Computing mean logarithmic mass from muon counts in air shower experiments

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I discuss the conversion of muon counts in air showers, which are observable by experiments, into mean logarithmic mass, an important variable to express the mass composition of cosmic rays. Stochastic fluctuations in the shower development and statistical fluctuations from muon sampling can subtly bias the conversion. A central theme is that the mean of the logarithm of the muon number is not identical to the logarithm of the mean. It is discussed how that affects the conversion in practice. Simple analytical formulas to quantify and correct such biases are presented, which are applicable to any kind of experiment.

I. INTRODUCTION

The mean logarithmic mass \( \langle \ln A \rangle \) is a common variable to summarize the mass composition of cosmic rays. Most ground-based experiments infer the mass by counting muons in cosmic-ray induced air showers [1]. This paper discusses the conversion of muon number to mean logarithmic mass from the point of view of the data analyst, with a focus on the effect of stochastic fluctuations in the shower development and the detector response on the conversion. The fluctuations can bias estimates of \( \langle \ln A \rangle \) in several ways. Biases here are defined in the usual statistical sense; if \( \hat{x} \) is an estimate of the true value \( x \) that fluctuates according to a probability density \( f(\hat{x}) \), then the bias is the expectation \( \langle \hat{x} - x \rangle = \int (\hat{x} - x) f(\hat{x}) d\hat{x} \). We generally want \( \hat{x} \) to have zero bias, so that the sample average \( \langle \hat{x} \rangle \) converges to \( x \) for large samples.

The results in this paper are not specific to a particular type of experiment. It is assumed throughout this paper that an experiment provides an unbiased estimate \( \hat{N}_\mu \) of the total number of muons \( N_\mu \) produced in an air shower and an estimate \( \hat{E} \) of the shower energy \( E \). This is far from trivial and much of the difficulty in running an experiment deals with this. The total number of muons \( N_\mu \) produced in an air shower cannot be directly measured, because experiments can only count muons that reach the ground, while some decay on the way. The experimental distinction between muons and other charged particles at the ground is not easy either [2–5]. But in principle, \( \hat{N}_\mu \) can be inferred for a given geometry and shower energy from the measurement by applying an average correction obtained from air shower simulations. Highly-inclined air showers recorded by Haverah Park and the Pierre Auger Observatory have been analyzed in this way [6–9]. Similarly, an estimate \( \hat{E} \) of the shower energy can be inferred from the number of electrons and photons that reach the ground, or by recording the longitudinal shower profile with telescopes.

The paper deals with the comparably easier part of the conversion of the unbiased estimates \( \hat{E} \), \( \hat{N}_\mu \) to \( \langle \ln A \rangle \). Fluctuations occur in the shower development and arise from the sampling of an air shower by a detector. It is important to distinguish between these two kinds of fluctuations, because they are approximately independent [10]. Both randomly shift the estimates \( \hat{E} \), \( \hat{N}_\mu \) away from their true values \( E \), \( N_\mu \), and these random shifts cause some subtle biases in the conversion to \( \langle \ln A \rangle \). We quantify these biases. Knowing their sizes allows one to safely neglect them if they are small, and to correct them otherwise.

II. FROM MUON NUMBER TO MASS

It is instructive to introduce fluctuations step-by-step. We start by ignoring fluctuations from detector sampling and consider only stochastic fluctuations in the shower development. The true muon number \( N_\mu \) and the shower energy \( E \) shall be exactly known and the energy \( E \) shall be same for all showers. Stochastic fluctuations in the hadronic interactions are still causing the muon number \( N_\mu \) to vary randomly.

The first point to make is that \( \langle \ln A \rangle \) is best computed from the mean logarithmic muon number \( \langle \ln N_\mu \rangle \), and not the mean of the muon number \( \langle N_\mu \rangle \). In either case, the average here is computed over many air showers with the same shower energy \( E \).

The following argument is similar to the one developed by the Pierre Auger collaboration for the depth of shower maximum [11]. The relationship between \( N_\mu \) and \( A \) can be understood within the Matthews-Heitler model of a hadronic shower [12]. The analytical model treats air showers in a simplified way, but describes surprisingly many features of air showers correctly. According to the model, the total muon number \( N_\mu \) for a cosmic ray with \( A \) nucleons scales with a power of the number of nucleons

\[
N_\mu = A^{1-\beta} N_\mu^p,
\]

where \( N_\mu^p \) is the number of muons in a proton-induced air shower, and \( \beta \approx 0.9 \) is a constant.

This behavior is well reproduced in full air shower simulations. In the Matthews-Heitler model, stochastic fluctuations in the shower development are neglected. To show that Eq.1 holds for the real showers, several sets of vertical showers with identical primary particles were simulated with CORSIKA [13] compiled with the

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Figure 1. Average logarithm of the number of muons \(\langle N_\mu \rangle\) (circles) and logarithm of the average number of muons \(\ln(\langle N_\mu \rangle)\) (squares) in simulated vertical air showers produced by primary particles with \(A\) nucleons. A fitted straight line (dashed) is shown for comparison. Solid markers stand for averages computed over showers from a single primary, open markers stand for an equal mix of proton and iron showers. Error bars indicate the statistical uncertainty of the finite sample (2200 showers per primary and energy).

CONEX option, using the hadronic interaction models SIBYLL-2.3 [14] and GHEISHA [15]. The showers were simulated in a US standard atmosphere until a slant depth of 1050 gcm\(^{-2}\). The number of muons \(N_\mu\) in each shower was taken from the maximum of the longitudinal muon profile. Proton, helium, nitrogen, silicon, and iron primaries were simulated. For each primary, the averages \(\ln(\langle N_\mu \rangle)\) and \(\langle N_\mu \rangle\) were computed. The two are subtly different, because the expectation is noncommutative with a non-linear mapping \(f(x), E[f(x)] \neq f(E[x])\). The dependence on \(A\) is shown in Fig. 1 for a wide range of primary energies. For a pure composition where all showers are initiated by a primary with the same mass \(A\), both \(\ln(\langle N_\mu \rangle)\) and \(\langle N_\mu \rangle\) scale with \(\ln A\) as predicted by Eq. 1. This result is independent of the hadronic interaction model and shower inclination.

To use Eq. 1 to get an estimate of \(\langle \ln A \rangle\) for real air showers, we consider the realistic case where the mass \(A\) is another stochastic variable that changes from shower to shower. For a pure composition, the simulations showed that \(\langle N_\mu \rangle = A^{1-\beta} \langle N_\mu^p \rangle\). If \(f_A\) is the fraction of primaries with \(A\) nucleons in a mixed composition, we have

\[
\sum_A f_A \langle N_\mu \rangle = \sum_A f_A A^{1-\beta} \langle N_\mu^p \rangle = \langle N_\mu^p \rangle \sum_A f_A A^{1-\beta}
\leftrightarrow \langle N_\mu \rangle = \langle N_\mu^p \rangle A^{1-\beta}.
\]

Unfortunately, we cannot convert \(\langle A^{1-\beta} \rangle\) to \(\langle \ln A \rangle\) or \(\langle \ln N_\mu \rangle\), because these are non-linear functions of \(A\). The solution is to start from \(\langle \ln N_\mu \rangle = (1-\beta) \ln A + \langle \ln N_\mu^p \rangle\) for a pure composition, which is also supported by the simulations. Then the result of the superposition is

\[
\langle \ln N_\mu \rangle = (1-\beta) \langle \ln A \rangle + \langle \ln N_\mu^p \rangle,
\]

where we used that \(\langle ax + by \rangle = a\langle x \rangle + b\langle y \rangle\) for constants \(a, b\) and stochastic variables \(x, y\).

Both \(\beta\) and \(\langle \ln N_\mu^p \rangle\) can be obtained from air shower simulations. If \(\langle A\rangle\) is available, it can be used to substitute \(\beta\). The two related formulas for \(\langle \ln A \rangle\) are

\[
\langle \ln A \rangle = \frac{\langle \ln N_\mu \rangle - \langle \ln N_\mu^p \rangle}{1-\beta}, \quad \langle \ln A \rangle = \frac{\langle \ln N_\mu \rangle - \langle \ln N_\mu^p \rangle}{\langle \ln N_\mu^{Fe} \rangle - \langle \ln N_\mu^p \rangle} \ln 56.
\]

This approach is very elegant, because the equations are true whatever the probability distributions are for \(A, N_\mu, N_\mu^p,\) and \(N_\mu^{Fe}\).

As previously stated, the mean of the logarithm is not the same as the logarithm of the mean, \(\ln(\langle N_\mu \rangle)\) is always higher than \(\langle \ln N_\mu \rangle\). Still, the two are quite close and the bias of substituting one for the other may be negligible in some situations. To judge when this is safe, a simple formula to compute the bias is given in section III. Some analyses [16] do not produce an estimate of the muon number event-by-event, only the average \(\langle N_\mu \rangle\) over many showers. In these cases, the formula can be used to correct the difference \((\langle \ln N_\mu \rangle - \langle \ln(\langle N_\mu \rangle) \rangle)\).

So far fluctuations introduced by detector sampling were neglected, but \(N_\mu\) is not known in practice, only an estimate \(\tilde{N}_\mu\) which fluctuates around \(N_\mu\). Some muons decay on the way to the ground, the detector does not count all muons that arrive, and so on. It is assumed that these losses are corrected on average, but they introduce additional fluctuations. Since the mean of the logarithm is not the logarithm of the mean, we find \(\langle \ln \tilde{N}_\mu \rangle \neq \langle \ln N_\mu \rangle\) even if \(\tilde{N}_\mu\) is an unbiased estimate of \(N_\mu\). How to correct for this effect is discussed in section IV.

Finally, one has to consider that the average \(\langle \ln \tilde{N}_\mu \rangle\) is not computed over showers with the same energy \(E\), only over showers from a single primary, open markers stand for an equal mix of proton and iron showers.
in practice, but for showers that fall into the same energy bin. The energy $E$ is also not known exactly, only an estimate $\hat{E}$ of it. The quantitative impact of that is calculated in section V.

### III. MUON NUMBER: MEAN LOGARITHM AND LOGARITHM OF MEAN

The difference $(\langle \ln N_\mu \rangle - \ln \langle N_\mu \rangle)$ can be calculated with a simple formula. To derive it, we use the following general substitution

$$N_\mu = \langle N_\mu \rangle (1 + \epsilon),$$

where $\epsilon = (N_\mu - \langle N_\mu \rangle)/\langle N_\mu \rangle$ is the relative random deviation of the muon number from its mean. By construction, $\langle \epsilon \rangle = 0$. The average logarithmic muon number is

$$\langle \ln N_\mu \rangle = \langle \ln [\langle N_\mu \rangle (1 + \epsilon)] \rangle = \ln \langle N_\mu \rangle + \langle \ln (1 + \epsilon) \rangle.$$  

For small relative fluctuations, $\epsilon \ll 1$, the second logarithm can be expanded into a Taylor series,

$$\langle \ln N_\mu \rangle = \ln \langle N_\mu \rangle + \langle \epsilon - \frac{1}{2} \epsilon^2 + O(\epsilon^3) \rangle = \ln \langle N_\mu \rangle - \frac{1}{2} \langle \epsilon^2 \rangle + O(\epsilon^3).$$

The second-order term $\langle \epsilon^2 \rangle$ is equal to the variance of the relative deviations from the mean,

$$\langle \epsilon^2 \rangle = \langle \epsilon \rangle^2 - \langle \epsilon \rangle = \text{Var}[\epsilon] = \text{Var}[(N_\mu - \langle N_\mu \rangle)/\langle N_\mu \rangle] = \text{Var}[N_\mu]/\langle N_\mu \rangle^2.$$  

Therefore, the offset can be computed for $\epsilon \ll 1$ as

$$\langle \ln N_\mu \rangle - \ln \langle N_\mu \rangle \approx -\frac{1}{2} \text{Var}[N_\mu - \langle N_\mu \rangle]/\langle N_\mu \rangle].$$

Table I lists $\langle N_\mu \rangle$, $\ln \langle N_\mu \rangle$, and $\text{Var}[(N_\mu - \langle N_\mu \rangle)/\langle N_\mu \rangle]$ for the air shower simulations described in the previous section. A useful empirical parametrization of the latter is shown in the appendix. The numbers confirm for single elements that $\text{Var}[\epsilon] \ll 1$, which implies $\epsilon \ll 1$. Eq. 10 is therefore a good approximation for single primaries above $10^{15}$ eV.

It also holds for any mix of primaries. The variance for a mix of primaries is larger than for a single primary, because the difference in the means $\langle N_\mu \rangle$ of different primaries contributes to the variance. With the data in Table I, $\text{Var}[\epsilon]$ was computed for all pairs of primaries. The largest value $\text{Var}[\epsilon] = 0.063$ is found at $10^{15}$ eV for a mix of proton and iron. This value is still small and thus Eq. 10 remains valid.

With these numbers, it is possible to address the question whether using $\ln(N_\mu)$ instead of $(\ln N_\mu)$ in Eq. 4 or 5 introduces a noticeable bias. In the most extreme case, the bias is $(\ln \langle N_\mu \rangle - \langle \ln N_\mu \rangle) \approx 0.03$. In the conversion to $(\ln A)$, this bias is multiplied by a factor $1/(1 - \beta)$, see Eq. 4. For $\beta \approx 0.9$, this is a factor of 10, so that the bias in $(\ln A)$ is 0.3. This is about 7% of the overall difference between proton and iron. Using the wrong mean makes the composition appear heavier than it truly is. The effect is small, but since the bias is easy to correct with Eq. 10, applying the correction is recommended.

### IV. BIAS FROM SAMPLING FLUCTUATIONS

The second type of difficulty in applying Eq. 4 or 5 is that $N_\mu$ is not known, only an estimate $\hat{N}_\mu$. To measure $N_\mu$, an experiment would have to collect and count all muons with perfect accuracy. In reality, detectors sample only a small fraction of all particles, and cannot perfectly distinguish between muons and other shower particles. They measure an event-wise estimate $\hat{N}_\mu$ of $N_\mu$, which differs by a random offset for each shower.

This paper is only concerned with the effect of fluctuations, so it is again assumed that the estimate is unbiased, $E[\hat{N}_\mu] = N_\mu$. It still follows that $(\ln \hat{N}_\mu) \neq (\ln N_\mu)$, because of the fluctuations and the non-linear mapping.

A simple formula for the size of this bias can be derived analog to the previous section. The relative offset $\epsilon = (\hat{N}_\mu - N_\mu)/N_\mu$ is introduced, which represents the additional random fluctuations introduced by the muon sampling. Typical values are again small, the Pierre Auger Observatory [9, 17] achieves resolutions better than 30%, so $\text{Var}[\epsilon] < 0.09$. An expansion in a Taylor
series for $\hat{e} \ll 1$ yields
\[
\langle \ln \hat{N}_\mu \rangle = \langle \ln [N_\mu (1 + \hat{e})] \rangle = \langle \ln N_\mu \rangle + \langle \ln (1 + \hat{e}) \rangle
\]
\[
= \langle \ln N_\mu \rangle - \frac{1}{2} \langle \hat{e}^2 \rangle + \mathcal{O}(\langle \hat{e}^3 \rangle). \tag{11}
\]

The term $\langle \hat{e} \rangle$ is zero, because $\hat{N}_\mu$ is unbiased. Values for $\langle \hat{e}^2 \rangle = \text{Var}[(\hat{N}_\mu - N_\mu)/N_\mu]$ can be obtained from Monte-Carlo simulations of the experiment.

To give an example, the previously quoted value $\text{Var}[(\hat{N}_\mu - N_\mu)/N_\mu] = 0.09$ results in a bias $\langle \ln \hat{N}_\mu \rangle - \langle \ln N_\mu \rangle = -0.045$. Using Eq. 4 and $\beta \simeq 0.9$, this translates into a bias in $\langle \ln A \rangle$ of -0.45 or 11% of the proton-iron distance, which makes the composition appear lighter.

V. BIAS FROM BINNING IN ENERGY

In the previous sections, it was discussed how stochastic fluctuations of $N_\mu$ from shower-to-shower, and the additional fluctuations in its estimate $\hat{N}_\mu$ make it difficult to compute (ln $N_\mu$), which is the natural quantity to convert to $\langle \ln A \rangle$. It was assumed throughout that averages over $\ln N_\mu$ and $\ln \hat{N}_\mu$ can be computed for air showers with the exact same shower energy $E$, which is not possible in practice. In the final section, the bias from binning showers in energy is investigated, which is orthogonal to the effects discussed before. We will reach a point in complexity that cannot be handled with simple formulas anymore. The general case should be treated numerically or via a full Monte-Carlo simulation of the experiment.

Complexity is again introduced step-by-step. The true energy $E$ of each shower shall be known, but it now varies randomly from shower to shower. Showers then need to be binned in energy to compute an average of $\ln \hat{N}_\mu$, called $\langle \ln \hat{N}_\mu \rangle$ for distinction. The offset $\langle \ln \hat{N}_\mu \rangle - \langle \ln N_\mu \rangle$ is investigated in the following.

Showers are sorted into a logarithmic energy interval $[\ln E_0, \ln E_1]$. The average $\langle \ln \hat{N}_\mu \rangle$ is compared with the true value at the bin center $\langle \ln E \rangle = \frac{1}{2} (\ln E_0 + \ln E_1)$. The cosmic ray flux has a steeply falling spectrum $E^{-\gamma}$, therefore the event distribution inside the bin is very uneven, with more events near $\ln E_0$. This leads to a bias, since $\langle \ln N_\mu \rangle$ depends on the logarithm of the energy, $\langle \ln N_\mu \rangle = \beta \ln E + c$, where $c$ is a constant and the value of $\beta$ is very close to the one in Eq. 4, although they are not strictly the same. For the calculation, it does not matter whether they are exactly the same.

Lafferty and Wyatt [18] offered a general discussion of binning biases. As a remedy, they propose to adjust the horizontal placement of the data point in the bin. In general, it is simpler and equivalent to just compute the bias and correct for it. We will follow that strategy.

To compute the average value $\langle \ln \hat{N}_\mu \rangle$ over an energy interval $\Delta \ln E = \ln E_1 - \ln E_0$, one has to integrate the argument over the interval weighted by the energy frequency $\propto E^{-\gamma}$. The result is expressed as a function of the expected value $\langle \ln \hat{N}_\mu \rangle = N_\mu^0 + \beta \langle \ln E \rangle$. With $x = \ln E$, $\Delta x = x_1 - x_0$, and $E^{-\gamma}dx = e^{(1-\gamma)x}dx$, I get
\[
\langle \ln \hat{N}_\mu \rangle = \int_{x_0}^{x_1} (\ln N_\mu^0 + \beta x) e^{(1-\gamma)x}dx. \tag{12}
\]

For $\Delta x \ll 1$, I can use Eq. B3 from the appendix to approximate the result
\[
\langle \ln \hat{N}_\mu \rangle = \int_{x_0}^{x_1} (\ln N_\mu^0 + \beta x + (1 - \gamma) \Delta x^2/12) + \mathcal{O}(\Delta x^3). \tag{13}
\]

A typical bin width of 0.1 in log$_{10}$ $E$ is equivalent to $\Delta x \approx 0.23$, so that the higher orders can be neglected. With a spectral index $\gamma = 2.7$, and $\beta \simeq 0.9$, the bias for $\langle \ln \hat{N}_\mu \rangle$ is $-0.007$. This translates into a bias of $-0.07$ for $\langle \ln A \rangle$, about 2% of the proton-iron distance.

Alternatively, Eq. 12 can be solved exactly by partial integration, but the resulting formula provides less insight. The point of this paper is to provide simple formulas to estimate the size of biases, therefore the Taylor expansion is shown here.

Finally, one has to consider that the shower energy $E$ is also only known to a finite resolution. In practice, one only has an estimate $\hat{E}$ that varies stochastically around $E$. As before, it is assumed that $\hat{E}$ is an unbiased estimate for the energy. Events are sorted into energy bins based on the energy estimate $\hat{E}$, therefore also events with true energies outside of the bin interval contribute to the computation of $\langle \ln \hat{N}_\mu \rangle$. Correcting for this effect is conceptually related to the unfolding of resolution effects from distributions [19].

To compute $\langle \ln \hat{N}_\mu \rangle$, we need to convolve the integrand in Eq. 12 with a energy resolution kernel. Usually, a normal distribution is appropriate
\[
\mathcal{N}(\hat{E}; \sigma) = \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{1}{2} (\hat{E} - E)^2/\sigma^2}, \tag{14}
\]
which describes the probability to observe an energy estimate $\hat{E}$ with resolution $\sigma$ for a given true energy $E$. This leads to
\[
\langle \ln \hat{N}_\mu \rangle = \int_{E_0}^{E_1} dE \int_{-\infty}^{\infty} dx \mathcal{N}(\hat{E}; e^x)(\ln N_\mu^0 + \beta x) e^{(1-\gamma)x} \int_{E_0}^{E_1} dE \int_{-\infty}^{\infty} dx \mathcal{N}(\hat{E}; e^x) e^{(1-\gamma)x}. \tag{15}
\]
The binning bias is comparable to the other biases previously considered. For a common bin width of 0.1 in $\log_{10} E$, an energy resolution of 15%, a spectral index $\gamma = 2.7$, and $\beta = 0.9$, the bias for $\langle \ln N_\mu \rangle$ is $-0.03$. This translates into a bias of $-0.3$ for $\langle \ln A \rangle$, about 7% of the proton-iron distance. This bias is making the composition appear lighter.

VI. CONCLUSIONS

The impact of stochastic fluctuations in the number of muons and the shower energy as well as in their experimental estimates on the computation of $\langle \ln A \rangle$ was discussed. Only $\langle \ln N_\mu \rangle$ has a straight-forward relationship to the mean logarithmic mass $\langle \ln A \rangle$ of cosmic rays. The biases calculated here are typically smaller than 10% of the proton-iron distance, but can be larger for detectors with poor resolution. Several may need to be added.

To get the smallest systematic uncertainty, the muon number should be measured event-by-event and the mean logarithmic muon number $\langle \ln N_\mu \rangle$ computed, correcting for resolution and binning effects. A computation based on the mean muon number $\langle N_\mu \rangle$ is possible, but requires a correction that depends on the size of the natural fluctuations of $N_\mu$ for showers of the same energy, more precisely on $\text{Var}[\langle N_\mu - \langle N_\mu \rangle \rangle / \langle N_\mu \rangle]$. This variance has to be measured or estimated from air shower simulations. If simulation results are reported, the variance $\text{Var}[\langle N_\mu - \langle N_\mu \rangle \rangle / \langle N_\mu \rangle]$ should generally be included.

VII. ACKNOWLEDGMENTS

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Figure 4. Variance of the relative deviation from the mean muon number as a function of logarithmic mass $\ln A$ of the primary cosmic ray, using the data from Table I. Solid lines are fits described in the text.

Figure 5. Energy dependence of the fluctuation parameters from Eq. A1. Lines represent the fits described in the text.

Appendix A: Parametrization of relative variance of muon number

The relative variance $\text{Var}[(N_\mu - \langle N_\mu \rangle)/\langle N_\mu \rangle]$ of the muon number for primary cosmic rays with energy $E$ and mass $A$ plays an important role in section III. It is useful to have a parameterization for this quantity. Based on the numbers in Table I, the evolution is shown as a function of $\ln A$ for several energies in Fig. 4. The simulations are well described by the model,

$$\text{Var}[(N_\mu - \langle N_\mu \rangle)/\langle N_\mu \rangle] = p_0(E) + p_1(E)/A, \quad (A1)$$

where $p_0$ and $p_1$ are energy-dependent parameters. The formula is motivated by the superposition model [12], which states that an air shower with $A$ nucleons approximately behaves like a superposition of $A$ showers with energy $E/A$. If the $A$ nucleons develop independently, the fluctuations in the individual sub-showers average out. This leads to a $1/A$ reduction in the variance. The other parameter $p_0$ summarizes correlated fluctuations which do not cancel, for example, fluctuations due to the depth of the first interaction.

The energy-dependence of the parameters $p_0$ and $p_1$ is shown in Fig. 5 and well described by a power law,

$$p_i(E) = a_i (E/10^{15} \text{eV})^{b_i} \quad (A2)$$

$$a_0 = 0.00317 \pm 0.00037 \quad b_0 = -0.295 \pm 0.033$$
$$a_1 = 0.0455 \pm 0.0032 \quad b_1 = -0.0727 \pm 0.0095. \quad (A3)$$

These numerical values are valid for a set of pure primary cosmic rays with mass $A$ of vertical incidence, simulated with SIBYLL-2.3 in a standard atmosphere. To compute the variance for mixtures of primaries, the mean $\langle N_\mu \rangle$ also needs to be parametrized as a function of $A$, which can be done with a power-law as well.
Appendix B: Taylor series

The following Taylor series are used in the paper:

\[
\frac{1}{e^{(1-\gamma)x_0}} \int_{x_0}^{x_0+\Delta x} e^{(1-\gamma)x} x^n dx \approx \\
1 + \frac{\Delta x}{2} \left( (1-\gamma) + \frac{n}{x_0} \right) \\
+ \frac{\Delta x^2}{6} \left( (1-\gamma)^2 + \frac{2n(1-\gamma)}{x_0^{n-1}} + \frac{n(n-1)}{x_0^{n-2}} \right) \\
+ O(\Delta x^3), \quad (B1)
\]

\[
\frac{a_0 + a_1 \Delta x + a_2 \Delta x^2}{b_0 + b_1 \Delta x + b_2 \Delta x^2} \approx \\
\frac{a_0 + x a_1 b_0 - a_0 b_1}{b_0} + x^2 \frac{a_2 b_0^2 - a_1 b_0 b_1 + a_0 (b_1^2 - b_0 b_2)}{b_0^3} + O(x^3). \quad (B2)
\]

Combining these two series, one gets

\[
\frac{1}{x_0^n} \int_{x_0}^{x_0+\Delta x} e^{(1-\gamma)x} x^n dx \approx \\
1 + \frac{\Delta x}{2 x_0} \cdot \frac{n}{x_0} + \frac{\Delta x^2}{6 x_0^2} \left( \frac{n(1-\gamma)}{2x_0} + \frac{n(n-1)}{x_0^2} \right) \\
+ O(\Delta x^3). \quad (B3)
\]