ISOLATED RATIONAL CURVES ON K3-FIBERED CALABI–YAU THREEFOLDS
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Abstract. In this paper we study 16 complete intersection K3-fibered Calabi–Yau
variety types in biprojective space $\mathbb{P}^n_1 \times \mathbb{P}^1$. These are all the CICY-types that are
K3 fibered by the projection on the second factor. We prove existence of isolated
rational curves of bidegree $(d,0)$ for every positive integer $d$ on a general Calabi–Yau
variety of these types. The proof depends heavily on existence theorems for
curves on K3-surfaces proved by S. Mori and K. Oguiso. Some of these varieties are
related to Calabi–Yau varieties in projective space by a determinantal contraction,
and we use this to prove existence of rational curves of every degree for a general
complete intersection Calabi–Yau variety in projective space.

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1. Introduction.

A complete intersection $F$ in a multiprojective space $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ is given by
polynomials $f_1, \ldots, f_m$. Each $f_j$ has a multidegree $(g_{1j}, \ldots, g_{kj})$. We can visualize
these numerical data of $F$ by the following:

\[
\begin{array}{c|ccc}
    n_1 & g_{11} & \cdots & g_{1m} \\
    \vdots & \vdots & & \vdots \\
    n_k & g_{k1} & \cdots & g_{km} \\
\end{array}
\]

We use $[n||g_{ij}]$ as a short form for this. We say that $F$ is of type $[n||g_{ij}]$.

By a general $F$ of type $[n||g_{ij}]$ we understand a generic choice of the defining
polynomials $f_j$ of multidegree $(g_{1j}, \cdots, g_{kj})$. A nonsingular complete intersection is
Calabi-Yau, abbreviated $CY$, if and only if

$$n_i + 1 = \sum_{j=1}^{m} g_{ij}$$

for every $i \in \{1, \ldots, k\}$. We then say that the threefold is of $CICY$-type, where
$CICY$ is an abbreviation for complete intersection Calabi-Yau threefold. We shall
use the notion of $CICY$-type for singular varieties as well. A permutation of the
polynomials $f_j$ will in general change the matrix $[g_{ij}]$, but define the same variety.
We are not interested in distinguishing between the different possibilities of indexing
of polynomials and projective spaces, and say that $[n||g_{ij}]$ and $[n^*||g_{ij}^*]$ represent
the same type if one can go from one to the other by row interchanges on $[n||g_{ij}]$
and column interchanges on $[g_{ij}]$. 

Let \( C \subset \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k} \) be an irreducible rational curve, and let \( f: \mathbb{P}^1 \to C \) be a parametrisation. We have a natural notion of multidegree \((d_1, \ldots, d_k)\) of the curve \( C \) by defining \( d_i \) to be the unique number that satisfies \( \mathcal{O}_{\mathbb{P}^1}(d_i) \cong (\pi_i \circ f)^* \mathcal{O}_{\mathbb{P}^{n_i}}(1) \) where \( \pi_i \) denotes the projection on factor \( i \).

Let \( Z \subseteq F \) be a closed subscheme. The tangent space of the Hilbert scheme \( \text{Hilb}_F \) at the point \([Z]\) is isomorphic to \( H^0(Z, \mathcal{N}_{Z/F}) \).

**Definition 1.1.** Let \( Z \) be a point in \( \text{Hilb}_F \). If \( H^0(Z, \mathcal{N}_{Z/F}) = 0 \), we say that \( Z \) is isolated in \( F \).

Note that a subscheme \( Z \) is isolated if the tangent space at the point of the Hilbert scheme corresponding to \( Z \) is zero dimensional. Recall also that \( Z \) on \( F \) is isolated if \( Z \) does not deform to first order on \( F \).

**Remark 1.2.** Let \( C \) be a nonsingular rational curve on a smooth CICY threefold \( F \). The normal sheaf \( \mathcal{N}_{C/F} \) is a rank 2 locally free sheaf on \( C \cong \mathbb{P}^1 \). The pullback of this sheaf to \( \mathbb{P}^1 \) is isomorphic to \( \mathcal{O}(a) \oplus \mathcal{O}(b) \), where \( \mathcal{O}(a) \) is short for \( \mathcal{O}_{\mathbb{P}^1}(a) \). Since \( F \) is CY we get \( a + b = -2 \) from the exact sequence

\[
0 \to T_C \to T_F|_C \to \mathcal{N}_{C/F} \to 0,
\]

where \( T_C \) is the tangent bundle of \( C \) and \( T_F|_C \) is the tangent bundle of \( X \) restricted to \( C \). Hence \( C \) is isolated if and only if \( \mathcal{N}_{C/F} \cong \mathcal{O}(-1) \oplus \mathcal{O}(-1) \).

The following theorem is the main result of the paper.

**Theorem 1.3.** There are exactly 16 types of CICY’s in biprojective spaces of the form \( \mathbb{P}^n \times \mathbb{P}^1 \), for positive integers \( n \). For a general (and hence smooth) CICY of each of these 16 types, and for all positive integers \( d \), there exists an isolated smooth rational curve \( C \) of bidegree \((d, 0)\).

Theorem 1.3 has a nice corollary.

**Corollary 1.4.** Let \( F \) be a general member of one of the 5 well-known CICY-types in projective spaces \( \mathbb{P}^n \), for positive integers \( n \). Then for all positive integers \( d \), there exists a smooth isolated rational curve \( C \) of degree \( d \) in \( F \).

**Remark 1.5.**

While we were working with the material in this paper, we obtained the manuscript ([12]). It is clear that our Corollary 1.4 can be obtained from the results of that paper. Theorems 5.1 and 5.2 of ([12]), in combination with Oguiso/Mori’s existence theorem on curves on K3-surfaces, give the conclusion of our Corollary 1.4. Our approach is independent of Kley’s, but share the idea, going back to Clemens, of using K3-surfaces to study curves on CICY’s.

Our paper is organized as follows. In paragraph 2 we recall some basic facts about K3-surfaces that will be essential in proving our result. The paragraph ends with Proposition 2.4, which will be an important tool in paragraph 4. In paragraph 3 we prove the first part of Theorem 1.3. We give a list of the 16 types of CICY-threefolds in biprojective spaces \( \mathbb{P}^n \times \mathbb{P}^1 \) for positive integers \( n \). Furthermore, we show how one can reduce the last part of Theorem 1.3 to showing the existence of one smooth isolated rational bidegree \((d, 0)\) curve \( C \) on one special CICY of each type.
In paragraph 4 we complete the proof of Theorem 1.3. We use the results from paragraph 3, and we show that for each positive integer \( d \), and each of the 16 CICY-types there exists an isolated smooth rational curve of bidegree \((d, 0)\) contained in a CICY of that type. The main point is to construct suitable \( F \) that are \( \mathbb{P}^1 \) fibrations of \( K3 \)-surfaces that are complete intersections in projective spaces. We use a result by S. Mori and K. Oguiso (for a more general statement, see Theorem 8.1 of [13]) that gives the existence of a smooth rational curve \( C \) of each degree, such that each \( C \) is isolated in a \( K3 \)-surface \( S \). For each of the 16 types we start with such a pair \((C, S)\), and we show that it is possible to choose the \( \mathbb{P}^1 \)-fibration such that \( F \) is smooth, and \( C \) is isolated, not only in \( S \), but also in \( F \). In this analysis the material from paragraph 2 will be essential. In paragraph 5 we describe determinantal contractions of CICY’s. We use this description to show how Corollary 1.4 follows from Theorem 1.3.

Corollary 1.4. was proved by H. Clemens for infinitely many \( d \) for the quintic in \( \mathbb{P}^4 \). S. Katz ([11]) gave a proof for the quintic in \( \mathbb{P}^4 \) for all \( d \) and hence proved Corollary 1.4 in this case. K. Oguiso proved Corollary 1.4 in case of the complete intersection \((4, 2)\) in \( \mathbb{P}^5 \). Our proof is inspired by ([17]). Results like Theorem 1.3 and Corollary 1.4 are interesting when one counts rational curves on Calabi–Yau threefolds. Mirror symmetry gives predictions on the number of rational curves of a given (multi)degree on a generic Calabi–Yau threefold. Nevertheless, it remains to prove rigorously that for a generic CICY of a given type, and for a given (multi)degree, there exists a nonempty set of isolated curves of this (multi)degree. For the quintic in \( \mathbb{P}^4 \) this is essentially Clemens’ conjecture. The finiteness has only been proved for \( d \leq 9 \) ([9],[11],[16]), while the nonemptiness is proved by S. Katz ([11]). The analogue of Clemens’ conjecture is wrong for general CICY’s in multiprojective spaces in general ([18]). Our results are a step towards proving the analogue of Clemens’ conjecture for complete intersections in \( \mathbb{P}^n \times \mathbb{P}^1 \) (and in \( \mathbb{P}^n \)). There is also a more philosophical side of our approach. The moduli space of Calabi–Yau varieties may be connected. This is popularly known as Reid’s fantasy. Candelas et al. proved connectedness for complete intersections in multiprojective spaces ([3]). The proof relies on existence of determinantal contractions that give a map from one Calabi–Yau variety in one multiprojective space to another Calabi–Yau in another multiprojective space. This map is an isomorphism outside strata of codimension 2 that are contracted to points. Curves that do not intersect these strata are mapped isomorphically to curves on another Calabi–Yau in another multiprojective space. Our philosophical point of view is therefore that studying rational curves on a Calabi–Yau variety in a multiprojective space is too restrictive. This paper is an illustration of this point. Constructing special \( K3 \)-fibrations in biprojective spaces does not only prove existence of isolated rational curves of every bidegree \((d, 0)\), but also proves existence of isolated rational curves of every degree \( d \) for the five complete intersection Calabi–Yau threefolds in projective spaces.

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2. Some basic theory of \( K3 \)-surfaces.

Let \( S \) be a \( K3 \)-surface. Study the following part of the long exact sequence...
obtained from the well-known exponential sequence:

\[
0 = H^1(S, \mathcal{O}) \longrightarrow \text{Pic}(S) = H^1(S, \mathcal{O}^*) \xrightarrow{\cdot c} H^2(S, \mathbb{Z}) \xrightarrow{\cdot \kappa} H^2(S, \mathcal{O}) = \mathbb{C}.
\]

All \(K3\)-surfaces \(S\) are diffeomorphic, and \(H^2(S, \mathbb{Z})\) is torsion free of rank 22 for all such \(S\). As usual, fix once and for all an isomorphism between the lattice \(H^2(S, \mathbb{Z})\) and a standard rank 22 lattice with a known intersection form. This gives an intersection matrix \(A = [a_{ij}]\) \(i, j = 1, ..., 22\) compatible with the cup-product form on \(H^2(S, \mathbb{Z})\). Then we complexify \(H^2(S, \mathbb{Z})\) and obtain \(H^2(S, \mathbb{C})\). We identify \(\mathbb{P}^{21}\) with \(\mathbb{P}(H^2(S, \mathbb{C}))\).

Let \(P\) be the well-known period map from the set of isomorphism classes of analytical \(K3\)-surfaces into \(\mathbb{P}^{21}\). In short each isomorphism class determines a unique holomorphic 2-form on \(S\) up to multiplicative constant. Integration of this form then gives a well-defined element of \(\mathbb{P}^{21} = \mathbb{P}(H^2(S, \mathbb{C}))\), and this element is the image of the isomorphism class by \(P\). We have:

**Definition 2.1.** Let the subset \(D\) of \(\mathbb{P}^{21}\) be defined by

\[
D = \{(\gamma_1, ..., \gamma_{22}) | \sum_{ij} a_{ij} \gamma_i \gamma_j = 0, \sum_{ij} a_{ij} \gamma_i \gamma_j > 0\}.
\]

\(D\) can then be viewed as a moduli space of analytical \(K3\)-surfaces (Locally this holds, even for a fixed identification between \(H^2(S, \mathbb{Z})\) and the standard rank 22 lattice as described above. Taking into account the possibility of different such identifications, one has to divide \(D\) with a discrete group to obtain a moduli space of analytical \(K3\)-surfaces, but we shall only be interested in the local theory). We see that \(D\) may be viewed as “half of a hyperquadric in \(\mathbb{P}^{21}\)."

The following result is a consequence of Theorem 14 in [13]:

**Proposition 2.2.** Let \(C_1, ..., C_s\), for some \(s\) in the range \(1, ..., 20\), be independent elements in \(\text{Pic}(S)\), for a given fixed (analytical structure on a ) \(K3\)-surface \(S\). We view \(\text{Pic}(S)\) as a subgroup of \(H^2(S, \mathbb{Z})\) via the map \(c\) above.

Let \(H_i\) be the embedded tangent space at \(P(S)\) in \(\mathbb{P}^{21}\) of the subset of \(D\) consisting of period points of \(K3\)-surfaces with \(C_i\) in the images of the respective \(c\)-maps, for \(i = 1, ..., s\). Then the intersection of the \(H_i\) is a space of dimension \(20 - s\).

**Remark 2.3.** This means that the set of \(K3\)-surfaces that ”keep \(C_1, ..., C_s\)” has dimension at most \(20 - s\), even infinitesimally. See also [8], page 594.

**Algebraic \(K3\)-surfaces in a fixed projective space**

Let us study the space \(R_n\) of embedded smooth \(K3\)-surfaces in \(\mathbb{P}^n\), for \(n = 3, 4\). For \(n = 3\) this is an open subset of \(\mathbb{P}^{34}\), parametrizing quartic surfaces in \(\mathbb{P}^3\). For \(n = 4\), \(R_n\) is an open subset of the set of complete intersections of type \((2, 3)\). For \(n = 5\) we define \(R_n\) to be the space of all complete intersection \(K3\)-surfaces (of type \((2, 2, 2)\)). Then \(R_5\) is dense in the set of all embedded smooth \(K3\)-surfaces in \(\mathbb{P}^5\), and also in the space of nets of quadric hypersurfaces in \(\mathbb{P}^5\). In each case it is well known, and easy to prove, by a simple dimension count, that there is an 19-dimensional set of projective equivalence classes of \(K3\)-surfaces as described. The spaces \(R_n\) are smooth of dimensions \(34, 43, 54\) for \(n = 3, 4, 5\) respectively. See [8], page 594.
p. 591-594. It is well known that projective equivalence in these cases amount to the same thing as abstract isomorphism, in the sense that two surfaces are mapped to the same point by the period map $P$ described above. Let $P_n$ be the pull-back of the period map $P$ to $R_n$, and set $D_n = \text{im}(P_n)$.

Maps of tangent spaces
Let $S$ be a fixed $K3$-surface, corresponding to a point $[S]$ in $R_n$, for $n = 3, 4, 5$. Assume that $S$ contains a smooth rational curve $C$ of degree $d$, for some positive integer $d$. Let $F$ inside $D_n$ be the locus of those $K3$-surfaces that “keep $C$” in the sense described above. Look at the following maps:

$$T_{R_n,[S]} \xrightarrow{dP_n} T_{D_n,[S]} \xrightarrow{f} T_{D_n,[S]}/TF,[S] = \mathbb{C}.$$ 

Here $f$ is the natural quotient map. The quotient to the right is isomorphic to $\mathbb{C}$, since $\dim T_{D_n,[S]} = 19$, and $\dim TF,[S] = 18$, by Proposition 2.2 (Put $C_1 = \text{hyperplane section of } S$ in $\mathbb{P}^n$, and $C_2 = C$). Clearly the composite map $f \circ dP_n$ is surjective and linear. Let us define and study natural surjective linear maps $W_n \xrightarrow{\phi_n} T_{R_n,[S]}$, where $W_n$ will be $A^{35}, A^{15} \times A^{35}, (A^{21})^3$, for $n = 3, 4, 5$, respectively. $W_3$ will parametrize homogeneous quartic polynomials in 4 variables, $W_4$ will parametrize pairs of homogeneous quadric and cubic polynomials in 5 variables, and $W_5$ will parametrize triples of homogeneous quadric polynomials in 6 variables. Let $S$ be a $K3$-surface in $\mathbb{P}^3$ with equation $q_4(x_0, ..., x_3) = 0$. We construct a surjective linear map $W_3 \longrightarrow T_{R_3,[S]}$ in the following manner,

$$\phi_3: p_4(x_0, ..., x_3) \longrightarrow \text{“the } K3\text{-surface with equation”:}$$

$$q_4(x_0, ..., x_3) + tp_4(x_0, ..., x_3) = 0.$$

Here the polynomial $q_4(x_0, ..., x_3)$ is fixed. We think of $t$ as an infinitesimal variable with $t^2 = 0$, thus the points of the tangent space $T_{R_3,[S]}$ can be given by such equations $q_4(x_0, ..., x_3) + tp_4(x_0, ..., x_3) = 0$ over $C[t]/t^2$. Clearly $\phi_3$ is surjective and essentially maps $p_4(x_0, ..., x_3)$ to its image in the vector space of all quartic polynomials in 4 variables modulo $q_4(x_0, ..., x_3)$. Let $S$ be a $K3$-surface in $\mathbb{P}^4$ with equation $f_2(x_0, ..., x_4) = f_3(x_0, ..., x_4) = 0$. In an analogous way we construct a linear map $W_4 \longrightarrow T_{R_4,[S]}$:

$$\phi_4: (p_2(x_0, ..., x_4), p_3(x_0, ..., x_4)) \longrightarrow \text{“the } K3\text{-surface with equation”:}$$

$$f_2(x_0, ..., x_4) + tp_2(x_0, ..., x_4) = 0$$

$$f_3(x_0, ..., x_4) + tp_3(x_0, ..., x_4) = 0.$$ 

Clearly $\phi_4$ is surjective and maps $(p_2(x_0, ..., x_4), p_3(x_0, ..., x_4))$ to the pair of images modulo $f_2(x_0, ..., x_4)$ and $(f_2(x_0, ..., x_4), f_3(x_0, ..., x_4))$, respectively. Clearly $T_{R_4,[S]}$ can be viewed as the set of such pairs of all such cosets of quadric and cubic polynomials, respectively. Let $S$ be a $K3$-surface in $\mathbb{P}^5$ with equation

$$q_5(x_0, ..., x_5) = h(x_0, ..., x_5) = i(x_0, ..., x_5) = 0.$$
In an analogous manner we construct a linear map $W_5 \rightarrow T_{R_5,[S]}$:

$$
\phi_5: (r_2(x_0, \ldots, x_5), s_2(x_0, \ldots, x_5), u_2(x_0, \ldots, x_5)) \rightarrow \text{"the K3-surface ":}
$$

$$
g_2(x_0, \ldots, x_5) + tr_2(x_0, \ldots, x_5) = 0 \qquad \text{h}_2(x_0, \ldots, x_5) + ts_2(x_0, \ldots, x_5) = 0 \qquad \text{i}_2(x_0, \ldots, x_5) + tu_2(x_0, \ldots, x_5) = 0.
$$

Clearly $\phi_5$ is surjective and maps $(r_2(x_0, \ldots, x_5), s_2(x_0, \ldots, x_5), u_2(x_0, \ldots, x_5))$ to the triple of images modulo $(g_2(x_0, \ldots, x_5), h_2(x_0, \ldots, x_5), i_2(x_0, \ldots, x_5))$. We see that $T_{R_5,[S]}$ can be viewed as the set of such triples of cosets of quadric polynomials. Study the composite surjective, linear map

$$
\eta_n = f \circ dP_n \circ \phi_n: W_n \rightarrow \mathbb{C}, \text{ for } n = 3, 4, 5.
$$

The surjectivity of the linear map $\eta_n$ gives:

**Proposition 2.4.** The kernel of the linear map $\eta_n$ is a proper subspace of codimension 1 in $W_n$.

In paragraph 5 we will produce some natural subcones of the $W_n$, arising from CICY’s that are $K3$-fibrations. We will prove that general points of these cones are not contained in ker($\eta_n$). It will be sufficient to prove that the linear spans of these cones are all of $W_n$.

**3. CICY threefolds in multiprojective spaces.**

In the first part of this paragraph we find all CICY-types in a biprojective space of the type $\mathbb{P}^n \times \mathbb{P}^1$. We want to discard those CICYs that are projected to a zero dimensional set by the second projection. Such threefolds are easily identified with a CICY in $\mathbb{P}^n$ by the first projection.

**Definition 3.1.** Let $F$ be a complete intersection CY-variety in $\mathbb{P}^n \times \mathbb{P}^1$. If the projection on the second factor is surjective, then $F$ is $\mathbb{P}^1$-surjective. We also use this notion for CICY-types, i.e. a CICY-type is $\mathbb{P}^1$-surjective if the general member is $\mathbb{P}^1$-surjective.

**Lemma 3.2.** There are 16 $\mathbb{P}^1$-surjective CICY-types and they are divided into the following three groups:

- **Group 1.**

  \[ \begin{array}{c|c}
    3 & 4 \\
    1 & 2 \\
  \end{array} \]

  \[ \begin{array}{c|c}
    4 & 4 1 \\
    1 & 0 2 \\
  \end{array} \]

  \[ \begin{array}{c|c}
    4 & 4 1 \\
    1 & 1 1 \\
  \end{array} \]

  \[ \begin{array}{c|c}
    5 & 4 1 1 \\
    1 & 0 1 1 \\
  \end{array} \]

- **Group 2.**
Group 3.

\[
\begin{array}{cccc}
4 & 3 & 2 \\
1 & 2 & 0 \\
5 & 3 & 2 & 1 \\
1 & 0 & 0 & 2 \\
6 & 3 & 2 & 1 & 1 \\
1 & 0 & 0 & 1 & 1 \\
\end{array}
\begin{array}{cccc}
4 & 3 & 2 \\
1 & 0 & 2 \\
5 & 3 & 2 & 1 \\
1 & 0 & 1 & 1 \\
5 & 3 & 2 & 1 \\
1 & 1 & 0 & 1 \\
\end{array}
\begin{array}{cccc}
4 & 3 & 2 \\
1 & 1 & 1 \\
5 & 3 & 2 & 1 \\
1 & 0 & 0 & 2 \\
6 & 3 & 2 & 1 & 1 \\
1 & 0 & 0 & 1 & 1 \\
\end{array}
\]

Proof. We want to determine:

\[
\begin{array}{c|cccc}
& g_{11} & \cdots & g_{1m} \\
\vdots & \vdots & \ddots & \vdots \\
n_k & g_{k1} & \cdots & g_{km} \\
\end{array}
\]

where \( k = 2 \), and \( \sum g_{1j} = n_1 + 1 \) and \( \sum g_{2j} = 2 \), and \( n_2 = 1 \). Note that a column with only one 1 and rest zeros gives a hyperplane in one of the factors and thus gives a natural identification with a CICY-type in a lower dimensional embedding space. Discarding those, gives \( n_1 \leq 7 \). To see this, take first \( n_1 = 8 \). Since we have 6 polynomials in \( \mathbb{P}^8 \times \mathbb{P}^1 \), at least three of them must have degree 1 in the first factor. Furthermore, since we have at most two nonzero entries on the second factor, at least one of the 1’s in the first row must have a zero in its column. Thus the variety can be identified with a variety of a CICY-type a lower dimensional embedding space. If \( n_1 > 8 \) then the first row must contain more than three 1’s and the same argument applies.

Since we are looking at 3-dimensional CY-varieties, we also have \( n_1 \geq 3 \).

Computation yields the list given above.

Remark 3.3. The reason for the partition of the 16 CICY-types into three groups will become apparent in paragraph 4. Note also that all these CICY-types are \( K3 \)-fibrations over \( \mathbb{P}^1 \). In the following we will also adopt a nonfigurative notation for the different CICY-types. For example we will write \( X_{4[41,1]} \) for

\[
\begin{array}{cccc}
4 & 4 & 1 \\
1 & 1 & 1 \\
\end{array}
\]

We close this paragraph by proving the following theorem, which will be crucial in paragraphs 4 and 5.
Theorem 3.4. Let \((d_1, \ldots, d_k) \in \mathbb{N}^k\) and not all \(d_i = 0\). Suppose there exists a CI threefold \(F\) of type \([n||g_{ij}]\) in \(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}\) containing an isolated nonsingular rational curve \(C\) with multidegree \((d_1, \ldots, d_k)\). Assume that \(F\) is smooth at all points of \(C\), then there exist isolated rational curves with this multidegree on a general CI of type \([n||g_{ij}]\).

Proof. Let \(M\) parametrise the space of all smooth rational curves of multidegree \((d_1, \ldots, d_k)\), and let \(G \subseteq \text{Hilb}(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k})\) be the parameter space of CI threefolds of type \([n||g_{ij}]\). Let
\[
J_0 = \{(C, F) \in M \times G | C \subseteq F, \text{ and } F \text{ is nonsingular along } C\}
\]
The incidence variety \(J_0\) is equipped with projection maps
\[
\pi_1: J_0 \rightarrow M \text{ and } \pi_2: J_0 \rightarrow G.
\]
A standard dimension count gives
\[
dim J_0 \geq \dim G
\]
Let \((C, F) \in J_0\) be as in the statement of the theorem, and study the map
\[
d\pi_2: T_{J_0,(C,F)} \rightarrow T_{G,F}.
\]
Then \(\ker d\pi_2 = H^0(N_{C/F}) = 0\). This implies that \(d\pi_2\) is injective. It is also surjective, since \(\dim J_0 \geq \dim G\) implies that \(\dim T_{J_0,(C,F)} \geq \dim J_0 \geq \dim G = \dim T_{G,F}\) (\(G\) is smooth at \(F\)). The surjectivity of \(d\pi_2\) implies that \(\pi_2\) is dominant since \(G\) is irreducible. (The injectivity implies that the component of \(J_0\) has dimension equal to the dimension of \(G\)).

4. Existence of isolated rational curves on \(\mathbb{P}^1\)-surjective CICYs.

A key tool in this paragraph will be

Proposition 4.1. Let \(d\) be a positive integer. Then:

i.) ([15]) There exists a nonsingular quartic \(K3\)-surface \(S_1\) in \(\mathbb{P}^3\), containing an isolated rational curve \(C_1\) of degree \(d\), for every positive integer \(d\).

ii.) ([17]) There exists a nonsingular \(K3\)-surface \(S_2\) of type \((2,3)\) in \(\mathbb{P}^4\), containing an isolated nonsingular rational curve \(C_2\) of degree \(d\), for every positive integer \(d\).

iii.) ([13]) There exists a nonsingular \(K3\)-surface \(S_3\) of type \((2,2,2)\) in \(\mathbb{P}^5\), containing an isolated nonsingular rational curve \(C_3\) of degree \(d\), for every positive integer \(d\).

Proof. See [15], [17], and Theorem 8.1 of [13].

Remark 4.2. One can prove that \(S_2\) can be taken not only to be smooth, but also to be the intersection of a smooth hyperquadric and a smooth hypercubic. It is easy to see that the hypercubic can be taken to be smooth, given that \(S_2\) is. From the way Oguiso constructed the surface \(S_2\), it is clear that the rank of the Picard group of this hypersurface is 2, with the rational curve, and a hyperplane section as generators. If the hyperquadric were singular, necessarily with a single singular point, then \(S_2\) would contain a plane cubic curve, with arithmetic genus 1. A study of such a curve contradicts the fact that this curve must be a linear integer combination of the generators of \(\text{Pic}(S_2)\). Hence \(S_2\) cannot be singular. We thank Kristian Ranestad for pointing this out to us. It is also obvious that \(S_3\) can be taken to be the intersection of three smooth hyperquadrics. We will assume this.

We will use Proposition 4.1 and some additional argument to prove the main result of this section, which is:
Theorem 4.3. Let \( d \) be a positive integer. For each of the 16 \( \mathbb{P}^1 \)-surjective CICY-types listed in Lemma 3.2, there exists a smooth 3-fold \( F \) of that type containing an isolated rational curve \( C \) of bidegree \( (d, 0) \).

Proof. Let a positive integer \( d \), and one of the 16 CICY-types be given. We will produce a smooth \( K3 \)-fibration \( F \) over \( \mathbb{P}^1 \) of this CICY-type such that the fibre over the point \((0, 1)\) is one of the \( K3 \)-surfaces \( S_i \) of Proposition 4.1, containing a curve \( C_i \) of degree \( d \) for one \( i \in \{1, 2, 3\} \). In addition the fibration will be chosen in such a way that \( C_i \) does not deform ”sideways” in \( F \), not even infinitesimally. In the language of paragraph 2: the induced infinitesimal deformation of \( S_i \) corresponds to an element outside \( \ker \eta_n \) for \( n = 3, 4, \) or 5. Assume that such an \( F \) has been found, and consider (following [17]) the exact sequence:

\[
0 \longrightarrow N_{C_i/F} = \mathcal{O}(-2) \xrightarrow{h} N_{C_i/F} \xrightarrow{g} N_{S_i/F}|_{C_i} = \mathcal{O} \longrightarrow 0
\]

Since \( h \) is injective, the bundle \( N_{C_i/F} \) is \( \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \) or \( \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2) \). Assume \( N_{C_i/F} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2) \). Then \( g \) gives an isomorphism \( g': H^0(N_{C_i/F}) \rightarrow H^0(N_{S_i/F}|_{C_i}) = \mathbb{C} \). On the other hand, since \( N_{S_i/F} = \mathcal{O}_{S_i} \), we have that the restriction map \( \text{Res}: H^0(N_{S_i/F}) \longrightarrow H^0(N_{S_i/F}|_{C_i}) = \mathbb{C} \) is an isomorphism. Looking at the combined isomorphism \( (g')^{-1} \circ \text{Res}: H^0(N_{S_i/F}) \rightarrow H^0(N_{C_i/F}) \), we see that any non-zero element of \( H^0(N_{S_i/F}) \) is mapped to a non-zero infinitesimal deformation of \( C_i \) along the deformation of \( S_i \) of \( F \). This means that \( H^0(N_{S_i/F}) \) is contained in \( \ker(f \circ dP_n) \), in the language of paragraph 2, or that \( H^0(N_{S_i/F}) \) is in the image \( \phi_n(\ker \eta_n) \) (Here a natural identification is made between \( H^0(N_{S_i/\mathbb{P}^n}) \) and \( T_{R_n,[S_i]} \), and \( H^0(N_{S_i/F}) \) is in a natural way a subset of \( H^0(N_{S_i/\mathbb{P}^n}) \)). This is a contradiction, and hence \( N_{C_i/F} \) must be \( \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \). In other words \( C_i \) is isolated in \( F \). Hence, in order to prove the theorem it is enough to construct \( K3 \)-fibrations in each of the 16 cases satisfying the following properties:

a. The fibre over the point \((0 : 1) \in \mathbb{P}^1 \) is one of the \( S_i \) from Proposition 4.1.

b. The induced element of \( H^0(N_{S_i/F}) \subseteq H^0(N_{S_i/\mathbb{P}^n}) \subseteq T_{R_n,[S_i]} \) is not contained in \( \ker(f \circ dP_n) \).

c. The constructed \( F \) is smooth.

In each of the 16 cases we start with a rational curve \( C_i \) inside one of the three \( K3 \)-surfaces \( S_i \) \((i \in \{1, 2, 3\}) \) constructed by Oguiso and others, depending on whether we are in Group 1, 2 or 3, using the terminology of paragraph 3. The curve \( C_i \) is now isolated in \( S_i \). Let the \( K3 \)-surfaces \( S_i \) be given by:

\[
S_1 = Z(q_0(x_0, \ldots, x_3)) \subseteq \mathbb{P}^3 \\
S_2 = Z(k_0(x_0, \ldots, x_4), c_0(x_0, \ldots, x_4)) \subseteq \mathbb{P}^4 \\
S_3 = Z(k_{01}(x_0, \ldots, x_5), k_{02}(x_0, \ldots, x_5), k_{03}(x_0, \ldots, x_5)) \subseteq \mathbb{P}^5
\]

where \( q_0 \) is a polynomial of degree 4, \( k_0 \) and \( c_0 \) are polynomials of degree 2 and 3 respectively, and \( k_{01}, k_{02}, k_{03} \) are polynomials of degree 2. In the following we will write \( \bar{x}_i \) as a short form for \( x_0, \ldots, x_i \) in order to make the exposition more transparent. The surface \( S_1 \) will after this convention be the zero set of \( (q_0(\bar{x}_3)) \). In general we will use \( l, h, o, e \) as names for polynomials of degree 1, 2, 3, 4 respectively,
in order to make the degrees manifest in the construction. As seen above we have reserved the subscript zero, for the polynomials defining a special $K3$ surface. A polynomial with a subscript without a zero will unless otherwise stated be a general polynomial of the assigned degree. We treat the three different groups appearing in Lemma 3.2 separately.

In 12 of the 16 cases we will see directly that the linear span of a relevant subset of $W_n$ is all of $W_n$. See the remark after Proposition 2.4. These cases include the 9 cases that will be mentioned in Remark 5.4. Five of these cases will be used to prove Corollary 1.4 in chapter 5. The remaining cases are irrelevant to the proof of Corollary 1.4, but in order to prove Theorem 1.3 we will show that in the 4 cases not among the 12 mentioned, the linear spans of the corresponding relevant subsets are not contained in ker $\eta_n$ even if they are not equal to all of $W_n$. A useful observation will then be: Different sets of equations sometimes give rise to the same $K3$-surface. The sets $f=g=0$ and $f=g+kg=0$, for homogeneous polynomials $f, g, k$ of degrees $d, d', d' - d$, with $d \leq d'$ give the same $K3$-surface. The first order deformation $f + tm = g + kf = 0$ of the latter set corresponds to the first order deformation $f + tm = g - tkm = 0$ of the first set. Therefore, if $(m, 0) \in \ker \eta_n$, then $(m, -km) \in \ker \eta_n$ and $(0, km) \in \ker \eta_n$ also, for any $k$. We treat the three different groups appearing in Lemma 3.2 separately.

Consider Group 1. As a typical case, we study:

$$X_{4|41,1|02} :$$

$$q_0(x_3) + x_4 c_1(x_4) = 0,$$

$$x_4 z^2 + l_1(x_4) y z + l_2(x_4) y^2 = 0,$$

The $K3$-surface $S_1$ now sits as the fibre of $F$ over $P = (0 : 1)$ in $\mathbb{P}^1 = \text{Proj } \mathbb{C}[y, z]$. A local parameter of $\mathbb{P}^1$ at $P$ is $t = y/z$, and letting the $K3$-surface vary in the family, we obtain as a first order deformation (modulo $t^2$):

$$q_0(x_3) - tl_1(x_3, 0)c_1(x_3, 0) = 0.$$

Recall the map $\eta_n = f \circ dP_n \circ \phi_n: W_n \to \mathbb{C}$ introduced at the end of paragraph 2. We study the case $n = 3$.

The deformations above correspond to applying the map $\phi_3$ (and $\eta_3$) to a subset of $W_3 = \mathbb{A}^{35}$ of the following form:

$$\{ l_1(x_3, 0)c_1(x_3, 0) \}$$

We see that the linear span of the subset inside $\mathbb{A}^{35}$ is all of this space. Recall: Any homogeneous polynomial is in the span of all monomials of the degree in question. We also see that the condition that $\eta_3$, restricted to some subdomain of $\mathbb{A}^{35}$ is zero, is equivalent to the condition that $\eta_3$, restricted (only) to all of the linear span of this subdomain of $\mathbb{A}^{35}$ is zero. This is true, simply since $\ker(\eta_3)$ is a linear subspace. But since the linear span of the set in question is all of $\mathbb{A}^{35}$, Proposition 2.4 gives that the set is not contained in the kernel of $\eta_3$. Hence, if $l_1, c_1$ are chosen generically enough $C_1$ will not survive in the first order deformations described. The three other cases in Group 1 are treated in a practically identical way. In all cases the linear span of the relevant subdomain is $\mathbb{A}^{35}$. 
In Group 2, recall that the equation of $S_2$ be $k_0(\vec{x}_4) = c_0(\vec{x}_4) = 0$ for smooth hypersurfaces $k_0(\vec{x}_4) = 0$ and $c_0(\vec{x}_4) = 0$. We only treat 3 of the 7 types in Group 2 in detail: Let the equations of $F$ be:

$$X_{4|32,1|20} :$$

$$k_0(\vec{x}_4) = 0$$
$$c_0(\vec{x}_4)z^2 + c_1(\vec{x}_4)yz + c_2(\vec{x}_4)y^2 = 0$$

$$X_{4|32,1|02} :$$

$$c_0(\vec{x}_4) = 0$$
$$k_0(\vec{x}_4)z^2 + k_1(\vec{x}_4)yz + k_2(\vec{x}_4)y^2 = 0$$

$$X_{5|321,1|002} :$$

$$k_0(\vec{x}_4) + x_5l_1(\vec{x}_5) = 0$$
$$c_0(\vec{x}_4) + x_5k_1(\vec{x}_5) = 0$$
$$x_5z^2 + l_2(\vec{x}_5)yz + l_3(\vec{x}_5)y^2 = 0.$$  

respectively. The remaining 3 types essentially behave as the third among these 3 listed types. Working locally, as in Group 1, we obtain:

$$c_0(\vec{x}_4) + tc_1(\vec{x}_4) = k_0(\vec{x}_4) = 0,$$

$$k_0(\vec{x}_4) + tk_1(\vec{x}_4) = c_0(\vec{x}_4) = 0,$$

and

$$k_0(\vec{x}_4) - tl_2(\vec{x}_4,0)l_1(\vec{x}_4,0) = 0$$
$$c_0(\vec{x}_4) - tl_2(\vec{x}_4,0)k_1(\vec{x}_4,0) = 0,$$

respectively.

We recall that the equation of $S_2$ is $k_0(\vec{x}_4) = c_0(\vec{x}_4) = 0$. In analogy with case Group 1 we apply the linear map $\phi_4$ from the parameter space $A^{15} \times A^{35}$ of pairs of homogeneous quadric polynomials and cubic polynomials in 5 variables $\vec{x}_4$ to the tangent space $T_{R_4,[S_2]}$. The deformations above correspond to applying $\phi_4$ on subsets of the forms:

$$\{(0,c_1(\vec{x}_4))\},$$  
$$\{(k_1(\vec{x}_4),0)\},$$  
$$\{((l_2(\vec{x}_4,0)l_1(\vec{x}_4,0),l_2(\vec{x}_4,0)k_1(\vec{x}_4,0)\}.$$

respectively. In the third case (and the 4 cases corresponding to the types not listed), one easily sees that the linear span of the subsets is all of $A^{15} \times A^{35}$. Put $k_1 = 0$, and thereafter; $l_1 = 0$. This gives $(l_2l_1,0)$ and $(0,l_2k_2)$. The rest is as in Group 1. Now consider the second of the first two cases where the linear span is not all $A^{15} \times A^{35}$. In fact the desired deformations only correspond to $A^{15} \times 0$. On the other hand, if all of $A^{15} \times 0$ were contained in ker($\eta$), then for all $k_1(\vec{x}_4)$, the
would give deformations such that $C_2$ deforms in the family. But then
\[
k_0(\bar{x}_4) + tk_1(\bar{x}_4)l_1(\bar{x}_4) = 0
\]
would give the same thing, for all (the same) $k_1(\bar{x}_4)$ and all linear forms $l_1(\bar{x}_4)$, and $0 \times \mathbb{A}^{35}$ would be contained in $\ker(\eta_4)$. Hence the linear span, $\mathbb{A}^{15} \times \mathbb{A}^{35}$, of the union of $\mathbb{A}^{15} \times 0$ and $0 \times \mathbb{A}^{35}$, would be contained in $\ker(\eta_4)$, again a contradiction.

The last case can be handled by a modified version of the strategy working for the other cases considered above. Look at the maps
\[
f \circ dP_4 \circ \phi_4 : \{0\} \times \mathbb{A}^{35} \to T_{R_4[S]} \to T_{D_4[S]} \to \mathbb{C}
\]
Here the first map is certainly not surjective, but the composite map is. This is true since all smooth quadrics in $\mathbb{P}^4$ are projectively equivalent.

That is: Infinitesimal deformations of $K3$-surfaces from $S$ of type:
\[
c_0(\bar{x}_4) + tc_1(\bar{x}_4) = k_0(\bar{x}_4) = 0,
\]
are as general as desired, when intrinsic properties, like containing a smooth rational curve, are concerned. See also [17] for another formulation of the argument.

Finally we treat Group 3. We only treat 3 of the 5 types in Group 3 in detail: Let the equations of $S_3$ be $k_{01}(\bar{x}_5) = k_{02}(\bar{x}_5) = k_{03}(\bar{x}_5) = 0$. Let the equations of $F$ be:

\begin{align*}
X_{5|222,1|200} : & \\
k_{01}(\bar{x}_5)z^2 + k_1(\bar{x}_5)yz + k_2(\bar{x}_5)y^2 = 0 \\
k_{02}(\bar{x}_5) = k_{03}(\bar{x}_5) = 0
\end{align*}

\begin{align*}
X_{5|222,1|110} : & \\
k_{01}(\bar{x}_5)z + k_1(\bar{x}_5)y = 0 \\
k_{02}(\bar{x}_5)z + k_2(\bar{x}_5)y = 0 \\
k_{03}(\bar{x}_5) = 0
\end{align*}

\begin{align*}
X_{7|22211,1|00011} : & \\
k_{01}(\bar{x}_5) + x_6l_1(\bar{x}_6) + x_7l_2(\bar{x}_7) = 0 \\
k_{02}(\bar{x}_5) + x_6l_3(\bar{x}_6) + x_7l_4(\bar{x}_7) = 0 \\
k_{03}(\bar{x}_5) + x_6l_5(\bar{x}_6) + x_7l_6(\bar{x}_7) = 0 \\
x_6z + l_7(\bar{x}_7)y = 0 \\
x_7z + l_8(\bar{x}_7)y = 0.
\end{align*}

The two remaining types are essentially treated as the last of the 3 types listed. From a local viewpoint, these cases turn into:
\[
k_{01}(\bar{x}_5) + tk_{1}(\bar{x}_5) = k_{02}(\bar{x}_5) = k_{03}(\bar{x}_5) = 0,
\]
and
\[
k_{01}(\bar{x}_5) + tk_{1}(\bar{x}_5) = k_{02}(\bar{x}_5) = tk_{1}(\bar{x}_5) = 0.
\]
and

\[
\begin{align*}
    k_{01}(\bar{x}_5) &= tl_7(\bar{x}_5, 0, 0)l_1(\bar{x}_5, 0) - tl_8(\bar{x}_5, 0, 0)l_2(\bar{x}_5, 0, 0) = 0 \\
    k_{02}(\bar{x}_5) &= tl_7(\bar{x}_5, 0, 0)l_3(\bar{x}_5, 0) - tl_8(\bar{x}_5, 0, 0)l_4(\bar{x}_5, 0, 0) = 0 \\
    k_{03}(\bar{x}_5) &= tl_7(\bar{x}_5, 0, 0)l_5(\bar{x}_5, 0) - tl_8(\bar{x}_5, 0, 0)l_6(\bar{x}_5, 0, 0) = 0.
\end{align*}
\]

We recall that the equation of \( S_3 \) is

\[
k_{01}(\bar{x}_5) = k_{02}(\bar{x}_5) = k_{03}(\bar{x}_5) = 0.
\]

In analogy with Group 1 we apply the linear map \( \phi_5 \) from the parameter space \( (\mathbb{A}^{21})^3 \) of triples of homogeneous quadric polynomials in 6 variables \( \bar{x}_5 \) to the tangent space \( T_{R_5[S_3]} \). The deformations above correspond to applying \( \phi_5 \) on the subsets:

\[
\{(k_1(\bar{x}_5), 0, 0), \{k_1(\bar{x}_5), k_2(\bar{x}_5), 0\}, \{-l_7(\bar{x}_5, 0, 0)l_1(\bar{x}_5, 0) - l_8(\bar{x}_5, 0, 0)l_7(\bar{x}_5, 0, 0)l_3(\bar{x}_5, 0) - l_5(\bar{x}_5, 0, 0)}
\]

The linear span of the third subset is \( (\mathbb{A}^{21})^3 \). In the first case the linear span is \( \mathbb{A}^{21} \times 0 \times 0 \). On the other hand, if both \( \mathbb{A}^{21} \times 0 \times 0 \), and \( 0 \times \mathbb{A}^{21} \times 0 \), and \( 0 \times 0 \times \mathbb{A}^{21} \), all were contained in \( \ker(\eta_5) \), then the linear span of the union of these 3 sets, i.e. \( \mathbb{A}^{21} \times \mathbb{A}^{21} \times \mathbb{A}^{21} \), would be, again a contradiction. Hence one of these 3 sets is not contained in the union, and by interchanging the roles of \( k_{01}(\bar{x}_5) \), \( k_{02}(\bar{x}_5) \), \( k_{03}(\bar{x}_5) \) if necessary we obtain an element outside \( \ker(\eta_5) \). This argument certainly applies to give a proof for the second case also, since a desired deformations in the first case also is included in the second case.

To complete the proof of Theorem 4.3 it is enough to prove that in each of the 16 cases listed above, a general \( F \) as constructed, is smooth. To do so we will use the following version (extension) of Bertini’s theorem, as given in [10].

**Lemma 4.4 ([10]).** On an arbitrary ambient variety \( V \), if a linear system has no fixed components, then the general member has no singular points outside of the union of the base locus of the system and the singular locus of the ambient variety.

Repeated usage of this result in each of the 16 cases will solve the problem. We will be able to put ourselves in a position where the ambient varieties always are smooth.

As an example take the case \( X_{4;32,1;20} \):

\[
c_0(\bar{x}_4)z^2 + c_1(\bar{x}_4)yz + c_2(\bar{x}_4)y^2 = k_0(\bar{x}_4) = 0.
\]

Here the fixed smooth hypersurfaces \( Z(k_0(\bar{x}_4)) \) and \( Z(c_0(\bar{x}_4)) \) cut out the smooth surface \( S_2 \) in \( \mathbb{P}^4 \). On the smooth ambient variety \( V_1 = Z(k_0(\bar{x}_4)) \) in \( \mathbb{P}^4 \times \mathbb{P}^1 \) we study the linear system

\[
S_1 = \{c_0(\bar{x})z^2 + c_1(\bar{x})yz + c_2(\bar{x})y^2\}, \text{and} \ c_0(\bar{x}), c_1(\bar{x}), \text{and} \ c_2(\bar{x}) \text{ arbitrary cubics}.
\]
This linear system has no fixed components, and the base locus of \( \mathcal{L}_1 \) is \( Z(c_0(\bar{x}_4), k_0(\bar{x}_4), y) \) in \( \mathbb{P}^4 \times \mathbb{P}^1 \), that is \( S_2 \). But since \( S_2 \) is the scheme-theoretical complete intersection in \( V_1 \) of any element \( l \) of \( \mathcal{L}_1 \) with the hypersurfaces \( c_0(\bar{x}_4) \) and \( y \), this element \( l \) cannot be singular at a point in \( S_2 \) without \( S_2 \) being so. Hence all elements of \( \mathcal{L}_1 \) are smooth at all points of the base locus of \( \mathcal{L}_1 \). Since a general element is smooth outside the base locus, by Lemma 4.4, a general element of \( \mathcal{L}_1 \) is smooth. Hence a general \( F \) of type \( X_{4|32,1|20} \) is smooth. This was a relatively easy type to handle. Other easy types, which can be handled in a similar way, are: \( X_{3|4,1|1}, X_{4|32,1|02}, X_{5|222,1|200} \). In these cases it suffices to apply Lemma 4.4. once.

As another (and slightly more involved) example take the case \( \mathcal{L}_1 \) given as \( \left\{ q_0(\bar{x}_4) + x_4 c_1(\bar{x}_4) + x_5 c_2(\bar{x}_5) \mid c_1(\bar{x}_4) \text{and} c_2(\bar{x}_5) \text{arbitrary cubics} \right\} \) and

\[
\mathcal{L}_0 = \left\{ q_0(\bar{x}_4) + x_4 c_1(\bar{x}_4) \mid c_1(\bar{x}_4) \text{arbitrary cubic} \right\}
\]

have no fixed components, and the base loci are \( q_0(\bar{x}_4) = x_4 = x_5 = 0 \) and \( q_0(\bar{x}_4) = x_4 = 0 \), respectively. These loci are isomorphic to \( S_1 \times \mathbb{P}^1 \), and a \( x_5 \)-cone over \( S_1 \times \mathbb{P}^1 \), respectively, for the smooth quartic \( S_1 \) in \( \mathbb{P}^3 \) from Proposition 4.1. Hence the base locus, say \( B_1 \), is smooth, and the second, say \( B_0 \), is singular only along \( (0, \ldots, 0, 1) \times \mathbb{P}^1 \). But \( B_1 \) is also the scheme-theoretical intersection of any member of \( \mathcal{L}_1 \) and \( Z(x_4, x_5) \). Since \( B_1 \) is smooth of dimension 3, no member of \( \mathcal{L}_1 \) can then be singular at any point of \( B_1 \), and the general members of \( \mathcal{L}_1 \) are then smooth everywhere, in virtue of Lemma 4.4. By an analogous argument a general member of \( \mathcal{L}_0 \) can only be singular along \( (0, \ldots, 0, 1) \times \mathbb{P}^1 \). Hence there exists an element \( q_0(\bar{x}_4) + x_4 c_1(\bar{x}_4) + x_5 c_2(\bar{x}_5) \) of \( \mathcal{L}_1 \), such that \( Z(q_0(\bar{x}_4) + x_4 c_1(\bar{x}_4) + x_5 c_2(\bar{x}_5)) \) is smooth, and \( Z(q_0(\bar{x}_4) + x_4 c_1(\bar{x}_4)) \) is singular only along \( (0, \ldots, 0, 1) \times \mathbb{P}^1 \). Fix \( Z(q_0(\bar{x}_4) + x_4 c_1(\bar{x}_4) + x_5 c_2(\bar{x}_5)) \) as the new ambient variety \( V_2 \). We denote \( Z(q_0(\bar{x}_4) + x_4 c_1(\bar{x}_4), x_5, y) \) by \( G \). This is smooth, since \( Z(q_0(\bar{x}_4) + x_4 c_1(\bar{x}_4)) \) is smooth for all points with \( x_5 = 0 \). From now on \( c_1(\bar{x}_4) \) and \( c_2(\bar{x}_5) \) are fixed. Next, consider the linear system on \( V_2 \):

\[
\mathcal{L}_2 = \{ x_5 z + l_2(\bar{x}_5) y \mid l_2(\bar{x}_5) \text{arbitrary} \}.
\]

This system has no fixed component, and the base locus is \( V_2 \cap Z(x_5, y) \). This is the smooth quartic hypersurface \( G \) in \( \mathbb{P}^4 \). It is clear that all members of \( \mathcal{L}_2 \) are smooth on \( G \) (On each point of \( G, y = 0 \), and hence \( z \) can be taken to be one, and hence we may express properly dehomogenized versions of \( q_0(\bar{x}_4) + x_4 c_1(\bar{x}_4) + x_5 c_2(\bar{x}_5) \) of any point of \( G \), such that these expressions are independent modulo the square of the maximal ideal of \( p \)). Hence a general element of \( \mathcal{L}_2 \) will give a smooth submanifold of \( V_2 \). Fix one such element, that is, fix \( l_2(\bar{x}_5) \). This gives a smooth submanifold \( V_3 \) given as

\[
a_3(\bar{x}_3) + a_2(\bar{x}_3) + a_1(\bar{x}_3) + a_0(\bar{x}_3) = x_5 z + l_2(\bar{x}_5) y = 0.
\]
Next, consider the linear system on $V_3$:

$$\mathcal{L}_3 = \{x_4z + l_1(\bar{x}_5)y| l_1(\bar{x}_5)\text{arbitrary}\}.$$

This linear system has no fixed component, and the basis locus is given as $Z(q_0(\bar{x}_3), x_4, x_5, y)$. This is $S_1$, a smooth surface. But $S_1 = V(y|V_3, l|V_3)$, scheme-theoretically, for each $l$ in $\mathcal{L}_3$, so no such $l$ in $\mathcal{L}_3$ can be singular on $S_1$. Hence a general member of $\mathcal{L}_3$ is smooth. This gives that a general $F$ as constructed, of the type $X_5|{411,1}|011$ is a smooth 3-fold. Other cases that can be treated in a very similar way are: $X_5|{321,1}|002, X_5|{321,1}|011, X_5|{321,1}|101$.

Intermediate cases (where we need to use Lemma 4.4 twice) are:

$$X_4|{41,1}|02, X_4|{41,1}|11, X_4|{32,1}|11, X_5|{222,1}|200.$$

The cases $X_6|{2221,1}|0002, X_6|{2221,1}|0011, X_7|{22211,1}|00011$ are similar to each other, but no essentially new technique is required. As an example we treat the case $X_7|{22211,1}|00011$:

$$k_{01}(\bar{x}_5) + x_6l_1(\bar{x}_6) + x_7l_2(\bar{x}_7) = 0$$
$$k_{02}(\bar{x}_5) + x_6l_3(\bar{x}_6) + x_7l_4(\bar{x}_7) = 0$$
$$k_{03}(\bar{x}_5) + x_6l_5(\bar{x}_6) + x_7l_6(\bar{x}_7) = 0$$

$$x_6z + l_7(\bar{x}_7)y = 0$$
$$x_7z + l_8(\bar{x}_7)y = 0.$$

Using Lemma 4.4 several times as in the first part of the proof in the case of type $X_5|{411,1}|011$, we obtain that for a generic fixed choice of $l_1(\bar{x}_6)$, $l_2(\bar{x}_7)$, $l_3(\bar{x}_6)$, $l_4(\bar{x}_7)$, $l_5(\bar{x}_6)$, $l_6(\bar{x}_7)$ both the $(2, 2, 2)$ four-fold

$$k_{01}(\bar{x}_5) + x_6l_1(\bar{x}_6) + x_7l_2(\bar{x}_7) = 0$$
$$k_{02}(\bar{x}_5) + x_6l_3(\bar{x}_6) + x_7l_4(\bar{x}_7) = 0$$
$$k_{03}(\bar{x}_5) + x_6l_5(\bar{x}_6) + x_7l_6(\bar{x}_7) = 0$$

in $\mathbb{P}^7$ and the $(2, 2, 2)$ threefold

$$k_{01}(\bar{x}_5) + x_6l_1(\bar{x}_6) = 0$$
$$k_{02}(\bar{x}_5) + x_6l_3(\bar{x}_6) = 0$$
$$k_{03}(\bar{x}_5) + x_6l_5(\bar{x}_6) = 0$$

in $\mathbb{P}^6$ are smooth, and so are their respective counterparts in $\mathbb{P}^7 \times \mathbb{P}^1$, given by the same equations. We stick to such a fixed generic choice of the 6 linear functions. Let the new ambient space $V_1$ be defined by:

$$k_{01}(\bar{x}_5) + x_6l_1(\bar{x}_6) + x_7l_2(\bar{x}_7) = 0$$
$$k_{02}(\bar{x}_5) + x_6l_3(\bar{x}_6) + x_7l_4(\bar{x}_7) = 0$$
$$k_{03}(\bar{x}_5) + x_6l_5(\bar{x}_6) + x_7l_6(\bar{x}_7) = 0$$

in $\mathbb{P}^7 \times \mathbb{P}^1$. Let $\mathcal{L}_1$ be the linear system

$$\{x_4z + l(\bar{x}_5)y| l(\bar{x}_5)\text{is arbitrary}\}.$$
on $V_1$. The base locus of $\mathcal{L}_1$ is $V_1$ is $Z(\bar{x}_7, y)$, which is the smooth 3-fold in $\mathbb{P}^6$ referred to. This is the scheme-theoretical intersection of $Z(\bar{x}_7, y)$ on $V_1$ and every element of $\mathcal{L}_1$; hence every element of $\mathcal{L}_1$ is smooth along this base locus, and consequently a generic element is smooth everywhere. Choose one such fixed element, that is fix $l_8(\bar{x}_7)$, and let the new ambient variety $V_2$, be cut out by this element on $V_1$. At last study the linear system $$\mathcal{L}_2 = \{x_6z + l_6(\bar{x}_7)y | l_6(\bar{x}_7) \text{ is arbitrary}\}$$ on $V_2$. The base locus is $$Z(y, k_{01}(\bar{x}_5), k_{02}(\bar{x}_5), k_{03}(\bar{x}_5)) = S_3,$$ which is also the scheme-theoretical complete intersection of $y$ and any element of on $V_2$. Hence any such section is smooth on $S_3$, and a generic section is smooth everywhere. Hence a generic 3-fold $F$ constructed of type $X_{7|22211,1|00011}$ is smooth.

5. Determinantal contractions.

In this section we will prove Corollary 1.4. First we will give a way to relate a certain CICY in one multiprojective space to a CICY in another multiprojective space. The method of determinantal contractions is introduced in [2]. The construction is used in [3] to show that the moduli space of CICY varieties is connected. We use the same notation as introduced in the paragraph 1.

Let $X$ be a CICY of type:

\begin{align}
N & \begin{array}{c}a_0 \cdots a_n \end{array} \\
n & \begin{array}{c}0 1 \cdots 1 \end{array}
\end{align}

Here we have introduced a block notation. The $N$ is a column consisting of the dimensions of the first $k - 1$ projective spaces. Likewise $M$ is a block matrix where the rows are the multidegrees of the first $m - n - 1$ polynomials in the $k - 1$ first projective spaces. The $a_i$’s are $(k - 1)$-dimensional vectors. The zero in the last row is a short way of writing $m - n - 1$ zeros. In the following we denote the last $n + 1$ polynomials by $q_0, \cdots, q_n$ We can write the $q_i$’s as

\begin{align*}
q_0 &= q_{00}x_0 + \cdots + q_{0n}x_n \\
& \vdots \\
q_n &= q_{n0}x_0 + \cdots + q_{nn}x_n
\end{align*}

where $x_0, \cdots, x_n$ are the projective coordinates in the projective space singled out in the last row of (5.1). We use $P$ for the product of the remaining $k - 1$ projective spaces in order to keep the notation simple. We can write this on matrix form

$$
\begin{pmatrix}
q_0 \\
\vdots \\
q_n
\end{pmatrix} = A
\begin{pmatrix}
x_0 \\
\vdots \\
x_n
\end{pmatrix}
$$
where

\[(5.2) \quad A = \begin{pmatrix} q_{00} & \cdots & q_{0n} \\ \vdots & & \vdots \\ q_{n0} & \cdots & q_{nn} \end{pmatrix} \]

That some point \((a, b) \in \mathbb{P} \times \mathbb{P}^n\) is in the common zero set of the \(q_i\)'s, is equivalent to \(a\) being in the zero set of \(\det(A)\), since not all the \(x_i\)'s can vanish. In other words, the projection of \(F\) on the factor \(P\) gives us a new variety \(F_s\) of type:

\[
\begin{array}{c|c|c}
N & M & \text{mult}|A| \\
\hline
\end{array}
\]

where \text{mult}|A| denotes the multidegree of the determinant of \(A\). This is a singular variety in general.

**Proposition 5.1.** Let \(F\) be a CICY of dimension \(m\) in a multiprojective space. Suppose that there exists a determinantal contraction \(f: F \to F_s\). Let

\[F^r_s = \{p| \ \dim f^{-1}(p) \geq r - 1\}.\]

Then \(F^r_s\) is the zero set of the \((n - r + 2)\) minors of \(A\), and the fiber over each point \(q \in F^r_s - F^{r+1}_s\) is isomorphic to \(\mathbb{P}^{r-1}\).

**Proof.** We use the notation introduced above, i.e., \(F\) is of type

\[
\begin{array}{c|c|c|c}
N & M & a_0 & \cdots & a_n \\
\hline
n & 0 & 1 & \cdots & 1 \\
\end{array}
\]

and \(F_s\) is of type

\[
\begin{array}{c|c|c|c}
N & M & \text{mult}|A| \\
\hline
\end{array}
\]

and is defined by the zero set of the first \(m - n - 1\) polynomials defining \(F\) and the determinant of the following matrix:

\[(5.3) \quad A = \begin{pmatrix} q_{00} & \cdots & q_{0n} \\ \vdots & & \vdots \\ q_{n0} & \cdots & q_{nn} \end{pmatrix} \]

We have the following commutative diagram:

\[
\begin{array}{ccc}
X & \longrightarrow & P \times \mathbb{P}^n \\
\downarrow{\bar{pr}_1} & & \downarrow{pr_1} \\
X_s & \longrightarrow & P
\end{array}
\]

where \(pr_1\) is the projection on the first factor and \(\bar{pr}_1\) its restriction.

This means that two points \(p = q \times s\) and \(p' = q' \times s'\) are mapped to the same point in \(F_s\) if and only if \(q = q'\). Let \(f \in \mathcal{P}_1 \times \cdots \times \mathcal{P}_k\) be polynomials defining \(F\), and let \(q\) be a point in \(F_s\). Then \(q \times v\) is in the fiber over \(q\) if and only if \(Av = 0\) where \(A\) is the matrix associated to \(f\) as above. In other words the points
in the fiber are naturally identified with the kernel of the map \( A: \mathbb{A}^{n+1} \to \mathbb{A}^{n+1} \), affinely. Hence, if \( q \in F^r_s - F'^r_{s+1} \) then the fiber has affine dimension \( r \). This gives \( pr^{-1}_1(q) \cong \mathbb{P}^{r-1} \). □

**Remark 5.2.** Let \( Y \) be a threefold with an isolated singularity \( p \). Assume that there exists a nonsingular variety \( Y' \) and a morphism

\[
Y' \to Y,
\]

such that \( \pi^{-1}(p) \cong \mathbb{P}^1 \), and \( Y' - \pi^{-1}(p) \cong Y - \{p\} \). Then we say that \( Y' \to Y \) is a small resolution of the singularity \( p \). Furthermore, if \( Y \) contains several isolated singularities, and we can find a \( Y' \) such that \( Y' \) is, locally around each singularity on \( Y \), a small resolution, then we say that \( Y' \to Y \) is a small resolution of \( Y \).

**Lemma 5.3.** Let \( V \subseteq \mathbb{P}^n \) be a smooth subvariety, and let \( S = Z(Q_0, L_0) \cap V \) be a smooth codimension 2 subvariety of \( V \), where \( \deg(Q_0) = s \), and \( \deg(L_0) = t \). Let \( \mathcal{L} \) be the linear system \( \{ Q_0L_1 - Q_1L_0 \} \) on \( V \), where \( L_1 \) varies through all \( t \)-ics and \( Q_1 \) varies through all \( s \)-ics. Then for a general element \( l \) of \( \mathcal{L} \), \( \text{Sing}(l) = Z(Q_0, Q_1, L_0, L_1) \). In general \( Z(Q_0, Q_1, L_0, L_1) \cap V \) will be a smooth manifold of degree \( s^2t^2\deg V \).

**Proof.** By Lemma 4.4 a general \( V \) has no singular points outside the base locus of \( \mathcal{L} \). This base locus is \( S \). For each point \( p \) in \( V \), let \( W \) be the subspace of \( \mathbb{A}^{n+1} \) spanned by the gradients of a set of generators of \( I(V) \). For \( p \) in \( V \) and \( l \) in \( \mathcal{L} \) we have that \( p \) belongs to \( \text{Sing}(l) \) iff

\[
\nabla l(p) = (\frac{\partial l}{\partial x_0}(p), \ldots, \frac{\partial l}{\partial x_n}(p)) \in W.
\]

Set \( l = Q_0L_1 - Q_1L_0 \). Then

\[
\frac{\partial l}{\partial x_i}(p) = Q_0(p)\frac{\partial L_1}{\partial x_i}(p) + L_1(p)\frac{\partial Q_0}{\partial x_i}(p) - Q_1(p)\frac{\partial L_0}{\partial x_i}(p) - L_0(p)\frac{\partial Q_1}{\partial x_i}(p),
\]

for \( i = 0, \ldots, n \).

For points in the base locus \( Z(q_0, L_0) \subseteq V \) this reduces to:

\[
\frac{\partial l}{\partial x_i}(p) = L_1(p)\frac{\partial Q_0}{\partial x_i}(p) - Q_1(p)\frac{\partial L_0}{\partial x_i}(p), \quad i = 0, \ldots, n
\]

Hence (5.4) reduces to \( [L_1(p)Q_1(p)]M \in W \), where the rows of \( M \) is a matrix with rows that are the gradients of \( Q_0 \) and \( -L_0 \) at \( p \), respectively. The subvariety \( S \) is smooth of codimension 2 at \( p \), so the rows are independent. Hence, for a general element \( l \) of \( \mathcal{L} \), \( \text{Sing}(l) = Z(Q_0, Q_1, L_0, L_1) \) The last assertion of the lemma is immediate.
Application to complete intersection Calabi–Yau manifolds in projective space.

In the previous section we constructed isolated rational curves of multidegree \((d,0)\) for every integer \(d\), and for every \(\mathbb{P}^1\)-surjective CICY-type in biprojective space. Some of these CICY-types allow a determinantal contraction to mildly singular Calabi-Yau varieties in a projective space of dimension 4, 5, 6 or 7. We will use this to show existence of isolated rational curves of every degree on generic CICYs in these projective spaces. There are 5 types \(X_{4|5}, X_{5|42}, X_{5|33}, X_{6|332}, X_{7|2222}\). Consider first a CICY \(F\) of the type \(X_{4|41,1|11}\) constructed in the previous section.

\[
\begin{pmatrix}
q_0(\bar{x}_3) + x_4c_1(\bar{x}_4) & q_1(\bar{x}_4) \\
x_4 & l_1(\bar{x}_4)
\end{pmatrix}
\begin{pmatrix}
z \\
y
\end{pmatrix} = 0
\]

In the following denote the \(2 \times 2\) matrix by \(A\). The threefold \(X_{4|41,1|11}\) has a determinantal contraction to the quintic \(F_s = Z(\det(A)) \subseteq \mathbb{P}^4\). Explicitly \(F_s\) is the zero-set of the polynomial: \((q(\bar{x}_3) + x_4c_1(\bar{x}_4))l_1(\bar{x}_4) - x_4q_1(\bar{x}_4)\). By letting \(Q_0 = q(\bar{x}_3) + x_4c_1(\bar{x}_4)\) and \(L_0 = x_4\) and \(V = \mathbb{P}^3 \times \mathbb{P}^1\) and using Lemma 5.3 we get that \(F_s\) has exactly the 16 singular points given by the vanishing of the four polynomials \(q_0(\bar{x}_3), x_4, q_1(\bar{x}_4), l_1(\bar{x}_4)\).

Let \(\pi: F \to F_s\) be the contraction map. As noted before the fiber over each singular point is a \(\mathbb{P}^1\). By construction \(F\) contains a rational curve \(C_1\) of bidegree \((d,0)\). Furthermore, if \(C_1\) does not intersect any of the \(\mathbb{P}^1\) contracted by \(\pi\), then the image of \(C_1\) by \(\pi\) is an isolated curve on \(F_s\). This is possible to achieve. First observe that the intersection between \(S_1\) in \(F\) and a \(\mathbb{P}^1\) that is contracted is transversal, i.e. gives a point in \(S\). Furthermore, varying \(q_1(\bar{x}_4)\) and \(l_1(\bar{x}_4)\) moves the 16 intersection points freely in \(S_1\). But \(q_1(\bar{x}_4)\) and \(l_1(\bar{x}_4)\) were chosen to be generic (by construction of \(F\) in the previous paragraph). Hence, \(C_1\) does not intersect any of the \(\mathbb{P}^1\)’s contracted by \(\pi\). Denote the image of \(C_1\) by \(D\). By construction \(N_{D/F_s} \cong N_{C_1/F} \cong \mathcal{O}(-1) \oplus \mathcal{O}(-1)\). Then Corollary 1.4 follows from Theorem 3.4 in case of quintic threefolds in \(\mathbb{P}^4\).

The other four CICY-types:

\(X_{5|33}\):
Here we consider the following \(F\) of type \(X_{5|321,1|011}\) with matrix \(A\) given by:

\[
A = \begin{pmatrix}
k_0(\bar{x}_4) + x_5l_1(\bar{x}_5) & k_1(\bar{x}_5) \\
x_5 & l_2(\bar{x}_5)
\end{pmatrix}
\]

\(X_{5|42}\):
Here we consider the following \(F\) of type \(X_{5|321,1|101}\) with matrix \(A\) given by:

\[
A = \begin{pmatrix}
c_0(\bar{x}_4) + x_5k_1(\bar{x}_5) & c_1(\bar{x}_5) \\
x_5 & l_2(\bar{x}_5)
\end{pmatrix}
\]

\(X_{6|322}\):
Here we consider the following \(F\) of type \(X_{6|2221,1|1001}\) with matrix \(A\) given by:

\[
A = \begin{pmatrix}
k_0(\bar{x}_5) + x_6l_1(\bar{x}_6) & k_1(\bar{x}_6) \\
x_6 & l_4(\bar{x}_5)
\end{pmatrix}
\]

\(X_{7|2222}\):
Here we consider the following \(F\) of type \(X_{7|2221,1|0001}\) with matrix \(A\) given by:
\[ A = \begin{pmatrix} x_6 & l_7(x_7) \\ x_7 & l_8(x_7) \end{pmatrix} \]

As another example, let us study the last case in detail. We will study the contraction \( \pi : F \rightarrow F_s \). Let \( V \) be defined by:

\[ X_{7|22211,1|00011} : \]

\[
\begin{align*}
  k_{01}(\bar{x}_5) + x_6l_1(\bar{x}_6) + x_7l_2(\bar{x}_7) &= 0 \\
  k_{02}(\bar{x}_5) + x_6l_3(\bar{x}_6) + x_7l_4(\bar{x}_7) &= 0 \\
  k_{03}(\bar{x}_5) + x_6l_5(\bar{x}_6) + x_7l_6(\bar{x}_7) &= 0
\end{align*}
\]

Take \( Q_0 = x_6 \) and \( L_0 = x_7 \). We see that \( Q_0 \) and \( L_0 \) cut out

\[ S_3 = Z(k_{01}(\bar{x}_5), k_{02}(\bar{x}_5), k_{03}(\bar{x}_5), x_6, x_7) \]

inside \( V \), and that \( S_3 \) is smooth of codimension 2 in \( V \). Hence Lemma 5.3 can be used. The threefold \( F_s = V \cap Z(x_6l_8(\bar{x}_7) - x_7l_7(\bar{x}_7) \) will only be singular at the 8 points given by the vanishing of the 7 polynomials

\[
\begin{align*}
  k_{01}(\bar{x}_5), k_{02}(\bar{x}_5), k_{03}(\bar{x}_5), l_7(\bar{x}_7), l_8(\bar{x}_7), x_6, x_7.
\end{align*}
\]

for general \( l_7(\bar{x}_5) \) and \( l_8(\bar{x}_5) \).

Furthermore, for general \( l_7(\bar{x}_7) \) and \( l_8(\bar{x}_7) \) the isolated bidegree \((d,0)\) curve \( C_3 \) in \( S_3 \) does not intersect any of the \( \mathbb{P}^1 \) contracted by \( \pi \). Hence, the image of \( C_3 \) by \( \pi \) is a curve \( D \) of degree \( d \). By construction \( N_{D/F_s} \cong N_{C_3/F} \cong O(1) \oplus O(-1) \).

Then Corollary 1.4 follows from Theorem 3.4 in this case.

**Remark 5.4.** In order to prove Corollary 1.4 we used five determinantal contractions. There are 9 of the 16 CICY-types of Lemma 3.2 that have determinantal contractions. The four “unused” CICY-types with such contractions are \( X_{5|411,1|011}, X_{5|222,1|110}, X_{4|32,1|11}, X_{6|3211,1|0011} \). The first two can be used to give additional proofs of Corollary 1.4. in case of \( X_{5|42} \). These two additional proofs use the rigid curves on \( S_2 \) and \( S_3 \) respectively. Hence the \( X_{5|42} \) case can be proved using any of the three surfaces \( S_1, S_2, S_3 \). The contraction \( X_{4|32,1|11} \) can be used to give an additional proof of Corollary 1.4 in the \( X_{4|5} \) case (using \( S_2 \)), while the contraction \( X_{6|3211,1|0011} \) can be used in the \( X_{6|322} \)-case. For the remaining CICY-types we have no choice of determinantal contraction.

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