Orbit-like trajectory of the vortex core in a magnetic nanodot

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In physics, conserved quantities are key to understanding and describing physical phenomena. These conserved quantities are related to Noether’s theorem and the Lagrangian description both in classical mechanics and in field theory. In this article we have found the equation of the vortex core trajectory in terms of two conserved physical quantities, namely the energy, \( E \), and a vector perpendicular to the orbit plane, \( \hat{r} \). These magnetic textures are very attractive not only from a fundamental point of view but also applied, because these magnetic vortices can contain four degenerate states: the clockwise (\( C = 1 \)) or counterclockwise (\( C = -1 \)) circularity and the upwards (\( p = 1 \)) or downward (\( p = -1 \)) polarity. If we consider that both circularity and polarity can store one bit data, then each magnetic vortex can store two bits of information, which makes them potential candidates for application in future information storage devices. Besides, they usually do not have significant magnetic stray-fields, reducing the effect of the magnetostatic interaction between dots, enabling close packing required for technological applications. However, it is important to note that vortices can interact strongly when they are out of equilibrium.

Considering future spintronic applications, it is important to understand the dynamic response of the vortex core excited by external magnetic fields or spin current. If the vortex core is displaced from its equilibrium position in the dot center, its dipolar energy can be modeled simply as a surface anisotropy and experiences a restoring force dependent on the effects of edge. The response to this force is the gyrotropic model, which is the lowest frequency excitation of a magnetic vortex and can be detected in resonance experiments. It is important to note that this mode is very different from the precessional mode generally seen in uniformly magnetized dots. In addition, the vortex core dynamics has been investigated considering magnetic impurities and notches on a disk. In this way, these magnetic textures have been proposed as frequency-tunable nanoscale microwave generators for future wireless communication and to be used in biological applications.

In order to realize the application of vortex-based devices, it would be convenient to handle the orbits of their cores. In this work we are interested in calculating the equation of the vortex core trajectory as a function of energy, \( E \), and of a vector perpendicular to the orbit plane, \( \hat{r} \). In the presence and absence of a dissipative term. Our results would not only help for deeper understanding on the dynamic properties of vortex core, but could also be interesting for high speed and high density magnetic memories.

II. THEORETICAL MODEL

The movement of the vortex core, assuming a rigid vortex (the spin structure does not vary in the \( z \)-direction), is associated with the topological charge of the magnetic vortex known as the gyrovectors \( \hat{G} \) and \( \hat{c} \) such that \( \hat{G} \times \hat{c} \), where \( \hat{G} = 2\pi q_{ph} M_0 / \gamma \), \( h \) is the dot thickness, \( M_0 \) is the saturated magnetization and \( \gamma = 1.76 \times 10^{11} \text{As/kg} \) is the gyromagnetic ratio. The winding number \( q = 1 \) and \( q = -1 \) corresponds to a vortex and an antivortex, respectively, while the polarity \( p = 1 \) and \( p = -1 \) refers to the vortex core pointing in the +\( z \) and −\( z \) direction, respectively. The gyrovectors is included in the Thiele’s equation that describes the movement of the vortex core, whose generalized form is given by:

\[
\mathcal{M} \frac{d^2 \vec{r}_c}{dt^2} - \hat{G} \times \frac{d\vec{r}_c}{dt} + \vec{V}_c \vec{c} = 0, \tag{1}
\]

where \( \vec{r}_c \) and \( \vec{c} \) are the position and the vertical component of the vortex core and the magnetic energy of the vortex, respectively, while \( \mathcal{M} \) is the vortex mass. In general, \( \mathcal{M} \) is a tensor; however, for our calculations it will be enough to use an \( \mathcal{M} \) scalar. As the vortex core moves in the dot plane (\( xy \)-plane), \( \vec{r}_c = x_i \hat{i} + y_j \hat{j} \) and thus the disturbance of the core will be \( s = |\vec{r}_c| = \sqrt{x_i^2 + y_j^2} \). Equation (1) is obtained from the well-known Euler-Lagrange equation for the Lagrangian function \( \mathcal{L} = \frac{1}{2} \mathcal{M} |\vec{r}_c|^2 - \vec{c} - \hat{G} \cdot \vec{V}_c \).
Figure 1(a) shows a vortex with a circularity \( C = 1 \) and a core deviated a distance \( s \) from the center (equilibrium position) of a circular disk of radius \( R = 100 \) nm. The scheme also shows the polar coordinates \((s, \theta)\) defined above. We can observe that \( \theta \) is a cyclic coordinate, so that \(-\frac{d}{ds} \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = 0\), which allows us to obtain an integral vector of motion perpendicular to the dot plane, \( \vec{A} = \frac{1}{2} \mathcal{G} \vec{r}_c \times \vec{r}_c = \mathcal{L} \vec{r}_c \times \vec{r}_c \), where \( \mathcal{L} = \mathcal{M} \vec{r}_c \times \vec{r}_c \) is the standard angular momentum of a particle with mass \( \mathcal{M} \) (it is considered that the vortex core acts as a soliton).

On the other hand, it is well known that any conserved physical magnitude can be related to preserved symmetries of the system through Noether’s theorem using infinitesimal transformations of the general coordinates of the system. In fact, an infinitesimal rotation of the \( \theta \) angle in the \( xy \)-plane leads us to the conserved vector \( \vec{A} \). As shown in reference[19], \( \vec{A} \) is completely analogous to the vector found in the Hall effect of electrons under the action of a uniform magnetic field applied perpendicular to the two-dimensional movement of electrons (in the absence of an electric field) by replacing the topological gyrovector \( \mathcal{G} \) by a magnetic field.

The conserved vector \( \vec{A} \) can be used to describe the orbit-like trajectory of the vortex core using the conservative energy \( E = \sum \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \mathcal{L} = \mathcal{M} (s^2 + s^2 \dot{\theta}^2)/2 + \varepsilon(s) \), through a strategy similar to that used to describe the orbit of a reduced mass in the well-known Kepler problem (see, for example, reference[19]). In our case, \( E = \mathcal{M} s^2 /2 + \varepsilon(s) \), with an effective potential \( \varepsilon_{eff}(s) = \varepsilon(s) + (G s^2 /2 - A)^2 / (2 \mathcal{M} s^2) \). Thus, the movement of the vortex core is equivalent to the movement of a particle of mass \( \mathcal{M} \) at an effective potential \( \varepsilon_{ef} \). This effective potential can be handled for small displacements of the vortex core. In fact, Guslienko et al.[26] showed that for small deviations of the vortex core, the magnetic energy could be approximated by a polynomial of second order, so that the effective potential for small deviations remains as \( \varepsilon_{eff}(s) = a_0 + a_1 s + a_2 s^2 + (G s^2 /2 - A)^2 / (2 \mathcal{M} s^2) \), where \( a_0 = \varepsilon(0) \) corresponds to the magnetic energy of a vortex without displacement of its core, while \( a_1 = -\pi \mathcal{M} \hbar K_c R /2 \) and \( a_2 = \pi \hbar (K_c R /2 - A_{ex}) /R^2 \). Here \( B \) is the intensity of the magnetic field applied in the plane of the disk, \( A_{ex} \) is the stiffness constant and \( K_c \) is the surface anisotropy constant of a magnetic dot of radius \( R \) and thickness \( h \). Since the first term of \( \varepsilon_{eff} \) does not depend on the deviation of the vortex core, then it is convenient to redefine \( E \rightarrow E - \varepsilon(0) \) and \( \varepsilon_{eff}(s) \rightarrow \varepsilon_{eff}(s) - \varepsilon(0) \).

It is interesting to note that we can obtain the condition for a bounded or unbounded orbit as a function of the constants \( E \) and \( A \) by studying the form of the effective potential. In fact, if the energy \( E \) is equal to an extremity of \( \varepsilon_{eff} \), then the vortex core will describe a circular, stable or unstable movement, depending on whether it is a minimum or a maximum of the effective potential. On the other hand, if the energy \( E \) is slightly above a minimum or a maximum, then the vortex core will describe a bounded or unbounded orbit. Figure 1b shows the effective potential of a vortex core displaced slightly from its equilibrium position in a cobalt dot of \( R = 100 \) nm radius and \( h = 10 \) nm thickness, for different magnitudes of the external magnetic field applied parallel to the dot plane. The magnetic parameters were extracted from reference[14]: \( M_0 = 1.4 \times 10^6 \) A/m, \( A_{ex} = 3 \times 10^{-11} \) J/m and \( K_c = 1.518 \times 10^{-20} \) J/mm². Besides, we used \( A = G s_0^2 /2 \approx 2.5 \times 10^{-17} \) pJ/s, with \( s_0 = 10 \) nm (deviation of the vortex core respect to the dot center at \( t = 0 \)) and circularity \( C = -1 \).

III. RESULTS AND DISCUSSION

A. Orbit-like trajectory of the vortex core for \( B = 0 \)

The extremity values of the effective potential are obtained from \( \frac{d}{ds} \varepsilon_{eff} = 0 \). We can go further analytically by studying the movement of the vortex core when the magnetic field is turned off \( (a_1 = 0) \), so that the extremities of the effective potential are obtained from the following equation \( s_x^2 = 2A /\sqrt{G^2 + 8 \mathcal{M} a_2} \), from which the bounded orbits satisfy the following condition \( \Delta = 2A^2 a_2 - AEG - E^2 \mathcal{M} \leq 0 \). In the particular case \( s_0 \geq 0 \) and \( v(t = 0) = 0 \), the bounded orbit condition is given by \( -\mathcal{M} a_2 s_0^2 /2 \leq 0 \), so that the vortex core orbit will always be bounded when \( v(t = \infty) = 0 \) and the magnetic field is turned off. The turning points \( s_+ \) and \( s_- \) of the orbit are

\[
 s_x^2 = \frac{2AG + 4E\mathcal{M} \pm 4\sqrt{-\mathcal{M} \Delta}}{G^2 + 8a_2 \mathcal{M}}. \\
(3)
\]

It is important to note that when \( v(t = 0) = 0 \), then \( s_+ = s_0 \) and \( s_- = s_0 \sqrt{G^2 / (G^2 + 8a_2 \mathcal{M})} \).

From the above conditions, energy \( E^* = ( -AG + A\sqrt{G^2 + 8\mathcal{M} a_2}/2\mathcal{M} \) defines a circular movement of the vortex core with a radius \( s^* = \sqrt{2A /G} \) and energy \( E^* = 2A a_2 /G \). In addition, through direct integration we can obtain \( s(t) \) and \( \theta(t) \), which can be expressed as standard functions.

![FIG. 1. (color online) (a) Scheme of a vortex with circularity \( C = 1 \) in a circular dot, with its core deviated a distance \( s \) from the dot center. (b) Effective potential as a function of the core deviation length for a vortex in a cobalt dot of \( R = 100 \) nm and thickness \( 10 \) nm for a vortex circularity \( C = -1 \), \( \mathcal{M} /\mathcal{M} = 20 \) kHz and \( A = 2.5 \times 10^{-17} \) pJ/s.](image-url)
when the magnetic field is turned off:
\[
s^2(t) = \frac{\mathcal{M}^2 A^2 \Omega^2 - (f(s_0) e^{\mathcal{M} t} - c_2)^2}{\mathcal{M}^2 \Omega^2 e^{\mathcal{M} t} f(s_0)}
\]
(4)

\[
\theta(t) - \theta_0 = \frac{G}{2\mathcal{M}} t - \frac{A}{\mathcal{M}} \int_0^t \frac{dt}{s^2(t)},
\]
(5)

where \(s_0\) and \(\theta_0\) are the deviation and the angle of the vortex core at \(t = 0\), respectively, \(\Omega = \sqrt{8\mathcal{M} a_2 + G^2 / \mathcal{M}}\), \(c_2 = 2\mathcal{M} E + GA\) and \(f(s_0) = c_2 - \mathcal{M}^2 \Omega^2 s_0^2 / 2 + \mathcal{M} \Omega \sqrt{A^2 + \mathcal{M}^2 \Omega^2 s_0^2 / 4 - c_2 s_0^2}\).

Figures 2a and 2b shows the orbit-like trajectory of the vortex core in a cobalt dot obtained both analytically (red line, using Eqs. (4) and (5)) and numerically (blue dots, solving Eq. (1)) for different \(G/\mathcal{M}\) values. We have used \(a_1 = 0\), \(s_0 = 10\) nm, \(\theta_0 = \pi/2\) and \(v_0 = 0\). From this figure we can see that the orbit-like trajectory is bounded between two concentric circles of radius \(s_+\) and \(s_-\) (represented with dashed lines, \(s_{\pm}\) were obtained using Eq. (3)). Figures 2a and 2b show the temporal evolution of \(A, G s^2 / 2\) and \(L_z\) as the vortex core moves in Figs. 2a and 2b, respectively.

FIG. 2. (color online) Orbit-like trajectory of the vortex core in a cobalt dot of \(R = 100\) nm radius and \(h = 10\) nm thickness in the absence of a magnetic field (\(B = 0\)) when (a) \(G/\mathcal{M} = 20\) kHz and (b) \(G/\mathcal{M} = 100\) kHz. We have used \(s_0 = 10\) nm, \(\theta_0 = \pi/2\) and \(v_0 = 0\). The temporal evolution of \(A, G s^2 / 2\) and \(L_z\) as the vortex core moves in Fig. 2a and 2b are shown in Figs. 2c and 2d, respectively.

B. Orbit-like trajectory of the vortex core for \(B \neq 0\)

When the magnetic field is non-zero, the coefficient \(a_1 \neq 0\), so that the orbit-like trajectory of the vortex core depends on its circularity. Although there is no analytical solution for \(B\) other than zero, it is possible to obtain the vortex core path by numerical integration of Eq. (1). Figures 3 show the orbit-like trajectory of the vortex core when a magnetic field of 30 mT is applied in the \(x\)-direction for an initial small deviation of \(s_0 = 10\) nm in the \(y\)-direction, for different values of \(G/\mathcal{M}\). Figures 3a and 3b consider a vortex with a circularity \(C = -1\), while Figs. 3c and 3d consider a circularity \(C = 1\). From the previous figures it can be seen that the orbit is still bounded, although its shape now clearly depends on its circularity.

FIG. 3. (color online) Orbit-like trajectory of the vortex core in a cobalt dot of \(R = 100\) nm radius and \(h = 10\) nm thickness in the presence of a magnetic field (\(B = 30\) mT) when (a,c) \(G/\mathcal{M} = 20\) kHz and (b,d) \(G/\mathcal{M} = 100\) kHz. (a) and (b) consider a circularity \(C = -1\), while (c) and (d) a circularity \(C = 1\).

Figure 4 shows the size of region \(|s_+ - s_-|\) where the orbit-like trajectory can move as a function of \(G/\mathcal{M}\) for different magnitudes of the external magnetic field applied in the \(x\)-direction, when the initial small deviation is \(s_0 = 10\) nm in the \(y\)-direction and \(v_0 = 0\) for (a) \(C = -1\) and (b) \(C = 1\). From the previous figure it can be seen that the orbit is bounded and that the region size depends on the circularity of the vortex. If \(C = -1\), then \(s_+ = s_0\), while for \(C = 1\), \(s_- = s_0\). It is also interesting to note that for \(B = 10\) mT and \(C = 1\), \(|s_+ - s_-| \ll 1\), since the energy \(E\) is slightly higher the minimum of the effective potential, and moreover, the initial small deviation \(s_0\) in that case is very close to the distance \(s\) which minimize the magnetic energy \(\varepsilon(s)\) of the deviated vortex core, as showed in Ref. 23, which explain why the vortex core describes almost the movement of a circle of radius close to \(s_0\) for that case.

In the case where \(E\) can satisfy the bounded orbit condition, it is important to note that the closed/unclosed orbit condition...
will be given by
\[
\int_{s_0}^{s_n} ds \frac{(G s^2/2 - A) / (\mathcal{M} s^2)}{\sqrt{\mathcal{M} (E - \epsilon_{\text{eff}}(s))}} = \alpha 2\pi ,
\]
so that the orbit will be closed if \( \alpha \) is a rational number, that is, \( \alpha = n_1/n_2 \) with \( n_1, n_2 \) integers numbers. \[\text{[Equation 6]}\]

C. Orbit-like trajectory of the vortex core with a dissipative term

The analysis carried out so far has been in the absence of a dissipative term. If we include a dissipative term in Eq. (1), the vortex core will be described by\[\text{[Equation 7]}\]
\[
\mathcal{M} \frac{d^2 \vec{r}_c}{dt^2} - \vec{G} \times \frac{d \vec{r}_c}{dt} + \vec{\nabla}_r \epsilon + D \frac{d \vec{r}_c}{dt} = 0 ,
\]
where the dissipation dyadic \( D \) is a tensor and its leading term is diagonal \( D = D_0 \), which is proportional to the damping coefficient \( \alpha d \). With this dissipative term, the energy \( E \) is no longer conservative and \( \vec{A} \) varies in time through \( d\vec{A}/dt = D_0 \vec{L}/\mathcal{M} \). If we turn off the external magnetic field, then the vortex core will evolve towards the dot center, as shown in Fig. 5 due to the dissipative term. A similar effect was observed in reference\[\text{[13]}\]by micromagnetic simulations.

On the other hand, if we maintain an applied external magnetic field then the movement of the vortex core will depend on the circularity of the vortex, as shown in Fig. 6. These figures show the orbit-like trajectory of the vortex core when \( B = 30 \) mT (applied in the \( x \)-direction), \( G/\mathcal{M} = 20 \) kHz when (a) \( C = -1 \) and (b) \( C = 1 \). The curves were obtained solving numerically Eq. (7). Figs. 6a and 6b show the temporal evolution of \( A, G s^2/2 \) and \( L_z \) as the vortex core moves in Figs. 6a and 6b, respectively.

D. Orbit-like trajectory of the vortex core for any initial deviation (not necessarily small)

In order to provide a complete study of the orbit-like trajectory of the vortex core, we will consider an arbitrary initial
deviation (which may be large enough), so that the $\varepsilon(s) = a_{1}s + a_{2}s^2$ approximation for magnetic energy is no longer valid and, therefore, the exact analytical expression for each energy term included in $\varepsilon(s)$ should be considered, including the contribution of the cylinder mantle to dipolar energy. Nevertheless, this last expression can be approximated with high accuracy by the surface energy term introduced in Ref. [23], which allow to study the orbit-like trajectory for any initial deviation and intense magnetic fields, without great computational effort. Fig. 7 shows the orbit-like trajectory of the vortex core for $G/\mathcal{M} = 20$ kHz, $s_0 = 80$ nm and a magnetic field in the $x$-direction of 50 mT. (a and b) correspond to $C = -1, 1$ (without dissipation) while (c and d) correspond to $C = -1, 1$ (for $D_0 = G/10$), respectively. From the previous figures it can be seen that in the absence of a dissipative term, the orbit is still bounded between two concentric circles of radii $s_-$ and $s_+$, while in the presence of a dissipative term, the orbit collapses towards the dot center if the initial deviation decreases the region of the magnetic moments aligned with the external field applied in the $x$-direction (Fig. 7d), or stops before reaching the dot center if the initial deviation increases the region of magnetic moments aligned with the magnetic field (Fig. 7b).

**FIG. 7.** (color online) Orbit-like trajectory of the vortex core in a cobalt dot of $R = 100$ nm radius and $h = 10$ nm thickness in the presence of a magnetic field in the $x$-direction ($B = 50$ mT) for (a,b) in absence of a dissipative term (for $C = -1, 1$, respectively), and (c,d) in presence of a dissipative term (for $C = -1, 1$, respectively). We have used $s_0 = 80$ nm, $\theta_0 = \pi/2$, $v_0 = 0$, $G/\mathcal{M} = 20$ kHz.

### IV. CONCLUSIONS

In the absence of a dissipative term, the orbit-like trajectory of the vortex core is governed by two conserved quantities: mechanical energy $E$ and vector $\vec{A}$. We have shown that the movement of the vortex core is equivalent to the movement of a quasi-particle of mass $\mathcal{M}$ at an effective potential $\varepsilon_{\text{eff}}$. The form of the effective potential depends on the magnetic energy of the vortex. Thus, for small deviations of the vortex core from the dot center, we show analytically that the orbit-like trajectory is bounded between two concentric circles of radius $s_-$ and $s_+$. On the other hand, the shape of the orbit described by the vortex core depends on the $G/\mathcal{M}$ frequency, the intensity of the external magnetic field, the initial deviation of the vortex core from its equilibrium position and the circularity of the vortex. Instead, by including a dissipation term both quantities $E$ and $\vec{A}$ are no longer conservatives and depending of the intensity of the magnetic field and vortex circularity, the core evolves into the dot center or moves away until it stops. The first hold if the magnetic field is turned off or the initial deviation decreases the region of magnetic moments pointing in the magnetic field direction, while the last hold if the region of magnetic moment aligned with the magnetic field direction is increased.

It is worth mentioning that the cycloidal orbit of the vortex core is maintained even for a pair of well separated vortices, that is, that the distance between the vortex cores is greater than the sum of the radii of their cores, which suggests that the analysis presented here can be extended to two well separated vortices. Moreover, cycloidal orbits have been observed both experimentally and numerically in skyrmions. In this way, it would be interesting to extend our analysis for deviated skyrmionic textures.
