The ideal relativistic rotating gas as a perfect fluid with spin

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We show that the ideal relativistic spinning gas at complete thermodynamical equilibrium is a fluid with a non-vanishing spin density tensor $\sigma_{\mu\nu}$. After having obtained the expression of the local spin-dependent phase space density $f(x,p)_{\sigma\tau}$ in the Boltzmann approximation, we derive the spin density tensor and show that it is proportional to the acceleration tensor $\Omega_{\mu\nu}$ constructed with the Frenet-Serret tetrad. We recover the proper generalization of the fundamental thermodynamical relation, involving an additional term $-(1/2)\Omega_{\mu\nu}\sigma^{\mu\nu}$. We also show that the spin density tensor has a non-vanishing projection onto the four-velocity field, i.e. $t^\mu = \sigma_{\mu\nu}u^\nu \neq 0$, in contrast to the common assumption $t^\mu = 0$, known as Frenkel condition, in the thus-far proposed theories of relativistic fluids with spin. We briefly address the viewpoint of the accelerated observer and inertial spin effects.

I. INTRODUCTION

The study of relativistic fluids with spin has a long history. Among the early major contributors to the subject are Mathisson [1] and Weyssenhoff [2]. The latter formulated a model enforcing the condition of vanishing projection of the spin density tensor over the four-velocity field, namely:

$$t^\mu = \sigma_{\mu\nu}u^\nu = 0 \quad (1)$$

also known as Frenkel condition. This auxiliary constraint onto the fluid motion did not seem justified by established physical principles and for a system of pointlike relativistic massive particles in general does not apply [3]. This condition was in fact critically reexamined in a later work by Bohm and Vigier [4] where they proposed to extend the theory to a more general motion allowing the four-vector $t^\mu \neq 0$, yet without obtaining a conclusive quantitative formulation. A major step in formulating a theory of relativistic fluid with spin was made in 1960 by F. Halbwachs [5] who worked out a Lagrangian variational approach, yet keeping the Frenkel condition as auxiliary equation of motion.

The hydrodynamics of relativistic fluids with spin has since become the subject of specialized literature in General Relativity [6], especially in the framework of Einstein-Cartan gravity theory [7]. Most of these studies assume the Frenkel condition (1), which implies a considerable simplification in the calculations. Since this condition, as has been mentioned, does not derive from any major physical principle, it would be interesting to test it in a simple situation.

Indeed, a system which allows to perform a theoretical independent test exists: the macroscopic ideal spinning gas of particles with spin at full thermodynamical equilibrium. The condition of full thermodynamical equilibrium implies that this study can be carried out within the framework of statistical mechanics, independently of any previous formulation of a hydrodynamical theory and freely from the danger of begging the question.

One of us studied in a previous work [8] the physics of the ideal rotating relativistic gas of particles with spin (in the Boltzmann statistics) at full thermodynamical equilibrium obtaining the expression of the polarization four-vector and momentum spectra. In this paper, we resume that study and derive the local expression of all thermodynamical quantities for this system seen as a fluid. Particularly, we will obtain the expression of the spin tensor and the spin density tensor and show that the latter enters in the general thermodynamical equation relating proper entropy density, proper energy density and pressure, coupled to the acceleration tensor. The nature of this additional term, which has been already extensively used in the aforementioned works [6, 7], is that of a quantum correction to thermodynamics owing to the presence of spin as a further degree of freedom of particles. We will provide the expression of the single-particle spin-dependent phase space density showing that it is a non-diagonal matrix in spin coordinates. We will finally demonstrate that even for this simple system (indeed the simplest among all possible relativistic fluids) the four-vector $t^\mu$ defined in (1) is non-vanishing, thus violating the Frenkel condition.

The paper is organized as follows: in Sect. II we summarize and further explain the main results obtained in the work [8]. In Sect. III we calculate the local thermodynamical densities of the rotating gas viewed as a macroscopic fluid. In Sect. IV we introduce and derive the angular velocity tensor for a rotating fluid and in Sect. V the spin tensor showing its proportionality to the acceleration tensor. In Sect. VI we briefly take the viewpoint of the accelerated fluid cell in the rotating gas.

Throughout this work, we will confine our attention to a special relativistic fluid, i.e. we will disregard gravitational effects and assume a flat Minkowski spacetime.
Notation

In this paper we adopt the natural units, with $\hbar = c = K = 1$. The Minkowskian metric tensor is $\text{diag}(1, -1, -1, -1)$; for the Levi-Civita symbol we use the convention $\epsilon^{0123} = 1$. We will use the relativistic notation with repeated greek indices assumed to be saturated. Contractions will also be denoted by a dot for both vectors and tensors; the saturated index in a contraction between a vector and a rank 2 tensor is specified by the position of the vector, either on the left or right. Double contraction of rank 2 tensors $A$ and $B$ will also be denoted by a colon, i.e. $A : B$.

Space-time linear transformations (translations, rotations, boosts) and SL(2,C) transformations are written in serif font, e.g. $R, L; [p]$ will denote the Lorentz or SL(2,C) transformation taking the time like vector $m\hat{t} = (m, 0, 0, 0)$ to four-momentum $p$. Operators in Hilbert space will be denoted by an upper hat, e.g. $\hat{R}$. Unit vectors will be denoted with a smaller hat e.g. $\hat{p}$; even though the notation is unambiguous, we will make explicit mention of either possibility whenever confusion may arise.

II. ROTATING RELATIVISTIC GAS AT EQUILIBRIUM

A macroscopic system with given angular momentum at complete thermodynamical equilibrium can only have a rigid velocity field:

$$\mathbf{v} = \mathbf{\omega} \times \mathbf{x}$$

where $\mathbf{\omega}$ is a constant vector, as seen from the inertial system where the system is at rest. This is a simple consequence of enforcing statistical equilibrium and is a well known result for non-relativistic systems [9] which can be easily extended to relativistic ones with the additional bound $\|\mathbf{\omega} \times \mathbf{x}\| < 1$ (see Appendix B in ref. [8]). Obviously, for an isolated system, $\mathbf{\omega}$ can be constant only if it is directed along one of the system’s principal axes. Hence, thermodynamical equilibrium of a rotating body requires $\mathbf{\omega}$ to be parallel to $\mathbf{J}$ and, therefore, that the system’s shape is symmetric for rotation around $\mathbf{\omega}$.

The whole physics of an equilibrated system with finite angular momentum $\mathbf{J}$, chemical potential $\mu$ and temperature $T$ is fully contained in the density operator $\hat{\rho}_J$:

$$\hat{\rho}_J = \frac{1}{Z_J} \exp\left[-\hat{H} + \mu \hat{Q}\right]/T \mathbf{P}_J \mathbf{P}_V$$

where $\mathbf{P}_J$ is the quantum Wigner projector onto a given state $|J, M\rangle$:

$$\mathbf{P}_J = \frac{1}{M(G)} \int_{SU(2)} dg \frac{D^J\{g^{-1}\}_M^M}{\tilde{g}}$$

$Z_J$ is the partition function:

$$Z_J = \text{tr} \exp\left[-\hat{H} + \mu \hat{Q} + \mathbf{\omega} \cdot \hat{\mathbf{J}}\right]/T \mathbf{P}_V$$

and $\mathbf{P}_V$ is the projector onto localized states $|h_V\rangle$

$$\mathbf{P}_V = \sum_{h_V} |h_V\rangle\langle h_V|$$

which form a complete set of quantum states for the system in its finite region $V$. In a first quantization scheme, these states are a complete set of functions which vanish outside the domain $V$. On the other hand, in a quantum field theory, there are some complications which have been discussed extensively in refs. [10, 11], yet they do not essentially modify the first-quantization scenario (see also Appendix A).

The density operator (3) defines a grand-canonical ensemble with fixed angular momentum. If both $V$ and $\mathbf{J}$ are large, it can be shown [8, 11] that this ensemble is equivalent (at least for first moments of statistical distributions) to the rotational grand-canonical ensemble whose density operator reads:

$$\hat{\rho}_\omega = \frac{1}{Z_\omega} \exp\left[-\hat{H} + \mu \hat{Q} + \mathbf{\omega} \cdot \hat{\mathbf{J}}\right]/T \mathbf{P}_V$$

being $\mathbf{\omega}$ the angular velocity vector and $Z_\omega$ the relevant partition function:

$$Z_\omega = \text{tr} \exp\left[-\hat{H} + \mu \hat{Q} + \mathbf{\omega} \cdot \hat{\mathbf{J}}\right]/T \mathbf{P}_V$$
The transition from (3) to (7), or from (5) to (8) can be carried out through a large $J$ limit of the Wigner rotation matrices in (4) and a subsequent saddle-point expansion in the angular variables defining the SU(2) parameters in the integral (4); for details see refs. [8, 11]. For studying macroscopic rotating systems, the ensemble (7) is thus the most natural choice. It can be shown that indeed, as long as $\omega/T \ll 1$, the partition function (8) can be written as the product of grand-canonical partition functions of infinitesimal cells moving with a velocity (2), according to the general wisdom that equilibrated macroscopic systems must be rigidly rotating.

It is worth showing this in some detail for an ideal relativistic gas of massive particles with spin $S$ in the limit of Boltzmann statistics, i.e. considering particle as distinguishable objects. In this case, we can simply write the partition function (8) as a product of single-particle partition functions and the calculation of (8) in fact reduces to compute matrix elements of single-particle compatible operators $\hat{h}, \hat{q}, \hat{j}$, like this:

$$\langle p, \tau \rfloor \exp[-(\hat{h} + \mu \hat{p} + \omega \cdot \hat{j})/T] | P_V \rangle [p, \sigma] = e^{\epsilon/T + \mu q/T} \langle p, \tau \lfloor \exp[\omega \cdot \hat{j}/T] | P_V \rangle [p, \sigma] \tag{9}$$

where $\sigma$ and $\tau$ label polarization states $^1$. The above equality requires that, as mentioned, $[\hat{h}, \hat{j}] = [\hat{q}, \hat{j}] = [\hat{q}, \hat{h}] = 0$.

The matrix element on the right hand side of eq. (9) can be calculated by means of an analytical prolongation from imaginary $\omega$ values. After replacing $\omega/T$ with $-i \phi$ we have a rotation $R_\omega(\phi)$ around the axis $\omega$ of an angle $\phi$ appearing in the eq. (9). We can then expand the matrix element on the right hand side as:

$$\langle p, \tau \lfloor R_\omega(\phi) | P_V \rangle [p, \sigma] = \sum_{\sigma'} \int d^3p' \langle p, \tau \lfloor R_\omega(\phi) | p', \sigma' \rangle \langle p, \sigma' \rfloor | P_V \rangle [p, \sigma] \tag{10}$$

Note that this expansion would have been impossible for the operator $\exp[\omega \cdot \hat{j}/T]$ itself because, for real $\omega$, all its matrix elements calculated with momentum eigenstates turn out to be infinite. In other words, the correct procedure to make the analytic continuation of the matrix element is, first, go to imaginary $\omega$ values and thereafter insert an identity resolution like in (10). The matrix element of the representation of a rotation is well known; it involves a Dirac’s delta and a Wigner rotation matrix:

$$\langle p, \tau \lfloor R_\omega(\phi) | p', \sigma' \rangle = \delta^3(p - R_\omega(\phi)p') D^S((R_\omega(\phi)p')^{-1}R_\omega(\phi)p')_{\tau\sigma} \tag{11}$$

where $[p]$ is defined as a Lorentz transformation (in fact its SL(2,C) correspondent) taking the timelike vector $m = (m, 0, 0, 0)$ to $p$ and $D^S$ denotes, as usual, the $(2S+1)$-dimensional SL(2,C) representation (what is commonly labelled as the $(S,0)$ in fact). Obtaining the matrix element of the projector $P_V$ over momentum eigenstates is not so straightforward. The calculation can be done in a quantum field theoretical framework [11]:

$$\langle p', \sigma' \rfloor | P_V \rangle [0] = \frac{1}{2} \sqrt{\frac{2}{\varepsilon}} \int_V d^3x \; e^{i\epsilon(p - R_\omega(\phi)^{-1}p')} \left( D^S([p]^{-1}R_\omega(\phi)[p]) + D^S([p][p]^{-1}R_\omega(\phi)[p]^{-1}) \right)_{\tau\sigma} \langle 0 | P_V | 0 \rangle \tag{12}$$

where the last factor $\langle 0 | P_V | 0 \rangle$, the vacuum expectation value of the projector $P_V$, is an immaterial factor [10, 11] and tends to 1 for large volumes $V$ as $P_V \rightarrow 1$; we henceforth neglect it. The formula (12) gives the correct infinite volume limit $P_V \rightarrow 1$, that is the momentum eigenstate normalization. For finite volumes, a striking feature is the presence of non-trivial matrix elements in spin coordinates; their particular form can be justified also by requiring $P_V$ to commute with rotation operators (see Appendix A).

Plugging (11) and (12) into (10), and neglecting $P_V$ vacuum expectation value, one obtains:

$$\langle p, \tau \lfloor R_\omega(\phi) | P_V \rangle [p, \sigma] = \int_V d^3x \; e^{i\epsilon(p - R_\omega(\phi)^{-1}p')} \left( \frac{1}{2} (D^S([p]^{-1}R_\omega(\phi)[p]) + D^S([p][p]^{-1}R_\omega(\phi)[p]^{-1}) \right)_{\tau\sigma} \tag{13}$$

In order to get to this expression, advantage needs to be taken of the unitarity of the Wigner rotation, i.e.:

$$D^S([R_\omega(\phi)p')^{-1}R_\omega(\phi)p']) = D^S([R_\omega(\phi)p']R_\omega(\phi)[p']^{-1}) \tag{14}$$

and the unitarity of $R$ itself as an SL(2,C) matrix. The equation (13) can now be analytically prolonged to imaginary angles yielding the final expression for the matrix element in (9):

$$\langle p, \tau \lfloor \exp[\omega \cdot \hat{j}/T] | P_V \rangle [p, \sigma] = \int_V d^3x \; e^{i\epsilon(p - R_\omega(\omega/T)^{-1}p')} \left( \frac{1}{2} (D^S([p]^{-1}R_\omega(i\omega/T)[p]) + D^S([p][p]^{-1}R_\omega(i\omega/T)[p]^{-1}) \right)_{\tau\sigma} \tag{15}$$

$^1$ For the spin quantum numbers $\sigma$ we follow the convention in [12], where they are defined as eigenvalues of the projection of the Pauli-Lubanski vector divided by the mass onto the third space-like four-vector orthogonal to the particle four-momentum. For more details see ref. [12] and [11].
The matrix element (15) is the key ingredient to calculate the equilibrium single-particle phase space distribution and thence, the thermodynamics of the rotating Boltzmann gas; it is then worth some comment.

Firstly, it should be noted that if the system’s shape is symmetric by rotation around the \( \hat{\omega} \) axis, as has been mentioned, the projector \( P_V \) commutes with the corresponding generator in the Hilbert space:

\[
[P_V, \hat{J} \cdot \hat{\omega}] = 0
\] (16)

Therefore, the operator on the left-hand side of eq. (15) is hermitian and so, the expectation value on the left hand side ought to be real. This is easily seen for the spin matrices on the right-hand-side as \( R_\omega(i\omega/T) = \exp[\omega \cdot J/T] \) is hermitian. Showing that the integral on the right-hand-side of eq. (15) is real requires a more elaborate calculation which is carried out in Appendix B.

Secondly, it should be pointed out that the matrix element in (9) has a spacial integral form which allows us to readily write a phase-space distribution:

\[
f(x, p)_{\tau \sigma} = \lambda e^{-\varepsilon/T} e^{i 4 \pi \cdot x (p - R_\omega(i\omega/T)^{-1}(p))} \frac{1}{2} \left( D^S(|p|^{-1}R_\omega(i\omega/T)|p|) + D^S(|p|^1R_\omega(i\omega/T)|p|^{-1}) \right)_{\tau \sigma}
\] (17)

where \( \lambda = \exp[\mu q/T] \) is the fugacity. Unlike its integral over the system’s region, the function in (17) has, strictly speaking, no straightforward physical meaning and it does not need to be real or positive. Eventually, one could could calculate from it some more meaningful function like the Wigner’s function. However, for very large systems, one expects the function (17) to become the actual phase-space single-particle distribution. For a cylindrically symmetric system to be macroscopic, the product \( TR \) being \( R \) its maximal transverse radius, is supposed to be large, namely:

\[
TR \gg 1
\]

Hence, since a relativistic bound \( \omega R < 1 \) is implied in eq. (2), for proper macroscopic systems (and indeed for most practical purposes) the ratio between \( \omega \) and \( T \) is small, i.e.:

\[
\frac{\hbar \omega}{KT} \ll 1
\] (18)

where we have purposely restored the natural constants. Consequently, the difference between momenta in the exponent of (17) can be well approximated by the lowest order term in \( \omega/T \), i.e.:

\[
p - R_\omega(i\omega/T)^{-1}(p) = p - \left[ \cosh \frac{\omega}{T} p - i \sinh \frac{\omega}{T} \hat{\omega} \times p + (1 - \cosh \frac{\omega}{T}) p \cdot \hat{\omega} \right] = \frac{\omega}{T} \hat{\omega} \times p
\] (19)

so that the phase-space distribution function (17) reduces to:

\[
f(x, p)_{\tau \sigma} = \lambda \exp \left[ -\varepsilon/T - x \cdot (\omega \times p)/T \right] \frac{1}{2} \left( D^S(|p|^{-1}R_\omega(i\omega/T)|p|) + D^S(|p|^1R_\omega(i\omega/T)|p|^{-1}) \right)_{\tau \sigma}
\]

\[
= \lambda \exp \left[ -\varepsilon/T + p \cdot (\omega \times x)/T \right] \frac{1}{2} \left( D^S(|p|^{-1}R_\omega(i\omega/T)|p|) + D^S(|p|^1R_\omega(i\omega/T)|p|^{-1}) \right)_{\tau \sigma}
\]

\[
= \lambda \exp \left[ -\varepsilon/T + p \cdot v/T \right] \frac{1}{2} \left( D^S(|p|^{-1}R_\omega(i\omega/T)|p|) + D^S(|p|^1R_\omega(i\omega/T)|p|^{-1}) \right)_{\tau \sigma}
\] (20)

where we have used eq. (2). The eq. (20) is the unnormalized single-particle phase-space distribution for the ideal rotating relativistic Boltzmann gas, seen by the inertial frame, and it turns out to be a non-diagonal matrix in the spin coordinates. The eq. (20) is one of the main results of this work and clearly shows that for a rotating gas the polarization states are not evenly populated. Furthermore, if the total number of particles is not fixed, (20) coincides with the density of particles in phase-space, including spin degrees of freedom. In order to obtain the properly called phase-space density in \( (x, p) \) we have to take the trace of the matrix (20):

\[
f(x, p) = \sum_{\sigma} f(x, p)_{\sigma \sigma} = \lambda e^{-\varepsilon/T + p \cdot v/T} \chi \left( \frac{\omega}{T} \right)
\] (21)

being:

\[
\chi \left( \frac{\omega}{T} \right) = \text{tr} D^S(R_\omega(i\omega/T)) = \frac{\sinh(S + \frac{1}{2}) \hbar \omega}{\sinh \left( \frac{\hbar \omega}{2T} \right)}
\] (22)
The exponent in (21) can be rewritten in a covariant form by introducing the temperature four-vector:

\[ \beta = \frac{1}{\gamma T} (\gamma, \gamma v) = \frac{1}{\gamma T} \mu = \frac{1}{T_0} \mu \]  

(23)

being \( \gamma = (1 - v^2)^{-1/2} \). Thereby, the invariant local temperature \( T_0 = 1/\sqrt{\beta} \) measured in the comoving frame differs by a factor \( \gamma \) from the global temperature measured by the inertial observer. Since \( \gamma \) increases as a function of the radial distance, in a rotating gas at equilibrium there will be a steady gradient of local temperature; since also \( \mu/T \) must be constant, there will also be a steady gradient of the local chemical potential, i.e. the local chemical potential will be given by

\[ \mu_0 = \gamma \mu \]  

(24)

both these effects have been pointed out by Israel [13]. Finally, the phase-space distribution can be written in the compact form:

\[ f(x, p) = \lambda e^{-\beta \cdot p} \frac{1}{2} (D^S([p]^{-1} R_0(i\omega/T)[p]) + D^S([p]^{-1} R_0(i\omega/T)[p])^{-1})_{\sigma \tau} \]  

(25)

which definitely shows that the ideal rotating gas is made of cells moving at the four-velocity \( u(x) \) pertaining to a rigid motion.

The rotational grand-canonical partition function of the ideal Boltzmann gas can be readily calculated by using the single-particle matrix element of eq. (15) for \( \sigma = \tau \), integrating over momenta, summing over polarization states and finally summing over all possible particle multiplicities \( N \), dividing by \( N! \), as usual. This eventually yields the simple result, in the approximation (18):

\[ Z_\omega = \exp \left[ \int_V d^3x \int d^3p f(x, p) \right] = \exp \left[ \lambda \chi \left( \frac{\omega}{T} \right) \int_V d^3x \int d^3p e^{-\beta \cdot p} \right] \]  

(26)

### III. THE IDEAL ROTATING GAS AS A PERFECT FLUID

We have shown in the previous section that the globally equilibrated rotating gas can be seen as a set of locally equilibrated cells moving with a rigid velocity field (2), at least in the limit \( \omega/T \ll 1 \). In this section, we will reinforce this interpretation providing the relevant stress-energy tensor and the local thermodynamic relation.

The stress-energy tensor \( T^{\mu \nu} \) can be calculated directly from particle phase space density \( f(x, p) \):

\[ T^{\mu \nu} = \int \frac{d^3p}{\varepsilon} p^\mu p^\nu f(x, p) \]  

(27)

thus, for the ideal relativistic Boltzmann gas, according to (21), and using the four-temperature vector \( \beta \):

\[ T^{\mu \nu} = \lambda \chi \left( \frac{\omega}{T} \right) \frac{\partial}{\partial \beta^\mu} \frac{\partial}{\partial \beta^\nu} \int \frac{d^3p}{\varepsilon} e^{-\beta \cdot p} \]  

(28)

The integral on the right-hand side of (28) is apparently Lorentz-invariant, therefore it can only depends on \( \sqrt{\beta^2} = 1/T_0 \); if we define it as \( F(\sqrt{\beta^2}) \) we get, after some algebra:

\[ T^{\mu \nu} = \lambda \chi \left( \frac{\omega}{T} \right) F''(\sqrt{\beta^2}) \frac{\beta^\mu \beta^\nu}{\beta^2} + \lambda \chi \left( \frac{\omega}{T} \right) F'(\sqrt{\beta^2}) \frac{g^{\mu \nu} \beta^2 - \beta^\mu \beta^\nu}{\sqrt{\beta^2}} \]  

\[ = \lambda \chi \left( \frac{\omega}{T} \right) F''(\sqrt{\beta^2}) u^\mu u^\nu + \lambda \chi \left( \frac{\omega}{T} \right) F'(\sqrt{\beta^2}) \frac{g^{\mu \nu} - u^\mu u^\nu}{\sqrt{\beta^2}} \]  

(29)
We are therefore led to identify the pressure and the proper energy density with \[ p = -\lambda \chi \left( \frac{\omega}{T} \right) \frac{F'(\sqrt{\beta^2})}{\sqrt{\beta^2}} = T \int d^3p \, f(x, p) \] \[ \rho = \lambda \chi \left( \frac{\omega}{T} \right) F''(\sqrt{\beta^2}) = \int d^3p \, (\epsilon - v \cdot p) f(x, p) \] (30)

and the stress-energy tensor thereby becomes:

\[ T^{\mu\nu} = (\rho + p) u^\mu u^\nu - pg^{\mu\nu} \] (31)

which is the required form for a fluid at full thermodynamical equilibrium [13]. The identification of pressure with the expression in eq. (30) dictates that \( \omega/T = \gamma \omega/T_0 \) is a Lorentz invariant quantity; this will be further discussed in Sect. V.

The correctness of the identification of pressure and proper energy density is confirmed by the analysis of the basic thermodynamical equation. For a general system in the rotational grand-canonical ensemble, entropy can be obtained from its definition:

\[ S = -\text{tr}[\hat{\rho}_\omega \log \hat{\rho}_\omega] \]

By using the expression of \( \hat{\rho}_\omega \) in (7), we readily arrive at the basic thermodynamical relation for this ensemble involving the expectation values of global quantities:

\[ TS = \langle E \rangle - \mu \langle Q \rangle + T \log Z\omega - \omega \cdot \langle J \rangle \] (32)

where \( T \) is the global temperature. The equation (32) can be transformed into a local relation by taking into account that \( \log Z\omega \) is in fact a spatial integral over the system's domain \( V \). Comparing (26) with the first of (30) we can write:

\[ \log Z\omega = \frac{1}{T} \int d^3x \, p(x) \] (33)

This equation has been derived here for the ideal Boltzmann gas, but it holds for a general rotating system at thermodynamical equilibrium, as we will see. The mean values of energy, charge and angular momentum can be obtained from the partition function (8) by taking derivatives with respect to temperature, chemical potential and angular velocity. As the operators \( \hat{Q}, \hat{H} \) and \( \hat{J} \) commute with each other, this gives:

\[ \frac{\partial \log Z\omega}{\partial T} = \frac{1}{T^2} \left( \langle \hat{H} \rangle - \mu \langle \hat{Q} \rangle + \omega \cdot \langle \hat{J} \rangle \right) \]

\[ \frac{\partial \log Z\omega}{\partial \mu} = \frac{1}{T} \langle \hat{Q} \rangle \]

\[ \frac{\partial \log Z\omega}{\partial \omega} = \frac{1}{T} \langle \hat{J} \cdot \hat{\omega} \rangle \] (34)

Thus, if \( \log Z\omega \) is an integral over the system’s domain of some function, the same happens for its derivatives and one can then re-write the entropy as an integral:

\[ TS = T \int_V d^3x \, \frac{dS}{d^3x} = \int_V d^3x \, \frac{d\langle E \rangle}{d^3x} - \mu \frac{d\langle Q \rangle}{d^3x} + \frac{d\log Z\omega}{d^3x} - \omega \cdot \frac{d\langle J \rangle}{d^3x} \] (35)

The global temperature \( T \) is independent of \( x \) and one can then assume that the above relation holds locally:

\[ T \frac{dS}{d^3x} = \frac{d\langle E \rangle}{d^3x} - \mu \frac{d\langle Q \rangle}{d^3x} + T \frac{d\log Z\omega}{d^3x} - \omega \cdot \frac{d\langle J \rangle}{d^3x} \] (36)

\[ \frac{\beta \mu}{\sqrt{\beta^2}} \frac{\partial}{\partial \beta^0} = \frac{d}{d\sqrt{\beta^2}} \]
In general, the angular momentum density can be written as the sum of an orbital and a spin part:

\[
\frac{d\mathbf{J}}{d^3x} = \mathbf{x} \times \frac{d\mathbf{P}}{d^3x} + \frac{d\mathbf{S}}{d^3x} \tag{37}
\]

where \(d\mathbf{P}/d^3x\) is the linear momentum density. For the present, such decomposition can be explicitly demonstrated, for the ideal relativistic Boltzmann gas, only for the component of the angular momentum along \(\omega\). Indeed, plugging in the eq. (34) the partition function (26) and using the (23) and (2):

\[
\langle \mathbf{J} \cdot \mathbf{\omega} \rangle = T \frac{\partial \log Z_\omega}{\partial \mathbf{\omega}} = \int d^3x \cdot \int d^3p \ (\mathbf{x} \times \mathbf{p}) f(\mathbf{x}, \mathbf{p}) + \frac{\chi'(\mathbf{p})}{\chi(\mathbf{p})} f(\mathbf{x}, \mathbf{p}) \tag{38}
\]

which allows us to clearly identify the orbital and spin part of the angular momentum density. The complete calculation of all the components will be presented in Sect. V.

By using the eq. (37) and recalling the relations \(T = T_0/\gamma\), \(\mu = \mu_0/\gamma\), the (36) can be rewritten as:

\[
T_0 \frac{1}{\gamma} \frac{dS}{d^3x} = \gamma \frac{1}{\gamma} \frac{d\langle E \rangle}{d^3x} - \gamma (\mathbf{\omega} \times \mathbf{x}) \frac{1}{\gamma} \frac{d\mathbf{P}}{d^3x} - \mu_0 \frac{1}{\gamma} \frac{d\langle Q \rangle}{d^3x} + T_0 \frac{1}{\gamma} \frac{d\log Z_\omega}{d^3x} - \gamma \mathbf{\omega} \cdot \frac{1}{\gamma} \frac{d\mathbf{S}}{d^3x} \tag{39}
\]

where we have multiplied and divided by \(\gamma = (1 - v^2)^{-1}\) where needed. Now:

\[
\gamma \frac{d\langle E \rangle}{d^3x} - \gamma (\mathbf{\omega} \times \mathbf{x}) \cdot \frac{d\mathbf{P}}{d^3x}
\]

is manifestly the energy of the local cell with volume \(d^3x\) seen in the cell’s comoving frame, i.e. its mass density, so that we can also write the (39) as:

\[
T_0 \frac{1}{\gamma} \frac{dS}{d^3x} = \frac{1}{\gamma} \frac{d\langle M \rangle}{d^3x} - \mu_0 \frac{1}{\gamma} \frac{d\langle Q \rangle}{d^3x} + T_0 \frac{1}{\gamma} \frac{d\log Z_\omega}{d^3x} - \gamma \mathbf{\omega} \cdot \frac{1}{\gamma} \frac{d\mathbf{S}}{d^3x} \tag{40}
\]

In the eq. (40) now appear densities rescaled by the Lorentz factor \(\gamma\), i.e. densities measured in the comoving frame or proper densities. We can therefore write the above equality in the more familiar form:

\[
T_0 s = \rho - \mu_0 q + p - \gamma \mathbf{\omega} \cdot \mathbf{\sigma} \tag{41}
\]

where \(s\) is the proper entropy density, \(\rho\) the proper energy density, \(q\) the proper charge density and \(\mathbf{\sigma}\) the proper spin density. The term \(T d\log Z_\omega/d^3x\) has been identified with the pressure on the basis of (33), which has been proved for the ideal Boltzmann gas. Nevertheless, since only through this identification the eq. (41) gets the form of the basic local thermodynamical relation, the (33) must have a general validity.

With respect to the usually known form, the (41) contains an additional term owing to the non-vanishing spin density. In the non-relativistic limit this term becomes \(\mathbf{\omega} \cdot \mathbf{\sigma}\) and is in fact responsible for the Barnett effect [14], i.e. the spontaneous magnetization of a body spun around its axis when reaching thermodynamical equilibrium [15]. Unlike all other terms, the spin-rotation term has a non-covariant form. Expressing it, as necessary, in a fully covariant form, requires the introduction of two tensors, the acceleration tensor \(\Omega\) and the spin density tensor \(\mathbf{\sigma}\). It will then become clear that the general covariant form of (41) is:

\[
T_0 s = \rho - \mu_0 q + p - \frac{1}{2} \Omega_{\mu\nu} \sigma^{\mu\nu} \tag{42}
\]

Deriving the tensors \(\Omega\) and \(\mathbf{\sigma}\) will be the subjects of the two forthcoming sections.

**IV. THE ACCELERATION TENSOR**

What is the right relativistic generalization of the \(\omega\) vector in a rigid velocity field (2)? This is the acceleration tensor \(\Omega\), which is discussed for instance in ref. [16], as well as in several papers quoted in the bibliography, e.g. [5]. For the ease of reading, we spend this Section to introduce and describe the main features of this tensor.

The starting point is the definition of an observer and a moving frame in the (Minkowski) spacetime, which is meant to be a generalization of an accelerated frame in non-relativistic physics. An observer is simply a world-line in spacetime. To this world-line one can attach a set of four unit vectors, called a tetrad, which are a basis of the tangent space in each point of the world-line are in fact a reference frame for the observer. The four tetrad vectors \(e_i\)
are labelled by indices with the convention that the time-like vector $e_0$ is the four-velocity $u$, i.e. the vector tangent to the world-line. The other three space-like unit vectors can be chosen at will. The tetrad is orthonormal in the tangent space, thus:

$$e_i \cdot e_j = e_i^\mu e_j^\mu = \eta_{ij}$$

being $\eta = \text{diag}(1, -1, -1, -1)$. Furthermore, any vector of the tangent space can be decomposed along the tetrad, hence the tetrad is complete:

$$l = \sum_i e_i \otimes e^i \implies \sum_i e_{i\mu} e_i^\mu = g_{\mu\nu} \quad (43)$$

Since the tetrads at different space-time points of the world-line are orthonormal, the evolution of a tetrad along the observer world-line can be described by means of the Lorentz transformation $\Lambda$ connecting the initial tetrad at the proper time $\tau_0$ with that at the proper time $\tau$. Taking advantage of the group structure of Lorentz transformations, it is also possible to write the tetrad at the time $\tau + d\tau$ as:

$$e_i(\tau + d\tau) = \Lambda(\tau + d\tau)(e_i(0)) = \Lambda'(d\tau)\Lambda(\tau)((e_i(0)) = \Lambda'(d\tau)((e_i(\tau))$$

where $\Lambda'$ is the infinitesimal Lorentz transformation connecting $e_i(\tau)$ to $e_i(\tau + d\tau)$. Using the generators of Lorentz group $J_{\mu\nu}$:

$$e_i(\tau + d\tau) = \left[ 1 - \frac{i}{2} d\phi^{\mu\nu} J_{\mu\nu} \right] ((e_i(\tau)) = e_i(\tau) - \frac{i}{2} \Omega^{\mu\nu} d\tau J_{\mu\nu}((e_i(\tau))$$

where, by definition $\Omega^{\mu\nu} d\tau$ are the parameters of that infinitesimal Lorentz transformation. Reminding that:

$$(J_{\mu\nu})^\alpha_\beta = i \left( \delta^\alpha_\mu g_{\nu\beta} - \delta^\alpha_\nu g_{\mu\beta} \right)$$

and that $\Omega^{\mu\nu}$ is an antisymmetric tensor, we can then rewrite (44) as:

$$\frac{De_i^\alpha}{d\tau} = \dot{e}_i^\alpha = -\frac{i}{2} \Omega^{\mu\nu} (J_{\mu\nu})^\beta_\delta e_i^\delta = \Omega^{\alpha\beta} e_i^\beta \quad (45)$$

or, in a more compact way:

$$\dot{e}_i = \Omega \cdot e_i \quad (46)$$

Although we are working in a flat spacetime, we emphasized with symbols needed for general curved spacetimes, that the derivative in (45) is indeed the covariant derivative of the tetrad vector along the world-line, what is also denoted with an upper dot. If we are dealing with a fluid, then we will have to consider a tetrad field $e_i(x)$, because to each hydrodynamical cell will be associated a world-line. In hydrodynamics, the derivative along the cell trajectory is called the convective derivative and can be expressed as $u \cdot \partial$.

The tensor $\Omega$ in eq. (45) is the correct generalization of the angular velocity and acceleration and will be henceforth referred to as the acceleration tensor, following ref. [17], as has been mentioned. The eq. (45) generalizes the classical equations of motion of the axes of a rigid frame in space, i.e.:

$$\frac{d\dot{e}}{d\tau} = \omega \times \dot{e} \quad (47)$$

We will make the relation between $\Omega^{\mu\nu}$ and $\omega$ explicit for the uniform rotating fluid shortly; for the general relation, see Appendix C. The equation (45) can be easily inverted to obtain an explicit expression of $\Omega^{\mu\nu}$. For this purpose, one has to multiply both sides of (45) by $e^i_\beta$, sum over the index $i$ and take advantage of the completeness relation (43). The result is:

$$\Omega^{\mu\nu} = \sum_i e_i^\mu e^\nu_i \quad (48)$$

In this form, the tensor $\Omega$ is not manifestly antisymmetric, but this can be readily shown by using again (43):

$$\sum_i \dot{e}_i^\mu e^\nu_i = \sum_i (e_i^\mu e^\nu_i) - \sum_i e_i^\mu \dot{e}_i^\nu = \dot{g}_{\mu\nu} - \sum_i e_i^{\mu\nu} e_i^\nu = -\sum_i e_i^{\mu\nu} \dot{e}_i^\nu$$
We can now move on to determine the acceleration tensor for the rigidly rotating fluid. Yet, before doing this, we face another fundamental question, that is how to attach a tetrad to a hydrodynamical cell. When introducing the tetrad, we have invoked the world-line of an “observer”, which is always tacitly associated with an inner reference frame made of three orthogonal unit vectors, whereas for a material fluid pointlike cell, such association is not so straightforward and could be done in several manners. For instance, it can be done by using purely geometrical considerations, like e.g. Fermi-Walker transport or considering the instantaneous Lorentz boost without rotation connecting the tetrad to the inertial frame axes; or, rather, it can be specified by directions relevant to actual physical quantities like e.g. the spin density tensor discussed in the next section. In general, we would like to have the most natural definition of the tetrad, possibly determined only by the four-velocity $u(\mathbf{x})$ without invoking additional physical fields. Such a natural tetrad exists and it is provided by the Frenet-Serret unit vectors associated to the world-lines of hydrodynamic cells [18], i.e. the four-velocity field tangent lines. Even if this definition depends solely on the field $u(\mathbf{x})$, it is not unique because, at it is well-known, the four-velocity can be associated with the motion of charge (Eckart frame) or energy (Landau frame). Notwithstanding, in the present case of full thermodynamical equilibrium, all velocities must be the same [13] and for a macroscopic fluid with finite angular momentum are given by (2). We will later show, in Sect. V, that this tetrad is precisely the one which makes the spin density tensor, at least for an ideal Boltzmann gas, proportional to the acceleration tensor. Other choices would result in an acceleration tensor not proportional to the spin density tensor.

For a uniformly rotating fluid, the Frenet-Serret vectors can be easily determined and are those corresponding to the axes of a rigidly rotating frame, that is, in the inertial frame components (see fig. 1):

\[ e_0 = u = (\gamma, \gamma v) = (\gamma, \gamma \gamma \hat{j}) \quad e_1 = (0, \hat{i}) \quad e_2 = (\gamma v, \gamma \hat{j}) \quad e_3 = (0, \hat{k}) \]

whence we can calculate the derivatives along the world-lines. For $i = 1, 2, 3$:

\[ \frac{d e_i}{d \tau} = \gamma \frac{d e_i}{d t} = (0, \gamma \omega \times e_i) \]

because $\gamma$ and $v$ are constant along the field lines. Therefore, the Frenet-Serret tetrad is such that the time derivative, in the fixed inertial frame, of the space vectors yields the simple law (47) involving the assigned $\omega$ vector. Taking the
derivative of $e_0$, one gets the acceleration four-vector $A$:

$$\frac{de_0}{d\tau} = \frac{du}{d\tau} \equiv A = \gamma \frac{du}{d\tau} = (\gamma^4 v \cdot a, \gamma^4 v \cdot a + \gamma^2 a) = (0, \gamma^2 a) = (0, \gamma^2 \omega \times v) \quad (51)$$

where, by definition, $a \equiv dv/dt$ and we have used the explicit form of the velocity field (2) implying $a = \omega \times v$. We are now in a position to calculate the acceleration tensor from the formula (48). The easiest method is to multiply $\Omega$ scalarly on both sides by the basis unit vectors of the inertial frame, that is:

$$\Omega_{\mu\nu} = E_\mu \cdot \sum_i \dot{e}_i \otimes e_i \cdot E_\nu \quad (52)$$

which leads us to, being $\omega = \omega \hat{k} = \omega \hat{k}'$, and using eqs. (51) and (50):

$$\Omega = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma \omega & 0 \\ 0 & -\gamma \omega & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (53)$$

It is worth pointing out that the double contraction of $\Omega$ with itself yields:

$$\Omega : \Omega \equiv \Omega_{\mu\nu} \Omega^{\mu\nu} = 2 \gamma^2 \omega^2 \quad (54)$$

and, therefore, $\gamma \omega$ must be a Lorentz invariant; this is further illustrated in Appendix C. If $\gamma \omega$ is a Lorentz invariant, so is the ratio $\omega/T$ as:

$$\frac{\omega}{T} = \frac{\omega}{\gamma T} = \sqrt{\frac{\Omega : \Omega}{I_0}} = \sqrt{\frac{1}{2} \beta^2 \Omega : \Omega} \quad (55)$$

As a consequence, pressure in eq. (30) can be written in a manifestly covariant form:

$$p = \lambda F'(\sqrt{\beta^2}) \chi \left( \sqrt{\frac{1}{2} \beta^2 \Omega : \Omega} \right) \quad (56)$$

as well as all other local thermodynamical quantities.

V. THE SPIN TENSOR

We now come to the central subject of this work, i.e. the determination of the spin density tensor for the ideal rotating gas and its relation with the acceleration tensor. Some preliminary definitions are necessary. For a fluid, angular momentum conservation is expressed by an equation like:

$$\partial_\lambda J_{\lambda,\mu\nu} = \partial_\lambda (S_{\lambda,\mu\nu} + x^\mu T^{\lambda\nu} - x^\nu T^{\lambda\mu}) = 0 \quad (57)$$

where $S_{\lambda,\mu\nu}$ is a rank 3 tensor antisymmetric in the $\mu\nu$ indices commonly known as spin tensor. In classical field theories, the spin tensor can be made vanishing by a suitable redefinition of the stress-energy tensor, well known as Belinfante symmetrization procedure. However, such procedure does not leave the thermodynamic relation (42) invariant even when $\Omega = 0$, because both energy density and pressure are affected by the Belinfante transformation (see Appendix D for a detailed calculation) and the transformed entropy would have a different local and global value, which is unacceptable. The profound reason of the lack of the freedom of changing spin tensor and stress-energy tensor in presence of thermodynamics, is the fact that entropy is an effective counting of degrees of freedom, a concept tightly related to the discrete and quantum nature of matter and definitely outside a purely classical field mechanical framework.

At full thermodynamical equilibrium we must have a symmetric stress-energy tensor in the form (31) and therefore the spin tensor must be conserved as well:

$$\partial_\lambda S_{\lambda,\mu\nu} = 0$$
Furthermore, to meet the requirement that all currents must be parallel at full thermodynamical equilibrium, the spin tensor must be factorizable into an antisymmetric spin density tensor $\sigma$ and the four-velocity:

$$S^{\lambda,\mu\nu} = \sigma^{\mu\nu} u^\lambda$$

(58)

To determine the spin density tensor for a rotating ideal Boltzmann gas, it is necessary to calculate total angular momentum density and subtract the orbital part. A separation between angular and spin part has already been encountered in eq. (38) for the component of the total angular momentum along $\omega$. The orbital angular momentum density must look like:

$$\frac{dL}{dx} = \int d^3 p \ (x \times p) f(x, p)$$

hence, if such a form is retrieved in the calculation of the total angular momentum, the rest can be unambiguously identified as the spin density. In the particular case of the Boltzmann ideal gas, particles can be handled independently of each other. This makes it easy to calculate the spin density tensor, what can be done by simply summing up single particle contributions. We then need to calculate the expectation value of the angular momentum tensor (i.e. the Lorentz group generators) over single particle states, that is:

$$\langle \langle \hat{J}^{\mu\nu} \rangle \rangle = \frac{1}{(N)} \sum_{\sigma} \int d^3 p \ p^{[\sigma} \hat{J}^{\mu\nu} \exp\left(\left(-\hat{h} + \mu \hat{q} + \hat{\omega} \cdot \hat{\mathbf{J}}\right)/T\right) \bar{P}_V |p, \sigma\rangle$$

where the normalizing factor

$$\langle N \rangle = \int_V d^3 x \int d^3 p \ f(x, p)$$

concides with the mean number of particles (see Sect. II). We will proceed separately for the space components of the angular momentum tensor, i.e. the properly called angular momentum $J$, and the time components, i.e. the boost generators $K_i = J^0$. Taking the $z$ axis as the $\omega$ direction, we can write the mean values in (59) as derivatives:

$$\langle \langle \hat{J}_i \rangle \rangle = \frac{1}{(N)} \sum_{\sigma} \int d^3 p \ i \frac{\partial}{\partial \phi_i} \left| \hat{R}_{\hat{e}} (\phi_i) \exp\left(\left(-\hat{h} + \mu \hat{q} + \hat{\omega} \cdot \hat{\mathbf{J}}\right)/T\right) \bar{P}_V |p, \sigma\rangle \right|_{\phi_i=0}$$

$$\langle \langle \hat{K}_i \rangle \rangle = \frac{1}{(N)} \sum_{\sigma} \int d^3 p \ i \frac{\partial}{\partial \xi_i} \left| \hat{L}_{\hat{e}} (\xi_i) \exp\left(\left(-\hat{h} + \mu \hat{q} + \hat{\omega} \cdot \hat{\mathbf{J}}\right)/T\right) \bar{P}_V |p, \sigma\rangle \right|_{\xi_i=0}$$

(60)

where $\hat{R}_{\hat{e}} (\phi_i) = \exp[-i\phi_i \hat{J}_i]$ is a rotation of axis $\hat{e}_i$ and angle $\phi_i$ and $\hat{L}_{\hat{e}} (\xi_i) = \exp[-i\xi_i \hat{K}_i]$ is a Lorentz boost of direction $\hat{e}_i$ and hyperbolic angle $\xi_i$. The matrix elements in the above equation can be calculated in a similar fashion as that for in eq. (9). One just needs to insert a resolution of the identity made of momentum-spin eigenstates between the operators $\hat{R}_{\hat{e}} (\phi_i) \hat{L}_{\hat{e}} (\xi_i)$ and the density operator and work out the obtained expression following the procedure drawn in Sect. II from eq. (9) to (15). This leads to:

$$\langle \langle \hat{J}_i \rangle \rangle = \frac{1}{(N)} \sum_{\sigma} \int d^3 p \ e^{-\epsilon/T + \mu q/T} \ i \frac{\partial}{\partial \phi_i} \int_V d^3 x \ e^{i\epsilon \hat{p} - R_{\hat{e}} (\sigma \omega / T - 1 \hat{R}_{\hat{e}} (\phi_i - 1)) \hat{p}}$$

$$\times \frac{1}{2} \left( D^S (|p|^{-1} \hat{R}_{\hat{e}} (\phi_i) \hat{R}_{\hat{e}} (\sigma \omega / T) |p\rangle) + D^S (|p| \hat{R}_{\hat{e}} (\phi_i) \hat{R}_{\hat{e}} (\sigma \omega / T) |p\rangle) \right) |_{\phi_i=0}$$

$$= \frac{1}{(N)} \int d^3 p \ e^{-\epsilon / T + \mu q / T} \ i \frac{\partial}{\partial \phi_i} \int_V d^3 x \ e^{i\epsilon \hat{p} - R_{\hat{e}} (\sigma \omega / T - 1 \hat{R}_{\hat{e}} (\phi_i - 1)) \hat{p}} \tr D^S (\hat{R}_{\hat{e}} (\phi_i) \hat{R}_{\hat{e}} (\sigma \omega / T)) |_{\phi_i=0}$$

(61)

and

$$\langle \langle \hat{K}_i \rangle \rangle = i \frac{\partial}{\partial \xi_i} \frac{1}{(N)} \sum_{\sigma} \int d^3 p \ e^{-L_{\hat{e}} (\xi_i - 1) \hat{p} / T + \mu q / T} \left| \xi_i \hat{p} \right| \int_V d^3 x \ e^{i \hat{p} \sum_{\hat{r}=1}^3 \hat{r} (p - R_{\hat{e}} (\sigma \omega / T - 1 \hat{L}_{\hat{e}} (\xi_i - 1))) \hat{r} - \hat{p} \sum_{\hat{r}=1}^3 \hat{r} (p - R_{\hat{e}} (\sigma \omega / T - 1 \hat{L}_{\hat{e}} (\xi_i - 1))) \hat{r}}$$

$$\times \frac{1}{2} \left( D^S (|p|^{-1} \hat{L}_{\hat{e}} (\xi_i) \hat{R}_{\hat{e}} (\sigma \omega / T) |p\rangle) + D^S (|p| \hat{L}_{\hat{e}} (\xi_i - 1) \hat{R}_{\hat{e}} (\sigma \omega / T) |p\rangle) \right) |_{\xi_i=0}$$

$$= \frac{1}{(N)} \int d^3 p \ i \frac{\partial}{\partial \xi_i} \left| \xi_i \hat{p} \right| \int_V d^3 x \ e^{i \hat{p} \sum_{\hat{r}=1}^3 \hat{r} (p - R_{\hat{e}} (\sigma \omega / T - 1 \hat{L}_{\hat{e}} (\xi_i - 1))) \hat{r} - \hat{p} \sum_{\hat{r}=1}^3 \hat{r} (p - R_{\hat{e}} (\sigma \omega / T - 1 \hat{L}_{\hat{e}} (\xi_i - 1))) \hat{r}}$$

$$\times \frac{1}{2} \left( \tr D^S (\hat{L}_{\hat{e}} (\xi_i) \hat{R}_{\hat{e}} (\sigma \omega / T)) + \tr D^S (\hat{L}_{\hat{e}} (\xi_i - 1) \hat{R}_{\hat{e}} (\sigma \omega / T)) \right) |_{\xi_i=0}$$

(62)
Note the difference between the traces of the finite dimensional representation matrices in (61) and (62) which is owing to the non-unitarity of the SL(2,C) representation of boosts, i.e. $D^S(L(\xi))^{-1} \neq D^S(L(\xi))$.

Let us first focus on the eq. (61). If we let $p' = R_0(\phi_t)^{-1}(p)$, the energy as well as the momentum integral measure does not change and the derivative with respect to $\phi_t$ can be worked out:

$$\langle \hat{H} \rangle = \frac{1}{\langle N \rangle} \int d^3p' e^{-\epsilon'/T+\mu_q/T} i \frac{\partial}{\partial \phi_t} \int_V d^3x e^{i\varphi_0(\phi_t)(p'-R_0(\phi_t))} \left[ D^S(R_0(\phi_t)R_0(\epsilon/\hbar T)) \right]_{\phi_t=0}$$

$$= \frac{1}{\langle N \rangle} \int d^3p' e^{-\epsilon'/T+\mu_q/T} \int_V d^3x e^{i\varphi_0(p'-R_0(\epsilon/\hbar T))} \left[ (x \times p') \cdot \hat{\epsilon} \chi \left( \frac{\omega}{T} \right) + trD^S(J x \hat{R}_0(\epsilon/\hbar T)) \right]$$

where we used the (22). The (63) can be rewritten in a more compact form recalling the calculations of Sect. 2:

$$\langle \hat{H} \rangle = \frac{1}{\langle N \rangle} \int d^3x \int d^3p f(x,p) \left[ (x \times p) + \frac{1}{\chi(\frac{\epsilon}{T})} tr[D^S(J x \hat{R}_0(\epsilon/\hbar T))] \right]$$

where we have redefined $p'$ as $p$. In this form, eq. (64) allows us to pin down the spacial part of the spin density tensor which is the second term in the integrand of (64) and more specifically:

$$\frac{dS}{dx} = \int d^3p f(x,p) \frac{1}{\chi(\frac{\omega}{T})} tr[D^S(J x \hat{R}_0(\epsilon/\hbar T))]$$

The $z$ component of the trace is readily calculated as:

$$trD^S(J_z \hat{R}_0(\epsilon/\hbar T)) = \frac{\partial}{\partial \epsilon/\hbar T} trD^S(\hat{R}_0(\epsilon/\hbar T)) = \frac{\partial}{\partial \epsilon/\hbar T} tr\exp[\omega D^S(J_z)/T] = \chi' \left( \frac{\omega}{T} \right)$$

As far as the $x, y$ components are concerned, we first note that for any analytic function $f(J)$ of the rotation generators, or a representation thereof, and for any operator $\hat{O}$ such that $[\hat{O}, J_z] = 0$, we have:

$$tr(J_{x,y} f(J_z) \hat{O}) = 0$$

To prove it, we just calculate the trace of $J_{x,y} J_z^{N-1} \hat{O}$: taking advantage of the ciclicity of trace and the Lie algebra properties:

$$tr(J_{x,y} J_z^{N-1} \hat{O}) = tr(J_{x,y} J_z^N \hat{O}) = tr(J_z J_{x,y} J_z^{N-1} \hat{O})$$

$$= tr(J_{x,y} J_z^{N-1} \hat{O}) \pm i tr(J_{y,z} J_z^{N-1} \hat{O}) = tr(J_{x,y} J_z^{N-1} \hat{O}) \pm i tr(J_{y,z} J_z^{N-1} \hat{O})$$

Comparing the first and last side of the above equation, we conclude that $tr[J_{y,z} J_z^N \hat{O}] = 0$ for any $N$ and therefore the (67). We now apply the above reasoning to our specific case where $f(J_z) = \exp[\omega J_z/T]$ and $\hat{O} = 1$, which both fulfill the aforementioned conditions and thus conclude that:

$$trD^S(J_{x,y} \hat{R}_0(\epsilon/\hbar T)) = 0$$

As a consequence, the spin density (65) turns out to be, in view of (66) and (69):

$$\frac{dS}{dx} = \int d^3p f(x,p) \frac{\chi'(\frac{\epsilon}{T})}{\chi(\frac{\epsilon}{T})} \omega = n(x) \frac{\chi'(\frac{\epsilon}{T})}{\chi(\frac{\epsilon}{T})} \omega$$

being $n(x) = \int d^3p f(x,p)$ the particle density. Similarly, the proper spin density (see Section III) turns out to be:

$$\sigma = \frac{1}{\gamma} \frac{dS}{dx} = \frac{1}{\gamma} n(x) \frac{\chi'(\frac{\epsilon}{T})}{\chi(\frac{\epsilon}{T})} \omega = \nu \frac{\chi'(\frac{\epsilon}{T})}{\chi(\frac{\epsilon}{T})} \omega$$

being $\nu$ the proper particle density:

$$\nu = \frac{1}{\gamma} n(x) = \frac{1}{\gamma} \int d^3p f(x,p) = \frac{T}{T_0} \int d^3p f(x,p) = \frac{p}{T_0}$$

where we have used the equation (30). We note in passing that the $x, y$ components of the orbital part of the angular momentum density vanish as well because the system (and the phase space distribution) is geometrically invariant by rotation around the $z$ axis.
We can now turn to the eq. (62). We have seen, for angular momentum, that the derivative with respect to group parameter produces two terms: the first, obtained by taking the derivative of the spacial integrand, gives rise to an "orbital" part, while the second term stems from the derivative of the matrices of the finite-dimensional SL(2, C) representation and gives rise to the actual "spin" part. Similarly, for boosts, the derivative of SL(2, C) representation matrices yields the contribution to the spin tensor density while the derivative of the spacial integral generates an orbital part:

\[
\langle \hat{k}_l \rangle_{\text{orb}} = \frac{1}{(N)} \lambda \chi \left( \frac{\omega}{T} \right) \int d^3 p \frac{\partial}{\partial \xi} e^{-L_{\xi}(\xi)^{-1}(p)^0/T} \left[ \sum_{i=1}^{3} x^i (p-R_\omega(\omega \xi/T)^{-1} L_{\xi}(\xi)^{-1}(p)^i)^{r} \right]_{\xi=0} \frac{\varepsilon}{V} \left[ \sum_{i=1}^{3} x^i (p-R_\omega(\omega \xi/T)^{-1} L_{\xi}(\xi)^{-1}(p)^i)^{r} \right]_{\xi=0} \varepsilon
\]

To go on, we need to work out the action of a Lorentz boost of hyperbolic angle $\xi$ along the direction $\hat{n}$ on the four-momentum $p$:

\[
\hat{L}_n(\xi)(p)^0 = p^0 \cosh \xi + \hat{n} \cdot \hat{p} \sinh \xi
\]

\[
\hat{L}_n(\xi)(p)^i = p^i + p^0 \sinh \xi \hat{n}^i + (\cosh \xi - 1) \hat{n} \cdot \hat{p} \hat{n}^i
\]

Let $p' = L_{\xi}(\xi)^{-1}(p)$ and implement this change of variable in the integral of eq. (73):

\[
\langle \hat{k}_l \rangle_{\text{orb}} = \frac{1}{(N)} \lambda \chi \left( \frac{\omega}{T} \right) \frac{\partial}{\partial \xi} \int d^3 p' \frac{1}{\varepsilon^3/2} e^{-\varepsilon'/T} \left[ \hat{L}_n(\xi)(p')^0 \right] \frac{1}{\varepsilon^3/2} e^{-\varepsilon'/T} \int d^3 x e^{i \sum_{r=1}^{3} x^r (L_{\xi}(\xi)(p')-R_\omega(\omega \xi/T)^{-1}(p'))^r} \left| \xi=0 \right|
\]

The derivative can be worked out by using (74):

\[
\langle \hat{k}_l \rangle_{\text{orb}} = \frac{1}{(N)} \lambda \chi \left( \frac{\omega}{T} \right) \frac{3i}{2} \int d^3 x f(x, p) \frac{1}{\varepsilon^3/2} e^{-\varepsilon'/T} \int d^3 x e^{i \sum_{r=1}^{3} x^r (L_{\xi}(\xi)(p')-R_\omega(\omega \xi/T)^{-1}(p'))^r} \left| \xi=0 \right|
\]

where, in the last expression, we have redefined $p'$ as $p$. It is now easy to realize that, in a rotating system, the last integral vanishes. In fact, we have:

\[
\frac{d\langle \hat{k}_l \rangle_{\text{orb}}}{d^3 p} = \frac{1}{(N)} \frac{3i}{2 \varepsilon} \int d^3 x f(x, p) - \frac{1}{(N)} \varepsilon \int d^3 x \frac{P}{\varepsilon} f(x, p)
\]

The rightmost integral yields the origin of the inertial frame, which is centered on the axis of the rotating system according to the (2). The leftmost integral, on the other hand, is imaginary and the integrand is proportional to particle velocity; once integrated over momentum, this term yields the mean particle velocity which vanishes. Thus, for any constant $C$:

\[
\langle \hat{k}_l \rangle_{\text{orb}} = \frac{1}{(N)} \int d^3 x \int d^3 p \left( C \frac{P}{\varepsilon} - \varepsilon x \right) f(x, p) = 0
\]

On the other hand, the "spin" part of the eq. (62) can be written as:

\[
\langle \hat{k}^* \rangle_{\text{spin}} = \frac{1}{(N)} \int d^3 p e^{-T+\mu q/T} \int d^3 x e^{i \mu q \cdot p-R_\omega(\omega \xi/T)^{-1}(p)}
\]

\[
\times \frac{1}{\varepsilon} \frac{1}{2} \left( \text{tr} \left[ D^S(L_{\xi}(\xi)R_\omega(\omega \xi/T)) \right] + \text{tr} \left[ D^S(L_{\xi}(\xi)^{-1}R_\omega(\omega \xi/T)) \right] \right) \left| \xi=0 \right|
\]

\[
= \frac{1}{(N)} \int d^3 x \int d^3 p f(x, p) \frac{1}{2} \text{tr} \left[ D^S(K_l R_\omega(\omega \xi/T)) + D^S(K_l^* R_\omega(\omega \xi/T)) \right]
\]

By using the known properties of representations, the last matrix can be written as:

\[
D^S(K_l)R_\omega(\omega \xi/T) + D^S(K_l R_\omega(\omega \xi/T)) = D^S(K_l)D^S(R_\omega(\omega \xi/T)) + D^S(K_l^*)D^S(R_\omega(\omega \xi/T))
\]
but in fact, for finite-dimensional $\text{SL}(2,\mathbb{C})$ representations $D^S(K)$ is anti-hermitian, hence $D^S(K^\dagger) = -D^S(K)$ and the above expression vanishes. Consequently, the contribution of boosts to spin density in a rotating ideal Boltzmann gas is zero, i.e.:

$$\frac{dS^0}{d^3x} = 0 \quad \text{or} \quad \frac{dK_{\text{spin}}}{d^3x} = 0$$  \hspace{1cm} (79)$$

Collecting the eqs. (71) and (79) we can finally write the spin density tensor $\sigma$ as:

$$\sigma_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \nu \chi(\Omega) \chi(\Omega) & 0 \\ 0 & -\nu \chi(\Omega) & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$  \hspace{1cm} (80)$$

The result (80) has three major implications. Firstly, for an ideal relativistic Boltzmann gas at thermodynamical equilibrium, the spin density tensor (80) is proportional to the acceleration tensor (53) defined with the Frenet-Serret tetrad, through a Lorentz scalar $\iota$ depending on the intensive invariant parameters $T_0$, $\mu_0$ and $\Omega: \Omega$:

$$\sigma_{\mu\nu} = \iota \Omega_{\mu\nu}$$  \hspace{1cm} (81)$$

with:

$$\iota = \nu \frac{1}{\sqrt{2}\Omega : \Omega} \chi'(\Omega) = \nu \frac{1}{\sqrt{2}\Omega : \Omega} \frac{p}{T_0} \frac{1}{\sqrt{\Omega : \Omega}} = \lambda \chi'(\sqrt{\frac{1}{2} \beta^2 \Omega : \Omega}) F'(\sqrt{\beta^2})$$  \hspace{1cm} (82)$$

where, in (82), we have used the eqs. (54), (55), (72) and (56). The last form of $\iota$ emphasizes its nature of Lorentz scalar depending on the Lorentz scalars $\mu_0$, $\beta^2$ and $\Omega: \Omega$. Also, since $\omega \ll T$ and using the (22):

$$\iota \approx \nu \frac{1}{\gamma T} \frac{S(S+1)}{3} = \nu \frac{S(S+1)}{T_0} \frac{1}{3} \Rightarrow \iota \approx h^2 \nu \frac{S(S+1)}{KT_0}$$  \hspace{1cm} (83)$$

where in the last expression we have restored the natural constants.

Secondly, the basic thermodynamical relation (41) can be written as:

$$T_0 s = \rho - \mu_0 q + p - \frac{1}{2} \Omega : \sigma = \rho - \mu_0 q + p - \frac{1}{2} \Omega_{\mu\nu} \sigma^{\mu\nu}$$  \hspace{1cm} (84)$$

as anticipated in eq. (42). For an ideal Boltzmann gas the additional term is:

$$\frac{1}{2} \Omega : \sigma = \frac{1}{2} \iota \Omega : \Omega = \nu \sqrt{\frac{1}{2} \Omega : \Omega} \chi'(\Omega) \approx \nu \gamma \omega^2 \frac{S(S+1)}{3} \Rightarrow \nu \gamma \frac{h^2 \omega^2 S(S+1)}{KT}$$  \hspace{1cm} (85)$$

using (82). Once again, in the eq. (85), we have restored the natural units, which allows us to conclude that for ordinary macroscopic gases the correction to thermodynamic relation owing to spin is extremely small, being $h\omega/KT \ll 1$ under all circumstances. For instance, for a mole of a monoatomic gas with $S = 1$ at 273 K, $p = 10^5$ Pa and $\omega = 100$ Hz, $h\omega/KT = \mathcal{O}(10^{-12})$ and the energy density associated to spin-angular velocity term is of the order of $10^{-19}$ J/m$^3$. However, in extreme situation, like e.g. in astrophysics or relativistic heavy ion collisions, this term could be much more important. Also, as has been mentioned at the end of Sect. III, the term coupling spin and angular velocity-acceleration is responsible for the magnetization observed in rotating bodies, the Barnett effect [14, 15].

Finally, the projection of the spin density tensor over the four-velocity field, is non-vanishing and proportional to the four-acceleration $A$ through the factor $\iota$:

$$t^\mu \equiv \sigma^{\mu\nu} u_\nu = \iota \Omega^{\mu\nu} u_\nu = \iota A^\mu$$  \hspace{1cm} (86)$$

Particularly, in the inertial frame, this vector has time and space components:

$$t = (0, \nu \gamma^2 \omega \times \nu)$$  \hspace{1cm} (87)$$
(see eq. (51) and it is thus orthogonal to both $\omega$ and the velocity. This result is in contrast with common assumptions in the hydrodynamics of fluids with spin, as has been mentioned in the Introduction, but it is not surprising as there is no principle reason for it to vanish in a system of many particles [3]. Indeed, while the vector $l$ for a quantum relativistic particle in a pure state vanishes, this is generally neither the case for a many-particle states nor for a single particle in a mixture of states (see Appendix E).

We end this Section by noting that the calculations herein performed provide an alternative way of determining the polarization four-vector of particles, which was computed explicitly for different spins in ref. [8]. The relevant proof is shown in Appendix F.

VI. VIEWPOINT OF THE ACCELERATED OBSERVER

Since the ideal gas we are dealing with is a macroscopic system, we can ideally attach to each point a physical observer (i.e. a tetrad) moving along the velocity flow lines. For him to observe local thermodynamical equilibrium, it is necessary that all (local) physical quantities are constant in his frame. If this is the case, and assuming the locality hypothesis [17, 19], stating that the accelerated observer measures the same quantities as the comoving observer (i.e. the inertial observer instantaneously coinciding with the accelerated one), the relation he finds among the local thermodynamical quantities are actual equations of state. Particularly, the relation (81) linking a kinematical quantity like the acceleration tensor and the spin density tensor is to be promoted to the rank of equation of state for a system in local thermodynamical equilibrium.

The condition of steadiness for the non-inertial observer means, for a scalar:

$$\frac{dX}{d\tau} = 0$$

where $\tau$ is the proper time. For vector quantities, this turns into a condition on the components with respect to the observer’s tetrad:

$$\frac{dX^i}{d\tau} = \frac{d(X \cdot e^i)}{d\tau} = 0$$

implying that the four-vector $X$ varies according to:

$$\dot{X} = \sum_i (X^i e_i) = \sum_i \dot{X}^i e_i + \sum_i X^i \dot{e}_i = \sum_i X^i \Omega \cdot e_i = \Omega \cdot X$$

Similarly, for rank 2 tensors, one should have:

$$\frac{dT^{ij}}{d\tau} = 0$$

implying:

$$\dot{T} = \sum_{ij} (T^{ij} e_i \otimes e_j) = \sum_{ij} T^{ij} (\dot{e}_i \otimes e_j + e_i \otimes \dot{e}_j) = \sum_{ij} T^{ij} (\Omega \cdot e_i \otimes e_j + e_i \otimes \Omega \cdot e_j) = \Omega \cdot T + T \cdot \Omega^T$$

where $^T$ stands for the transpose. Applying the above equation to the acceleration tensor itself, one obtains that the condition of steadiness amounts to enforcing that the convective derivative of $\Omega$ at thermodynamical equilibrium must vanish, i.e.:

$$\dot{\Omega} = 0$$  \hspace{1cm} (88)$$

For a rigidly rotating fluid, all above conditions are fulfilled provided that the tetrad is the Frenet-Serret one. The local temperature $T_0$, the local chemical potential $\mu_0$ and the acceleration tensor have vanishing convective derivatives, as it turns out from eqs. (23),(24) and (53). The spin tensor density is also stationary in the non-inertial frame because, for the ideal gas, it is proportional to the acceleration tensor. We stress that this does not happen for other tetrads, e.g. a Fermi-Walker transported one; in that frame, the acceleration tensor components and, consequently, the spin density tensor components would vary and therefore, one could not properly talk about local thermodynamical equilibrium. This is why, in a rigidly rotating fluid, the Frenet-Serret frame is a privileged reference.
Comparing the view of the inertial observer with that of the non-inertial one allows to identify inertial forces and helps elucidating the role of spin. To start with, it is useful to take the non-relativistic limit of phase-space distribution (20) seen by the inertial observer. At first order in $v$, for a gas of spinless particles, this reads:

$$f(x, p) \propto \exp \left( -\frac{p^2}{2mT} + \frac{v \cdot p}{T} \right)$$

which, once integrated, yields the invariant partition function. At each point $x$ the inertial observer has to integrate in momentum the integral (89) because he deals with a rotating gas and, in order to take angular momentum conservation into account, he has to calculate a rotational grand-canonical partition function with an $\omega \cdot L$ term [9]. Conversely, for the non-inertial local observer, in the non-relativistic usual view, the particles of the gas as living in external fields, the centrifugal and Coriolis fields. He starts writing a lagrangian:

$$L = \frac{1}{2} m v^2 + \frac{1}{2} m (\omega' \times x)^2 - m x \cdot (\omega' \times v')$$

with centrifugal and Coriolis potential energies. From (90) a hamiltonian can be derived:

$$H = \frac{1}{2} m v^2 - \frac{1}{2} m (\omega' \times x)^2 = \frac{p^2}{2m} - m p \cdot (\omega' \times x)$$

where, from (90):

$$p = m v' - m \omega' \times x$$

Since in the non-relativistic limit $\omega = \omega'$, the non-inertial observer then writes a single-particle partition function in the point $x$:

$$\int d^3p \, e^{-H/T} = \int d^3p \exp \left( -\frac{p^2}{2mT} + \frac{v \cdot p}{T} \right)$$

with the same temperature $T$, as there is no difference in the non-relativistic limit. The result is the same the inertial observer would get, according to (89) and this means that pressure, entropy density and all local thermodynamical quantities are the same for the two observers. Yet, the physical interpretation of the non-inertial observer is different: for his point of view, the coupling between $\omega$ and $x \times p$ arises from the presence of external fields, i.e. inertial forces, in his frame the angular momentum of the system is simply vanishing.

It is worth pointing out that, from a relativistic point of view, the effect of centrifugal potential in the hamiltonian (91) is already contained in the dependence of temperature on the distance from the rotation axis. The rotating observers at different distances $r$ measure different local temperatures $T_0(r)$ and write the single-particle partition function of an ideal gas without external fields in their comoving frame, where $\beta = (1/T_0, 0)$:

$$\int d^3p \, e^{-\beta \cdot p} = \int d^3p \, e^{-\varepsilon / T_0(r)} = \int d^3p \, e^{-\varepsilon / (\gamma(r) T)} \simeq m \int d^3v' \exp \left[ -\frac{\varepsilon}{T} + \frac{\varepsilon v'^2}{2T} \right]$$

The last term yields, in the non-relativistic limit and disregarding the mass term:

$$m \int d^3v' \exp \left[ -\frac{mv'^2}{2T} + \frac{1}{2} m (\omega' \times x)^2 \right]$$

which eventually, after the change of variable (92), turns out to be equal to the right hand side of eq. (93). This result was as a consequence, from the point of view of the non-inertial observer, of the presence of an apparent gravitational field introducing a position-dependent temperature [20], which is still an inertial effect.

Similarly, the non-vanishing polarization of particles that he measures will be attributed to an inertial effect, a coupling between the acceleration tensor and spin of the form:

$$-\frac{1}{2} \Omega_{\mu \nu} \tilde{S}^{\mu \nu}$$

which is, in his view, an additional term of the hamiltonian to be taken into account when calculating thermodynamical quantities. Clarifying the nature of this energy, its relation with Thomas precession and spin-rotation coupling [21] and calculating thermodynamical quantities in stationary accelerated frames will be the subjects of another work.
VII. SUMMARY AND OUTLOOK

We have studied the physics of the ideal relativistic rotating gas at full thermodynamical equilibrium, in the Boltzmann approximation. We have shown that, provided that it is sufficiently large, the gas is a rigidly rotating fluid with a non-vanishing spin density. We have obtained the expressions of the phase space distribution (see Sect. II):

\[ f(x,p)_{\tau \sigma} = \lambda e^{-\beta \cdot p} \frac{1}{2} \left( D^S\left([p]^{-1}R_{\omega}(i\omega/T)[p]\right) + D^S\left([p]^1R_{\omega}(i\omega/T)[p]^{-1}\right)\right)_{\tau \sigma} \]

as a matrix in the polarization states. From this distribution, we have inferred the spin density tensor \(\sigma_{\mu \nu}\) and found that, at equilibrium, is proportional to the acceleration tensor constructed with Frenet-Serret tetrad, i.e.:

\[ \sigma_{\mu \nu} = \iota \Omega_{\mu \nu} \]

where \(\iota\) is a Lorentz-scalar coefficient depending on the other thermodynamical variables. Furthermore, we have recovered the generalized fundamental thermodynamical relation

\[ T_{0s} = \rho - \mu_0 q + p - \frac{1}{2} \Omega_{\mu \nu} \sigma^{\mu \nu} \]

including an additional term involving the spin density tensor, already used in literature in fact. We have shown (see Appendix E) that this term cannot be made vanishing by using Belinfante transformations of the stress-energy and spin tensors because this would imply a change in the total entropy and so the spin density tensor has physical reality.

The condition of full equilibrium makes it possible to study a relativistic fluid without the prerequisite of a theory of perfect fluids with spin and therefore, it allows to carry out an independent test of assumptions made in the past about the dynamics of fluids with spin. Particularly, we have found that the projection of the spin density tensor onto the four-velocity

\[ t_\mu = \sigma_{\mu \nu} u^\nu \]

is not vanishing, unlike commonly assumed [6, 7], in agreement with general arguments of special relativity [3] and envisaged by Bohm and Vigier [4]. Therefore, an observer comoving with the spin density needs two non-vanishing space-like vectors, not just one, to describe it.

The non-vanishing of the four-vector \(t\) has a considerable impact on the general theory of a relativistic perfect fluid with spin, which needs to be revised from the formulation of Halbwachs [5] and Ray [22]. This revision may open a window on new and unexpected features of relativistic hydrodynamics and its phenomenological consequences in various fields where relativistic hydrodynamics is a fundamental tool, from relativistic heavy ion collisions to cosmology.

Acknowledgments

We gratefully acknowledge useful discussions with G. Longhi, H. Liu, L. Lusanna, K. Rajagopal.

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APPENDIX A - Projection onto localized states

The meaning of the projector $P_V$ is easily understood in a first-quantization framework for a single spinless particle, where it can be written a sum over a complete set of eigenfunctions vanishing outside the region $V$:

$$P_V = \sum_k |k\rangle \langle k|$$

with:

$$u_k(x) = \langle x, \sigma | k, \sigma \rangle$$

The functions $u_k$ are complete in the sense that:

$$\sum_k u_k^*(x') u_k(x) = \delta^3(x - x') \theta_V(x) \theta_V(x')$$

where $\theta_V(x) = 0$ if $x \notin V$. Therefore, one has:

$$\langle p'| P_V | p \rangle = \sum_k \langle p'| k \rangle \langle k| p \rangle = \int d^3x \int d^3x' e^{-i p' \cdot x'} \sum_k u_k^*(x') u_k(x) e^{i p \cdot x} = \int_V d^3x e^{i(p-p') \cdot x}$$

which accounts for the integral part of eq. (12).

In a quantum-field theoretical framework, the projector $P_V$ can still be defined and its matrix elements calculated [10, 11] giving rise to the eq. (12) for single-particle states. Although the derivation of (12) in refs. [10, 11] was partly based on an unproved conjecture, the non-trivial spin structure of eq. (12) can be indeed justified by the requirement that the projector onto localized states commute with rotation operators:

$$[P_V, \hat{R}_n(\phi)] = 0$$

for any angle $\phi$. Therefore, by applying single-particle relativistic states on both left and right hand side:

$$\langle p', \sigma' | P_V \hat{R}_n(\phi) | p, \sigma \rangle = \sum_{\tau} \langle p', \sigma' | P_V | R_n(\phi)(p), \tau \rangle D^{\tau S}([R_n(\phi)(p)]^{-1} R_n(\phi)[p])_{\tau \sigma}$$

(97)
whereas:

\[ \langle p', \sigma' | R_n(\phi) P_V | p, \sigma \rangle = \langle p, \sigma | P_V R_n(\phi)^\dagger | p', \sigma' \rangle^* = \sum_\tau \langle p, \sigma | P_V R_n(\phi)^{-1} | p', \tau \rangle^* D^S([R_n(\phi)^{-1} | p']^{-1} R_n(\phi)^{-1} | p'])_{\sigma' \tau} \]

The (97) and (98) ought to be equal, so if we define:

\[ M(p, q)_{\alpha, \beta} = \langle p, \alpha | P_V | q, \beta \rangle \]

from (97) and (98) ensues:

\[ M(p', R_n(\phi)(p)) D^S([R_n(\phi)(p)]^{-1} R_n(\phi)[p]) = D^S([p']^\dagger [R_n(\phi)(p)]^{-1} | p']) M^\dagger(p, R_n(\phi)^{-1} | p')) \]

The last equation tells us that the matrix \( M \) cannot be proportional to the identity, as far as the spin is concerned. In fact, the eq. (99) can be fulfilled if, according to eq. (12):

\[ M(p, q) = F(p - q) \frac{1}{2} [D^S([p]^{-1}[q]) + D^S([p]^{-1}[q]^\dagger)] \]

being \( F \) a scalar function. Thereby, the eq. (99) becomes:

\[ F(p' - R_n(\phi)(p)) \frac{1}{2} [D^S([p']^{-1}[R_n(\phi)(p)]) + D^S([p']^{-1}[R_n(\phi)(p)]^{-1})] D^S([R_n(\phi)(p)]^{-1} R_n(\phi)[p]) = D^S([p']^{-1} R_n(\phi)[p]) D^S([p']^{-1} R_n(\phi)[p]) D^S([R_n(\phi)(p)]^{-1} R_n(\phi)[p]) F(p - R_n(\phi)^{-1} | p'))^* \]

Taking advantage of the unitarity of Wigner rotations (see eq. (14)), both left and right hand side of the above equation yield the same matrix multiplied by two different scalar functions:

\[ F(p' - R_n(\phi)(p)) \frac{1}{2} (D^S([p']^{-1} R_n(\phi)[p]) + D^S([p']^{-1} R_n(\phi)[p])) = \frac{1}{2} D^S([p']^{-1} R_n(\phi)[p]) + D^S([p']^{-1} R_n(\phi)[p])) F(p - R_n(\phi)^{-1} | p'))^* \]

so that we are left with the simple requirement:

\[ F(p' - R_n(\phi)(p)) = F(p - R_n(\phi)^{-1} | p'))^* \]

which is fulfilled if, again according to eq. (12):

\[ F(p' - R_n(\phi)(p)) \propto \int_V d^3x \exp[i \mathbf{x} \cdot (\mathbf{p}' - R_n(\phi)(\mathbf{p}))] \]

and if the region \( V \) is invariant by rotation around the axis \( \mathbf{n} \).

**APPENDIX B - Reality of phase space distribution**

We want to show that the integral:

\[ \int_V d^3x \exp [i \mathbf{x} \cdot (\mathbf{p} - R_\omega(\mathbf{i} \omega/T)^{-1}(\mathbf{p}))] \]

is real. First, we note that:

\[ \mathbf{p} - R_\omega(\mathbf{i} \omega/T)^{-1}(\mathbf{p}) \]

has vanishing component along \( \omega \). Therefore, taking \( \mathbf{\hat{\omega}} \) as the unit vector of the \( z \) axis, we can write the above integral as:

\[ \int_V d^3x \exp [i \mathbf{x}_T \cdot (\mathbf{p}_T - R_\omega(\mathbf{i} \omega/T)^{-1}(\mathbf{p}_T))] \]
and, by using the explicit expression in (19):

\[
\int_V \, d^3x \, \exp \left[ i \mathbf{x}_T \cdot \left( (1 - \cosh \frac{\omega}{T}) \mathbf{p}_T - i \sinh \frac{\omega}{T} \mathbf{\omega} \times \mathbf{p}_T \right) \right]
\]

The imaginary part of the last integral reads:

\[
\int_V \, d^3x \, \sin \left[ (1 - \cosh \frac{\omega}{T}) \mathbf{x}_T \cdot \mathbf{p}_T \right] \exp \left[ i \sinh \frac{\omega}{T} \mathbf{x}_T \cdot (\mathbf{\omega} \times \mathbf{p}_T) \right]
\]

In order to show that this vanishes, we need to find a rotation of \( \mathbf{x}_T \) around \( \mathbf{\omega} \), i.e. in the transverse plane, which changes the sign of \( \mathbf{x}_T \cdot \mathbf{p}_T \) and, at the same time, leaves \( \mathbf{x}_T \cdot (\mathbf{\omega} \times \mathbf{p}_T) \) unchanged. This transformation in the transverse plane exists because \( \mathbf{p}_T \) and \( (\mathbf{\omega} \times \mathbf{p}_T) \) belong to that plane and are orthogonal. Once this transformation is performed, and the relevant change of variable \( x \rightarrow x' \) in the integral applied, the integral (103) is transmuted into its opposite as the region \( V \) is symmetric by rotation around \( \mathbf{\omega} \), hence (103) vanishes.

**APPENDIX C - The angular velocity vector in relativity**

To better understand the physical content of the antisymmetric tensor \( \Omega \), it is useful to decompose it into two four-vectors through the definitions:

\[
A_{\mu} = \Omega_{\mu\rho} u^\rho
\]

\[
\omega_{\mu} = -\frac{1}{2} \epsilon_{\mu\rho\sigma} \Omega^{\rho\sigma} u^\sigma
\]

which can be inverted to give [16]:

\[
\Omega_{\mu\nu} = \epsilon_{\mu\rho\sigma} \omega^\rho u^\sigma + A_{\mu} u_{\nu} - A_{\nu} u_{\mu}
\]

Both four-vectors \( A \) and \( \omega \) are orthogonal to the four-velocity and are therefore space-like. It is easy to verify, by using (48) and reminding that \( c_0 = u \), that \( A \) is just the four-acceleration vector \( du/d\tau \). In order to grasp the physical meaning of the four-vector \( \omega \), it is necessary to first look at motion of the tetrad when \( \omega = 0 \). In this case, the equation of motion of the tetrad four-vectors read, from (45) and (105):

\[
\dot{e}_i^\mu = A^\mu e_i - u^\mu A \cdot e_i
\]

This is the well known Fermi-Walker transport of the vector \( e_i \). Now, with respect to the Fermi-Walker transported tetrad, any other comoving spacial frame will be seen by that observer either as fixed with respect to the Fermi-Walker one or ”rotating” with an equation like (47) with the time \( t \) replaced by the observer’s proper time \( \tau \), that is:

\[
\dot{e}_i' = \omega \times e_i
\]

where the prime denotes that these are spacial vectors seen by the moving observer and not by the inertial one. In the Fermi-Walker transported frame the four-vector \( \omega \) reads:

\[
\omega_{\mu} = -\frac{1}{2} \epsilon_{\mu\rho\sigma} \Omega^{\rho\sigma} u^\sigma = -\frac{1}{2} \epsilon_{\mu\rho\sigma} \sum_{i=0}^{3} \dot{e}_i^{\mu} e_i^{\rho} u^\sigma = -\frac{1}{2} \epsilon_{\mu\rho\sigma} A^\nu u^\sigma - \frac{1}{2} \epsilon_{\mu\rho\sigma} \sum_{i=1}^{3} \dot{e}_i^{\mu} e_i^{\rho} u^\sigma
\]

\[
= -\frac{1}{2} \epsilon_{\mu\rho\sigma} \sum_{i=1}^{3} \dot{e}_i^{\mu} e_i^{\rho}
\]

being \( u^\sigma = \delta_0^\sigma \) in any comoving frame. The last equality, along with (107) implies that in the comoving Fermi-Walker transported frame

\[
\omega = (0, \omega')
\]

which makes it clear the meaning of the four-vector \( \omega \). Now, to get it expressed in the inertial frame, it suffices to apply a pure Lorentz boost (without rotations) to the components seen by the comoving observer, with an instantaneous velocity \( v \) with respect to the inertial observer. Therefore:

\[
\omega^0 = \gamma \omega'^0 + \gamma v \cdot \omega' = \gamma v \cdot \omega'
\]

\[
\omega^i = \omega'^i + \gamma \omega'^0 v^i + \frac{\gamma - 1}{v^2} v^j \omega' v^i = \omega''^i + \frac{\gamma - 1}{v^2} v^j \omega'^j v^i
\]
where we also used $\omega^0 = 0$.

The last step is to provide a definition of angular velocity vector $\omega$ in the inertial frame. We could, for instance, define it as the spacial part of the $\omega$ four-vector in (109), but it would not coincide with the angular velocity vector $\omega$ that has been defined for a uniformly spinning system related to the velocity with eq. (2). Instead, we will use the latter as definition of $\omega$ and thereafter derive its relation between $\omega'$. For a uniformly rotating system, we know the $\Omega$ tensor from (53) and this allows to readily determine the $\omega$ four-vector from (104):

$$\omega = (\gamma^2 \omega \cdot v, \gamma^2 \omega)$$  \hspace{1cm} (110)

We will keep this as a definition of $\omega$ in general. One one hand, the (110), along with the (51), allows us to write the $\Omega$ tensor in general as a function of the acceleration $a$ and the angular velocity $\omega$ seen by the inertial frame, by using the (105). This leads to, in the inertial frame:

$$\Omega = \left( \begin{array}{c}
0 \\
-\gamma^3 a - \gamma^3 (\omega \times v) \\
\gamma^4 \omega - \gamma^3 \omega \cdot vv + \gamma^3 a \times v
\end{array} \right)$$  \hspace{1cm} (111)

Finally, comparing (110) with (109) we obtain the transformation rule of the angular velocity vector between the inertial and comoving frame:

$$\omega = \frac{1}{\gamma^2} \omega' + \frac{\gamma - 1}{\gamma^2 v^2} v \cdot \omega' v = \frac{1}{\gamma^2} \omega' + \frac{1}{\gamma + 1} v \cdot \omega' v$$  \hspace{1cm} (112)

**APPENDIX D - Non-invariance of thermodynamics under Belinfante transformation**

The conservation of energy-momentum and angular momentum for a continuous system can be written as system of two partial differential equations:

$$\partial_\mu T^{\mu \nu} = 0$$
$$\partial_\lambda J^{\lambda \mu \nu} = \partial_\lambda (S^{\lambda \mu \nu} + x^\mu T^{\lambda \nu} - x^\nu T^{\lambda \mu}) = 0$$  \hspace{1cm} (113)

$T$ being the stress-energy tensor and $S$ the spin tensor. This set of two equations is invariant in form under a transformation [5]:

$$T'^{\mu \nu} = T^{\mu \nu} + \frac{1}{2} \partial_\lambda (\Phi^{\lambda \mu \nu} - \Phi^{\mu \lambda \nu} - \Phi^{\nu \lambda \mu})$$
$$S'^{\lambda \mu \nu} = S^{\lambda \mu \nu} - \Phi^{\lambda \mu \nu}$$  \hspace{1cm} (114)

where $\Phi$ is an arbitrary tensor of rank three which is antisymmetric in the last two indices.

Choosing $\Phi^{\lambda \mu \nu} = S^{\lambda \mu \nu}$, the new spin tensor $S'$ vanishes and the new stress-energy tensor becomes symmetric; this is the well-known Belinfante symmetrization procedure. However, this transformation does not leave, for a system in full thermodynamical equilibrium, the basic relation (42) invariant, whether the $\Omega : \sigma/2$ term is included or not, and, moreover, it also changes the total entropy. To show this, we first remind the definitions of proper energy density and pressure for a given stress-energy tensor:

$$\rho = T^{\mu \nu} u_\mu u_\nu$$
$$p = \frac{1}{3}(u_\mu u_\nu - g_{\mu \nu})T^{\mu \nu}$$  \hspace{1cm} (116)

For a system at full thermodynamical equilibrium, such as a rigidly rotating fluid, the stress-energy tensor must have [13] the form:

$$T^{\mu \nu} = (\rho + p)u_\mu u_\nu - pg^{\mu \nu}$$  \hspace{1cm} (117)

as well as a spin tensor which must factorize into a spin density second-order tensor and the four-velocity:

$$S^{\lambda \mu \nu} = \sigma^{\mu \nu} u^\lambda$$  \hspace{1cm} (118)
The stress-energy tensor in (117) is already symmetric, yet the spin tensor is not vanishing. We could then think of applying the Belinfante symmetrization procedure described above in order to make the new spin tensor $S'$ vanishing, thereby obtaining a new symmetric stress-energy tensor $T'$. Using eq. (115) and (118), this new tensor reads:

$$T'^{\mu\nu} = T^{\mu\nu} + \frac{1}{2} \partial_\lambda (\sigma^{\mu\nu} u^\lambda - \sigma^{\lambda\nu} u^\mu - \sigma^{\lambda\mu} u^\nu) = T^{\mu\nu} - \frac{1}{2} \partial_\lambda (\sigma^{\lambda\nu} u^\mu) - \frac{1}{2} \partial_\lambda (\sigma^{\lambda\mu} u^\nu) + \frac{1}{2} \sigma^{\mu\nu} + \frac{1}{2} \sigma^{\lambda\nu} \partial_\lambda u^\mu$$ (119)

At thermodynamical equilibrium the last two terms vanish: the second one, because the velocity field is rigid, and the first one because of the (88) and the vanishing of convective derivatives of $T_0$, $\mu_0$:

$$\dot{\sigma} = (i\dot{\Omega}) = i(T_0, \mu_0, \Omega : \Omega) + i\dot{\Omega} = 0$$

Thus, the (119) becomes:

$$T'^{\mu\nu} = T^{\mu\nu} - \frac{1}{2} \partial_\lambda (\sigma^{\lambda\nu} u^\mu) - \frac{1}{2} \partial_\lambda (\sigma^{\lambda\mu} u^\nu)$$ (120)

For the stress-energy tensor according (120), the new energy density and pressure read, applying the (116):

$$\rho' = T'^{\mu\nu} u_\mu u_\nu = \rho - u_\mu u_\nu \partial_\lambda (\sigma^{\lambda\nu} u^\mu) = \rho - u_\nu \partial_\lambda \sigma^{\lambda\nu}$$

$$p' = \frac{1}{3}(\rho' - T'^{\mu\mu}) = \frac{1}{3} (\rho - u_\nu \partial_\lambda \sigma^{\lambda\nu} - T^\mu_\mu + \partial_\lambda (\sigma^{\lambda\nu} u_\nu)) = p + \frac{1}{3} \sigma^{\lambda\nu} \partial_\lambda u_\nu$$ (121)

The thermodynamical relation (42) in the new variables should read, as $S' = 0$ and assuming the invariance of $q$:

$$T_0 s' = \rho' + p' - \mu_0 q' = \rho + p - \mu_0 q - u_\nu \partial_\lambda \sigma^{\lambda\nu} + \frac{1}{3} \sigma^{\lambda\nu} \partial_\lambda u_\nu$$ (122)

Hence, in order to keep the thermodynamic relation invariant, the entropy should be modified by a term depending on the spin density tensor:

$$T_0 s' = T_0 s - u_\nu \partial_\lambda \sigma^{\lambda\nu} + \frac{1}{3} \sigma^{\lambda\nu} \partial_\lambda u_\nu + \frac{1}{2} \Omega_{\lambda\nu} \sigma^{\lambda\nu} = T_0 s - \partial_\lambda (\sigma^{\lambda\nu} u_\nu) + \frac{1}{3} \sigma^{\lambda\nu} \partial_\lambda u_\nu + \frac{1}{2} \Omega_{\lambda\nu} \sigma^{\lambda\nu}$$ (123)

The last expression can also be rewritten as, by using the definition (86) and taking advantage of the antisymmetry of $\sigma$:

$$T_0 s' = T_0 s - \partial \cdot t + \frac{2}{3} \sigma^{\lambda\nu} (\partial_\lambda u_\nu - \partial_\nu u_\lambda) + \frac{1}{2} \Omega_{\lambda\nu} \sigma^{\lambda\nu}$$ (124)

It can be shown that, for a rotating gas, all terms added to entropy density on the right-hand-side of the above equation are strictly positive. Using eqs. (87), (82) as well as (54) and (53), the eq. (124) can be rewritten as:

$$T_0 (s' - s) = -\nabla \cdot (\nu \gamma^2 \omega \times \nu) + \frac{2}{3} \nu \gamma \omega \cdot \nabla \times \gamma \nu + \nu \gamma^2 \omega^2$$

The vector field $\nu \gamma^2 \omega \times \nu$ is proportional to the centripetal acceleration and therefore has a negative divergence; the curl of the vector field $\nabla \times \gamma \nu$ for the velocity (2) yields $2\gamma \omega$. Hence, all three terms on the right-hand are positive and total entropy should be changed by a finite positive amount.

The latter condition is clearly physically unacceptable. Since the rigidly rotating gas is a system in full thermodynamical equilibrium, the new entropy density obtained after the Belinfante transformation should at least have a vanishing integral over the system’s region, which is not the case.

Therefore, we conclude that making the spin tensor vanishing is impossible and it must have a physical reality.

**APPENDIX E - The four-vector $t$**

The four-vector $t$, for a single massive particle, is defined as the expectation value of the operator $\hat{\tau}^\mu = \hat{J}^{\mu\nu} \hat{P}_\nu$ divided by the mass. If the particle is in a pure state $|p, \sigma\rangle$ then the expectation value:

$$t^\mu \equiv \frac{1}{m} \langle p, \sigma | \hat{\tau}^\mu | p, \sigma \rangle = \frac{1}{m} \langle p, \sigma | \hat{J}^{\mu\nu} \hat{P}_\nu | p, \sigma \rangle$$
transforms as a vector because so does the relevant operator. In other words, by using the very definition of single particle states from the theory of Poincaré group representations [12, 23]:

\[ t^\mu = \frac{1}{m} \langle p^0, \sigma | \widehat{J}^{\mu \nu} \widehat{P}_\nu | p^0, \sigma \rangle = \frac{1}{m} \langle p^0, \sigma | \widehat{J}^{\mu \nu} \widehat{P}_\nu | p^0, \sigma \rangle = \frac{1}{m} \langle p^0, \sigma | \widehat{J}^{\mu \nu} \widehat{P}_\nu | p^0, \sigma \rangle \]  

(125)

where \( p^0 = (m, 0, 0, 0) \). Therefore, to prove that \( t = 0 \) we just need to show it for a particle at rest. In this case:

\[ \langle p^0, \sigma | \widehat{J}^{\mu \nu} \widehat{P}_\nu | p^0, \sigma \rangle = m \langle p^0, \sigma | \widehat{J}^{\mu 0} | p^0, \sigma \rangle \]

The boost generators \( \widehat{J}^{\mu 0} \) however anticommute with the parity operator and since the states of a single particle at rest have a definite parity (which is, by definition, the intrinsic parity of particles) the right hand side of the above equation ought to vanish. Thence, in view of (125):

\[ t^\mu = \frac{1}{m} \langle p, \sigma | \widehat{J}^{\mu \nu} \widehat{P}_\nu | p, \sigma \rangle = \frac{1}{m} \langle p, \sigma | \widehat{J}^{\mu \nu} \widehat{P}_\nu | p, \sigma \rangle = 0 \]  

(126)

The validity of eq. (126) is readily extended to any pure state. First we note that the above reasoning can be extended to a matrix element with different spin components and same momentum:

\[ \langle p, \sigma | \widehat{\tau}^{\mu \nu} | p, \sigma' \rangle = 0 \]

For a general pure state, it suffices to decompose single particle states onto momentum-spin eigenstates and show that:

\[ \langle p', \sigma' | \widehat{\tau}^{\mu \nu} | p, \sigma \rangle = 0 \]  

(127)

for \( p \neq p' \). By using standard Poincaré group commutation relations:

\[ [\widehat{J}^{\lambda \sigma}, \widehat{P}^{\nu}] = i \widehat{P}^{\lambda} g^{\nu \sigma} - i \widehat{P}^{\sigma} g^{\nu \lambda} \]

it is easy to show that:

\[ [\widehat{\tau}^{\mu \nu}, \widehat{P}^{\rho}] = i \widehat{P}^{\mu} \widehat{P}^{\nu} - ig^{\mu \nu} \widehat{P} \]

(128)

and therefore, if \( p \neq p' \):

\[ \langle p', \sigma' | \widehat{\tau}^{\mu \nu} \widehat{P}^{\rho} | p, \sigma \rangle = \langle p', \sigma' | \widehat{P}^{\rho} \widehat{\tau}^{\mu \nu} | p, \sigma \rangle = p^\nu \langle p', \sigma' | \widehat{\tau}^{\mu} | p, \sigma \rangle = p^\nu \langle p', \sigma' | \widehat{\tau}^{\mu} | p, \sigma \rangle \]

that is:

\[ (p^\nu - p'^\nu) \langle p', \sigma' | \widehat{\tau}^{\mu \nu} | p, \sigma \rangle = 0 \]

Since the last equation applies to all \( p \)'s components, and \( p \neq p' \), it turns out that the only possibility is that the matrix element of \( \widehat{\tau} \) vanishes, i.e. the eq. (127).

Altogether, all single-particle matrix elements of the operator \( \widehat{\tau} \) vanish, i.e.:

\[ \langle p', \sigma' | \widehat{\tau}^{\mu \nu} | p, \sigma \rangle = 0 \]

for any \( p, p', \sigma \) and \( \sigma' \), and thus the expectation value of \( \widehat{\tau} \), hence \( t \), vanishes for a generic pure state \( | \Phi \rangle \):

\[ t^\mu = \frac{1}{m} \langle \Phi | \widehat{\tau}^{\mu \nu} | \Phi \rangle = 0 \]

However, for a free multiparticle system this is no longer true [3] because:

\[ \widehat{\tau}^{\mu \nu} = \widehat{J}^{\mu \nu} \widehat{P}_\nu = \sum_i \widehat{J}^{\mu \nu}_i \sum_j \widehat{P}_\nu j \neq \sum_i \widehat{J}^{\mu \nu}_i \]

Also, for a single particle in a mixture of states the expectation value of \( \widehat{\tau} \), i.e. \( \text{tr}(\widehat{\rho} \widehat{\tau}) \) is not vanishing unless the density operator is a scalar operator, i.e. commutes with all unitary representations of Lorentz transformations. In fact, the operator \( \exp(\omega \cdot \widehat{J} / T) \), that we have dealt with in this work, does not.
APPENDIX F - Polarization four-vector

The results of Sect. V allow to determine the polarization four-vector of particles with momentum $p$ and general spin $S$, in an ideal Boltzmann gas. This determination is alternative with respect to that presented in ref. [8]. The polarization four-vector is defined as:

$$\Pi^\mu(p) \equiv \frac{1}{m} \sum_\sigma \langle p, \sigma | \hat{W}^\mu \hat{\rho} | p, \sigma \rangle \equiv \frac{1}{m} \langle \langle \hat{W}^\mu \rangle \rangle_p$$  \hspace{1cm} (129)

where $W$ is the Pauli-Lubanski four-vector:

$$W_\mu = -\frac{1}{2} \varepsilon_{\mu\nu\rho\tau} J_\nu^\rho P^\tau$$

and $\hat{\rho}$ is the single-particle spin density operator, which, for our present case reads:

$$\hat{\rho} = \frac{1}{n(p)} \exp[-(\varepsilon(p) + \mu \hat{q} + \omega \cdot \hat{j})/T] P_V$$

being

$$n(p) = \sum_\sigma \langle p, \sigma | \exp[-(\varepsilon(p) + \mu \hat{q} + \omega \cdot \hat{j})/T] P_V | p, \sigma \rangle = \int_V d^3x \ f(x, p)$$

Thus, in view of (129), the polarization four-vector can be written in the inertial frame as:

$$\Pi_0 = \frac{1}{n(p)} \langle \hat{j} \rangle_p \cdot \frac{P}{m}$$

$$\Pi = \frac{1}{n(p)} \langle \langle \hat{k} \rangle \rangle_p \times \frac{P}{m} + \frac{1}{n(p)} \langle \langle \hat{j} \rangle \rangle_p \frac{\varepsilon}{m}$$  \hspace{1cm} (130)

By using the results of eqs. (64),(65) and (77), we can readily find the various expression of the expectation values in the eq. (130), that is:

$$\langle \langle \hat{j} \rangle \rangle_p = \int_V d^3x \ (x \times p) f(x, p) + \int_V d^3x \ f(x, p) \frac{\chi'(\tilde{\omega})}{\chi(\tilde{\omega})} \tilde{\omega}$$

$$\langle \langle \hat{k} \rangle \rangle_p = \int_V d^3x \ (C p - \varepsilon x) f(x, p)$$  \hspace{1cm} (131)

By plugging (131) into (130) we get:

$$\Pi_0 = \frac{\varepsilon'(\tilde{\omega})}{\chi(\tilde{\omega})} \tilde{\omega} \cdot P$$

$$\Pi = \frac{\varepsilon'(\tilde{\omega})}{\chi(\tilde{\omega})} \frac{\varepsilon}{m}$$  \hspace{1cm} (132)

which is the same result found in ref. [8].