ON A PROBLEM BY ARENS, GOLDBERG, AND LUXEMBURG

RAYMOND REDHEFFER AND TERENCE TAO

Abstract. We construct a normed algebra $A$ with norm $N(\cdot)$ over the reals, which is quadrative in the sense that $N(A^2) \leq N(A)^2$ for all $A \in A$, but is not 3-bounded in the sense that $N(A^3) \leq N(A)^3$. This answers a question of Arens, Goldberg, and Luxemburg.

Let $A$ be a normed algebra over a field $F$, either $\mathbb{R}$ or $\mathbb{C}$. In [2] the norm $N$ of the algebra is called quadrative if $N(A^2) \leq N(A)^2$ for all $A \in A$, k-bounded for a positive integer $k$ if $N(A^k) \leq N(A)^k$ for all $A \in A$, and strongly stable if it is k-bounded for all $k = 1, 2, 3, \ldots$. It was seen in [1] that boundedness for a particular $k > 2$ does not ensure strong stability or even quadrativity. Let $W = (\omega_{ij})$ be a fixed $2 \times 2$ matrix of positive entries. Then for the $W$-weighted sup norm on $\mathbb{C}^{2 \times 2}$, the algebra of $n \times n$ complex matrices,

$$||A||_{W,\infty} = \max_{i,j} \omega_{ij} |\alpha_{ij}|, \quad A = (\alpha_{ij}) \in \mathbb{C}^{2 \times 2},$$

Arens and Goldberg proved:

**Theorem** [1, Theorem 2]. If $k \geq 3$, then there exists a $2 \times 2$ weight matrix $W$ for which $|| \cdot ||_{W,\infty}$ is $k$-bounded but not strongly stable, in fact not even quadrative on $\mathbb{C}^{2 \times 2}$.

Our main theorem gives a negative answer to the following question raised in [2]: Does quadrativity imply strong stability?

**Theorem 0.1.** There exists a commutative algebra $A$ of $2 \times 2$ matrices over $\mathbb{R}$ and a norm on $A$ such that $|A^2| \leq |A|^2$ for all $A \in A$ and $|A^3| > |A|^3$ for some $A \in A$.

**Proof.** In Theorem 0.1 the elements of the algebra are real matrices of the form

$$A = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} =: [[a, b]]$$

where the symbol on the right is introduced to save space. We will use the identity

$$[[a, b]]^k = [[a^k, kba^{k-1}]]$$

for any $[[a, b]]$ and any integer $k \geq 1$. 

1
We observe that the algebra $\mathcal{A}$ contains a multiplicative semi-group

$$G := \{ \exp[-t, t] : t \geq 0 \} = \{ [e^{-t}, te^{-t}] : t \geq 0 \}$$

In particular, if $A = [e^{-t}, te^{-t}]$ is an element of $G$, then so is $A^2 = [e^{-2t}, 2te^{-2t}]$ and $A^3 = [e^{-3t}, 3te^{-3t}]$.

We can write $G$ as a graph of $b$ over $a$; indeed, setting $a = e^{-t}$ we have

$$G := \{ [a, b] : 0 < a \leq 1, b = e^{-a} \}$$

where $f(a)$ is the function $f(a) := -a \log e$ on the interval $0 < a \leq 1$. We remark that on this interval the function $f$ is concave (since $f''(a) = -1/a$), non-negative and attains its maximum at the point $a = e^{-1}$, $f(a) = e^{-1}$. We define the modified function $g(a)$ on $0 < a \leq 1$ by setting $g(a) := f(a)$ when $e^{-1} \leq a \leq 1$ and $g(a) := e^{-1}$ when $0 < a \leq e^{-1}$; note that $g$ is still (weakly) concave.

Define a ball to be any non-empty bounded open convex subset of $\mathcal{A}$ which is symmetric around the origin. Then for every ball $\Omega$, we can define a norm $N_\Omega$ on $\mathcal{A}$ in the usual manner as

$$N_\Omega(A) := \inf \{ t : t > 0, A \in t\Omega \},$$

so that $\Omega$ is the unit ball of $A$. The fact that $\Omega$ is a ball ensures that $N_\Omega$ is indeed a norm.

Let $k \geq 2$ be an integer. We say that a norm $N(\cdot)$ on $\mathcal{A}$ is $k$-bounded if one has $N(A^k) \leq N(A)^k$ for all $A \in \mathcal{A}$. Also, we shall say that a ball $\Omega$ is $k$-bounded if one has $A^k \in \Omega$ whenever $A \in \Omega$. It is clear from homogeneity that $N_\Omega$ is $k$-bounded if and only if $\Omega$ is $k$-bounded. We say that $N$ or $\Omega$ is quadratic if it is 2-bounded.

As an example, consider the set

$$\Omega_0 := \{ [a, b] : |a| < 1; |b| < g(|a|) \}.$$

It is clear that this set is a ball. We now show that $\Omega_0$ is $k$-bounded for every integer $k \geq 2$. Let $[a, b] \in \Omega_0$; we have to show that $[a, b]^k = [a^k, kba^{k-1}]$ is also in $\Omega_0$. 

By reflection symmetry in the $a$ and $b$ axes, we may assume that we are in the first quadrant $a, b \geq 0$. There are two cases: $e^{-1} \leq a < 1$ and $0 \leq a \leq e^{-1}$.

First suppose that $e^{-1} \leq a < 1$. Then $b < g(a) = -a \log a$. Thus

$kba^{k-1} < -a^k \log a^k = f(a^k) \leq g(a^k)$,

and so $[\langle a^k, kba^{k-1} \rangle] \in \Omega_0$ as desired.

Now suppose that $0 \leq a \leq e^{-1}$. Then $b < g(a) = e^{-1}$. Thus

$kba^{k-1} < ke^{-k} \leq e^{-1} = g(a^k)$

since the function $te^{-t}$ attains its maximum at $t = 1$, and since $a^k$ is clearly bounded by $e^{-1}$. Thus $[\langle a^k, kba^{k-1} \rangle] \in \Omega_0$ as desired.

We identify three interesting points on the boundary of $\Omega_0$: $P_1 := [\langle e^{-1}, e^{-1} \rangle]$, $P_2 := [\langle e^{-1/2}, \frac{1}{2}e^{-1/2} \rangle]$, and $P_3 := [\langle e^{-1/3}, \frac{1}{3}e^{-1/3} \rangle]$. Note that $P_3^3 = P_1$ and $P_2^2 = P_1$. Also, $P_1$ is the point of $\Omega_0$ where the two constraints $|b| < f(|a|)$ and $|b| < e^{-1}$ intersect.

We now modify the ball $\Omega_0$ slightly, to prove
Proposition. There exists a ball $\Omega$ which is 2-bounded but not 3-bounded.

Proof. The idea is to chip a small amount away from $\Omega_0$, enough to destroy the 3-boundedness but not enough to destroy the 2-boundedness.

We shall need three small numbers $0 < \varepsilon_3 < \varepsilon_2 < \varepsilon_1 < 1$ to be chosen later. We define $\Omega$ to be the set of matrices $[a, b]$ in which $|a| < e^{-1/3}$ and $b$ satisfies all three of the inequalities

$$|b| < g(|a|), \quad |b| < e^{-1} - \varepsilon_3, \quad |b| < e^{-1/2} - T_2 \frac{1}{2} |a| - \varepsilon_1.$$ 

Note that the line $b = e^{-1/2} - \frac{3}{2} a$ is the tangent line to the curve $b = g(a)$ at the point $P_2$. Thus the restriction $|b| \leq e^{-1/2} - \frac{1}{2} |a| - \varepsilon_1$ cuts off a small sliver of $\Omega_0$ near the point $P_2$ (and similarly for the other three quadrants, by reflection symmetry). The restriction $|a| < e^{-1/3}$ cuts off everything in $\Omega_0$ to the right of $P_3$, while the restriction $|b| < e^{-1} - \varepsilon_3$ cuts off a very thin horizontal sliver from the straight portion of the boundary of $\Omega_0$, and in particular cuts off a small sliver near $P_1$.

It is clear that $\Omega$ is still a bounded open non-empty convex symmetric set, i.e. a ball. Also, it is clear that $\Omega$ is no longer 3-bounded, because one can get arbitrarily close to $P_3$ in $\Omega$, but one cannot get arbitrarily close to $P_1 = P_3^3$.

It remains to show that $\Omega$ is 2-bounded. To do this, we take any $[a, b] \in \Omega$; our task is to show that $|[a, b]|^2 = |a^2, 2ab|$ is also in $\Omega$. By symmetry we may assume that we are in the first quadrant $a, b \geq 0$. Since $a < e^{-1/3}$, we have $a^2 < e^{-2/3} < e^{-1/3}$, so we only have to show the three inequalities

$$2ab < g(a^2) \quad (1)$$
$$2ab < e^{-1} - \varepsilon_3 \quad (2)$$
$$2ab < e^{-1/2} - T_2 \frac{1}{2} a^2 - \varepsilon_1. \quad (3)$$

Recall that the line $y = e^{-1/2} - \frac{3}{2} x - \varepsilon_1$ was just a small perturbation of the tangent line $y = e^{-1/2} - \frac{1}{2} x$ of the curve $y = g(x)$ at the point $x = e^{-1/2}, y = \frac{1}{2} e^{-1/2}$. In particular we see from the concavity of $g$ that, if $\varepsilon_1$ is sufficiently small,

$$e^{-1/2} - T_2 \frac{1}{2} x - \varepsilon_1 > g(x)$$

for all $x < e^{-2/3} < e^{-1/2}$. Since $a^2 < e^{-2/3}$, we thus see that the condition (3) is redundant, being implied automatically by (1).

It remains to prove (1) and (2). To do this we divide into three cases.

Case 1. $e^{-1/2} + \varepsilon_2 \leq a < e^{-1/3}$. Then we have $b < g(a) = f(a) = -a \log a$. Thus

$$2ab < -a^2 \log a^2 = f(a^2) = g(a^2)$$

since

$$e^{-1} + 2\varepsilon_2 e^{-1/2} + \varepsilon_2^2 \leq a^2 < e^{-2/3}. \quad (4)$$
Figure 2. The ball $\Omega$ in the first quadrant. The vertical line is the condition $|a| < e^{-1/3}$; the horizontal line is the condition $|b| < e^{-1} - \varepsilon_3$; and the slanted line is the condition $|b| < e^{-1/2} - \frac{1}{3}|a| - \varepsilon_1$.

Note that $\Omega$ gets arbitrarily close to $P_3$ but not to $P_1$ or $P_2$; also the region removed near $P_2$ is larger than that near $P_1$ since it depends on $\varepsilon_1$ instead of $\varepsilon_3$.

This gives (1). If $\varepsilon_2$ is chosen sufficiently small compared to $\varepsilon_1$, and $\varepsilon_3$ is chosen sufficiently small compared to $\varepsilon_2$, then we see from (4) (and the monotonicity of $g(x)$ for $x > e^{-1}$) that

$$g(a^2) < g(e^{-1}) - \varepsilon_3 = e^{-1} - \varepsilon_3.$$  

This gives (2) as desired.

**Case 2.** $e^{-1/2} - \varepsilon_2 < a < e^{-1/2} + \varepsilon_2$. In this case we use the bound

$$b < e^{-1/2} - T\frac{1}{2}a - \varepsilon_1,$$

and so

$$2ab < 2e^{-1/2}a - a^2 - 2\varepsilon_1a.$$  

Since $a = e^{-1/2} + O(\varepsilon_2)$, we thus have

$$2ab < e^{-1} - 2\varepsilon_1e^{-1/2} + O(\varepsilon_2).$$
On the other hand, since $a^2 = e^{-1} + O(\varepsilon_2)$, we have
\[ g(a^2) = g(e^{-1}) + O(\varepsilon_2) = e^{-1} + O(\varepsilon_2). \]
Thus if $\varepsilon_2$ is sufficiently small compared to $\varepsilon_1$, we obtain (1). Using the above estimate for $2ab$, we see that (2) follows from
\[ e^{-1} - 2\varepsilon_1 e^{-1/2} + O(\varepsilon_2) < e^{-1} - \varepsilon_3. \]
This holds if both $\varepsilon_2$ and $\varepsilon_3$ are sufficiently small compared to $\varepsilon_1$.

**Case 3.** $0 < a \leq e^{-1/2} - \varepsilon_2$. In this case we use the bound $b < g(a)$, so that $2ab < 2ag(a)$. Since
\[ a^2 \leq e^{-1} - 2\varepsilon_2 e^{-1/2} + \varepsilon_2^2 \]
we have $g(a^2) = e^{-1}$ where $\varepsilon_2$ is small. Thus (1) follows from (2), and it suffices to show that
\[ 2ag(a) < e^{-1} - \varepsilon_3. \]
First suppose that $a \leq e^{-1}$. Then $2ag(a) \leq 2e^{-2}$, which is certainly acceptable if $\varepsilon_3$ is small enough. Thus we may take $a > e^{-1}$, in which case
\[ 2ag(a) = 2af(a) = -a^2 \log a^2 = f(a^2). \]
Since $f$ attains its maximum $e^{-1}$ at $e^{-1}$, we thus see from (5) that $f(a^2) < e^{-1} - \varepsilon_3$, if $\varepsilon_3$ is sufficiently small compared to $\varepsilon_2$. This concludes the proof of the Proposition in all three cases. \qed

One may try to improve this counterexample by adding another natural condition to the norm $N$, namely that the identity $[[1, 0]]$ have norm 1. This is equivalent to $[[1, 0]]$ lying on the boundary of $\Omega$. It is true that the counterexample constructed above does not obey this condition, but this is easily rectified by replacing the ball $\Omega$ constructed above with the convex hull $\text{hull}(\Omega, [[1, 0]], [[-1, 0]])$ of $\Omega$ with the points $[[-1, 0]]$. We omit the computation which shows that this ball remains 2-bounded and not 3-bounded.

**Acknowledgement.** It is a pleasure to express our appreciation to Professor Moshe Goldberg, who brought the problem to our attention and supplied the references.

**References**

[1] R. Arens and M. Goldberg, *Weighted $\ell_\infty$ norms for matrices*. Linear Algebra Appl. 201 (1998), 155–163.

[2] R. Arens, M. Goldberg and W.A.J. Luxemburg, *Stable Seminorms Revisited*. Math. Ineq. Appl. 1, No. 1 (1998), 31–40.
Department of Mathematics, UCLA, Los Angeles CA 90095-1555

E-mail address: rr@math.ucla.edu

Department of Mathematics, UCLA, Los Angeles CA 90095-1555

E-mail address: tao@math.ucla.edu