MONGE-AMÈRE MEASURES ON SUBVARIETIES

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ABSTRACT. In this article we address the question whether the complex Monge-Ampère equation is solvable for measures with large singular part. We prove that under some conditions there are no solution when the right-hand side is carried by a smooth subvariety in \( \mathbb{C}^n \) of dimension \( k < n \).

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1. Introduction

In this article we study the complex Monge-Ampère equation

\[
(dd^c u)^n = \mu
\]

(1.1)

where \( \mu \) is a given non-negative Radon measure and \( (dd^c \cdot)^n \) denotes the complex Monge-Ampère operator. Monge-Ampère techniques have an interesting history with applications ranging from algebraic and complex geometry to dynamics and theoretical physics (see e.g. [2, 6, 15, 17, 18]). For an historical account of the complex Monge-Ampère operator we refer to [20, 25].

In the seminal article [4], by Bedford and Taylor it was proved that if \( u \) is a continuous plurisubharmonic function defined on \( \Omega \subset \mathbb{C}^n \), then the left-hand side \((dd^c u)^n\) of the Monge-Ampère equation can not charge on any subvariety in \( \Omega \) of dimension \( k < n \). On the other hand, they show that \((dd^c u)^n\) can charge at a single point and that (1.1) have (in this case) no unique solution. Several authors have studied the case when \( \mu \) is given by a single point mass or a finite sum of such (see e.g. [8, 12, 21, 26, 27]). In [1], a measure \( \mu \) was constructed that do not have any atoms and it is supported by a pluripolar set such that the equation (1.1) have a solution with this given measure. Hence, there exists a measure \( \mu \) with large singular part for which equation (1.1) is solvable. The case when the measure \( \mu \) vanishes on all pluripolar subsets of \( \Omega \) was completed in [10] (see also [4]).

The growing use of complex Monge-Ampère techniques in applications imply a growing demand on knowledge of (1.1) with a large singular part of the given right-hand side (see e.g. [30, 31]). Therefore, we address in this article the following question:

**Aim:** Let \( \Omega \) be a bounded hyperconvex domain in \( \mathbb{C}^n \) and let \( S \) be smooth subvariety in \( \Omega \) of dimension \( k < n \). Assume that \( \mu \) is a non-negative Radon measure (not identically zero) defined on \( S \) with finite total mass. Do there exists a plurisubharmonic function such that \((dd^c u)^n = \mu^? (with suitable interpretation of dimensions)

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that this question is more involved. We construct a function \( u \) such that

\[ (dd^c u)^n = \mu \times \delta_{0}^{-k} \]

with \( \mu = (dd^c u)^k \). Here \( \delta_0 \) denotes the dirac measure at the origin of \( \Delta \). It should be emphasized that if \( u \in \mathcal{E}(\Omega) \) and \( u|_{\partial \Omega} \) is not identically \( -\infty \), then by Theorem 5.11 in [10], we have that \( (dd^c u)^n(\{u = -\infty\}) = 0 \), and therefore there exists a pluripolar Borel set \( E \) in \( S \) such that \( (dd^c u)^n(S \setminus E) = 0 \). Example 2.1 shows that this question is more involved. We construct a function \( u \in \mathcal{F}(\Omega) \) such that \( (dd^c u)^n = \delta_0 \), and \( \{u = -\infty\} = \Omega \). To show that the situation is even more intricate we construct in Example 4.0 an example of a non-positive plurisubharmonic function \( u \) with \( u(z) > -\infty \) for all \( z \), but \( (dd^c u)^n \) is not a well-defined Radon measure.

We end this article in section 5 by proving the following: Assume that \( \mu \) is a non-negative Radon measure defined on \( \Delta^k \) with finite total mass such that it vanishes on all pluripolar sets in \( \Delta^k \). Then there exists no function \( u \in \mathcal{E}(\Delta^n) \) such that \( u(z',z'') = u(z',|z_{k+1}|,...,|z_n|) \) and \( (dd^c u)^n = \mu \times \delta_0^{-k} \). Here we have that \( z' = (z_1,...,z_k) \).

For further information on pluripotential theory we refer to [22, 23, 24]

2. Preliminaries

Following the notation introduced by the second-named author in [9, 10] for a bounded hyperconvex domain \( \Omega \subset \mathbb{C}^n \) we define:

\[ \mathcal{E}_0(\Omega) = \left\{ \varphi \in \mathcal{PSH}(\Omega) \cap L^\infty(\Omega) : \lim_{z \to \partial \Omega} \varphi(z) = 0, \int_{\Omega} (dd^c \varphi)^n < \infty \right\}, \]

\[ \mathcal{F}(\Omega) = \left\{ \varphi \in \mathcal{PSH}(\Omega) : \exists \{u_j\} \subset \mathcal{E}_0(\Omega), \varphi_j \searrow \varphi, \sup_{j} \int_{\Omega} (dd^c \varphi_j)^n < \infty \right\}, \]

\[ \mathcal{E}(\Omega) = \left\{ \varphi \in \mathcal{PSH}(\Omega) : \forall \omega \in \Omega \exists \varphi_\omega \in \mathcal{F}(\Omega) \text{ such that } \varphi_\omega = \varphi \text{ on } \omega \right\}. \]

We also need the following generalization to a complex manifold \( X \):

\[ \mathcal{E}(X) = \left\{ u \in \mathcal{PSH}(X) : z \in X \text{ there exist a neighbourhood } W \text{ of } z \right. \]

\[ \left. \text{such that } u \in \mathcal{E}(W) \right\}. \]

In the following example we show that there exists a function \( u \in \mathcal{F}(\Omega) \) such that \( (dd^c u)^n = \delta_0 \), and \( \{u = -\infty\} = \Omega \).
Example 2.1. Let $\Omega \subset \mathbb{C}^n$ be a hyperconvex domain in $\mathbb{C}^n$. This example shows that there exists a function $u \in \mathcal{F}(\Omega)$ such that $(dd^c u)^n = \delta_0$, and $\{u = -\infty\} = \Omega$.

**Step 1:** For $j \geq 1, 1 \leq m \leq n$ let \( \{a_{mj}\}_{j \geq 1}, a_{mj} > 0 \), be sequences of real numbers such that
\[
\sum_{j=1}^{\infty} (a_{1j} \cdots a_{nj})^{1/j} < +\infty,
\]
and
\[
\sum_{j=1}^{\infty} \min(a_{1j}, \ldots, a_{m-1j}, a_{m+1j}, \ldots, a_{nj}) = \infty \quad \text{for all } 1 \leq m \leq n.
\]
To simplify the notation let $A(j) = \min(a_{1j}, \ldots, a_{m-1j}, a_{m+1j}, \ldots, a_{nj})$. Set
\[
u(z) = \sum_{j=1}^{\infty} \max (a_{1j} \ln |z_1|, \ldots, a_{nj} \ln |z_n|).
\]
Then we have that $u \in \mathcal{F}(\Delta^n)$, and $(dd^c u)^n = c\delta_0$ for some
\[
c \in \left[ \sum_{j=1}^{\infty} a_{1j} \cdots a_{nj}, \left( \left( \sum_{j=1}^{\infty} (a_{1j} \cdots a_{nj})^{1/j} \right)^n \right) \right].
\]
Furthermore, we have that
\[
u(z_1, \ldots, z_{m-1}, 0, z_{m+1}, \ldots, z_n) \\
\leq \sum_{j=1}^{\infty} A(j) \max (\log |z_1|, \ldots, \log |z_{m-1}|, \log |z_{m+1}|, \ldots, \log |z_n|) = -\infty.
\]
Hence,
\[\{u = -\infty\} = \{0\} \times \Delta^{n-1} \cup \cdots \cup \Delta^{n-1} \times \{0\}.
\]

**Step 2:** We can assume that the unit ball $B$ is contained in $\Omega$. Let $\{S_j\}$ be a family of hyperplanes such that $\bigcup_{j=1}^{\infty} (S_j \cap B) = \overline{B}$. By using step 1 together with changing coordinates we can choose $\varphi_j \in \mathcal{F}(B)$ such that
\[
(dd^c \varphi_j)^n = \frac{1}{2^j} \delta_0 \quad \text{and} \quad \varphi_j|_{S_j \cap \mathbb{B}} = -\infty.
\]
Set
\[
\psi = \sum_{j=1}^{\infty} \varphi_j.
\]
Then $\psi \in \mathcal{F}(B)$, $\psi|_{S_j \cap \mathbb{B}} = -\infty$ for all $j$, and $(dd^c \psi)^n = \delta_0$. Set
\[
\psi^r = \sup\{\Phi \in \mathcal{P}\mathcal{S}\mathcal{H}(B) : \Phi \leq 0 \text{ and } \Phi \leq \psi \text{ on } B(0, r)\}.
\]
Here $B(0, r) \subset \mathbb{C}^n$ is the ball with radius $r$. This construction yields that $\{\psi^r\}$ increases pointwise to a function $\varphi \in \mathcal{F}(B)$, and $(dd^c \varphi)^n = c\delta_0$, $c > 0$. From the fact that $\psi^r \leq \varphi_j$ on $B(0, r)$ and $(dd^c \varphi)^n = 0$ on $B \setminus \{0\}$, we get that $\varphi \leq \varphi_j$ on $B$ for all $j \geq 1$, which yields that $\varphi|_{S_j \cap \mathbb{B}} = -\infty$ for all $j \geq 1$. Finally, set
\[
u = \sup\{v \in \mathcal{P}\mathcal{S}\mathcal{H}(\Omega) : \nu \leq 0 \text{ and } \nu \leq \varphi \text{ on } \mathbb{B}\}.
\]
By Lemma 4.5 in [29], Theorem 2.2 in [11] and Lemma 4.1 in [1], we get $u \in \mathcal{F}(\Omega)$, $(dd^c u)^n = c\delta_0$ and $\{u = -\infty\} = \Omega$.

**Proposition 2.2.** Assume that $\Omega \subseteq \mathbb{C}^n$ is a bounded hyperconvex domain. Let $u \in \mathcal{F}(\Omega)$ and $v \in \mathcal{PSh}(\Omega)$, $v \leq 0$, and $w \in \mathcal{E}(\Omega)$ be such that $(dd^c w)^n$ vanishes on pluripolar sets. If $(dd^c u)^n(\{u > -\infty\}) = 0$ and $u \geq v + w$ on a neighborhood $D$ of $\{u = -\infty\}$ then $u \geq v$ on $\Omega$.

**Proof.** We have that

$$\max(u, v) + w = \max(u + w, v + w) \leq u,$$

and therefore by Lemma 4.1 in [1] we get that

$$(dd^c \max(u, v))^{n} \geq \chi_{\{u = -\infty\}}(dd^c u)^n = (dd^c u)^n.$$

Therefore, by Proposition 3.4 in [28] implies that $u = \max(u, v) \geq v$ on $\Omega$. \qed

3. A SUFFICIENT CONDITION ON $\mu$

**Lemma 3.1.** Assume that $\Omega_1 \subset \mathbb{C}^{n_1}$ and $\Omega_2 \subset \mathbb{C}^{n_2}$ are bounded hyperconvex domains. Let $u_1 \in \mathcal{E}(\Omega_1)$, $u_2 \in \mathcal{E}(\Omega_2)$ be such that

$$(dd^c u_1)^n(\{u_1 > -\infty\}) = (dd^c u_2)^n(\{u_2 > -\infty\}) = 0.$$

Then

$$(dd^c \max(u_1, u_2))^{n_1 + n_2} = (dd^c u_1)^{n_1} \land (dd^c u_2)^{n_2}. \quad (3.1)$$

**Proof.** Set $u_1^j = \max(u_1, -j)$ and $u_2^j = \max(u_2, -j)$. From [7] (see also [3]), we have that

$$(dd^c \max(u_1^j, u_2^j))^{n_1 + n_2} = (dd^c u_1^j)^{n_1} \land (dd^c u_2^j)^{n_2}.$$

By letting $j \to \infty$, we obtain that (3.1). \qed

**Lemma 3.2.** Let $\varphi \in \mathcal{E}(\Delta^k)$ be such that $(dd^c \varphi)^k(\{\varphi > -\infty\}) = 0$. Then

$$(dd^c \max(\varphi, \log |z_{k+1}|, \ldots, |z_n|))^n = (dd^c \varphi)^k \times \delta_0^{n-k}.$$  

**Proof.** It follows from Lemma 3.1. \qed

From Lemma 3.2 we have that

**Theorem 3.3.** Let $\Omega$ be a bounded hyperconvex domain in $\mathbb{C}^n$ and $S$ be a subvariety in $\Omega$ with dimension $k < n$. Assume that $\varphi \in \mathcal{E}(S)$ such that

$$(dd^c \varphi)^k(\{\varphi > -\infty\}) = 0.$$  

Then exists a function $u \in \mathcal{E}(\Omega)$ such that $(dd^c u)^n = (dd^c \varphi)^k$.  

4. A necessary condition to belong to \( E(\Omega) \)

In this section we start with introducing some notation. For \( z = (z_1, \ldots, z_n) \in \mathbb{C}^n \), we write \( z' = (z_1, \ldots, z_k) \) and \( z'' = (z_{k+1}, \ldots, z_n) \). Then we define
\[
\|z\| = \max(|z_1|, \ldots, |z_n|), \\
\|z'\| = \max(|z_1|, \ldots, |z_k|), \\
\|z''\| = \max(|z_{k+1}|, \ldots, |z_n|).
\]

With these notation we make the following definition

**Definition 4.1.** Let \( u \in \mathcal{PSh}(\Delta^n), u \leq 0 \). We define
\[
\phi_u(z', r) = \frac{\max_{\|z''\| = r} u(z', z'')}{|\log r|},
\]
and
\[
\phi_u(z') = (-\nu_u(z', \cdot)(0))^*,
\]
where \( \nu_u(z', \cdot)(0) \) is the Lelong number of the function \( u(z', \cdot) \) at \( 0 \).

From the construction in Definition 4.1, we get that \( \phi_u(\cdot, r) \in \mathcal{PSh}(\Delta^k), \phi_u(\cdot, r) \leq 0 \) and that \( \phi_u(z', r) \not\sim -\nu_u(z', \cdot)(0) \) as \( r \searrow 0 \). Thanks to [5], we have that \( \phi_u \in \mathcal{PSh}(\Delta^k) \), \( \phi_u \leq 0 \), and that the set
\[
\{ z' \in \Delta^k : \phi_u(z') \neq -\nu_u(z', \cdot)(0) \}
\]
is a pluripolar set in \( \Delta^k \). Furthermore, we get that
- if \( u \geq v \), then \( \phi_u \geq \phi_v \)
- \( \phi_{au+bv} = a\phi_u + b\phi_v \), for all \( u, v \in \mathcal{PSh}(\Delta^n) \), \( u, v \leq 0 \), and \( a, b \geq 0 \)
- \( \phi_{\max(u,v)} = \max(\phi_u, \phi_v) \).

**Theorem 4.2.** Let \( u \in \mathcal{PSh}(\Delta^n), u \leq 0 \). Then we have that \( \phi_u \) is a constant function.

**Proof.** Take \( z'_0 \in \Delta^k \). We will only need to prove that
\[
\phi_u(z') \leq \phi_u(z'_0) \quad \text{for all} \quad z' \in \Delta^k.
\]

Fix \( \epsilon > 0 \). We can choose \( r > 0 \) small enough such that
\[
\phi_u(z') \leq \phi_u(z'_0) + \epsilon \quad \text{for all} \quad \|z' - z'_0\| < r.
\]

This implies that
\[
\nu_u(z', \cdot)(0) \geq -\phi_u(z'_0) - \epsilon \quad \text{for all} \quad \|z' - z'_0\| < r.
\]

Therefore, we have that
\[
u_u(z', z'') \leq (-\phi_u(z'_0) - \epsilon) \log \|z''\| \quad \text{for} \quad \|z' - z'_0\| < r, \ z'' \in \Delta^{n-k}.
\]

Hence,
\[
\{ z' \in \Delta^k : \|z' - z'_0\| < r \} \times \{ 0 \}^{n-k} \subset \{ z \in \Delta^n : \nu_u(z) \geq -\phi_u(z'_0) - \epsilon \}.
\]

On the other hand, from Siu's theorem (see e.g. [32], [13]) we have that
\[
\{ z \in \Delta^n : \nu_u(z) \geq -\phi_u(z'_0) - \epsilon \}
\]
Lemma 3.3 in [1], we have that
\[ \{ z \in \Delta^n : \nu(z) \geq -\phi_u(z_0') - \epsilon \} = \Delta^k \times \{ 0 \}^{n-k}. \]
Thus,
\[ u(z', z'') \leq (-\phi_u(z'_0) - \epsilon) \log \| z'' \| \quad \text{for all } z \in \Delta^n. \]
Hence, \( \phi_u(z'_0) \leq \phi_u(z'_0) + \epsilon \) for all \( z' \in \Delta^k \). Let now \( \epsilon \to 0^+ \), and we finally get that
\[ \phi_u(z') \leq \phi_u(z'_0) \quad \text{for all } z' \in \Delta^k. \]

Remark. If \( k = n - 1 \), then \((u - \phi_u \log \| z'' \|) \in \mathcal{P}SH(\Delta^n)\).

Lemma 4.3. Let \( u \) be a pluriharmonic function, and let \( \{ u_j \} \) be a sequence of plurisubharmonic functions that converges to \( u \) in \( dV_{2n} \) on \( \Omega \) as \( j \to \infty \). Then \( \{ u_j \} \) converges to \( u \) in capacity, as \( j \to \infty \).

Proof. Let \( K \subseteq L \subseteq D \subseteq \Omega \), and \( \delta > 0 \). We shall prove that
\[ \text{Cap}_D(\{ |u_j - u| > \delta \} \cap K) \to 0, \quad \text{as } j \to +\infty, \]
Choose \( \phi \in \mathcal{E}_0(D) \) that satisfies \((dd^c \phi)^n = dV_{2n}\). Take \( A > 0 \) such that \( A\phi \leq -1 \) on \( L \). Let \( 0 < \varepsilon < \frac{\delta}{2} \). Hartog’s theorem yields that there exists a \( j_0 \) such that
\[ u_j \leq u + \varepsilon \quad \text{for all } z \in D, \quad j \geq j_0. \]

By Lemma 3.3 in [1], we have that
\[ \text{Cap}_D(\{ |u_j - u| > \delta \} \cap K) = \text{Cap}_D(\{ u - u_j > \delta \} \cap K) = \text{Cap}_D(\{ u_j < u - \delta \} \cap K) \]
\[ = \sup \left\{ \int_{\{ u_j < u - \delta \} \cap K} (dd^c \phi)^n : \phi \in \mathcal{P}SH(D), \ -1 \leq \phi \leq 0 \right\} \]
\[ = \sup \left\{ \int_{\{ u_j < u - \delta \} \cap K} (dd^c \phi)^n : \phi \in \mathcal{P}SH(D), \ h_{D,L}^n \leq \phi \leq 0 \right\} \]
\[ \leq \frac{1}{\delta} \sup \left\{ \int_D (u - u_j + \varepsilon)(dd^c \phi)^n : \phi \in \mathcal{P}SH(D), \ h_{D,L}^n \leq \phi \leq 0 \right\} \]
\[ \leq \frac{1}{\delta} \int_D (u - u_j + \varepsilon)(dd^c \phi)^n \leq \frac{A^n}{\delta} \int_D (u - u_j + \varepsilon)(dd^c \phi)^n \]
\[ = \frac{A^n}{\delta} \int_D (u - u_j + \varepsilon) dV_{2n} \leq \frac{A^n}{\delta} \left( \int_D |u - u_j| dV + \varepsilon V_{2n}(D) \right). \]
Hence,
\[ \limsup_{j \to +\infty} \text{Cap}_D(\{ |u_j - u| > \delta \} \cap K) \leq \varepsilon \frac{A^n V_{2n}(D)}{\delta} \quad \text{for all } \varepsilon > 0. \]
Thus,
\[ \text{Cap}_D(\{ |u_j - u| > \delta \} \cap K) \to 0, \quad \text{as } j \to +\infty, \]
\[ \square \]
**Theorem 4.4.** Let \( u \in \mathcal{PSH}(\Delta^n), \ u \leq 0 \). Then we have that
\[
\frac{u(z', rz'')}{|\log r|} \to \phi_u
\]
in capacity on \( \Delta^k \times (\Delta_\frac{1}{r})^{n-k} \), as \( r \to 0^+ \). Here \( \Delta_\frac{1}{r} \subseteq \mathbb{C} \) denotes the disc of radius \( \frac{1}{r} \).

**Proof.** From
\[
\phi_u(z', r) = \max_{\|z''\| = 1} \frac{u(z', rz'')}{|\log r|} \to \phi_u
\]
as \( r \to 0^+ \), and Theorem 3.2.12 in [19] we get that
\[
\frac{u(z', rz'')}{|\log r|} \to \phi_u \quad \text{on } dV_{2n} \quad \text{on } \Delta^k \times (\Delta_\frac{1}{r})^{n-k}
\]
as \( r \to 0^+ \). We complete this proof by using Lemma 4.3 and obtain that
\[
\frac{u(z', rz'')}{|\log r|} \to \phi_u
\]
in capacity on \( \Delta^k \times (\Delta_\frac{1}{r})^{n-k} \) as \( r \to 0^+ \). \( \square \)

**Theorem 4.5.** Let \( u \in \mathcal{E}(\Delta^n) \). Then \( \phi_u \) is identically 0.

**Proof.** Assume that \( \phi_u < 0 \). Hence
\[
u_u(z) = -\phi_u \log \|z''\| \quad \text{on } \Delta^n.
\]
Hence, \( \nu_u(z) \geq -\phi_u \) on \( \Delta^n \times \{0\}^k \). This is not possible, since \( u \in \mathcal{E}(\Delta^n) \). \( \square \)

Example 4.6 shows that the converse of Theorem 4.5 is in generally false.

**Example 4.6.** In this example we construct a function \( u \in \mathcal{PSH}(\Omega), \ u \leq 0 \) such that
\[
u(z) > -\infty \quad \text{for all } z \in \Omega
\]
but \( u \notin \mathcal{E}(\Omega) \). We can assume that \( \Omega \subseteq \Delta^n \). Let \( u \) be defined on \( \Delta^n \) as
\[
u(z) = \sum_{j=1}^{\infty} \max \left( \frac{1}{2j} \log \frac{|z_1 - \frac{1}{2^n}|}{1 - \frac{2^n}{|z_1|}}, \frac{2}{j} \log |z_2|, \log |z_3|, \ldots, \log |z_n|, -2j \right).
\]
We start by proving that \( \nu(z) > -\infty \) for all \( z \in \Delta^n \). If \( z_1 = 0 \), then we have that
\[
u(0) \geq \sum_{j=1}^{\infty} \frac{1}{2j} \log \frac{1}{2j} > -\infty.
\]
If \( z_1 \neq 0 \) we choose \( j_0 \) be such that \( |z_1| > \frac{1}{2j_0 - 1} \). Hence
\[
u(z) \geq \sum_{j=1}^{j_0} -2j + \sum_{j=j_0+1}^{\infty} \frac{1}{2j} \log \frac{|z_1 - \frac{1}{2^n}|}{1 - \frac{2^n}{|z_1|}} \geq \sum_{j=1}^{j_0} -2j + \log \frac{|z_1|}{4} \sum_{j=j_0+1}^{\infty} \frac{1}{2j} > -\infty.
\]
Next, we shall show that \( u \notin \mathcal{E}(W) \) for all neighbourhoods \( W \) of 0. Set
\[
u_k = \sum_{j=1}^{k} \max \left( \frac{1}{2j} \log \frac{|z_1 - \frac{1}{2^n}|}{1 - \frac{2^n}{|z_1|}}, \frac{2}{j} \log |z_2|, \log |z_3|, \ldots, \log |z_n|, -2j \right).
\]
We have $u_k \in \mathcal{E}_0(\Delta^n)$ and $\varphi_k \searrow u$ as $k \to \infty$, and
\[
(dd^c u_k)^n \geq \sum_{j=1}^k \left( dd^c \max \left( \frac{1}{2^j} \log \frac{|z_1 - \frac{1}{2^j}| - 2 |z_2|, \log |z_3|, ..., \log |z_n|, -2^j \right) \right)^n
\]
\[
= \sum_{j=1}^k \frac{1}{j} \sigma_{\{|z_1 - \frac{1}{2^j}| = e^{-4t}\}} \times \sigma_{\{|z_2| = e^{-t}\}} \times \sigma_{\{|z_3| = e^{-t}\}} \times \sigma_{\{|z_n| = e^{-2t}\}},
\]
where $\sigma_{\{|z_j| = r\}}$ is the normalized surface measure on $\{|z_j| = r\}$. Hence,
\[
\int_W (dd^c u_k)^n \to \infty
\]
as $k \to \infty$ for all neighbourhood $W$ of 0.

**Definition 4.7.** For each $u \in \mathcal{PSH}(\Omega_1 \times \Omega_2)$ and $w_2 \in \Omega_2$ we define
\[
E(u, t, w_2) = \{ z_1 \in \Omega_1 : u(z_1, z_2) \leq t \log \|z_2 - w_2\| + O(1), \text{ for every } z_2 \in \Omega_2 \}
\]
\[
= \{ z_1 \in \Omega_1 : \nu_{u(z_1)}(w_2) \geq t \}.
\]

**Theorem 4.8.** Let $u \in \mathcal{E}(\Omega_1 \times \Omega_2)$. Then
\[
\bigcup_{t>0} E(u, t, w_2)
\]
is a pluripolar set in $\Omega_1$ for all $w_2 \in \Omega_2$.

**Proof.** Since this problem is purely local we can without loss of generality assume that $\Omega_1 = \Delta^k$, $\Omega_2 = \Delta^{n-k}$ and $w_2 = 0$. Theorem 4.5 yields that $\phi_u \equiv 0$. We have that
\[
\bigcup_{t>0} E(u, t, 0) = \{ z' \in \Delta^k : \phi_u(z') \neq -\nu_u(z', 0) \},
\]
and therefore it follows that $\bigcup_{t>0} E(u, t, 0)$ is a pluripolar set in $\Delta^k$. \hfill \Box

5. The toric case

**Theorem 5.1.** Let $u \in \mathcal{E}(\Delta^n)$ be such that $u(z', z'') = u(z', |z_{k+1}|, ..., |z_n|)$. Then there exists a Borel pluripolar set $E$ in $\Delta^k$ such that
\[
(dd^c u)^n(\Delta^k \setminus E) \times \{ 0 \}^n-k = 0.
\]

**Proof.** Without loss generality we can assume that $u \in \mathcal{F}(\Delta^n)$. Theorem 6.3 in [10] yields that there exists a function $\varphi \in \mathcal{E}_0(\Delta^k)$, $0 \leq f \in L^1(\mathbb{R})$, a non-negative Radon measure $\nu$ defined on $\Delta^k$, and a Borel pluripolar set $E \subset \Delta^k$ such that
\[
1_{\Delta^k \times \{ 0 \}^n-k} (dd^c u)^n = f(dd^c \varphi)^k + \nu,
\]
and $\nu(\Delta^k \setminus E) = 0$. We shall prove that
\[
f(dd^c \varphi)^k = 0.
\]
Fix $t \in (0, 1)$. Thanks to Lemma 4.3 in [11], we can find a function $v \in \mathcal{F}(\Delta^n)$ such that $v \geq u$, $(dd^c v)^n = 1_{\Delta^k} f(dd^c \varphi)^k$ and
\[
v(z', z'') = v(z', |z_{k+1}|, ..., |z_n|).
Next choose a sequence \( \{ r_j \} \) with \( r_j \searrow 0 \). By the quasicontinuity of \( \phi_v(\cdot, r_j) \) (see e.g. [5]), we can find a decreasing sequence of open sets \( \{ G_m \}_{m \geq 1} \) in \( \Delta_k^j \) such that

\[
\text{Cap}_{\Delta^j_k}(G_m) \leq \frac{1}{m} \quad \text{and} \quad \phi_v(\cdot, r_j) \big|_{\Delta^j_k \setminus G_m} \text{ are continuous}.
\]

Furthermore, it can be chosen such that each element is continuous on \( \Delta_k^j \setminus G_m \) for all \( j, m \geq 1 \), and \( \nu_v(z, r_j)(0) = 0 \) on \( \Delta_k^j \setminus G_m \) for all \( m \geq 1 \). By Dini's theorem we have \( \phi_v(z', r_j) \) converges uniformly 0 on \( z' \in \Delta_k^j \setminus G_m \), as \( j \to \infty \), for all \( m \geq 1 \). Hence, for each \( m \) we can choose \( j_m \) such that

\[
\epsilon_m = - \min_{z' \in \Delta_k^j \setminus G_m} \phi_v(z', r_{j_m}) \searrow 0, \quad \text{as} \quad m \to \infty.
\]

Since \( v(z', z'') = v(z', \{ z_{k+1}, \ldots, z_n \}) \), we have that

\[
v(z', z'') \geq \epsilon_{j_m} (\log |z_{k+1}| + \ldots + \log |z_n|),
\]

for all \( (z', z'') \in (\Delta_k^j \setminus G_m) \times \Delta_k^{n-k} \). Set

\[
w_{j_m} = \max(v, \epsilon_{j_m} (\log |z_{k+1}| + \ldots + \log |z_n|)),
\]

and choose \( v_l \in E_0 \cap C(\Delta^n) \) such that \( v_l \searrow v \). Set

\[
h_{G_m, \Delta^k} = \text{sup} \{ \varphi \in \mathcal{P}SH(\Delta^k) : \varphi \leq -1 \text{ on } G_m \}
\]

We have that

\[
\int_{\Delta_k^j \setminus G_m \times \Delta_k^{n-k} \setminus j_m} (dd^c w_m)^n
\geq \lim_{l \to \infty} \int_{\Delta_k^j \setminus G_m \times \Delta_k^{n-k} \setminus j_m} (dd^c \max(v_l, \epsilon_{j_m} (\log |z_{k+1}| + \ldots + \log |z_n|) + \frac{1}{n}))^n.
\]

Since,

\[
\Delta_k^j \setminus G_m \times \Delta_k^{n-k} \setminus j_m \subset \{ v_l > \epsilon_{j_m} (\log |z_{k+1}| + \ldots + \log |z_n|) + \frac{1}{n} \}
\]

and \( h(dd^c v_l)^n \to h(dd^c v)^n \) weakly as \( l \to \infty \) for all \( h \in \mathcal{P}SH(\Delta^n) \cap L^\infty(\Delta^n) \), we have that

\[
\int_{\Delta_k^j \setminus G_m \times \Delta_k^{n-k} \setminus j_m} (dd^c w_m)^n \geq \lim_{l \to \infty} \int_{\Delta_k^j \setminus G_m \times \Delta_k^{n-k} \setminus j_m} (dd^c v_l)^n
\geq \lim_{l \to \infty} \int_{\Delta_k^j \times \Delta_k^{n-k} \setminus j_m} (1 + h_{G_m, \Delta^k})(dd^c v_l)^n \geq \int_{\Delta_k^j \times \Delta_k^{n-k} \setminus j_m} (1 + h_{G_m, \Delta^k})(dd^c v)^n
\]

\[
= \int_{\Delta_k^j} (1 + h_{G_m, \Delta^k}) f (dd^c \varphi)^k.
\]

From the fact that

\[
\int_{\Delta^k} (dd^c h_{G_m, \Delta^k})^n = \text{Cap}_{\Delta^k}(G_{j_m}) \searrow 0 \quad \text{as} \quad m \to \infty
\]

we get \( h_{G_m, \Delta^k} \not\searrow 0 \) a.e on \( \Delta^k \), as \( m \to \infty \). This yields that

\[
\lim_{m \to \infty} \int_{\Delta_k^j \setminus G_m \times \Delta_k^{n-k} \setminus j_m} (dd^c w_m)^n \geq \int_{\Delta_k^j} f (dd^c \varphi)^k.
\]
On the other hand, since $v \leq w_m \not\to 0$ as $m \to \infty$, we get that
\[
\lim_{m \to \infty} \int_{\Delta^k \times \Delta^{n-k}} (dd^c w_m)^n \leq \lim_{m \to \infty} \int_{\Delta^k \times \Delta^{n-k}} (dd^c w_m)^n \leq 0
\]
Thus,
\[
\int_{\Delta^k} f(\varphi)^k = 0
\]
To complete this proof let $t \to 1^-$. □

By combining Theorem 5.1 with Theorem 5.3 we get the following corollary

**Corollary 5.2.** Let $\mu$ be a non-negative Radon measure defined on $\Delta^k$ which vanish on every pluripolar sets in $\Delta^k$. Then there is exists no function $u \in \mathcal{E}(\Delta^n)$ such that
\[
u(z', z'') = u(z', |z_{k+1}|, \ldots, |z_n|) \quad \text{and} \quad (dd^c u)^n = \mu.
\]

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