Some eigenstates for a model associated with solutions of tetrahedron equation.

V. Two cases of string superposition

I.G. Korepanov

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Abstract

In paper IV (solv-int/9704013) we have considered a string living in the infinite lattice that was, in a sense, generated by a “particle”. Here we show how to construct multi-string eigenstates generated by several particles. It turns out that, at least in some cases, this allows us to bypass the difficulties of constructing multi-particle states. We also present and discuss the “dispersion relations” for our particles–strings.

Introduction

Let us recall that we have introduced in paper [1] some “one-particle” eigenstates for the model based upon solutions of the tetrahedron equation. In the same paper, we have also constructed some “two-particle” states. However, some special condition arised in this construction, and the superposition of two arbitrary one-particle states was not achieved. Even the “creation operators” of paper [2] did not give a clear answer concerning multi-particle states.

On the other hand, in paper [3] we have brought in correspondence to a one-particle state some new state that could be called “one open string”. It was done using some special “kagome transfer matrix”. Here we will show that the superposition of such one-string states is easier to construct, because of degeneracy of kagome transfer matrix: it turns into zero the “obstacles” that hampered constructing of multi-particle states.

The scheme of string—particle “marriage” in [3] was as follows: take a one-particle state from [1, 2], and apply to it a kagome transfer matrix with boundary
conditions corresponding to the presence of two string tails in the infinity, e.g. like this: 

\[ ✞ ✝ \]

In this paper, we are going to complicate this scheme in the following way: the boundary conditions will correspond to the presence of an even number of string tails at the infinity, and instead of a one-particle state, we will use some special multi-particle vector \( \Psi \). Its peculiarity will be in the fact that \( \Psi \) is \textit{no longer an eigenstate} of the hedgehog transfer matrix \( T \) defined in \[1\]. Instead, it will obey the condition

\[ T\Psi = \lambda \Psi + \Psi', \tag{1} \]

where \( \lambda = \text{const} \), and \( \Psi' \) is annulled by the kagome transfer matrix of paper \[4\] which we will denote \( K \).

Recall that we have defined \( T \) in such a way that its degrees could be described geometrically as “oblique slices” of the cubic lattice. The transfer matrix \( T \) can be passed through the transfer matrix \( K \):

\[ TK = KT, \tag{2} \]

the boundary conditions (such as the number and form of tails at the infinity) for \( K \) being intact. Define vector \( \Phi \) as

\[ \Phi = K\Psi. \]

This together with (1) and (2) gives

\[ T\Phi = \lambda \Phi, \tag{3} \]

exactly as needed for an eigenvector.

We also present in this paper the “dispersion relations” for our particles–strings in a workable form—something that was missing in papers \[1, 2\].

1 Eigenvectors of the “several open strings” type for the infinite lattice

Let there be \( n \) one-particle amplitudes \( \varphi_{A^{(1)}} \), \ldots, \( \varphi_{A^{(n)}} \) of the same type as those described in the work \[1\]. Let us compose an “\( n \)-particle vector” \( \Psi \), i.e. put in correspondence to each unordered \( n \)-tuple of vertices \( A^{(1)}, \ldots, A^{(n)} \) of the kagome lattice the symmetrized amplitude in the following way:

\[ \psi_{A^{(1)}, \ldots, A^{(n)}} = \sum_n \varphi_{A^{(1)}} \cdots \varphi_{A^{(n)}}, \tag{4} \]
where $s$ runs through the group of all permutations of the set \{1, \ldots, n\}.

As for the boundary conditions for the transfer matrix $K$ described in the Introduction, let us assume that there are exactly $2n$ string tails, and they all go in positive directions, that is between the east and the north. Thus, in each of the points $A^{(1)}, \ldots, A^{(n)}$ a string is created, and they are not annihilated.

The vector $\Psi$ is not an eigenvector of transfer matrix $T$ due to problems arising when two or more points $A^{(k)}$ get close to one another. Nevertheless, the vector $\Phi = K\Psi$ is an eigenvector, because for it those problems disappear due to the simple fact: creation of two or more strings within one triangle of the kagome lattice is geometrically forbidden.

2 Eigenvectors of the “closed string” type for the infinite lattice

In this section, we will put in correspondence to each unordered pair of vertices of the infinite kagome lattice an “amplitude” $\Psi_{AB}$ according to the following rules. If one of the vertices, say $A$, precedes the other one, say $B$, in the sense that they can be linked by a path—a broken line—going along lattice edges in positive directions, namely northward, eastward, or to the north-east, then let us put

$$\Psi_{AB} = \varphi_A \psi_B - \psi_A \varphi_B,$$

where $\varphi$ and $\psi$ are two one-particle amplitudes of the same type as in paper [1].

In the case if vertices $A$ and $B$ cannot be joined by a path of such kind, let us put

$$\Psi_{AB} = 0.$$

The values $\Psi_{AB}$ are components of the vector $\Psi$ that belong to the two-particle subspace of tensor product of two-dimensional spaces situated in all kagome lattice vertices. What prevents $\Psi_{AB}$ from being an eigenvector of the hedgehog transfer matrix is discrepancies arising near those pairs $A, B$ that lie at the “border” between such pairs where one of the vertices precedes the other (so to speak, “the interval $AB$ is timelike”), and such pairs where it does not (“the interval $AB$ is spacelike”).

Those discrepancies, however, disappear for the vector $\Phi = K\Psi$, where $K$ is the kagome transfer matrix described in the Introduction with the boundary conditions reading no string tails at the infinity. This is because if a string cannot, geometrically, be created at the point $A$ (or $B$) and then annihilated at the point $B$ (or $A$), then the amplitude $\Psi_{AB}$ doesn’t influence at all the vector $\Phi$. The only thing that remains to be checked for (3) to hold is a situation where $A$ and $B$ are in the same kagome lattice triangle that will be turned inside out by one of the hedgehogs.
The constructed eigenvectors of transfer matrix $T$ are of course eigenvectors for translation operators through periods of kagome lattice as well. Let us consider here relations between the corresponding eigenvalues, starting from the simplest one-particle eigenstate.

Consider once again some triangle $ABC$ of the kagome lattice, and its image $A'B'C'$ under the action of $S$-matrix-hedgehog, as in Figure 2. Let us write out some relations of the type (6), namely

\[
\begin{pmatrix}
\varphi_{A'} \\
\varphi_{B'}
\end{pmatrix} = \begin{pmatrix}
\alpha & \beta \\
\gamma & \delta
\end{pmatrix}
\begin{pmatrix}
\varphi_A \\
\varphi_B
\end{pmatrix},
\]

\[
\begin{pmatrix}
\psi_{A'} \\
\psi_{B'}
\end{pmatrix} = \begin{pmatrix}
\alpha & \beta \\
\gamma & \delta
\end{pmatrix}
\begin{pmatrix}
\psi_A \\
\psi_B
\end{pmatrix},
\]

(6)

where

\[
\alpha = -\delta, \quad \alpha\delta - \beta\gamma = -1.
\]

(7)

It follows from the formulas (6) and (7) that

\[
\varphi_A\psi_B - \varphi_B\psi_A = \varphi_{B'}\psi_{A'} - \varphi_{A'}\psi_B,
\]

i.e.

\[
\Psi_{AB} = \Psi_{B'A'},
\]

exactly what was needed to comply with the fact that an $S$-operator-hedgehog acts as a unity operator in the two-particle subspace.

### 3 Dispersion relations

The constructed eigenvectors of transfer matrix $T$ are of course eigenvectors for translation operators through periods of kagome lattice as well. Let us consider here relations between the corresponding eigenvalues, starting from the simplest one-particle eigenstate.

Consider once again some triangle $ABC$ of the kagome lattice, and its image $A'B'C'$ under the action of $S$-matrix-hedgehog, as in Figure 2. Let us write out some relations of the type (6), namely

\[
\begin{pmatrix}
\varphi_{A'} \\
\varphi_{B'}
\end{pmatrix} = \begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
\begin{pmatrix}
\varphi_A \\
\varphi_B
\end{pmatrix},
\]

(8)
where \( \varphi \) is any one-particle vector, and the numbers \( a, \ldots, \tilde{a} \) satisfy conditions of type (7), i.e.

\[
a = -d, \quad ad - bc = -1, \\
\tilde{a} = -\tilde{d}, \quad \tilde{a}d - \tilde{b}c = -1.
\]

From (8) follows

\[
\frac{\varphi_B}{\varphi_{B'}} = -\frac{a}{\varphi_A/\varphi_{A'}} + 1. \\
(\varphi_A/\varphi_{A'}) - a \tag{10}
\]

and from (9) follows

\[
\frac{\varphi_C}{\varphi_{C'}} = -\frac{\tilde{a}}{(\varphi_B/\varphi_{B'})} + 1. \\
(\varphi_B/\varphi_{B'}) - \tilde{a}
\]

Surely, the numbers \( a \) and \( \tilde{a} \) depend on an \( S \)-operator-hedgehog. On the other hand, this latter is parameterized by exactly two parameters. So, it seems that it makes sense to take \( a \) and \( \tilde{a} \) as those parameters.

We can take for eigenvalue of the hedgehog transfer matrix \( T \) either \( \varphi_A'/\varphi_A \), or \( \varphi_B'/\varphi_B \), or \( \varphi_C'/\varphi_C \). These variants correspond, strictly speaking, to different definitions of \( T \), but each of them is consistent with the requirement that the degrees of \( T \) must be represented graphically as “oblique layers” of cubic lattice (the difference being that, with the three different definitions, the action of transfer matrix \( T \) corresponds to the shifts through cubic lattice periods along three different axes). Our goal is to express the eigenvalues of translation operators acting within the kagome lattice for a given one-particle state through, say, \( \varphi_A'/\varphi_A \).

If we speak about translation through one lattice period to the right in the sense of Figures 2 and 3 then this eigenvalue is \( \varphi_D/\varphi_A \). It is clear that

\[
\frac{\varphi_D}{\varphi_B} = \frac{\varphi_A'}{\varphi_{B'}}
\]
the ratios of values \( \varphi \) in the triangle \( DBE \) are the same as in \( A'B'C' \). Thus,

\[
\frac{\varphi_D}{\varphi_A} = \frac{\varphi_{A'}}{\varphi_{B'}} = \frac{\varphi_{A'}}{\varphi_A} \cdot \frac{-a(\varphi_A/\varphi_{A'}) + 1}{(\varphi_A/\varphi_{A'}) - a}
\]

(11)

(we have used (10). A similar relation can be written out for the translation through one lattice period in upward direction in the sense of Figures 2 and 3, namely

\[
\frac{\varphi_C}{\varphi_E} = \frac{\varphi_{B'}}{\varphi_B} \cdot \frac{-\tilde{a}(\varphi_B/\varphi_{B'}) + 1}{(\varphi_B/\varphi_{B'}) - \tilde{a}},
\]

(12)

where one has to substitute the expression (10) for \( \varphi_B/\varphi_{B'} \).

It is clear that the “dispersion relations” of type (11–12) survive also for a string “created by a particle”, if we substitute the eigenvalue of transfer matrix \( T \) instead of \( \varphi_{A'}/\varphi_A \), and the eigenvalues of translation operators instead of \( \varphi_D/\varphi_A \) and \( \varphi_{C'}/\varphi_E \). As for the multi-string states, all of the eigenvalues are obtained for them as products of corresponding eigenvalues for each string.

4 Discussion

We have shown in this paper that the string—particle “marriage” from paper 4 makes possible a simple and clear construction of at least some multi-string states. Recall that, from all the corresponding multi-particle states, we only could explicitly construct some two-particle states 1, with an additional restriction that could be formulated as “the total momentum of two particles is zero”. As for
the present paper, the momenta of “particles” generating the multi-string states of Sections 1 and 2 can change independently.

These states have been constructed for the infinite kagome lattice. We have to recognize that constructing such states on a finite lattice remains an open problem.

It is also unclear how to combine the results of Sections 1 and 2, i.e. construct such states with string “creation” and “annihilation” where the total number of “creating” and “annihilating” particles would be more than two. Note that in Section 1 we use the symmetrized product of one-particle amplitudes, while in Section 2—the antisymmetrized one.

Concerning the dispersion relations of Section 3, let us remark that perhaps there are too many of them. It is probably caused by the fact that, for now, we managed to construct not all one-particle and/or one-string states.

References

[1] I.G. Korepanov, Some eigenstates for a model associated with solutions of tetrahedron equation, solv-int/9701016, 7p.

[2] I.G. Korepanov, Some eigenstates for a model associated with solutions of tetrahedron equation. II. A bit of algebraization, solv-int/9702004, 8p.

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