FROM DOUBLE QUANTUM SCHUBERT POLYNOMIALS TO $k$-DOUBLE SCHUR FUNCTIONS VIA THE TODA LATTICE

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Abstract. We show that the $k$-double Schur functions defined by the authors, and the quantum double Schubert polynomials studied by Kirillov and Maeno and by Ciocan-Fontanine and Fulton, can be obtained from each other by an explicit rational substitution. The main new ingredient is an explicit computation of Kostant’s solution to the Toda lattice in terms of equivariant Schubert classes.

1. Introduction

Our purpose is to establish that the quantum double Schubert polynomials studied in [KM] [CF] are related to the $k$-double Schur functions of the authors [LS11a] by an explicit rational substitution of variables. The quantum double Schubert polynomials are the Schubert basis of the torus-equivariant quantum cohomology ring $QH^T(SL_n/B)$ of the flag manifold $SL_n/B$ [AC LS11b]. The authors have shown [LS11a] that the $k$-double Schur functions are a symmetric function realization (see the map $\kappa$ in (1)) of the equivariant Schubert basis of the equivariant homology ring $H_T(Gr_{SL_n})$ where $Gr_{SL_n}$ is the affine Grassmannian of $SL_n$.

The context for our work is the following commutative diagram of isomorphisms, which is explained in the remainder of this introductory section. The isomorphisms going around the square are valid for any simply-connected semisimple algebraic group $G$.

\[
\begin{array}{ccc}
QH^T(G/B)_{(q)} & \xrightarrow{\psi} & H_T(Gr_G)_{T} \\
\downarrow{\rho} & & \downarrow{\phi} \\
\mathbb{C}[A^\circ \times h/W h] & \xrightarrow{\Psi} & \mathbb{C}[Z^\circ]
\end{array}
\]

The vertical isomorphisms may be viewed as providing explicit presentations for the top rings as coordinate rings of varieties. The map $\rho$ is Kim’s presentation of $QH^T(G/B)$ [Kim] and the map $\phi$ is an isomorphism due independently to Ginzburg [Gin] and Peterson [Pet]. Peterson’s quantum to affine isomorphism $\psi$ [Pet LS10 LL] is defined in terms of Schubert classes, while the map $\Psi$, which is Kostant’s solution to the Toda lattice [Ko79], is defined in terms of explicit rational substitutions of coordinates.

Forgetting equivariance, we recover the setting of [LS11b].

Our proof consists of computing images of variables in Kim’s presentation under Kostant’s isomorphism, and writing these elements in terms of affine Schubert classes. By Kostant’s work [Ko79 Ko96], the images of these generators are ratios of minors of the matrix entries of a centralizer family $Z$. In the nonequivariant setting [LS11b] these minors are directly recognizable as Schubert classes in $Gr_{SL_n}$, but equivariantly this does not hold. However the minors are

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Schubert classes in the larger ring $H_T(\text{Gr}_{PGL_n})$ and as such, have a nice determinantal formula (Theorem 20 and Proposition 27).

Peterson [Pet] has given a model for $H_T(\text{Gr}_G)$, where $G$ is simply-connected and semisimple, in terms of Kostant and Kumar’s nilHecke ring [KK]. To compute in $H_T(\text{Gr}_{G_{ad}})$ where $G_{ad}$ is the adjoint group of the same type as the simply-connected group $G$, we use Chaput, Manivel, and Perrin’s generalization [CMP] of Peterson’s construction to an extended affine nilHecke construction of $H_T(\text{Gr}_{G_{ad}})$. Computation of Schubert structure constants in the extended affine setting, appear to yield new positivity properties (Conjecture 20), despite the lack of a geometric interpretation coming from the positivity of the product in $QH^T(G/B)$ [Min].

Finally, we compare the minors of the centralizer with symmetric functions, using Molev’s Jacobi-Trudi formula [Mo] for dual Schur functions, and show that the quantum double Schubert polynomials correspond to $k$-double Schur functions up to a twist by an automorphism of the symmetric function ring.

1.1. Peterson’s quantum versus affine Theorem. We review Peterson’s results relating quantum and affine Schubert calculus [Pet], following the presentation in [LS10]. See also [LL].

Let $G ⊃ B ⊃ T$ be a semisimple simply-connected algebraic group $G$ over $\mathbb{C}$, a Borel subgroup $B$, and maximal torus $T$, $W = N(T)/T$ the Weyl group, and $I$ the set of Dynkin nodes.

Let $QH^T(G/B)$ be the $T$-equivariant small quantum cohomology ring of $G/B$ [Kim]. It contains a polynomial subring generated by $H_2(G/B) \cong \mathbb{Q}$ where $\mathbb{Q}$ is the coroot lattice. $H_2(G/B)$ has basis given by the quantum parameters $\{q_i \mid i \in I\}$; $q_i$ corresponds to the simple coroot $\alpha_i^\vee$. Let $S = H^T(\text{pt})$. $QH^T(G/B)$ has a basis over $S[q_i \mid i \in I]$ given by the quantum equivariant Schubert classes $\sigma^w$ for $w \in W$.

On the other hand, let $\text{Gr}_G = G(\mathbb{C}[[t]])/(G(\mathbb{C}[[t]]))$ be the affine Grassmannian of $G$. Since $\text{Gr}_G$ is an affine Kac-Moody homogeneous space, its equivariant homology $H_T(\text{Gr}_G)$ is a free $S$-module with equivariant Schubert basis $\xi_w$ for $w \in W_{af}$ where $W_{af}$ is the affine Weyl group and $W_{af}$ is the set of minimum length coset representatives in $W_{af}/W$ [KK]. $H_T(\text{Gr}_G)$ also carries an $S$-linear Pontryagin product since $\text{Gr}_G$ is weakly homotopy equivalent to the space of based loops into the compact form of $G$ and the latter space has a $T$-equivariant Pontryagin product coming from the product in the target of loops.

We have $W_{af} \cong W \times \mathbb{Q}^\vee$ where $\mu \mapsto t_\mu$ denotes the embedding $\mathbb{Q}^\vee \to W_{af}$. For $\lambda \in Q^\vee$ antidominant, $t_\lambda \in W_{af}$. The Schubert basis has the factorization property

$$\xi_w t_\mu = \xi_w t_\mu$$

for $w \in W_{af}$ and $\mu \in Q^\vee$ antidominant.

By (2) the set

$$T = \{\xi_\lambda \mid \lambda \in Q^\vee \text{ is antidominant}\} \subset H_T(\text{Gr})$$

is multiplicatively closed. Finally, for any $w \in W_{af}$ there is a sufficiently antidominant $\lambda \in Q^\vee$ such that $wt_\lambda \in W_{af}$.

Theorem 1. [Pet] [LS10] [LL] There is an $S$-algebra isomorphism

$$\psi : QH^T(G/B)_q \cong H_T(\text{Gr}_G)_T$$

$$\sigma^w q_\mu \mapsto \xi_w t_\lambda \xi_{t_\lambda - t_\mu}$$

for all $w \in W$ and $\mu \in Q^\vee$ and any $\lambda \in Q^\vee$ antidominant such that $wt_\lambda \in W_{af}$ and $\lambda - \mu$ is antidominant, where $QH^T(G/B)$ is localized at the quantum parameters and $H_T(\text{Gr}_G)$ is localized at the set $T$.

This is the map $\psi$ in (1).
1.2. Kim’s presentation. We recall a presentation [GK] [Kim] for $QH^T(SL_n/B)$ or more generally $QH^T(G/B)$. Consider the $n \times n$ matrix $C_n$ with diagonal entries $x_i$ for $1 \leq i \leq n$, superdiagonal entries all $-1$, and subdiagonal entries $q_i$ for $1 \leq i \leq n-1$.

$$
C_n = \begin{pmatrix}
x_1 & -1 & & & \\
q_1 & x_2 & -1 & & \\
 & q_2 & x_3 & -1 & \\
 & & & \ddots & \ddots \\
 & & & q_{n-2} & x_{n-1} & -1 \\
 & & & & q_{n-1} & x_n
\end{pmatrix}
$$

(4)

Let $g_{j,n}(x; q) \in \mathbb{C}[x; q]$ be the value of the $j$-th basic invariant evaluated at $C_n$:

$$
det(C_n - z \text{id}_n) = \sum_{j=0}^n (-z)^{n-j} g_{j,n}(x; q).
$$

(5)

We let $S \cong \mathbb{C}[a_1, \ldots, a_n]/(a_1 + \cdots + a_n)$; the $a_1$ are coordinates on the Cartan subalgebra of $\mathfrak{sl}_n$.

Let $J$ be the ideal in $S[x; q]$ generated by $g_{j,n}(x; q) - e_j(a)$ for $1 \leq j \leq n$ where $e_j(a) = e_j(a_1, \ldots, a_n)$ is the elementary symmetric polynomial.

**Theorem 2.** [GK] [Kim] There is an $S$-algebra isomorphism

$$
\rho = \rho_{SL_n} : QH^T(SL_n/B) \cong S[x; q]/J.
$$

For any simply-connected semisimple algebraic group $G$ we recall a formulation of Kim’s construction of $QH^T(G/B)$ (see also [Ric]). Let $g = \text{Lie}(G)$ with Cartan subalgebra $h = \text{Lie}(T) \subset g$. Let $G^\vee$ be the Langlands dual group, $g^\vee$ its Lie algebra, and $h^\vee$ the Cartan subalgebra. The complexified root lattice of $g$ (resp. $g^\vee$) is isomorphic to $h^\vee$ (resp. $h$). A perfect pairing of the root lattices of $g^\vee$ and $g$ is given by $\langle \alpha_i, \alpha_j \rangle = a_{ij}$ where $(a_{ij} \mid i, j \in I)$ is the Cartan matrix. By duality this induces a perfect pairing $h \times h^\vee \to \mathbb{C}$. There is an embedding $h \to g^\vee$ such that $h \in h$ maps to the functional on $g^\vee$ that vanishes on all the root subspaces of $g^\vee$, and on $h^\vee$ is given by $h' \mapsto \langle h', h \rangle$ for $h' \in h^\vee$.

$G^\vee$ acts on $g^\vee$ by the coadjoint action

$$
(g \cdot f)(x) = f(\text{Ad}(g^{-1}) \cdot x) \quad \text{for } g \in G^\vee, f \in g^\vee^* \text{ and } x \in g^\vee.
$$

Let $E = \sum_{i=1}^{n-1} f_i^* \in g^\vee^*$ where $f_i^* \in g^\vee^*$ is the functional taking the value 1 on the basis element $f_i^* \in \mathfrak{g}^\vee$ of the one-dimensional root subspace $\mathfrak{g}_{-\alpha_i}^\vee$ and zero on other weight spaces. Let $e_i^\vee \in g^\vee^*$ be defined similarly.

Consider the affine scheme and open subscheme

$$
\mathcal{A} = (-E + h) \oplus \bigoplus_{i \in I} C e_i^\vee \subset g^\vee^*
$$

(6)

$$
\mathcal{A}^0 = (-E + h) \oplus \bigoplus_{i \in I} \mathbb{C} e_i^\vee \subset g^\vee^*
$$

(7)

There is a morphism $\mathcal{A} \to h/W$ where $W$ is the Weyl group, defined by the inclusion $\mathcal{A} \subset g^\vee^*$ followed by the quotient map $g^\vee^* \to g^\vee^*/G^\vee \cong h/W$. Then Kim’s theorem [Kim] is that there is an $S$-algebra isomorphism

$$
\rho : QH^T(G/B) \cong \mathbb{C}[\mathcal{A} \times_{h/W} h].
$$

(8)

Let us consider the above construction for $g = \mathfrak{sl}_n(\mathbb{C})$. We have $G^\vee = PGL_n$. Consider the pairing $\mathfrak{gl}_n \times \mathfrak{gl}_n \to \mathbb{C}$ given by $(A, B) \to \text{trace}(AB)$. This induces a perfect pairing $\mathfrak{sl}_n \times \mathfrak{pgl}_n \to \mathbb{C}$ which extends $h \times h^\vee \to \mathbb{C}$. The pairing gives an isomorphism $\mathfrak{pgl}_n^\vee \cong \mathfrak{sl}_n$, under which $e_i^\vee$ (resp. $f_i^\vee$) is identified with the matrix in $\mathfrak{sl}_n$ having entry 1 in position $(i+1, i)$ (resp. $(i, i+1)$).
and zeroes elsewhere. The presentation of Theorem 2 is recovered by translating the general construction for this special case, where the $x$ variables are coordinates on $\mathfrak{h}$ in $\mathcal{A}$, the $q$ variables are the coordinates of the $e_i^{\vee}$, and the $a$ variables are the coordinates of the base scheme $\mathfrak{h}$; the right hand copy in $\mathcal{A} \times_{\mathfrak{h}/W} \mathfrak{h}$.

Localizing the isomorphism of Theorem 2 at the quantum parameters yields the map $\rho$ in (1) where $\mathcal{A}^0 \subset \mathcal{A}$ is the Zariski open subset on which the $q_i$ are nonvanishing.

1.3. Quantum double Schubert polynomials. For $i \in I$ let $s_i^q$ be the operator that exchanges $a_i$ and $a_{i+1}$ and $\partial_i^q = (a_i - a_{i+1})^{-1} (1 - s_i^q)$ the divided difference operator. For $w \in S_n$ the quantum double Schubert polynomial $\tilde{\mathcal{S}}_w(x; q) \in S[x; q]$ is defined by $[KM]$ if $w > w$.

\[
\tilde{\mathcal{S}}_{w_0}(x; a) = \prod_{i=1}^{n-1} \det(C_i - a_{n-i} \cdot id)
\]

\[
\tilde{\mathcal{S}}_w(x; a) = -\partial_i^q \tilde{\mathcal{S}}_{w}(x; a)
\]

Theorem 3. [AC] [LS11b] Under the isomorphism $\rho_{SL_n}$ of Theorem 2, for every $w \in S_n$ the $w$-th equivariant quantum Schubert class $\sigma^w \in QH^T(SL_n/B)$, maps to the coset of the quantum double Schubert polynomial $\tilde{\mathcal{S}}_w(x; a)$.

1.4. Centralizer family of Ginzburg and Peterson. For a semisimple algebraic group $G$, following Peterson [Pet] (cf. [Rie]) let

\[
\mathcal{Z} = \{(h, g) \in \mathfrak{h} \times G^\vee \mid g \cdot (-E + h) = -E + h\}
\]

\[
= \{(h, b) \in \mathfrak{h} \times B' \mid b \cdot (-E + h) = -E + h\}
\]

where $B' \subset G^\vee$ is the Borel in the Langlands dual group $G^\vee$.

Theorem 4. [Gin] [Pet] There is an $S$-Hopf algebra isomorphism

\[
\phi: \mathcal{C}[\mathcal{Z}] \to H_T(Gr_G).
\]

Let $\mathcal{Z}^0 \subset \mathcal{Z}$ be the open subset given by the non-vanishing locus of the set $\mathcal{T}$ in (9). Then

\[
\mathcal{C}[\mathcal{Z}^0] \cong H_T(Gr_G)_T.
\]

Alternatively, $(h, b) \in \mathcal{Z}^0$ if and only if $b \in B'_{w_0}B'^\vee$ is in the open opposite Bruhat cell. The localized isomorphism furnishes the map $\phi$ of (1) when $G$ is simply-connected.

In [4] we shall give an explicit description (and a new proof) of Theorem 4 in terms of Schubert classes for both the simply-connected group $G = SL_n$ and the adjoint group $G = PGL_n$.

1.5. Symmetric function realization of $H_T(Gr_{SL_n})$. In this section $G = SL_n$. For historical reasons we use the notation $k = n - 1$ here. In [LS11a] the $S$-Hopf algebra $H_T(Gr_{SL_n})$ and its equivariant Schubert basis is realized by symmetric functions.

Theorem 5. [LS11a] There is an $S$-Hopf algebra $\Lambda_{(n)}(y||a)$ of symmetric functions and an $S$-basis consisting of the $k$-double Schur functions $s^{(k)}_\lambda(y||a)$ and an isomorphism of $S$-Hopf algebras

\[
H_T(Gr_{SL_n}) \to \Lambda_{(n)}(y||a)
\]

sending the Schubert basis to the $k$-double Schur basis.

This theorem is explained, and made more precise, in [3.3]. After localizing this yields the map $\kappa$ in (1). The $k$-double Schur functions are double analogues [Mo] of Lapointe, Lascoux, and Morse’s $k$-Schur functions [LLM].

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1 Our quantum double Schubert polynomials $\tilde{\mathcal{S}}_w(x; a)$ differ from those in [KM] by negating the variables $a_i$. 
1.6. Kostant’s isomorphism. In Kostant’s solution of the generalized Toda lattice he showed:

**Theorem 6.** [Ko79] Thm. 2.4] There is an isomorphism $Z^o \to A^o \times h/W \ h$ over $h$.

Denote by $\Psi$ (see 11) the induced isomorphism

$$\Psi : \mathbb{C}[A^o \times h/W \ h] \to \mathbb{C}[Z^o].$$

Kostant also computes the images of various elements under this map. We recall this as well as the explicit description for $Z^o$ in the case $G = SL_n$, which we assume for the rest of the section. We have $G^o = PGL_n$. Let $\{z_{ij} \mid 1 \leq i \leq j \leq n\}$ be the projective matrix entry coordinates on $B^o$ and let $S = \mathbb{C}[h] = \mathbb{C}[a_1, \ldots, a_n]/(a_1 + \cdots + a_n)$. Using the various identifications at the end of 12 we see that $(h, b) \in Z$ if and only if

$$z_{ij} = z_{i-1,j-1} + (a_{i-1} - a_j)z_{i-1,j} \quad \text{for} \quad 2 \leq i \leq j \leq n. \tag{13}$$

Fix $k$ with $0 \leq k \leq n$. For a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ with $\lambda_1 \leq n - k$ let

$$z_{\lambda, k} = \det(z_{p, \lambda_{k+1-q}+q})_{1 \leq p, q \leq k}. \tag{14}$$

It is the minor of the matrix $(z_{ij})$ using the first $k$ rows and the columns given by the set of indices $(\lambda_1 + 1, \lambda_2 - 1 + 2, \ldots, \lambda_k + k)$.

Let $R_k$ be the rectangular partition with $n - k$ rows and $k$ columns. For any $0 \leq i \leq k$ let $R_k - i$ be the partition $R_k$ but with $i$ cells removed from the last column.

Let $D_i = z_{R_i,n-i}$ and $D'_i = z_{R_i,n-i-1}$ for $0 \leq i \leq n$. Let $Z^o \subset Z$ be the Zariski open subset on which all the $D_i$ are nonvanishing.

Since $z_{ij} \neq 0$ for all $1 \leq i \leq n$ we may use the affine coordinates $y_{ij} = z_{ij}/z_{i1}$ for $1 \leq i \leq j \leq n$. In particular we let $y_{11} = 1$. Let $y_{\lambda, k}$, $D_i^y$, and $(D'_i)^y$ be the quantities where $y$ variables are used instead of $z$ variables. The following is the coordinatized version of Theorem 6 for $G = SL_n$.

**Theorem 7.** [Ko79] Thm. 37, Cor. 28] There is an $S$-algebra isomorphism

$$(S[q; x]/I)[q_1^{-1}, \ldots, q_n^{-1}] \cong S[y_{ij}][y_{22}^{-1}, \ldots, y_{nn}^{-1}, (D_i^y)^{-1}, \ldots, (D_{n-1}^y)^{-1}] \tag{15}$$

defined by

$$x_1 + \cdots + x_i \mapsto a_1 + \cdots + a_i + \frac{D'_i}{D_i} \tag{16}$$

$$q_i \mapsto \frac{D_{i-1}D_{i+1}}{D_i^2}. \tag{17}$$

Note that $D'_i/D_i$ and $D_{i-1}D_{i+1}/D_i^2$ are equal to the analogous ratios involving $y$ variables; in fact any ratio $f/g$ of homogeneous polynomials $f$ and $g$ in $z_{ij}$ of the same degree, is equal to the corresponding ratio in the $y$ variables.

**Remark 8.** In Ko96 Thm. 37, Cor. 28] the respective formulae (10) and (17) are given for the zero fiber of the centralizer family, which is the nonequivariant case. However these computations work for the general fiber as well, although it requires additional work to compare these expressions with equivariant affine Schubert classes, as we shall do.

1.7. Main theorem. Recall from 1.6 the definition of the partitions $R_i$ and $R'_i = R_i - 1$. Let $(R_i)^\vee$ and $(R'_i)^\vee$ be the transpose partitions. Let $\text{Des}(w) = \{i \in I \mid ws_i < w\}$.

**Theorem 9.** The rational transformation

$$x_1 + \cdots + x_i \mapsto a_1 + \cdots + a_i + \frac{s_i(R_i)^\vee(y||a)}{s_i(R'_i)^\vee(y||a)} \tag{18}$$

$$q_i \mapsto \frac{s_i(R_{i-1})^\vee(y||a)s_i(R_{i+1})^\vee(y||a)}{s_i(R_i)^\vee(y||a)^2}. \tag{19}$$
 sends 

\[ \tilde{\Sigma}_w(x; a) \mapsto \frac{\lambda^{(k)}(y\|a)^{\omega_i}}{\prod_{i \in \Des(w)} \tilde{s}(R_i)^{\lambda(w)}(y\|a)} \]

where \( \lambda(w) \) is as described in [LS11b] and the automorphisms \( \eta \) and \( \tau \) are defined in [FJS].

**Remark 10.** Forgetting equivariance, the map of Theorem 9 is related to the isomorphism of [LS11b] by the “transpose” automorphism \( \omega \). The map \( \omega \) does not induce an isomorphism equivariantly. So perhaps a more geometrically correct definition of a usual \( k \)-Schur function would incorporate either \( \omega \) or the induced involution on the index set (induced by the affine Dynkin automorphism \( i \mapsto -i \) mod \( n \)). In fact one sees this “transposed” indexing in two geometric computations of homology Schubert classes in \( \text{Gr}_{SL_n} \) [Mag] [BS].

**1.8. Comments.** In principle, an analogue of Theorem 9 holds for any semisimple group \( G \), and one could try to use it to find symmetric function realizations of the Schubert basis of \( H_T(\text{Gr}_G) \), from the corresponding quantum double Schubert polynomials, or vice versa. However, it appears that neither the symmetric functions, nor the quantum double Schubert polynomials are known for any \( G \) outside of type \( A \).

In the nonequivariant setting, quantum Schubert polynomials were constructed in general type in [Mar]. Symmetric functions representing non-equivariant affine Schubert classes were constructed in [Lam] for \( SL_n \), [LSS] for \( Sp_{2n} \), and [Pon] for \( SO_n \). See also Magyar’s construction [Mag], which in principle can be applied for any \( G \).

In [LS11b] parabolic quantum double Schubert polynomials were defined and showed to represent Schubert classes in \( QH^T(SL_n/P) \) for a parabolic subgroup \( P \subset SL_n \). Moreover there is a parabolic analogue of Theorem 9. One might try to obtain a parabolic analogue of Theorem 9.

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### 2. Extended affine nilHecke and Peterson algebras

Let \( G \) be simply-connected and let \( G_{\text{ad}} \) be the group of adjoint type with the same Lie algebra. Let \( T \subset G \) be a maximal torus and let \( T \) act on \( G_{\text{ad}} \) via \( T \rightarrow G_{\text{ad}} \rightarrow G \). In this section we recall the extended Peterson algebra, an algebraic model for \( H_T(\text{Gr}_{G_{\text{ad}}}) \) first studied by Chaput, Manivel, and Perrin [CMP].

#### 2.1. Extended affine Weyl group

Let \( W_c = W \ltimes P^\vee \) be the extended affine Weyl group, where \( P^\vee = \bigoplus_{i \in I} \mathbb{Z} \omega_i^\vee \) is the coweight lattice and \( \omega_i^\vee \) are the fundamental coweights. Let \( \Sigma' \) be the group of automorphisms of the affine Dynkin diagram. Let \( I_{af} = I \cup \{0\} \) be the affine Dynkin node set. Let \( I^* = \Sigma' \cdot \{0\} \subset I_{af} \) be the set of special nodes. We have \( P^\vee/Q^\vee = \{ \omega_i^\vee + Q^\vee \mid i \in I^* \} \) where \( \omega_0^\vee := 0 \). For every \( i \in I^* \), the map \( \omega_i^\vee + Q^\vee \mapsto \omega_i^\vee - \omega_i^\vee + Q^\vee \) defines a permutation of the set \( I^* \) which extends uniquely to an element \( \tau_i \in \Sigma' \) called the special automorphism associated with \( i \in I^* \). It satisfies \( \tau_i(0) = 0 \). For \( \mu \in P \) let \( w_{0i}^\mu \in W \) be the shortest element such that \( w_{0i}^\mu \cdot \mu = w_0 \cdot \mu \). We have

\[ \tau_i = w_{0i}^\omega_i t_{-\omega_i^\vee} \quad \text{for } i \in I^*. \]

The special automorphisms \( \Sigma = \{ \tau_i \mid i \in I^* \} \) form a subgroup of \( \Sigma' \). We have \( W_c = \Sigma \ltimes W_{af} \) where \( \sigma \in \Sigma \) acts on \( W_{af} \) by the group automorphism denoted \( w \mapsto w^\sigma \) where \( s_i^\sigma = s_i^{\sigma(i)} \) for all \( i \in I_{af} \). Moreover \( W_c \) admits a length function extending that of \( W_{af} \) and \( \Sigma \) is the subgroup of length-zero elements in \( W_c \).

\( W_c \) acts on the weight lattice \( P \) of \( G \), the polynomial ring \( S = \text{Sym}(P) \), and the fraction field \( \mathbb{F} = \text{Frac}(S) \) via the level zero action. Explicitly, the translation elements \( t_\lambda \) for \( \lambda \in P^\vee \) act on \( P \) trivially. \( \sigma \in \Sigma \) acts on the affine root lattice \( \bigoplus_{i \in I_{af}} \mathbb{Z} \alpha_i \) by \( \alpha_i^\sigma = \alpha_{i^\sigma} \) for \( i \in I_{af} \) where
the superscript indicates the action of the automorphism $\sigma$. This induces an action of $\Sigma$ on the quotient of the affine root lattice by the one-dimensional subspace $\mathbb{Z}\delta$ where $\delta$ is the null root, and this quotient is isomorphic to the finite root lattice. This action of $\Sigma$ extends uniquely to $P$.

2.2. Extended small torus affine nilHecke algebra. Let $\mathbb{A}_F = F \otimes \mathbb{Q}[W]$ be the twisted group algebra. It has multiplication $(p \otimes v)(q \otimes w) = p(v \cdot q) \otimes vw$ for $p, q \in F$ and $v, w \in W$. The ring $\mathbb{A}_F$ acts on $\mathbb{F}$: the field $F$ acts on itself by left multiplication and $W_e$ acts on $\mathbb{F}$ as above.

For $i \in I_{af}$ define the divided difference element $A_i \in \mathbb{A}_F$ by

$$A_i = \alpha_i^{-1}(s_i - 1).$$

Since the $A_i$ satisfy the same braid relations as $W_{af}$, for $v \in W_{af}$ one may define $A_v = A_{s_i_1} \cdots A_{s_i_k}$ where $v = s_i_1 \cdots s_i_k$ is a reduced decomposition. For $\sigma \in \Sigma$ and $v \in W_{af}$ define $A_{\sigma v} = \sigma A_v \in \mathbb{A}_F$.

In particular $A_\sigma = \sigma$. Since $A_i^2 = 0$ for $i \in I_{af}$, one may show that for all $u, v \in W_e$,

$$A_u A_v = \begin{cases} A_{uv} & \text{if } \ell(u) + \ell(v) = \ell(uv) \\ 0 & \text{otherwise}. \end{cases}$$

The small torus extended affine nilCoxeter algebra $\mathbb{A}_0$ is the subring of $\mathbb{A}_F$ generated by \{\$A_i | i \in I_{af}\$\} and $\Sigma$. We have

$$\mathbb{A}_0 = \bigoplus_{w \in W_e} \mathbb{Z} A_w.$$  

The ring $\mathbb{A}_0$ acts on $S$: for $i \in I_{af}$, $\lambda \in P$, and $s, s' \in S$ we have

$$A_i \cdot \lambda = - (\alpha_i^\vee, \lambda)$$

and

$$A_i \cdot (ss') = (A_i \cdot s)s' + (s \cdot A_i)(s' \cdot A_i).$$

The small-torus finite (resp. affine, resp. extended affine) nilHecke algebra $\mathbb{A}_f$ (resp. $\mathbb{A}$, resp. $\mathbb{A}_e$) is by definition the subring of $\mathbb{A}_F$ generated by $S$ and \{\$A_i | i \in I\$\} (resp. \{\$A_i | i \in I_{af}\$\}, resp. $\mathbb{A}_0$). The action of $\mathbb{A}_F$ on $\mathbb{F}$ restricts to an action of $\mathbb{A}_e$ on $S$. We have

$$\mathbb{A}_e \cong \bigoplus_{w \in W_e} SA_w.$$  

Note that there is an embedding $W_e \to \mathbb{A}_e$ via

$$\sigma \mapsto \sigma$$

for $\sigma \in \Sigma$

and

$$s_i \mapsto 1 + \alpha_i A_i$$

for $i \in I_{af}$.

2.3. Coproduct. For $S$-modules $M$ and $N$ define $M \otimes_N N$ to be the quotient of $M \otimes \mathbb{Z} N$ by the submodule generated by the elements $sm \otimes n - m \otimes sn$ for $s \in S$, $m \in M$, and $n \in N$. Make a similar definition of $M \otimes_N N$ for $\mathbb{F}$-modules $M$ and $N$. Define the $\mathbb{F}$-submodule

$$\Delta(\mathbb{A}_F) = \bigoplus_{w \in W_e} Fw \otimes w \subset \mathbb{A}_F \otimes \mathbb{A}_F.$$  

The componentwise product on $\mathbb{A}_F \otimes \mathbb{A}_F$ induces an ill-defined product on $\mathbb{A}_F \otimes \mathbb{A}_F$ because elements of $\mathbb{F}$ do not intertwine the same way with different elements of $W_e$. However the componentwise product does induce a well-defined product on $\Delta(\mathbb{A}_F)$. This given, there is a ring and left $\mathbb{F}$-module homomorphism $\Delta: \mathbb{A}_F \to \Delta(\mathbb{A}_F)$ given by

$$\Delta(q) = q \otimes 1$$

for $q \in \mathbb{F}$

and

$$\Delta(w) = w \otimes w$$

for $w \in W_e$.

One may show that for all $i \in I_{af}$,

$$\Delta(A_i) = A_i \otimes 1 + s_i \otimes A_i = A_i \otimes 1 + 1 \otimes A_i + \alpha_i A_i \otimes A_i.$$
Let $M$ and $N$ be $\mathcal{A}_F$-modules. Then $M \otimes_T N$ is an $\mathcal{A}_F$-module induced by the componentwise action of $\Delta(\mathcal{A}_F)$ on $M \otimes_T N$:

\begin{align}
(29) \quad s \cdot (m \otimes n) &= \Delta(s) \cdot (m \otimes n) = (s \cdot m) \otimes n = m \otimes (s \cdot n) \\
(30) \quad A_i \cdot (m \otimes n) &= \Delta(A_i) \cdot (m \otimes n) = (A_i \cdot m) \otimes n + (s_i \cdot m) \otimes (A_i \cdot n).
\end{align}

One may make analogous definitions for $\mathcal{A}_e$ instead of $\mathcal{A}_F$. For $\mathcal{A}_e$-modules $M$ and $N$, $\mathcal{A}_e$ acts on $M \otimes_S N$ as above.

### 2.4. Extended Peterson subalgebra and $j$-basis

The Peterson subalgebra $\mathcal{B} \subset \mathcal{A}$ and its extended analogue $\mathcal{B}_e \subset \mathcal{A}_e$ are by definition the centralizer subalgebras

\[ \mathcal{P} = Z_\mathcal{A}(S) = \{ a \in \mathcal{A} \mid sa = as \text{ for all } s \in S \} \]

\[ \mathcal{P}_e = Z_{\mathcal{A}_e}(S) = \{ a \in \mathcal{A}_e \mid sa = as \text{ for all } s \in S \} . \]

We have

\[
\begin{align*}
\mathcal{F} \otimes_{\mathcal{S}} \mathcal{P} &= \bigoplus_{\mu \in Q^+} \mathcal{F} t_\mu \\
\mathcal{F} \otimes_{\mathcal{S}} \mathcal{P}_e &= \bigoplus_{\mu \in P^+} \mathcal{F} t_\mu .
\end{align*}
\]

Define the extended affine Grassmannian elements in $\mathcal{W}_e$ to be the subset

\[ \mathcal{W}_e^0 = \{ w \in \mathcal{W}_e \mid \ell(\mathcal{W}v) > \ell(w) \text{ for all } v \in \mathcal{W} \setminus \text{id} \} = \{ \sigma w \mid \sigma \in \mathcal{S}, w \in \mathcal{W}_e^0 \} . \]

**Lemma 11.** Let $w \in \mathcal{W}$ and $\lambda \in P^\vee$. Then $\mathcal{W}_e^0 \subseteq \mathcal{W}_e$ if and only if $\langle \lambda, \alpha_i \rangle \leq 0$ for all $i \in I$ and strict inequality for $i \in I$ such that $w_i < w$.

The following result is the natural extension of a theorem of Peterson [Pet, Lam] to the adjoint group setting, and its proof goes through the same way. Parts (2) and (3) were established in [CMP] and (1) is nearly immediate. See also [LSS] for a more algebraic approach.

**Theorem 12.**

1. The map $\Delta : \mathcal{A}_e \rightarrow \Delta(\mathcal{A}_e)$ restricts to an $S$-algebra homomorphism $\mathcal{P}_e \rightarrow \mathcal{P}_e \otimes_S \mathcal{P}_e$, making $\mathcal{P}_e$ into a commutative $S$-Hopf algebra.

2. There is a unique $S$-basis $\{ j_w \mid w \in \mathcal{W}_e^0 \}$ of $\mathcal{P}_e$ such that

\[
j_w = A_w + \sum_{x \in \mathcal{W}_e \setminus \mathcal{W}_e^0} j_x A_x
\]

for some elements $j_x \in S$.

3. There is an $S$-Hopf algebra isomorphism

\[
\begin{align*}
\theta : \mathcal{H}_T(\text{Gr}_{G_m}) &\cong \mathcal{P}_e \\
\xi_w &\mapsto j_w
\end{align*}
\]

where $\{ \xi_w \mid w \in \mathcal{W}_e^0 \}$ is the Schubert basis of $\mathcal{H}_T(\text{Gr}_{G_m})$.

Let $j : \mathcal{A}_e \rightarrow \mathcal{P}_e$ be the projection to the Peterson subalgebra; it is the left $S$-module homomorphism defined by

\[
j(\sum_{w \in \mathcal{W}_e} a_w A_w) = \sum_{w \in \mathcal{W}_e^0} a_w j_w \quad \text{for } a_w \in S.
\]

Alternatively, $j(a) \in \mathcal{P}_e$ is determined by the requirement that $j(a) - a \in \sum_{v \in \mathcal{W} \setminus \text{id}} \mathcal{A}_e A_v$.
Lemma 13. For all \(a, a' \in A_c\) and \(b \in \mathbb{P}_e\),

\[
\begin{align*}
(34) & \quad j(b) = b \\
(35) & \quad j(a'a) = j(a'j(a)) \\
(36) & \quad j(ba) = b j(a).
\end{align*}
\]

Proof. Equation (34) holds by the definitions. \(a - j(a)\) has no term \(A_v\) for \(v \in W_e^0\); therefore the same is true of \(a'(a - j(a))\). Therefore \(j(a'(a - j(a))) = 0\) from which (35) follows. Equation (36) follows from (34) and (35). \(\square\)

2.5. \(j\)-basis and Dynkin automorphisms.

Lemma 14. For any \(\sigma \in \Sigma\), conjugation by \(\sigma\) defines a group automorphism of \(W_{af}\) and \(W_e\) and a ring automorphism of \(A_c, \mathbb{A}, \mathbb{P}_e, \) and \(\mathbb{P}\).

Let \(i \mapsto i^*\) be the involutive automorphism of the affine Dynkin diagram such that \(w_0 \cdot \alpha_i = -\alpha_{i^*}\) for \(i \in I\) and \(0^* = 0\). Then for \(i \in I^*\) we have \(\tau_i^{-1} = \tau_{i^*}\) and \((w_0^{\alpha_i})^{-1} = w_0^{\alpha_{i^*}}\).

Lemma 15.

\[j_{\tau_i} = t_{w_{\tau_i}^{\alpha_i}} \quad \text{for } i \in I^*\]

Proof. We have

\[t_{w_{\tau_i}^{\alpha_i}} = t_{-w_0^{\alpha_i}} = w_0^{\alpha_i} t_{-w_0^{\alpha_i}} (w_0^{\alpha_i})^{-1} = \tau_i w_0^{\alpha_{i^*}}.\]

Since \(w_0^{\alpha_{i^*}} \in W\), its expansion in \(A_c\) into the \(A\) basis, has no Grassmannian terms other than \(A_{i^*}\). Therefore \(t_{w_{\tau_i}^{\alpha_i}} \in \mathbb{P}_e\) has no Grassmannian terms other than \(A_{i^*}\). The lemma follows. \(\square\)

For \(a \in \mathbb{P}\) and \(\tau \in \Sigma\), let \(a^\tau := \tau a \tau^{-1}\).

Lemma 16. For \(w \in W_{af}^0\) and \(i \in I^*\) we have

\[j_{\tau_i w} = j_{\tau_i} j_{w_i^T}^T.\]

Proof. By the proof of Lemma 15

\[j_{w_i^T} j_{\tau_i} = j_{w_i^T} \tau_i w_0^{\alpha_{i^*}} = \tau_i j_w w_0^{\alpha_{i^*}}.\]

Since \(w_0^{\alpha_{i^*}} \in W\), the right hand side is an element of \(\mathbb{P}_e\) with unique Grassmannian term \(A_{\tau_i w}\). \(\square\)

Lemma 17. Let \(\lambda \in P^\vee\) be antidominant. Then

\[j_{\tau_{\lambda}} = \sum_{\mu \in W \cdot \lambda} A_{\mu}.\]

Proof. The proof is the same as that for the case that \(\lambda \in Q^\vee\); see [Lam, Proposition 4.5]. \(\square\)

Lemma 18. Let \(w \in W_e^0\) and \(\mu \in P^\vee\) be antidominant. Then \(w t_{\mu} \in W_e^0\) and

\[j_{w t_{\mu}} = j_{w} j_{t_{\mu}}.\]

Proof. The only possible Grassmannian terms in the RHS are of the form \(A_{I_w \cdot \mu} A_{\mu}\) where \(u \in W\). But this product is length additive only when it is equal to \(A_{w t_{\mu}}\). \(\square\)

Remark 19. Specializing to \(G = SL_n\) and forgetting equivariance, the factorization of Lemma 18 is the \(k\)-rectangle factorization property for \(k\)-Schur functions [LLM].
2.6. **Type A.** For type \( A_{n-1}^{(1)} \), \( J_a = \mathbb{Z}/n\mathbb{Z} \), \( (i+n\mathbb{Z})^* = -i + n\mathbb{Z} \), and \( \tau_i = \tau^{-1} \) where \( \tau \) is the rotation \( j + n\mathbb{Z} \mapsto j + 1 + n\mathbb{Z} \), \( w^{\omega_i}_0 \in W = S_i \) is the permutation sending \( j \mapsto j + n - i \) for \( 1 \leq j \leq i \) and \( j \mapsto j - i \) for \( i + 1 \leq j \leq n \). Letting \( w^j_0 = s_is_{i+1} \cdots s_{i+j-1} \) and \( d^j_i = s_{i+j-1} \cdots s_{i+1}s_i = (u^j_i)^{-1} \) we have we have \( w^{\omega_i}_0 = w^{\omega_i}_{n-1} \cdots w^{\omega_i}_{1} \). Lemma \( 15 \) reads
\[
\hat{j}_r \rightarrow t^{\omega_i} \hat{j}_r \quad \text{for } 0 \leq k \leq n
\]
with the convention that \( \omega_i^0 = \omega_i^n = 0 \).

2.7. **Positivity.** Define the subset of Graham-positive elements by \( \mathbb{Z} \geq [\alpha_i \mid i \in I] \subset S \). The coefficients \( j^w_0 \) are also structure constants of the \( j^w \)-basis. For the non-extended Peterson algebra they coincide with equivariant 3-point Gromov-Witten invariants, which are known to be Graham-positive by work of Mihalcea [Mih].

**Conjecture 20.** *All the coefficients \( j^w_0 \) are Graham-positive.*

3. Symmetric functions

The main purpose of this section is to establish Proposition 23 which expresses certain “small” affine Schubert classes as a determinant of Dynkin-twists of special Schubert classes.

Let \( \mathbb{C}[a_i \mid i \in \mathbb{Z}] \) be a polynomial ring over the complex numbers in variables indexed by integers.

3.1. **A Hopf algebra of symmetric functions.** Let \( \hat{\Lambda}(y|a) \) be the \( \mathbb{C}[a]-\)Hopf algebra of symmetric series in the variables \( y = (y_1, y_2, \ldots) \) with coefficients in \( \mathbb{C}[a] \), with primitive elements \( p_r[y] = \sum_{i \geq 1} y_i^r \) which generate \( \hat{\Lambda}(y|a) \) in the sense that \( \hat{\Lambda}(y|a) = \mathbb{C}[a][p_r[y] \mid r \geq 1] \). \( \hat{\Lambda}(y|a) \) is a completion of the usual \( \mathbb{C}[a]-\)algebra of symmetric functions in the \( y \) variables.

The dual elementary symmetric functions \( \hat{e}_j(y|a) \in \hat{\Lambda}(y|a) \) are defined by
\[
\sum_{j \geq 0} \hat{e}_j(y|a)(t + a_0)(t + a_1) \cdots (t + a_{j-1}) = \prod_{i} \frac{1 + ty_i}{1 - a_0y_i}.
\]
Setting \( t = -a_r \) for \( r \in \mathbb{Z}_{\geq 0} \) yields
\[
\sum_{j = 0}^{r} \hat{e}_j(y|a)(a_0 - a_r)(a_1 - a_r) \cdots (a_{j-1} - a_r) = \prod_{i} \frac{1 - a_r y_i}{1 - a_0y_i}.
\]
One may solve for \( \hat{e}_j(y|a) \) in terms of \( \hat{e}_i(y|a) \) for \( i < j \); in particular the system \( 12 \) uniquely defines the \( \hat{e}_j(y|a) \). It also follows that \( \hat{e}_j(y|a) \) depends only on the parameters \( a_0, a_1, \ldots, a_j \).

Let \( \tau \) be the \( \mathbb{C}\)–algebra automorphism of \( \hat{\Lambda}(y|a) \) given by
\[
a_i^\tau = a_{i+1} \quad \text{for } i \in \mathbb{Z}
\]
\[
p_r[y]^\tau = p_r[y] \quad \text{for } r \geq 1.
\]
Let \( \eta \) be the \( \mathbb{C}[a]-\)algebra automorphism of \( \mathbb{C}[a] \) defined by
\[
a_i^\eta = -a_{1-i} \quad \text{for } i \in \mathbb{Z}.
\]
Let \( \omega \) be the \( \mathbb{C}[a]-\)algebra automorphism of \( \hat{\Lambda}(y|a) \) given by
\[
s_\lambda[y]^\omega = s_{\lambda'}[y] \quad \text{for } \lambda \in \mathcal{Y}
\]
where \( \mathcal{Y} \) is Young’s lattice of partitions and \( \lambda \mapsto \lambda' \) is the transpose or conjugate map. Define the dual homogeneous symmetric functions \( \hat{h}_i(y|a) \in \hat{\Lambda}(y|a) \) by
\[
\hat{h}_i(y|a) = \hat{e}_i(y|a)^\omega^0 \quad \text{for } i \geq 0.
\]

\[\text{It is possible to work over the integers. However we use \( \mathbb{C} \) since we work with coordinate rings of complex varieties.}\]
The dual Schur functions \( \hat{s}_\lambda(y||a) \in \hat{\Lambda}(y||a) \) may be defined by [Mo]

\[
\hat{s}_\lambda(y||a) = \det(\hat{h}_{\lambda_i-\lambda_j}^{\ell_{\lambda_i}}(y||a))_{1 \leq i, j \leq \ell(\lambda)}
\]

Then we have

\[
\hat{s}_\lambda(y||a)^{\omega_\eta} = \hat{s}_\lambda(y||a) \quad \text{for } \lambda \in \mathbb{Y}.
\]

3.2. A dual Hopf algebra of symmetric functions. Let \( \Lambda(x||a) \) be the \( \mathbb{C}[a] \)-Hopf algebra of symmetric functions given by the polynomial ring over \( \mathbb{C}[a] \) generated by the primitive elements \( p_r[x - a_+] = \sum_{i \geq 1} (x_i^r - a_i^r) \) for \( r \geq 0 \). Note that this coproduct involves the \( a_i \) variables nontrivially. We define a perfect pairing \( \langle \cdot , \cdot \rangle : \Lambda(x||a) \times \hat{\Lambda}(y||a) \to \mathbb{C}[a] \) by

\[
\langle p_\lambda[x - a_+], p_\mu[y] \rangle = z_\lambda \delta_{\lambda,\mu}
\]

where \( z_\lambda = \prod_i i^{m_i} m_i! \) and \( m_i \) is the number of times the part \( i \) occurs in the partition \( \lambda \). We have [Mo]

\[
\langle s_\lambda(x||a), \hat{s}_\mu(y||a) \rangle = \delta_{\lambda,\mu}
\]

where \( s_\lambda(x||a) \) is the double Schur function, which is essentially a limit involving double Schubert polynomials indexed by Grassmannian permutations.

**Proposition 21.** \( \Lambda(x||a) \) and \( \hat{\Lambda}(y||a) \) are Hopf dual over \( \mathbb{C}[a] \) with respect to \( \langle \cdot , \cdot \rangle \). In particular for \( f, g \in \Lambda(x||a) \) and \( h \in \hat{\Lambda}(y||a) \) we have

\[
\langle f \otimes g , \Delta(h) \rangle = \langle fg , h \rangle.
\]

3.3. The \( k \)-double Schur functions. Let \( k = n - 1 \). The ring \( S = \mathbb{C}[a_1, \ldots, a_n]/(a_1 + a_2 + \cdots + a_n) \) is a \( \mathbb{C}[a] \)-algebra via the \( \mathbb{C} \)-algebra homomorphism \( \mathbb{C}[a] \to S \) defined by \( a_{i+r} \mapsto a_i \) for all \( 1 \leq i \leq n \) and \( r \in \mathbb{Z} \).

The \( \mathbb{C} \)-algebra automorphism \( \eta \) of \( \mathbb{C}[a] \) induces a \( \mathbb{C} \)-algebra automorphism of \( S \) (also denoted \( \eta \)) via the \( \mathbb{C}[a] \)-action on \( S \).

In [LS11a] the authors introduced a family of symmetric functions \( s^{(k)}_\lambda(y||a) \in \hat{\Lambda}(y||a) \) for \( \lambda \in \mathbb{Y} \) with \( \lambda_1 < n \) called \( k \)-double Schur functions. They are linearly independent. Due to a nonstandard \( S \)-module structure used in [LS11a], we must apply the automorphism \( \eta \circ \omega \) to revert to the standard \( S \)-action.

**Theorem 22.** [LS11a] Let \( \mathcal{N}_{(n)}(y||a) = S \otimes_{\mathbb{C}[a]} \bigoplus_{\lambda < n} \mathbb{C}[a] s^{(k)}_\lambda(y||a)^{\omega_\eta} \). Then

1. \( \mathcal{N}_{(n)}(y||a) \) is an \( S \)-Hopf algebra with structure induced from \( \hat{\Lambda}(y||a) \).
2. The elements \( 1 \otimes s^{(k)}_\lambda(y||a)^{\omega_\eta} \) are an \( S \)-basis of \( \mathcal{N}_{(n)}(y||a) \).
3. There is an \( S \)-Hopf isomorphism

\[
\kappa : H_T(\text{Gr}_{SL_n}) \to \mathcal{N}_{(n)}(y||a)
\]

\[
\xi_{\omega_\lambda} \mapsto s^{(k)}_\lambda(y||a)^{\omega_\eta}
\]

where \( \lambda \mapsto \omega_\lambda^{\text{ad}} \) is the bijection sending the \((n-1)\)-bounded partition \( \lambda \in \mathbb{Y} \) with \( \lambda_1 < n \), to \( \omega_\lambda^{\text{ad}} \in W_\lambda^{\text{ad}} \) (see [LS11a] Section 2.2). In particular, for \( 0 \leq r \leq n - 1 \) we have

\[
\xi_{\omega_r} \mapsto \hat{e}_r(y||a) \quad \text{where} \quad c_r = s_{r-1} \cdots s_1 s_0.
\]

The effect of \( S \otimes_{\mathbb{C}[a]} \cdot \) is to identify subscripts of the \( a_i \) modulo \( n \). In [LS11a], we used the polynomial ring \( \mathbb{C}[a_1, \ldots, a_n] \) instead of \( S \), but the results there easily specialize to the current situation. The reader is also warned that the convention for simple roots in [LS11a] is nonstandard: it uses \( a_i = a_{-i} - a_{-i-1} \) for \( i \in \mathbb{Z} \). The automorphism \( \eta \) is applied so that the standard \( S \)-actions can be used. The automorphism \( \omega \) is applied for convenience, so that small
Schubert classes map to the known basis of dual Schur functions (but with transposed indexing partitions).

We denote by $\tau$ the special automorphism of the type $A^{(1)}_{n-1}$ affine Dynkin diagram, that sends $i$ to $i + 1 \pmod{n}$ for all $i$. In the notation of \[2.4\] $\tau = \tau_i^{-1}$.

This automorphism acts on $H_T(\text{Gr}_{SL_n})$ by conjugation. In terms of the isomorphism $\mathbb{P} \simeq H_T(\text{Gr}_{SL_n})$, $\tau$ acts via $a \mapsto a^\tau = \tau a \tau^{-1}$.

**Lemma 23.** The isomorphism of Theorem 22 intertwines the action of $\tau$ on $H_T(\text{Gr}_{SL_n})$ and that induced by $\tau$ on $\Lambda'_n(y||a)$.

**Proof.** In this proof we use notations from affine symmetric groups freely; see [LS11a, Section 2].

Let $f \in \Lambda(x||a)$ and $g \in \Lambda'_n(y||a)$. It is clear that $\langle f, \tau g \rangle = \tau(f, g)$. But there is also an evaluation map $\Lambda(x||a) \otimes \mathbb{P} \rightarrow S$, given by $\langle f, t_\lambda \rangle = \epsilon_\lambda(f)$ (see [LS11a]). Here for $w \in W_{af} = \tilde{S}_n$, the evaluation $\epsilon_w(f)$ is given by the substitution $x_i = -a_{w(1-i)}$ (this differs from the formula in [LS11a] by $\eta$). It is enough to show that $\epsilon_{\tau \lambda \tau^{-1}}(f) = \tau \epsilon_\lambda(f)$ for any $\lambda \in Q^\vee$ and any $f \in \Lambda(x||a)$. But $\tau t(\lambda_1, \lambda_2, ..., \lambda_n) \tau^{-1} = t(\lambda_{n}, \lambda_{n-1}, ..., \lambda_1)$ so this follows from the proof of [LS11a, Lemma 9], which states that $\epsilon_{\tau(\lambda_1, \lambda_2, ..., \lambda_n)}(pr[x - a > 0]) = -\sum_{i=1}^n \lambda_i a_i'$.

**Lemma 24.** For any $\lambda \in \mathbb{Y}$ with $\lambda_1 \leq n - k$ and $\ell(\lambda) \leq k$ for some $1 \leq k \leq n - 1$ we have

$$s_\lambda^{(k)}(y||a) = \hat{s}_\lambda(y||a).$$

**Proof.** We shall use the notations of [LS11a, Section 4]. The partitions $\lambda$ of the lemma are exactly those with main hook length less than or equal to $n - 1$. Let $\mu$ have main hook length greater than $n - 1$.

By Lemma 9, the function $\epsilon_{\text{Gr}}(s_\mu(x||a))$ lies in the GKM ring $\Phi_{\text{Gr}}$, and thus can be expanded in terms of $\xi^\vee$'s, where $v \in \tilde{S}_n^0$. Suppose $\lambda$ has main hook length less than or equal to $n - 1$. Since $\lambda$ is not contained in $\mu$, we have $\epsilon_\lambda(s_\mu(x||a)) = 0$ by [LS11a, Proposition 1]. But since $\lambda$ is small, we also have $\epsilon_{w_\lambda}(s_\mu(x||a)) = 0$. Thus the support of $\epsilon_{\text{Gr}}(s_\mu(x||a))$ does not contain any $w_\lambda$ for $\lambda$ with main hook length less than or equal to $n - 1$.

By duality, the expansion of $s_\lambda^{(k)}(y||a)$ in terms of dual Schur functions does not involve $\hat{s}_\mu(y||a)$. The lemma follows from this observation and [LS11a, Corollary 31].

**Proposition 25.** For any $\lambda \in \mathbb{Y}$ with $\lambda_1 \leq n - k$ and $\ell(\lambda) \leq k$ for some $1 \leq k \leq n - 1$ we have

$$\xi_{w_\lambda^{(k)}} = \det(\epsilon_{c_{\lambda_{i-1}, j-1}}^{i,j-1})_{1 \leq i,j \leq k}.$$

**Proof.** Note that $\tau \eta = \eta \tau^{-1}$ and $\tau \omega = \omega \tau$. We have

$$\kappa(\det(\epsilon_{c_{\lambda_{i-1}, j-1}}^{i,j-1})) = \det(\epsilon_{c_{\lambda_{i-1}, j-1}}^{i,j-1}(y||a)) = \hat{s}_\lambda(y||a).$$

On the other hand, we have

$$\kappa(\xi_{w_\lambda^{(k)}}) = s_\lambda^{(k)}(y||a)^{\omega \eta} = \hat{s}_\lambda(y||a)^{\omega \eta} = \hat{s}_\lambda(y||a)$$

by Theorem 22, Lemmata 23 and 24, and [48]. The Proposition follows since $\kappa$ is injective.

4. **CENTRALIZER FAMILY FOR $G^\vee = SL_n$**

Under the identification of $\mathfrak{gl}_n^\vee \cong \mathfrak{s}_n$ of [1.2] the element $E$ is mapped to the principal nilpotent (also denoted $E$) with entries 1 on the superdiagonal and zeroes elsewhere. With $\mathfrak{h}$ still the Cartan subalgebra of $\mathfrak{s}_n$, and using the adjoint action of $SL_n$ on $\mathfrak{s}_n$, define the family

$$\tilde{Z} = \{(h, b) \in \mathfrak{h} \times SL_n \mid b \cdot (-E + h) = -E + h\}$$

$$= \{(h, b) \in \mathfrak{h} \times B \mid b \cdot (-E + h) = -E + h\}.$$
Theorem 26. There is an \( S \)-Hopf algebra isomorphism \( \tilde{\phi} : \mathbb{C}[\tilde{Z}] \to \mathbb{P}_e \) defined by

\[
\tilde{\phi}(z_{k,k+p}) = \frac{j_{r+1}}{j_{r+1}}
\]

for \( 1 \leq k \leq k + p \leq n \). Thus composing with \( \theta^{-1} \) of Theorem 12 we obtain the isomorphism \( \phi : \mathbb{C}[\tilde{Z}] \to H_T(Gr_{G_{\omega}}) \).

Proof. Let us first check well-definedness. Since the diagonal entries \( z_{kk} \) are units in \( \mathbb{C}[\tilde{Z}] \) we must check that \( \phi(z_{kk}) \) is a unit in \( \mathbb{P}_e \). Using (40) we have

\[
\tilde{\phi}(z_{kk}) = t_{\omega_{kk}}^{\cdot 1} - t_{\omega_{kk}}^{\cdot k}.
\]

But \( t_{\mu} \in \mathbb{P}_e \) for all \( \mu \in P_{\omega} \) and is invertible in \( \mathbb{P}_e \). The elements \( j_{r,k} \) are invertible in \( \mathbb{P}_e \) by (10). This ensures that the division in the right hand side of (53) is well-defined.

For \( 0 \leq k \leq n \) we have

\[
\tilde{\phi}(z_{k1}z_{22} \cdots z_{kk}) = \prod_{i=1}^{k} t_{\omega_i^{\cdot 1}} - t_{\omega_i^{\cdot k}} = t_{\omega_i^{\cdot k}}.
\]

For \( k = n \) this shows that the relation \( z_{11} \cdots z_{nn} - 1 \) is in the kernel of \( \tilde{\phi} \).

To check that the images of the equations (13) are satisfied, proceeding by induction on \( p \) it suffices to show

\[
t_{-s\cdots s_1 s_k} j_{r+1} = j_{r+1} + (a_k - a + p) j_{r+1}.
\]

We have

\[
t_{-\omega_{kk}} = (u^{\cdot 1} + 1)^{-1} \tau_1 = s_1 \cdots s_{n-1} \tau^{-1} = \tau^{-1} s_1 \cdots s_{n-1} s_{n-180}
\]

and

\[
t_{-s\cdots s_1 s_k} = s_k - s_1 \tau^{-1} s_1 \cdots s_{k-2} = \tau^{-1} s_k - s_1 \tau^{-1} s_1 \cdots s_{k-2} = \tau^{-1} s_k \cdots s_{n-180} s_1 \cdots s_{k-2}.
\]
Using Lemma 13 repeatedly without further mention, we have
\[ t_{-s_{k-2} \cdots s_{1}} \cdot j_{\tau^k e_p} = j(t_{-s_{k-2} \cdots s_{1}} \cdot \tau^k A_{p-1} \cdots A_{1} A_{0}) \]
\[ = j(\tau^{-1} s_{k} \cdots s_{n-1} s_{0} s_{1} \cdots s_{k-2} \tau^k A_{p-1} \cdots A_{0}) \]
\[ = j(\tau^k s_{0} \cdots s_{n-2} A_{p-1} \cdots A_{0}) \]
\[ = j(\tau^k s_{0} \cdots s_{p-1} s_{p} A_{p-1} \cdots A_{0}) \]
\[ = j(\tau^k s_{0} \cdots s_{p-1} A_{p-1} \cdots A_{0}) + j(\tau^k s_{0} \cdots s_{p-1} A_{p-1} \cdots A_{0}) \]
\[ = j(\tau^k A_{p-1} \cdots A_{0}) + j(\tau^k (a_0 + \cdots + a_p) s_{0} \cdots s_{p-1} A_{p} A_{0}) \]
\[ = j(\tau^k A_{p-1} \cdots A_{0}) + j(\tau^k (a_0 + \cdots + a_p) s_{0} \cdots s_{p-1} A_{p} A_{0}) \]
\[ = j(\tau^k e_p) = \sum (a_{k-1} - a_{k+p}) j_{t^k e_{p+i+1}} \]
as required, since \( a_0 + \cdots + a_p = a_0 - a_{p+1} \) and applying \( \tau^k \) yields \( a_{k-1} - a_{p+k} \).

Therefore \( \hat{\phi} \) is well-defined.

Next we check surjectivity. To help with this we work with a localized map. Let \( \hat{Z} \) be the
locus in \( \hat{Z} \) on which the functions \( D_i \) are nonvanishing. By Proposition 27 \( \hat{\phi} \) induces an \( S \)-algebra
homomorphism
\[ \hat{\phi} : \mathbb{C}[\hat{Z}] \rightarrow (\mathbb{P}_e)_{(t)} \]
where \( (\mathbb{P}_e)_{(t)} \) (resp. \( \mathbb{P}_{(t)} \)) is the localization of \( \mathbb{P}_e \) (resp. \( \mathbb{P} \)) at the multiplicatively closed subset
consisting of the elements \( j_{t\lambda} \) for \( \lambda \in P' \) (resp. \( \lambda \in Q' \)) antidominant.

Let \( \psi : QH^T(G/B)_{(q)} \rightarrow (\mathbb{P}_{(t)}) \) be the composition of the map \( \psi \) of Theorem 11 with
the isomorphism \( H_T(G/B) \cong \mathbb{P}_t \) induced by the map (32).

For \( i \in I \), by Lemma 14 we have \( s_i t_{-\omega^\gamma_i} \in W_c^0 \). By the definition of \( \psi \) and Lemma 18 we have
\[ \hat{\psi}(\sigma^i) = j_{s_i t_{-\omega^\gamma_i}} j_{t_{-\omega^\gamma_i}}^{-1} \]
\[ \hat{\psi}(q_i) = j_{t_{-\omega^\gamma_i}^{-1}} j_{t_{-\omega^\gamma_i}^{-1}}^{-1} \]
By Proposition 27 we have
\[ \hat{\phi}(\Psi((x_1 + \cdots + x_i) - (a_1 + \cdots + a_i))) = \hat{\phi}(D_i D_i^{-1}) = \hat{\psi}(\sigma^i) \]
\[ \hat{\phi}(\psi(q_i)) = \hat{\phi}(D_{i-1} D_i D_i^{-2}) = \hat{\psi}(q_i). \]
Since \( \hat{\psi} \) is an isomorphism and the elements \( q_i \) and \( \sigma^i \) generate \( QH^T(SL_n/B)_{(q)} \), it follows that
the image of \( \hat{\phi} \) contains \( \mathbb{P}_{(t)} \).

By Lemmata 16 and 15, equation (59), and Lemma 14 it follows that \( \hat{\phi} \) is surjective. Using
Lemma 18 we deduce that \( \hat{\phi} \) is surjective. Injectivity of \( \hat{\phi} \) follows by dimension-counting.

To check that \( \hat{\phi} \) is a coalgebra morphism, we note that \( \Delta(z_{ij}) = \sum_{i \leq k \leq j} z_{ik} \otimes z_{kj} \). In particular
\( \Delta(z_{ii}) = z_{ii} \otimes z_{ii} \) for \( 1 \leq i \leq n \). On the other hand, \( \hat{\phi}(z_{ii}) = t_{\omega^\gamma_i \omega_{i-1}} \) is grouplike. Thus \( \hat{\phi} \) is a
coalgebra morphism. \( \square \)

**Proposition 27.**

\[ \hat{\phi}(z_{\lambda,k}) = j_{\tau \omega^\gamma_k} \]
for all partitions \( \lambda = (\lambda_1, \ldots, \lambda_k) \) with \( \lambda_1 \leq n - k \). In particular
\[ \hat{\phi}(D_i) = j_{\omega^\gamma_i} \]
\[ \hat{\phi}(D'_i) = j_{s_i t_{\omega^\gamma_i}^{-1}} \]
for \( 0 \leq i \leq n - 1 \).
Proof. Applying Lemma 16, reversing the rows and the columns of the determinant and then transposing it, and using Proposition 25 we have
\[
\hat{\phi}(z_{\lambda,k}) = \det(j_{\tau-1}^{j-1} j_{\tau p c_{\lambda k+1-q}}) = (j_{\tau} j_{\tau 2} \cdots j_{\tau p})^{-1} \det(j_{\tau p c_{\lambda k+1-q}}) = (j_{\tau} j_{\tau 2} \cdots j_{\tau p}) \det(j_{\tau p c_{\lambda k+1-q}}) = j_{\tau} \det(j_{\tau p c_{\lambda k+1-q}})
\]
\[
= \hat{\phi}(D^i) = j_{\tau-i}^{s_{2i}} w_{R_i}^{af} = j_{w_{R_i}^{af}}(\tau_{n-1}^{n-1}) = j_{w_{R_i}^{af}}^{\omega_{\tau_{n-1}^{n-1}}} = j_{\omega_{\tau_{n-1}^{n-1}}}
\]
Taking \(\lambda = R_i\) with \(n - i\) parts, we obtain
\[
\hat{\phi}(D_i) = j_{\tau-i}^{s_{2i}} w_{R_i}^{af} = s_{2i} j_{w_{R_i}^{af}} = j_{w_{R_i}^{af}}^{s_{2i}} = j_{t-i}^{\omega_{\tau_{n-1}^{n-1}}}
\]
For \(D'_i\), by direct computation we have
\[
w_{R_i}^{af} = s_{2i} j_{w_{R_i}^{af}}^{s_{2i}} = s_{2i} j_{t-i}^{\omega_{\tau_{n-1}^{n-1}}}
\]
as required. \(\square\)

5. PROOF OF THEOREM 9

Proof of Theorem 9. The stated rational transformation is the composition of the maps \(\Psi, \hat{\theta}\), the isomorphism \(\theta^{-1}\) of Theorem 12, and \(\kappa\).

The description of the map via the images of the \(x\) and \(q\) variables, follows immediately from Theorems 7, 26, 12, 22, Proposition 27, and Lemma 24.

The description in terms of Schubert classes follows immediately from Theorems 1, 3 and 22 together with the explicit correspondence between Schubert classes computed in [LS11b]. \(\square\)

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