Vanishing theorem and finiteness theorem for $p$-harmonic 1 form

XiangZhi Cao

Contents

1 Introduction 1
2 Preliminary 3
3 Liouville type theorem for $p$ harmonic 1 form 4
4 Finiteness Theorem 12

Abstract

In this paper, we will show vanishing theorem of $p$ harmonic 1 form on submanifold $M$ in $\bar{M}$ whose BiRic curvature satisfying $\text{BiRic}^\alpha \geq \Phi_a(H, S)$. As an corollary, we can get the corresponding theorem for $p$ harmonic function and $p$ harmonic map. We also investigate the finiteness problem of $p$ harmonic 1 form on submanifold $M$ in $\bar{M}$ whose BiRic curvature satisfying $\text{BiRic}^\alpha \geq -k^2$.

1 Introduction

Let $(M, g)$ be an Riemannian manifold. A vector bundled valued differential $k$ form $\omega$ is called $p$ harmonic $k$ form if it satisfy

$$d\omega = 0, \delta(|\omega|^{p-2}\omega) = 0.$$
When \( p = 2 \), it is reduced to harmonic \( k \) form. Liouville type property and finiteness of the vector space for differential form are two important topics in the research of differential forms.

Han [18] studied obtain some vanishing and finiteness theorems for \( L^p \) \( p \)-harmonic 1-forms on a locally conformally flat Riemmannian manifolds. In [15], Han studied Liouville theorem for \( p \) harmonic 1 form on submanifold in sphere. In [14], Han proved the finiteness theorem of the space \( p \) harmonic one form on submanifold in Hadmard manifold if the first eigenvalue satisfies is bounded below by some constants. In [8], Dung showed some vanishing type theorems for \( p \)-harmonic \( l \)-forms on such a manifold with a weighted Poincar inequality. Motivated by [8], in [7], Chao et al. investigated \( p \)-harmonic \( l \)-forms on Riemannian manifolds with a weighted Poincar inequality, and we get a vanishing type theorem. In [1], Afuni studied the monotonicity of vector bundle valued harmonic form. In [25], Zhang used the Moser iteration to obtain the vanishing theorem for \( p \) harmonic 1 form on manifold with nonnegative Ricci curvature.

The energy functional of \( p \) harmonic map is defined by

\[
E_p(u) = \int_M \frac{\left| \nabla u \right|^p}{2} dv_g,
\]

whose Euler-lagrange equation is as follows:

\[
\tau_p(u) = \text{div}(\left| du \right|^{p-2} du),
\tag{1.1}
\]

A map \( u \) is called \( p \)-harmonic map if \( \tau_p(u) = \text{div}(\left| du \right|^{p-2} du) = 0 \). Motivated by [23], in [4], we studied the Liouville theorem of \( p \) harmonic map with finite energy from complete noncompact submanifold in partially nonnegative curved manifold into nonpositive curved manifold, the conditions in our theorem is the index of the operator \( \Delta + \frac{1}{4}(S - nH^2) \) is zero or the small condition on \( \| S - nH^2 \|_2 ^2 \). In [16], Han obtained liouville type theorem for \( p \) harmonic function on submanifold in sphere. Once can refer to [2, 6, 10, 20, 21, 22, 24, 26] and reference therein for the researches on \( p \) harmonic map and \( p \) harmonic function.

In [12], the authors defined the tensor

\[
\overline{\text{BiRic}}(u, v) = \text{Ric}(u, u) + a \text{Ric}(v, v) - K(u, v),
\]

and obtained vanishing theorem of harmonic one form on submanifold \( M \) in \( \overline{M} \) whose BiRic curvature satisfies \( \overline{\text{BiRic}} \geq \Phi_a(H, S) \). They also proved that the space \( L^{2p} \) harmonic 1 forms on \( M \) is finite if \( \overline{\text{BiRic}} \geq -k^2 \), provided the first eigenvalue is bounded below by a suitable constant and \( p \) shall satisfy some conditions.

Motivated by the conditions in [12], in this paper, we will establish similar Theorem as [12, Theorem 1.1, Theorem 4.4] for \( p \) harmonic 1 form on submanifold \( M \) in \( \overline{M} \) whose BiRic
curvature satisfies $\text{BiRic} \geq \Phi_a(H, S)$. As an corollary, we can get the corresponding theorem for $p$ harmonic function and $p$ harmonic map.

2 Preliminary

Before stating our proof, we give some important formulas, definitions and some lemmas. We say a map $u$ is of $L^q$-finite energy if $\int_M |\nabla u|^q < \infty$. For $p$-harmonic maps, we have the Bochner formula (c.f. Lemma 2.4 in [4])

$$\frac{1}{2} \Delta |du|^{2p-2} = |\nabla |du|^{p-2}du|^2 - \langle |du|^{p-2}du, \Delta |du|^{p-2}du \rangle + |du|^{2p-4} \langle du(\text{Ric}^M(e_k), du(e_k) \rangle - |du|^{2p-4} R^N(du(e_i), du(e_k)),$$

(2.1)

Definition 1. Space $H^{k,p}(L^p(M)) = \{ \omega \in A^k(M) : d\omega = 0; \delta d(|\omega|^{p-2}\omega) = 0; \int_{B_R} |\omega|^p = o(R^\gamma); 0 < \gamma < 2, \forall R > 0. \}$

Lemma 2.1 (c.f. Lemma 2.7 in [17]). For any section $\omega \in \Gamma(A^p(M))$ which satisfies $d\omega = 0$ and any function $f$ on $M$, we have

$$|d(f\omega)| \leq |df||\omega|.$$

(2.2)

Let

$$A_{p,n,q} = \begin{cases} \frac{1}{\max\{q,n-q\}} & \text{if } p = 2 \\ \frac{1}{(p-1)^{n}} \min\{1, \frac{(p-1)^2}{n-1}\} & \text{if } p > 2 \text{ and } q = 1 \\ 0 & \text{if } p > 2 \text{ and } 1 < q \leq n - 1 \end{cases}$$

We can see that $A_{2,n,1} = \frac{1}{n-1}$ if $n \geq 2$. Hereafter, we denote $A_{p,n,1}$ by $A_{p,n}$.

Lemma 2.2 (Kato’s inequality, c.f. Lemma 2.2 in [11]). For $p \geq 2, q \geq 1$, let $\omega$ be an $p$-harmonic $q$-form on a complete Riemannian manifold $M^n$. The following inequality holds

$$|\nabla (|\omega|^{p-2}\omega)|^2 \geq (1 + A_{p,n,q})|\nabla |\omega|^{p-1}|^2.$$

Moreover, when $p = 2, q > 1$ then the equality holds if and only if there exists a 1-form $\alpha$ such that

$$\nabla \omega = \alpha \otimes \omega - \frac{1}{\sqrt{q+1}} \theta_1(\alpha \wedge \omega) + \frac{1}{\sqrt{n+1-q}} \theta_2(i_\alpha \omega).$$
3 Liouville type theorem for $p$ harmonic 1 form

Lemma 3.1 (c.f. [12]). Let $M^n$ be an immersed hypersurface in a Riemannian manifold $M^{n+1}$. Let $\text{Ric}, S, H$ denote the functions that assign to each point $p$ of $M$ the Ricci curvature, the square length of the second fundamental form, and the mean curvature respectively of $M$ at $p$, then for any tangent vector $X \in T_pM$, we have

$$\text{Ric}(X, X) \geq \left( \text{BiRic} - \Phi_a(H, S) - a(\text{Ric}(N, N) + S) \right) |X|^2,$$

where

$$\Phi_a(H, S) = \left( \frac{n-1}{n} - a \right) S - \frac{1}{n^2} \left\{ 2(n-1)H^2 - (n-2)H \sqrt{(n-1)(nS-H^2)} \right\}.$$

We generalize Theorem 3.1 in [12].

Theorem 3.2. Let $M^m (m \geq 2)$ be a complete, noncompact, connected, oriented, and stable hypersurface immersed in a Riemann manifold $\tilde{M}$. For any $p \geq 2$, if $\text{BiRic} \geq \Phi_a(H, S)$, for some positive constant $a$ satisfying

$$a < \frac{4}{(m-1)p^2} + \frac{4(p-1)}{p^2},$$

then there does not exist any nontrivial $L^p$ $p$-harmonic 1-form on $M$.

Proof. Without generality, we can assume $M - \{\omega(x) = 0, \text{for } \forall x \in M \} \neq \emptyset$. The proof below proceeds on $M^+ = M - \{x \in M, \omega(x) = 0\}$. Considering the integral on $M$ is identical to that of $M$, we prefer to integrate function about $s$ on $M$ in the subsequent computations.

According to Bochner formula for $p$ harmonpnic 1 form[18]:

$$\frac{1}{2}\Delta |\omega|^{2(p-1)} = |\nabla (|\omega|^{p-2}\omega)|^2 - \langle \delta d(|\omega|^{p-2}\omega), |\omega|^{p-2}\omega \rangle + |\omega|^{2p-4}\text{Ric}(\omega, \omega). \quad (3.1)$$

On the other hand,

$$\frac{1}{2}\Delta |\omega|^{2(p-1)} = |\omega|^{p-1}\Delta |\omega|^{p-1} + |\nabla |\omega|^{p-1}|^2. \quad (3.2)$$

Thus we have (c.f. [18, (3)])

$$|\omega|^{p-1}\Delta |\omega|^{p-1} = |\nabla (|\omega|^{p-2}\omega)|^2 - |\nabla |\omega|^{p-1}|^2 - \langle \delta d(|\omega|^{p-2}\omega), |\omega|^{p-2}\omega \rangle + |\omega|^{2p-4}\text{Ric}(\omega, \omega) \geq \frac{1}{(m-1)(p-1)^2} |\nabla |\omega|^{p-1}|^2 - \langle \delta d(|\omega|^{p-2}\omega), |\omega|^{p-2}\omega \rangle + |\omega|^{2p-4}\text{Ric}(\omega, \omega),$$

where we have used Kato inequality Ln lemma 2.3 in [17] in the first inequality.
Thus, we have
\[
|\omega|\Delta|\omega|^{p-1} \geq \frac{4}{(m-1)p^2} |\nabla|\omega|^{2}\|^2 - \left\langle \delta d(|\omega|^{p-2}\omega), \omega \right\rangle \\
+ |\omega|^p \left( \text{BiRic}^a \left( \frac{X}{|X|}, N \right) - \Phi_a(H, S) - a(\overline{\text{Ric}}(N, N) + S) \right).
\]

(3.3)

It is well known that we can choose cutoff function \( \phi \) on noncompact manifold \( M \) such that
\[
\begin{cases}
0 \leq \phi \leq 1 \\
\phi = 1, \text{ on } B_R(x_0) \\
\phi = 0, \text{ on } M - B_{2R}(x_0), \\
|\nabla \phi| < \frac{C}{R},
\end{cases}
\]

(3.4)

Multiplying both sides by \( \phi^2 \), and integrating over \( M \), we have
\[
\int_M \phi^2 |\omega|\Delta |\omega|^{p-1} \geq \int_M \phi^2 \frac{4}{(m-1)p^2} |\nabla|\omega|^{2}\|^2 - \int_M \phi^2 \left\langle \delta d(|\omega|^{p-2}\omega), \omega \right\rangle \\
+ \int_M \phi^2 |\omega|^p \left( \text{BiRic}^a \left( \frac{X}{|X|}, N \right) - \Phi_a(H, S) - a(\overline{\text{Ric}}(N, N) + S) \right).
\]

which can be rearranged as
\[
-\int_M \phi^2 \nabla |\omega| \nabla |\omega|^{p-1} - 2 \int_M \phi \nabla |\omega| \nabla |\omega|^{p-1} \\
\geq \int_M \phi^2 \frac{4}{(m-1)p^2} |\nabla|\omega|^{2}\|^2 - \int_M \phi^2 \left\langle \delta d(|\omega|^{p-2}\omega), \omega \right\rangle \\
+ \int_M \phi^2 |\omega|^p \left( \text{BiRic}^a \left( \frac{X}{|X|}, N \right) - \Phi_a(H, S) - aq \right).
\]

where \( q = (\overline{\text{Ric}}(N, N) + S) \).

Next, we need to deal with the last two terms on the right hand side of (3.5). By the stable condition, we have
\[
\int_M q\varphi^2 |\omega|^p \leq \int_M |\nabla \varphi|^2 |\omega|^p + \frac{p^2}{4} \int_M \varphi^2 |\omega|^{p-2} |\nabla|\omega|^2 + p \int_M \varphi |\omega|^{p-1} \langle \nabla |\omega|, \nabla \varphi \rangle.
\]

Thus, we have
\[
\frac{4}{(m-1)p^2} \int_M \phi^2 |\nabla|\omega|^{2}\|^2 \\
\leq -\int_M \phi^2 \nabla |\omega| \nabla |\omega|^{p-1} - 2 \int_M \phi |\omega|^{p-1} \nabla |\omega| \nabla \phi \\
+ \int_M \phi^2 \left\langle \delta d(|\omega|^{p-2}\omega), \omega \right\rangle \\
+ a \left[ \int_M |\nabla \varphi|^2 |\omega|^p + \frac{p^2}{4} \int_M \varphi^2 |\omega|^{p-2} |\nabla|\omega|^2 + p \int_M \varphi |\omega|^{p-1} \langle \nabla |\omega|, \nabla \varphi \rangle \right].
\]

(3.5)
Notice that $\phi^2 \nabla |\omega| \nabla |\omega|^{p-1} = \frac{4(p-1)}{p^2} |\nabla |\omega|^{\frac{p}{2}}|^2$, thus we get

$$
\frac{4}{(m - 1)p^2} \int_M \phi^2 |\nabla |\omega|^{\frac{p}{2}}|^2 \leq -\int_M \phi^2 \frac{4(p - 1)}{p^2} |\nabla |\omega|^{\frac{p}{2}}|^2 + (ap - 2) \int_M \phi |\omega|^{p-1} \nabla |\omega| \nabla \phi \\
+ \int_M \phi^2 \langle \delta d(|\omega|^{p-2} \omega), \omega \rangle \\
+ a \left[ \int_M |\nabla \phi|^2 |\omega|^p + \int_M \phi^2 |\nabla |\omega|^{\frac{p}{2}}|^2 \right].
$$

(3.6)

Now we need to estimate some terms on the right hand side of (3.6). By [17, (9)], we know that

$$
\int_M \left\langle d(|\omega|^{p-2} \omega), d\omega \right\rangle \leq \frac{4p - 2}{p} \int_M \left| \nabla |\omega|^{\frac{p}{2}} \phi |\nabla \phi | |\omega|^{\frac{p}{2}} \right|
$$

Using Young’s inequality, we get

$$
\frac{4}{(m - 1)p^2} \int_M \phi^2 |\nabla |\omega|^{\frac{p}{2}}|^2 + \int_M \phi^2 \frac{4(p - 1)}{p^2} |\nabla |\omega|^{\frac{p}{2}}|^2 \\
+ \int_M \phi^2 |\omega|^p \left( \overline{\text{BiRic}^2} \left( \frac{X}{|X|}, N \right) - \Phi_a (H, S) \right) \\
\leq \int_M \frac{(2 - ap)^2}{4\epsilon^2} |\nabla \phi|^2 |\omega|^p + \int_M \frac{4}{p^2} \phi^2 |\nabla |\omega|^{\frac{p}{2}}|^2 \\
+ \left[ (a + \frac{(p - 2)^2}{4\epsilon_3(p^2)} 16) \int_M |\nabla \phi|^2 |\omega|^p + (a + \epsilon_3) \int_M \phi^2 |\nabla |\omega|^{\frac{p}{2}}|^2 \right].
$$

(3.7)

which can be written as

$$
\int_M \phi^2 |\omega|^p \left( \overline{\text{BiRic}^a} \left( \frac{X}{|X|}, N \right) - \Phi_a (H, S) \right) \\
\leq \left( \epsilon \frac{4}{p^2} + (a + \epsilon_3) - \frac{4}{(m - 1)p^2} - \frac{4(p - 1)}{p^2} \right) \int_M \phi^2 |\nabla |\omega|^{\frac{p}{2}}|^2 \\
+ \left[ (a + \frac{(p - 2)^2}{4\epsilon_3(p^2)} 16) + \frac{(2 - ap)^2}{4\epsilon^2} \right] \int_M |\nabla \phi|^2 |\omega|^p.
$$

(3.8)

Noticing that $\overline{\text{BiRic}^a} \left( \frac{X}{|X|}, N \right) \geq \Phi_a (H, S)$, let $R \to \infty$, we have

$$
0 \leq \left( \epsilon \frac{4}{p^2} + (a + \epsilon_3) - \frac{4}{(m - 1)p^2} - \frac{4(p - 1)}{p^2} \right) \int_M \phi^2 |\nabla |\omega|^{\frac{p}{2}}|^2.
$$

Since $a - \frac{4}{(m - 1)p^2} - \frac{4(p - 1)}{p^2} < 0$, we can choose small $\epsilon, \epsilon_3$ such that $\epsilon \frac{4}{p^2} + (a + \epsilon_3) - \frac{4}{(m - 1)p^2} - \frac{4(p - 1)}{p^2} < 0$. so, $|\omega|$ is a constant, all the inequalities in (3.1) and (3.3) becomes equality. Suppose that $|\omega| \neq 0$, by (3.1) and (3.3), we have

$$
\text{Ric}(\omega^*, \omega^*) = 0.
$$
then, as in [12], we see that for any unit tangent vector \( X \),
\[
\text{Ric}(X, X) \geq 0.
\]
Then \( \text{Vol}(M) = \infty \), this gives a contradiction to that \( |\omega| \in L^p \).

\[
\square
\]

Similar to [12, Corollary 3.3], we can establish

**Corollary 3.3.** Let \( M^n \) be a complete, noncompact, stable, and minimal hypersurface immersed in a Riemannian manifold with non-negative \( \text{BiRic} \) curvature for some
\[
\frac{n - 1}{n} \leq a < \frac{4}{(n - 1)p^2} + \frac{4(p - 1)}{p^2},
\]
then there does not exist any nontrivial \( L^p \) p-harmonic 1-form on \( M \).

**Proof.** By the argument before [12, Corollary 3.3], we know that if \( H = 0 \), then \( \Phi_a(H, S) = \left( \frac{n - 1}{n} - a \right)S \)

\[
\square
\]

Similar to [12, Corollary 3.4], we can obtain

**Corollary 3.4.** Let \( M^n \) be a complete, noncompact, stable immersed hypersurface in a Riemannian manifold \( \tilde{M} \). If one of the following conditions holds, then there is no nontrivial \( L^p \) harmonic 1-form on \( M \):

1. \( \text{BiRic} \geq \frac{n - 5}{4}H^2 \) and \( p < 2 + \frac{2}{\sqrt{n - 1}} \);
2. \( \text{BiRic} \geq \Phi_1(H, S) \) and \( p < 2 + \frac{2}{\sqrt{n - 1}} \);
3. \( \text{BiRic} \geq 0, 2 \leq n \leq 5 \) and \( p < 2 + \frac{2}{\sqrt{n - 1}} \);
4. \( \text{BiRic} \geq 0, \frac{n - 1}{2} \leq a < \frac{4}{(n - 1)p^2} + \frac{4(p - 1)}{p^2} \),
then there does not exist any nontrivial \( L^p \) p-harmonic 1-form on \( M \).

**Proof.** By [12, (3.7)], we know that
\[
\Phi_1(H, S) \leq \frac{n - 5}{4}H^2.
\]
Then in Theorem 3.2, take \( a = 1 \), we can finish the proof as in [12, Corollary 3.4]. The statement follows from the fact that \( \Phi_a(H, S) \leq \left( \frac{n - 1}{2} - a \right)S \).

\[
\square
\]

Similar to [12, Corollary 3.5], we can obtain

**Corollary 3.5.** Let \( M^n, (n \geq 3) \) be a complete, compact, stable immersed hypersurface in a Riemannian manifold \( \tilde{M} \). Suppose that one of the following conditions holds
(1) $\text{BiRic}^{a} \geq \frac{n-5}{4} H^2$ and $a = 1, p < 2 + \frac{2}{\sqrt{n-1}}$;

(2) $\text{BiRic}^{a} \geq 0, \sqrt{n-1} \leq a < \frac{4(p-1)^2}{(m-1)p^2}.$

If $M$ admits a nontrivial $p$-harmonic 1-form $\omega$, then $\omega$ is parallel and $M$ has at least $n-1$ principal curvatures which are equal. Moreover, if $H = 0$ then $M$ is totally geodesic.

Proof. Follow the proof of [12, Corollary 3.5], we have

$$0 \leq \left( \epsilon \frac{4}{p^2} + (a + \epsilon_3) - \frac{4}{(m-1)p^2} - \frac{4(p-1)}{p^2} \right) \int_M |\nabla|\omega|^{p-2}|^2.$$ 

The rest proof is the same as that of [12, Corollary 3.5].

Corollary 3.6. Let $M^m(m \geq 2)$ be an $m$-be a complete, noncompact, connected, oriented, and stable hypersurface immersed in a Riemann manifold $\bar{M}$. For any $p > 0$, if $\text{BiRic}^{a} \geq \Phi_a(H, S)$, for some positive constant $a$ satisfying

$$a < \frac{4(p-1)^2}{(m-1)p^2} + \frac{4(p-1)}{p^2}.$$ 

Let $u$ be an $p$-harmonic function on $M$. For $p \geq 2$, if $u$ has finite $p$-energy, then $u$ must be a constant map.

Remark 3.7. One can use the conditions and methods of [10, Theorem 2.3] to conclude that every $p$ harmonic function $u$ with finite $L^{2\beta}$ energy is constant provided $\beta$ satisfying some certain conditions.

Proof. First, we recall the Bochner formula for $p$ harmonic function (c.f. [4] or [16])

$$\frac{1}{2}\Delta |du|^{p-2} = |\nabla (|du|^{p-2}du)|^2 - \langle |du|^{p-2}du, \Delta (|du|^{p-2}du) \rangle$$

$$+ |du|^{p-4}\langle du(\text{Ric}^M(e_k), du(e_k)) \rangle,$$  

the following Kato inequality for $p$-harmonic function (cf. lemma 2.4 in [16]):

$$\nabla (|du|^{p-2}du)|^2 \geq \frac{n}{n-1} |\nabla|du|^{p-1}|^2.$$  

In this case, (3.3) can be replaced by

$$|\omega|\Delta|\omega|^{p-1} \geq \frac{4(p-1)^2}{(m-1)p^2} |\nabla|\omega|^{p-2}|^2 - \left\langle \delta d(|\omega|^{p-2}\omega), \omega \right\rangle$$

$$+ |\omega|^p \left( \text{BiRic}^{a} \left( \frac{X}{|X|}, N \right) - \Phi_a(H, S) - a(\text{Ric}(N, N) + S) \right).$$ 

8
The rest proof is almost the same as in the proof of Theorem 3.2 below (3.3), once we replace \( \frac{4}{(m-1)p^2} \) by \( \frac{4(p-1)^2}{(m-1)p^2} \).

\[ \square \]

**Corollary 3.8.** Let \( M^n(n \geq 2) \) be an \( m \)-be a complete, noncompact, connected, oriented, and stable hypersurface immersed in a Riemann manifold \( \bar{M} \). For any \( p > 0 \), if \( \text{BiRic} \geq \Phi_a(H, S) \), for some positive constant \( a \) satisfying

\[ a < \frac{4(p-1)}{p^2} \]

Let \( u : (M, g, dv_g) \to (N, h) \) be an \( p \)-harmonic map from an oriented complete noncompact manifold into a Riemannian manifold and \( K^N \leq 0 \). For \( p \geq 2 \), if \( u \) has finite \( p \)-energy, then \( u \) must be a constant map.

**Proof.** It is well known that (c.f. [4])

\[
\frac{1}{2} \Delta |\omega|^{2p-2} = |\nabla (|\omega|^{p-2} \omega)|^2 - \langle |\omega|^{p-2} \omega, \Delta (|\omega|^{p-2} \omega) \rangle \\
+ |\omega|^{2p-4} \langle \text{Ric}^M(e_k), du(e_k) \rangle + |\omega|^{2p-4} \langle R^N(du(e_i), du(e_j), du(e_i), du(e_j)) \rangle,
\]

(3.12)

However, for \( p \) harmonic map, we only have (3)

\[ |\nabla (|\omega|^{p-2} \omega)|^2 \geq |\nabla |\omega|^{p-1}|^2. \]

In this case, in the proof of Theorem 3.2,(3.3) can be replaced by

\[
|\omega| \Delta |\omega|^{p-1} \geq - \left( \delta d(|\omega|^{p-2} \omega), \omega \right) \\
+ |\omega|^p \left( \text{BiRic}^a \left( \frac{X}{|X|}, N \right) - \Phi_a(H, S) - a(\text{Ric}(N, N) + S) \right).
\]

(3.13)

The rest proof is almost the same as in the proof of Theorem 3.2 below (3.3), once we replace \( \frac{4}{(m-1)p^2} \) by zero .

\[ \square \]

**Definition 2** (c.f. Definition 3.6 in [12]). An immersed hypersurface \( M^n \) in a Riemannian manifold \( M^{n+1} \) is said to have a Sobolev inequality if there exists a positive constant \( C_s \) such that

\[
\left( \int_M f^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} \leq C_s \int_M |\nabla f|^2
\]

(3.14)

for any nonnegative \( C^1 \)-functions \( f : M \to \mathbb{R} \) with compact support. Here, \( C_s \) is said to be the Sobolev constant.
Theorem 3.9. Let $M^n(n \geq 2)$ be an $n$-dimensional a complete, noncompact, connected, oriented, and stable minimal hypersurface immersed in a Riemann manifold $\bar{M}$. Assume that a Sobolev inequality holds on $M$. For any $p \geq 2$, if $\text{BiRic}^a \geq 0$, for some positive constant $a$ satisfying
\[
a < \min \left\{ \frac{4}{(n-1)p^2} + \frac{4(p-1)}{p^2}, \frac{n-1}{n} \right\},
\]
and
\[
\|S\|_\frac{n}{2} \leq \frac{4}{(n-1)p^2} + \frac{4(p-1)}{p^2} - a \cdot \frac{C_s(\frac{n-1}{n} - a)}{C_s}, \tag{3.15}
\]
where $C_s$ is the constant in (3.14), then there does not exist any nontrivial $L^p$ $p$-harmonic 1-form on $M$.

Proof. From (3.8) in the proof of Theorem 3.2, it is easy to see that
\[
\left( \epsilon \frac{4}{p^2} + (a + \epsilon_3) - \frac{4}{(n-1)p^2} - \frac{4(p-1)}{p^2} \right) \int_M \phi^2 |\nabla |\omega|^2|^2
\]
\[
+ \left[ (a + (p-2)^2) \frac{16}{4\epsilon_3(p^2)} + \frac{(2-2p)^2}{4\epsilon} \right] \int_M |\nabla \varphi|^2 |\omega|^p
\]
\[
\geq \int_M \phi^2 |\omega|^p \left( \text{BiRic}^a \left( \frac{X}{|X|}, N \right) - \Phi_a(H, S) \right)
\]
\[
\geq - C_{n,a} \int_M S \phi^2 |\omega|^p.
\]
By [12], we have
\[
\int_M S \phi^2 |\omega|^p \leq C_s |S|_{n/2} \left( \left( 1 + \frac{1}{\epsilon} \right) \int_M |\nabla \varphi|^2 |\omega|^p + \frac{(1+\epsilon)p^2}{4} \int_M \phi^2 |\omega|^{p-2} |\nabla |\omega|^2 \right).
\]
Thus ,we have
\[
0 \leq \left( \epsilon \frac{4}{p^2} + (a + \epsilon_3) - \frac{4}{(n-1)p^2} - \frac{4(p-1)}{p^2} + C_{n,a}C_s |S|_{n/2} (1 + \epsilon) \right) \int_M \phi^2 |\nabla |\omega|^2|^2
\]
\[
+ \left[ (a + (p-2)^2) \frac{16}{4\epsilon_3(p^2)} + \frac{(2-2p)^2}{4\epsilon} + C_{n,a}C_s |S|_{n/2} \left( 1 + \frac{1}{\epsilon} \right) \right] \int_M |\nabla \varphi|^2 |\omega|^p. \tag{3.17}
\]
Notice that Sobolev inequality holds on $M$, then $\text{vol}(M) = \infty$. Then by the same argument as in the proof of Theorem 3.2, we see that $\omega$ is trivial.

Corollary 3.10. Let $M^n(n \geq 2)$ be an $n$-dimensional a complete, noncompact, connected, oriented, and stable minimal hypersurface immersed in a Riemann manifold $\bar{M}$. Assume that
a Sobolev inequality holds on $M$. For any $p > \frac{n-2}{n-1}$, if $\text{BiRic}^a \geq 0$, for some positive constant $a$ satisfying
\[
a < \min \left\{ \frac{4}{(n-1)p^2} + \frac{4(p-1)}{p^2}, \frac{n-1}{n} \right\}
\]
and
\[
\|S\|_2 \leq \frac{4}{(n-1)p^2} + \frac{4(p-1)}{p^2} - a \quad \frac{C_s}{\sqrt{n-1}} - a)
\] (3.18)

Let $u$ be an $p$-harmonic function on $M$. For $p \geq 2$, if $u$ has finite $p$-energy, then $u$ must be a constant map.

**Corollary 3.11.** Let $M^n(n \geq 2)$ be an $n$-dimensional a complete, noncompact, connected, oriented, and stable minimal hypersurface immersed in a Riemann manifold $\bar{M}$. Assume that a Sobolev inequality holds on $M$. For any $p \geq 2$, if $\text{BiRic}^a \geq 0$, for some positive constant $a$ satisfying
\[
a < \min \left\{ \frac{4}{(n-1)p^2} + \frac{4(p-1)}{p^2}, \frac{n-1}{n} \right\}
\]
and
\[
\|S\|_2 \leq \frac{4}{(n-1)p^2} + \frac{4(p-1)}{p^2} - a \quad \frac{C_s}{\sqrt{n-1}} - a)
\] (3.19)

Let $u : (M, g, dv_g) \to (N, h)$ be an $p$-harmonic map from an oriented complete noncompact manifold into a Riemannian manifold and $K^N \leq 0$. For $p \geq 2$, if $u$ has finite $p$-energy, then $u$ must be a constant map.

**Proof.** As in the proof of Corollary 3.8, we replace $\frac{4}{(n-1)p^2}$ by zero in (3.3) and below. □

**Theorem 3.12.** Let $M^n(n \geq 2)$ be a complete, noncompact, oriented, and stable immersed hypersurface in a Riemann manifold $\bar{M}$. Assume that $M$ satisfies a Sobolev inequality,
\[
\left( \int_M |f|^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} \leq C_s \int_M |\nabla f|^2,
\]
for any smooth compactly supported function $f$ in $M$. For any $p \geq 2$, if $\text{BiRic}^a \geq 0$ for some positive constant $a$ satisfying
\[
a < \min \left\{ \frac{4}{(n-1)p^2} + \frac{4(p-1)}{p^2}, \frac{\sqrt{n-1}}{2} \right\}
\]
and
\[ \|S\|_{L^2} \leq \frac{4}{(n-1)p^2 + 4(p-1)} - a \] \hfill (3.20)
then there does not exist any nontrivial \( L^p \) \( p \)-harmonic 1-form on \( M \).

**Proof.** In (3.16), notice that
\[ \Phi(H, s) \geq C_{n,a} = \frac{\sqrt{n-1}}{2} - a \]
The proof is the same as in Theorem 3.12. \( \square \)

## 4 Finiteness Theorem

In this paper, we will use the method of [17] and [12] to prove that the space of \( L^p \) harmonic 1-forms on \( M \) is finite. Firstly, we consider the case where \( H \neq 0 \).

**Theorem 4.1.** Let \( \bar{M} \) be an \((n+1)\)-dimensional Hadamard manifold with \( \text{BiRic} \) satisfying
\[ -k^2 \leq \text{BiRic} \]
where \( a \) is a given nonnegative real number, \( k \) is a nonzero constant, and \( 3 \leq n \leq 4 \). Let \( M \) be a complete noncompact hypersurface with finite index that is immersed in \( \bar{M} \). If one of the following two conditions hold,

1. If \( \frac{2}{2 - \sqrt{n-1}} < p < \min\{1 + \sqrt{n-1}, 2\frac{n-2}{n-3}\} \), assume that
\[ \lambda_1(M) > \left( \frac{2-p}{p} + \frac{1}{n-1} \frac{2(p-1)}{p} - \epsilon \right)^{-1} \left( \frac{p}{2p-2} + \frac{1}{a+1} \right) k^2. \]
where \( \epsilon \) is small positive constant.

2. If \( \max\{\frac{2}{2 - \sqrt{n-1}}, 1 + \sqrt{n-1}\} \leq p < 3 \), assume that
\[ \lambda_1(M) > \left( \frac{2-p}{p} + \frac{2}{p(p-1)} - \epsilon \right)^{-1} \left( \frac{p}{2p-2} + \frac{1}{a+1} \right) k^2. \]
where \( \epsilon \) is small positive constant. Then,
\[ \dim \mathcal{H}^1(L^p(M)) < \infty, \]
where \( \mathcal{H}^1(L^p(M)) \) denotes the space of \( L^p \) harmonic 1-forms on \( M \).

**Remark 4.2.** The conclusion is different from that in [12] since we consider the space of \( L^p \) \( p \)-harmonic 1-forms on \( M \) instead of the space of \( L^{2p} \) harmonic 1-forms. Although in [17], Han consider also the finiteness problem of the space of \( L^p \) \( p \)-harmonic 1-forms, the assumptions in our theorem is different from that in [17].
Proof. By Lemma (2.2), it is easy to see that

\[
|\omega|^{p-1}\Delta|\omega|^{p-1} \geq A_{p,n} |\nabla|\omega|^{p-1}|^2 - \left\langle \delta d(|\omega|^{p-2}\omega), |\omega|^{p-2}\omega \right\rangle \\
+ |\omega|^{2p-4} \left( \text{BiRic}^a \left( \frac{X}{|X|}, N \right) - \Phi_a(H, S) - a(\text{Ric}(N, N) + S) \right) |\omega|^2.
\]

(4.1)

Computing directly, we have

\[
|\omega|^{\frac{p}{p}} \Delta |\omega|^{\frac{p}{p}} = \frac{2 - p}{p} |\nabla|\omega|^{\frac{p}{p}}|^2 + \frac{p}{2p - 2} |\omega|^{2-p} \left( |\omega|^{p-1}\Delta |\omega|^{p-1} \right)
\]

\[
= \frac{2 - p}{p} |\nabla|\omega|^{\frac{p}{p}}|^2 + \frac{p}{2p - 2} \left[ A_{p,n} |\nabla|\omega|^{p-1}|^2 - \left\langle \delta d(|\omega|^{p-2}\omega), |\omega|^{p-2}\omega \right\rangle \\
+ |\omega|^{2p-4} \left( \text{BiRic}^a \left( \frac{X}{|X|}, N \right) - \Phi_a(H, S) - a(\text{Ric}(N, N) + S) \right) |\omega|^2 \right]
\]

\[
= \frac{2 - p}{p} |\nabla|\omega|^{\frac{p}{p}}|^2 + \frac{p}{2p - 2} A_{p,n} \frac{4(p-1)^2}{p^2} |\nabla|\omega|^{\frac{p}{p}}|^2 - \frac{p}{2p - 2} \left\langle \delta d(|\omega|^{p-2}\omega), |\omega|^2 \right\rangle \\
+ \frac{p}{2p - 2} \left( -k^2 - \frac{\sqrt{n-1}}{2} S \right) |\omega|^p.
\]

(4.2)

By the stable condition, we have

\[
\int_{M \setminus B_R(o)} (\text{Ric}(N) + S) \varphi^2 |\omega|^p \leq \int_{M \setminus B_R(o)} \left| \nabla \left( \varphi |\omega|^{\frac{p}{p}} \right) \right|^2 \\
= \int_{M \setminus B_R(o)} \varphi^2 |\nabla|\omega|^{\frac{p}{p}}|^2 + \int_{M \setminus B_R(o)} |\nabla \varphi|^2 |\omega|^p \\
+ 2 \int_{M \setminus B_R(o)} \varphi |\omega|^{\frac{p}{p}} \left\langle \nabla \varphi, \nabla |\omega|^{\frac{p}{p}} \right\rangle.
\]

(4.3)

By (4.3) and divergence theorem, we have

\[
\int_{M \setminus B_R(o)} \left( \frac{-k^2}{a + 1} + S \right) \varphi^2 |\omega|^p \\
\leq \int_{M \setminus B_R(o)} |\nabla \varphi|^2 |\omega|^p - \int_{M \setminus B_R(o)} \varphi^2 |\omega|^{\frac{p}{p}} \Delta |\omega|^{\frac{p}{p}} \\
\leq \int_{M \setminus B_R(o)} |\nabla \varphi|^2 |\omega|^p - \int_{M \setminus B_R(o)} \varphi^2 \left( \frac{2 - p}{p} + \frac{p}{2p - 2} A_{p,n} \frac{4(p-1)^2}{p^2} \right) |\nabla|\omega|^{\frac{p}{p}}|^2 \\
+ \int_{M \setminus B_R(o)} \frac{p}{2p - 2} \varphi^2 \left\langle \delta d(|\omega|^{p-2}\omega), |\omega|^2 \right\rangle + \int_{M \setminus B_R(o)} \frac{p}{2p - 2} \varphi^2 \left( k^2 + \frac{\sqrt{n-1}}{2} S \right) |\omega|^p,
\]

(4.4)
where in the second inequality, we have used (4.2).

Thus, we get
\[
\int_{M \setminus B_R(o)} \varphi^2 \left( \frac{2-p}{p} + \frac{p}{2p-2} A_{p,n} \frac{4(p-1)^2}{p^2} \right) |\nabla |w|^2|^2
\leq \int_{M \setminus B_R(o)} |\nabla \varphi|^2 |\omega|^p + \int_{M \setminus B_R(o)} \varphi^2 \delta(|\omega|^{p-2}, \omega)
+ \int_{M \setminus B_R(o)} \varphi^2 \left( \frac{p}{2p-2} \frac{\sqrt{n-1}}{2} - 1 \right) S|\omega|^p + \int_{M \setminus B_R(o)} \left( \frac{p}{2p-2} + \frac{1}{a+1} \right) \varphi^2 k^2 |\omega|^p.
\] (4.5)

However, the second term on the right hand side, by Lemma 2.1, we get (c.f. [18])
\[
\frac{p}{2p-2} \int_M \langle d(|\omega|^{p-2}, d\omega) \rangle \leq \frac{2(p-2)}{p-1} \int_M |\nabla |w|^2|^2 |\varphi| |\nabla \varphi||\omega|^p.
\] (4.6)

Thus, we get
\[
\left( \frac{2-p}{p} + \frac{p}{2p-2} A_{p,n} \frac{4(p-1)^2}{p^2} - \epsilon \right) \int_{M \setminus B_R(o)} \varphi^2 |\nabla |w|^2|^2
\leq (1 + \frac{(p-2)^2}{\epsilon(p-1)^2}) \int_{M \setminus B_R(o)} |\nabla \varphi|^2 |\omega|^p + \int_{M \setminus B_R(o)} \varphi^2 |\omega|^p
+ \left( \frac{p}{2p-2} \frac{\sqrt{n-1}}{2} - 1 \right) \int_{M \setminus B_R(o)} \varphi^2 S|\omega|^p.
\] (4.7)

As in [12], we also have
\[
\int_{M \setminus B_R(o)} \varphi^2 |\omega|^p \leq \frac{1}{\lambda_1(M)} \int_{M \setminus B_R(o)} \left| \nabla \left( \varphi |\omega|^2 \right) \right|^2
\leq \frac{1}{\lambda_1(M)} \left( 1 + \frac{1}{\epsilon} \right) \int_{M \setminus B_R(o)} |\nabla \varphi|^2 |\omega|^p + \frac{1 + \epsilon}{\lambda_1(M)} \int_{M \setminus B_R(o)} \varphi^2 |\nabla |w|^2|^2.
\] (4.8)

So we get
\[
\left[ 1 - B^{-1} \left( \frac{p}{2p-2} + \frac{1}{a+1} \right) k^2 \frac{1 + \epsilon}{\lambda_1(M)} \right] \int_{M \setminus B_R(o)} \varphi^2 |\omega|^p
\leq \left[ \frac{1}{\lambda_1(M)} \left( 1 + \frac{1}{\epsilon} \right) + \frac{1 + \epsilon}{\lambda_1(M)} B^{-1}(1 + \frac{(p-2)^2}{\epsilon(p-1)^2}) \right] \int_{M \setminus B_R(o)} |\nabla \varphi|^2 |\omega|^p,
\] (4.9)

where \( B = \frac{2-p}{p} + \frac{p}{2p-2} A_{p,n} \frac{4(p-1)^2}{p^2} - \epsilon \). It suffice to prove that
\[
\frac{2-p}{p} + \frac{p}{2p-2} A_{p,n} \frac{4(p-1)^2}{p^2} > 0.
\]
which is
\[
\frac{2 - p}{p} + \frac{p}{2p - 2} \frac{4(p - 1)^2}{p^2} \frac{1}{(p - 1)^2} \min\{1, \frac{(p - 1)^2}{n - 1}\} > 0
\]
(4.10)

If \(2 \leq p \leq 1 + \sqrt{n - 1}\),
\[
\frac{2 - p}{p} + \frac{p}{2p - 2} \frac{1}{n - 1} \frac{4(p - 1)^2}{p^2} > 0.
\]
(4.11)
which holds if \(p < \frac{2n^2}{n - 3}\). Thus, we can choose \(\epsilon\) such that \(B > 0\).

If \(p \geq 1 + \sqrt{n - 1}\), we have
\[
\frac{2 - p}{p} + \frac{p}{2p - 2} \frac{4}{p^2} = \frac{2 - p}{p} + \frac{2}{p(p - 1)} > 0.
\]
(4.12)
which holds if \(p < 3\). Thus, we can choose \(\epsilon\) such that \(B > 0\).

Take \(r_0 > r_1\), we can choose cutoff function \(\phi\) with \(\text{supp} \phi \subseteq M - B_{x_0}(r_1)\), and sufficiently small \(\epsilon_3\), which implies that
\[
\int_M |\nabla|^{\frac{p}{2}} \omega^p \leq \tilde{D} \int_M |\nabla \phi|^2 |\omega|^p,
\]
where \(\tilde{D}\) is a constant depending on \(p\). Using the Young inequality and the Hoffman-Spruck inequality(cf.[19]), as in [12], we also have
\[
\left( \int_{M \setminus B_R(o)} \left( \varphi \omega^{\tilde{p}} \right) \right)^{\frac{2n}{n-2}} \leq C_s \int_{M \setminus B_R(o)} \left| \nabla \left( \varphi \omega^{\tilde{p}} \right) \right|^2 + C_s \int_{M \setminus B_R(o)} \varphi^2 |\omega|^p H^2
\]
\[
\leq (1 + s) C_s \int_{M \setminus B_R(o)} \varphi^2 |\nabla \omega^{\tilde{p}}|^2 + \left( 1 + \frac{1}{s} \right) C_s \int_{M \setminus B_R(o)} |\nabla \varphi|^2 |\omega|^p
\]
\[
+ C_s \int_{M \setminus B_R(o)} \varphi^2 |\omega|^p H^2
\]
(4.13)
where \(C_s\) is a constant in Sobolev inequality.

By (4.7) and (4.14),
\[
\left( \int_{M \setminus B_R(o)} \left( \varphi \omega^{\tilde{p}} \right) \right)^{\frac{2n}{n-2}} \leq A^{-1} (1 + s) C_s \left[ 1 + \left( \frac{p - 2}{2} \right) \frac{1}{4e(p - 1)^2} \int_{M \setminus B_R(o)} |\nabla \varphi|^2 |\omega|^p + \left( \frac{p}{2p - 2} + \frac{1}{a + 1} \right) k^2 \int_{M \setminus B_R(o)} \varphi^2 |\omega|^p
\]
\[+ \left( \frac{p}{2p - 2} \frac{\sqrt{n - 1}}{2} - 1 \right) \int_{M \setminus B_R(o)} \varphi^2 S |\omega|^p \right]
\]
\[+ \left( 1 + \frac{1}{s} \right) C_s \int_{M \setminus B_R(o)} |\nabla \varphi|^2 |\omega|^p + C_s \int_{M \setminus B_R(o)} \varphi^2 |\omega|^p H^2
\]
(4.15)
Using $|H|^2 \leq nS$, and

$$\frac{p}{2p-2} \frac{\sqrt{n-1}}{2} - 1 < 0$$

we can choose sufficiently large $s$ such that

$$A^{-1}(1 + s) \left( \frac{p}{2p-2} \frac{\sqrt{n-1}}{2} - 1 \right) + n \leq 0.$$ 

It follows from (4.15),

\[
\left( \int_{M \setminus B_R(o)} \left( \varphi |\omega|^\frac{p}{2} \right)^\frac{2m}{m-2} \right)^\frac{m-2}{m} \\
\leq A^{-1}(1 + s) C_s \left[ 1 + \frac{(p-2)^2}{4(p-1)^2} \int_{M \setminus B_R(o)} |\nabla \varphi|^2 |\omega|^p + \left( \frac{p}{2p-2} + \frac{1}{a+1} \right) k^2 \int_{M \setminus B_R(o)} \varphi^2 |\omega|^p \right] \\
+ \left( 1 + \frac{1}{s} \right) C_s \int_{M \setminus B_R(o)} |\nabla \varphi|^2 |\omega|^p
\]

By (4.9), we get

\[
\left( \int_{M \setminus B_R(o)} \left( \phi |\omega|^\frac{p}{2} \right)^\frac{2m}{m-2} d_{\text{m}} \right)^\frac{m-2}{m} \leq C_1 \int_{M \setminus B_R(o)} |\nabla \phi|^2 |\omega|^p.
\]

As in [12], we choose cutoff function $\phi$ on $M$ such that for $r > R + 1$

\[
\begin{cases}
0 \leq \phi \leq 1 \\
\phi = 1, \text{ on } B_r(o) \setminus B_{R+1}(o) \\
\phi = 0, \text{ on } B_R(o) \cup (M - B_{2r}(x_0)) \\
|\nabla \phi| < C_2, \text{ on } B_{R+1}(o) \setminus B_R(o), \\
|\nabla \phi| < \frac{C_2}{r}, \text{ on } B_{2r}(o) \setminus B_r(o)
\end{cases}
\]

According to the definition of $\phi$, we have

\[
\left( \int_{B_r(o) \setminus B_{R+1}(o)} \left( |\omega|^\frac{p}{2} \right)^\frac{2m}{m-2} dx \right)^\frac{m-2}{m} \leq P_1 \int_{B_{R+1}(o) \setminus B_R(o)} |\omega|^p + P_1 \int_{B_{2r}(o) \setminus B_r(o)} |\omega|^p.
\]

Then let $r \to \infty$, we have

\[
\left( \int_{M \setminus B_{R+1}(o)} \left( |\omega|^\frac{p}{2} \right)^\frac{2m}{m-2} dx \right)^\frac{m-2}{m} \leq P_1 \int_{B_{R+1}(o) \setminus B_R(o)} |\omega|^p.
\]
It follows from the Hölder inequality (cf. Formula (30) in [13] or [12, (4.16)(4.17)]) that
\[
\int_{B_{R+2}(o)} |\omega|^p \leq P_2 \int_{B_{R+1}(o)} |\omega|^p,
\]
where \( P_2 \) depends on \( \text{Vol}(B_{R+2}(o)), m, p \).

Recall we have proved the following inequality,
\[
|\omega| \Delta |\omega|^{p-1} = \frac{4}{(m-1)p^2} |\nabla |\omega|^{\frac{p}{2}}|^2 - \left\langle \delta d(|\omega|^{p-2}\omega), \omega \right\rangle + |\omega|^p \alpha.
\]
where \( \alpha = \text{BiRic}^a \left( \frac{X}{|X|}, N \right) - \Phi_a(H, S) - a(\text{Ric}(N, N) + S) \). Once we have (4.13)(4.18), the remained step is the same as that in [17] noticing that the difference of \( \alpha \) from that in [17, (32)] doesn’t affect the correctness of the statement. Finally, let \( V \) be any finite-dimensional subspace of \( H^1(L^p(M)) \).

By Lemma 2.2 in [17], there exists \( \omega \in V \) such that
\[
\dim(V) \int_{B_{R+1}(o)} |\omega|^p \leq \text{Vol}(B_{R+1}(o)) \min(C_p^{(m)}, \dim V) \sup_{B(R+1)(o)} |\omega|^p.
\]
Thus
\[
\dim(V) \leq C.
\]
where \( C \) depends on \( \text{Vol}(B_{R+1}(o)), m, p, H, \alpha \)

Next, we consider the case where \( H = 0 \).

**Theorem 4.3.** Let \( \bar{M} \) be an \((n+1)\)-dimensional Hadamard manifold with \( \text{BiRic}^a \) satisfying \(-k^2 \leq \text{BiRic}^a\), where \( a \) is a given nonnegative real number, \( k \) is a nonzero constant. Let \( M \) be a complete noncompact minimal hypersurface with finite index that is immersed in \( \bar{M} \). If one of the following two conditions hold,
\[
(1) \ 2 \leq p < \min\{1 + \sqrt{n-1}, 4 \frac{n}{n-1}, 2 \frac{n-2}{n-3}\}, \text{ assume that} \ 
\lambda_1 > \left( \frac{2-p}{p} + \frac{2(p-1)^2}{p(p-1)} - \left( \frac{p}{2p-2} \left( \frac{n-1}{n} \right) - 1 \right) \right)^{-1} \left( \frac{p}{2p-2} k^2 + \frac{p}{2p-2} \left( \frac{n-1}{n} \right) \frac{k^2}{a+1} \right) \]
\[
(2) \text{If} \ 1 + \sqrt{n-1} \leq p < 3, \text{ we assume that} \ 
\lambda_1 > \left( \frac{2-p}{p} + \frac{2}{p(p-1)} - \left( \frac{p}{2p-2} \left( \frac{n-1}{n} \right) - 1 \right) \right)^{-1} \left( \frac{p}{2p-2} k^2 + \frac{p}{2p-2} \left( \frac{n-1}{n} \right) \frac{k^2}{a+1} \right)
\]

Then,
\[
\dim \mathcal{H}^1(L^p(M)) < \infty,
\]
where \( \mathcal{H}^1(L^p(M)) \) denotes the space of \( L^p \) harmonic 1-forms on \( M \).
Proof. Recall the inequality we have proved
\[ |\omega|^{p-1} \Delta |\omega|^{p-1} \geq A_{p,n} |\nabla| |\omega|^{p-1}|^2 - \left\langle \delta d(|\omega|^{p-2}\omega), |\omega|^{p-2}\omega \right\rangle \]
\[ + |\omega|^{2p-4} \left( -k^2 - \frac{n-1}{n} S \right) |\omega|^2. \] (4.19)

As in (4.2), we have
\[ |\omega|^{\frac{p}{2}} \Delta |\omega|^{\frac{p}{2}} \]
\[ \geq \left( \frac{2-p}{p} + \frac{p}{2p-2} A_{p,n} \frac{4(p-1)^2}{p^2} \right) |\nabla| |\omega|^{\frac{p}{2}}|^2 - \frac{p}{2p-2} \left\langle \delta d(|\omega|^{p-2}\omega), \omega \right\rangle \]
\[ + \frac{p}{2p-2} \left( -k^2 - \frac{n-1}{n} S \right) |\omega|^p. \] (4.20)

Multiplying both sides by $\phi^2$
\[ \int_M \phi^2 \left( \frac{2-p}{p} + \frac{p}{2p-2} A_{p,n} \frac{4(p-1)^2}{p^2} \right) |\nabla| |\omega|^{\frac{p}{2}}|^2 - \frac{p}{2p-2} \int_M \phi^2 \left\langle \delta d(|\omega|^{p-2}\omega), \omega \right\rangle \]
\[ + \int_M \phi^2 \frac{p}{2p-2} \left( -k^2 - \frac{n-1}{n} S \right) |\omega|^p \]
\[ \leq \int_M \phi^2 |\omega|^{\frac{p}{2}} \Delta |\omega|^{\frac{p}{2}} \]
\[ = \int_M |\nabla \phi|^2 |\omega|^p - \int_M |\nabla (\phi |\omega|^p)|^2. \] (4.21)

However, by $-k^2 \leq \overline{BiRic}$, we know
\[ \int_{M \setminus B_R(o)} \left( -\frac{k^2}{a+1} + S \right) \phi^2 |\omega|^p \]
\[ \leq \int_{M \setminus B_R(o)} (\overline{\text{Ric}}(\mathcal{N}) + S) \phi^2 |\omega|^p \]
\[ \leq \int_{M \setminus B_R(o)} \left| \nabla \left( \phi |\omega|^{\frac{p}{2}} \right) \right|^2. \] (4.22)

Thus, we get
\[ \int_{M \setminus B_R(o)} (S \phi^2 |\omega|^p \leq \int_{M \setminus B_R(o)} \left| \nabla \left( \phi |\omega|^{\frac{p}{2}} \right) \right|^2 + \frac{k^2}{a+1} \int_{M \setminus B_R(o)} \phi^2 |\omega|^p. \] (4.23)
Combining (4.21) and (4.23), we have

\[
\int_M \phi^2 \left( \frac{2 - p}{p} + \frac{p}{2p - 2} A_{p,n} \frac{4(p - 1)^2}{p^2} \right) |\nabla |\omega|^{\frac{p}{2}}|^2
- \int_M \phi^2 \delta d(|\omega|^{p-2} \omega), \omega \right) - \int_M \frac{p}{2p - 2} k^2 \phi^2 |\omega|^p
\leq \int_M |\nabla \phi|^2 |\omega|^p - \int_M |\nabla (|\omega|^p)|^2
+ \frac{p}{2p - 2} \left( \frac{n - 1}{n} - 1 \right) \left[ \int_{M \setminus B_{R(o)}} \left| \nabla \left( |\omega|^{\frac{p}{2}} \right) \right|^2 + \frac{k^2}{a + 1} \int_{M \setminus B_{R(o)}} \phi^2 |\omega|^p \right],
\]

which can be written as

\[
\int_M \phi^2 \left( \frac{2 - p}{p} + \frac{p}{2p - 2} A_{p,n} \frac{4(p - 1)^2}{p^2} \right) |\nabla |\omega|^{\frac{p}{2}}|^2
- \left[ \frac{p}{2p - 2} k^2 + \frac{p}{2p - 2} \left( \frac{n - 1}{n} \right) \frac{k^2}{a + 1} \right] \int_M \phi^2 |\omega|^p - \int_M |\nabla \phi|^2 |\omega|^p
\leq \left[ \frac{p}{2p - 2} \left( \frac{n - 1}{n} \right) - 1 \right] \int_{M \setminus B_{R(o)}} \left| \nabla \left( |\omega|^{\frac{p}{2}} \right) \right|^2 + \frac{2p - 2}{p - 1} \int_M \left| \nabla |\omega|^{\frac{p}{2}} \right| \phi |\nabla \phi| |\omega|^{\frac{p}{2}}.
\]

Thus, we have

\[
\left( \frac{2 - p}{p} + \frac{p}{2p - 2} A_{p,n} \frac{4(p - 1)^2}{p^2} - \epsilon_1 \right) \int_M \phi^2 |\nabla |\omega|^{\frac{p}{2}}|^2
- \int_M |\nabla \phi|^2 |\omega|^p
\leq \left[ \frac{p}{2p - 2} \left( \frac{n - 1}{n} \right) - 1 \right] + \left[ \frac{p}{2p - 2} k^2 + \frac{p}{2p - 2} \left( \frac{n - 1}{n} \right) \frac{k^2}{a + 1} \right] \frac{1}{\lambda_1(M)} \int_{M \setminus B_{R(o)}} \left| \nabla \left( |\omega|^{\frac{p}{2}} \right) \right|^2 + \frac{2p - 2}{p - 1} \frac{1}{4 \epsilon_1} \int_M \left| \nabla |\omega|^{\frac{p}{2}} \right| |\omega|^p.
\]

(4.26)

Following [12], if \( \frac{p}{2p - 2} \left( \frac{n - 1}{n} \right) - 1 + \left[ \frac{p}{2p - 2} k^2 + \frac{p}{2p - 2} \left( \frac{n - 1}{n} \right) \frac{k^2}{a + 1} \right] \frac{1}{\lambda_1(M)} \leq 0 \), then by (4.26), we have

\[
\left( \frac{2 - p}{p} + \frac{p}{2p - 2} A_{p,n} \frac{4(p - 1)^2}{p^2} - \epsilon_1 \right) \int_M \phi^2 |\nabla |\omega|^{\frac{p}{2}}|^2
\leq \frac{2p - 2}{p - 1} \frac{1}{4 \epsilon_1} \int_M \left| \nabla |\omega|^{\frac{p}{2}} \right| |\omega|^p.
\]
By the conditions on \( p \) and (4.10)(4.11)(4.12), we have
\[
\frac{2 - p}{p} + \frac{p}{2p - 2}A_{p,n} \frac{4(p - 1)^2}{p^2} > 0.
\]

If \( \frac{p}{2p - 2} \left( \frac{n-1}{n} \right) - 1 + \left[ \frac{p}{2p - 2} k^2 + \frac{p}{2p - 2} \left( \frac{n-1}{n} \right) \frac{k^2}{a+1} \right] \frac{1}{\lambda_1(M)} > 0 \), then by (4.26), we have
\[
\left( \frac{2 - p}{p} + \frac{p}{2p - 2}A_{p,n} \frac{4(p - 1)^2}{p^2} - \epsilon_1 \right) \int_M |\nabla \phi|^2 |\omega|^p
\leq \left( 1 + 2 \frac{p - 2}{p - 1} \epsilon_1 \right) \int_M |\nabla \phi|^2 |\omega|^p + \frac{p}{2p - 2} \left( \frac{n-1}{n} \right) - 1 + \left[ \frac{p}{2p - 2} k^2 + \frac{p}{2p - 2} \left( \frac{n-1}{n} \right) \frac{k^2}{a+1} \right] \frac{1}{\lambda_1(M)}
\]
(4.27)

It follows that
\[
\left( \frac{2 - p}{p} + \frac{p}{2p - 2}A_{p,n} \frac{4(p - 1)^2}{p^2} - \epsilon_1 - B(1 + \epsilon_2) \right) \int_M |\nabla \phi|^2 |\omega|^p
\leq \left( 1 + 2 \frac{p - 2}{p - 1} \epsilon_1 \right) \int_M |\nabla \phi|^2 |\omega|^p + B(1 + \frac{1}{\epsilon_2}) \int_M |\nabla \phi|^2 |\omega|^p,
\]
(4.28)

where \( B = \left[ \frac{p}{2p - 2} \left( \frac{n-1}{n} \right) - 1 + \left[ \frac{p}{2p - 2} k^2 + \frac{p}{2p - 2} \left( \frac{n-1}{n} \right) \frac{k^2}{a+1} \right] \frac{1}{\lambda_1(M)} \right] \).

Note it suffices to prove that we can choose \( \epsilon_1, \epsilon_2 \) such that
\[
\frac{2 - p}{p} + \frac{p}{2p - 2}A_{p,n} \frac{4(p - 1)^2}{p^2} - \epsilon_1 - B(1 + \epsilon_2)
= \frac{2 - p}{p} + \frac{1}{2p - 2} - \left\{ \frac{p}{2p - 2} \left( \frac{n-1}{n} \right) - 1
\right. \\
+ \left. \left[ \frac{p}{2p - 2} k^2 + \frac{p}{2p - 2} \left( \frac{n-1}{n} \right) \frac{k^2}{a+1} \right] \frac{1}{\lambda_1(M)} \right\} - \epsilon_1 - B\epsilon_2 > 0.
\]

If \( p > 1 + \sqrt{n - 1} \),
\[
\frac{2 - p}{p} + \frac{1}{2p - 2} - \left( \frac{p}{2p - 2} \left( \frac{n-1}{n} \right) - 1 \right) > 0.
\]

which holds if \( p < \frac{4n}{n-1} \). Moreover,
\[
\lambda_1 > \left( \frac{2 - p}{p} + \frac{1}{2p - 2} - \left( \frac{p}{2p - 2} \left( \frac{n-1}{n} \right) - 1 \right) \right)^{-1}
\times \left( \frac{p}{2p - 2} k^2 + \frac{p}{2p - 2} \left( \frac{n-1}{n} \right) \frac{k^2}{a+1} \right).
\]
Thus in the case where \( p > 1 + \sqrt{n-1} \), we can choose \( \epsilon_1, \epsilon_2 \) such that
\[
\frac{2-p}{p} + \frac{p}{2p-2} A_{p,n} \frac{4(p-1)^2}{p^2} - \epsilon_1 - A(1 + \epsilon_2) > 0.
\]

If \( p \leq 1 + \sqrt{n-1} \), then
\[
\frac{2-p}{p} + \frac{1}{2p-2} \frac{(p-1)^2}{(m-1)p} \left( \frac{p}{2p-2} \left( \frac{n-1}{n} \right) - 1 + \frac{p}{2p-2} \frac{k^2}{n} \right) \frac{1}{1+\lambda_1(M)} - \epsilon_1 - A\epsilon_2.
\]

In this case, if
\[
\frac{2-p}{p} + \frac{1}{2p-2} \frac{(p-1)^2}{(n-1)p} - \left( \frac{p}{2p-2} \left( \frac{n-1}{n} \right) - 1 \right) > 0.
\]

which holds if \( p < \frac{4n}{n-1} \). In addition,
\[
\lambda_1 > \left( \frac{2-p}{p} + \frac{2(p-1)}{p(n-1)} - \left( \frac{p}{2p-2} \left( \frac{n-1}{n} \right) - 1 \right) \right)^{-1} \times \left( \frac{p}{2p-2} k^2 + \frac{p}{2p-2} \left( \frac{n-1}{n} \right) \frac{k^2}{a+1} \right).
\]

Thus in the case where \( p \leq 1 + \sqrt{n-1} \), we can choose \( \epsilon_1, \epsilon_2 \) such that
\[
\frac{2-p}{p} + \frac{p}{2p-2} A_{p,n} \frac{4(p-1)^2}{p^2} - \epsilon_1 - A(1 + \epsilon_2) > 0.
\]

Hence, from (4.28), we get
\[
\int_M |\nabla \omega|^2 |\phi|^2 \leq C \int_M |\nabla \phi|^2 |\omega|^p. \tag{4.29}
\]

The remained proof is the same as that in Theorem 4.1. \( \square \)

**References**

[1] Ahmad Afuni. Monotonicity for p-harmonic vector bundle-valued k-forms. *arXiv preprint arXiv:1506.03439*, 2015. 2
[2] Paul Baird and Sigmundur Gudmundsson. *p*-harmonic maps and minimal submanifolds. *Math. Ann.*, 294(4):611–624, 1992.  

[3] Pierre Bérard. A note on Bochner type theorems for complete manifolds. *Manuscr. Math.*, 69(3):261–266, 1990.  

[4] Xiangzhi Cao. Liouville type theorem about *p*-harmonic function and *p*-harmonic map with finite *L*<sup>*q*</sup>-energy. *Chinese Ann. Math. Ser. B*, 38(5):1071–1076, 2017.  

[5] Xiangzhi Cao. *Existence of generalized harmonic map and Liouville type theorems on Riemannian manifold*. PhD thesis, Wuhan university, 2018.  

[6] Jean-Baptiste Casteras, Esko Heinonen, and Ilkka Holopainen. Existence and non-existence of minimal graphic and *p*-harmonic functions. *Proc. Roy. Soc. Edinburgh Sect. A*, 150(1):341–366, 2020.  

[7] Xiaoli Chao, Aiying Hui, and Miaomiao Bai. Vanishing theorems for *p*-harmonic ℓ-forms on Riemannian manifolds with a weighted Poincaré inequality. *Differ. Geom. Appl.*, 76:13, 2021. Id/No 101741.  

[8] Nguyen Thac Dung. *p*-harmonic ℓ-forms on Riemannian manifolds with a weighted Poincaré inequality. *Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods*, 150:138–150, 2017.  

[9] Nguyen Thac Dung and Keomkyo Seo. Vanishing theorems for *L*<sup>2</sup> harmonic 1-forms on complete submanifolds in a Riemannian manifold. *Journal of Mathematical Analysis and Applications*, 423(2):1594–1609, 2015.  

[10] Nguyen Thac Dung and Keomkyo Seo. *p*-harmonic functions and connectedness at infinity of complete submanifolds in a riemannian manifold. *Annali di Matematica Pura ed Applicata (1923-)*, 196(4):1489–1511, 2017.  

[11] Nguyen Thac Dung and Pham Trong Tien. Vanishing properties of *p*-harmonic ℓ-forms on Riemannian manifolds. *J. Korean Math. Soc.*, 55(5):1103–1129, 2018.  

[12] Nguyen Thac Dung, Nguyen Van Duc, and Juncheol Pyo. Harmonic 1-forms on immersed hypersurfaces in a riemannian manifold with weighted bi-ricci curvature bounded from below. *Journal of Mathematical Analysis and Applications*, 484(1):123693, 2020.  

[13] Wenzhen Gan and Peng Zhu. *L*<sup>2</sup> harmonic 1-forms on minimal submanifolds in spheres. *Results in Mathematics*, 65(3-4):483–490, 2014.
[14] Yingbo Han. $p$-harmonic $l$-forms on complete noncompact submanifolds in sphere with flat normal bundle. *Bull. Braz. Math. Soc. (N.S.)*, 49(1):107–122, 2018.

[15] Yingbo Han. Vanishing theorem for $p$-harmonic 1-forms on complete submanifolds in spheres. *Bull. Iran. Math. Soc.*, 44(3):659–671, 2018.

[16] Yingbo Han and Shuxiang Feng. A Liouville type theorem for $p$-harmonic functions on minimal submanifolds in $\mathbb{R}^{n+m}$. *Mat. Vesn.*, 65(4):494–498, 2013.

[17] Yingbo Han and Hong Pan. $L^p$ $p$-harmonic 1-forms on submanifolds in a Hadamard manifold. *Journal of Geometry and Physics*, 107:79–91, 2016.

[18] Yingbo Han, Qianyu Zhang, and Mingheng Liang. $L^p$ $p$-harmonic 1-forms on locally conformally flat Riemannian manifolds. *Kodai Math. J.*, 40(3):518–536, 2017.

[19] David Hoffman and Joel Spruck. Sobolev and isoperimetric inequalities for Riemannian submanifolds. *Communications on Pure and Applied Mathematics*, 27(6):715–727, 1974.

[20] Jin Tang Li. $P$-harmonic maps for submanifolds with positive Ricci curvature. *Xiamen Daxue Xuebao Ziran Kexue Ban*, 40(6):1191–1195, 2001.

[21] Stefano Pigola, Marco Rigoli, and Alberto G Setti. Constancy of $p$-harmonic maps of finite $q$-energy into non-positively curved manifolds. *Mathematische Zeitschrift*, 258(2):347–362, 2008.

[22] Hiroshi Takeuchi. Stability and Liouville theorems of $p$-harmonic maps. *Japan. J. Math. (N.S.)*, 17(2):317–332, 1991.

[23] Qiaoling Wang. Complete submanifolds in manifolds of partially non-negative curvature. *Annals of Global Analysis and Geometry*, 37(2):113–124, 2010.

[24] Jian Feng Zhang and Yue Wang. A theorem of Liouville type for $p$-harmonic maps in weighted Riemannian manifolds. *Kodai Mathematical Journal*, 39(2):354–365, 2016.

[25] Xi Zhang. A note on $p$-harmonic 1-forms on complete manifolds. *Canadian mathematical bulletin*, 44(3):376–384, 2001.

[26] Zhen Rong Zhou. Stability and quantum phenomenon and Liouville theorems of $p$-harmonic maps with potential. *Kodai Mathematical Journal*, 26(1):101–118, 2003.