FINITE ITERATIVE ALGORITHM FOR THE COMPLEX GENERALIZED SYLVESTER TENSOR EQUATIONS*

Yifen Ke

Abstract A finite iterative algorithm is proposed to solve a class of complex generalized Sylvester tensor equations. The properties of this proposed algorithm are discussed based on a real inner product of two complex tensors and the finite convergence of this algorithm is obtained. Two numerical examples are offered to illustrate the effectiveness of the proposed algorithm.

Keywords Sylvester tensor equation, finite iterative algorithm, convergence.

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1. Introduction

A tensor is a multidimensional array of data whose elements are referred by using multiple indices. We use

$$
A = (a_{i_1 i_2 \ldots i_N}) \in \mathbb{C}^{I_1 \times I_2 \times \ldots \times I_N}
$$

to denote a complex tensor $A$, where $N$ is called the order of tensor $A$ and $(I_1, I_2, \ldots, I_N)$ is the dimension of $A$. As a special case, the vector is a 1-order tensor and the matrix is a 2-order tensor.

Let $\overline{A} = (\overline{a}_{i_1 i_2 \ldots i_N}) \in \mathbb{C}^{I_1 \times I_2 \times \ldots \times I_N}$ be a conjugate of tensor $A$, where $\overline{a}_{i_1 i_2 \ldots i_N}$ is the complex conjugate of $a_{i_1 i_2 \ldots i_N}$.

In this paper, we consider the iterative algorithm to solve the following complex generalized Sylvester tensor equation

$$
X \times_1 A_1 + \cdots + X \times_N A_N + \overline{X} \times_1 B_1 + \cdots + \overline{X} \times_N B_N = D,
$$

where the matrices $A_n, B_n \in \mathbb{C}^{I_n \times I_n}$ ($n = 1, 2, \ldots, N$) and the right-side tensor $D \in \mathbb{C}^{I_1 \times I_2 \times \ldots \times I_N}$ are known, and $X \in \mathbb{C}^{I_1 \times I_2 \times \ldots \times I_N}$ is the unknown tensor to be solved. For the tensor $X \in \mathbb{C}^{I_1 \times I_2 \times \ldots \times I_N}$ and the matrix $U \in \mathbb{C}^{I_n \times I_n}$, $X \times_n U \in \mathbb{C}^{I_1 \times I_2 \times \ldots \times I_N}$ is defined by

$$
(X \times_n U)_{i_1 \ldots i_n-1 j_{n+1} \ldots i_N} = \sum_{i_n=1}^{I_n} x_{i_1 \ldots i_{n-1} i_n} u_{i_n j_{n+1} \ldots i_N}.
$$

Email address: keyifen@fjnu.edu.cn

College of Mathematics and Informatics & FJKLMAA, Fujian Normal University, Fuzhou 350117, China

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When the complex generalized tensor equation (1.1) over the real field $\mathbb{R}$ and $B_n = 0$ for all $n = 1, 2, \ldots, N$, the tensor equation (1.1) reduces to the real Sylvester tensor equation

$$\mathcal{X} \times_1 A_1 + \mathcal{X} \times_2 A_2 + \cdots + \mathcal{X} \times_N A_N = D. \quad (1.2)$$

By using the Kronecker product $[1,13]$, the tensor equation (1.2) can be reformulated as follows:

$$Ax = b \quad (1.3)$$

with

$$A = I^{(I_N)} \otimes \cdots \otimes I^{(I_2)} \otimes A_1 + \cdots + A_N \otimes I^{(I_{N-1})} \otimes \cdots \otimes I^{(I_1)}, \quad (1.4)$$

and $x = \text{vec}(\mathcal{X})$, $b = \text{vec}(D)$. Here, $I^{(n)}$ stands for the identity matrix of order $n$ and the operator $\text{vec}(\cdot)$ stacks the columns of a tensor to form a vector. In the case that $\mathcal{X}$ is a 3-order tensor, the tensor equation (1.2) reduces to

$$\mathcal{X} \times_1 A + \mathcal{X} \times_2 B + \mathcal{X} \times_3 C = D, \quad (1.5)$$

which comes from the finite element [7] or spectral method [14,16–18] discretization of a linear partial differential equation in high dimension.

When $\mathcal{X}$ is a simple 2-order tensor, the complex generalized tensor equation (1.1) reduces to the extended Sylvester-conjugate matrix equation

$$A_1 X + X A_2^T + B_1 \mathcal{X} + \mathcal{X} B_2^T = D \quad (1.6)$$

and the Sylvester tensor equation (1.2) reduces to the Sylvester matrix equation

$$A_1 X + X A_2^T = D, \quad (1.7)$$

which arises frequently from the areas of systems and control theory [6] and has received much attention, e.g., see [5,8–10].

Nowadays, tensor equations have attracted much attention [1–4,7,11,13,15]. For example, Kressner and Tobler in [13] proposed a so-called tensor Krylov subspace method for solving (1.3) with the coefficient matrix $A$ as (1.4) and the right-hand side $b = b_1 \otimes b_2 \otimes \cdots \otimes b_N$, which is based on a tensorized Krylov subspace associated with $A$ and $b$, i.e., the Kronecker product of the usual Krylov subspaces. The tensorized Krylov subspace method transforms the system (1.3) to a system of smaller size. In [1], Ballani and Grasedyck presented an iterative scheme similarly to Krylov subspace method to solve the linear system (1.3) with

$$A = \sum_{j=1}^{k_A} \otimes_{n=1}^N A_{jn}, \quad A_{jn} \in \mathbb{R}^{I_n \times I_n}$$

and

$$b = \sum_{j=1}^{k_b} \otimes_{n=1}^N b_{jn}, \quad b_{jn} \in \mathbb{R}^{I_n},$$

which is based on the projection of the residual to a low dimensional subspace and all calculations are performed in hierarchical Tucker format. In particular, Beik, Movahed and Ahmadi-Asl in [2] proposed the conjugate gradient algorithm of tensor form (CG-BTF) to solving the Sylvester tensor equation (1.2) with $A_n \in$
\[ \mathbb{R}^{l_n \times l_n} (n = 1, 2, \ldots, N) \] being symmetric positive definite and the nested conjugate gradient algorithm of tensor form (NCG-BTF) for the Sylvester tensor equation \((1.2)\) with \(A_i \in \mathbb{R}^{l_n \times l_n} (n = 1, 2, \ldots, N) \) being nonsymmetric positive definite.

As a general form of the matrix equation \((1.6)\), that is

\[
\sum_{i_1=1}^{p} A_{i_1} X B_{i_1} + \sum_{i_2=1}^{q} C_{i_2} X D_{i_2} = F, \tag{1.8}
\]

where \(A_{i_1}, C_{i_2} \in \mathbb{C}^{m \times r}, B_{i_2}, D_{i_2} \in \mathbb{C}^{s \times n}(i_1 = 1, 2, \ldots, p; i_2 = 1, 2, \ldots, q), F \in \mathbb{C}^{r \times s} \) are the given known matrices and \(X \in \mathbb{C}^{r \times s} \) is the matrix to be determined. Wu, Lv and Hou [19] presented an iterative algorithm for solving \((1.8)\). After that, Zhang in [21] offered the finite iterative algorithm for solving the complex generalized coupled Sylvester matrix equations, that is

\[
\begin{aligned}
\sum_{j=1}^{q} (A_{i_1} X_{j} B_{i_1} + C_{i_2} X_{j} D_{i_2} + E_{i_1} X_{j} F_{i_2} + G_{i_2} X_{j} H_{i_2}) &= M_1, \\
\sum_{j=1}^{q} (A_{i_2} X_{j} B_{i_2} + C_{i_2} X_{j} D_{i_2} + E_{i_2} X_{j} F_{i_2} + G_{i_2} X_{j} H_{i_2}) &= M_2, \\
& \vdots \\
\sum_{j=1}^{q} (A_{p_j} X_{j} B_{p_j} + C_{p_j} X_{j} D_{p_j} + E_{i_1} X_{j} F_{p_j} + G_{p_j} X_{j} H_{p_j}) &= M_p,
\end{aligned} \tag{1.9}
\]

where \(A_{i_1}, E_{i_1} \in \mathbb{C}^{m_i \times r_j}, B_{i_2}, F_{i_2} \in \mathbb{C}^{s_j \times n_i}, C_{i_2}, G_{i_2} \in \mathbb{C}^{m_i \times s_j}, D_{i_2}, H_{i_2} \in \mathbb{C}^{r_j \times n_i}, M_i \in \mathbb{C}^{n_i \times n_i} (i = 1, 2, \ldots, p; j = 1, 2, \ldots, q) \) are the given known matrices and \(X_{j} \in \mathbb{C}^{r_j \times s_j} (j = 1, 2, \ldots, q) \) are the unknown matrices. In this paper, the finite iterative algorithm for solving the complex generalized Sylvester tensor equation \((1.1)\) will be discussed.

The remainder of this paper is organized as follows. In Section 2, we offers some symbols and preliminaries. In Section 3, we present a finite iterative algorithm for solving the complex generalized Sylvester tensor equation \((1.1)\) and constructs several results which be used in the convergence proof. Numerical examples are offered in Section 4 to illustrate the effectiveness of the proposed algorithm. Finally, some concluding remarks are given in Section 5.

### 2. Preliminaries

For a matrix \(A = (a_{ij}) \in \mathbb{C}^{l \times l}\), the symbols \(A^T\) and \(A^H\) denote the transpose and conjugate transpose of matrix \(A\), that is

\[
A^T = \begin{pmatrix}
  a_{11} & a_{21} & \cdots & a_{l1} \\
  a_{12} & a_{22} & \cdots & a_{l2} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{1l} & a_{2l} & \cdots & a_{ll}
\end{pmatrix}
\quad \text{and} \quad
A^H = \begin{pmatrix}
  \bar{a}_{11} & \bar{a}_{21} & \cdots & \bar{a}_{l1} \\
  \bar{a}_{12} & \bar{a}_{22} & \cdots & \bar{a}_{l2} \\
  \vdots & \vdots & \ddots & \vdots \\
  \bar{a}_{1l} & \bar{a}_{2l} & \cdots & \bar{a}_{ll}
\end{pmatrix}.
\]

Now, we recall the real inner product, which is given for complex spaces over the real field \(\mathbb{R}\). This inner product will play a very important role in this paper.
Definition 2.1 (\[20\]). A real inner product space is a vector space $V$ over the real field $\mathbb{R}$ together with an inner product, i.e., with a mapping

$$\langle \cdot, \cdot \rangle_r : V \times V \to \mathbb{R}$$

satisfying the following three axioms for all vectors $x, y, z \in V$ and all scalars $\eta \in \mathbb{R}$.

1. Symmetry: $\langle x, y \rangle_r = \langle y, x \rangle_r$.
2. Linearity in the first argument:
   $$\langle \eta x, y \rangle_r = \eta \langle x, y \rangle_r; \quad \langle x + y, z \rangle_r = \langle x, z \rangle_r + \langle y, z \rangle_r.$$
3. Positive-definiteness: $\langle x, x \rangle_r > 0$ for all $x \neq 0$.

Moreover, two vectors $x, y \in V$ are said to be orthogonal if $\langle y, x \rangle_r = 0$.

For the tensors $X, Y \in \mathbb{C}^{I_1 \times I_2 \times \cdots \times I_N}$, the inner product is defined as

$$\langle X, Y \rangle := \sum_{i_1, i_2, \ldots, i_N} x_{i_1i_2\cdots i_N} y_{i_1i_2\cdots i_N}$$

and a real inner product can be defined as

$$\langle X, Y \rangle_r := \text{Re}(\langle X, Y \rangle),$$

where $\text{Re}(\lambda)$ denotes the real part of the complex number $\lambda$. In fact, for the tensors $X, Y, Z \in \mathbb{C}^{I_1 \times I_2 \times \cdots \times I_N}$, it is easy to verify that

1. $\langle \eta X, Y \rangle_r = \eta \langle X, Y \rangle_r$ for any $\eta \in \mathbb{R}$;
2. $\langle X + Y, Z \rangle_r = \langle X, Z \rangle_r + \langle Y, Z \rangle_r$;
3. $\langle X, X \rangle_r > 0$ for all $X \neq 0$, where $0$ is the zero tensor in $\mathbb{C}^{I_1 \times I_2 \times \cdots \times I_N}$, i.e., all entries of $0$ are 0.

The induced-norm of a tensor $X$ is defined by the following formula

$$\|X\| := \sqrt{\langle X, X \rangle} = \sqrt{\langle X, X \rangle_r}. $$

For the linear operator $L$ from $\mathbb{C}^{I_1 \times I_2 \times \cdots \times I_N}$ to $\mathbb{C}^{I_1 \times I_2 \times \cdots \times I_N}$, the conjugate linear operator $L^*$ from $\mathbb{C}^{I_1 \times I_2 \times \cdots \times I_N}$ to $\mathbb{C}^{I_1 \times I_2 \times \cdots \times I_N}$ satisfies

$$\langle L(X), Y \rangle_r = \langle X, L^*(Y) \rangle_r$$

for any $X$ and $Y$ in $\mathbb{C}^{I_1 \times I_2 \times \cdots \times I_N}$.

Lemma 2.1. For the tensors $X, Y \in \mathbb{C}^{I_1 \times I_2 \times \cdots \times I_N}$ and $U \in \mathbb{C}^{I_n \times I_n}$, let

$$L_1(X) = X \times_n U \quad \text{and} \quad L_2(X) = X \times_n U,$$

according to the conjugate operator, we have

$$L_1^*(Y) = Y \times_n U^T \quad \text{and} \quad L_2^*(Y) = \overline{Y} \times_n U^T.$$
Proof. By the computation, we have

\[ \langle L_1(\mathcal{X}), \mathcal{Y} \rangle_r = \langle \mathcal{X} \times_n U, \mathcal{Y} \rangle_r \]

\[ = \text{Re} \left( \sum_{i_1, \ldots, i_{n-1}, j, i_{n+1}, \ldots, i_N} \left( \sum_{i_n=1}^{I_n} x_{i_1 \cdots i_{n-1} i_n j i_{n+1} \cdots i_N} y_{i_1 \cdots i_{n-1} j i_{n+1} \cdots i_N} u_{j i_n} \right) \right) \]

\[ = \text{Re} \left( \sum_{i_1, \ldots, i_{n-1}, j, i_{n+1}, \ldots, i_N} x_{i_1 \cdots i_{n-1} i_n j i_{n+1} \cdots i_N} \left( \sum_{j=1}^{J_n} y_{i_1 \cdots i_{n-1} j i_{n+1} \cdots i_N} u_{j i_n} \right) \right) \]

\[ = \text{Re} \left( \sum_{i_1, \ldots, i_{n-1}, j, i_{n+1}, \ldots, i_N} \left( \sum_{j=1}^{J_n} y_{i_1 \cdots i_{n-1} j i_{n+1} \cdots i_N} u_{j i_n} \right) x_{i_1 \cdots i_{n-1} i_n j i_{n+1} \cdots i_N} \right) \]

\[ = \langle \mathcal{X}, \mathcal{Y} \times_n U^T \rangle_r = \langle \mathcal{X}, L_1^* (\mathcal{Y}) \rangle_r \]

and

\[ \langle L_2(\mathcal{X}), \mathcal{Y} \rangle_r = \langle \mathcal{X} \times_n U, \mathcal{Y} \rangle_r \]

\[ = \text{Re} \left( \sum_{i_1, \ldots, i_{n-1}, j, i_{n+1}, \ldots, i_N} \left( \sum_{i_n=1}^{I_n} x_{i_1 \cdots i_{n-1} i_n j i_{n+1} \cdots i_N} y_{i_1 \cdots i_{n-1} j i_{n+1} \cdots i_N} u_{j i_n} \right) \right) \]

\[ = \text{Re} \left( \sum_{i_1, \ldots, i_{n-1}, j, i_{n+1}, \ldots, i_N} x_{i_1 \cdots i_{n-1} i_n j i_{n+1} \cdots i_N} \left( \sum_{j=1}^{J_n} y_{i_1 \cdots i_{n-1} j i_{n+1} \cdots i_N} u_{j i_n} \right) \right) \]

\[ = \text{Re} \left( \sum_{i_1, \ldots, i_{n-1}, j, i_{n+1}, \ldots, i_N} \left( \sum_{j=1}^{J_n} y_{i_1 \cdots i_{n-1} j i_{n+1} \cdots i_N} u_{j i_n} \right) x_{i_1 \cdots i_{n-1} i_n j i_{n+1} \cdots i_N} \right) \]

\[ = \langle \mathcal{X}, \mathcal{Y} \times_n U^T \rangle_r = \langle \mathcal{X}, L_2^* (\mathcal{Y}) \rangle_r. \]

This completes the proof. \( \Box \)

For simplicity, in the sequel, we use the following linear operator:

\[ L(\mathcal{X}) := \mathcal{X} \times_1 A_1 + \cdots + \mathcal{X} \times_N A_N + \overline{\mathcal{X}} \times_1 B_1 + \cdots + \overline{\mathcal{X}} \times_N B_N. \]  \hspace{1cm} (2.1)

Then, the complex generalized Sylvester tensor equation (1.1) can be reformulated by

\[ L(\mathcal{X}) = \mathcal{D}. \]  \hspace{1cm} (2.2)

According to Lemma 2.1, it is not difficult to verify that \( L^* \) is specified as follows:

\[ L^*(\mathcal{Z}) = \mathcal{Z} \times_1 A_1^H + \cdots + \mathcal{Z} \times_N A_N^H + \overline{\mathcal{Z}} \times_1 B_1^T + \cdots + \overline{\mathcal{Z}} \times_N B_N^T. \]  \hspace{1cm} (2.3)

With these results, we will present a finite iterative algorithm to solve the complex generalized Sylvester tensor equation (1.1) in the following section.

3. Algorithm and convergence

In this section, we will propose a finite iterative algorithm for solving the complex generalized Sylvester tensor equation (1.1) and establish the convergence result for the proposed iterative algorithm.
Algorithm 3.1. (Finite iterative algorithm for solving (1.1))

Step 1. Let the matrices $A_n, B_n \in \mathbb{C}^{I_1 \times I_n}$ for $n = 1, 2, \ldots, N$ and the right-hand side $D$. Choose any initial tensor $X_0 \in \mathbb{C}^{I_1 \times I_2 \times \ldots \times I_N}$, compute

\begin{align*}
R_0 &= D - L(X_0), \\
Q_0 &= L^*(R_0).
\end{align*}

Set $k := 0$.

Step 2. If $\|R_k\| = 0$ or $\|Q_k\| = 0$, then stop; else compute

\begin{align*}
\alpha_k &= \frac{\|Q_k\|^2}{\|R_k\|^2}, \\
X_{k+1} &= X_k + \alpha_k Q_k, \\
R_{k+1} &= D - L(X_{k+1}) = R_k - \alpha_k L(Q_k), \\
Q_{k+1} &= L^*(R_{k+1}) + \frac{\|R_{k+1}\|^2}{\|Q_k\|^2} Q_k.
\end{align*}

Step 3. Set $k := k + 1$, go to Step 2.

Now, we give some results for Algorithm 3.1.

Theorem 3.1. According to Algorithm 3.1, the tensor sequences $R_k$ and $Q_k$ for $k = 0, 1, 2, \ldots$ satisfy

\begin{align}
\langle R_k, R_l \rangle_r &= 0, \quad k, l = 0, 1, 2, \ldots, k \neq l, \\
\langle Q_k, Q_l \rangle_r &= 0, \quad k, l = 0, 1, 2, \ldots, k \neq l.
\end{align}

Proof. Using the mathematical induction. Firstly, we prove that

\begin{align*}
\langle R_0, R_1 \rangle_r &= 0 \quad \text{and} \quad \langle Q_0, Q_1 \rangle_r = 0.
\end{align*}

In fact, we have

\begin{align*}
\langle R_0, R_1 \rangle_r &= \langle R_0, R_0 - \alpha_0 L(Q_0) \rangle_r \\
&= \langle R_0, R_0 \rangle_r - \langle R_0, \alpha_0 L(Q_0) \rangle_r \\
&= \langle R_0, R_0 \rangle_r - \alpha_0 \langle L(Q_0), R_0 \rangle_r \\
&= \langle R_0, R_0 \rangle_r - \alpha_0 \langle Q_0, L^*(R_0) \rangle_r \\
&= \langle R_0, R_0 \rangle_r - \frac{\|R_0\|^2}{\|Q_0\|^2} \langle Q_0, Q_0 \rangle_r = 0
\end{align*}

and

\begin{align*}
\langle Q_0, Q_1 \rangle_r &= \langle Q_0, L^*(R_1) \rangle_r + \frac{\|R_1\|^2}{\|R_0\|^2} \langle Q_0, R_0 \rangle_r \\
&= \langle Q_0, L^*(R_1) \rangle_r + \frac{\|R_1\|^2}{\|R_0\|^2} \langle Q_0, Q_0 \rangle_r \\
&= \langle L(Q_0), R_1 \rangle_r + \frac{\|R_1\|^2}{\|R_0\|^2} \langle Q_0, Q_0 \rangle_r \\
&= \langle R_0 - R_1, R_1 \rangle_r + \frac{\|R_1\|^2}{\|R_0\|^2} \langle Q_0, Q_0 \rangle_r \\
&= \frac{1}{\alpha_0} \langle R_0, R_1 \rangle_r - \frac{1}{\alpha_0} \langle R_1, R_1 \rangle_r + \frac{\|R_1\|^2}{\|R_0\|^2} \langle Q_0, Q_0 \rangle_r \\
&= \frac{\|Q_0\|^2}{\|R_0\|^2} \langle R_1, R_1 \rangle_r + \frac{\|R_1\|^2}{\|R_0\|^2} \langle Q_0, Q_0 \rangle_r = 0.
\end{align*}
Now, we assume that the results of Theorem 3.1 are correct for \(k, l \leq s(s > 1)\). For \(l = s + 1\) and \(k = s\), we have

\[
\langle R_s, R_{s+1} \rangle_r = \langle R_s, R_s - \alpha_s L(Q_s) \rangle_r = \langle R_s, R_s \rangle_r - \alpha_s \langle L(Q_s), R_s \rangle_r = \langle R_s, R_s \rangle_r - \alpha_s \langle Q_s, L^*(R_s) \rangle_r
\]

\[
= \langle R_s, R_s \rangle_r - \alpha_s \langle Q_s, Q_s - \frac{\|R_s\|^2}{\|R_{s-1}\|^2} Q_{s-1} \rangle_r
\]

\[
= \langle R_s, R_s \rangle_r - \alpha_s \langle Q_s, Q_s \rangle_r + \frac{\alpha_s \|R_s\|^2}{\|R_{s-1}\|^2} \langle Q_s, Q_{s-1} \rangle_r
\]

\[
= \langle R_s, R_s \rangle_r - \frac{\|R_s\|^2}{\|Q_s\|^2} \langle Q_s, Q_s \rangle_r = 0
\]

and

\[
\langle Q_s, Q_{s+1} \rangle_r = \langle Q_s, L^*(R_{s+1}) + \frac{\|R_{s+1}\|^2}{\|R_s\|^2} Q_s \rangle_r
\]

\[
= \langle Q_s, L^*(R_{s+1}) \rangle_r + \frac{\|R_{s+1}\|^2}{\|R_s\|^2} \langle Q_s, Q_s \rangle_r
\]

\[
= \langle L(Q_s), R_{s+1} \rangle_r + \frac{\|R_{s+1}\|^2}{\|R_s\|^2} \langle Q_s, Q_s \rangle_r
\]

\[
= \left( \frac{R_s - R_{s+1}}{\alpha_s} \right)_{s+1} + \frac{\|R_{s+1}\|^2}{\|R_s\|^2} \langle Q_s, Q_s \rangle_r
\]

\[
= \frac{1}{\alpha_s} \langle R_s, R_{s+1} \rangle_r - \frac{1}{\alpha_s} \langle R_{s+1}, R_{s+1} \rangle_r + \frac{\|R_{s+1}\|^2}{\|R_s\|^2} \langle Q_s, Q_s \rangle_r
\]

\[
= -\frac{\|Q_s\|^2}{\|R_s\|^2} \langle R_{s+1}, R_{s+1} \rangle_r + \frac{\|R_{s+1}\|^2}{\|R_s\|^2} \langle Q_s, Q_s \rangle_r
\]

\[
= 0.
\]

Finally, we prove that the results of Theorem 3.1 are correct for \(l = s + 1\) and \(k \leq s - 1\). In fact, we have

\[
\langle R_k, R_{s+1} \rangle_r = \langle R_k, R_s - \alpha_s L(Q_s) \rangle_r
\]

\[
= \langle R_k, R_s \rangle_r - \alpha_s \langle L(Q_s), R_k \rangle_r
\]

\[
= -\alpha_s \langle Q_s, L^*(R_k) \rangle_r
\]

\[
= -\alpha_s \langle Q_s, Q_k - \frac{\|R_k\|^2}{\|R_{k-1}\|^2} Q_{k-1} \rangle_r
\]

\[
= -\alpha_s \langle Q_s, Q_k \rangle_r + \frac{\alpha_s \|R_k\|^2}{\|R_{k-1}\|^2} \langle Q_s, Q_{k-1} \rangle_r = 0
\]

and

\[
\langle Q_k, Q_{s+1} \rangle_r = \langle Q_k, L^*(R_{s+1}) + \frac{\|R_{s+1}\|^2}{\|R_s\|^2} Q_s \rangle_r
\]

\[
= \langle Q_k, L^*(R_{s+1}) \rangle_r + \frac{\|R_{s+1}\|^2}{\|R_s\|^2} \langle Q_k, Q_s \rangle_r
\]
If the tensor $\mathcal{X}_0$ is a solution of the complex generalized Sylvester tensor equation (1.1), for any initial iterative tensor $\mathcal{X}_0 \in \mathbb{C}^{I_1 \times I_2 \times \cdots \times I_N}$, the tensor sequences $\mathcal{R}_k$ and $\mathcal{Q}_k$ generated by Algorithm 3.1 satisfy

$$\langle \mathcal{Q}_k, \mathcal{X}_s - \mathcal{X}_k \rangle_r = \| \mathcal{R}_k \|^2, \quad k = 0, 1, 2, \ldots$$  \hspace{1cm} (3.3)

**Proof.** Using the mathematical induction. Firstly, we verify that

$$\langle \mathcal{Q}_0, \mathcal{X}_s - \mathcal{X}_0 \rangle_r = \| \mathcal{R}_0 \|^2.$$  

In fact, we have

$$\langle \mathcal{Q}_0, \mathcal{X}_s - \mathcal{X}_0 \rangle_r = (\mathcal{L}^*(\mathcal{R}_0), \mathcal{X}_s - \mathcal{X}_0)_r = (\mathcal{R}_0, \mathcal{L}(\mathcal{X}_s - \mathcal{X}_0))_r = (\mathcal{R}_0, \mathcal{L}(\mathcal{X}_s - \mathcal{X}_0))_r = (\mathcal{R}_0, \mathcal{D} - \mathcal{L}(\mathcal{X}_0))_r = \langle \mathcal{R}_0, \mathcal{R}_0 \rangle_r = \| \mathcal{R}_0 \|^2.$$

Now, we assume that the result of Theorem 3.2 is correct for $k = s(s \geq 1)$. For $k = s + 1$, we have

$$\langle \mathcal{Q}_{s+1}, \mathcal{X}_s - \mathcal{X}_{s+1} \rangle_r = \langle \mathcal{Q}_{s+1}, \mathcal{X}_s - \mathcal{X}_{s+1} \rangle_r,$$

$$= (\mathcal{L}^*(\mathcal{R}_{s+1}) + \frac{\| \mathcal{R}_{s+1} \|^2}{\| \mathcal{R}_{s} \|^2} \mathcal{Q}_s, \mathcal{X}_s - \mathcal{X}_{s+1})_r$$

$$= (\mathcal{L}^*(\mathcal{R}_{s+1}), \mathcal{X}_s - \mathcal{X}_{s+1})_r + \frac{\| \mathcal{R}_{s+1} \|^2}{\| \mathcal{R}_{s} \|^2} \langle \mathcal{Q}_s, \mathcal{X}_s - \mathcal{X}_{s+1} \rangle_r$$

$$= (\mathcal{R}_{s+1}, \mathcal{L}(\mathcal{X}_s - \mathcal{X}_{s+1}))_r + \frac{\| \mathcal{R}_{s+1} \|^2}{\| \mathcal{R}_{s} \|^2} \langle \mathcal{Q}_s, \mathcal{X}_s - \mathcal{X}_{s+1} \rangle_r$$

$$= (\mathcal{R}_{s+1}, \mathcal{D} - \mathcal{L}(\mathcal{X}_{s+1}))_r + \frac{\| \mathcal{R}_{s+1} \|^2}{\| \mathcal{R}_{s} \|^2} \langle \mathcal{Q}_s, \mathcal{X}_s - \mathcal{X}_{s+1} \rangle_r - \frac{\| \mathcal{R}_{s+1} \|^2}{\| \mathcal{R}_{s} \|^2} \langle \mathcal{Q}_s, \mathcal{Q}_s \rangle_r$$

$$= (\mathcal{R}_{s+1}, \mathcal{R}_{s+1})_r + \frac{\| \mathcal{R}_{s+1} \|^2}{\| \mathcal{R}_{s} \|^2} \langle \mathcal{Q}_s, \mathcal{Q}_s \rangle_r$$

According to the mathematical induction, the proof is completed. \hfill \Box

**Theorem 3.3.** If the complex generalized Sylvester tensor equation (1.1) is consistent, then a solution can be obtained within finite iteration steps by using Algorithm 3.1 for any initial tensor $\mathcal{X}_0 \in \mathbb{C}^{I_1 \times I_2 \times \cdots \times I_N}$. 

With Theorems 3.1 and 3.2, we can obtain the following conclusion.
At the end of this section, we present the NCG-BTF algorithm [2] for the Sylvester tensor equation (1.2) with $A_n \in \mathbb{R}^{I_n \times I_n}$ ($n = 1, 2, \ldots, N$) being real nonsymmetric positive definite. Let

$$H_n = \frac{A_n + A_n^T}{2} \quad \text{and} \quad S_n = \frac{A_n^T - A_n}{2}$$

(3.4)

for $n = 1, 2, \ldots, N$. Substituting $A_n = H_n - S_n$ into (1.2), we have

$$X \times_1 H_1 + X \times_2 H_2 + \cdots + X \times_N H_N = D + X \times_1 S_1 + X \times_2 S_2 + \cdots + X \times_N S_N.$$ 

We compute $X_{k+1}$ as the solution of the following tensor equation:

$$X \times_1 H_1 + X \times_2 H_2 + \cdots + X \times_N H_N = D + X_k \times_1 S_1 + X_k \times_2 S_2 + \cdots + X_k \times_N S_N,$$

where $X_k$ is the $k$-th approximate solution to the exact solution $X_\ast$ of (1.2). Notice that for the tensors $X, Y \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N}$, the inner product is reduced to

$$\langle X, Y \rangle := \sum_{i_1, i_2, \ldots, i_N} x_{i_1 i_2 \cdots i_N} y_{i_1 i_2 \cdots i_N}.$$ 

Then, the NCG-BTF algorithm for solving (1.2) is as follows.

**Algorithm 3.2.** (NCG_BTF algorithm for solving (1.2) [2])

**Step 1.** Let the matrices $A_n, B_n \in \mathbb{R}^{I_n \times I_n}$ for $n = 1, 2, \ldots, N$ and the right-hand side $D \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N}$. Given the tolerance $\sigma$. Choose any initial tensor $X_0 \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N}$, compute

$$R_0 = D - X_0 \times_1 A_1 - X_0 \times_2 A_2 - \cdots - X_0 \times_N A_N.$$ 

Set $X_{(0,0)} = X_0$ and $k := 0$.

**Step 2.** If $\|R_k\| = 0$, then stop; else compute

$$\bar{D} = D + X_{(k,0)} \times_1 S_1 + X_{(k,0)} \times_2 S_2 + \cdots + X_{(k,0)} \times_N S_N,$$

$$R_{(k,0)} = \bar{D} - X_{(k,0)} \times_1 H_1 - X_{(k,0)} \times_2 H_2 - \cdots - X_{(k,0)} \times_N H_N,$$

$$P_0 = R_{(k,0)},$$

for $\ell = 0, 1, 2, \ldots,$ do

$$W = P_{\ell} \times_1 H_{\ell+1} + P_{\ell} \times_2 H_{\ell+1} + \cdots + P_{\ell} \times_N H_{\ell+1},$$

$$\alpha_\ell = \frac{\langle R_{(k,\ell)}, R_{(k,\ell)} \rangle}{\langle W, P_{\ell} \rangle},$$

$$X_{(k,\ell+1)} = X_{(k,\ell)} + \alpha_\ell P_{\ell},$$

$$R_{(k,\ell+1)} = R_{(k,\ell)} - \alpha_\ell W,$$

if $\|R_{(k,\ell+1)}\| \leq \sigma \|R_{(k,0)}\|$, then compute

$$X_{k+1} = X_{(k,\ell+1)},$$

$$R_{k+1} = D - X_{k+1} \times_1 A_1 - X_{k+1} \times_2 A_2 - \cdots - X_{k+1} \times_N A_N,$$

else compute

$$\beta_\ell = \frac{\|R_{(k,\ell+1)}\|^2}{\|R_{(k,\ell)}\|^2},$$

$$P_{\ell+1} = R_{(k,\ell+1)} + \beta_\ell P_{\ell},$$

end if.

**Step 3.** Set $k := k + 1$ and $X_{(k,0)} = X_{k+1}$, go to Step 2.
4. Numerical examples

In this section, we use some test problems to examine the numerical effectiveness of Algorithm 3.1 for solving the complex generalized Sylvester tensor equation (1.1).

In actual computations, all runs are started from the initial tensor \( X_0 = 0 \), are terminated if the current iterations satisfy

(i) \( \| R_k \| \leq 10^{-6} \) or \( \| Q_k \| \leq 10^{-6} \) for Algorithm 3.1,

(ii) \( \| R_k \| \leq 10^{-6} \) for Algorithm 3.2,

or if the number of the prescribed iterative steps \( k_{\text{max}} = 4000 \) is exceeded, and are performed on a personal computer with 3.60 GHz central processing unit (Intel(R) Core(TM) i7-7700), 32.0 GB memory and Windows 10 operating system.

Here, ‘IT’ and ‘CPU’ denote the number of iteration steps and elapsed CPU time in seconds, respectively. In addition, \( \text{ERR} := \| R_k \| \). We comment here that the Tensor Toolbox [12] is utilized for solving the succeeding discussed problems.

Example 4.1. Consider the following convection-diffusion equation

\[
-\nu \Delta u + c^T \nabla u = f \quad \text{in} \quad \Omega = [0, 1] \times [0, 1] \times [0, 1],
\]

\[
u = 0, \quad \text{on} \quad \partial \Omega.
\]

By using a standard finite difference discretization on a uniform grid for the diffusion term and a second-order convergent scheme (Fromm’s scheme) for the convection term with the mesh-size \( h = 1/(p+1) \), the discrete system matrix of the form (1.2) is obtained such that

\[
A_n = \frac{\nu}{h^2} \begin{pmatrix}
2 & -1 \\
-1 & 2 & -1 \\
& & \ddots & \ddots & \ddots \\
-1 & 2 & -1 \\
-1 & 2 & -1
\end{pmatrix}_{p \times p} + \frac{c_n}{4h} \begin{pmatrix}
3 & -5 & 1 \\
1 & 3 & -5 & \ddots \\
& & \ddots & \ddots & \ddots \\
1 & 3 & -5 \\
1 & 3
\end{pmatrix}_{p \times p}, \quad n = 1, 2, 3.
\]

And the right-hand side \( D \) is constructed so that \( X_* = \text{ones}(p, p, p) \) is the exact solution of the real Sylvester tensor equation (1.2).

For Example 4.1, we compare Algorithms 3.1 and 3.2. The inner iteration of Algorithm 3.2 is terminated if the value \( \sigma = 10^{-3} \) or the number of the prescribed inner iterative steps \( \ell_{\text{max}} = 100 \). The notation ‘—’ denotes the case that the algorithm is stopped after \( k_{\text{max}} \) iterations without computing a suitable approximate solution.

The obtained results are presented in Table 1 with \( c_1 = c_2 = c_3 = 1 \) and Table 2 with \( c_1 = 2, c_2 = 4 \) and \( c_3 = 8 \), respectively. From both tables, we can see that the NCG BTF algorithm is stopped after \( k_{\text{max}} \) iterations while the proposed finite iterative algorithm performs well for the viscous parameter \( \nu = 0.01, 0.1 \).

Notice that when \( \nu = 1 \) and \( c_1 = c_2 = c_3 = 1 \), the NCG BTF algorithm surpasses the finite iterative algorithm in CPU time. However, the finite iterative algorithm outperforms the NCG BTF algorithm in terms of the required CPU time when \( \nu = 1, c_1 = 2, c_2 = 4 \) and \( c_3 = 8 \).
Table 1. Numerical results for Example 4.1 with $c_1 = c_2 = c_3 = 1$  

| Algorithm | $p$ | $10$ | $20$ | $30$ | $40$ |
|-----------|-----|-----|-----|-----|-----|
| $\nu = 0.01$ | Algorithm 3.1 | IT | 110 | 342 | 642 | 994 |
| | | CPU | 0.1802 | 0.8596 | 2.2244 | 5.4945 |
| | ERR | 8.3334e-05 | 9.8194e-05 | 9.9630e-05 | 9.6617e-05 |
| | $\|X_k - X^*\|$ | 3.5909e-06 | 2.8993e-06 | 2.333e-06 | 1.7522e-06 |
| | Algorithm 3.2 | IT | — | — | — | — |
| | CPU | — | — | — | — |
| | ERR | — | — | — | — |
| $\nu = 0.1$ | Algorithm 3.1 | IT | 119 | 429 | 934 | 1621 |
| | | CPU | 0.2086 | 1.0964 | 3.2363 | 8.5487 |
| | ERR | 8.3334e-05 | 9.8194e-05 | 9.9630e-05 | 9.6617e-05 |
| | $\|X_k - X^*\|$ | 3.5909e-06 | 2.8993e-06 | 2.333e-06 | 1.7522e-06 |
| | Algorithm 3.2 | IT | — | — | — | — |
| | CPU | — | — | — | — |
| | ERR | — | — | — | — |
| $\nu = 1$ | Algorithm 3.1 | IT | 118 | 458 | 1026 | 1823 |
| | | CPU | 0.2821 | 1.0776 | 2.4698 | 5.8402 |
| | ERR | 9.4749e-05 | 9.7837e-05 | 9.5953e-05 | 9.9879e-05 |
| | $\|X_k - X^*\|$ | 1.9364e-06 | 8.4346e-07 | 9.1760e-07 | 7.1202e-07 |
| | Algorithm 3.2 | IT | — | — | — | — |
| | CPU | — | — | — | — |
| | ERR | — | — | — | — |

Table 2. Numerical results for Example 4.1 with $c_1 = 2$, $c_2 = 4$, $c_3 = 8$  

| Algorithm | $p$ | $10$ | $20$ | $30$ | $40$ |
|-----------|-----|-----|-----|-----|-----|
| $\nu = 0.01$ | Algorithm 3.1 | IT | 167 | 612 | 670 | 963 |
| | | CPU | 0.2821 | 1.0776 | 2.4698 | 5.8402 |
| | ERR | 9.4749e-05 | 9.7837e-05 | 9.5953e-05 | 9.9879e-05 |
| | $\|X_k - X^*\|$ | 1.9364e-06 | 8.4346e-07 | 9.1760e-07 | 7.1202e-07 |
| | Algorithm 3.2 | IT | — | — | — | — |
| | CPU | — | — | — | — |
| | ERR | — | — | — | — |
| $\nu = 0.1$ | Algorithm 3.1 | IT | 185 | 547 | 1048 | 1604 |
| | | CPU | 0.3216 | 1.4728 | 3.5380 | 9.4980 |
| | ERR | 9.3391e-05 | 1.8217e-05 | 2.1200e-05 | 2.3934e-05 |
| | $\|X_k - X^*\|$ | 7.7475e-07 | 4.7910e-07 | 4.0709e-07 | 3.3172e-07 |
| | Algorithm 3.2 | IT | — | — | — | — |
| | CPU | — | — | — | — |
| | ERR | — | — | — | — |
| $\nu = 1$ | Algorithm 3.1 | IT | 211 | 767 | 1707 | 3018 |
| | | CPU | 0.3656 | 1.9861 | 5.8772 | 16.2717 |
| | ERR | 9.6083e-05 | 9.7733e-05 | 9.9412e-05 | 9.9650e-05 |
| | $\|X_k - X^*\|$ | 1.9164e-06 | 1.4820e-07 | 9.6294e-08 | 1.7522e-06 |
| | Algorithm 3.2 | IT | 91 | 89 | 89 | 89 |
| | CPU | 2.4500 | 6.5841 | 13.4773 | 26.1544 |
| | ERR | 8.7412e-05 | 9.7808e-05 | 9.2614e-05 | 9.1302e-05 |
| | $\|X_k - X^*\|$ | 1.0056e-06 | 1.1360e-06 | 1.0816e-06 | 1.7522e-06 |

Example 4.2. Consider the complex generalized Sylvester tensor equation (1.1) with  

$$A_1 = \begin{pmatrix} 7 - 3i & 3 - 18i & -5 - 21i \\ -17 - i & 4 + 4i & 6 + i \\ -1 & -11 + 13i & 3 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 8 - 5i & 4 - 7i & 13 + 8i \\ -6 + 12i & -1 + 30i & -4 + 9i \\ -3 - 18i & 11 + 6i & 5 - 3i \end{pmatrix},$$
Finite iterative algorithm...

\[
A_2 = \begin{pmatrix}
-4 & -1 - 4i & -6 + 8i \\
3 - 6i & -19 + 9i & 10 + 2i \\
3 - 16i & 10 - 18i & 5i
\end{pmatrix}, \quad B_2 = \begin{pmatrix}
6 + 19i & -2 - 7i & 4 + 8i \\
-7 + 26i & 5 & 17 + 16i \\
-9 - 17i & 2 + 15i & 12 + 5i
\end{pmatrix},
\]

\[
A_3 = \begin{pmatrix}
-7 - 3i & -5 + 2i & -10 - 4i \\
-4 - 4i & -3 + 7i & 11 - 6i \\
-2 - 2i & 0 & 4 - 5i
\end{pmatrix}, \quad B_3 = \begin{pmatrix}
5 + 11i & -7 - 36i & -14i \\
11 + 3i & -15 - 6i & 12 - 6i \\
12 - 6i & 13 - i & 7 - 5i
\end{pmatrix},
\]

\[
A_4 = \begin{pmatrix}
4 + 7i & 8 + 6i & 6 - i \\
-1 - 9i & -6 - 12i & 1 - 30i \\
22 + 4i & -7 - i & 7 - 6i
\end{pmatrix}, \quad B_4 = \begin{pmatrix}
9 - 4i & -3 - 2i & 2 - 2i \\
-17 + 8i & -5 + 9i & 6 - 24i \\
5i & -16 + 6i & 11 + 16i
\end{pmatrix},
\]

where \(i\) is imaginary unit. And the right-hand side \(\mathcal{D}\) is constructed so that \(\mathcal{X}_* = \text{ones}(3, 3, 3) + \text{ones}(3, 3, 3)i\) is the exact solution of the tensor equation (1.1).

For Example 4.2, we get the approximate solution as \(\mathcal{X}_*\) after 312 iterative steps, which the residual and error are

\[
\text{ERR} = 6.1095e-05 \quad \text{and} \quad \|\mathcal{X}_{312} - \mathcal{X}_*\| = 2.2969e-06.
\]

For more details, the convergence history of the finite iterative algorithm is plotted in Figure 1.

---

**Figure 1.** The convergence history of ERR for Example 4.2.
5. Conclusions

We have proposed a finite iterative algorithm to solve the complex generalized Sylvester tensor equation (1.1), which is generalized of Sylvester tensor equations (1.2) considered in [2]. It is proved that the algorithm is convergent within the finite iterative steps in the absence of the round-off error, which is based on the introducing real inner product. The applications of the proposed algorithm have been compared with the NCG-BTF algorithm when solves the Sylvester tensor equation (1.2). In addition, another example is offered to illustrate the effectiveness of the proposed algorithm for the complex generalized Sylvester tensor equation (1.1).

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