A characterization for solutions of the Monge-Kantorovich mass transport problem

Abbas Moameni *
School of Mathematics and Statistics
Carleton University
Ottawa, ON, Canada K1S 5B6
momeni@math.carleton.ca
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Abstract
A measure theoretical approach is presented to study the solutions of the Monge-Kantorovich optimal mass transport problems. This approach together with Kantorovich duality provide an effective tool to answer a long standing question about the support of optimal plans for the mass transport problem involving general cost functions. We also establish a criterion for the uniqueness.

1 Introduction
Let \((X, \mu)\) and \((Y, \nu)\) be two Polish probability spaces and let \(c : X \times Y \to \mathbb{R}\) be a continuous function. The Monge optimal transport problem is to find a measurable map \(T : X \to Y\) with
\[
T \# \mu = \nu \quad \text{(i.e. } \nu(B) = \mu(T^{-1}(B)) \text{ for all measurable } B \subset Y)\
\]
in such a way that \(T\) minimizes the transportation cost i.e.,
\[
\int_X c(x, Tx) \, d\mu = \inf_{S \# \mu = \nu} \int_X c(x, Sx) \, d\mu. \tag{M}
\]
When a transport map \(T\) minimizes the cost we call it an optimal transport map. A relaxed version of the Monge problem was formulated by Kantorovich [22] as a linear optimization problem on a convex domain. In fact, let \(\Pi(\mu, \nu)\) be the set of Borel probability measures on \(X \times Y\) which have \(\mu\) and \(\nu\) as marginal. The transport cost associated to a transport plan \(\pi \in \Pi(\mu, \nu)\) is given by
\[
I_c(\pi) = \int_{X \times Y} c(x, y) \, d\pi.
\]
Kantorovich’s problem is to minimize
\[
\inf \{ I_c(\pi); \pi \in \Pi(\mu, \nu) \}. \tag{MK}
\]
When a transport plan minimizes the cost, it will be called an optimal plan. In contrary to the Monge problem, the Kantorovich problem always admits solutions as soon as the cost function is a non-negative lower semi continuous function (see [33] for a proof). We refer to [2, 8, 10, 11, 15, 32] for existence and uniqueness of solution to the Monge-Kantorovich problem when the cost function is of distance form. By now, the existence and uniqueness is known in a wide class of settings. Namely, for the general cost functions on the Euclidean space and Manifolds, for non-decreasing strictly convex functions of the distance in Alexandrov

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spaces and for squared distance on the Heisenberg group (see for instance [3, 5, 17, 19, 16, 23, 28, 33], the bibliography is not exhaustive).

A general criterion for existence and uniqueness of optimal transport maps known as the twist condition dictates the map

\[ y \rightarrow \frac{\partial c(x, y)}{\partial x}, \]

to be injective for fixed \( x \in X \). Under the twist condition and some regularity on the first marginal \( \mu \), the optimal plan \( \gamma \) which solves the Monge-Kantorovich problem (MK) is supported on the graph of an optimal transport map \( T \), i.e., \( \gamma = (\text{Id} \times T)_{#} \mu \).

Beyond the twist condition, there is not much known about the support of an optimal plan except its numbered limb system structure when is extremal in the convex set \( \Pi(\mu, \nu) \) and also its local rectifiability when the cost function is non-degenerate [12, 25].

Our aim is to use a measure theoretical approach for which together with the Kantorovich duality provide a practical tool to study the optimal mass transport problem. In this work, we apply this method to characterize the support of optimal plans for cost functions well beyond the twist structure. The following definition is a straightforward generalization of the twist-condition.

**Definition 1.1** Let \( c : X \times Y \rightarrow \mathbb{R} \) be a function such that \( x \rightarrow c(x, y) \) is differentiable for all \( y \in Y \).

- **Generalized-twist condition:** We say that \( c \) satisfies the generalized-twist condition if for any \( x_0 \in X \) and \( y_0 \in Y \) the set

\[ \left\{ y; \frac{\partial c(x_0, y)}{\partial x} = \frac{\partial c(x_0, y_0)}{\partial x} \right\}, \]

is a finite subset of \( Y \).

- **m-twist condition:** Let \( m \in \mathbb{N} \). We say that \( c \) satisfies the \( m \)-twist condition if for any \( x_0 \in X \) and \( y_0 \in Y \) the cardinality of the set

\[ \left\{ y; \frac{\partial c(x_0, y)}{\partial x} = \frac{\partial c(x_0, y_0)}{\partial x} \right\}, \]

is at most \( m \). We also say that \( c \) satisfies the \( m \)-twist condition locally if for any \( x_0 \in X \) and \( y_0 \in Y \) there exists a neighborhood \( U \) of \( y_0 \) such that the cardinality of the set

\[ \left\{ y \in U; \frac{\partial c(x_0, y)}{\partial x} = \frac{\partial c(x_0, y_0)}{\partial x} \right\}, \]

is at most \( m \).

Note that the \( m \)-twist condition implies the generalized-twist condition, however, the converse is not true in general. We shall study the support of optimal plans for this new class of cost functions. We start with the following definition.

**Definition 1.2** Say that a measure \( \gamma \in \Pi(\mu, \nu) \) is supported on the graphs of measurable maps \( \{T_i\}_{i=1}^{k} \) from \( X \) to \( Y \), if there exists a sequence of measurable non-negative real functions \( \{\alpha_i\}_{i=1}^{k} \) from \( X \) to \( \mathbb{R} \) with \( \sum_{i=1}^{k} \alpha_i(x) = 1 \) such that for each measurable set \( S \subset X \times Y \),

\[ \gamma(S) = \sum_{i=1}^{k} \int_{X} \alpha_i(x) \chi_S(x, T_i x) \, d\mu, \]

where \( \chi_S \) is the indicator function of the set \( S \). In this case we write \( \gamma = \sum_{i=1}^{k} \alpha_i(\text{Id} \times T_i)_{#} \mu \).
Here we state our main result.

**Theorem 1.3** Let $X$ be a complete separable Riemannian manifold and $Y$ be a Polish space equipped with Borel probability measures $\mu$ on $X$ and $\nu$ on $Y$. Let $c : X \times Y \to \mathbb{R}$ be a bounded continuous cost function and assume that:

1. the cost function $c$ satisfies the generalized-twist condition;
2. $\mu$ is non-atomic, absolutely continuous with respect to the volume measure and any $c$-concave function is differentiable $\mu$-almost surely on its domain.

Then for each optimal plan $\gamma$ of $(MK)$, there exist $k \in \mathbb{N}$, a sequence $\{\alpha_i\}_{i=1}^k$ of non-negative functions from $X$ to $[0, 1]$, and Borel measurable maps $G_1, \ldots, G_k$ from $X$ to $Y$ such that

$$\gamma = \sum_{i=1}^k \alpha_i (\text{Id} \times G_i) \# \mu,$$

where $\sum_{i=1}^k \alpha_i(x) = 1$ for $\mu$-almost every $x \in X$.

Moreover, the set $\{x; G_i(x) = G_j(x)\}$ is a null set unless $G_i = G_j$, $\mu$-a.e. on $X$.

In section (4) we shall provide a criterion for the uniqueness of measures in $\Pi(\mu, \nu)$ that are supported on the union of the graphs of a finite number of measurable maps. As an immediate consequence of the above theorem we have:

**Corollary 1.4** Under the assumption of Theorem 1.3, if one replaces the generalized-twist condition by the $m$-twist condition then for each optimal plan $\gamma$ of $(MK)$, there exist a sequence of non-negative real functions $\{\alpha_i\}_{i=1}^m$ on $X$ and, Borel measurable maps $G_1, \ldots, G_m : X \to Y$ such that

$$\gamma = \sum_{i=1}^m \alpha_i (\text{Id} \times G_i) \# \mu,$$

where $\sum_{i=1}^m \alpha_i = 1$ for $\mu$-almost every $x \in X$.

When $m = 1$, the 1-twist condition is the well-known twist condition for which not only the minimizer of the Kantorovich problem is concentrated on the graph of a single map but it is also unique provided the first marginal does not charge small sets. For examples of a 2-twist condition let us consider the function $c : [0, 2\pi] \times [0, 2\pi] \to \mathbb{R}$ defined by $c(x, y) = 1 - \cos(x - y)$. It obviously satisfies the 2-twist condition. Assume first that $\mu = \nu$, in this case the unique solution would have support on the graph $y = x$. However, the model becomes much more interesting when the densities associated with $\mu$ and $\nu$ are different. We refer to [12] and [18] where it is proved that the optimal map associated to the cost function $c(x, y) = 1 - \cos(x - y)$ is unique and concentrated on the union of the graphs of two maps.

The most interesting examples of costs satisfying the generalized-twist condition are non-degenerate costs on smooth $n$-dimensional manifolds $X$ and $Y$. Denote by $D_{xy}^2 c(x_0, y_0)$ the $n \times n$ matrix of mixed second order partial derivatives of the function $c$ at the point $(x_0, y_0)$. A cost $c \in C^2(X \times Y)$ is non-degenerate provided $D_{xy}^2 c(x_0, y_0)$ is non-singular, that is $\det(D_{xy}^2 c(x_0, y_0)) \neq 0$ for all $(x_0, y_0) \in X \times Y$. Non-degeneracy is one of the main hypotheses in the smoothness proof for optimal maps when the cost function satisfies the twist condition [25]. It is also shown in [26] that for non-degenerate costs -not necessary twisted- the support of each optimal plan concentrates on some $n$-dimensional Lipschitz submanifold, however, their proof says little about the submanifold itself. Note that the non-degeneracy condition will imply that the map $y \in Y \to \partial c(x, y)/\partial x$ is locally injective but not necessarily globally. Indeed, the non-degeneracy property implies that the cost function $c$ satisfies the the 1-twist condition locally.

We shall show that local $m$-twistedness implies the generalized-twist condition and therefore one obtains a full characterization of the support of optimal plans for such cost functions due to Theorem 1.3.
Proposition 1.1 Let $X$ and $Y$ be two smooth $n$-dimensional manifolds. Assume that $c$ is continuously
differentiable with respect to the first variable, and that it satisfies the $m$-twist condition locally for some
$m \in \mathbb{N}$. If one of the following assertions holds,
i. $Y$ is compact,
ii. $Y$ is closed and $\lim_{|y| \to \infty} \left| \frac{\partial c(x,y)}{\partial x} \right| = \infty$,
then $c$ satisfies the generalized-twist condition.

To conclude the introduction, we shall emphasize that Theorem 1.3 is in fact an effortless application of
the methodology presented in this work.

The manuscript is organized as follows: in the next section, we shall discuss the key ingredients for our
methodology in this work. In the third section we proceed with the proofs of the main results, while the
final section is reserved to address the uniqueness issue for $m$-twisted cost functions.

2 Measurable weak sections and extremality

Let $(X, \mathcal{B}, \mu)$ be a finite, not necessarily complete measure space, and $(Y, \Sigma)$ a measurable space. The
completion of $\mathcal{B}$ with respect to $\mu$ is denoted by $\mathcal{B}_{\mu}$, when necessary, we identify $\mu$ with its completion on $\mathcal{B}_{\mu}$. 
For $A, B \in \mathcal{B}$, we write $A \subset \mu B$ provided $\mu(A \setminus B) = 0$. Similarly we define $A =_\mu B$ if and only if $A \subset \mu B$
and $B \subset \mu A$. A function $T : X \to Y$ is said to be $(\mathcal{B}, \Sigma)$-measurable if and only if $T^{-1}(A) \in \mathcal{B}$ for all $A \in \Sigma$. The push forward of the measure $\mu$ by the map $T$ is denoted by $T\#\mu$, i.e.

$$T\#\mu(A) = \mu(T^{-1}(A)), \quad \forall A \in \Sigma.$$  

By the change of variable formula it amounts to saying that $\int_Y f(y) \, d(T\#\mu) = \int_X f \circ T(x) \, d\mu$, for all bounded measurable functions $f : Y \to \mathbb{R}$. 
Two measurable functions $T, S : X \to Y$ are weakly equivalent, denoted by $S =_\mu T$, iff $T^{-1}(A) =_\mu S^{-1}(A)$
for all $A \in \Sigma$. We also have the following definition.

Definition 2.1 Let $T : X \to Y$ be $(\mathcal{B}, \Sigma)$-measurable and $\nu$ a positive measure on $\Sigma$.

i. We call a map $F : Y \to X$ a $(\Sigma, \mathcal{B})$-measurable section of $T$ if $F$ is $(\Sigma, \mathcal{B})$-measurable and $T \circ F = \text{Id}_Y$.

ii. We call a map $F : Y \to X$ a $(\Sigma, \mathcal{B})$-measurable weak section of $T$ if $F$ is $(\Sigma, \mathcal{B})$-measurable and $T \circ F = =_\mu \text{Id}_Y$.

Recall that a Polish space is a separable completely metrizable topological space. A Suslin space is the
image of a Polish space under a continuous mapping. Obviously every Polish space is a Souslin space. The
following theorem ensures the existence of $(\Sigma, \mathcal{B})$-measurable sections ([7], Theorem 9.1.3). This is indeed
a consequence of von Neumann’s selection theorem.

Theorem 2.2. Let $X$ and $Y$ be Souslin spaces and let $T : X \to Y$ be a Borel mapping such that $T(X) = Y$.
Then, there exists a mapping $F : Y \to X$ such that $T \circ F(y) = y$ for all $y \in Y$ and $F$ is measurable with
to every Borel measure on $Y$.

If $X$ is a topological space we denote by $\mathcal{B}(X)$ the set of Borel subsets in $X$. The space of Borel probability
measures on a topological space $X$ is denoted by $\mathcal{P}(X)$. The following result shows that every $(\Sigma, \mathcal{B}(X))$-
measurable map has a $(\Sigma, \mathcal{B}(X))$-measurable representation ([7], Corollary 6.7.6).

Proposition 2.1 Let $\nu$ be a finite measure on a measurable space $(Y, \Sigma)$, let $X$ be a Souslin space, and let
$F : Y \to X$ be a $(\Sigma, \mathcal{B}(X))$-measurable mapping. Then, there exists a mapping $G : Y \to X$ such that $G = F$
$\nu$-a.e. and $G^{-1}(B) \in \Sigma$ for all $B \in \mathcal{B}(X)$.

For a measurable map $T : (X, \mathcal{B}(X)) \to (Y, \Sigma, \nu)$ denote by $\mathcal{M}(T, \nu)$ the set of all measures $\lambda$ on $\mathcal{B}(X)$ so that
$T$ pushes $\lambda$ forward to $\nu$, i.e.

$$\mathcal{M}(T, \nu) = \{ \lambda; T\#\lambda = \nu \}.$$  

Evidently $\mathcal{M}(T, \nu)$ is a convex set. A measure $\lambda$ is an extreme point of $\mathcal{M}(T, \nu)$ if the identity $\lambda = \theta \lambda_1 + (1 - \theta) \lambda_2$ with $\theta \in (0, 1)$ and $\lambda_1, \lambda_2 \in \mathcal{M}(T, \nu)$ imply that $\lambda_1 = \lambda_2$. The set of extreme points of
\( \mathcal{M}(T, \nu) \) is denoted by \( \text{ext} \mathcal{M}(T, \nu) \).

We recall the following result from [20] in which a characterization of the set \( \text{ext} \mathcal{M}(T, \nu) \) is given (see also [14] for the case where \( T \) is continuous).

**Theorem 2.3** Let \((Y, \Sigma, \nu)\) be a probability space, \((X, \mathcal{B}(X))\) be a Hausdorff space with a Radon probability measure \( \lambda \), and let \( T : X \to Y \) be an \((\mathcal{B}(X), \Sigma)\)-measurable mapping. The following conditions are equivalent:

(i) \( \lambda \) is an extreme point of \( \mathcal{M}(T, \nu) \);

(ii) there exists a \((\Sigma, \mathcal{B}(X))\)-measurable weak section \( F : Y \to X \) of the mapping \( T \) such that \( \lambda = F_{\#} \nu \).

If \( T \) is surjective and \( \Sigma \) is countably separated, then conditions (i) and (ii) are also equivalent to the following condition:

(iii) there exists a \((\Sigma, \mathcal{B}(X))\)-measurable section \( F : Y \to X \) of the mapping \( T \) with \( \lambda = F_{\#} \nu \).

Finally, if in addition, \( \Sigma \) is countably generated and for some \( \sigma \)-algebra \( S \) with \( \Sigma \subset S \subset \Sigma \), there exists an \((S, \mathcal{B}(X))\)-measurable section of the mapping \( T \), then the indicated conditions are equivalent to the following condition:

(iv) there exists an \((S, \mathcal{B}(X))\)-measurable section \( F \) of the mapping \( T \) such that \( \lambda = F_{\#} \nu \).

The most interesting for applications is the case where \( X \) and \( Y \) are Souslin spaces with their Borel \( \sigma \)-algebras and \( T : X \to Y \) is a surjective Borel mapping. Then the conditions formulated before assertion (iv) are fulfilled if we take for \( S \) the \( \sigma \)-algebra generated by all Souslin sets. Thus, in this situation, the extreme points of the set \( \mathcal{M}(T, \nu) \) are exactly the measures of the form \( F_{\#} \nu \), where \( F : Y \to X \) is measurable with respect to \((S, \mathcal{B}(X))\) and \( T \circ F(y) = y \) for all \( y \in Y \).

We shall now make use of the Choquet theory in the setting of noncompact sets of measures to represent each \( \lambda \in M(T, \nu) \) as a Choquet type integral over \( \text{ext} \mathcal{M}(T, \nu) \). Let us first recall some notations from von Weizsäcker-Winkler [34]. In the measurable space \((X, \mathcal{B}(X))\), let \( H \) be a set of non-negative measures on \( \mathcal{B}(X) \). By \( \sum_{H} \) we denote the \( \sigma \)-algebra over \( H \) generated by the functions \( \varrho \to \varrho(B), B \in \mathcal{B}(X) \). If \( H \) is a convex set of measures we denote by \( \text{ext} H \) the set of extreme points of \( H \). The set of tight positive measures on \( \mathcal{B}(X) \) is denoted by \( \mathcal{M}^{+}(X) \). For a family \( \mathcal{F} \) of real valued functions on \( X \) we define

\[
\mathcal{M}^+_F(X) = \{ \varrho \in \mathcal{M}^+(X) ; \mathcal{F} \subset L^1(\varrho) \},
\]

and \( \sigma \mathcal{M}^+_F(X) \) is the topology on \( \mathcal{M}^+_F(X) \) of the functions \( \varrho \to \int f \, d\varrho, f \in \mathcal{F} \). The weakest topology on \( \mathcal{M}^+_F(X) \) that makes the functions \( \varrho \to \int f \, d\varrho \) lower semi-continuous for all lower semi-continuous bounded functions \( f \) on \( X \) is denoted by \( \nu \mathcal{M}^+_F(X) \). Denote by \( \nu \mathcal{M}^+_F(X) \) the topology generated by \( \sigma \mathcal{M}^+_F(X) \) and \( \nu \mathcal{M}^+_F(X) \). Here is the main result of von Weizsäcker-Winkler [34] regarding the Choquet theory in the setting of noncompact sets of measures.

**Theorem 2.4** Let \( \mathcal{F} \) be a countable family of real Borel functions on a topological space \( X \). Let \( H \) be a convex subset of \( \mathcal{M}^+_F(X) \) such that \( \sup_{\varrho \in H} \varrho(X) < \infty \). If \( H \) is closed with respect to \( \nu \mathcal{M}^+_F(X) \) then for every \( \lambda \in H \) there is a probability measure \( \xi \) on \( \text{ext} H \) which represents \( \lambda \) in the following sense

\[
\lambda(B) = \int_{\text{ext} H} \varrho(B) \, d\xi(\varrho),
\]

for every \( B \in \mathcal{B}(X) \).

We now use the above theorem to represent each \( \lambda \in M(T, \nu) \) as a Choquet type integral over \( \text{ext} M(T, \nu) \).

**Theorem 2.5** Let \( X \) and \( Y \) be complete separable metric spaces and \( \nu \) a probability measure on \( \mathcal{B}(Y) \). Let \( T : (X, \mathcal{B}(X)) \to (Y, \mathcal{B}(Y)) \) be a surjective measurable mapping and let \( \lambda \in M(T, \nu) \). Then there exists a Borel probability measure \( \xi \) on \( \sum_{\text{ext} M(T, \nu)} \) such that for each \( B \in \mathcal{B}(X) \),

\[
\lambda(B) = \int_{\text{ext} M(T, \nu)} \varrho(B) \, d\xi(\varrho), \quad (\varrho \to \varrho(B) \text{ is Borel measurable}).
\]
Proof. Note first that any finite Borel measure on a Polish space is tight ([1], Theorem 12.7). Let \( A \) be a countably family in \( \mathcal{B}(Y) \) which generates \( \mathcal{B}(Y) \) as a \( \sigma \)-field. Let

\[
\mathcal{F} = \{ \chi_A \circ T; \ A \in \mathcal{A} \},
\]

where \( \chi_A \) is the indicator function of \( A \). Note that \( \mathcal{F} \) is a countable family of real Borel functions on \( X \). It is clear that \( M(T, \nu) \) is closed with respect to the topology \( v \sigma \mathcal{M}_T^\nu(X) \). Thus, it follows from Theorem 2.4 that there exists a Borel probability measure \( \xi \) on \( \sum_{\text{ext} M(T, \nu)} \) such that for each \( B \in \mathcal{B}(X) \) the map \( \varrho \to \varrho(B) \) from \( \text{ext} M(T, \nu) \) to \( \mathbb{R} \) is Borel measurable and

\[
\lambda(B) = \int_{\text{ext} M(T, \nu)} \varrho(B) \, d\xi(\varrho).
\]

We refer the interested reader to [20] in which a more general version of the above result is considered. Indeed, in [20], S. Graf proved that the measurable sections of \( T \) can, modulo \( \nu \), be parameterized by the pre-image measures of \( \nu \). He has also shown that this parametrization can be done in a measurable way, i.e. if \( \Sigma \) is the \( \sigma \)-field of universally measurable subsets of \( Y \) then there exists an \( \sum_{\text{ext} M(T, \nu)} \otimes \Sigma - \mathcal{B}(X) \) measurable map \( L : \text{ext} M(T, \nu) \times Y \to X \) with the following properties:

i. For fixed \( \varrho \in \text{ext} M(T, \nu) \), the function \( L(\varrho, \cdot) \) is an \( \Sigma - \mathcal{B}(X) \) measurable section for \( T \).

ii. For every measurable section \( F \) for \( T \) there exists \( \varrho \in \text{ext} M(T, \nu) \) with \( L(\varrho, \cdot) = F(\cdot) \) for \( \nu \)-a.e. \( y \in Y \).

It then follows that for each bounded continuous function \( g \) on \( X \) and \( \lambda \) as in Theorem 2.5 one has

\[
\int_X g(x) \, d\lambda = \int_{\text{ext} M(T, \nu)} \int_X g(x) \, d\varrho(x) \, d\xi(\varrho) = \int_{\text{ext} M(T, \nu)} \int_Y g(L(\varrho, y)) \, d\nu(y) \, d\xi(\varrho).
\]

Finally, we recall the notion of measure isomorphisms and almost homeomorphisms.

Definition 2.6 Assume that \( X \) and \( Y \) are topological spaces with \( \mu \in \mathcal{P}(X) \) and \( \nu \in \mathcal{P}(Y) \). We say that \( (X, B(X), \mu) \) is isomorphic to \( (Y, B(Y), \nu) \) if there exists a one-to-one map \( T \) of \( X \) onto \( Y \) such that for all \( A \in B(X) \) we have \( T(A) \in B(Y) \) and \( \mu(A) = \nu(T(A)) \), and for all \( B \in B(Y) \) we have \( T^{-1}(B) \in B(X) \) and \( \mu(T^{-1}(B)) = \nu(B) \).

We shall also say that \( (X, B(X), \mu) \) and \( (Y, B(Y), \nu) \) are almost homeomorphic if there exists a one-to-one Borel mapping \( T \) from \( X \) onto \( Y \) such that \( \nu = \mu \circ T^{-1} \), \( T \) is continuous \( \nu \)-a.e., and \( T^{-1} \) is continuous \( \nu \)-a.e.

The following is due to Y. Sun [31].

Theorem 2.7 Let \( \mu \) be a Borel probability measure on a Polish space \( X \). Then the following assertions are true.

(i) There exist a Borel set \( Y \subset [0,1] \) and a Borel probability measure \( \nu \) on \( Y \) such that \( (X, \mu) \) and \( (Y, \nu) \) are almost homeomorphic.

(ii) If \( \mu \) has no atoms then \( (X, \mu) \) and \( ([0,1], \lambda) \), where \( \lambda \) is Lebesgue measure, are almost homeomorphic.

3 Properties of optimal plans.

In this section we shall proceed with the proofs of the statements in the introduction. We first state some preliminaries required for the proofs. Assume that \( \gamma \) is an optimal plan for \( (MK) \). It is standard that \( \gamma \in \mathcal{P}(\mu, \nu) \) is non-atomic if and only if at least one of \( \mu \) and \( \nu \) is non-atomic (see for instance [29]). Since \( \mu \) is non-atomic it follows from Theorem 2.7 that the Borel measurable spaces \( (X, B(X), \mu) \) and \( (X \times Y, B(X \times Y), \gamma) \) are isomorphic. Thus, there exists an isomorphism \( T = (T_1, T_2) \) from \( (X, B(X), \mu) \) onto \( (X \times Y, B(X \times Y), \gamma) \). It can be easily deduced that \( T_1 : X \to X \) and \( T_2 : X \to Y \) are surjective maps and

\[
T_1\#\mu = \mu \quad \& \quad T_2\#\mu = \nu.
\]

Consider the convex set

\[
\mathcal{M}(T_1, \mu) = \{ \lambda \in \mathcal{P}(X); T_1\#\lambda = \mu \},
\]

and note that \( \mu \in \mathcal{M}(T_1, \mu) \). The following result is essential in the sequel.
Lemma 3.1 Suppose $F_1, F_2$ are two distinct (weak) sections of $T_1$. Then the set
\[ \{ x \in X; F_1(x) = F_2(x) \} \]
is a null set with respect to the measure $\mu$.

Proof. Let $\lambda_i = F_i#\mu$ for $i = 1, 2$. It follows from Theorem 2.3 that $\lambda_1$ and $\lambda_2$ are extreme points of $\mathcal{M}(T_1, \mu)$. Let us assume that $\mu(\{ x; F_1(x) = F_2(x) \}) > 0$. We shall show that $\lambda_2$ can be written as a convex combination of two measures in $\mathcal{M}(T_1, \mu)$. This would then contradict the fact that $\lambda_2$ is an extreme point of $\mathcal{M}(T_1, \mu)$.

Define
\[ X_1 = \{ x \in X; F_2(x) = F_1(x) \} \quad \& \quad X_2 = \{ x \in X; F_2(x) \neq F_1(x) \}. \]

It $F_1$ and $F_2$ are distinct it follows that $0 < \mu(X_1) < 1$ from which we also obtain $0 < \mu(X_2) < 1$. Define measures $\eta_1$ and $\eta_2$ on $\mathcal{B}(X)$ as follows
\[ \eta_i(A) = \frac{\lambda_i(A \cap T_1^{-1}(X_i))}{\mu(X_i)} + \mu(A) - \frac{\mu(A \cap T_1^{-1}(X_i))}{\mu(X_i)}, \quad A \in \mathcal{B}(X), \quad i = 1, 2. \]

Note that $\eta_i \in \mathcal{M}(T, \mu)$. In fact,
\[ \eta_i(T_1^{-1}(A)) = \frac{\lambda_i(T_1^{-1}(A) \cap T_1^{-1}(X_i))}{\mu(X_i)} + \mu(T_1^{-1}(A)) - \frac{\mu(T_1^{-1}(A) \cap T_1^{-1}(X_i))}{\mu(X_i)} = \frac{\lambda_i(T_1^{-1}(A \cap X_i))}{\mu(X_i)} + \mu(T_1^{-1}(A)) - \frac{\mu(T_1^{-1}(A \cap X_i))}{\mu(X_i)} = \frac{\mu(T_1^{-1}(A \cap X_i))}{\mu(X_i)} + \mu(T_1^{-1}(A)) - \frac{\mu(T_1^{-1}(A \cap X_i))}{\mu(X_i)} = \mu(A), \]

where we have used the fact that $T_1#\lambda_1 = \mu = T_1#\mu$. We shall now show that $\mu(X_1)\eta_1 + \mu(X_2)\eta_2 = \lambda_2$. For $A \in \mathcal{B}(X)$ we have
\[ \sum_{i=1}^{2} \mu(X_i)\eta_i(A) = \sum_{i=1}^{2} \lambda_i(A \cap T_1^{-1}(X_i)) + \mu(A) - \sum_{i=1}^{2} \mu(A \cap T_1^{-1}(X_i)) = \sum_{i=1}^{2} \lambda_i(A \cap T_1^{-1}(X_i)) = \sum_{i=1}^{2} \mu(F_i^{-1}(A \cap T_1^{-1}(X_i))). \]

Note that $F_i^{-1}(A \cap T_1^{-1}(X_1)) = F_2^{-1}(A \cap T_1^{-1}(X_1))$. In fact,
\[ x \in F_i^{-1}(A \cap T_1^{-1}(X_1)) \iff F_1(x) \in A \cap T_1^{-1}(X_1) \iff x = T_1 \circ F_1(x) \in X_1 \& F_1(x) \in A \iff x = T_1 \circ F_2(x) \in X_1 \& F_2(x) \in A, \quad (\text{Since } F_1(x) = F_2(x) \text{ on } X_1) \iff x \in F_2^{-1}(A \cap T_1^{-1}(X_1)). \]
Therefore,
\[
\sum_{i=1}^{2} \mu(X_i) \eta_i(A) = \sum_{i=1}^{2} \mu\left(F_2^{-1}(A \cap T_1^{-1}(X_i))\right)
\]
\[
= \mu\left(\bigcup_{i=1}^{2} F_2^{-1}(A \cap T_1^{-1}(X_i))\right)
\]
\[
= \mu\left(F_2^{-1}(A) \cap \left(\bigcup_{i=1}^{2} T_1^{-1}(X_i)\right)\right)
\]
\[
= \mu(F_2^{-1}(A)) = \lambda_2(A).
\]

Thus, \(\lambda_2\) is a convex combination of \(\eta_1\) and \(\eta_2\). This completes the proof. \(\square\)

**Completion of the proof of Theorem 1.3.** Since \(\mu \in \mathcal{M}(T_1, \mu)\), it follows from Theorem 2.3 that there exists a Borel probability measure \(\xi\) on \(\sum_{M(T_1, \mu)}\) such that for each \(B \in \mathcal{B}(X)\),
\[
\mu(B) = \int_{\text{ext } M(T_1, \mu)} \varrho(B) d\xi(\varrho), \quad (\varrho \rightarrow \varrho(B) \text{ is measurable}).
\]

(3)

On the other hand, by Kantorovich duality (33), Theorem 5.10 there exists a pair of \(c\)-conjugate functions \(\varphi \in L^1(\mu)\) and \(\psi \in L^1(\nu)\) such that \(\varphi(x) + \psi(y) \leq c(x, y)\) for all \(x, y\) and
\[
\int_{X \times Y} c(x, y) d\gamma = \int_X \varphi(x) d\mu + \int_Y \psi(y) d\nu.
\]
As \(T = (T_1, T_2)\) is an isomorphism between \((X, \mathcal{B}(X), \mu)\) and \((X \times Y, \mathcal{B}(X \times Y), \gamma)\), it follows that
\[
\int_X c(T_1x, T_2x) d\mu = \int_X \varphi(T_1x) d\mu + \int_X \psi(T_2x) d\mu.
\]
from which together with the fact that \(c(x, y) \geq \varphi(x) + \psi(y)\) we obtain
\[
c(T_1x, T_2x) = \varphi(T_1x) + \psi(T_2x). \quad \mu - a.e.
\]
Since \(\varphi\) is \(\mu\) almost surely differentiable and \(T_1\# \mu = \mu\), it follows that
\[
D_1c(T_1x, T_2x) = \nabla \varphi(T_1x) \quad \mu - a.e.
\]
(4)

where \(D_1c\) stands for the partial derivative of \(c\) with respect to the first variable. Let \(A_\gamma \in \mathcal{B}(X)\) be the set with \(\mu(A_\gamma) = 1\) such that \(\mathbf{1}\) holds for all \(x \in A_\gamma\), i.e.
\[
D_1c(T_1x, T_2x) = \nabla \varphi(T_1x) \quad \forall x \in A_\gamma.
\]
(5)

Since \(\mu(X \setminus A_\gamma) = 0\), it follows from \(\mathbf{3}\) that
\[
\int_{\text{ext } M(T_1, \mu)} \varrho(X \setminus A_\gamma) d\xi(\varrho) = \mu(X \setminus A_\gamma) = 0,
\]
and therefore there exists a \(\xi\)-full measure subset \(K_\gamma\) of \(\text{ext } M(T_1, \mu)\) such that \(\varrho(X \setminus A_\gamma) = 0\) for all \(\varrho \in K_\gamma\).

Claim. The set \(K_\gamma\) is finite.

To prove the claim assume that \(K_\gamma\) contains an infinitely countable subset \(\{\varrho_n\}_{n \in \mathbb{N}}\). It follows from Theorem 2.3 part (iii) that there exist a sequence of \(\left(\mathcal{B}(X), \mathcal{B}(X)\right)\)-measurable sections \(\{F_n\}_{n \in \mathbb{N}}\) of the mapping \(T_1\) with \(\varrho_n = F_n\# \mu\). It follows from \(\mathbf{3}\) that
\[
D_1c(T_1 \circ F_n(x), T_2 \circ F_n(x)) = \nabla \varphi(T_1 \circ F_n(x)) \quad \forall x \in F_n^{-1}(A_\gamma).
\]
(6)

It follows that
\[
D_1c(x, T_2 \circ F_n(x)) = \nabla \varphi(x) \quad \forall x \in \bigcap_{n=1}^{\infty} F_n^{-1}(A_\gamma), \forall n \in \mathbb{N}.
\]
(7)
Since \( \theta_n(X \setminus A_\gamma) = 0 \) and \( \theta_n \) is a probability measure we have that \( \theta_n(A_\gamma) = 1 \) for every \( n \in \mathbb{N} \). Note that \( \theta_n(A_\gamma) = \mu\left( F_n^{-1}(A_\gamma) \right) \) and therefore \( \mu\left( \cap_{n=1}^\infty F_n^{-1}(A_\gamma) \right) = 1 \). This together with (7) yield that

\[
D_{1\gamma}(x, T_2 \circ F_n(x)) = \nabla \varphi(x) \quad \forall x \in \bar{A}_\gamma,
\]

where \( \bar{A}_\gamma = \cap_{n=1}^\infty F_n^{-1}(A_\gamma) \). Let

\[
A_{i,j} = \{ x \in X; F_i(x) = F_j(x) \}.
\]

It follows from Lemma 3.1 that \( \mu(A_{i,j}) = 0 \) provided \( i \neq j \). By setting \( B_\gamma = \cup_{i,j=1, i \neq j} A_{i,j} \) it follows that \( \mu(B_\gamma) = 0 \).

Take \( x \in \bar{A}_\gamma \setminus B_\gamma \). It follows from the generalized twist condition that the set

\[
L_x := \left\{ y \in Y; D_{1\gamma}(x, T_2 \circ F_1(x)) = D_{1\gamma}(x, y) \right\}
\]

is a finite set. On the other hand it follows from (8) that \( T_2 \circ F_n(x) \in L_x \) for all \( n \in \mathbb{N} \). Thus, there exist \( i, j \in \mathbb{N} \) with \( i \neq j \) such that \( T_2 \circ F_i(x) = T_2 \circ F_j(x) \). Since \( T_1 \circ F_1 = T_1 \circ F_j = Id_X \) and the map \( T = (T_1, T_2) \) is injective it follows that \( F_i(x) = F_j(x) \). This is a contradiction as \( x \notin B_\gamma \) from which the claim follows.

By the latter claim the cardinality of the set \( K_\gamma \) is a finite number. Let \( k = \text{Card}(K_\gamma) \) and assume that \( K_\gamma = \{ g_1, \ldots, g_k \} \). For every \( B \in \mathcal{B}(X) \) it follows from (3) that

\[
\mu(B) = \int_{\max M(T_1, \mu)} \varrho(B) d\xi(\varrho) = \int_{K_\gamma} \varrho(B) d\xi(\varrho).
\]

The latter representation shows that \( \mu \) is absolutely continuous with respect to the measure \( \sum_{i=1}^k \varrho_i \). It then follows that there exists a non-negative measurable function \( \alpha : X \to \mathbb{R} \cup \{ +\infty \} \) such that

\[
\frac{d\mu}{d\left( \sum_{i=1}^k \varrho_i \right)} = \alpha.
\]

Assume that \( F_1, \ldots, F_k \) are \( \mathcal{B}(X) \)-measurable sections of the mapping \( T_1 \) with \( \varrho_i = F_i \# \mu \). Define \( \alpha_i = \alpha \circ F_i \) for \( i = 1, \ldots, k \). We show that \( \sum_{i=1}^k \alpha_i(x) = 1 \) for \( \mu \)-almost every \( x \in X \). In fact, for each \( B \in \mathcal{B}(X) \) we have

\[
\mu(B) = \mu(T_1^{-1}(B)) = \sum_{i=1}^k \int_{T_1^{-1}(B)} \alpha(x) d\varrho_i = \sum_{i=1}^k \int_{F_i^{-1}(B)} \varrho_i(x) d\mu = \sum_{i=1}^k \int_B \alpha_i(x) d\mu,
\]

from which we obtain \( \mu(B) = \sum_{i=1}^k \int_B \alpha_i(x) d\mu \). Since this holds for all \( B \in \mathcal{B}(X) \) we have

\[
\sum_{i=1}^k \alpha_i(x) = 1, \quad \mu \text{ a.e.}
\]

It follows from Proposition 2.1 that each \( F_i \) is \( \mu \)-a.e. equal to a \( \mathcal{B}(X) \)-measurable function for which we still denote it by \( F_i \). For each \( i \in \{ 1, \ldots, k \} \), let \( G_i = T_2 \circ F_i \). We now show that \( \gamma = \sum_{i=1}^k \alpha_i(\text{Id} \times G_i) \# \mu \). For each bounded continuous function \( f : X \times Y \to \mathbb{R} \) it follows that

\[
\int_{X \times Y} f(x, y) d\gamma = \int_X f(T_1 x, T_2 x) d\mu = \sum_{i=1}^k \int_X \alpha(x) f(T_1 x, T_2 x) d\varrho_i
\]

\[
= \sum_{i=1}^k \int_X \alpha(F_i(x)) f(T_1 \circ F_i(x), T_2 \circ F_i(x)) d\mu
\]

\[
= \sum_{i=1}^k \int_X \alpha_i(x) f(x, G_i(x)) d\mu.
\]
Therefore,

$$\gamma = \sum_{i=1}^{k} \alpha_i (\text{Id} \times G_i) \# \mu.$$ 

To complete the proof we need to show that \( \{ x; G_i(x) = G_j(x) \} \) is a \( \mu \)-null unless \( G_i = G_j, \mu \)-a.e. on \( X \). Note that \( G_i = T_2 \circ F_i \) and therefore up to \( \mu \)-null set it follows that

\[
\{ x; G_i(x) = G_j(x) \} = \{ x; T_2 \circ F_i(x) = T_2 \circ F_j(x) \} = \{ x; T_2 \circ F_i(x) = T_2 \circ F_j(x) \& T_1 \circ F_i(x) = T_1 \circ F_j(x) \}, \quad (T_1 \circ F_i = T_1 \circ F_j = \text{Id})
\]

\[
= \{ x; T \circ F_i(x) = T \circ F_j(x) \}, \quad (T = (T_1, T_2))
\]

\[
= \{ x; F_i(x) = F_j(x) \}, \quad \text{(Since } T \text{ is injective)}
\]

By virtue of Lemma 3.1 \( \{ x; F_i(x) = F_j(x) \} \) is a \( \mu \)-null set from which the desired result follows. \( \square \)

Note that one can weaken the assumptions on the cost function in Theorem 1.3. Since, this does not require new ideas we do not elaborate.

**Proof of Corollary 1.4** The proof goes in the same lines as the proof of Theorem 1.3 and the only difference is that not only the set \( K_\gamma \) is finite but also its cardinality does not exceed \( m \) provided the cost function satisfies the \( m \)-twist condition. \( \square \)

We conclude this section by proving the generalized-twist property for locally \( m \)-twisted costs.

**Proof of Proposition 1.1** Fix \( x_0 \in X \) and \( y_0 \in Y \). We need to show that the set

\[
L_{(x_0,y_0)} = \left\{ y \in Y; D_1 c(x_0, y) = D_1 c(x_0, y_0) \right\},
\]

is finite. If \( L_{(x_0,y_0)} \) is not finite there exists an infinitely countable subset \( \{ y_n \}_{n \in \mathbb{N}} \subset L_{(x_0,y_0)} \). If either of the conditions (i) or (ii) in the statement of the proposition 1.1 holds then the sequence \( \{ y_n \}_{n \in \mathbb{N}} \) has an accumulation point \( \bar{y} \in Y \) and there exists a subsequence still denoted by \( \{ y_n \}_{n \in \mathbb{N}} \) such that \( y_n \to \bar{y} \). Since \( D_1 c \) is continuous it follows that \( \bar{y} \in L_{(x_0,y_0)} \). Since \( c \) is \( m \)-twisted locally, there exists a neighborhood \( U \) of \( \bar{y} \) such that the cardinality of the set

\[
\{ y \in U; D_1 c(x_0, y) = D_1 c(x_0, \bar{y}) \}
\]

is at most \( m \). This is a contradiction as \( \bar{y} \) is an accumulation point of the sequence \( \{ y_n \} \) and

\[
D_1 c(x_0, \bar{y}) = D_1 c(x_0, y_0) = D_1 c(x_0, y_n), \quad \forall n \in \mathbb{N}.
\]

This completes the proof. \( \square \)

## 4 Uniqueness

In this section we shall discuss some cases where we may have uniqueness in Theorem 1.3. Note first that uniqueness under the 1-twist condition has been extensively studied and it is known that if \( \mu \) is absolutely continuous with respect to the volume measure then uniqueness occurs. However, this can fail if \( \mu \) charges small sets. This observation makes it evident that under the \( m \)-twist condition on the \( x \) variable and 1-twist condition with respect to the \( y \) variable (\( x \to c_0(x,y) \) is injective) the uniqueness occurs provided both \( \mu \) and \( \nu \) do not charge small sets. There is also another uniqueness criterion known as the sub-twist property \( [2] \) i.e. for each \( y_1 \neq y_2 \in Y \) the map \( x \to c(x,y_1) - c(x,y_2) \) has no critical points, save at most one global maximum and one global minimum. Our approach is to study the extremality of the transport plans supported on the union of the graphs of finitely many functions in the convex set \( \Pi(\mu, \nu) \).

The following result shows that uniqueness may occur up to the support of optimal plans.
Proposition 4.1 Suppose that $c$ satisfies the $m$-twist condition and all the assumptions of Theorem 4.3 are fulfilled. Let $\gamma$ be an optimal plan such that

$$\gamma = \sum_{k=1}^{m} \alpha_k (Id \times G_k) \# \mu,$$

$$(\alpha_i(x) \geq 0 \text{ and } \alpha_1(x) \alpha_2(x) \ldots \alpha_m(x) \neq 0 \text{ for } \mu - a.e. \ x \in X ),$$

and for each $i \neq j$ the set $\{x; G_i(x) = G_j(x)\}$ is $\mu$-negligible. Then for any other optimal plan $\gamma$ we have

$$\text{Supp}(\gamma) \subseteq \text{Supp}(\gamma).$$

Proof. Take $\varphi$ and $\psi$ as in the proof of Theorem 4.3. It follows that

$$\int c(x,y) \, d\gamma = \int [\varphi(x) + \psi(y)] \, d\gamma,$$

from which we obtain

$$\sum_{k=1}^{m} \int X \alpha_i c(x,G_i x) \, d\mu = \sum_{k=1}^{m} \int X \alpha_i [\varphi(x) + \psi(G_i x)] \, d\mu.$$

It then follows that

$$\sum_{k=1}^{m} \int X \alpha_i [c(x,G_i x) - \varphi(x) - \psi(G_i x)] \, d\mu = 0.$$

Since each integrand in the latter expression is non-negative it yields that

$$c(x,G_i x) = \varphi(x) + \psi(G_i x) \quad \mu - a.e. \ \forall i \in \{1, \ldots, m\}.$$

Consequently we obtain,

$$D_1 c(x,G_i x) = \nabla \varphi(x) \quad \mu - a.e. \ \forall i \in \{1, \ldots, m\}.$$ (10)

Note also that for $i \neq j$ the set $\{x \in X; G_i(x) = G_j(x)\}$ is a null set with respect to the measure $\mu$. This together with (III) and the $m$-twist condition imply that the cardinality of the set $\{G_1x, \ldots, G_mx\}$ is $m$ for $\mu$-a.e. $x \in X$.

Now assume that $\gamma$ is also an optimal plan of $(MK)$. It follows from Theorem 4.3 that there exist a sequence of non-negative functions $\{\beta_i\}_{i=1}^{m}$ and, Borel measurable maps $T_1, \ldots, T_m : X \to Y$ such that

$$\gamma = \sum_{i=1}^{m} \beta_i (Id \times T_i) \# \mu$$

By a similar argument as above one obtains

$$\beta_i(x) [D_1 c(x,T_i x) - \nabla \varphi(x)] = 0 \quad \mu - a.e. \ \forall i \in \{1, \ldots, m\}.$$ For each $i$ define $\Omega_i = \{x \in X; \beta_i(x) \neq 0\}$. Since the cardinality of the set $\{G_1x, \ldots, G_mx\}$ is $m$ for $\mu$-a.e. $x \in X$ and since $c$ satisfies the $m$-twist condition we have that for each $i$, $\{T_i x\} \subseteq \{G_1x, \ldots, G_mx\}$ for $\mu$-a.e. $x \in \Omega_i$. This completes the proof. \(\square\)

We shall now provide a criterion for the uniqueness of measures in $\Pi(\mu, \nu)$ that are supported on the graphs of a finite number of measurable maps.

Theorem 4.1 Let $X$ and $Y$ be Polish spaces equipped with Borel probability measures $\mu$ on $X$ and $\nu$ on $Y$. Let $\{T_i\}_{i=1}^{k}$ be a sequence of measurable maps from $X$ to $Y$ such that $T_i$ is injective for each $i \in \{2, \ldots, k\}$ and $R(T_i) \cap R(T_j) = \emptyset$ for all $2 \leq i, j \leq k$ with $i \neq j$. Assume that the following assertions hold:

1. For each $x \in \text{Dom}(T_i)$ and each $2 \leq i \leq k$, if $T_i(x) \in R(T_i)$ and $T_i \circ T_i^{-1} \circ T_i(x) \in \bigcup_{j=2}^{k} R(T_j)$ then $T_1 \circ T_i^{-1} \circ T_i(x) \in R(T_i)$. 

We will use the following properties of Borel probability measures:

- Property 1
- Property 2

To prove the theorem, we will show that $\text{Supp}(\mu) \subseteq \text{Supp}(\nu)$ and $\text{Supp}(\nu) \subseteq \text{Supp}(\mu)$, which implies the uniqueness of the measures.

Proof. Since $\mu$ and $\nu$ are Borel probability measures, we have

$$\text{Supp}(\mu) = \bigcup_{x \in X} \overline{\{x\} \times Y} \text{ and } \text{Supp}(\nu) = \bigcup_{x \in Y} \overline{\{x\} \times X},$$

where $\overline{\{x\} \times Y}$ denotes the closure of the set $\{x\} \times Y$ in $\mathbb{R}^n$. Since $T_i$ is injective for each $i$, we have

$$\text{Supp}(\mu) = \bigcup_{x \in X} \overline{T_i(x) \times Y} = \bigcup_{x \in Y} \overline{T_i^{-1}(x) \times X} = \text{Supp}(\nu).$$

This shows that $\text{Supp}(\mu) \subseteq \text{Supp}(\nu)$ and $\text{Supp}(\nu) \subseteq \text{Supp}(\mu)$, thus proving the uniqueness of the measures. \(\square\)
2. There exists a bounded measurable function \( \theta : Y \to \mathbb{R} \) with the property that \( \theta(T_1 x) - \theta(T_2 x) \geq 0 \) on \( X \) and \( \theta(T_1 x) - \theta(T_2 x) = 0 \) if and only if \( T_1 x = T_2 x \).

Then there exists at most one \( \gamma \in \Pi(\mu, \nu) \) that is supported on the union of the graphs of \( T_1, T_2, ..., T_k \).

As an immediate consequence of the latter Theorem we recover the following result from [30].

**Corollary 4.4** Let \( X = Y = [0,1] \) and \( \mu = \nu \) is the Lebesgue measure. If \( T_1 \leq T_2 \) and one of \( T_1 \) or \( T_2 \) is injective on \( D = \{ x : T_1(x) \neq T_2(x) \} \) then there exists at most one \( \gamma \in \Pi(\mu, \nu) \) that is supported on the graphs of \( T_1, T_2 \).

**Proof** Suppose \( T_2 \) is injective on \( D \). One can define \( \theta : Y \to \mathbb{R} \) by \( \theta(y) = -y \). Since \( T_1 \leq T_2 \) then \( \theta(T_1(x)) - \theta(T_2(x)) \) is non-negative and \( \theta(T_1(y)) - \theta(T_2(y)) = 0 \) iff \( T_1(y) = T_2(y) \). The result then follows from Theorem 4.1 and Remark 4.6. \( \square \)

Here is another application of Theorem 4.1 for maps with disjoint ranges.

**Corollary 4.3** Let \( X \) and \( Y \) be Polish spaces equipped with Borel probability measures \( \mu \) on \( X \) and \( \nu \) on \( Y \). Let \( \{ T_i \}_{i=1}^k \) be a sequence of measurable maps from \( X \to Y \) such that \( T_i \) is injective for each \( i \in \{ 2, ..., k \} \) and \( R(T_i) \cap R(T_j) = \emptyset \) for all \( i \leq j \leq k \) with \( i \neq j \). If \( R(T_i) \) is measurable then there exists at most one \( \gamma \in \Pi(\mu, \nu) \) that is supported on the graphs of \( T_1, T_2, ..., T_k \).

**Proof.** Assumption (1) in Theorem 4.1 follows from the fact that \( R(T_i) \cap R(T_j) = \emptyset \) for each \( i, j \in \{ 1, ..., k \} \) with \( i \neq j \). By considering \( \theta(y) = \chi_{R(T_i)}(y) \), the indicator function of \( R(T_i) \), one can easily check that \( \theta \) satisfies the required properties in assumption (2) of Theorem 4.1. \( \square \)

We need some preliminaries before proving Theorem 4.4. For a map \( f \) from a set \( X \) to a set \( Y \) denote by \( \text{Dom}(f) \) the domain of \( f \) and by \( R(f) \) the range of \( f \). The graph of \( f \) is denoted by \( \text{Graph}(f) \) and defined by

\[
\text{Graph}(f) = \{(x, f(x)); x \in \text{Dom}(f)\}
\]

For a map \( g \) from \( Y \) to \( X \), the antigraph of \( g \) is denoted by \( \text{Antigraph}(g) \) and defined by

\[
\text{Antigraph}(g) = \{(g(y), y); y \in \text{Dom}(g)\}
\]

Here we recall the definition of aperiodic representations [4].

**Definition 4.4** Let \( X \) and \( Y \) be two sets and let \( f : X \to Y \) and \( g : Y \to X \). Define

\[
T(x) = \begin{cases} 
g \circ f(x), & x \in \text{Dom}(f) \cap f^{-1}(\text{Dom}(g)) = D(T), \\
x, & x \notin D(T). 
\end{cases}
\]

The maps \( f, g \) are aperiodic if \( x \in D(T) \) implies that \( T^n(x) \neq x \) for any \( n \geq 1 \).

If \( S = \text{Graph}(f) \cup \text{Antigraph}(g) \), \( \text{Graph}(f) \cap \text{Antigraph}(g) = \emptyset \) and \( f, g \) are aperiodic, then this is called an aperiodic decomposition of \( S \). Moreover, if \( (X, \Sigma(X)) \) and \( (Y, \Sigma(Y)) \) are measure spaces and the maps \( f \) and \( g \) are measurable we call the maps \( f, g \) measure-aperiodic if any \( T \)-invariant probability measure defined on \( \Sigma(X) \) is supported by \( X \setminus D(T) \).

It follows we say that \( \gamma \in \Pi(\mu, \nu) \) is concentrated on a set \( S \) if the outer measure of its complement is zero, i.e. \( \gamma^*(S^c) = 0 \). We recall the following result from [4] regarding doubly stochastic measures with aperiodic supports.

**Theorem 4.5 (Benes & Stepan 1987)** Let \( (X, \mathcal{B}(X), \mu) \) and \( (Y, \mathcal{B}(Y), \nu) \) be complete separable Borel metric spaces. Let \( f : X \to Y \) and \( g : Y \to X \) be aperiodic measurable maps and \( \text{Graph}(f) \cap \text{Antigraph}(g) = \emptyset \). Then there exists at most one \( \gamma \in \Pi(\mu, \nu) \) that is supported on \( S = \text{Graph}(f) \cup \text{Antigraph}(g) \) provided \( f \) and \( g \) are measure-aperiodic.
Beginning with the work of Lindenstrauss and Douglas \cite{13, 24}, Hestir and Williams \cite{21} provided an alternate proof of the latter Theorem while further refining the structure these graphs should take, and rewriting them in terms of measurable limb numbering systems. Chiappori, McCann and Nesheim \cite{12} further improved the result of Hestir and Williams by weakening the measurability requirement. For our purpose in this paper the result of Benes & Stepan seems to be more suitable.

**Proof of Theorem 4.4.** For each $i \geq 2$, since $T_i$ is injective we have that $R(T_i)$ is a measurable subset of $Y$. Define

$$g : \text{Dom}(g) = \bigcup_{i=2}^{k} R(T_i) \subset Y \rightarrow X,$$

by $g(y) = T_i^{-1}(y)$ for $y \in R(T_i)$ and note that $g$ is measurable. Define $f : \text{Dom}(f) \rightarrow Y$ by $f(x) = T_1(x)$ where

$$\text{Dom}(f) = \{ x \in \text{Dom}(T_1) ; \ T_1(x) \neq T_i(x) \text{ for all } 2 \leq i \leq k \}.$$

Note that $\text{Graph}(f) \cap \text{Antigraph}(g) = \emptyset$. In fact, if $\text{Graph}(f) \cap \text{Antigraph}(g) \neq \emptyset$ then there exists $x \in \text{Dom}(f)$ and $y \in \text{Dom}(g)$ with $(x, f(x)) = (g(y), y)$. It then follows that $y = f(x) = T_1(x)$ and $x = T_i^{-1}(y)$ for some $2 \leq i \leq k$. This is a contradiction as $T_i(x) \neq T_i(x)$ on $\text{Dom}(f)$. Define $T : X \rightarrow X$ as in Definition \cite{42} i.e.,

$$T(x) = \begin{cases} 
    g \circ f(x) , & x \in \text{Dom}(f) \cap f^{-1}(\text{Dom}(g)) = D(T), \\
    x , & x \notin D(T).
\end{cases}$$

We shall now proceed with the rest of the proof in two steps. In the first step we show that $f$ and $g$ are aperiodic and in the second step we show that $f$ and $g$ are measure-aperiodic. Then the result follows from Theorem \cite{43}.

**Step 1:** Assume that there exist $x \in D(T) = \text{Dom}(f) \cap f^{-1}(\text{Dom}(g))$ and $n \in \mathbb{N}$ such that $(g \circ f)^n(x) = x$. If $n = 1$ then $T_1(x) = T_1(x)$ for $x \in \text{Dom}(f)$ and some $i \geq 2$ which leads to a contradiction. Let us assume that $n > 1$. We have

$$(g \circ f)^{n-1} \circ g \circ f(x) = x.$$

It follows that $f(x) = T_1(x) \in \text{Dom}(g)$ and therefore $g \circ f(x) = T_i^{-1} \circ f(x)$ for some $2 \leq i \leq k$. Since $T_1(x) \in R(T_i)$ and $T_1 \circ T_i^{-1} \circ T_1(x) \in \text{Dom}(g)$, it follows from assumption (2) in the current Theorem that $T_1 \circ T_i^{-1} \circ T_1(x) \in R(T_i)$. This together with $g \circ f(x) = T_i^{-1} \circ f(x) \in \text{Dom}(f)$ imply that $T_1 \circ T_i^{-1} \circ T_1(x) = f \circ g \circ f(x)$ and

$$g \circ f \circ g \circ f(x) = T_i^{-1} \circ T_1 \circ T_i^{-1} \circ T_1(x).$$

Finally by induction we obtain that $(g \circ f)^n(x) = (T_i^{-1} \circ T_1)^n(x)$. On the other hand by assumption (2) in the Theorem there exists a function $\theta : Y \rightarrow \mathbb{R}$ with the property that $\theta(T_1(x)) - \theta(T_i(x))$ is non-negative and $\theta(T_1(x)) - \theta(T_i(x)) = 0$ if and only if $T_1(x) = T_i(x)$. It now follows from $(T_i^{-1} \circ T_1)^n(x) = x$ that

$$(T_1 \circ T_i^{-1})^{n-1} \circ T_1(x) = T_1(x),$$

and therefore,

$$\theta(T_1(x)) = \theta((T_1 \circ T_i^{-1})^{n-1} \circ T_1(x)). \quad (11)$$

We also have

$$\theta((T_1 \circ T_i^{-1})^{n-1} \circ T_1(x)) = \theta((T_1 \circ T_i^{-1}) \circ (T_1 \circ T_i)^{n-2} \circ T_1(x)) \geq \theta((T_1 \circ T_i^{-1} \circ (T_1 \circ T_i)^{n-2} \circ T_1(x)),$$

since $\theta \circ T_1 \geq \theta \circ T_i$.

from which we obtain

$$\theta((T_1 \circ T_i^{-1})^{n-1} \circ T_1(x)) \geq \theta((T_1 \circ T_i)^{n-2} \circ T_1(x)).$$

By repeating the latter argument we obtain that

$$\theta((T_1 \circ T_i^{-1})^{n-1} \circ T_1(x)) \geq \theta(T_1(x)). \quad (12)$$
It now follows from (11) and (12) that \( \theta(T_1(x)) \geq \theta(T_i(x)) \). On the other hand by the assumption we have that \( \theta(T_1(x)) - \theta(T_i(x)) \) is non-negative and therefore it must be zero, i.e.,

\[
\theta(T_1(x)) = \theta(T_i(x)).
\]

By the properties of \( \theta \) we have \( T_1(x) = T_i(x) \) that contradicts with the fact that \( x \in \Dom(f) \). This completes the proof of Step (1).

Step 2: To prove that \( f \) and \( g \) are measure-aperiodic we need to show that any \( T \)-invariant probability measure on \( \mathcal{B}(X) \) is supported in \( X \setminus \Dom(T) \) where \( \Dom(T) = \Dom(f) \cap f^{-1}(\Dom(g)) \). Suppose that \( \lambda \) is a probability measure on \( \mathcal{B}(X) \) with \( T_{\#} \lambda = \lambda \). Note first that since \( T(x) = x \) for each \( x \in X \setminus \Dom(T) \) we have that \( \lambda(T^{-1}(A)) = \lambda(A) \) for every measurable subset of \( D(T) \). It then implies that \( (g \circ f)_{\#} \lambda = \lambda \) on \( D(T) \). Let \( f_{|D(T)} \) be the restriction of \( f \) on \( D(T) \) and let \( \eta \) be the push forward of \( \lambda \) by \( f_{|D(T)} \). Since \( (g \circ f)_{\#} \lambda = \lambda \) on \( D(T) \), it follows that \( g_{\#} \eta = \lambda \). Let \( \mathcal{M}(\lambda) \) be the set of positive measures on \( \mathcal{B}(X) \cap \Dom(g) \) defined by

\[
\mathcal{M}(\lambda) = \{ \zeta; \ g_{\#} \zeta = \lambda \}.
\]

Note that \( \mathcal{M}(\lambda) \) is convex and \( \eta \in \mathcal{M}(\lambda) \). By Theorem 2.3 extreme points of the set \( \mathcal{M}(\lambda) \) are determined by the preimages of \( \lambda \). It follows from the construction of \( g \) that the preimages of \( \lambda \) are exactly the maps \( T_2, \ldots, T_k \). Thus, by Theorem 2.3 the extreme points of \( \mathcal{M}(\lambda) \) are exactly the measures, \( \zeta_i = T_i_{\#} \lambda \) for \( i = 2, \ldots, k \). It then follows that \( \eta \in \mathcal{M}(\lambda) \) can be written as a convex combination of these measures,

\[
\eta = \sum_{i=2}^{k} \beta_i \zeta_i,
\]

where \( \beta_2, \ldots, \beta_k \) are non-negative real numbers with \( \sum_{i=2}^{k} \beta_i = 1 \). Considering that \( \eta \) is the push forward of \( \lambda \) by the map \( f_{|D(T)} \) together with \( g_{\#} \eta = \lambda \) we have

\[
\int_{\Dom(T)} \theta(T_1(x)) \, d\lambda = \int_{\Dom(T)} \theta(f(x)) \, d\lambda = \int_{\Dom(g)} \theta(y) \, d\eta = \sum_{i=2}^{k} \beta_i \int_{\Dom(g)} \theta(y) \, d\zeta_i = \sum_{i=2}^{k} \beta_i \int_{D(T)} \theta(T_i(x)) \, d\lambda,
\]

where \( \theta \) is the function given by the assumption (2) in the current Theorem. Since \( \theta(T_1(x)) \geq \theta(T_i(x)) \) for \( 2 \leq i \leq k \), it follows from (12) that

\[
\int_{D(T)} \theta(T_1(x)) \, d\lambda = \sum_{i=2}^{k} \beta_i \int_{D(T)} \theta(T_i(x)) \, d\lambda \leq \sum_{i=2}^{k} \beta_i \int_{D(T)} \theta(T_1(x)) \, d\lambda = \int_{D(T)} \theta(T_1(x)) \, d\lambda.
\]

This in fact implies that

\[
\sum_{i=2}^{k} \beta_i \int_{D(T)} [\theta(T_i(x)) - \theta(T_1(x))] \, d\lambda = 0.
\]

Since each \( \beta_i \) is non-negative and \( \sum_{i=2}^{k} \beta_i = 1 \) at least one of them should be nonzero. Assuming that \( \beta_{i_0} \neq 0 \), we must have \( \theta(T_{i_0}(x)) = \theta(T_1(x)) \) for \( \lambda \) almost every \( x \in D(T) \). Therefore, by the properties of the function \( \theta \) we must have \( T_{i_0}(x) = T_1(x) \) for \( \lambda \) almost every \( x \in D(T) \). On the other hand, for each \( x \in D(T) \) we have \( T_{i_0}(x) \neq T_1(x) \) from which we obtain that \( \lambda \) must be zero on \( D(T) \). This indeed proves that \( \lambda \) must be supported in \( X \setminus D(T) \). This completes the proof of Step (2).
Remark 4.6  Theorem 4.1 still holds if one replaces the injectivity of $T_2, ..., T_k$ with the following assumption:

- For each $i \geq 2$, $T_i$ is injective on the set $D_i = \{ x; T_1(x) \neq T_i(x) \}$.

In fact, one just needs to redefine the domain of $f$ and $g$ as follows: $\text{Dom}(g) = \cup_{i=2}^{k} T_i(D_i)$ and $\text{Dom}(f) = \text{Dom}(T_1)$.

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