SHARP - III: FIRST USE OF ADAPTIVE OPTICS IMAGING TO CONSTRAIN COSMOLOGY WITH GRAVITATIONAL LENS TIME DELAYS

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ABSTRACT
Accurate and precise measurements of the Hubble constant are critical for testing our current standard cosmological model and revealing possibly new physics. With Hubble Space Telescope (HST) imaging, each strong gravitational lens system with measured time delays can allow one to determine the Hubble constant with an uncertainty of ∼7%. Since HST will not last forever, we explore adaptive-optics (AO) imaging as an alternative that can provide higher angular resolution than HST imaging but has a less stable point spread function (PSF) due to atmospheric distortion. To make AO imaging useful for time-delay-lens cosmography, we develop a method to extract the unknown PSF directly from the imaging of strongly lensed quasars. In a blind test with two mock data sets created with different PSFs, we are able to recover the important cosmological parameters (time-delay distance, external shear, lens mass profile slope, and total Einstein radius). Our analysis of the Keck AO image of the strong lens system RXJ 1131−1231 shows that the important parameters for cosmography agree with those based on HST imaging and modeling within 1-σ uncertainties. Most importantly, the constraint on the model time-delay distance by using AO imaging with 0.045″ resolution is tighter by ∼50% than the constraint of time-delay distance by using HST imaging with 0.09″ when a power-law mass distribution for the lens system is adopted. Our PSF reconstruction technique is generic and applicable to data sets that have multiple nearby point sources, enabling scientific studies that require high-precision models of the PSF.

Key words: gravitational lensing:strong – cosmology:distance scale – methods:data analysis – adaptive optics

1 INTRODUCTION
The discovery of the accelerated expansion of the Universe (Perlmutter et al. 1999; Riess et al. 1998) and obser-
vations of the Cosmic Microwave Background (CMB; e.g., Hinshaw et al. 2012; Planck Collaboration et al. 2015) have established a standard cosmological paradigm where our Universe is spatially flat and is dominated by cold dark matter (CDM) and dark energy: the so-called flat ΛCDM model, where Λ represents a constant dark energy density. While the CMB provides strong constraints on the parameters of this model, a relaxation of the assumptions in this model, such as spatial flatness or constant dark energy density, leads to a strong degeneracy between the cosmological parameters, particularly those with the Hubble constant $H_0$. Therefore, independent and accurate measurements of $H_0$ provide one of the most useful complements to the observations of the CMB in constraining the spatial curvature of the Universe, dark energy equation of state, and the number of neutrino species (e.g., Hu 2005; Riess et al. 2009, 2011; Freedman et al. 2012; ?). The recent inferred value of Hubble constant $H_0 = 67.8 \pm 0.9 \, \text{km} \, \text{s}^{-1} \, \text{Mpc}^{-1}$, based on the Planck satellite data of the CMB and the assumption of the flat ΛCDM model, is low in comparison to several direct measurements including those from the Cepheids distance ladder with $H_0 = 74.3 \pm 1.5 \, \text{(stat.)} \pm 2.1 \, \text{(sys.)} \, \text{km} \, \text{s}^{-1} \, \text{Mpc}^{-1}$ (Freedman et al. 2012) and $H_0 = 73.8 \pm 2.4 \, \text{km} \, \text{s}^{-1} \, \text{Mpc}^{-1}$ (Riess et al. 2011). If this indication of tension is not ruled out by systematic effects, then this could indicate new physics beyond the standard flat ΛCDM model. Therefore, pinning down the Hubble constant with independent methods is a key approach to better understand our Universe.

Strong gravitational lensing with time delays provides a one-step measurement of a cosmological distance in the Universe. The background source is composed of a centrally varying source, such as an active galactic nucleus (AGN), and its host galaxy. The time delays between the multiple images of the source, induced by the foreground lens, are given by $\Delta \tau = \frac{1}{2} D_{\Delta t} \Delta \tau$. Here, $\Delta \tau$ is dependent on the geometry and the gravitational potential of the lens system; $\Delta \tau$ can be tightly constrained by the spatially extended images (we usually call them “arcs”) of the lensed background galaxy (e.g., Kochanek et al. 2001; Suyu et al. 2009), together with stellar kinematics of the foreground lens galaxy (e.g., Treu & Koopmans 2002; Koopmans et al. 2003; Suyu et al. 2010, 2014) and studies of the lens environment combined with ray-tracing through numerical simulations (e.g., Hilbert et al. 2007, 2009; Suyu et al. 2010; Fassnacht et al. 2011; Greene et al. 2013; Collett et al. 2013). The stellar kinematics and lens environment studies are important for overcoming the mass-sheet degeneracy and source-position transformations in lensing (Falco et al. 1985; Schneider & Shulse 2013; 2014; Xu et al. 2015). Therefore, by measuring the time delays between the multiple images and modeling the lens and line-of-sight mass distributions, we can constrain $D_{\Delta t}$, which is the so-called time-delay distance that encompasses cosmological dependences and is particularly sensitive to the Hubble constant (e.g., Suyu et al. 2010). The time delays in combination with the stellar velocity dispersion measurements of the lens galaxy further allow us to infer the angular diameter distance to the lens galaxy (Paraficz & Hjorth 2009; Jee et al. 2014).

? have shown that for each lens system we can measure $H_0$ to $\sim7\%$ precision. Hubble Space Telescope (HST) imaging is imperative for this analysis because it not only provides high angular resolution but also a stable point spread function (PSF) for the lens mass modeling. However, HST’s lifetime is finite\(^1\), and the angular resolution is also limited by its aperture size. Given the dozens of time-delay lenses from COSMOGRAIL\(^2\) (e.g., Vuissoz et al. 2007, 2008; Courbin et al. 2011; Jee et al. 2013b,a; Ratna Kumar et al. 2013; Eulaers et al. 2013), and hundreds of new lenses to be discovered in the near future (e.g., Oguri & Marshall 2010; Agnello et al. 2015; Chan et al. 2015; Marshall et al. 2015; More et al. 2015), finding an alternative long-term solution for this promising method is timely.

One alternative approach is imaging from the ground via adaptive optics (AO), which is a technology used to improve the performance of optical systems by reducing the effect of wavefront distortions (e.g., Rousset et al. 1990; Beckers 1993; Watson 1997; Brase 1998). In other words, it aims at correcting the deformations of an incoming wavefront by deforming a mirror and thus compensating for the distortion. The advantages of using AO imaging are (1) the angular resolution obtained with telescopes that are larger than HST can be higher than that of HST since a perfect AO system would lead to a diffraction limited PSF, (2) ground-based telescopes are more accessible. The disadvantage is that we do not have a stable PSF model a priori, since the atmospheric distortion varies both temporally and spatially across the image. Lens targets typically do not have a nearby bright star within $\sim10$ arcseconds, and stars at further angular distance from the target may be insufficient in providing an accurate PSF model given the spatial variation of the PSF across the field.

In HST imaging, we can use the lensing arcs to constrain the lens mass model by using the stable PSF of HST to separate the arc from the bright AGN, but we cannot do so in AO imaging. The contamination of the AGN light on the lensing arcs in AO imaging makes it difficult to constrain the lens model, and consequently $H_0$. One therefore needs to obtain a good PSF model for the AO data, and there are recent studies that aim to do so directly from the AO imaging. Lagattuta et al. (2010) use three Gaussian components as the PSF model to subtract the AGN light which is sufficient to study the lensing galaxy and its substructures. However, the analytical model is not sufficient to describe the complexity of the PSF (see Figure 1 of Lagattuta et al. 2010) which could potentially impact the cosmographic measurements. Rusu et al. (2015) use either an analytic or a hybrid PSF to study the host galaxies of the lensed AGNs (see also Rusu et al. 2014). The hybrid PSF is built from elliptical Moffat profiles (Moffat 1969) with central parts iteratively tuned to match a single AGN image. While this hybrid PSF is useful for extracting properties of the AGN host galaxy, the central parts of the PSF model could manifest the noise pattern in the image (see Figure B.7 of Rusu et al. 2015) which also could potentially impact cosmographic measurements. Agnello et al. (2015) use an iterative method to reconstruct the PSF directly from lens imaging by averaging the doubly lensed AGN. This method is valid only when the lensed AGN are separated far enough from each other. For

\(^1\) And no equivalent optical space-based telescope might be forthcoming soon.

\(^2\) COSmolological MOonitoring of GRAVItational Lenses
Strong lensing AO imaging with time delays for cosmography

2 OBSERVATION

The RXJ 1131–1231 system was observed on the nights of UT 2012 May 16 and May 18 with the Near Infrared Camera 2 (NIRC2) on the Keck-2 Telescope (e.g., Wizinowich et al. 2003). This image was a part of SHARP data. The adaptive optics corrections were achieved through the use of a $R = 15.8$ tip-tilt star located 54.5 arcseconds from the lens system and a laser guide star. The system was observed in the “Wide Camera” mode, which provides a roughly $40'' \times 40''$ field of view and a pixel scale of 0.0397 arcseconds. This pixel scale slightly undersamples the point spread function (PSF), but the angular extent of the lens system and the distance from the tip-tilt star made the use of the Wide Camera the preferable approach.

The observations consisted of 61 exposures, each consisting of 6 coadded 10 s exposures, for a total on-source integration time of 3660 s. The data were reduced by a python-based pipeline that has steps that do the flat-field correction, subtract the sky, correct for the optical distortions in the raw images, and combine the calibrated data frames (for details, see Auger et al. 2011). The final image has a pixel-scale of 0.04 arcseconds and is shown in Figure 1.

Figure 1. Keck AO image (K′ band) of the gravitational lens RXJ1131-1231. The lensed AGN image of the spiral source galaxy are marked by A, B, C and D, and the star-forming regions in the background spiral galaxy form plentiful lensed features. The foreground main lens and the satellite are indicated by G and S, respectively.

3 BASIC THEORY

3.1 The Theory of Gravitational Lensing with Time Delay

In this section we briefly explain the relation between gravitational time delays and cosmology. When a light ray passes near a massive object, it experiences a deflection in its trajectory and acquires a time delay by the gravitational field with respect to the travel time without the massive object. Therefore, the time delay has two contributions: (1) the geometric delay, $\Delta t_{\text{geom}}$, which is caused by the bent trajectory being longer than the unbent one, and (2) the gravitational delay, $\Delta t_{\text{grav}}$, which is due to the fact that the space and time are affected around the gravitational field, so after integrating the gravitational potential along the path, a far away observer receives the light later by a Shapiro delay (Shapiro 1964;Refsdal 1964).

The combination of the two delays is

$$\Delta t = \frac{D_{\Delta t}}{c} \left[ \frac{1}{2} (\theta - \beta)^2 - \psi(\theta) \right],$$

where $\theta$, $\beta$, and $\psi(\theta)$ are the image coordinates, the source coordinates, and the lens potential respectively. The time-delay distance is defined as

$$D_{\Delta t} \equiv (1 + z_s) \frac{D_s D_{AB}}{D_{ls}} \propto H_0^{-1}.$$

where $D_s$, $D_l$, and $D_{ls}$ are the angular diameter distances to the lens, to the source, and between the lens and the source, respectively. Thus, we can measure $D_{\Delta t}$ via gravitational

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3 Strong-lensing High Angular Resolution Program (Fassnacht et al. in prep.)
lensing with time delays. Notice that the gradient of the term in the square brackets in Equation (1) vanishes at the positions of the lensed images and yields the lens equation

\[ \beta = \theta - \nabla \psi(\theta), \] (3)

which governs the deflection of light rays.

We refer the reader to, e.g., Schneider et al. (2006), Bartelmann (2010), Treu (2010), Suyu et al. (2010), Treu & Ellis (2014) for more details.

3.2 Probability Theory

A meaningful measurement should have an uncertainty as a reference and it is also the key to confirm or rule out possible models. Thus, we need to analyze our data based on a probability theory that can present this idea. Bayes’ theorem provides the conditional probability distribution, so we can obtain the posterior probability distribution of the model parameters given the data from Bayes’ rule. For example, if we are interested in the posterior of the parameters \( \pi \) of the hypothesis model \( H \) given the data \( d \), it can be expressed as

\[
P(\pi | d, H) = \frac{P(d | \pi, H) P(\pi | H)}{P(d | H)}, \quad (4)
\]

where the Bayesian evidence can be used to rank the model and our prior based on the data (e.g., MacKay 1992; Hobson et al. 2002; Marshall et al. 2002).

In addition, if we are interested in the posterior of a specific parameter, \( \pi_N \), the posterior distribution can be obtained by marginalizing over other parameters

\[
P(\pi_N | d, H) = \int P(\pi | d, H) \prod_{i=1}^{N-1} d\pi_i. \quad (5)
\]

3.3 Markov chain Monte Carlo

Obtaining the probability distribution function of the parameters in a model can be non-trivial, especially when the number of parameters is high. It is computationally infeasible to explore a high-dimensional parameter space on a regular grid since the number of the grid points for the task exponentially increases with the number of dimensions. Due to the fact that the parameter space is typically large in strong lensing analyses, one can bypass the use of grids by obtaining samples in the multi-dimensional parameter space that represent the probability distribution (i.e., the number density of the samples is proportional to the probability density). A Markov Chain Monte Carlo (MCMC) provides an efficient way to draw samples from the posterior probability density function (PDF) of the lens parameters, because of the approximately linear relation between the computational time and the dimension of the parameter space.

We use MCMC sampling that is implemented in GLEE, a strong lens modeling software developed by S. H. Suyu and A. Halkola (Suyu & Halkola 2010; Suyu et al. 2012b). It is based on Bayes’ theorem and follows Dunkley et al. (2005) to achieve efficient sampling and to test convergence. The pragmatic procedure for convergence is described in Suyu & Halkola (2010). We use Bayesian language in the following sections.

4 METHOD: PSF RECONSTRUCTION AND LENS MODELING

In this section, we describe a novel procedure to analyze the AO imaging without a PSF model a priori. Readers who are not planning to use this method may wish to proceed directly to Section 5 on the scientific results enabled by the method.

The assumption of this strategy is that the PSF does not change significantly within several arcseconds, which is typically valid in AO imaging (van Dam et al. 2006; Wizinowich et al. 2006). We show an overall flow chart in Figure 2 to illustrate how to obtain iteratively the PSF, background source intensity, the lens mass and light model.

In Section 4.1, we decompose the observed light from the lens system into three components (lens galaxy, lensed arcs of the background source galaxy, and the lensed background AGN) and introduce the notation that we will use in the subsequent discussion. In Section 4.2, we obtain the preliminary global structure of AGN light model, while separating the lens light and arc light. In Section 4.3, we obtain the fine structure of the AGN light and incorporate it into the preliminary AGN light model. This is accomplished by correcting the PSF model. In Section 4.4, we update the PSF and use it to model the lens mass and source intensity distributions. Since the lens galaxy light is quite smooth and less sensitive to the PSF model, we use the PSF built from the AGN light for the lens galaxy light model. The PSF updating and lens mass modeling are repeated until the corrections to the PSF become insignificant. (See the criteria in Section 4.3.3 and Section 4.4.3.)

4.1 Light components of the lens system

As shown in Figure 3, our model for the observed light in the lens system on the image plane has three contributions: the lens galaxy light, the arc light (the lensed background source, i.e., the host galaxy of the AGN), and the light of the multiple AGNs on the image plane. We define

\[
d = d' + n, \quad (6)
\]

where \( d \) is the vector of observed data (image pixel values),

\[
d' = Kg + KLs + Mu, \quad (7)
\]

and \( n \) is the noise in the data characterized by the covariance matrix \( C_D \) (we use subscript D to indicate “data”). The blurring matrix \( K \) accounts for the PSF convolution, the vector \( g \) is the image pixel values of the lens galaxy light, the matrix \( L \) maps source intensity to the image plane, the vector \( s \) describes the source surface brightness on a grid of pixels, the matrix \( M \) is composed using the positions and the intensities of the AGNs, and \( w \) is the vector of pixel values of the PSF grid. We refer to Treu & Koopmans (2004) for constructing \( K \) and \( L \), and illustrate the effect of \( M \) in Figure 3.
4.2 Determining the light components

The goal in this section is to obtain the preliminary model of each of the three light components. In step 1 of Figure 2, we input the observed image into the lens modeling software GLEE with a nearby star as our initial PSF model. If there is no nearby star, any star in the field can be used as the initial PSF or we can use one of the AGN images. A different initial PSF does not affect the final results, although we note that a good initial PSF would be helpful as they would require fewer iterations of PSF corrections. In step 2, we decompose the predicted total light sequentially into lens light, arc light, and AGN light. We detail this process in Section 4.2.1 to Section 4.2.3 below.

4.2.1 Lens Light Model (Step 2)

For modeling the light distribution of the lens galaxy, we use parametrized profiles, such as the elliptical Sérsic profile,

\[
I_S(\theta_1, \theta_2) = I_s \exp \left[ -k \left( \frac{\sqrt{\theta_1^2 + \theta_2^2}}{q_{s\text{e}} R_{\text{eff}}} \right)^{1/n_{\text{s\text{e}}}} - 1 \right],
\]  

(8)
Figure 3. Top panel: we decompose the image into lens light, arc light, and AGN light sequentially. Bottom panel: we model the AGN light by placing the PSF grid (described by vector $w$) at each of the AGN positions and scaling each PSF by its respective AGN amplitude. This procedure can be characterized by a matrix $M$, such that the AGN light model on the image plane can be expressed as $Mw$.

where $I_s$ is the amplitude, $k$ is a constant such that $R_{\text{eff}}$ is the effective radius, $q_L$ is the minor-to-major axis ratio, and $n_{\text{Sérsic}}$ is the Sérsic index (Sérsic 1968).

In order to get a preliminary model of the lens light, we mask out the arc light and AGN light region; that is, we boost the uncertainty of the region where the arc light and the AGN light are apparently dominant. Thus, in the fitting region, equation (7) becomes effectively

$$d^0 = Kg.$$

By Bayes’ rule, we have

$$P(\eta|d) \propto P(d|\eta)P(\eta),$$

where $\eta$ represents the parameters of lens light (such as $I_s, q_L, n_{\text{Sérsic}}, R_{\text{eff}}$). We assume uniform prior on the lens light parameters, so we want to obtain

$$P(d|\eta) = \frac{\exp[-E_{D,m\text{ArcAGN}}(d|\eta)]}{Z_{D,m\text{ArcAGN}}},$$

where

$$E_{D,m\text{ArcAGN}}(d|\eta) = \frac{1}{2}(d - Kg)^T C_{D,m\text{ArcAGN}}^{-1}(d - Kg)$$

$$= \frac{1}{2} \chi^2_{m\text{ArcAGN}},$$

and

$$Z_{D,m\text{ArcAGN}} = (2\pi)^{N_{\text{d}}/2} |\det C_{D,m\text{ArcAGN}}|^{1/2}$$

is the normalization for the probability. The covariance matrix, $C_{D,m\text{ArcAGN}}$, is the original covariance matrix with entries corresponding to the arc and AGN mask region boosted (see Appendix A), and $N_d$ is the number of image pixels. We denote $\tilde{\eta}$ as the maximum likelihood parameters (which maximizes equation 11).

Since the initial PSF is a prototype, usually there are some significant residuals in the lens light center when maximizing the posterior of lens light parameters. However, this does not affect the subsequent lens light prediction in the arc region, because the residuals are far from the arc regions. To recap, we can obtain $\tilde{\eta}$ by masking out the arc light and AGN light regions.

4.2.2 Arc Light Model (Step 2)

For modeling the arc light, we describe the source intensity on a grid of pixels on the source plane and map the source intensity values onto the image plane using a lens mass model (via the operation $KLs$ in equation (7)). We use elliptically symmetric power-law distributions to model the dimensionless surface mass density of lens galaxies,

$$\kappa_{\text{pl}}(\theta_1, \theta_2) = \frac{3 - \gamma'}{1 + q} \left( \frac{\theta_E}{\sqrt{\theta_1^2 + \theta_2^2/q^2}} \right)^{\gamma' - 1},$$

where $\gamma'$ is the radial power-law slope ($\gamma' = 2$ corresponding to isothermal), $\theta_E$ is the Einstein radius, and $q$ is the axis ratio of the elliptical isodensity contour. In addition to the lens galaxies, we include a constant external shear with the
following lens potential in polar coordinates $\theta$ and $\varphi$: 
\[ \psi_{\text{ext}}(\theta, \varphi) = \frac{1}{2} \gamma_{\text{ext}} \theta^2 \cos 2(\varphi - \phi_{\text{ext}}), \]  
where $\gamma_{\text{ext}}$ is shear strength and $\phi_{\text{ext}}$ is the shear angle. The shear position angle of $\phi_{\text{ext}} = 0^\circ$ corresponds to a shearing along $\theta_1$, whereas $\phi_{\text{ext}} = 90^\circ$ corresponds to shearing along $\theta_2$.\(^4\)

We model the arc light with the lens light fixed. Since the AGN light dominates near the AGN image positions, we mask out the region where the arc is hard to be seen; that is, we want to minimize the contribution to the source intensity reconstruction from the AGN light. Since the regions of the AGN are masked out, we temporarily\(^5\) drop the AGN component, $M\bar{w}$, in Equation (7), which given $\hat{\eta}$ becomes
\[ d^\circ = K\hat{g} + KLs, \]  
where $\hat{g} = g(\hat{\eta})$. The posterior based on the arc light is
\[ P(\zeta|d, \Delta t, \hat{\eta}) \propto P(d, \Delta t|\hat{\eta}, \zeta)P(\zeta), \]  
where $\zeta$ are the parameters of the lens mass distributions (such as $\gamma'$, $\theta_\gamma$, $\gamma_{\text{ext}}$). The likelihood of the data can be expressed as
\[ P(d, \Delta t|\hat{\eta}, \zeta, s) = \int ds P(d, \Delta t|\hat{\eta}, \zeta, s)P(s), \]  
where
\[ P(d, \Delta t|\hat{\eta}, \zeta, s) \]

\[ = \exp[-E_{D, \text{mAGN}}(\hat{d}|\hat{\eta}, \zeta, s)] Z_{D, \text{mAGN}}^{-1} \]

\[ \cdot \prod_{i=1}^{N_{\text{AGN}}} \frac{1}{\sqrt{2\pi} \sigma_{\text{AGN},i}} \exp\left(-\frac{\theta_{\text{AGN},i} - \theta_{\text{AGN},i}^p}{2\sigma_{\text{AGN},i}}\right) \]

\[ \cdot \prod_{i=1}^{N_{\text{AGN}}} \frac{1}{\sqrt{2\pi} \sigma_{\Delta t,i}} \exp\left(-\frac{(\Delta t_i - \Delta t_i^p)^2}{2\sigma_{\Delta t,i}^2}\right) \]  
\[ E_{D, \text{mAGN}}(\hat{d}|\hat{\eta}, \zeta, s) \]

\[ = \frac{1}{2}(d - K\hat{g} - KLs)^T C_{D, \text{mAGN}}^{-1}(d - K\hat{g} - KLs), \]  
and
\[ Z_{D, \text{mAGN}} = (2\pi)^{N_{\text{d}}/2}(\det C_{D, \text{mAGN}})^{1/2} \]

is the normalization for the probability. We discuss the “mAGN” regions in Appendix A. In the second term of Equation (19), $\theta_{\text{AGN},i}$ is the measured AGN image position and $\sigma_{\text{AGN},i}$ is the estimated positional uncertainty of AGN image $i$; in the third term, $\Delta t_i$ is the measured time delay with uncertainty $\sigma_{\Delta t,i}$ for image pair $i = AB, CB$, or $DB$. After we maximize the likelihood of the data, we obtain $\hat{\zeta}$, and also the predicted arc light of the reconstructed background source intensity, $s$, of the AGN host galaxy. Note that if there is no time-delay information, one can remove the last term in Equation (19).

4.2.3 AGN Light Model (Step 2)

In Equation (7), we use $M\bar{w}$ to represent the AGN light. In the next section, we further decompose the PSF, $\bar{w}$, into the global structure and the fine structure that are shown in Figure 4. In particular, we define
\[ \bar{w} = w_{[0]} + T_{[0]} \delta w_{[0]}, \]  
where $w_{[0]}$ is the vector of global structure, $\delta w_{[0]}$ is the vector of fine structure and the subscript $[0]$ represents the zero-th iteration. Since, in this section, we focus on the global structure of the PSF, we postpone the discussion of $T$ to Equation (29) and let
\[ \bar{w} = w_{[0]}. \]  

By using $\hat{\eta}$, $\hat{\zeta}$, and $\hat{s}$ from the previous two sections and keeping them fixed, we model the global structure of the PSF with Gaussian profiles, each of the form
\[ I_c(\theta_1, \theta_2) = I_\theta \exp\left(-\frac{\theta_1^2 + (\theta_2/q_\theta)^2}{2\sigma_\theta^2}\right), \]  
where $I_\theta$ is the amplitude, $q_\theta$ is the axis ratio, and $\sigma_\theta$ is the width. We find that a few Gaussians ($\sim 2 - 4$) with a common centroid are sufficient in describing the global structure.
of the PSF.\footnote{The different Gaussian components can vary their amplitudes, position angles and axis ratios.} Substituting Equation (23) into Equation (7), given \( \hat{\eta}, \hat{\zeta} \) and \( \hat{\delta} \), we obtain
\[
d'' = K\hat{g} + K\hat{L}\hat{\delta} + M\hat{w}_{0},
\]
where \( \hat{L} = L(\hat{\zeta}) \), which is kept fixed at this step. Note that the \( K \) matrix here is based on the initial PSF model, before the multi-Gaussian fitting. The posterior of the PSF and AGN parameters is given by
\[
P(\nu, \xi | \hat{\eta}, \hat{\zeta}) = \frac{P(d|\hat{\eta}, \xi, \nu, \xi)P(\nu, \xi)}{P(d|\hat{\eta}, \xi)},
\]
where \( \nu \) represents the parameters of the Gaussian profiles in Equation (24) that yield \( \hat{w}_{0} \), \( \xi \) are the amplitudes and the positions of the AGN, which are coded in \( M \). The likelihood of Equation (26) is
\[
P(d|\hat{\eta}, \xi, \nu, \xi) = \frac{\exp[-E_{D}(d|\hat{\eta}, \xi, \nu, \xi)]}{Z_{D}},
\]
where
\[
E_{D}(d|\hat{\eta}, \xi, \nu, \xi) = \frac{1}{2}(d - K\hat{g} - K\hat{L}\hat{\delta} - M\hat{w}_{0})^{T} \cdot C_{D}^{-1}(d - K\hat{g} - K\hat{L}\hat{\delta} - M\hat{w}_{0}),
\]
and \( Z_{D} = (2\pi)^{N_{d}/2}(\det C_{D})^{1/2} \). We denote \( \hat{\nu} \) and \( \hat{\xi} \) as the maximum likelihood parameters (that maximizes Equation (26)) from which we can obtain the optimal AGN light on the image, given the optimized source and lens mass models from the previous sections.

### 4.3 Pixelated Fine Structure of AGN light

In this section, we introduce the inner loop which aims at extracting the fine structure, \( \delta w_{0} \), in Equation (22) by using a correction grid. We show it visually in Figure 5. The goal of the inner loop is to incorporate most of the fine structures into the PSF model; then in the outer loop, we can use the updated PSF model obtained from the inner loop to remodel all the light components (which require a given PSF model). Since this section is the starting point of the inner loop and outer loop, the \( \hat{\eta}, \hat{\zeta}, \hat{\delta}, \hat{\nu}, \) and \( \xi \) we get by optimizing Equations (10), (17), and (26) are actually the zero-th outer loop iteration and the zero-th inner loop iteration, which we denote by \( \hat{\eta}_{[0]}, \hat{\zeta}_{[0]}, \hat{\delta}_{[0]}, \hat{\nu}_{[0,0]}, \) and \( \hat{\xi}_{[0,0]} \).

#### 4.3.1 PSF Correction for Each Iteration

(Inner Loop: Step 3)

In general, given \( \hat{\eta}_{[n]}, \hat{\zeta}_{[n]}, \hat{\delta}_{[n]}, \hat{\nu}_{[m,m]}, \) and \( \hat{\xi}_{[n,m]} \), where \( m \) is the iteration number of the inner loop and \( n \) is the

\[ \hat{\nu}_{[m,m]} \] is only present when \( n = m = 0 \), which corresponds to parameters of the Gaussian profiles in Equation (24).
iteration number of the outer loop, we can write out the equation as
\[ d^p = K(n)g_k(n) + K(n)L(n)\hat{z}(n) \]
\[ + M(n,m)(\hat{w}(n,m) + T(n,m)\delta w_t(n,m)) \equiv d_{\text{correction}}. \]  
(29)

where
\[ \hat{w}(n,m) = \begin{cases} w(n,m) & \text{if } n = m = 0 \\ \text{otherwise} \end{cases} \]
\[ g_k(n) = g(\eta(n)), \quad L(n) = L(\xi(n)), \quad M(n,m) = M(\xi(n,m)), \quad K(n) \]

is the n-th blurring matrix (we explain how to get the n-th blurring matrix in Section 4.4.1), T(n,m) is a matrix which makes \( \delta w_t(n,m) \) the same length as \( \hat{w}(n,m) \) by padding with zeros (we show it visually in Appendix B), and \( \delta w_t(n,m) \) is the fine structure we want to obtain by the end of this section.

The posterior of \( \delta w_t(n,m) \) is
\[ P(\delta w_t(n,m) | d, \eta(n), \nu(n,m), \xi(n,m), \lambda_{\delta w_t(n,m)}, \mathcal{R}) \]
\[ = \frac{P(d | \delta w_t(n,m), \eta(n), \nu(n,m), \xi(n,m)) \cdot P(\delta w_t(n,m) | \lambda_{\delta w_t(n,m)}, \mathcal{R})}{P(d | \lambda_{\delta w_t(n,m)}, \eta(n), \nu(n,m), \xi(n,m))} \]  
(30)

where \( P(\delta w_t(n,m) | \lambda_{\delta w_t(n,m)}, \mathcal{R}) \) is the prior on \( \delta w_t(n,m) \) given \( \{\lambda_{\delta w_t(n,m)}, \mathcal{R}\} \) with \( \mathcal{R} \) denoting a particular form of “regularization” on \( \delta w_t(n,m) \) and \( \lambda_{\delta w_t(n,m)} \) characterising the strength of the regularization. We can write the likelihood in Equation (30) as
\[ P(d | \delta w_t(n,m), \eta(n), \nu(n,m), \xi(n,m)) \]
\[ = \exp\left[ -E_{D,\text{mAC}}(\lambda_{\delta w_t(n,m)}) (d | \delta w_t(n,m), \eta(n), \nu(n,m), \xi(n,m)) \right] / Z_{D,\text{mAC}}(\lambda_{\delta w_t(n,m)}) \]  
(31)

where “mAC” stands for maskAGNcenter,
\[ E_{D,\text{mAC}}(\lambda_{\delta w_t(n,m)}) (d | \delta w_t(n,m), \eta(n), \nu(n,m), \xi(n,m)) \]
\[ = \frac{1}{2} (d - d_{\text{correction}})^T C_{D,\text{mAC}}^{-1} (d - d_{\text{correction}}), \]  
(32)

and \( Z_{D,\text{mAC}} = (2\pi)^{N/2} (\det C_{D,\text{mAC}})^{1/2} \) is the normalization for the probability. We discuss the mAC (maskAGNCenter) regions in Appendix A.

The prior/regularization we impose in Equation (30) on the correction grid (fine structure of PSF) is to prevent the correction grid from absorbing the noise in the observed image. We express the prior in the following form
\[ P(\delta w_t(n,m) | \lambda_{\delta w_t(n,m)}, \mathcal{R}) \]
\[ = \exp\left[ -\lambda_{\delta w_t(n,m)} E_{\delta w_t(n,m)}(\delta w_t(n,m) | \mathcal{R}) \right] / Z_{\delta w_t(n,m)}(\lambda_{\delta w_t(n,m)}) \]  
(33)

where \( \lambda_{\delta w_t(n,m)} \) is the regularization constant of correction, \( Z_{\delta w_t(n,m)}(\lambda_{\delta w_t(n,m)}) \]
\[ = \int dN_{\delta w_t(n,m)}(\delta w_t(n,m)) \exp(-\lambda_{\delta w_t(n,m)} E_{\delta w_t(n,m)}) \]

is the normalization of the prior probability distribution (note that the optimal \( \lambda_{\delta w_t(n,m)} \) is not determined yet), and \( N_{\delta w_t(n,m)} \) is the number of pixels of the correction grid. We use the curvature form for the function \( E_{\delta w_t(n,m)} \), which is discussed in Suyu et al. (2006).

Again, it is easy to understand that we want to maximize Equation (30). We obtain the most probable solution
\[ \delta w_t(n,m) = (F + \lambda_{\delta w_t(n,m)} H)^{-1} (M(n,m) T(n,m))^T C_{D,\text{mAC}}^{-1} u, \]  
(34)

where
\[ F = \nabla \nabla E_{D,\text{mAC}}, \]
\[ = T(n,m)^T (M(n,m) C_{D,\text{mAC}}^{-1} M(n,m) T(n,m))^T, \]  
(35)

\[ H = \nabla \nabla E_{\delta w_t(n,m)}, \]  
(36)

\[ u = d - K(n) g_k(n) - K(n) L(n) \hat{z}(n) - M(n,m) \hat{w}(n,m), \]  
(37)

and
\[ \nabla \equiv \frac{\partial}{\partial \delta w_t(n,m)}. \]  
(38)

Now, we go back to find the optimal regularization constant; that is, we want to maximize
\[ P(\lambda_{\delta w_t(n,m)} | d, \mathcal{R}) = \frac{P(d | \lambda_{\delta w_t(n,m)}, \mathcal{R}) P(\lambda_{\delta w_t(n,m)})}{P(d | \mathcal{R})} \]  
(39)

using Bayes’ rule. If we assume a flat prior in \( \log \lambda_{\delta w_t(n,m)} \), we want to maximize \( P(d | \lambda_{\delta w_t(n,m)}, \mathcal{R}) \), which is the evidence in equation (30). Following the results from Suyu et al. (2006), we get
\[ 2 \hat{\lambda}_{\delta w_t(n,m)} E_{\delta w_t(n,m)}(\delta w_t(n,m)) \]
\[ = N_{\delta w_t(n,m)} - \hat{\lambda}_{\delta w_t(n,m)} Tr[ (F + \hat{\lambda}_{\delta w_t(n,m)} H)^{-1} H], \]  
(40)

where \( Tr \) denotes the trace and \( \hat{\lambda}_{\delta w_t(n,m)} \) is the optimal regularization constant. If we set \( m = 0 \) (zeroth iteration of the fine structure), we obtain \( \delta \hat{w}(n,0) \). Due to the sharp intensity of the AGN center, the residuals there are much stronger than the peripheral area. If we directly extract the full correction grid, the regularization intends to under-regularize on the peripheral area and over-regularize on the center. To avoid this problem, at first, we extract the correction only around the AGN center; that is, we start from small \( N_{\delta w_t(n,m)} \) (half light radius or smaller) and increase it gradually (around 1.2 times previous size each time) as we obtain \( \delta w_t(n,m) \). We show the idea in Figure 5 (note that the indices on \( \delta w_t \) in the figure are labeling the pixels, rather than the iteration numbers).

Since every iteration of \( \delta w_t(n,m) \) has its own fine structure (correction) uncertainty, according to Suyu et al. (2006), we also take as estimates of the 1σ uncertainty on each pixel value the square root of the corresponding diagonal element of the covariance matrix given by
\[ C_{\delta w_t(n,m)} = (F + \hat{\lambda}_{\delta w_t(n,m)} H)^{-1}. \]  
(41)
4.3.2 Add Fine Structure into Global Structure

(Inner Loop: Step 4)

We start with the zeroth inner loop iteration, by setting \( m = 0 \), of the global structure, \( w_{[n,0]} \), and fine structure, \( \delta w_{[n,0]} \) (which we can obtain by following the previous two sections). We then add the fine structure into the global structure by defining

\[
w_{[n,1]} = w_{[n,0]} + T_{[n,0]} \delta w_{[n,0]},
\]

where \( w_{[n,1]} \) is the first iteration in inner loop. More generally, we define the \( m + 1^{\text{th}} \) iteration of the PSF as

\[
w_{[n,m+1]} = w_{[n,m]} + T_{[n,m]} \delta w_{[n,m]}.
\]

We recalculate the AGN parameters every time after getting a new \( w_{[n,m+1]} \), so given the same \( \tilde{\eta}_{[n]} \) and \( \tilde{\zeta}_{[n]} \) in Equation (29), the posterior of the AGN parameters is given by

\[
P(\xi_{[n,m+1]} | d, \tilde{\eta}_{[n]}, \tilde{\zeta}_{[n]}) = \frac{P(d | \tilde{\eta}_{[n]}, \tilde{\zeta}_{[n]}, \xi_{[n,m+1]})) P(\xi_{[n,m+1]}))}{P(d | \tilde{\eta}_{[n]}, \tilde{\zeta}_{[n]}))} \tag{44}
\]

(recall that \( \xi_{[n,m+1]} \) represents the relative amplitudes and the positions of the AGNs in the \( n^{\text{th}} \) outer loop iteration, and \( m + 1^{\text{th}} \) inner loop iteration). The likelihood of Equation (44) is

\[
P(d | \tilde{\eta}_{[n]}, \tilde{\zeta}_{[n]}, \xi_{[n,m+1]})) = \exp[-E_{D,[n,m+1]}(d | \tilde{\eta}_{[n]}, \tilde{\zeta}_{[n]}, \xi_{[n,m+1]}))]
\]

where

\[
E_{D,[n,m+1]}(d | \tilde{\eta}_{[n]}, \tilde{\zeta}_{[n]}, \xi_{[n,m+1]})) = \frac{1}{2}(d - \Omega)^\top C_{D}^{-1}(d - \Omega)
\]

and

\[
\Omega = K_{[n]} \tilde{g}_{[n]} + K_{[n]} \tilde{L}_{[n]} \tilde{\theta}_{[n]} + M_{[m,m+1]} w_{[n,m+1]},
\]

and \( Z_{D} \), as usual, is \( (2\pi)^{N \times 2/(\det C_{D})^{1/2}} \). After maximizing Equation (44), we obtain \( \tilde{\xi}_{[n,m+1]} \). We then replace the \( \tilde{\xi}_{[n,m]} \) from the previous iteration with the \( \tilde{\xi}_{[n,m+1]} \) we just obtained, and conduct the next inner loop iteration.

4.3.3 The Criteria to Stop the Inner Loop.

During every inner loop, we gradually increase the size, \( N_{\delta w_{[n,m]}} \), of the correction grid. Then, if (1) there is no residuals outside the correction grid, (2) Equation (34) has no intensity, and (3) Equation (46) no longer decreases, we stop the inner loop. Assuming we have \( N_{\text{inner}} \) iterations in the inner loop, we obtain \( w_{[n,N_{\text{inner}}]} \) and \( \tilde{\xi}_{[n,N_{\text{inner}}]} \). We then define

\[
w_{[n,N_{\text{inner}}]} \equiv w_{[n+1,0]} \equiv w_{[n+1]}
\]

and

\[
\tilde{\xi}_{[n,N_{\text{inner}}]} \equiv \tilde{\xi}_{[n+1,0]} \equiv \tilde{\xi}_{[n+1]}. \tag{49}
\]

4.4 Lens Modeling with updated PSF

The goal of the outer loop in Figure 2 is to remodel all the light components with the updated PSF; that is, we want to obtain a better lens light model and arc light model with the new blurring matrix, and the underlying fine structure can then be revealed.

4.4.1 Update the Blurring Matrix and the Image Covariance Matrix (Outer Loop: Step 6)

Blurring matrix: After obtaining the last version of the PSF from Section 4.3.3, we update the blurring matrix, \( K \), in Equation (7). In order to accelerate modeling speed, which highly depends on the size of the PSF for convolution of the extended images, we choose the central \( l_{[n]} \times l_{[n]} \) pixels of the updated PSF grid (that has \( N_{\delta w_{[n,N_{\text{inner}}]}} \) pixels) as the new PSF to construct \( K_{[n+1]} \) for the spatially extended images.\(^8\)

Image covariance matrix: We accumulate the uncertainty of the PSF pixel grid from every inner loop. The accumulated uncertainty is

\[
n_{[n+1]}^2 \equiv \sum_{m=0}^{N_{\text{inner}}} T_{[n,m]}.k_{i} C_{[\delta w_{[n,m]},1]} \delta_{ij} \tag{50}
\]

where \( T_{[n,m],k_{i}} \) is the element at \( k \) row and \( i \) column of \( T_{[n,m]}, C_{[\delta w_{[n,m]},i]} \) is the element at \( i \) row and \( j \) column of \( C_{\delta w_{[n,m]}}, \) and \( \delta_{ij} \) is the Kronecker delta. The element of the \( n + 1^{\text{th}} \) noise vector is defined as

\[
n_{[n+1],i} = \sqrt{n_{[n]}^2 + \sum_{j} M_{[n+1],nk} n_{[n+1],jk}} \tag{51}
\]

which is characterized by the covariance matrix \( C_{\delta w_{[n+1]}}.\)^9

Note that \( n_{[n]} \) is the element of the original data noise vector.

4.4.2 Lens Modeling with All Light Components

(Outer loop: Step 2)

In general, when executing the next iteration of outer loop, we can express Equation (7) as

\[
d^{e} = K_{[n+1]} g_{[n+1]} + K_{[n+1]} L_{[n+1]} s_{[n+1]} + M_{[n+1]} w_{[n+1]}
\]

\[
\equiv d^{\text{total}}\tag{52}
\]

The posterior can be written as

\[
P(\eta_{[n+1]}, \zeta_{[n+1]} | d, \Delta t) \propto P(\eta_{[n+1]}, \zeta_{[n+1]} | d, \Delta t) P(\eta_{[n+1]}, \zeta_{[n+1]} | d, \Delta t) \tag{53}
\]

\(^8\) For example, we choose \( l_{[n]} = n_{[n]} \).

\(^9\) The purpose of updating the the image covariance matrix is to speed up the modeling to the final answer since the correction uncertainty that we add into the image covariance is around AGN; that is, we weight the arc region more. However, in the end, if there is no “correction”, Equation (50) is close to zero.
The likelihood of the data can be expressed as
\[
P(d, \Delta t|\eta_{[n+1]}, \zeta_{[n+1]}, \xi_{[n+1]})
\]
\[
= \int ds_{[n+1]} P(d, \Delta t|\eta_{[n+1]}, \zeta_{[n+1]}, \xi_{[n+1]}, s_{[n+1]})P(s_{[n+1]}),
\]
where
\[
P(d, \Delta t|\eta_{[n+1]}, \zeta_{[n+1]}, \xi_{[n+1]}, s_{[n+1]})
\]
\[
= \exp\left[\frac{-E_{D,[n+1]}(d|\eta_{[n+1]}, \zeta_{[n+1]}, \xi_{[n+1]}, s_{[n+1]})}{Z_{D,[n+1]}}\right].
\]

In Section 4.4.3, we mention the Criteria to Stop the Outer Loop. We iterate the outer loop until Equation (54) holds true. The size of the PSF in AO image is the next set of inner loop iterations. If we have a total of \(N\) outer loop iterations, we obtain the final \(K_{[N]}\) and \(w_{[N]}\).

5 DEMONSTRATION AND BLIND TEST

In this section, we demonstrate the method using two mock data sets that are created with different PSFs, and show that we can recover the input parameters in both mocks by using the strategy in Section 4 together with GLEE. S.H.S. simulates AO images that mimic the strong lensing system, RXJ 1131 – 1231, with two foreground lens galaxies and a background source comprised of an AGN and its host galaxy. S.H.S. uses an elliptically symmetric power-law profile to describe the main lens mass distribution and a pseudothermal elliptic mass profile to describe the mass distribution of the satellite galaxy. The background host galaxy of the AGN is described by a Sérsic profile with additional star-forming regions superposed, and the lens light distribution is based on a composite of two Sérsic light profiles. The simulated lensed images and background sources are shown in the third and second column, respectively, of the first (mock #1) and third (mock #2) rows of Figure 6. The difference between the two mocks is their PSFs. In mock #1, the PSF is taken to be a star observed with Keck’s laser guide star adaptive optics system (LGS AO) that is relatively sharp and with a lot of structures (FWHM is \(\sim 0.03''\)). In mock #2, the PSF is relatively diffuse and without structures, which is similar to the PSF in the real data (FWHM is \(\sim 0.045''\)). We show them in the first column of the first and third rows of Figure 6. G.C.F.C. does a blind test of the PSF reconstruction method on mock #1; that is, G.C.F.C. does not know the input value at the beginning, and S.H.S. only reveals the input value when G.C.F.C. has completed the analysis of mock #1. Since the input value is the same in mock #2, G.C.F.C. models mock #2 by using the same strategy although the mock #2 test is performed after mock #1 and therefore is not blinded.

5.1 Mock #1: a sharp and rich structured PSF

The mock #1 image has 200 \(\times\) 200 surface brightness pixels as constraints. The pixel size is 0.04'' . The simulated time delays in days relative to image B are: \(\Delta t_{AB} = 1.5 \pm 1.5, \Delta t_{CB} = -0.5 \pm 1.5, \Delta t_{DB} = 90.5 \pm 1.5\).

We follow the procedure described in Section 4 and Figure 2. The reconstructions are shown in the second row of Figure 6. To demonstrate the iterative process visually, we show the process in Figure 7. The first column shows each PSF correction grid in different iteration, the second column shows the cumulative PSF correction from iteration 1 to iteration 18, the third column is the PSF model at each iteration, and the right-most column shows the best fitting residuals with current PSF model. It is obvious that we get better and better normalized residuals as the iterative PSF corrections proceed. We follow Section 4.3.3 and increase gradually the size of the PSF; the size of the final PSF is 85 \(\times\) 85 (which corresponds to 3.4'' \(\times\) 3.4''). However, since the PSF is very sharp in mock #1, the PSF size with 19 \(\times\) 19 (which corresponds to 0.76'' \(\times\) 0.76'') for the blurring matrix is enough. While 19 \(\times\) 19 is sufficient for the extended source/lens light, it is not for the AGNs; 85 \(\times\) 85 is needed for describing the AGNs.

We try a series of source resolutions from coarse to fine, and the parameter constraints stabilize starting at \(\sim 52 \times 52\) source pixels, corresponding to source pixel size of \(\sim 0.045''\). In order to quantify the systematic uncertainty, we consider the following set of source resolutions: 52 \(\times\) 52, 54 \(\times\) 54, 56 \(\times\) 56, 58 \(\times\) 58, 60 \(\times\) 60, and 62 \(\times\) 62. We weight each choice of the
Figure 6. The simulation (input), reconstruction (output), and normalized residuals of mock #1 and mock #2. The left column shows the input/output PSF, the middle left column shows the input/output sources (host galaxy of the AGN), the middle right column shows the input/output images, and the right column shows the normalized image residuals (in units of the estimated pixel uncertainties). Our PSF reconstruction method is able to reproduce both the global and fine structures in the PSF, yielding successful reconstructions of the background source intensity and the lensed images. Both reduced $\chi^2$ are $\sim 1$.

We weight the chains by the same weight because the source evidence are similar, and the lens parameterizations are the same.

5.2 Mock #2: a diffuse and smooth PSF

The mock #2 image has $300 \times 300$ surface brightness pixels as constraints (the larger dimensions of the image are necessary for modeling the diffuse PSF). The pixel size and time delays are the same as in mock #1. The size of the final PSF is $127 \times 127$ (which corresponds to $5.08'' \times 5.08''$). Since the PSF is very diffuse in mock #2, the PSF size with

\[ \int_0^{\theta_{E,\text{tot}}} \frac{2\pi}{\pi \theta_{E,\text{tot}}^2} \kappa_{\text{tot}}(\theta, \varphi) d\varphi d\theta = 1, \]  

$\kappa_{\text{tot}}$ is the total projected mass density including the main galaxy and its satellite, and $\varphi$ is the polar angle on the image plane. The total Einstein radius in here is only a circular approximation for the elliptical galaxy plus its satellite.
Figure 7. We demonstrate the iterative reconstruction process. From the left to the right, we show the PSF correction, cumulative PSF correction, current PSF model, and normalized residuals after using the current PSF model at iteration 1, 9, and 18. Since we sequentially increase the PSF correction grid as we iterate, the size of the PSF correction grid at iteration 1 is smaller than that of other iterations.

59 × 59 (which corresponds to 2.36′′ × 2.36′′) for the blurring matrix is needed to convolve the spatially extended images. We show the reconstruction in the fourth row of Figure 6.

We also try a series of source resolutions from coarse to fine, and the parameter constraints stabilize starting at ~ 59 × 59. To quantify systematic uncertainties due to source resolution, we consider the following set of source resolutions: 59 × 59, 60 × 60, 61 × 61, 62 × 62, and 63 × 63. We also weight each source resolution equally, and combine the Markov chains together. We show the constraints on the same important parameters as mock #1 for cosmography in the lower panel of Figure 8. The white dots represent the input values. The results show that we can recover the important parameters for cosmography. Again, although we cannot recover the individual Einstein radius due to the strong degeneracy between these two Einstein radii, we can still recover the total Einstein radius.

We use the source-intensity-weighted regularization in the source reconstruction to prevent the source from fitting to the noise. The noise-overfitting problem is due to the fact that the outer region of the source plane is under-regularized. We do two tests which show its negligible impact on cosmographic inference: (1) We test it by changing the image covariance, $C_D$, such that the uncertainties corresponding to low surface brightness areas are boosted (which is a similar effect as allowing the source to be more regularized at low surface brightness regions). The results show that the relative posteriors of lens/cosmological parameters are insensitive to such changes of $C_D$ (hence the source regularization); (2) We impose the source-intensity weighted regularization on the source plane, which can regularize more on the low surface bright area on the source plane (see e.g., Tagore & Keeton 2014, for another type of regularization based on analytic profile). Specifically, we obtain the first version of the source intensity distribution $s_i$ on a grid of pixels following the method of Suyu et al. (2006) with a constant regularization for all source pixels. We then repeat the source reconstruction but with the regularization constant $\lambda$ scaled inversely proportional to $s_i^4$, allowing high/low source intensity regions to be less/more regularized. The relative posteriors between the different MCMC samples in the chains are the same between the uniform and source-intensity-weighted source regularizations. Furthermore, even with different source reconstruction methods, the Einstein radius, which also plays an important role in cosmographic inference, is still robust.

6 REAL DATA MODELING

We apply our newly-developed PSF reconstruction method to the real AO imaging shown in Section 2, and use the time delays from Tewes et al. (2013b). For the lens light, we use two Sérsic profiles with common centroids and position angles for the main lens galaxy, and use one circular Sérsic profile for the small satellite (whereas in the mock data in...
Figure 8. **Upper panel**: the posterior probability distribution of the key lens model parameters for mock #1. We use the PSF size, $19 \times 19$, for convolution of the spatially extended images of the AGN host galaxy. We combine the different source resolutions: $52 \times 52$, $54 \times 54$, $56 \times 56$, $58 \times 58$, $60 \times 60$, and $62 \times 62$, and weight each chain equally. The contours/shades mark the 68.3%, 95.4%, and 99.7% credible regions. The white dots are the input values. **Lower panel**: the posterior probability distribution of the key lens model parameters for mock #2. We use the PSF size, $59 \times 59$, for convolution of the spatially extended images of the AGN host galaxy. We combine the different source resolutions: $59 \times 59$, $60 \times 60$, $61 \times 61$, $62 \times 62$, and $63 \times 63$ and weight each chain equally. We can recover the key lens parameters for cosmography such as the modeled time-delay distance, total Einstein radius, and external shear, despite the strong degeneracy between the Einstein radii of the main and satellite galaxies (which consequently we do not recover in mock #2).
Figure 9. RXJ1131−1231 AO image reconstruction of the most probable model with a source grid of 79 × 79 pixels and 69 × 69 PSF for convolution of spatially extended images. Top left: RXJ1131−1231 AO image. Top middle: predicted lensed image of the background AGN host galaxy. Top right: predicted light of the lensed AGNs, the bright compact region: lensed images of a bright compact region in the AGN host galaxy, and the lens galaxies. Bottom left: predicted image from all components, which is a sum of the top-middle and top-right panels. Bottom middle: image residual, normalized by the estimated 1-σ uncertainty of each pixel. Bottom right: the reconstructed host galaxy of the AGN in the source plane.

Figure 10. The left panel is the reconstructed AO PSF. The right panel is the radial average intensity of the PSF, which shows the core plus its wings.

Section 5 we describe the light of the satellite as a point source with PSF, \( w \). We find that, in this AO image, 4 concentric Gaussian profiles provide a good description of the initial global structure of the PSF\(^{12}\), which is the procedure we discussed in Section 4.2.3. For modeling the main lens mass, we use an elliptical symmetric distribution with a power-law profile and an external shear which are described in Section 4.2.2; for modeling the mass distribution of the satellite, we use a pseudo-isothermal mass distribution.

\(^{12}\) Due to unknown PSF, we do not have prior information on PSF. Thus, we test multiple concentric Gaussian profiles to fit the AGN. However, we find that the initial PSF model does not affect the final results which is shown in Section 5, because the iterative method will correct it in the end.
Figure 11. Posterior of the key lens model parameters for RXJ 1131−1231 and the time delays. We use the PSF size, 59 × 59, for convolution of the spatially extended lens and arcs. We show the constraints from Markov chains of different source resolutions: 71 × 71, 73 × 73, 75 × 75, 77 × 77, and 79 × 79. The contours mark the 68.3%, 95.4%, and 99.7% credible regions for each source resolution. The spread in the constraints from different chains allow us to quantify the systematic uncertainty due to the pixelated source resolution.

Figure 12. Posterior of the key lens model parameters for RXJ1131 and the time delays. We compare the PSF size, 59 × 59 and 69 × 69, for convolution of the spatially extended lens and arcs. The constraints correspond to the combination of Markov chains of different source resolutions (71 × 71, 73 × 73, 75 × 75, 77 × 77, and 79 × 79) in both PSF sizes. The contours mark the 68.3%, 95.4%, and 99.7% credible regions. The constraints of the two PSF sizes are in good agreement, indicating that PSF sizes larger than ∼ 59 × 59 are sufficient to capture the PSF features for convolving the spatially extended images.
Figure 13. Left panel: comparison of posterior of the key lens model parameters between AO imaging (dashed) and HST imaging (shades). The AO constraints are from the combination of both the 59 × 59 and 69 × 69 chains containing the series of source resolutions (e.g., Figure 12 for 59 × 59). The contours/shades mark the 68.3%, 95.4%, and 99.7% credible regions. The AO constraints are consistent with the HST constraints, and are in fact ~50% tighter on the modeled time-delay distance. Right panel: PDFs for $D_{\Delta t}$, showing the constraints from HST image and AO image.

We use the source-intensity-weighted regularization in the source reconstruction.

The mass model parameterization is the same as 7 except for a slight difference in the definition of $\theta_E$ due to ellipticity. In this paper, we compare the $\theta_E$ as defined in equation (14). Thus the $\theta_E$ shown in this paper is slightly different from that of 7.
The reference of the position is in Figure 9.
All the position angles are measured counterclockwise from positive θ2 (north).
The amplitude is in equation (8).

**Note:** There are total 39 parameters that are optimized or sampled. The optimal parameters have little effect on the key parameters for cosmology (such as $D_{\text{model}}^{\Delta t}$). For the lens light, two Sérsic profiles with common centroid and position angle are used to describe the main lens galaxy G. They are denoted as G1 and G2 above. The source pixel parameters (s) are marginalized and are thus not listed.

PSF is needed for convolution of the spatially extended lens and arcs. By performing MCMC sampling, we can recover the important parameters for cosmography (time-delay distance, external shear, slope, and total Einstein radius of the main galaxy plus its satellite). Although we cannot recover the individual Einstein radius, the effect on time-delay distance due to the presence of the satellite is less than 1% (7).

- We model the AO RXJ 1131–1231 image by the iterative PSF correction scheme. We compare the results of important parameters with the results from modeling the HST imaging in ?. Except for the highly degenerate Einstein radius of the main galaxy, other important parameters for cosmography agree with each other within 1–σ (Figure 13). Furthermore, the constraint of time-delay distance by using AO imaging with 0.045'' resolution is tighter than the constraint of time-delay distance by using HST imaging with 0.09'' by around 50%.

The iterative PSF reconstruction method that we have developed is general and widely applicable to studies that require high-precision PSF reconstruction from multiple nearby point sources in the field (e.g., the search of faint companions of stars in star clusters). For the case of gravitational lens time delays, this method lifts the restriction of using HST strong lensing imaging, and opens a new series of AO imaging data set to study cosmology. From the upcoming surveys, hundreds of new lenses are predicted to be discovered; this method not only can motivate more telescopes to be equipped with AO technology, but also facilitate the goal to reveal possible new physics by beating down the uncertainty on $H_0$ to 1% via strong lensing (?).

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APPENDIX A: ARC AND AGN MASK REGIONS

We show the three different mask regions, maskArcAGN (mArcAGN), maskArc (mArc), and maskAGNcenter (mAc) in Figure A1. For modeling the lens light in Section 4.2.1, we mask out the region which contains significant arc light and AGN light in the left panel. For modeling the arc light in Section 4.2.2, we mask out the region with significant AGN light shown in the middle panel. For extracting the PSF correction, we show the residuals in the right panel (which is the image with the lens light, arc light, and AGN light subtracted). When the size of the correction grid is small such that the correction grids do not overlap other AGN center, we only need to mask out the area where it comes obviously from the host galaxy of AGN. For instance, if the background AGN has compact bright blobs in its host galaxy, due to the limit of the resolution on the source plane, the predicted arc cannot reconstruct the compact blobs, so there are residuals around these compact blobs on the image plane (shown in the right panel with red arrows). In order to prevent the correction grid from absorbing the light due to the resolution problem and adding non-PSF features into the PSF, we mask them out. When the correction grid is enlarged and covers other AGN centers, we need to mask out both regions (AGN centers and lensed compact blob).

APPENDIX B: T[n,m] MATRIX

Since \( \delta w_{[n,m]} \) has different length in each iteration of inner/outer loop \([n,m]\), we use a matrix \( T_{[n,m]} \) to make \( \delta w_{[n,m]} \) the same length as \( w_{[n,m]} \) by padding the two-dimensional boundaries of the PSF correction grid with zeros, as illustrated in Figure A2.

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Figure A1. The three different mask regions which are circled in red, and the white arrows indicate the special area which need to be masked out (that is, we boost the uncertainty in that region) while we extract the PSF correction. The left panel shows the maskArcAGN region for fitting the lens light, and the middle panel shows the maskAGN region for fitting the arc light. When obtaining the PSF corrections, the white circles in the right panel need to be masked out when the PSF grid is small. As we increase the PSF grid around each AGN image such that the grid contains other AGN images (shown in the right panel of Figure 5), we mask out the red circles associated with these other AGN images and also the white circles.

Figure A2. The matrix $T_{[n,m]}$ for making $\delta w_{[n,m]}$ the same length as $w_{[n,m]}$. The indices of $\delta w_i$ and $0_k$ are for the pixels (rather than the PSF correction iterations), $T_{[n,m]}$ is a matrix at the $n^{th}$ outer loop and the $m^{th}$ inner loop.

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