Softening Transitions with Quenched 2D Gravity

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We perform extensive Monte Carlo simulations of the 10-state Potts model on quenched two-dimensional $\Phi^3$ gravity graphs to study the effect of quenched connectivity disorder on the phase transition, which is strongly first order on regular lattices. The numerical data provides strong evidence that, due to the quenched randomness, the discontinuous first-order phase transition of the pure model is softened to a continuous transition.

1. INTRODUCTION

Systems subject to quenched random disorder often show a completely different behaviour than the pure case. If the pure system has a continuous phase transition it is well known that quenched random disorder can drive the critical behaviour into a new universality class, or the transition can even be eliminated altogether \[1\]. In the case of a first-order phase transition in the pure system the effect of quenched random disorder can also be very dramatic, with the possibility for a softening to a continuous transition.

The paradigm for testing the latter prediction is the two-dimensional (2D) $q$-state Potts model which exhibits on regular lattices for $q \geq 5$ a first-order transition whose strength increases with $q$. In Ref. \[2\] the effect of quenched bond disorder was investigated for the 8-state model and it was found that the critical behaviour of the quenched model could be well described by the Onsager Ising model universality class. In Ref. \[3\] the effect of quenched connectivity disorder was studied by putting the 8-state model on 2D Poissonian random lattices showing that for this type of quenched disorder the transition stays first order.

A different sort of connectivity disorder appears in 2D gravity triangulations or their dual $\Phi^3$ graphs. In such models one is interested in the coupling of matter to 2D gravity, so the disorder is annealed rather than quenched. Motivated by Wexler’s mean field results for $q = \infty$ Potts models coupled to 2D gravity \[4\], simulations of the 10-state and 200-state Potts model coupled to 2D gravity gave convincing evidence \[5\] for a continuous transition.

As the only quenched connectivity disorder seriously investigated to date, 2D Poissonian random lattices, showed no sign of softening for first-order transitions it is interesting to enquire whether the salient feature for the softening in the 2D gravity simulations is the annealed nature of the connectivity disorder or whether it is some intrinsic features of the graphs themselves. We discuss here results of a Monte Carlo (MC) study of the 10-state Potts model on quenched random lattices drawn from the equilibrium distribution of pure 2D gravity triangulations (or, more exactly, their dual $\Phi^3$ graphs).

2. MODEL AND RESULTS

We used the standard definition of the $q$-state Potts model,

$$Z_{\text{Potts}} = \sum_{\{\sigma_i\}} e^{\beta \sum_{\langle ij \rangle} \delta_{\sigma_i \sigma_j}} ; \sigma_i = 1, \ldots, q, \quad (1)$$

where $\langle ij \rangle$ denotes the nearest-neighbour bonds of random $\Phi^3$ graphs with up to 10,000 sites. For each lattice size we generated 64 independent replica using the Tutte algorithm \[6\], and performed long simulations of the 10-state...
model near the transition point using the single-cluster update algorithm. After thermalization we recorded measurements of the energy $E$ and the magnetization $M$ in 64 time-series files. The corresponding quantities per site are denoted in the following by $e = E/N$ and $m = M/N$.

Given this raw data we employed standard reweighting techniques to compute, e.g., the specific heat, $C^{(i)}(\beta) = \beta^2 N \langle (e^2) - \langle e \rangle^2 \rangle$, for each replica labeled by the superindex $(i)$, and then performed the replica average, $C(\beta) \equiv \langle C^{(i)}(\beta) \rangle \equiv (1/64) \sum_{i=1}^{64} C^{(i)}(\beta)$, denoted by the square brackets. To perform the replica average at the level of the $C^{(i)}$ (and not at the level of energy moments) is motivated by the general rule that quenched averages should be performed at the level of the free energy and not the partition function. Finally, we determined the maximum, $C_{\text{max}} = C(\beta_{\text{max}})$, for each lattice size and studied the finite-size scaling (FSS) behaviour of $C_{\text{max}}$ and $\beta_{\text{max}}$. The error bars on the two quantities entering the FSS analysis are estimated by jack-kniving over the 64 replicas.

The analysis of the magnetic susceptibility, $\chi(\beta) = \beta N \langle [m^2] - \langle m \rangle^2 \rangle$ and the energetic Binder parameter $V(\beta) = 1 - \langle (e^4) \rangle / 3 \langle (e^2) \rangle^2$, proceeds exactly along the same lines, yielding $\chi_{\text{max}}$ and $\beta_{\text{max}}$ as well as $V_{\text{min}}$ and $\beta_{\text{min}}$. In order to be prepared for the possibility of a second-order phase transition, the magnetic Binder parameter, $U(\beta) = 1 - \langle (m^4) \rangle / 3 \langle (m^2) \rangle^2$, was also measured as, in this case, its crossing for different lattice sizes provides an alternative determination of the critical coupling $\beta_c$, and the FSS of either the maximum slopes or the slopes at $\beta_c$ can be used to extract the correlation length exponent.

If the transition was of first-order one would expect for large system sizes an asymptotic FSS behaviour of the form $C_{\text{max}} = a_C + b_C N^{\alpha_C} \ldots$, and $\chi_{\text{max}} = a_{\chi} + b_{\chi} N \ldots$. However a linear scaling with $N$ is not consistent with our data. Furthermore, at a first-order phase transition one would expect that the energetic Binder-parameter minima approach in the infinite-volume limit a non-trivial value related to the latent heat. Our data, however, approaches the trivial limit of $2/3$, also strong evidence for a continuous transition.

![Figure 1. Maxima of $C$ vs $N$. The fit line is discussed in the text.](image-url)

Being thus convinced that the transition is continuous, the next goal is to determine the critical exponents and the corresponding universality class. To this end we have tried to describe the scaling of the specific-heat and susceptibility maxima with the standard FSS ansatz $C_{\text{max}} = a_C + b_C N^{\alpha_C/D_\nu}$ and $\chi_{\text{max}} = a_{\chi} + b_{\chi} N^{\gamma/D_\nu}$, where $\alpha$, $\gamma$, and $\nu$ are the usual universal critical exponents at a continuous phase transition, $a_{C,\chi}$ and $b_{C,\chi}$ are non-universal amplitudes, and $D$ is the intrinsic Hausdorff dimension of the graphs. By performing a non-linear three-parameter fit to the specific-heat maxima we obtained $\alpha/D_\nu = 0.22(7)$, with a reasonable goodness-of-fit parameter $Q = 0.10$; see Fig. 1. Since the background term $a_C$ turned out to be consistent with zero, we also tried linear two-parameter fits with $a_C = 0$ kept fixed and the resulting exponent estimate over all data points gave a fully consistent value of $\alpha/D_\nu = 0.222(7)$ (with $Q = 0.15$) but, as expected, a much smaller error bar. The susceptibility maxima grow very fast with $N$, such that also here the constant term $a_{\chi}$ can safely be neglected. The linear fit over all data points yielded $\gamma/D_\nu = 0.732(10)$ (with $Q = 0.27$), and omitting the $N = 250$ point we obtained $\gamma/D_\nu = 0.719(14)$ (with $Q = 0.31$).

The pseudo-transition points $\beta_{C_{\max}}$, $\beta_{\chi_{\max}}$, and $\beta_{V_{\min}}$ were fitted to the standard FSS ansatz $\beta_{C_{\max}} = \beta_c + c C N^{-1/D_\nu}$, etc. By taking the aver-
age of the three estimates of $\beta_c$ we estimated the transition point to be $\beta_c = 2.2445(20)$. This value is consistent with the estimate obtained from the crossings of the magnetic Binder parameter $U$ for different lattice sizes. The estimate of $1/D\nu$ from the FSS of the pseudo-transition points is not very stable but a more precise estimate can be obtained by analyzing the FSS of the magnetic Binder-parameter slopes in the vicinity of $\beta_c$. Both the maximum slopes and the slopes at $\beta_c$ are expected to scale as $dU/d\beta \propto N^{1/D\nu}$.

The fit to the maximum slopes for $N \geq 500$ gave $1/D\nu = 0.616(29)$ (with $Q = 0.29$), and for the slopes at $\beta_c = 2.2445$ we obtained $1/D\nu = 0.614(30)$ (with $Q = 0.78$).

On annealed gravity graphs it was found that the 10-state Potts model appeared to display the exponents of the 4-state Potts model coupled to gravity, which are listed as “annealed $q = 4$” in Table 1 below. The values for the exponents measured in our simulations appear to be best fitted by the “quenched $q = 4$” exponents derived in [7] and also listed in Table 1. On the basis of the numerical evidence in this paper and the earlier results of [3], it would thus seem that the connectivity disorder of both quenched and annealed $\Phi^3$ gravity graphs has the effect of softening the first-order transition of the $q = 10$ Potts model to a continuous transition. Unlike the case of bond disorder, however, we find exponents associated with the 4-state Potts model on annealed 2D gravity graphs we see exponents associated with the 4-state Potts model, in this case the “quenched” exponents.

3. CONCLUSIONS

To summarize, we have obtained strong numerical evidence that due to connectivity disorder the phase transition in the 10-state Potts model on quenched random gravity graphs is softened to a continuous phase transition. This result is in contrast to a recent simulation of the 8-state model on Poissonian Delaunay/Voronoi random lattices where the transition stays first order as on regular lattices [3]. It is, however, in qualitative agreement with the quenched random bond case [3], and like the simulations in [3] of the 10-state Potts model on annealed 2D gravity graphs we see exponents associated with the 4-state Potts model, in this case the “quenched” exponents.

ACKNOWLEDGMENTS

Work supported in part by EC HCM network grant ERB-CHRX-CT930343 and NATO collaborative research grant CRG951253 (C.F.B. and D.A.J.). W.J. thanks the DFG for a Heisenberg fellowship. The simulations were performed on a T3D parallel computer of Zuse Institut Berlin.

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