Feedback Refinement Relations for the Synthesis of Symbolic Controllers

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Abstract

We present an abstraction and refinement methodology for the automated controller synthesis to enforce general predefined specifications. The designed controllers require quantized (or symbolic) state information only and can be interfaced with the system via a static quantizer. Both features are particularly important with regard to any practical implementation of the designed controllers and, as we prove, are characterized by the existence of a feedback refinement relation between plant and abstraction. Feedback refinement relations are a novel concept of system relations introduced in this paper. Our work builds on a general notion of system with set-valued dynamics and possibly non-deterministic quantizers to permit the synthesis of controllers that robustly, and provably, enforce the specification in the presence of various types of uncertainties and disturbances. We identify a class of abstractions that is canonical in a well-defined sense, and provide a method to efficiently compute canonical abstractions of perturbed nonlinear sampled systems. We demonstrate the practicality of our approach on two examples – a path planning problem for a mobile robot and an aircraft landing maneuver.

Index Terms

Discrete abstraction, symbolic model, nonlinear system, symbolic control, automated synthesis, robust synthesis; MSC: Primary, 93B51; Secondary, 93B52, 93C10, 93C30, 93C55, 93C57, 93C65

I. Introduction

A common approach to engineer reliable, robust, high-integrity hardware and software systems that are deployable in safety-critical environments, is the application of formal verification techniques to ensure the correct, error-free implementation of some given formal specifications. Typically, the verification phase is executed as a distinct step after the design phase, e.g. [1]. In case that the system fails to satisfy the specification, it is the engineer’s burden to identify the fault, adjust the system accordingly and return to the verification phase. A more appealing approach, especially in the context of intricate, complex dynamical systems such as cyber-physical systems, is to merge the design and verification phase and utilize automated correct-by-construction formal synthesis procedures, e.g. [2]. In our treatment of controller design problems we follow the latter approach. That is, given a mathematical system description and a formal specification which expresses the desired system behavior, we seek to synthesize a controller that provably enforces the specification on the system. Subsequently, we often refer to the system that is to be controlled as the plant.

For finite systems, which are described by transition systems with finite state, input and output alphabets, there exist a number of automata-theoretic schemes, known under the label of reactive synthesis, to algorithmically synthesize controllers that enforce complex specifications, possibly formulated in some temporal logic, see e.g. [3]–[7].

Those methods have been extended to infinite systems like cyber-physical systems within an abstraction and refinement framework which roughly proceeds in three steps, e.g. [2], [8]–[10]. In
the first step, the concrete infinite system (together with the specification) is lifted to an abstract domain where it is substituted by a finite system, which is often referred to as abstraction or symbolic model. In the second step, an auxiliary problem on the abstract domain (“abstract problem”) is solved using one of the previously mentioned methods for finite systems. In the third step, the controller that has been synthesized for the abstraction is refined to the concrete system.

The correctness of this controller design concept is usually ensured by relating the concrete system with its abstraction in terms of a system relation. One of the most common approach is based on alternating simulation relations and approximate variants thereof [2]. In this work, we address two shortcomings of the abstraction and refinement process based on (approximate) alternating simulation relations and related concepts. The first shortcoming, which we refer to as the state information issue, results from the fact that the refined controller requires the exact state information of the concrete system. However, usually, the exact state is not known and only quantized (or symbolic) state information is available, which constitutes a major obstacle to the practical implementation of the synthesized controllers. The second issue refers to the huge amount of dynamics added to the abstract controller in the course of its refinement, so that, effectively, the refined controller contains the abstraction as a building block. Given the fact that an abstraction may very well comprise millions of states and billions of transitions [10], [11], an implementation of the refined controller is often too expensive to be practical. We refer to this problem as the refinement complexity issue. We explicitly illustrate both issues by two examples in Section IV. See also [12].

In this paper, we propose a novel notion of system relation, termed feedback refinement relation, to resolve both issues. If the concrete system is related with the abstraction via a feedback refinement relation, then, as we shall show, the abstract controller can be connected to the plant via a static quantizer only, irrespective of the particular specification we seek to enforce on the plant. See Fig. 1. Moreover, the existence of a feedback refinement relation between plant and abstraction is not only

![Figure 1](image.png)

Figure 1. Closed loop resulting from the abstraction and refinement approach based on feedback refinement relations, proposed in this paper.

sufficient to ensure the simple structure of the closed loop in Fig. 1, but in fact also necessary.

Our work builds on a general notion of system with set-valued dynamics and possibly non-deterministic quantizers. This is particularly useful to model various types of disturbances, including plant uncertainties, input disturbances and state measurement errors. We demonstrate how to account for those perturbations in our framework so that the synthesized controllers robustly enforce the specification.

We also present results related to the actual construction of abstractions. First, we show that the set membership relation together with an abstraction whose state alphabet is given as a cover of the concrete state alphabet is canonical. That is, given any abstraction that leads to a solution of a
particular, concrete control problem, we can always find a canonical abstraction that yields a solution as well. Second, we provide a method to efficiently compute canonical abstractions of perturbed nonlinear sampled systems.

Finally, we demonstrate the practicality of our approach on two examples – a path planning problem for a mobile robot and an aircraft landing maneuver.

**Related Work.** Feedback refinement relations are based on the common principle of “accepting more inputs and generation fewer outputs” that is often encountered in component-based design methodologies, e.g. contract-based design [13] and interface theories [14]. Those theories are usually developed in a purely behavioral setting, see e.g. [13]–[15], and are therefore not immediately applicable in our framework which is based on stateful systems. This class of systems contains a great variety of system descriptions, including common models like transition systems [2], [16] as well as discrete-time control systems [17].

There exist a number of abstraction-based controller synthesis methods, based on stateful systems, that do not suffer from the state information issue nor from the refinement complexity issue [10], [18]–[24]. However, none of those approaches offers necessary and sufficient conditions for the controller refinement procedure to be free of the mentioned issues. In addition, the majority of these works are tailored to certain types of specifications or systems. Specifically, simple safety and reachability problems are considered in [22], [23] and [10], [18], [19], [22], respectively, and GR(1) specifications are considered in [21], while [20], [22], [23] is limited to piecewise affine, incrementally stable, and simple integrator dynamics, respectively. Moreover, plants are assumed to be non-blocking in [10], [18]–[24]. In contrast, our framework covers stateful systems with general, set-valued dynamics, including transitions systems and discrete-time control systems as special cases. We allow systems to be blocking, which is useful e.g. to model sampled control systems with finite escape time or in the context of discrete event systems whenever the occurrence of the next event is not guaranteed. Additionally, any linear time property can serve as a specification.

Besides the limited specifications and systems, often a lack of robustness further restricts the applicability of the methods. For example, [18], [20], [22] do not cover uncertainties in plant dynamics, while in [19]–[22] the quantizer is assumed to be deterministic which mandates the state measurement to be precise, without any error, see Section VI-B.

The synthesis scheme in [24] is, similarly to our work, based on a novel system relation. However, in contrast to the theory in [24], feedback refinement relations do not rely on a metric of the state alphabet, which is crucial in establishing the necessity as well as the canonicity result. Likewise, the authors of [24] consider perturbations, but assume that the effect of these perturbations is given as level sets of a metric.

Additionally to a general controller synthesis framework, we present a method to construct abstractions of perturbed nonlinear control systems. The abstractions are based on a cover of the state alphabet by non-empty compact hyper-intervals and the over-approximation of attainable sets of those hyper-intervals under the system dynamics. While the utilization of attainable sets for the construction of abstractions is a well-known concept [10], [11], [19], [25], [26], none of the aforementioned works accounts for uncertainties or perturbations. Moreover, while our method to over-approximate attainable sets is similar to those in [11], [26] in that it is based on a growth bound, we present several extensions that render the approach more efficient.

To summarize, our contribution is threefold. First, we introduce feedback refinement relations as a novel means to synthesize symbolic controllers. We show that feedback refinement relations are necessary and sufficient for the controller refinement that solves the state information issue and the refinement complexity issue. Our theory applies to a more general class of synthesis problems than previous research that addresses the mentioned issues, and in particular, any linear time property can serve as a specification. Second, our work permits the synthesis of controllers that robustly, and provably, enforce the specification in presence of various uncertainties and disturbances. Third, we identify a class of canonical abstractions and presented a method to compute such abstractions. Our
construction improves known methods in several directions and thereby, as we demonstrate by some numerical examples, facilitates a more efficient computation of abstractions of perturbed nonlinear control systems.

Some of the results we present have been announced in [12].

II. Notation

\( \mathbb{R}, \mathbb{R}_+, \mathbb{Z} \) and \( \mathbb{Z}_+ \) denote the sets of real numbers, non-negative real numbers, integers and non-negative integers, respectively, and \( \mathbb{N} = \mathbb{Z}_+ \setminus \{0\} \). We adopt the convention that \( \pm \infty + x = \pm \infty \) for any \( x \in \mathbb{R} \). \([a, b], [a, b[ , [a, b], \) and \([a, b] \) denote closed, open and half-open, respectively, intervals with end points \( a \) and \( b \). \([a; b], a; b[, [a; b[, \) and \([a; b] \) stand for discrete intervals, e.g. \([a; b] = [a, b] \cap \mathbb{Z} \), \([1; 4[ = \{1, 2, 3\} \), and \([0; 0[ = \emptyset \).

In \( \mathbb{R}^n \), the relations \( <, \leq, \geq, \succ \) are defined component-wise, i.e., \( a < b \) iff \( a_i < b_i \) for all \( i \in [1; n] \).

\( f : A \Rightarrow B \) denotes a set-valued map of \( A \) into \( B \), whereas \( f : A \rightarrow B \) denotes an ordinary map; see [27]. If \( f \) is set-valued, then \( f \) is strict and single-valued if \( f(a) \neq \emptyset \) and \( f(a) \) is a singleton, respectively, for every \( a \). The restriction of \( f \) to a subset \( M \subseteq A \) is denoted \( f|_M \). Throughout the text, we denote the identity map \( X \rightarrow X : x \mapsto x \) by \( \text{id} \). The domain of definition \( X \) will always be clear from the context.

We identify set-valued maps \( f : A \Rightarrow B \) with binary relations on \( A \times B \); i.e., \( (a, b) \in f \) iff \( b \in f(a) \). Moreover, if \( f \) is single-valued, it is identified with an ordinary map \( f : A \rightarrow B \). The inverse mapping \( f^{-1} : B \Rightarrow A \) is defined by \( f^{-1}(b) = \{a \in A | b \in f(a)\} \), and \( f \circ g \) denotes the composition of \( f \) and \( g \), \( (f \circ g)(x) = f(g(x)) \).

The set of maps \( A \rightarrow B \) is denoted \( B^A \), and the set of all signals that take their values in \( B \) and are defined on intervals of the form \( [0; T[ \) is denoted \( B^\infty \), \( B^\infty = \bigcup_{T \in \mathbb{Z}_+ \cup \{\infty\}} B^{[0; T[} \).

Given the sets \( A \) and \( \{a\} \), we identify the set \( A \times \{a\} \) with \( A \), whenever \( a \) is clear from the context or is not relevant in the current context. Similarly, we identify \( f : A \times \{a\} \Rightarrow B \) with \( f(\cdot, a) : A \Rightarrow B \).

III. Plants, Controllers, and Closed Loops

A. Systems

We consider dynamical systems of the form

\[
\begin{align*}
  x(t + 1) &\in F(x(t), u(t)) \\
y(t) &\in H(x(t), u(t)).
\end{align*}
\]

The motivation to use a set-valued transition function \( F \) and a set-valued output function \( H \) in our system description, originates from the desire to describe disturbances and other kinds of non-determinism in a unified and concise manner. Moreover, this description is sufficiently expressive to model the plant and the controller. However, in order to define a meaningful serial composition or a closed loop composed of two of this general type of systems, we need internal variables when we define the transition function and output map of the overall system. Consider e.g. the serial composition in Fig. 2. Here, the introduction of the internal variable \( u_2 \) (or, equivalently, \( y_1 \)) ensures that the constraint \( u_2 = y_1 \) is enforced simultaneously in \( F_2(x_2, y_1) \) and \( H_2(x_2, y_1) \) even if \( H_1 \) is multi-valued. As a result we consider a slightly more general system description given by

![Figure 2. Serial composition of two dynamical systems of the form (1). The symbol // denotes a delay.](image-url)
where \( v \) is used as internal variable to enforce those interconnection constraints. We formalize the system as follows.

**III.1 Definition.** A system \( S \) is a sextuple

\[
S = (X, U, V, Y, F, H),
\]

where \( X, U, V \) and \( Y \) are nonempty sets, \( H : X \times U \supseteq Y \times V \) is strict, and \( F : X \times V \supseteq X \).

A quadruple \((u, v, x, y) \in U^{[0; T]} \times V^{[0; T]} \times X^{[0; T]} \times Y^{[0; T]}\) is a solution of the system \((3)\) (on \([0; T]\), starting at \((0)\)) if \( T \in \mathbb{N} \cup \{\infty\}\), \((2a)\) holds for all \( t \in [0; T - 1] \), and \((2b)\) holds for all \( t \in [0; T] \).

We call the sets \( X, U, V, \) and \( Y \) the state, input, internal variable, and output alphabet, respectively. The functions \( F \) and \( H \) are, respectively, the transition function and the output function of \((3)\).

We recover system \((1)\) without internal variables, if \( V = U \) and for all \( y \in Y \) and \( x \in X \), the inclusion \((y, v) \in H(x, u)\) implies \( v = u \). In this case, we say that \( S \) is basic and drop the internal variable space in the definition of \( S \), i.e., \( S = (X, U, Y, F, H) \), as well as the second element of the image in the output function, i.e., \( H : X \times U \supseteq Y \). The definition of solution \((u, x, y)\) of \( S \) without internal variables is adapted accordingly.

We say that \( S \) is autonomous if its input alphabet is a singleton, and it is static if it is basic, its state alphabet is a singleton and its transition function is strict. Since the solutions of a static system \( S \) with state alphabet \( \{x\} \) is essentially determined by the set-valued output map \( H(x, \cdot) \) we identify \( S \) with its output function \( H \), and interchangeably refer to a set-valued map \( S : U \supseteq Y \) as system \( S \) and as a set-valued function.

We frequently denote solutions by \((v, x, y)\) (resp. \((x, y)\)) rather than by \((0, v, x, y)\) (resp. \((0, x, y)\)), provided the system \( S \) is autonomous (resp. autonomous and basic).

The system is *Moore* if its output does not directly depend on its input, i.e., if the following condition holds:

\[
(y, v) \in H(x, u) \land u' \in U \Rightarrow \exists_v (y, v') \in H(x, u').
\]

The system is Moore with state output if \( X = Y \) and \((y, v) \in H(x, u)\) implies \( y = x \). We remark that the system \( S \) is basic and Moore with state output iff \( H \) is the identity map on \( X \times U \).

**B. System composition**

In the following, we define the serial and feedback composition of two systems. We start with the serial composition.

**III.2 Definition.** Let \( S_i = (X_i, U_i, V_i, Y_i, F_i, H_i) \) be systems, \( i \in \{1, 2\} \), and assume that \( Y_1 \subseteq U_2 \).

Then \( S_1 \) is serial composable with \( S_2 \), and the serial composition of \( S_1 \) and \( S_2 \), denoted \( S_2 \circ S_1 \), is the sextuple

\[
(X_{12}, U_1, V_1, Y_2, F_{12}, H_{12}),
\]

where \( X_{12} = X_1 \times X_2 \), \( V_{12} = V_1 \times V_2 \), and \( F_{12} : X_{12} \times V_{12} \supseteq X_{12} \) and \( H_{12} : X_{12} \times U_1 \supseteq Y_2 \times V_{12} \) satisfy

\[
F_{12}(x, v) = F_1(x_1, v_1) \times F_2(x_2, v_2),
\]

\[
H_{12}(x, u_1) = \{(y_2, v) \mid \exists_{y_1} (y_1, v_1) \in H_1(x_1, u_1) \land (y_2, v_2) \in H_2(x_2, y_1)\}.
\]

We readily see that the output function \( H_{12} \) is strict which implies that \( S_2 \circ S_1 \) is a system.

We use the serial composition mainly to describe the interconnection of an input quantizer \( Q : U' \supseteq U \) or a state quantizer \( Q : X \supseteq X' \) with a system \( S \). We assume that \( Q \) is strict and interpret the
quantizer as a static system. Suppose that $U'$ is a non-empty set, then the serial composition $S \circ Q$ of $Q$ and $S$ is defined by

$$S \circ Q = (X, U', V, Y, F, H'),$$

where $H': X \times U' \Rightarrow Y \times V$ takes the form $H'(x, u') = H(x, Q(u'))$. Now suppose that $S$ is a basic Moore system with state output, then we may interpret $Q: X \Rightarrow X'$ as a measurement map that yields a quantized version of the state of the system $S$. This situation is modeled by the serial composition $Q \circ S$ of $S$ and $Q$,

$$Q \circ S = (X, U, X', F, H'),$$

where $H'$ takes the form $H'(x, u) = Q(x)$. Note that $Q \circ S$ is again a basic Moore system (though not with state output).

We turn our attention to the feedback composition of two systems as illustrated in Fig. 3.

**III.3 Definition.** Let $S_i = (X_i, U_i, V_i, Y_i, F_i, H_i)$ be systems, $i \in \{1, 2\}$, and assume that $S_2$ is Moore, $Y_2 \subseteq U_1$, and $Y_1 \subseteq U_2$, and that the following condition holds:

(Z) If $(y_2, v_2) \in H_2(x_2, y_1)$, $(y_1, v_1) \in H_1(x_1, y_2)$ and $F_2(x_2, v_2) = \emptyset$, then $F_1(x_1, v_1) = \emptyset$.

Then $S_1$ is **feedback composable** with $S_2$, and the **closed loop** composed of $S_1$ and $S_2$, denoted $S_1 \times S_2$, is the sextuple

$$(X_{12}, \{0\}, V_{12}, Y_{12}, F_{12}, H_{12}),$$

where $X_{12} = X_1 \times X_2$, $V_{12} = V_1 \times V_2$, $Y_{12} = Y_1 \times Y_2$, and $F_{12}: X_{12} \times V_{12} \Rightarrow X_{12}$ and $H_{12}: X_{12} \Rightarrow Y_{12} \times V_{12}$ satisfy

$$F_{12}(x, v) = F_1(x_1, v_1) \times F_2(x_2, v_2),$$

$$H_{12}(x) = \{(y, v) | (y_1, v_1) \in H_1(x_1, y_2) \land (y_2, v_2) \in H_2(x_2, y_1)\}.$$

![Figure 3. Closed loop $S_1 \times S_2$ of systems $S_1$ and $S_2$ according to Definition III.3, in which the system $S_2$ is required to be Moore.](image)

The requirement (Z), which has its analog in the theory developed in [2], is particularly important and will be needed later to ensure that if the concrete closed loop is non-blocking, then so is the abstract closed loop. The assumption that $S_2$ is additionally Moore is common [28] and ensures that the closed loop does not contain a delay free cycle.

We also point out that we do not include initial states in the system description but rather account for them in the specification, and we avoid the assumption that the controller is allowed to set the initial state of the plant, as appears e.g. in [2].

We conclude this section with a proposition that we use in several proofs throughout the paper.

**III.4 Proposition.** Let $S_1$ be feedback composable with $S_2$, and let $T \in \mathbb{N} \cup \{\infty\}$. Then the closed loop $S_1 \times S_2$ is an autonomous Moore system, and $(v, x, y)$ is a solution of $S_1 \times S_2$ on $[0; T]$ iff $(y_2, v_1, x_1, y_1)$ is a solution of $S_1$ on $[0; T]$ and $(y_1, v_2, x_2, y_2)$ is a solution of $S_2$ on $[0; T]$. 
IV. Motivation

In this section, we provide two simple examples that demonstrate the previously mentioned shortcomings that led to the development of the novel notion of feedback refinement relation. The first example demonstrates the state information issue, i.e., the refined controller requires full state information and cannot be symbolic. The second example illustrates the refinement complexity issue, e.g., a static controller for the abstraction cannot be refined to a static controller for the concrete system. Both examples show that the drawbacks do not depend on the specific refinement technique, but are intrinsic to the use of alternating simulation relations.

Let us consider two basic Moore systems $S_1$ and $S_2$ and two controllers $C_1$ and $C_2$, which are also basic systems,

$$S_i = (X_i, U, F_i, Y, H_i)$$
$$C_i = (X_{c,i}, Y, F_{c,i}, U, H_{c,i}),$$

in which we assume that the transition functions of the four systems are all strict and that $H_i(x) = \{x\}$ for all $x \in X_i$. Hence, we readily see that the controller $C_i$ is feedback composable with the system $S_i$, $i \in \{1, 2\}$. Subsequently, we interpret $S_1$ as the concrete system and $S_2$ as its abstraction.

Let $Q \subseteq X_1 \times X_2$ be a strict relation. Then $Q$ is an alternating simulation relation from $S_1$ to $S_2$ if the following holds for every pair $(x_1, x_2) \in Q$:

(ASR) If $u_2 \in U$, then there exists $u_1 \in U$ such that the condition

$$\emptyset \neq Q(x'_1) \cap F_2(x_2, u_2).$$

(7)

holds for every $x'_1 \in F_1(x_1, u_1)$.

Note that usually there is an additional condition on outputs of related states, which here would have required the notion of approximate rather than ordinary alternating simulation relation [2, Def. 9.6]. Since that subtlety is not essential to our discussion, we omit it here in favor of a clearer presentation.

As already mentioned, alternating simulation relations are often used to prove the correctness of a particular abstraction-based controller design procedure. The very center of any such argument is the reproducibility of the system behavior of the concrete closed loop $C_1 \times S_1$ by the abstract closed loop $C_2 \times S_2$, i.e., for every solution $((x_{c,1}, x_{s,1}), y_1)$ of $C_1 \times S_1$ on $\mathbb{Z}_+$ there exists a solution $((x_{c,2}, x_{s,2}), y_2)$ of $C_2 \times S_2$ on $\mathbb{Z}_+$ satisfying

$$(x_{s,1}(t), x_{s,2}(t)) \in Q \text{ for all } t \in \mathbb{Z}_+.$$  

(8)

This reproducibility property is then used to provide evidence that certain properties that the abstract closed loop $C_2 \times S_2$ satisfies, actually also hold for the concrete closed loop $C_1 \times S_1$.

In the first example, we show that (8) cannot hold if $C_1$ attains state information only through $Q$, i.e., if $C_1$ takes the form $C'_1 \circ Q$. In other words, it is shown that we are not able to refine a controller $C'_1$ from $C_2 \times S_2$ for the system $Q \circ S_1$.

IV.1 Example. We consider the systems $S_1$ and $S_2$ which we graphically illustrate by
The input and output alphabets of $S_1$ and $S_2$ are given by $U = \{0, 1\}$ and $Y = \{1, 2, 3\}$, respectively. The transition functions should be clear from the illustration, e.g. $F_1(2, 1) = \{1\}$ and $F_1(1, u) = \{1\}$ for any $u \in U$. It is also easily verified that the relation $Q$ given by $Q = \{(1, 1), (2, 3), (3, 3)\}$ is an alternating simulation relation from $S_1$ to $S_2$.

Let the abstract controller $C_2$ be static with $H_{c,2}(0, 3) = \{0\}$, i.e., $C_2$ enables exactly the control letter 0 at the abstract state 3. If the concrete controller $C_1$ is symbolic, then, at the initial time, the sets of control letters enabled at the plant states 2 and 3 coincide. Indeed, these sets must only depend on the associated abstract states, and $Q(2) = Q(3)$. In addition, by the symmetry of the plant $S_1$, we may assume without loss of generality that the control letter 0 is enabled at the initial time, so that there exists a solution $((x_{c,1}, x_{s,1}), y_1)$ of the closed loop $C_1 \times S_1$ satisfying $x_{s,1}(0) = x_{s,1}(1) = 2$. Then the condition (8) requires $x_{s,2}(0) = x_{s,2}(1) = 3$ to hold for any solution $((x_{c,2}, x_{s,2}), y_2)$ of $C_2 \times S_2$ — a requirement that contradicts the dynamics of $C_2 \times S_2$. This shows that the property of reproducibility cannot be attained using a symbolic controller for the plant $S_1$. The crucial point with this example is that the condition (ASR) cannot be satisfied if the choice of $u_1$ depends only on the abstract states associated with the plant state $x_1$, but not directly on $x_1$ itself. To see this, let $u_2 = 0$ and $x_2 = 3$. Then $u_1 = 0$ is required if $x_1 = 3$, whereas only $u_1 = 1$ is admissible if $x_1 = 2$.

In the next example we show that a static controller $C_2$ for the abstraction $S_2$ cannot be refined to a static controller $C_1$ for the concrete system $S_1$.

**IV.2 Example.** We consider the systems $S_1$ and $S_2$ with the transition functions illustrated graphically by

\[
\begin{align*}
S_1 : & \quad 3 \quad \text{0} \quad 1 \quad \text{1} \quad \text{0} \quad 2 \\
S_2 : & \quad 3 \quad \text{0} \quad 1 
\end{align*}
\]

The common input alphabet is given by $U = \{0, 1\}$ and the common output alphabet, by $Y = \{1, 2, 3, 4\}$. It is easily verified that the relation $Q$ given by $Q = \{(1, 1), (2, 2), (2, 3), (4, 4)\}$ is an alternating simulation relation from $S_1$ to $S_2$. In addition, in this example the relation $Q$ satisfies even the more restrictive requirement that $u_1 = u_2$ holds in (ASR).

Let a static abstract controller $C_2$ be given that enables exactly the control letters 0 and 1 at the abstract states 2 and 3, respectively. If the concrete controller $C_1$ is static, then the set of control letters enabled at the plant state 2 does not vary with time. By the symmetry of the plant $S_1$, we may again assume without loss of generality that the control letter 0 is enabled at the state 2, so that there exists a solution $((x_{c,1}, x_{s,1}), y_1)$ of the closed loop $C_1 \times S_1$ satisfying $x_{s,1}(0) = x_{s,1}(2) = 1$. Then the condition (8) asks for $x_{s,2}(0) = x_{s,2}(2) = 1$ for some solution $((x_{c,2}, x_{s,2}), y_2)$ of $C_2 \times S_2$ — a requirement that contradicts the dynamics of $C_2 \times S_2$. This shows that the property of reproducibility cannot be attained using a static controller for the plant $S_1$ despite the fact that the abstract controller is static. The crucial point with this example is that the condition (7) only mandates that for each transition from $x_1$ to $x_1'$ in $S_1$ there exists a state $x_2' \in Q(x_1')$ that is a successor of $x_2$ in $S_2$, but it is not required that every $x_2' \in Q(x_1')$ succeeds $x_2$; consider e.g. the case $x_1 = x_2 = 1$, $x_1' = x_2' = 2$. As a result, the state 1 and 4 cannot precede the state 2 and 3, respectively, in $S_2$, and so, implicitly, the static controller $C_2$ has some access to the history of the solution. In contrast, at the state 2 the dynamics of $S_1$ does not encode analogous information, which in fact could here only be provided by
a controller for $S_1$ that is dynamic rather than static.

As our examples show, alternating simulation relations are not adequate for the controller refinement, whenever i) the concrete controller has merely symbolic state information and ii) the complexity of the refined controller should not exceed the complexity of the abstract controller.

We conclude this section with two important remarks.

First, we would like to point out that in both examples the respective relation $Q$ is not merely an alternating simulation relation according to our definition in (ASR), but also an 1-approximate bisimulation relation and 1-approximate alternating bisimulation relation according to Definitions 9.5 and 9.8 in [2], respectively. Hence, the latter concepts also suffer from both issues described in this section.

Second, we would like to emphasize that we restricted our attention to basic controllers throughout this section for the sake of brevity of presentation only. The same arguments hold for the case that the controllers are given as systems with internal variables, i.e., $C_i = (X_{c,i}, Y_{c,i}, U_{c,i}, F_{c,i}, H_{c,i})$.

V. Feedback Refinement Relations

In this section, we introduce feedback refinement relations as a novel means to compare systems in the context of controller synthesis.

A. Definition and basic properties

We start by introducing the behavior of a system, where we follow the notion of infinitary completed trace semantics [29].

V.1 Definition. Let $S$ denote the system (3). The behavior of $S$, denoted $\mathcal{B}(S)$, is defined by

$$\mathcal{B}(S) = \{(u, y) | \exists v, x, T(u, v, x, y) \text{ is a solution of } S \text{ on } [0; T],$$

$$\text{and if } T < \infty, \text{ then } F(x(T - 1), v(T - 1)) = \emptyset\}.$$ (9)

Note that it often occurs that a system is non-continuable for a certain state-input pair, e.g. the terminating state of a terminating program. With our notion of system behavior, which possibly consists of finite signals as well as infinite signals, such signals are naturally included as valid elements of the system behavior.

In our definition of system relation below, we need a notion of state dependent admissible inputs. For any basic system $S$ of the form (3), we define the set $U_S(x)$ of admissible inputs at the state $x \in X$ by

$$U_S(x) = \{u \in U \mid F(x, u) \neq \emptyset\}.$$

V.2 Definition. Let $S_i = (X_i, U_i, X_i, F_i, id)$ be basic Moore systems with state output, $i \in \{1, 2\}$, and assume $U_2 \subseteq U_1$. A strict relation $Q \subseteq X_1 \times X_2$ is a feedback refinement relation from $S_1$ to $S_2$ if the following holds for all $(x_1, x_2) \in Q$:

(i) $U_{S_2}(x_2) \subseteq U_{S_1}(x_1)$;
(ii) $u \in U_{S_2}(x_2) \Rightarrow Q(F_1(x_1, u)) \subseteq F_2(x_2, u)$.

The fact that $Q$ is a feedback refinement relation from $S_1$ to $S_2$ will be denoted $S_1 \preceq_Q S_2$, and we write $S_1 \preceq_S S_2$ if $S_1 \preceq_Q S_2$ holds for some $Q$.

Intuitively, and similarly to simulation relations and their variants, a feedback refinement relation from a system $S_1$ to a system $S_2$ associates states of $S_1$ with states of $S_2$, and imposes certain conditions on the local dynamics of the systems in the associated states. However, while e.g. alternating simulation relations only require that for each input $u_2$ admissible for $S_2$ there exists an associated input $u_1$ admissible for $S_1$ [2], our definition above additionally mandates that $u_1 = u_2$. Moreover, the definition of (approximate) alternating simulation relation requires that for each transition from
$x_1$ to $x'_1$ in $S_1$ there exists a state $x'_2$ associated with $x'_1$ and a transition from $x_2$ to $x'_2$ in $S_2$; see condition (7). In contrast, feedback refinement relations require the existence of the latter transition for every state $x'_2$ associated with $x'_1$.

With the following proposition, we show that the feedback refinement relations are closed under union and that $\preceq$ is reflexive and transitive. The proof uses Definition V.2 in a straightforward way and is omitted.

**V.3 Proposition.** For all basic Moore systems $S_1$, $S_2$ and $S_3$ with state output, the following holds.

(i) $S_1 \preceq_{id} S_1$.
(ii) If $S_1 \preceq_Q S_2$ and $S_1 \preceq_R S_2$, then $S_1 \preceq_{Q \cup R} S_2$.
(iii) If $S_1 \preceq_Q S_2$ and $S_2 \preceq_R S_3$, then $S_1 \preceq_{R \cap Q} S_3$.

**B. Feedback composability and behavioral inclusion**

In the following, we present the main result of this section. We consider three systems $S_1$, $S_2$ and $C$ and assume that $C$ is feedback composable with $S_2$. We first prove that, given a feedback refinement relation $Q$ from $S_1$ to $S_2$, $Q \circ S_1$ and $S_1$ are, respectively, feedback composable with $C$ and $C \circ Q$. Subsequently, we show that the behavior of the closed loops $C \times (Q \circ S_1)$ and $(C \circ Q) \times S_1$ are both fully reproducible by the closed loop $C \times S_2$.

Even though we do not assign any particular role to the systems $S_1$, $S_2$ and $C$, in foresight of the next section, where we use our result to develop abstraction-based solutions of general control problems, we might regard $S_1$, $S_2$ and $C$ as the plant, the abstraction and controller for the abstraction, respectively. In this context, we might assume that the state of $S_1$ is accessible only through the measurement map $Q$. In that case, $Q \circ S_1$ actually represents the system for which we seek a controller and the behavior of $\mathcal{B}(C \times (Q \circ S_1))$ is of interest. Alternatively, we may start with the premise that a controller for $S_1$ needs to be realizable on a digital device and hence, can accept only a finite input alphabet. In that case, we may interpret $Q$ as an input quantizer for the discrete controller $C$ and the behavior of $\mathcal{B}((C \circ Q) \times S_1)$ is of interest. In any case, we show that both behaviors are reproduced by the abstract close loop $\mathcal{B}(C \times S_2)$.

**V.4 Theorem.** Let $Q$ be a feedback refinement relation from the basic system $S_1$ to the basic system $S_2$,

$$S_i = (X_i, U_i, X_i, F_i, id),$$

for $i \in \{1, 2\}$, and assume that the system $C$ is feedback composable with $S_2$. Then the following statements are true.

(i) $C$ is feedback composable with $Q \circ S_1$, and $C \circ Q$ is feedback composable with $S_1$.
(ii) $\mathcal{B}(C \times (Q \circ S_1)) \subseteq \mathcal{B}(C \times S_2)$.
(iii) For every $(u, x_1) \in \mathcal{B}((C \circ Q) \times S_1)$ there exists a map $x_2$ such that $(u, x_2) \in \mathcal{B}(C \times S_2)$ and $(x_1(t), x_2(t)) \in Q$ for all $t$ in the domain of $x_1$.

**Proof.** By our hypotheses, $S_1$ and $S_2$ are Moore with state output, $U_2 \subseteq U_1$, the serial composition $Q \circ S_1$ is Moore,

$$Q \circ S_1 = (X_1, U_1, X_2, F_1, H'_1),$$

where $H'_1$ takes the form $H'_1(x, u) = Q(x)$, and the systems $S_1$, $S_2$, and $Q \circ S_1$ are all basic. Let the system $C$ be of the form

$$C = (X_c, U_c, V_c, Y_c, F_c, H_c),$$

and observe that $Y_c \subseteq U_1$ and $X_2 \subseteq U_c$ as $C$ is feedback composable (f.c.) with $S_2$. Moreover, since $X_1 \neq \emptyset$ and $Q$ is strict, the serial composition $C \circ Q$ is well-defined,

$$C \circ Q = (X_c, X_1, V_c, Y_c, F_c, H'_c),$$
where $H'_c$ takes the form $H'_c(x_c, x_1) = H_c(x_c, Q(x_1))$.

To prove (i), we first observe that the conditions
\[ x_2 \in Q(x_1), (u, v) \in H_c(x_c, x_2), F_1(x_1, u) = \emptyset \] (13)
together imply $F_c(x_c, v) = \emptyset$. Indeed, it follows from (13) and the requirement (i) in Definition V.2 that $F_2(x_2, u) = \emptyset$, and our claim follows as $C$ is f.c. with $S_2$. This shows that $C$ is f.c. with $Q \circ S_1$. Similarly, let $x_1 \in X_1$, $(u, v) \in H'_c(x_c, x_1)$ and $F_1(x_1, u) = \emptyset$. Then, by the definition of $H'_c$, there exists $x_2 \in Q(x_1)$ such that $(u, v) \in H_c(x_c, x_2)$. Then (13) holds, and so $F_c(x_c, v) = \emptyset$ as we have already shown. Hence, $C \circ Q$ is f.c. with $S_1$, which completes the proof of (i).

To prove (ii), let $(u, x_2) \in \mathcal{B}(C \times (Q \circ S_1))$ be defined on $[0; T]$, $T \in \mathbb{N} \cup \{\infty\}$. Then there exist maps $x_c, x_1$ and $v$ such that $((v, u), (x_c, x_1), (u, x_2))$ is a solution of $C \times (Q \circ S_1)$ on $[0; T]$. Moreover, if additionally $T < \infty$, then we also have
\[ F_c(x_c(T - 1), v(T - 1)) = \emptyset \lor F_1(x_1(T - 1), u(T - 1)) = \emptyset. \] (14)
By Proposition III.4, $(u, u, x_1, x_2)$ is a solution of $Q \circ S_1$ on $[0; T]$, and $(x_2, v, x_c, u)$ is a solution of $C$ on $[0; T]$. The former fact implies the following.
\[
\begin{align*}
\forall t \in [0; T] & \quad x_2(t) \in Q(x_1(t)), \\
\forall t \in [0; T - 1] & \quad x_1(t + 1) \in F_1(x_1(t), u(t)).
\end{align*}
\] (15) (16)
We claim that $(u, u, x_2, x_2)$ is a solution of $S_2$, so that $((v, u), (x_c, x_2), (u, x_2))$ is a solution of $C \times S_2$ by Proposition III.4. First, we observe that $F_2(x_2(t), u(t)) \neq \emptyset$ for every $t \in [0; T - 1]$. Indeed, $(u(t), v(t)) \in H_c(x_c(t), x_2(t))$ for every such $t$ since $(x_2, v, x_c, u)$ is a solution of $C$ on $[0; T]$. Hence, $F_2(x_2(t), u(t)) = \emptyset$ for some $t \in [0; T - 1]$ implies $F_c(x_c(t), v(t)) = \emptyset$ as $C$ is f.c. with $S_2$. This is a contradiction as $x_c(t + 1) \in F_c(x_c(t), v(t))$, so $F_2(x_2(t), u(t)) \neq \emptyset$ for every $t \in [0; T - 1]$. Consequently, $u(t) \in U_{S_2}(x_2(t))$ for all $t \in [0; T - 1]$, so (15), (16) and the requirement (ii) in Definition V.2 imply that $x_2(t + 1) \in F_2(x_2(t), u(t))$ for all $t \in [0; T - 1]$. This shows that $((v, u), (x_c, x_2), (u, x_2))$ is a solution of $C \times S_2$ on $[0; T]$.

Finally, we see that if $T < \infty$ and $u(T - 1) \in U_{S_2}(x_2(T - 1))$, then (15) and the requirement (i) in Definition V.2 together imply $F_1(x_1(T - 1), u(T - 1)) \neq \emptyset$, and in turn, (14) shows that $F_c(x_c(T - 1), v(T - 1)) = \emptyset$. Thus, $(u, x_2) \in \mathcal{B}(C \times S_2)$, which proves (ii).

To prove (iii), let $(u, x_1) \in \mathcal{B}((C \circ Q) \times S_1)$ be defined on $[0; T]$, $T \in \mathbb{N} \cup \{\infty\}$. Then there exist maps $x_c$ and $v$ such that $((v, u), (x_c, x_1), (u, x_1))$ is a solution of $(C \circ Q) \times S_1$ on $[0; T]$. Moreover, if additionally $T < \infty$, then we also have
\[ F_c(x_c(T - 1), v(T - 1)) = \emptyset \lor F_1(x_1(T - 1), u(T - 1)) = \emptyset. \] (17)
By Proposition III.4, $(u, u, x_1, x_1)$ and $(x_1, v, x_c, u)$ is a solution of $S_1$ and $C \circ Q$, respectively. In particular, by the definition of $H'_c$, there exists a map $x_2:[0; T] \to X_2$ such that $x_2(t) \in Q(x_1(t))$ and $(u(t), v(t)) \in H_c(x_c(t), x_2(t))$ for all $t \in [0; T]$. Then $(x_2, v, x_c, u)$ is a solution of $C$ and $Q \circ S_1$, respectively, so $((v, u), (x_c, x_1), (u, x_2))$ is a solution of $C \times (Q \circ S_1)$ by Proposition III.4. We next observe that if $T < \infty$ and $F_1(x_1(T - 1), u(T - 1)) \neq \emptyset$, then (17) implies $F_c(x_c(T - 1), v(T - 1)) = \emptyset$. This shows that $(u, x_2) \in \mathcal{B}(C \times (Q \circ S_1))$, and so (iii) follows from (ii).

Next we show, that feedback refinement relations are not only sufficient, but indeed necessary for the controller refinement as considered in this paper.

**V.5 Theorem (Necessity).** Consider two basic Moore systems $S_i$ with state output, $i \in \{1, 2\}$, of the form (10) and a strict relation $Q \subseteq X_1 \times X_2$. If for any system $C$ that is feedback composable with $S_2$ follows that $C$ is feedback composable with $Q \circ S_1$ and $\mathcal{B}(C \times (Q \circ S_1)) \subseteq \mathcal{B}(C \times S_2)$ holds, then $Q$ is a feedback refinement relation from $S_1$ to $S_2$.

**Proof.** In the proof we consider systems $Q \circ S_1$ and $C$ of the form (11) and (12), respectively.
Let $C$ be such that $U_c = X_2$, $Y_c = U_2$, $X_c = V_c = \{0\}$, and $F_c(0, 0) = \emptyset$. Obviously, $C$ is feedback composable (f.c.) with $S_2$, and in turn, $C$ is f.c. with $Q \circ S_1$ by our hypothesis. Then $Y_c \subseteq U_1$, and so $U_2 \subseteq U_1$ as required in Definition V.2.

To prove that $Q$ satisfies the condition (i) in Definition V.2, let $(x_1, x_2) \in Q$ and $u \in U_2$, and assume that $F_2(x_2, u) \neq \emptyset$. In order to show that $F_1(x_1, u) \neq \emptyset$, let $C$ be such that $U_c = V_c = X_2$, $Y_c = U_2$, $X_c = \{0\}$, $F_c(0, x_2') = X_c$, $H_c(0, x_2') = \{(u, x_2')\}$ for all $x_2' \in X_2$, and $F_c(0, x_2') = \emptyset$ for all $x_2' \in X_2 \setminus \{x_2\}$. Then $C$ is f.c. with $S_2$. In particular, the condition (Z) in Definition III.3 reduces to $F_2(x_2, u) \neq \emptyset$. Then $C$ is also f.c. with $Q \circ S_1$ by our hypothesis, and here the condition (Z) implies $C(x_1, u) \neq 0$, which proves our claim.

To prove that $Q$ satisfies the condition (ii) in Definition V.2, we choose $C$ as basic system with $X_c = \{x_2\}$, $U_c = X_2$, $Y_c = U_2$, $H_c$ and $F_c$ implicitly by $u \in H_c(x_2, x_2)$ and $\{x_c\} = F_c(x_2, x_2)$ if $F_2(x_2, u) \neq \emptyset$ and $U_2 = H_c(x_2, x_2)$ and $\emptyset = F_c(x_2, x_2)$ if $F_2(x_2, u) = \emptyset$ holds for all $u \in U_2$. With this definition of $C$ condition (Z) holds and $C$ is f.c. with $S_2$, and by our hypothesis, $C$ is also f.c. with $Q \circ S_1$. If condition (ii) does not hold, then there exist $(x_1, x_2) \in Q$, $u \in U_2(x_2)$, $x_1' \in F_1(x_1, u)$ and $x_2' \in Q(x_2')$ such that $x_2' \notin F_2(x_2, u)$. Let $\bar{x}_1 = x_1 x_1'$ and $\bar{u} = uu'$ with $u' \in H_c(x_2, x_2')$. Then $(\bar{u}, \bar{x}_1, \bar{x}_1)$ is a solution of $S_1$ on $[0; 2]$. Define $\bar{x}_2 = x_2 x_2'$ and observe that $(\bar{u}, \bar{x}_1, \bar{x}_2)$ is a solution of $Q \circ S_1$. Let $\bar{x}_c = x_c x_2$, since $F_2(x_2, u) \neq \emptyset$, we see that $u \in H_c(x_2, x_2)$ and $\{x_c\} = F_c(x_2, x_2)$. Also $u' \in H_c(x_2, x_2')$ by construction of $u$ and thus $(\bar{x}_2, \bar{x}_c, \bar{u})$ is a solution of $C$. Hence by Proposition III.4 we see that $(0, (\bar{x}_c, \bar{x}_2))$ is a solution of $C \times (Q \circ S_1)$. Consider $(\bar{u}, \bar{x}_2) \in B(C \times (Q \circ S_1))$ with $\bar{u}|_{[0; 2]} = \bar{u}$ and $\bar{x}_2|_{[0; 2]} = \bar{x}_2$. Since $\bar{x}_2(1) \notin F_2(\bar{x}_2(0), \bar{u}(0))$ the sequence $(0, (\bar{x}_c, \bar{x}_2), (\bar{u}, \bar{x}_2))$ cannot be a solution of $C \times S_2$, and so $(\bar{u}, \bar{x}_2) \notin B(C \times S_2)$. This is a contradiction, which establishes condition (ii) in Definition V.2.

VI. Symbolic Controller Synthesis

In this section, we use Theorem V.4 to synthesize controllers with respect to general specifications. In the first part, we formally introduce the control problem and show how to solve it with the help of an abstraction. In the second part, utilizing our general framework based on set-valued transition functions and quantizers, we show how to synthesize controllers that are robust with respect to various disturbances, including model uncertainties, input disturbances and measurement errors.

A. Solution of control problems

We begin with the definition of the synthesis problem.

VI.1 Definition. Let $S$ denote the system (3). Given a set $Z$, any subset $\Sigma \subseteq Z^\infty$ is called a specification on $Z$. A system $S$ is said to satisfy a specification $\Sigma$ on $U \times Y$ if $B(S) \subseteq \Sigma$. Given a specification $\Sigma$ on $U \times Y$, the system $C$ solves the control problem $(S, \Sigma)$ if $C$ is feedback composable with $S$ and the closed loop $C \times S$ satisfies $\Sigma$.

It is clear that we can use linear temporal logic (LTL) to define a specification for a given system $S$. Indeed, suppose that we are given a finite set $P$ of atomic propositions, a labeling function $L : U \times Y \Rightarrow P$ and an LTL formula $\varphi$ defined over $P$, see e.g. [16, Chapter 5]. Then we can formulate the control problem $(S, \Sigma)$ to enforce the formula $\varphi$ on $S$ using the specification

$$\Sigma = \{(u, y) \in (U \times Y)^{Z^+} \mid L \circ (u, y) \text{ satisfies } \varphi\}.$$  

With our notion of specification we can additionally define specifications that are not expressible in LTL as e.g. the requirement that the closed loop is contractive.

We are now going to solve control problems using Theorem V.4. As we have already discussed, the concrete control problem $(S_1, \Sigma_1)$ will not be solved directly. Instead, we will consider an auxiliary problem for the abstraction ("abstract control problem"), whose solution will induce a solution of the concrete problem.
VI.2 Definition. Let the systems $S_1$ and $S_2$ take the form (10), let $\Sigma_1$ be a specification on $U_1 \times X_1$, and let $Q \subseteq X_1 \times X_2$ be a strict relation. A specification $\Sigma_2$ on $U_2 \times X_2$ is called an abstract specification associated with $S_1$, $S_2$, $Q$ and $\Sigma_1$, if the following condition holds. 

If $(u, x_2) \in \Sigma_2$, where $x_2$ and $u$ are defined on $[0; T]$ for some $T \in \mathbb{N} \cup \{\infty\}$, and if $x_1 : [0; T] \rightarrow X_1$ satisfies $(x_1(t), x_2(t)) \in Q$ for all $t \in [0; T]$, then $(u, x_1) \in \Sigma_1$.

For the sake of simplicity, we write $(S_1, \Sigma_1) \preceq_Q (S_2, \Sigma_2)$ whenever $S_1 \preceq_Q S_2$ and $\Sigma_2$ is an abstract specification associated with $S_1$, $S_2$, $Q$ and $\Sigma_1$. The result presented below shows that the technique of abstraction-based controller synthesis using feedback refinement relations is feasible and resolves the state information and refinement complexity issues as explained and illustrated in Sections I and IV.

VI.3 Theorem. If $(S_1, \Sigma_1) \preceq_Q (S_2, \Sigma_2)$ and the abstract controller $C$ solves the abstract control problem $(S_2, \Sigma_2)$, then the refined controller $C \circ Q$ solves the concrete control problem $(S_1, \Sigma_1)$.

Proof. As $C$ solves $(S_2, \Sigma_2)$, $C$ is feedback composable with $S_2$, and hence, $C \circ Q$ is feedback composable with $S_1$ by Theorem V.4.

It remains to show that $\mathcal{B}((C \circ Q) \times S_1) \subseteq \Sigma_1$. So, let $(u, x_1) \in \mathcal{B}((C \circ Q) \times S_1)$ be arbitrary and invoke Theorem V.4 again to see that there exists a map $x_2$ such that $(u, x_2) \in \mathcal{B}(C \times S_2)$ and $(x_1(t), x_2(t)) \in Q$ for all $t$ in the domain of $x_2$. Then $(u, x_2) \in \Sigma_2$ since $C$ solves $(S_2, \Sigma_2)$, and the definition of the abstract specification $\Sigma_2$ shows that $(u, x_1) \in \Sigma_1$. 

The above result shows how to use a solution of an auxiliary, abstract control problem to arrive at a solution of the concrete control problem. Details on how to solve the abstract control problems are beyond the scope of the present paper. Indeed, large classes of these problems can be solved efficiently using standard algorithms, e.g. [2], [5]–[7], [30], [31].

B. Uncertainties and disturbances

In this subsection, we show that it is an easy task in our framework to synthesize controllers that are robust with respect to various disturbances including plant uncertainties, input disturbances and measurement errors. In particular, we demonstrate that the synthesis of a robust controller can be reduced to the solution of an auxiliary, unperturbed control problem.

Let us consider the closed loop illustrated in Fig. 4 consisting of a plant given by a basic Moore system $S_1$ with state output,

$$S_1 = (X_1, U_1, X_1, F_1, \text{id}),$$  \hspace{1cm} (18)

the perturbation maps $P_1$, given by strict set-valued maps with non-empty domains

$$P_1 : \hat{U}_1 \ni U_1, \quad P_2 : X_1 \ni \hat{X}_1,$$

$$P_3 : \hat{U}_1 \ni Y_1, \quad P_1 : X_1 \ni Y_2,$$  \hspace{1cm} (19)

and a strict quantizer

$$Q : \hat{X}_1 \ni X_2.$$  \hspace{1cm} (20)

We seek to synthesize a controller given as a system

$$C = (X_c, X_2, V_c, \hat{U}_1, F_c, H_c),$$  \hspace{1cm} (21)

to robustly enforce a given specification $\Sigma_1$ on $Y_1 \times Y_2$.

The behavior of the closed loop in Fig. 4 is defined as the set of all sequences $(y_1, y_2) \in (Y_1 \times Y_2)^{[0; T]}$, $T \in \mathbb{N} \cup \{\infty\}$, for which there exist a solution $(u, x, x)$ of $S_1$ on $[0; T]$ and a solution $(u_c, v_c, x_c, y_c)$ of $C$ on $[0; T]$ that satisfy the following two conditions:
(i) For all $t \in [0; T]$ we have
\[
    u(t) \in P_1(y_c(t)), \quad u_c(t) \in Q(P_2(x(t))),
    y_1(t) \in P_3(y_c(t)), \quad y_2(t) \in P_4(x(t)).
\] (22)

(ii) If $T < \infty$, then
\[
    F_1(x(T - 1), u(T - 1)) = \emptyset, \quad \text{or} \quad F_c(x_c(T - 1), v_c(T - 1)) = \emptyset.
\] (23)

Figure 4. Various perturbations in the closed loop.

It is straightforward to observe, that the perturbations maps $P_1$ and $P_2$ may be used to model input disturbances and measurement errors, respectively. We assume that the uncertainties of the dynamics of $S_1$ have already been modeled by the set-valued transition function $F_1$. The controller $C$ and the quantizer $Q$, which will usually be discrete, are not subject to any additional perturbations either. The maps $P_3$ and $P_4$ are useful in the presence of output disturbances. For example, the plant $S_1$ might represent a sampled variant of a continuous-time control system and the specification of the desired behavior is naturally formulated in continuous time, rather than in discrete time. In that context, one can use $P_3$ and $P_4$ to “robustify” the specification like in [32] such that properties of the sampled behavior carry over to the continuous-time behavior.

Given some specifications $\Sigma_1$ on $Y_1 \times Y_2$ and $\Sigma_1$ on $\hat{U}_1 \times X_1$, we call $\Sigma_1$ a robust specification of $\Sigma_1$ w.r.t. $P_3$ and $P_4$ if for the functions $(u, x, y_1, y_2) \in (\hat{U}_1 \times X_1 \times Y_1 \times Y_2)^{|0; T|}$, $T \in \mathbb{N} \cup \{\infty\}$, we have that
\[
    (u, x) \in \hat{\Sigma}_1 \quad \text{and} \quad \forall t \in [0; T]\ y_1(t) \in P_3(u(t)), y_2(t) \in P_4(x(t))
\]
implies $(y_1, y_2) \in \Sigma_1$.

In the following result, we present sufficient conditions for a controller $C$ to robustly enforce a given specification $\Sigma_1$ on the perturbed closed loop illustrated in Fig. 4, in terms of the auxiliary basic Moore system $S_1$ with state output,
\[
    \hat{S}_1 = (X_1, \hat{U}_1, X_1, \hat{F}_1, \text{id}),
    \quad \hat{F}_1(x, u) = F_1(x, P_1(u)),
\] (24)

(24)
together with a robust specification $\hat{\Sigma}_1$ of $\Sigma_1$. We show in the subsequent corollary, which follows immediately by Theorem VI.3, how to use an abstraction $(S_2, \Sigma_2)$ to synthesize such a controller $C$.

VI.4 Theorem. Consider a system $S_1$, perturbation maps $P_i$, $i \in [1; 4]$, a quantizer $Q$, and a controller $C$ as illustrated in Fig. 4 and respectively defined in (18), (19), (20) and (21), and assume that $F_1$ is strict. Let $\Sigma_1$ be a specification on $Y_1 \times Y_2$. Let $(\hat{S}_1, \hat{\Sigma}_1)$ be an auxiliary control problem, where $\hat{S}_1$ follows from $S_1$ according to (24) and $\hat{\Sigma}_1$ is a robust specification of $\Sigma_1$ w.r.t. $P_3$ and $P_4$. 

If $C \circ \hat{Q}$, with $\hat{Q} = Q \circ P_2$, solves the control problem $(\hat{S}_1, \hat{\Sigma}_1)$, then the behavior of the perturbed closed loop in Fig. 4 is a subset of $\Sigma_1$.

Proof. Our assumptions imply that $C \circ \hat{Q}$ is feedback composable with $\hat{S}_1$. Using Definition III.3, Proposition III.4 and the properties (22)-(23), it is straightforward to show that $(y_1, y_2)$ is an element of the behavior of the closed loop in Fig. 4 iff there exists $(y_c, x) \in B((C \circ \hat{Q}) \times \hat{S}_1)$ satisfying $y_1(t) \in P_3(y_c(t))$ and $y_2(t) \in P_4(x(t))$ for all $t$. Here, the assumption that $F_1$ is strict excludes the case that $\hat{S}_1$ is non-blocking while $S_1$ is blocking. Consequently, if $(y_1, y_2)$ is an element of the behavior of the closed loop in Fig. 4, then there exist $(y_c, x) \in \Sigma_1$ satisfying $y_1(t) \in P_3(y_c(t))$ and $y_2(t) \in P_4(x(t))$ for all $t$, and so $(y_1, y_2) \in \Sigma_1$ by the definition of $\hat{\Sigma}_1$. □

VI.5 Corollary. In the context of Theorem VI.4, if $C$ solves an abstract control problem $(S_2, \Sigma_2)$ with $(\hat{S}_1, \hat{\Sigma}_1) \preceq_{\hat{Q}} (S_2, \Sigma_2)$, where $X_2$ is the state space of $S_2$, then the behavior of the closed loop in Fig. 4 is a subset of $\Sigma_1$.

In the following example we demonstrate that it is crucial to account for the measurement errors $P_2$ in terms of the auxiliary quantizer $\hat{Q} = Q \circ P_2$, as opposed to accounting for those type of disturbances in terms of an alternative auxiliary system $\hat{S}_1 = (X_1, U_1, X_1, F_1, \text{id})$ with $F_1$ given by

$$F_1(x, u) = P_2(F_1(x, P_1(u))).$$

(25)

VI.6 Example. We consider the system $S_1$ of the form (18) with the transition function illustrated graphically

![Diagram](image)

The state and input alphabet are given by $X_1 = \{a, b, c, d\}$ and $U_1 = \{0, 1\}$, respectively. Suppose we are given the specification $\Sigma_1$ on $U_1 \times X_1$ defined implicitly by $(u, x) \in \Sigma_1$ iff $d$ is in the image of $x$. Let us consider the quantizer $Q = \text{id}$ and the perturbation maps $P_1 = P_3 = P_4 = \text{id}$ and $P_2$ defined by $P_2(a) = \{a\}$, $P_2(b) = P_2(c) = \{b, c\}$ and $P_2(d) = \{d\}$. Let the auxiliary system $\hat{S}_1$ coincide with $S_1$ except the transition function is given by $\hat{F}_1(x, u) = P_2(F_1(x, u))$.

The controller $C \circ \hat{Q}$, with $C$ given as static system, with the output map $H_c : X_1 \rightarrow U_1$ defined by $H_c(a) = H_c(d) = U_1$, $H_c(b) = \{1\}$, $H_c(c) = \{0\}$ solves the control problem $(\hat{S}_1, \hat{\Sigma}_1)$. However, the sequence

$$(u, x) = ((0, a), (1, c), (1, c), (1, c), \ldots)$$

which violates the specification $\Sigma_1$ is a behavior of the closed loop according to Fig. 4. □

As the example demonstrates, we cannot rely on the auxiliary system with transition function (25) to synthesize a robust controller but we need a quantizer that is robust with respect to disturbances. That is essentially expressed by requiring that $C \circ \hat{Q}$ with $\hat{Q} = Q \circ P_2$ solves the auxiliary control problem $(\hat{S}_1, \hat{\Sigma}_1)$. Intuitively, we require that the controller $C$ “works” with any quantizer symbol $x_2 \in Q(P_2(x_1))$ no matter how the disturbance $P_2$ is acting on the state $x_1$. Note that in Example VI.6, the controller $C \circ (\text{id} \circ P_2)$ does not solve the control problem $(\hat{S}_1, \hat{\Sigma}_1)$ (which in this case equals $(S_1, \Sigma_1)$).

As a last remark we would like to mention that in the context of control systems, any symbolic controller synthesis procedure that is based on a deterministic quantizer is bound to be non-robust. Indeed, consider the context of Theorem VI.4 and suppose that $X_1 = \mathbb{R}^n$, $X_2$ is a partition of $X_1$ and let $P_2(x_1)$ equal the closed Euclidean ball with radius $\varepsilon \geq 0$ centered at $x_1$. Let us consider the deterministic quantizer $Q = \varepsilon$. Then $\hat{Q} = Q \circ P_2$ is deterministic only in the degenerate case $\varepsilon = 0$. 
VII. Canonical Feedback Refinement Relations

In this section, we show that the set membership relation $\in$, together with an abstraction whose state alphabet is a cover of the concrete state alphabet is canonical. A cover of a set $X$ is a set of subsets of $X$ whose union equals $X$.

We show that $(S_1, \Sigma_1) \preceq_Q (S_3, \Sigma_3)$ implies that there exist $(S_2, \Sigma_2)$, with $X_2$ being a cover of $X_1$ by non-empty subsets, together with a relation $R$ such that

$$(S_1, \Sigma_1) \preceq \in (S_2, \Sigma_2) \preceq_R (S_3, \Sigma_3)$$

holds. This implies that if we can solve the concrete control problem $(S_1, \Sigma_1)$ using some abstract control problem $(S_3, \Sigma_3)$, then we can equally use an abstract control problem $(S_2, \Sigma_2)$ with $X_2$ corresponding to a cover of $X_1$ by non-empty subsets. Moreover, $(S_2, \Sigma_2)$ can be derived from the problem $(S_3, \Sigma_3)$ and the quantizer $Q$ alone and is otherwise independent of $(S_1, \Sigma_1)$.

A. Canonical abstractions

VII.1 Proposition. Let $S_1$ and $S_2$ be basic Moore systems with state output, in which $S_1$ and $S_2$ are of the form (10) and $X_2$ is a cover of $X_1$ by non-empty subsets, and let $U_2 \subseteq U_1$. Then $S_1 \preceq \in S_2$ iff the following conditions hold.

(i) $x \in \Omega \in X_2$ implies $U_{S_1}(\Omega) \subseteq U_{S_2}(x)$.

(ii) If $\Omega, \Omega' \in X_2$, $u \in U_{S_1}(\Omega)$ and $\Omega' \cap F_1(\Omega, u) \neq \emptyset$, then $\Omega' \in F_2(\Omega, u)$.

The above result, whose straightforward proof we omit, will be used in our proof of the canonicity result, Theorem VII.2. It additionally indicates constructive methods to compute a canonical abstraction $S_2$ of a plant $S_1$ if the abstract state space $X_2$ and the input alphabet $U_2 \subseteq U_1$ are given. From condition (ii) it follows that, if $\Omega \in X_2$, $u \in U_2$ and $F_1(x, u) \neq \emptyset$ for every $x \in \Omega$, then we may either choose $F_2(\Omega, u)$ to be empty, which is of course not desirable, or ensure that the latter set contains every cell $\Omega'$ that intersects the attainable set $F_1(\Omega, u)$ of the cell $\Omega$ under the control letter $u$. This can be achieved by numerically over-approximating attainable sets, for which many efficient algorithms are available, see e.g. [10] and Section VIII.

On the other hand, condition (i) requires that $F_2(\Omega, u)$ is empty whenever $F_1(x, u)$ is so for some $x \in \Omega$. This raises the question of how to detect the phenomenon of blocking of the dynamics of the plant. If the transition function $F_1$ is explicitly given, we assume that its description directly facilitates the detection of blocking. In the case that the plant represents a sampled system, so that $F_1$ is the time-$\tau$-map of some continuous-time control system, blocking can usually be detected in the course of over-approximating attainable sets. For example, if an over-approximation $W$ of the attainable set $F_1(\Omega, u)$ is computed using interval arithmetic, and if $F_1(x, u) = \emptyset$ for some $x \in \Omega$, then $W$ will be unbounded, e.g. [33, Chapter II.3], which is easily detected.

B. Canonicity result

Before we state and prove the canonicity result, we introduce a technical condition that we impose on the feedback refinement relation $Q$ from $(S_1, \Sigma_1)$ to $(S_3, \Sigma_3)$, i.e.,

(C) if $\emptyset \neq Q^{-1}(x) = Q^{-1}(\tilde{x})$, $\emptyset \neq Q^{-1}(x') = Q^{-1}(\tilde{x}')$, $\tilde{x}' \in F_3(\tilde{x}, u)$, and $u \in U_{S_3}(x)$, then $x' \in F_3(x, u)$.

We point out that condition (C) is not an essential restriction and it actually holds for a great variety of different abstractions and relations. For example, it automatically holds if the abstraction $S_3$ is defined as a quotient system [2, Definition 4.17]. In that case, the elements of $X_3$ correspond to

1 One should always choose $F_q(\Omega, u) \neq \emptyset$, since it enlarges the set of control letters available to any abstract controller and thereby facilitates the solution of the abstract control problem.
the equivalence classes of an equivalence relation on $X_1$. Therefore, we have that $Q^{-1}(x) = Q^{-1}(\tilde{x})$ implies $x = \tilde{x}$ and condition (C) is trivially satisfied. Similarly, relations that are based on level sets of simulation functions $V : X_1 \times X_3 \to \mathbb{R}^+$ with $X_1, X_3 \subseteq \mathbb{R}^n$, see e.g. [34], for popular choices of simulation functions like $V(x_1, x_3) = \sqrt{(x_1 - x_3)^TP(x_1 - x_3)}$ with $P$ being a positive definite matrix, where $x^T$ denotes the transpose of $x$, satisfy (C). In this case, the relation is given by $Q = \{(x_1, x_3) \in X_1 \times X_3 \mid V(x_1, x_3) \leq \varepsilon\}$ and again $Q^{-1}(x) = Q^{-1}(\tilde{x})$ implies $x = \tilde{x}$ and we conclude that (C) holds. Lastly, the condition (C) also holds, for the case that $Q$ is given and the abstraction $S_3$ is computed using a deterministic algorithm to over-approximate attainable sets. This is immediate from the following reformulation of the condition (ii) in Definition V.2: If $x_2, x_2' \in X_2$, $u \in U_{S_2}(x_2)$, and $Q^{-1}(x_2') \cap F_1(Q^{-1}(x_2), u) \neq \emptyset$, then $x_2' \in F_2(x_2, u)$.

VII.2 Theorem. Let $(S_3, \Sigma_3)$ be a control problem, in which $S_3$ is of the form (10). Let $X_1$ be any set, and assume that $Q : X_1 \to X_3$ satisfies the condition (C).

Then there exist $S_2$ of the form (10), a relation $R \subseteq X_2 \times X_3$ and a specification $\Sigma_2$ on $U_2 \times X_2$ such that the following holds.

If $(S_1, \Sigma_1) \preceq_Q (S_3, \Sigma_3)$ and the system $S_1$ has state space $X_1$, then $(S_1, \Sigma_1) \preceq_\varepsilon (S_2, \Sigma_2) \preceq_R (S_3, \Sigma_3)$ and $X_2$ is a cover of $X_1$ by non-empty subsets.

Proof. We will prove that our claim holds for the following choices: $X_2 = \{\Omega \mid \emptyset \neq \Omega = Q^{-1}(x) \land x \in X_3\}$, $R(\Omega) = \{x \in X_3 \mid \Omega = Q^{-1}(x)\}$, $U_2 = U_3$, $F_2(\Omega, u) = R^{-1}(F_3(R(\Omega), u))$, and $(u, \Omega) \in (U_2 \times X_2)^\infty$ is an element of $\Sigma_2$ iff there exists $(u, x_3) \in \Sigma_3$ satisfying $(\Omega(t), x_3(t)) \in R$ for all $t$ in the domain of $u$.

Assume now that $(S_1, \Sigma_1) \preceq_Q (S_3, \Sigma_3)$, which directly implies our claim on $X_2$. It is also easy to see that $S_2$ is a basic Moore system with state output.

To prove $S_1 \preceq_\varepsilon S_2$, we first notice that the condition (i) in Proposition VII.1 is obviously satisfied; it remains to establish (ii). To this end, we assume that the system $S_1$ takes the form (10). Let $\Omega, \Omega' \in X_2$ and $u \in U_{S_2}(\Omega)$ and assume that $\Omega' \cap F_1(\Omega, u) \neq \emptyset$. By the latter fact there exist $x_1 \in \Omega$ and $x_1' \in \Omega' \cap F_1(x_1, u)$, and $u \in U_{S_2}(\Omega)$ implies that there exists $x_3$ such that $\Omega = Q^{-1}(x_3)$ and $u \in U_{S_2}(x_3)$. We pick $x_3'$ satisfying $\Omega' = Q^{-1}(x_3')$. Then $(x_1, x_3), (x_1', x_3') \in Q$, and so $S_1 \preceq_Q S_3$ implies $x_3' \in Q(x_1') \subseteq F_3(x_3', u)$, and hence, $\Omega' \in F_2(\Omega, u)$. This proves $S_1 \preceq_\varepsilon S_2$.

To prove $S_2 \preceq_R S_3$, let $(\Omega, x_3) \in R$ and $u \in U_{S_3}(x_3)$ and pick any $x_1 \in \Omega$. Then $(x_1, x_3) \in Q$, and using $S_1 \preceq_Q S_3$ we obtain $u \in U_{S_1}(x_1)$. The latter fact implies that there exists $x_1' \in F_1(x_1, u)$, and using $S_1 \preceq_Q S_3$ again we see that $Q(x_1') \subseteq F_3(x_3, u)$. Since $Q$ is strict we may pick $x_3' \in Q(x_1')$. Then $R^{-1}(x_3') \neq \emptyset$, and hence, $u \in U_{S_2}(\Omega)$, which proves the condition (i) in Definition V.2.

To prove (ii), let $(\Omega, x_3) \in R$, $u \in U_{S_3}(x_3)$ and $\Omega' \in F_2(\Omega, u)$. Then $\Omega' \in R^{-1}(F_3(\Omega, u))$, so there exist $\tilde{x}_3$ and $\tilde{x}_3' \in F_3(\tilde{x}_3, u)$ satisfying $\Omega = Q^{-1}(\tilde{x}_3)$ and $\Omega' = Q^{-1}(\tilde{x}_3')$. Then condition (C) implies $x_3' \in F_3(x_3, u)$, and in turn, $R(\Omega') \subseteq F_3(x_3, u)$, which proves (ii).

To complete the proof, we notice that, by the definition of $\Sigma_2$, $\Sigma_3$ is an abstract specification associated with $S_2$, $S_3$, $R$ and $\Sigma_2$, which shows $(S_2, \Sigma_2) \preceq_R (S_3, \Sigma_3)$. Finally, to prove $(S_1, \Sigma_1) \preceq_\varepsilon (S_2, \Sigma_2)$, let $(u, \Omega) \in \Sigma_1$, assume that $u$ is defined on $[0; T]$, and let $x_1 : [0; T] \to X_1$ satisfy $x_1(t) \in \Omega(t)$ for all $t \in [0; T]$. Then, by the definition of $\Sigma_2$, there exists $(u, x_3) \in \Sigma_3$ such that $R(\Omega(t)) = \{x_3(t)\}$ for all $t \in [0; T]$. The latter condition implies $(x_1(t), x_3(t)) \in Q$, and $(S_1, \Sigma_1) \preceq_Q (S_3, \Sigma_3)$ implies $(u, x_1) \in \Sigma_1$, which completes the proof. \[\square\]

VIII. Computation of Abstractions for Perturbed Sampled Control Systems

In the previous section we have seen that the computation of abstractions basically reduces to the over-approximation of attainable sets of the plant. A large number of over-approximation methods have been proposed which apply to different classes of systems, e.g. [2], [9], [10], [35]–[39]. In this section, we present an approach to over-approximate attainable sets of continuous-time perturbed control systems, given as differential inclusions, based on a matrix-valued Lipschitz inequality.
A. The sampled system

Let us consider a perturbed control system of the form
\[ \dot{x} \in f(x, u) + W \] (26)
with \( f : \mathbb{R}^n \times U \to \mathbb{R}^n, U \subseteq \mathbb{R}^m \) and \( W \subseteq \mathbb{R}^n \). We assume throughout this section that \( U \) is non-empty, \( W \) contains the origin, and that \( f(\cdot, u) \) is locally Lipschitz for all \( u \in U \). We use the set \( W \) to represent various uncertainties in the dynamics of the control system (26).

For \( \tau \in \mathbb{R}_+ \) and an interval \( I \subseteq [0, \tau] \), a solution of (26) on \( I \) with (constant) input \( u \in U \) is defined as an absolutely continuous function \( \xi : I \to \mathbb{R}^n \) that satisfies \( \xi(t) \in f(\xi(t), u) + W \) for almost every (a.e.) \( t \in I \). We say that \( \xi \) is continuous to \([0, \tau]\) if there exists a solution \( \hat{\xi} \) of (26) on \([0, \tau]\) with input \( u \in U \) such that \( \hat{\xi}|_I = \xi \).

We formulate a sampled variant of (26) as system as follows.

VIII.1 Definition. Let \( S_1 = (X_1, U_1, X_1, F_1, \text{id}) \) be a basic Moore system with state output and \( \tau > 0 \). We say that \( S_1 \) is the sampled system associated with the control system (26) and the sampling time \( \tau \), if \( X_1 = \mathbb{R}^n, U_1 = U \) and the following holds: \( x_1 \in F_1(x_0, u) \) if there exists a solution \( \xi \) of (26) on \([0, \tau]\) with input \( u \) satisfying \( \xi(0) = x_0 \) and \( \xi(\tau) = x_1 \).

In the sequel, \( \varphi \) denotes the general solution of the unperturbed system associated with (26) for constant inputs. That is, if \( x_0 \in \mathbb{R}^n, u \in U \), and \( f(\cdot, u) \) is locally Lipschitz, then \( \varphi(\cdot, x_0, u) \) is the unique non-continuable solution of the initial value problem \( \dot{x} = f(x, u), x(0) = x_0 \) [33].

Similar to other approaches [11], [26] to over-approximate attainable sets that are known for unperturbed systems, our computation of attainable sets of the perturbed system is based on an estimate of the distance of neighboring solutions of (26).

VIII.2 Definition. Consider the sets \( K \subseteq \mathbb{R}^n, U' \subseteq U \) and the sampling time \( \tau > 0 \). A map \( \beta : \mathbb{R}^n_+ \times U' \to \mathbb{R}^n_+ \) is a growth bound on \( K, U' \) associated with \( \tau \) and (26) if the following conditions hold for (26) and \( \beta \):

(i) \( \beta(r, u) \geq \beta(r', u) \) whenever \( r \geq r' \) and \( u \in U' \),
(ii) if \( \xi \) is a solution of (26) on \([0, \tau]\) with input \( u \in U' \) and \( \xi(0), p \in K \) then

\[ |\xi(\tau) - \varphi(\tau, p, u)| \leq \beta(|\xi(0) - p|, u) \] (27)

holds component-wise.

Let us emphasize some distinct features of the estimate (27). First of all, we formulate the inequality (27) component-wise, which allows to bound the difference of neighboring solutions for each state coordinate independently. Second, \( \beta \) is a local estimate, i.e., we require (27) to hold only for initial states in \( K \). Moreover, \( \beta \) is allowed to depend on the input, but these inputs are assumed to be constant, and we do not bound the effect of different inputs on the distance of the solutions. All those properties contribute to more accurate over-approximations of the attainable sets. This, in turn, leads to less conservative abstractions which our example in Section IX-A demonstrates. Note that it is also immediate to account for extensions like time varying inputs and using different sampling times.

B. The abstraction

We continue with the construction of an abstraction \( S_2 \) of the sampled system \( S_1 \). The state alphabet \( X_2 \) of the abstraction is defined as a cover of the state alphabet \( X_1 \) where the elements of the cover \( X_2 \) are non-empty, closed hyper-intervals, i.e., every element \( x_2 \in X_2 \) takes the form
\[ [a, b] = \mathbb{R}^n \cap ([a_1, b_1] \times \cdots \times [a_n, b_n]) \]
for some \( a, b \in (\mathbb{R} \cup \{\pm \infty\})^n, a \leq b \).
Our notion of hyper-intervals allows elements of $X_2$ to be unbounded. Nevertheless, in the computation of the abstraction $S_2$, we work with a subset $\bar{X}_2$ of compact elements of $X_2$. We interpret those elements as the “real” quantizer symbols and the remaining elements as overflow symbols, see [10, Sect III.A].

**VIII.3 Definition.** Consider the systems $S_1$ and $S_2$ of the form (10), a set $\bar{X}_2 \subseteq X_2$ and a function $\beta : \mathbb{R}_+^n \times U_2 \to \mathbb{R}_+^n$. Given $\tau > 0$, let $S_1$ be the sampled system associated with (26) and sampling time $\tau$. We call $S_2$ an abstraction of $S_1$ based on $\bar{X}_2$ and $\beta$, if

(i) $X_2$ is a cover of $X_1$ by non-empty, closed hyper-intervals and every element $x_2 \in \bar{X}_2$ is compact;
(ii) $U_2 \subseteq U_1$;
(iii) $F_2(x_2, u) = \emptyset$ whenever $x_2 \in X_2 \setminus \bar{X}_2$, $u \in U_2$, and
(iv) for $x_2 \in \bar{X}_2$, $x'_2 \in X_2$ and $u \in U_2$ we have

\[
(\varphi(\tau, c, u) + [-r', r']) \cap x'_2 \neq \emptyset \Rightarrow x'_2 \in F_2(x_2, u),
\]

where $[a, b] = x_2$, $c = \frac{b+a}{2}$, $r = \frac{b-a}{2}$ and $r' = \beta(r, u)$.

Note that the implicit definition of the transition function $F_2$ according to (iv) in Definition VIII.3 is equivalently expressible as follows. Let $u \in U_2$ and $[a, b] \in \bar{X}_2$, then $[a', b'] \in X_2$ has to be an element of $F_2([a, b], u)$ if

\[
a' - r' \leq \varphi(\tau, c, u) \leq b' + r'
\]

holds with $c = \frac{b+a}{2}$, $r = \frac{b-a}{2}$ and $r' = \beta(r, u)$.

We illustrate the transition function $F_2(x_2, u)$ of an abstraction in Fig. 5.

![Figure 5. Illustration of the transition function of an abstraction.](image)

**VIII.4 Theorem.** Consider two systems $S_1$ and $S_2$ of the form (10), $\bar{X}_2 \subseteq X_2$ and $\tau > 0$. Suppose that $S_1$ is the sampled system associated with (26) and sampling time $\tau$. Let $\beta$ be a growth bound on $\cup_{x_2 \in X_2} x_2$, $U_2$ associated with $\tau$ and (26). If $S_2$ is an abstraction of $S_1$ based on $\bar{X}_2$ and $\beta$, then $S_1 \leq S_2$.

**Proof.** To verify the condition (i) in Proposition VII.1 first note that $U_{S_2}(x_2) = \emptyset$ if $x_2 \in X_2 \setminus \bar{X}_2$ by our assumption on $S_2$. On the other hand, if $x_1 \in x_2 \in \bar{X}_2$, then $U_2 \subseteq U_{S_1}(x_1)$ by our assumption on $\beta$, so the condition (i) in Proposition VII.1 is satisfied. To verify the requirement (ii) in Proposition VII.1, assume that $x_2, x'_2 \in X_2$ and $u \in U_{S_2}(x_2)$. Then $x_2 \in \bar{X}_2$ by our assumption on $S_2$, so $x_2 = [c - r, c + r]$ for some $c$, $r$. Moreover, if additionally $x_1 \in x_2$ and $x'_2 \cap F_1(x_1, u) \neq \emptyset$, then by Definition VIII.1 there exists a solution $\xi : [0, \tau] \to \mathbb{R}^n$ of the system (26) with input $u$ satisfying $\xi(0) = x_1$ and $\xi(\tau) \in x'_2$. It follows that $|\xi(0) - c| \leq r$, and hence, $|\xi(\tau) - \varphi(\tau, c, u)| \leq r'$. Then (28) implies that $x'_2 \in F_2(x_2, u)$. An application of Proposition VII.1 completes the proof.

C. A growth bound

In this subsection we present a specific growth bound for the case that $f$ is continuously differentiable in its first argument and the perturbations are given by $W = [-w, w]$ for some $w \in \mathbb{R}_+^n$. In the
following proposition, we use $D_j f_i$ to denote the partial derivative with respect to the $j$th component of the first argument of $f_i$.

**VIII.5 Theorem.** Let $\tau > 0$ and let $f$, $U$ and $W$ be as in (26) with $W = [-w, w]$ for some $w \in \mathbb{R}^n$. Let $U' \subseteq U$ and assume in addition that $f(\cdot, u)$ is continuously differentiable for every $u \in U'$. Furthermore, let $K \subseteq K' \subseteq \mathbb{R}^n$ with $K'$ being convex, so that for any $u \in U'$, any $\tau' \in [0, \tau]$ and any solution $\xi$ on $[0, \tau']$ of (26) with input $u$ and $\xi(0) \in K$, we have $\xi(t) \in K'$ for all $t \in [0, \tau']$. Lastly, let the parametrized matrix $L: U' \rightarrow \mathbb{R}^{n \times n}$ satisfy

$$L_{i,j}(u) \geq \begin{cases} D_j f_i(x, u), & \text{if } i = j, \\ |D_j f_i(x, u)|, & \text{otherwise} \end{cases}$$

for all $x \in K'$ and all $u \in U'$. Then any $\xi$ as above is continuou}s to $[0, \tau]$, and the map $\beta$ given by

$$\beta(r, u) = e^{L(u)\tau} r + \int_0^\tau e^{L(u)s} w \, ds$$

is a growth bound on $K$, $U'$ associated with $\tau$ and (26).

Theorem VIII.5 can be applied quite easily for obtaining growth bounds. Firstly, the computation of an a priori enclosure $K'$ to solutions of (26) is standard, e.g. [40] and the references therein. Secondly, the parametrized matrix $L$ requires bounding partial derivatives on $K'$. Such bounds can be computed in an automated way using, e.g., interval arithmetic [41]. Finally, given $L$, the evaluation of the expression for $\beta$ is straightforward.

We emphasize that Theorem VIII.5 provides only one of several methods to over-approximate attainable sets. Any over-approximation method can be used to compute abstractions based on feedback refinement relations.

Having a growth bound at hand, the application of Theorem VIII.4 becomes a routine task. Examples are presented in the next section.

For the proof of Theorem VIII.5 we need the following auxiliary result, which appears in [42] without proof.

**VIII.6 Lemma.** Let $\tau > 0$ and $A \subseteq \mathbb{R}^n$. Let $\xi_i: [0, \tau] \rightarrow A$, $i \in \{1, 2\}$, be two perturbed solutions of a dynamical system with continuous right hand side $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$, i.e., the maps $\xi_i$ are absolutely continuous and satisfy

$$|\dot{\xi}_i(t) - f(\xi_i(t))| \leq w_i(t) \quad \text{for a.e. } t \in [0, \tau],$$

where $w_i: [0, \tau] \rightarrow \mathbb{R}^n_+$, $i \in \{1, 2\}$, are integrable. Consider a matrix $L \in \mathbb{R}^{n \times n}$ with $L_{i,j} \geq 0$ for $i \neq j$ and suppose that for all $x, y \in A$ we have

$$x_i \geq y_i \Rightarrow f_i(x) - f_i(y) \leq \sum_{j=1}^n L_{i,j} |x_j - y_j|.$$  

(29)

Let $\rho: [0, \tau] \rightarrow \mathbb{R}^n_+$ be absolutely continuous and satisfying

$$\dot{\rho}(t) = L\rho(t) + w_1(t) + w_2(t)$$

for a.e. $t \in [0, \tau]$. Then $|\xi_1(0) - \xi_2(0)| \leq \rho(0)$ implies $|\xi_1(t) - \xi_2(t)| \leq \rho(t)$ for every $t \in [0, \tau]$.

**Proof.** Let $\tilde{\rho}: [0, \tau] \rightarrow \mathbb{R}^n_+$ be absolutely continuous such that $\dot{\tilde{\rho}}(0) = \rho(0)$ and $\tilde{\rho}'(t) = L\tilde{\rho}(t) + w_1(t) + w_2(t) + \varepsilon$ for some $\varepsilon \in (\mathbb{R}_+ \setminus \{0\})^n$ and a.e. $t \in [0, \tau]$. We shall prove that

$$|\xi_1(t) - \xi_2(t)| \leq \tilde{\rho}(t)$$  

(30)

holds for all $t \in [0, \tau]$, so that the lemma follows from a limit argument. To this end, denote the function $\xi_1 - \xi_2 - \tilde{\rho}$ on $[0, \tau]$ by $z$ and let $t_0 = \sup\{t \in [0, \tau] : \forall s \in [0,t] \, z(s) \leq 0\}$. Then $t_0 \geq 0$ as
$|\xi_1(0) - \xi_2(0)| \leq \rho(0)$, and since we can interchange the roles of $\xi_1$ and $\xi_2$ if necessary, we may assume without loss of generality that (30) holds for all $t \in [0, t_0]$. It remains to show that $t_0 = \tau$.

Assume that $t_0 < \tau$. Using (30), a continuity argument shows that we may choose $t_2 \in [t_0, \tau]$ and $i \in [1; n]$ such that $z_i(t_2) > 0$, $z_i(t_0) = 0$ and

$$\varepsilon_i + \sum_{j=1}^{n} L_{i,j} \tilde{\rho}_j(t) \leq \sum_{j=1}^{n} L_{i,j}|\xi_1(t) - \xi_2(t)|$$

for all $t \in [t_0, t_2]$. Define $t_1 = \sup \{t \in [t_0, t_2] \mid z_i(t) \leq 0\}$ and note that $z_i(t_1) = 0$ as $z_i$ is continuous. The inequality $z_i'(t) \leq f_i(\xi(t)) - f_i(\xi(t)) + w_{1,i}(t) + w_{2,i}(t) - \tilde{\rho}_i(t)$ for a.e. $t \in [t_1, t_2]$ and the definition of $\tilde{\rho}$ then imply that

$$z_i(t_2) \leq \int_{t_1}^{t_2} (f_i(\xi(t)) - f_i(\xi(t))) - \sum_{j=1}^{n} L_{i,j} \tilde{\rho}_j(t) - \varepsilon_i) dt.$$

Thus, $z_i(t_2) \leq 0$ by (31) and (29). This contradicts our choice of $t_2$, and so $t_0 = \tau$. \hfill \Box

**Proof of Theorem VIII.5.** Fix $p \in K$, $u \in U'$ and note that $\beta(r, u) \geq \beta(r', u)$ if $r \geq r'$ as all entries of $e^{L(u)t}$ are non-negative [43]. Next, we show that condition (ii) in Definition VIII.2 holds. In order to apply Lemma VIII.6 we shall establish (29) for $K'$, $f(\cdot, u)$ and $L(u)$ in place of $A$, $f$ and $L$. Indeed, by the mean value theorem, there exists $z \in \{x + t(y - x) \mid t \in [0, 1]\}$ such that $f_i(x, u) - f_i(y, u) = \sum_{j=1}^{m} D_j f_i(z, u)(x_j - y_j)$. Hence, by the definition of $L$, we obtain (29). Now, let $\xi$ be a solution on $[0, \tau]$ of (26) with input $u$ such that $\xi(0) \in K$. By Filippov’s Lemma [44], there exists an integrable map $s: [0, \tau] \to W$ such that $\xi(t) = f(\xi(t), u) + \xi(t)$ for a.e. $t \in [0, \tau]$. So, apply Lemma VIII.6 to $f(\cdot, u)$, $K'$, $\varphi(\cdot, p, u)$, $\xi$, $0$, $w$ and $L(u)$ in place of $f$, $A$, $\xi_1$, $\xi_2$, $w_1$, $w_2$ and $L$, respectively, to obtain $|\xi(\tau) - \varphi(\tau, p, u)| \leq \beta(|\xi(0) - p|, u)$.

Finally, suppose there exists $\xi: [0, \tau] \to K'$ as in the statement of the theorem that is not continuable to $[0, \tau]$. Then, there exist $t_0 \in [0, \tau]$ and a solution $\xi: [0, t_0] \to \mathbb{R}^n$ of (26) with input $u$ such that $\xi|_{[0, t_0]} = \xi$ and $\xi(t)$ becomes unbounded as $t \in [0, t_0]$ approaches $t_0$ [45]. On the other hand, applying Lemma VIII.6 to $f(\cdot, u)$, $K'$, $\xi|_{[0, t]}$, $\xi(0)$, $w$, $|f(\xi(0), u)|$, $L(u)$ and $t$ in place of $f$, $A$, $\xi_1$, $\xi_2$, $w_1$, $w_2$, $L$ and $\tau$ we conclude that $|\xi(t) - \xi(0)|$ is uniformly bounded for $t \in [0, t_0]$, which is a contradiction. \hfill \Box

**IX. Examples.**

In this section, we demonstrate the practicality of our approach on two reach-avoid problems.

**IX.1 Definition.** Let $S = (X, U, X, F, \text{id})$ be given and let $A_0, A_a, A_r \subseteq X$. A specification $\Sigma$ on $U \times X$ is a reach-avoid specification associated with $S$, $A_0$, $A_a$ and $A_r$ if $\Sigma$ equals the union over $T \in \mathbb{N} \cup \{\infty\}$ of the sets

$$\{(u, x): [0; T] \to U \times X \mid x(0) \in A_0 \Rightarrow \exists s \in [0; T] \mid x(s) \in A_r \land \forall t \in [0; s] \mid x(t) \notin A_a\}.$$

In words, $S$ satisfies $\Sigma$ iff any state trajectory of $S$ starting from $A_0$ reaches the set $A_r$ in finite time while avoiding the set $A_a$.

Definition IX.1 gives rise to the following remark. Let $S_1$ and $S_2$ be systems of the form (10) satisfying $S_1 \preceq S_2$. Let $\Sigma_1$ be a reach-avoid specification associated with $S_1$, $A_{1,0}$, $A_{1,a}$ and $A_{1,r}$. Then, the reach-avoid specification $\Sigma_2$ associated with $S_2$ and

$$A_{2,0} = \{\Omega \in X_2 \mid \Omega \cap A_{1,0} \neq \emptyset\}$$

$$A_{2,a} = \{\Omega \in X_2 \mid \Omega \cap A_{1,a} \neq \emptyset\}$$

$$A_{2,r} = \{\Omega \in X_2 \mid \Omega \subseteq A_{1,r}\}$$

is an abstract specification associated with $S_1$, $S_2$, $\in$ and $\Sigma_1$. Indeed, it is readily seen that $\Sigma_2$ satisfies the condition in Definition VI.2.
A. A path planning problem for a mobile robot

We consider a control problem for a mobile robot as given in [11, Sec. V]. The dynamics of the system are of the form (26), where \( f: \mathbb{R}^3 \times U \rightarrow \mathbb{R}^3 \) is given by

\[
f(x, (u_1, u_2)) = \begin{pmatrix}
u_1 \cos(\alpha + x_3) \cos(\alpha)^{-1} \\
u_1 \sin(\alpha + x_3) \cos(\alpha)^{-1} \\
u_1 \tan(u_2)
\end{pmatrix}
\]

with \( U = [-1, 1] \times [-1, 1] \) and \( \alpha = \arctan(\tan(u_2))/2 \). Here, \((x_1, x_2)\) is the position and \( x_3 \) is the orientation of the robot in the 2-dimensional plane. The control inputs \( u_1 \) and \( u_2 \) are the velocity and the steering angle of the robot. Perturbations are not acting on the system dynamics, i.e., \( W = \{(0, 0, 0)\} \).

In [11], the concrete control problem, denoted by \((S_1, \Sigma_1)\) below, consists of the sampled system associated with (26) and sampling time \( \tau = 0.3 \), and of the objective to steer the robot through a maze. More formally, \( \Sigma_1 \) is a reach-avoid specification associated with \( S_1 \) and the following sets. For \( c = \pi + 0.2 \)

we have \( A_{1,0} = \{(0.4, 0.4, 0)\} \), \( A_{1,r} = [9, 9.5] \times [0, 0.5] \times [-c, c] \) and the set \( A_{1,a} \) is indicated in dark blue in Fig. 6. (The third component of \( A_{1,a} \) equals \([-c, c]\).)

We proceed with computing an abstraction for \( S_1 \). As in [11], we let the abstraction \( S_2 \) be a basic system of the form (10) as follows. \( X_2 \) is a cover of \( \mathbb{R}^3 \) such that \( X_2 \) in Definition VIII.3 is given by shifted copies of the hyper-rectangle

\[
[-\frac{1}{10}, \frac{1}{10}] \times [-\frac{1}{10}, \frac{1}{10}] \times [-\frac{65}{3}, \frac{65}{3}],
\]

whose the centers correspond to the set

\[
\frac{2}{10} \times [0; 50] \times \frac{2}{10} \times \frac{2}{35} \times [-17; 17]
\]

removing the copies intersecting \( A_{1,a} \). The input alphabet is given by \( U_2 = \{0, \pm 0.3, \pm 0.6, \pm 0.9\} \times \{0, \pm 0.3, \pm 0.6, \pm 0.9\} \).

According to Theorem VIII.5, a growth bound on \( \mathbb{R}^3 \), \( U_2 \) associated with \( \tau \) and (26) is given by \( \beta(r, u) = e^{L(u)\tau}r \), where \( L: U_2 \rightarrow \mathbb{R}^{3 \times 3} \) is given by \( L_{1,3}(u_1, u_2) = |u_1\sqrt{\tan^2(u_2)}|/4 + 1, \) and \( L_{i,j}(u_1, u_2) = 0 \) for \((i, j) \notin \{(1, 3), (2, 3)\} \). We apply Theorem VIII.4 to compute a transition function for \( S_2 \) such that \( S_1 \preceq S_2 \). The computation takes 2.33 seconds (Intel Core i7 2.9 GHz) resulting in an abstraction having 28398299 transitions. Next, we solve the abstract control problem \((S_2, \Sigma_2)\), where \( \Sigma_2 \) is the reach-avoid specification associated with \( S_2 \) and (32). The controller synthesis for \((S_2, \Sigma_2)\), which uses a standard technique [46], is successfully completed within 0.22 seconds. Thus, by Theorem VI.3 we solved the concrete control problem \((S_1, \Sigma_1)\). See Fig. 6. (The run times stated in [11] are 13509 seconds for an abstraction with 34020088 transitions and 535 seconds for the controller synthesis with 34020088 transitions and 535 seconds for the controller synthesis (Intel Core 2 Duo 2.4 GHz).)

We would like to discuss two of the advantages of the growth bounds we have introduced in Section VIII, on the example of \( \beta \). As we already mentioned, \( \beta \) bounds each component of neighboring solutions separately, which can be directly seen by the formula \( \beta(r, u) = r + r_3 \cdot L_{1,3}(u_1, u_2) \cdot (\tau, \tau, 0)^\top \). This distinguishes \( \beta \) from an estimate based on a norm. Moreover, \( \beta \) depends on the input, which is crucial for the present example. Indeed, the function \( e^{(\sup L)\tau}r \), where \( \sup L \in \mathbb{R}^{3 \times 3} \) is given by \( (\sup L)_{i,j} = \sup_{u \in U_2} L_{i,j}(u) \), is also a growth bound on \( \mathbb{R}^3 \), \( U_2 \) associated with \( \tau \) and (26). However, this growth bound results in an abstraction having 36002897 transitions which leads to an unsolvable abstract control problem.
Figure 6. Projection of the states of $S_1$ and $S_2$, respectively, to $\mathbb{R}^2 \times \{0\}$. The sets $A_{1,a}$ and $A_{1,r}$ are indicated in dark blue and red, respectively. The states in $A_{2,a}$ and $A_{2,r}$ are indicated in light blue and light red, respectively. A closed-loop trajectory of the concrete control problem is shown.

B. An aircraft landing maneuver

We consider an aircraft DC9-30 whose dynamics we model as follows. (For a detailed discussion of the modeling we refer the reader to [47] and the references therein.) We use $x_1, x_2, x_3$ to denote the state variables, which respectively correspond to the velocity, the flight path angle and the altitude of the aircraft. The input alphabet is given by $U = [0, 160 \cdot 10^3] \times [0^\circ, 10^\circ]$ and represents the thrust of the engines (in Newton) and the angle of attack. The dynamics are given by $f: \mathbb{R}^3 \times U \to \mathbb{R}^3$, $f(x, u) = \left( \begin{array}{c} \frac{1}{m} (u_1 \cos u_2 - D(u_2, x_1) - mg \sin x_2) \\ \frac{1}{m} (u_1 \sin u_2 + L(u_2, x_1) - mg \cos x_2) \\ x_1 \sin x_2 \end{array} \right)$, where $D(u_2, x_1) = (2.7 + 3.08 \cdot (1.25 + 4.2 \cdot u_2)^2) \cdot x_1^2$, $L(u_2, x_1) = (68.6 \cdot (1.25 + 4.2 \cdot u_2)) \cdot x_1^2$ and $mg = 60 \cdot 10^3 \cdot 9.81$ account for the drag, lift and gravity, respectively [47].

We consider the input disturbance $P_1: U \Rightarrow U$ given by $P_1(u) = (u + [-5 \cdot 10^3, 5 \cdot 10^3] \times [-0.25^\circ, 0.25^\circ]) \cap U$ and measurement errors of the form $P_2: \mathbb{R}^3 \Rightarrow \mathbb{R}^3$ given by $P_2(x) = x + \frac{1}{20} [-0.25, 0.25] \times \frac{1}{20} [-0.05^\circ, 0.05^\circ] \times \frac{1}{20} [-1, 1]$. We do not consider any further disturbances, i.e., we let $W = \{(0, 0, 0)\}$, $P_3 = \text{id}$, and $P_4 = \text{id}$.

Let $S_1 = (X_1, U_1, X_1, F_1, \text{id})$ be the sampled system associated with (26) and the sampling time $\tau = 0.25$. We aim at steering the aircraft from an altitude of 55 meters close to the ground with an appropriate total and horizontal touchdown velocity. More formally, we consider the reach-avoid
specification $\Sigma_1$ associated with $S_1$ and

$$A_0 = [80, 82] \times [-2^\circ, -1^\circ] \times \{55\},$$
$$A_a = \mathbb{R}^3 \setminus ([58, 83] \times I \times [0, 56]),$$
$$A_r = ([63, 75] \times I \times [0, 2.5]) \cap \{x \in \mathbb{R}^3 | x_1 \sin x_2 \geq -0.91\},$$

where $I = [-3^\circ, 0^\circ]$.

As detailed in Section VI-B, the perturbed control problem is solved through an auxiliary unperturbed control problem. To begin with, define the basic system $\hat{S}_1$ by (24) with $\hat{U}_1 = U$. Next, let $X$ be a cover of $\mathbb{R}^3$ formed by subdividing $\mathbb{R}^3 \setminus A_a$ into $210 \times 210 \times 210$ hyper-intervals, and suitable unbounded hyper-intervals. Define $X_2 = \{P_2^{-1}(\Omega) | \Omega \in X\}$ and let $\bar{X}_2$ be the subset of compact elements of $X_2$ that do not intersect $A_a$. Define the abstraction for $\hat{S}_1$ as the basic system $S_2$ given by (10), where $U_2 = \{0, 32000\} \times U'$, $U'$ contains precisely 10 inputs equally spaced in $[0^\circ, 8^\circ]$. We apply Theorem VIII.5 with $w = M(5000, 0.25^\circ)^\top \leq (0.108, 0.002, 0)^\top$ and a suitable a priori enclosure $K'$ to obtain a growth bound, where $M \in \mathbb{R}_{++}^{2\times3}$ satisfies $M_{i,j} \geq |D_{j,2}f_i(x,u)|$ for all $x \in K'$ and $u \in P_1(U_2)$. Here, $D_{j,2}f_i$ stands for the partial derivative with respect to the $j$th component of the second argument of $f_i$. Note that $w$ accounts for the perturbation $P_1$. Then, we use Theorem VIII.4 to compute $F_2$ such that $\hat{S}_1 \preceq S_2$. The computation takes 674 seconds resulting in an abstraction with about $9.38 \cdot 10^9$ transitions (Intel Xeon E5 3.1 GHz). Finally, it takes 26 seconds to solve the abstract control problem $(S_2, \Sigma_2)$ using a standard technique [46], where $\Sigma_2$ is the reach-avoid specification associated with $S_2$ and (32). By Corollary VI.5 the behavior of the perturbed closed loop is a subset of $\Sigma_1$. See Fig. 7.

We proceed to make some comments on solving perturbed control problems. At first, Theorem VIII.5 allows to deal with time-varying input perturbations, when the theorem is applied as in this example. Second, accounting for measurement errors only requires inflating the cells that would have been used if measurement errors were not present. To conclude, perturbed control problems can be solved in our framework by using canonical abstractions.

![Figure 7. Time evolution of the altitude of the aircraft in the closed loop. The aircraft pitch $u_2 + x_2$ is indicated for 8 instants of time.](image)

**X. Conclusions**

We have presented a novel approach to abstraction-based controller synthesis which builds on the concept of feedback refinement relation introduced in the present paper. Our framework incorporates several distinct features. Foremost, the designed controllers require quantized (or symbolic) state information only and are connected to the plant via a static quantizer, which is particularly important for any practical implementation of the controller. Our work permits the synthesis of robust correct-by-design controllers in the presence of various uncertainties and disturbances, and more generally, applies to a broader class of synthesis problems than previous research addressing the state information and refinement complexity issues as explained and illustrated in Sections I and IV. Moreover, we do
not assume that the controller is able to set the initial state of the plant, which is also important in the context of practical control systems.

We have additionally identified a class of canonical abstractions, and have presented a method to compute such abstractions for perturbed nonlinear control systems. We utilized numerical examples to illustrate that our construction is more efficient than similar approaches and, simultaneously, demonstrate the applicability of our synthesis framework.

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