Blow-up of solutions to Nakao’s problem via an iteration argument

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Abstract

In this paper, we consider a weakly coupled system for a semilinear damped wave equation and a semilinear wave equation in $\mathbb{R}^n$, which is the so-called Nakao’s problem proposed by Professor Mitsuhiro Nakao. By applying an iteration method for unbounded multiplier with a slicing procedure, we prove blow-up of solutions for Nakao’s problem even for small data. We improve the blow-up result and upper bound estimates for lifespan comparing with the previous research, especially, in higher dimensional cases.

Keywords: Semilinear hyperbolic system, wave equation, damped wave equation, blow-up.

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1 Introduction

We investigate blow-up of solutions in finite time to the Cauchy problem for the weakly coupled system of a semilinear damped wave equation and a semilinear wave equation, namely,

$$
\begin{aligned}
  u_{tt} - \Delta u + u_t &= |v|^p, & x \in \mathbb{R}^n, & t > 0, \\
  v_{tt} - \Delta v &= |u|^q, & x \in \mathbb{R}^n, & t > 0, \\
  (u, u_t, v, v_t)(0, x) &= \varepsilon (u_0, u_1, v_0, v_1)(x), & x \in \mathbb{R}^n,
\end{aligned}
$$

(1.1)

where $p, q > 1$ and $\varepsilon$ is a positive parameter describing the size of initial data. The problem of critical curve for the weakly coupled system (1.1) was proposed by Professor Mitsuhiro Nakao, Emeritus of Kyushu University (also see [15, 21]). Here, the “critical curve” stands for the threshold condition of a pair of exponents $(p, q)$ between global (in time) existence of small data solutions and blow-up of solutions even for small data. Recently, by employing the test function method (see, for example, [11, 23]) the author of [21] proved that if the following conditions:

$$
\alpha_{N,W} := \max \left\{ \frac{q/2 + 1}{pq - 1} + \frac{1}{2}, \frac{q + 1}{pq - 1}, \frac{p + 1}{pq - 1} \right\} \geq \frac{n}{2},
$$

(1.2)

and

$$
1 < p, q < \infty (n = 1, 2), \quad 1 < p, q \leq \frac{n}{n - 2} (n \geq 3),
$$

hold, then every local (in time) solution $(u, v)$ to (1.1) blows up in finite time. Nevertheless, the curve $\alpha_{N,W} = n/2$ in the $p - q$ plane for a pair of exponents $(p, q)$ seems optimal only when $n = 1$.

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Precisely, for the case $n = 1$, under the condition $\alpha_{N,W} \geq 1/2$ (or, equivalently, $1 < p, q < \infty$) every local (in time) solution blows up. But, in the case $n \geq 2$, the condition (1.2) seems not optimal. The main goal in this paper is to improve the blow-up result stated in [21], particularly, in higher dimensional cases.

We sketch now some historical background related to (1.1). Since Nakao’s problem (1.1) is related to a weakly coupled system of semilinear damped wave equations and semilinear wave equations, we next recall some results for these systems, respectively.

On one hand, the next weakly coupled system of semilinear wave equations:

$$\begin{cases}
  u_{tt} - \Delta u = |v|^p, & x \in \mathbb{R}^n, \ t > 0, \\
  v_{tt} - \Delta v = |u|^q, & x \in \mathbb{R}^n, \ t > 0, \\
  (u, u_t, v, v_t)(0, x) = (u_0, u_1, v_0, v_1)(x), & x \in \mathbb{R}^n,
\end{cases}$$

for $n \geq 1$ with $p, q > 1$, has been widely studied in recent years. The papers [3, 4, 5, 1, 8, 7, 6, 9] investigated that the critical curve for (1.3) is described by the condition

$$\alpha_W := \max \left\{ \frac{p + 2 + q^{-1}}{pq - 1}, \frac{q + 2 + p^{-1}}{pq - 1} \right\} = \frac{n - 1}{2}. \quad (1.4)$$

In other words, if $\alpha_W < (n - 1)/2$, then there exists a unique global (in time) solution for small data; else if $\alpha_W \geq (n - 1)/2$, in general, every local (in time) solution blows up. Especially, we should underline that the approach for proving blow-up results for (1.3) is mainly based on generalized Kato’s type lemma or an iteration argument.

On the other hand, let us turn to recall the results for the weakly coupled system of semilinear classical damped wave equations

$$\begin{cases}
  u_{tt} - \Delta u + u_t = |v|^p, & x \in \mathbb{R}^n, \ t > 0, \\
  v_{tt} - \Delta v + v_t = |u|^q, & x \in \mathbb{R}^n, \ t > 0, \\
  (u, u_t, v, v_t)(0, x) = (u_0, u_1, v_0, v_1)(x), & x \in \mathbb{R}^n,
\end{cases}$$

for $n \geq 1$ with $p, q > 1$. The critical curve for (1.5) is characterized by the condition

$$\alpha_{DW} := \max \left\{ \frac{p + 1}{pq - 1}, \frac{q + 1}{pq - 1} \right\} = \frac{n}{2}. \quad (1.6)$$

which has been investigated by the authors of [20, 12, 13, 14]. To derive nonexistence results for global solutions to (1.5), the authors applied the test function method, which is useful to deal with a semilinear model with effective damping.

From the above results of critical curves for (1.3) and (1.5), we may expect that the critical curve for (1.1) is between (1.4) and (1.6). However, we should underline that the critical curve for (1.1) is not a simple combination of (1.4) and (1.6) because the critical curve to (1.1) seems to be influenced by varying degrees between semilinear wave equation and semilinear damped wave equation. Let us now focus on the blow-up of solutions. From the technical point of view, quite different methods are applied to derive blow-up results for (1.3) and (1.5). Therefore, it is significant for us to find a suitably unified approach to deal with Nakao’s problem (1.1). As mentioned in the above, [21]
proved blow-up for (1.1) by the test function method, however, for higher dimension cases \((n \geq 4)\) the condition fulfills
\[
\alpha_{N,W} \geq \frac{n-1}{2} \iff \alpha_{DW} \geq \frac{n-1}{2}.
\]
In other words, for higher dimension cases, the blow-up result from [21] is the same as those for weakly coupled system of damped wave equations. In this paper, we mainly improve the blow-up to the iteration argument, the blow-up result from [21] is partially improved for \(2 \leq n \leq 3\) and completely improved for \(n \geq 4\). Simultaneously, some estimates for upper bound of lifespan can be derived.

### 1.1 Main result

Before stating the main results of this paper, let us introduce a suitable notion of energy solutions to Nakao’s problem (1.1).

**Definition 1.1.** Let \((u_0, u_1, v_0, v_1) \in (H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)) \times (H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n))\). We say that \((u, v)\) is an energy solution of (1.1) on \([0, T]\) if
\[
\begin{align*}
&u \in C \left( [0, T), H^1(\mathbb{R}^n) \right) \cap C^1 \left( [0, T), L^2(\mathbb{R}^n) \right) \cap L^q_{\text{loc}}([0, T) \times \mathbb{R}^n), \\
v \in C \left( [0, T), H^1(\mathbb{R}^n) \right) \cap C^1 \left( [0, T), L^2(\mathbb{R}^n) \right) \cap L^p_{\text{loc}}([0, T) \times \mathbb{R}^n),
\end{align*}
\]

satisfies \((u, v)(0, \cdot) = \varepsilon (u_0, v_0)\) in \(H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)\) and the following integral relations:
\[
\begin{align*}
\int^t_0 \int_{\mathbb{R}^n} (-u_t(s, x) \phi_s(s, x) + u_t(s, x) \phi(s, x) + \nabla u(s, x) \cdot \nabla \phi(s, x)) \, dx \, ds \\
+ \int_{\mathbb{R}^n} u_t(t, x) \phi(t, x) \, dx - \varepsilon \int_{\mathbb{R}^n} u_1(x) \phi(0, x) \, dx = \int^t_0 \int_{\mathbb{R}^n} |v(s, x)|^p \phi(s, x) \, dx \, ds 
\end{align*}
\tag{1.7}
\]
and
\[
\begin{align*}
\int^t_0 \int_{\mathbb{R}^n} (-v_t(s, x) \psi_s(s, x) + \nabla v(s, x) \cdot \nabla \psi(s, x)) \, dx \, ds \\
+ \int_{\mathbb{R}^n} v_t(t, x) \psi(t, x) \, dx - \varepsilon \int_{\mathbb{R}^n} v_1(x) \psi(0, x) \, dx = \int^t_0 \int_{\mathbb{R}^n} |u(s, x)|^q \psi(s, x) \, dx \, ds 
\end{align*}
\tag{1.8}
\]
for any test functions \(\phi, \psi \in C^\infty_0([0, T) \times \mathbb{R}^n)\) and any \(t \in (0, T)\).

Clearly, the application of further steps of integration by parts in (1.7) and (1.8), we obtain
\[
\begin{align*}
\int^t_0 \int_{\mathbb{R}^n} u(s, x) \left( \phi_{ss}(s, x) - \Delta \phi(s, x) - \phi_s(s, x) \right) \, dx \, ds \\
+ \int_{\mathbb{R}^n} u_t(t, x) \phi(t, x) + u(t, x) \phi(t, x) - u(t, x) \phi_s(t, x) \, dx \\
- \varepsilon \int_{\mathbb{R}^n} u_1(x) \phi(0, x) + u_0(x) \phi(0, x) - u_0(x) \phi_s(0, x) \, dx = \int^t_0 \int_{\mathbb{R}^n} |v(s, x)|^p \phi(s, x) \, dx \, ds
\end{align*}
\]
Then, there exists a positive constant $\varepsilon_0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ the solution $(u, v)$ blows up in finite time. Furthermore, the upper bound estimate for the lifespans $T(\varepsilon)$ satisfying

$$T(\varepsilon) \leq C\varepsilon^{-1/\max\{F_1(n,p,q),F_2(n,p,q)\}}$$

holds, where $C > 0$ is an independent of $\varepsilon$ constant and

$$F_1(n,p,q) := \frac{2 + p^{-1}}{pq - 1} - \frac{n - 1}{2}, \quad F_2(n,p,q) := \frac{1 + 2q^{-1}}{pq - 1} - \frac{n - 1}{q}.$$  \hfill (1.11)  \hfill (1.12)

Proposition 1.2. Let us consider $p, q > 1$ if $n = 1, 2$, and $1 < p, q \leq n/(n-2)$ if $n \geq 3$ such that

$$\alpha_1 := \max\left\{\frac{q/2 + 1}{pq - 1}, \frac{1/2 + p}{pq - 1} - \frac{1}{2}\right\} > \frac{n - 1}{2}.$$  \hfill (1.13)

Let us assume that $(u_0, u_1, v_0, v_1) \in (H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)) \times (H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n))$ be nonnegative and compactly supported functions with supports contained in $B_R$ for some $R > 0$ such that $u_0, v_1$ are not identically zero. Let $(u, v)$ be the energy solution to Nakao’s problem (1.1) according to Definition 1.1 with lifespans $T(\varepsilon)$ satisfying

$$\text{supp } u, \text{ supp } v \subset \{(t, x) \in [0, T) \times \mathbb{R}^n : |x| \leq R + t\}.$$  \hfill (1.10)

Then, there exists a positive constant $\varepsilon_0 = \varepsilon_0(u_0, u_1, v_0, v_1, n, p, q, R)$ such that for any $\varepsilon \in (0, \varepsilon_0)$ the solution $(u, v)$ blows up in finite time. Furthermore, the upper bound estimate for the lifespans

$$T(\varepsilon) \leq C\varepsilon^{-1/\max\{F_3(n,p,q),F_4(n,p,q)\}}$$

holds, where $C > 0$ is an independent of $\varepsilon$ constant and

$$F_3(n,p,q) := \frac{2 + q}{pq - 1} - n + 1, \quad F_4(n,p,q) := \frac{1 + 2p}{pq - 1} - n.$$  \hfill (1.14)  \hfill (1.15)
Summarizing the above two results, we may immediately obtain the following conclusion for blow-up of energy solution to Nakao’s problem (1.1).

**Theorem 1.1.** Let us consider $p, q > 1$ if $n = 1, 2$, and $1 < p, q \leq n/(n-2)$ if $n \geq 3$ such that

$$\alpha_N := \max \left\{ \frac{q/2 + 1}{pq - 1}, \frac{2 + p^{-1}}{pq - 1}, \frac{1/2 + p}{pq - 1} - \frac{1}{2} \right\} > \frac{n-1}{2}. \quad (1.16)$$

Let us assume that $(u_0, u_1, v_0, v_1) \in (H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)) \times (H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n))$ have the same assumption as Theorem 1.1. Let $(u, v)$ be the energy solution to Nakao’s problem (1.1) according to Definition 1.1 with lifespans $T(\varepsilon)$ satisfying (1.10). Then, there exists a positive constant $\varepsilon_0 = \varepsilon_0(u_0, u_1, v_0, v_1, n, p, q, R)$ such that for any $\varepsilon \in (0, \varepsilon_0]$ the solution $(u, v)$ blows up in finite time. Furthermore, the upper bound estimate for the lifespans

$$T(\varepsilon) \leq C \varepsilon^{-1/F(n,p,q)}$$

holds, where $C > 0$ is an independent of $\varepsilon$ constant and

$$F(n, p, q) := \max \{ F_1(n, p, q), F_2(n, p, q), F_3(n, p, q), F_4(n, p, q) \}.$$  

**1.2 Some remarks for the blow-up results**

Let us compare the condition of the exponents for power nonlinearities in blow-up results, i.e. (1.9) and (1.13). Under our assumption $p, q > 1$ and $p, q \leq n/(n-2)$ if $n \geq 3$, the condition (1.13) partly improves (1.9) only when $n = 2$, while for $n \geq 3$

$$\left\{ (p, q) : \alpha_1 > \frac{n-1}{2}, \quad 1 < p, q \leq \frac{n}{n-2} \right\} \subseteq \left\{ (p, q) : \alpha_0 > \frac{n-1}{2}, \quad 1 < p, q \leq \frac{n}{n-2} \right\}.$$  

We next show the curve $\alpha_N = (n-1)/2$ in different dimensions. Clearly, when $n = 1$ the energy solution blows up in finite time for all $1 < p, q < \infty$. For this reason, we just describe the case when $n \geq 2$.

![Figure 1: The curve $\alpha_N = (n-1)/2$ in the $p - q$ plane](image)

**Figure 1:** The curve $\alpha_N = (n-1)/2$ in the $p - q$ plane
From the above graphs, we may observe that in the case when \( n = 2 \), all components in \( \alpha_N \) have influence of the condition (1.16). However, in the case when \( n \geq 3 \), under the assumption
\[
1 < p, q \leq n/(n - 2),
\]
we may derive
\[
\alpha_N < \frac{n - 1}{2} \iff \frac{2 + p^{-1}}{pq - 1} < \frac{n - 1}{2}.
\]
The other components in \( \alpha_N \) do not influence of blow-up result. Finally, we should underline that the critical curve in the \( p - q \) plane for Nakao's problem (1.1) is still open, especially, the global (in time) existence of small data solutions.

In the remaining part of this subsection, we will give some remarks and explanations for our result in Theorem 1.2, especially, the condition
\[
\alpha_N = \max \left\{ \frac{q/2 + 1}{pq - 1}, \frac{2 + p^{-1}}{pq - 1}, \frac{1/2 + q^{-1}}{pq - 1} \right\} > \frac{n - 1}{2}
\]
by comparing the result from [21] that
\[
\alpha_N, W = \max \left\{ \frac{q/2 + 1}{pq - 1} + \frac{1}{2}, \frac{q + 1}{pq - 1}, \frac{p + 1}{pq - 1} \right\} \geq \frac{n}{2}.
\]
To guarantee the local (in time) existence of solutions and blow-up in finite time, we assume \( p, q > 1 \) and \( p, q < n/(n - 2) \) if \( n \geq 3 \) throughout this subsection.

- Concerning the case \( n = 1 \), we prove blow-up result of energy solutions for Nakao's problem when \( 1 < p, q < \infty \), which corresponds to the blow-up result stated in [21]. In other words, we may assert that the condition (1.9) is optimal when \( n = 1 \).

- Concerning the case \( n = 2, 3 \), our result \( \alpha_N > (n - 1)/2 \) partially improves those in [21] that \( \alpha_N, W \geq n/2 \). However, some parts of our result are worse than those in [21]. We may see the next figures. Particularly, we may divide the condition \( \alpha_N > (n - 1)/2 \) by three parts that they are similar in form to the previous results in the subcritical case as follows:
  - Part I: \( \frac{q/2 + 1}{pq - 1} + \frac{1}{2} > \frac{n}{2} \), which is coincide as first component of \( \alpha_N, W > \frac{n}{2} \);
  - Part II: \( \frac{2 + p^{-1}}{pq - 1} > \frac{n - 1}{2} \), which is similar as the second component of \( \alpha_W > \frac{n - 1}{2} \);
  - Part III: \( \frac{1/2 + p^{-1}}{pq - 1} > \frac{n}{2} \), which is similar as the first component of \( \alpha_{DW} > \frac{n}{2} \);
Moreover, the condition \( \alpha_N, W \geq n/2 \) can be rewritten by the following way: a pair of exponents \((p, q)\) satisfy
\[
\frac{q/2 + 1}{pq - 1} + \frac{1}{2} \geq \frac{n}{2} \quad \text{or} \quad \alpha_{DW} \geq \frac{n}{2},
\]
whose first part is almost the same as Part I (the difference is the limit case). Nevertheless, the main different between these two conditions is the second and third parts of the above statement.

- Concerning the case \( n \geq 4 \), our result completely improve those stated in [21]. Again, one may see the following figure.
Figure 2: Blow-up range in the $p-q$ plane

**Notation:** We give some notations to be used in this paper. We write $f \lesssim g$ when there exists a positive constant $C$ such that $f \leq Cg$. Moreover, $B_R$ denotes the ball around the origin with radius $R$ in $\mathbb{R}^n$.

# 2 Proof of Proposition 1.1

## 2.1 Iteration frame

To begin with the proof, we introduce the following time-dependent functionals:

$$U(t) := \int_{\mathbb{R}^n} u(t, x) \, dx \quad \text{and} \quad V(t) := \int_{\mathbb{R}^n} v(t, x) \, dx.$$ 

According to finite propagation speed for the solutions to wave equations and damped wave equations. We know that if the exponents $p, q$ satisfy $1 < p, q < \infty$ when $n = 1, 2$, and $1 < p, q \leq n/(n-2)$ when $n \geq 3$, and initial data $(u_0, u_1, v_0, v_1) \in (H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n))^2$ has compact support in $B_R$, then the local (in time) weak solutions belong to energy spaces and have compact support in $B_{R+t}$.

Let us choose test functions $\phi$ and $\psi$ in (1.7) and (1.8) such that $\phi \equiv 1$ and $\psi \equiv 1$ on $\{(s, x) \in [0, t] \times \mathbb{R}^n : |x| \leq R + s\}$. So, we may immediately derive

$$\int_0^t \int_{\mathbb{R}^n} u_t(s, x) \, dx \, ds + \int_{\mathbb{R}^n} u_t(t, x) \, dx - \varepsilon \int_{\mathbb{R}^n} u_1(x) \, dx = \int_0^t \int_{\mathbb{R}^n} |v(s, x)|^p \, dx \, ds,$$

$$\int_{\mathbb{R}^n} v_t(t, x) \, dx - \varepsilon \int_{\mathbb{R}^n} v_1(x) \, dx = \int_0^t \int_{\mathbb{R}^n} |u(s, x)|^q \, dx \, ds.$$
The last equations can be rewritten by using time-dependent functions such that
\[
U'(t) + U(t) = U'(0) + U(0) + \int_0^t \int_{\mathbb{R}^n} |v(s, x)|^p \, dx \, ds, \quad (2.1)
\]
\[
V'(t) = V'(0) + \int_0^t \int_{\mathbb{R}^n} |u(s, x)|^q \, dx \, ds. \quad (2.2)
\]
Clearly from (2.1) and (2.2), the functionals fulfill \(U(t) \geq 0\) and \(V(t) \geq 0\) for any \(t \geq 0\), where we used our assumptions for nonnegative initial data \(u_0, u_1, v_0, v_1\). Precisely, (2.1) shows
\[
\frac{d}{dt} \left( e^t U(t) \right) \geq e^t \left( U'(0) + U(0) \right)
\]
and by integrating over \([0, t]\), one gets
\[
U(t) \geq e^{-t} U(0) + \left( 1 - e^{-t} \right) (U'(0) + U(0)) = U(0) \left( 1 - e^{-t} \right) U'(0) + U(0) \geq 0. \quad (2.3)
\]
Concerning \(V(t) \geq 0\) for any \(t \geq 0\), we just need to integrate (2.2) once. Thanks to finite propagation speed of \(v\) and Hölder’s inequality, we multiply both sides of (2.1) by \(e^t\) and take the integration with respect to \(t\) to obtain
\[
U(t) \geq \int_0^t e^{\tau-t} \int_{\mathbb{R}^n} |v(s, x)|^p \, dx \, ds \, d\tau \geq C_0 \int_0^t e^{\tau-t} \int_0^\tau (R+s)^{-n(p-1)} (V(s))^p \, ds \, d\tau \quad (2.4)
\]
with a positive constant \(C_0 > 0\). Moreover, taking integration of (2.2) over \([0, t]\) and considering the finite propagation speed of \(u\) show
\[
V(t) \geq \int_0^t \int_{\mathbb{R}^n} |u(s, x)|^q \, dx \, ds \, d\tau \geq C_0 \int_0^t \int_0^\tau (R+s)^{-n(q-1)} (U(s))^q \, ds \, d\tau. \quad (2.5)
\]
All in all, the iteration frames are constructed in (2.4) and (2.5).

### 2.2 Lower bound for the functionals

The main approach of the proof is based on an iteration procedure, which requires iteration frames and first lower bound estimates for the functionals \(U\) and \(V\), respectively. To begin with deriving the estimates of functionals from the below, let us introduce the eigenfunction \(\Phi\) of the Laplace operator in \(\mathbb{R}^n\), namely,
\[
\Phi(x) := e^x + e^{-x} \quad \text{if } n = 1,
\]
\[
\Phi(x) := \int_{\mathbb{S}^{n-1}} e^{x \cdot \omega} \, d\sigma_\omega \quad \text{if } n \geq 2,
\]
where \(\mathbb{S}^{n-1}\) is the \(n-1\) dimensional sphere. Particularly, the function \(\Psi\) has been introduced in [22]. It satisfies \(\Delta \Phi = \Phi\) and has the asymptotic behavior
\[
\Phi(x) \sim |x|^{-\frac{n-1}{2}} e^{|x|} \text{ as } |x| \to \infty.
\]
By defining the test function with separate variables such that
\[
\Psi(t, x) := e^{-t} \Phi(x),
\]
obviously, the function $\Psi = \Psi(t, x)$ solves the wave equation $\Psi_{tt} - \Delta \Psi = 0$.

Let us derive lower bound estimates for the functional $U$ in the first place by defining the auxiliary functional

$$V_1(t) := \int_{\mathbb{R}^n} v(t, x) \Psi(t, x) \, dx.$$  

According to [16], by our nontrivial assumption on initial data satisfying $v_0 \neq 0$, there exists a constant $C_1 = C_1(v_0, v_1) > 0$ such that

$$V_1(t) \geq \frac{1}{2} \left(1 - e^{-2t}\right) \int_{\mathbb{R}^n} \left(v_0(x) + v_1(x)\right) \Phi(x) \, dx + e^{-2t} \int_{\mathbb{R}^n} v_0(x) \Phi(x) \, dx \geq C_1 \varepsilon$$

for any $t \geq 0$. Indeed, by the asymptotic behavior of the test function $\Psi$, the next inequality holds (see, for example, estimate (2.5) in [10]):

$$\int_{|x| \leq R+t} |\Psi(t, x)|^{\frac{p}{p-1}} \, dx \leq C_2(R + t)^{\frac{(n-1)(2-p')}{2}},$$

where $C_2 = C_2(n, R) > 0$. The application of Hölder’s inequality indicates

$$\int_{\mathbb{R}^n} |v(t, x)|^p \, dx \geq (V_1(t))^p \left(\int_{|x| \leq R+t} |\Psi(t, x)|^{\frac{p}{p-1}} \, dx\right)^{\frac{p}{p-1}} \geq C_3 \varepsilon^p (R + t)^{n-1 - \frac{(n-1)p}{2}},$$

where $C_3 := C_1^p C_2^{1-p} > 0$. By plugging (2.6) into (2.4), we may derive

$$U(t) \geq C_3 \varepsilon^p \int_0^t e^\tau - t \int_0^\tau (R + s)^{n-1 - \frac{(n-1)p}{2}} \, ds \, d\tau \geq C_3 \varepsilon^p (R + t - \frac{(n-1)p}{2}) \int_0^t e^{\tau - t} \int_0^t s^{n-1} \, ds \, d\tau \geq C_3 \varepsilon^p (R + t - \frac{(n-1)p}{2}) \int_0^t e^{\tau - t} \int_0^\tau s^{n-1} \, ds \, d\tau \geq C_3 (1 - e^{-t/2}) \varepsilon^p (R + t - \frac{(n-1)p}{2}) \int_0^t e^{\tau - t} \int_0^\tau s^{n-1} \, ds \, d\tau$$

for any $t \geq 1$, where $[t/2, t] \subset [0, t]$ has been used.

Let us now take the consideration of lower bound estimates for $V$. According to (2.2) and the nontrivial and nonnegative assumption on $v_1$, the lower bound for the functional $V$ is given by

$$V(t) \geq V(0) + V'(0)t \geq C_4 \varepsilon t$$

for any $t \geq 1$, with a positive constant $C_4 = C_4(v_0, v_1)$.

In conclusion, we have obtained first lower bound estimates for the functionals

$$U(t) \geq D_1 (R + t)^{-\alpha_1} t^{\beta_1} \text{ for any } t \geq 1,$$

$$V(t) \geq Q_1 (R + t)^{-\alpha_1} t^{\beta_1} \text{ for any } t \geq 1,$$

carrying the multiplicative constants

$$D_1 := \frac{C_3(1 - e^{-1/2}) \varepsilon^p}{n^{2n}}, \quad Q_1 := C_4 \varepsilon,$$
and the exponents
\[ \alpha_1 := \frac{(n-1)p}{2}, \quad a_1 := 0, \quad \beta_1 := n, \quad b_1 := 1. \]

2.3 Iteration argument

In this subsection, we will derive sequences of lower bounds for each functional by using the iteration frames (2.4) and (2.5). We remark that the iteration argument has been used for weakly coupled systems, for examples, [1, 17, 18, 19]. More precisely, we will show
\[ U(t) \geq D_j (R + t)^{-\alpha_j} (t - L_j)^{\beta_j} \text{ for any } t \geq L_j, \]
\[ V(t) \geq Q_j (R + t)^{-a_j} (t - L_j)^{b_j} \text{ for any } t \geq L_j, \]
where \( \{D_j\}_{j \geq 1}, \{Q_j\}_{j \geq 1}, \{\alpha_j\}_{j \geq 1}, \{a_j\}_{j \geq 1}, \{\beta_j\}_{j \geq 1} \) and \( \{b_j\}_{j \geq 1} \) are sequences of nonnegative real numbers that will be determined later in the iteration procedure. Moreover, motivated by the recent paper [2], we construct \( \{L_j\}_{j \geq 1} \) to be the sequence of the partial products of the convergent infinite product
\[ \prod_{k=1}^{\infty} \ell_k \quad \text{with} \quad \ell_k := 1 + (pq)^{-(k+1)/2} \quad \text{for any } k \geq 1, \]
so that
\[ L_j := \prod_{k=1}^{j} \ell_k \quad \text{for any } j \geq 1. \]

Here, we used the facts that
\[ \prod_{k=1}^{\infty} \ell_k = \exp \left( \sum_{k=1}^{\infty} \ln \ell_k \right) \]
and the ratio test for determining a convergence series
\[ \lim_{k \to \infty} \frac{\ln \ell_{k+1}}{\ln \ell_k} = \lim_{k \to \infty} \frac{\ln(1 + (pq)^{-k/2})}{\ln(1 + (pq)^{-(k+1)/2})} = \lim_{k \to \infty} \frac{(1 + (pq)^{-(k+1)/2})(pq)^{-k/2}}{(1 + (pq)^{-k})(pq)^{-(k+1)/2}} = (pq)^{-1/2} < 1. \]

Particularly, the estimates (2.11) and (2.12) have been proved when \( j = 1 \) in the last subsection.

Thus, in order to prove (2.11) and (2.12) by applying an inductive argument, it just remains to show the induction step. On the one hand, let us plug (2.11) into (2.5) and shrink the domain \([0, t]\) into \([L_j, t]\) to get
\[ V(t) \geq C_0 D_j^q \int_{L_j}^{t} \int_{L_j}^{\tau} (R + s)^{-n(q-1)-q\alpha_j} (s - L_j)^{q\beta_j} ds d\tau \]
\[ \geq C_0 D_j^q (R + t)^{-n(q-1)-q\alpha_j} \int_{L_j}^{t} \int_{L_j}^{\tau} (s - L_j)^{q\beta_j} ds d\tau \]
\[ \geq \frac{C_0 D_j^q}{(q\beta_j + 1)(q\beta_j + 2)} (R + t)^{-n(q-1)-q\alpha_j} (t - L_{j+1})^{q\beta_j+2} \]
for $t \geq L_{j+1}$, where we used $L_{j+1} > L_j$ in the last line of the chain inequality. On the other hand, we combine (2.12) with (2.4) and shrink the domain again. It shows that

\[
U(t) \geq C_0 Q_j^p \int_0^t e^{\tau-t} \int_0^\tau (R + s)^{-n(p-1)-pa_j} (s - L_j)^{pb_j} \, ds \, d\tau
\]

\[
= C_0 Q_j^p (R + t)^{-n(p-1)-pa_j} \int_0^t e^{\tau-t} \int_0^\tau (s - L_j)^{pb_j} \, ds \, d\tau
\]

\[
\geq \frac{C_0 Q_j^p}{pb_j + 1} (R + t)^{-n(p-1)-pa_j} \int_0^t e^{\tau-t} (\tau - L_j)^{pb_j+1} \, d\tau
\]

\[
\geq \frac{C_0 Q_j^p}{pb_j + 1} (R + t)^{-n(p-1)-pa_j} (t/\ell_{j+1} - L_j)^{pb_j+1} \int_0^t e^{\tau-t} (\tau - L_j)^{pb_j+1} \, d\tau
\]

\[
= \frac{C_0 Q_j^p}{(pb_j + 1) \ell_{j+1}^{pb_j+1}} (R + t)^{-n(p-1)-pa_j} (t - L_j)^{pb_j+1} \left(1 - e^{(t/\ell_{j+1} - 1)}\right)
\]

for any $t \geq L_{j+1}$, which implies $L_j \geq t/\ell_{j+1}$. Furthermore, due to the fact that

\[
t \geq L_{j+1} = \ell_{j+1} L_j \geq \ell_{j+1} > 1,
\]

we may estimate

\[
1 - e^{t(1/\ell_{j+1} - 1)} \geq 1 - e^{-(\ell_{j+1}-1)} = 1 - \left(1 - (\ell_{j+1} - 1) + \frac{(\ell_{j+1} - 1)^2}{2}\right)
\]

\[
= (\ell_{j+1} - 1) \left(1 - \frac{\ell_{j+1} - 1}{2}\right) = (pq)^{-j/2} \left(1 - \frac{1}{2(pq)^{j/2}}\right)
\]

\[
= (pq)^{-j} ((pq)^{j/2} - 1/2) \geq ((pq)^{1/2} - 1/2) (pq)^{-j}.
\]

In other words, we obtain

\[
U(t) \geq \frac{C_0 Q_j^p ((pq)^{1/2} - 1/2)(pq)^{-j}}{(pb_j + 1) \ell_{j+1}^{pb_j+1}} (R + t)^{-n(p-1)-pa_j} (t - L_{j+1})^{pb_j+1}
\]

for any $t \geq L_{j+1}$. Thus, (2.11) and (2.12) are valid if the following recursive relations

\[
D_{j+1} := \frac{C_0 Q_j^p ((pq)^{1/2} - 1/2)(pq)^{-j}}{(pb_j + 1) \ell_{j+1}^{pb_j+1}}, \quad \alpha_{j+1} := n(p - 1) + pa_j, \quad \beta_{j+1} := pb_j + 1,
\]

\[
Q_{j+1} := \frac{C_0 D_j^q}{(q\beta_j + 1)(q\beta_j + 2)}, \quad a_{j+1} := n(q - 1) + qa_j, \quad b_{j+1} := q\beta_j + 2,
\]

are fulfilled.

### 2.4 Upper bound estimate for the lifespan

In the last subsection, we determined the sequence of lower bound estimates for $U$ and $V$. Thus, we may show that the $j$-dependent lower bounds for $U$ and $V$ blows up in finite time when $j \to \infty$. Simultaneously, the blow-up result and upper bound estimates for the lifespan will be derived.

Let us first determine the explicit representations of $\alpha_j, \beta_j, \alpha_j, \beta_j$, which contribute to the determination of estimates for the multiplicative constants $D_j$ and $Q_j$. 
Concerning the representations of \( \alpha_j \) and \( a_j \), we just discuss the case when \( j \) is an odd integer. For the remaining case that \( j \) is an even number, it is unnecessary for the proof of the theorem. By employing the previous definitions for the exponents \( \alpha_j \) and \( a_j \), one has

\[
\alpha_j = n(p-1) + p\alpha_{j-1} = n(pq-1) + pq\alpha_{j-2} = n(pq-1) + \sum_{k=0}^{(j-3)/2} (pq)^k + (pq)^{\frac{j-1}{2}} \alpha_1
\]

\[
= (n + \alpha_1)(pq)^{\frac{j-1}{2}} - n = \left(n + \frac{(n-1)p}{2}\right)(pq)^{\frac{j-1}{2}} - n, \tag{2.15}
\]

and by the same approach,

\[
a_j = (n + a_1)(pq)^{\frac{j-1}{2}} - n = n(pq)^{\frac{j-1}{2}} - n. \tag{2.16}
\]

Let us now consider the explicit formulas and upper bound estimates for \( \beta_j \) and \( b_j \) for all \( j \geq 1 \). Combining the definitions of these exponents with an odd number \( j \), we claim that

\[
\beta_j = pb_{j-1} + 1 = pq\beta_{j-2} + 2p + 1 = (2p + 1) + \sum_{k=0}^{(j-3)/2} (pq)^k + (pq)^{\frac{j-1}{2}} \beta_1
\]

\[
= \left(2p + 1 + \beta_1\right)(pq)^{\frac{j-1}{2}} - 2p + 1 = \left(2p + 1 + \frac{1}{pq-1}\right)(pq)^{\frac{j-1}{2}} - 2p + 1, \tag{2.17}
\]

and similarly,

\[
b_j = q\beta_{j-1} + 2 = pqb_{j-2} + q + 2 = (q + 2) + \sum_{k=0}^{(j-3)/2} (pq)^k + (pq)^{\frac{j-1}{2}} b_1
\]

\[
= \left(q + 2 + b_1\right)(pq)^{\frac{j-1}{2}} - q + 2 = \left(q + 2 + \frac{1}{pq-1}\right)(pq)^{\frac{j-1}{2}} - q + 2. \tag{2.18}
\]

In the case when \( j \) is an even number (i.e. \( j-1 \) is an odd number), by the formulas stated in (2.17) and (2.18), we may see from the definitions of \( \beta_j \) and \( b_j \) again that

\[
\beta_j = pb_{j-1} + 1 = q^{-1} \left(\frac{q + 2}{pq - 1} + 1\right) (pq)^{\frac{j-1}{2}} - 2p + 1, \tag{2.19}
\]

\[
b_j = q\beta_{j-1} + 2 = p^{-1} \left(\frac{2p + 1}{pq - 1} + n\right) (pq)^{\frac{j-1}{2}} - q + 2. \tag{2.20}
\]

Summarizing the derived representations for odd and even number \( j \geq 1 \), one obtains

\[
\beta_j \leq B_0(pq)^{\frac{j-1}{2}} \quad \text{and} \quad b_j \leq \bar{B}_0(pq)^{\frac{j-1}{2}} \quad \text{for odd number} \ j,
\]

\[
\beta_j \leq B_0(pq)^{\frac{j}{2}} \quad \text{and} \quad b_j \leq \bar{B}_0(pq)^{\frac{j}{2}} \quad \text{for even number} \ j,
\]

where \( B_0 = B_0(p, q, n) \) and \( \bar{B}_0 = \bar{B}_0(p, q, n) \) are positive and independent of \( j \) constants.

Our next aim is to derive some estimates for \( D_j \) and \( Q_j \) from the below. It is obviously that

\[
pb_{j-1} + 1 = \beta_j \leq B_0(pq)^{\frac{j}{2}}, \quad (q\beta_{j-1} + 1)(q\beta_{j-2} + 2) \leq (q\beta_{j-1} + 2)^2 = b_j^2 \leq \bar{B}_0^2(pq)^{\frac{j-1}{2}}.
\]
Moreover, it holds by the application of L'Hôpital's rule
\[
\lim_{j \to \infty} \ell_j^{p_{j-1} + 1} = \lim_{j \to \infty} \ell_j^{\beta_j} \leq \lim_{j \to \infty} \exp \left( B_0 (pq)^j / 2 \log \left( 1 + (pq)^{-j/2} \right) \right) = \exp \left( B_0 (pq)^{1/2} \right),
\]
thus, we may find a suitable constant \( M = M(n,p,q) \) such that \( \ell_j^{\beta_j} \geq M \) for any \( j \geq 1 \). Hence, we can give the following form for lower bounds:

\[
D_j = \frac{C_0 ((pq)^{1/2} - 1/2) (pq)^{-j+1}}{(pb_{j-1} + 1) \ell_j^{p_{j-1} + 1}} Q_{j-1}^p \geq \frac{C_0 M ((pq)^{1/2} - 1/2)}{B_0} (pq)^{-j + 1/2} Q_{j-1}^p,
\]
\[
Q_j = \frac{C_0}{(q \beta_{j-1} + 1)(q \beta_{j-1} + 2)} D_j^q \geq \frac{C_0}{B_0^q} (pq)^{-j} D_j^q.
\]
The derived inequalities immediately lead to
\[
D_j \geq \frac{C_0^{p+1} M ((pq)^{1/2} - 1/2)}{B_0^p B_0^p} (pq)^{-j - (j-1)p + 1} D_{j-2}^p := E_0 (pq)^{-j - (j-1)p + 1} D_{j-2}^p, \tag{2.19}
\]
\[
Q_j \geq \frac{C_0^{q+1} M^q ((pq)^{1/2} - 1/2)^q}{B_0^q B_0^q} (pq)^{-j - \frac{(j-1)q}{2} - (j-1)q} Q_{j-2}^q := E_0 (pq)^{-j - \frac{(j-1)q}{2} - (j-1)q} Q_{j-2}^q. \tag{2.20}
\]
Considering (2.19) with an odd number \( j \), we deduce
\[
\log D_j \geq pq \log D_{j-2} - \left( \frac{3}{2} + p \right) j - (p + 1) \log(pq) + \log E_0 \\
\geq (pq)^2 \log D_{j-4} - \left( \frac{3}{2} + p \right) (j + (j - 2)p) \log(pq) \\
+ (p + 1)(1 + pq) \log(pq) + (1 + pq) \log E_0 \\
\geq (pq)^{j-1} \log D_1 - \left( \frac{3}{2} + p \right) \log(pq) \sum_{k=1}^{(j-1)/2} ((j + 2 - 2k)(pq)^{k-1}) \\
+ (p + 1) \log(pq) \sum_{k=1}^{(j-1)/2} (pq)^{k-1} + \log E_0 \sum_{k=1}^{(j-1)/2} (pq)^{k-1}.
\]
By an inductive argument, the next formula can be derive
\[
\sum_{k=1}^{(j-1)/2} (j + 2 - 2k)(pq)^{k-1} = \frac{1}{pq - 1} \left( \frac{2pq}{pq - 1} \left( \frac{3}{2} (pq)^{j-1} - \frac{1}{2} (pq)^{j-3} - 1 \right) - j \right).
\]
One has
\[
\log D_j \geq (pq)^{j-1} \left( \log D_1 + \frac{\log(pq)}{2(pq - 1)^2} \left( 1 - 7pq - 4p^2 q \right) + \frac{\log E_0}{pq - 1} \right) \\
+ \frac{\log(pq)}{pq - 1} \left( \left( \frac{3}{2} + p \right) \left( \frac{2pq}{pq - 1} + j \right) - (p + 1) \right) - \frac{\log E_0}{pq - 1}.
\]
Thus, for an smallest nonnegative odd number satisfying
\[
j \geq j_0 := \frac{2(p + 1)}{3 + 2p} + \frac{2 \log E_0}{(3 + 2p) \log(pq)} - \frac{2pq}{pq - 1},
\]
the lower bound can be estimated by

$$\log D_j \geq (pq)^{j-1} \left( \log D_1 + \frac{\log(pq)}{2(pq-1)^2} \left( 1 - 7pq - 4p^2q + \frac{\log E_0}{pq-1} \right) \right)$$

$$= (pq)^{j-1} \log \left( D_1(pq)^{-(4p^2q+7pq-1)/(2(pq-1)^2)} E_0^{1/(pq-1)} \right)$$

$$= (pq)^{j-1} \log (E_1 \varepsilon^p)$$

(2.21)

for a suitable constant $E_1 = E_1(n,p,q)$. By the same way, we may show

$$\log Q_j \geq pq \log Q_{j-2} - \left( \left( 1 + \frac{3}{2}q \right) j - \frac{5}{2}q \right) \log(pq) + \log \tilde{E}_0$$

$$\geq (pq)^2 \log Q_{j-2} - \left( 1 + \frac{3}{2}q \right) (j + (j-2)pq) \log(pq)$$

$$+ \frac{5}{2}q(1 + pq) \log(pq) + (1 + pq) \log \tilde{E}_0$$

$$\geq (pq)^{j-1} \log Q_1 - \left( 1 + \frac{3}{2}q \right) \log(pq) \sum_{k=1}^{(j-1)/2} (j + 2 - 2k)(pq)^{k-1}$$

$$+ \frac{5}{2}q \log(pq) \sum_{k=1}^{(j-1)/2} (pq)^{k-1} + \log \tilde{E}_0 \sum_{k=1}^{(j-1)/2} (pq)^{k-1}.$$

As a consequence, it yields

$$\log Q_j \geq (pq)^{j-1} \left( \log Q_1 + \frac{\log(pq)}{(pq-1)^2} \left( -2pq^2 - 3pq - q + 1 \right) + \frac{\log \tilde{E}_0}{pq-1} \right)$$

$$+ \frac{\log(pq)}{pq-1} \left( \left( 1 + \frac{3}{2}q \right) \left( \frac{2pq}{pq-1} + j \right) - \frac{5}{2}q \right) - \log \tilde{E}_0.$$

If for an odd number $j$ we assume

$$j \geq j_1 := \frac{5q}{2 + 3q} + \frac{2 \log \tilde{E}_0}{(2 + 3q) \log(pq)} - \frac{2pq}{pq-1},$$

then the estimate holds

$$\log Q_j \geq (pq)^{j-1} \left( \log Q_1 + \frac{\log(pq)}{(pq-1)^2} \left( -2pq^2 - 3pq - q + 1 \right) + \frac{\log \tilde{E}_0}{pq-1} \right)$$

$$\geq (pq)^{j-1} \log(\tilde{E}_1 \varepsilon)$$

(2.22)

for a suitable constant $\tilde{E}_1 = \tilde{E}_1(n,p,q)$.

Let us denote

$$L := \lim_{j \to \infty} L_j = \prod_{j=1}^{\infty} \ell_j > 1.$$

Note that thanks to $\ell_j > 1$, it holds $L_j \uparrow L$ as $j \to \infty$. It leads that (2.11) and (2.12) hold for any odd number $j \geq 1$ and any $t \geq L$. 
Let us now consider an odd number \( j \) such that \( j \geq \max \{ j_0, j_1 \} \). Combining with (2.11), (2.15), (2.17) and (2.21), we may observe that

\[
U(t) \geq \exp \left( (pq) \frac{j-1}{2} \log(E_1 \varepsilon^p) \right) (R + t)^{-\alpha_j - \beta_j (t - L)^{\beta_j}}
\]

\[
= \exp \left( (pq) \frac{j-1}{2} \left( \log(E_1 \varepsilon^p) - \left( \frac{n-1}{2} + \frac{n}{2} \right) \log(R + t) + \left( \frac{2p+1}{pq-1} + \frac{n}{2} \right) \log(t - L) \right) \right)
\]

\[
\times (R + t)^{n(t - L)^{-\frac{(2p+1)}{(pq-1)}}}
\]

for any odd number \( j \geq \max \{ j_0, j_1 \} \) and any \( t \geq L \). Considering \( t \geq \{ R, 2L \} \), since \( R + t \leq 2t \) and \( t - L \geq t/2 \), we have the lower bound estimate for the functional \( U \) as follows:

\[
U(t) \geq \exp \left( (pq) \frac{j-1}{2} \log \left( E_1 \varepsilon^p 2^{-\left( \frac{n-1}{2} + \frac{2p+1}{pq-1} \right)} (t - L)^{\frac{1}{pq-1}} \right) \right) (R + t)^{n(t - L)^{-\frac{(2p+1)}{(pq-1)}}}
\]

(2.23)

for any odd number \( j \geq \max \{ j_0, j_1 \} \). The exponent for \( t \) in (2.23) can be rewritten by

\[
-n + \left( \frac{q+2}{pq-1} + 1 \right) = q \left( \frac{1 + 2q^{-1} - \frac{n-1}{q}}{pq-1} \right) = qF_2(n, p, q),
\]

where \( F_1(n, p, q) \) is defined in (1.11). By our assumption (1.9), i.e. \( F_1(n, p, q) > 0 \), the exponent for \( t \) in the exponential term of (2.23) is positive.

In a similar way as the above, we may derive the next inequality for an odd number \( j \) fulfilling \( j \geq \max \{ j_0, j_1 \} \):

\[
V(t) \geq \exp \left( (pq) \frac{j-1}{2} \log \left( E_1 \varepsilon^p 2^{-\left( \frac{n-1}{2} + \frac{2p+1}{pq-1} \right)} (t + R)^{n(t - L)^{-\frac{(q+2)}{(pq-1)}}} \right) \right) (t + R)^{n(t - L)^{-\frac{(q+2)}{(pq-1)}}}
\]

(2.24)

for any \( j \geq \max \{ j_0, j_1 \} \) and any \( t \geq L \). Recalling the definition (1.12), the exponent for \( t \) in (2.24) is

\[
-n + \frac{q+2}{pq-1} + 1 = q \left( \frac{1 + 2q^{-1} - \frac{n-1}{q}}{pq-1} \right) = qF_2(n, p, q).
\]

Considering \( F_2(n, p, q) > 0 \) coming from our assumption (1.9), the exponent for \( t \) in the exponential term of (2.24) is positive.

In the case when \( F_1(n, p, q) > 0 \), we set \( \varepsilon_0 = \varepsilon_0(u_0, u_1, v_0, v_1, n, p, q, R) > 0 \) such that

\[
E^{-1} \left( \frac{n-1}{2} + \frac{2p+1}{pq-1} \right) ^{1/(pF_1(n, p, q))} := E_2 \geq \varepsilon_0^{1/F_1(n, p, q)}.
\]

Therefore, for \( \varepsilon \in (0, \varepsilon_0) \) and \( t > E_2 \varepsilon^{-1/F_1(n, p, q)} \) carrying \( t \geq \max \{ R, 2L \} \), letting \( j \to \infty \) in (2.23), we may conclude that the lower bound for \( U \) blows up. Similarly, in the remaining case when \( F_2(n, p, q) > 0 \), then we can find a positive constant \( \varepsilon_0 = \varepsilon_0(u_0, u_1, v_0, v_1, n, p, q, R) > 0 \) such that

\[
E^{-1} \left( \frac{n+q+2}{pq-1} + 1 \right) ^{1/(qF_2(n, p, q))} := \tilde{E}_2 \geq \varepsilon_0^{1/F_2(n, p, q)}.
\]

Therefore, for \( \varepsilon \in (0, \varepsilon_0) \) and \( t > \tilde{E}_2 \varepsilon^{-1/F_2(n, p, q)} \) carrying \( t \geq \max \{ R, 2L \} \), letting \( j \to \infty \) in (2.24), we may conclude that the lower bound for \( V \) blows up. In conclusion, these statements proved that the energy solution \( (u, v) \) is not globally in time defined and, simultaneously, the lifespan of local (in time) of \( (u, v) \) can be estimated by

\[
T(\varepsilon) \leq C \varepsilon^{-1/\max\{F_1(n, p, q), F_2(n, p, q)\}}.
\]

The proof of the proposition is complete.
3 Proof of Proposition 1.2

In this section, we will sketch the proof of Proposition 1.2 by using the same techniques as shown in those for Proposition 1.1. Nevertheless, we now may provide another lower bound estimates for the functionals $U(t)$ and $V(t)$, which are defined in the previous section.

To begin with, from (2.3) and (2.8), we know there exist positive constants $\tilde{C}_0$ and $\tilde{C}_1$ relaying on initial data such that

$$U(t) \geq \tilde{C}_0 \varepsilon \quad \text{and} \quad V(t) \geq \tilde{C}_1 \varepsilon t,$$

where the nonnegative and nontrivial assumptions on $u_0$ and $u_1$ were used, respectively. Consequently, by employing Hölder’s inequality, supports for solution and the above estimates, we may derive

$$\int_{\mathbb{R}^n} |u(t, x)|^q \, dx \geq \tilde{C}_2(R + t)^{-n(q-1)}U(t)^q \geq \tilde{C}_0^q \tilde{C}_2 \varepsilon^q(R + t)^{-n(q-1)},$$

$$\int_{\mathbb{R}^n} |v(t, x)|^p \, dx \geq \tilde{C}_3(R + t)^{-n(p-1)}V(t)^p \geq \tilde{C}_1^p \tilde{C}_3 \varepsilon^p(R + t)^{-n(p-1)} t^p,$$

where $\tilde{C}_2, \tilde{C}_3 > 0$ are suitable constants depending on $n, p, q, R$. They lead that from the estimates (2.4) and (2.5) as follows:

$$U(t) \geq \tilde{C}_1^p \tilde{C}_3 \varepsilon^p \int_0^t e^{\tau - t} \int_0^\tau (R + s)^{-n(p-1)} s^p \, ds \, d\tau \geq \frac{\tilde{C}_1^p \tilde{C}_3 (1 - e^{-1/2}) \varepsilon^p}{(p + 1)2^{p+1}}(R + t)^{-n(p-1)} t^{p+1},$$

$$V(t) \geq \tilde{C}_0^q \tilde{C}_2 \varepsilon^q \int_0^t \int_0^\tau (R + s)^{-n(q-1)} \, ds \, d\tau \geq \frac{\tilde{C}_0^q \tilde{C}_2 \varepsilon^q}{2}(R + t)^{-n(q-1)} t^2,$$

for any $t \geq 1$. Here, we should underline that the improvement for the lower bound estimates of the functionals in comparison with (2.7) and (2.8) for large time. The lower bound estimate for the functional $U(t)$ is improved in the case when $n = 1$ and $n = 2$ with $1 < p < 2$, moreover, the lower bound estimate for the functional $V(t)$ is improved in the case when $1 < q < 1 + 1/n$.

In other words, the first lower bound estimates (2.9), (2.10) hold, providing that the multiplicative constants satisfy

$$D_1 := \frac{\tilde{C}_1^p \tilde{C}_3 (1 - e^{-1/2}) \varepsilon^p}{(p + 1)2^{p+1}}, \quad Q_1 := \frac{\tilde{C}_0^q \tilde{C}_2 \varepsilon^q}{2}$$

and the exponents fulfill

$$\alpha_1 := n(p - 1), \quad a_1 := n(q - 1), \quad \beta_1 := p + 1, \quad b_1 := 2.$$

Then, following the same approach as those for Theorem 1.1, one may derive blow-up of solutions when a pair of exponent $(p, q)$ satisfies

$$-(n + \alpha_1) + \frac{2p + 1}{pq - 1} + \beta_1 = p \left( \frac{2 + q}{pq - 1} - (n - 1) \right) = pF_3(n, p, q) > 0,$$

or

$$-(n + a_1) + \frac{q + 2}{pq - 1} + b_1 = q \left( \frac{1 + 2p}{pq - 1} - n \right) = qF_4(n, p, q) > 0,$$
where the functions $F_3(n, p, q)$ and $F_4(n, p, q)$ are defined in (1.14) and (1.15), respectively, moreover, we use our assumption (1.13) to guarantee the mentioned functions are positive. Furthermore, the lifespan of local (in time) of $(u, v)$ can be estimated by

$$T(\varepsilon) \leq C\varepsilon^{-1/\max\{F_3(n,p,q), F_4(n,p,q)\}}.$$ 

The proof is complete.

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