Dynamics of bright solitons and soliton arrays in the nonlinear Schrödinger equation with a combination of random and harmonic potentials

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Abstract
We report results of systematic simulations of the dynamics of solitons in the framework of the one-dimensional nonlinear Schrödinger equation, which includes the harmonic oscillator potential and a random potential. The equation models experimentally relevant spatially disordered settings in Bose–Einstein condensates (BECs) and nonlinear optics. First, the generation of soliton arrays from a broad initial quasi-uniform state by the modulational instability (MI) is considered following a sudden switch of the nonlinearity from repulsive to attractive. Then, we study oscillations of a single soliton in this setting, which models a recently conducted experiment in a BEC. The basic characteristics of the MI-generated array, such as the number of solitons and their mobility, are reported as functions of the strength and correlation length of the disorder, and of the total norm. For the single oscillating soliton, its survival rate is found. The main features of these dependences are explained qualitatively.

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1. Introduction

The interplay of disorder and nonlinearity is a topic that has been attracting a great deal of interest for a long time; see reviews [1–5]. In particular, much work has been done on the analysis of the dynamics of solitons in disordered external potentials, chiefly in one-dimensional (1D) settings; see, e.g., [6–10] and references therein. These studies find an important application in the description of Bose–Einstein condensates (BECs) trapped in random potentials. The latter topic was addressed in many theoretical [11–17] and experimental [18–20] works.

A ubiquitous theoretical model used in this field is the nonlinear Schrödinger equation (NLSE; also called the Gross–Pitaevskii equation, in terms of BECs [21]) for the wave function \( u(x,t) \), which includes a regular trapping potential and a term accounting for a disordered potential. The normalized form of this equation, with scaled time \( t \) and space coordinate \( x \), is

\[
iu_t + \frac{1}{2}u_{xx} + g|u|^2u - \frac{1}{2}\Omega^2x^2u - V_{\text{dis}}(x)u = 0,
\]

where \( g = +1 \) and \( -1 \) correspond to self-attractive and self-repulsive nonlinearities, both signs being possible in BECs. \( \Omega \) characterizes the strength of the harmonic oscillator (HO) trapping potential and \( V_{\text{dis}}(x) \) is a random potential representing the spatial disorder. In particular, this...
model describes dipole oscillations of the trapped condensate, experimentally studied in the recent work [20], where strong effective dissipation induced by the disordered potential was discovered. On the other hand, equation (1) with \( t \) replaced by the propagation distance coordinate \( z \) is a model of a nonlinear optical waveguide with a random perturbation of the local refractive index, which is represented by the disordered potential [2]; hence the results reported below apply to spatial solitons in a nonlinear waveguide as well.

Our objective is to systematically investigate fundamental properties of solitons in the framework of the model based on equation (1). These include the formation and motion of soliton trains due to the modulational instability (MI) of the broadly distributed condensate and, as suggested by the experiment reported in [20], oscillations of a single soliton which is originally placed at a distance from the minimum of the trapping potential, \( x = 0 \). The results for these two types of dynamical behavior, based on systematic simulations of equation (1) and averaging over many realizations of the random setting, as well as on a qualitative analysis of the observed effects, are reported, respectively, in sections 2 and 3, and the conclusion is presented in section 4.

2. The formation of soliton arrays by the modulation instability in a spatially disordered environment

The topic of this section is the effect of spatial disorder on the formation of soliton chains from an initial quasi-uniform state, and the subsequent evolution of the chains. The numerical simulations were subject to the following specifications. The spatial domain was taken as \(-60\pi < x < +60\pi\), and the trapping strength as \( \Omega^2 = 25 \times 10^{-4} \), hence the respective HO length is much smaller than the size of the integration domain, which makes it possible to consider multi-soliton trains and persistent motion of the solitons. The split-step Fourier method with a spatial grid composed of 1024 points and time step \( \Delta t = 0.001 \) was employed for the spatial and temporal discretization of equation (1). The total integration time was \( T = 80 \). A fourth-order optimized implementation of the splitting was used, as in [22, 23]. Further, the disorder was represented by a spatially correlated random function, \( V_{\text{dis}}(x) = V_d f(x, \theta) \), where \( V_d \) is the strength of the disorder, and the marginal distribution of \( f(x, \theta) \) has an exponential form, its covariance function being \( C(x_1, x_2) = \exp(-2(x_1 - x_2)^2/V_c^2) \), in which \( V_c \) is the correlation length. Other types of random functions, as in [24], can be considered in a similar way.

The initial quasi-uniform distribution of the condensate is taken as the ground-state solution of equation (1) with the self-repulsive nonlinearity, i.e. with \( \varepsilon = -1 \), in the form of \( u = \exp(-i\mu t)U(x) \), where \( \mu \) is the real chemical potential, and the function \( U(x) \) was found to be a stationary solution of the associated nonlinear diffusion equation (the imaginary-time version [25] of equation (1)):

\[
u_{t} = \frac{1}{2}u_{xx} - |u|^2u - \frac{1}{2}\Omega^2x^2u - V_{\text{dis}}(x)u + \mu u.\tag{2}\]

The same split-step method was used to solve equation (2). Then, the evolution of the MI of this state was simulated in real time, following a sudden switch of the self-repulsion into self-attraction, i.e. replacement of \( g = -1 \) by \( g = +1 \) in equation (1), which exactly corresponds to reversal of the sign of the nonlinearity by means of the Feshbach resonance in the well-known BEC experiment of [26].

The MI splits the initial quasi-uniform state into a chain of solitons, which, generally speaking, is an obvious outcome of the evolution induced by the reversal of the nonlinearity sign. However, a new aspect of this outcome, which we focus on here, is the effect of the disordered environment on the resulting solitary wave chain. We have collected results that demonstrate the dependence of the characteristics of the emerging soliton chain on three control parameters: strength \( V_d \) and correlation length \( V_c \) of the random potential, and the total norm of the initial state, \( N = \int_{-\infty}^{+\infty} |u(x)|^2 \, dx \). The characteristics whose variation was monitored are (i) the number of solitons; (ii) the largest displacement of solitons, i.e. the largest distance that peaks of individual solitons can travel over the course of evolution (basically the displacement for each soliton is computed as the distance between its leftmost and rightmost positions; then the largest displacement is simply the largest value among all the solitons); (iii) the normalized average kinetic energy per soliton, which is defined as

\[
\bar{K} = \left( \sum M_j \right)^{-1} \left( \sum \frac{1}{2}M_j v_j^2 \right),
\]

where the summation is performed on the full set of solitons in the emerging configuration, \( v_j \) is the velocity of the \( j \)th soliton and \( M_j = \int |u(x)|^2 \, dx \) is its effective mass, with the integration performed in the vicinity of the solitary wave’s peak where the local amplitude of the field exceeds half of its peak value.

The results presented below were produced by averaging over 100 different random realizations. Note that this leads to a non-integer number of solitons for most cases. Typical examples of the realizations are displayed in figures 1 and 2 for small and large correlation lengths, \( V_c = 5 \) and \( V_c = 25 \), respectively. Individual solitons can be easily identified in the plots.

The results for the number of solitons in the emerging pattern and their largest displacement are summarized in figures 3–5. In each panel, we fix two of the above-mentioned control parameters \((V_d, V_c, N)\) and vary the third one. For example, the norm is varied in figure 3, each curve corresponding to a particular set of values of \( V_d \) and \( V_c \).

The following conclusions can be drawn from figures 3–5.

1. According to figure 3, the number of solitons in the chain increases linearly with the total norm, so that the mean norm per soliton, \( N_{\text{sol}} \), is approximately constant. This is a direct effect of the trapping of the wave field by the random potential: in free space, the entire condensate tends to coalesce into a single soliton, which corresponds to a minimum of the system’s Hamiltonian [27]. However, the sufficiently strong disorder pins portions of the fragmented condensate, forcing them to self-trap into separated solitons. A fundamental threshold condition for the self-trapping of an initial state into an NLS soliton is
known in the form of the condition imposed on the area of the initial configuration [27]:
\[ S \equiv \int_{-\infty}^{+\infty} |u_0(x)| \, dx > S_0 \equiv \ln \left( 2 + \sqrt{3} \right) \approx 1.32. \]  
(4)

For fixed parameters of the disorder, i.e. fixed average width \( \bar{L} \) of local potential wells trapping fragments of the condensate, the above-mentioned constancy of the norm-per-soliton, \( N_{\text{sol}} \), implies a constant average amplitude, \( \bar{A} \sim \sqrt{N_{\text{sol}}/\bar{L}} \); hence the average area per soliton is constant too, \( \bar{S} \sim \bar{A} \bar{L} \sim \sqrt{N_{\text{sol}} \bar{L}} \), which is consistent with condition (4) that also implies an approximately constant area per soliton. Thus, condition (4) may explain the results observed in figure 3.
2. Figures 3 and 6 demonstrate the increase of both the largest distance traveled by the solitons and their mean kinetic energy, with an increase of the total norm. This feature may be explained by the fact that an individual soliton, moving through the disordered potential, is strongly braked due to the emission of radiation [1]. The effectively dissipative character of the motion of coherent wavepackets in this setting was recently demonstrated in the experiment of [20] (see also the following section). However, if the system is filled with the trapped condensate, the moving soliton actually interacts with trapped segments of the condensate, i.e. with an effective pseudopotential [10], which is essentially smoother than the ‘bare’ random potential. This effect leads to suppression of radiation losses, allowing the solitons to be more mobile.

3. Figure 4 shows that the soliton number decreases, while the largest displacement increases, as the correlation length of the disorder, $V_z$, increases. This conclusion is consistent with the conclusions presented in the previous item, as the increase of $V_z$ implies transition to a less disordered potential.

4. Figure 5 clearly shows that both the soliton number and largest displacement change very rapidly when $V_d$ increases from zero to small finite values, i.e. the disorder starts to kick in when its strength is quite small. The observed jump of the soliton number to larger values and the simultaneous drop of the free-path length are consistent with the above argument stating that a deeper random potential splits the condensate into a large number of solitons and impedes their free motion.

3. Oscillations of the soliton in the combined potential

Our next objective is to simulate the oscillatory motion of a single soliton in the combined HO and random potentials, which parallels the recently reported experiment performed in the condensate of $^7$Li [20]. While the latter experiment focused on the repulsive interaction case of the (dissipative in the presence of disorder) dipolar motion of a full condensate,
it is straightforward to envision such dynamics for an attractive interaction condensate, namely a localized solitary wave. For this purpose, simulations of equation (1) were run with parameters selected as the rescaled version of those dealt with in the experiment, where the total number of atoms was $\approx 10,000$, the scattering length $a_s$ is taken as three times the Bohr radius of $^7$Li, the transverse trapping frequency is $\omega_\perp = 2\pi \times 260$ Hz and the longitudinal one is $\omega_\parallel = 2\pi \times 5.5$ Hz. The initial conditions were taken as $u_0(x) = \sqrt{2a_0} \text{sech}(\sqrt{2a_0}(x-x_0))$, where $x_0$ is the initial shift of the soliton from the bottom of the HO potential, and $a_0$ is determined by the total number of atoms. With the above physical values, $\langle \omega^2 \rangle = 4.4749 \times 10^{-3}$ and $a_0 = 0.2269$ were used in the simulations. The split-step Fourier method was employed in this case too, with a sufficiently small spacing in order to properly resolve the size of the soliton. Over the course of the simulations, the disorder in equation (1) was turned on after one and a quarter of the period of the oscillations of the soliton in the HO potential, i.e. when the soliton’s center was passing through the origin $x = 0$.

For a given initial shift $x_0$, we mainly varied two parameters (as before), the correlation length $V_z$ and disorder strength $V_d$. For each fixed value of $V_z$, we varied $V_d$ to infer whether the soliton would survive after ten periods of the oscillations. The goal was to compute the survival rate of the soliton under the influence of disorder, using ten realizations for this purpose. In each realization, the survival of the soliton was registered if its final norm, integrated over the full-width at half-maximum (FWHM) range around its center, exceeded 50\% of the initial value in the same range. The survival rate is then defined as the number of realizations featuring the surviving soliton, divided by 10 (the total number of realizations). Typical examples of the survival and destruction of the oscillating solitons are displayed in figure 7. (Note that $h$ in the horizontal axis label denotes Planck’s constant.)

The results for $x_0 = 0.6$ and 0.1 mm are displayed in figures 8 and 9. They correspond to the dimensionless values of $x_0 = 254.6$ and 42.4 in equation (1). These clearly indicate that the survival rate drops to zero as the disorder strengthens and the correlation length decreases, in agreement with the experimental observations for the repulsive case [20]. This is a natural consequence of the increasing rate of emission of radiation by the soliton oscillating across the random potential.

When the disorder becomes very strong, the soliton starts to survive again, as seen in figure 9 for $x_0 = 0.1$ mm. This is explained by the fact that a very strong random potential consists of local potential wells that quickly trap and immobilize the soliton, preventing its decay into radiation and, in addition, the strong random potential impedes the separation of the radiation waves from the parent soliton. The restabilization boundary is shown in figure 10. The approximately linear dependence of the minimum disorder strength, necessary for restabilization, on $x_0$ can be explained with the help of the above argument: the driving force acting on the soliton in the HO potential grows linearly with $x_0$, while the largest pinning force, induced by the random potential, is proportional to $V_d$. Therefore, the equilibrium between the two, which determines the restabilization threshold, implies $V_d \sim x_0$. 

Figure 7. The left and right panels display examples of solitons which, respectively, survive and get destroyed over the course of oscillations, for an initial shift of the soliton to $x_0 = 0.1$ mm, its FWHM width 9.2142 $\mu$m and a disorder correlation length of $V_z = 9.4264 \mu$m. The disorder strength corresponding to the left and right panels is, respectively, $V_d = h \cdot 5$ (Hz) and $h \cdot 180$ (Hz), where $h$ is Planck’s constant.

Figure 8. The survival rate of the oscillating soliton for the initial shift $x_0 = 0.6$ mm.

Figure 9. The survival rate of the oscillating soliton for the initial shift $x_0 = 0.1$ mm.

Figure 10. The survival rate of the oscillating soliton for the initial shift $x_0 = 0.1$ mm.

Figure 11. The survival rate of the oscillating soliton for the initial shift $x_0 = 0.1$ mm.
of the above-analyzed mechanisms with the potential collapse-type events of super-critical atomic blobs and hence the relevant phenomenology will be considerable richer. The 2D setting is also of interest for the repulsive interaction dynamical case, whereby the formation of dark solitons and trains thereof reported in [20] may be substituted by the formation of vortices and vortex streets.

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