Easy-boat
Easy-boat Formalism With easy-boat

https://youtu.be/bNqiwW4p6WE
Characterizing Sliding Surfaces of Cyber-Physical Systems

\( \psi \)

\( \theta \)

\( d \)

\( s \)
\[ \dot{d} = \sin u \]
\[ \cos (\psi - u) + \cos \frac{\pi}{5} > 0 \]

**Controller**

\[
\text{in: } (d, \psi, q) \quad \text{out: } u \\
\text{if } d^2 - 1 > 0 \text{ then } q := \text{sign}(d) \\
\text{if } \cos (\psi + \text{atan}(d)) + \cos \frac{\pi}{4} \leq 0 \lor (d^2 \leq 1 \land \cos \psi + \cos \frac{\pi}{4} \leq 0) \\
\text{then } u := \pi + \psi - q \frac{\pi}{4}. \\
\text{else } u := -\text{atan}(d).
\]
Simulation
Simulation in the $(t, d)$-space
Simulation in the \((\int^t \cos u, d)-space\)
Formalism
Given $\mathbb{Q}^-$, $\mathbb{Q}^+$ disjoint and closed, two smooth functions $f_a$, $f_b$. We define [3]

$$
S(\mathbb{A}) : \begin{cases} 
\dot{x} = f(x, q) = \begin{cases} 
f_a(x, q) & \text{if } x \in \mathbb{A} \\
f_b(x, q) & \text{if } x \in \mathbb{B} = \overline{\mathbb{A}} 
\end{cases} \\
q = -1 & \text{as soon as } x \in \mathbb{Q}^- \\
= +1 & \text{as soon as } x \in \mathbb{Q}^+
\end{cases}
$$
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Sliding surface
The *sliding surface* $S(A)$ for $S(A)$ is the subset of $\partial \mathbb{A}$ such that the state can slide inside for a non degenerated interval of time. Since $Q^-$, $Q^+$ are disjoint, we have

$$S(A) = S_{q=-1}(A) \cup S_{q=+1}(A)$$
We can thus assume that $q$ is fixed and find the sliding surfaces for

$$S(A) : \dot{x} = \begin{cases} 
  f_a(x) & \text{if } x \in A \\
  f_b(x) & \text{if } x \in B = \overline{A}
\end{cases}$$
The *Lie derivative* of $c : \mathbb{R}^n \rightarrow \mathbb{R}$ with respect to $f$ is

$$\mathcal{L}_f^c (x) = \frac{dc}{dx}(x) \cdot f(x).$$
If $A : c(x) \leq 0$, then

$$S(A) = \partial A \cap \{x \mid \mathcal{L}_a^c(x) \geq 0 \land \mathcal{L}_b^c(x) \leq 0\}.$$
Sliding set $\mathcal{S}(A)$ (red) for $A = \{x | c(x) \leq 0\}$
**Proposition.** If we have two closed sets $A_1$ and $A_2$. We have [3]

\[(i) \quad S(A_1 \cap A_2) = (S(A_1) \cap A_2) \cup (S(A_2) \cap A_1)\]

\[(ii) \quad S(A_1 \cup A_2) = (S(A_1) \cap \overline{A_2}) \cup (S(A_2) \cap \overline{A_1})\]
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\[ S(A_1 \cup (A_2 \cap A_3)) \]
With easy-boat
Three problems:

- Safety: the sailboat never goes against the wind. [1]
- Capture: The boat will be captured by its corridor [2]
- Characterize of the sliding surface.
We take \( x = (d, \psi) \),

| Function \( f(x, q) \) |
|-------------------------|
| if \( \cos(x_2 + \tan x_1) + \cos \frac{\pi}{4} \leq 0 \lor (x_1^2 - 1 \leq 0 \land \cos x_2 + \cos \frac{\pi}{4} \leq 0) \) |
| then \( u := \pi + x_2 - q \frac{\pi}{4} \) |
| else \( u := -\tan x_1 \). |
| Return \( (\sin u, 0) \) |
\[ \mathbf{x} = (d, \psi) \]
\[ \mathbf{f}_a (\mathbf{x}, q) = \begin{pmatrix} \sin (\pi + x_2 - q \frac{\pi}{4}) \\ 0 \end{pmatrix} \]
\[ \mathbf{f}_b (\mathbf{x}) = \begin{pmatrix} \sin (-\tan x_1) \\ 0 \end{pmatrix} \]
\[ A_1 = \{ \mathbf{x} | \cos (x_2 + \tan x_1) + \cos \frac{\pi}{4} \leq 0 \} \]
\[ A_2 = \{ \mathbf{x} | x_1^2 - 1 \leq 0 \} \]
\[ A_3 = \{ \mathbf{x} | \cos x_2 + \cos \frac{\pi}{4} \leq 0 \} \]
\[ A = A_1 \cup (A_2 \cap A_3) \]
\[ Q^- = \{ \mathbf{x} | x_1 + 1 \leq 0 \} \]
\[ Q^+ = \{ \mathbf{x} | 1 - x_1 \leq 0 \} \]
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\[ L_{a}^{c_1} (x, q) = \frac{dc_1}{dx} (x) \cdot f_a (x, q) = -\sin(\frac{q\pi}{4} - x_2) \cdot \sin(\tan(x_1) + x_2) \]
\[ L_{b}^{c_1} (x) = \frac{dc_1}{dx} (x) \cdot f_b (x, q) = \frac{\sin(\tan x_1 + x_2) \cdot x_1}{x_1^2 + 1} \]
\[ L_{a}^{c_2} (x, q) = \frac{dc_2}{dx} (x) \cdot f_a (x) = 2 \sin\left(\frac{q\pi}{4} - x_2\right) \cdot x_1 \]
\[ L_{b}^{c_2} (x) = \frac{dc_2}{dx} (x) \cdot f_b (x, q) = \frac{-2x_2^2}{\sqrt{x_1^2 + 1}} \]
\[ L_{a}^{c_3} (x, q) = \frac{dc_3}{dx} (x) \cdot f_a (x, q) = 0 \]
\[ L_{b}^{c_3} (x) = \frac{dc_3}{dx} (x) \cdot f_b (x, q) = 0 \]

\[ S(A_1) = \partial A_1 \cap L_1^b \cap \left( L_1^a (1) \cap Q^- \cup L_1^a (-1) \cap Q^+ \right) \]
\[ S(A_2) = \partial A_2 \cap L_2^b \cap \left( L_2^a (1) \cap Q^- \cup L_2^a (-1) \cap Q^+ \right) \]
\[ S(A_3) = \partial A_3 \]
Characterizing Sliding Surfaces of Cyber-Physical Systems
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Characterizing sliding surfaces of cyber-physical systems.
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