Effects of multiplicative noises on synchronization in finite $N$-unit stochastic ensembles: Augmented moment approach

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Abstract

We have studied the synchronization in finite $N$-unit FitzHugh-Nagumo neuron ensembles subjected to additive and multiplicative noises, by using the augmented moment method (AMM) which is reformulated with the use of the Fokker-Planck equation. It has been shown that for diffusive couplings, the synchronization may be enhanced by multiplicative noises while additive noises are detrimental to the synchronization. In contrast, for sigmoid coupling, both additive and multiplicative noises deteriorate the synchronization. The synchronization depends not only on the type of noises but also on the kind of couplings.

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1 INTRODUCTION

Nonlinear stochastic equations subjected to additive and/or multiplicative noises have been widely adopted for a study on real systems in physics, biology, chemistry, economy and networks. Interesting phenomena caused by both the noises have been intensively investigated (for a recent review, see Ref. 1, related references therein). It has been realized that the property of multiplicative noises is different from that of additive noises in some respects as follows. (1) Multiplicative noises induce the phase transition, creating an ordered state, while additive noises are against the ordering [2]-[6]. (2) Although the stochastic resonance is not realized in linear systems with additive noises, it may be possible with multiplicative color noise (but not with multiplicative white noise) [7, 8]. (3) Although the probability distribution in stochastic systems subjected to additive Gaussian noise follows the Gaussian, it is not the case for multiplicative Gaussian noises which generally yield non-Gaussian distribution [9]-[12]. (4) The scaling relation of the effective strength for additive noise given by $\beta(N) = \beta(1)/\sqrt{N}$ is not applicable to that for multiplicative noise: $\alpha(N) \neq \alpha(1)/\sqrt{N}$, where $\alpha(N)$ and $\beta(N)$ denote effective strengths of multiplicative and additive noises, respectively, in the $N$-unit system [13].

In order to show the above item (4), the present author has adopted the augmented moment method (AMM) in a recent paper [13]. The AMM was originally developed by expanding variables around their mean values in order to obtain the second-order moments both for local and global variables in stochastic systems [14]. The AMM has been successfully applied to a study on dynamics of coupled stochastic systems described by Langevin, FitzHugh-Nagumo and Hodgkin-Huxley models subjected only to additive noises with global, local or small-world couplings (with and without transmission delays) [15]. In Ref. [13], we have reformulated the AMM with the use of the Fokker-Planck equation (FPE), in order to avoid the difficulty due to the Ito versus Stratonovich calculus inherent for multiplicative noise. It has been pointed out that a naive approximation of the scaling relation for multiplicative noise: $\alpha(N) = \alpha(1)/\sqrt{N}$, as adopted by Muñoz, Colaiori and Castellano in their recent paper [6], leads to the result which violates the central-limit theorem and which is in disagreement with those of AMM and direct simulations.

The purpose of the present paper is two folds: (1) to reformulate AMM for FitzHugh-Nagumo (FN) model subjected to both additive and multiplicative noises with the use of FPE [13], and (2) to discuss the respective roles of the two noises on the synchronization. Our calculations
have shown that multiplicative noises may enhance the synchronization while additive noises work to destroy it. This is similar to the property in item (1) discussed above.

The paper is organized as follows. In Sec. II, we have applied the DMA to finite $N$-unit FN networks subjected to additive and multiplicative noises. Numerical calculations are presented in Sec. III. The final Sec. IV is devoted to conclusion and discussion.

2 Noisy FN neuron ensembles

2.1 Augmented moment method

We have adopted $N$-unit FN neurons subjected to additive and multiplicative noises. Dynamics of a neuron $i$ in a given FN neuron ensemble is described by the nonlinear differential equations (DEs) given by

\[
\frac{dx_i}{dt} = F(x_i) - c y_i + \alpha G(x_i) \eta_i(t) + \beta \xi_i(t) + I_i(t) + I^e(t), \quad (i = 1 \text{ to } N)
\]

\[
\frac{dy_i}{dt} = b x_i - d y_i + e,
\]

with

\[
I_i(t) = \frac{J}{Z} \sum_{j \neq i} (x_j - x_i).
\]

In Eq. (1)-(3), $F(x) = kx(x - a)(1 - x)$, $k = 0.5$, $a = 0.1$, $b = 0.015$, $d = 0.003$ and $e = 0$ [14][16]: $x_i$ and $y_i$ denote the fast (voltage) variable and slow (recovery) variable, respectively: $G(x)$ an arbitrary function of $x$: $I^e(t)$ an external input whose explicit form will be shown shortly [Eq. (41)]: $J$ expresses strengths of diffusive couplings, $Z = N - 1$: $\alpha$ and $\beta$ denote magnitudes of multiplicative and additive noises, respectively, and $\eta_i(t)$ and $\xi_i(t)$ express zero-mean Gaussian white noises with correlations given by

\[
\langle \eta_i(t) \eta_j(t') \rangle = \delta_{ij} \delta(t - t'), \quad (4)
\]

\[
\langle \xi_i(t) \xi_j(t') \rangle = \delta_{ij} \delta(t - t'), \quad (5)
\]

\[
\langle \eta_i(t) \xi_j(t') \rangle = 0. \quad (6)
\]

The Fokker-Planck equation $p(\{x_i\}, \{y_i\}, t)$ is expressed in the Stratonovich representation by[13][17]

\[
\frac{\partial}{\partial t} p = -\sum_k \frac{\partial}{\partial x_k} \{[F(x_k) - c y_k + I_k] p\} - \sum_k \frac{\partial}{\partial y_k} \{(b x_k - d y_k + e) p\}
\]

\[
+ \alpha^2 \sum_k \frac{\partial}{\partial x_k} \{G(x_k) \frac{\partial}{\partial x_k} [G(x_k) p]\} + \sum_k \beta^2 \frac{\partial^2}{\partial x_k^2} p, \quad (7)
\]
where $I_k = I_k^{(c)} + I^{(e)}$.

We are interested also in dynamics of global variables $X(t)$ and $Y(t)$ defined by

$$
X(t) = \frac{1}{N} \sum_i x_i(t),
$$

$$
Y(t) = \frac{1}{N} \sum_i y_i(t).
$$

The probability of $P(X, Y, t)$ is expressed in terms of $p(\{x_i\}, \{y_i\}, t)$ by

$$
P(X, Y, t) = \int \int dx_i dy_i \Pi_i p_i(x_i, y_i, t) \delta(X - \frac{1}{N} \sum_i x_i) \delta(Y - \frac{1}{N} \sum_i y_i).
$$

Moments of local and global variables are expressed by

$$
\langle x_i^k y_i^\ell \rangle = \int \int dx_i dy_i p_i(x_i, y_i, t) x_i^k y_i^\ell,
$$

$$
\langle X^k Y^\ell \rangle = \int \int dXdY P(X, Y, t) X^k Y^\ell.
$$

By using Eqs. (1), (2), (7) and (11), we get equations of motions for means, variances and covariances of local variables by

$$
\frac{d\langle x_i \rangle}{dt} = \langle F(x_i) \rangle - c\langle y_i \rangle + \frac{\alpha^2}{2} \langle G'(x_i) G(x_i) \rangle,
$$

$$
\frac{d\langle y_i \rangle}{dt} = b\langle x_i \rangle - d\langle y_i \rangle + e,
$$

$$
\frac{d\langle x_i x_j \rangle}{dt} = \langle x_i F(x_j) \rangle + \langle x_j F(x_i) \rangle - c(\langle x_i y_j \rangle + \langle x_j y_i \rangle)
$$

$$
\quad + \frac{J}{Z} \sum_k (\langle x_i x_k \rangle + \langle x_j x_k \rangle - \langle x_i^2 \rangle - \langle x_j^2 \rangle)
$$

$$
\quad + \frac{\alpha^2}{2} \left[ \langle x_i G'(x_j) G(x_j) \rangle + \langle x_j G'(x_i) G(x_i) \rangle \right]
$$

$$
\quad + [\alpha^2 \langle G(x_i) \rangle^2 + \beta^2] \delta_{ij},
$$

$$
\frac{d\langle y_i y_j \rangle}{dt} = b(\langle x_i y_j \rangle + \langle x_j y_i \rangle) - 2d\langle y_i y_j \rangle,
$$

$$
\frac{d\langle x_i y_j \rangle}{dt} = \langle y_j F(x_i) \rangle - c\langle y_i y_j \rangle + b\langle x_i x_j \rangle - d\langle x_i y_j \rangle
$$

$$
\quad + \frac{w}{Z} \sum_k (\langle x_k y_j \rangle - \langle x_i y_j \rangle) + \frac{\alpha^2}{2} \langle y_j G'(x_i) G(x_i) \rangle,
$$

where $G'(x) = dG(x)/dx$.

Equations of motions for variances and covariances of global variables are obtainable from Eqs. (8), (9) and (12):

$$
\frac{d\langle V_\kappa \rangle}{dt} = \frac{1}{N} \sum_v \langle v_{\kappa i} \rangle,
$$

$$
\frac{d\langle V_\kappa V_\lambda \rangle}{dt} = \frac{1}{N^2} \sum_i \sum_j \frac{d\langle v_{\kappa i} v_{\lambda j} \rangle}{dt},
$$

$(\kappa, \gamma = 1, 2)$
where we adopt a convention: \( v_{1i} = x_i, \ v_{2i} = y_i, \ V_1 = X \) and \( V_2 = Y \). Equations (13) and (14) are used for \( N = 1 \) FN neuron (\( \alpha = 0 \)) and for \( N = \infty \) FN neuron ensembles (\( \alpha = 0 \)) in the mean-field approximation [18]. Equations (13)-(17) are employed in the moment method for a single FN neuron subjected to additive noises [16]. We will show that Eqs. (18) and (19) play important roles in discussing finite \( N \)-unit FN ensembles.

In the AMM [14], we define the eight quantities given by

\[
\mu_\kappa = \langle V_\kappa \rangle = \frac{1}{N} \sum_i \langle v_{\kappa i} \rangle, \quad (20)
\]

\[
\gamma_{\kappa,\lambda} = \frac{1}{N} \sum_i \langle (v_{\kappa i} - \mu_\kappa) (v_{\lambda i} - \mu_\lambda) \rangle, \quad (21)
\]

\[
\rho_{\kappa,\lambda} = \langle (V_\kappa - \mu_\kappa) (V_\lambda - \mu_\lambda) \rangle, \quad (\kappa, \gamma = 1, 2) \quad (22)
\]

with \( \gamma_{1,2} = \gamma_{2,1} \) and \( \rho_{1,2} = \rho_{2,1} \). It is noted that \( \gamma_{\kappa,\lambda} \) expresses the averaged fluctuations in local variables while \( \rho_{\kappa,\lambda} \) denotes fluctuations in global variables. Expanding Eqs. (13)-(19) around means of \( \mu_\kappa \) as \( v_{\kappa i} = \mu_\kappa + \delta v_{\kappa i} \), we get equations of motions for the eight quantities:

\[
\frac{d\mu_1}{dt} = f_\circ + f_2 \gamma_{1,1} - c \mu_2 + \frac{\alpha^2 \mu_1}{2} + I(e), \quad (23)
\]

\[
\frac{d\mu_2}{dt} = b \mu_1 - d \mu_2 + e, \quad (24)
\]

\[
\frac{d\gamma_{1,1}}{dt} = 2(a \gamma_{1,1} - c \gamma_{1,2}) + \frac{2JN}{Z} (\rho_{1,1} - \gamma_{1,1}) + 2a^2 \gamma_{1,1} + \frac{\alpha^2 \mu_1^2}{2} + \beta^2, \quad (25)
\]

\[
\frac{d\gamma_{2,2}}{dt} = 2(b \gamma_{1,2} - d \gamma_{2,2}), \quad (26)
\]

\[
\frac{d\gamma_{1,2}}{dt} = b \gamma_{1,1} + (a - d) \gamma_{1,2} - c \gamma_{2,2} + \frac{JN}{Z} (\rho_{1,2} - \gamma_{1,2}) + \frac{\alpha^2 \gamma_{1,2}}{2}, \quad (27)
\]

\[
\frac{d\rho_{1,1}}{dt} = 2(a \rho_{1,1} - c \rho_{1,2}) + 2a^2 \rho_{1,1} + \frac{\alpha^2 \mu_1^2}{N} + \frac{\beta^2}{N}, \quad (28)
\]

\[
\frac{d\rho_{2,2}}{dt} = 2(b \rho_{1,2} - d \rho_{2,2}), \quad (29)
\]

\[
\frac{d\rho_{1,2}}{dt} = b \rho_{1,1} + (a - d) \rho_{1,2} - c \rho_{2,2} + \frac{\alpha^2 \rho_{1,2}}{2}, \quad (30)
\]

where \( a = f_1 + 3f_3 \gamma_{1,1}, \ f_\ell = (1/\ell!) F^{(\ell)}(\mu_1), \) and \( G(x) = x \) is adopted, relevant expressions for a general \( G(x) \) being given in the appendix. The original \( 2N \)-dimensional stochastic equations given by Eqs. (1) and (2) are transformed to eight-dimensional deterministic equations. Equations (23)-(30) with \( \alpha = 0 \) (additive noises only) reduce to those obtained previously [14].
2.2 N dependence of effective noise strength

Comparing the $\beta^2$ term in $d\gamma_{1,1}/dt$ of Eq. (25) to that in $d\rho_{1,1}/dt$ of Eq. (28), we note that the effective strength of additive noise is scaled by

$$\beta \rightarrow \frac{\beta}{\sqrt{N}}. \quad (31)$$

As for multiplicative noise, however, the situation is not so simple. A comparison between the $\alpha^2$ terms in Eq. (27) and (30) yield the two kinds of scalings:

$$\alpha \rightarrow \frac{\alpha}{\sqrt{N}}, \quad \text{for } \mu_1 \text{ term,} \quad (32)$$

$$\alpha \rightarrow \alpha, \quad \text{for } \gamma_{1,1} \text{ and } \rho_{1,1} \text{ terms,} \quad (33)$$

The relations given by Eqs. (31)-(33) hold also for $d\gamma_{1,2}/dt$ and $d\rho_{1,2}/dt$ given by Eqs. (27) and (30). Thus the scaling behavior of the effective strength of multiplicative noises is quite different from that of additive noises, as previously pointed out for Langevin model [13].

Nevertheless, we note that in the limit of $J = 0$, AMM equations lead to

$$\rho_{\kappa,\lambda} = \frac{\gamma_{\kappa,\lambda}}{N}, \quad (\kappa, \lambda = 1, 2) \quad (34)$$

which is nothing but the central-limit theorem describing the relation between fluctuations in local and average variables.

2.3 Synchronization ratio

In order to quantitatively discuss the synchronization, we first consider the quantity given by [14]

$$R(t) = \frac{1}{N^2} \sum_{ij} \langle [x_i(t) - x_j(t)]^2 \rangle = 2[\gamma_{1,1}(t) - \rho_{1,1}(t)]. \quad (35)$$

When all neurons are in the completely synchronous state, we get $x_i(t) = X(t)$ for all $i$, and then $R(t) = 0$ in Eq. (35). On the contrary, in the asynchronous state, we get $R(t) = 2(1 - 1/N)\gamma_{1,1} \equiv R_0(t)$ from Eq. (34). We have defined the synchronization ratio given by [14]

$$S(t) = 1 - \frac{R(t)}{R_0(t)} = \left( \frac{N\rho_{1,1}(t)/\gamma_{1,1}(t) - 1}{N - 1} \right), \quad (36)$$

which is 0 and 1 for completely asynchronous ($R = R_0$) and synchronous states ($R = 0$), respectively. We have studied the synchronization ratios at $t_f$ and $t_m$ as given by

$$S_f = S(t_f), \quad (37)$$

$$S_m = S(t_m), \quad (38)$$
with

\[
t_f = \{ t \mid X(t) = \theta, dX(t)/dt > 0 \}, \tag{39}
\]
\[
t_m = \{ t \mid dS(t)/dt = 0 \}, \tag{40}
\]

\( t_f \) denoting the firing time at which the global variable \( X(t) \) crosses the threshold \( \theta \) from below and \( t_m \) the time when \( S(t) \) has the maximum value. \( S_f \) and \( S_m \) depend on model parameters such as the noise intensities (\( \alpha \) and \( \beta \)), the coupling strength (\( J \)) and the size of cluster (\( N \)).

3 CALCULATED RESULTS

We have made numerical calculations, applying an external input given by

\[
I^{(e)}(t) = A \Theta(t - t_{in}) \Theta(t_{in} + t_w - t), \tag{41}
\]

where \( A = 0.1, t_{in} = 40 \) and \( t_w = 10 \) [14]. AMM equations given by Eqs. (23)-(30) have been solved by using the fourth-order Runge-Kutta method with a time step of 0.01. Direct simulations (DS) for the \( N \)-unit FN model given by Eqs. (1)-(3) have been performed by using the Heun method with a time step of 0.003. Results of DS are averaged over 100 trials. All quantities are dimensionless.

Figures 1(a)-(d) show time courses of \( \mu_1(t) \), \( \gamma_{1,1}(t) \), \( \rho_{1,1}(t) \) and \( S(t) \), respectively, calculated by AMM (solid curves) and DS (dashed curves) with \( \alpha = 0.01, \beta = 0.001 \), \( J = 1.0 \) and \( N = 100 \). When the external input \( I^{(e)}(t) \) is applied at \( t = 40 \) to the quiescent states which have been randomized by applied small additive noises of \( \beta = 0.001 \), FN neurons fire, and \( \gamma_{1,1}(t), \rho_{1,1}(t) \) and \( S(t) \) develop. Results calculated by AMM are in good agreement with those of DS.

Figure 1(d) shows that when an input signal is applied at \( t = 40 \), \( S(t) \) is suddenly decreased but has a peak at \( t \sim 60 \) where \( X(t) \) is in the refractory period. In order to see the behavior of \( S(t) \) in more detail, we show in Fig.2, its time course for the four cases:

(1) \( \alpha = 0.0 \), (2) \( \alpha = 0.002 \), (3) \( \alpha = 0.01 \) and (4) \( \alpha = 0.05 \) with \( \beta = 0.001 \), \( J = 1.0 \) and \( N = 100 \). In the case (1), the system is subjected only to additive noise, for which \( S(t) \) plotted by dashed curve is increased by an applied input for \( 40 \leq t < 50 \). It shows \( S_f = 0.30 \) at \( t_f = 44.5 \) and \( S_m = 0.44 \) at \( t = 60.35 \), and approaches the equilibrium value of \( S = 0.159 \) at \( t > 100 \). In the case (2) with \( \alpha = 0.002 \), \( S(t) \) shown by dotted curve yields \( S_f = 0.205 \) at \( t_f = 44.5 \) and \( S_m = 0.526 \) at \( t = 60.37 \). In the case (3) with \( \alpha = 0.01 \), \( S(t) \)}
shown by solid curve leads to $S_f = 0.05$ at $t_f = 44.5$ and $S_m = 0.838$ at $t = 60.55$. In the case (4) with stronger multiplicative noise of $\alpha = 0.05$, $S(t)$ plotted by chain curve yields $S_f = 0.03$ at $t_f = 44.5$ and $S_m = 0.910$ at $t = 60.6$. We note that with more increasing $\alpha$, $S_f$ is much decreased while $S_m$ is much increased.

This trend is more clearly seen in Fig. 3(a), where $S_f$ and $S_m$ are plotted as a function of $\alpha$ for $\beta = 0.001$, $0.01$ and $0.02$. In the case of $\beta = 0.001$, $S_f$ is rapidly decreased and $S_m$ is rapidly increased with increasing $\alpha$. For stronger additive noises of $\beta = 0.01$ and $\beta = 0.02$, $S_f$ ($S_m$) is gradually decreased (increased) with increasing $\alpha$. These results show that multiplicative noises enhance $S_m$ but deteriorate $S_f$.

Figure 3(b) shows $S_f$ and $S_m$ as a function of $\beta$ for $\alpha = 0.0$, $0.01$ and $0.05$. In the case of $\alpha = 0.0$, $S_f$ and $S_m$ are almost independent of $\beta$ though they are gradually decreased for larger $\beta$. In the cases of $\alpha = 0.01$ and $\alpha = 0.05$, $S_f$ and $S_m$ are increased and decreased, respectively, with increasing $\beta$. Although these results give an impression that additive noises enhance $S_f$ and deteriorate $S_m$, it is not true. Rather additive noises work to recover $S_f$ and $S_m$ to the values of $S_f = 0.30$ and $S_m = 0.44$ for the absence of multiplicative noise ($\alpha = 0$).

We have studied the effects of the noises on $S_f$ and $S_m$: the former expresses the synchronization ratio at the firing time at $X(t) = \theta$ and the latter denotes its maximum value when $X(t)$ is in the refractory period at $t \sim 60$. We may note that if multiplicative noises exits, the synchronization ratio $S(t)$ is once decreased when an input applied, and it soon rebounds, showing the enhanced value. This trend is more significant for a considerable multiplicative noises. In this sense, the synchronization may be enhanced by multiplicative noises.

By using our AMM, it is possible to study the dependence of the synchronization on the size of ensembles ($N$). Figure 4 shows the $N$ dependences of $S_f$ and $S_m$ in the two cases for (1) $\alpha = 0.01$ and $\beta = 0.001$ (multiplicative noise dominant) and (2) $\alpha = 0.0$ and $\beta = 0.01$ (additive noise only) with $J = 1.0$. Both $S_f$ and $S_m$ are increased with decreasing $N$, which shows that the synchronization becomes better for smaller systems.

4 CONCLUSION AND DISCUSSION

Although we have adopted the diffusive coupling given by Eq. (3), the sigmoid coupling given by

$$I^{(c)}_i(t) = \frac{K}{Z} \sum_{j \neq i} H(x_j(t)),$$

(42)
has been widely employed for discussing networks, where \( K \) expresses a coupling strength and \( H(x) \) is an arbitrary function of \( x \) [Eq. (51)]. Diffusive and sigmoid couplings model electrical and chemical synapses, respectively, in neuronal systems. It is worthwhile to study also the case of sigmoid coupling although such FN ensembles subjected only to additive noises were studied with the use of AMM [14].

A straightforward calculation using the AMM discussed in Sec. II leads to equations of motion given by

\[
\begin{align*}
\frac{d\mu_1}{dt} &= f_o + f_2\gamma_{1,1} - c\mu_2 + K(h_0 + h_2\gamma_{1,1}) + \frac{\alpha^2\mu_1}{2} + I^{(e)}, \\
\frac{d\mu_2}{dt} &= b\mu_1 - d\mu_2 + e, \\
\frac{d\gamma_{1,1}}{dt} &= 2(a\gamma_{1,1} - c\gamma_{1,2}) + Kh_1(\rho_{1,1} - \frac{\gamma_{1,1}}{N}) + 2\alpha^2\gamma_{1,1} + \alpha^2\mu_1^2 + \beta^2, \\
\frac{d\gamma_{1,2}}{dt} &= 2(b\gamma_{1,2} - d\gamma_{2,2}), \\
\frac{d\gamma_{2,2}}{dt} &= b\gamma_{1,1} + (a - d)\gamma_{1,2} - c\gamma_{2,2} + Kh_1(\rho_{1,2} - \frac{\gamma_{1,2}}{N}) + \frac{\alpha^2\gamma_{1,2}}{2}, \\
\frac{d\rho_{1,1}}{dt} &= 2(a\rho_{1,1} - c\rho_{1,2}) + 2Kh_1\rho_{1,1} + 2\alpha^2\rho_{1,1} + \frac{\alpha^2\rho_1^2}{N} + \frac{\beta^2}{N}, \\
\frac{d\rho_{2,2}}{dt} &= 2(b\rho_{1,2} - d\rho_{2,2}), \\
\frac{d\rho_{1,2}}{dt} &= b\rho_{1,1} + (a - d)\rho_{1,2} - c\rho_{2,2} + Kh_1\rho_{1,2} + \frac{\alpha^2\rho_{1,2}}{2},
\end{align*}
\]

where \( h_\ell = (1/\ell !)H^{(\ell)}(\mu_1) \). Comparing Eqs. (43)-(50) to Eqs. (23)-(30), we note the following points in coupling contributions between the two types of couplings: (i) the contributions in \( d\gamma_{1,1}/dt \) and \( d\gamma_{1,2}/dt \) terms of diffusive coupling are proportional to \( (\rho_{1,1} - \gamma_{1,1}) \) while those for sigmoid couplings are proportional to \( (\rho_{1,1} - \gamma_{1,1}/N) \), (ii) \( d\mu_1/dt, d\rho_{1,1}/dt \) and \( d\rho_{1,2}/dt \) terms in diffusive coupling have no contributions from the couplings in contrast to those in sigmoid coupling, and (iii) there are no differences in \( d\mu_2/dt, d\gamma_{2,2}/dt \) and \( \rho_{2,2}/dt \) terms. The item (i) mainly yields the difference between the effects of multiplicative noises on the synchronization for the diffusive and sigmoid couplings, as will be discussed shortly.

We have performed numerical calculations by using Eqs. (43)-(50) with

\[
H(x) = \frac{1}{\{1 + \exp[(x - \theta)/w]\}},
\]

(51)
θ and w denoting the threshold and width, respectively. Time courses of $\mu_1(t)$, $\gamma_{1,1}(t)$ and $\rho_{1,1}(t)$ for $\theta = 0.5$, $w = 0.1$, $K = 0.1$ and $N = 10$ are similar to those shown in Figs. 1 and 6 in Ref. [14]. Figure 5 shows time courses of $S(t)$ for three $\alpha$ values of $\alpha = 0.0$, $0.01$ and $0.05$ with $\beta = 0.001$. We note that $S(t)$ has two peaks: one after an input is applied and the other when $X(t)$ is in the refractory period. In the case of $\alpha = 0.0$, we get $S_f = 0.108$ at $t = 44.16$ and $S_m = 0.342$ at $t = 62.92$. In the case of $\alpha = 0.01$, we get $S_f = 0.073$ at $t = 44.16$ and $S_m = 0.287$ at $t = 64.35$. In the case of $\alpha = 0.05$, we get $S_f = 0.053$ at $t = 44.15$ and $S_m = 0.284$ at $t = 64.32$. Both $S_f$ and $S_m$ are decreased with increasing $\alpha$. This behavior is more clearly shown in Fig. 6 where $S_f$ and $S_m$ are plotted as a function of $\alpha$.

A comparison between Figs. 3(a) and 6 shows that with increasing $\alpha$, $S_m$ for diffusive couplings is increased while that for sigmoid couplings is decreased. This difference mainly arises from the item (i) for the $d\gamma_{1,1}/dt$ term discussed above. Namely, $\gamma_{1,1}$ for diffusive coupling is reduced by a negative contribution proportional to $(\rho_{1,1} - \gamma_{1,1})$ (i.e. $\rho_{1,1} < \gamma_{1,1}$) which yields an enhancement in $S(t)$ of Eq. (36). On the contrary, $\gamma_{1,1}$ for sigmoid coupling is slightly increased by a positive contribution proportional to $(\rho_{1,1} - \gamma_{1,1}/N)$ (i.e. $\rho_{1,1} > \gamma_{1,1}/N$) which reduces $S(t)$ of Eq. (36). In contrast, effects of multiplicative noise are not effective for $S_f$ because $x_i$ is not large at $t_m < t < t_f$. Then with increasing $\alpha$, $S_f$ is decreased for both the couplings. The main difference between the two couplings is the presence of the feedback (second) term of Eq. (3). Indeed, if we adopt the diffusive coupling without this term, which is equivalent to the sigmoid coupling with $H(x) = x$ in Eq. (42), the synchronization is decreased with increasing $\alpha$ (result not shown). This situation is similar to that of the synchronization in small-world networks [19]. It was shown in Ref. [15e] that when a small-world network is made by introducing randomness to a regular network, the synchronization in the small-world network with diffusive couplings may be increased while that with sigmoid couplings is decreased. Our calculation implies that synchronization depends not only on the type of noises but also on the kind of couplings. This may also suggest that an ordered state in the multiplicative noise-induced phase transition reported in Refs. [2]-[6], might partly owe the diffusive couplings employed in these studies: multiplicative noises could not yield an ordered state with sigmoid couplings.

In summary, we have studied the synchronization in FitzHugh-Nagumo neuronal ensembles subjected to additive and multiplicative noises, by reformulating AMM with the use of FPE [13, 14]. The property of the two noises in FN neuron ensembles is summarized.
as follows.
(a) the scaling relation: $\alpha(N) = \alpha(1)/\sqrt{N}$ is not hold for multiplicative noises although the relation: $\beta(N) = \beta(1)/\sqrt{N}$ is valid for additive noises,
(b) multiplicative noises may enhance the synchronization ($S_m$) for diffusive couplings though both the two noises are generally detrimental to it, and
(c) for both the additive and multiplicative noises, the synchronization is more increased in smaller $N$ systems.

The item (a) supplements the result for Langevin model [13]. The item (b) is similar to the item (1) of an ordered state created by multiplicative noises [2]-[6] mentioned in the introduction.

A disadvantage of our AMM is that its applicability is limited to weak-noise cases. For multiplicative Gaussian noises, the probability distribution become non-Gaussian yielding divergent second and higher moments for a large $\alpha$, to which the AMM cannon be applied. On the contrary, an advantage of the AMM is that we can easily discuss dynamical property of the finite $N$-unit stochastic systems. We have solved the eight-dimensional ordinary differential equations for FitzHugh-Nagumo neuronal ensembles. In contrast, within direct simulation and the FPE approach, we have to solve the $2N$-dimensional stochastic equations and the $(2N + 1)$-dimensional partial differential equations, respectively, which are much laborious than AMM. Our AMM may be applied to a wide class of coupled stochastic models subjected to additive and/or multiplicative noises.

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**Appendix: Equations of motions for a general $G(x)$**

Although Eqs. (23)-(30) express equations of motion for $G(x) = x$, we present the result for a general form of $G(x)$:

$$
\frac{d\mu_1}{dt} = f_0 + f_2\gamma_{1,1} - c\mu_2 + \frac{\alpha^2}{2} [g_0g_1 + 3(g_1g_2 + g_0g_2)\gamma_{1,1}] + I^{(e)}, \tag{52}
$$

$$
\frac{d\mu_2}{dt} = b\mu_1 - d\mu_2 + e, \tag{53}
$$

$$
\frac{d\gamma_{1,1}}{dt} = 2(\sigma_{1,1} - c\gamma_{1,2}) + \frac{2wN}{Z}(\rho_{1,1} - \gamma_{1,1}) + 2\alpha^2(g_1^2 + 2g_0g_2)\gamma_{1,1}
$$
\[ \frac{d\gamma_{1,2}}{dt} = 2(b\gamma_{1,2} - d\gamma_{2,2}), \]  
(54)  
\[ \frac{d\gamma_{1,1}}{dt} = b\gamma_{1,1} + (a - d)\gamma_{1,2} - c\gamma_{2,2} + \frac{wN}{Z}(\rho_{1,2} - \gamma_{1,2}) + \frac{\alpha^2}{2}(g_1^2 + 2g_0g_2)\gamma_{1,1}, \]  
(56)  
\[ \frac{d\rho_{1,1}}{dt} = 2(a\rho_{1,1} - c\rho_{1,2}) + 2\alpha^2(g_1^2 + 2g_0g_2)\rho_{1,1} + \frac{\alpha^2 g_0^2}{N} + \frac{\beta^2}{N}, \]  
(57)  
\[ \frac{d\rho_{2,2}}{dt} = 2(b\rho_{1,2} - d\rho_{2,2}), \]  
(58)  
\[ \frac{d\rho_{1,2}}{dt} = b\rho_{1,1} + (a - d)\rho_{1,2} - c\rho_{2,2} + \frac{\alpha^2}{2}(g_1^2 + 2g_0g_2)\rho_{1,2}, \]  
(59)  
where \( g_\ell = (1/\ell !)G^{(\ell)}(\mu_1) \). For \( G(x) = x \), we get \( g_0 = \mu_1 \), \( g_1 = 1 \), and \( g_2 = g_3 = 0 \), with which Eqs. (52)-(59) reduce to Eqs. (23)-(30).
References

[1] B. Lindner, J. García-Ojalvo, A. Neiman, and L. Schimansky-Geier, Physics Report 392, 321 (2004).

[2] C. Van den Broeck, J. M. R. Parrondo, and R. Toral, Phys. Rev. Lett. 73 (1994) 3395.

[3] C. Van den Broeck, J. M. R. Parrondo, R. Toral, and R. Kawai: Phys. Rev. E 55 (1997) 4084.

[4] T. Birner, K. Lippert, R. Müller, A. Kühnel, and U. Behn, Phys. Rev E 65 (2002) 046110.

[5] R. Kawai, X. Sailer, L. Scimansky-Geier, and C. Van den Broeck: Phys. Rev E 69 (2004) 051104.

[6] M. A. Muñoz, F. Colaioiri, and C. Castellano: Phys. Rev. E 72 (2005) 056102.

[7] V. Berdichevsky and M. Gitterman, Europhys. Lett. 36, 161 (1996).

[8] A. V. Bazykin and K. Seki, Europhys. Lett. 40, 117 (1997).

[9] H. Sakaguchi: J. Phys. Soc. Jpn. 70 (2001) 3247.

[10] C. Anteneodo and C. Tsallis: J. Math. Phys. 44 (2003) 5194.

[11] T. S. Biro and A Jakovác, Phys. Rev. Lett. 94, 132302 (2005).

[12] H. Hasegawa, cond-mat/0506301 [Physica A (in press)].

[13] H. Hasegawa, cond-matt/0512429 [J. Phys. Soc. Jpn. (in press) ].

[14] H. Hasegawa: Phys. Rev E 67 (2003) 041903;

[15] H. Hasegawa: Phys. Rev E 68 (2003) 041909; ibid. 70 (2004) 021911; ibid. 70 (2004) 021912; ibid. 70 (2004) 066107; ibid. 72 (2005) 056139.

[16] R. Rodriguez and H. C. Tuckwell, Phys. Rev. E 54, 5585 (1996).

[17] H. Haken: Advanced Synergetics (Springer-Verlag, Berlin, 1983).

[18] J. A. Acebrón, A. R. Bulsara, and W. -J. Rappel, Phys. Rev. E 69, 026202 (2004).
[19] D. J. Watts and S. H. Strogatz, Nature 393, 440 (1998).
Figure 1: (color online). Time courses of (a) $\mu_1(t)$, (b) $\gamma_{1,1}(t)$, (c) $\rho_{1,1}(t)$ and (d) $S(t)$ for $\alpha = 0.01$, $\beta = 0.001$ with diffusive coupling (DC) of $J = 1.0$ and $N = 100$, solid and dashed curves denoting results of AMM and direct simulations, respectively. At the bottom of (a), an input signal is plotted. Vertical scales of (b) and (c) are multiplied by factors of $10^{-4}$ and $10^{-5}$, respectively.

Figure 2: (color online). Time courses of the synchronization ratio $S(t)$ for $\alpha = 0.0$ (dashed curve), $\alpha = 0.002$ (dotted curve), $\alpha = 0.01$ (solid curve) and $\alpha = 0.05$ (chain curve) with diffusive couplings of $J = 1.0$, $\beta = 0.001$ and $N = 100$.

Figure 3: (color online). (a) $\alpha$ dependences of $S_f$ and $S_m$ for $\beta = 0.001$ (circles), $\beta = 0.01$ (squares) and $\beta = 0.02$ (triangles), and (b) $\beta$ dependences of $S_f$ and $S_m$ for $\alpha = 0.0$ (circles), $\alpha = 0.01$ (squares) and $\alpha = 0.05$ (triangles) with diffusive coupling of $J = 1.0$ and $N = 100$: results of $S_m$ and $S_a$ of AMM are expressed by solid and chain curves, respectively, and those of DS by filled and open marks, respectively.

Figure 4: (color online). $N$ dependences of $S_f$ (solid curve) and $S_m$ (chain curve) for two sets of parameters: (1) $\alpha = 0.01$ and $\beta = 0.001$ and (2) $\alpha = 0.0$ and $\beta = 0.01$ with diffusive coupling of $J = 1.0$ and $N = 100$, calculated by AMM and DS (circles and squares).

Figure 5: (color online). Time courses of the synchronization ratio $S(t)$ for $\alpha = 0.0$ (dashed curve), $\alpha = 0.01$ (solid curve) and $\alpha = 0.05$ (chain curve) with sigmoid couplings (SC) of $K = 0.1$, $\beta = 0.001$ and $N = 10$.

Figure 6: (color online). $\alpha$ dependences of $S_f$ and $S_m$ for $\beta = 0.001$ (circles) and $\beta = 0.01$ (squares) with sigmoid coupling of $K = 0.1$ and $N = 10$: results of $S_m$ and $S_a$ of AMM are expressed by solid and chain curves, respectively, and those of DS by filled and open marks, respectively.