Global $L^r$-estimates and regularizing effect for solutions to the $p(t, x)$-Laplacian systems

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Abstract - We consider the initial boundary value problem for the $p(t, x)$-Laplacian system in a bounded domain $\Omega$. If the initial data belongs to $L^{r_0}$, $r_0 \geq 2$, we give a global $L^{r_0}(\Omega)$-regularity result uniformly in $t > 0$ that, in the particular case $r_0 = \infty$, implies a maximum modulus theorem. Under the assumption $1 < p_- = \inf_{\Omega} p(t, x) \leq p_+(t) := \sup_{\Omega} p(t, x) < \infty$, for $r \geq r_0$, we also state $L^{r_0} - L^r$ estimates for the solution. Complete proofs of the results presented here are given in the paper [12].

1. Introduction

The aim of this note is to present a global $L^{r_0}$-regularity result, uniformly in $t$, and to study the regularizing effect for solutions of the following $p(t, x)$-Laplacian system

$$u_t - \nabla \cdot (|\nabla u|^{p(t, x)-2}\nabla u) = 0, \quad \text{in } (0, \infty) \times \Omega,$$

$$u(t, x) = 0, \quad \text{on } (0, \infty) \times \partial \Omega,$$

$$u(0, x) = u_o(x), \quad \text{in } \Omega. \quad (1.1)$$

Here $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a bounded domain, $u : (0, \infty) \times \Omega \to \mathbb{R}^N$, $N \geq 1$ is a scalar or vector field, $u_t$ is the derivative of $u$ with respect to time, and the exponent $p := p(t, x)$ is bounded

$$1 < p_- := \inf_{\Omega} p(t, x) \leq p_-(t) := \inf_{\Omega} p(t, x) \leq p(t, x) \leq p_+(t) := \sup_{\Omega} p(t, x) \leq p_+ := \sup_{\Omega} p(t, x) < \infty, \quad (1.2)$$

and log-Hölder continuous in $(t, x)$, which means that there exists a constant $c_1$ such that

$$|p(t, x) - p(\tau, y)| \leq \frac{c_1}{\log(e + \frac{1}{|x - y|})} \quad (1.3)$$

is satisfied for all $(t, x)$ and $(\tau, y)$ in $(0, \infty) \times \Omega$.

System (1.1) belongs to the class of partial differential equations with non-standard growth. One of the principal reasons of its interest is the connection with the more intricate system modeling the motion of electro-rheological fluids with shear dependent...
viscosity, introduced in [23, 24]. For results on these fluids in the evolutionary case we refer to [10, 16, 20, 25, 26].

The well-posedness of problem (1.1) is rather recent. The first existence result, which allows to study this system by using classical tools of monotone operators theory, appeared in [6]. Subsequently in [18] and in the same period in [3], the existence was extended by removing the lower bound \( p_- > \frac{2n}{n+2} \) and, in [3], also the log-Hölder continuity condition. The technique of these two papers are completely different and the solutions obtained seems to be not comparable, unless a log-Hölder continuity condition is assumed, as proved in [4].

The following main theorem shows a global \( L^r \)-regularity result, uniformly in \( t \), for a weak solution (for definition see next section).

**Theorem 1.1** Let \( p \) satisfy assumptions (1.2)-(1.3) and let \( r_0 \in [2, \infty) \). Then, for all \( u_0 \in L^r(\Omega) \) the following estimate holds for the unique weak solution \( u \) of (1.1)

\[
\|u(t)\|_{r_0} \leq \|u_0\|_{r_0}, \quad \forall t \geq 0.
\]

Moreover, for all \( T \in (0, \infty) \), \( u \in C([0,T]; L^r(\Omega)) \), for \( r = 2 \) and for any \( r \in [2, r_0) \) if \( r_0 \in (2, \infty) \). Finally, if \( r_0 \in [2, \infty) \), then \( \lim_{t \to 0} \|u(t) - u_0\|_{r_0} = 0 \).

In the case \( r_0 = \infty \), analogous results are proved, with a constant exponent, in [13, 14, 15] for equations, in [9] locally for systems and, finally, globally for systems in [11]. Again for equations but with a variable exponent \( p(t, x) \), a similar result is proved in [6], where a more general non-linear parabolic equation is considered. Finally, still for the \( p(t, x) \)-system, the result is proved locally in the recent paper [7].

In [12] we prove Theorem 1.1 in a direct way, via a duality technique that makes use of a suitable, still quasi-linear, “adjoint problem”. This means that we reach our result without the investigation on high regularity properties of solutions, that could ensure boundedness by embedding. On the other hand this last question is still an interesting open problem. Actually in the framework of parabolic equations with non constant exponents, the unique regularity results known are the one in the pioneering paper [1], where a partial Hölder continuity of the spatial gradient is proved, and the result in [7], where a local Hölder continuity of the spatial gradient is proved. The results in [1] and [7] are proved under the assumption, quite natural in the regularity theory, of \( p_- > \frac{2n}{n+2} \).

Note that we employ the duality as we work on weak solutions, so we do not have enough regularity to give sense to an expression of the kind \( <u_t, u|u|^{r-2}> \), \( r \in [2, \infty) \). Once Theorem 1.1 is obtained, the previous quantity makes always sense if the initial data is in \( L^\infty(\Omega) \), and then for any initial data throughout a limit procedure.

Our next result is Theorem 1.2 where we get informations on the behavior of the \( L^r(\Omega) \)-norm of the solution, corresponding to an initial data in \( L^{r_0}(\Omega) \), \( 2 \leq r_0 \leq r \leq \infty \), in a right neighborhood of \( t = 0 \) as well as for \( t \to \infty \). This kind of property is known as regularizing effect.

**Theorem 1.2** Let \( u_0 \in L^{r_0}(\Omega) \) with \( r_0 \in [2, \infty) \), and let \( p \) satisfy assumptions (1.2)-(1.3) with \( p_- > \frac{2n}{n+2}r_0 \). Then, for all \( r \in [r_0, \infty) \), there exist nonnegative constants \( \gamma_-(r), \gamma_+(r) \) such that the weak solution \( u \) of (1.1) satisfies the following estimate

\[
\|u(t)\|_{r} \leq c(\|u_0\|_{r_0})(t^{-\gamma_-(r)} + t^{-\gamma_+(r)}), \quad \forall t > 0.
\]

\[
(1.5)
\]
If \( p_- > n \) then \( r = \infty \) is allowed.

**Remark 1.1** In the proof of Theorem 1.2 in [12] we obtain a precise expression of \( \gamma_-(r) \) and \( \gamma_+(r) \) if \( p_- \geq n \). Actually for \( r \in [r_0, \infty) \) \( \gamma_-(r) := \frac{(r-r_0)np_-}{rp_-(r_0p_- - 2n + np_-)} \) and \( \gamma_+(r) := \frac{(r-r_0)np_-}{rp_+(r_0p_- - 2n + np_-)} \). If \( p_- > n \) then \( r = \infty \) is allowed and \( \gamma_- := \frac{n}{r_0p_- - 2n + np_-} \) and \( \gamma_+ := \frac{n}{r_0p_- - 2n + np_-} \). Note that these exponents completely agree with those given in [19, 27, 21, 22] for the constant exponent case.

If \( p_- < n \) we could also give the expression of \( \gamma_-(r) \) and \( \gamma_+(r) \). On the other hand, as we treat this case by an iterative argument, it would become very tedious.

**Corollary 1.1** Let the assumptions of Theorem 1.2 be satisfied. Then \( u \in C([0, T]; L^{r_0}(\Omega)) \).

Further

\[
\|u(t)\|_{r_0} \leq \|u(s)\|_{r_0}, \quad \forall t > s \geq 0.
\]

**Remark 1.2** Having the result of Corollary 1.1 at disposal, we can state estimate (1.5) in the form of parabolic semigroup as

\[
\|u(t)\|_r \leq c(\|u(s)\|_{r_0})((t-s)^{-\gamma_-(r)} + (t-s)^{-\gamma_+(r)}), \quad \forall t > s > 0.
\]

2. **Ideas of the proofs**

Let us introduce few notation related to function spaces. We use standard notation for Sobolev and Bochner spaces. For spaces with variable exponents we mainly refer to the notation in [6] and [17]. Given a bounded and log-Hölder continuous exponent \( q \in \Omega_T \), with \( q_- > 1 \), for any \( t > 0 \) we introduce the Banach space

\[
V^q_t(\Omega) := \{ u \in L^2(\Omega) \cap W^{1,1}_0(\Omega), \nabla u \in L^{q(t, \cdot)}(\Omega) \},
\]

with norm

\[
\|u\|_{V^q_t(\Omega)} := \|u\|_{2, \Omega} + \|\nabla u\|_{q(t, \cdot), \Omega},
\]

and we denote by \((V^q_t)'\) its dual. We further introduce, for all \( T \in (0, \infty) \), the Banach space

\[
X^q(\Omega_T) := \{ u \in L^2(\Omega_T), u(t, \cdot) \in V^q_t(\Omega) \text{ a.e.} \ t \in [0, T], \nabla u \in L^{q(t)}(\Omega_T) \},
\]

with norm

\[
\|u\|_{X^q(\Omega_T)} := \|u\|_{2, \Omega_T} + \|\nabla u\|_{q(\cdot), \Omega_T},
\]

and we denote by \((X^q(\Omega_T))'\) its dual. Last we introduce the space

\[
W^q(\Omega_T) := \{ u \in X^q(\Omega_T), u_t \in (X^q(\Omega_T))' \}.
\]

We are in position to give the notion of weak solution.

**Definition 2.1** Let \( u_0 \in L^2(\Omega) \). A field \( u : (0, \infty) \times \Omega \to \mathbb{R}^N \) is said a weak solution of system (1.1) if, for all \( T \in (0, \infty) \), \( u \in W^p(\Omega_T) \),

\[
\int_0^T \left< u_t, \psi >_V^q(\Omega) + (|\nabla u|^{p-2}\nabla u, \nabla \psi) \right> dt = 0, \quad \forall \psi \in X^p(\Omega_T),
\]

\[
\lim_{t \to 0^+} \|u(t) - u_0\|_2 = 0.
\]
In order to prove Theorem 1.1 we introduce the approximating systems

\begin{equation}
  u_t - \nabla \cdot \left( \mu + |\nabla u|^2 \right)^{\frac{\mu - 2}{2}} \nabla u = 0, \quad \text{in } (0, \infty) \times \Omega,
  
  u(t, x) = 0, \quad \text{on } (0, \infty) \times \partial \Omega,
  
  u(0, x) = u_0(x), \quad \text{in } \Omega,
\end{equation}

with \( \mu \in (0, 1) \), and

\begin{equation}
  v_t - \nu \Delta v - \nabla \cdot \left( \mu + |\nabla v|^2 \right)^{\frac{\mu - 2}{2}} \nabla v = 0, \quad \text{in } (0, \infty) \times \Omega,
  
  v(t, x) = 0, \quad \text{on } (0, \infty) \times \partial \Omega,
  
  v(0, x) = v_0(x), \quad \text{in } \Omega.
\end{equation}

**Definition 2.2** Let \( \mu \in (0, 1) \). Let \( u_0 \in L^2(\Omega) \). A field \( u : (0, \infty) \times \Omega \to \mathbb{R}^N \) is said a weak solution of system (2.1) if, for all \( T \in (0, \infty) \), \( u \in W^p(\Omega_T) \),

\( \int_0^T \left[ <u_\tau, \psi>_{_{V^2(\Omega)}} + (a(\mu, v) \nabla u, \nabla \psi) \right] d\tau = 0, \quad \forall \psi \in X^p(\Omega_T), \)

\( \lim_{t \to 0^+} ||u(t) - u_0||_2 = 0. \)

Set \( q = q(t, x) := \max\{2, p(t, x)\} \).

**Definition 2.3** Let \( \mu \in (0, 1), \nu > 0 \). Let \( v_0 \in L^2(\Omega) \). A field \( v : (0, \infty) \times \Omega \to \mathbb{R}^N \) is said a weak solution of system (2.2) if, for all \( T \in (0, \infty) \), \( v \in L^2(0, T; W^{1,2}_0(\Omega)) \cap W^q(\Omega_T), \)

\( \int_0^T \left[ <v_\tau, \psi>_{_{V^2(\Omega)}} + (\nu \nabla v, \nabla \psi) + (a(\mu, v) \nabla v, \nabla \psi) \right] d\tau = 0, \quad \forall \psi \in X^q(\Omega_T), \)

\( \lim_{t \to 0^+} ||v(t) - v_0||_2 = 0. \)

System (2.1) is introduced in order to deal with a non-singular system. The introduction of system (2.2) is connected with the idea of applying a duality technique to prove Theorem 1.1. Let us explain in what sense.

In order to apply the duality, one aims at a reciprocity relation between the solution \( u \) and the solution of a local adjoint problem. Both should be in the sets of solutions, that is the space \( W^p(\Omega_T) \) and \( W^p(\Omega_t) \) respectively, \( p = p(s, x) := p(t - s, x) \). The “natural” adjoint of system (2.1) should be deduced, on \((0, t), \) from the following

\begin{equation}
  \psi_s - \nabla \cdot (B(u)(s, x) \nabla \psi) = 0, \quad \text{in } (0, t) \times \Omega,
\end{equation}

where, for all \( t > 0 \), we define for a.e. in \( s \in (0, t), \)

\begin{equation}
  B(u)(s, x) = (B(u)(s, x))_{\alpha \beta j} := \delta_{ij} \delta_{\alpha \beta} (\mu + |\nabla u(t - s, x)|^2)^{\frac{\mu - 2 - \alpha \beta}{2}},
\end{equation}

with \( \mu \in (0, 1) \). This system is not suitable, as \( \psi \not\in X^p(\Omega_t) \). An approximation of the adjoint which ensures the membership to \( X^p(\Omega_t) \) is

\begin{equation}
  \theta_s - \nabla \cdot (B(u)(s, x) \nabla \theta) - \varepsilon \nabla \cdot (|\nabla \theta|^{p-2} \nabla \theta) = 0, \quad \text{in } (0, t) \times \Omega,
\end{equation}

where, for all \( t > 0 \), we define for a.e. in \( s \in (0, t), \)

\begin{equation}
  B(u)(s, x) = (B(u)(s, x))_{\alpha \beta j} := \delta_{ij} \delta_{\alpha \beta} (\mu + |\nabla u(t - s, x)|^2)^{\frac{\mu - 2 - \alpha \beta}{2}},
\end{equation}

with \( \mu \in (0, 1) \). This system is not suitable, as \( \psi \not\in X^p(\Omega_t) \). An approximation of the adjoint which ensures the membership to \( X^p(\Omega_t) \) is

\begin{equation}
  \theta_s - \nabla \cdot (B(u)(s, x) \nabla \theta) - \varepsilon \nabla \cdot (|\nabla \theta|^{p-2} \nabla \theta) = 0, \quad \text{in } (0, t) \times \Omega,
\end{equation}
where $\varepsilon > 0$. In this case $\theta \in X\overline{T}(\Omega_t)$ but unfortunately $\theta_* \not\in (X\overline{T}(\Omega_t))'$. In order to overcome this impasse, we are led to modify the same system (2.1), approximating, in turn, with system (2.2). Now, the solution of (2.2) belongs to $W^q(\Omega_T)$, with $q = q(t, x) := \max\{2, p(t, x)\}$. Then, the same arguments that led us to system (2.5) now lead us to the following one as approximation of the adjoint of (2.2)

$$\varphi_\varepsilon - \nu \Delta \varphi - \nabla \cdot (B(v)(s, x)\nabla \varphi) - \varepsilon \nabla \cdot (|\nabla \varphi|^{q-2}\nabla \varphi) = 0, \text{ in } (0, t) \times \Omega,$$

where $\varphi = \overline{T}(s, x) := q(t - s, x)$. The solution of such system are showed to belong to the right space $W^r (\Omega_t)$, so that the following reciprocity relation can be obtained

$$(v(t), \varphi_\varepsilon) = (v(0), \varphi_\varepsilon(t)) - \varepsilon \int_0^t (|\nabla \varphi_\varepsilon(\tau)|^{q-2}\nabla \varphi_\varepsilon(\tau), \nabla v)d\tau. \quad (2.6)$$

This relation enables to get global $L^r_0$ estimates for solutions of system (2.2). Since this system depends on two parameters, $\mu > 0$ and $\nu > 0$, and in the limit as $\mu \to 0$ and $\nu \to 0$ leads to system (1.1), the last step is to perform such limit procedures.

The proof of Theorem 1.2 and its corollary is based, as in [19, 27], on the study of the following differential inequality

$$\frac{1}{r_0} \frac{d}{dt}\|u(\tau)\|_{r_0}^2 + \int_\Omega |\nabla u(\tau, x)|^{p(t, x)}|u(\tau, x)|^{r_0 - 2}dx \leq 0, \text{ a.e. in } (0, T), \quad (2.7)$$

together with an iterative argument. In order to get (2.7) firstly we approximate the data, and hence the solution, with a sequence of initial data in $C^0_0(\Omega)$. Since $u_0 \in C^0_0(\Omega)$ implies $u_0 \in L^\infty(\Omega)$, it follows from Theorem 1.1 that $u(t) \in L^\infty(\Omega)$, for any $t \geq 0$, and $u \in C([0, T]; L^\infty(\Omega))$ for any $\bar{r} < \infty$. Then one readily recognizes that $|u|^{r_0 - 2}u \in X^p(\Omega_T)$, for any $r_0 \in [2, \infty)$, so that $u|u|^{r_0 - 2}$ can be used as test function in the weak formulation of (1.1), and one readily gets (2.7). If $u_0$ is just in $L^{r_0}(\Omega)$, denoting by $\{u^n_0\}$ a sequence of functions in $C^0_0(\Omega)$ strongly converging to $u_0$ in $L^{r_0}(\Omega)$ and such that $\|u^n_0\|_{r_0} \leq c\|u_0\|_{r_0}$, and denoting by $\{u^n\}$ the corresponding sequence of weak solutions to problem (1.1), for the sequence $\{u^n\}$ one finds inequality (1.5) with $u^n$ in place of $u$. Then, as the estimate is uniform in $n \in \mathbb{N}$, we can take a subsequence converging to a limit function $u(t)$, that satisfies the same estimate. Finally, that the limit $u$ is solution of (1.1) corresponding to the initial data $u_0 \in L^{r_0}(\Omega)$ is standard.

We like to point out that a different approach could be the study of suitable integral inequalities, as made for instance in [21, 22] in order to prove the regularizing effect in the scalar case with a constant exponent $p$. However, due to the vectorial nature of our problem and to the variable exponent $p(t, x)$, an extension of the quoted technique to our setting seems to be not straightforward.

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