THE SPINLESS RELATIVISTIC KINK-LIKE PROBLEM

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Abstract

We constrain the possible bound-state solutions of the spinless Salpeter equation (the most obvious semirelativistic generalization of the nonrelativistic Schrödinger equation) with an interaction between the bound-state constituents given by the kink-like potential (a central potential of hyperbolic-tangent form) by formulating a bunch of very elementary boundary conditions to be satisfied by all solutions of the eigenvalue problem posed by a bound-state equation of this type, only to learn that all results produced by a procedure very much liked by some quantum-theory practitioners prove to be in severe conflict with our expectations.

Keywords: relativistic bound states, Bethe–Salpeter formalism, spinless Salpeter equation, kink-like potential

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1 Introduction

Within quantum physics, nonrelativistic bound states of spinless particles are described by the time-independent Schrödinger equation. A generalization of such bound-state equation towards a relativistic treatment of bound states is found if replacing the nonrelativistic free energy by its relativistically correct counterpart. The outcome of the improvement is called the spinless Salpeter equation. This name of the latter derives from the fact that it emerges in the course of the three-dimensional reduction of the Bethe–Salpeter formalism [1], which constitutes a commonly accepted formalism for the Lorentz-covariant description of bound states within quantum field theory: assuming in the homogeneous Bethe–Salpeter equation all bound-state constituents to propagate freely and interact instantaneously simplifies this equation to the so-called Salpeter equation [2]; disregarding, moreover, the negative-energy contributions and ignoring the spin degrees of freedom of all bound-state constituents leads finally to the spinless Salpeter equation. By construction, such spinless Salpeter equation is the eigenvalue equation of a Hamiltonian

\[ H \equiv T(p) + V(x), \quad T(p) \equiv \sqrt{p^2 + m_1^2} + \sqrt{p^2 + m_2^2}. \]  

(1)

Hamiltonians of the above type are, in general, nonlocal operators. Hence, finding exact and, in particular, analytic solutions to spinless-Salpeter problems is a definitely nontrivial task. However, one may try to deduce rigorous constraints on the predicted energy spectra. Here, we do this for what is called the kink-like potential, a spherically symmetric potential \( V(r) \) \((r \equiv |x|)\), depending on a range parameter \( \rho \) and a dimensionless coupling constant \( \kappa \):

\[ V(x) = V_K(r) \equiv \kappa \rho \tanh(\rho r) \equiv \kappa \rho \frac{\exp(\rho r) - \exp(-\rho r)}{\exp(\rho r) + \exp(-\rho r)}, \quad \rho > 0, \quad \kappa \geq 0. \]  

(2)

Clearly, this potential function \( V_K(r) \) is strictly increasing from \( V_K(0) = 0 \) to \( V_K(\infty) = \kappa \rho \):

\[ V_K(0) = 0 \leq V_K(r) \leq V_K(\infty) = \kappa \rho. \]

So, this potential is evidently bounded from below and, therefore, also the Hamiltonian (1). We get constraints on both energy spectra (Sec. 2) and number of bound states (Secs. 3, 4).

2 Eigenvalue Constraints from Operator Inequalities

“Very frequently,” that is, in fact, in almost all instances, it proves impossible to determine, by analytic means, energy levels of given bound-state problems in quantum physics exactly. In such situation, the location of the energy eigenvalues may be constrained by comparison with bound-state problems for which knowledge about their eigenvalue spectra is available. A valuable tool in any endeavour of this kind is the following spectral comparison theorem: \(^1\)

For any pair of self-adjoint semibounded operators \( A \) (with eigenvalues \( a_k, k \in \mathbb{N}_0 \), ordered by \( a_0 \leq a_1 \leq a_2 \leq \cdots \)) and \( B \) (with eigenvalues \( b_k, k \in \mathbb{N}_0 \), ordered by \( b_0 \leq b_1 \leq b_2 \leq \cdots \)) satisfying the inequality \( A \leq B \), the eigenvalues below onset of the essential spectrum fulfil

\[ a_k \leq b_k, \quad k \in \mathbb{N}_0. \]

\(^1\)For a proof, see Refs. [3, Sec. III], [4, Sec. 3], [5, Sec. 2], [6, Appendix], [7, Subsec. 3.1], or [8, Appendix].
We find numerous possibilities for applying this theorem to a semirelativistic Hamiltonian, in general, or to the particular instance of bound-state problems with a kinky potential (2).

By definition, the relativistic free-energy term $T(p)$ defined in Eq. (1) and — because of $V_K(r) \geq 0$ — the Hamiltonian $H$ with interaction potential (2) satisfy trivial lower bounds:

$$T(p) \geq m_1 + m_2 \geq 0, \quad H \geq m_1 + m_2 \geq 0.$$ 

Therefore, all eigenvalues $E_k, k = 0, 1, 2, \ldots$, of our operator $H$ are bounded from below by

$$E_k \geq m_1 + m_2 \geq 0, \quad k = 0, 1, 2, \ldots;$$

hence, its binding energies $B_k \equiv E_k - m_1 - m_2, k = 0, 1, 2, \ldots, \geq 0$.

A trivial upper bound to each semirelativistic Hamiltonian (1) arises from the fact that, regarded as functions of $p^2$, the Schrödinger free term $T_{NR}(p)$ found as nonrelativistic (NR) limit is linear and tangent to the sum of square roots in the relativistic kinetic energy $T(p)$:

$$T(p) \leq T_{NR}(p) \equiv m_1 + m_2 + \frac{p^2}{2m_1} + \frac{p^2}{2m_2} \quad \implies \quad H \leq H_{NR} \equiv T_{NR}(p) + V(x).$$

Our spectral comparison theorem then tells us that any Schrödinger energy eigenvalue $E_{S,k}$ is an upper bound to its spinless-Salpeter counterpart $E_k$: $E_k \leq E_{S,k}, k = 0, 1, 2, \ldots$. Thus, for a given potential, there is a bound state of $H$ below each bound state of $H_{NR}$. Hence, the number of bound states of $H, N$, is not lower than the number $N_{NR}$ of bound states of $H_{NR}$:

$$N \geq N_{NR}.$$ 

For our quest, it is convenient and advantageous to define the shifted kink-like potential

$$\tilde{V}_K(r) \equiv V_K(r) - \kappa \rho = \kappa \rho [\tanh(\rho r) - 1] = -\frac{2\kappa \rho}{1 + \exp(2\rho r)},$$

which is negative and monotonically increasing for $r < \infty$, and approaches zero for $r \to \infty$:

$$\tilde{V}_K(0) = -\kappa \rho \leq \tilde{V}_K(r) \leq \tilde{V}_K(\infty) = 0, \quad \lim_{r \to \infty} \tilde{V}_K(r) = 0.$$ 

Such shift has no implications on basic characteristics of bound-state problems, such as the number of bound states, and evident ones, easily taken into account, on energy eigenvalues. Finding simple potentials that form either an upper or a lower bound to the (shifted or not) kink-like potential does not pose a particularly big challenge. We discuss but a few of these:

- As border case, a candidate suggesting itself for comparison is the Coulomb potential

  $$V_C(r) \equiv -\frac{\alpha}{r}, \quad 0 \leq \alpha < \alpha_c = \frac{4}{\pi} = 1.273239 \ldots.$$ 

  Its critical coupling $\alpha_c$ arises from demanding the relativistic Coulomb problem to be bounded from below [9]. This potential is obviously a lower bound to $\tilde{V}_K(r)$ if $\alpha = \kappa$:

  $$V_C(r) \leq \tilde{V}_K(r) \quad \text{for } \alpha = \kappa.$$
This lower bound can be optimized by diminishing the value of the Coulomb coupling from $\alpha = \kappa$ until Coulomb potential and shifted kinky potential $\tilde{V}_K(r)$ get in contact. Requesting, at the point of contact, equality of the potentials and of their derivatives, we obtain a class of Coulomb lower bounds to $\tilde{V}_K(r)$ for all Coulomb couplings $\alpha \geq \tilde{\alpha}$, where the touching Coulomb coupling $\tilde{\alpha}$ is the solution of the $\rho$-independent equation

$$1 + \frac{\tilde{\alpha}}{\kappa} = \log \frac{\kappa}{\tilde{\alpha}} \quad \implies \quad \tilde{\alpha} \leq \kappa.$$ 

For the relativistic Coulomb problem, in turn, rigorous lower bounds to the spectrum $\sigma(H)$ of the operator $H$ could be given [9,10] for the equal-mass case $m_1 = m_2 = m$:

$$\sigma(H) \geq 2m \sqrt{1 - \left(\frac{\alpha}{\alpha_c}\right)^2} = 2m \sqrt{1 - \left(\frac{\pi \alpha}{4}\right)^2} \quad \text{for } \alpha < \alpha_c ,$$

$$\sigma(H) \geq 2m \sqrt{1 + \sqrt{1 - \alpha^2}} \quad \text{for } \alpha \leq 1 .$$

- An exponential potential with the same potential parameters as in the kink-like case,

$$V_E(r) \equiv -\kappa \rho \exp(-\rho r) ,$$

constitutes a lower bound to the kink-like potential (2), as is straightforward to show:

$$V_E(r) \leq \tilde{V}_K(r) .$$

- Likewise, it is an easy task to convince oneself that the exponential-squared potential

$$V_{E2}(r) \equiv -\kappa \rho \exp(-2\rho r) ,$$

i.e., with slope twice as large as in Eq. (2), is an upper limit to the kink-like potential:

$$\tilde{V}_K(r) \leq V_{E2}(r) .$$

### 3 Number of Bound States of Schrödinger Problems

Estimating the maximum number $N_{NR}$ of bound states accommodated by a nonrelativistic bound-state problem is, in view of the pertinent results available, not exorbitantly difficult. Among the first results in this respect is the bound by Bargmann [11]: For the Hamiltonian

$$H = \frac{p^2}{2\mu} + V(r) , \quad \mu > 0 ,$$

with $\mu$ given either by $\mu = m$, for a single bound particle of mass $m$, or by the reduced mass

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

of a system of two bound particles of masses $m_1, m_2$, Bargmann finds, as constraint to $N_{NR}$,

$$N_{NR} \leq \frac{I(I + 1)}{2} , \quad I \equiv 2\mu \int_0^\infty dr r |V_-(r)| ,$$

where just all negative potential parts matter: $V_-(r) \equiv -\max[0, -V(r)] = V(r) \theta(-V(r))$. For a shifted kinky potential (3), $I$ has to be proportional to $\mu \kappa / \rho$ on dimensional grounds:

$$I = 2\mu \int_0^\infty dr r |\tilde{V}_K(r)| = \frac{\pi^2 \mu \kappa}{12 \rho} , \quad N_{NR} \leq \frac{\pi^2 \mu \kappa}{24 \rho} \left(\frac{\pi^2 \mu \kappa}{12 \rho} + 1 \right).$$
4 Maximal Number of Semirelativistic Bound States

For a class of bound-state equations including the spinless Salpeter equation, I. Daubechies has found an easy-to-apply upper limit to the total number $N$ of bound states [12]. Assume that the Hamiltonian controlling the system under consideration, $H \equiv K(|p|) + V(x)$, is an $L^2(\mathbb{R}^3)$ operator composed of two sufficiently restricted ingredients: a kinetic energy $K(|p|)$ that is a strictly increasing, differentiable function of only the modulus of $p$ and a potential $V(x)$ that is a smooth function of compact support, $V \in C^\infty_0(\mathbb{R}^3)$, satisfying the conditions

$$
K(|p|) \geq 0, \quad K(0) = 0, \quad \lim_{|p| \to \infty} K(|p|) = \infty, \quad V(x) \leq 0.
$$

Under these conditions, the total number $N$ of bound states of $H$ is bounded from above by

$$
N \leq \frac{C}{6 \pi^2} \int d^3x \left[ K^{-1}(|V(x)|) \right]^3; \quad (4)
$$

the constant $C$, converting semiclassical into quantum limit [12], can be found numerically:

$$
C = \inf_{b > 0} \left\{ e^b \int_0^\infty \frac{dy}{y^2} e^{-by} [g(y)]^3 \right\} \left\{ b \int_0^\infty \frac{dy}{y + 1} e^{-by} \right\}^{-1}, \quad g(y) \equiv \sup_{x > 0} \frac{K^{-1}(xy)}{K^{-1}(x)}.
$$

For two particles of arbitrary masses $m_1, m_2$, the relativistic kinetic term, $K(|p|)$, becomes

$$
K(|p|) = \sqrt{|p|^2 + m_1^2} + \sqrt{|p|^2 + m_2^2} - m_1 - m_2.
$$

The inverse of this free-energy function, $K^{-1}(x)$, required by the bound (4), is easily found:

$$
K^{-1}(x) = \frac{\sqrt{x(x + 2m_1)(x + 2m_2)(x + 2m_1 + 2m_2)}}{2(x + m_1 + m_2)}, \quad x \geq 0.
$$

For the special case of equal masses, i.e., if $m_1 = m_2 = m$, this inverse function simplifies to

$$
K^{-1}(x) = \frac{\sqrt{x(x + 4m)}}{2}, \quad x \geq 0.
$$

So, the total number of bound states of the two-particle spinless-Salpeter equation satisfies

$$
N \leq \frac{C}{12 \pi} \int_0^\infty dr r^2 [V(r) (|V(r)| + 4m)]^{3/2},
$$

with the conversion factor $C = 14.107590867$ for $m > 0$ or $C = 6.074898097$ for $m = 0$ [13].

5 Application to Approximate Bound-State Solution

The kink-like potential (2) has met interest in the context of the Dirac equation [14,15], the Klein–Gordon equation [16], and also the spinless Salpeter equation [17]. So, let us examine whether the outcomes of Ref. [17] fit to the general restrictions collected in Secs. 2, 3, and 4.

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2For relativistic kinetic terms, the bound (4) holds for any potential $V(x) \leq 0$ in $L^{3/2}(\mathbb{R}^3) \cap L^3(\mathbb{R}^3)$ [12].
In order to define a given spinless Salpeter problem under consideration unambiguously and completely, the numerical values of the relevant mass and potential parameters have to be specified. Basically for illustrative purposes and simplicity of notation, below the case of bound-state constituents of equal masses, \( m_1 = m_2 \), will be in the focus of our interest. In Ref. [17], the numerical results for the binding energies are presented in form of one table and three figures, each of these relying on a (partly) different choice of the numerical values of the involved parameters required as input; we reproduce these parameter sets in Table 1. To facilitate comparison, we consider parameter values compatible with the sets in Table 1.

Table 1: Numerical values adopted in Ref. [17, Table 1 and Figs. 1–3] for the masses \( m_1, m_2 \) of the bound-state constituents and the parameters \( \rho \) and \( \kappa \) of the “kink-like” potential (2).

| Set of results | \( m_1 \) (arb. unit) | \( m_2 \) (arb. unit) | \( \rho \) (arb. unit) | \( \kappa \) |
|----------------|----------------------|----------------------|----------------------|-----|
| Table 1        | 2.0                  | 0.5                  | 0.01                 | 0.01|
| Figure 1       | 0.5                  | 0.5                  | 0.0001               | 0–0.1|
| Figure 2       | 0–2.0                | 0.5                  | 0.0001               | 0.1 |
| Figure 3       | 2.0                  | 0–2.0                | 0.0001               | 0.1 |

Having succeeded to formulate a well-defined bound-state problem, one of the very first questions that arise is that of the mere number of bound states to expect as solutions of this problem. To give but an idea, we list in Table 2 the maximal numbers of bound states of the relativistic kink-like problem for specific parameter values within their intervals in Table 1: we obtain fairly large numbers of possible bound states, primarily owing to the fact that for the parameter values adopted in Ref. [17], particularly that of the range \( \rho \) in the figures, the kink-like potential is extremely shallow. All binding energies can vary only over the interval

\[
V_K(0) = 0 \leq B_k \leq V_K(\infty) = \kappa \rho , \quad k = 0, 1, 2, \ldots .
\]

For \( \rho = 0.0001 \) and \( \kappa = 0.1 \), this is really tiny, compared to masses of \( O(1) \): \( 0 \leq B_k \leq 10^{-5} \).

Table 2: Upper bounds \( \overline{N} \geq N \) (\( \overline{N}_{NR} \geq N_{NR} \)), computed along the lines discussed in Sec. 4 (Sec. 3), to the number \( N \) (\( N_{NR} \)) of bound states caused by the relativistic (nonrelativistic) Hamiltonian \( H \) (\( H_{NR} \)) defined in Sec. 1 (Sec. 2) with the kink-like potential (2), for selected numerical values of mass and potential parameters used by Ref. [17, Table 1 and Figs. 1–3].

| Set of results | \( m_1 \) (arb. unit) | \( m_2 \) (arb. unit) | \( \rho \) (arb. unit) | \( \kappa \) | \( \overline{N} \) | \( \overline{N}_{NR} \) |
|----------------|----------------------|----------------------|----------------------|-----|----------------|----------------|
| Table 1        | 2.0                  | 0.5                  | 0.01                 | 0.01| —              | 0              |
| Figure 1       | 0.5                  | 0.5                  | 0.0001               | 0.01| 171            | 221            |
| Figure 2       | 0.5                  | 0.5                  | 0.0001               | 0.1 | 5419           | 21241          |
| Figure 3       | 2.0                  | 2.0                  | 0.0001               | 0.1 | 43356          | 338637         |
Our logically next move must be to narrow down the location of the energy levels of the problem under study with a sufficient degree of rigour. This can be achieved, among others, by exploitation of variational techniques or envelope theory, reviewed in, e.g., Refs. [18–20]; for recent use of Rayleigh–Ritz methods in spinless-Salpeter problems, see Refs. [13,21–23]. The (asserted) ultimate outcomes of the sequence of simplifying assumptions applied in Ref. [17] in order to be able to find a kind of approximate solution to the spinless relativistic kink-like problem, for bound states of vanishing orbital angular momentum only, consist of

- a quadratic relation, Eq. (26) of Ref. [17], for the bound states’ binding energies, with the explicit expressions for its pair of solutions given in Eq. (27) of Ref. [17], as well as
- an associated set of (unnormalized) bound-state wave functions, Eq. (37) of Ref. [17].

However, the supposedly exact solutions to the spinless Salpeter equation with kink-like potential proposed in Ref. [17] give rise to considerable concerns. Although — as recalled in Sec. 2 — all binding energies emerging from a spinless Salpeter equation with non-negative potential (which holds for the kink-like potential, cf. Sec. 1) are, already by definition of the potential, non-negative, strangely enough the numerical values of all binding energies given in Ref. [17, Table 1 and Figs. 1–3] are (upon reinstalling obviously missing negative signs in the labels of the ordinate of Fig. 3 in Ref. [17]) negative. Consequently, among the results of Ref. [17] there is nothing left for us to compare with, since acceptable binding energies have to be strictly positive. As a matter of fact, already the most cursory inspection reveals that, irrespective of parameters used, both roots of Eq. (26) of Ref. [17] are bound to be negative.

6 Summary and Concluding Remarks

Refraining from dealing with just numerical or analytical but just approximate solutions to the spinless Salpeter equation for which the achieved accuracy is not entirely under control, we formulated for the semirelativistic bound-state problem posed by such spinless Salpeter equation with hyperbolic-tangent-shaped “kink-like” central potential a couple of rigorous, but still elementary boundary conditions each corresponding exact solution should respect. Astonishingly, a recent study of precisely this problem [17], following a rather popular (and thus in these surroundings frequently adopted) route of approximations comes forth with a tentative solution that fails in satisfying already the most trivial of our general constraints.

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