Counterexamples to a conjecture by Gun and co-workers, its correct reformulation and the transcendence of some series

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Abstract

In a recent work, Gun, Murty and Rath have introduced a ‘theorem’ asserting that \( \sum_{n=-\infty}^{\infty} (n + \alpha)^{-k} \) yields a transcendental number for all \( \alpha \in \mathbb{Q} \setminus \mathbb{Z} \), \( k \) being an integer greater than 1. I show here in this short paper that this conjecture is false whenever \( k \) is odd and \( \alpha \) is a half-integer. I also prove that these are the only counterexamples, which allows for a correct reformulation. The resulting theorem implies the transcendence of both the polygamma function at rational entries and certain zeta series.

Key words: Dirichlet series, Cotangent derivatives, Transcendental numbers, Polygamma function, Zeta series

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1. Introduction

In a recent work on the transcendence of the log-gamma function and some discrete periods, Gun, Murty and Rath (GMR) have presented the following ‘theorem’ on the transcendence of certain Dirichlet series (see Theorem 4.1 in Ref. [1]):

**Conjecture 1 (GMR).** Let $\alpha$ be a non-integral rational number and $k > 1$ be a natural number. Then $\sum_{n=-\infty}^{+\infty} \frac{1}{(n+\alpha)^k}$ is a transcendental number.

In a recent work, I have shown that even the main ‘theorem’ in that work (namely, their Theorem 3.1) has an incorrect proof [2]. So, by suspecting that the above series could converge to an algebraic number for some non-integral rational $\alpha$, I have considered the possibility of finding a counterexample to their assertion. After some computational tests, I have found a simple counterexample: the series is **null** (hence an algebraic number) for $\alpha = \frac{1}{2}$ and $k = 3$. Of course, this implies that Conjecture 1 above, is **false**.

Here in this work, I show that the original proof by Gun and co-workers is **invalid** when $\alpha$ is a half-integer and $k$ is an odd integer. By repairing their defective proof, I show that these values of $\alpha$ and $k$ compose an (infinite) set containing all possible counterexamples, which allows for a correct reformulation of their assertion. This yields a theorem that determines whether $\sum_{n=-\infty}^{+\infty} 1/(n+\alpha)^k$ is an algebraic or a transcendental number, for all rational $\alpha$, $\alpha \notin \mathbb{Z}$, and every $k \in \mathbb{Z}$, $k > 1$. My proof is based only upon the periodicity and some basic properties of this series. I also show that

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1The short proof provided in Ref. [1] is, at first sight, quite convincingly.
the theorem has interesting consequences for the transcendence of both the polygamma function at rational entries and certain zeta series.

2. A first simple counterexample

By testing the validity of the GMR conjecture for $k = 3$ and some rational numbers $\alpha \in (0, 1)$, I have found the following counterexample.

**Lemma 1 (First counterexample).** The series

$$
\sum_{n=-\infty}^{+\infty} \frac{1}{(n + \frac{1}{2})^3}
$$

is null, thus an algebraic number.

**Proof.** By writing the above series as the sum of two series, one for the non-negative values of $n$ and the other for the negative ones, which is a valid procedure since the series converges absolutely, one has

$$
\sum_{n=-\infty}^{+\infty} \frac{1}{(n + \frac{1}{2})^3} = \sum_{n=-\infty}^{-1} \frac{1}{(n + \frac{1}{2})^3} + \sum_{n=0}^{+\infty} \frac{1}{(n + \frac{1}{2})^3} = \sum_{n=-\infty}^{-1} \frac{2^3}{(2n + 1)^3} + \sum_{n=0}^{+\infty} \frac{2^3}{(2n + 1)^3}.
$$

Now, by substituting $n = -m$ in the series for $n < 0$ and $n = j - 1$ in the series for $n \geq 0$, one has

$$
\sum_{n=-\infty}^{+\infty} \frac{1}{(n + \frac{1}{2})^3} = -8 \sum_{m=1}^{+\infty} \frac{1}{(2m - 1)^3} + 8 \sum_{j=1}^{+\infty} \frac{1}{(2j - 1)^3} = 0.
$$

\[\square\]
The existence of a counterexample to the GMR conjecture implies that its original statement (see Theorem 4.1 (2) of Ref. [1]) is false. In fact, by scrutinizing the proof furnished there in Ref. [1], I have found some defective points. Their proof reads [1]:

**Proof (GMR incorrect proof).** We know that

\[ \sum_{n=-\infty}^{+\infty} \frac{1}{(n+\alpha)^k} = \frac{1}{\alpha} + \frac{(-1)^k}{(k-1)!} D^{k-1}(\pi \cot \pi z) \big|_{z=\alpha}, \]

where \( D := \frac{d}{dz} \). It is a consequence of a result of Okada [3] that \( D^{k-1}(\pi \cot \pi z) \big|_{z=\alpha} \) is non-zero. But then it is \( \pi^k \) times a non-zero linear combination of algebraic numbers of the form \( \csc(\pi \alpha), \cot(\pi \alpha) \). Thus we have the result. \( \square \)

There in the Okada’s cited work [3], one finds, in its only theorem, the following (correct) linear independence result.

**Lemma 2 (Okada’s theorem).** Let \( k \) and \( q \) be integers with \( k > 0 \) and \( q > 2 \). Let \( T \) be a set of \( \varphi(q)/2 \) representatives mod \( q \) such that the union \( \{T, -T\} \) is a complete set of residues prime to \( q \). Then the real numbers \( D^{k-1}(\cot \pi z)\big|_{z=a/q}, a \in T, \) are linearly independent over \( \mathbb{Q} \).\(^2\)

See Ref. [3] for a detailed proof of this lemma based upon the partial fraction decomposition of \( D^{k-1}(\pi \cot \pi z) \), valid for all \( z \notin \mathbb{Z} \), as well as a theorem by Baker-Birch-Wirsing on cyclotomic polynomials (see his Corollary 1). Note, however, that this lemma says nothing about the cotangent derivatives at \( z = a/q \) with \( q = 2 \). Then, the linear independence over \( \mathbb{Q} \) is

\(^2\)Here, \( \varphi(q) \) is the Euler totient function.
not guaranteed if \( z \) is a half-integer, which is a source of potential counterexamples to the GMR conjecture. Moreover, the proof of Okada’s theorem is based upon the following partial fraction decomposition for his function

\[
F_k(z) = \frac{k}{(-2\pi i)^k} D^{k-1}(\pi \cot \pi z),
\]

valid for all \( z \notin \mathbb{Z} \) (see Eq. (1) of Ref. [3]):

\[
-\frac{k!}{(2\pi i)^k} \sum_{n=-\infty}^{+\infty} \frac{1}{(n + z)^k} = \frac{k}{(-2\pi i)^k} D^{k-1}(\pi \cot \pi z),
\]

which simplifies to

\[
\sum_{n=-\infty}^{+\infty} \frac{1}{(n + z)^k} = \frac{(-1)^{k-1}}{(k-1)!} D^{k-1}(\pi \cot \pi z).
\]

This identity, by itself, shows that the expression for \( \sum_{n=-\infty}^{+\infty} 1/(n + \alpha)^k \) taken into account by Gun and co-workers in their proof [1], as reproduced in our Eq. (3), is incorrect.

3. An infinity of counterexamples

As we are only interested on integer values of \( k, k > 1 \), let us define the following countable set of real functions:

\[
S_k(z) := \sum_{n=-\infty}^{+\infty} \frac{1}{(n + z)^k},
\]

defined for all real \( z \notin \mathbb{Z} \). From this definition, it is easy to deduce the following mathematical properties.

**Lemma 3.** The functions \( S_k(z), k \in \mathbb{Z}, k > 1 \), have the following properties, valid for all \( z \) in its domain (i.e., \( z \in \mathbb{R} \setminus \mathbb{Z} \)):

\footnote{Note that, for all \( m \in \mathbb{Z}, \lim_{z \to m} |S_k(z)| = \infty.\]
(i) All functions $S_k(z)$ are periodic, with an unitary period;

(ii) For even values of $k$, $S_k(z) > 0$;

(iii) All functions $S_k(z)$ are differentiable;

(iv) For odd values of $k$, $S_k(z)$ are strictly decreasing functions.

**Proof.** The property (i) follows from the fact that, $\forall k \in \mathbb{Z}$, $k > 1$, and for all real $z \notin \mathbb{Z}$, $S_k(z+1) = \sum_{n=-\infty}^{+\infty} 1/ [n + (z + 1)]^k = \sum_{m=-\infty}^{+\infty} 1/ (m + z)^k = S_k(z)$. Note that $n$ has been substituted by $m-1$ in the first series. Property (ii) is an obvious consequence of the fact that, for any positive even $k$, every term $1/(n + z)^k$ of the series that defines $S_k(z)$, see Eq. (5), is positive. Property (iii) follows from the fact that the series for $S_k(z)$ in Eq. (5) is term-by-term differentiable with respect to $z$, without restrictions for $z \in (0, 1)$, and that differentiation does not affect the convergence for any $z$ in this interval. This differentiability for $z \in (0, 1)$ can then be extended to all real $z \notin \mathbb{Z}$ — i.e., all points in the domain of $S_k(z)$ — by making use of their periodicity, as established in property (i). Property (iv) follows from a less direct argument. Firstly, from Eq. (4) we deduce that, for any positive integer $m$,

$$S_{2m+2}(z) = \frac{-1}{(2m + 1)!} D^{2m+1} (\pi \cot \pi z)$$

and

$$\frac{d}{dz} S_{2m+1}(z) = \frac{1}{(2m)!} D^{2m+1} (\pi \cot \pi z).$$

This property also follows from the representation of $S_k(z)$ as a sum/difference of two polygamma functions $\psi_k(z)$ with $z \in (0, 1)$, since each $\psi_k(z)$ is differentiable at all points of this interval. This representation will be considered in the next section.
By isolating the cotangent derivative at the right-hand side of Eq. (7) and then substituting it on Eq. (6), one finds that

\[
\frac{d}{dz} S_{2m+1}(z) = -(2m + 1) S_{2m+2}(z) .
\]  

(8)

By property (ii), \( S_{2m+2}(z) > 0 \), thus \( dS_{2m+1}/dz < 0 \) for all real \( z \not\in \mathbb{Z} \). Then, \( S_{2m+1}(z) \) is a strictly decreasing function in all points of its domain. \( \square \)

An immediate consequence of the periodicity of \( S_k(z) \) is the periodic repetition of the null result established for \( \alpha = \frac{1}{2} \) in Lemma 1. This leads to an infinite set of counterexamples to the GMR assertion, as establishes the following theorem.

**Lemma 4 (More counterexamples for \( k = 3 \)).** For every integer \( m \), the series

\[
\sum_{n=-\infty}^{+\infty} \frac{1}{(n + m + \frac{1}{2})^3}
\]

(9)
is null.

**Proof.** From the fact that \( S_3 \left( \frac{1}{2} \right) = 0 \) (see Lemma 1) and from the periodicity of \( S_3(z) \), with an unitary period, as established in Lemma 3, one has \( S_3 \left( m + \frac{1}{2} \right) = 0 \).

\( \square \)

All counterexamples to the GMR original assertion presented hitherto are particular cases of a more general set of counterexamples, as establishes the next lemma.
Lemma 5 (Counterexamples for odd values of $k$). For every odd integer $k$, $k > 1$, and every $m \in \mathbb{Z}$, the series
\[
\sum_{n=-\infty}^{+\infty} \frac{1}{(n + m + \frac{1}{2})^k}
\] (10)
is null.

**Proof.** The proof for $S_k(\frac{1}{2})$ (i.e., for $m = 0$), valid for any odd integer $k$, $k > 1$, is analogue to that developed in Lemma for $k = 3$. By writing the corresponding series as the sum of two series, one for $n < 0$ and the other for $n \geq 0$, one has
\[
S_k(\frac{1}{2}) = \sum_{n=-\infty}^{+\infty} \frac{1}{(n + \frac{1}{2})^k} = \sum_{n=-\infty}^{-1} \frac{1}{(n + \frac{1}{2})^k} + \sum_{n=0}^{\infty} \frac{1}{(n + \frac{1}{2})^k} = \sum_{n=-\infty}^{-1} \frac{2^k}{(2n + 1)^k} + \sum_{n=0}^{\infty} \frac{2^k}{(2n + 1)^k}.
\]

By substituting $n = -m$ in the series for $n < 0$ and $n = j - 1$ in the series for $n \geq 0$, one finds the following null result:
\[
S_k(\frac{1}{2}) = -2^k \sum_{m=1}^{\infty} \frac{1}{(2m - 1)^k} + 2^k \sum_{j=1}^{\infty} \frac{1}{(2j - 1)^k} = 0.
\] (11)
The extension of this null result to all other half-integer values of $z$ follows from the fact that $S_k(z)$ is periodic, with an unitary period (see property (i) of Lemma 3). Then, $S_k(m + \frac{1}{2}) = 0$.

With these properties and counterexamples in hands, we can now reformulate the GMR conjecture.
4. Reformulating the GMR conjecture

Let us present and prove a theorem which determines whether $S_k(\alpha)$ is an algebraic number or not for non-integral rational values of $\alpha$.

**Theorem 1 (Main result).** For any integer $k, k > 1$, and every $\alpha \in \mathbb{Q} \setminus \mathbb{Z}$, the series

$$
\sum_{n=\infty}^{+\infty} \frac{1}{(n+\alpha)^k}
$$

is either an algebraic multiple of $\pi^k$ or null. It is null if and only if $k$ is an odd integer and $\alpha$ is a half-integer.

**Proof.** From Eqs. (4) and (5), we know that, for any integer $k, k > 1$,

$$
S_k(\alpha) = \sum_{n=-\infty}^{+\infty} \frac{1}{(n+\alpha)^k} = \frac{(-1)^{k-1}}{(k-1)!} D^{k-1} (\pi \cot \pi z) \big|_{z=\alpha}.
$$

From Okada’s theorem, as reproduced in Lemma 2, one knows that, for all rational $\alpha \not\in \mathbb{Z}$ and any integer $k, k > 1$, $D^{k-1} (\pi \cot \pi z) \big|_{z=\alpha}$ is non-zero, the only possible exceptions being the half-integer values of $\alpha$ (i.e., non-integer fractions with a denominator equal to 2). From the usual rules for derivatives, it is easy to note that when the above cotangent derivative is non-zero, it is $\pi^k$ times a non-zero linear combination (with integer coefficients) of algebraic numbers of the form $\csc (\pi \alpha)$ and $\cot (\pi \alpha)$. When that derivative is null, the resulting equation $S_k(\alpha) = 0$ has no real roots if $k$ is even, according to property (ii) of Lemma 3. For odd values of $k$, on the other hand, all half-integer values of $\alpha$ are roots of $S_k(\alpha) = 0$, as establishes Lemma 5.

All that rests is to prove that the half-integers are the only roots of $S_{2\ell+1}(\alpha) = 0, \ell$ being any positive integer. For this, let us restrict our
attention to the open interval \((0,1)\). Note that \(\alpha = \frac{1}{2}\) belongs to the interval \((0,1)\) and is a root of \(S_{2\ell+1}(\alpha) = 0\) for all positive integer \(\ell\), as guaranteed by Lemma 5. From properties (iii) and (iv) of Lemma 3, we know that \(S_{2\ell+1}(\alpha)\) is a strictly decreasing differentiable function, for all real \(\alpha \in (0,1)\). Then, \(S_{2\ell+1}(\alpha_1) \neq S_{2\ell+1}(\alpha_2)\) for all distinct \(\alpha_1, \alpha_2 \in (0,1)\). In particular, \(S_{2\ell+1}(\alpha)\) cannot be null for two distinct values of its argument, both belonging to \((0,1)\). It follows that there is at most one real root in \((0,1)\). Therefore, \(\alpha = \frac{1}{2}\) is the only root in the interval \((0,1)\). Finally, the periodicity of \(S_{2\ell+1}(\alpha)\) guarantees that \(\alpha = \frac{1}{2} + m\) (i.e., the half-integers) are the only real solutions for \(S_{2\ell+1}(\alpha) = 0, \alpha \notin \mathbb{Z}\).

\[\blacksquare\]

From the fact that \(\pi\) is a transcendental number, as first proved by Lindemann (1882), it follows that

**Corollary 1 (Reformulation of the GMR conjecture).** For any integer \(k, k > 1\), and every \(\alpha \in \mathbb{Q} \setminus \mathbb{Z}\), the series

\[
\sum_{n=-\infty}^{+\infty} \frac{1}{(n+\alpha)^k}
\]  

(14)

is either null or a transcendental number. It is null if and only if \(k\) is an odd integer and \(\alpha\) is a half-integer.

Another interesting consequence of Theorem 1 comes from the fact that we can easily write the cotangent derivatives in Eq. (4) in terms of the polygamma function \(\psi_k(z) := \psi^{(k)}(z) = d^k \psi(z)/dz^k\), where \(\psi(z) := \frac{d}{dz} \ln \Gamma(z)\)
is the so-called digamma function. From the reflection formula for $\psi_k(z)$, namely \[ \psi_k(1 - z) - (-1)^k \psi_k(z) = (-1)^k D^k(\pi \cot \pi z), \] valid for all non-negative integers $k$, with $\psi_0(z) := \psi(z)$, and $z \not\in \mathbb{Z}$, and taking into account Eqs. (4) and (5), one finds \[ S_k(\alpha) = \frac{\psi_{k-1}(1 - \alpha) + (-1)^k \psi_{k-1}(\alpha)}{(k - 1)!}. \] (16)

From Theorem 1, we know that, for any integer $k > 1$ and every $\alpha \in \mathbb{Q} \setminus \mathbb{Z}$, $S_k(\alpha)$ is either null or an algebraic multiple of $\pi^k$. Of course, the same conclusion is valid for $(k - 1)! S_k(\alpha)$. By taking Eq. (16) into account, we have the following result.

**Corollary 2 (Transcendence of the polygamma function).** For any positive integer $k$ and every $\alpha \in \mathbb{Q} \setminus \mathbb{Z}$, \[ \psi_k(1 - \alpha) - (-1)^k \psi_k(\alpha) \] (17) is either null or an algebraic multiple of $\pi^{k+1}$. It is null if and only if $\alpha$ is a half-integer and $k$ is even.

Finally, let us make use of the above corollaries for establishing a result on the algebraic nature of certain zeta series. In Eq. (7) of Sec. 1.41 of Ref. [6], one finds the Taylor series expansion \[ z \cot z = \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n} B_{2n}}{(2n)!} z^{2n}, \] (18)
which converges for all real \( z \) with \(|z| < \pi\). By exchanging \( z \) by \( \pi z \) and making use of the Euler’s formula for even zeta values, namely

\[
\zeta(2n) = (-1)^{n-1} \frac{2^{2n-1} B_{2n} \pi^{2n}}{(2n)!},
\]

where \( \zeta(m) := \sum_{n=1}^{\infty} 1/n^m \) is the Riemann zeta function, it is easy to deduce that

\[
\pi \cot (\pi z) = 1/z - 2 \sum_{n=1}^{\infty} \zeta(2n) z^{2n-1},
\]

with the series at the right-hand being convergent for all real \( z \) with \(|z| < 1\).

From this equation, by calculating the successive derivatives on both sides, it is easy to derive the following general formulae for the derivatives of order \( m \), \( m \) being any non-negative integer:

\[
D^{2m}(\pi \cot \pi z) = \frac{(2m)!}{z^{2m+1}} - 2 \sum_{n=m+1}^{\infty} \zeta(2n) \cdot (2n-1) \cdot \cdots \cdot (2n-2m) z^{2n-2m-1}
\]

(21)

and

\[
D^{2m+1}(\pi \cot \pi z) = -\frac{(2m+1)!}{z^{2m+2}} - 2 \sum_{n=m+1}^{\infty} \zeta(2n) \cdot (2n-1) \cdot \cdots \cdot (2n-2m-1) z^{2n-2m-2}
\]

(22)

From Eq. (1) and Corollary 1, one readily deduces that

**Corollary 3 (Transcendence of certain zeta series).** For any positive integer \( m \) and every rational \( z \in (-1, 1) \setminus \{0\} \), both the zeta series

\[
\sum_{n=m+1}^{\infty} \zeta(2n) \cdot (2n-1) \cdot \cdots \cdot (2n-2m) z^{2n-2m-1}
\]

(23)

For \( z = 0 \), the series is null, but Eq. (20) is not valid due to an obvious division by zero.
and
\[ \sum_{n=m+1}^{\infty} \zeta(2n) \cdot (2n - 1) \cdot \ldots \cdot (2n - 2m - 1) \cdot z^{2n-2m-2} \quad (24) \]
 converge to some algebraic multiple of \( \pi^{2m+1} \) and \( \pi^{2m+2} \), respectively, the only exceptions being for the former series with \( z = \pm \frac{1}{2} \), for which it is null.

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