On one method of finding optimal control in construction industry

T. Titova
Moscow State University of Civil Engineering, Yaroslavskoe shosse, 26, Moscow, 129337, Russia
E-mail: tat_titova@mail.ru

Abstract. In this paper we consider options for solving optimal control problems with a quadratic quality criterion. There are optimal management tasks in the construction industry also. For example, in optimizing the construction technology of large complexes, construction work cycles at the facilities. A quadratic quality criterion leads to Riccati matrix differential equations that are generally not solvable in quadratures. In this paper we consider solution options in the case of a stationary system. We find a symmetric solution of algebraic matrix Riccati equation. We offer some new analytical methods for solving this nonlinear equation in various cases.

1. Introduction
When solving many problems in various fields of technology, it is necessary to obtain optimal process control. We get such tasks in the construction industry when optimizing the construction work of large complexes, managing human and material resources. Let \( \mathbf{x} \) be the vector of input construction objects, \( \mathbf{y} \) be the labor productivity vector, and \( u(y) \) be construction management. We consider the optimal control problem:

\[
\frac{d\mathbf{x}}{dt} = \mathbf{B}_1 \mathbf{y} + \mathbf{S} u
\]

by quadratic quality criterion

\[
J = \int_{0}^{\infty} (\mathbf{y}^T \mathbf{A}_1 \mathbf{y} + u^T \mathbf{R}_1 u) dt,
\]

where \( \mathbf{A}_1, \mathbf{B}_1 \) are square matrices of order \( n \); \( \mathbf{R}_1 \) is square matrix of order \( m \); \( \mathbf{S} \) is \( n \)-by-\( m \) matrix; \( \mathbf{A}_1, \mathbf{R}_1 \) are symmetric matrices; \( \mathbf{y} \) is \( n \)-dimensional vector; \( \mathbf{u} \) is \( m \)-dimensional vector. In this problem, it is required to find the control \( \mathbf{u} = f(\mathbf{y}) \), which minimizes the functional specified above. The quality criterion is fundamental in solving optimal control problems. In the case of a linear structure of this criterion, there are no difficulties. In the case of a quadratic quality criterion, the solution of the optimal control problem leads to the Riccati differential matrix equation. In the paper [1], it is proved that the Riccati matrix differential equation in the general case is not solvable in quadratures and has movable singular points. The method for solving such equations [2 - 6] is also successfully used for other nonlinear differential equations. In the article, we consider a stationary version of the optimal
control problem corresponding to a constant number of input objects. In this case, the Riccati differential matrix equation is reduced to the Riccati algebraic matrix equation. The minimal value of a functional is reached by the control \[ u = -R^{-1}(S)^TXy, \]
where \( X \) is a symmetric solution of matrix Riccati equation:
\[ XB_1 + (B_1)^TX - XSR^{-1}S^TX + A_1 = O \]

Let us introduce the following designations:
\[ A = -A_1, B = -(B_1)^T, C = SR^{-1}S^T \]

We obtain the following matrix Riccati equation:
\[ XCX + XB^T + BX + A = O \]
(1)

Further we also consider the Hamiltonian \( 2n \)-by-\( 2n \) matrix corresponding to this equation:
\[ V = \begin{pmatrix} B^T & C \\ -A & -B \end{pmatrix} \]
(2)

In the papers [13, 14] the matrix Riccati equation solution is proposed, which is based on the finding the eigenvectors and generalized eigenvectors of the corresponding Hamiltonian matrix. The matrix is suggested as nonsingular. In the paper [15], another solution is proposed, which is not based on the finding the eigenvectors of the Hamiltonian matrix. The authors impose limitation on the Hamiltonian matrix: its eigenvalues are different and should not lie on the imaginary axis of a complex plane. In the paper [16], the technique for finding the symmetric positive definite solution to the matrix Riccati equation is set forth, on condition that such solution exists. The existence of the solution was proved with the additional requirements: the matrix \( C \) is positive semi-definite, the basic system is uniformly completely controllable and uniformly completely observable. The technique is based on the factorization of the Hamiltonian matrix’s characteristic polynomial. In the papers [17 - 19], the research for the solution of the matrix Riccati equation is carried out, which is related to the question of controllability and observability. The different numeric solution techniques for the matrix Riccati equation (1) also exist [10, 20 - 22]. In this paper, we propose a new version of the analytical method for solving the matrix Riccati equation (1).

2. Solution of the matrix Riccati equation
Let us consider the matter of finding the symmetric solution of the matrix Riccati equation (1) with certain limitation on the Hamiltonian matrix (2).

Case 1. Suppose the Hamiltonian matrix (2) has no more than one pair of zero eigenvalues. Suppose \( \pm \lambda_i, i = 1, n \), are the matrix eigenvalues. Then it can be transformed to the following normal form [7, 8]:
\[ \Phi = \begin{pmatrix} U & I \\ O & -U^T \end{pmatrix}, \]
(3)
where
\[ U = \begin{pmatrix} \lambda_1 & \varepsilon_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \varepsilon_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & \varepsilon_{n-1} \\ 0 & 0 & 0 & \ldots & \lambda_n \end{pmatrix}, \]
\[ I = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & \varepsilon \end{pmatrix}, \]
Suppose \( T \) is a transforming matrix, reducing Hamiltonian matrix \( V \) to a normal form (3):

\[
T = \begin{pmatrix} F & H \\ G & W \end{pmatrix}, \quad T^{-1}VT = \Phi,
\]

where \( \det F \neq 0 \). Then the solution of the matrix Riccati equation (1) has the following form [7, 8]:

\[
X = GF^{-1}
\]

(4)

The symmetry of solution is proved in the papers [7, 8] only for nonsingular matrix \( V \). Let's prove, that obtained solution (4) is symmetric, provided that one pair of zero eigenvalues is available. In fact, on the basis of the equation \( VT = T\Phi \) we obtain the following equation system for the matrices \( F, G \):

\[
\begin{align*}
B^T F + CG &= FU \\
AF + BG &= -GU
\end{align*}
\]

(5)

Let's multiply the first and the second equations of the obtained system on the left respectively by \( G^T \) and \( F^T \) and then sum it up. As a result we obtain the following equation:

\[
G^T B^T F + F^T B G + F^T A F + G^T C G = (G^T F - F^T G) U
\]

(6)

Assume, that \( Q = G^T F - F^T G \). Note, that \( Q \) is a skew-symmetric matrix. In the left part of the equation (6) there is a symmetric matrix. Therefore, the matrix \( (G^T F - F^T G) U = QU \) is also symmetric, and, consequently, satisfies the following equation:

\[
QU + U^T Q = 0.
\]

In such case, the entries of the matrix \( Q = (q_{ij}) \) satisfy the following equation system:

\[
(\lambda_i + \lambda_j)q_{ij} + \epsilon_i \cdot q_{i-1,j} + \epsilon_j \cdot q_{i,j-1} = 0
\]

\[
q_{ij} = -q_{ji}, \quad \epsilon_0 = q_{0,0} = 0; \quad i, j = 1, n
\]

If we exclude the last equation from this system, then we obtain the system of linear homogenous equations with the determinant:

\[
\prod_{2 \leq i+j < 2n} (\lambda_i + \lambda_j) \neq 0
\]

Therefore, all \( q_{ij} = 0 \) if \( i+j < 2n \). However, \( q_{nn} \) also equals zero, since the matrix \( Q \) is skew-symmetric. Therefore we proved, that

\[
Q = G^T F - F^T G = 0
\]

(7)

From the equation (7) it follows that the matrix \( X = GF^{-1} \) is a symmetric solution of the matrix Riccati equation (1). So we have spread the formula, obtained in the papers [13, 14], in such case, when Hamiltonian matrix (2) can have one pair of zero eigenvalues. Such case occurs, for example, in the problem of stabilizing an airplane’s flight at a given height for some values of the parameters included in the system [11].

Case 2. Let’s consider the equation (1) if \( C = E \):

\[
X^2 + XB^T + BX + A = 0
\]

(8)
From the first system’s equation (5) we express the matrix \( G = -B^TF + FU \). We substitute the matrix \( G \) in the formula (4) and in the second equation of the system (5). We obtain:

\[
X = -B^TF + FU^{-1} \tag{9}
\]

\[
(A - BB^T)F + (B - B^T)FU + FU^2 = O \tag{10}
\]

Having determined the matrix \( F \) from equation (10), we obtain the solution of the matrix Riccati equation in the form (9). Using a symmetric solution of this equation, one can find a transformation which normalizes the Hamiltonian system with a quadratic Hamiltonian \( [23 - 25] \). This problem arises when we investigate the stability of system solutions.

3. The problem of ballistic entry of a satellite into the atmosphere

Let us consider the second application of the theory of optimal control for the problem of the ballistic entry of a satellite into the atmosphere, followed by the landing of an artificial satellite. Ballistic entry is a flight of a satellite in the upper atmosphere along a path close to circular. The task of aerodynamic drag is to reduce, as far as possible, small deviations from a circular path. Let us introduce the following designations: \( r \) is the Earth’s radius; \( y \) is the satellite’s height above the Earth’s surface; \( \theta \) is an angle between the tangent to the satellite’s trajectory and the tangent to a circle, passing through the given point of the trajectory and having the center in the Earth’s center; \( Q \) is a resistance, which is regulated by changing the area of the aerodynamic brake.

The program motion is circular to radius \( r + y \) with the velocity of \( v = \sqrt{g(r + y)} \). In such case we assume, that the program angle is sufficiently small, \( \theta \approx -1^\circ \), the program velocity \( v \approx 7.800 \text{ m/s} \), the total change of altitude is \( |\Delta y| < 30 \text{ km} \), which is negligible in comparison to \( r \).

In this case, the equations of perturbed motion have the form [11]:

\[
\Delta \dot{y} = -g \Delta \theta - g \Delta \left( \frac{Q}{mg} \right)
\]

\[
\Delta \dot{\theta} = \frac{2 \Delta \nu}{r + y}
\]

\[
\Delta \dot{y} = \nu \Delta \theta
\]

Let’s take a control function:

\[
\xi = \Delta \left( \frac{Q}{mg} \right)
\]

Having excluded \( \Delta \nu \) and \( \Delta \theta \), we obtain the following equation:

\[
\Delta \ddot{y} = \frac{2g}{r + y} \Delta \dot{y} - \frac{2vg}{r + y} \xi
\]

Let’s introduce new variables and a new control function:

\[
y_1 = \Delta y; \quad y_2 = \Delta \dot{y}; \quad y_3 = \Delta \ddot{y}; \quad u = -\frac{2g}{r + y} \Delta \dot{y} - \frac{2vg}{r + y} \xi
\]

In this case, we obtain the following system of equations:

\[
\dot{y}_1 = y_2; \quad \dot{y}_2 = y_3; \quad \dot{y}_3 = u
\]

It is necessary to find the control \( u = f(y) \), which minimizes the functional:
In this case, the control function is:
\[ u = - S^T \mathbf{y}, \]  
where \( X \) is a symmetric solution of the matrix Riccati equation:
\[ -XSS^T X + XB_1 + (B_1)^T X + A_1 = O \]  
In the equation (12)
\[ S = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} \]  
Note, that in the equation (12) the matrix \( SS^T \) is singular, and the matrix \( A_1 \) is positive definite. Multiply the equation (12) on the left and on the right by the matrix \( X^{-1} \). Then, we replace
\[ Y = \sqrt{A_1}X^{-1}\sqrt{A_1}, \quad B = \sqrt{A_1}B_1(\sqrt{A_1})^{-1}, \quad A = -\sqrt{A_1}SS^T \sqrt{A_1}. \]  
We obtain the following matrix Riccati equation of the form (8):
\[ Y^2 + YB^T + BY + A = O \]  
After finding nonsingular solution of the last equation, we obtain the solution of the equation (12):
\[ X = \sqrt{A_1}Y^{-1}\sqrt{A_1} \]  
We obtain the solution of the equation (13) in accordance with the formula (9):
\[ Y = -B^T + FUF^{-1} \]  
where the matrix \( F \) is a nonsingular solution of the equation (10),
\[ U = \begin{pmatrix} \lambda_1 & \varepsilon_1 & 0 \\ 0 & \lambda_2 & \varepsilon_2 \\ 0 & 0 & \lambda_3 \end{pmatrix} \]
\[ \varepsilon_1, \varepsilon_2 = 0; 1; \lambda_1, \lambda_2, \lambda_3 \] are eigenvalues of the Hamiltonian matrix, corresponding to this problem, \( \lambda_i + \lambda_j \neq 0 \).
In the equation (13)
\[ B = \begin{pmatrix} 0 & \frac{1}{\sqrt{a_2}} & 0 \\ 0 & 0 & \sqrt{a_2} \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & b_1 & 0 \\ 0 & 0 & b_2 \\ 0 & 0 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -a_3 \end{pmatrix} \]  
Let’s introduce the matrix
\[ Q(\lambda) = \lambda E + \lambda(B-B^T) + A - BB^T = \]
The characteristic equation has the following form [23]:
\[
\det(Q(\lambda)) = \lambda^6 - \lambda^4a_3 + \lambda^2a_2 - 1 = 0
\]

After finding solution of the characteristic equation, we obtain the eigenvalues \( \pm \lambda_1, \pm \lambda_2, \pm \lambda_3 \).

Let us write out the detailed solution for the case of different eigenvalues. Furthermore, let it be that \( a_2a_3 \neq 1 \). In this case, the columns of the matrix \( F \) can be found from the following system:
\[
Q(\lambda_i)f_i = 0, \quad i = 1, 2, 3
\]

From this system, we obtain:
\[
f_i = \begin{pmatrix}
\lambda_i b_1 \\
b_i^2 - \lambda_i^2 \\
\lambda_i b_2 (b_i^2 - \lambda_i^2)(\lambda_i^2 - a_3)
\end{pmatrix}, \quad i = 1, 2, 3
\]

As a result
\[
F = (f_1, f_2, f_3), \quad Y = -B^T + FUF^{-1}, \quad X = \sqrt{A_3}Y^{-1}\sqrt{A_3}, \quad u = -S^T X y.
\]

As an example, we find the control by means of the suggested algorithm for the case, when \( a_2 = a_3 = 5 \). We obtain the approximate value of the matrix \( X \):
\[
X \approx \begin{pmatrix}
3.45 & 3.45 & 1 \\
3.45 & 10.9 & 3.45 \\
1 & 3.45 & 3.45
\end{pmatrix}
\]

As a result, the control shall be
\[
u = -S^T X y = -y_1 - 3.45y_2 - 3.45y_3.
\]

4. Conclusions

We suggest the convenient method for solving the matrix Riccati equation under certain restrictions on the Hamiltonian matrix corresponding to this equation. We considered the problem of ballistic entry of a satellite into the atmosphere. The obtained method is applied in the problem of optimizing the system of differential equations of perturbed satellite motion. The results obtained in this paper can be used in the theory of optimal control in the analytical construction of optimal linear systems by a quadratic quality criterion.

Acknowledgments

The author would like to thank associate professor Matseevich T.A. for helpful discussions and valuable comments concerning this work.

References

[1] Orlov V N 2012 Method for the approximate solution of scalar and matrix Riccati differential equations (Cheboksary: Perfectum)

[2] Orlov V N, Kovalchuk O A 2018 Research of one class of nonlinear differential equations of third order for mathematical modelling the complex structures IOP Conference Series: Materials Science and Engineering 365 DOI:10.1088/1757-899X/365/4/042045

[3] Orlov V N, Kovalchuk O.A., Linnik E.P. and Linnik I.I. 2018 Research into a Class of Third-
Order Nonlinear Differential Equations in the Domain of Analyticity *Herald of the Bauman Moscow State Tech. Univ.* - *Nat.Sci.* 4 pp 24–35 DOI: 10.18698/1812-3368-2018-4-24-35

[4] Orlov V N, Kovalchuk O A 2018 Mathematical modeling of complex structures and nonlinear differential equations with movable points *IOP Conf. Series: Materials Science and Engineering* 456 012122 DOI:10.1088/1757-899X/456/1/012122

[5] Orlov V N, Kovalchuk O A 2019 Mathematical problems of reliability assurance the building constructions *E3S Web Conf.* 97 03031 DOI:https://doi.org/10.1051/e3sconf/20199703031

[6] Orlov V N 2019 Features of mathematical modelling in the analysis of console-type structures *E3S Web Conf.* 97 03036 DOI:https://doi.org/10.1051/e3sconf/20199703036

[7] Bryson A E and Yu-Chi H 1972 *Applied optimal control* (Moscow: Nauka)

[8] Kalman R E and Bucy R S 1961 *New Results in Linear Filtering and Prediction Theory* *Transactions of the ASME — Journal of Basic Engineering* 83 pp 95–107

[9] Athans M S and Falb P L 1968 *Optimal Control* (Moscow: Mashinostroyeniye)

[10] Kwakernaak K, Sivan R 1971 *Linear optimal control systems* (Moscow: Mir)

[11] Letov A M 1969 *Flight Dynamics and Control* (Moscow: Nauka)

[12] Alexandrov V V 2005 *Optimal motion control* (Moscow: Fizmatlit)

[13] Potter J E 1966 *Matrix quadratic solutions* *SIAM Journal on Applied Mathematics* 14 3 pp 496–501

[14] Martensson K 1971 *On the matrix Riccaty equation* *Information Science* 3 1 pp 17–23

[15] Alev F A and Larin V B 2014 *On the algorithms for solving discrete periodic Riccati equation* *Appl. Comput. Math* 13 1 pp 46–54

[16] Bucy R S and Joseph P D. 2005 *Filtering for Stochastic Processes with Applications to Guidance* (AMS Chelsea Publishing Series)

[17] Wimmer H K 1976 *On the algebraic Riccati equation* *Bull. Austral. Math. Soc.* 14 pp 457-461

[18] Willems J C 1971 *Least squares stationary optimal control and the algebraic Riccati equation* *IEEE Trans. Automat Control* AC 16 pp 621–634

[19] Coppel C A 1974 *Matrix quadratic equations* *Bull. Austral. Math. Soc.* 10 pp 377–401

[20] Bini D A, Meini B, Poloni F 2010 *Transforming Algebraic Riccati Equations into Unilateral Quadratic Matrix Equations* *Numer. Math.* 116 pp 553–578

[21] Fan H Y, Chu E K W 2017 *Projected nonsymmetric algebraic Riccati equations and refining estimates of invariant and deflating subspaces* *Journal of Computational and Applied Mathematics* 315 pp 80–86

[22] Larin V B 2009 *Solution of Matrix Equations in Problems of the Mechanics and Control* *Int. Appl. Mech.* 45 8 pp. 847–872

[23] Titova T N 2015 *Properties of Hamiltonian matrices* *Natural and Technical Sciences* 6 84 pp 65 – 66

[24] Bryuno A D and Petrov A G 2006 *Calculation of the Hamiltonian normal form* *Academy of Sciences Reports* 410 4 pp 474 – 478

[25] Titova T N 2017 *On linear Hamiltonian systems* *MATEC Web of Conferences* 106 04014