Generalized almost para-contact manifolds

Bayram Şahin and Fulya Şahin

Department of Mathematics, Inonu University, 44280, Malatya-Turkey.
(emails:bayram.sahin@inonu.edu.tr, fulya.sahin@inonu.edu.tr)

Abstract. In this paper, we introduce generalized almost para-contact manifolds and obtain normality conditions in terms of classical tensor fields. We show that such manifolds naturally carry certain Lie bialgebroid/quasi-Lie algebroid structures on them and we relate this new generalized manifolds with classical almost para-contact manifolds. The paper contains several examples.

1. Introduction

As a unification and extension of usual notions of complex manifolds and symplectic manifolds, the notion of generalized complex manifolds was introduced by Hitchin [8]. This subject has been studied widely in [7] by Gualtieri. In particular, the notion of a generalized Kähler structure was introduced and studied by Gualtieri in the context of the theory of generalized geometric structures. Later such manifolds and their submanifolds have been studied in many papers. For instance, in [3], Crainic gave necessary and sufficient conditions for a generalized almost complex manifold to be generalized complex manifold in terms of classical tensor fields. Generalized pseudo-Kähler structures have been also studied recently in [4]. Generalized geometry has received a reasonable amount of interest due to possible several relations with mathematical physics. Indeed, generalized Kähler structures describe precisely the bi-Hermitian geometry arising in super-symmetric $\sigma$– models [5].

A central idea in generalized geometry is that $TM \oplus T^\ast M$ should be thought of as a generalized tangent bundle to manifold $M$. If $X$ and $\xi$ denote a vector field and a dual vector field on $M$ respectively, then we write $(X, \alpha)$ (or $X + \alpha$) as a typical element of $TM \oplus T^\ast M$. The space of sections of the vector bundle $TM \oplus T^\ast M$ is endowed with two natural $\mathbb{R}^-$ bilinear operations: for the sections $(X, \alpha), (Y, \beta)$ of $TM \oplus T^\ast M = TM$, a symmetric bilinear form $\langle, \rangle$ is defined by

$$\langle X + \alpha, Y + \beta \rangle = \frac{1}{2}(i_X \beta + i_Y \alpha), \hspace{1cm} (1.1)$$


and the Courant bracket of two sections is defined by

$$\llbracket (X, \alpha), (Y, \beta) \rrbracket = [X, Y] + L_X \beta - L_Y \alpha - \frac{1}{2} d(i_X \beta - i_Y \alpha),$$

(1.2)

where $d$, $L_X$, and $i_X$ denote exterior derivative, Lie derivative and interior derivative with respect to $X$, respectively. The Courant bracket is antisymmetric but, it does not satisfy the Jacobi identity. In this paper we adapt the notions

$$\beta(\pi^* \alpha) = \pi(\alpha, \beta)$$

and

$$\omega_\pi(X)(Y) = \omega(X, Y)$$

(1.3)

which are defined as $\pi^* : TM^* \to TM$, $\omega_\pi : TM \to TM^*$ for any 1-forms $\alpha$ and $\beta$, 2-form $\omega$ and bivector field $\pi$, and vector fields $X$ and $Y$. Also we denote by $[,]_\pi$, the bracket on the space of 1-forms on $M$ defined by

$$[\alpha, \beta]_\pi = L_{\pi^* \alpha} \beta - L_{\pi^* \beta} \alpha - d\pi(\alpha, \beta).$$

(1.4)

As an analogue of generalized complex structures on even dimensional manifolds, the concept of generalized almost subtangent manifolds were introduced in [14] and such manifolds have been studied in [14] and [16]. On the other hand, the notion of a generalized contact pair on a manifold $M$ was introduced by Foon and Wade in [11], see also [13], [15] and [17]. As we mention above, the framework of generalized almost complex structures puts almost symplectic structures and almost complex structures on an equal footing. Similarly the notion of a generalized almost contact structure unifies almost cosymplectic structures and almost contact structures.

In this paper, we introduce generalized almost para-contact structures/manifolds and show that such manifolds include para-contact manifolds as a subclass. Then we investigate normality conditions for generalized almost para-contact manifolds in terms of classical tensor fields and obtain certain Lie algebroid structures (Courant algebroid, quasi-Lie bialgebroid) on such manifolds. We give various examples and show that classical almost para-contact manifolds can be described in this generalized geometry.

The paper is organized as follows. In section 2, we recall some basic notions needed for the paper. In section 3, we define generalized almost para-contact manifold, give examples and obtain normality conditions for generalized almost para-contact manifolds in terms of classical tensor fields. In section 4, we investigate Lie algebroid structures on generalized almost para-contact manifolds. For this aim, we construct several subbundles of big tangent bundle and show that all these subbundles are isotropic and then we use this information to obtain a characterization for quasi-Lie bialgebroid structure on a almost generalized para-contact manifold. We also introduce to the notion of strong generalized para-contact manifold and provide an example. In section 5, we show that classical normal almost para-contact manifolds are strong generalized para-contact manifolds.
2. Preliminaries

An \((2n + 1)\)-dimensional smooth manifold \(M\) has an almost para-contact structure \((\varphi, E, \eta)\) if it admits a tensor field \(\varphi\) of type \((1, 1)\), a vector field \(E\) and a \(1\)-form \(\eta\) satisfying the following compatibility conditions

\[
\varphi(E) = 0, \quad \eta \circ \varphi = 0 \quad (2.1)
\]

\[
\eta(E) = 1, \quad \varphi^2 = id - \eta \otimes E \quad (2.2)
\]

and the tensor field \(\varphi\) induces an almost paracomplex structure on each fibre on the distribution \(\mathcal{D}\) generated by \(\eta\). An immediate consequence of the definition of the almost para-contact structure is that the endomorphism \(\varphi\) has rank \(2n\).

Let \(M(2n+1)\) be an almost para-contact manifold with structure \((\varphi, E, \eta)\) and consider the manifold \(M(2n+1) \times \mathbb{R}\). We denote a vector field on \(M(2n+1) \times \mathbb{R}\) by \((X, f \frac{d}{dt})\), where \(X\) is tangent to \(M(2n+1)\), \(t\) is the coordinate on \(\mathbb{R}\), and \(f\) is a \(C^\infty\) function on \(M(2n+1) \times \mathbb{R}\). An almost paracomplex structure \(J\) on \(M(2n+1) \times \mathbb{R}\) is defined by

\[
J(X, f \frac{d}{dt}) = (\varphi X + fE, \eta(X) \frac{d}{dt}).
\]

If \(J\) is integrable which means that the Nijenhuis tensor of \(J\), \(N_J\), vanishes, we say that the almost para-contact structure \((\varphi, E, \eta)\) is normal. For details, see: [9] and [18].

We now recall the notion of generalized almost para-complex structure on \(T\mathcal{M} = TM \oplus TM^*\). A generalized almost paracomplex structure on \(M\) is a vector bundle automorphism \(J\) on \(T\mathcal{M}\) such that \(J^2 = I, J \neq I\) and \(J\) is orthogonal with respect to \(\langle,\rangle\), i.e.,

\[
\langle J e_1, e_2 \rangle + \langle e_1, Je_2 \rangle = 0, \quad e_1, e_2 \in \Gamma(T\mathcal{M}). \quad (2.3)
\]

A generalized almost paracomplex structure can be represented by classical tensor fields as follows:

\[
J = \begin{bmatrix}
a & \pi^4 \\
\theta & -a^*
\end{bmatrix}
\]

where \(\pi\) is a bivector on \(M\), \(\theta\) is a \(2\)-form on \(M\), \(a : TM \to TM\) is a bundle map, and \(a^* : TM^* \to TM^*\) is dual of \(a\), for almost para-complex structures see: [13] and [16].

A generalized almost paracomplex structure is called integrable (or just para-complex structure) if \(J\) satisfies the following condition

\[
[[J \alpha, J \beta]] - J([J \alpha, \beta] + [\alpha, J \beta]) + [\alpha, \beta] = 0, \quad (2.5)
\]

for all sections \(\alpha, \beta \in \Gamma(T\mathcal{M})\).
3. Generalized almost para-contact manifolds

In this section, we are going to define generalized almost para-contact structure and obtain the normality conditions. We first propose the following definition.

**Definition 1.** A generalized almost para-contact structure \((F, Z, \xi)\) on a smooth odd dimensional manifold \(M\) consists of a bundle endomorphism \(F\) from \(T\mathcal{M}\) to itself and a section \(Z + \xi\) of \(T\mathcal{M}\) such that

\[
\begin{align*}
F + F^* &= 0, \quad \xi(Z) = \mathcal{I} \\
F(\xi) &= 0, \quad F(Z) = 0 \\
F^2 &= \mathcal{I} - Z \circ \xi
\end{align*}
\]

where \(Z \circ \xi\) is defined by

\[
(Z \circ \xi)(X + \alpha) = \xi(X)Z + \alpha(Z)\xi.
\]

The bundle map \(F : T\mathcal{M} \rightarrow T\mathcal{M}\) is given by

\[
F = \begin{bmatrix} F & \pi^2 \\ \sigma^2 & -F^* \end{bmatrix},
\]

where \(F : TM \rightarrow TM\) is a bundle map. The following example shows that generalized almost para-contact structure is really a generalization of almost para-contact structure.

**Example 1.** Associated to any almost para-contact structure, we have a generalized almost para-contact structure by setting

\[
F = \begin{bmatrix} \varphi & 0 \\ 0 & -\varphi^* \end{bmatrix},
\]

with the given vector field \(E\) and 1-form \(\eta\).

We give another example of generalized almost para-contact manifolds.

**Example 2.** Let \(H_3\) be the three-dimensional Heisenberg group and \(\{X_1, X_2, X_3\}\) a basis for its algebra \(h_3\) so that \([X_1, X_2] = -X_3\). Let \(\{\alpha^1, \alpha^2, \alpha^3\}\) be a dual frame. Then \(d\alpha^3 = \alpha^1 \wedge \alpha^2\). Now for some real number \(\vartheta\), we define

\[
\begin{align*}
F &= \cosh \vartheta (X_2 \otimes \alpha^2 + X_3 \otimes \alpha^3), \quad \xi = \alpha^1, \quad Z = X_1 \\
\sigma &= \sinh \vartheta (\alpha^2 \wedge \alpha^3), \quad \pi = \sinh \vartheta (X_2 \wedge X_3).
\end{align*}
\]

We also, as given in (3.5), define

\[
F = \begin{bmatrix} F & \pi^2 \\ \sigma^2 & -F^* \end{bmatrix}.
\]
Then \((\xi, Z, \pi, \sigma, F)\) is a generalized almost para-contact structure on \(H_3\).

We now obtain the normality conditions for a generalized almost para-contact structure. For this aim, we make the following definition.

**Definition 2.** A generalized almost para-complex structure \(J\) on \(M \times \mathbb{R}\) is said to be \(M\)-adapted if it has the following three properties

(i) \(J\) is invariant by translation along \(\mathbb{R}\).

(ii) \(J(T\mathbb{R} \oplus 0) \subseteq 0 \oplus TM^*\).

(iii) \(J(0 \oplus T\mathbb{R}^*) \subseteq TM \oplus 0\).

The invariance of \(J\) by translations means that the Lie derivatives \(\frac{d}{dt}\) of the classical tensor fields of \(J\) defined by (2.4) vanish. If conditions (ii) and (iii) are also imposed, it follows that the classical tensor fields of an \(M\)-adapted, generalized almost para-complex structure are of the form

\[
a = F, \quad \pi = P + Z \wedge \frac{d}{dt}, \quad \theta = \sigma + \xi \wedge dt
\]

where \(P\) is a bivector on \(M\), \(\sigma\) is a 2-form on \(M\), \(F: TM \to TM\) is a bundle map. A generalized, almost para-contact structure will be called normal if the corresponding \(M\)-adapted, generalized almost para-complex structure on \(M \times \mathbb{R}\) is integrable. The following theorem gives necessary and sufficient conditions for a generalized almost para-contact manifold.

**Theorem 3.1.** A generalized, almost para-contact structure is normal if and only if the following conditions are satisfied.

(A1) \(P\) satisfies the equation

\[
[P^\sharp \alpha_1, P^\sharp \beta_1] = P^\sharp ([\alpha_1, \beta_1]_P).
\]  

(3.7)

(A2) \(P\) and \(F\) are related by the following two formulas

\[
FP^2 = P^\sharp F^*,
\]

(3.8)

\[
F^\sharp ([\alpha_1, \beta_1]_P) = L_P(\alpha_1) \cdot F^\sharp \beta_1 - L_P(\beta_1) \cdot F^\sharp \alpha_1 + dP(\beta_1, F^\sharp \alpha_1).
\]  

(3.9)

(A3) \(P, \sigma\) and \(F\) are related by the following four formulas

\[
i_Z\sigma = 0, i_\xi P = 0, \quad F^2 = Id - P^\sharp \sigma^b - Z^f \otimes \xi
\]

(3.10)

\[
N_F(X, Y) = P^\sharp(i_X \wedge Y (d\sigma^y)) - (d\xi(X, Y))^Z.
\]  

(3.11)

(A4) \(\xi, Z, F\) and \(\sigma^y\) are related by the following formulas

\[
F(Z) = 0, \xi \circ F = 0, \quad (L_{FX}\xi)Y - (L_FY\xi)X = 0
\]

(3.12)

\[
d\sigma_F(X, Y, Z) = d\sigma(FX, Y, Z) + d\sigma(X, FY, Z)
\]

\[
+ d\sigma(X, Y, FZ).
\]  

(3.13)
(A5) \( \xi, P, F \) and \( Z \) satisfy the following equations

\[
L_Z \xi = 0, \quad L_Z P = 0, \quad L_Z F = 0, \quad L_Z \sigma^\xi = 0, \quad L_{P^\xi} \alpha_1 = 0 \quad (3.14)
\]

for \( X, Y, Z \in \Gamma(TM) \) and \( \alpha_1, \beta_1 \in \Gamma(TM^*) \).

**Proof.** The condition \( J^2 = J \) implies the first equations appearing in (A2), (A3) and (A4). For remaining parts, we need to check the integrability of \( J \), i.e.

\[
\mathcal{N}(\mathcal{X}, \mathcal{Y}) = [J \mathcal{X}, J \mathcal{Y}] - J[\mathcal{X}, \mathcal{JY}] - J[\mathcal{JX}, \mathcal{Y}] + [\mathcal{X}, \mathcal{Y}] = 0
\]

for \( \mathcal{X}, \mathcal{Y}, \mathcal{Z} \in \Gamma(TM \oplus \mathbb{R}) \). First, if \( \mathcal{X} = (0, 0, \alpha_1, 0) \) and \( \mathcal{Y} = (0, 0, \beta_1, 0) \) are 1-forms, then we have, as the vector field part,

\[
[P^\xi \alpha_1, P^\xi \beta_1] = P^\xi (L_{P^\xi} \alpha_1 - L_{P^\xi} \beta_1 - dP(\alpha_1, \beta_1))
\]

which is \( (3.14) \) due to \( (3.10) \). For the 1-form part, we get \( (3.11) \). Now for vector fields \( \mathcal{X} = (X, 0, 0, 0) \) and \( \mathcal{Y} = (Y, 0, 0, 0) \) we have

\[
[FX, FY] - F[FX, Y] - F[X, FY] + [X, Y] = P^\xi (L_X \sigma^\xi(Y) - L_Y \sigma^\xi(X) + d\sigma(X, Y)) = (L_Y \xi(X))(Z) + (L_Y \xi(X))(Z) = 0.
\]

But using the following formula

\[
i_{X \wedge Y}(d\sigma) = L_X(i_Y(\sigma)) - L_Y(i_X(\sigma)) + d(i_{X \wedge Y}\sigma) - d(i_{X, Y}\sigma)
\]

we arrive at

\[
N_F(X, Y) = P^\xi(i_{X \wedge Y}(d\sigma)) - (d\xi(X, Y))Z
\]

which is \( (3.11) \). For 1-form part, we have

\[
\begin{align*}
L_F X \sigma^\xi(Y) - L_F Y \sigma^\xi(X) & = \frac{1}{2}d\sigma(Y, FX) + \frac{1}{2}d\sigma(X, FY) \\
+ F^\ast(L_X \sigma^\xi(Y) - L_Y \sigma^\xi(X)) & + d\sigma(X, Y) - \sigma^\xi([X, FY]) \\
& - \sigma^\xi([X, FY]) = 0
\end{align*}
\]

and

\[
FX(\xi(Y)) - FY(\xi(X)) - \xi([FX, Y]) - \xi([X, FY]) = 0. \quad (3.15)
\]

Now using the following formula

\[
d\sigma(X, Y, Z) = L_X \sigma(Y, Z) + L_Y \sigma(Z, X) + L_Z \sigma(X, Y) \\
- \sigma([X, Y], Z) - \sigma([Z, X], Y) - \sigma([Y, Z], X)
\]

then, \( (3.15) \) and \( (3.16) \) give \( (3.12) \) and \( (3.13) \). For \( \mathcal{X} = (X, 0, 0, 0) \) and \( \mathcal{Y} = (0, 0, \alpha_1, 0) \), we have, as 1-form part,

\[
\begin{align*}
L_X \alpha_1 - L_{FX} F^\ast \alpha_1 & - L_{P^\xi} \alpha_1 \sigma^\xi(X) + F^\ast(L_{FX} \alpha_1 - L_X F^\ast \alpha_1) \\
+ \frac{1}{2}d(\alpha_1(F^2 X) + \sigma(X, P^\xi \alpha_1)) & - \alpha_1(X) + \alpha_1(Z) \xi(X) \\
& - \sigma^\xi([X, P^\xi \alpha_1]) = 0.
\end{align*}
\]
Then from the third equation of (3.10) we arrive at
\[
L_X\alpha_1 - L_{FX}F^* \alpha_1 - L_{P^2 \alpha_1} \sigma^b(X) + F^*(L_{FX} \alpha_1 - L_X F^* \alpha_1)
- d\alpha_1(P^2 \sigma^b(X)) - \sigma^b([X, P^2 \alpha_1]) = 0.
\] (3.17)

And for the vector field part, we obtain
\[
[FX, P^2 \alpha_1] - F[X, P^2 \alpha_1] - P^2(L_{FX} \alpha_1 - L_X F^* \alpha_1) = 0
\]
which is equivalent to (3.19). For \(X = (0, 0, 0)\) and \(Y = (0, 0, 0, dt)\), we obtain the first equation in (3.14). In a similar way, we derive the other equations.

Now by using \(F, \sigma, P\), we can write the endomorphism \(F\) of \(\mathcal{T}M\) as
\[
F \begin{pmatrix} X \\ \alpha \end{pmatrix} = \begin{pmatrix} F & P^\sharp \\ \sigma^t & -F^* \end{pmatrix} \begin{pmatrix} X \\ \alpha \end{pmatrix}.
\] (3.18)

Also using \(Z\) and \(\xi\), we can define the endomorphism \(\mathfrak{z}\) of \(\mathcal{T}M\) given by
\[
\mathfrak{z} \begin{pmatrix} X \\ \alpha \end{pmatrix} = \begin{pmatrix} Z \otimes \xi \\ 0 & (Z \otimes \xi)^t \end{pmatrix} \begin{pmatrix} X \\ \alpha \end{pmatrix}.
\] (3.19)

Now we give a characterization for normality in terms of \((F, \mathfrak{z}, (Z, \xi))\).

**Theorem 3.2.** Let \(M\) be an odd dimensional manifold. Then \((F, \mathfrak{z}, (Z, \xi))\) defines generalized normal contact structure on \(M\) if and only if the following conditions are satisfied.

(i) For \((X, \alpha) \in \Gamma(\mathcal{T}M)\), \(F\) and \(\mathfrak{z}\) are related by the following formulas
\[
F^2 = I - \mathfrak{z}
\] (3.20)
\[
\mathcal{N}_F((X, \alpha), (Y, \beta)) = \mathfrak{z}[(X, \alpha), (Y, \beta)].
\] (3.21)

(ii) For \((X, \alpha) \in \Gamma(im F)\), \(Z, \xi\) and \(F\) are related by the following formulas
\[
\|Z \oplus \xi\|_g = 1
\] (3.22)
\[
[(Z, 0), F((X, \alpha))] = F(L_Z X, L_Z \alpha),
\] (3.23)

where \(g\) is the neutral metric of \(\mathcal{T}M\) given by
\[
g((X, \alpha), (Y, \beta)) = \alpha(Y) + \beta(X).
\]

(iii) For \((X, \alpha) \in \Gamma(im F)\), \(Z, \xi\) and \(F\) are related by the following formulas
\[
\mathcal{g}_g \circ F + F^t \circ \mathcal{g}_g = 0, F \circ \mathfrak{z} = 0
\] (3.24)
\[
[F(X, \alpha), (0, \xi)] = F(0, L_X \xi).
\] (3.25)
Proof. Using (3.6) and $J^2 = I$ we have

\[
(F^2 + P^t \sigma^\flat)(X) + (FP^t - P^t F^*) (\alpha)
\]

\[
g FZ + \xi(X)Z + (\sigma^\flat(X) - F^* \alpha + f \xi(Z)) \frac{d}{dt}
\]

\[
(\sigma^\flat \circ F - F^* \circ \sigma^\flat)(X) + (\sigma^\flat \circ P^t + F^* \sigma^\flat)(\alpha)
\]

\[
g \sigma^\flat(Z) - f F^* \xi + \alpha(Z) \xi + (\xi(FX + P^t \alpha + g Z)) dt
\]

\[
= X + f \frac{d}{dt} \alpha + \alpha + g dt.
\]  

(3.26)

Then (3.20), (3.22) and (3.24) follow from (3.26). On the other hand we have

\[
J((X + f \frac{d}{dt}, \alpha + g dt) = F(X, \alpha) + g(Z, 0) + f(0, \xi)
\]

\[
(\alpha(Z) \frac{d}{dt}, \xi(X)) dt.
\]  

(3.27)

It is clear that $(X, \alpha) \in \Gamma(imF)$ if and only if $\xi(X) = 0$ and $\alpha(Z) = 0$. Now we check the normality conditions. Since

\[
T \mathcal{M} = \text{im} F \oplus Sp\{(Z, 0)\} \oplus Sp\{(0, \xi)\} \oplus Sp\{\frac{d}{dt}\} \oplus Sp\{(0, dt)\},
\]

it will be enough to consider combinations $(X, \alpha) \in \Gamma(imF)$, $(0, \xi)$, $(\frac{d}{dt}, 0)$, $(0, dt)$. For $(\frac{d}{dt}, 0)$, $(0, dt)$, since $J(\frac{d}{dt}, 0) = (0, \xi)$ and $J(0, dt) = (Z, 0)$, we obtain

\[
N_J((\frac{d}{dt}, 0), (0, dt)) = 0 \iff L_Z \xi = 0.
\]  

(3.28)

For $(Z, 0)$, $(0, \xi)$, we get

\[
N_J((Z, 0), (0, \xi)) = 0 \iff L_Z \xi = 0.
\]  

(3.29)

For $(X, \alpha) \in \Gamma(imF)$, $(0, dt)$, using (3.27) we have

\[
N_J((X, \alpha), (0, dt)) = 0 \iff [F(X, \alpha), (Z, 0)] = F(L_X Z, L_Z \alpha).
\]  

(3.30)

For $(X, \alpha) \in \Gamma(imF)$, $(\frac{d}{dt}, 0)$, using (3.27) we obtain

\[
N_J((X, \alpha), (\frac{d}{dt}, 0)) = 0 \iff [F(X, \alpha), (0, \xi)] = F(0, L_X \xi).
\]  

(3.31)

For $(X, \alpha), (Y, \beta) \in \Gamma(imF)$, again from (3.27) we derive

\[
N_J((X, \alpha), (Y, \beta)) = 0 \iff N_J((X, \alpha), (Y, \beta)) - 3[F(X, \alpha), (Y, \beta)] = 0.
\]  

(3.32)

Remaining combinations give the conditions obtained in (3.28)-(3.31). Note that the condition $L_Z \xi = 0$ in (3.28) is equivalent to (3.21).
4. Lie bialgebroid structures on generalized almost para-contact manifolds

In this section we are going to investigate necessary and sufficient conditions for the existence of Lie bialgebroid structures on a generalized para-contact manifold. We first recall some notions needed for this section. A Lie algebroid structure on a real vector bundle $A \to M$ is defined by a vector bundle map $\rho_A : A \to TM$, the anchor of $A$, and an $\mathbb{R}$-Lie algebra bracket on $\Gamma(A), [\cdot, \cdot]_A$ satisfying the Leibnitz rule

$$[\alpha, f\beta]_A = f[\alpha, \beta]_A + L_{\rho_A(\alpha)}(f)\beta$$

for all $\alpha, \beta \in \Gamma(A), f \in C^\infty(M)$, where $L_{\rho_A(\alpha)}$ is the Lie derivative with respect to the vector field $\rho_A(\alpha)$. Suppose that $A \to P$ is a Lie algebroid, and that its dual bundle $A^* \to P$ also carries a Lie algebroid structure. Then $(A; A^*)$ is a Lie bialgebroid if for any $X, Y \in \Gamma(A)$,

$$d^*_X[Y] = L_X d^*_Y - L_Y d^*_X,$$

where $d^*_\cdot$ is the exterior derivative associated to $A^*$. On the other hand, a Courant algebroid $E \to M$ is a vector bundle on $M$ equipped with a non-degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$, a vector bundle map $\rho : E \to TM$, a bilinear bracket $\circ$ on $\Gamma(E)$ satisfying

(c1) $e_1 \circ (e_2 \circ e_3) = (e_1 \circ e_2) \circ e_3 + e_2 \circ (e_1 \circ e_3),$

(c2) $e \circ e = \rho^* d \langle e, e \rangle,$

(c3) $L_{\rho(e)} < e_1, e_2 >= < e \circ e_1, e_2 > + < e_1, e \circ e_2 >,$

(c4) $\rho(e_1 \circ e_2) = \rho[e_1, e_2],$

(c5) $e_1 \circ f e_2 = f(e_1 \circ e_2) + L_{\rho(e_1)} f e_2$

for all $e, e_1, e_2, e_3 \in \Gamma(E)$ and $f \in C^\infty(M, \mathbb{R})$. It is known that $(TM \oplus TM^*)$ with the non-degenerate metric $\langle \cdot, \cdot \rangle$ given in (1.1) and Courant bracket form is a Courant algebroid. The anchor is the natural projection from the direct sum to the summand $TM$. The following result is well known.

**Theorem 4.1.** If $(A, A^*)$ is a Lie bialgebroid, then $A \oplus A^*$ together with $([X, Y]_A, [\cdot, \cdot])$ is a Courant algebroid, where $[\cdot, \cdot]$ denotes the Courant bracket given in (1.2). Conversely, in a Courant algebroid $(E, \rho, [\cdot, \cdot], \circ)$, suppose that $L_1$ and $L_2$ are Dirac subbundles transversal to each other, then $(L_1, L_2)$ is a Lie bialgebroid.

A real, maximal isotropic sub-bundle $L \subset TM \oplus TM^*$ is called an almost Dirac structure. If $L$ is involutive, then the almost Dirac structure is said to be integrable, or simply a Dirac structure. Similarly, a maximal isotropic and involutive para-complex sub-bundle $L \subset (TM \oplus TM^*)_\mathbb{C}$ is called a para-complex
A quasi-Lie bialgebroid \([12]\) is a Lie algebroid \((A, [\cdot, \cdot]_A, \rho)\) equipped with a degree-one derivation \(d^*\) of the Gerstenhaber algebra \(\Gamma(\wedge^\bullet, [\cdot, \cdot], \wedge)\) and a 3-section of \(A\), \(X_A \in \Gamma(\wedge^3 A)\) such that \(d^*_A X_A = 0\) and \(d^2_A = [X_A, \cdot]\). If \(X_A\) is the null section, then \(d^*_A\) defines a structure of Lie algebroid on \(A^*\) such that \(d^*_A\) is a derivation of \([\cdot, \cdot]\_A\).

Let \((A, [\cdot, \cdot]_A, \rho)\) be a Lie algebroid and consider any closed 3-form \(\phi\). Equipping \(A^*\) with the null Lie algebroid structure, \((A^*, d^*_A, \pi)\) is canonically a quasi-Lie bi-algebroid. We note that Lie algebroid structures have been also extended to Poisson-Nijenhuis structures \([6]\).

Let \(M\) be a generalized almost para-contact manifold and \(F\) the bundle map given in (3.5). Then it has one real eigenvalue, namely, 0. The corresponding eigenbundle is trivialized by \(Z\) and \(\xi\), and we denote these bundles by \(L_Z\) and \(L_\xi\), respectively. Let \(ker\xi\) be the distribution on the manifold \(M\) defined by the point-wise kernel of the 1-form \(\xi\). Similarly, \(ker\overline{Z}\) is the sub-bundle of \(TM^*\) defined by the point-wise kernel of the vector field \(Z\) with respect to its evaluation on differential 1-forms. On the paracomplexified bundle \((TM \oplus TM^*)_C\), we have that \(F\) has three eigenvalues, namely, 0, 1 and \(-1\). We now define

\[
E^{(1,0)} = \{X + eF\mathcal{X} \mid \mathcal{X} \in \Gamma(ker\overline{Z} \oplus ker\xi)\}
\]

\[
E^{(0,1)} = \{X - eF\mathcal{X} \mid \mathcal{X} \in \Gamma(ker\overline{Z} \oplus ker\xi)\}
\]

where \(e^2 = 1\). The extension \(\mathcal{F}\) of the endomorphism \(\mathcal{F}\) with eigenvalues \(\mp 1\) to \((TM \oplus TM^*)_C\) has eigenvalues \(\mp e\), see \([1]\) for para-complex structures on a real vector space and other notions. Then \(L_Z \oplus L_\xi\) is the 0-eigenbundle, \(E^{(1,0)}\) is the 1-eigenbundle and \(E^{(0,1)}\) is the \((-1)\)-eigenbundle. We have natural decomposition \((TM \oplus TM^*)_C = L_Z \oplus L_\xi \oplus E^{(1,0)} \oplus E^{(0,1)}\). We now have the following four different paracomplex vector bundle which play different roles

\[
L = L_Z \oplus E^{(1,0)} \quad , \quad \overline{L} = L_Z \oplus E^{(0,1)}
\]

\[
L^* = L_\xi \oplus E^{(0,1)} \quad , \quad \overline{L}^* = L_\xi \oplus E^{(1,0)}.
\]

Since \(L_Z\) is a real line bundle, its paraconjugation is itself. Therefore paracomplex conjugation sends \(L\) to \(\overline{L}\). All of these bundles are independent of the choice of representatives of a generalized almost contact structure.

**Lemma 4.1.** The bundles \(L, L^*, \overline{L}, \overline{L}^*, E^{(1,0)}\) and \(E^{(0,1)}\) are isotropic with respect to \(<,>\) given in \((1.1)\).

**Proof.** For \(X + \alpha\), using \((3.5)\) we have

\[
\mathcal{F}(X + \alpha) = FX + \pi^2\alpha + \sigma_\xi(X) - F^*\alpha.
\]

Now if \(X + \alpha \in \Gamma(ker\xi \oplus ker\overline{Z})\), from \((1.1)\) we get

\[
< Z, \mathcal{F}(X + \alpha) > = \frac{1}{2} i_Z \sigma_\xi(X) - \alpha(F\overline{Z}).
\]
Then using (3.2) and (3.10) we have
\[ <Z, \mathcal{F}(X + \alpha)> = 0. \quad (4.1) \]
In a similar way we also have
\[ <\xi, \mathcal{F}(X + \alpha)> = 0. \quad (4.2) \]
Thus if \( X + \alpha \in \Gamma(\ker \xi \oplus \ker Z) \), then \( \mathcal{F}(X + \alpha) \in \Gamma(\ker \xi \oplus \ker Z) \). For \( X + \alpha, Y + \beta \in \Gamma(\ker \xi \oplus \ker Z) \), using (3.5) we have
\[ <X + \alpha + e\mathcal{F}(X + \alpha), Y + \beta + e\mathcal{F}(X + \beta)> = \beta(X) + e\beta(FX) + e\sigma(Y)(X) + e\sigma(X)(Y) - eF^*\beta(X) - F^*\beta(FX) + e\sigma(\pi Y)(X) + e\sigma(\pi X)(Y), \quad (4.3) \]
Since \( \sigma \) is symmetric and
\[ \sigma(Y, FX) + \sigma(X, FY) = F^*(\sigma(Y)(X) + \sigma(X)(Y)) \]
we have
\[ \sigma(Y, FX) + \sigma(X, FY) = 0. \quad (4.4) \]
On the other hand, using (3.10) and (1.3) we derive
\[ -\alpha(F^2Y) + \sigma(L)(\pi^2\alpha) + \alpha(Y) = 0. \quad (4.5) \]
Now putting (4.4) and (4.5) in (4.3) we obtain
\[ <X + \alpha + e\mathcal{F}(X + \alpha), Y + \beta + e\mathcal{F}(X + \beta)> = 0 \]
which shows that \( E^{(1,0)} \) is isotropic. Taking the paracomplex conjugation in above computation, we see that \( E^{(0,1)} \) is also isotropic. Also from (1.11) and (1.22) we obtain that the pairings between \( L_\xi \) or \( L_Z \) with those \( E^{(1,0)} \) and \( E^{(0,1)} \) are equal to zero. Thus \( L \) is isotropic. Similarly, we find that \( L^* \) is isotropic.

We now present the following notion.

**Definition 3.** Given a generalized almost para-contact structure, if the space \( \Gamma(L) \) of sections of the associated bundle \( L \) is closed under the Courant bracket, then the generalized almost para-contact structure is simply called a generalized para-contact structure.

Thus by considering the notions of Dirac structures, quasi-Lie bialgebroids and Definition 3, we have the following result.

**Corollary 4.1.** When \( \mathcal{J} = (\mathcal{Z}, \xi, \pi, \sigma, F) \) represents a generalized para-contact structure, the associated bundle \( L \) is a Dirac structure. In addition, the bundle \( L^* \) is a transversal isotropic complement of \( L \) in the Courant algebroid.
Thus and \(X\) if Isotropic \(\bar{\mathcal{J}}\) jugation we have \((1,1)\) structure. The pair \(L\) and \(L^*\) forms a Lie bialgebroid.

\[
\text{Proof. } \text{We first note that the inclusion of } L \text{ in } (TM \oplus TM^*)_C \text{ followed by the natural projection onto the first summand is an anchor map. When } L \text{ is closed under the Courant bracket, the restriction of the Courant bracket to } L \text{ completes the construction of a Lie algebroid structure on } L. \text{ Since } J = (Z, \xi, \pi, \sigma, F) \text{ is a generalized para-contact structure, } L \text{ is closed under the Courant bracket. From (7, Proposition 3.27), we know that a maximal isotropic sub-bundle } \bar{E} \text{ of } (TM \oplus TM^*)_C \text{ is involutive if and only the Nijenhuis operator } Nij \text{ given by}
\[
Nij(A, B, C) = \frac{1}{3}([A, B], C) + [B, C], A > + [C, A], B > \tag{4.6}
\]
vanishes on \(\bar{E}\). Since \(L\) is closed with respect to the Courant bracket, by conjugation we have
\[
[\Gamma(E^{(0,1)}), \Gamma(E^{(0,1)})] \subseteq \Gamma(L_Z \oplus E^{(0,1)}) = \Gamma(\bar{L}).
\]
Isotropic \(\bar{L}\) implies that \(< L_F, L_F \oplus E^{(0,1)} >= E^{(0,1)}, L_F \oplus E^{(0,1)} >= 0\). Thus if \(X_1, X_2\) and \(X_3\) are all sections of \(E^{(0,1)}\), then \(Nij(X_1, X_2, X_3) = 0\). Thus \(Nij = 0\) if and only if
\[
Nij(X_1, X_2, \xi) = 0
\]
for \(X_1, X_2 \in \Gamma(E^{(0,1)})\). Now for \(X_1 \in \Gamma(ker\xi)\) and \(\alpha_1 \in \Gamma(kerZ)\), by taking \(X_1 = X_1 + \alpha_1 - eF(X_1 + \alpha_1)\) and \(X_2 = X_2 + \alpha_2 - eF(X_2 + \alpha_2)\) we find
\[
X_1 = X_1 + \alpha_1 - eF(X_1 - e\pi^*\alpha_1 - e\sigma_6(X_1) + eF*(\alpha_1)
\]
and
\[
\rho(X_1) = X_1 - eF(X_1 - e\pi^*\alpha_1),
\]
where \(\rho : L^* \rightarrow TM\) is the anchor map. Thus we have
\[
Nij(X_1, X_2, \xi) = \frac{1}{3}(<\rho[X_1, X_2], \xi > + < \rho[X_1, \xi], \rho X_2 > + < \rho[\xi, X_1], \rho X_2 >}.
\]
Hence we get
\[
Nij(X_1, X_2, \xi) = \frac{1}{6} \xi([\rho X_1, \rho X_2]) + \frac{1}{6} (L_{\rho(X_2)} \xi)(\rho(X_1)) - \frac{1}{12} (d(i_{\rho(X_2)} \xi)(\rho(X_1)) - \frac{1}{6} (L_{\rho(X_1)} \xi)(\rho(X_1)) + \frac{1}{12} (d(i_{\rho(X_1)} \xi)(\rho(X_1)). \tag{4.7}
\]
Since $X_2 \in \Gamma(\ker \xi)$, by using the second equation of (3.10) we derive
\[ d(i_{\rho(X_2)}\xi)(\rho(X_1)) = 0. \tag{4.8} \]
Using this in (4.7) we obtain
\[ N_{ij}(X_1, X_2, \xi) = \frac{1}{6} \{\xi(\rho(X_1), \rho(X_2)) + (L_{\rho(X_2)}\xi)(\rho(X_1)) - (L_{\rho(X_1)}\xi)(\rho(X_2))\}. \]
Then exterior derivative and (4.8) imply that
\[ N_{ij}(X_1, X_2, \xi) = \frac{1}{3} \{-d\xi(\rho(X_1), \rho(X_2)) + d\xi(\rho(X_2), \rho(X_1)) - d\xi(\rho(X_1), \rho(X_2))\}. \]
Thus we arrive at
\[ N_{ij}(X_1, X_2, \xi) = -d\xi(\rho(X_1), \rho(X_2)) \]
which gives
\[ N_{ij} = -\mathcal{Z} \wedge \rho^*(d\xi)^{(2,0)}, \tag{4.9} \]
where $\rho^*(d\xi)^{(2,0)}$ is the $\wedge E^{(1,0)}$ component of the pullback of the $d\xi$ via the anchor map $\rho : L^* \to TM$. Then proof follows from Theorem 4.1, Lemma 4.1 and (4.9).

The above result motivates us to give the following definition.

**Definition 4.** An generalized almost para-contact structure is called a strong generalized para-contact structure if both $L$ and $L^*$ are closed under the Courant bracket.

Here is an example for strong generalized para-contact structure.

**Example 3.** Let $H_3$ be the three-dimensional Heisenberg group and $\{X_1, X_2, X_3\}$ a basis for its algebra $\mathfrak{h}_3$ so that $[X_1, X_2] = -X_3$. Let $\{\alpha^1, \alpha^2, \alpha^3\}$ be a dual frame. Then $d\alpha^3 = \alpha^1 \wedge \alpha^2$. Now for $t = r \cosh \vartheta + er \sinh \vartheta$, we define
\[ F_t = \frac{2r \sinh \vartheta}{(1-r^2)} (X_2 \otimes \alpha^2 + X_3 \otimes \alpha^3), \quad \xi = \alpha^1, \mathcal{Z} = X_1 \]
\[ \sigma_t = \frac{-r^2 + 2r \cosh \vartheta - 1}{(1-r^2)} (\alpha^2 \wedge \alpha^3), \quad \pi_t = \frac{r^2 + 2r \cosh \vartheta + 1}{(1-r^2)} (X_2 \wedge X_3). \]
Then $(\xi, \mathcal{Z}, \pi_t, \sigma_t, F_t)$ is a family of generalized almost para-contact structures. There are two subfamilies of this family, determined by $|t|^2 = r^2 < 1$ and $|t|^2 = r^2 > 1$. The corresponding bundle $L_t$ and $L^*_t$ are trivialized as follows:
\[ L_t = \text{span}\{X_1, X_2 + e\alpha^3 + eF(X_2 + e\alpha^3), X_3 - e\alpha^2 + eF(X_3 - e\alpha^2)\} = \text{span}\{X_1, -(r^2 + r \cosh v)X_2 + e\sinh vX_2 + e(-r^2 + r \cosh v)\alpha^3 - r \sinh \alpha^3, -(r^2 + r \cosh v)X_3 + r \sinh \alpha^2 + e\sinh vX_3 + e(r^2 - r \cosh v)\alpha^2\} \]
\[ L_t' = \text{span}\{\alpha^1, \alpha^2 + eX_3 - eF(\alpha^2 + eX_3), \alpha^3 - eX_2 - eF(\alpha^3 - eX_2)\} \]
\[ = \text{span}\{\alpha^1, -r \sinh vX_3 + (-r^2 + r \cosh v)\alpha^2 - e(r^2 + r \cosh v)X_3 + eX_3 - eF(\alpha^2 + eX_3), \alpha^3 - r \sinh vX_3 + (-r^2 + r \cosh v)\alpha^3 + e(r^2 + r \cosh v)X_3 + r \sinh vX_3\}. \]

It is easy to see that all Courant brackets on \( L_t' \) are trivial. The only non-zero Courant bracket on \( L_t \) is
\[ [X_1, -(r^2 + r \cosh v)X_2 + er \sinh vX_2 + e(-r^2 + r \cosh v)\alpha^3 - r \sinh vX_3, \alpha^3 + e(r^2 + r \cosh v)X_3 + r \sinh vX_3]. \]
which is belong to \( L_t \). Thus \((\xi, Z, \pi_t, \sigma_t, F_t)\) is a family of strong generalized para-contact structures.

### 5. Almost para-contact structures as an example of strong generalized para-contact structures

In this section we are going to show that an almost para-contact structure has a natural strong generalized para-contact structure on it. Let \((\varphi, E, \eta)\) be an almost para-contact structure on a manifold \(M\). An almost para-contact structure is a normal almost para-contact structure \cite{18} if
\[ N_{\varphi}(X, Y) = 2d\eta(X, Y)E, L_E\eta = 0, L_E\varphi = 0, \] (5.1)
where \( N_{\varphi} \) is defined by
\[ N_{\varphi}(X, Y) = [\varphi X, Y] + \varphi^2 [X, Y] - \varphi[X, Y] - \varphi[X, Y] \] (5.2)
for any vector fields \( X, Y \).

**Lemma 5.1.** Let \((\varphi, E, \eta)\) be a normal almost para-contact structure on a manifold \(M\). If \(X\) is a section of \(E^{1,0}\), then \([F, X]\) is again a section of \(E^{1,0}\).

**Proof.** For \(X = X + \alpha \in \Gamma(ker E \oplus ker \eta)\), using (1.2) we have
\[ [[E, X + eF X]] = L_E X + L_E \alpha + e(L_E \varphi X - L_E \varphi^* \alpha) \] (5.3)
due to \(X \in \Gamma(ker \eta)\) and \(\alpha \in \Gamma(Ker E)\). Since \(M\) is a normal almost para-contact manifold we have (5.1). Replacing \(X\) by \(E\) in (5.1) we have
\[ \varphi^2 [E, Y] = -\eta([E, Y])E. \]
Then we get
\[ L_E Y = \varphi L_E \varphi Y. \] (5.4)
\[ [5.1] \text{ and } [5.4] \text{ imply that } \varphi L_E Y = L_E \varphi Y. \] (5.5)
Thus using (5.5) in (5.3) and considering dual almost para-contact structure $\varphi^*$, we get
\[ [E, \mathcal{X} + eFX] = L_E X + L_E \alpha + e(\varphi L_E X - \varphi^* L_E \alpha) \]
which proves assertion.

In the sequel we show that a normal almost para-contact manifold is actually a strong generalized para-contact manifold.

**Theorem 5.1.** If $\mathcal{J} = (\mathcal{Z}, \xi, \pi, \sigma, F)$ represents a generalized almost para-contact structure associated to a classical normal almost para-contact structure on an odd-dimensional manifold $M$, then it is a strong generalized para-contact structure.

**Proof.** For $X, Y \in \Gamma(ker\eta)$, applying $\eta$ to (5.2) we have
\[ \eta(\varphi X, \varphi Y) = 2d\eta(X, Y). \tag{5.6} \]
Since $\varphi X$ and $\varphi Y$ are also sections of $ker\eta$, we derive
\[ -d\eta(X, Y) = d\eta(\varphi X, \varphi Y). \tag{5.7} \]
Now for $X, Y \in \Gamma(ker\eta)$, we have
\[ [[X + e\varphi X, Y + e\varphi Y] = [X, Y] + [\varphi X, \varphi Y] + e([\varphi X, Y] + [X, \varphi Y]). \]
Hence we get
\[ [[X + e\varphi X, Y + e\varphi Y] = [X, Y] + \varphi^2[\varphi X, \varphi Y] + \eta([\varphi X, \varphi Y])E + e([\varphi X, Y] + [X, \varphi Y]). \tag{5.8} \]
Since $M$ is a normal almost para-contact manifold, from (5.1), (5.2) and (5.6) we obtain
\[ \eta([\varphi X, \varphi Y])E + [X, Y] + \varphi^2[\varphi X, \varphi Y] = \varphi([\varphi X, Y] + [X, \varphi Y]). \tag{5.9} \]
Putting (5.9) in (5.8) we arrive at
\[ [[X + e\varphi X, Y + e\varphi Y] = \varphi([\varphi X, Y] + [X, \varphi Y]) + e([\varphi X, Y] + [X, \varphi Y]). \]
Hence we have
\[ [[X + e\varphi X, Y + e\varphi Y] = \varphi([\varphi X, Y] + [X, \varphi Y]) + e(\varphi^2([\varphi X, Y] + [X, \varphi Y]) + \eta([\varphi X, Y] + [X, \varphi Y])E). \]
On the other hand, from (5.7) we find
\[ \eta([\varphi X, Y] + [X, \varphi Y]) = 0. \]
Using this in above equation, we get

\[ [X + e\varphi X, Y + e\varphi Y] = \varphi([\varphi X, Y] + [X, \varphi Y]) \]

which shows that \([X + e\varphi X, Y + e\varphi Y] \) is a section of \(E^{(1,0)}\). Now for \(X \in \Gamma(ker\eta)\) and \(\alpha \in \Gamma(kerE)\), from (5.1)-(5.3), we obtain

\[ [X + e\varphi X, \alpha - e\varphi^*\alpha] = L_X\alpha - L_{\varphi X}\varphi^*\alpha - e(L_X\varphi^*\alpha - L_{\varphi X}\alpha). \]

Evaluating the above expression on the Reeb field \(E\), and for any vector field \(Y\), see that both parts of the above expression are sections of \(kerE\). Then observing that \(\varphi\), which shows that

\[ \llbracket X + e\varphi X, \alpha - e\varphi^*\alpha \rrbracket \in \ker E\alpha. \]

Using this in above equation, we get

\[ -\varphi^*(L_X\varphi^*\alpha - L_{\varphi X}\alpha)(Y) = -(L_X\varphi^*)(\varphi Y) + (L_{\varphi X}\alpha)(\varphi Y). \]

Then observing that \(\varphi^2 X = X\) on \(\ker\eta\), we obtain

\[ -\varphi^*\alpha = -\varphi X(\alpha(Y) + \varphi X\alpha(\varphi Y)) + \alpha([X, \varphi Y] - [\varphi X, \varphi Y]). \]

From (5.1) and (5.2) we have

\[ [\varphi X, \varphi Y] - \varphi[X, Y] = \varphi[\varphi X, Y] - [X, Y]. \]

Using this expression in above equation, we derive

\[ -\varphi^*\alpha = -\varphi X(\alpha(Y) + \varphi X\alpha(\varphi Y)) + \alpha([X, \varphi Y] - [\varphi X, \varphi Y]). \]

Hence we arrive at

\[ -\varphi^*\alpha = -L_X\alpha + L_{\varphi X}\varphi^*\alpha \]

on \(kerE\). Applying \(\varphi^*\) to (5.12) we have

\[ -(L_X\varphi^*\alpha - L_{\varphi X}\alpha) + (L_X\varphi^*\alpha - L_{\varphi X}\alpha)(E)\eta = \varphi^*(-L_X\alpha + L_{\varphi X}\varphi^*\alpha) \]

due to the transpose of the formula in (2.2). Since \(M\) is a normal almost para-contact manifold, we obtain \((L_X\varphi^*\alpha - L_{\varphi X}\alpha)(E)\eta = 0\), thus we have

\[ (L_X\varphi^*\alpha - L_{\varphi X}\alpha) = \varphi^*(L_X\alpha - L_{\varphi X}\varphi^*\alpha). \]

Using (5.13) in (5.11) we find

\[ [X + e\varphi X, \alpha - e\varphi^*\alpha] = L_X\alpha - L_{\varphi X}\varphi^*\alpha - e\varphi^*(L_X\alpha - L_{\varphi X}\varphi^*\alpha) \]

which shows that \([X + e\varphi X, \alpha - e\varphi^*\alpha] \in E^{(1,0)}\). Since the Courant bracket between two forms are zero, this shows that we have proved that the Courant bracket between two sections of \(E^{(1,0)}\) is again a section of \(E^{(1,0)}\). Since the bundle \(E^{(1,0)}\) is \(E\) invariant, the bundle is closed under the Courant bracket. Also we see that (5.7) shows that \(\rho^*(d\eta)^{(2,0)} = 0\). Therefore \(L^*\) is also closed with respect to the Courant bracket. Thus proof is completed by Lemma 5.1 and Definition 4.
References

[1] D. V. Alekseevsky, C. Medori and A. Tomassini, Homogeneous para-Kähler Einstein manifolds, Russian Math. Surveys 64:1 (2009), 1-43.

[2] G. R. Cavalcanti and M. Gualtieri, Generalized complex geometry and T-duality, arXiv:1106.1747.

[3] M. Crainic, Generalized complex structures and Lie brackets, Bull. Braz. Math. Soc., New Series 42(4), (2011), 559-578.

[4] J. Davidov, G. Grantcharov, O. Mushkarov, M. Yotov, Generalized pseudo-Kähler structures, Comm. Math. Phys. 304(1), (2011), 49-68.

[5] S. J. Gates Jr, C. M. Hull, M. Röcek, Twisted multiplets and new supersymmetric nonlinear σ-models, Nuclear Phys. B, 248(1), (1984), 157-186.

[6] J. Grabowski and P. Urbanski, Lie algebroids and Poisson Nijenhuis structures, Rep. Math. Phy. 40, (1997), 195-208.

[7] M. Gualtieri, Generalized complex geometry, Ph.D. thesis, Univ. Oxford, arXiv:math.DG/0401221, (2003).

[8] N. Hitchin, Generalized Calabi-Yau manifolds, Q. J. Math., 54, (2003), 281-308.

[9] S. Kaneyuki, F. L. Williams, Almost paracontact and parahodge structures on manifolds. Nagoya Math. J. 99, (1985), 173-187.

[10] Z-J. Liu, A. Weinstein, P. Xu, Manin triples for Lie bialgebroids, J. Differential Geom. Volume 45(3), (1997), 547-574.

[11] Y. S. Poon, A. Wade, Generalized contact structures, J. Lond. Math. Soc. 83(2), (2011), 333-352.

[12] D. Roytenberg, Quasi-Lie bialgebroids and twisted Poisson manifolds. Lett. Math. Phys. 61 (2002), 123-137.

[13] I. Vaisman, Dirac structures and generalized complex structures on $TM \times \mathbb{R}^h$. Adv. Geom. 7(3), (2007), 453-474.

[14] I. Vaisman, Reduction and submanifolds of generalized complex manifolds, Dif. Geom. and its appl., 25, (2007), 147-166.

[15] I. Vaisman, From generalized Kähler to generalized Sasakian structures. J. Geom. Symmetry Phys. 18 (2010), 63-86.

[16] A. Wade, Dirac structures and paracomplex manifolds, C. R. Acad. Sci. Paris, Ser. I, 338, (2004), 889-894.

[17] A. Wade, Local structure of generalized contact manifolds. Differential Geom. Appl. 30(1), (2012), 124-135.
[18] S. Zamkovoy, Canonical connections on paracontact manifolds, Ann. Glob. Anal. Geom., 36, (2009), 37-60.