Network Critical Slowing Down: Data-Driven Detection of Critical Transitions in Nonlinear Networks

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Abstract—Scheffer et al. (2009) presented a novel data-driven framework to predict critical transitions in complex systems. These transitions, which may stem from failures, degradation, or adversarial actions, have been attributed to bifurcations in the nonlinear dynamics. Their approach was built upon the phenomenon of critical slowing down, i.e., slow recovery in response to small perturbations near bifurcations. We extend their approach to detect and localize critical transitions in nonlinear networks. By introducing the notion of network critical slowing down, the objective of this article is to detect that the network is undergoing a bifurcation only by analyzing its signatures from measurement data. We focus on two classes of widely used nonlinear networks: Kuramoto model for the synchronization of coupled oscillators, and attraction–repulsion dynamics in swarms, each of which presents a specific type of bifurcation. Based on the phenomenon of critical slowing down, we study the asymptotic behavior of the perturbed system away and close to the bifurcation and leverage this fact to develop a deterministic method to detect and identify critical transitions in nonlinear networks. Furthermore, we study the state covariance matrix subject to a stochastic noise process away and close to the bifurcation and use it to develop a stochastic framework for detecting critical transitions. Our simulation results show the strengths and limitations of the methods.

Index Terms—Attraction–repulsion dynamics, bifurcation, critical slowing down, nonlinear coupled oscillators.

I. INTRODUCTION

A. Motivation

In complex systems, transitions from stable to unstable operating modes due to the change in system’s parameter are seen in a broad range of real-world applications, from biological and ecological systems [1] to infrastructure systems [2]. Examples include the following:

1) cluster synchronization in complex networks as a result of the infrastructure degradation over time [3];
2) the onset of Alzheimer’s disease caused by changes in brain functional network connectivity affected by aging [4];
3) the onset of the epilepsy, which is correlated to the gradual changes in the functional connectivity in the brain’s temporal lobe [5].

In some applications, these transitions play a crucial role in the operation of the network. However, in many other applications, they can endanger network’s safety and potentially result in catastrophic events. From a mathematical perspective, these transitions are described by bifurcations, changes in the stability of an equilibrium due to parameter variations. Detecting a system undergoing a bifurcation is challenging as it typically necessitates awareness of the system model and the values of the bifurcating parameters. Nevertheless, it has been demonstrated that bifurcations tend to leave discernible marks on the system’s measurements. One such mark is referred to as the phenomenon of critical slowing down [6]. Inspired by those observations, we introduce data-driven algorithms to detect and localize critical transitions in two widely studied nonlinear networks.

1) Qualitative Representation of the Phenomenon: critical slowing down is a phenomenon that can be intuitively understood using the example of a ball in a well, depicted in Fig. 1. Initially, the ball is in stable equilibrium at the bottom of a well. When perturbed, it quickly returns to its equilibrium position due to the steep slope of the well. However, as the slope decreases, the ball’s recovery rate decreases. Eventually, with a flat surface, the ball no longer recovers and remains displaced. This absence of recovery is due to the lack of a slope guiding it back to equilibrium. Further decreasing the slope leads to an unstable equilibrium point and bifurcation. This simple example provides insight into the system’s response near a bifurcation.

Near a bifurcation, slowing down leads to increased autocorrelation and variance in the system’s fluctuations. The intrinsic rates of change within the system decrease, causing the current state to resemble its past states more closely. This heightened autocorrelation results in an increased variance, which is a stochastic signature of critical slowing down (see Fig. 1, left).
Qualitative representation of critical slowing down. Right: Slowing down the recovery in response to perturbations near the bifurcation point. Left: Increase in autocorrelation and variance of the state close to the bifurcation.

B. Literature Review

The dynamics representing the evolution of natural systems, e.g., biological, ecological, and climate systems, represent different types of nonlinear phenomena, including the bifurcation. Thus, first attempts to use data in analyzing the behavior of large-scale networks and detecting their critical transitions belong to the research works in natural sciences and medicine [7], [8]. One of the signatures of complex systems near bifurcation is the phenomenon of critical slowing down; a slow recovery in response to perturbations near a bifurcation. critical slowing down was first introduced in statistical mechanics [9], and was revisited in a variety of complex systems [7], [10]. The effect of critical slowing down on the autocorrelation of the system’s state, its variance, and higher order moments of data distribution was discussed in [6] and [11], which became useful in detecting tipping points in complex systems using data. However, these approaches are usually based on the analysis of an aggregated scalar dynamical system and do not consider the role of network structure in these critical transitions. In a separate line of research, detection and identification of abrupt changes in dynamical systems has been extensively studied in engineering systems, see [12] and the references therein. These methods mainly consist of two steps: 1) generating a residual, i.e., a signal showing the deviation of the measurement from the system’s output in the normal condition, and 2) designing a decision rule based upon these residuals. The above steps require an extensive knowledge about the system’s dynamics in the normal and faulty conditions and are usually applicable to linear systems.

In nonlinear networks, not only the bifurcation can cause instability in the system, the underlying network topology can facilitate the propagation of this failure throughout the system. This phenomenon is known as cascading failure that can potentially lead to the collapse of the entire infrastructure [13], [14]. Hence, early detection and localization of failures in a network can help preventing their dissemination to the healthy parts. In this article, we focus on two well-known nonlinear dynamic networks, as discussed below.

1) Coupled Oscillator Networks: Coupled oscillators are ubiquitous in natural and engineering world, and their collective behaviors play a critical role in their real-world applications, including the operation of biological systems [15], the brain networks [16], the design of infrastructure networks [17], and in circuits and radio technology [18]. One of the simplest model for studying synchronization in coupled oscillator networks is the Kuramoto model. Kuramoto model is known to be the locally canonical model for studying weakly coupled oscillators [19] and has been successfully used to model a wide range of real-world systems, including power grids [20], robotics networks [21], and associative memory systems [22]. It is well-known that Kuramoto oscillators can undergo a bifurcation from synchrony to incoherency and synchronization is determined by a tradeoff between coupling strength and oscillators’ heterogeneity [20]. A large body of work in the literature is devoted to estimating or characterizing this tradeoff [20], [23], [24]. One of the most visible examples is the power grids, which, as a result of the unprecedented penetration of renewable energy resources and large increase in power demand, are being pushed toward their maximum capacity [25]. In this context, understanding the threshold of bifurcation can help system operators to ensure safety and security of the frequency synchronization. Another example is the brain network where computing the onset of bifurcation is important to understand different modes of functionality of the brain.

2) Attraction–Repulsion Networks: The attraction–repulsion functions are widely used in the formation and swarming of robots [26], [27]. They are inspired by animal grouping and biological systems, with attraction having a longer range than repulsion [28], [29]. The dynamics of these swarms are governed by an interplay between these two forces, and the network reaches local stability [30]. In these systems, a bifurcation can happen when the repulsion dominates the attraction, resulting in a change in the shape of the network. An example is the cluster formation of particles in granular flows, where the cluster size is the resultant of the two forces [31]. Another example is tissue growth in multicellular organisms, where density-dependent inhibition acts as a repulsion force between cells in a tissue and inhibits them from growing in a bounded environment [32]. This density-dependent inhibition is usually applied by cell–cell contacts. Tumor cells often lose this density-dependent inhibition, resulting in nonstopping cell production. Hence, understanding the logic behind bifurcations in these systems can help us to detect and identify them faster.

C. Contributions

In this article, we develop a deterministic and a stochastic data-driven method to detect critical transitions in oscillator networks and attraction–repulsion networks. The contributions of this article are as follows.

1) We study bifurcation for coupled oscillators and attraction–repulsion dynamics with two agents. The main motivation for choosing these classes of nonlinear system is their applications to engineering systems discussed above. Using suitable reduced-order models, we characterize the behavior of these systems at bifurcation and their bifurcation type. Following [6], we observe the phenomenon of critical slowing down near bifurcations...
and applied deterministic and stochastic methods to detect
the onset of bifurcation.

2) For networks of coupled oscillators and networks of
atraction–repulsion dynamics, we investigate asymptotic
behaviors of parameterized nonlinear networks and pro-
vide sufficient conditions, in terms of network parameters
and structure, under which a bifurcation happens (Propo-
sitions 1 and 2).

3) For networks of coupled oscillators and networks of
attraction–repulsion dynamics, we study the signature of
the bifurcation in these dynamics using a deterministic
and a stochastic approach. In the deterministic approach,
we investigate the asymptotic behavior of nonlinear net-
works away and close to the bifurcation (Theorem 1).
Building upon this result, we introduce the notion of
network critical slowing down and develop a completely
data-driven heuristic algorithm to detect bifurcations in
the network and localize the bifurcating edges. We further
demonstrate a dichotomy between the asymptotic behav-
ior of the state covariance before and at the bifurcation
(Theorem 2) and use it as a stochastic indicator of the
bifurcation in the network.

4) We demonstrate the efficiency of our algorithms using
several numerical simulations and compare the abilities,
limitations, and application domains of the proposed de-
deterministic and stochastic methods.

Our contributions distinguish this article from existing litera-
ture in three ways. First, we analyze multidimensional dynam-
ical systems over networks, whereas previous work focuses on
scalar dynamical systems. Second, we extend the application of
critical slowing down to new domains, such as robotic systems
and power grids, whereas previous studies mainly focused on
statistical physics and ecological systems. Third, we develop
data-driven methods for detecting and localizing bifurcations
without presuming knowledge of system parameters, avoiding
the need for system identification techniques or specific condi-
tions on exogenous inputs.

II. MODELING AND PROBLEM STATEMENT

In this section, we introduce two classes of nonlinear dy-
namics, namely, attraction–repulsion dynamics and nonlinear
coupled oscillators, and then, state the problem.

A. Notation

Given a function \( f : \mathbb{R}^n \rightarrow \mathbb{R}^m \), we denote the Jacobian of
\( f \) at point \( \mathbf{x} \) in \( \mathbb{R}^n \) by \( D_f(\mathbf{x}) \). For a vector \( \mathbf{v} \) in \( \mathbb{R}^n \), we define
the diagonal matrix \( \text{diag}(\mathbf{v}) \in \mathbb{R}^{n \times n} \) by \( \text{diag}(\mathbf{v})_{ii} = v_i \), for every
\( i \in \{1, \ldots, n\} \). Given a matrix \( \mathbf{M} \in \mathbb{R}^{n \times m} \), the Moore–Penrose
pseudoinverse of \( \mathbf{M} \) is denoted by \( \mathbf{M}^\dagger \). We denote an undirected
weighted graph by \( \mathcal{G} = \{ \mathcal{V}, \mathcal{E}, \mathcal{A} \} \), where \( \mathcal{V} = \{1, 2, \ldots, n\} \) is
a set of nodes and \( \mathcal{E} \subset \mathcal{V} \times \mathcal{V} \) is the set of edges with \( |\mathcal{E}| = m \),
and \( \mathcal{A} \in \mathbb{R}^{m \times m} \) is a diagonal weight matrix, where \( \mathcal{A}_{ee} = a_{ij} \)
is the weight of the edge \( e = (i, j) \) in \( \mathcal{E} \). An orientation of an
undirected weighted graph \( \mathcal{G} \) is defined by assigning a di-
rection to each edge in \( \mathcal{E} \). For an undirected weighted graph \( \mathcal{G} \)
with an arbitrary orientation and with \( m \) edges, numbered as
e_1, e_2, \ldots, e_m, its node-edge incidence matrix \( \mathbf{B}(\mathcal{G}) \in \mathbb{R}^{n \times m} \),
or simply \( \mathbf{B} \), is defined as

\[
[B]_{kl} = \begin{cases} 
1, & \text{if node } k \text{ is the head of edge } l \\
-1, & \text{if node } k \text{ is the tail of edge } l \\
0, & \text{otherwise.}
\end{cases}
\]

The Laplacian matrix of the graph \( \mathcal{G} \) is given by \( \mathbf{L} = \mathbf{B} \mathbf{A} \mathbf{B}^\top \).
We denote the eigenvalues of the Laplacian matrix \( \mathbf{L} \) by \( 0 = \lambda_1(\mathcal{G}) \leq \lambda_2(\mathcal{G}) \leq \cdots \leq \lambda_n(\mathcal{G}) \). Given a set of nodes \( S \subseteq \mathcal{V} \), the
cutset of \( S \) is defined as \( \partial S = \{ (u, v) \in \mathcal{E} | u \in S, v \in \mathcal{V} \setminus S \} \).
A 2-cutset is a cutset whose removal splits the graph into two
connected components.\(^1\) Given a set \( S \subseteq \mathcal{V} \), its indicator vector \( \chi^S \in \mathbb{R}^n \) is defined by

\[
\chi^S_i = \begin{cases} 
+1 & i \in S \\
-1 & i \notin S.
\end{cases}
\]

Let \( \mathcal{G}(\alpha) = \{ \mathcal{V}, \mathcal{E}, \mathcal{A}(\alpha) \} \) be a family of undirected weighted
graphs parameterized by a real-valued variable \( \alpha \in \mathbb{R} \). We define
the associated lower bound graph \( \mathcal{G}^L = \{ \mathcal{V}, \mathcal{E}, \mathcal{A}^L \} \), where
\( \mathcal{A} = \{ a_{ij} \}_{(i,j) \in \mathcal{E}} \) is defined by \( a_{ij} = \inf_{\alpha \in \mathbb{R}} a_{ij}(\alpha) \) for every
\( (i, j) \in \mathcal{E} \).

B. Nonlinear Coupled Oscillators

One of the simplest models for studying network of nonlinear
coupled oscillators is the Kuramoto model. The Kuramoto model
on an undirected weighted graph \( \mathcal{G} = \{ \mathcal{V}, \mathcal{E}, \mathcal{A} \} \) is described by the
following first-order dynamics:

\[
\dot{\theta}_i = \omega_i(\alpha) - \sum_{(i,j) \in \mathcal{E}} a_{ij} \sin(\theta_i - \theta_j), \quad i = 1, 2, \ldots, n
\tag{1}
\]

where \( \theta_i \in \mathbb{R} \) is the phase of agent \( i \) and \( a_{ij} \) is the coupling
strength between agents \( i \) and \( j \) given by \( a_{ij} = \mathcal{A}_{ee} \), for every
\( (i, j) = e \in \mathcal{E} \). In addition, we assume that \( \omega_i(\alpha) \) is the natural
frequency of agent \( i \) and it is parameterized by \( \alpha \in \mathbb{R} \).\(^2\) Let \( \mathbf{B} \in \mathbb{R}^{n \times m} \) be the incidence matrix of the graph \( \mathcal{G} \) with an arbitrary
orientation. Then, the coupled oscillator dynamics (1) can be
written in the matrix form

\[
\dot{\mathbf{θ}} = \omega(\alpha) - \mathbf{B} \mathbf{A} \sin(\mathbf{B}^\top \mathbf{θ})
\tag{2}
\]

where \( \theta = (\theta_1, \ldots, \theta_n)^\top \in \mathbb{R}^n \) and \( \omega(\alpha) = (\omega_1(\alpha), \ldots, \omega_n(\alpha))^\top \in \mathbb{R}^n \) are the vector of states and natural frequencies,
respectively.

C. Attraction–Repulsion Dynamics in Swarms

A swarm of \( n \) agents over a undirected unweighted graph
\( \mathcal{G} = \{ \mathcal{V}, \mathcal{E} \} \) is described by the following dynamics:

\[
\dot{x}_i = \sum_{(i,j) \in \mathcal{E}} g(x_i - x_j), \quad i = 1, 2, \ldots, n
\tag{3}
\]

\(^{1}\)In this article, for the sake of brevity, we focus on 2-cutsets. However, our
results can be generalized to general cutsets.

\(^{2}\)As an example, in power systems the natural frequencies are power demands
or power supplies and can change due to degradation or variation in power
distribution profile of the grid \([23]\).
where \( x_i \in \mathbb{R}^k \) and \( g : \mathbb{R}^k \rightarrow \mathbb{R}^k \) is a nonlinear function modeling the attraction and repulsion forces between the individuals. Among several classes of attraction–repulsion functions \( g(\cdot) \), linear long-range attraction and exponential short-range repulsion is widely used in the literature, specifically in swarm robotics [26], [33], inspired from natural swarms with long-range attraction and short-range repulsion. Using this model, the dynamics of each agent become

\[
\dot{x}_i = -\sum_{(i,j) \in E} (x_i - x_j) \left( w_{ij}^a(\alpha) - w_{ij}^r(\alpha) \exp\left(-\frac{\|x_i - x_j\|^2}{c}\right)\right)
\]

where \( w_{ij}^a \) is the attraction coefficient and \( w_{ij}^r \) is the repulsion coefficient and they represent the strength of attraction and repulsion forces between the node pair \((i, j)\), respectively. Here, \( w_{ij}^a \) is parameterized by the parameter \( \alpha \in \mathbb{R} \). One can write (4) in the following matrix form:

\[
\dot{x} = -BW(x, \alpha)B^T x
\]

where \( B \) is the incidence matrix of the graph \( G \) for an arbitrary orientation and \( \mathcal{V}(x, \alpha) = \text{diag}(\eta(x, \alpha)) \in \mathbb{R}^{|E| \times |E|} \) with

\[
\eta(x, \alpha) = \left( w_{ij}^a(\alpha) - w_{ij}^r(\alpha) \exp\left(-\frac{\|x_i - x_j\|^2}{c}\right)\right).
\]

D. Problem Statement

Consider a dynamical system on the graph \( G \) with the parameterized vector field \( f : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n \), in which the states of nodes evolve as follows:

\[
\dot{x} = f(x, \alpha)
\]

where \( x \in \mathbb{R}^n \) is the state of system, \( \alpha \in \mathbb{R} \) is the bifurcation parameter, and the parameterized vector field \( f \) takes either form of (2) or (5). We also assume the following.

1) We have access to the full state measurements \( x \).
2) We have access to the structure of the network, i.e., the node set \( \mathcal{V} \) and the edge set \( E \) of graph \( G \).
3) We have no knowledge of the network parameters, i.e., the weight matrix \( A \), the natural frequencies \( \omega_i \), the attraction and repulsion coefficients \( w_{ij}^a \) and \( w_{ij}^r \), and the bifurcation parameter \( \alpha \).

We know that parameter \( \alpha \) is changing and a bifurcation happens at some parameter value \( \alpha^* \). The problem is to detect when the system is close to bifurcation, i.e., \( \alpha \) is close to \( \alpha^* \). This aspect is a subtle point in our analysis because we can determine whether the system is close to bifurcation or not solely by studying the perturbed state. We address the above problem by studying the response of the system (6) to perturbations. The perturbations can be considered to be deterministic (i.e., disturbance signals with known magnitude) or stochastic (i.e., Gaussian noise with a known mean and covariance).

III. BIFURCATION ANALYSIS FOR 2-D SYSTEMS

For the sake of simplicity in the exposition, in this section and Section IV, we start with analyzing the bifurcation in networks with two agents. Our 2-D analysis can be also useful when studying aggregation of agents in a network or model-order reduction [34].

A. Nonlinear Coupled Oscillators

Consider a network consisting of two oscillators

\[
\begin{align*}
\dot{\theta}_1 &= \omega_1(\alpha) - k \sin(\theta_1 - \theta_2) \\
\dot{\theta}_2 &= \omega_2(\alpha) - k \sin(\theta_2 - \theta_1)
\end{align*}
\]

where \( \omega_i(\alpha) \) is the natural frequency of node \( i \), parameterized by the bifurcation parameter \( \alpha \in \mathbb{R} \), and \( k \) is a fixed coupling gain. We define \( \phi = \theta_1 - \theta_2 \) and \( \bar{\omega}(\alpha) = \omega_1(\alpha) - \omega_2(\alpha) \).

We show that when the ratio between the difference in natural frequencies and the coupling, i.e., \( \frac{\bar{\omega}(\alpha)}{2k} \), exceeds a threshold, a saddle node bifurcation happens.\(^3\) To show this, we subtract the two equations in (7) to get the reduced-order coupled oscillator dynamics as

\[
\dot{\phi} = \bar{\omega}(\alpha) - 2k \sin \phi.
\]

For \( \frac{\bar{\omega}(\alpha)}{2k} > 1 \), the reduced-order dynamics does not have any equilibrium point, as shown in Fig. 2(a). When \( \frac{\bar{\omega}(\alpha)}{2k} = 1 \), a saddle node bifurcation happens at \( \phi = \frac{\pi}{2} \) [Fig. 2(b)]. For \( \frac{\bar{\omega}(\alpha)}{2k} < 1 \), there are two equilibria, one stable and one unstable [Fig. 2(c)].

B. Attraction–Repulsion Dynamics

Bifurcations in different classes of attraction–repulsion dynamics has been studied in [35]. Consider two agents with scalar states \( x_1 \) and \( x_2 \). The attraction–repulsion dynamics are given by

\[
\begin{align*}
\dot{x}_1 &= -(x_1 - x_2) \left( w^a(\alpha) - w^r \exp\left(-\frac{\|x_1 - x_2\|^2}{c}\right)\right) \\
\dot{x}_2 &= -(x_2 - x_1) \left( w^a(\alpha) - w^r \exp\left(-\frac{\|x_2 - x_1\|^2}{c}\right)\right).
\end{align*}
\]

Subtracting the two equations and defining \( \phi = x_1 - x_2 \) give the reduced-order attraction–repulsion dynamics as follows:

\[
\dot{\phi} = -2\phi \left( w^a(\alpha) - w^r \exp\left(-\frac{\|\phi\|^2}{c}\right)\right).
\]

It can be shown that when the ratio between the repulsion coefficient and the attraction coefficient exceeds a threshold, a supercritical pitchfork bifurcation happens and the system transitions from one fixed point to three fixed points. For \( \frac{w^r}{w^a(\alpha)} \leq 1 \), the origin is a stable equilibrium point. For \( \frac{w^r}{w^a(\alpha)} > 1 \), the origin becomes unstable with two symmetrical stable equilibria at \( \phi_e = \pm (-c \ln \left(\frac{w^r}{w^a(\alpha)}\right))^\frac{1}{2} \), as shown in Fig. 2(e)–(g). In this figure, stable and unstable equilibrium points are shown with black and white circles, respectively. Fig. 2(h) shows the bifurcation diagram of (10).

IV. DETECTING CRITICAL TRANSITIONS FOR 2-D SYSTEMS

We focus on two different approaches to detect bifurcations in nonlinear systems: 1) Deterministic approach by considering \( ^3 \)See [1] for details about several types of bifurcations in nonlinear systems.
the recovery rate after perturbations, and 2) stochastic approach by considering the autocorrelation and variance of the state. Our framework is built upon the phenomenon of critical slowing down. It refers to the tendency of a system to take longer to return to equilibrium after perturbations, as discussed in [6].

### A. Deterministic Method

In this method, we consider the recovery rate of the system subject to a known and deterministic perturbation to detect bifurcations. For a given parameter $\alpha \in \mathbb{R}$, consider the dynamical system (6) and suppose that $\overline{\pi}_s(\alpha)$ is a locally asymptotically stable equilibrium point of (6). We study the behavior of (6) around $\overline{\pi}_s(\alpha)$ under a small perturbation $\epsilon$ and get

$$\frac{d(\overline{\pi}_s(\alpha) + \epsilon)}{dt} = f(\overline{\pi}_s(\alpha) + \epsilon).$$

Therefore, the dynamics for the perturbation $\epsilon$ around the equilibrium point $\overline{\pi}_s(\alpha)$ can be approximated by the following linear perturbation dynamics:

$$\frac{d\epsilon}{dt} = D_\overline{\pi}f(\overline{\pi}_s(\alpha)) \epsilon.$$  \hspace{1cm} (12)

Now we study the perturbation dynamics (12) for the reduced-order models of coupled oscillators dynamics (8) and attraction–repulsion dynamics (10) and propose algorithms for detection of critical transitions in these systems.

**Nonlinear coupled oscillators:** For the reduced-order coupled oscillator (8), bifurcation occurs when $\frac{\dot{\omega}(\alpha)}{2k} = 1$. For $\frac{\dot{\omega}(\alpha)}{2k} < 1$, the dynamical system (8) has two equilibrium points $\overline{\phi}_s = \arcsin\left(\frac{\dot{\omega}(\alpha)}{2k}\right)$ and $\overline{\phi}_u = \pi - \arcsin\left(\frac{\dot{\omega}(\alpha)}{2k}\right)$, as shown in Fig. 2(c). It is easy to see that $\overline{\phi}_s = \arcsin\left(\frac{\dot{\omega}(\alpha)}{2k}\right)$ is the locally asymptotically stable equilibrium point of (8). Using (12), the perturbation dynamics around $\overline{\phi}_s$ can be approximated by

$$\frac{d\epsilon}{dt} = -\cos\left(\arcsin\left(\frac{\dot{\omega}(\alpha)}{2k}\right)\right) \epsilon.$$  \hspace{1cm} (13)

For $\frac{\dot{\omega}(\alpha)}{2k} < 1$, $\epsilon = 0$ is an exponentially stable equilibrium point of (13) and, thus, the perturbation $\epsilon$ converges to zero. According to (13), by approaching to the bifurcation, i.e., $\frac{\dot{\omega}(\alpha)}{2k} \rightarrow 1$, the rate of the recovery of perturbation $\epsilon$ diminishes. Note that $\alpha$ is a parameter and remains independent of the perturbation state $\epsilon$. In other words, any changes in $\epsilon$ do not impact the value of $\alpha$.

**Attraction–repulsion dynamics:** For the reduced-order attraction–repulsion dynamics (10) bifurcation occurs when $\frac{w^r}{\omega^w(\alpha)} = 1$. For $\frac{w^r}{\omega^w(\alpha)} \leq 1$, the only equilibrium point of (10) is $\overline{\phi} = 0$ and, thus, (12) becomes

$$\frac{d\epsilon}{dt} = 2\left(w^r - \omega^w(\alpha)\right)\epsilon.$$  \hspace{1cm} (14)

For $\frac{w^r}{\omega^w(\alpha)} < 1$, based on (14), the perturbation goes exponentially to zero, i.e., $\epsilon(t) = (\epsilon(0))e^{2\left(w^r - \omega^w(\alpha)\right)t}$. By approaching to the bifurcation, i.e., $\frac{w^r}{\omega^w(\alpha)} \rightarrow 1$, the rate of the recovery after perturbation $\epsilon$ diminishes.

**Example 1:** Consider the coupled oscillator model (8) with the coupling $k = 2$. Two constant perturbation signal with value $\epsilon = 0.5$ is applied separately to state $\phi$ of the system from $t = 6 \, \text{s}$ to $t = 8 \, \text{s}$. The perturbation signal before bifurcation for different values of $\frac{\dot{\omega}}{2k}$ are shown in Fig. 3, (left). As shown in this figure, the closer the value of $\frac{\dot{\omega}}{2k}$ to the bifurcation point, i.e., $\frac{\dot{\omega}}{2k} = 1$, the larger the time required to return the perturbation to zero. The response at the bifurcation, $\frac{\dot{\omega}}{2k} = 1$, is shown in Fig. 3.
To show the type of bifurcation and the nature of the equilibrium point, we apply two perturbation signals with different signs. The perturbation with negative sign is recovered and the perturbation with a positive sign is no longer recovered. This shows that the origin is a saddle point.

Now, consider attraction–repulsion dynamics (10). The evolution of the perturbation signal $\epsilon$ for the attraction–repulsion dynamics is shown in Fig. 4 for different values of $\frac{w}{\alpha}$ and $c = 1$. The bifurcation occurs at $\frac{w}{\alpha} = 1$. Similar to the coupled oscillator dynamics, by approaching the bifurcation, the tendency of the system to recover the perturbation decreases. However, during and after bifurcation, the attraction–repulsion dynamics present a different behavior. In the bifurcation, the origin is still a globally stable equilibrium (unlike saddle-point equilibrium in the coupled oscillator dynamics). Thus, as shown in Fig. 4, (left), the perturbation will eventually be recovered to the origin. After bifurcation, Fig. 4, (right), the origin becomes unstable and the perturbation is recovered to one of the emerging stable equilibria, i.e., those shown in Fig. 2(g). The perturbation with negative sign is recovered to the positive stable equilibrium point and the perturbation with a positive sign is recovered to the negative stable equilibrium point. This shows that the origin (which was a stable before bifurcation) looses stability and two symmetric stable equilibria emerge around the origin. Hence, this is the supercritical pitchfork bifurcation.

B. Stochastic Method

In this section, we look at the autocorrelation of system’s state subject to a stochastic noise to detect bifurcations. The intuition is that near the bifurcation, the rates of change of the system’s states decrease. As a result, the state of the system at any given moment becomes more similar to its past state, which leads to an increase in the autocorrelation of system’s states. To formally show this, we can approximate the dynamics of the system close to the equilibrium $\mathbf{x}_s(\alpha)$ using the following linear system:

$$\frac{dx}{dt} = f(\mathbf{x}_s(\alpha)) + D_xf(\mathbf{x}_s(\alpha))(x - \mathbf{x}_s(\alpha)).$$  \hspace{1cm} (15)

Assume that a stochastic disturbance is applied in specific time steps $\Delta t$, for $\Delta t \in \mathbb{Z}_{>0}$ to the above dynamics. The state of the system after applying the stochastic white noise $\{\xi_t\}_{t=1}^{\infty}$ with variance $\sigma$ is given by

$$x_{t+1} - \mathbf{x}_s(\alpha) = e^{D_xf(\mathbf{x}_s(\alpha))\Delta t}(x_t - \mathbf{x}_s(\alpha)) + \xi_t$$  \hspace{1cm} (16)

for every $\alpha \in \mathbb{R}$. Following the argument in Section IV-A, for systems (8) and (10), the (approximated) exponential rates of recovery are:

$$D_xf(\mathbf{x}_s(\alpha)) = -\cos\left(\arcsin\left(\frac{\omega(\alpha)}{2k}\right)\right)$$

attraction–repulsion.

We define $\epsilon_t = x_t - \mathbf{x}_s(\alpha)$. If $\Delta t$ is independent of $x_t$, one can interpret (16) as a lag-1 autoregressive process

$$\epsilon_{t+1} = \gamma(\alpha)\epsilon_t + \xi_t$$  \hspace{1cm} (17)

where $\gamma(\alpha) = e^{D_xf(\mathbf{x}_s(\alpha))\Delta t}$. If the autocorrelation parameter $\gamma(\alpha)$ is close to zero, the state $x_t$ inherits the white noise characteristics of $\xi_t$ and if it is close to one, $\epsilon_{t+1}$ becomes closer to a red (autocorrelated) noise [36]. Since $|\gamma(\alpha)| < 1$, for stable system, the mean $E(\epsilon_t)$ is identical for all values of $t$ by the definition of wide sense stationarity and the variance is calculated as [36]

$$\lim_{t \to \infty} \text{var}(\epsilon_t) = \frac{\sigma^2}{1 - \gamma(\alpha)^2}.$$  

Close to the bifurcation, the recovery rate to the equilibrium is small, implying that $D_xf(\mathbf{x}_s(\alpha))$ approaches zero. Thus, the autocorrelation $\gamma(\alpha)$ tends to one and the variance of the error tends to infinity.

Example 2: Consider nonlinear coupled oscillator (8) with the coupling $k = 2$. A zero-mean white noise with variance $\sigma = 5$ is applied to the system at every integer instance of time $t \in \mathbb{Z}_{\geq 0}$. The moving variance of the state is shown in Fig. 5.
A. Nonlinear Coupled Oscillators

Consider the coupled oscillator dynamics (2) over a network $G = \{V, E, A\}$ with Laplacian matrix $L = BAB^\top$. We suppose that the vector of natural frequencies $\omega(\alpha) = (\omega_1(\alpha), \omega_2(\alpha), \ldots, \omega_n(\alpha))^\top$ is parameterized by a real-valued parameter $\alpha \in \mathbb{R}$ satisfying the following assumption.

**Assumption 1:** There exist a 2-cutset $\partial S$ for some $S \subseteq V$ and a bifurcation parameter value $\alpha^* \in \mathbb{R}$ such that the following holds:

i) for every $\alpha < \alpha^*$ we have $\|B^\top L^1 \omega(\alpha)\|_\infty < 1$

ii) if $\alpha = \alpha^*$, then $|(B^\top L^1 \omega(\alpha^*))_e| = 1$, for every $e \in \partial S$;

iii) for every $\alpha > \alpha^*$,

\[
\frac{1}{1 - (B^\top L^1 \omega(\alpha^*))_e} > 1, \text{ for every } e \in \partial S
\]

\[
\frac{1}{1 - (B^\top L^1 \omega(\alpha^*))_e} < 1, \text{ for every } e \in E \setminus \partial S.
\]

For acyclic undirected graphs, any single edge $e$ is an edge cut and the vector $B^\top L^1 \omega(\alpha) \in \mathbb{R}^m$ can be interpreted as the normalized flow on the cuts, i.e., $|(B^\top L^1 \omega(\alpha^*))_e|$, for $e \in \partial S$, increases throughout the bifurcation. The next proposition shows that the nonlinear coupled oscillator (2) undergoes a bifurcation at $\alpha = \alpha^*$.

**Proposition 1 (Bifurcation in oscillator networks):** Consider the nonlinear coupled oscillators (2) over a weighted undirected acyclic graph $G = \{V, E, A\}$. If the parameterized natural frequencies satisfy Assumption 1 for the singleton 2-cutset $\partial S = \{e\}$, then a bifurcation occurs in (2) such that the following holds:

i) For $\alpha < \alpha^*$, the dynamical system (2) has a locally asymptotically stable equilibrium manifold $\overline{\theta}_s(\alpha) + \text{span}\{1_n\}$ given by

\[
\overline{\theta}_s(\alpha) = L^1 B A \arcsin(B^\top L^1 \omega(\alpha))
\]

and an unstable equilibrium manifold $\overline{\theta}_u(\alpha) + \text{span}\{1_n\}$ where $\overline{\theta}_u = L^1 B A n$ with $n \in \mathbb{R}^|E|$ defined by

\[
n_i = \begin{cases} 
\arcsin(B^\top L^1 \omega(\alpha))_i & i \neq e \\
-\arcsin(B^\top L^1 \omega(\alpha))_i & i = e.
\end{cases}
\]

ii) For $\alpha > \alpha^*$, the dynamical system (2) has no equilibrium manifold.

**Remark 1:** The following remarks are in order.

i) In Proposition 1(i), as $\alpha$ converges to $\alpha^*$, the stable and unstable equilibrium manifold of (2) converges to each other, creating a saddle equilibrium manifold at $\alpha = \alpha^*$. As a result, this bifurcation can be considered a multidimensional version of the saddle-node bifurcation.

ii) Proposition 1(i) implies that, for $\alpha < \alpha^*$, there exist a locally stable and a locally unstable equilibrium manifolds for the coupled oscillators network (2). However, it does not exclude the existence of other equilibrium manifolds.

B. Attraction–Repulsion Dynamics

Consider the attraction–repulsion dynamics (5) over a undirected unweighted $G = \{V, E\}$, and suppose that the attraction coefficients $w^a = \{w^a_{ij}\}_{(i,j) \in E}$ are parameterized by a real-valued parameter $\alpha \in \mathbb{R}$ satisfying the following assumption.

**Assumption 2:** There exist a 2-cutset $\partial S$ for some $S \subseteq V$ and a bifurcation parameter value $\alpha^* \in \mathbb{R}$ such that the following holds:

i) If $\alpha < \alpha^*$, then $w^a_{ij}(\alpha) > w^r_{ij}$, for every $(i,j) \in E$.

ii) If $\alpha = \alpha^*$, then $w^a_{ij}(\alpha^*) = w^r_{ij}$ for all $(i,j) \in \partial S$.

iii) If $\alpha > \alpha^*$, then

\[
w^a_{ij}(\alpha) < w^r_{ij}, \text{ for every } (i,j) \in \partial S
\]

\[
w^a_{ij}(\alpha) > w^r_{ij}, \text{ for every } (i,j) \in E \setminus \partial S.
\]

If Assumption 2 holds, then at $\alpha = \alpha^*$, the attraction coefficients and the repulsion coefficients on the 2-cutset $\partial S$ are equal. The next proposition shows that, under Assumption 2, a bifurcation happens at $\alpha = \alpha^*$ for the attraction–repulsion dynamics (5), which leads to instability of the consensus. Due to space limitations, the proof is not presented here and can be found in [37].

**Proposition 2 (Bifurcation in attraction–repulsion dynamics):** Consider the attraction–repulsion dynamics (5) over an undirected unweighted network $G = \{V, E\}$. Then, if the parameterized attraction function satisfies Assumption 2 for a 2-cutset $\partial S$ for some $S \subseteq V$, then the following holds:

i) for $\alpha < \alpha^*$, the system (5) converges globally to the $\pi_s = \frac{1}{n} \underbrace{x, x, \ldots, x}_{n}$;

ii) for $\alpha > \alpha^*$, the subspace span{1n} is an unstable invariant manifold for the system (5).

**Remark 2:** Proposition 2 does not exclude the emergence of other contracting attractors after bifurcation. Indeed, in [30], it is shown that for attraction–repulsion dynamics (5) states of the agents converge to a positive invariant set defined as

\[
B_c = \{x : \|x - \pi_s\| \leq \beta\}, \text{ where } \beta = \frac{w^r}{w^a} \sqrt{\frac{c}{2}} \exp\left(\frac{-1}{2}\right).
\]

VI. DETECTION AND LOCALIZATION OF CRITICAL TRANSITIONS

Now, we extend the deterministic and the stochastic detection algorithms in Section IV to nonlinear networks.

A. Coupled Oscillator Networks

Consider the coupled oscillator dynamics (2) over a network $G = \{V, E, A\}$. Let $\overline{\theta}_s(\alpha) + \text{span}\{1_n\}$ be the locally asymptotically stable equilibrium manifold of (2) as described in Proposition 1. Using the change of coordinate $\epsilon = \theta - \overline{\theta}_s(\alpha)$, we can approximate the perturbation dynamics with the following linear system:

\[
\frac{d\epsilon}{dt} = -BA_\omega(\alpha)B^\top \epsilon
\]

where $A_\omega(\alpha) = A \text{ diag}(\cos(B^\top \overline{\theta}_s(\alpha)))$. We define the parameterized family of graphs $G(\alpha) = \{V, E, A_\omega(\alpha)\}$. Thus, the
matrix $B\mathcal{A}_\alpha(\alpha)B^\top$ is the Laplacian matrix of the graph $G(\alpha)$ and is positive semidefinite. We denote the eigenvalues of this Laplacian matrix by $0 = \lambda_1(\alpha) \leq \lambda_2(\alpha) \leq \ldots \leq \lambda_n(\alpha)$ and their associated normalized eigenvectors by $v_1(\alpha) = \frac{1}{\sqrt{n}}1_n, v_2(\alpha), \ldots, v_n(\alpha)$.

B. Attraction–Repulsion Networks

Consider the attraction–repulsion dynamics (5) over a undirected unweighted graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. Using the change of coordinate $x = x - \hat{x}_n$, where $\hat{x}_n = \frac{1}{n}1_n^\top x(0)$, we can approximate the perturbation dynamics with the following linear system:

$$\frac{d\epsilon}{dt} = -\frac{\partial}{\partial x} (BW(x, \alpha)B^\top x)$$

For every $x \in \mathbb{R}^n$ and every $\alpha \in \mathbb{R}$, we compute

$$\frac{\partial}{\partial x} (BW(x, \alpha)B^\top x) = -\frac{\partial}{\partial x} (BW(x, \alpha)) B^\top x - BW(x, \alpha)B^\top$$

For every $\alpha \in \mathbb{R}$, the first term in the right-hand side (RHS) of (20) is always zero when $x = \hat{x}_n$ because $B^\top \hat{x}_n = 0$. Moreover, using the fact that $\text{tr}(\hat{x}_n, \alpha^*) = 0_{n \times n}$, one can show the second term in the RHS of (20) is zero at the equilibrium point $x = \hat{x}_n$ at the bifurcation, $\alpha = \alpha^*$. Therefore, the perturbation dynamics (19) can be written as follows:

$$\frac{d\epsilon}{dt} = -B\mathcal{A}_\alpha(\alpha)B^\top \epsilon$$

where $\mathcal{A}_\alpha(\alpha) = \text{tr}(\hat{x}_n, \alpha)$. We define the parameterized family of graphs $G(\alpha) = (\mathcal{V}, \mathcal{E}, \mathcal{A}_\alpha(\alpha))$. Thus, the matrix $B\mathcal{A}_\alpha(\alpha)B^\top$ is the Laplacian matrix of the graph $G(\alpha)$ and is positive semidefinite. For the simplicity of notations, similar to the case of nonlinear oscillators, we denote the eigenvalues of this Laplacian matrix by $0 = \lambda_1(\alpha) \leq \lambda_2(\alpha) \leq \ldots \leq \lambda_n(\alpha)$ and their associated normalized eigenvectors by $v_1(\alpha) = \frac{1}{\sqrt{n}}1_n, v_2(\alpha), \ldots, v_n(\alpha)$.

The spectral properties of the Laplacian matrix $B\mathcal{A}_\alpha(\alpha^*)B^\top$ (respectively, $B\mathcal{A}_\alpha(\alpha^*)B^\top$) plays a critical role in our detection algorithm. Before we state the next lemma, recall that $\mathcal{G}$ is the lower bound graph associated with the parameterized graphs $G(\alpha)$ for $\alpha \in \mathbb{R}$ defined in Section II-A. The following lemma plays a crucial role in our detection and localization algorithms. The proof is not provided here due to space constraints and can be found in [37].

**Lemma 1**: Consider the perturbed coupled oscillator dynamics (18) [respectively, perturbed attraction–repulsion dynamics (21)] with bifurcation parameter $\alpha \in \mathbb{R}$ satisfying Assumption 1 (respectively, Assumption 2) with a 2-cutset $\partial S$ for some $S \subseteq \mathcal{V}$. Then, the following statements hold:

i) $\lim_{\alpha \to \alpha^*} \lambda_2(\alpha) = \lambda_2(\alpha^*) = 0$;

ii) $\lim_{\alpha \to \alpha^*} v_2(\alpha) = v_2(\alpha^*) = \frac{1}{\sqrt{n}}S$

where $\chi^S$ is the indicator vector of $S$ defined in Section II-A.

C. Network Critical Slowing Down: Deterministic Approach

Our key observation is that the system’s recovery rate converges to zero as it approaches the bifurcation. Moreover, the structure of the persisting perturbation determines the location of the bifurcation in the network. Now we can state the main result of this article.

**Theorem 1 (Convergence of perturbation dynamics)**: Consider, the nonlinear coupled oscillator network (2) [respectively, attraction–repulsion network (5)] satisfying Assumption 1 on an acyclic graph (respectively, satisfying Assumption 2 on an arbitrary graph). For every $\alpha < \alpha^*$ and every solution of $\epsilon : \mathcal{R}_{\geq 0} \to \mathcal{R}^n$ of the perturbation dynamics (18) [respectively, perturbation dynamics (21)], the following statements hold:

i) we have

$$\lim_{t \to \infty} \epsilon(t) = \frac{1}{n}1_n 1_n^\top \epsilon(0)$$

with the rate of convergence $\lambda_2(\alpha)$;

ii) we have

$$\lim_{t \to \infty} \left\| \epsilon(t) - \left(\frac{1}{n}1_n 1_n^\top + e^{-\lambda_2(\alpha)t} v_2(\alpha) v_2^\top(\alpha) \right) \epsilon(0) \right\|_2 = 0$$

uniformly in $\alpha$ with rate of convergence $\lambda_3(\mathcal{G})$.

**Remark 3 (Deterministic interpretation of network critical slowing down)**: Lemma 1 and Theorem 1(i) show that the rate of convergence of $\epsilon(t)$ to the consensus decreases to zero as the system gets closer to the bifurcation. This phenomenon, termed as “network critical slowing down,” is the generalization of the slowing down phenomenon discussed in Section IV. The uniform convergence in Theorem 1(ii) can be used to detect the bifurcation, as explained in the following section.

The proposed bifurcation detection and localization algorithm discussed in this section is based upon the phenomenon of critical slowing down discussed above. Using the uniform convergence result in Theorem 1(ii), for every $\alpha < \alpha^*$ the solution $\epsilon : \mathcal{R}_{\geq 0} \to \mathcal{R}^n$ of the perturbation dynamics satisfies

$$\left\| \epsilon(t) - \left(\frac{1}{n}1_n 1_n^\top + e^{-\lambda_2(\alpha)t} v_2(\alpha) v_2^\top(\alpha) \right) \epsilon(0) \right\|_2 \leq M e^{-\lambda_3(\mathcal{G})t}$$

where $M \in \mathcal{R}_{\geq 0}$ is a constant independent of $\alpha$. By Lemma 1, we know that $\lim_{\alpha \to \alpha^*} \lambda_2(\alpha) = 0$ and $\lambda_3(\mathcal{G}) > 0$. Thus, for $\alpha$ close enough to $\alpha^*$, we have $\lambda_2(\alpha) \leq \frac{1}{2} \lambda_3(\mathcal{G})$. Given a detection time scale $\zeta < 1$, and using $t \geq \frac{\ln(M \zeta^2)}{\lambda_3(\mathcal{G})}$, we get

$$Me^{-\lambda_3(\mathcal{G})t} \in O(\zeta^2)$$

$$e^{-\lambda_2(\alpha)t} v_2(\alpha) v_2^\top(\alpha) \epsilon(0) \in O(\zeta).$$

Thus, (22) shows that for a considerable time period, the term $e^{-\lambda_2(\alpha)t} v_2(\alpha) v_2^\top(\alpha) \epsilon(0)$ persists and can be used to detect the bifurcation.

To localize the bifurcating edge, according to Theorem 1, we define the vectors of residual measurement $r(t) := \epsilon(t) - \frac{1}{n}1_n 1_n^\top \epsilon(0)$ to approximate the direction of the Fiedler vector of $G(\alpha)$, denoted by $v_2(\alpha)$. It is well-known that the sign pattern of the Fiedler vector can be used to detect the graph’s edge cut [38]. Therefore, we can use the sign structure of the residual.
Algorithm 1: Bifurcation Detection and Localization.

Input: The initial perturbation $\epsilon(0)$, the detection time-scale $\zeta < 1$, and the residual threshold $\delta$.

Output: Detecting bifurcation and localizing bifurcating edge cut $\partial S$.

1: Set $t^* = \frac{\ln(\zeta^{-2})}{\kappa_3(\theta)}$
2: Collect $\{\epsilon(t)\}_{t \geq t^*}$
3: Compute the residual vector $r(t) := \epsilon(t) - \frac{1}{n} \bar{x}_n \bar{x}_n^\top \epsilon(0)$
4: if $\|r(t)\|_2 \geq \delta$ then
5: An edge cut undergoes a bifurcation.
6: $S := \{ j \in V : r_j(t) > 0 \}$
7: $\partial S$ are the bifurcating edges.

Fig. 6. Topology of the oscillator network in Example 3.

Remark 4: The following remarks are in order.

i) To identify the bifurcating edges based on Algorithm 1, we need to have a complete knowledge of the edge cut set $\partial S$. For that, a prior knowledge about the binary network topology is required to localize the bifurcating edges.

ii) Note that the values of the detection time-scale ($\zeta < 1$), the residual threshold ($\delta$), as well as the magnitude and structure of the initial perturbation $\epsilon(0)$ must be determined (or learned) based on the specification of the physical system, precision of measurements, and the application of interest.

iii) Algorithm 1 proposes a completely data-driven method for detection and localization of bifurcations. We can obtain the equilibrium manifold by having the state measurements (i.e., $\bar{x}$) over a long period of time. Then, $\epsilon = x - \bar{x}(\alpha)$ in Algorithm 1 is used for detection and localization of bifurcating edges.

iv) In many applications, the initial perturbation is the last instance of an exogenous signal $u_{\text{exg}} : [0, t_0] \to \mathbb{R}^n$ applied on the dynamical network. In this case, we directly use Theorem 1 for the initial condition $\epsilon(t_0)$, instead of $\epsilon(0)$. We use this setup for our simulations in the next example.

Example 3: Consider a coupled oscillator network (2) over an acyclic graph, shown in Fig. 6. The objective is to show the performance of Algorithm 1 on the nonlinear oscillator network in detecting and identifying the bifurcating edge; here, the edge (1,3). A perturbation signal $u_{\text{exg}} = [2 \ 0 \ 0 \ 0 \ 0 \ 0]^\top$ is applied on the network from $t = 2$ s to $t = 3$ s. The components of the residual signal $r$, defined in Algorithm 1, for a system away from and close to the bifurcation are shown in Fig. 7(a) and (b), respectively. By looking at the residual at $t \approx 5$ s, we see that its magnitude for the system close to the bifurcation is more than ten times larger than that of the system away from bifurcation.

D. Network Critical Slowing Down: Stochastic Approach

We now revisit the stochastic detection method discussed in Section 4-B and extend it to networks. Similar to the scalar case, we assume that stochastic disturbances are applied in specific time steps $\Delta t$, for $\Delta t \in \mathbb{Z}_{\geq 0}$ to the dynamical system. In this case, for close enough states to the $\bar{x}_s(\alpha)$, the dynamics of the unperturbed systems is approximated by $\bar{x}(\alpha)$, where $\bar{x}(\alpha)$ is given by

$$ x_{t+1} = \bar{x}(\alpha) + \Delta t \frac{\partial f}{\partial x}(\bar{x}(\alpha)) + \xi_t. $$

Thus, by introducing $e_t := x_t - \bar{x}(\alpha)$, we get

$$ e_{t+1} = \Gamma(\alpha)e_t + \xi_t $$

where $\Gamma(\alpha)$ is given by $\frac{\partial f}{\partial x}(\bar{x}(\alpha)) \Delta t$, Equation (24) can be considered as a multivariable autoregressive model (17). We need some prepossessing on the state data before further analysis. Let $Q \in \mathbb{R}^{m \times m}$ be

Authorized licensed use limited to the terms of the applicable license agreement with IEEE. Restrictions apply.
Evolution of the residual signal $r$ for attraction–repulsion dynamics (5) over the network in Fig. 6, (a) far from bifurcation, (b) when edge (2,3) becomes close to bifurcation, (c) when edge (3,4) becomes close to bifurcation.

Let $\mathbb{R}^{n \times n}$ be a matrix such that the following holds:

$$QQ^\top = I_{n-1}, \quad Q^\top Q = I_n - \frac{1}{\sigma^2} \mathbb{1}_n \mathbb{1}_n^\top.$$  

(25)

The proof of the following lemma is omitted due to space limitations and can be found in [37].

Lemma 2: Defining $\mathbb{E}_t = Qe_t$ and $\mathbb{E}(\alpha) = Q\mathbb{E}(\alpha)Q^\top$, we have

$$\lim_{t \to \infty} \text{tr}(\text{Cov}[\mathbb{E}_t]) = \sum_{i=1}^{n} \frac{\sigma^2}{1 - \lambda_i^2(\mathbb{E}(\alpha))}.$$  

The following two theorems characterize the error covariance of nonlinear coupled oscillators and attraction–repulsion dynamics at bifurcation.

Theorem 2: Consider the nonlinear coupled oscillator network (2) with parameterized natural frequencies satisfying Assumption 1 and attraction–repulsion dynamics (5) with attraction parameters satisfying Assumption 2. Both dynamics are subjected to a stochastic white noise process $\{\mathbb{E}_t\}_{t \geq 1}$ with covariance $\sigma I$. Then, the following statements hold.

i) For $\alpha < \alpha^*\text{,}$ for both dynamics, we have

$$\lim_{t \to \infty} \text{tr}(\text{Cov}[\mathbb{E}_t]) < \infty.$$  

ii) At bifurcation, i.e., $\alpha = \alpha^*$, for both dynamics, we have

$$\lim_{t \to \infty} \text{tr}(\text{Cov}[\mathbb{E}_t]) = \infty.$$  

Remark 5 (Stochastic interpretation of network critical slowing down): On the verge of a bifurcation, the impacts of shocks decay very slowly and their accumulating effect increases the variance of the state variable as shown in Theorem 2. In principle, critical slowing down could reduce the ability of the system to track the fluctuations, and thereby, produce an opposite effect on the variance.

Remark 6 (Decentralized detection): Theorem 2 naturally leads to a stochastic algorithm for detection of bifurcations in coupled oscillator networks and attraction–repulsion dynamics. One of the key features of this algorithm is that it can be implemented in a completely decentralized fashion. Indeed, each agent has access to its own state and can compute the corresponding variance. The trace of the covariance goes to infinity if and only if the variance of at least one node goes to infinity. Hence, if a node finds its variance going to infinity, without getting information from other nodes, it can raise an alarm for the bifurcation in the network.

VII. DISCUSSION: COMPARING DETERMINISTIC AND STOCHASTIC APPROACHES

Each of the above deterministic and stochastic method has limitations and advantages compared with the other.

A. Nonintrusive Nature and Measurement Precision

One of the advantages of stochastic method (compared with the deterministic method) is its nonintrusive nature. Indeed, the stochastic approach to detect critical transitions relies on the additive noise, which usually exists in most physical systems. Moreover, in the stochastic approach, the detection of the bifurcation is solely based on the growth of the state variance. Hence, even a coarse threshold for the variance is enough to detect the critical transition. However, in the deterministic method, according to Algorithm 1, some prior knowledge about the system, i.e., the values of $\delta$ and $\zeta$ are required to detect and localize the bifurcation.

B. Localization

While the deterministic method can precisely localize the bifurcating edges, the stochastic approach can only detect the occurrence of a bifurcation without localizing it in the network. In fact, this extra feature of the deterministic method, in both detecting and localizing the bifurcation in networks, comes with the price of being intrusive and requiring precise measurements.

C. Decentralized Implementation

Unlike the deterministic approach, the stochastic detection method can be implemented in a decentralized manner. The mechanism was discussed in Remark 6.

The advantages and limitations of deterministic and stochastic methods are summarized in Table I.

VIII. CONCLUSION

This article presents a new approach to detect and identify critical transitions in nonlinear networks. We propose deterministic and stochastic data-driven methods based on critical slowing down. The deterministic method detects slow recovery rates...
after perturbations, whereas the stochastic method measures state variance with noise. We also develop a heuristic algorithm to identify critical links. Comparative analysis is conducted to evaluate the effectiveness of both methods.

Our work can be extended in multiple ways. Generalizing our methods to a broader range of nonlinear systems, including well-known ones like Duffing and Van der Pol oscillators, is a future direction. Exploring other network topologies, such as digraphs, is also worth investigating. Quantifying indices based on the distance to bifurcation and analyzing higher order moments, like Skewness and Kurtosis [11], in nonlinear networks are additional interesting avenues for research.

APPENDIX

A. Proof of Proposition 1

Proof: For every \( \alpha \in \mathbb{R} \), the equilibrium points of (2) are the solutions of the following algebraic equation:

\[
\omega(\alpha) = BA \sin(B^T \theta).
\]

Thus, by multiplying both sides of the algebraic equation (26) in \( B^T L \), we get

\[
B^T L^T \omega(\alpha) = B^T L^T BA \sin(B^T \theta) = \sin(B^T \theta) \tag{27}
\]

where the last equality holds, because we have \( B^T L^T BA = I_n \) for acyclic graphs [24, Th. 5]. Regarding part (i), for \( \alpha < \alpha^* \), by Assumption 1, we get \( ||B^T L^T \omega(\alpha)||_\infty < 1 \). Therefore, for a solution \( \omega(\alpha) \) of (27)

\[
\begin{align*}
(B^T \bar{\theta}(\alpha))_i &= \arcsin(B^T L^T \omega(\alpha))_i \\
(B^T \bar{\theta}(\alpha))_i &= \pi - \arcsin(B^T L^T \omega(\alpha))_i
\end{align*}
\]

for every \( i \in \mathcal{E} \). Using some algebraic manipulations, one can find two equilibrium manifolds \( \bar{\theta}_s(\alpha) + \text{span}\{1_n\} \) and \( \bar{\theta}_u(\alpha) + \text{span}\{1_n\} \) for the dynamics (2) given by

\[
\bar{\theta}_s(\alpha) = L^T BA \sin^{-1}(B^T L^T \omega(\alpha))
\]

\[
\bar{\theta}_u(\alpha) = L^T BA \cos^{-1}(B^T L^T \omega(\alpha))
\]

where \( n \in \mathbb{R}[\mathcal{E}] \) is defined as follows:

\[
n_i = \begin{cases} 
\arcsin(B^T L^T \omega(\alpha))_i & i \neq e \\
\pi - \arcsin(B^T L^T \omega(\alpha))_i & i = e.
\end{cases}
\]

Regarding the local stability of the manifolds \( \bar{\theta}_s(\alpha) + \text{span}\{1_n\} \) and \( \bar{\theta}_u(\alpha) + \text{span}\{1_n\} \), we compute the Jacobian of the system (2)

\[
J_\theta = -BA \text{ diag}(\cos(B^T \theta)) B^T.
\]

At \( \theta = \bar{\theta}_s(\alpha) \), we have \( \frac{\pi}{2} 1_m < B^T \bar{\theta}_s(\alpha) < \frac{\pi}{2} 1_m \) and, therefore, \( \cos(B^T \bar{\theta}_s(\alpha)) > 0 \). This implies that \( \text{diag}(\cos(B^T \bar{\theta}_s(\alpha))) \) is a positive diagonal matrix and, therefore, \( J_{\bar{\theta}_s(\alpha)} \) is a negative semidefinite matrix with the kernel \( \text{span}\{1_n\} \). This means that \( \bar{\theta}_s(\alpha) + \text{span}\{1_n\} \) is locally asymptotically stable equilibrium manifold of the system.

At \( \theta = \bar{\theta}_u(\alpha) \), we have \( \frac{\pi}{2} < (B^T \bar{\theta}_u(\alpha))_e < \pi \). This implies that \( \cos(B^T \bar{\theta}_u(\alpha))_e < 0 \). Let \( e_e \in \mathbb{R}[\mathcal{E}] \) be the \( e \)th standard basis of \( \mathbb{R}^{[\mathcal{E}]} \). Since the graph \( G = \{V, E, A\} \) is acyclic, there exists \( v \in \mathbb{R}^n \), such that \( B^T v = e_e \). Therefore

\[
v^T J_{\bar{\theta}_u(\alpha)} v = -e_e^T A \text{ diag}(B^T \bar{\theta}_u(\alpha)) e_e
\]

\[
= -e_e^T A \cos((B^T \bar{\theta}_u(\alpha)) e_e) > 0.
\]

As a result, \( \bar{\theta}_u(\alpha) + \text{span}\{1_n\} \) is an unstable equilibrium manifold of the system.

Regarding part (ii), suppose that \( \bar{\theta}(\alpha) \) is an equilibrium point of the system, for some \( \alpha > \alpha^* \). Then, by (27), we have \( B^T L^T \omega(\alpha) = \sin(B^T \bar{\theta}(\alpha)) \), for some \( \alpha > \alpha^* \). This implies that \( ||B^T L^T \omega(\alpha)||_\infty \leq 1 \) and, therefore, we get \( ||(B^T L^T \omega(\alpha))_e||_1 \leq 1 \), for some \( \alpha > \alpha^* \). However, this is in contradiction with Assumption (1). Thus, the network (2) has no equilibrium manifold. □

B. Proof of Theorem 1

Proof: First, note that for coupled-oscillator network (2) [respectively, attraction–repulsion network (5)], for every \( \alpha \in \mathbb{R} \) and \( t \geq 0 \), the solution of the perturbation dynamics (18) [respectively, the perturbation dynamics (21)] is given by

\[
\epsilon(t) = \sum_{i=1}^{n} e^{-\lambda_i(\alpha)t} v_i(0) v_i^T(0) \epsilon(0) \tag{29}
\]

where \( 0 = \lambda_1(\alpha), \ldots, \lambda_n(\alpha) \) are the eigenvalues with the associated normalized eigenvectors \( \frac{1}{\sqrt{n}} 1_n = v_1(\alpha), \ldots, v_n(\alpha) \) for the matrix \( B A \text{ diag}(\cos(B^T \theta(\alpha))) B^T \) (respectively, the matrix \( BW(x, \alpha) B^T \)).

First, we focus on the coupled-oscillator network (2) and its associated perturbation dynamics (18). Regarding part (i), let \( \epsilon : \mathbb{R}_{\geq 0} \to \mathbb{R}^n \) be the solution of (18) for \( \alpha < \alpha^* \). Using Proposition 1(i), for every \( \alpha < \alpha^* \)

\[
B^T \bar{\theta}_s(\alpha) = \arcsin(B^T L^T \omega(\alpha)) \in (-\frac{\pi}{2} 1_m, \frac{\pi}{2} 1_m).
\]

This implies that \( \cos(B^T \bar{\theta}_s(\alpha)) > 0 \) and, therefore, \( 0 = \lambda_1(\alpha) < \lambda_2(\alpha) \leq \cdots \leq \lambda_n(\alpha), \) for every \( \alpha < \alpha^* \). As a result, using the formula (29), we get \( \lim_{t \to \infty} \epsilon(t) = \frac{1}{n} 1_n 1_n^T \epsilon(0) \). Regarding part (ii), using the formula (29), for every \( t \geq 0 \)

\[
n(0) + \sum_{i=3}^{n} e^{-\lambda_i(\alpha)t} v_i(0) v_i^T(0) \epsilon(0).
\]

(30)

Since the singleton \( \{\epsilon\} \) in a 2-cutset and using Lemma 1, we can deduce that \( 0 < \lambda_3(\alpha), \) for every \( \alpha < \alpha^* \). By continuity of \( \lambda_3(\alpha) \) on the parameter \( \alpha [39, \text{Lemma 4.3}], \) we get

\[
0 < \lambda_3(\overline{G}) \leq \lambda_3(\alpha) \leq \cdots \leq \lambda_n(\alpha), \quad \text{for all } \alpha < \alpha^*.
\]

Therefore, for every \( t \geq 0 \), we get

\[
\|\epsilon(t) - \frac{1}{n} 1_n 1_n^T \epsilon(0) - e^{-\lambda_2(\alpha)t} v_2(0) v_2^T(0) \epsilon(0)\|_2
\]

\[
\leq e^{-\lambda_3(\overline{G})t} \sum_{i=3}^{n} ||v_i(0) v_i^T(0)||_2 \|\epsilon(0)\|_2.
\]
Note that \( v_i(\alpha) \) is normalized for every \( \alpha < \alpha^* \). This implies that \( \|v_i(\alpha)v_i^\top(\alpha)\|_2 = 1 \). As a result, we get

\[
\|\varepsilon(t) - \frac{1}{n} n_i \mathbb{1}_{n_i} v_i(0) - e^{-\lambda_2(\alpha)} t v_2(\alpha) \varepsilon(0)\|_2 \\
\leq \eta e^{-\lambda_2(\mathcal{Q})} \|\varepsilon(0)\|_2.
\]

Thus, it is clear that as we take \( t \to \infty \), \( \varepsilon(t) \) converges to \( \frac{1}{n} n_i \mathbb{1}_{n_i} v_i(0) + e^{-\lambda_2(\alpha)} t v_2(\alpha) \varepsilon(0) \) uniformly in \( \alpha \) with exponential convergence rate \( \lambda_2(\mathcal{Q}) \).

Second, we focus on attraction–repulsion networks (5) and the associated perturbed dynamics (21). Regarding part (i), using Proposition 2, we have \( \lim_{t \to \infty} \varepsilon(t) = \frac{1}{n} n_i \mathbb{1}_{n_i} \varepsilon(0) \). For part (ii), the proof is similar to the coupled oscillator network case and we omit it for the sake of brevity.

\[\Box\]

**C. Proof of Theorem 2**

Regarding the nonlinear coupled oscillator network (2). Considering \( D_i \mathcal{f}(\mathcal{X}_s) = -BA_{co}(\alpha) B^\top \), we can write

\[
\Gamma(\alpha) = \frac{1}{n} n_i \mathbb{1}_{n_i} \mathbb{1}_{n_i}^\top + \sum_{i=2}^n e^{-\lambda_i(\alpha)} \Delta t v_i(\alpha) v_i^\top(\alpha)
\]

\[
\Gamma(\alpha) = \sum_{i=2}^n e^{-\lambda_i(\alpha)} \Delta t Q v_i(\alpha) v_i^\top(\alpha) Q^\top.
\]

Based on Lemma 1, we have \( \lim_{\alpha \to \alpha^*} \lambda_2(\alpha) = \lambda_2(\alpha^*) = 0 \). Moreover, as \( t \to \infty \), the matrix \( \Gamma(\alpha) \) is given by

\[
\lim_{t \to \infty} \Gamma(\alpha) = Q v_2(\alpha^*) v_2^\top(\alpha^*) Q^\top.
\]

Using this equation along with (25), we can verify that \( \Gamma(\alpha^*) \) has an eigenvalue \( \lambda_0 = 1 \) corresponding to eigenvector \( Q v_2 \).

Therefore, according to Lemma 2, at \( \alpha = \alpha^* \) we have

\[
\lim_{t \to \infty} \text{tr} (\text{Cov}(\mathcal{X}_t)) = \sum_{i=2}^n \frac{\sigma^2}{1 - \lambda_i^2(\Gamma(\alpha^*))} = \infty.
\]

For attraction–repulsion dynamics (5), the proof follows the same procedure as above.

\[\Box\]

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