The Higgs resonance in fermionic pairing

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The Higgs boson in fermionic condensates with the BCS pairing interaction describes the dynamics of the pairing amplitude. I show that the existence and properties of this mode are sensitive to the energy dispersion of the interaction. Specifically, when the pairing is suppressed at the Fermi level, the Higgs mode may become unphysical (virtual) state or a resonance with finite lifetime, depending on the details of interaction. Conversely, the Higgs mode is discrete for the pairing interaction enhanced at the Fermi level. This work illustrates conceptual difficulties associated with introducing collective variables in the many-body pairing dynamics.

The low-energy properties of many-body systems with spontaneously broken symmetries are encoded in the collective modes associated with the structure of the order parameter. The dynamics is usually dominated by the gapless Goldstone modes that restore the symmetry [1], which makes them easily observable in variety of systems. In contrast, the massive Higgs modes [2] resulting from the amplitude variations of the order parameter are hard to detect. Moreover, they do not necessarily exist when the order parameter is generated dynamically due to collective effects. Understanding how these modes emerge in a microscopic framework is essential for constructing field-theoretical descriptions operating with a partial set of dynamical variables.

This peculiarity of the Higgs mode becomes evident in the BCS model of superconductivity [3], where pairing of fermions with opposite spins and momenta is described by a complex-valued order parameter. Several collective modes distinct from the Higgs mode have been considered in superconductors: the excitonic states [4, 5], the high-temperature phase modes [6], and also a phonon gap mode observed in materials with coexisting superconductivity and charge-density waves [7, 8]. At the same time, no direct signatures of the intrinsic amplitude mode have been found so far. The main difficulty has to do with the structure of quasi-particle excitations in the system.

The role of quasi-particle states is clarified in the response to a small pairing perturbation. The induced dynamics can be described as the dephasing of individual quasi-particle states excited in the vicinity of the Fermi energy, with the rate of dephasing proportional to the excitation energy. As a result one finds a power-law decay of the perturbation [9]. Equivalently, this behavior can be linked to the square-root singularity of the density of states at the gap energy [9]. Recent extension of the analysis to non-linear regimes [10, 11] has also demonstrated the absence of the intrinsic collective excitations of the pairing amplitude. One of the implications of this behavior is the lack of an effective description of the pairing dynamics such as time-dependent Ginzburg-Landau theory in gapped superconductors [12, 13]. However, the results of the analysis rely on the key assumption of constant interaction between fermions and change qualita-

FIG. 1: The Higgs mode energy sketched as a function of dispersion: the mode is discrete within the spectral gap for positive dispersion (regime A); the mode is non-physical (virtual state) for a smooth negative dispersion (regime B); and it becomes a resonance inside the continuum of the BCS quasi-particles for sharp and negative dispersion (regime C).
lifetime of the Higgs mode is small compared to the quasiparticle relaxation time, making the dynamics analogous to the Landau damping \[^{14}\] in collisionless plasma.

Our system is described by the BCS Hamiltonian

$$\mathcal{H} = \sum_{p,\sigma} \epsilon_p a_p^\dagger a_p + \sum_{pq} \lambda_{pq} a_p^\dagger a_{-q}^\dagger a_{-q} a_q - \mu \sum_p a_p^\dagger a_p,$$

(1)

where \(a_p^\dagger\) is the creation operator of a spin-1/2 fermion with momentum state \(p\) and spin state \(\sigma = \uparrow, \downarrow\), \(\epsilon_p = p^2/(2m) - \mu\) is the energy of free fermions with respect to the Fermi energy \(\mu\). The summation is conventionally performed over the states within the band \(|\epsilon_p| < \omega_D\) of finite width related to e.g. the Debye energy.

The energy dispersion of interaction \(\lambda_{pq}\) in Eq. (1) leads to the energy-dependence of the pairing gap. The analytical treatment is simplified in the case when it is taken in a factorized form \(\lambda_{pq} = \lambda f_p f_q\), such that \(f_{|p|=|p'|} = 1\). Diagonalizing the BCS Hamiltonian (1), one obtains the quasi-particle excitation spectrum \(E_p = (\epsilon_p^2 + \Delta_F^2)^{1/2}\) characterized by the pairing amplitude

$$\Delta_p = f_p \Delta_F,$$

(2)

where the energy dependence is provided by \(f_p\), and \(\Delta_F\) is the equilibrium value of the pairing gap at the Fermi energy. The self-consistency condition takes on the form

$$2/\lambda_0 = \sum_p f_p^2 / E_p$$

and defines \(\Delta_F\) as a function of \(f_p\) and the coupling constant \(\lambda_0\). The factorized form of energy-dependent interaction simplifies the analysis. Nonetheless, the qualitative results obtained in this work also apply to a generic form of interaction.

We analyze the dynamics induced by a small perturbation of pairing interaction, \(\delta \lambda(t) = \lambda(t) - \lambda_0\). It is characterized by the expectation value of the pairing operator

$$\hat{\Delta} = \lambda_0 \sum_p f_p a_p^\dagger a_{-p}^\dagger,$$

(3)

calculated with respect to time-dependent many-body state. The problem is solved by employing the time-dependent mean-field approach \[^{15}\] valid at time scales smaller than the quasi-particle relaxation time. The linear response of the pairing amplitude \(\delta \Delta_p = \delta \Delta(t)\) is given by

$$\delta \Delta(t) = -i \int_{-\infty}^t d\tau \langle \hat{\Delta}(t), \hat{V}(\tau) \rangle,$$

(4)

where the operators evolve according to the unperturbed Hamiltonian (1), and the average \(\langle \ldots \rangle\) is taken with respect to its ground state. The perturbation operator

$$\hat{V}(t) = -\frac{\Delta_F}{\lambda_0} \left( \frac{\delta \lambda(t)}{\lambda_0} + \frac{\delta \Delta(t)}{\Delta_F} \right) \hat{\Delta}(t) + \text{h.c.}$$

(5)

includes a term \(\propto \delta \Delta(t)\) obtained from the averaging of the interaction energy within the mean-field approximation.

The solution of Eq. (4) simplifies if one assumes the particle-hole symmetry. This condition implies that the energy-dependent pairing gap is symmetric with respect to the Fermi energy. In this case, the phase and the amplitude modes of \(\Delta\) decouple and can be analyzed separately. Concentrating on the amplitude mode \(\delta \Delta'(t)\) excited by the perturbation (5), we obtain the solution of the inhomogeneous integral equation by the Fourier transformation:

$$\delta \Delta'(\omega) = [G(\omega) - 1] \frac{\Delta_F}{\lambda_0} \delta \lambda(\omega),$$

(6)

where \(G(\omega)\) is a retarded spectral function describing the many-body response. By the relation \(G^{-1}(\omega) = 1 + K(\omega)\) it is expressed through a correlation function \(K(\omega)\) that reads in real time as

$$iK(t) = \frac{\theta(t)}{\lambda_0} \left( \langle \hat{\Delta}(t), \hat{\Delta}^\dagger(0) \rangle + \langle \hat{\Delta}(t), \hat{\Delta}(0) \rangle \right).$$

(7)

Calculating \(K(t)\) we derive the expression for the spectral function

$$G^{-1}(\omega) = g \int_0^\omega d\epsilon \left( \frac{\Delta_F}{\omega} \right)^2 \frac{\omega^2/4 - \Delta_F^2}{E_N(E_N - \omega^2/4)},$$

(8)

where \(g = \nu \lambda_0\) is dimensionless coupling, and the particle-hole symmetry is enforced by energy-independent density of states \(\nu\). The states in (8) are labeled by energy \(\epsilon\) instead of momentum \(p\), and according to the usual rule \(\omega \rightarrow \omega + i0\) for the retarded function. It is worth noting that Eq. (8) is in agreement with Ref. \[^{16}\].

The analytic structure of \(G(\omega)\) in the complex \(\omega\)-plane defines the time-dependence of the pairing amplitude \(\delta \Delta(t)\). In particular, we are interested in locating the poles \(\omega_\Delta = \omega_0 - i\tau\) of \(G(\omega) \propto (\omega - \omega_\Delta)^{-1}\), or equivalently the simple roots of \(G^{-1}(\omega)\) on the real axis or in the lower-half plane. In the dynamics the former correspond to non-decaying oscillations with frequency \(\omega_0\), while the latter are characterized by finite lifetime \(\tau = 1/\Gamma\).

Let us first consider the case when the pairing gap

$$\Delta_F^2 \approx \Delta_F^2 - \beta \epsilon^2$$

(9)

has a smooth maximum at the Fermi energy, i.e. the curvature is small \(0 < \beta \ll 1\), and the characteristic energy scale \(\gamma\) of the maximum is large, i.e. \(\gamma \gg \Delta_F\). Although in a realistic situation these two parameters can be related to one another, for the sake of the argument we consider \(\gamma\) and \(\beta\) as independent parameters. In this case the root of \(G^{-1}(\omega) = 0\) is close to the quasi-particle continuum \(\omega_\Delta \lesssim 2\Delta_F\). By expanding Eq. (8) in small parameter \(z = 1 - \omega^2/(2\Delta_F)^2\), we obtain

$$G^{-1}(z) \approx -\frac{9\pi}{2} \left( \sqrt{z - \frac{2}{\pi}} \beta \ln \frac{2\gamma}{\Delta_F} \right).$$

(10)
In the dispersionless case \( \beta = 0 \), the spectral function reduces to \( G^{-1}(z) \approx -(g\pi/2)^2 \sqrt{z} \) at \( z \gtrless 0 \), and we recover the square-root singularity that leads to a power-law dephasing discussed in Ref. [9]. When \( \beta > 0 \), a simple pole of \( G(\omega) \) appears on the real axis of \( \omega \) at

\[
\omega^2 \approx 4\Delta_F^2 \left(1 - \frac{4}{\pi^2} \beta^2 \ln^2 \left[ \frac{2\gamma}{\Delta_F} \right] \right) \quad (11)
\]

inside the spectral gap of quasi-particle excitations. It is identified as the discrete Higgs mode in regime A of Fig. 1. Close to the pole the spectral function

\[
G(\omega) \approx \frac{8}{\pi^2 g} \beta \ln \left[ \frac{2\gamma}{\Delta_F} \right] \frac{(2\Delta_F)^2}{\omega^2 - \omega^2_\Delta} \quad (12)
\]

defines the spectral weight of the Higgs mode in the dynamics of \( \delta \Delta(t) \). As expected, it vanishes in the dispersionless limit \( \beta = 0 \). The pairing dynamics at \( \beta > 0 \) is characterized by non-decaying oscillations of \( \delta \Delta(t) \) with frequency \( (11) \).

By inspecting Eq. (11) we find that a virtual state is formed at smooth negative dispersion. The analytical continuation to negative \( \beta < 0 \) shows that the mode seemingly has the same energy as for \( \beta > 0 \). In fact it belongs to a non-physical sheet of \( \omega \) (regime B in Fig. 1) as it follows from careful analysis of Eq. (10). Indeed the decrease of \( \beta \) from positive values to zero, brings the energy of the mode to the branch point at \( \omega = 2\Delta_F \). Upon further reduction of \( \beta \) to negative values, the mode enters the non-physical sheet of the Riemann surface parameterized by \( \omega \) from the branch cut and moves away from this point along the real axis as a virtual state (see Fig. 2).

In the pairing dynamics, the virtual state leads to a power-law decay of \( \delta \Delta(t) \) similar to the dispersionless case of Ref. [9].

In the opposite limit of sharp dispersion, \( \gamma \ll \Delta_F \) and \( |\beta| \gg 1 \), this virtual state becomes a resonance within the quasi-particle spectrum with finite lifetime, \( \omega_\Delta = \omega_0 - i\Gamma \).

To demonstrate this effect we study a specific example of the energy-dependent pairing gap:

\[
\Delta^2 = \Delta_F^2 - \beta \frac{e^2}{1 + e^2/\gamma^2}, \quad \epsilon_\Delta = \Delta_\epsilon/\Delta_F, \quad (13)
\]

where \( \beta \) is a curvature at the Fermi energy, and \( \gamma \) is a characteristic energy scale. We assume \(|\beta|\gamma^2 \ll \Delta_F^2\), so that the asymptotic value of the pairing gap \( \Delta_\epsilon = \Delta_F(1 - \beta\gamma^2/\Delta_F^2)^{1/2} \) is close to \( \Delta_F \). In the case of smooth dispersion, \( 0 < \beta \ll 1 \), we obtain Eqs. (11) and (12). In the opposite limit of sharp energy dispersion \( |\beta| \gg 1 \), the energy of the Higgs mode is close to the asymptotic value of the spectral gap \( 2\Delta_\epsilon \), as it follows from the expansion of Eq. (8):

\[
\omega^2_\Delta \approx 4\Delta^2_\epsilon + 4 \int^\infty_0 \frac{d\epsilon}{\epsilon^2 + \Delta^2_F - \Delta^2_\epsilon}. \quad (14)
\]

In this expression, the upper limits of the integrals are substituted with infinity, by using their fast convergence. Explicit calculation for model (13) reveals a square-root singularity in the energy of the Higgs mode

\[
\omega^2_\Delta \approx 4\Delta^2_\epsilon + 4\gamma^2 \sqrt{\beta}, \quad (15)
\]

as a function of \( \beta \). This singularity is related to the form of interaction in Eq. (13), and is not universal.

The spectral weight of the Higgs mode in the dynamics is extracted from the residue of the pole in the spectral function

\[
G(\omega) \approx \frac{2}{\pi g} \beta^{1/4}(\gamma/\Delta_F)^{(2\Delta_F)^2/\omega^2 - \omega^2_\Delta}, \quad (16)
\]

vanishing in the dispersionless limit \( \gamma = 0 \) (see Eq. (13)).

The analytic structure of Eq. (15) is different from Eq. (11): as the sign of \( \beta \) changes from positive to negative, the former acquires an imaginary part with the real part simultaneously pushed in the quasi-particle continuum, since \( \Delta_\epsilon > \Delta_F \). To verify this qualitative observation one needs to continue \( G(\omega) \) analytically in the lower-half of complex \( \omega \)-plane and locate the pole.

The analytical continuation of \( G(\omega) \) is achieved by isolating the singularities of the integrand in Eq. (8). We notice that wherever the imaginary part of frequency \( \Im \omega \) changes from positive to negative value, provided its real part satisfies \( Re \omega > 2\Delta_F \), a singularity occurs in the integrand of \( G^{-1}(\omega) \) at the energy \( \epsilon \) that solves the equation \( E_\epsilon = Re \omega/2 \). By deforming the contour of integration
over the energy to enclose the pole, as Fig. 2(a) illustrates, we find the analytical continuation.

Next we apply this method to calculate the energy of the Higgs resonance in the case of negative dispersion at \( |\beta| \gg 1 \). The explicit calculation of \( G(\omega) \) for model interaction (15) is straightforward but tedious. Solving for the zeros of the analytic function, we define the energy of the Higgs resonance

\[
\omega_\Delta^2 \approx 4 \Delta_s^2 - 4i\gamma^2 \sqrt{|\beta'|}. \tag{17}
\]

As we have anticipated, the energy of the mode coincides with Eq. (15) analytically continued to negative values of \( \beta \). The spectral function is obtained from Eq. (16) by substituting \( \beta^{1/4} \) with \( |\beta|^{1/4}e^{\pi i/4} \).

The analytical structure of \( G(\omega) \) for the energy dispersion (15) is shown schematically in Fig. 2. At smooth and negative dispersion below a critical value, \(|\beta| \leq |\beta_c(\gamma)| \lesssim 1\), the Higgs mode belongs to the non-physical sheet (regime B). The critical point \( \beta_c(\gamma) \) depends weakly on the energy scale \( \gamma \ll \Delta_F \). In this parameter range there is an additional critical point \( |\beta(\gamma)| < |\beta_c(\gamma)| \) such that at \(|\beta| < |\beta_c|\) the energy of the mode is real. Conversely at \(|\beta| > |\beta_c|\) the energy develops imaginary part increasing until the mode enters the physical sheet at \( \beta = \beta_c \) where it becomes a resonance (regime C).

The change from analytic to non-analytic dependence on \( \beta \) in the Higgs mode energy (15) and (16) is accompanied by a singular behavior of the quasi-particle dephasing. Indeed, by expanding the quasi-particle energy with Eq. (15) analytically continued to negative values \( \beta \), the spectral function is obtained from Eq. (16) by

\[
\omega_{qp}^2 = \omega_0^2 - i\Gamma \tag{18}
\]

This singularity affects the quasi-particle dephasing in the pairing amplitude \( \delta \Delta_{qp}(t) \approx \Re \int d\epsilon u^* e^{-i \omega t}, \) where \( u^* \) is a smooth function finite at the Fermi energy. Upon substitution of the expansion of \( E_c \) to the phase factor, we obtain a 1/2-power-law decay \( \delta \Delta_{qp}(t) \approx \Re e^{-\frac{1}{2} \Delta_F t^2} [(1 - \beta t) t]^{-1/2} \) valid at long times \( t \gg 1/\Delta_F \). Function \( \delta \Delta_{qp} \) has a square-root singularity with a branch cut along the positive real axis at \( \beta = 1 \). At smaller values the function is analytic. There is no direct correspondence between this non-analyticity controlled by the low-energy expansion and the non-analyticity of the Higgs mode (15) defined by the high-energy asymptotics of the interaction. The explicit calculation in Eqs. (15) and (17) illustrates that the character of the latter is specific to the model.

The dynamics of the pairing amplitude \( \delta \Delta(t) \) is controlled by \( G(\omega) \). The two contributions, from the Higgs pole \( \omega_\Delta = \omega_0 - i\Gamma \) located on the physical sheet, and from the quasi-particle branch cut, compete with each other. At relatively small times \( t \gg 1/\Delta_F \) the quasi-particle states provide the main contribution leading to the algebraic decay of the pairing amplitude. At intermediate time scales, \( 1/\Delta_F \ll t \ll 1/\Gamma \), the Higgs mode dominates the dynamics characterized by the pairing amplitude oscillating with frequency \( \omega_0 \) and decaying exponentially with the decrement \( \gamma = 1/\Gamma \) (infinite for a discrete Higgs mode). Eventually, at even longer times \( t \gg \gamma \) the quasi-particle contribution prevails, although having by that time a smaller amplitude. When the Higgs mode turns into a virtual state, the power of algebraic decay is increased compared to the power 1/2 discussed above.

The analysis of the pairing dynamics within the BCS model (11) is restricted to time scales small compared to quasi-particle relaxation time \( \tau_{qp} \) (10). At weak coupling and at small temperatures it is exponentially suppressed, which validates our result for the lifetime of the Higgs mode (17) provided \( \gamma^2 \gg \Delta_F/\tau_{qp} \).

In conclusion, the energy dispersion of pairing interaction qualitatively changes collective dynamics of fermions. The suppression of the interaction at the Fermi level generically leads to formation of a virtual Higgs state with energy on a non-physical sheet of Riemann surface. On the other hand, the enhancement of interaction at the Fermi level makes this mode discrete with energy located in the spectral gap of quasi-particle excitations. Depending on the details of interaction, the mode may also become a resonance with finite lifetime within the quasi-particle continuum.

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