Continuous wavelet transform of Schwartz tempered distributions

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Abstract: The continuous wavelet transform of Schwartz tempered distributions is investigated and derive the corresponding wavelet inversion formula (valid modulo a constant-tempered distribution) interpreting convergence in $S'(\mathbb{R})$. But uniqueness theorem for the present wavelet inversion formula is valid for the space $S'_f(\mathbb{R})$ obtained by filtering (deleting) (i) all non-zero constant distributions from the space $S'(\mathbb{R})$, (ii) all non-zero constants that appear with a distribution as a union. As an example, in considering the distribution $\frac{x^2}{1+x^2} = 1 - \frac{1}{1+x^2}$ we would omit 1 and retain only $-\frac{1}{1+x^2}$. The wavelet kernel under consideration for determining the wavelet transform are those wavelets whose all the moments are non-zero. As an example, $(1 + kx - 2x^2)e^{-x^2}$ is such a wavelet. $k$ is an arbitrary constant. There exist many other classes of such wavelets. In our analysis, we do not use a wavelet kernel having any of its moments zero.

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1. Background results

The Schwartz testing function space $S(\mathbb{R})$ of rapid descent consists of infinitely differentiable functions $\phi$ defined on $\mathbb{R}$ such that

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The Continuous wavelet transform on Schwartz tempered distributions was firstly introduced by Holschneider in 1995. Pathak(2004) studied the various properties of continuous wavelet transform on Schwartz tempered distributions. Pandey and Upadhyay(2015) introduced the continuous wavelet transform using the concept of window functions. Motivated from the above results, the authors extend the continuous wavelet transform to Schwartz tempered distributions and investigate the corresponding wavelet inversion formula (valid modulo a constant tempered distribution) interpreting convergence in the weak distributional sense. This theory is true, when the wavelet kernel under consideration for determining the wavelet transform are those wavelets, whose all the moments are non-zero. The example of such types of wavelet kernel is also given in this work.

PUBLIC INTEREST STATEMENT

In the present work, the authors studied the continuous wavelet transform to Schwartz tempered distributions and found its inversion formula. This theory is useful for many researchers who are doing research work in image processing and signal processing. Researchers will be advantageous, who are using different types of integral transforms. The aforesaid theory is applicable, where many differential equations can be solved by exploiting the theory of Fourier transform. This work can be played an important role to study different types of integral equations. So the approach of this research paper is multidisciplinary in nature, which are applicable in mathematics, physics, and engineerings.
The topology on \( S(\mathbb{R}) \) is generated by two-parameter family of separating collection of seminorms

\[
\gamma_{m,n}(\phi) = \sup_{t \in \mathbb{R}} |t^m \phi^{(n)}(t)|, \quad \phi \in S(\mathbb{R}), \quad m,n = 0,1,2,\ldots
\]

The topology of \( S(\mathbb{R}) \) can as well be generated by the separating collection of the one-parameter family of seminorms

\[
\rho_m(\phi) = \sup_{t \in \mathbb{R}} \left( \frac{1}{1 + t^2} \right)^m |\phi^{(n)}(t)|, \quad \phi \in S(\mathbb{R}), \quad m = 0,1,2,\ldots
\]

It has been proved by Zemanian (1965, p. 111) that the topology generated by the sequence of seminorms \((\gamma_{m,n})\) on \( S(\mathbb{R}) \) is the same as that generated by the sequence of seminorms \((\rho_m)\). It has also been proved by him that

\[
\rho_0 \leq \rho_1 \leq \rho_2 \leq \cdots \rho_m \leq \cdots
\]

(Zemanian, 1965, pp. 111–112), i.e.

\[
\rho_0(\phi) \leq \rho_1(\phi) \leq \rho_2(\phi) \leq \cdots \rho_m(\phi) \leq \cdots, \quad \forall \phi \in S(\mathbb{R}).
\]

So a sequence of functions \( \{\phi_n\}_{n=1}^{\infty} \) in \( S(\mathbb{R}) \) converges to a function \( \phi \in S(\mathbb{R}) \) if and only if \( \rho_k(\phi_n - \phi) \to 0 \) as \( k \to \infty \)

for each \( k = 0,1,2,\ldots \). For example, the sequence of functions \( \left\{ \frac{x^n}{n!} \right\}_{n=1}^{\infty} \to 0 \) as \( n \to \infty \) in the topology of \( S(\mathbb{R}) \). Here 0 stands for the identically zero function in \( S(\mathbb{R}) \).

We say that a sequence \( \{\phi_n\}_{n=1}^{\infty} \) in \( S(\mathbb{R}) \) is a Cauchy sequence in \( S(\mathbb{R}) \) if \( \rho_k(\phi_n - \phi_m) \to 0 \) as \( n,m \to \infty \) for each \( k = 0,1,2,\ldots \). The topology on \( S(\mathbb{R}) \) is defined by the countable set of pseudonorms given by \((\rho_k)\) and with respect to this topology, \( S(\mathbb{R}) \) is sequentially complete (Yosida, 1995; Zemanian, 1965, 1968).

The result stated in the following paragraph is well known and can be found in many books (Gelfand & Shilov, 1968, pp. 21–23; Robewicz, 1972), p.21. We however state these facts to make the reading of this paper easy and interesting for many readers who may not know this result.

The space \( S(\mathbb{R}) \) is obviously metrizic by the metric \( \beta \) defined by

\[
\beta(\phi,\psi) = \sum_{k=0}^{\infty} \frac{1}{2^{k+1}} \frac{\rho_k(\phi - \psi)}{1 + \rho_k(\phi - \psi)}
\]

The fact that \( \beta \) is a metric is proved by using the fact that the function \( f(x) = \frac{1}{1+e^{-x}} \) is an increasing function of \( x \). It is well known that the topology generated by the metric \( \beta \) on \( S(\mathbb{R}) \) is the same as that generated by the sequence of seminorms \((\rho_k)\). Since the locally convex topological vector space \( S(\mathbb{R}) \) is complete and metrizable, it is a Fréchet space.

**Definition 1.** A function \( f \in L^2(\mathbb{R}) \) is said to be a window function if \( xf(x) \in L^2(\mathbb{R}) \) (Boggess & Narcowich, 2001; Chui, 1992).
It is proved in (Chui, 1992) that this window function $f$ also belongs to $L^1(\mathbb{R})$. A more general result in $n$-dimensions is proved in (Pandey & Upadhyay, 2015).

(1.4) Definition 2. A function $f \in L^2(\mathbb{R})$ is called a basic wavelet if the following admissibility condition is satisfied

$$
\int_{-\infty}^{\infty} \frac{|f(\lambda)|^2}{|\lambda|} d\lambda < \infty,
$$

from (Chui, 1992; Daubechies, 1990; Lebedeva & Postinikov, 2014; Postnikov, Lebedeva, & Lavrova, 2016).

We denote the constant defined in (1.4) by $C_f$. Here, $\hat{f}(\lambda)$ is the Fourier transform of $f$ which is given as

$$
\hat{f}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ix\lambda} dx.
$$

From Definition 2 it follows that the function $\psi = xe^{-x^2}$ is a basic wavelet belonging to $S(\mathbb{R})$. This is because $\hat{\psi}(\lambda) = \frac{d}{d\lambda} e^{-\lambda^2}$ is the Fourier transform of $\psi \in S(\mathbb{R})$. Therefore, $\int_{-\infty}^{\infty} \frac{|\hat{\psi}(\lambda)|^2}{|\lambda|^2} d\lambda = \frac{1}{4}$, which is bounded. The theorem stated below helps us in constructing wavelets in various testing function spaces very simply.

Theorem 1.1. A window function $f \in L^2(\mathbb{R})$ is a basic wavelet if and only if

$$
\int_{-\infty}^{\infty} f(x) dx = 0.
$$

A more general theorem in $n$-dimensions, $n \geq 1$ is proved in (Pandey & Upadhyay, 2015). Since $\phi \in D \subset L^2(\mathbb{R})$ is a window function, an element $\phi \in D$ is a basic wavelet if and only if

$$
\int_{-\infty}^{\infty} \phi(x) dx = 0.
$$

So

$$
\psi(x) = \begin{cases} 
xe^{-x^2} & |x| < 1 \\
0 & |x| \geq 1
\end{cases}
$$

is a basic wavelet in $D$. For a similar reason, a function $\phi \in S(\mathbb{R})$ is a basic wavelet in $S(\mathbb{R})$ if and only if

$$
\int_{-\infty}^{\infty} \phi(x) dx = 0.
$$

So the function $\psi(x) = xe^{-x^2}$ is a basic wavelet in $S(\mathbb{R})$. We have already verified this fact by direct calculation in the paragraph preceding Theorem 1.1.

2. Introduction

Let $S(\mathbb{R})$ be the Schwartz testing function space of rapid descent and let $s(\mathbb{R})$ be a subspace of $S(\mathbb{R})$ so that every element $\phi \in s(\mathbb{R})$ satisfies

$$
\int_{-\infty}^{\infty} \phi(x) dx = 0,
$$

i.e., every element of $s(\mathbb{R})$ is a basic wavelet. The subspace $s(\mathbb{R})$ of $S(\mathbb{R})$ is equipped with the topology induced by $S(\mathbb{R})$ on $s(\mathbb{R})$. One can verify that the restriction of $f \in S'(\mathbb{R})$ to $s(\mathbb{R})$ is in $s'(\mathbb{R})$ and, therefore, in the following
discussion the wavelet inversion formula that is valid for \( f \in S'(\mathbb{R}) \) restricted to \( S(\mathbb{R}) \modulo{} \) a constant distribution, is also valid for elements of \( S'(\mathbb{R}) \) restricted to \( s(\mathbb{R}) \). More clearly, we have

\[
\lim_{A,B \to \infty} \langle f, \phi \rangle = \langle f, \phi \rangle, \quad \forall \phi \in S(\mathbb{R}).
\]

We extend the continuous wavelet transform to the Schwartz tempered distribution space \( S'(\mathbb{R}) \), exploiting the structure formula

\[
\langle f, \phi \rangle = \left< g, (1 + x^2)^{m-1} \psi^{(m+1)}(x) + m2x(1 + x^2)^m \psi^{(m)}(x) \right>,
\]

(2.1)

Here \( f \in S'(\mathbb{R}) \) and \( \phi \in S(\mathbb{R}) \), and \( g \) is a function belonging to \( L^2(\mathbb{R}) \) depending upon \( f \) and not on \( \phi \). The structure formula (2.1) follows from the boundedness property of \( S'(\mathbb{R}) \), i.e., for \( f \in S'(\mathbb{R}) \) there exists a nonnegative integer \( m \) and a constant \( C > 0 \) such that

\[ |\langle f, \phi \rangle| \leq C \|f\|_{S'(\mathbb{R})} \|\phi\|_{S(\mathbb{R})}, \quad \forall \phi \in S(\mathbb{R}). \]

(2.2)

This is derived by virtue of the fact that \( \rho_0 \leq \rho_1 \leq \rho_2 \ldots \) and the method of contradiction; using (2.2) we get for a non-negative integer \( m \) satisfying

\[
|\langle f, \phi \rangle| \leq C \sup_{t \in \mathbb{R}} \left| \frac{\pi}{2}^m (1 + t^2)^m \phi^{(m)}(t) \right|, \quad \forall \phi \in S(\mathbb{R})
\]

\[
\leq C \frac{\pi}{2}^m \sup_{t \in \mathbb{R}} \left| \int_{-\infty}^{\infty} \frac{d}{dx} \left( (1 + x^2)^m \phi^{(m)}(x) \right) dx \right|
\]

\[
\leq C \frac{\pi}{2}^m \sup_{t \in \mathbb{R}} \left| \int_{-\infty}^{\infty} \left( (1 + x^2)^m \phi^{(m+1)}(x) + (1 + x^2)^{m-1} 2mx\phi^{(m)}(x) \right) \right|
\]

\[
= C \frac{\pi}{2}^m \sup_{t \in \mathbb{R}} \left| \int_{-\infty}^{\infty} \left( (1 + x^2)^{m-1} \phi^{(m+1)}(x) + (1 + x^2)^m 2mx\phi^{(m)}(x) \right) \right| \frac{1}{1 + x^2} dx.
\]

Therefore,

\[
|\langle f, \phi \rangle| \leq C \left( \frac{\pi}{2}^m \right) \left( 1 + x^2 \right)^{\frac{m}{2}} \left( 1 + x^2 \right)^{\frac{m+1}{2}} \|\phi\|_{S(\mathbb{R})} \|f\|_{S'(\mathbb{R})}
\]

using Holder’s inequality.

Now, using the Hahn Banach theorem (Yosida, 1995, p. 102, 105, 106), \( f \) can be extended to \( L^1(\mathbb{R}) \). Since \( S \) is dense in \( L^2(\mathbb{R}) \), this extension is unique.

By Riesz theorem, the dual of \( L^2(\mathbb{R}) \) is homeomorphic to \( L^1(\mathbb{R}) \) we get a function \( g \in L^2(\mathbb{R}) \) such that

\[
\langle f, \phi \rangle = \langle g(x), (1 + x^2)^{m-1} \psi^{(m+1)}(x) + (1 + x^2)^m 2mx\psi^{(m)}(x) \rangle
\]

from (Akhiezer & Glazman, 1961, p. 33). This justifies the structure formula (2.1).

**Fact 1:** If \( \psi \) is a wavelet belonging to \( S(\mathbb{R}) \), the continuous wavelet transform \( W_f(a,b) \) of \( f \in S'(\mathbb{R}) \) in view of the relation (2.1) can be proved to be

\[
W_f(a,b) = \left< g(x), \frac{(1 + x^2)^{m+1}}{a^{m+1} \sqrt{|a|}} \psi^{(m+1)} \left( \frac{x - b}{a} \right) \right> + \left< g(x), \frac{(1 + x^2)^m 2mx \psi^{(m)} \left( \frac{x - b}{a} \right)}{a^{m} \sqrt{|a|}} \right>
\]
Using the classical wavelet inversion formula for $L^2(\mathbb{R})$ functions as proved in (Boggess & Narcowich, 2001; Chui, 1992; Daubechies, 1990; Lebedeva & Postinikov, 2014), we will prove in the next section that

$$\lim_{M,N \to \infty} \left( \int_{-M}^{M} \int_{-N}^{N} W_f(a,b) \frac{1}{\sqrt{|a|}} \psi \left( \frac{x-b}{a^2} \right) db da \right) \cdot \frac{1}{a^2} = f$$

in the weak topology of $S'({\mathbb{R}})$, i.e.,

$$\left\langle \left( P \right), \frac{1}{\sqrt{|a|}} W_f(a,b) \psi \left( \frac{x-b}{a^2} \right), \phi(x) \right\rangle \to \left\langle f, \phi \right\rangle, \forall \phi \in S({\mathbb{R}}) \text{ as } M,N \to \infty.$$  

Since two tempered distributions having the same continuous wavelet transform may differ by a constant distribution, our inversion formula will be valid modulo a constant distribution. The structure formula for $f \in S'({\mathbb{R}})$ reduces a functional analytic problem to a classical problem of analysis, i.e. a $L^2(\mathbb{R})$ function theory.

Pathak (Pathak, 2004) extended the wavelet transform to Schwartz tempered distributions in the year 2004 using the method of adjoints, i.e.

$$\langle WT, \phi \rangle = \langle T, W\phi \rangle, \quad \phi \in S({\mathbb{R}}^n).$$

but he did not prove an inversion formula. Here $WT$ stands for the generalized wavelet transform of $T \in S'$ the dual of $S({\mathbb{R}}^n)$ and $W\psi(a,b) \in S({\mathbb{R}}^n \times {\mathbb{R}}_\alpha)$

(Pathak, 2004, 2009, Chapter III).

He defined the test function space $\tilde{S}({\mathbb{R}}^n \times {\mathbb{R}}_\alpha)$ containing the Schwartz testing function space $S({\mathbb{R}}^n \times {\mathbb{R}}_\alpha)$ whose topology is generated by a sequence of semi-norms $\gamma_{\ell,\kappa,\alpha,\beta}(\psi), \forall \alpha, \beta \in \mathbb{N}_0^n$ and $\ell, \kappa \in \mathbb{N}_0$, (Pathak, 2004, p. 413).

From (2.3) it follows that the wavelet transform of a constant distribution is zero as the wavelet $\psi$ belonging to the space $S({\mathbb{R}}^n)$ satisfies the condition

$$\psi(w)|_{w=0} = \int_{\mathbb{R}^n} \psi(x)dx = 0 \text{ and } \int_{\mathbb{R}^n} |\psi(w)|^2 dw > 0.$$  

Therefore, the wavelet transform of a constant distribution $\kappa$ is

$$\int_{\mathbb{R}^n} \kappa \psi \left( \frac{x-b}{a^2} \right) \frac{1}{\sqrt{|a|}} dx = 0.$$  

Thus, two wavelets having the same wavelet transform may differ by a constant.

Pathak (2004) was motivated to give the definition (2.3) for the wavelet transform of tempered distribution by the Parseval's type of relation for the wavelet transform

$$\langle WT, \phi \rangle = \langle T, W\phi \rangle, \quad f \in L^2(\mathbb{R}).$$

He strengthened his result (definition 2.2 (Pathak, 2004)) further by proving some continuity results and boundedness property; but he did not derive the corresponding wavelet inversion formula. Since
the wavelet transform of a constant distribution is zero the uniqueness theorem for the wavelet inversion formula will not be true; it will be valid modulo a constant distribution. In order that the uniqueness theorem may be valid we have to delete all non-zero constants distribution from the space $S'(\mathbb{R}^n)$. In addition, we have to delete a non-zero constant distribution from a tempered distribution which is contained in it as a sum or difference. For example, in considering the distribution

$$\frac{|x|^2}{1 + |x|^2} = 1 - \frac{1}{1 + |x|^2},$$

We delete the constant 1 and retain the tempered distribution $\frac{1}{1 + |x|^2}$ only.

The space $S'(\mathbb{R}^n)$ filtered this way is represented by the symbol $S'_F(\mathbb{R}^n)$, then the uniqueness theorem for the wavelet inversion formula will be valid for this space $S'_F(\mathbb{R}^n)$.

During the last five years several good results on the continuous wavelet transform of functions appeared. Notable amongst them is the work of Postnikov et al. (2016), Lebedeva & Postnikov (2014), who proved the wavelet inversion formula for functions in the year 2016 without a requirement of the admissibility condition.

Weisz (2013) proved the norm and a.e. convergence of inversion formula in $L_p$ and Wiener amalgam spaces. In 2014 he proved the inverse wavelet transform to summability means of Fourier transforms and obtained norm and almost everywhere convergence of the inversion formula for functions from the $L_p$ and Wiener amalgam spaces (Weisz, 2014). In 2015, Weisz (2015) also proved, using the summability methods of Fourier transform, norm convergence and convergence at Lebesgue points of the inverse wavelet transform for functions from the $L_p$ and Wiener amalgam spaces.

Our objective is to extend the continuous wavelet transform to Schwartz space $S'(\mathbb{R})$ and prove an inversion formula modulo a constant distribution and then extend the uniqueness theorem for the continuous wavelet transform of distributions to the space $S_F'(\mathbb{R})$; the space $S_F'(\mathbb{R})$ is a subspace of the space $S'(\mathbb{R})$.

Our spaces $S'(\mathbb{R})$ and $S_F'(\mathbb{R})$ are big spaces and they contain the spaces $L^1(\mathbb{R})$, $L^p(\mathbb{R})$ as considered by these authors.

The wavelets that we use as a kernel of the wavelet transform will not be any element of $s(\mathbb{R})$ will be those elements of $s(\mathbb{R})$ whose moments of any order will be non-zero. An example of one such wavelet is $(1 + x - 2x^2)e^{-x^2}$. Many more such wavelets can be constructed by assigning arbitrary values to the constant $k$ in the expression $(1 + kx - 2x^2)e^{-x^2}$. Another set of such wavelet kernels can be constructed by assigning appropriate values to the constants $k$ and $b$ in the expression $(1 + kx - bx^2)(e^{-x^2} - e^{-x^4})$. We first select $b$ such that $\int_{-\infty}^{\infty} (1 + kx - bx^2)(e^{-x^2} - e^{-x^4})dx = 0$. The number $b$ will be independent of $k$ and therefore $k$ can be assigned arbitrary real values, thereby proving the existence of wavelet kernels in $s(\mathbb{R})$ whose any moment will be non-zero. Many more such wavelets (unaccountably many of them) can be constructed.

Our reason to avoid wavelets whose every moment is zero is that wavelet transform of every polynomial function will be zero, and our inversion formula will break. This situation is already dealt with by Holschneider (1995). He quotients out the space of tempered distributions by the space of all polynomials.

2.1. Comparison of our results with that of Pathak

(1) Pathak (2004) followed the method of adjoints, whereas we have followed the method of embedding to define the wavelet transform of tempered distributions; but he did not prove the inversion formula.
2. Pathak took $a>0$ whereas we took $a \in \mathbb{R}$, $a \neq 0$, a more general result in this sense.

3. We have proven the inversion formula for the wavelet transform of distributions giving the situation where our inversion formula has unique results and where it does not.

4. Calculation of the wavelet transform is far easier by our method whereas calculation of the wavelet transform by Pathak’s method is not quite as easy.

5. We have proven the uniqueness theorem for the inversion formula for the wavelet transform for the space $S_c(\mathbb{R})$, $n = 1$ and the result can be extended for $n>1$.

Our objective is to prove the wavelet inversion formula for tempered distributions in the weak distributional sense and this will be accomplished in Chapter 3.

3. An integral wavelet transforms of schwarz tempered distributions in $\mathbb{R}$ and its inversion

3.1. Integral wavelet transform

In this section we will use three symbols $F_b$, $C_\psi$ and $F$; the symbols $F_b$ and $F$ stand for the Fourier transform of functions of $b$ and $t$ respectively, and the symbol $C_\psi$ stands for the admissibility constant which is defined in (1.4).

We require that $C_\psi$ be finite as in the derivation of the wavelet inversion formula. The expression $C_\psi$ appears in the denominator of the related expression, that is useful in our derivation of the inversion formula using the Fourier transform technique. But there exists an alternative reconstruction formula for the continuous wavelet transform, which is applicable even if the admissibility condition is violated see (Holschneider, 1995; Postnikov et al., 2016).

Fact 2: Let $s(\mathbb{R})$ be a subspace of $S(\mathbb{R})$ such that $\psi \in s(\mathbb{R})$ implies $\int_{-\infty}^{\infty} \psi(x)dx = 0$ and so $\psi$ is a basic wavelet. It is a simple exercise to show that $\psi\left(\frac{x-b}{a}\right)$ also belongs to $s'(\mathbb{R})$ as a function of $x$ for fixed $b$ and $a \neq 0$. Therefore, for $f \in s'(\mathbb{R})$, the integral wavelet transform $W_f(a,b)$ of $f$ is defined as

$$W_f(a,b) = \begin{cases} \frac{1}{\sqrt{a}} \langle f(x), \psi\left(\frac{x-b}{a}\right) \rangle, & a \neq 0, a,b \in \mathbb{R} \\ 0, & a = 0. \end{cases}$$

Since any constant distribution in $s'(\mathbb{R})$ can be identified as a zero distribution, the uniqueness theorem for the wavelet inversion formula in $s'(\mathbb{R})$ is valid.

Theorem 3.1. Let $s(\mathbb{R})$ be a subspace of the Schwartz testing function space $S(\mathbb{R})$ of rapid descent such that every $\psi(x) \in s(\mathbb{R}) \subset S(\mathbb{R})$ satisfies the condition

$$\int_{-\infty}^{\infty} \psi(x)dx = 0.$$ 

Then $\psi$ is a basic wavelet, i.e., it satisfies the conditions

(i) $\psi(x) \in L^2(\mathbb{R})$.

(ii) $\int_{-\infty}^{\infty} \frac{|\psi(\lambda)|^2}{|\lambda|} d\lambda < \infty$.

Proof. Note that (i) is trivially satisfied and (ii) follows as $\int_{-\infty}^{\infty} \frac{|\psi(\lambda)|^2}{|\lambda|} d\lambda \leq \|f\|_2^2 + 2^1 \|xf\|_2^2 < \infty$ by taking $n = 1$ in (Pandey & Upadhyay, 2015, Theorem 3.1.).

Theorem 3.2. Let $f \in L^2(\mathbb{R})$ and $\psi \in s(\mathbb{R}) \subset S(\mathbb{R})$ where $s(\mathbb{R})$ and $S(\mathbb{R})$ are spaces of functions as defined in Theorem 3.1. Then

$$F_b \left\{ \frac{1}{\sqrt{|a|}} \int_{-\infty}^{\infty} \psi\left(\frac{x-b}{a}\right) f(x)dx \right\} = \frac{1}{\sqrt{|a|}} \int_{-\infty}^{\infty} F_\psi\left(\frac{x-b}{a}\right) (\lambda^2) f(x)dx.$$
Proof. In (Pandey & Upadhyay, 2015, Theorem 3.1) the proof of above theorem is given.

Theorem 3.3. Let $\psi \in s(\mathbb{R}) \subset S(\mathbb{R})$, and $f \in L^2(\mathbb{R})$ then

$$f(x) = \frac{1}{C_{\psi}} \int_{-\infty}^{\infty} |a|^2 \frac{1}{a} \psi \left( \frac{x-b}{a} \right) W_f(a,b) \frac{db da}{a^2}$$

where $C_{\psi}$ denotes the admissibility constant (1.4).

Proof. It is a special case of (Pandey & Upadhyay, 2015, Theorem 3.1) see also (Daubechies, 1990).

Theorem 3.4. For $f \in S'(\mathbb{R})$, define the continuous wavelet transform or integral wavelet transform $W_f(a,b)$ of $f \in S(\mathbb{R})$ as

$$W_f(a,b) = \begin{cases} 
\left< f(x), \frac{1}{\sqrt{|a|}} \psi \left( \frac{x-b}{a} \right) \right>, & a \neq 0 \\
0, & a = 0.
\end{cases}$$

Then $W_f(a,b)$, $a \neq 0$, as a function of $b$, belongs to $L^2(\mathbb{R})$.

Proof. Using (2.1) for $f \in S'(\mathbb{R})$ and $\phi \in S(\mathbb{R})$ we have

$$\langle f, \phi \rangle = \left< g(t), (1 + t^2)^{m+1} \phi^{(m+1)}(t) \right> + \left< g(t), 2tm(1 + t^2)^m \phi^{(m)}(t) \right> \quad (3.1)$$

where $g \in L^2(\mathbb{R})$. Now,

$$\psi(x) \in s(\mathbb{R}) \subset S(\mathbb{R}), \quad \psi \left( \frac{x-b}{a} \right) \in s(\mathbb{R}) \subset S(\mathbb{R})$$

for fixed $a$ and $b$; $a \neq 0$. Therefore, replacing $\phi(t)$ by $\frac{1}{\sqrt{|a|}} \psi \left( \frac{t-b}{a} \right)$ in (3.1), we have

$$W_f(a,b) = \left< f, \frac{1}{\sqrt{|a|}} \psi \left( \frac{x-b}{a} \right) \right> = \left< g(t), \frac{1}{\sqrt{|a|}} (1 + t^2)^{m+1} \psi^{(m+1)} \left( \frac{t-b}{a} \right) \right>$$

$$+ \left< g(t), \frac{1}{\sqrt{|a|}} 2tm(1 + t^2)^m \frac{1}{a^m} \psi^{(m)} \left( \frac{t-b}{a} \right) \right>.$$

Now, by Plancherel’s formula we get

$$\left< g(t), \frac{1}{\sqrt{|a|}} \left( 1 + t^2 \right)^{m+1} \psi^{(m+1)} \left( \frac{t-b}{a} \right) \right>$$

$$= \int_{-\infty}^{\infty} |a| \sqrt{|a|} g(\lambda) \left\{ \left( 1 - \frac{d^2}{dx^2} \right)^{m+1} \left[ (-i \lambda)^{m+1} \overline{\psi} (a \lambda) \right] \right\} e^{i n a \lambda} d\lambda, \psi \in S(\mathbb{R}).$$

The function $\overline{\psi}(a \lambda) \in S(\mathbb{R})$. Hence the expression in the above integral which is in curly bracket is bounded. Therefore, the coefficient of $e^{i n a \lambda}$ in the integrand in the above integral belongs to $L^2(\mathbb{R})$ as a function of $\lambda$, which implies that the above integral as a function of $b$ belongs to $L^2(\mathbb{R})$. Similarly, we can show that

$$\left< g(t), \frac{1}{\sqrt{|a|}} 2tm(1 + t^2)^m \frac{1}{a^m} \psi^{(m)} \left( \frac{t-b}{a} \right) \right> \in L^2(\mathbb{R})$$

as a function of $b$ and is infinitely differentiable with respect to $b$ and each of its derivatives also belongs to $L^2(\mathbb{R})$. 

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Therefore, $W_f(a, b) \in L^2(\mathbb{R})$ as a function of $b$, $a \neq 0$. The form of the structure formula we have chosen is valid for $m \geq 0$. In fact, the structure formula for $f$ when $m = 0$ can also be derived from (2.1) by setting $m = 0$.

**Corollary.** Let $W_f(a, b), a \neq 0$ be the function defined in theorem 3.4, then $\frac{\partial f}{\partial b} W_f(a, b)$ belong to $L^2(\mathbb{R})$ as a function of $b$ for each $k = 1, 2, 3, \ldots$.

**Theorem 3.5.** [Inversion formula]: For $f \in S'(\mathbb{R})$ and $\psi \in s(\mathbb{R}) \subset S(\mathbb{R})$, define the Wavelet transform $W_f(a, b)$ of $f$ as defined in Theorem 3.4. Then

\[
\begin{align*}
(i) \frac{\partial \psi}{\partial b} &= \left( f(t), \frac{\psi}{\sqrt{|a|}} \psi\left( \frac{t-b}{a} \right) \right) \\
(ii) \frac{\partial \psi}{\partial a} &= \left( f(t), \frac{\psi}{\sqrt{|a|}} \psi\left( \frac{t-b}{a} \right) \right) . \\
(iii) W_f(a, b), \frac{\partial \psi}{\partial a} \text{ and } \frac{\partial \psi}{\partial b} \text{ are uniformly bounded in a compact neighbourhood of the point (a, b), a} \neq 0. \\
(iv) W_f(a, b) \text{ is a continuous function of (a, b) everywhere on } \mathbb{R}^2 \text{ except possibly at } a = 0 \text{ (the b-axis)}.
\end{align*}
\]

**3.2 Proof.** Using the structure formula (2.1) for $f$ we have

\[
W_f(a, b) = \left( f(t), \frac{1}{\sqrt{|a|}} \psi\left( \frac{x-b}{a} \right) \right) \\
= \left( g(t), \frac{1}{\sqrt{|a|}} (1 + t^2)^{m+1} \psi^{(m+1)} \left( \frac{t-b}{a} \right) \right) \\
+ \left( g(t), \frac{1}{\sqrt{|a|}} 2tm(1 + t^2)^m \psi^{(m)} \left( \frac{t-b}{a} \right) \right), \quad g \in L^2(\mathbb{R}).
\]

Therefore

\[
W_f(a, b) = \int_{-\infty}^{\infty} g(t)(1 + t^2)^{m+1} \frac{1}{\sqrt{|a|} a^{m+1} \psi^{(m+1)} \left( \frac{t-b}{a} \right)} dt \\
+ \int_{-\infty}^{\infty} g(t)2tm(1 + t^2)^m \frac{1}{\sqrt{|a|} a^m \psi^{(m)} \left( \frac{t-b}{a} \right)} dt. \quad m \geq 0.
\]

(3.2)

Therefore, using a standard result in analysis we have

\[
\begin{align*}
\frac{\partial^k W_f(a, b)}{\partial a^k} &= \int_{-\infty}^{\infty} g(t)(1 + t^2)^{m+1} \frac{1}{\sqrt{|a|} a^{m+1} \psi^{(m+1)} \left( \frac{t-b}{a} \right)} dt \\
&\quad + \int_{-\infty}^{\infty} g(t)2tm(1 + t^2)^m \frac{1}{\sqrt{|a|} a^m \psi^{(m)} \left( \frac{t-b}{a} \right)} dt. \quad (3.3)
\end{align*}
\]

A similar result for differentiation with respect to $b$ can be proved. Therefore, we obtain

\[
(i) \frac{\partial^k}{\partial a^k} W_f(a, b) = \left( f(t), \frac{\partial^k}{\partial a^k} \frac{1}{\sqrt{|a|}} \psi\left( \frac{t-b}{a} \right) \right)
\]
\( (ii) \frac{\partial^k}{\partial x^k} W_f(a, b) = \left< f(t), \frac{1}{\sqrt{|a|}} \psi \left( \frac{t-b}{a} \right) \right> \) \quad k = 1, 2, 3, \ldots \quad (3.4)

The results (3.3 and 3.4) are valid for \( m \geq 0 \).

\( (iii) W_f(a, b) = \left< f(t), \frac{1}{\sqrt{|a|}} \psi \left( \frac{t-b}{a} \right) \right> \)

Using the boundedness property of \( f \) we get,

\[
|W_f(a, b)| \leq C \frac{1}{|a|^{m+2}} \sup_{t, \phi} \left| \frac{1}{\phi^m} \left( 1 + t^2 \right)^m \psi^{(m)} \left( \frac{t-b}{a} \right) \right|
\]

\[
\leq C \frac{1}{|a|^{m+2}} \sup_{y, \phi} \left| \frac{1}{\phi^m} \left( 1 + (b + ay)^2 \right)^m \psi^{(m)}(y) \right|
\]

\[
\leq C \frac{1}{|a|^{m+2}} |P(|a|, |b|)|.
\]

where \( P \) is a polynomial of degree 2m. The non-negative integer \( m \) is the least possible value conforming to the boundedness property of \( f \). These polynomials will be uniformly bounded in a compact neighborhood of \( (a, b) \). Since \( a \neq 0 \), \( \frac{1}{|a|^{m+2}} \) will be also finite. Therefore, \( W_f(a, b) \) is bounded in a compact neighborhood of \( (a, b) \), \( a \neq 0 \). Similar bounds can be established for the first partial derivatives of \( W_f(a, b) \) with respect to \( a \) and \( b \).

\( (iv) \) To prove \( (iv) \), we assume \( (i) \) and \( (ii) \) for \( k = 1 \) and \( (iii) \). Now

\[
W_f(a + \Delta a, b + \Delta b) - W_f(a, b) = W_f(a + \Delta a, b + \Delta b) - W_f(a, b + \Delta b) + W_f(a, b + \Delta b) - W_f(a, b)
\]

\[
= \Delta a \frac{\partial W_f}{\partial a} (a + \theta \Delta a, b + \Delta b) + \Delta b \frac{\partial W_f}{\partial b} (a, b + \Delta b), \quad \theta < \alpha, \phi < 1.
\]

This is valid if \( W_f \) is real-valued. If it is complex-valued, then we apply the mean value theorem of differential calculus separately for the real and imaginary part of \( W_f \).

The first partial derivatives of \( W_f(a, b) \) are bounded in a compact neighborhood of \( (a, b) \) where \( a \neq 0 \) and \( (a + \Delta a, b + \Delta b) \) lies in the neighborhood of \( (a, b) \) during the limiting process. Therefore, from (3.6) \( W_f(a + \Delta a, b + \Delta b) - W_f(a, b) \to 0 \) as \( \Delta a, \Delta b \to 0 \).

Note that differentiability results proved by us apply to the wavelet transform of tempered distributions, whereas Pathak’s differentiability results apply to the wavelet transform of functions. Our results are a lot more general (Pathak, 2004).

**Theorem 3.6** [Inversion Formula]: Let \( f \) be a tempered distribution belonging to \( S'(\mathbb{R}) \) and \( \psi(x) \in S(\mathbb{R}) \subset S'(\mathbb{R}) \), and define \( W_f(a, b) \) of \( f \) with respect to the wavelet \( \psi \) by

\[
W_f(a, b) = \left< f(t), \frac{1}{\sqrt{|a|}} \psi \left( \frac{t-b}{a} \right) \right>.
\]

Then following inversion formula holds

\[
\lim_{N_1, N_2, N_3 \to \infty} \frac{1}{C_\psi} \int_{-N_3}^{N_3} f(t) \int_{-N_2}^{N_2} W_f(a, b) \frac{1}{\sqrt{|a|}} \psi \left( \frac{t-b}{a} \right) \frac{dadb}{a^2} = f.
\]

Above limit is interpreted in the weak distributional sense, i.e. in \( S'(\mathbb{R}) \).
\[
\lim_{N_1, M_1, N_2, M_2 \to \infty} \left\langle \frac{1}{C_\psi}(P) \right\rangle_{-N_1}^{N_1} \int_{-M_2}^{M_2} W_f(a, b) \frac{1}{\sqrt{|a|}} \psi \left( \frac{x - b}{a^2} \right) dbda, \phi(x) \right\rangle = \langle f, \phi \rangle, \quad \forall \phi \in S(\mathbb{R})
\] (3.8)

or
\[
\left\langle \frac{1}{C_\psi}(P) \right\rangle_{-\infty}^{\infty} \int_{-\infty}^{\infty} W_f(a, b) \frac{1}{\sqrt{|a|}} \psi \left( \frac{x - b}{a^2} \right) dbda, \phi(x) \right\rangle = \langle f(x), \phi(x) \rangle.
\]

When we use a structure formula for \( f \), the distributional problem is converted into the classical one and so all lower limits and upper limits of the integral will be \(-\infty\) and \(\infty\), respectively.

**Proof.** From (2.1), (3.7) can be written as
\[
W_f(a, b) = \int_{-\infty}^{\infty} g(t)(1 + t^2)^{m+1} \frac{1}{\sqrt{|a|} a^{m+1}} \psi^{(m+1)} \left( \frac{t - b}{a} \right) dt
\]
\[
+ \int_{-\infty}^{\infty} g(t)2tm(1 + t^2)^m \frac{1}{a^{m+1}} \psi^{(m)} \left( \frac{t - b}{a} \right) dt.
\]

Our aim is to find the inversion formula
\[
\frac{1}{C_\psi}(P) \left\langle \int_{-\infty}^{\infty} W_f(a, b) \frac{1}{\sqrt{|a|}} \psi \left( \frac{x - b}{a^2} \right) dbda \right\rangle = f
\]
interpreting convergence in the weak toplogy of \( S'(\mathbb{R}) \), i.e., as in (3.8)
\[
\left\langle \frac{1}{C_\psi}(P) \right\rangle_{-\infty}^{\infty} \int_{-\infty}^{\infty} W_f(a, b) \frac{1}{\sqrt{|a|}} \psi \left( \frac{x - b}{a^2} \right) dbda, \phi(x) \right\rangle = \langle f, \phi \rangle, \quad \forall \phi \in S(\mathbb{R}).
\]

For the sake of convenience, let us represent
\[
\frac{1}{C_\psi}(P) \left\langle \int_{-\infty}^{\infty} W_f(a, b) \frac{1}{\sqrt{|a|}} \psi \left( \frac{x - b}{a^2} \right) dbda \right\rangle \text{ by } F(x).
\]

In the following integrations we delete the region \(|a| \leq \varepsilon\) and make changes in the order of integration and after that let \( \varepsilon \to 0 \). Therefore, in view of (3.6) and Theorems 3.4 and 3.5, the integral in (3.11) is meaningful (it exists) and now when operated against \( \phi \in S(\mathbb{R}) \), (3.11) becomes:
\[
\langle F(x), \phi(x) \rangle = \lim_{\varepsilon \to 0} \frac{1}{C_\psi} \left\langle \int_{|a| \geq \varepsilon} \left\langle \int_{-\infty}^{\infty} g(t)(1 + t^2)^{m+1} \frac{1}{\sqrt{|a|} a^{m+1}} \psi^{(m+1)} \left( \frac{t - b}{a} \right) \frac{1}{\sqrt{|a|}} \psi \left( \frac{x - b}{a^2} \right) \frac{dt}{a^2} \right\rangle dbda \right\rangle
\]
\[
+ \frac{1}{C_\psi} \left\langle \int_{|a| \geq \varepsilon} \left\langle \int_{-\infty}^{\infty} g(t)2tm(1 + t^2)^m \frac{1}{a^{m+1}} \psi^{(m)} \left( \frac{t - b}{a} \right) \frac{dt}{a^2} \right\rangle dbda, \phi(x) \right\rangle.
\]

[The above angular brackets represent integration with respect to \( x \) in \( \mathbb{R} \).]

Therefore, we have
\[
\langle F(x), \phi(x) \rangle = \lim_{\varepsilon \to 0} \frac{1}{C_\psi} \left\langle \int_{|a| \geq \varepsilon} \left\langle \int_{-\infty}^{\infty} g(t)(1 + t^2)^{m+1} \frac{1}{\sqrt{|a|} a^{m+1}} (-1)^{m+1} \frac{\partial^{m+1}}{\partial b^{m+1}} \psi \left( \frac{t - b}{a} \right) \frac{dt}{a^2} \right\rangle dbda \right\rangle
\]
\[
\left\langle \psi \left( \frac{x - b}{a^2} \right) \frac{dbda}{a^2}, \phi(x) \right\rangle + \lim_{\varepsilon \to 0} \frac{1}{C_\psi} \left\langle \int_{|a| \geq \varepsilon} \left\langle \int_{-\infty}^{\infty} g(t)(1 + t^2)^m \frac{1}{\sqrt{|a|} a^m} (-1)^m \frac{\partial^m}{\partial b^m} \psi \left( \frac{t - b}{a} \right) \frac{dt}{a^2} \right\rangle dbda \right\rangle
\]
\[
\left\langle \psi \left( \frac{x - b}{a^2} \right) \frac{dbda}{a^2}, \phi(x) \right\rangle
\]
\[
\left\langle \frac{dbda}{a^2}, \phi(x) \right\rangle
\]
\[
(3.12)
\]
\[
\frac{\partial}{\partial b} \left[ \psi \left( \frac{t - b}{a} \right) \right] = - \frac{\partial}{\partial t} \left[ \psi \left( \frac{t - b}{a} \right) \right].
\]

so that

\[
\langle F(x), \phi(x) \rangle = \lim_{r \to \infty} \left\{ \int_{|x| > r} \int_{-\infty}^{\infty} g(t)(1 + t^2)^{m+1} \frac{1}{\sqrt{|a|}} \psi \left( \frac{t - b}{a} \right) \int_{\mathbb{R}^{m+1}} \phi \left( x - \frac{b}{a} \right) \frac{1}{\sqrt{|a|}} \frac{dt}{\partial x^{m+1}} \frac{dt}{\partial x^{m+1}} \frac{dt}{\partial x^{m+1}} \right\}.
\]

using integration by parts, note that the integral of the terms enclosed in the curly brackets with respect to \( t \) as a function of \( b \) is infinitely differentiable and belongs to \( L^2_{\psi}(\mathbb{R}) \) for each \( m = 0, 1, 2, 3, \ldots \). So to evaluate the integral term we use integration by parts with the limit terms being zero.

\[
\langle F(x), \phi(x) \rangle = \lim_{r \to \infty} \left\{ \int_{|x| > r} \int_{-\infty}^{\infty} g(t)(1 + t^2)^{m+1} \frac{1}{\sqrt{|a|}} \psi \left( \frac{t - b}{a} \right) \int_{\mathbb{R}^{m+1}} \phi \left( x - \frac{b}{a} \right) \frac{1}{\sqrt{|a|}} \frac{dt}{\partial x^{m+1}} \frac{dt}{\partial x^{m+1}} \right\}.
\]

In Equations (3.11)–(3.13) we could have taken finite limits of integration – \( M, N \) and then after integration by parts let \( M, N \to \infty \); the same results as shown above would have been obtained. Thus, there is no error involved in setting the lower and upper limits of the foregoing integrals as \(-\infty \) and \( \infty \). We also make use of the fact that

\[
\frac{\partial}{\partial b} \left[ \psi \left( \frac{x - b}{a} \right) \right] = - \frac{\partial}{\partial x} \left[ \psi \left( \frac{x - b}{a} \right) \right].
\]

Finally by distributional differentiation, (3.13) and (3.14) become

\[
\langle F(x), \phi(x) \rangle = \lim_{r \to \infty} \left\{ \int_{|x| > r} \int_{-\infty}^{\infty} g(t)(1 + t^2)^{m+1} \frac{1}{\sqrt{|a|}} \psi \left( \frac{t - b}{a} \right) \int_{\mathbb{R}^{m+1}} \phi \left( x - \frac{b}{a} \right) \frac{1}{\sqrt{|a|}} \frac{dt}{\partial x^{m+1}} \frac{dt}{\partial x^{m+1}} \right\}.
\]

If we express the expressions in (3.15) as a fourfold iterated integral by removing the angular brackets, the two expressions will be fourfold iterated integrals in the order \( dt \, db \, da \, dx \). We wish to express them in the order \( dx \, db \, da \) by switching the order of integrations. We cannot apply the Fubini-Tonelli theorem (Yosida, 1995), p.18 at this stage as none of the above iterated integrals is absolutely convergent. We therefore proceed as follows to apply Fubini’s theorem to switch the order of integration. We number the above integrands in (3.15)

\[
\frac{1}{C^2} \phi^{m+1} \left( x - \frac{b}{a} \right) \psi \left( \frac{t - b}{a} \right) g(t) \frac{(1 + t^2)^{m+1}}{|a|^2}
\]

and

\[
\frac{1}{C^2} \phi^{m} \left( x - \frac{b}{a} \right) \psi \left( \frac{t - b}{a} \right) g(t) \frac{(1 + t^2)^{m}2tm}{|a|^2}.
\]

Let \( K \) be a compact set of the “XBAT-space” given by

\[ [(x, b, a, t) : |x| \leq X_1, |b| \leq B_1, |e| \leq |a| \leq A_1, \text{and } |t| \leq T_1]. \]
Then the fourfold iterated integrals of the integrands (3.16) and (3.17) by (Yosida, 1995), p.18, with respect to the measure $dxdbdadt$ are absolutely convergent over compact set $K$ and so are integrable. Therefore, switches in the order of integration over $K$ can be done in $4!$ ways and all these $4!$ iterated integrals of integrands (3.16) and (3.17) are equal in view of Fubini’s theorem. Our concern for the time being is the equality of the fourfold iterated integrals $dxdbdadt$ and $dt db db dxdbda$ of the above mentioned integrands over the compact set $K$, which is valid in view of Fubini’s theorem. We now let $X_1, B_1, A_1$ and $T_1$ all tend to $\infty$ and then we let $\epsilon \to 0$; the fact that the fourfold iterated infinite integrals $dxdbdadt$ of integrands (3.16) and (3.17) are now convergent is proved by using the Plancherel theorem with respect to the Fourier transform $F_{\phi}$ (Boggess & Narcowich, 2001, p. 107; Pandey & Upadhyay, 2015). Hence,

\begin{equation}
\langle F(x), \phi(x) \rangle = \frac{1}{C_{\psi}} \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi(x) \phi(x) \psi(t) \psi(t) \right) dx db da dt + \frac{1}{C_{\psi}} \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi(x) \phi(x) \psi(t) \psi(t) dx db da dt
\end{equation}

(3.18)

Therefore,

\begin{equation}
\langle F(x), \phi(x) \rangle = \frac{1}{C_{\psi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi(x) \phi(x) \psi(t) \psi(t) dx db da dt + \frac{1}{C_{\psi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi(x) \phi(x) \psi(t) \psi(t) dx db da dt
\end{equation}

Now using the wavelet inversion formula the triple integrals

\begin{align*}
\frac{1}{C_{\psi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi(x) \phi(x) \psi(t) dx db da dt \\
and \frac{1}{C_{\psi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi(x) \phi(x) \psi(t) dx db da dt
\end{align*}

converge to $\psi^{(m+1)}(t), \psi^{(m)}(t)$, respectively,

(i) in $L^2(\mathbb{R})$

(ii) pointwise $\forall \ t \in \mathbb{R}$

(iii) uniformly for all $t \in (-\infty, \infty)$.

The results (i), (ii) and (iii) are all proved in (Pandey, Jha, & Singh, 2016), but the proofs of (i) and (ii) can also be found in (Boggess & Narcowich, 2001, p.258). This implies (in view of (iii)) that

\begin{equation}
\langle F(x), \phi(x) \rangle = \int_{\mathbb{R}} \psi^{(m+1)}(t) \phi(t)(1 + t^2)^{m+1} dt + \int_{\mathbb{R}} \psi^{(m)}(t) \phi(t)(1 + t^2)^m 2tm dt
\end{equation}

using continuous wavelet inversion formula

\begin{align*}
= \langle g(t), \phi^{(m+1)}(t)(1 + t^2)^{m+1} \rangle + \langle g(t), \phi^{(m)}(t)2tm(1 + t^2)^m \rangle \text{ using duality notation} \\
= \langle f(x), \phi(x) \rangle \forall \phi \in S(\mathbb{R}) \text{ and } f \in S'(\mathbb{R}),
\end{align*}

using the structure formula (2.1) for $f$.

We have proven that
\[
\langle \frac{1}{C_{\psi}} (P), \int_{-\infty}^{\infty} W_{\psi}(a,b) \frac{1}{\sqrt{|a|}} \psi \left( \frac{x-b}{a} \right) \frac{dbda}{a^2}, \phi(x) \rangle = \langle f, \phi \rangle, \quad f \in S_{\mathcal{F}}(\mathbb{R}). \tag{3.19}
\]

As explained earlier, this inversion formula is valid uniquely if \( f \in S_{\mathcal{F}}(\mathbb{R}) \). If \( f \in S'(\mathbb{R}) \) then \( f = h + c \) where \( h \in S_{\mathcal{F}}(\mathbb{R}) \) and \( c \) is a constant. Therefore, the wavelet transform of \( f \) is
\[
\langle f(x), \frac{1}{\sqrt{|a|}} \psi \left( \frac{x-b}{a} \right) \rangle = \langle h(x), \frac{1}{\sqrt{|a|}} \psi \left( \frac{x-b}{a} \right) \rangle + \int_{-\infty}^{\infty} c \frac{1}{\sqrt{|a|}} \psi \left( \frac{x-b}{a} \right) dx = \langle h(x), \frac{1}{\sqrt{|a|}} \psi \left( \frac{x-b}{a} \right) \rangle.
\]

Hence
\[
W_{\psi}(a,b) = W_{\psi}(a,b)
\]

and so, using (3.19) we get
\[
\langle \frac{1}{C_{\psi}} (P), \int_{-\infty}^{\infty} W_{\psi}(a,b) \frac{1}{\sqrt{|a|}} \psi \left( \frac{x-b}{a} \right) \frac{dbda}{a^2}, \phi(x) \rangle = \langle h, \phi \rangle
\]
\[
= \langle f - c, \phi \rangle
\]
\[
= \langle f, \phi \rangle + \langle -c, \phi \rangle.
\]

This explains the ambiguity in our inversion formula. For the validity of the uniqueness in our inversion formula \( f \) must belong to \( S_{\mathcal{F}}(\mathbb{R}) \).

4. Conclusion
In this present paper authors introduced the continuous wavelet transform on Schwartz tempered distributions and proved the corresponding wavelet inversion formula (valid modulo a constant distribution) interpreting convergence in the weak topology of \( S'(\mathbb{R}) \).

We have observed that our aforesaid investigations are true when the wavelet kernel under consideration for determining the wavelet transform are those wavelets whose all the moments are non-zero. Our entire results and facts are stated and proved as Lemmas and Theorems.

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