Abstract

Let $p$ be a monic complex polynomial of degree $n$ and let $K$ a measurable subset of the complex plane. We show that the area of $p(K)$, counted with multiplicity, is at least $\pi n (\text{Area}(K)/\pi)^n$ and that

$$\frac{\text{Area}(p^{-1}(K))}{\pi} \leq \left(\frac{\text{Area}(K)}{\pi}\right)^{1/n}.$$ 

Both bounds are sharp. The special case of the latter result in which $K$ a disc was proved by Pólya in 1928. We use Carleman's isoperimetric inequality relating the conductance and area for plane condensers. We include a summary of the necessary potential theory.

1 Introduction and Statement of Results

When $p$ is a polynomial of degree $n$ over $\mathbb{C}$, the set

$$E(p, r) = \{z \in \mathbb{C} : |p(z)| = r^n\}$$

is called a lemniscate, after Bernoulli's lemniscate $\{z \in \mathbb{C} : |z^2 - 1| = 1\}$. It is natural to ask how large the set enclosed by a lemniscate can be. We need to make some normalisation for this to be meaningful; the simplest is to ask for $p$ to be monic. Perhaps surprisingly, the area of the lemniscate $E(p, 1)$ is then bounded independently of $p$. In fact the following sharp inequality was proved by Pólya in 1928. [5].

*Supported by an EPSRC research studentship
Theorem 1 (Pólya’s inequality).
Let $p$ be a monic polynomial of degree $n$ over $\mathbb{C}$ and let $D$ be a disc in $\mathbb{C}$. Then the Euclidean area of $p^{-1}(D)$ is at most $\pi \left( \frac{\text{Area}(D)}{\pi} \right)^{1/n}$, with equality only when $p: z \mapsto a(z - b)^n + c$ and the center of $D$ is $c$, the unique critical value of $p$.

It is natural to ask whether we can get a larger preimage by fixing the area of $D$ but allowing its shape to vary. The main theorem of this paper is that we cannot.

Theorem 2. Let $p$ be a monic polynomial of degree $n$ over $\mathbb{C}$. Let $K$ be any measurable subset of the plane. Then

$$\text{Area}(p^{-1}(K)) \leq \pi \left( \frac{\text{Area}(K)}{\pi} \right)^{1/n},$$

with equality if and only if $K$ is (up to sets of measure zero) a disc and $p$ has a unique critical value at the center of that disc.

Theorem 2 is a simple consequence of the following stronger theorem:

Theorem 3. Let $p$ be a monic polynomial of degree $n$ over $\mathbb{C}$, and $K$ be any measurable subset of the plane. Define the multiplicity $n(z, p, K)$ to be the number of $p$–preimages of $z$ in $K$, counted according to their valency. Then the area of $p(K)$ counted with multiplicity satisfies

$$\int_{\mathbb{C}} n(z, p, K) \, dA(z) = \int_{K} |p'(z)|^2 \, dA \geq n \pi \left( \frac{\text{Area}(K)}{\pi} \right)^n,$$

with equality if and only if $K$ is (up to sets of measure zero) a disc and $p$ has a unique critical value at the center of that disc.

For a compact set $K \subset \mathbb{C}$, define

$$\rho(K) = \frac{\text{Area}(K)}{\pi \text{cap}(K)^2},$$

where $\text{cap}(K)$ is the logarithmic capacity of $K$. Then $\rho(K)$ is a measure of the roundness of $K$: we will see that $\rho(K) \in [0, 1]$, and $\rho(K) = 1$ if and only if $K$ is a full–measure subset of a disc. We use $\rho$ to formulate the following scale-invariant version of theorem 2.
**Theorem 4.** If $p$ is any complex polynomial of degree $n$, not necessarily monic, and $K$ is any compact subset of the plane, then

$$\rho(p^{-1}(K)) \leq \rho(K)^{1/n}.$$ 

This corollary is sharp for each value of $\rho$. To see this we can take $K$ to be the union of the unit disc with a radial line segment, and $p : z \mapsto z^n$, so that we still get equality in theorem 2.

In section 2 we give a quick introduction to the potential theory that we will need. In section 3 we discuss some isoperimetric inequalities and their relationship with Pólya’s inequality (theorem 1), which we prove since it is an important ingredient in the proof of theorem 2. Theorems 2, 3 and 4 are proved in section 4.

A survey of area estimates for lemniscates has recently been given by Lubinsky [4], with a view towards applications in the convergence theory of Padé approximation. In [2], Eremenko and Hayman made progress on the related problem of bounding the length of $E(p, r)$. Fryntov and Rossi [3] have obtained the sharp analogue of Pólya’s inequality (theorem 1) bounding the hyperbolic area of the preimage of a hyperbolic disc under a finite Blaschke product. This raises the question of finding the sharp Blaschke product analogues of theorems 2 and 3.

The author thanks Assaf Naor and Ben Green for posing the question that led to this paper, and his PhD supervisor Keith Carne for useful conversations.

## 2 Capacity of plane subsets and condensers

**Definition 1.** A plane condenser is a pair $(E, B)$ of subsets of $\mathbb{C}$, where $E \neq \mathbb{C}$ is open and $B$ is a non-empty closed subset of $E$.

The terminology arises from the fact that a pair of conducting cylinders with cross-section $\partial E$ and $B$ respectively could be used as a condenser (or capacitor). The capacity of the condenser $(E, B)$ is physically the capacitance per unit length of an infinitely long pair of such cylinders. The same quantity describes the conductance between $\partial E$ and $B$ of an isotropic resistor consisting of a plate in the shape of $E \setminus B$. We compute this conductance by considering the electrical potential $f$ that would be induced in
If we were to connect $\partial E$ to an electrical potential 0 and $B$ to potential +1. Given $f$, we can compute the current $-\nabla f$ that would flow in response to the potential $f$, and the power consumed is proportional to

$$L(f) = \int_{E \setminus B} |\nabla f|^2 dA,$$

We expect a physical potential $f$ to minimise $L(f)$ over all possible potential functions satisfying the given boundary conditions. Calculus of variations tells us that if there is an extremal $f$, it must be harmonic on $E \setminus B$. We call such an $f$ a Green’s function for the condenser. A Green’s function only exists if the boundary $\partial E \cup \partial B$ is regular for the Dirichlet problem, but we avoid this difficulty by defining

$$\text{cap}(E,B) = \frac{1}{4\pi} \inf L(f),$$

where the infimum is taken over all continuously differentiable $f : \mathbb{C} \to \mathbb{R}$ such that $f = 0$ on $\mathbb{C} \setminus E$ and $f = 1$ on $B$. We call such functions admissible for the condenser $(E,B)$. Note that cap$(E,B)$ may be zero, as it is when $B$ is a finite set. From the definition it is immediate that capacity is monotonic, i.e.

$$E \subseteq F \text{ and } B \supseteq C \implies \text{cap}(E,B) \geq \text{cap}(F,C).$$

**Lemma 1.** Suppose that on some open set $U \subset \mathbb{C}$ we have an analytic function $\psi$ such that each point of $E$ has exactly $n$ preimages in $U$, counted according to valency. Then

$$\text{cap}(\psi^{-1}(E),\psi^{-1}(B)) = n \text{cap}(E,B).$$

**Proof.** Suppose that $f : \mathbb{C} \to \mathbb{R}$ is any admissible function for $(E,B)$. The hypothesis implies that the restriction of $\psi$ to $\psi^{-1}(E)$ is a proper map, so we can extend $f \circ \psi$ to get an admissible function for $(\psi^{-1}(E),\psi^{-1}(B))$ by giving it the value 0 outside $U$. Since $\psi$ is almost everywhere conformal,

$$L(f \circ \psi) = \int_{\psi^{-1}(E \setminus B)} |\nabla (f \circ \psi)(z)|^2 d\text{Area}(z)$$

$$= \int_{\psi^{-1}(E \setminus B)} |(\nabla f)(\psi(z))||\psi'(z)|^2 d\text{Area}(z)$$

$$= n \int_{E \setminus B} |\nabla f(w)|^2 d\text{Area}(w) = n L(f).$$

$\square$
In particular the capacity is a **conformal invariant** of condensers: if \( \varphi : E \to \mathbb{C} \) is a univalent function then
\[
\text{cap}(E, B) = \text{cap}(\varphi(E), \varphi(B)).
\]
For example, if \( (E \setminus B) \) is a ring domain then its modulus is \( 1/(4\pi \text{cap}(E, B)) \).

Let \( K \) be a compact set in the plane. A Green’s function for \( K \) is a continuous function \( f : \mathbb{C} \to \mathbb{R} \), zero on \( K \) and harmonic on \( \mathbb{C} \setminus K \), with \( f(z) = \log |z| - \log t + o(1) \) as \( z \to \infty \). If \( K \) has a Green’s function then the logarithmic capacity of \( K \) is defined to be \( \text{cap}(K) = t \). For general \( K \) we define \( \text{cap}(K) = \inf \text{cap}(J) \) over all compact sets \( J \supset K \) with regular boundary for the Dirichlet problem on \( \mathbb{C} \setminus J \). By pulling back Green’s functions, it is easy to verify that if \( p \) is a monic polynomial of degree \( n \) then
\[
\text{cap}(p^{-1}(B)) = \text{cap}(B)^{1/n}.
\]

### 3 Isoperimetric Inequalities

A relationship between capacity and 2-dimensional Lebesgue measure is given by the following ‘isoperimetric’ inequality:

**Theorem 5.** (Carleman, 1918)

\[
\frac{1}{\text{cap}(E, B)} \leq \log \left( \frac{\text{Area}(E)}{\text{Area}(B)} \right),
\]

with equality iff \( E \) and \( B \) are concentric discs.

The proof of Carleman’s inequality uses the fact that the Dirichlet integral \( L(f) \) does not increase when \( f \) is replaced by its Schwarz symmetrization, the function \( S(f) \) whose superlevel sets are concentric discs with the same area as the corresponding level sets of \( f \). For details, see the classic book of Pólya and Szegő, [6], or [1] for a more modern account.

Taking \( E = B(0, R) \) and then letting \( R \to \infty \) in Carleman’s isoperimetric inequality yields the following well-known isoperimetric theorem for logarithmic capacity. For a simple proof, including the equality case, see theorem 5.3.5 in [7].

**Theorem 6.** For any compact set \( K \subset \mathbb{C} \),
\[
\text{Area}(K) \leq \pi \text{cap}(K)^2,
\]
with equality if and only if $K$ is a disc.

We have now collected everything we need to prove Pólya’s inequality, theorem 1. The capacity of the disc $D$ is precisely the radius of $D$, so

$$\text{cap}(D) = \left( \frac{\text{Area}(D)}{\pi} \right)^{1/2},$$

$$\text{cap}(p^{-1}(D)) = \left( \frac{\text{Area}(D)}{\pi} \right)^{1/2n},$$

and, applying theorem 6

$$\text{Area}(p^{-1}(D)) \leq \pi \left( \frac{\text{Area}(D)}{\pi} \right)^{1/n},$$

as required. In view of the strong link between logarithmic capacity and polynomials, theorems 1 and 6 are virtually equivalent. In [4], Pólya’s inequality is proved using Gronwall’s area formula, and used to deduce the isoperimetric inequality for logarithmic capacity.

### 4 Proof of theorems 2 and 3

**Lemma 2.** For any complex polynomial $g$ of degree $d$,

$$\int_{C} |g(w)| 1_{|g(w)| \leq x} \, dA \geq \frac{2x}{d + 2} \text{Area}\{w \in \mathbb{C} : |g(w)| \leq x\}.$$

**Proof.** By lemma 1 we have

$$\text{cap}(g^{-1}(B(0, x)), g^{-1}(B(0, s))) = \frac{d}{2 (\log x - \log s)}.$$

Theorem 5 gives

$$\frac{\text{Area}\{w \in \mathbb{C} : s \leq |g(w)| \leq x\}}{\text{Area}\{w \in \mathbb{C} : |g(w)| \leq x\}} \geq 1 - \left( \frac{s}{x} \right)^{2/d},$$

so

$$\int_{C} |g(w)| 1_{|g(w)| \leq x} \, dA = \int_{0}^{x} \text{Area}\{w \in \mathbb{C} : s \leq |g(w)| \leq x\} \, ds$$

$$\geq \text{Area}\{w \in \mathbb{C} : |g(w)| \leq x\} \int_{0}^{x} 1 - \left( \frac{s}{x} \right)^{2/d} \, ds$$

$$= \frac{2x}{d + 2} \text{Area}\{w \in \mathbb{C} : |g(w)| \leq x\}.$$
Now fix a monic polynomial $p$ and $A > 0$. Among all measurable sets $K$ with $\text{Area}(K) = A$, the Dirichlet integral

$$\int_K |p'(w)|^2 d\text{Area}(w)$$

is minimised when $K$ is the sublevel set

$$K_t = \{ w \in \mathbb{C} : |p'(w)|^2 \leq t \}.$$

Here $t$ is determined uniquely by the condition $\text{Area}(K_t) = A$. The polynomial $z \mapsto (p'(z)/n)^2$ is monic, with degree $2n - 2$, so theorem 1 gives

$$A = \text{Area}(K_t) \leq \pi \left( \frac{\pi (t/n^2)^2}{\pi} \right)^{1/(2n-2)}.$$

Rearranging this we have

$$t \geq n^2 \left( \frac{A}{\pi} \right)^{n-1}.$$

Now we apply lemma 2 to the polynomial $g = (p')^2$ to obtain

$$\int_{K_t} |p'(w)|^2 d\text{Area}(w) = \int_{\mathbb{C}} |p'(w)|^2 1_{|p'(w)|^2 \leq t} d\text{Area}(z) \geq \frac{2t}{2n} \text{Area}(K_t) = \frac{tA}{n} \geq n \pi \left( \frac{A}{\pi} \right)^n.$$

For equality, we must have equality in our application of Pólya’s inequality, so $p$ must be $p : z \mapsto (z - b)^n + c$, and $K$ can differ from disc $K_t$ at most by a set of 2-dimensional Lebesgue measure zero. This completes the proof of theorem 3.

To obtain theorem 2, observe that a monic polynomial $p$ maps $p^{-1}(K)$ onto $K$ with multiplicity $n$ everywhere, so

$$\text{Area}(K) = \frac{1}{n} \int_{p^{-1}(K)} |p'(w)|^2 d\text{Area}(w).$$

Finally, theorem 4 is obtained by dividing both sides of the inequality of theorem 2 by $\text{cap}(K)^2 = \text{cap}(p^{-1}(K))^2$. 

7
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