KMS STATES AND CONFORMAL MEASURES

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ABSTRACT. From a non-constant holomorphic map on a connected Riemann surface we construct an étale second countable locally compact Hausdorff groupoid whose associated groupoid C*-algebra admits a one-parameter group of automorphisms with the property that its KMS states corresponds to conformal measures in the sense of Sullivan. In this way certain quadratic polynomials give rise to quantum statistical models with a phase transition arising from spontaneous symmetry breaking.

1. INTRODUCTION

It was shown by D. Sullivan, [S], that for any rational map \( R \) on the Riemann sphere there is a Borel probability measure \( m \) on the Julia set \( J_R \) and an exponent \( \delta \in [0, 2] \) such that

\[
m(R(A)) = \int_A |R'(z)|^\delta \, dm(z)
\]

for every Borel subset \( A \subseteq J_R \) where \( R \) is injective. Such measures were subsequently called conformal and many results have been obtained about them, in particular results on their uniqueness or non-uniqueness and on the values of the exponent \( \delta \) for various classes of rational maps. Nonetheless it seems fair to say that they remain rather mysterious in general. The main purpose with this paper is to relate these measures to KMS states of a one-parameter group of automorphisms on a C*-algebra naturally associated to the rational map. Both in terms of intention and tools the approach we take is much in the spirit of D. Ruelle who did a similar thing for Gibbs measures of hyperbolic diffeomorphisms in [Ru]. Thus we associate to a rational map, and in fact to any non-constant holomorphic map on a connected Riemann surface, an étale groupoid such that its (reduced) C*-algebra can be defined as described by Renault in [R]. The same construction works for any totally invariant subset which is locally compact in the relative topology and has no isolated points. For rational maps on the Riemann sphere this means that as far as the construction of the groupoid and its C*-algebra is concerned, we can treat the Julia set or the Fatou set in the same way as the whole sphere. By construction the groupoid comes equipped with a natural real-valued homomorphism and the corresponding one-parameter group of automorphisms has the property that there is a one-to-one correspondence between the non-atomic conformal measures with exponent \( \delta \) for the holomorphic map and a face in the weak* closed set of its \( \delta \)-KMS states. The correspondence extends to atomic measures, but for them the relation is more complicated, and in particular the map from KMS states to measures is generally not injective. As an illustration of the general results we show how certain quadratic polynomials in this way give rise to quantum statistical models with a phase transition arising from spontaneous symmetry breaking in the sense of Bost and Connes, [BoC].

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The main tool for the identification of the KMS states is a recent result of Neshveyev, [N], in which he extends results of Renault to obtain a general description of the KMS states for the one-parameter group of automorphisms arising from a real-valued homomorphism on a second countable locally compact étale groupoid. Since the groupoids we construct have the property that all isotropy groups are abelian and the points in the unit space with non-trivial isotropy group are at most countable, the results of Neshveyev can be transferred from the full to the reduced groupoid $C^*$-algebra and be given slightly more detailed formulations. We do this in the first section of the paper before we move to the main part where we construct the étale groupoid of a holomorphic map and the relevant one-parameter group of automorphisms, which we call the \textit{conformal action}, on its $C^*$-algebra. We obtain a general description of the $\beta$-KMS states for the conformal action when $\beta \neq 0$ and illustrate it by considering the restriction to the Julia set $J_R$ of a quadratic polynomial $R$ which satisfies the Collet-Eckmann condition. Thanks to results of Graczyk and Smirnov, [GS1], [GS2], we can in this case give a complete description of the $\beta$-KMS states for the conformal action for positive $\beta$. When the critical point is pre-periodic there is only one KMS state, corresponding to the Sullivan measure with exponent equal to the Hausdorff dimension $HD(J_R)$ of $J_R$. When the critical point is not pre-periodic there is no $\beta$-KMS state for $0 < \beta < HD(J_R)$, a unique one for $\beta = HD(J_R)$, corresponding again to the Sullivan measure, and then two extremal $\beta$-KMS states for any $\beta > HD(J_R)$. The presence of the latter KMS states is caused by the summability of the Poincaré series for the critical point which was established in [GS2].

The notion of conformality for measures related to dynamical systems has been generalised in various ways, and we show in the last section that some of these generalisations can also be covered by the approach taken here. In particular, when the map in question is a rational map on the Riemann sphere we obtain a complete description of the KMS states for the gauge action which comes naturally from the construction of the $C^*$-algebra. This allows a direct comparison between our construction and that of Kajiwara and Watatani in [KW] where they construct $C^*$-algebras from rational maps on the Riemann sphere via Hilbert modules and the Cuntz-Pimsner construction. The KMS states of the gauge action on their algebras were described in [IKW], and there are both similarities and differences in the structure of the KMS states when compared to the findings in this paper. The differences show that there is generally no natural (gauge-preserving) way to pass from their $C^*$-algebras to the ones constructed here.

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\section{Groupoid $C^*$-algebras and KMS states}

\subsection{Groupoid $C^*$-algebras and one-parameter automorphism groups}

Let $G$ be an étale second countable locally compact Hausdorff groupoid with unit space $G(0)$. Let $r : G \to G(0)$ and $s : G \to G(0)$ be the range and source maps, respectively. For $x \in G(0)$ put $G^x = r^{-1}(x)$, $G_x = s^{-1}(x)$ and $G^x = s^{-1}(x) \cap r^{-1}(x)$. Note that $G^x$ is a group, the \textit{isotropy group} at $x$. The space $C_c(G)$ of continuous compactly
For each $x$ automorphism where the supremum is taken over all $C^*$-reduced groupoid space. To define the continuous one-parameter group of automorphisms of the full groupoid $G$ on $C^*(G)$ let $x \in G^{(0)}$. There is a representation $\pi_x$ of $C_c(G)$ on the Hilbert space $l^2(G_x)$ of square-summable functions on $G_x$ given by

$$\pi_x(f)\psi(g) = \sum_{h \in G^{(0)}} f(h)\psi(h^{-1}g). \quad (2.1)$$

$C_r^*(G)$ is the completion of $C_c(G)$ with respect to the norm

$$\|f\|_r = \sup_{x \in G^{(0)}} \|\pi_x(f)\|.$$ 

Since $\|f\|_r \leq \|f\|$ there is a canonical surjection $\lambda : C^*(G) \to C_r^*(G)$.

We are mainly interested in the reduced groupoid $C^*$-algebra and have primarily introduced the full version in order to use the results of Neshveyev, [N]. The inclusion $C_c(G^{(0)}) \subseteq C_c(G)$ extends to embeddings of $C_0(G^{(0)})$ into both $C^*(G)$ and $C_r^*(G)$ and $\lambda$ restricts to an isomorphism between the two copies of $C_0(G^{(0)})$ which we therefore identify. Note that the map $C_c(G) \to C_r(G^{(0)})$ which restricts functions to $G^{(0)}$ extends to a conditional expectation $P : C_r^*(G) \to C_0(G^{(0)})$.

Let $c : G \to \mathbb{R}$ be a continuous homomorphism, i.e. $c$ is continuous and $c(gh) = c(g)c(h)$ when $s(g) = r(h)$. For each $t \in \mathbb{R}$ we can then define an automorphism $\sigma^c_t$ of $C_c(G)$ such that

$$\sigma^c_t(f)(g) = e^{itec(g)}f(g).$$

For each $x \in G^{(0)}$ the same expression defines a unitary $u_t$ in $l^2(G_x)$ such that $u_t\pi_x(f)u^*_t = \pi_x(\sigma^c_t(f))$ and it follows therefore that $\sigma^c_t$ extends by continuity to an automorphism $\sigma_t^c$ of $C_r^*(G)$. It is easy to see that $\sigma^c = (\sigma^c_t)_{t \in \mathbb{R}}$ is a continuous one-parameter group of automorphisms of $C_r^*(G)$. In the same way $c$ gives also rise to a continuous one-parameter group of automorphisms of the full groupoid $C^*$-algebra $C^*(G)$ and $\lambda : C^*(G) \to C_r^*(G)$ is then equivariant.

2.2. **KMS states.** Let $A$ be a $C^*$-algebra and $\alpha_t, t \in \mathbb{R}$, a continuous one-parameter group of automorphisms of $A$. Let $\beta \in \mathbb{R}$. A state $\omega$ of $A$ is a $\beta$-KMS state when

$$\omega(aa_{i\beta}(b)) = \omega(ba)$$

for all elements $a, b$ of a dense $\alpha$-invariant $*$-algebra of $\alpha$-analytic elements, cf. [BR].

In this section we use the results of Neshveyev from [N] to study the KMS-states of $\sigma^c$ on $C_r^*(G)$ under a couple of additional assumptions which will hold for the groupoid constructed in the following section. Since it simplifies several things we will only consider the case $\beta \neq 0$.

For each $x \in G^{(0)}$ the subset $G^x_x$ is a closed discrete subgroup of $G$. When $G^x_x$ is amenable the full and reduced group $C^*$-algebras of $G^x_x$ coincide, i.e. the
canonical homomorphism $\lambda : C^*(G^x_z) \to C^*_r(G^x_z)$ is an isomorphism. This leads to the following

**Lemma 2.1.** Assume that $G^x_z$ is amenable for all $x \in G^{(0)}$. It follows that every state $\omega$ of $C^*(G)$ with $C_0(G^{(0)})$ in its centraliser (i.e. $\omega(af) = \omega(fa)$ for all $f \in C_0(G^{(0)})$, $a \in C^*(G)$) factorises through $C^*_r(G)$.

**Proof.** It suffices to show that $|\omega(f)| \leq \|f\|_r$ for all $f \in C_c(G)$. It follows from Theorem 1.1 of [N] that there is a Borel probability measure $\mu$ on $G^{(0)}$ and for each $x \in G^{(0)}$ a state $\omega_x$ on $C^*_r(G^x_z)$ such that $G^{(0)} \ni x \mapsto \omega_x(f|_{G^x_z})$ is Borel and

$$\omega(f) = \int_{G^{(0)}} \omega_x(f|_{G^x_z}) \, dm(x) \quad (2.2)$$

for all $f \in C_c(G)$. Note that $|\omega_x(f|_{G^x_z})| \leq \|f|_{G^x_z}\|_{C^*_r(G^x_z)} = \|f|_{G^x_z}\|_{C^*_r(G^x_z)}$ since $G^x_z$ is amenable. It follows from the definition of $\pi_x$, cf. (2.1), that $\|f|_{G^x_z}\|_{C^*_r(G^x_z)} = \|P_x \pi_x(f) P_x\|$ where $P_x : l^2(G_x) \to l^2(G^x_z)$ is the orthogonal projection. Therefore $|\omega_x(f|_{G^x_z})| \leq \|f|_{G^x_z}\|_{C^*_r(G^x_z)} \leq \|\pi_x(f)\|$ for each $x$ and then (2.2) shows that $|\omega(f)| \leq \|f\|_r$ as desired. \hfill $\Box$

Note that the measure $m$ in (2.2) is determined by the condition that $\omega(f) = \int_{G^{(0)}} f \, dm(x)$ for all $C_c(G^{(0)})$. We say that $m$ is the measure associated with $\omega$.

Since every KMS state for $\sigma^c$ has $C_0(G^{(0)})$ in its centraliser it follows from Lemma 2.1 that the map $\lambda : C^*(G) \to C^*_r(G)$ induces a bijection from the KMS states of the one-parameter group $\sigma^c$ on $C^*_r(G)$ onto its KMS states on $C^*(G)$. The following definition characterises the Borel probability measures which arise from KMS states in this way. Let $W \subseteq G$ be an open bi-section, i.e. an open subset such that $r : W \to G^{(0)}$ and $s : W \to G^{(0)}$ are both injective. Then $r : W \to r(W)$ is a homeomorphism and we denote its inverse by $r^{-1}_W$. Let $\beta \in \mathbb{R}\setminus\{0\}$. We say that a finite Borel measure $m$ on $G^{(0)}$ is $(G, c, \beta)$-conformal with exponent $\beta$ when

$$m(s(W)) = \int_{r(W)} e^{-\beta c(r^{-1}_W(x))} \, dm(x) \quad (2.3)$$

for every open bi-section $W$ of $G$. Note that in the terminology used in [N] this condition means that $m$ is quasi-invariant with Radon-Nikodym cocycle $e^{-\beta c}$.

Assume now that

a) $G^x_z$ is abelian for all $x \in G^{(0)}$ and

b) $G^x_z = \{x\}$ for all but at most countably many $x \in G^{(0)}$.

In the following we call a finite Borel measure $m$ on $G^{(0)}$ non-atomic when $m(\{x\}) = 0$ for all $x \in G^{(0)}$ and purely atomic when there is a Borel set $A \subseteq G^{(0)}$ such that $m(A) = m(G^{(0)})$ and $m(\{a\}) > 0$ for all $a \in A$. Similarly we will say that a KMS state for $\sigma^c$ is non-atomic when its associated measure is non-atomic, and purely atomic when it is purely atomic.

**Lemma 2.2.** Let $m$ be a finite Borel measure on $G^{(0)}$ and let $m = m^c + m^a$ be a decomposition of $m$ into the sum of the non-atomic measure $m^c$ and the purely atomic measure $m^a$. It follows that $m$ is $(G, c, \beta)$-conformal with exponent $\beta$ if and only if $m^c$ and $m^a$ both are.
Proof. Assume that $m$ is $(G, c)$-conformal with exponent $\beta$. Let $W \subseteq G$ be an open bi-section. When $V$ is an open subset of $r(W)$, the set $r_W^{-1}(V)$ is an open bi-section so it follows that $m \left( s(r_W^{-1}(V)) \right) = \int_V e^{\beta c(r_W^{-1}(x))} \, dm(x)$ for every open subset $V \subseteq r(W)$.

By (outer) regularity of the measures on $r(W)$ given by $B \mapsto m \left( s \left( r_W^{-1}(B) \right) \right)$ and $B \mapsto \int_B e^{\beta c(r_W^{-1}(x))} \, dm(x)$ it follows that

$$m \left( s(r_W^{-1}(B)) \right) = \int_B e^{\beta c(r_W^{-1}(x))} \, dm(x)$$

(2.4)

for every Borel subset $B \subseteq r(W)$. Let $E$ be the set of atoms for $m$. Since $G$ is covered by bi-sections it follows from (2.4) that $s \left( r^{-1}(E) \right) = E = r \left( s^{-1}(E) \right)$; a conclusion which can be put back into (2.4) to give

$$m^c(s(W)) = m \left( s(W) \setminus E \right) = m \left( s(r_W^{-1}(r(W) \setminus E)) \right)$$

$$= \int_{r(W) \setminus E} e^{\beta c(r_W^{-1}(x))} \, dm(x) = \int_{r(W)} e^{\beta c(r_W^{-1}(x))} \, dm^c(x).$$

Similarly, $m^a(s(W)) = \int_{r(W)} e^{\beta c(r_W^{-1}(x))} \, dm^a(x)$ and we conclude that both $m^c$ and $m^a$ are $(G, c)$-conformal with exponent $\beta$. The converse is trivial. \hfill \Box

Let $x \in G^{(0)}$. The $G$-orbit $Gx$ of $x$ is the set $Gx = r(G_x)$. We say that $Gx$ is consistent when $c(G_x^x) = 0$. When this holds we can define a map $l_x : Gx \to [0, \infty[$ such that $l_x(z) = e^{-c(g)}$ where $g$ is any element of $r^{-1}(z) \cap G_x$. We say that $Gx$ is $\beta$-summable when it is consistent and

$$\sum_{z \in Gx} l_x(z)^\beta < \infty.$$

Lemma 2.3. Let $m$ be a finite Borel measure on $G^{(0)}$ which is $(G, c)$-conformal with exponent $\beta$. Assume that $m(\{x\}) > 0$. It follows that the $G$-orbit $Gx$ is consistent and $\beta$-summable. Furthermore, $m(\{y\}) > 0$ for all $y \in Gx$.

Proof. Let $g \in G_x^x$. There is an open bi-section $W \subseteq G$ such that $g \in W$. It follows therefore from (2.4) that $m(\{x\}) = e^{\beta c(g)} m(\{x\})$ which implies that $c(g) = 0$. Hence $Gx$ is consistent. Similarly, we find that $m(\{x\}) = e^{\beta c(g)} m(\{y\}) = l_y(x)^{-\beta} m(\{y\})$ when $y = r(\xi x)$, proving that $m(\{y\}) > 0$ for all $y \in Gx$. Finally, observe that

$$m(\{x\}) \sum_{z \in Gx} l_x(z)^\beta = m(Gx)$$

which implies that $\sum_{z \in Gx} l_x(z)^\beta = m(Gx) / m(\{x\}) < \infty$. \hfill \Box

Consider then a consistent and $\beta$-summable $G$-orbit $O = Gx$. When we denote the Dirac measure at $z$ by $\delta_z$ we can define a Borel probability measure $m_O$ on $G^{(0)}$ such that

$$m_O = \left( \sum_{z \in Gx} l_x(z)^\beta \right)^{-1} \sum_{z \in Gx} l_x(z)^\beta \delta_z.$$

(2.5)

It is straightforward to check that $m_O$ is $(G, c)$-conformal with exponent $\beta$. Let $\varphi$ be a state on $C^* (G_x^x)$. For each $z \in O$ choose an element $\xi_z \in G$ such that $r(\xi_z) = z$ and $s(\xi_z) = x$. Then $\xi_z G_x^x \xi_z^{-1} = G_x^x$. Define a state $\varphi_z$ on $C^* (G_x^x)$ such that

$$\varphi_z(f) = \varphi \left( f^{\xi_z} \right),$$

where $f^{\xi_z} \in C_c(G_x^x) \subseteq C^* (G_x^x)$ is the function $f^{\xi_z}(y) = f(\xi_z y \xi_z^{-1})$. By using that $C^* (G_x^x)$ is abelian for all $x$ by assumption a), a direct calculation as in the proof of
Theorem 1.3 in [N] shows that there is a $\beta$-KMS state $\omega_{\beta}^G$ for $\sigma^c$ such that

$$\omega_{\beta}^G(h) = \left(\sum_{z \in \mathcal{O}} l_z(z)\beta\right)^{-1} \sum_{z \in \mathcal{O}} \varphi_z(h|G_z) l_z(z)\beta$$

for all $h \in C_c(G)$.

**Theorem 2.4.** Assume that a) and b) hold. Let $m$ be a non-atomic Borel probability measure on $G^{(0)}$. Assume that $m$ is $(G, c)$-conformal with exponent $\beta$, and let $\mathcal{O}_i$, $i \in I$, be a finite or countably infinite collection of consistent and $\beta$-summable $G$-orbits. Choose for each $i \in I$ an element $x_i \in \mathcal{O}_i$ and let $\varphi_i$ be a state on $C^*(G_x^i)$. Finally, let $t_0, t_i \in [0, 1], i \in I$, be numbers such that $t_0 + \sum_{i \in I} t_i = 1$.

There is then a $\beta$-KMS state $\omega$ for $\sigma^c$ on $C^*_c(G)$ such that

$$\omega(a) = t_0 \int_{G^{(0)}} P(a) \, dm + \sum_{i \in I} t_i \omega_{\varphi^i}^G(a)$$

for all $a \in C^*_c(G)$. Conversely, any $\beta$-KMS state $\omega$ for $\sigma^c$ on $C^*_c(G)$ admits a unique decomposition of the form (2.6).

**Proof.** It follows from Theorem 1.3 of [N] that $a \mapsto \int_{G^{(0)}} P(a) \, dm$ is a $\beta$-KMS state. Since the same is true for each $\omega_{\varphi^i}^G$ the fact that the $\beta$-KMS states constitute a weak*-closed convex set implies that $\omega$ is a $\beta$-KMS state.

Conversely, let $\omega$ be a $\beta$-KMS state. It follows from Theorem 1.3 of [N] that there is a Borel probability measure $m$ on $G^{(0)}$ and a Borel measurable field of states $\varphi_x$ on $C^*(G_x)$, $x \in G^{(0)}$, such that

$$\varphi_x(f) = \varphi_x(f|G_x) \left(f^{\xi^{-1}}\right), \quad f \in C_c(G_x),$$

for all $\xi \in G_x$ and $\mu$-almost every $x$, and

$$\omega(f) = \int_{G^{(0)}} \varphi_x(f|G_x) \, dm(x)$$

for all $f \in C_c(G)$. When $m$ is non-atomic it follows from Corollary 1.2 of [N] that $\omega(a) = \int_{G^{(0)}} P(a) \, dm$ for all $a$, and we are done. Assume that $m$ is purely atomic. It follows then from Lemma 2.3 that there is a countable collection $\mathcal{O}_i\, i \in I$, of consistent and $\beta$-summable $G$-orbits such that $m = \sum_{i \in I} m(\mathcal{O}_i) m_{\mathcal{O}_i}$. Choose an element $x_i \in \mathcal{O}_i$. It follows from (2.7) and (2.8) that $\omega = \sum_{i \in I} m(\mathcal{O}_i) \omega_{\varphi^i}$.

Finally, when $m$ is neither non-atomic nor purely atomic we apply Lemma 2.2 to get an $s \in [0, 1]$ and $(G, c)$-conformal Borel probability measures $m^c$ and $m^a$ with exponent $\beta$ such that $m^c$ is non-atomic, $m^a$ is purely atomic and $m = sm^c + (1-s)m^a$. Since $m^c$ is non-atomic it follows from assumption b) that

$$\omega(f) = s \int_{G^{(0)}} f(x) \, dm^c(x) + (1-s) \int_{G^{(0)}} \varphi_x(f|G_x) \, dm^a(x)$$

for all $f \in C_c(G)$. By repeating the previous argument with $m$ replaced by $m^a$ we get a countable collection $\mathcal{O}_i\, i \in I$, of consistent and $\beta$-summable $G$-orbits and elements $x_i \in \mathcal{O}_i$ such that

$$\int_{G^{(0)}} \varphi_x(f|G_x) \, dm^a(x) = \sum_{i \in I} m^a(\mathcal{O}_i) \omega_{\varphi^i}(f)$$
for all \( f \in C_c(G) \). It follows then that

\[
\omega(a) = s \int_{G(0)} P(a) \, dm' + (1 - s) \sum_{i \in I} m^a_i \omega_{\mathcal{O}_i}^{\varphi_i}(a)
\]

for all \( a \in C^*_r(G) \). This shows that \( \omega \) can be decomposed as in (2.6).

To see why the decomposition (2.6) is unique, let \( \{x_k\} \) be a numbering of the elements of \( \mathcal{O}_i \). For each \( l \in \mathbb{N} \) let \( \{f_{lk}\} \) be a real and bounded sequence in \( C_c(G(0)) \) converging point wise to the characteristic function of \( \{x_1, x_2, x_3, \ldots, x_l\} \). Let \( \{g_n\} \) be an approximate unit for \( C^*_r(G) \) contained in \( C_c(G(0)) \) and let \( a \in C^*_r(G) \). Since the Cauchy-Schwarz inequality implies that

\[
\omega_{\mathcal{O}_i}^{\varphi_j}(f_{lk}g_na) \leq \omega_{\mathcal{O}_i}^{\varphi_j}(a^*a) \int_{G(0)} f_{lk}^2 g_n^2 \, dm_{\mathcal{O}_i},
\]

we see that \( \lim_{k \to \infty} \omega_{\mathcal{O}_i}^{\varphi_j}(f_{lk}g_na) = 0 \) for all \( j \neq i \). It follows in the same way that \( \lim_{k \to \infty} \int_{G(0)} P(f_{lk}g_na) \, dm = 0 \) for all \( l \), and the estimate

\[
|\omega_{\mathcal{O}_i}^{\varphi_j}(g_na) - \omega_{\mathcal{O}_i}^{\varphi_j}(f_{lk}g_na)| \leq \omega_{\mathcal{O}_i}^{\varphi_j}(a^*a) \int_{G(0)} (1 - f_{lk}^2) g_n^2 \, dm_{\mathcal{O}_i}
\]

shows that \( \lim_{n \to \infty} \lim_{k \to \infty} \omega_{\mathcal{O}_i}^{\varphi_j}(f_{lk}g_na) = \omega_{\mathcal{O}_i}^{\varphi_j}(g_na) \). Thus

\[
\lim_{n \to \infty} \lim_{k \to \infty} \omega_{\mathcal{O}_i}^{\varphi_j}(f_{lk}g_na) = \lim_{n \to \infty} t_i \omega_{\mathcal{O}_i}^{\varphi_j}(g_na) = t_i \omega_{\mathcal{O}_i}^{\varphi_j}(a).
\]

This shows that \( t_i \omega_{\mathcal{O}_i}^{\varphi_j} \) is determined by \( \omega \), and hence so is \( t_i \) and \( \omega_{\mathcal{O}_i}^{\varphi_j} \). The same is then automatically true for \( t_0 \) and \( m \).

The uniqueness part of the statement makes it easy to identify the extremal \( \beta \)-KMS states. In particular it follows that they are either non-atomic or purely atomic.

3. An Amended Transformation Groupoid

Let \( X \) be a locally compact Hausdorff space and \( \psi : X \to X \) a map. Let \( \mathcal{P} \) be a pseudo-group on \( X \). More specifically, \( \mathcal{P} \) is a collection of local homeomorphisms \( \eta : U \to V \) between open subsets of \( X \) such that

i) for every open subset \( U \) of \( X \) the identity map \( \text{id} : U \to U \) is in \( \mathcal{P} \),

ii) when \( \eta : U \to V \) is in \( \mathcal{P} \) then so is \( \eta^{-1} : V \to U \), and

iii) when \( \eta : U \to V \) and \( \eta_1 : U_1 \to V_1 \) are elements in \( \mathcal{P} \) then so is \( \eta_1 \circ \eta : U \cap \eta^{-1}(V \cap U_1) \to \eta_1(V \cap U_1) \).

For each \( k \in \mathbb{Z} \) we denote by \( T_k(\psi) \) the elements \( \eta : U \to V \) of \( \mathcal{P} \) with the property that there are natural numbers \( n, m \) such that \( k = n - m \) and

\[
\psi^n(z) = \psi^m(\eta(z)) \quad \forall z \in U.
\]

The elements of \( \mathcal{T} = \bigcup_{k \in \mathbb{Z}} T_k(\psi) \) will be called local transfers for \( \psi \). We denote by \([\eta]_x\) the germ at a point \( x \in X \) of an element \( \eta \in T_k(\psi) \). Set

\[
\mathcal{G}_\psi = \{(x, k, \eta, y) \in X \times \mathbb{Z} \times \mathcal{P} \times X : \eta \in T_k(\psi), \eta(x) = y\}.
\]

We define an equivalence relation \( \sim \) in \( \mathcal{G}_\psi \) such that \( (x, k, \eta, y) \sim (x', k', \eta', y') \) when

i) \( x = x', \eta = \eta', k = k' \) and

ii) \( [\eta]_x = [\eta']_x \).
Let \([x, k, \eta, y]\) denote the equivalence class represented by \((x, k, \eta, y) \in \mathcal{G}_\psi\). The quotient space \(G_\psi = \mathcal{G}_\psi / \sim\) is a groupoid such that two elements \([x, k, \eta, y]\) and \([x', k', \eta', y']\) are composable when \(y = x'\) and their product is
\[
[x, k, \eta, y] [y, k', \eta', y'] = [x, k + k', \eta \circ \eta, y'].
\]
The inversion in \(G_\psi\) is defined such that \([x, k, \eta, y]^{-1} = [y, -k, \eta^{-1}, x]\). The unit space of \(G_\psi\) can be identified with \(X\) via the map \(x \mapsto [x, 0, \text{id}, x]\), where \(\text{id}\) is the identity map on \(X\). When \(\eta \in \mathcal{T}_k(\psi)\) and \(U\) is an open subset of the domain of \(\eta\) we set
\[
U(\eta) = \{[z, k, \eta, \eta(z)] : z \in U\}.
\] (3.2)
It is straightforward to verify that by varying \(k, \eta\) and \(U\) the sets (3.2) constitute a base for a topology on \(G_\psi\). In general this topology is not Hausdorff and to amend this we now make the following additional assumption.

**Assumption 3.1.** When \(x \in X\) and \(\eta(x) = \xi(x)\) for some \(\eta, \xi \in \mathcal{T}_k(\psi)\), then the implication
\[
x \text{ is not isolated in } \{y \in X : \eta(y) = \xi(y)\} \Rightarrow [\eta]_x = [\xi]_x
\] holds.

Then \(G_\psi\) is Hausdorff: Let \([x, k, \eta, y]\) and \([x', k', \eta', y']\) be different elements of \(G_\psi\). There are then open neighbourhoods’ \(U\) of \(x\) and \(U'\) of \(x'\) such that \(U(\eta) = \{[z, k, \eta, \eta(z)] : z \in U\}\) and \(U'(\eta') = \{[z, k', \eta', \eta'(z)] : z \in U'\}\) are disjoint. This is trivial when \((x, k, y) \neq (x', k', y')\) while it is a straightforward consequence of Assumption 3.1 when \((x, k, y) = (x', k', y')\).

Since the range and source maps are homeomorphisms from \(U(\eta)\) onto \(U\) and \(\eta(U)\), respectively, it follows that \(G_\psi\) is a locally compact Hausdorff space because \(X\) is. It is also straightforward to show that the groupoid operations are continuous so that we can conclude the following.

**Theorem 3.2.** Let Assumption 3.1 be satisfied. Then \(G_\psi\) is an étale locally compact Hausdorff groupoid.

Assumption 3.1 may be satisfied because of properties of the map \(\psi\), regardless of which pseudo-group \(\mathcal{P}\) is used; it holds for example trivially when \(\psi\) is locally injective on \(X\). When \(\psi\) is a local homeomorphism and the pseudo-group \(\mathcal{P}\) consists of all local homeomorphisms on \(X\), the groupoid \(G_\psi\) is the same as the one considered in increasing generality by Renault, Deaconu and Anantharaman-Delaroche, [Re], [De], [An]. It is therefore also a generalisation of the classical crossed product construction for homeomorphisms.

### 4. Étale Groupoids from Holomorphic Maps

Let \(S\) be a connected Riemann surface and \(H : S \to S\) a non-constant holomorphic map. Let \(X \subseteq S\) be a subset which is locally compact in the topology inherited from \(S\). Assume that no points are isolated in \(X\) and that \(X\) is totally \(H\)-invariant, i.e. that \(H^{-1}(X) = X\), and let \(H|_X : X \to X\) denote the restriction of \(H\) to \(X\). Let \(\mathcal{P}\) be the pseudo-group on \(X\) of local homeomorphisms \(\xi : U \to V\) between open subsets of \(X\) with the property that there are open subsets \(U_1, V_1\) in \(S\) and a bi-holomorphic map \(\xi_1 : U_1 \to V_1\) such that \(U_1 \cap X = U\), \(V_1 \cap X = V\) and \(\xi = \xi_1\) on \(U\). Then Assumption 3.1 is satisfied. This follows from the well-known fact that
holomorphic maps defined on the same open connected subset of the complex plane must be identical if they agree on a set with a limit point in their domain. Therefore Theorem 3.2 implies that $G_{H|x}$ is an étale locally compact Hausdorff groupoid. To simplify notation we denote it by $G_X$.

4.1. $G_X$ is second countable. To show that there is a countable base for the topology of $G_X$ we need some preparations that are also going to be instrumental in determining the isotropy groups.

Let $x \in S$. A conformal germ at $x$ is a holomorphic and injective map $\eta : U \to S$ where $U$ is an open neighbourhood of $x$. As is well-known a conformal germ is invertible in the sense that $\eta(U)$ is open and $\eta^{-1} : \eta(U) \to U$ is holomorphic. The set of conformal germs at $x$ will be denoted by $\mathcal{H}_x$. Two conformal germs $\eta, \eta' \in \mathcal{H}_x$ are equivalent when they agree in an open neighbourhood of $x$. We occasionally identify $\mathcal{H}_x$ with $\mathcal{H}_x/\sim$; hopefully it will be clear from the context when this happens. Let $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. For every $x \in S$ there are neighbourhoods $U$ and $V$ of $x$ and $H(x)$, respectively, and homeomorphisms $\varphi : U \to \Delta$, $\psi : V \to \Delta$ such that $H(U) \subseteq V$, $\varphi(x) = \psi(H(x)) = 0$, and there is an $n \in \mathbb{N}$ such that

\[
\begin{array}{ccc}
U & H & V \\
\varphi \downarrow & \Delta & \psi \\
\Delta & z \mapsto z^n & \Delta
\end{array}
\]

commutes. This it is trivial when $H'(x) \neq 0$ and it follows from Böttchers theorem, Theorem 9.1 in [Mi], when $H'(x) = 0$. The number $n$ is called the valency of $H$ at $x$ and we denote it by $\text{val}(H, x)$. The points in

\[C = \{x \in S : \text{val}(H, x) \geq 2\} = \{x \in S : H'(x) = 0\}\]

are the critical points of $H$.

Lemma 4.1. Let $H : S \to S$, $K : S \to S$ be non-constant holomorphic maps. Let $x, y \in S$ be points such that $H(x) = K(y)$. It follows that there is a conformal germ $\eta \in \mathcal{H}_x$ such that

a) $\eta(x) = y$ and

b) $K(\eta(z)) = H(z)$ for all $z$ in a neighbourhood of $x$

if and only if $\text{val}(H, x) = \text{val}(K, y)$, in which case there are exactly $\text{val}(H, x)$ elements of $\mathcal{H}_x$ with these properties, up to equivalence of germs. Furthermore, two elements in $\mathcal{H}_x$ which both satisfy a) and b) are equivalent if and only if they have the same derivative at $x$.

Proof. Assume first that there is a conformal germ $\eta \in \mathcal{H}_x$ such that a) and b) hold. Then

$$\text{val}(H, x) = \text{val}(K \circ \eta, x) = \text{val}(K, y) \text{val}(\eta, x) = \text{val}(K, y),$$

proving the necessity of the condition. Assume next that $\text{val}(H, x) = \text{val}(K, y) = j$. When $j = 1$, $H \in \mathcal{H}_x$ and $K \in \mathcal{H}_y$ and hence $\eta$ is the only element of $\mathcal{H}_x$, up to equivalence, which satisfies a) and b). Assume $j \geq 2$. Working locally using local charts at $x$, $y$ and $K(y) = H(x)$ we may assume that $S = \Delta$ and that $x = y = 0$. It follows from Böttchers theorem, Theorem 9.1 in [Mi], that there are conformal germs $\delta \in \mathcal{H}_0$ and $\mu \in \mathcal{H}_0$ such that $\delta(0) = \mu(0) = 0$ and

$$K(\delta(z)) - K(0) = \delta(z^j)$$
and

\[ H(\mu(z)) - H(0) = \mu\left(z^j\right) \]

for all \( z \) in a neighbourhood of \( 0 \). Then \( \eta \mapsto \delta^{-1} \circ \eta \circ \mu \) is a bijection from the set of elements \( \eta \in H_0 \) which satisfy \( \text{a)} \) and \( \text{b)} \) onto the set of elements \( \kappa \in H_0 \) such that

\[ \text{a')} \quad \kappa(0) = 0 \quad \text{and} \]

\[ \text{b')} \quad \kappa(z^j) = \delta^{-1} \circ \mu(z^j) \quad \text{for all } z \text{ in a neighbourhood of } 0. \]

The Taylor expansion of \( z \mapsto \delta^{-1} \circ \mu(z^j) \) at 0 has the form \( z^j \sum_{n=0}^{\infty} b_n z^n \) where \( b_0 = (\delta^{-1} \circ \mu)'(0) \neq 0 \). By using a holomorphic logarithm near \( b_0 \) we get a holomorphic function \( \kappa_0 \) such that \( \kappa_0(z^j) = \sum_{n=1}^{\infty} b_n z^n \) for all \( z \) in a neighbourhood of 0. Set \( \kappa(z) = z \kappa_0(z) \). Then \( \kappa \in H_0 \) since \( \kappa'(0) = \kappa_0(0) \neq 0 \), and we have proved existence. Since elements \( \kappa \) of \( H_0 \) that satisfy \( \text{a')} \) and \( \text{b')} \) agree up to multiplication by a \( j \)th root of unity, we see that their number is exactly \( j \), and that their equivalence class in \( H_0 \) is determined by their derivative at 0. \( \square \)

By specialising to the case where \( K = H \) and \( x = y \) the preceding proof also shows the following.

**Lemma 4.2.** Let \( H : S \to S \) be a non-constant holomorphic map and let \( x \in S \). It follows that the equivalence classes of conformal germs \( \eta \in H_x \) which satisfy that

\[ \begin{align*}
\text{i)} & \quad \eta(x) = x \quad \text{and} \\
\text{ii)} & \quad H(\eta(z)) = H(z) \quad \text{for all } z \text{ in a neighbourhood of } x \text{ in } S
\end{align*} \]

form a cyclic group of order \( \text{val}(H,x) \) under composition.

**Proposition 4.3.** \( G_X \) has a countable base for its topology, i.e. \( G_X \) is a second countable étale locally compact Hausdorff groupoid.

**Proof.** To construct a countable base for the topology of \( G_X \), fix \( n, m \in \mathbb{N} \). Since \( H \) is not constant the set of points that are critical for either \( H^n \) or \( H^m \) is a discrete subset of \( S \). Furthermore, \( S \) is second countable by \( \S \) in [Mi]. There is therefore a base \( \{U_i\}_{i=1}^{\infty} \) for the topology of \( S \) such that each \( U_i \) is connected and contains at most one critical point of \( H^n \) or \( H^m \). Furthermore, we can arrange that \( H^m \) is injective on \( U_j \) unless there is critical point of \( H^m \) in \( U_j \). Let \( i,j \in \mathbb{N} \). When \( U_i \) contains no critical points of \( H^n \) and \( U_j \) no critical point of \( H^m \) it follows that there is at most one conformal germ \( \mu \) with \( U_i \) as domain of definition and satisfying that

\[ \mu(U_i) \subseteq U_j \quad \text{and} \quad H^n(z) = H^m(\mu(z)) \quad \forall z \in U_i. \tag{4.1} \]

When \( U_i \) contains a critical point \( x_i \) of \( H^n \) it follows from Lemma 4.1 that for each \( j \in \mathbb{N} \) the number of conformal germs \( \eta \) with \( U_i \) as domain such that (4.1) holds is at most \( \text{val}(H^n,x_i) \). Hence the collection \( A(n,m,i,j) \) of conformal germs \( \mu \) with the properties that \( \mu \) is defined on \( U_i \) such that (4.1) holds, and \( U_j \) does not contain a critical point of \( H^m \) when \( U_i \) does not contain one of \( H^n \), is finite. By definition of the pseudo-group \( \mathcal{P} \), and our assumptions on the subset \( X \), it follows that for any \( \eta \in \mathcal{T}_k(H|X) \) the set \( U(\eta) \) from (3.2) is a union of sets of the form

\[ \{[x,k,\mu|_X,\mu(x)] : x \in U_i \cap X \} \]

for some \((n,m,i,j)\) and some \( \mu \in A(n,m,i,j) \). Such sets therefore form a countable base for the topology of \( G_X \). \( \square \)
4.2. The isotropy groups. We say that a point \(x \in X\) is pre-periodic when there is a \(p \in \mathbb{N}\) such that \(H^{p+n}(x) = H^n(x)\) for all large \(n\) and that \(x\) is pre-critical when there is an \(n \in \mathbb{N} \cup \{0\}\) such that \(H^n(x)\) is critical for \(H\).

**Proposition 4.4.** Let \(x \in X\) and let \((G_X)_x^z\) be the isotropy group at \(x\) in \(G_X\).

a) When \(x\) is neither pre-critical nor pre-periodic, \((G_X)_x^z \simeq 0\).

b) When \(x\) is pre-periodic and not pre-critical, \((G_X)_x^z \simeq \mathbb{Z}\).

c) When \(x\) is pre-critical and not pre-periodic \((G_X)_x^z\) is isomorphic to a non-zero subgroup of \(\mathbb{Q}/\mathbb{Z}\).

d) Assume that \(x\) is both pre-periodic and pre-critical. Let \(n \in \mathbb{N}\) be the least number such that \(H^n(x)\) is periodic.

d1) When \(H^j(x) \in C\) for some \(j \geq n\) the isotropy group \((G_X)_x^z\) is isomorphic to an infinite subgroup of \(\mathbb{Q}/\mathbb{Z}\), and

d2) when \(H^j(x) \notin C\) for all \(j \geq n\) the isotropy group \((G_X)_x^z\) is isomorphic to \(\mathbb{Z} \oplus \mathbb{Z}_d\) where \(d = \text{val}(H^n, x)\).

**Proof.**

a) Assume that \([x, k, \eta, x] \in (G_X)_x^z\) is not in the unit space. If \(k \neq 0\) the point \(x\) is pre-periodic. If \(k = 0\) and \([\eta]_x \neq [\text{id}]_x\) it follows from Lemma 4.2 that \(2 \leq \text{val}(H^n, x) = \text{val}(H, H^{n-1}(x)) \text{val}(H, H^{n-2}(x)) \cdots \text{val}(H, x)\) for all large \(n\); that is, \(x\) is pre-critical.

b) Let \(p \in \mathbb{N}\) be the least natural number such that \(H^{p+n}(x) = H^n(x)\) for all large \(n\). If \(\eta\) is a holomorphic germ at \(x\) such that \(H^{p+n}(z) = H^n(\eta(z))\) for all \(z\) in a neighbourhood of \(x\), it follows that \(H^n \in \mathcal{H}_x\) since \(\text{val}(H^n, x) = \text{val}(H^{n+p}, x) = 1\) and hence that \(\eta = H^{-n} \circ H^{p+n}\) in \(\mathcal{H}_x\). Thus

\[
(G_X)_x^z = \{[x, kp, \eta^k, x] : k \in \mathbb{Z}\} \simeq \mathbb{Z}.
\]

c) Since \(x\) is not pre-periodic

\[
(G_X)_x^z = \bigcup_n \{[x, 0, \eta, x] : \eta \in \mathcal{H}_x(n)\}
\]

where \(\mathcal{H}_x(n)\) is the set

\[
\{\eta \in \mathcal{H}_x : \eta(x) = x\text{ and }H^n(\eta(z)) = H^n(z)\text{ for all }z\text{ in a neighbourhood of }x\}.
\]

It follows from Lemma 4.2 that \(\mathcal{H}_x(n)\) is a finite cyclic group which is non-zero for all large \(n\) since \(x\) is pre-critical. Hence \((G_X)_x^z\) is a non-zero union of an increasing sequence of finite cyclic groups and hence a non-zero subgroup of \(\mathbb{Q}/\mathbb{Z}\).

d1) Let \(p\) be the period of \(H^n(x)\). Let \([x, k, \eta, x] \in G_X\) and assume that \(k \neq 0\). Then \(k = n_0 p\) for some \(n_0 \in \mathbb{Z} \setminus \{0\}\). Let \(l \in \mathbb{N}\) be so large that \(n_0 + l > 0\). Let \(\eta \in T^k(H|_X)\) satisfy \(\eta(x) = x\) and \(H^{n_0 p + lp + n}(z) = H^{lp + n}(\eta(z))\) for all \(z\) in an neighbourhood of \(x\) and some \(n \in \mathbb{N}\). Then

\[
\text{val}(H^p, H^n(x))^{n_0 + 1} \text{val}(H^n, x) = \text{val}(H^{n_0 p + lp + n}, x)
\]

\[
= \text{val}(H^{lp + n}, x) = \text{val}(H^p, H^n(x))\]

and we conclude that

\[
\text{val}(H^p, H^n(x))^{n_0 + 1} = \text{val}(H^p, H^n(x))^l.
\]

Since \(n_0 \neq 0\) this implies that \(\text{val}(H^p, H^n(x)) = 1\) and hence by periodicity that \(\text{val}(H, H^j(x)) = 1\) for all \(j \geq n\). This contradicts that we are in case d1) and we conclude therefore that \(k = 0\) when \([x, k, \eta, x] \in G_X\), i.e. the equality (4.2) holds.

To see that \((G_X)_x^z\) is infinite note that there is an \(m_0 \in \mathbb{N}\) such that \(H^{m_0}(x)\) is
p-periodic and \( \text{val}(H, H^m(x)) \geq 2 \). Then \( \text{val}(H^l, H^m(x)) \geq 2^l \) for all \( l \in \mathbb{N} \) and hence \( \text{val}(H^m + l, x) \geq 2^l \) for all \( l \in \mathbb{N} \). It follows that \( \lim_{n \to \infty} \#\mathcal{H}_x(n) = \infty \).

2. In this case \( d = \text{val}(H^n, x) \geq 2 \), and by Lemma 4.2 the elements \( \eta \in \mathcal{H}_x \) which satisfy that \( \eta(x) = x \) and that \( H^n(\eta(z)) = H^n(z) \) is a neighbourhood of \( x \) form a cyclic group \( F \) of order \( d \). By assumption \( \text{val}(H^p, H^n(x)) = 1 \); i.e. \( H^p \in \mathcal{H}_{H^n(x)} \). Since \( \text{val}(\gamma^{-1} \circ H^n, x) = \text{val}(H^n, x) \), it follows from Lemma 4.1 that there is a conformal germ \( \xi \) at \( x \) such that \( \xi(x) = x \) and \( \gamma^{-1} \circ H^n(z) = H^n(\xi(z)) \) in a neighbourhood of \( x \). Then \[
[x, kp, \mu \circ \xi^{-k}, x] \in (G_x)^x
\]
for all \( k \in \mathbb{Z} \) and all \( \mu \in F \). When \( [x, kp, \mu, x], [x, kp, \mu_1, x] \in (G_x)^x \) we have that \( H^n(\mu(z)) = H^n(\mu_1(z)) \) in a neighbourhood of \( x \) and hence \( \mu \circ \mu_1 \in F \). Thus \( (G_x)^x = \{ [x, kp, \mu \circ \xi^{-k}, x] : k \in \mathbb{Z}, \ \mu \in F \} \)
which is clearly an extension of \( \mathbb{Z} \) by \( \mathbb{Z}_d \). It follows from the chain rule and the last statement of Lemma 4.1 that \( \xi \circ \mu \circ \xi^{-1} |_x = [\mu]_x \) for all \( \mu \in F \). This shows that \( (G_x)^x \) is abelian and therefore isomorphic to \( \mathbb{Z} \oplus \mathbb{Z}_d \).

5. The conformal action and its KMS states

5.1. The conformal action. Consider \( S \) as a 1-dimensional complex manifold with a given metric \( g \), by which we mean that \( g \) is a continuous choice of norms on the tangent spaces. By definition of \( G_X \) we can then define a map \( L : G_X \to \mathbb{R} \) such that
\[
L[x, k, \eta, y] = \log |\eta'(x)|_g
\]
where \( \eta' \) denotes the differential of \( \eta \) and \( |\eta'(x)|_g \) the norm of \( \eta'(x) \) calculated with respect to the metric \( g \). Then \( L[x, k, \eta, y][y, l, \mu, z] = L[x, k, \eta, y] + L[y, l, \mu, z] \), i.e. \( L \) is a homomorphism \( L : G_X \to \mathbb{R} \) which is obviously continuous. The corresponding one-parameter group \( \gamma = (\gamma_t) \) of \( C^*_r(G_X) \) is determined by the condition that
\[
\gamma_t(f)[x, k, \eta, y] = |\eta'(x)|^u_{g} f[x, k, \eta, y]
\]
when \( f \in C_c(G_X) \). We call \( \gamma \) the conformal action.

By construction the conformal action depends on the chosen metric. However, the choice of another metric does not change the structure of the KMS states when \( X \) is compact. To see this let \( g^1 \) be another metric on \( S \). There is then a strictly positive function \( r : S \to [0, \infty[ \) such that \( g^1 = rg \). It follows that when we use \( g^1 \) instead of \( g \) in (5.1) we get a continuous homomorphism \( L^1 : G_X \to \mathbb{R} \) such that
\[
L^1[x, k, \eta, y] = L[x, k, \eta, y] + \log r(y) - \log r(x).
\]
Let \( \gamma^1 \) be the automorphism group obtained from \( L^1 \). Set \( u_t(x) = e^{-it\log r(x)} \). Then \( \{u_t\}_{t \in \mathbb{R}} \) is a strictly continuous unitary one-parameter group in \( M(C^*_r(G_X)) \) - the multiplier algebra of \( C^*_r(G_X) \) - and it is easy to check that
\[
u_t\gamma_t(a)u^*_t = \gamma^1_t(a)
\]
for all \( t \) and \( a \). Note that the \( u_t \)’s are in the fixed point algebra of both \( \gamma \) and \( \gamma^1 \). In particular, it follows therefore from (5.3) that \( \gamma \) and \( \gamma^1 \) are exterior equivalent, cf. 8.11.3 of [Fa]. It also gives the following.

Lemma 5.1. Assume that \( X \) is compact. Let \( \beta \in \mathbb{R} \). There is a bijective map from the \( \beta \)-KMS states of \( \gamma \) onto the \( \beta \)-KMS states of \( \gamma^1 \).
Proof. Since $X$ is compact, $r$ is bounded away from both 0 and $\infty$. Then $h(x) = e^{\frac{1}{\beta} \log r(x)} \in C^*_r(G_X)$. When $\omega$ is a $\beta$-KMS state for $\gamma$ the state
\[ a \mapsto \frac{\omega(hah)}{\omega(h^2)} \]
will be $\beta$-KMS state for $\gamma^1$, giving us a bijection with the obvious inverse. \qed

When $X$ is not compact the conformal action and its KMS states depends very much on the metric. See Remark 5.7

5.2. The KMS states of the conformal action. It follows from Theorem 2.4 that the $\beta$-KMS states for the conformal action can be described in terms of non-atomic $(G_X, L)$-conformal measures and atomic measures supported on consistent and $\beta$-summable $G_X$-orbits, as defined in Section 2.2. We describe in this section what these notions become in the present setting and formulate the according version of Theorem 2.4.

Let $\beta \in \mathbb{R} \setminus \{0\}$. Note that a finite Borel measure $m$ on $X$ is $(G_X, L)$-conformal with exponent $\beta$, as defined in Section 2.2, when the equality
\[ m(\eta(U)) = \int_U |\eta'(x)|^\beta_g \, dm(x) \tag{5.5} \]
holds for all local transfers $\eta : U \to V$ for $H|_X$. We shall compare this to the more conventional notion of conformality which goes back to the work of Sullivan in [S]. Thus a finite Borel measure $m$ on $X$ is $\beta$-conformal when
\[ m(H(A)) = \int_A |H'(x)|^\beta_g \, dm(x) \tag{5.6} \]
for every Borel subset $A \subseteq X$ for which $H : A \to X$ is injective.

Lemma 5.2. Let $m$ be a $\beta$-conformal measure. Let $U, V$ be open subsets of $X$ and $n, l \in \mathbb{N}$ natural numbers such that $H^n(U) = H^l(V)$. Assume that $H^n$ and $H^l$ are injective on $U$ and $V$, respectively. Then
\[ m(V) = m(H^{-l}(H^n(U))) = \int_U |(H^{-l} \circ H^n)'(x)|^\beta_g \, dm(x) \tag{5.7} \]
where $H^{-l}$ denotes the inverse of $H^l : V \to H^l(V)$.

Proof. By regularity of the Borel measures on $U$ given by $B \mapsto m(H^{-l}(H^n(B))$ and $B \mapsto \int_B |(H^{-l} \circ H^n)'(x)|^\beta_g \, dm(x)$ it suffices to prove that
\[ m(H^{-l}(H^n(K))) = \int_K |(H^{-l} \circ H^n)'(x)|^\beta_g \, dm(x) \tag{5.8} \]
when $K \subseteq U$ is compact.

In the following we write $t \overset{\delta}{\sim} s$ when $s, t, \delta \in \mathbb{R}$, $\delta > 0$ and $|s - t| \leq \delta$, and the inverses $H^{-1}$ which occur all come from the branch
\[ H^l(V) \overset{H^{-1}}{\longrightarrow} H^{l-1}(V) \overset{H^{-1}}{\longrightarrow} H^{l-2}(V) \overset{H^{-1}}{\longrightarrow} \cdots \overset{H^{-1}}{\longrightarrow} V. \]
Let $\epsilon \in [0, 1]$. We can then find a finite Borel partition $K = \sqcup_i K_i$ of $K$ such that $\left| H'(a) \right|_g^\beta \lesssim \left| H'(b) \right|_g^\beta$ when $a, b \in H^{-k}(H^n(K_i))$, $0 \leq k \leq l$, or $a, b \in H^{j}(K_i)$, $0 \leq j \leq n$. Furthermore, we can arrange that

$$\left| (H^{-l} \circ H^n)'(x) \right|_g^\beta \lesssim \left| (H^{-l} \circ H^n)'(y) \right|_g^\beta$$

(5.9)

when $x, y \in K_i$. For each $i$ we choose a point $a_i \in K_i$. Set

$$L = \sup \left\{ \left| H'(x) \right|_g^{-\beta} + \left| H'(x) \right|_g^\beta : x \in \bigcup_{0 \leq i \leq l} H^{-i}(H^n(K)) \cup \bigcup_{0 \leq j \leq n} H^{j}(K) \right\}.$$  

Consider now an arbitrary $i$ and set $\epsilon_1 = \epsilon m \left( H^{-l}(H^n(K_i)) \right)$. It follows then from (5.6) that

$$m \left( H^{-l+1}(H^n(K_i)) \right) \left| H'(H^{-l}(H^n(a_i))) \right|_g^\beta \lesssim m \left( H^{-l+1}(H^n(K_i)) \right)$$

(5.10)

and hence, since $\left| H'(H^{-l}(H^n(a_i))) \right|_g = \left| (H^{-1})'(H^{-l+1}(H^n(a_i))) \right|^{-1}_g$, that

$$m \left( H^{-l+1}(H^n(K_i)) \right) \lesssim \left| (H^{-1})'(H^{-l+1}(H^n(a_i))) \right|_g^\beta m \left( H^{-l+1}(H^n(K_i)) \right).$$

It follows from (5.10) that

$$m \left( H^{-l+1}(H^n(K_i)) \right) \leq (L + 1)m \left( H^{-l}(H^n(K_i)) \right).$$

In the same way we get the estimates

$$m \left( H^{-l+2}(H^n(K_i)) \right) \leq (L + 1)m \left( H^{-l+1}(H^n(K_i)) \right),$$

and

$$m \left( H^{-l+1}(H^n(K_i)) \right) \lesssim \left| (H^{-1})'(H^{-l+2}(H^n(a_i))) \right|_g^\beta m \left( H^{-l+2}(H^n(K_i)) \right)$$

where $\epsilon_2 = \epsilon m \left( H^{-l+1}(H^n(K_i)) \right)$. After $l$ repetitions of these arguments we find that

$$m \left( H^{-l}(H^n(K_i)) \right) \lesssim \prod_{j=0}^{l-1} \left| (H^{-1})'(H^{-j}(H^n(a_i))) \right|_g^\beta m \left( H^{n}(K_i) \right)$$

(5.11)

where $\epsilon' = \epsilon L^l(L + 1)m \left( H^{-l}(H^n(K_i)) \right)$.

Now attack $H^n(K_i)$ in a similar way. It follows from (5.6) that

$$m(H^n(K_i)) \lesssim \left| H'(H^{n-1}(a_i)) \right|_g^\beta m(H^{n-1}(K_i))$$

where $\epsilon_4 = \epsilon m(H^{n-1}(K_i))$. In particular,

$$m(H^n(K_i)) \leq (L + 1)m(H^{n-1}(K_i)).$$

Similarly,

$$m(H^{n-1}(K_i)) \lesssim \left| H'(H^{n-2}(a_i)) \right|_g^\beta m(H^{n-2}(K_i)),$$

where $\epsilon_5 = \epsilon m(H^{n-2}(K_i))$, and

$$m(H^{n-1}(K_i)) \leq (L + 1)m(H^{n-2}(K_i)).$$
After \( n \) steps we find that
\[
m(H^n(K_i)) \lesssim \prod_{j=0}^{n-1} |H'(H^j(a_i))|^\beta_g m(K_i) = |(H^n)'(a_i)|^\beta_g m(K_i)
\]
where \( \epsilon'' = \epsilon nL^n(L + 1)^n m(K_i) \). Combining with \((5.11)\) we find that
\[
m(H^{-1}(H^n(K_i))) \lesssim \sum_i \left| (H^{-1} \circ H^n)'(a_i) \right|^\beta_g m(K_i)
\]
where \( \delta = \epsilon' + L^l \epsilon'' \). By summing over \( i \) this gives us the estimate
\[
m(H^{-1}(H^n(K_i))) \sim \sum_i \left| (H^{-1} \circ H^n)'(a_i) \right|^\beta_g m(K_i)
\]
where \( \delta' = \epsilon (lL^j(L + 1)^l + L^l nL^n(L + 1)^n) \). It follows from \((5.9)\) that
\[
\sum_i \left| (H^{-1} \circ H^n)'(a_i) \right|^\beta_g m(K_i) \sim \int_K \left| (H^{-1} \circ H^n)'(x) \right|^\beta_g dm(x),
\]
which gives us \((5.8)\) because \( \epsilon > 0 \) was arbitrary.

Let \( \mathcal{C} \) be the set of critical points of \( H \) contained in \( X \).

**Lemma 5.3.** Let \( m \) be a finite Borel measure on \( X \).

a) Assume that \( m \) is \((G_X, L)\)-conformal with exponent \( \beta \) and that \( m(H(\mathcal{C})) = 0 \).
It follows that \( m \) is \( \beta \)-conformal.

b) Assume that \( m \) is \( \beta \)-conformal and that \( m(\mathcal{C}) = 0 \). It follows that \( m \) is a
\((G_X, L)\)-conformal with exponent \( \beta \).

**Proof.** a) Let \( A \subseteq X \) be a Borel subset such that \( H \) is injective on \( A \). Since \( H(A) \setminus H(A \cap \mathcal{C}) = H(A \setminus \mathcal{C}) \), it follows that \( m(H(A)) = m(H(A \setminus \mathcal{C})) \) because \( m(H(\mathcal{C})) = 0 \) by assumption. Write \( X \setminus \mathcal{C} = \bigcup_{i=1}^\infty U_i \cap X \) where \( U_i \) is open in \( S \) and \( H \) is univalent on \( U_i \) for all \( i \). There is a partition \( \mathcal{C} = \bigcup_{i=1}^\infty A_i \) of \( A \setminus \mathcal{C} \) into Borel sets such that \( A_i \subseteq U_i \). Let \( V \subseteq U_i \) be an open subset. Since \( H \) is univalent on \( U_i \) and \( H^{-1} \) is \( \mathcal{C} \)-conformal, \( H \cap X \) it follows that \( H : V \cap X \to H(V) \cap X \) is a local transfer for \( H|_X \) and hence that
\[
m(H(V \cap X)) = \int_{V \cap X} |H'(z)|^\beta_g dm(z)
\]
since \( m \) is \((G_X, L)\)-conformal. The regularity of the Borel measures on \( U_i \cap X \) given by \( D \mapsto m(H(D)) \) and \( D \mapsto \int_D |H'(z)|^\beta_g dm(z) \) now implies that \( m(H(A_i)) = \int_{A_i} |H'(z)|^\beta_g dm(z) \). Summing over \( i \) we find that
\[
m(H(A)) = m(H(A \setminus \mathcal{C})) = \int_{A \setminus \mathcal{C}} |H'(z)|^\beta_g dm(z) = \int_A |H'(z)|^\beta_g dm(z).
\]

b) Since \( m \) is \( \beta \)-conformal with no atoms at points in \( \mathcal{C} \) it follows that
\[
m\left( \bigcup_{j \in \mathbb{N}} H^{-j}(\mathcal{C}) \right) = 0.
\]

Let \( \eta : U \to V \) be a local transfer for \( H|_X \) and let \( n, l \in \mathbb{N} \) be such that \( H^n(z) = H^l(\eta(z)) \) for all \( z \in U \). To establish the equality \((5.5)\) we may assume that \( U = \)
Let \( X \cap U_0 \) where \( U_0 \subseteq S \) is open and relatively compact in \( S \). Set \( E = U \cap \bigcup_{0 \leq j \leq n} H^{-j}(C) \) which is then a finite set. Let \( W \) be an open subset of \( U \setminus E \) such that \( H^n|_W \) is injective. Then \( H^1 \) is injective on \( \eta(W) \) and \( \eta = H^{-1} \circ H^n \) where \( H^{-1} \) denotes the inverse of \( H^1 : \eta(W) \to H^n(W) \). It follows therefore from Lemma [5.2] that

\[
m(\eta(W)) = \int_W |\eta'(x)|^\beta_x \, dm(x). \tag{5.14}
\]

The same equality holds, for the same reason, for any open subset of \( W \) and therefore the regularity of the Borel measures on \( W \) given by \( B \mapsto m(\eta(B)) \) and \( B \mapsto \int_B |\eta'(x)|^\beta_x \, dm(x) \) shows that (5.14) holds when \( W \) is substituted by any Borel subset of \( W \). Since \( U \setminus E \) is the union of a countable partition of Borel sets each of which is a subset of an open set on which \( H^n \) is injective, we conclude that (5.14) holds when \( W \) is replaced by \( U \setminus E \). Since \( \eta(E) \) and \( E \) are both subsets of \( \bigcup_{j \geq 0} H^{-j}(C) \) it follows from [5.13] that \( m(V) = \int_U |\eta'(x)|^\beta_x \, dm(x) \).

\( \square \)

It follows from Lemma [5.3] that except for measures with an atom at a critical point or a critical value, the conformal measures are the same as the \((G_X, L)\)-conformal measures. This conclusion cannot be improved; there are examples with purely atomic \((G_X, L)\)-conformal measures that are not conformal, and also examples with purely atomic conformal measures that are not \((G_X, L)\)-conformal. See Remark [5.3]. However, it follows from Lemma [5.3] that the two notions of conformality agree for non-atomic measures, and this conclusion suffices for the present purposes.

We turn next to a description of the consistent \( G_X \)-orbits.

**Lemma 5.4.** Let \( K : S \to S \) be a holomorphic map and \( x \in S \) a point such that \( \text{val}(K, x) = j \geq 2 \). Let \( \kappa \in H_{K(x)} \) such that \( \kappa(K(x)) = K(x) \). There is then a conformal germ \( \delta \in H_x \) such that \( \kappa \circ K(z) = K \circ \delta(z) \) for all \( z \) in a neighbourhood of \( x \), \( \delta(x) = x \) and \( \delta'(x)^j = \kappa'(K(x)) \).

**Proof.** We prove this first when \( S \) is an open neighbourhood of 0 in \( \mathbb{C} \), \( x = 0 \) and \( K(0) = 0 \). In this case it follows from Böttcher’s theorem that there is conformal germ \( \varphi \in H_0 \) such that \( \varphi(0) = 0 \) and \( \varphi \circ K \circ \varphi^{-1}(z) = z^j \) in a neighbourhood of 0. The Taylor series of \( \varphi \circ K \circ \varphi^{-1}(z) \) at 0 has the form \( z^j \left( \sum_{n=0}^{\infty} b_n z^n \right) \) with \( b_0 \neq 0 \). Using a holomorphic logarithm near \( b_0 \) we get a holomorphic map \( \delta_1 \) such that \( \delta_1(z)^j = \sum_{n=0}^{\infty} b_n z^n \) in a neighbourhood of 0. Set \( \delta_2(z) = z \delta_1(z) \). Then \( \delta_2 \in H_0 \), \( \delta_2(z)^j = \varphi \circ K \circ \varphi^{-1}(z) \) in a neighbourhood of 0 and \( \delta_2'(0)^j = \delta_1'(0)^j = b_0 = (\varphi \circ K \circ \varphi^{-1})'(0) = \kappa'(0) \). Set \( \delta = \varphi^{-1} \circ \delta_2 \circ \varphi \).

The general case: There are local charts \( \varphi, \psi \) defined in an open neighbourhood of \( x \) and \( K(x) \), respectively, such that \( \psi(K(x)) = 0 \) and \( \varphi(x) = 0 \). It follows from the case dealt with above that there is a conformal germ \( \delta_3 \) at 0 such that \( \delta_3'(0)^j = (\psi \circ K \circ \psi^{-1})'(0) = \kappa'(K(x)) \) and \( \psi \circ K \circ \psi^{-1} \circ \psi \circ K \circ \varphi^{-1}(z) = \psi \circ K \circ \varphi^{-1} \circ \delta_3(z) \) for all \( z \) in a neighbourhood of 0. Then \( \delta = \varphi^{-1} \circ \delta_3 \circ \varphi \) will have the required properties.

\( \square \)

Recall that the orbit of a periodic point \( y \) of period \( p \) is neutral when \( |(H^p)'(y)|_g = 1 \) and critical when \( (H^p)'(y) = 0 \).

**Lemma 5.5.** Let \( x \in X \). The \( G_X \)-orbit \( G_X x \) is consistent if and only if \( x \) is either not pre-periodic or is pre-periodic to a critical or neutral periodic orbit.
Proof. We must show that \( L((G_X)^+_x) \neq \{0\} \) if and only if \( x \) is pre-periodic to a periodic orbit which is neither critical nor neutral. It follows from Proposition 4.4 that \((G_X)^+_x\) is a torsion group unless \( x \) is pre-periodic to a non-critical periodic orbit. So what remains is to prove that when \( x \) is pre-periodic to a non-critical periodic orbit, \( L((G_X)^+_x) = \{0\} \) if and only if that periodic orbit is neutral. Assume therefore that \( x \) is pre-periodic to non-critical periodic orbit. Let \( n \in \mathbb{N} \) be the least natural number such that \( H^n(x) \) is periodic, say of period \( p \), and assume that \( \text{val}(H,H^j(x)) = 1 \) for all \( j \geq n \). Let \( \eta \) be a local transfer such that \( \eta(x) = x \) and \( H^{k+1}(z) = H^m(\eta(z)) \) in a neighborhood of \( x \) for some \( k \in \mathbb{N} \) and some \( m \geq n \). Then \( (H^k)^j(H^p(z)) = (H^m)^j(\eta(x)) \eta'(x) \) which implies that \( |\eta'(x)|_g = |(H^k)^j(H^p(z))|_g \) when \( (H^m)^j(x) \neq 0 \). So when \( x \) is not pre-critical we conclude that \( L((G_X)^+_x) = \{0\} \) if and only if \( x \) is pre-neutral.

It remains to consider the case when \( \text{val}(H^n,x) \geq 2 \). It follows from Lemma 5.4 that there is a conformal germ \( \delta \in \mathcal{H}_s \) such that \( H^{k+1}(z) = H^m(\delta(z)) \) in a neighborhood of \( x \), \( \delta(x) = x \) and \( \delta'(x)^i = (H^k)^j(H^p(z)) \) where \( j = \text{val}(H^n,x) = \text{val}(H^n,x) \). Then \( H^m \circ \eta^{-1} \circ \delta(z) = H^m(z) \) for all \( z \) in a neighborhood of \( x \) and it follows from Lemma 4.2 that \( |(\eta^{-1} \circ \delta')^i(x)|_g = 1 \). Hence \( |\eta'(x)|_g = |\delta'(x)|_g = |(H^k)^j(H^p(z))|_g \). It follows that \( |\eta'(x)|_g = 1 \) if and only if \( x \) is pre-neutral.

We summarise now the results on the KMS states for the conformal action which we obtain by specialising Theorem 2.4 to \( G_X \). For \( x \in X \) let \( G_X x \) be the \( G_X \)-orbit of \( x \), i.e. \( G_X x = \{r(\xi) : \xi \in (G_X)^+_x\} \). Assume that \( x \) is either not pre-periodic or is pre-periodic to a critical or neutral orbit. Then \( G_X x \) is consistent in the sense of Section 2.2 by Lemma 5.5. For each \( z \in G_X x \) we choose a local transfer \( \eta \) such that \( \eta(x) = z \) and set \( l_z(x) = |\eta'(x)|_g \). As in Section 2.2 we will say that \( G_X x \) is \( \beta \)-summable when \( \sum_{z \in G_X x} l_z(x)^{\beta} \) finite. Assume that this is the case and let \( \varphi \) be a state of \( C^*(G_X^+_x) \). For each \( z \in G_X x \) choose an element \( \xi_z \in G_X \) such that \( r(\xi_z) = z \) and \( s(\xi_z) = x \). Then \( \xi_z(G_x)^+_x \xi_z^{-1} = (G_X)^+_x \). Define a state \( \varphi_z \) on \( C^*((G_X)^+_x) \) such that \( \varphi_z(f) = \varphi(f^{\xi_z}) \), where \( f^{\xi_z}(y) = f(\xi_z y) \xi_z^{-1} \). As in Section 2.2 we define a state \( \omega^\varphi_z \) on \( C^r(G_X) \) such that

\[
\omega^\varphi_z(f) = \left( \sum_{z \in \mathcal{O}} l_z(x)^\beta \right)^{-1} \sum_{z \in \mathcal{O}} \varphi_z \left( f \mid (G_X)^+_x \right) |l_z(x)|_g^{\beta}
\]

for all \( f \in C_c(G_X) \).

We can now combine Lemma 5.3 and Lemma 5.5 with Theorem 2.4 to get the following.

Theorem 5.6. Let \( m \) be a non-atomic \( \beta \)-conformal probability measure and let \( x_i, i \in I \), be a finite or countably infinite collection of points in \( X \) each of which is either not pre-periodic or is pre-periodic to a critical or neutral orbit. Let \( \mathcal{O}_i = G_X x_i \) be the \( G_X \)-orbit of \( x_i \) and assume that they are all \( \beta \)-summable. Finally, let \( t_0, t_i \in [0,1], i \in I \), be numbers such that \( t_0 + \sum_{i \in I} t_i = 1 \).

There is then a \( \beta \)-KMS state \( \omega \) for the conformal action on \( C^r(G_X) \) such that

\[
\omega(a) = t_0 \int_X P(a) \ dm + \sum_{i \in I} t_i \omega^\varphi_{x_i} (a) \quad (5.15)
\]
for all $a \in C^*_r(G_X)$. Conversely, any $\beta$-KMS state $\omega$ for the conformal action on $C^*_r(G)$ admits a unique decomposition of the form (5.13).

Remark 5.7. We can now show by example that the structure of the KMS states of the conformal action depends on the chosen metric when $X$ is not compact. Let $S = \mathbb{C}, H(z) = e^z$ and $X = \mathbb{C}\{0\}$. When the conformal action on $C^*_r(G_X)$ is defined by use of the usual ‘flat’ metric $|\cdot|$ there are no $\beta$-KMS states for the conformal action. To see this note that there are no critical points in this case so that $(G_X, L)$-conformality is the same as ordinary conformality by Lemma 5.3. By Theorem 5.6 it suffices therefore to show that for any $\beta \neq 0$ there can be no Borel probability measure $m$ on $\mathbb{C}\{0\}$ such that

$$m(e^A) = \int_A |e^z|^{\beta} \ dm(z)$$

(5.16)

for every Borel subset $A$ of $\mathbb{C}\{0\}$ on which $e^z$ is injective. To see that this is so, assume that $m$ is such a measure. Let $w \in \mathbb{C}\{0\}$ and choose an element $z \in \mathbb{C}\{0\}$ such that $e^z = w$. There is an open neighborhood $U$ of $z$ of diameter $< \pi$ such that $e^z$ is injective on $U$. Then $(U + 2\pi ni) \cap (U + 2\pi ki) = \emptyset$ when $n \neq k$ in $\mathbb{N}$ and (5.16) yields the conclusion that

$$Nm(e^U) = \sum_{k=1}^N \int_{U+2\pi ki} |e^z|^{\beta} \ dm(z) \leq \sup_{z \in U} |e^z|^{\beta}$$

for all $N \in \mathbb{N}$, which implies that $m(e^U) = 0$. Thus every $w \in \mathbb{C}\{0\}$ has an open neighborhood of zero $m$-measure. This is impossible by regularity of $m$.

Consider instead the metric $g(z) = \sqrt{f(z)} |\cdot|$ where $f : \mathbb{C}\{0\} \to ]0, \infty[$ is any continuous function such that $\int_{\mathbb{C}\{0\}} f(z) \ dz = 1$ when $dz$ is the Lebesgue measure. Using this metric the requirement of $\beta$-conformality for a Borel probability measure $m$ on $\mathbb{C}\{0\}$ is that

$$m(e^A) = \int_A |e^z|^{\beta} f(e^z)^{\frac{\beta}{2}} f(z)^{-\frac{\beta}{2}} \ dm(z)$$

(5.17)

for every Borel subset $A$ of $\mathbb{C}\{0\}$ on which $e^z$ is injective. It is easy to check that (5.17) is satisfied when $\beta = 2$ and $m = f(z) \ dz$. Thus the conformal action, defined with the metric $g(z) = \sqrt{f(z)} |\cdot|$ has a non-atomic 2-KMS state.

6. Phase transition with spontaneous symmetry breaking from quadratic maps

We now specialise to the case where $S$ is the Riemann sphere $\overline{\mathbb{C}}$ and $H$ is a rational map on $\overline{\mathbb{C}}$ of degree at least 2. We change the notation accordingly by setting $H = R$. In this case there are three obvious candidates for the set $X$; namely, $\overline{\mathbb{C}}$, the Fatou set $F_R$ and the Julia set $J_R$. In this section we consider some examples with $X = J_R$ where we can give a complete description of the KMS states for the conformal action on $C^*_r(G_{J_R})$ thanks to results of Graczyk and Smirnov, [GS1], [GS2].

Up to conjugation by a conformal automorphism of $\overline{\mathbb{C}}$, any polynomial of degree 2 on the Riemann sphere has the form

$$R_c(z) = z^2 + c$$
for some \( c \in \mathbb{C} \). Since \( J_{Rc} \subseteq \mathbb{C} \) we can assume that the metric defining the conformal action is the usual ‘flat’ metric \( | \cdot | \). We say that \( R_c \) satisfies the Collet-Eckmann condition when there is a \( C > 0 \) and a \( \lambda > 1 \) such that
\[
|R_n^c(c)| \geq C\lambda^n
\]
for all \( n \in \mathbb{N} \). For example \( z^2 - 2 \) satisfies the Collet-Eckmann condition because \( R_{-2}(c) = R_{-2}(-2) = 2 \) is a repelling fixed point for \( R_{-2} \). It was shown by Benedicks and Carleson in [BC] that the set of the \( a \)'s in \((0,2]\) for which the polynomial \( R_{-a} \) satisfies the Collet-Eckmann condition is of positive Lebesgue measure.

**Lemma 6.1.** Assume that \( R_c \) satisfies the Collet-Eckmann condition and let \( HD(J_{Rc}) \) be the Hausdorff dimension of the Julia set \( J_{Rc} \).

a) When the critical point \( 0 \) is pre-periodic there are no \( \beta \)-summable \( G_{J_{Rc}} \)-orbits for any \( \beta > 0 \).

b) When \( 0 \) is not pre-periodic there are no \( \beta \)-summable \( G_{J_{Rc}} \)-orbits when \( 0 < \beta \leq HD(J_{Rc}) \), and when \( \beta > HD(J_{Rc}) \),
\[
G_{J_{Rc}}0 = \bigcup_{n \geq 0} R_{-c}^{-n}(0)
\]
is the only \( \beta \)-summable \( G_{J_{Rc}} \)-orbit.

**Proof.** Assume that \( \mathcal{O} \) is a \( \beta \)-summable \( G_{J_{Rc}} \)-orbit, \( \beta > 0 \), and let \( m_{\mathcal{O}} \) be the corresponding purely atomic \( (G_{J_{Rc}}, L) \)-conformal measure (2.5). Since \( \lim_{n \to \infty} |(R_{c}^n)'(c)|^\beta = \infty \) we conclude that \( c \notin \mathcal{O} \), and hence that \( m_{\mathcal{O}} \) is \( \beta \)-conformal by Lemma 5.3. It follows therefore from item 3 of Corollary 5.1 in [GS2] that \( \mathcal{O} = \bigcup_{n \geq 0} R_{-c}^{-n}(0) \), i.e. this is the only possible \( \beta \)-summable \( G_{J_{Rc}} \)-orbit when \( \beta > 0 \).

In case a) \( 0 \) is pre-periodic, and it clearly can not be pre-periodic to a critical periodic orbit. Furthermore, it follows from Lemma 10 in [GS1] that the Collet-Eckmann condition rules out the presence of a periodic neutral orbit in \( J_{Rc} \). It follows therefore from Lemma 5.3 that \( \mathcal{O} = G_{J_{Rc}}0 \) is not consistent. There is therefore no \( \beta \)-summable \( G_{J_{Rc}} \)-orbit when \( 0 \) is pre-periodic.

In case b) it follows from item 1 and 2 of Corollary 5.1 in [GS2] that \( \beta > HD(J_{Rc}) \) since \( m_{\mathcal{O}} \) is \( \beta \)-conformal and purely atomic, and then from item 3 of the same corollary that \( \mathcal{O} = \bigcup_{n \geq 0} R_{-c}^{-n}(0) \) actually is \( \beta \)-summable when \( \beta > HD(J_{Rc}) \).

**Theorem 6.2.** Assume that \( R_c \) satisfies the Collet-Eckmann condition and let \( HD(J_{Rc}) \) be the Hausdorff dimension of the Julia set \( J_{Rc} \). Consider the conformal action \( \gamma \) on \( C^*_r(G_{J_{Rc}}) \). Let \( \beta > 0 \).

a) When \( 0 \) is pre-periodic for \( R_c \) there is a \( \beta \)-KMS state for \( \gamma \) if and only if \( \beta = HD(J_{Rc}) \). It is given by
\[
\omega(a) = \int_{J_{Rc}} P(a) \, dm,
\]
where \( m \) is the \( HD(J_{Rc}) \)-conformal measure of [GS2].

b) When \( 0 \) is not pre-periodic there are
- no \( \beta \)-KMS states when \( 0 < \beta < HD(J_{Rc}) \),
- a unique \( \beta \)-KMS state \( \omega \), given by (6.2), when \( \beta = HD(J_{Rc}) \), and
• two extremal $\beta$-KMS states when $\beta > \text{HD}(J_{R_c})$. The corresponding measure on $J_{R_c}$ is purely atomic and supported on $\bigcup_{n \geq 0} R^{-n}(0)$.

Proof. Note that the isotropy group of 0 in $G_{J_{R_c}}$ is $\mathbb{Z}_2$ by c) of Proposition 4.4. Therefore a $\beta$-summable $G_{J_{R_c}}$-orbit will give rise to two extremal $\beta$-KMS states by Theorem 5.6. Other than that all the statements follow by combining Lemma 6.1 and Theorem 5.6 with Corollary 5.1 of [GS2]. □

Thanks to the results from [GS2], Theorem 6.2 can be extended to general rational maps $R$ satisfying the Collet-Eckmann condition of [GS1], provided the critical points in $J_R$ have the same valency. The only difference is that the number of extreme $\beta$-KMS states increases, depending on the number and the valency of the critical points in $J_{R_c}$, and on whether or not their orbits intersect. We have here restricted attention to the quadratic case for concreteness and because it simplifies the statement.

If we adopt the terminology of Bost and Connes from [BoC], the phase transition which occurs in Theorem 6.2 at the inverse temperature $\beta = \text{HD}(J_{R_c})$ when 0 is not pre-periodic, is caused by spontaneous symmetry breaking. To see this note that we can define an automorphism $\xi$ of $C^*_r(G_{J_{R_c}})$ such that

$$\xi(f)[x,k,\eta,y] = \eta'(x) f[x,k,\eta,y]$$

when $f \in C_c(G_{J_{R_c}})$. This automorphism commutes with the conformal action and interchanges its two extremal $\beta$-KMS states when $\beta > \text{HD}(J_{R_c})$ and 0 is not pre-periodic. Hence for all $\beta \geq \text{HD}(J_{R_c})$ there is exactly one $\xi$-invariant $\beta$-KMS state for the conformal action.

Remark 6.3. Note that in case a) of Theorem 6.2 there is a purely atomic $\beta$-conformal measure for $\beta > \text{HD}(J_{R_c})$ which is supported on the backward orbit of the critical point 0, cf. [GS2], and that this measure is not $(G_{J_{R_c}}, L)$-conformal because the $G_{J_{R_c}}$-orbit of 0 is not consistent. To show that there are also rational maps with a $(G_{J_{R_c}}, L)$-conformal measure which is not conformal, consider the polynomial

$$R(z) = z \left( 1 + \frac{z^2}{2} \right)^2$$

which has a critical point at $z = -2$. The critical value 0 is a fixed point and $R^{-1}(0) = \{0, -2\}$. It follows that the point 0 is its own $G_{J_{R_c}}$-orbit, i.e. $G_{J_{R_c}}0 = \{0\}$. Note that 0 is a parabolic fixed point since $R'(0) = 1$. It follows that the Dirac measure at 0 is $(G_{J_{R_c}}, L)$-conformal for any exponent $\beta \in \mathbb{R}$. Since it is supported by a critical value it can not be $\beta$-conformal for any $\beta \neq 0$.

7. Generalized gauge actions and their KMS states

7.1. Generalized gauge actions. Consider again the setting of Section 4 and let $f : X \to \mathbb{R}$ be a continuous function. We can then define a homomorphism $c_f : G_X \to \mathbb{R}$ such that

$$c_f[x,k,\eta,y] = \lim_{N \to \infty} \left( \sum_{i=0}^{N} f(H^i(x)) - \sum_{i=0}^{N-k} f(H^i(y)) \right).$$

(7.1)

c_f is continuous and we can consider the corresponding one-parameter group $\alpha_f^t = \sigma_t^f$ of automorphisms on $C^*_r(G_X)$. Following the general scheme laid out in Section
2.2 the KMS states of $\alpha^f$ can be determined from the corresponding non-atomic $(G_X,c_f)$-conformal measures and summable $G_X$-orbits.

Let $\beta \in \mathbb{R}$. Following [DU] we say that a finite Borel measure $m$ on $X$ is $e^{\beta f}$-conformal when

$$m(H(A)) = \int_A e^{\beta f(x)} \, dm(x)$$

for all Borel subsets $A$ of $X$ such that $H : A \to X$ is injective. It is then easy to adopt the proofs of Lemma 5.2 and Lemma 5.3 to obtain the following.

**Lemma 7.1.** Let $m$ be a finite Borel measure on $X$ such that $m$ has no mass at the critical points or critical values of $H$ in $X$. Let $\beta \in \mathbb{R}\setminus\{0\}$. Then $m$ is $(G_X,c_f)$-

conformal with exponent $\beta$ if and only if $m$ is $e^{\beta f}$-conformal.

In particular, the non-atomic $(G_X,c_f)$-conformal measures with exponent $\beta$ coincide with the non-atomic $e^{\beta f}$-conformal Borel measures.

Which $G_X$-orbits are $\beta$-summable depends of course very much on the behaviour of $f$. However, the case $f = 1$ is easy to handle as we do in the following section.

7.2. The gauge action. The gauge action on $C^*_r(G_X)$ is the generalised gauge action $\alpha^1$ obtained from the homomorphism (7.1) when $f$ is the constant function $1$. It is determined by the condition that

$$\alpha^1(f)[x,k,\eta,y] = e^{ikt} f[x,k,\eta,y]$$

when $f \in C_c(G_X)$.

Assume now that $S$ is the Riemann sphere and $H$ a rational map $R$ of degree at least 2. Assume also that $X = J_R$, the Julia set of $R$. In this case we can describe all $\beta$-KMS states for the gauge action on $C^*_r(G_{J_R})$ when $\beta \neq 0$.

**Lemma 7.2.** Let $m$ be a $e^{\beta}$-conformal Borel probability measure. Assume that $m$ is non-atomic. Then $\beta = \log d$ and $m$ is the Lyubich measure of maximal entropy, cf. [L].

**Proof.** Let $C$ be the critical points of $R$. Every point of $J_R \setminus R(C)$ is contained in an open set $U \subseteq J_R \setminus R(C)$ such that $R^{-1}(U)$ is a disjoint union $R^{-1}(U) = \sqcup_{i=1}^n V_i$ where each $V_i$ is open and $R : V_i \to U$ is a homeomorphism. Let $B \subseteq U$ be a Borel subset and set $B_i = R^{-1}(B) \cap V_i$. Since $m(B_i) = e^{-\beta} m(B) = m(B_1)$ we conclude that $m(R^{-1}(B)) = dm(B_1) = de^{-\beta} m(B)$. Every Borel subset of $J_R \setminus R(C)$ is the disjoint union of a countable collection of Borel sets each of which is a subset of an open set $U \subseteq J_R \setminus R(C)$ as above. It follows therefore that

$$m\left(R^{-1}(B)\right) = de^{-\beta} m(B) \quad (7.2)$$

for every Borel set $B \subseteq J_R$ because $m\left(R(C)\right) = 0$. Taking $B = J_R$ yields the conclusion that $\beta = \log d$ as asserted. Once this is established it follows from (7.2) that $m$ is $R$-invariant and then from the theorem in [PLM] and [L] that $m$ is the Lyubich measure. \hfill $\Box$

**Lemma 7.3.** Let $O$ be a $\beta$-summable $G_{J_R}$-orbit, $\beta \neq 0$. It follows that there is a critical point $c$ for $R$ which not pre-periodic such that $O = G_{J_R}c$.

**Proof.** Let $x \in O$. Since $J_R$ does not contain any critical periodic orbits the d1)-case in Proposition 4.4 does not arise. It follows therefore from Proposition 4.4 that the homomorphism $c_1[x,k,\eta,y] = k$ which defines the gauge action $\alpha^1$ does not
annihilate \((G_{JR})_x^x\) when \(x\) is pre-periodic. Thus \(O\) is is not consistent and hence not \(\beta\)-summable when \(x\) is pre-periodic. Assume therefore that \(x\) is not pre-periodic. To finish the proof it suffices to show that \(G_{JR}x\) is not summable unless \(x\) is pre-critical. Assume therefore that no element of the forward orbit \(\{R^n(x) : n = 0, 1, 2, \ldots\}\) is critical. Since \(R^n(x) \in O\) for all \(n\) and \(l_x(R^n(x)) = e^n\) we see that \(\sum_{z \in O} l_x(z) = \infty\), unless \(\beta < 0\). Consider therefore now the case \(\beta < 0\). Since \(R^{-1}(x)\) is not critical, and the degree of \(R\) at least 2, there is an element \(x_n \in R^{-1}(R^n(x))\) other than \(R^{-1}(x)\) when \(n \geq 1\). Note that

\[ N = \{n \geq 1 : x_n \text{ is not critical}\} \]

must be infinite because \(x\) is not pre-periodic. Assume to reach a contradiction that there is a critical point in the backward orbit of \(x_n\) for every \(n \in N\). Since there are only finitely many critical points there must then be some \(m < n\) in \(N\) such that the same critical point is contained in both the backward orbit of \(x_m\) and the backward orbit of \(x_n\). Note that \(c\) can not be pre-periodic since \(x\) is not. There is therefore a unique \(k \in \mathbb{N}\) such that \(R^k(c) = R^n(x)\) and a unique \(k' \in \mathbb{N}\) such that \(R^{k'}(c) = R^m(x)\). Then \(R^{n-m+k'}(c) = R^m(c)\) and hence \(n-m+k' = k\). Since \(R^{n-m+k'}(c) = R^{-1}(x)\) while \(R^{k}(c) = x_n \neq R^{-1}(x)\), this is a contradiction. There must therefore be an \(n \in N\) such that the backward orbit of \(x_n\) does not contain any critical point. Choose \(z_k \in R^{-k}(x_n)\) and note that \(z_k\) is an element of \(O\) for all \(k\). Since

\[ l_x(z_k)^\beta = e^{\beta(n-k-1)} \]

we conclude that \(\sum_{z \in O} l_x(z)^\beta = \infty\), also when \(\beta < 0\). \(\square\)

Let \(C_0\) be the (possibly empty) set of critical and not pre-periodic points in \(J_R\). Since the elements are not pre-periodic the limit

\[ \text{VAL}_\infty(c) = \lim_{n \to \infty} \text{val}(R^n, c) \]

exists for every \(c \in C_0\). We define an equivalence relation \(\sim\) on \(C_0\) such that \(x \sim y\) if and only if \(\text{VAL}_\infty(x) = \text{VAL}_\infty(y)\) and \(R^n(x) = R^m(y)\) for some \(n, m \in \mathbb{N}\). It follows from Lemma 7.3 that two elements \(x, y \in C_0\) are in the same \(G_{JR}\)-orbit if and only if \(x \sim y\). For each \(\xi \in C_0/\sim\) choose a representative \(c_\xi \in C_0\) and set

\[ [\xi] = G_{JR}c_\xi. \]

It follows from Lemma 7.3 that these sets constitute the only possible \(\beta\)-summable \(G_{JR}\)-orbits, for any \(\beta \neq 0\).

**Lemma 7.4.** Let \(d\) be the degree of \(R\). When \(\beta > \log d\) the set \([\xi]\) is \(\beta\)-summable for all \(\xi \in C_0/\sim\). When \(\beta \leq \log d\) the set \([\xi]\) is \(\beta\)-summable if and only if it is finite.

**Proof.** Note that

\[ [\xi] \subseteq \bigcup_{c \in C_0, n \in \mathbb{N}} R^{-n}(c) \]

and \(l_c(z) = e^{-n}\) when \(z \in R^{-n}(c) \cap G_{JR}c\). There is then a \(K_\beta > 0\) such that

\[ K_\beta^{-1}e^{-\beta n} \leq l_c(z)^\beta \leq K_\beta e^{-\beta n} \]

for all \(z \in [\xi] \cap R^{-n}(C_0)\) and all \(n\). Since an element \(c \in C_0\) is not periodic, there is also a \(C > 0\) such that

\[ C^{-1}d^m \leq \#[\xi] \cap R^{-n}(C_0) \leq Cd^n\]
for all \( n \), provided that \( [\xi] \) is not finite. Hence
\[
K^{-1}_\beta C^{-1} e^{n(\log d - \beta)} \leq \sum_{z \in [\xi]\cap R^{-n}(C_0)} l_{c_\xi}(z)^\beta \leq CK\beta e^{n(\log d - \beta)}
\]
for all \( n \) when \( [\xi] \) is infinite. This proves the lemma.

Let \( C_{00} \) be the set of critical and not pre-periodic points \( c \) with the property that \( G_{J_R}c \) is finite. For each \( \xi \in C_{00}/\sim \), set \( \text{VAL}_\infty(\xi) = \text{VAL}_\infty(c_\xi) \). Note that it follows from the proof of Proposition 4.4 that \( \text{VAL}_\infty(\xi) \) is the order of the cyclic isotropy groups \( (G_{J_R})^x_x, x \in [\xi] \). We can then summarise our findings as follows.

**Theorem 7.5.** Let \( R \) be a rational map of degree \( d \geq 2 \) on the Riemann sphere with Julia set \( J_R \) and consider the gauge action \( \alpha^1 \) on \( C^*_r(G_{J_R}) \).

- When \( 0 \neq \beta < \log d \) there are exactly
  \[
  \sum_{\xi \in C_{00}/\sim} \text{VAL}_\infty[\xi]
  \]
  extremal \( \beta \)-KMS states for \( \alpha^1 \), all purely atomic. In fact, the corresponding measures on \( J_R \) have finite support.
- When \( \beta = \log d \) there are exactly
  \[
  1 + \sum_{\xi \in C_{00}/\sim} \text{VAL}_\infty[\xi]
  \]
  extremal \( \beta \)-KMS states for \( \alpha^1 \). One is non-atomic and the associated measure is the Lyubich measure. The others are all purely atomic and the associated measures have finite support.
- When \( \log d < \beta < \infty \) there are exactly
  \[
  \sum_{\xi \in C_{00}/\sim} \text{VAL}_\infty[\xi]
  \]
  extremal \( \beta \)-KMS states for \( \alpha^1 \), all purely atomic.

**Example 7.6.** To give an example where there are KMS-states for which the associated measures have finite support we use the work of M. Rees. She shows in Theorem 2 of [R] that for ‘many’ \( \lambda \in \mathbb{C}\setminus\{0\} \) the rational map
\[
R(z) = \lambda \left(1 - \frac{2}{z}\right)^2
\]
has a dense critical forward orbit. In particular, the Julia set \( J_R \) is the entire sphere. The critical points are 0 and 2, and \( R^{-1}(0) = \{2\} \). Hence the \( G_{J_R} \)-orbit of 0 consists only of the point 0. Since \( \text{VAL}_\infty(0) = 2 \) there are for all \( \beta \neq 0 \) exactly two extremal \( \beta \)-KMS states for the gauge action on \( C^*_r(G_{J_R}) \) such that the associated measure is the Dirac measure at 0.

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