THE CHARACTER TABLE OF $GL_n(\mathbb{F}_q) \rtimes \langle \sigma \rangle$

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Abstract. We compute the character table of $GL_n(\mathbb{F}_q) \rtimes \langle \sigma \rangle$, $\sigma$ being an order 2 exterior automorphism. After giving the parametrisations of $\sigma$-stable irreducible characters of $GL_n(q)$ and the conjugacy classes contained in $GL_n(q) \sigma$, we study how the extension of a $\sigma$-stable character of $GL_n(q)$ to $GL_n(q) \sigma$ decomposes into induced cuspidal functions, the most difficult part of which has been solved by J-L. Waldspurger. The result is a purely combinatorial formula that only involves various Weyl groups and the generalised Green functions of classical groups. Finally, we explicitly determine the table for $n = 2$ and 3.

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Introduction

General Notations.
We fix an odd prime number $p$ and denote by $q$ a fixed power of $p$, then we fix a prime number $\ell$ prime to $p$. We choose an algebraic closure $k$ of the finite field $\mathbb{F}_q$ with $q$ elements. Denote by $N_{F'/F_{\ell'}} : \mathbb{F}_{q'} \rightarrow \mathbb{F}_{q'}$ the norm map whenever $s$ divides $r$. Let $G/k$ be an arbitrary linear algebraic group defined over $\mathbb{F}_q$ with Frobenius endomorphism $F$. We denote by $G^F$ the set of fixed points of $F$, which is a finite group. We may also denote this group by $G(q)$ if $F$ is clear from the context. We only consider the $\mathbb{Q}_\ell$-representations of the finite groups $G^F$, and notably $GL_n(q)$ for some non negative integer $n$. We will always assume that $q > n$. 

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Throughout the article, we will denote by $Z_H = n$ where for any linear algebraic group $H$. We put $N_m = C_H(n)$ extending characters to $GL_n$ of the irreducible characters due to Lusztig and Srinivasan, which is fit for the problem of extending characters to $GL_n(q) \rtimes \langle \sigma \rangle$ (see below).

Since we will be primarily interested in a non-connected group $G$ with two connected components, the hypothesis on $p$ implies that the unipotent elements of $G$ are all contained in $G^\circ$. If $G^1$ is the connected component of $G$ other than $G^\circ$ with some fixed element $u \in G^1$, we will often write the Jordan decomposition of an element of $G^1$ as $so u$, with $so$ being semi-simple and $u$ unipotent. Note that in this expression $so$ should be regarded as a whole and $s$ may well not be semi-simple.

**Irreducible Characters of $GL_n(q)$.**

Suppose $G = GL_n(q)$ over $k$. Given a Frobenius $F$ of $G$, the associated finite group $G^F = G(q)$ is denoted by $GL_n(q)$ or $GL_n^-(q)$ according to whether the action of $F$ on the Dynkin diagram is trivial or not. The character table of $GL_n(q)$ has been well known since the work of Green [Gr]. Instead of the combinatorial point of view of Green, we present below a parametrisation of the irreducible characters due to Lusztig and Srinivasan, which is fit for the problem of extending characters to $GL_n(q) \rtimes \langle \sigma \rangle$ (see below).

For each $F$-stable Levi subgroup $L$, we denote by $\text{Irr}_{\text{reg}}(L^F)$ the set of regular linear characters of $L^F$ (see [LS, §3.1]), and denote by $\text{Irr}(W_L)^F$ the set of $F$-stable irreducible characters of the Weyl group $W_L = W_L(T)$, with $T \subset L$ being an $F$-stable maximal torus. We take $\theta \in \text{Irr}_{\text{reg}}(L^F)$ and $\varphi \in \text{Irr}(W_L)^F$. For each $\varphi$, we denote by $\tilde{\varphi}$ an extension of $\varphi$ to $W_L \rtimes \langle F \rangle$. We put

$$R^G_\varphi \theta = \epsilon_{G} \epsilon_{L^F} |W_L|^{-1} \sum_{w \in W_L} \tilde{\varphi}(wF) R^G_{1_w} \theta,$$

where for any linear algebraic group $H$, $\epsilon_H := (-1)^{rk_H}$ and $rk_H$ is the $\mathbb{F}_q$-rank of $H$, and $R^G_{1_w} \theta$ is the Deligne-Lusztig induction ([DL]) of $(T, \theta)$.

**Theorem 1.** (Lusztig, Srinivasan, [LS] Theorem 3.2) Let $G = GL_n^+(q)$. For some choice of $\tilde{\varphi}$, the virtual character $R^G_\varphi \theta$ is an irreducible character of $G^F$. Moreover, all irreducible characters of $G^F$ are of the form $R^G_\varphi \theta$ for a triple $(L, \varphi, \theta)$. The characters associated to the triples $(L, \varphi, \theta)$ and $(L', \varphi', \theta')$ and distinct if and only if one of the following conditions is satisfied

- $(L, \theta)$ and $(L', \theta')$ are not $G^F$-conjugate;
- $(L, \theta) = (L', \theta')$ and $\varphi \neq \varphi'$.

Therefore, the calculation of the values of the irreducible characters of $GL_n(q)$ is reduced to the calculation of the values of Deligne-Lusztig characters, i.e. virtual characters of the form $R^G_{1_w} \theta$. 
Clifford Theory.

Let $\sigma$ be an automorphism of order 2 of $\text{GL}_n$ that is compatible with the given Frobenius. It defines a semi-direct product $\text{GL}_n(q) \rtimes \mathbb{Z}/2\mathbb{Z}$. This group will be denoted by $\text{GL}_n(q) \rtimes \langle \sigma \rangle$ (or simply $\text{GL}_n(q)\langle \sigma \rangle$) in order to specify the action of $1 \in \mathbb{Z}/2\mathbb{Z}$. We will assume that $\sigma$ is an exterior automorphism. Regarded as element of this non-connected group, $\sigma = (\text{Id}, 1)$ satisfies $\sigma^2 = 1$ and $\sigma g \sigma^{-1} = \sigma(g)$, for all $g \in \text{GL}_n(q)$.

The representations of $\text{GL}_n(q) \rtimes \langle \sigma \rangle$ are related to the representations of $\text{GL}_n(q)$ by the Clifford theory in the following way. Let $H$ be a finite group and let $N$ be a normal subgroup of $H$ such that $H/N = \mathbb{Z}/r\mathbb{Z}$ with $r$ prime, and let let $\chi$ be an irreducible character of $H$. We denote by $\chi_N$ the restriction of $\chi$ to $N$. Then

- Either $\chi_N$ is irreducible;
- Or $\chi_N = \bigoplus \theta_i$, where $\theta_i \in \text{Irr}(N)$ are some distinct irreducible characters.

Moreover, the $\theta_i$'s form an orbit under the action of $H/N$ on $\text{Irr}(N)$. Conversely, $\chi_N \in \text{Irr}(N)$ extends to an irreducible character of $H$ if and only if it is invariant under the action of $H/N$ by conjugation. If $\chi$ is such an extension, we obtain all other extensions by multiplying $\chi$ by a character of $H/N$.

Denote by $\text{Irr}(\text{GL}_n(q))^{\sigma}$ the set of $\sigma$-stable irreducible characters, i.e. those satisfying $\chi = \sigma^* \chi := \chi \circ \sigma$. The irreducible characters of $\text{GL}_n(q) \rtimes \langle \sigma \rangle$ are either an extension of a character $\chi \in \text{Irr}(\text{GL}_n(q))^{\sigma}$ or an extension of $\chi \oplus \sigma \chi$ with $\chi$ a non-$\sigma$-stable irreducible character of $\text{GL}_n(q)$. Note that the extension of $\chi \oplus \sigma \chi$ for $\chi$ non-$\sigma$-stable vanishes on the component $\text{GL}_n(q).\sigma$, whereas two extensions of $\chi \in \text{Irr}(\text{GL}_n(q))^{\sigma}$ differ by a sign on $\text{GL}_n(q).\sigma$. Their values on $\text{GL}_n(q)$ are then given by the character table of $\text{GL}_n(q)$. Once we fix an extension $\tilde{\chi}$ for each $\chi \in \text{Irr}(\text{GL}_n(q))^{\sigma}$, it remains for us to calculate the restriction of $\tilde{\chi}$ on $\text{GL}_n(q).\sigma$. If no confusion arises, we will also say that $\tilde{\chi}|_{\text{GL}_n(q).\sigma}$ is an extension of $\chi$ to $\text{GL}_n(q).\sigma$.

The conjugacy classes of $\text{GL}_n(q).\langle \sigma \rangle$ consist of the conjugacy classes of $\text{GL}_n(q)$ that are stable under $\sigma$, of the unions of pairs of conjugacy classes of the form $(C, \sigma(C))$, with $C \subset \text{GL}_n(q)$ non-$\sigma$-stable, and of the conjugacy classes contained in $\text{GL}_n(q).\sigma$. From the equality

$$
\frac{1}{2} \#\text{[\sigma-stable classes]} + \frac{1}{2} \#\text{[non-\sigma-stable classes]} + \#\text{[classes in } \text{GL}_n(q).\sigma] = \#\text{[classes of } \text{GL}_n(q).\langle \sigma \rangle]$$

and from the fact that $\#\text{[\sigma-stable classes]} = \#\text{[\sigma-stable characters]}$, we deduce that

$$
\#\text{[classes contained in } \text{GL}_n(q).\sigma] = \#\text{[\sigma-stable characters]}.
$$

So the table that we are going to calculate is a square table, its lines and columns being indexed by the $\sigma$-stable irreducible characters of $\text{GL}_n(q)$ and the conjugacy classes in $\text{GL}_n(q).\sigma$ respectively. However, there is no natural bijection between the classes and the characters.
Deligne-Lusztig Induction for $\text{GL}_n(q) \rtimes <\sigma>$. 

Let $\sigma$ be as above and let $\rho : \text{GL}_n(q) \to V$ be a $\sigma$-stable irreducible representation (meaning that $\rho$ and $\rho \circ \sigma$ are isomorphic). Defining an extension of $\rho$, say $\tilde{\rho}$, is to define an action of $\sigma$ on $V$ in such a way that $\tilde{\rho}(\sigma)^2 = \text{Id}$ and that $\tilde{\rho}(\sigma)\rho(g)\tilde{\rho}(\sigma)^{-1} = \rho(\sigma(g))$ for all $g \in \text{GL}_n(q)$. Except in some particular cases, we do not know how to do it. However, when $\sigma$ is quasi-central, we have a natural action of $\sigma$ on the Deligne-Lusztig varieties $X_w$ associated to $w \in W^\sigma$, the subgroup of $\sigma$-fixed elements of $W = W_G(T_0)$, with $T_0$ being a $\sigma$-stable and $F$-stable maximal torus of $G$. This allows us to define the extensions of the Deligne-Lusztig characters $R^G_{T_w} 1$ to $\text{GL}_n(q) <\sigma>$, where $T_w$ is an $F$-stable maximal torus corresponding to the $F$-conjugacy class of $w$. By expressing a unipotent character of $\text{GL}_n(q)$ as a linear combination of these Deligne-Lusztig characters, we can obtain an extension of this unipotent character. More concretely, if we take an $F$-stable and $\sigma$-stable Borel subgroup $B_0 \subset G$, the variety $X_w$ consists of the Borel subgroups $B$ such that $(B, F(B))$ are conjugate to $(B_0, \tilde{w}B_0\tilde{w}^{-1})$ by $G$, where $\tilde{w}$ is a representative of $w \in W^\sigma$ in $G$ which can be chosen to be $\sigma$-stable (one needs $\sigma$ to be quasi-central here). The action of $\sigma$ on $X_w$ is just $B \mapsto \sigma(B)$, which induces an action on the cohomology. The character $R^G_{T_w} 1$ thus extends to the function 

$$g\sigma \mapsto \text{Tr}(g\sigma|H^*_c(X_w, \bar{Q}_e)),$$

denoted by $R^G_{T_w, \sigma} 1$. This is a particular case of the Deligne-Lusztig induction for non-connected reductive groups developed by Digne and Michel [DM94]. More generally, given an $F$-stable and $\sigma$-stable Levi factor of a $\sigma$-stable parabolic subgroup, we have the maps $R^G_{L, \sigma}$, that sends $L^F$-invariant functions on $L^F.\sigma$ to $G^F$-invariant functions on $G^F.\sigma$.

Each irreducible character $\chi$ of $\text{GL}_n(q)$ is induced from an irreducible character $\chi_L$ of $L^F$, where $L$ is an $F$-stable Levi subgroup as in the setting of Theorem [1]. If $L$ is moreover an $\sigma$-stable factor of some $\sigma$-stable parabolic subgroup, and $\chi_L$ is a $\sigma$-stable character of $L^F$, then $\chi$ is also $\sigma$-stable. Suppose that we know how to calculate $\tilde{\chi}_L$, an extension of $\chi_L$ to $L^F.\sigma$, then the character formula will allow us to calculate the values of $R^G_{L, \sigma} \tilde{\chi}_L$, which coincides with the values of an extension of $\chi$ to $G^F.\sigma$.

Quadratic-Unipotent Characters.

However, there exist some $\sigma$-stable characters of $G^F$ that cannot be obtained by the above procedure. Let us look at some examples for $n = 2, 3$ and $4$. We take as $\sigma$ the automorphism $g \mapsto \eta g^{-1} \sigma^{-1} \eta$ with $g \in \text{GL}_n(k)$, where

$$J_2 = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad J_4 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

If $G = \text{GL}_2(k)$, and $T$ is the maximal torus consisting of diagonal matrices, 1 the trivial character of $F^*_q$, $\eta$ the order 2 irreducible character of $F^*_q$, and if we denote by $\theta$ the character
(1, η) of $T^F \cong \mathbb{F}_q^r \times \mathbb{F}_q^{s}$, then one can verify that $R_{L, \theta}^G$ is an \( \sigma \)-stable irreducible character while \( \theta \in \text{Irr}(T^F) \) is not \( \sigma \)-stable.

Besides, a priori, the map $R_{L, \theta}^G$ is not defined for $L^F, <\sigma>$, but for the normaliser $N_{G, <\sigma>}(L, P)$ (the set of elements that simultaneously normalise $L$ and $P$, with $L$ being a Levi factor of $P$). If $L$ is a \( \sigma \)-stable Levi factor of a \( \sigma \)-stable parabolic subgroup $P$, then $N_{G, <\sigma>}(L, P) = L, <\sigma>$, otherwise, the two groups are not the same, as the following examples show. In fact, what really matters is whether $N_{G, <\sigma>}(L, P)$ meets the connected component $G, \sigma$.

If $G = GL_4(k)$ and $L = C_G(t)$ with $t = \text{diag}(1, -1, -1, 1)$, then $L \cong GL_2(k) \times GL_2(k)$ and \( \theta := (\text{Id} \circ \text{det}, \eta \circ \text{det}) \) is a \( \sigma \)-stable irreducible character of $L^F$ and thus induces a \( \sigma \)-stable irreducible character of $GL_4(q)$. However, $L$ is not a \( \sigma \)-stable Levi factor of a \( \sigma \)-stable parabolic subgroup, because otherwise the \( \sigma \)-fixed part $L'' \cong SL_2(k) \times SL_2(k)$ would be a Levi subgroup of $G'' \cong Sp_4(k)$ (cf. Proposition 2.1.2). In this case, $N_{G, <\sigma>}(L, P) = L \cup Ln\sigma$, where $n \in N_G(L)$ permutes the two direct factors of $L$. So we are essentially in the same situation as the previous example, i.e. \( \theta \) is not \( n\sigma \)-stable.

Now we take $L = GL_3(k) \times k^r$ and a character with semi-simple part $(\text{Id}, \eta)$ with respect to this direct sum. In this case, $L$ itself is not \( \sigma \)-stable. In fact, it is not conjugate to any \( \sigma \)-stable Levi subgroup. So $N_{G, <\sigma>}(L, P) \subset G$, no matter which parabolic subgroup is $P$.

The above examples are typical. Let $L = G_1 \times G_2$ be a Levi subgroup of $G$, where $G_1 \cong GL_m(k)$ and $G_2 \cong GL_{n-m}(k)$. Let $\chi_1$(resp. $\chi_2$) be a unipotent irreducible character of $G_1^F$(resp. $G_2^F$). The character $\chi_L := \chi_1 \otimes \chi_2 \eta$ (or $\chi_1 \eta \otimes \chi_2$) always induces a \( \sigma \)-stable irreducible character of $G^F$, where we regard $\eta$ as a central character of $G_1^F$ or $G_2^F$. But $L$ does not fit into Deligne-Lusztig theory for non-connected groups: either $L$ is not conjugate to any \( \sigma \)-stable Levi factor of \( \sigma \)-stable parabolic subgroups, or $\chi_L$ is not a \( \sigma \)-stable character of $L^F$.

The irreducible characters of $GL_n(q)$ of the form $\chi_L$ as above are called quadratic-unipotent. They are parametrised by the 2-partitions of $n$. Their extensions to $GL_n(q), <\sigma>$ have been computed by J.-L. Waldspurger by using character sheaves for non-connected groups. The main result is as follows.

Let $(\mu_+, \mu_-)$ be a 2-partition of $n$, to which is associated the data $(\varphi_+, \varphi_-, h_1, h_2)$, where $\varphi_+$ (resp. $\varphi_-$) is an irreducible character, determined by the 2-quotient of $\mu_+$ (resp. $\mu_-$), of the Weyl group $\mathfrak{W}_+$ (resp. $\mathfrak{W}_-$) of type $C_{N_+}$ (resp. $C_{N_-}$), while $h_1$ and $h_2$ are two non negative integers related to the 2-cores of $\mu_+$ and $\mu_-$. We have $n = 2N_+ + 2N_- + h_1(h_1 + 1) + h_2^2$.

**Theorem 2** (Waldspurger). The extension of the quadratic-unipotent character of $GL_n(q)$ associated to $(\mu_+, \mu_-)$ is given up to a sign by

\[
R_{L, \theta}^{G, \sigma} := \frac{1}{|\mathfrak{W}_+|} \frac{1}{|\mathfrak{W}_-|} \sum_{w_+ \in \mathfrak{W}_+, w_- \in \mathfrak{W}_-} \varphi_+(w_+)^{\varphi_-}(w_-)R_{L, w, \sigma}^{G, \sigma} \phi_w.
\]

In the above expression, $L_w$ is a \( \sigma \)-stable and $F$-stable Levi factor, isomorphic to $GL_{h_1}(h_1 + 1), h_2^2(k) \times T_{w_1} \times T_{w_-}$, with $w_\pm \in \mathfrak{W}_\pm$ of a \( \sigma \)-stable parabolic subgroup, and $\phi_w$ is some kind of "tensor product" of the characters $\phi(h_1, h_2)$, $\tilde{\eta}$ and $\tilde{\eta}$, where $\phi(h_1, h_2)$ is a cuspidal function on
Levi subgroup corresponds to such an $I$ small general linear groups. It has at most one $\sigma(\chi)$ in a way compatible with the action of $B$ subgroup containing $G$ diagram of direct factors of $L$. Extensions of the simple roots with the Dynkin diagram of $G$ then $L_w$ becomes a $\sigma$-stable maximal torus and $W_\sigma$ is isomorphic to $W_\sigma$, the $\sigma$-fixed subgroup of the Weyl group of $G$.

**Parametrisation of $\sigma$-Stable Characters of $\text{GL}_n(q)$**

A general $\sigma$-stable irreducible character is the product of a quadratic-unipotent component and a component that looks like induced from an $\sigma$-stable Levi factor of a $\sigma$-stable parabolic subgroup. See Proposition [5.2.2] and Proposition [5.2.3] for the details.

Suppose that $\chi$ is a $\sigma$-stable irreducible character corresponding to $(M, \theta, \varphi)$ as in Theorem [I].

We write $\theta = (\alpha_i)_i$ with respect to the the decomposition of $M^F$ into a product of some $\text{GL}_n(q')$'s, where $\alpha_i$ are some characters of $F_{q_i}$ and we have omitted the determinant map from the notation. It is easy to see that the action of $\sigma$ sends $\chi$ to the character associated to $(\sigma(M), \sigma, \theta, \sigma, \varphi)$. According to the parametrisation of the irreducible characters of $\text{GL}_n(q)$, there exists some

$$g \in N_{G^F}(\sigma(M), M) = \{x \in G^F | x \sigma(M) x^{-1} = M\}$$

such that $\sigma, \theta = \text{ad}^* g \theta$. Note that the value of $\alpha_i$ only depends on the determinant of the corresponding factor. The action of $\sigma$ inverts the determinant while the conjugation by $g$ does not change the determinant. We can then conclude that for each $\alpha_i$, its inverse $\alpha_i^{-1}$ is also a factor of $\theta$. The factors satisfying $\alpha_i^{-1} = \alpha_i$ form the quadratic-unipotent part.

Fix a $\sigma$-stable maximal torus $T$ contained in a $\sigma$-stable Borel subgroup $B \subset G$ and identify the simple roots with the Dynkin diagram of $G$. If $I$ is a $\sigma$-stable subdiagram of the Dynkin diagram of $G$, then it defines a $\sigma$-stable Levi factor $L_I$ of some $\sigma$-stable standard parabolic subgroup containing $B$. We fix an isomorphism between $L_I$ and a direct product of some smaller general linear groups. It has at most one $\sigma$-stable direct factor, denoted by $L_0$, and so $L_I \cong L_0 \times L_1$ so that $\sigma$ non-trivially permutes the direct factors of $L_1$. All $\sigma$-stable standard Levi subgroup corresponds to such an $I$. We associate a quadratic-unipotent character $\chi_0$ to $L_0^F$, and a pair of characters, with semi-simple parts $\alpha_i$ and $\alpha_i^{-1}$ respectively, to each pair of direct factors of $L_1$ that are exchanged by $\sigma$. By defining the unipotent parts of the character in a way compatible with the action of $\sigma$, we obtain a $\sigma$-stable character of $L_1^F$, denoted by $\chi_1$. Then $R_L^G(\chi_0 \boxtimes \chi_1)$ is a $\sigma$-stable irreducible character of $G^F$ (for suitable $\alpha_i$'s), and all $\sigma$-stable irreducible characters of $\text{GL}_n(q)$ are obtained this way. We will calculate the extension of $\chi_{L_I} = \chi_0 \boxtimes \chi_1$ to $L^F, \sigma$, and then apply the map $R_{L_0}^G, \sigma$. Note that if we regard $\chi$ as induced from $(M, \theta, \varphi)$ following Theorem [I] then $M$ is not necessarily $\sigma$-stable.

**Extensions of $\sigma$-Stable Characters.**

The extension of the quadratic-unipotent part determined, the problem is reduced to the following.

\[ \text{GL}_n(q) \]
\textbf{Problem.} Put }L_1 = G_0 \times G_0, G_0 = \text{GL}_m(k), \text{ and let } \sigma_0 \text{ be an automorphism of } G_0 \text{ of order } 2. \text{ Denote by } F_0 \text{ the Frobenius of } \text{GL}_m(k) \text{ that sends each entry to its } q \text{-th power. Define an automorphism } \sigma \text{ of } L_1 \text{ by}
\begin{equation}
(g, h) \mapsto (\sigma_0(h), \sigma_0(g)),
\end{equation}

and a Frobenius } F \text{ by}
- \text{ Linear Case: } (g, h) \mapsto (F_0(g), F_0(h)),
- \text{ Unitary Case: } (g, h) \mapsto (F_0(h), F_0(g)).

The problem is to decompose the extension of a } \sigma \text{-stable irreducible character of } L_1^F \text{ to } L_1^F, \langle \sigma \rangle \text{ as a linear combination of Deligne-Lusztig characters.}

Let us first look at the linear case. We have } L_1^F = G_0^F \times G_0^F. \text{ Let } \chi \text{ be a unipotent character of } G_0^F. \text{ Then } \chi \otimes \chi \in \text{Irr}(L_1^F) \text{ is } \sigma \text{-stable. In order to calculate its extension, we separate } \sigma \text{ into two automorphisms, one sending } (g, h) \to (\sigma_0(g), \sigma_0(h)), \text{ the other one, denoted by } \tau, \text{ sending } (g, h) \to (h, g). \text{ Denote by } \tilde{\chi} \text{ an extension of } \chi \text{ to } G_0^F \sigma_0 F_0. \text{ Consider the } \tau \text{-stable character } \tilde{\chi} \otimes \tilde{\chi} \text{ of } G_0^F \sigma_0^2 \times G_0^F \sigma_0 \langle \sigma_0 \rangle. \text{ Its extension to } (G_0^F \sigma_0 \langle \sigma_0 \rangle \times G_0^F \sigma_0 \langle \sigma_0 \rangle) \times \langle \tau \rangle \text{ restricts to an irreducible character } \chi \text{ of } L_1^F, \langle \sigma \rangle, \text{ regarded as a subgroup of } (G_0^F \sigma_0 \langle \sigma_0 \rangle \times G_0^F \sigma_0 \langle \sigma_0 \rangle) \times \langle \tau \rangle. \text{ This gives an extension of } \chi \otimes \chi. \text{ Some linear algebra calculation shows that } \tilde{\chi}((g, h) \sigma) = \chi(g \sigma_0(h)). \text{ The latter is the value of a character of } \text{GL}_m(q).

The unitary case is a little more complicated and relies on the result of the linear case. In this case, } L_1^F \cong G_0^F, \text{ and the action of } \sigma \text{ on } G_0^F \text{ is given by } g \mapsto \sigma_0 F_0(g), \text{ which can be thought of as another Frobenius endomorphism. That is where the Shintani descent intervenes, which relates the functions on } G_0^F \sigma_0 F_0 \text{ to the functions on } G_0^F \sigma_0 F_0. \text{ Note that } \sigma_0 F_0 = F_0 \text{ acts trivially on } G_0^F \sigma_0 F_0. \text{ We know how to calculate the characters of } G_0^F \sigma_0 F_0 \cong \text{GL}_m(q), \text{ which extends trivially to } G_0^F \sigma_0 F_0. \text{ Thus, we obtain the extension of a character to } L_1^F, \sigma. \text{ The result is as follows.}

Let } \theta_1 \text{ be a } \sigma \text{-stable linear character of } L_1^F \text{ and let } \varphi \in \text{Irr}(W_{L_1}^\sigma)^F. \text{ Note that a } \sigma \text{-stable linear character extends trivially to } L_1^F, \sigma, \text{ and that } W_{L_1}^\sigma \text{ is in fact a product of symmetric groups.}

\textbf{Theorem 3.} \text{ Let } \chi_{L_1} \text{ be a } \sigma \text{-stable irreducible character of } L_1^F \text{ defined by } (\theta, \varphi). \text{ Then, for some choice of } \tilde{\varphi}, \text{ the extension of } \chi_{L_1} \text{ to } L_1^F, \sigma \text{ is given up to a sign by}
\begin{equation}
R_{\tilde{\varphi}}^{L_1, \sigma} \tilde{\theta}_1 = |W_{L_1}^\sigma|^{-1} \sum_{w \in W_{L_1}^\sigma} \varphi(w F) R_{w \sigma, \sigma}^{L_1, \sigma} \tilde{\theta}_1.
\end{equation}

Combined with the preceding theorem, it gives the main theorem below.

According to the parametrisation of } \sigma \text{-stable characters, each } \chi \in \text{Irr}(\text{GL}_n(q))^\sigma \text{ is of the form } R_{\tilde{\chi}}^G(\chi_0 \otimes \chi_1), \text{ where } L \cong L_0 \times L_1 \text{ is a } \sigma \text{-stable and } F \text{-stable Levi factor of some } \sigma \text{-stable parabolic subgroup, } \chi_0 \text{ is a quadratic-unipotent character of } L_0^F \text{ and } \chi_1 \text{ is a } \sigma \text{-stable irreducible character of } L_1^F \text{ whose semi-simple part and unipotent part are defined by } \theta_1 \in \text{Irr}_{\text{reg}}(L_1^F) \text{ and } \varphi \in \text{Irr}(W_{L_1}^\sigma) \text{ respectively. The notations of Theorem 2 are used in the following theorem.
Main Theorem. For some choice of $\tilde{\phi}$, the extension of $\chi$ is up to a sign given by

$$\tilde{\chi}|_{G^{f},\sigma} = |W_{L_{1}}^{\sigma} \times _{\mathbb{W}_{+}}^{\mathbb{W}_{-}} \mathbb{W}_{-}^{-1} \sum_{(w, w_{+}, w_{-}) \in W_{L_{1}}^{\sigma} \times \mathbb{W}_{+} \times \mathbb{W}_{-}} \tilde{\phi}(wF)\phi_{+}(w_{+})\phi_{-}(w_{-})R_{G,\sigma}^{G,\sigma}(\tilde{\theta}_{1}\tilde{\mathcal{E}}_{w}).$$

The symbol $\tilde{\mathcal{E}}$, which we define in §9.1, signifies some kind of tensor product on the component $(T_{w} \times L_{w})\sigma$.

Green Functions.

We remark that in $R_{L_{w},\sigma}^{L_{w},\sigma}\phi_{w}$ appears the generalised Green functions defined on the centraliser of a semi-simple element in $L_{0,\sigma}$, which is in general a product of $GL_{m}(q)$, $Sp_{2m}(q)$, $SO_{2m+1}(q)$ and $SO_{2m}^{\pm}(q)$, where the negative sign means that the Frobenius is twisted by a graph automorphism of order 2. The Green functions of finite classical groups were first tackled in [S83]. In [L85, V], Lusztig gives an algorithm of calculating the generalised Green functions of classical groups. The values of the (generalised) Green functions have since been computed by various people, and we will not make explicit theirs values except in the examples.

Organisation of the Article.

In Section 1, we recall many results on finite classical groups that will be used in the computation of the character table. In Section 2 and 3, we recall some results on non-connected linear algebraic groups and the theory of Deligne-Lusztig induction for the finite groups that arise from non-connected groups. These results will be used throughout the article. In Section 4, we specialise to the group $GL_{n}(q)$ and give some explicit information on this group. In Section 5 and 6, we give the parametrisation of $\sigma$-stable irreducible characters of $GL_{n}(q)$ and the conjugacy classes contained in $GL_{n}(q)\sigma$. The parametrisations are given in terms of some very explicit combinatorial data. In section 7, we collect some information on the eigenvalues of the Frobenius endomorphism on intersection cohomology groups of Deligne-Lusztig varieties and recall some results on Shintani descent. In Section 8 we recall some general results on character sheaves and give some examples in the case that we are interested in. The contents of Section 7 and 8 will only be used in Section 9, where we essentially prove the main theorem, which is stated in Section 11. In Section 10, we tackle a particular problem in the computation of Deligne-Lusztig inductions, namely the determination of the set $\{h \in GL_{n}(q) \mid hsah^{-1} \in M\sigma \}$ for some semi-simple element $s$ and some Levi subgroup $M$. We give a procedure to reduce the problem to some computations in Weyl groups. This allows us to write the formula in the main theorem in terms of purely combinatorial data. In the last section, we explicitly determine the complete character table of $GL_{n}(q)$ for $n = 2$ and 3.

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1. Finite Classical Groups

1.1. Partitions and Symbols.

1.1.1. We denote by \( \mathcal{P}_n \) the set of all partitions of the integer \( n \geq 0 \) and by \( \mathcal{P} \) the union \( \bigcup_n \mathcal{P}_n \). A partition is written as \( \lambda = (\lambda_1 \geq \lambda_2 \geq \cdots) \), a decreasing sequence of positive integers, or as \( \lambda = (1^{m_1}, 2^{m_2}, \ldots) \) where \( m_i \) is the multiplicity of \( i \) that appears in \( \lambda \). Each \( \lambda_i \) is called a part of \( \lambda \). We denote by \( |\lambda| \) the size of \( \lambda \) and \( l(\lambda) \) the length of \( \lambda \). For any integer \( e \geq 1 \), we denote by \( \mathcal{P}_n(e) \) the set of all \( e \)-partitions of \( n \), i.e. the \( e \)-tuples of partitions \( (\lambda^{(1)}, \ldots, \lambda^{(e)}) \in \mathcal{P}^e \) (direct product of \( e \) copies) satisfying \( \sum_i |\lambda^{(i)}| = n \). In the cases that concern us, \( e = 2 \).

1.1.2. A partition \( \lambda = (1^{m_1}, 2^{m_2}, \ldots) \) is called symplectic if \( m_i \) is even for every odd \( i \). To each symplectic partition \( \lambda \) we associate a finite set \( \kappa(\lambda) := \{ i \text{ even} | m_i > 0 \} \) and put \( \kappa(\lambda) = |\kappa(\lambda)| \). We denote by \( \mathcal{P}_n^{sp} \subset \mathcal{P}_n \) the subset of symplectic partitions. A partition \( \lambda = (1^{m_1}, 2^{m_2}, \ldots) \) is called orthogonal if \( m_i \) is even for every even \( i \). To each orthogonal partition \( \lambda \) we associate a finite set \( \kappa(\lambda) := \{ i \text{ odd} | m_i > 0 \} \) and put \( \kappa(\lambda) = |\kappa(\lambda)| \). We denote by \( \mathcal{P}_n^{ort} \subset \mathcal{P}_n \) the subset of orthogonal partitions. We may omit \( \lambda \) from the notation \( \kappa(\lambda) \) or \( \kappa(\lambda) \) if no confusion arises. The orthogonal partitions with \( \kappa = 0 \) are called degenerate. The subset of non degenerate partitions of \( n \) is denoted by \( \mathcal{P}_n^{ort, nd} \) and that of degenerate partitions is denoted by \( \mathcal{P}_n^{ort, d} \).

1.1.3. Given a partition \( \lambda \), we take \( r \geq l(\lambda) \), and we put \( \delta_r = (r - 1, r - 2, \ldots, 1, 0) \). Let \( (2y_1 > \cdots > 2y_{l_0}) \) and \( (2y_1' + 1 > \cdots > 2y_{l_1}' + 1) \) be the even parts and the odd parts of \( \lambda + \delta_r \), where the sum is taken term by term and \( \lambda \) is regarded as an decreasing sequence of integers \( (\lambda_i)_i \) with \( \lambda_i = 0 \) for \( i > l(\lambda) \). Denote by \( \lambda^{(0)} \) the partition defined by \( \lambda^{(0)}_k = y_k - l_0 + k \) and denote by \( \lambda^{(1)} \) the partition defined by \( \lambda^{(1)}_k = y_k' - l_1 + k \). Then \( (\lambda^{(0)}, \lambda^{(1)}) \) is a 2-partition that depends on \( r \). Changing the value of \( r \) will permutes \( \lambda^{(0)} \) and \( \lambda^{(1)} \). The 2-quotient of \( \lambda \) is then the unordered pair of partitions \( (\lambda^{(0)}, \lambda^{(1)}) \).

Denote by \( \lambda' \) the partition that has as its parts the numbers \( 2s + t \), \( 0 \leq s \leq l_i - 1 \), \( t = 0, 1 \). We have \( l(\lambda') = l(\lambda) \). The 2-core of \( \lambda \) is the partition defined by \( (\lambda'_k - l(\lambda) + k)_{1 \leq k \leq l(\lambda)} \). It is independent of \( r \) and, if non trivial, is necessarily of the form \( (d, d - 1, \ldots, 2, 1) \), for some \( d \in \mathbb{Z}_{>0} \). Fixing \( r \), the above constructions give a bijection between the partitions of \( n \) with the same 2-core and the 2-partitions of \( (n - r)/2 \).

1.1.4. We refer to [L84a] for the notion of symbols.

Fix an even positive integer \( N \). A symbol of symplectic type is the equivalence class of ordered pairs \((A, B)\) with \( A \) a finite subset of \( \{0, 1, 2, \ldots\} \) and \( B \) a finite subset of \( \{1, 2, 3, \ldots\} \) satisfying the following conditions.
THE CHARACTER TABLE OF GL_n(F_q) \rtimes \langle \sigma \rangle

1.2.1. Fix a positive integer \( n \).

(a) For any integer \( i \), the set \([i, i+1] \) is contained neither in \( A \) nor in \( B \);

(b) \(|A| + |B| \) is odd;

(c) \( \sum_{a \in A} a + \sum_{b \in B} b = \frac{1}{2}N + \frac{1}{2}(|A| + |B|)(|A| + |B| - 1) \);

under the equivalence that identifies \((A, B)\) and \((\{0\} \cup (A + 2), \{1\} \cup (B + 2))\). The set of these symbols is denoted by \( \Psi^p_{N} \).

Fix a positive integer \( N > 2 \). A symbol of orthogonal type is the equivalence class of the unordered pairs \((A, B)\) of finite subsets of \([0, 1, 2, \ldots]\) satisfying the following conditions

(a) For any integer \( i \), \([i, i+1] \) is contained neither in \( A \) nor in \( B \);

(c) \( \sum_{a \in A} a + \sum_{b \in B} b = \frac{1}{2}N + \frac{1}{2}(|A| + |B| - 1)(|A| + |B| - 1) \);

under the equivalence that identifies \((A, B)\) and \((\{0\} \cup (A + 2), \{0\} \cup (B + 2))\). The set of these symbols is denoted by \( \Psi_{N}^{\text{ort}} \). A symbol of orthogonal type is called degenerate if it is of the form \((A, A)\). The subset of non degenerate symbols is denoted by \( \Psi_{N}^{\text{ort, nd}} \) and that of degenerate symbols is denoted by \( \Psi_{N}^{\text{ort, d}} \).

Two symbols are similar if they admit representatives \((A, B)\) and \((A', B')\) such that \( A \cup B = A' \cup B' \) and \( A \cap B = A' \cap B' \).

1.1.5. To each symplectic partition is associated a similarity class of symbols of symplectic type in the following manner. Let \( \lambda \) be such a partition and let \( r \) be an integer such that \( 2r \geq \ell(\lambda) \). Denote by \((2y_1 > \cdots > 2y_r)\) and \((2y'_1 + 1 > \cdots > 2y'_r + 1)\) the even parts and the odd parts of \( \lambda + \delta_{2r} \). One can verify that there are indeed \( r \) even parts and \( r \) odd parts. Put \( A = \{0\} \cup \{y'_k + r + 2 - k \mid 1 \leq k \leq r\} \) and put \( B = \{y_k + r + 1 - k \mid 1 \leq k \leq r\} \). Then \((A, B)\) is a symplectic symbol, whose similarity class is independent of \( r \).

To each orthogonal partition is associated a similarity class of symbols of orthogonal type in the following manner. Let \( \lambda \) be such a partition and let \( r \) be an integer such that \( r \geq \ell(\lambda) \). Denote by \((2y_1 > \cdots > 2y_{[r/2]} \) and \((2y'_1 + 1 > \cdots > 2y'_{([r+1]/2)} + 1\) the even parts and the odd parts of \( \lambda + \delta_{r} \). One can verify that there are indeed \( [r/2] \) even parts and \( ([r+1]/2) \) odd parts. Put \( A = \{0\} \cup \{y'_k + ([r+1]/2) - k \mid 1 \leq k \leq r\} \) and put \( B = \{y_k + [r/2] - k \mid 1 \leq k \leq r\} \). Then \((A, B)\) is a symbol of orthogonal type, whose similarity class is independent of \( r \).

The above constructions define a bijection between \( \Phi^p_{N} \) \((\text{resp. } \Phi^{\text{ort}}_{N})\) and the similarity classes in \( \Psi^p_{N} \) \((\text{resp. } \Psi_{N}^{\text{ort}})\). (cf. \cite{L84a} §11.6, §11.7.)

1.2. Weyl Groups. Some basic facts about Weyl groups of type \( B_m, C_m \) and \( D_m \).

1.2.1. Fix a positive integer \( m \).

Denote by \( w_0 \) the permutation \((1, -1)(2, -2) \cdots (m, -m)\) of the set

\[ I = \{1, \ldots, m, -m, \ldots, -1\}. \]

The set of permutations of \( I \) that commute with \( w_0 \) is identified with \((\mathbb{Z}/2\mathbb{Z})^m \rtimes \mathbb{Z}_m \). This is the Weyl group of type \( B_m \) and \( C_m \), which will be denoted by \( \Psi^c_m \). We will identify \( \mathbb{Z}/2\mathbb{Z} \) with \( \mu_2 \), the multiplicative 2-element group, and denote its elements by \( \pm 1 \). An element of
$\mathfrak{W}_m^C$ can be written as
\[(1.2.1.1) \quad w = ((\epsilon_1, \ldots, \epsilon_m), \tau) \in (\mathbb{Z}/2\mathbb{Z})^n \times \mathfrak{Z}_m,\]
with $((1, \ldots, 1, \epsilon_i, 1, \ldots, 1), \epsilon_i = -1$ being the permutation $(i, -i)$, and $((1, \ldots, 1, \tau)$ being the permutation
\[i \mapsto \tau(i), \quad -i \mapsto -\tau(i).\]

1.2.2. The permutation $\tau$ is decomposed into cycles $\tau = c_{l_1} \cdots c_{l_r}$, where the disjoint subsets $l_r \subset \{1, \ldots, m\}$ form a partition of $\{1, \ldots, m\}$ and $c_{l_r}$ is a circular permutation of the indices in $l_r$. The permutation $\tau$ determines a partition $(\tau_1, \ldots, \tau_l)$ of $m$, also denoted by $\tau$, with $\tau_r = |l_r|$ for any $1 \leq r \leq l$, where $l = l(\tau)$ is the length of the partition.

For all $1 \leq r \leq l$, put $\xi_r = \prod_{k \in l_r} \epsilon_k$ and $\xi_r = (\epsilon_k)_{k \in l_r}$. Define the permutations
\[(1.2.2.1) \quad \tau^{(0)} = \prod_{\xi_r = 1} c_{l_r}, \quad \tau^{(1)} = \prod_{\xi_r = -1} c_{l_r},\]
so that $\tau = \tau^{(0)}\tau^{(1)}$. Also denote by $\tau^{(0)} = (\tau_r^{(0)})$ and $\tau^{(1)} = (\tau_r^{(1)})$ the associated partitions. We then have a 2-partition $(\tau^{(0)}, \tau^{(1)})$, which determines the conjugacy class of $w$. We sometimes call it a signed partition of $n$.

The conjugacy classes and irreducible characters of $\mathfrak{W}_m^C$ are both parametrised by the 2-partitions of size $n$. Write $l_0 = l(\tau^{(0)})$, and $l_1 = l(\tau^{(1)})$.

1.2.3. The Weyl group of type $D_m$, denoted by $\mathfrak{W}_m^D$, is the subgroup of $\mathfrak{W}_m^C$ consisting of the elements $((\epsilon_1, \ldots, \epsilon_m), \tau)$ such that $\prod \epsilon_i = 1$. For any positive integer $a$, let
\[\text{sgn} : \mathfrak{W}_m^C \to \{\pm 1\}\]
be the map whose kernel is $\mathfrak{W}_m^D$. The parametrisation of the conjugacy classes of $\mathfrak{W}_m^D$ is given as follows. (See [Ca, Proposition 25]) Let $\tau$ be a signed partition. If each part of $\tau$ is even and the $\xi_r$'s are all equal to 1, then the conjugacy class of $\mathfrak{W}_m^C$ corresponding to $\tau$ splits into two classes of $\mathfrak{W}_m^D$. Otherwise, this conjugacy class restricts to one single class of $\mathfrak{W}_m^D$ if it is non-empty.

Suppose we have fixed a maximal torus contained in a Borel subgroup of $SO_{2m}(k)$, which defines a set of simple roots $\{e_1 - e_2, \ldots, e_{m-1} - e_m, e_{m-1} + e_m\}$. The element $((1, \ldots, 1, -1, 1)$, which belongs to $\mathfrak{W}_m^C \setminus \mathfrak{W}_m^D$, can be realised as an element of $O_{2m}(k) \setminus SO_{2m}(k)$. Its conjugation action on $SO_{2m}(k)$ permutes the two simple roots $e_{m-1} - e_m$ and $e_{m-1} + e_m$, and thus induces a non-trivial graph automorphism.

1.2.4. Let $m_0$ and $m$ be some non negative integers and put $n = 2m + m_0$. Let $L$ be a Levi subgroup of $G = Sp_n(k)$ (resp. $SO_n(k)$) which is isomorphic to $Sp_{m_0}(k) \times (k^*)^m$ (resp. $SO_{m_0}(k) \times (k^*)^m$), then $N_G(L)/L$ is isomorphic to $\mathfrak{W}_m^C$ if $m_0 > 0$. If $m_0 = 0$, then $N_G(L)/L \cong \mathfrak{W}_m^D$.

1.3. Unipotent Classes and Centralisers. The parametrisation of unipotent conjugacy classes of finite classical groups is well known. We refer to ([LiSe, Chapter 3, Chapter 7]) for a more
complete survey. In this paragraph, \( G \) will be one of the groups \( \text{GL}_n(k) \), \( \text{Sp}_n(k) \), \( \text{SO}_n(k) \) and \( \text{O}_n(k) \). If the Frobenius \( F \) is split, we denote by \( \text{GL}_n(q) \), \( \text{Sp}_n(q) \), \( \text{SO}_n(q) \) and \( \text{O}_n(q) \) the associated finite groups, and if \( F \) induces a graph automorphism of order 2, we denote by \( \text{GL}_n^{-}(q) \), \( \text{SO}_n^{-}(k) \) and \( \text{O}_n^{-}(k) \) (only when \( n \) is even for the orthogonal groups) the associated finite groups. We will use the notations \( \text{SO}^{+}_n(k) \) and \( \text{O}^{+}_n(k) \) for all \( n \), understanding that \( \epsilon \) can be "-" only if \( n \) is even and that "+" corresponds to the split groups.

1.3.1. The unipotent classes of \( \text{GL}_n(k) \) are parametrised by \( \mathcal{P}_n \), with the sizes of Jordan blocks given by the corresponding partition. The unipotent classes of \( \text{Sp}_n(k) \) are parametrised by \( \mathcal{P}_n^{\text{sp}} \), and so in bijection with the similarity classes of \( \Psi_n^{\text{sp}} \). These are represented by the Jordan matrices in \( \text{GL}_n(k) \) that belong to \( \text{Sp}_n(k) \). The unipotent classes of \( \text{O}_n(k) \) are parametrised by \( \mathcal{P}_n^{\text{ort}} \), and so in bijection with the similarity classes of \( \Psi_n^{\text{ort}}(k) \). These are represented by the Jordan matrices in \( \text{GL}_n(k) \) that belong to \( \text{O}_n(k) \). A unipotent class of \( \text{O}_n(k) \) is called degenerate if the corresponding partition is degenerate. A unipotent class of \( \text{O}_n(k) \) splits into two \( \text{SO}_n(k) \)-classes if and only if it is degenerate, and restricts to one single \( \text{SO}_n(k) \)-class if otherwise. A degenerate partition alone forms a similarity class. ([L84a, 13.4(c)].)

1.3.2. Let \( u \) be a unipotent element in \( G \), associated to the partition \( \lambda = (1^{m_1}, 2^{m_2}, \ldots) \). In general, we have \( C_G(u) = VR \), where \( V \) is the unipotent radical of \( C_G(u) \), whence an affine space, and \( R \) is a reductive group given as follows. For any symplectic or orthogonal partition \( \lambda \), let \( \kappa \) be the \( \kappa(\lambda) \) defined in §1.1.2.

If \( G = \text{GL}_n(k) \), then

\[
R \cong \prod_{i|m_i>0} \text{GL}_m.
\]

In particular, \( C_G(u) \) is connected.

If \( G = \text{Sp}_n(k) \), then

\[
R \cong \prod_{i \text{ odd}} \text{Sp}_m \times \prod_{i \text{ even}} \text{O}_m.
\]

So \( C_G(u)/C_G(u)^{\circ} \cong (\mathbb{Z}/2\mathbb{Z})^{\kappa} \).

If \( G = \text{O}_n(k) \), then

\[
R \cong \prod_{i \text{ odd}} \text{O}_m \times \prod_{i \text{ even}} \text{Sp}_m.
\]

So \( C_G(u)/C_G(u)^{\circ} \cong (\mathbb{Z}/2\mathbb{Z})^{\kappa} \). If \( G = \text{SO}_n(k) \), then we have \( R \cap \text{SO}_n(k) \) instead, with the same \( V \) (cf. [LiSe, Lemma 3.11]).

In the case \( G = \text{O}_n(k) \), an element \( \varpi \in C_G(u) \) belongs to \( \text{SO}_n(k) \) if and only if its equivalence class \( \varpi = (e_1, \ldots, e_\kappa) \in C_G(u)/C_G(u)^{\circ} \) satisfies \( \sum_i e_i = 0 \). So \( C_{\text{SO}}(u) \cong (\mathbb{Z}/2\mathbb{Z})^{\kappa-1} \) if \( \kappa > 0 \). (See [LiSe, §7.1] for these assertions.)

In either of the cases \( \text{Sp}_n(q) \), \( \text{SO}_n^{+}(q) \), if we fix an \( F \)-stable representative \( u \) of an arbitrary \( F \)-stable unipotent class, then \( F \) acts trivially on \( C_G(u)/C_G(u)^{\circ} \) ([SO7, §4.1]). It follows that if the \( G \)-conjugacy class of \( u \) contains an \( F \)-stable element, then it contains one single \( G^F \)-class.
if $G = \text{GL}_n^\epsilon(k)$, $2^\kappa G^F$-classes if $G = \text{Sp}_n(k)$, and $2^{n-1} G^F$-classes if $G = \text{SO}_n^\epsilon(k)$ as long as $\kappa > 0$. The $G$-conjugacy class of $u$ does not contain any element of $G^F$ only when $G = \text{O}_n^+(q)$ and $\kappa(\lambda) = 0$. If $G = \text{O}_n^-(q)$ and $\kappa(\lambda) = 0$, the corresponding $G$-conjugacy class contains one single $G^F$-class, which splits into two classes for the conjugation by $\text{SO}_n(q)$. These two classes are called *degenerate*.

Fixing an $F$-stable unipotent conjugacy class which corresponds to the partition $\lambda$, then the centraliser of an $F$-stable element in this conjugacy class is of the form $V^F R^F$, where $V^F \cong \mathbb{F}_q^{\dim V}$ and $R^F$ is given as follows.

If $G = \text{GL}_n^\epsilon(q)$, then

$$R^F \cong \prod \text{GL}_m^\epsilon(q).$$

If $G = \text{Sp}_n(q)$, then

$$R^F \cong \prod_{i \text{ odd}} \text{Sp}_m^i(q) \times \prod_{i \text{ even}} \text{O}_m^\epsilon(q),$$

for some $\kappa$-tuple of signs $(\epsilon_i)_{i \in \kappa}$.

If $G = \text{O}_n^+(q)$, then

$$R^F \cong \prod_{i \text{ odd}} \text{O}_m^i(q) \times \prod_{i \text{ even}} \text{Sp}_m(q),$$

for some $\kappa$-tuple of signs $(\epsilon_i)_{i \in \kappa}$, subject to the condition that $\sum_i \epsilon_i = \epsilon$.

1.3.3. If we fix an $F$-stable representative in each $F$-stable unipotent $G$-conjugacy class, then the unipotent conjugacy classes in $G^F$ are in bijection with one of the sets:

$$\Psi_n^{sp} := \bigsqcup_{\lambda \in \mathcal{F}_n^{sp}} (\mathbb{Z}/2\mathbb{Z})^{\kappa(\lambda)}, \text{ if } G^F = \text{Sp}_n(q);$$

$$\Psi_n^{ort, +} := \bigsqcup_{\lambda \in \mathcal{F}_n^{ort, even}} (\mathbb{Z}/2\mathbb{Z})^{\kappa(\lambda) - 1} \sqcup \bigsqcup_{\lambda \in \mathcal{F}_n^{ort, odd}} (\{\text{pt}\} \sqcup \{\text{pt}\}), \text{ if } G^F = \text{SO}_n^+(q);$$

$$\Psi_n^{ort, -} := \bigsqcup_{\lambda \in \mathcal{F}_n^{ort, odd}} (\mathbb{Z}/2\mathbb{Z})^{\kappa(\lambda) - 1}, \text{ if } G^F = \text{SO}_n^-(q).$$

Write $\Psi_n^{sp} = \bigsqcup_{n \in \mathbb{Z}_{\geq 0}, n \text{ even}} \Psi_n^{sp}$ and $\Psi_n^{ort, \eta} = \bigsqcup_{n \in \mathbb{Z}_{\geq 0}} \Psi_n^{ort, \eta}$, where $n$ must be even if $\eta = -$.

1.3.4. We take $u$ and $\lambda$ as above. We have $\dim \text{Sp}_n(k) = n(n + 1)/2$, $\dim \text{SO}_n(k) = n(n - 1)/2$, and the following formulas.

(1.3.4.1) $\dim C_{\text{GL}}(u) = \sum_{i} im_{i}^2 + 2 \sum_{i < j} im_{i}m_{j}$

(1.3.4.2) $\dim C_{\text{Sp}}(u) = \frac{1}{2} \sum_{i} im_{i}^2 + \sum_{i < j} im_{i}m_{j} + \frac{1}{2} \sum_{i \text{ odd}} m_{i}$

(1.3.4.3) $\dim C_{\text{SO}}(u) = \frac{1}{2} \sum_{i} im_{i}^2 + \sum_{i < j} im_{i}m_{j} - \frac{1}{2} \sum_{i \text{ odd}} m_{i}$
For the unipotent radicals, we have,

\[(1.3.4.4) \quad \dim V(u) = \sum_i (i - 1)m_i^2 + 2 \sum_{i < j} im_i m_j, \text{ if } G = \text{GL}_n(k),\]

\[(1.3.4.5) \quad \dim V(u) = \frac{1}{2} \sum_i (i - 1)m_i^2 + \sum_{i < j} im_i m_j + \frac{1}{2} \sum_{i \text{ even}} m_i, \text{ if } G = \text{Sp}_n(k),\]

\[(1.3.4.6) \quad \dim V(u) = \frac{1}{2} \sum_i (i - 1)m_i^2 + \sum_{i < j} im_i m_j + \frac{1}{2} \sum_{i \text{ even}} m_i, \text{ if } G = \text{SO}_n(k).\]

The cardinality of finite classical groups is as follows.

\[(1.3.4.7) \quad |\text{GL}_n(q)| = q^n(q^{n-1})/2 \prod_{i=1}^n (q^{i} - 1),\]

\[(1.3.4.8) \quad |\text{GL}_n^{-}(q)| = q^n(q^{n-1})/2 \prod_{i=1}^n (q^{i} - (-1)^i),\]

\[(1.3.4.9) \quad |\text{Sp}_{2m}(q)| = q^{m^2} \prod_{i=1}^m (q^{2i} - 1),\]

\[(1.3.4.10) \quad |\text{SO}_{2m+1}(q)| = q^{m^2} \prod_{i=1}^m (q^{2i} - 1),\]

\[(1.3.4.11) \quad |\text{SO}_{2m}(q)| = q^{m(m-1)}(q^{m} - 1) \prod_{i=1}^{m-1} (q^{2i} - 1),\]

\[(1.3.4.12) \quad |\text{SO}_{2m}^{-}(q)| = q^{m(m-1)}(q^{m} + 1) \prod_{i=1}^{m-1} (q^{2i} - 1).\]

1.4. Cuspidal Local Systems.

1.4.1. Let \(G\) be a connected reductive group. Denote by \(N_G\) the set of pairs \((C, E)\), which consists of a unipotent \(G\)-conjugacy class and an irreducible \(G\)-equivariant local system \(E\) on \(C\). This is a finite set. We refer to [L84a] Definition 2.4 for the definition of cuspidal pair. If \((C, E)\) is a cuspidal pair, then \(E\) is called a cuspidal local system. The subset of \(N_G\) consisting of cuspidal pairs is denoted by \(N_G^{(0)}\).

If \(G = \text{Sp}_{2n}(k)\), then there is a bijection [L84a] (11.6.1)]

\[(1.4.1.1) \quad N_G \longleftrightarrow \Psi_{2n}^{sp},\]

and if \(G = \text{SO}_N(k)\) with \(N\) being either even or odd, then there is a natural map [L84a] (11.7.3)]

\[(1.4.1.2) \quad N_G \rightarrow \Psi_{N}^{ort},\]

which is a bijection over the subset \(\Psi_{N}^{ort,nd}\).
1.4.2. Recall the classification of cuspidal local systems on classical groups.

Theorem 1.4.1. ([L84a, Corollary 12.4], [L84a, Corollary 13.4])

(i) Let $G = \text{Sp}_{2n}(k)$. There exists a cuspidal pair on $G$ if and only if $2n = d(d + 1)$ for some nonnegative integer $d$. In this case, the unique element of $\mathcal{N}_G^{(0)}$ corresponds under (1.4.1.1) to the symbol

$$((0, 2, 4, \ldots, 2d), \emptyset) \in \Psi_{2n}^{sp}, \text{ if } d \text{ is even},$$

and corresponds to the symbol

$$((0, 1, 3, 5, \ldots, 2d + 1)) \in \Psi_{2n}^{sp}, \text{ if } d \text{ is odd}.$$

In either case, the underlying conjugacy class of the cuspidal pair corresponds to the symplectic partition $(2d, 2d - 2, \ldots, 4, 2)$.

(ii) Let $G = \text{SO}_N(k)$. There exists a cuspidal pair on $G$ if and only if $2n = d^2$ for some nonnegative integer $d$. In this case, the unique element of $\mathcal{N}_G^{(0)}$ corresponds under (1.4.1.2) to the symbol

$$((0, 2, 4, \ldots, 2d - 2), \emptyset) \in \Psi_N^{ort}.$$

The underlying conjugacy class of the cuspidal pair corresponds to the orthogonal partition $(2d - 1, 2d - 3, \ldots, 3, 1)$.

1.4.3. Let $\lambda$ be a symplectic or an orthogonal partition, then the irreducible $G$-equivariant local systems on the conjugacy class corresponding to $\lambda$ is parametrised by the irreducible representations of $(\mathbb{Z}/2\mathbb{Z})^{\kappa(\lambda)}$ or $(\mathbb{Z}/2\mathbb{Z})^{\kappa(\lambda) - 1}$ according to whether $\lambda \in \Psi_{2n}^{sp}$ or $\lambda \in \Psi_N^{ort, nd}$. Let $\varrho$ be the nontrivial irreducible character of $\mathbb{Z}/2\mathbb{Z}$.

Let $\lambda$ be the symplectic partition of case (i) of the above theorem. There is a natural bijection between the similarity class corresponding to $\lambda$ and the set of subsets of $\kappa(\lambda)$ (cf. [L84a §11.5, §11.6]), whence the set of irreducible representations of $(\mathbb{Z}/2\mathbb{Z})^{\kappa(\lambda)}$. If we write an element of $(\mathbb{Z}/2\mathbb{Z})^{\kappa(\lambda)}$ as $(e_i)_{i \in \kappa(\lambda)} = (e_2, e_4, \ldots, e_{2d})$, then the unique cuspidal local system corresponds to the representation $(1, \varrho, 1, \varrho, \ldots)$ of $(\mathbb{Z}/2\mathbb{Z})^{\kappa(\lambda)}$.

Let $\lambda$ be the orthogonal partition of case (ii) of the above theorem. By [L84a §11.5, §11.7], there is a natural bijection between the similarity class corresponding to $\lambda$ and the set of irreducible representations of the subgroup of $(\mathbb{Z}/2\mathbb{Z})^{\kappa(\lambda)}$ consisting of elements $(\bar{e}_i)$ satisfying $\sum_i \bar{e}_i = 0$. Denote by $((\mathbb{Z}/2\mathbb{Z})^{\kappa(\lambda)})^\vee$ the group of irreducible representations of $(\mathbb{Z}/2\mathbb{Z})^{\kappa(\lambda)}$ and let $\Delta \subset ((\mathbb{Z}/2\mathbb{Z})^{\kappa(\lambda)})^\vee$ be the subgroup generated by the element $(e_i)_{i \in \kappa(\lambda)}$ with $e_i = \varrho$ for all $i$. Then this similarity class is in bijection with $((\mathbb{Z}/2\mathbb{Z})^{\kappa(\lambda)})^\vee / \Delta$. If we write an arbitrary element of $(\mathbb{Z}/2\mathbb{Z})^{\kappa(\lambda)}$ as $(e_i)_{i \in \kappa(\lambda)} = (e_1, e_3, \ldots, e_{2d - 1})$, then the unique cuspidal local system is represented by the representation $(\varrho, 1, \varrho, 1, \ldots)$ of $(\mathbb{Z}/2\mathbb{Z})^{\kappa(\lambda)}$.

2. Non-Connected Algebraic Groups
2.1. **Quasi-Semi-Simple Elements.** We say that a not necessarily connected algebraic group \(G\) is reductive if \(G^\circ\) is reductive. In this section we denote by \(G\) such a group and denote by \(F\) the Frobenius endomorphism.

2.1.1. An automorphism of \(G^\circ\) is **quasi-semi-simple** if it leaves stable a pair consisting of a maximal torus and a Borel subgroup containing it. An element of \(G\) is **quasi-semi-simple** if it induces by conjugation a quasi-semi-simple automorphism of \(G^\circ\). Let \((T^\circ, B^\circ)\) be a pair consisting of a maximal torus and a Borel subgroup containing it. Put \(B = N_C(B^\circ)\) and \(T = N_C(B^\circ, T^\circ)\) to be the normalisers. By definition, an element of \(G\) is quasi-semi-simple if and only if it belongs to \(T\) for some \(B^\circ\) and \(T^\circ\).

Every semi-simple element is quasi-semi-simple ([St, Theorem 7.5]). Every element of \(G\) normalises some Borel subgroup of \(G^\circ\). Let \(s \in G\) be a quasi-semi-simple element, every \(s\)-stable (for the conjugation) Borel subgroup contains some \(s\)-stable maximal torus. Every \(s\)-stable parabolic subgroup of \(G^\circ\) contains some \(s\)-stable Levi factor ([DM94, Proposition 1.11]). However, an \(s\)-stable Levi subgroup of \(G^\circ\) is not necessarily an Levi factor of some \(s\)-stable parabolic subgroup.

Let \(G^1 \neq G^\circ\) be a connected component of \(G\), and let \(s \in G^1\) be a quasi-semi-simple element. Fix an \(s\)-stable maximal torus \(T^\circ\) contained in some \(s\)-stable Borel subgroup of \(G^\circ\). The quasi-semi-simple \(G^\circ\)-conjugacy classes in \(G^1\) are then described as follows.

**Proposition 2.1.1.** ([DM18], [Proposition 1.16]) Every quasi-semi-simple \(G^\circ\)-conjugacy class in \(G^1\) has a representative in \(C_T(s)^\circ\). Two elements \(t\) and \(t'\) with \(t, t' \in C_T(s)^\circ\), represent the same class if and only if \(t\) and \(t'\), when passing to the quotient \(T^\circ/(T^\circ, s)\), belong to the same \(W^s\)-orbit, where \((T^\circ, s)\) is the commutator, which is preserved by \(W^s := \{w \in W_{G^\circ}(T^\circ) \mid sws^{-1} = w\}\).

2.1.2. Let \(L^\circ\) be a Levi factor of some parabolic subgroup \(P^\circ \subset G^\circ\). Put \(P = N_C(P^\circ)\) and \(L = N_C(P^\circ, L^\circ)\) to be the normalisers. According to [Spr, Lemma 6.2.4], \(P\) is a parabolic subgroup of \(G\), in the sense that \(G/P\) is proper. Suppose that the Levi decomposition of \(P^\circ\) is given by \(P^\circ = U \rtimes L^\circ\), where \(U\) is the unipotent radical of \(P^\circ\), then \(P = U \rtimes L\). (See [DM94, Proposition 1.5]) In particular, \(L\) is a Levi factor of \(P\). (For an arbitrary linear algebraic group \(G\), a Levi factor \(H\) of \(G\) is a closed subgroup such that \(G = R_u(G) \rtimes H\).)

2.1.3. If \(s \in G\) is a quasi-semi-simple element, then the connected centraliser \(H = C_G(s)^\circ\) is reductive. ([Spr, §1.17]) If the pair \((T^\circ, B^\circ)\) consists of an \(s\)-stable maximal torus and an \(s\)-stable Borel subgroup of \(G^\circ\) containing it, then \(C_B(s)^\circ\) is a Borel subgroup of \(H\) containing the maximal torus \(C_T(s)^\circ\) ([DM94, Théorème 1.8(iii)]). More generally, we have

**Proposition 2.1.2.** For \(s\) and \(H\) as above, we have

(i) If \((L^\circ, P^\circ)\) is a pair consisting of an \(s\)-stable Levi subgroup and an \(s\)-stable parabolic subgroup containing it as a Levi factor, then \(C_{P^\circ}(s)^\circ\) is a parabolic subgroup of \(H\) with \(C_{L^\circ}(s)^\circ\) as a Levi factor.
Remark 2.1.7. The groups $P$ and $L$ in this proposition are not necessarily unique in general. See however Proposition 2.2.1.

Proof. The first part is [DM94, Proposition 1.11]. Given $L'$ and $P'$, there exists a cocharacter of $H$, say $\lambda$, such that $L' = L'_\lambda$ and $P' = P'_\lambda$, where $L'_\lambda$ and $P'_\lambda$ are the Levi subgroup and parabolic subgroup associated to $\lambda$. Regarded as a cocharacter of $G^\circ$, it defines a Levi subgroup and a parabolic subgroup $L_\lambda \subset P_\lambda$ of $G^\circ$. They are s-stable since the image of $\lambda$ commutes with $s$. It is clear that $L'_\lambda = L_\lambda \cap H$ and $P'_\lambda = P_\lambda \cap H$. □

Proposition 2.1.4. Given $L'$ as in the preceding proposition, we write $L = C_{G^\circ}(Z_{L'}^\circ)$. It is an s-stable Levi factor of some s-stable parabolic subgroup of $G^\circ$, such that $L' = C_L(s)^\circ$. If $M \subset G^\circ$ is an s-stable Levi factor of some s-stable parabolic subgroup, such that $L' \subset M$, then $L \subset M$.

The proof is completely analogous to [L03, §2.1] where the assertion is proved for $s$ semi-simple.

Proof. We can find a cocharacter $\chi : k' \to Z_{L'}^\circ$ such that $L = C_{G^\circ}(\chi(k'))$. As in the preceding proposition, we see that $L$ is an s-stable Levi factor of some s-stable parabolic subgroup. Since $L' = C_H(Z_{L'}^\circ)$, we have $C_L(s)^\circ = L'$.

Note that $M' := (M \cap H)^\circ = C_M(s)^\circ$ is a Levi subgroup of $H$, and that $L'$ is a Levi subgroup of $H$ contained in $M'$. Since $(Z_M^\circ \cap H)^\circ \subset Z_{M'}^\circ$, whence $(Z_M^\circ \cap H)^\circ \subset Z_{L'}^\circ$, whence $C_G(Z_M^\circ \cap H)^\circ \subset C_G((Z_M^\circ \cap H)^\circ)$. According to [L03, §1.10], $C_G((Z_M^\circ \cap H)^\circ)^\circ = M$, so $L \subset M$. □

Remark 2.1.5. In particular, if $T' \subset C_G(s)^\circ$ is a maximal torus, then $T := C_G(T')$ is the unique maximal torus of $G^\circ$ containing $T'$. It is s-stable and contained in an s-stable Borel subgroup, and we have $C_T(s)^\circ = T'$.

Remark 2.1.6. Let $M$ be an s-stable Levi factor of some s-stable parabolic subgroup $Q \subset G^\circ$ such that $C_M(s)^\circ = L'$. Suppose in the proof of the above proposition that the equality $(Z_M^\circ \cap H)^\circ = Z_{L'}^\circ$ holds, i.e. the s-fixed part of the centre of $M$ coincides with the centre of the s-fixed part of $M$. Then we have $M = L$ by [L03, §1.10]. We will see in Proposition 2.1.8 that this equality can be satisfied only if $s$ is an isolated element of $N_G(Q) \cap N_G(M)$.

Remark 2.1.7. It follows from the definition of $L$ that if an element of $G$ normalises $L'$, then it normalises $L$.

2.1.4. A quasi-semi-simple automorphism $\sigma$ of $G^\circ$ is quasi-central if it satisfies the following condition.

There exists no quasi-semi-simple automorphism of the form $\sigma' = \sigma \circ \text{ad } g$ with $g \in G^\circ$ such that $\dim C_G(\sigma)^\circ < \dim C_G(\sigma')^\circ$. 

A quasi-semi-simple element of $G$ is quasi-central if it induces by conjugation a quasi-central automorphism of $G^\circ$.

A quasi-semi-simple element $\sigma \in G$ is quasi-central if and only if there exists a $\sigma$-stable maximal torus $T$ contained in a $\sigma$-stable Borel subgroup of $G^\circ$ such that every $\sigma$-stable element of $N_G^\sigma(T)/T$ has a representative in $C_T^\sigma$ (DM94 Théorème 1.15). Considering the natural map $N_G^\sigma(C_T^\sigma) \rightarrow N_G^\sigma(T)$, this simply means that $W_G^\sigma(T) \cong W_G(C_T^\sigma)$.

If $\sigma$ is quasi-central, we will often denote $C_H^\sigma$ by $H^\sigma$.

2.1.5. Let $g = g_s g_u$ be the Jordan decomposition of an element of $G$. Write $L'(g) = C_G(g_s)^\circ$ and $L(g) = C_G(Z_{L'}^\circ)$. We say that $g$ is isolated in $G$ if $L(g) = G^\circ$. The conjugacy class of an isolated element will be called isolated. The isolated elements can be characterised as follows.

Proposition 2.1.8. [L03 §2.2] Let $g \in G$, and put $L' = L'(g)$ and $L = L(g)$. Then the following assertions are equivalent.

(i) $L = G^\circ$;
(ii) $Z_{L'}^\circ = C_{Z_{G^\circ}}(g)^\circ$;
(iii) There is no $g_s$-stable proper parabolic subgroup $Q \subset G^\circ$ with $g_s$-stable Levi factor $M$ such that $L' \subset M$.

Our definition of isolated element agrees with the definitions in the literatures, due to the following result, which is not obvious.

Proposition 2.1.9. ([L03 IV. Proposition 18.2]) Let $s \in G$ be a semi-simple element and $u \in G$ a unipotent such that $su = us$. Then $su$ is isolated in $G$ (for the definition in [L03 §2]) if and only if $s$ is isolated in $G$.

Therefore, the definition of isolated semi-simple elements coincides with [DM18 Definition 3.1], where one fixes a maximal torus $T$, a Borel subgroup $B \subset G^\circ$ containing $T$ and a quasi-central element $\sigma \in N_G(T,B)$, and says that $t \sigma \in T.\sigma$ is isolated if $C_G(t \sigma)^\circ$ is not contained in a $\sigma$-stable Levi factor $M$ of a $\sigma$-stable proper parabolic subgroup $Q$ of $G^\circ$. Note that in this definition, $M$ necessarily contains $T$ because $C_M(t \sigma)^\circ$ contains $C_T(t \sigma)^\circ$. Let us also recall that (cf. [DM18 Definition 3.12]), an element $t \sigma$ is quasi-isolated if $C_G(t \sigma)$ is not contained in a $\sigma$-stable Levi factor of a $\sigma$-stable proper parabolic subgroup of $G^\circ$.

2.2. Parabolic Subgroups and Levi Subgroups.

2.2.1. Recall that in the setting of Proposition 2.1.2 one does not have a bijection in general.

Proposition 2.2.1. ([DM94 Corollaire 1.25]) Let $\sigma$ be a quasi-central automorphism of $G^\circ$.

1) The map $P \mapsto (P^\sigma)^\circ$ defines a bijection between the $\sigma$-stable parabolic subgroups of $G^\circ$ and the parabolic subgroups of $(G^\circ)^\circ$.
(2) Then map \( L \mapsto (L^{\circ})^{\circ} \) defines a bijection between the \( \sigma \)-stable Levi factors of \( \sigma \)-stable parabolic subgroups of \( G^{\circ} \) and the Levi subgroups of \( (G^{\circ})^{\circ} \).

Considering the fact that \( W_{G^{\circ}}(T^{\circ}) = W_{(G^{\circ})^{\circ}}((T^{\circ})^{\circ}) \), the bijection is obtained at the level of Weyl groups.

2.2.2. The following propositions will be useful.

**Proposition 2.2.2.** ([DM94] Proposition 1.6) Let \( \sigma \) be a quasi-semi-simple element of \( G \) and let \((L^{\circ}, P^{\circ})\) be a pair consisting of a Levi subgroup of \( G^{\circ} \) and a parabolic subgroup that contains it as a Levi factor. If the \( G^{\circ} \)-conjugacy class of \((L^{\circ}, P^{\circ})\) is \( \sigma \)-stable, then there exists \( x \in G^{\circ} \) such that \((xL^{\circ}x^{-1}, xP^{\circ}x^{-1})\) is \( \sigma \)-stable.

**Proposition 2.2.3.** ([DM94] Proposition 1.38) Let \( \sigma \) be an \( F \)-stable quasi-central element of \( G \) and let \((L^{\circ}, P^{\circ})\) be a pair consisting of an \( F \)-stable Levi factor and a parabolic subgroup containing it as a Levi factor. If the \( G^{\circ F} \)-conjugacy class of \((L^{\circ}, P^{\circ})\) is \( \sigma \)-stable, then there exists \( x \in G^{\circ F} \) such that \((xL^{\circ}x^{-1}, xP^{\circ}x^{-1})\) is \( \sigma \)-stable.

Let \( L^{\circ} \) be a Levi factor of some parabolic subgroup \( P^{\circ} \) of \( G^{\circ} \), put \( L = N_{G}(L^{\circ}, P^{\circ}) \). Let \( G^{1} \) be a connected component of \( G \). It acts by conjugation on the \( G^{\circ} \)-conjugacy classes of the pairs \((L^{\circ}, P^{\circ})\). Then \( L \) meets \( G^{1} \) if and only if the class of \((L^{\circ}, P^{\circ})\) is \( \sigma \)-stable for this action. According to the above propositions, there is a conjugate of \((L^{\circ}, P^{\circ})\) that is \( \sigma \)-stable. This means that \( L \) contains \( \sigma \) and so \((L^{\circ})^{\circ}\) is a Levi subgroup of \((G^{\circ})^{\circ}\).

**Proposition 2.2.4.** ([DM94] Proposition 1.40) Assume that \( \sigma \in G \) is quasi-central, \( F \)-stable, and \( G/G^{\circ} \) is generated by the component of \( \sigma \). Then the \( G^{F} \)-conjugacy classes of the \( F \)-stable groups \( L = N_{G}(L^{\circ}, P^{\circ}) \) meeting the connected component \( G^{\circ} \) are in bijection with the \(((G^{\circ})^{\circ})^{F} \)-conjugacy classes of the \( F \)-stable Levi subgroups of \((G^{\circ})^{\circ}\) in the following manner. Each \( L \) has a \( G^{F} \)-conjugate \( L_{1} \) containing \( \sigma \), and the bijection associates the \(((G^{\circ})^{\circ})^{F} \)-class of \(((L_{1})^{\circ})^{\circ}\) to the \( G^{F} \)-class of \( L \).

This gives in particular the classification of the \( G^{F} \)-conjugacy classes of the \( F \)-stable groups of the form \( T = N_{G}(T^{\circ}, B^{\circ}) \).

3. Generalised Deligne-Lusztig Induction

3.1. **Induction for Connected Groups.** We recall some generalities on the Deligne-Lusztig induction for connected reductive groups. In this section we assume \( G \) to be connected. If \( X \) is an algebraic variety over \( k \), we denote by \( H^{i}_{c}(X) \) the \( i \)-th cohomology group with compact support with coefficient in \( \mathbb{Q}_{c} \), and we denote by \( H^{i}_{c}(X) = \bigoplus (-1)^{j}H^{j}_{c}(X) \) the virtual vector space. For a finite group \( H \), denote by \( C(H) \) the set of the invariant \( \mathbb{Q}_{c} \)-valued functions on \( H \).
3.1.1. Let $L$ be an $F$-stable Levi factor of some parabolic subgroup $P \subset G$ not necessarily $F$-stable. The Levi decomposition writes $P = L U$. Put $L_G^{-1}(U) = \{ x \in G | x^{-1} F(x) \in U \}$. Then $G^F$ acts on $L_G^{-1}(U)$ by left multiplication and $L^F$ acts by right multiplication. This induces a $G^F \times (L^F)^\sigma$-module structure on $H_c^*(L_G^{-1}(U))$ for any $i$. Let $\theta \in C(L^F)$, then the Deligne-Lusztig induction of $\theta$, denoted by $R_G^L \theta$, is the isomorphism on $G^F$ defined by

\[ R_G^L \theta(g) = |L^F|^{-1} \sum_{l \in L^F} \theta(l^{-1}) \text{Tr}((g, l)H_c^*(L^{-1}(U))), \quad \text{for any } g \in G^F. \]

It does not depend on the choice of $P$ if $q \neq 2$ (cf. [DM20] §9.2). The functions of the form $R_G^L \theta$ with $L$ being a maximal torus are called Deligne-Lusztig characters.

3.1.2. The Green function is defined on the subset of unipotent elements in the following manner.

\[ Q_C^G(-, -) : G_u^F \times L_u^F \longrightarrow \mathbb{Z} \]

\[ (u, v) \mapsto \text{Tr}((u, v)H_c^*(L_G^{-1}(U))). \]

The calculation of the Deligne-Lusztig induction is reduced to the calculation of the Green functions. If $g = su$ is the Jordan decomposition of $g \in G^F$, we have the character formula ([DM91] Proposition 12.2),

\[ R_G^L \theta(g) = |L^F|^{-1}|C_G(s)^{\sigma F}|^{-1} \sum_{\{ h \in G^F | h^{-1} l \in L \} \subset C_G(s)_0^{\sigma F}} \sum_{v \in C_G(s)_0^{\sigma F}} Q_C^G(s^{-1}h)(u, \sigma^{-1}h \theta(sv)), \]

where $hL = h^{-1}Lh$ and $h \theta(sv) = \theta(sv)h^{-1})$. The Green functions are sometimes normalised in such a way that the factor $|L^F|^{-1}$ is removed from the above formula.

3.1.3. We will need the following simple lemmas. Let $\sigma$ be an automorphism of $G$ that commutes with $F$. If $L \subset G$ is an $F$-stable Levi subgroup, then we also denote by $\sigma$ the isomorphism $L^F \rightarrow \sigma(L)^F$ and the isomorphism $W_L(T) \rightarrow W_{\sigma(L)}(\sigma(T))$ for an $F$-stable maximal torus $T$.

**Lemma 3.1.1.** Let $M \subset G$ be an $F$-stable Levi subgroup and let $\theta$ be a character of $M^F$. Then the character $(R_M^G \theta) \circ \sigma^{-1}$ is equal to $R_{\sigma(M)}^{\sigma} \sigma \theta$.

**Proof.** Let $Q$ be a parabolic subgroup containing $M$ such that $Q = MU_Q$. Then $\sigma(L^{-1}(U_Q)) = L^{-1}(\sigma(U_Q))$ as $F$ commutes with $\sigma$, and so

\[ \text{Tr}((\sigma(g), \sigma(l))H_c^*(L^{-1}(\sigma(U_Q)))) = \text{Tr}((g, l)H_c^*(L^{-1}(U_Q))) \]

for any $g \in G^F$ and $l \in M^F$. The assertion then follows from the definition of $R_M^G \theta$. \qed

**Lemma 3.1.2.** Assume that $\chi \in \text{Irr}(\text{GL}_n(q))$ is of the form $R_{\chi}^G \theta$ for a triple $(M, \varphi, \theta)$ as in Theorem 7. Then, the character $\chi \circ \sigma^{-1}$ is of the form $R_{\chi \circ \sigma}^\varphi \sigma, \theta$ for a triple $(\sigma(M), \sigma, \varphi, \sigma, \theta)$. 

\[ \text{THE CHARACTER TABLE OF GL}_n(F_q) \times <\sigma> \]
Proof. Since $F$ commutes with $\sigma$, we can define $T_{\sigma(w)}$ to be $\sigma(T_w)$ and so by the definition of $R^G_{\sigma(T_w)}\theta$, the lemma follows from the equality $R^G_{\sigma(T_w)}\sigma_!\theta(g) = R^G_{T_w}\theta(\sigma^{-1}(g))$. □

3.2. Induction for Non Connected Groups. We will assume that $G/G^o$ is cyclic, and fix $\sigma \in G$ quasi-central and $F$-stable such that $G/G^o$ is generated by the component of $\sigma$. We may write $G = G^o.<\sigma>$.\hfill\(\Box\)

3.2.1. Given an $F$-stable Levi factor $L^o$ of some parabolic subgroup $P^o$ not necessarily $F$-stable that is decomposed as $P^o = L^oU$, we put $L$ and $P$ to be the normalisers defined in §2.1.1.\hfill\ref{2.1.1}

Put $\mathcal{L}^{-1}_G(U) = \{x \in G|\chi_1F(x) \in U\}$ and $\mathcal{L}^{-1}_G(U) = \{x \in G^o|\chi_1F(x) \in U\}$. Then $G^F \times (L^F)^{op}$ acts on $\mathcal{L}^{-1}_G(U)$, and $H^F(U\mathcal{L}^{-1}_G(U))$ is thus a $G^F \times (L^F)^{op}$-module. For $\theta \in C(L^F)$, the (generalised) Deligne-Lusztig induction of $\theta$ is defined by,

$$R^G_L\theta(g) = |L^F|^{-1}\sum_{l \in L^F} \theta(l^{-1}) \text{Tr}((g,l)H^F(\mathcal{L}^{-1}_G(U))), \quad \text{for any } g \in G^F.$$ \hfill\ref{3.2.1.1}

It does not depend on the choice of $P^o$ if $q \neq 2$.\hfill\ref{3.2.1.1}

According to Proposition 2.2.4, the generalised Deligne-Lusztig induction are parametrised by the pairs $(L^o,P^o)$ consisting of an $F$-stable and $\sigma$-stable Levi factor of some $\sigma$-stable parabolic subgroup. Since only those $L$ that meets $G^o\sigma$, interest us, we can assume that $L = L^o.<\sigma>$. The restriction of $R^G_L$ to $L^o\sigma$ is a map $C(L^o\sigma) \rightarrow C(G^o\sigma)$, that we denote by $R^G_{L^o\sigma}$. To simplify the terminology, we may also call these maps Deligne-Lusztig inductions.

3.2.2. The following lemma shows that the induction thus defined is compatible with that defined for connected groups.

**Lemma 3.2.1.** We keep the notations as above. Denote by $\text{Res}$ the usual restriction of functions. If $G^F = L^F.G^oF$, then,

$$\text{Res}_{G^oF}^G \circ R^G_L = R_{L^o}^G \circ \text{Res}_{L^oF}^F.$$ \hfill\ref{3.2.2.1}

*Proof.* See [DM94, Corollaire 2.4 (i)]. □

3.2.3. The Green function is defined by

$$Q^G_L(-,-): G^u \times L^u \rightarrow \mathbb{Z}$$

$$\begin{cases} 0 & \text{if } uv \notin G^o \\ \text{Tr}(u,v)H^F(\mathcal{L}^{-1}_G(U)) & \text{otherwise.} \end{cases}$$ \hfill\ref{3.2.3.1}

Note that $\mathcal{L}^{-1}_G(U)$ is the usual Deligne-Lusztig variety.

\(\Box\)

\(\Box\)

As is pointed out to be by F. Digne, for $G = G^o.<\sigma>$ and $\sigma$ inducing a non trivial automorphism, this is reduced to the case of connected groups.
Proposition 3.2.2. (Character Formula, [DM94] Proposition 2.6) Let \( L \) be the normaliser of the pair \( (L^s \subset P^o) \) as above, and let \( \theta \) be a character of \( L^F \). Then for any \( g \in G^F \) with Jordan decomposition \( g = su \),

\[
R^G_L \theta (g) = |L^F|^{-1} |C_G(s)^{o_F}|^{-1} \sum_{\ell \in \mathbb{Z}^L} \sum_{\varepsilon \in C_{h_L}(s)^{o_F}} Q_{C_{h_L}(s)^{o_F}}^{C_E(s)^{o_F}} (u, v^{-1})^h \theta (sv).
\]

This formula will only be used in the following form.

Proposition 3.2.3. ([DM94] Proposition 2.10) We keep the notations as above, except that \( \theta \) is now a \( \sigma \)-stable character of \( L^{o_F} \). Denote by \( \tilde{\theta} \) an extension of \( \theta \) to \( L^{o_F, \sigma} \), and let \( su \) be the Jordan decomposition of \( g^o, g \in G^{o_F} \). Then,

\[
R^G_L \tilde{\theta} (g^o) = |L^{o_F}|^{-1} |C_G(s)^{o_F}|^{-1} \sum_{\ell \in \mathbb{Z}^L} \sum_{\varepsilon \in C_{h_L}(s)^{o_F}} Q_{C_{h_L}(s)^{o_F}}^{C_E(s)^{o_F}} (u, v^{-1})^h \tilde{\theta} (sv).
\]

The Green functions are sometimes normalised in such a way that the above two formulas should be multiplied by \( |C_{h_L}(s)^{o_F}| \).

3.3. Uniform Functions.

3.3.1. The irreducible characters of \( G^F \) for connected \( G \) can be expressed as linear combinations of the Deligne-Lusztig inductions of cuspidal functions on various \( L^F \). For \( G = \text{GL}_n(F_q) \), we have Theorem 1 in Introduction. It shows that in this particular case one only needs the functions \( R^G_L \theta \) induced from \( F \)-stable tori, and the transition matrix is given by the characters of the Weyl group of \( G \). In general, the transition matrix could be more complicated and the functions induced from the characters of tori are not sufficient. The invariant functions on \( G^F \) that are linear combinations of the \( R^G_L \theta \)'s are called uniform functions.

Recall that for each \( w \in W_G \), the Weyl group of \( G \), and some \( \tilde{w} \in G \) representing \( w \), one can find \( g \in G \) such that \( g^{-1} F(g) = \tilde{w} \), then \( T_w := g T g^{-1} \) is an \( F \)-stable maximal torus such that \( T^F_{\tilde{w}} \cong T^F_w \) with \( F_w := \text{ad} \tilde{w} \circ F \). Then \( w \mapsto T_w \) defines a bijection between the \( F \)-conjugacy classes of \( W_G \) and the \( G^F \)-conjugacy classes of \( F \)-stable maximal tori.

3.3.2. Now assume that \( G \) is non connected. Let \( T \subset G^o \) be an \( F \)-stable and \( \sigma \)-stable maximal torus contained in a \( \sigma \)-stable Borel subgroup. If \( \theta \in \text{Irr}(T^F) \) extends into \( \tilde{\theta} \in \text{Irr}(T^F, <\sigma>) \), then \( R^G_{T^F} \tilde{\theta} \) belongs to \( C(G^{o_F, \sigma}) \), the set of \( \tilde{\theta} \)-valued functions on \( G^{o_F, \sigma} \) that are invariant under \( G^{o_F} \). A function in \( C(G^{o_F, \sigma}) \) is called uniform if it is a linear combination of functions of the form \( R^G_{T^F} \tilde{\theta} \).

3.3.3. Let \( \tilde{\chi} \) be an irreducible character of \( G^F \). It is called unipotent if \( \chi := \text{Res}^G_{G^{o_F}} \tilde{\chi} \) contains a unipotent character as a direct summand. In this case, \( \text{Res}^G_{G^{o_F}} \chi \) is a sum of unipotent characters, as its summands are \( G^F \)-conjugate. Denote by

\[
E(G^{o_F, \sigma}(1)) = \{ \tilde{\chi} \mid \text{Res}^G_{G^{o_F}} \tilde{\chi} \text{ contains a unipotent character} \}.
\]
An element of $C(G^\sigma F, \sigma)$ is called **unipotent** if it is a linear combination of some elements of $E(G^\sigma, (1))$, and we denote by $C(G^\sigma, (1))$ this subspace. It is clear that the characters $R^G_{T_{w\sigma}} 1$ are unipotent functions, and they are parametrised by the $F$-conjugacy classes of $W^\sigma$. An element of $C(G^\sigma, \sigma)$ is called **uniform-unipotent** if it is a linear combination of the functions $R^G_{T_{w\sigma}} 1$.

3.3.4. A natural question is to identify those elements of $E(G^\sigma F, (1))$ that are uniform. We have,

**Theorem 3.3.1.** (DM94, Théorème 5.2) Let $G = GL_n^F(k)$ and let $W_G$ be the Weyl group defined by some $F$-stable and $\sigma$-stable maximal torus of $G$, and so $F$ acts trivially on the $\sigma$-fixed part $W^\sigma_G$. For any $\varphi \in \text{Irr}(W^\sigma_G)$, put

\[(3.3.4.1) R^G_{\varphi} 1 := \frac{1}{|W^\sigma_G|} \sum_{w \in W^\sigma_G} \varphi(w) R^G_{T_{w\sigma}} 1.\]

Then, $R^G_{\varphi} 1$ is an extension of a principal series unipotent representation of $G(q)$.

This gives all uniform-unipotent functions on $GL_n^F(F_q)$. It follows that an element of $E(G^\sigma, (1))$ is either uniform or orthogonal to the space of uniform(-unipotent) functions.

3.3.5. (DM15, Proposition 6.4) The characteristic functions of quasi-semi-simple classes are uniform. Consequently, all non uniform characters vanish on the quasi-semi-simple classes.

4. The Group $GL_n(k) \rtimes \mathbb{Z}/2\mathbb{Z}$

In what follows, we write $G = GL_n(k)$.

4.1. **Automorphisms of** $GL_n(k)$.

4.1.1. Let $\vartheta_n$ be the matrix

\[(4.1.1.1) (\vartheta_n)_{ij} = \delta_{ij(n+1-j)} = \begin{pmatrix} 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots \\ 1 & & 1 \end{pmatrix} \]

Put $t_0 = \text{diag}(a_1, \ldots, a_n)$ with $a_i = 1$ if $i \leq \lfloor (n + 1)/2 \rfloor$ and $a_i = -1$ otherwise. Put $\vartheta_n' = t_0 \vartheta_n$.

Define the matrices

\[(4.1.1.2) \vartheta = \begin{cases} \vartheta_n' & \text{if } n \text{ is even} \\ \vartheta_n & \text{if } n \text{ is odd} \end{cases}, \quad \vartheta' = \begin{cases} \vartheta_n & \text{if } n \text{ is even} \\ \vartheta_n' & \text{if } n \text{ is odd} \end{cases} \]

The automorphism $\sigma \in \text{Aut}(GL_n(k))$ that sends $g$ to $\vartheta g^{-t} \vartheta^{-1}$ will be called the standard automorphism. We will denote by $\sigma'$ the automorphism defined by replacing $\vartheta$ with $\vartheta'$.
in the definition of \( \sigma \). They are quasi-semi-simple automorphisms because the maximal torus consisting of the diagonal matrices and the Borel subgroup consisting of the upper triangular matrices are stable under the action of \( \sigma \) or \( \sigma' \). Moreover, \( \sigma \) is a quasi-central involution regardless of the parity of \( n \), while \( \sigma' \) is not an involution if \( n \) is odd and is not quasi-central if \( n \) is even.

4.1.2. The classification of the involutions and the quasi-central automorphisms is well known. ([LiSe, Lemma 2.9] and [DM94, Proposition 1.22]).

The conjugacy classes of involutions are described as follows.
- If \( n = 2m + 1 \), the exterior involutions (exterior automorphisms of order 2) are all \( G \)-conjugate and their centralisers are of type \( B_m \).
- If \( n = 2m > 2 \), there are two \( G \)-conjugacy classes of exterior involutions, with centralisers of type \( C_m \) and \( D_m \) respectively. If \( n = 2 \), the connected centralisers are \( \text{SL}_2(k) \) and \( k^* \) respectively.

The conjugacy classes of quasi-central automorphisms are described as follows.
- If \( n = 2m + 1 \), there are two classes of exterior quasi-central automorphisms, with centralisers of type \( B_m \) and of type \( C_m \) respectively.
- If \( n = 2m \), there is one single class of exterior quasi-central automorphisms, with centraliser of type \( C_m \).

Explicitly,

\[
(4.1.2.1) \quad \begin{cases} 
(G^\sigma)^c \cong \text{Sp}_{2m}(k) & \text{if } n = 2m, \\
(G^\sigma)^c \cong \text{SO}_{2m+1}(k) & \text{if } n = 2m + 1.
\end{cases}
\]

Put \( t = \text{diag}(a_1, \ldots, a_m, a_{m+1}, \ldots, a_{2m}) \) or \( \text{diag}(a_1, \ldots, a_m, 1, a_{m+1}, \ldots, a_{2m}) \) according to the parity of \( n \), with \( a_i = i \) for \( i \leq m \) and \( a_i = -i \) for \( i > m \), we have

\[
(4.1.2.2) \quad \begin{cases} 
(G^{t_0})^c = \text{SO}_{2m}(k) & \text{si } n = 2m, \\
(G^{t_0})^c = \text{Sp}_{2m}(k) & \text{si } n = 2m + 1.
\end{cases}
\]

We say that an automorphism is of symplectic type or orthogonal type according to the type of its centraliser.

4.1.3. We will encounter another type of quasi-central automorphism. Let \( \tau_0 \) be an involution of \( G \), not necessarily an exterior automorphism. Define an automorphism \( \tau \) of \( G \times G \) by \( (g, h) \mapsto (\tau_0(h), \tau_0(g)) \). It is easy to see from the definition that \( \tau \) is quasi-central and \( C_{G \times G}(\tau) \cong G \).

4.2. The Group \( \bar{G} \).
4.2.1. Let $H$ be an abstract group and let $\tau$ be an automorphism of finite order of $H$. By a $\tau$-conjugacy class of $H$, we mean an orbit in $H$ under the action $h \mapsto hx\tau(h^{-1})$, $x, h \in H$. By a $\tau$-class function on $H$, we mean a function that is constant on the $\tau$-conjugacy classes. We denote by $C(H, \tau)$ the set of $\tau$-class functions.

On the other hand, the conjugacy classes of $H \rtimes <\tau>$ contained in $H.\tau$ are identified with the $H$-conjugacy classes in $H.\tau$, as $\tau(h)\tau = h^{-1}(h\tau)h$, which are in turn identified with the $\tau$-conjugacy classes of $H$. This justifies the notation $C(H, \tau)$.

4.2.2. The semi-direct product $\bar{G} := G \rtimes \mathbb{Z}/2\mathbb{Z}$ is defined by an exterior involution. If $n$ is odd, there is only one class of involutions. It is the class of $\sigma$. If $n$ is even, there are two classes of involutions. Whenever it is necessary to distinguish the two non isomorphic resulting semi-direct products, we will denote by $\hat{\sigma}$ the one defined by the symplectic type automorphism, i.e. by $\sigma$, and denote by $\hat{\sigma}'$ the one defined by the orthogonal type automorphisms, i.e. by $\sigma'$. We will often denote by $\sigma$ the element $(e, 1) \in \bar{G}$, although $\sigma$ is not actually an element of $\bar{G}$.

By definition, $\sigma$ is a quasi-central element in $G.\sigma$.

4.2.3. The character tables of $\hat{\sigma}\bar{G}^F$ and of $\hat{\sigma}'\bar{G}^F$ are related in the following manner.

The $G^F$-conjugacy classes in $\hat{\sigma}\bar{G}^F \setminus \bar{G}^F$ are in bijection with the $\sigma$-conjugacy classes in $G^F$, which are in bijection with the $t_0\alpha'$-conjugacy classes in $G^F$ (See §4.1.1 for $t_0$), which are in bijection with the $G^F$-conjugacy classes in $\hat{\sigma}' \bar{G}^F \setminus \bar{G}^F$. More specifically, for any $g \in G^F$, the $G^F$-class of $g\sigma \in \hat{\sigma}\bar{G}^F$ corresponds to the $G^F$-class of $gt_0\alpha' \in \hat{\sigma}'\bar{G}^F$.

Since $\sigma$ and $\sigma'$ differ by an inner automorphism, the set of $\sigma$-stable characters coincides with that of $\sigma'$-stable characters. However, the extension of a $\sigma$-stable character to $\bar{G}^F$ behaves differently for $\hat{\sigma}\bar{G}^F$ and $\hat{\sigma}'\bar{G}^F$. Let $\rho : G^F \to \text{GL}(V)$ be a $\sigma$-stable irreducible representation of $G^F$. To find an extension $\bar{\rho} \in \text{Irr}(\hat{\sigma}\bar{G}^F)$ of $\rho$ is to define $\bar{\rho}(\sigma)$ in such a way that $\bar{\rho}(\sigma)^2 = \rho(\sigma^2) = \text{Id}$ and $\bar{\rho}(\sigma)\bar{\rho}(g)\bar{\rho}(\sigma)^{-1} = \rho(\sigma(g))$ for all $g \in G^F$. Suppose that we have defined such an extension, and we would like to define $\bar{\rho}' \in \text{Irr}(\hat{\sigma}'\bar{G}^F)$ by $\bar{\rho}'(t_0\alpha') = \rho(\alpha)$. For $\hat{\sigma}\bar{G}^F$, if $\rho(\text{Id}) \neq \text{Id}$, the equality $\bar{\rho}(\sigma)^2 = \text{Id}$ would be violated. Consequently, we define instead $\bar{\rho}'(t_0\alpha') = \rho(\alpha) \sqrt{\rho(-1)}$, where $\rho(-1)$ has as value $\pm \text{Id}$ regarded as a scalar. Replacing $\sqrt{\rho(-1)}$ by $-\sqrt{\rho(-1)}$ defines another extension of $\rho$ to $\hat{\sigma}'\bar{G}^F$. We denote by $\hat{\chi}$ and $\hat{\chi}'$ the characters of $\bar{\rho}$ et $\bar{\rho}'$ respectively. Then, for all $g \in G^F$,

\begin{equation}
\hat{\chi}'(gt_0\alpha') = \hat{\chi}(g\sigma) \sqrt{\rho(-1)}.
\end{equation}

Convention 4.2.1. Because of the above discussion, we will also denote by $\sigma$ the element $t_0\alpha' \in \hat{\sigma}'\bar{G}^F$. We will later parametrise the conjugacy classes in $\hat{\sigma}'\bar{G}^F \setminus \bar{G}^F$ with respect to $\sigma$ (following Proposition 2.1.1).

Remark 4.2.2. We have $\rho(-1) = -\text{Id}$ if and only if $\chi(-\text{Id}) = -\chi(\text{Id})$. Let $m$ be the "multiplicity" of $\eta$ in the semi-simple part of $\chi$. From Theorem 1 and the character formula (3.1.2.2),
combined with the parametrisation of \( \sigma \)-stable irreducible characters of \( \text{GL}_n(q) \) that we will prove later (Proposition 5.2.2 and Proposition 5.2.3), we see that \( \chi(\text{Id}) = \eta(-1)^m \chi(\text{Id}) \), which is equal to \(-\chi(\text{Id})\) only if \( m \) is odd and \( q \equiv 3 \mod 4 \). In particular, if \( \tilde{\chi} \) is a uniform function on \( G^\sigma \), in which case \( m \) must be even by Corollary 11.1.2, then \( \rho(-1) \) always equals to \( \text{Id} \).

**Question 4.2.3.** If \( q \equiv 1 \mod 4 \), then the character table of \( \bar{G}_F \) and that of \( \tilde{G}_F \) coincide under the bijections of characters and conjugacy classes described above. Are these groups isomorphic? Working with finite groups, there might be isomorphisms that are not deduced from the underlying algebraic groups.

4.3. **Quasi-Semi-Simple Elements.**

4.3.1. We have said in §2.1 that all semi-simple elements are quasi-semi-simple. For \( \bar{G} \), we have

**Lemma 4.3.1.** An element of \( \bar{G} \) is quasi-semi-simple if and only if it is semi-simple.

More generally, if \( G \) is a reductive algebraic group and \( G/G^o \) is semi-simple, then all quasi-semi-simple elements are semi-simple. (See [DM94, Remarque 2.7]) In positive characteristic, this is to require that \( \text{char} \ k \nmid |G/G^o| \). We give a short proof below in the particular case of \( \bar{G} \).

**Proof.** It suffices to show that each quasi-semi-simple element \( s\sigma \in G.\sigma \) is semi-simple. We see that \( s\sigma \) is semi-simple if and only if \( (s\sigma)^2 = s\sigma(s)\sigma^2 \) is semi-simple, as we have assumed \( \text{char} \ k \) to be odd. Let \( (T, B) \) be a pair consisting of a maximal torus and a Borel subgroup containing it, both normalised by \( \sigma \). Then every quasi-semi-simple element is conjugate to an element of \((T^\sigma)^\sigma \) (Proposition 2.1.1), and its square, lies in \( T \), and so is semi-simple. \( \square \)

**Remark 4.3.2.** That \( s\sigma \) is semi-simple does not imply that \( s \) is so. Let us fix \( T \) and \( B \) as above, and write \( B = TU \), where \( U \) is the unipotent radical of \( B \). If we take \( u \in U \), then \( u\sigma(u^{-1}) \) is a unipotent element, whereas \( u\sigma u^{-1} \) is semi-simple.

**Remark 4.3.3.** There is no unipotent element in \( G.\sigma \) because an odd power of \( s\sigma \) lies in \( G.\sigma \).

4.3.2. **Isolated Elements.** Define the diagonal matrix

\[
(t(j)) = \text{diag}(i, \ldots, i, 1, \ldots, 1, -i, \ldots, -i) \in \text{GL}_n(k)
\]

The elements \( t(j)\sigma \), \( 0 \leq j \leq [n/2] \), are the representatives of the isolated elements (§2.1.5), except when \( n \) is even and \( j = 1 \), in which case \( t(j)\sigma \) is quasi-isolated ([DM18, Proposition 4.2]). We have,

\[
C_G(t(j)\sigma) \cong O_{2j}(k) \times \text{Sp}_{n-2j}(k), \text{ if } n \text{ is even;}
\]

\[
C_G(t(j)\sigma) \cong \text{Sp}_{2j}(k) \times O_{n-2j}(k), \text{ if } n \text{ is odd.}
\]
In particular, when \( n \) is even, \( t(j)\sigma \) is quasi-central only if \( j = 0 \), and when \( n = 2m + 1 \), \( t(j)\sigma \) is quasi-central only if \( j = m \). Note that our choice of \( \sigma \) for odd \( n \) is different from that of [DM18]. In this article, we will not encounter quasi-isolated elements that are not isolated.

4.4. Parabolic Subgroups and Levi Subgroups.

4.4.1. The parabolic subgroups \( P \subset G \) such that \( N_G(P) \) meets \( G.\sigma \) are described as follows.

Let \( P \subset G \) be a parabolic subgroup. The normaliser \( N_G(P) \) meets \( G.\sigma \) if and only if there exists \( g \in G \) such that \( g\sigma \) normalises \( P \), if and only if the \( G \)-conjugacy classes of \( P \) is \( \sigma \)-stable. According to Proposition 2.2.2, this means that there exists a \( G \)-conjugate of \( P \) that is \( \sigma \)-stable. We can then assume that \( P \) is \( \sigma \)-stable. Then, there exist a maximal torus and Borel subgroup containing this torus, both being \( \sigma \)-stable and contained in \( P \). A standard parabolic subgroup with respect to \( T \) and \( B \) is \( \sigma \)-stable if and only if the Dynkin subdiagram that defines it is \( \sigma \)-stable. We conclude that if \( T \) is the maximal torus of the diagonal matrices and \( B \) is the Borel subgroup of the upper triangular matrices, then every standard parabolic subgroup with respect to \( T \subset B \) such that \( N_G(P) \) meets \( G.\sigma \) has as its unique Levi factor containing \( T \) the group of the form \([5.2.2.1]\).

5. Parametrisation of Characters

5.1. F-Stable Levi Subgroups. Recall the parametrisation of the \( F \)-stable Levi subgroups of \( G = \text{GL}_n(k) \).

5.1.1. Notations. Denote by \( T \subset G \) the maximal torus consisting of the diagonal matrices and denote by \( B \subset G \) the Borel subgroup consisting of the upper triangular matrices. The Frobenius \( F \) of \( G \) sends each entry of an matrix to its \( q \)-th power. Denote by \( \Phi \) the root system defined by \( T \) and \( \Delta \subset \Phi \) the set of simple roots determined by \( B \). Denote by \( W := W_G(T) = N_G(T)/T \) the Weyl group of \( G \). The Frobenius acts trivially on \( W \). Given a subset \( I \subset \Delta \), we denote by \( P_I \) the standard parabolic subgroup defined by \( I \), and \( L_I \) the unique Levi factor of \( P_I \) containing \( T \). A Levi subgroup of the form \( L_I \) is called a standard Levi subgroup. Every Levi subgroup is conjugated to a standard Levi subgroup.

**Proposition 5.1.1.** The set of the \( G^F \)-conjugacy classes of the \( F \)-stable Levi subgroups of \( G = \text{GL}_n(k) \) is in bijection with the set of the unordered sequence of pairs of positive integers \((r_1,d_1) \cdots (r_s,d_s)\), satisfying \( \sum_i r_i d_i = n \).

We sketch the proof in order to introduce some notations that will be used later.

**Sketch of proof.** The \( G \)-conjugacy classes of the Levi subgroups are in bijection with the equivalence classes of the subsets \( I \subset \Delta \). Two subsets \( I \) and \( I' \) are equivalent if there is an element \( w \in W \) such that \( I' = wI \). It suffices for us to fix \( I \subset \Delta \) and only consider the \( G^F \)-conjugacy classes of \( L_I \). The \( G^F \)-conjugacy classes of the \( G \)-conjugates of \( L_I \) are in bijection with the \( F \)-conjugacy classes of \( N_G(L_I)/L_I \). Note that \( F \) acts trivially on \( N_G(L_I)/L_I \).
Let $\Gamma_I$ be a finite set such that $|\Gamma_I| = \dim Z_{L_I}$ and we fix an isomorphism $Z_{L_I} \cong (k')^{|\Gamma_I|}$. There are positive integers $n_i$, $i \in \Gamma_I$, such that $L_I \cong \prod_{i \in \Gamma_I} \operatorname{GL}_{n_i}$. For any $r \in \mathbb{Z}_{\geq 0}$, put $\Gamma_{I,r} := \{ i \in \Gamma_I \mid n_i = r \}$ and put $N_r = |\Gamma_{I,r}|$. Then the equivalence class of $I \subset \Delta$ is specified by the sequence $(N_r)_{r \geq 0}$. It is known that $N_G(L_I) / L_I \cong \prod_r \Xi(\Gamma_{I,r})$, where $\Xi(\Gamma_{I,r})$ denotes the permutation group of the set $\Gamma_{I,r}$. The conjugacy classes of $N_G(L_I) / L_I$ are then parametrised by the partitions of $N_r$, $r \in \mathbb{Z}_{\geq 0}$. The desired sequence $(r_1, d_1) \cdots (r_s, d_s)$ will then be defined in such a way that $(d_i)_{i \in [1 \leq i \leq s, r_i = r]}$ forms the corresponding partition of $N_r$.

Let $nL_I \subseteq N_G(L_I) / L_I$ be a class representing a $G^F$-class of $F$-stable Levi subgroup. Then $n$ acts on $Z_{L_I}$ by a permutation $w$ of $\Gamma_I$. For each $r \in \mathbb{Z}_{\geq 0}$, denote by $\Lambda_i(n)$ the set of the orbits in $\Gamma_{I,r}$ under the action of $n$ and write $\Lambda_i(n) = \bigcup_r \Lambda_{i,r}(n)$. For each $i \in \Lambda_i(n)$, put $r_i = r$ if $i \in \Lambda_{i,r}(n)$ and define $d_i$ to be the cardinality of $i$. The $d_i$'s, for $i \in \Lambda_i(n)$, form a partition of $|\Gamma_{I,r}|$. The integer $s$ in the statement of the proposition is equal to $|\Lambda_i(n)|$. \hfill \square

If $L$ is an $F$-stable Levi subgroup corresponding to $(n_1, d_1) \cdots (n_s, d_s)$, then $(L, F)$ is isomorphic to a standard Levi subgroup

$$L_I \cong \prod_i \operatorname{GL}_{n_i}(k)^{d_i},$$

equipped with a Frobenius acting on each factor $\operatorname{GL}_{n_i}(k)^{d_i}$ in the following manner,

$$\operatorname{GL}_{n_i}(k)^{d_i} \rightarrow \operatorname{GL}_{n_i}(k)^{d_i};$$

$$\begin{pmatrix} g_1, g_2, \ldots, g_{d_i} \end{pmatrix} \mapsto \begin{pmatrix} F_0(g_{d_i}), F_0(g_1), \ldots, F_0(g_{d_i-1}) \end{pmatrix},$$

where $F_0$ is the Frobenius of $\operatorname{GL}_{n_i}(k)$ that sends each entry to its $q$-th power.

5.2. $\sigma$-Stable Characters. Fix $T$ and $B$ as above, then standard Levi subgroups and standard parabolic subgroups are always taken with respect to $T \subset B$. In this section, we will consider the automorphism $\sigma$ defined in §4.1.1. The operation $\chi \mapsto \chi \circ \sigma^{-1}$ defines an involution of $\operatorname{Irr}(\operatorname{GL}_n(q))$. Denote by $^\sigma \chi$ the image of $\chi$ under this involution.

5.2.1. Quadratic-Unipotent Characters. Given a 2-partition $(\mu_1, \mu_2) \in \mathcal{P}_n(2)$, we define an irreducible character $\chi(\mu_1, \mu_2) \in \operatorname{Irr}(\operatorname{GL}_n(q))$ as follows. Put $m_1 = |\mu_1|$ and $m_2 = |\mu_2|$ and so $m_1 + m_2 = n$. Let $M \subseteq \operatorname{GL}_n(k)$ be a standard Levi subgroup isomorphic to $\operatorname{GL}_{m_1}(k) \times \operatorname{GL}_{m_2}(k)$, i.e.

$$M = \begin{pmatrix} \operatorname{GL}_{m_1} & 0 \\ 0 & \operatorname{GL}_{m_2} \end{pmatrix}.$$ 

Denote by $W_M := W_M(T)$ the Weyl group of $M$. It is isomorphic to $\Xi_{m_1} \times \Xi_{m_2}$. The Frobenius $F$ acts trivially on $W_M$. With respect to the isomorphism $M^F \cong \operatorname{GL}_{m_1}(q) \times \operatorname{GL}_{m_2}(q)$, we define a linear character $\theta \in \operatorname{Irr}(M^F)$ to be $(1 \circ \det, \eta \circ \det)$, where $1$ is the trivial character of $F_q^*$.
and η is the order 2 irreducible character of \( \mathbb{F}_q^* \). It is regular in the sense of [LS, §3.1]. We define \( \varphi \in \text{Irr}(W_M) \) by a 2-partition \((\mu_1, \mu_2)\) in such a way that the factor corresponding to \( \Xi_{m_1} \) is defined by the partition \( \mu_1 \), and the other by \( \mu_2 \). According to Theorem [1] the triple \((M, \theta, \varphi)\) gives an irreducible character of \( \text{GL}_n(q) \), which we denote by \( \chi_{(\mu_1, \mu_2)} \). The irreducible characters of \( \text{GL}_n(q) \) thus obtained are called quadratic-unipotents.

**Lemma 5.2.1.** For any \((\mu_1, \mu_2) \in \mathcal{P}_n(2)\), the character \( \chi_{(\mu_1, \mu_2)} \) is \( \sigma \)-stable.

**Proof.** If \( \chi_{(\mu_1, \mu_2)} \) is of the form \( R^G_{\varphi} \theta \) for a triple \((M, \theta, \varphi)\), then \( \sigma \chi_{(\mu_1, \mu_2)} \) is of the form \( R^G_{\varphi, \theta, \sigma \cdot \varphi} \) for the triple \((\sigma(M), \sigma \cdot \theta, \sigma \cdot \varphi)\) according to Lemma [3.1.2] Explicitly, \( \sigma(M) \) is the Levi subgroup \((5.2.1.1)\) with \( \text{GL}_{m_1} \) and \( \text{GL}_{m_2} \) exchanged, \( \sigma \cdot \theta \) associates to the factor \( \text{GL}_{m_1} \) the trivial character of \( \mathbb{F}_q \) and \( \eta \) to the other factor, and \( \sigma \cdot \varphi \) associates to the factor \( \text{GL}_{m_1} \) the character of \( \Xi_{m_1} \) corresponding to \( \mu_1 \) and to the other factor the character corresponding to \( \mu_2 \). Then the conjugation by \( \beta_M \) sends \((\sigma(M), \sigma \cdot \theta, \sigma \cdot \varphi)\) to \((M, \theta, \varphi)\), and so \( \sigma \chi_{(\mu_1, \mu_2)} = \chi_{(\mu_1, \mu_2)} \) according to Theorem [1].

### 5.2.2
Now we construct some more general \( \sigma \)-stable irreducible characters. If \( I \subset \Delta \) is a \( \sigma \)-stable subset, then it defines a \( \sigma \)-stable standard Levi subgroup. Every \( \sigma \)-stable standard Levi subgroup is of this form. We are going to use the notations of §5.1.1 and of Proposition [5.1.1] Denote by 0 the unique element of \( \Gamma_I \) fixed by \( \sigma \). For any \( i \in \Gamma_I \), denote by \( i^\ast \) its image under \( \sigma \). The standard Levi subgroup \( L_i \) is decomposed as \( \prod_{i \in \Gamma_I} L_i \). For any \( i \in \Gamma_I \), write \( n_i = r \) if \( i \in \Gamma_{I,r} \), and so \( L_i \cong \text{GL}_{n_i}(k) \). Schematically, \( L_i \) is equal to

\[
\begin{pmatrix}
L_{i_1} \\
\vdots \\
L_{i_s}
\end{pmatrix}
\]

\[
\begin{array}{c}
onumber\text{GL}_{n_0} \\
L_{i_r} \\
\vdots \\
L_{i_{t_r}}
\end{array}
\]

### 5.2.3
Recall that \( W_G(L_i) := N_G(L_i)/L_i \) is isomorphic to \( \prod_{r \in \mathbb{Z}_{N_r}} \Xi_{N_r} \). Write \( N_r' = N_r/2 \) if \( n_0 \neq r \) and write \( N_r' = (N_r - 1)/2 \) if \( n_0 = r \). Then, \( W_G(L_i)^a = \prod_r \Xi_{N_r} \). Regarded as a block permutation...
matrix an element of \( \mathcal{W}_{N_t}^C \) typically acts on \( \prod_{i|n_i=r} L_i \) in the following manners.

\[
\begin{pmatrix}
L_{i_1} \\
\vdots \\
L_{i_k} \\
\vdots \\
L_{i_l}
\end{pmatrix}
\begin{pmatrix}
L_{i_1} \\
\vdots \\
L_{i_l} \\
\vdots \\
L_{i_1'}
\end{pmatrix}
\]

(5.2.3.1)

corresponding to a cycle of positive sign and a cycle of negative sign respectively.

If \( \bar{w} \) is an element of \( W_G(L_i)^\sigma \), then a block permutation matrix that represents it, denoted by \( \bar{w} \), obviously can be chosen to be \( \sigma \)-stable, and so there is some \( g \in (C^\sigma)^\circ \) such that \( g^{-1}F(g) = \bar{w} \). Put \( M_{\bar{w}} := gL_i g^{-1} \). It is an \( F \)-stable and \( \sigma \)-stable Levi factor of a \( \sigma \)-stable parabolic subgroup, and there is an isomorphism \( \text{ad} \colon L_i^F \cong M_{\bar{w}}^F \). A character of \( M_{\bar{w}}^F \) is \( \sigma \)-stable if and only if it is identified with a \( \sigma \)-stable character of \( L_i^F \) by \( \text{ad} g \) since \( g \) is \( \sigma \)-stable. Let us fix such a \( w \) and construct some \( \sigma \)-stable irreducible characters of \( L_i^F \).

Write \( \Lambda = \Lambda_1(\bar{w}) \), following the proof of Proposition 5.1.1. The action of \( \sigma \) on \( \Gamma_i \) induces an action on \( \Lambda \) as \( w \) commutes with \( \sigma \), which allows us to use the notation \( i^* \) for \( i \in \Lambda \). If \( i, j \in \Gamma_i \) belong to the same orbit of \( w \), then \( n_i = n_j \), which justifies the notation \( n_i \) for \( i \in \Lambda \). Also note that \( n_i = n_i' \). For any \( i \in \Lambda \), denote by \( d_i \) the cardinality of the orbit \( i \).

(5.2.3.2) 
\[ \Lambda_1 = \{ i \in \Lambda | i^* \neq i \} / \sim \]

where the equivalence identifies \( i \) to \( i^* \). We will say that \( i \) belongs to \( \Lambda_1 \) if \( i \neq i^* \) and the equivalence class of \( i \) belongs to \( \Lambda_1 \). Write

(5.2.3.3) 
\[ \Lambda_2 = \{ i \in \Lambda | i = i^* \} \setminus \{0\}. \]

We can choose an isomorphism

(5.2.3.4) 
\[ L_i \cong \prod_{i \in \Lambda_1} (\text{GL}_{n_i} \times \text{GL}_{n_i^*})^{d_i} \times \prod_{i \in \Lambda_2} (\text{GL}_{n_i} \times \text{GL}_{n_i^*})^{d_i/2} \times \text{GL}_{n_0} \]

such that \( F\bar{w} \) and \( \sigma \) act on \( L_i \) in the following manner.

\(^\circ\) By block permutation matrix, we always mean a block matrix with each block equal to an identity matrix or a zero matrix, which itself is also a permutation matrix. Of course this is more restrictive than a block matrix that is also a permutation matrix. 
The action of $F_w$ is given by:

\[
i = 0 : \quad GL_{n_0} \longrightarrow GL_{n_0} \\
A \mapsto F_0(A);
\]

\[
i \in \Lambda_1 : \quad (GL_{n_i} \times GL_{n_r})^{d_i} \longrightarrow (GL_{n_i} \times GL_{n_r})^{d_i} \\
(A_1, B_1, A_2, B_2, \ldots, A_{d_i}, B_{d_i}) \mapsto \\
(F_0(A_{d_i}), F_0(B_{d_i}), F_0(A_1), F_0(B_1) \ldots F_0(A_{d_i-1}), F_0(B_{d_i-1}));
\]

\[
i \in \Lambda_2 : \quad (GL_{n_i} \times GL_{n_r})^{d_i/2} \longrightarrow (GL_{n_i} \times GL_{n_r})^{d_i/2} \\
(A_1, B_1, A_2, B_2, \ldots, A_{d_i/2}, B_{d_i/2}) \mapsto \\
(F_0(B_{d_i/2}), F_0(A_{d_i/2}), F_0(A_1), F_0(B_1) \ldots F_0(A_{d_i/2-1}), F_0(B_{d_i/2-1})).
\]

where $F_0$ is the Frobenius of $GL_r(k)$ that sends each entry to its $q$-th power, for any $r$.

The action of $\sigma$ is given by:

\[
i = 0 : \quad GL_{n_0} \longrightarrow GL_{n_0} \\
A \mapsto \sigma_0(A);
\]

\[
i \neq 0 : GL_{n_i} \times GL_{n_r} \longrightarrow GL_{n_i} \times GL_{n_r} \\
(A, B) \mapsto (\sigma_i(B), \sigma_i(A))
\]

where $\sigma_0$ is the standard automorphism of $GL_{n_0}$ (§4.1.1) and $\sigma_i$ is the automorphism of $GL_{n_i}$ that sends $g$ to $g^{-1}d_i^{-1}g^{-1}$ regardless of the parity of $n_i$ (§4.1.1). We have

\[
L_i^{F_w} \cong \prod_{i \in \Lambda_1} (GL_{n_i}(q^{d_i}) \times GL_{n_r}(q^{d_i})) \times \prod_{i \in \Lambda_2} GL_{n_i}(q^{d_i}) \times GL_{n_0}(q)
\]

and $\sigma$ acts on it in the following manner,

\[
i = 0 : \quad GL_{n_i}(q) \longrightarrow GL_{n_i}(q) \\
A \mapsto \sigma_0(A);
\]

\[
i \in \Lambda_1 : GL_{n_i}(q^{d_i}) \times GL_{n_r}(q^{d_i}) \longrightarrow GL_{n_i}(q^{d_i}) \times GL_{n_r}(q^{d_i}) \\
(A, B) \mapsto (\sigma_i(B), \sigma_i(A));
\]

\[
i \in \Lambda_2 : \quad GL_{n_i}(q^{d_i}) \longrightarrow GL_{n_i}(q^{d_i}) \\
A \mapsto \sigma_i F_0^{d_i/2}(A).
\]

Denote by $L_1$ the product of the direct factors of $L_1$ except $L_0$. If no confusion arises, we may also denote by $F_w$ and $\sigma$ their restrictions on $L_1$. With respect to the decomposition of $L_i^{F_w}$ as above, a linear character $\theta_1$ of $L_i^{F_w}$ can be written as

\[
\prod_{i \in \Lambda_1} (\alpha_i, \alpha_r) \prod_{i \in \Lambda_2} \alpha_i \in \prod_{i \in \Lambda_1} (\text{Irr}(F_{q_i}^r) \times \text{Irr}(F_{q_i}^r)) \times \prod_{i \in \Lambda_2} \text{Irr}(F_{q_i}^r).
\]
The set $\text{Irr}(W_{L_1})^{F_w}$ is in bijection with the irreducible characters of

\[(5.2.3.15) \prod_{i \in \Lambda_1} (\mathcal{Z}_{n_i} \times \mathcal{Z}_{n_r}) \times \prod_{i \in \Lambda_2} \mathcal{Z}_{n_i}.\]

Such a character can be written as $\varphi_1 = \prod_{i \in \Lambda_1} (\varphi_i, \varphi_i^*) \prod_{i \in \Lambda_2} \varphi_i$.

5.2.4. Suppose that the factors of $\theta_1$ and $\varphi_1$ satisfy

for any $i \in \Lambda_1 \sqcup \Lambda_2$, $\alpha_i \neq 1$ or $\eta \circ N_{F_{\varphi^i}}/F_q$,

\[(5.2.4.1)\]

for any $i \in \Lambda_1$, $\alpha_r = \alpha_i^{-1}$,

\[(5.2.4.2)\]

for any $i \in \Lambda_2$, $\alpha_r^{\theta/2} = \alpha_i^{-1}$,

\[(5.2.4.3)\]

for any $i \in \Lambda_1$, $\varphi_i^* = \varphi_i$.

\[(5.2.4.4)\]

We choose $\tilde{\varphi}_1$, an extension of $\varphi_1$ to $W_{L_1}^{<F_w>}$, in such a way that

\[(5.2.4.5)\]

\[\chi_1 = R_{\tilde{\varphi}_1}^{G} \theta_1 = |W_{L_1}|^{-1} \sum_{v \in W_{L_1}} \tilde{\varphi}_1(vF_{w}) R_{L_1}^{L_1} \theta_1,\]

is an irreducible character of $L_1^{F_w}$.

**Proposition 5.2.2.** Let $\chi_0$ be a quadratic-unipotent character of $L_{0}^{F_0}$. Then, $\chi_1 \boxtimes \chi_0$ is a $\sigma$-stable irreducible character of $L_{1}^{F_w}$. Identified with a character of $M_{1,w}$, its induction $R_{M_{1,w}}^{G} (\chi_1 \boxtimes \chi_0)$ is a $\sigma$-stable irreducible character of $GL_n(q)$.

**Proof.** By the hypothesis on $\theta_1$ and $\varphi_1$, $\chi_1$ is $\sigma$-stable, and so $\chi_1 \boxtimes \chi_0$ is $\sigma$-stable. It follows from the definition of $R_{M_{1,w}}^{G}$ that if $\chi_1 \boxtimes \chi_0$ is $\sigma$-stable, then $R_{M_{1,w}}^{G} (\chi_1 \boxtimes \chi_0)$ is $\sigma$-stable.

\[\square\]

5.2.5. The rest of this section is to prove the following proposition.

**Proposition 5.2.3.** Every $\sigma$-stable irreducible character of $GL_n(q)$ is of the form given by Proposition [5.2.2]

5.2.6. We will need the following lemma.

**Lemma 5.2.4.** Let $M$ be a Levi subgroup of $G = GL_n(k)$. Let $x \in G$ be such that $x^{-1} \sigma$ preserves $M$, i.e. $x^{-1} \sigma(M)x = M$. Then the action of $x^{-1} \sigma$ on $Z_M$ is the composition of $z \mapsto z^{-1}$ and an automorphism induced by an element of $N_{GL_n}(M)$.

**Proof.** Let $L_f$ be a standard Levi subgroup and $g \in G$ be such that $M = gL_f g^{-1}$. We have the maps

\[L_f \xrightarrow{\text{ad}_g} M \xrightarrow{\text{ad}_x} \sigma(M),\]

and it is equivalent to considering $Z_{L_f}$. The automorphism

\[(\text{ad}_g)^{-1} \circ \sigma^{-1} \circ (\text{ad}_x) \circ (\text{ad}_g)(l),\]
can be written as $\tau^{-1} \circ \text{ad} \, n$, with $\tau$ being the transpose-inverse automorphism and $n = j^{-1} \sigma(g)^{-1} x g$. Since $L_j$ is automatically $\tau$-stable, we have $n \in N_G(L_j)$. Note that $\tau$ acts on the centre by inversion.

Suppose that $\chi \in \text{Irr}(\text{GL}_n(q))$ is an irreducible character induced from the triple $(M, \varphi, \theta)$ following Theorem 5.2.7. Let $L_j$ be a standard Levi subgroup that is $G$-conjugate to $M$. If the $G^F$-conjugacy class of $M$ is represented by $(n_1, d_1) \cdots (n_s, d_s)$, then $M \cong \prod_{i \in \Lambda} \text{GL}_{n_i}(k)^{d_i}$, and $M^F \cong \prod_{i \in \Lambda} \text{GL}_{n_i}(q^{d_i})$, where $\Lambda := \Lambda_f(n)$ with $n \in N_G(L_j)$ following the notations in the proof of Proposition 5.1.1. With respect to this decomposition, we write $\theta = (\alpha_i)_{i \in \Lambda}$ with $\alpha_i \in \text{Irr}(F^*_{q^{d_i}})$, where we have abbreviated $\alpha_i \circ \det$ as $\alpha_i$. Denote by $\hat{\alpha}_i$ the $F$-orbit of $\alpha_i$ for each $i$, and write $\hat{\alpha}_i^{-1} = \{a^{-1} \mid a \in \hat{\alpha}_i\}$. The semi-simple part of $\chi$ can then be represented by the sequence $(n_1, \hat{\alpha}_i) \cdots (n_s, \hat{\alpha}_s)$.

**Proposition 5.2.5.** In order for $\chi$ to be $\sigma$-stable, it is necessary that for each $i \in \Lambda$, there exists a unique $i^* \in \Lambda$ such that $n_i = n_{i^*}$ and $\hat{\alpha}_i^{-1} = \hat{\alpha}_{i^*}$.

**Proof.** Assume that $\chi$ is $\sigma$-stable. Theorem 5.2.7 implies that there exists $x \in G^F$ such that

\begin{align}
(5.2.6.1) \quad & \text{ad} \, x(M) = \sigma(M), \\
(5.2.6.2) \quad & (\text{ad} \, x)^* \sigma, \theta = \theta,
\end{align}

so $x$ satisfies the assumption in the previous lemma. Put $Z_M := Z_M/[M, M]$ and regard $\theta$ as a character of $Z_M^F$. Choose an isomorphism $Z_M \cong (k^r)^r$ for some $r$. Let $v \in \mathfrak{z}_r$ be such that the action of $x^{-1} \sigma$ on $Z_M$ is the composition of $v$ and the inversion. Since $x$ and $\sigma$ are $F$-stable, the action of $v$ is compatible with the action of $F$ and so induces an action on $Z_{M^F}$. If we express the action of $F$ as the composition of some permutation $\tau \in \mathfrak{z}_r$ and raising each entry to the $q$-th power, then $v$ commutes with $\tau$. The centraliser of $\tau$ is described by its orbits in $[1, \ldots, r]$: an element commuting with $\tau$ induces a permutation among the orbits of the same size and a cyclic permutation within each orbit. If we write $\theta = (\alpha_i)$, then the action of $x^{-1} \sigma$ on $Z_{M^F}$ induces $(\alpha_i) \mapsto (\alpha_i^{-\varphi})$ for some $0 \leq c < d_i$, with $i^* := v^{-1}(i)$, whence the assertion.

For any $i \in \Lambda$, the $i^*$ as claimed in the proposition is unique. Suppose $i^*$ and $i'''$ with $i^* \neq i'''$ both satisfy the assertion in the proposition, then $\hat{\alpha}_{i^*} = \hat{\alpha}_{i'''}$, which contradicts the regularity of $\theta$.

**Remark 5.2.6.** If $i^* = i$, then it is necessary that $d_i$ is an even number, and $\hat{\alpha}_i^{-1} = \hat{\alpha}_i^d$.

**Remark 5.2.7.** There are at most two $i \in \Lambda$ such that $\alpha_i^2 = 1$ in order for $\theta$ to be regular. We denote them by $\pm$. It is necessary that $d_+ = d_- = 1$. Also denote by $\pm$ the corresponding two elements of $\Gamma$. 

\[ \text{THE CHARACTER TABLE OF } \text{GL}_n(F_q) \times \langle \tau \rangle \]
5.2.7. The previous proposition allows us to define the sets

\[(5.2.7.1) \quad \Lambda_1 = \{i \in \Lambda | i^* \neq i\} \sim \]

\[(5.2.7.2) \quad \Lambda_2 = \{i \in \Lambda | i = i^* \} \setminus \{\pm\},\]

in a way similar to (5.2.3.2) and (5.2.3.3). Since the \(G^F\)-conjugacy class of \(M\) only depends on the coset of \(n\), we can assume that \(n\) is a block permutation matrix. There exists a block permutation matrix \(x\) such that \(L' := xLx^{-1}\) is of the form

\[
\begin{pmatrix}
L_{i_1} \\
\vdots \\
L_{i_k} \\
L_+ \\
L_- \\
L_{i'} \\
\vdots \\
L_{i'}
\end{pmatrix}
\]

where \(L'_{ij} \cong L_{ij}\) for any \(j\). From this, we can recover the induction data of 5.2.4 as follows.

On \(L'\), the Frobenius concerned is \(F_{xnx^{-1}}\), which fixes \(L_+\) and \(L_-\) as \(d_+ = d_- = 1\). We can further conjugate by a permutation matrix that normalises \(L'\), say \(y\), in such a way that the action of \(F_{yxn^{-1}x^{-1}}\) on \(L'\) is built up from (5.2.3.1) and moreover the \(\alpha_i\)'s satisfy the hypothesis 5.2.4. Such \(y\) exists because a conjugacy class in \(N_G(L_i)/L_i\) is uniquely determined by the \(d_i\)'s. Denote by \(\theta'\) and \(\varphi'\) the characters associated to \(L_+\) that are transferred from \(\theta\) and \(\varphi\) via \(\text{ad}\, g\), \(\text{ad}\, x\) and \(\text{ad}\, y\).

Put \(v = ynx^{-1}y^{-1}\), it is a \(\sigma\)-stable block permutation matrix by the choice of \(y\). Let \(h \in (G^F)^\circ\) be such that \(h^{-1}F(h) = v\). Then \(M_{\theta', \varphi} = hL_xh^{-1}\) is an \(F\)-stable Levi subgroup that is \(G^F\)-conjugate to \(M\), because \(nL_x\) is conjugate to \(vL_x\). If we regard \(\theta'\) and \(\varphi'\) as some characters associated to \(M_{\theta', \varphi}\) via the isomorphism \(\text{ad}\, h\), then \(\chi\) is equal to the induction \(R_{\varphi'}^G, \theta'\) for a triple \((M_{\theta', \varphi}, \varphi', \theta')\).

Define \(L_1\) to be the standard Levi subgroup of the form (5.2.2.1) such that \(n_0 = n_+ + n_-\) and that \(L_1\) coincides with \(L'\) away from \(\text{GL}_{n_0}\). We see that \(M_{\theta', \varphi} : = hL_xh^{-1}\) is a \(\sigma\)-stable and \(F\)-stable Levi factor of a \(\sigma\)-stable parabolic subgroup. Moreover, it contains \(M_{\theta', \varphi}\) and \(\sigma(M_{\theta', \varphi})\).

Note that \(\chi\) is defined by the triple \((\sigma(M_{\theta', \varphi}), \sigma, \varphi', \sigma, \theta')\) and \(\sigma, \theta'\) is already equal to \(\theta'\) away from \(\text{GL}_{n_0}\), by construction. By Theorem 1 in order for \(\chi\) to be \(\sigma\)-stable, it is necessary that \(\sigma, \varphi' = \varphi'\) away from \(\text{GL}_{n_0}\). And it suffices since in the component \(\text{GL}_{n_0}\), we have a quadratic-unipotent character. This completes the proof of Proposition 5.2.3.
The type of a $\sigma$-stable irreducible character consists of some non negative integers $n_{+},$ and some positive integers $n_{i}, d_{i}, n'_{j},$ and $d'_{j}$ parametrised by some possibly empty finite sets $\Lambda_{1}$ and $\Lambda_{2},$ denoted by

\[ t = n_{+}n_{-}(n_{i}, d_{i})_{i \in \Lambda_{1}}(n'_{j}, d'_{j})_{j \in \Lambda_{2}}, \]

satisfying

\[ n = n_{+} + n_{-} + \sum_{i} 2n_{i}d_{i} + \sum_{j} 2n'_{j}d'_{j}. \]

If $\bar{t} = \bar{n}_{+}\bar{n}_{-}(\bar{n}_{i}, \bar{d}_{i})_{i \in \Lambda_{1}}(\bar{n}'_{j}, \bar{d}'_{j})_{j \in \Lambda_{2}}$ is another sequence of integers, we regard it as the same as $t$ if and only if there exist some bijections $\Lambda_{1} \cong \tilde{\Lambda}_{1}$ and $\Lambda_{2} \cong \tilde{\Lambda}_{2}$ such that the integers are matched (and $n_{+} = \bar{n}_{+}$ and $n_{-} = \bar{n}_{-}$). We denote by $\tilde{\mathcal{T}}_{X}$ the set of the types of the $\sigma$-stable irreducible characters of $GL_{n}(q)$.

Given $t \in \mathcal{T}_{X},$ denote by $\tilde{\mathcal{T}}_{X}(t)$ the set of the data

\[ \bar{t} = \lambda_{+}\lambda_{-}(\lambda_{i}, \hat{\alpha}_{i})_{i \in \Lambda_{1}}(\lambda'_{j}, \hat{\alpha}'_{j})_{j \in \Lambda_{2}} \]

satisfying

1. $\lambda_{+} \in \mathcal{P}_{n_{+}}, \lambda_{i} \in \mathcal{P}_{n_{i}}, \lambda'_{j} \in \mathcal{P}_{n'_{j}},$ for the integers $n_{+}, n_{-}, n_{i}$ and $n'_{j}$ associated to $t;$
2. $\hat{\alpha}_{i} \subset \text{Irr}(\mathbb{F}_{q}^{d_{i}})$ is an $F$-orbit of order $d_{i}$ that is not stable under inversion, with $\hat{\alpha}_{i}$ identified with $\hat{\alpha}_{i}^{-1};$
3. $\hat{\alpha}'_{j} \subset \text{Irr}(\mathbb{F}_{q}^{2d'_{j}})$ is an $F$-orbit of order $2d'_{j}$ that is stable under inversion;
4. $\hat{\alpha}_{i} \neq \hat{\alpha}_{i}^{-1}$ if $i \neq i'$ and $\hat{\alpha}'_{j} \neq \hat{\alpha}'_{j}$ if $j \neq j'$.

If $\bar{\lambda}_{+}\bar{\lambda}_{-}(\bar{\lambda}_{i}, \hat{\beta}_{i})_{i \in \Lambda_{1}}(\bar{\lambda}'_{j}, \hat{\beta}'_{j})_{j \in \Lambda_{2}}$ is another such datum, we regard it as the same as $\bar{t}$ if and only if $\bar{\lambda}_{+} = \lambda_{+}, \bar{\lambda}_{-} = \lambda_{-},$ and there exist bijections $\Lambda_{1} \cong \tilde{\Lambda}_{1}$ and $\Lambda_{2} \cong \tilde{\Lambda}_{2}$ such that the partitions and orbits are matched.

Write $\tilde{\mathcal{T}}_{X} = \sqcup_{t \in \mathcal{T}_{X}} \tilde{\mathcal{T}}_{X}(t).$ By Proposition 5.2.2 and Proposition 5.2.3, the $\sigma$-stable irreducible characters are in bijection with $\tilde{\mathcal{T}}_{X}.$

Equivalently, the set $\tilde{\mathcal{T}}_{X}$ can be described as follows. Write

\[ k^{\gamma} = \lim_{\rightarrow} \text{Irr}(\mathbb{F}_{q^{n}}), \]

as $n$ runs over positive integers. Denote by $\sigma$ the inversion $a \mapsto a^{-1}$ on this set. Denote by $\Phi^{\sigma}$ the set of $<F> \times <\sigma>$-orbits in $k^{\gamma}$. For any map of sets $f : \Phi^{\sigma} \rightarrow \mathcal{P},$ write

\[ ||f|| = \sum_{x \in \Phi^{\sigma}} |f(x)| \cdot |x|, \]

where $|x|$ is the size of the orbit. Then $\tilde{\mathcal{T}}_{X}$ is identified with the set

\[ \{ f : \Phi^{\sigma} \rightarrow \mathcal{P}; ||f|| = n \}. \]
The $<F>\times<\sigma>$-orbits are either the single points $[1]$, $[\eta]$, or the union of two $F$-orbits that are not stable under inversion, or a single $F$-orbit that is stable under inversion. One easily sees how $\xi_x$ is recovered from these maps.

6. Parametrisation of Conjugacy Classes

6.1. $F$-Stable Quasi-Semi-Simples Classes. A $G$-conjugacy class contains some $G^F$-conjugacy classes if and only if it is $F$-stable. We will first give the parametrisation of the $F$-stable quasi-semi-simple conjugacy classes in $G.\sigma$. Recall that in $G^F$, we denote by $\sigma$ the element $t_0\sigma'$ (cf. Convention [4.2.1]).

6.1.1. We begin with the parametrisation of the quasi-semi-simple $G$-conjugacy classes. We take for $T$ the maximal torus consisting of the diagonal matrices, then $(T^\circ)$ consists of the matrices

(6.1.1.1) $\text{diag}(a_1,a_2,\ldots,a_m,a_m^{-1},\ldots,a_2^{-1},a_1^{-1})$, if $n = 2m$,

(6.1.1.2) $\text{diag}(a_1,a_2,\ldots,a_m,1,a_m^{-1},\ldots,a_2^{-1},a_1^{-1})$, if $n = 2m + 1$,

with $a_i \in k^*$ for all $i$, and the commutator $[T,\sigma]$ consists of the matrices

(6.1.1.3) $\text{diag}(b_1,b_2,\ldots,b_m,b_m,\ldots,b_2,b_1)$, if $n = 2m$,

(6.1.1.4) $\text{diag}(b_1,b_2,\ldots,b_m,b_{m+1},b_m,\ldots,b_2,b_1)$, if $n = 2m + 1$,

with $b_i \in k^*$ for all $i$. So the elements of $S := [T,\sigma] \cap (T^\circ)$ are the matrices

(6.1.1.5) $\text{diag}(e_1,e_2,\ldots,e_m,e_m,\ldots,e_2,e_1)$, if $n = 2m$,

(6.1.1.6) $\text{diag}(e_1,e_2,\ldots,e_m,1,e_m,\ldots,e_2,e_1)$, if $n = 2m + 1$,

with $e_i = \pm 1$ for all $i$. We index the entries of a diagonal matrix by the set

\[\{1,2,\ldots,m,-m,\ldots,-2,-1\}\]

or the set

\[\{1,2,\ldots,m,0,-m,\ldots,-2,-1\}\]

according to the parity of $n$ so that every matrix in $(T^\circ)^\circ$ satisfies $a_{-i} = a_i^{-1}$ for all $i$. We will abbreviate an element of $(T^\circ)^\circ$ as $[a_1,\ldots,a_m]$ regardless of the parity of $n$.

We have the following proposition.

Proposition 6.1.1. ([DM18 Proposition 1.16]) The quasi-semi-simple classes in $G.\sigma$ are in bijection with the $W^0$-orbits in $T/[T,\sigma] \cong (T^\circ)^\circ/S$.

That is to say, the class of $[a_1,\ldots,a_m]\sigma$ is invariant under the following operations,

- Permutation of the $a_i$’s;
- $a_i \mapsto a_i^{-1}$, for any $i$;
- $a_i \mapsto -a_i$, for any $i$. 

THE CHARACTER TABLE OF $GL_n(F_q) \rtimes <\sigma>$
and \([b_1, \ldots, b_m]\sigma\) belongs to the same class if it only differs from \([a_1, \ldots, a_m]\sigma\) by these operations. For another description of these conjugacy classes, see also [DM15] Example 7.3.

**Remark 6.1.2.** The parametrisation given by this proposition relies on the choice of \(\sigma\). In either \(\tilde{\mathcal{G}}\) or \(\hat{\mathcal{G}}\), we will use the symplectic type quasi-central element \(\sigma\) to parametrise quasi-semi-simple conjugacy classes.

6.1.2. Denote by \(\hat{k}\) the quotient of \(k^*\) by the action of \(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}\):

\[
(1, 0) : a \mapsto a^{-1}, \quad (0, 1) : a \mapsto -a.
\]

For any \(a \in k\), denote by \(\hat{a}\) the set \([a, -a, a^{-1}, -a^{-1}]\). Write \(S = (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})^n\). The torus \((T^o)^\sigma\) is isomorphic to \((k^*)^m\). Let \(S\) act on it componentwise, i.e. each component \(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}\) acts on \(k^*\) as above. Then the quasi-semi-simple conjugacy classes in \(G, \sigma\) are parametrised by the \(\mathcal{Z}_m\)-orbits in \(\hat{k}^m\). The conjugacy class of \(t\sigma = [a_1, \ldots, a_m]\sigma\) is determined by the multiset \(\{a_1, \ldots, a_m\}\). We may regard its elements as the eigenvalues of \(t\sigma\).

The action of \(F\) on \(k\) induces an action on \(\hat{k}\), given by

\[
\hat{a} \mapsto \hat{a}^d := \{a^d, -a^d, a^{-d}, -a^{-d}\}.
\]

Denote by \(\Phi\), the set of \(F\)-orbits in \(\hat{k}\). Let \(\hat{a} \in \Phi\), and take \(\hat{a} \in \hat{\alpha}\). Write \(d = |\hat{a}|\). Denote by \(e\) and \(\epsilon\) the signs such that \(a^d = ea^{\epsilon}\), for any \(a \in \hat{a}\). Note that \(e\) and \(\epsilon\) are independent of the choice of \(\hat{a} \in \hat{\alpha}\) or the choice of \(a \in \hat{a}\). We say that \(\hat{a}\) is an orbit of type \((d, e, \epsilon)\).

There is some ambiguity with the type thus defined. Let \(\hat{\alpha}_1\) be an orbit of type \((d_1, e_1, \epsilon_1)\) and let \(\hat{\alpha}_2\) be an orbit of type \((d_2, e_2, \epsilon_2)\). Obviously if \(d_1 > d_2\), then \(\hat{\alpha}_1\) and \(\hat{\alpha}_2\) are distinct orbits. Suppose \(d_1 = d_2 = d\). If \(x \in k^*\) satisfies both of the two equations \(x^{d_1} = e_1x^{e_1}\) and \(x^{d_2} = e_2x^{e_2}\), then \(e_1e_2x^{\epsilon_1-\epsilon_2} = 1\). In order for this equation to be solvable, either \(e_1 = e_2, \epsilon_1 = \epsilon_2\), that is, \(\hat{\alpha}_1\) and \(\hat{\alpha}_2\) coincide, or \(e_1 = e_2, \epsilon_1 = -\epsilon_2\), which gives \(x^2 = 1\), or \(e_1 = -e_2, \epsilon_1 = -\epsilon_2\), which gives \(x^2 = -1\). The latter ones are the orbits \([1, -1]\) and \([i, -i]\), so \(d = 1\), and they are said to be of type \((1)\) and of type \((i)\) respectively. Except these cases, orbits of different types are all distinct. In the following, these two orbits are treated separately and so there will be no confusion among types.

6.1.3. Let us define some combinatorial data that parametrise the \(F\)-stable quasi-semi-simple conjugacy classes.

We call the type of an \(F\)-stable quasi-semi-simple conjugacy class the data consisting of some non negative integers \(n_+, n_-\), with the parity of \(n_+\) being that of \(n\), some positive integers \(n_i, d_i\) and some signs \(e_i\) and \(\epsilon_i\), parametrised by a possibly empty finite set \(\Lambda\), denoted by

\[
\mathfrak{t} = n_+n_-(n_i, d_i, e_i, \epsilon_i)_{i \in \Lambda}.
\]
Proposition 6.1.3. The F-stable quasi-semi-simple conjugacy classes in \( G \) are in bijection with the set \( \mathcal{I}_{C, s} \) defined as follows.

If \( \tilde{t} = n_+ n_-(n_i, \hat{a}_i)_{i \in \Lambda} \) is another sequence of integers, we regard it as the same as \( t \) if and only if there exists a bijection \( \Lambda \sim \tilde{\Lambda} \) such that the integers and the signs are matched, and moreover, \( n_+ = n_+ \) and \( n_- = n_- \). We will denote by \( \mathcal{I}_{C, s} \) the set of the types of the \( F \)-stable quasi-semi-simple conjugacy classes.

Given \( t \in \mathcal{I}_{C, s} \), denote by \( \mathcal{I}_{C, s}(t) \) the set of the data

\[
(6.1.3.3) \quad \tilde{t} = n_+ n_-(n_i, \hat{a}_i)_{i \in \Lambda},
\]

satisfying
- for any \( i, \hat{a}_i \) is an orbit of type \((d_i, e_i, \epsilon_i) \neq (1) \) or \( (0) \);
- if \( i \neq i' \), then \( \hat{a}_i \neq \hat{a}_{i'} \).

If \( s = m_+ n_-(m_{ij}, \hat{b}_{ij})_{i \in \Lambda} \) is another such datum, we regard it the same as \( \tilde{t} \) if and only if there exists a bijection \( \Lambda \sim \tilde{\Lambda} \) such that the integers and the orbits are matched, and moreover, \( n_+ = m_+ \) and \( n_- = m_- \). We will denote by \( \mathcal{I}_{C, s} \) the union of \( \mathcal{I}_{C, s}(t) \) as \( t \) runs over \( \mathcal{I}_{C, s} \).

It will sometimes be convenient to distinguish between the \( i \)'s with \( \epsilon_i = 1 \) and the \( i \)'s with \( \epsilon_i = -1 \). Put \( \Lambda_1 \subset \Lambda \) to be the subset of the \( i \)'s such that \( \epsilon_i = 1 \) and put \( \Lambda_2 = \Lambda \setminus \Lambda_1 \). The following notations will also be used,

\[
(6.1.3.4) \quad t = n_+ n_-(n_i, d_i, e_i)_{i \in \Lambda_1}, \quad \tilde{t} = n_+ n_-(n_i, \hat{a}_i)_{i \in \Lambda}.
\]

There is another equivalent description of \( \mathcal{I}_{C, s} \). Write \( \Phi^c = \Phi \setminus (\hat{1} \sqcup \hat{i}) \). For any map of sets \( \Phi \rightarrow \mathbb{Z}_{\geq 0} \), define

\[
\|f\| = f(\hat{1}) + f(\hat{i}) + \sum_{x \in \Phi^c} 2f(x) \cdot |x|,
\]

where \( |x| \) is the size of the orbit. Then \( \mathcal{I}_{C, s} \) can be identified with the set

\[
\{ f : \Phi \rightarrow \mathbb{Z}_{\geq 0} ; \|f\| = n \}.
\]

Under this identification, \( n_+ \) is the image of the orbit of type \((1) \) and \( n_- \) is the image of the orbit of type \((0) \) and \( \Lambda \) parametrises the inverse image of \( \mathbb{Z}_{>0} \) in \( \Phi ^c \).

Proposition 6.1.3. The F-stable quasi-semi-simple conjugacy classes in \( G, \sigma \) are in bijection with \( \mathcal{I}_{C, s} \).

Proof. Define a map

\[
(6.1.3.5) \quad \psi : \mathcal{I}_{C, s} \rightarrow \{ F\text{-stable quasi-semi-simple classes in } G, \sigma \} \subset k^n / \Xi_m
\]
as follows.
Write $\tilde{t} = n_* n_-(n_*, \check{\hat{t}})_{i \in \Lambda}$ with $\check{\hat{t}}_i$ of type $(d_i, e_i, e_i)$. We are going to define an element of $\tilde{t}^m$ from $\tilde{t}$, regarding the elements of $\check{\hat{t}}_i$'s as eigenvalues and the $n_-'$s as their multiplicities. We will write $\psi(\tilde{t}) = (\check{\hat{t}})_1 \leqsm m$.

(i). Take $[n_*/2]$ subsets of $\{1, \ldots, m\}$, each consisting of a point, which will be called of type $(1)$, and then $(n_-/2)$ subsets, each consisting of a point, which will be called of type $(i)$, and take for each $i \in \Lambda$, $n_i$ subsets of cardinality $d_i$. These subsets, combined with $\{0\}$ if $n$ is odd (we require that $\{0\}$ is of type $(1)$), form a partition of $\{1, \ldots, m\}(\cup\{0\})$ and we denote it by $(I_r)_r$.

(ii). Choose for each $r$ an identification $I_r \cong \mathbb{Z}/d_r \mathbb{Z}$, where $d_r := d_i$ if $I_r$ comes from $i$.

(iii). If $I_r$ comes from each $i \in \Lambda$ by the procedure (i), and $\check{\hat{t}}_i = \{\check{\hat{t}}_i \check{\hat{t}}_i, \ldots, \check{\hat{t}}_i^{d_i-1}\}$, define for all $k \in I_r$, $\check{\hat{t}}_k := \check{\hat{t}}_i^{d_i}$, under the identification $I_r \cong \mathbb{Z}/d_r \mathbb{Z}$. If $I_r$ is of type $(1)$ (resp. $(i)$), we define the only entry of $t$ corresponding to $I_r$ to be $\tilde{t}(\text{resp. } \check{\hat{t}})$. Thus, we have defined an element of $\tilde{k}_m$, whence a quasi-semi-simple conjugacy class in $G_{\sigma}$.

The class $\psi(\tilde{t})$ does not depend on the choices of the subsets $I_r$ or the identifications $I_r \cong \mathbb{Z}/d_r \mathbb{Z}$ due to the conjugation by $\varnothing_m$. Observe that the class of $\psi(\tilde{t})$ is $F$-stable if and only if $[\check{\hat{t}}]_1 \leqsm m$ and $[\check{\hat{t}}^{d_i}]_1 \leqsm m$ coincide as multisets. This is satisfied by $\psi(\tilde{t})$ since to each $I_r$ is associated an $F$-orbit. The map $\psi$ is surjective because the map $\check{\hat{t}}_r \mapsto \check{\hat{t}}_r^{d_i}$ identifies $[\check{\hat{t}}]_1 \leqsm m$ with itself, and so we can choose a bijection $f$ from $[1, \ldots, m]$ to itself such that $\check{\hat{t}}_{f(\check{\hat{t}})} = \check{\hat{t}}_r^{d_i}$. Then $f$ defines a permutation of $[1, \ldots, m]$, which can be expressed as a product of some cyclic permutations, so the above constructions can be reversed. Injectivity of $\psi$ is obvious. \hfill $\Box$

6.2. Centralisers and $G^F$-Classes. We will see that the centraliser of a quasi-semi-simple element is in general a product of a symplectic group, an orthogonal group and some linear groups.

6.2.1. Let $C$ be an $F$-stable quasi-semi-simple conjugacy class corresponding to $\tilde{t} = n_+ n_- (n_+, \check{\hat{t}})_{i \in \Lambda}(n_-, \check{\hat{t}})_{j \in \Lambda_2}$ following Proposition [6.1.3] and we denote its type by $t = n_+ n_-(n_+, d_i, e_i)_{i \in \Lambda}(n_-, d_j, e_j)_{j \in \Lambda_2}$. Let $t\sigma$, with $t \in (T^{\sigma})^-$ be a representative of $C$ as defined in the proof of Proposition [6.1.3].

Lemma 6.2.1. We have,

$$C_G(t\sigma) \cong \text{Sp}_n(k) \times \text{O}_n(k) \times \prod_{i \in \Lambda_1} \text{GL}_{n_i}(k) \times \prod_{j \in \Lambda_2} \text{GL}_{n_j}(k), \quad \text{if } n \text{ is even},$$

$$C_G(t\sigma) \cong \text{Sp}_n(k) \times \text{O}_n(k) \times \prod_{i \in \Lambda_1} \text{GL}_{n_i}(k) \times \prod_{j \in \Lambda_2} \text{GL}_{n_j}(k), \quad \text{if } n \text{ is odd}.$$

In particular, the isomorphism class of $C_G(t\sigma)$ only depends on the type of $C$.

The numbers $n_+$ and $n_-$ are exchanged only because we have chosen different types of $\sigma$ for even and odd $n$. 

Proof. If \( z \in G \) commutes with \( t \sigma \), then it commutes with \( t \sigma \sigma = t \sigma(t) \sigma^2 \), with \( \sigma^2 = \pm 1 \) being central. Let us calculate \( C_{G(t \sigma(t))}(t \sigma) \). That the \( \hat{\alpha}_i \)'s are pairwise distinct means that for \( a_i \in \alpha_i \), \( a_j \in \alpha_j \), \( i \neq j \), we have \( a_i^\tau \neq \pm a_j^{\tau} \), for all \( c \), so \( a_i^2 \neq a_j^2 \), for all \( c \). Besides, the integers \( n_+ \) and \( n_- \) become the multiplicities of 1 and \(-1 \) in \( t \sigma(t) \) respectively. Consequently, the centraliser of \( t \sigma(t) \) is a Levi subgroup \( L_0 := C_G(t \sigma(t)) \) isomorphic to

\[
\prod_{i \in \Lambda_1} (\text{GL}_{n_i} \times \text{GL}_{n_i})^{d_i} \times \prod_{j \in \Lambda_2} (\text{GL}_{n_j'} \times \text{GL}_{n_j'})^{d_j} \times \text{GL}_{n_+} \times \text{GL}_{n_-}
\]

with the action of \( \sigma \) given by

\[
\sigma : \text{GL}_{n_i} \times \text{GL}_{n_i} \rightarrow \text{GL}_{n_i} \times \text{GL}_{n_i}
\]

\[
(g, h) \mapsto (\sigma_0(h), \sigma_0(g)),
\]

for all \( i \in \Lambda_1 \), and similarly for \( j \in \Lambda_2 \), where \( \sigma_0(g) = J_i'g^{-1}J_i^{-1} \), with \( (J_i')_{ab} = \delta_{a,n_i+1-b} \), for any \( i \) or \( j \).

The action induced by \( t \sigma \) on each \( \text{GL}_{n_i} \times \text{GL}_{n_i} \) coincides with that of \( \sigma \). If \( n \) is even, and \( \hat{G} = \hat{G}_r \), the action induced by \( t \sigma \) on \( \text{GL}_{n_+} \) and \( \text{GL}_{n_-} \) are respectively the automorphisms associated to \( J_r \) or \( J'_r \) defined in §4.1.1. It follows that in \( \hat{G} = \hat{G}_r \),

\[
L_0 := C_G(t \sigma) \cong \text{Sp}_{n_+}(k) \times \text{O}_{n_-}(k) \times \prod_{i \in \Lambda_1} \text{GL}_{n_i}(k) \times \prod_{j \in \Lambda_2} \text{GL}_{n_j'}(k).
\]

The case of \( \hat{G} = \hat{G}_r \) and that of odd \( n \) are similar. \( \square \)

6.2.2. Let us introduce a sign \( \eta \in \{\pm 1\} \) that can be \(-1 \) only if

- \( n \) is even and \( n_- > 0 \), or
- \( n \) is odd and \( n_+ > 0 \),

or rather, if the orthogonal factor of the centraliser is non trivial.

Proposition 6.2.2. The quasi-semi-simple conjugacy classes in \( G^F, \sigma \) are parametrised by the data

\[
\{(\eta, \hat{\tau}) \subset \{\pm 1\} \times \hat{\Xi}_{C,s}.
\]

Proof. Clear. \( \square \)

To simplify the notation, we will write \( \eta \hat{\tau} \) instead of \( (\eta, \hat{\tau}) \). One should be careful however as to which \( G^F \)-class corresponds to \( \eta = 1 \) in this parametrisation. This is explained below.

If the centraliser of a semi-simple element has two connected components, then the corresponding two \( G^F \)-classes can be distinguished by the homomorphism

\[
\text{GL}_n(q) \times<\sigma> \longrightarrow \mathbb{F}_q/(\mathbb{F}_q^*)^2 \longrightarrow \mu_2.
\]

The first map sends \( g \in G(q) \) to \( \text{det}(g) \mod (\mathbb{F}_q^*)^2 \) and sends \( \sigma \) to 1, and the second map is the nontrivial homomorphism. The value of \( \eta \) is identified with the image of the corresponding \( G^F \)-class under this homomorphism. In fact, the above homomorphism is the only nontrivial
character of GL_n(q).<σ> that is non-vanishing on GL_n(q).σ, and extends the character η o det of GL_n(q), with η being the order 2 irreducible character of \( \mathbb{F}_q \). This explains the notation η.

To see that the above homomorphism can distinguish the two \( G^F \)-conjugacy classes contained in the same \( G \)-conjugacy class, we argue as follows. Let \( tσ \in G^F.σ \) be such that \( C_G(tσ) \) has two connected component. Then according to our concrete description of \( C_G(tσ) \), its two connected components are distinguished by the values of the determinant, which is ±1, corresponding to the two connected components of the orthogonal factors. Let \( g \in G \) be such that \( g^{-1}F(g) = z \in C_G(tσ) \setminus C_G(tσ)^o \), then \( gσg^{-1} \) is a representative of another \( G^F \)-conjugacy class. Applying the determinant to the equality \( g^{-1}F(g) = z \) gives \( \det(g)g^{-1} = -1 \), so that \( \det(g)^2 \in \mathbb{F}_q^* \setminus (\mathbb{F}_q^*)^2 \); applying the above homomorphism to the element \( gσg^{-1} \) gives \( \det(s) \det(g)^2 \), whence the claim.

### 6.2.3. The centraliser of each quasi-semi-simple element of \( G^F.σ \) is given as below. Let \( tσ \in G^F.σ \) be a quasi-semi-simple element corresponding to

\[ ηd_+d_-(n_j, \hat{λ}_j)_{i∈Λ_1}(n_j', \hat{λ}_j')_{j∈Λ_2}.\]

If \( n \) is even, then its centraliser in \( G^F \) is

\[ Π \overset{Gl}{ѓ} \times \overset{Gl^+}{O^-}_n(q) × \prod_{i∈Λ_1} GL_n(q^{d_i}) × \prod_{j∈Λ_2} GL_n^-(q^{d_j}).\]

If \( n \) is odd, then its centraliser in \( G^F \) is

\[ Π \overset{Gl}{ѓ} \times \overset{Gl^+}{O^-}_n(q) × \prod_{i∈Λ_1} GL_n(q^{d_i}) × \prod_{j∈Λ_2} GL_n^-(q^{d_j}).\]

Note that for odd \( n_+ \), \( O^+_n(q) \) is isomorphic to \( O^-_n(q) \).

### 6.2.4. We refer to \[1.3.3\] for the parametrisation of the unipotent classes of finite classical groups. Let \( C \) be a semi-simple \( G^F \)-conjugacy class corresponding to

\[ ηI = ηd_+d_-(n_j, \hat{λ}_j)_{i∈Λ_1}(n_j', \hat{λ}_j')_{j∈Λ_2}.\]

For odd \( n \), the \( G^F \)-classes which have \( C \) as semi-simple part are parametrised by

\[ Λ^{d_+}_nA^{d_-}_n(λ_i)_{i∈Λ_1}(λ'_j)_{j∈Λ_2},\]

where \( Λ^{d_+}_n ∈ \Psi_{n_+}^{PP} \), \( Λ^{d_-}_n ∈ \Psi_{n_+}^{ort,+} \), \( λ_i ∈ \mathcal{P}_n \), \( λ'_j ∈ \mathcal{P}_n' \).

For even \( n \), the \( G^F \)-classes which have \( C \) as semi-simple part are parametrised by

\[ Λ^{d_+}_nA^{d_-}_n(λ_i)_{i∈Λ_1}(λ'_j)_{j∈Λ_2},\]

where \( Λ^{d_+}_n ∈ \Psi_{n_+}^{PP} \), \( Λ^{d_-}_n ∈ \Psi_{n_+}^{ort,+} \), \( λ_i ∈ \mathcal{P}_n \), \( λ'_j ∈ \mathcal{P}_n' \).

These data can equivalently described as follows. Recall that each element of \( \Psi^{PP} \) (resp. \( \Psi^{ort,+} \)) is associated with a partition of symplectic type (resp. orthogonal type). For any
\[ \Lambda \in \Psi^p \sqcup \Psi^{prt, \pm}, \text{ denote by } |\Lambda| \text{ the size of the corresponding partition. For any } \eta \in \{\pm\} \text{ and any triple } f = (\Lambda^\varepsilon, \Lambda^o, f^\circ) \text{ with } \eta \in \{\pm\}, \Lambda^\varepsilon \in \Psi^p, \Lambda^o \in \Psi^{prt, \eta}, \text{ and } f^\circ : \Phi^o \to \mathcal{P}, \] being a map of sets, write
\[ ||f|| = |\Lambda^\varepsilon| + |\Lambda^o| + \sum_{x \in \Phi^o} 2|f(x)| \cdot |x|. \]

Finally, put \( \mathfrak{X}_C = \{(\eta, f); ||f|| = n\} \) (with \( f \) and \( -f \) identified if \( |\Lambda^\varepsilon| = 0 \)). This set parametrises the set of \( G^F \)-conjugacy classes in \( G^F \). Given any such 4-tuple \( f = (\eta, \Lambda^\varepsilon, \Lambda^o, f^\circ) \), the semi-simple part of the corresponding conjugacy class, described by \( \Phi^o \to \mathbb{Z}_{\geq 0} \), is obtained by taking the sizes of \( \Lambda^\varepsilon, \Lambda^o \) and the images of \( f^\circ \).

### 7. Shintani Descent

Now \( G \) denote a connected reductive group over \( k \).

#### 7.1. Eigenvalues of the Frobenius

In this part, we collect some results on the eigenvalues of the Frobenius endomorphism acting on the intersection cohomology of the Deligne-Lusztig variety \( X_\bar{w} \). We will write \( X_{w,F} \) if it is necessary to specify the Frobenius that is involved.

Recall that \( X_\bar{w} \) is the subvariety of the flag variety \( \mathcal{B} \) consisting of the Borel subgroups \( B \) such that \((B,F(B))\) are conjugate to \((B_0, \bar{w}B_0\bar{w}^{-1})\) by \( G \), where \( \bar{w} \in G \) is a representative of \( w \in W_G \) and \( B_0 \) is some fixed \( F \)-stable Borel subgroup.

#### 7.1.1. The Deligne-Lusztig character \( R_{\Gamma_w}^G 1 \)

is realised by the virtual representation
\[ \bigoplus_i (-1)^i H_i^\ell(X_{\bar{w}}, \bar{\mathbb{Q}}_\ell). \]

Recall that the Lusztig series \( \mathcal{E}(G^F, (1)) \) consists of the irreducible representations that appear as a direct summand of some \( R_{\Gamma_w}^G 1 \), or equivalently of a vector space \( H_i^\ell(X_{w,F}, \bar{\mathbb{Q}}_\ell) \). Denote by \( H^\ell(X_{w,F}, \bar{\mathbb{Q}}_\ell) \) the intersection cohomology of \( X_{\bar{w}} \). By [L84b Corollary 2.8], each element of \( \mathcal{E}(G^F, (1)) \) is also an irreducible \( G^F \)-subrepresentation of \( H^\ell(X_{w,F}, \bar{\mathbb{Q}}_\ell) \) for some \( i \) and \( \bar{w} \). We write

\[ \begin{align*}
M_i(w,F) &:= H^\ell(X_{w,F}, \bar{\mathbb{Q}}_\ell) \\
H_i(w,F) &:= \text{End}_{G^F}(M_i(w,F))
\end{align*} \]

If \( w = 1 \), \( M_i(1,F) \) is just the \( \ell \)-adic cohomology of \( X_1 \) and its simple factors are the principal series representations, which are in bijection with the irreducible representations of \( H_i(1,F) \). If \( F \) is split, \( H_i(1,F) \cong \bar{\mathbb{Q}}_\ell[W] \), and if \( F \) is twisted by a graph automorphism \( \sigma \), \( H_i(1,F) \cong \bar{\mathbb{Q}}_\ell[W^\sigma] \).

In what follows, we fix the Frobenius \( F \) and \( w \in W \), and write \( M_i \) and \( H_i \) instead of \( M_i(w,F) \) and \( H_i(w,F) \).

#### 7.1.2. Denote by \( F_0 \) a split Frobenius over \( \bar{F}_{q^u} \) and denote by \( F \) the Frobenius defining the \( \bar{F}_q \)-structure of \( G \). Assume that some power of \( F_0 \) is a power of \( F \). Let \( b \) be the smallest integer such that \( F_0^b \) is a power of \( F \). In the case that interests us, \( b = 1 \) or \( 2 \). Note that \( F \) and \( F_0 \)
commute. The action of $F_0$ on $\mathcal{B}$ induces an isomorphism of $M_i$ as a vector space. Moreover $F_0$ induces by conjugation an algebra automorphism of $H_i$, still denoted by $F_0$, which is unipotent ([L84b Theorem 2.18]). Let $\rho$ be an irreducible representation of $G^F$ that appears in $M_i$. Denote by $M_{i,\rho}$ the isotypic component corresponding to $\rho$. By [L84b Proposition 2.20], the action of $F_0$ respects the isotypic decomposition, i.e. $F_0(M_{i,\rho}) = M_{i,\rho}$. The algebra $H_i$ is decomposed into some simple algebras $H_{i,\rho} = \text{End}_{G^F}(M_{i,\rho})$.

7.1.3. Now assume that $\rho$ is fixed. Denote by $[\rho]$ the vector space on which $G^F$ acts by the representation $\rho$. There exists a $\mathbb{Q}_\ell$-space $V$ such that $M_{i,\rho} \cong [\rho] \otimes V$ and that $H_{i,\rho} \cong \text{End}_{\mathbb{Q}_\ell}(V)$. Since $H_{i,\rho}$ is a simple algebra, we have $F_0 = \phi_G \otimes \phi_H$ with $\phi_G \in \mathbb{Q}_\ell[G^F]$ and $\phi_H \in H_{i,\rho}$, which are invertible as $F_0$ is. Consider the adjoint representation,

$$GL(V) \longrightarrow GL(H_{i,\rho})$$

$$\phi_H \longmapsto \text{ad} \phi_H.$$  

Since $\text{ad} \phi_H = \text{ad} F_0$, which is unipotent, we see that $\phi_H$ is a unipotent endomorphism up to a scalar. Modifying $\phi$ if necessary, we can assume that $\phi_H$ is unipotent. Choose a basis $\{e_1, \ldots, e_s\}$ of $V$ in such a way that in this basis $\phi_H$ is triangular. Each $M_{i,\rho,\tau} = [\rho] \otimes e_\tau$ provides the representation $\rho$. By [L84b Proposition 2.20], the representation $\rho : G^F \rightarrow GL(M_{i,\rho,\tau})$ is $F_0$-stable and extends into a representation of $G^F.\langle F_0 \rangle$, denoted by $\bar{\rho}$, with $F_0^b$ acting trivially on $G^F$. The action of $F_0$, regarded as an element of $G^F.\langle F_0 \rangle$, on $M_{i,\rho,\tau}$ is defined by $(\lambda'_\rho)^{-1}q^{-i/2}e_\tau$, where $\lambda'_\rho$ is a root of unity. Another choice of $\bar{\rho}$ corresponds to a multiple of $\lambda'_\rho$ by a $b$-th root of unity. The value of $\lambda'_\rho$ only depends on $\rho$ and a choice of $\bar{\rho}$, and does not depend on $w$ or $i$. In other words, $F_0 = \bar{\rho} \otimes q'$, where $q'$ is a unipotent endomorphism multiplied by $\lambda'_\rho q'^{i/2}q'$. 

7.1.4. If we consider a Frobenius $F'_0$ that is not necessarily split, it may happen that the action of $F'_0$ does not respect the isotypic components of $M_i$. However, we can nevertheless consider those components that are preserved by $F'_0$. In fact, a component $M_{i,\rho}$ is $F'_0$-stable if and only if the character $\theta_\rho \in \text{Irr}(H_i)$ associated is $F'_0$-stable. Let $M_{i,\rho}$ be such a component, we still have $M_{i,\rho} \cong [\rho] \otimes V$ and $\rho$ extends into a representation of $G^F.\langle F'_0 \rangle$.

7.1.5. Now we consider the action of $F_0$ on the $\ell$-adic cohomology. By [S85 Lemma 1.4], the eigenvalues of $F_0$ on $H_i(X_{\overline{\mathbb{Q}}_\ell})$ are $\lambda'_\rho$ times a power of $q'^{ib/2}$ which is not necessarily $q^{ib/2}$. Let $\mu = \lambda'_\rho q^{ib/2}$ be such an eigenvalue. Then, the subspace $M_{i,\rho,\mu} \subset M_{i,\rho}$ of eigenvalue $\mu$ is $F_0$-stable and there exists a decomposition $M_{i,\rho,\mu} \cong [\rho] \otimes V_\mu$ such that the action of $F_0$ on $M_{i,\rho,\mu}$ is decomposed as $\bar{\rho} \otimes q'_\mu$ where $q'_\mu$ is $\lambda'_\rho q'^{ib/2}$ times a unipotent endomorphism of $V_\mu$. Once again, $\lambda'_\rho$ only depends on $\rho$ and a choice of $\bar{\rho}$.

7.1.6. There are two particular cases that interest us.
Theorem 7.1.1. ([L77 Theorem 3.34]) Let $i \in \mathbb{Z}$ and $w \in W_G$ be arbitrary. If $(G,F)$ is of type $A_n$, $n \geq 1$, then all of the eigenvalues of $F$ on $H^i_c(X_w,F_2)$ are powers of $q$. If $(G,F)$ is of type $^2A_n$, $n \geq 2$, then all of the eigenvalues of $F^2$ on $H^i_c(X_w,F_2)$ are powers of $(-q)$.

7.2. Shintani Descent.

7.2.1. Let $F_1$ and $F_2$ be two commuting Frobenius endomorphism. Denote by $\mathcal{K}(G^{F_1}, F_2)$ the $F_2$-conjugacy classes of $G^{F_1}$ and by $\mathcal{K}(G^{F_2}, F_1)$ the $F_1$-conjugacy classes of $G^{F_2}$, and we denote by $C(G^{F_2}, F_1)$ and $C(G^{F_1}, F_2)$ the set of functions that are constant on the $F_1$-conjugacy classes of $G^{F_2}$ and the functions that are constant on the $F_2$-conjugacy classes of $G^{F_1}$ respectively.

Define a map $N_{F_1/F_2} : \mathcal{K}(G^{F_1}, F_2) \to \mathcal{K}(G^{F_2}, F_1)$ as follows. For $g \in G^{F_1}$, there exists $x \in G$ such that $xF_2(x^{-1}) = g$. Then $g' := x^{-1}F_1(x) \in G^{F_2}$, and its $F_1$-conjugacy class is well defined. This defines a bijection from $N_{F_1/F_2} : \mathcal{K}(G^{F_1}, F_2) \to \mathcal{K}(G^{F_2}, F_1)$. We write $g' = N_{F_1/F_2}(g)$ by abus of notation. Denote by $\text{Sh}_{F_2/F_1} : C(G^{F_2}, F_1) \to C(G^{F_1}, F_2)$ the induced bijection. It is easy to check that $\text{Sh}_{F_2/F_1} \circ \text{Sh}_{F_1/F_2} = \text{Id}$ and that $\text{Sh}_{F_1/F}$ is an involution that may not be the identity.

7.3. Action on the Irreducible Characters.

7.3.1. Let $U$ be a unipotent character of $G^{F_1}$ that extends to $G^{F_1, <F_2>}$. We denote the restriction to $G^{F_1, F_2}$ of its extension by $E_{F_2}(U) \in C(G^{F_1}, F_2)$. The Shintani descent sends it into $C(G^{F_2}, F_1)$. On the other hand, the unipotent irreducible characters of $G^{F_2}$ that extends to $G^{F_1, F_2}$, which we denote by $E(C(G^{F_2}, (1))^{F_1})$, have as extensions some elements of $C(G^{F_2}, F_1)$. We will see that the functions $\text{Sh}_{F_1/F_2} E_{F_2}(U)$ can be expressed as linear combinations of the extensions of the elements of $E(C(G^{F_2}, (1))^{F_1})$.

7.3.2. Let $B$ be an $F_1$-stable and $\sigma$-stable Borel subgroup. Put

\[ (3.2.3.1) \quad H := \text{End}_{G^{F_1}}(\text{Ind}_{B^{F_1}} G^{F_1} 1) \cong \text{End}_{G^{F_1}}(\hat{Q}_1[B^{F_1}]) \cong \hat{Q}_1[W^{F_1}]. \]

The irreducible characters of $H$ are in bijection with the principal series representations of $G^{F_1}$. For $\psi \in \text{Irr}(H)$, denote by $U_\psi \in \text{Irr}(G^{F_1})$ the corresponding character. By $\text{[7.1.4]}$ $U_\psi$ extends to $G^{F_1, <F_2>}$ if $\psi$ is $F_2$-stable, in which case $\psi$ itself extends to $G^{F_1, <F_2>}$ in such a way that the action of $F_2$ on $\text{Ind}_{B^{F_1}} G^{F_1} 1$ is decomposed into $F_2 = E_{F_2}(U_\psi)(F_2) \otimes E_{F_2}(\psi)(F_2)$, where we denote by $E_{F_2}(U_\psi)$ and $E_{F_2}(\psi)$ the extensions of $U_\psi$ and $\psi$ respectively.

7.3.3. Let $\rho \in \text{Irr}(G^{F_2})$ be unipotent. The $\rho$-isotypic component of $H^i_c(X_w,F_2)$ is of the form $[\rho] \otimes V$. The action of the split Frobenius $F_0$ on this component can be written as $\rho(F_0) \otimes \varphi$, where $\varphi$ is $\lambda'_{\rho}$ times a power of $q^{1/2}$ and a unipotent endomorphism, according to $\text{[7.1.5]}$. We denote by $\Omega_{F_2}$ the isomorphism of the space $C(G^{F_2}, (1))$ that multiplies $\rho$ by $\lambda'_{\rho}$. Denote by $E_{F_0}(\rho)$ the restriction of $\rho$ to $G^{F_2, F_0}$.
7.3.4. Fix the split Frobenius $F_0$ and the order 2 quasi-central automorphism $\sigma$. In what follows, we only consider $(F_1, F_2) = (\sigma F_0^m, \sigma_2 F_0)$, where $m \in \mathbb{Z}_{>0}$ and $\sigma_i = 1$ or $\sigma$. Take $\rho \in \mathcal{E}(G^F, (1)^{\nu_1})$, i.e. a $\sigma_1$-stable representation, and denote by $E_{\sigma_1}(\rho)$ an extension of $\rho$ to $G^{F_2}$. Since $F_0$ acts as $\sigma_2^{-1}$ on $G^F$, we can define the extension $E_{F_0}(\rho)(F_0)$ to be an extension $E_{\sigma_1}(\rho)(\sigma_2^{-1})$, which commutes with $E_{\sigma_1}(\rho)(\sigma_1)$ because either one of $\sigma_1$ and $\sigma_2$ is 1 or they are equal. This allows us to define an extension $E_{F_1}(\rho)$ of $\rho$ to $G^{F_2} : F_1$ by requiring $E_{F_1}(\rho)(\sigma_1 F_0^m) = E_{\sigma_1}(\rho)(\sigma_1) E_{F_0}(\rho)(F_0^m)$. It is well defined. In addition, $E_{\sigma_1}$ defines an isomorphism of vector spaces

$\mathcal{Q}_l[\mathcal{E}(G^F, (1)^{\nu_1})] \to C(G^F, \sigma_1, (1))$

$\rho \mapsto E_{\sigma_1}(\rho)$.

7.3.5. The following theorem makes explicit the transition matrix.

**Theorem 7.3.1.** ([DM94 Théorème 5.6]) We keep the above notations. For any $m \in \mathbb{Z}_{>0}$, and any $\psi \in \text{Irr}(W^{\sigma_1})^{\sigma_2}$ we have

$\text{Sh}_{\sigma_1, F_0^m/\sigma_2 F_0} E_{\sigma_2 F_0}(U_{\psi}) = \sum_{\rho \in \mathcal{E}(G^{F_0^m}, 1)^{\nu_1}} \langle R_{\psi}^{G^F, \sigma_1}, E_{\sigma_1}(\rho) \rangle_{G^{F_0^m}, \sigma_1} \lambda^m_\rho E_{F_0^m}(\rho)$

(7.3.5.1)

$= E_{\sigma_1, \sigma_2^{-m}}(Q_{\sigma_2 F_0}^{m, \sigma_1} E_{\sigma_1}(R_{\psi}^{G^{F_0^m}, \sigma_1}))$.

(See [DM94 Définition 5.1] for the definition of $R_{\psi}^{G^{F_0^m}, \sigma_1}$ or more generally in 9.2.2.2)

7.4. Commutation with the Deligne-Lusztig Induction.

7.4.1. The following proposition due to Digne shows how the Deligne-Lusztig induction commutes with Shintani Descent.

**Proposition 7.4.1.** ([D1 Proposition 1.1]) Let $G$ be a connected reductive group defined over $\mathbb{F}_q$, equipped with the Frobenius endomorphism $F$ and let $\sigma$ be a quasi-central automorphism of $G$. Let $L \subset P$ be an $F$-stable and $\sigma$-stable Levi factor of a $\sigma$-stable parabolic subgroup. Then

$\text{Sh}_{\sigma F/F} \circ R_{L^{F, \sigma^{-1}}}^{G^{F, \sigma^{-1}}} = R_{L^{F, \sigma^{-1}}}^{G^{F, \sigma}} \circ \text{Sh}_{\sigma F/F}$.

8. Character Sheaves

In this section, $G$ denotes a not necessarily connected reductive group. By local system, we mean a local system of $\mathcal{O}_f$-vector spaces. If $X$ is a variety over $k$, we denote by $\mathcal{D}(X)$ the bounded derived category of constructible $\mathcal{O}_f$-sheaves on $X$. For any $g \in G$, denote by $g_s, g_u$ the Jordan decomposition of $g$, with $g_s$ being semi-simple and $g_u$ unipotent.

8.1. Character Sheaves on Groups Not Necessarily Connected.
8.1.1. If $G^{1}$ is a connected component of $G$, define

$$Z^{o}_{G^{o}, G^{1}} := Cz^{o}_{G^{o}}(g)^{o}, \quad \text{for any } g \in G^{1}.$$ 

It does not depend on the choice of $g \in G^{1}$. An isolated stratum of $G^{1}$ is an orbit of isolated elements under the action of $Z^{o}_{G^{o}, G^{1}} \times G^{o}$ given by

$$(z, x) : g \longmapsto zgx^{-1}.$$ 

(See \cite{L03} I, §1.21 (d), §3.3 (a))

**Example 8.1.1.** For the group $\tilde{G}$ defined in §4.2.2, we have $Z^{o}_{G^{o}, G, o} = \{1\}$, and so an isolated stratum of $G, o$ is an isolated G-conjugacy class.

Given a stratum $S$, denote by $S(S)$ the category of local systems on $S$ invariant under the action of $Z^{o}_{G^{o}, G^{1}} \times G^{o}$ given by

$$(z, x) : g \longmapsto z^{n}xgx^{-1},$$

for some integer $n > 0$. We refer to \cite{L03} I, §6 for the definition of cuspidal local system (for $G$). If $E \in S(S)$ is cuspidal, we say that $(S, E)$ is a cuspidal pair (for $G$).

8.1.2. Let $S$ be an isolated stratum of $G$ contained in some connected component $G^{1}$, let $E \in S(S)$, and let $S_{s}$ be the set of semi-simple parts of the elements of $S$. We assume that $Z^{o}_{G^{o}, G^{1}} = \{1\}$ until the lemma below.

Let $s \in S_{s}$ be an arbitrary element and put

$$U_{s} := \{u \in C_{G}(s) \mid u \text{ is unipotent such that } su \in S\}.$$ 

Then any $C_{G}(s)^{o}$-conjugacy class in $U_{s}$ is a stratum of $C_{G}(s)$ (cf. \cite{L03} §6.5). Note that the map $U_{s} \to S, u \mapsto su$, defines an embedding.

**Lemma 8.1.2.** (\cite{L03} Lemma 6.6) The pair $(S, E)$ is cuspidal for $G$ if and only if for any $C_{G}(s)^{o}$-conjugacy class $S' \subset U_{s}$, the pair $(S', E|_{S'})$ is cuspidal for $C_{G}(s)$.

8.1.3. Let $L$ be a Levi subgroup of $G^{o}$ and let $S$ be an isolated stratum of $N_{G}(L, P)$ for a parabolic subgroup $P$ with Levi factor $L$. (See \cite{L03} I, §2.2 (a), §3.5) Define,

$$S_{reg} = \{g \in S \mid C_{G}(g_{s})^{o} \subset L\}.$$ 

Define

$$Y_{L,S} = \bigcup_{x \in G^{o}} xS_{reg}x^{-1},$$ 

and

$$\hat{Y}_{L,S} = \{(g, xL) \in G \times G^{o}/L \mid x^{-1}gx \in S_{reg}\},$$

equipped with the action by $G^{o}$, $h : (g, xL) \mapsto (hgh^{-1}, hxL)$, and

$$\hat{Y}_{L,S} = \{(g, x) \in G \times G^{o} \mid x^{-1}gx \in S_{reg}\},$$
equipped with the action by $G^\circ \times L$, $(h, l) : (g, x) \mapsto (hgh^{-1}, hxl^{-1})$. Consider the morphisms
$$\tilde{\mathcal{Y}}_{L,S} \xleftarrow{\alpha} Y_{L,S} \xrightarrow{\beta} Y_{L,S},$$
where $\alpha(g, x) = x^{-1}gx$, $\beta(g, x) = (g, xL)$ and $\pi(g, xL) = g$. Put
$$W_S := \{ n \in N_{G^\circ}(L) \mid nSn^{-1} = S \},$$
and $W_S^\circ = \tilde{W}_S/L$. It is a finite group. Then $\beta$ is a principal $L$-bundle and $\pi$ is a principal $W_S$-bundle. (See [L03, I, §3.13]) If $E \in S(S)$ is irreducible and cuspidal for $N_G(L, P)$, put
$$\tilde{W}_E := \{ n \in \tilde{W}_S \mid \text{ad}(n)^\ast E \cong E \},$$
and $W_E = \tilde{W}_E/L$.

**Example 8.1.3.** Let $G = \text{GL}_n(k)$ and $\tilde{G} = G \rtimes \langle \sigma \rangle$ as in §3.2.2. Let $T$ be the maximal torus consisting of diagonal matrices and $B$ the Borel subgroup consisting of upper triangular matrices. Let $P$ be a $\sigma$-stable standard parabolic subgroup with respect to $B$, such that the unique Levi factor $L$ containing $T$ is isomorphic to $\text{GL}_m(k) \times (k')^{2N}$ for some non negative integers $m$ and $N$. Put $\tilde{L} = N_{\tilde{G}}(L, P) = L \cup L_\sigma$ and let $S \subset L_\sigma$ be an isolated stratum. Then $N_G(L)/L \cong \mathbb{Z}_{2N}$ and $W_S \cong \mathbb{I}_N$. We consider $W_E$ for some particular $E$ in the following.

Let $S' = S/Z_{\tilde{L}, L, \sigma}$ and let $\pi' : S \to S'$ be the projection. Let $M = L/[L, L]$ and $\tilde{M} = L/([L, L])$. Then $\text{ad} \sigma$ induces an action on $M$. Define the map $\mathcal{L}_\sigma : M \to M$, $m \mapsto m^{-1} \sigma(m)$. Then put $L'' = \tilde{M}/\text{Im}(\mathcal{L}_\sigma)$ and let $\pi'' : \tilde{L} \to L''$ be the projection. Let $S''$ be the image of $S$ under $\pi''$. Suppose that $E' \in S(S')$ is irreducible and $L \in S(S'')$ satisfies $L^\circ \cong \mathbb{Q}_L$. Consider the local system $E := \pi''^\ast E' \otimes \pi''^\ast L$ on $S$, which lies in $S(S)$ by [L03, §5.3]. Note that $S''$ is a connected component of $\tilde{L}''$, and since the identity component $L''$ is central, the equivariant local systems on $S''$ can be identified with those on $L''$, which is a torus isomorphic to $(k')^N$.

The action of $W_S$ on $L$ induces an action on $L''$. Let $T_+ \subset L''$ be the largest $W_E$-stable subtorus such that $L|_{T_+} \cong \mathbb{Q}_T$. Let $T_- \subset L''$ be the unique $W_E$-stable subtorus such that $L'' \cong T_+ \times T_-$. Let $N_+ = \dim T_+$ and $N_- = \dim T_-$, then $N_+ + N_- = N$. Since the factor $\pi''^\ast E'$ is automatically $W_S$-invariant, we have $W_E \cong \mathbb{I}_{N_+} \times \mathbb{I}_{N_-}$.

8.1.4. Fix $E \in S(S)$. There exists a $G^\circ$-equivariant local system $\tilde{E}$ on $\tilde{Y}_{L,S}$ such that $\beta^\ast \tilde{E} \cong \alpha^\ast E$. Denote by $E = \text{End}(\pi_\tilde{E})$, the endomorphism algebra of $\pi_\tilde{E}$. We have a canonical decomposition ([L03, II, §7.10 (a); IV, §21.6])
$$E = \bigoplus_{w \in W_S} E_w,$$
where the factors $E_w := \text{Hom}(\text{ad}(n_w)^\ast E, E)$, each one defined by some representative $n_w$ of $w$, are of dimension 1 and satisfy $E_w E_v = E_{wv}$. Choose base $\{ b_w \mid w \in W_S \}$ of $E$ with $b_w \in E_w$ for any $w$.

Define
$$K = \text{IC}(\tilde{Y}_{L,S}, \pi_\tilde{E}),$$
where $\tilde{Y}_{L,S}$ is the closure of $Y_{L,S}$ in $G$. There exists a canonical isomorphism $E \cong \text{End}(K)$. Let $\Lambda'$ be a finite set parametrising the isomorphism classes of the irreducible representations of $E$ and for each $i \in \Lambda'$, we denote by $V_i$ a corresponding representation. Then, we have the canonical decompositions

$$\pi_1 \tilde{E} \cong \bigoplus_{i \in \Lambda'} V_i \otimes (\pi_1 \tilde{E})_i, \quad K \cong \bigoplus_{i \in \Lambda'} V_i \otimes K_i,$$

where

$$(\pi_1 \tilde{E})_i = \text{Hom}_E(V_i, \pi_1 \tilde{E}), \quad K_i = \text{Hom}_E(V_i, K)$$

are the simple factors. Moreover, $K_i \cong \text{IC}(\tilde{Y}_{L,S}, (\pi_1 \tilde{E})_i)$.

8.1.5. Assume that $F(L) = L$, $F(S) = S$ and $F^*E \simeq E$, where $F$ is the Frobenius of $G$. We fix an isomorphism $\phi_0 : F^*E \simeq E$. It induces an isomorphism $\tilde{\phi} : F^*\pi_1 \tilde{E} \simeq \pi_1 \tilde{E}$ and an isomorphism $\phi : F^*K \simeq K$. Recall that, given a variety $X/k$ equipped with the Frobenius $F$, a complex $A \in D(X)$ and an isomorphism $\phi : F^*A \simeq A$, the characteristic function $\chi_{A,\phi} : X^F \to \tilde{Q}_\ell$ is defined by

$$(8.1.5.1) \quad \chi_{A,\phi}(x) = \sum_{i \in \mathbb{Z}} (-1)^i \text{Tr}(\phi, H^i_x A),$$

where $H^i_x A$ is the stalk at $x$ of the cohomology sheaf in degree $i$ of $A$.

**Theorem 8.1.4.** ([L03], III, Theorem 16.14, §16.5, §16.13]) Let $s$ and $u \in G^F$ be a semi-simple element and a unipotent element such that $su = us \in \tilde{Y}_{L,S}$. Then,

$$(8.1.5.2) \quad \chi_{K,\phi}(su) = \sum_{h \in G^F, \ h^{-1}\text{sh} S_h} |L^F_h| \frac{[L^F_h]}{|C_G(s)^0|} Q_{L_h, C_G(s)^0, \epsilon_h, F_h, \phi_h}(u),$$

where $S_s$ is the set of the semi-simple parts of the elements of $S$, and $Q_{L_h, C_G(s)^0, \epsilon_h, F_h, \phi_h}$ is the generalised Green function (See §8.1.6 below) associated to the data $L_h, C_G(s)^0, \epsilon_h, F_h, \phi_h$ defined by

- $L_h := hLh^{-1} \cap C_G(s)^0$;
- $\epsilon_h := \{v \in C_G(s) \mid v \text{ unipotent, } h^{-1}svh \in S\}$;
- $F_h$: inverse image of $E$ under the embedding $\epsilon_h \to S, v \mapsto h^{-1}svh$;
- $\phi_h : F^*F_h \simeq F_h$, an isomorphism induced from $\phi_0$ under the above embedding.

Denote by $L^1$ (resp. $G^1$) the connected component of $N_G(L)$ (resp. $G$) containing $S$. If we define

- $\Sigma_h := hZ^0_L, h^{-1}\epsilon_h$;
- $\Sigma_h$: inverse image of $C_L(s)$ by the embedding $\Sigma_h \to S, v \mapsto h^{-1}svh$;
- $\phi_h^* : F^*E_h \simeq E_h$ an isomorphism induced from $\phi_0$ under the above embedding,

then the above embedding $\epsilon_h \to S$ factors through the inclusion $\epsilon_h \to \Sigma_h$, $F_h$ is the inverse image of $E_h$ and $\phi_h$ is induced from $\phi_h^*$ under this inclusion. The point is that, $\Sigma_h$ is a finite
union of isolated strata, which has \( \zeta_h \) as the subset of unipotent elements, so that these data fit into the following definition of generalised Green functions.

8.1.6. Generalised Green Functions. Given

- \( G \) a reductive algebraic group,
- \( L \subset G^\circ \) an \( F \)-stable Levi subgroup,
- \( \Sigma^u \) the set of the unipotent elements of a finite union of isolated strata \( \Sigma \) of \( N_G(L) \) satisfying \( F(\Sigma) = \Sigma, F(\Sigma^u) = \Sigma^u \),
- \( \mathcal{F} \) an \( L \)-equivariant local system on \( \Sigma^u \), and
- \( \phi : F^* \mathcal{F} \simeq \mathcal{F} \) an isomorphism,

we choose a local system \( \mathcal{E} \) on \( \Sigma \) that restricts to \( \mathcal{F} \) under the inclusion \( \Sigma^u \to \Sigma \) and an isomorphism (which always exists) \( \phi' : F^* \mathcal{E} \simeq \mathcal{E} \) that induces \( \phi_1 \). Define \( K = IC(Y_{L,\Sigma}, \pi_1 \mathcal{E}) \), where \( Y_{L,\Sigma}, \tilde{Y}_{L,\Sigma} \) and \( \pi : \tilde{Y}_{L,\Sigma} \to Y_{L,\Sigma} \) are defined by the procedure \([8.1.3]\) and denote by \( \phi : F^* K \simeq K \) the isomorphism induced from \( \phi' \).

The generalised Green function associated to \( G, L, \Sigma^u, \mathcal{F} \) and \( \phi_1 \), denoted by \( Q_{L,G,\Sigma^u,\mathcal{F},\phi_1} \), is defined by \(([L03] \text{ III, } \S 15.12)\)

\[
G^F_u = \{ \text{unipotent elements of } G^F \} \to \mathbb{Q}_\ell
\]

\[
u \mapsto \chi_{K,\phi}(\nu).
\]

It does not depend on the choice of \( \mathcal{E} \) and \( \phi' \).

8.1.7. Let \( G, L, \Sigma^u, \mathcal{F}, \mathcal{E}, \phi_1 \) and \( \phi'_1 \) be as in \([8.1.6]\) and let \( Q^G_L(-,-) \) be the two variable Green function as in \([3.2.3]\).

Theorem 8.1.5. \(([L90] \text{ Theorem 1.14})\) There exists a constant \( q_0 > 1 \) which only depends on the Dynkin diagram of \( G \), such that if \( q > q_0 \), then for any \( u \in G^F_u \),

\[
Q_{L,G,\Sigma^u,\mathcal{F},\phi_1}(u) = (-1)^{\dim \Sigma} |L^F|^{-1} \sum_{\nu \in L^F_u} Q^G_L(u, \nu) \chi_{\mathcal{E},\phi'_1}(\nu).
\]

In the context of Theorem 8.1.4 the isomorphism \( \phi_h \) is induced from \( \phi_0 \) via the embedding \( \zeta_h \to S \), \( \nu \mapsto h^{-1} s v h \), and so \( \chi_{\mathcal{E},\phi'_1}(\nu) = \chi_{\mathcal{E},\phi_0}(h^{-1} s v h) \). The above theorem then says

\[
Q_{L_h,\Sigma^u,\mathcal{F},h,\phi_1}(u) = (-1)^{\dim \Sigma} |L^F|^{-1} \sum_{\nu \in (L^F_u)^h} Q^G_{L_h}(u, \nu) \chi_{\mathcal{E},\phi_0}(h^{-1} s v h).
\]

Comparing Proposition 3.2.3 and Theorem 8.1.4, this shows that \( \chi_{K,\phi} \) is equal to \( R^{G^1}_{L_1} \chi_{\mathcal{E},\phi_0} \) up to a sign.

8.1.8. The isomorphism \( \phi \) of \([8.1.5]\) induces an algebra isomorphism \( \iota : \mathcal{E} \simeq \mathcal{E} \). There exists a subset \( \Lambda \subset \Lambda' \) and some isomorphisms \( \iota_i : V_i \simeq V_i, \phi_i : F^* K_i \simeq K_i \), for \( i \in \Lambda \), such that the isomorphism \( b_{\iota,\phi} : F^* K \simeq K \), with respect to the decomposition \( K = \bigoplus_{i \in \Lambda'} V_i \otimes K_i \), is of the form
8.1.10. Write classes ([L03, IV, §20.4]), and write the isomorphism
(8.1.8.1) \( \phi \) is the contragradient representation.

8.1.9. Take a representative \( n_w \in \mathcal{W}_E \) of \( w \), and an element \( g_w \in G \) such that \( g_w^{-1}F(g_w) = n_w \). Define \( L_w = g_w L g_w^{-1}, S_w = g_w S g_w^{-1} \) and \( E_w = \text{ad}(g_w^{-1})^* E \). Then \( L_w \) and \( S_w \) are \( F \)-stable and the isomorphism \( \phi_0 : F^*E \cong E \) induces an isomorphism \( \phi_{0,w} : F^*E_w \cong E_w \). These allow us to define \( Y_{L_w,S_w}, \tilde{Y}_{L_w,S_w}, \pi_w : \tilde{Y}_{L_w,S_w} \to Y_{L_w,S_w}, \mathcal{E}_w, K_w \) and \( \phi_w : F^*K_w \cong K_w \) by the same procedure. It can be checked that (See [L03, IV, §21.6])

(8.1.9.1) \( \chi_{K,b_{w},\phi} = \chi_{K_w,\phi_w}. \)

8.1.10. Write \( \mathcal{W} = \mathcal{W}_E \) denote by \( \mathcal{W} \) a set of representatives of the effective \( F \)-conjugacy classes ([L03, IV, §20.4]), and write \( \mathcal{W}_w = \{ v \in \mathcal{W} \mid F^{-1}(v)wv^{-1} = w \} \). If \( w \) is not in some effective \( F \)-conjugacy class and \( i \in \Lambda \), then \( \text{Tr}(b_{w_i}t_i, V_i) = 0 \) ([L03, §20.4 (a)]). We have for all \( i, j \in \Lambda \) ([L03, IV, §20.4 (c)])

(8.1.10.1) \( \sum_{w \in \mathcal{W}_w} |\mathcal{W}_w|^{-1} \text{Tr}(b_{w_i}t_i, V_i) \text{Tr}(t_j^{-1}b_w^{-1}, V_j) = \delta_{ij}, \)

where \( \delta_{ij} = 1 \) if \( i = j \) and \( \delta_{ij} = 0 \) otherwise. In fact, \( |\mathcal{W}| = |\Lambda| \) and \( (\text{Tr}(b_{w_i}t_i, V_i))_{i \in \Lambda, w \in \mathcal{W}} \) is an invertible square matrix ([L03] §20.4 (e), (f), (g)).

This, combined with the equalities (8.1.8.1) and (8.1.9.1), gives

(8.1.10.2) \( \chi_{K,o,\phi} = \frac{1}{|\mathcal{W}_E|} \sum_{w \in \mathcal{W}_w \text{ effective}} \text{Tr}(b_{w_i}t_i, V_i^\vee) \chi_{K_w,\phi_w} \)

where \( V_i^\vee \) is the contragradient representation.

Example 8.1.6. Let \( \mathcal{G}, L, S, \) and \( \mathcal{E} \) be as in Example 8.1.3. Let \( K_i \) be the character sheaf corresponding to the irreducible representation \( V_i^\vee \) and let \( \phi_i \) be chosen in such a way that (8.1.10.2) holds. Since the action of \( F \) on \( \mathcal{W}_E \) is trivial, we may require that \( t_i = 1d \) and \( b_w \) is the element \( w \) in the group algebra of \( \mathcal{W}_E \). Then

\[
\chi_{K_i,\phi_i} = \frac{1}{|\mathcal{W}_E|^r} \sum_{w \in \mathcal{W}_w} \text{Tr}(w, V_i^\vee) R_{L \times \mathcal{O}}^{\mathcal{G}} \chi_{L \times \mathcal{O}, \phi_0,\phi}. \]

9. Extensions of \( \sigma \)-Stable Characters

9.1. Some Elementary Lemmas. When dealing with a Levi subgroup \( L \) of \( \text{GL}_n \), one often regards it as a direct product of smaller \( \text{GL}_{n'} \)'s and reduces the problem to these direct factors. However, if \( \sigma \) is an automorphism of \( L \), then \( L \rtimes \langle \sigma \rangle \) is not actually the direct product of
groups of the form $\text{GL}_{q'} \rtimes \langle \sigma' \rangle$. We give some lemmas that allow us to apply arguments in the same spirit. Let $H$ denote a finite group in this part, which could either be a finite group of Lie type or a Weyl group. Let $\sigma$ be an automorphism of $H$, which could be induced from the automorphism of an algebraic group or the Frobenius. Denote by $H.\langle \sigma \rangle$ the semi-direct product of $H$ and the cyclic group generated by $\sigma$, with the generator acting as $\sigma$ on $H$.

**Lemma 9.1.1.** Let $H = H_1 \times \cdots \times H_s$ be a product of finite groups and let $\sigma = \sigma_1 \times \cdots \times \sigma_s$ be the product of some automorphisms of the direct factors. Then $H.\langle \sigma \rangle$ is a subgroup of $\prod_i H_i.\langle \sigma_i \rangle$.

Moreover, if the characters $\chi_i \in \text{Irr}(H_i)$ extend to $\bar{\chi}_i \in \text{Irr}(H_i.\langle \sigma_i \rangle)$, then $\bar{\chi} := (\bar{\chi}_i)_{\mid H.\langle \sigma \rangle}$ is irreducible and restricts to $\mathcal{S}_i \bar{\chi}_i \in \text{Irr}(H)$.

**Proof.** We define a map by

$$H.\langle \sigma \rangle \longrightarrow \prod_i H_i.\langle \sigma_i \rangle$$

(9.1.0.1)

$$(h_1, \ldots, h_s)\sigma^i \longmapsto (h_1\sigma_1^i, \ldots, h_s\sigma_s^i),$$

which is obviously a homomorphism and is injective. Then the assertion on $\bar{\chi}$ is immediate. □

We define an exterior tensor product that is "twisted" by $\sigma$.

**Definition 9.1.2.** Let $H = H_1 \times H_2$ be a product of finite groups and $\sigma = \sigma_1 \times \sigma_2$ a product of automorphisms. For $i = 1$ and 2, let $f_i$ be a function on $H_i.\sigma_i$ that is invariant under the conjugation by $H_i$. The function $f_1 \cong f_2$ on $H.\sigma$ is defined as the restriction of

$$f_1 \cong f_2 = (\text{pr}_1 f_1 \cdot \text{pr}_2 f_2) \in C(H_1.\langle \sigma_1 \rangle \times H_2.\langle \sigma_2 \rangle)$$

to $H.\langle \sigma \rangle$.

**Lemma 9.1.3.** Let $H = K \times \cdots \times K$ be the direct product of $d$ copies of a finite group $K$. Let $\psi$ be an automorphism of $K$, let $\zeta = (i_1, \ldots, i_d) \in \mathcal{S}_d$ be a circular permutation, and let $(n_1, \ldots, n_d)$ be a $d$-tuple of integers. With these data, we can define an automorphism $\Psi$ of $H$ by

$$\Psi : H \longrightarrow H$$

(9.1.0.2)

$$(k_1, \ldots, k_d) \longmapsto (\psi^{n_1}(k_{\zeta(1)}), \ldots, \psi^{n_d}(k_{\zeta(d)})).$$

Denote by $\mathcal{H}$ the direct product $K \rtimes \langle \psi \rangle \times \cdots \times K \rtimes \langle \psi \rangle$, and let $\zeta$ act by permuting the components:

$$\zeta : (k_1, \ldots, k_d) \longmapsto (k_{\zeta(1)}, \ldots, k_{\zeta(d)}), \quad k_i \in K \rtimes \langle \psi \rangle,$$

Let $\chi$ be a $\psi$-stable irreducible character of $K$ and denote by $\bar{\chi}$ an extension of $\chi$ to $K \rtimes \langle \psi \rangle$. Then,

(i) $H \rtimes \langle \psi \rangle$ is a subgroup of $\mathcal{H}$;

(ii) The character $\bar{\chi} \otimes \cdots \otimes \bar{\chi}$ of $\mathcal{H}$ extends to a character of $\mathcal{H} \rtimes \langle \zeta \rangle$. Its restriction $\bar{\chi}$ to $H \rtimes \langle \psi \rangle$ is irreducible;

(iii) For all $h = (k_1, \ldots, k_d) \in H$, we have

$$\bar{\chi}(h\Psi) = \bar{\chi}(k_1 \psi^{n_1} k_2^{n_2} \psi \cdots k_d \psi^{n_d}).$$
Proof. For each $i \in \{1, \ldots, d\}$ and $r \in \mathbb{Z}_{>0}$ put $n_i(r) = \sum_{0 \leq s \leq r-1} n_i(s)$. Define a map

$$H. \langle \psi \rangle \longrightarrow K \rtimes \langle \psi \rangle \times \cdots \times K \rtimes \langle \psi \rangle \rtimes \langle \zeta \rangle$$

(9.1.0.3)

$$(k_1, \ldots, k_d) \mapsto (k_1, \ldots, k_d)$$

$$(k_1, \ldots, k_d) \Psi^r \mapsto (k_1 \psi^{n_1(r)}, \ldots, k_d \psi^{n_d(r)} \zeta^r).$$

One can verify that it is an injective group homomorphism. The character $\bar{\chi}$ is irreducible as its restriction to $H$ is so.

Let us compute the value of $\bar{\chi}$. Let $\rho : K \to \text{GL}(V)$ be a representation that realises the character $\chi$. Let $\tilde{\rho}$ denote its extension to $K. \langle \psi \rangle$. Then $V^{\otimes d}$ is a representation of $H$, defining the action of $\zeta$ by $\psi_1 \otimes \cdots \otimes \psi_d \mapsto \psi_{\zeta(1)} \otimes \cdots \otimes \psi_{\zeta(d)}$. We use an argument of linear algebra. Take $A^{(i)}$, $\ldots$, $A^{(d)} \in \text{GL}(V)$ and let $\zeta$ act on $V^{\otimes d}$ as above. Then we have

$$\text{Tr}(A^{(1)} \otimes \cdots \otimes A^{(d)} \circ \zeta | V \otimes \cdots \otimes V) = \text{Tr}(A^{(i)} \cdots A^{(i)}) | V).$$

We conclude the proof by taking $\tilde{\rho}(k_i \psi^{n_i})$ for $A^{(i)}$. \hfill \square

9.2. Uniform Extensions.

9.2.1. Denote by $L$ the algebraic group defined over $\mathbb{F}_q$

$$L = \prod_{i \in \Lambda_1} (\text{GL}_{n_i} \times \text{GL}_{n_i})^{d_i} \times \prod_{i \in \Lambda_2} (\text{GL}_{n_i} \times \text{GL}_{n_i})^{d_i}$$

for some finite sets $\Lambda_1$ and $\Lambda_2$, endowed with the Frobenius $F$ acting on it as $F_w$ in (5.2.3.6) and (5.2.3.7), and with the automorphism $\sigma$ acting on it as in (5.2.3.9). Let $T \subset L$ be an $F$-stable and $\sigma$-stable maximal torus and we wirte $W_L := W_L(T)$. For all $i \in \Lambda_1 \cup \Lambda_2$, we write $L_i := (\text{GL}_{n_i} \times \text{GL}_{n_i})^{d_i}$, and denote by $T_i$ the corresponding direct factor of $T$. Write $W_i := W_{L_i}(T_i)$. Then we have

$$W_i \cong (\mathbb{Z}_{n_i} \times \mathbb{Z}_{n_i})^{d_i}, \quad W_i^\sigma \cong \mathbb{Z}_{n_i}^{d_i}$$

for all $i \in \Lambda_1 \cup \Lambda_2$. These are some direct factors of $W_L$ and of $W_L^\sigma$ that are stable under $F$.

Define an injection

$$\text{Irr}(W_L^\sigma)^F \hookrightarrow \text{Irr}(W_L)^F$$

(9.2.1.3)

$$\varphi \mapsto [\varphi]$$

in the following manner.

For each $i \in \Lambda_1$, we have the bijections

$$\text{Irr}(W_i)^F \cong \mathcal{P}_{n_i} \times \mathcal{P}_{n_i}, \quad \text{Irr}(W_i^\sigma)^F \cong \mathcal{P}_{n_i}.$$ 

We define $\text{Irr}(W_i^\sigma)^F \to \text{Irr}(W_i)^F$ to be sending $\varphi$ to $(\varphi, \varphi)$.

For each $i \in \Lambda_2$, we have

$$\text{Irr}(W_i)^F \cong \mathcal{P}_{n_i}, \quad \text{Irr}(W_i^\sigma)^F \cong \mathcal{P}_{n_i}.$$ 

(9.2.1.5)
We define \( \text{Irr}(W_L^F) \to \text{Irr}(W_L^F) \) to be sending \( \varphi \) to \( \varphi \).

For any \( \varphi \in \text{Irr}(W_L^F) \), we denote by \( \tilde{\varphi} \) an extension of \( \varphi \) to \( W_L \times <F> \) and denote by \( [\varphi] \) an extension of \( \varphi \) to \( W_L \times <F> \), where \( F \) is regarded as an automorphism of finite order of the Weyl group. Eventually, these choices need to be specified.

Denote by \( \text{Irr}^\sigma(L^F) \) the set of the \( \sigma \)-stable linear characters of \( L^F \). For any \( \varphi \in \text{Irr}^\sigma(L^F) \), we denote by \( \tilde{\varphi} \) an extension of \( \varphi \) to \( W_L \times <F> \) and denote by \( \tilde{[\varphi]} \) an extension of \( [\varphi] \) to \( W_L \times <F> \), where \( F \) is regarded as an automorphism of finite order of the Weyl group. Eventually, these choices need to be specified.

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\[ \tilde{\varphi} : \text{Irr}(L^F) \to \text{Irr}(L^F) \]

Denote by \( \tilde{\varphi} \) its trivial extension to \( L^F \times <\sigma> \). We also denote by the same letter the restriction of \( \tilde{\varphi} \) to \( L^F \).

\[ \tilde{\varphi} : \text{Irr}(L^F) \to \text{Irr}(L^F) \]

9.2.2. Given \( \varphi \in \text{Irr}(W_L^F) \) and \( \theta \in \text{Irr}^\sigma(L^F) \), by Theorem \( \Box \), there is a particular choice of the extension \( \tilde{\varphi} \) such that

\[ \chi_1 := R_{[\varphi]}^L \theta = |W_L|^{-1} \sum_{w \in W_L} [\varphi](wF)R_{[\varphi]}^L \theta. \]

is a character. Obviously, it is a \( \sigma \)-stable character of \( L^F \). Denote by \( \tilde{\chi}_1 \) an extension of \( \chi_1 \) to \( L^F \times <\sigma> \).

For any choice of the extension \( \tilde{\varphi} \), put

\[ R_{[\varphi]}^L \tilde{\varphi} = |W_L^\sigma|^{-1} \sum_{w \in W_L^\sigma} \tilde{\varphi}(wF)R_{[\varphi]}^L \tilde{\varphi}. \]

It is an \( L^F \)-invariant function on \( L^F \times <\sigma> \).

**Theorem 9.2.1.** For a particular choice of the extension \( \tilde{\varphi} \), we have

\[ \tilde{\chi}_1|_{L^F \times <\sigma>} = \pm R_{[\varphi]}^L \tilde{\varphi}. \]

We will prove this theorem in the following section.

9.3. **The Proof.** The proof is to reduce the problem to smaller and smaller factors of \( L \), until we can apply the known results on \( \text{GL}_{n'}(q) \), for various \( n' \). The choice of the extension \( \tilde{\varphi} \) will also be reduced to the smaller components until the choices are clear.

9.3.1. **Reduction to the Unipotent Characters.** Let \( \tilde{\chi}_1 = R_{[\varphi]}^L \theta \) be an irreducible character of \( L^F \), which is necessarily \( \sigma \)-stable. Denote by \( \tilde{\chi}_1 \in \text{Irr}(L^F \times <\sigma>) \) such that \( \tilde{\chi}_1|_{L^F} = R_{[\varphi]}^L \theta \). Assume that for some choice of \( \tilde{\varphi} \),

\[ \tilde{\chi}_1|_{L^F \times <\sigma>} = R_{[\varphi]}^L \tilde{\varphi} \]

where we denote by \( 1 \) the trivial extension of the trivial character. Since \( (R_{[\varphi]}^L \theta) \otimes \theta = R_{[\varphi]}^L \theta \) and \( (R_{[\varphi]}^L \tilde{\varphi}) \otimes \tilde{\varphi} = R_{[\varphi]}^L \tilde{\varphi} \), we have

\[ (\tilde{\chi}_1 \otimes \tilde{\varphi})|_{L^F} = R_{[\varphi]}^L \theta, \quad (\tilde{\chi}_1 \otimes \tilde{\varphi})|_{L^F \times <\sigma>} = R_{[\varphi]}^L \tilde{\varphi}. \]

So it suffices to prove the theorem for the unipotent characters.
9.3.2. Reduction with Respect to the Action of $F$ and $\sigma$. We have decomposed $L$ into a product of the $L_i$'s for $i \in \Lambda_1 \cup \Lambda_2$, each one being $F$-stable and $\sigma$-stable. Let us show that it suffices to prove the theorem for the $L_i$'s.

Write $F = (F_i)_{i \in \Lambda_1 \cup \Lambda_2}$ and $\sigma = (\sigma_i)_{i \in \Lambda_1 \cup \Lambda_2}$ where for each $i$, $F_i$ and $\sigma_i$ are respectively a Frobenius and an automorphism of the corresponding direct factor. The given $\varphi$ can be written as $(\varphi_i)_{i \in \Lambda_1 \cup \Lambda_2}$ with $\varphi_i \in \text{Irr}(W_i^{\varphi_i}),$ then $[\varphi] = ([\varphi_i])$. Suppose that $R_{W_i^{\varphi_i},1}^{L_i^{\varphi_i}}$ are some irreducible characters, denoted by $\chi_i$, each one being $\sigma$-stable, and they extend to $\tilde{\chi}_i \in \text{Irr}(L_i^{\varphi_i},<\sigma_i>)$. We will show that if for some choices of the extensions $\tilde{\varphi}_i \in \text{Irr}(W_i^{\varphi_i},<F_i>)$, the following equality holds

$$\tilde{\chi}_i |_{L_i^{\varphi_i},1} = R_{W_i^{\varphi_i},1}^{L_i^{\varphi_i}},$$

then there is some choice of $\tilde{\varphi} \in \text{Irr}(W_{L_i}^{\varphi_i},<F>)$ such that

$$\tilde{\chi}_1 |_{L_{F_i},1} = R_{W_{L_i}^{\varphi_i},1}^{L_i^{\varphi_i}}.$$  

Given the extensions of the factors $\tilde{\chi}_i$, we can obtain an extension $\tilde{\chi}_1$ following Lemma 9.1.1 By definition, for any $l \sigma \in L_{F_i,1}$ which can be identified with $\prod_i l_i \sigma_i \in \prod_i L_i^{\varphi_i},<\sigma_i>$, we have,

$$\tilde{\chi}_1(l \sigma) = \prod_i R_{W_i^{\varphi_i},1}^{L_i^{\varphi_i}}(l_i \sigma_i).$$

On the one hand,

$$\prod_i R_{W_i^{\varphi_i},1}^{L_i^{\varphi_i}}(l_i \sigma_i) = \prod_i \text{Tr}(l_i \sigma_i | H_c^*(X_{w_i}))$$

$$= \text{Tr}(\prod_i l_i \sigma_i | \bigotimes_i H_c^*(X_{w_i}))$$

$$= \text{Tr}(l \sigma | H_c^*(X_w)) = R_{W_{L_i}^{\varphi_i},1}^{L_i^{\varphi_i}}(l \sigma).$$

where the $T_{w_i}$ and $T_w$ are defined with respect to $T_i$ and $T$.

On the other hand, applying Lemma 9.1.1 to the Weyl groups and the Frobenius, we obtain an extension $\tilde{\varphi}$, such that for any $w = \prod_i w_i \in W_{L_i}^{\varphi_i}$, we have,

$$\tilde{\varphi}(wF) = (\prod_i \tilde{\varphi}_i) |_{W_{L_i}^{\varphi_i}}(\prod_i w_i F_i) = \prod_i \tilde{\varphi}_i(w_i F_i).$$

Consequently, $\tilde{\chi}_1 |_{L_{F_i},1} = R_{W_{L_i}^{\varphi_i},1}^{L_i^{\varphi_i}}$. (One may also check that $\boxtimes_i R_{W_i^{\varphi_i},1}^{L_i^{\varphi_i}} = R_{W_{L_i}^{\varphi_i},1}^{L_i^{\varphi_i}}$ with some similar but simpler arguments.)

9.3.3. The Linear Part, I. In this part we fix $i \in \Lambda_1$. Write $M = \text{GL}_{n_i} \times \text{GL}_{n_i}$, equipped with the Frobenius

$$F_M : (g,h) \mapsto (F_0(g), F_0(h))$$

with $F_0$ being the standard Frobenius of $\text{GL}_n$ and the automorphism

$$\tau(g,h) = (\sigma_0(h), \sigma_0(g))$$
of the form \([9.1.3]\) with \(\sigma_0\) commuting with \(F_0\). Then \(L_i = M \times \cdots \times M\) is a direct product of \(d_i\) copies of \(M\) equipped with the automorphism \(\sigma = \tau \times \cdots \times \tau\) consisting of \(d_i\) copies of \(\tau\). We will fix \(i\) and write \(d = d_i\). We may assume that the maximal torus \(T_i \subset L_i\), which is \(F_i\)-stable and \(\sigma_i\)-stable, is of the form \(T_M \times \cdots \times T_M\), where \(T_M \subset M\) is an \(F_M\)-stable and \(\tau\)-stable maximal torus. Note that \(F_M\) acts trivially on \(W_M := W_M(T_M)\). The Frobenius \(F_i\) acts on \(L_i\) in the following manner

\[
M \times \cdots \times M \rightarrow M \times \cdots \times M
\]

\[(m_1, \ldots, m_d) \mapsto (F_M(m_d), F_M(m_1), \ldots, F_M(m_{d-1})).\]

We have a natural commutative diagram

\[
\begin{array}{ccc}
\text{Irr}(W_i^\tau) & \longrightarrow & \text{Irr}(W_i^\sigma) \\
\downarrow & & \downarrow \\
\text{Irr}(W_i^\tau) & \longrightarrow & \text{Irr}(W_i^\tau)
\end{array}
\]

where the upper horizontal bijective map \(\varphi_i \mapsto \varphi_i = (\varphi_M, \ldots, \varphi_M)\) identifies each element of \(\text{Irr}(W_i^\sigma)\) with \(d\) identical copies of an element of \(\text{Irr}(W_i^\tau)\). Denote by \([\varphi_M]\) and \([\varphi_i]\) the images of the vertical maps defined as in \((9.2.1.3)\), which are matched under the lower horizontal map.

Endow \(M\) with the Frobenius \(F_M^d\). Suppose that for some choice of \(\tilde{\varphi}_M\),

\[
R_{\varphi_M}^{M, \tau} 1 = |W_M^\tau|^{-1} \sum_{w \in W_i^\tau} \tilde{\varphi}_M(wF_M^d) R_{\varphi_i}^{M, \tau} 1
\]

where \(T_w\) is defined with respect to \(T_M\), is an extension of the irreducible character \(R_{[\varphi_M]}^M 1\) of \(M_{T_M}^d\). Let us show that

\[
R_{\varphi_i}^{L_i, \sigma_1} 1 = |W_i^\sigma_1|^{-1} \sum_{w \in W_i^\sigma_1} \tilde{\varphi}_i(wF_i) R_{\varphi_i}^{L_i, \sigma_1} 1
\]

is an extension of the irreducible character \(R_{[\varphi_i]}^{L_i, \sigma_1} 1\) of \(L_i^F\). In fact, there is a natural isomorphism \(M_{T_M}^d \cong L_i^F\) compatible with the action of \(\tau\) and \(\sigma\). We are going to show that \(R_{\varphi_i}^{M, \tau} 1\) coincides with \(R_{\varphi_i}^{L_i, \sigma_1} 1\) under this isomorphism and they are an extension of the same character of \(M_{T_M}^d = L_i^F\) corresponding to \([\varphi] = [\varphi_i]\).

Applying Lemma \(9.1.3\) to \(K = W_i^\tau, H = W_i^\sigma, \psi = F_M\), and \(\zeta = (d, \ldots, 2, 1) \in \Sigma_d\), we deduce from \(\tilde{\varphi}_M\) an extension \(\tilde{\varphi}_i\) such that for \(w = (w_1, \ldots, w_d) \in (W_i^\tau)^d \cong W_i^{\sigma_1},

\[
\tilde{\varphi}_i(wF_i) = \tilde{\varphi}_M(w_1F_Mw_{d-1}F_M \cdots w_1F_M)
\]

\[
= \tilde{\varphi}_M(w_dF_Mw_{d-1} \cdots w_1F_M).
\]

Write \(w'_1 = w_dw_{d-1} \cdots w_1\). Then, \(w\) is \(F_{L_i}\)-conjugate to \(w' = (w'_1, 1, \ldots, 1)\), so for any \(l \in L_i^F\), we have

\[
R_{[\varphi_i]}^{L_i, \sigma_1} 1(l\sigma) = R_{[\varphi_i]}^{L_i, \sigma_1} 1(l\sigma) = \text{Tr}(l\sigma|H_i^*(X_{w'}))
\]
We can write \( l = (m, F_M(m), \ldots, F_M^{d-1}(m)) \) with \( m \) satisfying \( F^d_M(m) = m \). Since the two varieties
\[
X_{w'} = \{ B \in B_L | (B, F_i(B)) \in O(w') \},
\]
\[
X_{w'_1} = \{ B \in B_M | (B, F_M(B)) \in O(w'_1) \}
\]
are isomorphic, and the actions of \( l \sigma_i \) and of \( m \tau \) on the two varieties are compatible, we have
\[
\text{Tr}(l \sigma_i | H^*_i(X_{w'})) = \text{Tr}(m \tau | H^*_i(X_{w'_1})) = R_{W^*_M}^{M \tau, \sigma_i} 1 (m \tau),
\]
Consequently, the value of \( \phi_i(wF_i)R_{W^*_M}^{L_i, \sigma_i} 1 \) only depends on \( w'_1 \in W^*_M \) and is equal to \( \phi_M(w' F_M)R_{W^*_M}^{M \tau, \sigma_i} 1 \), This, together with the fact that \( |W^*_M| = |W^*_M| \), shows that \( R_{W^*_M}^{M \tau} = R_{W^*_M}^{L_i, \sigma_i} 1 \).
(Similar arguments show that \( R_{[\varphi]}^{M} 1 = R_{[\varphi]}^{L_i} 1 \) .)

9.3.4. **The Unitary Part, I.** In this part we require that \( i \in \Lambda_2 \). We keep the same notations as above except that \( F_i \) acts on \( L_i \) in the following manner
\[
M \times \cdots \times M \longrightarrow M \times \cdots \times M
\]
(9.3.4.1)
\[
(m_1, \ldots, m_d) \longmapsto (F'_M(m_d), F_M(m_1), \ldots, F_M(m_{d-1})),
\]
where
\[
F'_M : (g, h) \mapsto (F_0(h), F_0(g)).
\]
Denote by \( F''_M : M \rightarrow M \) the Frobenius
\[
F''_M : (g, h) \mapsto (F'_0(h), F'_0(g)).
\]
We still have a natural identification
\[
\text{Irr}(W^*_M)^{F''_M} \longrightarrow \text{Irr}(W^*_M)^{F_i}
\]
(9.3.4.2)
\[
\varphi \mapsto \varphi_{F_i} = (\varphi, \ldots, \varphi)
\]
\( (F'_M \) acts trivially on \( \text{Irr}(W^*_M) \) since \( \sigma_0 \) induces an inner automorphism on the Weyl group) and an isomorphism \( M^{F''_M} \cong L_i^F \). In a way similar to \( 9.3.3 \) from the equality \( (9.3.3.3) \), with \( F_M^d \) replaced by \( F''_M \), one deduces the equality \( (9.3.3.4) \), with \( F_i \) defined in the present setting.

9.3.5. From now on we write \( M = G \times G, G = GL_n^\epsilon(q) \) with \( \epsilon = \pm \). Denote by \( F_0 \) the split Frobenius of \( G \) and by \( F'_0 \) the Frobenius of \( G \) corresponding to \( \epsilon \), and denote by \( \sigma_0 \) an order 2 automorphism of \( G \), which commutes with the Frobenius endomorphisms.

9.3.6. **The Linear Part, II.** It is essential that we allow \( G \) to be \( GL_n^\epsilon(q) \), which will be applied to the unitary part later. Define
\[
\sigma : M \rightarrow M \quad F : M \rightarrow M
\]
\[
(g, h) \mapsto (\sigma_0(h), \sigma_0(g)) \quad (g, h) \mapsto (F'_0(g), F'_0(h)).
\]
Let \( \chi_G \) be a unipotent irreducible character of \( G_F^0 \) corresponding to some \( \varphi \in \text{Irr}(W_G) \), it defines a character \( \chi_M = \chi_G \otimes \chi_G \in \text{Irr}(M^F) \) which is invariant under the action of \( \sigma \) and so extends to \( M^F. \langle \sigma \rangle \), denoted by \( \check{\chi}_M \). Every \( \sigma \)-stable irreducible unipotent character of \( M^F \) is of the form \( \chi_G \otimes \chi_G \). Regarding \( \varphi \) as a character of \( W_M^\sigma \), we show that up to a sign,

(9.3.6.1) \[
\check{\chi}_M|_{M^F.\sigma} = R_{\varphi}^{M.\sigma} 1 := |W_M^\sigma|^{-1} \sum_{w \in W_M^\sigma} \check{\varphi}(wF) R_{T_w.\sigma}^{M.\sigma} 1,
\]

for some choice of the extension \( \check{\varphi} \).

We apply Lemma 9.1.3 by taking \( G_F^0 \) for \( K \) and obtain

\[
\check{\chi}_M((g, h)\sigma) = \chi_G(g\sigma_0(h))
\]

for any \((g, h) \in M^F\). By Theorem 1, the irreducible unipotent character of \( G_F^0 \) can be expressed as

\[
|W_G|^{-1} \sum_{w \in W_G} \check{\varphi}_G(wF') R_{T_w}^G 1,
\]

for some choice of \( \check{\varphi}_G \). The extension \( \check{\varphi} \) is then defined by \( \check{\varphi}_G \) under the isomorphism \( W_G \cong W_{M'}^\sigma \) noticing that the action of \( F \) is compatible with the action of \( F_0^\sigma \) under this isomorphism. Comparing this expression with \( R_{\varphi}^{M.\sigma} 1 \), we are reduced to show that for any \((g, h) \in M^F\)

(9.3.6.2) \[
R_{T_w.\sigma}^{M.\sigma} 1((g, h)\sigma) = R_{T_w}^G 1(g\sigma_0(h))
\]

with \( w_M = (w, \sigma_0(w)) \in W_M^\sigma \). Observe that \((g, h)\sigma \mapsto R_{T_w}^G 1(g\sigma_0(h)) \) defines a function on \( M^F.\sigma \) invariant under the conjugation by \( M^F \).

9.3.7. Let us prove a more general assertion. Let \( I \) be an \( F_0^\sigma \)-stable Levi subgroup of \( G \). Then \( J := I \times \sigma_0(I) \) is a \( \sigma \)-stable Levi factor of a \( \sigma \)-stable parabolic subgroup of \( M \), which justifies the functor \( R_{J_{<\sigma>}}^{M_{<\sigma>}} \). Let \( \chi_I \) be an irreducible character of \( I_F^0 \), it defines a character \( \chi_I = \chi_I \otimes \sigma_0(\chi_I) \in \text{Irr}(I^F) \) which is invariant under the action of \( \sigma \) and thus extends to \( I^F. \langle \sigma \rangle \). Denoted by \( \check{\chi}_I \) a choice of such extension. The following lemma with \( I = T_w \) and \( \chi_I = 1 \) proves the above assertion.

**Lemma 9.3.1.** We keep the notations as above and write \( \bar{I} = I.\langle \sigma \rangle \). Assume that for any \((g', h') \in I^F \), we have \( \check{\chi}_I((g', h')\sigma) = \chi_I(g' \sigma_0(h')) \). Then, for any \((g, h) \in M^F \), we have

(9.3.7.1) \[
R_{J_{<\sigma>}}^{M_{<\sigma>}} \check{\chi}_I((g, h)\sigma) = R_{T_w}^G \chi_I(g\sigma_0(h)).
\]

**Proof.** Let \((g, h)\sigma = \zeta \mu \) be the Jordan decomposition, with \( \zeta \) semi-simple and \( \mu \) unipotent, and we write \( \mu = (u, w) \) and \( \zeta = (s, t)\sigma \). Beware that neither \( s \) nor \( t \) is necessarily semi-simple. Also let \( g\sigma_0(h) = \tilde{s}u \) be the Jordan decomposition, with \( \tilde{s} \) semi-simple and \( \tilde{u} \) unipotent. The
proof will simply be comparing the following two formulas term by term.

\[ R^M_{L,<\sigma>} \chi_j ((g, h) \sigma) = |F|^{-1} |C_M(\zeta)^{\sigma^F}|^{-1} \sum_{\{x \in M^F\}} \sum_{x \chi x^{-1} \in I \sigma} Q_{C_{\chi^{-1}}(\zeta), \mu, \sigma^{-1}}^{|C_M(\zeta)^{\sigma^F}|} \chi_j (\zeta \sigma) \]

\[ R^G_{L} \chi_I (g \sigma_0 (h)) = |F|^F \chi_I (g \sigma_0 (h)) = |F|^F |C_G(\xi)^{\sigma^F}|^{-1} \sum_{\{y \in C_G(\xi)^{\sigma^F}\}} \sum_{y \sigma y^{-1} \in I \sigma} Q_{C_{\xi^{-1}}(\xi), \mu, \sigma^{-1}}^{|C_G(\xi)^{\sigma^F}|} \chi_I (g \sigma_0 (h)). \]

An element \((z_1, z_2) \in M\) commutes with \((s, t) \sigma\) if and only if \(z_2 = t \sigma_0(z_1) s^{-1}\) and \(z_1 = s \sigma_0(z_2) s^{-1}\), if and only if \(z_1 \in C_G(\sigma_0(t))\) and \(z_2 = t \sigma_0(z_1) s^{-1}\), whence an isomorphism \(C_M((s, t) \sigma) \cong C_G(\sigma_0(t))\). An element of \(M. <\sigma>\) is semi-simple if and only if its square is semi-simple since the characteristic is odd. The equality \((s, t) \sigma^2 = (s \sigma_0(t), t \sigma_0(s))\) shows that \(s \sigma_0(t)\) is semi-simple. The unipotent part \((u, w)\) commutes with \((s, t) \sigma\), so \(u \in C_G(\sigma_0(t))\). Considering the equality

\[(g \sigma_0(h), h \sigma_0(g)) = ((g, h) \sigma)^2 = ((s, t) \sigma)^2 (u, w)^2 = (s \sigma_0(t), t \sigma_0(s))(u^2, w^2),\]

we have \(g \sigma_0(h) = s \sigma_0(t) u^2\). This gives the Jordan decomposition of \(g \sigma_0(h)\) because \(u\) commutes with \(s \sigma_0(t)\) and \(u^2\) is unipotent. Therefore, \(s = s \sigma_0(t), \bar{u} = u^2\) and \(|C_M(\zeta)^{\sigma^F}| = |C_G(\xi)^{\sigma^F}|\).

Write \(x = (x_1, x_2) \in M^F\). The condition \(x \zeta x^{-1} \in I \sigma\) means that \(x_1 \sigma_0(x_2^{-1}) \in I\) and that \(x_2 t \sigma_0(x_1^{-1}) \in \sigma_0(I)\), which implies that \(x_1 \sigma_0(t) x_1^{-1} \in I\). Fix \((x_1, x_2)\) satisfying \(x \zeta x^{-1} \in I \sigma\), then every element of \((x_1, \sigma_0(\xi^F) x_2)\) also satisfies it. Let \((x_1, x_2)\) and \((x_1, x'_2)\) be two elements satisfying \(x \zeta x^{-1} \in I \sigma\), then the conditions \(x_1 \sigma_0(x_2^{-1}) \in I\) and \(x_1 \sigma_0(x_2^{-1}) \in I\) implies that \(x_2 x_2^{-1} \in \sigma_0(\xi^F)\). We thus obtain a bijection of sets

\[ \{x \in M^F | x \zeta x^{-1} \in I \sigma\} \cong \{y \in C_G(\xi)^{\sigma^F} | y \sigma y^{-1} \in I\} \times I \sigma, \]

with \(y\) corresponding to the factor \(x_1\) of \(x\). We will see that the sum over \(x\) in the character formula is invariant under multiplying \(x_2\) by an element of \(\sigma_0(I \sigma)\) on the left, which cancels a factor \(|F|^F|\) from \(|F|^F|\). Then

\[ (9.3.7.2) \quad x \zeta x^{-1} = (x_1 \sigma_0(x_2) \sigma_0(x_2)^{-1}, x_2 t \sigma_0(t \sigma_0(v_1) \sigma_0(x_1)^{-1})). \]

Taking into account the equality \(v_2 = t \sigma_0(v_1) s^{-1}\), we have

\[ (9.3.7.3) \quad \left( x_1 \sigma_0(x_2) \sigma_0(x_2)^{-1} \right) \sigma_0 \left( x_2 t \sigma_0(t \sigma_0(v_1) \sigma_0(x_1)^{-1}) \right) = x_1 \sigma_0(t) v_2^2 \sigma_0 (x_1)^{-1}. \]

By assumption, \(\bar{\chi}_j ((g', h') \sigma) = \chi_I (g' \sigma_0(h'))\), for any \((g', h') \in I \sigma\), whence

\[ (9.3.7.4) \quad x \bar{\chi}_j (\xi \sigma v) = \chi_I (x_1 \sigma_0(t) v_2^2 \sigma_0 (x_1)^{-1}) = x_1 \bar{\chi}_I (\xi \sigma v). \]
where we also see that multiplying $x_2$ by an element of $\sigma_0(F^0)$ on the left does not change the value. Since $v \mapsto v^2$ defines a bijection of $C_{p^{-1}A}^\sigma$ into itself, it only remains to show the first of the following equalities of Green functions

\[(9.3.7.5) \quad Q_{\sigma_0(F^0)}^{(\phi)}(u^2, v^{-1}) = Q_{\sigma_0(F)}^{(\phi)}(u, v^{-1}) = Q_{\sigma_0(F^0)}^{(\phi)}(\mu, v^{-1}),\]

which follows from the fact that the value of the Green function only depends on the associated partition and a power prime to $p$ does not change the Jordan blocks of a unipotent matrix.

\[\square\]

9.3.8. The Unitary Part, II. Define

\[\sigma : M \to M \quad F : M \to M\]

\[(g, h) \mapsto (\sigma_0(h), \sigma_0(g)) \quad (g, h) \mapsto (F_0(h), F_0(g)).\]

Now, $M^F$ is isomorphic to $G^{F_0}$ under the map $(g, F_0(g)) \mapsto g$, and $M^I$ is isomorphic to $G$ under the map $(g, \sigma_0(g)) \mapsto g$. The Frobenius $F$ acts on $M^I = G$ by $g \mapsto \sigma_0 F_0(g)$. The automorphism $\sigma$ acts on $M^F$ by

\[(g, F_0(g)) \mapsto (\sigma_0 F_0(g), \sigma_0(g)) = (\sigma_0 F_0(g), \sigma_0 F_0^2(g)),\]

or in other words, $\sigma$ acts on $G^{F_0}$ as $\sigma_0 F_0$.

Let $\chi_M$ be a unipotent irreducible character of $M^F$, and let $\tilde{\chi}_M$ be an irreducible character of $M^F <\sigma>$ that extends $\chi_M$. We are going to show that up to a sign,

\[(9.3.8.1) \quad \tilde{\chi}_M|_{M^F, \sigma} = R_{\sigma_0 F_0}^{M^F, \sigma} = \sum_{w \in W_{M^F}} \phi(w F) R_{\sigma_0 F_0}^{M^F, \sigma} 1\]

for some choice of $\phi$. Under the isomorphism $W_M^\sigma \cong W_G$, the Frobenius $F$ acts as $\sigma_0$ on $W_G$, and so an $F$-stable character of $W_M^\sigma$ is just a character of $W_G$. We are reduced to show that if $\chi_G$ is an irreducible unipotent character of $G^{F_0}$ corresponding to $e \in \text{Irr}(W_G)$, then its extension $\tilde{\chi}_G$ to $G^{F_0} <\sigma_0 F_0>$ is given by the above formula up to a sign.

We need the Shintani descent. Suppose that $\chi_G \in \text{Irr}(G^{F_0} <\sigma_0 F_0>$ is an unipotent character corresponding to $e \in \text{Irr}(W_G)^F$. We apply Theorem 7.3.1 with $(\sigma_1 F_0^m, \sigma_2 F_0) = (F_0^2, \sigma_0 F_0)$, i.e. $m = 2, \sigma_1 = 1$ and $\sigma_2 = \sigma_0$, and deduce that

\[(9.3.8.2) \quad \tilde{\chi}_G = E_{\sigma_0 F_0}(\chi_G) = \text{Sh}_{\sigma_0 F_0}^{F_0} \Omega^{2} \phi_{\sigma_0 F_0}^{G^{F_0}} R_{\sigma_0 F_0}^{G^{F_0}} 1\]

\[= \pm \text{Sh}_{\sigma_0 F_0}^{F_0} \Omega^{2} \phi_{\sigma_0 F_0}^{G^{F_0}} R_{\sigma_0 F_0}^{G^{F_0}} 1 = \pm \text{Sh}_{\sigma_0 F_0}^{F_0} \Omega^{2} \phi_{\sigma_0 F_0}^{G^{F_0}} R_{\sigma_0 F_0}^{G^{F_0}} 1\]

since $\Omega^2 \phi_{\sigma_0 F_0}^{G^{F_0}} 1$ is an irreducible unipotent character $G^{\sigma_0 F_0}$ on which $\Omega^{2} \phi_{\sigma_0 F_0}$ acts as a scalar, whose value is given by $\$7.1.6$. For example, $\Omega_{\sigma_0 F_0}^{F_0} = 1$ on principal series representations and $\Omega_{\sigma_0 F_0}^{F_0} = -1$ on cuspidal unipotent characters according to ([L77 Table I]). The sign ($\pm$) does not matter since the two extensions of $\chi_G$ only differ by a sign. It remains to show that $\text{Sh}_{\sigma_0 F_0}^{F_0} \Omega_{\sigma_0 F_0}^{G^{F_0}} 1 = R_{w_M}^{M_\sigma} 1$, where $w_M$ is as in (9.3.6.2) and $M$ is equipped with
the Frobenius \( F \). The function \( R^{G_0 F_0}_{T_w} 1 \) is invariant under \( F_0^2 \)-conjugation as \( F_0^2 \) acts trivially on \( G_0 F_0 \), which justifies \( \text{Sh}_{o_0 F_0 / F_0^2} \circ R^{G_0 F_0}_{T_w} 1 \).

Proposition 7.4.1 gives

\[
(9.3.8.3) \quad \text{Sh}_{o F / F} \circ R^{M_{o F} \sigma}_{T_{w M} \sigma} 1 = R^{M_{o F} \sigma}_{T_{w M} \sigma} 1.
\]

(One checks that with respect to a fixed \( F \)-stable and \( \sigma \)-stable maximal torus \( T \subset M \), the maximal torus \( T_{w M} \), of type \( w M \) with respect to \( F \) is also of type \( w M \) with respect to \( \sigma F \), using the fact that for \( \sigma \) quasi-central, \( w M \) has a representative in \( M^\sigma \).) Since \( F \) acts as \( \sigma \) on \( M_{o F} \), the function \( R^{M_{o F} \sigma}_{T_{w M} \sigma} 1 \) is invariant under the \( F \)-conjugation of \( M_{o F} \), and its Shintani descent \( \text{Sh}_{o F / F} \circ R^{M_{o F} \sigma}_{T_{w M} \sigma} 1 \) belongs to \( C(M_{o F}) \). There is a natural bijection \( C(M_{o F}) \not\cong C(G^{F_0^2}_{o F_0}) \). Let us show that

\[
\text{Sh}_{o_0 F_0 / F_0^2} \circ R^{G_0 F_0}_{T_w} 1 = \text{Sh}_{o F / F} \circ R^{M_{o F} \sigma}_{T_{w M} \sigma} 1,
\]

which concludes the proof.

For \( g \in G^{F_0^2}_{o F_0} \), there exists \( x \in G \) such that \( x o_0 F_0(x)^{-1} = g \), and so

\[
N_{F_0^2 / o_0 F_0} (g) = x^{-1} F_0^2 (x) \in G^{o_0 F_0}.
\]

We also have

\[
(x, F_0(x)) \sigma F (x^{-1}, F_0(x))^{-1} = (g, F_0(g)),
\]

and so

\[
N_{F / o F}((g, F_0(g)) = (x^{-1}, F_0(x)) F(x, F_0(x)) = (x^{-1} F_0^2(x), 1) \in M_{o F}.
\]

Therefore,

\[
(9.3.8.4) \quad \text{Sh}_{o F / F} \circ R^{M_{o F} \sigma}_{T_{w M} \sigma} 1((g, F_0(g)) \overset{\circ}{=} R^{M_{o F} \sigma}_{T_{w M} \sigma} 1((x^{-1} F_0^2(x), 1)) = \text{Sh}_{o_0 F_0 / F_0^2} \circ R^{G_0 F_0}_{T_w} 1(g).
\]

Equality \( \circ \) is the definition of Shintani descent and we have identified the functions on \( M_{o F} \) to the functions on \( M_{o F} \) invariant under the \( F \)-conjugation. We have equality \( \circ \) by (§9.3.6.2) with the automorphism \( \sigma(g, h) = (\sigma_0(g), \sigma_0(h)) \) and the Frobenius \( \sigma F(g, h) = (\sigma_0 F_0(g), \sigma_0 F_0(h)) \).

Equality \( \overset{\circ}{=} \) is again the definition of Shintani descent.

9.4. **Extensions of Quadratic-Unipotent Characters.** In this section, we focus on \( L_0 \cong \text{GL}_{n_0}(k) \), equipped with the Frobenius \( F_0 \) which sends each entry to its \( q \)-th power. Let \( \sigma_0 \) and \( \sigma_0' \) be the automorphisms defined for \( \text{GL}_{n_0}(k) \) in the same way \( \sigma \) and \( \sigma' \) are defined for \( \text{GL}_n(k) \). Now the semi-direct product of \( \text{GL}_{n_0}(k) \) by \( \sigma_0 \) (resp. \( \sigma_0' \)) is denoted by \( \sigma_0 \Gamma_0 \) (resp. \( \sigma_0' \Gamma_0 \)). We may regard \( \sigma_0 \) also as an element of \( \sigma_0 \Gamma_0 \), acting as \( \sigma_0 \) on \( \text{GL}_{n_0}(k) \) but satisfying \( \sigma_0^2 = -\text{Id} \). The point is that, we want to fix a quasi-central element in \( \sigma_0 \Gamma_0 \) to work with, and \( \sigma_0 \) is a convenient choice. When we need to distinguish \( \sigma_0 \Gamma_0 \) and \( \sigma_0' \Gamma_0 \), it is useful to adopt the following point of view: in either case, we express an element not in the identity component as \( g \sigma_0 \) with \( g \in L_0 \), and regard an invariant function in \( g \sigma_0 \) as a function in \( g \) that is invariant under the \( o_0 \)-conjugation.
9.4.1. Let \((\mu_+, \mu_-)\) be a 2-partition of \(n_0\) and write \(n_+ = |\mu_+|\) and \(n_- = |\mu_-|\). Let \(m_+\) and \(m_-\) be some non-negative integers such that \((m_+, \ldots, 2, 1)\) and \((m_-, \ldots, 2, 1)\) are the 2-cores of \(\mu_+\) and of \(\mu_-\) respectively, and write \(N_\pm = (n_\pm - m_\pm (m_\pm + 1))/2\). There exists a unique pair \((h_1, h_2) \in \mathbb{N} \times \mathbb{Z}\) such that

\[
\begin{align*}
m_+ &= \sup(h_1 + h_2, -h_1 - h_2 - 1) \quad \text{(9.4.1.1)} \\
n_- &= \sup(h_1 - h_2, h_2 - h_1 - 1).
\end{align*}
\]

Note that exchanging \(\mu_+\) and \(\mu_-\) sends \((h_1, h_2)\) to \((h_1, -h_2)\). Write

\[m = m_+(m_+ + 1)/2 + m_-(m_- + 1)/2,\]

and so \(n_0 = m + 2N_+ + 2N_-\). We have \(m = h_1(h_1 + 1) + h_2^2\). Fix some integers \(r_+ > l(\mu_+)\) and \(r_- > l(\mu_-)\) satisfying:

**Assumption.** \(r_- \equiv h_2 \mod 2; r_+ \equiv h_2 + 1 \mod 2.\)

Let \((\alpha_+, \beta_+)_{r_+}\) and \((\alpha_-, \beta_-)_{r_-}\) be the 2-partitions associated to \(\mu_+\) and to \(\mu_-\) respectively (See §1.1.3), such that the unordered 2-partitions \((\alpha_+\beta_+)\) and \((\alpha_-\beta_-)\) are the corresponding 2-quotients. Each of the two 2-partitions \((\alpha_+\beta_+)_{r_+}\) determines an (isomorphism class of) irreducible representation of \(\mathbb{C}^\times\) respectively, denoted by \(\rho_+\) and \(\rho_-\).

Then with the fixed \(r_+\) and \(r_-\), the 2-partitions \((\mu_+, \mu_-)\) are in bijection with the data \((h_1, h_2, \rho_+, \rho_-)\), identifying \(\rho_+\) and \(\rho_-\) with the 2-partitions.

9.4.2. We also consider the data \((h_1, h_2, w_+, w_-)\), with \(w_+ \in \mathbb{C}^\times\) and \(w_- \in \mathbb{C}^\times\), where \(h_1, h_2, N_+\) and \(N_-\) are as above. To simplify, we write \(\mathcal{W}_+ = \mathbb{C}^\times_{N_+}\), \(\mathcal{W}_- = \mathbb{C}^\times_{N_-}\) and \(w = (h_1, h_2, w_+, w_-)\) instead. To each \(w\) is associated an \(F_0\)-stable \(\sigma_0\)-stable Levi factor of a \(\sigma_0\)-stable parabolic subgroup, isomorphic to \(L_w \cong T_{w_+} \times T_{w_-} \times \text{GL}(m)\), each factor preserved by \(\sigma_0\), where \(T_{w_+}\) is isomorphic to \((k^\times)^{2N_+}\) equipped with the Frobenius twisted by \(w_+\). We write \(\sigma_0 = \sigma_+ \times \sigma_- \times \sigma_0\) and \(F_0 = F_+ \times F_- \times F_0\) with respect to this decomposition.

9.4.3. We describe the cuspidal local systems on \(\text{GL}_m(k)\). \(\sigma_0\). Let \((h_1, h_2) \in \mathbb{N} \times \mathbb{Z}\) be as above. By §4.3.2 there is a unique semi-simple isolated conjugacy class on \(\text{GL}_m(k)\). \(\sigma_0\) with connected centralizer isomorphic to \(\text{Sp}_{h_1(h_1+1)}(k) \times \text{SO}_{h_2}(k)\). Let \(s_0\) be an \(F\)-stable element representing this conjugacy class. By Theorem 1.4.1 there is a unique cuspidal local system on \(\text{Sp}_{h_1(h_1+1)}(k) \times \text{SO}_{h_2}(k)\), which is supported on the unipotent conjugacy class whose symplectic (resp. orthogonal) component corresponds to the symplectic partition \(\lambda_1 := (2h_1, 2h_1 - 2, \ldots, 2)\) (resp. orthogonal partition \(\lambda_2 := (2|h_2| - 1, 2|h_2| - 3, \ldots, 1)\)). Let \(u = (u_1, u_2) \in \text{Sp}_{h_1(h_1+1)}(k) \times \text{SO}_{h_2}(k)\) be an \(F\)-stable element representing this unipotent conjugacy class.

By Lemma 8.1.2 the cuspidal local systems on \(\text{GL}_m(k)\). \(\sigma_0\) is supported on the conjugacy class of \(s\sigma_0u\), which itself is an isolated stratum according to Example 8.1.1. The irreducible equivariant local systems on this conjugacy class are parametrised by \(((\mathbb{Z}/2\mathbb{Z})^\times)^{h_1}) \times \text{SO}_{h_2}(k)\).
The GL$(m)$ isomorphism $u$ is given by choosing $a \in CGL_m(k)(s00u)\circ$ is determined by the natural quotient map

\[(9.4.3.1) \quad ((Z/2Z)_{\kappa(1)})^\forall \times ((Z/2Z)_{\kappa(2)})^\forall \to ((Z/2Z)_{\kappa(1)})^\forall \times ((Z/2Z)_{\kappa(2)})^\forall /\Delta.\]

Therefore by Lemma 8.1.2 again, there are two cuspidal local systems supported on the class of $s00u$ in the unique cuspidal local system supported on $Sp_{\mathbb{H}((l+1))}(k) \times SO_{\mathbb{H}2}(k)$. These exploit all cuspidal local systems on $GL_{ml}(k),\sigma_{00}$.

9.4.4. We may choose $s00$ in such a way that $CGL_m(q)(s00)$ is split. Denote by $C_1 \subset Sp_{\mathbb{H}((l+1))}(k)$ (resp. $C_2 \subset SO_{\mathbb{H}2}(k)$) the conjugacy class of $u_1$ (resp. $u_2$), and by $\mathcal{E}_1$ (resp. $\mathcal{E}_2$) the unique cuspidal local system supported on $C_1$ (resp $C_2$), which is necessarily $F$-stable, i.e. there is an isomorphism $\psi_1 : F\mathcal{E}_1 \simeq \mathcal{E}_1$ (resp. an isomorphism $\psi_2 : F\mathcal{E}_2 \simeq \mathcal{E}_2$). Recall that $\sigma$ is the nontrivial irreducible character of $Z/2Z$.

The $Sp_{\mathbb{H}((l+1))}(q)$-conjugacy classes contained in $C^F_1$ are parametrised by the elements of $(Z/2Z)_{\kappa(1)}$. For any $a = (e_i)_{i \in \kappa(1)} \in (Z/2Z)_{\kappa(1)},$ denote by $C_a$ the corresponding conjugacy class. We choose $u_1$ to be the split element [S07, §2.9] of $C_1$ so that if $\psi_1$ is normalised in such a way that it induces the identity map on the stalk ($\mathcal{E}_1_{u_1}$), then the characteristic function $\phi_1$ of $\mathcal{E}_1$ is given by

\[\phi_1(C_a) = \prod_{i \in \{1,\ldots,l_1\}, \ i \text{ is even}} \sigma(e_{2i}) ,\]

according to §1.4.3.

The $SO_{\mathbb{H}2}(q)$-conjugacy classes contained in $C^F_1$ are parametrised by the elements of $(Z/2Z)_{\kappa(2)}^{-1},$ identified with the subgroup of elements $e_i \in \kappa(2)$ and with $\sum_i e_i = 0$. For any $a = (e_i)_{i \in \kappa(2)} \in (Z/2Z)_{\kappa(2)}^{-1},$ denote by $C_a$ the conjugacy class. Again, we choose $u_2$ to be the split element of $C_2$ so that if $\psi_2$ is normalised in such a way that it induces the identity map on the stalk ($\mathcal{E}_2_{u_2}$), then the characteristic function $\phi_2$ of $\mathcal{E}_2$ is given by

\[\phi_2(C_a) = \prod_{i \in \{1,\ldots,l_2\}, \ i \text{ is odd}} \sigma(e_{2i-1}) ,\]

according to §1.4.3.

We choose $s00$ and $u = (u_1, u_2)$ as above and denote by $C$ the conjugacy class of $s00u$. The $GL_m(q)$-conjugacy classes contained in $C^F$ are parametrised by $a, b \in (Z/2Z)_{\kappa(1)} \times (Z/2Z)_{\kappa(2)}$, and we denote by $C_{a,b}$ the corresponding conjugacy class. If we normalise the isomorphism $\psi : F\mathcal{E} \simeq \mathcal{E}$ for a cuspidal local system $\mathcal{E}$ supported on the conjugacy of $s00u$ in such a way that the induced map on the stalk at $s00u$ is the identity, then the characteristic
function \( \phi \) is either
\[
\phi(C_{a,b}) = \prod_{i \in \{1, \ldots, h_1\}, \ i \ \text{is even}} \varrho(e_i) \prod_{j \in \{1, \ldots, |h_2|\}, \ j \ \text{is odd}} \varrho(e_{2j-1}'),
\]
with \((a, b) = ((e_i), (e'_j))\), or
\[
\phi(C_{a,b}) = \prod_{i \in \{1, \ldots, h_1\}, \ i \ \text{is even}} \varrho(e_i) \prod_{j \in \{1, \ldots, |h_2|\}, \ j \ \text{is even}} \varrho(e_{2j-1}'),
\]
since the two elements in the fibre of \((9.4.3.1)\) differ by \(\Delta = (\varrho, \ldots, \varrho)\). We call these functions cuspidal functions.

9.4.5. To each \((h_1, h_2) \in \mathbb{N} \times \mathbb{Z}\) is associated a unique cuspidal function \(\phi(h_1, h_2)\) supported on \(C\). Put
\[
s(h_2) = \begin{cases} 0, & \text{if } h_2 \geq 0, \\ 1, & \text{if } h_2 < 0, \end{cases}
\]
and put
\[
\delta(h_1) = \dim C_1 = \frac{|h_2^3 - h_2|}{3},
\]
\[
\delta(h_2) = \dim C_2 = \frac{h_1(2h_1 + 1)(h_1 + 1)}{6},
\]
\[
\delta(h_1, h_2) = \delta(h_1) + \delta(h_2).
\]

Then, \(\phi(h_1, h_2)\) is defined as
\[
(9.4.5.1) \quad \phi(h_1, h_2)(C_{a,b}) = q^{\delta(h_1, h_2)/2} \prod_{i \in \{1, \ldots, h_1\}, \ i \ \text{is even}} \varrho(e_i) \prod_{j \in \{1, \ldots, |h_2|\}, \ j \equiv h_1 + 1 + s(h_2) \mod 2} \varrho(e_{2j-1}'),
\]
with \((a, b) = ((e_i), (e'_j))\).

**Remark 9.4.1.** We have in fact simultaneously produced two functions, one on \(GL_m(q) \wr <\sigma_0>\setminus GL_m(q)\) and the other on \(GL_m(q) \wr <\sigma'_0>\setminus GL_m(q)\) with \(\sigma'_0\) being the orthogonal type automorphism. Regarded as functions on \(GL_m(q)\) that are invariant under \(\sigma_0\)-conjugation, they are identical.

9.4.6. Let \(Id\) be the trivial character of \(T_{w^+}^F\). It trivially extends to \(T_{w^+}^F \wr <\sigma_0>\), and so we can regard \(Id\) as a function on \(T_{w^+}^F \wr <\sigma_0>\). Similarly, composing \(\eta\) with the homomorphism \(T_{w^-}^F \to \mathbb{F}_q^3\) defined by the product of norm maps, we can regard \(\eta\) as an invariant function on \(T_{w^-}^F \wr <\sigma_\eta>\), whose value at \(\sigma_\eta\) is equal to 1. Then, \(Id \equiv \tilde{\otimes} \tilde{\eta} \tilde{\otimes} \phi(h_1, h_2)\) is an invariant function on \(L_{w^-}^F \wr \sigma_0\), denoted by \(\phi_w\).

**Remark 9.4.2.** In the case of \(\tilde{\sigma_0} = \tilde{\sigma_0}\), the element \(\sigma_0\) satisfies \(\sigma_0^2 = -1\) and so is each of its component: \(\sigma_+, \sigma_-\) and \(\sigma_0\). We can nevertheless extend \(\eta\) in such a way that its value at \(\sigma_-\) is equal to 1, because the value of \(\eta\), regarded as a character of \(T_{w^-}^F\), is always equal to 1 at
−1 ∈ T_{F_w}^R. Consequently, we have defined functions \( \phi_w \) both in \( \hat{\mathcal{G}}_0 \) and in \( \check{\mathcal{G}}_0 \). Regarded as functions on \( L_w^F \) that are invariant under the \( \sigma_0 \)-conjugation, they are identical.

Denote by \( \varphi_+ \) and \( \varphi_- \) the characters of \( \rho_+ \) and of \( \rho_- \) respectively. We have then the invariant functions on \( L_{F_0,\sigma_0}^R \) defined by

\[
R_{\rho}^{L_{F_0,\sigma_0}} := \frac{1}{|\mathcal{R}_+|} \frac{1}{|\mathcal{R}_-|} \sum_{w_+ \in \mathcal{R}_+, w_- \in \mathcal{R}_-} \varphi_+(w_+)\varphi_-(w_-)R_{L_{w,\sigma_0}}^{L_{F_0,\sigma_0}} \phi_w.
\]

This is the characteristic function of character sheaf obtained in Example 8.1.6. Again, this definition makes sense in \( \hat{\mathcal{G}}_0 \) and in \( \check{\mathcal{G}}_0 \).

9.4.7. We keep the notations as above and assume that \( p \neq 2 \) and \( q > n_0 \). Let \( \chi_{(\mu_+,\mu_-)} \) be a quadratic-unipotent character, which extends to a character \( \tilde{\chi}_{(\mu_+,\mu_-)} \in \operatorname{Irr}(L_{F_0,\sigma_0}^R) \). Recall that to each pair of partitions \((\mu_+,\mu_-)\) is associated a unique tuple \((h_1,h_2,\rho_+,\rho_-)\) as in §9.4.1 which defines the function \( R_{\rho}^{L_{F_0,\sigma_0}} \) above.

**Theorem 9.4.3.** (\[W\] §17) Suppose \( L_{F_0,\sigma_0}^R \) if \( n_0 \) is even. Then for any \((\mu_+,\mu_-) \in \mathcal{P}_{n_0}(2)\), we have,

\[
\tilde{\chi}_{(\mu_+,\mu_-)}^{L_{F_0,\sigma_0}} = \pm R_{\rho}^{L_{F_0,\sigma_0}}.
\]

Given a quadratic-unipotent character \( \chi_0 \) of \( \GL_{n_0}(q) \), let \( \rho : \GL_{n_0}(q) \to \GL(V) \) be a representation that realises it. Then \( \rho(-\text{Id}) = \pm \text{Id}_V \), with \( \rho(-\text{Id}) = -\text{Id}_V \) exactly when \( \chi(-\text{Id}) = -\chi(\text{Id}) \). Define the indicator

\[
\gamma_\chi = \begin{cases} 
1 & \text{if } \chi(-\text{Id}) = -\chi(\text{Id}); \\
1 & \text{otherwise.}
\end{cases}
\]

**Corollary 9.4.4.** Suppose that \( n_0 \) is even and \( L_{F_0,\sigma_0}^R \) if \( n_0 \) is even. Then for any \((\mu_+,\mu_-) \in \mathcal{P}_{n_0}(2)\), we have,

\[
\tilde{\chi}_{(\mu_+,\mu_-)}^{L_{F_0,\sigma_0}} = \pm \gamma_\chi R_{\rho}^{L_{F_0,\sigma_0}}.
\]

**Proof.** This follows from §4.2.3. One sees from Proposition 3.2.3 that the induced function \( R_{L_{w,\sigma_0}}^{L_{F_0,\sigma_0}} \phi_w \) regarded as a function on \( L^F \) invariant under \( \sigma_0 \)-conjugation, does not depend on whether we work with \( \hat{\mathcal{G}}_0 \) or with \( \check{\mathcal{G}}_0 \). \( \square \)

10. Computation of the Character Formula

In order to determine the character table of \( \GL_n(q,<\sigma>) \), we need to explicitly calculate the induced functions of the form \( R_{M,\sigma}^{\mathcal{G}} \phi_M \) for some invariant function \( \phi_M \) defined on some \( \sigma \)-stable and \( F \)-stable Levi factor \( M \) of some \( \sigma \)-stable parabolic subgroup.

10.1. Connected Groups.
10.1.1. Let us recall how this is done for a split connected group $G$. Fix a split maximal torus $T_0$ and let $W_G$ denote the Weyl group defined by $T_0$. Let us simplify the situation and assume that $M = T$, $\tau \in W_G$, is a maximal torus. The character formula (cf. §3.1.2) reads

$$R_{T_\tau}^G \theta(g) = |T_\tau^F|^{-1} |C_G^c(s)^F|^{-1} \sum_{h \in G^F [s] \cap T_\tau} Q_{C_G^c(s)}^G(t)^h \theta(s),$$

for the Jordan decomposition $g = su$. Assume that $s \in T^F_\tau$, and put

$$A(s, \tau) := \{ h \in G \mid hsh^{-1} \in T_\tau \}.$$

We have to determine the set

$$A^F(s, \tau) := \{ h \in A(s, \tau) \mid F(h) = h \}.$$

10.1.2. Write $L = C_G^c(s)^0$. Define

$$B(s, \tau) := \{ \text{The } L^F\text{-conjugacy classes of the } F\text{-stable maximal tori of } L \}$$

that are $G^F$-conjugate to $T_\tau$. It parametrises a subset of the Green functions of $L^F$. We fix $s$ and $\tau$ and write $A$, $A^F$ and $B$ in what follows. Observe that there is a surjective map $A^F \to B$ which sends $h$ to the class of $h^{-1}T_\tau h$. It factors through $\iota : A^F / L^F \to B$. The value of the Green function only depends on $\iota(h)$ while $h^{-1} \theta(s)$ is constant on each right $L^F$-coset of $A^F$. The calculation is eventually reduced to evaluating $h^{-1} \theta(s)$ on the fiber of $\iota$ over an element $\varphi \in B$. We may regard $\varphi$ as the $F$-conjugacy class of some $\nu \in W_L(T_\tau)$.

10.1.3. We have $A = N_G(T_\tau) L$, that is, the set of the elements $nl$ with $n \in N_G(T_\tau)$ and $l \in L$, since for each $h$, there exists $l \in L$ such that $h^{-1}T_\tau h = lT_\tau l^{-1}$. We deduce from it an isomorphism

$$A^F / L^F \cong (A / L)^F \cong (N_G(T_\tau) / N_L(T_\tau))^F \cong (W_G(T_\tau) / W_L(T_\tau))^F,$$

which sends $h = nl$ to the class of $n$. This does not depend on the choice of the $n$ and $l$ such that $h = nl$. We choose some $g \in G$ such that $T_\tau = gT_0g^{-1}$, and put $L_0 = g^{-1}Lg$. We can further identify the above set to $(W_G(T_0) / W_{L_0}(T_0))^F$. Write $W_{L_0} = W_{L_0}(T_0)$. The conjugation by $\tau$ preserves $L_0$ since $L$ is $F$-stable. Now, a coset $wL_0$ is $\tau$-stable if and only if

$$w^{-1}Tw^{-1} \in W_{L_0}$$

and $\iota(wL_0) = \varphi$ if and only if

$$w^{-1}Tw^{-1} \in \varphi,$$

regarding $\varphi$ as a $\tau$-conjugacy class of $W_{L_0}(T_0)$. Indeed, if $hL^F$ corresponds to $wL_0$, then the $L^F$-conjugacy class of $h^{-1}T_\tau h = l^{-1}T_\tau l$ is represented by $lF(l)^{-1} = n^{-1}F(n)$, which is none other than $w^{-1}Tw^{-1}$ under $\text{ad } g^{-1}$. The computation of the $w$'s is completely combinatorial.
Moreover, from Proposition 2.1.8 (i), we deduce that

\[ s \]

Lemma 10.2.1. Suppose that \( s \) isomorphic to \( h \) \( \mu \) of \( M \) which is a Levi factor of an

We will give a combinatorial description of this set.

Define

\[ W \] isomorphic to \( GL_n(k) \times SO_{n_2}(k) \) and a torus.

Define

\[ A^F(s, \tau, h_1, h_2) = \{ h \in A(s, \tau, h_1, h_2) | F(h) = h \}. \]

We will give a combinatorial description of this set.

Write \( L' = C_G(s)\). If \( K' \subset L' \) is an \( F \)-stable Levi subgroup, put \( K = C_G(Z_{K'}) \). By Proposition 2.1.4 it is the smallest \( F \)-stable and \( s \)-stable Levi subgroup of \( G \) such that \( (K \cap L')^0 = K' \), which is a Levi factor of an \( s \)-stable parabolic subgroup, say \( Q \). So \( N_G(K) \cap N_G(Q) = K.<s\sigma>. \) Moreover, from Proposition 2.1.8(i), we deduce that \( s \sigma \) is isolated in \( K.<s\sigma>. \)

Lemma 10.2.1. Suppose that \( s \sigma \) is an \( F \)-stable semi-simple element with connected centraliser

isomorphic to \( Sp_{h_1(h_1+1)}(k) \times SO_{h_2^2}(k) \) and a torus.

Proof. We use the notations above the lemma. By assumption, there exists an \( F \)-stable Levi

subgroup \( K' \subset L' \) that is isomorphic to the product of \( Sp_{h_1(h_1+1)}(k) \times SO_{h_2^2}(k) \) and a torus.
Then $K$ is isomorphic to the product of $\text{GL}_m(k)$ and a torus, therefore $G$-conjugate to $M_0$. By Proposition 2.2.4, $K$.<s>o</s> is $G^F$-conjugate to some $M_\mu$.<s>o</s>. □

Define

$$B(s_0, \tau, h_1, h_2) = \{ \text{The } L^F\text{-conjugacy classes of the } F\text{-stable Levi subgroups } K' \subset L' \}
$$

isomorphic to the product of $\text{Sp}_{h_1(h_1+1)}(k) \times \text{SO}_{h_2}^0(k)$ and a torus

such that $K$.<s>o</s> is $G^F$-conjugate to $M_\tau$.<s>o</s>.

In what follows, we fix $s_0, \tau, h_1$ and $h_2$, then $A(s_0, \tau, h_1, h_2)$ and $B(s_0, \tau, h_1, h_2)$ may be denoted by $A$ and $B$.

10.2.3. We will assume that $s_0 \in M_\mu$ for a fixed $\mu$. Write $s_0\sigma = s_0^{-1}s_\mu$, $L_0' = C_{G}(s_0\sigma)^0$ and $L' = C_{G}(\sigma)^0$. Then $s_0\sigma \in M_0.\sigma$ is isolated with connected centraliser isomorphic to the product of $\text{Sp}_{h_1(h_1+1)}(k) \times \text{SO}_{h_2}^0(k)$ and a torus.

**Lemma 10.2.2.** There is a natural bijection:

(10.2.3.1) \[
A \xrightarrow{\sim} N_G(M_0.\sigma).L_0'
\]

(10.2.3.2) \[
h \longmapsto g^{-1}_1h_{s_\mu}.
\]

**Proof.** For $h \in A$, write $x = g_1^{-1}h_{s_\mu}$. By definition, $xs_0\sigma x^{-1} \in S$. There exists $l \in L_0'$ such that $x^{-1}M_0x \cap L_0' = l(M_0 \cap L_0')l^{-1} = lM_0l^{-1} \cap L_0'$. By assumption, $s_0\sigma$ is isolated in $x^{-1}M_0.\sigma x$ and in $M_0.\sigma$, and so Remark 2.1.6 implies that $x^{-1}M_0x = lM_0l^{-1}$. So there exists $n \in N_G(M_0)$ such that $x = nl^{-1}$. From the fact $xs_0\sigma x^{-1} \in S$ we see that $n\sigma(n)^{-1} \in M_0$.

Let us determine the set $N_G(M_0.\sigma)$. If $n$ normalises $M_0.\sigma$, then it normalises $M_0.\sigma.M_0.\sigma$, so it normalises $M_0$. Then, $nM_0.\sigma n^{-1} = nM_0\sigma n^{-1}n\sigma(n)^{-1}\sigma \in M_0.\sigma$, so $n\sigma(n)^{-1} \in M_0$. We see that $N_G(M_0.\sigma)$ consists of those components of $N_G(M_0)$ that are $\sigma$-stable. Finally, we note that $nl$ belongs to $A$, for any $n \in N_G(M_0.\sigma)$ and $l \in L_0'$.

Since $F(h) = F(g_1, s_\sigma g_1^{-1}) = g_1^{\tau}F(x)\mu^{-1}g_1^{-1}$, the Frobenius is transferred to the map $F_{\tau,\mu} : x \mapsto \tau F(x)\mu^{-1}$ via the above bijection. The set $N_G(M_0.\sigma).L_0'$ is mapped into itself by $F_{\tau,\mu}$.

Write $M_\nu' = C_{M_\nu}(s_\sigma)^0$. It is a Levi subgroup of $L_0'$ isomorphic to the product of $\text{Sp}_{h_1(h_1+1)}(k) \times \text{SO}_{h_2}^0(k)$ and a torus. Similarly, for any $v \in W^\sigma$ such that $s_\sigma \in M_\nu.\sigma$, write $M_\nu' = C_{M_\nu}(s_\sigma)^0$.

**Corollary 10.2.3.** There is a natural bijection:

(10.2.3.3) \[
A^F/L^F \cong (N_G(M_0.\sigma)/N_{L_0'}(M_\nu')).\]

**Remark 2.1.7** implies that $N_{L_0'}(M_\nu')$ is indeed a subgroup of $N_G(M_0.\sigma)$.

**Proof.** Since $L_0'$ is connected, $A^F/L^F \cong (A/L')^F$. The bijection is then induced from the bijection of Lemma 10.2.2. □
10.2.4. Let us point out that the identity component of \( N_G(M_0, \sigma) \) is \( M_0 \) whereas that of \( N_{L'_0}(M'_0) \) is \( M'_0 \), so we cannot directly reduce the problem to a purely combinatorial one as in §10.1.1.

Write \( N'_0 = N_{L'_0}(M'_0) \) and \( \bar{N}'_0 := N'_0M_0 \). It is the union of the connected components of \( N_{G}(M_0, \sigma) \) that meet \( L'_0 \).

**Lemma 10.2.4.** Each connected component of \( \bar{N}'_0 \) contains exactly one connected component of \( N'_0 \).

**Proof.** It suffices to consider the identity components of the two groups. Note that the connected component of an element of \( N'_0 \) are determined by its action on \( Z^\sigma_{M'_0} \). An element of the identity component of \( N'_0 \) must induce trivial action on this torus because \( M_0 = C_G(Z^\sigma_{M'_0}) \).

We deduce from this lemma an isomorphism \( N'_0/M'_0 \approx \bar{N}'_0/M_0 \). We can then regard \( W'_0 := N_{L'_0}(M'_0)/M'_0 \) as a subgroup of \( W^\sigma \).

**Lemma 10.2.5.** Let \( n \in N_G(M_0, \sigma) \) and let \( w \) be the class of \( n \) in \( W^\sigma \). Then

(i) The coset \( nN'_0 \) is \( F_{\tau, \mu} \)-stable if and only if

\[
(10.2.4.1) \quad n^{-1} \tau F(n) \mu^{-1} \in N'_0.
\]

(ii) In the case of (i), we have

\[
(10.2.4.2) \quad w^{-1} \tau w \mu^{-1} \in W'_0.
\]

**Proof.** Obvious.

**Lemma 10.2.6.** Suppose that there exists some \( w \in W^\sigma \) satisfying (10.2.4.2). Then there exists some \( n \in N_G(M_0, \sigma) \) that represents \( w \) and satisfies (10.2.4.1).

**Proof.** Let \( \dot{w} \in N_G(M_0, \sigma) \) be an \( F \)-stable element representing \( w \) then we want to find an element \( t \in M_0 \) such that

\[ \dot{w}^{-1} t^{-1} \tau F(t) \dot{w} \mu^{-1} \in N'_0. \]

Write

\[ \dot{w}^{-1} t^{-1} \tau F(t) \dot{w} \mu^{-1} = \left( \dot{w}^{-1} t^{-1} \tau F(t) \dot{w} \right) \left( \dot{w}^{-1} t \dot{w} \mu^{-1} \right). \]

Since \( t \mapsto t^{-1} \tau F(t) \) is surjective onto \( M_0 \), the desired \( t \) exists and \( t \dot{w} \) is the sought-for \( n \).

**Proposition 10.2.7.** Let \( \sigma \) be an \( F \)-stable semi-simple element with connected centraliser isomorphic to \( \text{Sp}_{n_0}(k) \times SO_{n_1}(k) \times \prod \text{GL}_{n_i}(k) \) for some non-negative integers \( n_s, n_o \) and \( n_i \)'s. Then the set \( A^F(\sigma, \tau, h_1, h_2) \) is non-empty if and only if

(i) \( h_1(h_1 + 1) \leq n_s, h_2^2 \leq n_o \) and,

(ii) the solution set of equation (10.2.4.2) is non-empty.

**Proof.** This is a combination of Corollary 10.2.3, Lemma 10.2.1, Lemma 10.2.11 and Lemma 10.2.6.
Remark 10.2.8. Condition (ii) in the above Proposition is actually independent of the choice of $\mu$. Indeed, from the proof of Lemma 10.2.1 one sees that the choice of $M_\mu$ comes from a choice of an $F$-stable Levi subgroup $K' \subset L'$ and a different choice of $K'$ results in multiplying $\mu$ on the left by an element of $W'_0$.

10.2.5. We assume from now on that $A^F$ is non-empty.

Lemma 10.2.9. The map

$$
\iota : A^F / L'^F \longrightarrow B
$$

(10.2.5.1)

$$
hL'^F \longmapsto \text{the class of } C_{h^{-1}M_\mu}(\sigma^0).
$$

is well defined and surjective.

Proof. If $h \in A^F$, then $hs_0h^{-1}$ normalises $M_\tau$ and a parabolic subgroup containing it, so $\sigma$ normalises $h^{-1}M_\tau h$ and a parabolic subgroup containing it. It follows that $K' := C_{h^{-1}M_\tau}(\sigma)$ is an $F$-stable Levi subgroup of $L'$. We then obtain the Levi subgroup $K$ as above. From the fact that $C_{h^{-1}M_\tau}(\sigma) = h^{-1}C_{M_\tau}(hs_0h^{-1})h$ and from the assumption on $hs_0h^{-1}$, we deduce that $\sigma \in K,<\sigma>$ is isolated with centraliser isomorphic to the product of $Sp_{h_0(h_1+1)}(k) \times O_{h_2}(k)$ and a torus. Since $\sigma$ is also isolated in $h^{-1}M_\tau.<\sigma>h$, by Remark 2.1.6 we have $K = h^{-1}M_\tau h$, and so $K,<\sigma> = h^{-1}M_\tau,<\sigma>h$. We see that the $L'^F$-class of $K'$ indeed belongs to $B$. Obviously this map factors through the quotient $A^F / L'^F$. Given $K' \in B$ with $h \in G^F$ such that $hK,<\sigma>h^{-1} = M_\tau,<\sigma>$, the same argument shows that $h \in A^F$, whence surjectivity. □

10.2.6. We can in fact further assume that $\sigma \in M_\tau, \sigma$, i.e. $\mu = \tau$ as long as $A^F$ is non-empty. The equation (10.2.4.2) then becomes

$$
w^{-1}\tau w^{-1} \in W'_0,
$$

with $W'_0$ being stable under conjugation by $\tau$, and we will denote by $W(\sigma, \tau)$ the set of those $w \in W^\sigma$ satisfying this condition. Then $B$ is the set of $\tau$-conjugacy classes of $W'_0$ that meet $\{w^{-1}\tau w^{-1} \mid w \in W(\sigma, \tau)\}$.

Remark 10.2.10. With $\tau = \mu$, the condition (10.2.4.2) simply means that the $\tau$-conjugacy class of $w$ meets $W'_0$. This is what one would guess without any computation. But it is not useful in determining whether $A^F$ is empty or not, because one would have to first find a $G$-conjugate $s_0\sigma$ of $\sigma$ such that the associated $W'_0$ is $\tau$-stable, which is an equally difficult problem. In practice, it is easier to find some $\mu \in W^\sigma$ and some $s_0\sigma$ such that $W'_0$ is $\mu$-stable, then we can see if $A^F$ is empty using (10.2.4.2).

Let $\psi$ be a $\tau$-conjugacy class of $W'_0$.

Lemma 10.2.11. Let $n \in N_G(M_0, \sigma)$ and let $w$ be the class of $n$ in $W^\sigma$. Suppose that $n$ satisfies (10.2.4.1). Then under the bijection (10.2.3.3) between the classes of $h$ and the classes of $n$, the
$F_{\tau}$-stable Levi subgroup $C_{h^{-1}M_{\tau}h}(so)^{\circ} \subset L'$ lies in the class $\tilde{v}$ if and only if

\[(10.2.6.1) \quad w^{-1}\tau w\tau^{-1} \in \tilde{v}.\]

**Proof.** Write $h = nl$ with $n \in N_{G}(M_{\tau},\sigma)$ and $l \in L'$. The $L'^{\tau}$-class of the Levi subgroup $C_{h^{-1}M_{\tau}h}(so)^{\circ} = l^{-1}M_{\tau}l \subset L'$ is given by $IF(l)^{-1} = n^{-1}F(n) \in N_{L'}(M_{\tau}')$, or rather $n_{0}^{-1}lF(n_{0})l^{-1} \in N_{L_{0}'}(M_{\tau}'_{0})$, with $n_{0} = g_{\tau}^{-1}n_{\tau}g_{\tau}$. Then the class of $n_{0}$ is $\tilde{v}$. \hfill $\Box$

Denote by $W_{\tau,v}$ the subset of $W(so,\tau)$ consisting of elements $w$ satisfying (10.2.6.1).

**Lemma 10.2.12.** We have:

(i) $W_{\tau,v}$ is a union of $W'_{0}$-cosets;

(ii) $W_{\tau,v}$ surjects onto $\tilde{v}$ via $w \mapsto w^{-1}\tau w\tau^{-1}$;

(iii) The fibre in $W_{\tau,v}$ over each element of $\tilde{v}$ is in bijection with $C_{W'}(\tau)$.

**Proof.** (i) and (ii): If $w^{-1}\tau w\tau^{-1} = \nu$ and $w_{0} \in W'_{0}$, then $(\nu w_{0})^{-1}\tau w_{0}\tau^{-1} = w_{0}^{-1}\nu(\tau w_{0}\tau^{-1})$. (iii): If $w_{1}$ and $w_{2}$ are mapped to the same element of $\tilde{v}$ under the map of (ii), then $w_{2} \in C_{W'}(\tau)w_{1}$. \hfill $\Box$

The combinatorial aspect of $A_{\tau}^{F}$ is now completely understood.

10.2.7. Let $h \in A_{\tau}^{F}$ and suppose that $h = nl$ for some $n \in N_{G}(M_{\tau},\sigma)$ and $l \in L'$. Denote by $w$ the class of $n$ in $W^{\sigma}$ (which is now defined with respect to $M_{\tau}$) and denote $\nu = w^{-1}\tau w\tau^{-1} \in W'_{0}$ and let $\tilde{v}$ be the $\tau$-conjugacy class of $\nu$. We fix an isomorphism $M_{\tau} \cong \text{GL}_{m}(k) \times T_{\tau}$ with $T_{\tau} \cong (k^{\times})^{2N}$, and decompose the action of $\sigma$ as $\sigma_{0} \times \sigma_{1}$. The automorphism $\sigma_{0}$ of $\text{GL}_{m}(k)$ is defined in the same way as $\sigma$ is defined for $\text{GL}_{n}(k)$ and if we index the direct factors of $T_{\tau}$ by the set

$$\{1, \ldots, N_{\tau}, -N_{\tau}, \ldots, -1\},$$

then $\sigma_{1}$ acts on $T_{\tau}$ by $(t_{i}, t_{-i}) \mapsto (t_{-i}^{-1}, t_{i}^{-1})$. For any $(h_{1}, h_{2}) \in N \times Z$ such that $h_{1}(h_{1} + 1) + h_{2}^{2} = m$, there is a unique isolated semi-simple conjugacy class in $\text{GL}_{m}(k)$. If $h_{2}^{2} > 0$, then there are two $\text{GL}_{m}(q)$-conjugacy classes contained in this isolated class.

Recall (§1.2.3) that $\text{sgn} : \mathbb{F}_{\tilde{v}}^{\circ} \to \{\pm 1\}$ is the map whose kernel is $\mathbb{F}^{D}$.

**Proposition 10.2.13.** Fix $so$, $\tau$, $h_{1}$ and $h_{2}$. Then the $\text{GL}_{m}(q)$-conjugacy class of the direct factor of $hs_{\sigma}h^{-1}$ corresponding to $\text{GL}_{m}(k)$. If $\sigma_{0}$ only depends on $\tilde{v}$.

**Proof.** According to the proof of Lemma [10.2.11] the $L'^{\tau}$-conjugacy class of the $F$-stable Levi subgroup $C_{h^{-1}M_{\tau}h}(so)^{\circ} \subset L'$ corresponds the $\tau$-conjugacy class $\tilde{v}$ with respect to $C_{M_{\tau}}(so)^{\circ}$. We know that

$$L' \cong \text{Sp}_{m_{1}}(k) \times \text{SO}_{n_{0}}(k) \times \prod_{i} \text{GL}_{n_{i}}(k)$$

for some non negative integers $n_{i}$, $n_{0}$ and $n_{i}'s$, and $L'^{\tau}$ is isomorphic to $\text{Sp}_{m_{1}}(q) \times \text{SO}_{n_{0}}^{0}(q) \times \prod_{i} \text{GL}_{n_{i}}^{\pm}(q)$ with $\eta \in \{\pm\}$ depending on the $\text{GL}_{m}(q)$-conjugacy class of $so$. Let $M_{\sigma}$ be the direct factor of $C_{h^{-1}M_{\tau}h}(so)^{\circ}$ corresponding to $\text{SO}_{n_{0}}(k)$ with respect to the above isomorphism. Then
Proposition 10.2.13. And with the same notations in Proposition 10.2.14, if we denote $z(i)$ is equal to $hL$, By hypothesis, $h$ is regarded as a subgroup of $W$, then the cardinality of the inverse image of $M_\tau$, then the cardinality of the inverse image of $M_\tau$ is twisted according to the value of $\text{sgn}(v_\text{so})$. We then remark that the two $GL_m(q)$-conjugacy classes contained in the same isolated $GL_m(k)$-class are exactly distinguished by the action of $F$ on the connected centraliser.

\[ \square \]

10.2.8. We now come to the last step in evaluating the induction $R^G_{M_\tau}$.

Proposition 10.2.14. Fix $\sigma_0, h_1, h_2, \tau, \nu$ and $\phi_\tau$. Then,

(i) The inverse image under the natural map (cf. Lemma 10.2.2)

$$\pi: A^F \longrightarrow (N_G(M_\tau, \sigma).L'/L')^F \longrightarrow (W^\nu/W)^\tau$$

of a $W'$-coset in $W_{\tau,\nu}$ if non-empty, is $M_\tau hL'^F$, for some $h \in A^F$. Its cardinality is equal to $|M_\tau^F|.|L'^F|.|C_G(M_\tau, \text{hso}(\nu))|^F$.

(ii) If we decompose $M_\tau \cong GL_m(k) \times T_\tau$ and write $\sigma = \sigma_0 \times \sigma_1$ as in §10.2.7 then

$$C_{M_\tau}(\text{hso}(\nu)) = SO_{h_2}^{\text{sgn}(\nu_\text{so})}(q) \times \text{Sp}_{h_1(h_1+1)}(q) \times T_\tau^F,$$

where $T_\tau := C_{T_\tau}(\sigma_1)^\circ$.

Remark 10.2.15. The surjection $A^F \rightarrow B$ factors through $\pi$.

Proof. (i). Suppose $h$ and $h' \in A^F$ have the same image. According to the way in which $W'$ is regarded as a subgroup of $W$, $h' \in M_\tau hL'$. If we write $h' = xh'$ with $x \in M_\tau$ and $l' \in L'$. By hypothesis, $h$ and $h'$ define $F$-stable $L'$-cosets, so $h^{-1}F(h) \in L'$ and $h^{-1}x^{-1}F(xh) \in L'$. We deduce that $x^{-1}F(x)$ lies in $hL'h^{-1}$. We then notice that the map $a \mapsto a^{-1}F(a)$ preserves $hL'h^{-1}$, so it is a surjective map from $hL'h^{-1}$ onto itself. We see that $x \in M_\tau(hL'h^{-1})$. Therefore $h' \in M_\tau hL'$ and it is necessary that $h' \in M_\tau hL'^F$. We have $|M_\tau^F hL'^F| = |M_\tau^F|.|L'^F|.|M_\tau^F hL'^F|.$

Note that $M_\tau^F \cap hL'^F h^{-1} = (M_\tau \cap hL'^F h^{-1})^F = C_{M_\tau}(\text{hso}(\nu)) \cap hL'^F h^{-1}$.

(ii) To calculate $C_{M_\tau}(\text{hso}(\nu))$, we can write $\text{hso}(\nu) = (l_0 l_0) \in \text{GL}_m(k)$ and $l_1 \in T_\tau$. Then $C_{\text{GL}_m(k)}(l_0 l_0)^F \cong \text{SO}_{h_2}^{\text{sgn}(\nu_\text{so})}(q) \times \text{Sp}_{h_1(h_1+1)}(q)$ according to the proof of Proposition 10.2.13 and $C_{T_\tau}(l_1)^\circ = C_{T_\tau}(l_1)^\circ$ since $l_1$ acts trivially on $T_\tau$.

Corollary 10.2.16. With the same notations in Proposition 10.2.14, if we denote $z_\tau = |C_{W'}(\tau)|$ and $z_\nu = |C_{W}(\nu)|$, then the cardinality of the inverse image of $W_{\tau,\nu}$ under the map of Proposition 10.2.14 (i) is equal to

$$|L'^F||M_\tau^F||T_\tau^F|^{-1}|\text{Sp}_{h_1(h_1+1)}(q)|^{-1}|\text{SO}_{h_2}^{\text{sgn}(\nu_\text{so})}(q)|^{-1}z_\tau z_\nu^{-1}.$$

In particular, if $h_1 = 0$ and $h_2 = 0$ or 1, in which case $M_\tau$ is a maximal torus of $G$, then this is equal to

$$(10.2.8.1) |L'^F||M_\tau^F||T_\tau^F|^{-1}z_\tau z_\nu^{-1}.$$
11. The Formula

11.1. Decomposition into Deligne-Lusztig Inductions.

11.1.1. By Proposition 5.2.2 and Proposition 5.2.3 every $\sigma$-stable irreducible character $\chi$ of $\GL_n(q)$ is of the form $R_{M_{1,L}}^{C}(\chi_1 \boxtimes \chi_0)$, for an $F$-stable Levi subgroup $M_{1,L}$ isomorphic to the $\sigma$-stable standard Levi subgroup $L_1$ of the form (5.2.2.1) equipped with the Frobenius $F_w$ given by (5.2.3.5), (5.2.3.6) and (5.2.3.7) and with the action of $\sigma$ given by (5.2.3.8) and (5.2.3.9). Decomposing $L_1$ into $L_1 \times L_0$ following (5.2.3) then $\chi_1$ and $\chi_0$ are identified with some $\sigma$-stable characters of $L_1^{F_w}$ and $L_0^{F_0}$ respectively, where we also denote by the same letter the restriction of $F_w$ to $L_1$ and by $F_0$ its restriction to $L_0$. We decompose $\sigma$ into $(\sigma_1 \sigma_0)$ with respect to $L_1 \times L_0$. Recall that $\chi_0$ is defined by a 2-partition $(\mu_+, \mu_-)$ and that $\chi_1$ is defined by $[\varphi_1] \in \Irr(W_{L_1}^{F_w})$ and $\theta_1 \in \Irr_{reg}(L_1^{F_w}, \sigma_1)$ satisfying the assumptions of (5.2.4) where $\varphi_1 \in \Irr(W_{L_1}^{F_w})$.

By Lemma 3.2.1 an extension of $\chi$ to $G^F, \sigma$ is obtained by first extending $\chi_1 \boxtimes \chi_0$ to $M_{1,L}^{F_w, \sigma}$ and then taking the induction $R_{M_{1,L}}^{C, \sigma}$. One can equally extend $\chi_1 \boxtimes \chi_0$, regarded as a character of $L_1^{F_w}$, to $L_1^{F_w, \sigma}$. The extension of $\chi_1$ to $L_1^{F_w, \sigma}$ is given by Theorem 9.2.1 and the extension of $\chi_0$ to $L_0^{F_0, \sigma_0}$ is given by Theorem 9.4.3. Explicitly, we have

$$\tilde{\chi}_1|_{L_1^{F_w, \sigma_1}} = |W_{L_1}^{\sigma_1}|^{-1} \sum_{w_1 \in W_{L_1}^{\sigma_1}} \tilde{\varphi}_1(w_1 F_w) R_{T_{w_1}, \sigma_1}^{L_1, \sigma_1} \tilde{\vartheta}_1,$$

$$\tilde{\chi}_0|_{L_0^{F_0, \sigma_0}} = \frac{1}{|W_{L_1}^{\sigma_1}|} \frac{1}{|W_{L_1}^{\sigma_0}|} \sum_{w_+ \in W_{L_1}^{\sigma_1}} \varphi_+(w_+) \varphi_-(w_-) R_{L_0, \sigma_0}^{L_0, \sigma_0} \tilde{\phi}_w,$$

and in the second equality there is an extra $\gamma_{\chi_0}$ if $\sigma_0$ is of symplectic type. Now put

$$\gamma_{\chi} := \begin{cases} \gamma_{\chi_0} & \text{if } \sigma \text{ is of symplectic type}, \\ 1 & \text{otherwise}. \end{cases}$$

Write $L_{w_1,w} = T_{w_1} \times L_w$. It is an $F_w$-stable and $\sigma$-stable Levi factor of a $\sigma$-stable parabolic subgroup of $L_1$. Combining the above two formulas gives

$$\tilde{\chi}_1 \boxtimes \tilde{\chi}_0|_{L_1^{F_w, \sigma}} = \gamma_{\chi} |W_{L_1}^{\sigma_1}| \times |W_{L_1}^{\sigma_0}| \times R_{L_{w_1}, \sigma}^{L_1, \sigma} (\tilde{\partial} \Box \tilde{\phi}_w),$$

Suppose that under the isomorphism $L_1, <\sigma> \cong M_{1,L} \times <\sigma>$, which is compatible with the action of $\sigma$, the subgroups $L_{w_1,w}$ become $M_{w_1,w}$, and that the decomposition $L_1 \times L_0$ induces the decomposition $M_{1,L} \cong M_1 \times M_0$. We deduce
Theorem 11.1.1. The extension of $\chi$ is given by the following formula:

$$\tilde{\chi}|_{G^T, \sigma} = \gamma^T_{\chi}|W^G_{M_1} \times W_+ \times W_-|^{-1} \sum_{(w_1, w_+, w_-) \in W^G_{M_1} \times W_+ \times W_-} \tilde{\phi}(w_1 F) \phi_+(w_+) \phi_-(w_-) R_{M_{w_1 \sigma}}^{G, \sigma} (\tilde{\theta}) \tilde{\phi}_w.$$  

11.1.2. Recall §5.2.8 that the $\sigma$-stable irreducible characters of $GL_n(q)$ are parametrised by $\hat{\mathfrak{X}}_\chi$. Denote by $\hat{\mathfrak{X}}_\chi^0 \subset \hat{\mathfrak{X}}_\chi$ the subsets of the elements

$$\lambda_+ \lambda_- (\lambda_i, \alpha_i)_{i \in A_1} (\lambda'_j, \alpha'_j)_{j \in A_2}$$

in which at most one of $|\lambda_+|$ and $|\lambda_-|$ is odd, and $\lambda_\pm$ is a partition with trivial 2-core or with 2-core (1) according to the parity.

Corollary 11.1.2. The $\sigma$-stable irreducible characters of $GL_n(q)$ that extends to uniform functions on $GL_n(q, \sigma)$ are in bijection with $\hat{\mathfrak{X}}_\chi^0$, and the extensions of these characters constitute a base (identifying two extensions of the same character) of the space of the uniform functions on $GL_n(q, \sigma)$.

Proof. We have seen in Theorem 11.1.1 that the extension of a general $\sigma$-stable irreducible character is decomposed into a linear combination of cuspidal functions induced from

$$M^F_{w_1, w} \cong T^F_{w_1} \times T^F_{w_+} \times T^F_{w_-} \times GL_m(q),$$

for various $w_1$ and $w$. Cuspidal functions induced from $M^F_{w_1, w}$ with $m > 1$ can not be uniform (see §3.3.2 for the definition of uniform functions). Now the condition $m \leq 1$ is equivalent to the condition that the sum of the 2-cores of $\lambda_+$ and $\lambda_-$ is either empty or (1). We see that $\lambda_+$ and $\lambda_-$ satisfy the assumption in the definition of $\hat{\mathfrak{X}}_\chi^0$, whence the first assertion.

For each $\tilde{\chi} \in \hat{\mathfrak{X}}_\chi^0$, denote by $\chi_\tilde{\chi}$ the corresponding character, and choose an extension $\tilde{\chi} \in C(GL_n(q, \sigma))$. Then, $\{ \chi_\tilde{\chi} | \tilde{\chi} \in \hat{\mathfrak{X}}_\chi^0 \}$ is a set consisting of functions orthogonal to each other. Theorem 11.1.1 gives a transition matrix between the set of (generalised) Deligne-Lusztig characters and that of the $\tilde{\chi}$'s.

Remark 11.1.3. The extension of an irreducible character is then either uniform, or orthogonal to all uniform functions. Since the characteristic function of a quasi-semi-simple conjugacy class is uniform, the extension of a character corresponding to an element of $\hat{\mathfrak{X}}_\chi \setminus \hat{\mathfrak{X}}_\chi^0$ vanishes on every quasi-semi-simple element.

12. Examples

We give the character tables of $GL_2 \times <\sigma>$ and $GL_3 \times <\sigma>$. We assume that $q \equiv 1 \mod 4$ and write $i = \sqrt{-1}$. Denote by $\mu_2$ the 2-elements group, identified with $\{\pm 1\}$. Let $\eta \in \text{Irr}(\mathbb{F}_q^\times)$ denote the order 2 character. The $\sigma$-stable irreducible characters are specified in terms of $(L, \varphi, \theta)$ as in Theorem 11. Denote by $T$ the maximal torus consisting of the diagonal matrices and denote by $w$ the unique nontrivial element of $W_G(T)$, either for $GL_2(k)$ or $GL_3(k)$, that is fixed by $\sigma$. Denote by $T_w$ a $\sigma$-stable and $F$-stable maximal torus corresponding to the conjugacy class of $w$. In the following, we will freely use the formulas in §11.3.4.
12.1. \(\sigma\)-stable Irreducible Characters. We specify the \(\sigma\)-stable irreducible characters of \(\text{GL}_2(q)\) and \(\text{GL}_3(q)\), and compute the numbers of these characters and of the quadratic-unipotent characters of \(\text{GL}_4(q)\) and \(\text{GL}_5(q)\).

12.1.1. Suppose \(G = \text{GL}_2(k)\). There are \(q + 3\) \(\sigma\)-stable irreducible characters of \(\text{GL}_2(q)\), and 5 of them are quadratic-unipotent, among which one extends into a non-uniform function.

The quadratic-unipotent characters induced from \(L = G\) are the following.

\[
\begin{array}{cccc}
\text{Id} & \eta \text{Id} & \text{St} & \eta \text{St} \\
\end{array}
\]

The only one quadratic-unipotent character induced from \(L = T\) is the following.

\[R^G_T(1, \eta)\]

It is the unique \(\sigma\)-stable irreducible character with non-uniform extension.

Other \(\sigma\)-stable irreducible characters are either of the form,

\[R^G_T(\alpha, \alpha^{-1})\]

with \(\alpha \in \text{Irr}(\mathbb{F}_q^*)\) satisfying \(\alpha^q = \alpha, \alpha \neq 1 \text{ or } \eta\). There are \((q - 3)/2\) of them; or of the form,

\[R^G_T(\omega)\]

with \(\omega \in \text{Irr}(\mathbb{F}_q^*)\) satisfying \(\omega^q = \omega^{-1}, \omega \neq 1 \text{ or } \eta\). There are \((q - 1)/2\) of them.

12.1.2. Suppose \(G = \text{GL}_3(k)\). There are \(2q + 6\) \(\sigma\)-stable irreducible characters of \(\text{GL}_3(q)\), and 10 of them are quadratic-unipotent, among which two have non-uniform extensions.

The quadratic-unipotent characters induced from \(L = G\) are the following.

\[
\begin{array}{cccc}
\text{Id} & \eta \text{Id} & \chi_2 & \eta \chi_2 \chi_3 & \eta \chi_3 \\
\end{array}
\]

The characters \(\chi_2\) and \(\eta \chi_2\) are associated to the sign character of \(\mathbb{S}_3\). The characters \(\chi_3\) and \(\eta \chi_3\) are associated to the degree 2 character of \(\mathbb{S}_3\), and these two characters have non-uniform extensions.

The quadratic-unipotent characters induced from \(L \cong \text{GL}_2(k) \times k^*\) are the following.

\[
\begin{array}{cccc}
R^G_T(\text{Id}, \eta) & R^G_T(\eta \text{Id}_2, \text{Id}) & R^G_T(\text{St}, \eta) & R^G_T(\eta \text{St}, \text{Id}) \\
\end{array}
\]

Other \(\sigma\)-stable irreducible characters are either of the form,

\[R^G_T(\alpha, 1, \alpha^{-1}) \quad R^G_T(\alpha, \eta, \alpha^{-1})\]

with \(\alpha \in \text{Irr}(\mathbb{F}_q^*)\) satisfying \(\alpha^q = \alpha, \alpha \neq 1 \text{ or } \eta\). There are \(q - 3\) of them; or of the form,

\[R^G_T(\omega, 1, \omega^{-1}) \quad R^G_T(\omega, \eta, \omega^{-1})\]

with \(\omega \in \text{Irr}(\mathbb{F}_q^*)\) satisfying \(\omega^q = \omega^{-1}, \omega \neq 1 \text{ or } \eta\). There are \(q - 1\) of them.

12.1.3. Suppose \(G = \text{GL}_4(k)\). There are

\[\frac{1}{2}(q - 2)(q - 3) + 7(q - 2) + 20\]

\(\sigma\)-stable irreducible characters of \(\text{GL}_4(k)\), and 20 of them are quadratic-unipotent.
The Levi subgroups $L = GL_4(k)$, $L = GL_3(k) \times k'$ and $L = GL_2(k) \times GL_2(k)$ give rise to $(5 + 3 + 2) \times 2$ quadratic-unipotent characters, knowing that $|\text{Irr}(\mathbb{Z}_4)| = 5$.

The Levi subgroup $L \cong GL_2(k) \times (k' \times k')$ gives

$$|\text{Quad. Unip. of } GL_2| \times (q - 2) = 5(q - 2)$$

noticing that $q - 2 = (q - 3)/2 + (q - 1)/2$ as in the case of $G = GL_2(k)$.

The Levi subgroup $L \cong GL_2(k) \times GL_2(k)$ gives $(q - 2) \times 2$ with $2 = |\text{Irr}(\mathbb{Z}_2)|$.

The maximal torus $(k')^2 \times (k')^2$ gives $\frac{1}{2}(q - 2)(q - 3)$.

12.1.4. Suppose $G = GL_5(k)$. There are

$$(q - 2)(q - 3) + 14(q - 2) + 36$$

$\sigma$-stable irreducible characters, and 36 of them are quadratic-unipotent.

The Levi subgroups $L = GL_5(k)$, $L \cong GL_4(k) \times k'$ and $L \cong GL_3(k) \times GL_2(k)$ give $(7 + 5 + 3 \times 2) \times 2$ quadratic-unipotent characters, knowing that $|\text{Irr}(\mathbb{Z}_5)| = 7$.

The Levi subgroup $L \cong GL_3(k) \times (k')^2$ gives

$$|\text{Quad. Unip. of } GL_3| \times (q - 2) = 10(q - 2).$$

The Levi subgroup $L \cong k' \times (GL_2(k) \times GL_2(k))$ gives $2 \times 2 \times (q - 2)$, with one factor $2 = |\text{Irr}(\mathbb{Z}_2)|$ and the other factor $2 = ||1, \eta||$.

The maximal torus $L \cong k' \times (k')^2 \times (k')^2$ gives $2 \times \frac{1}{2}(q - 2)(q - 3)$.

12.2. Conjugacy Classes. We present the conjugacy classes of $GL_2(q).\sigma$ and of $GL_3(q).\sigma$, and count the isolated classes of $GL_4(q).\sigma$ and of $GL_5(q).\sigma$. Denote by $su$ the Jordan decomposition of an element of the conjugacy class concerned. Note that $O_1(k) \cong {\mu}_2$ and $O_2(k) \cong k' \times <\tau>$ with $\tau(x) = x^{-1}$.

12.2.1. Suppose $G = GL_2(k)$. There are $q + 3$ conjugacy classes, and 5 of them are isolated.

- $s = (1, 1).\sigma$, $C_G(s) = SL_2(k)$.

The unipotent parts are given by the partitions defined by Jordan blocks. Then the centralisers and the $G^2$-classes are specified accordingly as below,

| (1^2) | (2) |
|-------|------|
| $SL_2(k)$ | $O_1(k)V$ |
| $C_1$ | $C_2$ | $C_3$ |

where $V \cong A^1$ is the unipotent radical. If we use the unit element of a root subgroup of $SL_2(k)$ to represent $(2)$, then $C_2$ corresponds to the identity component of the centraliser. The two components of $O_1(k)V$ have as representatives the scalars $\pm \text{Id}$.

- $s = (i, -i).\sigma$, $C_G(s) = O_2(k)$.

Denote by $C_4$ the $G(q)$-class corresponding to the identity component, and $C_5$ the
other class. The two components of $O_2(k)$ have as representatives
\[
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]
respectively, and so induce the Frobenius $x \mapsto x^q$ and $x \mapsto x^{-q}$ respectively. In other words, the centralisers of $C_4$ and $C_5$ are $O_2^+(q)$ and $O_2^-(q)$ respectively.

- $s = (a, a^{-1})\sigma$, $C_G(s) = k^s$.

For any value of $a$, the corresponding $G$-class contains a unique $G(q)$-class. The classes are as follows.
\[
\begin{array}{cccc}
C_6(a) & C_7(a) & C_8(a) & C_9(a) \\
(1^{q-1}) & (1^{q+1}) & (1^{q+1}) & (1^{q+1})
\end{array}
\]
The Frobenius on $C_G(s) \cong k^s$ with $s \in C_6$ or $C_8$ is $x \mapsto x^q$, while the Frobenius on $C_G(s) \cong k^s$ with $s \in C_7$ or $C_9$ is $x \mapsto x^{-q}$.

We have
- $|C_1| = |G(q)|/|SL_2(q)| = q - 1$;
- $|C_2| = |C_3| = |G(q)|/2|V(q)| = \frac{1}{2}(q - 1)^2(q + 1)$;
- $|C_4| = |G(q)|/|O_2^+(q)| = \frac{1}{2}q(q + 1)(q - 1)$; $|C_5| = |G(q)|/|O_2^-(q)| = \frac{1}{2}q(q - 1)^2$;
- $|C_6| = |C_8| = |G(q)|/(q - 1) = q(q + 1)(q - 1)$;
- $|C_7| = |C_9| = |G(q)|/(q + 1) = q(q - 1)^2$.

12.2.2. Suppose $G = GL_3(k)$. There are $2q + 6$ conjugacy classes, and 10 of them are isolated. Now each semi-simple G-conjugacy class contains two $G(q)$-conjugacy classes, distinguished by the sign $\eta$ (cf. (6.2.2.2)). Depending on the value of $\eta$, we will write $C^+$ or $C^-$ to represent the corresponding conjugacy class contained in a given $G$-conjugacy class.

**Notation 12.2.1.** In what follows, we write $\epsilon$ instead of $\eta$ to avoid clashing with the character of $F_q^\times$.

- $s = (1, 1, 1)\sigma$, $C_G(s) = O_3(k)$.

The unipotent parts are given by the partitions defined by Jordan blocks. Then the centralisers and the $G^L$-classes are specified accordingly as below,
\[
\begin{array}{cccc}
(1^3) & (3) \\
O_3(k) & O_1(k).V \\
C_1^+ & C_1^- & C_2^+ & C_2^-
\end{array}
\]
where $V \cong A^1$ is the unipotent radical.

- $s = (i, 1, -i)\sigma$, $C_G(s) \cong SL_2(k) \times O_1(k)$.

The unipotent parts are given by the partitions defined by Jordan blocks. Then the centralisers and the $G^L$-classes are specified accordingly as below,
12.2.3 Suppose we have $s \in G$. Then the unipotent parts are given by the partitions defined by Jordan blocks. Then the connected components of $C_G(s)$ are as follows.

The conjugacy classes are as follows.

- $s = (a, 1, a^{-1})$, identified with \{\text{diag}(x, \pm 1, x^{-1}); x \in k^*\}.

The Frobenius on $C_G(s) \cong k^*$ with $s \in C_6^\pm$ or $C_8^\pm$ is $x \mapsto x^q$, while the Frobenius on $C_G(s) \cong k^*$ with $s \in C_7^\pm$ or $C_9^\pm$ is $x \mapsto x^{-q}$.

We have

- $|C_4^\pm| = |C_5^\pm| = |G(q)|/|O_3(q)|$;
- $|C_4^\pm| = |C_5^\pm| = |G(q)|/2|V(q)|$;
- $|C_4^\pm| = |C_5^\pm| = |G(q)|/2|SL_2(q)|$;
- $|C_4^\pm| = |C_5^\pm| = |G(q)|/4|V(q)|$;
- $|C_6^\pm| = |C_8^\pm| = |G(q)|/2(q-1)$;
- $|C_7^\pm| = |C_9^\pm| = |G(q)|/2(q+1)$.

12.2.3 Suppose $G = GL_4(k)$. There are 20 isolated conjugacy classes.

- $s = (1, 1, 1, 1)$, $C_G(s) = Sp_4(k)$.

The unipotent parts are given by the partitions defined by Jordan blocks. Then the reductive parts of the centralisers are specified accordingly as below.

This gives 7 classes.

- $s = (i, 1, 1, -i)$, $C_G(s) = SL_2(k) \times O_2(k)$.

The unipotent parts are given by the partitions defined by Jordan blocks. Then the reductive parts of the centralisers are specified accordingly as below.

This gives 6 classes.
12.2.4. Suppose $G = GL_5(k)$. There are 36 isolated conjugacy classes.
- $s = (1, 1, 1, 1, 1) \sigma$, $C_G(s) = O_5(k)$.
The unipotent parts are given by the partitions defined by Jordan blocks. Then the reductive parts of the centralisers are specified accordingly as below,

\[
\begin{array}{c|c|c|c}
1 & 4 & 13 & 2^3 \\
O_5(k) & O_2(k) \times O_1(k) & O_1(k) \times SL_2(k) & O_1(k)
\end{array}
\]

This gives 10 classes.
- $s = (1, 1, 1, -i, -i) \sigma$, $C_G(s) = O_3(k) \times SL_2(k)$.
The unipotent parts are given by the partitions defined by Jordan blocks. Then the reductive parts of the centralisers are specified accordingly as below,

\[
\begin{array}{c|c|c|c|c}
1 & 3 & 2 & 2^2 & 2^3 \\
O_3(k) & O_2(k) \times O_1(k) & O_1(k) \times SL_2(k) & SL_2(k) & O_1(k)
\end{array}
\]

This gives $2 \times 12 = 24$ classes.
- $s = (i, i, -i, -i) \sigma$, $C_G(s) = O_4(k)$.
The unipotent parts are given by the partitions defined by Jordan blocks. Then the reductive parts of the centralisers are specified accordingly as below,

\[
\begin{array}{c|c|c|c|c}
1 & 4 & 2^2 & 2^3 \\
O_4(k) & O_1(k) \times O_1(k) & O_1(k) \times SL_2(k) & SL_2(k) & O_1(k)
\end{array}
\]

This gives 14 classes.

12.3. **The Tables.** The calculation of the values of the uniform characters is reduced to the determination of the sets

\[ A = A(s \sigma, T_w) = \{ h \in G^F | h s o h^{-1} \in T_{w \sigma} \} \]

for various $G^F$-conjugacy classes of $F$-stable and $\sigma$-stable maximal tori $T_w$ contained in some $\sigma$-stable Borel subgroups, and semi-simple $G^F$-conjugacy classes of elements $s \sigma$.

12.3.1. The procedure (cf. [10]) for computing $A$ can be summarised as follows.

Suppose $s \sigma$ is an $F$-fixed element contained in $T.\sigma$, and $T_w$ can be written as $g T g^{\sigma}$ for some $g \in C_G(\sigma)$. If $h \in G^F$ conjugates $s \sigma$ into $T_{w \sigma}$, then there exists some $l \in C_G(s) \cap C_G(\sigma)$ such that $n := g^{-1} h l$ lies in $N_G(T.\sigma)$. Recall that $N_G(T.\sigma) \subset N_G(T)$ consists of the connected components that are stable under $\sigma$. Then $g^{-1} h s o h^{-1} g = n s o n^{-1}$ is an $F_w$-fixed element
of \(T\sigma\). If \(s\sigma\) is an \(F\)-fixed element contained in \(T_w\sigma\), then similar arguments show that 
\(g^{-1}\text{hs}^{-1} = n g^{-1}s \sigma g^{-1}n^{-1} = h l\). The conjugation by \(n\) can be separated into a 
permutation of the "eigenvalues" and a conjugation by an element of \(T\). For each \(s\sigma\) and \(T_w\sigma\)
we will first find some \(t \in T\) such that \(tsot^{-1}\) (or \(tg^{-1}s \sigma g^{-1}\) if we start with some 
\(s\sigma \in T_w\sigma\)) is fixed by \(F_w\), then evaluate any character of \(T_w, <\sigma>\) under the isomorphism 
\(T_w, <\sigma> \cong T^F, <\sigma>\). The value does not depend on the choice of \(t\), and the permutation of the "eigenvalues" is 
simple.

We will use the following observation. Let \(\omega \in \text{Irr}(\mathbb{F}^s_{\sigma})\) be such that \(\omega^f = \omega^{-1}\), and let 
\(a \in \mathbb{F}^s_{\sigma}\*\) be such that \(a^f = a\). Then \(a = b^{f+1}\) for some \(b \in \mathbb{F}^s_{\sigma}\*\), so
\[\omega(a) = \omega(b^{f+1}) = \omega^{f+1}(b) = 1.\]

12.3.2. Suppose \(G = GL_2(k)\). Consider the characters \(R^G_T(1, \eta), R^G_T(\alpha, \alpha^{-1})\) and \(R^G_T(\omega)\).

The calculation of the extensions of \(R^G_T(1, \eta)\) is a direct application of the theorem of 
Waldspurger. Following the notations of (9.4.1), we have \((\mu_+, \mu_-) = ((1), (1))\). The 2-cores are 
(1) and (1), and so \(m_+ = m_- = 1\). We deduce that \(h_1 = 1\) and \(h_2 = 0\). So the cuspidal function 
is supported on the class of \(su\) with \(C_G(s)^{\sigma} = SL_2(k)\) and \(u\) corresponding to the partition (2).
The value does not depend on the support on whose \(\pm \sqrt{q}\) and vanish on all other classes.

If \(s\sigma = \sigma\) and so \(C_G(s\sigma) = SL_2(k)\), then \(h \sigma h^{-1} = \text{det}(h) \sigma\) (regarding \(\text{det}(h)\) as a scalar matrix),
which belongs to \(T\sigma\) or \(T_w\sigma\) for any \(h\). So \(A = G^F\) and \(\tilde{\theta}(\text{hs}^{-1}) = \theta(\sigma) = 1\) for any \(h\) as \(\theta\)
has trivial value on the scalars.

If \(s\sigma = (i, -i)\sigma\) and so \(C_G(s\sigma) = O_2(k)\), then the elements of \(A\) are exactly those \(h \in G^F\)
such that \((h^{-1}Th \cap C_G(s\sigma))^\sigma\) is a maximal torus of \(C_G(s\sigma)^\sigma = SO_2(k)\) which itself is a torus whose 
centraliser in \(G\) is \(T\) or \(T_w\) according to whether \(s\sigma \in C_4\) or \(s\sigma \in C_5\). Consequently, 
\(A(C_5, T) = A(C_4, T_w) = \emptyset\), while \(A(C_4, T)\) and \(A(C_5, T_w)\) are the normalisers of \(T\) and \(T_w\)
respectively. It is easy to check that
\[\tilde{\theta}(\text{hs}^{-1}) = \alpha(i)\alpha^{-1}(-i) = \alpha(-1)\]
if \(s\sigma \in C_4\). If \(s\sigma \in C_5\), then we use the method at the beginning of this section. It suffices 
to find some \(t \in T\) such that \(tsot^{-1}\) is fixed by \(F_w\). Indeed, we can take \(t = \text{diag}(\lambda, 1)\) with 
\(\lambda^f = -\lambda\) so that \((i\lambda)^f = -i\lambda\). We get \(\theta(s\sigma) = \omega(i\lambda)\). The value is independent of the choice 
of \(\lambda\). We can also do it directly and explicitly, and obtain the same result. The elements of 
\(O_2(k) \setminus SO_2(k)\) are of the form
\[
\begin{pmatrix}
0 & x \\
x^{-1} & 0
\end{pmatrix},
\]
so they do not belong to \(SL_2(k)\). Let us describe \(T_s\) explicitly by choosing \(g_2 \in SL_2(k)\) such that
\[
g_2^{-1}F(g_2) = \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
\]
and putting $T_w = g_2 T g_2^{-1}$. We choose $\lambda \in k^*$ such that $\lambda^q = -\lambda$. Put

$$g_1 = g_2 \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix},$$

such that $g_1^{-1} F(g_1) \in O_2(k)$. Then the representative $s \sigma \in C_5$ is given by

$$g_1 \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} g_1^{-1} = g_2 \begin{pmatrix} i\lambda & 0 \\ 0 & -i\lambda \end{pmatrix} g_2^{-1} \sigma \in T_w \sigma,$$

and so $\tilde{\theta}(s \sigma) = \omega(i\lambda)$. If $\omega = \eta$, then taking the norm gives $\lambda^2$ and evaluating $\eta$ gives $-1$.

If $s \sigma = (a, a^{-1}) \sigma$, and so $C_5(s \sigma) = T$ or $T_w$ according to whether $a^{q} = \pm a$ or $a^{q} = \pm a^{-1}$, then $A$ is equal to the normaliser of $T$ or $T_w$ or empty according to the $G^F$-class of $C_5(s \sigma)$. If $s \sigma \in C_8$, the $F$-stable conjugate of $s \sigma$ in $T \sigma$ is given by $\text{diag}(a\lambda, a^{-1}\lambda) \sigma$ with $\lambda^q = -\lambda$. If $\theta = \eta \circ \det|T$, then $\tilde{\theta}((a\lambda, a^{-1}\lambda) \sigma) = \eta(\lambda^2) = -1$. If $s \sigma \in C_9$, the representative of $C_9(a)$ is given by

$$g \begin{pmatrix} a\lambda & 0 \\ 0 & a^{-1}\lambda \end{pmatrix} g^{-1} \sigma \in T^F_s \sigma.$$ 

Again, if $\theta = \eta \circ \det|T$, then $\tilde{\theta}((a\lambda, a^{-1}\lambda) \sigma) = \eta(\lambda^2) = -1$.

12.3.3. Suppose $G = \text{GL}_3(k)$. Consider the characters $\chi_3, R_T^G(\alpha, 1, \alpha^{-1}), R_T^G(\omega, 1, \omega^{-1}), R_T^G(\eta \chi_3), R_T^G(\alpha, \eta, \alpha^{-1})$ and $R_T^G(\omega, \eta, \omega^{-1})$.

For $\chi_3$, we use the theorem of Waldspurger. We have $(\mu_+, \mu_-) = ((1,3), \varnothing)$. The 2-cores are $(2,1)$ and $\varnothing$, and so $m_+ = 2$ and $m_- = 0$. We deduce that $h_1 = 1$ and $h_2 = 1$. So the cuspidal function is supported on the class of $su$ with $C_5(su) \cong \text{SL}_2(k) \times O_1(k)$ and $u$ corresponding to the partition $(2)$. We find $\delta(h_1, h_2) = 1$ and so the values of this character are $\pm \sqrt{7}$.

If $s \sigma = (1, 1, 1) \sigma$ and so $C_5(s \sigma) = O_3(k)$, then one has to understand the set $A^F = (N_G(T, \sigma) L)^F$, with the notations of Lemma[10.2.2]. If $h = nl \in A^F$, then the $L^F$-conjugacy class of $h^{-1} T \sigma n \cap L'$ corresponds to the $F$-class of $n^{-1} F(n) \in N_{L'}(T \cap L')$. But $N_G(T, \sigma) \cong W_G(T)^{\sigma} \cong \Sigma_2$, so $n^{-1} F(n)$ necessarily belongs to $T \cap L' = (T^\sigma)^0$, i.e. $h^{-1} T \sigma n \cap L'$ is always $L^F$-conjugate to $T \cap L'$ and the only Green function that appears in the formula of $R_T^G(\theta(\sigma) u) = Q_{(T^\sigma)^0}(u)$. We also have a similar result for $T_w$. Expressing the elements $h = nl$ as some explicit matrices, we find that $\tilde{\theta}((h \sigma) h^{-1})$ does not depend on $h$. It remains to calculate

\begin{equation}
|N_G(T, \sigma) L^F| = |L^F| |N_G(T, \sigma)| |N_{L^F}(C_T(\sigma)^0)|^{-1}
= |SO_3(q)| \cdot 2(q-1)^3 \cdot 2(q-1),
\end{equation}

For $T_w$, we have

\begin{equation}
|N_G(T_w, \sigma) L^F| = |SO_3(q)| \cdot 2(q-1)(q^2 - 1) \cdot 2(q + 1).
\end{equation}

The other $G(q)$-class contained in the $G$-class of $\sigma$ has as representative $(1, \lambda^2, 1) \sigma$ with $\lambda^q = -\lambda$, so for example the value of $R_T^G(\alpha, \eta, \alpha^{-1})(C_2)$ differs from $R_T^G(\alpha, 1, \alpha^{-1})(C_2)$ by a sign.
The main difference between \( GL_3(q) \) and \( GL_2(q) \) is the class \((i, 1, -i)\sigma \) (as opposed to \((i, -i)\sigma \) for \( GL_2(k) \)). We have \( C_G(s\sigma) \cong SL_2(k) \times O_1(k) \) so in particular it contains representatives of each element of \( W_G(T)^\sigma \). Therefore, the sets \( A^F \) are not empty either for \( T \) or for \( T_w \).

Suppose that \( s\sigma \) represents \( C^e_3 \) and we want to evaluate \( R^G_T(a, \eta, a^{-1}) \) at \( s\sigma \). Let \( t \in T \) be such that \( tsat^{-1} \) is fixed by \( F \). Then \( tsat^{-1} \) can be written as diag\((ix, y, -ix)\sigma \). It is necessary that \( x^q = x \) and \( y^q = y \). So \( \alpha(i)\alpha^{-1}(-i) = \alpha(-1) \). Applying \( \epsilon \) gives \( \eta(y) = \epsilon(C^e_3) \). Therefore \( (a, \eta, a^{-1})(hsoh^{-1}) \) evaluates \( \epsilon\alpha(-1) \). Now we evaluate \( R^G_T(\omega, \eta, \omega^{-1}) \) at \( s\sigma \). Again we write \( tsat^{-1} \) as diag\((ix, y, -ix)\sigma \), but which is \( F_w \)-stable. It is necessary that \( x^q = -x \) and \( y^q = y \). Applying \( \epsilon \) gives \( \eta(y) = -\epsilon(C^e_3) \) since \( x^2 \notin (F^*_q)^2 \). Therefore \( (\omega, \eta, \omega^{-1})(hsoh^{-1}) \) evaluates \( -\epsilon\omega(i\lambda) \). If \( s\sigma \in C^e_8 \), then an \( F \)-stable element diag\((ax, y, a^{-1}x)\sigma = tsat^{-1} \) satisfies \( x^q = -x \) and \( y^q = y \). Therefore \( (a, \eta, a^{-1})(hsoh^{-1}) \) evaluates \( -\epsilon\alpha(a^{\pm 2}) \), where the \( \pm 2 \) power is due to permutation of "eigenvalues". For \( (\omega, \eta, \omega^{-1}) \) at \( C^e_9 \) the calculation is similar.
Table 1. The Character Table of $\text{GL}_2(q). \langle \sigma \rangle$

|                | $(1, 1)\sigma$ | $(i, -i)\sigma$ | $(a, a^{-1})\sigma$ |
|----------------|----------------|------------------|---------------------|
| $(1^2)$        | $(2)$          | 1                | $\sigma^{i+1} = 1$ |
| $C_1$          | 0              | $-\sqrt{q}$     | $0$                 |
| $C_2$          | 0              | $\sqrt{q}$      | $0$                 |
| $C_3$          | $\eta$Id      | 1                | 1                   |
| $C_4$          | 1              | 1                | 1                   |
| $C_5$          | 1              | 1                | 1                   |
| $C_6(\alpha)$  | $q$            | 0                | $\alpha(\lambda^2) + \alpha(\lambda^{-2})$ |
| $C_7(\alpha)$  | 0              | $2\alpha(-1)$   | 0                   |
| $C_8(\alpha)$  | $q + 1$        | 1                | 1                   |
| $C_9(\alpha)$  | $1 - q$        | 1                | $\omega(\lambda) + \omega(\lambda^{-1})$ |

$\text{R}^G_{\eta}(1, \eta)$

$\text{R}^G_{\eta}(\alpha, \alpha^{-1})$

$\text{R}^G_{\eta}(\omega)$
Table 2. The Character Table of \( \text{GL}_3(F_q) \times \langle \sigma \rangle \)

| Character | \( \chi_0 \) | \( \chi_3 \) | \( \chi_5 \) | \( \chi_7 \) | \( \chi_{10} \) | \( \chi_{15} \) |
|-----------|--------------|--------------|--------------|--------------|--------------|--------------|
| \( a, 1, a^{-1} \) | 0 | 0 | 0 | 0 | 0 | 0 |
| \( a, 1, -a \) | \( a, 1, a^{-1} \) | \( a, 1, -a \) | \( a, 1, a^{-1} \) | \( a, 1, -a \) | \( a, 1, a^{-1} \) | \( a, 1, -a \) |
| \( a, 1, -a \) | \( a, 1, a^{-1} \) | \( a, 1, -a \) | \( a, 1, a^{-1} \) | \( a, 1, -a \) | \( a, 1, a^{-1} \) | \( a, 1, -a \) |
| \( a, 1, -a \) | \( a, 1, a^{-1} \) | \( a, 1, -a \) | \( a, 1, a^{-1} \) | \( a, 1, -a \) | \( a, 1, a^{-1} \) | \( a, 1, -a \) |

Note: The table entries represent the character values for different conjugacy classes in \( \text{GL}_3(F_q) \times \langle \sigma \rangle \). The entries are placeholders for actual values which depend on the specific character \( \chi \) and the conjugacy class.
Table 3. The Character Table of $\text{GL}_3(q).<\sigma>$, (ii)

|      | $C_1^+$ | $C_1^-$ | $C_2^+$ | $C_2^-$ | $C_3^+$ | $C_3^-$ | $C_4^+$ | $C_4^-$ | $C_5^+$ | $C_5^-$ |
|------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| $\text{Id}$ | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       |
| $\eta \text{Id}$ | 1     | -1     | 1      | -1      | 1      | -1      | 1      | 1     | -1     | -1     |
| $\chi_2$ | $q$     | $q$     | 0       | 0       | $q$     | $q$     | 0       | 0       | 0       | 0       |
| $\eta \chi_2$ | $q$   | $-q$   | 0       | 0       | $q$   | $-q$   | 0       | 0       | 0       | 0       |
|      | $\epsilon$ | $\epsilon$ | $\epsilon$ | $\epsilon$ | $\epsilon$ | $\epsilon$ | $\epsilon$ | $\epsilon$ | $\epsilon$ | $\epsilon$ |
Table 4. The Character Table of $\text{GL}_3(q).\langle \sigma \rangle$, (iii)

|                | $C_1^+$ | $C_1^-$ | $C_2^+$ | $C_2^-$ | $C_3^+$ | $C_3^-$ | $C_4^+$ | $C_4^-$ | $C_5^+$ | $C_5^-$ |
|----------------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| $R_{L}^{C}(\text{Id}_2, \eta)$ | 1       | -1      | 1       | -1      | $q$     | $-q$    | 0       | 0       | 0       | 0       |
| $R_{L}^{C}(\eta \text{Id}, \text{Id})$ | 1       | 1       | 1       | 1       | $q$     | $q$     | 0       | 0       | 0       | 0       |
| $R_{L}^{C}(\text{St}, \eta)$ | $q$     | $-q$    | 0       | 0       | 1       | $-1$    | 1       | 1       | $-1$    | $-1$    |
| $R_{L}^{C}(\eta \text{St}, \text{Id})$ | $q$     | $q$     | 0       | 0       | 1       | 1       | 1       | 1       | 1       | 1       |
| $\epsilon$    | $\epsilon$ | $-\epsilon$ | $-\epsilon$ | $\epsilon$ | $-\epsilon$ | $\epsilon$ | $-\epsilon$ | $\epsilon$ | $-\epsilon$ | $\epsilon$ |
Table 5. The Character Table of GL_3(q).<σ>, (iv)

|          | $C_1^+$ | $C_1^-$ | $C_2^+$ | $C_2^-$ | $C_3^+$ | $C_3^-$ | $C_4^+$ | $C_4^-$ | $C_5^+$ | $C_5^-$ |
|----------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| $(\alpha, 1, \alpha^{-1})$ | $q + 1$ | $q + 1$ | $1$     | $1$     | $(q + 1)\alpha(-1)$ | $(q + 1)\alpha(-1)$ | $\alpha(-1)$ | $\alpha(-1)$ | $\alpha(-1)$ | $\alpha(-1)$ |
|          | $\alpha(\alpha^2) + \alpha(\alpha^{-2})$ | $0$     |         | $\alpha(\alpha^2) + \alpha(\alpha^{-2})$ | $0$     |         |         |         |         |         |
| $(\alpha, \eta, \alpha^{-1})$ | $q + 1$ | $-q - 1$ | $1$     | $-1$    | $(q + 1)\alpha(-1)$ | $-(q + 1)\alpha(-1)$ | $\alpha(-1)$ | $\alpha(-1)$ | $-\alpha(-1)$ | $-\alpha(-1)$ |
|          | $\epsilon(\alpha(\alpha^2) + \alpha(\alpha^{-2}))$ | $0$     |         | $-\epsilon(\alpha(\alpha^2) + \alpha(\alpha^{-2}))$ | $0$     |         |         |         |         |         |
| $(\omega, 1, \omega^{-1})$ | $-q + 1$ | $-q + 1$ | $1$     | $1$     | $(-q + 1)\omega(\bar{i})$ | $(-q + 1)\omega(\bar{i})$ | $\omega(\bar{i})$ | $\omega(\bar{i})$ | $\omega(\bar{i})$ | $\omega(\bar{i})$ |
|          | $0$     | $\omega(\alpha) + \omega(\alpha^{-1})$ | $0$     |         | $\omega(\alpha\lambda) + \omega(\alpha^{-1}\lambda)$ |         |         |         |         |         |
| $(\omega, \eta, \omega^{-1})$ | $-q + 1$ | $q - 1$  | $1$     | $-1$    | $(q - 1)\omega(\bar{i})$ | $(-q + 1)\omega(\bar{i})$ | $-\omega(\bar{i})$ | $-\omega(\bar{i})$ | $\omega(\bar{i})$ | $\omega(\bar{i})$ |
|          | $0$     | $\epsilon(\omega(\alpha) + \omega(\alpha^{-1}))$ | $0$     |         | $-\epsilon(\omega(\alpha\lambda) + \omega(\alpha^{-1}\lambda))$ |         |         |         |         |         |
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