The $p$-primary subgroups of the cohomology of $BPU_n$ in dimensions less than $2p+5$

Xing Gu, Yu Zhang, Zhilei Zhang, and Linan Zhong

Abstract. Let $PU_n$ denote the projective unitary group of rank $n$ and $BPU_n$ be its classifying space. For an odd prime $p$, we extend previous results to a complete description of $H^s(BPU_n;\mathbb{Z})_p$ for $s<2p+5$ by showing that the $p$-primary subgroups of $H^s(BPU_n;\mathbb{Z})$ are trivial for $s=2p+3$ and $s=2p+4$.

1. Introduction

Let $U_n$ denote the group of $n \times n$ unitary matrices. The unit circle $S^1$ can be viewed as the normal subgroup of scalar matrices of $U_n$. We let $PU_n$ denote the quotient group of $U_n$ by $S^1$, and $BPU_n$ be the classifying space of $PU_n$. In this paper we consider $H^*(BPU_n;\mathbb{Z})$, the ordinary cohomology of $BPU_n$ with coefficients in $\mathbb{Z}$.

A review of the literature. The ordinary and generalized cohomology of $BPU_n$ for special $n$ has been the subject of various works such as Kono-Mimura [15], Kameko-Yagita [14], Kono-Yagita [16], Toda [19], and Vavpetić-Viruel [21]. Vezzosi [22] and Vistoli [23] studied the Chow ring of the classifying space (in the sense of Totaro [20]) of $BPGL_3(\mathbb{C})$ and $BPGL_p(\mathbb{C})$ for $p$ an odd prime, respectively. Much of their results applies to the ordinary cohomology of $BPU_p$.

None of the works above dealt with $H^*(BPU_n;\mathbb{Z})$ for $n$ not a prime number. The first named author considered $H^*(BPU_n;\mathbb{Z})$, as well as the Chow ring of $BPGL_n(\mathbb{C})$ for an arbitrary $n$ in [10], [12] and [13]. In particular, in [12], the first named author determined the ring structure of $H^*(BPU_n;\mathbb{Z})$ in dimensions less than or equal to $10$.

Other related works include Duan [6], in which the integral cohomology of $PU_n$ is fully determined, and Crowley-Gu [5], in which the image of the canonical map $H^*(BPU_n;\mathbb{Z}) \rightarrow H^*(BU_n;\mathbb{Z})$ is studied.

The cohomology of $BPU_n$ plays significant roles in the study of the topological period-index problem ([1], [2], [9] and [11]), and in the study of anomalies in particle physics ([4], [8]).

Notations. Throughout the rest of this paper, $H^*(-)$ denotes $H^*(-;\mathbb{Z})$. For an abelian group $A$ and a prime number $p$, let $A_{(p)}$ be the localization of $A$ at $p$, and let $pA$ denotes the $p$-primary subgroup of $A$, i.e., the subgroup of $A$ of all torsion elements with torsion order a power of $p$. In particular, we have a canonical isomorphism $\mu H^*(-) \cong p[H^*(-)]_{(p)}$, and we will not distinguish the two throughout this paper. Tensor products of $\mathbb{Z}_{(p)}$-modules are always taken over $\mathbb{Z}_{(p)}$.

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The main theorem and some remarks. We review a basic fact on the cohomology of $BPU_n$. Consider the short exact sequence of Lie groups

$$1 \to \mathbb{Z}/n \to SU_n \to PSU_n \simeq PU_n \to 1,$$

which induces a fiber sequence of their classifying spaces

$$(1.1) \quad B(\mathbb{Z}/n) \to BSU_n \to BPU_n$$

When $p \nmid n$, the space $B(\mathbb{Z}/n)$ is $p$-locally contractible and we have

$$(1.2) \quad H^*(BPU_n; \mathbb{Z}/p) \simeq H^*(BSU_n; \mathbb{Z}/(p))$$

Since $\mathbb{Z}/(p)$ is a flat $\mathbb{Z}$-module, and in particular, $H^*(-; \mathbb{Z}/(p)) \simeq H^*(-)(p)$, we have an isomorphism of $\mathbb{Z}/(p)$-algebras

$$(1.3) \quad H^*(BPU_n)(p) \simeq H^*(BSU_n)(p) = \mathbb{Z}(p)[c_2, c_3, \ldots, c_n],$$

which shows $H^*(BPU_n)(p)$ is torsion-free for $p \nmid n$. In other words, we have the following

**Proposition 1.1.** Suppose $x \in H^*(BPU_n)$ is a torsion class. Then there exists some $i \geq 0$ such that $n^i x = 0$.

Therefore, to determine the graded abelian group structure of $H^*(BPU_n)$, it suffices to consider the $p$-primary subgroup $pH^*(BPU_n)$ for $p \mid n$.

**Remark 1.2.** In the case of Chow rings, Vezzosi [22] proved the stronger result that all torsion classes in the Chow ring of $BPGL_n(\mathbb{C})$ are $n$-torsion.

To state the main theorem, recall that, as shown in [12], the integral cohomology group $H^3(BPU_n)$ is generated by a class denoted by $x_1$. In addition, $P^i$ will denote the $i$th Steenrod reduced power operation, and

$$\delta : H^*(-; \mathbb{Z}/p) \to H^{*+1}(-)$$

will denote the connecting homomorphism. Finally, a bar over an integral cohomology class will denote the mod $p$ reduction of this class. For instance, $\bar{x}_1$ denotes the mod $p$ reduction of $x_1$, which is in $H^3(BPU_n; \mathbb{Z}/p)$.

**Theorem 1.** Let $p > 2$ be a prime number, and $n = p^m$ for a positive integer $m$ co-prime to $p$. Then the $p$-primary subgroup of $H^*(BPU_n)$ in dimensions less than $2p + 5$ is as follows:

1. For $r > 0$, we have

$$pH^r(BPU_n) \cong \begin{cases} \mathbb{Z}/p^r, & s = 3, \\ \mathbb{Z}/p, & s = 2p + 2, \\ 0, & s < 2p + 5, s \neq 3, 2p + 2. \end{cases}$$

The group $pH^{2p+2}(BPU_n)$ is generated by $\delta \bar{P}^1(\bar{x}_1)$.

2. For $r = 0$, we simply have $pH^*(BPU_n) = 0$ for all $s \geq 0$.

**Remark 1.3.** Note $pH^*(BPU_n) \cong pH^*(BPU_n)(p)$. By the discussion preceding Remark 1.2, Theorem 1 completely determines $H^*(BPU_n; \mathbb{Z}/p)$ for $0 \leq s < 2p + 5$.

For $s \leq 3$, the groups $pH^*(BPU_n)$ are well known and are part of Theorem 1.1 of [12]. For $3 < s < 2p + 2$, they are given in Theorem 1.2 of [12]. Therefore, what remains to show is

$$(1.4) \quad pH^{2p+2}(BPU_n) \cong \mathbb{Z}/p, \quad pH^{2p+3}(BPU_n) = pH^{2p+4}(BPU_n) = 0.$$
Remark 1.4. For \( p = 2 \), it was shown by the first named author \cite{12} that the 2-torsion subgroup of \( H^*(BPU_n) \) in dimension \( s = 2p + 3 = 7 \) is \( \mathbb{Z}/2 \) if \( n \equiv 2 \mod 4 \), and is 0 otherwise. In particular, Theorem 1 does not generalize to the case \( p = 2 \).

Remark 1.5. For \( p = 3 \), (1.4) follows immediately from the computation in \cite{12} of \( H^*(BPU_n) \) in dimensions 8, 9 and 10.

Organization of the paper. In Section 2 we discuss some preliminary results of the Serre spectral sequence \( U^E \) associated to the fiber sequence

\[
\Omega B \rightarrow BU_n \rightarrow BPU_n \rightarrow K(\mathbb{Z}, 3).
\]

This will be our main tool for computing the \( p \)-primary subgroup \( \rho H^*(BPU_n) \). We will also show that (1.4) can be deduced from Theorem 1.2 of \cite{12} and Proposition 2.10 which says that certain chain complex \( M \) constructed from the differentials in \( U^E \) is exact.

In Section 3 we prove Proposition 2.10. The proof is based on the explicit computation of some relevant differentials in \( U^E \). This section finishes our proof of Theorem 1.

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2. The spectral sequences

The Serre spectral sequence \( U^E \). We follow the strategy employed in \cite{12} to compute the cohomology of \( BPU_n \). The short exact sequence of Lie groups

\[
1 \rightarrow S^1 \rightarrow U_n \rightarrow PU_n \rightarrow 1
\]

induces a fiber sequence of their classifying spaces

\[
BS^1 \rightarrow BU_n \rightarrow BPU_n.
\]

Notice that \( BS^1 \) is of the homotopy type of the Eilenberg-Mac Lane space \( K(\mathbb{Z}, 2) \) and indeed we obtain another fiber sequence

\[
U : BU_n \rightarrow BPU_n \xrightarrow{\chi} K(\mathbb{Z}, 3).
\]

Remark 2.1. In general, it is not always possible to obtain a fiber sequence of the form \( F \rightarrow E \rightarrow B \) from a fiber sequence \( \Omega B \rightarrow F \rightarrow E \). See Ganea \cite{7} for more.

We will use the Serre spectral sequence associated to the last fiber sequence to compute the cohomology of \( BPU_n \). For notational convenience, we denote this spectral sequence by \( U^E \). The \( E_2 \) page of \( U^E \) has the form

\[
U^E_{2^{a,b}} = H^s(K(\mathbb{Z}, 3); H^t(BU_n)) \Rightarrow H^{s+t}(BPU_n).
\]
In principle, Cartan and Serre \[3\] determined the cohomology of $K(A, n)$ for all finitely generated abelian groups $A$. Also see Tamanoi \[18\] for a nice treatment.

We summarize the $p$-local cohomology of $K(\mathbb{Z}, 3)$ in low dimensions as follows.

**Proposition 2.2.** Let $p > 2$ be a prime. In degrees up to $2p + 5$, we have

\[
H^s(K(\mathbb{Z}, 3))_{(p)} = \begin{cases} 
\mathbb{Z}(p), & s = 0, 3, \\
\mathbb{Z}/p, & s = 2p + 2, 2p + 5, \\
0, & s < 2p + 5, s \neq 0, 3, 2p + 2.
\end{cases}
\]

where $x_1$, $y_{p, 0}$, $x_1y_{p, 0}$ are generators on degree $3, 2p + 2, 2p + 5$ respectively. In addition, we have $y_{p, 0} = \delta \mathbb{P}^1(\overline{x}_1)$.

Here we use the same notations for the generators as in \[12\]. Sometimes we abuse notations and let $x_1, y_{p, 0}$ denote $\chi^*(x_1), \chi^*(y_{p, 0})$, where $\chi : BPU_n \to K(\mathbb{Z}, 3)$ is defined in \[2.4\]. For instance, we have

**Proposition 2.3** (Theorem 1.2, \[12\]). Let $p$ be a prime. In $H^{2p + 2}(BPU_n)$, we have $y_{p, 0} \neq 0$ of order $p$ when $p \mid n$, and $y_{p, 0} = 0$ otherwise. Furthermore, the $p$-torsion subgroup of $H^k(BPU_n)$ is 0 for $3 < k < 2p + 2$.

Also recall

\[
H^*(BU_n) = \mathbb{Z}[c_1, c_2, \ldots, c_n], \quad |c_i| = 2i.
\]

In particular, $H^*(BU_n)$ is torsion-free. We have

\[
U^E_{2} \cong H^*(K(\mathbb{Z}, 3)) \otimes H^1(BU_n).
\]

**The auxiliary fiber sequences and spectral sequences.** To determine some of the differentials in $U^E$, we consider two more fiber sequences.

Let $T^n$ be the maximal torus of $U^n$ with the inclusion denoted by

\[
\psi : T^n \to U_n.
\]

Passing to quotients over $S^1$, we have another inclusion of maximal torus

\[
\psi' : PT^n \to PU_n.
\]

The quotient map $T^n \to PT^n$ fits in an exact sequence of Lie groups

\[
1 \to S^1 \to T^n \to PT^n \to 1,
\]

which induces a fiber sequence

\[
T : BT^n \to BPT^n \to K(\mathbb{Z}, 3).
\]

Notice that we have

\[
H^*(BT^n) = \mathbb{Z}[v_1, v_2, \ldots, v_n], \quad |v_i| = 2.
\]

The next fiber sequence is simply the path fibration for the space $K(\mathbb{Z}, 3)$

\[
K : K(\mathbb{Z}, 2) \to \ast \to K(\mathbb{Z}, 3)
\]

where $\ast$ denotes a contractible space. We denote their associated Serre spectral sequences as $^T E$ and $^K E$ respectively.

We denote the corresponding differentials of $^U E$, $^T E$, and $^K E$ by $d_\ast^T$, $d_\ast^T$, and $d_\ast^K$, respectively, if there are risks of ambiguity. Otherwise, we simply denote the differentials by $d_\ast^\ast$. 
These fiber sequences fit into the following homotopy commutative diagram:

\[
\begin{array}{c}
K : & K(\mathbb{Z}, 2) & \longrightarrow & * & \longrightarrow & K(\mathbb{Z}, 3) \\
\downarrow \phi & \downarrow B\varphi & \downarrow & & \downarrow = \\
T : & BT^n & \longrightarrow & BPT^n & \longrightarrow & K(\mathbb{Z}, 3) \\
\downarrow \phi & \downarrow B\psi & \downarrow & & \downarrow = \\
U : & BU_n & \longrightarrow & BPU_n & \longrightarrow & K(\mathbb{Z}, 3) \\
\end{array}
\]

(2.5)

Here, the map \(B\varphi : K(\mathbb{Z}, 2) \simeq BS^1 \to BT^n\) is the de-looping of the diagonal map \(S^1 \to T^n\). The induced homomorphism between cohomology rings is as follows:

\[B\varphi^* : H^*(BT^n) = \mathbb{Z}[v_1, v_2, \ldots, v_n] \to H^*(BS^1) = \mathbb{Z}[v], \ v_i \mapsto v.\]

The map \(B\psi : BT^n \to BU_n\) induces the injective ring homomorphism

\[B\psi^* : H^*(BU_n) = \mathbb{Z}[c_1, \ldots, c_n] \to H^*(BT^n) = \mathbb{Z}[v_1, \ldots, v_n],
\]

\[c_i \mapsto \sigma_i(v_1, \ldots, v_n),\]

where \(\sigma_j(t_1, t_2, \cdots, t_n)\) be the \(j\)th elementary symmetric polynomial in variables \(t_1, t_2, \cdots, t_n\):

\[
\begin{align*}
\sigma_0(t_1, t_2, \cdots, t_n) &= 1, \\
\sigma_1(t_1, t_2, \cdots, t_n) &= t_1 + t_2 + \cdots + t_n, \\
\sigma_2(t_1, t_2, \cdots, t_n) &= \sum_{i<j} t_i t_j, \\
& \vdots \\
\sigma_p(t_1, t_2, \cdots, t_n) &= t_1 t_2 \cdots t_n.
\end{align*}
\]

(2.6)

We will use the associated maps of spectral sequences to compute the differentials in \(U E\). This is possible because we have a good understanding of the corresponding differentials in \(T E\) and \(K E\). In particular, we have the following results.

**Proposition 2.4** ([12], Corollary 2.16). The higher differentials of \(K E_\ast^\ast\) satisfy

\[
d_3(v) = x_1,
\]

\[
d_{2p-1}(x_1 v^{p-1}) = v^{p-1} - (p-1) y_{p,0}, \quad e > 0, \ \gcd(l, p) = 1,
\]

\[
d_r(x_1) = d_r(y_{p,0}) = 0, \quad \text{for all } r,
\]

and the Leibniz rule.

**Remark 2.5.** Proposition 2.4 is a special case of Corollary 2.16, [12]. Here, we take the opportunity to correct a typo in the original Corollary 2.16, [12], where the condition \(k \geq e\) should be replaced by \(e > k\).

**Proposition 2.6** ([12], Proposition 3.2). The differential \(T d_\ast^\ast\), is partially determined as follows:

\[
T d_r^\ast 2t(v^t \xi) = (B\pi_i)^* (K d_r^\ast 2t(v^t \xi)),
\]

(2.7)

where \(\xi \in T E_\ast^0\), a quotient group of \(H^*(K(\mathbb{Z}, 3))\), and \(\pi_i : T^n \to S^1\) is the projection of the \(i\)th diagonal entry. In plain words, \(T d_r^\ast 2t(v^t \xi)\) is simply \(K d_r^\ast 2t(v^t \xi)\) with \(v\) replaced by \(v_i\).
Remark 2.7. Here we correct another typo in the original Proposition 3.2 in [12], in which “$\xi \in T E^{0,*}_r$” should be replaced by “$\xi \in T E^{*,0}_r$.”

Proposition 2.8 ([12], Proposition 3.3).  

(1) The differential $T d^0_{3,t}$ is given by the “formal divergence”

$$\nabla = \sum_{i=1}^n (\partial/\partial v_i) : H^t(BT^n; R) \to H^{t-2}(BT^n; R),$$

in such a way that $T d^0_{3} = \nabla(-) \cdot x_1$. For any ground ring $R = \mathbb{Z}$ or $\mathbb{Z}/m$ for any integer $m$, the spectral sequence degenerates at $T E^{0,*}_4$. Indeed, we have $T E^{0,*}_\infty = T E^{0,*}_4 = \text{Ker} T d^0_{3} = \mathbb{Z}[v_1 - v_n, \ldots, v_{n-1} - v_n]$. 

(2) The spectral sequence degenerates at $T E^{0,*}_r$. Indeed, we have $T E^{0,*}_\infty = T E^{0,*}_4 = \text{Ker} T d^0_{3} = \mathbb{Z}[v_1 - v_n, \ldots, v_{n-1} - v_n]$.

Corollary 2.9 ([12], Corollary 3.4).  

$U d^0_{3,*}(c_k) = \nabla(c_k)x_1 = (n - k + 1)c_{k-1}x_1$.

Computations in the spectral sequence $U E$. In order to study

$$p H^*(BPU_n) \cong p[H^*(BPU_n)(p)],$$

it suffices to look at the $p$-localized spectral sequence, where the $E_2$ page becomes

(2.8)  

$$U E^{s,t}_2(p) = H^s(K(\mathbb{Z}, 3))(p) \otimes H^t(BU_n) = H^s(K(\mathbb{Z}, 3)) \otimes H^t(BU_n)(p).$$

By abuse of notation, for the rest of this paper, we let $U E, T E$ and $K E$ denote the corresponding $p$-localized Serre spectral sequences.

By Proposition 2.2 and (2.3), in the range $s \leq 2p + 5$, the only cases in which $U E^{s,t}_2$ could be nonzero are when $s = 0, 3, 2p + 2, 2p + 5$ and $t$ is even. To simplify the notations, we let

$$M^0 = U E^{0,2p-2}_2, \quad M^1 = U E^{3,2p}_2, \quad M^2 = U E^{2p+2,2}_2, \quad M^3 = U E^{2p+5,0}_2.$$

Inspection of degrees shows that $U E^{3,2p}_s$ can receive only the $d_3$ differential and support the $d_{2p-1}$ differential. Similarly, $U E^{2p+2,2}_s$ can receive only the $d_{2p-1}$ differential and support the $d_3$ differential. In addition, all $d_2$’s are trivial and therefore we have $U E^{3,2p}_s = U E^{3,2p}_s$. We let $\delta^0$ be the map

$$\delta^0 : M^0 = U E^{0,2p+2}_3 \xrightarrow{d_3} U E^{3,2p}_3 = M^1.$$

We let $\delta^1$ be the composition

$$\delta^1 : M^1 = U E^{3,2p}_3 \to U E^{3,2p}/\text{Im} d_3 = U E^{3,2p}_2 \xrightarrow{d_{2p-1}} U E^{2p+2,2}_2 = \text{Ker} d_3 \subset M^2.$$

We let $\delta^2$ be the map

$$\delta^2 : M^2 = U E^{2p+2,2}_3 \xrightarrow{d_3} U E^{2p+5,0}_3 = M^3.$$

One immediately sees that

$$M^0 \xrightarrow{\delta^0} M^1 \xrightarrow{\delta^1} M^2 \xrightarrow{\delta^2} M^3$$

is a chain complex of $\mathbb{Z}_{(p)}$-modules, which we denote by $M$. We will show later that Theorem 1 is a consequence of the following

Proposition 2.10. Let $p \geq 3$ be a prime number such that $p \mid n$. The chain complex $M$ defined above is exact.
Proof of Theorem 2.1 assuming Proposition 2.10. Let $n = p^r m$. For $r = 0$, the theorem follows from Proposition 1.1. In the rest of the proof we assume $r > 0$. First, we prove

$$pH^{2p+2}(BU_n) \cong \mathbb{Z}/p.$$  

By Proposition 2.3, $y_{p,0} \in U_{2p+2,0}$ survives to a nonzero element in $H^{2p+2}(BU_n)$ of order $p$. Therefore, we have

$$U_{2p+2,0} \cong \mathbb{Z}/p.$$  

Since the only nontrivial entries in $U^{*,*}_{2p+2}$ of total degree $2p + 2$ are $U_{2p+2,0}$ and $U^{0,2p+2}$, we have a short exact sequence of $\mathbb{Z}(p)$-modules

$$0 \to U_{2p+2,0} \to H^{2p+2}(BU_n) \to U^{0,2p+2} \to 0.$$  

Since $U^{0,2p+2} \subset U_{2p+2,0}$ is a free $\mathbb{Z}(p)$-module, the above short exact sequence splits and we have

$$H^{2p+2}(BU_n) \cong U_{2p+2,0} \oplus U^{0,2p+2},$$  

from which we deduce

$$pH^{2p+2}(BU_n) \cong U_{2p+2,0} \cong \mathbb{Z}/p.$$  

Since the row $E^*,0$ is the image of $\chi^*$, the above implies

(2.9) $$pH^{2p+2}(BU_n) = \chi^*(H^{2p+2}(K(Z,3))).$$  

From (2.9) and Proposition 2.2, it follows that $pH^{2p+2}(BU_n)$ is generated by $\delta P^1(\bar{x}_1)$.

Next, we prove

$$pH^{2p+3}(BU_n) = H^{2p+3}(BU_n) \cong 0.$$  

The exactness of $M$ at $M^1$ implies $U_{2p}^{3,2} = 0$. On the other hand, $U_{2}^{3,2}$ is the only nontrivial entry in $U_{2}^{*,*}$ of total degree $2p + 3$. Hence, we have

$$pH^{2p+3}(BU_n) \subset H^{2p+3}(BU_n) = U_{2p}^{3,2} = 0.$$  

Finally, we prove

$$pH^{2p+4}(BU_n) = 0.$$  

The exactness of $M$ at $M^2$ implies $U_{2p+2}^{2,2} = 0$. Since $U_{2}^{0,2p+4}$ and $U_{2}^{2p+2,2}$ are the only nontrivial entries in $U_{2}^{*,*}$ of total degree $2p + 4$, we have

$$H^{2p+4}(BU_n) \cong U_{2p+4}^{0,2}.$$  

which is torsion-free. In particular, we have $pH^{2p+4}(BU_n) = 0$.  

The proof of Proposition 2.10 occupies Section 3.
3. The proof of Proposition \textbf{2.10}

From \textbf{(2.8)}, we can write out the \( \mathbb{Z}(p) \)-modules \( M^0, M^1, M^2, M^3 \) more explicitly:

\[
M^0 = H^0(K(\mathbb{Z}, 3)) \otimes H^{2p+2}(BU_n)(p) \cong H^{2p+2}(BU_n)(p)
\]

is the free \( \mathbb{Z}(p) \)-module generated by monomials in \( c_1, \ldots, c_{p+1} \) in dimension \( 2p+2 \), and

\[
M^1 = H^3(K(\mathbb{Z}, 3)) \otimes H^{2p}(BU_n)(p) \cong H^{2p}(BU_n)(p)
\]

is the free \( \mathbb{Z}(p) \)-module generated by elements of the form \( cx_1 \) where \( c \) is a monomial in \( c_1, \ldots, c_p \) in dimension \( 2p \). Furthermore, we have

\[
M^2 = H^{2p+2}(K(\mathbb{Z}, 3)) \otimes H^2(BU_n)(p) = \mathbb{Z}(p)\{c_1y_{p,0}\}/p \cong \mathbb{Z}/p
\]

and

\[
M^3 = H^{2p+5}(K(\mathbb{Z}, 3)) \otimes H^0(BU_n)(p) = \mathbb{Z}(p)\{x_1y_{p,0}\}/p \cong \mathbb{Z}/p.
\]

The exactness of \( M \) at \( M^2 \).

**Lemma 3.1.** In the spectral sequence \( T E \), we have

\[
\begin{cases}
  v_n^k x_1 \in \text{Im} T d_3, & 0 \leq k \leq p-2 \text{ or } k = p, \\
  T d_{3,p-1}^3(v_n^{p-1}x_1) = y_{p,0}.
\end{cases}
\]

**Proof.** When \( p \mid k+1 \), the first formula in Proposition \textbf{2.4} together with Proposition \textbf{2.6} imply that

\[
v_n^k x_1 = \frac{1}{k+1} T d_3(v_n^{k+1})
\]

is in the image of \( T d_3 \). This completes the proof for the case \( 0 \leq k \leq p-2 \) or \( k = p \).

The remaining case is proved by applying the second formula in Proposition \textbf{2.4} taking \( e = l = 1 \), and then Proposition \textbf{2.6}. \( \square \)

**Lemma 3.2.** The map \( \delta^1 : M^1 \to M^2 \cong \mathbb{Z}/p \) is surjective.

**Proof.** Recall the morphism of fiber sequences \( \Psi \) introduced in \textbf{(2.5)}, and the induced morphism \( \Psi^* : U E \to T E \) of spectral sequences.

For \( 1 \leq i \leq n \), let \( v'_i = v_i - v_n \). It follows from (2) of Proposition \textbf{(2.8)} that the \( v'_i \)'s are permanent cycles. To determine the value of \( \delta^1 \) at \( c_p x_1 \in M^1 \), we have

\[
\Psi^* \delta^1(c_p x_1) = \Psi^* U \mu_{2p-1}(c_p x_1) = T d_{2p-1}^3 \Psi^*(c_p x_1)
\]

\[
= T d_{2p-1}^3 \sum_{n \geq i_1 > i_2 > \cdots > i_p \geq 1} v_{i_1} v_{i_2} \cdots v_{i_p} x_1
\]

\[
(3.2)
\]

\[
= T d_{2p-1}^3 \sum_{n \geq i_1 > i_2 > \cdots > i_p \geq 1} (v'_i + v_n)(v'_2 + v_n) \cdots (v'_p + v_n) x_1
\]

\[
= T d_{2p-1}^3 \sum_{n \geq i_1 > i_2 > \cdots > i_p \geq 1} \sum_{j=0}^p \sigma_j(v'_1, \cdots, v'_p) v_n^{p-j} x_1).
\]

where \( \Psi^* : U E \to T E \) is the morphism of spectral sequences induced by the inclusions of maximal tori \( T^n \to U_n \) and \( PT^n \to PU_n \), as introduced in \textbf{(2.5)}, and \( \sigma_i \) the elementary symmetric polynomials in \( p \) variables, as in \textbf{(2.6)}.****
By Lemma 3.1 we simplify (3.2) and obtain

\[ \Psi^* \delta^1(c_p x_1) = T d_{2p-1} \left( \sum_{n \geq i_1 > i_2 > \ldots > i_p \geq 1} \sigma_1(v'_{i_1}, \ldots, v'_{i_p}) v_n^{p-1} x_1 \right). \]

To proceed, we evaluate the expression

\[ \sum_{n \geq i_1 > i_2 > \ldots > i_p \geq 1} \sigma_1(t_{i_1}, \ldots, t_{i_p}) \]

for variables \( t_i, 1 \leq i \leq n \). Since it is multi-linear and symmetric in the variables \( t_1, \ldots, t_n \), we have

\[ \sum_{n \geq i_1 > i_2 > \ldots > i_p \geq 1} \sigma_1(t_{i_1}, \ldots, t_{i_p}) = \lambda \sum_{i=1}^{n} t_i \]

for some \( \lambda \in \mathbb{Z} \). Taking the substitution \( t_1 = \ldots = t_n = 1 \) and comparing both sides of the above, we obtain

\[ \lambda = \frac{p}{n} \left( \frac{n}{p-1} \right) = \left( \frac{n-1}{p-1} \right) \not\equiv 0 \pmod{p} \]

and

\[ \sum_{n \geq i_1 > i_2 > \ldots > i_p \geq 1} \sigma_1(t_{i_1}, \ldots, t_{i_p}) = \left( \frac{n-1}{p-1} \right) \sum_{i=1}^{n} t_i. \]

Consider the following commutative diagram:

\[
\begin{array}{ccc}
M^1 &=& \Psi^* E_2^{3,2p} \\
\downarrow && \downarrow \\
U E_2^{3,2p} & \xrightarrow{\Psi^*} & T E_2^{3,2p} \\
\downarrow && \downarrow \\
U E_{2p-1}^{3,2p} & \xrightarrow{\Psi^*} & T E_{2p-1}^{3,2p} \\
\downarrow && \downarrow \\
U E_{2p-1}^{2p+2,2} & \xrightarrow{\Psi^*} & T E_{2p-1}^{2p+2,2} \\
\downarrow && \downarrow \\
M^2 &=& \Psi^* E_2^{2p+2,2} \\
\end{array}
\]

where the composition of the left vertical maps is \( \delta^1 \) and we resume the computation of \( \Psi^* \delta^1(c_p x_1) \) started in (3.3):

\[ \Psi^* \delta^1(c_p x_1) = T d_{2p-1} \left( \frac{n-1}{p-1} \sum_{i=1}^{n} v'_{i} v_n^{p-1} x_1 \right) \] (by \( 3.4 \))

\[ = \left( \frac{n-1}{p-1} \right) \sum_{i=1}^{n} v'_{i} y_{p,0} \] (since \( v'_{i} \)'s are permanent cocycles)

\[ = \left( \frac{n-1}{p-1} \right) \sum_{i=1}^{n} v_{i} y_{p,0} \] (since \( y_{p,0} \) is \( p \)-torsion)

\[ = \Psi^* \left( \left( \frac{n-1}{p-1} \right) c_1 y_{p,0} \right). \]
By the injectivity of
\[ \Psi^*: M^2 = U_{E_2}^{2p+2,2} \to T_{E_2}^{2p+2,2} \]
together with (3.5), we have
\[ \delta^1(c_p x_1) = \left( \frac{n-1}{p-1} \right) c_1 y_{p,0} \neq 0 \]
and we conclude. \qed

**Lemma 3.3.** The chain complex $\mathcal{M}$ is exact at $M^2$.

**Proof.** By Lemma 3.2 and the fact that $\mathcal{M}$ is a chain complex, we have $\delta^2 = 0$ and the lemma follows. Alternatively, one may compute $\delta^2 = d_3^{2p+2,2}$ directly with Corollary 2.9 and obtain the same result. \qed

**The exactness of $\mathcal{M}$ at $M^1$.** Recall that the $\mathbb{Z}(p)$-module $M^1$ is freely generated by elements of the form $cx_1$ for
\[ c \in S' := \{ c_1^i c_2^j \cdots c_p^k | i_k \geq 0, \sum_k k i_k = p \}. \]
Indeed, $S'$ is simply the set of monomials in $c_1, c_2, \ldots, c_n$ in $H^{2p}(BU_n)$. We define a total ordering $\mathcal{O}$ on monomials in $c_1, c_2, \ldots, c_n$ as follows. We assert
\[ c_1^{i_1} c_2^{i_2} \cdots c_p^{i_p} > c_1^{j_1} c_2^{j_2} \cdots c_p^{j_p} \]
if and only if
1. there is at least one $k$ such that $i_k \neq j_k$, and
2. for the smallest such $k$, we have $i_k > j_k$.

Let $S := S' - \{ c_p \}$. Then $\mathcal{O}$ defines total orderings on $S$, $S'$ and $S'x_1$ as well. To compare $cx_1, c'x_1 \in S'x_1$, we assert $cx_1 > c'x_1$ if and only if $c > c'$.

Let $L$ be the $\mathbb{Z}(p)$-submodule of $H^{2p}(BU_n)(p)$ spanned by $S$. We define a $\mathbb{Z}(p)$-linear map
\[ \tau: L \to M^0 = H^{2p+2}(BU_n)(p) \]
as follows. Each element in $S$ is of the form $c_1^{i_1} c_2^{i_2} \cdots c_p^{i_p}$ such that $k < p$ and $i_k > 0$, and we define
\[ \tau(c_1^{i_1} c_2^{i_2} \cdots c_p^{i_p}) := (c_1^{i_1} c_2^{i_2} \cdots c_{k-1}^{i_{k-1}})(c_k^{i_k-1} c_{k+1}). \]

**Lemma 3.4.** Let $\bar{\tau}: L/pL \to M^0/pM^0$ and $\delta^0: M^0/pM^0 \to M^1/pM^1$ denote the mod $p$ reductions of $\tau$ and $\delta^0$, respectively. Then the image of the composition
\[ L/pL \xrightarrow{\bar{\tau}} M^0/pM^0 \xrightarrow{\delta^0} M^1/pM^1 \]
is $Lx_1/pLx_1$. In particular, we have
\[ \text{Im} \delta^0 \tau \subset W := Lx_1 + (pc_p x_1) \subset M^1. \]

**Proof.** Consider the $\mathbb{Z}(p)$-basis $S$, $S'x_1$ for $L$ and $M^1$, respectively, both in the descending order with respect to the ordering $\mathcal{O}$. Notice that $c_p x_1$ is the smallest element in $S'$. We study the $(N + 1) \times N$ matrix $A$ of the map
\[ \delta^0 \tau: L \to M^1 \]
with respect to these basis, where $N$ is the cardinality of $S$. 

Consider an arbitrary element
\[ c := c_1^{i_1} \cdots c_k^{i_k} \in S \]
with \( k < p \) and \( i_k > 0 \). By Corollary 2.9 and the Leibniz’s formula, we have
\[
\delta^0(\tau(c) = \delta^0(c_1^{i_1} \cdots c_k^{i_k-1} c_{k+1}^{i_k})
\]
\[
= \begin{cases} (n-k)cx_1 + ni_1c_1^{i_1-1}c_2^{i_2} \cdots c_k^{i_k-1}c_{k+1}x_1 + \text{(higher order terms)}, & i_1 > 0, \\ (n-k)cx_1 + \text{(higher order terms)}, & i_1 = 0. \end{cases}
\]
In both cases, we have
\[
\delta^0(\tau(c)) \equiv (n-k)cx_1 + \text{(higher order terms)} \pmod{p}.
\]
Therefore, the matrix \( A \) satisfies
\[
A \equiv \begin{pmatrix} \lambda_1 & * & \cdots & * \\ 0 & \lambda_2 & * & \vdots \\ 0 & 0 & \ddots & * \\ 0 & 0 & \cdots & 0 \end{pmatrix} \pmod{p},
\]
where the \( \lambda_i \)'s are of the form \( n-k \) for \( k < p \), which are invertible in \( \mathbb{Z}_p \), and we have verified that the image of the composition
\[
L/pL \xrightarrow{\delta} M^0/pM^0 \xrightarrow{\delta_0} M^1/pM^1
\]
is \( Lx_1/pLx_1 \). The equation (3.6) follows from the above and the fact
\[
M^1 = Lx_1 + (c_px_1).
\]

\[ \square \]

**Lemma 3.5.** Consider the \( \mathbb{Z}_p \)-submodule \( V = \tau(L) + (c_1c_p - c_{p+1}) \) of \( M^0 \). We have \( \delta^0(V) \subset W \) where
\[
W := Lx_1 + (pc_px_1) \subset M^1
\]
is the \( \mathbb{Z}_p \)-submodule of \( M^1 \) defined in Lemma 3.4.

**Proof.** By Lemma 3.3, we have \( \delta^0(\tau(L)) \subset W \). On the other hand, we have
\[
(3.7) \quad \delta^0(c_1c_p - c_{p+1}) = (n-p+1)c_1c_{p-1}x_1 + pc_px_1 \in W,
\]
and we conclude.

\[ \square \]

**Lemma 3.6.** The chain complex \( \mathcal{M} \) is exact at \( M^1 \).

**Proof.** By Lemma 3.3, the restriction of \( \delta^0 \) to \( V \) has image in \( W \). Therefore, we write \( \delta^0_V := \delta^0|_V : V \to W \) and consider its mod \( p \) reduction
\[
\delta^0_V : V/pV \to W/pW = Lx_1/pLx_1 + (pc_px_1)/(p^2c_px_1).
\]
By Lemma 3.3, we have \( Lx_1/pLx_1 \subset \text{Im} \delta^0_V \).

By \( Lx_1/pLx_1 \subset \text{Im} \delta^0_V \) and (3.7), we have \( [pc_px_1] \in \text{Im} \delta^0_V \), where \( [pc_px_1] \) is the class in \( W/pW \) represented by \( pc_px_1 \). Therefore, \( \delta^0_V : V/pV \to W/pW \) is surjective. By Nakayama’s lemma in commutative algebra (Theorem 2.2, Chapter 1, [17]), \( \delta^0_V : V \to W \) is surjective.

Therefore, we have
\[
(3.8) \quad \text{Im} \delta^0 \supset \text{Im} \delta^0_V = W = Lx_1 + (pc_px_1).
\]
On the other hand, we have $\text{Ker}\delta^1 \supset \text{Im}\delta^0$, and therefore $\text{Ker}\delta^1 \supset W$. Now, by Lemma 3.2 we have
\[
\mathbb{Z}/p \cong M^1/(L + (pc_p x_1)) = M^1/W \to M^1/\text{Ker}\delta^1 \cong \mathbb{Z}/p,
\]
where the arrow is the tautological quotient map, which is surjective. Therefore, the above composition is a bijection. It follows that we have
\[
W = \text{Ker}\delta^1 \supset \text{Im}\delta^0,
\]
and the lemma follows from (3.8) and (3.9). \qed

Lemma 3.6 and Lemma 3.3 complete the proof of Proposition 2.10.

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