Spectral analysis of the Neumann–Poincaré operator on the crescent-shaped domain and touching disks and analysis of plasmon resonance *

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Abstract

We consider the Neumann–Poincaré operator on a planar domain enclosed by two touching circular boundaries. This domain, which is a crescent-shaped domain or touching disks, has a cusp at the touching point of two circles. We analyze the operator via the Fourier transform on the boundary circles of the domain. In particular, we define a Hilbert space on which the operator is bounded, self-adjoint. We then obtain the complete spectral resolution of the Neumann–Poincaré operator. On both the crescent-shaped domain and touching disks, the Neumann–Poincaré operator has only absolutely continuous spectrum on the closed interval $[-1/2, 1/2]$. As an application, we analyze the plasmon resonance on the crescent-shaped domain and touching disks.

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Key words. Neumann–Poincaré operator; Touching disks; Spectral resolution; Resonance

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1 Introduction

Spectral analysis of the Neumann–Poincaré (NP) operator has received much attention in recent years due to its applications to electromagnetic problems in metamaterials, such as localized surface plasmon resonance of nanoparticles and invisibility cloaking [1, 5, 12, 13, 19, 20, 32, 33]. The NP operator is a singular integral operator which naturally appears when one solves interface problems for the Laplacian by using the layer potentials. More precisely, given a simply connected bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$, the NP operator $K_{\partial \Omega}$ is defined by

$$K_{\partial \Omega}(\psi)(x) = \text{p.v.} \frac{1}{2\pi} \int_{\partial \Omega} \frac{\langle x - y, \nu_x \rangle}{|x - y|^2} \psi(y) d\sigma(y), \quad x \in \partial \Omega,$$

for a density function $\psi \in L^2(\partial \Omega)$, where p.v. denotes the Cauchy principal value, and $\nu_x$ denotes the outward unit normal vector to $\partial \Omega$ at $x \in \partial \Omega$. The NP operator is similarly defined in three dimensions [25, 39]. In this paper, we investigate the spectrum of the Neumann–Poincaré operator on a planar domain that is enclosed by two touching circular boundaries: a crescent-shaped domain and touching disks (see Figure 1.1). This domain has a cusp at the touching point of the two boundary circles. To the best knowledge of the authors, this is the first article providing the spectral resolution of the NP operator on a planar domain with a cusp.

![Figure 1.1: A crescent-shaped domain (left) and touching disks (right).](image)

For the case of a Lipschitz domain (without a cusp point on its boundary), the spectral property of the NP operator has been studied in many literatures. Let us review some essential
results. We refer to a review article [7] and the references therein for more results. The NP operator is not symmetric on $L^2(\partial\Omega)$ unless $\Omega$ is a disk or a ball [30]. However, it can be realized as a self-adjoint operator on $H^{0,-1/2}(\partial\Omega)$ with a new inner-product which is differently defined but equivalent to the original inner-product, based on Plemelj’s symmetrization principle [4, 22, 26]. Here, $H^{0,-1/2}(\partial\Omega)$ is the Sobolev space $H^{-1/2}(\partial\Omega)$ with the mean-zero condition. We denote by $\mathcal{H}^*$ the space $H^{0,-1/2}(\partial\Omega)$ equipped with the new inner product. Since $\mathcal{K}^s_{\partial\Omega}$ is self-adjoint on $\mathcal{H}^*$, its spectrum $\sigma(\mathcal{K}^s_{\partial\Omega})$ is a closed set contained in the real line. In fact, the spectrum of the NP operator on $L^2_0(\partial\Omega)$ (or $\mathcal{H}^*$) is contained in $(-1/2, 1/2)$ [16, 17, 25] (see also [24, 27] for the permanence of the spectrum for the NP operator with different norms).

For a $C^{1,\alpha}$ domain, $\mathcal{K}^s_{\partial\Omega}$ on $\mathcal{H}^*$ is compact as well as symmetric so that it admits only discrete real eigenvalues, namely $\lambda_j$, as its spectrum, where zero is the only possible accumulation point. The NP operator admits the spectral decomposition

$$\mathcal{K}^s_{\partial\Omega} = \sum_{j=1}^{\infty} \lambda_j \psi_j \otimes \psi_j,$$

(1.2)

where $\psi_j$ are eigenvectors corresponding to eigenvalues $\lambda_j$. We refer to [6, 21, 34] for the decay estimates for the eigenvalues of the NP operator on a planar domain (see also [9] for the symmetricity of the spectrum on a planar domain). For simple shapes such as disks or ellipses, the complete sets of eigenvalues are known [2]. We refer to [2, 18] for the eigenvalues of $\mathcal{K}^s_{\partial\Omega}$ of an ellipse or an ellipsoid and to [3] for the eigenvalue property of the NP operator of tori. The NP operator can also be defined for domains with separated components. For example, the eigenvalues of a domain consisting of two separated unit disks were explicitly expressed in terms of the distance between the disks [12] (see also [13, 31]).

For a Lipschitz domain with corners, the NP operator admits a continuous spectrum as well as eigenvalues, and it can be decomposed into three parts: the absolutely continuous spectrum, singularly continuous spectrum, and pure point spectrum (namely, $\sigma_{ac}(\mathcal{K}^s_{\partial\Omega})$, $\sigma_{sc}(\mathcal{K}^s_{\partial\Omega})$ and $\sigma_{pp}(\mathcal{K}^s_{\partial\Omega})$, respectively). It holds the following integral expression by the spectral theorem on a bounded, self-adjoint operator on a Hilbert space (see, for instance, [38, 40]):

$$\mathcal{K}^s_{\partial\Omega} = \int_{\sigma(\mathcal{K}^s_{\partial\Omega})} t d\mathbb{E}(t),$$

(1.3)

where $\mathbb{E}(t)$ is a resolution of identity for $\mathcal{K}^s_{\partial\Omega}$. Various studies have investigated the spectral properties of the NP operator for cornered domains [20, 23, 29, 36, 37]. It was shown that the essential spectrum on a bounded planar domain with corners is an interval determined by the corner angles [20, 23, 35, 36, 37]. For intersecting disks, the complete spectral resolution of the NP operator was obtained [23], where $\sigma(\mathcal{K}^s_{\partial\Omega})$ consists of only the absolutely continuous spectrum. In [20], a numerical method to determine the spectrum was developed by using its relation to the plasmon resonance rate. We refer to see [4, 7, 10, 11, 14, 29] for more results on the spectral properties of the NP operator and plasmon resonance. Numerical examples obtained in [20] show that all three kinds of spectrums (that is, $\sigma_{ac}(\mathcal{K}^s_{\partial\Omega})$, $\sigma_{sc}(\mathcal{K}^s_{\partial\Omega})$ and $\sigma_{pp}(\mathcal{K}^s_{\partial\Omega})$) can appear depending on the domains. The existence of embedded eigenvalues within the essential spectrum was verified numerically [20] and analytically [29]. Also, it was shown that infinitely many embedded eigenvalues appear for a perturbed sphere by smoothly attaching a conical singularity [25].
Touching disks can be considered as the limiting geometry of two different types of shapes. One is the limiting geometry of two separated disks as the distance between them tends to zero, and the other is that of intersecting disks whose external corner angle tends to zero. The spectrum \( \sigma(K_{0\partial\Omega}^*) \) for the domain consisting of two separated unit disks with the distance \( \delta > 0 \) is a sequence of eigenvalues given by (see [12])

\[
\lambda^\pm_n = \pm \frac{(1 + \delta - \sqrt{\delta(2 + \delta)})2n}{2}, \quad n = 1, 2, \ldots. \quad (1.4)
\]

As \( \delta \) tends to zero, \( \lambda^\pm_n \) become densely located in \([-1/2, 1/2]\). For intersecting disks with external angle \( \theta \in (0, \pi) \) at the corner points, it holds that (see [23])

\[
\sigma(K_{0\partial\Omega}^*) = \sigma_{ac}(K_{0\partial\Omega}^*) = [-b, b], \quad b = |1/2 - \theta/\pi|. \quad (1.5)
\]

As the external angle \( \theta \) tends to zero, the spectral bound \( b \) tends to 1/2. The extension of the results on the spectrum of the separated disks and intersecting disks, (1.4) and (1.5), to touching disks is not straightforward as it involves the layer potential technique on a domain with a cusp point, which, to the best knowledge of the authors, has not been established. We need to generalize the NP operator defined on a Lipschitz domain to the considered domains with a cusp point in a suitable function space.

In the present paper, for \( \Omega \) a crescent-shaped domain and touching disks, we first define the NP operator on \( L^2_0(\partial\Omega) \) as a boundary integral operator similarly to the Lipschitz domain case. We then further investigate the NP operator via the Fourier transform on the boundary circles of the domain. More precisely, two touching circular boundaries of the domain are mapped to two parallel lines with a Möbius transformation, and the Fourier transform is applied on the two parallel lines. By using the expression of \( K_{0\partial\Omega}^* \) on \( L^2_0(\partial\Omega) \) in terms of the Fourier transform, we define a Hilbert space (denoted by \( K_0^{-1/2} \)), which is analogous to \( H^* \) of the Lipschitz domain case, and generalize \( K_{0\partial\Omega}^* \) on \( L^2_0(\partial\Omega) \) to \( K_0^{-1/2} \) such that \( K_{0\partial\Omega}^* \) is bounded, self-adjoint. As a main result, we derive the complete spectral resolution of the NP operator on the space \( K_0^{-1/2} \). It turns out that the spectrums of the NP operators on a crescent–shaped domain and touching disks are both

\[
\sigma(K_{0\partial\Omega}^*) = \sigma_{ac}(K_{0\partial\Omega}^*) = [-1/2, 1/2].
\]

Note that this is identical to the limit of the spectrum of the separated disks in (1.4) and that of intersecting disks in (1.5) as \( \delta \) and \( \theta \) tend to zero, respectively. The analyses for the crescent-shaped domain and touching disks go similarly. We provide the full details for the crescent-shaped domain and briefly state the results for touching disks without a detailed proof.

As an application of the spectral resolution of the NP operator, we compute the order of plasmon resonance on a crescent-shaped domain. It is worth remarking that the plasmon resonance for the crescent-shaped domain was studied in [8] but without mathematical rigor.

The remainder of this paper is organized as follows. In Section 2, we briefly explain the properties of the layer potential operators on a Lipschitz domain and plasmon resonance. Section 3, Section 4, and Section 5 address the crescent-shaped domain case. Section 3 is devoted to deriving expressions of the layer potential operators by using a Möbius transformation and the Fourier transform. In Section 4, we define the Hilbert space \( K_0^{-1/2} \) and derive the spectral resolution of the NP operator. We then analyze the plasmon resonance in Section 5. In Section 6, we derive the spectral resolution of the NP operator for touching disks.
2 Preliminary: layer potential operators on a Lipschitz domain

Let $D$ be a simply connected bounded Lipschitz domain in $\mathbb{R}^2$. For a density function $\psi \in L^2(\partial D)$, the single-layer potential $S_{\partial D}[\psi]$ is defined by

$$S_{\partial D}[\psi](x) = \int_{\partial D} \Gamma(x - y)\psi(y)\, d\sigma(y), \quad x \in \mathbb{R}^2,$$

where $\Gamma(x)$ is the fundamental solution to the Laplacian, that is $\Gamma(x) = \frac{1}{4\pi} \ln |x|$. The single-layer potential is harmonic in $\mathbb{R}^2 \setminus \partial D$ and satisfies the jump relations [39]:

$$S_{\partial D}[\psi]|^+(x) = S_{\partial D}[\psi]|^-(x) \quad \text{a.e. } x \in \partial D,$$

$$\frac{\partial}{\partial \nu} S_{\partial D}[\psi]|^\pm(x) = \left( \pm \frac{1}{2} I + K_{\partial D}^* \right) \psi(x) \quad \text{a.e. } x \in \partial D,$$

(2.1)

where the NP operator $K_{\partial D}^*$ is given by $\Gamma[\partial D]$, the symbol $+$ and $-$ stand for the limit to $\partial D$ from the outside and inside from $\partial \Omega$, respectively, and $p.v$ denotes the Cauchy principal value.

The NP operator $K_{\partial D}^*$ is in general not self-adjoint on $L^2(\partial D)$; however, it can be symmetrized using Plemelj’s symmetrization principle (see [26]):

$$S_{\partial D} K_{\partial D}^* = K_{\partial D} S_{\partial D},$$

(2.2)

where $K_{\partial D}$ is the $L^2$ adjoint of $K_{\partial D}^*$. We denote by $\mathcal{H}^*$ the space $H_0^{-1/2}(\partial D)$ equipped with the inner product

$$\langle \varphi, \psi \rangle_{\mathcal{H}^*} := -\int_{\partial D} \varphi S_{\partial D}[\psi] \, d\sigma \quad \text{for } \varphi, \psi \in H_0^{-1/2}(\partial D)$$

(2.3)

and let $||| \cdot |||_{\mathcal{H}^*}$ be the corresponding norm, which is equivalent to the $H^{-1/2}$ norm, i.e.,

$$C_1 ||| \varphi |||_{H^{-1/2}} \leq ||\varphi||_{\mathcal{H}^*} \leq C_2 ||\varphi||_{H^{-1/2}}$$

for all $\varphi \in H_0^{-1/2}(\partial D)$ with some positive constants $C_1$ and $C_2$. From (2.2), $K_{\partial D}^*$ is self-adjoint on $\mathcal{H}^*$. As a result, the spectrum of $K_{\partial D}^*$ lies on the real axis. In fact, the spectrum of $K_{\partial D}^*$ on $\mathcal{H}^*$ lies in $(-1/2, 1/2)$ [15, 39].

When the domain $D$ is the disk of radius $r_0 > 0$ centered at $c$, it holds that

$$K_{\partial D}^* [\varphi] = \frac{1}{4\pi r_0} \int_{\partial D} \varphi \, d\sigma \quad \text{on } \partial D$$

(2.4)

and that the spectrum of $K_{\partial D}^*$ consists of eigenvalues $0$ and $\frac{1}{2}$. The eigenspace that corresponds to the eigenvalue $0$ is $\mathcal{H}^*(\partial D)$, and a constant function is an eigenfunction of $K_{\partial D}^*$ corresponding to the eigenvalue $\frac{1}{2}$. The single-layer potential for the constant function $\frac{1}{r_0}$ is

$$S_{\partial D} \left[ \frac{1}{r_0} \right](x) = \begin{cases} \frac{\ln r_0}{2} & \text{if } x \in D, \\ \ln |x - c| & \text{if } x \in \mathbb{R}^2 \setminus D. \end{cases}$$

(2.5)

Suppose that the domain $D$ is occupied by a homogeneous material with the dielectric constant $\epsilon_c + i\delta$ ($\delta$ is the dissipation factor) and that the matrix $\mathbb{R}^2 \setminus D$ has the dielectric constant $\epsilon_m$. We assume that $\epsilon_m = 1$. We express the dielectric constant of the entire space as

$$\epsilon = (\epsilon_c + i\delta) \chi_D + \chi_{\mathbb{R}^2 \setminus D},$$

(2.6)
where $\chi_A$ means the characteristic function of a set $A$. We now consider the potential problem
\begin{equation}
\begin{cases}
\nabla \cdot \epsilon \nabla u_\delta = f & \text{in } \mathbb{R}^2, \\
u_\delta(z) = O(|z|^{-1}) & \text{as } |z| \to \infty,
\end{cases}
\end{equation}
where $f$ is a source function that is compactly supported in $\mathbb{R}^2 \setminus D$ satisfying $\int_{\mathbb{R}^2} fdz = 0$. An example of the source function is a polarized dipole $f(z) = p \cdot \nabla \delta_q(z)$, where $\delta_q$ is the Dirac mass at $q$, and $p, q$ are constant vectors. The solution $u_\delta$ can be expressed as
\begin{equation}
u_\delta = F + S_{\partial D}[\varphi_\delta] \text{ in } \mathbb{R}^2,
\end{equation}
where $F$ denotes the Newtonian potential of $f$, i.e., $F(z) = \int_{\mathbb{R}^2} \Gamma(z - y) f(y) dy$, and the density function $\varphi_\delta$ satisfies
\begin{equation}
(\lambda_\delta I - K_{\partial D}^*)[\varphi_\delta] = \partial_\nu F \text{ on } \partial D
\end{equation}
with $\lambda_\delta$ given by
\begin{equation}
\lambda_\delta = \frac{\epsilon_c + 1 + i\delta}{2(\epsilon_c - 1) + 2i\delta} = \frac{\epsilon_c + 1}{2(\epsilon_c - 1)} + O(\delta).
\end{equation}
The plasmon resonance
\begin{equation}
\| \nabla u_\delta \|_{L^2(D)} \to \infty \text{ as } \delta \to 0,
\end{equation}
may occur depending on $\epsilon_c$ and the spectrum of $K_{\partial D}^*$. The blow-up rate (or, the resonance rate) in (2.10) is essentially related to the blow-up feature of the norm of the density function $\varphi_\delta$ (see, for instance, [23, Section 5] and (5.9) in Section 5).

One can classify the spectrum of the NP operator on a Lipschitz domain by the resonance rate [20]. For $g \in \mathcal{H}^*(\partial D)$, let $\varphi_{t,\delta}$ be the solution to
\begin{equation}
((t + i\delta)I - K_{\partial D}^*)[\varphi_{t,\delta}] = g \text{ on } \partial D
\end{equation}
and define an indicator function
\begin{equation}
\alpha_g(t) := \sup\{\alpha \mid \limsup_{\delta \to 0} \delta^\alpha \| \varphi_{t,\delta} \|_{\mathcal{H}^*} = \infty\}, \ t \in (-1/2, 1/2).
\end{equation}
Then, $0 \leq \alpha_g(t) \leq 1$ for all $t$.

**Theorem 2.1** ([20]). Let $g \in \mathcal{H}^*$. For $t \in (-1/2, 1/2)$, the following holds.

(a) If $\alpha_g(t) > 0$, then $t \in \sigma(K_{\partial D}^*)$.

(b) If $\alpha_g(t) = 1$ and $t$ is isolated, then $t \in \sigma_{pp}(K_{\partial D}^*)$.

(c) If $\frac{1}{2} \leq \alpha_g(t) < 1$, then $t \in \sigma(K_{\partial D}^*)$.

### 3 Layer potential operators on a crescent-shaped domain

We consider a crescent-shaped domain $\Omega$ that is the region enclosed by the boundaries of two touching disks $B_R$ and $B_r$ such that $B_r \subset B_R$. In other words, $\Omega = B_R \setminus \overline{B_r}$. The boundary of $\Omega$ is composed of two circles $\partial B_R$ and $\partial B_r$ that are tangent at the origin point; see the left figure in Figure 3.1. As $\Omega$ has a cusp on its boundary, one cannot apply the results of the layer potential operators of Lipschitz domains. Instead, we will generalize the concepts of the single-layer potential and the NP operator to the crescent-shaped domain by using a Möbius transformation. We will then derive the integral expressions of the layer potential operators via the Fourier transform on $\mathbb{R}$.
3.1 Möbius transformation

We identify \( z = (z_1, z_2) \in \mathbb{R}^2 \) with \( z = z_1 + iz_2 \in \mathbb{C} \). Let \( \Psi : \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\} \) be the Möbius transformation, that is,

\[
w = \Psi(z) = \frac{1}{z}.
\]  

(3.1)

Obviously, \( \Psi \) is a conformal mapping. We set \( w = x + iy, (x, y) \in \mathbb{R}^2 \). Note that \( \Psi^{-1}(w) = \Psi(w) = \frac{1}{w} \). The scale factors of the mapping \( w \mapsto \Psi^{-1}(w) \) with respect to \( x \) and \( y \) coincide. We denote the scale factor by

\[
h(x, y) = |\Psi'(w)| = \frac{1}{x^2 + y^2}.
\]

The Möbius transformation \( \Psi \) maps the left half-plane to the left half-plane, and maps the right half-plane to the right half-plane. In particular, \( \Psi \) maps a disk of radius \( |a| \) centered at \((a, 0)\) to the half-plane determined by \( x > \frac{1}{2a} \) if \( a > 0 \), and to the half plane determined by \( x < \frac{1}{2a} \) if \( a < 0 \). We set for \( a \neq 0 \in \mathbb{R} \) that

\[
B_a = \{ (x, y) \in \mathbb{R}^2 \mid |x + iy - a| \leq |a| \}, \quad h_a(y) = h\left(\frac{1}{2a}, y\right) = \frac{1}{(\frac{1}{2a})^2 + y^2}.
\]  

(3.2)

The disks \( B_R \) and \( B_r \) are defined in this sense. The crescent-shaped domain \( \Omega = B_R \setminus B_r \) is mapped via the Möbius transformation onto the vertical strip (see Figure 3.1)

\[
S := \Psi(\Omega) = \left(\frac{1}{2R}, \frac{1}{2r}\right) \times (-\infty, \infty),
\]

and vice versa. For later use, we denote by \( q \) the width of \( S \), i.e.,

\[
q = \frac{1}{2r} - \frac{1}{2R} > 0.
\]  

(3.3)

We denote by \( \nu \) the outward unit normal vector to \( \partial \Omega \), except at the touching point of \( \partial B_R \) and \( \partial B_r \). Note that \( \nu \) is directed toward the exterior of \( B_R \) on \( \partial B_R \), but toward the interior of

![Figure 3.1: A crescent-shaped domain \( \Omega = B_R \setminus B_r \) (gray region in the left figure) and the vertical strip \( S = \Psi(\Omega) \) (gray region in the right figure). Arrows indicate the outward normal vectors to \( \partial \Omega \) or \( \partial S \).](image)
Following our normal vector convention, we then define the normal derivative of a function \( v \) at \( z \in \partial \Omega \) with \( \Psi(z) = x + iy \),

\[
\frac{\partial v}{\partial \nu} = -\frac{1}{h_R(y)} \frac{\partial (u \circ \Psi)}{\partial x} \quad \text{for } z \in \partial B_R \setminus \{0\},
\]

\[
\frac{\partial v}{\partial \nu} = \frac{1}{h_r(y)} \frac{\partial (u \circ \Psi)}{\partial x} \quad \text{for } z \in \partial B_r \setminus \{0\}.
\]

Recall that the value of an integrand function at a set of measure zero doesn’t affect the integral value. We disregard the origin point, where two normal vectors are defined, in the layer potential formulation on the crescent-shaped domain \( \Omega \) in the following subsection.

### 3.2 Generalization of the layer potential operators to the crescent-shaped domain

A density function \( \varphi \in L^2(\partial \Omega) \) can be decomposed as

\[
\varphi = \varphi_R \chi_{\partial B_R} + \varphi_r \chi_{\partial B_r} =: \varphi_R + \varphi_r.
\]

The normal vector \( \nu \) of \( \partial \Omega \) points toward the exterior of \( B_R \) on \( \partial B_R \) and toward the interior of \( B_r \) on \( \partial B_r \), as mentioned before. We define \( K_{\partial B_r}^* \) and \( K_{\partial B_R}^* \) by the integral expression (1.1) with this normal vector convention. We now define the single-layer potential and the NP operator on the crescent-shaped domain as follows.

**Definition 1.** For \( \varphi = \varphi_R + \varphi_r \in L^2(\partial \Omega) \), we define

\[
S_{\partial \Omega}[\varphi] := S_{\partial B_R}[\varphi_R] + S_{\partial B_r}[\varphi_r] \quad \text{on } \mathbb{R}^2
\]

and

\[
K_{\partial \Omega}^*[\varphi] := \left( K_{\partial B_R}^*[\varphi_R] + \frac{\partial}{\partial \nu} S_{\partial B_r}[\varphi_r] \right) \chi_{\partial B_R}
\]

\[
+ \left( \frac{\partial}{\partial \nu} S_{\partial B_R}[\varphi_R] \right) \chi_{\partial B_r} + K_{\partial B_r}^*[\varphi_r] \quad \text{on } \partial \Omega.
\]

**Lemma 3.1.** We have

\[
S_{\partial \Omega}[\psi]^+(x) = S_{\partial \Omega}[\psi]^-(x) \quad \text{a.e. } x \in \partial \Omega,
\]

\[
\frac{\partial}{\partial \nu} S_{\partial \Omega}[\varphi] \bigg|_{\partial \Omega}^\pm(x) = \left( \pm \frac{1}{2} I + K_{\partial \Omega}^* \right)[\varphi](x) \quad \text{a.e. } x \in \partial \Omega.
\]

**Proof.** The continuity of the single-layer potential \( S_{\partial \Omega}[\varphi] \) across \( \partial \Omega \) directly follows from the continuity of \( S_{\partial B_R}[\varphi_R] \) and \( S_{\partial B_r}[\varphi_r] \).

On the boundary circles \( \partial B_R \) and \( \partial B_r \), we can apply the results of the layer potential operators on Lipschitz domains that are described in Subsection 2. Applying the jump relation (2.1), we have

\[
\frac{\partial}{\partial \nu} S_{\partial B_R}[\varphi_R] \bigg|_{\partial B_R}^\pm = \left( \pm \frac{1}{2} I + K_{\partial B_R}^* \right)[\varphi_R] \quad \text{on } \partial B_R,
\]

\[
\frac{\partial}{\partial \nu} S_{\partial B_r}[\varphi_r] \bigg|_{\partial B_r}^\pm = \left( \pm \frac{1}{2} I + K_{\partial B_r}^* \right)[\varphi_r] \quad \text{on } \partial B_r.
\]
where \( \nu \) is the outward normal vector to \( \partial \Omega \), the interior and exterior limits are defined corresponding to the direction of \( \nu \), and \( K^s_{\partial B_r} \) and \( K^s_{\partial B_R} \) are also defined with this normal vector convention. Also, we have

\[
\frac{\partial}{\partial \nu} S_{\partial B_r}[\varphi_r] \bigg|_{\partial B_R} \pm \left( \pm \frac{1}{2} I + K^s_{\partial B_r} \right) [\varphi_r] \bigg|_{\partial B_R} \left\{ \begin{array}{ll}
\varphi_r \bigg|_{\partial B_R} \\
0
\end{array} \right.
\]

and

\[
\frac{\partial}{\partial \nu} S_{\partial B_R}[\varphi_R] \bigg|_{\partial B_r} \pm \left( \pm \frac{1}{2} I + K^s_{\partial B_R} \right) [\varphi_R] \bigg|_{\partial B_r} \left\{ \begin{array}{ll}
\varphi_R \bigg|_{\partial B_r} \\
0
\end{array} \right.
\]

Hence, we prove the lemma. \( \square \)

We emphasize that it is necessary to analyze the mapping properties of \( \frac{\partial}{\partial \nu} S_{\partial B_r}[\varphi_r] \bigg|_{\partial B_R} \) and \( \frac{\partial}{\partial \nu} S_{\partial B_R}[\varphi_R] \bigg|_{\partial B_r} \) to understand the spectral structure of the NP operator on the crescent-shaped domain.

As in the previous subsection, we set

\[
z = \frac{1}{x + iy} \quad \text{for} \ z \neq 0.
\]

Let \( z_t = \Psi^{-1}(\frac{1}{2r} + it) \) on \( \partial B_r \) and \( z_t = \Psi^{-1}(\frac{1}{2R} + it) \) on \( \partial B_R \); then we have \( |z - z_t| = \frac{|x - \frac{1}{2r} + iy - t|}{|x + iy + \frac{1}{2r} + it|} \) on \( \partial B_r \) and a similar relation on \( \partial B_R \), respectively. We identify \( \varphi_R, \varphi_r \) in (3.5) with the functions on \( \mathbb{R} \) given by

\[
\bar{\varphi}_R(y) = (\varphi_R \circ \Psi^{-1}) \left( \frac{1}{2R} + iy \right),
\]

\[
\bar{\varphi}_r(y) = (\varphi_r \circ \Psi^{-1}) \left( \frac{1}{2r} + iy \right) \quad \text{for} \ y \in \mathbb{R}.
\]

Then, the single-layer potential (3.6) satisfies

\[
S_{\partial \Omega}[\varphi](z) = \frac{1}{2\pi} \int_{\partial B_r} \ln |z - z_t| \varphi_r(z_t) d\sigma(z_t) + \frac{1}{2\pi} \int_{\partial B_R} \ln |z - z_t| \varphi_R(z_t) d\sigma(z_t)
\]

\[
= \frac{1}{4\pi} \int_{-\infty}^{\infty} \left( \ln \left[ \left( x - \frac{1}{2r} \right)^2 + (y - t)^2 \right] - \ln \left[ \left( \frac{1}{2r} \right)^2 + t^2 \right] \right) \bar{\varphi}_r(t) h_r(t) dt
\]

\[
+ \frac{1}{4\pi} \int_{-\infty}^{\infty} \left( \ln \left[ \left( x - \frac{1}{2R} \right)^2 + (y - t)^2 \right] - \ln \left[ \left( \frac{1}{2R} \right)^2 + t^2 \right] \right) \bar{\varphi}_R(t) h_R(t) dt
\]

\[
- \frac{1}{4\pi} \ln(x^2 + y^2) \int_{\partial \Omega} \varphi(z) d\sigma(z)
\]

(3.10)
and
\[
\frac{\partial}{\partial \nu} S_{\partial B_R} [\varphi_R] \bigg|_{\partial B_R} (z) = \frac{1}{2\pi} \int_{\partial B_R} \frac{\partial}{\partial \nu} \ln |z - z_t| \varphi_R(z_t) \, d\sigma(z_t)
\]
\[
= \frac{1}{2\pi} \frac{1}{h_R(y)} \int_{-\infty}^{\infty} \left( \frac{q}{q^2 + (y - t)^2} + \frac{2\pi}{(2\pi)^2 + y^2} \right) \tilde{\varphi}_R(t) h_R(t) \, dt
\]
\[
= \frac{1}{2\pi} \frac{1}{h_R(y)} \int_{-\infty}^{\infty} \frac{q}{q^2 + (y - t)^2} \tilde{\varphi}_R(t) h_R(t) \, dt + \frac{1}{4\pi R} \int_{-\infty}^{\infty} \tilde{\varphi}_R(t) h_R(t) \, dt,
\]
\[
\frac{\partial}{\partial \nu} S_{\partial B_R} [\varphi_R] \bigg|_{\partial B_r} (z) = \frac{1}{2\pi} \frac{1}{h_R(y)} \int_{-\infty}^{\infty} \frac{q}{q^2 + (y - t)^2} \tilde{\varphi}_R(t) h_R(t) \, dt - \frac{1}{4\pi r} \int_{-\infty}^{\infty} \tilde{\varphi}_R(t) h_R(t) \, dt,
\]
where \( q \) denotes the width of the strip \( \Psi(\Omega) \) (that is, \( q = \frac{1}{2r} - \frac{1}{2R} \)). Furthermore, it holds from (2.3) that
\[
K^s_{\partial B_R} [\varphi_R] = \frac{1}{4\pi R} \int_{\partial B_R} \varphi_R \, d\sigma \quad \text{on} \quad \partial B_R,
\]
\[
K^s_{\partial B_r} [\varphi_R] = -\frac{1}{4\pi r} \int_{\partial B_r} \varphi_r \, d\sigma \quad \text{on} \quad \partial B_r.
\]
Hence, for \( \varphi = \varphi_R + \varphi_r \in L^2(\partial \Omega) \), it holds that
\[
K^s_{\partial \Omega} [\varphi](z) = \begin{cases} 
\frac{1}{2\pi h_R(y)} \int_{-\infty}^{\infty} \frac{q}{q^2 + (y - t)^2} \tilde{\varphi}_R(t) h_R(t) \, dt + \frac{1}{4\pi R} \int_{\partial \Omega} \varphi \, d\sigma & \text{for } x = \frac{1}{2R}, \\
\frac{1}{2\pi h_r(y)} \int_{-\infty}^{\infty} \frac{q}{q^2 + (y - t)^2} \tilde{\varphi}_R(t) h_R(t) \, dt - \frac{1}{4\pi r} \int_{\partial \Omega} \varphi \, d\sigma & \text{for } x = \frac{1}{2r}
\end{cases}
\]
(3.11)

with \( \tilde{\varphi}_R \) and \( \tilde{\varphi}_r \) given by (3.9).

3.3 Layer potential operators in terms of the Fourier transform

The Fourier transform and its inversion in \( \mathbb{R} \) are defined as
\[
\mathcal{F}[f](k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) e^{-iky} \, dy,
\]
\[
\mathcal{F}^{-1}[f](k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) e^{iky} \, dy.
\]
Recall that we identify \( \varphi \in L^2(\partial \Omega) \) with two functions \( \tilde{\varphi}_R, \tilde{\varphi}_r \) given by (3.9). We can further identify \( \varphi \), via the Fourier transform, with
\[
U[\varphi] := \begin{bmatrix} \mathcal{F}[h_R \tilde{\varphi}_R] \\ \mathcal{F}[h_r \tilde{\varphi}_r] \end{bmatrix}.
\]
(3.12)
The inversion of the operator \( U \) is
\[
U^{-1} \begin{bmatrix} f_R \\ f_r \end{bmatrix}(z) = \begin{cases} 
\frac{1}{h_R(y)} \mathcal{F}^{-1}[f_R](y) & \text{for } z \in \partial B_R, \\
\frac{1}{h_r(y)} \mathcal{F}^{-1}[f_r](y) & \text{for } z \in \partial B_r,
\end{cases}
\]
where \( z \) and \( y \) satisfy the relation \((3.8)\).

We now express the single-layer potential and the NP operator for \( \varphi \) in terms of \( U[\varphi] \) as follows.

**Lemma 3.2.** Let \( \varphi = \varphi_R + \varphi_r \in L_0^2(\partial \Omega) \). For \( z = \Psi^{-1}(x + iy) \in \partial \Omega \), we have

\[
S_{\partial \Omega}[\varphi](z) = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{2|k|} \left( e^{-|x|/2\pi|k|} - e^{-|y|/2\pi|k|} \right) e^{iky} dk + C,
\]

where \( C \) is the constant given by

\[
C = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{2|k|} \left( e^{-|x|/2\pi|k|} + e^{-|y|/2\pi|k|} \right) dk.
\]

**Proof.** The assumption \( \varphi \in L_0^2(\partial \Omega) \) implies that \( h_R|h_R|^2, h_r|h_r|^2 \in L^1(\mathbb{R}) \). Since \( h_R \) and \( h_r \) are bounded and integrable on \( \mathbb{R} \), we have

\[
h_R|h_R|^2, h_r|h_r|^2 \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}). \tag{3.13}
\]

Note that for fixed \( a, b, A, B \), the function \( \ln(A^2 + (a - t)^2) - \ln(B^2 + (b - t)^2) \) is square integrable on any bounded interval of \( t \). Furthermore, we have

\[
\ln(A^2 + (a - t)^2) - \ln(B^2 + (b - t)^2) = (-2a + 2b) \frac{1}{t} + (-a^2 + A^2 + b^2 - B^2) \frac{1}{t^2} + O\left(\frac{1}{t^3}\right) \text{ as } t \to \infty, \tag{3.14}
\]

where \( O(\frac{1}{t}) \) is uniformly bounded with respect to small \( |b| \geq 0 \) (with fixed \( a, A, B \)). Applying the dominated convergence theorem to \((3.10)\), we obtain

\[
S_{\partial \Omega}[\varphi](\Psi^{-1}(x + iy)) = \lim_{c \to y} \left[ \frac{1}{4\pi} \int_{-\infty}^{\infty} \left( \ln \left( \left( x - \frac{1}{2R} \right)^2 + (y - t)^2 \right) - \ln \left( \left( \frac{1}{2R} \right)^2 + (t - y + c)^2 \right) \right) \tilde{\varphi}_R(t)h_R(t) dt \right.
\]
\[
+ \left. \frac{1}{4\pi} \int_{-\infty}^{\infty} \left( \ln \left( \left( x - \frac{1}{2r} \right)^2 + (y - t)^2 \right) - \ln \left( \left( \frac{1}{2r} \right)^2 + (t - y + c)^2 \right) \right) \tilde{\varphi}_r(t)h_r(t) dt \right].
\]

The last term in \((3.10)\) vanishes assuming the mean-zero condition on \( \varphi \). The Fourier transform of \( \ln(y^2 + a^2) \) is

\[
\mathcal{F}[\ln(y^2 + a^2)](k) = -\frac{1}{\sqrt{2\pi}} \left( \frac{-a||k|}{|k|} + 2\gamma_E \delta(k) \right), \tag{3.15}
\]

where \( 1/|k| \) is defined in the sense of principal value and \( \gamma_E \) denotes Euler’s constant. The convolution theorem of the Fourier transform, i.e., \( \mathcal{F}[f_1 \ast f_2] = \sqrt{2\pi} \mathcal{F}[f_1] \mathcal{F}[f_2] \), leads to the relation

\[
S_{\partial \Omega}[\varphi](z) = \lim_{c \to y} \frac{1}{\sqrt{2\pi}} \left[ \int_{-\infty}^{\infty} \frac{1}{2|k|} \left( e^{-|x|/2\pi|k|} - e^{-|y|/2\pi|k|} \right) e^{-iky} dk \right.
\]
\[
+ \left. \frac{1}{2\pi} \left( e^{-|x|/2\pi|k|} - e^{-|y|/2\pi|k|} \right) e^{-iky} dk \right]. \tag{3.16}
\]
From the mean-zero assumption on $\varphi$, we have

$$\int_{-\infty}^{\infty} \varphi_R h_R \, dy + \int_{-\infty}^{\infty} \varphi_r h_r \, dy = 0 \quad (3.17)$$

and, hence,

$$F[h_R \varphi_R](k) + F[h_r \varphi_r](k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\varphi_R h_R)(y)(e^{-iky} - 1) \, dy + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\varphi_r h_r)(y)(e^{-iky} - 1) \, dy.$$

From (3.13) and the fact that $h_R$ is uniformly bounded, we have

$$\left| \int_{-\infty}^{\infty} (\varphi_R h_R)(y)(e^{-iky} - 1) \, dy \right| \leq \|\varphi_R(h_R)^{1/2}\|_{L^2(\mathbb{R})} \|(h_R)^{1/2}(e^{-iky} - 1)\|_{L^2(\mathbb{R})}$$

for some positive constant $C$, and a similar relation holds for $\varphi_r h_r$. Thus, for any constants $a$ and $b$, it holds that

$$e^{-|a||k|} F[h_R \varphi_R](k) + e^{-|b||k|} F[h_r \varphi_r](k) = O(|k|^{1/2}) \quad \text{as } |k| \to 0. \quad (3.18)$$

Then, we can apply the dominated convergence theorem to (3.16) and, as a result, change the order of the limit and integration. Hence, we prove the lemma. \qed

Recall that $q = \frac{1}{2r} - \frac{1}{2R} > 0$. We set

$$P := \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}. \quad (3.19)$$

This matrix satisfies $P = P^{-1} = P^T$ and, for any $d_1, d_2$,

$$\frac{1}{2} \begin{bmatrix} d_1 + d_2 & -d_1 + d_2 \\ -d_1 + d_2 & d_1 + d_2 \end{bmatrix} = P^{-1} \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} P. \quad (3.20)$$

**Lemma 3.3.** For $\varphi = \varphi_R + \varphi_r \in L_0^2(\partial \Omega)$, it holds that

$$U[K_{\partial \Omega}[\varphi]] = P^{-1} \left( \frac{1}{2} e^{-q|k|} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} P \right) U[\varphi]. \quad (3.21)$$

**Proof.** It holds that

$$F \left[ \frac{q}{q^2 + y^2} \right](k) = \sqrt{\frac{\pi}{2}} e^{-q|k|}.$$
From the assumption $\varphi = \varphi_R + \varphi_r \in L^2_0(\partial \Omega)$, we have $\int_{\partial \Omega} \varphi \, d\sigma = 0$. It then follows by applying the convolution theorem of the Fourier transform to (3.11) that

$$U[K^*_R \varphi] = \frac{1}{2} e^{-q|k|} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \mathcal{F} [h_R \varphi_R] = \frac{1}{2} e^{-q|k|} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} U[\varphi].$$

From (3.20), we obtain (3.21).

**Lemma 3.4.** For $\psi, \varphi \in L^2_0(\partial \Omega)$, we have

$$\int_{\partial \Omega} \psi(z) \overline{S_{\partial \Omega}[\varphi]}(z) \, d\sigma(z) = \int_{-\infty}^{\infty} \left( U[\psi](k) \right)^T \mathcal{P}^{-1} \left( -\frac{1}{2|k|} \begin{bmatrix} 1 - e^{-q|k|} & 0 \\ 0 & 1 + e^{-q|k|} \end{bmatrix} \mathcal{P} \right) U[\varphi](k) \, dk.$$

**Proof.** Set $\psi = \psi_R + \psi_r$ and $\varphi = \varphi_R + \varphi_r$. From Lemma 3.2 we have

$$\int_{\partial \Omega} \psi \overline{S_{\partial \Omega}[\varphi]} \, d\sigma = \int_{-\infty}^{\infty} \left( \psi_R(y) \overline{S_{\partial \Omega}[\varphi]}(y) + \psi_r(y) \overline{S_{\partial \Omega}[\varphi]}(y) \right) h_R(y) \, dy + \int_{-\infty}^{\infty} \psi_r(y) \overline{S_{\partial \Omega}[\varphi]}(y) h_r(y) \, dy$$

$$= -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left( \psi_R(h_R(y)) \overline{S_{\partial \Omega}[\varphi]}(y) \left( \frac{1}{1/(2R) + iy} \right) + \psi_r(h_r(y)) \overline{S_{\partial \Omega}[\varphi]}(y) \left( \frac{1}{1/(2r) + iy} \right) \right) h_R(y) \, dy$$

$$-\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left( \psi_r(h_r(y)) \overline{S_{\partial \Omega}[\varphi]}(y) \left( \frac{1}{1/(2r) + iy} \right) \right) h_r(y) \, dy.$$

By changing the order of integrations, one obtains

$$-\int_{\partial \Omega} \psi \overline{S_{\partial \Omega}[\varphi]} \, d\sigma = \int_{-\infty}^{\infty} \left( U[\psi](k) \right)^T \left[ -\frac{1}{2|k|} e^{-q|k|} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \mathcal{P} \right] U[\varphi](k) \, dk.$$

In view of (3.20), this completes the proof. \qed

4 Spectral resolution of the NP operator on a crescent-shaped domain

We denote the two matrix-valued functions in Lemma 3.3 and Lemma 3.4 as follows:

$$S = -\frac{1}{2|k|} \begin{bmatrix} 1 - e^{-q|k|} & 0 \\ 0 & 1 + e^{-q|k|} \end{bmatrix}, \quad k \in \mathbb{R} \setminus \{0\}, \quad (4.1)$$

$$K = \frac{1}{2} e^{-q|k|} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad k \in \mathbb{R}.$$

In terms of these matrix-valued functions, we define a Hilbert space $K^{-1/2}_0$ that extends $L^2_0(\partial \Omega)$. We then generalize the layer potential operators to be defined on $K^{-1/2}_0$ by using the integral expressions in Subsection 3.3. We finally obtain the spectral resolution of the NP operator on $K^{-1/2}_0$. 

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4.1 Hilbert space $K_{0}^{-1/2}$

For $\varphi \in L_{0}^{2}(\partial \Omega)$, it holds that $\varphi = U^{-1}P\hat{\varphi}$ with

$$
\hat{\varphi} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \left[ \mathcal{F}[hR\tilde{\varphi}_R] \right].
$$

From (3.18), we have

$$
\int_{-\infty}^{\infty} \hat{\varphi}^{T} (-\mathcal{S}) \hat{\varphi} dk < \infty.
$$

Based on these relations, we define a Hilbert space:

**Definition 2.** We define

$$
K_{0}^{-1/2} := \left\{ \varphi = U^{-1}P\hat{\varphi} \bigg| \hat{\varphi} = \begin{bmatrix} \hat{\varphi}_1 \\ \hat{\varphi}_2 \end{bmatrix} \text{ satisfying } \int_{-\infty}^{\infty} \hat{\varphi}^{T} (-\mathcal{S}) \varphi dk < \infty \right\}, \quad (4.2)
$$

where $\hat{\varphi}_1$ and $\hat{\varphi}_2$ are measurable functions on $\mathbb{R}$, and $P$ is given by (3.19).

The inverse Fourier transform $U^{-1}$ in (4.2) is defined in the tempered distribution sense. Indeed, for any function $f$ on $\mathbb{R}$ satisfying

$$
\int_{-\infty}^{\infty} \frac{1}{1 + |k|} |f(k)|^2 dk < \infty, \quad (4.3)
$$

it holds that, for any $\psi$ in the Schwartz class $\mathcal{S}$,

$$
\left| \int_{-\infty}^{\infty} f \psi dk \right| \leq \left( \int_{-\infty}^{\infty} \frac{1}{1 + |k|} |f|^2 dk \right)^{1/2} \left( \int_{-\infty}^{\infty} (1 + |k|) |\psi|^2 dk \right)^{1/2}
$$

$$
\leq C \left( \int_{-\infty}^{\infty} (1 + |k|)^4 |\psi|^2 \frac{1}{(1 + |k|)^2} dk \right)^{1/2}
$$

$$
\leq C \| (1 + |k|)^2 \psi(k) \|_{\infty}
$$

$$
\leq C \sum_{\alpha \leq 2} \| \hat{\varphi}_1 \|_{\alpha,0} < \infty,
$$

where $\| \phi \|_{\alpha,\beta} := \sup_{x \in \mathbb{R}} |x^\alpha \partial^\beta \phi(x)|$. Therefore, we have $f \in \mathcal{S}'(\mathbb{R})$, where $\mathcal{S}'(\mathbb{R})$ denotes the class of tempered distributions. The Fourier transform and its inversion on $L^2(\mathbb{R})$ can be extended to $\mathcal{S}'(\mathbb{R})$, where the inversion formula still holds. Since $\hat{\varphi}_1$ and $\hat{\varphi}_2$ satisfy the decay condition (4.3), we can define the inverse transform for $P\hat{\varphi}$.

As the $(2,2)$-component of $\mathcal{S}$ blows up as $|k|^{-1}$ near $k = 0$, the condition $\int_{-\infty}^{\infty} \hat{\varphi}^{T} (-\mathcal{S}) \varphi dk < \infty$ implies decay of $\hat{\varphi}_2$ near $k = 0$; we highlight this property by adding the subscript 0 in $K_{0}^{-1/2}$. On the other hand, we add the superscript $-1/2$ in $K_{0}^{-1/2}$ since we define an inner product in a similar way to (2.3) (see (4.5) below).

It is straightforward to obtain the following.

**Lemma 4.1.** We have

$$
L_{0}^{2}(\partial \Omega) \subset K_{0}^{-1/2}. \quad (4.4)
$$
Definition 3. Let $\varphi \in K_0^{-1/2}$ be given by $\varphi = U^{-1}P\hat{\varphi}$.

(a) We define the NP operator $K^*_{0\Omega} : K_0^{-1/2} \rightarrow K_0^{-1/2}$ by

$$K^*_{0\Omega}[\varphi] := U^{-1}P\hat{\varphi}. \quad (4.8)$$

(b) We define the single-layer potential of $\varphi$: for $z = \Psi^{-1}(x + iy) \in \mathbb{C}$,

$$S_{0\Omega}[\varphi](z) := -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{2|k|} \left(e^{-|x-\frac{i}{2}\pi}|k|} \hat{\varphi}_R(k) + e^{-|x-\frac{i}{2}\pi}|k|} \hat{\varphi}_r(k)\right)e^{iky} dk
$$

$$+ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{2|k|} \left(e^{-\frac{|k|}{2\pi}} \hat{\varphi}_R(k) + e^{-\frac{|k|}{2\pi}} \hat{\varphi}_r(k)\right) dk; \quad (4.9)$$

where $\hat{\varphi}_R$ and $\hat{\varphi}_r$ is given by

$$\begin{bmatrix} \hat{\varphi}_R \\ \hat{\varphi}_r \end{bmatrix} = P \begin{bmatrix} \hat{\varphi}_1 \\ \hat{\varphi}_2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -\hat{\varphi}_1 + \hat{\varphi}_2 \\ \hat{\varphi}_1 + \hat{\varphi}_2 \end{bmatrix}. \quad (4.10)$$

It is worth emphasizing that (4.8) and (4.9) hold for $\varphi \in L_0^2(\partial\Omega)$. In other words, (4.8) and (4.9) are natural extensions of the NP operator and the single-layer potential on $L_0^2(\partial\Omega)$ to $K_0^{-1/2}$.

We observe that

$$\|K^*_{0\Omega}[\varphi]\|_{-1/2}^2 = -\int_{-\infty}^{\infty} (\mathbf{K}\hat{\varphi})^T S(\mathbf{K}\varphi) dk \leq \frac{1}{4}\|\varphi\|_{-1/2}^2 \quad \text{for any } \varphi \in K_0^{-1/2}. \quad (4.11)$$
Hence, $K^*_\partial \Omega$ is a bounded linear operator on $K^{-1/2}_0$ and its operator norm is bounded by $\frac{1}{\pi}$. Since $\mathbb{S}$ and $\mathbb{K}$ are real diagonal matrices, we have $\mathbb{S}^T \mathbb{K} = \mathbb{K}^T \mathbb{S}$. This induces that

$$<(\psi, K^*_\partial \Omega [\varphi])_{-1/2} = (K^*_\partial \Omega [\psi], \varphi)_{-1/2} \text{ for all } \varphi, \psi \in K^{-1/2}_0.$$ 

In other words, $K^*_\partial \Omega$ is self-adjoint on $K^{-1/2}_0$. In view of (4.8), $K^*_\partial \Omega$ is identical to the matrix $\mathbb{K}$ via the transformation $U^{-1} P$. From this, one can infer that the spectrum of $K^*_\partial \Omega$ on $K^{-1/2}_0$ lies in the interval $[-1/2, 1/2]$, that is the spectrum of $\mathbb{K}$. In Subsection 4.3, we will prove it by deriving the spectral resolution of the NP operator on $K^{-1/2}_0$.

In the remainder of this subsection, we obtain properties of the layer potential operators by assuming a decay condition on the density function as $k$ tends to zero.

**Lemma 4.2.** Let $\varphi \in K^{-1/2}_0$ be given by $\varphi = U^{-1} P \hat{\varphi}$ satisfying $\hat{\varphi}_1, \hat{\varphi}_2 \in L^1(\mathbb{R})$ and $\hat{\varphi}_2(k) = O(|k|^{1/2})$ as $|k| \to 0$. Set $z = z_1 + iz_2 = \Psi(x + iy) \in \mathbb{C}$. Then, we have the following.

(a) The single-layer potential $\mathcal{S}_\partial \Omega [\varphi](z)$ is continuous and uniformly bounded in $\mathbb{C}$ and $\mathcal{S}_\partial \Omega [\varphi](z) = O(|z|^{-1})$ as $|z| \to \infty$.

(b) The partial derivatives $\partial_x \mathcal{S}_\partial \Omega [\varphi](z), \partial_y \mathcal{S}_\partial \Omega [\varphi](z)$ are uniformly bounded for $x \neq \frac{1}{2\pi}, \frac{1}{2\pi}$ and

$$\frac{\partial}{\partial z_1} \mathcal{S}_\partial \Omega [\varphi](z), \frac{\partial}{\partial z_2} \mathcal{S}_\partial \Omega [\varphi](z) = (x^2 + y^2) O(|x|^{-\frac{3}{2}}) \text{ as } |x| \to \infty,$$

where $O(|x|^{-\frac{3}{2}})$ is uniform with respect to $y$.

(c) The single-layer potential is harmonic, i.e., $\Delta_{(z_1, z_2)} \mathcal{S}_\partial \Omega [\varphi](z) = 0$ in $\mathbb{C} \setminus \partial \Omega$.

**Proof.** From the assumption that $\hat{\varphi}_2(k) = O(|k|^{1/2})$, the integral (4.9) is finite for any $z \in \mathbb{C}$. One can also show that $\mathcal{S}_\partial \Omega [\varphi](z)$ is continuous in the whole complex plane by applying the dominated convergence theorem. We can rewrite (4.9) as

$$\mathcal{S}_\partial \Omega [\varphi](z) = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{2|k|} \left( (e^{-|x|/2\pi|k|} - e^{-|k|/2\pi}) e^{iky} + e^{-|k|/2\pi} (e^{iky} - 1) \right) \hat{\varphi}_R(k) dk$$

$$- \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{2|k|} \left( (e^{-|x|/2\pi|k|} - e^{-|k|/2\pi}) e^{iky} + e^{-|k|/2\pi} (e^{iky} - 1) \right) \hat{\varphi}_r(k) dk. \quad (4.11)$$

It then follows that

$$\mathcal{S}_\partial \Omega [\varphi](z) = \int_{-\infty}^{\infty} O(|x| + |y|) |\hat{\varphi}_R(k)| dk + \int_{-\infty}^{\infty} O(|x| + |y|) |\hat{\varphi}_r(k)| dk \text{ as } |x + iy| \to 0,$$

where $O(|x| + |y|)$ terms are uniform with respect to $k$. This proves (a).

From (4.9), we have

$$\frac{\partial}{\partial y} \mathcal{S}_\partial \Omega [\varphi](z) = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{ik}{2|k|} \sqrt{2} e^{-|x|/2\pi|k|} |\hat{\varphi}_2(k)| e^{iky} dk$$

$$- \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{ik}{2|k|} \left( e^{-|x|/2\pi|k|} - e^{-|x|/2\pi|k|} \right) \hat{\varphi}_r(k) e^{iky} dk. \quad (4.12)$$

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and

\[
\frac{\partial}{\partial x} S_{\partial \Omega}[\varphi](z) = \begin{cases}
\frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} \left( e^{-\left(x - \frac{1}{2r}\right)^2} \hat{\varphi}_R(k) + e^{-\left(x - \frac{1}{2r}\right)^2} \hat{\varphi}_r(k) \right) e^{i ky} dk & \text{for } x > \frac{1}{2r}, \\
\frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} \left( e^{-\left(x - \frac{1}{2r}\right)^2} \hat{\varphi}_R(k) - e^{-\left(x - \frac{1}{2r}\right)^2} \hat{\varphi}_r(k) \right) e^{i ky} dk & \text{for } \frac{1}{2R} < x < \frac{1}{2r}, \\
\frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} \left( -e^{-\left(x - \frac{1}{2r}\right)^2} \hat{\varphi}_R(k) - e^{-\left(x - \frac{1}{2r}\right)^2} \hat{\varphi}_r(k) \right) e^{i ky} dk & \text{for } x < \frac{1}{2R}.
\end{cases}
\]

Because of \( \hat{\varphi}_1, \hat{\varphi}_2 \in L^1(\mathbb{R}) \), \( \frac{\partial}{\partial x} S_{\partial \Omega}[\varphi](z) \) and \( \frac{\partial}{\partial y} S_{\partial \Omega}[\varphi](z) \) are uniformly bounded in \( \mathbb{C} \). We have

\[
\frac{\partial}{\partial x} S_{\partial \Omega}[\varphi](z), \quad \frac{\partial}{\partial y} S_{\partial \Omega}[\varphi](z) = \int_{-\infty}^{\infty} e^{-\left|x - \frac{1}{2r}\right|^2} O(\left|k\right|^\frac{1}{2}) (1 + \hat{\varphi}_r(k)) dk = O(\left|x\right|^{-\frac{3}{2}}) \text{ as } \left|x\right| \to \infty,
\]

where the second equality can be derived by splitting the integral into \( |k| < \left|x - \frac{1}{2r}\right|^{-\frac{1}{2}} \) and \( |k| > \left|x - \frac{1}{2r}\right|^{-\frac{1}{2}} \). This proves (b).

Recall that \( \Psi \) is a conformal mapping. By taking the Laplacian for the right-hand side of (4.9) (switching the order of differentiation and integration), we observe (c). \( \square \)

**Lemma 4.3.** Let \( \varphi \in K_0^{-1/2} \) be given by \( \varphi = U^{-1} P \hat{\varphi} \) satisfying \( \hat{\varphi}_1, \hat{\varphi}_2 \in L^1(\mathbb{R}) \) and \( \hat{\varphi}_2(k) = O(|k|^{\frac{1}{4}}) \) as \( |k| \to 0 \). We have

\[
\| \nabla S_{\partial \Omega}[\varphi] \|_{L^2(\Omega)}^2 = \left\langle \varphi, \left( \frac{1}{2} I - K_{\partial \Omega}^* \right) [\varphi] \right\rangle_{-1/2} < \infty, \tag{4.13}
\]

\[
\| \nabla S_{\partial \Omega}[\varphi] \|_{L^2(\partial \Omega)}^2 = \left\langle \varphi, \left( \frac{1}{2} I + K_{\partial \Omega} \right) [\varphi] \right\rangle_{-1/2} < \infty. \tag{4.14}
\]

**Proof.** We define \( \varphi_R \) and \( \varphi_r \) as in (4.10). Note that

\[
\varphi_R(k) + \varphi_r(k) = \sqrt{2} \varphi_2(k) = O(|k|^{\frac{1}{4}}) \text{ as } |k| \to 0. \tag{4.15}
\]

For fixed \( s > 0 \), we set

\[
\Omega_s := \left\{ z = \Psi(w) \mid w = x + iy \text{ with } \frac{1}{2R} < |x| < \frac{1}{2r}, \ |y| < s \right\}.
\]

Then, \( \Omega_s \) is a Lipschitz domain. We identify \( z = z_1 + iz_2 \) with \( z = (z_1, z_2) \in \mathbb{R}^2 \). Applying the divergence theorem, we obtain

\[
\int_{\Omega_s} \| \nabla S_{\partial \Omega}[\varphi] \|_{\mathbb{R}^2}^2 \ dx = \int_{\partial \Omega_s} \frac{\partial}{\partial \nu} S_{\partial \Omega}[\varphi] = \int_{\partial \Omega_s} \frac{\partial}{\partial \nu} S_{\partial \Omega}[\varphi] \ h_R(y) \ dy + \int_{\partial \Omega_s} \frac{\partial}{\partial \nu} S_{\partial \Omega}[\varphi] \ h_r(y) \ dy + \int_{\partial \Omega_s} \frac{\partial}{\partial \nu} S_{\partial \Omega}[\varphi] \ h(x, y) \ dx =: I + II + III. \tag{4.16}
\]
We first estimate $III$ as $s \to \infty$. For $(x, y)$ in the domain of the integral $III$, we have

$$
\frac{\partial}{\partial \nu} S_{\partial \Omega} | \left. \frac{\partial}{\partial y} S_{\partial \Omega} \right| \frac{\partial}{\partial x} S_{\partial \Omega} \bigg|_{y=\pm s}
= \frac{1}{\sqrt{2\pi}} \left( \int_{-\infty}^{\infty} \frac{ik}{2|k|} e^{-|x-\frac{h}{2r}|^2} \hat{\varphi}_R(k) + e^{-\frac{|k|^2}{4}} \hat{\varphi}_R(k) e^{\pm iks} dk + \int_{-\infty}^{\infty} \frac{ik}{2|k|} e^{-|x-\frac{h}{2r}|^2} \hat{\varphi}_R(k) e^{\mp iks} dk \right),
$$

where the first equality holds similarly to (3.4) and the second one is from (4.9). Applying the Riemann–Lebesgue lemma to (4.17), we obtain

$$
\frac{\partial}{\partial \nu} S_{\partial \Omega} \bigg|_{y=\pm s} h(x, y) \to 0 \quad \text{as} \quad s \to \infty.
$$

Note that $S_{\partial \Omega} [\varphi](z)$ and $h(x, y) \nabla_{(z_1, z_2)} S_{\partial \Omega} [\varphi](z)$ are uniformly bounded independent of $s$ for $(x, y)$ satisfying $\frac{-h}{2r} < x < \frac{h}{2r}$, $|y| \geq s$. It then holds by applying the dominated convergence theorem that

$$
III \to 0 \quad \text{as} \quad s \to \infty.
$$

Now, we estimate $I + II$. From (3.4) and (4.9), we have

$$
S_{\partial \Omega} [\varphi](z) = -\mathcal{F}^{-1} \left[ \frac{1}{2|k|} \left( \hat{\varphi}_R(k) + e^{-|k|^2} \hat{\varphi}_r(k) \right) \right](y) + C \quad \text{in} \quad I,
$$

$$
S_{\partial \Omega} [\varphi](z) = -\mathcal{F}^{-1} \left[ \frac{1}{2|k|} \left( e^{-|k|^2} \hat{\varphi}_R(k) + \hat{\varphi}_r(k) \right) \right](y) + C \quad \text{in} \quad II,
$$

where $C$ is the constant given by $C = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{2|k|} \left( e^{-\frac{|k|^2}{4}} \hat{\varphi}_R(k) + e^{-\frac{|k|^2}{4}} \hat{\varphi}_r(k) \right) dk$. We also have

$$
\frac{\partial}{\partial \nu} S_{\partial \Omega} \bigg|_{x=\frac{h}{2r}} h_R(y) = \frac{\partial}{\partial x} S_{\partial \Omega} \bigg|_{x=\frac{h}{2r}} = \frac{1}{2} \mathcal{F}^{-1} \left[ -\hat{\varphi}_R(k) + e^{-|k|^2} \hat{\varphi}_r(k) \right] (y) \quad \text{in} \quad I,
$$

$$
\frac{\partial}{\partial \nu} S_{\partial \Omega} \bigg|_{x=\frac{h}{2r}} h_r(y) = \frac{\partial}{\partial x} S_{\partial \Omega} \bigg|_{x=\frac{h}{2r}} = -\frac{1}{2} \mathcal{F}^{-1} \left[ -e^{-|k|^2} \hat{\varphi}_R(k) + \hat{\varphi}_r(k) \right] (y) \quad \text{in} \quad II.
$$

In other words,

$$
\begin{bmatrix}
\mathcal{F} \left[ S_{\partial \Omega} [\varphi] \bigg|_{x=\frac{h}{2r} - C} \right] T \\
\mathcal{F} \left[ S_{\partial \Omega} [\varphi] \bigg|_{x=\frac{h}{2r} - C} \right]
\end{bmatrix} = \begin{bmatrix} \hat{\varphi}_R & \hat{\varphi}_r \end{bmatrix}^T P^{-1} S P
$$

and

$$
\begin{bmatrix}
\mathcal{F} \left[ \frac{\partial}{\partial \nu} S_{\partial \Omega} [\varphi] \bigg|_{x=\frac{h}{2r}} \right] \bigg|_{h_R} \\
\mathcal{F} \left[ \frac{\partial}{\partial \nu} S_{\partial \Omega} [\varphi] \bigg|_{x=\frac{h}{2r}} \right] \bigg|_{h_r}
\end{bmatrix} = -P^{-1} \left( \frac{1}{2} I - \mathbb{K} \right) P \begin{bmatrix} \hat{\varphi}_R & \hat{\varphi}_r \end{bmatrix}.
$$

Applying the Plancherel theorem, we derive that as $s \to \infty$,

$$
I + II \to \int_{-\infty}^{\infty} \begin{bmatrix} \hat{\varphi}_1 & \hat{\varphi}_2 \end{bmatrix}^T \left( -S \right) \left( \frac{1}{2} I - \mathbb{K} \right) \begin{bmatrix} \hat{\varphi}_1 & \hat{\varphi}_2 \end{bmatrix} dk + C'
$$

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This implies that $C' = 0$. Let $\tilde{g} \in L^1(\mathbb{R})$ and $\tilde{g}(k) = O(|k|^\frac{3}{2})$ as $|k| \to 0$. It then holds by Fubini’s theorem and the dominated convergence theorem that

$$\int_{-s}^{s} \mathcal{F}^{-1}(\tilde{g}) \, dy = \int_{-s}^{s} \int_{-\infty}^{\infty} \tilde{g}(k) e^{iky} \, dk \, dy = \int_{-\infty}^{\infty} \tilde{g}(k) \int_{-s}^{s} e^{iky} \, dy \, dk = 2 \int_{-\infty}^{\infty} \tilde{g}(k) \sin(ks) \, dk.$$

By the Riemann–Lebesgue lemma, the last term converges to 0 as $s \to \infty$ and, thus,

$$\lim_{s \to \infty} \int_{-s}^{s} \mathcal{F}^{-1}(\tilde{g}) \, dy = 0.$$

This implies that $C' = 0$. From (4.13), (4.18), (4.16) and (4.18), we prove (4.13).

Note that

$$\Omega^c = \left\{ z = \Psi(w) \bigg| w = x + iy \text{ with } x \in \left(\frac{1}{2r}, \infty\right) \cup (-\infty, \frac{1}{2R}) \right\}.$$

From Lemma 4.2(b), we obtain

$$\int_{\Omega^c} |\nabla S_{\partial \Omega}[^{\varphi}]|^2 \, dx \leq C \int_{x \in (-\infty, \frac{1}{2r}) \cup (\frac{1}{2r}, \infty)} \int_{-\infty}^{\infty} |x|^{-\frac{3}{2}} \frac{1}{x^2 + y^2} \, dy \, dx < \infty$$

and

$$\int_{\Omega^c} |\nabla S_{\partial \Omega}[^{\varphi}]|^2 \, dz = - \int_{\partial \Omega^c} S_{\partial \Omega}[^\varphi] \frac{\partial}{\partial u} S_{\partial \Omega}[^\varphi] \, d\sigma = - \int_{x = \frac{1}{2r}, y \in \mathbb{R}} S_{\partial \Omega}[^\varphi] \frac{\partial}{\partial u} S_{\partial \Omega}[^\varphi] \, h_R(y) \, dy - \int_{x = \frac{1}{2r}, y \in \mathbb{R}} S_{\partial \Omega}[^\varphi] \frac{\partial}{\partial u} S_{\partial \Omega}[^\varphi] \, h_r(y) \, dy =: I + II.$$

Note that

$$\frac{\partial}{\partial u} S_{\partial \Omega}[^\varphi] \big| h_R(y) = - \frac{\partial}{\partial x} S_{\partial \Omega}[^\varphi] \big|_{x = \frac{1}{2r}} = \frac{1}{2} \mathcal{F}^{-1} \left[ \tilde{\varphi}_R(k) + e^{-|k|q} \tilde{\varphi}_r(k) \right] (y) \quad \text{in } I,$$

$$\frac{\partial}{\partial u} S_{\partial \Omega}[^\varphi] \big| h_r(y) = \frac{\partial}{\partial x} S_{\partial \Omega}[^\varphi] \big|_{x = \frac{1}{2r}} = - \frac{1}{2} \mathcal{F}^{-1} \left[ - e^{-|k|q} \tilde{\varphi}_r(k) - \tilde{\varphi}_r(k) \right] (y) \quad \text{in } II,$$

which implies that

$$\left[ \mathcal{F} \left[ \frac{\partial}{\partial u} S_{\partial \Omega}[^\varphi] \big| h_R \right] \right] = \mathcal{P}^{-1} \left[ \frac{1}{2} I + \mathbb{K} P \left[ \tilde{\varphi}_R \right] - \tilde{\varphi}_r \right].$$

(4.21)
Applying also (4.19), one can derive that

\[ I + II = \text{const.} + \int_{-\infty}^{\infty} \left[ \frac{\partial}{\partial x} \phi_1 \right] T (-S) \left( \frac{1}{2} I + K \right) \frac{1}{2} \phi_1 \frac{1}{2} \phi_2 dk. \]

One can show that the constant term is zero similarly to the proof of \( C' = 0 \). Hence, we prove (4.14).

4.3 Spectral resolution of \( K_{0}^{1/2} \) on \( K_{0}^{-1/2} \)

To derive the spectral resolution of the NP operator on \( K_{0}^{-1/2} \), we define a pair of orthogonal projection operators \( P_1(s) \) and \( P_2(s) \) on \( K_{0}^{-1/2} \) for each \( s \in \mathbb{R} \cup \{ \infty \} \). Let \( \varphi \in K_{0}^{-1/2} \) be given by \( \varphi = U^{-1} P \hat{\varphi} \) and \( \hat{\varphi} = [\hat{\varphi}_1, \hat{\varphi}_2]^T \). We define

\[ P_1(s) \varphi = U^{-1} P \left[ \chi(-\infty, s) \hat{\varphi}_1 \right], \quad P_2(s) \varphi = U^{-1} P \left[ \chi(-\infty, s) \hat{\varphi}_2 \right] \]

for \( s \in \mathbb{R} \) and

\[ P_1(\infty) \varphi = U^{-1} P \left[ \hat{\varphi}_1 \right], \quad P_2(\infty) \varphi = U^{-1} P \left[ \hat{\varphi}_2 \right]. \]

Note that \( P_1(\infty) + P_2(\infty) = I \), where \( I \) is the identity operator on \( K_{0}^{-1/2} \). We then define a family of projection operators \( E(t) \) on \( K_{0}^{-1/2} \), \( t \in [-1/2, 1/2] \), by

\[ E(t) := \begin{cases} 
P_1(-\ln(2tq)) - P_1\left( \frac{\ln(-2tq)}{q} \right), & t \in [-1/2, 0), \\
\quad P_1(\infty), & t = 0, \\
- P_2\left( -\frac{\ln(2tq)}{q} \right) + P_2\left( \frac{\ln(2tq)}{q} \right) + I, & t \in (0, 1/2].
\end{cases} \tag{4.22} \]

It is straightforward to obtain that, for \( s, t \in [-1/2, 1/2] \),

\[ \begin{cases} 
E(t) E(s) = E(\min(t, s)), \\
\lim_{t \to s^+} E(t) = E(s), \\
E(-\frac{1}{2}) = 0, \quad \text{and} \quad E(\frac{1}{2}) = I.
\end{cases} \tag{4.23} \]

The limit in (4.23) is in the sense of strong convergence, i.e., \( \lim_{t \to s^+} E(t) \psi = E(s) \psi \) for all \( \psi \in K_{0}^{-1/2} \), which holds from the dominated convergence theorem and the fact that the integral in (4.7) is finite. In short, we have the following lemma.

**Lemma 4.4.** The family of operators \( \{E(t)\}_{t \in [-\frac{1}{2}, \frac{1}{2}]} \) is a resolution of identity on \( K_{0}^{-1/2} \) and satisfies

\[ \lim_{t \to s} E(t) = E(s) \tag{4.24} \]

in the sense of strong convergence.
We have
\[
\langle \psi, \mathcal{P}_1(s) \varphi \rangle_{-1/2} = \int_{-\infty}^{s} \left[ \hat{\psi}_1 \hat{\psi}_2 \right]^{T} (-\mathcal{S}) \begin{bmatrix} \hat{\varphi}_1 \\ 0 \end{bmatrix} dk,
\]
which is almost everywhere differentiable in \( s \) from the Lebesgue differentiation theorem. Similarly, \( \langle \psi, \mathcal{E}(t) \varphi \rangle_{-1/2} \) is almost everywhere differentiable in \( t \).

**Lemma 4.5.** For all \( \varphi, \psi \in K_0^{-1/2} \), it holds that
\[
\langle \psi, \varphi \rangle_{-1/2} = \int_{-\frac{1}{2}}^{\frac{1}{2}} d\langle \psi, \mathcal{E}(t) \varphi \rangle_{-1/2}.
\]  
(4.25)

**Proof.** Let \( \varphi, \psi \) be given by (4.22). It then holds from (4.5) that
\[
\langle \psi, \mathcal{P}_1(\pm k) \varphi \rangle_{-1/2} = \frac{1}{2} \int_{-\infty}^{\pm k} \hat{\psi}_1(s) \frac{1 - e^{-q|s|}}{|s|} ds \quad \text{for} \quad k > 0.
\]  
(4.26)

From (4.26) and a similar derivation, we obtain
\[
\frac{d}{dk} \langle \psi, (\mathcal{P}_1(k) - \mathcal{P}_1(-k)) \varphi \rangle_{-1/2} = \frac{1}{2} \hat{\psi}_1(k) \frac{1 - e^{-q|k|}}{|k|} + \frac{1}{2} \hat{\psi}_1(-k) \frac{1 - e^{-q|k|}}{|k|},
\]
\[
\frac{d}{dk} \langle \psi, (\mathcal{P}_2(k) - \mathcal{P}_2(-k)) \varphi \rangle_{-1/2} = \frac{1}{2} \hat{\psi}_2(k) \frac{1 + e^{-q|k|}}{|k|} + \frac{1}{2} \hat{\psi}_2(-k) \frac{1 + e^{-q|k|}}{|k|}.
\]

We then obtain that, letting \( t = -\frac{1}{2} e^{-qk} \in [-1/2, 0) \),
\[
\frac{1}{2} \int_{-\infty}^{\infty} \hat{\psi}_1(k) \frac{1 - e^{-q|k|}}{|k|} dk = \int_{0}^{\infty} \frac{d}{dk} \langle \psi, (\mathcal{P}_1(k) - \mathcal{P}_1(-k)) \varphi \rangle_{-1/2} dk
\]
\[
= \int_{0}^{0} d\langle \psi, \left( \mathcal{P}_1 \left( -\frac{\ln(-2t)}{q} \right) - \mathcal{P}_1 \left( \frac{\ln(-2t)}{q} \right) \right) \varphi \rangle_{-1/2}
\]
and that, letting \( t = \frac{1}{2} e^{-qk} \in (0, 1/2] \),
\[
\frac{1}{2} \int_{-\infty}^{\infty} \hat{\psi}_2(k) \frac{1 + e^{-q|k|}}{|k|} dk = \int_{0}^{\infty} \frac{d}{dk} \langle \psi, (\mathcal{P}_2(k) - \mathcal{P}_2(-k)) \varphi \rangle_{-1/2} dk
\]
\[
= \int_{0}^{0} d\langle \psi, \left( \mathcal{P}_2 \left( -\frac{\ln(2t)}{q} \right) - \mathcal{P}_2 \left( \frac{\ln(2t)}{q} \right) \right) \varphi \rangle_{-1/2}
\]
Since \( \langle \psi, \mathbb{I} \varphi \rangle_{-1/2} \) is constant, we have \( d\langle \psi, \mathbb{I} \varphi \rangle_{-1/2} = 0 \). Hence, we complete the proof. \( \square \)

**Theorem 4.6.** Let \( \{ \mathcal{E}(t) \}_{t \in [-1/2, 1/2]} \) be the resolution of the identity on \( K_0^{-1/2} \) given by (4.22).
We have the spectral resolution of \( \mathcal{K}^{\ast}_{\mathcal{D}\Omega} \) on \( K_0^{-1/2} \) as
\[
\mathcal{K}^{\ast}_{\mathcal{D}\Omega} = \int_{-\frac{1}{2}}^{\frac{1}{2}} t d\mathcal{E}(t).
\]
In other words, it holds that
\[
\langle \psi, \mathcal{K}^{\ast}_{\mathcal{D}\Omega} \varphi \rangle_{-1/2} = \int_{-\frac{1}{2}}^{\frac{1}{2}} t d\langle \psi, \mathcal{E}(t) \varphi \rangle_{-1/2} \quad \text{for all} \quad \psi, \varphi \in K_0^{-1/2}.
\]  
(4.27)
Proof. Let \( \varphi, \psi \) be given by (4.6). From the definition of the NP operator and the inner product on \( K_0^{-1/2} \), we have

\[
\langle \psi, \mathcal{K}_{\partial \Omega}^*[\varphi] \rangle_{-1/2} = -\frac{1}{4} \int_{-\infty}^{\infty} e^{-q|k|} \widehat{\psi_1} \varphi_1 \frac{1 - e^{-q|k|}}{|k|} dk + \frac{1}{4} \int_{-\infty}^{\infty} e^{-q|k|} \widehat{\varphi_2} \varphi_2 \frac{1 + e^{-q|k|}}{|k|} dk \\
= : I + II.
\]

Following the derivation in the proof of Lemma 4.5, we obtain

\[
I = \int_{0}^{\infty} \left( \frac{1}{2} e^{-q|k|} \right) \frac{d}{dk} \langle \psi, (\mathcal{P}_1(k) - \mathcal{P}_1(-k)) \varphi \rangle_{-1/2} \, dk = \int_{-\frac{1}{2}}^{0} t \frac{d}{dt} \langle \psi, (\mathcal{P}_1(t) - \mathcal{P}_1(t)) \varphi \rangle_{-1/2} \, dt
\]

and

\[
II = \int_{0}^{\infty} \left( \frac{1}{2} e^{-q|k|} \right) \frac{d}{dk} \langle \psi, (\mathcal{P}_2(k) - \mathcal{P}_2(-k)) \varphi \rangle_{-1/2} \, dk = \int_{0}^{\frac{1}{2}} t \frac{d}{dt} \langle \psi, (\mathcal{P}_2(t) - \mathcal{P}_2(t)) \varphi \rangle_{-1/2} \, dt.
\]

This proves the theorem. \( \square \)

The condition \( \lim_{t \to s^-} E(t) \neq E(s) \) characterizes the point spectrum of \( \mathcal{K}_{\partial \Omega}^* \). By Lemma 4.4, we conclude that \( \mathcal{K}_{\partial \Omega}^* \) has only a discrete spectrum. We need the following lemma to prove \( \mathcal{K}_{\partial \Omega}^* \) has only an absolutely continuous spectrum.

**Lemma 4.7.** We have

\[
\frac{d}{dt} \langle \psi, E(t) \varphi \rangle_{-1/2} = \begin{cases} 
\frac{1}{2} \left( 1 + \frac{2t}{|2|t\ln(2|t|)} \right) \left( \widehat{\psi_1}(k) \overline{\varphi_1(k)} + \widehat{\psi_1}(-k) \overline{\varphi_1}(-k) \right), & t \in [-1/2, 0), \\
\frac{1}{2} \left( 1 + \frac{2t}{|2|t\ln(2|t|)} \right) \left( \widehat{\psi_2}(k) \overline{\varphi_2(k)} + \widehat{\psi_2}(-k) \overline{\varphi_2}(-k) \right), & t \in (0, 1/2]
\end{cases}
\]

with \( k = -\frac{1}{q} \ln(2|t|) \).

**Proof.** First, we consider the case \( t \in [-1/2, 0) \). From the derivation in the proof of Lemma 4.5, we have \( E(t) = \mathcal{P}_1(k) - \mathcal{P}_1(-k) \) with \( t = -\frac{1}{2} e^{-qk} \) \( (k > 0) \) and

\[
\langle \psi, E(t) \varphi \rangle_{-1/2} = \frac{1}{2} \int_{-k}^{k} \widehat{\psi_1}(s) \overline{\varphi_1}(s) \frac{1 - e^{-q|s|}}{|s|} ds,
\]

\[
\frac{d}{dt} \langle \psi, E(t) \varphi \rangle_{-1/2} = \frac{1}{2} \left( \widehat{\psi_1}(k) \overline{\varphi_1}(k) + \widehat{\psi_1}(-k) \overline{\varphi_1}(-k) \right) \frac{1 - e^{-q|k|}}{|k|} \frac{dk}{dt}. \tag{4.28}
\]

Now, let \( t \in (0, 1/2] \). We have \( E(t) = -\mathcal{P}_2(k) + \mathcal{P}_2(-k) + I \) with \( t = \frac{1}{2} e^{-qk} \) \( (k > 0) \) and

\[
\langle \psi, E(t) \varphi \rangle_{-1/2} = \frac{1}{2} \int_{-k}^{k} \widehat{\psi_2}(s) \overline{\varphi_2}(s) \frac{1 + e^{-q|s|}}{|s|} ds + \text{const.},
\]

\[
\frac{d}{dt} \langle \psi, E(t) \varphi \rangle_{-1/2} = \frac{1}{2} \left( \widehat{\psi_2}(k) \overline{\varphi_2}(k) + \widehat{\psi_2}(-k) \overline{\varphi_2}(-k) \right) \frac{1 + e^{-q|k|}}{|k|} \frac{dk}{dt}. \tag{4.29}
\]

From (4.28) and (4.29), we prove the lemma. \( \square \)
For any \( \varphi \in K_0^{-1/2} \), it holds from (4.28) and (4.29) that
\[
\int_{-\frac{1}{2}}^{\frac{1}{2}} |\frac{d}{dt}(\varphi, E(t)\varphi)|_{-\frac{1}{2}} \, dt = \frac{1}{2} \int_{-\infty}^{\infty} \left( \frac{1-e^{-q|k|}}{|k|} |\hat{\varphi}_1(k)|^2 + \frac{1+e^{-q|k|}}{|k|} |\hat{\varphi}_2(k)|^2 \right) \, dk < \infty. \tag{4.30}
\]
In other words, \( \frac{d}{dt}(\varphi, E(t)\varphi)_{-\frac{1}{2}} \) is integrable for any \( \psi \in K_0^{-1/2} \), which implies that \( \frac{d}{dt}(\psi, E(t)\varphi)_{-\frac{1}{2}} \) is integrable for any \( \psi, \varphi \in K_0^{-1/2} \). In view of (4.27), we obtain the following theorem.

**Theorem 4.8.** The NP operator \( K_{0\Omega}^* \) on \( K_0^{-1/2} \) has only the absolutely continuous spectrum \( [-\frac{7}{8}, \frac{7}{8}] \).

### 5 Plasmon resonance on a crescent-shaped domain

In this section, we analyze the plasmon resonance on a crescent-shaped domain \( \Omega = B_R \setminus \overline{B}_r \).

We consider the transmission problem (2.7) with \( \epsilon \) given by
\[
\epsilon = (\epsilon_c + i\delta)\chi_\Omega + \chi_{\mathbb{R}^2 \setminus \Omega}. \tag{5.1}
\]
Let \( f \) be compactly supported away from \( \Omega \) so that the Newtonian potential \( F \) is smooth in a neighborhood of \( \overline{\Omega} \). It then holds from Lemma 4.1 that \( \partial_\nu F \in K_0^{-1/2} \).

Set \( \lambda_\delta \) as in (2.8). Because of \( \lambda_\delta \in \mathbb{R}^2 \setminus [-\frac{1}{2}, \frac{1}{2}] \), the problem
\[
(\lambda_\delta I - K_{0\Omega}^*)[\varphi^\delta] = \partial_\nu F \tag{5.2}
\]
is solvable in \( K_0^{-1/2} \). The solution \( \varphi^\delta \in K_0^{-1/2} \) is of the form (see (4.2))
\[
\varphi^\delta = U^{-1} P \varphi^\delta \quad \text{with} \quad \varphi^\delta = \begin{bmatrix} \hat{\varphi}_1^\delta \\ \hat{\varphi}_2^\delta \end{bmatrix},
\]
which, by Theorem 4.6, admits the following integral expression:
\[
\varphi^\delta = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\lambda_\delta - t} \, dE(t)[\partial_\nu F]. \tag{5.3}
\]
Recall the definition of \( U \) in (3.12). For any fixed positive integer \( n \), if \( g \in L^1(\mathbb{R}) \) satisfies \( \partial^\alpha g \in L^1(\mathbb{R}) \) and \( \partial^\alpha g(x) \to 0 \) as \( |x| \to \infty \) for all \( |\alpha| \leq n \), then \( \|F[g](k)\| \leq \frac{C}{(1+|k|)^n} \) with some positive constant \( C \). From the smoothness of \( \partial_\nu F \) on \( \partial B_R \) and \( \partial B_r \), each component of \( U[\partial_\nu F] \) then belongs to \( L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \) so that
\[
PU[\partial_\nu F] \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}). \tag{5.4}
\]
From (4.1), (4.8) and (5.2), \( \varphi^\delta \) satisfies
\[
(\lambda_\delta I - \mathbb{K})[\varphi^\delta] = PU[\partial_\nu F],
\]
which leads to
\[
\varphi^\delta = \begin{bmatrix} (\lambda_\delta - \frac{1}{2}e^{-|k|})^{-1} \\ 0 \end{bmatrix} PU[\partial_\nu F] \quad \text{and} \quad \varphi^\delta = \begin{bmatrix} (\lambda_\delta + \frac{1}{2}e^{-|k|})^{-1} \\ 0 \end{bmatrix} PU[\partial_\nu F]. \tag{5.5}
\]
Because of $\partial_r F \in L^2_0(\partial\Omega)$, from (3.18), the second entry of $PU[\partial_r F]$ satisfies $O(|k|^{-1})$ as $k \to 0$. From the assumption that $\lambda_3 \in \mathbb{R}^2 \setminus [-\frac{1}{2}, \frac{1}{2}]$, it then holds that

$$\varphi^\delta_1, \varphi^\delta_2 \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \quad \text{and} \quad \varphi^\delta \in O(|k|^{\frac{1}{2}}).$$

Then, from Lemma 4.2 (a), $S_{\Omega}[\varphi^\delta]$ is continuous in $\mathbb{R}^2$, harmonic in $\mathbb{R}^2 \setminus \partial\Omega$, and harmonic at infinity. Furthermore, from Lemma 4.3, $|\nabla S_{\Omega}[\varphi]| \in L^2(\mathbb{R}^2)$. As $\varphi^\delta$ satisfies (5.2), the solution to (2.7) with $\epsilon$ given by (5.1) satisfies

$$u_\delta(z) = F(z) + S_{\Omega}[\varphi^\delta](z).$$

Since $F$ is smooth in a region containing $\overline{\Omega}$, (2.10) is equivalent to $\|\nabla S_{\Omega}[\varphi^\delta]\|_{L^2(\Omega)} \to \infty$ as $\delta \to 0$. From (5.3) and (2.9), we have

$$\|\varphi^\delta\|_{-1/2}^2 = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{|\lambda_3 - t|^2} \, d\langle \partial_r F, \mathbb{E}(t)\partial_r F \rangle_{-1/2}$$

with

$$\lambda_3 - t = \frac{(\lambda_0 - t) + \frac{1-2\beta}{2(\epsilon_c-1)} \epsilon \delta}{1 + \frac{1}{2(\epsilon_c-1)} \epsilon \delta} \quad \text{with} \quad \lambda_0 = \frac{\epsilon_c + 1}{2(\epsilon_c-1)}. \quad (5.7)$$

Note that $\lambda$ converges to $\lambda_0$ as $\delta \to 0$. We see from Lemma 4.3 and Theorem 4.6 that

$$\|\nabla S_{\Omega}[\varphi^\delta]\|_{L^2(\Omega)}^2 = \int_{-\frac{1}{2}}^{\frac{1}{2}} \left( \frac{1}{2} - t \right) \frac{1}{|\lambda_3 - t|^2} \, d\langle \partial_r F, \mathbb{E}(t)\partial_r F \rangle_{-1/2}$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \left( \frac{1}{2} - t \right) \frac{1}{|\lambda_3 - t|^2} \, d\langle \partial_r F, \mathbb{E}(t)\partial_r F \rangle_{-1/2}.$$

For any $\varphi \in K_0^{-1/2}$, we have $\frac{d}{dt}\langle \varphi, \mathbb{E}(t)\varphi \rangle_{-1/2} \geq 0$ from Lemma 4.7. Then, (5.6) leads us to

$$\|\nabla S_{\Omega}[\varphi^\delta]\|_{L^2(\Omega)}^2 \leq \|\varphi^\delta\|_{-1/2}^2. \quad (5.8)$$

We now assume that $\lambda_0 < \frac{1}{2}$. Then, we can take $t_1, t_2$ independent of $\delta$ such that $\lambda_0 < t_1 < t_2 < \frac{1}{2}$. Then, we have

$$\|\nabla S_{\Omega}[\varphi^\delta]\|_{L^2(\Omega)}^2 = \left( \int_{-\frac{1}{2}}^{t_1} + \int_{t_2}^{\frac{1}{2}} \right) \left( \frac{1}{2} - t \right) \frac{1}{|\lambda_3 - t|^2} \, d\langle \partial_r F, \mathbb{E}(t)\partial_r F \rangle_{-1/2}$$

$$\geq \left( \frac{1}{2} - t_2 \right) \int_{-\frac{1}{2}}^{t_2} \frac{1}{|\lambda_3 - t|^2} \, d\langle \partial_r F, \mathbb{E}(t)\partial_r F \rangle_{-1/2}$$

$$= \left( \frac{1}{2} - t_2 \right) \left( \|\varphi^\delta\|_{-1/2}^2 - \int_{t_2}^{\frac{1}{2}} \frac{1}{|\lambda_3 - t|^2} \, d\langle \partial_r F, \mathbb{E}(t)\partial_r F \rangle_{-1/2} \right)$$

$$\geq \left( \frac{1}{2} - t_2 \right) \left( \|\varphi^\delta\|_{-1/2}^2 - \frac{1}{|\lambda_3 - t_2|^2} \|\partial_r F\|_{-1/2}^2 \right).$$

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Hence, we have

\[ C_1 \| \varphi^\delta \|_{-1/2} - C_2 \leq \| \nabla S_{\varphi^\delta} \|_{L^2(\Omega)} \leq \| \varphi^\delta \|_{-1/2} \quad \text{for } \lambda_0 < \frac{1}{2}, \quad (5.9) \]

where \( C_1, C_2 \) are some positive constants independent of \( \delta \). Hence, for \( \lambda_0 < \frac{1}{2} \), the resonance condition \( \| \nabla S_{\varphi^\delta} \|_{L^2(\Omega)} \to \infty \) as \( \delta \to 0 \) is equivalent to that \( \| \varphi^\delta \|_{-1/2} \to \infty \) as \( \delta \to 0 \). In the following, we characterize the resonance depending on \( \lambda_0 \).

Lemma 4.7 leads us to

\[ d(\partial_r F, E(t)\partial_r F)_{-1/2} = Q(t) \, dt \quad (5.10) \]

with

\[ Q(t) = \begin{cases} \frac{1}{2} \int_0^t \left( |\tilde{f}_1(k)|^2 + |\tilde{f}_1(-k)|^2 \right) \, dt, & \text{for } t \in [-1/2, 0), \\ \frac{1}{2} \int_0^t \left( |\tilde{f}_2(k)|^2 + |\tilde{f}_2(-k)|^2 \right) \, dt, & \text{for } t \in (0, 1/2], \end{cases} \quad (5.11) \]

where \( k = -\frac{1}{2} \log(2|t|) \) and \( [\tilde{f}_1, \tilde{f}_2]^T = PU(\partial_r F) \). From (4.30), \( Q \) belongs to \( L^1(\mathbb{R}) \). The two functions \( \tilde{f}_1, \tilde{f}_2 \) are bounded, continuous, and vanishing at infinity since \( \partial_r F \) is smooth on \( \partial B_R \) and \( \partial B_r \). Since \( \partial_r F \) has a zero mean on each \( \partial B_r \) and \( \partial B_R \), the two components of \( U(\partial_r F) \) are both zero at \( k = 0 \). Hence, \( Q(t) \) is continuous for \( t \in (-\frac{1}{2}, 0) \cup (0, \frac{1}{2}) \). For \( t = \pm \frac{1}{2} \), we have

\[ \lim_{t \to (-\frac{1}{2})^+} Q(t) = 2|\tilde{f}_1(0)|^2 = 0, \quad \lim_{t \to \frac{1}{2}^-} Q(t) = 2|\tilde{f}_2(0)|^2 = 0. \quad (5.12) \]

In view of (5.9), we can determine the order of resonance from the following propositions:

**Proposition 5.1.** For \( \lambda_0 \in [-\frac{1}{2}, 0) \cup (0, \frac{1}{2}) \), let \( \varphi^\delta \) be the solution to (5.2) and \( [\tilde{f}_1, \tilde{f}_2]^T = PU(\partial_r F) \). Then, it holds that

\[ \lim_{\delta \to 0} \delta \| \varphi^\delta \|_{-1/2}^2 = \| \epsilon_c - 1 \| \begin{cases} \frac{Q(\lambda_0)}{t - \lambda_0} & \text{if } \lambda_0 \in (-\frac{1}{2}, 0) \cup (0, \frac{1}{2}), \\ 0 & \text{if } \lambda_0 = \pm \frac{1}{2}, \end{cases} \]

where \( Q \) is given by (5.11).

**Proof.** Let \( \lambda_0 \in [-\frac{1}{2}, 0) \cup (0, \frac{1}{2}) \). Then, we can assume that \( \epsilon_c \) is bounded. From (5.10) and (5.11), we have

\[ \frac{1}{|\lambda_0 - t|^2} = \frac{1}{(\lambda_0 - t)^2 + \left( \frac{1}{2} - t \right)^2 (\epsilon_c - 1)^2} = \frac{1 + \tilde{\delta}^2}{(\lambda_0 - t)^2 + \left( \frac{1}{2} - t \right)^2 \delta^2} = \frac{1}{\left( t - \frac{2\lambda_0 + \tilde{\delta}^2}{2(1 + \tilde{\delta}^2)} \right)^2 + \frac{\delta^2(1-2\lambda_0)^2}{4(1+\delta^2)^2}}, \quad \tilde{\delta} = \left| \frac{\delta}{\epsilon_c - 1} \right|. \]

We then have from (5.6) and (5.10) that

\[ \| \varphi^\delta \|_{-1/2}^2 = \frac{\pi (1 + \tilde{\delta}^2)}{\delta (\frac{1}{2} - \lambda_0)} \int_{-\infty}^{\infty} \frac{\tilde{\delta}(1-2\lambda_0)}{2(1+\tilde{\delta}^2)} \frac{Q(t) \chi(-\frac{1}{2}, \frac{1}{2})}{\left( t - \frac{2\lambda_0 + \tilde{\delta}^2}{2(1+\tilde{\delta}^2)} \right)^2 + \frac{\delta^2(1-2\lambda_0)^2}{4(1+\delta^2)^2}} \, dt. \quad (5.13) \]
Now, we simplify (5.13) using the property for the Poisson kernel of the upper half plane: for \( g \in L^1(\mathbb{R}) \) and \( x \in \mathbb{R} \),

\[
\lim_{y \to 0^+} \frac{1}{\pi} \int_0^\infty \frac{y}{(x-t)^2 + y^2} g(t) dt = \frac{g(x^-) + g(x^+)}{2}
\]

if \( g(x^-) \) and \( g(x^+) \) exist. Here, \( g(x^-) \) and \( g(x^+) \) indicate the limit from the left and the right, respectively. From (5.12) and the continuity of \( Q(t) \) on \( t \in (-\frac{1}{2}, 0) \cup (0, \frac{1}{2}) \), we complete the proof of the lemma.

**Proposition 5.2.** Suppose \( \lambda_0 = 0 \) (or, equivalently, \( \epsilon_c = -1 \)). Let \( \varphi_\delta \) be the solution to (5.2). Then, it holds that

\[
\lim_{\delta \to 0} \delta^2 \|\varphi_\delta\|_{-1/2}^2 = 0.
\]

**Proof.** One can derive that \( \lim_{\delta \to 0} \delta^2 \|\varphi_\delta\|_{-1/2}^2 = 0 \) by following the proof of Proposition 4 in [23] with the fact that \( Q \in L^1(\mathbb{R}) \).

---

### 6 Analysis of the NP operator on touching disks

In this section, we provide the spectral resolution of the NP operator on the touching disks \( \Omega = B_R \cup B_{-r} \), following the notation in (3.2). The two disks \( B_R \) and \( B_{-r} \) are tangent to each other at the origin (see the left figure in Figure 6.1). We follow the derivations for the result on a crescent-shaped domain in the previous sections. We omit most proofs since the analysis is almost the same as for the crescent-shaped domain case.

The touching disks \( \Omega \) is mapped onto the region

\[
S := \Psi(\Omega) = \left( -\infty, -\frac{1}{2r} \right) \times (-\infty, \infty) \cup \left( \frac{1}{2r}, \infty \right) \times (-\infty, \infty)
\]

via the Möbius mapping \( \Psi \) defined in (3.1), and vice versa. The outward normal vector convention is described in Figure 6.1. For a function \( v \), the normal derivative at \( z \in \partial \Omega \) with

![Diagram of touching disks and Möbius mapping](image)

**Figure 6.1:** Touching disks \( \Omega = B_R \cup B_{-r} \) (gray region in the left figure) and \( S = \Psi(\Omega) \) (gray region in the right figure). Arrows indicate the outward normal vectors to \( \partial \Omega \) or to \( \partial S \).
\[ \Psi(z) = x + iy \]
\[
\frac{\partial u}{\partial \nu} = -\frac{1}{h_R(y)} \frac{\partial (u \circ \Psi)}{\partial x} \quad \text{for } z \in \partial B_R \setminus \{0\},
\]
\[
\frac{\partial u}{\partial \nu} = \frac{1}{h_{-r}(y)} \frac{\partial (u \circ \Psi)}{\partial x} \quad \text{for } z \in \partial B_{-r} \setminus \{0\},
\]
where \( h_R(y) \) and \( h_{-r}(y) \) are defined as in (6.2). Similar to the crescent-shaped domain case, we denote by \( \tilde{q} \) the distance between the two boundary lines of \( S \), i.e.,
\[
\tilde{q} := \frac{1}{2R} + \frac{1}{2r}.
\]

### 6.1 Generalization of the layer potential operators on the crescent-shaped domain

A density function \( \varphi \in L^2(\partial \Omega) \) can be decomposed as
\[
\varphi = \varphi \chi_{\partial B_R} + \varphi \chi_{\partial B_{-r}}
\]
\[
= : \varphi_R + \varphi_{-r}.
\]

We identify \( \varphi_R, \varphi_{-r} \) with the functions on \( \mathbb{R} \) given by
\[
\tilde{\varphi}_R(y) = (\varphi_R \circ \Psi^{-1})(\frac{1}{2R} + iy),
\]
\[
\tilde{\varphi}_{-r}(y) = (\varphi_{-r} \circ \Psi^{-1})(-\frac{1}{2r} + iy) \quad \text{for } y \in \mathbb{R}.
\]

We now define the single-layer potential and the NP operator on touching disks. One can easily find that the jump relations in Lemma 3.1 holds for touching disks.

**Definition 4.** For \( \varphi = \varphi_R + \varphi_{-r} \in L^2(\partial \Omega) \), we define
\[
\mathcal{S}_{\partial \Omega}[\varphi] := \mathcal{S}_{\partial B_R}[\varphi_R] + \mathcal{S}_{\partial B_{-r}}[\varphi_{-r}] \quad \text{on } \mathbb{R}^2
\]
and
\[
\mathcal{K}^*_{\partial \Omega}[\varphi] := \left( \mathcal{K}^*_{\partial B_R}[\varphi_R] + \frac{\partial}{\partial \nu} \mathcal{S}_{\partial B_{-r}}[\varphi_{-r}] \bigg|_{\partial B_R} \right) \chi_{\partial B_R}
\]
\[
+ \left( \frac{\partial}{\partial \nu} \mathcal{S}_{\partial B_R}[\varphi_R] \bigg|_{\partial B_{-r}} + \mathcal{K}^*_{\partial B_{-r}}[\varphi_{-r}] \right) \chi_{\partial B_{-r}} \quad \text{on } \partial \Omega.
\]

We set \( z = \frac{1}{2 + iy} \) for \( z \neq 0 \). Let \( z_t = \Psi^{-1}(\frac{1}{2R} + it) \) on \( \partial B_{-r} \) and \( z_t = \Psi^{-1}(\frac{1}{2R} + it) \) on \( \partial B_R \); then the single-layer potential (6.4) satisfies
\[
\mathcal{S}_{\partial \Omega}[\varphi](z) = \frac{1}{2\pi} \int_{\partial B_{-r}} \ln |z - z_t| \varphi_{-r}(z_t) d\sigma(z_t) + \frac{1}{2\pi} \int_{\partial B_R} \ln |z - z_t| \varphi_R(z_t) d\sigma(z_t)
\]
\[
= \frac{1}{4\pi} \int_{-\infty}^{\infty} \left( \ln \left[ \left( x + \frac{1}{2r} \right)^2 + (y - t)^2 \right] - \ln \left[ \left( \frac{1}{2r} \right)^2 + t^2 \right] \right) \tilde{\varphi}_{-r}(t) h_{-r}(t) dt
\]
\[
+ \frac{1}{4\pi} \int_{-\infty}^{\infty} \left( \ln \left[ \left( x - \frac{1}{2R} \right)^2 + (y - t)^2 \right] - \ln \left[ \left( \frac{1}{2R} \right)^2 + t^2 \right] \right) \tilde{\varphi}_R(t) h_R(t) dt
\]
\[
- \frac{1}{4\pi} \ln(x^2 + y^2) \int_{\partial \Omega} \varphi(z) d\sigma(z).
\]
Thus, we have
\[
\begin{align*}
\frac{\partial}{\partial \nu} S_{\delta B_{-r}}[\varphi_{-r}]_{\partial B_R}(z) &= -\frac{1}{2 \pi h_R(y)} \int_{-\infty}^{\infty} \frac{\bar{q}}{q^2 + (y-t)^2} \bar{\varphi}_{-r}(t) h_{-r}(t) \, dt + \frac{1}{4 \pi R} \int_{-\infty}^{\infty} \bar{\varphi}_{-r}(t) h_{-r}(t) \, dt, \\
\frac{\partial}{\partial \nu} S_{\delta B_R}[\varphi_R]_{\partial B_r}(z) &= -\frac{1}{2 \pi h_r(y)} \int_{-\infty}^{\infty} \frac{\bar{q}}{q^2 + (y-t)^2} \bar{\varphi}_{-r}(t) h_{-r}(t) \, dt + \frac{1}{4 \pi r} \int_{-\infty}^{\infty} \bar{\varphi}_{-r}(t) h_{-r}(t) \, dt
\end{align*}
\]
with \( \bar{q} \) given by (6.1). The remaining terms of \( K_{\partial \Omega}^* \) are
\[
K_{\partial \Omega}^*[\varphi_{-r}] = \frac{1}{4 \pi R} \int_{\partial B_R} \varphi_R \, d\sigma \quad \text{on } \partial B_R,
\]
\[
K_{\partial B_{-r}}^*[\varphi_{-r}] = \frac{1}{4 \pi r} \int_{\partial B_{-r}} \varphi_{-r} \, d\sigma \quad \text{on } \partial B_{-r}.
\]
As a result, we obtain that
\[
K_{\partial \Omega}^*[\varphi](z) = \begin{cases} 
-\frac{1}{2 \pi h_R(y)} \int_{-\infty}^{\infty} \frac{\bar{q}}{q^2 + (y-t)^2} \bar{\varphi}_{-r}(t) h_{-r}(t) \, dt + \frac{1}{4 \pi R} \int_{\partial \Omega} \varphi \, d\sigma & \text{for } x = \frac{1}{2 R}, \\
-\frac{1}{2 \pi h_r(y)} \int_{-\infty}^{\infty} \frac{\bar{q}}{q^2 + (y-t)^2} \bar{\varphi}_{-r}(t) h_{-r}(t) \, dt + \frac{1}{4 \pi r} \int_{\partial \Omega} \varphi \, d\sigma & \text{for } x = -\frac{1}{2 r}.
\end{cases}
\]
Similar to the crescent-shaped domain case, we define
\[
U[\varphi] = \begin{bmatrix} F[h_R \bar{\varphi}_R] \\ F[h_{-r} \bar{\varphi}_{-r}] \end{bmatrix}, \tag{6.7}
\]
where \( F \) is the Fourier transform (see Subsection 3.3 for the definition). By applying the Fourier transform to (6.5) and (6.6), we express the layer potential operators as follows.

**Lemma 6.1.** Let \( \varphi = \varphi_R + \varphi_{-r} \in L_0^2(\partial \Omega) \). For \( z = \Psi^{-1}(x + iy) \in \partial \Omega \), we have
\[
S_{\partial \Omega}[\varphi](z) = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{2|k|} \left( e^{-|x-\frac{1}{2\pi}||k|} F[h_R \bar{\varphi}_R](k) + e^{-|x+\frac{1}{2\pi}||k|} F[h_{-r} \bar{\varphi}_{-r}](k) \right) e^{iky} \, dk + C,
\]
where \( C \) is the constant given by
\[
C = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{2|k|} \left( e^{-\frac{1}{2\pi}||k|} F[h_R \bar{\varphi}_R](k) + e^{-\frac{1}{2\pi}||k|} F[h_{-r} \bar{\varphi}_{-r}](k) \right) \, dk.
\]

**Lemma 6.2.** For \( \varphi \in L_0^2(\partial \Omega) \), it holds that
\[
U[K_{\partial \Omega}^*[\varphi]] = P^{-1} \left( \frac{1}{2} e^{-|k|\bar{q}} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} P \right) U[\varphi]
\]
with \( \bar{q} \) given by (6.1).
6.2 Spectral resolution of the NP operator on touching disks

We define two matrix-valued functions in Lemma 6.1 and Lemma 6.2 as

\[
\tilde{\mathbf{S}} = -\frac{1}{2|k|} \begin{bmatrix} 1 - e^{-q|k|} & 0 \\ 0 & 1 + e^{-q|k|} \end{bmatrix}, \quad k \in \mathbb{R} \setminus \{0\},
\]

\[
\tilde{\mathbf{K}} = \frac{1}{2} e^{-q|k|} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad k \in \mathbb{R}.
\]

(6.8)

Note that the definition of the matrix \(\tilde{\mathbf{S}}\) is identical to \(\mathbf{S}\) of the crescent-shaped domain case in (4.1), except that \(q\) is now replaced by \(\tilde{q}\). Note also that the signs of the diagonal entries of \(\tilde{\mathbf{K}}\) are changed from the diagonal entries of \(\mathbf{K}\) of the crescent-shaped domain case (see (4.1)).

Definition 5. We define \(K_{0}^{-1/2}\) as the same as Definition 4 with \(\mathbf{S}\) replaced by \(\tilde{\mathbf{S}}\). In other words,

\[
K_{0}^{-1/2} := \left\{ \varphi = U^{-1} P \hat{\varphi} \left| \begin{array}{c} \hat{\varphi}_1 \\ \hat{\varphi}_2 \end{array} \right. \text{ satisfying } \int_{-\infty}^{\infty} \hat{\varphi}_1^T (\tilde{\mathbf{S}}(k) \hat{\varphi}) dk < \infty \right\},
\]

where \(\hat{\varphi}_1\) and \(\hat{\varphi}_2\) are measurable functions on \(\mathbb{R}\), and \(P\) is given by (3.16).

Also, we define the inner product \(\langle \cdot, \cdot \rangle_{-1/2}\) and the norm \(\|\cdot\|_{-1/2}\) on \(K_{0}^{-1/2}\) as the same as in Section 4.1 with \(\mathbf{S}\) replaced by \(\tilde{\mathbf{S}}\).

We now naturally extend the single-layer potential and the NP operator on \(L^2_0(\partial \Omega)\) to \(K_{0}^{-1/2}\) by generalizing the formulas in Lemma 6.1 and Lemma 6.2, respectively.

Definition 6. Let \(\varphi \in K_{0}^{-1/2}\) be given by \(\varphi = U^{-1} P \hat{\varphi}\).

(a) We define the NP operator \(\mathcal{K}_{0}^* : K_{0}^{-1/2} \to K_{0}^{-1/2}\) by

\[
\mathcal{K}_{0}^*[\varphi] := U^{-1} P \tilde{\mathbf{K}} \hat{\varphi}.
\]

(6.9)

(b) We define the single-layer potential of \(\varphi\): for \(z = \Psi^{-1}(x + iy) \in \mathbb{C}\),

\[
\mathcal{S}_{0\Omega}[\varphi](z) := -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{2|k|} \left( e^{-\frac{|x - k|}{\sqrt{2}} |k|} \hat{\varphi}_R(k) + e^{-\frac{|x + k|}{\sqrt{2}} |k|} \hat{\varphi}_r(k) \right) e^{iky} dk
\]

\[
+ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{2|k|} \left( e^{-\frac{|x + k|}{\sqrt{2}} \frac{\partial}{\partial k}} \hat{\varphi}_R(k) + e^{-\frac{|x - k|}{\sqrt{2}} \frac{\partial}{\partial k}} \hat{\varphi}_r(k) \right) dk,
\]

(6.10)

where \(\hat{\varphi}_R\) and \(\hat{\varphi}_r\) is given by

\[
\begin{bmatrix} \hat{\varphi}_R \\ \hat{\varphi}_r \end{bmatrix} = P \begin{bmatrix} \hat{\varphi}_1 \\ \hat{\varphi}_2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -\hat{\varphi}_1 + \hat{\varphi}_2 \\ \hat{\varphi}_1 + \hat{\varphi}_2 \end{bmatrix}.
\]

(6.11)

From arguments similar to those for the crescent-shaped domain case, it can be shown that \(\mathcal{K}_{0}^*\) is a bounded linear operator on \(K_{0}^{-1/2}\) whose operator norm is bounded by \(\frac{1}{2}\). In addition, \(\mathcal{K}_{0}^*\) is self-adjoint on \(K_{0}^{-1/2}\), and the spectrum of \(\mathcal{K}_{0}^*\) on \(K_{0}^{-1/2}\) lies in the interval \([-1/2, 1/2]\). In the remainder of this subsection, we derive the spectral resolution of the operator.
To derive the spectral resolution of the NP operator on $K_0^{-1/2}$, we define a pair of orthogonal projection operators $P_1(s)$ and $P_2(s)$ on $K_0^{-1/2}$ for each $s \in \mathbb{R} \cup \{\infty\}$. Let $\varphi \in K_0^{-1/2}$ be given by $\varphi = U^{-1}P\hat{\varphi}$ and $\hat{\varphi} = [\hat{\varphi}_1, \hat{\varphi}_2]^T$. We define

$$P_1(s)\varphi = U^{-1}P \begin{bmatrix} \chi(-\infty, s) \hat{\varphi}_1 \\ 0 \end{bmatrix}, \quad P_2(s)\varphi = U^{-1}P \begin{bmatrix} 0 \\ \chi(-\infty, s) \hat{\varphi}_2 \end{bmatrix}$$

for $s \in \mathbb{R}$ and

$$P_1(\infty)\varphi = U^{-1}P \begin{bmatrix} \hat{\varphi}_1 \\ 0 \end{bmatrix}, \quad P_2(\infty)\varphi = U^{-1}P \begin{bmatrix} 0 \\ \hat{\varphi}_2 \end{bmatrix}.$$

Note that $I = P_2(\infty) + P_1(\infty)$, where $I$ is the identity operator on $K_0^{-1/2}$. Now, we define a family of projection operators $E(t)$ on $K_0^{-1/2}$, $t \in [-1/2, 1/2]$, as in (4.22). We then define

$$\tilde{E}(t) := I - E(-t). \quad (6.12)$$

In other words,

$$\tilde{E}(t) := \begin{cases} P_2 \left( -\frac{\ln(-2t)}{q} \right) - P_2 \left( \frac{\ln(-2t)}{q} \right), & t \in [-1/2, 0), \\ P_2(\infty), & t = 0, \\ -P_1 \left( -\frac{\ln(2t)}{q} \right) + P_1 \left( \frac{\ln(2t)}{q} \right) + I, & t \in (0, 1/2]. \end{cases} \quad (6.13)$$

Then, the family of operators $\{\tilde{E}(t)\}_{t \in [-1/2, 1/2]}$ is a resolution of identity on $K_0^{-1/2}$ and satisfies

$$\lim_{t \to s} \tilde{E}(t) = \tilde{E}(s) \quad (6.14)$$

in the sense of strong convergence.

Following the derivations of (4.25) and (4.27) for the case of the crescent-shaped domain, it can be shown that, for all $\psi, \varphi \in K_0^{-1/2}$,

$$\langle \psi, \varphi \rangle_{-1/2} = \int_{-1/2}^{1/2} \langle \psi, E(t)\varphi \rangle_{-1/2},$$

$$\langle \psi, \mathbf{K}^*_0[\hat{\varphi}] \rangle_{-1/2} = -\int_{-1/2}^{1/2} t \langle \psi, \mathbf{E}(t)\varphi \rangle_{-1/2}. \quad (6.15)$$

The right-hand side of (6.15) has different sign of from that of (4.27) because the signs of the diagonal entries of $\tilde{K}$ are changed from the diagonal entries of $\mathbf{K}$ in (4.1). In view of (6.12), we obtain the following.

**Lemma 6.3.** For all $\varphi, \psi \in K_0^{-1/2}$, it holds that

$$\langle \psi, \varphi \rangle_{-1/2} = \int_{-1/2}^{1/2} \langle \psi, \tilde{E}(t)\varphi \rangle_{-1/2}.$$

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Theorem 6.4. Let \( \{ \tilde{E}(t) \}_{t \in [-1/2, 1/2]} \) be the resolution of the identity on \( K_{0}^{-1/2} \) given by (6.13). Then we have the following spectral resolution of \( K_{\partial \Omega}^{*} \) on \( K_{0}^{-1/2} \):

\[
K_{\partial \Omega}^{*} = \int_{-1/2}^{1/2} t \, d \tilde{E}(t).
\]

In other words, it holds that

\[
\langle \psi, K_{\partial \Omega}^{*}[\phi] \rangle_{-1/2} = \int_{-1/2}^{1/2} t \, d \langle \psi, \tilde{E}(t)\phi \rangle_{-1/2} \quad \text{for all } \psi, \phi \in K_{0}^{-1/2}.
\] (6.16)

The condition \( \lim_{t \to s} \tilde{E}(t) \neq \tilde{E}(s) \) characterizes the point spectrum of \( K_{\partial \Omega}^{*} \). From (6.14), we conclude that \( K_{\partial \Omega}^{*} \) has only a continuous spectrum. In view of (6.12) and Lemma 4.7, we have

\[
\frac{d}{dt} \langle \psi, \tilde{E}(t)\phi \rangle_{-1/2} = \frac{d}{ds} \langle \psi, \tilde{E}(s)\phi \rangle_{-1/2} \bigg|_{s=-t}
\]

and

\[
\frac{d}{dt} \langle \psi, \tilde{E}(t)\phi \rangle_{-1/2} = \begin{cases} 
\frac{1 - 2t}{2|t\ln(2|t|)|} \left( \tilde{\psi}_{1}(k) \overline{\tilde{\varphi}_{1}(k)} + \tilde{\psi}_{1}(-k) \overline{\tilde{\varphi}_{1}(-k)} \right), & t \in [-1/2, 0), \\
\frac{1 - 2t}{2|t\ln(2|t|)|} \left( \tilde{\psi}_{2}(k) \overline{\tilde{\varphi}_{2}(k)} + \tilde{\psi}_{2}(-k) \overline{\tilde{\varphi}_{2}(-k)} \right), & t \in (0, 1/2]
\end{cases}
\]

with \( k = -\frac{1}{q} \ln(2|t|) \). Similar to (4.29), it holds that

\[
\int_{-1/2}^{1/2} \left| \frac{d}{dt} \langle \varphi, \tilde{E}(t)\varphi \rangle_{-1/2} \right| \, dt = \frac{1}{2} \int_{-\infty}^{\infty} \left( 1 - e^{-|k|} \right) \frac{|\tilde{\varphi}_{1}(k)|^{2} + 1 + e^{-|k|}}{|k|} \frac{|\tilde{\varphi}_{2}(k)|^{2}}{|k|} \, dk < \infty.
\]

Finally, we obtain the following theorem.

Theorem 6.5. The NP operator \( K_{\partial \Omega}^{*} \) on \( K_{0}^{-1/2} \) has only the absolutely continuous spectrum \([ -1/2, 1/2 ] \).

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