Non-Relativistic Holography
– A Group-Theoretical Perspective

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We give a review of some group-theoretical results related to non-relativistic holography. Our main playgrounds are the Schrödinger equation and the Schrödinger algebra. We first recall the interpretation of non-relativistic holography as equivalence between representations of the Schrödinger algebra describing bulk fields and boundary fields. One important result is the explicit construction of the boundary-to-bulk operators in the framework of representation theory, and that these operators and the bulk-to-boundary operators are intertwining operators. Further, we recall the fact that there is a hierarchy of equations on the boundary, invariant w.r.t. Schrödinger algebra. We also review the explicit construction of an analogous hierarchy of invariant equations in the bulk, and that the two hierarchies are equivalent via the bulk-to-boundary intertwining operators. The derivation of these hierarchies uses a mechanism introduced first for semi-simple Lie groups and adapted to the non-semisimple Schrödinger algebra. These requires development of the representation theory of the Schrödinger algebra which is reviewed in some detail. We also recall the q-deformation of the Schrödinger algebra. Finally, the realization of the Schrödinger algebra via difference operators is reviewed.

Keywords: Schrödinger equation, Schrödinger algebra, invariant operators

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1. Introduction

The role of nonrelativistic symmetries in theoretical physics was always important. Currently one of the most popular fields in theoretical physics - string theory, pretending to be a universal theory - encompasses together relativistic quantum field theory, classical gravity, and certainly, nonrelativistic quantum mechanics, in such a way that it is not even necessary to separate these components.

Since the cornerstone of quantum mechanics is the Schrödinger equation then it is not a surprise that the Schrödinger group - the group that is the maximal group of symmetry of the Schrödinger equation - is playing recently more and more a prominent role in theoretical physics. Especially important are the recent developments to non-relativistic conformal holography, cf., e.g., [1–41].

The latter is natural since originally the Schrödinger group, actually the Schrödinger algebra, was introduced by Niederer [42] and Hagen [43] as a nonrelativistic limit of the vector-field realization of the conformal algebra (see also [44]).

Recently, Son [3] proposed another method of identifying the Schrödinger algebra in d+1 space-time. Namely, Son started from anti de Sitter (AdS) space in d+3 dimensional space-time with metric that is invariant under the corresponding conformal algebra so(d+1,2) and then deformed the AdS metric to reduce the symmetry to the Schrödinger algebra.

In view of the relation of the conformal and Schrödinger algebra there arises the natural question. Is there a nonrelativistic analogue of the AdS/CFT correspon-
dence (CFT stands for Conformal Field Theory), in which the conformal symmetry is replaced by Schrödinger symmetry. Indeed, this is to be expected since the Schrödinger equation should play a role both in the bulk and on the boundary.

Thus, we review some nonrelativistic aspects of the AdS/CFT correspondence. Since the literature on the subject is immense we mention some aspects that will not be covered in this review:

— Applications of supersymmetry to non-relativistic holography, cf., e.g., [45–61].
— Other approaches to non-relativistic holography, cf., e.g., [62–80].
— Applications of non-relativistic holography to condensed matter systems, cf., e.g., [81–89].
— Approaches using the Galilean subgroup of the Schrödinger group, cf., e.g., [90–97].
— Applications to the problem of ageing using various subgroups of the Schrödinger group, cf., e.g., [98–105].

Returning to our main topic we first remind the two ingredients of the AdS/CFT correspondence [106–108]:

1. the holography principle, which is very old, and means the reconstruction of some objects in the bulk (that may be classical or quantum) from some objects on the boundary;
2. the reconstruction of quantum objects, like 2-point functions on the boundary, from appropriate actions on the bulk.

The realization of the first ingredient is reviewed in Section 3 in the simplest case of the (3+1)-dimensional bulk. It is shown that the holography principle is established using representation theory only, that is, no action is specified. (Such representation-theoretic intertwining operator realization of the AdS/CFT correspondence in the conformal case was given in [109].)

For the implementation of the first ingredient in the Schrödinger algebra context in [22] was used a method that is standard in the mathematical literature for the construction of discrete series representations of real semisimple Lie groups [110, 111], and which method was applied in the physics literature first in [112] in exactly an AdS/CFT setting, though that term was not used then.

The method utilizes the fact that in the bulk the Casimir operators are not fixed numerically. Thus, when a vector-field realization of the algebra in consideration is substituted in the Casimir it turns into a differential operator. In contrast, the boundary Casimir operators are fixed by the quantum numbers of the fields under consideration. Then the bulk/boundary correspondence forces an eigenvalue equation involving the Casimir differential operator. That eigenvalue equation is used to find the two-point Green function in the bulk which is then used to construct the boundary-to-bulk integral operator. This operator maps a boundary field to a bulk field similarly to what was done in the conformal context by Witten [108] (see also [109]). This is the first main result of [22].

The second main result of [22] is that this operator is an intertwining operator, namely, it intertwines the two representations of the Schrödinger algebra acting in
the bulk and on the boundary. This also helps us to establish that each bulk field has actually two bulk-to-boundary limits. The two boundary fields have conjugated conformal weights $\Delta, 3 - \Delta$, and they are related by a boundary two-point function.

In Section 4 we review the second ingredient of the non-relativistic version of the AdS/CFT correspondence. Namely, we review the connection of the results [3,4,17] with the formalism of [22].

In Section 5 is reviewed the Schrödinger equation as an invariant differential equation on the (1+1)-dimensional case. On the boundary this was done in [113] (extending the approach in the semi-simple group setting [114]), constructing actually an infinite hierarchy of invariant differential equations, the first member being the free heat/Schrödinger equation (see also [115]). In Section 5.4 is reviewed the extension of this construction to the bulk combining techniques from [22] and [113].

In Section 6 is reviewed the Schrödinger equation as an invariant differential equation in the general (n+1)-dimensional case following [116,117]. The general situation is very complicated and requires separate study of the cases: $n = 2N$ and $n = 2N + 1$. In Section 7 is reviewed separately and in more detail the (3+1)-dimensional case, since it is most important for physical applications. In Section 8 is reviewed the $q$-deformation of the Schrödinger algebra in the (1+1)-dimensional case, cf. [118]. In Section 9 are reviewed the difference analogues of the Schrödinger algebra in the (n+1)-dimensional case, cf. [119].

2. Preliminaries

2.1. Schrödinger algebra $\hat{S}(n)$

The Schrödinger algebra $S(n)$, $(n \geq 1)$, in (n+1)-dimensional space-time has $(n^2 + 3n + 6)/2$ generators with the following non-trivial commutation relations, cf., e.g., [120]:

\begin{align*}
[J_{ij}, J_{kl}] &= \delta_{ik}J_{j\ell} + \delta_{j\ell}J_{ik} - \delta_{il}J_{jk} - \delta_{jk}J_{il} \\
[J_{ij}, P_k] &= \delta_{ik}P_j - \delta_{jk}P_i \\
[J_{ij}, G_k] &= \delta_{ik}G_j - \delta_{jk}G_i \\
[P_t, G_i] &= P_i \\
[K, P_i] &= -G_i \\
[D, G_i] &= G_i \\
[D, P_i] &= -2P_t \\
[D, K] &= 2K \\
[P_t, K] &= D
\end{align*}

where $J_{ij} = -J_{ji}$, $i,j = 1,2,\ldots,n$, are the generators of the rotation subalgebra $so(n)$, $P_i$, $i = 1,2,\ldots,n$, are the generators of the abelian subalgebra $t(n)$ of space translations, $G_i$, $i = 1,2,\ldots,n$, are the generators of the abelian subalgebra
$G(n)$ of special Galilei transformations, $P_t$ is the generator of time translations, $D$ is the generator of dilatations (scale transformations), $K$ is the generator of Galilean conformal transformations.

Actually, most often we shall work with the central extension of the Schrödinger algebra $\hat{S}(n)$, obtained by adding the central element $M$ to $S(n)$ which enters the additional commutation relations:

$$[P_k, G_\ell] = \delta_{k\ell} M. \quad (2.2)$$

Note that the centre is one-dimensional. Of course, (2.2) gives also a central extension of the Galilei subalgebra $G(n)$, however, for $n = 1, 2$ this is not the full central extension $\hat{G}(n)$ of $G(n)$, since the centre is $(n+1)$-dimensional in these cases, cf. e.g., [120].

The centrally extended Schrödinger algebra for $n = 3$ was introduced in [42,43] (see also [44]) by deformation and extension of the standard vector field realization of the conformal algebra $C(3)$ in $3 + 1$-dimensional space-time. The resulting vector field realization for arbitrary $n$ is:

\[
\begin{align*}
P_j &= \partial_j & \quad (2.3a) \\
G_j &= t\partial_j + Mx_j & \quad (2.3b) \\
P_t &= \partial_t & \quad (2.3c) \\
D &= 2t\partial_t + x_j\partial_j + \Delta & \quad (2.3d) \\
K &= t^2\partial_t + tx_j\partial_j + \frac{M}{2}x_j^2 + t\Delta & \quad (2.3e) \\
J_{jk} &= x_k\partial_j - x_j\partial_k + \Sigma_{jk} & \quad (2.3f)
\end{align*}
\]

where $\partial_t \equiv \partial/\partial t$, $\partial_j \equiv \partial/\partial x_j$, summation over repeated indices is assumed, $\Delta$ is a number called the conformal weight (more about will be said below). We note that (2.3f) may be extended by the matrices of a finite-dimensional representation $\Sigma_{jk}$ of $so(n)$ which satisfies (2.1a) as follows:

$$J_{jk} = x_k\partial_j - x_j\partial_k + \Sigma_{jk} \quad (2.3f')$$

Now we list the important subalgebras of the Schrödinger algebra $S(n)$:

The generators $J_{ij}, P_i$ form the $((n+1)n/2)$-dimensional Euclidean subalgebra $E(n)$.

The generators $J_{ij}, P_i, D$ form the $((n+1)n/2+1)$-dimensional Euclidean Weyl subalgebra $W(n)$.

The subalgebras $\tilde{E}(n)$ ann $\tilde{W}(d)$ generated by $J_{ij}, G_i$ and by $J_{ij}, G_i, D$, resp., are isomorphic to $E(n), W(n)$, resp.

The generators $J_{ij}, P_i, G_i, P_t$ form the $(n(n+2)/2)$-dimensional Galilei subalgebra $G(n)$. The generators $J_{ij}, P_i, G_i, K$ form another $(n(n+2)/2)$-dimensional subalgebra $\hat{G}(n)$ which is isomorphic to the Galilei subalgebra.

The isomorphic pairs mentioned above are conjugated to each other in a sense explained below.
For the structure of $\hat{S}(n)$ it is also important to note that the generators $D,K,P_t$ form an $sl(2,\mathbb{R})$ subalgebra.

Obviously $S(n)$ is not semisimple and has the following Levi–Malcev decomposition (for $n \neq 2$):

$$S(n) = N(n) \oplus M(n)$$

where $M(n) = sl(2,\mathbb{R}) \oplus so(n)$, with $M(n)$ acting on $N(n)$, where the maximal solvable ideal $N(n)$ is abelian, while the semisimple subalgebra (the Levi factor) is $M(n)$.

For $n = 2$ the maximal solvable ideal $t(n) \oplus g(n) \oplus so(2)$ is not abelian, while the Levi factor $sl(2,\mathbb{R})$ is simple. Note, however, that many statements below will hold for arbitrary $n$, incl. $n = 2$, if we extend the definition of $N(n)$ and $M(n)$ to the case $n = 2$, (which is natural since $M(2)$ is then reductive, and it is well known that many semisimple structural and representation-theoretic results hold for the reductive case).

The commutation relations (2.1) are graded if we define:

$$\begin{align*}
\text{deg } D &= 0 \\
\text{deg } G_j &= 1 \\
\text{deg } K &= 2 \\
\text{deg } P_j &= -1 \\
\text{deg } P_t &= -2 \\
\text{deg } M &= 0 \\
\text{deg } J_{jk} &= 0
\end{align*}$$

As expected the corresponding grading operator is $D$.

For future reference we record also the following involutive antiautomorphism of the Schrödinger algebra:

$$\omega(P_t) = K, \quad \omega(P_j) = G_j, \quad \omega(J_{jk}) = -J_{jk} \quad \omega(D) = D, \quad \omega(M) = M.$$  

Note that this conjugation is transforming the isomorphic pairs of subalgebras introduced above, namely, we have:

$$\omega(\mathcal{E}) = \hat{\mathcal{E}}, \quad \omega(\mathcal{W}) = \hat{\mathcal{W}}, \quad \omega(\mathcal{G}) = \hat{\mathcal{G}}.$$  

We end this discussion of the general structure of $\hat{S}(n)$ with the question of invariant scalar products. Since the Schrödinger algebra is not semisimple its Cartan-Killing form is degenerate. More than that - the Schrödinger algebra does not have any nondegenerate ad-invariant symmetric bilinear form [113]. For the discussion of non-semisimple Lie algebras with nondegenerate ad-invariant symmetric bilinear form we refer to [121].
Matrix representation

It is useful to have a representation of $\hat{S}(n)$ by $(2n + 2) \times (2n + 2)$ matrices:

$$
D_{ab} = \sum_{\mu=1}^{n} (-\delta_{a,2\mu}\delta_{b,2\mu} + \delta_{a,2\mu+1}\delta_{b,2\mu+1}), \quad M_{ab} = -2\delta_{a1}\delta_{b,2n+2},
$$

$$(G_k)_{ab} = \delta_{a1}\delta_{b,2k} - \delta_{a,2n+3-2k}\delta_{b,2n+2},
$$

$$(P_k)_{ab} = -\sum_{\mu=1}^{n} \delta_{a,2\mu}\delta_{b,2n+3-2\mu}, \quad (K)_{ab} = \sum_{\mu=1}^{n} \delta_{a,2\mu+1}\delta_{b,2n+2-2\mu},
$$

$$(J_{kl})_{ab} = \delta_{a,2l}\delta_{b,2k} - \delta_{a,2k}\delta_{b,2l} - \delta_{a,2n+3-2k}\delta_{b,2n+3-2l} + \delta_{a,2n+3-2l}\delta_{b,2n+3-2k},$$

where $X_{ab}$ denotes the $(a,b)$ element of matrix $X$. In this representation $D$ is diagonal: $D = diag(0, -1, 1, -1, 1, \cdots, -1, 1, 0)$, while $P_t$ and $K$ are minor-diagonal.

For $n = 3$ we give the above explicitly. Positions of non-zero entries are indicated by the name of the generators as follows:

$$
\begin{bmatrix}
0 & G_1 & P_3 & G_2 & P_2 & G_3 & P_1 & M \\
D & J_{12} & J_{13} & P_t & P_1 & & & \\
J_{12} & D & J_{23} & K & J_{13} & G_3 & & \\
J_{23} & K & D & J_{12} & G_2 & & & \\
P_t & J_{23} & D & P_3 & & & & \\
K & J_{13} & J_{12} & D & G_1 & & & \\
0 & & & & & & & 0
\end{bmatrix}
$$

The first column and eighth row are empty.

2.2. Triangular decomposition of $\hat{S}(n)$

The grading (2.5) can be viewed as extension of the triangular decomposition of the algebra $sl(2, \mathbb{R}) = sl(2, \mathbb{R})^+ \oplus sl(2, \mathbb{R})^0 \oplus sl(2, \mathbb{R})^-$, where $sl(2, \mathbb{R})^+$ is spanned by $K$, the Cartan subalgebra $sl(2, \mathbb{R})^0$ is spanned by $D$, and $sl(2, \mathbb{R})^-$ is spanned by $P_t$. Taking into account also the triangular decomposition: $so(n) = so(n)^+ \oplus so(n)^0 \oplus so(n)^-$, (more precisely of its complexification $so(n, \mathbb{C})$), we can introduce the following triangular decomposition:

$$
\hat{S}(n) = \hat{S}(n)^+ \oplus \hat{S}(n)^0 \oplus \hat{S}(n)^-
$$

$$
\hat{S}(n)^+ = g(n) \oplus sl(2, \mathbb{R})^+ \oplus so(n)^+
$$

$$
\hat{S}(n)^0 = sl(2, \mathbb{R})^0 \oplus so(n)^0 \oplus \text{lin.span } M
$$

$$
\hat{S}(n)^- = t(n) \oplus sl(2, \mathbb{R})^- \oplus so(n)^-
$$

(Clearly, for $n = 1$ the $so(n)$ factors are missing, while for $n = 2$ only the Cartan subalgebra $so(n)^0$ survives.)
3. Non-relativistic holography

3.1. Choice of bulk and boundary

In the beginning of this section we review Son [3]. To realize the Schrödinger symmetry in \((n + 1)\) dimensions geometrically, Son takes the AdS metric, which is invariant under the conformal group \(O(n + 2, 2)\) in \((n + 2)\) dimensions, and then deforms it to reduce the symmetry down to the Schrödinger group. The AdS space, in Poincaré coordinates, is

\[
 ds^2 = \frac{\eta_{\mu\nu} dx^\mu dx^\nu + dz^2}{z^2}, \quad \mu, \nu = 0, 1, \ldots, n + 1, \quad \eta_{\mu\nu} = \text{diag}(-1, 1, \ldots, 1) .
\]  

(3.1)

The generators of the conformal group correspond to the following infinitesimal coordinates transformations that leave the metric unchanged,

\[
 P^\mu : \quad x^\mu \rightarrow x^\mu + a^\mu, \\
 D : \quad x^\mu \rightarrow (1 - a) x^\mu, \quad z \rightarrow (1 - a) z, \\
 K^\mu : \quad x^\mu \rightarrow x^\mu + a^\mu (z^2 + x \cdot x) - 2 x^\mu (a \cdot x)
\]  

(3.2)

(3.3)

(here \(x \cdot x \equiv \eta_{\mu\nu} x^\mu x^\nu\)).

Then Son deforms the above metric so to reduce the symmetry to the Schrödinger group. The resulting metric is [3]:

\[
 ds^2 = -\frac{2(dx^+)^2}{z^4} + \frac{-2dx^+ dx^- + dx^i dx^i + dz^2}{z^2}, \quad i = 1, \ldots, n .
\]  

(3.3)

It is straightforward to verify that the metric (3.3) exhibits a full Schrödinger symmetry. Indeed, the generators of the Schrödinger algebra correspond to the following isometries of the metric:

\[
 P_i : \quad x_i \rightarrow x_i + a_i, \quad H : \quad x_+ \rightarrow x_+ + a, \quad M : \quad x_- \rightarrow x_- + a, \\
 G_i : \quad x_i \rightarrow x_i - a_i x_+, \quad x_- \rightarrow x_- - a_i x_i, \\
 D : \quad x_i \rightarrow (1 - a)x_i, \quad z \rightarrow (1 - a)z, \quad x_+ \rightarrow (1 - a) x_+, \quad x_- \rightarrow x_-, \\
 K : \quad z \rightarrow (1 - a x_+) z, \quad x_i \rightarrow (1 - a x_+) x_i, \quad x_+ \rightarrow (1 - a x_+) x_+, \quad x_- \rightarrow x_- - \frac{a}{2}(x_i x_i + z^2),
\]  

(3.4)

while the generators \(J_{jk}\) of \(so(n)\) rotate the coordinates \(x_j\) as before. We require that the Schrödinger algebra is an isometry of the above metric. We also need to replace the central element \(M\) by the derivative of the variable \(x_-\) which is chosen so that \(\frac{\partial}{\partial x_-}\) continues to be central. Note the variable \(x_-\) does not scale w.r.t. \(D\). Such variables are called ultralocal.

Thus, a vector-field realization of the Schrödinger algebra \(\hat{S}(n)\) in the \((n + 3)\)-
dimensional bulk space \((t, x_i, x_-, z)\) is:

\[
\begin{align*}
    P_j &= \partial_j, \\
    G_j &= t \partial_j + mx_j, \\
    P_t &= \partial_t, \\
    D &= 2t \partial_t + x_j \partial_j + z \frac{\partial}{\partial z}, \\
    K &= t^2 \partial_t + tx_j \partial_j + tz \frac{\partial}{\partial z} + \frac{1}{2}(x_j^2 + z^2) M, \\
    J_{jk} &= x_k \partial_j - x_j \partial_k, \\
    M &= \frac{\partial}{\partial x_-}
\end{align*}
\]

We would like to treat the realization (2.3) as vector-field realization on the boundary of the bulk space \((t, x_i, x_-, z)\). Obviously, the variable \(z\) is the variable distinguishing the bulk, namely, the boundary is obtained when \(z = 0\). The exact map will be displayed below but heuristically, passing from (3.5) to (2.3) one first replaces \(z \frac{\partial}{\partial z}\) with \(\Delta\) and then sets \(z = 0\).

### 3.2. One-dimensional case

Here and below we review the paper [22]. Now we restrict to the 1+1 dimensional case, \(n = 1\). In this case the centrally extended Schrödinger algebra has six generators:

- time translation: \(P_t\)
- space translation: \(P_x\)
- Galilei boost: \(G\)
- dilatation: \(D\)
- conformal transformation: \(K\)
- mass: \(M\)

with the following non-vanishing commutation relations:

\[
\begin{align*}
    [P_t, D] &= 2P_t, \quad [D, K] = 2K, \quad [P_t, K] = D, \\
    [P_t, G] &= P_x, \quad [P_x, D] = P_x, \\
    [P_x, K] &= G, \quad [D, G] = G, \\
    [P_x, G] &= M.
\end{align*}
\]

Further we need also the Casimir operator. It turns out that the lowest order nontrivial Casimir operator is the 4-th order one [122]:

\[
\tilde{C}_4 = (2MD - \{P_x, G\})^2 - 2\{2MK - G^2, 2MP_t - P_x^2\}
\]

In fact, there are many cancellations, and the central generator \(M\) is a common linear multiple. (This is seen immediately by setting \(M = 0\), then \(\tilde{C}_4 \to 0\).)
The metric (3.3) of the four-dimensional bulk space \((t, x, x_-, z)\) now reads:

\[
ds^2 = -\frac{2(dt)^2}{z^4} + \frac{-2dtdx_- + (dx)^2 + dz^2}{z^2}
\]

(3.8)

Accordingly, the vector-field realization of the Schrödinger algebra is given by:

\[
\begin{align*}
P_t &= \frac{\partial}{\partial t}, \quad P_x = \frac{\partial}{\partial x}, \quad M = \frac{\partial}{\partial x_-}, \\
G &= t \frac{\partial}{\partial x} + xM, \\
D &= x \frac{\partial}{\partial x} + z \frac{\partial}{\partial z} + 2t \frac{\partial}{\partial t}, \\
K &= t \left( x \frac{\partial}{\partial x} + z \frac{\partial}{\partial z} + t \frac{\partial}{\partial t} \right) + \frac{1}{2} (x^2 + z^2)M.
\end{align*}
\]

(3.9)

and it generates an isometry of (3.8). This vector-field realization of the Schrödinger algebra acts on the bulk fields \(\phi(t, x, x_-, z)\).

In this realization the Casimir becomes:

\[
\tilde{C}_4 = M^2 C_4,
\]

\[
C_4 = \hat{Z}^2 - 4\hat{Z} - 4z^2 \hat{S} = 4z^2 \frac{\partial^2}{\partial z^2} - 8z \frac{\partial}{\partial z} + 5 - 4z^2 \hat{S},
\]

(3.10)

\[
\hat{S} = 2 \frac{\partial}{\partial t} \frac{\partial}{\partial x_-} - \frac{\partial^2}{\partial x^2},
\]

(3.11)

\[
\hat{Z} = 2z \frac{\partial}{\partial z} - 1
\]

Note that (3.11) is the pro-Schrödinger operator.

The vector-field realization (2.3) of the Schrödinger algebra on the boundary becomes:

\[
\begin{align*}
P_t &= \frac{\partial}{\partial t}, \quad P_x = \frac{\partial}{\partial x}, \quad M = \frac{\partial}{\partial x_-}, \\
G &= t \frac{\partial}{\partial x} + xM, \\
D &= x \frac{\partial}{\partial x} + \Delta + 2t \frac{\partial}{\partial t}, \\
K &= t \left( x \frac{\partial}{\partial x} + \Delta + t \frac{\partial}{\partial t} \right) + \frac{1}{2} x^2 M
\end{align*}
\]

(3.12)

where \(\Delta\) is the conformal weight.

\textsuperscript{a}This metric was given first in [123, 124], prior to [3], albeit without relation to Schrödinger symmetry.
Thus, the vector-field realization of the Schrödinger algebra (3.12) acts on the boundary field $\phi(t, x, x_-)$ with fixed conformal weight $\Delta$.

In this realization the Casimir becomes:

$$\tilde{C}_4^0 = M^2 C_4^0, \quad C_4^0 = (2\Delta - 1)(2\Delta - 5)$$

As expected $C_4^0$ is a constant which has the same value if we replace $\Delta$ by $3 - \Delta$:

$$C_4^0(\Delta) = C_4^0(3 - \Delta)$$

This already means that the two boundary fields with conformal weights $\Delta$ and $3 - \Delta$ are related, or in mathematical language, that the corresponding representations are (partially) equivalent.

### 3.3. Boundary-to-bulk correspondence

As we explained in the Introduction we concentrate on one aspect of AdS/CFT [107, 108], namely, the holography principle, or boundary-to-bulk correspondence, which means to have an operator which maps a boundary field $\varphi$ to a bulk field $\phi$, cf. [108], also [109].

Mathematically, this means the following. We treat both the boundary fields and the bulk fields as representation spaces of the Schrödinger algebra. The action of the Schrödinger algebra in the boundary, resp. bulk, representation spaces is given by formulae (3.12), resp. by formulae (3.9). The boundary-to-bulk operator maps the boundary representation space to the bulk representation space.

The fields on the boundary are fixed by the value of the conformal weight $\Delta$, correspondingly, as we saw, the Casimir has the eigenvalue determined by $\Delta$:

$$C_4^0 \varphi(t, x, x_-) = \lambda \varphi(t, x, x_-),$$

$$\lambda = (2\Delta - 1)(2\Delta - 5)$$

Thus, the first requirement for the corresponding field on the bulk $\phi(t, x, x_-)$ is to satisfy the same eigenvalue equation, namely, we require:

$$C_4 \phi(t, x, x_-) = \lambda \phi(t, x, x_-),$$

$$\lambda = (2\Delta - 1)(2\Delta - 5)$$

where $C_4$ is the differential operator given in (3.10). Thus, in the bulk the eigenvalue condition is a differential equation.

The other condition is the behaviour of the bulk field when we approach the boundary:

$$\phi(t, x, x_-, z) \rightarrow z^\alpha \varphi(t, x, x_-),$$

$$\alpha = \Delta, 3 - \Delta$$

Let us denote by $\hat{C}^\alpha$ the space of bulk functions $\phi(t, x, x_-, z)$ satisfying (3.16) and (3.17).
To find the boundary-to-bulk operator we first find the two-point Green function in the bulk solving the differential equation:

\[(C_4 - \lambda) G(\chi, z; \chi', z') = z'^4 \delta^3(\chi - \chi') \delta(z - z') \] (3.18)

where \(\chi = (t, x_-, x)\).

It is important to use an invariant variable which here is:

\[u = \frac{4zz'}{(x - x')^2 - 2(t - t')(x_- - x'_-) + (z + z')^2} \] (3.19)

The normalization is chosen so that for coinciding points we have \(u = 1\).

In terms of \(u\) the Casimir becomes:

\[C_4 = 4u^2(1 - u) \frac{d^2}{du^2} - 8u \frac{d}{du} + 5 \] (3.20)

The eigenvalue equation can be reduced to the hypergeometric equation by the substitution:

\[G(\chi, z; \chi', z') = G(u) = u^\alpha \hat{G}(u) \] (3.21)

and the two solutions are:

\[\hat{G}(u) = \alpha, (\alpha - 1; 2(\alpha - 1); u), \quad \alpha = \Delta, 3 - \Delta \] (3.22)

where \(F = \alpha F_1\) is the standard hypergeometric function.

As expected at \(u = 1\) both solutions are singular: by [125], they can be recast into:

\[G(u) = \frac{u^\Delta}{1 - u} F(\Delta - 2, \Delta - 1; 2(\Delta - 1); u), \quad \alpha = \Delta, \]

\[G(u) = \frac{u^{3 - \Delta}}{1 - u} F(1 - \Delta, 2 - \Delta; 2(2 - \Delta); u), \quad \alpha = 3 - \Delta . \]

Now the boundary-to-bulk operator is obtained from the two-point bulk Green function by bringing one of the points to the boundary, however, one has to take into account all info from the field on the boundary.

More precisely, we express the function in the bulk with boundary behaviour (3.17) through the function on the boundary by the formula:

\[\phi(\chi, z) = \int d^3\chi' S_\alpha(\chi - \chi', z) \varphi(\chi'), \] (3.23)

where \(d^3\chi' = dx'_+ dx'_- dx'\) and \(S_\alpha(\chi - \chi', z)\) is defined by

\[S_\alpha(\chi - \chi', z) = \lim_{z' \to 0} z'^{-\alpha} G(u) = \left[ \frac{4z}{(x - x')^2 - 2(t - t')(x_- - x'_-) + z^2} \right]^{\alpha} \] (3.24)

An important ingredient of this approach is that the bulk-to-boundary and boundary-to-bulk operators are actually intertwining operators. To see this we need some more notation.
Let us denote by $L_\alpha$ the bulk-to-boundary operator:

$$(L_\alpha \phi)(\chi) \doteq \lim_{z \to 0} z^{-\alpha} \phi(\chi, z), \quad (3.25)$$

where $\alpha = \Delta, 3 - \Delta$ consistently with (3.17). The intertwining property is:

$$L_\alpha \circ \hat{X} = \tilde{X}_\alpha \circ L_\alpha, \quad X \in \mathcal{S}(1), \quad (3.26)$$

where $\tilde{X}_\alpha$ denotes the action of the generator $X$ on the boundary (3.12) (with $\Delta$ replaced by $\alpha$ from (3.17)), $\hat{X}$ denotes the action of the generator $X$ in the bulk (3.9).

Let us denote by $\tilde{L}_\alpha$ the boundary-to-bulk operator in (3.23):

$$\varphi(\chi, z) = (\tilde{L}_\alpha \varphi)(\chi, z) \doteq \int d^3\chi' S_\alpha(\chi - \chi', z) \varphi(\chi') \quad (3.27)$$

The intertwining property now is:

$$\tilde{L}_\alpha \circ \tilde{X}_{3-\alpha} = \hat{X} \circ \tilde{L}_\alpha, \quad X \in \mathcal{S}(1). \quad (3.28)$$

Next we check consistency of the bulk-to-boundary and boundary-to-bulk operators, namely, their consecutive application in both orders should be the identity map:

$$L_{3-\alpha} \circ \tilde{L}_\alpha = 1_{\text{boundary}}, \quad (3.29)$$

$$\tilde{L}_\alpha \circ L_{3-\alpha} = 1_{\text{bulk}}. \quad (3.30)$$

Checking (3.29) in [22] was obtained:

$$(L_{3-\alpha} \circ \tilde{L}_\alpha \varphi)(\chi) = 2^{3\alpha} \pi^{3/2} \frac{\Gamma(\alpha - \frac{3}{2})}{\Gamma(\alpha)} \varphi(\chi) \quad (3.31)$$

Thus, in order to obtain (3.29) exactly, we have to normalize, e.g., $\tilde{L}_\alpha$.

We note the excluded values $\alpha - 3/2 \notin \mathbb{Z}_-$ for which the two intertwining operators are not inverse to each other. This means that at least one of the representations is reducible. This reducibility was established [113] for the associated Verma modules with lowest weight determined by the conformal weight $\Delta$ and is reviewed below.

Checking (3.30) is now straightforward, but also fails for the excluded values.

Note that checking (3.29) we used (3.25) for $\alpha \to 3-\alpha$, i.e., we used one possible limit of the bulk field (3.23). But it is important to note that this bulk field has also the boundary as given in (3.25). Namely, we can consider the field:

$$\varphi_0(\chi) \doteq (L_\alpha \phi)(\chi) = \lim_{z \to 0} z^{-\alpha} \phi(\chi, z), \quad (3.32)$$

where $\phi(\chi, z)$ is given by (3.23). We obtain immediately:

$$\varphi_0(\chi) = \int d^3\chi' G_\alpha(\chi - \chi') \varphi(\chi'), \quad (3.33)$$
where
\[ G_\alpha(\chi) = \left[ \frac{4}{x^2 - 2tx} \right]^\alpha. \] (3.34)

If we denote by \( G_\alpha \) the operator in (3.33) then we have the intertwining property:
\[ \tilde{X}_\alpha \circ G_\alpha = G_\alpha \circ \tilde{X}_{3-\alpha}. \] (3.35)

Thus, the two boundary fields corresponding to the two limits of the bulk field are equivalent (partially equivalent for \( \alpha \in \mathbb{Z} + 3/2 \)). The intertwining kernel has the properties of the conformal two-point function.

Thus, for generic \( \Delta \) the bulk fields obtained for the two values of \( \alpha \) are not only equivalent - they coincide, since both have the two fields \( \phi_0 \) and \( \phi \) as boundaries.

Remark: For the relativistic AdS/CFT correspondence the above analysis relating the two fields in (3.33) was given in [109]. An alternative treatment relating these two fields via the Legendre transform was given later in [126].

As in the relativistic case there is a range of dimensions when both fields \( \Delta, 3-\Delta \) are physical:
\[ \Delta_0^0 \equiv 1/2 < \Delta < 5/2 \equiv \Delta_+^0. \] (3.36)

At these bounds the Casimir eigenvalue \( \lambda = (2\Delta - 1)(2\Delta - 5) \) becomes zero.

4. Non-relativistic reduction

In this Section we review the connection of [3, 4, 17] with the formalism of [22] reviewed in the previous Section. For this we consider the action for a scalar field in the background (3.8):
\[ I(\phi) = -\int d^3\chi dz \sqrt{-g} \left( \partial^\mu \phi^* \partial_\mu \phi + m_0^2|\phi|^2 \right). \] (4.1)

By integrating by parts, and taking into account a non-trivial contribution from the boundary, one can see that \( I(\phi) \) has the following expression:
\[ I(\phi) = \int d^3\chi dz \sqrt{-g} \phi^*(\partial^\mu \partial_\mu \phi - m_0^2\phi) - \lim_{z \to 0} \int d^3\chi \frac{1}{z^3} \phi^* z \partial z \phi. \] (4.2)

The second term is evaluated using (3.23). For \( z \to 0 \), one has
\[ z \partial z \phi \sim \alpha(4z)^\alpha \int d^3\chi' \frac{\varphi(\chi')}{[(x-x')^2 - 2(x_+ - x_+')(x_- - x_-')]^\alpha} + O(z^{\alpha+2}). \] (4.3)

It follows that
\[ \lim_{z \to 0} \int d^3\chi \frac{1}{z^3} \phi^* z \partial z \phi = \lim_{z \to 0} \alpha \int d^3\chi d^3\chi' z^{\alpha-3} \phi^*(\chi, z) \left( \frac{4}{A} \right)^\alpha \varphi(\chi'). \] (4.4)
The equation of motion being read off from the first term in (4.2) can be expressed in terms of the differential operator (3.10):

\[
(\partial^\mu \partial_\mu - m_0^2) \phi = \left( \frac{C_4}{4} - \frac{5}{4} + 2 \partial_\xi^2 - m_0^2 \right) \phi = 0.
\]  

(4.5)

The fields in the bulk (3.23) do not solve the equation of motion. Now we set an Ansatz for the fields on the boundary:

\[ \varphi(\chi) = e^{Mx - \phi(x_+, x)} \]

Further we compactify the \( x_- \) coordinate:

\[ x_- + a \sim x_- \] as in, e.g., \([5, 66]\). This leads to a separation of variables for the fields in the bulk in the following way:

\[
\varphi(\chi, z) = e^{Mx - \int dx_+ dx' \int_0^a d\xi \left( 4z \left( \frac{x - x'}{2(x_+ - x'_+)} \xi + z^2 \right) \right)^\alpha e^{-M\xi} \varphi(x_+, x').
\]

Thus we are allowed to make the identification

\[
\partial_{x_-} \equiv M
\]

both in the bulk and on the boundary \([3, 4]\). We remark that under this identification the operator (3.11) becomes the Schrödinger operator. Integration over \( \xi \) turns out to be incomplete

\[
\varphi(\chi, z) = e^{Mx - \phi(x_+, x, z)},
\]

(4.6)

\[
\varphi(x_+, x, z) = (-2z)^\alpha M^{\alpha-1} \gamma(1-\alpha, Ma)
\]

\[
\times \int \frac{dx_+ dx'}{(x_+ - x'_+)^\alpha} \exp \left( -\frac{(x - x')^2 + z^2}{2(x_+ - x'_+)} M \right) \varphi(x_+, x').
\]

(4.7)

This formula was obtained first in \([17]\). The equation of motion (4.5) now reads

\[
\left( \frac{\lambda - 5}{4} - m^2 \right) \phi(x_+, x, z) = 0,
\]

(4.8)

where \( m^2 = m_0^2 - 2M^2 \). Requiring \( \phi(x_+, x, z) \) to be a solution to the equation of motion makes the connection between the conformal weight and mass:

\[
\Delta_\pm = \frac{1}{2} (3 \pm \sqrt{9 + 4m^2}).
\]

(4.9)

This result is identical to the relativistic AdS/CFT correspondence \([107, 108]\). The action (4.2) evaluated for this classical solutions has the following form (\( \alpha = \Delta_\pm \)):

\[
I(\phi) = -(-2)^\alpha M^{\alpha-1} \gamma(1-\alpha, Ma)
\]

\[
\times \int dx dx_+ dx'dx'_+ \left( \frac{(x - x')^2}{2(x_+ - x'_+)} M \right) \varphi(x_+, x)^* \varphi(x'_+, x').
\]

(4.10)

The two-point function of the operator dual to \( \phi \) computed from (4.10) coincides with the result of \([3, 4, 127–129]\). We remark that the Ansatz for the boundary fields \( \varphi(\chi) = \exp(Mx_- - \omega x_+ + ikr) \) used in \([3, 4]\) is not necessary to derive (4.10).

One can also recover the solutions in \([3, 4]\) rather simply in the group theoretical context of \([22]\). We use again the eigenvalue problem of the differential operator (3.10):

\[
C_4 \phi(x_+, x, z) = \lambda \phi(x_+, x, z).
\]

(4.11)
but make separation of variables $\phi(x_+, x, z) = \psi(x_+, x)f(z)$. Then (4.11) is written as follows:

$$\frac{1}{f(z)} \left( \frac{\partial^2}{\partial z^2} - \frac{2}{z} \frac{\partial}{\partial z} + \frac{5 - \lambda}{4z^2} \right) f(z) = \frac{1}{\psi(x_+, x)} \hat{S}\psi(x_+, x) = p^2 \text{ (const)}$$

Schrödinger part is easily solved: $\psi(x_+, x) = \exp(-\omega x_+ + ikx)$ which gives

$$p^2 = -2M\omega + k^2.$$  \hfill (4.12)

The equation for $f(z)$ now becomes

$$\frac{\partial^2}{\partial z^2} f(z) - \frac{2}{z} \frac{\partial}{\partial z} f(z) + \left( 2M\omega - k^2 - \frac{m^2}{z^2} \right) f(z) = 0.$$  \hfill (4.13)

This is the equation given in [3, 4] for $d = 1$. Thus solutions to equation (4.13) are given by modified Bessel functions: $f_\nu(z) = z^{3/2}K_{\nu}(pz)$ where $\nu$ is related to the effective mass $m$ [3, 4]. In the group theoretic approach one can see its relation to the eigenvalue of $C_4 : \nu = \sqrt{\lambda + 4}/2$ [22].

We close this section by giving the expression of (4.10) for the alternate boundary field $\varphi_0$. To this end, we again use the Ansatz $\varphi(\chi) = e^{M_x-}\varphi(x_+, x)$ for (3.33). Then performing the integration over $x_-'$ it is immediate to see that:

$$\varphi_0(x_+, x) \sim e^{M_{x-}} \int \frac{dx'dx'}{(x_+ - x_+')^\alpha} \exp \left( -\frac{(x - x')^2}{2(x_+ - x_+')}M \right) \varphi(x_+', x').$$  \hfill (4.14)

One can invert this relation since $G_{3-\alpha} \circ G_\alpha = 1_{\text{boundary}}$. Substitution of (4.14) and its inverse to (4.10) gives the following expression:

$$I(\phi) \sim \int \frac{dx dx' dx'' dx'}{(x_+ - x')^\alpha} \exp \left( -\frac{(x - x')^2}{2(x_+ - x_+')}M \right) \varphi_0(x_+, x)^* \varphi_0(x_+', x').$$  \hfill (4.15)

5. Non-relativistic invariant differential equations for $n = 1$

5.1. Canonical procedure

In this subsection, we briefly outline the method of [114] that shall be used subsequently. Let $G$ be a complex semisimple Lie group and $\mathfrak{g}$ its Lie algebra. Let $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_-$ be the standard triangular decomposition. We consider representations of $\mathfrak{g}$ whose representation spaces $C_\Lambda$ are $\mathbb{C}^\infty$ functions $\mathcal{F}$ on $G$ with the property called right covariance

$$\mathcal{F}(gxg_+) = e^{\Lambda(H)}\mathcal{F}(g), \quad g \in G, \ x = e^H \in G_0, \ H \in \mathfrak{g}_0, \ g_- \in G_-,$$  \hfill (5.1)

where $\Lambda \in \mathfrak{g}_0^*$, $\Lambda(H) \in \mathbb{Z}$, $G_0 \equiv \exp \mathfrak{g}_0$, $G_\pm \equiv \exp \mathfrak{g}_\pm$. Thus the functions of $C_\Lambda$ are actually functions on the coset $G/B$, $B \equiv G_0G_-$ being a Borel subgroup, or equivalently [114], on $G_+$ which is dense in $G/B$. The restricted representation spaces of functions $f$ on $G_+$, such that $f = \mathcal{F}|_{G_+}$, we denote by $C_\Lambda$. We introduce the left $\pi_L(X)$ and right $\pi_R(X)$ actions of $\mathfrak{g}$ on $C_\Lambda$ by the standard formulae

$$\pi_L(X)\mathcal{F}(g) \equiv \frac{d}{dt} \mathcal{F}(e^{-\tau X}g) \bigg|_{\tau = 0}, \quad \pi_R(X)\mathcal{F}(g) \equiv \frac{d}{dt} \mathcal{F}(ge^{\tau X}) \bigg|_{\tau = 0}.$$  \hfill (5.2)
where $X \in \mathfrak{g}$. The left action gives representations of $\mathfrak{g}$ by differential operators. It may be considered for arbitrary $\Lambda(H) \in \mathbb{C}$ and it may be restricted to $C_{\Lambda}$. On the other hand, the space $C_{\Lambda}$ has a lowest weight structure with respect to the right action, since one can show, using the right covariance, that
\[
\pi_R(H)\mathcal{F}(g) = \Lambda(H)\mathcal{F}(g), \quad \pi_R(X)\mathcal{F}(g) = 0, \quad (5.3)
\]
where $H \in \mathfrak{g}_0$ and $X \in \mathfrak{g}_-$. Thus we are prompted to employ properties of the Verma module $V^\Lambda$ with the lowest weight $\Lambda$, such that $V^\Lambda \simeq U(\mathfrak{g}_+)v_0$, where $v_0$ is the lowest weight vector, $U(\mathfrak{g}_+)$ is the universal enveloping algebra of $\mathfrak{g}_+$. When a Verma module is reducible, it has (at least one) singular vector $v_s$ such that:
\[
Hv_s = \Lambda'(H), \quad \Lambda' \neq \Lambda, \quad H \in \mathfrak{g}_0, \quad Xv_s = 0, \quad X \in \mathfrak{g}_-.
\]
It has the general structure $v_s = P(\mathfrak{g}_+)v_0$, where $P(\mathfrak{g}_+)$ is a polynomial of the generators of $\mathfrak{g}_+$. Then it is shown that the same polynomial $P(\mathfrak{g}_+)$ gives rise to a $\mathfrak{g}$-invariant differential equation given explicitly by:
\[
P(\pi_R(\mathfrak{g}_+))\psi = 0, \quad \psi = \mathcal{F}, \quad f.
\]

Below the procedure of [114] shall be used in our non-semisimple Schrödinger setting.

### 5.2. Verma modules and singular vectors

In this subsection we follow [113], see also [115–117]. We consider lowest weight modules (LWM) over $\hat{S}(n)$, in particular, Verma modules, which are standard for semisimple Lie algebras (SSLA) and their $q$-deformations. For more information on representations of the Schrödinger algebra we refer to [141–161].

A lowest weight module (LWM) $M^\Lambda$ over $\hat{S}(n)$ is given by the lowest weight $\Lambda \in \mathcal{H}^*$ ($\mathcal{H}^*$ is the dual of $\hat{S}(n)^0$) and a lowest weight vector $v_0$ such that $Xv_0 = 0$ if $X \in \hat{S}(n)^-$, $Hv_0 = \Lambda(H)v_0$ if $H \in \hat{S}(n)^0$. In particular, we use the Verma modules $V^\Lambda$ over $\hat{S}(n)$ which are the lowest weight modules induced from a one-dimensional representation of the analogue of a Borel subalgebra $\mathcal{B}^0 \oplus \hat{S}(n)^-$ spanned by $v_0$. The Verma module is given explicitly by $V^\Lambda \cong U(\hat{S}(n)^+) \otimes v_0$, where $U(\hat{S}(n)^+)$ is the universal enveloping algebra of $\hat{S}(n)^+$. Further, for brevity we shall omit the sign $\otimes$, i.e., we shall write instead of $\otimes v_0$ just $v_0$.

Now we restrict to the case $n = 1$. Then the Cartan subalgebra $\hat{S}(1)^0$ is generated by $D, M$ and we can write all above mentioned properties as:
\[
Dv_0 = \Delta v_0, \quad Mv_0 = Mv_0, \quad (5.4)
\]
\[
P_xv_0 = 0, \quad P_tv_0 = 0
\]
where $\Delta \in \mathbb{R}$ is the (conformal) weight.

The Borel subalgebra $\mathcal{B}$ is generated by the nonpositively graded generators $D, M, P_x, P_t$. Now we denote the Verma module as $V^\Lambda$ since $M$ is constant.

Clearly, $U(\hat{S}^+)$ is abelian and has basis elements $p_{k,\ell} = G^kK^\ell$. The basis vectors of the Verma module are $v_{k,\ell} = p_{k,\ell} \otimes v_0$, (with $v_{0,0} = v_0$). The action of $\hat{S}$ on this basis is derived easily from (3.6):
\[
Dv_{k,\ell} = (k + 2\ell + \Delta)v_{k,\ell}
\]
\[ G \ v_{k,\ell} = v_{k+1,\ell} \]
\[ K \ v_{k,\ell} = v_{k,\ell+1} \]
\[ P_x \ v_{k,\ell} = \ell \ v_{k+1,\ell-1} + M \ k \ v_{k-1,\ell} \]
\[ P_t \ v_{k,\ell} = \ell (k + \ell - 1 + \Delta) \ v_{k,\ell-1} + M \frac{k(k-1)}{2} \ v_{k-2,\ell} \quad (5.5) \]

Because of (5.5) we notice that the Verma module \( V^\Delta \) can be decomposed in homogeneous (w.r.t. \( D \)) subspaces as follows:
\[ V^\Delta = \bigoplus_{n=0}^{\infty} V^\Delta_n \]
\[ V^\Delta_n = \text{lin.span.} \{ v_{k,\ell} \mid k + 2\ell = n \} \]
\[ \dim V^\Delta_n = 1 + \left[ \frac{n}{2} \right] \quad (5.6) \]

Next we analyze the reducibility of \( V^\Delta \) through the so-called singular vectors. In analogy to the SSLA situation (cf., e.g., [114]) a singular vector \( v_s \) here is a homogeneous element of \( V^\Delta \), such that \( v_s \notin C v_0 \), and
\[ P_x \ v_s = 0, \quad P_t \ v_s = 0 \quad (5.7) \]

All possible singular vectors were given explicitly in [113], where was proved:

**Proposition 1.** The singular vectors of the Verma module \( V^\Delta \) over \( \hat{S} \) are given as follows:
\[ v_s^p = a_0 \sum_{\ell=0}^{p/2} (-2M) \ell \left( \frac{p/2}{\ell} \right) v_{p-2\ell,\ell} \]
\[ = a_0 \left( G^2 - 2MK \right)^{p/2} \otimes v_0, \quad \Delta = \frac{3-p}{2}, \quad p \in 2\mathbb{N}, \ M a_0 \neq 0 \]
\[ v_s^p = a_0 v_p = a_0 G^p \otimes v_0, \quad \Delta \text{ arbitrary}, \ p \in \mathbb{N}, \ M = 0, \ a_0 \neq 0. \quad \diamond \]

**Remark:** We stress the very different character of the representations for \( M \neq 0 \) and \( M = 0 \) from one another and furthermore from the semisimple case. For \( M \neq 0 \) and fixed lowest weight at most one singular vector may exist and that vector can be only of **even** grade. For \( M = 0 \) an infinite number of singular vectors exist - one for each positive grade - and there is no restriction on the weight. This difference is because the value \( M = 0 \) changes the algebra - it is not a centrally extended one anymore. Both cases **differ from the semisimple case**. To compare we take, e.g., the algebra \( sl(2) \) since it also has only one Cartan generator as \( \hat{S}(1) \). For \( sl(2) \) for a fixed lowest weight only one singular vector is possible, however, for any ‘grade’ \( n\beta \), where \( n \in \mathbb{N}, \ \beta \) the positive root of \( sl(2) \), and not just for even \( n \). \quad \diamond

Whenever there is a singular vector the Verma module is reducible. We could analyze this reducibility also via an analogue of the Shapovalov form [130] used in the semisimple case. This is a bilinear form which we define using the involutive antiautomorphism of the Schrödinger algebra (cf. (2.6)):
\[ \omega(P_t) = K, \quad \omega(P_x) = G, \quad \omega(D) = D, \quad \omega(M) = M \quad (5.9) \]
Explicitly, the form here is given by:

\[
(v_{k\ell}, v'_{k'\ell'}) = (p_{k\ell} \otimes v_0, p_{k'\ell'} \otimes v_0) = \left(v_0, \omega(p_{k\ell}) p_{k'\ell'} \otimes v_0\right) = \left(v_0, P^e_\ell P^k_x G^k K^\ell v_0\right)
\]

(5.10)

supplemented by the normalization condition \((v_0, v_0) = 1\). Clearly, subspaces with different weights are orthogonal w.r.t. to this form:

\[
(v_{k\ell}, v_{k'\ell'}) \sim \delta_{k+2\ell, k'+2\ell'}
\]

(5.11)

To show this for \(k + 2\ell > k' + 2\ell'\) we move all \(P_t\) and \(P_x\) operators to the right until there are no \(G\) and \(K\) operators left, while for \(k + 2\ell < k' + 2\ell'\) we first rewrite the LHS of (5.11) as:

\[
(v_{k\ell}, v_{k'\ell'}) = (p_{k\ell} \otimes v_0, p_{k'\ell'} \otimes v_0) = \left(v_0, \omega(p_{k'\ell'}) p_{k\ell} \otimes v_0\right) = \left(v_0, P^e_\ell' P^k'_x G^k K^\ell' v_0\right)
\]

(5.12)

and then again move all \(P_t\) and \(P_x\) operators to the right until there are no \(G\) and \(K\) operators left. The above also shows that the form is symmetric. In the case \(k + 2\ell = k' + 2\ell'\) we have the following explicit expression:

\[
(v_{k,\ell}, v_{k+2a,\ell-a}) = \frac{\Gamma^a_{k+a}(k+1)}{\Gamma^a_{k}(k+1)} \times F_3 \left(\frac{a}{2}, \frac{a+k}{2}, a-k; 1, 0; y\right)
\]

(5.13)

where \(a \in \mathbb{Z}_+, (a)_p = \Gamma(a+p)/\Gamma(a)\) is the Pochhammer symbol and \(F_3(a, b, c; a', b', y)\) is a generalized hypergeometric series:

\[
F_3(a, b, c; a', b', y) = \sum_{s \in \mathbb{Z}_+} \frac{(a)_s(b)_s}{(c)_s(a')_s(b')_s} y^s
\]

(5.14)

which for \(a\), or \(b\), or \(c \in \mathbb{Z}_-\) reduces to a polynomial.

A singular vector is orthogonal to any other vector w.r.t. to the form (5.10). Thus we expect (as in the SSLA case) to obtain the same reducibility results analyzing the so-called determinant formula. The determinant formula is the determinant of the matrix \(\mathcal{M}_p\) of all Shapovalov forms at a fixed grade \(p\). In [113] was made the following conjecture for \(\det \mathcal{M}_p\):

\[
\det \mathcal{M}_p = \text{const.}(p) m^\alpha_p \prod_{i=0}^{[p/2]^{-1}} (2\Delta - 1 + 2i)\left[\frac{\Delta}{2}\right]^{-i},
\]

(5.15)

\[
\alpha_p = \begin{cases} 
\frac{p^2 + 2p}{4} & \text{for } p \text{ even} \\
\frac{(p+1)^2}{4} & \text{for } p \text{ odd}
\end{cases}
\]

which was verified for \(p \leq 6\). We see that the determinant has zeroes exactly in the cases when we have singular vectors. The above conjecture was proved in [131].
Further we consider the consequences of the reducibility of the Verma modules. We start with \( M \neq 0 \) and consider the subspace of \( V^{(3-p)/2} \):

\[
I^{(3-p)/2} = U(S^+) v^p_s
\]

(5.16)

It is invariant under the action of the Schrödinger algebra. Indeed, all vectors have grade \( \geq \deg_{\text{min}} = p \) : lower grades can not be achieved since the negative grade generators annihilate \( v^p_s \). Furthermore this subspace is isomorphic to a Verma module \( V^{d'} \) with shifted weight \( \Delta' = \Delta + p = (p + 3)/2 \). The latter Verma module has no singular vectors, since its weight is restricted from below : \( \Delta' \geq 5/2 \), while by (5.8) the necessary weight is \( \leq 1/2 \).

Let us denote the factor–module \( V^{(3-p)/2}/I^{(3-p)/2} \) by \( L^{(3-p)/2} \). Let us denote by \( |p\rangle \) the lowest weight vector of \( L^{(3-p)/2} \). It satisfies the following conditions:

\[
P_x |p\rangle = 0 \quad (5.17a) \\
P_t |p\rangle = 0 \quad (5.17b) \\
\left(G^2 - 2MK^p\right)^{p/2} |p\rangle = 0 \quad (5.17c)
\]

Consider in more detail the simplest example \( p = 2 \). In this case the last condition (5.17c) is:

\[
G^2 |2\rangle = 2MK |2\rangle
\]

(5.18)

i.e., \( K \) can be replaced by \( G^2/2M \). Thus, all vectors of a fixed grade are proportional:

\[
G^k K^\ell |2\rangle = \frac{1}{(2M)^\ell} G^{k+2\ell} |2\rangle , \quad \Delta = 1/2
\]

(5.19)

so all graded subspaces are one-dimensional, i.e., we have a *singleton* basis:

\[
\dim V_n^{1/2} = 1 , \quad \forall n
\]

(5.20)

which is given only in terms of \( G \). (The term 'singleton' was used first for two special representations of the algebra \( so(3,2) \) [132–137].)

Analogously, for arbitrary \( p \in 2\mathbb{N} \) and \( \Delta = (3-p)/2 \), from (5.17c) we see that:

\[
K^{p/2} |p\rangle = -\sum_{\ell=0}^{p/2-1} \frac{1}{(-2M)^{p/2-\ell}} \binom{p/2}{\ell} G^{p-2\ell} K^\ell |p\rangle
\]

(5.21)

Applying repeatedly this relation to the basis one can get rid of all powers of \( K \) which are \( \geq p/2 \). Thus the basis of \( L^{(3-p)/2} \) will be quasi–singleton if \( p \geq 4 \), namely,

\[
\dim V_n^{(3-p)/2} = 1 , \quad \text{for } n = 0, 1 \text{ or } n \geq p
\]

(5.22)

and it is given by:

\[
v^p_{k\ell} \equiv G^k K^\ell |p\rangle , \quad p \in 2\mathbb{N} , \ k, \ell \in \mathbb{Z}_+ , \ \ell \leq p/2 - 1 , \ \Delta = \frac{3-p}{2}
\]

(5.23)
The transformation of this basis is easily obtained from (5.5):

\[ \begin{align*}
D v^p_{k,\ell} &= \left( k + 2\ell + \frac{3-p}{2} \right) v^p_{k,\ell} \quad (5.24a) \\
G v^p_{k,\ell} &= v^p_{k+1,\ell} \quad (5.24b) \\
K v^p_{k,\ell} &= \left\{ \begin{array}{ll}
\frac{v^p_{k,\ell+1}}{\ell} - \sum_{s=0}^{(p/2)-1} \frac{1}{(-2M)^{p/2-s}} (p/2) v^p_{k+p-2s,s} & \ell < \frac{p}{2} - 1 \\
\frac{v^p_{k,\ell+1}}{\ell} + mk v^p_{k-1,\ell} & \ell = \frac{p}{2} - 1 \\
\frac{v^p_{k,\ell+1}}{\ell} + mk v^p_{k-1,\ell} & \ell = \frac{p}{2} - 1
\end{array} \right. \quad (5.24c) \\
P_x v^p_{k,\ell} &= \ell v^p_{k+1,\ell} + mk v^p_{k-1,\ell} \quad (5.24d) \\
P_t v^p_{k,\ell} &= \ell \left( k + \ell + \frac{1-p}{2} \right) v^p_{k,\ell+1} + m \frac{k(k-1)}{2} v^p_{k-2,\ell} \quad (5.24e)
\end{align*} \]

From the transformation rules we see that \( \mathcal{L}^{(3-p)/2} \) is irreducible. It is also clear that in the simplest case \( p = 2 \) the irrep \( \mathcal{L}^{1/2} \) is also an irrep of the centrally extended Galilean subalgebra \( \hat{G}(1) \) spanned by \( P_x, P_t, G \).

For \( M = 0 \) we consider the subspaces of \( V^\Delta \):

\[ I^\Delta_p = U(S^+) G^p \otimes v_0 , \quad p \in \mathbb{N} \quad (5.25) \]

They are invariant under the action of the Schrödinger algebra, which is shown as in the case \( M \neq 0 \). The corresponding singular vectors are:

\[ v^p_s = G^p \otimes v_0 \quad (5.26) \]

Furthermore the subspace \( I^\Delta_p \) is isomorphic to a Verma module \( V^{\Delta'} \) with shifted weight \( \Delta' = \Delta + p \). The latter Verma module again has an infinite number of singular vectors, and an infinite number of subspaces \( I^{\Delta+p}_{p'} = U(S^+) G^{p'} \otimes v_0 \), \( p' \in \mathbb{N} \), isomorphic to Verma modules \( V^{\Delta+p+p'} \). Furthermore, the original Verma module \( V^\Delta \) is itself a submodule of an infinite number of Verma modules \( V^{\Delta+p''} \), \( p'' \in \mathbb{N} \). Altogether, for each \( \Delta \) there exists a doubly infinite sequence of Verma modules:

\[ \cdots \supset V^{\Delta-1} \supset V^{\Delta} \supset V^{\Delta+1} \supset \cdots , \quad M = 0, \ \Delta \text{ arbitrary} \quad (5.27) \]

Of course, all \( V^{\Delta} \) whose weights differ by an integer are in one and the same sequence. Such embedding diagrams were called multiplets in [138].

For each \( V^{\Delta} \) the submodule \( I^\Delta_1 \cong V^{\Delta+1} \) contains as submodules all other submodules of \( V^\Delta \), i.e., in (5.26) only the singular vector with \( p = 1 \) is relevant.

Consider the factor space \( \hat{\mathcal{L}}^\Delta_0 = V^\Delta/V^{\Delta+1} \) and denote by \( \tilde{0} \) its lowest weight vector. The latter satisfies the following conditions:

\[ \begin{align*}
P_x \tilde{0} &= 0 \quad (5.28a) \\
P_t \tilde{0} &= 0 \quad (5.28b) \\
G \tilde{0} &= 0 \quad (5.28c)
\end{align*} \]

Consequently, the basis of \( \hat{\mathcal{L}}^d_0 \) is given by:

\[ v^0_{\ell} = K^\ell \tilde{0} \quad (5.29) \]
This is another example of an even smaller than singleton basis, since the odd-graded levels are empty. Thus, we call the basis in (5.29) a \textit{singleton–void} basis. Its transformation rules are (cf. (5.5)):

\begin{align*}
D \hat{\psi}_\ell^0 &= (2\ell + \Delta) \hat{\psi}_\ell^0 \\
G \hat{\psi}_\ell^0 &= 0 \\
K \hat{\psi}_\ell^0 &= \hat{\psi}_{\ell+1}^0 \\
P_x \hat{\psi}_\ell^0 &= 0 \\
P_t \hat{\psi}_\ell^0 &= \ell(\ell - 1 + \Delta) \hat{\psi}_{\ell-1}^0
\end{align*}

Clearly, \( \tilde{L}_{\Delta} \) is in fact a Verma module over the \( \mathfrak{sl}(2, \mathbb{R}) \) subalgebra spanned by \( D, K, P_t \). It is well known that such a Verma module is reducible (cf., e.g., [114]) iff \( \Delta \in \mathbb{Z}^- \). In this case there exists a singular vector given by:

\[ v_s^0 = \hat{\psi}_{1-\Delta}^0 = K^{1-\Delta} |0\rangle, \quad \Delta \in \mathbb{Z}^- \] (5.31)

Thus the invariant subspace \( \tilde{L}_{\Delta} \) of \( \tilde{L}_{\Delta} \) is spanned by \( K^{1-\Delta} |0\rangle \), \( \ell \in \mathbb{Z}_+ \), and is isomorphic to another Verma module \( \tilde{L}_{\Delta-2} \) which is irreducible.

Let us stress that \( v_s^0 \) is not a singular vector of the original Verma module \( V^{\Delta} \), but of its factor module \( \tilde{L}_{\Delta} \). Vectors, which become singular vectors only in factor modules, are called \textit{subsingular} vectors.

Consider now the factor space \( \mathcal{L}_{\Delta} = \tilde{L}_{\Delta} / \tilde{L}_{\Delta-2} \), denoting by \(|0\rangle\) its lowest weight vector. It satisfies the following conditions:

\begin{align*}
P_x |0\rangle &= 0 \\
P_t |0\rangle &= 0 \\
G |0\rangle &= 0 \\
K^{1-\Delta} |0\rangle &= 0
\end{align*}

Consequently, the basis of \( \mathcal{L}_{\Delta} \) is given by:

\[ v_{\Delta}^\ell = K^\ell |0\rangle, \quad -\Delta, \ell \in \mathbb{Z}_+, \ell \leq -\Delta \] (5.33)

Hence \( \mathcal{L}_{\Delta} \) is finite-dimensional: \( \dim \mathcal{L}_{\Delta} = 1 - \Delta \), and in fact, when \( \Delta \) runs through \( \mathbb{Z}_- \) one obtains all irreducible finite-dimensional representations of \( \mathfrak{sl}(2, \mathbb{R}) \). The latter are not unitary, except in the trivial one-dimensional case obtained for \( \Delta = 0 \).

The transformation rules for \( v_{\Delta}^\ell \) are as for \( \hat{\psi}_{\ell}^0 \) in (5.30), except that \( K v_{\Delta}^0 = 0 \).

Summarizing the above in [113] was proved:

\textbf{Theorem 1.} The list of the irreducible lowest weight modules over the (centrally extended) Schrödinger algebra is given by:

\begin{itemize}
  \item \( V^d \), when \( \Delta \neq (3 - p)/2, \ p \in 2\mathbb{N} \) and \( M \neq 0 \);
  \item \( \mathcal{L}^{(3-p)/2} \), when \( \Delta = (3 - p)/2, \ p \in 2\mathbb{N} \) and \( M \neq 0 \);
  \item \( \tilde{L}_{\Delta}^0 \), when \( \Delta \notin \mathbb{Z}_- \) and \( M = 0 \);
  \item \( \mathcal{L}_{\Delta}^d \), when \( d \in \mathbb{Z}_- \) and \( M = 0 \).
\end{itemize}

In the last case one has: \( \dim \mathcal{L}_{\Delta}^d = 1 - \Delta \); in all other cases the irreps are infinite-dimensional. The representation \( \mathcal{L}^{1/2} \) is also an irrep of the centrally extended
Galilean subalgebra $\hat{G}(1)$. The irreps in the last two cases are also irreps of the subalgebra $sl(2,\mathbb{R})$.

### 5.3. Generalized Schrödinger equations from a vector–field realization of the Schrödinger algebra

Now we shall employ vector–field representation (3.12) as in [113]. This realization was used to construct a polynomial realization of the irreducible lowest weight modules considered in the previous Subsection. For this realization we represent the lowest weight vector by the function $1$. Indeed, the constants in (3.12) are chosen so that (5.4) is satisfied:

$$D 1 = \Delta , \quad M 1 = M , \quad P_x 1 = 0, \quad P_t 1 = 0 \quad (5.34)$$

Applying the basis elements $p_{k,\ell} = G^k K^\ell$ of the universal enveloping algebra $U(\hat{S}^+)$ to $1$ we get polynomials in $x,t$.

Let us introduce notation for these polynomials by $f_{k,\ell} \equiv p_{k,\ell} 1$. (In partial cases we have explicit expressions for $f_{k,\ell}$ from [113] but we shall not need them here.)

Let us denote by $C^\Delta$ the spaces spanned by the elements $f_{k,\ell}$, and by $L^\Delta$ the irreducible subspace of $C^\Delta$. Now in [113] was shown:

**Theorem 2.** The irreducible spaces $L^\Delta$ give a realization of the irreducible lowest weight representations of $\hat{S}(1)$ given in Theorem 1. \hfill $\Diamond$

We consider now in more detail the most interesting cases of the representations $L^{(3-p)/2}$ with $M \neq 0$ and $p \in 2\mathbb{N}$.

We first introduce an operator by the polynomial $G^2 - 2MK \in U(\hat{S}^+)$ expressed this polynomial in the vector–field realization:

$$S \doteq G^2 - 2MK = t^2 (\partial_x^2 - 2M \partial_t) + 2Mt(\frac{1}{2} - \Delta) \quad (5.35)$$

In these case we have [113]:

**Proposition 2.** Each basis polynomial $f_{k,\ell}$ of $L^{(3-p)/2}$ satisfies:

$$S^{p/2} f_{k,\ell} = (t^2(\partial_x^2 - 2M \partial_t) + (p-2)Mt)^{p/2} f_{k,\ell} = t^p (\partial_x^2 - 2M \partial_t)^{p/2} f_{k,\ell} = 0, \quad \Delta = \frac{3-p}{2} \quad (5.36)$$

Thus we have obtained in (5.36) an infinite hierarchy of PDO’s $S^{p/2}$ which give rise to differential equations we call **free generalized heat/Schrödinger equations**. The equations are obtained by substituting the vector field realization in the singular vectors (thus extending the procedure of [114]). This substitution gives:

$$S^{p/2} = (G^2 - 2MK)^{p/2} = (t^2(\partial_x^2 - 2M \partial_t) + (p-2)mt)^{p/2} = t^p (\partial_x^2 - 2M \partial_t)^{p/2} \quad (5.37)$$
Thus, the hierarchy of equations is:

\[ t^p \left( \partial_x^2 - 2M \partial_t \right)^{p/2} f = 0 \]  \hspace{1cm} (5.38)

In the case of function spaces with elements which are polynomials in \( t \) (as our representation spaces) or singular at most as \( t^{-p/2} \) for \( t \to 0 \), the hierarchy is:

\[ \left( \partial_x^2 - 2M \partial_t \right)^{p/2} f = 0 \]  \hspace{1cm} (5.39)

The above Proposition also shows that the representation spaces are comprised from solutions of the corresponding equations (5.39). The case \( p = 2 \) and \( M \) real is the ordinary heat or diffusion equation and for \( p = 2 \) and \( M \) purely imaginary we get the free Schrödinger equation. So the members of the hierarchy of equations which are invariant under the Schrödinger group have generically higher orders of derivatives in \( t \). This shows that the Schrödinger symmetry is not necessarily connected with first order (in \( t \)) differential operators.

We can further extend [114] to the non-semisimple situation by considering equations with non-zero RHS. However, invariance w.r.t. the Schrödinger algebra requires that the RHS is an element of the irreducible representation space \( C^{(p+3)/2} \), while the functions in the LHS are not restricted to the solution subspace of (5.39). Thus, using the operator in (5.38) we obtained the following hierarchy of generalized heat/Schrödinger equations:

\[ t^p \left( \partial_x^2 - 2M \partial_t \right)^{p/2} f = j, \quad f \in C^{(3-p)/2}, \quad j \in C^{(3+p)/2} \]  \hspace{1cm} (5.40)

**Remark:** It is interesting to note that (5.40) looks similar to an hierarchy of equations involving the d’Alembert operator and conditionally invariant w.r.t. conformal algebra \( su(2,2) \):

\[ \square^n \varphi(x) = \varphi'(x), \quad n \in \mathbb{N} \]  \hspace{1cm} (5.41)

where \( \varphi, \varphi' \) are scalar fields of different fixed conformal weights depending on \( n \), \( x = (x_0, x_1, x_2, x_3) \) denotes the Minkowski space-time coordinates, and \( \square \) is the d’Alembert operator: \( \square = \partial^\mu \partial_\mu = (\vec{\partial})^2 - (\partial_0)^2 \), cf., e.g., [139].

Of course, we may consider more general function spaces of the variables \( t, x \), say \( C^\infty(\mathbb{R}^2, \mathbb{R}) \), on which the centrally extended Schrödinger algebra is acting by formulae (3.12). For the example of the ordinary heat equation one may find also solutions of the type:

\[ \left( \alpha \cosh(\sqrt{\lambda}x) + \beta \sinh(\sqrt{\lambda}x) \right) \exp \left( \frac{\lambda t}{2M} \right), \quad \alpha, \beta, \lambda \in \mathbb{R} \]  \hspace{1cm} (5.42)

which results from separation of the \( t \) and \( x \) variables, while the polynomial solutions above may be obtained also by separation of the variables \( t \) and \( Mx^2/2t \), cf. [113].
5.4. Generalized Schrödinger equations in the bulk

In this subsection we review [140]. Now we shall employ the bulk vector–field representation (3.9) trying similarly to the previous subsection to construct generalized Schrödinger equations in the bulk. We start with the operator (distinguishing bulk operators by hats):

\[ \hat{S} = \hat{G}^2 - 2 \partial_- \hat{K} = t^2 (\partial_x^2 - 2 \partial_x \partial_t) + 2t(\frac{1}{2} - z \partial_z) \partial_- - z^2 \partial_x^2 \] (5.43)

We could use the one-point invariant variable obtained from \( u \) by setting in (3.19)

\[ t' = 0, \quad x' = 0, \quad x'_\perp = 0, \quad z' = 1, \] i.e., we use

\[ \tilde{u} = \frac{4z}{x^2 - 2tx_\perp + (z + 1)^2} \] (5.44)

Substituting this change in (5.43) we obtain:

\[ \hat{S} = \frac{t^2}{z} \left( \tilde{u}^4 \frac{\partial}{\partial \tilde{u}} + \tilde{u}^4 \frac{\partial^2}{\partial \tilde{u}^2} \right). \] (5.45)

We shall elaborate on the use of (5.45) elsewhere.

Now we set an Ansatz for the fields in the bulk:

\[ \phi(t, x, x_\perp, z) = \exp(Mx - \phi(t, x, z)) \]

which leads to the identification \( \frac{\partial}{\partial x_\perp} = M \) both in the bulk and on the boundary. Thus, we shall use:

\[ \hat{S}_0 \equiv \hat{G}^2_0 - 2M \hat{K}_0 = t^2 (\partial_x^2 - 2M \partial_t) + 2tM(\frac{1}{2} - z \partial_z) - z^2 M^2 \] (5.46)

Thus, we obtain the following Schrödinger-like equation in the bulk:

\[ \hat{S}_0 \phi = \phi', \quad \phi \in \hat{C}^{1/2}, \quad \phi' \in \hat{C}^{5/2}. \] (5.47)

The relation to the Schrödinger equation on the boundary is seen by the following commutative diagram:

\[ \begin{array}{ccc}
\hat{C}^{1/2} & \xrightarrow{S_0} & \hat{C}^{5/2} \\
\downarrow L_{1/2} & & \downarrow L_{1/2} \\
C^{1/2} & \xrightarrow{S} & C^{5/2}
\end{array} \] (5.48)

where \( L_{1/2} \) is the bulk-to-boundary operator defined in (3.25), and (5.48) may be re-written as the intertwining relation:

\[ S \circ L_{1/2} = L_{1/2} \circ \hat{S}_0, \] acting as operator \( \hat{C}^{1/2} \rightarrow C^{5/2} \) (5.49)

The relation (5.49) (and so (5.48)) follows by substitution of the definitions.

As expected, we have a Schrödinger-like hierarchy of equations in the bulk:

\[ (\hat{S}_0)^p/2 \phi = \phi', \quad \phi \in \hat{C}^{(3-p)/2}, \quad \phi' \in \hat{C}^{(3+p)/2}, \quad p \in 2\mathbb{N} \] (5.50)
They are equivalent to the Schrödinger hierarchy of equations on the boundary (5.40) which is proved by showing the analogues of (5.48) and (5.49):
\[
\hat{C}^\Delta \rightarrow (\hat{S}_0)^{p/2} \hat{C}^{3-\Delta} \\
\downarrow L_\Delta \quad \downarrow L_\Delta \\
C^\Delta \rightarrow (\hat{S}_0)^{p/2} C^{3-\Delta}
\]
(5.51)
\[
S^{p/2} \circ L_\Delta = L_\Delta \circ (\hat{S}_0)^{p/2} , \text{ acting as operator } \hat{C}^\Delta \rightarrow C^{3-\Delta}, \quad (5.52)
\]
\[
\Delta = (3-p)/2, \quad p \in 2\mathbb{N}
\]

6. Non-relativistic invariant differential equations for arbitrary \( n \)

In this Section we review the papers [116, 117].

6.1. Gauss decomposition of the Schrödinger group

6.1.1. Triangular decomposition of \( \hat{S}(n) \) for \( n = 2N \)

\[
\hat{S}(2N) = \hat{S}(2N)^+ \oplus \hat{S}(2N)^0 \oplus \hat{S}(2N)^-,
\]
\[
\hat{S}(2N)^+ = \text{l.s.}\{ G_a, K, E_{t,\pm t_j} \},
\]
\[
\hat{S}(2N)^0 = \text{l.s.}\{ D, M, H_k \},
\]
\[
\hat{S}(2N)^- = \text{l.s.}\{ P_a, P_t, E_{-(t,\pm t_j)} \},
\]
(6.1)

where l.s. stands for linear span, \( a, k, i, j \) are integers of \( 1 \leq a \leq 2N, 1 \leq k \leq N, 1 \leq i < j \leq N \). For \( N > 1 \) the generators \( \{ H_k \}, \{ E_{t,\pm t_j} \}, \{ E_{-(t,\pm t_j)} \} \) are the Cartan subalgebra generators, the positive root vectors and the negative root vectors of \( \text{so}(2N) \), respectively. They are related to the antisymmetric generators \( \{ J_{ij} \} \) as follows (only the first line remaining for \( N = 1 \))
\[
H_k = -iJ_{k N+k}, \quad (k = 1, 2, \cdots N)
\]
(6.2)
\[
E_{\pm(t_j+\ell_k)} = -\frac{1}{2}(J_{jk} \mp iJ_j N+k \pm iJ_k N+j - J_{N+j N+k}),
\]
\[
(1 \leq j < k \leq N)
\]
\[
E_{t_j-t_k} = -\frac{1}{2}(J_{jk} + iJ_j N+k + iJ_k N+j + J_{N+j N+k}),
\]
\[
(1 \leq j \neq k \leq N)
\]
6.1.2. Triangular decomposition of $\hat{S}(n)$ for $n = 2N + 1$

\[ \hat{S}(2N + 1) = \hat{S}(2N + 1)^+ \oplus \hat{S}(2N + 1)^0 \oplus \hat{S}(2N + 1)^- , \]
\[ \hat{S}(2N + 1)^+ = \text{ls.}\{ G_a, K, E_{\ell_k}, E_{\ell_j \pm \ell_k} \}, \]
\[ \hat{S}(2N + 1)^0 = \text{ls.}\{ D, M, H_k, \}, \]
\[ \hat{S}(2N + 1)^- = \text{ls.}\{ P_a, P_t, E_{-(\ell_k)} , E_{-(\ell_j)} \}, \]

where $a, k, i, j$ are integers of $1 \leq a \leq 2N + 1, 1 \leq k \leq N, 1 \leq i < j \leq N$. The generators $\{ H_k, \} \{ E_{\ell_k}, E_{\ell_j \pm \ell_k} \}, \{ E_{-(\ell_k)} , E_{-(\ell_j)} \}$ are the Cartan subalgebra generators, the positive root vectors and the negative root vectors of $so(2N + 1)$, respectively. They are related to the antisymmetric generators $\{ J_{ij} \}$ as in the case $n = 2N$ for $H_k, E_{\pm(\ell_j + \ell_k)}$, and $E_{\ell_j - \ell_k}$, while $E_{\pm \ell_k}$ is defined by:

\[ E_{\pm \ell_k} = -\frac{1}{\sqrt{2}}(J_k 2N+1 \mp iJ_{N+k} 2N+1), \quad (k = 1, 2, \cdots , N) \quad (6.4) \]

6.1.3. Gauss decomposition of the Schrödinger group $\hat{S}(n)$

Let $g \in \hat{S}(n)$ be an element with Gauss decomposition: $g = g_+ g_0 g_-$. For even $n = 2N$ we have:

\[ g_+ = \left( \prod_{a=1}^{2N} e^{x_a G_a} \right) e^{tK} \left( \prod_{1 \leq j < k \leq N} e^{\xi_{jk} E_{\ell_j + \ell_k}} \right) \left( \prod_{1 \leq j < k \leq N} e^{\eta_{jk} E_{\ell_j - \ell_k}} \right) \]
\[ g_0 = e^{D} e^{mM} \prod_{k=1}^{N} e^{h_k H_k} , \quad (6.5) \]
\[ g_- = \left( \prod_{a=1}^{2N} e^{w_a P_a} \right) e^{\kappa P_t} \left( \prod_{1 \leq j < k \leq N} e^{\xi_{jk} E_{-(\ell_j - \ell_k)}} \right) \left( \prod_{1 \leq j < k \leq N} e^{\eta_{jk} E_{-(\ell_j + \ell_k)}} \right) \]
For odd \( n = 2N + 1 \) we have:

\[
g_+ = \left( \prod_{a=1}^{2N+1} e^{x_a G_a} \right) e^{iK} \left( \prod_{k=1}^{N} e^{\varphi_k E_{\ell_k}} \right) \times \left( \prod_{1 \leq j < k \leq N} e^{\xi_{jk} E_{\ell_j + \ell_k}} \right) \left( \prod_{1 \leq j < k \leq N} e^{\eta_{jk} E_{\ell_j - \ell_k}} \right)
\]

\[
g_0 = e^{\delta D} m M \prod_{k=1}^{N} e^{h_k H_k}
\]

\[
g_- = \left( \prod_{a=1}^{2N+1} e^{u_a P_a} \right) e^{s P_s} \left( \prod_{k=1}^{N} e^{\varphi_k E_{-\ell_k}} \right) \times \left( \prod_{1 \leq j < k \leq N} e^{\xi_{jk} E_{-\ell_j - \ell_k}} \right) \left( \prod_{1 \leq j < k \leq N} e^{\eta_{jk} E_{-\ell_j + \ell_k}} \right)
\]

\[
6.2. Representations of \( \hat{S}(n) \)
\]

Let briefly recall the SSLG setting [114] adapted to the Schrödinger setting above:

- \( C_\Lambda \): the space of \( C^\infty \) functions \( F \) on \( \hat{S}(n) \) with right covariance property
  \[
  F(gxg') = e^{\Lambda(H)} F(g),
  \]
  where \( x = e^H \in g_0 \), \( H \in s^0(\hat{n}) \), \( g' \in g_- \), \( \Lambda \in (\check{s}^0(\hat{n}))^* \). We use the following notation for the values of \( \Lambda(H) \): \( \Lambda(M) = -m \), \( \Lambda(D) = \Delta \), \( \Lambda(H_k) = -h_k \in \mathbb{Z} \).

- \( C_\Lambda \): space of restricted functions \( \psi = F \mid_{g_+} \).

- \( \pi_L(X) \): left action of \( X \in \hat{s}(n) \) on \( C_\Lambda \)
  \[
  \pi_L(X) F(g) = \frac{d}{d\tau} F(e^{-\tau X} g) \bigg|_{\tau=0}
  \]

- \( \pi_R(X) \): right action of \( X \in \hat{s}(n) \) on \( C_\Lambda \)
  \[
  \pi_R(X) F(g) = \frac{d}{d\tau} F(g e^{\tau X}) \bigg|_{\tau=0}
  \]

6.2.1. Representations for \( n = 2N \)

Define

\[
G_k^\pm = G_k \pm iG_{N+k}, \quad x_k^\pm = \frac{1}{2}(x_k \mp i x_{N+k}), \quad k = 1, 2, \cdots, N
\]

then

\[
\prod_{a=1}^{2N} e^{x_a G_a} = \prod_{k=1}^{N} e^{x_k^+ G_k^+} e^{x_k^- G_k^-}.
\]
Sometimes the use of (6.11) is more convenient. We denote each factor of $g_+$ by

$$g_+ = \Gamma x e^{tK} \Gamma \Gamma_\eta,$$

$$\Gamma_x = \prod_{a=1}^{2N} e^{x_a G_a}, \quad \Gamma_\xi = \prod_{1 \leq j < k \leq N} e^{\xi_{jk} E_{\ell_j + \ell_k}}, \quad \Gamma_\eta = \prod_{1 \leq j < k \leq N} e^{\eta_{jk} E_{\ell_j - \ell_k}}$$ (6.12)

**Action of $\hat{S}(2N)^0$ :**
Following the general procedure of [114] and results of [116, 117] we obtain the following formulae for the vector-field representation:

$$\pi_L(M) = -\Lambda(M), \quad \pi_L(D) = -\Lambda(D) - \sum_{a=1}^n x_a \frac{\partial}{\partial x_a} - 2t \frac{\partial}{\partial t}$$ (6.13)

$$\pi_L(H_k) = -\Lambda(H_k) + x_k^+ \frac{\partial}{\partial x_k} - x_k^- \frac{\partial}{\partial x_k} - \sum_{j=1}^{k-1} (\xi_{jk} \frac{\partial}{\partial \xi_{jk}} - \eta_{jk} \frac{\partial}{\partial \eta_{jk}})$$ (6.14)

The $x$-dependent parts of (6.14) can be rewritten as:

$$i \left( x_k \frac{\partial}{\partial x_{N+k}} - x_{N+k} \frac{\partial}{\partial x_k} \right)$$ (6.15)

**Action of $\hat{S}(2N)^+$ [116] :**

$$\pi_L(G_a) = -\frac{\partial}{\partial x_a}; \quad \pi_L(K) = -\frac{\partial}{\partial t}$$ (6.16)

$$\pi_L(E_{\ell_j + \ell_k}) = x_j^+ \frac{\partial}{\partial x_k} - x_k^+ \frac{\partial}{\partial x_j} - \frac{\partial}{\partial \xi_{jk}}$$ (6.17)

$$\pi_L(E_{\ell_j - \ell_k}) = x_j^+ \frac{\partial}{\partial x_k} - x_k^- \frac{\partial}{\partial x_j} - \sum_{s=k+1}^N \xi_{ks} \frac{\partial}{\partial \xi_{js}} - \sum_{r=1}^{j-1} \xi_{rk} \frac{\partial}{\partial \xi_{jr}} + \sum_{r=j+1}^{k-1} \xi_{rk} \frac{\partial}{\partial \xi_{jr}}$$ (6.18)

**Action of $\hat{S}(2N)^-$ :** Standardly, the left action is determined from the formula

$$e^{-\tau X} g_+ g_0 g_- = g'_+ g'_0 g'_-, \quad (6.19)$$

where $X \in g_-$. For us, it will be enough to find $g'_+$ and $g'_0$ because the vector field representation is restricted to $g_+$. 


We introduce the generators: $P_k^\pm \equiv P_k \pm iP_{N+k}$. We have:

$$e^{-\tau P_k^+} g = \left( \prod_{r=1}^{N} e^{\delta^+ G_r^+ e^{x_r^+ G_r^-} e^{x_r^- G_r^+}} \right) e^{tK} \left( \prod_{r<s} e^{\xi_r E_{r+s}} \right) \left( \prod_{r<s} e^{\eta_r \xi_{r+s}} \right) \times$$

$$\times e^{\delta^D e^{m'M} \left( \prod_{r=1}^{N} e^{h_r H_r} \right) g'_-} ,$$

$$x^+_r = x^+_r - \tau t , \quad x^-_r = x^-_r \quad \text{(otherwise),} \quad m' = m - 2\tau x^-_k ,$$

$$e^{-\tau P_k^-} g = \left( \prod_{r=1}^{N} e^{\delta^- G_r^+ e^{x_r^- G_r^-} e^{x_r^+ G_r^+}} \right) e^{t'K} \left( \prod_{r<s} e^{\xi_r E_{r+s}} \right) \left( \prod_{r<s} e^{\eta_r \xi_{r+s}} \right) \times$$

$$\times e^{\delta^D e^{m'M} \left( \prod_{r=1}^{N} e^{h_r H_r} \right) g'_-} ,$$

$$x^-_r = x^-_r - \tau t , \quad x^-_r = x^-_r \quad \text{(otherwise),} \quad m' = m - 2\tau x^+_k ,$$

$$e^{-\tau P_k^0} g = \left( \prod_{r=1}^{N} e^{\delta^0 G_r^+ e^{x_r^- G_r^-} e^{x_r^+ G_r^+}} \right) e^{t'K} \left( \prod_{r<s} e^{\xi_r E_{r+s}} \right) \left( \prod_{r<s} e^{\eta_r \xi_{r+s}} \right) \times$$

$$\times e^{\delta^D e^{m'M} \left( \prod_{r=1}^{N} e^{h_r H_r} \right) g'_-} ,$$

$$x^+_j = x^+_j - \tau t x^+_j , \quad x^-_j = x^-_j - \tau t x^-_j \quad \text{t' = t - \tau t^2 ,}$$

$$\delta' = \delta - \tau t , \quad m' = m - 2\tau \sum_{j=1}^{N} x^+_j x^-_j ,$$

$$e^{-\tau E_{-(t_1+t_2)}} g = \left( \prod_{r=1}^{N} e^{\delta^+ G_r^+ e^{x_r^+ G_r^-} e^{x_r^- G_r^+}} \right) e^{tK} \left( \prod_{r<s} e^{\xi_r E_{r+s}} \right) \left( \prod_{r<s} e^{\eta_r \xi_{r+s}} \right) \times$$

$$\times e^{\delta^D e^{mM} \left( \prod_{r=1}^{N} e^{h_r H_r} \right) g'_-} ,$$

$$x^+_i = x^+_i - \tau (\delta_{ij} x^-_k - \delta_{ik} x^-_j) , \quad x^-_i = x^-_i ,$$

$$\xi'_{rs} = \xi_{rs} - \delta_{js}\delta_{ks}\xi^2_{rs} , \quad \xi'_{sr} = \xi_{sr} - \delta_{is}\delta_{kr}\xi^2_{sr} ,$$

$$\eta'_{rs} = \eta_{rs} - \tau \{ \delta_{ks}(\xi_{ir} - \xi_{ri}) + \delta_{sj}(\xi_{rk} - \xi_{kr}) \} ,$$

$$h'_l = h_l - \tau (\delta_{jl} + \delta_{kl}) \xi_{jk} ,$$
\[ e^{-\tau E_{(t_j-t_k)}}g = \left( \prod_{r=1}^{N} e^{x_{r}^{+}G_{r}^{+}}e^{x_{r}^{-}G_{r}^{-}} \right) e^{tK} \left( \prod_{r<s} e^{\xi_{rs}E_{t_r-t_s}} \right) \left( \prod_{r<s} e^{h'_{rs}E_{t_r-t_s}} \right) \times \]
\[ \times e^{\delta D} e^{mM} \left( \prod_{r=1}^{N} e^{h'_{r},H_{r}} \right) g'_{-}, \]
\[ (6.24) \]
\[ x_{r}^{+} = x_{r}^{+} + \tau \delta_{ij} x_{k}^{+}, \quad x_{r}^{-} = x_{r}^{-} - \tau \delta_{ik} x_{j}^{-}, \]
\[ \xi'_{rs} = \xi_{rs} - \tau \delta_{kr} \xi_{is} - \tau \delta_{sk} \xi_{ri} \], \quad h'_{l} = h_{l} - \tau (\delta_{kl} - \delta_{jl}) \xi_{jk}, \]
\[ \eta'_{rs} = \eta_{rs} - \tau \delta_{rk} \eta_{js} + \tau \delta_{js} \eta_{rk} - \delta_{kj} \delta_{sk} \eta_{rs}^{2}. \]

The above formulae are correct up to terms quadratic in \( \tau \). This is enough since these formulae are used only to obtain the left action of the corresponding generators of the algebra (see formulae (6.8)).

Thus, from the infinitesimal left action we obtain the vector-field representation:
\[ \pi_{L}(P_{r}^{+}) = -i \frac{\partial}{\partial x_{r}^{+}} - 2\Lambda(M)x_{r}^{-}, \]
\[ \pi_{L}(P_{r}^{-}) = -i \frac{\partial}{\partial x_{r}^{-}} - 2\Lambda(M)x_{r}^{+}, \]
\[ \pi_{L}(P_{t}) = -i \tau \frac{\partial}{\partial t} - i \sum_{j=1}^{N} \left( x_{j}^{+} \frac{\partial}{\partial x_{j}^{+}} + x_{j}^{-} \frac{\partial}{\partial x_{j}^{-}} \right) - 2\Lambda(M) \sum_{j=1}^{N} x_{j}^{+} x_{j}^{-} - \Lambda(D)t, \]
\[ \pi_{L}(E_{-}(t_j+t_k)) = -(\Lambda(H_{j}) + \Lambda(H_{k})) \xi_{jk} + x_{j}^{+} \frac{\partial}{\partial x_{j}^{+}} - x_{j}^{-} \frac{\partial}{\partial x_{j}^{-}} - \xi_{ik}^{2} \frac{\partial}{\partial \xi_{ik}} - \sum_{p=j+1}^{k-1} \xi_{jp} \frac{\partial}{\partial \eta_{pk}} \sum_{p=1}^{j-1} \xi_{pk} \frac{\partial}{\partial \eta_{pj}} + \sum_{p=1}^{j-1} \xi_{pj} \frac{\partial}{\partial \eta_{pk}}, \]
\[ \pi_{L}(E_{-}(t_j-t_k)) = -(\Lambda(H_{k}) - \Lambda(H_{j})) \xi_{jk} + x_{k}^{+} \frac{\partial}{\partial x_{k}^{+}} - x_{j}^{-} \frac{\partial}{\partial x_{j}^{-}} + \eta_{jk}^{2} \frac{\partial}{\partial \eta_{jk}} - \sum_{s=j+1}^{k-1} \xi_{js} \frac{\partial}{\partial \xi_{sk}} + \sum_{s=k+1}^{N} \xi_{js} \frac{\partial}{\partial \xi_{ks}} - \sum_{r=1}^{j-1} \xi_{rj} \frac{\partial}{\partial \xi_{rk}} - \sum_{q=k+1}^{N} \eta_{rq} \frac{\partial}{\partial \eta_{kq}}. \]

6.2.2. Representations for \( n = 2N + 1 \)

In addition to \( G_{k}^{\pm}, x_{k}^{\pm} \) given by (6.10), let us introduce
\[ G_{0} = \sqrt{2}G_{2N+1}, \quad x_{0} = \frac{1}{\sqrt{2}}x_{2N+1}, \]
\[ (6.26) \]
then
\[ \prod_{a=1}^{2N+1} e^{x_{a}G_{a}} = e^{x_{0}G_{0}} \prod_{k=1}^{N} e^{x_{k}G_{k}^{+}}e^{x_{k}G_{k}^{-}}. \]
\[ (6.27) \]
The difference from $n = 2N$ is the existence of $G_0$ and $E_{\ell_k}$.

**Action of $\hat{S}(2N + 1)^0$**:

Since $[D, G_0] = D_0$, $[D, E_{\ell_k}] = 0$, the left representation of $M, D$ are the same as in the $n$-even case - cf. (6.13), while for $H_k$ we have:

$$\pi_L(H_k) = \pi_L(H_k)^{\text{even}} - \varphi_k \frac{\partial}{\partial \varphi_k}$$

(6.28)

where $\pi_L(H_k)^{\text{even}}$ is the RHS of formula (6.14).

**Action of $\hat{S}(2N + 1)^+$**:

The representations of $G_a$, $K$, $E_{\ell_j + \ell_k}$ are the same as in the $n$-even case - cf. (6.16) with $a = 1, 2, \ldots, n = 2N + 1$, while for the others we have:

$$\pi_L(E_{\ell_j}) = \pi_L(E_{\ell_j})^{\text{even}} - \varphi_k \frac{\partial}{\partial \varphi_k}$$

(6.29)

$$\pi_L(E_{\ell_j - \ell_k}) = \pi_L(E_{\ell_j - \ell_k})^{\text{even}} - \varphi_k \frac{\partial}{\partial \varphi_j}$$

(6.30)

where $\pi_L(E_{\ell_j - \ell_k})^{\text{even}}$ is the RHS of (6.18).

**Action of $\hat{S}(2N + 1)^-$**:

Since $P^\pm_k$ commute with $G_0$ and $[P^+_k, E_{\ell_k}] = P_0$ gives rise only in changing of $g_-$, the left action of $P^\pm_k$ is the same as in the even case. Next,

$$[P_t, G_0^0] = nG_0^{n-1} + n(n-1)G_0^{n-2}M, \quad [P_t, E_{\ell_k}] = -P^+_k$$

(6.31)

provide only

$$m' = m_0^{\text{even}} - \tau x_0^2,$$

from which follows

$$\pi_L(P_t) = \pi_L(P_t)^{\text{even}} - \Lambda(M)x_0^2.$$ 

(6.32)

For the left action of $E_{-(\ell_j - \ell_k)}$ the unique non-vanishing commutator is

$$[E_{-(\ell_j - \ell_k)}, E_{\ell_k}] = E_{\ell_k}$$

which means that

$$\varphi_z' = \varphi_z - \tau \delta_{z\ell_k} \varphi_j$$

and consequently

$$\pi_L(E_{-(\ell_j - \ell_k)}) = \pi_L(E_{-(\ell_j - \ell_k)})^{\text{even}} - \varphi_j \frac{\partial}{\partial \varphi_k}.$$ 

(6.33)
For the rest of the generators we first obtain:

\[ e^{-\tau P_0} g = e^{x_0 G_0} \left( \prod_{r=1}^{N} e^{x_r^+ G_r^+} e^{x_r^- G_r^-} \right) e^{tK} \left( \prod_{r<s} e^{\xi_{rs} E_{r+s}} \right) \left( \prod_{r<s} e^{\eta_{rs} E_{r-s}} \right) \times \]

\[ \times e^{\delta D e^{m'M}} \prod_{r=1}^{N} e^{h_r H_r} g'_+, \]

(6.34)

\[ x_0' = x_0 - \tau t, \quad m' = m - 2\tau x_0, \]

\[ e^{-\tau E_{-\ell_k}} g = e^{x_0^+ G_0} \left( \prod_{r=1}^{N} e^{x_r^+ G_r^+} e^{x_r^- G_r^-} \right) e^{tK} \left( \prod_{r<s} e^{\xi_{rs} E_{r+s}} \right) \times \]

\[ \times \left( \prod_{r<s} e^{\eta_{rs} E_{r-s}} \right) e^{\delta D e^{m'M}} \left( \prod_{r=1}^{N} e^{h_r^+ H_r} \right) g'_+, \]

(6.35)

\[ x_0' = x_0 + \tau x_k^-, \quad x_k'^+ = x_k^+ - \tau \delta_{rk} x_0, \]

\[ \varphi_z = \varphi_z - \tau \delta_{zk} \frac{\varphi_k^2}{2} + \tau (\delta_{zk} \varphi_k + \delta_{r} \xi_{rk} - \delta_{z} \eta_{q}) , \]

\[ \eta_{pq} = \eta_{pq} - \tau \delta_{zp} \delta_{kq} \varphi_z, \quad h_z' = h_z - \tau \delta_{zk} \varphi_z , \]

\[ e^{-\tau E_{-(\ell_1 + \ell_2)}} g = e^{x_0^+ G_0} \left( \prod_{r=1}^{N} e^{x_r^+ G_r^+} e^{x_r^- G_r^-} \right) e^{tK} \left( \prod_{r<s} e^{\xi_{rs} E_{r+s}} \right) \times \]

\[ \times \left( \prod_{r<s} e^{\eta_{rs} E_{r-s}} \right) e^{\delta D e^{m'M}} \left( \prod_{r=1}^{N} e^{h_r^+ H_r} \right) g'_+, \]

(6.36)

\[ x_0'^+ = (x_0'^+)_{even} , \quad x_0'^- = (x_0'^-)_{even} , \]

\[ \xi_{rs}(\xi_{rs})_{even} , \quad \eta_{rs}'(\eta_{rs})'_{even} \quad h_1' = (h_1')_{even} , \]

\[ \varphi_z' = \varphi_z - \tau (\delta_{z} \varphi_z \xi_{zk} - \delta_{zk} \varphi_z \xi_{zk}) . \]

From the above follow the vector-field representation:

\[ \pi_L(P_0) = -t \frac{\partial}{\partial x_0} - 2x_0 \Lambda(M) , \]

(6.37)

\[ \pi_L(E_{-\ell_k}) = -\varphi_k \Lambda(H_k) + x_k \frac{\partial}{\partial x_0} - x_0 \frac{\partial}{\partial x_k} - \]

\[ - \frac{\varphi_k^2}{2} \frac{\partial}{\partial \varphi_k} + \sum_{s=k+1}^{N} \xi_{qs} \frac{\partial}{\partial \varphi_s} + \sum_{r=1}^{k-1} \xi_{rk} \frac{\partial}{\partial \varphi_r} \]

\[ - \sum_{q=k+1}^{N} \eta_{kq} \frac{\partial}{\partial \varphi_q} - \sum_{p=1}^{k-1} \varphi_p \frac{\partial}{\partial \eta_{pk}} , \]

(6.38)

\[ \pi_L(E_{-(\ell_1 + \ell_2)}) = \pi_L(E_{-(\ell_1 + \ell_2)})_{even} - \xi_{jk} (\varphi_j \frac{\partial}{\partial \varphi_j} - \varphi_k \frac{\partial}{\partial \varphi_k}) \]

(6.39)
6.3. Singular vectors and invariant equations for $\hat{S}(2N)$

6.3.1. Singular vectors

Let $v_0$ be the lowest weight vector of the Verma module $V^\Lambda$ with lowest weight $\Lambda$:

$$Hv_0 = \Lambda(H)v_0, \quad Xv_0 = 0, \quad \forall H \in g^0, \quad \forall X \in g^- \quad (6.40)$$

A singular vector is a vector $v_s$ in the Verma module $V^\Lambda$ such that:

$$Hv_s = \Lambda'(H)v_s, \quad Xv_s = 0, \quad \forall H \in g^0, \quad \forall X \in g^-, \quad \Lambda' \neq \Lambda \quad (6.41)$$

Since the singular vectors are in $V^\Lambda \simeq U(g)v_0$, let us introduce $[116]

$$v_{\alpha\beta\gamma\lambda\rho} = \left\{ \prod_{r=1}^{N} (G_r^+)^{\alpha_r} \right\} \left\{ \prod_{r=1}^{N} (G_r^-)^{\beta_r} \right\} K^\gamma \left\{ \prod_{1 \leq r \leq N} (E_{\ell_{r}-\ell_{r}'})^{\rho_{rs}} \right\} \times \left\{ \prod_{1 \leq r \leq N} (E_{\ell_{r}})^{\alpha_r} \right\} v_0, \quad (6.42)$$

where $\alpha, \beta, \lambda$, and $\rho$ are abbreviation of sets of non-negative integers

$$\begin{align*}
\alpha &= [\alpha_1, \alpha_2, \ldots, \alpha_N], \\
\beta &= [\beta_1, \beta_2, \ldots, \beta_N] \\
\lambda &= [\lambda_{12}, \lambda_{13}, \ldots, \lambda_{N-1 \, N}], \\
\rho &= [\rho_{12}, \rho_{13}, \ldots, \rho_{N-1 \, N}] \quad (6.43)
\end{align*}$$

and $\gamma$ is also a non-negative integer.

Elementary computation shows that the action of $g^0$ on $v_{\alpha\beta\gamma\lambda\rho}$ is given by

$$Mv_{\alpha\beta\gamma\lambda\rho} = \Lambda(M)v_{\alpha\beta\gamma\lambda\rho} = -mv_{\alpha\beta\gamma\lambda\rho}$$

$$Dv_{\alpha\beta\gamma\lambda\rho} = (\Lambda(D) + p)v_{\alpha\beta\gamma\lambda\rho} = (p - d)v_{\alpha\beta\gamma\lambda\rho}$$

$$H_k v_{\alpha\beta\gamma\lambda\rho} = (\Lambda(H_k) + r_k)v_{\alpha\beta\gamma\lambda\rho} = (r_k - h_k)v_{\alpha\beta\gamma\lambda\rho} \quad (6.44)$$

$$p = \sum_{r=1}^{N} (\alpha_r + \beta_r) + 2\gamma \quad (6.45)$$

$$r_k = -\alpha_k + \beta_k + \sum_{s=k+1}^{N} (\lambda_{ks} + \rho_{ks}) + \sum_{r=1}^{k-1} (\lambda_r - \rho_{r\,k})$$

The singular vectors have the following form of

$$v_s = \sum_{\alpha,\lambda,\rho} a_{\alpha\lambda\rho} v_{\alpha\beta\gamma\lambda\rho} \quad (6.46)$$

where we fix the values of $p, r_k \ (k = 1, 2, \ldots, N)$ to make $v_{\alpha\beta\gamma\lambda\rho}$ homogeneous. Let us look for the singular vectors with non-zero central element: $\Lambda(M) \neq 0$.

Applying $Xv_s = 0$ (6.41) for $X = P_k^{\pm}$, $X = P_t$, $X = E_{-(\ell_{i} \pm \ell_{j})}v_s \ (i < j)$, in [117] was obtained the general explicit formula for the singular vectors:

$$v_s = c \left( \sum_{a=1}^{2N} G_a^{2} - 2\Lambda(M) K \right)^{p/2} \sum_{\lambda,\rho} \left( \prod_{i < s} E_{\ell_{i} + \ell_{s}}^{\lambda_{i,s}} \prod_{p < q} E_{\ell_{p} - \ell_{q}}^{\rho_{pq}} \right) v_0. \quad (6.47)$$
The **Example N=1** is trivial since \( (6.47) \) reduces just to:

\[
v_s = c \left( \sum_{a=1}^{2} G_a^2 - 2 \Lambda(M) K \right)^{p/2} v_0 .
\]

**Example: N=2**

We have \( i = 1, j = 2, \) \( r_1 = \lambda_{12} + \rho_{12}, \) \( r_2 = \lambda_{12} - \rho_{12} \) are fixed, and

\[
\lambda_{12} = (1 + h_1 + h_2)/3, \quad \rho_{12} = (h_1 - h_2 - 1)/3.
\]

Consequently the form of the singular vectors is:

\[
v_s = c \left( \sum_{a=1}^{4} G_a^2 - 2 \Lambda(M) K \right)^{p/2} \left( 1 + E_{\ell_1+\ell_2}^{\lambda_{12}} + E_{\ell_1-\ell_2}^{\rho_{12}} + E_{\ell_1+\ell_2}^{\lambda_{12}} E_{\ell_1-\ell_2}^{\rho_{12}} \right) v_0 .
\]

**Example: N=3**

1: \( i = 1, j = 2 \)

From the extra conditions listed above, it follows: \( \lambda_{13} = 0, \rho_{13} = 0. \)

2: \( i = 1, j = 2 \)

In this case we have \( \lambda_{12} = 0. \)

3: \( i = 2, j = 3 \)

From the extra conditions listed above, it follows: \( \lambda_{12} = 0, \rho_{13} = 0. \)

This leads to

\[
r_1 = \rho_{12}, \quad r_2 = \lambda_{23} + \rho_{23} - \rho_{12}, \quad r_3 = \lambda_{23} - \rho_{23}
\]

On the other hand

\[
\rho_{12} = 1 + h_1 - h_2 + r_2 - r_1, \quad \lambda_{23} = 1 + h_2 + h_3 - r_2 - r_3, \quad \rho_{23} = 1 + h_2 - h_3 + r_3 - r_2
\]

It follows

\[
\rho_{12} = (3h_1 - h_2 + 5)/7, \quad \rho_{23} = (h_1 + 2h_2 + 4)/7 - h_3/3, \quad \lambda_{23} = (h_1 + 2h_2 + 4)/7 + h_3/3.
\]

The most general form of the singular vector for \( n = 2N = 6 \) is:

\[
v_s = c \left( \sum_{a=1}^{6} G_a^2 - 2 \Lambda(M) K \right)^{p/2} \sum_{q=0, \lambda_{23}}^{\rho_{12}, \rho_{23}} E_{\ell_2+\ell_3}^{q} E_{\ell_1-\ell_2}^{r} E_{\ell_2-\ell_3}^{s} v_0 .
\]

**6.3.2. Invariant equations**

Following the general procedure of [114] to obtain the invariant operators and equations we must substitute any generator \( X \) in the expression \( (6.47) \) for the singular
vector with the right action \( \pi_R(X) \) given in [116]. Thus, we obtain the invariant equations
\[
\left( \sum_{\alpha=1}^{2N} \frac{\partial^2}{\partial x_\alpha^2} + 2M \frac{\partial}{\partial t} \right)^{p/2} \sum_{\lambda, \rho} \left( \prod_{\ell_r < s} \pi_R(E_{\ell_r + \ell_s})^{\lambda_{rs}} \prod_{p < q} \pi_R(E_{\ell_p - \ell_q})^{\rho_{pq}} \right) \psi = \psi'
\]
where \( \pi_R(E_{\ell_r + \ell_s}), \pi_R(E_{\ell_p - \ell_q}) \) are given by formulae (4.27), (4.28) of [116], \( m = -\Lambda(M) \), and \( \psi' \) belongs to the representation \( \Lambda' \) such that \( \Lambda'(M) = \Lambda(M) \), \( \Lambda'(H_k) = \Lambda(H_k) + r_k \) and \( \Lambda'(D) = \Lambda(D) + p = \Delta + p = \frac{1}{2} p + N + 1 \).

In particular for \( N = 2 \) one obtains:
\[
\left( \sum_{\alpha=1}^{4} \frac{\partial^2}{\partial x_\alpha^2} + 2M \frac{\partial}{\partial t} \right)^{p/2} \sum_{\lambda=0, \rho=0} \left( \frac{\partial}{\partial \xi_{12}} \right)^{\lambda} \left( \frac{\partial}{\partial \eta_{12}} \right)^{\rho} \psi = \psi'
\]
while for \( N = 3 \) the invariant equation is written in the form
\[
\left( \sum_{\alpha=1}^{6} \frac{\partial^2}{\partial x_\alpha^2} + 2M \frac{\partial}{\partial t} \right)^{p/2} \sum_{\eta, \rho} \left( \frac{\partial}{\partial \xi_{23}} + \eta_{13} \frac{\partial}{\partial \xi_{12}} \right)^{q} \left( \frac{\partial}{\partial \eta_{12}} + \eta_{23} \frac{\partial}{\partial \eta_{13}} \right)^{r} \left( \frac{\partial}{\partial \eta_{23}} \right)^{s} \psi = \psi'
\]

where \( q = 0, \lambda_{23} ; r = 0, \rho_{12} ; s = 0, \rho_{23} \).

### 6.4. Singular vectors and invariant equations for \( \hat{S}(2N + 1) \)

#### 6.4.1. Singular vectors

In comparison to the \( n = 2N \) case we have additional elements \( G_0 \) and \( E_{\ell_k} \). Let us introduce
\[
v_{\omega, \alpha, \gamma, \lambda, \rho} = G_0 \left\{ \prod_{r=1}^{N} (G_r^+)^{\alpha_r} \right\} \left\{ \prod_{r=1}^{N} (G_r^-)^{\beta_r} \right\} K^{\gamma} \times \left\{ \prod_{r=1}^{N} E_{\ell_r}^{\sigma_r} \right\} \left\{ \prod_{1 \leq r < s \leq N} (E_{\ell_r + \ell_s})^{\lambda_{rs}} \right\} \left\{ \prod_{1 \leq r < s \leq N} (E_{\ell_r - \ell_s})^{\rho_{rs}} \right\} v_0
\]
where \( \sigma = [\sigma_1, \sigma_2, \ldots, \sigma_N] \) is a set of non-negative integers. The action of \( M, D, H_k \) is given as in the \( n = 2N \) case, i.e., by (6.44), but the values of \( p \) and \( r_k \) now are given by:
\[
p = \omega + \sum_{r=1}^{N} (\alpha_r + \beta_r) + 2\gamma,
\]
\[
r_k = -\alpha_k + \beta_k + \sigma_k + \sum_{r=k+1}^{N} (\lambda_{kr} + \rho_{kr}) + \sum_{r=1}^{k-1} (\lambda_{rk} - \rho_{rk})
\]
The singular vectors have the form of
\[
v_s = \sum_{\omega, \alpha, \sigma, \lambda, \rho} a_{\omega, \alpha, \sigma, \lambda, \rho} v_{\omega, \alpha, \sigma, \lambda, \rho}.
\]
Substituting the right actions into the singular vectors (6.59) and performing similar computation to 6.4.2. Invariant equations

Thus, the general form of the singular vector for \( \lambda \) Under the conditions, which must be satisfied, it follows that \( \sigma_1 = r_1 = (2h_1 + 1)/3 \) and consequently, cf. [116],

\[
v_s = c \left( \sum_{a=1}^{2N+1} G_a^2 - 2\Lambda(M)K \right)^{p/2} \sum_{\sigma=0,\ell_N:\lambda: \rho} \left( E_{\ell_N}^{\sigma} \prod_{r<s} E_{\ell_r+\ell_s}^{\lambda_{rs}} \prod_{p<q} E_{\ell_p-\ell_q}^{\rho_{pq}} \right) v_0.
\]

(6.59)

Example: \( N=1 \)

We have: \( r_1 = \sigma_1 = 2h_1 - 2r_1 \). It follows that \( \sigma_1 = r_1 = (2h_1 + 1)/3 \) and consequently, cf. [116],

\[
v_s = c \left( \sum_{a=1}^{3} G_a^2 - 2\Lambda(M)K \right)^{p/2} \sum_{\sigma=0,\sigma_1} E_{\ell_1}^{\sigma} v_0
\]

(6.60)

Example: \( N=2 \)

We have: \( i = 1, j = 2 \) and \( \sigma_1 = 0, \sigma_2 = 2h_2 - 2r_2 + 1 \)

Under the conditions, which must be satisfied, it follows that \( \lambda_{12} = 0 \) and \( \rho_{12} = r_2 - r_1 + h_1 - h_2 + 1 \). On the other hand \( r_1 = \rho_{12}, \ r_2 = \sigma_2 - \rho_{12}. \) This provide

\[
\rho_{12} = (6h_1 - h_2 + 3)/5, \quad \sigma_2 = 2(2h_1 + h_2 + 1)/5
\]

Thus, the general form of the singular vector for \( n = 2N + 1 = 5 \) is:

\[
v_s = c \left( \sum_{a=1}^{5} G_a^2 - 2\Lambda(M)K \right)^{p/2} \sum_{\sigma=0,\sigma_2: \rho=0,\rho_{12}} E_{\ell_2}^{\sigma} E_{\ell_1-\ell_2}^{\rho} v_0
\]

(6.61)

6.4.2. Invariant equations

Substituting the right actions into the singular vectors (6.59) and performing similar computation to \( n = 2N \) case, we obtain the invariant equations

\[
\left( \sum_{a=1}^{2N+1} \frac{\partial^2}{\partial x_a^2} + 2M \frac{\partial}{\partial t} \right)^{p/2} \sum_{\sigma,\lambda, \rho} \left( \prod_{r<s} \pi_R(E_{\ell_r+\ell_s})^{\lambda_{rs}} \prod_{p<q} \pi_R(E_{\ell_p-\ell_q})^{\rho_{pq}} \right) \psi = \psi'
\]

(6.62)

where \( \pi_r(E_{\ell_N}), \pi_R(E_{\ell_r+\ell_s}), \pi_R(E_{\ell_p-\ell_q}) \) are given in [116], \( m = -\Lambda(M) \), and \( \psi' \) belongs to the representation \( \Lambda' \) such that \( \Lambda'(M) = \Lambda(M), \ \Lambda'(H_{\ell}) = \Lambda(H_{\ell}) + r_{\ell} \) and \( \Lambda'(D) = \Lambda(D) + p = \Delta + p = \frac{1}{2}(p + 2N + 3). \)

In particular for \( N = 1 \) one obtains [116], cf. (6.60):

\[
\left( \sum_{a=1}^{3} \frac{\partial^2}{\partial x_a^2} + 2M \frac{\partial}{\partial t} \right)^{p/2} \sum_{\sigma=0,\sigma_1} \left( \frac{\partial}{\partial \varphi_1} \right)^{\sigma} \psi = \psi',
\]

(6.63)
while for $N = 2$ the invariant equation is obtained from (6.61):

$$\left( \sum_{a=1}^{5} \frac{\partial^2}{\partial x_a^2} + 2M \frac{\partial}{\partial t} \right)^{\rho/2} \sum_{\sigma, \rho} \left( \frac{\partial}{\partial \varphi_2} + \eta_{12} \frac{\partial}{\partial \varphi_1} + \frac{\partial}{\partial \xi_{12}} \right)^{\sigma} \left( \frac{\partial}{\partial \eta_{12}} + \eta_{23} \frac{\partial}{\partial \eta_{13}} \right)^{\rho} \psi = \psi'$$

(6.64)

where $\sigma = 0, \sigma_2; \rho = 0, \rho_{12}$.

7. Non-relativistic invariant equations for $\hat{S}(3)$

7.1. Algebraic structure and actions

We consider the case $n = 3$ (which is the $N = 1$ odd case of the previous section) separately since it is most relevant for the physical applications.

**Triangular decomposition:** $\hat{S}(3) = \hat{S}(3)^+ \oplus \hat{S}(3)^0 \oplus \hat{S}(3)^-$

$$\hat{S}(3)^+ = \{G_0, G_+, G_-, K, E_+\}$$

$$\hat{S}(3)^0 = \{D, M, H\}$$

$$\hat{S}(3)^- = \{P_0, P_+, P_-, P_t, E_-\}.$$  

(7.1)

Here we have simplified the notations as follows:

$$E_+ = E'_{t_0} = -\frac{1}{\sqrt{2}} (J_{13} - i J_{23})$$

$$E_- = E'_{-t_0} = -\frac{1}{\sqrt{2}} (J_{13} + i J_{23}), \ H_1 = H = -J_{12}$$

$$G_0 = \sqrt{2} G_3, \ G_+ = G_1 + i G_2, \ G_- = G_1 - i G_2$$

$$P_0 = \sqrt{2} P_3, \ P_+ = P_1 + i P_2, \ P_- = P_1 - i P_2.$$  

(7.2)

We recall that this complexification reflects on the coordinates which read

$$x_0 = \frac{1}{\sqrt{2}} x_3 \quad x_+ = x_1 - i x_2 \quad x_- = x_1 + i x_2.$$  

(7.3)

Taking into account (2.1, 7.2) the non-vanishing commutation relations are:

$$[P_1, D] = 2P_t, \ [P_0, \pm, D] = P_{0,\pm}, \ [P_1, G_{0,\pm}] = P_{0,\pm}$$

$$[P_1, K] = D, \ [P_0, \pm, K] = G_{0,\pm}, \ [P_0, G_0] = [P_{\pm}, G_+]= 2M$$

$$[D, G_{0,\pm}] = G_{0,\pm}, \ [D, K] = 2K$$

$$[P_0, E_{\pm}] = -P_\mp, \ [P_{\pm}, E_{\pm}] = P_0, \ [P_\pm, H] = \pm P_\mp$$

$$[G_0, E_{\pm}] = -G_{\mp}, \ [G_{\pm}, E_{\pm}] = G_0, \ [G_{\pm}, H] = \pm G_{\mp}$$

$$[E_{\pm}, H] = \mp E_{\mp}, \ [E_+, E_-] = -H.$$  

(7.4)
**Gauss decomposition**: Let \( g \in \mathcal{S} \) be the element of the Schrödinger group, than its triangular decomposition is 
\[
g = g_+ g_0 g_-
\]
where 
\[
g_+ = e^{\tau_0 G_0} e^{x_+ G_+} e^{x_- G_-} e^{t K} e^{\varphi_+ E_+}
g_0 = e^{h M} e^{h H}
g_- = e^{\tau_0 P_0} e^{x_+ P_+} e^{x_- P_-} e^{k P} e^{\varphi_- E_-}.
\]
(7.5)

**Remark**: The explicit expressions for \( g_+, g_0, g_- \) can be easily obtained from the following relations valid in the matrix representation (2.8) (in terms of \( 8 \times 8 \) matrices):

\[
M^q = 0, \quad q > 1
\]
\[
D^{2q+1} = D, \quad H^{2q+1} = H, \quad q \geq 0 \tag{7.6}
\]
\[
D^{2q} = D^2, \quad H^{2q} = H^2, \quad q > 0
\]
\[
G^q_0 = G^q_0 = G^q_- = G_0 G_+ = G_0 G_- = 0, \quad q > 1 \tag{7.7}
\]
\[
P^q_0 = P^q_+ = P^q_- = 0, \quad q > 1
\]
\[
P^q_0 = 0, \quad q > 1
\]
\[
K^q = 0, \quad q > 1
\]
\[
E^q_+ = 0, \quad q > 2
\]
\[
E^q_- = 0, \quad q > 2, \tag{7.8}
\]

where in the above expressions \( q \) is an integer number. Furthermore the matrix realization of the algebra \( \mathcal{S}(3) \) may also be used to obtain the left(right) action.

**Left action:**

- **Action of \( \mathcal{S}(3)^0 \).** Using the commutation relations and matrix realization of \( \mathcal{S}(3) \), up to terms quadratic in \( \tau \) it follows 
  \[
e^{-\tau M} g_+ = g_+ e^{-\tau M}
e^{-\tau D} g_+ = \prod_{a=1}^{3} \exp \left( x_a e^{-\tau G_a} \right) \exp \left( t e^{-2\tau K} \right) e^{\varphi_+ E_+} e^{-\tau D}
e^{-\tau H} g_+ = e^{x_0 G_0} e^{x_+ e^{-\tau G_+}} e^{x_- e^{-\tau G_-}} e^{\varphi_+ e^{-\tau E_+}} e^{-\tau H}.
\]
  (7.10)

The above expressions provide the following representation by the left action:

\[
\pi_L(M) = -\Lambda(M)
\]
\[
\pi_L(D) = -\Lambda(D) - \sum_{a=1}^{3} x_a \frac{\partial}{\partial x_a} - 2t \frac{\partial}{\partial t}
\]
\[
\pi_L(H) = -\Lambda(H) + x_+ \frac{\partial}{\partial x_+} - x_- \frac{\partial}{\partial x_-} - \varphi_+ \frac{\partial}{\partial \varphi_+}.
\]
(7.11)
Action of $\hat{S}(3)^+$

\[
e^{-\tau G_0} g_+ = e^{(x_0 - \tau)G_0} e^{x_+G_+} e^{x_-G_-} e^{tK} e^{\varphi+E_+}
\]
\[
e^{-\tau G_+} g_+ = e^{x_0G_0} e^{(x+ - \tau)G_+} e^{x_-G_-} e^{tK} e^{\varphi+E_+}
\]
\[
e^{-\tau G_-} g_+ = e^{x_0G_0} e^{x_+G_+} e^{(x- - \tau)G_-} e^{tK} e^{\varphi+E_+}
\]
\[
e^{-\tau K} g_+ = e^{x_0G_0} e^{x_+G_+} e^{x_-G_-} e^{(t- \tau)K} e^{\varphi+E_+}
\]
\[
e^{-\tau E_+} g_+ = e^{(x_0 + \tau x_0)G_0} e^{x_+G_+} e^{(x- - \tau x_0)G_-} e^{tK} e^{(\varphi + \tau)E_+}
\]

(7.12)

provide

\[
\pi_L(G_0) = -\frac{\partial}{\partial x_0}, \quad \pi_L(G_+) = -\frac{\partial}{\partial x_+}, \quad \pi_L(G_-) = -\frac{\partial}{\partial x_-}
\]
\[
\pi_L(K) = -\frac{\partial}{\partial t}, \quad \pi_L(E_+) = x_+ \frac{\partial}{\partial x_0} - x_0 \frac{\partial}{\partial x_+} - \frac{\partial}{\partial \varphi_+}
\]

(7.13)

Action of $\hat{S}(3)^-$:

\[
\pi_L(P_0) = -\Lambda(M)x_0 - t \frac{\partial}{\partial x_0}
\]
\[
\pi_L(P_+) = -\Lambda(M)x_+ - t \frac{\partial}{\partial x_+}
\]
\[
\pi_L(P_-) = -\Lambda(M)x_- - t \frac{\partial}{\partial x_-}
\]
\[
\pi_L(P_\tau) = -\Lambda(D)t - \frac{1}{4} \Lambda(M)(x_0^2 + x_+^2 + x_-^2) - t \sum_{a=1}^3 x_a \frac{\partial}{\partial x_a} - t^2 \frac{\partial}{\partial t}
\]
\[
\pi_L(E_-) = -\Lambda(H)\varphi_+ + x_- \frac{\partial}{\partial x_0} - x_0 \frac{\partial}{\partial x_+} - \frac{1}{4} \varphi_+^2 \frac{\partial}{\partial \varphi_+}
\]

(7.14)

Right action:

- By definition

\[
\pi_R(M) = \Lambda(M), \quad \pi_R(D) = \Lambda(D), \quad \pi_R(H) = \Lambda(H)
\]

(7.15)

- The right action of $\hat{S}(3)^+$ is given by (again up to terms quadratic in $\tau$):

\[
g_+ e^{\tau G_0} = e^{(x_0 + \tau)G_0} e^{x_+G_+} e^{(x_- + \tau \varphi_+ - \tau G_-)G_-} e^{tK} e^{\varphi_+ E_+}
\]
\[
g_+ e^{\tau G_+} = e^{(x_0 + \tau \varphi_+)G_0} e^{x_+G_+} e^{(x_- - \tau \varphi_+)G_-} e^{tK} e^{\varphi_+ E_+}
\]
\[
g_+ e^{\tau G_-} = e^{x_0G_0} e^{x_+G_+} e^{(x_- + \tau G_-)G_-} e^{tK} e^{\varphi_+ E_+}
\]
\[
g_+ e^{\tau K} = e^{x_0G_0} e^{x_+G_+} e^{x_-G_-} e^{(t + \tau K)} e^{\varphi_+ E_+}
\]
\[
g_+ e^{\tau E_+} = e^{x_0G_0} e^{x_+G_+} e^{x_-G_-} e^{tK} e^{(\varphi_+ + \tau)E_+}
\]

(7.16)
which provides
\[ \pi_R(G_0) = \frac{\partial}{\partial x_0} + \varphi + \frac{\partial}{\partial x} \]
\[ \pi_R(G_+) = \frac{\partial}{\partial x} - \varphi + \frac{\partial}{\partial x_0} - \frac{1}{2} \varphi^2 \frac{\partial}{\partial x} \]
\[ \pi_R(G_-) = \frac{\partial}{\partial x}, \quad \pi_R(K) = \frac{\partial}{\partial t}, \quad \pi_R(E_+) = \frac{\partial}{\partial \varphi}. \]  
(7.17)

7.2. **Singular vectors**

In this case a basis in the Verma module \( V^\Lambda \cong U(\hat{S}(3)^+) \) is

\[ v_{\omega,\alpha,\beta,\gamma,\sigma} = G_0^\omega G_+^\alpha G_-^\beta K^\gamma E_+^\sigma v_0, \]  
(7.18)

where \( \omega, \alpha, \beta, \gamma, \sigma \) are nonnegative integers. Using the properties of lowest weight vector

\[ M v_0 = \Lambda(M) v_0 = M v_0 \]
\[ D v_0 = \Lambda(D) v_0 = \Delta v_0 \]
\[ H v_0 = \Lambda(H) v_0 = -h v_0, \]  
(7.19)

we first calculate the action of the Cartan generators on the basis (7.18)

\[ M v_{\omega,\alpha,\beta,\gamma,\sigma} = \Lambda(M) v_{\omega,\alpha,\beta,\gamma,\sigma} \]
\[ D v_{\omega,\alpha,\beta,\gamma,\sigma} = (p + \Delta) v_{\omega,\alpha,\beta,\gamma,\sigma}, \quad p = \omega + \alpha + \beta + 2\gamma \]
\[ H v_{\omega,\alpha,\beta,\gamma,\sigma} = (r - h) v_{\omega,\alpha,\beta,\gamma,\sigma}, \quad r = -\alpha + \beta + \sigma. \]  
(7.20)

In the same manner we calculate the action of generators from \( \hat{S}(3)^- \)

\[ P_0 v_{\omega,\alpha,\beta,\gamma,\sigma} = 2\omega \Lambda(M) v_{\omega-1,\alpha,\beta,\gamma+1,\sigma} + \gamma v_{\omega+1,\alpha,\beta,\gamma-1,\sigma} \]  
(7.21)

Next we apply the requirement \( v_{\omega,\alpha,\beta,\gamma,\sigma} \) to be homogeneous, from which follows that \( r \) and \( p \) are to be fixed. Thus, from five variables \( \omega, \alpha, \beta, \gamma, \sigma \) only three are independent. We choose \( \omega, \alpha, \sigma \) as independent variables, which means that the singular vectors must have the form

\[ v_s = \sum_{\omega,\alpha,\sigma} a_{\omega,\alpha,\sigma} v_{\omega,\alpha,\beta,\gamma,\sigma}, \]  
(7.22)

the sum obeying the following:

The replacement \( \alpha \to \alpha \pm 1 \) causes the change of dependent variables

\[ \beta \to \beta \pm 1, \quad \gamma \to \gamma \mp 1 \]
The replacement $\omega \rightarrow \omega \pm 2$ causes the change $\gamma \rightarrow \gamma \mp 1$ and do not reflect on the others.

Next we use the definition of a singular vector

$$ Hv_s = \Lambda'(H)v_s , \ H \in \hat{S}(3)^0 ; \ Xv_s = 0 ; \ X \in \hat{S}(3)^-, $$

(7.23)

to obtain

$$ P_0 v_s = 0 $$

(1)

$$ 2\omega \Lambda(M)a_{\omega \sigma}\nu_{\omega -1,\alpha \beta \sigma} + \gamma a_{\omega \sigma}\nu_{\omega +1,\alpha \beta,\gamma -1,\sigma} = $$

$$ = (2\omega \Lambda(M)a_{\omega \sigma} + (\gamma + 1)a_{\omega -2,\alpha \sigma}) \nu_{\omega -1,\alpha \beta \gamma \sigma} = 0 $$

$$ a_{\omega \sigma} = -\frac{\gamma + 1}{2\Lambda(M)\omega}a_{\omega -2,\alpha \sigma} $$

(7.24)

$$ P_+ v_s = 0 $$

(2)

$$ a_{\omega \sigma} = -\frac{\gamma + 1}{2\Lambda(M)\beta}a_{\omega,\alpha -1,\sigma} $$

(7.25)

$$ P_- v_s = 0 $$

(3)

$$ a_{\omega \sigma} = -\frac{\gamma + 1}{2\Lambda(M)\alpha}a_{\omega,\alpha -1,\sigma} $$

(7.26)

$$ P_t v_s = 0 $$

(4)

$$ (\omega + 1)(\omega + 2)\Lambda(M)a_{\omega +2,\alpha \sigma} + 2(\alpha + 1)^2\Lambda(M)a_{\omega,\alpha +11,\beta -1,\sigma} + \gamma (p - \gamma + \Delta - 1)a_{\omega \alpha \sigma} = 0 $$

(7.28)

$$ E_- v_s = 0 $$

(5)

$$ (\omega a_{\omega,\alpha -1,\sigma} - \alpha a_{\omega -2,\alpha \sigma}) \nu_{\omega -1,\alpha \beta -1,\gamma +1,\sigma} + \frac{1}{2}\sigma(\sigma - 1 - 2h)a_{\omega \alpha \sigma}\nu_{\omega \alpha \beta \gamma \sigma -1} = 0, $$

(7.27)

We note that from (7.24) the substitution $\omega = 1$ leads to $a_{\omega \sigma} = 0$ for $\omega$ odd, while from (7.25, 7.26) follows that $\alpha = \beta$. Further substitution of (7.24, 7.25) in (7.27) gives

$$ \gamma (p/2 + \Delta - 5/2) = 0 $$

$$ p = \omega + 2\alpha + 2\gamma = (2\Delta + 5)/2. $$

(7.28)

Then $p$ must be also even.

Solving the above recurrence relations was done in [116]. The result for the most general form of the singular vector is:

$$ v^{p,\sigma}_s = c \left( \frac{1}{2}G_0^2 + G_+ - 2\Lambda(M)K \right)^{p/2} E^\sigma_+ v_0, $$

(7.29)

where $\sigma$ has two values: $\sigma = 0, 1 + 2h \in \mathbb{N}$, $p = 5 - 2\Delta \in 2\mathbb{N}$, $c$ arbitrary nonzero constant.

Furthermore, one can verify that $v^\sigma_0 = E^\sigma_+ v_0 = v^{0,\sigma}_s$ is also a singular vector. Really the relations

$$ P_0 v^\sigma_s = P_+ v^\sigma_s = P_- v^\sigma_s = P_t v^\sigma_s = 0 $$

(7.30)
are automatically satisfied. The last condition
\[ E - v^\sigma_s = \frac{1}{2} \sigma (\sigma - 1 - 2h) v^\sigma_s = 0 \]  
(7.31)
is satisfied again for \( \sigma = 1 + 2h = 1 - 2\Lambda(H) \).

Thus, \( v^p,\sigma_s \) is a composite singular vector and we have a quartet to Verma modules with weights \( \Lambda = (\Lambda(M) = M, \Lambda(D) = \Delta, \Lambda(H) = -h), \) \( \Lambda_p = (M, p + \Delta, -h), \) \( \Lambda'_s = (M, \Delta, \sigma - h), \) \( \Lambda_{p,\sigma} = (M, p + \Delta, \sigma - h) \), which can be given in an embedding commutative diagram:

\[
\begin{array}{ccc}
V^\Lambda & \to & V^{\Lambda_p} \\
\downarrow & & \downarrow \\
V^{\Lambda'_s} & \to & V^{\Lambda_{p,\sigma}} 
\end{array}
\]  
(7.32)
where each arrow points to the embedded module. By construction, \( V^\Lambda \) has singular vectors \( v^p,0_s \) and \( v^0_s,\sigma \), \( V^{\Lambda'_s} \) has singular vector \( v^p,0_s \), \( V^{\Lambda_p} \) has singular vector \( v^0_s,\sigma \).

All above properties follow from the following relations:
\[
\begin{align*}
M v^p,\sigma_s &= \Lambda(M) v^p,\sigma_s \\
D v^p,\sigma_s &= (\Lambda(D) + p) v^p,\sigma_s \\
H v^p,\sigma_s &= (\Lambda(H) + \sigma) v^p,\sigma_s
\end{align*}
\]  
(7.33)

### 7.3. Non-relativistic equations

The formula (7.29) provides (taking the right action) the following form of invariant differential operator and correspondingly invariant equation
\[
D_{p,\sigma} \psi(t, x_1, x_2, x_3; \varphi_+) = \psi'(t, x_1, x_2, x_3; \varphi_+). 
\]  
(7.34)
where \( \psi \) belongs to the representation \( C^\Lambda \) characterized by \( \Lambda \), while \( \psi' \) belongs to the representation \( C^{\Lambda_{p,\sigma}} \) characterized by \( \Lambda_{p,\sigma} \).

Of course, analogously to the Verma module embedding picture (7.32) there is a quartet commutative diagram:

\[
\begin{array}{ccc}
C^\Lambda & \to & C^{\Lambda_p} \\
\downarrow & & \downarrow \\
C^{\Lambda'_s} & \to & C^{\Lambda_{p,\sigma}} 
\end{array}
\]  
(7.35)
where each vertical arrow depicts the differential operator \( D_{0,\sigma} = \left( \frac{\partial}{\partial \varphi_+} \right)^\sigma \), while each horizontal arrow depicts the differential operator \( D_{p,0} \).
8. \(q\)-Schrödinger algebra

8.1. \(q\)-deformation of the Schrödinger algebra

In this Section we review [118]. For other approaches to \(q\)-deformations of the Schrödinger algebra we refer to [162–183].

We use the following \(q\)-number notations:

\[
[a]_q = \frac{q^a - q^{-a}}{q - q^{-1}}, \quad [a]_{q^{1/2}}' = [a]_{q^{1/2}} = \frac{q^{a/2} - q^{-a/2}}{q^{1/2} - q^{-1/2}} = [a/2]_q \tag{8.1}
\]

and similarly for any diagonal operator instead of \(a\).

Now we construct a \(q\)-deformation \(\hat{S}_q(1)\) of the Schrödinger algebra under the following conditions:

- 1. A realization of the generators \(P_t, P, G, K\) in terms of \(q\)-difference operators and multiplication operators should be available.
- 2. In the limit \(q \to 1\) we should have the classical relations (3.6).
- 3. The subalgebra structure should be preserved by the deformation and especially the \(d\)-deformed \(sl(2, \mathbb{C})\) subalgebra generated by \(D, K, P_t\) should coincide with the usual Drinfeld-Jimbo deformation \(U_q(sl(2, \mathbb{C}))\).

With these conditions we get for \(\hat{S}_q(1)\) the following nontrivial relations instead of (3.6):

\[
\begin{align*}
P_t \ G - q \ G \ P_t &= P_x \tag{8.2a} \\
[P_x, K] &= G \ q^{-D} \tag{8.2b} \\
[D, G] &= G \tag{8.2c} \\
[D, P_x] &= -P_x \tag{8.2d} \\
[D, P_t] &= -2P_t \tag{8.2e} \\
[D, K] &= 2K \tag{8.2f} \\
[P_t, K] &= [D]_q \tag{8.2g} \\
P_x \ G - q^{-1} \ G \ P_x &= m \tag{8.2h} \\
P_t \ P_x - q^{-1} \ P_x \ P_t &= 0 \tag{8.2i}
\end{align*}
\]

Conditions 2. and 3. can now be checked directly, (8.2e,f,g) are the standard commutation relations of the Drinfeld-Jimbo deformation \(U_q(sl(2, \mathbb{C}))\). Moreover we obtain a \(q\)-deformed centrally extended Galilei subalgebra generated by \(P_t, P, G\). The deformation is a "mild" one, in the sense that commutators are turned into \(q\)-commutators, cf. (8.2a,h,i) and it differs from the Galilei algebra \(q\)-deformation given in [163], which is not a surprise taking into account that the latter is not a subalgebra of a \((q\)-deformed) Schrödinger algebra. Condition 1. will be discussed later.

The commutation relations (8.2) are graded as the undeformed ones (3.6) if we define the grading as in (2.5).
For real \( q \) the commutation relations (8.2) are preserved also by the involutive antiautomorphism (2.6) supplemented by the condition \( \omega(q) = q \). With this conjugation the subalgebra generated by \( P_t + K, i(P_t - K), D \), (the fixed points of \( \omega \)), is \( U_q(sl(2, \mathbb{R})) \).

### 8.2. Lowest weight modules of \( \hat{\mathcal{S}}_q(1) \)

We consider lowest weight modules (LWM) of \( \hat{\mathcal{S}}_q(1) \), in particular, Verma modules, in complete analogy with the undeformed case \( q = 1 \). In particular, the lowest weight vector fulfills (5.4). The Verma module \( V^\Delta \) is given explicitly by

\[
V^\Delta = U_q(S^+) \otimes v_0, \quad \text{where} \quad U_q(S^+) \quad \text{is the} \ q \text{-deformed universal enveloping algebra of} \ S^+ = S(1)^+. \quad \text{Clearly,} \quad U_q(S^+) \quad \text{has the basis elements} \quad p_{k,\ell} = G^k K^\ell.
\]

The basis vectors of the Verma module are \( v_{k,\ell} = p_{k,\ell} \otimes v_0 \), (with \( v_{0,0} = v_0 \)). The action of the \( q \)-Schrödinger algebra on this basis is derived easily from (8.2):

\[
D \ v_{k,\ell} = (k + 2\ell + \Delta) \ v_{k,\ell} \tag{8.3a}
\]

\[
G \ v_{k,\ell} = v_{k+1,\ell} \tag{8.3b}
\]

\[
K \ v_{k,\ell} = v_{k,\ell+1} \tag{8.3c}
\]

\[
P_x v_{k,\ell} = q^{\frac{1-k}{2}} M [k]_q \ v_{k-1,\ell} + q^{1-\Delta-\ell-k} [\ell]_q v_{k+1,\ell-1} \tag{8.3d}
\]

\[
P_t \ v_{k,\ell} = [\ell]_q [k + \ell - 1 + \Delta]_q \ v_{k,\ell-1} + M \frac{[k]_q [k-1]_q}{[2]_q^2} v_{k-2,\ell} \tag{8.3e}
\]

For the derivation of (8.3) the following relations (which follow from (8.2)) are usefull:

\[
P_x G^k - q^{-k} G^k P_x = M q^{(1-k)/2} [k]_q^2 G^{k-1} \tag{8.4a}
\]

\[
P_x K^\ell - K^\ell P_x = q^{1-\ell} [\ell]_q G K^{\ell-1} q^{-D} \tag{8.4b}
\]

\[
P_t G^k - q^k G^k P_t = [k]_q G^{k-1} P_x + M \frac{[k]_q [k-1]_q}{[2]_q} G^{k-2} \tag{8.4c}
\]

\[
P_t K^\ell - K^\ell P_t = [\ell]_q K^{\ell-1} D + \ell - 1)_q \tag{8.4d}
\]

Because of (8.3)a we notice that the Verma module \( V^\Delta \) can be decomposed in homogeneous subspaces w.r.t. grading operator \( D \) as in (5.6).

Next we analyze the reducibility of \( V^\Delta \) through singular vectors. The considerations are exactly similar to the undeformed case. All possible singular vectors were given explicitly in [118] as follows. Fix the grade \( p > 0 \) and denote the singular vector as \( v^p_k \). Consider the case of \( \text{even} \) grade, \( p \in 2\mathbb{N} \). Since \( v^p_k \in V^\Delta_p \) we have:

\[
v^p_k = \sum_{\ell=0}^{p/2} a_\ell \ v_{p-2\ell,\ell} = Q^p(G, K) \otimes v_0, \quad \text{\( p \) \text{even}} \tag{8.5}
\]

Applying (5.7) we obtain that a singular vector exists only for \( \Delta = -\frac{3p}{2} \) (as for \( q = 1 \)) and is given explicitly for arbitrary \( q \) by the formula:

\[
v^p_k = a_0 \sum_{\ell=0}^{\ell} \left( -M[2]_q^\ell \right)^{\ell} v_{p-2\ell,\ell} = a_0 \left( G^2 - M [2]_q^\ell K \right)^{\ell} \otimes v_0
\]
\[
Q^p(G, K) = a_0 \left( G^2 - M \left[ 2 \right]_q^\prime K \right)^{\frac{p}{q}} \tag{8.6}
\]

where
\[
\binom{p}{s}_q = \frac{[p]_q!}{[s]_q![p-s]_q!}, \quad [n]_q! = [n]_q[n-1]_q \ldots [1]_q \tag{8.7}
\]

For odd grade there are no singular vectors as for \( q = 1 \).

To analyze the consequences of the reducibility of our Verma modules we take the subspace of \( V^{(3-p)/2} \):
\[
I^{(3-p)/2} = U_q(S^+) \nu^p_n \tag{8.8}
\]

It is invariant under the action of the \( q \)-deformed Schrödinger algebra, and is isomorphic to a Verma module \( V^{\Delta'} \) with shifted weight \( \Delta' = \Delta + p = (p+3)/2 \). The latter Verma module has no singular vectors.

Let us denote by \( L^{(3-p)/2} \) the factor–module \( V^{(3-p)/2}/I^{(3-p)/2} \) and by \( |p\rangle \) the lowest weight vector of \( L^{(3-p)/2} \). As a consequence of (5.7) and (8.6) \( |p\rangle \) satisfies:
\[
P_x |p\rangle = 0 \tag{8.9a}
\]
\[
P_t |p\rangle = 0 \tag{8.9b}
\]
\[
\sum_{\ell=0}^\frac{p}{2} ( - m \left[ 2 \right]_q^\prime )^\ell \binom{\frac{p}{2}}{\ell}_q G^{p-2\ell} K^\ell |p\rangle = 0 \tag{8.9c}
\]

Now from (8.9c) we see that:
\[
K^{p/2} |p\rangle = - \sum_{\ell=0}^{p/2-1} \frac{1}{(-m\left[ 2 \right]_q^\prime)^{p/2-\ell}} \binom{p/2}{\ell}_q G^{p-2\ell} K^\ell |p\rangle \tag{8.10}
\]

By a repeated application of this relation to the basis one can get rid of all powers \( \geq p/2 \) of \( K \). Thus the basis of \( \mathcal{L}^{(3-p)/2} \) will be a singleton basis for \( p = 2 \), and a quasi–singleton basis for \( p \geq 4 \):
\[
dim V^{(3-p)/2}_n = 1, \quad \text{for } n = 0, 1 \text{ or } n \geq p \tag{8.11}
\]

and it is given by:
\[
\nu^p_{k\ell} \equiv G^k K^\ell |p\rangle, \quad p \in 2\mathbb{N}, \ k, \ell \in \mathbb{Z}_+, \ \ell \leq p/2 - 1, \ \Delta = \frac{3-p}{2} \tag{8.12}
\]

The transformation rules of this basis are (8.3) except (8.3c) for \( \ell = p/2 - 1 \), when we have:
\[
K \nu^p_{k,p/2-1} = - \sum_{s=0}^{p/2-1} \frac{1}{(-m\left[ 2 \right]_q^\prime)^{p/2-s}} \binom{p/2}{s}_q \nu^p_{k+p-2s,s} \tag{8.3c'}
\]

From the transformation rules we see that \( \mathcal{L}^{(3-p)/2} \) is irreducible. In the simplest case \( p = 2 \) the irrep \( \mathcal{L}^{1/2} \) is also an irrep of the \( q \)-deformed centrally extended Galilean subalgebra \( \hat{\mathfrak{g}}_q(1) \) generated by \( P_x, P_t, G \).

Hence, the complete list of the irreducible lowest weight modules over the \( q \)-deformed centrally extended Schrödinger algebra is given by [118]:

\[
Q^p(G, K) = a_0 \left( G^2 - M \left[ 2 \right]_q^\prime K \right)^{\frac{p}{q}} \tag{8.6}
\]
8.3. Vector–field realization of $\hat{S}_q(1)$ and generalized $q$-deformed heat equations

Let us introduce the "number" operator $N_y$ for the coordinate $y = x, t$, i.e.,

$$N_y y^k = k y^k, \quad (8.13)$$

and the $q$-difference operators $D_y, D'_y$, which admit a general definition on a larger domain than polynomials, but on polynomials are well defined as follows:

$$D_y = \frac{1}{y}[N_y]_q, \quad D'_y = \frac{1}{y[\frac{1}{2}]_q} [N_y]_{\frac{1}{2}}_q = \frac{1}{y}[N_y]'_q, \quad (8.14a)$$

so that for any suitable function $f$ we obtain as a consequence of (8.13):

$$D_y f(y) = \frac{f(qy) - f(q^{-1}y)}{y(q-q^{-1})}, \quad (8.15a)$$

$$D'_y f(y) = \frac{f(q^{\frac{1}{2}}y) - f(q^{-\frac{1}{2}}y)}{y(q^{\frac{1}{2}} - q^{-\frac{1}{2}})}. \quad (8.15b)$$

For $q \to 1$ one has: $N_y \to y \partial_y, \ D_y, D'_y \to \partial_y$.

With this notation there exists [118] a five-parameter realization of (8.2) via $q$-difference operators (or vector–field realization for short):

$$P_t = q^{c_1-1} \mathcal{D}_t \ q^{(c_1-c_4)N_t+(1-c_4)N_x} \quad (8.16a)$$

$$P_x = q^{c_2} \mathcal{D}'_x \ q^{-(c_4+c_5+\frac{3}{2})N_x} \quad (8.16b)$$

$$D = 2N_t + N_x + \Delta \quad (8.16c)$$

$$G = q^{c_5-c_1-c_4+c_5} t \mathcal{D}'_x \ q^{(c_5-c_4)N_t+(c_5+c_4-\frac{3}{2})N_x}$$

$$+ q^{c_1-3c_5-c_3-\frac{1}{2}} mx \ q^{c_4N_t+(c_5+1)N_x} \quad (8.16d)$$

$$K = q^{-c_1+c_5-1-\Delta} t^2 \mathcal{D}_t \ q^{(c_5-1)N_t+c_4N_x}$$

$$+ q^{-c_1+c_5-1-\Delta} tx \mathcal{D}_x \ q^{(c_5-2)N_t+(c_4-1)N_x}$$

$$+ q^{-c_1+c_5-1} [\Delta]_q t \ q^{c_5N_t+c_4N_x}$$

$$+ q^{-2c_2-3c_3-\frac{3}{2}\Delta} \ q^{(1)_{c_3}^2} \ q^{2(c_4-1)N_t-2(c_5+1)N_x} \quad (8.16e)$$

where $c_1, c_2, c_3, c_4, c_5$ are arbitrary parameters. (There might be other vector–field realizations that are not equivalent to the one just given.)

For $q = 1$ we recover the standard vector–field realization of $\hat{S}(1)$ (3.12).

Our realization (8.16) may be used to construct a polynomial realization of the irreducible lowest weight modules considered in Section 3. For that case we represent...
the lowest weight vector by the function 1. Indeed, the constants in (8.16) are chosen so that (5.4) is satisfied:

\[ D1 = \Delta, \quad P_x 1 = 0, \quad P_t 1 = 0 \quad (8.17) \]

Applying the basis elements \( p_{k,\ell} = G_k^{\ell}K_\ell \) of the universal enveloping algebra \( U_q(S^+) \) to 1 we get polynomials in \( x, t \) which will be denoted by \( f_{k,\ell} \equiv p_{k,\ell} 1 \). For the explicit expressions we refer to [118]. There it was also shown that the basis \( f_{k,\ell} \) is a realization of the irreducible lowest weight representations of \( \hat{S}_q(1) \) listed at the end of the previous section. Indeed, there is 1-to-1 correspondence between the states \( v_{k,\ell} \) of the Verma modules over \( \hat{S}_q(1) \) and the polynomials \( f_{k,\ell} \). The irreducible lowest weight representations of \( \hat{S}_q(1) \) are factor–modules of Verma modules, with factorization over the invariant subspaces generated by singular vectors. This statement is trivial if there is no singular vector. When a singular vector exists, i.e., for the representations \( V^{(3-p)/2} \), we first obtain a \( q \)-difference operator by substituting in \( Q_p(G, K) \) (cf. (8.5), (8.6)) each generator with its vector–field realization. For the irreducibility of \( L^{(3-p)/2} \) it is enough to show that the \( q \)-difference operator \( Q_p(G, K) \) vanishes identically when applied to 1. This contains more information as \( Q_p(G, K) \) gives also a \( q \)-difference equation invariant under the action of \( \hat{S}_q(1) \). Because of this invariance the solutions of this equation are elements of \( L^{(3-p)/2} \). Thus we have an infinite family of \( q \)-difference equations, the family members being labelled by \( p \in 2\mathbb{N} \), i.e., we have one equation for each representation space \( V^{(3-p)/2} \). These equations may be called generalized \( q \)-deformed heat equations (\( m \) real) or generalized \( q \)-deformed Schrödinger equations (\( m \) imaginary). The case \( p = 2 \) is a \( q \)-difference analog of the ordinary heat/Schrödinger equation.

Before making the last example explicit we make a choice of constants in (8.16) and set for simplicity \( c_1 = c_2 = c_3 = c_4 = c_5 = 0 \) so that to work with simpler expressions for the generators:

\[ P_t = D_t \ q^{N_t+N_x} \quad (8.18a) \]
\[ P_x = D_x \ q^{\frac{1}{2}N_x} \quad (8.18b) \]
\[ D = 2N_t + N_x + \Delta \quad (8.18c) \]
\[ G = t \ D'_t \ q^{-\frac{1}{2}N_x} + q^{-\frac{1}{2}} M x \ q^{-N_x} \quad (8.18d) \]
\[ K = q^{-\Delta-1} t^2 \ D_t \ q^{-N_t} + q^{-\Delta-1} tx \ D_x \ q^{-2N_t-N_x} + q^{-1} [\Delta] q \ t + q^{-\Delta+\frac{2}{2}} [\frac{1}{2}] q \ M x^2 q^{-2N_t-2N_x} \quad (8.18e) \]

The operator \( S_q = Q = G^2 - [2]_q' M \ K \) determining the singular vectors
reads:
\[
S_q = t^2 q^\frac{1}{2} \left( D_x'^2 q^{-N_x} - q^{-\Delta - \frac{3}{2}} [2]_q'' M D_t q^{-N_t} \right) + \\
+ Mtx D_x' q^{-2N_x} \left( [2] q'' - (1 + q^{N_x}) q^{-\Delta - 1 - 2N_t} \right) q^{-\frac{3}{2}N_x} + \\
+ q^{-1} M t \left( q^{-2N_x} - [2\Delta]_q' \right) + \\
+ q^{-2} M^2 x^2 \left( 1 - q^{-\Delta + \frac{1}{2} - 2N_t} \right) q^{-2N_x}
\]
(8.19)

which for \( q = 1 \) gives:
\[
S = t^2 \left( \partial_x^2 - 2M \partial_t \right) + M t \left( 1 - 2\Delta \right)
\]
(8.20)

Hence we interpret for \( \Delta = \frac{1}{2} \) (which corresponds to the lowest singular vector \( (p = 2) \)) the equation \( S_q f = 0 \) as a \( q \)-deformed heat/Schrödinger equation as we motivated in the Introduction. The explicit form of this equation is:
\[
S_q f = 0
\]
\[
S_q = t^2 q^\frac{1}{2} \left( D_x'^2 q^{-N_x} - q^{-2} [2]_q'' M D_t q^{-N_t} \right) + \\
+ Mtx D_x' q^{-2N_x} \left( [2] q'' - (1 + q^{N_x}) q^{-\frac{3}{2} - 2N_t} \right) q^{-\frac{3}{2}N_x} - \\
- \lambda q^{-1} Mtx D_x q^{-N_x} + \lambda q^{-2} M^2 x^2 t D_t q^{-N_t - 2N_x}
\]
(8.21)

where \( \lambda \equiv q - q^{-1} \).

This is the proposal of [118] for a \( q \)-deformed heat equation. For \( q \rightarrow 1 \ (\lambda \rightarrow 0) \) this equation leads to the ordinary heat/Schrödinger equation.

**Remark:** We note that there exists another \( q \)-deformation of the vector–field realization of \( \tilde{S}(1) \) given by Floreanini and Vinet [164]. They start with a special \( q \)-deformed heat equation and look for a \( q \)-symmetry algebra on its solution variety. The resulting \( q \)-deformation of the Schrödinger algebra in [164], which we call \( on \ shell \) deformation is different from the one of [118] and is valid only on the solutions of the \( q \)-deformed heat equation under consideration. 

9. Difference analogues of the free Schrödinger equation

9.1. Motivations

In this Section we review [119]. For other approaches to difference equations with Schrödinger algebra symmetry we refer to [164, 184–186].

The time evolution of physical systems is generically described via (partial) differential equations, especially via the Schrödinger equation for the case of non-relativistic quantum mechanics. The use of such equations, however, is an idealization, because the infinitesimal structure inherent in the definition of differential
operators,
\[
\partial_x f(x_0) = \lim_{\zeta \to 0} \frac{f(x_0 + \zeta) - f(x_0 - \zeta)}{2\zeta},
\]
(9.1)
cannot be reproduced in physical measurements. A realistic measurement of a "differential quantity" such as velocity etc. actually involves measurements at two distinct points in the physical space-time, i.e., it is based on measurements at \(x\) and \(x + \zeta\) with \(\zeta\) finite and not infinitesimal. Hence, in a realistic physical setting, only finite difference quotients of non-infinitesimal quantities should occur. The physical space-time could be either continuous or discrete for the space and/or time coordinates. On very small length scales (Planck scale) it is likely that some "grained" or "lattice" structure is more appropriate as an "arena" for physical theories than a continuous space-time.

Apart from such more fundamental considerations there are also practical reasons for the use of finite difference equations. Generically, one has to use approximations in order to get a quantitative description of a physical system, and sometimes lattice models are useful in this context. In these models the objects of the theory are allowed to "live" only on discrete points of a lattice. It is obvious that in this setting finite difference operators involving the lattice spacing are basic quantities.

There are various types of (finite) difference operators. The usual choice, used also in the context of lattices, are operators of additive type:
\[
D_x f(x) = \frac{f(x + \zeta) - f(x - \zeta)}{2\zeta},
\]
(9.2)
i.e., functions at \(x \pm \zeta\) are compared. (Note, that \(\lim_{\zeta \to 0} D_x = \partial_x\).) Another type which was recently being discussed in the context of generalized symmetries are difference operators of multiplicative type, the so called \(q\)-difference operators:
\[
D_x^{(q)} f(x) = \frac{f(qx) - f(q^{-1}x)}{x(q - q^{-1})},
\]
(9.3)
which compare functions at \(qx\) and \(q^{-1}x\). They appear naturally in the representation theory of quantum groups or more generally for \(q\)-deformed symmetries as we discussed in the previous Section.

If one wants to model physical systems through difference operators one has to derive or to motivate, e.g., evolution equations as difference equations. A formal attempt uses of a kind of correspondence principle. This means to replace the usual differential operator by a difference operator so that in the continuum limit the "usual" theory is reproduced.

A more generic method is the following: If a differential equation or relation is derived from first principles, e.g., from a symmetry group or an algebra, one can try to formulate already the principles in terms of difference operators. If necessary, one has to change the derivation and possibly also further assumptions are needed. Certainly also here we will get in the limiting case a differential equation, however, the result may differ from the one obtained from the correspondence principle.
Here we discuss the free quantum mechanical Schrödinger equation (SE) without spin on $\mathbb{R}_+^n \times \mathbb{R}_t$ based on first principles: The SE is physically characterized through representation theory of the central extension of the $(n+1)$-dimensional Schrödinger algebra $\hat{S}$. We gave in previous sections a purely algebraic construction for the family of $\hat{S}$ invariant Schrödinger equations from a family of singular vectors in Verma modules over $\hat{S}$. (We used also $q$-difference operators to derive a $q$-analogue of the Schrödinger equation, cf. [118] and the previous Section.) Now, we use the general method and in the realization of $\hat{S}$ through vector fields we replace the vector fields with additive difference vector fields, i.e., vector fields with difference operators instead of differential operators. Because the construction of Verma modules and the construction of invariant differential equation is completely algebraic we can apply this method also in this case.

To relate the Schrödinger algebra invariance as a first principle for a quantum mechanical evolution equation with difference operators we need the definitions and some properties of these operators and a realization of $\hat{S}$ through difference vector fields. This construction is not unique: additional assumptions which are physically well motivated are necessary. The essential point in our derivation is the observation made in [113] (and in previous Sections) that the construction of invariant equations from Verma modules is independent of the realization of the generators of the corresponding algebra. The reason for this is that the construction of [113] is completely algebraic, i.e., uses only the commutation relations of the algebra.

9.2. Definition and notation

As we discussed above the natural candidate for the formulation of difference analogues of differential equations are additive finite difference operators as in (9.2). In the representation of $\hat{S}$ one needs (partial) finite difference operators with respect to space-coordinates $x_i$ and the time-coordinate $t$. We use $\tau$ as “fundamental (time-)length” for the time-coordinate and $\xi$ as “fundamental (space-) length” for the space coordinate, i.e.

$$D_t f(t, x) \equiv \frac{f(t+\tau, x) - f(t-\tau, x)}{2\tau},$$

$$D_x f(t, x) \equiv \frac{f(t, x+\xi) - f(t, x-\xi)}{2\xi}. \quad (9.4)$$

The generalization to the $n+1$-dimensional case is obvious. Note that (9.2) is not the only possibility for the introduction of a finite difference operator with the “correct” limit; one could, e.g., use any expression of the form,$^b$

$$D_z^{(a,b)} \equiv \frac{f(z + a\xi) - f(z - b\xi)}{(a + b)\xi}, \quad a, b \in \mathbb{Z}, \quad a \neq -b.$$ 

$^b$The requirement $a, b \in \mathbb{Z}$ reflects the fact that only entire multiples of the fundamental length $\xi$ are considered measurable.
Thus, \( D_z = D_z^{(1,1)} \).

In the context of finite difference operators **shift operators** \( T_z^a \), defined as

\[
T_z^a f(z) \equiv f(z + a\zeta), \quad a \in \mathbb{Z}
\]  

are useful. We have

\[
D_z^{(a,b)} = \frac{T_z^a - T_z^{-b}}{(a + b)\zeta}.
\]  

Apart from the symmetric difference operator \((9.2)\), we will also use a "forward difference operator"

\[
D_z^+ \equiv D_z T_z = \frac{T_z^2 - 1}{2\zeta}
\]  

and a "backward difference operator"

\[
D_z^- \equiv D_z T_z^{-1} = \frac{1 - T_z^{-2}}{2\zeta}.
\]  

For our purposes it is important that \( D_z^{(a,b)} \) and \( T_z^a \) are, by definition, **linear operators**.

### 9.3. Construction of a realization of \( \hat{\mathcal{S}} \)

We construct a realization of \( \hat{\mathcal{S}} \) with finite difference and shift operators. As explained above this is possible only if one imposes additional assumptions which we choose to be the following:

1. In the zero-limit of the “fundamental lengths” the representation \((2.3)\) should be recovered.
2. The number of additional terms with vanishing limit in the representation should be “as small as possible”.
3. The generators of space translations \( P_i \) (and of time translations \( P_t \) when we consider representations with difference operators in \( t \)) should take a “simple form”, i.e., they should be realized by terms:

\[
P_i = D_i T_i^a, \quad a \in \mathbb{Z}.
\]

\(^c\)We use the shorthand notation \( T_i \equiv T_z \) and \( D_i \equiv D_z \).
It turns out that the following operators constitute a realization of $\hat{S}$ obeying
the above mentioned assumptions, cf. [119]

$$P_t = D_t^+, \quad \text{(9.8)}$$

$$P_i = D_i^+, \quad \text{(9.8)}$$

$$D = 2tD_t^+ + \sum_{i=1}^{n} x_i D_i^- + \Delta,$$

$$J_{ij} = x_j D_i^+ T_j^{-2} - x_i D_j^+ T_i^{-2}$$

$$G_i = tD_i^+ T_i^{-2} + M x_i T_i^{-2},$$

$$K = (t^2 - 2\tau t) D_t^- T_t^{-2} + t \left( \sum_{i=1}^{n} x_i D_i^- \right) T_t^{-2}$$

$$+ \frac{M}{2} \sum_{i=1}^{n} (x_i T_i^{-2})^2 + t\Delta T_t^{-2}$$

By setting $\tau \to 0$, i.e.

$$D_t^\pm \to \partial_t, \quad T_t^a \to 1$$

of $\xi \to 0$, i.e

$$D_i^\pm \to \partial_i, \quad T_i^a \to 1$$

one obtains realizations of $\hat{S}$ in which only space (respectively time) differentials are replaced by difference operators. We stress the fact that all these realizations of $\hat{S}$ are linear.

9.4. Invariant finite difference equations

Above we obtained $\hat{S}$-invariant partial differential equations by inserting the $\hat{S}$-
realization (2.3) into the expression (5.8) which determines the singular vectors of
the corresponding Verma modules. We mentioned already that the validity of this
result does not depend on whether one has a realization with differential
operators or not. In fact any linear realization leads to invariant equations. Thus, we can
insert (9.8) into (5.8) and obtain $\hat{S}$-invariant finite difference equations. As a result,
we find that the equations

$$\left( D_t^+ - \frac{1}{2M} \left( \sum_{i=1}^{n} D_i^+ \right)^2 \right)^{\frac{p}{2}} \psi(t, x) = 0, \quad p \text{ even.} \quad \text{(9.9)}$$
are invariant under the $\hat{S}$-realization (9.8) with $\Delta = \frac{n+2-p}{2}$.

As a special case ($p = 2$) we obtain a finite difference analogue of the free Schrödinger equation in $n$ space dimensions:

$$
\left( D_i^+ - \frac{1}{2M} \left( \sum_{i=1}^{n} D_i^+ \right)^2 \right) \psi(t, x) = 0.
$$

(9.10)

Setting $\tau \to 0$ or $\xi \to 0$ in (9.8), respectively, we obtain discrete-continuous analogues of the free Schrödinger equation in which only space- or time-differentiation is replaced with the corresponding difference operators, i.e.

$$
\left( \partial_t - \frac{1}{2M} \left( \sum_{i=1}^{n} D_i^+ \right)^2 \right) \psi(t, x) = 0.
$$

(9.11)

or respectively

$$
\left( D_i^+ - \frac{1}{2M} \left( \sum_{i=1}^{n} \partial_i \right)^2 \right) \psi(t, x) = 0.
$$

(9.12)

We stress that all these analogues of the free Schrödinger equation are not postulated but derived by our algebraic construction. Especially the appearance of $D_i^+$ – instead of another $D^{(a,b)}$ or linear combinations of several such operators – is forced by the assumptions given above and the construction of the equations.

In the case $n = 1$ equation (9.10) was considered in [164, 184], equations (9.11),(9.12) - in [184]. These authors looked for the symmetry of these equations employing difference or differential-difference operators, and they found that on the solution set of the equations these operators satisfy the $n = 1$ Schrödinger algebra, though the explicit expressions of the operators are different from ours (and for (9.10) between the two papers mentioned).

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