Competition evolution of Rayleigh-Taylor bubbles

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Material mixing induced by a Rayleigh-Taylor instability occurs ubiquitously in either nature or engineering when a light fluid pushes against a heavy fluid, accompanying with the formation and evolution of chaotic bubbles. Its general evolution involves two mechanisms: bubble-merge and bubble-competition. The former obeys a universal evolution law and has been well-studied, while the latter depends on many factors and has not been well-recognized. In this paper, we establish a theory for the latter to clarify and quantify the longstanding open question: the dependence of bubbles evolution on the dominant factors of arbitrary density ratio, broadband initial perturbations and various material properties (e.g., viscosity, miscibility, surface tensor). Evolution of the most important characteristic quantities, i.e., the diameter of dominant bubble \(D\) and the height of bubble zone \(h\), is derived: (i) the \(D\) expands self-similarly with steady aspect ratio \(\beta \equiv D/h \approx (1+A)/4\), depending only on dimensionless density ratio \(A\), and (ii) the \(h\) grows quadratically with constant growth coefficient \(\alpha \equiv h/(Ag^2) \approx [2\phi/\ln(2\eta_0)]^2\), depending on both dimensionless initial perturbation amplitude \(\eta_0\) and material-property-associated linear growth rate ratio \(\phi \equiv \Gamma_{\text{actual}}/\Gamma_{\text{ideal}} \leq 1\). The theory successfully explains the continued puzzle about the widely varying \(\alpha \in (0.02, 0.12)\) in experiments and simulations, conducted at all value of \(A \in (0, 1)\) and widely varying value of \(\eta_0 \in [10^{-7}, 10^{-2}]\) with different materials. The good agreement between theory and experiments implies that majority of actual mixing depends on initial perturbations and material properties, to which more attention should be paid in either natural or engineering problems.

Key words: Rayleigh-Taylor instability, bubbles competition, turbulent mixing

1. Introduction

When two fluids are separated by an irregular perturbed interface and are accelerated in a direction opposite to that of the density gradient, Rayleigh-Taylor (RT) instability occurs and develops rapidly into the turbulent regime (Cheng et al. 2002) consisting of a bubble mixing zone (formed when a light fluid penetrates a heavy fluid) and a spike mixing zone (formed when a heavy fluid penetrates a light fluid). The mixing occurs ubiquitously in systems extending from micro to astrophysical scales (Livescu 2013). As the simplest and primary descriptor of mixing, quantitative knowledge of the evolution of the structure and height of the mixing zone plays a fundamental role (Cheng et al. 2002; Dimonte et al. 2004; Dimonte 2004; Zhang et al. 2016) for understanding many natu-
ral phenomena (e.g., supernova explosions) and engineering applications (e.g., inertial confinement fusion).

Up to now, it is well-known that the height of the bubble mixing zone $h$ grows quadratically with constant quadratic growth coefficient $\alpha \equiv h/(Ag^2)$ (Read 1984; George et al. 2002; Kadau et al. 2004; Lim et al. 2010; Youngs 2017), and the diameter of the dominant bubble $D$ expands self-similarly with quasi-steady aspect ratio $\beta \equiv D/h$ (Alon et al. 1995; Dimonte & Schneider 2000; Dimonte et al. 2004), where $g$ is acceleration, $A \equiv (R - 1)/(R+1) \in [0,1]$ is dimensionless Atwood number defined with density ratio $R \equiv \rho_{\text{heavy}}/\rho_{\text{light}}$. Due to the nearly stationary center of mass of mixing zone, the evolution of the spike mixing zone can be determined by that of the bubble mixing zone (Cheng et al. 1999, 2000; Zhang et al. 2016). Consequently, knowledge of the values of $\alpha$ and $\beta$ becomes extremely important, but is still an open question (Dimonte & Schneider 2000; Dimonte et al. 2004; Dimonte 2004). This puzzle may be attributed to the continued lack of a unified theory to regularise the observed $\alpha$ and $\beta$ with the dominant factors affecting mixing evolution, including the density ratio, initial perturbation amplitude and material properties (e.g., viscosity, surface tensor, miscibility or diffusivity) (Read 1984; Linden & Redondo 1991; Dalziel et al. 1999; Dimonte 2004; Kadau et al. 2004; Ramaprabhu et al. 2005; Mueschke et al. 2009; Banerjee & Andrews 2009; Lim et al. 2010).

In earlier studies, possibly influenced by the facts that all measured $\alpha \in (0.05, 0.07)$ in different apparatus are independent of $A$ (Read 1984; Youngs 1989; Kucherenko et al. 1991; Dimonte & Schneider 2000), researchers tended to find a universal $\alpha$ (Alon et al. 1994, 1995; Oron et al. 2001). However, except for the comparable results predicted with Front-Tracking method (George et al. 2002), majority of shortwave-perturbation simulations over the past several decades predicted a much smaller $\alpha \approx 0.025$ (Dimonte et al. 2004, 2005; Cabot & Cook 2006; Youngs 2013, 2017). Moreover, a recent shortwave-perturbation experiment (Olson & Jacobs 2009) with miscible fluids indirectly validated previous numerical simulations (Dimonte et al. 2004) and excluded the possibility of universal $\alpha$. Now the observed $\alpha$ has changed widely from 0.02 to 0.12 (Dimonte et al. 2004, 2005; Youngs 2013, 2017). Although many factors would affect the value of $\alpha$ and $\beta$ (Dimonte et al. 2004), we argue that the the major factors can be classified as two categories: (i) the initial perturbations at the interface and (ii) the material properties (e.g., density, viscosity, diffusivity, thermal diffusivity, surface tensor). This classification can be easily understood from the viewpoint of direct numerical simulation. The former determines the initial condition, and the latter determined the dimensionless parameters of governing equation, i.e. the Atwood, Reynolds, Schmidt, Prandtl and Weber Number (Cook & Dimotakis 2001; George et al. 2002). However, up to now, a quantitative dependence of either $\alpha$ or $\beta$ on these factors has not been established.

In the other hand, now it is clear that self-similar evolution of RT-mixing can be achieved through two limiting and distinct mechanisms: bubble-merger and bubble-competition (Dimonte 2004; Dimonte et al. 2005; Youngs 2013). If the interface is perturbed entirely by random combined waves with individual wavelengths $\lambda$ much shorter than the system width $L$, bubbles will expand self-similarly via merging with their smaller neighbours (Alon et al. 1994, 1995), leading to a universal lower bound $\alpha \approx 0.025$ (Dimonte et al. 2004, 2005; Youngs 2013, 2017). If perturbation involves some longer wavelengths $\lambda$ comparable to $L$, the mixing at a later time evolves dominantly via the competition between the individual growth of the long waves seeded initially, leading to a larger $\alpha$ (Dimonte 2004; Ramaprabhu et al. 2005; Dimonte et al. 2005; Youngs 2013). In the latter situation, since the growth of individual wave closely relates to initial perturbation amplitude $h_0$ and material-property-associated linear growth rate $\Gamma$, the
corresponding $\alpha$ may thus depend on dimensionless $\eta_0 \equiv h_0/\lambda$ and $\phi = \Gamma_{\text{actual}}/\Gamma_{\text{ideal}}$. Because the latter situation dominates in actual flow scenarios (Haan 1989; Dimonte 2004; Ramaprabhu et al. 2005; Youngs 2013), formulating $\alpha(A, \eta_0, \phi)$ and $\beta(A, \eta_0, \phi)$ thus becomes extremely important but no self-consistent or satisfactory (Zmitrenko et al. 1997; Ramaprabhu et al. 2005) theory has yet been established.

In this paper, a theory is established yielding analytic relations of $\beta \approx (1 + A)^4$ and $\alpha \approx \left[2\phi/\ln(2\eta_0)\right]^2$, which successfully reproduce the observed results (Read 1984; Youngs 1989; Kucherenko et al. 1991; Dimonte & Schneider 2000; Ramaprabhu et al. 2005; Youngs 2013) and formulate the disordered data (Dimonte et al. 2004, 2005).

2. Theory

In this section, we present current theory by progressively clarifying the problems evolving from single-wave, wavepacket, and broadband-wave perturbations as follows. For the sake of conciseness, all mathematical derivations are given in Appendix.

2.1. Evolution from single-wave perturbation

Up to now, until time $t_{Re}$, corresponding to the possible appearance of a reacceleration stage, the development of an instability starting from a linear stage with exponentially growing $h(t)$ and transitioning into a quasi-steady stage with linearly growing $h(t)$ has been widely recognised and well formulated (Zhang 1998; Mikaelian 1998; Sohn 2003; Mikaelian 2003; Abarzhi et al. 2003; Goncharov 2002; Ramaprabhu & Dimonte 2005; Zhang & Guo 2016). In the two-dimensional (2D) problem, Zhang & Guo (2016) obtained a universal analytical expression $h(t, A, h_0, \dot{h}_0)$ for an arbitrary $A$ and initial perturbation until $t_{Re}$, where the dot and subscript 0 denote, respectively, the derivative with respect to time and the value at the initial time. In the three-dimensional (3D) problem, one can obtain a similar expression by following Zhang’s procedure (Zhang & Guo 2016) and by referring to previous analytical solutions (Sohn 2003; Mikaelian 2003) (see appendix A). The 3D solution also works until $t_{Re}$, but its form is slightly complex. To simplify the solution, corresponding to the famous concept of boundary layer thickness $\delta_99$ introduced originally to divide the spatial-dependent velocity profile (Tani 1977), we define a time boundary layer thickness $t_99$ to divide the time-dependent velocity evolution $\dot{h}(t)$ as follows: at time $t_99$, the instantaneous velocity asymptotically approaches the time-independent terminal velocity $\dot{h}_\infty$ of the quasi-steady stage, i.e., $\dot{h}_99 = 0.99\dot{h}_\infty \rightarrow \dot{h}_\infty$. Consequently, for $t \in [t_99, t_{Re}]$, a linearly growing $h(t)$ is obtained (see appendix B)

$$h(t) = -\frac{\lambda}{\chi(A, \eta_0)} + \dot{h}_\infty(A, g, \lambda) t, \quad t \in [t_99, t_{Re}],$$

(2.1)

where the dimensionless initial perturbation amplitude $\eta_0$ is very small in general problem (Dimonte 2004; Ramaprabhu et al. 2005; Youngs 2013). Moreover, the introduction of $t_99$ directly leads to the following important findings (see appendix B): (a) the instability enters the quasi-steady stage when $h(t)$ grows to $h_99(t_99) \approx C_2 \Theta \lambda$, (b) $t_99$ is universal scaled as $t_99 \propto \sqrt{\lambda \Theta}/(Ag)$, and (c) independent of $A, \lambda, g, \eta_0, \dot{\eta}_0$ and $\Theta$, $\dot{h}_\infty$ is always proportional to the average velocity $h_99 \equiv (h_99 - h_0)/t_99$ as $\dot{h}_\infty/h_99 = \text{const} = C$, where $\Theta(A)$ arises from potential theory and differs among different theories (Goncharov 2002; Sohn 2003) (see appendix A). These findings work for an arbitrary density ratio and initial perturbations and will be used below.
2.2. Evolution from wavepacket perturbations

For an interface perturbed by a narrow wavepacket gathering around the dominant $\lambda_d$ , $h(t)$ evolves differently than problem 2.1. However, previous studies show that the root-mean-square amplitude $h_{rms} \equiv [(2\pi)^{-1}L^2 \int h^2 k dk]^{1/2}$ grows similarly to that of $h(t)$ in problem 2.1 until some autocorrelation time after $t_{99}$ (Haan 1989; Dimonte 2004), where $k \equiv 2\pi/\lambda$. Therefore, if we treat $h$ and $\lambda$ as $h_{rms}$ and $\lambda_d$, respectively, equation (2.1) still applies, and for conciseness hereafter this treatment is adopted. However, due to the introduction of a newly defined amplitude, the asymptotical velocity $h_\infty$ in equation (2.1) becomes an unknown and is determined, with the aid of the important findings emphasised in problem 2.1 as follows: (i) $h_\infty$ is determined by using finding (c), i.e., $\dot{h}_\infty = C\dot{h}_{99}^9$, (ii) the constant $C$ is viewed as a unique unknown parameter in the current theory and is determined with experimental data, (3) $\dot{h}_{99}^9$ is determined by combining findings (a), (b) and the exponential growth of $h(t)$ in the linear stage, thus relating it to the initial perturbation $\eta_0$ and material-property-associated $\phi$ (see appendix C). After obtaining $h_\infty$, the final solution derived from equation (2.1) gives

$$h(t) = -\lambda/\chi(C, A, \eta_0) + Fr(C, A, \eta_0, \phi)\sqrt{Ag\lambda/(1 + A)}t, \ t \in [t_{99}, t_{Re}],$$

(2.2)

where $Fr$ (see appendix C for the expression) is called Froude number following the literature (Dimonte 2004; Ramaprabhu et al. 2005; Dimonte et al. 2005), and $\phi \equiv \Gamma_{actual}/\Gamma_{ideal} \leq 1$ is defined as the ratio of the actual linear growth rate $\Gamma_{actual}$ to the ideal linear growth rate $\Gamma_{ideal} \equiv \sqrt{Ak\phi}$, quantifying the relative decrement of the linear growth rate caused by various material properties (Dimonte 2004).

2.3. Evolution from broadband wave perturbations

Considering an initial perturbation with arbitrary amplitude spectrum $h(\lambda)$, the dimensionless $\eta_0(\lambda)$, evaluated from $h_{rms}$ and $\lambda_d$ of the wavepacket, obviously depends on $\lambda$. However, numerical studies (Ramaprabhu et al. 2005; Youngs 2013) show that the shape of the amplitude spectrum has little effect on either $\alpha$ or $\beta$. Moreover, in most actual problems (Haan 1989; Dimonte 2004; Youngs 2013), $h(\lambda) \propto \lambda^2/L$, for which spectrum the corresponding $\eta_0$ is independent of $\lambda$ (Dimonte 2004). Therefore, hereafter we only describe a theory for $\lambda$-independent $\eta_0$. As for problems evolving from an interface perturbed by a broadband wave, the bubble-competition mechanism means that the bubble mixing zone grows as the competition of a series of amplitudes of individual wavepackets. Following Birkhoff and Dimonte (2004), the self-similar evolution of the dominant bubble can be obtained by seeking the dominant wavelength $\lambda_d$ that maximises equation (2.2). This is accomplished by requiring $\partial h/\partial \lambda = 0$ in equation (2.2) to give $\lambda_d = \chi^2 Fr^2 Agt^2/[4(1 + A)]$, which when substituting into equation (2.2) yields $h = \lambda_d/\chi$. Naturally, $\alpha \equiv h/(Agt^2) = \chi Fr^2/[4(1 + A)]$, and $\beta \equiv D/h = \chi$ if we use the approximation relation $\lambda_d \propto D$ (Goncharov 2002; Dimonte 2004; Dimonte et al. 2005; Ramaprabhu & Dimonte 2005). The unique unknown constant $C \approx 8.843$ in $Fr(C)$ is determined with the observed $\beta[C, \Theta(A)] \approx 0.5$ at $A = 1$, the only value of $A$ for which there is no disagreement (Dimonte et al. 2005) regarding the value of $\Theta(A) = 1$. As for $\Theta(A)$, the analytical results $\Theta(A) = 2/(1 + A)$ (Goncharov 2002) agrees very well with numerical simulations (Dimonte 2004; Dimonte et al. 2005; Ramaprabhu & Dimonte 2005) that are thus used in current theory. Based on the above results, we finally obtain (see appendix D)

$$\beta \approx (1 + A)/4, \ \alpha \approx [2\phi/\ln(2\eta_0)]^2.$$

(2.3)
that all the experiments involve comparable initial perturbation. Following this logic, we may explain why the $\alpha$ et al. Kucherenko in miscible experiment are smaller than that of immiscible experiments (Youngs 1989; et al. than that of predicted with miscible code (George $\phi$ the $\phi <$ rate is also considerable, and one must set $\phi$ without involving extreme acceleration, the contribution of viscosity to linear growth rate is neglectable (Read 1984), and thus we have $\Gamma_{\text{ideal}} \rightarrow \Gamma_{\text{ideal}}$ and $\phi \approx 1$. However, for immiscible experiment conducted with $g \gg g_0$ (gravitation acceleration), the contribution of miscibility to the linear growth rate is considerable, and $\phi$ is smaller than 1. This may explain why the observed $\alpha$ in miscible experiment are smaller than that of immiscible experiments (Youngs 1989; Kucherenko et al. 1991; Linden & Redondo 1991; Dimonte & Schneider 2000), provided that all the experiments involve comparable initial perturbation. Following this logic, we may explain why the $\alpha$ predicted by the immiscible Front-Tracking code is larger than that of predicted with miscible code (George et al. 2002). Similarly, in simulations without involving extreme acceleration, the contribution of viscosity to linear growth rate is also considerable, and one must set $\phi < 1$. Here we give an example to determine the $\phi$ in large eddy simulations (Ramaprabhu et al. 2005; Youngs 2013), where either numerical or modeled viscosity is considerable. Following Dimonte (2004) viewpoint, the value of $\phi$ are dominantly determined by the monotonously increased $\phi(\lambda)$ from $\phi(\lambda_u) \approx 0.65$ (Ramaprabhu et al. 2005) to $\phi(\lambda_{\text{max}}) = 1$ (Dimonte 2004), where $\lambda_u$ and $\lambda_{\text{max}}$ denote, respectively, the most unstable and longest wavelength. Therefore, we set $\phi = (1 + 0.65)/2 = 0.825$ by considering the simplest linear average.

Taking account of the discussions above, our current theory was validated systematically by reproducing all the available experiments and simulations, but only the series of Rocket Rig (RR) (Read 1984; Youngs 1989) and Linear Electronic Motor (LEM) (Dimonte & Schneider 2000) experiments and the series of simulations using moderate (Ramaprabhu et al. 2005) and fine (Youngs 2013) grids are presented here. In figures

**Figure 1.** Comparison between theories and numerical simulations (Ramaprabhu et al. 2005; Youngs 2013). According to the definitions, $\eta_0 \approx \varepsilon$ and $\eta_0 = k h_{\text{rms}}/(2\pi)$, where $\varepsilon$ and $kh_{\text{rms}}$ are the original dimensionless initial perturbation amplitudes given by Youngs (2013) and Ramaprabhu et al. (2005), respectively.
Figure 2. Comparison between theories and experiments (Read 1984; Youngs 1989; Dimonte & Schneider 2000) for (a) scaled $\alpha_{\text{scl}} \equiv \alpha/\alpha_{\text{average}}$ and (b) $\beta$. In figure (a), regardless of the experiment or theory, $\alpha_{\text{average}}$ was set to 0.05 and 0.063 for LEM and RR data, respectively (Dimonte et al. 2005). In figure (1), we validate current theory with a series of simulations. Dimonte predicted that $\alpha(A, \eta_0)$ depends on both $A$ and $\eta_0$, so in the figure (1.a) we plotted the possible zone bounded by the two limiting curves with $A = 0$ and $A = 1$. However, as shown in this figure, scant data was located in this zone, and Youngs’ simulation also do not support Dimonte’s prediction that $\alpha$ decreases with increasing $A$. In figure (1.b), a significant deviation of Dimonte’s predictions from the simulation is observed. In fact, the simulations implied that $\alpha$ and $\beta$ depend, respectively, only on $\eta_0$ and $A$, which is consistent with our predictions.

In figure 2, we validate current theory with a series of experiments. For the LEM experiment, the current theory was exactly validated in the sense that equation (2.3) predicted the observed $\alpha_{\text{average}} = 0.05$ with the measured $\eta_{0\text{average}} = 4/(2\pi) \times 10^{-4}$, giving $\alpha_{\text{scl theory}} = 1$. For RR experiments, due to the lack of $\eta_0$, an averaged $\eta_0 = 1.73 \times 10^{-4}$ was estimated using the current theory to produce the measured $\alpha_{\text{average}} = 0.063$. We also used this $\eta_0$ to check Dimonte’s theory. As shown in the figure (2.a), the deviation of Dimonte’s prediction from experiments increases with increasing $\eta_0$ and $A$, while our predictions agree very well. In figure (2.b), the solid and dashed thick lines denote, respectively, our predictions using the approximate relation $\lambda_d = D$ and exact relation $\lambda_d = 1.07D$ (Goncharov 2002; Dimonte 2004) from Goncharov’s solution. As shown in this figure, the prediction with the exact relation agrees better.

From these comparisons, we can conclude that our predictions agree very well with both experiments and numerical simulations conducted at all density ratios $A \in (0, 1)$ and widely varying $\eta_0 \in (10^{-7}, 10^{-2})$ with different materials. Noting that we do not need to adjust parameters to fit experiments or simulations or to fit $\alpha$ or $\beta$, so our reproduction is self-consistent.
4. Conclusions

Now it is clear that general evolution of turbulent RT mixing via two mechanisms: merging adjacent waves (Alon et al. 1994, 1995) and amplifying the individual waves presented in initial perturbations (Dimonte 2004; Ramaprabhu et al. 2005). The former is well studied previously and obeys a universal law (Alon et al. 1994, 1995), and the latter is clarified in this paper and its dependence on density ratio, initial perturbation and material properties are formulated too. The theoretical results are verified for all published results (Read 1984; Youngs 1989; Dimonte & Schneider 2000; Ramaprabhu et al. 2005).

Our theory implies that most actual mixing depends on initial perturbations and evolves dominantly via the competition of individual wavepacket amplitudes, comprised of a narrow single-wave band. In addition, we also point out that in actual problems bubble-merge and bubble-competition exist simultaneously, so the two mechanisms should be considered simultaneously. However, this is beyond the scope of this paper, and will be addressed in another paper aiming to explain the observed transition (Ramaprabhu et al. 2005) at \( \eta_0 \approx 5 \times 10^{-5} \). We wish that current theory could promote the understanding of associated astronomical phenomena and the development of controllable thermonuclear fusion.

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Appendix A. Analytical solution for \( t \in [t_0, t_{Re}] \)

Using Goncharov’s (Goncharov 2002) solution for the 3D problem, Mikaelian (1998, 2003) obtained an explicit \( h(t) \) for a special initial condition until \( t_{Re} \). We noticed that Mikaelian’s 3D solution has the same form as Zhang’s 2D universal solution (Zhang & Guo 2016). However, Zhang obtained a universal solution without requiring a special initial condition. Instead he used a reasonable (Zhang 1998; Zhang & Guo 2016) assumption that the curvature of the bubble tip is steady. Therefore, Mikaelian’s special solution (Mikaelian 1998, 2003) can be viewed as a natural consequence of the universal solution, as noted by Zhang (1998) for \( A = 1 \). Based on this logic and the previous 3D analytical solution (Goncharov 2002; Abarzhi et al. 2003; Sohn 2003), we can write the 3D universal solution with the same form as the 2D universal solution as follows:

\[
\begin{align*}
\dot{h} &= b(\dot{h}_{\infty}^2 - \dot{h}^2)/\dot{h}_{\infty} \quad (A \ 1) \\
\dot{h} &= \dot{h}_{\infty} f(bt, \varepsilon) \\
\dot{h} &= h_0 + b^{-1}\dot{h}_{\infty} \ln[\cosh(bt) + \varepsilon \sinh(bt)] \\
\end{align*}
\]

where \( b \equiv \sqrt{2\beta_1 gA/(\lambda \Theta)} \), \( \dot{h}_{\infty} \equiv \Theta Ag\lambda/(2\beta_1) \), \( f(bt, \varepsilon) \equiv [\sinh(bt)+\varepsilon \cosh(bt)]/[\cosh(bt)+\varepsilon \sinh(bt)] \geq f(bt,0) = \tanh(bt) \), and \( \varepsilon \equiv \dot{h}_0/\dot{h}_{\infty} \) are very small in the general problem, \( \beta_1 \approx 3.832 \) is the first zero of the Bessel function \( J_1(x) \), \( \Theta(A) \) can be \( 2/(1+A) \), \( 1 \) and so on by following Goncharov’s (Goncharov 2002), Sohn’s (Sohn 2003) and others (Abarzhi et al. 2003) theories.

Appendix B. Simplified linear solution for \( t \in [t_{99}, t_{Re}] \)

Considering a special time \( t = t_{99} = C_1/b \) with \( C_1 = \tanh^{-1}(0.99) = \ln \sqrt{199} \), we obtained \( \dot{h}_{99}(t_{99}) \geq 0.99\dot{h}_{\infty} \) and \( h_{99}(t_{99}) = (\eta_0 + C_2\Theta)\lambda \) by substituting \( t_{99} \) into equation
and neglecting $\varepsilon$, with $C_2 \approx \ln(\sqrt{199}/2)/(2\beta_1)$. If we further ignore $\eta_0$, we obtain $h_{99} \approx C_2 \Theta \lambda$—finding (a). Furthermore, the above definition gives $t_{99} \propto \sqrt{\lambda \Theta/(gA)}$—finding (b). According to the definition, $h_{99}^\infty = (h_{99} - h_0)/t_{99} = C_2 \Theta \lambda / t_{99} = \dot{h}_{\infty}/C$ with $C = C_1/(2\beta_1 C_2)$—finding (c). For $t > t_{99}$, $h(t) \geq 0.99 \dot{h}_{\infty}$ and tends to steady $\dot{h}_{\infty}$, thus the evolution of $h(t)$ can be approximated (error $\leq 1\%$) as

$$h = \left[ \eta_0 + \Theta C_2 \lambda + \dot{h}_{\infty} (t - t_{99}) \right] = -\lambda/\chi + \dot{h}_{\infty} t, \quad t \in [t_{99}, t_{Re}],$$

(B1)

where $\chi \equiv -[\eta_0 + \Theta C_2 (1 - C)]^{-1}$ and the relation $\dot{h}_{\infty} = Ch_{99}^\infty = C C_2 \Theta \lambda / t_{99}$ is used.

**Appendix C. Determination of $h_{99}^\infty$**

According to $h_{99}^\infty \equiv C_2 \Theta \lambda / t_{99}$ derived in the appendix [B] the key is to determine $t_{99}$. Because $\Theta(A)$ differs in different theories except for $\Theta(A = 1) = 1$ [Dimonte et al. 2005] (see appendix [A]), we first determine $t_{99}$ for $A = 1$. For $A = 1$, $h_{99} \approx C_2 \lambda \approx \lambda/4$ (see the first equality of equation (B1)), which is near the end of the linear stage. In the linear stage, $h(t) = h_0 \cosh(\Gamma_{actual} t)$ [Dimonte 2004], where $\Gamma_{actual} = \phi \Gamma_{ideal}$ is the actual linear growth rate. Thus, $t_{99}$ can be solved from the requirement that $C_2 \lambda = h_0 \cosh(\Gamma_{actual} t_{99})$ yielding $t_{99}(A) = \ln F(\eta_0)/\sqrt{\lambda \Theta(A)/(Ag)}$, where $F(\eta_0) = C_2/\eta_0 + (C_2^2/\eta_0^2 - 1) \approx 2C_2/\eta_0$. Noting that for $A = 1$ the determined $t_{99}(A = 1) \propto \sqrt{\lambda \Theta/(Ag)}$ is self-consistent with the important finding (b), we thus assume that the above $t_{99}(A)$ applies for all $A$. Thus,

$$t_{99}(A) = CC_2 \Theta \lambda / t_{99} = Fr \sqrt{Ag} \lambda/(1 + A),$$

(C1)

where $Fr \equiv CC_2 \sqrt{2\pi C/(1 + A)} / \ln F(\eta_0)$ is the Froude number.

**Appendix D. Final results.**

Substituting $Fr$ and $\chi$ to the solution $\beta = \chi$ and $\alpha = \chi Fr^2/[4(1 + A)]$ gives

$$\begin{align*}
\beta &= \frac{1}{\Theta C_2 (C - 1) - \eta_0} \approx \frac{C_2 (C - 1) \Theta}{\eta_0} \approx \frac{\pi C_2 C^2 \Theta^2}{2(\pi C - 2)} / \ln^2 F(\eta_0), \\
\alpha &= \frac{1}{\Theta C_2 (C - 1) - \eta_0} \approx \frac{\pi C_2 C^2 \Theta^2}{2(\pi C - 2)} / \ln^2 F(\eta_0),
\end{align*}$$

(D1)

where the last relation is obtained by neglecting $\eta_0$ when it is a small quantity. When $A = 1$, $\Theta(1) = 1$, and the constant $C \approx 8.843$ is determined to meet the observed [Dimonte & Schneider 2000] quantity $\beta(A = 1) \approx 0.5$. Substituting $C \approx 8.843$ and $\Theta = 2/(1 + A)$ into equation (D1) immediately gives equation (2.3).

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