GENERALISED BARGMANN SUPERALGEBRAS

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In memory of Winifred Whittle

ABSTRACT. The Bargmann algebra and centrally-extended Newton-Hooke algebras describe the non-relativistic symmetries of massive particles in flat and curved spacetimes, respectively. These three algebras all arise as deformations of the universal central-extension of the static kinematical Lie algebra. In this paper, we classify the $N=1$ super-extensions for each of these algebras in $(3+1)$-dimensions, up to isomorphism. We then identify the non-empty branches of the algebraic variety describing the $N=2$ super-extensions of these algebras. We find 9 isomorphism classes in the $N=1$ case and 22 branches in the $N=2$ case. We then give a brief discussion on some applications of these Lie superalgebras, including their possible uses for non-relativistic supergravity and holography.

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1. Introduction

1.1. Background. The idea of modelling spacetime geometrically was pioneered by Albert Einstein in his theory of general relativity. The equivalence principle underlyng this theory requires the spacetime geometry to be pseudo-Riemannian. Thus, this model builds upon special relativity; namely, it has local Lorentz symmetry at its core. However, not all physical phenomena of interest appear to us in an intrinsically Lorentzian relativistic way. A large body of literature is emerging, which aims to generalise Einstein’s idea to allow other kinematical symmetries to dictate the geometry of spacetime.

The question of which symmetries may describe physical systems has lead to the recent classification of kinematical Lie algebras [1–4]. This classification divides into four distinct types: Riemannian, pseudo-Riemannian, Galilean and Carrollian. The former two are those traditionally posited to describe kinematical systems, but it is the latter two that are receiving renewed interest.

One reason for this interest has its roots in the search for a quantum theory of gravity. Conventional approaches try to access the quantum gravity corner of the Bronstein cube via general relativity or relativistic quantum field theory. However, a possible third approach to relativistic quantum gravity may lie in first producing a non-relativistic quantum gravity theory.

As is often the case when exploring new territory, it is best to start from somewhere familiar. Beginning from a Lorentzian system in $(D+2)$-dimensions, a null reduction down to $(D+1)$-dimensions produces a
non-relativistic counterpart. This process has led not only to non-relativistic versions of gravity [5–8] but non-relativistic versions of string theory [9, 10], holography [11, 12] and supersymmetric quantum field theories [13, 14]. There are several other approaches to forming non-relativistic (super)gravity theories currently under investigation, including gauging known symmetry algebras, such as the Bargmann algebra [6, 15–18] and super-Bargmann algebra [18, 19], producing new algebras to gauge and Lagrangians through expansion procedures [10, 20, 21] and finding extensions that may allow for the production of an action [22–31]. Applying these approaches to Maxwellian extensions of Newtonian gravity is also an active field of research [32–35], and non-relativistic symmetries and geometries have even found their place in double field theory [36–39].

It may be thought that since the Newton-Cartan (NC) geometry one obtains by gauging the Bargmann algebra may be found via a null reduction of general relativity in flat space [6], a similar story may be true for the centrally-extended Newton-Hooke algebras, which are the curved equivalents. Namely, that the non-relativistic geometries obtained through the gauging of the centrally-extended Newton-Hooke algebras may be obtained via null reductions of anti-de Sitter (AdS) and de Sitter (dS) space. However, this is not the case. Seeking a unified description of these algebras, we note that the Galilean and Newton-Hooke spacetimes are described by NC structures $(\tau, h)$ in which the clock one-form is closed, $d\tau = 0$ [40]. In a recent work by José Figueroa-O’Farrill, it is shown that these NC structures arise via a null reduction of a particular type of Bargmann space [41]. More explicitly, these Bargmann spaces are described by a triple $(B, g, \xi)$ consisting of a $(D + 2)$-dimensional Lorentzian manifold $B$, a Lorentzian metric $g$, and a null, nowhere-vanishing vector field $\xi$, such that $\nabla \xi = 0$ [45]. In other words, $g$ is a Brinkmann metric, or $(B, g)$ is a pp-wave [44]. Using light-cone coordinates, we may write

$$g = 2du dv + A(u, x) du^2 + B_i(u, x) du dx^i + g_{ij}(u, x) dx^i dx^j \quad \text{and} \quad \xi = \frac{\partial}{\partial v}, \quad (1.1.1)$$

where $u$ and $v$ are our light-cone coordinates, and $i, j = 1, \ldots, D – 1$. Now, taking the quotient of $B$ with respect to the isometry subgroup generated by $\xi$, one obtains an NC space, $N = B/\xi$. Using the musical isomorphism

$$\xi^a = g(\xi, -) = du \quad (1.1.2)$$

produces the clock one-form on the base space, while the spatial metric $h$ is defined

$$h(\alpha, \beta) = g^{-1}(\pi^*(\alpha), \pi^*(\beta)), \quad (1.1.3)$$

where $\alpha, \beta \in \Omega^1(N)$, and $\pi : B \to N$ is the projection of this trivial principle $\mathbb{R}$ bundle. The Galilean and Newton-Hooke spacetimes then arise through different choices of $h$ for our NC base space $(N, \tau, h)$. In this picture, the central-extended Galilean and Newton-Hooke algebras may be thought of as the Lie algebras for the isometry groups of the total spaces, or, equivalently, in the following way. Taking the flat case as our example, the Galilean algebra $g$ arises as the centraliser of the null Killing vector $\xi$ in the Lie algebra of Killing vector fields $\mathfrak{K}$:

$$\mathfrak{K} = \{X \in \mathfrak{X}(B)| \mathcal{L}_x g = 0\}$$

$$\mathfrak{g} = \{X \in \mathfrak{K}| [X, \xi] = 0\}. \quad (1.1.4)$$

This Lie algebra may admit a central-extension $\langle \xi \rangle$ fitting into the short exact sequence

$$0 \to \langle \xi \rangle \to \mathfrak{g} \to \mathfrak{g}/\langle \xi \rangle \to 0. \quad (1.1.5)$$

The Bargmann algebra is then the non-trivial central extension, i.e. the extension that does not admit a splitting such that $g = \mathfrak{g}/\langle \xi \rangle + \langle \xi \rangle$ as a Lie algebra. The centrally-extended Newton-Hooke algebras then appear in an identical manner when $g$ is replaced in this story by $\mathfrak{n}_+$.

The primary aim of the present paper is to provide a set of super-extensions for the Bargmann and centrally-extended Newton-Hooke algebras in $(3 + 1)$-dimensions to facilitate future investigations into non-relativistic physics. It may be read as a case study for the development of a formalism for classifying kinematical Lie superalgebras (KLsAs) which began in [46]. We will first address the issue of introducing a central extension, $Z$, before describing how to generalise the construction for extended supersymmetry.

As far as we are aware, this is the first classification of super-extensions for these algebras. Several papers have asked what the possible "super-kinematics" are, and discussed either $N = 1$ [47] or $N = 2$ [48] super-extensions of the kinematical Lie algebras first identified by Bacry and Lévy-LeBlond in their pioneering paper [49]. Others consider contractions of the anti-de Sitter (AdS) superalgebra $osp(1|4)$ [50–52]. However, we believe this is the first attempt to classify these centrally-extended kinematical Lie algebras.

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1These results are also shown in [42–44].

2Thank you to José Figueroa-O’Farrill for this interpretation.
In the $N = 1$ case, we will provide a full classification, analogous to that of the non-centrally extended kinematical Lie superalgebras in [46]. However, the $N = 2$ case is considerably more involved. Therefore, in this instance, we will stop short of explicitly identifying each super-extension of the algebras and instead highlight which brackets are allowed to be non-vanishing. Thus we are able to identify important features, such as which generators appear in the bracket $\{Q, Q\}$ and which act on the supercharges, without burdening the reader with excessive detail. However, should the reader wish to investigate a particular type of super-extension further, we will provide the required tools.

For the present paper, we will use the classification of non-trivial central extensions of kinematical Lie algebras in $(3 + 1)$-dimensions presented in [4] as our starting point (see Table 1). This list includes the Bargmann $\mathfrak{g}$ and centrally-extended Newton-Hooke algebras $\mathfrak{n}_{\pm}$, as well as the centrally-extended static kinematical Lie algebra $\mathfrak{a}$, of which the former three are all deformations. ² From now on, we will refer to these algebras collectively as generalised Bargmann algebras.

Table 1. Generalised Bargmann algebras in $D = 3$

| GBA | Nonzero Lie brackets in addition to $[J,J] = J$, $[J,B] = B$, $[J,P] = P$ | Comments |
|-----|-------------------------------------------------|----------|
| 1   | $[B, P] = Z$                                    | $\mathfrak{a}$ |
| 2   | $[B, P] = Z$ $[H,B] = B$ $[H,P] = -P$          | $\mathfrak{n}_-$ |
| 3   | $[B, P] = Z$ $[H,B] = P$ $[H,P] = -B$          | $\mathfrak{n}_+$ |
| 4   | $[B, P] = Z$ $[H,B] = -P$                      | $\hat{0}$ |

1.2. Outline of Paper. In section 2, we will introduce the objects of interest and describe the classification problem we wish to solve. In particular, section 2.1 will introduce kinematical Lie algebras and their central extensions, and setup the universal generalised Bargmann algebra, which plays a crucial role in ensuring we do not duplicate our calculations later on. In section 2.2, we give a brief definition of kinematical Lie superalgebras and define the classification problem. The exact details of the quaternionic formalism in the $N = 1$ and $N = 2$ cases are left until sections 3.1 and 4.1, respectively.

In section 3, we classify the $N = 1$ super-extensions of the generalised Bargmann algebras. As mentioned above, section 3.1 sets up the quaternionic formalism for the $N = 1$ super-extensions. There are also some useful preliminary results presented here, and a discussion on the possible basis transformations $G \subset GL(s_0) \times GL(s_1)$ for the $N = 1$ kinematical Lie superalgebras. In section 3.2, we classify the $N = 1$ super-extensions of the four algebras before summarising our findings in section 3.3. This final section includes a discussion on unpacking the quaternionic formalism to a notation that may be more familiar to the reader.

Section 4 investigates the $N = 2$ super-extensions of the generalised Bargmann algebras. Following the same pattern as section 3, section 4.1 introduces the quaternionic formalism for the $N = 2$ case. It includes some preliminary results and the definition of the group of basis transformations $G$ for these superalgebras. Section 4.2 identifies four different branches of possible super-extension for the investigation in section 4.3. The results are summarised in section 4.4, where there is also a brief discussion on unpacking the $N = 2$ quaternionic formalism. Finally, in section 5, we give some concluding remarks and discuss possible future directions for this work.

The classifications in this paper adopt an unconventional formalism, and the details of how we arrive at our results can be quite involved, particularly for the $N = 2$ case. For those pressed for time or interested solely in the superalgebras in the classification, the following sections present the required reading to access the results. For the quaternionic formalism in the $N = 1$ case, see section 3.1, and, in the $N = 2$ case, see section 4.1. Sections 3.3 and 4.4 include discussions on unpacking the notation to something more conventional in the $N = 1$ and $N = 2$ cases, respectively. Table 3 contains the results of the $N = 1$ classification, and Table 6 contains the results of the $N = 2$ analysis.

2. Introduction to Formalism

In this section, we will introduce the concept of a kinematical Lie (super)algebra and set up the classification problem we wish to solve in this paper. Of particular importance is the definition of our universal generalised Bargmann algebra, which will be used throughout both the $N = 1$ and $N = 2$ classification problems. All the statements made in this section assume that we are working in three spatial dimensions, $D = 3$.

²In this paper, we will not use the standard notation for the static kinematical Lie algebra $s$; instead, we will use $a$ for Aristotelian, since this is the sole example of an Aristotelian algebra in this paper. This notational change allows us to use $a$ when referring to kinematical Lie superalgebras.
2.1. Kinematical Lie Algebras. A kinematical Lie algebra (KLA) $\mathfrak{t}$ is a 10-dimensional real Lie algebra containing a rotational subalgebra $\mathfrak{r}$ isomorphic to $\mathfrak{so}(3)$ such that, under the adjoint action of $\mathfrak{r}$, it decomposes as $\mathfrak{t} = \mathfrak{r} \oplus 2\mathfrak{V} \oplus \mathbb{R}$, where $\mathfrak{V}$ is a three-dimensional $\mathfrak{so}(3)$ vector module and $\mathbb{R}$ is a one-dimensional $\mathfrak{so}(3)$ scalar module. We will denote the real basis for this Lie algebra as $[J_1, B_1, P_1, H]$, where $J_1$ is the generator for the subalgebra $\mathfrak{r}$, $B_1$ and $P_1$ span our two copies of $\mathfrak{V}$, and $H$ spans the $\mathfrak{so}(3)$ scalar module. The Lie brackets common to all kinematical Lie algebras are

$$[J_1, J_1] = \epsilon_{ijk} J_k \quad [J_1, B_1] = \epsilon_{ijk} B_k \quad [J_1, P_1] = \epsilon_{ijk} P_k \quad [J_1, H] = 0. \quad (2.1.1)$$

The classification of these algebras was first presented in [53], completing the work of [49], in which the kinematical Lie algebras admitting parity and time-reversal automorphisms were classified. Throughout this paper, we will frequently use the following abbreviated notation for the Lie brackets of these algebras.

$$[J_1, B_1] = \epsilon_{ijk} B_k \quad \text{is equivalent to} \quad [J, B] = B$$

$$[H, B_1] = P_1 \quad \text{is equivalent to} \quad [H, B] = P$$

$$[B_1, P_1] = \delta_{ij} H \quad \text{is equivalent to} \quad [B, P] = H,$$

*et cetera.* The static kinematical Lie algebra, of which all other KLAs are deformations, admits a universal central extension $[\mathfrak{g}] = \mathfrak{so}(3)$ scalar module in the underlying vector space. Let $\mathfrak{Z}$ span this extra copy of $\mathbb{R}$. Our centrally-extended static kinematical Lie algebra is spanned by $J, B, P, H$, and $\mathfrak{Z}$, with non-vanishing brackets

$$[J, J] = J \quad [J, B] = B \quad [J, P] = P \quad [B, P] = \mathfrak{Z}. \quad (2.1.3)$$

All other centrally-extended kinematical Lie algebras are deformations of this algebra; therefore, these brackets are common to all such algebras. For a complete classification of the centrally-extended kinematical Lie algebras see Table 2, taken from [4]. The three sections of this table, starting from the top, are the non-trivial central extensions, the trivial central extensions, and, finally, the non-central extensions of kinematical Lie algebras.

### Table 2. Centrally-extended kinematical Lie algebras in $D = 3$

| KLA | Nonzero Lie brackets in addition to $[J, J] = J, [J, B] = B, [J, P] = P$ | Comments |
|-----|------------------------------------------------------------------|---------|
| 1   | $[B, P] = \mathfrak{Z}$ | â |
| 2   | $[B, P] = \mathfrak{Z}$ | $[H, B] = B$ | $[H, P] = -P$ | $[B, B] = \epsilon \oplus \mathbb{R} \mathbb{Z}$ |
| 3   | $[B, P] = \mathfrak{Z}$ | $[H, B] = B$ | $[H, P] = -B$ | $[B, B] = -J$ | $p \oplus \mathbb{R} \mathbb{Z}$ |
| 4   | $[B, P] = \mathfrak{Z}$ | $[H, B] = -P$ | $[B, B] = -J$ | $so(4, 1) \oplus \mathbb{R} \mathbb{Z}$ |
| 5   | $[B, P] = \mathfrak{Z}$ | $[H, B] = \mathfrak{Z}$ | $[B, B] = \mathfrak{Z}$ | $so(5) \oplus \mathbb{R} \mathbb{Z}$ |
| 6   | $[B, P] = \mathfrak{Z}$ | $[H, B] = \mathfrak{Z}$ | $[B, B] = \mathfrak{Z}$ | $so(3, 2) \oplus \mathbb{R} \mathbb{Z}$ |
| 7   | $[B, P] = \mathfrak{Z}$ | $[H, B] = \mathfrak{Z}$ | $[B, B] = \mathfrak{Z}$ | $so(3, 1) \oplus \mathbb{R} \mathbb{Z}$ |
| 8   | $[B, P] = \mathfrak{Z}$ | $[H, B] = \mathfrak{Z}$ | $[B, B] = \mathfrak{Z}$ | $so(3, 1) \times \mathbb{R}^4$ |
| 9   | $[B, P] = \mathfrak{Z}$ | $[H, B] = \mathfrak{Z}$ | $[B, B] = \mathfrak{Z}$ | $so(3, 1) \times \mathbb{R}^{3,1}$ |
| 10  | $[B, P] = \mathfrak{Z}$ | $[H, B] = \mathfrak{Z}$ | $[B, B] = \mathfrak{Z}$ | $so(3, 1) \times \mathbb{R}^{3,1}$ |
| 11  | $[B, P] = \mathfrak{Z}$ | $[H, B] = \mathfrak{Z}$ | $[B, B] = \mathfrak{Z}$ | $so(3, 1) \times \mathbb{R}^{3,1}$ |
| 12  | $[B, P] = \mathfrak{Z}$ | $[H, B] = \mathfrak{Z}$ | $[B, B] = \mathfrak{Z}$ | $so(3, 1) \times \mathbb{R}^{3,1}$ |
| 13  | $[B, P] = \mathfrak{Z}$ | $[H, B] = \mathfrak{Z}$ | $[B, B] = \mathfrak{Z}$ | $so(3, 1) \times \mathbb{R}^{3,1}$ |
| 14  | $[B, P] = \mathfrak{Z}$ | $[H, B] = \mathfrak{Z}$ | $[B, B] = \mathfrak{Z}$ | $so(3, 1) \times \mathbb{R}^{3,1}$ |
| 15  | $[B, P] = \mathfrak{Z}$ | $[H, B] = \mathfrak{Z}$ | $[B, B] = \mathfrak{Z}$ | $so(3, 1) \times \mathbb{R}^{3,1}$ |

In the present paper, we will focus solely on the first of these sections, and it shall be exclusively the algebras of this section that we are referring to when using the term *generalised Bargmann algebras.* It is useful for our later calculations to define a *universal* generalised Bargmann algebra. In addition to the standard kinematical brackets given in (2.1.1), this algebra has non-vanishing brackets

$$[B, P] = Z \quad [H, B] = \lambda B + \mu P \quad [H, P] = \eta B + \varepsilon P, \quad (2.1.4)$$

where $\lambda, \mu, \eta, \varepsilon \in \mathbb{R}$. Setting these four parameters to certain values allows us to reduce to the four cases of interest. For example, $\hat{g}$ is given by setting $\lambda = \eta = \varepsilon = 0$ and $\mu = -1$. By beginning with the universal algebra, and only picking our parameters, and, thus, our algebra, when we can no longer make progress in the universal case, we reduce the amount of repetition in our calculations.

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4Note, the universal generalised Bargmann algebra is not a Lie algebra for arbitrary $\lambda, \mu, \eta, \varepsilon$. It is used here simply as a computational tool.
2.2. Kinematical Lie Superalgebras. An $N$-extended kinematical Lie superalgebra (KLSA) $\mathfrak{s}$ is a real Lie superalgebra $\mathfrak{s} = \mathfrak{s}_0 \oplus \mathfrak{s}_1$, such that $\mathfrak{s}_0 = \mathfrak{t}$ is a kinematical Lie algebra, and $\mathfrak{s}_1$ consists of $N$ copies of $\mathfrak{s}$, the real four-dimensional spinor module of the rotational subalgebra $\tau \cong \mathfrak{so}(3)$.

Under the isomorphism $\tau \cong \mathfrak{sp}(1)$, $\mathfrak{s}$ may be described as a copy of the quaternions where $\tau$ acts via left quaternion multiplication. We will denote the quaternions by $\mathbb{H}$ and the quaternion units by $1, j, k$, where $jj = k$, and $\bar{\beta} = -\bar{\beta}$. Now, to incorporate the $s_0$ generators into this formalism, we introduce the following injective $\mathbb{R}$-linear maps:

\[
\begin{align*}
J : \text{Im}(\mathbb{H}) &\to s_0 \quad \text{such that} \quad J(\omega) = \omega_1 J_1 \quad \text{where} \quad \omega = \omega_1 j + \omega_2 j + \omega_3 k \in \text{Im}(\mathbb{H}) \\
B : \text{Im}(\mathbb{H}) &\to s_0 \quad \text{such that} \quad B(\beta) = \beta_1 B_1 \quad \text{where} \quad \beta = \beta_1 j + \beta_2 j + \beta_3 k \in \text{Im}(\mathbb{H}) \\
P : \text{Im}(\mathbb{H}) &\to s_0 \quad \text{such that} \quad P(\pi) = \pi_1 P_1 \quad \text{where} \quad \pi = \pi_1 \bar{j} + \pi_2 j + \pi_3 k \in \text{Im}(\mathbb{H}).
\end{align*}
\]

We may now write the kinematical brackets of (2.1.1) using these maps and the standard quaternion calculus as follows.

\[
\begin{align*}
[J, J] &= J \implies [J(\omega), J(\omega')] = \frac{1}{2} J([\omega, \omega']) \\
[J, B] &= B \implies [J(\omega), B(\beta)] = \frac{1}{2} B([\omega, \beta]) \\
[J, P] &= P \implies [J(\omega), P(\pi)] = \frac{1}{2} P([\omega, \pi]) \\
[J, H] &= 0 \implies [J(\omega), H] = 0,
\end{align*}
\]

where $\omega = \beta, \pi \in \text{Im}(\mathbb{H})$, $[\omega, \beta] := \omega\bar{\beta} - \beta\omega$, and $\omega\bar{\beta}$ is given by quaternion multiplication. The additional brackets for the universal generalised Bargmann algebra may also be written in this quaternionic notation as

\[
\begin{align*}
[B, P] &= Z \implies [B(\beta), P(\pi)] = \text{Re}(\bar{\beta}\pi)Z \\
[H, B] &= \lambda B + \mu P \implies [H, B(\beta)] = \lambda B(\beta) + \mu P(\beta) \\
[H, P] &= \eta B + \epsilon P \implies [H, P(\pi)] = \eta B(\pi) + \epsilon P(\pi).
\end{align*}
\]

To write the $s_1$ generators in this language, we will use the injective $\mathbb{R}$-linear map

\[
Q : \mathbb{H}^N \to s_1 \quad \text{such that} \quad Q(\theta) = \theta_0 Q_0 \quad \text{where} \quad \theta \in \mathbb{H}^N,
\]

and $\{Q_0\}$ form a real basis for $s_1$.

The exact form of the brackets involving $Q$ will depend on whether we are considering the $N = 1$ or $N = 2$ case; therefore, we will leave this discussion to sections 3.1 and 4.1, respectively. The only important point, for now, is that the bracket $[J, Q]$ is fixed from the outset in both these instances.

A super-extension $s$ of one of our generalised Bargmann algebras $\mathfrak{t}$ will be a kinematical Lie superalgebra such that $s_0 = \mathfrak{t}$. To determine the super-extensions of the generalised Bargmann algebras, we, therefore, begin by letting $s_0$ be our universal generalised Bargmann algebra. We then need to specify the Lie brackets $[H, Q], [Z, Q], [B, Q], [P, Q]$, and $[Q, Q]$. Each of the $s_0, s_1$ components of the bracket must be an $\tau$-equivariant endomorphism of $s_1$, while the $s_1$ component must be an $\tau$-equivariant map $\tau^s s_1 \to s_0$. The space of possible brackets will be a real vector space $\mathcal{J}$. We then use the super-Jacobi identities to cut out an algebraic variety $\mathcal{J} \subset \mathcal{J}$. Since we are exclusively interested in supersymmetric extensions, we restrict ourselves to those Lie superalgebras for which $[Q, Q]$ is non-vanishing, which define a sub-variety $\mathcal{J} \subset \mathcal{J}$. This sub-variety may be unique to each generalised Bargmann algebra; therefore, it is at this stage we start to set the parameters of the universal algebra, where applicable. The isomorphism classes of the remaining kinematical Lie superalgebras are then in one-to-one correspondence with the orbits of $\mathcal{J}$ under the subgroup $G \subset \text{GL}(s_0) \times \text{GL}(s_1)$. The group $G$ contains the automorphisms of $s_0 = \mathfrak{t}$ and additional transformations which are induced by the endomorphism ring of $s_1$. The form of this subgroup will be discussed in the $N = 1$ and $N = 2$ cases in sections 3.1.2 and 4.1.2, respectively.

In the $N = 1$ case, we will identify each orbit of $\mathcal{J}$ explicitly, giving a full classification of the generalised Bargmann superalgebras in this instance. However, in the $N = 2$ case, we will only identify the non-empty branches of $\mathcal{J}$. Each branch will have a unique set of $[s_0, s_1]$ and $[s_1, s_1]$ brackets for the associated generalised Bargmann algebra. Thus we are able to highlight the form of the possible super-extensions without spending too much time pinpointing exact coefficients.

\[^{5}\text{So long as to keep the notation consistent, when referring to the } \mathfrak{so}(3) \text{ scalar module basis elements } H \text{ and } Z \text{ in this formalism, we will use } H \text{ and } Z. \text{ Therefore, if we consider } J(\omega) = \omega_1 J_1 \text{ to be the map between } J \text{ and } J, \text{ the map between } H \text{ and } H \text{ is } H = H. \text{ Similarly, } Z = Z.\]

\[^{6}\text{It may be important to note at this stage that } \theta = \theta_1 j + \theta_2 j + \theta_3 k \text{ is just a quaternion with real components } \theta_1 \text{; there are no Grassmann variables here.}\]
3. $N = 1$ Extensions of the Generalised Bargmann Algebras

Our investigation into generalised Bargmann superalgebras begins with the simplest case, $N = 1$. Section 3.1 will complete our set up for this case by specifying the precise form of the $[s_0, s_1]$ brackets for the universal generalised Bargmann superalgebra are

$$[B(\bar{\beta}), P(\pi)] = \text{Re}(\bar{\beta} \pi)Z \quad [H, B(\bar{\beta})] = \lambda B(\bar{\beta}) + \mu P(\bar{\beta}) \quad [H, P(\pi)] = \eta B(\pi) + \epsilon P(\pi),$$

where $\bar{\beta}, \pi \in \text{Im}(H)$ and $\lambda, \mu, \eta, \epsilon \in \mathbb{R}$. We now want to specify the possible $[s_0, s_1]$ and $[s_1, s_1]$ brackets. From ([46]), we have

$$[J(\omega), Q(\theta)] = \frac{\omega}{2} Q(\theta \omega) \quad [B(\beta), Q(\theta)] = Q(\beta \theta) \quad [P(\pi), Q(\theta)] = Q(\pi \theta) \quad [H, Q(\theta)] = Q(\theta h),$$

where $\omega, \pi, \beta \in \text{Im}(H)$, $\theta, \bar{\theta}, \eta, \epsilon \in \mathbb{R}$. Since $Z$ is just another $\mathfrak{so}(3)$ scalar module, and, therefore, the analysis of the bracket $[Z, Q]$ will be identical to that of $[H, Q]$, we know we can write

$$[Z, Q(\theta)] = Q(\theta z),$$

where $z \in \mathbb{H}$. Having added this additional generator, the possible $[s_1, s_1]$ brackets are now $\mathfrak{so}(3)$-equivariant elements of $\text{Hom}_{\mathbb{R}}(\mathbb{R}^2, \mathbb{R}) = \text{Hom}_{\mathbb{R}}(3V \oplus \mathbb{R}, 3V \oplus 2\mathbb{R}) = 9 \text{Hom}_{\mathbb{R}}(V, V) \oplus 2 \text{Hom}_{\mathbb{R}}(\mathbb{R}, \mathbb{R})$. As may have been expected, given that we did not alter the vectorial sector of the algebra, the number of $\text{Hom}_{\mathbb{R}}(V, V)$ elements does not change. However, now that we have an additional $\mathfrak{so}(3)$ scalar module, we have an additional scalar map, so

$$[Q(\theta), Q(\theta')] = n_0 |\theta|^2 H + n_1 |\theta|^2 Z - J(\theta n_2 \bar{\theta}) - B(\theta n_3 \overline{\theta}) - P(\theta n_4 \overline{\theta}),$$

where $n_0, n_1 \in \mathbb{R}$ and $n_2, n_3, n_4 \in \text{Im}(H)$. This expression polarises to

$$[Q(\theta), Q(\theta')] = n_0 \text{Re}(\theta \overline{\theta}') H + n_1 \text{Re}(\theta \overline{\theta}') Z - J(\theta' n_2 \overline{\theta} + \theta n_2 \bar{\theta}') - B(\theta' n_3 \overline{\theta} + \theta n_3 \overline{\theta}') - P(\theta' n_4 \overline{\theta} + \theta n_4 \overline{\theta}').$$

3.1.1. Preliminary Results. Following the example of ([46]), we will now consider the super-Jacobi identities for this super-extension of the generalised Bargmann algebra and use them to define our variety $\mathcal{J}$. We have three types of Jacobi identity to consider

1. $[s_0, s_0, s_1]$,
2. $[s_0, s_1, s_1]$, and
3. $[s_1, s_1, s_1]$.

We do not need to consider the $(s_0, s_0, s_0)$ case as we already know that these are satisfied by the generalised Bargmann algebras. Equally, we do not need to include $J$ in our investigations as the identities involving the rotational subalgebra $\mathfrak{r}$ impose the $\mathfrak{so}(3)$-equivariance of the brackets, which we already have by construction. Now, let us consider each type of identity in turn. In the following discussions, we will only write down explicitly those identities which are not trivially satisfied.

$[s_0, s_0, s_1]$.

By imposing these super-Jacobi identities, we ensure that $s_1$ is an $s_0$ module, not just an $\mathfrak{so}(3)$ module. The $(s_0, s_0, s_1)$ identities can be summarised as follows.
Lemma 3.1. The following relations between $H, Z$ implied by the corresponding t-brackets:

$$[H, Z] = \lambda H + \mu Z \implies [x, \hbar] = \lambda x + \mu z$$
$$[H, B] = \lambda B + \mu P \implies [\hbar, \hbar] = \lambda \hbar + \mu \bar{\hbar}$$
$$[H, P] = \lambda B + \mu P \implies [\hbar, \hbar] = \lambda \hbar + \mu \bar{\hbar}$$
$$[Z, B] = \lambda B + \mu P \implies [\hbar, z] = \lambda \hbar + \mu \bar{\hbar}$$
$$[Z, P] = \lambda B + \mu P \implies [\hbar, z] = \lambda \hbar + \mu \bar{\hbar}$$

$$(3.1.6)$$

$${\bf Substituting in the relevant brackets, we find}$$

$$Q(\theta x h) = \lambda Q(\theta h) + \mu Q(\theta x) + Q(\theta x z).$$

$$(3.1.8)$$

Using the injectivity of $Q$, we arrive at

$$[x, \hbar] = \lambda \hbar + \mu z.$$  

$$(3.1.9)$$

Finally, the $[B, P, Q]$ identity is

$$[B(\beta), [P(\pi), Q(\theta)]]] = [[B(\beta), P(\pi)], Q(\theta)] + [P(\pi), [B(\beta), Q(\theta)]].$$  

$$(3.1.10)$$

Substituting in the brackets from (3.1.2), we arrive at

$$Q(\theta \pi \theta \pi) = \text{Re}(\bar{\beta} \pi)(\lambda Q(\theta \hbar) + \mu Q(\theta x)) + \bar{Q}(\pi \beta \theta \bar{\hbar}).$$

$$(3.1.11)$$

Letting $\beta = \pi = \bar{i}$, we find

$$[\hbar, \bar{\hbar}] = \lambda \hbar + \mu z.$$  

$$(3.1.12)$$

Now, let $\beta = \bar{i}$ and $\pi = \bar{j}$. In this case, the $\lambda$ and $\mu$ terms vanish and we are left with

$$0 = \bar{\hbar} + \bar{\bar{\hbar}} = 0.$$  

$$(3.1.13)$$

By imposing these super-Jacobi identities, we ensure that the $[Q, \bar{\theta}]$ bracket is an $s_i$-equivariant map $\bigotimes s_i \rightarrow s_i$. The $(s_0, s_1, s_3)$ identities can be difficult to study if we are trying to be completely general; however, we know that all four algebras in Table 1 can be written as specialisations of the universal generalised Bargmann algebra:

$$[B, P] = Z \quad [H, B] = \lambda B + \mu P \quad [H, P] = \eta B + \epsilon P,$$  

$$(3.1.14)$$

where $\lambda, \mu, \eta, \epsilon \in \mathbb{R}$. Therefore, we may use the brackets of this algebra to obtain the following result.

Lemma 3.2. The $[H, Q]$ identity produces the conditions

$$0 = n_i \text{Re}(\hbar) \quad \text{where} \quad i \in \{0, 1\},$$
$$0 = \text{Re}(\bar{\hbar} + m_2 \bar{\hbar})$$
$$0 = n_3 + m_4 = n_3 + m_3 \bar{\hbar}$$
$$0 = n_4 + m_4 = n_4 + m_4 \bar{\hbar}. 

$$

$$(3.1.15)$$

The $[Z, Q]$ identity produces the conditions

$$0 = n_i \text{Re}(\hbar) \quad \text{where} \quad i \in \{0, 1\},$$
$$0 = \text{Re}(\bar{\hbar} + m_2 \bar{\hbar})$$

$$0 = n_j + m_j \bar{\hbar} \quad \text{where} \quad j \in \{2, 3, 4\}. 

$$

$$(3.1.16)$$

The $[B, Q]$ identity produces the conditions

$$0 = n_0 \text{Re}(\theta \beta \theta \bar{\hbar})$$
$$0 = \text{Re}(\bar{\theta} \beta (\theta \pi + 2 \theta m_2 \bar{\hbar}))$$
$$0 = \text{Re}(\theta \beta \theta \bar{\hbar} + \beta \theta \bar{m}_2 \bar{\hbar})$$
$$\lambda n_0 |\theta|^2 \beta + \frac{1}{4} |\theta \beta m_2 \bar{\hbar}| = 0 = \text{Re}(\theta \beta \bar{\theta} \beta) + \beta \theta \bar{m}_2 \bar{\hbar}$$
$$\mu n_0 |\theta|^2 \beta = 0 = \text{Re}(\theta \beta \theta \bar{\hbar} + \beta \theta \bar{m}_2 \bar{\hbar}) 

$$(3.1.17)$$
The \([P, Q, Q]\) identity produces the conditions
\[
0 = n_0 \Re(\bar{\theta} \eta \eta_0)
\]
\[
0 = \Re(\bar{\eta}(\eta_3 - 2n_1 \bar{\theta}))
\]
\[
0 = \eta_1 \Re(\bar{\theta} \eta_2 + \pi \theta \eta_4)
\]
(3.1.18)

Proof. The \([H, Q, Q]\) super-Jacobi identity is written
\[
[H, [Q(0), Q(0)]] = 2[H, Q(0)], Q(0)].
\]
(3.1.19)

Using (3.1.2) and (3.1.4), we find
\[
- B(\theta(\lambda n_3 + \eta n_4) - P(\theta(\mu n_3 + \epsilon n_4)) = 2n_0 \Re(\bar{\theta} \eta_0) H + 2n_1 \Re(\bar{\theta} \eta_0) Z
\]
\[
- J(\theta n_2 \bar{\omega} \bar{\eta} + \theta \eta n_2 \bar{\eta}) - B(\theta n_3 \bar{\omega} \bar{\eta} + \theta \eta n_3 \bar{\eta}) - P(\theta n_4 \bar{\omega} \bar{\eta} + \theta \eta n_4 \bar{\eta}).
\]
(3.1.20)

Comparing \(H, Z, J, B,\) and \(P\) coefficients, and using the injectivity and linearity of the maps \(J, B,\) and \(P,\) we find
\[
0 = n_1 \Re(\bar{\eta} h) \quad \text{where} \quad i \in \{0, 1\}
\]
\[
0 = \eta n_2 + n_4 \bar{h}
\]
(3.1.21)

The calculations for the \([Z, Q, Q]\) identity follows in an analogous manner. The key difference in this case is that the L.H.S. vanishes in all instances since \(Z\) commutes with all basis elements. The \([B, Q, Q]\) identity is
\[
[B(\beta), (Q(\theta), Q(\theta))]) = 2[B(\beta), Q(\theta)], Q(\theta)].
\]
(3.1.22)

Substituting in the relevant brackets, the L.H.S. becomes
\[
\text{L.H.S.} = -\lambda n_0 |\bar{\theta}|^2 B(\beta) - \frac{1}{2} B(\theta n_2 \bar{\eta}) - \Re(\bar{\theta} \eta n_4 \bar{\eta}) Z,
\]
(3.1.23)

and the R.H.S. becomes
\[
\text{R.H.S.} = 2n_0 \Re(\bar{\theta} \eta_0 \bar{\beta}) H + 2n_1 \Re(\bar{\theta} \eta_0 \bar{\beta}) Z
\]
\[
- J(\theta n_2 \bar{\omega} \eta_0 + \theta \eta n_2 \eta_0) - B(\theta n_3 \bar{\omega} \eta_0 + \theta \eta n_3 \eta_0) - P(\theta n_4 \bar{\omega} \eta_0 + \theta \eta n_4 \eta_0).
\]
(3.1.24)

Again, comparing coefficients and using the injectivity and linearity of our maps, we get
\[
0 = n_0 \Re(\bar{\theta} \eta_0 \bar{\beta})
\]
\[
0 = \Re(\bar{\eta} \theta n_4 + n_2 \bar{\eta})
\]
\[
0 = 0 \Re(\bar{\theta} \eta_2 + \theta \eta n_2 \bar{\eta})
\]
(3.1.25)

The \([P, Q, Q]\) results follow in identical fashion by replacing \(b\) with \(\eta_1\) and \(\bar{\eta}\) with \(\pi\).

The last super-Jacobi identity to consider is the \((s_1, s_1, s_1)\) case, \([Q, Q, Q]\).

Lemma 3.3. The \([Q, Q, Q]\) identity produces the condition
\[
n_0 h + n_1 z = \frac{1}{4} n_2 + n_4 \bar{b} + n_4 \bar{p}.
\]
(3.1.26)

Proof. The \([Q, Q, Q]\) identity is
\[
0 = [[Q(\theta), Q(\theta)], Q(\theta)].
\]
(3.1.27)

Using (3.1.4), and the brackets in (3.1.2), we find
\[
0 = [n_0 |\bar{\theta}|^2 H + n_1 |\bar{\theta}|^2 Z - J(\theta n_2 \bar{\eta}) - B(\theta n_3 \bar{\eta}) - P(\theta n_4 \bar{\eta}), Q(\theta)]
\]
\[
= n_0 |\bar{\theta}|^2 Q(\theta \bar{h}) + n_1 |\bar{\theta}|^2 Q(\theta \bar{z}) - \frac{1}{2} Q(\theta n_2 \bar{\eta}) - Q(\theta n_3 \bar{\eta} \bar{b}) - Q(\theta n_4 \bar{\eta} \bar{p})
\]
(3.1.28)

Since \(Q\) is injective, this gives us
\[
n_0 h + n_1 z = \frac{1}{4} n_2 + n_4 \bar{b} + n_4 \bar{p}.
\]
(3.1.29)
3.1.1. **Basis Transformations.** As well as modifying the super-Jacobi identities presented in [46], the new \( so(3) \) scalar also impacts the subgroup \( G \subset GL(s_3) \times GL(s_1) \) of basis transformation for kinematical Lie superalgebras. All the automorphisms in \( G \) generated by \( so(3) \) remain the same for \( b, p, \) and \( h \), but we may now add how \( z \) transforms. These automorphisms act by rotating the three imaginary quaternionic bases \( i, j, \) and \( k \) by an element of \( SO(3) \). In particular, we have a homomorphism \( Ad : Sp(1) \to \text{Aut}(H) \) defined such that for \( u \in Sp(1) \) and \( s \in H, \text{Ad}_u(s) = u s u^\dagger \). The map \( \text{Ad}_u \) then acts trivially on the real component of \( s \) and via \( SO(3) \) rotations on \( \text{Im}(H) \). Therefore, \( \tilde{B} = B \circ \text{Ad}_u, \tilde{P} = P \circ \text{Ad}_u, H = H, \tilde{Q} = Q \circ \text{Ad}_u \). Since \( Z \) is an \( so(3) \) scalar, \( \tilde{Z} = Z \). Substituting this with \( \bar{Q} = Q \circ \text{Ad}_u \) into the \( [Q, Q] \) bracket, we find that \( \tilde{z} = u z u \). Additionally, substituting these transformations into the \( [Q, Q] \) bracket, we see that \( c_1 \) remains inert. The other type of transformations to consider are the \( so(3) \)-equivariant maps \( s \to s \). Since we now have two \( so(3) \) scalars, we can have

\[
\hat{H} = a H + b Z \\
\tilde{Z} = c H + d Z
\]

where \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}(2, \mathbb{R}) \). (3.1.30)

The \( so(3) \) vector and spinor maps remain unchanged from those given in [46]. In particular, \( \hat{Q}(s) = Q(s q) \) for \( q \in H^x \). Substituting \( H, \tilde{Z} \) and \( \bar{Q} \) into the brackets

\[
[\hat{H}, \hat{Q}(\theta)] = \hat{Q}(\theta \hat{h}) \\
[\tilde{Z}, \hat{Q}(\theta)] = \hat{Q}(\theta \tilde{z})
\]

we find

\[
\hat{h} = q(a h + b z) q \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\
\tilde{z} = q(c h + d z) q \begin{pmatrix} 1 \\ -1 \end{pmatrix}
\]

These means that the transformation in \( G \) produce the following basis changes

\[
\begin{align*}
J & \mapsto J \circ \text{Ad}_u \\
B & \mapsto e B \circ \text{Ad}_u + f P \circ \text{Ad}_u \\
P & \mapsto h B \circ \text{Ad}_u + i P \circ \text{Ad}_u \\
H & \mapsto a H + b Z \\
Z & \mapsto c H + d Z \\
Q & \mapsto Q \circ \text{Ad}_u \circ R_n
\end{align*}
\]

These transformations may be summarised by \( \{ A = (a \ b \ c \ d), C = (\xi \ \xi'), q, u \} \in \text{GL}(\mathbb{R}^2) \times \text{GL}(\mathbb{R}^2) \times \mathbb{H}^x \times \mathbb{H}^x \).

3.2. **Classification.** The calculations for classifying the super-extensions of \( \hat{h} \) and \( \hat{g} \) all follow identically. It will, therefore, only be stated once below. However, the central extension of the static kinematical Lie algebra is a little different, so will be treated first.

3.2.1. \( \hat{h} \). Using the preliminary results from Lemma 3.1 in section 3.1.1, we find \( b = p = x = 0 \). Substituting these quaternions into the \( [B, Q, Q] \) and \( [P, Q, Q] \) identities with the relevant brackets, we get \( n_2 = 0, n_3 = 0 \) and \( n_4 = 0 \). Then, wanting \( [Q, Q] \neq 0 \), the \( [H, Q, Q] \) conditions tells us that \( \text{Re}(h) = 0 \). Finally, Lemma 3.3 reduces to \( n_0 h = 0 \). Therefore, we have two possible cases: one in which \( n_0 = 0 \) and \( h \in \text{Im}(H) \) and another in which \( h = 0 \) and \( n_0 \) is unconstrained. In the former case, the subgroup \( G \subset GL(s_3) \times GL(s_1) \) can be used to set \( h = k \) and \( n_1 = 1 \), such that the only non-vanishing brackets involving \( Q \) are

\[
[H, Q(\theta)] = Q(\theta k) \quad \text{and} \quad [Q(\theta), Q(\theta)] = \theta^2 Z.
\]

Notice, however, that this case also allows for \( h = 0 \), leaving only

\[
[Q(\theta), Q(\theta)] = \theta^2 Z.
\]

In the latter case, we can use \( G \) to scale \( n_0 \) and \( n_1 \), so the non-vanishing brackets are

\[
[Q(\theta), Q(\theta)] = \theta^2 H + \theta^2 Z.
\]

3.2.2. \( \hat{g} \). Using the preliminary results of Lemmas 3.1 and 3.3, we instantly find \( b = p = x = 0 \), and, subsequently, \( n_0 h = \frac{1}{4} n_2 \). The super-Jacobi identity \( [B, Q, Q] \) then tells us that \( n_2 = n_3 = n_4 = 0 \) and the identity \( [P, Q, Q] \) gives us \( n_3 = 0 \). Thus, the \( (s_1, s_1, s_1) \) condition is trivially satisfied. The only remaining condition is from \( [H, Q, Q] \), which tells us \( n_1 \text{Re}(h) = 0 \). Since we want \( [Q, Q] \neq 0 \), we must have \( n_1 \neq 0 \), therefore, \( h \in \text{Im}(H) \). Using \( G \) to set \( h = k \) and \( n_1 = 1 \), we have non-vanishing brackets

\[
[H, Q(\theta)] = Q(\theta k) \quad \text{and} \quad [Q(\theta), Q(s)] = \theta^2 Z.
\]

Similar to the \( \hat{h} \) case, the restriction \( h \in \text{Im}(H) \) does not remove the choice \( h = 0 \). Therefore, we may also have

\[
[Q(\theta), Q(s)] = \theta^2 Z.
\]
as the only non-vanishing bracket.

3.3. **Summary.** Table 3 lists all the $N = 1$ generalised Bargmann superalgebras we have classified. Each Lie superalgebra is an $N = 1$ super-extension of one of the generalised Bargmann algebras given in Table I, taken from [4]. It is interesting to compare this list of $N = 1$ super-extensions of centrally-extended kinematical Lie algebras to the list of centrally-extended $N = 1$ kinematical Lie superalgebras given in Table 4. This table is a reduced and adapted version of one given in [46], where we have only kept those extensions built upon the static, Newton-Hooke, and Galilean algebras.

Notice that only one of the generalised Bargmann superalgebras is present in the classification of centrally-extended kinematical Lie superalgebras, S2. Although it does not match exactly, we can use the basis transformations in $G \subset GL(s_l) \times GL(s_l)$ to bring it into the same form as the second superalgebra in Table 4. This result shows us that, in general, super-extending and centrally-extending kinematical Lie algebras are not commutative processes.

![Table 3](image)

The first column gives each generalised Bargmann superalgebra a unique identifier, and the second column tells us the underlying generalised Bargmann algebra $t$. The next four columns tell us how the $s_l$ generators $H, Z, B,$ and $P$ act on $Q$. Recall, the action of $J$ is fixed, so we do not need to state this explicitly. The final column then specifies the $[Q, Q]$ bracket.

![Table 4](image)

The first column identifies the kinematical Lie algebra $t$ underlying the extensions. The second column indicates whether the central extension has been introduced in the $[B, P]$ bracket. The next four columns show the $[s_{ij}, s_{ij}]$ brackets for the KLSA. As we can see, the only non-vanishing case is $[H, Q(\theta)] = Q(\theta h)$, where $\theta \in H$ and $h \in \operatorname{Im}(H)$. The final column then tells us whether the central extension enters the $[Q, Q]$ bracket.

3.3.1. **Unpacking the Notation.** Although the quaternionic formulation of these superalgebras is convenient for our purposes, it is perhaps unfamiliar to the reader. Therefore, in this section, we will convert the $N = 1$ superextension for the Bargmann algebra into a more conventional format. The brackets for this algebra, excluding the $s_{ij}$ brackets which have already been discussed in section 2.2, take the form

$$[H, Q(\theta)] = Q(\theta h) \quad \text{and} \quad [Q(\theta), Q(\theta)] = [\theta]^2 Z.$$  

(3.3.1)
Letting \( Q(\theta) = \sum_{a=1}^{4} \theta_a Q_a \), where \( \theta = \theta_4 + \theta_1 \beta + \theta_2 \bar{\beta} + \theta_3 \kappa \), we can rewrite these brackets as
\[
[H, Q_a] = \sum_{b=1}^{4} Q_b \Sigma^b \quad \text{and} \quad [Q_a, Q_b] = \delta_{ab} Z, \tag{3.3.2}
\]
where, for \( \sigma_2 \) being the second Pauli matrix,
\[
\Sigma = \begin{pmatrix} 0 & i\sigma_2 \\ -i\sigma_2 & 0 \end{pmatrix}. \tag{3.3.3}
\]

4. **N = 2 Extensions of the Generalised Bargmann Algebras**

Having established the introduction of the central extension \( Z \) into our classification problem in section 3, we now look to introduce an additional \( so(3) \) spinor module. Section 4.1 will describe the setup for the \( N = 2 \) generalised Bargmann superalgebras, before extending the preliminary results from [46] to this case. It is also in this section that the group of basis transformations \( G \) will be adapted for extended supersymmetry. The number of additional parameters in this case means that there are several branches of super-extension for each generalised Bargmann algebra. In section 4.2, we use the preliminary results from the \( \{ s_0, s_\beta, s_1 \} \) super-Jacobi identities to identify four possible branches. Each branch is then explored in detail in section 4.3 where we identify the non-empty sub-branches, which are summarised in section 4.4.

4.1. **Setup for the N = 2 Calculation.** For completeness, recall that, in addition to the standard kinematical Lie brackets, the brackets for the universal generalised Bargmann superalgebra are
\[
\begin{align*}
[B(\beta), P(\pi)] &= \text{Re}(\bar{\beta}\pi)Z \\
[H, B(\beta)] &= \lambda B(\beta) + \mu P(\beta) \\
[H, P(\pi)] &= \eta B(\pi) + \epsilon P(\pi),
\end{align*}
\]
where \( \beta, \pi \in \text{Im}(H) \) and \( \lambda, \mu, \eta, \epsilon \in \mathbb{R} \). Because we now have two spinor modules, the brackets involving \( s_1 \) need to be adapted from those given in [46]. We will continue to use the map \( Q \) for the odd dimensions; however, it now acts on \( \theta \), a vector in \( H^2 \). We will choose \( H^2 \) to be a left quaternionic vector space such that \( H \) acts linearly from the left and all \( 2 \times 2 \) \( H \) matrices act on the right. Therefore, writing \( \theta \) out in its components, we have
\[
\theta = (\theta_1, \theta_2), \tag{4.1.2}
\]
where \( \theta_1, \theta_2 \in H \). The \( \{ s_0, s_1 \} \) brackets are again the \( so(3) \)-equivariant endomorphisms of \( s_1 \). Since we choose \( so(3) \) to act via left quaternionic multiplication, the commuting endomorphisms are all those that may act on the right. In the present case, these are elements of \( \text{Mat}_2(H) \). Thus the brackets containing the \( so(3) \) scalars are
\[
\begin{align*}
[H, Q(\theta)] &= Q(\theta H) \\
[Z, Q(\theta)] &= Q(\theta Z),
\end{align*}
\]
where \( H, Z \in \text{Mat}_2(H) \). In [46], \( [J, Q], [B, Q] \) and \( [P, Q] \) were described by Clifford multiplication \( V \otimes S \to S \), which is given by left quaternionic multiplication by \( \text{Im}(H) \). The space of such maps is isomorphic to the space of \( \tau \)-equivariant maps of \( S \), which is a copy of the quaternions. Now with \( s_1 = S \oplus S \), we have four possible endomorphisms of this type. Labelling the two spinor modules \( S_1 \) and \( S_2 \), we may use the Clifford action to map \( S_1 \) to \( S_1 \), \( S_1 \) to \( S_2 \), \( S_2 \) to \( S_1 \), or \( S_2 \) to \( S_2 \). All of these maps may be summarised as follows
\[
\begin{align*}
[J(\omega), Q(\theta)] &= \frac{1}{2} Q(\omega \theta) \\
[B(\beta), Q(\theta)] &= Q(\beta \theta B) \\
[P(\pi), Q(\theta)] &= Q(\pi \theta P),
\end{align*}
\]
Here, \( \omega, \beta, \pi \in \text{Im}(H) \) and \( B, P \in \text{Mat}_2(H) \). Finally, consider the \( [Q, Q] \) bracket. This will consist of the \( so(3) \)-equivariant \( \mathbb{R} \)-linear maps \( \bigotimes^2 s_1 \to s_0 \). To write down these maps, we make use of the \( so(3) \)-invariant inner product on \( s_1 \)
\[
\langle \theta, \theta' \rangle = \text{Re}(\theta \theta^\dagger) \quad \text{where} \quad \theta, \theta' \in H^2 \quad \theta^\dagger = \bar{\theta}^T. \tag{4.1.5}
\]
This bracket’s \( so(3) \)-invariance is clear on considering left multiplication by \( u \in \text{sp}(1) \) and noting \( \text{sp}(1) \cong so(3) \). We can now use this bracket to identify \( \bigotimes^2 s_1 \) with the symmetric \( \mathbb{R} \)-linear endomorphisms of \( s_1 \cong S^2 \cong H^2 \), i.e. the maps \( \mu : H^2 \to H^2 \) such that \( \langle \mu(\theta), \theta' \rangle = \langle \theta, \mu(\theta') \rangle \). A general \( \mathbb{R} \)-linear map of \( H^2 \) may be written
\[
\mu(\theta) = \phi(\theta) M \quad \text{where} \quad \phi \in \text{Im}(H) \quad \text{and} \quad M \in \text{Mat}_2(H). \tag{4.1.6}
\]
Now, inserting this definition into the condition for a symmetric endomorphism, we obtain the following two cases:
1. \( \phi \in \mathbb{R} \) and \( M = M^\dagger \)
2. \( \phi \in \text{Im}(H) \) and \( M = -M^\dagger \).
The first instance gives us our $\mathfrak{so}(3)$ scalar modules in $\mathbb{O}^2 S^2$; therefore, these will map to either $H$ or $Z$ in $s_0$ to ensure we have $\mathfrak{so}(3)$-equivariance. The condition on $M$ states that it must be of the form

$$M = \begin{pmatrix} a & b + m \\ b - m & c \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + m \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

(4.1.7)

where $a, b, c \in \mathbb{R}$ and $m \in \text{Im}(H)$. We can make sense of this result using the decomposition of the odd part of the superalgebra, $s_1 \cong S^2 \cong S \otimes \mathbb{R}^2$, and our knowledge of the maps in $\text{Hom}(\mathbb{O}^2 S, s_0)$ derived from [46]. Symmetrising the decomposed $s_1$, we get $\mathbb{O}^2 S^2 \cong \mathbb{O}^2 S \otimes \mathbb{O}^2 R^2 \otimes \Lambda^2 S \otimes \Lambda^2 R^2$. Notice that the coefficients for the $\mathbb{O}^2 \mathbb{R}^2$ basis elements result in multiplying $\theta \in S^2$ by $\mathbb{R}$ on both the left and right. Thus we find three copies of the scalar map in $\mathbb{O}^2 S \rightarrow \mathbb{R} \oplus 3\mathbb{V}$, one for each of the $\mathbb{O}^2 \mathbb{R}^2$ basis. Similarly, the coefficient for the $\Lambda^2 \mathbb{R}^2$ basis element produces the three scalar maps in $\Lambda^2 S \rightarrow 3\mathbb{R} \oplus \mathbb{V}$. Thus the above matrix accounts for the $\mathfrak{so}(3)$ scalar maps in both the symmetric and anti-symmetric components of the decomposition.

Now, the second case gives us our $\mathfrak{so}(3)$ vector modules in $\mathbb{O}^2 S^2$; therefore, these will map to $B, P$, or $J$ to ensure $\mathfrak{so}(3)$-equivariance. The condition on $M$ in this case produces

$$M = \begin{pmatrix} r & -d + l \\ -d - l & r \end{pmatrix} = r \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + l \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + r \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + d \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

(4.1.8)

where $r, l, d \in \text{Im}(H)$ and $d \in \mathbb{R}$. Again, we can understand this result through the decomposition of $s_1$. From [46], we know that the $\mathfrak{so}(3)$ vectors in $\text{Hom}(\mathbb{O}^2 S, s_0)$ come from simultaneous left and right quaternionic multiplication by $\text{Im}(H)$. These are precisely the maps we find in the coefficients for the $\mathbb{O}^2 \mathbb{R}^2$ basis, as expected through the decomposition of $\mathbb{O}^2 S^2$. Finally, we recover the $\mathfrak{so}(3)$ vector in $\Lambda^2 S \rightarrow 3\mathbb{R} \oplus \mathbb{V}$ as the coefficient to the $\Lambda^2 \mathbb{R}^2$ basis.

Putting all this together, we may write

$$[Q(\theta), Q(\theta')] = (\theta, \theta N_0) H + (\theta, \theta N_1) Z + (\theta, J\theta N_1) + (\theta, B\theta N_3) + (\theta, P\theta N_4),$$

(4.1.9)

where $N_0, N_1$ are quaternion Hermitian, and $N_2, N_3, N_4$ are quaternion skew-Hermitian, as stated above, and

$$J = J_{i_0} + J_{i_1} + J_{i_2}, \quad B = B_{i_0} + B_{i_1} + B_{i_2}, \quad P = P_{i_0} + P_{i_1} + P_{i_2}.$$

(4.1.10)

Using the fact that $\text{Re}(\omega J) = J(\omega)$ and $N_i = N_i^\dagger$, for $i \in \{0, 1\}$, we can write

$$[Q(\theta), Q(\theta')] = \text{Re}(\theta N_0 \theta^\dagger H) + \text{Re}(\theta N_1 \theta^\dagger Z) - J(\theta N_2 \theta^\dagger) - B(\theta N_3 \theta^\dagger) - P(\theta N_4 \theta^\dagger).$$

(4.1.11)

This polarises to

$$[Q(\theta), Q(\theta^\dagger)] = \frac{1}{2} \left( \text{Re}(\theta N_0 \theta^\dagger H) + \text{Re}(\theta N_1 \theta^\dagger Z) - J(\theta N_2 \theta^\dagger) - B(\theta N_3 \theta^\dagger) - P(\theta N_4 \theta^\dagger) \right).$$

(4.1.12)

4.1.1. Preliminary Results. As in the $N = 1$ case, we can form a number of universal results that will help us when investigating the super-extensions of the generalised Bargmann algebras. The following subsections will cover the $(s_0, s_0, s_0), (s_0, s_1, s_0), \text{and } (s_1, s_1, s_1)$ identities, respectively.

$$\{s_0, s_0, s_0\}.$$

In the $N = 1$ case, $h, z, b, p \in \mathbb{H}$, and in the $N = 2$ case $H, Z, B, P \in \text{Mat}_2(\mathbb{H})$. Since $\mathbb{H}$ and $\text{Mat}_2(\mathbb{H})$ are both associative, non-commutative algebras, the algebraic manipulations are the same in both cases. Therefore, the $N = 1$ results generalise to the $N = 2$ case; the only difference being that the variables are $2 \times 2$ matrices rather than $\mathbb{H}$ elements.

**Lemma 4.1.** The following relations between $H, Z, B, P \in \text{Mat}_2(\mathbb{H})$ are implied by the corresponding $\varepsilon$-brackets:

$$[H, Z] = hH + \mu Z \implies [Z, H] = \lambda H + \mu Z.$$

$$[H, B] = \lambda B + \mu P \implies [B, H] = \lambda B + \mu P.$$

$$[H, P] = \lambda B + \mu P \implies [P, H] = \lambda B + \mu P.$$

$$[Z, B] = \lambda B + \mu P \implies [B, Z] = \lambda B + \mu P.$$

$$[Z, P] = \lambda B + \mu P \implies [P, Z] = \lambda B + \mu P.$$

$$[B, B] = AB + \mu P + vJ \implies B^2 = \frac{1}{2} AB + \frac{1}{2} \mu P + \frac{1}{4} v.$$

$$[P, P] = AB + \mu P + vJ \implies P^2 = \frac{1}{2} AB + \frac{1}{2} \mu P + \frac{1}{4} v.$$

$$[B, P] = \lambda H + \mu Z \implies BP + PB = 0 \text{ and } [B, P] = \lambda H + \mu Z.$$

**Proof.** See the proof of Lemma 3.1 for the algebraic manipulations that produce the above results. \qed
(s_0, s_1, s_1).

As in the N = 1 case, we use the universal generalised Bargmann algebra to simplify our analysis here. Recall the brackets for this algebra are

\[ [B, P] = Z, \quad [H, B] = \lambda B + \mu P, \quad [H, P] = \eta B + \varepsilon P, \]

where \( \lambda, \mu, \eta, \varepsilon \in \mathbb{R} \). Using these brackets, we obtain the following result.

**Lemma 4.2.**

The \([H, Q, Q]\) identity produces the conditions

\[ \begin{align*}
0 &= HN_i + N_iH^\dagger & \text{where} & & i \in \{0, 1, 2\} \\
\lambda N_3 + \eta N_4 &= HN_3 + N_3H^\dagger \\
\mu N_3 + \varepsilon N_4 &= HN_4 + N_4H^\dagger.
\end{align*} \]

The \([Z, Q, Q]\) identity produces the conditions

\[ 0 = ZN_i + N_iZ^\dagger \quad \text{where} \quad i \in \{0, 1, 2, 3, 4\}. \]

The \([B, Q, Q]\) identity produces the conditions

\[ \begin{align*}
0 &= BN_0 - N_0B^\dagger \\
N_4 &= BN_1 - N_1B^\dagger \\
0 &= \beta \theta BN_2\theta^\dagger + \theta N_2(\beta \theta B)^\dagger \\
\lambda \text{Re}(\theta N_0\theta^\dagger)\beta + \frac{1}{2}[\beta, \theta N_2\theta^\dagger] &= \beta \theta BN_4\theta^\dagger + \theta N_4(\beta \theta B)^\dagger \\
\mu \text{Re}(\theta N_0\theta^\dagger)\beta &= \beta \theta BN_4\theta^\dagger + \theta N_4(\beta \theta B)^\dagger.
\end{align*} \]

The \([P, Q, Q]\) identity produces the conditions

\[ \begin{align*}
0 &= PN_0 - N_0P^\dagger \\
-N_3 &= PN_1 - N_1P^\dagger \\
0 &= \pi \theta PN_2\theta^\dagger + \theta N_2(\pi \theta P)^\dagger \\
\eta \text{Re}(\theta N_0\theta^\dagger)\pi &= \pi \theta PN_4\theta^\dagger + \theta N_4(\pi \theta P)^\dagger \\
\varepsilon \text{Re}(\theta N_0\theta^\dagger)\pi + \frac{1}{2}[\pi, \theta N_2\theta^\dagger] &= \pi \theta PN_4\theta^\dagger + \theta N_4(\pi \theta P)^\dagger,
\end{align*} \]

where \( \beta, \pi \in \text{Im}(\mathbb{H}) \) and \( \theta \in \mathbb{H}^\dagger \).

**Proof.** Beginning with the \([H, Q, Q]\) identity, we have

\[ \text{L.H.S.} = -[H, B(\theta N_4\theta^\dagger)] - [H, P(\theta N_4\theta^\dagger)] \]

\[ = -B(\theta(\lambda N_3 + \eta N_4)\theta^\dagger) - P(\theta(\mu N_3 + \varepsilon N_4)\theta^\dagger). \]

Substituting \([H, Q(\theta)] = Q(\theta H)\) into the R.H.S. and using the polarised form of the \([Q, Q]\) bracket, we find

\[ \text{R.H.S.} = B(\theta(\lambda N_3 + \eta N_4)\theta^\dagger) + P(\theta(\mu N_3 + \varepsilon N_4)\theta^\dagger)Z \]

\[ - J(\theta(\lambda HN_2 + N_2H^\dagger)\theta^\dagger) - B(\theta(\mu HN_3 + N_3H^\dagger)\theta^\dagger) - P(\theta(\varepsilon HN_4 + N_4H^\dagger)\theta^\dagger). \]

Comparing coefficients and using the injectivity and linearity of the maps \( J, B \) and \( P \), we get the desired conditions. The \([Z, Q, Q]\) result follows in an analogous manner. Consider the \([B, Q, Q]\) Jacobi identity

\[ [B(\beta), [Q(\theta), Q(\theta)]] = 2[[B(\beta), Q(\theta)], Q(\theta)]. \]

Since \( B \) commutes with \( Z \) and \( B \), the L.H.S. takes the following form

\[ \text{L.H.S.} = B(\beta), Re(\theta N_0\theta^\dagger)J - J(\theta N_2\theta^\dagger) - P(\theta N_4\theta^\dagger) \]

\[ = -\text{Re}(\theta N_0\theta^\dagger)(\lambda B(\beta) + \mu P(\beta)) + \frac{1}{2}B(\theta N_2\theta^\dagger, \beta) - \text{Re}(\theta N_4\theta^\dagger)Z. \]

Turning attention to the R.H.S., we find

\[ \text{R.H.S.} = B(\beta), \theta N_0\theta^\dagger + \theta N_0B^\dagger \theta^\dagger \theta\beta)H + \text{Re}(\theta BN_1\theta^\dagger + \theta N_1B^\dagger \theta^\dagger \beta)Z \]

\[ - J(\beta BN_2\theta^\dagger + \theta N_2B^\dagger \theta^\dagger \beta) - B(\beta BN_3\theta^\dagger + \theta N_3B^\dagger \theta^\dagger \beta) - P(\beta BN_4\theta^\dagger + \theta N_4B^\dagger \theta^\dagger \beta). \]
Using the property $\tilde{\beta} = -\beta$, since $\beta \in \text{Im}(H)$, and the cyclic property of $\text{Re}$, the first two terms can have their coefficients written in the form

$$\text{Re}(\beta \theta BN_i \theta^\dagger + \theta N_i B^\dagger \theta^\dagger \tilde{\beta}) = \text{Re}(\beta \theta (BN_i - N_i B^\dagger) \theta^\dagger),$$

(4.1.22)

for $i \in \{0, 1\}$. Again, comparing coefficients we obtain the desired results. The $[P, Q, Q]$ case follows identically. \qed

\begin{itemize}
\item \{s_1, s_1, s_1\}
\end{itemize}

The last super-Jacobi identity to consider is the $\{s_1, s_1, s_1\}$ case, $[Q, Q, Q]$.

**Lemma 4.3.** The $[Q, Q, Q]$ identity produces the condition

$$\text{Re}(\theta N_i \theta^\dagger)\theta H + \text{Re}(\theta N_i \theta^\dagger)\theta Z = \frac{1}{2} \theta N_2 \theta^\dagger \theta + \theta N_3 \theta^\dagger \theta B + \theta N_4 \theta^\dagger \theta P.$$  

(4.1.23)

**Proof.** The $[Q, Q, Q]$ identity is written

$$0 = [[Q(\theta), Q(\theta)], Q(\theta)].$$  

(4.1.24)

Substituting in the $[Q, Q]$ bracket, this becomes

$$0 = [\text{Re}(\theta N_0 \theta^\dagger)H + \text{Re}(\theta N_2 \theta^\dagger)Z - J(\theta N_2 \theta^\dagger) - B(\theta N_3 \theta^\dagger) - P(\theta N_4 \theta^\dagger), Q(\theta)].$$  

(4.1.25)

Finally, using the brackets in (4.1.3) and (4.1.4) and the injectivity of $Q$, we obtain the desired result. \qed

4.1.2. **Basis Transformations.** We will investigate the subgroup $G \subset \text{GL}(s_0) \times \text{GL}(s_1)$ by first looking at the transformations induced by the adjoint action of the rotational subalgebra $r \cong \mathfrak{so}(3)$. We will then look at the $\mathfrak{so}(3)$-equivariant maps transforming the basis of the underlying vector space. These will act via Lie algebra automorphisms in $s_0$ and endomorphisms of the $\mathfrak{so}(3)$ module $S^2$ in $s_1$. Note, in the former case, where the automorphism is induced by $ad_{\lambda}$, each $\mathfrak{so}(3)$ module will transform into itself, while, in the latter case, when the transformation is some $\mathfrak{so}(3)$-equivariant map, the modules transform into one another. For completeness, at the end of the section, we determine the automorphisms of each generalised Bargmann algebra.

Recall that $\text{Sp}(1)$ is the double-cover of $\text{Aut}(H)$, and $\text{Aut}(H) \cong \text{SO}(3)$. We, therefore, write $\lambda \in \text{Aut}(H)$ as $\lambda(s) = usu$ for some $u \in \text{Sp}(1)$, which will act trivially on the real component of $s$ and rotate the imaginary components. Using this result, we can represent the action of $\text{Aut}(H)$ on the $\mathfrak{so}(3)$ vector modules in $s_0$ by pre-composing the linear maps $J$, $B$, and $P$ with $\text{Ad}_u$, for $u \in \text{Sp}(1)$. To preserve the kinematical brackets in $[s_0, s_0]$, we must pre-compose with the same $u$ for each map. Note, $\mathfrak{so}(3)$ acts trivially on $H$ and $Z$, so these basis elements will be left invariant under these automorphisms. For $s_1$, we restrict to the individual copies of $S$ through diagonal matrices. To preserve the $[J, Q]$ bracket, we must pre-compose with the same $u$ as above.

Therefore, we write $\tilde{Q}(\theta) = Q(\theta)\tilde{u}$. We can now investigate how these automorphisms affect our brackets

$$[J(\omega), Q(\theta)] = \frac{1}{2} Q(\omega \theta), \quad [B(\beta), Q(\theta)] = Q(\beta \theta B), \quad [P(\pi), Q(\theta)] = Q(\pi \theta P).$$  

(4.1.26)

$$[Q(\theta), Q(\theta)] = \text{Re}(\theta N_0 \theta^\dagger)H + \text{Re}(\theta N_2 \theta^\dagger)Z - J(\theta N_2 \theta^\dagger) - B(\theta N_3 \theta^\dagger) - P(\theta N_4 \theta^\dagger).$$

Transforming the basis, we have

$$[\tilde{J}(\omega), \tilde{Q}(\theta)] = \frac{1}{2} \tilde{Q}(\omega \theta), \quad [\tilde{B}(\beta), \tilde{Q}(\theta)] = \tilde{Q}(\beta \theta \tilde{B}), \quad [\tilde{P}(\pi), \tilde{Q}(\theta)] = \tilde{Q}(\pi \theta \tilde{P}).$$  

(4.1.27)

$$[\tilde{Q}(\theta), \tilde{Q}(\theta)] = \text{Re}(s_0 \theta N_0 \theta^\dagger)\tilde{H} + \text{Re}(s_0 \theta N_2 \theta^\dagger)\tilde{Z} - \tilde{J}(\theta N_2 \theta^\dagger) - \tilde{B}(\theta N_3 \theta^\dagger) - \tilde{P}(\theta N_4 \theta^\dagger),$$

with $\tilde{H} = H$, $\tilde{Z} = Z$, $\tilde{J} = J \circ \text{Ad}_u$, $\tilde{B} = B \circ \text{Ad}_u$, $\tilde{P} = P \circ \text{Ad}_u$, and $\tilde{Q} = Q \circ \text{Ad}_u$, where it is understood that $\text{Ad}_u$ acts diagonally on the $s_1$ basis, $Q$. The transformed matrices are

$$\tilde{H} = DHD^{-1}, \quad \tilde{B} = DBD^{-1}, \quad \tilde{Z} = DZD^{-1}, \quad \tilde{P} = DPD^{-1}, \quad \tilde{N}_i = D_N i D^{-1}.$$  

(4.1.28)

where $D = u\mathbb{1}$ for $u \in \text{Sp}(1)$ and $i \in \{0, 1, \ldots, 4\}$. Therefore, $D^{-1} = D^\dagger = u\mathbb{1}$. These automorphisms simultaneously rotate all quaternions, all the components of the matrices $H, Z, B, P$ and $N_i$, by the same $\text{Sp}(1)$ element.
Next, we want to consider the \( \mathfrak{so}(3) \)-equivariant linear maps which leave the rotational subalgebra invariant: \( (J, B, P, H, Z, Q) \rightarrow (J, \tilde{B}, \tilde{P}, \tilde{H}, \tilde{Z}, \tilde{Q}) \). These take the general form

\[
\begin{align*}
\tilde{H} &= aH + bZ \\
\tilde{Z} &= cH + dZ \\
\tilde{B}(\beta) &= eB(\beta) + fP(\beta) + gJ(\beta) \\
\tilde{P}(\pi) &= hB(\pi) + iP(\pi) + jJ(\pi) \\
\tilde{Q}(\theta) &= Q(\theta M),
\end{align*}
\]  

(4.1.29)

where \( a, ..., j \in \mathbb{R} \) and \( M \in GL(H^2) \). Crucially,

\[
A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{R}) \quad \text{and} \quad C = \begin{pmatrix} e & f & g \\ h & i & j \\ 0 & 0 & 1 \end{pmatrix} \in GL(3, \mathbb{R}),
\]  

(4.1.30)

act on \( (H, Z)^T \) and \( (B, P, J)^T \), respectively. Each of the generalised Bargmann algebra allows different transformations of this type; however, there are some important general results. Therefore, we will begin by working through the analysis of these maps with the universal generalised Bargmann algebra before focussing on each algebra separately.

As in the \( N = 1 \) case, the checking of brackets that include \( J \) is really verifying that the above maps are \( \mathfrak{so}(3) \)-equivariant, so this does not give us any information not already presented. The first bracket we will consider is \( [B, P] = Z \). Substituting in the maps of (4.1.29), we find the following important results:

\[
d = ei - fh, \quad c = 0, \quad \text{and} \quad g = j = 0.
\]  

(4.1.31)

The vanishing of \( c \) tells us that \( d \neq 0 \) if we are to have \( A \in GL(2, \mathbb{R}) \). Also, the vanishing of \( g \) and \( j \) shows that we can reduce \( C \) to an element of \( GL(2, \mathbb{R}) \),

\[
C = \begin{pmatrix} e & f \\ h & i \end{pmatrix},
\]

acting on \( (B, P)^T \). The remaining \([s_0, s_1]\) brackets are \([H, B]\) and \([H, P]\), which produce

\[
0 = \lambda e(a - 1) + \eta af - \mu h \quad 0 = \eta(e - ai) + \epsilon h - \lambda ah \\
0 = \lambda f - \epsilon af + \mu(i - ea) \quad 0 = \eta f + \epsilon i(1 - a) - \mu ah,
\]

(4.1.33)

respectively. Clearly, these conditions are dependent on the exact choice of generalised Bargmann algebra, so we will leave these results in this form for now.

Now, since the \([s_0, s_1]\) and \([s_1, s_1]\) brackets are so far independent of the chosen algebra, the following results will hold for all the generalised Bargmann algebras. Reusing (4.1.27), in this instance we find

\[
\begin{align*}
\tilde{H} &= M(aH + bZ)M^{-1} \\
\tilde{Z} &= dMZM^{-1} \\
\tilde{B} &= M(eB + fP)M^{-1} \\
\tilde{P} &= M(hB + iP)M^{-1}
\end{align*}
\]

(4.1.34)

Putting the two types of transformation in \( G \) together, we have

\[
\begin{align*}
J &\mapsto J \circ Ad_u \\
B &\mapsto eB \circ Ad_u + fP \circ Ad_u \\
P &\mapsto hB \circ Ad_u + iP \circ Ad_u \\
H &\mapsto aH + bZ \\
Z &\mapsto dZ \\
Q &\mapsto Q \circ Ad_u \circ R_M.
\end{align*}
\]  

(4.1.35)

These transformations may be summarised by \( (A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, C = \begin{pmatrix} e & f & g \\ h & i & j \\ 0 & 0 & 1 \end{pmatrix}, M, u) \in GL(\mathbb{R}^2) \times GL(\mathbb{R}^2) \times GL(H^2) \times \mathbb{R}^\infty \). Now that we have the most general element of the subgroup \( G \subset GL(s_0) \times GL(s_1) \) for \( s_0 = t \) the universal generalised Bargmann algebra, we can restrict ourselves to the automorphisms of \( s_0 \) and set the parameters \( \lambda, \mu, \eta, \epsilon \in \mathbb{R} \) to determine the automorphism group for each of the generalised Bargmann algebras. The results of this investigation are presented in Table 5.
In this instance, all the conditions vanish as \( \lambda = \mu = \eta = \epsilon = 0 \); therefore, the matrices \( A \) and \( C \) are left as stated above.

\[ \hat{a} \]

Having \( \lambda = -\epsilon = 1 \) and \( \mu = \eta = 0 \), the conditions in (4.1.33) become

\[
\begin{align*}
0 &= e(a-1) \quad \text{and} \quad 0 = h(1+a) \\
0 &= f(1+a) \quad \text{and} \quad 0 = i(1-a).
\end{align*}
\]

(4.1.36)

Notice, if \( a \notin \{\pm 1\} \) then \( C \) must vanish, which cannot happen if we are to retain the basis elements \( B \) and \( P \). Therefore, we are left with two cases: \( a = 1 \) and \( a = -1 \). In the former instance, we have automorphisms with

\[
A = \begin{pmatrix} 1 & b \\ 0 & e^2 + h^2 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} e & h \\ 0 & e \end{pmatrix}.
\]

(4.1.37)

In the latter instance, we have

\[
A = \begin{pmatrix} -1 & b \\ 0 & -eh \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 0 & h \\ f & 0 \end{pmatrix}.
\]

(4.1.38)

In this case, \( \lambda = \epsilon = 0 \) and \( \mu = -\eta = 1 \). Therefore, our constraints become

\[
\begin{align*}
0 &= h + af \\
0 &= e - ai \\
0 &= i - ae \quad \text{and} \quad 0 = f + ah.
\end{align*}
\]

(4.1.39)

Taking the expressions for \( h \) and \( i \) from the conditions on the left and substituting them into the conditions on the right, we find

\[
0 = (1 - a^2)f \quad \text{and} \quad 0 = (1 - a^2)e.
\]

(4.1.40)

If \( a^2 \neq 1 \), we would need both \( f \) and \( e \) to vanish, which contradicts our assumption that \( C \in \text{GL}(2, \mathbb{R}) \). Therefore, we need \( a^2 = 1 \), which presents two cases: \( a = 1 \) and \( a = -1 \). In the former instance, we find automorphisms of the form

\[
A = \begin{pmatrix} 1 & b \\ 0 & e^2 + h^2 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} e & h \\ 0 & e \end{pmatrix}.
\]

(4.1.41)

In the latter instance, we get

\[
A = \begin{pmatrix} -1 & b \\ 0 & -e^2 - h^2 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} e & h \\ h & -e \end{pmatrix}.
\]

(4.1.42)

Finally, we have \( \lambda = \eta = \epsilon = 0 \) and \( \mu = -1 \), which, when substituted into (4.1.33), produces

\[
0 = h \quad \text{and} \quad i = ae.
\]

(4.1.43)

Therefore, automorphisms for the Bargmann algebra take the form

\[
A = \begin{pmatrix} a & b \\ 0 & ie \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} e & f \\ 0 & ae \end{pmatrix}.
\]

(4.1.44)

**Table 5. Automorphisms of the generalised Bargmann algebras**

| \( \hat{a} \) | \( (a \ b), (e \ f) \) |
| \( \hat{n}_- \) | \( \begin{pmatrix} 1 & b \\ 0 & ei \end{pmatrix}, \begin{pmatrix} e & 0 \\ 0 & i \end{pmatrix} \cup \begin{pmatrix} -1 & b \\ 0 & -h \end{pmatrix}, \begin{pmatrix} 0 & h \\ f & 0 \end{pmatrix} \) |
| \( \hat{n}_+ \) | \( \begin{pmatrix} 1 & b \\ 0 & e^2 + h^2 \end{pmatrix}, \begin{pmatrix} e & h \\ -h & e \end{pmatrix} \cup \begin{pmatrix} -1 & b \\ 0 & -e^2 - h^2 \end{pmatrix}, \begin{pmatrix} e & h \\ h & -e \end{pmatrix} \) |
| \( \hat{g} \) | \( \begin{pmatrix} a & b \\ 0 & ie \end{pmatrix}, \begin{pmatrix} e & f \\ 0 & ae \end{pmatrix} \) |

| \( \hat{a} \) | \( (a \ b), (e \ f) \) |
| \( \hat{n}_- \) | \( \begin{pmatrix} 1 & b \\ 0 & ei \end{pmatrix}, \begin{pmatrix} e & 0 \\ 0 & i \end{pmatrix} \cup \begin{pmatrix} -1 & b \\ 0 & -h \end{pmatrix}, \begin{pmatrix} 0 & h \\ f & 0 \end{pmatrix} \) |
| \( \hat{n}_+ \) | \( \begin{pmatrix} 1 & b \\ 0 & e^2 + h^2 \end{pmatrix}, \begin{pmatrix} e & h \\ -h & e \end{pmatrix} \cup \begin{pmatrix} -1 & b \\ 0 & -e^2 - h^2 \end{pmatrix}, \begin{pmatrix} e & h \\ h & -e \end{pmatrix} \) |
| \( \hat{g} \) | \( \begin{pmatrix} a & b \\ 0 & ie \end{pmatrix}, \begin{pmatrix} e & f \\ 0 & ae \end{pmatrix} \) |

(4.1.44)
4.2. Establishing Branches. Before proceeding to the discussion in which the non-empty sub-branches are identified, we first establish the possible \([s_0, s_1]\) brackets. More specifically, we establish the possible forms for \(Z, H, B, P \in \text{Mat}_2(\mathbb{H})\). In this section, we focus solely on the results of Lemma 4.1 concerning the \((s_0, s_0, s_1)\) super-Jacobi identities. Using the universal generalised Bargmann algebra, we find that \(B, P \in \text{Mat}_2(\mathbb{H})\), which encode the brackets \([B, Q]\) and \([P, Q]\), respectively, form a double complex. Analysing this structure, we identify four possible cases:

1. \(B = 0\) and \(P = 0\)
2. \(B = 0\) and \(P \neq 0\)
3. \(B \neq 0\) and \(P = 0\)
4. \(B \neq 0\) and \(P \neq 0\).

Taking each of these cases in turn, we find forms for \(Z\) and \(H\) to establish four branches. These branches will form the basis for our investigations into the possible super-extensions for each of the generalised Bargmann algebras in section 4.3.

Using the results of Lemma 4.1, we notice that \(B^2 = P^2 = 0\) and \(BP + PB = 0\); therefore, \(B\) and \(P\) are the differentials of a double complex in which the modules are \(s_1\). What does this mean for the form of \(B\) and \(P\)? Notice that we could simply set \(B\) and \(P\) to zero. However, assuming at least one component of these matrices is non-vanishing, we find the following cases. Take \(P\) as our example and let

\[
P = \begin{pmatrix} p_1 & p_2 \\ p_3 & p_4 \end{pmatrix}.
\]

The fact that this squares to zero tells us

\[
p_1^2 + p_2p_3 = 0 \quad \text{and} \quad p_1p_2 + p_2p_4 = 0 \quad \text{and} \quad p_3p_1 + p_4p_3 = 0 \quad \text{and} \quad p_3p_2 + p_4^2 = 0.
\]

There are two cases, \(p_3 = 0\) and \(p_3 \neq 0\), which we shall now consider in turn.

In the \(p_3 = 0\) case, the constraints in (4.2.2) become

\[
p_1^2 = 0 \quad \text{and} \quad p_1p_2 + p_2p_4 = 0 \quad \text{and} \quad p_2^2 = 0.
\]

Therefore, \(p_1 = p_4 = 0\) and \(p_2\) is unconstrained, leaving the matrix

\[
P = \begin{pmatrix} 0 & p_2 \\ 0 & 0 \end{pmatrix}.
\]

In the \(p_3 \neq 0\) case, we can use the first and third constraints of (4.2.2) to get \(p_2 = -p_2^2p_3^{-1}\) and \(p_4 = -p_3^2p_4^{-1}\), respectively. These choices trivially satisfy the second and fourth constraints such that we arrive at

\[
P = \begin{pmatrix} p_1 & -p_2^2p_3^{-1} \\ p_3 & -p_3^2p_4^{-1} \end{pmatrix}.
\]

In a completely analogous manner, we find

\[
B = \begin{pmatrix} 0 & b_2 \\ b_3 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_1 & -b_2^2b_3^{-1} \\ b_3 & -b_3b_1b_3^{-1} \end{pmatrix}.
\]

Now, what does the anti-commuting condition tell us about the non-vanishing matrices? Notice, we have four options:

1. \(p_3 \neq 0\), \(b_3 \neq 0\),
2. \(p_3 \neq 0\), \(b_3 = 0\),
3. \(p_3 = 0\), \(b_3 \neq 0\), and
4. \(p_3 = 0\), \(b_3 = 0\).

**Option 1.** Here we will find three distinct sub-options. Interestingly, these three sub-options are equivalent to options 2, 3, and 4 above. Substituting the matrices associated with \(p_3 \neq 0\) and \(b_3 \neq 0\) into \(BP + PB = 0\) gives us

\[
0 = b_1p_1 - b_2^2b_3^{-1}p_3 + p_1b_1 - p_2^2p_3^{-1}b_3
\]

\[
= -b_1^2p_3p_3^{-1} + b_2^2b_3^{-1}p_3p_1p_3^{-1} - p_1b_2^2b_3^{-1} + p_2^2p_3^{-1}b_3b_3^{-1}
\]

\[
0 = b_3p_1 - b_2b_3b_3^{-1}p_3 + p_3b_1 - p_3p_1p_3^{-1}b_3
\]

\[
0 = -b_3^2p_3p_3^{-1} + b_2b_3b_3^{-1}p_3p_1p_3^{-1} - p_3b_2^2b_3^{-1} + p_3p_1p_3^{-1}b_3b_3^{-1}.
\]

Multiplying the first of these conditions on the right by \(b_1b_3^{-1}\) and adding it to the second condition, we obtain

\[
0 = b_1(p_1 - b_2b_3^{-1}p_3)(b_1b_3^{-1} - p_1p_3^{-1}).
\]

Since the quaternions have no zero-divisors, one of these terms must vanish. The vanishing of the second is equivalent to the vanishing of the third, so we have two sub-options:

1.1 \(b_1 = 0\), and
In the latter case, the third and fourth conditions of (4.2.7) are trivially satisfied, but in the former case, a little more work is required. Setting \( b_1 = 0 \), we obtain

\[ 0 = b_3 p_1^{-1} p_3^{-1} \quad \text{and} \quad 0 = b_3 p_1 - p_3 p_1 p_3^{-1} b_3. \]  

(4.2.9)

Again, using the fact the quaternions have no zero-divisors, these conditions mean this sub-option further divides into two sub-options:

1.1.1 \( b_3 = 0 \), and
1.1.2 \( p_1 = 0 \),

with \( p_3 \) left free. Recall that to arrive at these options we first made a choice to multiply the first condition of (4.2.7) by \( b_1 b_3^{-1} \). We could equally have multiplied by \( p_1 p_3^{-1} \) such that case 1.1 above read \( p_1 = 0 \). (Notice, the second case is symmetric, so would remain the same in this instance.) Analogous subsequent calculations would lead to sub-options \( p_3 = 0 \) and \( b_1 = 0 \). Putting all of this together, we have four sub-options to consider:

Sub-option 1: \( B = 0 \)

\[ P = \begin{pmatrix} p_1 & -p_1^2 p_3^{-1} \\ p_3 & -p_3 p_1 p_3^{-1} \end{pmatrix} \]

Sub-option 2: \( B = \begin{pmatrix} 0 & 0 \\ b_3 & 0 \end{pmatrix} \)

\[ P = \begin{pmatrix} 0 & 0 \\ p_3 & 0 \end{pmatrix} \]

(4.2.10)

Sub-option 3: \( B = \begin{pmatrix} b_1 & -b_1^2 b_3^{-1} \\ b_3 & -b_3 b_1 b_3^{-1} \end{pmatrix} \)

\[ P = 0 \]

Sub-option 4: \( B = \begin{pmatrix} b_1 & -b_1^2 b_3^{-1} \\ b_3 & -b_3 b_1 b_3^{-1} \end{pmatrix} \)

\[ P = \begin{pmatrix} p_1 & -p_1^2 p_3^{-1} \\ p_3 & -p_3 p_1 p_3^{-1} \end{pmatrix} \] where \( b_1 b_3^{-1} = p_1 p_3^{-1} \).

In fact, this list can be simplified further. For all generalised Bargmann algebras, we can choose a transformation \((1, 1, M, 1)\), where, \( M \) takes the form

\[ M = \begin{pmatrix} 1 & -b_1 b_3^{-1} \\ 0 & 1 \end{pmatrix} \]  

(4.2.11)

such that sub-option 4 becomes sub-option 2. In summary, the \( p_3 \neq 0 \) and \( b_3 \neq 0 \) assumption lead to three separate sub-options.

Sub-option 1: \( B = 0 \)

\[ P = \begin{pmatrix} p_1 & -p_1^2 p_3^{-1} \\ p_3 & -p_3 p_1 p_3^{-1} \end{pmatrix} \]

Sub-option 2: \( B = \begin{pmatrix} 0 & 0 \\ b_3 & 0 \end{pmatrix} \)

\[ P = \begin{pmatrix} 0 & 0 \\ p_3 & 0 \end{pmatrix} \]  

(4.2.12)

Sub-option 3: \( B = \begin{pmatrix} b_1 & -b_1^2 b_3^{-1} \\ b_3 & -b_3 b_1 b_3^{-1} \end{pmatrix} \)

\[ P = 0 \]

Option 2. Letting \( p_3 \neq 0 \) and \( b_3 = 0 \), the anti-commuting condition tells us

\[ 0 = b_2 p_3 \quad \text{and} \quad p_1 b_2 = b_2 p_3 p_1 p_3^{-1}. \]  

(4.2.13)

Using the first condition, \( b_2 = 0 \), and, with \( b_3 = 0 \), we are left with sub-option 1 above.

Option 3. Now, consider \( p_3 = 0 \) and \( b_3 \neq 0 \). Substituting the relevant forms of \( P \) and \( B \) into the anti-commuting condition, \( BP + PB = 0 \), we find

\[ 0 = p_2 b_3 \quad \text{and} \quad b_1 p_2 = p_2 b_3 b_1 b_3^{-1}. \]  

(4.2.14)

This is identical to option 2 only \( b \) and \( p \) have been swapped. Therefore, we have a similar result: \( p = 0 \) such that we have sub-option 3 above.

Option 4. The final case to consider is \( p_3 = 0 \) and \( b_3 = 0 \), where

\[ P = \begin{pmatrix} 0 & p_2 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & b_2 \\ 0 & 0 \end{pmatrix}. \]  

(4.2.15)

These strictly upper-triangular matrices are equivalent to the strictly lower-triangular matrices of sub-option 2 above. Thus, again, we find no new cases to carry forward.

To simplify the rest of the calculations, we will choose to use the transformation in (4.2.11) for all generalised
Bargmann algebras and all options. Combining the case in which both B and P vanish with the non-vanishing options, we find

Case 1: \( B = 0 \) \( \Rightarrow \) \( P = 0 \)

Case 2: \( B = 0 \) \( \Rightarrow \) \( P = \begin{pmatrix} 0 & 0 \\ \varepsilon P_3 & 0 \end{pmatrix} \)

Case 3: \( B = \begin{pmatrix} 0 & 0 \\ \varepsilon B_3 & 0 \end{pmatrix} \) \( \Rightarrow \) \( P = \begin{pmatrix} 0 & 0 \\ \varepsilon P_3 & 0 \end{pmatrix} \)

Case 4: \( B = \begin{pmatrix} 0 & 0 \\ B_3 & 0 \end{pmatrix} \) \( \Rightarrow \) \( P = 0 \).

In all cases, it is a straightforward computation to show that \([B, P] = Z\) tells us that \( Z = 0 \). Therefore, we are left with only \( H \) to determine. From the results in Lemma 4.1, the conditions we have including \( B, P \) and \( H \) are

\[
[B, H] = \lambda B + \mu P \quad \text{and} \quad [P, H] = \eta B + \varepsilon P. \tag{4.2.17}
\]

**Case 1.** The vanishing of \( B \) and \( P \) in this instance, when substituted into (4.2.17), means we do not obtain any conditions on \( H \). Thus, we find a branch with matrices

\[
B = P = Z = 0 \quad \text{and} \quad H \text{ unconstrained.} \tag{4.2.18}
\]

**Case 2.** Notice that the vanishing of \( B \) means that the second condition in (4.2.17) becomes

\[
\varepsilon \begin{pmatrix} 0 & 0 \\ \varepsilon P_3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \varepsilon P_3 & 0 \end{pmatrix} \begin{pmatrix} h_1 & h_2 \\ h_3 & h_4 \end{pmatrix} - \begin{pmatrix} h_1 & h_2 \\ h_3 & h_4 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \varepsilon P_3 & 0 \end{pmatrix}. \tag{4.2.19}
\]

This gives us two constraints

\[
0 = h_2 p_3, \quad \text{and} \quad \varepsilon p_3 = p_3 h_1 - h_4 p_3. \tag{4.2.20}
\]

The first constraint here tells us that either \( h_2 \) or \( p_3 \) must vanish. In the latter instance, we recover the matrices from case 1. In the former instance, we can use the second constraint to write \( h_4 \) in terms of \( h_1 \) and obtain the matrices

\[
B = Z = 0 \quad P = \begin{pmatrix} 0 & 0 \\ \varepsilon p_3 & 0 \end{pmatrix} \quad H = \begin{pmatrix} h_1 & 0 \\ p_3 h_1 - h_4 p_3 & -\varepsilon \end{pmatrix}. \tag{4.2.21}
\]

The first condition in (4.2.17) does not add any new branches to those already given as, with \( B = 0 \), it reduces to \( 0 = \mu P \). Therefore, for those generalised Bargmann algebras with \( \mu \neq 0 \), it gives the branch identified in case 1, and, for those with \( \mu = 0 \), it leaves \( p_3 \) free to fix \( h_4 \) as prescribed for the branch presented in this case.

**Case 3.** Substituting the \( B \) and \( P \) associated with this case into (4.2.17), we get the following constraints

\[
0 = h_2 B_3, \quad 0 = h_2 p_3, \quad \lambda B_3 + \mu p_3 = b_3 h_1 - h_4 B_3, \quad \eta B_3 + \varepsilon p_3 = p_3 h_1 - h_4 p_3. \tag{4.2.22}
\]

The first two constraints above tell us that if \( h_2 \neq 0 \), then we again arrive at the branch with \( B = P = Z = 0 \) and \( H \) unconstrained. Letting \( h_2 = 0 \), we focus on the second two constraints. Notice, for this branch to be distinct from the other two, we require \( b_3 \neq 0 \) and \( p_3 \neq 0 \). These assumptions allow us to take inverses of both \( b_3 \) and \( p_3 \) in the following calculations. Multiplying the third constraint on the right by \( b_3^{-1} \), we can rearrange for \( h_4 \) and substitute this into the fourth constraint to get

\[
\eta B_3 + \varepsilon p_3 = p_3 h_1 - b_3 h_1 b_3^{-1} p_3 + \mu p_3 b_3^{-1} p_3 + \lambda p_3. \tag{4.2.23}
\]

Multiplying this expression by \( b_3^{-1} \) on the left and rearranging, we find

\[
[u, h_1] = -\mu u^2 + (\varepsilon - \lambda) u + \eta \tag{4.2.24}
\]

where \( u = b_3^{-1} p_3 \). Alternatively, we could have chosen to multiply the fourth condition on the right by \( p_3^{-1} \) to get our expression for \( h_4 \) and substituted this into the third constraint. Multiplying this on the left by \( p_3^{-1} \) produces the similar condition

\[
[v, h_1] = \eta v^2 + (\lambda - \varepsilon) v + \mu \tag{4.2.25}
\]

where \( v = p_3^{-1} b_3 \). Depending on the generalised Bargmann algebra in question, one of these will prove more useful than the other. We will leave these constraints in this form to be analysed separately for each generalised Bargmann algebra.
4.2.17

\[ 0 = \mathfrak{h}_2 \mathfrak{b}_3 \quad \text{and} \quad \lambda \mathfrak{b}_3 = \mathfrak{b}_3 \mathfrak{h}_1 - \mathfrak{h}_4 \mathfrak{b}_3. \quad (4.2.26) \]

From the first expression above, if \( \mathfrak{b}_2 \neq 0 \), we recover the branch presented in case 1. However, setting \( \mathfrak{b}_2 = 0 \), \( \mathfrak{b}_3 \) is general, and we can use the second constraint to write \( \mathfrak{h}_4 \) in terms of \( \mathfrak{h}_1 \) and \( \mathfrak{b}_3 \):

\[ P = Z = 0 \quad B = \begin{pmatrix} 0 & 0 \\ \mathfrak{b}_3 & 0 \end{pmatrix} \quad H = \begin{pmatrix} \mathfrak{h}_1 & 0 \\ \mathfrak{h}_3 & \mathfrak{b}_3 \mathfrak{h}_1 \mathfrak{b}_3^{-1} - \lambda \end{pmatrix}. \quad (4.2.27) \]

The second constraint in (4.2.17) does not produce any new branches for \( B, P, \) and \( H \). Substituting in \( P = 0 \), it becomes \( 0 = \eta \mathfrak{b} \). Therefore, if \( \eta \neq 0 \), \( B \) must vanish leaving the branch from case 1; and, if \( \eta = 0 \), \( \mathfrak{b}_3 \) is left free so we can write \( \mathfrak{b}_4 \) as prescribed for the branch presented here.

In summary, we have the following three branches for all generalised Bargmann algebras

1. \( B = P = Z = 0 \) and \( H \) unconstrained
2. \( B = Z = 0 \) \( P = \begin{pmatrix} 0 & 0 \\ \mathfrak{b}_3 & 0 \end{pmatrix} \) \( H = \begin{pmatrix} \mathfrak{h}_1 & 0 \\ \mathfrak{h}_3 & \mathfrak{b}_3 \mathfrak{h}_1 \mathfrak{b}_3^{-1} - \epsilon \end{pmatrix} \)
3. \( P = Z = 0 \) \( B = \begin{pmatrix} 0 & 0 \\ \mathfrak{b}_3 & 0 \end{pmatrix} \) \( H = \begin{pmatrix} \mathfrak{h}_1 & 0 \\ \mathfrak{h}_3 & \mathfrak{b}_3 \mathfrak{h}_1 \mathfrak{b}_3^{-1} - \lambda \end{pmatrix} \)

There is also a possible fourth branch depending on the generalised Bargmann algebra:

\[ Z = 0 \quad H = \begin{pmatrix} \mathfrak{h}_1 & 0 \\ \mathfrak{b}_3 & \mathfrak{h}_4 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 0 \\ \mathfrak{b}_3 & 0 \end{pmatrix} \quad P = \begin{pmatrix} 0 & 0 \\ \mathfrak{b}_3 & 0 \end{pmatrix}, \quad (4.2.28) \]

subject to

\[ [u, \mathfrak{h}_1] = -\mu u^2 + (\epsilon - \lambda)u + \eta \quad \text{and} \quad [v, \mathfrak{h}_1] = \eta v^2 + (\lambda - \epsilon)v + \mu \quad (4.2.29) \]

where \( u = \mathfrak{b}_3^{-1} \mathfrak{b}_3 \) and \( v = \mathfrak{b}_3^{-1} \mathfrak{b}_3 \).

4.3. \textbf{Classification.} In this section, we complete the story started in section 4.2. Each branch we identified in section 4.2 encodes the possible \([\mathfrak{s}_i, \mathfrak{s}_j]\) brackets for a generalised Bargmann superalgebra \( \mathfrak{s} \). Here, we take each branch in turn and find corresponding \([\mathfrak{Q}, \mathfrak{Q}]\) brackets. Since our interests are in supersymmetry, we will always impose the condition that \( [\mathfrak{Q}, \mathfrak{Q}] \neq 0 \); therefore, we are only interested in branches for which at least one of the \( N_i \) matrices does not vanish. Note, the imposition of this condition means that the various branches identified here belong to the sub-variety \( \mathcal{Y} \) of the real algebraic variety cut out by the super-Jacobi identities \( \mathcal{J} \subset \mathcal{Y} \).

We will begin our investigation into each branch by stating the associated matrices, \( B, P, H, Z \in \text{Mat}_2(\mathfrak{h}) \). These matrices are then substituted into the conditions from Lemmas 4.2 and 4.3, which use the Lie brackets of the universal generalised Bargmann algebra. This process produces a system of equations containing \( B, P, H, Z \) encoding the \([\mathfrak{s}_i, \mathfrak{s}_j]\) components of the bracket, the matrices \( N_i \) for \( i \in \{0, 1, \ldots, 4\} \) encoding the \([\mathfrak{s}_1, \mathfrak{s}_j]\) components of the bracket, and the four parameters of the universal generalised Bargmann algebra, \( \lambda, \mu, \eta, \epsilon \in \mathbb{R} \). Any conditions which do not contain one of the parameters \( \lambda, \mu, \eta, \epsilon \) are analysed and possible dependencies among the \( N_i \) matrices are found. Once these dependencies have been established, we start setting parameters to consider the various generalised Bargmann algebras. In branches 1 and 2, we will see that multiple generalised Bargmann algebras produce the same set of conditions. In these instances, we will highlight the relevant algebras but only analyse the system once to avoid repetition.

In branches 2, 3 and 4, we find that the vanishing of certain matrices \( N_i \) imposes the vanishing of other \( N_i \). Thus, we end up with a chain of dependences, which lead to different sub-branches. These sub-branches will be labelled such that sub-branches with a larger branch number will have more non-vanishing matrices \( N_i \). For example, sub-branch 2.2 may have non-vanishing \( N_0 \) and \( N_1 \), but sub-branch 2.3 may additionally have non-vanishing \( N_3 \). Within each sub-branch, we regularly find two options: one in which \( N_0 \) vanishes, leaving \( H \) free, and one in which \( H = 0 \) such that \( N_0 \) is unconstrained. Using sub-branch 2.2 as our example, the former instance, with \( N_0 = 0 \), will be labelled 2.2.i, and the latter instance will be labelled 2.2.ii. In branch 4, we will find some instances in which both \( N_0 \) and \( H \) can be non-vanishing. Using sub-branch 4.3 as an example, we will label these cases as 4.3.iii.

Each sub-branch is designed to have a unique set of non-vanishing matrices. However, the components within the matrices are not completely fixed by the super-Jacobi identities. Therefore, each sub-branch is given as a tuple \( \{ M, \mathcal{C}, X \} \), where \( M \) labels the underlying generalised Bargmann algebra, and \( X \) will be the branch number. This tuple consists of \( M \), the subset of matrices in \( \{ B, P, H, Z, N_0, N_1, N_2, N_3, N_4 \} \) describing the branch, and \( \mathcal{C} \), the set of constraints on the components of the matrices. After stating \( \{ M, \mathcal{C} \} \) for a given sub-branch, we proceed to a discussion on possible parameterisations of the super-extensions in the sub-branch. In particular, the
Since the second condition must hold for all \( \theta \in \mathbb{H} \), therefore, we will define its components as necessary. Throughout this section, we will use the following forms for the quaternion Hermitian matrices:

\[
N_0 = \begin{pmatrix} a & c \\ \bar{d} & b \end{pmatrix} \quad \text{and} \quad N_1 = \begin{pmatrix} c & r \\ \bar{f} & d \end{pmatrix},
\]

where \( a, b, c, d \in \mathbb{R} \), and \( e, f, g \in \mathbb{H} \). The quaternion skew-Hermitian matrices will be defined

\[
N_3 = \begin{pmatrix} e & f \\ -\bar{f} & \bar{e} \end{pmatrix} \quad \text{and} \quad N_4 = \begin{pmatrix} \eta & m \\ -\bar{m} & \bar{\eta} \end{pmatrix},
\]

where \( e, g, \eta, \bar{\eta} \in \text{Im}(\mathbb{H}) \) and \( f, m \in \mathbb{H} \). We will only briefly need to consider parts of the \( N_2 \) matrix explicitly; therefore, we will define its components as necessary.

### 4.3.1. Branch 1.

\[
B = P = Z = 0 \quad \text{and} \quad H \text{ unconstrained.}
\]

Using the remaining conditions from the \( (s_0, s_1, s_2, s_3) \) and \( (s_1, s_2, s_3) \) super-Jacobi identities, we can look to find some expressions for the matrices \( N_i \). The conditions derived from the \( [B, Q, Q] \) identity in Lemma 4.2 immediately give \( N_4 = 0 \) due to the vanishing of \( B \). Similarly, the \( [P, Q, Q] \) conditions give us \( N_3 = 0 \) due to the vanishing of \( P \). We are thus left with

\[
0 = \mu \text{Re}(\theta N_0 \bar{\theta}^\dagger) \\
0 = \eta \text{Re}(\theta N_0 \bar{\theta}^\dagger) \\
0 = \lambda \text{Re}(\theta N_0 \bar{\theta}^\dagger) + \beta + \frac{\beta}{2} |\beta| \theta \\
0 = \epsilon \text{Re}(\theta N_0 \bar{\theta}^\dagger) + \frac{\beta}{2} |\beta| \theta
\]

Since \( [\epsilon, d] \) is perpendicular to both \( e \) and \( d \) for \( e, d \in \mathbb{H} \), the final two conditions can be reduced to

\[
0 = \lambda \text{Re}(\theta N_0 \bar{\theta}^\dagger) \\
0 = \beta + \frac{\beta}{2} |\beta| \theta
\]

Substituting \( \theta = (1, 0), \beta = (0, 1), \) and \( \beta = (1, 1) \) into the \( N_2 \) conditions above, we find that

\[
N_2 = \begin{pmatrix} 0 & e \\ -e & 0 \end{pmatrix},
\]

where \( e \in \mathbb{R} \). Now substituting \( \theta = (1, \bar{\beta}) \) into the \( N_2 \) conditions, we find

\[
0 = -2e|\beta| \bar{\beta}.
\]

We can choose any \( \beta \in \text{Im}(\mathbb{H}) \); therefore, we may choose \( \beta = \bar{\beta} \). Thus we find that \( e \) must vanish, making \( N_2 = 0 \). This result reduces the conditions further:

\[
0 = \mu \text{Re}(\theta N_0 \bar{\theta}^\dagger) \\
0 = \eta \text{Re}(\theta N_0 \bar{\theta}^\dagger) \\
0 = \lambda \text{Re}(\theta N_0 \bar{\theta}^\dagger) |\beta| \\
0 = \epsilon \text{Re}(\theta N_0 \bar{\theta}^\dagger) |\beta|
\]

Focussing on the conditions common to all generalised Bargmann algebras, i.e. those conditions which do not contain \( \lambda, \mu, \eta, \) or \( \epsilon \), we have only

\[
0 = H N_i + N_i H^\dagger \quad i \in \{0, 1\} \\
0 = \text{Re}(\theta N_0 \bar{\theta}^\dagger) |\theta|
\]

Since the second condition must hold for all \( \theta \in \mathbb{H} \), we find that either

(i) \( N_0 = 0 \) and \( H \neq 0 \), or
(ii) \( N_0 \neq 0 \) and \( H = 0 \).
We can now split this analysis in two depending on the generalised Bargmann algebra of interest. First, we will discuss the algebras in which at least one of the parameters $\lambda, \mu, \eta, \varepsilon$ are non-vanishing. Subsequently, we will consider the algebras in which all of these parameters vanish. The former instance encapsulates $\mathfrak{n}_\pm$ and $\mathfrak{g}$, and the latter encapsulates $\hat{a}$.

$\mathfrak{n}_\pm$ and $\mathfrak{g}$.

All of these algebras have non-vanishing values for at least one of the parameters $\lambda, \mu, \eta, \varepsilon$. Therefore, all have the conditions for branch 1 reduce to

\begin{align}
0 &= \mathbf{HN}_i + \mathbf{N}_i \mathbf{H}^\dagger \quad i \in \{0, 1\} \\
0 &= \text{Re}(\theta \mathbf{N}_0 \mathbf{N}_0^\dagger) \\
0 &= \text{Re}(\theta \mathbf{N}_0 \mathbf{N}_0^\dagger)\theta \mathbf{H}.
\end{align}

(4.3.11)

Substituting $\theta = (1, 0)$, $\theta = (0, 1)$, and $\theta = (1, 1)$ into the second condition above, we find that

\[ \mathbf{N}_0 = \begin{pmatrix} 0 & \text{Im}(\mathbf{q}) \\ -\text{Im}(\mathbf{q}) & 0 \end{pmatrix}. \]

(4.3.12)

Now substitute $\theta = (1, i)$ into this condition using the convention that $\mathbf{q} = \mathbf{q}_1 \mathbf{i} + \mathbf{q}_2 \mathbf{j} + \mathbf{q}_3 \mathbf{k}$ to find

\[ 0 = \text{Re}(\theta \mathbf{q}_1) = \mathbf{q}_1. \]

(4.3.13)

Using $\theta = (1, j)$ and $\theta = (1, k)$, we get analogous expressions for $\mathbf{q}_2$ and $\mathbf{q}_3$, so $\mathbf{q} = 0$. Therefore, $\mathbf{N}_0 = 0$, and we cannot produce a super-extension in sub-branch 1.ii for these generalised Bargmann algebras.

The only remaining matrices are $\mathbf{H}$ and $\mathbf{N}_1$, such that

\[ 0 = \mathbf{HN}_1 + \mathbf{N}_1 \mathbf{H}^\dagger, \]

(4.3.14)

with no constraints on $\mathbf{H}$ and $\mathbf{N}_1 = \mathbf{N}_1^\dagger$. So far, we have not used any basis transformations for this branch; therefore, we can choose $\mathbf{N}_1$ to be the canonical quaternion Hermitian form, 1. The above condition then states that $\mathbf{H}^\dagger = -\mathbf{H}$. Thus, this branch produces one non-empty sub-branch for $\mathfrak{n}_\pm$ and $\mathfrak{g}$, with the set of non-vanishing matrices given by

\[ \mathcal{M}_{\mathfrak{n}_\pm} = \{ \mathbf{H} = \begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_2 \\ -\mathbf{b}_2 & \mathbf{b}_3 \end{pmatrix}, \quad \mathbf{N}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \}. \]

(4.3.15)

Although already explicit in the forms of $\mathbf{H}$ and $\mathbf{N}_1$, we note that the set of constraints for this sub-branch is

\[ \mathcal{C}_{\mathfrak{n}_\pm} = \{ \mathbf{H}^\dagger = -\mathbf{H}, \quad \mathbf{N}_1 = \mathbf{N}_1^\dagger \}. \]

(4.3.16)

Our only comment on $\mathbf{H}$ going into the analysis of this branch was that it was unconstrained; therefore, we may choose to have $\mathbf{H} = 0$. Thus there is certainly a super-extension in this sub-branch, one with only $\mathbf{N}_1 = \mathbb{I}$ non-vanishing. However, wanting to introduce some more parameters, we may let $\mathbf{b}_1$, $\mathbf{b}_2$ and $\mathbf{b}_3$ be non-vanishing. These quaternions can be fixed using the group of basis transformations $\mathbf{G} \subset \text{GL}(\mathfrak{g}_3) \times \text{GL}(\mathfrak{g}_1)$ by noticing that $\mathbf{H}^\dagger = -\mathbf{H}$ tells us that $\mathbf{H} \in \text{sp}(2)$. Therefore, the residual $\text{Sp}(2) \subset \text{GL}(\mathfrak{g}_1)$ which fixes $\mathbf{N}_1 = \mathbb{I}$ acts on $\mathbf{H}$ via the adjoint action of $\text{Sp}(2)$ on its Lie algebra. Thus, we can make $\mathbf{H}$ diagonal and choose the two imaginary quaternions parameterising it, arriving at

\[ \mathbf{H} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \mathbf{N}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \]

(4.3.17)

\[ \hat{a}. \]

Since $\hat{a}$ has $\lambda = \mu = \eta = \varepsilon = 0$, the conditions in (4.3.9) become

\begin{align}
0 &= \mathbf{HN}_i + \mathbf{N}_i \mathbf{H}^\dagger \quad i \in \{0, 1\} \\
0 &= \text{Re}(\theta \mathbf{N}_0 \mathbf{N}_0^\dagger)\theta \mathbf{H}.
\end{align}

(4.3.18)

Unlike the $\mathfrak{n}_{\pm}$ and $\mathfrak{g}$ case, these conditions do not instantly set $\mathbf{N}_0 = 0$; therefore, we may have super-extensions with either (i) $\mathbf{N}_0 = 0$ and $\mathbf{H} \neq 0$, or (ii) $\mathbf{N}_0 \neq 0$ and $\mathbf{H} = 0$. First, setting $\mathbf{H} \neq 0$, we know this imposes $\mathbf{N}_0 = 0$, and, as in the $\mathfrak{n}_{\pm}$ and $\mathfrak{g}$ case, we may use the basis transformations to set $\mathbf{N}_1 = \mathbb{I}$, such that $\mathbf{H}^\dagger = -\mathbf{H}$. Therefore, one of the possible super-extensions for $\hat{a}$ has non-vanishing matrices

\[ \mathcal{M}_{\hat{a}, 1, i} = \{ \mathbf{H} = \begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_2 \\ -\mathbf{b}_2 & \mathbf{b}_3 \end{pmatrix}, \quad \mathbf{N}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \}. \]

(4.3.19)

As before, the set of conditions for this super-extension is

\[ \mathcal{C}_{\hat{a}, 1, i} = \{ \mathbf{H}^\dagger = -\mathbf{H}, \quad \mathbf{N}_1^\dagger = \mathbf{N}_1 \}. \]

(4.3.20)
and we can use $G$ to fix the quaternions in $\mathfrak{h}$. Alternatively, setting $N_0 \neq 0$, we need $H = 0$. Thus the second possible super-extension in this branch has

$$M_{a,1,ii} = \{N_0 = \begin{pmatrix} a & b \\ \bar{a} & \bar{b} \end{pmatrix}, \quad N_1 = \begin{pmatrix} c & r \\ \bar{r} & d \end{pmatrix}\} \text{ and } \bar{e}_{a,1,ii} = \{N_0^0 = N_0, \quad N_1^0 = N_1\}. \quad (4.3.21)$$

Since the primary constraint on these matrices is that both be non-vanishing, we can choose to have $b, a, c$ and $r$ vanish. Using the scaling symmetry of the $s_0$ basis elements present in $G \subset GL(s_0) \times GL(s_1)$, we can write down the super-extension

$$N_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad N_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \quad (4.3.22)$$

Therefore, this sub-branch is not empty. Additionally, we may choose to keep all the parameters in the matrices of $M_{a,1,ii}$ and use the basis transformations to fix them. In particular, we can let $N_0 = 1$. This choice leaves us with a residual $Sp(2)$ action with which to fix the parameters of $N_1$, which may give us $N_1 = 1$.

### 4.3.2. Branch 2.

$$B = Z = 0 \quad P = \begin{pmatrix} 0 & 0 \\ \bar{p}_3 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} p_1 & 0 \\ \bar{p}_3 & \bar{p}_3 \bar{p}_1 \bar{p}_3^{-1} - \epsilon \end{pmatrix}. \quad (4.3.23)$$

As above, it is useful to exploit the vanishing matrices of the branch to simplify the conditions from Lemmas 4.2 and 4.3. In particular, the $[B, Q, Q]$ super-Jacobi identity tells us $N_4 = 0$ due to the vanishing of $B$. The rest of the $[B, Q, Q]$ conditions tells us that

$$0 = \lambda \text{Re}(\theta N_0 \theta^\dagger) + \frac{1}{2} [\beta, \theta N_2 \theta^\dagger]$$

$$0 = \mu \text{Re}(\theta N_0 \theta^\dagger) \beta. \quad (4.3.24)$$

The $[P, Q, Q]$ conditions become

$$0 = PN_0 - N_0 P^\dagger$$

$$-N_3 = PN_1 - N_1 P^\dagger$$

$$0 = \pi \text{Re}(\theta N_0 \theta^\dagger) + \theta N_3 (\pi \theta P)^\dagger$$

$$\eta \text{Re}(\theta N_0 \theta^\dagger) \pi = \pi \text{Re}(\theta N_3 \theta^\dagger) + \theta N_3 (\pi \theta P)^\dagger$$

$$0 = \epsilon \text{Re}(\theta N_0 \theta^\dagger) \pi + \frac{1}{2} [\pi, \theta N_2 \theta^\dagger]. \quad (4.3.25)$$

Since the conditions from the $[Z, Q, Q]$ identity are all satisfied due to $Z = 0$, the final conditions are

$$0 = H N_i + N_i H^\dagger \quad \text{where } \quad i \in \{0, 1, 2\}$$

$$\lambda N_3 = H N_3 + N_3 H^\dagger$$

$$0 = \mu N_3, \quad (4.3.26)$$

from $[H, Q, Q]$. The result from Lemma 4.3 then gives us

$$\text{Re}(\theta N_0 \theta^\dagger) \theta H = \frac{1}{2} \theta N_2 \theta^\dagger \theta. \quad (4.3.27)$$

As in branch 1, the conditions

$$0 = \lambda \text{Re}(\theta N_0 \theta^\dagger) + \frac{1}{2}[\beta, \theta N_2 \theta^\dagger] \quad \text{and} \quad 0 = \epsilon \text{Re}(\theta N_0 \theta^\dagger) \pi + \frac{1}{2} [\pi, \theta N_2 \theta^\dagger], \quad (4.3.28)$$

tell us that $N_2 = 0$. Therefore, the conditions reduce further to

$$0 = \mu N_3$$

$$0 = H N_i + N_i H^\dagger \quad \text{where } \quad i \in \{0, 1\}$$

$$0 = PN_0 - N_0 P^\dagger$$

$$\lambda N_3 = H N_3 + N_3 H^\dagger$$

$$0 = \lambda \text{Re}(\theta N_0 \theta^\dagger) \beta$$

$$-N_3 = PN_1 - N_1 P^\dagger$$

$$\forall \beta, \pi \in \text{Im}(H), \forall \theta \in H^2. \quad (4.3.29)$$

$$0 = \mu \text{Re}(\theta N_0 \theta^\dagger) \beta$$

$$\eta \text{Re}(\theta N_0 \theta^\dagger) \pi = \pi \text{Re}(\theta N_3 \theta^\dagger) + \theta N_3 (\pi \theta P)^\dagger$$

$$0 = \epsilon \text{Re}(\theta N_0 \theta^\dagger) \pi$$

$$0 = \text{Re}(\theta N_0 \theta^\dagger) \theta H$$

We can now use the following two conditions common to all generalised Bargmann algebras to highlight the possible sub-branches:

$$-N_3 = PN_1 - N_1 P^\dagger \quad \text{and} \quad 0 = \text{Re}(\theta N_0 \theta^\dagger) \theta H. \quad (4.3.30)$$

Substituting the $N_3$ from (4.3.2) and the $N_3$ from (4.3.3) into the first condition here, we can write

$$N_3 = \begin{pmatrix} 0 & c p_3 \\ -c \bar{p}_3 & \bar{r} p_3^\dagger - r \bar{p}_3 \end{pmatrix}. \quad (4.3.31)$$
This result tells us that \( N_3 \) is dependent on \( N_1 \); if \( N_1 = 0 \) then \( N_3 = 0 \). Therefore, we may organise our investigation into the possible super-extensions by considering each of the following sub-branches in turn

1. \( N_1 = 0 \) and \( N_3 = 0 \),
2. \( N_1 \neq 0 \) and \( N_3 = 0 \),
3. \( N_1 \neq 0 \) and \( N_3 \neq 0 \).

Next, consider the condition from the \([Q, Q, Q] \) identity:

\[ 0 = \text{Re}(\theta N_0 \theta^\dagger) \theta H. \]  

(4.3.32)

Notice, this is identical to the condition from the \([Q, Q, Q] \) identity we found in branch 1. Therefore, as before, we have two cases to consider in each sub-branch:

1. \( N_0 = 0 \) and \( H \neq 0 \), and
2. \( N_0 \neq 0 \) and \( H = 0 \).

We will now consider each generalised Bargmann algebra in turn to determine whether they have super-extensions associated to these sub-branches.

\( \hat{a} \).

In addition to the conditions already discussed in producing the possible sub-branches,

\[ -N_3 = \pi N_1 - N_1 \pi^\dagger \quad \text{and} \quad 0 = \text{Re}(\theta N_0 \theta^\dagger) \theta H, \]

(4.3.33)

substituting \( \lambda = \mu = \eta = \varepsilon = 0 \) into (4.3.29) leaves us with

\[ 0 = HN_i + N_i H^\dagger \quad \text{where} \quad i \in \{0, 1, 3\} \]

\[ 0 = PN_0 - N_0 \pi^\dagger \]

\[ 0 = \pi \theta \pi N_3 \theta^\dagger + \theta N_3 (\pi \theta \pi)^\dagger. \]

(4.3.34)

None of these conditions force the vanishing of any more \( N_i \); therefore, \textit{a priori} we may find super-extensions in each of the sub-branches. The only restriction to the matrices so far has been the re-writing of \( N_3 \):

\[ N_3 = \begin{pmatrix} 0 & c\bar{p}_3 \\ -c\bar{p}_3 & \bar{r}p_3 - \bar{p}_3 x \end{pmatrix}. \]

(4.3.35)

**Sub-branch 2.1.** Setting \( N_1 = N_3 = 0 \), we are left with only \( N_0 \), subject to

\[ 0 = HN_0 + N_0 H^\dagger \quad \text{and} \quad 0 = PN_0 - N_0 \pi^\dagger. \]

(4.3.36)

We know that we may have two possible cases for this sub-branch: either (i) \( N_0 = 0 \) and \( H \neq 0 \), or (ii) \( N_0 \neq 0 \) and \( H = 0 \). Since we need \( N_0 \neq 0 \) for a supersymmetric extension, we must have the latter case. This leaves only the second condition above with which to restrict the form of \( N_0 \). Since \( p_3 \neq 0 \), this tells us

\[ 0 = a \quad \text{and} \quad 0 = p_3 q - \bar{a} \bar{p}_3. \]

(4.3.37)

Thus the sub-branch is given by

\[ M_{a,2.1.ii} = \{ p = \begin{pmatrix} 0 \\ p_3 \\ 0 \\ 0 \end{pmatrix}, \quad N_0 = \begin{pmatrix} 0 \\ q \end{pmatrix} \} \quad \text{and} \quad E_{a,2.1.ii} = \{ 0 = p_3 q - \bar{a} \bar{p}_3 \}. \]

(4.3.38)

This sub-branch is parameterised by two collinear quaternions \( p_3 \) and \( q \), and a single real scalar \( b \), such that it defines an 8-dimensional space in the sub-variety \( \mathbb{S} \). Notice that we can choose either \( q = 0 \) or \( b = 0 \) and this sub-branch remains supersymmetric. Choosing the former case, we can use the endomorphisms of \( s_1 \) to set \( p_3 = \mathbb{I} \) and the scaling symmetry of \( H \) to produce

\[ P = \begin{pmatrix} 0 & 0 \\ \mathbb{I} & 0 \end{pmatrix} \quad \text{and} \quad N_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \]

(4.3.39)

In the latter case, we can still choose \( p_3 = \mathbb{I} \), and the condition in \( E_{a,2.1.ii} \) will impose that \( q \) must also lie along \( \mathbb{I} \). Again using the scaling symmetry of \( H \) in \( G \subset \text{GL}(s_1) \times \text{GL}(s_1) \), we arrive at

\[ P = \begin{pmatrix} 0 & 0 \\ \mathbb{I} & 0 \end{pmatrix} \quad \text{and} \quad N_0 = \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix}. \]

(4.3.40)

These two examples turn out to be the only super-extensions in this sub-branch. Keeping both \( b \) and \( q \) at the outset, we can use the endomorphisms of \( s_1 \) to set \( b = 0 \) while imposing that \( p_3 \) and \( q \) lie along \( \mathbb{I} \). Thus, in this case, we could always retrieve the second example above.
Sub-branch 2.2. Setting $N_1 \not= 0$ but keeping $N_3 = 0$, the conditions in (4.3.33) and (4.3.34) become

$$0 = PN_1 - N_1P^\dagger \quad \text{where} \quad i \in \{0, 1\}. \quad (4.3.31)$$

Importantly, we can now have super-extensions in either of the two cases: (i) $N_0 = 0$ and $H \not= 0$, or (ii) $N_0 \not= 0$ and $H = 0$. In the former case, in which $N_0 = 0$, (4.3.33) and (4.3.34) become

$$0 = PN_1 - N_1P^\dagger \quad \text{and} \quad 0 = HN_1 + N_1H^\dagger. \quad (4.3.32)$$

The first of these conditions tells us that

$$N_1 = \begin{pmatrix} 0 & r \\ \overline{r} & d \end{pmatrix}, \quad (4.3.33)$$

such that $0 = p_3r - \overline{r}\overline{p}_3$. Substituting this $N_1$ into the latter condition, we find

$$0 = b_1r + r[p_3b_1p_3^{-1}] \quad \text{and} \quad 0 = \text{Re}(h_3r) + d \text{Re}(h_1). \quad (4.3.34)$$

Assuming $r \not= 0$ and $b_1 \not= 0$, take the real part of the first constraint to get $\text{Re}(b_1) = 0$. Alternatively, with $r = 0$, $d \not= 0$ for $N_1 \not= 0$; therefore, the second constraint would also impose $\text{Re}(h_1) = 0$. This result allows us to simply the constraints to

$$0 = \text{Re}(b_1) \quad 0 = [b_1, r p_3] \quad 0 = \text{Re}(h_3r). \quad (4.3.35)$$

In fact, the second constraint above is satisfied by

$$0 = p_3r - \overline{r}\overline{p}_3, \quad (4.3.36)$$

so the set of constraints on this sub-branch becomes

$$\mathcal{G}_{a, 2.2.1} = \{0 = \text{Re}(b_1), \quad 0 = \text{Re}(h_3r), \quad 0 = p_3r - \overline{r}\overline{p}_3\}. \quad (4.3.37)$$

Subject to these constraints, we have the following non-vanishing matrices

$$M_{a, 2.2.1} = \left\{P = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} b_1 & 0 \\ p_3 & p_3b_1p_3^{-1} \end{pmatrix}, \quad N_1 = \begin{pmatrix} 0 & r \\ \overline{r} & d \end{pmatrix} \right\}. \quad (4.3.38)$$

This sub-branch consists of two collinear quaternions $p_3$ and $r$, one quaternion $b_1$ that is perpendicular to these two in $\text{Im}(H)$, and one imaginary quaternion $h_1$. In addition, there is a single real scalar, $d$. Notice that if $H$ vanishes, we produce a system that is equivalent to the one found in sub-branch 2.1.ii; therefore, this sub-branch is certainly non-empty. However to investigate the role of $H$, we will require at least one of its components to be non-vanishing. To simplify $H$ as far as possible, let $b_1 = 0$. Now we can choose either $r$ or $d$ to vanish while maintaining supersymmetry. Letting $r \not= 0$, we can use the endomorphisms of $s_{ij}$ on $p_3$ and $b_3$, and employ the scaling of $Z$ on $N_1$ to arrive at

$$P = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 0 & 0 \\ 0 & -\delta \end{pmatrix}, \quad N_1 = \begin{pmatrix} 0 & r \\ \overline{r} & d \end{pmatrix}. \quad (4.3.39)$$

Thus, there exist super-extensions in this sub-branch for which $H \not= 0$. Wanting to be more be a little more general, we can choose for only $b_3$ to vanish. Then, using the endomorphisms in $\text{GL}(s_{ij})$, we can set $r = \delta d$ such that $p_3$ also lies along $\delta$. Utilising the scaling symmetry of $P$ and $Z$ in $\text{GL}(s_{ij})$, we can remove the constants from the matrices $P$ and $N_1$ to get

$$P = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad N_1 = \begin{pmatrix} 0 & \delta \\ -\delta & 1 \end{pmatrix}. \quad (4.3.40)$$

Employing the residual endomorphisms of $s_{ij}$, we can now choose $b_1$ to lie along $I$. This change allows us to use the scaling symmetry of $H$ in $\text{GL}(s_{ij})$ such that $H$ becomes

$$H = \begin{pmatrix} \delta & 0 \\ 0 & \delta \end{pmatrix}. \quad (4.3.41)$$

Now, returning to the latter case, in which $H$ vanishes, we have only

$$0 = PN_1 - N_1P^\dagger \quad \text{where} \quad i \in \{0, 1\}, \quad (4.3.42)$$

which tells us that

$$N_0 = \begin{pmatrix} 0 & q \\ \bar{q} & b \end{pmatrix} \quad \text{and} \quad N_1 = \begin{pmatrix} 0 & r \\ \overline{r} & d \end{pmatrix}, \quad (4.3.43)$$

where

$$0 = p_3q - \bar{q}\overline{p}_3 \quad \text{and} \quad 0 = p_3r - \overline{r}\overline{p}_3. \quad (4.3.44)$$

Therefore, the set of non-vanishing matrices is given by

$$M_{a, 2.2.ii} = \left\{P = \begin{pmatrix} 0 & 0 \\ q & 0 \end{pmatrix}, \quad N_0 = \begin{pmatrix} 0 & q \\ \bar{q} & b \end{pmatrix}, \quad N_1 = \begin{pmatrix} 0 & r \\ \overline{r} & d \end{pmatrix} \right\}. \quad (4.3.45)$$
subject to
\[ \mathcal{C}_{a,2.2,i} = \{0 = p_3 q - q p_3 \quad \text{and} \quad 0 = p_3 r - r p_3\}. \] (4.3.56)

Notice that the matrices \( N_0 \) and \( N_1 \) and the constraints on their components take the same form as the matrix \( N_0 \) and its constraints in sub-branch 2.1.ii. However, this sub-branch is distinct. Notice that, using the endomorphisms of \( s_1 \) and the conditions in \( \mathcal{C}_{a,2.2,i} \), we can make all the quaternions parameterising this sub-branch of \( \mathcal{S} \) lie along \( \hat{i} \). The scaling symmetry of \( P \) may then be employed to set \( p_3 = \hat{i} \), leaving only \( b \) and \( d \) unfixed. The last of the endomorphisms of \( s_1 \) may set one of these parameters to zero, but not both; therefore, we cannot have \( N_0 = N_1 \), which would be a necessary condition for this sub-branch to be equivalent to \( \{ M_{a,2.1,ii}, \mathcal{C}_{a,2.1,ii} \} \). However, we can fix all the parameters of this sub-branch. Had we chosen \( q = b \hat{i} \) with the initial \( s_1 \) endomorphism and set \( d = 0 \), we could scale \( H \) and \( Z \) to find
\[ P = \begin{pmatrix} 0 & 0 \\ \hat{i} & 0 \end{pmatrix}, \quad N_0 = \begin{pmatrix} 0 & \hat{i} \\ -\hat{i} & 1 \end{pmatrix}, \quad N_1 = \begin{pmatrix} 0 & \hat{i} \\ -\hat{i} & 0 \end{pmatrix}. \] (4.3.57)

Thus, this sub-branch is non-empty and we can fix all parameters in each super-extension it contains.

**Sub-branch 2.3.** Finally, with \( N_1 \neq 0 \) and \( N_3 \neq 0 \), we can substitute \( \theta = (0, 1) \) into
\[ 0 = \pi \theta P N_3 \theta^\dagger + \theta N_3 (\pi \theta P) \dagger \] (4.3.58)
to find \( c = 0 \). Therefore, \( N_1 \) and \( N_3 \) are reduced to
\[ N_1 = \begin{pmatrix} 0 & \frac{r}{d} \\ \frac{r}{d} & d \end{pmatrix} \quad \text{and} \quad N_3 = \begin{pmatrix} 0 & 0 \\ 0 & \mp p_3 - p_3 \frac{r}{d} \end{pmatrix}. \] (4.3.59)

Recall, the condition
\[ 0 = \text{Re} (\theta N_0 \theta^\dagger) \theta H \] (4.3.60)
tells us that either (i) \( N_0 = 0 \) and \( H \neq 0 \), or (ii) \( N_0 \neq 0 \) and \( H = 0 \). Letting \( N_0 = 0 \), the final conditions for this sub-branch are
\[ 0 = H N_i + N_i H^\dagger \quad \text{where} \quad i \in \{1, 3\}. \] (4.3.61)

From the discussion in sub-branch 2.2.i, we know that the \( N_1 \) case produces the constraints
\[ 0 = \text{Re} (q_1), \quad 0 = [h_1, r p_3], \quad \text{and} \quad 0 = \text{Re} (h_3 r). \] (4.3.62)

Interestingly, the \( N_3 \) condition adds no new constraints to this set; therefore, we have
\[ \mathcal{C}_{a,2.3,i} = \{0 = \text{Re} (q_1), \quad 0 = [h_1, r p_3], \quad 0 = \text{Re} (h_3 r)\}. \] (4.3.63)

The corresponding matrices for this sub-branch are given by
\[ M_{a,2.3,i} = \{ P = \begin{pmatrix} 0 & 0 \\ p_3 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} h_1 & 0 \\ h_3 & p_3 h_1 p_3^{-1} \end{pmatrix}, \quad N_1 = \begin{pmatrix} 0 & \frac{r}{d} \\ \frac{r}{d} & d \end{pmatrix}, \quad N_3 = \begin{pmatrix} 0 & 0 \\ 0 & \mp p_3 - p_3 \frac{r}{d} \end{pmatrix} \}. \] (4.3.64)

To establish the existence of super-extensions in this sub-branch, begin by setting \( H = 0 \) and \( d = 0 \). The endomorphisms of \( s_1 \) may be used to set \( p_3 \) to lie along \( \hat{i} \) and scale \( r \) such that \( r \in \text{Sp}(1) \). We can then utilise the automorphisms of \( \hat{i} \) and the scaling symmetry of \( P \) and \( B \) in \( \text{GL}(s_0) \) to set \( r \) and fix the parameters in \( P \) and \( N_3 \). This leaves us with a super-extension whose matrices are written
\[ P = \begin{pmatrix} 0 & 0 \\ \hat{i} & 0 \end{pmatrix}, \quad N_1 = \begin{pmatrix} 0 & 1 + j \\ 1 - j & 0 \end{pmatrix}, \quad N_3 = \begin{pmatrix} 0 & 0 \\ 0 & k \end{pmatrix}. \] (4.3.65)

Using this parameterisation, we can also introduce \( h_1 \). Substituting \( p_3 = \hat{i} \) and \( r = 1 + j \) into the constraints of \( \mathcal{C}_{a,2.3,ii} \), we find
\[ P = \begin{pmatrix} 0 & 0 \\ \hat{i} & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 - k & 0 \\ 0 & 1 + k \end{pmatrix}, \quad N_1 = \begin{pmatrix} 0 & 1 + j \\ 1 - j & 0 \end{pmatrix}, \quad N_3 = \begin{pmatrix} 0 & 0 \\ 0 & k \end{pmatrix}. \] (4.3.66)

Looking to include \( h_3 \) or \( d \) leads to the introduction of parameters that cannot be fixed using the basis transformations \( G \subset \text{GL}(s_0) \times \text{GL}(s_1) \) and the constraints.

In the latter case, for which \( N_0 \neq 0 \), the only remaining condition is
\[ 0 = P N_0 - N_0 P^\dagger, \] (4.3.67)
which we know from the previous sub-branches, tells us that \( p_3 \) and \( q \) are collinear, and that \( a = 0 \). Therefore, the non-vanishing matrices for this sub-branch are
\[ M_{a,2.3,ii} = \{ P = \begin{pmatrix} 0 & 0 \\ p_3 & 0 \end{pmatrix}, \quad N_0 = \begin{pmatrix} 0 \quad q \end{pmatrix}, \quad N_1 = \begin{pmatrix} 0 & \frac{r}{d} \\ \frac{r}{d} & d \end{pmatrix}, \quad N_3 = \begin{pmatrix} 0 & 0 \\ 0 & \mp p_3 - p_3 \frac{r}{d} \end{pmatrix} \}. \] (4.3.68)

and the constraints are given by
\[ \mathcal{C}_{a,2.3,ii} = \{0 = p_3 q - q p_3\}. \] (4.3.69)
This sub-branch of $\mathcal{S}$ has 13 real parameters, being parameterised by two collinear quaternions $p_3$ and $r$, an additional quaternion $r$ and two real scalars, $b$ and $d$. Letting $d = 0$ and $q = 0$, we can use the same transformations as in sub-branch 2.3.i to fix

$$P = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad N_1 = \begin{pmatrix} 0 & 1 + i \\ 1 - i & 0 \end{pmatrix}, \quad N_3 = \begin{pmatrix} 0 & 0 \\ 0 & \bar{k} \end{pmatrix}.$$  \hspace{1cm} (4.3.70)

Subsequently employing the scaling symmetry of $H$ in $GL(s_\mathbb{H})$, we can fix $b$ such that

$$N_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$  \hspace{1cm} (4.3.71)

Therefore, there are certainly super-extensions in this sub-branch. We can introduce either $q$ or $d$ while continuing to fix all the parameters of the super-extension; however, attempting to include both leads to the inclusion of a parameter that we cannot fix with the constraints of $\mathcal{C}_{a,.2.3.i}$ and basis transformations in $G$.

### n-

Setting $\mu = \eta = 0$, $\lambda = 1$ and $\epsilon = -1$, the conditions reduce to

$$0 = \Re(0N_0\theta^\dagger)$$

where $i \in \{0, 1\}$

$$0 = \Re(0N_0\theta^\dagger)$$

Using this result, the first condition above tells us

$$0 = \pi \theta PN_3 \theta^\dagger + \theta N_3 (\pi \theta P)^\dagger$$

Hence, we cannot have any solutions along sub-branches satisfying case (ii) for $\mathcal{n}$. We are left with

$$0 = \Re(0N_0\theta^\dagger)$$

$$0 = \Re(0N_0\theta^\dagger)$$

Using this result, the first condition above tells us

$$0 = \Re(0N_0\theta^\dagger)$$

$$0 = \Re(0N_0\theta^\dagger)$$

Sub-branch 2.1. Since $N_1$ and $N_3$ are the only possible non-vanishing matrices encoding the $[Q, Q]$ bracket, we cannot have a super-extension in this branch.

### Sub-branch 2.2.

With $N_3 = 0$, we are left with

$$0 = \Re(0N_0\theta^\dagger)$$

$$0 = \Re(0N_0\theta^\dagger)$$

The latter condition tells us that $p_3$ and $r$ are collinear and $0 = cp_3$. Since we must have $p_3 \neq 0$ in this branch, we have $c = 0$. Using this result, the first condition above tells us

$$0 = \Re(0N_0\theta^\dagger)$$

In fact, utilising the collinearity of $p_3$ and $r$, the first of these constraints becomes

$$0 = (2 \Re(\bar{h}_1) + 1) \Re(p_3 r).$$  \hspace{1cm} (4.3.76)

Thus, we have

$$\mathcal{C}_{a,.2.2.i} = \{0 = \bar{r} p_3 - p_3 r, \quad 0 = (2 \Re(\bar{h}_1) + 1) \Re(p_3 r), \quad 0 = \Re(\bar{h}_3 r) + d(\Re(\bar{h}_1) + 1)\}.  \hspace{1cm} (4.3.77)

The non-vanishing matrices in this instance are

$$N_{a,.2.2.i} = \{P = \begin{pmatrix} 0 & 0 \\ p_3 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} h_1 & 0 \\ h_3 & 0 \end{pmatrix}, \quad N_1 = \begin{pmatrix} 0 & r \\ d & 0 \end{pmatrix}\}.$$  \hspace{1cm} (4.3.78)

Therefore, the sub-branch in $\mathcal{S}$ for these super-extensions of $\mathcal{n}$. is parameterised by two collinear quaternions $p_3$ and $r$, two quaternions encoding the action of $H$ on $s_1$, $h_1$ and $h_3$, and one real scalar $d$. Notice, this is the first instance in which setting some parameters to zero imposes particular values for other parameters in the extension. In particular, the vanishing of $r$ imposes $\Re(\bar{h}_1) = -1$ by the third constraint in $\mathcal{C}_{a,.2.2.i}$, since $d \neq 0$ in this instance. However, if $r = 0$, the second constraint implies $2 \Re(\bar{h}_1) = -1$. In the former case, we can set $b_3$ and the imaginary part of $h_1$ to zero. Using the endomorphisms of $s_1$ to set $p_3 = i$, we can subsequently employ the scaling symmetry of $H$ and $Z$ to obtain a super-extension with matrices

$$P = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad N_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$  \hspace{1cm} (4.3.79)

Therefore, there exist super-extensions in this sub-branch for which $r = 0$. Letting $r \neq 0$, we may again use the endomorphisms of $s_1$ to impose that $p_3$ lies along $\bar{r}$; however, due to the first constraint in $\mathcal{C}_{a,.2.2.i}$ this
also means that \( r \) lies along \( \hat{i} \). Utilising the scaling symmetry of the \( s_9 \) basis elements, we may write down the matrices

\[
P = \begin{pmatrix} 0 & 0 \\ \hat{i} & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad N_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]

(4.3.80)

Thus, super-extensions for which \( r \neq 0 \) exist in this sub-branch. In both cases, residual \( s_1 \) endomorphisms may be used to set \( h_3 \) and the imaginary part of \( h_1 \) should we choose to include them.

**Sub-branch 2.3.** Setting \( N_3 \neq 0 \), we must now consider

\[
0 = \pi \theta P N_3 \theta^\dagger + \theta N_3 (\pi \theta P)^\dagger,
\]

(4.3.81)

which, on substituting in \( \theta = (0, 1) \), tells us that \( c = 0 \). Therefore, as in sub-branch 2.2, the first condition of (4.3.73) tells us

\[
0 = \hat{h}_1 r + r \hat{p}_3 \hat{h}_3 \hat{p}_3^{-1} + 1
\]

\[
0 = \text{Re}(\hat{h}_3 r) + d(\text{Re}(h_1) + 1).
\]

(4.3.82)

However, unlike sub-branch 2.2, \( r \) and \( \hat{p}_3 \) are not collinear since the imaginary part of \( \hat{p}_3 r \) makes up the only non-vanishing component of \( N_3 \):

\[
N_3 = \begin{pmatrix} 0 & 0 \\ 0 & \bar{r} \hat{p}_3 - \hat{p}_3 r \end{pmatrix}.
\]

(4.3.83)

Substituting this \( N_3 \) into its condition from the \([H, Q, Q]\) identity, we find

\[
(1 - 2 \text{Re}(\hat{h}_3)) \text{Im}(l) = [\text{Im}(\hat{h}_4), \text{Im}(l)],
\]

(4.3.84)

where \( \hat{h}_4 = \hat{p}_3 \hat{h}_3 \hat{p}_3^{-1} + 1 \) and \( 0 = \bar{r} \hat{p}_3 - \hat{p}_3 r \). Since \( \text{Im}(l) \) is perpendicular to \([\text{Im}(\hat{h}_4), \text{Im}(l)]\), both sides of this expression must vanish separately. Substituting \( h_4 \) and \( l \) into the above expressions, we find

\[
0 = (1 + 2 \text{Re}(\hat{h}_1)) \text{Im}(\hat{p}_3 r) \quad \text{and} \quad 0 = [\hat{h}_1, r \hat{p}_3].
\]

(4.3.85)

As stated above, \( r \) and \( \hat{p}_3 \) are not collinear; therefore, the first constraint here tells us that

\[
2 \text{Re}(\hat{h}_1) = -1.
\]

(4.3.86)

Substituting this result into the second constraint in (4.3.82), we find

\[
2 \text{Re}(\hat{h}_3 r) = -d.
\]

(4.3.87)

Putting all these results together, the constraints are

\[
\varepsilon_{\text{a..2.3.1}} = \{ 2 \text{Re}(\hat{h}_1) = -1, \quad 2 \text{Re}(\hat{h}_3 r) = -d, \quad 0 = [\hat{h}_1, r \hat{p}_3] \},
\]

(4.3.88)

for the non-vanishing matrices

\[
M_{\text{a..2.3.1}} = \{ P = \begin{pmatrix} 0 & 0 \\ \hat{i} & 0 \end{pmatrix}, \quad H = \begin{pmatrix} \hat{h}_1 & \hat{p}_3 \hat{h}_3 \hat{p}_3^{-1} + 1 \\ \hat{p}_3 \hat{h}_3 \hat{p}_3^{-1} + 1 \end{pmatrix}, \quad N_1 = \begin{pmatrix} 0 & \bar{r} \\ -1 & 0 \end{pmatrix}, \quad N_3 = \begin{pmatrix} 0 & 0 \\ 0 & \bar{r} \hat{p}_3 - \hat{p}_3 r \end{pmatrix} \}.
\]

(4.3.89)

Notice, the sub-branch in \( \mathcal{F} \) describing these super-extensions of \( \hat{a}_- \) is parameterised by four quaternions \( \hat{p}_3, \hat{h}_1, \hat{h}_3 \), and \( r \), and one real scalar \( d \). Wanting to establish the existence of super-extensions in this sub-branch, we can choose to set \( h_3, d, \) and the imaginary part of \( h_1 \) to zero. Then, utilising the endomorphisms of \( s_j \), we can impose that \( \hat{p}_3 \) must lie along \( \hat{i} \) and that \( r \) must have unit norm. Subsequently employing \( \text{Aut}(\mathbb{H}) \) to fix \( r \), we can finally scale \( H, Z, P, \) and \( B \) to get the super-extension

\[
P = \begin{pmatrix} 0 & 0 \\ \hat{i} & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad N_1 = \begin{pmatrix} 0 & 1 + \bar{r} \\ 0 \end{pmatrix}, \quad N_3 = \begin{pmatrix} 0 & 0 \\ 1 - j & 0 \end{pmatrix}.
\]

(4.3.90)

Having established that this sub-branch is not empty, we may look to introduce the components we have set to zero for this example. Notably, we may introduce the imaginary part of \( \hat{h}_1 \) while still fixing all parameters using the basis transformations \( G \subset \text{GL}(s_9) \times \text{GL}(s_1) \). However, the inclusion of either \( h_3 \) or \( d \) will introduce parameters that cannot be fixed.
\[ \hat{n}_i \text{ and } \hat{\varpi}. \]

Substituting \( \lambda = \epsilon = 0, \mu = \pm 1 \) into the conditions of (4.3.29),\(^7\) we instantly have \( N_3 = 0 \) and
\[
\begin{align*}
0 &= \bar{H}N_i + N_i H^\dagger \quad \text{where } i \in \{0, 1\} \\
0 &= \bar{P}N_i - N_i P^\dagger \quad \text{where } i \in \{0, 1\} \\
0 &= \text{Re}(\theta N_0 \theta^\dagger) \theta H \\
0 &= \pm \text{Re}(\theta N_0 \theta^\dagger) \varpi.
\end{align*}
\]

(4.3.92)

The final condition here states that \( N_0 = 0 \); therefore, \( N_1 \) is the only possible non-vanishing matrix of those encoding \( [Q, Q] \). This result tells us there will be no sub-branch 2.1 or 2.3 for these algebras and no sub-branch satisfying case \((ii)\), in which \( N_0 \neq 0 \). Therefore, the conditions reduce to
\[
0 = \bar{H}N_1 + N_1 H^\dagger \quad \text{and} \quad 0 = \bar{P}N_1 - N_1 P^\dagger.
\]

(4.3.93)

Under the assumption that \( p_3 \neq 0 \) for this branch of super-extensions, the latter condition tells us that \( \epsilon = 0 \) and that \( p_3 \) and \( \varpi \) are collinear:
\[
0 = \bar{\varpi}(\bar{r}p_3 - p_3 \varpi).
\]

(4.3.94)

Substituting these results into the first condition, we find
\[
0 = \text{Re}(\bar{h}_1), \quad 0 = [\bar{h}_1, \sigma p_3] \quad \text{and} \quad 0 = \text{Re}(\bar{h}_3 \varpi).
\]

(4.3.95)

Notice that since \( p_3 \) and \( \varpi \) are collinear, the second constraint is instantly satisfied. Thus, our constraints reduce to
\[
\hat{\theta}_{\bar{h}_3, \sigma} = [0 = \text{Re}(\bar{h}_1), \quad 0 = \text{Re}(\bar{h}_3 \varpi), \quad 0 = \bar{\varpi}(\bar{r}p_3 - p_3 \varpi)].
\]

(4.3.96)

The non-vanishing matrices in this instance are
\[
\hat{M}_{\bar{h}_3, \sigma} = \{P = \begin{pmatrix} 0 & 0 \\ \bar{p}_3 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} \bar{h}_1 & 0 \\ \bar{h}_3 & \varpi \bar{h}_1 \bar{p}_3^{-1} \end{pmatrix}, \quad N_1 = \begin{pmatrix} 0 & \varpi \\ \bar{r} & 0 \end{pmatrix} \}.
\]

(4.3.97)

This sub-branch has identical \( (\hat{M}, \hat{\theta}) \) to sub-branch 2.2.1 for \( \hat{\alpha} \). Therefore, for a discussion on the existence of such super-extensions, we refer the reader to the discussion found there.

### 4.3.3. Branch 3.

\[ P = Z = 0 \quad B = \begin{pmatrix} 0 & 0 \\ \bar{b}_3 & 0 \end{pmatrix} \quad H = \begin{pmatrix} \bar{h}_1 & 0 \\ \bar{h}_3 & \bar{b}_3 \bar{h}_1 \bar{b}_3^{-1} - \lambda \end{pmatrix}. \]

(4.3.98)

Exploiting the vanishing of \( Z \) and \( P \), we can reduce the conditions from Lemmas 4.2 and 4.3. In particular, the vanishing of \( P \), when substituted into the conditions from the \( [P, Q, Q] \) super-Jacobi identity, tells us that \( N_3 = 0 \) and
\[
0 = \eta \text{Re}(\theta N_0 \theta^\dagger) \pi \\
0 = \epsilon \text{Re}(\theta N_0 \theta^\dagger) \pi + \frac{1}{2}[\pi, \theta N_2 \theta^\dagger].
\]

(4.3.99)

The \( [B, Q, Q] \) identity then produce
\[
0 = BN_0 - N_0 B^\dagger \\
N_4 = BN_1 - N_1 B^\dagger \\
0 = \beta \theta B N_2 \theta^\dagger + \theta N_2 (\beta \theta B)^\dagger \\
0 = \lambda \text{Re}(\theta N_0 \theta^\dagger) \beta + \frac{1}{2}[\beta, \theta N_2 \theta^\dagger] \\
\mu \text{Re}(\theta N_0 \theta^\dagger) \beta = \beta \theta B N_4 \theta^\dagger + \theta N_4 (\beta \theta B). \]

(4.3.100)

The conditions from the \( [Z, Q, Q] \) identity are satisfied since \( Z = 0 \), and, lastly, the \( [H, Q, Q] \) super-Jacobi identity produces
\[
0 = \bar{H}N_i + N_i H^\dagger \quad \text{where } i \in \{0, 1, 2\} \\
0 = \eta N_4 \\
\epsilon N_4 = HN_4 + N_4 H^\dagger.
\]

(4.3.101)

From Lemma 4.3, we get
\[
\text{Re}(\theta N_0 \theta^\dagger) \theta H = \frac{1}{2} \theta N_2 \theta^\dagger \theta.
\]

(4.3.102)

\(^7\)Whether we are in the \( \hat{n}_+ \) or \( \hat{n}_- \) case makes no difference: the distinction between the two is the value of \( \eta \), which, if non-vanishing, would add the condition
\[
0 = \text{Re}(\theta N_0 \theta^\dagger) \pi.
\]

(4.3.91)

This condition sets \( N_0 = 0 \), but we already have this result from another condition. Therefore, the super-extensions are the same for both of these generalised Bargmann algebras.
As in both previous branches, the conditions
\[ 0 = \lambda \text{Re}(\theta N_0 \theta^\dagger) \beta + \frac{1}{2} [\beta, \theta N_2 \theta^\dagger] \]
\[ 0 = \varepsilon \text{Re}(\theta N_0 \theta^\dagger) \pi + \frac{1}{2} [\pi, \theta N_2 \theta^\dagger], \]
(4.3.103)
tell us \( N_2 = 0 \), such that, putting everything together, we have
\[ 0 = \eta N_4 \]
\[ 0 = H N_i + N_i H^\dagger \quad \text{where} \quad i \in \{0, 1\} \]
\[ 0 = B N_0 - N_0 B^\dagger \]
\[ 0 = \eta \text{Re}(\theta N_0 \theta^\dagger) \pi \]
\[ 0 = \lambda \text{Re}(\theta N_0 \theta^\dagger) \beta \]
\[ 0 = \varepsilon \text{Re}(\theta N_0 \theta^\dagger) \pi \]
\[ 0 = \text{Re}(\theta N_0 \theta^\dagger) \theta H \]
(4.3.104)

We can now use some of the conditions common to all generalised Bargmann algebras to identify possible sub-branches with which we can organise our investigations. Substituting the \( N_4 \) from (4.3.2) and the \( N_i \) from (4.3.3) into the condition
\[ N_4 = B N_1 - N_1 B^\dagger, \]
(4.3.105)
we can write \( N_4 \) in terms of the parameters in \( N_1 \) and \( B \):
\[ N_4 = \begin{pmatrix} 0 & -c b_3 \\ c b_3 & -b_3 \end{pmatrix}. \]
(4.3.106)

Notice that this means \( N_4 \) is completely dependent on \( N_1 \): if \( N_1 = 0 \) then \( N_4 = 0 \). Therefore, in general, we have the following sub-branches:

1. \( N_1 = 0 \) and \( N_4 = 0 \),
2. \( N_1 \neq 0 \) and \( N_4 = 0 \),
3. \( N_1 \neq 0 \) and \( N_4 \neq 0 \).

Also, as in branches 1 and 2, the condition derived from the \([Q, Q, Q] \) identity tells us that either \( N_0 \) or \( H \) vanishes. We will consider both of these cases within each sub-branch, identifying them as

(i) \( N_0 = 0 \) and \( H \neq 0 \), and
(ii) \( N_0 \neq 0 \) and \( H = 0 \).

Setting \( \lambda = \mu = \eta = \varepsilon = 0 \), the conditions in (4.3.104) reduce to
\[ 0 = H N_i + N_i H^\dagger \quad \text{where} \quad i \in \{0, 1\} \]
\[ 0 = B N_0 - N_0 B^\dagger \]
\[ 0 = \beta \theta B N_1 \theta^\dagger + \theta N_4 (\beta \theta B)^\dagger \]
\[ 0 = \text{Re}(\theta N_0 \theta^\dagger) \theta H \]
\[ N_4 = B N_1 - N_1 B^\dagger. \]
(4.3.107)

As in branch 2, none of these conditions force the vanishing of any more \( N_i \); therefore, super-extensions may be found in each of the sub-branches. In fact, because of the symmetry of the generators \( B \) and \( P \) in this generalised Bargmann algebra, we may use automorphisms to transform the above conditions into those in (4.3.33) and (4.3.34), which describe the super-extensions of \( \hat{a} \) in branch 2. More explicitly, substitute the transformation with matrices
\[ A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \]
(4.3.108)
and the quaternion \( u = 1 \), into (4.3.34). Putting the transformed matrices into the conditions of (4.3.107), we recover the conditions of (4.3.33) and (4.3.34). Therefore, all the super-extensions of \( \hat{a} \) in this branch are equivalent to the super-extensions of branch 2. Thus, for this particular generalised Bargmann algebra, this branch produces no new super-extensions.
Therefore, the constraints in this instance are given by
\[ 0 = \Re(\theta N_0^{\dagger} \beta) \quad \text{and} \quad 0 = -\Re(\theta N_0^{\dagger}) \pi \] (4.3.110)

The conditions
\[ 0 = \Re(\theta N_0^{\dagger} \beta) \quad \text{and} \quad 0 = -\Re(\theta N_0^{\dagger}) \pi \] (4.3.111)

Notice, this result tells us that we cannot have any sub-branches satisfying case (ii); therefore, all sub-branches \( [M, \bar{C}] \) discussed below will have a subscript ending in \( i \). Like the \( \hat{a} \) case, this generalised Bargmann algebra allows for an automorphism which transforms the conditions for this branch into the conditions for branch 2. However, in this instance, this branch will produce some distinct super-extensions. This result is a consequence of the parameters \( \epsilon \) and \( \lambda \) and their appearance in \( H \). In branch 2, the matrix \( H \) is written as
\[ H = \begin{pmatrix} \hat{h}_1 & 0 \\ \hat{h}_3 & 0 \end{pmatrix} \] (4.3.112)

and in this branch, it is written
\[ H = \begin{pmatrix} \hat{h}_1 & 0 \\ \hat{h}_3 & 0 \end{pmatrix} \] (4.3.113)

Since \( \tilde{a} \) has \( \epsilon = -1 \) and \( \lambda = 1 \), this matrix differs in these branches, if only be a sign. Thus, although the investigations into the super-extensions of \( \tilde{a} \) in this branch will be very similar to those in the previous branch, we will give a partial presentation of them here to demonstrate any consequences of this change in sign. In particular, we will omit the discussions on the existence of super-extensions and parameter fixing as these require only trivial adjustments from those found in branch 2.

**Sub-branch 3.1.** As \( N_0 = 0 \), we cannot have both \( N_1 \) and \( N_4 \) vanish; therefore, there is no super-extension in this sub-branch.

**Sub-branch 3.2.** Letting \( N_1 \neq 0 \) and \( N_4 = 0 \), we are left with only the conditions
\[ 0 = \Re(\theta N_0^{\dagger} \beta) \quad \text{and} \quad 0 = \Re(\theta N_0^{\dagger}) \pi \] (4.3.114)

The second condition above tells us that
\[ 0 = \Re(\theta N_0^{\dagger} \beta) \quad \text{and} \quad 0 = \Re(\theta N_0^{\dagger}) \pi \] (4.3.115)

As \( \mathbb{B}_3 \neq 0 \) by assumption, \( c = 0 \). Substituting this result into the first condition above, we find
\[ 0 = \Re(\theta N_0^{\dagger} \beta) \quad \text{and} \quad 0 = \Re(\theta N_0^{\dagger}) \pi \] (4.3.116)

Using the collinearity of \( \mathbb{B}_3 \) and \( r \), the first of these constraints tells us that
\[ 0 = (2 \Re(\theta_1) - 1) \Re(\mathbb{B}_3 r) \] (4.3.117)

Therefore, the constraints in this instance are given by
\[ \mathcal{E}_{\tilde{a}, 3.2.1} = \{ 0 = \mathbb{B}_3 r - \bar{r} \mathbb{B}_3, \quad 0 = (2 \Re(\theta_1) - 1) \Re(\mathbb{B}_3 r), \quad 0 = \Re(\mathbb{B}_3 r) + d(\Re(\theta_1) - 1) \} \] (4.3.118)

The non-vanishing matrices in this instance are
\[ \mathcal{M}_{\tilde{a}, 3.2.1} = \left\{ B = \begin{pmatrix} 0 & 0 \\ \bar{r} \mathbb{B}_3 & \bar{r} \mathbb{B}_3 \end{pmatrix}, \quad H = \begin{pmatrix} \hat{h}_1 & 0 \\ \hat{h}_3 & \Re(\theta_1) - 1 \end{pmatrix} \right\} \] (4.3.119)

This sub-branch of \( \mathcal{S} \) is parameterised by two collinear quaternions \( \mathbb{B}_3 \) and \( r \), two quaternions encoding the action of \( H \) on \( \mathbb{B}_3, \hat{h}_1 \) and \( \hat{h}_3 \), and one real scalar \( d \). Notice that the real component of \( \hat{h}_1 \) varies depending on whether \( r \) vanishes. Together with the super-extensions in sub-branch 2.2.1 for \( \tilde{a} \), these are the only super-extensions that demonstrate this type of dependency. If \( r = 0 \), the first two constraints of \( \mathcal{E}_{\tilde{a}, 3.2.1} \) are trivial, and the third condition tell us that \( \Re(\theta_1) = 1 \), since \( d \neq 0 \) for \( N_1 \neq 0 \). However, if \( r \neq 0 \), the second constraint
requires \(2 \text{Re}(\mathfrak{h}_1) = 1\). In this instance, the third constraint then becomes \(2 \text{Re}(\mathfrak{h}_3\mathfrak{r}) = d\). As the matrices and conditions for this sub-branch are so similar to those in 2.2.i, we refer the reader the discussion on existence of super-extensions and parameter fixing presented there.

**Sub-branch 3.3.** Finally, let \(N_1 \neq 0\) and \(N_4 \neq 0\). The condition
\[
0 = \beta \theta N_4 \theta^\dagger + \theta N_4 (\beta \theta)^\dagger
\]
(4.3.120)

imposes \(c = 0\), such that
\[
N_1 = \begin{pmatrix} 0 & r \\ \bar{r} & d \end{pmatrix} \quad \text{and} \quad N_4 = \begin{pmatrix} 0 & 0 \\ 0 & \bar{b}_3 \mathfrak{r} - \bar{r} \bar{b}_3 \end{pmatrix}.
\]
(4.3.121)

This result reduces the conditions in (4.3.111) to
\[
0 = H N_1 + N_1 H^\dagger \\
-N_4 = H N_4 + N_4 H^\dagger.
\]
(4.3.122)

Substituting the \(N_4\) from (4.3.121) into the second condition above, we have
\[
-l = [\mathfrak{h}_4, \mathfrak{r}],
\]
(4.3.123)

where \(l = \bar{b}_3 \mathfrak{r} - \bar{r} \bar{b}_3\) and \(\mathfrak{h}_4 = [\bar{b}_3 \mathfrak{h}_1 \bar{b}_3^{-1} - 1\). We can rewrite this condition as
\[
(1 + 2 \text{Re}(\mathfrak{h}_4))l = [l, \mathfrak{h}_4].
\]
(4.3.124)

Notice that the R.H.S. of this expression must lie in \(\text{Im}(H)\) and be orthogonal to \(l\), which is imaginary by construction. Therefore, both sides of this expression must vanish independently:
\[
0 = (1 + 2 \text{Re}(\mathfrak{h}_4))l \quad 0 = [l, \mathfrak{h}_4].
\]
(4.3.125)

Substituting \(l\) and \(\mathfrak{h}_4\) into these constraints, we find
\[
0 = (2 \text{Re}(\mathfrak{h}_1) - 1)(\bar{b}_3 \mathfrak{r} - \bar{r} \bar{b}_3) \quad \text{and} \quad 0 = [\mathfrak{h}_1, \mathfrak{r} \bar{b}_3],
\]
(4.3.126)

respectively. For \(N_4\) to not vanish, we must have \(\text{Im}(\bar{b}_3 \mathfrak{r}) \neq 0\), so, by the first constraint above, we need \(2 \text{Re}(\mathfrak{h}_1) = 1\). The first condition in (4.3.122) produces the same constraints as in sub-branch 3.2; namely,
\[
0 = \bar{h}_1 \mathfrak{r} + r \bar{b}_3 \mathfrak{h}_1 \bar{b}_3^{-1} - 1 \\
0 = \text{Re}(\bar{h}_3 \mathfrak{r}) + d(\text{Re}(\mathfrak{h}_1) - 1).
\]
(4.3.127)

Notice that the requirement of setting \(2 \text{Re}(\mathfrak{h}_1) = 1\) makes the second constraint here \(2 \text{Re}(\bar{h}_3 \mathfrak{r}) = d\), and that the first constraint is equivalent to \(0 = [\mathfrak{h}_1, \mathfrak{r} \bar{b}_3]\). Therefore, the constraints on this sub-branch are given by
\[
\mathcal{M}_{\mathfrak{h}_1, 3.3.} = \{ d = 2 \text{Re}(\bar{h}_3 \mathfrak{r}), \quad 1 = 2 \text{Re}(\mathfrak{h}_1) \quad \text{and} \quad 0 = [\mathfrak{h}_1, \mathfrak{r} \bar{b}_3] \},
\]
(4.3.128)

and the non-vanishing matrices are
\[
\mathcal{M}_{\mathfrak{h}_1, 3.3.} = \{ B = \begin{pmatrix} 0 & 0 \\ \bar{b}_3 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} \mathfrak{h}_1 & 0 \\ \mathfrak{h}_3 & \bar{b}_3 \mathfrak{h}_1 \bar{b}_3^{-1} - 1 \end{pmatrix}, \quad N_1 = \begin{pmatrix} 0 & r \\ \bar{r} & d \end{pmatrix}, \quad N_4 = \begin{pmatrix} 0 & 0 \\ 0 & \bar{b}_3 \mathfrak{r} - \bar{r} \bar{b}_3 \end{pmatrix} \}.
\]
(4.3.129)

For the discussion on existence of super-extensions and how to fix the parameters of the matrices describing this sub-branch of \(\mathcal{S}\), we refer the reader to sub-branch 2.3.i. The application of the discussion to the present case requires only minor adjustments.

\(\mathfrak{n}_+\).

Substituting \(\lambda = \epsilon = 0, \mu = 1, \text{and} \eta = -1\) into the results for Lemmas 4.2 and 4.3, we find
\[
0 = -N_4 \\
0 = H N_1 + N_1 H^\dagger \quad \text{where} \quad i \in \{0, 1, 4\} \\
0 = B N_0 - N_0 B^\dagger \\
0 = -\text{Re}(\theta N_0 \theta^\dagger) \pi \\
0 = \text{Re}(\theta N_0 \theta^\dagger) \theta H.
\]
(4.3.130)

Therefore, \(N_4\) vanishes, and \(N_0\) vanishes by \(0 = -\text{Re}(\theta N_0 \theta^\dagger) \pi\). This leaves us with
\[
0 = H N_1 + N_1 H^\dagger \quad \text{and} \quad 0 = B N_1 - N_1 B^\dagger.
\]
(4.3.131)

Notice that these conditions are similar to those of (4.3.93), which describe the super-extensions of \(\mathfrak{n}_+\) in branch 2. In fact, we can utilise the automorphisms of \(\mathfrak{n}_+\) to transform the above conditions into those in (4.3.93). Unlike the \(\mathfrak{n}_-\) case, since \(\mathfrak{n}_+\) has vanishing \(\epsilon\) and \(\lambda\), there is no discrepancy between the transformed matrices and those of branch 2; therefore, the super-extensions of \(\mathfrak{n}_+\) in branches 2 and 3 are equivalent. Thus, we have no
new super-extensions here.
\( \hat{\eta} \).

Substituting \( \lambda = \eta = \epsilon = 0 \) and \( \mu = -1 \) into (4.3.104), we have
\[
0 = HN_i + N_i H^\dagger \quad \text{where} \quad i \in \{0, 1, 4\} \\
0 = BN_0 - N_0 B^\dagger \\
0 = \text{Re}(\theta N_0 \theta^\dagger) \theta H
\]

With these conditions, we can now investigate the three sub-branches.

**Sub-branch 3.1.** We cannot have \( N_1 = N_4 = 0 \), since the vanishing of \( N_4 \) means \( N_0 = 0 \) through
\[
-\text{Re}(\theta N_0 \theta^\dagger) \beta = \beta \theta BN_4 \theta^\dagger + \theta N_4 (\beta \theta B)^\dagger.
\]

This would cause all \( N_i \) to vanish such that \( [Q, Q] = 0 \). Therefore, there is no super-extension in this sub-branch.

**Sub-branch 3.2.** With only \( N_1 \neq 0 \), the conditions reduce to
\[
0 = HN_1 + N_1 H^\dagger \quad \text{and} \quad 0 = BN_1 - N_1 B^\dagger.
\]

Notice that this is the same set of conditions as the \( \hat{\eta}_1 \) case above. Therefore, we may expect the analysis for this generalised Bargmann algebra to be analogous. However, there is a very important distinction. In the \( \hat{\eta}_1 \) case, we were able to use the automorphisms to transform the conditions into those of sub-branch 2.2. This automorphism is not permitted by the generalised Bargmann algebra \( \hat{\eta} \). Therefore, although the analysis will be the same *mutatis mutandis* as that of sub-branch 2.2, the resulting super-extensions will be distinct.

Now, since \( N_1 \) is the only possible non-vanishing matrix in the \( [Q, Q] \) bracket, it must have non-zero components. The latter condition above tells us that \( c = 0 \) and \( b_3 \) and \( r \) are collinear quadrilaterals, while the former condition imposes
\[
0 = h_1 r + r \bar{b}_3 h_1 \bar{b}_3^{-1} \\
0 = \text{Re}(h_1 r) + d \text{Re}(h_1).
\]

Notice that if \( r = 0 \), we need \( d \neq 0 \) for the existence of a super-extension; therefore, the final constraint above would impose \( \text{Re}(h_1) = 0 \). Similarly, if \( r \neq 0 \), the first constraint would also enforce \( \text{Re}(h_1) = 0 \). Thus, in all super-extensions, we require \( \text{Re}(h_1) = 0 \). Using this result, these two constraints simplify to
\[
0 = [h_1, r \bar{b}_3] \quad \text{and} \quad 0 = \text{Re}(h_3 r).
\]

However, since \( b_3 \) and \( r \) are collinear and it is only the imaginary part of \( r \bar{b}_3 \) that will contribute to \( [h_1, r \bar{b}_3] \), the first of these constraints is already satisfied. Therefore, the final set of constraints on this sub-branch is
\[
\mathcal{C}_{\hat{\eta}, 3.2.1} = \{ 0 = b_3 r - \bar{r} \bar{b}_3, \quad 0 = \text{Re}(h_1), \quad 0 = \text{Re}(h_3 r) \}.
\]

Subject to these constraints, we have non-vanishing matrices are
\[
\mathcal{M}_{\hat{\eta}, 3.2.1} = \begin{Bmatrix} B = \begin{pmatrix} 0 & 0 \\ b_3 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} h_1 & 0 \\ b_3 & b_3 b_1 b_3^{-1} \end{pmatrix}, \quad N_1 = \begin{pmatrix} 0 & r \\ \frac{d}{r} & 0 \end{pmatrix} \end{Bmatrix}.
\]

Since this \( (M, C) \) is analogous to the one found in branch 2 for \( \hat{\eta}_1 \) and \( \hat{\eta} \), we will omit the discussion on existence of super-extensions and parameter fixing.

**Sub-branch 3.3.** Finally, with \( N_4 \neq 0 \), we can think of setting \( N_0 \neq 0 \) and \( H = 0 \). But first, try setting \( N_0 = 0 \) to allow \( H \neq 0 \). The conditions become
\[
0 = HN_i + N_i H^\dagger \quad \text{where} \quad i \in \{1, 4\} \\
N_4 = BN_1 - N_1 B^\dagger \\
0 = \beta \theta BN_4 \theta^\dagger + \theta N_4 (\beta \theta B)^\dagger.
\]

Notice that the second condition above allows us to write \( N_4 \) in terms of \( B \) and \( N_1 \):
\[
N_4 = \begin{pmatrix} 0 & c \bar{b}_3 \\ c b_3 & -c \bar{b}_3 \end{pmatrix}.
\]

The third condition then imposes \( c = 0 \), since \( b_3 \neq 0 \), leaving us with
\[
N_1 = \begin{pmatrix} 0 & r \\ \frac{d}{r} & 0 \end{pmatrix} \quad \text{and} \quad N_4 = \begin{pmatrix} 0 & 0 \\ 0 & -r b_3 - \bar{r} \bar{b}_3 \end{pmatrix}.
\]
Using these matrices in the final conditions,  
\[ 0 = H N_i + N_i H^\dagger \quad \text{where} \quad i \in \{1, 4\}, \]  
(4.3.142)  
produces  
\[ 0 = [h_1, r \hat{b}_3], \]  
(4.3.143)  
when \( i = 4 \), and, when \( i = 1 \), we obtain  
\[ 0 = h_1 r + r \hat{b}_3 b_1 \hat{b}_3^{-1} \]  
\[ 0 = \text{Re}(h_3 r) + d \text{Re}(h_1). \]  
(4.3.144)  
Since \( r \neq 0 \) for \( N_4 \neq 0 \), the first condition here states that \( \text{Re}(h_1) = 0 \). Therefore, the constraints on the parameters of this super-extension are given by  
\[ e_{\phi, 3.3;i} = \{0 = \text{Re}(h_1), \ 0 = [h_1, r \hat{b}_3], \ 0 = \text{Re}(h_3 r)\}. \]  
(4.3.145)  
The non-vanishing matrices associated with this sub-branch are  
\[ M_{\phi, 3.3;i} = \{ B = \begin{pmatrix} 0 & 0 \\ \hat{b}_3 & 0 \end{pmatrix}, \ N_0 = \begin{pmatrix} 0 & 0 \\ -2c |\hat{b}_3|^2 & 0 \end{pmatrix}, \ N_1 = \begin{pmatrix} c & r \\ \frac{r}{d} & d \end{pmatrix}, \ N_4 = \begin{pmatrix} 0 & -\hat{c} \hat{b}_3 \\ \hat{b}_3 r - r \hat{b}_3 \end{pmatrix} \}. \]  
(4.3.146)  
The \((M, C)\) of this sub-branch is the same mutatis mutandis as that of sub-branch 2.3.i for \( \hat{a} \); therefore, we refer the reader to the discussion found there on existence of super-extensions and parameter fixing.

Finally, let \( N_0 \neq 0 \) such that \( H = 0 \). The conditions remaining from (4.3.132) are  
\[ 0 = BN_0 - N_0 B^\dagger \]  
\[ N_4 = BN_1 - N_1 B^\dagger \]  
\[ -\text{Re}(\theta N_0 \theta^\dagger) = \beta \theta N_4 \theta^\dagger + \theta N_4 (\theta N B)^\dagger. \]  
(4.3.147)  
We know how the second condition acts from the discussion at the beginning of this branch. The first of these conditions tells us  
\[ 0 = a \quad \text{and} \quad 0 = b \hat{b}_3 q - \hat{q} b_3, \]  
(4.3.148)  
and the third, substituting in \( \theta = (0, 1) \), produces  
\[ -2 \text{Re}(s \hat{b}_3) - b |\hat{q}|^2 = 2c |\hat{b}_3|^2. \]  
(4.3.149)  
Now substituting \( \theta = (1, s) \) into the third condition, we find  
\[ -2 \text{Re}(s \hat{b}_3) - b |\hat{q}|^2 = 2c |\hat{b}_3|^2 |\hat{b}_3|^2. \]  
(4.3.150)  
Therefore, using the previous result and letting \( s = 1, s = 1, s = \hat{j}, \text{ and } s = \hat{k} \), we see that all components of \( q \) must vanish. We thus have non-vanishing matrices  
\[ M_{\phi, 3.3;i} = \{ B = \begin{pmatrix} 0 & 0 \\ \hat{b}_3 & 0 \end{pmatrix}, \ N_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \ N_1 = \begin{pmatrix} c & r \\ \frac{r}{d} & d \end{pmatrix}, \ N_4 = \begin{pmatrix} 0 & -\hat{c} \hat{b}_3 \\ \hat{b}_3 r - r \hat{b}_3 \end{pmatrix} \}. \]  
(4.3.151)  
Interestingly, there are no additional constraints to the parameters of this sub-branch; therefore, \( e_{\phi, 3.3;i} \) is empty. Notice the sub-branch of \( \mathcal{X} \) for this type of super-extension is parameterised by two quaternions \( \hat{b}_3 \) and \( r \), and two real scalars \( c \) and \( d \). To demonstrate that this sub-branch is not empty, we begin by setting both \( r \) and \( d \) to zero. This choice allows us to utilise the endomorphisms of \( \hat{s}_1 \) to set \( \hat{b}_3 = i \) and \( c = 1 \). Employing the scaling symmetry of the basis elements, we arrive at  
\[ B = \begin{pmatrix} 0 & 0 \\ \hat{b}_3 & 0 \end{pmatrix}, \ N_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \ N_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \ N_4 = \begin{pmatrix} 0 & \hat{i} \hat{b}_3 \\ \hat{b}_3 r - r \hat{b}_3 \end{pmatrix}. \]  
(4.3.152)  
We may now look to introduce \( r \) and \( d \). Again using the endomorphisms of \( \hat{s}_1 \), we can impose that \( \hat{b}_3 \) must lie along \( i \), set \(|r|^2 = 1 \), and choose \( \sqrt{2c} = 1 \). This choice for \( r \) imposes that \( r \in \text{Sp}(1) \), and we may utilise \( \text{Aut}(H) \) to fix \( \sqrt{2}r = 1 + i \). Having chosen \( r \neq 0 \), we can always employ the residual endomorphisms of \( \hat{s}_1 \) to set \( d = 0 \).

Using the only remaining symmetry, the scaling of \( H, Z, B, \text{ and } P \), we find  
\[ B = \begin{pmatrix} \frac{0}{\sqrt{2}} & 0 \\ 0 & 0 \end{pmatrix}, \ N_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \ N_1 = \begin{pmatrix} 1 & 1 + \hat{i} \\ 1 - \hat{i} & 0 \end{pmatrix}, \ N_4 = \begin{pmatrix} 0 & \hat{i} \\ \hat{i} & 2i \end{pmatrix}. \]  
(4.3.153)
4.3.4. Branch 4.

\[
Z = 0 \quad H = \begin{pmatrix} \h_1 & 0 \\ \h_3 & h_4 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 0 \\ b_3 & 0 \end{pmatrix} \quad P = \begin{pmatrix} 0 & 0 \\ p_3 & 0 \end{pmatrix},
\]

(subject to
\[
[u, h_1] = -\mu u^2 + (\lambda - \epsilon)u + \eta \quad \text{or} \quad [v, h_1] = \eta v^2 + (\lambda - \epsilon)v + \mu,
\]

where \(0 \neq u = \bar{\mathbb{D}}_3^{-1}p_3\) and \(0 \neq v = p_3^{-1}\bar{\mathbb{D}}_3\). Recall, we keep both of these constraints as, depending on the generalised Bargmann algebra under investigation, one of them will prove more useful than the other. We still need to determine the generalised Bargmann algebras for which this branch could provide a super-extension. Therefore, we will consider each algebra in turn, and analyse those for which the above constraints may hold.

\[\hat{\mathfrak{a}}.\]

Setting \(\lambda = \mu = \eta = \epsilon = 0\) in (4.3.155), we could still get a super-extension, as long as we impose

\[0 = [u, h_1].\]

Throughout this section, we will choose to write parameters in terms of \(\mathbb{D}_3\); therefore, we write \(p_3 = \mathbb{D}_3 u\) and \(\bar{h}_4 = \mathbb{D}_3 h_3\mathbb{D}_3^{-1}\), where \(u \in H\). Notice that the significance of \(u\) is only manifest when \(h_1 \neq 0\): when \(h_1\) vanishes, we are simply replacing \(p_3\) with \(u\). However, since \(u\) will be important is several instances, we will always use this notation.

Since neither \(B\) nor \(P\) vanish, there are no immediate results as in the three previous branches: all the conditions of Lemmas 4.2 and 4.3 must be taken into consideration. However, as with branches 2 and 3, we can organise our investigations based on dependencies. In particular, the conditions

\[
N_4 = BN_1 - N_1 B^\dagger \\
-N_3 = PN_1 - N_1 P^\dagger \\
\frac{1}{2}[\beta, \theta N_2 \theta^\dagger] = \beta \theta BN_3 \theta^\dagger + \theta N_3 (\beta \theta B)^\dagger \\
\frac{1}{2}[\pi, \theta N_2 \theta^\dagger] = \pi \theta PN_4 \theta^\dagger + \theta N_4 (\pi \theta P)^\dagger,
\]

show us that if \(N_1\) vanishes, so must \(N_2, N_3,\) and \(N_4\). Additionally, the vanishing of either \(N_3\) or \(N_4\) means we must have \(N_2 = 0\). Therefore, we can divide our investigations into the following sub-branches.

1. \(N_1 = N_2 = N_3 = N_4 = 0\)
2. \(N_1 \neq 0\) and \(N_2 = N_3 = N_4 = 0\)
3. \(N_1 \neq 0, N_3 \neq 0,\) and \(N_2 = N_4 = 0\)
4. \(N_1 \neq 0, N_4 \neq 0,\) and \(N_2 = N_3 = 0\)
5. \(N_1 \neq 0, N_3 \neq 0, N_4 \neq 0,\) and \(N_2 = 0\)
6. \(N_1 \neq 0, N_2 \neq 0, N_3 \neq 0,\) and \(N_4 \neq 0\).

Unlike branches 1, 2 and 3, the \([Q, Q, Q]\) super-Jacobi identity will not always result in the cases (i), in which \(N_0 = 0\) and \(H \neq 0\), or (ii), in which \(N_0 \neq 0\) and \(H = 0\). There are instances in which both \(N_0\) and \(H\) may not vanish. These cases, will be labelled (iii).

**Sub-Branch 4.1.** With only \(N_0\) left available, it cannot vanish for a supersymmetric extension to exist. Therefore, the \([Q, Q, Q]\) identity,

\[
\text{Re}(\theta N_0 \theta^\dagger) \theta H = 0,
\]

tells us we must have \(H = 0\). The remaining conditions are then

\[0 = BN_0 - N_0 B^\dagger \quad \text{and} \quad 0 = PN_0 - N_0 P^\dagger,
\]

which tell us

\[0 = a, \quad 0 = \bar{\mathbb{D}}_3 a - \bar{q} \bar{a} \quad \text{and} \quad 0 = \bar{\mathbb{D}}_3 uq - \bar{q} \bar{u} \bar{b}.
\]

This sub-branch thus has non-vanishing matrices

\[
B = \begin{pmatrix} 0 & 0 \\ b_3 & 0 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 0 \\ p_3 & 0 \end{pmatrix}, \quad N_0 = \begin{pmatrix} 0 & q \\ q & b \end{pmatrix},
\]

subject to the constraints

\[0 = \bar{\mathbb{D}}_3 a - \bar{q} \bar{a}, \quad 0 = \bar{\mathbb{D}}_3 uq - \bar{q} \bar{u} \bar{b}.
\]

Notice that these matrices and constraints are very similar to \((\mathbb{M}_{6.2.1.a}, \mathbb{C}_{6.2.1.b})\). In fact, employing the automorphisms of \(\hat{\mathfrak{a}}\), we can show that the above system is equivalent to sub-branch 2.1.ii. Using the endomorphisms of \(s_1\) and the constraints above, we can set \(b_3, p_3, a, q\) to lie along \(\mathbb{E}\), and set \(b = 0\). In particular, this
means that \( u \in \mathbb{R} \). Scaling \( B, P, \) and \( H \), we find the matrices
\[
B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 0 \\ \bar{u} & 0 \end{pmatrix}, \quad N_0 = \begin{pmatrix} 0 & \bar{u} \\ -\bar{u} & 0 \end{pmatrix},
\]
which under the basis transformation with
\[
C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]
recovers the maximal super-extension of sub-branch 2.1.ii. Thus, this sub-branch does not contribute any new super-extensions to \( \hat{a} \).

**Sub-branch 4.2.** The \( \{ Q, Q, Q \} \) identity still imposes that either \( N_0 \) or \( H \) must vanish in this sub-branch; however, we can now consider the case where \( N_0 = 0 \) as we have \( N_1 \neq 0 \). First, consider case (i), with \( N_0 = 0 \) such that \( H \neq 0 \). The conditions remaining are
\[
0 = HN_1 + N_1 H^\dagger
\]
\[
0 = BN_1 - N_1 B^\dagger
\]
\[
0 = PN_1 - N_1 P^\dagger,
\]
The latter two conditions tell us that \( c = 0 \) and \( b_3 \) is collinear with \( b_{3u} \) and \( r \). Substituting these results into the first condition, we find
\[
0 = \bar{h}_1 r + r \bar{b}_3 h_1 \bar{b}_3^{-1}
\]
\[
0 = \text{Re}(\bar{h}_1 r) + d \text{Re}(h_1).
\]
We know from the analysis of branch 3 that demanding \( N_1 \neq 0 \) under these conditions imposes \( \text{Re}(h_1) = 0 \); and, that having the condition
\[
0 = \bar{b}_3 r - \bar{r} \bar{b}_3
\]
means we always satisfy the imaginary part of
\[
0 = h_1 \bar{r} + r \bar{b}_3 \bar{h}_1 \bar{b}_3^{-1}.
\]
Putting all this together, we find the constraints on this sub-branch to be
\[
0 = \text{Re}(h_1), \quad 0 = \text{Re}(h_3 r), \quad 0 = \bar{b}_3 r - \bar{r} \bar{b}_3, \quad 0 = \bar{b}_3 ur - \bar{r} \bar{u} \bar{b}_3, \quad 0 = \{ u, h_1 \}.
\]
The non-vanishing matrices are then
\[
B = \begin{pmatrix} 0 & 0 \\ \bar{b}_3 & 0 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 0 \\ \bar{b}_3 u & 0 \end{pmatrix}, \quad H = \begin{pmatrix} \bar{h}_1 \\ \bar{b}_3 \end{pmatrix} \begin{pmatrix} 0 & \bar{b}_3 \\ 0 & 0 \end{pmatrix}^{-1} \quad \text{and} \quad N_1 = \begin{pmatrix} 0 & r \\ \bar{r} & d \end{pmatrix}.
\]
Notice that \( \bar{b}_3, r \) and \( \bar{b}_3 u \) all being collinear implies that \( u \in \mathbb{R} \). Thus the final constraint is satisfied, and, as in sub-branch 4.1, we can use the endomorphisms of \( s_1 \) and the automorphisms of \( \hat{a} \) to rotate \( B \) and \( P \) such that we only have the matrix \( P \), in which \( g_3 = \bar{1} \). The resulting matrices and constraints are then equivalent to those found in sub-branch 2.2.ii, and, therefore, this sub-branch does not produce any new super-extensions for \( \hat{a} \).

Now, considering case (ii), let \( H = 0 \). The remaining conditions are
\[
0 = BN_1 - N_1 B^\dagger \quad \text{where} \quad i \in \{ 0, 1 \}
\]
\[
0 = PN_1 - N_1 P^\dagger \quad \text{where} \quad i \in \{ 0, 1 \}.
\]
Therefore, \( N_0 \) and \( N_1 \) take the same form in this instance: both \( a \) and \( c \) vanish, with \( q \) and \( r \) being collinear to both \( b_3 \) and \( \bar{b}_3 \). In summary, the constraints are
\[
0 = \bar{b}_3 q - \bar{q} \bar{b}_3, \quad 0 = \bar{b}_3 u q - \bar{q} \bar{u} \bar{b}_3, \quad 0 = \bar{b}_3 r - \bar{r} \bar{b}_3, \quad 0 = \bar{b}_3 ur - \bar{r} \bar{u} \bar{b}_3,
\]
and the non-vanishing matrices are
\[
B = \begin{pmatrix} 0 & 0 \\ \bar{b}_3 & 0 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 0 \\ \bar{b}_3 u & 0 \end{pmatrix}, \quad N_0 = \begin{pmatrix} 0 & q \\ \bar{q} & b \end{pmatrix}, \quad \text{and} \quad N_1 = \begin{pmatrix} 0 & r \\ \bar{r} & d \end{pmatrix}.
\]
Through the same use of the subgroup \( G \subset \text{GL}(s_{\bar{0}}) \times \text{GL}(s_{\bar{1}}) \) as discussed for case (ii), we find that this sub-branch is equivalent to 2.2.ii for \( \hat{a} \).
Sub-branch 4.3. Now with $N_3 \neq 0$, we can use

$$-N_3 = PN_4 - N_1 P^\dagger \quad \text{and} \quad 0 = \pi \theta P N_3 \theta^\dagger + \Theta N_3 (\pi \theta P)^\dagger$$  \hspace{1cm} (4.3.174)

to first write $N_4$ in terms of $P$ and $N_1$ before setting $c = 0$ by substituting $\theta = (0, 1)$ into the latter condition. This produces the matrix

$$N_3 = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{r}u_b \end{pmatrix}. \hspace{1cm} (4.3.175)$$

Since $N_3$ and $B$ are non-vanishing, the condition from the $[Q, Q, Q] \text{ identity no longer states that we must set}$

$$\text{either } N_0 \text{ or } H \text{ to zero. We have} \hspace{2cm} (4.3.176)$$

$$\text{Substituting } \theta = (0, 1) \text{ into the above condition, we find} \hspace{2cm} (4.3.177)$$

$$b \bar{b}_3 = (\tilde{r}u_b \bar{e}_3 - \bar{b}_3 \bar{u}r) b_3 \quad \text{and} \quad b \bar{b}_1 = 0. \hspace{2cm} (4.3.178)$$

which tell us $a$ and $c$ vanish, and

$$0 = \bar{b}_3 q - \bar{q} \bar{b}_3, \quad 0 = \bar{b}_3 u_q - \bar{q} \bar{u} b_3, \quad \text{and} \quad 0 = \bar{b}_3 r - \bar{r} \bar{b}_3. \hspace{2cm} (4.3.179)$$

Using these results and the rewriting of $\bar{b}_3$ in (4.3.177), the conditions from the $[H, Q, Q] \text{ identity are instantly}$

$$\text{satisfied. Therefore, the constraints on the parameters of this sub-branch are} \hspace{2cm} (4.3.180)$$

$$0 = \bar{b}_3 q - \bar{q} \bar{b}_3, \quad 0 = \bar{b}_3 u_q - \bar{q} \bar{u} b_3, \quad 0 = \bar{b}_3 r - \bar{r} \bar{b}_3, \quad \text{and} \quad b \bar{b}_3 = (\tilde{r}u_b \bar{e}_3 - \bar{b}_3 \bar{u}r) b_3. \hspace{2cm} (4.3.181)$$

Notice that the first three constraints here tell us that $\bar{b}_3$ is collinear with both $q$ and $r$ and that $\bar{b}_3 u$ is collinear

$$\text{with } q. \text{ In particular, were we to use the endomorphisms of } e_1 \text{ to set } q \text{ to lie along } \bar{i}, \text{ both of these}$

$$\text{constraints would all lie along } \bar{i} \text{ as well. Thus, } \bar{b}_3 u \in \Re, \text{ such that } N_3 = 0. \text{ Therefore, this sub-branch is empty.} \hspace{2cm} (4.3.182)$$

Sub-branch 4.4. This sub-branch will be very similar to the one above due to the similarity in the conditions

$$\text{the super-Jacobi identities imposes on } N_3 \text{ and } N_4. \text{ Using} \hspace{2cm} (4.3.183)$$

$$N_4 = BN_1 - N_1 B^\dagger \quad \text{and} \quad 0 = \beta \theta B N_4 \theta^\dagger + \Theta N_4 (\beta \theta B)^\dagger, \hspace{2cm} (4.3.184)$$

we know $N_4$ may be written

$$N_4 = \begin{pmatrix} 0 & 0 \\ 0 & b \bar{b}_3 - \bar{r} \bar{b}_3 \end{pmatrix}. \hspace{2cm} (4.3.185)$$

Lemma 4.3 then tells us that

$$\text{Substituting } \theta = (0, 1) \text{ into this condition produces} \hspace{2cm} (4.3.186)$$

$$b \bar{b}_3 = (\bar{b}_3 r - \bar{r} \bar{b}_3) b_3 u \quad \text{and} \quad b \bar{b}_1 = 0. \hspace{2cm} (4.3.187)$$

Since $\bar{b}_3 u \neq 0$ and $\bar{b}_3 r - \bar{r} \bar{b}_3 \neq 0$ by assumption, $b$ cannot vanish; therefore, $\bar{b}_1 = 0$. The conditions

$$0 = PN_4 - N_1 P^\dagger \quad \text{where} \quad i \in \{0, 1\} \hspace{2cm} (4.3.188)$$

tell us that both $a$ and $c$ vanish, and

$$0 = \bar{b}_3 q - \bar{q} \bar{b}_3, \quad 0 = \bar{b}_3 u_q - \bar{q} \bar{u} b_3, \quad \text{and} \quad 0 = \bar{b}_3 r - \bar{r} \bar{b}_3. \hspace{2cm} (4.3.189)$$

Finally, we have the conditions from the $[H, Q, Q] \text{ identity, which impose} \hspace{2cm} (4.3.190)$$

$$0 = \text{Re}(\bar{b}_3 q) \quad \text{and} \quad 0 = \text{Re}(\bar{b}_3 u r). \hspace{2cm} (4.3.191)$$

However, using the form of $\bar{b}_3$ in (4.3.184) and the collinearity of $\bar{b}_3 u$ with $q$ and $r$, both of these constraints are already satisfied. Therefore, the final set of constraints on this sub-branch is

$$0 = \bar{b}_3 q - \bar{q} \bar{b}_3, \quad 0 = \bar{b}_3 u_q - \bar{q} \bar{u} b_3, \quad 0 = \bar{b}_3 r - \bar{r} \bar{b}_3, \quad b \bar{b}_3 = (\bar{b}_3 r - \bar{r} \bar{b}_3) b_3 u. \hspace{2cm} (4.3.192)$$

Notice that the first three constraints tell us that $\bar{b}_3, \bar{b}_3 u, q$, and $r$ are collinear. This tells us that $\bar{b}_3 r \in \Re$; therefore, significantly, $N_4 = 0$. Thus this sub-branch is empty.
Sub-branch 4.5. Now with non-vanishing $N_3$ and $N_4$, we can begin by using
\[
-N_3 = PN_1 - N_1 P^\dagger \quad \text{and} \quad N_4 = BN_1 + N_1 B^\dagger,
\]
and
\[
0 = \pi \theta PN_3 \theta^\dagger + \theta N_3 (\pi \theta P)^\dagger \quad \text{and} \quad 0 = \beta \theta BN_3 \theta^\dagger + \theta N_4 (\beta \theta B)^\dagger,
\]
which
\[
N_1 = \begin{pmatrix} 0 & r \\ d & 0 \end{pmatrix}, \quad N_3 = \begin{pmatrix} 0 & 0 \\ \bar{r}\bar{b}_3 - \bar{b}_3 ur & 0 \end{pmatrix} \quad \text{and} \quad N_4 = \begin{pmatrix} 0 & 0 \\ 0 & \bar{b}_3 r - \bar{r}\bar{b}_3 \end{pmatrix}.
\]
Using these results, substitute $\theta = (0, 1)$ into the condition from the $[Q, Q, Q]$ identity to find
\[
\bar{b}b_3 = (\bar{r}\bar{b}_3 - \bar{b}_3 ur)\bar{b}_3 + (\bar{b}_3 r - \bar{r}\bar{b}_3)\bar{b}_3 u \quad \text{and} \quad \bar{b}b_3 = 0.
\]
As in all previous sub-branches, the $[P, Q, Q]$ and $[B, Q, Q]$ conditions on $N_0$ tell us
\[
0 = a, \quad 0 = \bar{b}_3 a - \bar{a}\bar{b}_3 \quad \text{and} \quad 0 = \bar{b}_3 u a - \bar{a}\bar{b}_3.
\]
Finally, the $[H, Q, Q]$ identities tell us
\[
0 = \text{Re}(h_3 q) + b \text{Re}(h_1) \quad \text{and} \quad 0 = \text{Re}(h_3 r) + d \text{Re}(h_1)
\]
\[
0 = h_1 q + q \bar{b}_3 h_1 \bar{b}_3, \quad \text{and} \quad 0 = h_1 r + r \bar{b}_3 h_1 \bar{b}_3.
\]
Since, by assumption, $N_1 \neq 0$, these constraints mean we must have $\text{Re}(h_1) = 0$. If $r = 0$, we would need $d \neq 0$, which, when substituted into $0 = d \text{Re}(h_1)$, mean $\text{Re}(h_1) = 0$. Alternatively, if $r \neq 0$, we multiply
\[
0 = h_1 r + \bar{r} \bar{b}_3 h_1 \bar{b}_3
\]
on the right by $r^{-1}$ and take the real part to obtain $\text{Re}(h_1) = 0$. Knowing this, we can use the fact $\bar{b}_3 h_1 \bar{b}_3 \in \text{Im}(H)$ to rewrite the remaining imaginary part of this constraint as
\[
0 = [h_1, r \bar{b}_3].
\]
Additionally, since $\text{Re}(h_1) = 0$, we can use $0 = \bar{b}_3 q - \bar{q}\bar{b}_3$ to instantly satisfy the condition
\[
0 = h_1 q + q \bar{b}_3 h_1 \bar{b}_3^\dagger.
\]
These results leave us with
\[
\mathcal{C}_{a,4.5.iii} = \{0 = \bar{b}_3 q - \bar{q}\bar{b}_3, \quad 0 = \bar{b}_3 u a - \bar{a}\bar{b}_3, \\
0 = \text{Re}(h_1 q), \quad 0 = \text{Re}(h_3 r), \quad 0 = \text{Re}(h_1), \\
0 = [h_1, r \bar{b}_3], \quad 0 = b h_1, \quad \bar{b}h_3 = -2 \text{Im}(b_3 ur) \bar{b}_3 + 2 \text{Im}(b_3 r) \bar{b}_3 u\}.
\]
Subject to these constraints, the non-vanishing matrices are
\[
\mathcal{M}_{a,4.5.iii} = \left\{ B = \begin{pmatrix} 0 & 0 \\ \bar{b}_3 & 0 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 0 \\ \bar{b}_3 u & 0 \end{pmatrix}, \quad H = \begin{pmatrix} h_1 & 0 \\ \bar{b}_3 & \bar{b}_3 h_1 \bar{b}_3^{-1} \end{pmatrix}, \\
N_0 = \begin{pmatrix} 0 & q \\ \bar{q} & b \end{pmatrix}, \quad N_1 = \begin{pmatrix} 0 & r \\ \bar{r} & d \end{pmatrix}, \quad N_3 = \begin{pmatrix} 0 & 0 \\ \bar{r}\bar{b}_3 - \bar{b}_3 ur & 0 \end{pmatrix}, \quad N_4 = \begin{pmatrix} 0 & 0 \\ 0 & \bar{b}_3 r - \bar{r}\bar{b}_3 \end{pmatrix}\right\}.
\]
The wealth of parameters describing this sub-branch mean we will only highlight one parameterisation of these super-extensions here, though many more may exist. In particular, we will choose to set $b$, $d$ and $\bar{b}_3$ to zero. We will also utilise the subgroup $G < \text{GL}(\mathfrak{s}_3) \times \text{GL}(\mathfrak{s}_3)$ to impose that $\bar{b}_3 \mathfrak{s}_3$ and $\bar{b}u \mathfrak{s}_3$ lie along $\mathfrak{t}$. The residual endomorphisms of $\mathfrak{s}_3$ may then scale $r$ such that its norm becomes 1. Employing $\text{Aut}(\mathfrak{h})$, we can set $h_1$ to lie along $\mathfrak{h}$ as well. Having made these choices, the constraint
\[
0 = [h_1, r \bar{b}_3]
\]
tells us $r \in \mathbb{R}(1, i)$. Notice that for $N_3$ and $N_4$ to be non-vanishing $r$ must have a real component; therefore, to simplify the form of the matrices in our example, we will choose $r = 1$. The remaining constraints in $\mathcal{C}_{a,4.5.iii}$ are then satisfied, and we can use the scaling symmetry of the $\mathfrak{s}_3$ basis elements to produce
\[
B = \begin{pmatrix} 0 & 0 \\ \mathfrak{h} & 0 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 0 \\ \mathfrak{h} & 0 \end{pmatrix}, \quad H = \begin{pmatrix} \mathfrak{h} & 0 \\ 0 & \mathfrak{h} \end{pmatrix}.
\]
\[
N_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad N_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad N_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad N_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.
\]
Sub-branch 4.6. We find that this sub-branch is empty using the analysis from the previous sub-branch. Again, we use

\[
-N_3 = PN_1 - N_1 P^\dagger \quad \text{and} \quad N_4 = BN_1 + N_1 B^\dagger, \tag{4.3.201}
\]

to write

\[
N_3 = \begin{pmatrix}
0 & 0 \\
0 & \bar{\xi} + \bar{\eta} + \bar{\delta} + \bar{\nu}
\end{pmatrix}, \quad N_4 = \begin{pmatrix}
0 & 0 \\
0 & \bar{\xi} - \bar{\eta} - \bar{\delta} - \bar{\nu}
\end{pmatrix}. \tag{4.3.202}
\]

Substituting these matrices into

\[
\begin{align*}
\frac{1}{2}[\beta, \theta N_2 \theta^\dagger] &= \beta \theta BN_4 \theta^\dagger + \theta N_4 (\beta \theta B)^\dagger, \\
\frac{1}{2}[\pi, \theta N_2 \theta^\dagger] &= \pi \theta PN_4 \theta^\dagger + \theta N_4 (\pi \theta P)^\dagger,
\end{align*}
\]

the R.H.S. of both of these constraints vanishes, setting \(N_2 = 0\). Therefore, this sub-branch is empty.

\(\hat{n}_-\).

Setting \(\mu = \eta = 0, \lambda = 1\) and \(\varepsilon = -1\), the first condition in (4.3.155) becomes

\[
[u, \hat{h}_1] = 2u. \tag{4.3.204}
\]

Since \([u, \hat{b}_1]\) is perpendicular to \(u\) in \(\text{Im}(H)\) this branch cannot provide a super-extension for \(\hat{n}_-\).

\(\hat{n}_+\).

In this case, for which \(\lambda = \varepsilon = 0, \mu = 1\), and \(\eta = -1\), the first constraint in (4.3.155) gives us

\[
[h_1, u] = u^2 + 1. \tag{4.3.205}
\]

Taking the real part of (4.3.205) produces

\[
\text{Re}(u^2) = -1, \tag{4.3.206}
\]

therefore, \(u \in \text{Im}(H)\), such that \(|u|^2 = 1\), i.e. it is a unit-norm vector quaternion, or right versor. The imaginary part of (4.3.205) imposes

\[
[u, \hat{b}_1] = 0. \tag{4.3.207}
\]

Thus, we could get a super-extension of \(\hat{n}_+\) in this branch. Wishing to write our parameters in terms of \(\hat{b}_3\), we have \(\hat{g}_3 = \hat{b}_3 u\) and \(\hat{b}_3 = \hat{b}_3 (\hat{b}_1 - u)\hat{b}_3^{-1}\), where \(u \in \text{Im}(H)\), such that \(|u|^2 = -1\).

As with the \(\hat{a}\) case above, all of the conditions of Lemmas 4.2 and 4.3 must be taken into account. The conditions

\[
N_4 = BN_1 - N_1 B^\dagger \quad \text{and} \quad -N_3 = PN_1 - N_1 P^\dagger \tag{4.3.208}
\]

tell us that if \(N_1 = 0, N_3 = 0\) and \(N_4 = 0\). Substituting these results into

\[
\begin{align*}
-\text{Re}(\theta N_0 \theta^\dagger) &= \pi \theta PN_3 \theta^\dagger + \theta N_3 (\pi \theta P)^\dagger, \\
\text{Re}(\theta N_0 \theta^\dagger) &= \beta \theta BN_4 \theta^\dagger + \theta N_4 (\beta \theta B)^\dagger, \\
\frac{1}{2}[\beta, \theta N_2 \theta^\dagger] &= \beta \theta BN_4 \theta^\dagger + \theta N_4 (\beta \theta B)^\dagger, \\
\frac{1}{2}[\pi, \theta N_2 \theta^\dagger] &= \pi \theta PN_4 \theta^\dagger + \theta N_4 (\pi \theta P)^\dagger,
\end{align*}
\]

we see that if \(N_3\) or \(N_4\) vanish, so must \(N_0\) and \(N_2\). Equally, if \(N_3\) vanishes \(N_4\) necessarily vanishes and vice versa due to

\[
-N_4 = HN_3 + N_3 H^\dagger \quad \text{and} \quad N_3 = HN_4 + N_4 H^\dagger. \tag{4.3.210}
\]

Therefore, based on these dependencies, our investigation into this branch of possible super-extensions of \(\hat{n}_+\) divides into the following sub-branches.

1. \(N_1 \neq 0, N_0 = N_2 = N_3 = N_4 = 0\)
2. \(N_1 \neq 0, N_3 \neq 0, N_4 \neq 0, N_0 = N_2 = 0\)
3. \(N_1 \neq 0, N_3 \neq 0, N_4 \neq 0, N_0 \neq 0 \text{ and } N_2 = 0\)
4. \(N_1 \neq 0, N_3 \neq 0, N_4 \neq 0, N_0 = 0 \text{ and } N_2 \neq 0\)
5. \(N_1 \neq 0, N_3 \neq 0, N_4 \neq 0, N_0 \neq 0 \text{ and } N_2 \neq 0\)

Unlike the super-extensions of \(\hat{n}_+\) found in branches 1, 2 and 3, the \([Q, Q, Q]\) identity will not impose that either \(N_0\) or \(H\) must vanish. In the first sub-branch above, we instantly see that \(N_0 = 0\); therefore, the super-extensions found here are extensions satisfying (i). However, all other sub-branches have either non-vanishing \(N_3\) or non-vanishing \(N_4\). Since \(B \neq 0\) and \(P \neq 0\), the \([Q, Q, Q]\) identity will now form relationships between \(N_0, N_3\) and \(N_4\) with, in general, \(H \neq 0\). Therefore, these super-extensions, for which \(N_0 \neq 0 \text{ and } H \neq 0\), will be labelled (iii) to distinguish them from the cases (i) and (ii).
Sub-branch 4.1. With only $N_1 \neq 0$, the conditions from Lemmas 4.2 and 4.3 reduce to
\[
0 = HN_3 + N_1 H^\dagger \\
0 = PN_1 - N_1 P^\dagger \\
0 = BN_1 - N_1 B^\dagger.
\] (4.3.211)
The latter two conditions tell us
\[
0 = c, \quad 0 = b_3 r - \bar{r} \bar{b}_3 \quad \text{and} \quad 0 = b_3 u r - \bar{r} \bar{u} \bar{b}_3,
\] (4.3.122)
which, when substituted into the first conditions, produce
\[
0 = \text{Re}(h_1) \quad \text{and} \quad 0 = \text{Re}(h_3 r).
\] (4.3.213)
Therefore, the non-vanishing matrices for this super-extension are
\[
B = \begin{pmatrix} 0 & 0 \\ b_3 & 0 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 0 \\ b_3 u & 0 \end{pmatrix}, \quad H = \begin{pmatrix} h_1 & b_3 (h_1 - u) b_3^{-1} \\ h_3 & b_3 (h_1 - u) b_3^{-1} \end{pmatrix}, \quad N_1 = \begin{pmatrix} 0 & r \\ d & 0 \end{pmatrix},
\] (4.3.214)
subject to the constraints
\[
0 = [u, h_1], \quad 0 = \text{Re}(h_1), \quad 0 = \text{Re}(h_3 r), \quad 0 = b_3 r - \bar{r} \bar{b}_3, \quad 0 = b_3 u r - \bar{r} \bar{u} \bar{b}_3, \quad \psi^2 = -1.
\] (4.3.215)
However, notice that the final three constraints listed above require one of $b_3, u$ or $r$ to vanish. Since neither $b_3$ or $u$ can vanish in this sub-branch, it must be that $r = 0$. Therefore, the final set of matrices is
\[
M_{a,.4.1} = \{ B = \begin{pmatrix} 0 & 0 \\ b_3 & 0 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 0 \\ b_3 u & 0 \end{pmatrix}, \quad H = \begin{pmatrix} h_1 & 0 \\ h_3 & b_3 (h_1 - u) b_3^{-1} \end{pmatrix}, \quad N_1 = \begin{pmatrix} 0 & 0 \\ d & 0 \end{pmatrix} \},
\] (4.3.216)
and the final set of constraints is
\[
\mathcal{E}_{a,.4.1} = \{ [u, h_1], \quad 0 = \text{Re}(h_1), \quad \psi^2 = -1 \}.
\] (4.3.217)
To demonstrate that this sub-branch of $\mathcal{S}$ is not empty, choose to set $h_1$ and $h_3$ to zero. Using the endomorphisms of $\mathcal{S}$ and $\text{Aut}(H)$ on $b_3$ and $u$, respectively, we may write $b_3 = i$ and $u = j$. Employing the scaling symmetry of $Z$, we arrive at the super-extension
\[
B = \begin{pmatrix} 0 & 0 \\ i & 0 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 0 \\ b_3 u & 0 \end{pmatrix}, \quad H = \begin{pmatrix} h_1 & 0 \\ h_3 & b_3 (h_1 - u) b_3^{-1} \end{pmatrix}, \quad N_1 = \begin{pmatrix} 0 & 0 \\ d & 0 \end{pmatrix}.
\] (4.3.218)
Thus, this sub-branch is not empty. We may then introduce $h_1$ while continuing to fix all the parameters of the super-extension; however, this cannot be achieved on introducing $h_3$.

Sub-branch 4.2. Now with $N_3 \neq 0$ and $N_4 \neq 0$ as well as $N_1 \neq 0$, we can use the conditions
\[
-N_3 = PN_1 - N_1 P^\dagger \quad 0 = \beta B N_3 \theta^\dagger + \theta N_3 (\beta B)^\dagger \quad 0 = \beta B N_4 \theta^\dagger + \theta N_4 (\beta B)^\dagger \\
N_3 = BN_1 - N_1 B^\dagger \quad 0 = \pi \theta P N_3 \theta^\dagger + \theta N_3 (\pi \theta P)^\dagger \quad 0 = \pi \theta P N_4 \theta^\dagger + \theta N_4 (\pi \theta P)^\dagger,
\] (4.3.219)
and the analysis of branches 2 and 3 to write
\[
N_1 = \begin{pmatrix} 0 & r \\ \bar{r} & d \end{pmatrix}, \quad N_3 = \begin{pmatrix} 0 & 0 \\ 0 & -r \bar{b}_3 - b_3 u r \end{pmatrix}, \quad N_4 = \begin{pmatrix} 0 & 0 \\ 0 & b_3 r - \bar{r} \bar{b}_3 \end{pmatrix}.
\] (4.3.220)
This leaves only the $[H, Q, Q]$ conditions:
\[
0 = HN_3 + N_3 H^\dagger \\
-N_4 = HN_3 + N_3 H^\dagger \\
N_3 = HN_4 + N_4 H^\dagger.
\] (4.3.221)
We know from sub-branch 4.1.i that, since $c = 0$, the first of these produces
\[
0 = \text{Re}(h_1) \quad \text{and} \quad 0 = \text{Re}(h_3 r).
\] (4.3.222)
Writing $\bar{r} \bar{u} \bar{b}_3 - b_3 u r = -2 \text{Im}(b_3 u r)$ and $b_3 r - \bar{r} \bar{b}_3 = 2 \text{Im}(b_3 r)$ to simplify our expressions, the second and third conditions give us
\[
\text{Im}(b_3 r) = h_4 \text{Im}(b_3 u r) + \text{Im}(b_3 u r) b_3^{-1} \\
\text{Im}(b_3 r) = h_4 \text{Im}(b_3 u r) + \text{Im}(b_3 u r) b_3^{-1},
\] (4.3.223)
respectively, where $h_4 = b_3 (h_1 - u) b_3^{-1}$. Notice that since $h_1, u \in \text{Im}(H)$, and $h_4$ is written in terms of the adjoint action of $b_3 \in H$, $h_4 \in \text{Im}(H)$. Therefore, using $h_4 = -h_4$, we find
\[
\text{Im}(b_3 r) = -[h_4, [h_4, \text{Im}(b_3 r)]] \quad \text{and} \quad \text{Im}(b_3 u r) = -[h_4, [h_4, \text{Im}(b_3 u r)]]].
\] (4.3.224)
This imposes the constraint that \( \text{Im}(b_{32}^*) \) and \( \text{Im}(b_{33}^*) \) must be perpendicular to \( h_4 \) in \( \text{Im}(h) \). The constraints for this sub-branch are summarised as follows.

\[
\mathcal{E}_{a_{1,4.2.3}} = \{ 0 = [b_1, u], \quad -1 = u^2 \quad 0 = \text{Re}(h_1), \quad 0 = \text{Re}(b_{32}^*), \\
\text{Im}(b_{32}^*) = -[h_4, \text{Im}(b_{33}^*)], \quad \text{Im}(b_{33}^*) = -[h_4, \text{Im}(b_{33}^*)] \}.
\]

\( (4.3.225) \)

The non-vanishing matrices are then

\[
\begin{align*}
\mathcal{M}_{a_{1,4.2.3}} &= \{ B = \begin{pmatrix} 0 & 0 \\ b_{32} & 0 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 0 \\ b_{33} & 0 \end{pmatrix}, \quad H = \begin{pmatrix} h_1 & 0 \\ b_{33} & b_{32}(h_1 - u)b_{33}^{-1} \end{pmatrix}, \\
N_1 &= \begin{pmatrix} 0 & r \\ d & 0 \end{pmatrix}, \quad N_3 = \begin{pmatrix} 0 & 0 \\ 0 & -2 \text{Im}(b_{32}^*) \end{pmatrix}, \quad N_4 = \begin{pmatrix} 0 & 0 \\ 0 & 2 \text{Im}(b_{32}^*) \end{pmatrix}. \end{align*}
\]

\( (4.3.226) \)

To demonstrate the existence of super-extensions in this sub-branch, we will begin by simplifying our parameter set as much as possible. In particular, we begin by setting both \( h_3 \) and \( d \) to zero. We then utilise \( \text{Aut}(H) \) and the endomorphisms of \( s_1 \) to set \( u = \frac{2}{3} \) and impose that \( b_{33} \) lies along \( \bar{l} \). Notice that with \( u \) along \( \bar{l} \), the first constraint in \( \mathcal{E}_{a_{1,4.2.3}} \) tells us that \( h_1 \) must also lie along \( \bar{l} \), as must \( h_4 = b_{33}(h_1 - u)b_{33}^{-1} \). With these choices, the two constraints involving \( h_4 \) impose \( r \in \mathbb{R}(1, \bar{j}) \), and that \( |h_1| = \frac{2}{3} \) or \( |h_1| = \frac{2}{3} \). Residual endomorphisms then allow us to scale \( r \) such that it has unit norm. Finally, we can scale the \( s_{(j)} \) basis elements to arrive at

\[
\begin{align*}
B &= \begin{pmatrix} 0 & 0 \\ \bar{l} & 0 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 0 \\ k & 0 \end{pmatrix}, \quad H = \begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix}, \\
N_1 &= \begin{pmatrix} 0 & r \\ 1 - j & 0 \end{pmatrix}, \quad N_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad N_4 = \begin{pmatrix} 0 & 0 \\ 0 & k + j \end{pmatrix}. \end{align*}
\]

\( (4.3.227) \)

**Sub-branch 4.3.** The beginning of the investigation of this sub-branch is identical to that of the previous sub-branch. The \([P, Q, Q]\) and \([B, Q, Q]\) identities produce

\[
\begin{align*}
-N_3 &= P N_1 - N_1 P^\dagger \quad 0 = \beta \theta B N_3 \theta^\dagger + \theta N_3 (\beta \theta B)^\dagger, \\
N_4 &= B N_1 - N_1 B^\dagger \quad 0 = \pi \theta P N_4 \theta^\dagger + \theta N_4 (\pi \theta P)^\dagger,
\end{align*}
\]

where the first two conditions give \( N_3 \) and \( N_4 \) the form

\[
N_3 = \begin{pmatrix} \frac{1}{\sqrt{2}} \text{c}_{b_{32} u} & -\text{c}_{b_{33} u} \\ -\text{c}_{b_{33} u} & \text{c}_{b_{32} u} \end{pmatrix} \quad \text{and} \quad N_4 = \begin{pmatrix} 0 & \text{c}_{b_{33} \bar{u}} \\ \text{c}_{b_{33} \bar{u}} & -\text{c}_{b_{32} \bar{u}} \end{pmatrix}.
\]

\( (4.3.228) \)

Substituting this \( N_3 \) with \( 0 = (0, 1) \) into

\[
0 = \beta \theta B N_3 \theta^\dagger + \theta N_3 (\beta \theta B)^\dagger,
\]

we acquire

\[
0 = 2 \text{c}_{b_{32} u}^2 \text{Im}(\beta u) \quad \forall \beta \in \text{Im}(H).
\]

\( (4.3.229) \)

As, by assumption, \( b_{33} \neq 0 \) and \( u \neq 0 \), this imposes \( c = 0 \), such that

\[
N_1 = \begin{pmatrix} 0 & r \\ d & 0 \end{pmatrix}, \quad N_3 = \begin{pmatrix} 0 & 0 \\ 0 & -\text{c}_{b_{33} \bar{u}} \end{pmatrix}, \quad N_4 = \begin{pmatrix} 0 & 0 \\ 0 & \text{c}_{b_{33} \bar{u}} \end{pmatrix}.
\]

\( (4.3.230) \)

With this form of \( N_3 \) and \( N_4 \),

\[
-\text{Re}(\theta N_0 \theta^\dagger) \pi = \pi \theta P N_3 \theta^\dagger + \theta N_3 (\pi \theta P)^\dagger \\
\text{Re}(\theta N_0 \theta^\dagger) \beta = \beta \theta B N_4 \theta^\dagger + \theta N_4 (\beta \theta B)^\dagger,
\]

have a vanishing R.H.S., showing that \( N_0 \neq 0 \). This result contradicts our assumption that \( N_0 \neq 0 \); therefore, this sub-branch does not contain any super-extensions.

**Sub-branch 4.4.** Letting \( N_0 = 0 \) and \( N_2 \neq 0 \), we can use

\[
-N_3 = P N_1 - N_1 P^\dagger \quad 0 = \beta \theta B N_4 \theta^\dagger + \theta N_4 (\beta \theta B)^\dagger \\
N_4 = B N_1 - N_1 B^\dagger \quad 0 = \pi \theta P N_4 \theta^\dagger + \theta N_3 (\pi \theta P)^\dagger,
\]

\( (4.3.231) \)

to again write

\[
N_1 = \begin{pmatrix} 0 & r \\ d & 0 \end{pmatrix}, \quad N_3 = \begin{pmatrix} 0 & 0 \\ 0 & -\text{c}_{b_{33} \bar{u}} \end{pmatrix}, \quad N_4 = \begin{pmatrix} 0 & 0 \\ 0 & \text{c}_{b_{33} \bar{u}} \end{pmatrix}.
\]

\( (4.3.232) \)

Substituting these \( N_1 \) into

\[
\frac{1}{2}[\theta, \theta N_2 \theta^\dagger] = \beta \theta B N_3 \theta^\dagger + \theta N_3 (\beta \theta B)^\dagger \\
\frac{1}{2}[\pi, \theta N_2 \theta^\dagger] = \pi \theta P N_4 \theta^\dagger + \theta N_4 (\pi \theta P)^\dagger,
\]

the R.H.S. vanishes for both, showing \( N_2 = 0 \), contradicting our initial assumption in this sub-branch.
Sub-branch 4.5. With none of the $N_1$ vanishing, we start again by writing $N_3$ and $N_4$ in terms of $N_1$ using conditions from the $[P, Q, Q]$ and $[B, Q, Q]$ identities:

$$
N_3 = \begin{pmatrix} 0 & -\bar{c}_3^* u \\ -c_3 u & \bar{r}\bar{b}_3 - \bar{b}_3 ur \end{pmatrix} \quad \text{and} \quad N_4 = \begin{pmatrix} 0 & -\bar{c}_3^* \\ c_3 u & \bar{b}_3 r - \bar{r}\bar{b}_3 \end{pmatrix}.
$$

(4.3.237)

Letting

$$
N_2 = \begin{pmatrix} n & m \\ -\bar{m} & \bar{n} \end{pmatrix}
$$

where $n, \bar{l} \in \text{Im}(\mathcal{H})$, $m \in \mathbb{H}$,

(4.3.238)

we can use

$$
\frac{1}{2}[\beta, \theta N_2 \theta^\dagger] = \theta \mathcal{B} \mathbf{N}_3 \theta^\dagger + \theta \mathbf{N}_4 (\beta \theta \mathcal{B})^\dagger
$$

(4.3.239)

to write $N_2$ in terms of $b_3$ and $u$. First let $\theta = (0, 1)$ to find

$$
\frac{1}{2}[\beta, l] = -c[\beta, \bar{b}_3 u b_3] \quad \forall \beta \notin \text{Im}(\mathcal{H}).
$$

(4.3.240)

Therefore,

$$
l = -2c b_3 u \bar{b}_3.
$$

(4.3.241)

Next, substitute in $\theta = (1, 1)$ to get

$$
[\beta, 2 \text{Im}(m)] + \frac{1}{2}[\beta, l] = -c[\beta, b_3 u \bar{b}_3].
$$

(4.3.242)

Using the previous result, this tells us that $\text{Im}(m) = 0$. Analogous calculations with $\theta = (0, 1)$ and $\theta = (1, 1)$ show that, in fact, all of $m$ must vanish. Finally, substituting in $\theta = (1, 0)$ into (4.3.239), we find $n = 0$ since the R.H.S. vanishes. Therefore, we are left with

$$
N_2 = \begin{pmatrix} 0 & 0 \\ 0 & -2c b_3 u \bar{b}_3 \end{pmatrix}.
$$

(4.3.243)

We would have arrived at the same expression had we used $N_3$ and

$$
\frac{1}{2}[\pi, \theta N_2 \theta^\dagger] = \pi \mathcal{B} \mathbf{N}_3 \theta^\dagger + \theta \mathbf{N}_4 (\pi \theta \mathcal{B})^\dagger.
$$

(4.3.244)

This form of $N_2$ automatically satisfies all other conditions it is involved in from both the $[B, Q, Q]$ and $[P, Q, Q]$ identities. Finally, we can put this $N_2$ into

$$
0 = \mathcal{H} N_2 + N_2 \mathcal{H}^\dagger
$$

(4.3.245)

to get

$$
0 = \bar{b}_3 \mathcal{H} + \mathcal{H} \bar{b}_3,
$$

(4.3.246)

where $b_3 = b_3(b_1 - u) \bar{b}_3^{-1}$ and $l = -2c b_3 u \bar{b}_3$. Working through some algebra, noting $\text{Re}(u^2) = -1$ and the fact $c \neq 0$ for $N_2 \neq 0$, we arrive at $b_1 u = \bar{b}_3 u$. Since $u \in \text{Im}(\mathcal{H})$, this forces $b_1 \in \text{Im}(\mathcal{H})$ such that $u$ and $b_1$ are collinear.

Now turn to $N_0$ and consider

$$
-\text{Re}(\theta \mathbf{N}_0 \theta^\dagger) = \pi \mathcal{B} \mathbf{N}_3 \theta^\dagger + \theta \mathbf{N}_4 (\pi \theta \mathcal{B})^\dagger
$$

(4.3.247)

$$
\text{Re}(\theta \mathbf{N}_0 \theta^\dagger) \beta = \theta \mathbf{N}_4 (\beta \theta \mathcal{B})^\dagger.
$$

(4.3.248)

Letting $\theta = (1, 0)$ in either of these conditions, we find that $\alpha = 0$. Next, substituting $\theta = (0, 1)$ into the second condition produces

$$
-b = 2c |b_3|^2.
$$

(4.3.249)

We would have arrived at the same result had we substituted into the first condition and used the fact $|u|^2 = 1$.

Now substituting $\theta = (1, s)$ into the second condition, we find

$$
-2 \text{Re}(\bar{s} a) - b |b|^2 = 2c |b|^2 |b_3|^2.
$$

(4.3.250)

Therefore, using the previous result and letting $s = 1$, $s = 0$ and $s = k$, we see that all components of $q$ must vanish. All other conditions on $N_0$ are now automatically satisfied, leaving

$$
N_0 = \begin{pmatrix} 0 & 0 \\ 0 & -2c |b_3|^2 \end{pmatrix}.
$$

(4.3.251)

Equipped with these $N_i$, we can now analyse the condition from Lemma 4.3:

$$
\text{Re}(\theta \mathbf{N}_0 \theta^\dagger) \theta \mathcal{H} = \frac{1}{2} \mathbf{N}_2 \theta^\dagger \theta + \theta \mathbf{N}_3 \theta^\dagger \theta \mathcal{B} + \theta \mathbf{N}_4 \theta^\dagger \theta \mathcal{P}.
$$

(4.3.252)

Letting $\theta = (0, 1)$:

$$
-2c |b_3|^2 (b_3 \bar{b}_3 (h_1 - u) \bar{b}_3^{-1}) = -c b_3 u \bar{b}_3 (0 \quad -\text{Im}(\bar{b}_3 u r) \quad (b_3 \quad 0) + \text{Im}(b_3 r) \quad (b_3 u \quad 0)).
$$

(4.3.253)

Concentrating on the second component, we have

$$
-2c |b_3|^2 b_3 (h_1 - u) \bar{b}_3^{-1} = -c b_3 u \bar{b}_3.
$$

(4.3.254)
Using the fact $\|b_3\|^2b_3 = b_3$ and cancelling relevant terms leaves

$$2b_3h_1b_3 = 0.$$ \hfill (4.3.254)

Since, by assumption $b_3 \neq 0$, we get $h_1 = 0$. The first component of (4.3.252) gives us a prescription for $h_3$,

$$-2c|b_3|^2b_3 = -2\text{Im}(b_3\bar{u}r)b_3 + 2\text{Im}(b_3r)b_3u,$$ \hfill (4.3.255)

therefore, we can fully describe $H$ in terms of $B$, $P$, and $N_1$.

The final conditions to consider are those from the $[H, Q, Q]$ super-Jacobi identity for $N_1$, $N_3$ and $N_4$. First, the $N_1$ condition tell us

$$0 = c\bar{b}_3 + r\bar{d}b_3^{-1} \quad 0 = \text{Re}(b_3r).$$ \hfill (4.3.256)

Notice that the second constraint here is automatically satisfied by the first, since $c \neq 0$ for a non-vanishing $N_0$. Substituting this expression for $b_3$ into the previous prescription, we find

$$|r|^2|b_3|^2b_3\bar{d}b_3^{-1} = [\text{Im}(b_3\bar{r}), \text{Im}(b_3u\bar{r})] + \text{Re}(b_3\bar{u}r)\text{Im}(b_3\bar{r}) - \text{Re}(b_3r)\text{Im}(b_3u\bar{r}).$$ \hfill (4.3.257)

Now, the constraints that $N_3$ and $N_4$ conditions produce are the ones given in sub-branch 4.2.iii:

$$\text{Im}(b_3\bar{r}) = -[h_4, [h_4, \text{Im}(b_3\bar{r})]] \quad \text{and} \quad \text{Im}(b_3u\bar{r}) = -[h_4, [h_4, \text{Im}(b_3u\bar{r})]].$$ \hfill (4.3.258)

These tell us that $\text{Im}(b_3\bar{r})$ and $\text{Im}(b_3u\bar{r})$ are perpendicular to $h_4$ in $\text{Im}(H)$. Therefore, the expression in (4.3.257) becomes

$$0 = \text{Re}(b_3\bar{r}), \quad 0 = \text{Re}(b_3u\bar{r}) \quad \text{and} \quad |r|^2b_3\bar{u}b_3 = [\text{Im}(b_3\bar{r}), \text{Im}(b_3u\bar{r})].$$ \hfill (4.3.259)

Putting all of these constraints together, we have

$$c_{a_{a_{a_{a_{a_{, 4.5.iii}}}}}} = [u^2 = -1, \quad \text{Im}(b_3\bar{r}) = -[h_4, [h_4, \text{Im}(b_3\bar{r})]], \quad \text{Im}(b_3u\bar{r}) = -[h_4, [h_4, \text{Im}(b_3u\bar{r})]], \quad 0 = \text{Re}(b_3\bar{r}), \quad 0 = \text{Re}(b_3u\bar{r}), \quad |r|^2b_3\bar{u}b_3 = [\text{Im}(b_3\bar{r}), \text{Im}(b_3u\bar{r})]],$$ \hfill (4.3.260)

for non-vanishing matrices

$$M_{a_{a_{a_{a_{a_{a_{, 4.5.iii}}}}}}} = \left\{ B = \begin{pmatrix} 0 & 0 \\ \bar{d}b_3 & 0 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 0 \\ b_3u & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 0 & 0 \\ -c^{-1}b_3\bar{u}b_3^{-1} & 0 \end{pmatrix}, \quad N_0 = \begin{pmatrix} 0 & -2[c\bar{b}_3] \\ 0 & 0 \end{pmatrix}, \quad N_1 = \begin{pmatrix} c \bar{v} \\ \bar{v} \end{pmatrix}, \quad N_2 = \begin{pmatrix} 0 & 0 \\ 0 & -2c\bar{b}_3u\bar{b}_3 \end{pmatrix}, \quad N_3 = \begin{pmatrix} 0 & c\bar{b}_3 \bar{u} \\ -c\bar{b}_3u \end{pmatrix}, \quad N_4 = \begin{pmatrix} c\bar{b}_3 \bar{u} \bar{v} - c\bar{b}_3 \\ 2\text{Im}(b_3\bar{r}) \end{pmatrix} \right\}.$$ \hfill (4.3.261)

To demonstrate that there are super-extensions in this sub-branch, we will first simplify this system by letting parameters vanish where possible. In particular, $v$ and $d$ in $N_1$ may be set to zero. This reduces $c_{a_{a_{a_{a_{a_{a_{, 4.5.iii}}}}}}}$ to contain only $u^2 = -1$. Now we can use the endomorphisms of $e_1$ to impose $b_3 = i$, $u = j$ and $c = 1$. With these choices, the matrices become

$$B = \begin{pmatrix} 0 & 0 \\ \frac{i}{k} & 0 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 0 \\ k & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 0 & 0 \\ 0 & j \end{pmatrix}, \quad N_0 = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}, \quad N_1 = \begin{pmatrix} 0 \frac{i}{k} \\ 0 \frac{j}{k} \end{pmatrix}, \quad N_2 = \begin{pmatrix} 0 & 0 \\ 0 & -2k \end{pmatrix}, \quad N_3 = \begin{pmatrix} 0 & 0 \\ -k & 0 \end{pmatrix}, \quad N_4 = \begin{pmatrix} 0 \frac{i}{k} \\ \frac{j}{k} \frac{i}{k} \end{pmatrix}.$$ \hfill (4.3.262)

As there are no restrictions on the parameter $d$, we may introduce it without affecting our other parameter choices; however, this is not the case for $v$. There are several constraints in $c_{a_{a_{a_{a_{a_{a_{, 4.5.iii}}}}}}}$ involving $v$; therefore, we need to examine these constraints to determine whether new parameters must be chosen. Interrogating

$$\text{Im}(b_3\bar{r}) = -[h_4, [h_4, \text{Im}(b_3\bar{r})]] \quad \text{and} \quad \text{Im}(b_3u\bar{r}) = -[h_4, [h_4, \text{Im}(b_3u\bar{r})]]$$ \hfill (4.3.263)

with the parameter choices stated above, we find that $v$ must vanish. In particular, due to $h_4 = b_3u\bar{b}_3^{-1}$ having unit length, we cannot replicate the analysis of sub-branch 4.2.iii, where the magnitude of $h_4$ was necessarily either $\frac{\sqrt{3}}{2}$ or $-\frac{\sqrt{3}}{2}$. Thus, we cannot produce a super-extension in this sub-branch for which $v \neq 0$. This simplifies the above ($M, \mathcal{C}$), such that the remaining constraints are

$$c_{a_{a_{a_{a_{a_{a_{, 4.5.iii}}}}}} = [u^2 = -1],$$ \hfill (4.3.264)

and the non-vanishing matrices are now

$$M_{a_{a_{a_{a_{a_{a_{, 4.5.iii}}}}}} = \left\{ B = \begin{pmatrix} 0 & 0 \\ \bar{d}b_3 & 0 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 0 \\ b_3u & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 0 & 0 \\ 0 & \bar{b}_3u\bar{b}_3^{-1} \end{pmatrix}, \quad N_0 = \begin{pmatrix} 0 & 0 \\ 0 & -2c|b_3|^2 \end{pmatrix}, \quad N_1 = \begin{pmatrix} c \bar{v} \\ \bar{v} \end{pmatrix}, \quad N_2 = \begin{pmatrix} 0 & 0 \\ 0 & -2c\bar{b}_3u\bar{b}_3 \end{pmatrix}, \quad N_3 = \begin{pmatrix} 0 & c\bar{b}_3 \bar{u} \bar{v} - c\bar{b}_3 \\ 0 \end{pmatrix}, \quad N_4 = \begin{pmatrix} c\bar{b}_3 \bar{u} \bar{v} - c\bar{b}_3 \\ 0 \end{pmatrix} \right\}. $$ \hfill (4.3.265)
Finally, substitute \( \lambda = \eta = \epsilon = 0 \) and \( \mu = -1 \) into the second constraint in (4.3.155) to investigate the \( \hat{g} \) case. We find

\[ [v, h_1] = -1, \]

which, since \([v, h_1] \in \text{Im}(H)\), is inconsistent. Therefore, we cannot get a super-extension of \( \hat{g} \) in this branch.

4.4. Summary. Table 6 lists all the sub-branches of \( \mathcal{N} \) we found that contain \( \mathcal{N} = 2 \) generalised Bargmann superalgebras. Each Lie superalgebra in one of these branches is an \( \mathcal{N} = 2 \) super-extension of one of the generalised Bargmann algebras given in Table 1, taken from [4]. It is interesting that \( Z \) only appears in

\[ [B, P] = Z \quad \text{and} \quad [Q, Q] = Z. \quad (4.4.1) \]

Therefore, in all instances, \( Z \) remains central after the super-extension. In particular, this means that we may always find a kinematical Lie superalgebra (without a central-extension) by taking the quotient of our generalised Bargmann superalgebra \( s \) by the \( \mathbb{R} \)-span of \( Z, s/\langle Z \rangle \).

Table 6. Sub-branches of \( \mathcal{N} = 2 \) generalised Bargmann superalgebras (with \( [Q, Q] \neq 0 \))

| (S)B | \( \mathfrak{f} \) | \( H \) | \( Z \) | \( B \) | \( P \) | \( [Q, Q] \) |
|------|---------|------|-------|------|-----|----------------|
| 1.i  | \( a \) | \checkmark | \( Z \) |
| 1.ii | \( a \) | \checkmark | \( H + Z \) |
| 2.1.ii | \( a \) | \checkmark | \( H \) |
| 2.2.i | \( a \) | \checkmark | \checkmark | \( Z \) |
| 2.3.i | \( a \) | \checkmark | \checkmark | \( Z + B \) |
| 2.3.ii | \( a \) | \checkmark | \checkmark | \( H + Z + B \) |
| 4.5.iii | \( a \) | \checkmark | \checkmark | \( H + Z + B + P \) |
| 1.i | \( a_- \) | \checkmark | \( Z \) |
| 2.2.i | \( a_- \) | \checkmark | \( Z \) |
| 2.3.i | \( a_- \) | \checkmark | \checkmark | \( Z + P \) |
| 3.2.i | \( a_- \) | \checkmark | \checkmark | \( Z \) |
| 3.3.i | \( a_- \) | \checkmark | \checkmark | \( Z + P \) |
| 1.i | \( a_+ \) | \checkmark | \( Z \) |
| 2.2.i | \( a_+ \) | \checkmark | \( Z \) |
| 4.1.i | \( a_+ \) | \checkmark | \checkmark | \( Z \) |
| 4.2.iii | \( a_+ \) | \checkmark | \checkmark | \( Z + B + P \) |
| 4.5.iii | \( a_+ \) | \checkmark | \checkmark | \( H + Z + B + P \) |
| 1.i | \( \hat{a} \) | \checkmark | \( Z \) |
| 2.2.i | \( \hat{a} \) | \checkmark | \( Z \) |
| 3.2.i | \( \hat{a} \) | \checkmark | \checkmark | \( Z \) |
| 3.3.i | \( \hat{a} \) | \checkmark | \checkmark | \( Z + P \) |
| 3.3.ii | \( \hat{a} \) | \checkmark | \checkmark | \( H + Z + P \) |

The first column indicates the sub-branch of generalised Bargmann superalgebras, so that the reader may navigate back to find the conditions on the non-vanishing parameters of these superalgebras. The second column then tells us the underlying generalised Bargmann algebra \( \mathfrak{f} \). The next four columns tells us which of the \( s_0 \) generators \( H, Z, B, \) and \( P \) act on \( Q \). Recall, \( J \) necessarily acts on \( Q \), so we do not need to state this explicitly. The final column shows which \( s_0 \) generators occur in the \([Q, Q]\) bracket.

4.4.1. Unpacking the Notation. Although the formalism employed in this classification was useful for our purposes, it may be unfamiliar to the reader. Therefore, in this section, we will convert one of the \( \mathcal{N} = 2 \) super-extensions of the Bargmann algebra in sub-branch 3.3.ii into a more standard notation. The \( s_0 \) brackets have already been discussed in section 2.2, so we will concentrate solely on the \([s_0, s_1]\) and \([s_1, s_1]\) brackets,

\[ [B(\beta), Q(\theta)] = Q(\beta B) \quad \text{and} \quad [Q(\theta), Q(\theta)] = \text{Re}(\theta N_0 \theta^\dagger) H + \text{Re}(\theta N_1 \theta^\dagger) Z - P(\theta N_1 \theta^\dagger), \quad (4.4.2) \]
where
\[
B = \begin{pmatrix} 0 & 0 \\ \frac{1}{b_3} & 0 \\ 0 & 0 \end{pmatrix}, \quad N_0 = \begin{pmatrix} 0 & 0 \\ 0 & -2c|b_3|^2 \end{pmatrix}, \quad N_1 = \begin{pmatrix} c & r \\ -f & d \end{pmatrix}, \quad N_4 = \begin{pmatrix} 0 & -c\bar{b}_3 \\ c\bar{b}_3 & b_3r - \bar{r}\bar{b}_3 \end{pmatrix}.
\]

Let \([Q_a^1]\) be a real basis for the first \(\mathfrak{so}(3)\) spinor module in \(s_1 = S^1 \oplus S^2\) where \(a \in \{1, 2, 3, 4\}\), and \([Q_a^2]\) be a basis for the second \(\mathfrak{so}(3)\) spinor module. Letting \(\theta = (\theta_1, \theta_2)\), and substituting the above matrices into the \([s_{\theta_1}, s_{\theta_2}]\) bracket, we get
\[
[B(\beta), Q^1(\theta_1)] = 0 \quad \text{and} \quad [B(\beta), Q^2(\theta_2)] = Q^1(\beta \theta_2 b_3).
\]
Substituting \(\theta = \theta' = (\theta_1, 0), \theta = (\theta_1, 0)\) and \(\theta' = (0, \theta_2)\), and \(\theta = \theta' = (0, \theta_2)\) into the \([s_{\theta_1}, s_{\theta_2}]\) bracket we find
\[
[B(\beta), Q^1(\theta_1)] = c\theta_1^3 Z
\]
\[
(Q^1(\theta_1), Q^2(\theta_2)) = R_{c \theta_1} \bar{Z} - \frac{4}{P}(\theta_2 b_3 \bar{\theta}_1 - \theta_1 \bar{b}_3 \theta_2)
\]
\[
[Q^2(\theta_2), Q^2(\theta_2)] = -2c|b_3|^2[\theta_2 |H + d|\theta_2|^2Z - P(\theta_2 \bar{b}_3 r - \bar{r}\bar{b}_3) \bar{\theta}_2).
\]

For the purposes of the present example, we will set the parameters of this super-extension as specified in (4.3.152); therefore, we have \([s_{\theta_1}, s_{\theta_2}]\) brackets
\[
[B(\beta), Q^1(\theta_1)] = Q^2(\beta \theta_2 b_3) \quad \text{and} \quad [B(\beta), Q^2(\theta_2)] = 0,
\]
and \([s_{\theta_1}, s_{\theta_2}]\) brackets
\[
(Q^1(\theta_1), Q^1(\theta_1)) = \theta_1^2 Z, \quad (Q^1(\theta_1), Q^2(\theta_2)) = -\frac{4}{P}(\theta_2 \theta_1 \theta_1 + \theta_1 \theta_2^2) \quad \text{and} \quad [Q^2(\theta_2), Q^2(\theta_2)] = \theta_2 |H.
\]
Now, we can write
\[
[B_1, Q_{a_1}^1] = \sum_{b=1}^{4} Q_{a_1}^1 b_1 b_a \quad [Q_{a_1}^1, Q_{b_1}^1] = \delta_{ab} Z \quad [Q_{a_1}^1, Q_{a_2}^2] = \sum_{i=1}^{3} P_i \Gamma^i_{ab}, \quad [Q_{a_1}^1, Q_{b_1}^2] = \delta_{ab} H.
\]
Our brackets then produce the \(\beta_i\) matrices
\[
\beta_1 = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}, \quad \beta_2 = \begin{pmatrix} 0 & -\sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}, \quad \beta_3 = \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix},
\]
and the symmetric \(\Gamma^1\) matrices
\[
\Gamma^1 = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}, \quad \Gamma^2 = \begin{pmatrix} 0 & -\sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}, \quad \Gamma^3 = \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix},
\]
where \(\sigma_1\) and \(\sigma_3\) are the first and third Pauli matrix, respectively. This \(N = 2\) Bargmann superalgebra takes the same form as the \((2 + 1)\)-dimensional Bargmann superalgebra utilised in [19].

5. DISCUSSION

In this paper, we classified the \(N = 1\) super-extensions of the generalised Bargmann algebras with three-dimensional spatial isotropy up to isomorphism. We also presented the non-empty sub-branches of the variety \(\mathcal{J}\) describing the \(N = 2\) super-extensions of the generalised Bargmann algebras. To simplify this classification problem, we utilised a quaternionic formalism such that \(\mathfrak{so}(3)\) scalar modules were described by copies of \(F\), \(\mathfrak{so}(3)\) vector modules were described by copies of \(\mathfrak{l}(\mathfrak{h})\), and \(\mathfrak{so}(3)\) spinor modules were described by copies of \(\mathfrak{h}\). We began by defining a universal generalised Bargmann algebra, which, under the appropriate setting of some parameters, may be reduced to the centrally-extended static kinematical Lie algebra \(\mathfrak{a}\), the centrally-extended Newton-Hooke algebras \(\mathfrak{a}_{ij}\), or the Bargmann algebra \(\mathfrak{b}\). The most general form for the \([s_{\theta_1}, s_{\theta_2}]\) and \([s_{\theta_1}, s_{\theta_2}]\) bracket components were found before substituting them into the super-Jacobi identities and finding the constraints on the parameters for these maps. Because of the formalism in use, solving these constraints amounted to some linear algebra over the quaternions, and paying attention to the allowed basis transformations \(G \subset \text{GL}(s_0) \times \text{GL}(s_1)\). Since we are only interested in supersymmetric extensions of these algebras, we limited ourselves to those branches which allow for non-vanishing \([Q, Q]\). The results of the \(N = 1\) and \(N = 2\) analyses are in Tables 3 and 6, respectively. We found 9 isomorphism classes in the \(N = 1\) case, and 22 non-empty sub-branches in the \(N = 2\) case.

These classifications have a few interesting features. The \(N = 1\) classification showed that if we centrally-extended a kinematical Lie algebra before finding its super-extensions, we will generally obtain very different results than if we super-extended the algebra before finding its central extensions. It would be interesting to investigate whether there are any special properties of those generalised Bargmann superalgebras which can arise through both procedures.

Although, not particularly interesting in the \(N = 1\) case, the double complex structure found in which \(B\) and \(P\) act as the differentials on modules \(s_1\) may be interesting to study for \(N \geq 2\). Notice that this interpretation is possible due to the kinematical Lie algebras having \([B, B] = [P, P] = 0\); therefore, we will have a double complex
on any Lie superalgebra with these brackets. In particular, all non-relativistic and ultra-relativistic kinematical Lie algebras have these brackets: this structure will be present for all investigations into extended supersymmetry for Galilean and Carrollian spacetimes. Thus a full understanding of this structure may shed some light on extended supersymmetry in these regimes.

As with the classification of the $N = 1$ super-extensions of the non-centrally-extended kinematical Lie algebras in [46], both the $N = 1$ classification and $N = 2$ branch analysis presented here demonstrate that each generalised Bargmann algebra can have many possible super-extensions. With the exception of the superalgebras of sub-branch 2.1.ii for the centrally-extended static KLA $\tilde{\alpha}$, every super-extension has the central extension $Z$ in the $[Q, Q]$ component of the bracket. Therefore, understanding the significance of the supersymmetry containing this internal component appears to be vital for a full understanding of these generalised Bargmann superalgebras and their phenomenology.

A clear next step in this project would be to determine the automorphism groups for each of the generalised Bargmann superalgebras to determine their admissible Lie super pairs as was done for the kinematical Lie superalgebras without central extension in [46]. With this information, we could then classify the possible generalised Bargmann superspaces in $(3+1)$-dimensions, and, "superising" the work in [40], find the geometric properties of these spaces, such as the invariant structures and their associated symmetries.

In this paper, we restricted ourselves to only the first section in Table 2; therefore, another obvious extension to the current work would be to classify the super-extensions for the algebras in the other two sections. One further direction of investigation would be to classify the generalised Bargmann superalgebras for different dimensions $D$ and higher $N$. Notice that the formalism introduced in section 4.1 could easily be recycled for investigations into $D = 3$ kinematical Lie superalgebras with $N > 2$. Wanting to explore $N$-extended super-extensions, we would only change the size of the $s_1$ vector space and the quaternionic matrices, ensuring we were working with $s_1 = \bigoplus_{i=1}^{N} S^i$, where $S^i$ is a copy of the four-dimensional real $so(3)$ spinor module, and $H, Z, B, P \in \text{Mat}_{N}(\mathbb{H})$. It may also be interesting to look at $D = 2$ due to the connection with Chern-Simons theories (see [23, 26]), and to determine the invariant tensors for the Lie superalgebras presented in this paper and try to map them down to the invariant tensors for the Lie superalgebras in one dimension lower.

In addition to introducing more $so(3)$ spinor modules into the $s_1$ vector space as suggested above, we could look to extend the $s_0$ vector space. Having demonstrated how one may incorporate the extra $so(3)$ scalar generator $Z$ into the underlying vector space, we could introduce the additional generators to consider classifying Maxwell superalgebras, or graded conformal Lie superalgebras, extending the work of [56] in the latter case.

For future research, it is interesting that many of these superalgebras can be gauged to produce supergravity theories in $(3+1)$-dimensions. Following the relativistic case, for a superalgebra to gauge to a non-trivial supergravity theory, we must have the commutator of two (local) supersymmetry transformations producing a (local) spacetime translation; therefore, $[Q, Q]$ must have $H, P$, or $H$ and $P$ on the right-hand-side. A list of the sub-branches in which such generalised Bargmann superalgebras are found is given in Table 7.

Another exciting application of the superalgebras classified here lies in possible holographic dualities. There is an extensive literature investigating non-relativistic holography (see [16, 57-59] for a few examples), in which torsional Newton-Cartan (TNC) geometries are dual to Hořava-Lifshitz gravity. On the Newton-Cartan side of this duality, we have some geometry that may be obtained by gauging a non-relativistic algebra [60]; therefore, we may consider taking any of the generalised Bargmann algebras as our starting point. To obtain the dual theory, we may use the following procedure. Let $\phi$ be an isomorphism which exchanges the two $so(3)$ scalar modules $H$ and $Z$, and exchanges the two $so(3)$ vector modules $B$ and $P$. To be more explicit, define $\phi$ as

$$\phi : s \rightarrow s' \text{ such that } \phi(H) = Z, \phi(B_i) = P_i, \phi(Z) = H, \phi(P_i) = B_i, \phi(Q^A_a) = Q^A_a,$$

where $i$ runs over $so(3)$ vector indices $\{1, 2, 3\}$, $a$ runs over $so(3)$ spinor indices $\{1, 2, 3, 4\}$, and $A$ labels our spinor module $\{1, 2\}$. Focussing solely on the generalised Bargmann superalgebras described in Table 8, the $N = 1$ cases have two brackets in common:

$$[B(\beta), P(\pi)] = \text{Re}(\beta\pi)Z \text{ and } [Q(s), Q(s)] = |s|^2Z.$$

For the $N = 2$ superalgebras, these two brackets are very similar:

$$[B(\beta), P(\pi)] = \text{Re}(\beta\pi)Z \text{ and } [Q(s), Q(s)] = \text{Re}(sN_1s^5)Z,$$

Non-relativistic superalgebras with $[Q, Q] = Z$ have been considered in papers such as [54, 55].
Table 7. Supergravity Algebras

| S | t  | (S)B | H | Z | B | P | [Q, Q] |
|---|----|------|---|---|---|---|--------|
| 3 | ˆa | -   | ✓ |   | H | Z |        |
| - | ˆa | 1.ii |   |   | H |   | H + Z  |
| - | ˆa | 2.1.i | ✓ |   |   |   | H      |
| - | ˆa | 2.3.i | ✓ | ✓ | Z | B |        |
| - | ˆa | 2.3.i | ✓ | ✓ | H | Z | B + P  |
| - | ˆa | 4.5.iii | ✓ | ✓ | H | Z | B + P |
| - | ˆa- | 2.3.i | ✓ | ✓ | Z | P |        |
| - | ˆa- | 3.3.i | ✓ | ✓ | Z | P |        |
| - | ˆa+ | 4.2.iii | ✓ | ✓ | Z | B | P      |
| - | ˆa+ | 4.5.iii | ✓ | ✓ | H | Z | B + P |
| - | ˆg | 3.3.i | ✓ | ✓ | Z | P |        |
| - | ˆg | 3.3.ii | ✓ |   | H | Z | P      |

The first column gives the unique identifier for the \( N = 1 \) generalised Bargmann superalgebras \( s_i \), and the second column tells us the underlying generalised Bargmann algebra \( t \). For the \( N = 2 \) superalgebras, the third column indicates the sub-branch the generalised Bargmann superalgebra comes from, so that the reader may navigate back to find the conditions on the non-vanishing parameters of the superalgebra. The next four columns tell us which of the \( s_0 \) generators \( H, Z, B, \) and \( P \) act on \( Q \). Recall, \( J \) necessarily acts on \( Q \), so we do not need to state this explicitly. The final column shows which \( s_0 \) generators occur in the \([Q, Q]\) bracket.

where \( N^i_1 = N^1_1 \). Notice that by sending \( Z \) to \( H \), the two \( s \) brackets including the mass generator \( Z \) become \([B, P] = H\) and \([Q, Q] = H\); thus, under this isomorphism, we obtain Carroll superalgebras from these generalised Bargmann superalgebras.\(^9\) By exchanging the \( \mathfrak{so}(3) \) vector modules \( B \) and \( P \) as well, we have the following interpretation for the generalised Bargmann superalgebra. The centrally-extended static kinematical Lie algebra \( \hat{a} \), with only \([B, P] = Z\), becomes the Carroll algebra. The centrally-extended dS Galilean algebra \( \hat{a}_- \) becomes

\[
[B, P] = H, \quad [Z, B] = -B, \quad [Z, P] = P.
\]

Interpreting \( Z \) as a dilatation, these are the brackets for the Carroll Lifshitz algebra first discussed in \([61]\). In fact, our \( N = 1 \) extension of this algebra is very similar to the superalgebra found to describe the Carroll superparticle in \([62]\). The only difference being that, in our context, \([Z, Q] = Q\), whereas they have the dilatation acting trivially on \( Q \). That we have such a bracket may not be too surprising given the two \( N = 1 \) Galilean superalgebras found in \([46]\). There, we found the Galilean superalgebra obtained through a contraction of the Poincaré superalgebra, in which the only bracket involving \( s_1 \) was \([Q, Q] = -P\); but, we also found a super-extension of the Galilean algebra which included \([H, Q] = Q\). Analogously, it may be that since the super-extension of the Carroll-Lifshitz algebra presented in \([62]\) comes from a limit, this bracket is not present. The centrally-extended AdS Galilean algebra \( \hat{a}_+ \) and the Bargmann algebra become

\[
[B, P] = H, \quad [Z, B] = P, \quad [Z, P] = -B,
\]

and

\[
[B, P] = H, \quad [Z, B] = P,
\]

respectively. For now, we do not have any interpretation for these Lie algebras; however, it may be of interest to explore which theories require an extension of the Carroll algebra by a generator that acts on the \( \mathfrak{so}(3) \) vector modules through a rotation (the \( \hat{a}_+ \) case) or a nilpotent matrix (the \( \hat{g} \) case). Using this isomorphism, we may systematically produce Carroll superalgebras corresponding to each of the generalised Bargmann superalgebras. With these connections to non-relativistic holography, it would be interesting to use the methods applied in this classification to the generalised Lifshitz and Schrödinger algebras presented in \([56]\).

If interested in extending this work to Lifshitz algebras, it is important to note that papers concentrating on supersymmetric Lifshitz field theories such as \([63, 64]\) utilise a super-extension of the Lifshitz algebra in which the modules of \( s_1 \) transform as scalars under \( \mathfrak{so}(3) \). Therefore, the classification method highlighted in this paper would need to be modified to take this change of \( \mathfrak{so}(3) \) action into account.

\(^9\)This duality was also recognised in the non-supersymmetric context in \([45]\).
Table 8. Generalised Bargmann superalgebras with $[Q, Q] = Z$

| S | t | (S) | H | Z | B | P | [Q, Q] |
|---|---|-----|---|---|---|---|--------|
| 1 | â | –  | $\varnothing$ | Z |
| 2 | â | –  | $\checkmark$ | $\varnothing$ | Z |
|    | â | 1.i | $\checkmark$ | $\varnothing$ | Z |
|    | â | 2.2.i | $\checkmark$ | $\checkmark$ | Z |
| 4 | n_0 | –  | Z |
| 5 | n_1 | –  | $\checkmark$ | Z |
|    | n_2 | 1.i | $\checkmark$ | Z |
|    | n_2 | 2.2.i | $\checkmark$ | $\checkmark$ | Z |
|    | n_2 | 3.2.i | $\checkmark$ | $\checkmark$ | Z |
| 6 | n_3 | –  | Z |
| 7 | n_3 | –  | $\checkmark$ | Z |
|    | n_3 | 1.i | $\checkmark$ | Z |
|    | n_3 | 2.2.i | $\checkmark$ | $\checkmark$ | Z |
|    | n_3 | 4.1.i | $\checkmark$ | $\checkmark$ | Z |
| 8 | i_0 | –  | Z |
| 9 | i_0 | –  | $\checkmark$ | Z |
|    | i_0 | 1.i | $\checkmark$ | Z |
|    | i_0 | 2.2.i | $\checkmark$ | $\checkmark$ | Z |
|    | i_0 | 3.2.i | $\checkmark$ | $\checkmark$ | Z |

The first column gives the unique identifier for each of the $N = 1$ generalised Bargmann superalgebras $S$, and the second column tells us the underlying generalised Bargmann algebra $t$. For the $N = 2$ superalgebras, the third column indicates the sub-branch the generalised Bargmann superalgebra comes from, so that the reader may navigate back to find the conditions on the non-vanishing parameters of the superalgebra. The next four columns tell us which of the $s_{\mathbb{C}}$ generators $H$, $Z$, $B$, and $P$ act on $Q$. Recall, $J$ necessarily acts on $Q$, so we do not need to state this explicitly. The final column shows which $s_{\mathbb{C}}$ generators occur in the $[Q, Q]$ bracket.

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