Coarse Cohomology with Twisted Coefficients

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Abstract

To a coarse structure we associate a Grothendieck topology which is determined by
coarse covers. A coarse map between coarse spaces gives rise to a morphism of Grothendieck
topologies. This way we define sheaves and sheaf cohomology on coarse spaces. We obtain
that sheaf cohomology is a functor on the coarse category: if two coarse maps are close they
induce the same map in cohomology. There is a coarse version of a Mayer-Vietoris sequence
and for every inclusion of coarse spaces there is a coarse version of relative cohomology.
Cohomology with constant coefficients can be computed using the number of ends of a
coarse space.

Contents

0 Introduction .......................................................... 2
  0.1 Approach .......................................................... 2
  0.2 What is Coarse Geometry? ....................................... 3
  0.3 Background and related Theories ............................... 3
  0.4 Main Contributions ................................................ 6
  0.5 Outline ............................................................. 7

1 The Coarse Category ................................................ 8
  1.1 Coarse Spaces ...................................................... 8
  1.2 Coarse Maps ....................................................... 10

2 Coentourages ........................................................ 10
  2.1 Definition ......................................................... 10
  2.2 A Discussion/ Useful to know ................................... 12
  2.3 On Maps .......................................................... 14

3 Limits and Colimits .................................................. 18
  3.1 The Forgetful Functor .......................................... 18
  3.2 Limits .............................................................. 18
  3.3 Colimits ............................................................ 20

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In this paper we introduce a new invariant on coarse spaces called coarse sheaf cohomology. This cohomology theory is not based on the topology of open sets of a metric space but on coarse covers. Coarse covers are finite families of subsets of a metric space which satisfy a boundedness condition. The collection of coarse covers of a coarse space determine a Grothendieck topology, coarse maps serve as morphisms of Grothendieck topologies. This way we define sheaves on coarse spaces and coarse maps transfer sheaves between spaces.

Of particular interest to us are locally constant sheaves since cohomology with constant coefficient an abelian group $A$ is a functor on coarse spaces. Remarkable is the connection between the number of ends of a coarse space and its constant coefficient.

Without more advanced methods (which has been done in a follow-up research) cohomology of non-trivial examples is hard to compute. If $A$ is an infinite group, $\mathbb{Z}$ for example, then cohomology with coefficient $\mathbb{Z}$ on $\mathbb{Z}_+$, the positive integers is highly nontrivial [Kee94]. This problem does not occur if $A$ is finite. Finite coefficients produce interesting cohomology groups.

Our purpose is to pursue an algebraic geometry approach to coarse geometry. We present sheaf cohomology on coarse spaces and study coarse spaces by coarse cohomology with twisted coefficients. The method is based on the theory on Grothendieck topologies.

Note that sheaves on Grothendieck topologies and sheaf cohomology theory have been applied in a number of areas and have lead to many breakthroughs on previously unsolved problems. As stated in [McL07] one can understand a mathematical problem by

1. finding a mathematical world natural for the problem.
2. Expressing your problem cohomologically.
3. The cohomology of that world may solve your problem.

That way we can apply general theory on sheaf cohomology for tackling previously unsolved problems and studying notions which are quite well known.
0.2  What is Coarse Geometry?

The topic coarse geometry studies metric spaces from a large scale point of view. We want to examine the global structure of metric spaces. One way to approach this problem is by forgetting small scale structure. The coarse category consists of coarse spaces as objects and coarse maps modulo closeness as morphisms.

Now coarse maps preserve the coarse structure of a space in the coarse category. A coarse structure is made of entourages which are surroundings of the diagonal. For us metric spaces are the main objects of study. If \( X \) is a metric space a subset \( E \subseteq X^2 \) is an entourage if

\[
\sup_{(x,y) \in E} d(x, y) < \infty.
\]

The exact opposite of a coarse space and coarse geometry of metric spaces are uniform spaces and the uniform topology of a metric space. Like coarse spaces uniform spaces are defined via surroundings of the diagonal. Uniform entourages get smaller though while coarse entourages get larger the sharper the point of view.

Many algebraic properties of infinite finitely generated groups are hidden in the geometry of their Cayley graph. To a finitely generated group is associated the word length with regard to a generating set. Note that the metric of the group depends on the choice of generating set while the coarse structure associated to the word length metric is independent of the choice of generating set. Note that group homomorphisms are instances of coarsely uniform maps between groups and a group isomorphism is an instance of a coarse equivalence between groups. It is very fruitful to group theory to consider infinite finitely generated groups as coarse objects; these will be a source of examples for us.

Note the examples \( \mathbb{R}^n \) and \( \mathbb{Z}^n \) both are coarse spaces induced by a metric, for \( \mathbb{R}^n \) it is the euclidean metric and for \( \mathbb{Z}^n \) the metric is induced by the group \((\mathbb{Z}^n, +)\). Now \( \mathbb{Z}^n \) and \( \mathbb{R}^n \) look entirely different on small scale they are the same on large scale though. There is a coarse equivalence \( \mathbb{Z}^n \rightarrow \mathbb{R}^n \).

0.3  Background and related Theories

Nowadays it is hard to embrace all cohomology theory and other theories in the coarse category because of the diversity of the toolsets used.

A cohomology theory assigns an abelian group with a space, in a functorial manner. There are classical examples like Čech cohomology, simplicial homology, ... etc. which all fit in a general framework. The standard choice in the topological category are the Eilenberg-Steenrod axioms. They consist of 5 conditions which characterize singular cohomology on topological spaces. A generalized cohomology theory is a sequence of contravariant functors \((H^n)_n\) from the category of pairs of topological spaces \((X, A)\) to the category of abelian groups equipped with natural transformations

\[
\delta : H^n(A, \emptyset) \rightarrow H^{n+1}(X, A)
\]

for \( n \in \mathbb{N} \), such that

1. Homotopy: If \( f_1, f_2 : (X, A) \rightarrow (Y, B) \) are homotopic morphisms then they induce isomorphic maps in cohomology.

2. Excision: If \( (X, A) \) is a pair and \( U \subseteq A \) a subset such that \( \bar{U} \subseteq A^\circ \) then the inclusion

\[
i : (X \setminus U, A \setminus U) \rightarrow (X, A)
\]

induces an isomorphism in cohomology.
3. **Dimension:** The cohomology of the point is concentrated in degree 0.

4. **Additivity:** If $X = \bigsqcup_{\alpha} X_{\alpha}$ is a disjoint union of topological spaces then

$$H^n(X, \emptyset) = \prod_{\alpha} H^n(X_{\alpha}, \emptyset).$$

5. **Exactness:** Every pair of topological spaces $(X, A)$ induces a long exact sequence in cohomology:

$$\cdots \rightarrow H^n(X, A) \rightarrow H^n(X, \emptyset) \rightarrow H^n(A, \emptyset) \rightarrow H^{n+1}(X, A) \rightarrow \cdots.$$  

We are interested in theories that are functors on coarse spaces and coarse maps. Let us first recall the standard theories.

There are a number of cohomology theories in the coarse category we present two of them which are the most commonly used ones. We first present the most basic facts about *controlled operator $K$-theory* and Roe’s *coarse cohomology*.

We begin with a covariant invariant $K_*(C^*(\cdot))$ on proper metric spaces called *controlled $K$-theory*. Note that if a proper metric space $B$ is bounded then it is compact. Then [HR00, Lemma 6.4.1] shows

$$K_p(C^*(B)) = \begin{cases} \mathbb{Z} & p = 0 \\ 0 & p = 1. \end{cases}$$

There is a notion of flasque spaces for which controlled $K$-theory vanishes. An exemplary example is $\mathbb{Z}_+$; in [HR00, Lemma 6.4.2] it is shown that

$$K_*(C^*(\mathbb{Z}_+)) = 0.$$

The above is used in order to compute the controlled $K$-theory of $\mathbb{Z}^n$:

$$K_p(C^*(\mathbb{Z}^n)) = \begin{cases} \mathbb{Z} & p \equiv n \mod 2 \\ 0 & p \equiv n + 1 \mod 2 \end{cases}$$

which is [HR00, Theorem 6.4.10]. The notion of Mayer-Vietoris sequence is adapted to this setting: If there are two subspaces $A, B$ of a coarse space and if they satisfy the coarse excisive property which is introduced in [HRY93], then [HRY93, Lemmas 1,2; Section 5] combine to a Mayer-Vietoris sequence in controlled $K$-theory. There is a notion of homotopy for the coarse category which is established in [HR94]. Then [HR94, Theorem 5.1] proves that controlled $K$-Theory is a coarse homotopy invariant.

Let us now consider *coarse cohomology* $HX^*(:, A)$ which for $A$ an abelian group is a contravariant invariant on coarse spaces. The [Roe03, Example 5.13] notes that if a coarse space $B$ is bounded then

$$HX^q(B; A) = \begin{cases} A & q = 0 \\ 0 & \text{otherwise}. \end{cases}$$

Now the space $\mathbb{Z}^n$ reappears as an example in [Roe03, Example 5.20]:

$$HX^q(\mathbb{R}^n; \mathbb{R}) = \begin{cases} 0 & q \neq n \\ \mathbb{R} & q = n \end{cases}$$
Whereas another example is interesting: the \cite{Roe03} Example 5.21 shows that if $G$ is a finitely generated group then there is an isomorphism

$$HX^*(G; \mathbb{Z}) = H^*(G; \mathbb{Z}[G]).$$

Here the right side denotes group cohomology. In order to compute coarse cohomology there is one method: We denote by $H^*_c(X; A)$ the cohomology with compact supports of $X$ as a topological space. There is a character map

$$c : HX^q(X; A) \rightarrow H^*_c(X; A)$$

By \cite{Roe03} Lemma 5.17 the character map $c$ is injective if $X$ is a proper coarse space which is topologically path-connected. Now \cite{Roe03} Theorem 5.28 states: If $R$ is a commutative ring and $X$ is a uniformly contractible proper coarse space the character map for $R$-coefficients is an isomorphism.

In the course of this article we will design a new cohomology theory on coarse spaces. It has all the pros of the existing coarse cohomology theories and can be compared with them. The main purpose of this work is to design computational tools for the new theory and compute cohomology of a few exemplary examples.

Our main tool will be sheaf cohomology theory, which we now recall. If $X$ is a coarse space then $\text{Sheaf}(X)$ denotes the abelian category of sheaves of abelian groups on $X$. Note that $\text{Sheaf}(X)$ has enough injectives. Then the global sections functor

$$\mathcal{F} \mapsto \Gamma(X, \mathcal{F})$$

is a left exact functor between abelian categories $\text{Sheaf}(X)$ and $\mathbb{Ab}$, the category of abelian groups. The right derived functors are the sheaf cohomology functors. If $\mathcal{F}$ is a sheaf on $X$ then $\check{H}^*(X, \mathcal{F})$ denotes coarse cohomology with twisted coefficients with values in $\mathcal{F}$.

There are many ways to compute sheaf cohomology. One of them uses acyclic resolutions. Now every sheaf $\mathcal{F}$ on a coarse space $X$ has an injective resolution and injective sheaves are acyclic. Thus there exists a resolution

$$0 \rightarrow \mathcal{F} \rightarrow I_0 \rightarrow I_1 \rightarrow I_2 \rightarrow \cdots$$

with acyclics $I_q, q \geq 0$. Then the sheaf cohomology groups $\check{H}^q(X, \mathcal{F})$ are the cohomology groups of the following complex of abelian groups

$$0 \rightarrow I_0(X) \rightarrow I_1(X) \rightarrow I_2(X) \rightarrow \cdots.$$

We can also compute sheaf cohomology by means of Čech cohomology. If $(U_i)_{i \in I}$ is a coarse cover of a subset $U \subseteq X$ and $\mathcal{F}$ an abelian presheaf on $X$ then the group of $q$-cochains is

$$C^q([U_i \rightarrow U]_i, \mathcal{F}) = \prod_{(i_0, ..., i_q) \in I^{q+1}} \mathcal{F}(U_{i_0} \cap \cdots \cap U_{i_q})$$

The coboundary operator $d^q : C^q([U_i \rightarrow U]_i, \mathcal{F}) \rightarrow C^{q+1}([U_i \rightarrow U]_i, \mathcal{F})$ is defined by

$$(d^qs)_{i_0, ..., i_{q+1}} = \sum_{\nu=0}^{q+1} (-1)^\nu s_{i_0, ..., \hat{i}_\nu, ..., i_{q+1}}|_{i_0, ..., i_{q+1}}$$

Then $C^*([U_i \rightarrow U]_i, \mathcal{F})$ is a complex and $\check{H}^*([U_i \rightarrow U]_i, \mathcal{F})$ is defined to be its cohomology. Now sheaf cohomology can be computed:

$$\check{H}^q(U, \mathcal{F}) = \lim_{\longleftarrow \{U_i \rightarrow U\}} \check{H}^q([U_i \rightarrow U]_i, \mathcal{F}).$$
In good circumstances we can compute sheaf cohomology using an acyclic cover. If \((U_i)_{i \in I}\) is a coarse cover of a coarse space \(X\) and \(F\) a sheaf on \(X\) and if for every nonempty \(\{i_1, \ldots, i_n\} \subseteq I\), \(q > 0\) we have that
\[
\check{H}^q(U_{i_1} \cap \cdots \cap U_{i_n}, F) = 0
\]
then already
\[
\check{H}^q(X, F) = \check{H}^q(\{U_i \to U\}_{i \in I}, F)
\]
for every \(q \geq 0\).

Note that homotopy also plays an important part when computing sheaf cohomology.

0.4 Main Contributions

The general idea of this work is to transfer toolsets from other topics like algebraic topology and algebraic geometry and use them in the coarse category. The cohomology theory we are aiming at has its roots in algebraic geometry. First let us note a few aspects which distinguishes the new theory.

There has been much effort in establishing axioms for cohomology theories in the coarse category. In [BE17] has been proposed a choice of axioms for coarse cohomology theories. Now we will test our theory against the Eilenberg-Steenrod axiom system. The new theory satisfies similar properties which are going to be discussed in the following list

1. **Homotopy:** The relation close on coarse maps can be regarded as a notion of homotopy on the coarse category. Sheaf cohomology on coarse spaces is an invariant modulo close.

2. **Excision:** Subsection 4.5 presents local cohomology in the coarse category.

3. **Dimension:** The space \(\mathbb{Z}_+\) can be understood as the coarse equivalent of a point. It is however not acyclic for general coefficients. If the spaces \(\mathbb{Z}^n\) are understood as representatives for dimension then coarse cohomology with twisted coefficients sees dimension.

4. **Additivity:** Sheaf cohomology sees coproducts, see subsection 5.2.

5. **Exactness:** Subsection 4.4 presents a coarse version of the Mayer-Vietoris sequence.

Now why are there so many powerful results is one of the most natural questions we can ask. The main reason is, that typically sheaf cohomology is a powerful tool in a number of areas. Examples are de Rham cohomology in differential geometry, singular cohomology for nice enough spaces in algebraic topology and étale cohomology in algebraic geometry.

A Grothendieck topology is the least amount of data needed to define sheaves and sheaf cohomology. And that is where we start. We define the Grothendieck topology of coarse covers associated to a coarse space in Definition 58. Then we discover in Lemma 62 that coarse maps give rise to a morphism of topologies. That is all the information that we need to use the powerful machinery of sheaf cohomology.

Then we obtain the first important result: if two coarse maps are close then they induce isomorphic maps in cohomology with twisted coefficients. This is Theorem 72.

**Theorem A.** Coarse cohomology with twisted coefficients is a functor on coarse spaces and coarse maps modulo closeness.

Thus coarsely equivalent coarse spaces have the same cohomology.
The coarse equivalent of a trivial space is either the empty set or a bounded space or both. If $B$ is a bounded space then for every coefficient $F$ on $B$: 
\[ \check{H}^*(B, F) = 0 \]
which is a result of Example 64.

Some computational tools we recognize from algebraic topology can be adopted for our setting. The Chapter 4.4 presents a coarse version of Mayer-Vietoris:

**Theorem B. (Mayer-Vietoris)** Let $X$ be a coarse space and $A, B$ two subsets that coarsely cover $X$. Then there is an exact sequence in cohomology
\[
\cdots \to \check{H}^{i-1}(A \cap B, F) \to \check{H}^i(A \cup B, F) \to \check{H}^i(A, F) \times \check{H}^i(B, F) \to \check{H}^i(A \cap B, F) \to \cdots
\]
for every sheaf $F$ on $X$.

The Chapter 4.5 discusses relative cohomology in the coarse category.

**Theorem C.** Let $Z \subseteq X$ be a subspace of a coarse space and let $Y = X \setminus Z$. Then there is a long exact sequence
\[
0 \to \check{H}^0(U, \Gamma_Z(F)) \to \check{H}^0(U, F) \to \check{H}^0(U, F|_Y) \to \check{H}^1(U, \Gamma_Z(F)) \to \cdots
\]
for every subset $U \subseteq X$ and every sheaf $F$ on $X$.

In Chapter 5 constant coefficients on coarse spaces are introduced. If $A$ is an abelian group and $X$ is a coproduct of $n$ unbounded coarse spaces then its number of ends is at least $n$ and $A(X) \geq A^n$. In fact if $X$ does not have finitely many ends then $A(X) = A^\infty$. This is discussed in Theorem 86.

First let us note that $\mathbb{Z}^+$ is imperfect as a coarse version of a point as it is not a final object and does not have trivial cohomology. While $\check{H}^q(\mathbb{Z}^+, A) = 0$ for $q \geq 2$ and every constant coefficient $A$, the cohomology in degree 1,
\[ \check{H}^1(\mathbb{Z}^+, \mathbb{Z}) \neq 0 \]
is nontrivial for $\mathbb{Z}$-coefficients. By Lemma 87 every unbounded subset $U \subseteq \mathbb{Z}^+$ has either infinitely many ends or the inclusion $U \to \mathbb{Z}^+$ is coarsely surjective.

Note if $X$ is any metric space we can find a sequence $(x_i)_i \subseteq X$ with $d(x_i, x_j) > i$ for $j < i$. This space is discrete which means every entourage on $(x_i)_i$ has finitely many offdiagonal points. Its Higson compactification is homeomorphic to the Stone-Čech compactification of the natural numbers.

This makes it extremely hard to compute cohomology of specific examples. The cohomology of a bounded metric space is trivial of course since every sheaf on it vanishes. If $X$ is discrete or $\text{asdim}(X) = 0$ then the cohomology of $X$ is acyclic.

### 0.5 Outline

Now we indicate an outline of the chapters that are going to appear.

- Chapters 1, 2 serve as an introduction.
- In Chapter 3 we construct new spaces out of old ones.
- Chapter 4 studies the coarse cohomology theory with twisted coefficients.
- Chapter 5 presents cohomology with constant coefficients.
1 The Coarse Category

The following chapter introduces coarse spaces and coarse maps between coarse spaces. It has been kept as short as possible, giving only the most basic definitions needed for understanding this paper. All this information can be found in [Roe03, chapter 2].

1.1 Coarse Spaces

Definition 1. (inverse, product) Let $X$ be a set and let $E$ be a subset of $X^2$. Then the inverse $E^{-1}$ is defined by

$$E^{-1} = \{(y,x) | (x,y) \in E\}.$$ 

A set $E$ is called symmetric if $E = E^{-1}$.

For two subsets $E_1, E_2 \subseteq X^2$ the product $E_1 \circ E_2$ is given by

$$E_1 \circ E_2 = \{(x,z) | \exists y : (x,y) \in E_1, (y,z) \in E_2\}.$$ 

Definition 2. (coarse structure) Let $X$ be a set. A coarse structure on $X$ is a collection of subsets $E \subseteq X^2$ which will be referred as entourages which follow the following axioms:

1. the diagonal $\Delta_X = \{(x,x) | x \in X\}$ is an entourage;
2. if $E$ is an entourage and $F \subseteq E$ a subset then $F$ is an entourage;
3. if $F, E$ are entourages then $F \cup E$ is an entourage;
4. if $E$ is an entourage then the inverse $E^{-1}$ is an entourage;
5. if $E_1, E_2$ are entourages then their product $E_1 \circ E_2$ is an entourage.

The set $X$ together with the coarse structure on $X$ will be called a coarse space.

Definition 3. (connected) A coarse space $X$ is connected if

6. for every points $x, y \in X$ the set $\{(x,y)\} \subseteq X^2$ is an entourage.

In the course of this paper all coarse spaces are assumed to be connected unless said otherwise.

Definition 4. (bounded set) Let $X$ be a coarse space. A subset $B \subseteq X$ is called bounded if $B^2$ is an entourage.

Definition 5. Let $X$ be a set and let $K \subseteq X$ and $E \subseteq X^2$ be subsets. One writes

$$E[K] = \{x | \exists y \in K : (x,y) \in E\}.$$ 

In case $K$ is just a set containing one point $p$, we write $E_p$ for $E[\{p\}]$ (called a section).

Lemma 6. Let $X$ be a coarse space.

- If $B_1, B_2 \subseteq X$ are bounded then $B_1 \times B_2$ is an entourage and $B_1 \cup B_2$ is bounded.
- For every bounded subset $B \subseteq X$ and entourage $E$ the set $E[B]$ is bounded.
Proof. • Fix two points \( b_1 \in B_1 \) and \( b_2 \in B_2 \) then \( (b_1, b_2) \) is an entourage in \( X \). Thus
\[
B_1^2 \circ (b_1, b_2) \circ B_2^2 = B_1 \times B_2
\]
is an entourage. Now
\[
(B_1 \cup B_2)^2 = B_1^2 \cup B_1 \times B_2 \cup B_2 \times B_1 \cup B_2^2
\]
is an entourage, thus \( B_1 \cup B_2 \) is bounded.
• Note that
\[
E \circ B^2 = E[B] \times B
\]
is an entourage.

Remark 7. Note that an intersection of coarse structures is again a coarse structure.

• If \( X \) is a set and \( \delta \) a collection of subsets of \( X^2 \) then the smallest coarse structure \( \varepsilon \) that contains each element of \( \delta \) is called the coarse structure that is generated by \( \delta \). Then \( \delta \) is called a subbase for \( \varepsilon \).

• If \( \varepsilon \) is a coarse structure and \( \varepsilon' \subseteq \varepsilon \) a subset such that \( E \in \varepsilon \) implies there is some \( E' \in \varepsilon' \) with \( E \subseteq E' \) then \( \varepsilon' \) is called a base for \( \varepsilon \).

Example 8. If \( X \) is a set there are two trivial coarse structures on \( X \):

1. the discrete coarse structure consists of subset of the diagonal and finitely many off-diagonal points.

2. the maximal coarse structure is generated by \( X^2 \). Note that in this case each subset of \( X \) and in particular \( X \) itself is bounded.

Example 9. If \( X \) is a metric space with metric \( d \) then the bounded coarse structure of \( X \) consists of those subsets \( E \subseteq X^2 \) for which
\[
\sup_{(x, y) \in E} d(x, y) < \infty.
\]
A coarse space \( X \) is called metrizable if there is a metric \( d \) that can be defined on it such that \( X \) carries the bounded coarse structure associated to \( d \). Note that by [Roe03, Theorem 2.55] a coarse space is metrizable if and only if it has a countable base.

Metric spaces which are of particular interest in the topic of coarse geometry are Riemannian manifolds and finitely generated groups. Let \( G \) be a finitely generated group equipped with a finite generating set \( S \). Then the word length \( l : G \to \mathbb{N} \) according to \( S \) assigns an element \( g \in G \) with the minimal length of a word written in the alphabet \( S \cup S^{-1} \) that represents \( g \). Then the map
\[
d : G \times G \to G \\
(g, h) \mapsto l(g^{-1}h)
\]
is a metric on \( G \). Note the metric depends on the generating set, the coarse structure associated to \( d \) does not though.
Example 10. If $X$ is a paracompact and locally compact Hausdorff space and $\tilde{X}$ a compactification of $X$ with boundary $\partial X$ then the topological coarse structure associated to the given compactification consists of subsets $E \subseteq X^2$ such that
\[
\partial E \cap \partial X^2 \setminus \Delta_{\partial X} = \emptyset.
\]
If the compactification is second countable then by [Roe03, Example 2.53] the topological coarse structure on $X$ is not metrizable.

1.2 Coarse Maps

Definition 11. (close) Let $S$ be a set and let $X$ a be coarse space. Two maps $f, g : S \to X$ are called close if
\[
\{(f(s), g(s)) | s \in S\} \subseteq X^2
\]
is an entourage.

Definition 12. (maps) Let $f : X \to Y$ be a map between coarse spaces. Then $f$ is called
- coarsely proper if for every bounded set $B$ in $Y$ the inverse image $f^{-1}(B)$ is bounded in $X$;
- coarsely uniform if every entourage $E$ of $X$ is mapped by $f^\times_2 = f \times f : X^2 \to Y^2$ to an entourage $f^\times_2(E)$ of $Y$;
- a coarse map if it is both coarsely proper and coarsely uniform;
- a coarse embedding if $f$ is coarsely uniform and for every entourage $F \subseteq Y^2$ the inverse image $(f^\times_2)^{-1}(F)$ is an entourage.

Definition 13. (coarsely equivalent)
- A coarse map $f : X \to Y$ between coarse spaces is a coarse equivalence if there is a coarse map $g : Y \to X$ such that $f \circ g : Y \to Y$ is close to the identity on $Y$ and $g \circ f : X \to X$ is close to the identity on $X$.
- two coarse spaces $X, Y$ are coarsely equivalent if there is a coarse equivalence $f : X \to Y$.

Definition 14. We denote by $\text{Coarse}$ the category with objects coarse spaces and morphisms coarse maps modulo close. Then coarse equivalences are the isomorphisms in the coarse category.

2 Coentourages

In this chapter coentourages are introduced. We study the dual characteristics of coentourages to entourages.

2.1 Definition

This is a special case of [Roe03, Definition 5.3, p. 71]:

Definition 15. Let $X$ be a coarse space. A subset $C \subseteq X^2$ is called a coentourage if for every entourage $E$ there is a bounded set $B$ such that
\[
C \cap E \subseteq B^2.
\]
The set of coentourages in $X$ is called the cocoarse structure of $X$. 
Lemma 16. The following properties hold:

1. Finite unions of coentourages are coentourages.
2. Subsets of coentourages are coentourages.
3. If \( f : X \to Y \) is a coarse map between coarse spaces then for every coentourage \( D \subseteq Y^2 \) the set \( (f^\times 2)^{-1}(D) \) is a coentourage.

Proof. 1. Let \( C_1, C_2 \) be coentourages. Then for every entourage \( E \) there are bounded sets \( B_1, B_2 \) such that
\[
(C_1 \cup C_2) \cap E = C_1 \cap E \cup C_2 \cap E \\
\subseteq B_1 \times B_1 \cup B_2 \times B_2 \\
\subseteq (B_1 \cup B_2)^2.
\]
Now \( B_1 \cup B_2 \) is bounded because \( X \) is connected.

2. Let \( C \) be a coentourage and \( D \subseteq C \) a subset. Then for every entourage \( E \) there is some bounded set \( B \) such that
\[
D \cap E \subseteq C \cap E \\
\subseteq B^2.
\]

3. This is actually a special case of [Roe03, Lemma 5.4]. For the convenience of the reader we include the proof anyway.

Let \( E \) be an entourage in \( X \). Then there is some bounded set \( B \subseteq Y \) such that
\[
f^2((f^\times 2)^{-1}(D) \cap E) \subseteq D \cap f^2(E) \\
\subseteq B^2.
\]
But then
\[
(f^\times 2)^{-1}(D) \cap E \subseteq (f^\times 2)^{-1} \circ f^\times 2((f^\times 2)^{-1}(D) \cap E) \\
\subseteq (f^\times 2)^{-1}(B^2) \\
= f^{-1}(B)^2.
\]

Example 17. If \( G \) is an infinite countable group then there is a canonical coarse structure on \( G \): A subset \( E \subseteq G^2 \) is an entourage if the set
\[
\{ g^{-1}h : (g,h) \in E \}
\]
is finite. Note for finitely generated group this definition agrees with Example [9]. If \( U, V \subseteq G \) are two subsets of \( G \) then \( U \times V \) is a coentourage if \( U \cap V g \) is finite for every \( g \in G \).

Example 18. In the coarse space \( Z \) one can see three examples:

- the even quadrants are a coentourage: \( \{(x,y) : xy < 0\} \).
- For \( n \in \mathbb{Z} \) the set perpendicular to the diagonal with foot \( (n,n) \) is a coentourage: \( \{(n-x,n+x) : x \in \mathbb{Z}\} \).
Here is another example: \( \{(x, 2x) : x \in \mathbb{Z}\} \) is a coentourage.

**Example 19.** Look at the infinite dihedral group which is defined by

\[ D_\infty = \langle a, b : a^2 = 1, b^2 = 1 \rangle. \]

In \( D_\infty \) the set

\[ \{(ab)^n, (ab)^n a : n \in \mathbb{N}\} \times \{(ba)^n, (ba)^n b : n \in \mathbb{N}\} \]

is a coentourage.

### 2.2 A Discussion/ Useful to know

**Lemma 20.** Let \( X \) be a coarse space. Then for a subset \( B \subseteq X \) the set \( B^2 \) is a coentourage if and only if \( B \) is bounded.

**Proof.** If \( B \) is bounded then it is easy to see that \( B^2 \) is a coentourage. Conversely suppose \( B^2 \) is a coentourage. Then

\[ \Delta_X \cap B^2 \subseteq B^2 \]

and \( B^2 \) is the smallest squared subset of \( X^2 \) which contains

\[ \{(b, b) : b \in B\} \]

which is \( \Delta_X \cap B^2 \). Thus \( B \) is bounded.

**Definition 21.** (dual structure) If \( X \) is a coarse space let \( \varepsilon \) and \( \gamma \) be collections of subsets of \( X^2 \). Denote by \( \beta \) the set of bounded sets. We say that \( \varepsilon \) detects \( \gamma \) if

1. for every \( D \notin \gamma \) there is some \( E \in \varepsilon \) such that \( D \cap E \not\subseteq B \times B \) for every \( B \in \beta \).
2. a subset \( D \subseteq X \times X \) is contained in \( \gamma \) if for every \( E \in \varepsilon \) there is some \( B \in \beta \) such that \( D \cap E \subseteq B \times B \).

Then we say that \( \varepsilon \) is dual to \( \gamma \) if \( \varepsilon \) detects \( \gamma \) and \( \gamma \) detects \( \varepsilon \). By definition the collection of entourages detects the collection of coentourages.

**Proposition 22.** Let \( X \) be a coarse space with the bounded coarse structure of a metric space then the coarse structure of \( X \) is dual to the cocoarse structure of \( X \).

**Proof.** Let \( F \subseteq X^2 \) be a subset which is not an entourage. Then for every entourage there is a point in \( F \) that is not in \( E \). Now we have a countable basis for the coarse structure:

\[ E_1, E_2, \ldots, E_n, \ldots \]

ordered by inclusion. Then there is also a sequence \( (x_i, y_i) \subseteq X^2 \) with \( (x_i, y_i) \notin E_i \) and \( (x_i, y_i) \in F \). Denote this set of points by \( f \). Then for every \( i \) the set

\[ E_i \cap f \]

is a finite set of points, thus \( f \) is a coentourage. But \( F \cap f = f \) is not an entourage, specifically it is not contained in \( B^2 \) if \( B \) is bounded.

---

\[ ^1 \text{In what follows coarse spaces with the bounded coarse structure of a metric space will be referred to as metric spaces.} \]
Proposition 23. Let $X$ be a paracompact and locally compact Hausdorff space. Let $\bar{X}$ be a compactification of $X$ and equip $X$ with the topological coarse structure associated to the given compactification. Then

1. a subset $C \subseteq X^2$ is a coentourage if $\bar{C} \cap \Delta_{\partial X}$ is empty.
2. if $U, V$ are subsets of $X$ then $U \times V$ is a coentourage if $\partial U \cap \partial V = \emptyset$.
3. the coarse structure of $X$ is dual to the cocoarse structure of $X$.

Proof. easy.

Lemma 24. Let $X$ be a coarse space. If $C \subseteq X^2$ is a coentourage and $E \subseteq X^2$ an entourage then $C \circ E$ and $E \circ C$ are coentourages.

Proof. Let $F \subseteq X^2$ be any entourage. Without loss of generality $E$ is symmetric and contains the diagonal. Now $C$ being a coentourage implies that there is some bounded set $B \subseteq X$ such that

$$C \cap E^{-1} \circ F \subseteq B^2$$

Then

$$E \circ C \cap F \subseteq E \circ (C \cap E^{-1} \circ F) \subseteq E \circ B^2 \subseteq (E[B] \cup B)^2$$

If $X$ is a set a collection $\beta$ of subsets of $X$ is called a bornology if

- $X = \bigcup_{B \in \beta} B$;
- if $A \in \beta$ and $A' \subseteq A$ then $A' \in \beta$.
- if $A, B \in \beta$ then $A \cup B \in \beta$.

If $X$ is a metric space then the set of bounded subsets of $X$ are a bornology. We will use this property in Theorem 25.

Now we are going to characterize coentourages axiomatically.

Theorem 25. If $X$ is a set let $\gamma$ be a collection of subsets of $X^2$ such that

1. $\gamma$ is closed under taking subsets, finite unions and inverses;
2. we say a subset $B \subseteq X$ is bounded if $B \times X \in \gamma$ and require $X = \bigcup_{B \in \beta} B$;
3. for every $C \in \gamma$ there is some bounded set $B \subseteq X$ such that $C \cap \Delta_X \subseteq B^2$;
4. If $E$ is detected by $\gamma$ and $C \in \gamma$ then $E \circ C \in \gamma$.
Then $\gamma$ detects a coarse structure.

Proof. Denote by $\beta$ the collection of bounded sets of $X$. Note that by points 1 and 2 the system $\beta$ is a bornology. Now we show that $\gamma$ detects a coarse structure by checking the axioms in Definition 2:

1. Point 3 guarantees that the diagonal is an entourage.
2. That is because $\beta$ is a bornology.
3. Same.
4. By point 1 the inverse of an entourage is an entourage.
5. Suppose $E, F \subseteq X^2$ are detected by $\gamma$. Without loss of generality $E$ is symmetric and contains the diagonal. Then there is some bounded set $B$ such that

\[ F \cap E^{-1} \circ C \subseteq B^2. \]

But then

\[
E \circ F \cap C \subseteq E \circ (F \cap E^{-1} \circ C) \\
\subseteq E \circ B^2 \\
\subseteq (E|B \cup B)^2
\]

and that is bounded because of the first point.

6. This works because of point 2.

\[\square\]

**Definition 26.** (coarsely disjoint) If $A, B \subseteq X$ are subsets of a coarse space then $A$ is called coarsely disjoint to $B$ if

\[ A \times B \subseteq X^2 \]

is a coentourage. Being coarsely disjoint is a relation on subsets of $X$.

### 2.3 On Maps

Note that in this chapter every coarse space is assumed to have the property that the coarse structure is dual to the cocoarse structure.

**Lemma 27.** Two coarse maps $f, g : X \to Y$ are close if and only if for every coentourage $D \subseteq Y^2$ the set $(f \times g)^{-1}(D)$ is a coentourage.

Proof. Denote by $\beta$ the collection of bounded sets. Suppose $f, g$ are close. Let $C \subseteq Y^2$ be a coentourage and $E \subseteq X^2$ an entourage. Set

\[ S = (f \times g)^{-1}(C) \cap E. \]

Then there is some bounded set $B$ such that

\[
(f \times g)(S) = (f \times g) \circ ((f \times g)^{-1}(C) \cap E) \\
\subseteq (f \times g) \circ (f \times g)^{-1}(C) \cap (f \times g)(E) \\
\subseteq C \cap (f \times g)(E) \\
\subseteq B^2.
\]
But \( f \) and \( g \) are coarsely proper thus

\[
S \subseteq (f^{-1} \times g^{-1}) \circ (f \times g)(S) \\
\subseteq f^{-1}(B) \times g^{-1}(B)
\]

is in \( \beta^2 \).

Now for the reverse direction: Let \( C \subseteq Y^2 \) be a coentourage. There is some bounded set \( B \subseteq X^2 \) such that

\[
\Delta_X \cap (f \times g)^{-1}(C) \subseteq B^2.
\]

Then

\[
(f \times g)(\Delta_X) \cap C = (f \times g)(\Delta_X) \cap (f \times g) \circ (f \times g)^{-1}(C) \\
= (f \times g)(\Delta_X \cap (f \times g)^{-1}(C)) \\
\subseteq (f \times g)(B^2).
\]

But \( f, g \) are coarsely uniform thus \( (f \times g)(B^2) \in \beta^2 \).

\[\text{Proposition 28.} \quad \text{A map } f : X \to Y \text{ between coarse spaces is coarse if and only if} \]

- for every bounded set \( B \subseteq X \) the image \( f(B) \) is bounded in \( Y \)
- and for every coentourage \( C \subseteq Y^2 \) the reverse image \( (f \times g)^{-1}(C) \) is a coentourage in \( X \)

\[\text{Proof.} \quad \text{Suppose } f \text{ is coarse. By Lemma } [16] \text{ point 3 the second point holds and by coarsely uniformness the first point holds.} \]

Suppose the above holds. Let \( E \subseteq X^2 \) be an entourage. For every coentourage \( D \subseteq Y^2 \) there is some bounded set \( B \) such that

\[
E \cap (f \times g)^{-1}(D) \subseteq B^2.
\]

Then

\[
f^{\times 2}(E) \cap D = f^{\times 2}(E) \cap f^{\times 2} \circ (f^{\times 2})^{-1}(D) \\
= f^{\times 2}(E \cap (f^{\times 2})^{-1}(D)) \\
\subseteq f(B)^2.
\]

Because of point 1 we have \( f^{\times 2}(B) \in \beta \). By point 2 the reverse image of every bounded set is bounded.

\[\text{Definition 29.} \quad \text{A map } f : X \to Y \text{ between coarse spaces is called coarsely surjective if one of the following equivalent conditions applies:} \]

- There is an entourage \( E \subseteq Y^2 \) such that \( E[\text{im } f] = Y \).
- there is a map \( r : Y \to \text{im } f \) such that

\[
\{(y, r(y)) : y \in Y\}
\]

is an entourage in \( Y \).
- The inclusion \( \text{im } f \to Y \) is a coarse equivalence.
We will refer to the above map \( r \) as the retract of \( Y \) to \( \text{im} \ f \). Note that it is a coarse equivalence.

**Lemma 30.** *Every coarse equivalence is coarsely surjective.*

*Proof.* Let \( f : X \to Y \) be a coarse equivalence and \( g : Y \to X \) its inverse. Then \( f \circ g : Y \to \text{im} \ f \) is the retract of Definition 29.

**Lemma 31.** *Coarsely surjective coarse maps are epimorphisms in the category of coarse spaces and coarse maps modulo close.*

*Proof.* Suppose \( f : X \to Y \) is a coarsely surjective coarse map between coarse spaces. Then there is an entourage \( E \subseteq Y^2 \) such that \( E[\text{im} \ f] = Y \). We show \( f \) is an epimorphism. Let \( g_1, g_2 : Y \to Z \) be two coarse maps to a coarse space such that \( g_1 \circ f, g_2 \circ f \) are close. Then the set \( H := g_1 \circ f \times g_2 \circ f(\Delta_X) \) is an entourage. Then
\[
g_1 \times g_2(\Delta_Y) \subseteq g_1^x(E) \circ H \circ g_2^x(E^{-1})
\]
is an entourage. Thus \( g_1, g_2 \) are close.

**Definition 32.** A map \( f : X \to Y \) between coarse spaces is called *coarsely injective*\(^2\) if for every entourage \( E \subseteq Y^2 \) the set \( f^{-1}(E) \) is an entourage.

**Remark 33.** Let \( f : X \to Y \) be a map between coarse spaces. If \( f \) is coarsely injective and maps bounded sets to bounded sets then \( f^x(C) \) is a coentourage for every coentourage \( C \subseteq X^2 \).

If on the other hand \( f^x \) maps coentourages to coentourages, the space \( X \) is metric and \( f \) is coarsely proper then \( f \) is coarsely injective.

*Proof.* Suppose \( f \) is coarsely injective and maps bounded sets to bounded sets. Let \( C \subseteq X^2 \) be a coentourage and \( E \subseteq Y^2 \) be an entourage. Then there exists a bounded set \( B \subseteq X \) such that \( C \cap (f^x)^{-1}(E) \subseteq B^2 \). Then
\[
f^x(B^2) \supseteq f^x(C \cap (f^x)^{-1}(E)) = f^x(C) \cap E.
\]
Since \( E \) was an arbitrary entourage this implies that \( f^x(C) \) is a coentourage in \( Y \).

Now suppose \( f^x \) maps coentourages to coentourages, \( X \) is a metric space and \( f \) is coarsely proper. Let \( E \subseteq Y^2 \) be an entourage and \( C \subseteq X^2 \) be a coentourage. Then there exists a bounded set \( B \subseteq Y \) such that \( f^x(C) \cap E \subseteq B^2 \). Then
\[
(f^x)^{-1}(B^2) \supseteq (f^x)^{-1}(f^x(C) \cap E) = (f^x)^{-1} \circ f^x(C) \cap (f^x)^{-1}(E) \supseteq C \cap (f^x)^{-1}(E).
\]
Since \( C \) was an arbitrary coentourage the set \( (f^x)^{-1}(E) \) is an entourage by Proposition 22.

**Lemma 34.** *Let \( f : X \to Y \) be a coarse equivalence. Then \( f \) is coarsely injective.*

\(^2\)Note that every coarsely injective coarse map is called a coarse embedding. Although term 'coarse embedding' is in general use and describes the notion more appropriately we will use the former term 'coarsely injective' because adjectives are easier to handle.
Proof. Let \( g : Y \to X \) be a coarse inverse of \( f \). Then there is an entourage
\[
F = \{(g \circ f(x), x) : x \in X\}
\]
in \( X \). But then \( g \circ f \) is coarsely injective because for every coentourage \( C \subseteq X^2 \) we have
\[
g \circ f^x(C) \subseteq F \circ C \circ F^{-1}
\]
and \( F \circ C \circ F^{-1} \) is again a coentourage by Lemma 24. But
\[
f^x(C) \subseteq (g^x)^{-1} \circ g^x \circ f^x(C)
\]
is a coentourage, thus \( f \) is coarsely injective.

**Lemma 35.** Coarsely injective coarse maps are monomorphisms in the category of coarse spaces and coarse maps modulo closeness.

Proof. Suppose \( f : X \to Y \) is a coarsely injective coarse map between coarse spaces. We show \( f \) is a monomorphism. Let \( g_1, g_2 : Z \to X \) be two coarse maps such that \( f \circ g_1, f \circ g_2 : Z \to Y \) are close. Then
\[
H := f \circ g_1 \times f \circ g_2(\Delta Z)
\]
is an entourage. Now
\[
g_1 \times g_2(\Delta Z) \subseteq (f^x)^{-1}(H)
\]
is an entourage. Thus \( g_1, g_2 \) are close.

**Remark 36.** Every coarse map can be factored into an epimorphism followed by a monomorphism.

**Proposition 37.** If a coarse map \( f : X \to Y \) is coarsely surjective and coarsely injective then \( f \) is a coarse equivalence.

Proof. We just need to construct the coarse inverse. Note that the map \( r : Y \to \text{im } f \) from the second point of Definition 29 is a coarse equivalence which is surjective. Without loss of generality we can replace \( f \) by \( \hat{f} = r \circ f \). Now define \( g : \text{im } f \to X \) by mapping \( y \in \text{im } f \) to some point in \( \hat{f}^{-1}(y) \) where the choice is not important. Now we show:

1. \( g \) is a coarse map: Let \( E \subseteq (\text{im } f)^2 \) be an entourage. Then
\[
g^x(E) \subseteq (f^x)^{-1}(E)
\]
is an entourage. And if \( B \subseteq X \) is bounded then
\[
g^{-1}(B) \subseteq f(B)
\]
is bounded.

2. \( \hat{f} \circ g = \text{id}_{\text{im } f} \)

3. \( g \) is coarsely injective: Let \( D \subseteq (\text{im } f)^2 \) be a coentourage. Then
\[
g^x(D) \subseteq (f^x)^{-1}(D)
\]
is a coentourage.

4. \( g \circ \hat{f} \sim \text{id}_X \): we have \( g \circ \hat{f} : X \to \text{im } g \) is coarsely injective and thus the retract of Definition 29 with coarse inverse the inclusion \( i : \text{im } g \to X \). But
\[
g \circ \hat{f} \circ i = \text{id}_{\text{im } g}.
\]
3 Limits and Colimits

The category Top of topological spaces is both complete and cocomplete. In fact the forgetful functor Top → Sets preserves all limits and colimits that is because it has both a right and left adjoint. We do something similar for coarse spaces.

Note that the following notions generalize the existing notions of product and disjoint union of coarse spaces.

3.1 The Forgetful Functor

Definition 38. Denote the category of connected coarse spaces and coarsely uniform maps between them by DCoarse.

Theorem 39. The forgetful functor η : DCoarse → Sets preserves all limits and colimits.

Proof. There is a functor δ : Sets → DCoarse that sends a set X to the coarse space X with the discrete coarse structure. Then every map of sets induces a coarsely uniform map.

There is a functor α : Sets → DCoarse which sends a set X to the coarse space X with the maximal coarse structure. Again every map of sets induces a coarsely uniform map.

Let X be a set and Y a coarse space. Then

\[ \text{Hom}_{\text{Sets}}(X, \eta Y) = \text{Hom}_{\text{DCoarse}}(\delta X, Y) \]

and

\[ \text{Hom}_{\text{Sets}}(\eta Y, X) = \text{Hom}_{\text{DCoarse}}(Y, \alpha X) \]

Thus the forgetful functor is right adjoint to δ and left adjoint to α.

An application of the [Wei94, Adjoints and Limits Theorem 2.6.10] gives the result. □

3.2 Limits

The following definition is a generalization of [Gra06, Definition 1.21]:

Definition 40. Let X be a set and f_i : X → Y_i a family of maps to coarse spaces. The pullback coarse structure of (f_i)_i on X is generated by \( \bigcap_i (f_i^{\times 2})^{-1}(E_i) \) for \( E_i \subseteq Y_i \) an entourage for every i. That is, a subset \( E \subseteq X^2 \) is an entourage if for every i the set \( f_i^{\times 2}(E) \) is an entourage in \( Y_i \).

Lemma 41. The pullback coarse structure is indeed a coarse structure; the maps \( f_i : X → Y_i \) are coarsely uniform.

Proof. We check the axioms of a coarse structure:

1. \( \Delta_X \subseteq (f_i^{\times 2})^{-1}(\Delta_{Y_i}) \) for every i.

2. If \( E \subseteq X \times X \) is an entourage and \( F \subseteq E \) a subset then \( f_i^{\times 2}(F) \subseteq f_i^{\times 2}(E) \) is an entourage in \( Y_i \) for every i.

3. if \( E_1, E_2 \) are entourages in X then for every i there are entourages \( F_1, F_2 \subseteq Y_i^2 \) such that \( E_1 \subseteq (f_i^{\times 2})^{-1}(F_1) \) and \( E_2 \subseteq (f_i^{\times 2})^{-1}(F_2) \). But then

\[
E_1 \cup E_2 \subseteq (f_i^{\times 2})^{-1}(F_1) \cup (f_i^{\times 2})^{-1}(F_2) = (f_i^{\times 2})^{-1}(F_1 \cup F_2)
\]

in which every entourage is the union of a subset of the diagonal and finitely many off-diagonal points.
4. if $E$ is an entourage in $X$ then for every $i$ there is an entourage $F$ in $Y_i$ such that $E \subseteq (f_i^{x^2})^{-1}(F)$. But then

$$E^{-1} \subseteq (f_i^{x^2})^{-1}(F^{-1})$$

5. If $E_1, E_2$ are as above then

$$E_1 \circ E_2 \subseteq (f_i^{x^2})^{-1}(F_1 \circ F_2)$$

6. If $(x, y) \in X$ then for every $i$

$$f_i^{x^2}(x, y) = (f_i(x), f_i(y))$$

is an entourage.

\[\Box\]

Remark 42. Note that it would be ideal if the pullback coarse structure is well-defined up to coarse equivalence and if there is a universal property. We cannot use naively the limit in $\text{Sets}$ and equip it with the pullback coarse structure as the following example shows:

Denote by $\phi: \mathbb{Z} \to \mathbb{Z}$ the map that maps $i \mapsto 2i$ and by $\psi: \mathbb{Z} \to \mathbb{Z}$ the map that maps $i \mapsto 2i + 1$. Then both $\phi, \psi$ are isomorphisms in the coarse category. The pullback of

\[
\begin{array}{c}
\mathbb{Z} \\
\phi \downarrow \\
\mathbb{Z} \\
\psi \downarrow \\
\mathbb{Z}
\end{array}
\]

is $\emptyset$ in $\text{Sets}$ but should be an isomorphism if the diagram is supposed to be a pullback diagram in $\text{Coarse}$.

Proposition 43. Let $X$ have the pullback coarse structure of $(f_i: X \to Y_i)_i$. A subset $C \subseteq X^2$ is a coentourage if for every $i$ the set $f_i^{x^2}(C)$ is a coentourage in $Y_i$. Note that the converse does not hold in general.

Proof. Let $C \subseteq X^2$ have the above property. If $F \subseteq X^2$ is a subset such that

$$S = C \cap F$$

is not bounded then there is some $i$ such that $f_i^{x^2}(S)$ is not bounded. Then

$$f_i^{x^2}(C) \cap f_i^{x^2}(F) \supseteq f_i^{x^2}(C \cap F)$$

is not bounded but $f_i^{x^2}(C)$ is a coentourage in $Y_i$. Thus $f_i^{x^2}(F)$ is not an entourage in $Y_i$, thus $F$ does not belong to the pullback coarse structure on $X$. Thus $C$ is detected by the pullback coarse structure. \[\Box\]

Example 44. (Product) The pullback coarse structure on products agrees with [Gra06, Definition 1.32]: If $X, Y$ are coarse spaces the product $X \times Y$ has the pullback coarse structure of the two projection maps $p_1, p_2$:

- A subset $E \subseteq (X \times Y)^2$ is an entourage if and only if $p_1^{x^2}(E)$ is an entourage in $X$ and $p_2^{x^2}(E)$ is an entourage in $Y$.
- A subset $C \subseteq (X \times Y)^2$ is a coentourage if and only if $p_1^{x^2}(C)$ is a coentourage in $X$ and $p_2^{x^2}(C)$ is a coentourage in $Y$. 

19
3.3 Colimits

Proposition 45. If $f_i : Y_i \to X$ is a finite family of injective maps from coarse spaces then the subsets

$$f_i^{x^2}(E_i)$$

for $i$ an index and $E_i \subseteq Y_i^2$ an entourage are a subbase for a coarse structure; the maps $f_i : Y_i \to X$ are coarse maps.

Proof. Suppose $E_i \subseteq Y_i^2$ is an entourage. Let $C \subseteq X^2$ be an element of the pushout cocoarse structure. Denote

$$S = f_i^{x^2}(E_i) \cap C.$$

Then

$$(f_i^{x^2})^{-1}(S) = (f_i^{x^2})^{-1} \circ f_i^{x^2}(E_i) \cap (f_i^{x^2})^{-1}(C) = E_i \cap (f_i^{x^2})^{-1}(C)$$

implies that $f_i^{x^2}(E_i)$ is an entourage.

Now $E \subseteq X^2$ is an entourage if for every $i$

$$E \cap (\text{im } f_i)^2$$

is an entourage and if $E \cap (\bigcup (\text{im } f_i)^2)^c$ is bounded.

We show that this is indeed a coarse structure by checking the axioms of Definition 2:

1. We show the diagonal in $X$ is an entourage. Let $C \subseteq X^2$ be a subset such that

$$(f_i^{x^2})^{-1}(C) \subseteq Y_i^2$$

is a coentourage. Denote

$$S = \Delta_X \cap C.$$

Then

$$(f_i^{x^2})^{-1}(\Delta_X \cap C) = (f_i^{x^2})^{-1}(\Delta_X) \cap (f_i^{x^2})^{-1}(C) = \Delta_X \cap (f_i^{x^2})^{-1}(C) \subseteq B_i^2$$

is bounded.

2. easy

3. easy

4. easy

5. If $E_1, E_2 \subseteq X^2$ have the property that for every element $C \subseteq X^2$ of the pushout cocoarse structure and every $i$:

$$(f_i^{x^2})^{-1}(E_1) \cap (f_i^{x^2})^{-1}(C)$$

and

$$(f_i^{x^2})^{-1}(E_2) \cap (f_i^{x^2})^{-1}(C)$$
are bounded in \( Y_i \) we want to show that \( E_1 \circ E_2 \) has the same property. Now without loss of generality we can assume that there are \( ij \) such that \( E_1 \subseteq (\text{im } f_i)^2 \) and \( E_2 \subseteq (\text{im } f_j)^2 \) the other cases being trivial or they can be reduced to that case. Then
\[
E_1 \circ (E_2 \cap (\text{im } f_i)^2) \subseteq (\text{im } f_i)^2
\]
and
\[
(E_1 \cap (\text{im } f_j)^2) \circ E_2 \subseteq (\text{im } f_j)^2
\]
are entourages and the other cases are empty.

6. If \((x_1, x_2) \in X^2\) then for every \(i\)
\[
(f_i^{x^2})^{-1}(x_1, x_2)
\]
is either one point or the empty set in \( Y_i \), both are entourages.

\(\Box\)

**Definition 46.** Let \( X \) be a set and \( f_i : Y_i \to X \) a finite family of injective maps from coarse spaces. Then define the **pushout cocoarse structure** on \( X \) to be those subsets \( C \) of \( X^2 \) such that for every \( i \) the set
\[
(f_i^{x^2})^{-1}(C) \subseteq Y_i^2
\]
is a coentourage.

**Example 47.** Let \( A, B \) be coarse spaces and \( A \sqcup B \) their disjoint union. The cocoarse structure and the coarse structure of \( A \sqcup B \) look like this:

- A subset \( D \subseteq (A \cup B)^2 \) is a coentourage if \( D \cap A^2 \) is a coentourage in \( A \) and \( D \cap B^2 \) is a coentourage in \( B \).
- A subset \( E \subseteq (A \cup B)^2 \) is an entourage if \( E \cap A^2 \) is an entourage of \( A \) and \( E \cap B^2 \) is an entourage of \( B \) and \( E \cap (A \times B \cup B \times A) \) is contained in \( S \times T \cup T \times S \) where \( S \) is bounded in \( A \) and \( T \) is bounded in \( B \). This definition actually agrees with [Mit01, Definition 2.12, p. 277].

**Example 48.** Let \( G \) be a countable group that acts on a set \( X \). We require that for every \( x, y \in X \) the set
\[
\{g \in G : g \cdot x = y\}
\]
is finite. Then the pushout cocoarse structure of the orbit maps
\[
i_x : G \to X
\]
\[
g \mapsto g \cdot x
\]
for \( x \in X \) is dual to the minimal connected \( G \)-invariant coarse structure of [Roe03, Example 2.13].

**Proof.** Note that by the above requirement a subset \( B \subseteq X \) is bounded if and only if it is finite. Fix an element \( x \in X \) and denote by \( X_x \subseteq X \) the orbit of \( x \).

For every \( C \subseteq G^2 \) coentourage
\[
E \cap i_x^2(C)
\]
being bounded implies that
\[
(i_x^{x^2})^{-1}(E) \cap C \subseteq (i_x^{x^2})^{-1}(E \cap i_x^{x^2}(C))
\]

21
is bounded. Thus if $E \subseteq X^2$ is an entourage then $(i^x_{x^2})^{-1}(E)$ is an entourage.
If $(i^x_{x^2})^{-1}(E)$ is an entourage then $E = i^x_{x^2} \circ (i^x_{x^2})^{-1}(E)$. For every $C \subseteq G^2$ coentourage
$$(i^x_{x^2})^{-1}(E) \cap C$$
being bounded implies that
$$E \cap i^x_{x^2}(C)$$
is bounded. Thus $E$ is an entourage.

The $i^x_{x^2}(E)$ for $E \subseteq G^2$ an entourage are a coarse structure on $X'$ because $i_x$ is surjective on $X'$.
If $x, y$ are in the same orbit $X'$ then $i_x, i_y$ induce the same coarse structure on $X'$.

4 Coarse Cohomology with twisted Coefficients

We define a Grothendieck topology on coarse spaces and describe cohomology with twisted coefficients on coarse spaces and coarse maps. We have a notion of Mayer-Vietoris and a notion of relative cohomology.

4.1 Coarse Covers

Definition 49. Let $X$ be a coarse space and let $(U_i)_i$ be a finite family of subspaces of $X$. It is said to coarsely cover $X$ if the complement of
$$\bigcup_i U_i^2$$
is a coentourage.

Example 50. The coarse space $\mathbb{Z}$ is coarsely covered by $\mathbb{Z}_-$ and $\mathbb{Z}_+$. An example for a decomposition that does not coarsely cover $\mathbb{Z}$ is $\{x \in \mathbb{Z} : x \text{ is even}\} \cup \{x \in \mathbb{Z} : x \text{ is odd}\}$.

Remark 51. The finiteness condition is important, otherwise $(\{x, y\})_{x, y \in X}$ would coarsely cover $X$, but if $X$ is not bounded we don’t want $X$ to be covered by bounded sets only.

Lemma 52. A nonbounded coarse space $X$ is coarsely covered by one element $U$ if and only if $X \setminus U$ is bounded.

Proof. By definition $U$ coarsely covers $X$ if and only if $(U^2)^{\circ}$ is a coentourage; now $(U^\circ)^2 \subseteq (U^2)^{\circ}$ thus $U^\circ$ is bounded by Lemma 20.

Conversely, if $U^\circ$ is bounded then
$$(U^2)^{\circ} = X \times U^\circ \cup U^\circ \times X$$
is a coentourage, thus $U$ coarsely covers $X$.

Remark 53. If $X$ is coarsely covered by $(U_i)_i$ and they cover $X$ (as sets) then it is the colimit (see Definition 16) of them:
$$X \cong \bigcup_i U_i$$
as a coarse space.

This is going to be useful later:
Proposition 54. A finite family of subspaces $(U_i)_i$ coarsely covers a metric space $X$ if and only if for every entourage $E \subseteq X^2$ the set
\[ E[U_1^i] \cap \ldots \cap E[U_n^i] \]
is bounded.

Remark 55. This appeared already in [DKU98] Definition 2.1]; wherein $U_1^i, \ldots, U_n^i$ is a finite system of subsets of $X$ that diverges.

Proof. We proceed by induction on the number $X$ system of subsets of $X$.

If there is one piece $U_1$, then by Lemma [52] one subset $U_1 \subseteq X$ coarsely covers $X$ if and only if $U_1^i$ is bounded. By this and Lemma [6] for every entourage $E \subseteq X^2$ the set $E[U_1^i]$ is bounded.

Conversely if $E[U_1^i]$ is bounded for every entourage $E \subseteq X^2$ then $U_1^i$ itself is bounded which implies that $U_1$ coarsely covers $X$.

Consider next the case of two subsets $U_1, U_2$. We first claim that they form a coarse cover if and only if $U_1^i \times U_2^i$ is a coentourage. Indeed $X^2 \setminus (U_1^i \cup U_2^i) = U_1^i \times U_2^i \cup U_2^i \times U_1^i$, so $X^2 \setminus (U_1^i \cup U_2^i)$ is a coentourage if and only if both of $U_1^i \times U_2^i$ and $U_2^i \times U_1^i$ are coentourages. Let $E \subseteq X^2$ be an entourage. Now by Lemma [53] this implies that $U_1^i \times E[U_2^i]$ is a coentourage, namely we have that the set $E[U_1^i] \cap E[U_2^i]$ is bounded.

Conversely from the assumption that $E[U_1^i] \cap E[U_2^i]$ is bounded for every entourage $E \subseteq X^2$, we deduce $E[U_1^i] \cap U_2^i$ is a bounded set. This implies that $U_1^i \times U_2^i$ is a coentourage.

Now we consider the inductive step. Suppose $n \geq 3$. Subsets $U_1, \ldots, U_n$ form a coarse cover if and only if for every $i < j$ the sets $\{U_i \cup U_j\} \cup \{U_k : k \neq i, j\}$ form a coarse cover of $X$. Let $E$ be an entourage. By the induction hypothesis
\[ E[(U_i \cup U_j)^c] \cap E[U_{i,j}^i] \cap \ldots \cap E[U_n^i] \cap \ldots \cap E[U_n^i] \]
is bounded for every $i < j$. Since $E[U_{i,j}^i] \supseteq E[U_i^i \cap U_j^i]$ we obtain
\[ E[U_1^i] \cap \ldots \cap E[U_n^i] = \bigcap_{i<j} E[(U_i \cup U_j)^c] \cap \ldots \cap E[U_i^i] \cap \ldots \cap E[U_j^i] \cap \ldots \cap E[U_n^i] \]
is bounded.

Conversely let $U_1, \ldots, U_n \subseteq X$ be subsets with $E[U_1^i] \cap \ldots \cap E[U_n^i]$ bounded. Then since $E[U_i^i \cap U_j^i] \subseteq E[U_i^i] \cap E[U_j^i]$ we obtain that
\[ E[(U_i \cup U_j)^c] \cap E[U_{i,j}^i] \cap \ldots \cap E[U_i^i] \cap \ldots \cap E[U_j^i] \cap \ldots \cap E[U_n^i] \]
is bounded for every $i < j$. By induction hypothesis $\{U_i \cup U_j\} \cup \{U_k : k \neq i, j\}$ is a coarse cover for every $i < j$. Thus $U_1, \ldots, U_n$ is a coarse cover.

Proposition 56. If $r : X \to Y$ is a surjective coarse equivalence then $(V_i)_i$ is a coarse cover of $Y$ if and only if $(r^{-1}(V_i))_i$ is a coarse cover of $X$.

Proof. Suppose $(V_i)_i$ is a coarse cover of $X$. Then $(\bigcup_1 f_x^{c})^c$ is a coentourage in $Y$ thus
\[ \bigcup_1 r^{-1}(V_i)^c = (f_x^{c})^{-1}(\bigcup_1 f_x^{c}) \]
is a coentourage. Thus $(f^{-1}(V_i))_i$ is a coarse cover of $X$.

Conversely suppose $(f^{-1}(V_i))_i$ is a coarse cover of $X$ then
\[ (\bigcup_1 f_x^{c})^c = (f^{c})^{-1}(f^{-1}(V_i))^c \]
is a coentourage in $Y$. 

\[ \Box \]
4.2 The Coarse Site

Remark 57. In what follows we define a Grothendieck topology on the category of subsets of a coarse space $X$. What we call a Grothendieck topology is sometimes called a Grothendieck pretopology. We stick to the terminology of \cite{Art62}. If $\mathcal{C}$ is a category a Grothendieck topology $T$ on $\mathcal{C}$ consists of

- the underlying category $\text{Cat}(T) = \mathcal{C}$
- the set of coverings $\text{Cov}(T)$ which consists of families of morphisms in $\mathcal{C}$ with a common codomain. We write
  \[ \{ U_i \to U \}_i \]
  where $i$ stands for the index. They comply with the following rules:

1. Every isomorphism is a covering.
2. Local character: If $\{ U_i \to U \}_i$ is a covering and for every $i$ the family $\{ V_{ij} \to U_i \}_j$ is a covering then the composition $\{ V_{ij} \to U_i \to U \}_ij$ is a covering.
3. Stability under base change: For every object $U \in \text{Cat}(T)$, morphism $V \to U$ and covering $\{ U_i \to U \}_i$ all fibre products $U_i \times_U V$ exist and the family $\{ U_i \times_U V \to V \}$ is a covering.

In the course of this paper we will mostly (but not always) apply the theory of Grothendieck topologies as portrayed in \cite{Tam94, parts I,II].

Definition 58. To a coarse space $X$ is associated a Grothendieck topology $X_{ct}$ where the underlying category of $X_{ct}$ consists of subsets of $X$, there is an arrow $U \to V$ if $U \subseteq V$. A finite family $(U_i)_i$ covers $U$ if they coarsely cover $U$, that is, if

\[ U^2 \cap \left( \bigcup_i U_i^2 \right)^c \]

is a coentourage in $X$.

Lemma 59. The construction $X_{ct}$, is indeed a Grothendieck topology.

Proof. We check the axioms for a Grothendieck topology:

1. if $U \subseteq X$ is a subset the identity $\{ U \to U \}$ is a covering
2. Let \( \{ U_i \to U \} \) be a covering and suppose for every \( i \) there is a covering \( \{ U_{ij} \to U_i \} \), then:

\[
\begin{align*}
U^2 \cap ( \bigcup_{ij} U_{ij}^2 )^c &= U^2 \cap \bigcap_i \bigcap_j U_{ij}^{2c} \\
&= \bigcap_i (U^2 \cap \bigcap_j U_{ij}^{2c}) \\
&= \bigcap_i [(U^2 \cap U_i^2 \cap \bigcap_j U_{ij}^{2c}) \cup (U^2 \cap U_i^{2c} \cap \bigcap_j U_{ij}^{2c})] \\
&\subseteq \bigcap_i [(U_i^2 \cap \bigcap_j U_{ij}^{2c}) \cup (U_i^{2c})] \\
&\subseteq \bigcup_i (U_i^2 \cap \bigcap_j U_{ij}^{2c}) \cup \bigcap_i (U_i^{2c}) \\
&= \bigcup_i [U_i^2 \cap (\bigcup_{ij} U_{ij}^2)] \cup [U^2 \cap (\bigcup_{ij} U_{ij}^2)^c]
\end{align*}
\]

Therefore \( U^2 \cap ( \bigcup_{ij} U_{ij}^2 )^c \) is a finite union of coentourages, since the index set is finite; so it is a coentourage by Lemma 16.

3. Let \( \{ U_i \to U \} \) be a covering and let \( V \subseteq U \) be an inclusion. Then

\[
\begin{align*}
V^2 \cap ( \bigcup_i (V \cap U_i)^2 )^c &= V^2 \cap \bigcap_i (V \cap U_i)^{2c} \\
&= V^2 \cap \bigcap_i (U_i^{2c} \cup V^{2c}) \\
&= V^2 \cap \bigcap_i U_i^{2c} \\
&= V^2 \cap (\bigcup_i U_i^2)^c \\
&\subseteq U^2 \cap (\bigcup_i U_i^2)^c
\end{align*}
\]

So \( \{ V \cap U_i \to V \} \) is a covering of \( X_{ct} \).

\[\square\]

Remark 60. If \( T, T' \) are two Grothendieck topologies a functor \( f : \text{Cat}(T) \to \text{Cat}(T') \) is called a morphism of topologies if

1. if \( \{ \varphi_i : U_i \to U \} \) is a covering in \( T \) then \( \{ f(\varphi_i) : f(U_i) \to f(U) \} \) is a covering in \( T' \).

2. if \( \{ U_i \to U \} \in \text{Cov}(T) \) and \( V \to U \) a morphism in \( \text{Cat}(T) \) then the canonical morphism

\[
f(U_i \times_U V) \to f(U_i) \times_{f(U)} f(V)
\]

is an isomorphism for every \( i \).

Definition 61. Let \( f : X \to Y \) be a coarse map between coarse spaces. Then we define a functor

\[
f^{-1} : \text{Cat}(Y_{ct}) \to \text{Cat}(X_{ct}) \\
U \mapsto f^{-1}(U)
\]
Lemma 62. The functor \( f^{-1} \) induces a morphism of Grothendieck topologies \( f^{-1} : Y_{ct} \to X_{ct} \).

Proof. We check the axioms for a morphism of Grothendieck topologies:

1. Let \( \{U_i \to U\}_i \) be a covering in \( Y \). Then
   \[
   f^{-1}(U)^2 \cap (\bigcup_i f^{-1}(U_i)^2)^c = (f^\times)^{-1}(U^2 \cap (\bigcup_i U_i^2)^c)
   \]
   is a coentourage. Thus \( \{f^{-1}(U_i) \to f^{-1}(U)\}_i \) is a covering in \( X \).

2. for every \( U,V \) subsets of \( X \) we have
   \[
   f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V)
   \]

Remark 63. Let \( T \) be a Grothendieck topology.

- A presheaf on \( T \) with values in \( \mathcal{C} \) is defined as a contravariant functor \( F : \text{Cat}(T) \to \mathcal{C} \).
- A morphism \( \eta : F \to G \) of presheaves with values in \( \mathcal{C} \) is a natural transformation of contravariant functors.
- A presheaf is a sheaf on \( T \) if for every covering \( \{U_i \to U\} \in \text{Cov}(T) \) the diagram
  \[
  F(U) \to \prod_i F(U_i) \Rightarrow \prod_{ij} F(U_i \times_U U_j)
  \]
  is an equalizer diagram in \( \mathcal{C} \). Exactness at \( F(U) \) means that the first arrow \( s \mapsto (s|_{U_i})_i \) is injective (global axiom) and exactness at \( \prod_i F(U_i) \) means that the image of the first arrow is equal to the kernel of the double arrow, hence consists of all \( (s_i)_i \) such that \( s_i|_{U_j} = s_j|_{U_i} \) (gluing axiom).

- A morphism of sheaves is a morphism of the underlying presheaves.

Example 64. Let \( B \) be a space with the indiscrete (maximal) coarse structure. Then \( B \) is already covered by the empty covering. But then the equalizer diagram for that covering is
   \[
   F(B) \to \prod_0 \Rightarrow \prod_0
   \]
   Thus every sheaf on \( B \) vanishes.

Proposition 65. (Sheaf of Functions) If \( X,Y \) are coarse spaces then the assignment \( U \subseteq X \to (\text{coarse maps } U \to Y \text{ modulo closeness}) \) is a sheaf on \( X_{ct} \).

Proof. We check the sheaf axioms:

1. global axiom: Let \( f,g : U \to Y \) be two coarse maps and suppose \( U \) is coarsely covered by \( U_1,U_2 \) and \( f|_{U_1} \sim g|_{U_1} \) and \( f|_{U_2} \sim g|_{U_2} \). Then
   \[
   f \times g(\Delta_U) = f \times g(\Delta_{U_1}) \cup f \times g(\Delta_{U_2}) \cup f \times g(\Delta_{U \setminus (U_1 \cup U_2)})
   \]
   The first two terms of the union are entourages because \( f,g \) are close on \( U_1 \) and \( U_2 \). The last term is an entourage because \( U \setminus (U_1 \cup U_2) \) is bounded. Therefore \( (f \times g)(\Delta_U) \) is a union of three entourages, so is itself an entourage. Thus \( f,g \) are close on \( U \).
2. gluing axiom: Suppose \( U \subseteq X \) is coarsely covered by \( U_1, U_2 \) and \( f_1 : U_1 \to Y \) and \( f_2 : U_2 \to Y \) are coarse maps such that \( f_1|_{U_2} \sim f_2|_{U_1} \).

Then there is a global map \( f : U \to Y \) defined in the following way:

\[
    f(x) = \begin{cases} 
        f_1(x) & x \in U_1, \\
        f_2(x) & x \in U_2 \setminus U_1, \\
        p & x \in U \setminus (U_1 \cup U_2).
    \end{cases}
\]

Here \( p \) denotes some point in \( Y \). Now we show \( f \) is a coarse map:

We show \( f \) is coarsely uniform: If \( E \subseteq U^2 \) is an entourage then

(a) \( f^* (E \cap U_1^2) = f_1^* (E \cap U_1^2) \) is an entourage;

(b) \[
    f^* (E \cap (U_1 \cap U_2) \times (U_2 \setminus U_1)) = f_1 \times f_2 (E \cap (U_1 \cap U_2) \times (U_2 \setminus U_1)) \\
    \subseteq f_1 \times f_2 (\Delta_{U_1 \cap U_2}) \circ f_2^* (E \cap (U_1 \cap U_2) \times (U_2 \setminus U_1))
\]

is an entourage;

(c) \( f^* (E \cap (U_2 \setminus U_1)^2) = f_2^* (E \cap (U_2 \setminus U_1)^2) \) is an entourage;

(d) \( E \cap U_1 \times U_2^c \) and \( E \cap U_2 \times U_1^c \) are already bounded. Now \( f \) maps bounded sets to bounded sets because \( f_1, f_2 \) and the constant map to \( p \) do.

Since

\[
    U^2 = (U_1 \cap U_2) \times (U_2 \setminus U_1) \cup (U_2 \setminus U_1) \times (U_1 \cap U_2) \cup (U_2 \setminus U_1)^2 \cup (U \setminus (U_1 \cup U_2))^2
\]

the set \( f^* (E) \) is a finite union of entourages and therefore itself an entourage. Thus \( f \) is coarsely uniform.

We show \( f \) is coarsely proper: If \( B \subseteq Y \) is bounded then

\[
    f^{-1} (B) \subseteq f_1^{-1} (B) \cup f_2^{-1} (B) \cup (U \setminus (U_1 \cup U_2))
\]

is bounded.

Thus we showed \( f \) is a coarse map.

\[\Box\]

4.3 Sheaf Cohomology

Sheaves on the Grothendieck topology \( X_{ct} \) give rise to a cohomology theory on coarse spaces and coarse maps:

Remark 66. If \( T \) is a Grothendieck topology denote by \( \text{Presheaf}(T) \) the category of abelian presheaves on \( T \) and by \( \text{Sheaf}(T) \) the category of abelian sheaves on \( T \). The category \( \text{Sheaf}(T) \) is a full subcategory of \( \text{Presheaf}(T) \), denote by \( i : \text{Sheaf}(T) \to \text{Presheaf}(T) \) the inclusion functor. The functor \( i \) is left exact by [Tam94, Theorem I.3.2.1]. If \( U \in \text{Cat}(T) \) then denote by \( \Gamma(U, \cdot) : \text{Presheaf}(T) \to \text{Ab} \) the section functor which is an exact functor by [Tam94, Proposition I.2.1.1]. Then \( \Gamma(U, \cdot) \circ i \) is additive and a composition of a left exact functor and an exact functor
and therefore left exact. The category \( \text{Sheaf}(T) \) is an abelian category with enough injectives therefore the right derived functor

\[
\check{H}^q(U, \mathcal{F}) = R^q(\Gamma(U, \cdot) \circ i)(\mathcal{F})
\]

exists for \( \mathcal{F} \) an abelian sheaf on \( T \). See [Tam94, Definition I.3.3.1].

**Remark 67.** (coarse cohomology with twisted coefficients) Let \( \mathcal{F} \) be a sheaf of abelian groups on a coarse space \( X \), let \( U \subseteq X \) be a subset and let \( q \geq 0 \) be a number. Then the \( q \)th coarse cohomology group of \( U \) with values in \( \mathcal{F} \) is

\[
\check{H}^q(U, \mathcal{F}),
\]

the \( q \)th sheaf cohomology of \( U \) in \( X \)ct with coefficient \( \mathcal{F} \).

**Remark 68.** (functoriality) Let \( f : X \rightarrow Y \) be a coarse map between coarse space. There is a direct image functor

\[
f_* : \text{Sheaf}(X_{ct}) \rightarrow \text{Sheaf}(Y_{ct})
\]

\( \mathcal{F} \mapsto f_*\mathcal{F} \)

where

\[
f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}(V))
\]

for every \( V \subseteq Y \). The left adjoint functor to \( f_* \) exists by [Tam94, Proposition I.3.6.2] and is denoted inverse image functor

\[
f^* : \text{Sheaf}(Y) \rightarrow \text{Sheaf}(X).
\]

Note that \( f^* \) is exact. Then there is an edge homomorphism of the Leray spectral sequence\(^4\) of \( f_* \) which will also be denoted by \( f_* \): let \( U \subseteq Y \) be a subset and let \( \mathcal{F} \) be a sheaf on \( X \); then there is a homomorphism

\[
f_* : \check{H}^*(f^{-1}U, \mathcal{F}) \rightarrow \check{H}^*(U, f_*\mathcal{F}).
\]

**Remark 69.** Let \( T \) be a Grothendieck topology. By [Tam94, Theorem I.3.1.1] the adjoint to the inclusion functor \( i : \text{Sheaf}(T) \rightarrow \text{Presheaf}(T) \) exists and is denoted by \( \# \). If \( \mathcal{F} \) is a presheaf then \( \mathcal{F}\# \) is the sheaf associated to the presheaf \( \mathcal{F} \), also called the sheafification of \( \mathcal{F} \).

Define for an abelian presheaf \( \mathcal{F} \) on \( T \):

\[
\mathcal{F}^!(U) = \lim_{\{U_i \rightarrow U\} \in \text{Cov}(T)} H^0(\{U_i \rightarrow U\}, \mathcal{F})
\]

for \( U \in \text{Cat}(T) \). Here the right side, the term \( H^0(\{U_i \rightarrow U\}, \mathcal{F}) \), denotes the 0th Čech cohomology associated to the covering \( \{U_i \rightarrow U\}_i \) with values in \( \mathcal{F} \). The functor \( \mathcal{F}^! \) is a presheaf and

\[
\mathcal{F}^\# = (\mathcal{F}^!)^!
\]

is the sheaf associated to the presheaf \( \mathcal{F} \).

**Lemma 70.** Let \( X \) be a coarse space and denote by \( p : X \times \{0,1\} \rightarrow X \) the projection to the first factor. Then

\[
R^q p_* = 0
\]

for \( q > 0 \).

\(^4\)This is [Tam94, Theorem I.3.7.6, p. 71]
Proof. In a general setting if $\mathcal{F}$ is a sheaf on a coarse space denote by $\mathcal{H}^q(\mathcal{F})$ the presheaf

$$ U \mapsto \check{H}^q(U, \mathcal{F}). $$

Then [Tam94, Proposition I.3.4.3] says that

$$ \mathcal{H}^q(\mathcal{F})^\dagger = 0 $$

for $q > 0$.

Now [Tam94, Proposition I.3.7.1] implies that for every coarse map $f : X \to Y$ and sheaf $\mathcal{F}$ on $X$

$$ R^q f_*(\mathcal{F}) \cong (f_* \mathcal{H}^q(\mathcal{F}))^\#.$$ 

Define

$$ H = \{((x, i), (x, 0)) : (x, i) \in X \times \{0, 1\} \subseteq (X \times \{0, 1\})^2 \} $$

as a subset of $X \times \{0, 1\}$ which is an entourage. We identify $X \times 0$ with $X$. Then $(U_i)_i$ coarsely covers $U \subseteq X$ if and only if $(H[U_i])_i$ coarsely covers $H[U]$.

Let $V_1, V_2$ be a coarse cover of $U \times \{0, 1\}$. Write

$$ V_1 = V_0^0 \times 0 \cup V_1^1 \times 1 $$

and

$$ V_2 = V_2^0 \times 0 \cup V_2^1 \times 1. $$

Note that

$$ V_i^c = (V_i^0 \times 0)^c \cap (V_i^1 \times 1)^c $$

$$ = (V_i^0)^c \times 0 \cup (V_i^1)^c \times 1 $$

for $i = 1, 2$. But then

$$ ((V_i^0)^c \cup (V_i^1)^c) \times ((V_i^0)^c \cup (V_i^1)^c) $$

is a coentourage in $U$. Thus

$$ (V_1^0 \cap V_1^1) \times \{0, 1\}, (V_2^0 \cap V_2^1) \times \{0, 1\} $$

is a coarse cover that refines $V_1, V_2$.

We show that $p_*$ and $\#$ commute for presheaves $\mathcal{G}$ on $X$: Let $U \subseteq X$ be a subset then

$$ (p_* \mathcal{G})^\dagger(U) = \lim_{\{U_i \to U\}_i \in \text{Cov}(X)} H^0(\{U_i \to U\}_i, p_* \mathcal{G}) $$

$$ = \lim_{\{U_i \to U\}_i \in \text{Cov}(X)} H^0(\{H[U_i] \to H[U]\}_i, \mathcal{G}) $$

$$ = \lim_{\{V_i \to H[U]\}_i \in \text{Cov}(X \times \{0, 1\})} H^0(\{V_i \to H[U]\}_i, \mathcal{G}) $$

$$ = \mathcal{G}^\dagger(H[U]) $$

$$ = p_* \mathcal{G}^\dagger(U). $$

\[\square\]

Remark 71. Note that two coarse maps $f, g : X \to Y$ are close if the map $h : X \times \{0, 1\} \to Y$ agreeing with $f$ on $X \times 0$ and with $g$ on $X \times 1$ is a coarse map.
Proof. Suppose \( h \) is a coarse map we show \( f, g \) are close. The set
\[
f \times g(\Delta_X) = \{ f(x), g(x) : x \in X \}
\]
\[
= \{ h((x, 0), (x, 1)) : x \in X \}
\]
\[
= h^{\times 2}(\Delta_X \times \{0, 1\})
\]
is an entourage in \( Y \).

**Theorem 72.** (close maps) If two coarse maps \( f, g : X \rightarrow Y \) are close the induced homomorphisms \( f_*, g_* \) of coarse cohomology with twisted coefficients are isomorphic.

Proof. Define a coarse map
\[
h : X \times \{0, 1\} \rightarrow Y
\]
by \( h|_{X \times 0} = f \) and \( h|_{X \times 1} = g \). But the inclusions \( i_0 : X \times 0 \rightarrow X \times \{0, 1\} \) and \( i_1 : X \times 1 \rightarrow X \times \{0, 1\} \) are both sections of the projection \( p : X \times \{0, 1\} \rightarrow X \) which by Lemma 70 induces an isomorphism in coarse cohomology with twisted coefficients. Hence the maps \( f = h \circ i_0 \) and \( g = h \circ i_1 \) induce maps \( f_* = h_* \circ i_{0*} \) and \( g_* = h_* \circ i_{1*} \) which is the same map followed by isomorphisms.

**Corollary 73.** (coarse equivalence) Let \( f : X \rightarrow Y \) be a coarse equivalence. Then \( f \) induces an isomorphism in coarse cohomology with twisted coefficients.

### 4.4 Mayer-Vietoris Principle

In [Sha96, Section 4.4, p. 24] a Mayer-Vietoris principle for sheaf cohomology on topological spaces is described. It can be translated directly to a Mayer-Vietoris principle for coarse spaces.

**Theorem 74.** (Mayer-Vietoris) Let \( X \) be a coarse space and \( A, B \) two subsets that coarsely cover \( X \). Then there is an exact sequence in cohomology
\[
\cdots \rightarrow \check{H}^{i-1}(A \cap B, \mathcal{F}) \rightarrow \check{H}^i(A \cup B, \mathcal{F}) \rightarrow \check{H}^i(A, \mathcal{F}) \times \check{H}^i(B, \mathcal{F})
\]
\[
\rightarrow \check{H}^i(A \cap B, \mathcal{F}) \rightarrow \cdots
\]
for every sheaf \( \mathcal{F} \) on \( X \).

Proof. First note that a sheaf \( \mathcal{G} \) on a coarse space \( X \) is called flabby if the restriction map associated to an inclusion \( U \hookrightarrow X \) is surjective for every \( U \subseteq X \). This implies that Čech cohomology \( \check{H}^q(\{U_i \rightarrow U\}, \mathcal{G}) = 0 \) for \( q > 0 \) and every coarse cover \( (U_i)_i \) of \( U \subseteq X \). Thus flabby sheaves are acyclic for coarse cohomology with twisted coefficients. Note also that every injective sheaf on a coarse space is flabby, thus given a sheaf \( \mathcal{F} \) there always exists a flabby resolution of \( \mathcal{F} \).

If \( \mathcal{G} \) is a flabby sheaf on \( X \) the sequence
\[
0 \rightarrow \mathcal{G}(A \cup B) \rightarrow \mathcal{G}(A) \times \mathcal{G}(B) \xrightarrow{\mathcal{G}(\varphi)} \mathcal{G}(A \cap B) \rightarrow 0
\]
is an exact sequence. Here \( \varphi \) sends a pair \( (s_1, s_2) \) to \( s_1|_{A \cap B} - s_2|_{A \cap B} \). Thus if \( \mathcal{F} \) is an arbitrary sheaf on \( X \) there is an exact sequence of flabby resolutions of \( \mathcal{F}(A \cup B), \mathcal{F}(A) \times \mathcal{F}(B) \) and \( \mathcal{F}(A \cap B) \). This way we obtain the desired exact sequence in cohomology. \( \square \)
4.5 Local Cohomology

Let us define a version of relative cohomology for twisted coarse cohomology. There is already a similar notion for sheaf cohomology on topological spaces described in [Har67, chapter 1] which is called local cohomology. We do something similar:

**Definition 75. (support of a section)** Let $s \in \mathcal{F}(U)$ be a section. Then the support of $s$ is contained in $V \subseteq U$ if

$$s|_{V \cap U} = 0$$

Let $X$ be a coarse space and $Z \subseteq X$ a subspace. Then

$$\Gamma_Z(\mathcal{F}) : U \mapsto \ker(\mathcal{F}(U) \to \mathcal{F}(U \cap Z^c))$$

is a sheaf on $X$.

**Theorem 76.** Let $Z \subseteq X$ be a subspace of a coarse space and let $Y = X \setminus Z$. Then there is a long exact sequence

$$0 \to \check{H}^0(U, \Gamma_Z(\mathcal{F})) \to \check{H}^0(U, \mathcal{F}) \to \check{H}^0(U, \mathcal{F}|_Y) \to \check{H}^1(U, \Gamma_Z(\mathcal{F})) \to \cdots$$

for every subset $U \subseteq X$ and every sheaf $\mathcal{F}$ on $X$.

**Proof.** First we have an exact sequence

$$0 \to \Gamma_Z(\mathcal{F}) \to \mathcal{F} \to \mathcal{F}|_Y$$

and if $\mathcal{F}$ is flabby we can write 0 on the right.

Let $I = 0 \to \mathcal{F} \to I_0 \to I_1 \to \cdots$ be an injective resolution of $\mathcal{F}$. Note that every injective sheaf is flabby. Then there is an exact sequence of complexes

$$0 \to \Gamma_Z(I) \to I \to I|_Y \to 0$$

which shows what we wanted to show. \(\square\)

5 Constant Coefficients

Before introducing a new coefficient for coarse cohomology with twisted coefficients we introduce a numerical invariant of coarse spaces which will be of interest when studying the coefficient.

5.1 Number of Ends

If a space is the coarse disjoint union of two subspaces we have a special case of a coarse cover. In [Sta68] the number of ends of a group were studied; this notion can be generalized in an obvious way to coarse spaces.

This notation can also be found in [Mei08]:

**Definition 77.** A coarse space $X$ is called **one-ended** if for every coarse disjoint union $X = \bigsqcup_i U_i$ all but one of the $U_i$ are bounded.

**Lemma 78.** The coarse space $\mathbb{Z}_+$, which consists of the non-negative integers, is one-ended.
Thus we can assume \(e\) is a coarse disjoint union with nonboundeds. Thus

\[
(U^2 \cup V^2) \cap E(\mathbb{Z}_+, 1) = (U \times V \cup V \times U) \cap E(\mathbb{Z}_+, 1)
\]

which implies that \(U, V\) are not coarsely disjoint. \(\square\)

**Definition 79.** Let \(X\) be a coarse space. Its number of ends \(e(X)\) is at least \(n \geq 0\) if there is a coarse cover \((U_i)\) of \(X\) such that \(X\) is the coarse disjoint union of the \(U_i\) and \(n\) of the \(U_i\) are not bounded.

**Lemma 80.** If \(A, B\) are two coarse spaces and \(X = A \sqcup B\) their coarse disjoint union then

\[e(X) = e(A) + e(B).\]

**Proof.** Suppose \(e(A) = n\) and \(e(B) = m\). Then there are coarse disjoint unions \(A = A_1 \sqcup \ldots \sqcup A_n\) and \(B = B_1 \sqcup \ldots \sqcup B_m\) with nonboundeds. But then

\[X = A_1 \sqcup \ldots \sqcup A_n \sqcup B_1 \sqcup \ldots \sqcup B_m\]

is a coarse disjoint union with nonboundeds. Thus \(e(X) \geq e(A) + e(B)\).

Suppose \(e(X) = n\). Then there is a coarse disjoint cover \((U_i)_{i=1,\ldots,n}\) with nonboundeds of \(X\). Thus \((U_i \cap A)_i\) is a coarse disjoint union of \(A\) and \((U_i \cap B)_i\) is a coarse disjoint union of \(B\). Then for every \(i\) one of \(U_i \cap A\) and \(U_i \cap B\) is not bounded. Thus

\[e(X) \leq e(A) + e(B).\]

\(\square\)

**Example 81.** \(e(\mathbb{Z}) = 2\).

**Theorem 82.** Let \(f : X \to Y\) be a coarsely surjective coarse map and suppose \(e(Y)\) is finite. Then

\[e(X) \geq e(Y).\]

**Proof.** First we show that \(e(X) \geq e(\text{im} f)\): Regard \(f\) as a surjective coarse map \(X \to \text{im} f\). Suppose that \(e(\text{im} f) = n\). Then \(\text{im} f\) is coarsely covered by a coarse disjoint union \((U_i)_{i=1,\ldots,n}\) where none of the \(U_i\) are bounded. But then \((f^{-1}(U_i))_i\) is a coarse disjoint union of \(X\) and because \(f\) is a surjective coarse map none of the \(f^{-1}(U_i)\) are bounded.

Now we show that \(e(Y) = e(\text{im} f)\): Note that there is a surjective coarse equivalence \(r : Y \to \text{im} f\). By Proposition 56 a finite family of subsets \((U_i)_i\) is a coarse cover of \(\text{im} f\) if and only if \((r^{-1}(U_i))_i\) is a coarse cover of \(Y\). if \((U_i)_i\) is a coarse disjoint union so is \((r^{-1}(U_i))_i\). \(\square\)

**Corollary 83.** The number \(e(\cdot)\) is a coarse invariant.
5.2 Definition

**Definition 84.** Let $X$ be a coarse space and $A$ an abelian group. Then $A_X$ (or just $A$ if the space $X$ is known from context) is the sheafification of the constant presheaf which associates to every subspace $U \subseteq X$ the group $A$.

**Lemma 85.** A coarse disjoint union $X = U \sqcup V$ of two coarse spaces $U, V$ is a coproduct in \textit{Coarse}.

**Proof.** Denote by $i_1 : U \to X$ and $i_2 : V \to X$ the inclusions. We check the universal property: Let $Y$ be a coarse space and $f_1 : U \to Y$ and $f_2 : V \to Y$ coarse maps. But $U, V$ are a coarse cover of $X$ such that $U \cap V$ is bounded. Now we checked that already in Proposition[65] The existence of a map $f : X \to Y$ with the desired properties would be the gluing axiom and the uniqueness modulo closeness would be the global axiom. 

**Theorem 86.** Let $X$ be a coarse space and $A$ an abelian group. If $X$ has finitely many ends then

$$A(X) = A^c(X)$$

and if $X$ does not have finitely many ends then

$$A(X) = \bigoplus_{N} A.$$ 

Here $A(X)$ means the evaluation of the constant sheaf $A$ on $X$ at $X$.

**Proof.** By the equalizer diagram for sheaves a sheaf naturally converts finite coproducts into finite products. If $X$ is one-ended and $U, V$ a coarse cover of $X$ with nonboundeds then $U, V$ intersect nontrivially. Thus $A(X) = A$ in this case. If $X$ has infinitely many ends then there is a directed system

$$\cdots \to U_1 \sqcup \cdots \sqcup U_n \to U_1 \sqcup \cdots \sqcup U_{n+1} \to$$

in the dual category of $\mathcal{I}_X$ which is the category of coarse covers of $X$. Here the $U_i$ are non-bounded and constitute a coarse disjoint union in $X$. Now we use [Tam94, Definition 2.2.5] by which

$$\hat{H}^0(X, A) = \lim_{\to} H^0((U_i)_i, A).$$

Then we take the direct limit of the system

$$\cdots \to A^n \to A^{n+1} \to A^{n+2} \to \cdots.$$ 

Thus the result.

**Lemma 87.** If a subset $U \subseteq \mathbb{Z}_+$ is one-ended then the inclusion

$$i : U \to \mathbb{Z}_+$$

is coarsely surjective.

**Proof.** If the inclusion $i : U \to \mathbb{Z}_+$ is not coarsely surjective then there is an increasing sequence $(v_i)_i \subseteq \mathbb{Z}_+$ such that for every $u \in U$:

$$|u - v_i| > i.$$
Now define
\[ A := \{ u \in U : v_{2i} < u < v_{2i+1}, i \in \mathbb{N} \} \]
and
\[ B := \{ u \in U : v_{2i+1} < u < v_{2i}, i \in \mathbb{N} \}. \]

Then for every \( a \in A, b \in B \) there is some \( j \) such that \( a < v_j < b \). Then
\[ |a - b| = |a - v_j| + |b - v_j| > 2j. \]

If \( i \in \mathbb{N} \) then \( |a - b| \leq i \) implies \( a, b \leq v_i \). Thus \( A, B \) are a coarsely disjoint decomposition of \( U \).

\[ \Box \]

6 Remarks

The starting point of this research was the idea to define sheaves on coarse spaces as presented in \[\text{[Sch99]}\]. And then we noticed that cocontrolled subsets of \( X^2 \) which have first been studied in \[\text{[Roe03]}\] have some topological features.

Finally, after defining coarse covers which depend on the notion of coentourages, we came up with the methods of this paper. Note that coarse cohomology with twisted coefficients is basically just sheaf cohomology on the Grothendieck topology determined by coarse covers.

It would be possible, conversely, after a more thorough examination that other cohomology and homology theories in the coarse category can be computed using sheaf cohomology tools. By \[\text{[Har17]}\] a modified version of controlled operator \( K \)-theory is a cosheaf on proper metric spaces. We obtain a new Mayer-Vietoris sequence using coarse covers. We also examined coarse cohomology by Roe and looked for sheaf properties. As of now we showed coarse cohomology in dimension 2 is a sheaf on coarse spaces. It would be interesting to explore if coarse cohomology in dimension \( 2 + q \) is a derived functor.

We wonder if the new sheaf cohomology will be of any help with understanding coarse spaces. We investigated in which way coarse covers determine a topology on a boundary of a coarse space. In \[\text{[Har19]}\] we introduce a functor which assigns a uniform space with a coarse space. The uniformity is generated by coarse covers and coarse maps are mapped to uniformly continuous maps.

However, as of yet, we do not know many interesting sheaves on coarse spaces besides the constant sheaf and the sheaf of functions. It would be interesting to find another class of sheaves which can be defined for coarse spaces.

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