Equivalent quantum equations with effective gravity in a system inspired by bouncing droplets experiments

Christian Borghesi *
Außerfern, Tirol, Austria

Abstract

It has been found in a classical system, mostly inspired by bouncing droplets experiments, that it is possible to write: an equivalent free Schrödinger equation; a covariant guidance formula (which leads to the equivalent de Broglie-Bohm guidance formula in the low-velocity approximation); the energy and momentum of the ‘particle’, which have a direct counterpart as well in special relativity (only valid when the ‘particle’ is in its reference state) as for the ones attributed to the whole system in quantum mechanics. These results were found in a previous paper without external potential. In the current paper we show that these results hold in presence of an external potential. With regard to the considered system, a general covariant formulation is used and a natural interpretation between the proper time and the ‘world time’ (as in general relativity) is suggested. Hence, the system behaves as if an effective gravitational field (generated by an experimenter) exists. This natural interpretation and the effective gravitational field only involve pulsations.

Introduction

Bouncing droplets experiments, in which droplets bounce and ‘walk’ on a vibrating liquid substrate (initiated in [1] and see [2] for a review) have shown for the first time that classical and macroscopic systems exhibit quantum-like phenomena. The fact that droplets are guided by the wave that they have generated is reminiscent of the pilot wave suggested by de Broglie [3] (see [4, 2, 5] for a discussion in this context) and more generally of the double solution program [6, 7] (see [8, 9] for current and well-thought-over views).

Nevertheless, it seems tricky to mathematically formalise bouncing droplet problems in order to obtain equations close to the ones in corresponding quantum systems. To deal with more convenient systems, from the mathematical point of view, we have recently suggested a classical and macroscopic system [10] - also inspired by a sliding bead on a vibrating string experiment [11] and, with more success, [12].

The latter toy system consists of (i) an elastic medium, which carries transverse waves governed by a Klein-Gordon-like equation, and (ii) one high elastic medium density, considered as a point of mass \( m_0 \) and called concretion. This system is wave monistic and, also, invariant by the Lorentz-Poincaré transformation specific to the elastic medium (in particular to the propagation speed of waves). This approach was very encouraging. For instance it has been found: a strictly equivalent free Schrödinger equation; a covariant guidance formula (which leads to the equivalent de Broglie-Bohm guidance formula in the low-velocity approximation); the energy and momentum of the ‘particle’ concretion have a direct counterpart in special relativity (only valid when the concretion is in its reference state) as well as in quantum mechanics with the ones commonly attributed to the so-called quantum system. However these results only concern the special case without external potential. It is then natural to wonder if these results hold in presence of an external potential. The aim of this study is to answer this question.

Since this article is a direct continuation of the previous one [12], we do not again explain and discuss the toy system, neither the manner it is inspired by bouncing droplets experiments, etc. Nevertheless, for the sake of clarity, let us very briefly recall the notation and the Lagrangian density used in [12]. This reads

\[
\mathcal{L} = \frac{1}{2} \mathcal{T} \left( 1 + \frac{\rho_0(\vec{r}, t)}{\mu_0} \right) \left[ \partial_\mu \varphi \partial^\mu \varphi - \frac{\Omega^2}{c_m^2} \varphi^2 \right],
\]

where \( \varphi \) denotes a (real-valued) transverse wave (as the interface height of the liquid bath in bouncing droplets experiments), \( \mathcal{T} \) a ‘tension’ of the elastic medium, \( \mu_0 \) its mass per element of volume, \( c_m \), the propagation speed of the wave (such that \( \mathcal{T} = \mu_0 c_m^2 \)), \( \Omega_m \) the reference transverse pulsation of the elastic medium (i.e. the elastic medium tends to support transverse standing vibrations at this pulsation, as at the Faraday pulsation in bouncing droplets experiments) and \( \rho_0 \) the very high elastic medium density corresponding to the concretion of mass \( m_0 \) \( (\rho_0(\vec{r}_0, t_0) = m_0 \delta(\vec{r}_0 - \vec{\xi}_0) \) in its proper reference frame, in which \( \delta \) denotes the Dirac delta function and \( \vec{\xi} \) the location of the concretion). Here, \( \partial_\mu \varphi \partial^\mu \varphi \) means \( (\nabla \varphi)^2 - (\nabla \varphi)^2 \).

*christian.borghesi@protonmail.com
The ‘particle’ concretion, which is just a point-like high density elastic medium, has the same properties (per unit mass) as the homogeneous elastic medium itself – as shown in the above Lagrangian density. We note that this classical system is wave monistic and, in addition, Lorentz-Poincaré covariant with respect to the elastic medium (in particular to \(c_m\)). Finally, this elastic medium shares two properties of bouncing droplets systems: (i) to carry propagative waves (here à la d’Alembert) and (ii) the tendency of the elastic medium to support standing waves (here at pulsation \(\Omega_m\)).

Let us now very briefly comment this toy system. The concretion is assumed to be a stable particle (as for walkers’ dynamics in bouncing droplet experiments) and can be seen as a simplification of more complicated phenomena (for instance as peaked solitons developed in very interesting studies \([13, 14]\) also inspired by bouncing droplets experiments and de Broglie’s double solution program, due to a non-linear self-focusing potential of gravitational nature). In contrast to de Broglie’s double solution program (in which particles consist of a peaked concentration of energy), the ‘particle’ concretion is depicted by a very high elastic medium density rather than a very high amplitude of the wave. Moreover, as seen in \([12]\) the transverse wave \(\varphi\) – continuous at the location of the concretion – also plays the role of a “pilot wave”. Finally, this system can be interpreted as a simplification (with the modification of the particle’s description) of de Broglie’s double solution program in a classical system.

According to the nature of the concretion (i.e. a point-like high density elastic medium), we seek in the following an external potential which acts in the same manner on the homogeneous elastic medium as on the concretion (but naturally \(\rho_0/\mu_0\) times more). This facilitates to retrieve a potential-like energy able to act on the concretion from a wave equation valid at any point – the latter being physically related to the mass per element of volume \(\mu_0\).

1 Modifying natural transverse pulsations

By using a cell with different fluid depths in bouncing droplets experiments, authors have for instance studied tunnel-like effects \([13]\) (see \([16]\) for a theoretical model) and non-specular reflection of walking droplets \([17]\). A barrier with a different thickness in the cell bath modifies the Faraday pulsation (or threshold) at the location of the barrier. This allows them to generate a kind of potential barrier acting both on walkers and on surface waves. We are inspired by these phenomena in this section.

1.1 Framework

For the concretion without external potential \([12]\), the elastic medium has the tendency to support transverse standing wave at pulsation \(\Omega_m\). (Recall that his property was inspired by bouncing droplets experiments, in which some quantum-like phenomena occur when each transverse perturbation at the surface of the bath tends to generate a harmonic oscillation at the Faraday pulsation at the location of the perturbation. This property is mimicked in our toy system by a quadratic/harmonic potential associated with every element of the elastic medium, which tends to support a transverse oscillation at the pulsation \(\Omega_m\).) We call in the following the natural transverse pulsation at a point \(\vec{r}\) of the elastic medium, the pulsation of the transverse standing wave that the elastic medium tends to support at this point. Up to now, the natural transverse pulsation of the elastic medium, \(\Omega_m\), was uniform and constant in time. We assume in this section that the natural transverse pulsation is no longer uniform and can also change in time. Let \(\omega_h(\vec{r}, t)\) an additional natural transverse pulsation to the reference one, \(\Omega_m\). (We believe that \(\omega_h(\vec{r}, t)\) could easily be tunable by an experimenter – as soon as this toy system would be experimentally performed.) The natural transverse pulsation of the medium, at position \(\vec{r}\) and time \(t\), then reads:

\[
\Omega_m(\vec{r}, t) = \Omega_m + \omega_h(\vec{r}, t).
\]  

This means that the elastic medium at position \(\vec{r}\) and time \(t\) tends to support a transverse standing oscillation at pulsation \(\Omega_m(\vec{r}, t)\).

We assume through this paper that \(\omega_h(\vec{r}, t)\) (i) can be positive or negative, (ii) is in magnitude much lower than \(\Omega_m\) (i.e. \(|\omega_h| \ll \Omega_m\)) and (iii) is very smooth comparatively to characteristics related to \(\Omega_m\) (i.e. \(\omega_h(\vec{r}, t)\) has a characteristic time evolution much longer than \(1/\Omega_m\) and has a characteristic length evolution much greater than \(c_m/\Omega_m\)).

The inhomogeneous natural transverse pulsation of the elastic medium, \(\Omega_m(\vec{r}, t)\) instead of \(\Omega_m\), constitutes the only difference from the toy system suggested in \([12]\). The Lagrangian density of the system then becomes

\[
\mathcal{L} = \frac{1}{2} \mathcal{T} \left(1 + \frac{\rho_0(\vec{r}, t)}{\mu_0} \right) \left[ \partial_{\mu} \varphi \partial^{\mu} \varphi - \frac{\Omega_m^2(\vec{r}, t)}{c_m^2} \varphi^2 \right].
\]  

Note that this Lagrangian density is Lorentz-Poincaré invariant with respect to the considered elastic medium. It is furthermore important to note that the additional pulsation, \(\omega_h(\vec{r}, t)\), acts on the homogeneous elastic medium as well as on the concretion (but \(\rho_0/\mu_0\) times more). This is due to the fact that the concretion has the same properties as the homogeneous elastic medium per unit mass.
1.2 From the wave equation to the equivalent Schrödinger equation

The wave equation stems from the principle of least action, when the wave field is subjected to a small change while the 4-position of the concretion is fixed. Calculations, which are very similar to the ones detailed in Appendix A2, lead to:

\[ \Box m \psi + \frac{\Omega_m^2}{c_m^2}(\vec{r},t) \psi = -\frac{\rho_0}{\mu_0} \left( \Box m \psi + \frac{\Omega_m^2}{c_m^2}(\vec{r},t) \right) \psi - \frac{1}{\mu_0} [\partial_\rho (\rho_0) \partial^\mu \psi] , \]

where \( \Box_m \) denotes the d’Alembert operator specific to the elastic medium (i.e. taking into account the propagation speed of the wave).

The concretion is not the source of the wave any longer (a state related to an intimate harmony between the wave and the concretion and previously called symbiosis): (i) when the wave \( \psi \) obeys a Klein-Gordon-like equation with a heterogeneous natural transverse pulsation \( \Omega_m(\vec{r},t) \) and (ii) when the following equation

\[ \partial_\rho (\rho_0) \partial^\mu \psi = 0 , \]

called the symbiosis equation, is satisfied. We will discuss this equation below.

It was very convenient to introduce the modulating wave, \( \psi \), which modulates the ‘natural’ wave of the medium without any ‘particle’. As described in Appendix A4.1, \( \psi \) is specifically associated to the presence of a particle in the elastic medium. Using the complex notation and the reference pulsation of the elastic medium, \( \Omega_m \), the (complex-valued) modulating wave, \( \psi \), is defined from the (real-valued) transverse wave, \( \varphi \), as

\[ \varphi(\vec{r},t) = \text{Re} \left[ \psi(\vec{r},t) e^{-i\Omega_m t} \right] , \]

where \( \text{Re}[\cdots] \) denotes the real part. When the wave is in symbiosis with the concretion and in the low-velocity approximation (more precisely at first-order approximation with respect to \( \nu^2/c_m^2, \omega/\Omega_m \) and \( \omega_h/\Omega_m \)), the wave equation (1) leads to

\[ \frac{i}{\hbar} \frac{\partial \psi}{\partial t} = -\frac{c_m^2}{2 \Omega_m} \Delta \psi + \omega_h \psi , \]

where \( \Delta \) denotes the Laplace operator. (Calculations are very similar to the ones detailed in Appendix A4.)

It is now very tempting to define a potential-like energy able to act on the concretion, \( V_h(\vec{r},t) \), due to the additional natural transverse pulsation at position \( \vec{r} \) and time \( t \) and to the mass \( m_0 \) of the concretion, such that

\[ V_h(\vec{r},t) = \hbar_{\text{exp}} \omega_h(\vec{r},t) ; \]

in which we have used the coefficient

\[ \hbar_{\text{exp}} = \frac{m_0 c_m^2}{\Omega_m} , \]

which is specific to the studied system. (\( \hbar_{\text{exp}} \) has been introduced in Appendix A4.1 as a proportionality coefficient between wave characteristics and particle characteristics, but naturally appears in the energy of the concretion.) Now, by using the coefficient \( \hbar_{\text{exp}} \) and the potential-like energy \( V_h(\vec{r},t) \), Eq. (7) becomes:

\[ i \frac{\hbar_{\text{exp}}}{\hbar} \frac{\partial \psi}{\partial t} = -\frac{\hbar_{\text{exp}}^2}{2 m_0} \Delta \psi + V_h(\vec{r},t) . \]

This equation has the same form as the Schrödinger equation with an external potential \( V_h \).

1.3 From the symbiosis equation to a contradiction of the system

The symbiosis equation (3) is the same as in Appendix A4.1. Moreover the mass continuity equation (related to the concretion) is also unchanged by adding \( \omega_h(\vec{r},t) \) to \( \Omega_m \). Thus, the symbiosis equation associated with the conservation of the mass of the concretion lead to the same results as in Appendix A4.1 (cf. Eq. (6) therein), namely: (i) a covariant guidance formula (which provides an equivalent de Broglie-Bohm guidance formula in the low-velocity limit) and (ii) the velocity of the concretion remains constant in time. The latter constitutes an incoherence. Indeed, the velocity of the concretion under the action of a potential \( V_h \) should normally be able to vary.

The toy system suggested in this section then exhibits an inconsistency. Something else in this system or formalisation should be missing or wrong. But before overcoming this contradiction, let us study another way to generate external potentials.

2 Accelerating frame

One more time, bouncing droplets experiments will inspire us. By means of a rotating liquid bath (with respect to the laboratory reference frame) authors have observed quantisation of classical orbits of walkers (see [15] for more technical details and information). As evoked in Introduction, by using the rotating frame they have been able to generate external potentials acting on both the liquid bath and walkers.

1Note this definition is qualitatively in agreement with bouncing droplets experiments. For instance Fig. 1 in Appendix A2, shows that the barrier in the cell bath increases the acceleration Faraday threshold. This means that the Faraday pulsation increases at the location of the barrier, because the forcing vibration amplitude is uniform over the cell bath. This corresponds in our system to an increasing potential-like energy at the location of the cell barrier.

2In this vein, we are not inspired by a potential which only acts on walkers, as for example this resulting from a magnetised droplet in a magnetic field [22].
2.1 Framework

Let $\mathcal{R}_X$ a reference frame where the (Minkowski) Klein-Gordon-like equation without source (Eq. (4) in [12]) governs the transverse wave, $\varphi$, in symbiosis with the concretion. For instance in bouncing droplets experiments in a rotating frame with respect to the laboratory reference frame, $\mathcal{R}_X$ is the rotating frame itself, in which the liquid bath seems at rest but with a parabolic shape, and $\mathcal{R}$ the laboratory reference frame. (See Appendix A1 for an illustration and calculations corresponding to this paragraph.) In the rotating frame, Faraday waves are indeed the same as in the absence of any rotation ([20] §5.2.3). In parallel it could be very interesting to notice that a stationary bouncing droplet remains stationary in the rotating frame ([20] §5.2.3 and [21] §4.1.1) – as in an inertial frame. (Note the difference with the usual representation: in classical mechanics the laboratory reference frame is commonly considered as a Galilean reference frame while the rotating frame as a non-Galilean one.) Anyway, we are now seeking an expression of the wave equation in the reference frame $\mathcal{R}$. This can be carried out by means of the coordinate transformation, from the spatio-temporal coordinates of $\mathcal{R}_X$ (written as $(\vec{R},T)$ or as the 4-position $X^\mu$) to the ones of $\mathcal{R}$ (written as $(\vec{r},t)$ or $x^\mu$). It is then particularly convenient to use the metric $g_{\mu\nu}$ related to the elastic medium in the reference frame $\mathcal{R}$. We must keep in mind that this metric is specific to the elastic medium, and, more precisely, to the wave equation of $\varphi$. In $\mathcal{R}_X$ the elastic-medium metric is the Minkowski one – because $\varphi$ obeys the (Minkowski) Klein-Gordon-like equation (without source). Thus, $g_{\mu\nu}$ of $\mathcal{R}$ is easily derived from the equivalent Minkowski metric associated with the elastic medium in $\mathcal{R}_X$.

In [12] the study was Lorentz-Poincaré invariant with respect to the considered elastic medium (i.e. the propagation speed of the wave, $c_m$) because a (Minkowski) Klein-Gordon-like equation governed the wave dynamics in the elastic medium. Now, even more generally, we seek to study our monistic system in any reference frame $\mathcal{R}$. It is then particularly convenient to directly use a general covariant formulation valid in any reference frame. As in relativity, this general covariant formulation uses the metric associated with $\mathcal{R}$. (But recall that, here, as mentioned above, the metric is associated with the elastic medium itself.) In this way, the Lagrangian density (1) of the toy system is written in any reference frame $\mathcal{R}$ as $\sqrt{-g_m} \mathcal{L}$, where

$$\mathcal{L} = \frac{1}{2} T \left( 1 + \frac{\rho_0(\vec{r},t)}{\mu_0} \right) \left[ g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - \frac{\Omega^2}{c_m^2} \right]$$

and $\sqrt{-g_m}$ denotes the square root of the negative of the determinant of the elastic-medium metric tensor $g_{\mu\nu}$. (The action reads $S = \int \mathcal{L} \sqrt{-g_m} dt dx^1 dx^2 dx^3.)$

2.2 From the wave equation to the equivalent Schrödinger equation

The wave equation stems from the principle of least action (see Appendix A2 for more details) and is written as

$$\Box_m \varphi + \frac{\Omega^2}{c_m^2} \varphi = \frac{\rho_0}{\mu_0} \left( \Box_m \varphi + \frac{\Omega^2}{c_m^2} \varphi \right) - \frac{1}{\mu_0} g^{\mu\nu}_m \partial_\mu (\rho_0) \partial_\nu \varphi.$$  

(12)

where $\Box_m$ explicitly denotes the d’Alembert operator (specific to the elastic medium) in curvilinear coordinates, i.e. $\Box_m \varphi = \frac{1}{c_m} \partial_\mu (\sqrt{-g_m} g^{\mu\nu}_m \partial_\nu \varphi)$. (In this paper we are not interested in the equation of motion for the concretion because, as seen in [12] and in the low-velocity approximation, this equation could very weakly perturb the velocity and trajectory of the concretion given by the guidance formula, i.e. by the wave equation.)

The concretion does not generate waves any longer: (i) when the wave is governed by the Klein-Gordon-like equation in curvilinear coordinates ($\Box_m \varphi + \frac{\Omega^2}{c_m^2} \varphi = 0$) and, in addition, (ii) when the following symbiosis equation is satisfied:

$$g^{\mu\nu}_m \partial_\mu (\rho_0) \partial_\nu \varphi = 0.$$  

(13)

As expected, the corresponding results seen in [12] are generalised.

From now on we only study the case in which the elastic-medium metric, $g_{\mu\nu}$, is a Newtonian one, constant in time. The simplest case corresponds to a uniformly accelerated frame, $\mathcal{R}$, with respect to $\mathcal{R}_X$, which is also called the Rindler metric (see Appendix A3 for an illustration). The elastic-medium metric is then written as

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

in which $g_{00}$ is constant in time and conveniently written as

$$\sqrt{g_{00}} = 1 + \epsilon(\vec{r}).$$

(15)

In general relativity, $\epsilon(\vec{r})$ of a Newtonian metric constant in time is equal to the gravitational potential energy per unit mass and divided by $c^2$ (cf. e.g. [23] [87]), i.e. $m_0 c^2 \epsilon(\vec{r})$ is equal to the gravitational potential energy per unit mass and divided by $c^2$ (cf. e.g. [23] [87]), i.e. $m_0 c^2 \epsilon(\vec{r})$ is equal to the gravitational

As in [12], throughout this paper $x^0 = c_m t$ and $x^i$ ($i = 1, 2, 3$) are the spatial location, $\vec{r}$, of a point in the elastic medium at rest, i.e. without considering its transverse displacement $\varphi(\vec{r},t)$. Moreover we use the metric signature $(+,−,−,−).$  

In Cartesian coordinates and in a 3D elastic medium the (Minkowski) Klein-Gordon-like equation without source reads:

$$\frac{1}{c_m^2} \partial^2_{TT} \varphi - \partial^2_{XX} \varphi - \partial^2_{YY} \varphi - \partial^2_{ZZ} \varphi + \frac{\Omega^2}{c_m^2} \varphi = 0.$$
potential acting on a point mass \( m_0 \) at position \( \vec{r} \). Here, for example when \( \mathcal{R} \) is uniformly accelerated with respect to \( \mathcal{R}_X \) along the \((OX)\) axis with the uniform and constant acceleration \( a \) (which corresponds to the Rindler metric), we get \( \epsilon(\vec{r}) = \frac{2a}{c_m} \). Then \( m_0 c_m^2 \epsilon(\vec{r}) = m_0 a x \), which is equal in classical mechanics to the potential energy associated with the fictitious force due to the acceleration. More generally we define the (fictitious) potential energy acting on the concretion (if it was located at point \( \vec{r} \)) as

\[
V(\vec{r}) = m_0 c_m^2 \epsilon(\vec{r}),
\]

(16)

In this Section we do not necessarily consider that \( |\epsilon| \ll 1 \) – and write this case only when it occurs. Finally \( \rho_0 \), the mass density of the concretion expressed in its proper reference frame, is written (cf. e.g. \( 23 \) §90) in \( \mathcal{R} \) as

\[
\rho_0 = m_0 \sqrt{1 - V^2/c_m^2} \delta(\vec{r} - \vec{\xi}(t))
\]

(17)

where \( V \) denotes the velocity of the concretion measured in terms of the proper time, that is, by an observer located at the given point (cf. e.g. \( 23 \) §88) – which implies that \( \dot{V} = \frac{1}{\sqrt{\rho_0 m_0}} \ddot{\vec{r}} \), where the velocity of the concretion measured in \( \mathcal{R} \) is \( \ddot{\vec{r}} = \frac{\ddot{\vec{r}}}{\dot{\rho}_0} \).

As seen in \( 10 \) \( 12 \), in the low-velocity approximation, the (Minkowski) Klein-Gordon-like equation without source (applying to the transverse wave \( \varphi \)) leads to an equivalent free Schrödinger equation (applying to the modulating wave \( \psi \)). Here, in the Newtonian elastic-medium metric in the small-acceleration approximation (\( |\epsilon| \ll 1 \)) and in the low-velocity approximation, the wave equation \( 12 \) without source leads to 

\[
-21 \frac{\partial^2}{\partial_\xi^2} \frac{\partial \varphi}{\partial t} - \Delta \varphi - \ddot{\vec{v}} \cdot \nabla \varphi + 2 \epsilon \frac{\partial \varphi}{\partial t} = 0.
\]

(The calculation is very similar to the one detailed in \( 12 \) Appendix A4, in which the definition of the modulating wave \( \psi \)) is used and the term \( \frac{\partial^2 \varphi}{\partial_\xi^2} \) is neglected in the low-velocity approximation as well as a term with \( \epsilon \frac{\partial \varphi}{\partial t} \).) When the characteristic length evolution of \( \psi \) (say the wavelength of \( \psi \)) is much longer than the one of \( \epsilon(\vec{r}) \) (equal to \( c_m^2/a \) in Rindler coordinates), in other words when the variation of \( \epsilon(\vec{r}) \) is very small over a wavelength of \( \psi \), the term \( \ddot{\vec{v}} \cdot \nabla \psi \) is negligible. This leads to

\[
\frac{i}{\hbar} \frac{\partial \psi}{\partial t} = - \frac{\hbar^2}{2m_0} \Delta \psi + \epsilon \Omega_m \psi.
\]

By using \( \hbar_{\text{exp}} \), defined in \( 9 \), and the potential energy \( V \), defined in \( 16 \), related to the concretion in \( \mathcal{R} \), we get

\[
\frac{i}{\hbar_{\text{exp}}} \frac{\partial \psi}{\partial t} = - \frac{\hbar_{\text{exp}}^2}{2m_0} \Delta \psi + V \psi.
\]

This equation has the exact form as the Schrödinger equation with the potential \( V \).

As an illustration, let us consider a concretion in symbiosis studied in a reference frame \( \mathcal{R} \) uniformly accelerated with respect to \( \mathcal{R}_X \), as illustrated in Appendix A3. For the sake of simplicity we consider a concretion at rest (i.e. only with transverse oscillations) in \( \mathcal{R}_X \). The corresponding transverse wave is then \( \varphi(\vec{R}, T) = A \cos(\Omega_m T) \), which satisfies the (Minkowski) Klein-Gordon-like equation without source. By using the coordinate transformation \( \vec{\xi} \), a Taylor polynomial expansion and the modulating wave \( \psi \), we get

\[
\varphi(\vec{r}, t) = A \cos \left( \frac{\Omega_m c_m a}{a} \left( 1 + \frac{a x}{c_m} \right) \sinh \left( \frac{a t}{c_m} \right) \right) \quad \text{and} \quad \psi(\vec{r}, t) = A e^{-i \Omega_m \left( a x + \frac{a^2}{c_m^2} \right) \frac{t^2}{2}}.
\]

(20)

It is easy to check that \( \psi \) satisfies the equivalent Schrödinger equation \( 18 \) or \( 19 \), in which \( \epsilon(\vec{r}) = \frac{2a}{c_m} \) or \( V = m_0 a x \).

### 2.3 From the symbiosis equation to the concretion guidance formula

As seen in \( 12 \), the symbiosis equation in addition to the conservation of the mass \( m_0 \) of the concretion lead to a covariant guidance formula, which implies the equivalent de Broglie-Bohm guidance formula in the low-velocity approximation. Here, by using Eq. \( 16 \) and the mass continuity equation (see Appendix A4 for more details) we get

\[
\frac{1}{c_m^2 g_{m0}} \sqrt{1 - V^2/c_m^2} \frac{\partial \varphi}{\partial t} \left[ \frac{d}{dt} \sqrt{g_{m0}} \frac{\partial \varphi}{\partial t} \right] \rho_0 + \left[ \ddot{\vec{v}} \cdot \nabla \varphi + \nabla \ddot{\vec{v}} \varphi \right] \cdot \nabla \rho_0 = 0,
\]

(21)

(This relation only concerns a concretion studied in \( \mathcal{R} \) with a Newtonian elastic-medium metric \( 14 \) constant in time.) This implies that both terms are zero. This leads to the two following equations:

\[
\frac{d}{dt} \sqrt{g_{m0}} \frac{\partial \varphi}{\partial t} = 0,
\]

(22)

\[
\frac{\ddot{\vec{v}}}{c_m^2 g_{m0}} \frac{\partial \varphi}{\partial (\vec{\xi}, t)} + \nabla \varphi(\vec{\xi}, t) = 0.
\]

(23)
Eq. (22) has a direct counterpart in general relativity: the energy conservation of a point mass $m_0$ in a static gravitational field (cf. e.g. [23] §88), i.e. in a Newtonian metric constant in time. On the other side, Eq. (23) yields (see below), in the low-velocity approximation, an equivalent de Broglie-Bohm guidance formula in quantum mechanics.

Eq. (23) (called the $\varphi$-guidance formula) generalises in any reference frame (with a Newtonian elastic-medium metric constant in time) the corresponding one in [12], which meant that the concretion in symbiosis is located in its proper reference frame at a local extremum of the transverse displacement wave $\varphi$. Furthermore, it here appears more clearly that the symbiosis equation also leads to an energy conservation (cf. below Eq. (29)). It is very interesting to note that, on the contrary of the previous section of the paper and by using a general covariant formulation, the problem concerning the guidance formula and the velocity of the concretion under the influence of a potential (cf. Section 1.3) is, at this step, solved.

For the concretion in a uniformly accelerated frame, in which $\varphi$ is given by Eq. (20), the $\varphi$-guidance formula yields $\vec{v} \psi(\xi,t) = \frac{\hat{c}^2}{\Omega_m} \grad \psi(\xi,t)$. It is convenient to write $\psi$ as

$$\psi = F e^{i\Phi},$$

where the magnitude $F$ and the phase $\Phi$ are two real functions and, in addition

$$\omega(\vec{r},t) = -\frac{\partial \Phi(\vec{r},t)}{\partial t}, \quad \vec{k}(\vec{r},t) = \grad \Phi(\vec{r},t).$$

Thus, the velocity of the concretion reads

$$\vec{v} = \frac{\hat{c}^2}{\Omega_m} \grad \Phi(\xi,t).$$

This equation has a direct counterpart in quantum mechanics: the de Broglie-Bohm guidance formula [3, 24]. We note that the corresponding results seen in [12] are recovered in the same form. Moreover, similarly as in [12], the amplitude of the transverse wave at the location of the concretion is such that: $\grad F(\xi,t) = 0$ and $\grad \Phi(\xi,t) = 0$. The amplitude of transverse oscillation of the concretion remains constant in time and the concretion is located at a local extremum of the vibration amplitude field, $F(\vec{r},t)$.

The equivalent guidance formula (20) in addition to the equivalent Schrödinger equation (11) (both based on the symbiosis between the transverse wave $\varphi$ and the concretion) yield

$$m_0 \frac{d\vec{v}}{dt} = -\grad \left[ Q(\xi,t) + V(\xi,t) \right], \quad \text{where} \quad Q = -\frac{\hat{c}^2}{2m_0} \frac{\Delta F}{F}.$$

(The calculation is very similar to the one in [12] Appendix A3.) $Q$ is the same wave potential as in [12] – an equivalent of the de Broglie-Bohm quantum potential.

For the concretion in a uniformly accelerated frame, where $\psi$ is given by Eq. (20), the equivalent guidance formula (23) directly leads to $\vec{v} = -a \tilde{t} \hat{e}_x$, as seen above for the low-velocity and small-acceleration limit. The wave potential potential, $Q (27)$, is here zero. The concretion behaves then as a point mass in classical mechanics moving in a potential $V$.

### 2.4 Energetic considerations

The energy, $W_{\text{conc}}$, and momentum, $\vec{p}_{\text{conc}}$, of the concretion are naturally defined from the time-averaged value over one transverse oscillation of the energy and momentum densities in the oscillating elastic medium. These densities are associated with the stress-energy tensor of the system, which results from the Lagrangian density (cf. Eq. (11)). We again consider the case in which the wave and the concretion are in symbiosis – the velocity of the concretion is then given by the $\varphi$-guidance formula (23).

We first consider that the concretion is in its reference state. This state has been defined in [12] when the concretion is in symbiosis with the wave and has a transverse oscillation in its proper reference frame at the pulsation $\Omega_m$. Note that this reference state makes zero the time-averaged value over one transverse period of the part of the Lagrangian density accounting for the concretion. Eq. (11) in [12] related to this reference state is then generalised here as

$$\left\langle \left( g^{\mu \nu} \frac{\partial \rho \phi}{\partial x^\mu} \frac{\partial \rho \phi}{\partial x^\nu} - \frac{\Omega_m^2}{c_m^2} x^2 \right) \xi(t) \right\rangle = 0,$$

where $\xi(t)$ is the position of the concretion.
where \((\cdots)_{\xi(t)}\) denotes the time-averaged value over one transverse period at the location of the concretion. For instance, a concretion at rest in \(\mathcal{R}\) oscillates at the pulsation \(\sqrt{2m_0c^2_\text{m}}\) with its reference state – which is indeed in agreement with an oscillation at pulsation \(\Omega_m\) measured in its proper time.

When the concretion is in its reference state (see Appendix A5 for more details) the energy and momentum of the concretion are written as

\[
W_{\text{conc}} = m_0 \frac{\sqrt{2m_0c^2_\text{m}}}{{\sqrt{1 - \frac{\Omega^2}{c^2_\text{m}}}}} \langle \varphi^2(\vec{\xi}, t) \rangle
\]

\[
\vec{p}_{\text{conc}} = m_0 \frac{1}{\sqrt{2m_0c^2_\text{m}}} \frac{\Omega^2_\text{m}}{c^2_\text{m}} \langle \varphi^2(\vec{\xi}, t) \rangle \vec{v}.
\]  

(29)

Here again, it is very tempted to assume that the experimenter can perform the condition related to the transverse oscillation amplitude of the concretion such that

\[
\Omega^2_\text{m} \langle \varphi^2(\vec{\xi}, t) \rangle = c^2_\text{m}.
\]  

(30)

(Recall that this conditions means in the proper reference frame of the concretion that the average quadratic transverse oscillation velocity of the concretion is equal to \(c_\text{m}\) and/or the energy density in the very close neighbourhood to the concretion is equal to the ‘tension’ \(T\) proper to the elastic medium.) Provided that the condition (30) is fulfilled, the energy and momentum of the concretions are written in the same form as in general relativity for a point mass \(m_0\) in a static gravitational field (cf. e.g. [23] §88 or [25] §112), i.e. in a Newtonian metric constant in time. In addition, it is interesting to notice that Eq. (22) (deduced from the symposium equation) implies that the energy of the concretion, \(W_{\text{conc}}\), is constant in time.

Now, we consider the general case for which the concretion is not necessarily in its reference state. In the low-velocity and small-acceleration approximation (see Appendix A5 for more details), by using the pulsation \(\Omega\), defined in Eq. (26), at the location of the concretion and provided that condition (30) is fulfilled, the energy and momentum of the concretion are written as

\[
W_{\text{conc}} = m_0 c^2_\text{m} + \hbar \exp \omega(\vec{\xi}, t)
\]

\[
\vec{p}_{\text{conc}} = m_0 \vec{v}.
\]  

(31)

(32)

Here again, the total energy of the concretion is equal to the equivalent rest mass energy of the concretion \((m_0 c^2_\text{m})\), similarly to relativity plus an additional energy, \(E_{\text{conc}} = \hbar \exp \omega(\vec{\xi}, t)\) (similarly to quantum mechanics). (It is worth noting that both the equivalent rest mass energy \((m_0 c^2_\text{m})\) and the coefficient \(\hbar \exp\) come naturally from the same condition (30).)

To conclude, the corresponding results for \(W_{\text{conc}}\) and \(\vec{p}_{\text{conc}}\) seen in [12] are here generalised, as expected. We note nevertheless that expressions in the low-velocity and small-acceleration approximation have here the same form as in the absence of external potential. Let us now briefly write again expression of \(W_{\text{conc}}\) and \(\vec{p}_{\text{conc}}\) in the light of quantum mechanics.

**Other expressions in the low-velocity and small-acceleration approximation**

In the same manner as in [12], in the low-velocity and small-acceleration approximation the additional energy of the concretion reads

\[
E_{\text{conc}} = -\hbar \exp \frac{\partial \Phi(\vec{\xi}, t)}{\partial t} + i \hbar \exp \frac{\partial \psi(\vec{\xi}, t)}{\partial t} = E_{\text{conc}} \psi(\vec{\xi}, t)
\]  

(33)

and the linear momentum of the concretion

\[
\vec{p}_{\text{conc}} = \hbar \exp \vec{\nabla} \Phi(\vec{\xi}, t)
\]

\[
\frac{\hbar \exp}{2} \vec{\nabla} \psi(\vec{\xi}, t) = \vec{p}_{\text{conc}} \psi(\vec{\xi}, t).
\]  

(34)

As in the absence of external potential, the additional energy and the momentum of the concretion are then in exact agreement with the energy and momentum of an equivalent quantum system – apart from they specifically concern \(\psi\) at the location of the concretion. As has been suggested by Louis de Broglie in quantum mechanics (see e.g. [26], §11), in our toy system the ‘particle’ concretion accounts for quantities (here energy and momentum) commonly attributed to the wave-like nature of the system.

Finally, similarly to quantum mechanics with a particle point of view (see e.g. [26]), the energy of the concretion is also written as

\[
E_{\text{conc}} = \frac{1}{2} m_0 v^2 + Q(\vec{\xi}, t) + V(\vec{\xi}, t).
\]  

(35)

Moreover, let us mention that the quantum potential is responsible for the deviations of the Bohmian trajectories from the classical behaviour in classical mechanics (see e.g. [27]). In our toy system, it is interesting to notice that the wave potential, \(Q\), also indicates a non-reference state degree of the concretion.
Indeed, in the low-velocity and small-acceleration approximation, Eqs. (6), (33), (34), (35) and condition (30) lead to

$$\frac{1}{2} \left( \left( g^{\mu \nu}_{\text{conc}} \partial_\mu \varphi \partial_\nu \varphi - \frac{\Omega_m^2}{c_m^2} \nu^2 \right) \xi(t) \right) = \frac{Q(\xi(t))}{m_0 c_m^2}. \tag{36}$$

When $Q(\xi(t)) = 0$, the concretion is in its reference state.

Let us now illustrate these results for the concretion in a uniformly accelerated frame, where $\psi$ is given by Eq. (29), i.e. the phase of $\psi$ is $\Phi(\vec{r}, t) = -\Omega_m (a x t/c_m^2 + a^2 t^2/c_m^2)$. The wave vector $\vec{k}_{\text{conc}}$ at the location of the concretion is $\vec{k}(\xi, t) = -\frac{\Omega_m}{c_m} a t \vec{e}_z$. This yields $\vec{F}_{\text{conc}} = -m_0 a t \vec{e}_z$ – which is in agreement with the second equation of (34). The pulsation $\omega(\vec{r}, t) = -\frac{\partial \Phi(\vec{r}, t)}{\partial t}$ at the location of the concretion is $\omega(\xi(t)) = \frac{\Omega_m}{c_m} (a \xi + \frac{1}{2} a^2 t^2)$. Since $\xi(t) = -\frac{1}{2} a t^2$ (as seen above), $\omega(\xi(t)) = 0$. The additive energy of the concretion, $E_{\text{conc}}$, is then zero and remains constant in time.

Here, the wave potential, $Q$, is zero. We then recover $E_{\text{conc}} = \frac{1}{2} m_0 t^2 + V(\xi, t) = 0$. Furthermore $Q(\xi, t) = 0$ implies that the concretion is in its reference state, as expected. (Indeed, for the sake of clarity the concretion has been considered at rest with a transverse oscillation at pulsation $\Omega_m$ in the reference frame $R_X$ shown in Appendix (A3).

3 General formulation

In the same manner as in general relativity – in which a gravitational field is locally equivalent to an accelerated frame (with respect to an inertial frame) and vice versa – we would like to associate an equivalent potential with an accelerated frame. By comparison with equivalent Schrödinger equations (Eqs. (7) and (13)) and/or with potential energies (Eqs. (8) and (16)) in Sections 1 and 2, it is very tempting to assume that $\omega_1$ in Section 1 is related to $\epsilon$ in the Newtonian elastic-medium metric in Section 2. More precisely we assume

$$\frac{\omega_1}{\Omega_m} \equiv \epsilon, \tag{37}$$

which implies that

$$\sqrt{g_{00}} = \frac{\Omega_m}{\Omega_m} = 1 + \frac{\omega_1(\vec{r})}{\Omega_m}. \tag{38}$$

(From now on, we only consider the case in which $\omega_1$ is constant in time, as $\epsilon$ in Section 2.) Recall in general relativity that $\epsilon$ (from $\sqrt{g_{00}} = 1 + \epsilon$) of a Newtonian metric is associated with the gravitational potential energy per unit mass divided by $c^2$. Here, $\epsilon$ (from $\sqrt{g_{00}(\vec{r})}$) now accounts for the acceleration of the reference frame $R$ with respect to $R_X$ in Section 2 is now associated with the additional natural transverse pulsation $\omega_1$ at point $\vec{r}$. For instance, the system in which $R$ is uniformly accelerated with respect to $R_X$ along the $(OX)$ axis with the uniform and constant acceleration $a$ (i.e. $\epsilon(\vec{r}) = \frac{a}{c_m^2}$) now becomes equivalent to a system studied in $R$ at rest with respect to $R_X$, but with $\omega_1(\vec{r}) = \frac{a}{c_m^2}$. (Note that the potential-like energy (5) able to act on the concretion, $V_0(\vec{r}) = \frac{a}{c_m^2}$, becomes in this case $V_0(\vec{r}) = m_0 a x$, as in classical mechanics for a point mass.) It is nevertheless important to keep in mind that the Newtonian elastic-medium metric only concerns in general relativity the low-velocity and small-potential/acceleration limit. By analogy with this, from now on we only discuss and consider our toy system under this approximation.

All results seen in Section 2 can be applied to a system with an heterogeneous natural transverse pulsation, $\Omega_m(\vec{r}) = \Omega_m + \omega_1(\vec{r})$, as soon as the equivalence between $\epsilon$ and $\omega_1$ in Eq. (37) (i.e. the Newtonian elastic-medium metric (13) is written with $g_{00}$ given in (25)) is assumed. Hence, the limitation of the formalisation seen in Section 1 is now overcome.

This raises then the following question: what is wrong with the formalisation suggested in Section 1? To answer this question, let us come back to the relation between the proper time $\tau$ and the ‘world time’ $t$ in $R$ (cf. e.g. 23, 88). At point $\vec{r}$, $d\tau = \sqrt{g_{00}} c_m^2 dt$. By assuming Eq. (38) (i.e. by assuming the equivalence between $\epsilon$ and $\omega_1$ (37)) the relation between the proper time $\tau$ and the ‘world time’ $t$ at point $\vec{r}$ becomes

$$d\tau = \frac{\Omega_m}{\Omega_m} d\tau. \tag{39}$$

This allows us to answer the previous question: in Section 1 the formalisation has not fully taken into account different proper times due to the inhomogeneous natural transverse pulsation. In particular the Lagrangian density in Section 1 has not taken into account the factor $\sqrt{-g_{00}}$ due to this effect. This missing led to a wrong symbiosis equation and also a wrong energy of the concretion. The wrong symbiosis equation yields a velocity of the concretion constant in time in any potential-like energy. The wrong energy of the concretion (that it would have without the modification of proper times) is no longer in agreement with the equivalent energy of the corresponding quantum system, neither with general relativity for a concretion in its reference state. [1]

[1] The study of systems with harmonic potentials, Coulomb-like potentials, etc. is very similar – conceptually speaking – to examples analysed in [12]. Indeed, in the low-velocity and small-acceleration approximation the guidance formula, expressions of $E_{\text{conc}}$ and $\vec{F}_{\text{conc}}$ with respect to $\psi$ are the same as without external potential.

[2] Following calculations very similar to the ones in [12] and according to condition (30) the energy of the concretion at rest in $R$ and in its reference state (i.e. oscillating at the pulsation $\Omega_m(\vec{r})$) would have been $W_{\text{conc}} = m_0 c_m^2 + 2V_0(\vec{r})$ and not $m_0 c_m^2 + V_0(\vec{r})$ as expected. For the general case but in the low-velocity and small-potential approximation, $W_{\text{conc}}$ would have been $m_0 c_m^2 + h_{\text{exp}} - V_0(\vec{r})$. 

8
In general relativity the relation between the proper time $\tau$ and the ‘world time’ $t$ depends on the gravitational field (cf. e.g. [23] §88). However relation (39) seems particularly easy and natural to interpret. The proper time is indeed related to the natural transverse pulsation, $\Omega_m'$, with regard to the reference one, $\Omega_m$. For a concretion in its reference state and at rest at point $\vec{r}$, the period of a transverse oscillation expressed with the ‘world time’ is equal to the reference period of a transverse oscillation expressed with the proper time. (Recall that at rest and in its reference state, the concretion oscillates in $\sqrt{g_{m00}} \Omega_m$.) More generally, the elastic medium at point $\vec{r}'$ tends to support standing transverse oscillations at the pulsation $\Omega_m'(\vec{r}')$. In short, the natural transverse period in ‘world time’ is equal to the reference one in proper time. (This also explains why the (Minkowski) Klein-Gordon-like equation, valid in $\mathcal{R}_X$ with the reference pulsation $\Omega_m$, played a crucial role in Section 2.)

To conclude, an experimenter could then generate any external potential by modifying natural transverse pulsations. Furthermore, modifying natural transverse pulsations implies that proper time elapses differently at different points of space in $\mathcal{R}$. More precisely, this means that external potentials generated by modifying natural transverse pulsations are gravitational-like fields.

4 Conclusion

We have sought in this paper two different manner to generate a potential-like energy acting both on the homogeneous elastic medium and on the concretion (but, for the latter, $\rho_0/\mu_0$ times more). The first case consists in modifying the natural transverse pulsation, i.e. the pulsation of transverse standing oscillations that a given point of the elastic medium tends to support; as for instance, in bouncing droplets experiments, by using a cell with different fluid depths (which locally modifies the Faraday pulsation/threshold) [15, 17]. The second case consists in studying the system in an accelerated frame with respect to one in which the transverse wave obeys a (Minkowski) Klein-Gordon-like equation; as for instance, in bouncing droplets experiments, by means of a rotating liquid bath [18, 19]. A general covariant formulation in a Newtonian elastic-medium metric constant in time (as the one in general relativity to deal with a point mass in a static gravitational field) is used. This covariant formalisation includes the two different previous manners to generate external potentials (considered as constant in time in this paper), but is only valid under the low-velocity and small-potential approximation. By using this general covariant formulation, all results which are valid in a system without external potential [12] are recovered and/or generalised.

We nevertheless note that the symbiosis equation (35) plays a role more important than in [12]. Indeed, the symbiosis equation combined with the mass conservation of the concretion not only lead to the general covariant guidance formula (23) but also to Eq. (22), which implies the conservation of the energy of the concretion in its reference state. It is also interesting to note that, here again, the energy and momentum of the concretion have the same equivalent form: (i) as in general relativity (in a static gravitational field or in a Newtonian metric constant in time) for a concretion in its reference state (ii) as well as the ones attributed to the so-called quantum system (in the low-velocity and small-potential approximation). Here again, the condition concerning the oscillation amplitude of the concretion (39) allows us to write both the equivalent rest mass energy, $m_0 \, c^2$, and the coefficient $h_{\exp}$ in the additional energy – equal to $h_{\exp} \, \omega$. We also note that expressions in the low-velocity and small-potential approximation for the guidance formula, $E_{\text{conc}}, \, \bar{E}_{\text{conc}}$, and the wave potential $Q$ with respect to the modulating wave, $\psi$, are the same as in [12], without external potential. (Another interpretation of $Q$, the equivalent de Broglie-Bohm quantum potential, is given: $Q$ also indicates a non-reference state degree of the concretion.) Finally, $\psi$ is governed by a strictly equivalent Schrödinger equation – of course with an external potential.

However the more interesting seems to concern the meaning of the external potential. An experimenter could generate any external potential by modifying natural transverse pulsations: $\Omega_m \to \Omega_m'(\vec{r}) = \Omega_m + \omega_h(\vec{r})$. This external potential is written in the equivalent Schrödinger equation. Furthermore, it is important to stress that modifying natural transverse pulsations not only generates an external potential but also changes proper time at different points of the elastic medium – as a static gravitational field in a Newtonian metric in general relativity. Without taking account the modification of proper times, the system is inconsistent. Indeed the velocity of the concretion in symbiosis would have been constant in time, whatever the external potential; and, in addition, the energy of the concretion would have been different to the equivalent energy of the corresponding quantum system as well as in general relativity for a concretion in its reference state. (For the sake of non contradiction, the toy system behaves then as if an equivalent static gravitational field exists. Hence, $g_{m00}$ of the Newtonian elastic-medium is such that $\sqrt{g_{m00}} = 1 + \omega_h(\vec{r})$. This corresponds to a potential acting on the concretion (and written in the equivalent Schrödinger equation) equal to

$$V(\vec{r}) = h_{\exp} \, \omega_h(\vec{r}) . \tag{40}$$

Hence, the effective gravitational acceleration, $\vec{g}$, generated by the modification of natural transverse pulsations of the elastic medium, reads

$$\vec{g}(\vec{r}) = -\sqrt{c_m^2} \, \omega_h(\vec{r}) \Omega_m . \tag{41}$$

For instance, let a (flat) 2D elastic medium, $(Oxy)$. (Transverses oscillations are directed along the vertical axis.) If $\omega_h(x, y) = \frac{\alpha \, c_m}{\epsilon_5}$ (as $\epsilon(\vec{r}) = \epsilon_0^2$ in Section 2), where $\alpha$ is constant and has the dimension of acceleration, the effective gravitational acceleration, associated with this modification of natural transverse pulsations, is uniform and reads $\vec{g} = -a \, \vec{e}_z$. In short and more generally, in our toy system an effective gravitational field results from a natural transverse pulsations ‘field’.
In this paper, the elastic medium and the concretion only evolve under the influence of an effective gravity. A possible kind of effective gravity field generated by the concretion has not been investigated in this article. (It could also be interesting in forthcoming studies to be inspired by another approach with quantum solitons \[13, 14\], also inspired by bouncing droplets experiments and de Broglie’s double solution program, in which the appearance of an effective gravitation is predicted.)

Finally, an accurate formalisation of a system with inhomogeneous natural transverse pulsations necessitates to take into account that proper time elapses differently at different points of space. In general relativity and in a Newtonian metric constant in time, the gravitational field implies a relation between the proper time and the ‘world time’. Here, by using the reference natural transverse pulsation, $\Omega_m$ (the one considered without external potential), this relation is very easy to interpret: the natural transverse period of the elastic medium expressed in ‘world time’ is equal to the reference one expressed in proper time.

Appendix

A1 The wave equation with a rotating elastic medium

Let $\mathcal{R}_X$ a reference frame which has a uniform rotation (with the angular velocity of rotation, $\omega$) with respect to the laboratory reference frame, $\mathcal{R}$. Using the spherical coordinates, the (Minkowski) Klein-Gordon-like equation without source is written in $\mathcal{R}_X$ as

$$
\frac{1}{c^2_m} \frac{\partial^2 \varphi}{\partial t^2} - \frac{1}{R^2} \frac{\partial}{\partial R} \left( \rho^2 \frac{\partial \varphi}{\partial \rho} \right) - \frac{1}{R^2 \sin \Theta} \frac{\partial}{\partial \Theta} \left( \sin \Theta \frac{\partial \varphi}{\partial \Theta} \right) - \frac{1}{R^2 \sin^2 \Theta} \frac{\partial^2 \varphi}{\partial \varphi^2} = 0
$$

(A1)

The coordinate transformation from $\mathcal{R}_X$ to $\mathcal{R}$, and using spherical coordinates, is written as

$$
\begin{align*}
& t = T \\
& r = R \\
& \theta = \Theta \\
& \phi = \Phi + \omega T
\end{align*}
$$

(A2)

This leads to $\frac{\partial}{\partial T} = \frac{\partial}{\partial t}$, $\frac{\partial}{\partial \rho} = \frac{\partial}{\partial r} + \frac{\omega}{c_m} \frac{\partial}{\partial \varphi}$, $\frac{\partial}{\partial \rho} = \frac{\partial}{\partial r}$, $\frac{\partial}{\partial \Theta} = \frac{\partial}{\partial \theta}$ and $\frac{\partial}{\partial \varphi} = \frac{\partial}{\partial \varphi}$. Putting into Eq. (A1), the wave equation becomes in the laboratory reference frame $\mathcal{R}$:

$$
\frac{1}{c^2_m} \frac{\partial^2 \varphi}{\partial t^2} - \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \varphi}{\partial r} \right) - \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \varphi}{\partial \theta} \right) - \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \varphi}{\partial \varphi^2} + \frac{\omega^2}{c^2_m} \varphi = \frac{\omega}{c_m} \frac{\partial^2 \varphi}{\partial t \partial \varphi} + \frac{\omega^2}{c^2_m} \varphi^2 = 0
$$

(A3)

Let us now again writing the previous wave equation by means of the metric $g_{m\mu}$ associated with the elastic medium in the laboratory reference frame $\mathcal{R}$. According to the fact that the metric related to the elastic medium in $\mathcal{R}_X$ is Minkowski and by using the coordinate transformation (A2), the space-time interval related to the elastic medium reads

$$
ds^2_m = c^2_m \, dt^2 - r^2 \, dr^2 - r^2 \, d\theta^2 - r^2 \sin^2 \theta \left( d\phi - \frac{\omega}{c_m} \, dt \right)^2.
$$

(A4)

This leads to the elastic-medium metric:

$$
g_{m\mu} = \begin{pmatrix}
1 - r^2 \omega^2 c^{-2} \sin^2 \theta & 0 & 0 & r^2 \omega^2 c^{-2} \sin^2 \theta \\
0 & -1 & 0 & 0 \\
0 & 0 & -r^2 & 0 \\
r^2 \omega^2 c^{-2} \sin^2 \theta & 0 & 0 & -r^2 \sin^2 \theta
\end{pmatrix},
$$

(A5)

since $ds^2_m = g_{m\mu} dx^\mu dx^\nu$. The generalised Klein-Gordon-like equation written in the reference frame $\mathcal{R}$ is written as $\square_m (\sqrt{-g_m} g_{m\mu} \partial_\mu \varphi) + \frac{r^2}{c^2_m} \partial_\nu \varphi = 0$ (cf. Eq. (12)), where $g_m$ denotes the determinant of the elastic-medium metric tensor $g_{m\mu}$. This leads again to Eq. (A3).

We take advantage of the wave equation (A3) to give a solution of $\varphi$ in $\mathcal{R}$ by comparison with the one in $\mathcal{R}_X$. Let the transverse wave expressed in $\mathcal{R}_X$ as $\varphi = f(r, \theta) \cos(\Omega t - m \phi)$. Since $\partial_\rho^2 \varphi + 2 \omega \partial_\varphi \partial_\phi + \omega^2 \partial_\varphi^2 = -(\Omega - m \omega) \varphi$, the solution of Eq. (A3) is the same as for differential equation (A1) but with $\varphi = f(r, \theta) \cos((\Omega - m \omega) t - m \phi)$. Thus, the solution in $\mathcal{R}_X$ such that $\varphi = f(R, \Theta) \cos(\Omega t - m \phi)$ leads to the following solution in $\mathcal{R}$: $\varphi = f(r, \theta) \cos(\Omega t + m \omega \phi)$. The latter expression is also directly deduced from $\varphi = f(R, \Theta) \cos(\Omega t - m \phi)$ established in $\mathcal{R}_X$ and by using the coordinate transformation (A2), as performed by de Broglie when he deals with the Zeeman effect (cf. [28] §I.1). Note that a rotational splitting is also observed in bouncing droplets experiments [29], as well as, for instance, in asteroseismology with oscillations in rotating stars [30].
A2 Calculation of the wave equation

The wave equation comes from a principle of least action, when the wave field is subjected to a small change, \( \varphi \rightarrow \varphi + \delta \varphi \), while the 4-position of the concretion is fixed.

\( \varphi \rightarrow \varphi + \delta \varphi \) leads to the wave equation (12). (Note that the generalised Euler-Lagrange equation in curvilinear coordinates, \( \partial \phi \) values are fixed, are written as

\[
\frac{\partial}{\partial t} \left( 1 + \frac{\rho_0(T,t)}{\rho_0} \right) \left[ g^\mu_\nu \partial_\nu \partial_\mu (\delta \varphi) - \frac{\delta \varphi}{c_m^2} \varphi \delta \varphi \right] \sqrt{-g_m} \, dt \, d^3r = 0.
\]

(A6)

Integrating by parts the term with \( \partial_\mu (\delta \varphi) \), with fixed end points, and since the small change \( \delta \varphi \) is arbitrary, lead to the wave equation (13). (Note that the generalised Euler-Lagrange equation in curvilinear coordinates, \( \frac{\delta \mathcal{L}}{\delta \partial \phi} = \frac{1}{\sqrt{-g_m}} \partial_\mu \left( g^\mu_\nu \partial_\nu \phi \right) \), leads to the same result, as expected.)

A3 Illustration of Rindler coordinates

Let a reference frame, \( \mathcal{R} \), uniformly accelerated along the \((OX)\) axis of a reference frame \( \mathcal{R}_X \), with the acceleration \( a \). In the case of Rindler coordinates, the coordinate transformation is written as

\[
\begin{align*}
    c_m T &= \left( x + \frac{c^2}{a} \right) \sinh \left( \frac{a}{c_m} \right) \\
    X + \frac{c^2}{a} &= \left( x + \frac{c^2}{a} \right) \cosh \left( \frac{a}{c_m} \right) \\
    Y &= y \\
    Z &= z
\end{align*}
\]

(A7)

In the main text the concretion is assumed to be at rest in \( \mathcal{R}_X \), for instance in \( O_X \).

A4 Combining the symbiosis equation with the mass continuity equation

The mass continuity equation for the concretion is written in the considered elastic medium as

\[
\sqrt{-g_m} \frac{\partial}{\partial t} \left( \rho_0 U^\mu \right) = 0,
\]

where \( U^\mu = c_m \frac{d\vec{r}}{dt} \) is the 4-velocity of the concretion. In the Newtonian elastic-medium metric, \( -g_m = g_{m00} \) and \( ds_m = \sqrt{g_{m00}} \sqrt{1 - \frac{c^2}{c_m^2}} \, dt \). This leads to

\[
\frac{1}{\sqrt{1 - \frac{c^2}{c_m^2}}} \left( \frac{\partial}{\partial t} \rho_0 + v^i \partial_i \rho_0 \right) + \rho_0 \left( \frac{\partial}{\partial t} \frac{1}{\sqrt{1 - \frac{c^2}{c_m^2}}} + v^i \partial_i \frac{1}{\sqrt{1 - \frac{c^2}{c_m^2}}} + \frac{1}{\sqrt{1 - \frac{c^2}{c_m^2}}} \partial_1 v^1 \right) = 0,
\]

(A8)

in which \( v^i \) is the \( i \)-th component of the vector velocity of the concretion, \( \vec{v} = \frac{dt}{d\tau} \), expressed in the laboratory reference frame \( \mathcal{R} \). It is convenient to use the vector gradient, \( \nabla \), whose \( i \)-th component is \( \partial_i \), and the particle derivative \( \frac{d}{dt} = \frac{d}{d\tau} + v^i \partial_i \). Moreover, in the Newtonian elastic-medium metric: \( \partial_i v^i = \frac{d}{d\tau} \sqrt{g_{m00}} \cdot \sqrt{g_{m00}} \). This result is for instance obtained by considering a Galilean elastic-medium reference frame \( \mathcal{R}_X \) which coincides with \( \mathcal{R} \) at a given time. (Indeed in \( \mathcal{R}_X \): \( dT = \sqrt{g_{m00}} \, dt \), \( dx^X = dx^i \), \( V^i = \frac{dx^i}{\sqrt{g_{m00}}} \) and \( \frac{\partial V^i}{\partial x^i} = 0 \). Hence, the continuity equation becomes

\[
\frac{\partial}{\partial t} \rho_0 + \vec{v} \cdot \nabla \rho_0 + \rho_0 \left( \sqrt{1 - \frac{c^2}{c_m^2}} \frac{d}{dt} \frac{1}{\sqrt{1 - \frac{c^2}{c_m^2}}} + \vec{v} \cdot \nabla \sqrt{g_{m00}} \right) = 0
\]

(A9)

The symbiosis equation (13) becomes in the Newtonian elastic-medium metric:

\[
\frac{1}{c_m^2 g_{m00}} \frac{\partial}{\partial t} \frac{\partial \rho_0}{\partial \varphi} = - \vec{v} \cdot \nabla \rho_0 = 0.
\]

(A10)

By substituting the term \( \frac{\partial m}{\partial t} \) in Eq. (A10), we get

\[
\frac{1}{c_m^2 g_{m00}} \frac{\partial \varphi}{\partial t} \left[ \sqrt{1 - \frac{c^2}{c_m^2}} \frac{d}{dt} \frac{1}{\sqrt{1 - \frac{c^2}{c_m^2}}} + \vec{v} \cdot \nabla \sqrt{g_{m00}} \right] \rho_0 + \left[ \frac{1}{c_m^2 g_{m00}} \frac{\partial \varphi}{\partial t} \vec{v} + \nabla \varphi \right] \cdot \nabla \rho_0 = 0.
\]

(A11)

Since the acceleration is constant in time, i.e. \( \frac{\partial m}{\partial t} g_{m00} = 0 \), \( \vec{v} \cdot \nabla \sqrt{g_{m00}} = \frac{d}{dt} \sqrt{g_{m00}} \). Then, Eq. (A11) leads to Eq. (21) in the main text.
A5 Energy and linear momentum of the concretion

As usual, the energy density and the momentum density are evaluated from the stress-energy tensor density, \( \sqrt{-g} T \) of the system. \( T_{\alpha\beta} = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\alpha} \left( \sqrt{-g} T^{\mu\nu} \frac{\partial}{\partial x^\mu} \right) \) (cf. e.g. \( [23] \) §94). According to Eq. (11) and in the Newtonian elastic-medium metric (and using \( \frac{\partial}{\partial x^\alpha} \sqrt{-g} = -\frac{1}{2} \sqrt{-g} g_{\alpha\beta} \), cf. e.g. \( [23] \) §86) we get

\[
T_{00} = T \left( 1 + \frac{\rho_0}{\mu_0} \right) \left( \frac{\partial \varphi}{\partial \alpha} \frac{\partial \varphi}{\partial \alpha} - \frac{1}{2} g_{00} \left( \frac{\partial \varphi}{\partial \varphi} g^{\mu\nu} - \frac{\Omega^2_m}{c_m^2} \varphi^2 \right) \right),
\]

(A12)

where \( \rho_0 \) is given by Eq. (17). Without \( \rho_0 \), i.e. without the concretion, this results is in agreement with the stress-energy tensor of the Klein-Gordon field in a metric \( g_{\mu\nu} \) (cf. e.g. \( [31] \) §6.7), as expected.

The energy and the momentum of a particle moving in a Newtonian metric constant in time is given

\[
W = \frac{1}{2} m_0 \sqrt{1 - \frac{v^2}{c_m^2}} \left( \frac{\partial \varphi(\vec{\xi},t)}{\partial t} \right)^2 + g_{00} \Omega^2_m \left( \varphi^2(\vec{\xi},t) \right),
\]

(A13)

in which \( \vec{V} = \frac{1}{\sqrt{g_{00}}} \frac{\partial \varphi}{\partial \varphi} \frac{\partial \varphi}{\partial \varphi} \) (Recall that \( \vec{V} \) denotes the velocity of the concretion measured in terms of the proper time.)

We first evaluate the case for which the concretion is in its reference state, i.e. condition \( [28] \) is satisfied (in addition to the symbiosis between the wave and the concretion). This condition and the \( \varphi \)-guidance formula \( [23] \) lead to

\[
\left( \frac{\partial \varphi(\vec{\xi},t)}{\partial t} \right)^2 = \left( \frac{g_{00} \Omega^2_m}{1 - v^2/c_m^2} \right) \left( \varphi^2(\vec{\xi},t) \right).
\]

(It is interesting to notice that \( \frac{g_{00} \Omega^2_m}{1 - v^2/c_m^2} \) remains constant in time, because Eq. \( [22] \) is satisfied for the concretion in symbiosis with the wave.) Hence Eqs. (A13) yield Eqs. (29).

We now consider the general case in which the concretion is not necessarily in its reference state (but it is in symbiosis with the wave). However the calculation is carried out under the low-velocity and small-acceleration approximation. By using Eqs. (9), (24) and (25), we get

\[
\left( \frac{\partial \varphi(\vec{\xi},t)}{\partial t} \right)^2 = \left( \Omega_m + \omega(\vec{\xi},t) \right)^2 \varphi^2(\vec{\xi},t)
\]

– because the magnitude \( F \) at the location of the concretion is constant in time. In first-order approximation in \( \frac{v^2}{c_m^2} \), \( \frac{\partial \varphi}{\partial \varphi} \) (cf. Appendix A6 in \( [12] \)) and \( \epsilon \), Eqs. (A14) become

\[
W_{\text{conc}} = m_0 \Omega^2_m \varphi^2(\vec{\xi},t) \left( 1 + \frac{\omega(\vec{\xi},t)}{\Omega_m} \right)
\]

\[
\vec{p}_{\text{conc}} = m_0 \Omega^2_m \varphi^2(\vec{\xi},t) \frac{\vec{V}}{c_m},
\]

(A14)

Taking into account condition \( [30] \) and \( h_{\text{exp}} = \frac{m_0 c^2_m}{\Omega^2_m} \), these equations become Eqs. (31) and (32).

References

[1] Y. Couder, S. Protière, E. Fort, A. Boudaoud, Walking and orbiting droplets. Nature 437, 208 (2005).
[2] J. W. M. Bush, Pilot-Wave Hydrodynamics. Annu. Rev. Fluid Mech. 47, 269-292 (2015).
[3] L. de Broglie, La mécanique ondulatoire et la structure atomique de la matière et du rayonnement. J. de Phys. Radium, série VI, t. VIII, n°5 (1927).
[4] Y. Couder and E. Fort, Probabilities and trajectories in a classical wave-particle duality probabilities and trajectories in a classical wave-particle duality. J. of Phys., Conf. Series 361, 012001, (2012).
[5] J. W. M. Bush, The new wave of pilot-wave theory. Physics Today 68(8), 47 (2015).
[6] L. de Broglie, Une tentative d’interprétation causale et non linéaire de la mécanique ondulatoire. Gauthier-Villars Ed., Paris, 1956. English translation: Non-linear Wave mechanics – A causal interpretation. Elsevier Ed., Amsterdam, 1960.
[7] L. de Broglie, *Interpretation of quantum mechanics by the double solution theory.* Ann. fond. de Broglie **12**, n°4 (1987). English translation from a paper originally published in the book *Foundations of Quantum Mechanics – Rendiconti della Scuola Internazionale di Fisica “Enrico Fermi”*; Course 49 (1970) ed. by B. d’Espagnat, Academic Press N.Y. 1972.

[8] D. Fargue, *Louis de Broglie’s “double solution” a promising but unfinished theory.* Ann. Fond. de Broglie **42**(1), 9 (2017).

[9] S. Colin, T. Durt and R. Willox, *L. de Broglie’s double solution program: 90 years later.* Ann. Fond. de Broglie **42**(1), 19 (2017).

[10] C. Borghesi, *Dualité onde-corpuscule formée par une masselotte oscillante dans un milieux élastique.* Ann. Fond. de Broglie **42**(1), 9 (2017). English translation: *Wave-particle duality coming from a bead oscillator in an elastic medium*; [arXiv:1609.09260v3 [physics.class-ph]]

[11] A. Boudaoud, Y. Couder, and M. Ben Amar, *A self-adapative oscillator.* Eur. Phys. J. B **9**, 159-165 (1999).

[12] C. Borghesi, *Equivalent quantum equations with an external potential in a system inspired by bouncing droplets experiments.* Found. (Phys. 2017).

[13] T. Durt, *Generalized guidance equation for peaked quantum solitons and effective gravity.* Europhys. Lett. **114**, No. 1 (2016).

[14] T. Durt, *L. de Broglie’s double solution and self-gravitation.* Ann. Fond. de Broglie **42**(1), 73 (2017).

[15] A. Eddi, E. Fort, F. Moisy and Y. Couder, *Unpredictable Tunneling of a Classical Wave-Particle Association.* Phys. Rev. Lett. **102**, 240401, (2009).

[16] A. Nachbin, P. A. Milewski and J. W. M. Bush, *Tunneling with a Hydrodynamic Pilot-Wave Model.* Phys. Rev. Fluids **2**, 034801 (2017).

[17] G. Pucci, P. J. Sáenz, L. M. Faria and J. W. M. Bush, *Non-specular reflection of walking droplets.* J. Fluid Mech. **804**, R3 (2016).

[18] E. Fort, A. Eddi, A. Boudaoud , J. Moukhtar and Yves Couder, *Path-memory induced quantization of classical orbits,* Proc. Natl. Acad. Sci. USA **107**, vol. 41, 17515-17520 (2010).

[19] D. M. Harris and J. W. M. Bush, *Droplets walking in a rotating frame: from quantized orbits to multimodal statistics.* J. Fluid Mech. **739**, 444–464 (2014).

[20] A. Eddi, *Marcheurs, Dualité onde-particle et mémoire de chemin.* PhD thesis, Université Paris VII, 2011. [https://tel.archives-ouvertes.fr/tel-00575626/](https://tel.archives-ouvertes.fr/tel-00575626/)

[21] D. Harris, *The pilot-wave dynamics of walking droplets in confinement.* PhD thesis, Massachusetts Institute of Technology, 2015. [https://dspace.mit.edu/handle/1721.1/99068](https://dspace.mit.edu/handle/1721.1/99068)

[22] S. Perrard, M. Labousse, M. Miskin, E. Fort and Y. Couder, *Self-organization into quantized eigenstates of a classical wave-driven particle.* Nature Comm. **5**, 3219 (2014).

[23] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields.* Pergamon Press, 1971.

[24] D. Bohm, *A Suggested Interpretation of the Quantum Theory in Terms of “Hidden” Variables.* I. Phys. Rev. **85**, 166-179 (1952).

[25] L. de Broglie, *Ondes et mouvements.* Ed. J. Gabay, Paris, 1988.

[26] P. R. Holland, *The quantum theory of motion.* Cambridge U. Press, 1995.

[27] X. Oriols, J. Mompart, *Overview of Bohmian Mechanics.* In X. Oriols, J. Mompart, *Applied Bohmian Mechanics: From Nanoscale Systems to Cosmology*; Pan Stanford Ed., 2012.

[28] L. de Broglie, *Eléments de théorie des quanta et de mécanique ondulatoire.* Ed. J. Gabay, Paris, 2012.

[29] A. Eddi, J. Moukhtar, S. Perrard, E. Fort and Y. Couder, *Level Splitting at Macrosopic Scale.* Phys. Rev. Lett. **108**, 264503 (2012).

[30] R. G. Deupree and W. Beslin, *Rotational splitting modes of pulsation modes.* The Astrophys. J. **721**, 1900-1907 (2010).

[31] Matthias Blau *Lecture Notes on General Relativity.* [http://www.blau.itp.unibe.ch/Lecturenotes.html](http://www.blau.itp.unibe.ch/Lecturenotes.html)

[32] Joel Franklin, *Lectures Notes; Stress Tensors, Particles and Fields.* [http://www.reed.edu/physics/courses/Physics411/html/page2/files/Lecture.19.pdf](http://www.reed.edu/physics/courses/Physics411/html/page2/files/Lecture.19.pdf)