Topological Properties

Weyl fermions with nonlinear dispersion have appeared in real world systems, such as in the Weyl semi-metals and topological insulators. We consider the most general form of Dirac operators, and study its topological properties embedded in the chiral anomaly, in the index theorem, and in the odd-dimensional partition function, by employing the heat kernel. We find that all of these topological quantities are enhanced by a winding number defined by the Dirac operator in the momentum space, regardless of the spacetime dimensions. The chiral anomaly in $d = 3 + 1$, in particular, is also confirmed via the conventional Feynman diagram. These interconnected results allow us to clarify the relationship between the chiral anomaly and the Chern number of the Berry connection, under dispute in some recent literatures, and also lead to a compact proof of the Nielsen-Ninomiya theorem.
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1 Introduction

In quantum field theories with fermions, we often encounter topological properties, with the chiral anomaly being perhaps the best known such effect. Although initially derived from Feynman diagram [1][2], its topological nature became quickly apparent via Fujikawa’s alternative explanation [3] as the failure of the measure to be invariant under chiral rotations, which in turn translates to the Atiyah-Singer index density of the Dirac operator.

In recent years, chiral anomalies and other topological aspects emerged as relevant and useful concepts in condensed matter systems as well, notably in Weyl semi-metals and topological insulators [4][6]. In the study of such systems, one encounters fermion systems of more general kind than those familiar to high energy physics. Instead of the usual Dirac operator, linear in the spacetime derivatives, a modified Hamiltonian
of type

\[ H \sim \sigma^+(-iD_+)^n + \sigma^-(iD_-)^n + \sigma^3(-iD_3) , \]  

(1.1)
on a flat spatial \( \mathbb{R}^3 \) has appeared in the context of the Weyl semi-metal. A chiral two-component fermion with such a generalized Dirac operator as the Hamiltonian is expected to suffer "\( n \)" times the usual chiral anomaly. This was initially motivated by merging of a pair of chiral Dirac cones in the Brillouin zone, while more direct demonstrations via the Fujikawa method were recently given for \( n = 2 \) and \( n = 3 \) [7,8]. Although this Hamiltonian is natural from coalescence of several Weyl cones in the Brillouin zone, its topological equivalence to multiple Weyl fermions, as manifest in the anomaly, is hardly immediate from the usual continuum field theory viewpoint.

This begs for general inquiries into the anomaly and other topological aspects for fermions whose spacetime Dirac operator takes the most general form

\[ \gamma^\mu \mathcal{P}_\mu(-iD) , \]  

(1.2)
where \( -iD_\mu \) is the covariant momentum operator and \( \mathcal{P}_\mu \) polynomials, or even an arbitrary smooth functions thereof. In the end, we will compute the index density, the anomaly, and also the anomalous phase of the partition function in odd spacetime dimensions, and find that all of these are minimally modified by the winding number of the map \( K_\mu \to \mathcal{P}_\mu(K) \). Apart from this overall factor, the structure of the anomaly and the phase of the partition function remains intact. The same computation can be manipulated to show that this winding number is alternatively computed by counting of the critical points \( K_* \), defined by \( \mathcal{P}(K_*) = 0 \), weighted by parities.

The latter should be reminiscent of the Morse theory [9] for those who are familiar with index theorems, but at the same time, this alternative picture shows how the winding number information of the generalized Dirac operator is connected to the Dirac/Weyl cones in the momentum space, in a way that has been fruitfully used in the condensed matter literatures. In particular, the two alternative interpretations via the winding number and the Morse counting represent, respectively, the ultraviolet and the infrared viewpoint of one and the same quantity. The former viewpoint will connect to topological objects known in the momentum space as the Berry monopole, whose quantized flux can be also related to the \( d = 2 + 1 \) topological insulator in the condensed matter literatures [10–16].
Extending the discussion to odd spacetime dimensions, one finds a similar modification of the anomalous phase of the partition function. Given an odd-dimensional Dirac operator, this phase is computed by the eta-invariant which in turn are related to the Chern-Simons action. We will also see how this Chern-Simons effective action is also multiplicatively enhanced by the same kind of the winding number as in even dimensions. If we consider this odd-dimensional spacetime as a flat boundary of an even dimensional half space-time, an Atiyah-Patodi-Singer index theorem holds, again with the new overall multiplicative factor by the same winding number. This also means that the connection via APS index theorem \[17\] between the \(d = 2 + 1\) boundary fermions and a bulk \(d = 3 + 1\) topological field theory carries over verbatim: much as in even dimensional anomalies, the odd-dimensional anomalous phase does not distinguish between \(N\) ordinary Dirac fermions and a generalized Dirac fermion with the winding number \(N\).

It is our aim to derive these general results, and to explore their physical consequences. The starting point of this investigation is the chiral anomaly for such generalized Weyl fermions, which is one of the most robust handles we have in all of quantum fermions with continuous classical symmetries. The chiral anomaly, after many decades of its initial discovery, can be still mysterious. On the one hand, it is an infrared phenomenon of anomalous particle creation and annihilation at zero energy (level crossing point), in background field configurations where both parity and time-reversal symmetries are broken. On the other hand, its topological nature makes it computable also in ultraviolet scales, leading to its expression in terms of local topological density of background fields. This infrared-ultraviolet connection is a profound characteristic of chiral anomaly, which, when formulated in Euclidean space, leads to its deep connection to the index theorems in mathematics.

The infrared-ultraviolet connection of the chiral anomaly may also manifest itself in momentum space. In the infrared view point, the anomaly should be given by contributions from local level crossing points, where in/out-flows of particle numbers happen. Since the particle number is conserved away from these points due to the Liouville theorem \[18\], the same anomaly may also be seen in the ultraviolet region of large momenta, captured by some topology of the theory in consideration.

Can we prove the existence of such an infrared-ultraviolet connection in momentum space? If yes, what topology of the fermion theory in large momenta contains the information of the infrared chiral anomaly? One of our main results in this work is
to provide a rigorous answer to this question. We show the existence of an infrared-ultraviolet connection of chiral anomaly in momentum space for a general class of theories, where the Dirac operator is an arbitrary polynomial of covariant derivatives. In particular, we prove in Section 3.1 that the topology of Berry’s curvature of projected chiral spinor in the asymptotically large momentum region precisely carries the same information of chiral anomaly in the infrared.

We hope that our work answers some of the questions raised in [19–21] regarding the connection between chiral anomaly and the Berry’s curvature in momentum space. The Berry’s curvature of chiral spinors is an essential ingredient of the kinetic description of chiral particles in phase space, the chiral kinetic theory [22, 26], where semi-classical approximation is justified at large momenta. It is responsible for many novel transport phenomena in real-time dynamics of (pseudo) chiral fermion systems, in both condensed matter physics of Dirac/Weyl semi-metals [22, 27, 32] and the physics of quark-gluon plasma in relativistic heavy-ion collisions [33, 34].

This includes most notably the Chiral Magnetic Effect [35–37], the Chiral Vortical Effect [38, 39], and the Anomalous Hall Effect [40–43]. Within the kinetic theory description, it has been argued that chiral anomaly may also be explained by the same Berry’s curvature [22, 23, 49]. Since the kinetic theory description as well as the concept of Berry’s phase breaks down near level crossing points where chiral anomaly happens, a more rigorous treatment is needed to justify such a relation between chiral anomaly and the Berry’s phase in momentum space [19, 21]. The infrared-ultraviolet connection we show in this work fills the missing logical gap between the two.

For the most part of this note, we will employ the heat kernel methods [50], as it is universally applicable to all spacetime dimensions and is very effective for extracting topological information. In Section 4, however, we will also resort to the usual triangular Feynman diagram for $d = 4$ chiral anomaly, where the modification of the current operators, on top of the higher inverse power of the propagator, plays a crucial role.

* See [44–46] for the experimental observation of Chiral Magnetic Effect in Weyl semimetals, and see [47, 48] for the recent status of experimental search of Chiral Magnetic Effect in relativistic heavy-ion experiments.
2 Generalized Spinors and Dirac Operators

We would like to consider a Dirac index problem with the operator generalized as

\[ Q = \gamma^\mu P_\mu (-iD) \]  

(2.1)

with smooth functions \( P_\mu \). Let us take the Dirac matrices in the chiral basis,

\[ \gamma^a = \begin{pmatrix} 0 & \sigma^a \\ \sigma^a & 0 \end{pmatrix}, \quad \gamma^4 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \]  

(2.2)

and we use the covariant derivative

\[ D_\mu = \partial_\mu + A_\mu , \]  

(2.3)

with anti-hermitian gauge field \( A_\mu \). The Dirac operator has the form

\[ Q = \begin{pmatrix} 0 & \mathcal{D} \\ \mathcal{D}^\dagger & 0 \end{pmatrix} \]  

(2.4)

with

\[ \mathcal{D} = (\sigma^a P_a - iP_4) , \quad \mathcal{D}^\dagger = (\sigma^a P_a + iP_4) . \]  

(2.5)

We are interested in the index theorem of \( Q \), and the chiral anomaly associated with a Weyl fermion with the kinetic operator \( \mathcal{D} \).

2.1 Index Density and Chiral Anomaly

As is well known from the Fujikawa method, the failure of the chiral rotation of the path integral for the relevant two-component Weyl fermion,

\[ \int \left[ D\bar{\psi}D\psi \right] e^{\int \bar{\psi}D\psi} \]  

(2.6)

is measured by the index density, which can be written formally as

\[ \text{Tr} (\Upsilon) \equiv \lim_{s \to 0} \text{Tr} \left( \Gamma e^{-sQ^2} \right) \]  

(2.7)
with \( \Gamma = -\gamma^1\gamma^2\gamma^3\gamma^4 \). Note that the trace here is over the 4-component Dirac spinors even though the physical system is that of a Weyl spinor. One can understand this from the well-known fact that, in the Euclidean signature, \( \bar{\psi} \) has to be treated as independent and transforms oppositely to \( \psi \) under the chiral rotation. In practice, \( \psi \) and \( \bar{\psi} \) together define a Dirac spinor, for which the formal index problem follows. For a most comprehensive study of anomaly and the connection to the index theorem, we refer readers to Ref. [51].

Here, we proceed to compute this quantity by modifying the usual heat kernel method, or

\[
\lim_{s \to 0} \text{Tr} \left( \Gamma e^{-sQ^2} \right) = \lim_{s \to 0} \int d^4x \, \text{tr} \left( \Gamma G_s(x;x) \right) ,
\]

(2.8)

where \( G_s(y; x) \equiv \langle y|e^{-sQ^2}|x \rangle \) obeys

\[
-\partial_s G_s(y; x) = Q^2 G_s(y; x) , \quad \lim_{s \to 0} G_s(y; x) = \delta^{(4)}(y - x) .
\]

(2.9)

So, the problem boils down to how one computes \( G_s(x; x) \). For this, we start with

\[
Q^2 = \mathcal{P}_\mu \mathcal{P}_\mu + \frac{1}{4} [\gamma^\mu \gamma^\nu] [\mathcal{P}_\mu, \mathcal{P}_\nu] ,
\]

(2.10)

which we further split as

\[
Q^2 = Q^2_0 + \delta Q^2 , \quad Q^2_0 \equiv \mathcal{P}_\mu (-i\partial) \mathcal{P}_\mu (-i\partial) .
\]

(2.11)

With the latter, we can perform the usual heat kernel expansion

\[
G_s(y; x) = \sum_{l=0} \mathcal{G}^{(l)}(y; x)
\]

\[
G^{(l+1)}_s(y; x) = - \int_0^s dt \int d^4z \, G^{(0)}_{s-t} (y; z) \delta Q^2 G^{(l)}_t (z; x) ,
\]

(2.12)

where the free heat kernel

\[
G^{(0)}_s(z; x) = \langle z|e^{-sQ^2_0}|x \rangle ,
\]

(2.13)
which is easily found

\[ G_s^{(0)}(x + X; x) = \int \frac{d^4K}{(2\pi)^4} e^{iK \cdot X} e^{-sP(K)^2} \]  

(2.14)

in the momentum space \( \tilde{R}^4 \) of \( K_\mu \).

For the index density, the crucial step is the power counting of small \( s \) in (2.12). Each iteration brings down a factor of \( s \) given the \( s \)-integral, but further fractional factors of \( s \) arises from the \( z \)-integral combined with operators in \( \delta \mathcal{Q}^2 \). Note, in particular, that each derivative in \( \delta \mathcal{Q}^2 \) will cost some inverse fractional power of \( s \). One key identity will be

\[ \frac{1}{\sqrt{\pi}} \int d^4K \det \left( \frac{\partial P_\mu}{\partial K_\alpha} \right) e^{-sP(K)^2} = s^{-2}N_P, \]  

(2.15)

where \( N_P \) is the asymptotic winding number of the map, \( K \to P(K) \). In other words, \( N_P \) measures the multiplicity of the map over the target \( \mathbb{R}^4 \) with the orientation taken into account. With (2.10) and (2.11), and with the insertion of \( \Gamma \) in (2.7), it is clear that the first nontrivial expression out of (2.12) will occur at the second iteration, where we expect to find something like (2.15) times \( s^2 \), leading us to \( N_P \) in the end. Let us now track how this occurs.

The relevant contribution can be found from the further expansion of the squared Dirac operator

\[ \frac{1}{4} [\gamma^\mu, \gamma^\nu][P_\mu, P_\nu] = \frac{1}{2} \gamma^\mu \gamma^\nu F_{\alpha\beta} \frac{\partial P_\mu(K)}{\partial K_\alpha} \frac{\partial P_\nu(K)}{\partial K_\beta} + \cdots, \]  

(2.16)

where the ellipsis denotes terms that come with less free standing derivatives, i.e., less factors of \( K \)'s, or more \( A \)'s. These cost less power of \( s^{-1} \) and effectively disappear as \( s \to 0 \) limit is taken in the end. Because of the \( \Gamma \) insertion in (2.7), the first nontrivial term arises in the second order of the iteration of (2.12), when one pulls down \( \gamma \gamma F \) in (2.16). This will be accompanied effectively by a factor of \( s^2/2 \), due to the two \( s \) integrals, producing a term like (2.15). This shows how all the subsequent terms in (2.16) become irrelevant for the purpose of computing the index density. In fact, all interaction pieces in \( \mathcal{P}(-iD)^2 \) belong to the latter category, so for the purpose of computing the index density, all that matter is the first term on the right hand side of (2.16) in place of \( \delta \mathcal{Q}^2 \).
Let us trace this process more explicitly. Since an explicit factor of \( x \)'s in \( \delta Q^2 \), such as in Taylor expansion of \( F \)'s, cost positive factors \( s \)'s, relative to the one in (2.16), we only need to worry about how the free-standing derivatives in \( \delta Q^2 \) works in the heat kernel expansion. With

\[
\Pi(K, F) \equiv \frac{1}{2} \gamma^\mu \gamma^\nu F_{\alpha \beta} \frac{\partial P_\mu(K)}{\partial K_\alpha} \frac{\partial P_\nu(K)}{\partial K_\beta}
\]

one finds

\[
G_s^{(1)}(x + X; x) = \int_0^s dt \int d^4 Y \left( G_s^{(0)}(x + X; x + Y) \Pi(-i \partial; F) G_t^{(0)}(x + Y; x) + \cdots \right)
\]

\[
= \int_0^s dt \int d^4 Y \int \frac{d^4 K}{(2\pi)^4} e^{iK \cdot (X - Y)} e^{-(s - t) P(K)^2} \times \Pi(W; F) \int \frac{d^4 W}{(2\pi)^4} e^{iW \cdot Y} e^{-t P(W)^2} + \cdots
\]

\[
= s \int \frac{d^4 K}{(2\pi)^4} e^{iK \cdot X} e^{-s P(K)^2} \times \Pi(K; F(x)) + \cdots . \quad (2.19)
\]

The above is from an expansion of the operator \( Q^2 \) around a generic point \( x \), and the momentum \( K \) is conjugate to the “small” displacement \( X \).

It is clear that the momentum factors pile up through the iteration, and since the position-dependence of \( F_{\mu \nu}(x) \) does not enter in the small \( s \) limit, the iteration can be performed in the momentum space straightforwardly. Repeating one more time, the same computation gives

\[
G_s^{(2)}(x + X; x) = \frac{s^2}{2} \int \frac{d^4 K}{(2\pi)^4} e^{iK \cdot X} e^{-s P(K)^2} \times (\Pi(K; F(x)))^2 + \cdots . \quad (2.20)
\]

Now we are ready to compute the index density:

\[
\lim_{s \to 0} \text{Tr} \left( \Gamma e^{-sQ^2} \right) = \lim_{s \to 0} \int d^4 x \text{ tr} \left( \Gamma G_s(x; x) \right)
\]

\[
= \lim_{s \to 0} \frac{s^2}{2} \int d^4 x \text{ tr} \left( \frac{1}{4} \Gamma \gamma^\mu \gamma^\nu \gamma^\mu' \gamma^\nu' \right) F_{\alpha \beta} F_{\alpha' \beta'}
\]

\[\text{\footnote{Here we used}} \]

\[
\int d^4 Y e^{i(W - K) \cdot Y} = (2\pi)^4 \delta^{(4)}(W - K) . \quad (2.18)
\]
\[ \times \int \frac{d^4 K}{(2\pi)^4} \frac{\partial P_\mu(K)}{\partial K_\alpha} \frac{\partial P_\nu(K)}{\partial K_\beta} \frac{\partial P_{\mu'}(K)}{\partial K_{\alpha'}} \frac{\partial P_{\nu'}(K)}{\partial K_{\beta'}} e^{-sP(K)^2}. \quad (2.21) \]

With \( \Gamma = -\gamma^1 \gamma^2 \gamma^3 \gamma^4 \) inserted, we need to collect four distinct \( \gamma \)'s to ensure nonzero result, and then the fermionic trace above gives \(-4\epsilon^{\mu\nu\mu'\nu'}\). Then, we may invoke (2.15) and find

\[ \lim_{s \to 0} \text{Tr} \left( \Gamma e^{-sQ^2} \right) = -\frac{N_P}{32\pi^2} \int d^4 x \; \epsilon^{\alpha\beta\alpha'\beta'} F_{\alpha\beta} F_{\alpha'\beta'} = N_P \cdot \left( -\frac{1}{8\pi^2} \int F \wedge F \right), \quad (2.22) \]

which shows the usual index density, and hence the chiral anomaly is enhanced by a factor of \( N_P \), the winding number associated with the map \( K \to \mathcal{P}(K) \).

Although we have computed the index density in four dimensions, the generalization to arbitrary even dimensions, \( d \), is immediate, and gives

\[ \lim_{s \to 0} \text{Tr} \left( \Gamma e^{-sQ^2} \right) = N_P \cdot \left( \frac{1}{(d/2)!(2\pi i)^{d/2}} \int F \wedge \cdots \wedge F \right), \quad (2.23) \]

with \( d/2 \) number of the field strength 2-form \( F = F_{\mu\nu} dx^\mu \wedge dx^\nu/2 \), anti-hermitian as before. The winding number of the map \( K \to \mathcal{P}(K) \) enters this formula via\( ^4 \)

\[ N_P \equiv \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}^d} d^d K \det \left( \tilde{\partial}_a P_\mu \right) e^{-P(K)^2}, \quad (2.24) \]

Each oriented copy of \( \mathbb{R}^d \) in the image gives 1 times the sign of the Jacobian, so this measures how many times the map \( \mathcal{P}(K) \) covers the target \( \mathbb{R}^d \).

### 2.2 Infrared Interpretation

Note that this integral produces an integer, as long as \( \mathcal{P}^2 \) is asymptotically unbounded, since

\[ N_P = \frac{1}{\sqrt{\pi}} \int d^d \mathcal{P} \; e^{-\mathcal{P}^2}, \quad (2.25) \]

where \( N_P \) is now hidden in the integration domain; The integral on the right hand side gives 1 for each integration domain of \( \mathbb{R}^d \), but this is multiplied by \( N_P \) since

\[ \text{We introduced } \tilde{\partial} \text{ to emphasize that it is a partial derivative in the momentum space.} \]
the map $K \to P$ is $N_P$-fold cover of $\hat{\mathbb{R}}^d$. This topological characterization may be considered an ultraviolet description since the winding number is defined via the asymptotic behavior of the map $P(K)$.

On the other hand, the index and the anomaly are fundamentally infrared phenomena, so should be equally visible in the small $|P|$ limit. For this, note that the expression is invariant under $P \to C \cdot P$ for any positive real number $C$, which we already used to scale away $s$ above to reach (2.24). Going back to $K$-space integral and taking a limit of $C \to \infty$, however, we see that (2.24) localizes at the critical points, $P = 0$, and the winding number has an alternative form, as a sum over the critical points, weighted by $\pm 1$, depending on the sign of the determinant there,

$$\sum_{\{K_\ast|P(K_\ast) = 0\}} 1 \cdot \text{sgn} \left[ \det(\tilde{\partial}^\alpha P_\mu) \right]_{K = K_\ast}, \quad (2.26)$$

provided that all the critical points are non-degenerate, i.e., provided that the determinants there do not vanish. In fact, it is easy to see how this generalizes to a case with degenerate critical points,

$$\sum_{\{K_\ast|P(K_\ast) = 0\}} N_P(K_\ast), \quad (2.27)$$

where $N_P(K_\ast)$ is the local winding number near such (degenerate) critical points. This is a Morse theory counting if $P_\mu = \tilde{\partial}_\mu W(K)$ for some Morse function $W(K)$ [9], although we do not really need the latter here.

Although we started with the expression (2.24) that has a natural interpretation as a winding number, measured at the asymptotic region of $K$-space, this alternative description counts the critical points, $P = 0$, where the Dirac operator $Q$ may be approximated by a linear form

$$Q \simeq \tilde{\partial}^\alpha P_\mu(K_\ast) (-i \gamma^\mu D_\alpha).$$

One merely counts how many approximate Dirac cones appear in the infrared end of the dynamics, whose chiralities are dictated by the matrix $\tilde{\partial}^\alpha P_\mu(K_\ast)$. If the latter has negative determinant, this can be translated to a chirality flip, relative to others with positive determinant.
3 Non-Relativistic Iso-Spinor

For condensed matter systems, one sometimes encounters such generalized Weyl fermions where the two components actually refer to flavors or “isospin” rather than the real spin associated with the angular momentum. In recent years, the chiral anomaly in this isospin context has surfaced as important issues, so let us apply what we have developed in the previous section to these real systems.

Since these are all non-relativistic fermions, the direction 4 plays a special role as the genuine (Euclidean) time direction and as such we will be content with the single derivative there. As such, we may specialize to the case

\[ \mathcal{P}_4 = -iD_4 + \Delta(-i\vec{D}) , \quad \mathcal{P}_a = P_a(-i\vec{D}) , \quad (3.1) \]

where we split the four vector to the Euclidean time component and a 3-vector distinguished by the arrow. This will correspond to, in Lorentzian time, a two-component Weyl Hamiltonian of type

\[ H = \sigma^a P_a - \Delta . \quad (3.2) \]

The necessary Wick rotation prescription will become clearer when we compute the anomaly by a Feynman diagram in next section, but for now we will stick to this Euclidean viewpoint.

We could simply rely on the results of the previous section, whereby the anomaly can be seen not affected by the presence of \( \Delta \) at all. Still, let us retrace part of these steps for an illustration, with a simplifying assumption of \( \Delta = 0 \). In other words, let us consider

\[ \mathcal{D} = \sigma^a P_a(-i\vec{D}) - D_4 , \quad \mathcal{D}^\dagger = \sigma^a P_a(-i\vec{D}) + D_4 , \quad (3.3) \]

for some smooth functions \( P_a \) of \(-iD_a=1,2,3\). In addition, we will assume that \( |\vec{P}(\vec{k})|^2 \) grows indefinitely at large \( \vec{k} \)'s; this would be the case, for example, if \( P \)'s are generic polynomials. Our experience above suggests that the anomaly must be again dictated by a topology of the map \( k_a \rightarrow P_a(\vec{k}) \).
The zero-th order heat kernel is

\[
G_s^{(0)}(x + X; x) = \frac{1}{\sqrt{4\pi s}} e^{-X^2/4s} \times \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k} \cdot \vec{X}} e^{-s\vec{P}^2},
\]  

where we now separate out the 3-vector $\vec{X}$ from 4-vector $X$, etc. The pieces in $Q^2$ that can contribute to the index density are, as before

\[
\frac{1}{4} \{ \gamma^\mu, \gamma^\nu \} [P_\mu, P_\nu] = \sum_{a=1}^3 \gamma^4 \gamma^a F_{4h} \partial^h P_a(\vec{k}) + \cdots 
+ \sum_{b>c} \gamma^b \gamma^c F_{fg} \partial^f P_b(\vec{k}) \partial^g P_c(\vec{k}) + \cdots
\]

expressed in the momentum space, and $F$ is considered slowly varying. The structure of the index density we have seen implies that the integrand will contain the multiplicative factor

\[
\frac{1}{6} \epsilon^{abc} \epsilon_{hfg}(\partial^h P_a)(\partial^f P_b)(\partial^g P_c) = \det \left( \frac{\partial P_a(\vec{k})}{\partial k_h} \right),
\]

and we can deduce the expression that plays the role of $N_P$ in (2.22),

\[
N_{\vec{P}} \equiv \frac{s^{3/2}}{\sqrt{\pi}} \int d^3 k \det \left( \frac{\partial P_a(\vec{k})}{\partial k_h} \right) e^{-s\vec{P}^2} = \frac{1}{\sqrt{\pi}} \int d^3 P e^{-\vec{P}^2},
\]

where the factor $s$ on the left hand side is scaled away without affecting the result by the change of the integration variable $s^{1/2} \vec{P} \rightarrow \vec{P}$ on the right hand side.

Again, the map $\vec{k} \rightarrow \vec{P}(\vec{k})$ is in general a multiple cover of the target $\hat{\mathbb{R}}^3$, which translates to the domain of the integral on the right hand side being several copies of $\hat{\mathbb{R}}^3$: The integral measures precisely this multiplicity. Alternatively, this can be viewed as the winding number of the map $\vec{k}/|\vec{k}| \rightarrow \vec{P}(\vec{k})/|P(\vec{k})|$, from $\hat{S}^2$ to $\hat{S}^2$, in the large $|\vec{k}|$ limit. We conclude that for the most general non-relativistic Weyl fermion in 3+1 dimensions, (3.7) is the right coefficient to the chiral anomaly.
3.1 The Berry Monopoles in the Momentum Space

We saw that the chiral anomaly of a general two-component isospinor is given by the winding number of the map $\vec{k} \rightarrow \vec{P}(\vec{k})$ in momentum space. We will show that the same topology underlies the Berry’s connection of projected chiral spinor in momentum space. For this, one considers a two-level problem with the Hamiltonian

$$\sigma^a P_a(\vec{k}),$$

whose two eigenvalues are $\pm|\vec{P}(\vec{k})|$. Denoting the two respective eigenvectors by $|\pm\rangle$, the Berry connection is

$$\mathcal{A}^\pm = -\langle \pm | \frac{\partial}{\partial k_a} | \pm \rangle \, dk_a$$

A well-known result is that, when $P_a = k_a$, the Berry connection carries the unit magnetic flux. Let us recapitulate this simpler case first. In the spherical coordinates,

$$\sigma^a k_a = |\vec{k}| \begin{pmatrix} \cos \theta & \sin \theta e^{-i\phi} \\ \sin \theta e^{i\phi} & -\cos \theta \end{pmatrix},$$

the two eigenvectors are

$$|+\rangle_{\sigma^a k_a} = \begin{pmatrix} \cos(\theta/2)e^{-i\phi/2} \\ \sin(\theta/2)e^{i\phi/2} \end{pmatrix}, \quad |-\rangle_{\sigma^a k_a} = \begin{pmatrix} -\sin(\theta/2)e^{-i\phi/2} \\ \cos(\theta/2)e^{i\phi/2} \end{pmatrix}.$$  

Therefore, the two Berry connections are common up to a sign,

$$\mathcal{A}^\pm \bigg|_{\sigma^a k_a} = \pm \frac{i}{2} \cos \theta d\phi ,$$

which carries a unit $2\pi$ magnetic flux over $\hat{S}^2$, or the unit Chern number. As usual this is up to $U(1)$ gauge transformations under the innocuous phase rotations,

$$|\pm\rangle \rightarrow e^{i\Lambda_\pm(\vec{k})}|\pm\rangle.$$  

What we outline here is a well-known computation in the context of $d = 2 + 1$ topological insulators generally. See for example Ref. [16]. One way to realize the latter is by imagining a two-dimensional Brillouin zone as a slice in $d = 3 + 1$ Brillouin zone of Weyl semi-metal, between a pair of chiral Weyl and anti-chiral Weyl points.
Coming back to $H = \sigma^a P_a$ is merely a matter of replacing $\vec{k}$ by $\vec{P}(\vec{k})$, so we can express

$$A^\pm \bigg|_{\sigma^a P_a} = \pm \frac{i}{2} \cos \Theta \, d\Phi$$

(3.14)

with the spherical angles $\Theta$ and $\Phi$ of $\vec{P}$, which needs to be pulled back to $\vec{k}$ space. In other words, denoting the standard $2\pi$ Wu-Yang monopole field \[52\] in $\vec{P}$ space by $A$ and $F$ its field strength, the Berry connection and the field strength thereof in the momentum space are

$$A^\pm = \pm A_a \frac{\partial P^a}{\partial k_f} \, dk_f, \quad F^\pm = \pm \frac{1}{2} \, F_{ab} \frac{\partial P^a}{\partial k_f} \frac{\partial P^b}{\partial k_g} \, dk_f \wedge dk_g.$$  (3.15)

Although (3.14) seemingly gives a unit $2\pi$ flux, this is not generally the case because the map $\vec{k} \to \vec{P}$ can be a multi-cover of the target $\tilde{\mathbb{R}}^3$.

The actual integral that computes the Chern number is, with $P^*$ being the pullback of the map $\vec{P}(\vec{k})$

$$\frac{1}{2\pi i} \int_{S_2} F^\pm = \pm \frac{1}{2\pi} \int_{S_2} P^*(F) = \pm \frac{1}{2\pi} \int_{B_3} P^*(dF),$$  (3.16)

where $S_2 = \partial B_3$ is an arbitrary 2-surface in the $\vec{k}$ space. In particular, if one takes $B_3 = \tilde{\mathbb{R}}^3$ the entire momentum space,

$$\frac{1}{2\pi i} \int_{\tilde{\mathbb{R}}^3} P^*(dF) = \frac{1}{2\pi i} \int_{\tilde{S}_2} P^*(F) = N_{\vec{P}},$$  (3.17)

since at the asymptotic two-sphere $\tilde{S}_2^2$ the total flux is multiplied by the net winding number $N_{\vec{P}}$.

On the other hand, $dF = 0$ everywhere except at the origin $\vec{P} = 0$, so we may rewrite this integral as

$$\frac{1}{2\pi} \sum_{\{\vec{k}_*\mid \vec{P}(\vec{k}_*) = 0\}} \int_{B_3(\vec{k}_*)} P^*(dF),$$  (3.18)
where $B^3_{\epsilon}(\vec{k}_*)$ is an infinitesimal 3-ball centered at the $\vec{k}_*$. This in turn becomes,

$$\frac{1}{2\pi i} \sum_{\{\vec{k}_*|P(\vec{k}_*)=0\}} \int_{\partial B^3_{\epsilon}(\vec{k}_*)} P^*(F),$$

(3.19)

and, since $F$ is a unit monopole in $\vec{P}$ space, we find

$$\sum_{\{\vec{k}_*|P(\vec{k}_*)=0\}} N_{\vec{P}}(\vec{k}_*) = N_{\vec{P}},$$

(3.20)

where $N_{\vec{P}}(\vec{k}_*)$ is the winding number of $\vec{P}$, measured by the infinitesimal neighborhood around $\vec{k}_*$, which of course sum up to $N_{\vec{P}}$.

The total flux is computed by taking $S_2$ equal to the asymptotic $\tilde{S}^2$ of $\vec{k}$ space $\tilde{\mathbb{R}}^3$, which maps to $N_{\vec{P}}$ times the asymptotic $\tilde{S}^2$ of the target $\tilde{\mathbb{R}}^3$ if the map $\vec{P}$ covers the target $\tilde{\mathbb{R}}^3$ $N_{\vec{P}}$ times. As such, this Berry connection carries precisely $2\pi N_{\vec{P}}$ total flux, where $N_{\vec{P}}$ may be also computed as in (3.7), provided that $\vec{P}^2$ is divergent everywhere asymptotically. This shows that the chiral anomaly is precisely given by the same winding number $N_{\vec{P}}$, offering a rigorous logical link between the chiral anomaly and the Berry’s connection in momentum space. As should be clear from how we arrive at this connection via our generalized index theorem, we emphasize that this is a non-trivial, yet understandable manifestation of infrared-ultraviolet connection, which is a fundamental characteristic of chiral anomaly.

### 3.2 Infrared View and Real Material

As we have already seen in Section 2, one can compute the same winding number alternatively by rescaling $P_a \rightarrow C \cdot P_a$, and going back to $d^3k$ integral. This localizes to small neighborhoods around $\vec{k} = \vec{k}_*$ where $\vec{P}(\vec{k}_*) = 0$, and as such really counts inverse images of $\vec{P} = 0$ with weights $\pm 1$,

$$\sum_{\{\vec{k}_*|\vec{P}(\vec{k}_*)=0\}} 1 \cdot \text{sgn} \left[ \det(\partial^b P_a) \right] \bigg|_{\vec{k} = \vec{k}_*}. \quad (3.21)$$

This alternative form is viable when the critical points are isolated and non-degenerate, where one is counting Weyl cones with $\vec{P}$ locally linear $\vec{k}$. If the sign happens to be negative, it has the same effect as flipping signs of odd number of $\gamma^a$’s, resulting
in anti-Weyl fermions instead of Weyl fermions. When we allow degenerate critical points, we find

$$\sum_{\{\vec{k}_* | \vec{P}(\vec{k}_*) = 0\}} N_{\vec{P}(\vec{k}_*)},$$

the same as (3.20).

In real material the momentum $\vec{k}$ lives in a compact Brillouin zone, $\tilde{\mathbb{R}}^3 / \Lambda$, where $\Lambda$ is the dual lattice to the crystalline lattice of the material. On the flip side, $\vec{P}$ does not extend indefinitely into $\hat{\mathbb{R}}^3$ either. Instead, $\vec{P}(\vec{k}) = \vec{P}(\Lambda \vec{k})$. One immediate question is how the story so far is affected by such a compact Brillouin zone. Consider a pair of energy bands that meets at one or more Bloch momentum $\vec{k} = \vec{k}_*$, which for our purpose means a continuous map from $\vec{k}$ in the Brillouin zone $\tilde{\mathbb{R}}^3 / \Lambda$ to a $2 \times 2$ Hamiltonian $H(\vec{k})$ such that

$$H(\vec{k}) = \vec{\sigma} \cdot \vec{P}(\vec{k}) - \Delta(\vec{k}) , \quad \vec{P}(\vec{k}_*) = 0 .$$

The appearance $\Delta$ is of course necessary for real material, but its connection to the same symbol in the above Euclidean computation might look a bit unclear, since the naive Wick rotation would render $\Delta$ in (3.1) to become pure imaginary. The answer to this is that one really starts with Lorentzian signature and rotates the contour of $k_0$ such that one reaches (3.1) in the end. More on this would be elaborated in the next section.

In both the Euclidean anomaly computation and the Berry phase computation, we have seen that only the winding number associated with $\vec{P}(\vec{k})$ matters for these topological characterization. In the target, one starts with a monopole of Berry connection near $\vec{P} = 0$, which we must pull-back to the Brillouin zone $\tilde{\mathbb{R}}^3 / \Lambda$. One puzzling aspect is that now $\vec{P}^2$ must be bounded as it is defined on a compact Brillouin zone, to begin with, and in real material the energy eigenvalues $-\Delta \pm |\vec{P}|$ must be bounded above and below: the integral formulae such as (3.7) appear to have no reason to produce an integer. This quandary is saved by the observation that under such circumstances (3.7) always produces zero, as we see below.

Let us replace $\tilde{\mathbb{R}}^3$ by $\tilde{\mathbb{R}}^3 / \Lambda$ and consider a surface $S_2 = \partial B_3$ that encloses all the
critical points \( P(\vec{k}_*) = 0 \), whereby we have

\[
N_{\vec{P}} = \frac{1}{2\pi i} \int_{B_3} dP^*(F) = \frac{1}{2\pi i} \int_{S_2} P^*(F) .
\]  
(3.24)

On the other hand, since \( \tilde{\mathbb{R}}^3/\Lambda \) is closed, \( -S_2 \) is also a boundary to the complement \( B_3^c \). Given that no magnetic monopole exists in \( B_3^c \), we have

\[
N_{\vec{P}} = -\frac{1}{2\pi i} \int_{-S_2} P^*(F) = -\frac{1}{2\pi i} \int_{B_3^c} dP^*(F) = 0 ,
\]  
(3.25)

and further may as well shrink \( B_3^c \) to nothing, so that \( B_3 = \tilde{\mathbb{R}}^3/\Lambda \), and find

\[
N_{\vec{P}} = \frac{1}{2\pi} \int_{\tilde{\mathbb{R}}^3/\Lambda} d(P^*(F)) = 0 .
\]  
(3.26)

In view of our infrared alternative, the same can be expressed as

\[
\sum_{\{\vec{k}_* | \vec{P}(\vec{k}_*)=0\}} N_{\vec{P}}(\vec{k}_*) = 0 ,
\]  
(3.27)

or

\[
\sum_{\{\vec{k}_* | \vec{P}(\vec{k}_*)=0\}} 1 \cdot \text{sgn} \left[ \det(\partial^h P_a) \right] \bigg|_{\vec{k}=\vec{k}_*} = 0 ,
\]  
(3.28)

if all the critical points are non-degenerate.

In the end, the number of Weyl points and the number of anti-Weyl points are always equal, for any system on a real space lattice, provided that \( \vec{P} \) is smooth enough in such a compact Brillouin zone. This is, of course, nothing but the Nielsen-Ninomiya theorem \[53\].

### 3.3 Multiple and Degenerate Weyl Semi-Metal

Problems of this kind has been dealt with in recent literature for a simple power-like \( \vec{P} \). The main prototype is

\[
\mathcal{D} = (\sigma^+(-iD_+)^2 + \sigma^-(-iD_-)^2 + \sigma^3(-iD_3) - D_4) ,
\]  
(3.29)
where $D_{\pm} = D_1 \mp iD_2$ and $\sigma^\pm = (\sigma^1 \pm i\sigma^2)/2$. This has been motivated by a merging of a pair of Dirac cones, initially separated in the Brillouin zone. A slight generalization of this can be achieved by elevating the power to an arbitrary positive $n$, and also replacing $-iD_3$ by

$$P_3 = C_l(-iD_3)^l + \cdots ,$$

(3.30)

where the ellipsis denotes lower order monomials of $-iD_3$. As is clear from the general formalism, these subleading pieces are irrelevant for the problems at hand, so we may as well consider

$$\mathcal{D} = \left( \sigma^+(-iD_+)^n + \sigma^-(-iD_-)^n + \sigma^3C_l(-iD_3)^l - D_4 \right),$$

(3.31)

whose associated Dirac operator is

$$Q = \left( \begin{array}{cc} 0 & \mathcal{D} \\ \mathcal{D}^\dagger & 0 \end{array} \right) = \left( \begin{array}{c} \gamma^+(-iD_+)^n + \gamma^-(-iD_-)^n + \gamma^3C_l(-iD_3)^l + \gamma^4(-iD_4) \end{array} \right)$$

(3.32)

with $\gamma^\pm = (\gamma^1 \pm i\gamma^2)/2$. The anomaly computation for $n = 2, 3$ and $l = 1$ with $C_1 = 1$ has been performed in recent literatures [7,8]. Although this class of examples clearly fall under the general formalism above, we repeat the exercise in part as an concrete example of our general formulation but also to show explicitly how the additional power $l$ enters the story.

The zero-th order part of the squared Dirac operator is

$$-Q^2_0 = ((\partial_1)^2 + (\partial_2)^2)^n + C^2_l(\partial_3)^{2l} + (\partial_4)^2 ,$$

(3.33)

so that, now with $X = (Z; U, V)$ and $K = (p; q, \tilde{q})$

$$G^{(0)}_s(x + X; x) = \frac{1}{\sqrt{4\pi s}} e^{-V^2/4s} \times \int \frac{d^2p \, dq}{(2\pi)^3} e^{i(p \cdot Z + qU)} e^{-s(p_1^2 + p_2^2)^n - sC^2_lq^{2l}}.$$  

(3.34)

Again the key quantity to compute is the second pieces in

$$Q^2 = \mathcal{P}_\mu \mathcal{P}_\mu + \frac{1}{4} [\gamma^\mu, \gamma^\nu][\mathcal{P}_\mu, \mathcal{P}_\nu].$$

(3.35)
and the relevant terms in $\delta Q^2$ are
\begin{align*}
\frac{1}{4}[\gamma^\mu, \gamma^\nu][P_\mu, P_\nu] & = \gamma^1 \gamma^2 (n^2 F_{12})(p_+ p_-)^{n-1} + \cdots \\
& + \gamma^\pm \gamma^3 (-n l (F_{13} \mp i F_{23})) C_l q^{l-1} (p_+)^{n-1} + \cdots \\
& + \gamma^\pm \gamma^4 (-n (F_{14} \mp i F_{24}))(p_+)^{n-1} + \cdots \\
& + \gamma^3 \gamma^4 F_{34} l C_l q^{l-1} + \cdots .
\end{align*}
(3.36)

As before, the momentum accumulates through each iteration, and the relevant part of the heat kernel can be found at the second order of iteration,
\begin{align*}
G_s(x + X; x) & = \cdots - \Gamma s^{3/2} \sqrt{4\pi} (l \times f(F(x))) e^{-V^2/4s} \\
& \times \int \frac{d^2p dq}{(2\pi)^3} (p_+ p_-)^{n-1} q^{l-1} e^{ip Z + iq U} e^{-s(p^2)^n + C_l^2(q^2)^l} + \cdots
\end{align*}
(3.37)
with
\begin{equation}
f(F) = n^2 (F_{12} F_{34} + F_{31} F_{24} + F_{23} F_{14}).
\end{equation}
(3.38)

For even $l$ the $q$-integral vanishes since the integrand is odd under $q \to -q$. For odd $l$, on the other hand, the momentum integral gives in the coincident limit,
\begin{equation}
\int \frac{d^2p}{(2\pi)^2} (p^2)^{n-1} e^{-s(p^2)^n} \times \int \frac{dq}{2\pi} C_l q^{l-1} e^{-s C_l^2(q^2)^l} = \frac{1}{4\pi s} \frac{1}{n} \frac{\text{sgn}(C_l)}{l}.
\end{equation}
(3.39)

This brings us to
\begin{equation}
N_{\vec{P}} = \begin{cases} 
\text{sgn}(C_l) \cdot n & \text{for odd } l \\
0 & \text{for even } l
\end{cases},
\end{equation}
(3.40)
where $l$ does not add to the winding number but rather turn it on or off, depending on its value modulo 2.
4 Diagrammatic Computation of the Anomaly

One of the points that remains unresolved in the above anomaly computation via the Fujikawa viewpoint was the nature of the Wick rotation. Although the computation stands well-defined as a generalized index problem for a spinor valued in a vector bundle over \( \mathbb{R}^d \), its relation to quantum theories in real world with the Lorentzian signature needs to be clarified further when the spatial momentum mixes in with the frequency. The usual Wick rotation \(-it \to \tau\) no longer works, because of such a mix, so it is necessary to readdress the issue from Lorentzian viewpoint by computing the usual Feynman diagram and see how its result is connected to those above. In particular, this will give us a clearer picture of exactly what Wick rotation we have effectively performed and also clarify why the general form of the anomaly remains intact, modulo a multiplicative constant, despite the higher inverse power of the momentum in the propagators.

For this, we consider the generalized fermion action \( S = \int_x \mathcal{L} \) where

\[
\mathcal{L} = -\bar{\psi} \gamma^\mu \mathcal{P}_\mu (-iD) \psi + iM \bar{\psi} \psi. \tag{4.1}
\]

It is a massive fermion of mass \( M \), and the true object we need is a subtraction between \( M = 0 \) fermion and the Pauli-Villars fermion of \( M \) in \( M \to \infty \) limit. Once we have this subtraction, the loop integral is finite and we are free to do shift/change of variable of loop variables. When we do such operations in the following, it is understood that we do the same simultaneously for both \( M = 0 \) fermion and Pauli-Villars fermion contributions in the regularized integrand, so that it is justified.

We assume that \( \mathcal{P}_\mu (-iD) \) is expandable in power series, and we show details for a particular order \( n \) term explicitly, and wherever we can replace the order \( n \) expression with the general \( \mathcal{P}_\mu (k) \) we will do so. The action with an order \( n \) term is

\[
-\bar{\psi} \gamma^\mu \mathcal{P}_\mu (-iD) \psi + iM \bar{\psi} \psi
= -C_{\alpha_1 \alpha_2 \cdots \alpha_n} \bar{\psi} \gamma^\mu (-iD)_{\alpha_1} (-iD)_{\alpha_2} \cdots (-iD)_{\alpha_n} \psi + iM \bar{\psi} \psi. \tag{4.2}
\]

Either by the Noether method or by introducing auxiliary chiral gauge field \( A_5 \) in \( D = \partial + A + A_5 \gamma^5 \) and differentiating the action with respect to \( A_5 \), the chiral current
is obtained as
\[
J_{A}^{\mu} = \sum_{s=1}^{n} C_{\nu}^{\alpha_{1} \cdots \alpha_{s-1} \mu \alpha_{s+1} \cdots \alpha_{n}} ((iD)_{\alpha_{s-1}} \cdots (iD)_{\alpha_{1}} \bar{\psi}) \gamma^\nu \gamma^5 (-iD)_{\alpha_{s+1}} \cdots (-iD)_{\alpha_{n}} \psi . \tag{4.3}
\]

Note that \( D\psi = (\partial + A)\psi \) and \( D\bar{\psi} = (\partial - A)\bar{\psi} \). Although we don’t use the conserved vector current \( j^\mu \), it has the same form as the above except \( \gamma^5 \). Using the classical equation of motion, it is easily seen that
\[
\partial_{\mu} J_{A}^{\mu} = 2M \bar{\psi} \gamma^5 \psi . \tag{4.4}
\]

If there was no UV divergence in the correlation functions involving \( J_{A}^{\mu} \), the (4.4) would imply a Ward identity for the correlation functions,
\[
\partial_{\mu} \langle J_{A}^{\mu} \cdots \rangle = 2M \langle \bar{\psi} \gamma^5 \psi \cdots \rangle \tag{4.5}
\]

which could be seen diagrammatically order by order in the loop expansion. We have checked this explicitly at 1-loop up to second order in the background gauge fields, that is relevant to the chiral anomaly. In showing this, one needs to shift or change the loop momenta, which is valid only when the integrand is UV finite. For a divergent diagram, such as the one we need to compute for chiral anomaly, these operations are allowed only after subtracting the regularization contribution to make it finite, in our case, the Pauli-Villars fermion. The regularized chiral current is therefore
\[
J_{A, \text{reg}}^{\mu} = J_{A, M=0}^{\mu} - J_{A, M}^{\mu} , \tag{4.6}
\]

which has now the valid Ward identity
\[
\partial_{\mu} \langle J_{A, \text{reg}}^{\mu} \cdots \rangle = -2M \langle \bar{\psi}_{M} \gamma^5 \psi_{M} \cdots \rangle , \tag{4.7}
\]

where \( \psi_{M} \) is the Pauli-Villars fermion. The right-hand side of the above will be seen to be finite in the 1-loop order that is sufficient to derive our chiral anomaly\footnote{If the right-hand side remains divergent, although the degree of divergence should be reduced, one needs to introduce the second pair of Pauli-Villars fields to subtract such divergence. One continues to introduce the necessary Pauli-Villars fields until it becomes finite.} and the chiral anomaly, whatever it is, is obtained by
\[
\mathcal{A} = \lim_{M \to \infty} -2M \langle \bar{\psi}_{M} \gamma^5 \psi_{M} \rangle . \tag{4.8}
\]
There is another viewpoint that leads to the same formula for chiral anomaly. Consider a physical massive fermion of mass $M$. Its chiral current has the Ward identity

$$\partial_{\mu} j_{\mu}^A = 2M \bar{\psi} \gamma^5 \psi + A.$$  \hfill (4.9)

As $M \rightarrow \infty$, the fermion should decouple from the low energy regime, and the symmetry current and its Ward identity should be provided and saturated by other low energy degrees of freedom in $M \rightarrow \infty$ limit. This means that the right-hand side of the above must vanish in $M \rightarrow \infty$ limit.

In the diagrammatic evaluation of (4.8) up to second order in the background gauge fields, there are three diagrams to compute as shown in Figure 1. Since a non-vanishing $\gamma^5$ trace requires at least four $\gamma$ matrices, and each vertex and propagator contains at most one $\gamma$ matrix, the diagrams (b) and (c) vanish trivially, and we only consider the triangle diagram (a). The propagator in momentum space as a spinor matrix is

$$\langle \psi(k) \bar{\psi}(k') \rangle = (2\pi)^4 \delta^{(4)}(k + k') \frac{(-i)}{\gamma^\mu P_\mu(k) - iM}.$$  \hfill (4.10)

\footnote{We denote the 4-momentum in the Lorentzian signature by $k$, to be distinguished from the Euclidean 4-momentum $K$ or from the spatial 3-momentum $\vec{k}$. The precise relation between the Euclidean $K$ we used in Section 2 and Lorentzian $k$ is more subtle than for the usual Dirac fermions. See the last part of this section.}
We need the interaction vertex only up to first order in the gauge field,

\[
iS \sim - \sum_{s=1}^{n} C_{\mu_1 \ldots \mu_n}^{\alpha_1 \ldots \alpha_n} ((i\partial)_{\alpha_1} \ldots (i\partial)_{\alpha_s} \bar{\psi}) \gamma^\mu A_{\alpha_1} (i\partial)_{\alpha_{s+1}} \ldots (i\partial)_{\alpha_n} \psi. \quad (4.11)
\]

The Feynman rule for this vertex as shown in Figure 2 is given by

\[
- \sum_{s=1}^{n} C_{\mu_1 \ldots \mu_n}^{\alpha_1 \ldots \alpha_n} k_{\alpha_1} \ldots k_{\alpha_{s-1}} (k-q)_{\alpha_{s+1}} \ldots (k-q)_{\alpha_n} \gamma^\mu A_{\alpha_s} \equiv \Gamma(k, q) \cdot A(q), \quad (4.12)
\]

where \( \psi \) carries a momentum \( k-q \), \( \bar{\psi} \) a momentum \( -k \), and the momentum of the gauge field \( A \) is \( q \). Note that

\[
\Gamma(k, 0) \cdot A(q) = -\gamma^\mu \frac{\partial P_{\mu}(k)}{\partial k_\nu} A_\nu(q), \quad (4.13)
\]

which will be used later.

It is easy to write down the expression of \(-2M\langle \bar{\psi}_M \gamma^5 \psi_M \rangle \) of momentum \( p \) as (where \( \psi \) carries a momentum \( k \) and \( \bar{\psi} \) a momentum \( p-k \))

\[
2M \int_q \int_k \text{Tr} \left( \gamma^5 \frac{(-i)}{\gamma \cdot P(k) - iM} \Gamma(k, q) \cdot A(q) \frac{(-i)}{\gamma \cdot P(k-q) - iM} \right. \\
\times \Gamma(k-q, p-q) \cdot A(p-q) \left. \frac{(-i)}{\gamma \cdot P(k-p) - iM} \right),
\]

\[
(4.14)
\]
where $\int_k = \int \frac{d^4 k}{(2\pi)^4}$ is the loop integration. This is equal to

$$2iM \int_q \int_k \frac{\text{Tr}(\mathcal{I})}{(P(k)^2 + M^2 - i\epsilon)(P(k-q)^2 + M^2 - i\epsilon)(P(k-p)^2 + M^2 - i\epsilon)},$$

(4.15)

where $\mathcal{P}^2 = \mathcal{P}_\mu \mathcal{P}^\mu$, and

$$\text{Tr}(\mathcal{I}) = \text{Tr}\left( \gamma^5 (\gamma \cdot \mathcal{P}(k) + iM) \Gamma(k,q) \cdot A(q)(\gamma \cdot \mathcal{P}(k-q) + iM) \right.\
\times \left. \Gamma(k-q,p-q) \cdot A(p-q)(\gamma \cdot \mathcal{P}(k-p) + iM) \right).$$

(4.16)

Note that we introduced $-i\epsilon$ for the time ordered correlation functions.

The non-zero $\gamma^5$ trace in the above requires precisely four $\gamma$ matrices, and

$$\text{Tr}(\mathcal{I}) = iM(\text{Tr}(\mathcal{I}_1) + \text{Tr}(\mathcal{I}_2) + \text{Tr}(\mathcal{I}_3)),
$$

(4.17)

where

$$\begin{align*}
\text{Tr}(\mathcal{I}_1) &= \text{Tr}\left( \gamma^5 (\Gamma(k,q) \cdot A(q))(\gamma \cdot \mathcal{P}(k-q))(\Gamma(k-q,p-q) \cdot A(p-q))(\gamma \cdot \mathcal{P}(k-p)) \right), \\
\text{Tr}(\mathcal{I}_2) &= \text{Tr}\left( \gamma^5 (\gamma \cdot \mathcal{P}(k))(\Gamma(k,q) \cdot A(q))(\Gamma(k-q,p-q) \cdot A(p-q))(\gamma \cdot \mathcal{P}(k-p)) \right), \\
\text{Tr}(\mathcal{I}_3) &= \text{Tr}\left( \gamma^5 (\gamma \cdot \mathcal{P}(k))(\Gamma(k,q) \cdot A(q))(\gamma \cdot \mathcal{P}(k-q))(\Gamma(k-q,p-q) \cdot A(p-q)) \right).
\end{align*}$$

(4.18)

Since $\gamma^5$ trace is totally anti-symmetric with respect to four $\gamma$ matrices in each $\mathcal{I}_{1,2,3}$, we can shuffle four factors inside each $\mathcal{I}_{1,2,3}$ up to sign changes. By the same reason, we can add any multiple of one factor to the other factor without changing the result.

Adding $\text{Tr}(\mathcal{I}_2)$ and $\text{Tr}(\mathcal{I}_3)$, we have

$$\text{Tr}(\mathcal{I}_2) + \text{Tr}(\mathcal{I}_3) = \text{Tr}\left( \gamma^5 (\gamma \cdot \mathcal{P}(k))(\Gamma(k,q) \cdot A(q))(\Gamma(k-q,p-q) \cdot A(p-q)) \right.\
\times \left. (\gamma \cdot \mathcal{P}(k-p) - \gamma \cdot \mathcal{P}(k-q)) \right).$$

(4.19)
Subtracting the second factor from the last factor in $\text{Tr}(I_1)$,
\[
\text{Tr}(I_1) = \text{Tr} \left( \gamma^5 (\Gamma(k, q) \cdot A(q)) (\gamma \cdot \mathcal{P}(k-q)) (\Gamma(k-q, p-q) \cdot A(p-q)) \times (\gamma \cdot \mathcal{P}(k-p) - \gamma \cdot \mathcal{P}(k-q)) \right). \tag{4.20}
\]

Adding these, we have
\[
\text{Tr}(I_1) + \text{Tr}(I_2) + \text{Tr}(I_3) = \text{Tr} \left( \gamma^5 (\gamma \cdot \mathcal{P}(k) - \gamma \cdot \mathcal{P}(k-q)) \times (\gamma \cdot \mathcal{P}(k-p) - \gamma \cdot \mathcal{P}(k-q)) \times (\Gamma(k, q) \cdot A(q)) (\Gamma(k-q, p-q) \cdot A(p-q)) \right). \tag{4.21}
\]

The first factor vanishes linearly in small $q$ limit,
\[
\gamma \cdot \mathcal{P}(k) - \gamma \cdot \mathcal{P}(k-q) \approx \gamma^\mu \frac{\partial \mathcal{P}_\mu(k)}{\partial k_\nu} q_\nu + \cdots, \tag{4.22}
\]
which, together with $A(q)$, gives a first order derivative of $A$ in coordinate space. The same is true for the second factor,
\[
\gamma \cdot \mathcal{P}(k-p) - \gamma \cdot \mathcal{P}(k-q) \approx -\gamma^\mu \frac{\partial \mathcal{P}_\mu(k)}{\partial k_\nu} (p-q)_\nu + \cdots, \tag{4.23}
\]
that combines with $A(p-q)$ to give another derivative of $A$ in coordinate space. If we truncate higher order derivatives of $A$ in coordinate space, we need to put $q$ and $(p-q)$ in the other factors to zero. Then the third and fourth factors become
\[
\Gamma(k, q) \cdot A(q) \rightarrow \Gamma(k, 0) \cdot A(q) = -\gamma^\mu \frac{\partial \mathcal{P}_\mu(k)}{\partial k_\nu} A_\nu(q), \tag{4.24}
\]

**We can make it less ad hoc by the observation that the expansion parameter is either $q/k$ or $(p-q)/k$. In $M \rightarrow \infty$ limit, our final result of chiral anomaly, which is finite, is dominated by the integration region of $k \sim M$, and the higher derivative terms are suppressed by additional powers of $\partial/M$, which vanish in $M \rightarrow \infty$.**
and
\[ \Gamma(k-q,p-q) \cdot A(p-q) \to -\gamma^\mu \frac{\partial P_\mu(k)}{\partial k_\nu} A_\nu(p-q). \] (4.25)

Computing the trace, we obtain
\[ \text{Tr}(I) = 4M \det \left( \frac{\partial P_\mu}{\partial k_\nu} \right) \epsilon^{\alpha\beta\alpha'\beta'} (iq_\alpha) A_\beta(q)(i(p-q)_{\alpha'}) A_{\beta'}(p-q), \] (4.26)

and in the remaining loop integral of (4.15) in \( M \to \infty \) limit, we can neglect \( q \) and \( p \) in the denominator since it is dominated by \( k \sim M \gg p, q \) region, and we arrive at the result for the chiral anomaly
\[ 8iM^2 \int_q \epsilon^{\alpha\beta\alpha'\beta'} (iq_\alpha) A_\beta(q)(i(p-q)_{\alpha'}) A_{\beta'}(p-q) \times \int_k \det \left( \frac{\partial P_\mu(k)}{\partial k_\nu} \right) \frac{1}{(P(k)^2 + M^2 - i\epsilon)^3}, \] (4.27)

which is in coordinate space,
\[ \mathcal{A} = \epsilon^{\alpha\beta\alpha'\beta'} F_{\alpha\beta} F_{\alpha'\beta'} \int_k \det \left( \frac{\partial P_\mu(k)}{\partial k_\nu} \right) \frac{2iM^2}{(P(k)^2 + M^2 - i\epsilon)^3}. \] (4.28)

The above \( k \) integration can be done in \( P \) space up to the winding number \( N_P \) explained before,
\[ \int_k \det \left( \frac{\partial P_\mu(k)}{\partial k_\nu} \right) \frac{2iM^2}{(P(k)^2 + M^2 - i\epsilon)^3} = N_P \int \frac{d^4P}{(2\pi)^4} \frac{2iM^2}{(P^2 + M^2 - i\epsilon)^3}, \] (4.29)

and the Wick rotation \( P^0 \to iP^0 \) gives
\[ \int \frac{d^4P}{(2\pi)^4} \frac{2iM^2}{(P^2 + M^2 - i\epsilon)^3} = - \int \frac{d^4P_E}{(2\pi)^4} \frac{2M^2}{(P_E^2 + M^2)^3} = - \frac{1}{16\pi^2}, \] (4.30)

which finally gives the chiral anomaly
\[ \mathcal{A} = -N_P \frac{1}{16\pi^2} \epsilon^{\alpha\beta\alpha'\beta'} F_{\alpha\beta} F_{\alpha'\beta'}. \] (4.31)

This result is the same as (2.22) except an overall factor 2, which has a well-known ori-
gin: Here we are computing the anomaly of a single 4-component Dirac-like fermions while, in Sections 2 and 3, we computed for 2-component chiral fermions. Note that, again, the winding number $N_P$ factors out cleanly; the final form of the integral is expressed in terms of $P$, in place of $K$, such that the only surviving information about $P$ is how many times $P$ covers the momentum space $\mathbb{R}^4$.

Specializing to the case of a non-relativistic Weyl semimetal, where

$$\mathcal{P}_0(k) = k_0 + \Delta(\vec{k}), \quad \mathcal{P}_a(k) = P_a(\vec{k}), \quad a = 1, 2, 3,$$

we have

$$\int_k \det \left( \frac{\partial \mathcal{P}_\mu(k)}{\partial k_\nu} \right) \frac{2iM^2}{(\mathcal{P}(k)^2 + M^2 - i\epsilon)^3}$$

$$= \int \frac{d^3 k}{(2\pi)^3} \det \left( \frac{\partial \mathcal{P}_a(\vec{k})}{\partial k_b} \right) \int \frac{dk^0}{(2\pi)} \frac{2iM^2}{(-k^0 - \Delta(\vec{k}))^2 + \vec{P}(\vec{k})^2 + M^2 - i\epsilon)^3}.$$

Note that $\Delta(\vec{k})$ can be simply removed by a shift of $k^0$ integration, and the chiral anomaly is not affected by $\Delta(\vec{k})$. Since $\Delta(\vec{k})$ determines the shape of the Fermi surface by $|\vec{P}(\vec{k})| = -\Delta(\vec{k})$, the chiral anomaly is independent of occupation number of states, but is determined only by the topology of the level crossing point.

Performing the Wick rotation of $k^0$ after shifting to remove $\Delta(\vec{k})$, the $k^0$ integration is easily done to give

$$-\frac{3}{8}M^2 \int \frac{d^3 k}{(2\pi)^3} \det \frac{1}{\mathcal{P}(\vec{k})^2 + M^2}.$$

Again, up to the winding number $N_{\vec{P}}$ of the map $\vec{k} \to \vec{P}(\vec{k})$, the above integral can be done in $\vec{P}$ space,

$$-N_{\vec{P}} \frac{3}{8}M^2 \int \frac{d^3 \vec{P}}{(2\pi)^3} \frac{1}{(\vec{P}^2 + M^2)^{5/2}} = -N_{\vec{P}} \frac{1}{16\pi^2},$$

which reproduces the previous result by the index theorem.

One important point of this computation is the Wick rotation, $\mathcal{P}^0 \to i\mathcal{P}^0_E$, which is not the same as Wick rotation of the Lorentzian time to the Euclidean time. In

\[ \text{The Hamiltonian for the right-handed Weyl component is } \vec{P}(\vec{k}) \cdot \vec{\sigma} + \Delta(\vec{k}). \]
the presence of $\Delta(\vec{k})$, the latter cannot be justified as the rotation of the contour $k_0 \to ik_4$ can encounter a pole along the way. On the other hand, as far as the diagrammatic computation goes, the Wick rotation is merely a trick that allows efficient computation. Therefore, a different contour choice in the complex plane of $k_0$ such that $\mathcal{P}^0 \to i\mathcal{P}_E^0$ occurs is perfectly acceptable, as long as it is consistent with the idea of the Feynman propagator.

5 Generalized Spinors in Odd Dimensions

In odd spacetime dimensions, neither the usual $\mathbb{Z}$-valued index nor the chiral anomaly exists. Instead one finds discrete anomalies, which have been the focus of active investigations recently in the context of the topological insulators. In all such investigations, the main physical quantity of interest is the phase of the partition function. When the fermion is massless, the reality of the action naively implies that the partition function is real, yet actual path integral generically produces a phase factor. And, much as in the anomaly computation in even dimensions, this becomes apparent under the inevitable regularization of the path integral.

This phase has been computed in Ref. [54] and shown to be proportional to the eta-invariant,

$$\eta \equiv \lim_{s \to 0} \sum \text{sgn}(\lambda)|\lambda|^{-s}$$

with the sum over the eigenvalues of the Dirac operator, which measures some notion of the asymmetry of the eigenvalues of the Dirac operators under a sign flip. In the physics community, this is sometimes misrepresented by its cousin, namely the Chern-Simons action,

$$\frac{\pi}{2} \eta \left|_{-i\sigma^a(\partial_+ A_a)} \right. = \frac{1}{2} S_{CS}(A) + \cdots .$$

The ellipsis denote certain non-local terms, which makes the entire expression gauge-invariant even if the Chern-Simons coefficient is not properly quantized: this has to be since the eigenvalues that define the eta-invariant are always gauge invariant no matter what [54].

In this section, we will extend our investigation using the modified heat kernel to
Dirac fermions in odd spacetime dimensions, with high-derivative Dirac operators as before. We will explicitly work out $d = 3$ Dirac fermion with the higher-derivative action

$$S_3 = - \int d^3x \bar{\psi} Q_3 \psi ,$$

for which the relevant Dirac operator is

$$Q_3 = \sigma^a P_a (-i\vec{D}) ,$$

again with an arbitrary smooth functions $P_a$.

We are interested in the purely imaginary piece of

$$W(A) = - \log \left( \int [D\bar{\psi} D\psi] e^{-S_3} \right) ,$$

which we will denote as $W^{\text{odd}}$. We will find that this phase, or equivalently, the eta-invariant of this modified Dirac operator, is such that

$$W^{\text{odd}} = \pm \frac{i\pi}{2} \eta \bigg|_{\sigma^a P_a} = N_{\vec{P}} \cdot \left( \pm \frac{i}{2} S_{\text{CS}}(A) + \cdots \right)$$

with the same winding number $N_{\vec{P}}$ of the map $P_a$, as in (3.7). The same notation $P_a$ as in Section 3 is used here because eventually we will connect to $d = 4$ system with the 4-th direction treated as the normal to a $d = 3$ boundary. For Majorana fermions, the discussions below carry over straightforwardly by multiplying the effective action by $1/2$.

### 5.1 Eta-Invariant for Generalized Spinors

As has been studied thoroughly for $\vec{P} = -i\vec{D}$ in Ref. [54], and also reviewed more recently in the context of topological insulators in Ref. [17], the path integral of such a fermion would lead to an effective action whose imaginary part is particularly simple to compute. We start with the Pauli-Villars regularized partition function

$$\prod \frac{\lambda}{\lambda + iM}$$

(5.7)
with eigenvalues $\lambda$ and the regulator mass $M \rightarrow \pm \infty$. The phase of individual pieces are $-i(\pi/2)\text{sgn}(M)(\lambda)$, leading to [54]

$$W(A)^{\text{odd}} = \pm i\frac{\pi}{2} \eta(A)$$

with (5.1).

Exactly the same reasoning goes through even if we replace the usual first order Dirac operator by $Q_3 = \sigma^a P_a(-i\vec{D})$,

$$W(A^{(R)})_{\vec{P}}^{\text{odd}} = \pm i\frac{\pi}{2} \eta_{\vec{P}}(A^{(R)})$$

where we put the subscript $\vec{P}$ to remember that the Dirac operator is replaced by $Q_3$. The eta-invariant has the usual definition,

$$\eta_{\vec{P}}(A^{(R)}) = \lim_{s \rightarrow 0} \sum \text{sgn}(\lambda_Q)|\lambda_Q|^{-s},$$

where the sum is now over all eigenvalues of the operator $Q_3$. We labeled the gauge field by the representation $R$ of $\psi$ explicitly for the purpose of clarifying the normalization.

As noted already, the local part of this effective action for the usual $P_a = -iD_a$, on the other hand, is the well-known Chern-Simons action,

$$\frac{\pi}{2} \eta_{\vec{P}=-i\vec{D}}(A^{(R)}) = t_2(R) \cdot \frac{1}{2} S_{CS}(A) + \cdots,$$

where $S_{CS}(A)$ is the properly quantized Chern-Simons action such that its half appears when we integrate out complex unit-charge $\psi$ under $U(1)$ or $\psi$ in the fundamental representation in $SU(N)$ for example. $t_2(R)$ is the quadratic invariant that keeps track of the gauge representation $R$. When $R$ is the adjoint representation, for example, it happens to be equal to the twice the dual Coxeter number, $t_2(R = \text{adjoint}) = 2h^\vee$.

Here, we wish to show that, with general $\vec{P}$, the above relations is modified simply as

$$\frac{\pi}{2} \eta_{\vec{P}=-i\vec{D}} = N_{\vec{P}} \cdot t_2(R) \cdot \frac{1}{2} S_{CS}(A) + \cdots$$

with the same winding number $N_{\vec{P}}$ that appears in $d = 4$ chiral anomaly. Upon
introducing a Pauli-Villars regulator with $M$, again, we write the imaginary part of the effective action as

$$W(A^{(R)})^{\text{odd}}_{\vec{P}} = \frac{1}{2} \sum_{\text{finite}} \left[ \log(\lambda_{Q_3} + i M) - \log(\lambda_{Q_3} - i M) \right]_{M \to \pm \infty}. \quad (5.13)$$

The way to the relation of type (5.11) comes from how this effective action varies as we vary the gauge field,

$$\delta W(A^{(R)})^{\text{odd}}_{\vec{P}} = \frac{1}{2} \sum \left[ \frac{\delta \lambda_{Q_3}}{\lambda_{Q_3} + i M} - \frac{\delta \lambda_{Q_3}}{\lambda_{Q_3} - i M} \right]_{M \to \pm \infty} = -i M \cdot \text{Tr} \left[ \frac{\delta Q_3}{(Q_3)^2 + M^2} \right]_{M \to \pm \infty} = -i M \cdot \int_0^{\infty} ds \text{ Tr} \left[ \delta Q_3 e^{-s((Q_3)^2 + M^2)} \right]_{M \to \pm \infty}. \quad (5.14)$$

Although the scaling of $s$ looks different from those we used in the anomaly computation, the content is no different because of $e^{-sM^2}$ term in the integrand. The large $M$ limit confines $s$ integral effectively to a region of $s < 1/M^2$, so again the small $s$ expansion of the heat kernel becomes sufficient. And the computation again boils down to a heat kernel one in the coincident limit.

As such, in $d = 3$, the first iteration suffices,

$$G^{(1)}_s(y; x) = \int_0^{s} d\tau \int d^3 z \ G^{(0)}_{s-t}(y; z) \times ((Q_3)^2 - \bar{P}(-i\partial)^2) G^{(0)}_{t}(z; x) \quad (5.15)$$

with

$$G^{(0)}(x + X; x) \equiv \langle x + X| e^{-s\bar{P}(-i\partial)^2}|x \rangle = \int \frac{d^3 k}{(2\pi)^3} e^{i \vec{k} \cdot \vec{X}} e^{-s\bar{P}(\vec{k})^2}, \quad (5.16)$$

as we can see explicitly below.

With nontrivial $\bar{P}$, we find again in the momentum space an expansion around a generic point $x$

$$\delta Q_3 = \sigma^a \bar{\partial}^b P_a(\vec{k}) \delta A_b(x) + \cdots$$

$$(Q_3)^2 = \bar{P}(\vec{k})^2 + i \epsilon^{abc} \sigma^a F_{fg}(x) \delta^f P_b(\vec{k}) \partial^g P_c(\vec{k}) + \cdots, \quad (5.17)$$
where the ellipsis denotes again those terms that are suppressed by small \( s \), or equivalently large \( M \), scaling. Since \( \delta Q_3 \) carries a single \( \sigma^a \), the trace over the spinor requires another \( \sigma^a \), which can be supplied by the field strength term in \( Q_3^2 \) shown above.

The trace over the two-component spinor indices leaves behind,

\[
\delta_{A(x)} W(A^{(R)})^{\text{odd}}_{\vec{P}} = i M \cdot \int_0^\infty ds \; s^{-sM^2} \int \frac{d^3k}{(2\pi)^3} e^{-s\vec{P}(\vec{k})^2} \det \left( \frac{\partial P_f(\vec{k})}{\partial k_g} \right) \left. \text{tr}_R \left( \epsilon^{abc} \delta_{A^a}(x) F_{bc}(x) \right) \right|_{M \rightarrow \pm \infty} 
\]

at a generic point \( x \), \( \vec{k} \)-integrals and an \( s \)-integral, and \( N_{\vec{P}} \) is the same winding number we encountered in Section 3.

Let us count the factor of \( M \) to ensure one ends up with a finite quantity: the three \( P \) integrations will generate \( s^{-3/2} \), so the final \( s \) integral will be of the form

\[
M \cdot \int_0^\infty ds \; s^{-1/2} e^{-sM^2} = \text{sgn}(M) \cdot \int_0^\infty d\tilde{s} \; \tilde{s}^{-1/2} e^{-\tilde{s}} = \text{sgn}(M) \cdot \sqrt{\pi} \cdot \tilde{s}^{1/2} e^{-\tilde{s}}. 
\]

\( M \) is scaled out leaving behind only its sign, and as in the previous anomaly computation, other terms in \((Q_3)^2\) can at most contribute pieces that scale inversely with the large \( M \).

This way, the quantity inside the large parenthesis remains finite in the limit and produces the variation of the imaginary part of the effective action due to a single two-component spinor with \( \vec{P} = -i\vec{D} \). For \( \vec{P} = -i\vec{D} \), in fact, this is precisely how one shows the relation (5.11) by starting with the above and integrating over \( \delta\vec{A}(x) \) back to the Chern-Simons action. Therefore, we obtain at the end of the computation

\[
W(A^{(R)})^{\text{odd}}_{\vec{P}} = N_{\vec{P}} \cdot \left( \pm i t_2(R) \cdot \frac{1}{2} S_{CS}(A) + \cdots \right), 
\]

or,

\[
W(A^{(R)})^{\text{odd}}_{\vec{P}} = N_{\vec{P}} \cdot \left( \pm \frac{\pi}{2} \eta_{\vec{P} = -i\vec{D}}(A^{(R)}) \right) = N_{\vec{P}} \cdot W(A^{(R)})^{\text{odd}}_{\vec{P} = -i\vec{D}}, 
\]

given the general relation between the Chern-Simons action and the eta-invariant in
(5.11), and also from how the effective action has to be gauge invariant.

Generalization to higher dimension $d = 2n - 1$ is also straightforward. It is clear that the variation of the effective action (5.18) can be easily extended to $d = (2n - 1)$ and produce,

$$\delta_{A(x)} W^{\text{odd}}_{Q_{2n-1}} \sim N_{\vec{\beta}} \cdot \text{tr} \left( \delta A \wedge F^{n-1} \right),$$  \hspace{1cm} (5.22)

now expressed as the $d$-form, for the generalized Dirac operator

$$Q_{2n-1} = \gamma^a P_a (-\vec{D})$$ \hspace{1cm} (5.23)

with $2n - 1$ dimensional Dirac matrices $\gamma^a$'s. On the other hand, the Chern-Simons density is defined via

$$dS_{CS} = \frac{1}{n!(2\pi i)^n} \int \text{tr} F^n$$ \hspace{1cm} (5.24)

so its variation is such that

$$d \left[ \delta_{A(x)} S_{CS} \right] = \frac{1}{(n - 1)! (2\pi i)^n} \cdot \text{tr} \left( (d(\delta A) + A\delta A + \delta AA) F^{n-1} \right)$$

$$= d \left[ \frac{1}{(n - 1)! (2\pi i)^n} \cdot \text{tr} \left( (\delta A(x) \wedge F^{n-1}(x)) \right) \right]$$ \hspace{1cm} (5.25)

which brings us back to

$$W(A^{(R)})^{\text{odd}}_{\vec{\beta}} = N_{\vec{\beta}} \cdot \left( \pm \frac{i}{2} S_{CS} (A^{(R)}) + \cdots \right) = N_{\vec{\beta}} \cdot \left( \pm \frac{i}{2} \eta_{\vec{\beta} = -i\vec{D}} (A^{(R)}) \right)$$ \hspace{1cm} (5.26)

following the same pattern we saw for $d = 3$.

### 5.2 Boundary Fermions and APS-like Index Theorem

When $d$ dimensional manifold $\mathcal{M}_d$ has a boundary $\Sigma_{d-1}$, an Atiyah-Patodi-Singer (APS) index problem on $\mathcal{M}_d$ can be formulated via the extension of the manifold by attaching a semi-infinite cylinder with the cross section $\Sigma_{d-1}$. Imposing the square normalizability condition for the ground states, one finds

$$\mathcal{I}_{\mathcal{M}_d}^{\text{APS}} = \mathcal{I}_{\mathcal{M}_d}^{\text{bulk}} - \frac{\eta_{\Sigma_{d-1}}}{2},$$ \hspace{1cm} (5.27)
where $\eta_{\Sigma_{d-1}}$ is the usual eta-invariant of $\Sigma_{d-1}$.

Is there a way to extend this result to the generalized Dirac problem of our kind? Extending the covariant derivative to include the spin connection, say, $\nabla_\mu$, is no big deal. However, one also needs a covariantly constant tensor $C^a_{\mu_1,\cdots,\mu_l}$ such that operators such as

$$\gamma^a C^a_{\mu_1,\cdots,\mu_l} \nabla_{\mu_1} \cdots \nabla_{\mu_l}$$

(5.28)
can be used in place of the ordinary $\gamma^a e^\mu_a \nabla_\mu$ with the vielbein $e^\mu_a$. Such a tensor $C$ implies reduced holonomy, yet the latter is classified completely and known to be rather sparse. As such, general higher-derivative Dirac operator with the curved geometry is generally difficult to construct.

When both $\mathcal{M}_d$ and $\Sigma_{d-1}$ are flat, one other hand, our discussions so far does imply an APS-like index theorem. The easiest is $\mathcal{M}_d = \mathbb{T}^{d-1} \times \mathbb{R}_+$ with the boundary $\Sigma_{d-1} = \mathbb{T}^{d-1}$ at the origin of $\mathbb{R}_+$. With the generalized Dirac operator as in (3.1)

$$Q_{d-1} - D_d = \gamma^a P_a (-i\vec{D}) - D_d,$$

(5.29)
and thus

$$Q_d = \gamma^a P_a (-i\vec{D}) + \gamma^d (-iD_d) = \begin{pmatrix} 0 & Q_{d-1} - D_d \\ Q_{d-1} + D_d & 0 \end{pmatrix},$$

(5.30)
as in (3.3), with the direction $d$ considered as the normal to $\Sigma_{d-1}$. We have already seen that the bulk part of the index is

$$I_{Q_d}^{\text{bulk}} = N_P \cdot I^{\text{bulk}},$$

(5.31)
while the eta-invariant is similarly enhanced as

$$\frac{\eta_{Q_{d-1}}}{2} = N_P \cdot \frac{\eta}{2}.$$  

(5.32)

With these, we can easily retrace the steps found in the appendix of Ref. [54] with the modified heat kernel we have discussed so far, and arrive at a generalized APS
index theorem,

\[ \mathcal{I}_{Q_d} = N_{\vec{P}} \cdot \left( \frac{1}{n!(2\pi i)^n} \int_{T^{2n-1} \times \mathbb{R}_+} \text{tr} \left( F^{(R)} \wedge \cdots \wedge F^{(R)} \right) - \frac{\eta^{T^{2n-1}}(A^{(R)})}{2} \right) \]  \hspace{1cm} (5.33)

for any even dimensions \( d = 2n \).

This form of the APS-like index theorem would suffice for understanding interface between topological insulators and ordinary insulator, by considering physics very near the boundary. As before, the factor \( N_{\vec{P}} \) can be realized either as \( N_{\vec{P}} \) many flavors of ordinary boundary fermions, or a single fermions with higher order Dirac operator with the winding number \( N_{\vec{P}} \).

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