On the conformal anomaly from higher derivative gravity in AdS/CFT correspondence

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abstract

We follow to Witten proposal [1] in the calculation of conformal anomaly from $d + 1$-dimensional higher derivative gravity via AdS/CFT correspondence. It is assumed that some $d$-dimensional conformal field theories have a description in terms of above $d + 1$-dimensional higher derivative gravity which includes not only Einstein term and cosmological constant but also curvature squared terms. The explicit expression for two-dimensional and four-dimensional anomalies is found, it contains higher derivative corrections. In particular, it is shown that not only Einstein gravity but also theory with the Lagrangian $L = aR^2 + bR_{\mu\nu}R^{\mu\nu} + \Lambda$ (even when $a = 0$ or $b = 0$) is five-dimensional bulk theory for $d = 4$ $\mathcal{N} = 4$ super Yang-Mills theory in AdS/CFT correspondence. Similar $d + 1 = 3$ theory with (or without) Einstein term may describe $d = 2$ scalar or spinor CFTs. That gives new versions of bulk side which may be useful in different aspects. As application of our general formalism we find next-to-leading corrections to the conformal anomaly of $\mathcal{N} = 2$ supersymmetric theory from $d = 5$ AdS higher derivative gravity (low energy string effective action).

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1 Introduction. Conformal anomaly from dilatonic gravity.

Recently so-called AdS/CFT (anti-de Sitter space vs. conformal field theory) correspondence attracted a lot of attention [2, 3, 1]. In particular, the correspondence between correlation functions in AdS and those of the boundary manifold was discussed.

$D$ dimensional anti-de Sitter space can be realized by imposing a constraints on $D + 1$ coordinates:

$$- x_0^2 + x_1^2 + \cdots x_{D-1}^2 - x_D^2 = -L^2.$$  (1)

From this realization, it is easy to see that this space has $SO(D - 1, 2)$ symmetry as isometry. The algebra of $SO(D - 1, 2)$ symmetry is the same as the algebra of conformal transformations acting on $D - 1$ dimensional Minkowski space.

In an adequate coordinate choice, the metric on the $D$-dimensional anti-de Sitter space is given by

$$ds_{\text{AdS}}^2 = \rho^{-2} d\rho^2 + \rho^{-1} \sum_{i,j=0}^{D-2} \eta_{ij} dx^i dx^j.$$  (2)

Here $\eta$ is the metric on the flat $D - 1$-dimensional Minkowski space. We should note that there is a boundary when $\rho$ vanishes. The topology of the boundary is almost that of the $D - 1$ dimensional Minkowski space, or more exactly, Minkowski space with a point at infinity, that is, topologically compactified Minkowski space. On the boundary manifold, $SO(D - 1, 2)$ acts exactly as usual conformal transformation. When we consider the surface with fixed finite $\rho$, there is a correction proportional $\rho$ only in the conformal boost but the correction vanishes just on the boundary, that is, in the limit that $\rho$ vanishes.

AdS/CFT correspondence is conjectured in ref. [2]. When $N$ p-branes in superstring theory or so-called M-theory coincide with each other and the coupling constant is small, the classical supergravity on AdS$_{D=d+1=p+2}$, which is the low energy effective theory of superstring or M-theory, is, in some sense, dual to large $N$ conformal field theory on $M^d$, which is the boundary of the AdS. For example, $d = 2$ case corresponds to $(4,4)$ superconformal
field theory, $d = 4$ case corresponds to $U(N)$ or $SU(N)$ $\mathcal{N} = 4$ super Yang Mills theory and $d = 6$ case to $(0,2)$ superconformal field theory.

The conjecture tells that partition function in $d$-dimensional conformal field theory is given in terms of the classical action in $d + 1$-dimensional gravity theory:

$$Z_d(\phi_0) = e^{-S_{\text{AdS}}(\phi_{\text{classical}}(\phi_0))}. \quad (3)$$

Here $\phi_0$ is the value of the field $\phi$ on the boundary and $\phi_{\text{classical}}(\phi_0)$ is a field on bulk background, which is AdS, given by solving the equations of motion with the boundary value $\phi_0$ on $M^d$. $S_{\text{AdS}}(\phi_{\text{classical}}(\phi_0))$ is the classical gravity action on AdS. When we substitute the classical solution into the action, the action, in general, contains infrared divergences coming from the infinite volume of AdS. Then we need to regularize the infrared divergence. It is known that as a result of the regularization and the renormalization (for the introduction to the renormalization in background field method see ref.\[10\]) there often appear anomalies. In ref.\[1\] Witten made the proposal how to calculate the conformal anomaly from classical gravity (bulk) side. This proposal has been worked out in detail in ref.\[4\] (for Einstein gravity) where it was shown that the conformal (Weyl) anomaly may be recovered after regularizing the above infrared divergence (from bulk side- Einstein theory). The usual result for conformal anomaly of boundary QFT thus may be reproduced. This is a kind of IR-UV duality.

Since the calculation of ref.\[4\] was done without dilaton background, in the previous paper \[5\], the authors investigate the conformal anomaly in the non-trivial dilaton background. We start from the action of $d+1$-dimensional dilatonic gravity with boundary terms

$$S = \frac{1}{16\pi G} \left[ \int_{M_{d+1}} d^{d+1}x \sqrt{-\hat{G}} \left\{ \hat{R} + X(\phi)(\nabla \phi)^2 + Y(\phi)\Delta \phi + 4\lambda^2 \right\} + \int_{M_d} d^d x \sqrt{-\hat{g}} \left( 2\hat{\nabla}_\mu n^\mu + \alpha \right) \right]. \quad (4)$$

Here $M_{d+1}$ is $d + 1$ dimensional manifold, which is identified with AdS$_{d+1}$ and $n_\mu$ is the unit vector normal to the boundary manifold $M_d$. $\alpha$ is a parameter which is chosen properly. The boundary terms play a role for cancellation of the leading infrared divergences which guarantee the system depends only on the boundary value. Note that dilatonic gravity makes the beatiful realization
of two-boundaries AdS/CFT correspondence [15] (for other examples of such two-boundaries correspondence see refs. [16, 17]).

$X(\phi), Y(\phi)$ can be arbitrary functions of $\phi$. The arbitrariness is not real but apparent. In fact, by the redefinition,

$$\varphi \equiv \int d\phi \sqrt{2V(\phi)} \ , \ V(\phi) \equiv X(\phi) - Y'(\phi)$$

(5)

the action can be rewritten as

$$S = \frac{1}{16\pi G} \left[ \int d^{d+1}x \sqrt{-\hat{G}} \left\{ \hat{R} + 2(\hat{\nabla} \varphi)^2 + 4\lambda^2 \right\} 
+ \int_{M_d} d^d x \sqrt{-\hat{g}} \left( 2\hat{\nabla}_{\mu} n^\mu + \alpha + Y(\phi) n^\mu \partial_\mu \phi \right) \right].$$

(6)

The last term on $M_d$ finally does not contribute to anomaly.

Following the calculation in [4], the explicit expression for the conformal anomaly has been obtained

$$T = \frac{l}{16\pi G} \left\{ R(0) + X(\phi(0))(\nabla \phi(0))^2 + Y(\phi(0))\Delta \phi(0) \right\}$$

(7)

for $d = 2$. This result could be compared with the UV-calculation of the conformal anomaly of matter dilaton coupled $N$ scalars [4, 5, 9] and $M$ Majorana dilaton coupled spinors [18] whose action is given by

$$S = \frac{1}{2} \int d^2 x \sqrt{-g} \left\{ f(\phi) g^{\mu\nu} \sum_{i=1}^N \partial_\mu \chi_i \partial_\nu \chi_i + g(\phi) \sum_{i=1}^M \bar{\psi}_i \gamma^\mu \partial_\mu \psi_i \right\}.$$

(8)

The results in (7) correspond to

$$\frac{l}{16\pi G} = \frac{2N + M}{48\pi},$$

$$\frac{l}{16\pi G} X(\phi(0)) = -\frac{N}{4\pi} \left( \frac{f''}{f'} - \frac{f'^2}{4f^2} \right) - \frac{M}{24\pi} \left( \frac{g''}{g} - \frac{g'^2}{g^2} \right),$$

$$\frac{l}{16\pi G} Y(\phi(0)) = -\frac{N}{4\pi} \frac{f'}{2f} - \frac{M}{24\pi} \frac{g'}{g}.$$

(9)

For $d = 4$, the expression of the conformal anomaly from bulk dilatonic gravity has the following form:

$$T = \frac{l^3}{8\pi G} \left[ \frac{1}{8} R(0)_{i\bar{j}} R_{(0)}^{i\bar{j}} - \frac{1}{24} R^2(0) \right]$$
\[ + \frac{1}{2} R_{ij}^{(0)} \partial_i \varphi^{(0)} \partial_j \varphi^{(0)} - \frac{1}{6} R_{(0)} g_{ij}^{(0)} \partial_i \varphi^{(0)} \partial_j \varphi^{(0)} \\
+ \frac{1}{4} \left\{ \frac{1}{\sqrt{-g^{(0)}}} \partial_i \left( \sqrt{-g^{(0)}} g_{ij}^{(0)} \partial_j \varphi^{(0)} \right) \right\}^2 + \frac{1}{3} \left( g_{ij}^{(0)} \partial_i \varphi^{(0)} \partial_j \varphi^{(0)} \right)^2 \]  

(10)

In [8], it has been calculated the conformal anomaly coming from the multiplets of \( \mathcal{N} = 4 \) supersymmetric \( U(N) \) or \( SU(N) \) Yang-Mills coupled with \( \mathcal{N} = 4 \) conformal supergravity (in a covariant way). If we choose the length parameter \( l \) as

\[ \frac{l^3}{16\pi G} = \frac{2N^2}{(4\pi)^2} \]  

(11)

and consider the background where only gravity and the real part of the scalar field \( \varphi \) are non-trivial and other fields vanish, IR-calculation (10) from the bulk exactly reproduces the result of UV-calculation by Liu and Tseytlin. Since dilaton field always appears in string theory, the conformal anomaly gives further check of AdS/CFT correspondence.

In this paper, we consider \( R^2 \)-gravity, which contains the squares of the curvatures in the action, as an extension of Einstein gravity. This is our bulk theory (for a general review and list of refs. on quantum higher derivative gravity see [10]). Moreover for some combinations of higher derivative terms it corresponds to different compactifications of superstring and heterotic string effective actions.

The paper is organised as follows. In the next section we present general formalism for the calculation of conformal anomaly from bulk higher derivative gravity. As an example we give the explicit result for two-dimensional CFT anomaly as obtained from three-dimensional higher derivative gravity. Section three is devoted to the same calculation in four-dimensional case. We find conformal \( d = 4 \) QFT (boundary \( \mathcal{N} = 4 \) super Yang-Mills theory) which has description in terms of five-dimensional higher derivative gravity. The application of results of section 3 to evaluation of next-to-leading correction to \( \mathcal{N} = 2 \) supersymmetric theory conformal anomaly from bulk AdS higher derivative gravity (low energy string effective action) is done in section 4. The last section is devoted to brief description of some perspectives.
2 Conformal anomaly from bulk higher
derivative gravity. General formalism and d2 case.

As an extension of the Einstein gravity, we consider $R^2$ gravity (bulk theory), which contains the squares of the curvatures in the action:

$$S = \int d^{d+1}x \sqrt{-\hat{G}} \left\{ a\hat{R}^2 + b\hat{R}_{\mu\nu}\hat{R}^{\mu\nu} + c\hat{R}_{\mu\nu\rho\sigma}\hat{R}^{\mu\nu\rho\sigma} + \frac{1}{\kappa^2}\hat{R} - \Lambda \right\} + S_B$$

(12)

Here $M_{d+1}$ is $d + 1$ dimensional manifold whose boundary is $d$ dimensional manifold $M_d$. The boundary term $S_B$ is necessary to make the variational principle to be well-defined but we do not need the explicit expression of $S_B$ since it does not contribute to the anomaly. Note that in Einstein gravity the boundary terms may be used to present the action as functional of fields and their first derivatives [12]. One can also add higher derivative boundary terms.

Note that for some values of parameters above theory can be related with superstring or heterotic string effective action. Superstring effective action does not include $R^2$ terms in $d + 1 = 10$ dimensions. However, making Calabi-Yau compactification in lower dimensions one can get also $R^2$ terms (see for example [11]). We may consider as particular example of higher derivative gravity the effective action of heterotic string [13, 14] which we take for torus compactification and supposing all fields except the metric tensor to be constant.

$$S_{het} = \int d^{d+1}x \sqrt{-g} \left\{ \frac{1}{\kappa^2}R - \Lambda + \frac{\alpha'}{8}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \right\}.$$  

(13)

The cosmological constant may be non-zero, for example for covariantly constant gauge fields.

The conventions of curvatures are given by

$$\begin{align*}
R & = g^{\mu\nu}R_{\mu\nu} \\
R_{\mu\nu} & = -\Gamma^\lambda_{\mu\lambda,\nu} + \Gamma^\lambda_{\mu\nu,\lambda} - \Gamma^\eta_{\mu\nu,\lambda} + \Gamma^\eta_{\mu\lambda,\nu} \\
\Gamma^\eta_{\mu\lambda} & = \frac{1}{2}g^{\nu\rho}(g_{\mu\nu,\lambda} + g_{\lambda\nu,\mu} - g_{\mu\lambda,\nu})
\end{align*}.$$ 

(14)
First we investigate if the equations of motion for the action (12) have a solution which describes anti de Sitter space, whose metric is given by

\[ ds^2 = \hat{G}^{(0)}_{\mu\nu} dx^\mu dx^\nu = \frac{l^2}{4} \rho^{-2} d\rho d\rho + \sum_{i=1}^{d} \rho^{-1} \eta_{ij} dx^i dx^j . \] (15)

When we assume the metric in the form (15), the scalar, Ricci and Riemann curvatures are given by

\[
\hat{R}^{(0)} = -\frac{d^2 + d}{l^2} , \quad \hat{R}^{(0)}_{\mu\nu} = -\frac{d}{l^2} G^{(0)}_{\mu\nu} , \quad \hat{R}^{(0)}_{\mu\nu\rho\sigma} = -\frac{1}{l^2} \left( G^{(0)}_{\mu\rho} G^{(0)}_{\nu\sigma} - G^{(0)}_{\mu\sigma} G^{(0)}_{\nu\rho} \right) ,
\] (16)

which tell that these curvatures are covariantly constant. Then in the equations of motion from the action (12), the terms containing the covariant derivatives of the curvatures vanish and the equations have the following forms:

\[
0 = -\frac{1}{2} G^{(0)}_{\xi\xi} \left( a \hat{R}^{(0)} + b \hat{R}^{(0)}_{\mu\nu} \hat{R}^{(0)}_{\mu\nu\rho\sigma} + c \hat{R}^{(0)}_{\mu\nu\rho\sigma} \hat{R}^{(0)}_{\mu\nu\rho\sigma} + \frac{1}{\kappa^2} \hat{R}^{(0)} - \Lambda \right) + 2a \hat{R}^{(0)}_{\xi\xi} + 2b \hat{R}^{(0)}_{\mu\nu} \hat{R}^{(0)}_{\mu\nu\rho\sigma} + 2c \hat{R}^{(0)}_{\mu\nu\rho\sigma} \hat{R}^{(0)}_{\mu\nu\rho\sigma} + \frac{1}{\kappa^2} \hat{R}^{(0)}_{\xi\xi} .
\] (17)

Then substituting Eqs.(16) into (17), we find

\[
0 = \frac{a}{l^4} (d + 1) d^2 (d - 3) + \frac{b}{l^4} d^2 (d - 3) + \frac{2c}{l^4} d (d - 3) - \frac{d(d - 1)}{\kappa^2 l^2} - \Lambda .
\] (18)

The equation (18) can be solved with respect to \( l^2 \) if

\[
\frac{d^2 (d - 1)^2}{\kappa^4} - 4d(d - 3) \left\{ (d + 1)da + db + 2c \right\} \Lambda \geq 0
\] (19)

which can been found from the determinant in (18). Then we obtain

\[
l^2 = -\frac{d(d + 1)}{\kappa^2} \pm \frac{\sqrt{d^2 (d - 1)^2} - 4d(d - 3) \left\{ (d + 1)da + db + 2c \right\} \Lambda}{2\Lambda} .
\] (20)

The sign in front of the root in the above equation may be chosen to be positive what corresponds to the Einstein gravity \((a = b = c = 0)\).
The existence of the solution where the sign is negative tells that there can be anti de Sitter solution even if the cosmological constant $\Lambda$ is positive since $l^2$ can be positive if $(d - 3) \{(d + 1)da + db + 2c\} < 0$. The solution describes the de Sitter or anti de Sitter space, depending on the overall sign of $l^2$. $l^2$ can be negative or positive depending from the choice of the parameters and sign in above equation.

In order to calculate the conformal anomaly, we consider the fluctuations around the anti de Sitter space (15). As in [4, 5], we assume the metric has the following form:

$$ds^2 \equiv \hat{G}_{\mu\nu} dx^\mu dx^\nu = \frac{l^2}{4} \rho^{-2} d\rho d\rho + \sum_{i=1}^{d} \hat{g}_{ij} dx^i dx^j$$

$$\hat{g}_{ij} = \rho^{-1} g_{ij}.$$  \hspace{1cm} (21)

We should note that there is a redundancy in the expression of (21). In fact, if we reparametrize the metric:

$$\delta \rho = \delta \sigma \rho, \quad \delta g_{ij} = \delta \sigma g_{ij}.$$  \hspace{1cm} (22)

by a constant parameter $\delta \sigma$, the expression (21) is invariant. The transformation (22) is nothing but the scale transformation on $M_d$.

In the parametrization in (21) we find the expressions for the scalar curvature $\hat{R}$$$

$$\hat{R} = -\frac{d^2 + d}{l^2} + \rho R + \frac{2(d - 1)\rho}{l^2} g^{ij} g'_{ij} + \frac{3\rho^2}{l^2} g^{ij} g^{kl} g'_{ik} g'_{jl}$$

$$\quad - \frac{4\rho^2}{l^2} g^{ij} g'''_{ij} + \frac{\rho^2}{l^2} g^{ij} g^{kl} g'_{ij} g'_{kl}.$$  \hspace{1cm} (23)

for Ricci tensor $\hat{R}_{\mu\nu}$

$$\hat{R}_{\rho\rho} = -\frac{d}{4\rho^2} - \frac{1}{2} g^{ij} g'''_{ij} + \frac{1}{4} g^{ik} g^{lj} g'_{ik} g'_{lj}$$

$$\hat{R}_{ij} = R_{ij} - \frac{2\rho}{l^2} g''_{ij} + \frac{2\rho}{l^2} g^{kl} g'_{ik} g'_{jl} - \frac{\rho}{l^2} g'_{ij} g'_{kl}$$

$$\quad - \frac{2 - d}{l^2} g'_{ij} + \frac{1}{l^2} g_{ij} g^{kl} g'_{kl} - \frac{d}{l^2 \rho} g_{ij}$$

$$\hat{R}_{i\rho} = \hat{R}_{\rho i}.$$  

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\[ \hat{R}_{ijkl} = \frac{1}{\rho} R_{ijkl} \]  
\[ \hat{R}_{ij} = \hat{R}_{ij} = -\hat{R}_{ij} = \hat{R}_{ij} = 0 \]  
\[ \hat{R}_{ij} = \hat{R}_{ij} = -\hat{R}_{ij} = \hat{R}_{ij} = 0 \]  
\[ \hat{R}_{ijkl} = \frac{1}{\rho} R_{ijkl} \]  
\[ -\frac{1}{\rho^2 l^2} \left\{ (g_{jl} - \rho g_{jl}) (g_{ik} - \rho g_{ik}) - (g_{jk} - \rho g_{jk}) (g_{il} - \rho g_{il}) \right\} \]

Here “’” expresses the derivative with respect to \( \rho \) and \( R, R_{ij}, R_{ijkl} \) are scalar curvature, Ricci and Riemann tensors, respectively, on \( M_d \).

As in the previous papers [4, 5], we expand the metric \( g_{ij} \) as a power series with respect to \( \rho \),
\[ g_{ij} = g_{(0)} + \rho g_{(1)} + \rho^2 g_{(2)} + \cdots \]  
Substituting (26) into (23), (24), and (25), we find the following expansions with respect to \( \rho \):
\[ \sqrt{-\hat{g}} = \frac{1}{2} \rho^{-\frac{d}{2} - 1} \sqrt{-g_{(0)}} \left\{ 1 + \frac{\rho}{2} g_{(0)} g_{(1)} g_{ij} + \rho^2 \left( \frac{1}{2} g_{(0)} g_{(2)} g_{ij} - \frac{1}{4} g_{(0)} g_{(1)} g_{ik} g_{(1)} g_{jl} 
+ \frac{1}{8} \left( g_{(0)} g_{(1)} g_{ij} \right)^2 \right) + O(\rho^3) \right\} \]
$$\sqrt{-\hat{G}\hat{R}} = \frac{l}{2}\rho^{-\frac{d}{2}-1}\sqrt{-g(0)} \left\{ -\frac{d^2 + d}{l^2} \right. $$

$$+ \rho \left( R_{(0)} + \frac{-d^2 + 3d - 4}{2l^2} g_{(0)}^{ij} g_{(1)ij} \right) $$

$$+ \rho^2 \left( -g_{(1)ij} R_{(0)} + \frac{1}{2} R_{(0)} g_{(0)}^{ij} g_{(1)ij} \right) $$

$$+ \frac{-d^2 + 7d - 24}{2l^2} g_{(0)}^{ij} g_{(2)ij} $$

$$+ \frac{d^2 - 7d + 20}{4l^2} g_{(0)}^{ij} g_{(1)ik} g_{(1)jl} $$

$$+ \frac{-d^2 + 7d - 16}{8l^2} \left( g_{(0)}^{ij} g_{(1)ij} \right)^2 + O(\rho^3) \} $$

$$\sqrt{-\hat{G}\hat{R}^2} = \frac{l}{2}\rho^{-\frac{d}{2}-1}\sqrt{-g(0)} \left\{ \frac{d^2(d + 1)^2}{l^2} \right. $$

$$+ \rho \left( -\frac{2d(d + 1)}{l^2} R_{(0)} + \frac{d^4 - 6d^3 + d^2 + 8d}{2l^4} g_{(0)}^{ij} g_{(1)ij} \right) $$

$$+ \rho^2 \left( R_{(0)}^2 + \frac{2d(d + 1)}{l^2} g_{(1)ij} R_{(0)}^{ij} \right) $$

$$+ \frac{-d^2 + 3d - 4}{l^2} R_{(0)} g_{(0)}^{ij} g_{(1)ij} $$

$$+ \frac{d^4 - 14d^3 + 33d^2 + 48d}{2l^4} g_{(0)}^{ij} g_{(2)ij} $$

$$+ \frac{-d^4 + 14d^3 - 25d^2 - 40d}{4l^4} g_{(0)}^{ij} g_{(0)ik} g_{(1)jl} $$

$$+ \frac{d^4 - 14d^3 + 49d^2 - 32d + 32}{8l^4} \left( g_{(0)}^{ij} g_{(1)ij} \right)^2 + O(\rho^3) \} $$

$$\sqrt{-\hat{G}\hat{R}_{\mu\nu}\hat{R}^{\mu\nu}} = \frac{l}{2}\rho^{-\frac{d}{2}-1}\sqrt{-g(0)} \left\{ \frac{d^2(d + 1)}{l^2} \right. $$

$$+ \rho \left( -\frac{2d}{l^2} R_{(0)} + \frac{d^3 - 7d^2 + 8d}{2l^4} g_{(0)}^{ij} g_{(1)ij} \right) $$

$$+ \rho^2 \left( R_{(0)}^{ij} R_{(0)ij} + \frac{4d - 4}{l^2} g_{(1)ij} R_{(0)}^{ij} \right) $$
\[
\sqrt{-\hat{G}} \hat{R}_{\mu\nu\rho\sigma} \hat{R}^{\mu\nu\rho\sigma} = \\
\frac{1}{2} \rho^{-\frac{d}{2}-1} \sqrt{-g(0)} \left\{ \frac{2d(d+1)}{l^4} \\
+ \rho \left( -\frac{4}{l^2} R(0) + \frac{d^2 - 7d + 8}{2l^4} g^{ij}(0) g_{ij}(0) \\
+ \rho^2 \left( R^{ijkl}(0) R_{ijkl}(0) + \frac{4}{l^2} g_{ij}(1) R^{ij}(0) - \frac{2}{l^2} R(0) g_{ij}(0) g_{ij}(1) \\
+ \frac{d^2 - 15d + 48}{4l^4} g^{ij}_{(0)} g^{(2)ij} \\
+ \frac{-d^2 + 23d - 56}{2l^4} g^{ij}_{(0)} g^{ij}_{(0)} g^{(1)ij} \\
+ \frac{d^2 - 15d + 48}{4l^4} \left( g^{ij}_{(0)} g^{ij}_{(1)} \right)^2 + O(\rho^3) \right) \right\} .
\]

(27)

We regard \(g_{(0)ij}\) as independent field on \(M_d\). We can solve \(g_{(l)ij}\) \((l = 1, 2, \ldots)\) with respect to \(g_{(0)ij}\) using equations of motion. When substituting the expression (26) or (27) into the classical action (12), the action diverges in general since the action contains the infinite volume integration on \(M_{d+1}\). We regularize the infrared divergence by introducing a cutoff parameter \(\epsilon\) (following to Witten [1]):

\[
\int d^{d+1}x \rightarrow \int d^dx \int d\rho , \quad \int_{M_d} d^dx (\cdots) \rightarrow \int d^dx (\cdots)|_{\rho=\epsilon} .
\]

(28)

Then the action (12) can be expanded as a power series of \(\epsilon\):

\[
S = S_0(g_{(0)ij}) \epsilon^{-\frac{d}{2}} + S_1(g_{(0)ij}, g_{(1)ij}) \epsilon^{-\frac{d}{2}-1} \\
+ \cdots + S_n \ln \epsilon + S_{\frac{d}{2}} + O(\epsilon^{\frac{d}{2}}).
\]

(29)

The term \(S_n\) proportional to \(\ln \epsilon\) appears when \(d = \text{even}\). In (29), the terms proportional to the inverse power of \(\epsilon\) in the regularized action are invariant
under the scale transformation
\[ \delta g_{(0)\mu\nu} = 2\delta\sigma g_{(0)\mu\nu} , \quad \delta\epsilon = 2\delta\sigma\epsilon . \]  
(30)

The invariance comes from the redundancy \( \text{(22)} \). The subtraction of these terms proportional to the inverse power of \( \epsilon \) does not break the invariance. When \( d \) is even, however, there appears the term \( S_{\ln} \) proportional to \( \ln\epsilon \). The subtraction of the term \( S_{\ln} \) breaks the invariance under the transformation \( \text{(30)} \). The reason is that the variation of the \( \ln\epsilon \) term under the scale transformation \( \text{(30)} \) is finite when \( \epsilon \to 0 \) since \( \ln\epsilon \to \ln\epsilon + \ln(2\delta\sigma) \). Therefore the variation should be canceled by the variation of the finite term \( S_{\frac{d}{2}} \) (which does not depend on \( \epsilon \))

\[ \delta S_{\frac{d}{2}} = -\ln(2\sigma)S_{\ln} \]  
(31)

since the original total action \( \text{(12)} \) is invariant under the scale transformation. Since the action \( S_{\frac{d}{2}} \) can be regarded as the action renormalized by the subtraction of the terms which diverge when \( \epsilon \to 0 \), the \( \ln\epsilon \) term \( S_{\ln} \) gives the conformal anomaly \( T \) of the renormalized theory on the boundary \( M_d \):

\[ S_{\ln} = -\frac{1}{2} \int d^2x \sqrt{-g_{(0)}} T . \]  
(32)

First we consider \( d = 2 \) case. When \( d = 2 \), substituting \( \text{(26)} \) or \( \text{(27)} \) into the action \( \text{(12)} \) and using the regularization \( \text{(28)} \), we obtain

\[ S_{\ln} = \int d^2x \sqrt{-g_{(0)}} \left[ g^{ij}_{(0)}g_{(1):ij} \left\{ -\frac{3a}{l^3} - \frac{b}{l^3} - \frac{c}{2l\kappa^2} - \frac{1\Lambda}{4} \right\} \right. \\
+ \left. R_{(0)} \left\{ \frac{6a}{l} - \frac{2b}{l} - \frac{2c}{l} + \frac{1}{2\kappa^2} \right\} \right] . \]  
(33)

Since Eq.\( \text{(18)} \) gives the following equation for \( d = 2 \)

\[ 0 = -\frac{12a}{l^4} - \frac{4b}{l^4} - \frac{4c}{l^4} - \frac{2}{\kappa^2l^2} - \Lambda , \]  
(34)

we find the term proportional to \( g^{ij}_{(0)}g_{(1):ij} \) in \( \text{(33)} \) vanishes. Then using \( \text{(32)} \), we find the expression of Weyl anomaly \( T \)

\[ T = \left\{ 12a + 4b + 4c - \frac{l}{\kappa^2} \right\} R_{(0)} . \]  
(35)
When \( a = b = c = 0 \), the above equation reproduces the result of refs.\([4, 5]\):

\[
T = -\frac{l}{\kappa^2} R_{(0)} .
\]  

(36)

For the heterotic string effective action \([13]\), we obtain

\[
T_{\text{string}} = \left\{ \frac{\alpha'}{2} - \frac{l}{\kappa^2} \right\} R_{(0)} .
\]

(37)

The important result which follows from above consideration is the following: two-dimensional CFT may have description not only in terms of three-dimensional Einstein gravity (bulk theory) but also in terms of three-dimensional higher derivative gravity where Einstein term may be absent (but cosmological term is present). Choosing the coefficients \( a, b, c \) in Eq.\((35)\) so that the correspondent conformal anomaly would give the anomaly for two-dimensional scalar or spinor CFT we get new examples of AdS/CFT correspondence. It is interesting that in this case new theory appears on the bulk side. Note that one can consider higher derivative gravity with Einstein term but without of cosmological term as the bulk theory.

3 Conformal anomaly for \( d = 4 \) theory

In this section, we consider \( d = 4 \) case. Substituting the expressions in \((27)\) into the action \((12)\) and using the regularization in \((28)\), we find

\[
S_{\text{in}} = -\frac{1}{2} \int dx^4 \sqrt{-g_{(0)}} \left[ l \left( a R_{(0)}^2 + b R_{(0)ij} R_{(0)ij} + c R_{(0)ijkl} R_{(0)ijkl} \right) 
\right.
\]

\[
+ \left( \frac{4a}{l^3} + \frac{8b}{l^3} + \frac{4c}{l^3} - \frac{6}{l \kappa^2} - \frac{l \Lambda}{2} \right) g_{(0)ij} g_{(2)ij}
\]

\[
+ \left( \frac{4a}{l} + \frac{12b}{l} + \frac{12c}{l} - \frac{l \kappa^2}{2} \right) g_{(1)ij} R_{(0)}^{ij}
\]

\[
+ \left( - \frac{8a}{l} - \frac{2b}{l} - \frac{2c}{l} + \frac{l}{2 \kappa^2} \right) R_{(0)} g_{(0)ij} g_{(1)ij}
\]

\[
+ \left( \frac{20a}{l^3} + \frac{8b}{l^3} + \frac{10c}{l^3} + \frac{2}{l \kappa^2} + \frac{l \Lambda}{4} \right) g_{(0)ij} g_{(0)kl} g_{(1)ijkl}
\]

\[
+ \left( \frac{6a}{l^3} + \frac{2b}{l^3} + \frac{c}{l^3} - \frac{1}{2l \kappa^2} - \frac{l \Lambda}{8} \right) \left( g_{(0)ij} g_{(1)ij} \right)^2 \right]
\]

(38)
Other terms proportional $\epsilon^{-1}$ or $\epsilon^{-2}$ which diverge when $\epsilon \to 0$ can be subtracted without loss of the general covariance and scale invariance. Since Eq. (18) has the following form in $d = 4$,

$$0 = \frac{80a}{l^4} + \frac{16b}{l^4} + \frac{8c}{l^4} - \frac{12}{k^2 l^2} - \Lambda,$$  \hspace{1cm} (39)

we find that the term in (38) proportional to $g^{ij}_{(0)} g^{(2)ij}$ vanishes. The equation given by the variation over $g^{(1)ij}$ is given by

$$0 = AR^{ij}_{(0)} + Bg^{ij}_{(0)} R_{(0)} + 2Cg^{lk}_{(0)} g^{jl}_{(0)} g^{(1)kl} + 2Dg^{ij}_{(0)} g^{kl}_{(0)} g^{(1)kl}$$

Here we used (39) to re-write the expressions of $C$ and $D$ and to remove the cosmological constant $\Lambda$. Multiplying $g^{(1)ij}$ with (40), we obtain

$$g^{ij}_{(1)ij} = - \frac{A + 4B}{2(C + 4D)} R_{(0)}.$$  \hspace{1cm} (41)

Substituting (11) into (10), we can solve (10) with respect to $g^{(1)ij}$ as follows:

$$g^{(1)ij} = - \frac{A}{2C} R^{ij}_{(0)} + \frac{AD - BC}{2C(C + 4D)} R_{(0)} g^{(0)ij}.$$  \hspace{1cm} (42)

Substituting (12) into (38) (and using (39)), we find the following expression for the anomaly $T$

$$T = \left( al - \frac{A^2 D - 4B^2 C - 2ABC}{4C(C + 4D)} \right) R^2_{(0)}$$

\[ + \left( bl - \frac{A^2}{4C} \right) R^{ij}_{(0)} R^{ij}_{(0)} + cr R^{ijkl}_{(0)} R^{ijkl}_{(0)}. \]  \hspace{1cm} (43)
If we put \( a = b = c = 0 \), the above expression reproduces the result in [4, 5]:

\[
T = -\frac{l^3}{12\kappa^2} R^2_{(0)} + \frac{l^3}{4\kappa^2} R_{(0)ij} R^{ij}_{(0)}. \tag{44}
\]

For general case, by substituting (40) into (43), we obtain

\[
T = \left( -\frac{l^3}{8\kappa^2} + 5al + bl \right) (G - F) + \frac{cl}{2} (G + F). \tag{45}
\]

Here we used the Gauss-Bonnet invariant \( G \) and the square of the Weyl tensor \( F \), which are given by

\[
G = R^2_{(0)} - 4R_{(0)ij} R^{ij}_{(0)} + R_{(0)ijkl} R^{ijkl}_{(0)},
\]

\[
F = \frac{1}{3} R^2_{(0)} - 2R_{(0)ij} R^{ij}_{(0)} + R_{(0)ijkl} R^{ijkl}_{(0)}. \tag{46}
\]

This is really beautiful and compact result which shows that apparently any variant of higher derivative AdS gravity has correspondence with some \( d = 4 \) conformal field theory. It is very general and it maybe used in various contexts as we see in next section. It maybe also used to search new examples of bulk/boundary correspondence with various higher derivative supergravities on bulk side.

Consider one particular example:

\[
d = 4, \quad \frac{l^3}{\kappa^2} - 40al - 8bl = \frac{2N^2}{(4\pi)^2}, \tag{47}
\]

Then, the anomaly of \( D = 4 \) \( \mathcal{N} = 4 \) super Yang-Mills is reproduced.

It is very interesting that last relation is still true even in case of absence the Einstein term there. In fact, in this case we found new example of AdS/CFT correspondence where boundary QFT is well-known \( \mathcal{N} = 4 \) super Yang-Mills theory and bulk side is described by higher derivative five-dimensional gravity with cosmological term and without (or with) the Einstein term. Moreover, one can consider more narrow class of theories with \( a = 0 \) or \( b = 0 \) in initial classical action.

It should be also noted that one cannot consider pure higher derivative gravity in above context because without cosmological and Einstein terms the theory is scale invariant. Then it presumably cannot be useful as bulk
side in AdS/CFT correspondence. In our approach the manifestation of this fact is that for pure $R^2$-gravity the conformal anomaly is zero.

Finally we note that when only $c$ and cosmological constant are not zero among the parameters of AdS higher derivative gravity as it follows from general Eq. (45) bulk side does not correspond to $\mathcal{N} = 4$ super Yang-Mills theory.

4 Next-to-leading corrections to $\mathcal{N} = 2$ superconformal theory trace anomaly from AdS/CFT correspondence

Our main motivation in this work was to find new higher derivative versions of bulk theory in AdS/CFT correspondence via calculation of holographic conformal anomaly. However, our general result maybe applied perfectly well for another formulations (or motivation): when higher derivatives terms in AdS gravity appear as next-to-leading corrections in low-energy string effective action (after compactification). Let us demonstrate this on the example of $\mathcal{N} = 2$ SCFT (and corresponding bulk AdS higher derivative gravity).

Recently (after the first version of this paper appeared in hep-th ), the trace anomaly of $d = 4$, $\mathcal{N} = 2$ and $\mathcal{N} = 4$ SCFT has been investigated up to next to leading order in the $1/N^2$ expansion by Blau, Narain and Gava [22]. For an $\mathcal{N} = 2$ SCFT theory with $n_V$ vector multiplets and $n_H$ hypermultiplets, the trace anomaly is, by usual UV calculations, given by

$$T = \frac{1}{24 \cdot 16\pi^2} \left[ -\frac{1}{3}(11n_V + n_H)R^2 
+ 12n_V R_{ij} R^{ij} + (n_H - n_V) R_{ijkl} R^{ijkl} \right], \quad (48)$$

and especially, when the gauge group is $Sp(N)$,

$$T = \frac{1}{24 \cdot 16\pi^2} \left[ - \left( 8N^2 + 6N - \frac{1}{3} \right) R^2 
+ (24N^2 + 12N) R_{ij} R^{ij} + (6N - 1) R_{ijkl} R^{ijkl} \right]. \quad (49)$$

The $\mathcal{N} = 2$ theory with the gauge group $Sp(N)$ arise as the low-energy theory on the world volume on $N$ D3-branes sitting inside 8 D7-branes at an
The string theory dual to this theory has been conjectured to be type IIB string theory on $\text{AdS}_5 \times X^5$ where $X_5 = S_5/Z_2$, whose low energy effective action is given by (see the corresponding derivation in ref. [22])

$$S = \int_{\text{AdS}_5} d^5x \sqrt{G} \left\{ \frac{N^2}{4\pi^2} (R - 2\Lambda) + \frac{6N}{24 \cdot 16\pi^2} R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \right\}. \quad (50)$$

In order to use the general result for the conformal anomaly $T$ in (45), we compare (50) with (24) and find

$$\frac{1}{\kappa^2} = \frac{N^2}{4\pi^2}, \quad a = b = 0, \quad c = \frac{6N}{24 \cdot 16\pi^2}. \quad (51)$$

Furthermore we need to choose

$$\Lambda = -\frac{12}{\kappa^2} = -\frac{12N^2}{4\pi^2}, \quad (52)$$

which corresponds to that the length scale is chosen to be unity in [22]. We should note, however, the length scale $l$ in our paper has a correction. Substituting (51) and (52) into (39), we find

$$\frac{1}{l^2} = 1 + \frac{2ck^2}{3l^2} \quad = 1 + \frac{2ck^2}{3} + \mathcal{O}\left((ck^2)^2\right). \quad (53)$$

In the second line in (53), we have assumed $ck^2 = \frac{1}{16N}$ is small or $N$ is large. Then we find the following expression for $T$:

$$T = \left( -\frac{1}{8\kappa^2} + \frac{c}{8} \right) (G - F) + \frac{c}{2} (G + F) + \frac{1}{\kappa^2} \mathcal{O}\left((ck)^2\right)$$

$$= \frac{N^2}{16\pi^2} \left( -\frac{1}{3} R_{(0)}^2 + R_{(0)ij} R_{(0)}^{ij} \right)$$

$$+ \frac{6N}{24 \cdot 16\pi^2} \left( \frac{3}{4} R_{(0)}^2 - \frac{13}{4} R_{(0)ij} R_{(0)}^{ij} + R_{(0)ijkl} R_{(0)}^{ijkl} \right) + \mathcal{O}(1). \quad (54)$$

In the first line in (54), we have substituted (53) and put $a = b = 0$ and in the second line, (51), (52) and the explicit expression for $G$ and $F$ in (46) are substituted. Eq. (54) exactly reproduces the result in (48) as in [22].
Hence, we demonstrated how to apply our general formalism to calculate next-to-leading order corrections to conformal anomaly from bulk AdS side. Similarly, one can get conformal anomaly from bulk side for any other low-energy string effective action.

5 Discussion

In summary, we studied higher derivative gravity as bulk theory in AdS/CFT correspondence. We explicitly calculated the conformal anomaly from such theory following to Witten proposal and assuming that they describe some boundary QFTs. This approach opens new lines in AdS/CFT correspondence as it presents new versions of bulk theory. For example, three-dimensional $R^2$ gravity with cosmological constant and with (or without) Einstein term may describe $d = 2$ scalar or spinor CFT. In the same way, five-dimensional $R^2$ gravity with cosmological constant and with (or without) Einstein term may be considered as bulk theory for $\mathcal{N} = 4$ super Yang-Mills quantum theory. Pure $R^2$ gravity seems to be not good as bulk theory due to its scale invariance (no mass scale). We also applied our general result to calculation of next-to-leading corrections to conformal anomaly of $\mathcal{N} = 4$ SCFT from bulk AdS higher derivative gravity. Perfect agreement with results of very recent work [22] is found.

There are also interesting possibilities to extend the results of present work. In the recent paper [19] it was studied the conformal anomaly of submanifold observables in AdS/CFT correspondence. Such investigation is expected to be useful for the study of Wilson loops in large $N$ theories in frames of bulk-boundary correspondence[20]. It is clear that modification of bulk theory (say of the form we discuss in this work) will also modify the conformal anomaly on submanifolds.

In the recent paper [21] the form of Witten proposal has been applied to study the boundary stress tensor of anti-de Sitter Einstein gravity in $d+1 = 3$ and $d + 1 = 5$ dimensions via AdS/CFT correspondence. As by-product the conformal anomaly in $d = 2$ and $d = 4$ is also recovered. It would be of interest to re-consider this problem for our version of higher derivative gravity as it may lead to acceptable definition of the gravitational energy and stress tensor even in such case.

For example, the Casimir energy of $d = 4$ $SU(N) \mathcal{N} = 4$ super Yang-Mills
theory on $S^3 \times R$ which is global boundary of five-dimensional AdS is given by

$$E_{\text{Casimir}} = \frac{3(N^2 - 1)}{16r}.$$  \hspace{1cm} (55)

Here $r$ is the radius of $S^3$. By multiplying $\sqrt{-g_{tt}} = r/l$, the Casimir mass is given by

$$M_{\text{Casimir}} = \frac{3(N^2 - 1)}{16l},$$  \hspace{1cm} (56)

which gives the non-vanishing ground state energy for AdS$_5$, i.e. corresponding BH mass \textsuperscript{[21]}. Using eq.(47) as the conversion formula to gauge theory variables from higher derivative gravity we get AdS mass for higher derivative gravity suggesting the way to construct stress-energy tensor in such theory.

As the final remark we would like to point out that Eq.(45) maybe used in the constructive way in order to find new examples of bulk/boundary relations. Let us imagine that we got some variant of low-energy string effective action (after some compactification) as particular version of $d = 5$ AdS higher derivative gravity. It may also come in another way as bosonic sector of some higher derivative AdS supergravity. Eq.(47) gives holographic conformal anomaly (up to next-to-leading terms) from bulk side. Having such conformal anomaly maybe often enough to suggest new boundary CFT.

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