I. INTRODUCTION

Thermalization is the process towards thermal equilibrium between physical bodies through mutual interaction. It has a central importance in non-equilibrium thermodynamics. Thermalization is, in general, a highly non-linear and complicated process that may not be described by a few parameters. Bodies in thermal contact reach a temperature that maximizes the total entropy. For quantum systems, the following question arises: How can a system achieve an equilibration even though time evolution is linear and unitary and the final stationary state depends on the initial conditions? The eigenstate thermalization [1] is a hypothesis about when and why quantum states will undergo thermalization. The canonical typicality [2] is another approach. For this purpose, one can also use equilibration to dephase quantum states [3]. For the review of the eigenstate thermalization, see [4]. In terms of quantum information language, when the system thermalizes, its quantum state shrinks to a thermal distribution in which no memory of the initial state remains. Therefore, no unitary transformation describes this process even though the total system evolves unitarily. Thus studying thermalization is closely related to the study of quantum decoherence [5–7]. The results in this work will find their applications in the quantum computation [8] naturally.

In this work, we divide the whole body into a small system and the rest, which plays the role of a heat bath. We assume that the heat bath is ideal so that the system’s action on the heat bath does not react to itself. During thermalization, the quantum states of the system entangle to those of the heat bath. This entanglement erases the local memory about the initial conditions and makes the system evolve to equilibrate with the heat bath. This fact reminds us of the essence of thermodynamics, describing the state of a system by using a few parameters that maximizes the total entropy. For quantum systems, the following question arises: How can a system achieve an equilibration thermodynamically.

Thermalization...
Because $\text{tr}\hat{\rho}_I(t) = 1$, the Kraus operators should satisfy: $\sum_p \hat{K}_p^\dagger \hat{K}_p = 1$. Then, one can write the time evolution of the state of the system as

$$\hat{\rho}_I(t) = U_I(t)\hat{\rho}_I U^{-1}_I(t), \quad U_I(t) = U_I(\hat{K}_p(t)),$$

where $U_I(t)$ is an effective time evolution operator. Denoting the effective Hamiltonian for the evolution at time $t$ as $\hat{H}_I(t)$, the time evolution operator $U_I(t)$ can be written as:

$$U_I(t) = \lim_{N \to \infty} \prod_{k=0}^{N-1} e^{-i\hat{H}_I(k \hat{x}) \frac{t}{N}},$$

where the production is in descending order of $k$. Note that $U_I$ needs not be unitary thus $\hat{H}_I$ needs not be Hermitian in general. From the form of time evolution operator $U_I(t)$, the consistency of the relation holds automatically:

$$U_I(t + t_1 + t_2) = U_I(t_2)U_I(t + t_1) = U_I(t_1 + t_2)U_I(t),$$

$$\hat{\rho}_I(t + t_1 + t_2) = \hat{\rho}_I(t + t_1 + t_2) = \hat{\rho}_I(t + (t_1 + t_2)).$$

The density matrix $\hat{\rho}_I$ satisfies a differential equation:

$$\frac{d\hat{\rho}_I}{dt} = -i[\hat{H}_I, \hat{\rho}_I].$$

Here, as a probe to study thermalization, we consider a quantum oscillator undergoing thermalization. A quantum harmonic oscillator is a useful model in a variety of fields in physics due to its simplicity and broad applicability [10–13]. An example of the above interaction Hamiltonian for the combined system is $\hat{H}_{\text{coupling}} = \epsilon m \hat{x} \sum_k \hat{x}_k$, where a set of an infinite number of coupled quantum oscillators approximates a heat bath [10]. There have been many other oscillator models of heat baths [14–16]. Another example of an ideal bath is a massless quantum field coupled to a harmonic oscillator [17]. In this case, the effects of the oscillator on the bath radiate away to infinity.

For the analysis of thermalization, we first approximate the quantum state of the oscillator with a Gaussian-type density matrix. In numerous literature [18, 19], the reduced density matrix of a subsystem which belongs to an infinite system was shown to be described by Gibbs distribution or generalized Gibbs ensemble. Secondly, we use a Liouville-type equation to describe the time evolution of the density matrix.

Henceforth, in this article, we begin with a frequency-modulated harmonic oscillator with the Hamiltonian

$$\hat{H}_{\text{osc}}(t) = \frac{\hat{p}^2}{2m} + \frac{1}{2} m \omega^2(t) \hat{x}^2,$$

where the mass $m$ is constant, but the frequency $\omega$ changes with time.

The paper is organized as follows: In Sec. II, we review the ‘invariant’ prescription in our previous work, which allow us to study nonadiabatic processes of quantum oscillators in a simple, consistent viewpoint. In Sec. III, we introduce an interaction picture that appropriately describes the oscillator undergoing thermalization. This picture enables us to generalize the ‘invariant’ prescription in physically reasonable conditions. In addition to this, we suggest a solvable ansatz equation for the thermal time evolution. In Sec. IV, we solve the ansatz and present an example. In Sec. V, we study the non-equilibrium thermodynamics of the oscillator. We write the corresponding non-equilibrium first law. As a natural result, we show that the time derivative of entropy contributes to the system’s energy. Finally, we summarize results and discuss their implications and possible applications.

II. REVIEW OF THE ERMAKOV-LEWIS-RIESENFELD INVARIANT METHOD AND NONADIABATICITY

In Ref. [20], we studied nonadiabatic thermodynamics of quantum oscillators. There, we restrict our interest only to unitary time evolution and excluded thermalization. We briefly review the method to give readers ideas of generalizing it for thermalization in the next section.

The ELR invariant denotes the invariant suggested by Ermakov, Lewis, and Riesenfeld [21, 22]. The method has been widely used [23, 24]. Later, in Refs. [25–27] the authors developed an equivalent invariant creation/annihilation operator.
A. An ELR-invariant state

Consider and operator $\hat{A}$ and a quantum state given by the initial density matrix $\hat{\rho}_i$. The expectation value of the operator $A$ in the Heisenberg picture is given by

$$\langle \hat{A}(t) \rangle_{\text{Hei}} = \text{Tr} \left[ \hat{\rho}_i \hat{A}_{\text{Hei}}(t) \right].$$

(9)

The time evolution of an operator $\hat{A}_{\text{Hei}}$ for the oscillator (8) is given by

$$\hat{A}_{\text{Hei}}(t) = U_{\text{osc}}^\dagger(t) \hat{A}(0) U_{\text{osc}}(t), \quad U_{\text{osc}} = \lim_{N \to \infty} \prod_k e^{-i\hat{H}_{\text{osc}}(k \frac{\pi}{N}) \frac{\pi}{N}}.$$

(10)

Note that the time evolution operator $U_{\text{osc}}$ is unitary and the Hamiltonian operator $\hat{H}_{\text{osc}}$ is Hermitian.

Because $\hat{\rho}_i$ is invariant (constant) under the Heisenberg picture, we suggested replacing the initial state $\hat{\rho}_i$ with an ELR invariant state $\hat{\rho}_I$ as follows:

$$\hat{\rho}_i = \hat{\rho}_0 e^{-\hat{H}_I/T_0} \to \hat{\rho}_I(t) = \hat{\rho}_0 e^{-\hat{I}_{\text{osc}}((t)/T_0)} = \frac{dt}{\partial t} = \frac{\partial I_{\text{osc}}}{\partial t} + i[\hat{H}_{\text{osc}}, \hat{I}_{\text{osc}}] = 0,$$

(11)

where $T_0$ and $\hat{\rho}_0$ denote the initial temperature and the normalization constant of the state, respectively. The invariant $\hat{I}_{\text{osc}}(t)$ is normalized so that $\hat{I}_{\text{osc}}(0) = \hat{H}_{\text{osc}}(0)$. The invariant $\hat{I}_{\text{osc}}(t)$ has the form [22]:

$$\hat{I}_{\text{osc}}(t) = \hat{g}_- (t) \frac{\hat{p}^2}{2m} + \hat{g}_0 (t) \frac{\hat{p}^2 + \hat{x} \hat{p}}{2} + \hat{g}_+ (t) \frac{\hat{m} \hat{x}^2}{2}.$$

(12)

The invariant condition gives one independent equation for $\hat{g}_-$. In terms of $h$, $\hat{h}^2 = \hat{g}_-(t)$, the equation is written as:

$$\hat{h} = -\omega^2 \hat{h} + \omega^2 \frac{\hat{h}^2}{h^3}.$$

(13)

Given a solution $\hat{g}_-(t)$ to this equation, $\hat{g}_0, \hat{g}_+$ are determined to be

$$\hat{g}_0 = -\frac{\hat{g}_-}{2}, \quad \hat{g}_+ = \frac{\omega^2}{\hat{g}_-} \left( 1 + \frac{\hat{g}_-^2}{4\omega^2} \right).$$

(14)

For a frequency-modulated quantum harmonic oscillator, a function $\hat{g}_-(t)$ is sufficient to determine $\hat{I}_{\text{osc}}(t)$. Hence, the use of the invariant state gives clear advantages over traditional analysis for studying nonadiabatic, time-dependent processes because the knowledge of a scalar function $\hat{g}_-(t)$ determines the whole time evolution of all operators.

The invariant $\hat{I}_{\text{osc}}$ has an equivalent form [25]:

$$\hat{I}_{\text{osc}}(t) = \omega_I \left[ \hat{b}_I(t) \hat{b}_I(t) + \frac{1}{2} \right],$$

(15)

where $\omega_I$ is an initial frequency, the annihilation operator $\hat{b}_{\text{osc}}(t)$ is given by

$$\hat{b}_I \equiv e^{i\theta_I(t)} \hat{b}_I(t), \quad \hat{b}_I(t) \equiv \left( 1 + \frac{\hat{g}_0}{\omega_I} \right) \sqrt{\frac{m \omega_I}{2g_-}} \hat{x}(t) + i \sqrt{\frac{g_-}{2m \omega_I}} \hat{p}(t),$$

(16)

and the creation operator $\hat{b}_I(t)\dagger$ is given by the adjoint operation. The invariant $\hat{I}_{\text{osc}}(t)$ differs from the Hamiltonian $\hat{H}_{\text{osc}}$ in (8) by

$$\hat{I}_{\text{osc}}(t) - \hat{g}_- \hat{H}_{\text{osc}} = \frac{m \hat{g}_-}{4} \hat{x}^2 - \frac{\hat{g}_-}{4} (\hat{x} \hat{p} + \hat{p} \hat{x}).$$

(17)

Note that $\hat{I}_{\text{osc}}(t) - \hat{g}_- \hat{H}_{\text{osc}} = 0$ in the adiabatic limit.

Based on these operators, we construct the invariant vacuum and the corresponding quantum states

$$\hat{b}_I |0\rangle_I = 0, \quad |n\rangle_I = \frac{1}{\sqrt{n!}} (\hat{b}_I^\dagger)^n |0\rangle_I.$$

(18)

These linear invariants determine the time evolution of the position $\hat{x}(t)$ and the momentum $\hat{p}(t)$ in the Heisenberg picture (9). In this picture, the state $\hat{\rho}_I(t)$ does not evolve. On the other hand, an operator $\hat{A}$ evolves as $\hat{A}(t, p(t), q(t))$ and be spanned to $\sum_{n,m} |n\rangle_I A_{nm}(t)_I|m\rangle$ where $A_{nm}(t) = \langle n|\hat{A}(t, p(t), q(t)) |m\rangle$. 
B. Nonadiabaticity

From the equation (13) we obtained a quantity called a ‘squeezing factor’:

\[ \mathcal{J}_I = \frac{\dot{h}_I^2}{2\omega_I^2} + \frac{1}{2} \left( \frac{1}{\hbar} - \frac{\omega(t)}{\omega_I}\hbar \right)^2. \]  

(19)

This quantity is time-dependent but conserved in adiabatic changes. With an appropriate initial condition, we defined the factor \( \mathcal{J}_I \) as:

\[ \mathcal{J}_I(t) = \frac{1}{2\omega_I^2} \int_{-\infty}^{t} \left( h_I^2(t') - \frac{\omega_I}{\omega} \right) \frac{d\omega^2(t')}{dt'} dt'. \]  

(20)

With the above ‘invariant’ prescription, we could analyze nonadiabatic thermodynamics satisfactorily and defined a measure for ‘nonadiabaticity’, \( \mathcal{A}_I = d\mathcal{J}_I/dt \):

\[ \mathcal{A}_I = \frac{1}{2\omega_I^2} \left( \bar{g}_-(t) - \frac{\omega_I}{\omega} \right) \frac{d\omega^2(t)}{dt}. \]  

(21)

Note that if there are nonadiabatic durations, the effects make differences from the adiabatic results and the information is stored in \( \mathcal{J}_I \).

C. Time evolution of the thermal state

The ‘invariant thermal state’ in (11) is, explicitly,

\[ \hat{\rho}_I = 2 \sinh \frac{\epsilon_I}{2} e^{-\epsilon_I/T_0} = \left( 1 - e^{-\epsilon_I} \right) \sum_{n=0}^{\infty} e^{-n\epsilon_I} |n\rangle_{II} \langle n|, \quad \epsilon_I \equiv \frac{\omega_I}{T_0}, \]  

(22)

where \( T_0 \) denotes the initial temperature. This density introduces a parameter \( \epsilon_I(S) \) as a function of the entropy \( S \) through

\[ S \equiv -\text{Tr}(\hat{\rho}_I \log \hat{\rho}_I) = \frac{\epsilon_I}{e^{\epsilon_I} - 1} - \log[1 - e^{-\epsilon_I}], \]  

(23)

which is a monotonically decreasing function from \( \infty \) to zero of \( \epsilon_I \in [0, \infty) \).

Then, we consider the time evolution of an initial thermal state \( \hat{\rho}_I \propto e^{-H_{\text{osc}}(0)/T_0} \) with temperature \( T_0 \). Since \( \hat{H}_{\text{osc}}(0) = H_{\text{osc}}(0) \), we have \( \hat{\rho}_I = \hat{\rho}_I \). The energy \( E = \text{Tr}[\hat{\rho}_I H_{\text{osc}}(t)] \) for the ‘invariant thermal state’ is time-dependent and has a value:

\[ E = \frac{\omega_{\text{eff}}}{2} \coth \frac{\epsilon_I}{2}, \]  

(24)

where \( \omega_{\text{eff}} \) denotes

\[ \omega_{\text{eff}} \equiv \omega + \Omega_I, \quad \Omega_I \equiv \omega_I \mathcal{J}_I. \]  

(25)

As seen here, the energy is composed of two parts: the original frequency part and the contribution of squeezing.

We also obtained temperatures of the oscillator in a nonadiabatic process:

\[ T(\omega_{\text{eff}}, S) \equiv \left( \frac{\partial E}{\partial S} \right)_{\omega_{\text{eff}}} = \frac{\omega_{\text{eff}}}{\epsilon_I} = \frac{\omega + \Omega_I}{\epsilon_I}. \]  

(26)

In the (adiabatic) limit \( \mathcal{J}_I \to 0 \), the temperature goes to:

\[ T(t) = \frac{T_0}{\bar{g}_-(t)} = \frac{\omega(t)}{\omega_I} T_0, \]  

(27)

showing that the temperature changes with the \( (\bar{g}_-)^{-1} \) factor.
D. Thermodynamics in the presence of a nonadiabaticity

The thermal state depends on parameters $m$, $\omega$ in the Hamiltonian, $\epsilon$ in the density, and $(\omega \tilde{g}_- / \omega_I)$, $\tilde{g}_- / \omega_I$ in the operator $\hat{b}_I$. Fortunately, the mass $m$ does not affect on the thermodynamics and the parameters $(\omega \tilde{g}_- / \omega_I)$ and $\tilde{g}_- / \omega_I$ affect on the energy only through their combination, $\omega I \mathcal{S}_I$. Consequently, the parameter space of thermal states is sufficiently described by the three parameters, $(\omega, \epsilon_I, \omega I \mathcal{S}_I)$.

The $\mathcal{S}_I$-factor plays the role of a thermodynamic quantity similarly to the temperature, the entropy and the energy. A nonadiabatic thermal state $\hat{\rho}_I$ for a quantum harmonic oscillator is given by

$$\hat{\rho}_I \propto \exp \left[-\epsilon_I \left(\hat{b}_I^\dagger \hat{b}_I + \frac{1}{2}\right)\right]. \quad (28)$$

With the parameters $(\omega, \epsilon_I, \omega I \mathcal{S}_I)$, the extended first law of thermodynamics reads as

$$\delta E(\omega_{\text{eff}}, S) = -F_\omega \delta \omega_{\text{eff}} + T \delta S, \quad (29)$$

where $\omega_{\text{eff}}$ is an effective frequency in Eq. (25) and

$$F_\omega(S) = -\frac{\partial E}{\partial \omega_{\text{eff}}} \bigg|_S = -\frac{1}{2} \coth \frac{\epsilon_I}{2}, \quad T(\omega_{\text{eff}}, S) = \frac{\partial E}{\partial S} \bigg|_{\omega_{\text{eff}}} = \frac{\omega_{\text{eff}}}{\epsilon_I} = \frac{\omega + \Omega_I}{\epsilon_I}. \quad (30)$$

E. Physical constraints

The variation of $\omega_{\text{eff}}$ is composed of two parts:

$$\delta \omega_{\text{eff}} = \delta \omega + \delta \Omega_I. \quad (31)$$

The variations $\delta \omega$ and $\delta \Omega_I$ must be related by physical constraints to present a specific thermodynamic process. For example, $\delta \Omega_I = \omega_I \delta \mathcal{S}_I$ in (19) is determined by Eq. (13). As an example, when $\Omega_I$ is just a function of $\omega$ only, the nonadiabaticity is automatically determined as

$$\Omega'_I \delta \omega = \delta \Omega_I \rightarrow A_I = \frac{1}{\omega_I} \frac{d\Omega_I}{dt} = \frac{\dot{\omega}}{\omega_I} \Omega'_I. \quad (32)$$

Putting this into Eq. (31), the variations of effective frequency is

$$\delta \omega_{\text{eff}} = (1 + \Omega'_I) \delta \omega. \quad (33)$$

Conversely, given $\Omega'_I$ at an instance, one can determine the frequency change $\delta \omega$ from $\delta \Omega_I$. The relation (32) plays the role of an equation of state for the oscillator thermal system. For example, one can choose $\Omega'_I$ to satisfy

$$\frac{\omega g}{\omega_I} = 1 + \Omega'_I. \quad (34)$$

As seen here, specifying $\Omega_I$ and $\Omega'_I$ fully determines the nonadiabatic time evolution of an initial thermal state of the oscillator.

III. TIME EVOLUTION IN THERMALIZATION

To analyze thermalization processes, we take advantage of the simplicity of the prescription in the previous section and seek a generalization of the method. After that, we suggest an ansatz equation that describes the time evolution of the oscillator system undergoing thermalization.

A. The interaction picture

As we mentioned earlier, the effects of entanglement between the systems during thermalization give rise to a non-unitary time evolution in general. Therefore, the time evolution of the system is generated by a non-Hermitian
operator $\hat{H}_I(t)$ which effectively describes the whole effects of $\hat{H}_{\text{tot}}$ done on the system $I$. For the effective description of thermalization of the oscillator, we divide the effective Hamiltonian into an oscillator part (Hermitian) and the other part (non-Hermitian):

$$\hat{H}_I = \hat{H}_{\text{osc}} + \Delta H, \quad \Delta H \equiv \hat{H}_I - \hat{H}_{\text{osc}}.$$  \hfill (35)

where $\hat{H}_{\text{osc}}$ is the Hamiltonian for the frequency-modulated harmonic oscillator in Eq. (8). Then, in an interaction picture, the state evolves as

$$\hat{\rho}_{\text{int}} = U_{\text{osc}}^\dagger(t)\hat{\rho}_{\text{Sch}}(t)U_{\text{osc}},$$  \hfill (36)

where the time evolution operator $U_{\text{osc}}$ is given in Eq. (10). The density matrix $\hat{\rho}_{\text{Sch}} = \hat{\rho}_I$ evolves as

$$\hat{\rho}_{\text{Sch}}(t) = U_I(t)\hat{\rho}_I U_I^{-1}(t),$$  \hfill (37)

where $U_I(t)$ is related to $\hat{H}_I$ as in Eq. (5).

An operator in this picture also evolves as

$$\hat{A}_{\text{int}}(t) = U_{\text{osc}}^\dagger(t)\hat{A}_{\text{Hei}}(t)U_{\text{osc}}.$$  \hfill (38)

Since $\hat{A}_{\text{int}} = \hat{A}_{\text{osc}}^{\text{Hei}}$, the expectation value of an operator $\hat{A}$ in this picture can be written as

$$\langle \hat{A}(t) \rangle_{\text{int}} = \text{Tr} \left[ \hat{\rho}_{\text{int}} \hat{A}_{\text{Hei}}(t) \right].$$  \hfill (39)

When $\Delta H \to 0$, this expectation value goes to:

$$\langle \hat{A}(t) \rangle_{\text{Hei}} = \text{Tr} \left[ \hat{\rho}_I \hat{A}_{\text{Hei}}(t) \right],$$  \hfill (40)

where $\hat{\rho}_I$ is the ‘invariant’ density matrix described in the previous section. Note that in the above choice of division based on $\hat{H}_{\text{osc}}$ as in Eq. (35), all the information about thermalization in $\Delta H$ moves to the state $\hat{\rho}_{\text{int}}$. Equipped with the same operators in the previous section, to study thermalization, all we need to know is the time evolution of the state $\hat{\rho}_{\text{int}}$.

B. A prescription in the interaction picture

In most of thermalization processes, the deviation $|\text{det}(\hat{\rho}_{\text{int}} - \hat{\rho}_I)|$ could be large. But when $|\text{det}(\Delta H)|$ is small, $\hat{\rho}_{\text{int}}$ can have a similar functional form to $\hat{\rho}_I$. The resemblance between the two expectation values (39) and (40) inspires us to approximate $\hat{\rho}_{\text{int}}$ to have the same functional form as $\hat{\rho}_I$. Because we loosen the condition for the functions $\tilde{g}_0$ and $\tilde{g}_\pm$, the density matrix $\hat{\rho}_{\text{int}}$ is not invariant anymore.

The generator of the time evolution, $\hat{\rho}_I \to \hat{\rho}_{\text{Sch}}$, is $\hat{H}_I$. Therefore, from Eq. (36), the difference between $\hat{\rho}_{\text{int}}$ and $\hat{\rho}_I$ comes from a commutator $[\hat{H}_{\text{osc}}, \hat{H}_I] = [\hat{H}_{\text{osc}}, \Delta H]$. Since $\hat{H}_{\text{osc}}$ is a quadratic operator of $\hat{x}$ and $\hat{p}$, if $\Delta H$ is quadratic, the resulting density matrix becomes an exponential of a quadratic operator of $\hat{x}$ and $\hat{p}$. In general, the state undergoes continuous squeezing operations and is represented as a Gaussian type density matrix. Even when $\Delta H$ is not quadratic but made of even functions of $\hat{x}$ and $\hat{p}$ only, the Gaussian approximation may work.

In line with this consideration, we suggest a prescription for this case, where we replace the invariant operator $\hat{I}_{\text{osc}}$ for $\hat{\rho}_I$ to a functional operator $\hat{I}$ for $\hat{\rho}_{\text{int}}$ ($\equiv \hat{\rho}_{\text{int}}$):

$$\hat{\rho}_I(t) = \hat{\rho}_0 e^{-\hat{I}_{\text{osc}}(t)/T_0} \to \hat{\rho}_I = \hat{\rho}_0 e^{-\hat{I}/T_0},$$  \hfill (41)

and the ansatz for the operator $\hat{I}(t)$ with the same functional form as $\hat{I}_{\text{osc}}$:

$$\hat{I}(t) = g_-(t) \hat{p}^2 + 2m \left[ g_0(t) \frac{\hat{p} \hat{x} + \hat{x} \hat{p}}{2} + g_+(t) \frac{m \hat{x}^2}{2} \right].$$  \hfill (42)

where $g_0(t)$ and $g_\pm(t)$ are temporal functions. For the case of the ELR invariant $\hat{I}_{\text{osc}}$, only $\tilde{g}_-$ is independent. However, at the present case, more general time dependencies for $g_0$ and $g_\pm$ are allowed to describe thermalization process. Thus, $\hat{I}(t)$ is not an invariant anymore.
One can rewrite the operator $\hat{I}(t)$ equivalently as

$$\hat{I}(t) = \omega_I(t) \left( \hat{b}^\dagger(t)\hat{b}(t) + \frac{1}{2} \right),$$  \hspace{1cm} (43)$$

where $\omega_I$ denotes the ELR-like frequency and $\hat{b}/\hat{b}^\dagger$ denotes a canonical annihilation/creation operator:

$$\hat{b}(t) = i\sqrt{\frac{g_-}{2m\omega_I}} \hat{p} + \left(1 + \frac{ig_0}{\omega_I}\right) \sqrt{\frac{m\omega_I}{2g_-}} \hat{x}, \quad \hat{b}^\dagger(t) = -i\sqrt{\frac{g_-}{2m\omega_I}} \hat{p} + \left(1 - \frac{ig_0}{\omega_I}\right) \sqrt{\frac{m\omega_I}{2g_-}} \hat{x},$$  \hspace{1cm} (44)$$

with $[\hat{b}(t), \hat{b}^\dagger(t)] = 1$ by taking the form from the ELR invariant $\hat{I}(t)$. Note that $\omega_I$ replaces $\omega_J$. One can construct a complete set of basis $\{|n\rangle_I\}$ from the vacuum state $|0\rangle_I$ satisfying $\hat{b}(t)|0\rangle_I = 0$. Based on this state, the natural thermal state $\hat{\rho}_I$ in Eq. (41) can be constructed as

$$\hat{\rho}_I \propto \exp \left[ -\epsilon \left( \hat{b}^\dagger\hat{b} + \frac{1}{2} \right) \right].$$  \hspace{1cm} (45)$$

C. The form of the effective Hamiltonian

For $\hat{H}_{\text{tot}} \equiv \Sigma_{ij} \hat{H}^{(1)}_i \otimes \hat{H}^{(2)}_j$, the time evolution of a density matrix is written as

$$\hat{\rho}_I(t + \delta t) = \text{tr}_B \left[ e^{-i\hat{H}_{\text{tot}}\delta t} (\hat{\rho}_I(t) \otimes \hat{\rho}_B) e^{i\hat{H}_{\text{tot}}\delta t} \right] \approx \text{tr}_B \left[ (\hat{\rho}_I(t) \otimes \hat{\rho}_B) - i\delta t \left[ \Sigma_{ij} \hat{H}^{(1)}_i \otimes \hat{H}^{(2)}_j, (\hat{\rho}_I(t) \otimes \hat{\rho}_B) \right] \right] \approx \hat{\rho}_I(t) \left\{ 1 - i\delta t [\hat{I}(t), \hat{H}_I]/T_0 \right\} \approx \hat{\rho}_0 e^{-i\hat{I}(t) - i\delta t [\hat{H}_I, \hat{I}(t)]}/T_0,$$  \hspace{1cm} (46)$$

up to the first order in $\delta t$ and where $\hat{H}_I$ is defined

$$\hat{H}_I \equiv \Sigma_{ij} \left[ \text{tr}_B (\hat{H}^{(2)}_j \hat{\rho}_B) \right] \hat{H}^{(1)}_i.$$  \hspace{1cm} (47)$$

Then, from Eq. (46), we have

$$\hat{\rho}_I(t + \delta t) \sim e^{-i\hat{I}(t+\delta t)/T_0}, \quad \hat{I}(t + \delta t) \equiv \hat{I}(t) - i\delta t [\hat{H}_I, \hat{I}(t)],$$  \hspace{1cm} (48)$$

up to the first order in $\delta t$ and $\epsilon(t)$. It is possible that a higher order contributions may grow non-linearly and spoil this result. In this work, we confine ourselves to the linear regime. Time evolution of $\hat{I}(t)$ can be read off formally from (48) as

$$i \frac{d\hat{I}(t)}{dt} = [\hat{H}_I, \hat{I}(t)],$$  \hspace{1cm} (49)$$

where $\hat{H}_I$ is given in Eq. (47).

D. Minimal ansatz for the time evolution equation

The time evolution of the system undergoing thermalization is determined by the equation (49). Thus, $\hat{H}_I$ plays the role of a Hamiltonian operator for $\hat{I}(t)$. Hence, in order to solve $\hat{I}(t)$, we need to know the terms $\text{tr}_B (\hat{H}^{(2)}_j \hat{\rho}_B)$ in (47), which requires exact information about the heat bath. However, in many cases, we do not need to know the exact information. This is because an ideal heat bath will re-organize itself in a subtle way so that the action of the system to the bath may not react on it. This lack of information precludes us from solving the time evolution of the system in general. Although the condition imposed on $\Delta H$ discussed earlier greatly simplifies the problem, it is not enough to solve it practically.
Therefore, we need to have a solvable functional form \( D(\hat{H}_I, \hat{I}) \) which effectively represents the commutator \([\hat{H}_I, \hat{I}]\):

\[
i\frac{d\hat{I}(t)}{dt} \equiv D(\hat{H}_I, \hat{I}). \tag{50}
\]

The functional \( D \) vanishes when \( \Delta H \to 0 \). That is, because \( \Delta H \to 0 \) leads to \( \hat{I}(t) \to \hat{I}_{osc}(t) \), we have

\[
i\frac{d\hat{I}(t)}{dt} \to i\frac{d\hat{I}_{osc}(t)}{dt} = 0. \tag{51}
\]

As we notice in Eq. (17), \( \hat{I}_{osc} - \bar{g}_- \hat{H}_{osc} = 0 \) in the adiabatic limit. Thus a plausible candidate for \( D \) can be:

\[
D(\hat{H}_I, \hat{I}(t)) = D(\hat{I} - g\hat{H}_{osc}), \tag{52}
\]

where \( g \) is a function that \( g \to \bar{g}_- \) in the limit.

As we noted before, in our interaction picture, any physical operators are the same as those used in the frequency-modulated oscillator. Hence, we present a solvable ansatz in a minimal form:

\[
D(\hat{H}_I, \hat{I}(t)) = -\alpha \left( \hat{I}(t) - g\hat{H}_{osc}(t) \right). \tag{53}
\]

Here \( g \) is interpreted as the temperature scale similar in Eq. (27). The quantity \( \alpha \) represents how fast thermalization happens and has the dimension of frequency. Explicitly,

\[
\frac{d\hat{I}(t)}{dt} = \frac{\partial \hat{I}(t)}{\partial t} - i[\hat{H}_{osc}, \hat{I}(t)] = -\alpha \left( \hat{I}(t) - g\hat{H}_{osc}(t) \right), \tag{54}
\]

where, \( \partial \hat{I}/\partial t \) represents the explicit time derivative of \( \hat{I} \). The purpose of this equation is to describe thermalization effectively as a time evolution of globally measurable parameters. Later just below Eq. (72) in Sec. VB, we show that the terminal value of the function \( g(t) \) is determined by the ratio between the initial temperature of the oscillator and the temperature of the bath

\[
g = \frac{T_0}{T_{bath}}. \tag{55}
\]

E. An interpretation

The ansatz is ‘minimal’ in the sense that the equation above is self-sufficient to solve without the knowledge on the exact form of \( \hat{H}_I \) (or \( \Delta H \)). Nevertheless, it is sufficient for us to probe generic properties that emerge in thermalization.

The state \( \hat{\rho}_I \) is a scaled/squeezed state of a ‘Hamiltonian thermal state’:

\[
\hat{\rho}_H \propto e^{-\beta(t)\hat{H}_{osc}(t)}. \tag{56}
\]

Here \( \beta(t) \) is an unspecified inverse temperature which contains the information of the temperature changes from the initial to that of the thermal bath. The difference \( \Delta \hat{I}(t) \equiv \hat{I}(t) - g(t)\hat{H}_{osc}(t) \), where \( g(t) \equiv \beta(t)T_0 \), contributes to the scaling/squeezing.

In the thermodynamic limit, it is well known that the density matrix, \( \hat{\rho}_I \), will be described by the diagonal elements of the Hamiltonian eigenstates [3]. It is to be noted that the operator \( \hat{I}(t) \) will evolve toward the Hamiltonian with time because of thermalization. An instability may appear when \( \omega_I \leq 0 \) at any moment, because the density matrix could have negative eigenvalues. Therefore, we restrict our attentions to the case \( \omega_I > 0 \) always.

IV. EFFECTIVE DESCRIPTION OF THERMALIZATION

Let us solve the ansatz equation (54) by using the relation (42). The equation gives a set of differential equations for \( g_k \):

\[
\dot{g}_- = -2g_0 - \alpha(g_- - g), \quad \dot{g}_0 = \omega^2 g_- - g_+ - \alpha g_0, \quad \dot{g}_+ = 2\omega^2 g_0 - \alpha(g_+ - \omega^2 g). \tag{57}
\]
Given \( g, \alpha \) and \( \omega(t) \), the equation presents a whole time evolution \( \{g_k(t), k = 0, \pm \} \) from a set of initial data \( \{g_k(0), k = 0, \pm \} \). Once we get \( g_k \), the quantum state at time \( t \) takes the form,

\[
\hat{\rho}(t) = 2 \sinh \left( \frac{\epsilon(t)}{2} e^{-\hat{I}(t)/T_0} \right), \quad \epsilon(t) \equiv \frac{\omega_I(t)}{T_0}. \tag{58}
\]

Note that the operator \( \hat{I} \) evolves with time as thermalization goes on, but \( T_0 \) does not.

In reality, thermalization happens when the oscillator contacts a heat bath of different temperatures. The energy exchange with the bath modifies both the effective frequency and the entropy of the oscillator in general. The change of the effective frequency induces the change of the temperature too. Therefore, the time-dependence of frequency \( \omega(t) \) must not be a prerequisite but be determined through some equation of state. We cover these thermodynamic changes in the next section. In this section, we observe how the temperature changes when the frequency varies in a predetermined way. For this purpose, we solve Eq. (54) explicitly. Note that when \( g_k \) are independent of time with \( \omega = \omega_0 \), they satisfy

\[
g_- = g, \quad g_0 = 0, \quad g_+ = \omega_0^2 g_- . \tag{59}
\]

In this case, we have \( \omega_I^2 = \omega_0^2 g_-^2 \). If we choose the mode solution as a simple oscillation of the form \( f \propto e^{-i\omega_0 t} \), we get \( \omega_I = \omega_0 \) with \( g_- = 1 \). We determine the time evolution of \( g_k \) numerically. We present solutions for the case \( g(t) = g_-(t) \) in Fig. 1 with \( \alpha = 0.1 \) and 0.5. The frequency squared increases monotonically from \( \omega_0 \) to \( \sqrt{2}\omega_0 \). Note that the value of \( g_-^{-1} \) deviates from that without thermalization, \( \omega(t)/\omega_0 \), but goes to \( \omega(t)/\omega_I(t) \).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1}
\caption{The time evolution of \( g_- \) for \( \omega^2(t) = \omega_0^2 [1 + \tanh(\omega_0 t)] \) for \( t > 0 \) and \( \alpha = 0.1 \) (L) and 0.5 (R). The blue, the black, and the red curves correspond \( \omega(t)/\omega_0 \), \( 1/g_- \), and \( \omega(t)/\omega_I(t) \), respectively.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig2}
\caption{The time evolution of \( \mathcal{S} \) for \( \omega^2(t) = \omega_0^2 [1 + \tanh(\omega_0 t)] \) for \( t > 0 \) and \( \alpha = 0.1 \) (L) and 0.5 (R).}
\end{figure}

The ‘squeezing’ factor \( \mathcal{S} \) for the state (58) that generalizes \( \mathcal{S}_I \) in (19) is written as,

\[
\mathcal{S} \equiv \frac{g_0^2}{2\omega_I^2 g_-} + \frac{1}{2} \left( \frac{1}{\sqrt{g_-}} - \frac{\omega}{\omega_I} \sqrt{g_-} \right)^2 ,
\tag{60}
\]
where $\omega_I$ and $g_0$ replace the invariant frequency $\omega_f$ and the time derivative $-\dot{g}_-/2$, respectively. Here, $\omega_f$ and $-\dot{g}_-/2$ are the corresponding values without thermalization. Because of the differential relation (57), $g_0 = 0$ does not lead to $\dot{g}_- = 0$ unless $g_- = g$. The time evolution of the nonadiabaticity for the models in Fig. 1 is plotted in Fig. 2. As shown in the figure, the quick change of the frequency makes the nonadiabaticity increase. Asymptotically, the nonadiabaticity decreases to zero because of thermalization.

Note that $J$-factor defined above generalizes $J_1$-factor to include thermalization. Hence, unlike $J_1$, it must be different from the $Q$-factor of Husimi [28] or the parameter $Q^*$ in the irreversible work [29] because the definition works even in the presence of thermalization.

V. THE NON-EQUILIBRIUM THERMODYNAMICS OF A HARMONIC OSCILLATOR

In this section, we consider the thermodynamics of thermalization of a harmonic oscillator thermal state (45). We derive the non-equilibrium first law of thermodynamics for the oscillator.

A. Non-equilibrium thermodynamics

The density matrix and the Hamiltonian have three independent parameters $(m, \omega, \epsilon)$, two from the Hamiltonian and one from the density matrix, respectively. Here, the creation/annihilation operator $\hat{b}$ and $\hat{b}^\dagger$ given in Eq. (44) have two additional independent parameters $(\omega g_-/\omega_I)$ and $(g_0/\omega_I)$. The energy expectation value of the thermal state takes the same form as that in Eq. (24) with the effective frequency, $\omega_{\text{eff}} \equiv \omega + \Omega_I$, where $\Omega_I \equiv \omega_I J$. Fortunately, the two parameters $(\omega g_-/\omega_I)$ and $(g_0/\omega_I)$ contribute to the energy expectation value only through $\Omega_I$. Following the same reasoning as in Sec. IID, the mass does not contribute to the thermodynamics. Therefore, the thermal quantum state is described by the parameters $(\omega, S, \Omega_I)$.

Because the energy is formally the same as Eq. (24), the thermodynamics will also be described by the same manner once we introduce an effective frequency $\omega_{\text{eff}}$. The variation of the energy gives

$$\delta E(\omega_{\text{eff}}, S) = -F_\omega(S) \delta \omega_{\text{eff}} + T(\omega_{\text{eff}}, S) \delta S,$$

where the force $F_\omega$ and the temperature $T$ are the same as those in Eq. (30) with the replacement $\Omega_I \to \Omega_S$.

A crucial point of thermalization distinguished from the thermodynamics in Sec. IID is that the ELR frequency $\omega_I$ is not invariant any more but changes with time. This naturally implies that the thermodynamics must be dependent on the ELR frequency. Let us first observe how the dependence happens. The time-derivative of $\omega_I$ is calculated in Eq. (A1) in the appendix to give

$$\delta \omega_I = \alpha [g \omega_{\text{eff}} - \omega_I] \delta t = \alpha \omega_I \left( \frac{T}{T_{\text{bath}}} - 1 \right) \delta t.$$

This relation shows how/why the ELR frequency varies with time. The ELR frequency $\omega_I$ is modified by the thermalization ($\alpha \neq 0$). In the subsequent discussions, we show how the change contributes to the first law of thermodynamics. $\omega_I$ increases/decreases when $T \gtrless T_{\text{bath}}$, which signifies what happens during thermalization.

Now, we observe the first law in Eq. (61). The variation of the effective frequency becomes

$$\delta \omega_{\text{eff}} = \delta \omega + \delta \Omega_I.$$

In the absence of thermalization, the time evolution is achieved by specifying an equation of state $\Omega'_I = (\partial \Omega_I / \partial \omega)$ as in Eq. (33) which relates $\delta \omega$ and $\delta \Omega_I$ and determines $\dot{g}_-$ through Eq. (21). However, in the presence of the thermalization, it is not enough to specify $\Omega'_I$. The variations of $\Omega_I$ and $\omega$ are independent formally. However, thermalization forces some of the variations to be related through Eqs. (A1) and (A2), which present a differential relation,

$$\delta \Omega_I = \left( \frac{\omega}{\omega_I} g_--1 \right) \delta \omega - \alpha g \frac{\Omega_I}{\omega_I} (\Omega_I + 2\omega) \delta t,$$

from Eq. (33). The second term in Eq. (64) represents the temporal development of the nonadiabaticity, which does not originate from the frequency change. This term prevents the equation from being interpreted as a thermodynamic relation, which requires the first law to consist of measurable global quantities written as functions that appear explicitly in the Hamiltonian or density. As a resolution, we replace the $\delta t$ term with a $\delta \omega_I$ term by using Eq. (62).
Effectively, this change makes the ELR frequency $\omega_T$ a new parameter describing the harmonic oscillator thermal state under thermalization. Now, the variation of the nonadiabaticity becomes integrable with respect to $\omega$ and $\omega_T$:

$$\delta \Omega_T(\omega, \omega_T) = \frac{\partial \Omega_T}{\partial \omega} \delta \omega - \frac{g}{g_{\text{eff}}} \frac{\omega_T^2 - \omega^2}{\omega_T} \frac{\delta \omega_T}{\omega_T}. \quad (65)$$

The first term is nothing but the formula in Eq. (34). Given $\Omega_T(\omega, \omega_T)$, we can specify $g_-$ through

$$\Omega'_T(\omega, \omega_T) \equiv \frac{\partial \Omega_T}{\partial \omega} \omega_T \Rightarrow \frac{\omega}{\omega_T} g_-= 1 + \Omega'_T. \quad (66)$$

This equation also determines $g_0/\omega_T$ from the definition of $\Omega_T$ in Eq. (25) and Eq. (60). Given $\omega_T$, one can determine the last unknown quantity, $g_+$. The relation (65) associates the three variations $\delta \Omega_T$, $\delta \omega$, and $\delta \omega_T$. A curve in the space $(\omega, \omega_T, \Omega_T)$ represents a thermodynamic process of the oscillator. To specify the curve, we need another relation that associates the three parameters:

$$\Omega'_\omega(\omega, \omega_T) \equiv \frac{\partial \Omega_T}{\partial \omega_T} \omega \equiv -\frac{g}{g_{\text{eff}}} \frac{\omega_T^2 - \omega^2}{\omega_T} \frac{\delta \omega_T}{\omega_T}. \quad (67)$$

Note that $\Omega'_\omega$ vanishes when $g = 0$ and describes the nonadiabaticity creation from the thermalization ($\delta \omega_T \neq 0$).

To find physical implication of $\omega_T$, we define a standard temperature $T_0$:

$$T_0 = T_{\omega_{\text{eff}}} = \frac{\omega_T}{\epsilon}. \quad (68)$$

Then, we may write the ELR frequency in terms of $T_0$ and $\epsilon$ as $\omega_T = T_0 \epsilon = \omega_{\text{eff}} T_0 / T$. The variation of $\omega_T$ becomes

$$\delta \omega_T = T_0 \delta \epsilon + \epsilon \delta T_0 = T_0 \left( \frac{\partial S}{\partial \epsilon} \right)^{-1} \delta S + \epsilon \delta T_0. \quad (69)$$

Therefore, once the standard temperature is invariant in time, the change of $\omega_T$ is completely determined by the change of entropy of the oscillator.

**B. Time evolution**

Consider a time evolution of the oscillator under the thermalization process. Note that the ELR frequency $\omega_T$ is a temporal function. On the other hand, $T_0$ is a kinematical constant (58) for the thermal system defined at the moment when $\omega_{\text{eff}} = \omega_T$. Therefore, the creation rate of the entropy becomes

$$\Gamma_s = \frac{dS}{dt} = \left( \frac{\partial S}{\partial \log \epsilon} \right) \frac{d \log \omega_T}{dt} = -\frac{\epsilon^2}{4 \sinh^2 \frac{\epsilon}{2}} \frac{\dot{\omega_T}}{\omega_T}. \quad (70)$$

This relates the derivative $\dot{\omega_T}$ with the physical creation rate of the entropy. Note that when the entropy decreases, the ELR frequency $\omega_T$ increases. However it does not imply that $\omega(t)$ also increases, which will be determined by the functional forms of $\Omega_T$ and $\Omega'_\omega$.

Let us examine the first law (61) in detail. By using Eq. (67), the first law becomes

$$\delta E = -F_\omega \delta \omega - F_T \delta \omega_T + T \delta S. \quad (71)$$

Here,

$$F_\omega \equiv -\left( \frac{\partial E}{\partial \omega} \right)_{\omega_T, S} = (1 + \Omega'_\omega) F_\omega, \quad F_T \equiv -\left( \frac{\partial E}{\partial \omega_T} \right)_{\omega, S} = -\frac{g}{g_{\text{eff}}} \frac{\omega_T^2 - \omega^2}{\omega_T (g_{\text{eff}} - \omega)} F_\omega = \Omega'_\omega F_\omega.$$

Note that $\delta \omega_T = 0$ when the thermalization is absent. The force $F_T$ vanishes when $\Omega_T = 0$ and diverges at

$$\omega_T = \delta \omega_T \equiv g \omega_{\text{eff}}. \quad (72)$$
When $\omega_T > \bar{\omega}_T$, $F_T < 0$ because $F_\omega < 0$ and $\omega_{\text{eff}} > \omega$. This makes $\delta \omega_T > 0$ for positive $\delta E$. On the other hand, when $\omega_T < \bar{\omega}_T$, $F_T > 0$ making $\delta \omega_T < 0$ for positive $\delta E$. Because a system tends to have a minimum energy, $\omega_T = \bar{\omega}_T$ becomes an attractor. Therefore, asymptotically, $\omega_T \rightarrow \bar{\omega}_T$. This implies that the asymptotic value of $\omega_T$ must be determined by the formula (71) to get

$$\lim_{t \rightarrow \infty} \omega_T = \lim_{t \rightarrow \infty} \frac{g_\omega(t)}{1 - g^T} = g_\omega \omega, \quad$$

where we use $\lim_{t \rightarrow \infty} T = 0$.

The temperature of the system under thermalization becomes

$$T \equiv \left( \frac{\partial E}{\partial S} \right)_{\omega_T} = \frac{\omega_{\text{eff}}}{\epsilon} = \frac{T_0}{g} \left( 1 + \frac{1}{\alpha} \frac{\dot{\omega}_T}{\omega_T} \right). \quad (72)$$

At the initial time, $\left( 1 + \frac{1}{\alpha} \frac{\omega_T}{\omega} \right)_{t=0} = g$, which is consistent with Eq. (62). On the other hand, $\dot{\omega}_T = 0$ for equilibrium configuration at $t = \infty$. This consideration gives $T_\infty = T_0/g$, which must be the temperature of the heat bath, which we have mentioned in Eq. (55). When $\alpha = 0$, one can immediately notice from Eq. (62) that $\omega_T$ does not change in time in the absence of a thermalization making $T = T_\text{bath}$. Note that, we do not use the condition $\alpha = \text{constant}$ until now. Therefore, the results hold even if $\alpha$ depends on time without loss of generality.

As we have mentioned in Eq. (68), the entropy change $\delta S$ can be written as the change of $\delta \omega_T$. From Eq. (62), we find that $\omega_T + \alpha \dot{\omega}_T = \omega_{\text{eff}} \epsilon$. Therefore, we get

$$\delta \omega_{\text{eff}} = \delta \omega + \delta \Omega = \frac{\omega g}{\omega} \delta \omega + \Omega_\omega \dot{\omega}_T = \frac{1}{\alpha} \left[ \delta \omega_T + \alpha \delta \omega_T \right].$$

This relates the variation $\delta \omega$ with $\delta \omega_T$ and $\dot{\omega}_T$. Therefore, putting these into the first law (61), we may find that the energy can be expressed as a function of $\omega_T$ and $\dot{\omega}_T$. Explicitly, by using $\omega_T$ as in (A4), the energy in Eq. (24) is written as

$$E(\omega_T, \dot{\omega}_T) = \frac{\omega_T}{g} \left( 1 + \frac{1}{\alpha} \frac{\dot{\omega}_T}{\omega_T} \right) \coth \frac{\omega_T}{2T_0}. \quad (73)$$

Because the entropy $S$ is just a function of $\epsilon = \omega_T/T_0$, where $T_0$ is a constant, $\omega_T$ denotes the temporal change of the entropy. Therefore, one may regard the energy as a function of the entropy and its time-derivative: $E = E(S, \dot{S})$.

VI. SUMMARY AND DISCUSSION

In this work, we have studied the thermalization of a quantum harmonic oscillator when it contacts an ideal heat bath. To describe the thermalization process, we generalized the Ernmark-Lewis-Riesenfeld invariant method used in our previous work by using the interaction picture adequate for our purpose. After imposing appropriate conditions on the thermalization process, we introduce an ansatz equation that describes the time evolution effectively. Our analysis is ‘minimal’ in the sense that the thermal state is described by a squeezed state of the quantum oscillator.

We have also studied the first law of thermodynamics during thermalization. Thermalization process can be described formally by a similar first law to that of the ordinary thermodynamics,

$$\delta E = -F_\omega \delta \omega - F_T \delta \omega_T + T \delta S,$$

where $\omega_T$ is an ELR frequency of which time derivative develops the thermalization. Here $T$ and $F_\omega, F_T$ denote the oscillator’s temperature and external forces from the bath, respectively. A crucial difference in this first law from the ordinary one is the additional contribution from $\delta \omega_T$.

As we have seen in our previous work [20], the functions $F_\omega, F_T$ and $T$ are measurable by using the geometric phase in Ref. [30]. Obtaining an exact form of $V_T(\dot{F}_T)$ in the geometric phase change $\Delta_{\text{th}} = \int \frac{1}{2} V_T(\dot{F}_T) d\omega_T$ is an interesting study. In Refs. [31, 32], a response function defined by a squared commutator of two operators was used to signify the presence of thermalization. The thermalization in this work is also determined from the difference between $\bar{T}$ and $H_{\text{osc}}$. If we adopt the two operators as $\bar{T}$ and $H_{\text{osc}}$, the non-vanishing response function implies the existence of thermalization. To demonstrate the effect of thermalization, we wrote the energy as a function of $\omega_T$ and $\dot{\omega}_T$,

$$E(\omega_T, \dot{\omega}_T) = \frac{\omega_T}{g} \left( 1 + \frac{1}{\alpha} \frac{\dot{\omega}_T}{\omega_T} \right) \coth \frac{\omega_T}{2T_0}. \quad (73)$$
This equation vividly shows how the ELR frequency $\omega_I$ contributes to the energy of the oscillator. The presence of $\dot{\omega}_I$ term in the energy expectation value indicates that $\omega_I$ and $\dot{\omega}_I$ become dynamic quantities that play the roles of a position and a velocity in classical mechanics. It has deep implications on thermodynamics if one expresses the energy as a function of entropy and its time derivative. Notice that the entropy $S$ is just a function of $\omega_I$ and $\dot{\omega}_I \propto \dot{S}$.

Therefore, we can rewrite the energy in terms of dynamical variables $S$ and $\dot{S}$,

$$E = E(S, \dot{S}),$$

when thermalization is undergoing. Regarding the energy formula as a Hamiltonian function in classical mechanics, a corresponding Lagrangian $L = L(S, \dot{S})$ exists. This resemblance makes us consider a generalization of Carter’s formulation of the thermodynamics in which matter Lagrangian is a function of entropy and number densities. The generalized Lagrangian would be a function of the rates of the entropy and the number densities additionally that entail the information about thermalization.

This observation tells more than it appears. Thermodynamics can be generalized, at least partly, to include thermalization, making the thermodynamics truly dynamical. However, one should keep in mind the following: i) The general thermalization may not be described by squeezing only. ii) Anyone interested in the detailed physical state of the oscillator should be careful about the mode mixing between the harmonic oscillator and the heat bath.

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**Appendix A: The time-derivatives**

Let us calculate the time derivatives of the ELR frequency $\omega_I$ explicitly. From Eq. (57), we get the time-derivative of $\omega_I$ to be

$$\frac{d\omega_I}{dt} = \frac{1}{2\omega_I} \frac{d(g_+g_- - g_0^2)}{dt} = -\alpha \omega_I \left(1 - \frac{g_+ + \omega^2 g_-}{2\omega_I^2}\right) = \alpha \left[g_+ (\omega + \omega_I \mathcal{S}) - \omega_I\right]. \quad (A1)$$

Next, we calculate the time derivative of the squeezing factor to give

$$\frac{d\mathcal{S}}{dt} = \frac{g_- - \dot{\omega}_I}{\omega_I} \left(\frac{\omega}{\omega_I} - \frac{1}{g_-}\right) - \alpha g \mathcal{S} \left(2\mathcal{S} + \frac{3\omega}{\omega_I} - \frac{1}{g_-}\right). \quad (A2)$$

Here we use Eqs. (57), (A1), and (60). Combining the two equations, we get

$$\frac{d\Omega_I}{dt} = \left(\frac{\omega}{\omega_I} - 1\right) \frac{d\omega}{dt} - \alpha g \frac{\Omega_I}{\omega_I} (\Omega + 2\omega). \quad (A3)$$

From (A1), one can obtain $\omega_{\text{eff}} = \omega + \omega_I \mathcal{S}$:

$$\frac{d\omega_I}{dt} = \alpha \left[g_+ (\omega + \omega_I \mathcal{S}) - \omega_I\right] \implies \omega_{\text{eff}} = \frac{\omega_I}{g} \left(1 + \frac{1}{\alpha} \frac{\dot{\omega}_I}{\omega_I}\right), \quad (A4)$$

which gives Eq. (73).

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