FINITE DEGREE CLONES ARE UNDECIDABLE

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Abstract. A clone of functions on a finite domain determines and is determined by its system of invariant relations (=predicates). When a clone is determined by a finite number of relations, we say that the clone is of finite degree. For each Minsky machine $M$ we associate a finitely generated clone $C$ such that $C$ has finite degree if and only if $M$ halts, thus proving that deciding whether a given clone has finite degree is impossible.

1. Introduction

A clone is a set of operations on a domain which is closed under composition and contains all projections. Emil Post [34] in 1941 famously classified all clones on a 2-element domain (the Boolean clones), of which there are countably many. In contrast to this, there are continuum many clones over even a 3-element domain, as proven in 1959 by Janov and Mučnik [18]. The problem under consideration in this paper has its roots in investigations in the 1970s of the structure of the lattices of clones over domains of more than 2 elements. Before discussing the history of the problem, however, it will be useful to establish some background.

There are two common methods of finitely specifying a clone of operations. The first is to generate the clone from a finite set of functions via composition and variable manipulations. The second method is to specify the clone as all operations preserving a given finite set of relations. A relation $R$ on domain $D$ is said to be preserved by an operation $f : D^n \rightarrow D$ if $f(r_1, \ldots, r_n) \in R$ whenever $r_1, \ldots, r_n \in R$. The polymorphism clone on a set $R$ of relations over domain $D$ is

$$\text{Pol}(R) = \bigcup_{n \in \mathbb{N}} \{ f : D^n \rightarrow D \mid f \text{ preserves each relation in } R \}.$$ 

We say that a clone $C$ is determined by $R$ if $C = \text{Pol}(R)$. The supremum of the arities of the relations contained in $R$ is the degree of $R$, written

$$\text{deg}(R) = \sup \{ \text{arity}(R) \mid R \in R \},$$

and the degree of a clone $C$ is the infimum of the degrees of all sets of relations which determine $C$,

$$\text{deg}(C) = \inf \{ \text{deg}(R) \mid C = \text{Pol}(R) \}.$$ 

Both of these values can be infinite, and we regard them as total functions. Of course, since there are uncountably many clones on domains of more than 2 elements, there is no enumeration of them and hence no standard sense in which $\text{deg}(\cdot)$ can be computable. We resolve this complication by considering only those clones...
which have finite domain and are generated by finitely many operations (i.e. the clones of finite algebras). The clone generated by the algebra $\mathbb{A} = \langle A; f_1, \ldots, f_n \rangle$ is the smallest clone with domain $A$ containing all the $f_i$. The problem that we consider in this paper is the following, which we call the \textit{Finite Degree Problem}:

\textbf{Input}: clone $\mathcal{C}$ generated by the algebra $\mathbb{A} = \langle A; f_1, \ldots, f_n \rangle$

\textbf{Output}: whether $\deg(\mathcal{C}) < \infty$.

We show that the Finite Degree Problem is undecidable by constructing for each Minsky machine $M$ a finite algebra $\mathbb{A}(M)$ such that if $\mathcal{C}$ is the clone generated by $\mathbb{A}(M)$ then $\deg(\mathcal{C}) < \infty$ if and only if $M$ halts.

It is difficult to determine the precise origin of the Finite Degree Problem. Questions surrounding the algorithmic computation of the degree of a clone date back to the 1970s with papers by Romov [37, 36] and Jablonskii [17]. The closely related question of deciding whether an algebra admits a natural duality has been open since the late 1970s, but apparently first appears in print in 1991 with Davey [12]. The Finite Degree Problem is likely a contemporary of this problem, but does not appear in print until 2006 in [5] in which it is credited by Ralph McKenzie to Miklós Maróti in 2004.

Investigations into which structures have finite degree and under what conditions have yielded a host of results over the years, which we now give a brief overview of. All of the following structures \textit{on a finite domain} have finite degree:

- all bands [15];
- many semigroups, but not all [24, 11];
- semilattices, and more generally any clone containing a semilattice operation that commutes with the other operations [13, 10];
- clones containing the lattice operations of $\land$ and $\lor$, and more generally algebras with a near unanimity term (if the algebra belongs to a congruence distributive variety then this is an equivalence) [2, 3];
- groups, rings, and more generally algebras with a cube term (if the algebra belongs to a congruence modular variety then this is an equivalence) [1, 4].

Aside from results on specific structures, necessary conditions for a clone to have finite degree have also been established. Rosenberg and Szendrei [38] and Davey and Pitkethly [14] both establish general algebraic conditions which imply finite degree.

The technique of encoding a model of computation into an algebraic structure was pioneered by McKenzie [25, 26], where it was proven that it is undecidable whether an algebra is finitely axiomatizable (this is famously known as Tarski’s Problem). Since then, a handful of other authors have used a similar approach to prove that other algebraic properties are undecidable. Maróti [22] proves that it is undecidable whether an algebra has a near unanimity term defined on all but 2 elements of a finite domain (it was later discovered that this is decidable without this restriction, see Maróti [23]). McKenzie and Wood [27] prove certain “omitting types” statements about algebras are undecidable. The author [32] proves that the technical property of DPSC is undecidable, thus giving an alternate proof of the undecidability of Tarski’s Problem. Most recently, Nurakunov and Stronkowski [33] prove that profiniteness is undecidable.
We begin in Section 2 with a discussion of a very simple model of computation, the Minsky machine, before continuing on to a brief survey of the necessary algebraic background and some of the notation used in the paper in Section 3. The algebra $\mathcal{A}(M)$ mentioned above is precisely defined in Section 4 and the exact manner in which it encodes the computation of the Minsky machine $M$ is proven in Section 5. In Section 6 we show that $\deg(\mathcal{A}(M)) = \infty$ when $M$ does not halt. The converse is quite a bit more complicated. Tools necessary for the analysis are developed in Section 7 and the main argument is divided into cases and addressed in Section 8. Lastly, Section 9 contains a statement of the main theorem and a discussion of related open problems.

A great deal of effort was spent in constructing the algebra $\mathcal{A}(M)$ so that the entire argument would be as straightforward as possible. Much of this effort took the form of computer experimentation and verification, allowing for rapid iteration of the definitions. Significant portions of many of the lemmas and theorems can be verified computationally. The framework that was used was built specifically for this task, but the majority of it is suited to general algebraic structures. This computational framework as well as several examples are available online at the URL below.

\[ \text{http://ittc.ku.edu/~moore/preprints/2018_AM.zip} \]

2. Minsky machines

Minsky machines are a very simple model of computation for which the halting problem is undecidable, and were defined in 1961 by Marvin Minsky [30, 31]. A Minsky machine has states $\{0, 1, \ldots, N\}$, where 0 is the halting state and 1 is the initial state, registers $A$ and $B$ which hold non-negative values, and a finite set of instructions. Minsky machine instructions come in two types:

- $(i, R, j)$, interpreted as “in state $i$, increase register $R$ by one and enter state $j$”;
- $(i, R, k, j)$, interpreted as “in state $i$, if register $R$ is 0 then enter state $k$, otherwise decrease $R$ by one and enter state $j$”.

In order for the instructions to unambiguously describe a Minsky machine there must be an instruction of the form $(1, \ldots)$ and for each state $i$ there must be at most one instruction of the form $(i, \ldots)$.

A Minsky machine configuration is a triple of the form $(s, \alpha, \beta)$, where $s$ indicates the state of the machine, $\alpha$ the value of the $A$ register, and $\beta$ the value of the $B$ register. Formally, a Minsky machine with states $\mathcal{S} = \{0, \ldots, N\}$ and instructions $\mathcal{I}$ is an operation on the space of all possible configurations

$\mathcal{M}: \mathcal{S} \times \mathbb{N} \times \mathbb{N} \rightarrow \mathcal{S} \times \mathbb{N} \times \mathbb{N}$
defined by

\[ \mathcal{M}(i, \alpha, \beta) = \begin{cases} 
(j, \alpha + 1, \beta) & \text{if } (i, A, j) \in \mathcal{I}, \\
(j, \alpha, \beta + 1) & \text{if } (i, B, j) \in \mathcal{I}, \\
(j, \alpha - 1, \beta) & \text{if } (i, A, k, j) \in \mathcal{I}, \alpha \neq 0, \\
(j, \alpha, \beta - 1) & \text{if } (i, B, k, j) \in \mathcal{I}, \beta \neq 0, \\
(k, \alpha, \beta) & \text{if } (i, A, k, j) \in \mathcal{I}, \alpha = 0, \\
(k, \alpha, \beta) & \text{if } (i, B, k, j) \in \mathcal{I}, \beta = 0, \\
(0, \alpha, \beta) & \text{if } i = 0.
\]

Since the function \( \mathcal{M} \) is determined by \( \mathcal{I} \), it is usual to use the same symbol for both. That is, we indicate that \( \mathcal{M} \) has some instruction, say \((1, A, 2)\), by simply writing \((1, A, 2) \in \mathcal{M}\).

A single application of the function \( \mathcal{M} \) to a configuration represents a single computational step of the Minsky machine. To indicate multiple steps in the computation, we can compose \( \mathcal{M} \):

\[ \mathcal{M}^n(i, \alpha, \beta) = \mathcal{M} \circ \cdots \circ \mathcal{M}(i, \alpha, \beta). \]

We say that a Minsky machine \( \mathcal{M} \) halts on input \( A = \alpha, B = \beta \) if there is some \( n \) such that \( \mathcal{M}^n(1, \alpha, \beta) = (0, \alpha', \beta') \). We say that a Minsky machine \( \mathcal{M} \) halts (without reference to input) if it halts on input \( A = B = 0 \). By replacing the halting state with a new state \( k \) and appending instructions \((k, A, k + 1, k)\) and \((k + 1, B, 0, k + 1)\) to the list of instructions, a Minsky machine can be made to return the registers to 0 before halting. Thus, we can assume without loss of generality that all halting machines return both registers to 0 before halting. Furthermore, any Minsky machine can be converted to an equivalent machine with first instruction of the form \((1, R, s)\).

If a given Minsky machine with states \( \{0, \ldots, N\} \) does not have an instruction of the form \((k, \ldots)\) for some state \( k \) then without changing the halting status of the machine we may add an instruction of the form \((k, R, k)\) to \( \mathcal{M} \). We therefore assume throughout that Minsky machines have exactly one instruction for each state.

Let \( \Sigma(\mathcal{M}) \) be the directed graph with vertices \([N]\) and an edge \( i \rightarrow j \) if and only if \( \mathcal{M}(i, \alpha, \beta) = (j, \alpha', \beta') \) for some \( \alpha, \beta, \alpha', \beta' \in \mathbb{N} \). We call this the state graph of \( \mathcal{M} \). If state \( \ell \) is reachable from state \( k \) along a (possibly length 0) directed path then we write \( k \leadsto \ell \). If there is a state \( k \in \Sigma(\mathcal{M}) \) such that \( 1 \not\leadsto k \) then we can eliminate state \( k \) from \( \mathcal{M} \) without changing the halting status of \( \mathcal{M} \). We therefore assume that all states are reachable from 1 in the state graph.

A further modification of \( \mathcal{M} \) allows us to assume that every state has a path to the halting state 0. If we have \( 1 \leadsto \ell \not\leadsto 0 \) then there is at least one pair of states \( i, k \in \Sigma(\mathcal{M}) \) such that \( i \leadsto k, i \not\leadsto 0, \) and \( k \not\leadsto 0 \). This is only possible if \((i, R, k, j) \in \mathcal{M} \) or \((i, R, j, k) \in \mathcal{M} \) for some register \( R \) and \( j \not\leadsto 0 \). For all such pairs \( i, k \), we do the following:

- add a new state \( n^k_i \),
- if \((i, R, k, j) \in \mathcal{M} \) then replace this instruction with \((i, R, n^k_i, j)\) and add the instruction \((n^k_i, R, i, i)\) to \( \mathcal{M} \),
- if \((i, R, j, k) \in \mathcal{M} \) then replace this instruction with \((i, R, j, n^k_i)\) and add the instruction \((n^k_i, R, i)\) to \( \mathcal{M} \).
The new instructions cause \( M \) to loop upon entering state \( n_i \). Since \( k \not\rightarrow 0 \), modifying \( M \) in this manner does not change its halting status. After performing this procedure for all \( i, k \) as described above, we next eliminate any states which are not reachable from the initial state 1. After performing this procedure, we will have \( 1 \not\rightarrow k \not\rightarrow 0 \) for all states \( k \in \Sigma(M) \).

Summarizing, we assume the following about every Minsky machine \( M \) we consider in this paper:

- \( M \) returns both registers to 0 before halting,
- \( M \) has exactly one instruction for each state \( k \),
- \( M \) begins with an instruction of the form \((1, R, s)\), and
- for every state \( k \) of \( M \), there are paths in the state graph leading from the initial state 1 to \( k \), and from \( k \) to the halting state 0.

By the discussion in the paragraphs above, the halting problem restricted to the set of Minsky machines satisfying these is still undecidable.

3. Algebraic background and notation

In this section we give a brief background of the algebraic notions used in the proof. Good references for additional details are McKenzie, McNulty, Taylor [28] and Burris [7].

An algebra \( \mathbb{A} \) consists of a non-empty set \( A \), called the universe of \( \mathbb{A} \), and a set of operations \( \mathcal{F} \) on \( A \), called the fundamental operations of \( \mathbb{A} \). This is typically shortened to \( \mathbb{A} = \langle A; \mathcal{F} \rangle \). From the operations in \( \mathcal{F} \) we can generate new operations by composition and variable identification. These together with the projections are the term operations of \( \mathbb{A} \). A subset \( B \subseteq A \) which is closed under all operations from \( \mathcal{F} \) is called a subuniverse. If \( B \neq \emptyset \) then \( B \) together with the operations from \( \mathcal{F} \) restricted to \( B \) form a subalgebra of \( \mathbb{A} \), written \( B \leq \mathbb{A} \).

The operations of \( \mathbb{A} \) extend coordinate-wise to operations of \( \mathbb{A}^m \) for any \( m \in \mathbb{N} \). A subuniverse \( C \subseteq \mathbb{A}^m \) is called a relation (or subpower) of \( \mathbb{A} \). If \( D \subseteq \mathbb{A}^m \) is a subset then the smallest relation containing \( D \) is called the subalgebra generated by \( D \), written \( \text{Sg}_{\mathbb{A}^m}(D) \). We denote by \( \text{Rel}(\mathbb{A}) \) the set of all finitary relations of \( \mathbb{A} \). This set is closed under intersection (of equal arity relations), product, permutation of coordinates, and projection onto a subset of coordinates. Another way of saying this is that if relations are viewed as predicates and \( p(x_1, \ldots, x_n) \) is a primitive-positive formula in this language of these predicates then the set of values in \( \mathbb{A}^n \) for which \( p \) is true forms a relation.

From the discussion above, it is clear that the operations of \( \mathbb{A} \) determine the relations. The opposite is also true for finite \( A \): \( t \) is a term operation of \( \mathbb{A} \) if and only if it preserves all the relations of \( \mathbb{A} \). We can formalize this by introducing two new operations on sets of term operations and relations. Let \( E \) be a domain, \( \mathcal{G} \) be a set of operations on \( E \), and \( \mathcal{R} \) a set of subsets of powers of \( E \). Define

\[
\text{Rel}(\mathcal{G}) = \bigcup_{m \in \mathbb{N}} \left\{ R \subseteq E^m \mid R \text{ is closed under each operation from } \mathcal{G} \right\}
\]

and

\[
\text{Pol}(\mathcal{R}) = \bigcup_{m \in \mathbb{N}} \left\{ f : E^m \to E \mid f \text{ preserves each set in } \mathcal{R} \right\}.
\]
These two operations form a Galois connection:

$$R \subseteq \text{Rel}(G) \quad \text{if and only if} \quad G \subseteq \text{Pol}(R).$$

This relationship is quite famous and was first discovered by Geiger [16] and by Bodnárčuk, Kalužnin, Kotov, and Romov [6]. Every Galois connection defines two closure operators. For Rel and Pol these are $\text{Clo} = \text{Pol} \circ \text{Rel}$ and $\text{RClo} = \text{Rel} \circ \text{Pol}$, the clone and relational clone, respectively. If $A$ is an algebra with fundamental operations $F$ then the set of term operations of $A$ is $\text{Clo}(F)$ and the set of relations of $A$ is $\text{Rel}(F) = \text{Rel}(A)$.

If $R$ is a set of relations on $A$ and $R \in \text{RClo}(R)$ (that is, $R$ is preserved by every operation which preserves relations in $R$) then we say that $R$ entails $R$ and write $R \models R$. It is not difficult to prove that $R \models R$ if and only if $R$ can be built from the relations in $R \cup \{=\}$, in finitely many steps, by applying the following constructions:

1. intersection of equal arity relations,
2. (cartesian) product of finitely many relations,
3. permutation of the coordinates of a relation, and
4. projection of a relation onto a subset of coordinates.

We call these entailment constructions. Similarly, for an operation $f$ on $A$ we write $R \models f$ if $f \in \text{Pol}(R)$. We define the degree of $R$ to be the supremum of the arities of the relations in $R$,

$$\text{deg}(R) = \sup \{ \text{arity}(S) \mid S \in R \}.$$ 

For a clone $C$, we define the degree to be the infimum of the degrees of all sets of relations which determine $\text{Rel}(C)$,

$$\text{deg}(C) = \inf \{ \text{deg}(R) \mid R \models \text{Rel}(C) \} = \inf \{ \text{deg}(R) \mid \text{Pol}(R) = C \}.$$ 

Finally, for an algebra $A$ we define the degree of $A$ to be the degree of its clone,

$$\text{deg}(A) = \text{deg}(\text{Clo}(A)).$$

In general, any of these quantities may be infinite. An algebra $A$ has finite degree (or is said to be finitely related) if $\text{deg}(A) < \infty$.

Lastly, we adopt a convention for projections of elements and subsets of powers intended to increase readability. If $m \in \mathbb{N}$ and $I \subseteq [m]$ then

- for $B \subseteq A$, define $a^{-1}(B) = \{ i \in [m] \mid a(i) \in B \}$ and for $b \in A$ define $a^{-1}(b) = a^{-1}\{b\}$,
- denote the projection of $a \in A^m$ to coordinates $I$ by $a(I) \in A^I$,
- denote the projection of $S \subseteq A^m$ to coordinates $I$ by $S(I) \subseteq A^I$, and
- define $a(\neq i) = a([m] \setminus i)$ and likewise $a(\neq i, j) = a([m] \setminus \{i, j\})$.

It is possible to confuse this notation for projection with the notation for function application, but we will take special care to avoid ambiguous situations.
4. The algebra \( \mathbb{A}(\mathcal{M}) \)

We begin by defining the underlying set of \( \mathbb{A}(\mathcal{M}) \). Let \( \mathcal{M} \) be a Minsky machine with states \{0, \ldots, N\} and define

\[
M_i = \{ (i, c) \mid c \in \{\bullet, \times, 0, A, B\} \}
\]

and

\[
A(\mathcal{M}) = \bigcup_{i=0}^{N} M_i.
\]

We next define several important subsets of \( A(\mathcal{M}) \). Let

\[
X = \{ (0, \times), \ldots, (N, \times) \}, \quad Y = A(\mathcal{M}) \setminus X,
\]

\[
D = \{ (0, \bullet), \ldots, (N, \bullet) \}, \quad E = A(\mathcal{M}) \setminus D,
\]

\[
C = A(\mathcal{M}) \setminus (X \cup D).
\]

An easy way to keep these straight is that \( X \) contains elements with second coordinate \( \times \), \( D \) contains elements with second coordinate \( \bullet \) (“dot”), and \( C \) contains elements with neither. The set \( Y \) is “not \( X \)” and \( E \) is “not \( D \).” We will now define the operations of \( \mathbb{A}(\mathcal{M}) \). It will be convenient in the operation definitions which follow to make use three “helper” functions which are not operations of \( \mathbb{A}(\mathcal{M}) \). Let

\[
\mathbf{X}(i, c) = (i, \times), \quad \mathbf{st}(i, c) = i, \quad \text{and} \quad \mathbf{con}(i, c) = c.
\]

The second two of these are referred to as the state and content of an element. We extend both of these functions to elements of \( A(\mathcal{M})^m \) in different ways: for \( m > 1 \) and \( \alpha \in A(\mathcal{M})^m \) define

\[
\mathbf{st}(\alpha) = (\mathbf{st}(\alpha(1)), \ldots, \mathbf{st}(\alpha(m))) \quad \text{and} \quad \mathbf{con}(\alpha) = \{ \mathbf{con}(\alpha(i)) \mid i \in [m] \}.
\]

The algebra has a semilattice reduct with meet defined as

\[
\langle i, c \rangle \land \langle j, d \rangle = \begin{cases} 
\langle i, c \rangle & \text{if } \langle i, c \rangle = \langle j, d \rangle, \\
\min(i, j), \times) & \text{otherwise.}
\end{cases}
\]

The semilattice operation defines an order: we write \( x \leq y \) if and only if \( x \land y = x \). The next two operations encode the computation of \( \mathcal{M} \) on elements of powers of \( \mathbb{A}(\mathcal{M}) \). Let

\[
M(x, y) = \begin{cases}
\langle j, R \rangle & \text{if } x = \langle i, \bullet \rangle, \ y = \langle i, 0 \rangle, \ (i, R, j) \in \mathcal{M}, \\
\langle j, 0 \rangle & \text{if } x = \langle i, \bullet \rangle, \ y = \langle i, R \rangle, \ (i, R, k, j) \in \mathcal{M}, \\
\langle j, \bullet \rangle & \text{if } x = \langle i, 0 \rangle, \ y = \langle i, \bullet \rangle, \ (i, R, j) \in \mathcal{M}, \\
\langle j, \bullet \rangle & \text{if } x = \langle i, R \rangle, \ y = \langle i, \bullet \rangle, \ (i, R, k, j) \in \mathcal{M}, \\
\langle j, c \rangle & \text{if } x = y = \langle i, c \rangle, \ \left[ (i, R, j) \in \mathcal{M} \text{ or } (i, R, k, j) \in \mathcal{M} \right], \\
\langle j, \times \rangle & \text{elif } \mathbf{st}(x) = \mathbf{st}(y) = i, \ \left[ (i, R, j) \in \mathcal{M} \text{ or } (i, R, k, j) \in \mathcal{M} \right], \\
X(y) & \text{otherwise,}
\end{cases}
\]

and

\[
M'(x) = \begin{cases}
\langle k, c \rangle & \text{if } x = \langle i, c \rangle, \ (i, R, k, j) \in \mathcal{M}, \ c \neq R, \\
\langle k, \times \rangle & \text{elif } \mathbf{st}(x) = i, \ (i, R, k, j) \in \mathcal{M}, \\
X(x) & \text{otherwise.}
\end{cases}
\]
The next operations are involved with the representation of initial and halting states of \( \mathcal{M} \) in \( \mathcal{A}(\mathcal{M}) \). Define

\[
I(x, y) = \begin{cases} 
(1, \bullet) & \text{if } x \in D, \\
(1, 0) & \text{elif } y \in C, \\
(1, \times) & \text{otherwise},
\end{cases} \quad H(x) = \begin{cases} 
(0, 0) & \text{if } x \in \{ (0, 0), (0, \bullet) \}, \\
(0, \times) & \text{otherwise}.
\end{cases}
\]

The next several operations are technical, but are intimately involved in entailment and enforce a certain regularity on the structure of subpowers of \( \mathcal{A}(\mathcal{M}) \). Let

\[
N_0(x, y, z) = \begin{cases} 
y & \text{if } x = (0, \bullet), \text{ st}(y) = \text{st}(z), \\
z & \text{if } x = (0, 0), \text{ st}(y) = \text{st}(z), \\
X(y \land z) & \text{otherwise},
\end{cases} \quad S(x, y, z) = \begin{cases} 
(1, 0) & \text{if } x = (1, 0), \text{ st}(y) = \text{st}(z), \\
(1, \times) & \text{otherwise}, \\
\end{cases}
\]

\[
N_\ast(u, x, y, z) = \begin{cases} 
x & \text{if } x = y \notin X, \text{ st}(x) = \text{st}(y) = \text{st}(z), \\
x & \text{if } u \in D, \ y \in X, \text{ st}(x) = \text{st}(y) = \text{st}(z), \\
y & \text{elif } u \in D, \ x \in X, \text{ st}(x) = \text{st}(y) = \text{st}(z), \\
z & \text{elif } u \in D, \ z \in \{ x, y \}, \text{ st}(x) = \text{st}(y) = \text{st}(z), \\
X(x \land y \land z) & \text{otherwise},
\end{cases} \quad P(u, v, x, y) = \begin{cases} 
x & \text{if } \text{st}(u) = \text{st}(v), \\
y & \text{otherwise}.
\end{cases}
\]

The algebra \( \mathcal{A}(\mathcal{M}) \) is

\[
\mathcal{A}(\mathcal{M}) = \langle A(\mathcal{M}); \land, \lor \rangle.
\]

This completes the definition of \( \mathcal{A}(\mathcal{M}) \).

Each operation of \( \mathcal{A}(\mathcal{M}) \) plays an important role in the argument, and each has been defined to be as simple as possible. Though the argument is technical, we now attempt to give rough description of the role that each operation plays.

- The semilattice operation \( \land \) induces an order on the algebra that is “flat” modulo \( X \). That is, if \( a \land b \notin X \) then \( a = b \).
- The operations \( \land, \lor, I \) encode the computation of \( \mathcal{M} \) in the relations of \( \mathcal{A}(\mathcal{M}) \). See Example 5.7.
- The operation \( S \) is technical, and is used to produce special term operations \( z_i(x) \) used elsewhere in the argument. See Lemma 5.3.
- The operations \( H \) and \( N_0 \) are responsible for ensuring the entailment of certain relations when \( \mathcal{M} \) halts. See Theorem 5.12 Corollary 5.13 and Theorem 8.3.
- The operations \( N_\ast \) and \( P \) are responsible for the entailment of relations which are “non-computational”. See Definition 5.5 and Theorem 8.3.
5. The encoding of computation

In this section we build the tools necessary to prove that the relations of $\mathcal{A}(\mathcal{M})$ encode the computation of $\mathcal{M}$ in a "faithful" manner.

**Definition 5.1.** An $n$-ary operation $f$ of $\mathcal{A}(\mathcal{M})$ is said to be $X$-absorbing if for all $a_1, \ldots, a_n \in A(\mathcal{M})$, if $a_i \in X$ for some $i$ then $f(a_1, \ldots, a_n) \in X$.

**Definition 5.2.** An element $s \in A(\mathcal{M})^n$ is said to be synchronized if $st(s(i))$ is constant over all $i \in [n]$; we refer to the common value as $st(s)$ or "the state of $s$". A subset $S \subseteq A(\mathcal{M})^n$ is said to be synchronized if all of its elements are.

**Lemma 5.3.** Each of the following hold for $\mathcal{A}(\mathcal{M})$.

1. The state map $st$ is a homomorphism of $\mathcal{A}(\mathcal{M})$. Equivalently, if $t$ is a term operation of $\mathcal{A}(\mathcal{M})$ and $s_1, \ldots, s_m \in A(\mathcal{M})^n$ are synchronized then $t(s_1, \ldots, s_m)$ is synchronized as well.
2. $\land, M, M', H, S$ are $X$-absorbing.
3. If $a \in A(\mathcal{M}) \setminus D = E$ then the unary function $I(a, x)$ is $X$-absorbing.
4. For all states $i$ there is a term operation of $\mathcal{A}(\mathcal{M})$ defined by
   \[
   z_i(x) = \begin{cases} 
   \langle i, 0 \rangle & \text{if } x \in C, \\
   \langle i, x \rangle & \text{otherwise.}
   \end{cases}
   \]
5. If $(x, y)$ is a term in the operations $\{M, M'\}$ and $t(a, b) = c \notin X$ for some $a, b, c \in A(\mathcal{M})$ then $st(a) = st(b)$ and
   \[
   t(\langle st(a), 0 \rangle, \langle st(a), 0 \rangle) = \langle st(c), 0 \rangle.
   \]

**Proof.** (1)–(3): These can be proven by carefully examining the definitions of the operations of $\mathcal{A}(\mathcal{M})$.

(4): If $(i, R, j) \in \mathcal{M}$ or $(i, R, k, j) \in \mathcal{M}$ then
   \[
   M(\langle i, 0 \rangle, \langle i, 0 \rangle) = \langle j, 0 \rangle,
   \]
and if $(i, R, k, j) \in \mathcal{M}$ then
   \[
   M'(\langle i, 0 \rangle) = \langle k, 0 \rangle.
   \]

From the assumptions at the end of Section 2 every state can be reached from 1 in the state graph. Hence, there is a way to compose operations from $\{M, M'\}$ to obtain a term operation $f$ such that
   \[
   f(\langle 1, 0 \rangle, \ldots, \langle 1, 0 \rangle) = \langle i, 0 \rangle.
   \]

Let $T(x) = S(I(x, x), I(x, x), I(x, x))$. It follows that $z_i(x) = f(T(x), \ldots, T(x))$.

(5): The proof is by induction on the complexity of $t$. For the base case where $t$ is a projection, the conclusion clearly holds. For the inductive step, there are two cases: $t(x, y) = M(t_1(x, y), t_2(x, y))$ or $t(x, y) = M'(t_1(x, y))$. Suppose that $t(x, y) = M(t_1(x, y), t_2(x, y))$. Let $t_1(a, b) = c_1$ and $t_2(a, b) = c_2$. From the definition of $M$, if $t(a, b) = c \notin X$ then $c_1, c_2 \notin X$. $st(c_1) = st(c_2)$, and $\mathcal{M}$ has an instruction of the form $(st(c_1), R)$ or $(st(c_1), R, k, st(c))$. By the inductive
hypothesis, these observations, and the definition of \( M \), it follows that
\[
\begin{align*}
t \left( \langle \text{st}(a), 0 \rangle, \langle \text{st}(a), 0 \rangle \right) &= M \left( t_1 \left( \langle \text{st}(a), 0 \rangle, \langle \text{st}(a), 0 \rangle \right), t_2 \left( \langle \text{st}(a), 0 \rangle, \langle \text{st}(a), 0 \rangle \right) \right) \\
&= M \left( \langle \text{st}(c_1), 0 \rangle, \langle \text{st}(c_1), 0 \rangle \right) = \langle \text{st}(c), 0 \rangle,
\end{align*}
\]
as claimed. The case when \( t(x, y) = M'(t_1(x, y)) \) is similar. \( \square \)

**Definition 5.4.** We say that the Minsky machine \( M \) has
- \( k \)-step capacity \( C \) if
\[
C \geq \max \{ \alpha + \beta \mid M^n(1, 0, 0) = (i, \alpha, \beta) \text{ for some } n \leq k \},
\]
- capacity \( C \) if
\[
C \geq \max \{ \alpha + \beta \mid M^n(1, 0, 0) = (i, \alpha, \beta) \text{ for some } n \in \mathbb{N} \},
\]
- and halts with capacity \( C \) if it has capacity \( C \) and halts.

We say that the relation \( R \leq A(M)^m \) has
- capacity \( C \) if
\[
\left| \left\{ i \mid \exists r \in R \cap Y^m \ r(i) \in D \right\} \right| > C
\]
- and weak capacity \( C \) if
\[
\left| \left\{ i \mid \exists r \in R \ r(i) \in D \right\} \right| > C.
\]

We say that some elements \( \sigma_1, \ldots, \sigma_{C+1} \in R \) witness \( R \) having capacity (resp. weak capacity) \( C \) if there are distinct elements
\[
i_1, \ldots, i_{C+1} \in \left\{ i \mid \exists r \in R \ r(i) \in D \right\}
\]
such that \( \sigma_j(i_j) \in D \) and \( \sigma_j \in R \cap Y^m \) (resp. \( \sigma_j \in R \)).

Observe that for each \( m \in \mathbb{N} \) the halting problem is decidable for Minsky machines with capacity \( m \) since there are a finite (though quite large) number of configurations. At first glance, the definitions of capacity for machines and relations seem to be at odds. As we will see, however, a relation with capacity \( C \) can encode any Minsky machine computation with capacity \( C \).

**Definition 5.5.** If the relation \( R \leq A(M)^m \) is synchronized and
\[
\left| r^{-1}(D) \right| = \left| \left\{ i \in [m] \mid r(i) \in D \right\} \right| \leq 1
\]
(i.e. \( r \) does not have two coordinates with content \( \bullet \)) for all \( r \in R \) then we call \( R \) computational.

**Definition 5.6.** \( R \subseteq A(M)^m \) is halting if it contains an element \( r \in R \) such that
\[
r \notin \left\{ \langle 0, 0 \rangle, \langle 0, \bullet \rangle \right\}^m \setminus \left\{ \langle 0, 0 \rangle \right\}^m.
\]
Such an \( r \) is called a halting vector of \( R \). If \( R \) is not halting then we say that \( R \) is non-halting.

The easiest way to see how relations of \( A(M) \) encode computation is to work through an example.
Example 5.7. Consider the Minsky machine

\[ M = \{ (1, A, 2), (2, B, 3), (3, A, 4, 3), (4, B, 0, 4) \} \]

Recall that a configuration of \( M \) is a triple \((k, \alpha, \beta)\) where \( k \) is a state, \( \alpha \in \mathbb{N} \) is the value of register \( A \), and \( \beta \in \mathbb{N} \) is the value of register \( B \). We regard \( M \) as an operation on the set of configurations. Running through the computation on the initial configuration \((1, 0, 0)\), we have the table below.

| \( n \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| \( M^n(1,0,0) \) | \((1,0,0)\) | \((2,1,0)\) | \((3,1,1)\) | \((3,0,1)\) | \((4,0,1)\) | \((4,0,0)\) | \((0,0,0)\) |

(We assumed in Section 2 that all \( M \) zeroed out the registers on halting.) Let us see how this is encoded in \( \mathcal{A}(M)^3 \). For \( i \in [3] \) define elements \( \sigma_i \in A(M)^3 \) and subalgebra \( S \leq \mathcal{A}(M)^3 \) by

\[
\sigma_i(j) = \begin{cases} 
\{1, \bullet\} & \text{if } i = j \\
\{1, 0\} & \text{otherwise},
\end{cases}
\]

and \( S = S_{\mathcal{A}(M)^3} \{\sigma_1, \sigma_2, \sigma_3\} \)

The relation \( S \) is computational and has capacity 2. Let \( s \in S \cap Y^3 \). The value \( \mathbf{st}(s) \) will correspond to the state of the computation, and the values

\[
\alpha = |s^{-1}(\langle \mathbf{st}(s), A \rangle)| \quad \text{and} \quad \beta = |s^{-1}(\langle \mathbf{st}(s), A \rangle)|
\]

will correspond to the value of registers \( A \) and \( B \), respectively. Observe that the elements \( \sigma_1, \sigma_2, \sigma_3 \) correspond to the configuration \((1,0,0)\). We will need some notation. For \( k \) a state and distinct indices \( i_1, i_2, i_3 \in [3] \), define elements of \( A(M)^3 \)

\[
\left[ \begin{array}{c|c|c|c} k & i_1 & i_2 & i_3 \\ \hline \mathbf{st} & A & B \\ \end{array} \right](j) = \begin{cases} 
\langle k, \bullet \rangle & \text{if } j = i_1 \\
\langle k, A \rangle & \text{if } j = i_2 \\
\langle k, B \rangle & \text{if } j = i_3 \\
\langle k, 0 \rangle & \text{otherwise},
\end{cases}
\]

Additionally, define

\[
\left[ \begin{array}{c|c|c} k & i_1 & i_2 \\ \hline \mathbf{st} & \emptyset & \emptyset \\ \end{array} \right](j) = \begin{cases} 
\langle k, \bullet \rangle & \text{if } j = i_1 \\
\langle k, A \rangle & \text{if } j = i_2 \\
\langle k, 0 \rangle & \text{otherwise},
\end{cases}
\]

and define \( \left[ \begin{array}{c|c|c} k & i_1 & i_3 \\ \hline \mathbf{st} & \emptyset & \emptyset \\ \end{array} \right] \) and \( \left[ \begin{array}{c|c} k & i_1 \\ \hline \mathbf{st} & \emptyset \\ \end{array} \right] \) similarly. In the computations to follow below, the coordinates \( i, j, k \in [3] \) are all distinct.

First, observe that \( \left[ \begin{array}{c} 1 \\ \hline \mathbf{st} \end{array} \right] \) \( = \sigma_i \). We have

\[
\left[ \begin{array}{c} 2 \\ \hline \mathbf{st} \end{array} \right] \] = \( M(\left[ \begin{array}{c} 1 \\ \hline \mathbf{st} \end{array} \right], \left[ \begin{array}{c} 1 \\ \hline \mathbf{st} \end{array} \right]) \), e.g.,

\[
\left[ \begin{array}{c} 2 \\ \hline \mathbf{st} \end{array} \right] = \left( \begin{array}{c} \langle 2, 0 \rangle \\ \hline \langle 2, A \rangle \end{array} \right) = \begin{pmatrix} M(\langle 1, 0 \rangle, \langle 1, 0 \rangle) & \langle 1, 0 \rangle, \langle 1, 0 \rangle \end{pmatrix},
\]

\[
\left[ \begin{array}{c} 2 \\ \hline \mathbf{st} \end{array} \right] = \left( \begin{array}{c} \langle 2, 0 \rangle \\ \hline \langle 2, A \rangle \end{array} \right) = \begin{pmatrix} M(\langle 1, 0 \rangle, \langle 1, 0 \rangle) & \langle 1, 0 \rangle, \langle 1, 0 \rangle \end{pmatrix},
\]
corresponding to the configuration $M^1(1, 0, 0) = (2, 1, 0)$. Next,

$$\left[ 3 \mid i \mid j \mid k \right] = M \left( \left[ 2 \mid k \mid j \mid \emptyset \right], \left[ 2 \mid i \mid j \mid \emptyset \right] \right),$$

e.g.,

$$\left[ 3 \mid 1 \mid 3 \mid 2 \right] = \begin{pmatrix} (3, \bullet) \\ (3, B) \end{pmatrix} = M \begin{pmatrix} (2, 0), & (2, \bullet) \\ (2, A), & (2, A) \end{pmatrix},$$

corresponding to the configuration $M^2(1, 0, 0) = (3, 1, 1)$. Next,

$$\left[ 3 \mid i \mid \emptyset \mid j \right] = M \left( \left[ 3 \mid k \mid i \mid j \right], \left[ 3 \mid i \mid k \mid j \right] \right),$$

e.g.,

$$\left[ 3 \mid 2 \mid \emptyset \mid 1 \right] = \begin{pmatrix} (3, B) \\ (3, \bullet) \end{pmatrix} = M \begin{pmatrix} (3, A), & (3, \bullet) \\ (3, A), & (3, \bullet) \end{pmatrix},$$

corresponding to the configuration $M^3(1, 0, 0) = (3, 0, 1)$. Next,

$$\left[ 4 \mid i \mid \emptyset \mid j \right] = M' \left( \left[ 3 \mid i \mid \emptyset \mid j \right] \right),$$

e.g.,

$$\left[ 4 \mid 3 \mid \emptyset \mid 2 \right] = \begin{pmatrix} (4, 0) \\ (4, B) \end{pmatrix} = M' \begin{pmatrix} (3, 0) \\ (3, B) \end{pmatrix},$$

corresponding to the configuration $M^4(1, 0, 0) = (4, 0, 1)$. Next,

$$\left[ 4 \mid i \mid \emptyset \mid \emptyset \right] = M \left( \left[ 4 \mid j \mid \emptyset \mid i \right], \left[ 4 \mid i \mid \emptyset \mid j \right] \right),$$

e.g.,

$$\left[ 4 \mid 1 \mid \emptyset \mid \emptyset \right] = \begin{pmatrix} (4, \bullet) \\ (4, 0) \end{pmatrix} = M \begin{pmatrix} (4, B), & (4, \bullet) \\ (4, 0), & (4, 0) \end{pmatrix},$$

corresponding to the configuration $M^5(1, 0, 0) = (4, 0, 0)$. Finally, we have

$$\left[ 0 \mid i \mid \emptyset \mid \emptyset \right] = M' \left( \left[ 4 \mid i \mid \emptyset \mid \emptyset \right] \right),$$

e.g.,

$$\left[ 0 \mid 2 \mid \emptyset \mid \emptyset \right] = \begin{pmatrix} (0, 0) \\ (0, \bullet) \end{pmatrix} = M' \begin{pmatrix} (4, 0) \\ (4, 0) \end{pmatrix},$$

corresponding to the halting configuration $M^6(1, 0, 0) = (0, 0, 0)$.

Since $S$ can witness the halting of $M$, the relation $S$ will have a lot of “non-computational” vectors in $Y^3$. In general, if a relation does not witness the halting of $M$ then this will not be the case.

Now that we have some intuition for how computation is encoded, let us continue exploring the structure of the relations of $A(M)$.

**Lemma 5.8.** Let $R \leq A(M)^m$ be a relation and let $a, b, c, d \in R$.

1. If $R$ is computational non-halting then $N_0(a, b, c) \not\in Y^m$ or $N_0(a, b, c) = c$.
2. If $R$ is computational then $N_\bullet(a, b, c, d) \leq b$ or $N_\bullet(a, b, c, d) \leq c$.
3. If $R$ is synchronized then $P(a, b, c, d) \in \{c, d\}$.
4. $S(a, b, c) \leq I(a, a)$ and if $S(a, b, c) \not\in X^m$ then $S(a, b, c) \leq a$. 
\((5) \bigcap \{ \mathbb{R} | \mathbb{R} \leq \mathbb{A}(\mathcal{M})^m \text{ synchronized} \} = \{ r \in X^m | r \text{ is synchronized} \} \).

**Proof.** (1)–(4): These items follow directly from the definitions. The most complicated one is item (2), so we will leave the others to the reader. If \(b, c, d\) do not share the same state then \(N_\ast(a, b, c, d) = X(b \wedge c \wedge d) \leq b\). Assume now that \(b, c, d\) share the same state. If \(\bullet \notin \text{con}(a)\) then \(N_\ast(a, b, c, d) = b\) or \(N_\ast(a, b, c, d) = X(b \wedge c \wedge d) \leq b\), so also assume that there is \(k\) with \(a(k) \in D\) and \(a(\neq k) \in E^{m-1}\) (we use \(R \) being computational here). Hence \(N_\ast(a, b, c, d)(\neq k) \leq (b \wedge c)(\neq k)\), so we just need to show that \(N_\ast(a, b, c, d)(k)\) is less than equal to \(b\) or \(c\). The possibilities are

\[
N_\ast(a, b, c, d)(k) = \begin{cases} 
  b = c & \text{if } b = c \notin X, \\
  c & \text{if } b \in X, \\
  b & \text{if } c \notin X, \\
  b = d & \text{if } d \notin X, \\
  c = d & \text{if } c = d \notin X, \\
  X(b \wedge c \wedge d) & \text{otherwise.}
\end{cases}
\]

In all cases we have \(N_\ast(a, b, c, d)(k)\) less than equal to \(b\) or \(c\), so we are finished.

(5): Let \(r \in R\) and let \(s = H(I(r, r))\). It is not hard to see that \(s(i) = \langle 0, \times \rangle\) for all \(i\). We also have that \(z_k(s)(i) = \langle k, \times \rangle\) for all \(i\), from Lemma 5.3 item (4). The conclusion follows immediately. \(\square\)

**Definition 5.9.** Define elements \(\sigma_i \in A(\mathcal{M})^m\) for \(i \in [m]\) by

\[
\sigma_i(j) = \begin{cases} 
  \langle 1, \bullet \rangle & \text{if } i = j, \\
  \langle 1, 0 \rangle & \text{otherwise.}
\end{cases}
\]

Let \(\Sigma_m = \{\sigma_1, \ldots, \sigma_m\}\) and define the \(m\)-th sequential relation of \(\mathbb{A}(\mathcal{M})\) to be

\(S_m = S_{\mathbb{A}(\mathcal{M})^m}(\Sigma_m)\).

**Definition 5.10.** Let \(k\) be a state of \(\mathcal{M}\) and \(\alpha, \beta, m \in \mathbb{N}\) be such that \(\alpha + \beta < m\). Let \(P_m\) be the set of permutations on \([m]\) and define \(c(k, \alpha, \beta) \subseteq A(\mathcal{M})^m\) to be

\[
c(k, \alpha, \beta) = \bigcup_{p \in P_m} \left\{ p \left( \langle \langle k, \bullet \rangle, \langle k, A \rangle, \ldots, \langle k, A \rangle, \langle k, B \rangle, \ldots, \langle k, B \rangle, \langle k, 0 \rangle, \ldots, \langle k, 0 \rangle \rangle \right) \right\}
\]

(the permutation \(p\) acts on a tuple by permuting coordinates). This is the set of vectors encoding the Minsky machine configuration \((k, \alpha, \beta)\).

**Lemma 5.11.** Let \(S_m \leq \mathbb{A}(\mathcal{M})^m\) be as in Definition 5.7

(1) \(S_m\) is computational and has capacity \(m - 1\).

(2) If \(p\) is a permutation on \([m]\) then \(p(S_m) = S_m\).

(3) \(c(k, \alpha, \beta) \cap S_m \neq \emptyset\) if and only if \(c(k, \alpha, \beta) \subseteq S_m\).

(4) If \(S_m \cap C^m \neq \emptyset\) then \(S_m\) is halting.

**Proof.** (1): The generators \(\Sigma_m\) are witnesses to \(S_m\) having capacity \(m - 1\). Furthermore, \(\Sigma_m\) is synchronized, so by Lemma 5.3 item (1), \(S_m\) must be as well. Let \(s \in S_m\) be such that \(|s^{-1}(D)| \geq 2\). Examining the operations of \(A(\mathcal{M})\), we can
see that any such element must have been generated by elements of $\Sigma_m$ with more than one coordinate in $D$. $\Sigma_m$ contains no such vectors.

(2), (3): The generators $\Sigma_m$ are closed under $p$, so $S_m$ must be as well. Applying item (2) for all permutations of $[m]$ proves item (3).

(4): The generating set $\Sigma_m$ contains no vectors in $C^m$. A careful analysis of the operations of $A(M)$ shows that the least complexity term operation generating an element in $S_m \cap C^m$ from $\Sigma_m$ is of the form $H(t_1(\overline{v}))$ or $N_0(t_1(\overline{v}), t_2(\overline{v}), t_3(\overline{v}))$, where $t_1(\overline{v})(\ell) \in D$ for some $\ell$. Looking at the definitions, we can see that $t_1(\overline{v})$ is a halting vector in either of these cases. \hfill $\square$

We are now ready to prove the main result of this section, the Coding Theorem, which proves that $S_m$ encodes the computation of $M$.

**Theorem 5.12** (The Coding Theorem). Let $M$ be a Minsky machine.

1. If $\mathcal{M}^n(1, 0, 0) = (k, \alpha, \beta)$ and $M$ has $n$-step capacity $m - 1$ then $c(k, \alpha, \beta) \subseteq S_m$.

2. If $c(k, \alpha, \beta) \subseteq S_m$ and $M$ does not halt with capacity $m - 1$ then for some $n$ we have $\mathcal{M}^n(1, 0, 0) = (k, \alpha, \beta)$ and $M$ has $n$-step capacity $m - 1$.

**Proof.** For the first item, we refer the reader to Example 5.7 and Lemma 5.11.

For the second item, suppose that $c(k, \alpha, \beta) \subseteq S_m$ and $M$ does not halt with capacity $m - 1$. We will analyze the generation of $S_m = S_{\mathcal{A}(M)}(\Sigma_m)$. Let $G_0 = \Sigma_m$ and

$$G_n = \left\{ F(\overline{v}) \mid F \text{ a fundamental } \ell\text{-ary operation, } \overline{v} \in G_{n-1}^\ell \right\} \cup G_{n-1}.$$

Observe that $S_m = \bigcup G_n$, so $c(k, \alpha, \beta) \cap G_n \neq \emptyset$ for some least $n$. A key observation for what follows is that since $\Sigma_m$ is closed under coordinate permutation, so is $G_n$, so $c(k, \alpha, \beta) \cap G_n \neq \emptyset$ implies $c(k, \alpha, \beta) \subseteq G_n$. After proving the next claim, we will be done.

**Claim.** If $n$ is minimal such that $c(k, \alpha, \beta) \subseteq G_n$ then $\mathcal{M}^n(1, 0, 0) = (k, \alpha, \beta)$ and $M$ has $n$-step capacity $m - 1$.

**Proof of claim.** The proof shall be by induction on $n$. Observe that $c(1, 0, 0) = \Sigma_m = G_0$, $\mathcal{M}^0(1, 0, 0) = (1, 0, 0)$, and $M$ has $0$-step capacity $m - 1 \geq 0$. This establishes the basis of the induction.

Suppose now that $n > 0$ and let $s \in c(k, \alpha, \beta) \subseteq G_n$. This implies that $s = F(\overline{v})$ for some $\ell$-ary fundamental operation $F$ and $\overline{v} \in G_{n-1}^\ell$. We break into cases depending on which fundamental operation $F$ is.

**Case $F \in \{ \wedge, N, P \}$:** These operations have the property that if $s = F(\overline{v})$ then $s \leq g_i$ for some $g_i$ amongst the $\overline{v}$. Since $s \in Y^m$, this implies $s = g_i$, so $s \in G_{n-1}$ and hence $c(k, \alpha, \beta) \subseteq G_{n-1}$, a contradiction.

**Case $F \in \{ H, S \}$:** These operations have ranges contained entirely in $E$. Since $\bullet \in \mathsf{con}(s)$, $s$ cannot be the output of such an operation.

**Case $F = I$:** If $s = I(a, b)$ then $s \in c(1, 0, 0)$, and we are back in the base case.

**Case $F = N_0$:** If $s = N_0(a, b, c)$ then $N_0(a, b, c) \in Y^m$. If $a$ is not a halting vector then we have $s = c$ by Lemma 5.8 item (1), so $s$ and hence $c(k, \alpha, \beta)$ are contained in $G_{n-1}$, a contradiction. If $a$ is a halting vector then from the definition of $N_0$ we have that $a \in c(0, 0, 0)$, so $c(0, 0, 0) \subseteq G_{n-1}$, and by the inductive hypothesis we
have that \( \mathcal{M}^{n-1}(1,0,0) = (0,0,0) \). Hence \( \mathcal{M} \) halts in \( n-1 \) steps with capacity \( m-1 \), contradicting the hypotheses.

**Case** \( F \in \{ M, M' \} \): Let \( s = M(a,b) \). If \( a \in C^m \) then by Lemma 5.11 item (4), we have that \( G_{n-1} \) contains a halting vector. This gives rise to a contradiction as in the case for \( F = N_0 \). If \( a \not\in C^m \) then since \( s \in Y^m \) we have that \( a(\ell) \in D \) for some \( \ell \), from the definition of \( M \). Also from the definition, there is some instruction \( (i, R, k) \in \mathcal{M} \) or \( (i, R, j, k) \in \mathcal{M} \) such that \( a, b \in c(i, \alpha + \varepsilon, \beta + \tau) \) where the different possibilities for \( (\varepsilon, \tau) \) correspond to the different possibilities for the instruction. In any case, by the inductive hypothesis we have that \( \mathcal{M}^{n-1}(1,0,0) = (i, \alpha + \varepsilon, \beta + \tau) \) and \( \mathcal{M} \) has \( (n-1) \)-step capacity \( m-1 \). We therefore have

\[ \mathcal{M}^{n}(1,0,0) = \mathcal{M}(i, \alpha + \varepsilon, \beta + \tau) = (k, \alpha, \beta). \]

Since \( \alpha + \beta \leq m-1 \) and \( \mathcal{M} \) has \( (n-1) \)-step capacity \( m-1 \), it follows that \( \mathcal{M} \) has \( n \)-step capacity \( m-1 \). The analysis for \( M' \) is similar.

**Corollary 5.13.** The following are equivalent.

1. \( \mathcal{M} \) halts with capacity \( m-1 \),
2. \( S_m \) is halting,
3. every computational \( \mathbb{R} \leq A(\mathcal{M})^f \) with capacity \( m-1 \) is halting.

**Proof.** We begin by proving the equivalence of the first two items. Suppose that \( \mathcal{M} \) halts with capacity \( m-1 \). By Theorem 5.12 this implies that \( c(0,0,0) \subseteq S_m \) (recall that we assumed in Section 2 that \( \mathcal{M} \) would zero the registers before halting). Any element of \( c(0,0,0) \) is a halting vector, so \( S_m \) is halting. For the converse, suppose that \( S_m \) has a halting vector \( s \). It follows that \( s \in c(0,0,0) \) and hence, by Lemma 5.11 item (3), that \( c(0,0,0) \subseteq S_m \). Towards a contradiction assume that \( \mathcal{M} \) does not halt with capacity \( m-1 \). By Theorem 5.12 for some \( n \) we have \( \mathcal{M}^n(1,0,0) = (0,0,0) \) and \( \mathcal{M} \) has \( n \)-step capacity \( m-1 \). This is a contradiction.

We next prove the equivalence of items (2) and (3). Suppose that \( S_m \) is halting and that \( \mathbb{R} \leq A(\mathcal{M})^f \) has capacity \( m \) witnessed by \( (\sigma'_i)_{i \in \mathcal{I}} \) with \( |\mathcal{I}| = m \), say \( \sigma'_i(i) \in D \) and \( \sigma'_i(\neq i) \in C^{f-1} \). Let \( \sigma_i = I(\sigma'_i, \sigma'_i) \) and observe that \( (\sigma_i)_{i \in \mathcal{I}} \) satisfies Equation (5.1) from the definition of \( S_m \):

\[ \sigma_i(j) = \begin{cases} (1, \bullet) & \text{if } i = j, \\ (1, 0) & \text{otherwise}. \end{cases} \]

(5.1 redux)

Let \( S = S_{\mathcal{A}(\mathcal{M})^f}(\{ \sigma_i \mid i \in \mathcal{I} \}) \). We have that \( S(\mathcal{I}) = S_m \) and if \( s \in S \cap Y^f \) then \( \text{con}(s(i)) = 0 \) for all \( i \in [f] \setminus [m] \). Combining these yields a halting vector for \( S \). We have that \( S \leq \mathbb{R} \), so \( \mathbb{R} \) must also be halting. The converse is clear since \( S_m \) has capacity \( m-1 \), by Lemma 5.11 item (1).

If \( \mathcal{M} \) halts then \( S_m \) is halting for some \( m \). Projecting on a single coordinate, it follows that

\[ T = S_{\mathcal{A}(\mathcal{M})^f}(\{ (1, \bullet), (1,0) \} \]

is also halting (i.e. \( (0, \bullet) \in T \)). Independent of the halting status of \( \mathcal{M} \), let us consider this relation. Whether or not \( T \) is halting is a decidable property. If \( T \) is non-halting then it is not possible for \( \mathcal{M} \) to halt (the converse does not hold, of course). We therefore assume from this point onward that \( \mathcal{M} \) is such that \( T \) is halting.
6. **If M Does Not Halt**

Recall from Section 3 that Rel(A(M)) is the set of all finitary relations of A(M). If \( R \subseteq \text{Rel}(A(M)) \) is a set of relations then we have \( R \models R \) if and only if \( R \) can be obtained from relations in \( R \cup \{=\} \), in finitely many steps, by applying the following constructions:

1. intersection of equal arity relations,
2. (cartesian) product of finitely many relations,
3. permutation of the coordinates of a relation, and
4. projection of a relation onto a subset of coordinates.

A close analysis of various projections of relations is called for, so we remind the reader of the convention for projections adopted in Section 3: for \( m \in \mathbb{N} \) and \( I \subseteq [m] \),

- denote the projection of \( a \in A(M)^m \) to coordinates \( I \) by \( a(I) \in A(M)^I \),
- denote the projection of \( S \subseteq A(M)^m \) to coordinates \( I \) by \( S(I) \subseteq A(M)^I \), and
- define \( a(\neq i) = a([m] \setminus i) \) and likewise \( a(\neq i, j) = a([m] \setminus \{i, j\}) \).

Finally, for \( n \in \mathbb{N} \) we define \( \text{Rel} \leq n(A(M)) \) to be the set of at most \( n \)-ary relations of \( A(M) \).

The next theorem shows that the relations built using the entailment constructions above must have a certain form. This theorem is essentially Theorem 3.3 from Zadori [39] and we refer the interested reader to that paper for the proof.

**Theorem 6.1.** Let \( A \) be an algebra and let \( R \) be a set of relations on \( A \). Then \( R \models S \) if and only if

\[
S = \pi \left( \bigcap_{i \in I} \mu_i \left( \prod_{j \in J_i} R_{ij} \right) \right)
\]

for some finite index sets \( I \) and \( (J_i)_{i \in I} \), where the \( R_{ij} \in R \cup \{=\} \), \( \pi \) is a coordinate projection, and the \( \mu_i \) are coordinate permutations.

We now take a close look at relations of this form.

**Lemma 6.2.** Suppose that

\[
\sigma_1, \ldots, \sigma_m \in \pi \left( \bigcap_{i \in I} \mu_i \left( \prod_{j \in J_i} R_{ij} \right) \right) = S \leq A(M)^m,
\]

where \( \sigma_1, \ldots, \sigma_m \) are the generators of \( S_m \) (Definition 3,9), \( \pi \) is a projection, the \( \mu_i \) are permutations, the \( R_{ij} \) are a finite collection of members of \( \text{Rel} \leq n(A(M)) \), and \( n < m \). Then \( S \cap C^m \neq \emptyset \).

**Proof.** We begin by establishing some notation. Let

\[
\mathbb{E} = \bigcap_{i \in I} \mu_i \left( \prod_{j \in J_i} R_{ij} \right) \leq A(M)^M.
\]

Without loss of generality assume that \( P = [m] \) is the set of coordinates that \( \pi \) projections onto and let \( Q = [M] \setminus P = \{m + 1, \ldots, M\} \). Define \( K_{ij} \subseteq [M] \) to be the coordinates of \( R_{ij} \) in the permuted product \( \mu_i \left( \prod_{j \in J_i} R_{ij} \right) \). Let \( K^P_{ij} = K_{ij} \cap P \) and \( K^Q_{ij} = K_{ij} \cap Q \). Observe that
\begin{itemize}
\item $|M| = P \sqcup Q$ (the disjoint union),
\item $|K_{ij}| \leq n < m$ and $K_{ij} = K_{ij}^P \sqcup K_{ij}^Q$, and
\item for every $i \in I$ we have $P = \bigsqcup_{i \in J_i} K_{ij}^P$ and $Q = \bigsqcup_{i \in J_i} K_{ij}^Q$.
\end{itemize}

We have $\sigma_1, \ldots, \sigma_m \in S$, and since $B(P) = S$ there must be elements $\tau_1, \ldots, \tau_m \in B$ such that $\tau_\ell(P) = \sigma_\ell$ for all $\ell$. Take each $\tau_\ell$ to be minimal (under the semilattice order) with this property and such that $\text{st}(\tau_\ell) = 1$ (such $\tau_\ell$ exist — just use operation $I$).

\textbf{Claim.} Let $q \in Q$. Either
\begin{enumerate}
\item $\tau_\ell(q) = \tau_k(q) \in \{(1,0), (1,\times)\}$ for all $\ell, k \in [m]$ or
\item there is a unique $\ell \in [m]$ such that $\tau_\ell(q) = (1,\bullet)$ and for all $k \in [m] \setminus \{\ell\}$, $\tau_k(q) = (1,0)$.
\end{enumerate}

\textbf{Proof of claim.} Observe that if $\text{con}(\tau_\ell(q)) \in \{A,B\}$ then the element
\[ \tau_\ell \land I(\tau_\ell, \tau_\ell) \]
will be properly less than $\tau_\ell$ while still having projection on $P$ to $\sigma_\ell$, contradicting the minimality of $\tau_\ell$. Hence it must be that $\text{con}(\tau_\ell(q)) \in \{\bullet, 0, \times\}$.

Suppose that $\tau_\ell(q) \neq \tau_k(q)$ and assume without loss of generality that $\tau_\ell(q) \notin X$. Let
\[ t_\ell' = M(\tau_k, \tau_\ell) \quad \text{and} \quad t_\ell = I(t_\ell', t_\ell'). \]
The machine $M$ begins with an instruction of the form $(1,R,s)$. From the definition of $M$ and $I$ it therefore follows that $t_\ell(P) = \sigma_\ell$. We have assumed that $\tau_\ell(q) \neq \tau_k(q)$, so it also follows that either $\{\text{con}(\tau_k(q)), \text{con}(\tau_\ell(q))\} = \{\bullet, 0\}$ or $t_\ell(q) \notin X$. The first of these two possibilities is item (2) from the claim. In the second possibility where $t_\ell(q) \in X$, we have $t_\ell < \tau_\ell$, contradicting the minimality of $\tau_\ell$.

From the above claim, we can partition $Q$ into two pieces,
\[ Q_\cap = \{q \in Q \mid \text{claim 1 item (1) holds}\}, \quad Q_\cup = \{q \in Q \mid \text{claim 1 item (2) holds}\}. \]

Let $K_{ij}^{Q_\cup} = K_{ij} \cap Q_\cup$. Fix an $i \in I$ and for each $j \in J_i$ choose an $\ell_j \in [m] \setminus K_{ij}^P$ (such $\ell_j$ exist for all $j$ since $|K_{ij}| < m$). Let $L = \{\ell_j \mid j \in J_i\}$ be the set of these choices. Using projections of the $\tau_j$ on $K_{ij}$, there must be an element $\alpha_i^L \in \mu_i(\prod_{j \in J_i} R_{ij})$ such that
\begin{itemize}
\item $\alpha_i^L(K_{ij}^Q) = \tau_{\ell_j}(K_{ij}^Q)$ and
\item for $p \in P$ we have $\alpha_i^L(p) = \tau_{\ell_j}(p) = \sigma_{\ell_j}(p) = (1,0)$ for the unique $j$ such that $p \in K_{ij}^P$.
\end{itemize}

It follows from this that $\alpha_i^L(P) \in C^m$, so it suffices to show that there is some system of choices $(L_i)_{i \in I}$ such that for all $i,i' \in I$ we have $\alpha_i^{L_i} = \alpha_{i'}^{L_i'}$. That is, the element $\alpha_i^{L_i}$ does not depend on $i$ and thus lies in the intersection $\bigcap_{i \in I} \mu_i(\prod_{j \in J_i} R_{ij})$.

\textbf{Claim.} Fix an $i \in I$. For all $q \in Q_\cap$, all choices of $L$ as above, and all $\ell \in [m]$, we have $\alpha_i^L(q) = \tau_\ell(q) \in \{(1,0), (1,\times)\}$.
Proof of claim. For each \( q \in Q_- \) there is a unique \( j \) such that \( q \in K_{ij}^{Q_-} \), so by the construction of \( \alpha^L_i \) we have \( \alpha^L_i(q) = \tau_{\ell_j}(q) \) for some unique \( \ell_j \in L \). Since \( q \in Q_- \), by the first claim the conclusion follows.

For each \( q \in Q_\neq \), by the first claim there is a unique \( \ell_q \) such that \( \tau_{\ell_q}(q) = (1, \bullet) \). It follows that \( \mathbb{Q}_\neq \) can be partitioned,

\[
\mathbb{Q}_\neq = \bigsqcup_{\ell \in [m]} \mathbb{Q}_\neq^\ell \quad \text{where} \quad \mathbb{Q}_\neq^\ell = \{ q \in \mathbb{Q}_\neq \mid \tau_{\ell}(q) = (1, \bullet) \}.
\]

Claim. For all \( \ell \in [m] \) and all \( j \in J_i \) such that \( Q_{\neq}^\ell \cap K_{ij} \neq \emptyset \) there exists \( k \in \mathbb{N} \) \( \setminus K_{ij} \) such that

\[
\operatorname{con}(\tau_k(K_{ij}^P \cup K_{ij}^{Q_{\neq}})) = \{0\}.
\]

Proof of claim. Let \( K = [m] \setminus K_{ij}^P \) and observe that for every \( k \in K \) we have \( \operatorname{con}(\tau_k(K_{ij}^P)) = \{0\} \) and \( \operatorname{con}(\tau_k(K_{ij}^{Q_{\neq}})) \subseteq \{\bullet, 0\} \) by the previous claims. As mentioned before the statement of the claim, for each \( q \in \mathbb{Q}_\neq \) there is a unique \( \ell_q \) such that \( \tau_{\ell_q}(q) = (1, \bullet) \) and \( \tau_{\ell_q}(q) = (1, 0) \) for all \( \ell \neq \ell_q \). Towards a contradiction, let us assume that for all \( \ell \in K \) we have \( \bullet \in \operatorname{con}(\tau_k(K_{ij}^{Q_{\neq}})) \). It follows that

\[
K = \{ \ell_q \in [m] \mid q \in K_{ij}^{Q_{\neq}} \}.
\]

We therefore have \( |K| = m - |K_{ij}^P| \) (from the start of the proof of the claim) and \( |K| \leq |K_{ij}^{Q_{\neq}}| \). Hence \( m - |K_{ij}^P| \leq |K_{ij}^{Q_{\neq}}| \), so

\[
m \leq |K_{ij}^P| + |K_{ij}^{Q_{\neq}}| \leq |K_{ij}^P| + |K_{ij}^{Q_{\neq}}| = |K_{ij}| \leq n < m,
\]

a contradiction.

Consider \( \alpha^L_i \) for some fixed \( i \in I \) and fixed \( L \). Suppose that for some \( \ell_h \in L \) we have \( \bullet \in \operatorname{con}(\alpha^L_i(K_{ij}^{Q_{\neq}})) \). The set \( Q_{\neq}^\ell \) has a covering

\[
Q_{\neq}^\ell \subseteq \bigsqcup_{j \in J_i} K_{ij}^{Q_{\neq}} \quad \text{where} \quad J_i = \{ j \in J_i \mid Q_{\neq}^\ell \cap K_{ij}^{Q_{\neq}} \neq \emptyset \}.
\]

For each \( K_{ij}^{Q_{\neq}} \) in this covering, replace \( \ell_j \) in \( L \) with some \( k_j \) satisfying the conclusion of the previous claim. \( \ell_h \) will be replaced in this process, along with possibly others. Repeat this procedure on the newly obtained \( \alpha^L_i \) until \( \bullet \not\in \operatorname{con}(\alpha^L_i(Q_{\neq})) \) and call the final result \( \alpha_i \). For a fixed \( i \), we have constructed an element \( \alpha_i \) such that

\[
\bullet \alpha_i(p) = (1, 0) \quad \text{for all} \quad p \in P,
\]

\[
\alpha_i(Q_-) = \tau_1(Q_-) = \cdots = \tau_m(Q_-), \quad \text{and}
\]

\[
\alpha_i(q) = (1, 0) \quad \text{for all} \quad q \in Q_{\neq}.
\]

The description of \( \alpha_i \) above does not depend on \( i \), so \( \alpha_i \) is a common element in the intersection \( \bigcap_{i \in I} \mu_i(\prod_{j \in J_i} \mathbb{R}_{ij}) \). It follows that \( \alpha_i(P) \in S \cap C_m \).

\[ \square \]

Theorem 6.3. The following hold for any Minsky machine \( \mathcal{M} \).

1. If \( \mathcal{M} \) does not halt with capacity \( m \) then \( m < \deg(\mathcal{A}(\mathcal{M})) \).
2. If \( \mathcal{M} \) does not halt then \( \mathcal{A}(\mathcal{M}) \) is not finitely related.
Proof. For item (1), suppose that \( \deg(\mathcal{A}(\mathcal{M})) \leq m \). This implies in particular that
\[
\text{Rel}_{\leq m}(\mathcal{A}(\mathcal{M})) \models \Sigma_{m+1}.
\]
By Theorem 6.1 there is some projection \( \pi \), permutations \( \mu_i \), and a finite collection of relations \( R_{ij} \in \text{Rel}_{\leq m}(\mathcal{A}(\mathcal{M})) \) such that
\[
\Sigma_{m+1} = \pi \left( \bigcap_{i \in I} \mu_i \left( \prod_{j \in J_i} R_{ij} \right) \right).
\]
By Lemma 6.2 this implies that \( \Sigma_{m+1} \cap C_{m+1} \neq \emptyset \), and by Lemma 6.11 and Corollary 5.13 this implies that \( \mathcal{M} \) halts with capacity \( m \), a contradiction. Item (2) follows from item (1).

7. If \( \mathcal{M} \) Halts — Tools

The argument showing that \( \mathcal{A}(\mathcal{M}) \) is finitely related when \( \mathcal{M} \) halts is quite long and intricate. This section develops the tools necessary. Throughout this section and the next (Section 8), we assume that \( \mathcal{M} \) halts with capacity \( \kappa \). We begin by highlighting some important relations of \( \mathcal{A}(\mathcal{M}) \).

The strategy for the main proof is to show that for some suitably chosen \( k \), we have \( \text{Rel}_{\leq k}(\mathcal{A}(\mathcal{M})) \models \text{Rel}_{\leq n}(\mathcal{A}(\mathcal{M})) \) for all \( n \). We therefore consider an arbitrary \( m \)-ary operation \( f \) which preserves \( \text{Rel}_{\leq k}(\mathcal{A}(\mathcal{M})) \), arbitrary \( \mathcal{R} \leq \mathcal{A}(\mathcal{M})^m \), and arbitrary \( r_1, \ldots, r_m \in R \) and endeavor to show that \( f(r_1, \ldots, r_m) \in R \). The relations which we define below will play an important role in analyzing the behavior of \( f \) on \( R \), and following each definition we attempt to give the reader some intuition for how they can be used.

Definition 7.1. Let
\[
\mu = \left\{ \begin{pmatrix} a \\ a \end{pmatrix}, \begin{pmatrix} a \\ X(a) \end{pmatrix} \mid a \in E \right\} \subseteq A(\mathcal{M})^2,
\]
\[
\chi = \left\{ \begin{pmatrix} a_1 \\ a_2 \\ a_2 \end{pmatrix}, \begin{pmatrix} X(a_1) \\ a_2 \\ X(a_2) \end{pmatrix} \mid (a_1, a_2) \in E^2 \text{ synchronized} \right\} \subseteq A(\mathcal{M})^3.
\]
Operations which preserve \( \mu \) are monotone on \( E \) (see Lemma 7.6). The property that \( \chi \) describes is more subtle. Let \( f \) be an operation and consider an evaluation of the form
\[
f \begin{pmatrix} a_1 \\ b_1 \\ \cdots \\ b_k \\ \cdots \\ b_m \\ a_m \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}
\]
where each \( (a_i, b_i) \in E^2 \) is synchronized and \( \alpha_1 \notin X \). If \( f \) preserves \( \mu \) and \( \chi \) then we can conclude that replacing \( b_k \) with \( X(b_k) \) in the second line of input does not change the output of \( f \):
\[
f \begin{pmatrix} a_1 \\ b_1 \\ \cdots \\ b_k \\ X(b_k) \\ \cdots \\ b_m \\ a_m \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}
\]
(the input vectors are elements of \( \chi \), so the output is in \( \chi \) as well). The details of this are contained in Lemma 8.8.
Definition 7.2. Define three subsets of $A(M)^3$,

\[ \Delta_\emptyset = \left\{ \begin{array}{c} (z, a, b), (z, a, b), (a, a, b), (a, a, b), (b, b, b) \\ (z, z, z), (z, z, z), (z, z, z), (z, z, z), (z, z, z) \end{array} \right\}, \]

\[ \Delta_{\exists A} = \left\{ \begin{array}{c} (z, a, b), (z, a, b), (a, a, b), (a, a, b), (b, b, b) \\ (z, z, z), (z, z, z), (z, z, z), (z, z, z), (z, z, z) \end{array} \right\}, \]

\[ \Delta_{\exists B} = \left\{ \begin{array}{c} (z, a, b), (z, a, b), (a, a, b), (a, a, b), (b, b, b) \\ (z, z, z), (z, z, z), (z, z, z), (z, z, z), (z, z, z) \end{array} \right\}. \]

As an example of how $\Delta_{\exists A}$ can be used, consider an evaluation of an operation $f$,

\[ f \left( \begin{array}{c} (i, A), (i, B), (i, 0), (i, a), (i, a), (i, A) \\ (i, 0), (i, B), (i, A), (i, A) \end{array} \right) = \left( \begin{array}{c} (j, A) \\ (j, A) \end{array} \right). \]

We can add a row to this evaluation in such a way that the input vectors are in $\Delta_{\exists A}$, and if $f$ preserves $\Delta_{\exists A}$ then the output will be in $\Delta_{\exists A}$ and therefore equal to $(j, A)$:

\[ f \left( \begin{array}{c} (i, A), (i, B), (i, 0), (i, B), (i, a), (i, a), (i, A) \\ (i, 0), (i, B), (i, A), (i, A) \end{array} \right) = \left( \begin{array}{c} (j, A) \\ (j, A) \end{array} \right). \]

Let us call this new third row the “added row for $\Delta_{\exists A}$”. Similar manipulations can be performed using $\Delta_\emptyset$ and $\Delta_{\exists B}$. Doing this for the 2-line evaluation at the start and writing just the “added” rows, we obtain

\[ f \left( \begin{array}{c} (i, 0), (i, B), (i, 0), (i, B), (i, 0), (i, A) \\ (i, A), (i, B), (i, A), (i, A), (i, A), (i, A) \end{array} \right) = \left( \begin{array}{c} (j, A) \\ (j, A) \end{array} \right). \]

The first row is the added row for $\Delta_\emptyset$, the second for $\Delta_{\exists A}$, and the third for $\Delta_{\exists B}$. The subpower $\Gamma$ defined next can be used to further manipulate the input. This technique is discussed in detail in the proof of Theorem 8.10.
Definition 7.3. Define a subset of $A(M)^4$,

$$
\Gamma = \left\{ \begin{pmatrix}
    z \\
    a \\
    b \\
    \alpha
\end{pmatrix}, \begin{pmatrix}
    a \\
    b \\
    z \\
    \beta
\end{pmatrix}, \begin{pmatrix}
    b \\
    z \\
    a \\
    \gamma
\end{pmatrix}, \begin{pmatrix}
    c_1 \\
    c_2 \\
    c_3 \\
    c_1 \cap c_2 \cap c_3
\end{pmatrix} \right\},
$$

$$
\left\{ \begin{pmatrix}
    (a, b, z, c_1, c_2, c_3) \in E^6 \text{ synchronized,} \\
    \alpha \in \{z, a\}, \beta \in \{z, b\}, \gamma \in \{z, a, b\}
\end{pmatrix} \right\}.
$$

As an example of how $\Gamma$ can be used, consider the “added row” evaluation that we ended the discussion of the $\Delta_\psi, \Delta_{3A}, \Delta_{3B}$ relations with:

$$
f\left( \begin{array}{cccc}
    (i, 0), & (i, B), & (i, 0), & (i, A) \\
    (i, A), & (i, B), & (i, 0), & (i, A), & (i, A) \\
    (i, 0), & (i, B), & (i, B), & (i, A), & (i, A)
\end{array} \right) = \left( \begin{array}{c}
    (j, A) \\
    (j, A) \\
    (j, A)
\end{array} \right).
$$

If $f$ preserves $\Gamma$ then a row can be added to this evaluation so that the input vectors will be in $\Gamma$ and the output will remain unchanged:

$$
f\left( \begin{array}{cccc}
    (i, 0), & (i, B), & (i, 0), & (i, A) \\
    (i, A), & (i, B), & (i, 0), & (i, A), & (i, A) \\
    (i, 0), & (i, B), & (i, B), & (i, A), & (i, A) \\
    (i, \alpha), & (i, \beta), & (i, \gamma), & (i, A), & (i, A)
\end{array} \right) = \left( \begin{array}{c}
    (j, A) \\
    (j, A) \\
    (j, A) \\
    (j, A)
\end{array} \right),
$$

where $\alpha \in \{0, A\}$, $\beta \in \{0, B\}$, and $\gamma \in \{0, A, B\}$. Note that different choices of $\alpha, \beta, \gamma$ result in the first three rows of the original evaluation of $f$ in Definition 7.2.

As a result, if $f$ preserves $\Gamma$ then the behavior of $f$ on the three rows above determines the behavior of $f$ on many other rows. This technique is discussed in detail in the proof of Theorem 8.10.

Lemma 7.4. The subpowers $\mu, \chi, \Delta_\psi, \Delta_{3A}, \Delta_{3B}$ of Definitions 7.1 and 7.2 are relations of $A(M)$.

Proof. It is a straightforward (though tedious) procedure to verify that these are all relations. We will sketch the proof for $\Delta_{3A}$ and leave the others to the reader.

It suffices to show that if $F$ is an $\ell$-ary fundamental operation and $g_1, \ldots, g_\ell \in \Delta_{3A}$ then

$$
\alpha = F(g_1, \ldots, g_\ell) \in \Delta_{3A}.
$$

There are a few observations that we can make.

- $\Delta_{3A} \subseteq E^3$ (i.e. $\Delta_{3A}$ has no elements with content $\bullet$). This simplifies the definitions of many of the operations of $A(M)$.
- If $d \in \Delta_{3A}$ has $d(1, 2) \in Y^2$ then $d \in Y^3$.
- If $\times \in \{\text{con}(\alpha(1)), \text{con}(\alpha(2))\}$ then $\alpha \in \Delta_{3A}$ since the elements $c_1$ and $c_2$ are unconstrained. Hence, we may assume that $\alpha(1, 2) \in Y^2$.
- If $d \in \Delta_{3A} \cap Y^3$ then $d(1, 2)$ uniquely determines $d(3)$. 

The proof can be done by cases depending on which operation $F$ is. All of these cases are quite straightforward using these observations. □

**Lemma 7.5.** The subpower $\Gamma$ of Definition 7.3 is closed under all operations of $\mathcal{A}(\mathcal{M})$ except for $I$.

**Proof.** As in the previous lemma, the proof is straightforward after making a few observations. We will therefore provide only a sketch of it. It is sufficient to show that if $F$ is an $\ell$-ary fundamental operation and $g_1, \ldots, g_\ell \in \Gamma$ then
\[ \alpha = F(g_1, \ldots, g_\ell) \in \Gamma. \]
There are a few observations that we can make.

- $\Gamma \subseteq E^4$ (i.e. $\Gamma$ has no elements with content ●). This simplifies the definitions of many of the operations of $\mathcal{A}(\mathcal{M})$.
- If $d \in \Gamma$ has $d(4) \in X$ and $d(1, 2, 3) \in Y^3$ then
  \[ |\{ \text{con}(d(i)) \mid i \in \{1, 2, 3\} \}| \geq 2. \]
  In particular, if $d(1) = d(2) = d(3) \in Y$ then $d(4) = d(1)$.
- If $d \in \Gamma \cap Y^4$ then $d(4) \in \{d(1), d(2), d(3)\}$.

The proof can be done by cases depending on which operation $F$ is. All of these cases are straightforward using these observations. □

**Lemma 7.6.** Assume that there is $\ell$ such that

- $\text{Rel}_{\leq 2}(\mathcal{A}(\mathcal{M})) \models f$ and $f$ is $n$-ary,
- $G = \{g_1, \ldots, g_n\} \subseteq E$ and $g_\ell \in C$,
- $f(g_1, \ldots, g_\ell, \ldots, g_n) = \alpha \in Y$, and
- $f(g_1, \ldots, x(g_\ell), \ldots, g_n) \in Y$.

Then $f(g_1, \ldots, x(g_\ell), \ldots, g_n) = \alpha$.

**Proof.** The function $f$ respects binary relations, so in particular it respects $\mu$ from Definition 7.1. Consider
\[ f \left( \left( \begin{array}{c} g_1 \\ g_1 \\ \vdots \\ g_\ell \\ x(g_\ell) \\ \vdots \\ g_n \\ g_n \end{array} \right) \right) = \left( \begin{array}{c} \alpha \\ \beta \end{array} \right). \]
The hypotheses on $G$ mean that all the argument vectors are in $\mu$, so the output must be as well. By hypothesis $\beta \not\in X$, so the only possibility for $(\alpha, \beta) \in \mu$ is if $\beta = \alpha$, as claimed. □

We next analyze some metrics which can be defined on relations. A major component of the argument in Section 8 is proving that entailment by lower arity relations is guaranteed when these metrics are small or large enough.

**Definition 7.7.** Let $\mathcal{R} \leq \mathcal{A}(\mathcal{M})^m$ be computational and define
\begin{align*}
\mathcal{D}(\mathcal{R}) &= \{ i \in [m] \mid \mathcal{R}(i) \cap D \neq \emptyset \}, \\
\mathcal{H}(\mathcal{R}) &= \{ i \in [m] \mid \mathcal{R}(\neq i) \text{ is halting} \}.
\end{align*}
We call $\mathcal{D}(\mathcal{R})$ the dot part of $\mathcal{R}$ and $\mathcal{H}(\mathcal{R})$ the approximately halting part of $\mathcal{R}$. When the relation is clear, we will sometimes use $\mathcal{D}$ for $\mathcal{D}(\mathcal{R})$ and $\mathcal{H}$ for $\mathcal{H}(\mathcal{R})$.

**Lemma 7.8.** Let $\mathcal{R} \leq \mathcal{A}(\mathcal{M})^m$ be computational.
Let \( \sigma_i(j) = \begin{cases} 
1, \bullet & \text{if } i = j, \\
1, 0 & \text{otherwise.} 
\end{cases} \) \((\ref{eq:theta})\)

(1) Let \( I = \mathcal{H}(\mathbb{R}) \cap D(\mathbb{R}) \). There are vectors \((\sigma_i)_{i \in I}\) in \( R \) satisfying Equation \((\ref{eq:theta})\).

(2) If \( D(\mathbb{R}) \neq \emptyset \) then \( R \) is halting if and only if \( R \cap C^m \neq \emptyset \).

(3) \( R \) has capacity \( |D(\mathbb{R}) \cap \mathcal{H}(\mathbb{R})| - 1 \) and this is the largest capacity it has.

(4) If \( R \) is non-halting then \( |D(\mathbb{R}) \cap \mathcal{H}(\mathbb{R})| \leq \kappa \).

(5) If \( R \cap C^m \neq \emptyset \) and \( R \) has weak capacity \( k \) then \( R \) has capacity \( k \).

Proof. (1): Let \( \tau_i' \in R \) be such that \( \tau_i' \neq i \) is a halting vector and let \( \sigma_i' \in R \) be such that \( \sigma_i'(i) \in D \). Define \( \sigma_i = I(\sigma_i', H(\tau_i')) \). It is easy to check that \( \sigma_i \) satisfies Equation \((\ref{eq:theta})\).

(2): If \( R \) is halting then there is some vector \( r \in R \) such that \( r(i) = \langle 0, \bullet \rangle \) and \( r(\neq i) \in \{(0, 0)\}^{m-1} \). It follows that \( H(r) = \{(0, 0), \ldots, (0, 0)\} \in C^m \). For the other direction, if \( c' \in R \cap C^m \) then let \( c = I(c', c') = \{(1, 0), \ldots, (1, 0)\} \). Since \( D(\mathbb{R}) \neq \emptyset \), \( R \) has non-negative weak capacity (see Definition \((\ref{eq:theta})\)). Let \( \sigma' \) be a witness to \( R \) having weak capacity 0, say \( \sigma'(i) \in D \). Let \( \sigma = I(\sigma', c) \) so that \( \sigma(i) = \langle 1, \bullet \rangle \) and \( \sigma(\neq i) = \{(1, 0), \ldots, (1, 0)\} \). We assumed at the end of Section \((\ref{eq:theta})\) that \( T = S_{\mathcal{H}(\mathcal{M})} \{ \{(1, 0), \{1, \bullet\}\} \} \) was halting, so \( \mathbb{R}(i) \), containing this subalgebra, must halt. This means that there is a term \( t \) in the operations \( \{M, M'\} \) such that \( t(\sigma, c)(i) = \langle 0, \bullet \rangle \). From the definitions of \( \sigma \) and \( c \) and by Lemma \((\ref{eq:theta})\) item (5), this implies

\[ t(\sigma, c)(j) = t\left( \langle 1, 0 \rangle, \langle 1, 0 \rangle \right) = \langle 0, 0 \rangle \]

for all \( j \neq i \). Hence \( t(\sigma, c)(i) = \langle 0, \bullet \rangle \) and \( t(\sigma, c)(\neq i) \in \{(0, 0)\}^{m-1} \), so \( t(\sigma, c) \) is a halting vector and \( R \) is therefore halting.

(3): Item (1) implies that \( R \) has capacity \( |D(\mathbb{R}) \cap \mathcal{H}(\mathbb{R})| - 1 \) (the \( \sigma_i \) are witnesses). Suppose now that we have a vector \( r \in R \cap Y^m \) such that \( r(j) \in D \). It follows that \( j \in D(\mathbb{R}) \) and that \( r(\neq j) \in C^m \). By item (2) we have that \( R(\neq j) \) is halting and thus \( j \in H(\mathbb{R}) \). Therefore \( j \in D(\mathbb{R}) \cap H(\mathbb{R}) \).

(4): This follows from item (3) (recall that \( \mathcal{M} \) halts with capacity \( \kappa \)).

(5): Let \( c \in R \cap C^m \) and let \( \tau_i \) be a witness to \( R \) having weak capacity 0, say \( \tau_i(i) \in D \). Define \( \sigma_i = I(\tau_i, c) \) and observe that \( \sigma_i \in Y^m \) satisfies equation \((\ref{eq:theta})\).

Doing this for all \( k \) witnesses of \( R \)'s weak capacity yields witnesses to \( R \) having capacity \( k \).

The set \( \Gamma \) from Definition \((\ref{eq:theta})\) will play an important role in the argument for entailment. Since \( \Gamma \) is closed under all operations except for \( I \) by Lemma \((\ref{eq:theta})\) it will be necessary to understand a bit about how \( I \) can interact with the other operations. The next lemma and proposition are our first steps in this direction.

**Definition 7.9.** Let \( \mathbb{R} \leq \mathcal{A}(\mathcal{M})^m \). Define \( \mathbb{R}_I = S_{\mathcal{H}(\mathcal{M})} \{ I(R \cap Y^m, R \cap Y^m) \} \).

**Lemma 7.10.** Let \( \mathbb{R} \leq \mathcal{A}(\mathcal{M})^m \) be computational.

1. If \( p \) is a permutation on \([m]\) which restricts to a permutation on \( K = \{ i \mid \exists r \in R \cap Y^m \text{ such that } r(i) \in D \} \)

then \( p(R_I) = R_I \).
follows that  

For the reverse inclusion, suppose that  

\[ \text{proof.} \quad (1): \quad \text{Let } a, b \in R \cap Y^m. \]  

From the definition of  

\[ I(a, b)(j) = \begin{cases} \langle 1, \bullet \rangle & \text{if } a(j) \in D, \\ \langle 1, \times \rangle & \text{if } a(j) \neq b(j) \in D, \\ \langle 1, 0 \rangle & \text{otherwise.} \end{cases} \]  

This is a typical element of  

\[ I(R \cap Y^m, R \cap Y^m). \]  

Observe that the first two cases in the equation imply  

\[ j \in K. \]  

For all pairs  

\[ i, j \in K, \]  

choose  

\[ r_i, r_j \in R \cap Y^m \]  

such that  

\[ r_i(i), r_j(j) \in D \]  

and define elements  

\[ s_{ij} = I(r_i, r_j). \]  

From the description of elements of  

\[ I(R \cap Y^m, R \cap Y^m), \]  

we have  

\[ \{ s_{ij} \mid i, j \in K \} = I(R \cap Y^m, R \cap Y^m). \]  

The set on the left is closed under the permutation  

\[ p, \]  

so  

\[ I(R \cap Y^m, R \cap Y^m) \]  

must be as well. There are the generators of  

\[ R_i, \]  

so the conclusions follows.

(2): Since  

\[ R_i \leq R \text{ and } D(R_i) \subseteq D(R), \]  

if  

\[ R_i \text{ is halting then so is } R. \]  

Conversely, if  

\[ R \text{ is halting then } R \cap C^m \neq \emptyset \]  

by Lemma 7.8 item (5). Pick any  

\[ r \in R \cap C^m. \]  

It follows that  

\[ I(r, r) \in I(R \cap Y^m, R \cap Y^m), \]  

so  

\[ R_i \cap C^m \neq \emptyset. \]  

Therefore  

\[ R_i \text{ is halting.} \]  

(3): Suppose that  

\[ i \in D(R_i). \]  

Since  

\[ R_i \leq A(M)^m, \]  

it follows that there is a generator  

\[ g = I(a, b), a, b \in R \cap Y^m, \]  

with  

\[ g(i) \in D. \]  

This implies that  

\[ a(i) \in D, \]  

so  

\[ i \in D(R) \text{ and } a(\neq i) \in C^{m-1}. \]  

Since  

\[ |D(R)| \geq 2, \]  

we have  

\[ D(R(\neq i)) \neq \emptyset \]  

and  

\[ R(\neq i) \cap C^{m-1} \neq \emptyset. \]  

By Lemma 7.8 item (2)  

\[ R(\neq i) \]  

must halt, so  

\[ i \in H(R). \]  

For the reverse inclusion, suppose that  

\[ i \in D(R) \text{ and } i \in H(R). \]  

By Lemma 7.8 item (1), we have that there is an element  

\[ \sigma_i \in R \]  

such that  

\[ \sigma_i(i) = \langle 1, \bullet \rangle \]  

and  

\[ \sigma_i(\neq i) \in \{ (1, 0) \}^{m-1}. \]  

Hence  

\[ \sigma_i \in R_i, \]  

so  

\[ i \in D(R_i). \]  

(4): This follows from items (2) and (3) above, Lemma 7.8 item (1), and the definition of  

\[ \mathcal{S}. \]  

\[ \text{Proposition 7.11.} \quad \text{Let } R \leq A(M)^m \text{ be computational non-halting. If } t \text{ is a } k\text{-ary term operation and } \tau \in R^k \text{ is such that } t(\tau) \in Y^m \text{ then} \]

(1)  

\[ t(\tau) \in R_i \]  

or

(2)  

there is a term operation  

\[ s \]  

without operation  

\[ I \]  

in its term tree such that

\[ s(\tau) = t(\tau). \]  

\[ \text{Proof.} \quad \text{Let } t(\tau) = \alpha \text{ and assume that (2) is not the case, so if we have } s(\tau) = \alpha \text{ then } s \text{ has } I \text{ in its term tree. We will prove that } \alpha \in R_i. \]  

The proof shall be by induction on the complexity of  

\[ t. \]  

If we have  

\[ \alpha = I(a, b) \]  

for some  

\[ a, b \in R \cap Y^m \]  

then  

\[ \alpha \in R_i \]  

by definition. This establishes the basis of the induction. Assume now that  

\[ t \]  

is not a projection, so  

\[ t \]  

can be written as

\[ t(\tau) = F(f_1(\tau), \ldots, f_k(\tau)), \]
where $F$ is an $\ell$-ary fundamental operation and the $f_i$ are other $k$-ary term operations. We will proceed by cases depending on which operation $F$ is.

**Case** $F \in \{\land, N_0, N_*, P\}$: Since $\mathbb{R}$ is computational and non-halting, such $F$ have the property that $F(\overline{\pi}) \leq a_i$ for some $a_i$ amongst the $\pi$, by the various parts of Lemma 5.8. Therefore, if $\alpha = F(f_1(\overline{\pi}), \ldots, f_n(\overline{\pi}))$ then $\alpha \leq f_j(\overline{\pi})$ for some $j$. Since $\alpha \in Y^m$, this implies that $f_j(\overline{\pi}) = \alpha$. As (2) does not hold, $f_j$ must have $I$ in its term tree, so by the inductive hypothesis we have that $\alpha = f_j(\overline{\pi}) \in R_I$.

**Case** $F \in \{M', H\}$: In this case, $F$ is $X$-absorbing and unary, by Lemma 5.3 item (2). It follows that $F(f_1(\overline{\pi})) = \alpha \in Y^m$ implies $f_1(\overline{\pi}) \in R_I$ and that $I$ is in the term tree of $f_1$. Therefore the inductive hypothesis applies and $f_1(\overline{\pi}) \in R_I$. Hence $\alpha \in R_I$.

**Case** $F = M$: Since $\alpha \in Y^m$, by Lemma 5.3 item (2) we have $f_1(\overline{\pi}), f_2(\overline{\pi}) \in Y^m$. The term operation $t$ has $I$ in its term tree, so one of the $f_i$ does as well. By the inductive hypothesis, one of $f_1(\overline{\pi})$ is in $R_I$. If $D(\mathbb{R}) = \emptyset$ then $f_1(\overline{\pi}) = f_2(\overline{\pi})$, so both belong to $R_I$. If $D(\mathbb{R}) \neq \emptyset$ then $R \cap C^m = \emptyset$ by Lemma 7.8 item (2). It follows from this and the definition of $M$ that there are coordinates $j, k$ such that

- $f_1(\overline{\pi})(k), f_2(\overline{\pi})(j), \alpha(j) \in D$,
- $f_1(\overline{\pi})(j) = f_2(\overline{\pi})(k)$, and
- $f_1(\overline{\pi})(\ell) = f_2(\overline{\pi})(\ell)$ for $\ell \notin \{j, k\}$.

That is, $f_1(\overline{\pi})$ and $f_2(\overline{\pi})$ equal under the coordinate transposition swapping $j$ and $k$. By Lemma 7.10 item (1), one of them being in $R_I$ implies the other is in $R_I$ as well. Therefore $\alpha \in R_I$.

**Case** $F \in \{I, S\}$: From the definitions and Lemma 5.8 item (4), we have that $\alpha = I(\alpha, \alpha)$ in this case. Thus $\alpha \in R_I$.

This completes the case analysis, the induction, and the proof. □

The next proposition and subsequent definition establishes the biggest tool we have for analyzing the halting status of a relation. It is absolutely essential to the proofs in the next section.

**Proposition 7.12.** Suppose that the relation $\mathbb{R} \leq \mathbb{A}(\mathbb{M})^m$ is computational non-halting. There exists $\mathcal{N} \subseteq [m]$ such that

1. $\mathbb{R}(\mathcal{N})$ is non-halting,
2. $|\mathcal{N} \cap D(\mathbb{R})| \leq \kappa$,
3. if $D(\mathbb{R}) \neq \emptyset$ then $\mathcal{N} \cap D(\mathbb{R}) \neq \emptyset$, and
4. $(\mathbb{N} \setminus D(\mathbb{R})) \subseteq \mathcal{N}$.

**Proof.** If $D(\mathbb{R}) = \emptyset$ then take $\mathcal{N} = [m]$. It is not hard to see that $\mathcal{N}$ satisfies (1)–(4). Assume now that $D(\mathbb{R}) \neq \emptyset$ and let $\mathcal{N}'$ be minimal such that $\mathcal{N}' \cap D(\mathbb{R}) \neq \emptyset$ and $\mathbb{R}(\mathcal{N}')$ is non-halting. Since $\mathbb{R}$ is already non-halting, there is at least one such $\mathcal{N}'$. We begin by proving that $|\mathcal{N}' \cap D(\mathbb{R})| \leq \kappa$.

Suppose that we have distinct $i_1, \ldots, i_{\kappa+1} \in \mathcal{N}' \cap D(\mathbb{R})$. By the minimality of $\mathcal{N}'$, we have that $\mathbb{R}(\mathcal{N}' \setminus \{i_k\})$ is halting for each $k$. Therefore

$$i_1, \ldots, i_{\kappa+1} \in \mathcal{H}(\mathbb{R}(\mathcal{N}')) \cap D(\mathbb{R}(\mathcal{N}')),$$
so $\mathcal{R}(\mathcal{N}')$ is halting by Lemma 7.8 item (4), a contradiction. Hence $|\mathcal{N}' \cap D(\mathcal{R})| \leq \kappa$. Let

$$\mathcal{N} = \mathcal{N}' \cup ([m] \setminus D(\mathcal{R})).$$

It is easy to see that $|\mathcal{N} \cap D(\mathcal{R})| = |\mathcal{N}' \cap D(\mathcal{R})| \leq \kappa$. Suppose towards a contradiction that $\mathcal{R}(\mathcal{N})$ is halting. By Lemma 7.8 item (2) we have $\mathcal{R}(\mathcal{N}) \cap C^{\mathcal{N}} \neq \emptyset$. It follows that $\mathcal{R}(\mathcal{N}') \cap C^{\mathcal{N}} \neq \emptyset$, and so $\mathcal{R}(\mathcal{N}')$ is halting, contradicting the choice of $\mathcal{N}'$. Therefore $\mathcal{R}(\mathcal{N})$ is non-halting, and we are done. \qed

**Definition 7.13.** For each $\mathcal{R}$ that is computational non-halting we fix a set of indices $\mathcal{N}(\mathcal{R})$ satisfying the conclusion of Proposition 7.12. We call $\mathcal{N}(\mathcal{R})$ the *inherently non-halting* part of $\mathcal{R}$. As with $D$ and $\mathcal{H}$, if the relation is clear then we will sometimes use $\mathcal{N}$ instead of $\mathcal{N}(\mathcal{R})$.

**Lemma 7.14.** Let $\mathcal{R} \subseteq A(\mathcal{M})^m$ be computational non-halting and suppose that $r \in R$.

1. If $D(\mathcal{R}) \neq \emptyset$ then there is $j \in \mathcal{N}(\mathcal{R})$ with $r(j) \in D \cup X$.
2. If $i \notin \mathcal{N}(\mathcal{R})$ and $r(i) \in D$ then there is $j \in \mathcal{N}(\mathcal{R})$ with $r(j) \in X$.
3. $D(\mathcal{R}_i) \subseteq \mathcal{N}(\mathcal{R}) \cap D(\mathcal{R})$.
4. $\mathcal{H}(\mathcal{R}) \subseteq \mathcal{N}(\mathcal{R})$.
5. If $|D(\mathcal{R})| \leq 1$ then $\mathcal{N}(\mathcal{R}) = [m]$. \[Proof.\] (1): We have that $\mathcal{R}(\mathcal{N}(\mathcal{R}))$ is non-halting. Since $\mathcal{N}(\mathcal{R}) \cap D(\mathcal{R}) \neq \emptyset$, this implies $\mathcal{R}(\mathcal{N}(\mathcal{R})) \cap C^{\mathcal{N}(\mathcal{R})} = \emptyset$ by Lemma 7.8 item (2). Therefore $r(\mathcal{N}(\mathcal{R})) \notin C^{\mathcal{N}(\mathcal{R})}$. The conclusion follows.

(2): This follows from item (1). If we have $r(i) \in D$ and $r(j) \in D \cup X$ for $j \neq i$ then $r(j) \in X$ since $\mathcal{R}$ is computational.

(3): We already have $D(\mathcal{R}_i) \subseteq D(\mathcal{R})$, so we only need to show $D(\mathcal{R}_i) \subseteq \mathcal{N}(\mathcal{R})$. Let $i \in D(\mathcal{R}_i)$. The only way this is possible is if there is a generator $g = I(a, b)$, $a, b \in R \cap Y^m$, with $g(i), a(i) \in D$. If $i \notin \mathcal{N}(\mathcal{R})$ then by item (2) above there is $j \in \mathcal{N}(\mathcal{R})$ with $a(j) \in X$, contradicting $a \in Y^m$.

(4): Let $i \in \mathcal{H}(\mathcal{R})$, so that $\mathcal{R}(\neq i)$ is halting. If $i \notin \mathcal{N}(\mathcal{R})$ then $\mathcal{R}(\mathcal{N}) = \mathcal{R}(\neq i)(\mathcal{N})$, so we have that $\mathcal{R}(\mathcal{N})$ is halting as well, contradicting Proposition 7.12 item (1). Hence $i \in \mathcal{N}(\mathcal{R})$.

(5): If $D(\mathcal{R}) = \emptyset$ then $\mathcal{N}(\mathcal{R}) = [m]$ from the proof of Proposition 7.12. If $D(\mathcal{R}) = \{i\}$ then $i \in \mathcal{N}(\mathcal{R})$ since $D(\mathcal{R}) \cap \mathcal{N}(\mathcal{R}) \neq \emptyset$. Since we also have $([m] \setminus D(\mathcal{R})) \subseteq \mathcal{N}(\mathcal{R})$, the conclusion follows. \qed

We have now built enough tools to attack the main problem.

8. If $\mathcal{M}$ halts — entailment

As with the previous section, we assume throughout that $\mathcal{M}$ halts with capacity $\kappa$. The overall structure of the argument will be to consider a relation $\mathcal{R} \in \text{Rel}_{\leq m}(A(\mathcal{M}))$, and proceed by cases. These cases are laid out in the proof of the main entailment theorem, which we begin the section with (after introducing some notation). The proof references the theorems later in this section, but it is useful at the outset to see the overall strategy.

**Definition 8.1.** Let $\mathcal{R} \subseteq A(\mathcal{M})^m$ and $\alpha \in A(\mathcal{M})^m$. 
• We say that \( A(\alpha, i) \) holds for \( R \) if \( \alpha(\neq i) \in R(\neq i) \).

• We say that \( A_1(\alpha, i) \) holds for \( R \) if \( A(\alpha, i) \) holds for \( R_1 \).

• If \( \mathbb{R} \) has \( A(\alpha, i) \) then fix an element \( \alpha_i \in R \) such that \( \alpha_i(\neq i) = \alpha(\neq i) \), and likewise if \( A_1(\alpha, i) \) holds. If \( \mathbb{R} \) has both \( A(\alpha, i) \) and \( A_1(\alpha, i) \) then take \( \alpha_i \in R_I \subseteq R \).

If the relation \( \mathbb{R} \) is clear, we will use \( A(\alpha, i) \) and \( A_1(\alpha, i) \) without reference to the relation.

**Corollary 8.2.** If \( \mathcal{M} \) halts then \( \deg(\mathcal{A}(\mathcal{M})) \leq \kappa + 15 \).

**Proof.** We will show that \( \text{Rel}_{\leq \kappa + 15}(\mathcal{A}(\mathcal{M})) \models \text{Rel}_{\leq m}(\mathcal{A}(\mathcal{M})) \) by induction on \( m \). The base case of \( m = \kappa + 15 \) is included in the hypotheses. Suppose now that \( m \geq \kappa + 16, \mathbb{R} \leq \mathcal{A}(\mathcal{M}) \), \( m \), \( \text{Rel}_{\leq m-1}(\mathcal{A}(\mathcal{M})) \models f \), and \( f(r_1, \ldots, r_n) = \alpha \) for some \( r_1, \ldots, r_n \in R \). We endeavor to prove \( \alpha \in R \). Let \( G = \{ r_1, \ldots, r_n \} \). Without loss of generality we may assume that \( \mathbb{R} = S_{\mathcal{A}(\mathcal{M})}(G) \).

If \( \mathbb{R} \) is not computational or is halting then Theorem 8.3 yields \( \text{Rel}_{\leq m-1} \models \mathbb{R} \), so \( \alpha \in R \). Therefore we assume that

1. \( \mathbb{R} \) is both computational and non-halting, so \( |\mathcal{N}(\mathbb{R}) \cap \mathcal{D}(\mathbb{R})| \leq \kappa \) by Proposition 7.12.

If \( \kappa \in \text{con}(\alpha) \) then Theorem 8.11 yields \( \alpha \in R \). Therefore we assume that

2. \( \alpha \in \mathcal{Y}^m \).

By the inductive hypothesis \( \mathbb{R} \) has \( A(\alpha, k) \) for all \( k \in [m] \). If there are distinct \( i, j \notin \mathcal{N} \) such that \( \mathbb{R} \) has \( A_1(\alpha, i) \) and \( A_1(\alpha, j) \) then Theorem 8.6 yields \( \alpha \in R \). Therefore we assume that

3. there is at most one \( i \notin \mathcal{N}(\mathbb{R}) \) such that \( A_1(\alpha, i) \).

If \( |[m] \setminus \mathcal{D}(\mathbb{R})| \geq 11 \) then Theorem 8.10 yields \( \alpha \in R \). Therefore we assume that

4. \( |[m] \setminus \mathcal{D}(\mathbb{R})| \leq 10 \), so \( |\mathcal{N}| \leq \kappa + 10 \) by Proposition 7.12.

Finally, our list of assumptions agrees with the hypotheses of Theorem 8.10, so \( \alpha \in R \).

Having established the overall strategy we will be pursuing, we prove our first entailment theorem — entailment for non-computational or halting relations.

**Theorem 8.3.** If \( m \geq 3 \) and \( \mathbb{R} \leq \mathcal{A}(\mathcal{M}) \) fails to be computational or is halting then \( \text{Rel}_{\leq m-1}(\mathcal{A}(\mathcal{M})) \models \mathbb{R} \).

**Proof.** Towards a contradiction, suppose that \( \text{Rel}_{\leq m-1}(\mathcal{A}(\mathcal{M})) \not\models \mathbb{R} \). This implies that there is some \( n \)-ary function \( f \) and \( \tau \in R^n \) such that \( \text{Rel}_{\leq m-1}(\mathcal{A}(\mathcal{M})) \models f \) and \( f(\tau) = \alpha \notin \mathbb{R} \). Since \( \mathbb{R}(\neq i) \in \text{Rel}_{\leq m-1}(\mathcal{A}(\mathcal{M})) \), we have that \( \alpha(\neq i) \in \mathbb{R}(\neq i) \), so \( A(\alpha, i) \) holds and we have elements \( \alpha_i \in R \) for all \( i \). There are three cases to consider: \( \mathbb{R} \) is not synchronized, there is \( r \in R \) with \( |r^{-1}(D)| \geq 2 \), or \( \mathbb{R} \) is halting.

**Case \( \mathbb{R} \) is not synchronized:** In this case there is an \( r \in R \) with a non-constant state. For each state \( i \) of \( \mathcal{M} \) let

\[
K_i = \{ j \mid \text{st}(r(j)) = i \} \quad \text{and} \quad L_i = [m] \setminus K_i
\]

and pick some \( K_k \neq \emptyset \). Let \( s = z_k(r) \) from Lemma 5.3, item (4). It follows that for \( a, b \in R \)

\[
P(r, s, a, b)(j) = \begin{cases} a(j) & \text{if } j \in K_k, \\ b(j) & \text{otherwise.} \end{cases}
\]
That is, $\mathbb{R}$ is obtained by some permutation of the coordinates of $\mathbb{R}(K_k) \times \mathbb{R}(L_k)$, so $\mathbb{R}$ is entailed by lower-arity relations.

**Case $\exists r \in R \mid r^{-1}(D) \geq 2$:** Assume that $\mathbb{R}$ is synchronized. Let us choose distinct $i, j$ such that $r(i), r(j) \in D$ and let $k$ be distinct from $i$ and $j$ (we use $m \geq 3$ here). From the definition of $N\bullet$ it follows that

$$N\bullet(r, \alpha_i, \alpha_j, \alpha_k)(\ell) = \begin{cases} 
\alpha(i) & \text{if } \ell = i, \\
\alpha(j) & \text{if } \ell = j, \\
\alpha(\ell) & \text{otherwise},
\end{cases}$$

so $\alpha = N\bullet(r, \alpha_i, \alpha_j, \alpha_k)$ and hence $\alpha \in R$ (we use $m \geq 3$ here).

**Case $\mathbb{R}$ is halting:** Let us assume that $\mathbb{R}$ is computational and that $r \in R$ is a halting vector. That is, $r(\ell) = (0, \bullet)$ for some $\ell$ and $r(\ell) \in \{(0,0)\}^{m-1}$. It is not possible for there to be two coordinates $i$ at which $\alpha(i) \in D$ or $r(i) \in D$ since $\mathbb{R}$ is computational. If $\alpha(\ell) \in D$ or $\alpha \in C^m$ then by definition of $N_0$,

$$\alpha = N_0(r, \alpha_i)$$

for some $i \neq \ell$, and hence $\alpha \in R$. The other possibility is that there is some $k \neq \ell$ with $\alpha(k) \in D$. Let $s = H(r)$ and $\beta' = I(\alpha_k, s)$. We have $s \in \{(0,0)\}^m$, $\beta'(k) = (1, \bullet)$, and $\beta'(j) = (1, 0)$ for all $j \neq k$. From $\beta'$ and $s$ we can obtain a halting vector $r'$ such that $r'(k) \in D$ and $r'(j) = (0, 0)$ (we use that $\mathbb{T}$ from the end of Section 5 is halting here). As before,

$$\alpha = N_0(r', \alpha_i)$$

for some $i \neq k$, so $\alpha \in R$. 

\[\square\]

### 8.1. Entailment for $\mathbb{R}_I$

We now prove an entailment theorem for the relation $\mathbb{R}_I$.

By Proposition 7.11, this will allow us to make use of the set $\Gamma$ from Definition 7.3 in certain situations.

**Lemma 8.4.** Suppose that $\mathbb{R} \leq \mathcal{A}(\mathcal{M})^m$ is computational and let $D_1 = D(\mathbb{R}_I)$ and $L = [m] \setminus D_1$. Then $\alpha \in R_I$ if and only if

$$\alpha(D_1) \in R_I(D_1) \quad \text{and} \quad \alpha(L) \in \left\{ \begin{array}{c}
\langle \text{st}(\alpha), 0 \rangle \\
\vdots \\
\langle \text{st}(\alpha), 0 \rangle \\
\langle \text{st}(\alpha), x \rangle \\
\vdots \\
\langle \text{st}(\alpha), x \rangle
\end{array} \right\}.$$

**Proof.** We begin by building some tools. Define

$$Q = \bigcup_{i \text{ a state of } \mathcal{M}} \left\{ \langle i, 0 \rangle, \ldots, \langle i, 0 \rangle \right\} \subseteq \mathcal{A}(\mathcal{M})^L.$$

Examining the definitions of the operations of $\mathcal{A}(\mathcal{M})$, observe that $Q$ is a subuniverse of $\mathcal{A}(\mathcal{M})^L$. From Lemma 7.10 item (4) we have that $\mathbb{R}_I(D_1) = S_{|D_1|}$ and there are elements $(\sigma_i)_{i \in D_1}$ satisfying Equation 5.1:

$$(5.1 \text{ redux}) \quad \sigma_i(j) = \begin{cases} 
(1, \bullet) & \text{if } j = i, \\
(1, 0) & \text{otherwise}
\end{cases}$$
that generate \( R_I \). Let \( x = z_1(\sigma_i) \wedge z_2(\sigma_i) \) for the term operations \( z_j \) defined in Lemma 5.3 item (4) and observe that \( x = ((1, x), \ldots, (1, x)) \). Suppose that \( D_I = \{i_1, \ldots, i_k\} \). For \( t \in D_I \) and \( j \leq k \) define the sequence of elements \( \tau_t^j \in R_I \) by

\[
\tau_t^0 = x \quad \text{and} \quad \tau_t^j = N_\star(\sigma_{i_j}, \tau_t^{j-1}, \sigma_{i_j}, \sigma_t).
\]

Let \( \tau_t = \tau_t^k \). It is not hard to see that \( \tau_t(D_I) = \sigma_t(D_I) \) and \( \tau_t(L) = x(L) \in X_L \).

We are now ready to prove the lemma.

Suppose that \( \alpha \in R_I \). It is immediate that \( \alpha(D_I) \in R_I(D_I) \). Furthermore, if \( \sigma_i \) is one of the generators of \( R_I \) then \( \sigma_i(L) \in Q \). Since \( Q \) is a subuniverse, this implies that \( \alpha(L) \in Q \). This completes the “only if” portion of the proposition. The “if” portion will give us more difficulty.

Suppose that \( \alpha(D_I) \in R_I(D_I) \) and \( \alpha(L) \in Q \). It follows that there is a term operation \( t \) such that

\[
\alpha(D_I) = t(\overline{\sigma})(D_I),
\]

where \( \overline{\sigma} \) are the generators of \( R_I \). Let \( \beta = t(\overline{\sigma}) \). Clearly \( \beta \in R_I \), so if \( \alpha \not\in R_I \) then \( \alpha \neq \beta \). Since the \( \overline{\sigma} \) are all equal with content 0 on coordinates \( L \), \( \alpha \neq \beta \) implies that one of \( \text{con}(\alpha(L)), \text{con}(\beta(L)) \) is \( \{ \times \} \) and the other is \( \{ 0 \} \). Thus there are two cases to consider.

For the first case, suppose that \( \text{con}(\alpha(L)) = \{ \times \} \) and \( \text{con}(\beta(L)) = \{ 0 \} \). By Lemma 5.3 item (5) we have

\[
\alpha(L) = \overline{1}(\alpha(L)) = \overline{1}(t(\overline{\sigma})(L)) = t(\overline{1}(\overline{\sigma})(L)) = t(\overline{\sigma})(L)
\]

for the elements \( \overline{\sigma} = (\tau_t^0)_{t \in D_I} \) defined at the start of the proof. Since \( \overline{\sigma}(D_I) = \sigma_t(D_I) \), we have that \( \alpha = t(\overline{\sigma}) \) and hence \( \alpha \in R_I \).

For the second case, suppose that \( \text{con}(\alpha(L)) = \{ 0 \} \) and \( \text{con}(\beta(L)) = \{ \times \} \). After proving the next claim, we will be done.

**Claim.** If \( a \) and \( b \) are such that \( b \in R_I \), \( a(D_I) = b(D_I) \), \( \text{con}(b(L)) = \{ \times \} \), and \( \text{con}(a(L)) = \{ 0 \} \) then \( a \in R_I \).

**Proof of claim.** Let \( G_0 = \{ \sigma_i \mid i \in D_I \} \) be the generators of \( R_I \) and

\[
G_n = \left\{ F(\overline{\sigma}) \mid F \text{ a fundamental } \ell \text{-ary operation, } \overline{\sigma} \in G_{n-1}^\ell \right\} \cup G_{n-1}.
\]

Suppose towards a contradiction that the claim is false. Choose a counterexample \( a, b \) with \( b \in G_n \) such that \( n \) is minimal. When \( b \in G_0 \) the claim’s hypothesis fails, so it holds vacuously. Assume that \( n > 0 \), so

\[
b = F(g_1, \ldots, g_\ell)
\]

for some \( \ell \)-ary operation \( F \) and elements \( g_1, \ldots, g_\ell \in G_{n-1} \). If one of the \( g_i \) has \( g_i(L) \in X^k \) then by the inductive hypothesis there is an element \( g_i' \in G_{n-1} \) with \( g_i'(D_I) = g_i(D_I) \) and \( \text{con}(g_i'(L)) \in \{ 0 \} \). Let \( b' \) be the result of replacing \( g_i \) with \( g_i' \) in the arguments of \( F \). There are two possibilities for \( b'(L) \): either \( \text{con}(b'(L)) = \{ 0 \} \) (and so \( b' = a \)) or \( b'(L) = b(L) \in X^\ell \). In the first possibility we conclude that \( a \in R_I \), a contradiction, and in the second possibility we conclude \( b' = b \). We may therefore assume without loss of generality that \( \text{con}(g_i)(L) \in \{ 0 \} \) for all \( i \).

Looking through the definitions of the operations, we can see that if \( \text{con}(g_i)(L) \in \{ 0 \} \) for all \( i \) and \( b(L) = F(\overline{\sigma})(L) \in X^\ell \) then it must be that \( b(D_I) = F(\overline{\sigma})(D_I) \in
Lemma 8.8. Assume that

Note that $K$ is compatible over $D_I$. We now have

\[ a = \bigcap_{i \in D_I} z_{\text{st}(a)}(\sigma_i), \]

where $z_{\text{st}(a)}$ is the term operation from Lemma 5.3 item (4). It follows that $a \in R_I$, and we are done.

Proposition 8.5. Let $\mathbb{R} \leq A(M)^m$ be computational non-halting. If $r \in R_I$, $i \not\in N(\mathbb{R}) \cap D(\mathbb{R})$, and $\text{con}(r(i)) \neq 0$ then $r(j) \in X$ for all $j \not\in N(\mathbb{R}) \cap D(\mathbb{R})$.

Proof. This follows immediately from the observation that $D(R_I) \subseteq N(\mathbb{R}) \cap D(\mathbb{R})$ (Lemma [7.14] item (3)) and an application of Lemma 8.4.

Theorem 8.6. Let $\mathbb{R} \leq A(M)^m$ be computational non-halting. If $\alpha \in Y^m$ is such that

- $\lambda(\alpha, i)$ for all $i$, and
- there are distinct $k, \ell \not\in N(\mathbb{R})$ such that $\lambda_k(\alpha, k)$ and $\lambda_l(\alpha, \ell)$

then $\alpha \in R_I$.

Proof. We have $\alpha_k, \alpha_\ell \in R_I$. Proposition 8.5 implies that

\[ \text{con}(\alpha_k(\ell)) = \text{con}(\alpha(\ell)) = 0 \quad \text{and} \quad \text{con}(\alpha_\ell(k)) = \text{con}(\alpha(k)) = 0. \]

Furthermore, if $\text{con}(\alpha_k(k)) \neq 0$ then $\alpha_k(\ell) \in X$ by the same proposition, a contradiction. It follows that $\alpha = \alpha_k$.

8.2. Entailment when $|D|$ is small. We next show how relations with small $|D|$ are entailed. The key to the argument is to first prove that the generating set of such relations has a very specific form, and then to use the relations from Definitions 7.1, 7.2, and 7.3.

Definition 8.7. Let $G \subseteq A(M)^m$ and $\mathbb{R} = S_{\text{Ax}(\mathbb{M})}(G)$. We say that $G$ is $\chi$-compatible over $K \subseteq [m]$ if

\[ G(K) \subseteq \left\{ \begin{pmatrix} (\alpha, \chi) \\ \vdots \\ (\alpha, a_m) \end{pmatrix} \mid a_1, \ldots, a_m \in \{A, B, 0\}, \alpha \text{ a state} \right\}. \]

If $K$ is not specified then we take $K = [m] \setminus D(\mathbb{R})$ (the non-dot coordinates of $\mathbb{R}$). Note that $K = \emptyset$ is allowed.

Lemma 8.8. Assume that

- $\text{Rel}_{\leq 3}(A(M)) \models f$ and $f$ is $n$-ary,
- $G = \{g_1, \ldots, g_n\} \subseteq E^2$ and $\mathbb{R} = S_{\text{Ax}(\mathbb{M})^2}(G)$ is synchronized, and
- $f(g_1, \ldots, g_n) = \alpha \in Y^2$.

If $G$ is not $\chi$-compatible then there is $g_\ell \in G$ such that $g_\ell \not\in X^2$ and

\[ f(g_1, \ldots, \chi(g_\ell), \ldots, g_n) = \alpha. \]
Proof; $G$ is not \( \chi \)-compatible, so there is \( \ell \) such that (modulo permuting coordinates) we have \( g_\ell(1) \in X \) while \( g_\ell(2) \in Y \). Therefore \( X(g_\ell(1)) = g_\ell(1) \) and \( X(g_\ell(2)) \neq g_\ell(2) \). Consider

\[
f \left( \begin{pmatrix} g_1(1) \\ g_1(2) \end{pmatrix}, \ldots, \begin{pmatrix} g_\ell(1) \\ g_\ell(2) \end{pmatrix}, \ldots, \begin{pmatrix} g_n(1) \\ g_n(2) \end{pmatrix} \right) = \begin{pmatrix} \alpha(1) \\ \alpha(2) \end{pmatrix}.
\]

Each of the input vectors lies in the relation \( \chi \) from Definition 7.1, so the output lies in \( \chi \) as well. The relation \( \chi \) has the property that if \( r \in \chi \) and \( r(1), r(2) \notin X \) then \( r(3) \notin X \). Since \( \alpha(1), \alpha(2) \notin X \) we have \( \beta \notin X \), so by the definition of \( \chi \) we now have \( \alpha(2) = \beta \). Projecting the above equality onto coordinates \( \{1, 3\} \) yields the conclusion of the lemma. \( \square \)

**Proposition 8.9.** Assume that \( m \geq 4 \) and

- \( \text{Rel}_{\leq m-1}(A(\mathcal{M})) \models f \) and \( f \) is \( n \)-ary,
- \( G = \{ g_1, \ldots, g_n \} \subseteq A(\mathcal{M})^m \) and \( \mathbb{R} = Sg_{\mathcal{A}(\mathcal{M})}(G) \) is computational,
- \( f(g_1, \ldots, g_n) = \alpha, \) and
- \( K \subseteq [m] \setminus \mathcal{D}(\mathbb{R}) \) and \( \alpha(K) \in Y^K \).

If \( G \) is not \( \chi \)-compatible over \( K \) then \( \alpha \in R \).

Proof. Observe that \( G \) being \( \chi \)-compatible over \( K \) means that \( G(K) \) is \( \chi \)-compatible and that \( G(K) \subseteq E^K \). If \( |K| \leq 1 \) then \( G \) is always \( \chi \)-compatible. Assume therefore that \( |K| \geq 2 \).

The proof is by induction on the number of coordinates which are \( Y \) (i.e. not in \( X \)) in \( G \):

\[
\sum_{i=1}^{m} \left| \{ k \mid g_k(i) \in Y, g_k \in G \} \right|.
\]

If this quantity is 0 then \( G(K) \subseteq X^K \). Choose some \( k \in K \). Since \( X \leq A(\mathcal{M}) \) we have \( f(g_1, \ldots, g_n)(k) = \alpha(k) \in X \), contradicting \( \alpha(K) \in Y^K \). This establishes the basis of the induction.

If \( G \) fails to be \( \chi \)-compatible then there is \( g_\ell \in G \) and coordinates \( j, k \in K \) such that \( g_\ell(j) \in X \) while \( g_\ell(k) \in Y \). Define

\[
\hat{g}_{\ell}(i) = \begin{cases} X(g_{\ell}(k)) & \text{if } i = k, \\
g_\ell(i) & \text{otherwise}, \end{cases} \quad \text{and} \quad \mathcal{E} = \{ g_1, \ldots, \hat{g}_{\ell}, \ldots, g_n \}.
\]

Since \( G(\{ j, k \}) \) is not \( \chi \)-compatible, Lemma 8.8 implies that

\[
f(g_1, \ldots, \hat{g}_{\ell}, \ldots, g_n) = \alpha.
\]

The arguments have 1 fewer coordinates in \( Y \), so \( \alpha \in Sg_{\mathcal{A}(\mathcal{M})}(\mathcal{E}) \). Hence there is a term operation \( t \) that generates \( \alpha \) from \( \mathcal{E} \). Consider the equation

\[
t \left( \begin{pmatrix} g_1(j) \\ g_1(k) \\ g_2(j) \\ g_2(k) \\ g_3(j) \\ g_3(k) \\ g_n(j) \\ g_n(k) \end{pmatrix}, \begin{pmatrix} X(g_1(j)) \\ X(g_1(k)) \\ g_\ell(j) \\ g_\ell(k) \\ X(g_\ell(j)) \\ X(g_\ell(k)) \end{pmatrix}, \begin{pmatrix} g_1(j) \\ g_1(k) \\ g_2(j) \\ g_2(k) \\ g_3(j) \\ g_3(k) \\ g_n(j) \\ g_n(k) \end{pmatrix} \right) = \begin{pmatrix} \alpha(j) \\ \alpha(k) \end{pmatrix}.
\]

Projecting the arguments on coordinates \( \{1, 2\} \) yields \( G(\{ j, k \}) \) and on \( \{1, 3\} \) yields \( \mathcal{E}(\{ j, k \}) \). Since \( t \) is a term operation and all the input vectors lie in \( \chi \), the output
must as well. The relation \( \chi \) has the property that if \( r \in \chi \) and \( r(3) \notin X \) then \( r(2) \notin X \). Since \( \alpha(k) \notin X \), we have \( \gamma \notin X \), and by the definition of \( \chi \) we conclude that \( \alpha(k) = \gamma \). Projecting on coordinates \( \{1, 2\} \) now yields \( t(g_1, \ldots, g_n) = \alpha \), so \( \alpha \in R \). This completes the induction and the proof.

**Theorem 8.10.** Assume that \( m \) is such that

- \( \text{Rel}_{\leq m-1}(A(M)) \models f \) and \( f \) is \( n \)-ary,
- \( G = \{g_1, \ldots, g_n\} \subseteq A(M)^m \) and \( R = \Sigma_{\leq m}(G) \) is computational,
- \( f(g_1, \ldots, g_n) = \alpha \in Y^m \),
- \( |[m] \setminus \mathcal{D}(R)| \geq 11 \), and
- there is at most one \( k \notin \mathcal{N}(R) \) such that \( A_1(\alpha, k) \).

Then \( \alpha \in R \).

**Proof.** By Proposition 8.9 if \( G \) is not \( \chi \)-compatible then \( \alpha \in R \). Assume therefore that \( G \) is \( \chi \)-compatible and let

\[
K = [m] \setminus \left\{ k \mid k \in \mathcal{D} \text{ or } k \notin \mathcal{N} \text{ and } A_1(\alpha, k) \right\}.
\]

The hypotheses of the theorem mean that \( |K| \geq 10 \) and that \( G \) is \( \chi \)-compatible on \( K \). Since \( \alpha(K) \in Y^K \), it follows that one of the sets

\[
\alpha^{-1}(\{j\}) \cap K, \quad \alpha^{-1}(\{A\}) \cap K, \quad \alpha^{-1}(\{B\}) \cap K
\]

contains 4 elements. Let us suppose that \( \alpha^{-1}(\{A\}) \cap K \) has 4 elements, call them \( 1, 2, 3, 4 \). The argument that follows applies equally well to the other possibilities. We will closely examine \( f \) evaluated on these coordinates.

We have \( f(g_1, \ldots, g_n) = \alpha \). Evaluation at a coordinate is just evaluation on a “row” of this equation. For \( i \in [n] \), define the length \( n \) tuples \([i] = (g_1(i), \ldots, g_n(i))\) and note that \([i](j) = g_j(i)\). For distinct \( i_1, \ldots, i_k \in [n] \), define the length \( n \) tuples

\[
[i_1] \cdots [i_k]_V(j) = \begin{cases} g_j(i_1) & \text{if } g_j(i_1) = \cdots = g_j(i_k), \\ \langle \text{st}(g_j), 0 \rangle & \text{otherwise}, \end{cases}
\]

\[
[i_1] \cdots [i_k]_A(j) = \begin{cases} g_j(i_1) & \text{if } g_j(i_1) = \cdots = g_j(i_k), \\ g_j(i_\ell) & \text{if } \text{con}(g_j(i_\ell)) = A \text{ for some } i_\ell \in \{i_1, \ldots, i_k\}, \\ \langle \text{st}(g_j), 0 \rangle & \text{otherwise}, \end{cases}
\]

\[
[i_1] \cdots [i_k]_B(j) = \begin{cases} g_j(i_1) & \text{if } g_j(i_1) = \cdots = g_j(i_k), \\ g_j(i_\ell) & \text{if } \text{con}(g_j(i_\ell)) = B \text{ for some } i_\ell \in \{i_1, \ldots, i_k\}, \\ \langle \text{st}(g_j), 0 \rangle & \text{otherwise}. \end{cases}
\]

We claim that

\[
f \begin{pmatrix} [1|2|3|4]_V \\ [1|2|3|4]_A \\ [1|2|3|4]_B \end{pmatrix} = \begin{pmatrix} \langle \text{st}(\alpha), A \rangle \\ \langle \text{st}(\alpha), A \rangle \\ \langle \text{st}(\alpha), A \rangle \end{pmatrix} = \alpha(\{1, 2, 3\}).
\]

Using the relations \( \Delta_V \), \( \Delta_A \), and \( \Delta_B \) it is not difficult to see that this is true. It is, however, most easily seen by working through an example. See Figure 4 for an example showing \( f([1|2|3|4]_A) = \langle \text{st}(\alpha), A \rangle \).
Define vectors \( h_i \in A(M)^{m-1} \) by

\[
h_i(j) = \begin{cases} 
[1|2|3|4]|_v(i) & \text{if } j = 1, \\
[1|2|3|4]|_{3A}(i) & \text{if } j = 2, \\
[1|2|3|4]|_{3B}(i) & \text{if } j = 3, \\
g_i(j) & \text{if } j \notin \{1, 2, 3, 4\},
\end{cases}
\]

let \( E = \{h_1, \ldots, h_n\} \), and let \( S = S_{\mathbb{G}(A(M))^{m-1}}(E) \). From the previous paragraph, we can see that

\[
f(h_1, \ldots, h_n)(j) = \begin{cases} 
\alpha(j) & \text{if } j \in \{1, 2, 3\}, \\
\alpha(j) & \text{if } j \notin \{1, 2, 3, 4\},
\end{cases}
\]

so \( f(h_1, \ldots, h_n) = \alpha(\neq 4) \). Since \( \text{Rel}_{\leq m-1}(A(M)) \models f \), we have that \( f \) preserves \( S \). Therefore \( \alpha(\neq 4) \in S \), so there is a term operation \( t \) such that \( t(h_1, \ldots, h_n) = \alpha(\neq 4) \).

We chose 1, 2, 3, 4 from \( K \), and \( K \) does not include any coordinates \( k \) for which \( A_k(\alpha, k) \) holds for \( \mathbb{R} \). Since \( K \) is also disjoint from \( D(\mathbb{R}) \), we have that \( S_I = \mathbb{R}_I(\neq 4) \). Therefore \( \alpha(\neq 4) \notin R_I \), and so by Proposition \( \ref{prop:universality} \) we can assume that the term operation \( t \) does not have \( I \) in its term tree and hence respects the relation \( \Gamma \) from Definition \( \ref{def:universality} \) by Lemma \( \ref{lem:universality} \).

We will use \( \Gamma \) to show that \( t(\{j\}) = \alpha(j) \) for \( j \in \{1, 2, 3, 4\} \). As \( t(h_1, \ldots, h_n)(j) = \alpha(j) \) for \( j \notin \{1, 2, 3, 4\} \) already, this will finish the proof. Again, this is most easily seen by example — see Figure \( \ref{fig:example} \). The vectors \( h_1(\{1, 2, 3\}), \ldots, h_n(\{1, 2, 3\}) \) make up the first three rows of typical elements of \( \Gamma \). Carefully examining \( \Gamma \), we see that we can complete the \( h_i(\{1, 2, 3\}) \) to elements of \( \Gamma \) in many ways while keeping \( t \) constant on this new row. Due to how \( [1|2|3|4]|_v, [1|2|3|4]|_{3A}, \) and \( [1|2|3|4]|_{3B} \) were defined, there are completions that equal each of \( [1], [2], [3], \) and \( [4] \). Thus \( t(\{j\}) = (\mathfrak{st}(\alpha), A) \) and hence \( t(g_1, \ldots, g_n) = \alpha \), so \( \alpha \in R \), as claimed. \( \square \)
Figure 2. The argument showing $t(\begin{bmatrix} 1 \\ 2 \end{bmatrix}) = \langle \text{st}(\alpha), A \rangle$. For brevity, we show only the content of the vectors. We have $c = A$ because the argument columns are in $\Gamma$ and the term operation $t$ preserves $\Gamma$ since it does not have $I$ in its term tree.

The same approach used to prove the above theorem can also be used to prove entailment when $\times \in \text{con}(\alpha)$. We do this in the next theorem.

**Theorem 8.11.** Assume that $m \geq 11$,

- $\text{Rel}_{\leq m-1}(A(M)) \models f$ and $f$ is $n$-ary,
- $G = \{g_1, \ldots, g_n\} \subseteq A(M)^m$,
- $R = \text{Sg}_{h(M)^m}(G)$ is computational non-halting,
- $f(g_1, \ldots, g_n) = \alpha$ and $\times \in \text{con}(\alpha)$.

Then $\alpha \in R$.

**Proof.** If $G$ is not $\chi$-compatible then $\alpha \in R$ by Proposition 8.8. Assume therefore that $G$ is $\chi$-compatible and assume towards a contradiction that $\alpha \not\in R$. We have that $\text{Rel}_{\leq m-1}(A(M)) \models f$ and $R \leq h(M)^m$, so $R$ has $A(\alpha, i)$ for all $i$. In Definition 8.1 we fixed elements $\alpha_i \in R$ witnessing this. We will make use of these elements in the argument to follow.

Suppose that there are two distinct coordinates $k, \ell$ such that $\alpha(k), \alpha(\ell) \in X$. In this case

$$\alpha = \alpha_k \land \alpha_\ell,$$

so $\alpha \in R$. Therefore there must be a unique coordinate $k$ such that $\alpha(k) \in X$ and $\alpha_k \in Y^m$. We will use this coordinate in the following analysis.

Suppose that there is $\ell$ such that $\alpha(\ell) \in D$. It follows from the definition that

$$\alpha = N_a(\alpha_k, \alpha_\ell, \alpha_k, \alpha_k),$$

so $\alpha \in R$. Therefore $\alpha \in E^m$. Since $\alpha_k \in Y^m$ and $R$ is non-halting, it must be that $D(R(\neq k)) = \emptyset$, by Lemma 7.8 item (2).

Suppose that $R(k) \cap D \neq \emptyset$. Choose $d' \in R$ such that $d'(k) \in D$ and let $d = I(d', \alpha_k)$. It follows that $d(k) \in D$ and $d(\neq k) \in C^{m-1}$, so $z_0(d)(k) \in X$ and $z_0(d)(\neq k) = (0, 0)$ by Lemma 5.3 item (4). Using $N_0$ we now have

$$N_0(z_0(d), \alpha_k, \alpha_k(j)) = \begin{cases} \alpha_k(j) & \text{if } j \neq k \\ \langle \text{st}(\alpha), x \rangle & \text{otherwise.} \end{cases}$$
Since \( \alpha(k) \in X \), it follows that \( N_0(z_0(d), \alpha_k, \alpha_k) = \alpha \) and hence \( \alpha \in R \). Therefore it must be that \( R(k) \cap D = \emptyset \). Combining this with the previous paragraph, we have \( D(\mathbb{R}) = \emptyset \).

At this point, the analysis becomes quite similar to that performed in Theorem \( \text{S.10} \). Let \( K = |m| \setminus \{ k \} \) and find 4 distinct values, call them \( 1, 2, 3, 4 \in K \), such that \( \alpha \) has a common value on these coordinates (we use \( |K| \geq 10 \) here). Using \( f \) and \( G \), produce the row vectors \( \left[ 1 | 2 | 3 | 4 \right]_y \), \( \left[ 1 | 2 | 3 | 4 \right]_A \), and \( \left[ 1 | 2 | 3 | 4 \right]_B \). As before, we have

\[
\begin{bmatrix}
\left[ 1 | 2 | 3 | 4 \right]_y \\
\left[ 1 | 2 | 3 | 4 \right]_A \\
\left[ 1 | 2 | 3 | 4 \right]_B
\end{bmatrix}
= \begin{bmatrix}
(\alpha(1)) \\
(\alpha(1)) \\
(\alpha(1))
\end{bmatrix}
\]

Form \( h_i \in A(M)^m-1 \) from the \( g_i \) by

\[
h_i(j) = \begin{cases}
\left[ 1 | 2 | 3 | 4 \right]_y(i) & \text{if } j = 1, \\
\left[ 1 | 2 | 3 | 4 \right]_A(i) & \text{if } j = 2, \\
\left[ 1 | 2 | 3 | 4 \right]_B(i) & \text{if } j = 3, \\
g_i(j) & \text{if } j \not\in \{1, 2, 3, 4\},
\end{cases}
\]

let \( E = \{h_1, \ldots, h_n\} \), and let \( S = S_{g_{\Delta(A)}}(E) \). From the previous paragraph, we have

\[
f(h_1, \ldots, h_n)(j) = \begin{cases}
\alpha(j) & \text{if } j \in \{1, 2, 3\}, \\
\alpha(j) & \text{if } j \not\in \{1, 2, 3, 4\},
\end{cases}
\]

so \( f(h_1, \ldots, h_n) = \alpha(\neq 4). \) Since \( \text{Rel}_{\Delta_{m-1}}(A(M)) \models f \), we have that \( f \) preserves \( \mathbb{S} \). Therefore \( \alpha(\neq 4) \in S \), so there is a term operation \( t \) such that \( t(h_1, \ldots, h_n) = \alpha(\neq 4) \). There is a difficulty in continuing as we did in the proof of Theorem \( \text{S.10} \) however: we cannot assume that \( I \) does not appear in the term tree of \( t \) since \( \alpha \not\in Y^m \), and so we cannot make use of the relation \( \Gamma \). It turns out that this difficulty is not insurmountable, however.

**Claim.** There is a term operation \( s \) without \( I \) in its term tree such that

\[
s(h_1, \ldots, h_n)(\neq k) \in C^{m-2} \quad \text{and} \quad s(h_1, \ldots, h_n)(k) \in X.
\]

**Proof of claim.** We begin by building some tools. In Section \( \text{S.2} \) we assumed that for each state \( k \) there is a directed path in the state graph to the halting state 0. Similarly to the proof of Lemma \( \text{S.3} \) item (4), for each state \( k \) there is a term \( f \) in the operations \( \{M, M'\} \) such that

\[
f(\langle k, 0 \rangle, \ldots, \langle k, 0 \rangle) = \langle 0, 0 \rangle.
\]

Let \( w_k(x) = H(f(x, \ldots, x)) \). From the definitions, for \( c \neq \bullet \) we have

\[
w_k(\langle k, c \rangle) = \begin{cases}
\langle 0, 0 \rangle & \text{if } c = 0, \\
\langle 0, x \rangle & \text{otherwise},
\end{cases}
\]

(\( D(\mathbb{S}) = \emptyset \), so the value of \( w_k(k, \bullet) \) is immaterial here). Furthermore, from Lemma \( \text{S.4} \) we have that

\[
S_I = \bigcup_{i \text{ a state}} \left\{ \begin{bmatrix}
\langle i, 0 \rangle \\
\vdots \\
\langle i, 0 \rangle
\end{bmatrix}, \begin{bmatrix}
\langle i, x \rangle \\
\vdots \\
\langle i, x \rangle
\end{bmatrix} \right\}.
\]
We will say that the element \( a \in S \) avoids \( I \) if there is a term operation \( s \) without \( I \) in its term tree such that \( s(\overline{b}) = a \). From Proposition \( \ref{prop:avoidance} \) and our observation about \( S_I \) above, we have that if \( b \in S \cap Y^{m-1} \) and \( b \) does not avoid \( I \) then \( \text{con}(b) = \{0\} \). We are now ready to prove the claim.

As usual, we will proceed by induction. Let \( G_0 = S \) be the generators of \( S \) and

\[
G_n = \left\{ F(\overline{b}) \mid F \text{ a fundamental } \ell \text{-ary operation, } \overline{b} \in G_{n-1}^\ell \right\} \cup G_{n-1}.
\]

Choose \( n \) minimal such that there is \( a \in G_n \) with \( a(\neq k) \in C^{m-1} \) and \( a(k) \in X \) (from the paragraph prior to the claim, we know that \( t(\overline{b}) \) is such an element). If \( a \) avoids \( I \) then we are done, so assume that \( a \) does not avoid \( I \). We will prove that there exists an element \( a' \in S \) which avoids \( I \) and has \( a'(\neq k) \in C^{m-2} \) and \( a'(k) \in X \). If \( a \in G_0 = S \) then \( a \) avoids \( I \), so we are done. Assume that \( n > 0 \), so

\[
a = F(b_1, \ldots, b_\ell)
\]

for some \( \ell \)-ary operation \( F \) and elements \( b_1, \ldots, b_\ell \in G_{n-1} \). We proceed by cases depending on which operation \( F \) is. The cases for \( F = I \) and \( F = P \) are quite straightforward (using \( D(S) = \emptyset \) for \( F = I \)), and so we omit them.

**Case** \( F \in \{\lambda, M, N_\bullet\} \): If \( a = b \wedge c \) then \( b(\neq k) = c(\neq k) = a(\neq k) \in C^{m-1} \), so by minimality of \( n \) we have \( b(k), c(k) \in C \) and \( b(k) \neq c(k) \). Hence \( b, c \in Y^{m-1} \). The element \( a \) does not avoid \( I \), so one of \( b \) or \( c \) does not. Without loss of generality, suppose that \( b \) does not avoid \( I \). From Proposition \( \ref{prop:avoidance} \) and the observation about \( S_I \) above, it follows that \( \text{con}(b) = \{0\} \). Since \( c(\neq k) = b(\neq k) \) and \( c(k) \neq b(k) \), it must be that \( \text{con}(c(\neq k)) = \{0\} \) and \( \text{con}(c(k)) \neq 0 \). It follows that \( c \notin S_I \) and thus \( c \) avoids \( I \). The element \( a' = w_{\text{ext}(c)}(c) \) therefore avoids \( I \) and has \( a'(\neq k) \in C^{m-2} \) and \( a'(k) \in X \). The analysis for \( F = M \) is almost identical and \( F = N_\bullet \) similarly reduces since \( D(S) = \emptyset \).

**Case** \( F \in \{M', H\} \): If \( a = M'(b) \) then \( b \) does not avoid \( I \) and by minimality of \( n \) we have \( b \in C^{m-1} \). By Proposition \( \ref{prop:avoidance} \) and the observations about \( S_I \) above, it must be that \( \text{con}(b) = \{0\} \), but then \( M'(b)(\neq k) \in C^{m-1} \) and \( M'(b)(k) \in X \) is impossible. The case for \( F = H \) is similar.

**Case** \( F = N_0 \): If \( a = N_0(b, c, d) \) then \( b(\neq k) \in \{0, 1\}^{m-1} \), \( d(\neq k) \in C^{m-2} \), and either \( d(k) \in X \) or \( b(k) \neq 0, 1 \). The possibility where \( d(k) \in X \) contradicts the minimality of \( n \), so it must be that \( b(k) \neq 0, 1 \). It follows that \( b \notin S_I \). If \( b(k) \in X \) then the minimality of \( n \) is contradicted again, so it must be that \( b \in Y^{m-1} \) and hence avoids \( I \). The element \( a' = w_0(b) \) therefore avoids \( I \) and has \( a'(\neq k) \in C^{m-2} \) and \( a'(k) \in X \).

**Case** \( F = S \): If \( a = S(b, c, d) \) then \( b(\neq k) = c(\neq k) = d(\neq k) = a(\neq k) \in \{0, 1\}^{m-2} \) and one of \( b(k), c(k), d(k) \) is not equal to \( 1, 0 \). Suppose that \( b(k) \neq 1, 0 \). By the minimality of \( n \) we have \( b(k) \notin X \), so \( b \in Y^{m-1} \) and \( \text{con}(b(k)) \neq 0 \). Thus \( b \notin S_I \) and we have that \( b \) avoids \( I \). The element \( a' = w_1(b) \) satisfies the claim.

In all cases, we have produced an element \( a' \in S \) which avoids \( I \) and has \( a'(\neq k) \in C^{m-2} \) and \( a'(k) \in X \), proving the claim.

Apply the above claim to the term operation \( t \) to produce a new term operation \( s \) without \( I \) in its term tree such that \( s(h_1, \ldots, h_n)(\neq k) \in C^{m-2} \) and \( s(h_1, \ldots, h_n)(k) \in X \). Since \( s \) does not have \( I \) in its term tree, it respects \( \Gamma \), and
so as in the proof of Theorem 8.10 we obtain
\[ s(g_1, \ldots, g_n)(\neq k) \in C^{m-1} \quad \text{and} \quad s(g_1, \ldots, g_n)(k) \in X. \]
Let \( r = s(g_1, \ldots, g_n) \). As in the fourth paragraph of the proof, it follows that
\[ N_0(z_0(r), \alpha_k, \alpha_k) = \alpha, \] so \( \alpha \in R \). This completes the proof. \( \square \)

8.3. Entailment for everything else. Finally, we prove that relations not ruled out by the previous entailment theorems are also entailed. This is the result that we have been building towards. We begin by proving an extension of Proposition 7.11.

Proposition 8.12. Let \( \mathbb{R} \leq \mathbb{A}(\mathcal{M})^m \) be computational non-halting. If \( \alpha \in Y^m \) is such that
- \( K = N(\mathbb{R}) \cup \{ i \mid \mathbb{R} \text{ has } A_1(\alpha, i) \} \), and
- \( |m \setminus K| \geq 3 \)
then for all \( k \notin K \) there is an \( \ell \notin K \cup \{ k \} \) such that \( \alpha(\neq k, \ell) \notin (\mathbb{R}(\neq k, \ell))_I \).

Proof. Let \( k \notin K, D_I = D(\mathbb{R}), \) and \( L = [m] \setminus D_I \). By Lemma 7.14 item (3), we have that \( k \notin D_I \). Furthermore, \( \alpha \notin R_I \) since otherwise we would have \( |m| = K \). Since \( \alpha \notin R_I \) and \( \alpha \in Y^m \), by a slight modification of Lemma 8.3 we have that either
- \( \alpha(K) \notin R_I(K) \) or
- \( \text{con}(\alpha(i)) \notin \{0, x\} \) for some \( i \notin K \).

If we are in the first situation then for all \( k, \ell \notin K \) we have \( \alpha(\neq k, \ell) \notin R(\neq k, \ell)_I \). Assume therefore that \( \alpha(K) \in R_I(K) \) and that we are in the second situation. Fix \( i \notin K \) such that \( \text{con}(\alpha(i)) \notin \{0, x\} \). From the definition of \( K \), it is not possible for \( \mathbb{R} \) to have \( A_1(\alpha, i) \), so it must be that there is some \( j \notin K \) distinct from \( i \) such that \( \text{con}(\alpha(j)) \notin \{0, x\} \). We have that \( |m \setminus K| \geq 3 \), so it follows that for every \( k \notin K \) there is an \( \ell \notin K \) distinct from \( k \) such that \( \text{con}(\alpha(\ell)) \notin \{0, x\} \) (just choose \( \ell = i \) or \( \ell = j \)). By Lemma 8.3 and since \( k, \ell \notin K \), this is enough to give us \( \alpha(\neq k, \ell) \notin R(\neq k, \ell)_I \). \( \square \)

The next three lemmas are technical, but form the core of the argument in the entailment theorem in this section. The first of these technical lemmas is a kind of extension of Lemma 7.6.

Lemma 8.13. Assume the following
- \( t \) is an \( n \)-ary term operation,
- \( G = \{g_1, \ldots, g_n\} \subseteq A(\mathcal{M})^m \) and \( \mathcal{E} = \{e_1, \ldots, e_n\} \subseteq A(\mathcal{M})^m \),
- \( \mathbb{R} = S_{g_{k(M)}^m}(G) \) is computational non-halting,
- \( \mathbb{S} = S_{g_{\mathbb{A}(\mathcal{M})}^m}(\mathcal{E}) \) is computational,
- \( k \in [m] \),
- for each \( i \in [n] \) we have \( e_i(\neq k) = g_i(\neq k) \) and \( e_i(k) \leq g_i(k) \), and
- \( t(\overline{a}) = \alpha \) and \( \alpha(k) \in Y \).

Then there exists a term operation \( s \) such that \( \alpha \leq s(\overline{f}) \).

Proof. We begin with a less formal statement of the lemma. View \( G \) and \( \mathcal{E} \) as \( m \times n \) matrices. We obtain \( \mathcal{E} \) from \( G \) by replacing the content of some entries in the \( k \)-th
row with \( \times \). The lemma asserts that if \( \alpha \in S \) has row \( k \) in \( Y \) then it is less than or equal to some element in \( \mathbb{R} \).

Observe that \( \mathbb{R} \) being non-halting implies \( S \) is non-halting. As usual, the proof shall be by induction on the complexity of \( t \). If \( t \) is a projection then \( \alpha = e_i \) for some \( i \), so \( \alpha \leq g_i \). Assume now that

\[
t(\overline{x}) = F(f_1(\overline{x}), \ldots, f_t(\overline{x}))
\]

for some \( \ell \)-ary fundamental operation \( F \) and \( n \)-ary term operations \( f_i \). We will proceed by cases depending on \( F \).

**Case** \( F \in \{ \land, N_0, N_\ast, P \} \): Such \( F \) have the property that \( F(\overline{a}) \leq a_i \) for some \( a_i \) among the \( \overline{a} \), by Lemma 5.8 and since \( S \) and \( \mathbb{R} \) are computational non-halting. If \( \alpha = F(f_1(\overline{x}), \ldots, f_t(\overline{x})) \leq f_i(\overline{x}) \) then \( f_i(\overline{x})(k) = \alpha(k) \in Y \), so the inductive hypothesis applies. Thus there is \( h_i \) such that \( \alpha \leq f_i(\overline{x}) \leq h_i(\overline{y}) \).

**Case** \( F \in \{ M, M', H, S \} \): Such \( F \) are \( X \)-absorbing, by Lemma 5.6 item (2). Therefore, if \( \alpha = F(f_1(\overline{x}), \ldots, f_t(\overline{x})) \) then \( f_i(\overline{x})(k) \in Y \) for all \( i \). The inductive hypothesis applies, so there are \( h_i \) such that \( f_i(\overline{x}) \leq h_i(\overline{y}) \). It follows that \( \alpha \leq F(h_1(\overline{y}), \ldots, h_t(\overline{y})) \).

The remaining (and most complicated) case is \( F = I \). Suppose that \( \alpha = I(f_1(\overline{x}), f_2(\overline{x})) \). Since \( \alpha(k) \in Y \), either \( \alpha(k) \in D \) or \( \alpha(k) \in C \). We will examine these possibilities in their own cases.

**Case** \( F = I, \alpha(k) \in D \): If \( \alpha(k) \in D \) then \( f_1(\overline{x})(k) \in D \), so there is a term operation \( h_1 \) such that \( f_1(\overline{x}) \leq h_1(\overline{y}) \). This implies \( h_1(\overline{y})(k) \in D \). Since \( \mathbb{R} \) is computational and \( I \) depends on its first input only at those coordinates with content \( \bullet \), we have \( \alpha = I(h_1(\overline{y}), f_2(\overline{x})) \).

**Case** \( F = I, \alpha(k) \in C \): If \( \alpha(k) \in C \) then \( f_2(\overline{x})(k) \in C \), so there is a term operation \( h_2 \) such that \( f_2(\overline{x}) \leq h_2(\overline{y}) \). If \( \bullet \notin \text{con}(\alpha) \) then \( \alpha \leq I(h_2(\overline{y}), h_2(\overline{y})) \), and we are done. If, on the other hand, \( \bullet \in \text{con}(\alpha) \) then there is \( j \neq k \) such that \( \alpha(j) \in D \). This implies that \( f_1(\overline{x})(j) \in D \). It follows that \( f_1(\overline{y})(j) \notin D \) since \( \mathbb{R} \) is computational. From the definition of \( I \) we now have \( \alpha \leq I(f_1(\overline{y}), h_2(\overline{y})) \). \( \square \)

**Lemma 8.14.** Assume the following

- \( G = \{ g_1, \ldots, g_n \} \subseteq A(M)^p-1 \),
- \( \mathbb{R} = S_{g_{\mathcal{A}(M)^p-1}(G)} \) is computational non-halting,
- \( t \) is an \( n \)-ary term operation without \( I \) in its term tree,
- \( t(g_1, \ldots, g_n) = \alpha \in Y^{p-1} \), and
- \( k \notin \mathcal{N}(\mathbb{R}) \) is such that \( \alpha(k) \notin D \).

Define elements of \( e_i \in A(M)^p \) for \( i \in [n] \) by

\[
e_i(j) = \begin{cases} 
  g_i(j) & \text{if } j \in [p-1], \\
  g_i(k) & \text{if } j = p \text{ and } c_{\text{on}}(g_i(k)) \in \{ \text{con}(\alpha(k)), \bullet \}, \\
  \chi(g_i(k)) & \text{otherwise},
\end{cases}
\]

\[
\beta(j) = \begin{cases} 
  \alpha(j) & \text{if } j \in [p-1], \\
  \alpha(k) & \text{if } j = p.
\end{cases}
\]

Then \( t(e_1, \ldots, e_n) = \beta \).
Proof. We begin with a less formal statement of the lemma. View $G$ as a $(p-1) \times n$ matrix. Copy row $k$ of this matrix and put it at the bottom, making a $p \times n$ matrix. In row $p$ (the new row), for each entry with content not either $\bullet$ or $\text{con}(\alpha(k))$, replace that content with $\times$. Call the resulting vectors $e_1, \ldots, e_n$. The Lemma asserts that if the copied row $k$ is not in $\mathcal{N}(\mathbb{R})$, $\alpha \in Y^{p-1}$, and $\alpha(k) \notin D$ then $t(\overline{t})$ is just the vector $\alpha$ with the $k$-th row copied to the bottom.

Let $\mathcal{E} = \{e_i \mid i \in [n]\} \subseteq A(M)^p$ and $\mathcal{S} = \text{S}_{g_k(M)^p}(\mathcal{E})$. Let us make some observations about $\mathcal{S}$:

- $\mathcal{S}(\neq p) = \mathbb{R}$,
- $\mathcal{S}$ need not be computational, but it is synchronized,
- $\mathcal{S}(\mathcal{N}(\mathbb{R})) = \mathbb{R}(\mathcal{N}(\mathbb{R}))$ is non-halting, so $\mathcal{S}$ is non-halting as well,
- for each $s \in \mathcal{S}$ there is an $\ell \in \mathcal{N}(\mathbb{R})$ such that $s(\ell) \in D \cup X$, by Lemma \ref{lem:finite-deg-2} item (1).

Now let us examine $\alpha$ and $\beta$. Let $\beta_p = t(e_1, \ldots, e_n)$ and note that $\beta_p(\neq p) = \beta(\neq p) = \alpha$ and $\bullet \in \text{con}(\beta_p(\mathcal{N}(\mathbb{R})))$ by the last item above.

Since $t$ does not have $I$ in its term tree, we will analyze the subset of $\mathcal{S}$ generated by $\mathcal{E}$ without using $I$ in the generation. Call this subset $S'$. Let $G_0 = \mathcal{E}$ and

$$G_n = \left\{ F(\overline{t}) \mid F \text{ a fundamental } k\text{-ary operation, } F \neq I, \overline{t} \in G_{n-1}^k \right\} \cup G_{n-1}$$

Note that $S' = \bigcup G_n$. Since $\beta_p \in S'$, there is a least $n$ such that $\beta_p \in G_n$. We will show that $\beta_p \in G_n$ implies $\beta \in S'$ by induction on $n$. The set $G_0 = \mathcal{E}$ has this property by definition of the $e_i$, establishing the base case. Suppose now that $n > 0$, so

$$\beta_p = F(h_1, \ldots, h_\ell)$$

for some $\ell$-ary fundamental operation $F$ and $h_1, \ldots, h_\ell \in G_{n-1}$. We break into cases based on $F$.

Case $F \in \{\wedge, N_n, P\}$: In this case, by the various parts of Lemma \ref{lem:finite-deg-2} we have that $\beta_p = F(\overline{h}) \leq h_i$ for some $h_i$. Since $\beta_p(\neq p) \in Y^{p-1}$, this implies that $\beta_p(\neq p) = h(\neq p)$, so the inductive hypothesis yields $\beta = h_i \in S'$.

Case $F \in \{H, S\}$: Recall that $\bullet \in \text{con}(\beta_p(\mathcal{N}(\mathbb{R})))$. Since the range of $H$ and $S$ are disjoint from $D$, $\beta_p$ cannot be the output of one of them.

Case $F = M$: Say $\beta_p = M(a, b)$. We have that $a(\neq p), b(\neq p) \in Y^{p-1}$. If $a(k) \in D$ then $\times \in a(\mathcal{N}(\mathbb{R}))$ by Lemma \ref{lem:finite-deg-2} item (1), contradicting $a(\neq p) \in Y^{p-1}$. Hence $a(k) \notin D$, and so from the definition of $M$ we have $\text{con}(a(k)) = \text{con}(b(k)) = \text{con}(\beta_p(k))$. Since $\beta_p(k) = a(k)$, we can use the inductive hypothesis to conclude that $\text{con}(a(p)) = \text{con}(b(p)) = \text{con}(\beta(p))$. Evaluating $M(a, b)$ yields $\beta = M(a, b)$, so $\beta \in S'$.

Case $F = M'$: Say $\beta_p = M'(a)$. From the definition, we have $a(\neq p) \in Y^{p-1}$ and $\text{con}(a(k)) = \text{con}(\beta_p(k))$. Immediately before the start of the induction we observed that $\bullet \in \text{con}(\beta_p(\mathcal{N}(\mathbb{R})))$. Since $k \notin \mathcal{N}(\mathbb{R})$, it follows that $a(k) \notin D$. The inductive hypothesis therefore applies to $a$, and we get $\beta = M'(a)$, so $\beta \in S'$.

Case $F = N_2$: Let $\beta_p = N_2(a, b, c, d)$. If $|a^{-1}(D)| \leq 1$ then $\beta_p \leq b$ or $\beta_p \leq c$ by Lemma \ref{lem:finite-deg-2}. Without loss of generality say $\beta_p \leq b$. Since $\beta_p(\neq p) \in Y^{p-1}$, we have that $\beta_p(\neq p) = b(\neq p)$ and so the inductive hypothesis gives us $\beta = b \in S'$. From the construction of $\mathcal{S}$, the only other possibility is that $a(k) = a(p) \in D$. 

\[ \text{finite degree clones are undecidable} \]
From the definition of $N_\bullet$, we have that $b(\neq k, p) = c(\neq k, p) = \beta_p(\neq k, p)$. Every element of $S$ has content at coordinate $p$ in $\{\text{con}(\beta(k)), \bullet, \times\}$. If $\beta_p(p) \in D$ then $\mathbf{x} \in \text{con}(\beta_p(N(\mathbb{R})))$ by Lemma 8.14 item (1), a contradiction. Let us assume that $\beta_p(p) \in X$, since otherwise $\beta_p = \beta$. By similar logic we have $b(p), c(p) \notin D$, so $b(p) = c(p) \in X$. By the contrapositive of the inductive hypothesis, both $\text{con}(b(k))$ and $\text{con}(c(k))$ are distinct from $\text{con}(\beta_p(k))$. From the definition of $N_\bullet$, this is only possible if $\beta_p(k) \in X$, a contradiction. \hfill \square

**Lemma 8.15.** Assume the following

- $G = \{g_1, \ldots, g_n\} \subseteq A(M)^p$,  
- $\mathbb{R} = \text{Sg}_{\mathbb{R}}(G)$ is computational non-halting,  
- $K$ is a set with $N(\mathbb{R}) \subseteq K$ and $[p-1] \setminus K \geq 2$,  
- $t$ is an $n$-ary term operation without $I$ in its term tree,  
- $t(g_1, \ldots, g_n) = \alpha \in Y^{p-1}$, and  
- for all $i \in [n]$ and each $j \notin K$ we have $\text{con}(g_i(j)) \in \{\text{con}(\alpha(j)), \bullet, \times\}$.

Fix two distinct elements $\ell_1, \ell_2 \notin K$ and define elements of $e_i \in A(M)^p$ for $i \in [n]$ by

$$e_i(j) = \begin{cases} 
g_i(j) & \text{if } j \in [p-1], 
g_i(\ell_1) & \text{if } j = p \text{ and } g_i(\ell_1), g_i(\ell_2) \notin D \cup X, 
g_i(\ell_1) & \text{if } j = p \text{ and } g_i(\ell_1) \in D, 
X(g_i(\ell_1)) & \text{otherwise,} 
\end{cases}$$

$$\beta(j) = \begin{cases} 
\alpha(j) & \text{if } j \in [p-1], 
\alpha(\ell_1) & \text{if } j = p.
\end{cases}$$

Then $t(e_1, \ldots, e_n) = \beta$.

**Proof.** The proof is quite similar to the proof of Lemma 8.14. The less formal statement of the lemma is similar as well. View $G$ as a $(p-1) \times n$ matrix and fix a set of coordinates $K$ such that outside of $K$ the content of the rows of $G$ is always in $\{\text{con}(\alpha(j)), \bullet, \times\}$. Pick two such rows, $\ell_1$ and $\ell_2$. Copy row $\ell_1$ to the bottom of the matrix, so that it is now $p \times n$. For each entry in the new row (row $p$), if above that entry at rows $\ell_1$ and $\ell_2$ we have content $\text{con}(\alpha(k)) \neq \bullet$ at row $\ell_1$ and content $\times$ or $\bullet$ at row $\ell_2$ then replace the content of that entry in row $p$ with $\times$. Call the resulting vectors $e_1, \ldots, e_n$. The lemma asserts that if $\alpha \in Y^{p-1}$ then $t(\mathbf{e})$ is just the vector $\alpha$ with the $\ell_1$-th row copied to the bottom.

Let $\mathcal{E} = \{e_i \mid i \in [n]\} \subseteq A(M)^p$ and $S = \text{Sg}_{\mathbb{R}^p}(\mathcal{E})$. The same observations made in the proof of Lemma 8.14 about $S$ are true here. The most salient are that $S$ need not be computational and that for each $s \in S$ there is $i \in N(\mathbb{R})$ such that $s(i) \in D \cup X$. Let us now examine $\alpha$ and $\beta$. Let $\beta_p = t(e_1, \ldots, e_n)$ and note that $\beta_p(\neq p) = \beta(\neq p) = \alpha$.

Since $t$ does not have $I$ in its term tree, we will analyze the subset of $S$ generated by $\mathcal{E}$ without using $I$ in the generation. Call this subset $S'$. Let $G_0 = \mathcal{E}$ and $G_n = \left\{F(\overline{g}) \mid F \text{ a fundamental } k\text{-ary operation, } F \neq I, \overline{g} \in G_{n-1}^k \right\} \cup G_{n-1}$.

Note that $S' = \bigcup G_n$. Since $\beta_p \in S'$, there is a least $n$ such that $\beta_p \in G_n$. We will show that $\beta_p \in G_n$ implies $\beta \in S'$ by induction on $n$. The proof is very similar to
Lemma 8.14. As in the proof of that lemma, the base case is done by inspection of $E$. The crux in the inductive step is when $F = N_\ast$, so we will leave the other cases for the reader.

Case $F = N_\ast$: Let $\beta_p = N_\ast(a, b, c, d)$. If $|a^{-1}(D)| \leq 1$ then $\beta_p \leq b$ or $\beta_p \leq c$. In either case the inductive hypothesis gives us $\beta \in S'$. If $|a^{-1}(D)| \geq 2$ then the only possibility is that $a(\ell_1) = a(p) \in D$. From the definition of $N_\ast$ we have that $b(\neq \ell_1, p) = c(\neq \ell_1, p) = \beta_p(\neq \ell_1, p)$. Every element of $S'$ has content at coordinate $p$ in $\text{con}(\beta(\ell_1)), \bullet, \times$. If $\beta_p(p) \in D$ then $\times \in \text{con}(\beta_p(\mathcal{N}(\mathbb{R})))$ by Lemma 7.14 item (1), a contradiction. Let us assume that $\beta_p(p) \in X$, since otherwise $\beta_p = \beta$. By similar logic we have $b(p), c(p) \notin D$, so $b(p) = c(p) \in X$. By the inductive hypothesis, we must have $b(\ell_1) = c(\ell_1) \in X$. From the definition of $N_\ast$, this forces $\beta_p(\ell_1) \in X$ as well, a contradiction. $\square$

**Theorem 8.16.** Assume that $m \geq \kappa + 16$ and

- $\text{Rel}_{\leq m-1}(A(\mathcal{M})) \models f$ and $f$ is $n$-ary,
- $G = \{g_1, \ldots, g_n\} \subseteq A(\mathcal{M})^m$,
- $J = \mathsf{Sg}_{\mathcal{A}(\mathcal{M})^m}(G)$ is computational non-halting and $|[m] \setminus D(\mathbb{R})| \leq 10$,
- $f(g_1, \ldots, g_n) = \alpha \in Y^m$,
- there is at most one $k \notin \mathcal{N}(\mathbb{R})$ such that $A_1(\alpha, k)$.

Then $\alpha \in R$.

**Proof.** The proof is by induction on the number of positions in $G$ which are in $Y$:

$$\sum_{i=1}^{m} |\{k \mid g_k(i) \in Y, g_k \in G\}|.$$ 

If this quantity is 0 then $G \subseteq X^m$. Since $X$ is a subuniverse of $A(\mathcal{M})$, we have that $f(\overline{\gamma})(i) \in X$, contradicting $\alpha \in Y^m$. This establishes the basis of the induction. The next claim is the main tool we will use in the induction.

**Claim.** Let $E = \{e_1, \ldots, e_n\} \subseteq A(\mathcal{M})^m$ be synchronized and such that for all $i \in [n]$, some $k \in [m]$, and some $\ell \in [n]$ we have

- $e_i(\neq k) = g_i(\neq k)$,
- $e_i(k) \leq g_i(k)$, and
- $e_\ell(k) < g_\ell(k)$ (i.e. $e_\ell(k) \in X$ and $g_\ell(k) \in Y$).

If $f(\overline{\gamma}) = \alpha$ then $\alpha \in R$.

**Proof of claim.** Let $\mathcal{S} = \mathsf{Sg}_{\mathcal{A}(\mathcal{M})^m}(E)$ and observe that $\mathcal{E}$ has fewer positions in $Y$ than $G$ does. The generators $\mathcal{E}$ are synchronized and (from the hypotheses of the theorem) $|D(\mathbb{R})| \geq \kappa + 6$, so $\mathcal{S}$ is computational. Furthermore, a slight extension of Lemma 8.13 gives us that $R$ being non-halting implies $\mathcal{S}$ is as well. It follows that the inductive hypothesis applies, so we have $\alpha \in \mathcal{S}$ and there is a term operation $t$ such that $t(\overline{\gamma}) = \alpha$. By Lemma 8.13 we obtain a term operation $s$ such that $\alpha \leq s(\overline{\gamma})$. Since $\alpha \in Y^m$, this implies $\alpha = s(\overline{\gamma})$, so $\alpha \in R$. \hfill $\Box$
Let
\[ K' = \mathcal{N}(\mathbb{R}) \cup \{ k \mid \mathbb{R} \text{ has } A_I(\alpha, k) \} \]
and
\[ K = K' \cup \left\{ k \notin K' \mid \text{for all } \ell \notin K' \alpha(\neq k, \ell) \notin R(\neq k, \ell)_I \right\}. \]

Since \( \mathbb{R} \) is non-halting, \( |\mathcal{N}(\mathbb{R}) \cap \mathcal{D}(\mathbb{R})| \leq \kappa \). By hypothesis \( |[m] \setminus \mathcal{D}(\mathbb{R})| \leq 10 \). It follows from these that \( |\mathcal{N}(\mathbb{R})| \leq \kappa + 10 \), so combined with the last hypothesis of the theorem we have \( |K'| \leq \kappa + 11 \). By Proposition 8.12, we also have that \( |K| \leq \kappa + 12 \). Observe that if \( \ell \notin K \) then
- \( \ell \notin \mathcal{N}(\mathbb{R}) \),
- \( \alpha_\ell \notin R_I \) since \( \mathbb{R} \) does not have \( A_I(\alpha, \ell) \), and
- for each \( \ell' \notin K \) there is a set \( L \) with \( |L| = m - 2 \), \( K \cup \{ \ell, \ell' \} \subseteq L \), and \( \alpha(L) \notin R(L)_I \) (by Proposition 8.12 and the construction of \( K \)).

We are now ready to proceed with the proof.

Claim. \( \bullet \in \text{con}(\alpha(\mathcal{N})). \)

Proof of claim. Suppose that \( \bullet \notin \text{con}(\alpha) \) and note that the hypotheses of the theorem imply \( |\mathcal{D}(\mathbb{R})| > \kappa + 1 \). If \( \alpha_i \in C^{m} \) for some \( i \in [m] \) then \( \mathbb{R} \) is halting, by Lemma 7.8 item (2). Therefore for each \( i \) we have \( \alpha_i(i) \in \mathcal{D} \cup \mathcal{X} \) and \( \alpha_i(\neq i) \in C^{m} \), so \( i \in \mathcal{H}(\mathbb{R}) \). It follows that \( \mathcal{H}(\mathbb{R}) = [m] \) and so \( \mathcal{H}(\mathbb{R}) \cap \mathcal{D}(\mathbb{R}) = |\mathcal{D}(\mathbb{R})| > \kappa + 1 \). Thus \( \mathbb{R} \) is halting by Lemma 7.8 item (4), a contradiction. Suppose now that \( \alpha(\iota) \in \mathcal{D} \) but \( \iota \notin \mathcal{N}(\mathbb{R}) \). This implies that \( \alpha(\mathcal{N}(\mathbb{R})) \subseteq C^{[\mathcal{N}(\mathbb{R})]} \cap \mathcal{N}(\mathcal{N}(\mathbb{R})) \), contradicting \( \mathbb{R}(\mathcal{N}(\mathbb{R})) \) being non-halting (Proposition 7.12 item (1)).

Claim. If \( \alpha \notin R \) then for every row \( \ell \notin K \), the content of all the entries is in \( \{\text{con}(\alpha(\ell)), \bullet, \times\} \).

Proof of claim. It follows from the previous claim that \( \alpha(\iota) \in \mathcal{D} \) for some \( i \in \mathcal{N} \). Pick some \( \ell \notin K \). By the observations after the first claim above, there is a set \( L \) such that \( \ell \notin L \), \( |L| = m - 2 \), and \( \alpha(L) \notin R_I(L) \) (we take \( \ell = \ell' \) in the observation). Construct the elements \( e_1, \ldots, e_n \in A(\mathcal{M})^{m-1} \) (on coordinates \( L \)) so that \( e_i(L) = g_i(L) \) for all \( i \), but the \( e_i \) have an “extra” row \( p \notin [m] \). Let \( \mathcal{I} = \{ e_1, \ldots, e_n \} \). Since \( \mathcal{R}_{\leq m-1}(A(\mathcal{M})) = f, \mathcal{I} \subseteq A(\mathcal{M})^{m-1} \), and \( \alpha(L) \notin R_I(L) \), by Proposition 7.14 there is a term operation \( t \) without \( I \) in its term tree such that \( t(\mathcal{I}) = f(\mathcal{I}) \). Apply Lemma 8.14 with this term operation \( t \) to obtain \( f(\mathcal{I})(p) = t(\mathcal{I})(p) = \alpha(\ell) \). The \( p \)-th row of \( \mathcal{I} \) will have at most the same number of \( Y \) entries as the \( \ell \)-th row of \( G \). Let \( \mathcal{E} = \{ h_1, \ldots, h_n \} \) be obtained by replacing the \( \ell \)-th row of \( G \) with the \( p \)-th row of \( \mathcal{I} \). It follows that \( f(\mathcal{E}) = \alpha \), and if \( \mathcal{E} \) has fewer \( Y \) entries than \( G \) then \( \alpha \in R \), by the first claim. The only way for \( \alpha \notin R \) is if for every row \( \ell \notin K \), the content of all the entries is in \( \{\text{con}(\alpha(\ell)), \bullet, \times\} \).

Claim. If \( \alpha \notin R \) then for every distinct \( \ell_1, \ell_2 \notin K \) and \( g_j \in G \), if \( g_j(\ell_1) \in X \cup D \) then \( g_j(\ell_2) \in X \cup D \).

Proof of claim. This is similar to the previous claim, but for Lemma 8.15. Towards a contradiction, pick distinct \( \ell_1, \ell_2 \notin K \) such that \( g_j(\ell_1) \in X \cup D \) and \( g_j(\ell_2) \notin X \cup D \) for some \( j \). As in the proof of the previous claim, we have \( \alpha(\iota) \in D \) for some \( i \in \mathcal{N} \) and there is a set \( L \) such that \( \ell_1, \ell_2 \in L \), \( |L| = m - 2 \), and \( \alpha(L) \notin R_I(L) \) by the observations after the first claim above. Construct the elements \( e_1, \ldots, e_n \in A(\mathcal{M})^{m-1} \) (on coordinates \( L \)) as in Lemma 8.15 so that \( e_i(L) = g_i(L) \) for all \( i \), but
Theorem 9.1. The following are equivalent.

(1) \( \mathcal{M} \) halts,

(2) \( \deg(\mathcal{A}(\mathcal{M})) < \infty \) (i.e. \( \mathcal{A}(\mathcal{M}) \) is finitely related),

(3) \( \mathcal{M} \) halts with capacity \( \deg(\mathcal{A}(\mathcal{M})) \).

Many standard results follow from this theorem. We detail a couple of the more interesting ones below.

- There exists infinitely many Minsky machines \( \mathcal{M} \) such that the halting status of \( \mathcal{M} \) is independent of ZFC (see Chaitin [8] or Kolmogorov [21]).
- As a consequence of the theorem above, there are finite algebras \( \mathcal{A} \) whose finite-relatedness is independent of ZFC.
Let $\sigma$ be a fixed finite algebraic signature (name and arity specification of the functions) and define

$$\text{maxdeg}_\sigma(n) = \sup \left\{ \deg(A) \mid A \text{ has signature } \sigma, \text{ is finite degree, and } |A| \leq n \right\}$$

If we remove the requirement that the algebras have signature $\sigma$ then it isn’t too hard to show that $\text{maxdeg}(n)$ is infinite. Let $\tau$ be the signature of $A(M)$ and observe that $\tau$ does not depend on $M$. It follows from Theorem 9.1 that $\text{maxdeg}_\sigma(n)$ is not computable, and so $\text{maxdeg}_\sigma(n)$ is not, in general, computable. This is essentially the Busy Beaver function of Radó [35].

There are several related problems which are conjectured to be undecidable as well. We have shown that given a finite set of operations $\mathcal{F}$, it is undecidable whether there is finite $\mathcal{R}$ such that $\text{Rel}(\mathcal{F}) = \text{RClo}(\mathcal{R})$. The dual of this problem is also suspected to be undecidable.

**Problem.** Decide if a clone is finitely generated: given finite $\mathcal{R}$, decide whether there is a finite $\mathcal{F}$ such that $\text{Pol}(\mathcal{R}) = \text{Clo}(\mathcal{F})$.

The most sweeping result on finitely related algebras is the following theorem. The “if” portion is due to Aichinger, Mayr, McKenzie [11] and the “only if” portion is due to Barto [4].

**Theorem 9.2.** A finite algebra in a congruence modular variety is finitely related if and only if it has a cube term.

The existence of a cube term is a weak Maltsev condition, but it is a decidable property. This follows independently from Kazda and Zhuk [19] and Kearnes and Szendrei [20]. As a consequence of Theorem 9.1 there can be no decidable property which characterizes finitely related meet-semidistributive algebras (of which $A(M)$ is one). It is still possible, however, that there is an undecidable weak Maltsev condition which does.

**Problem.** Is there a weak Maltsev condition that characterizes finite relatedness for finite algebras in congruence meet-semidistributive varieties?

The next two problems concern the theory of Natural Dualities, and date back to the start of the field in the 1970s (see McNulty [29] section 3 for a history of the problem). A good reference for the background is Clark and Davey [9]. We produce the duality entailment constructions from constructions (1)–(4) of Section 8 by replacing (4) by replacing (4) by

$$\text{(4') bijective projection of a relation onto a subset of coordinates.}$$

A set of relations $\mathcal{R}$ duality entails a relation $\mathcal{R}$ if and only if $\mathcal{R}$ can be constructed in finitely many steps from the entailment constructions (1)–(3) of Section 8 and (4') above. If this is the case then we write $\mathcal{R} \models_\partial \mathcal{R}$ and we refer to the set of all such $\mathcal{R}$ as $\text{RClo}^\partial(\mathcal{R})$.

**Problem.** Decide if every relation of an algebra is duality entailed by a finite subset of them. That is, given algebra $A$, decide whether there is finite $\mathcal{R}$ such that $\text{Rel}(A) = \text{RClo}^\partial(\mathcal{R})$.

It should be clear from the constructions that $\text{RClo}^\partial(\mathcal{R}) \subseteq \text{RClo}(\mathcal{R})$, so if $A$ is finitely duality related then it is finitely related. It follows that $A(M)$ is not finitely duality related if $M$ does not halt.
**Problem.** If $\mathcal{M}$ halts, is $\mathcal{A}(\mathcal{M})$ finitely duality related?

A positive answer to this problem would prove the undecidability of the duality entailment problem. If an algebra is finitely duality related then it is dualizable. The converse does not follow, however. This leads us to a more general (and more important) version of the above problem.

**Problem.** Decide whether a finite algebra is dualizable.

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