No semistability at infinity for Calabi-Yau metrics asymptotic to cones

Song Sun¹ · Junsheng Zhang¹

Received: 26 September 2022 / Accepted: 16 February 2023 / Published online: 18 April 2023
© The Author(s), under exclusive licence to Springer-Verlag GmbH Germany, part of Springer Nature 2023

Abstract
We discover a “no semistability at infinity” phenomenon for complete Calabi-Yau metrics asymptotic to cones, which is proved by eliminating the possible appearance of an intermediate K-semistable cone in the 2-step degeneration theory developed by Donaldson and the first author. It is in sharp contrast to the setting of local singularities of Kähler-Einstein metrics. A byproduct of the proof is a polynomial convergence rate to the asymptotic cone for such manifolds, which bridges the gap between the general theory of Colding-Minicozzi and the classification results of Conlon-Hein.

1 Introduction
Throughout this paper we denote by \((X, p, g, J, \omega, \Omega_1)\) a pointed complete Calabi-Yau manifold of complex dimension \(n\). This means that \(p \in X\), \((g, J, \omega)\) is a complete Kähler metric, \(\Omega_1\) is a holomorphic volume form and the following complex Monge-Ampère equation is satisfied

\[
\omega^n = (\sqrt{-1})^n \Omega \wedge \overline{\Omega}.
\]

We will always impose the following two conditions

- Euclidean volume growth: there exists a \(\kappa > 0\) such that \(\text{Vol}(B_g(p, R)) \geq \kappa R^{2n}\) for all \(R > 0\).
- Quadratic curvature decay: there exists a \(C > 0\) such that \(|\text{Rm}_g(q)| \leq C d_g(p, q)^{-2}\) for all \(q \neq p\).

We also denote by \((\mathcal{C}, O, g_C, J_C, \omega_C, \Omega_C)\) a Calabi-Yau cone. This means that \(O\) is a distinguished point called the vertex, \(\mathcal{C} \setminus \{O\}\) is a smooth manifold diffeomorphic to

---

¹ Department of Mathematics, University of California, Berkeley, CA 94720, USA

S. Sun
sosun@berkeley.edu

J. Zhang
jszhang@berkeley.edu
$\mathbb{R}_{>0} \times \mathcal{L}$ for some compact manifold $\mathcal{L}$, $(g_C, J_C, \omega_C, \Omega_C)$ is a Calabi-Yau metric on $\mathcal{C} \setminus \{O\}$ and $g_C = dr^2 + r^2 g_L$ is a Riemannian cone. A Calabi-Yau cone is naturally a normal affine algebraic cone.

It is known by [15, 24] that any $X$ as above is asymptotic to a unique Calabi-Yau cone $\mathcal{C}$ at infinity. Moreover, $X$ is naturally a quasi-projective variety and there is an algebro-geometric 2-step degeneration from $X$ to $\mathcal{C}$, via a possible intermediate $K$-semistable cone $W$. The main result of this paper is

**Theorem 1.1 (No semistability at infinity)** $W$ is isomorphic to $\mathcal{C}$ as normal affine algebraic cones.

This comes out of our expectation, since the 2-step degeneration theory of [15] works symmetrically with respect to asymptotic cones at infinity and local tangent cones at a singularity and it is known that $W$ can not always be eliminated in the local setting. Theorem 1.1 reveals a sharp contrast between the two cases. Our proof makes a novel use of the Tian-Yau construction of complete Calabi-Yau metrics and the Bishop-Gromov volume monotonicity in Riemannian geometry. As a byproduct, we also show that $X$ is polynomially asymptotic to $\mathcal{C}$ at infinity:

**Theorem 1.2 (Polynomial rate to the asymptotic cone)** There exists a diffeomorphism $\Psi$ from the open subset $\{r > 1\} \subset \mathcal{C}$ to the complement of a compact set in $X$ such that for some $\delta > 0$ and for all $k \geq 0$,

$$
\left| \nabla^k_{g_C} (\Psi^* g - g_C) \right|_{g_C} + \left| \nabla^k_{g_C} (\Psi^* J - J_C) \right|_{g_C} + \left| \nabla^k_{g_C} (\Psi^* \Omega - \Omega_C) \right|_{g_C} = O(r^{-\delta-k}),
$$

$r \to \infty$.

(1.1)

Theorem 1.2 bridges the gap between two previous results. On the one hand, on the level of Riemannian metrics the result of Colding-Minicozzi [6], which uses the Łojasiewicz-Simon theory, implies that $X$ is asymptotic to a unique cone at a logarithmic rate. On the other hand, Conlon-Hein [11] classified complete Calabi-Yau manifolds asymptotic to a Calabi-Yau cone, however under the stronger extra assumption of polynomial rate (as given exactly by (1.1)).

**2 Outline of the proof**

**2.1 2-step degeneration theory**

Fix $\lambda = 1/\sqrt{2}$. Denote by $(X_j, J_j, g_j, \Omega_j, p_j)$ the rescaling of $X$ by a factor $\lambda^j$. This means that we take $J_j = J$, $p_j = p$, $g_j = \lambda^2 j g$, $\Omega_j = \lambda^j \Omega$. It follows from the Bishop-Gromov volume monotonicity that after passing to a subsequence $(X_j, p_j)$ converges in the pointed Gromov-Hausdorff sense to a Calabi-Yau cone. We summarize below the theory developed in [15] and [24]. In [15] a more general singular setting is considered but under a technical assumption that $X$ is a rescaled Gromov-Hausdorff limit of polarized Kähler-Einstein manifolds. This assumption was later
removed in [24] for complete Calabi-Yau manifolds with Euclidean volume growth and quadratic curvature decay, so the results of [15] apply in our setting.

One conclusion of [15, 24] is that the asymptotic cone is unique as a Calabi-Yau cone, i.e., it does not depend on the choice of subsequences involved in the definition. This may also follow from the more general result of Colding-Minicozzi [6] (which does not concern the complex structure), but the complex geometric proof in [15, 24] more importantly gives an algebro-geometric description of the asymptotic cone. This is crucial for us so below we review some relevant statements.

Let \((\mathcal{C}, g_{\mathcal{C}}, J_{\mathcal{C}}, \omega_{\mathcal{C}}, \Omega_{\mathcal{C}}, O)\) be the unique asymptotic Calabi-Yau cone of \(X\). Since \(\mathcal{C}\) is a Kähler cone, there is a Reeb vector field \(\xi = J_{\mathcal{C}}(r \partial_r)\), which is holomorphic and Killing. It generates a holomorphic isometric action of a compact torus \(T\), and \(\mathcal{C}\) is naturally a normal affine variety. The coordinate ring \(R(\mathcal{C})\) can be intrinsically characterized as the space of holomorphic functions on \(\mathcal{C} \setminus \{O\}\) that have polynomial growth at infinity. More explicitly, there is a holomorphic embedding \(\Phi_{\infty}: (\mathcal{C}, O) \rightarrow (\mathbb{C}^N, 0)\) such that the Reeb vector field \(\xi\) extends to a linear vector field on \(\mathbb{C}^N\) of the form \(Re(\sqrt{-1} \sum d_j z_j \partial z_j)\), where \(d_j \in \mathbb{R}_{>0}\) for all \(j\). Let \(\Lambda: \mathbb{C}^N \rightarrow \mathbb{C}^N\) be the diagonal linear transformation of \(\mathbb{C}^N\) given by

\[\Lambda(z_1, \ldots, z_N) = (\lambda^{d_1} z_1, \ldots, \lambda^{d_N} z_N),\]

and let \(T_{\xi}\) be the one parameter group of linear transformations generated by \(\xi\). Denote by \(G_{\xi}\) the subgroup of \(GL(N; \mathbb{C})\) consisting of elements that commute with \(T_{\xi}\) and denote \(K_{\xi} = G_{\xi} \cap U(N)\). Notice \(T_{\xi}\) acts on \(\mathcal{C}\).

Denote by \(B_j\) the unit ball around \(p_j\) in \(X_j\), which we can identify with the ball of radius \(\lambda^{-j}\) around \(p\) in \(X\). Denote by \(B_{\infty}\) the unit ball around the vertex \(O\) in \(\mathcal{C}\). Then we know that \(B_{\infty}\) is the Gromov-Hausdorff limit of \(B_j\) as \(j \rightarrow \infty\). Fix a distance function on \(B_j \cup B_{\infty}\) (for each \(j\)) that realizes the Gromov-Hausdorff convergence. We have the following

(a) The ring \(R(X)\) of holomorphic functions on \(X\) with polynomial growth is finitely generated and \(\text{Spec}(R(X))\) defines a normal affine algebraic variety \(X'\) with isolated singularities. There is a natural map \(\pi: (X, p) \rightarrow (X', p')\) which is a crepant resolution of singularities.

(b) For all \(j \geq 1\), there are holomorphic embeddings \(\Phi_j: (X', p') \rightarrow (\mathbb{C}^N, 0)\) and \(F_j \in G_{\xi}\) satisfying \(F_j = \Lambda + \tau_j\) for linear maps \(\tau_j \rightarrow 0\), such that

(1) \(\Phi_{j+1} = F_{j+1} \circ \Phi_j\).

(2) For any subsequence of \(\{j\}\) tends to infinity, passing to a further subsequence \(\Phi_j(\pi(B_j))\) converges to \(h.\Phi_{\infty}(B_{\infty})\) in the Hausdorff sense in \(\mathbb{C}^N\) for some \(h \in K_{\xi}\). This convergence is compatible with the Gromov-Hausdorff convergence in the sense that given any sequence \(x_j \in B_j\) converging to \(x_{\infty} \in B_{\infty}\), we have \(\Phi_j(\pi(x_j))\) converges to \(h.\Phi_{\infty}(x_{\infty})\).

(3) We will always identify \((X', p')\) with \((\Phi_1(X'), 0)\). Denote \(Y_j \equiv \Lambda^{-1} X'\). Then the weighted asymptotic cone \(W = \lim_{j \rightarrow \infty} Y_j\) is a normal affine algebraic variety in \(\mathbb{C}^N\). In particular, \(W\) is also invariant under the \(T_{\xi}\) action. This limit is to be understood in a strong sense: there exists a generating set \(\{P_1, \ldots, P_j\}\) of the ideal \(\mathcal{I}_X\) of polynomials on \(\mathbb{C}^N\) vanishing on \(X'\), such
that the leading terms \( \{ \tilde{P}_1, \ldots, \tilde{P}_l \} \) generate the ideal \( I_W \). Here the leading term of a polynomial is defined via the weighted degree associated to \( \Lambda \).

(4). The coordinate ring \( R(W) \) can be intrinsically described as the graded ring associated to a filtration on \( R(X) \) determined by a degree function associated to the Calabi-Yau metric \( g \).

(5). \( \mathcal{C} \) is in the closure of the \( G_\xi \) orbit of \( W \) in a suitable multi-graded Hilbert scheme. In particular, \( W \) has only one isolated singularity.

The motivation in [15] was to relate the metric scaling on \( X \) to the algebro-geometric scaling \( \Lambda \). The main discovery there is that they are almost but not exactly the same, and one may realize the asymptotic cone \( \mathcal{C} \) as a 2-step degeneration from \( X' \) through the intermediate cone \( W \). From the technical aspect the central issue is that as \( j \to \infty \), even though \( F_j \) converges to \( \Lambda \) the accumulated error in the composition \( F_j \circ F_{j-1} \circ \cdots \circ F_2 \) may still diverge away from \( \Lambda^{j-1} \). Notice \( W \) does not inherit any canonical metric from \( X \) or \( \mathcal{C} \). In [15] a parallel result is also proved for tangent cones at local singularities of Kähler-Einstein metrics. However there are important distinctions between the two settings.

One such distinction was already pointed out in [15], which shows that the local setting is more rigid and the asymptotic setting is more flexible. In the case of local tangent cones, it is conjectured in [15] that there is a local notion of (K-)stability which characterizes both \( (W, \xi) \) and \( \mathcal{C} \) as invariants of the underlying algebraic singularity (by analogy with the Harder-Narasimhan-Seshadri filtration for sheaves, one should think of \( W \) as a semistable object and \( \mathcal{C} \) as a polystable object). Later Li [19] related this picture to a generalized volume minimization in Sasaki geometry and reformulated this conjecture using more algebro-geometric terminologies. There has been active research on this topic. The uniqueness part of the conjecture in [15] was confirmed by Li-Wang-Xu [20]. By way of contrast as suggested at the end of [15], for asymptotic cones of complete Calabi-Yau metrics one should not expect either \( W \) or \( \mathcal{C} \) is a canonical object associated to the underlying algebraic variety \( X \). Explicit examples have been constructed on \( \mathbb{C}^n \) if we allow \( W \) and \( \mathcal{C} \) to have singular cross sections (see [12, 18, 23, 28]).

The main result of this paper shows that the asymptotic setting is however more rigid from a different perspective. In the case of tangent cones at singularities of Kähler-Einstein metrics, it is known that both of the 2 steps are necessary in order to degenerate the singularity to the metric tangent cone \( \mathcal{C} \). In the case of asymptotic cones at infinity we will show that \( W \) is always isomorphic to \( \mathcal{C} \), in other words, one can degenerate the variety to \( \mathcal{C} \) in one step. Theorem 1.1 can be reformulated as

**Theorem 1.1 (No semistability at infinity)** \( W \) is isomorphic to \( \mathcal{C} \) as affine varieties with \( T_\xi \) action.

### 2.2 Outline of the proof

Now we explain the key ideas in the proof of Theorem 1.1. Theorem 1.2 will be a consequence. Our strategy is to construct a Calabi-Yau metric \( \omega_W \) on \( W \setminus \{0\} \) so that

1. the asymptotic cone of \( \omega_W \) at infinity is given by \( \mathcal{C} \);
(2) the metric completion of $\omega_W$ is homeomorphic to $W$ and is a pointed Gromov-Hausdorff limit of complete Calabi-Yau metrics. Furthermore the argument in [15] can be extended to study the singular point $0 \in W$.

Now the Bishop-Gromov inequality applying on $W$ yields that the volume ratio

$$\nu(R) = \frac{\text{Vol}(B(O, R))}{R^{2n}}$$

is a non-increasing function in $R$. On a Calabi-Yau cone $C$ we denote by $\kappa(C) = \text{Vol}(B(O, 1))$ the volume density of $C$. Now as $R \to \infty$, $\nu(R)$ converges to $\kappa(C)$, while as $R \to 0$ it converges to the volume density of the metric tangent cone at 0. By [15] and the generalized volume minimization in [21] we know the latter is not bigger than $\kappa(C)$. So $\nu(R)$ is constant in $R$, which implies that $(W, \omega_W)$ must be a metric cone hence is isometric to $C$. With more work one can show that $W$ and $C$ are $T_\xi$-equivariantly isomorphic.

The existence of $\omega_W$ is therefore the crux of the matter. To achieve this we will make use of the Tian-Yau construction [30] which produces complete Calabi-Yau metrics out of lower dimensional compact Kähler-Einstein metrics. In [30] one starts with a Fano manifold $X$ and a smooth divisor $D$ which is $\mathbb{Q}$-linearly equivalent to $K_X^{-1}$, then assuming that $D$ admits a Kähler-Einstein metric with positive Ricci curvature, one can find a complete Calabi-Yau metric on $X \setminus D$. The rough idea is to first use the Calabi ansatz to write down a background complete Kähler metric which is approximately Ricci-flat and then solve the complex Monge-Ampère equation on the noncompact manifold $X \setminus D$. The work of [30] has been extended widely to produce more examples of Calabi-Yau metrics.

In the proof of Theorem 1.1 we will apply the Tian-Yau construction in a non-traditional manner. That is, we will use the unknown Calabi-Yau metric $\omega$ on $X$ as a model at infinity to construct a Calabi-Yau metric $\omega_W$ on $W$. This is achieved by constructing a diffeomorphism between the ends of $X$ and $W$, then grafting $\omega$ to a Kähler form on $W$ that is slowly asymptotic to the cone $C$, but satisfies the complex Monge-Ampère equation with polynomially decaying error. Most of the work in this paper is devoted to making this strategy rigorous. There is an extra difficulty caused by the fact that $W$ has an isolated singularity. To get around the issue, we notice that $W$ is the limit of $Y_j$ in $\mathbb{C}^N$, and $Y_j$ admits a crepant resolution $\hat{Y}_j$ (which is isomorphic to $X$). We will first construct a family of Calabi-Yau metrics $\hat{\omega}_{j, \epsilon}$ on $\hat{Y}_j$ with uniform estimates, where $\epsilon$ roughly measures the size of the exceptional set, and then take a double limit by first letting $\epsilon \to 0$ and then letting $j \to \infty$. The first limiting step follows similarly the line of argument in [7] which constructs certain asymptotically conical (in the sense of [10]) Calabi-Yau metrics with singularities, under more restrictive assumptions. In the course of the proof we will bring in more robust techniques, such as the Hörmander $L^2$ estimate, which is of independent interest.

In the rest of this paper we will present a detailed proof of the main results. In Sect. 3 we construct a family of background Kähler metrics on $\hat{Y}_j$ which are asymptotic to the cone $C$ and are approximate solutions to the complex Monge-Ampère equation at infinity. In Sect. 4 we solve the complex Monge-Ampère equation with respect to these background metrics and derive uniform estimates. In Sect. 5 we complete the proof of Theorem 1.1 and 1.2.
We make a convention on the notations throughout this paper: \( K \) will denote a compact set in some space; \( C \) will denote a positive constant; \( C_\sharp \) will denote a constant that depends only on \( \sharp \); in particular when \( j \) does not appear in the subscript it means that the constant is uniform in \( j \). The precise meaning of these objects may vary between lines.

\section{Construction of background metrics}

\subsection{Weak conical approximation at infinity}

We denote by \( r = d(O, \cdot) \) the radial function on the asymptotic Calabi-Yau cone \( C \), by \( Q_R \) the subset \( \{ r \geq R \} \) and by \( A_{a,b} \) the annulus \( \{ a \leq r \leq b \} \). The following lemma is standard so we only give a sketch of proof.

\begin{lemma}
There exist an \( R_0 > 0 \), a compact set \( K \subset X \) and a diffeomorphism \( \Phi : Q_{R_0} \to X \setminus K \) such that for all \( l \geq 0 \),
\[
\lim_{s \to \infty} \sup_{\partial Q_s} s^l \left( \left| \nabla^l_{g_C} (\Phi^* g - g_C) \right|_{g_C} + \left| \nabla^l_{g_C} (\Phi^* J - J_C) \right|_{g_C} \right) = 0.
\]
\end{lemma}

\begin{proof}
Since \( C \) is the unique asymptotic cone of \( X \), for \( i \) large we may find a diffeomorphism \( \Phi_i \) from \( A_{4^i, 8 \cdot 4^i} \) onto \( B(p, 8 \cdot 4^i (1 + \epsilon_i)) \setminus B(p, 4^i (1 + \epsilon_i)) \) such that
\[
4^i \sup_{A_{4^i, 8 \cdot 4^i}} \left( \left| \nabla^l_{g_C} (\Phi_i^* g - g_C) \right|_{g_C} + \left| \nabla^l_{g_C} (\Phi_i^* J - J_C) \right|_{g_C} \right) \leq \epsilon_{i, l},
\]
where \( \epsilon_i \to 0 \) and for each \( l \) fixed, \( \epsilon_{i, l} \to 0 \) as \( i \to \infty \). Moreover, the rescaled maps \( \Phi_i \circ 4^i \) realize the Gromov-Hausdorff convergence to the cone \( C \). It suffices to glue together \( \Phi_i \)'s to get the desired \( \Phi \). This can be done in a straightforward way. By a contradiction argument one can prove that for \( i \geq i_0 \), in the intersection \( \text{Im}(\Phi_i) \cap \text{Im}(\Phi_{i+1}) \), the composition \( 4^{-i-1} \circ \Phi_{i-1} \circ \Phi_i \circ 4^{i+1} \) is \( \epsilon_i \) (for \( \epsilon_i \to 0 \)) close to a holomorphic isometric embedding \( \Psi_i \) of the annulus \( A_{1.2} \) into itself with \( \Psi_i^* r = r \).

It follows that \( \Psi_i \) preserves the Reeb vector field on \( C \), hence it extends to a global holomorphic isometry of \( C \) which fixes the vertex. Then we replace \( \Phi_{i+1} \) by \( \Phi_{i+1} \circ \Psi_i \circ \cdots \circ \Psi_{i_0} \) and perform an obvious interpolation between \( \Phi_i \) and \( \Phi_{i+1} \). \qed

Notice that at this point we know the metric \( \Phi^* g \) is asymptotic to \( g_C \) at infinity, but without any quantitative rate of decay. Nevertheless, the difference of the Levi-Civita connections does decay at the order \( o(r^{-1}) \). The general result of Colding-Minicozzi [6] yields a logarithmic decay rate of the Riemannian metric but we will not need it in this paper. Using the diffeomorphism constructed above, we may naturally view \( r \) as a function on the end of \( X \) which satisfies
\[
C^{-1} d_g(p, \cdot) \leq r(\cdot) \leq C d_g(p, \cdot).
\]
Given a constant \( \tilde{R} > R_0 + 1 \), we can further replace \( r \) by the regularized maximum \( \max\{r, \tilde{R}\} \) (see [13, Chap. 1, Lemma 5.18]), so it can be viewed as a smooth function on \( X \). Moreover, by Lemma 3.1, we can fix an \( \tilde{R} \) large so that \( r^2 \) is a smooth plurisubharmonic exhaustion function on \( X \) which is strictly plurisubharmonic outside a compact set. Then we know that for any sufficiently large number \( R \), the sub-level set

\[
X_R := \{ r < R \}
\]  

is 1-convex. By the Grauert-Riemenschneider vanishing theorem [16] and triviality of the canonical bundle \( K_X \), we get

\[
H^1(X_R, \mathcal{O}) = H^1(X_R, K_{X_R}) = 0.
\]  

This will be used in the proof of Theorem 3.9.

### 3.2 Weighted elliptic estimates

Here we collect some facts on weighted elliptic theory for ends of Riemannian manifolds asymptotic to cones. These are not sharp results but suffice for our applications in this paper.

For any \( k \in \mathbb{Z}_{\geq 0}, \alpha \in (0, 1), \delta \in \mathbb{R}, R > 0 \), we define the Banach space \( C^{k,\alpha}_{\delta} (\mathcal{Q}_R) \) using the following weighted norm on a function \( f \) defined on \( \mathcal{Q}_R \subset \mathcal{C} \):

\[
\| f \|_{C^{k,\alpha}_{\delta} (\mathcal{Q}_R)} = \sup_{s \geq R} \left\{ \sum_{m=0}^{k} s^{-\delta + m} \| \nabla^m g \|_{C^0} f \|_{C^m_{\delta} (A_s, 2s)} + s^{-\delta + k + \alpha} [ f ]_{C^{k,\alpha}_{\delta} (A_s, 2s)} \right\},
\]

where

\[
[f]_{C^{k,\alpha}_{\delta} (A_s, 2s)} = \sup_{y_1, y_2 \in A_s, \text{inj} \leq 1} \left\{ \frac{| \nabla^k g \|_{C^{\alpha}_{\delta}} (y_1) - \nabla^k g \|_{C^{\alpha}_{\delta}} (y_2) |}{d_{\delta} (y_1, y_2)^{\alpha}} \right\}. 
\]

Fix \( k \geq 2n + 1 \) and \( \alpha \in (0, 1) \). The following can be proved in exactly the same way as [27, Proposition 6.7], so we omit the proof here.

**Lemma 3.2** There exists a finite set \( \Gamma \subset (0, 1) \) such that for all \( \delta \in (0, 1) \setminus \Gamma \) and for all \( R \geq 1 \), one can find a bounded linear map \( S_R : C^{k,\alpha}_{\delta} (\mathcal{Q}_R) \rightarrow C^{k+2,\alpha}_{-\delta} (\mathcal{Q}_R) \) with \( \Delta_{\delta} S_R = \text{Id} \) and with the operator norm \( \| S_R \| \leq C \) for some \( C \) depending only on \( C, \delta, k, \alpha \) (but not on \( R \)).

Given any \( R_1 > 1 \) and suppose there is another Riemannian metric \( g \) on \( \mathcal{Q}_{R_1} \) such that for all \( l \geq 0 \) and \( s \geq R_1 \),

\[
s^l \sup_{\partial \mathcal{Q}_s} | \nabla^l g \|_{C^{\alpha}_{\delta}} (g - g_{\mathcal{C}}) | \leq e_l(s)
\]

for some function \( e_l \) defined on \([R_1, \infty)\) with \( \lim_{s \to \infty} e_l(s) = 0 \).
**Proposition 3.3**  Given $\delta \in (0, 1) \setminus \Gamma$, there exist $R_2 \geq R_1$ and $C > 0$ depending only on $C$, $R_1$, $k$, $\delta$, $\alpha$ and the functions $e_1, \ldots, e_k$, and a bounded linear map $T : C^{k,\alpha} (Q_{R_2}) \to C^{k,\alpha+2} (Q_{R_2})$ such that $\Delta g \circ T = \text{Id}$ and $\|T\| \leq C$.

**Proof**  One can write $\Delta g = \Delta_{g_C} + (g - g_C) \ast \nabla^2_{g_C} + \nabla_{g_C} g \ast \nabla_{g_C} g$. Then on $Q_R (R \geq R_1)$, we have

$$\|\Delta g - \Delta_{g_C}\| \leq C \sup_{s \geq R, 0 \leq l \leq k+2} e_l(s).$$

The right hand side is small when $R \gg 1$. Then the conclusion follows from standard functional analysis. $\square$

The following is an application of the standard elliptic regularity by a straightforward rescaling argument. We omit the proof.

**Proposition 3.4**  Let $g$ be a Riemannian metric on $Q_{R_1}$ satisfying (3.3). Suppose $u$ is a function defined on $Q_{R_1}$ such that for some $s, s' \in \mathbb{R}$ and for all $l \geq 0$,

$$\left| \nabla^l g (\Delta g u) \right| \leq C_l r^{-l+2+s},$$

and $\int_{Q_{R_1}} u^2 r^{-2s' - 2n} d\text{Vol}_g \leq \hat{C}$. Then for any $l > 0$ and $\gamma > \max(s, s')$, there exist constants $A_{\gamma l} > 0$ depending only on $\gamma$, $C$, $R_1$, $C_0, C_1, \ldots, C_l$, $\hat{C}$ and the functions $e_1, \ldots, e_l$, such that

$$\left| \nabla^l g u \right| \leq A_{\gamma l} r^{\gamma - l} \text{ on } Q_{R_1}.$$

### 3.3 Rough algebraic approximation of the Calabi-Yau metric

We continue to use the notations in Sect. 2. Recall that $X'$ is the affine variety defined by the ring $R(X)$ of holomorphic functions on $X$ with polynomial growth and we identify it with its image in $\mathbb{C}^N$ under the holomorphic embedding $\Phi_1$. We also have the crepant resolution map $\pi : X \to X' \subset \mathbb{C}^N$. In the following outside a compact $K$ of $X$ containing the exceptional set of $\pi$, we also identify $X$ with $X'$. In particular, $X \setminus K$ is holomorphically embedded in $\mathbb{C}^N$.

Since the coordinate functions on $\mathbb{C}^N$ have positive weight with respect to $\xi$, we can find a Kähler cone metric $\omega_\xi$ on $\mathbb{C}^N$ with Reeb vector field $\xi$. Fix a choice of such $\omega_\xi$. Then we have $\Lambda^* \omega_\xi = \lambda^2 \omega_\xi$. Denote by $r_\xi = d_{\omega_\xi} (0, \cdot)$ the radial function on $\mathbb{C}^N$ defined by $\omega_\xi$. Notice we are not assuming $\omega_\xi$ is Calabi-Yau. The following result shows that we may view the restriction of $\omega_\xi$ to $X$ as a rough approximation of the unknown Calabi-Yau metric $\omega$.

**Proposition 3.5**  There exists a compact set $K \subset X$ such that for all $\delta > 0$ and $k \geq 1$, on $X \setminus K$ we have

$$C_\delta^{-1} r^{-\delta/2} \omega \leq \omega_\xi \leq C_\delta r^{\delta/2} \omega, \quad (3.4)$$

$$\left| \nabla^k g \omega_\xi \right| \leq C_{\delta, k} r^{\delta - k}, \quad (3.5)$$

$$C_\delta^{-1} r^{1-\delta} \leq r_\xi \leq C_\delta r^{1+\delta}. \quad (3.6)$$
Proof Since we identify $X'$ with $\Phi_1(X')$ and outside a compact set identify $X$ with $X'$ using $\pi$, we have $\Phi_j = F_j \circ \cdots \circ F_2 : X \to \mathbb{C}^N$. By Item (b) in Sect. 2 we know that for all $j \geq 1$ and $k \geq 0$, the following holds on $B_{j+1} \setminus B_j$:

$$C^{-1} \lambda^{-2j} r \leq \Phi_j^* r^\lambda \leq C \lambda^{2j} r,$$

$$|\nabla^k \omega (\lambda^{-2j} \Phi_j^* \omega)_{\omega} |_{\omega} \leq C k \lambda^{kj}. \quad (3.7)$$

By construction we have $F^{-1}_j = \Lambda^{-1} \circ \sigma_j$ for a linear map $\sigma_j \in G_{\xi}$ with $\|\sigma_j - \text{Id} \| \to 0$, where $\| \cdot \|$ is a fixed norm on the space of $N \times N$ matrices. Since $\sigma_j$ commutes with $\Lambda$ we may write

$$F^{-1}_j \circ \cdots \circ F^{-1}_2 = \Lambda^{-j} \sigma_2 \circ \cdots \circ \sigma_j.$$

For any $\sigma \in G_{\xi}$ we have $L_{r_{\xi}} \partial_{r_{\xi}} \sigma^* \omega_{\xi} = 2 \sigma^* \omega_{\xi}$. It follows that $\sigma^* \omega_{\xi}$ is homogeneous of degree 2 on $\mathbb{C}^N$. So one can find a small neighborhood $U$ of $\text{Id}$ such that for all $\sigma \in U$ and all $k \geq 0$, the following holds on $\mathbb{C}^N$:

$$|\nabla^k \omega_{\xi} (\sigma^* \omega_{\xi} - \omega_{\xi}) |_{\omega_{\xi}} \leq C k r_{\xi}^{-k} \| \sigma - \text{Id} \|. \quad (3.8)$$

In particular we can also assume

$$|\nabla^k \omega_{\xi} (\nabla \sigma^* \omega_{\xi} - \nabla \omega_{\xi}) |_{\omega_{\xi}} \leq C k r_{\xi}^{-k-1} \| \sigma - \text{Id} \|, \quad (3.9)$$

where as usual we view the difference of the two connections as a tensor.

Since $\|\sigma_j - \text{Id} \| \to 0$, for $j \gg 1$ we have $\sigma_j \in U$. We notice the elementary fact that for all $\delta > 0$,

$$\prod_{l=2}^j \left( 1 + C \| \sigma_l - \text{Id} \| \right) \leq C \delta \lambda^{-j \delta}.$$

From this, iterating (3.8) and using (3.7) yield (3.4). (3.6) follows from a similar argument.

We now prove the case $k = 1$ in (3.5). The case $k > 1$ can be proved similarly by induction. Notice for any $\sigma \in U$ and 2-form $\gamma$, we have

$$\nabla \omega_{\xi} \sigma^* \gamma = \nabla \sigma^* \omega_{\xi} \sigma^* \gamma + (\nabla \omega_{\xi} - \nabla \sigma^* \omega_{\xi}) \sigma^* \gamma$$

$$= \sigma^* (\nabla \omega_{\xi} \gamma) + (\nabla \omega_{\xi} - \nabla \sigma^* \omega_{\xi}) \sigma^* \gamma.$$

Therefore,

$$|\nabla \omega_{\xi} (\sigma^*_j \cdots \sigma^*_2 \omega_{\xi}) |_{\omega_{\xi}} = |\sum_{l=2}^j \sigma_{l+1}^* \cdots \sigma_2^* (\nabla \omega_{\xi} - \nabla \sigma_{l-1}^* \omega_{\xi}) \sigma_{l-1}^* \cdots \sigma_2^* \omega_{\xi} |_{\omega_{\xi}}.$$

Using (3.8) and (3.9) we obtain that on the annulus $\{ \lambda \leq r_{\xi} \leq \lambda^{-1} \} \subset \mathbb{C}^N$ for all $\delta > 0$,

$$|\nabla \omega_{\xi} (F^{-1}_j \circ \cdots \circ F^{-1}_2 \omega_{\xi}) |_{\omega_{\xi}} \leq C \lambda^{-2j} \prod_{l=2}^j (1 + C \| \sigma_l - \text{Id} \|)$$
\[ \leq C_\delta \lambda^{-(2+\delta)} j. \]  
(3.10)

By definition \( \omega_\xi = \Phi_j^* \left( (F_2^{-1} \circ \cdots \circ F_j^{-1})^* \omega_\xi \right) \), so we have

\[
|\nabla_\omega \omega_\xi|_\omega = |\Phi_j^*(\nabla_{\omega_\xi}(F_2^{-1} \circ \cdots \circ F_j^{-1})^* \omega_\xi)|_\omega \\
+ |(\nabla_\omega - \nabla_{\Phi_j^* \omega_\xi}) \Phi_j^*(F_2^{-1} \circ \cdots \circ F_j^{-1})^* \omega_\xi|_\omega.
\]

The first term of the right hand side can be handled using (3.10) and (3.7), and the second term can be dealt with using (3.7) and (3.4). Together we get (3.5) when \( k = 1 \).

Although not needed in this paper, we remark that the above \( \omega_\xi \) is not unique and we can find one with explicit algebraic formula. We emphasize that from the proof we can not get rid of the \( \delta \) in the estimates. In particular, we can not claim the uniform equivalence between \( \omega \) and \( \omega_\xi \). Nonetheless we do know the error is smaller than any polynomial order which suffices for our applications. Moreover, the differences between the covariant derivatives do decay at infinity. Namely,

**Corollary 3.6** There exists a compact set \( K \subset X \) such that on \( X \setminus K \), for all \( \delta > 0 \) and \( k \geq 0 \) we have

\[ |\nabla^k_\omega (\nabla_\omega - \nabla_{\omega_\xi})|_\omega \leq C_\delta k \delta^{-k-1}. \]  
(3.11)

**Proof** Two Riemannian metrics \( g_1 \) and \( g_2 \) on a manifold determine a section \( T \) of \( \text{End}(TM) \) via

\[ g_2(\cdot, \cdot) = g_1(T \cdot, \cdot), \]

which is positive and self-adjoint with respect to both \( g_1 \) and \( g_2 \). Using Koszul formula, one can check directly that

\[ \nabla_{g_2} - \nabla_{g_1} = T^{-1} * \nabla_{g_1} T \]

where we view both sides as (1,2)-tensors and \( * \) denotes some algebraic operations. Therefore to estimate \( |\nabla^k_{g_1} (\nabla_{g_2} - \nabla_{g_1})|_{g_1} \) it suffices to estimate \( T^{-1} \) and \( \nabla^k_{g_1} T \). Then (3.11) follows from Proposition 3.5.

**3.4 Plurisubharmonic weight functions**

We construct here some plurisubharmonic functions that will be used later. As mentioned in Sect. 2 we always identify \( (X', p') \) with \( (\Phi_1(X'), 0) \) in \( \mathbb{C}^N \). Then we can naturally restrict functions and forms defined on \( \mathbb{C}^N \) to \( X' \) and then pull-back to \( X \) via the crepant resolution map \( \pi \). When there is no confusion we will omit the various pull-back notations.

\[ \copyright \ Springer \]
Lemma 3.7 (1) Let \((K, \omega_K)\) be a Kähler cone and let \(\rho\) denote the distance function from the vertex of the cone. Then the function \(\varphi = \log(\rho^2 + 1) - \frac{1}{100}(\log(\rho^2 + 3))^{\frac{3}{2}}\) is a strictly plurisubharmonic function and there exists a compact set \(K \subset K\) such that on \(K \setminus K\),

\[
\sqrt{-1} \partial \bar{\partial} \varphi \geq C \rho^{-2} (\log \rho)^{-\frac{3}{2}} \omega_K.
\]

(2) The function on \(X\) defined by \(\varphi = \log(r^2 \xi + 1) - \frac{1}{100}(\log(r^2 \xi + 3))^{\frac{3}{2}}\) satisfies \(\sqrt{-1} \partial \bar{\partial} \varphi > 0\) away from the exceptional set of \(\pi\). Moreover outside a compact set \(K\) it satisfies that for all \(\epsilon > 0\),

\[
(1 - \epsilon) \log(r^2 + 1) - C \epsilon \leq \varphi \leq (1 + \epsilon) \log(r^2 + 1) + C \epsilon,
\]

\[
\sqrt{-1} \partial \bar{\partial} \varphi \geq C \epsilon^{-1} r^{-2-\epsilon} \omega,
\]

\[
|\nabla_\omega \varphi|_\omega \leq C \epsilon r^{-1+\epsilon}.
\]

Proof (1) follows from a straightforward computation: the first term is strictly plurisubharmonic but is decaying at the rate \(\rho^{-4}\) along the radial direction; the second term is used to strengthen the positivity along the radial direction. Applying (1) to the Kähler cone \((\mathbb{C}^N, \omega_\xi)\) and then using Proposition 3.5, we obtain (2).

3.5 Decomposition of the Kähler form

We will use the following version of Hörmander’s \(L^2\) estimate, which follows from for example [13, Chapter VIII, Theorem 6.1].

Theorem 3.8 Let \(M\) be an \(n\)-dimensional complex manifold admitting a complete Kähler metric and \(\omega_M\) be a Kähler metric on \(M\) which is not necessarily complete. Suppose \(\varphi_M\) is a smooth function with \(\sqrt{-1} \partial \bar{\partial} \varphi_M \geq \Upsilon \omega_M\) for a continuous non-negative function \(\Upsilon\). Let \(q\) be a positive integer. Then we have

(1) for any \((n, q)\) form \(\eta\) on \(M\) with \(\bar{\partial} \eta = 0\) and \(\int_M \Upsilon^{-1} |\eta|^2_{\omega_M} e^{-\varphi_M} \omega_M^n < \infty\), there exists an \((n, q - 1)\) form \(\xi\) satisfying \(\bar{\partial} \xi = \eta\) and with estimate

\[
\int_M |\xi|^2_{\omega_M} e^{-\varphi_M} \omega_M^n \leq \int_M q^{-1} \Upsilon^{-1} |\eta|^2_{\omega_M} e^{-\varphi_M} \omega_M^n;
\]

(2) if the Ricci curvature of \(\omega_M\) is non-negative, then the conclusion in (1) holds for \((0, q)\) forms.

The main result of this subsection is

Theorem 3.9 For every \(\epsilon > 0\), there exist a Kähler form \(\beta\) and a real-valued smooth function \(\psi\) on \(X\) such that

(1) \(\omega = \beta + \sqrt{-1} \partial \bar{\partial} \psi\),

(2) \(|\nabla_\omega^k \beta|_\omega = O(r^{-2+\epsilon-k})\) for all \(k \geq 0\),
\( |\nabla_k^\omega \psi|_\omega = O(r^{2+\varepsilon-k}) \) for all \( k \geq 0 \).

**Proof** Let \( \varphi \) be the plurisubharmonic function on \( X \) obtained in Lemma 3.7-(2). Fix \( \varepsilon > 0 \).

**Step 1.** Using the diffeomorphism \( \Phi \) constructed in Lemma 3.1, for \( R \) large we can identify \( X \setminus X_R \) with \( Q_R \) and we may assume \( X_R \) is 1-convex and contains the exceptional set of \( \pi \). Then one can write \( \omega = \tilde{\beta} + d\tilde{\eta} \) with \( |\nabla_k^\omega \tilde{\beta}|_\omega = O(r^{-2-k}) \) for all \( k \geq 0 \) and \( \tilde{\eta} = 0 \) on \( X_R \).

**Step 2.** We decompose \( \tilde{\beta} \) according to types: 
\[ \tilde{\beta} = \tilde{\beta}^{2,0} + \tilde{\beta}^{1,1} + \tilde{\beta}^{0,2} \] 
Since \( d\tilde{\beta} = 0 \), we have \( \partial \tilde{\beta}^{0,2} = 0 \). Notice that \( \tilde{\beta}^{0,2} \) is of type \( (0,2) \), \( \tilde{\beta}^{2,0} \) is of type \( (2,0) \), \( \tilde{\beta}^{1,1} \) is of type \( (1,1) \). The decay of \( \tilde{\beta}^{0,2} \) implies \( |d\eta_2|_\omega < C \) on \( X \), so we have \( |\lambda| = O(r^{1+\delta}) \) for all \( \delta > 0 \). It follows that \( d\eta_2 = d\gamma \), where \( \gamma = \chi \eta_2 + \lambda \), with \( |\gamma| = O(r^{1+\delta}) \) for all \( \delta > 0 \).

**Step 3.** Let \( \chi \) be a smooth compactly supported function on \( X \) which equals 1 in a neighborhood of \( X_R \). Then \( \eta_2 = \chi \eta_2 + \eta_2' \), where \( \eta_2' \) vanishes in a neighborhood of \( X_R \). Writing \( d\eta_2 = dr \wedge \omega_1 + \omega_2 \), where \( \partial_r \omega_1 = \partial_r \omega_2 = 0 \). We define \( \lambda = \int_0^r \omega_1 dr \). Then \( \lambda \) vanishes in a neighborhood of \( X_R \) so extends to a 1-form on \( X \). Moreover, one can check that \( d\eta_2' = d\lambda \). The decay of \( \beta_2 \) implies \( |d\eta_2|_\omega < C \) on \( X \), so we have \( |\lambda| = O(r^{1+\delta}) \) for all \( \delta > 0 \). It follows that \( d\eta_2 = d\gamma \), where \( \gamma = \chi \eta_2 + \lambda \), with \( |\gamma| = O(r^{1+\delta}) \) for all \( \delta > 0 \).

**Step 4.** Since \( d\gamma = \omega - \beta_2 \) is of type \((1,1)\), we have \( \bar{\partial} \gamma^{0,1} = 0 \). Since \( H^1(X_R, \mathcal{O}) = 0 \) by (3.2), there exists a smooth function \( f \) on \( X_R \) such that \( \bar{\partial} f = \gamma^{0,1} \) on \( X_R \). Choose a cut off function \( \chi_0 \) which equals 1 in a neighborhood of the exceptional set of \( \pi \) and equals 0 outside \( X_{R-\varepsilon'} \) for some small positive number \( \varepsilon' \). We now apply Theorem 3.8 (with \( \varphi_M = (n + 2 + \frac{\varepsilon}{2}) \varphi \)) to obtain a function \( u \) satisfying \( \bar{\partial} u = \gamma^{0,1} - \bar{\partial}(\chi_0 f) \) and
\[ \int_X |u|^{2} e^{-(n+2+\frac{\varepsilon}{2})}\omega^n < \infty. \]
Then \( d\gamma = \sqrt{-1} \bar{\partial} \bar{\partial} \psi_1 \), where \( \psi_1 = \sqrt{-1}(u + \chi_0 f - \bar{u} - \bar{\chi}_0 \bar{f}) \). Again by Proposition 3.4 we see \( |\nabla_k^\omega \psi_1|_\omega = O(r^{2+\varepsilon-k}) \) for all \( k \geq 0 \).
Step 5. We have achieved that $\omega = \beta_2 + \sqrt{-1} \partial \bar{\partial} \psi_1$ with estimates on both $\beta_2$ and $\psi_1$. It remains to modify $\beta_2$ to be a Kähler form. By the asymptotics for the complex structure and the Riemannian metric obtained in Lemma 3.1, we know that there exists a large constant $R > R$ such that on the region $\{ r > R \}$ we have $\sqrt{-1} \partial \bar{\partial} r^\epsilon \geq c r^{-2+\epsilon} \omega$ for some positive number $c > 0$. Choose a smooth non-decreasing convex function $u : \mathbb{R} \to \mathbb{R}$ such that $u(t) = 1$ for $t \leq 2 R^\epsilon$ and $u(t) = t$ for $t \gg 1$ and let $h = u \circ r^\epsilon$. Then $h$ can be naturally viewed as a smooth plurisubharmonic function on $X$ and outside a compact set $K'$ we have

$\sqrt{-1} \partial \bar{\partial} h \geq c r^{-2+\epsilon} \omega$

for some positive number $c > 0$. Choose a smooth compactly supported function $\chi$ on $X$ which equals 1 on a neighborhood of $K'$. Then we define

$\beta = \beta_2 + \sqrt{-1} \partial \bar{\partial} (\chi \psi_1 + Ah),$

$\psi = (1 - \chi) \psi_1 - Ah,$

where $A$ is a constant. One can choose $A$ large so that $\beta$ is Kähler. Then $\omega = \beta + \sqrt{-1} \partial \bar{\partial} \psi$. It follows from the above construction that we have the desired estimates. $\square$

3.6 Construction of background Kähler forms

Recall some notations in Sect. 2: $X' \subset \mathbb{C}^N$ is the affine variety defined by the ring $R(X)$ of holomorphic functions on $X$ with polynomial growth, $W$ is the weighted asymptotic cone of $X'$ with respect to $\Lambda$ and $Y_j = \Lambda^{-1}.X'$. So in particular away from 0 we know that $Y_j$ converges to $W$ as smooth submanifolds in $\mathbb{C}^N$. For notational convenience we also denote $Y_\infty \equiv W$. We denote by $\text{Sing}(Y_j)$ the singular set of $Y_j$, which consists of finitely many points, and denote $Y_j^\circ = Y_j \setminus \text{Sing}(Y_j)$. We also define $\mathcal{S} \equiv \cup_{1 \leq j \leq \infty} \text{Sing}(Y_j)$.

Proposition 3.10 There exist a constant $\delta_0 > 0$, a compact set $K \subset \mathbb{C}^N$ which contains $\mathcal{S}$, and for each $j \in \mathbb{Z}_{\geq 1} \cup \{ \infty \}$ a diffeomorphism $p_j : X' \setminus K \to Y_j \setminus K$ such that for all $k \geq 0$

$|\nabla^k_{\omega_\xi} (p_j - \text{Id})|_{\omega_\xi} \leq C_k r_\xi^{-\delta_0 - k},$

where we view both $p_j$ and $\text{Id}$ as maps into $\mathbb{C}^N$, and

$|\nabla^k_{\omega_\xi} \left( J_{Y_j} (p_j) - J_{X'} \right) |_{\omega_\xi} \leq C_k r_\xi^{-\delta_0 - k}.$

Remark 3.11 Here the definite rate $\delta_0$ arises from the algebraicity of $W$. It is of crucial importance in our argument as it beats all the (arbitrarily small polynomial order) error when we compare $X$ to the asymptotic cone $C$ using the 2-step degeneration theory (Proposition 3.5).
Proof By definition we can find finitely many holomorphic polynomials \( F_1, \ldots, F_m \) on \( \mathbb{C}^N \) which generate the ideal of \( X' \), such that \( G_1, \ldots, G_m \) generate the ideal of \( W \). Here \( G_\alpha \) is the leading term of \( F_\alpha \) with respect to the \( \Lambda \) action. Then we know that for all \( l \in \mathbb{N} \) we have

\[
F_\alpha (\Lambda^{-l} z) = \lambda^{-l c_\alpha} (G_\alpha (z) + \lambda^{l k_\alpha} E_{\alpha l} (z))
\]

for some constants \( \delta_\alpha, c_\alpha > 0 \) and some polynomials \( E_{\alpha l} \) whose coefficients are uniformly bounded independent of \( l \). Denote \( F_{\alpha, l} = (\Lambda^{-l})^* F_\alpha \), then \( F_{\alpha, 1}, \ldots, F_{\alpha, l} \) generate the ideal of \( Y_l \). Let \( \delta_0 = \min_\alpha \delta_\alpha \).

We define \( p_\infty \) to be the normal projection map from \( X' \) to \( W \), i.e., for \( x \in X' \), we let \( p_\infty (x) \) to the unique point in \( W \) that is closest to \( X' \) with respect to the metric \( \omega_\xi \). We need to show this is well-defined outside a compact set in \( X \) and satisfies the desired properties. For \( l \geq 1 \) we denote by \( A_l \) the annulus in \( \mathbb{C}^N \) defined by \( \lambda^{-l+1} \leq r_\xi \leq \lambda^{-l-1} \). Then \( \Lambda \) maps \( X' \cap A_l \) to \( Y_l \cap A_1 \). By the conical nature of \( \omega_\xi \), it suffices to consider the normal projection from \( Y_l \cap A_1 \) to \( W \) for \( l \geq 1 \).

Claim. There is a covering of \( W \cap A_1 \) by finitely many open sets of the form \( U_\gamma \cap W \) where \( U_\gamma \subset \mathbb{C}^N \) such that for \( l \) large there are holomorphic functions \( h_{\gamma, l} : U_\gamma \cap W \to \mathbb{C}^N \) with \( |\nabla^k \omega_\xi (h_{\gamma, l} - \text{Id})| \leq C_k \lambda^{l k_0} \) for all \( k \geq 0 \) and \( Y_l \cap A_1 \subset U_\gamma \cap \text{Im}(h_{\gamma, l}) \).

To prove the Claim, we fix a point \( w \in W \cap A_1 \). Then we can find a neighborhood \( U \subset \mathbb{C}^N \), such that \( W \cap U \) is given by the zero set of \( N - n \) number of \( G_\alpha \)'s (in the scheme-theoretic sense). For simplicity of notation we may assume these are \( G_1, \ldots, G_{N-n} \). Shrinking \( U \) if necessary we may find local holomorphic coordinates \( \{ \xi_1, \ldots, \xi_N \} \) such that \( \xi_\alpha = G_\alpha \) for \( \alpha = 1, \ldots, N-n \). Now we have \( F_{\alpha, l} = \lambda^{-l c_\alpha} (G_\alpha (z) + \lambda^{l k_\alpha} E_{\alpha l} (z)) \). It follows that for \( l \) large the common zero set of \( F_{\alpha, l} (\alpha = 1, \ldots, N-n) \) is a smooth complex submanifold in \( U \), so in particular it agrees with \( Y_l \cap U \). Using the local coordinates \( \{ \xi_1, \ldots, \xi_N \} \) it is easy to see that \( Y_l \cap U \) is contained in the image of a holomorphic function \( h_l : U \to \mathbb{C}^N \) such that \( |\nabla^k \omega_\xi (h_l - \text{Id})|_{\omega_\xi} \leq C_k \lambda^{l k_0} \) for all \( k \geq 0 \). Since \( A_1 \) is compact the Claim follows.

Since the normal injectivity radius of \( W \cap A_1 \) is uniformly bounded, there is a tubular neighborhood \( \mathcal{N} \) of \( W \cap A_1 \) such that the normal projection map \( \Pi : \mathcal{N} \to W \) is smooth. It follows from the Claim that for \( l \) large, \( Y_l \cap A_1 \subset \mathcal{N} \). So \( p_\infty \) is well-defined and smooth outside a compact set in \( X' \). It is straightforward to check that it satisfies the desired derivative bounds. The estimates on the complex structures follow from the fact that \( h_{\gamma, l} \) is holomorphic.

Now for each \( j < \infty \) we can perform the above arguments uniformly (noticing that the closeness between \( Y_j \) and \( W \) gets improved as \( j \) gets larger) to construct the corresponding projection maps \( q_j : Y_j \setminus K \to W \) with uniform estimates. Then we let \( p_\infty = q_j^{-1} \circ p_\infty \).

Now we fix \( \epsilon_0 \ll \delta_0 \) and apply Theorem 3.9 with \( \epsilon = \epsilon_0 \) to get a decomposition on \( X \)

\[
\omega = \beta + \sqrt{-1} \partial \bar{\partial} \psi.
\]
Let $\psi'_j = \psi \circ p_j^{-1}$ and $r_j = r \circ p_j^{-1}$. Denote $\gamma_j \equiv \sqrt{-1} \partial \bar{\partial} \psi'_j$. By Lemma 3.1, Proposition 3.5, Corollary 3.6 and Proposition 3.10, we have

**Corollary 3.12** There exists a compact set $K \subset \mathbb{C}^N$ such that for all $j \in \mathbb{Z}_{\geq 1} \cup \{\infty\}$, $\gamma_j$ is a Kähler form on $Y_j \setminus K$. Moreover,

1. for all $k \geq 0$ and $\delta > 0$,

   $$C^{-1}_\delta r_j^{1-\delta} \leq r_\xi \leq C_\delta r_j^{1+\delta} \quad \text{and} \quad |\nabla_{\gamma_j}^k (\gamma_j - \omega_\xi)|_{\gamma_j} \leq C_{\delta,k} r_j^{-\delta-k},$$

2. for all $k \geq 0$ and $\delta < \delta_0 - \epsilon_0$,

   $$|\nabla_{\omega_\xi}^k (p_j^* \gamma_j - \omega)|_{\omega} \leq C_{\delta,k} r_j^{-\delta-k}. \quad (3.13)$$

Next we extend $\gamma_j$ to a global Kähler form on $Y_j^\circ$ in a uniformly controlled way. In the following, we denote by $\omega_0 = \sqrt{-1} \partial \bar{\partial} \psi_0$ the Kähler form corresponding to the standard Euclidean metric on $\mathbb{C}^N$, where $r_0$ is the radius function for the Euclidean metric.

**Lemma 3.13** There exist compact sets $K_0 \subset K_1 \subset \mathbb{C}^N$ containing a neighborhood of $\mathcal{S}$, a constant $A_0$ and Kähler forms $\omega_j = \sqrt{-1} \partial \bar{\partial} \psi_j$ on $Y_j^\circ$, such that for all $j \in \mathbb{Z}_{\geq 1} \cup \{\infty\}$,

1. $\omega_j = A_0 \omega_0$ on $K_0 \cap Y_j^\circ$,
2. $\psi_j = \psi'_j$ on $Y_j \setminus K_1$.

**Proof** By Proposition 3.5 and Proposition 3.10, we know that there exist constants $c, T_1 > 0$ such that $\sqrt{-1} \partial \bar{\partial} \psi'_j \geq cr_\xi^{-\frac{1}{2}} \omega_\xi$ on $\{r_\xi \geq T_1/2\} \cap Y_j$. Using a cut-off function on $\mathbb{C}^N$ we may extend $\psi'_j$ smoothly to $Y_j^\circ$, so that it vanishes when $r_\xi \leq T_1/2$, equals $\psi \circ p_j^{-1}$ when $r_\xi \geq T_1$ and satisfies $\sqrt{-1} \partial \bar{\partial} \psi'_j \geq -A^2/2 \omega_0$ on $\{r_\xi \leq T_1\} \cap Y_j$ for some constant $A > 0$ independent of $j$.

Choose a smooth increasing and convex function $u : \mathbb{R} \to \mathbb{R}$ such that $u(t) = T_1^2/3$ for $t \leq T_1^2/2$ and $u(t) = t$ for $t \geq T_1^2$. Define the function $r_1^2 = u \circ r_\xi^2$ on $\mathbb{C}^N$. Then $\sqrt{-1} \partial \bar{\partial} r_1^2$ is globally non-negative and equals $\omega_\xi$ on $\{r_\xi \geq T_1\}$. Choose a cutoff function $\rho : \mathbb{R} \to \mathbb{R}$ which equals 1 when $t \leq 2T_1^2$ and equals 0 when $t \geq 4T_1^2$. Then it is straightforward to check that there exists a small positive constant $\epsilon$ such that $\phi = \epsilon (\rho \circ r_1^2 \cdot r_0^2 + r_1^2)$ satisfies that $\sqrt{-1} \partial \bar{\partial} \phi$ is a Kähler form on $\mathbb{C}^N$. By construction there exists a constant $T > 1$ such that $\{r_\xi \leq T_1\} \subseteq \{\phi \leq T\}$.

It is easy to find a smooth function $\chi : [0, \infty) \to \mathbb{R}$ such that the following hold

- $\chi(t) = A_0 t$ if $t \leq T$, where $A_0$ is a constant bigger than $A \epsilon^{-1}$;
- $\chi(t) = 0$ for $t \gg 1$;
- for all $t > 0$ we have $\chi'' \geq -\zeta_8 t^{-\frac{3}{2}}$ and $\chi' \geq -\zeta_8 t^{-\frac{1}{2}}$.

Then $\psi_j = \chi \circ \phi + \psi'_j$ satisfies the desired properties. $\square$
Recall that we have the crepant resolution map $\pi : X \to X'$, thus for $j \in \mathbb{Z}_{\geq 1}$ the composition map $\Lambda^j \circ \pi$ is a crepant resolution of $Y_j$. We denote the latter by $\pi_j : \hat{Y}_j = X \to Y_j$. For all $j \in \mathbb{Z}_{\geq 1}$ and $\epsilon \in (0, \lambda^j)$ (we emphasize here that the domain of $\epsilon$ depends on $j$), we have the following family of Kähler forms on $\hat{Y}_j$

$$\tilde{\omega}_{j,\epsilon} \equiv \epsilon \beta + \pi_j^* \omega_j = \epsilon \beta + \sqrt{-1} \partial \bar{\partial} \pi_j^* \psi_j,$$

where $\beta$ is given in (3.12). Notice that $\beta$ and $\tilde{\omega}_{j,\epsilon}$ can also be naturally viewed as forms on $Y_j$ via $(\pi_j^{-1})^*$. The following lemma (together with Corollary 3.12-(2)) shows that they are uniformly asymptotic to the cone $C$.

**Lemma 3.14** There exists a compact set $K \subset \mathbb{C}^N$ such that for all $j \in \mathbb{Z}_{\geq 1}$ and $k \geq 0$ on $Y_j \setminus K$ we have

$$\lambda^{4j} \left| \frac{1}{\omega_j} \nabla^k (\pi_j^{-1})^* \beta \right|_\omega_j \leq C_k r_j^{- \frac{1}{2} - k}.$$

**Proof** Outside a compact set, we identify $X$ with $X'$ and therefore we may just view $\pi_j$ as $\Lambda^j$. Let us first prove the case $k = 0$. By Corollary 3.12 and the fact $\Lambda^* \omega_\xi = \lambda^j \omega_\xi$, we know that for any $\epsilon_1, \epsilon_2 > 0$, there exists constants $C_{\epsilon_1}$ and $C_{\epsilon_2}$ such that

$$\left| (\Lambda^{-j})^* \beta \right|_\omega_j \leq C_{\epsilon_1} \left| (\Lambda^{-j})^* \beta \right|_\omega_\xi \leq C_{\epsilon_1} (\Lambda^{-j})^* (|\beta|_{(\Lambda^j)^* \omega_\xi}) r_\xi^\epsilon_1 \leq C_{\epsilon_1} (\Lambda^{-j})^* (|\beta|_{\omega} r_\xi^\epsilon_2) r_\xi^\epsilon_1 \leq C_{\epsilon_1} C_{\epsilon_2} \lambda^{-2j} (\Lambda^{-j})^* (r^{-1} r_\xi^{\epsilon_1 + \epsilon_2}), \quad (3.14)$$

where for the last inequality we used the estimate of $\beta$ with respect to the metric $\omega$ obtained in Theorem 3.9. By choosing $\epsilon_1$ and $\epsilon_2$ small enough, one can directly show that there exists a positive constant $C_0$ such that

$$\lambda^{4j} \left| (\Lambda^{-j})^* \beta \right|_\omega_j \leq C_0 r_j^{- \frac{1}{2}}.$$ This proves the case $k = 0$. Note that suppose we have two Riemannian metrics $g_1$ and $g_2$ then for any tensor $\eta$ and another fixed background metric $g_0$, we have

$$\left| \nabla^{k_1} \eta \right|_{g_0} \leq C_k \sum_{k_1 + k_2 = k \atop k_i \geq 0} \left| \nabla_{g_1} - \nabla_{g_2} \right|_{g_0} \left| \nabla_{g_2}^{k_1} \eta \right|_{g_0}$$

$$+ C_k \sum_{k_1 + k_2 + k_3 = k - 1 \atop k_i \geq 0, k_i \geq 1} \left| \nabla_{g_2}^{k_1} (\nabla_{g_1} - \nabla_{g_2}) \right|_{g_0} \left| \nabla_{g_1} - \nabla_{g_2} \right|_{g_0} \left| \nabla_{g_2}^{k_2} \eta \right|_{g_0}, \quad (3.15)$$

where we view $\nabla_{g_1} - \nabla_{g_2}$ as a tensor and make a convention that $\nabla_{g_2}^0 \eta = \eta$. Then the proof for $k \geq 1$ follows from the same argument as we did in (3.14) by using (3.15) together with Corollary 3.6 and Theorem 3.9. \qed

\@ Springer
Recall that $\Omega$ is the given holomorphic volume form on $X$. Since $\Omega$ is holomorphic and satisfies the equation $\omega^n = (\sqrt{-1})^n \Omega \wedge \overline{\Omega}$, we obtain $\nabla_\omega \Omega = 0$. We can view $\Omega$ naturally as a holomorphic volume form on $X' = Y_1$ (notice the canonical bundle of each $Y_j$ is trivial).

**Lemma 3.15** For any $j \in \mathbb{Z}_{\geq 1}$, there exists a nowhere vanishing holomorphic volume form $\Omega_j$ on $Y_j$ satisfying:

1. There exist a constant $\delta_1 > 0$ and a compact set $K \subset \mathbb{C}^N$ such that on $Y_j \setminus K$, for all $k \geq 0$ the function $h_j = \log \left( \frac{\Omega_j \wedge \overline{\Omega}_j}{(p_j^{-1} \ast (\Omega \wedge \overline{\Omega}))} \right)$ satisfies
   $$|\nabla^k h_j| \leq C k r_j^{-\delta_1 - k}.$$

2. For every compact set $K_1 \subset \mathbb{C}^N$ we have
   $$\sup_{j \in \mathbb{Z}_{\geq 1}} \int_{K_1 \cap Y_j} (\sqrt{-1})^n \Omega_j \wedge \overline{\Omega}_j < \infty.$$

**Remark 3.16** It follows from the proof that for $j \in \mathbb{Z}_{\geq 1}$, $\Omega_j$ coincides with $(\Lambda^{-j})^* \Omega$ up to a nonzero multiplicative constant depending on $j$. For our purpose we need the precise asymptotics of $\Omega_j$ and the stated properties of $h_j$ essentially determine this constant. Moreover from the proof, we also obtain
   $$\left| \nabla^k_{\omega_j} (p_j^{-1} \ast \Omega - \Omega_j) \right|_{\omega_j} \leq C k r_j^{-\delta_1 - k} \text{ on } Y_j \setminus K, \text{ for all } k \geq 0. \quad (3.16)$$

Note that (1) and (2) in the proposition are invariant if we multiply $\Omega_j$ by a complex number with norm 1 and the estimate $(3.16)$ can be used to eliminate this freedom.

**Proof** Let $\varphi = \log(r_{\xi}^2 + 1) - \frac{1}{100} (\log(r_{\xi}^2 + 3))^2$ and denote $\varphi_j = \varphi|_{Y_j}$. Then by Lemma 3.7-(1) and Corollary 3.12, we know that outside a compact set $K$ for every $\epsilon > 0$, there is a positive constant $c_\epsilon$ such that $\sqrt{-1} \partial \overline{\partial} \varphi_j \geq c_\epsilon r_j^{-2-\epsilon} \omega_j$. Let $p_j$ be given in Proposition 3.10. For all $\delta \in (0, \delta_0 - \epsilon_0)$ we have
   $$\left| \left( p_j^{-1} \ast \Omega \right)^{n,0} \right|_{\omega_j} - 1 = O(r_j^{-\delta}), \quad (3.17)$$

and for all $k \geq 1$, $\left| \nabla^k_{\omega_j} \left( p_j^{-1} \ast \Omega \right)^{n,0} \right|_{\omega_j} \leq C_{\delta,k} r_j^{-\delta - k} \text{ since } \nabla_\omega \Omega = 0.$

Fix $T$ large so that $K \subset \{ r_{\xi} \leq T \}$, and fix a cut off function $\chi$ on $\mathbb{C}^N$ which equals 0 when $r_{\xi} \leq T$, equals 1 when $r_{\xi} > 2T$. Recall that we use $Y_\circ = Y_j \setminus \text{Sing}(Y_j)$ to denote the smooth locus of $Y_j$. Since $Y_j$ is affine and $\text{Sing}(Y_j)$ consists of finitely many points, we know $Y_\circ$ admits a complete Kähler metric. Then for any $\tau > 0$ small, we can apply Theorem 3.8 on $M = Y_\circ$ with $\omega_M = \omega_j$ and $\varphi_M = (n-1 + \tau) \varphi_j$, to
obtain an \((n, 0)\) form \(v_j\) satisfying \(\bar{\partial} v_j = \bar{\partial} \left( \chi \cdot ((p_j^{-1})^* \Omega)^{n,0} \right)\), such that

\[
\int_{Y_j^e} e^{-(n-1+\tau)\varphi_j} (\sqrt{-1})^{n} v_j \wedge \bar{v}_j \leq C.
\]

By Proposition 3.4, when \(r_j \geq 2T\), we have \(|\nabla_{\omega_j} v_j| \leq C_k r_j^{-\delta_1-k}\) for some constant \(\delta_1 > 0\) and all \(k \geq 0\).

Now let \(\Omega_j = \chi((p_j^{-1})^* \Omega)^{n,0} - v_j\). It is holomorphic on \(Y_j^\circ\) and is non-zero by (3.17) and the decay of \(v_j\). Notice that the function \(\frac{(\Lambda_j)^s \Omega_j}{\Omega}\) is a holomorphic function on \(X\) which is \(O(r^s)\) for all \(\epsilon > 0\), so must be a nonzero constant. In particular, \(\Omega_j\) is a nowhere vanishing holomorphic volume form on \(Y_j\). The desired estimates follow from straightforward computation.

In the following proposition, we omit the notation \((\pi_j^{-1})^*\) and view \(\tilde{\omega}_{j,\epsilon}\) as a Kähler form on \(Y_j^\circ \subset \mathbb{C}^N\). We also view \(r_j\) (which is a priori only defined on the end of \(Y_j\) using \(r \circ p_j^{-1}\)) as a positive smooth function on \(Y_j\) which is a constant on \(K_0 \cap Y_j\) where \(K_0 \subset \mathbb{C}^N\) is a fixed compact set independent of \(j\), containing \(\mathcal{G} = \bigcup_{1 \leq j \leq \infty} \text{Sing}(Y_j)\).

**Proposition 3.17** There exist a \(\delta_2 > 0\), a compact set \(K \subset \mathbb{C}^N\) containing a neighborhood of \(\mathcal{G}\) and for all \(j \in \mathbb{Z}_{\geq 1}\) and \(\epsilon \in (0, \lambda^{4j})\) a smooth function \(u_j,\epsilon\) on \(Y_j^\circ\) satisfying

1. for all \(k \geq 0\), \(|\nabla_{\tilde{\omega}_{j,\epsilon}} u_j,\epsilon|_{\tilde{\omega}_{j,\epsilon}} \leq C_k r_j^{-\delta_2+2-k}\) on \(Y_j \setminus K\);
2. let \(e^{-f_j,\epsilon} = \left( \frac{\tilde{\omega}_{j,\epsilon} + \sqrt{-1} \partial \bar{\partial} u_{j,\epsilon}}{(1)^\epsilon \Omega_j \wedge \bar{\Omega}_j} \right)^n\), then on \(Y_j \setminus K\) for all \(k \geq 0\), \(|\nabla_{\tilde{\omega}_{j,\epsilon}} f_j,\epsilon|_{\tilde{\omega}_{j,\epsilon}} \leq C_k r_j^{-\delta_2-2-k}\);
3. \(u_j,\epsilon\) is supported on \(Y_j \setminus K\);
4. \(\omega_j + \sqrt{-1} \partial \bar{\partial} u_{j,\epsilon} \geq \frac{1}{2} \omega_j\) and for all \(k \geq 0\), \(|\nabla_{\omega_j} u_{j,\epsilon}|_{\omega_j} \leq C_k (r_j + 1)^{-\delta_2+2-k}\) on \(Y_j^\circ\).

**Proof** We first identify the end of \(X\) differentiably with \(Q_{R_0}\) via Lemma 3.1. Then using the maps \(p_j\) constructed in Proposition 3.10 we may identify the end of all \(Y_j\) with \(Q_{R_1}\) for some \(R_1 > R_0\). In particular we may view the metrics \(\tilde{\omega}_{j,\epsilon}\) (associated to \(\tilde{\omega}_{j,\epsilon}\)) on \(Y_j^\circ\) as a family of Riemannian metrics on \(Q_{R_1}\). By Lemma 3.1, Corollary 3.12 and Lemma 3.14 we know that for all \(j\),

\[
s^j \sup_{\partial Q_s} |\nabla_{\partial Q_s} (\tilde{\omega}_{j,\epsilon} - g_C)| g_C \leq e_l(s)
\]

for some \(e_l(s)\) with \(\lim_{s \to \infty} e_l(s) = 0\). Then we can apply Proposition 3.3 to solve the Poisson equation with uniform estimates for the metric \(\tilde{\omega}_{j,\epsilon}\) on \(Q_{R_2}\), where \(R_2 \geq R_1\) is given in Proposition 3.3. More precisely, we solve the equation in a fixed Hölder space for \(k\) large and then apply elliptic regularity as Proposition 3.4 to gain the
decay for all derivatives. Next we can argue as in [10, Sect. 2.4] to obtain functions $u_{j,\epsilon}$ defined only on the end of $Y_j$ satisfying item (1) and (2) in the proposition. Fix a smooth function $H(t):[0, \infty) \to [0, 1]$, which equals 0 for $t \leq 1$ and equals 1 for $t \geq 2$. Combining the estimate in (1) with Lemma 3.13 and Lemma 3.14, it is direct to show that we can choose $R_3$ sufficiently large independent of $j$ and $\epsilon$, such that $H\left(\frac{r_j}{R_3}\right)u_{j,\epsilon}$ is a global defined function on $Y_j$ satisfying all (1)-(4) in the proposition. □

For all $j \in \mathbb{Z}_{\geq 1}$ and $\epsilon \in (0, \lambda^{4j})$, we define Kähler forms on $\hat{Y}_j$

$$\omega_{j,\epsilon} \equiv \tilde{\omega}_{j,\epsilon} + \sqrt{-1} \partial \bar{\partial} \pi_j^* u_{j,\epsilon},$$

and we denote by $g_{j,\epsilon}$ and $J_{j,\epsilon}$ the corresponding Riemannian metric and complex structure. It follows from Lemma 3.1, Proposition 3.10, Corollary 3.12, Lemma 3.14 and Proposition 3.17 that $(g_{j,\epsilon}, J_{j,\epsilon})$ is still uniformly asymptotic to the cone $\mathcal{C}$. More precisely, we have

**Proposition 3.18** There exist a compact set $K \subset \mathbb{C}^N$ and a large positive number $R_2$ such that for all $j \in \mathbb{Z}_{\geq 1}$, the diffeomorphism $P_j = p_j \circ \Phi : \mathcal{Q}_{R_2} \to Y_j \setminus K$ satisfies that for all $l \geq 0$,

$$\lim_{s \to \infty} \sup_{j \geq 1} \sup_{s \in \mathcal{Q}_s} \left| \nabla^l_{g_j,\epsilon} (\pi_{j,\epsilon}^{-1} \omega_{j,\epsilon}) \right|_{g_{j,\epsilon}} = 0,$$

where we omit $(\pi_{j,\epsilon}^{-1})_*$ again and view $g_{j,\epsilon}$ and $J_{j,\epsilon}$ as tensors on $Y_j$.

### 4 Uniform estimates for the complex Monge-Ampère equation

From now on for $j \in \mathbb{Z}_{\geq 1}$ we will abuse notations and identify $\omega_j$, $\Omega_j$ and $r_j$ with their pull-backs under $\pi_j : \hat{Y}_j \to Y_j$. We now look for Calabi-Yau metrics on $\hat{Y}_j$ which are asymptotic to $\omega_{j,\epsilon}$. To do this we need to solve on $\hat{Y}_j$ the equation

$$\left( \omega_{j,\epsilon} + \sqrt{-1} \partial \bar{\partial} \varphi_{j,\epsilon} \right)^n = (\sqrt{-1})^{n^2} \Omega_j \wedge \bar{\Omega}_j. \quad (4.1)$$

Let $\mu_j$ denote the volume form $(\sqrt{-1})^{n^2} \Omega_j \wedge \bar{\Omega}_j$ which is also equal to $e^{f_{j,\epsilon}} \omega_{j,\epsilon}^n$. It is proved in Proposition 3.17 that there exist a $\delta_2 > 0$ and a compact set $K \subset \mathbb{C}^N$ such that on $\hat{Y}_j \setminus \pi_j^{-1}(K \cap Y_j)$ for all $k \geq 0$,

$$\left| \nabla^k_{\omega_{j,\epsilon}} f_{j,\epsilon} \right|_{\omega_{j,\epsilon}} \leq C_k r_j^{-b_2 - 2 - k}. \quad (4.2)$$

The following theorem can be derived from Tian-Yau [30, Sect. 4]. See also Hein [17, Sect. 4].

**Theorem 4.1** ([17, 30]) For fixed $j$ and $\epsilon$, (4.1) admits a smooth solution $\varphi_{j,\epsilon}$ on $\hat{Y}_j$ such that
(1) there exists a constant $p_0$ depending only on $\delta_2$ such that $\|\varphi_{j,\epsilon}\|_{L^{p_0}(\mu_j)} < \infty$ and $\varphi_{j,\epsilon}(x) \to 0$ as $r_j(x) \to \infty$.

(2) $\|\nabla^k_{\omega_{j,\epsilon}} \varphi_{j,\epsilon}\|_{L^\infty(\omega_{j,\epsilon})} < \infty$ for all $k \geq 0$.

Remark 4.2 The proof of the $L^\infty$ bound of $\varphi_{j,\epsilon}$ is via the Moser iteration, thus indeed we know that $\|\varphi_{j,\epsilon}\|_{L^p(\mu_j)} < \infty$ for all $p \gg 1$. In the following, we fix a large $p_0$ such that $q_0 = \frac{n(p_0 + 1)}{n + p_0}$ satisfies

$$(2 + \delta_2)q_0 > 2n. \quad (4.3)$$

Since the background metrics $\omega_{j,\epsilon}$ are asymptotic to the cone metric on $C$, by the method in [30, Sect. 5] and using $r_j^{-2\delta}$ as a barrier to apply the maximum principle, we can prove the following decay estimates for $\varphi_{j,\epsilon}$. We remark that alternatively, one can also use the method of [7, 10] to solve the complex Monge-Ampère equation in a suitable weighted space (since we have Lemma 3.2), then the decay property will be a direct by-product.

Lemma 4.3 There exist a constant $\delta_3 > 0$ depending only on $\delta_2$, compact sets $K_{j,\epsilon} \subset \mathbb{C}^N$ depending only on $\|\varphi_{j,\epsilon}\|_{L^\infty}$ and constants $Q_{k,j,\epsilon}$ depending only on $k$ and $\|\varphi_{j,\epsilon}\|_{L^\infty}$ such that on $Y_j \setminus K_{j,\epsilon}$ for all $k \geq 0$,

$$|\nabla^k_{\omega_{j,\epsilon}} \varphi_{j,\epsilon}|_{\omega_{j,\epsilon}} \leq Q_{k,j,\epsilon} r_j^{-\delta_3 - k}.$$ 

Proof We focus on the $C^0$ decay property since the higher order estimates follow from a standard rescaling argument and elliptic estimates (see [30, Proposition 5.1]). Fix a positive constant $\delta < \min\left\{\frac{\delta_2}{2}, \frac{1}{2}\right\}$ and then we can prove the lemma holds for $\delta_3 = 2\delta$.

Claim. There is a large constant $R_*$ independent of $j$ and $\epsilon$ such that for every $R \geq R_*$, on the region $\{r_j \geq R\}$ we have

$$\left(\omega_{j,\epsilon} + R\sqrt{-1}\partial\bar{\partial} r_j^{-2\delta}\right)^n < e^{f_{j,\epsilon}} \omega^n_{j,\epsilon}$$

and

$$\left(\omega_{j,\epsilon} - R\sqrt{-1}\partial\bar{\partial} r_j^{-2\delta}\right)^n > e^{f_{j,\epsilon}} \omega^n_{j,\epsilon}.$$ 

The decay of $\varphi_{j,\epsilon}$ follows from this claim by applying the maximal principle to the region $\{r_j \geq R_{j,\epsilon}\}$ with $R_{j,\epsilon} = \max\{R_*, \|\varphi_{j,\epsilon}\|_{L^\infty}\}$. Let us prove the Claim. We only prove the first inequality since the second inequality follows from the same argument. By (4.2) and Proposition 3.18, we know that outside a compact set $K_0$, there exists a constant $C_0 > 0$ such that

$$1 - C_0 r_j^{-2 - \delta_2} \leq e^{f_{j,\epsilon}} \leq 1 + C_0 r_j^{-2 - \delta_2},$$

$$\left|\sqrt{-1}\partial\bar{\partial} r_j^{-2\delta}\right|_{\omega_{j,\epsilon}} \leq C_0 r_j^{-2(\delta + 1)},$$

$$\Delta_{\omega_{j,\epsilon}} r_j^{-2\delta} \leq -\delta r_j^{-2(\delta + 1)}. \quad (4.4)$$
The last inequality here is due to the fact that on the cone $C$, for the radial function $r$ we have

$$\Delta_{\omega^C} r^{-2\delta} = 4\delta (-n + \delta + 1) r^{-2(\delta+1)}.$$ 

Choose $R_\ast$ large such that $K_0 \subseteq \{ r_j \leq R_\ast \}$ and by the second inequality in (4.4) and the choice of $\delta$ we may also assume that for all $R \geq R_\ast$, on $\{ r_j \geq R \}$ we have

$$\left( \frac{\omega_{j,\epsilon} + R \sqrt{-1} \partial \bar{\partial} r_j^{-2\delta}}{\omega_{j,\epsilon}^n} \right)^n \leq 1 + R \Delta_{\omega_{j,\epsilon}} r_j^{-2\delta} + \frac{\delta}{2} r_j^{-2(\delta+1)},$$

(4.5)

and

$$\left( -R \delta + \frac{\delta}{2} \right) r_j^{-2(\delta+1)} \leq -2C_0 r_j^{-2-\delta_2}.$$ 

(4.6)

Then the claim follows from (4.4), (4.5) and (4.6). \hfill \Box

Denote

$\hat{\omega}_{j,\epsilon} = \omega_{j,\epsilon} + \sqrt{-1} \partial \bar{\partial} \varphi_{j,\epsilon}.$

Below we will derive uniform estimates on $\varphi_{j,\epsilon}$. Then we will take a double limit, first as $\epsilon \to 0$ and then as $j \to \infty$, to obtain a Calabi-Yau metric on $W \setminus \{0\}$. The following arguments are essentially standard, given the large body of literature in this field; see for example the recent work by Collins-Guo-Tong [7] which deals with the case when the background metric is asymptotically conical in the stronger sense (with polynomial decay rate). Notice that we do not need uniform estimates for all $j$ and $\epsilon \in (0, \lambda^{4j})$. Instead we first fix $j$ and derive uniform estimates as $\epsilon \to 0$, which allows us to take a limit $\varphi_j$. Then we will derive uniform estimates for $\varphi_j$ as $j \to \infty$. Notice we are free to pass to subsequences when taking limits and we do not need uniqueness of the limits.

Let

$$A_{j,\epsilon} = \sup_{q \geq q_0} \| e^{-f_{j,\epsilon}} - 1 \|_{L^q(\mu_j)}, \quad A_j = \limsup_{\epsilon \to 0} A_{j,\epsilon}.$$

By (4.2) and (4.3) we know for each $j$, $\epsilon$, $A_{j,\epsilon} < \infty$. The following lemma shows that $A_j < \infty$ and moreover we have a uniform bound on $A_j$. 

**Lemma 4.4** There exists a constant $C > 0$ such that $A_j \leq C$ for all $j$.

**Proof** Let $K$ be the union of the compact sets obtained in Lemma 3.15-(1) and Proposition 3.17, then by (4.2), (4.3) and Lemma 3.15, we obtain there exists a constant $C$ independent of $j$ and $\epsilon$ such that

$$A_{j,\epsilon} \leq C \left( \| e^{-f_{j,\epsilon}} \|_{L^\infty(\pi_j^{-1}(K))} + 1 \right).$$

(4.7)
By the construction in Lemma 3.13 and Proposition 3.17, \( \omega_j + \sqrt{-1} \partial \bar{\partial} u_{j,\epsilon} \leq C_1 \omega_j \leq C_2 \omega_0 \) on \( Y_j^\circ \cap K \), where \( \omega_0 \) denotes the Kähler form for the Euclidean metric on \( \mathbb{C}^N \), for some constants \( C_1 \) and \( C_2 \) independent of \( j \) and \( \epsilon \). Therefore by the definition of \( f_{j,\epsilon} \) we obtain

\[
\limsup_{\epsilon \to 0} \left\| e^{-f_{j,\epsilon}} \right\|_{L^\infty(\pi_j^{-1}(K))} \leq C_2 \left\| \frac{\omega_0^n}{(\sqrt{-1})^n \Omega_j \wedge \Omega_j} \right\|_{L^\infty(K \cap Y_j^\circ)}.
\]

(4.8)

By the Gauss-Codazzi equation we know the Ricci curvature of \( \omega_0 |_{Y_j} \) is non-positive, it follows that the function \( \frac{\omega_0^n}{(\sqrt{-1})^n \Omega_j \wedge \Omega_j} \) is plurisubharmonic on \( Y_j^\circ = Y_j \setminus \text{Sing}(Y_j) \). Pulling back to \( \hat{Y}_j \) it is a smooth plurisubharmonic function. By the maximal principle and Lemma 3.15, we conclude that the supremum of \( \frac{\omega_0^n}{(\sqrt{-1})^n \Omega_j \wedge \Omega_j} \) on \( K \cap Y_j^\circ \) has a uniform upper bound independent of \( j \). Then the lemma follows from (4.7) and (4.8).

\( \Box \)

As observed by [7] (following an earlier idea of Tosatti) it is more convenient to use the Calabi-Yau metrics \( \hat{\omega}_{j,\epsilon} \) as background metrics when we do Moser iteration. By Lemma 4.3 each \( \hat{\omega}_{j,\epsilon} \) is weakly asymptotic to the cone \( C \), so by Bishop-Gromov inequality we know the volume of a metric ball of radius \( R \) in \( (\hat{Y}_j, \hat{\omega}_{j,\epsilon}) \) is bounded below by \( \kappa(C) R^{2n} \), where \( \kappa(C) \) is the volume density of \( C \). So we have a uniform Sobolev inequality, namely, there exists \( S > 0 \) independent of \( j \) and \( \epsilon \) such that for all \( v \in C_0^\infty(\hat{Y}_j) \),

\[
\| v \|^2_{L^2(\hat{Y}_j, \hat{\omega}_{j,\epsilon})} \leq S \left\| \nabla \hat{\omega}_{j,\epsilon} v \right\|_{L^2(\hat{Y}_j, \hat{\omega}_{j,\epsilon})}^2.
\]

Lemma 4.5 There exists a continuous function \( \mathcal{P} : (0, \infty) \to (0, \infty) \) such that for all \( j \in \mathbb{Z}_{\geq 1} \) and \( \epsilon \in (0, \lambda^{4j}) \),

(1). \( \| \varphi_{j,\epsilon} \|_{L^\infty} \leq \mathcal{P}(A_{j,\epsilon}) \),

(2). \( \mathcal{P}(A_{j,\epsilon})^{-1} \omega_j \leq \hat{\omega}_{j,\epsilon} \leq \mathcal{P}(A_{j,\epsilon}) \frac{\mu_j}{\omega_j} \omega_j \).

Proof The proof is standard, see for example [7, Sect. 4]. For completeness, we include some details.

(1). We have \( \hat{\omega}_{j,\epsilon}^n - \omega_{j,\epsilon}^n = \sqrt{-1} \partial \bar{\partial} \varphi_{j,\epsilon} \wedge \sum_{k=0}^{n-1} \left( \hat{\omega}_{j,\epsilon}^k \wedge \omega_{j,\epsilon}^{n-1-k} \right) \). For any exponent \( p \geq p_0 \), multiplying both sides by \( |\varphi_{j,\epsilon}|^{p-1} \varphi_{j,\epsilon} \) and integrating by parts (the estimates obtained in Theorem 4.1 justify this), we get

\[
\int_X |\varphi_{j,\epsilon}|^{p-1} \varphi_{j,\epsilon} (e^{-f_{j,\epsilon}} - 1) \mu_j = \frac{4p}{(p+1)^2} \int_X \sqrt{-1} \partial |\varphi_{j,\epsilon}|^{p+1} \wedge \bar{\partial} |\varphi_{j,\epsilon}|^{p+1} \wedge \sum_{k=0}^{n-1} \left( \hat{\omega}_{j,\epsilon}^k \wedge \omega_{j,\epsilon}^{n-1-k} \right) .
\]
In the following $C$ always denote a constant depending only on $n$ and $p_0$. By Sobolev inequality, we get
\[
\left( \int_X |\varphi_{j,\varepsilon}|^{(p+1) \frac{n}{n-1} \mu_j} \right)^{\frac{n-1}{n}} \leq C Sp \int_X |\varphi_{j,\varepsilon}|^p e^{-f_{j,\varepsilon}} - 1 |\mu_j|.
\]
Applying the Hölder’s inequality to the right hand side, we obtain
\[
\left\| \varphi_{j,\varepsilon} \right\|_{L^{n(p_0+1)}(\mu_j)} \leq C \left\| e^{-f_{j,\varepsilon}} - 1 \right\|_{L^{p_0}(\mu_j)}^{\frac{1}{p_0}} \left\| \varphi_{j,\varepsilon} \right\|_{L^{p+1}(\mu_j)}^{\frac{1}{p+1}} \left( \mu_j \right)
\]
for all $p \geq p_0$.

Then a standard Moser iteration argument completes the proof.

(2). Note that the second inequality follows from the first one and the elementary inequality that $\tr_\beta(\alpha) \leq \frac{\alpha^n}{\beta^n} \left( \tr_\alpha(\beta) \right)^{n-1}$ for any two positive $(1, 1)$-forms $\alpha$ and $\beta$. The first inequality here is a consequence of Chern-Lu’s inequality. More precisely, by construction the bisectional curvature of $\omega_j$ has a uniform upper bound independent of $j$. Then Chern-Lu’s inequality implies that there is a constant $B$ independent of $j$ and $\varepsilon$ such that,
\[
\Delta \hat{\omega}_{j,\varepsilon} \log(\tr_{\hat{\omega}_{j,\varepsilon}} \omega_j) \geq -B \tr_{\hat{\omega}_{j,\varepsilon}} \omega_j.
\]
By the definition of $\hat{\omega}_{j,\varepsilon}$ and the estimates for $u_{j,\varepsilon}$ in Proposition 3.17, we have
\[
\tr_{\hat{\omega}_{j,\varepsilon}} (\sqrt{-1} \bar{\partial} \partial \varphi_{j,\varepsilon}) \leq n - \frac{1}{2} \tr_{\hat{\omega}_{j,\varepsilon}} \omega_j.
\]
Therefore,
\[
\Delta \hat{\omega}_{j,\varepsilon} \left( \log(\tr_{\hat{\omega}_{j,\varepsilon}} \omega_j) - 2(B + 1)\varphi_{j,\varepsilon} \right) \geq \tr_{\hat{\omega}_{j,\varepsilon}} \omega_j - 2(B + 1)n.
\]
Now we apply the maximum principle to the function $F = \log(\tr_{\hat{\omega}_{j,\varepsilon}} \omega_j) - 2(B + 1)\varphi_{j,\varepsilon}$. Since $\varphi_{j,\varepsilon}$ tends to zero at infinity, $F$ converges to $\log n$ at infinity. Notice $\pi_j^* \omega_j$ is degenerate along the exceptional set, so $F$ tends to $-\infty$ on $\pi_j^{-1}(\Sing(Y_j))$. If the supremum of $F$ is attained at infinity, then we automatically have a uniform upper bound for $\log(\tr_{\hat{\omega}_{j,\varepsilon}} (\omega_j))$. If the maximum is achieved on $\hat{Y}_j$, then $\log(\tr_{\hat{\omega}_{j,\varepsilon}} (\omega_j)) \leq C$, where $C$ depends only on $\left\| \varphi_{j,\varepsilon} \right\|_{L^\infty}$.

Using this we can also improve Lemma 4.3, namely, we can make the dependence of $K_{j,\varepsilon}$ and $Q_{k,j,\varepsilon}$ on $j$ and $\varepsilon$ to be through $A_{j,\varepsilon}$. Since we are free to pass to subsequences when taking limits and we do not need uniqueness of the limits, we may assume in the following that $A_j = \lim_{\varepsilon \to 0} A_{j,\varepsilon}$ for every $j$. For fixed $j$ since as $\varepsilon \to 0$ we have $A_{j,\varepsilon} \to A_j$, passing to a subsequence $\varepsilon_{j,l}$, as $l \to \infty$ we can make $\varphi_{j,\varepsilon_{j,l}}$ converge locally smoothly to a limit $\varphi_j$ and $u_{j,\varepsilon_{j,l}}$ converge locally smoothly to a limit $u_j$ on $Y_j^\circ$. The metric $\hat{\omega}_j = \omega_j + \sqrt{-1} \bar{\partial} \partial (u_j + \varphi_j)$ defines a Calabi-Yau metric on $Y_j \setminus \Sing(Y_j)$, i.e. $\hat{\omega}_j^n = (\sqrt{-1})^n \Omega_j \wedge \bar{\Omega}_j$. Moreover, we have
On \( Y_j^\circ \),
\[
|\varphi_j| \leq \mathcal{P}(A_j).
\] (4.9)

This implies \( \hat{\omega}_j \) extends as a closed positive \((1, 1)\) current on \( Y_j \).

On \( Y_j^\circ \),
\[
\hat{\omega}_j \geq \mathcal{P}(A_j)^{-1}\omega_j.
\] (4.10)

There exist compact sets \( K_j \) depending only on \( A_j \) and continuous functions \( Q_k : (0, \infty) \to (0, \infty) \) for all \( k \geq 0 \) such that on \( Y_j \setminus K_j \)
\[
|\nabla^k_{\omega_j} \varphi_j| \leq Q_k(A_j) r_j^{-\delta_3-k}.
\]

There exists a compact set \( K \subset \mathbb{C}^N \) such that \( u_j \) is a constant on \( K \cap Y_j \) and on \( Y_j^\circ \)
\[
|\nabla^k_{\omega_j} u_j|_{\omega_j} \leq C_k (r_j + 1)^{-\delta_2+2-k}.
\]

Moreover, a standard argument using the Bishop-Gromov volume comparison (see [7, Lemma 4.14]) implies that for any compact set \( K \subset \mathbb{C}^N \), there is a constant \( C_K > 0 \) independent of \( j \) and \( \epsilon \) such that
\[
\text{diam}(\pi_j^{-1}(K \cap Y_j), \hat{\omega}_{j,\epsilon}) \leq C_K.
\]

So the diameter of \( K \cap Y_j^\circ \) with respect to \( \hat{\omega}_j \) is also uniformly bounded independent of \( j \).

By construction we know that as \( j \to \infty \), \( \omega_j \) converges smoothly to \( \omega_\infty = \sqrt{-1}\partial\bar{\partial}\psi_\infty \) on \( W \setminus \{0\} \), and \( \psi_\infty \) is continuous across 0. Using the above estimates and Lemma 4.4 by passing to a subsequence we may assume that \( \varphi_j \) converges locally smoothly to a smooth function \( \varphi_\infty \) and \( u_j \) converges locally smoothly to a smooth function \( u_\infty \) and on \( W \setminus \{0\} \), such that \( \hat{\omega}_\infty = \omega_\infty + \sqrt{-1}\partial\bar{\partial}(u_\infty + \varphi_\infty) \) is a Kähler Ricci flat metric on \( W \setminus \{0\} \). Moreover \( u_\infty \) is a constant near 0, \( \varphi_\infty \) is globally bounded, \( \hat{\omega}_\infty \geq C^{-1}\omega_\infty \) and there exists a compact set \( K \) such that on \( W \setminus K \)
\[
|\nabla^k_{\omega_\infty} u_\infty| \leq C_k r_\infty^{-\delta_2+2-k}, \quad |\nabla^k_{\omega_\infty} \varphi_\infty| \leq C_k r_\infty^{-\delta_3-k},
\] (4.11)
where we recall \( r_\infty = r \circ \rho_\infty^{-1} \). Notice that since \( \Omega_j \) is parallel with respect to \( \hat{\omega}_j \), we may assume \( \Omega_j \) converges to a holomorphic volume form \( \Omega_\infty \) on \( W \setminus \{0\} \) such that \( \hat{\omega}_\infty^\flat = (\sqrt{-1})^n \Omega_\infty \wedge \bar{\Omega}_\infty \). Moreover the metric \( \hat{\omega}_\infty \) has finite diameter on \( K \cap W \), and it is asymptotic to the cone \( C \) at infinity.

5 Proof of the main results

We now investigate the singular behavior of \( \hat{\omega}_\infty \) at 0. To this end we need to study the singular behavior of \( \hat{\omega}_j \), so we first fix \( j \) and consider the family \( \hat{\omega}_{j,l} \equiv \hat{\omega}_{j,\epsilon_j,l} \).
as $l \to \infty$. Passing to a further subsequence we let $(Z_j, dZ_j)$ be a pointed Gromov-Hausdorff limit of $(\tilde{Y}_j, \hat{\omega}_{j,l}, p_j)$ (where $p_j$ is a point on the exceptional set of $\pi_j$) as $l \to \infty$. As mentioned before, the metrics $\hat{\omega}_{j,l}$ are uniformly volume non-collapsing, so the general convergence theory leads to a decomposition $Z_j = \mathcal{R} \cup S$, where the regular set $\mathcal{R}$ is a smooth open manifold with $d_{Z_j}|_{\mathcal{R}}$ induced by a Calabi-Yau metric $(\omega_{\mathcal{R}}, J_\mathcal{R})$, and the singular set $S$ is closed and has Hausdorff codimension at least 4.

**Proposition 5.1** $Z_j$ agrees with the metric completion of $(Y_j^o, \hat{\omega}_j)$ and is naturally homeomorphic to $Y_j$.

This would likely follow from the arguments of J. Song [26] (and its generalization in [7]). We outline a different and more direct proof here, which may be of independent interest. It is not hard to see that we can naturally view $(Y_j^o, \hat{\omega}_j)$ as embedded in $(\mathcal{R}, \omega_{\mathcal{R}})$ as an open Kähler manifold. Moreover, since $\omega_j \leq \mathcal{P}(A_j) \hat{\omega}_{j,l}$, by passing to a further subsequence, one can take the limit of the holomorphic maps $\pi_j : (\tilde{Y}_j, \hat{\omega}_{j,l}) \to Y_j \subset \mathbb{C}^N$ and obtain a surjective continuous map $\pi_\infty : Z_j \to Y_j \subset \mathbb{C}^N$, which is the identity on $Y_j^o$ and holomorphic on $\mathcal{R}$. It then follows that $\mathcal{R} \setminus Y_j^o$ is a complex analytic set in $\mathcal{R}$, given by $\pi_\infty^{-1}(\text{Sing}(Y_j)) \cap \mathcal{R}$. By the discussion in Sect. 4, on $Y_j^o$ we may write $\hat{\omega}_j = \sqrt{-1} \partial \bar{\partial} \hat{\varphi}_j$, where $\hat{\varphi}_j$ is locally bounded across $\text{Sing}(Y_j)$. A key point is the following

**Lemma 5.2** The function $\hat{\varphi}_j$ extends smoothly to $\mathcal{R}$. Moreover, for any compact set $K \subset Z_j$ we have

$$\sup_{K \cap \mathcal{R}} |\nabla_{\omega_{\mathcal{R}}} \hat{\varphi}_j|_{\omega_{\mathcal{R}}} < \infty.$$ 

**Proof** On $Y_j^o$, the form $\sqrt{-1} \partial \bar{\partial} \hat{\varphi}_j = \hat{\omega}_j = \omega_{\mathcal{R}}$ is smooth. Since $\mathcal{R} \setminus Y_j^o$ is a closed pluripolar set and $\hat{\varphi}_j$ is locally bounded across $\text{Sing}(Y_j)$, by well-known removable singularity theorem for plurisubharmonic functions (see [13, Chapter I, Theorem 5.24]), we obtain that the same equation holds on $\mathcal{R}$ in the sense of currents, so elliptic regularity gives a smooth extension of $\hat{\varphi}_j$ to $\mathcal{R}$.

To prove the gradient bound we fix an open subset $U \subset Y_j$ containing $\text{Sing}(Y_j)$ and with $\partial U$ smooth. Denote $\hat{U} = \pi_j^{-1}(U)$. Its boundary $\partial \hat{U}$ is naturally identified with $\partial U$. We can solve the Poisson equation $\Delta_{\hat{\omega}_{j,l}} \eta_l = n$ on $\hat{U}$ with boundary condition given by $\hat{\varphi}_j|_{\partial \hat{U}}$. Then by the standard Moser iteration and Cheng-Yau gradient estimate we have $|\nabla_{\hat{\varphi}_j,l} \eta_l| \leq C$ for all $l$. So passing to a subsequence $\eta_l$ converges uniformly to a limit $\eta$ on $\pi_\infty^{-1}(U) \subset Z_j$, with $|\nabla_{\hat{\omega}_j} \eta| \leq C, \Delta_{\hat{\varphi}_j} \eta = n$ on $\pi_\infty^{-1}(U) \cap \mathcal{R}$ and $\eta|_{\partial \hat{U}} = \hat{\varphi}_j|_{\partial \hat{U}}$.

Now consider the function $f = \hat{\varphi}_j - \eta$ on $\pi_\infty^{-1}(U) \cap \mathcal{R}$. It is harmonic and $f|_{\partial \hat{U}} = 0$. One can conclude that $f$ is identically zero from the standard removable singularity theorem for harmonic functions on Ricci limit spaces. For example, using the fact that the Hausdorff codimension of $S$ is at least 4, for $\epsilon > 0$ small, one can construct (see [14, Proposition 3.5]) a cut-off function $\chi_\epsilon : Z_j \to \mathbb{R}$ which is equal to 1 when $d_{Z_j}(x, S) \geq \epsilon$ and vanishes in a neighborhood of $S$, such that
\[ \int_{Z_j} d\omega_j(\cdot, S)^{-1}\left| \nabla_{\omega_j} \chi_{\epsilon} \omega_j^n \right| < \epsilon. \]

By the local Cheng-Yau gradient estimate we have \[ \left| \nabla_{\omega_j} \hat{\phi}_j \right| \leq C d\omega_j(\cdot, S)^{-1}. \]

Then by a straightforward integration by parts argument one sees that \( f \equiv 0. \)

Given this Lemma, one can follow the strategy of [14, 15] to construct holomorphic functions by the Hörmander \( L^2 \) technique (see a related observation in [29] for studying the asymptotic cone of complete \( \sqrt{-1} \partial \bar{\partial} \)-exact Calabi-Yau manifolds with Euclidean volume growth). Here one can work on the trivial holomorphic line bundle \( L \) over \( Y_j^\circ \) endowed with the hermitian metric \( |\cdot|_{\hat{\omega}_j} \equiv e^{-\hat{\phi}_j} \). Notice \( Y_j^\circ \) admits a complete Kähler metric, so one can apply Theorem 3.8 here. Let \( \nabla_L \) be the corresponding Chern connection on \( L \). For \( k \geq 1 \), denote by \( V_k \) the space of holomorphic sections \( f \) of \( L^k \) over \( Y_j^\circ \) such that

\[ \| f \|_k^2 \equiv \int_{Y_j^\circ} |f|_{k\hat{\omega}_j}^2 (k\hat{\omega}_j)^n < \infty. \]

Since \( Y_j \) is normal it is clear that any holomorphic function \( f \) on \( Y_j^\circ \) extends to a holomorphic function on \( Y_j \). In particular, for any compact set \( K \subset \mathbb{C}^N \), \( |f| \) and \( |df|_{\omega_j} \) are uniformly bounded over \( Y_j^\circ \cap K \). By (4.9), (4.10) and Lemma 5.2, we know that \( |f|_{k\hat{\omega}_j} \) and \( |\nabla_L f|_{k\hat{\omega}_j} \) are also uniformly bounded over \( Y_j^\circ \cap K \). Viewed in \( Z_j \), it follows that \( f \) extends to a holomorphic function on \( \mathcal{R} \) and the above norms are locally bounded near \( S \).

To apply the idea in [14] we need to show that for any compact set \( K \subset Y_j \), there exist constants \( K_0, K_1 > 0 \) depending on \( K \) such that for all \( k \geq 1 \) and any \( f \in V_k \), we have

\[ \sup_K |f|_{k\hat{\omega}_j} \leq K_0 \| f \|_k, \quad \sup_K |\nabla_L f|_{k\hat{\omega}_j} \leq K_1 \| f \|_k. \]

These were proved by applying the Moser iteration to the corresponding differential inequalities

\[ \Delta_{k\hat{\omega}_j} |f|_{k\hat{\omega}_j} \geq -C |f|_{k\hat{\omega}_j}, \quad \Delta_{k\hat{\omega}_j} |\nabla_L f|_{k\hat{\omega}_j} \geq -C |\nabla_L f|_{k\hat{\omega}_j}, \]

which a priori only hold on \( \mathcal{R} \), but again using the existence of a good cut-off function as in [14, Proposition 3.5], and noticing that there is a Sobolev inequality on \( Z_j \) (since \( Z_j \) is a volume non-collapsing Ricci limit space), one can make the usual arguments go through.

Using the existence of tangent cones at points in \( Z_j \), it is straightforward to adapt the arguments in [14, 15] to construct holomorphic functions in \( V_k \) that separate points. Namely, given any \( x_1 \neq x_2 \in Z_j \), one can find \( k \geq 1 \) and \( f_1, f_2 \in V_k \) such that \( |f_\alpha(x_\alpha)|_{k\hat{\omega}_j} \geq 1 \), and \( |f_\alpha(x_\beta)|_{k\hat{\omega}_j} \leq 1/10 \) for \( \alpha \neq \beta \).

**Proof of Proposition 5.1** We first prove \( \pi_\infty \) is injective (hence is a homeomorphism). Suppose \( x_1 \neq x_2 \in Z_j \). Then we can construct the functions \( f_1, f_2 \) as above. But both can be viewed as holomorphic functions on \( Y_j \), so it follows that \( \pi_\infty(x_1) \neq \pi_\infty(x_2) \).
Since \( \text{Sing}(Y_j) \) consists of finitely many points, we see \( Z_j \backslash Y_j^o = \pi^{-1}_\infty(\text{Sing}(Y_j)) \) is also a finite set. Clearly this implies that \( Z_j \) is indeed the metric completion of \((Y_j^o, \tilde{\omega}_j)\).

Now we further let \( j \to \infty \). Passing to a subsequence we may take a pointed Gromov-Hausdorff limit \((Z_j, d_{Z_j}) \to (Z_\infty, d_\infty)\). The same argument as above combined with the uniform estimates in Sect. 4 gives

**Proposition 5.3** \( Z_\infty \) agrees with the metric completion of \((W \backslash \{0\}, \hat{\omega}_\infty)\) and is naturally homeomorphic to \( W \).

In particular, we may naturally identify \( Z_\infty \) with \( W \).

**Proof of Theorem 1.1** As is proved in Sect. 4, \( Z_\infty \) has a unique asymptotic cone given by \( C \). From the above discussion it is straightforward to extend the arguments of [15] to show that there is a unique metric tangent cone \( C' \) at 0. Furthermore, by the minimizing property of K-semistable valuations proved by Li-Xu [21], we know the volume densities satisfy \( \kappa(C') \leq \kappa(C) \), where the right hand side can be interpreted as the normalized volume associated to the natural valuation of \( W \) at 0. Notice \( Z_\infty \) is a pointed Gromov-Hausdorff limit of complete Ricci-flat manifolds, so the Bishop-Gromov inequality applies to \( Z_\infty \), which gives that \( \kappa(C') \geq \kappa(C) \). Hence the equality holds and then \( Z_\infty \) must be a Kähler cone itself, hence is isomorphic to \( C \) as a Kähler cone. Denote by \( \xi' \) the Reeb vector field on \( Z_\infty \) and by \( \tilde{r} \) the radial function on \( Z_\infty \).

It remains to show that \( \xi' = \xi \). For this we need to use some results in [15]. The algebraic structure of \( Z_\infty \) can be intrinsically characterized in terms of the ring \( R(Z_\infty) \) holomorphic functions on \( Z_\infty \) of polynomial growth (measured using the Kähler cone metric). The cone structure yields a decomposition

\[
R(Z_\infty) = \bigoplus_{\mu \geq 0} R_\mu(Z_\infty), \tag{5.2}
\]

where \( R_\mu(Z_\infty) \) consists of \( f \) which are of homogeneous of degree \( \mu \), i.e., \( L_{\xi'} f = -\mu f \). Conversely, \( \xi' \) can be recovered from this decomposition.

The above decomposition is determined by a filtration associated with a degree function. For any nonzero \( f \in R(Z_\infty) \), the following is well-defined (see [15])

\[
d_{Z_\infty}(f) \equiv \lim_{R \to \infty} \sup_{\tilde{r}(x) = R} \frac{\log |f(x)|}{\log R} < \infty.
\]

Then the decomposition in (5.2) is determined by the filtration of \( R(W) \) with respect to \( d_{Z_\infty} \).

On the other hand, the coordinate ring \( R(W) \) of \( W \) consists of restrictions of polynomial functions on \( \mathbb{C}^N \). We define

\[
d_W(f) \equiv \lim_{R \to \infty} \sup_{\tilde{r}(x) = R} \frac{\log |f(x)|}{\log R}.
\]
There is a decomposition $R(W) = \bigoplus_{\mu \geq 0} R_\mu(W)$, where $R_\mu(W)$ is the space of polynomials $f$ with $\mathcal{L}_{J_\xi} f = -\mu f$. It can be checked that this decomposition is also determined by the filtration of $R(W)$ with respect to $d_W$.

Notice that by the construction of $\hat{\omega}_\infty$ we know for all $\epsilon > 0$, $C_{\epsilon}^{-1} r_\xi^{1-\epsilon} \leq \tilde{r} \leq C \epsilon r_\xi^{1+\epsilon}$ on $W \setminus K$ for a fixed compact set $K$. It follows that $R(Z_\infty) = R(W)$ and $d_W(f) = d_{Z_\infty}(f)$ for any $f \in R(W)$. So we must have $R_\mu(Z_\infty) = R_\mu(W)$ for all $\mu \geq 0$. In particular, $\xi' = \xi$. Hence we have constructed a $T_\xi$-equivariant isomorphism between $W$ and $C$. This completes the proof of Theorem 1.1.

Proof of Theorem 1.2 We simply notice that by the estimate of $\varphi_\infty$ and $u_\infty$ in (4.11), the background metric $\omega_\infty$ is polynomially asymptotic to the Calabi-Yau cone metric $\hat{\omega}_\infty$ on $W$ (which is the same as $\mathcal{C}$). Moreover by the construction of $\omega_\infty$ (see Corollary 3.12 and Lemma 3.13), it is polynomially asymptotic to $\omega$ under the diffeomorphism $p_\infty$. Then $p_\infty$ provides the diffeomorphism that shows $\omega$ and $J$ are asymptotic to $\hat{\omega}_\infty$ and $J_W$ in a polynomial rate. The asymptotics of $\Omega$ follows from (3.16) by letting $j \to \infty$.

Remark 5.4 Using the same proof one can weaken the Calabi-Yau condition in Theorem 1.1 and 1.2 to be Ricci-flat Kähler. Alternatively, one can show this by working on the universal cover since it is well known [2, 22] that Riemannian manifolds with non-negative Ricci curvature and Euclidean volume growth have finite fundamental groups.

6 Discussions

We point out some further related directions that one can explore. The guiding problem is

Problem 6.1 (Algebraization for complete Calabi-Yau metrics) Give an algebro-geometric characterization of all complete Calabi-Yau metrics with Euclidean volume growth.

Remark 6.2 Without the volume growth condition, the situation is more complicated and extra assumptions are needed in order to make connections with algebraic geometry. In complex dimension 2 under a natural finite energy condition, the models at infinity are completely classified by Sun-Zhang [27], and there is a complete classification of the Calabi-Yau metrics in terms of algebro-geometric data by the work of many people; see [27] for references.

Given Theorem 1.2, the results of Conlon-Hein [11] provide an answer to Problem 6.1 under the extra quadratic curvature decay condition. Namely, such Calabi-Yau metrics are always constructed as follows (ignoring uniqueness issues for the moment), where each step is essentially algebro-geometric.

(1) Choose a Calabi-Yau cone $(\mathcal{C}, \xi)$. This is equivalent to choosing a K-polystable Fano cone [9].
Choose a normal affine variety $X'$ with $(\mathcal{C}, \xi)$ as a weighted asymptotic cone at infinity.

(3) Choose a crepant Kähler resolution $\pi : X \to X'$.

(4) Choose a Kähler class on $X$. This is a numerical condition according to [8, 11].

We want to ask how much of the above picture holds in general, if we allow singularities in the relevant objects. There are several technical obstacles to realize this program, mostly related to the possible appearance of singularities on the asymptotic cone. First one needs to extend the 2-step degeneration theory of [15] to this setting. In particular, we would like to construct abundant holomorphic functions with polynomial growth on $X$. This is related to a special case of Yau’s compactification conjecture, which we reiterate as follows

**Conjecture 6.3 (Yau’s Compactification conjecture)** A complete Calabi-Yau manifold with Euclidean volume growth is naturally a quasi-projective variety.

There has been partial progress towards this. For example, it is known that any asymptotic cone is naturally a normal affine variety [25]. It is also observed in [29] that the results of [15] automatically extend to the setting when the Calabi-Yau metric is $\bar{\partial}\partial$-exact so Conjecture 6.3 holds in this case. Now suppose the 2-step degeneration theory works. The following is natural to expect

**Conjecture 6.4 (No semistability at infinity, the general version)** Theorem 1.1 holds without the quadratic curvature decay assumption.

**Remark 6.5** One can also formulate a conjectural general version of Theorem 1.2, where one uses the Gromov-Hausdorff distance to measure the rate of convergence to the asymptotic cone.

One difficulty in proving Conjecture 6.4 using the strategy of this paper is related to the generalization of the Tian-Yau construction, which is itself an interesting question. A first step would be

**Question 6.6 (Prescribing asymptotic cone)** Given a singular Calabi-Yau cone $(\mathcal{C}, \xi)$ and a normal affine variety $X'$ with $(\mathcal{C}, \xi)$ as a weighted asymptotic cone at infinity, when does $X'$ admit a (possibly singular) Calabi-Yau metric asymptotic to $(\mathcal{C}, \xi)$ in the Gromov-Hausdorff sense?

There are some recent extensions of the Tian-Yau construction with singular asymptotic cones for special examples; see [4, 12, 18, 23, 28].

As mentioned in Sect. 2 the classification of complete Calabi-Yau metrics with Euclidean volume growth on a fixed underlying algebraic variety is a subtle problem. Even on $\mathbb{C}^n (n \geq 3)$ the situation can be complicated; the recent works by Székelyhidi [29] and Chiu [4] make progress in this direction. We make an attempt here to formulate some general questions. To minimize technical issues we restrict to the case of $\partial\bar{\partial}$-exact metrics on smooth affine varieties. As mentioned above the results of [15]
apply here. Fix a smooth affine variety $X$ with the coordinate ring $R(X)$. A complete $\partial\bar{\partial}$-exact Calabi-Yau metric $\omega$ on $X$ with Euclidean volume growth is said to be a compatible metric if the space of holomorphic functions on $X$ with polynomial growth with respect to $\omega$ coincides with $R(X)$. A compatible Calabi-Yau metric $\omega$ defines a degree function $d_\omega : R(X) \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ satisfying
\begin{itemize}
  \item $d_\omega(f) = \infty$ if and only if $f = 0$;
  \item $d_\omega(f) = 0$ if and only if $f$ is a nonzero constant;
  \item $d_\omega(fg) = d_\omega(f) + d_\omega(g)$;
  \item $d_\omega(f + g) \leq \max(d_\omega(f), d_\omega(g))$.
\end{itemize}
Furthermore, $d_\omega$ gives rise to a filtration of $R(X)$ whose associated graded ring is the coordinate ring $R(W)$ of the intermediate K-semistable cone $W$, and $W$ degenerates to the unique asymptotic cone $C$ as affine algebraic cones. The result of Li-Wang-Xu [20] implies that $(C, \xi)$ is uniquely determined by $d_\omega$. Clearly, if $\omega$ is compatible, then for any $\lambda > 0$ and any algebraic automorphism $F$ of $X$, $\lambda F^*\omega$ is also compatible. Furthermore, $d_\omega = d_{\lambda \omega}$ and $d_{F^*\omega} = F^*d_\omega$.

In terms of algebro-geometric language one notices that $\nu_\omega \equiv -d_\omega$ is a valuation on $R(X)$ with values in $\mathbb{R}_{\leq 0} \cup \{-\infty\}$. We call such a valuation a negative valuation. We say a negative valuation $\nu$ is semistable (resp. polystable) if it defines a filtration whose associated graded ring is finitely generated and defines a K-semistable (resp. K-polystable) Fano cone $C_\nu$ (in the sense of [9]). Geometrically, the cone is a weighted asymptotic cone at infinity of $X$ under some affine embedding. Conjecture 6.4 implies that the negative valuation induced by a compatible Calabi-Yau metric on $X$ is always polystable.

Given a semistable valuation $\nu$ on $X$, we define $G_\nu$ as the group of algebraic automorphisms of $X$ that preserve $\nu$ and define $\tilde{G}_\nu$ as the group of algebraic automorphisms of $C_\nu$ that fixes the vertex and preserves the cone structure. Both groups are finite-dimensional. Notice that $\tilde{G}_\nu$ contains the algebraic torus $\mathbb{T}_\nu$ generated by the cone vector field. There is a natural homomorphism $\varphi : G_\nu \to \tilde{G}_\nu$.

To classify the space of all compatible Calabi-Yau metrics on $X$, assuming Conjecture 6.4, one can proceed as follows:

**Step 1.** Determine the space $\mathcal{V}$ of all possible polystable negative valuations on $X$ up to algebraic automorphism. This is essentially an algebro-geometric question. One can also study the related question with polystability replaced by semistability. In general there may be infinitely many polystable negative valuations on a given $X$. Indeed, given any polystable cone $V$ in $\mathbb{C}^n$ which is a hypersurface, one can find a polystable negative valuation $\nu$ on $X = \mathbb{C}^n$ with $C_\nu = \mathbb{C} \times V$. The result of Székelyhidi [28] implies that if $V$ has only isolated singularities, then the associated negative valuation actually arises from a compatible Calabi-Yau metric on $\mathbb{C}^n$. Still, it seems interesting to investigate the structure of the space $\mathcal{V}$.

**Step 2.** Now we fix a polystable negative valuation $\nu$ on $X$. Denote by $\mathcal{M}_\nu$ the space of all compatible Calabi-Yau metrics $\omega$ on $X$ with $\nu = \nu_\omega$. Is $\mathcal{M}_\nu$ nonempty? This is related to Question 6.6.

**Step 3.** Suppose $\mathcal{M}_\nu \neq \emptyset$. We know that all $\omega \in \mathcal{M}_\nu$ have the same asymptotic cone at infinity given by $C_\nu$. But we should expect more and one would like to have
Conjecture 6.7 (Uniform equivalence) There exists a constant $C > 0$ such that for all $\omega_1, \omega_2 \in \mathcal{M}_v$, we have $C^{-1}\omega_1 \leq \omega_2 \leq C\omega_1$.

It is easy to see the converse is true: if two compatible Calabi-Yau metrics $\omega_1$ and $\omega_2$ are uniformly equivalent, then $v_{\omega_1} = v_{\omega_2}$. We remark that one can also ask an analogous question for local singularities of Kähler-Einstein metrics (on algebraic varieties with klt singularities), that is, whether any two such metrics are indeed locally uniformly equivalent. The answer to this question, currently not known, is also related to Conjecture 6.7. The recent work by Chiu-Székelyhidi [5] made progress in understanding local singular behavior of polarized Kähler-Einstein metrics.

Step 4. Denote by $\mathcal{N}_v$ the space of Calabi-Yau cone metrics on $C_v$ (with the given Reeb vector field determined by $v$). Notice $C_v$ can be realized as a weighted asymptotic cone at infinity for some embedding $X \subset \mathbb{C}^N$. One is attempted to define a natural map $\mathcal{C} : \mathcal{M}_v \to \mathcal{N}_v$ given by taking the (appropriate) rescaled limit of the Kähler form under the weighted asymptotic cone construction.

Conjecture 6.8 (Generalized uniqueness) The map $\mathcal{C}$ is well-defined and is bijective.

We explain Conjecture 6.7 and 6.8 indeed hold if we assume $C_v$ has only an isolated singularity at the vertex. In this case any Calabi-Yau metric in $\mathcal{M}_v$ has quadratic curvature decay so we may apply the arguments of this paper. Since we have fixed the negative valuation $v$, this fixes the algebro-geometric degeneration from $X$ to $C_v$ in terms of a weighted asymptotic cone. Then by Sect. 3.6 we can fix the diffeomorphism between the end of $X$ and the end of $C_v$. For every $\omega_\infty \in \mathcal{N}_v$, by [10] (or the arguments in Sect. 4 of this paper) one gets a compatible Calabi-Yau metric on $X$ which is asymptotic to $\omega_\infty$ at a polynomial rate. Conversely, given a compatible Calabi-Yau metric $\omega \in \mathcal{M}_v$, using the arguments in this paper, we can get a Calabi-Yau cone metric on $C_v$ which is the asymptotic cone of $\omega$ at a polynomial rate. One can see this is exactly the $\mathcal{C}$ map that we want. Furthermore, it is independent of the auxiliary data chosen. The uniqueness theorem [10, Theorem 3.1] proved by Conlon-Hein implies that $\mathcal{C}$ is bijective.

Notice even in this case it is not clear whether the action of $G_v$ is transitive on $\mathcal{M}_v$ (modulo scaling): we know that the action of $\tilde{G}_v$ on $\mathcal{N}_v$ is transitive by the generalized Bando-Mabuchi theorem (see for example, [15, Proposition 4.8]). Indeed, $\mathcal{N}_v$ can be identified with the homogeneous space $\tilde{G}_v/K_v$ for a maximal compact subgroup $K_v$ whose complexification is $\tilde{G}_v$. But the map $\varphi : G_v \to \tilde{G}_v$ could be complicated in general: neither the injectivity nor the surjectivity seems obvious. It seems to us that Conjecture 6.8 is a more natural formulation than a naive uniqueness statement. We mention that the above conjecture is compatible with the result and conjecture in [4, 29] for complete Calabi-Yau metrics on $\mathbb{C}^3$ with specific asymptotic cones.

The above four steps altogether would lead to a satisfactory classification of compatible Calabi-Yau metrics on a fixed smooth affine variety. More generally, when the metrics are not necessarily $\bar{\partial}\bar{\partial}$-exact, one needs to further involve Kähler classes in the discussion.

From another perspective, one can also try to classify complete Calabi-Yau metrics asymptotic to a given Calabi-Yau cone, without fixing the underlying algebraic
variety. Notice again this should contrast with the local situation. As is well known (see [15, 18, 28]), there is no boundedness of local singularities of Calabi-Yau metrics with a given tangent cone if the latter has nonisolated singularities. One example is given by the $A_k$ singularities in dimension $n \geq 3$ for $k \geq 4$, whose local tangent cones are all given by the product of an $n - 1$ dimensional $A_1$ singularity with $\mathbb{C}$. However, for asymptotic cones, there are more constraints. For example, the Bishop-Gromov inequality ensures a strong uniform non-collapsing condition, which gives a preliminary compactness in this situation. In particular, one would expect a well-behaved moduli space when the volume density of the asymptotic cone is bounded away from zero.

**Problem 6.9 (Moduli problem)** Given a positive number $\lambda > 0$, study the structure of the moduli space of (possibly singular) $\partial\bar{\partial}$-exact Calabi-Yau metrics on affine varieties whose asymptotic cones have volume density at least $\lambda$.

It is straightforward to formulate an algebro-geometric counterpart of the above problem. That is, one can study the moduli space $\mathcal{M}_{\geq \lambda}$ of pairs $(X, v)$, where $X$ is an affine variety and $v$ is a polystable negative valuation on $X$ with $C_v$ having volume density at least $\lambda$. It seems to be an approachable question given recent advances in birational algebraic geometry. However, as pointed out by Yuji Odaka there are some complications. For simplicity we fix a $K$-polystable cone $C$, then the deformation space $\text{Def}(C)$ of $C$ as affine varieties is in general infinite dimensional, but it is natural to expect that the subspace $\text{Def}_-(C)$ which consists of those deformations to the above pairs $(X, v)$ with $C_v = C$ is finite dimensional, and there is an action of $\tilde{G}_v$ on $\text{Def}_-(C)$. Now since $\tilde{G}_v$ is noncompact the quotient is in general not Hausdorff, so any meaningful construction of such moduli space will have to address this issue. On the other hand, if we accept the above picture, then the space of $\partial\bar{\partial}$-exact Calabi-Yau metrics asymptotic to $C$ (up to isomorphism) is given by the quotient $\text{Def}_-(C)/K_v$. It seems an interesting question to us to explore the structure here in more detail.

Finally, we point out that while Theorem 1.2 is a Riemannian geometric statement, our proof in this paper hinges on complex geometry. One can ask whether the polynomial convergence rate is a more general phenomenon in geometric analysis, or is it only special in the complex geometric world. Notice this is a question involving *global* geometry, not only the end at infinity. Indeed, we expect that the technique of Adams-Simon [1] can be used to show that given a Calabi-Yau cone $C$ with obstructed deformations in an appropriate sense, one can produce an end of Calabi-Yau metric which is asymptotic to $C$ at only a logarithmic rate; see [3] for a related work on isolated singularities of $G_2$ metrics. Such an end is also interesting in terms of complex geometry since it can not be filled in as a complete Kählerian manifold – otherwise using the Tian-Yau construction one would get a contradiction with Theorem 1.2.

**Question 6.10 (Polynomial rate to the asymptotic cone)** Is a complete Ricci-flat Riemannian manifold with Euclidean volume growth always asymptotic to a unique metric cone at a polynomial rate (in the Gromov-Hausdorff distance)?

Even the uniqueness of asymptotic cones is not known to date without the quadratic curvature decay condition. An answer to Question 6.10, either in the positive or the negative, would be interesting. A special setting is given by manifolds...
with special holonomy, i.e., $G_2$ and $Spin(7)$ manifolds. One can also explore similar questions for global solutions of other geometric PDEs, such as minimal submanifolds and Yang-Mills connections. These will be important in classifying singularity models.

Acknowledgements The first author would like to thank Xuemiao Chen for many discussions on related questions in the context of Hermitian-Yang-Mills connections. The second author thanks Ronan Conlon for his lectures at UC Berkeley. We thank Lorenzo Foscolo for helpful discussions, and Ronan Conlon, Yuji Odaka for comments. We also thank the anonymous referee for useful suggestions which improved the presentation. Both authors are partially supported by NSF Grant DMS-2004261. The first author is partially supported by the Simons Collaboration Grant in special holonomy.

References

1. Adams, D., Simon, L.: Rates of asymptotic convergence near isolated singularities of geometric extrema. Indiana Univ. Math. J., 37(2), 225–254 (1988)
2. Anderson, M.T.: On the topology of complete manifolds of non-negative Ricci curvature. Topology 29(1), 41–55 (1990)
3. Chen, G.: Rate of asymptotic convergence near an isolated singularity of $G_2$ manifold. J. Geom. Anal. 28(4), 3139–3170 (2018)
4. Chiu, S.-K.: Nonuniqueness of Calabi-Yau metrics with maximal volume growth. arXiv:2206.08210 (2022)
5. Chiu, S.-K., Székelyhidi, G.: Higher regularity for singular Kähler-Einstein metrics. arXiv:2202.11083 (2022)
6. Holck Colding, T., Minicozzi, W.P.: On uniqueness of tangent cones for Einstein manifolds. Invent. Math. 196(3), 515–588 (2014)
7. Collins, T.C., Guo, B., Tong, F.: On the degeneration of asymptotically conical Calabi–Yau metrics. Math. Ann. 383, 867–919 (2022)
8. Collins, T.C., Tosatti, V.: A singular Demailly–Păun theorem. C. R. Math. 354(1), 91–95 (2016)
9. Collins, T., Székelyhidi, G.: Sasaki–Einstein metrics and K–stability. Geom. Topol. 23(3), 1339–1413 (2019)
10. Conlon, R.J., Hein, H.-J.: Asymptotically conical Calabi–Yau manifolds, I. Duke Math. J. 162(15), 2855–2902 (2013)
11. Conlon, R.J., Hein, H.-J.: Classification of asymptotically conical Calabi-Yau manifolds. arXiv:2201.00870 (2022)
12. Conlon, R.J., Rochon, F.: New examples of complete Calabi-Yau metrics on $\mathbb{C}^n$ for $n \geq 3$. arXiv:1705.08788 (2017)
13. Demailly, J.-P.: Complex Analytic and Differential Geometry. Citeseer (1997)
14. Donaldson, S., Sun, S.: Gromov–Hausdorff limits of Kähler manifolds and algebraic geometry. Acta Math. 213(1), 63–106 (2014)
15. Donaldson, S., Sun, S.: Gromov–Hausdorff limits of Kähler manifolds and algebraic geometry. II. J. Differ. Geom. 107(2), 327–371 (2017)
16. Riemensneider, O., Grauert, H.: Verschwindungssätze für analytische Kohomologiegruppen auf komplexen Räumen. Invent. Math. 11, 263–292 (1970)
17. Hein, H.-J.: On gravitational instantons. Phd thesis, Princeton University (2010)
18. Hein, H.-J.: Ricci-flat metrics on $A_k$ singularities. Oberwolfach report 31 (2015)
19. Li, C.: Minimizing normalized volumes of valuations. Math. Z. 289(1–2), 491–513 (2018)
20. Li, C., Wang, X., Xu, C.: Algebraicity of the metric tangent cones and equivariant K-stability. J. Am. Math. Soc. 34(4), 1175–1214 (2021)
21. Li, C., Xu, C.: Stability of valuations: higher rational rank. Peking Math. J. 1(1), 1–79 (2018)
22. Li, P.: Large time behavior of the heat equation on complete manifolds with non-negative Ricci curvature. Ann. Math. 124(1), 1–21 (1986)
23. Li, Y.: A new complete Calabi–Yau metric on $\mathbb{C}^3$. Invent. Math. 217(1), 1–34 (2019)
24. Liu, G.: Compactification of certain Kähler manifolds with nonnegative Ricci curvature. Adv. Math. 382, 107652 (2021)
25. Liu, G., Székelyhidi, G.: Gromov-Hausdorff limits of Kähler manifolds with Ricci curvature bounded below II. Commun. Pure Appl. Math. 74(5), 909–931 (2021)
26. Song, J.: Riemannian geometry of Kähler-Einstein currents. arXiv:1404.0445 (2014)
27. Sun, S., Zhang, R.: Collapsing geometry of hyperkaehler 4-manifolds and applications. arXiv:2108.12991 (2021)
28. Székelyhidi, G.: Degenerations of $\mathbb{C}^n$ and Calabi–Yau metrics. Duke Math. J. 168(14), 2651–2700 (2019)
29. Székelyhidi, G.: Uniqueness of some Calabi–Yau metrics on $\mathbb{C}^n$. Geom. Funct. Anal. 30(4), 1152–1182 (2020)
30. Tian, G., Tung Yau, S.: Complete Kähler manifolds with zero Ricci curvature II. Invent. Math. 106(1), 27–60 (1991)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.