \text{GL}_{2n+1}(E)$ to the one for $H = \text{U}_{2n+1}$ via the local Langlands correspondence established by Mok [32]. More precisely, one can prove the multiplicity one result in tempered $L$-packets (Theorem 4.3). Combining it with Theorem 1.1 (1), we obtain Theorem 1.1 (2).

An advantage for using Theorem 1.2 is that we do not need huge computations of elements in Hecke algebras as in [39, Section 6] and [42, Chapter 8]. In fact, after reducing Theorem 1.2 to comparing topologically semisimple conjugacy classes (Lemma 2.7), we only need a result of Kottwitz [21, Proposition 7.1] and its twisted analogue (Proposition 3.2) together with easy linear algebraic calculations (Proposition 3.5). On the other hand, the relation between Rankin–Selberg integrals and newforms is not clear in this paper.

This paper is organized as follows. In Section 2, we review the theory of endoscopy. Especially, we state our second main result as Theorem 2.5, and reduce it to comparing certain conjugacy classes in Lemma 2.7. This comparison is investigated in Section 3. Finally, we prove Theorem 1.1 in Section 4. In Appendix A, we generalize Hilbert’s Theorem 90 by using the faithfully flat descent.
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Notation. Let $E/F$ be an unramified quadratic extension of non-archimedean local fields of characteristic 0 and of residue characteristic $p > 2$. The non-trivial element in $\text{Gal}(E/F)$ is denoted by $x \mapsto \overline{x}$. Set $\mathfrak{o}_E$ (resp. $\mathfrak{o}_F$) to be the ring of integers of $E$ (resp. $F$), and $\mathfrak{p}_E$ (resp. $\mathfrak{p}_F$) to be its maximal ideal. Fix a uniformizer $\varpi$ of $F$, which is also a uniformizer of $E$. Write $q = |\mathfrak{o}_F/\mathfrak{p}_F|$ so that $q^2 = |\mathfrak{o}_E/\mathfrak{p}_E|$.

The identity matrix of size $N$ is denoted by $I_N$. Set $J_N = \begin{pmatrix} 1 & & & & \\ & -1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & (-1)^{N-1} & & & \\ & & & & \end{pmatrix}$. When $N = 2n$, we also put $J'_{2n} = \begin{pmatrix} 0 & J_n & \\ t & J_n & 0 \\ \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 1 \\ \vdots \\ (-1)^{n-1} \\ 1 \\ \end{pmatrix} & \begin{pmatrix} 1 \\ \vdots \\ (-1)^{n-1} \\ 1 \\ \end{pmatrix} \\ \begin{pmatrix} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \end{pmatrix} & \begin{pmatrix} 1 \\ \vdots \\ (-1)^{n-1} \\ 1 \\ \end{pmatrix} \end{pmatrix}$.

For an algebraic variety $X$ over $F$, we denote the set of its $F$-valued points by $X = X(F)$.

2. Matching of orbital integrals and descent to centralizers

In this section, we review several notions concerning transfers, and we reduce our problem to a correspondence of certain semisimple conjugacy classes.

2.1. Groups. Let $G = \text{Res}_{E/F}(\text{GL}_N,E)$ be the Weil restriction of $\text{GL}_N$ over $E$. In particular, $G = G(F) = \text{GL}_N(E)$. Define an involution $\theta$ on $G$ by

$$\theta(x) = J_N \overline{x} J_N^{-1}.$$

Let $\tilde{G} = G \rtimes \theta$ be the non-neutral component of the group $G \rtimes \langle \theta \rangle$. Then $\tilde{G}$ is a twisted space over $G$. Let $H = G_{\theta} = \{ h \in G \mid \theta(h) = h \}$ be a unitary group. It is an endoscopic group for $\tilde{G}$.

For $\gamma \in H$, write $H_{\gamma}$ for its centralizer. Similarly, for $\tilde{\delta} = \delta \rtimes \theta \in \tilde{G}$, one can consider its conjugate

$$g \tilde{\delta} g^{-1} = g(\delta \rtimes \theta) g^{-1} = (g \delta \theta(g)^{-1}) \rtimes \theta$$

for $g \in G$. Set

$$G_{\tilde{\delta}} = \{ g \in G \mid g \tilde{\delta} g^{-1} = \tilde{\delta} \}$$

to be the centralizer of $\tilde{\delta}$. We sometimes write $N(\delta) = \delta \theta(\delta) = \tilde{\delta}^2$. Note that $G_{\tilde{\delta}} \subset G_{N(\delta)}$. 

Assume until the end of this subsection that \( N = 2n+1 \) is odd. Let \( G' = \text{Res}_{E/F}(\text{GL}_{2n,E}) \) with an inclusion \[
\iota: G' \hookrightarrow G, \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} A & 0 \\ 0 & B \\ C & 0 \\ D \end{pmatrix}
\] for matrices \( A, B, C, D \) of size \( n \times n \). Note that if we set \( H' = \iota^{-1}(H) \), then \( H' = G'_{\theta'} \) with \[
\theta'(x) = J_{2n}^{1} \iota^{-1} J_{2n}^{-1}.
\]
If we fix \( \epsilon \in o_E^\times \) with \( \tau = -\epsilon \), and if we set \( \kappa = \text{diag}(\epsilon, \ldots, \epsilon, 1, \ldots, 1) \in G' \), then \[
\epsilon J_{2n}^{1} = \kappa J_{2n}^{1} \kappa^{-1}
\] so that \( \theta'(x) = \kappa \cdot \theta(\kappa^{-1} x \kappa) \cdot \kappa^{-1} \). In particular, we have \( G'_{\theta'} = \kappa G'_{\theta'} \kappa^{-1} \).

Set \( \mathbb{K}_0 = \text{GL}_{2n+1}(o_E) \), which is a hyperspecial maximal compact subgroup of \( G \). For a fixed positive integer \( m > 0 \), we define \( \mathbb{K}_m \) by the compact open subgroup of \( G \) consisting of matrices \( k \) of the form \[
\begin{pmatrix}
\mathbb{O}_E & o_E & p_E^{-m} \\
1 & p_E m & o_E \\
n & p_E m & o_E
\end{pmatrix}
\]
with \( \det(k) \in o_E^\times \). Note that \( \theta(\mathbb{K}_m) = \mathbb{K}_m \). Moreover, \( \iota^{-1}(\mathbb{K}_m) \) (resp. \( \iota^{-1}(\mathbb{K}_m) \cap G'_{\theta'} \)) is a hyperspecial maximal compact subgroup of \( G' \) (resp. \( G'_{\theta'} \)). Finally, we define \( \mathbb{K}_{m,H} = \mathbb{K}_m \cap H \) for \( m \geq 0 \), which is a compact open subgroup of \( H \).

### 2.2. Norm correspondence

For a subgroup \( J \subset G(F) \) (resp. \( J_H \subset H(F) \)), we say that elements \( \delta, \delta' \in G(F) \) (resp. \( \gamma, \gamma' \in H(F) \)) are \( J \)-conjugate (resp. \( J_H \)-conjugate) if there exists \( g \in J \) (resp. \( h \in J_H \)) such that \( \delta' = g \delta g^{-1} \) (resp. \( \gamma' = h \gamma h^{-1} \)).

**Remark 2.1.** For a general connected reductive group \( H \) over \( F \), the \( H(F) \)-conjugacy might be different from the so-called stably conjugacy, which is a bit more complicated. Although they are the same in our setting, we do not use the terminology of the stably conjugacy in this paper.

Let \( Cl_{ss}(G) \) (resp. \( Cl_{ss}(H) \)) be the set of semisimple \( G(F) \)-conjugacy classes in \( G(F) \) (resp. \( H(F) \)-conjugacy classes in \( H(F) \)). Here, see [23, Section 3.2] for the definition of the semisimplicity for elements of \( G \). As in [23, Theorem 3.3A], one can define a map \( \mathcal{A}: Cl_{ss}(H) \to Cl_{ss}(G) \) as follows. Let \( [\gamma] \in Cl_{ss}(H) \) and \( [\tilde{\delta}] \in Cl_{ss}(G) \). As representatives, we may take diagonal elements \( \gamma = \text{diag}(\zeta_1, \ldots, \zeta_N) \) and \( \tilde{\delta} = \text{diag}(x_1, \ldots, x_N) \times \theta \) with \( \zeta_1, \ldots, \zeta_N, x_1, \ldots, x_N \in E \otimes_F \mathbb{F} \). Then \( \mathcal{A}([\gamma]) = [\tilde{\delta}] \) if and only if \( \{\zeta_1, \ldots, \zeta_N\} = \{x_1/\mathbb{F} N, \ldots, x_N/\mathbb{F} N\} \) as multi-sets. Note that \( \mathcal{A} \) is bijective in our setting (see [34, Section 4.2]). For semisimple elements \( \gamma \in H \) and \( \tilde{\delta} \in G \), we call \( \gamma \) is a norm of \( \tilde{\delta} \) if \( \mathcal{A}([\gamma]) = [\tilde{\delta}] \). In other words, \( \gamma \) is a norm of \( \tilde{\delta} = \delta \times \theta \) if and only if \( \gamma \) is \( G(F) \)-conjugate to \( N(\delta) \).
Definition 2.2. (1) An element $\tilde{\delta} = \delta \times \theta \in \tilde{G}$ is strongly regular semisimple if its centralizer $G_{\tilde{\delta}}$ is abelian. Write $G_{\text{srs}}$ for the set of strongly regular semisimple elements in $G$.

(2) An element $\gamma \in H$ is strongly regular semisimple if its centralizer $H_{\gamma}$, is a maximal torus of $H$. Write $H_{\text{srs}}$ for the set of strongly regular semisimple elements in $H$.

(3) An element $\gamma \in H_{\text{srs}}$ is strongly $G$-regular semisimple if there is $\tilde{\delta} \in G_{\text{srs}}$ such that $\gamma$ is a norm of $\tilde{\delta}$.

The rationality of semisimple conjugacy classes via the norm correspondence is a delicate problem in general. However, in our setting, we can prove the following.

Proposition 2.3. (1) For every semisimple element $\tilde{\delta} \in \tilde{G}$, there exists a semisimple element $\gamma \in H$ such that $\gamma$ is a norm of $\tilde{\delta}$. Moreover, $G_{\tilde{\delta}}(F) \cong H_{\gamma}(F)$.

(2) For every semisimple element $\gamma \in H$, there exists a semisimple element $\tilde{\delta} \in \tilde{G}$ such that

- $\gamma$ is a norm of $\tilde{\delta}$; and
- $G_{\tilde{\delta}} = H_{\gamma}$ as algebraic subgroups of $G$.

In particular, every $\gamma \in H_{\text{srs}}$ is strongly $G$-regular semisimple.

Proof. We prove (1). Let $\tilde{\delta} = \delta \times \theta \in \tilde{G}$ be a semisimple element. We regard $N(\delta) = \delta \theta(\delta)$ as an element in $H(E)$. Since $\theta(N(\delta)) = \delta^{-1}N(\delta)\delta$, the $H(F)$-conjugacy class of $N(\delta)$ in $H(F)$ is $\text{Gal}(F/F)$-invariant. Since $H$ is quasi-split and its derived group is simply connected, by [20, Theorem 4.1], there exist $\gamma \in H = H(F)$ and $h \in H(F) = GL_N(F)$ such that $\gamma = hN(\delta)h^{-1}$. Especially, $\gamma$ is semisimple and is a norm of $\tilde{\delta}$. Since $E \otimes F \cong \overline{E} \times \overline{F}$ with $\text{Gal}(E/F)$ acting on the right hand side by switching the two factors, we have $G(F) \cong GL_N(F) \times GL_N(F)$ with $\theta$ acting on the right hand side by $\theta(g_1, g_2) = (J_N^t g_2^{-1} J_N^{-1}, J_N^t g_1^{-1} J_N^{-1})$. Hence

$$G_{\tilde{\delta}}(F) \cong \{ (g_1, g_2) \in GL_N(F) \times GL_N(F) \mid \delta J_N^t g_2^{-1} J_N^{-1} = g_1 \delta, \delta J_N^t g_1^{-1} J_N^{-1} = g_2 \tilde{\delta} \}$$

$$\cong \{ (g_1, g_2) \in GL_N(F) \times GL_N(F) \mid g_2 = \delta J_N^t g_1^{-1} J_N^{-1} \tilde{\delta}^{-1}, N(\delta)g_1N(\delta)^{-1} = g_1 \}$$

$$\cong \{ g_1 \in GL_N(F) \mid N(\delta)g_1N(\delta)^{-1} = g_1 \}$$

$$\cong \{ h_1 \in GL_N(F) \mid \gamma h_1 \gamma^{-1} = h_1 \} = H_{\gamma}(F).$$

Next, we show (2). Let $\gamma \in H$ be a semisimple element. Since $[E^\times : F^\times] = \infty$, there exists $\alpha \in E^\times$ such that $\delta = \alpha I_N + \overline{\alpha} \gamma$ is an invertible matrix so that $\delta \in G$. Since $\gamma$ is semisimple in $H$, we see that $\tilde{\delta} = \delta \times \theta$ is semisimple in $G$. We claim that $\tilde{\delta}$ satisfies the desired conditions.

Remark that the map $x \mapsto \theta(x)^{-1} = J_N^t \overline{\gamma} J_N^{-1}$ can be extended to an anti-$E$-linear endomorphism of $M_N(E)$. In particular, $\theta(\delta)^{-1} = \overline{\alpha} I_N + \alpha \gamma^{-1}$ since $\theta(\gamma) = \gamma$. Hence $\gamma \theta(\delta)^{-1} = \delta$ so that $\gamma = \delta \theta(\delta)$. This means that $\gamma$ is a norm of $\tilde{\delta}$.

Note that $\tilde{\delta}^2 = \delta \theta(\delta) = \gamma$. Therefore, both $G_{\tilde{\delta}}$ and $H_{\gamma}$ are subgroups of the centralizer $G_{\gamma}$ of $\gamma$ in $G$. Moreover, for $g \in G_{\gamma}$, since $g \delta = \delta g$ by the construction of $\delta$, we have

$$g \in G_{\tilde{\delta}} \iff g(\delta \times \theta) = (\delta \times \theta) g$$

$$\iff g \delta \times \theta = \theta(g) \times \theta$$

$$\iff g = \theta(g) \iff g \in H_{\gamma}.$$

Hence we conclude that $G_{\tilde{\delta}} = H_{\gamma}$. \qed
2.3. Matching of orbital integrals. For \( \tilde{\delta} \in \tilde{G}^{\text{hrs}} \) and \( f \in C_c^\infty(\tilde{G}) \), we define the orbital integral of \( f \) at \( \tilde{\delta} \) by

\[
O_{\tilde{\delta}}(f) = \int_{\tilde{G}/G_{\tilde{\delta}}} f(g\tilde{\delta}g^{-1})dg.
\]

Similarly, for \( \gamma \in H^{\text{hrs}} \) and \( f^H \in C_c^\infty(H) \), the orbital integral of \( f^H \) at \( \gamma \) is defined by

\[
O_{\gamma}(f^H) = \int_{H/H_\gamma} f^H(h\gamma h^{-1})dh.
\]

Here, \( dg \) (resp. \( dh \)) is a left \( G \)-invariant (resp. \( H \)-invariant) measure on \( G/G_{\tilde{\delta}} \) (resp. on \( H/H_\gamma \)) induced by Haar measures on \( G \) and \( G_{\tilde{\delta}} \) (resp. \( H \) and \( H_\gamma \)). Finally, we define the stable orbital integral of \( f^H \) at \( \gamma \) by

\[
SO_{\gamma}(f^H) = \sum_{\gamma' \sim \gamma/\sim} O_{\gamma'}(f^H),
\]

where the sum is over the set of \( H \)-conjugacy classes within the \( (\overline{F}) \)-conjugacy class of \( \gamma \).

As explained in [34, Section 4.1], \( \mathbf{H} \) is an endoscopic group for \( \tilde{G} \) (with respect to the standard base change \( L \)-embedding from \( L\mathbf{H} \) to \( L\tilde{G} \)). Fix a \( \theta \)-stable \( F \)-splitting of \( G \). Let \( \Delta(\gamma, \tilde{\delta}) \) be the Kottwitz–Shelstad transfer factor with respect to \( (G, H) \) (see [23, Sections 4, 5]). By [34, Proposition 4.3], we know that \( \Delta(\gamma, \tilde{\delta}) = 1 \) for \( \gamma \in H^{\text{hrs}} \) and \( \tilde{\delta} \in \tilde{G}^{\text{hrs}} \) such that \( \gamma \) is a norm of \( \tilde{\delta} \).

**Definition 2.4.** We say that \( f \in C_c^\infty(\tilde{G}) \) and \( f^H \in C_c^\infty(H) \) have matching orbital integrals if, for every strongly \( G \)-regular semisimple element \( \gamma \in H^{\text{hrs}} \), the identity

\[
SO_{\gamma}(f^H) = \sum_{\tilde{\delta} \sim \gamma/\sim} \Delta(\gamma, \tilde{\delta})O_{\tilde{\delta}}(f) = \sum_{\tilde{\delta} \sim \gamma/\sim} O_{\tilde{\delta}}(f)
\]

holds, where the sum is taken over the set of \( G \)-conjugacy classes of \( \tilde{\delta} \in \tilde{G}^{\text{hrs}} \) such that \( \gamma \) is a norm of \( \tilde{\delta} \). Here, we choose measures as in the manner of [23, Section 5.5]. In this situation, we say that \( f^H \) is a transfer of \( f \).

When \( N \) is odd, we have defined compact open subgroups \( \mathbb{K}_m \subset G \) and \( \mathbb{K}_{m,H} \subset H \) in Section 2.1. In this and next sections, we will prove the following.

**Theorem 2.5.** Suppose that \( N \) is odd. For \( m > 0 \), the function \( \text{vol}(\mathbb{K}_{m,H}; dh)^{-1}\mathbf{1}_{\mathbb{K}_{m,H}} \in C_c^\infty(H) \) is a transfer of \( \text{vol}(\mathbb{K}_m; dg)^{-1}\mathbf{1}_{\mathbb{K}_m} \in C_c^\infty(\tilde{G}) \).

2.4. Descent. To compare orbital integrals, we introduce the following notions.

**Definition 2.6.**

1. An element \( u \in G \) (resp. \( v \in H \)) is called topologically unipotent if \( \lim_{n \to \infty} u^{p^n} = 1 \) (resp. \( \lim_{n \to \infty} v^{p^n} = 1 \)).

2. An element \( s \times \theta \in G \) (resp. \( t \in H \)) is called topologically semisimple if \( s \times \theta \) (resp. \( t \)) is of finite order prime to \( p \). Note that \( s \times \theta \in G \) is topologically semisimple if and only if \( N(s) = s\theta(s) \in G \) is topologically semisimple, i.e., \( N(s) \) is of finite order prime to \( p \).

3. A topological Jordan decomposition (TJD) of \( \tilde{\delta} = \delta \times \theta \in \tilde{G} \) (resp. \( \gamma \in H \)) is a decomposition \( \tilde{\delta} = s\delta u = u\tilde{s} \) (resp. \( \gamma = t\gamma v = v\gamma t \)) with topologically semisimple \( \tilde{s} = s \times \theta \in \tilde{G} \) (resp. \( t \in H \)) and topologically unipotent \( u \in G \) (resp. \( v \in H \)). In this situation, we write \( \tilde{s} = \tilde{s}_u \) and \( u = \tilde{u} \) (resp. \( t = \gamma_s \) and \( v = \gamma_u \)).
By [41], the TJD of $\delta$ uniquely exists if $\delta$ is in a compact subgroup of $G \ltimes \langle \theta \rangle$, or equivalently, $N(\delta) = \delta^2$ is in a compact subgroup of $G$. This is similar for the TJD of $\gamma$. Moreover, in this case, $\delta_s$ and $\delta_u$ (resp. $\gamma_s$ and $\gamma_u$) belong to the closure of the subgroup generated by $\delta$ (resp. $\gamma$).

The following is the first step of the proof of Theorem 2.5.

**Lemma 2.7.** Fix $\gamma \in H^\mathrm{st}$. Suppose that $\gamma$ has a TJD $\gamma = \gamma_s \gamma_u$.

1. For a compact open subgroup $K \subset G$ satisfying $\theta(K) = K$, we have
   \[
   \sum_{\delta \sim \gamma / \sim} O_\delta(\mathrm{vol}(K; dg)^{-1}1_K) = \sum_{\tilde{s} \in I_\gamma(K)} SO_u(\mathrm{vol}(K \cap G_{\tilde{s}}; dg_{\tilde{s}})^{-1}1_{K \cap G_{\tilde{s}}}),
   \]
   where $I_\gamma(K)$ is a set of representatives $\tilde{s}$ of the $K$-conjugacy classes satisfying the following conditions:
   - $\tilde{s} \in \tilde{K}$;
   - $\tilde{s} = s \ltimes \theta$ is topologically semisimple;
   - $\gamma_{\tilde{s}}$ is a norm of $\tilde{s}$;
   - there exists $u \in K \cap G_{\tilde{s}}$ such that $\gamma$ is a norm of $\tilde{s} u$.

2. For a compact open subgroup $K_H \subset H$, we have
   \[
   SO_\gamma(\mathrm{vol}(K_H; dh)^{-1}1_{K_H}) = \sum_{t \in J_\gamma(K_H)} SO_u(\mathrm{vol}(K_H \cap H_t; dh_t)^{-1}1_{K_H \cap H_t}),
   \]
   where $J_\gamma(K_H)$ is a set of representatives $t$ of the $K_H$-conjugacy classes satisfying the following conditions:
   - $t \in K_H$;
   - $t$ is topologically semisimple;
   - $t$ is $H(\bar{F})$-conjugate to $\gamma_s$;
   - there exists $v \in K_H \cap H_t$ such that $tv$ is $H(\bar{F})$-conjugate to $\gamma$.

Here, the right hand sides are sums of stable orbital integrals for $G_{\tilde{s}}$ and $H_t$, which are independent of the choices of $u \in G_{\tilde{s}}$ and $v \in H_t$.

**Proof.** First, we prove (2). The proof is the same as the one of [19, Lemma 6.3.1]. Since $\mathrm{vol}(K_H; dh)^{-1}1_{K_H}$ is bi-$K_H$-invariant, we have
\[
SO_\gamma(\mathrm{vol}(K_H; dh)^{-1}1_{K_H})
= \sum_{\gamma' \sim \gamma / \sim} \int_{H/H_\gamma'} \mathrm{vol}(K_H; dh)^{-1}1_{K_H}(h\gamma'h^{-1})dh
= \sum_{\gamma' \sim \gamma / \sim} \int_{K_H \backslash H/H_\gamma'} \int_{K_H \cap H_{h\gamma'h^{-1}}} \mathrm{vol}(K_H; dh)^{-1}1_{K_H}(kh\gamma'h^{-1}k^{-1})dkdh
= \sum_{\gamma''} \mathrm{vol}(K_H \cap H_{\gamma''}; dh_{\gamma''})^{-1}1_{K_H}(\gamma''),
\]
where $\gamma'' \in K_H$ runs over a set of representatives of the $K_H$-conjugacy classes which are $H(\bar{F})$-conjugate to $\gamma$. Here, in the last equation, we set $\gamma'' = h\gamma'h^{-1}$, and we note that
\[
\mathrm{vol}(K_H/K_H \cap H_{\gamma''}; dk) = \frac{\mathrm{vol}(K_H; dh)}{\mathrm{vol}(K_H \cap H_{\gamma''}; dh_{\gamma''})}.
\]
By considering the TJD of \( \gamma'' \), the sum \( \sum_{\gamma''} \) above can be replaced with \( \sum_{t \in J_{\gamma}(K_H)} \sum_{\gamma''} \), where

- \( \gamma'' \in K_H \) runs over a set of representatives of the \( K_H \)-conjugacy classes which are \( H(\mathbb{F}) \)-conjugate to \( \gamma \) such that \( \gamma_s'' = t \).

For each \( t \in J_{\gamma}(K_H) \), we fix \( \gamma'' \) as above. If we write \( \gamma'' = tv \) for its TJD, we can rewrite

\[
SO_{\gamma}(\text{vol}(K_H; dh)^{-1}1_{K_H}) = \sum_{t \in J_{\gamma}(K_H)} \sum_{v'} \text{vol}(K_H \cap H_{tv'}; dh_{tv'})^{-1}1_{K_H}(tv'),
\]

where \( v' \in K_H \cap H_t \) runs over a set of representatives of the \( (K_H \cap H_t) \)-conjugacy classes which are \( H(\mathbb{F}) \)-conjugate to \( v \).

By a similar computation as above, since \( H_t \cap H_{v'} = H_{tv'} \), we have

\[
SO_v(\text{vol}(K_H \cap H_t; dh_t)^{-1}1_{K_H \cap H_t}) = \sum_{v' \sim \gamma v \sim} \int_{H_t \cap H_{tv'}} \text{vol}(K_H \cap H_t; dh_t)^{-1}1_{K_H \cap H_t}(h v' h^{-1}) d\bar{h}
\]

\[
= \sum_{v'} \text{vol}(K_H \cap H_{tv'}; dh_{tv'})^{-1}1_{K_H \cap H_t}(v'),
\]

where in the last sum, \( v' \in K_H \cap H_t \) runs over a set of representatives of the \( (K_H \cap H_t) \)-conjugacy classes which are \( H(\mathbb{F}) \)-conjugate to \( v \). Namely, the sum \( \sum_{v'} \) is the same as in the expression of \( SO_{\gamma}(\text{vol}(K_H; dh)^{-1}1_{K_H}) \) above. By noting that \( 1_{K_H}(tv') = 1_{K_H \cap H_t}(v') = 1 \), we conclude that

\[
SO_{\gamma}(\text{vol}(K_H; dh)^{-1}1_{K_H}) = \sum_{t \in J_{\gamma}(K_H)} SO_v(\text{vol}(K_H \cap H_t; dh_t)^{-1}1_{K_H \cap H_t}).
\]

The proof of (1) is essentially the same as the one of (2). Since \( \theta(K) = K \), we see that \( \text{vol}(K; dg)^{-1}1_{\hat{K}} \) is bi-\( K \)-invariant. Hence we have

\[
\sum_{\tilde{\delta} \sim \gamma \sim} O_{\tilde{\delta}}(\text{vol}(K; dg)^{-1}1_{\hat{K}}) = \sum_{\tilde{\delta} \sim \gamma \sim} \int_{G/G_{\tilde{\delta}}} \text{vol}(K; dg)^{-1}1_{\hat{K}}(g \tilde{g} g^{-1}) d\tilde{g}
\]

\[
= \sum_{\tilde{\delta} \sim \gamma \sim} \int_{\tilde{\delta} \cap G_{\tilde{\delta}} \sim} \int_{K/K \cap G_{\tilde{\delta}} g_{\tilde{\delta}} g^{-1}} \text{vol}(K; dg)^{-1}1_{\hat{K}}(kg \tilde{g} g^{-1} k^{-1}) dk d\tilde{g}
\]

\[
= \sum_{\tilde{\delta}'} \text{vol}(K \cap G_{\tilde{\delta}'}; dg_{\tilde{\delta}'}^{-1})^{-1}1_{\hat{K}}(\tilde{\delta}'),
\]

where \( \tilde{\delta}' \in \hat{K} \) runs over a set of representatives of the \( K \)-conjugacy classes such that \( \gamma \) is a norm of \( \tilde{\delta}' \). By considering the TJD of \( \tilde{\delta}' \), the sum \( \sum_{\tilde{\delta}'} \) above can be replaced with \( \sum_{\tilde{s} \in I_{\gamma}(K)} \sum_{\tilde{\delta}''} \), where

- \( \tilde{\delta}' \in \hat{K} \) runs over a set of representatives of the \( K \)-conjugacy classes such that \( \gamma \) is a norm of \( \tilde{\delta}' \), and such that \( \tilde{\delta}'' \equiv \tilde{s} \).
For each $\tilde{s} \in I_\gamma(K)$, we fix $\tilde{\delta}'$ as above. If we write $\tilde{\delta}' = \tilde{s}u$ for its TJD, we can rewrite
\[
\sum_{\tilde{\delta} \leftrightarrow \gamma/\sim} O_\delta(\text{vol}(K; dg)^{-1}1_K) = \sum_{\tilde{s} \in I_\gamma(K)} \sum_{u'} \text{vol}(K \cap G_{\tilde{s}u'}; dg_{\tilde{s}u'})^{-1}1_K(\tilde{s}u'),
\]
where $u' \in K \cap G_{\tilde{s}u'}$ runs over a set of representatives of the $(K \cap G_{\tilde{s}})$-conjugacy classes which are $G_{\tilde{s}}(F)$-conjugate to $u$.

By the same computation as in the proof of (2), since $(G_{\tilde{s}})_{u'} = G_{s_{\tilde{s}u'}}$, we have
\[
SO_u(\text{vol}(K \cap G_{\tilde{s}}; dg_{\tilde{s}})^{-1}1_{K \cap G_{\tilde{s}}}) = \sum_{u' \sim u/\sim} \int_{G_{\tilde{s}/(G_{\tilde{s}})_{u'}}} \text{vol}(K \cap G_{\tilde{s}}; dg_{\tilde{s}})^{-1}1_{K \cap G_{\tilde{s}}}(gu'g^{-1})dg
= \sum_{u'} \text{vol}(K \cap G_{\tilde{s}u'}; dg_{\tilde{s}u'})^{-1}1_{K \cap G_{\tilde{s}}}(u'),
\]
where $\sum_{u'}$ is the same as in the expression of $\sum_{\tilde{\delta} \leftrightarrow \gamma/\sim} O_\delta(\text{vol}(K; dg)^{-1}1_K)$ above. By noting that $1_K(\tilde{s}u') = 1_{K \cap G_{\tilde{s}}}(u') = 1$, we conclude that
\[
\sum_{\tilde{\delta} \leftrightarrow \gamma/\sim} O_\delta(\text{vol}(K; dg)^{-1}1_K) = \sum_{\tilde{s} \in I_\gamma(K)} SO_u(\text{vol}(K \cap G_{\tilde{s}}; dg_{\tilde{s}})^{-1}1_{K \cap G_{\tilde{s}}}).
\]
This completes the proof. \qed

We will prove the following key proposition in the next section.

**Proposition 2.8.** Suppose that $N$ is odd. For $m > 0$, set $\mathbb{K}_m = \mathbb{K}_m \times \theta \subset \tilde{G}$ and $\mathbb{K}_{m,H} = \mathbb{K}_m \cap H$. Then for $\gamma \in H_{\text{str}}$, there exists a bijection
\[ J_{\gamma}(\mathbb{K}_{m,H}) \longleftrightarrow I_\gamma(\mathbb{K}_m), \ t \leftrightarrow \tilde{s}. \]
Moreover, by replacing $I_\gamma(\mathbb{K}_m)$ if necessary, we can assume that
- $t$ is a norm of $\tilde{s}$;
- $H_t = G_{\tilde{s}}$;
- if $tv$ is $H(F)$-conjugate to $\gamma$, and if $\gamma$ is a norm of $\tilde{s}u$, then one can take $v = u^2$.

Using this proposition, we can prove Theorem 2.5.

**Proof of Theorem 2.5.** By Lemma 2.7 and Proposition 2.8, it suffices to show that
\[ SO_u^2(1_{\mathbb{K}_{m,H} \cap H_t}) = SO_u(1_{\mathbb{K}_{m,H} \cap H_t}) \]
for topologically unipotent elements $u \in H_t$ such that both $u$ and $u^2$ are strongly regular semisimple. This follows from the fact that the map $u \mapsto u^2$ is a homeomorphism from the subset of topologically unipotent elements in $H_t$ to itself (see [11, Lemma 3.2.7]). \qed
3.1. Topologically semisimple elements in hyperspecial maximal compact subgroups.

As in Section 2.1, let $G = \text{Res}_{E/F}(\text{GL}_{N,E})$ and $H = G_\theta$. We fix a hyperspecial maximal compact subgroup $\mathbb{K}$ of $G$ satisfying $\theta(\mathbb{K}) = \mathbb{K}$ such that $\mathbb{K}_H = \mathbb{K} \cap H$ is also a hyperspecial maximal compact subgroup of $H$. In this subsection, we consider conjugacy classes of topologically semisimple elements in $\mathbb{K} = \mathbb{K} \rtimes \theta$ or in $\mathbb{K}_H$. To do this, we take a smooth integral model $\mathcal{G}$ of $G$ over $\mathcal{O}_F$ such that $\mathcal{G}(\mathcal{O}_F) = \mathbb{K}$. Since $\theta(\mathbb{K}) = \mathbb{K}$, one can extend $\theta$ to an automorphism of $\mathcal{G}$ over $\mathcal{O}_F$. Moreover $\mathcal{H} = \{ h \in \mathcal{G} \mid \theta(h) = h \}$ is a hyperspecial smooth integral model of $H$ over $\mathcal{O}_F$.

**Lemma 3.1.** Let $\tilde{s} = s \rtimes \theta \in \bar{\mathbb{K}}$ be a topologically semisimple element. Then there exists $t \in \mathbb{K}_H$ such that $t$ is a norm of $\tilde{s}$.

**Proof.** We denote the images of $s$ and $\theta(s)$ in $\mathcal{H}(\mathcal{O}_F/\mathcal{P}_F)$ by $\bar{s}$ and $\bar{\theta}(s)$, respectively. Let $\bar{\theta}$ be the Frobenius map of $\mathcal{H}(\mathcal{O}_F/\mathcal{P}_F)$, i.e., $\bar{\theta}((h_{ij})_{i,j}) = J_N^{-t}(h_{ij}^q)_{i,j}J_N^{-1}$. Then $\bar{\theta}(\bar{s}) = \bar{\theta}(s)$ and $\bar{\theta}(\bar{\theta}(s)) = \bar{s}$ (although $\bar{\theta}$ is not an involution). By Lang's theorem, there exists $h \in \mathcal{H}(\mathcal{O}_F/\mathcal{P}_F)$ such that $h^{-1}\bar{\theta}(h) = \bar{s}$. If we set $\bar{\gamma} = h\bar{\theta}(s)h^{-1}$, then $\bar{\theta}(\bar{\gamma}) = \bar{\theta}(h) \cdot \bar{\theta}(s) \bar{\theta}(h)^{-1} = h\bar{s} \cdot \bar{\theta}(s) \bar{s} \cdot \bar{s}^{-1} = \bar{\gamma}$.

Hence $\bar{\gamma} \in \mathcal{H}(\mathcal{O}_F/\mathcal{P}_F)$. Take an arbitrary representative $\gamma \in \mathcal{H}(\mathcal{O}_F)$ of $\bar{\gamma}$. Write $\gamma = tv$ for its TJD, and $t, \bar{\tau}$ for the images of $t, v$ in $\mathcal{H}(\mathcal{O}_F/\mathcal{P}_F)$. Since both $\bar{\gamma}$ and $\bar{\gamma}^{-1}$ commute with $\bar{\tau}$, we see that $\bar{\tau}$ is trivial, i.e., $\bar{\tau} = \tau$. Especially, $t$ has the same eigenpolynomial as $N(s) = s\theta(s)$. This means that $t$ is a norm of $\tilde{s}$. \qed

Let $T$ be the diagonal maximal torus of $G$ and $\mathcal{T}$ be its standard integral model over $\mathcal{O}_F$. Then $T_\theta = T \cap H$ is the diagonal maximal torus of $H$ with an integral model $\mathcal{T}_\theta = \mathcal{T} \cap \mathcal{H}$. For simplicity, from now on, we assume that $\mathbb{K} = t_1\text{GL}_N(\mathcal{O}_E)t_1^{-1}$ for some $t_1 \in T$. In particular, $\mathcal{G}_0 = t_1^{-1}\mathcal{G}t_1$ is a smooth integral model of $G$ such that $\mathcal{G}_0(\mathcal{O}_F) = \text{GL}_N(\mathcal{O}_E)$. Moreover, if we put $t_2 = J_N^{-t_1^{-1}}J_N^{-1}$ so that $t_2 = \bar{\theta}(t_1)$, since $\theta(\mathbb{K}) = \mathbb{K}$, we have $t_2\text{GL}_N(\mathcal{O}_E)t_2^{-1} = t_1^{-1}\text{GL}_N(\mathcal{O}_E)t_1^{-1}$.

The following is a key fact for the proof of Proposition 2.8.

**Proposition 3.2.**

(1) Let $\tilde{s}, \tilde{s}' \in \bar{\mathbb{K}}$ be topologically semisimple elements. If $\tilde{s}$ and $\tilde{s}'$ are $G(\bar{F})$-conjugate, then $\tilde{s}$ and $\tilde{s}'$ are $\mathbb{K}$-conjugate.

(2) Let $t, t' \in \mathbb{K}_H$ be topologically semisimple elements. If $t$ and $t'$ are $H(\bar{F})$-conjugate, then $t$ and $t'$ are $\mathbb{K}_H$-conjugate.

The assertion (2) is [21, Proposition 7.1]. The proof of (1) might be similar. For the sake of completeness, we give a detail of the proof of (1) in the rest of this subsection.

For a finite extension $L$ of $F$, we denote the ring of integers of $L$ by $\mathcal{O}_L$.

**Lemma 3.3.** Let $L$ be a finite Galois extension of $F$ containing $E$. If $\tilde{s} \in \mathcal{T}(\mathcal{O}_L) \rtimes \theta$ is topologically semisimple, then the centralizer subgroup scheme $\mathcal{G}_{\tilde{s},\mathcal{O}_L}$ of $\tilde{s}$ in $\mathcal{G}_{\mathcal{O}_L}$ is a smooth group scheme over $\mathcal{O}_L$ with connected reductive fibers.

**Proof.** We put $t = \bar{s}^2$. This is an element of $\mathcal{H}(\mathcal{O}_L)$ which belongs to $\mathcal{T}_\theta(\mathcal{O}_L)$. By the same argument as in the proof of Proposition 2.3 (1), we see that the centralizer subgroup schemes $\mathcal{G}_{\tilde{s},\mathcal{O}_L}$ and $\mathcal{H}_{\mathcal{O}_L}$ are isomorphic over $\mathcal{O}_L$. Since $t$ is topologically semisimple, by the argument
in the second paragraph of the proof of [21, Proposition 7.1], we see that $H_{t, \theta_L}$ is a smooth group scheme over $\sigma_L$ with connected reductive fibers. (Note that the derived group of $H$ is simply-connected.)

When $L$ is a finite extension of $F$ containing $E$, the isomorphism $E \otimes_F L \cong L \times L$ induces an isomorphism $G(L) \cong GL_N(L) \times GL_N(L)$ such that

- $G(0_L)$ is mapped to $t_1GL_N(0_L)t_1^{-1} \times t_1GL_N(0_L)t_1^{-1} = t_1GL_N(0_L)t_1^{-1} \times t_2GL_N(0_L)t_2^{-1};$
- $T(0_L)$ (resp. $T(0_L)$) is mapped to $T_N(0_L) \times T_N(0_L)$ (resp. $T_N(L) \times T_N(L)$), where $T_N$ is the diagonal maximal torus of $GL_N;$
- the automorphism $\theta$ is given by $(g_1, g_2) \mapsto (J_N^t g_2^{-1} J_N^{-1}, J_N^t g_2^{-1} J_N^{-1}).$

Note that, if we put $t_0 = (t_1, t_2)$, then $G(0_L) = t_0G(0_L)t_0^{-1}$. Moreover, since $t_0$ is $\theta$-invariant, we have $t_0(T(0_L) \times \theta)t_0^{-1} = T(0_L)$ and $t_0(G(0_L) \times \theta)t_0^{-1} = G(0_L) \times \theta$.

**Lemma 3.4.** Suppose that $L$ is a finite Galois extension of $F$ containing $E$. Let $\tilde{s} \in T(0_L) \times \theta$ and $\tilde{s}' \in G(0_L) \times \theta$ be topologically semisimple elements. If $\tilde{s}$ and $\tilde{s}'$ are $G(L)$-conjugate, then $\tilde{s}$ and $\tilde{s}'$ are $G(0_L)$-conjugate.

**Proof.** We claim that it suffices to show the assertion when $G = G_0$. Indeed, if $\tilde{s} \in T(0_L) \times \theta$ and $\tilde{s}' \in G(0_L) \times \theta$ are $G(L)$-conjugate, then $t_0^{-1} \tilde{s} t_0 \in T(0_L) \times \theta$ and $t_0^{-1} \tilde{s}' t_0 \in G(0_L) \times \theta$ are $G(L)$-conjugate. If we were to find $y \in G_0(0_L)$ such that $t_0^{-1} \tilde{s} t_0 = y(t_0^{-1} \tilde{s}' t_0) y^{-1}$, then $\tilde{s}' = (t_0 y t_0^{-1})^\theta(t_0 y t_0^{-1})^{-1} \in G(0_L)$. Thus we may assume that $G = G_0$ so that $t_0 = (t_1, t_2)$ is trivial.

Let us suppose that $x \in G(L)$ satisfies $\tilde{s}' = x \tilde{s} x^{-1}$. Let $B = TU$ be the upper triangular Borel subgroup of $G$ with the unipotent radical $U$. By the Iwasawa decomposition, we may write $x = kut$, where $k \in G(0_L)$, $u \in U(L)$ and $t \in T(L)$. By replacing $\tilde{s}'$ with $k \tilde{s}' k^{-1}$, we may assume that $k$ is trivial.

Since $ut \tilde{s}^{-1} u^{-1} = \tilde{s}' \in G(0_L)$, by looking at the diagonal entries, we see that $t$ must belong to $T(0_L) \cdot T_\theta(L)$. If we write $t = t' t''$ with $t' \in T(0_L)$ and $t'' \in T_\theta(L)$, since $\tilde{s} \in T(0_L) \times \theta$ is commutative with $t''$, we have

$$\tilde{s}' = ut \tilde{s}^{-1} u^{-1} = ut' \tilde{s}' t' u^{-1} = t' \cdot t'' \cdot \tilde{s}' \cdot t''^{-1} \cdot t''^{-1}.$$ 

Hence, by replacing $\tilde{s}'$ with $t' \tilde{s}' t''$ and $u$ with $t'' \tilde{s}' t''^{-1}$, respectively, we may assume that $t$ is trivial, i.e., $\tilde{s}' = u \tilde{s} u^{-1}$ for $u \in U(L)$.

To get the assertion, it is enough to show that $u$ belongs to $U(\sigma_L) G_\tilde{s}(L)$, where $U$ is the standard integral model of $U$. Since both $\tilde{s}' = u \tilde{s} u^{-1}$ and $\tilde{s}$ belong to $G(\sigma_L)$, we have $u \tilde{s} u^{-1} \tilde{s}^{-1} \in G(\sigma_L)$. If we write $\tilde{s} = s \times \theta$ with $s \in T(\sigma_L)$, then we have

$$u \tilde{s} u^{-1} \tilde{s}^{-1} = u \cdot s \theta(u^{-1}) s^{-1} \in G(\sigma_L).$$

Since $L$ contains $E$, we have an isomorphism $G(\sigma_L) \cong GL_N(\sigma_L) \times GL_N(\sigma_L)$ as above. Note that it furthermore satisfies that $U(\sigma_L)$ is mapped to $U_N(\sigma_L) \times U_N(\sigma_L)$, where $U_N$ is the upper triangular unipotent subgroup of $GL_N$. Then, by letting $s = (s_1, s_2)$ and $u = (u_1, u_2)$, we have

$$u \cdot s \theta(u^{-1}) s^{-1} = (u_1 \cdot s_1 J_N^t u_2 J_N^{-1} s_1^{-1}, u_2 \cdot s_2 J_N^t u_1 J_N^{-1} s_2^{-1}).$$
Write \( u_1 = (u_{1,i,j})_{i,j} \) and \( u_2 = (u_{2,i,j})_{i,j} \). Then, for \( 1 \leq k \leq N - 1 \), the \((k, k + 1)\)-entry of \( u_1 \cdot s_1 J_N t u_2 J_N^{-1} s_1^{-1} \) is given by

\[
 u_{1,k,k+1} - \alpha_{k,k+1}(s_1)u_{2,N-k,N+1-k},
\]

where \( \alpha_{k,k+1} \) denotes the root of \( T_N \) corresponding to the \((k, k + 1)\)-entry. Similarly, the \((N - k, N + 1 - k)\)-entry of \( u_2 \cdot s_2 J_N t u_1 J_N^{-1} s_1^{-1} \) is given by

\[
 u_{2,N-k,N+1-k} - \alpha_{N-k,N+1-k}(s_2)u_{1,k,k+1}.
\]

Thus the condition that \( u s u^{-1} s^{-1} \in \mathcal{G}(\mathfrak{o}_L) \) implies that

1. \( u_{1,k,k+1} - \alpha_{k,k+1}(s_1)u_{2,N-k,N+1-k} \in \mathfrak{o}_L \); and
2. \( u_{2,N-k,N+1-k} - \alpha_{N-k,N+1-k}(s_2)u_{1,k,k+1} \in \mathfrak{o}_L \).

Since \( \tilde{s} \) is topologically semisimple, \( \tilde{s}^2 = (s_1 J_N t s_2 J_N^{-1}, s_2 J_N t s_1^{-1} J_N^{-1}) \) is of finite order prime to \( p \). Hence \( \alpha_{k,k+1}(s_1)\alpha_{N-k,N+1-k}(s_2) \) is a root of unity of order prime to \( p \). In particular, one of

- \( \alpha_{k,k+1}(s_1)\alpha_{N-k,N+1-k}(s_2) = 1 \); or
- \( 1 - \alpha_{k,k+1}(s_1)\alpha_{N-k,N+1-k}(s_2) \in \mathfrak{o}_L^\times \)

holds.

In the latter case, the matrix

\[
\begin{pmatrix}
1 & -\alpha_{N-k,N+1-k}(s_2) \\
-\alpha_{N-k,N+1-k}(s_1) & 1
\end{pmatrix}
\]

belongs to \( \text{GL}_2(\mathfrak{o}_L) \). Hence the above two conditions (1) and (2) imply that both \( u_{1,k,k+1} \) and \( u_{2,N-k,N+1-k} \) belong to \( \mathfrak{o}_L \). Now we consider the former case, i.e., we assume that \( \alpha_{k,k+1}(s_1)\alpha_{N-k,N+1-k}(s_2) = 1 \). In this case, if we define \( v_1 = (v_{1,i,j})_{i,j}, v_2 = (v_{2,i,j})_{i,j} \in U_N(L) \) by \( v_{1,i,j} = v_{2,N+1-j,N+1-i} = \delta_{i,j} \) unless \( (i, j) = (k, k + 1) \) (where \( \delta_{i,j} \) is the Kronecker delta) and by

\[
 v_{1,k,k+1} = u_{1,k,k+1}; \quad v_{2,N-k,N+1-k} = \alpha_{N-k,N+1-k}(s_2)u_{1,k,k+1},
\]

then \( v = (v_1, v_2) \in \mathcal{G}_\mathfrak{s}(L) \). Moreover, the \((k, k + 1)\)-entry of \( u_1 v_1^{-1} \) equals 0, and the \((N - k, N + 1 - k)\)-entry of \( u_2 v_2^{-1} \) is given by \( u_{2,N-k,N+1-k} - \alpha_{N-k,N+1-k}(s_2)u_{1,k,k+1} \), which lies in \( \mathfrak{o}_L \) by (2).

By this observation, we see that \( u \) belongs to \( U(\mathfrak{o}_L) \cdot [U(L), U(L)] \cdot \mathcal{G}_\mathfrak{s}(L) \), where \([U(L), U(L)]\) denotes the commutator subgroup of \( U(L) \). By applying the same argument to the entries of \( u \) appearing in \([U(L), U(L)]/[U(L), U(L), U(L), U(L)]\), we next see that \( u \) belongs to \( U(\mathfrak{o}_L) \cdot [U(L), U(L), U(L)] \cdot \mathcal{G}_\mathfrak{s}(L) \). Repeating this procedure, we eventually conclude that \( u \) is in fact an element of \( U(\mathfrak{o}_L) \mathcal{G}_\mathfrak{s}(L) \). This completes the proof.

Now we are ready to prove Proposition 3.2 (1).

Proof of Proposition 3.2 (1). Let \( \tilde{s}, \tilde{s}' \in \mathcal{G}(\mathfrak{o}_F) \rtimes \theta \) be topologically semisimple elements. Suppose that \( \tilde{s} \) and \( \tilde{s}' \) are \( \mathcal{G}(\mathcal{F}) \)-conjugate. We take a finite Galois extension \( L \) of \( F \) such that

- \( \tilde{s} \) and \( \tilde{s}' \) are \( \mathcal{G}(L) \)-conjugate;
- \( L \) contains \( E \); and
- we can find an element \( \tilde{s}'' \in \mathcal{T}(\mathfrak{o}_L) \rtimes \theta \) which is \( \mathcal{G}(L) \)-conjugate to both \( \tilde{s} \) and \( \tilde{s}' \).
Then, by Lemma 3.4, \( \tilde{s}, \tilde{s}' \) and \( \tilde{s}'' \) are \( \mathcal{G}(\mathfrak{o}_L) \)-conjugate to each other.

Let us consider the closed subscheme \( \mathcal{Y} \) of \( \mathcal{G} \) defined over \( \mathfrak{o}_F \) whose set of \( R \)-valued points is given by

\[
\mathcal{Y}(R) = \{ g \in \mathcal{G}(R) \mid g\tilde{s}g^{-1} = \tilde{s}' \}
\]

for any \( \mathfrak{o}_F \)-algebra \( R \). Since \( \tilde{s}, \tilde{s}' \) and \( \tilde{s}'' \) are \( \mathcal{G}(\mathfrak{o}_L) \)-conjugate to each other, we see that \( \mathcal{Y}_{\mathfrak{o}_L} \) and \( \mathcal{G}_{\tilde{s},\mathfrak{o}_L} \) are isomorphic to \( \mathcal{G}_{\tilde{s}'',\mathfrak{o}_L} \) as schemes over \( \mathfrak{o}_L \). Since \( \mathcal{G}_{\tilde{s}'',\mathfrak{o}_L} \) is smooth over \( \mathfrak{o}_L \) with connected reductive fibers by Lemma 3.3, so are \( \mathcal{Y}_{\mathfrak{o}_L} \) and \( \mathcal{G}_{\tilde{s},\mathfrak{o}_L} \). Thus the faithfully-flatness of \( \mathfrak{o}_L/\mathfrak{o}_F \) implies that \( \mathcal{Y} \) and \( \mathcal{G}_{\tilde{s},\mathfrak{o}_F} \) are smooth over \( \mathfrak{o}_F \) with connected reductive fibers.

We write \( \tilde{s} = s \times \theta \) and \( \tilde{s}' = s' \times \theta \) with \( s, s' \in \mathcal{G}(\mathfrak{o}_F) \). Let \( \overline{s} \) and \( \overline{s}' \) be the images of \( s \) and \( s' \) under the reduction map \( \mathcal{G}(\mathfrak{o}_F) \to \mathcal{G}(\mathfrak{o}_F/\mathfrak{p}_F) \), respectively. Since \( \tilde{s} \) and \( \tilde{s}' \) are \( \mathcal{G}(\mathfrak{o}_L) \)-conjugate, we see that \( s \times \theta \) and \( s' \times \theta \) are \( \mathcal{G}(\mathfrak{o}_L/\mathfrak{p}_L) \)-conjugate. Thus the standard argument via Lang’s theorem on the vanishing of the first Galois cohomology of \( \mathcal{G}_{\tilde{s},\mathfrak{o}_F/\mathfrak{p}_F} \) implies that \( \overline{s} \times \theta \) and \( \overline{s}' \times \theta \) are \( \mathcal{G}(\mathfrak{o}_F/\mathfrak{p}_F) \)-conjugate. In other words, \( \mathcal{Y}(\mathfrak{o}_F/\mathfrak{p}_F) \) is non-empty. Hence, by the smoothness of \( \mathcal{Y} \) over \( \mathfrak{o}_F \), we see that \( \mathcal{Y}(\mathfrak{o}_F) \) is also non-empty. Therefore \( \tilde{s} \) and \( \tilde{s}' \) are \( \mathcal{G}(\mathfrak{o}_F) \)-conjugate. This completes the proof of Proposition 3.2 (1). \( \square \)

### 3.2. Conjugacy classes of topologically semisimple elements

Suppose from now that \( N = 2n + 1 \) is odd. Fix a positive integer \( m > 0 \). Recall that \( \mathbb{K}_m \) is the subgroup of \( G \) consisting of matrices \( k \) of the form

\[
\begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{pmatrix}
\]

with \( \det(k) \in \mathfrak{o}_E^\times \). Note that \( \theta(\mathbb{K}_m) = \mathbb{K}_m \). Set \( \mathbb{K}_{m,H} = \mathbb{K}_m \cap H \).

**Proposition 3.5.**

1. Let \( \tilde{s} \in \tilde{\mathbb{K}}_m \) be a topologically semisimple element. Then there exists \( k \in \mathbb{K}_m \) such that

\[
k^{-1}\tilde{s}k \in \begin{pmatrix}
\mathfrak{o}_E & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{pmatrix} \times \theta.
\]

2. Let \( t \in \mathbb{K}_{m,H} \) be a topologically semisimple element. Then there exists \( k \in \mathbb{K}_{m,H} \) such that

\[
k^{-1}tk \in \begin{pmatrix}
\mathfrak{o}_E & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{pmatrix}.
\]

**Proof.** To show (1), we prepare some notation. Let \( \rho : G \rtimes \langle \theta \rangle \hookrightarrow \text{GL}_{2N}(E) \) be a faithful representation given by

\[
\rho(g) = \begin{pmatrix}
g & 0 \\
0 & \theta(g)
\end{pmatrix}, \quad \rho(g \rtimes \theta) = \begin{pmatrix}
0 & g \\
\theta(g) & 0
\end{pmatrix}.
\]

Set \( \tilde{V} = E^{2N} \) to be the representation space equipped with the canonical basis denoted by \( e_{-n}, \ldots, e_n, f_n, \ldots, f_{-n} \). Let \( \tilde{L} \) be the \( \mathfrak{o}_E \)-lattice of \( \tilde{V} \) spanned by

\[
\omega^{-m} e_{-n}, \ldots, \omega^{-m} e_{-1}, \epsilon_0, \ldots, \epsilon_n, \omega^{-m} f_n, \ldots, \omega^{-m} f_1, f_0, \ldots, f_{-n}.
\]
Then \( \rho(\mathbb{K}_m \rtimes \langle \theta \rangle) \) preserves \( \tilde{L} \). We denote by \( V \) (resp. by \( V' \)) the subspace of \( \tilde{V} \) generated by \( e_{-n}, \ldots, e_n \) (resp. by \( f_n, \ldots, f_{-n} \)). Set \( L = \tilde{L} \cap V \) and \( L' = \tilde{L} \cap V' \). Hence \( \tilde{V} = V \oplus V' \) and \( \tilde{L} = L \oplus L' \). Define a hermitian pairing \( \langle \cdot, \cdot \rangle : \tilde{V} \times \tilde{V} \to E \) so that \( V \) and \( V' \) are totally isotropic subspaces, and so that

\[
\left\langle \sum_{i=-n}^{n} a_i e_i, \sum_{i=-n}^{n} b_i f_i \right\rangle = \sum_{i=-n}^{n} (-1)^{i-n} a_i \overline{b_i}
\]

for \( a_i, b_i \in E \). Then \( \rho(G) \) is characterized by the subgroup of \( \text{GL}(\tilde{V}) \) consisting of linear operators \( f : \tilde{V} \to \tilde{V} \) such that \( f(V) \subset V \), \( f(V') \subset V' \) and such that \( \langle f(v), f(v') \rangle = \langle v, v' \rangle \) for \( v, v' \in \tilde{V} \). Moreover, one can check that \( \langle \rho(g \rtimes \theta)v, \rho(g \rtimes \theta)v' \rangle = \langle v, v' \rangle \) for \( v, v' \in \tilde{V} \).

Let \( s = s \rtimes \theta \in \mathbb{K}_m \) be a topologically semisimple element. We consider \( N(s) = s \theta(s) = (s \rtimes \theta)^2 \). It gives an \( \mathfrak{o}_E \)-linear automorphism of \( L \). Note that

\[
\rho(N(s)) e_0 \equiv e_0 \mod \varpi^m L.
\]

Let \( V_1 \) (resp. \( (L/\varpi^m L)_1 \)) be the eigenspace of \( \rho(N(s)) \) (resp. \( \rho(N(s)) \mod \varpi^m L \)) with respect to the eigenvalue \( 1 \). Since \( N(s) \) is of finite order prime to \( p \), we see that \( (L/\varpi^m L)_1 \) is the image of \( V_1 \cap L \) by the canonical projection \( L \to L/\varpi^m L \). In particular, there exists \( v_0 \in V_1 \cap L \) such that \( v_0 \equiv e_0 \mod \varpi^m L \).

Set \( v'_0 = \rho(s \rtimes \theta)v_0 \). Then we have \( v'_0 \in L' \) and \( v'_0 \equiv f_0 \mod \varpi^m L' \). In particular, \((-1)^n \langle v_0, v'_0 \rangle \in 1 + \mathfrak{p}^m_E \) since \( \langle e_0, f_0 \rangle = (-1)^n \). On the other hand, since \( \langle \cdot, \cdot \rangle \) is \( \rho(s \rtimes \theta) \)-invariant, we have

\[
\langle v_0, v'_0 \rangle = \langle \rho(N(s))v_0, \rho(s \rtimes \theta)v_0 \rangle = \langle \rho(s \rtimes \theta)v_0, v_0 \rangle = \langle v'_0, v_0 \rangle = \overline{\langle v_0, v'_0 \rangle}.
\]

Hence \((-1)^n \langle v_0, v'_0 \rangle \in (1 + \mathfrak{p}^m_E) \cap F = 1 + \mathfrak{p}^m_E = N_{E/F}(1 + \mathfrak{p}^m_E) \). In particular, we can find \( z \in 1 + \mathfrak{p}^m_E \) such that if we set \( e'_0 = zv_0 \) and \( f'_0 = zv'_0 \), then

- \( e'_0 \in \mathfrak{o}_E e_{-n} \oplus \cdots \oplus \mathfrak{o}_E e_{-1} \oplus (1 + \mathfrak{p}^m_E) e_0 \oplus \mathfrak{p}^m_E e_1 \oplus \cdots \oplus \mathfrak{p}^m_E e_n \);
- \( f'_0 \in \mathfrak{o}_E f_n \oplus \cdots \oplus \mathfrak{o}_E f_1 \oplus (1 + \mathfrak{p}^m_E) f_0 \oplus \mathfrak{p}^m_E f_{-1} \oplus \cdots \oplus \mathfrak{p}^m_E f_{-n} \);
- \( \langle e'_0, f'_0 \rangle = (-1)^n \langle e_0, f_0 \rangle \);
- \( f'_0 = \rho(s \rtimes \theta)e'_0 \) and \( e'_0 = \rho(s \rtimes \theta)f'_0 \).

Fix \( \epsilon \in \mathfrak{o}_E^{\times} \) such that \( \overline{\epsilon} = -\epsilon \). For \( a \in \mathfrak{o}_E^{\times} \), set

\[
z_a = \frac{1 + a \epsilon \varpi^m}{1 - a \epsilon \varpi^m}.
\]

Then \( N_{E/F}(z_a) = 1 \) and \( z_a = 1 + 2a \epsilon \varpi^m \mod \mathfrak{p}^{2m}_E \). If necessary, by replacing \( e'_0 \) and \( f'_0 \) by \( z_a e'_0 \) and \( z_a f'_0 \) simultaneously for a suitable \( a \in \mathfrak{o}_E^{\times} \), we may furthermore assume that

- \( \langle e_0, f'_0 - f_0 \rangle, \langle f_0, e'_0 - e_0 \rangle \in \varpi^m \mathfrak{o}_E^{\times} \).

Based on an argument of Witt’s theorem, for \(-n \leq j \leq n \) with \( j \neq 0 \), we set

\[
e'_j = e_j + \frac{\langle e_j, f'_0 - f_0 \rangle}{\langle e_0, f'_0 - f_0 \rangle}(e'_0 - e_0),
\]

\[
f'_j = f_j + \frac{\langle f_j, e'_0 - e_0 \rangle}{\langle f_0, e'_0 - e_0 \rangle}(f'_0 - f_0).
\]
We claim that \( \langle e_i', f_j' \rangle = \langle e_i, f_j \rangle \) for any \(-n \leq i, j \leq n\). Indeed, if both \(i\) and \(j\) are not equal to \(0\), then
\[
\langle e_i', f_j' \rangle - \langle e_i, f_j \rangle = (e_i, f_j) = 0.
\]
Similarly, one can easily check that \( \langle e_i', f_j' \rangle = \langle e_0', f_0' \rangle = 0 \) for \(j \neq 0\). Therefore, the change-of-basis matrix from \((e_{-n}, \ldots, e_n, f_{-n}, \ldots, f_n)\) to \((e_{-n}', \ldots, e_n', f_{-n}', \ldots, f_n')\) is given by \(\rho(k)\) for some \(k \in G\). Moreover, by construction, we see that \(k \in \mathbb{K}_m\). Since the orthogonal complement of \(\{e_i', f_0'\}\) in \(\bar{V}\), which is spanned by \(\{e_j', f_j'\}_{j \neq 0}\), is preserved by \(\rho(s \times \theta)\), we have
\[
k^{-1}(s \times \theta)k \in \begin{pmatrix} n & 1 & n \\ n & 0 & 1 \\ n & p_E^m & 0 & 0 \end{pmatrix} \times \theta,
\]
as desired.

The proof of (2) is similar and easier. Let us give a brief sketch.

- We regard \(V = E^N\) as a hermitian space so that \(H = U(V)\). Write \(e_{-n}, \ldots, e_n\) for the canonical basis.
- By the same argument as in the proof of (1), one can take
\[
e_0' \in o_E e_{-n} \oplus \ldots \oplus o_E e_{-1} \oplus (1 + p_E^m) e_0 \oplus p_E^m e_1 \oplus \cdots \oplus p_E^m e_n
\]
such that \(te_0' = e_0'\). Moreover, one can normalize \(e_0'\) so that \(\langle e_0', e_0' \rangle = \langle e_0, e_0 \rangle\).
- In addition, by replacing \(e_0'\) with
\[
\frac{1 + \epsilon \varpi^m}{1 - \epsilon \varpi^m} e_0'
\]
if necessary, we may furthermore assume that \(\langle e_0, e_0' - e_0 \rangle \in \varpi^m o_E^\times\).
- For \(-n \leq j \leq n\) with \(j \neq 0\), define \(e_j'\) by
\[
e_j' = e_j + \frac{\langle e_j, e_0' - e_0 \rangle}{\langle e_0, e_0' - e_0 \rangle} (e_0' - e_0).
\]
It is easy to check that \(\langle e_i', e_j' \rangle = \langle e_i, e_j \rangle\) for \(-n \leq i, j \leq n\).
- The change-of-basis matrix from \((e_{-n}, \ldots, e_n)\) to \((e_{-n}', \ldots, e_n')\) gives an element \(k \in \mathbb{K}_{m,H}\). This element satisfies the desired condition.

This completes the proof.

**Corollary 3.6.** For \(\gamma \in H_{\text{str}}\), we have \(|I_\gamma(\mathbb{K}_m)| \leq 1\) and \(|J_\gamma(\mathbb{K}_{m,H})| \leq 1\).

**Proof.** The assertion \(|J_\gamma(\mathbb{K}_{m,H})| \leq 1\) immediately follows from Propositions 3.5 (2) and 3.2 (2). Similarly, the assertion \(|I_\gamma(\mathbb{K}_m)| \leq 1\) is proven by using Propositions 3.5 (1) and 3.2 (1), but we have to check it carefully.
Let $\hat{s}_1 = s_1 \times \theta, \hat{s}_2 = s_2 \times \theta \in J_\gamma(\mathbb{K}_m)$. We will show that $\hat{s}_1$ is $\mathbb{K}_m$-conjugate to $\hat{s}_2$. By Proposition 3.5 (1), we may assume that

$$
\hat{s}_1, \hat{s}_2 \in 1 \begin{pmatrix} n & 1 & n \\ 0 & 0 & 0 \\ n^{-1}p_E^m & 0 & a_E \end{pmatrix} \times \theta
$$

after taking $\mathbb{K}_m$-conjugations if necessary.

Recall from Section 2.1 that we have an inclusion $\iota : G' = \text{GL}_{2n}(E) \hookrightarrow G = \text{GL}_{2n+1}(E)$. If we set $\mathbb{K}_m' = \iota^{-1}(\mathbb{K}_m) \subset G'$, then we can write $s_1 = \iota(s_1'), s_2 = \iota(s_2')$ for some $s_1', s_2' \in \mathbb{K}_m'$. Although $G$ and $G'$ have involutions both denoted by $\theta$, they are not compatible with respect to $\iota$. Indeed, $G'$ has another involution $\theta'$ such that $\iota(\theta'(x)) = \theta(\iota(x))$ for $x \in G'$. Two involutions $\theta$ and $\theta'$ on $G'$ are related by $\theta'(x) = \kappa \cdot \theta(\kappa^{-1} x \kappa) \cdot \kappa^{-1}$ for $x \in G'$, where $\kappa \in \mathbb{K}_m'$ is as in Section 2.1. In particular,

$$(s_1' \times \theta')^2 = s_1' \theta'(s_1') = s_1' \kappa \theta(\kappa^{-1} s_1' \kappa) \kappa^{-1} = \kappa^{-1} s_1' \kappa \times \theta)^2 \kappa^{-1}.$$ 

Since $\iota((s_1' \times \theta')^2) = (s_1 \times \theta)^2$, we see that $\kappa^{-1} s_1' \kappa \times \theta$ is a topologically semisimple element in $G' \times \theta$. Moreover, by considering the norm correspondence, there exists $g \in G'((\mathcal{F}))$ such that $s_2' \times \theta' = g(s_1' \times \theta')g^{-1}$. It is equivalent to saying that

$$s_2' = gs_1' \theta'(g^{-1}) = gs_1' \kappa \theta(\kappa^{-1} g^{-1} \kappa) \kappa^{-1}$$

$$\iff \kappa^{-1} s_2' \kappa = (\kappa^{-1} g \kappa) \cdot \kappa^{-1} s_1' \kappa \cdot \theta(\kappa^{-1} g^{-1} \kappa).$$

Hence $\kappa^{-1} s_2' \kappa \times \theta$ is $G'((\mathcal{F}))$-conjugate to $\kappa^{-1} s_1' \kappa \times \theta$. By Proposition 3.2 (1), there exists $k' \in \mathbb{K}_m'$ such that $\kappa^{-1} s_2' \kappa \times \theta = k'(\kappa^{-1} s_1' \kappa \times \theta)k'^{-1}$. By the same calculation as above, it is equivalent to saying that $s_2' \times \theta' = (kk' \kappa^{-1}) \cdot (s_1' \times \theta') \cdot (kk' \kappa^{-1})^{-1}$. If we set $k = \iota(kk' \kappa^{-1}) \in \mathbb{K}_m$, then we have $s_2 \times \theta = k(s_1 \times \theta)k^{-1}$, as desired.

**Lemma 3.7.** If $J_\gamma(\mathbb{K}_m, H) \neq \emptyset$, then $I_\gamma(\mathbb{K}_m) \neq \emptyset$. Moreover, for $t \in J_\gamma(\mathbb{K}_m, H)$, we can assume that $\tilde{s} \in J_\gamma(\mathbb{K}_m)$ satisfies the three conditions in Proposition 2.8.

**Proof.** Let $t \in J_\gamma(\mathbb{K}_m, H)$. In Lemma A.3 below, we will show that there exists $\tilde{s} = s \times \theta \in \mathbb{K}_m$ with $s \in \mathfrak{o}_E[t]$ such that $t = N(s)$. In particular, $t$ is a norm of $s = s \times \theta$. Since every $g \in G_t$ commutes with $s$, by the same argument as in the proof of Proposition 2.3 (2), we have $G_{\tilde{s}} = H_t$ as algebraic subgroups of $G$. Let $v \in \mathbb{K}_m \cap H_t$ be a topologically unipotent element such that $tv$ is $H((\mathcal{F}))$-conjugate to $\gamma$. By the same argument as in [11, Lemma 3.2.7], there exists a unique topologically unipotent element $u \in H_t = G_{\tilde{s}}$ such that $v = u^2$. Moreover, since $\mathbb{K}_m$ is closed in $G$ and $p$ is odd, we have $u \in \mathbb{K}_m \cap G_{\tilde{s}}$. Since $(\tilde{s}u)^2 = s\theta(u)\theta(s)u = N(s)u^2 = tv$, we see that $tv$ is a norm of $\tilde{s}u$. Therefore, we conclude that $\tilde{s} \in I_\gamma(\mathbb{K}_m)$.

**Remark 3.8.** For the proof of Lemma A.3, we will generalize Hilbert’s Theorem 90 by using the faithfully flat descent (Proposition A.1). However, if $q = [\mathfrak{o}_E/\mathfrak{p}_E] > N$, one can easily show the existence of $\tilde{s} = s \times \theta \in \mathbb{K}_m$ with $s \in \mathfrak{o}_E[t]$ such that $t = N(s)$ as follows.

As in the proof of Proposition 2.3 (2), we consider $s = (\alpha I_N + \overline{\alpha t})/ (\alpha + \overline{\alpha})$ for $\alpha \in \mathfrak{o}_E^\times$ with $\alpha + \overline{\alpha} \neq 0$. It suffices to find $\alpha$ such that $\det(s) \in \mathfrak{o}_E^\times$. If we denote the eigenpolynomial of $t$ by $\Phi_t$, we have $\det(s) = (-\overline{\alpha}/(\alpha + \overline{\alpha}))^N \Phi_t(-\alpha/\overline{\alpha})$. Hence it is enough to find $\alpha \in \mathfrak{o}_E^\times$ such
that \( \alpha + \overline{\alpha} \neq 0 \mod p_E \) and \( \Phi_f(-\alpha/\overline{\alpha}) \neq 0 \mod p_E \). Since

\[
|\{-\alpha/\overline{\alpha} \mid \alpha \in (\sigma_E/p_E)\times, \alpha + \overline{\alpha} \neq 0\}| = \frac{(q^2 - 1) - (q - 1)}{q - 1} = q > N,
\]

can be found \( \alpha \in \sigma_E^{\times} \) satisfying the desired conditions.

**Lemma 3.9.** If \( I_\gamma(\mathbb{K}_m) \neq \emptyset \), then \( J_\gamma(\mathbb{K}_{m,H}) \neq \emptyset \).

**Proof.** Take \( \tilde{s} = s \times \theta \in I_\gamma(\mathbb{K}_m) \). By Proposition 3.5 (1), we may assume that \( s = \iota(s') \) for some \( s' \in \mathbb{K}'_m = \iota^{-1}(\mathbb{K}_m) \subset G' \). As we have seen in the proof of Corollary 3.6, \( \kappa^{-1}s'K \times \theta \) is topologically semisimple, where \( \kappa \in \mathbb{K}'_m \) is as in Section 2.1. By Lemma 3.1, there is \( t' \in \mathbb{K}'_m \cap G'_{\theta} \) such that \( t' \) is a norm of \( \kappa^{-1}s'K \times \theta \). This means that there exists \( g' \in G'(\mathbb{F}) \) such that \( g't'g'^{-1} = \kappa^{-1}s'K\theta(\kappa^{-1}s'K) \). Since \( \theta'(x) = \kappa \cdot \theta(\kappa^{-1}xK) \cdot \kappa^{-1} \), we have \( (kg'\kappa^{-1}) \cdot (kt'\kappa^{-1}) \cdot (kg'\kappa^{-1})^{-1} = s\theta'(s') \). If we set \( t = \iota(kt'\kappa^{-1}) \) and \( g = \iota(kg'\kappa^{-1}) \), then we have \( t \in \mathbb{K}_{m,H} \) and \( gtg^{-1} = s\theta(s) \). Hence \( t \) is a norm of \( \tilde{s} \), which is equivalent to saying that \( t \) is \( H(\mathbb{F}) \)-conjugate to \( \gamma_s \).

By applying Lemma A.3 (or Remark 3.8 if \( q > N \)) to \( t \in \mathbb{K}_{m,H} \), one can obtain \( \tilde{s}_0 = s_0 \times \theta \in \mathbb{K}_m \) such that \( t = s_0 \theta(s_0) \) and \( G_{\tilde{s}_0} = H_t \). Since \( \tilde{s}, \tilde{s}_0 \in \mathbb{K}_m \), by the argument in Corollary 3.6, we have \( k \in \mathbb{K}_m \) such that \( \tilde{s}_0 = k\tilde{s}k^{-1} \). Then \( H_t = G_{\tilde{s}_0} = kG_{\tilde{s}}k^{-1} \).

Suppose now that \( \gamma \) is a norm of \( \tilde{s}u \) with \( u \in \mathbb{K}_m \cap G_{\tilde{s}} \). Set \( u_0 = kuk^{-1} \in \mathbb{K}_m \cap G_{\tilde{s}_0} \) and \( v = u_0^2 \in \mathbb{K}_{m,H} \cap H_t \). Then we have

\[
 tv = s_0 \theta(s_0)u_0^2 = (\tilde{s}_0u_0)^2 = (k\tilde{s}uk^{-1})^2 = k(\tilde{s}u)^2k^{-1}.
\]

It means that \( tv \) is a norm of \( \tilde{s}u \), which is equivalent to saying that \( tv \) is \( H(\mathbb{F}) \)-conjugate to \( \gamma \). Therefore, we conclude that \( t \in J_\gamma(\mathbb{K}_{m,H}) \).

Now Proposition 2.8 follows from Corollary 3.6 together with Lemmas 3.7 and 3.9.

4. LOCAL LANGLANDS CORRESPONDENCE AND LOCAL NEWFORMS

In this section, we prove Theorem 1.1 as an application of Theorem 2.5. To do this, we recall the local Langlands correspondence.

4.1. Local Langlands correspondence for \( U_{2n+1} \). Let \( W_E \subset W_F \) be the Weil groups of \( E \) and \( F \), respectively. Fix \( s \in W_F \setminus W_E \). Set \( WD_E = W_E \times SL_2(\mathbb{C}) \) to be the Weil–Deligne group of \( E \). We call a representation \( \phi : WD_E \to GL_N(\mathbb{C}) \) conjugate orthogonal if there exists a non-degenerate bilinear form \( B : \mathbb{C}^N \times \mathbb{C}^N \to \mathbb{C} \) such that

\[
\begin{align*}
B(\phi(w)v, \phi(sws^{-1})v') &= B(v, v'), \\
B(v', v) &= B(v, \phi(s^2)v')
\end{align*}
\]

for \( v, v' \in \mathbb{C}^N \) and \( w \in WD_E \).

Set \( H = U_{2n+1} \) to be the quasi-split unitary group with \( 2n + 1 \) variables as in the previous sections. Let \( \Phi_{\text{temp}}(H) \) be the set of equivalence classes of conjugate orthogonal representations \( \phi \) of \( WD_E \) of dimension \( 2n + 1 \) such that \( \phi(W_E) \) is bounded, and let \( \text{Irr}_{\text{temp}}(H) \) be the set of equivalence classes of irreducible tempered representations of \( H \).
For \( \pi \in \text{Irr}_{\text{temp}}(H) \), we have the Harish-Chandra character \( \Theta_\pi : C_c^\infty(H) \to \mathbb{C} \), which is defined by
\[
\Theta_\pi(f^H) = \text{tr} \left( v \mapsto \int_H f^H(h) \pi(h) v dh \right), \quad f^H \in C_c^\infty(H).
\]
On the other hand, for \( \phi \in \Phi_{\text{temp}}(H) \), let \( \pi_\phi \) be the irreducible tempered representation of \( G = \text{GL}_{2n+1}(E) \) associated with \( \phi \) by the local Langlands correspondence for \( \text{GL}_{2n+1}(E) \).
Since \( \pi_\phi \) is conjugate self-dual, we have a nonzero intertwining operator \( I : \pi_\phi \to \pi_\phi^0 \) with \( I^2 = \text{id} \). Note that \( I \) is unique up to \( \{ \pm 1 \} \). We normalize \( I \) by using the fixed \( \theta \)-stable \( F \)-splitting of \( G \). Then we can consider the twisted trace
\[
\Theta_{\pi_\phi, \theta}(f) = \text{tr} \left( v \mapsto \int_G f(g \times \theta)(\pi(g) \circ I)v dg \right), \quad f \in C_c^\infty(\hat{G}).
\]

The local Langlands correspondence for \( H \) is as follows.

**Theorem 4.1** ([32, Theorem 3.2.1]). For \( \phi \in \Phi_{\text{temp}}(H) \), there exists a finite subset \( \Pi_\phi \) of \( \text{Irr}_{\text{temp}}(H) \) such that
\[
\text{Irr}_{\text{temp}}(H) = \bigcup_{\phi \in \Phi_{\text{temp}}(H)} \Pi_\phi.
\]
Moreover, \( \Pi_\phi \) is characterized by the endoscopic character relation
\[
\Theta_{\pi_\phi, \theta}(f) = \sum_{\pi \in \Pi_\phi} \Theta_\pi(f^H)
\]
for any \( f \in C_c^\infty(\hat{G}) \) and \( f^H \in C_c^\infty(H) \) such that \( f^H \) is a transfer of \( f \).

We call \( \Pi_\phi \) the \( L \)-packet associated with \( \phi \). When \( \pi \in \Pi_\phi \), we say that \( \phi \) is the \( L \)-parameter for \( \pi \).

4.2. **Multiplicity one in \( L \)-packets.** Fix a non-trivial additive character \( \psi_E : E \to \mathbb{C}^\times \) such that \( \psi_E|_{\mathfrak{O}_E} = 1 \) but \( \psi_E|_{\mathfrak{p}_E} \neq 1 \). For \( \phi \in \Phi_{\text{temp}}(H) \), let \( \varepsilon(s, \phi, \psi_E) \) be the \( \varepsilon \)-factor associated with \( \phi \) and \( \psi_E \). We define the conductor \( c(\phi) \) of \( \phi \) by the non-negative integer satisfying
\[
\varepsilon(s, \phi, \psi_E) = q^{c(\phi)(1-2s)} \varepsilon(1/2, \phi, \psi_E).
\]
Here, we note that \( |\mathfrak{O}_E/\mathfrak{p}_E| = q^2 \), and that \( c(\phi) \) is independent of the choice of \( \psi_E \).

**Remark 4.2.** By [9, Proposition 5.2 (2)], if \( \psi_E \) is trivial on \( F \), then \( \varepsilon(1/2, \phi, \psi_E) = 1 \).

As an application of Theorem 2.5, we prove the following theorem.

**Theorem 4.3.** For \( \phi \in \Phi_{\text{temp}}(H) \), we have
\[
\sum_{\pi \in \Pi_\phi} \dim(\pi^{K_m,H}) = \begin{cases} 0 & \text{if } m < c(\phi), \\ 1 & \text{if } m = c(\phi). \end{cases}
\]
Moreover, if \( m > c(\phi) \), then the left hand side is nonzero.

**Proof.** Note that for \( \pi \in \text{Irr}_{\text{temp}}(H) \), we have
\[
\dim(\pi^{K_m,H}) = \Theta_\pi(\text{vol}(K_m,H; dh)^{-1} 1_{K_m,H}).
\]
Hence by Theorems 2.5 and 4.1, we have

$$\sum_{\pi \in \Pi_{\phi}} \dim(\pi^{K_m,H}) = \Theta_{\pi,\theta}(\vol(K_m;dg)^{-1}1_{K_m}).$$

Since $$\pi(g) \circ I = I \circ \pi(\theta(g))$$ and $$\theta(K_m) = K_m$$, we see that I preserves $$\pi^{K_m}$$. Moreover, the image of the operator

$$v \mapsto \vol(K_m;dg)^{-1} \int_G 1_{K_m}(g)(\pi(g) \circ I)v dg$$

is equal to $$\pi^{K_m}$$, and the restriction of this operator to $$\pi^{K_m}$$ coincides with I. Hence

$$\Theta_{\pi,\theta}(\vol(K_m;dg)^{-1}1_{K_m}) = \text{tr}(I;\pi^{K_m}).$$

Since

$$K_m = \begin{pmatrix} I_n & 1 \\ \sigma_n I_n & \sigma_n \end{pmatrix} \begin{pmatrix} \sigma_E & \sigma_E & \sigma_E \\ p_E & \sigma_E & \sigma_E \\ 0 & \sigma_E & \sigma_E \end{pmatrix}^{-1} \begin{pmatrix} I_n \\ \sigma_n I_n \end{pmatrix},$$

by [18, (5.1) Théorème], we have

$$\dim(\pi^{K_m}) = \begin{cases} 0 & \text{if } m < c(\phi), \\ 1 & \text{if } m = c(\phi). \end{cases}$$

In particular, if $$m < c(\phi)$$, then

$$\sum_{\pi \in \Pi_{\phi}} \dim(\pi^{K_m,H}) = \text{tr}(I;\pi^{K_m}) = 0.$$ 

On the other hand, if $$m = c(\phi)$$, then

$$\sum_{\pi \in \Pi_{\phi}} \dim(\pi^{K_m,H}) = \text{tr}(I;\pi^{K_m}) \in \{\pm 1\}.$$ 

Since the left hand side is a non-negative integer, it should be equal to 1.

Similarly, since $$\dim(\pi^{K_{c(\phi)+1}}) = 2n + 1$$ by [38, (2.2) Theorem 1], and since $$I^2 = \text{id}$$, we see that $$\text{tr}(I;\pi^{K_{c(\phi)+1}}) \neq 0$$ so that $$\sum_{\pi \in \Pi_{\phi}} \dim(\pi^{K_{c(\phi)+1},H}) > 0$$. Since

$$\begin{pmatrix} \varpi I_n & 0 \\ 0 & \varpi^{-1} I_n \end{pmatrix} K_{m+2,H} \begin{pmatrix} \varpi I_n & 0 \\ 0 & \varpi^{-1} I_n \end{pmatrix}^{-1} \subset K_{m,H},$$

we have $$\dim(\pi^{K_{m+2,H}}) \geq \dim(\pi^{K_{m,H}})$$ for any $$m \geq 0$$. Therefore, we conclude that

$$\sum_{\pi \in \Pi_{\phi}} \dim(\pi^{K_{c(\phi)+m,H}}) > 0$$

for any $$m > 0$. □
4.3. **Generic representations.** Let $U \subset H$ be the subgroup consisting of upper triangular unipotent matrices. By abuse of notation, we denote the character

$$U \ni u = (u_{i,j})_{i,j} \mapsto \psi_E(u_{1,2} + \cdots + u_{n,n+1}) \in \mathbb{C}^\times$$

by $\psi_E$. We say that an irreducible representation $\pi$ of $H$ is **generic** if $\text{Hom}_U(\pi, \psi_E) \neq 0$.

Recall that if $\phi \in \Phi_{\text{temp}}(H)$, the $L$-packet $\Pi_\phi$ contains a unique generic representation (see [32, Corollary 9.2.4] and [1, Theorem 3.1]). Now Theorem 1.1 follows from Theorem 4.3 and the following result.

**Theorem 4.4.** Let $\pi \in \text{Irr}_{\text{temp}}(H)$. If $\pi^{\mathcal{K}_{m,H}} \neq 0$ for some $m \geq 0$, then $\pi$ is **generic**.

**Proof.** As in Section 2.1, we consider another unramified unitary group

$$H' = G'_{2n} = \{ g' \in \text{GL}_{2n}(E) \mid g' J_{2n}^0 \mathcal{F} = J_{2n}^0 \}.$$ 

Via the inclusion $\iota : H' \hookrightarrow H$ defined in that subsection, we regard $H'$ as a subgroup of $H$. Then $\mathcal{K}_{m,H} \cap H'$ is a hyperspecial maximal compact subgroup of $H'$.

Let $\pi \in \text{Irr}_{\text{temp}}(H)$ be such that $\pi^{\mathcal{K}_{m,H}} \neq 0$. We claim that there exists an irreducible tempered unramified representation $\pi'$ of $H'$ such that $\text{Hom}_{H'}(\pi \otimes \pi', \mathcal{C}) \neq 0$. If this claim were to be shown, by the local Gan–Gross–Prasad conjecture ([9, Conjecture 17.3]) proven by Beuzart-Plessis [3, 4, 5], $\pi$ would be uniquely determined by the $L$-parameters for $\pi$ and $\pi'$. For the $L$-parameter for $\pi'$, see e.g., [9, Sections 10]. Since $\pi'$ is unramified, and since there exists only one unramified character $\chi$ of $E^\times$ such that $\chi|_{F^\times}$ is equal to the quadratic character associated to $E/F$, the $L$-parameter $\phi'$ for $\pi'$ should be of the form $\phi_1 \oplus \phi_1^\vee$ for some representation $\phi_1$ of $W_{DE}$, where $\phi_1^\vee$ is the conjugate dual of $\phi_1$. Using [9, Proposition 5.1 (2)] and [32, Corollary 9.2.4], we could conclude that $\pi$ is generic.

The claim is essentially the same as a lemma of Gan–Savin ([10, Lemma 12.5]). Fix $v \in \pi^{\mathcal{K}_{m,H}}$ such that $v \neq 0$. Let $\langle \cdot, \cdot \rangle$ be an $H$-invariant inner product on $\pi$. Then $f_\pi(h) = \langle \pi(h) v, v \rangle$ is a matrix coefficient of $\pi$. Moreover, it is bi-$\mathcal{K}_{m,H}$-invariant and $f_\pi(I_{2n+1}) \neq 0$. Since $f_\pi|_{H'} \neq 0$, by the same argument as in the proof of [10, Lemma 12.5], we can find an irreducible tempered representation $\pi'$ of $H'$ and a matrix coefficient $f_{\pi'}$ of $\pi'$ such that

$$\int_{H' \setminus H} f_\pi(h') f_{\pi'}(h') dh' \neq 0.$$

Here, the absolutely convergence of double integrals which we need was proven in [16, Proposition 2.1]. However, since $f_\pi$ is bi-$\mathcal{K}_{m,H}$-invariant, we can take $f_{\pi'}$ so that $f_{\pi'}$ is bi-$((\mathcal{K}_{m,H} \cap H'))$-invariant. Then $\pi'$ must be unramified and $\text{Hom}_{H'}(\pi \otimes \pi', \mathcal{C}) \neq 0$, as desired. □

**Appendix A. Faithfully flat descent and Hilbert’s Theorem 90**

In this appendix, we generalize Hilbert’s Theorem 90 (Proposition A.1) by using the faithfully flat descent. Using this proposition, we will prove Lemma A.3, which was used in the proof of Lemma 3.7.

**A.1. A generalization of Hilbert’s Theorem 90.** Now we prove a generalization of Hilbert’s Theorem 90.

**Proposition A.1.** Let $f : A \to B$ be a ring homomorphism between commutative rings $A$ and $B$ with the unit element 1, and let $G$ be a cyclic group of order $n$ with a generator $\sigma \in G$. Suppose that $G$ acts on $B$ from the left, and that
isomorphism. Therefore, we conclude that the case where $G$ is a finite group.

By the same argument, one would furthermore generalize Proposition A.1 to Remark A.2.

Then for any $x \in B$ with $\prod_{g \in G} g(x) = 1$, there exists $y \in B^\times$ such that $x = y/\sigma(y)$.

**Proof.** Remark that the homomorphism $\phi$ in (3) is well-defined by (2). Consider

$$M = \{y \in B \mid \sigma(y) \cdot x = y\}.$$ 

It is an $A$-submodule of $B$. We have to show that $M \cap B^\times$ is not empty.

Since $B$ is a flat $A$-module, we have

$$M \otimes_A B = \{y_B \in B \otimes_A B \mid (\sigma \otimes \text{id}_B)(y_B) \cdot (x \otimes 1) = y_B\}.$$ 

We translate it using the isomorphism $\phi: B \otimes_A B \to \prod_{g \in G} B$. If we define an action of $\sigma$ on $\prod_{g \in G} B$ by $(b_g)_{g \in G} \mapsto (\sigma(b_{\sigma^{-1}g}))_{g \in G}$, then one can check that the diagram

$$
\begin{array}{ccc}
B \otimes_A B & \xrightarrow{\phi} & \prod_{g \in G} B \\
\sigma \otimes \text{id}_B & & \downarrow \sigma \\
B \otimes_A B & \xrightarrow{\phi} & \prod_{g \in G} B
\end{array}
$$

is commutative. Since $\phi(x \otimes 1) = (x)_{g \in G}$, if we write $\phi(y_B) = (y_{B,g})_{g \in G}$, then the equality $(\sigma \otimes \text{id}_B)(y_B) \cdot (x \otimes 1) = y_B$ holds if and only if $\sigma(y_{B,\sigma^{-1}g})x = y_{B,g}$ for any $g \in G$. Such a $(y_{B,g})_{g \in G}$ is uniquely determined by $y_{B,1} \in B$. In fact, if we set $b = y_{B,1}$, then

$$y_{B,\sigma^n} = \sigma^k(b) \prod_{i=0}^{k-1} \sigma^i(x)$$

for $k > 0$. Here, we note that $y_{B,\sigma^n} = b$ by the 1-cocycle condition on $x$. Hence we deduce that $M_B \cong B$ as $B$-modules.

Since $f: A \to B$ is faithfully flat, we see that $M$ is a locally free $A$-module of rank one (see [13, Exposé VIII, Corollaire 1.11]). However, since Pic($A$) = 0, we conclude that $M$ is a free $A$-module of rank one. Choose an $A$-basis $y$ of $M$. Then $y_B = y \otimes 1$ is a $B$-basis of $M \otimes_A B$.

If we write $\phi(y_B) = (y_{B,g})_{g \in G}$, then $y_{B,1}$ is a $B$-basis of $B$, i.e., $y_{B,1} \in B^\times$. Since $y_{B,\sigma^n} = \sigma^k(y_{B,1}) \prod_{i=0}^{k-1} \sigma^i(x)$, we see that $(y_{B,g})_{g \in G} \in (\prod_{g \in G} B)^\times$, hence $y_B = y \otimes 1 \in (B \otimes_A B)^\times$.

Consider a homomorphism $T_y: B \to B$ defined by $T_y(b) = yb$. Then $T_y \otimes \text{id}_B: B \otimes_A B \to B \otimes_A B$ is an isomorphism. Since $f: A \to B$ is faithfully flat, we deduce that $T_y$ is already an isomorphism. Therefore, we conclude that $y \in B^\times$.

**Remark A.2.** By the same argument, one would furthermore generalize Proposition A.1 to the case where $G$ is a finite group.
A.2. A lemma. Now we prove the following lemma, which was used in the proof of Lemma 3.7.

Lemma A.3. For any semisimple element $t \in \mathbb{K}_{m,H}$, there exists $\tilde{s} = s \times \theta \in \mathbb{K}_m$ with $s \in \mathfrak{o}_E[t]$ such that $t = s\theta(s)$.

Proof. Write $B = \mathfrak{o}_E[t]$. Then $B$ is a commutative $\mathfrak{o}_E$-algebra with an involution $\sigma(x) = J_N\bar{x}J_N^{-1}$. Note that if $x \in \text{GL}_N(E)$, then $\theta(x) = \sigma(x)^{-1}$. Let $A$ be the $\mathfrak{o}_F$-subalgebra of $B$ consisting of $\sigma$-fixed elements in $B$. Since $A$ is a finitely generated $\mathfrak{o}_F$-module, and since $\mathfrak{o}_F$ is a complete local ring, hence henselian, we see that $A$ is the direct product of finitely many local rings (see [37, Chapitre I, §1, Définition 1 and §2, (3)]). Hence Pic$(A) = 0$ (see e.g., [46, Proposition 4.31.11]).

Note that the canonical homomorphism $A \otimes_{\mathfrak{o}_F} \mathfrak{o}_E \to B$ is isomorphism. Indeed, we have $\epsilon \in \mathfrak{o}_E^\times$ with $\bar{\epsilon} = -\epsilon$ such that $\{1, \epsilon\}$ is an $\mathfrak{o}_F$-basis of $\mathfrak{o}_E$. Then $\{1, \epsilon\}$ is also a basis of $B$ as an $A$-module. Therefore $B$ is a faithfully flat $A$-module. Moreover, since $E$ is unramified over $F$, the homomorphism

$$\phi: B \otimes_A B \to B \times B, \quad b_1 \otimes b_2 \mapsto (b_1b_2, b_1\sigma(b_2))$$

is isomorphism.

Therefore, we can apply Proposition A.1. Since $t = \theta(t) = \sigma(t)^{-1}$, we can find $s_0 \in B^\times$ such that $t = s_0/\sigma(s_0)$. Since $s_0$ and $s_0^{-1}$ are in $B = \mathfrak{o}_E[t]$ with $t \in \mathbb{K}_{m,H}$, both of them are of the form

$$\begin{pmatrix}
  n & 1 & n \\
  n & \mathfrak{o}_E & \mathfrak{o}_E^{-m} \\
  1 & \mathfrak{p}_E^m & \mathfrak{o}_E \\
  n & \mathfrak{p}_E^m & \mathfrak{p}_E^m & \mathfrak{o}_E
\end{pmatrix}.$$

Let $u, u' \in \mathfrak{o}_E$ be the $(n + 1, n + 1)$-entries of $s_0, s_0^{-1}$, respectively. Then we have $uu' \equiv uu' \equiv 1 \bmod \mathfrak{p}_E^m$ since $t \in \mathbb{K}_{m,H}$. Hence $u \in \mathfrak{o}_E^\times$, and $u \bmod \mathfrak{p}_E^m$ belongs to $(\mathfrak{o}_F/\mathfrak{p}_E^m)^\times$. Therefore, we can find $z \in \mathfrak{o}_F^\times$ such that $s = zs_0$ satisfies $t = s/\sigma(s) = s\theta(s)$ and $s \in \mathbb{K}_m$. \qed

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