Generators and defining relations for ring of differential operators on smooth affine algebraic variety in prime characteristic

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Abstract

For the ring of differential operators $\mathcal{D}(\mathcal{O}(X))$ on a smooth affine algebraic variety $X$ over a perfect field of characteristic $p > 0$, a set of algebra generators and a set of defining relations are found explicitly. A finite set of generators and a finite set of defining relations are given explicitly for the module $\text{Der}_K(\mathcal{O}(X))$ of derivations on the algebra $\mathcal{O}(X)$ of regular functions on the variety $X$. For an arbitrary irreducible affine algebraic variety $X$, it is proved that each term $\mathcal{D}(\mathcal{O}(X))_i$ of the order filtration $\mathcal{D}(\mathcal{O}(X)) = \bigcup_{i \geq 0} \mathcal{D}(\mathcal{O}(X))_i$ is a finitely generated left $\mathcal{O}(X)$-module. The same results are true for ring of differential operators on regular algebra of essentially finite type.

Mathematics subject classification 2000: 13N10, 16S32, 13N15, 14J17.

1 Introduction

In prime characteristic, differential operators and their modules are a more difficult and less developed area of Mathematics than in characteristic zero. The main difficulty is that in prime characteristic algebras of differential operators are not finitely generated, not left or right Noetherian, and contain a lot of nilpotent elements. As a result methods of affine Algebraic Geometry are not applicable (at least in the way they are in characteristic zero), and this is a principal unavoidable problem.

Key ingredients of the theory of (algebraic) $\mathcal{D}$-modules in characteristic zero are the Gelfand-Kirillov dimension, multiplicity, Hilbert polynomial, the inequality of Bernstein, and holonomic modules. In prime characteristic, straightforward generalizations of these either do not exist or give ‘wrong’ answers (as in the case of the Gelfand-Kirillov dimension: $\text{GK}(\mathcal{D}(P_n)) = n$ in prime characteristic rather than $2n$ as it ‘should’ be and it is in characteristic zero where $P_n$ is a polynomial algebra in $n$ variables).

In 70’s and 80’s, for rings of differential operators in prime characteristic natural questions were posed (see, for example, questions 1-4 in [24]) [some of them are still open] that can be summarized as to find generalizations of the mentioned concepts and results (that results in ‘good theory’ expectation of which was/is high, see, the remark of Björk in [24]). One of the questions in the paper of P. Smith [24] is to give a definition of holonomic module in prime characteristic. In characteristic zero, holonomic modules have remarkable homological properties based on which Mebkhout and Narvaez-Macarro [19] gave a definition of holonomic module. Another approach (based on the Cartier Lemma) was taken by Bogvad [9] who defined, so-called, filtration holonomic modules. This one is more close to the original idea of holonomicity in characteristic zero. Note that the two mentioned concepts of holonomicity
in prime characteristic appeared before analogues of the Gelfand-Kirillov dimension and the inequality of Bernstein have been recently found, \[6\]. In \[6\], analogues of the Gelfand-Kirillov dimension, multiplicity, the inequality of Bernstein, and holonomic modules are found in prime characteristic (using very different methods and ideas from characteristic zero) and classical properties of holonomic modules were proved. In prime characteristic, \(D\)-modules were studied by Haastert \[13\], Alvarez-Montaner, Blickle and Lyubeznik \[1\], Bezrukavnikov, Mirkovic, and Rumynin \[8\]; and they were used in study of (local) cohomology by Huneke and Sharp \[14\], Kashiwara and Lauritzen \[15\], Andersen and Kanea \[2\], and Lyubeznik \[16\]. \(D\)-modules were applied to the theory of tight closure by K. Smith \[20\] and to the ring of invariants by K. Smith and van den Bergh \[21\].

In this paper, module means a left module, \(N := \{0, 1, 2, \ldots \}\) is the set of natural numbers. The following notation will remain fixed throughout the paper (if it is not stated otherwise):

- \(K\) is a perfect field of characteristic \(p > 0\) (not necessarily algebraically closed);
- \(P_n := K[x_1, \ldots, x_n]\) is a polynomial algebra over \(K\);
- \(\partial_1 := \frac{\partial}{\partial x_1}, \ldots, \partial_n := \frac{\partial}{\partial x_n} \in \text{Der}_K(P_n)\);
- \(I := \sum_{i=1}^n P_n f_i\) is a prime but not a maximal ideal of the polynomial algebra \(P_n\) with a set of generators \(f_1, \ldots, f_m\);
- \(A := S^{-1}(P_n/I)\) (the localization of the algebra \(P_n/I\) at a multiplicatively closed subset \(S\) of \(P_n/I\)) which is a domain with the field of fractions \(Q := \text{Frac}(A)\), i.e. \(A\) is an arbitrary algebra of essentially finite type which is a domain; 
- \(\text{Der}_K(A)\) is the left \(A\)-module of \(K\)-derivations of the algebra \(A\);
- \(\text{HS}_K(A)\) is the set of higher derivations (Hasse-Schmidt derivations) of the algebra \(A\);
- \(\mathcal{D}(A)\) is the ring of \(K\)-linear differential operators on the algebra \(A\). The action of a differential operator \(\delta \in \mathcal{D}(A)\) on an element \(a \in A\) is denoted either by \(\delta(a)\) or \(\delta \cdot a\);
- the homomorphism \(\pi : P_n \rightarrow A, p \mapsto \overline{p}\), to make notation simpler we sometime write \(x_i\) for \(\overline{x_i}\) (if it does not lead to confusion);
- the Jacobi \(m \times n\) matrices \(J = (\frac{\partial f_i}{\partial x_j}) \in M_{m,n}(P_n)\) and \(\overline{J} = (\frac{\partial f_i}{\partial x_j}) \in M_{m,n}(A) \subseteq M_{m,n}(Q)\); \(r := \text{rk}_Q(J)\) is the rank of the Jacobi matrix \(J\) over the field \(Q\);
- \(a_r\) is the Jacobian ideal of the algebra \(A\) which is (by definition) generated by all the \(r \times r\) minors of the Jacobi matrix \(J\) (\(A\) is regular iff \(a_r = A\), it is the Jacobian criterion of regularity, \[11\], 16.20);

For \(i = (i_1, \ldots, i_r)\) such that \(1 \leq i_1 < \cdots < i_r \leq m\) and \(j = (j_1, \ldots, j_r)\) such that \(1 \leq j_1 < \cdots < j_r \leq n\), \(\Delta(i,j)\) denotes the corresponding minor of the Jacobi matrix \(J = (J_{ij})\), that is \(\det(J_{i',j'})\), \(i', j' \in \{1, \ldots, r\}\); and the element \(i\) (resp. \(j\)) is called non-singular if \(\Delta(i,j') \neq 0\) (resp. \(\Delta(i',j) \neq 0\)) for some \(j'\) (resp. \(i'\)). We denote by \(I_r\) (resp. \(J_r\)) the set of all the non-singular \(r\)-tuples \(i\) (resp. \(j\)).

Since \(r\) is the rank of the Jacobi matrix \(J\), it is easy to show that \(\Delta(i,j) \neq 0\) iff \(i \in I_r\) and \(j \in J_r\) (Lemma \[21\]). Denote by \(J_{r+1}\) the set of all \((r+1)\)-tuples \(j = (j_1, \ldots, j_{r+1})\) such
that $1 \leq j_1 < \cdots < j_{r+1} \leq n$ and when deleting some element, say $j_\nu$, we have a non-singular $r$-tuple $(j_1, \ldots, \hat{j_\nu}, \ldots, j_{r+1}) \in J_r$ where hat over a symbol here and everywhere means that the symbol is omitted from the list. The set $J_{r+1}$ is called the critical set and any element of it is called a critical singular $(r+1)$-tuple.

Let us describe the main results of the paper. The next theorem gives a set of generators and a set of defining relations for the left $A$-module $\text{Der}_K(A)$ when $A$ is a regular algebra.

**Theorem 1.1** Let the algebra $A$ be a regular algebra. Then the left $A$-module $\text{Der}_K(A)$ is generated by derivations $\partial_{i,j}$, $i \in I_r$, $j \in J_{r+1}$, where

$$\partial_{i,j} = \partial_{i_1, \ldots, i_r, j_1, \ldots, j_{r+1}} := \det \left( \begin{array}{ccc} \frac{\partial f_{i_1}}{\partial x_{j_1}} & \cdots & \frac{\partial f_{i_1}}{\partial x_{j_{r+1}}} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_{i_r}}{\partial x_{j_1}} & \cdots & \frac{\partial f_{i_r}}{\partial x_{j_{r+1}}} \\ \frac{\partial f_{j_1}}{\partial x_{j_1}} & \cdots & \frac{\partial f_{j_1}}{\partial x_{j_{r+1}}} \\ \cdots & \ddots & \cdots \\ \frac{\partial f_{j_{r+1}}}{\partial x_{j_1}} & \cdots & \frac{\partial f_{j_{r+1}}}{\partial x_{j_{r+1}}} \end{array} \right)$$

that satisfy the following defining relations (as a left $A$-module):

$$\Delta(i,j) \partial_{i',j'} = \sum_{l=1}^{s} (-1)^{r+1+\nu_l} \Delta(i'; j_1', \ldots, \hat{j_\nu_l}, \ldots, j_{r+1}') \partial_{i,j'}$$

for all $i, i' \in I_r$, $j = (j_1, \ldots, j_r) \in J_r$, and $j' = (j_1', \ldots, j_{r+1}') \in J_{r+1}$ where $\{j_\nu_1', \ldots, j_{\nu_s}'\} = \{j_1', \ldots, j_{r+1}'\} \setminus \{j_1, \ldots, j_r\}$.

For elements $i = (i_1, \ldots, i_r) \in I_r$ and $j = (j_1, \ldots, j_r) \in J_r$, Theorem 2.6 gives commuting iterative higher derivations $\{d_{i,j}^{[k]}\}_{k \geq 0}$, $\nu = r+1, \ldots, n$ of the localization $A_\Delta$ of the algebra $A$ at the powers of the element $\Delta := \Delta(i,j)$ where $\{j_{r+1}, \ldots, \nu\} = \{1, \ldots, n\} \setminus \{j_1, \ldots, j_r\}$. These iterative higher derivations can be found explicitly, i.e. their actions on the generators $x_i$ can be found explicitly (Theorem 2.8 and Theorem 2.12). Note that each higher derivation is uniquely determined by its action on algebra generators since the corresponding automorphism does.

Choose a function $N \to \mathbb{N}$, $k \mapsto n(k)$ such that $n(0) = 0$,

$$n(k+l) \geq n(k) + n(l) \text{ for all } k, l \in \mathbb{N};$$

$$d_{i,j}^{[k]} := \Delta^{n(k)} d_{i,j}^{[k]} \in \mathcal{D}(A), \quad k \geq 0,$$

for all $i \in I_r$, $j \in J_r$, and $\nu = r+1, \ldots, n$. Note that $d_{i,j}^{[0]} = \text{id}_A$, the identity map on $A$. The function $n(k)$ can be found explicitly (see Section 2). Moreover, any fast growing function satisfies the conditions (2) and (3).

The next result gives a set of generators and a set of defining relations for the $K$-algebra $\mathcal{D}(A)$ of differential operators on $A$ (it is well known and is not difficult to show that the algebra $\mathcal{D}(A)$ is not finitely generated and does not satisfy finitely many defining relations).

**Theorem 1.2** Let the algebra $A$ be a regular algebra. Then the ring of differential operators $\mathcal{D}(A)$ on $A$ is a simple algebra generated over $K$ by the algebra $A$ and the elements

$$d_{i,j}^{[k]} := \Delta^{n(k)} d_{i,j}^{[k]} \in \mathcal{D}(A), \quad k \geq 1, \quad i \in I_r, \quad j \in J_r, \quad \nu = r+1, \ldots, n.$$
where \( \{ j_{r+1}, \ldots, j_{n} \} = \{ 1, \ldots, n \} \setminus \{ j_1, \ldots, j_r \} \) and \( \Delta := \Delta(i, j) \). These elements satisfy the defining relations (R1)–(R5) over \( A \): for all \( \nu, \mu = r + 1, \ldots, n \) and natural numbers \( k, l \geq 1 \),

\[
\Delta^{n(l)} \partial^{[k]}_{i_{j_{i_{\mu}}} j_{\nu}} \sum_{s=0}^{l} a^{[s]}_{i_{j_{i_{\mu}}} j_{\nu}} (0, s) \partial^{[s]}_{i_{j_{i_{\mu}}} j_{\nu}} \Delta^{n(k)+k} = \Delta^{n(k)} \partial^{[l]}_{i_{j_{i_{\mu}}} j_{\nu}} \sum_{t=0}^{k} a^{[k]}_{i_{j_{i_{\mu}}} j_{\nu}} (0, t) \partial^{[k]}_{i_{j_{i_{\mu}}} j_{\nu}} \Delta^{n(l)+l}
\]

for some elements \( a^{[s]}_{i_{j_{i_{\mu}}} j_{\nu}} (0, s), a^{[k]}_{i_{j_{i_{\mu}}} j_{\nu}} (0, t) \in A \);

(R2)

\[
\Delta^{n(k+l)−n(k)} \partial^{[k]}_{i_{j_{i_{\mu}}} j_{\nu}} \sum_{s=0}^{l} a^{[s]}_{i_{j_{i_{\mu}}} j_{\nu}} (0, s) \partial^{[s]}_{i_{j_{i_{\mu}}} j_{\nu}} = \binom{k+l}{l} \partial^{[k+l]}_{i_{j_{i_{\mu}}} j_{\nu}} \Delta^{n(l)+l},
\]

(R3)

\[
[a^{[k]}_{i_{j_{i_{\mu}}} j_{\nu}}, x_{\rho}] = \delta_{j_{i_{\mu}} j_{\nu}} \Delta^{n(k)−n(k−1)} \partial^{[k−1]}_{i_{j_{i_{\mu}}} j_{\nu}}
\]

where \( \delta_{j_{i_{\mu}} j_{\nu}} \) is the Kronecker delta;

(R4)

\[
[a^{[k]}_{i_{j_{i_{\mu}}} j_{\nu}}, x_{\rho}] = \sum_{t=1}^{k} \Delta^{n(k)−n(t)−n(k−t)} \partial^{[t]}_{i_{j_{i_{\mu}}} j_{\nu}} (x_{\rho}) \partial^{[k−t]}_{i_{j_{i_{\mu}}} j_{\nu}}, \ s = 1, \ldots, r;
\]

(R5)

\[
\Delta^{m(l)} \partial^{[l]}_{i_{j_{i_{\mu}}} j_{\nu}} \Delta^{\sum_{\rho=0}^{\rho} k_{\rho}−k_{\rho}} = \Delta^{m(l)} \sum_{|k| \leq l} c_{i_{k_{\rho}}} \prod_{\rho=1}^{\rho} \sum_{t_{\rho}=0}^{k_{\rho}} a^{[k_{\rho}]}_{i_{j_{i_{\mu}}} j_{\nu}} (\sum_{\rho=0}^{\rho} t_{\rho}) \partial^{[t_{\rho}]}_{i_{j_{i_{\mu}}} j_{\nu}}
\]

for all \( i' \in I_r, j' \in J_r, j'' \in \{ 1, \ldots, n \} \setminus \{ j'_1, \ldots, j'_r \} \) and some \( c_{i_{k_{\rho}}} = c_{i_{k_{\rho}}}(i', j', j'', i, j) \in A \), and \( m(l) \in \mathbb{N} \) where \( \Delta := \Delta(i', j') \), \( \Sigma_{0} := 0 \), and

\[
\Sigma_{\rho} := \sum_{\nu=1}^{\rho} (n(k_{\rho}−\nu) + k_{\rho}−\nu), \ \rho \geq 1, \ k := (k_{r+1}, \ldots, k_{n}) \in \mathbb{N}^{n−r}, |k| := k_{r+1} + \cdots + k_{n}.
\]

Remark. All the elements of the algebra \( A \) in the defining relations (R1)–(R5) are found explicitly, see [14], [17] and [16].

When \( S = \{ 1 \} \), the algebra \( A = P_n/I \) is the algebra of regular functions on the irreducible affine algebraic variety \( X = \text{Spec}(A) \), therefore we have the explicit algebra generators for the ring of differential operators \( D(X) = D(A) \) on an arbitrary smooth irreducible affine algebraic variety \( X \). Any regular affine algebra \( A' \) is a finite direct product of regular affine domains, \( A' = \prod_{i=1}^{m} A_i \). Since \( D(A') \cong \prod_{i=1}^{m} D(A_i) \), Theorem 1.2 gives algebra generators and defining relations for the ring of differential operators on arbitrary smooth affine algebraic variety. Since \( \text{Der}_K(A') \cong \bigoplus_{i=1}^{m} \text{Der}_K(A_i) \), Theorem 1.7 gives generators and defining relations for the left \( A' \)-module of derivations \( \text{Der}_K(A') \) of the algebra \( A \).

In characteristic zero, analogues of Theorems 1.1 and 1.2 were proved in [5], Theorems 1.1 and 1.2 respectively. Theorem 1.1, [5] coincides with Theorem 1.1 but Theorem 1.2, [5] is much more simpler than Theorem 1.2 (the main difference is that the defining relations in Theorem 1.2, [5] are of first order in derivations).
Theorem 1.3 (Theorem 1.2, [5]) Let the algebra $A$ be a regular algebra over a field $K$ of characteristic zero. Then the ring of differential operators $D(A)$ on $A$ is generated over $K$ by the algebra $A$ and the derivations $\partial_{i,j}$, $i \in I_r$, $j \in J_{r+1}$ that satisfy the defining relations (1) and
\[
\partial_{i,j}(x_k) = x_k \partial_{i,j} + \partial_{i,j}(x_k), \quad i \in I_r, \quad j \in J_{r+1}, \quad k = 1, \ldots, n.
\]

Definition. The higher derivation algebra $\Delta(A)$ is a subalgebra of the ring of differential operators $D(A)$ generated by the algebra $A$ and the higher derivations $\text{HS}_K(A)$ of the algebra $A$.

Theorem 1.4 (Criterion of regularity of the algebra $A$ via $\Delta(A)$) The following statements are equivalent.

1. $A$ is a regular algebra.
2. $\Delta(A)$ is a simple algebra.
3. $A$ is a simple $\Delta(A)$-module.

In characteristic zero, the same criterion was proved in [18], 15.3.8 where $\Delta(A)$ is the derivation algebra, it is a subalgebra of the ring of differential operators $D(A)$ generated by the algebra $A$ and the set $\text{Der}_K(A)$ of all the $K$-derivations of the algebra $A$.

If the field $K$ has characteristic zero and the algebra $A$ is regular then the ring of differential operators $D(A)$ is a finitely generated Noetherian algebra. If $A$ is not regular then, in general, the algebra $D(A)$ need not be a finitely generated algebra nor a left or right Noetherian algebra, [7]: the algebra $D(A)$ can be finitely generated and right Noetherian yet not left Noetherian, [22] (so, in characteristic zero for a non-regular algebra $A$ the ring $D(A)$ behaves similarly as the ring $D(A)$ for a regular algebra $A$ in prime characteristic). Though, a kind of finiteness still holds for a singular algebra $A$ in characteristic zero.

Theorem 1.5 (Theorem 1.5, [5]) Let $K$ be a field of characteristic zero and $D(A) = \bigcup_{i \geq 0} D(A)_i$ be the order filtration of $D(A)$. Then, for each $i \geq 0$, $D(A)_i$ is a finitely generated left $A$-module.

In Section 3 we prove the same result in prime characteristic.

Theorem 1.6 Let $K$ be a perfect field of characteristic $p > 0$ and $D(A) = \bigcup_{i \geq 0} D(A)_i$ be the order filtration of $D(A)$. Then, for each $i \geq 0$, $D(A)_i$ is a finitely generated left $A$-module.

2 Generators and defining relations for ring of differential operators on regular algebra of essentially finite type

In this section, Theorems 1.1, 1.2 and 1.4 are proved.

Let $B$ be a commutative $K$-algebra. The ring of ($K$-linear) differential operators $D(B)$ on $B$ is defined as a union of $B$-modules $D(B) = \bigcup_{i=0}^{\infty} D(B)_i$ where $D(B)_0 = \text{End}_{R}(B) \simeq B$, $\left((x \mapsto bx) \leftrightarrow b\right)$,
\[
D(B)_i = \{u \in \text{End}_K(B) : [r,u] := ru - ur \in D(B)_{i-1} \text{ for each } r \in B\}.
\]
Then a direct computation shows that $K$ is a higher derivation since $\delta_{ij} = \delta_i \delta_j$. These conditions are equivalent to saying that the map $\delta : A \to A[[t]]$ is a Hasse-Schmidt derivation (or a Hasse-Schmidt derivation) over $K$ from $A$ to $A$ if, for each $k \geq 0$, $\delta_k(xy) = \sum_{i+j=k} \delta_i(x)\delta_j(y)$ for all $x, y \in A$. \hfill \square

**Theorem 2.2** Let $i = (i_1, \ldots, i_r) \in I_r$, $j = (j_1, \ldots, j_r) \in J_r$, and $\{1, \ldots, n\} \setminus \{j_1, \ldots, j_r\} = \{j_{r+1}, \ldots, j_n\}$. Then $\text{Der}_K(A) = \{(\Delta(i, j))^{-1} \sum_{k=r+1}^n a_{j_k}\delta_{i_1, \ldots, i_r; j_{r+1}, \ldots, j_k} \mid \text{where the elements } a_{j_{r+1}, \ldots, a_{jn}} \in A \text{ satisfy the following system of inclusions:}\}$

$$\sum_{k=r+1}^n \Delta(i; j_1, \ldots, j_{r-1}, r) a_{j_k} \in A \Delta(i, j), \ n = 1, \ldots, r.$$ \hfill \square

**Proof.** Repeat the proof of the same statement in characteristic zero, Theorem 2.12, p. 5. \hfill \square

Let us recall basic facts about higher derivations. For more detail the reader is referred to [17], Sec. 27.

A sequence $\delta = (1 := \text{id}_A, \delta_1, \delta_2, \ldots)$ of $K$-linear maps from a $K$-algebra $A$ to itself is called a higher derivation (or a Hasse-Schmidt derivation) over $K$ from $A$ to $A$ if, for each $k \geq 0$, $\delta_k(xy) = \sum_{i+j=k} \delta_i(x)\delta_j(y)$ for all $x, y \in A$. \hfill (5)

These conditions are equivalent to saying that the map $e : A \to A[[t]], \ x \mapsto \sum_{i \geq 0} \delta_i(x)t^i,$ is a $K$-algebra homomorphism where $A[[t]]$ is a ring of power series with coefficients from $A$. Let $\text{HS}_K(A)$ be the set of higher derivations on $A$. In general, a higher derivation $\delta = (\delta_i)$ is not determined by the derivation $\delta_1$.

Let $\delta = (\delta_i) \in \text{HS}_K(A)$. By [5], $\delta_1 \in \text{Der}_K(A)$ and $\delta_i \in \mathcal{D}(A)_i, \ i \geq 0$, \hfill (6)

since $\delta_i(x) - x\delta_i = \sum_{j=0}^{i-1} \delta_{i-j}(x)\delta_j$ for all $x \in A$ and the result follows by induction on $i$.

A higher derivation $\delta = (\delta_i) \in \text{HS}_K(A)$ is called iterative if $\delta_i\delta_j = (i+j)\delta_{i+j}$ for all $i, j \geq 0$. Then a direct computation shows that $\delta^p_1 = 0$ for all $i \geq 1, \hfill (7)$

$$\delta^p_1 = \delta_1 \cdots \delta_i = (2^{i-1})_{i \choose 1} (3^{i-2})_{i \choose 2} \cdots (p^{i-p})_{p \choose i} \delta_{pi} = 0 \delta_{pi} = 0.$$ For $i = 1$, we have $\delta^p_1 = 0$. \hfill 6
The higher derivations \( (1, \frac{\partial}{\partial t}, \frac{\partial^2}{\partial t^2}, \ldots) \in \text{HS}_K(P_n), \ i = 1, \ldots, n \). The \( K \)-algebra homomorphism \( P_n \to P_n[[t]], \ f(x_1, \ldots, x_n) \mapsto f(x_1, \ldots, x_i + t, x_{i+1}, \ldots, x_n) = \sum_{i \geq 0} \frac{\partial^i}{\partial t^i}(f) t^i \), gives the higher derivation \( (1, \frac{\partial}{\partial t}, \frac{\partial^2}{\partial t^2}, \ldots) \in \text{HS}_K(P_n) \). If \( \text{char}(K) = 0 \) then \( \frac{\partial^i}{\partial t^i} \) means \((k!)^{-1} \partial^i_k\), but if \( \text{char}(K) = p > 0 \) then
\[
\frac{\partial^i_k}{k!}(x_j^l) = \delta_{ij} \left( \frac{1}{k} \right) x_j^{l-k}
\]
for all \( k \geq 1, l \geq 0 \) and \( 1 \leq i, j \leq n \), where \( \delta_{ij} \) is the Kronecker delta.

The action of the higher derivations \( \partial_i^{[k]} := \frac{\partial^k}{\partial t^k} \) on the polynomial algebra \( P_n = K \otimes \mathbb{Z} \mathbb{Z}[x_1, \ldots, x_n] \) should be understood as the action of the element \( 1 \otimes \frac{\partial^k}{\partial t^k} \). The higher derivations \( \{\partial_i^{[k]} \}_{k \geq 0} \) are iterative and they commute, \( \partial_i^{[k]} \partial_j^{[l]} = \partial_j^{[l]} \partial_i^{[k]} \). For each element \( \alpha = (\alpha_i) \in \mathbb{N}^n \), let \( \partial^{[\alpha]} := \prod_{i=1}^n \partial_i^{[\alpha_i]} \).

**Theorem 2.3** Let the algebra \( A \) be a regular algebra. Then \( \mathcal{D}(A) \to \prod_{i \in \mathcal{I}_r, j \in \mathcal{J}_r} \mathcal{D}(A) \Delta(i,j) \) is a left and right faithfully flat extension of algebras where \( \mathcal{D}(A) \Delta(i,j) \) is the localization of the algebra \( \mathcal{D}(A) \) at the powers of the element \( \Delta(i,j) \).

**Proof.** The algebra \( A \) is regular, so \( A = \mathfrak{a}_r = (\Delta(i,j)) = (\Delta(i,j))_{i \in \mathcal{I}_r, j \in \mathcal{J}_r} \), hence the ideal of \( A \) generated by any power of the elements \( \{\Delta_{ij} \mid i \in \mathcal{I}_r, j \in \mathcal{J}_r\} \) is also equal to \( A \). The extension is a flat monomorphism. Suppose that the extension is not, say left faithful, then there exists a proper left ideal, say \( L \), of \( \mathcal{D}(A) \) such that \( \prod_{i \in \mathcal{I}_r, j \in \mathcal{J}_r} \mathcal{D}(A) \Delta(i,j) \otimes \mathcal{D}(A) (\mathcal{D}(A)/L) = 0 \), equivalently, there exists a sufficiently large natural number \( k \) such that \( \Delta(i,j)^k \in L \) for all \( i \in \mathcal{I}_r, j \in \mathcal{J}_r \). Since \( A = (\Delta(i,j)^k)_{i \in \mathcal{I}_r, j \in \mathcal{J}_r} \subseteq L \), we must have \( L = \mathcal{D}(A) \), a contradiction. \( \square \)

Let \( R \) be a (not necessarily commutative) algebra over a field \( K \), and let \( \delta \) be a \( K \)-derivation of the algebra \( R \). For any elements \( a, b \in R \) and a natural number \( n \), an easy induction argument gives the Leibniz formula
\[
\delta^n(ab) = \sum_{i=0}^n \binom{n}{i} \delta^i(a) \delta^{n-i}(b).
\]
It follows that the kernel \( C(\delta, R) := \ker \delta \) of \( \delta \) is a subalgebra (of constants for \( \delta \)) of \( R \) (since \( \delta(ab) = \delta(a)b + a\delta(b) = 0 \) for \( a, b \in C(\delta, R) \)), and the union of the vector spaces \( N(\delta, R) = \bigcup_{i \geq 0} N(\delta, i, R) \) is a positively filtered algebra (so-called, the nil-algebra of \( \delta \)) where \( N(\delta, i, R) := \{a \in R \mid \delta^i(a) = 0\} \), that is
\[
N(\delta, i, R)N(\delta, j, R) \subseteq N(\delta, i+j, R), \text{ for all } i, j \geq 0.
\]
Clearly, \( N(\delta, 0, R) = C(\delta, R) \) and \( N(\delta, R) := \{a \in R \mid \delta^n(a) = 0 \text{ for some natural } n\} \).

A \( K \)-derivation \( \delta \) of the algebra \( R \) is a locally nilpotent derivation if for each element \( a \in R \) there exists a natural number \( n \) such that \( \delta^n(a) = 0 \). A \( K \)-derivation \( \delta \) is locally nilpotent iff \( R = N(\delta, R) \). A derivation of \( R \) of the type \( \text{ad}(r) : x \mapsto [r, x] := rx - xr \) is called an inner derivation of \( R \) where \( r \in R \).

**Lemma 2.4** (Lemma 2.1, [7]) Let \( R \) be an algebra over an arbitrary field \( K \), \( \delta \) be a \( K \)-derivation of \( R \) such that \( \delta(x_i) = x_{i-1}, i \geq 0 \) for some elements \( x_i \in R \) such that \( x_{-1} = 0 \) and \( x_1 = 1 \). Then \( N(\delta, R) = \bigoplus_{i \geq 0} Cx_i = \bigoplus_{i \geq 0} x_i C \) where \( C := \ker \delta \), and \( N(\delta, i, R) = \bigoplus_{j=0}^i Cx_j = \bigoplus_{j=0}^i x_j C \) for all \( i \geq 0 \).
As the first application of Lemma 2.4 we find generators and defining relations for the algebra of differential operators with polynomial coefficients. The results are known but the proof is new. We have included the proof since it is short and similar patterns will appear later in the proofs of similar results in the general situation (Theorems 2.6 and 1.2).

Corollary 2.5

1. The algebra $\mathcal{D}(P_n)$ of differential operators with polynomial coefficients $P_n$ is generated (as an abstract $K$-algebra) by the elements $x_i$, $\partial_i^{[k]}$, $i = 1, \ldots, n$ and $k \geq 1$, that satisfy the following defining relations: for all $i, j = 1, \ldots, n$ and $k, l \geq 0$,

$$[x_i, x_j] = [\partial_i^{[k]}, \partial_j^{[l]}] = 0, \quad \partial_i^{[k]} \partial_j^{[l]} = \left(\frac{k + l}{k}\right) \partial_i^{[k+l]} \quad \partial_i^{[k]} x_j = \delta_{ij} \partial_i^{[k-1]},$$

where $\delta_{ij}$ is the Kronecker delta.

2. $\mathcal{D}(P_n) = \bigoplus_{\alpha, \beta \in \mathbb{N}^n} P_n x_\alpha \partial_\beta = \bigoplus_{\alpha, \beta \in \mathbb{N}^n} P_n \partial_\beta x_\alpha$ and $\mathcal{D}(P_n)_i = \bigoplus_{|\alpha| + |\beta| \leq 1} P_n x_\alpha \partial_\beta = \bigoplus_{|\alpha| + |\beta| \leq 1} P_n x_\alpha \partial_\beta$ for $i \geq 0$.

3. The algebra $\mathcal{D}(P_n)$ is a central simple algebra generated by the polynomial algebra $P_n$ and its higher derivations $HS_K(P_n)$.

4. The map $\mathcal{D}(P_n) \to \mathcal{D}(P_n)^o$, $x_i \mapsto x_i$, $\partial_i^{[k]} \mapsto (-1)^k \partial_i^{[k]}$, is a $K$ algebra isomorphism where $\mathcal{D}(P_n)^o$ is the opposite algebra. So, the algebra $\mathcal{D}(P_n)$ is self-dual.

Proof. It is obvious that the elements $x_i$, $\partial_i^{[k]}$ where $i = 1, \ldots, n$ and $k \geq 1$ satisfy the given relations. Then applying Lemma 2.4 several times to the algebra $E := \text{End}_K(P_n)$ and the set of commuting inner derivations $\text{ad} x_1, \ldots, \text{ad} x_n$, we obtain the algebra

$$N := N(\text{ad} x_1, \ldots, \text{ad} x_n; E) := \cap_{i=1}^n N(\text{ad} x_i; E) = \bigoplus_{\alpha \in \mathbb{N}^n} C \partial_\alpha = \bigoplus_{\alpha \in \mathbb{N}^n} \partial_\alpha C$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\partial_\alpha := \prod_{i=1}^n \partial_i^{\alpha_i}$, and $C := \cap_{i=1}^n \ker(\text{ad} x_i) = \text{End}_{P_n}(P_n) \simeq P_n$. Therefore,

$$N = \bigoplus_{\alpha, \beta \in \mathbb{N}^n} P_n x_\alpha \partial_\beta = \bigoplus_{\alpha, \beta \in \mathbb{N}^n} P_n \partial_\beta x_\alpha.$$

It follows that $N \subseteq \mathcal{D}(P_n)$, the inverse inclusion follows at once from the definitions of the algebra $N$ and $\mathcal{D}(P_n)$. Therefore, $N = \mathcal{D}(P_n)$. It follows that

$$\mathcal{D}(P_n)_i = \bigoplus_{|\alpha| + |\beta| \leq i} P_n x_\alpha \partial_\beta = \bigoplus_{|\alpha| + |\beta| \leq i} P_n \partial_\beta x_\alpha$$

for $i \geq 0$. This proves statement 2 and statement 3 apart from simplicity of the algebra $\mathcal{D}(P_n)$.

To prove simplicity of the algebra $\mathcal{D}(P_n)$, let $a = \sum a_{\alpha, \beta} x_\alpha \partial_\beta$ be a nonzero element of the algebra $\mathcal{D}(P_n)$ where $a_{\alpha, \beta} \in K$. We have to show that the ideal $(a)$ generated by the element $a$ is equal to the algebra $\mathcal{D}(P_n)$. To prove this we use induction on the degree $d := \text{deg}(a) = \max\{|\alpha| + |\beta| | a_{\alpha, \beta} \neq 0\}$ of the element $a$. The case $d = 0$ is obvious. Suppose that the result is true for all nonzero elements of degree $d < d$. If there exists a coefficient $a_{\alpha, \beta} \neq 0$ for some $\beta \neq 0$, i.e. $\beta_i \neq 0$ for some $i$, then applying the inner derivation $\text{ad} x_i$ to the element $a$ we have a nonzero element $[x_i, a]$ of degree $< d$, then induction gives the result.
Now, we are in a situation where $a_{\alpha\beta} = 0$ for all $\beta \neq 0$, that is $a \in P_n$ is a polynomial of degree $d > 0$. Then there exists a variable, say $x_i$, such that $\deg x_i(a) = m > 0$ (the degree in $x_i$). Then applying the inner derivation $\text{ad}(\partial_i^{[m]})$ to the element $a$ we have a nonzero element of degree $< d$, and induction finishes the proof of simplicity of the algebra $D(P_n)$. So, we have proved statement 3.

To prove statement 1, recall that the generators satisfy the relations from the first statement. They are defining relations since as it can be easily seen they guarantee that the following equality holds, $D = \sum Kx^i\partial_i^{[\beta]}$, where $D$ is an algebra generated by $x_i, \partial_i^{[k]}$, $i = 1, \ldots, n$ and $k \geq 1$ that satisfy the given relations. Since the algebra $D(P_n)$ is a factor algebra of $D$, the sum must be a direct sum. Then the relations must be defining relations.

Statement 4 follows from statement 1. □

**Theorem 2.6** Let $i = (i_1, \ldots, i_r) \in I_r$ and $j = (j_1, \ldots, j_r) \in J_r$, i.e. $\Delta = \Delta(i, j) \neq 0$, and $\{j_{r+1}, \ldots, j_n\} = \{1, \ldots, n\}\setminus\{j_1, \ldots, j_r\}$ and let $A_{\Delta}$ be the localization of the algebra $A$ at the powers of the element $\Delta$. Then

1. the algebra $D(A_{\Delta})$ of differential operators on $A_{\Delta}$ is a simple algebra such that

$$D(A_{\Delta}) = \bigoplus_{k_{r+1}, \ldots, k_n \geq 0} A_{\Delta} \delta^{[k_{r+1}]}_{i_1,j_{r+1}}, \ldots, \delta^{[k_n]}_{i_1,j_n} = \bigoplus_{k_{r+1}, \ldots, k_n \geq 0} \delta^{[k_{r+1}]}_{i_1,j_{r+1}}, \ldots, \delta^{[k_n]}_{i_1,j_n} A_{\Delta}$$

where $(\delta^{[k]}_{i_1,j_1}, k \geq 0)$ is the unique iterative higher derivation that is attached to the derivation $\Delta(i, j)^{-1}\partial_{i,j}$, and all the elements $\delta^{[k]}_{i_1,j_1}$, $k \geq 0$, $s = r + 1, \ldots, n$ commute. Therefore, the algebra $D(A_{\Delta})$ is generated by $A$ and $HS_K(A_{\Delta})$.

2. $\text{Der}_K(A_{\Delta}) = \bigoplus_{r+1} A_{\Delta} \partial_{i,j}$.

3. For each $l \geq 0$,

$$D(A_{\Delta})_l = \bigoplus_{k_{r+1}, \ldots, k_n \leq l} A_{\Delta} \delta^{[k_{r+1}]}_{i_1,j_{r+1}}, \ldots, \delta^{[k_n]}_{i_1,j_n} = \bigoplus_{k_{r+1}, \ldots, k_n \leq l} \delta^{[k_{r+1}]}_{i_1,j_{r+1}}, \ldots, \delta^{[k_n]}_{i_1,j_n} A_{\Delta}.$$

**Proof.** The second statement follows from Theorem 2.2. Without loss of generality we can assume that $i = (1, 2, \ldots, r)$ and $j = (1, 2, \ldots, r)$. Let

$$\partial_{r+1} := \Delta^{-1}\partial_{i,j}, \ldots, \partial_n := \Delta^{-1}\partial_{i,j,n}.$$ 

Then $\partial_i(x_j) = \delta_{ij}$ for all $i, j = r + 1, \ldots, n$. By the second statement, the commutator of the derivations

$$[\partial_i, \partial_j] = \sum_{k=r+1}^n a^k_{ij} \partial_k \in \text{Der}_K(A_{\Delta}), \quad a^k_{ij} \in A_{\Delta},$$

annihilates the elements $x_{r+1}, \ldots, x_n$. Therefore, all $a^k_{ij} = 0$ since

$$a^k_{ij} = \sum_{l=r+1}^n a^l_{ij}\partial_l(x_k) = [\partial_i, \partial_j](x_k) = 0;$$

and it follows that the derivations $\partial_i$ commute.
The functor of taking derivations commutes with localizations, therefore $\text{Der}_K(A_\Delta) = \oplus_{i=r+1}^n A_\Delta \partial_i$ implies $\text{Der}_K(Q) = \oplus_{i=r+1}^n Q \partial_i$. By [10], Theorem 1, Ch. V, Sec. 9, the field $Q$ is a finite separable extension of its subfield $Q' := K(x_{r+1}, \ldots, x_n)$, hence $\text{Der}_Q(Q) = 0$. In characteristic $p > 0$, a $p$'th power of a $K$-derivation is again a $K$-derivation, hence $\partial_i^p \in \text{Der}_K(A_\Delta) \subseteq \text{Der}_K(Q)$. Since $Q' \subseteq \ker(\partial_i^p)$ and $Q$ is algebraic and separable over $Q'$, we must have $\partial_i^p = 0$. Recall that over a perfect field any field extension is separable ([17], Theorem 26.3). Now, by [17], Theorem 27.4, each derivation $\partial_i \in \text{Der}_K(Q)$ can be extended to an iterative higher derivation $\partial_i^r := (\text{id}_Q, \partial_i := \partial_i^{[1]}, \partial_i^{[2]}, \ldots) \in HS_K(Q)$, and it is unique by [17], Theorem 27.2. Now, considering the derivation $\partial_i$ as an element of $\text{Der}_K(Q')$, by the same arguments the derivation $\partial_i$ can be uniquely extended to an iterative higher derivation

$$(\text{id}_{Q'}, \partial_i, \frac{\partial_i^2}{2!}, \ldots, \frac{\partial_i^k}{k!}, \ldots) \in HS_K(Q'),$$

then this iterative higher derivation has a unique extension to an iterative higher derivation of $Q$ (by [17], Theorem 27.2). By uniqueness, it must coincide with $\partial_i^r$, i.e. $\partial_i^r|_{Q'} = \frac{\partial_i^k}{k!}$ for all $i$ and $k$. A direct calculation gives,

$$-(\text{ad } x_i)(\partial_i^r) = [\partial_i^r, x_i] = \left[\frac{\partial_i^k}{k!}, x_i\right] = \frac{\partial_i^{k-1}}{(k-1)!} = \partial_i^{[k-1]}.$$ 

Clearly, the inner derivations $\text{ad}(x_{r+1}), \ldots, \text{ad}(x_n)$ of the algebra $E := \text{End}_K(A_\Delta)$ commute. Applying several times Lemma 2.4 we obtain the algebra

$$N = N(\text{ad}(x_{r+1}), \ldots, \text{ad}(x_n); E) := \bigcap_{i=r+1}^n N(\text{ad}(x_i), E) = \bigoplus_{\alpha \in \mathbb{N}^{n-r}} C \partial^{[\alpha]} = \bigoplus_{\alpha \in \mathbb{N}^{n-r}} \partial^{[\alpha]} C$$

where $\alpha = (\alpha_{r+1}, \ldots, \alpha_n)$, $\partial^{[\alpha]} := \partial_{\alpha_{r+1}} \cdots \partial_{\alpha_n}$, and $C := \bigcap_{i=r+1}^n \ker(\text{ad}(x_i))$ is the subalgebra of $E$. So, any element $u$ of $N$ is uniquely written as a sum $u = \sum_{\alpha \in \mathbb{N}^{n-r}} c_{\alpha} \partial^{[\alpha]}$, $c_{\alpha} \in C$. The algebra $N = \bigcup_{i \geq 0} N_i$ has a natural filtration by the total degree of the $\partial_i^r$'th, that is $N_i = \bigoplus_{|\alpha| \leq i} C \partial^{[\alpha]}$ where $|\alpha| := \alpha_{r+1} + \cdots + \alpha_n$. Clearly, $\mathcal{D}(A_\Delta) \subseteq N$ and $\mathcal{D}(A_\Delta)_i \subseteq N_i$ for each $i \geq 0$. Let us prove, by induction on $i$, that

$$\mathcal{D}(A_\Delta)_i = D_i := \sum_{|\alpha| \leq i} A_\Delta \partial^{[\alpha]}, \quad i \geq 0.$$ 

The case $i = 0$ is true, $\mathcal{D}(A_\Delta) = A_\Delta = D_0$. Suppose that $i > 0$, and by induction $\mathcal{D}(A_\Delta)_{i-1} = D_{i-1}$. Take $u \in \mathcal{D}(A_\Delta)_i$. Since $\mathcal{D}(A_\Delta)_i \subseteq N_i$, the element $u$ can be written a sum $u = \sum_{|\alpha| \leq i} c_{\alpha} \partial^{[\alpha]}$ for some elements $c_{\alpha} \in C$. For each $j = r+1, \ldots, n$,

$$-\text{ad}(x_j)(u) = \sum_{|\alpha| \leq i} c_{\alpha} \partial^{[\alpha-e_j]} \in \mathcal{D}(A_\Delta)_{i-1} = D_{i-1},$$

(where the set $e_{r+1}, \ldots, e_n$ is the obvious ‘free basis’ for $N^{n-r} := \mathbb{N}e_{r+1} \oplus \cdots \oplus \mathbb{N}e_n$), therefore all $c_{\alpha} \in A_\Delta$ with $\alpha \not= 0$. Since $c_0 = u - \sum_{\alpha \not= 0, |\alpha| \leq i} c_{\alpha} \partial^{[\alpha]} \in C \cap \mathcal{D}(A_\Delta)_i$, it follows from the claim below that $c_0 \in A_\Delta$. Therefore, $\mathcal{D}(A_\Delta) = D_i$, and so $\mathcal{D}(A_\Delta) = A_\Delta(\partial_{r+1}^{[k]}, \ldots, \partial_n^{[k]})_{k\geq 1}$.

Claim. $C \cap \mathcal{D}(A_\Delta)_i = A_\Delta$ for all $i \geq 0$. 

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We use induction on \( i \). The case \( i = 0 \) is trivial, \( C \cap \mathcal{D}(A_\Delta)_0 = C \cap A_\Delta = A_\Delta \). Note that the algebra \( C \) is an \( A_\Delta \)-bimodule, and so it is invariant under the inner derivation \( \text{ad}(a) \) for each element \( a \in A_\Delta \). If the intersection \( I_i := C \cap \mathcal{D}(A_\Delta)_i \neq A_\Delta \) for some \( i \geq 1 \), then obviously \( I_1 \neq A_\Delta \), and since

\[
\mathcal{D}(A_\Delta)_1 = A_\Delta + \text{Der}_K(A_\Delta) = A_\Delta + \sum_{i=r+1}^{n} A_\Delta \partial_i,
\]

one can choose an element \( u = a_0 + \sum_{i=r+1}^{n} a_i \partial_i \in I_1 \setminus A_\Delta \) for some elements \( a_i \in A_\Delta \) such that \( a_j \neq 0 \) for some \( j \geq r + 1 \). We have a contradiction: \( 0 = [u, x_j] = a_j \neq 0 \), which proves the claim.

Next, let us prove by induction on \( s = k + \ell \) that the elements \( \partial_i^{[k]}, \partial_j^{[\ell]} \), \( k, l \geq 0 \), \( i, j = r + 1, \ldots, n \) commute. The case \( s = 0 \) is obviously true as \( \partial_i^{[0]} = \text{id}_{A_\Delta} \) for all \( i \). Suppose that \( s > 0 \) and the result is true for all \( s' < s \). For each \( t = r + 1, \ldots, n \),

\[
[[\partial_i^{[k]}, \partial_j^{[\ell]}], x_t] = [[\partial_i^{[k]}, x_t], \partial_j^{[\ell]}] + [\partial_i^{[k]}, [\partial_j^{[\ell]}, x_t]] = \delta_{it}[\partial_i^{[k-1]}, \partial_j^{[\ell]}] + \delta_{jt}[\partial_i^{[k]}, \partial_j^{[\ell-1]}] = 0,
\]

by induction. By the claim, \( [\partial_i^{[k]}, \partial_j^{[\ell]}] \in A_\Delta \), and so \( [\partial_i^{[k]}, \partial_j^{[\ell]}] = [\partial_i^{[k]}, \partial_j^{[\ell]}] * 1 = 0 \).

To prove that the algebra \( \mathcal{D}(A_\Delta) \) is simple, let \( L \) be a nonzero ideal of \( \mathcal{D}(A_\Delta) \). It remains to prove that \( L = \mathcal{D}(A_\Delta) \). Take a nonzero element, say \( u \), of \( L \). Applying several times maps of the type \( \text{ad}(x_{jk}) \), \( r + 1 \leq k \leq n \), to the element \( u \) we have a nonzero element, say \( u_1 \in L \cap A_\Delta \). Since \( \text{Kdim}(A_\Delta) = \text{Kdim}(K[x_{r+1}, \ldots, x_n]) \), we must have \( A_\Delta u_1 \cap K[x_{r+1}, \ldots, x_n] \neq 0 \). Pick a nonzero element, say \( u_2 \), of the intersection, then \( u_2 \in \mathcal{D}(K[x_{r+1}, \ldots, x_n]) \subseteq \mathcal{D}(A_\Delta) \). The algebra \( \mathcal{D}(K[x_{r+1}, \ldots, x_n]) \) is a simple algebra (Corollary 2.7), hence \( L = \mathcal{D}(A_\Delta) \). This proves that the algebra \( \mathcal{D}(A_\Delta) \) is a simple algebra. The rest is obvious. \( \square \)

It is clear from Theorem 2.6 that a differential operator from the algebra \( \mathcal{D}(A_\Delta) \) (or from the algebra \( \mathcal{D}(A) \subseteq \mathcal{D}(A_\Delta) \)) is uniquely determined by its action on the polynomial subalgebra \( P := K[x_{j_{r+1}}, \ldots, x_{j_n}] \), that is the restriction map

\[
\mathcal{D}(A_\Delta) \to \text{Hom}_K(P, A_\Delta), \; \delta \mapsto \delta|_P,
\]

is an injective map. Then the second statement of Corollary 2.7 follows, the first statement of Corollary 2.7 was already proved in the proof of Theorem 2.6.

**Corollary 2.7** Let \( i = (i_1, \ldots, i_r) \in I_r \) and \( j = (j_1, \ldots, j_r) \in J_r \), i.e. \( \Delta(i,j) \neq 0 \), and \( \{j_{r+1}, \ldots, j_n\} = \{1, \ldots, n\} \setminus \{i_1, \ldots, i_r\} \). Then

1. The field \( Q \) is a finite separable field extension of the field \( K(x_{j_{r+1}}, \ldots, x_{j_n}) \).

2. The derivations \( \Delta(i,j)^{-1} \partial_{i,j_{r+1}}, \ldots, \Delta(i,j)^{-1} \partial_{i,j_{j_n}} \) from Theorem 2.6 are respectively the partial derivations \( \partial_{j_{r+1}} := \frac{\partial}{\partial x_{j_{r+1}}}, \ldots, \partial_{j_n} := \frac{\partial}{\partial x_{j_n}} \) of the algebra \( A_\Delta \) (and of the field of fractions of \( A \)).

**Remarks.** 1. Statement 1 of Corollary 2.7 is a strengthening of the following well known result – Theorem 26.2, [17] (and the Remark after Theorem 26.2, [17]): Let \( K \) be a field of characteristic \( p \) and \( L = K(y_1, \ldots, y_t) \) be a finitely generated field extension of \( K \) which is
separably generated over $K$; then there exists a subset of the set of generators, say $y_1, \ldots, y_d$, $d = \text{tr.deg}_K(L)$, such that $L$ is separable over $K(y_1, \ldots, y_d)$.

2. The equality $\partial_{x_{r+k}} := \frac{\partial}{\partial x_{r+k}}$ means that the derivation $\partial_{x_{r+k}}$ is a unique extension of the partial derivative $\frac{\partial}{\partial x_{r+k}}$ of the polynomial algebra $K[x_{r+1}, \ldots, x_n]$ to the algebra $A_{\Delta}$.

**Theorem 2.8** Keep the assumption of Theorem 2.6. Then the algebra $\mathcal{D}(A_{\Delta})$ is generated by the algebra $A_{\Delta}$ and the elements $\delta_{ij,j_{\nu}}^{[k]}$ where $\nu = r+1, \ldots, n$ and $k \geq 0$, that satisfy the following defining relations: for all $\nu, \mu = r+1, \ldots, n$ and $k, l \geq 1$ (where $\delta_{ij,j_{\nu}}^{[0]} := 1$)

$$\left[\delta_{ij,j_{\nu}}^{[k]}, \delta_{ij,j_{\mu}}^{[l]}\right] = 0, \quad \delta_{ij,j_{\nu}}^{[k]} \delta_{ij,j_{\mu}}^{[l]} = \binom{k+l}{k} \delta_{ij,j_{\nu}}^{[k+l]}, \quad \left[\delta_{ij,j_{\nu}}^{[k]}, x_{\nu}\right] = \delta_{ij,j_{\nu}}^{[k-1]},$$

and, for all $\nu = r+1, \ldots, n$ and $s = 1, \ldots, r$,

$$\left[\delta_{ij,j_{\nu}}^{[k]}, x_{\nu}\right] = \sum_{t=1}^{k} \delta_{ij,j_{\nu}}^{[t]} (x_{\nu}) \delta_{ij,j_{\nu}}^{[k-t]},$$

where $\delta_{ij,j_{\nu}}^{[t]} (x_{\nu}) = \left(\Delta^{-1} \partial_{x_{j_{\nu}}}^{t}\right)(x_{\nu}) \in A_{\Delta}$.

**Remark.** The elements $\delta_{ij,j_{\nu}}^{[t]} (x_{\nu})$ can be found explicitly by combining Corollary 2.7 (1) and Theorem 2.12.

**Proof.** Clearly, the generators satisfy the given relations. Suppose that $D$ is an algebra generated by the given elements that satisfy the given defining relations. One can easily see that

$$D = \bigoplus_{k_r+1, \ldots, k_n \geq 0} A_{\Delta} \delta_{ij,j_{r+1}}^{[k_r+1]}, \ldots, \delta_{ij,j_{n}}^{[k_n]}.$$

By Theorem 2.6 (1), the sum above must be direct. Therefore, $D = \mathcal{D}(A_{\Delta})$, which implies that the relations are defining relations for the algebra $\mathcal{D}(A_{\Delta})$. □

**Corollary 2.9** Let $Q$ be the field of fractions of the algebra $A$. Under the assumption of Theorem 2.6,

1. the algebra $\mathcal{D}(Q)$ of differential operators on $Q$ is a simple algebra such that

$$\mathcal{D}(Q) = \bigoplus_{k_r+1, \ldots, k_n \geq 0} Q \delta_{ij,j_{r+1}}^{[k_r+1]}, \ldots, \delta_{ij,j_{n}}^{[k_n]} = \bigoplus_{k_r+1, \ldots, k_n \geq 0} \delta_{ij,j_{r+1}}^{[k_r+1]}, \ldots, \delta_{ij,j_{n}}^{[k_n]} Q$$

where $(\delta_{ij,j_{\nu}}^{[k]})_{k \geq 0} \in HS_K(Q)$ is the iterative higher derivation which is a unique extension of the $(\delta_{ij,j_{\nu}}^{[k]})_{k \geq 0} \in HS_K(A_{\Delta})$ from Theorem 2.6, they also commute. Therefore, the algebra $\mathcal{D}(Q)$ is generated by the field $Q$ and $HS_K(Q)$.

2. $\text{Der}_K(Q) = \bigoplus_{\nu = r+1}^{n} Q \partial_{i,j_{\nu}}$.

3. For each $l \geq 0$,

$$\mathcal{D}(Q)_l = \bigoplus_{k_r+1 + \cdots + k_n \leq l} Q \delta_{ij,j_{r+1}}^{[k_r+1]}, \ldots, \delta_{ij,j_{n}}^{[k_n]} = \bigoplus_{k_r+1 + \cdots + k_n \leq l} \delta_{ij,j_{r+1}}^{[k_r+1]}, \ldots, \delta_{ij,j_{n}}^{[k_n]} Q.$$
Proof. Note that $\mathcal{D}(Q) \simeq Q \otimes_{A_\Delta} \mathcal{D}(A_\Delta)$ and $\text{Der}_K(Q) \simeq Q \otimes_{A_\Delta} \text{Der}_K(A_\Delta)$, and the results follow from Theorem 2.6. □

**Theorem 2.10** Let the algebra $A$ be a regular algebra. Then the algebra $\mathcal{D}(A)$ of differential operators on $A$ is a simple algebra generated by $A$ and $HS_K(A)$, that is $\mathcal{D}(A) = \Delta(A)$.

Proof. Let $\Delta = \Delta(A)$ be the subalgebra of $\text{End}_K(A)$ generated by $A$ and $HS_K(A)$. By Theorem 2.6, $\mathcal{D}(A)_{\Delta(i,j)} = \Delta_{\Delta(i,j)}$ for all non-singular $i$ and $j$, or equivalently $\prod_{i,j} \mathcal{D}(A)_{\Delta(i,j)} \mathcal{D}(A) = 0$. By Theorem 2.3 we must have $\mathcal{D}(A) = \Delta$. By Theorems 2.3 and 2.6 $\mathcal{D}(A)$ is a simple algebra. □

**Proposition 2.11** Let the algebra $A$ be a regular algebra. Then $\mathcal{D}(A) = \bigcap_{i \in I_r} \mathcal{D}(A)_{\Delta(i,i)}$ where the intersection is taken in the algebra $\mathcal{D}(Q)$.

Proof. We denote by $\mathcal{D}'$ the intersection. Then $\mathcal{D}(A) \subseteq \mathcal{D}'$ and $\mathcal{D}(A)_{\Delta(i,j)} = \mathcal{D}'_{\Delta(i,j)}$ for all $i \in I_r$ and $j \in J_r$. Then $\mathcal{D}(A) = \mathcal{D}'$ since the extension $\mathcal{D}(A) \to \prod_{i \in I_r, j \in J_r} \mathcal{D}(A)_{\Delta(i,j)}$ is faithfully flat. □

**Theorem 2.12** Let $k \subseteq L \subseteq L'$ be fields such that $L' = L(x)$ for a separable element $x \in L'$ over $L$ and $f(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_0 \in L[t]$ be a minimal polynomial for $x$. Then each higher derivation $\{\delta^{[i]}, i \geq 0\} \in HS_k(L)$ can be uniquely extended to a higher derivation $\{\delta^{[i]}, i \geq 0\} \in HS_k(L')$. Moreover,

$$\delta^{[i]}(x) = -\frac{1}{f'(x)} \sum_{m=0}^s \sum_{i_0 + \cdots + i_m = i, i, \neq i, \nu \geq 1} \delta^{[i_0]}(a_m)\delta^{[i_1]}(x)\cdots\delta^{[i_m]}(x), \ i \geq 1, \ (10)$$

where $f' = \frac{df}{dx}$.

Proof. The $k$-algebra homomorphism $\sigma(\cdot) = \sum_{i \geq 0} \delta_i(\cdot)t^i : L \to L[[t]]$ can be extended to a $k$-algebra homomorphism $\sigma(\cdot) = \sum_{i \geq 0} \delta_i(\cdot)t^i : L \to L'[[t]]$ iff $0 = \sigma(f(x)) = \sum_{i \geq 0} \delta_i(f(x))t^i$ iff $\delta_i(f(x)) = 0$, $i \geq 1$ iff the equalities of the theorem hold (for each $i \geq 1$, the equality $\delta_i(f(x)) = 0$ can be rewritten as (10)). □

**Corollary 2.13** Let $k \subseteq L \subseteq L'$ be fields such that $L'$ is separable over $L$. Then $HS_k(L) \subseteq HS_k(L')$.

**Corollary 2.14** Let $K \subseteq Q \subseteq Q'$ be fields finitely generated over the field $K$ and let the field $Q'$ be separable over $Q$. Then $\mathcal{D}_K(Q') = Q'\mathcal{D}_K(Q)$.

Proof. We may assume that the field $Q$ is as in Corollary 2.9 (by inverting, if necessary, one of the nonzero minors $\Delta(i,j)$). By Corollary 2.7 (1), the field $Q$ is a separable extension of its subfield $L := K(x_{j+1}, \ldots, x_{j_n})$, then so is $Q'$ since $Q'$ is separable over $Q$. The higher derivation $\{\delta^{[k]}_{i j}, k \geq 0\} \in HS_K(Q)$ can be extended uniquely to a higher derivation of the field $Q'$, by Corollary 2.13. When we write down statement 1 of Corollary 2.9 for the fields $Q$ and $Q'$ the equality $\mathcal{D}_K(Q') = Q'\mathcal{D}_K(Q)$ follows at once. □

Note that in Corollary 2.14 the inclusion $\mathcal{D}_k(Q) \subseteq \mathcal{D}_K(Q')$ follows from Corollary 2.9 (1), Corollary 2.13 and the simplicity of the algebra $\mathcal{D}_k(Q)$. Then the inclusion $Q'\mathcal{D}_K(Q) \subseteq \mathcal{D}_K(Q')$ is obvious.
Proposition 2.15 Let \( i, j' \in I_r, j = (j_1, \ldots, j_r) \in J_r, j' = (j'_1, \ldots, j'_{r+1}) \in J_{r+1} \), and \( \{j_{r+1}, \ldots, j_n\} = \{1, \ldots, n\}\setminus\{j_1, \ldots, j_r\} \). Then

1. \[
\partial_{i,j'} = \Delta(i,j)^{-1} \sum_{t=1}^s (-1)^{r+1+\nu_t} \Delta(i';j'_1, \ldots, j'_{\nu_t}, \ldots, j'_{r+1}) \partial_{j'_1j'_i} \tag{11}
\]

where \( j'_{\nu_t}, \ldots, j'_{\nu_s} \) are the elements of the set \( \{j'_1, \ldots, j'_{r+1}\}\setminus\{j_1, \ldots, j_r\} \).

2. \[
\partial_{i,j'} = (-1)^{r+1+k} \frac{\Delta(i';\ldots,j'_{\nu_1},\ldots,j'_{\nu_s})}{\Delta(i';\ldots,j'_{\nu_1},\ldots,j'_{\nu_s})} \partial_{ij'} \text{ provided } \Delta(i;j'_1, \ldots, j'_{\nu_1}, \ldots, j'_{\nu_s}) \neq 0.
\]

Proof. 1. By Theorem 2.2, \( \partial_{i,j'} = \Delta(i,j)^{-1} \sum_{k=r+1}^n \lambda_k \partial_{i,kj} \) for some \( \lambda_k \in A \). For each \( k' = r+1, \ldots, n \), \( \partial_{i,kj}(x_{j_{k'}}) = \delta_{k,k'} \Delta(i,j) \). Then evaluating the equality above at \( x_{j_{k'}} \), we get the equality \( \lambda_k = \partial_{i,j'}(x_{j_{k'}}) \). So, \( \lambda_k = 0 \) if \( j_k \notin \{j'_{\nu_1}, \ldots, j'_{\nu_s}\} \), and if \( j_k = j'_{\nu_s} \) then \( \lambda_k = (-1)^{r+1+\nu_s} \Delta(i,j'_1, \ldots, j'_{\nu_s}, \ldots, j'_{r+1}) \). This finishes the proof of the first statement.

2. By the first statement where we put \( j = (j'_1, \ldots, j'_{\nu_1}, \ldots, j'_{\nu_s}) \), we have \( \partial_{i,j'} = (-1)^{r+1+k} \frac{\Delta(i';\ldots,j'_{\nu_1},\ldots,j'_{\nu_s})}{\Delta(i';\ldots,j'_{\nu_1},\ldots,j'_{\nu_s})} \partial_{ij'} \). □

Remark. Let us fix elements \( i \in I_r \) and \( j \in J_r \). Then, for each \( i' \in I_r \) and \( j' \in J_{r+1} \) (as above), let \( a(i',j') \) be the vector of coefficients \( (\lambda_{r+1}, \ldots, \lambda_n) \) from the Proposition 2.15. If \( A \) is a regular algebra, the vectors \( a(i',j') \) form a generating set for the \( A \)-module of solutions to the system of inclusions from Theorem 2.2 (by Theorem 1.1).

Proof of Theorem 1.2 Let \( D = D(A) \) be the algebra generated by the algebra \( A \) and the elements \( d_{i,j}^{[k]} \) that satisfy the defining relations (R1)–(R5). Theorem 1.2 follows easily from the Claim, and the Claim follows from Theorem 2.6 and Theorem 2.8.

Claim: There is an algebra homomorphism \( D \to D := D(A) \) such that \( D_{\Delta(i,j)} \simeq D_{\Delta(i,j)} \) for all \( i \in I_r \) and \( j \in J_r \) where \( D_{\Delta(i,j)} \) is the (left and right) Ore localization of the algebra \( D \) at the powers of the element \( \Delta(i,j) \).

Indeed, we have the commutative diagram of algebra homomorphisms

\[
\begin{array}{ccc}
D & \longrightarrow & \prod_{i \in I_r, j \in J_r} D_{\Delta(i,j)} \\
\downarrow & & \downarrow \\
D & \longrightarrow & \prod_{i \in I_r, j \in J_r} D_{\Delta(i,j)}
\end{array}
\]

where the horizontal maps are faithfully flat extensions (since \( A = (\Delta(i,j))_{i \in I_r, j \in J_r} \), as \( A \) is regular) and the right vertical map is an isomorphism, by the Claim. By faithful flatness, the left vertical map is an isomorphism, i.e. \( D \simeq D \), and the ring \( D \) is simple since each ring \( D_{\Delta(i,j)} \) is so.

It remains to prove the Claim. By Theorem 2.8, for the elements \( i \in I_r \) and \( j \in J_r \), the algebra \( D_\Delta \) (where \( \Delta := \Delta(i,j) \)) is generated by the algebra \( A_\Delta \) and the elements \( \{d_{k,j}^{[k]} : k \geq 0\} \), \( \nu = r+1, \ldots, n \), that satisfy the four types of the defining relations of Theorem 2.8. It is obvious that the algebra \( D_\Delta \) is generated by the algebra \( A_\Delta \) and the elements \( \{d_{k,j}^{[k]} : k \geq 0\} \),
\[\nu = r+1, \ldots, n.\] When we multiply the fourth relation of Theorem 2.8 by the invertible element \(\Delta^{n(k)+k}\) on the left we obtain the relation (R4) which is equivalent to the original one. Using the relation (R4), we see that

\[
\Delta^s \delta_{i,j}^{[k]} \Delta^{n(k)+k+s} \in \sum_{t=0}^k A_{d_{i,j}}^{[t]}, \quad s \geq 0. \tag{12}
\]

In more detail, let \(d_{i,j}^{[t]} := d_{i,j}^{[t]}.\) By (R4), \((\text{ad}(\Delta))^j (d_{i,j}^{[k]}) \in \sum_{t=0}^{k-j} A_{d_{i,j}}^{[t]}.\) Then

\[
\Delta^s \delta_{i,j}^{[k]} \Delta^{n(k)+k+s} = \Delta^{s-n(k)} d_{i,j}^{[k]} \Delta^{n(k)+k+s}
\]

\[
= \Delta^{s-n(k)} \sum_{j=0}^k \binom{n(k)+k+s}{j} \Delta^{n(k)+k+s-j} (-\text{ad}(\Delta))^j (d_{i,j}^{[k]})
\]

\[
\in \sum_{t=0}^k A_{d_{i,j}}^{[k]}.
\]

By (12),

\[
\Delta^s \delta_{i,j}^{[k]} \Delta^{n(k)+k+s} = \sum_{t=0}^k a_{i,j}^{[k]}(s,t) d_{i,j}^{[t]}, \quad s \geq 0,
\tag{13}
\]

for some elements \(a_{i,j}^{[k]}(s,t) \in A.\) By Theorem 3.1.(5), [14],

\[
a_{i,j}^{[k]}(s,t) = \psi_{i,j}((-\text{ad}(x_{j}^r))^j (\Delta^s \delta_{i,j}^{[k]} \Delta^{n(k)+k+s})) \cdot \Delta^{-n(t)},
\tag{14}
\]

where \(\psi_{i,j}(\cdot) := \sum_{j \geq 0} (\text{ad}(x_{j})^j(\cdot)) \delta_{i,j}^{[k]} : D_\Delta \rightarrow D_\Delta.\) When we multiply the first relation of Theorem 2.8 by the element \(\Delta^{n(k)+n(l)}\) on the left and by the element \(\Delta^{n(k)+n(l)+k+l}\) on the right we obtain the relation (R1) using (13). Similarly, multiplying the second relation of Theorem 2.8 by the element \(\Delta^{n(k+l)}\) on the left and by the element \(\Delta^{n(l)+l}\) on the right we obtain the relation (R2) using (13). When we multiply the third relation of Theorem 2.8 by the element \(\Delta^{n(k)}\) on the left we obtain the relation (R3).

It is obvious that the algebra \(D_\Delta\) is generated by the algebra \(A_{\Delta}\) and the elements \(\{d_{i,j}^{[k]}, k \geq 0\}, \nu = r+1, \ldots, n\) that satisfy the defining relations (R1)–(R4). Recall that, for \(i' \in I_r\) and \(j' \in J_r,\) we have \(\Delta' := \Delta(i', j').\) By Theorem 2.6

\[
\delta_{i,j}^{[l]}_{\nu; \nu'} = \sum_{i, k, j, j'} b_{i, k}^{[l]} \delta_{i,j}^{[k]}_{\nu; \nu'} \cdot \delta_{j,j'}^{[l]}
\tag{15}
\]

for some elements \(b_{i, k} = b_{i, k}(i', j', j'; \nu) \in A_{\Delta}\) where the sum is taken over the vectors \(k := (k_{r+1}, \ldots, k_n) \in \mathbb{N}^n\) such that \(|k| \leq l.\) A formula for the elements \(b_{i, k}\) is given by Theorem 3.1.(5), [14],

\[
b_{i, k} = \psi_{j, r+1} \cdots \psi_{j, n} \prod_{\nu=r+1}^n (-\text{ad}(x_{j}^\nu))^{k_{r+\nu}} (\delta_{i,j}^{[l]}_{\nu; \nu'})
\tag{16}
\]

where \(\psi_{i,j}(\cdot) := \sum_{j \geq 0} (\text{ad}(x_{j})^j(\cdot)) \delta_{i,j}^{[k]} : D_\Delta \rightarrow D_\Delta.\)

For each natural number \(l,\) choose a natural number \(m(l)\) such that

\[c_{l, k} := \Delta^{m(l)} b_{i, k} \in A\] 
\tag{17}

The proof of Theorem 1.2 is complete. □

subject to the defining relations (1). We have the commutative diagram of left $3$ Ring of differential operators on singular irreducible affine $D$

In this Section, we prove Theorem 1.6 (see Proposition 3.3.(2)), the local finiteness of the ring $\Delta$ of differential operators on the algebra $A$.

Claim is obvious where $a$ is regular and the vertical maps are natural $\mathrm{Der}_K(A)$ isomorphism (by faithfully flatness), i.e. $\mathrm{Der}_K(A) \cong \mathrm{Der}_K(A)$. □

Proof of Theorem 1.1 Let $\mathrm{Der}(A)$ be a left $A$-module generated by symbols $\partial_{i,j}$ subject to the defining relations (1). We have the commutative diagram of left $A$-modules:

$$
\begin{array}{ccc}
\mathrm{Der}(A) & \longrightarrow & \prod_{i \in I_r, j \in J_r} \mathrm{Der}(A) \Delta(i,j) \\
\downarrow & & \downarrow \\
\mathrm{Der}_K(A) & \longrightarrow & \prod_{i \in I_r, j \in J_r} \mathrm{Der}_K(A) \Delta(i,j)
\end{array}
$$

where the horizontal maps are faithfully flat $A$-module monomorphisms as $A = (\Delta(i,j))_{i \in I_r, j \in J_r}$ $(A$ is regular) and the vertical maps are natural $A$-module epimorphisms. By Proposition 2.15, Theorem 2.2 and (1), each epimorphism $\mathrm{Der}(A) \Delta(i,j) \twoheadrightarrow \mathrm{Der}_K(A) \Delta(i,j)$ is an isomorphism. So, the right vertical map must be an isomorphism, and so the left vertical map must be an isomorphism (by faithfully flatness), i.e. $\mathrm{Der}(A) \cong \mathrm{Der}_K(A)$. □

Theorem 2.16 The set $\mathrm{HS}_K(A)$ of higher derivations of the algebra $A$ leaves invariant the Jacobian ideal $a_r$ of the algebra $A$.

Proof of Theorem 1.4 (1 $\Rightarrow$ 2) Theorem 2.10

(2 $\Rightarrow$ 3) Suppose that the $\Delta(A)$-module $A$ is not simple then it contains a proper ideal, say $a$, stable under $\mathrm{HS}_K(A)$. Then $a \Delta(A)$ is a proper ideal of the algebra $\Delta(A)$ since $0 \neq a \Delta(A)(A) \subseteq a$, a contradiction.

(3 $\Rightarrow$ 1) By Theorem 2.16 the Jacobian ideal $a_r$ is a nonzero $\Delta(A)$-submodule of $A$, therefore $A = a_r$ since $A$ is a simple $\Delta(A)$-module. So, $A$ is a regular algebra. □

3 Ring of differential operators on singular irreducible affine algebraic variety

In this Section, we prove Theorem 1.6 (see Proposition 3.3(2)), the local finiteness of the ring $D(A)$ of differential operators on the algebra $A$. 

Lemma 3.1 Let $R = K(x_1, \ldots, x_n)$ be a commutative finitely generated algebra over the field $K$, and $D(R)$ be the ring of differential operators on $R$. Each element $\delta \in D(R)_i$ is completely determined by its values on the elements $x^\alpha$, $\alpha \in \mathbb{N}^n$, $|\alpha| \leq i$.

Proof. It suffices to prove that if an element $\delta \in D(R)_i$ satisfies $\delta(x^\alpha) = 0$ for all $\alpha$ such that $|\alpha| \leq i$ then $\delta = 0$. We use induction on $i$. The case $i = 0$ is trivial: $\delta \in D(R)_0 = R$ and $0 = \delta \cdot 1 = \delta$. Suppose that $i \geq 1$ and the statement is true for all $i' < i$. For each $x_j$, $[\delta, x_j] \in D(R)_{i-1}$ and, for each $x^\alpha$ with $|\alpha| \leq i - 1$, $[\delta, x_j](x^\alpha) = \delta(x_j x^\alpha) - x_j \delta(x^\alpha) = 0$. By induction, $[\delta, x_j] = 0$. Now, for any $x^\alpha$,

$$\delta(x^\alpha) = \delta(x_j x^{\alpha-e_j}) = x_j \delta(x^{\alpha-e_j}) + [\delta, x_j](x^{\alpha-e_j}) = x_j \delta(x^{\alpha-e_j}) = \cdots = x^\alpha \delta(1) = 0,$$

and so $\delta = 0$, as required. $\square$

Let $S_1$ be a multiplicatively closed subset of the algebra $A$. Let us consider a natural inclusion $D(A) \subseteq S_1^{-1}D(A)$ of filtered algebras (by the total degree of derivations). $D(A) = \{ \delta \in S_1^{-1}D(A) | \delta(A) \subseteq A \}$ and $D(A)_i = \{ \delta \in S_1^{-1}D(A)_i | \delta(A) \subseteq A \}$, $i \geq 0$.

Lemma 3.2 Let $\delta \in S_1^{-1}D(A)_i$. Then $\delta \in D(A)$ iff $\delta(x^\alpha) \in A$ for all $\alpha$ such that $|\alpha| \leq i$.

Proof. ($\Rightarrow$) Trivial.

($\Leftarrow$) We use induction on $i$. When $i = 0$, $\delta \in S_1^{-1}D(A)_0 = S_1^{-1}A$ and $\delta = 1 = \delta(1) = 1 \in A$. Suppose that $i \geq 1$ and the statement is true for all $i' < i$. Let $\delta \in S_1^{-1}D(A)_i$, satisfy $\delta(x^\alpha) \in A$ for all $\alpha$ such that $|\alpha| \leq i$. For each $j$, $[\delta, x_j] \in S_1^{-1}D(R)_{i-1}$ and, for each $x^\alpha$ with $|\alpha| \leq i - 1$, $[\delta, x_j](x^\alpha) = \delta(x_j x^\alpha) - x_j \delta(x^\alpha) \in A$, and so, by induction, $[\delta, x_j] \in D(A)_{i-1}$. Now, for any $x^\alpha$,

$$\delta(x^\alpha) = \delta(x_j x^{\alpha-e_j}) = x_j \delta(x^{\alpha-e_j}) + [\delta, x_j](x^{\alpha-e_j}) \equiv x_j \delta(x^{\alpha-e_j}) \mod A$$

$$\equiv \cdots \equiv x^\alpha \delta(1) \equiv 0 \mod A,$$

and so $\delta \in D(A)_i$. $\square$

Proposition 3.3 Let $i = (i_1, \ldots, i_r) \in I_r$, $j = (j_1, \ldots, j_r) \in J_r$, $\{1, \ldots, n\} \backslash \{j_1, \ldots, j_r\} = \{j_{r+1}, \ldots, j_n\}$, and we keep the notation of Theorem 2.6. Then

1. For each $i \geq 0$, $D(A)_i = \{ \delta \in \sum_{|k| \leq i} A \delta[k] | \delta(x^\beta) \in A$ for all $\beta \in \mathbb{N}^n$ such that $|\beta| \leq i \}$

where $k = (k_{r+1}, \ldots, k_n) \in \mathbb{N}^{n-r}$ and $\delta[k] := \prod_{s=r+1}^{n} \delta_{k_{i_{s}}, j_{i_{s}}}^{[k_s]}$.

2. For each $i \geq 0$, $D(A)_i$ is a finitely generated left $A$-module.

Proof. 1. Clearly, $D(A)_i = \{ \delta \in \sum_{|k| \leq i} A \delta[k] | \delta(A) \subseteq A \}$. Then, by Lemma 3.2, $D(A)_i = \{ \delta \in \sum_{|k| \leq i} A \delta[k] | \delta(x^\beta) \subseteq A$ for all $\beta \in \mathbb{N}^n$ such that $|\beta| \leq i \}$. Then the conditions that $\delta(x_{j_{1+r}}, \ldots, x_{j_{n}}) \in A$ for all $\gamma = (\gamma_{r+1}, \ldots, \gamma_{n}) \in \mathbb{N}^{n-r}$ with $|\gamma| \leq i$ are equivalent to $\delta \in \sum_{|k| \leq i} A \delta[k]$. This gives the first statement.

2. $D(A)_i$ is the Noetherian left $A$-module as a submodule of the Noetherian $A$-module $\sum_{|k| \leq i} A \delta[k]$, and so $D(A)_i$ is a finitely generated left $A$-module. $\square$
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