DENSITY PROBLEMS FOR SECOND ORDER SOBOLEV SPACES AND CUT-OFF FUNCTIONS ON MANIFOLDS WITH UNBOUNDED GEOMETRY

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Abstract. We consider complete non-compact manifolds which satisfy either a sub-quadratic growth of the norm of the Riemann curvature, or a sub-quadratic growth of both the norm of the Ricci curvature and the squared inverse of the injectivity radius. We show the existence on such a manifold of a distance-like function with bounded gradient and mild growth of the Hessian. As a main application, we prove that smooth compactly supported functions are dense in $W^{2,p}$. The result can be improved for $p = 2$ avoiding both the upper bound on the Ricci tensor, and the injectivity radius assumption. As a further application we prove a new disturbed Sobolev inequality on manifolds with possibly unbounded curvature.

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1. Introduction and main results

Let $(M^m, g)$ be a smooth, complete, possibly non-compact, Riemannian manifold without boundary. For $p \in [1, \infty)$ and $k \geq 2$, denote by $W^{k,p}(M)$ the space of functions on $M$ whose (weak) derivatives of order 0 to $k$ have a finite $L^p$ norm. Moreover, let $W^{k,p}_0(M)$ be the closure of $C_\infty(M)$ in $W^{k,p}(M)$.

A classical result in geometric analysis states that for any complete Riemannian manifold, $W^{1,p}_0(M) = W^{1,p}(M)$ for any $p \in [1, \infty)$. In this paper, we are interested in the following

Problem 1.1. Under which (geometric) assumptions on $M$ does one have that $W^{2,p}_0(M) = W^{2,p}(M)$?

Classical results on this topic can be found in [4], [21] and references therein. In the following proposition we collect the most up-to-date achievements: point (I) was shown by E. Hebey, [20]; point (II) was proved by B. Güneysu in [15, Proposition III.18]; point (III) is due to L. Bandara, [5] (for an alternative proof see also [15, Proposition III.18]).

Proposition 1.2. Let $(M^m, g)$ be a complete Riemannian manifold.

(I) If $|\text{Ric}_g| \leq C$ for some constant $C \geq 0$ and $\text{inj}_g(M) > 0$, then for every $p \in [1, \infty)$ we have $W^{2,p}_0(M) = W^{2,p}(M)$.

(II) If $|\text{Riem}_g| \leq C$ for some constant $C \geq 0$, then for every $p \in [1, \infty)$ we have $W^{2,p}_0(M) = W^{2,p}(M)$.

2010 Mathematics Subject Classification. 46E35, 53C21.
Key words and phrases. Density problems, Sobolev spaces on manifolds, cut-off functions, disturbed Sobolev inequalities.
(III) If \( \text{Ric}_g \geq -C \) for some constant \( C \geq 0 \) (no assumptions on the injectivity radius!) then \( W^{2,2}_0(M) = W^{2,2}(M) \).

Often in the applications it is useful to relax the assumptions on the geometry of the manifold, allowing to the bounds on the curvature and on the injectivity radii to be more flexible. The main purpose of this paper is to investigate density problems for second order Sobolev spaces under not necessarily constant bounds on the curvature and (when it is the case) letting the injectivity radii suitably decay at infinity. In particular we obtain the following

**Theorem 1.3.** Let \((M, g)\) be a complete Riemannian manifold and \( o \in M \) a fixed reference point. Set \( r(x) = \text{dist}_g(x, o) \). Suppose that one of the following curvature assumptions holds

(a) for some \( i_0 > 0 \) and \( D > 0 \),
\[
|\text{Ric}_g|(x) \leq D^2(1 + r(x)^2), \quad \text{inj}_g(x) \geq \frac{i_0}{D(1 + r(x))} > 0 \quad \text{on } M.
\]

(b) for some \( D > 0 \),
\[
|\text{Sect}_g|(x) \leq D^2(1 + r(x)^2).
\]

Then, for every \( p \in [1, \infty) \), we have \( W^{2,p}_0(M) = W^{2,p}(M) \).

Moreover, in the special case \( p = 2 \), we can obtain the following improvement of [5, Theorem 1.1], where neither an upper bound on the Ricci curvature nor the assumption on the injectivity radii are required. As it is customary in this case the Bochner formula plays a key role.

**Theorem 1.4.** Let \((M, g)\) be a complete Riemannian manifold, \( o \in M \), \( r(x) = \text{dist}_g(o, x) \), and suppose that for some \( D > 0 \)
\[
\text{Ric}_g(x) \geq -D^2(1 + r(x)^2).
\]

Then \( W^{2,2}_0(M) = W^{2,2}(M) \).

To prove our density results we employ the method introduced in [14], [17]. The key step in the proof is the construction on the manifold of special sequences of cut-off functions, with a suitable control on the gradient and on second order derivatives. In this regard let us notice that, as a matter of fact, Proposition 2.2 can be seen as a consequence of the existence of such sequences of cut-off functions under the assumptions at hand. More precisely, the result in point (II) uses point (iii) of Proposition 2.3 below, while the result in point (III) can be seen as a consequence of the general criterion (point (a)) given in Proposition 2.4; Theorem B in [17], and point (i) in Proposition 2.3. Finally, also point (I) can be proved using the cut-off functions given by point (iv) in Proposition 2.3 together with Proposition 2.4 (b)\(^1\).

Note that such cut-off functions can be tailored starting from suitable smooth exhaustion functions whose gradient and Hessian are controlled in terms of explicit functions of the distance from a fixed reference point. In this direction, in this paper we prove the following

**Theorem 1.5.** Let \((M, g)\) be a complete Riemannian manifold and \( o \in M \) a fixed reference point, \( r(x) = \text{dist}_g(x, o) \). Suppose that one of the following curvature assumptions holds

(a) for some \( 0 < \eta \leq 1 \), some \( D > 0 \) and some \( i_0 > 0 \),
\[
|\text{Ric}_g|(x) \leq D^2(1 + r(x)^2)\eta, \quad \text{inj}_g(x) \geq \frac{i_0}{D(1 + r(x))}\eta > 0 \quad \text{on } M.
\]

(b) for some \( 0 < \eta \leq 1 \) and some \( D > 0 \),
\[
|\text{Sect}_g|(x) \leq D^2(1 + r(x)^2)\eta.
\]

Then there exists an exhaustion function \( H \in C^\infty(M) \) such that for some positive constant \( C > 1 \) independent of \( x \) and \( o \), we have on \( M \) that

(i) \( C^{-1}r(x) \leq H(x) \leq C \max \{r(x), 1\} \);

(ii) \( |\nabla H|(x) \leq C \);

(iii) \( |\text{Hess } H|(x) \leq C \max \{r(x)\eta, 1\} \).

\(^1\)Note however that the original proof by Hebey uses a different argument based on a delicate covering technique.
To obtain distance-like exhaustion functions with controlled gradient and Hessian, the previous strategy introduced in [26] and adopted also in [21] was the following. One starts with a distance-like function with bounded gradient (which always exists on complete manifolds, [12]) and let it evolve under the heat flow on \( M \). The evolution at a fixed positive time (say \( t = 1 \)) preserves the linear growth of the initial datum, as well as the boundedness of the gradient. Moreover Euclidean parabolic Schauder estimates, applied in harmonic coordinates charts of fixed radius centered at any \( x \in M \), permit to control the \( L^\infty \)-norm of the Hessian. However, when the Ricci curvature is unbounded and the injectivity radius is possibly null, the estimates in the heat flow method are difficult to implement, and the parabolic method apparently does not permit to get Theorem 1.5 in its more general assumptions. Accordingly, we use here a different strategy.

The starting point is a recent result established by D. Bianchi and A. G. Setti, [6], where exhaustion functions with controlled gradient and Laplacian are constructed on manifolds with Ricci curvature bounded from below by a possibly unbounded non-positive function of the distance from a fixed reference point, without any assumption on the injectivity radius. As in Tam’s result, our strategy is then, roughly speaking, to use harmonic coordinates in order to gain a control on the whole Hessian of these exhaustion functions. An application of Schauder estimates, Sobolev embeddings and a local Calderón-Zygmund inequality permits then to conclude the proof. Note that this latter part is technically more involved than in the parabolic case, since we have to estimates solutions of a semilinear (elliptic) equation instead of a homogenous (parabolic) equation. To deal with the non-uniform bounds on \( \text{Ric} \) and \( \text{inj} \), everything is done locally, in a suitable ball, with radius decaying at infinity, where we can guarantee the existence of harmonic coordinates with respect to which we have a good control on the metric.

We mention that these techniques can be naturally extended to study density problems for higher order Sobolev spaces on manifolds with unbounded geometry. These results will be presented in a forthcoming paper.

As a further application, the distance-like function \( H \) exhibited in Theorem 1.5 permits to deduce the validity of a disturbed Sobolev inequality on non-compact manifolds with possibly unbounded Ricci curvature and possibly vanishing global injectivity radius. It is well known that on a complete non-compact manifold with Ricci curvature bounded from below and a lower bound on \( \text{vol}(B_1(x)) \) uniform in \( x \), one has the continuous embedding \( W^{1,p}(M) \subset L^{p/(n-p)}(M) \). [27]. By a result of Croke, the assumption on the volumes of unitary balls is implied by a positive lower bound on the injectivity radius, [10]. Under a conformal change of the Riemannian metric the Ricci curvature modifies following an equation which involves the gradient and the Hessian of the conformal factor. Accordingly, in the assumption of Theorem 1.5 we can use the distance-like function \( H \) to get a metric in the same conformal class of \((M,g)\) with bounded Ricci curvature and a lower bound on the volumes of unitary balls\footnote{Because of [23], this result is true without curvature and injectivity radius assumptions. The main achievement here is the second order control of the conformal factor.} Finally one uses conformal Laplacians and the control on the scalar curvatures to deduce a Sobolev-type inequality on \((M,g)\).

**Theorem 1.6.** Let \((M^m,g)\) be a smooth, complete non-compact Riemannian manifold without boundary of dimension \( m \geq 3 \). Let \( o \in M \), \( r(x) = \text{dist}_g(x,o) \) and suppose that for some \( 0 < \eta \leq 1 \), \( D > 0 \) and some \( i_0 > 0 \),

\[
|\text{Ric}_g|(x) \leq D^2(1 + r(x)^2)^\eta \equiv \lambda^2(r(x)), \quad \text{inj}_g(x) \geq \frac{i_0}{D(1 + r(x))^\eta}.
\]

Then there exists constants \( A_1 > 0 \), \( A_2 > 0 \) such that for all \( \varphi \in C^\infty_c(M) \) it holds

\[
\left( \int_M \varphi^{2m-2}d\text{vol}_g \right)^{m-2} \leq A_1 \int_M |\nabla \varphi|^2d\text{vol}_g + A_2 \int_M r^{2\eta}\varphi^2d\text{vol}_g \tag{1}
\]

**Remark 1.7.** Essentially the same proof permits to deduce the validity of (1) provided that \( |\text{Sect}_g|(x) \leq D^2(1 + r(x)^2)^\eta \) and

\[
\text{vol}_g(B_{r^{-\eta}(x)})(x) \geq \frac{E}{(1 + r(x))^{m\eta}}. \tag{2}
\]

As a further application, the distance-like function \( H \) exhibited in Theorem 1.5 permits to deduce the validity of a disturbed Sobolev inequality on non-compact manifolds with possibly unbounded Ricci curvature and possibly vanishing global injectivity radius. It is well known that on a complete non-compact manifold with Ricci curvature bounded from below and a lower bound on \( \text{vol}(B_1(x)) \) uniform in \( x \), one has the continuous embedding \( W^{1,p}(M) \subset L^{p/(n-p)}(M) \). [27]. By a result of Croke, the assumption on the volumes of unitary balls is implied by a positive lower bound on the injectivity radius, [10]. Under a conformal change of the Riemannian metric the Ricci curvature modifies following an equation which involves the gradient and the Hessian of the conformal factor. Accordingly, in the assumption of Theorem 1.5 we can use the distance-like function \( H \) to get a metric in the same conformal class of \((M,g)\) with bounded Ricci curvature and a lower bound on the volumes of unitary balls\footnote{Because of [23], this result is true without curvature and injectivity radius assumptions. The main achievement here is the second order control of the conformal factor.} Finally one uses conformal Laplacians and the control on the scalar curvatures to deduce a Sobolev-type inequality on \((M,g)\).
It is a natural question whether \( \overline{\text{Ric}} \geq -r^{2n} \) would suffice to prove Theorem 1.6. Note that both the upper bound on Ricci and the lower bound on the injectivity radius are used to get harmonic radius estimates. On the other hand, the weight \( r^{2n} \) in (1) probably cannot be avoided since an unweighted Sobolev inequality would imply a lower bound on \( \text{vol}_g(B_1(x)) \), [27].

**Remark 1.8.** Disturbed Sobolev inequalities were already obtained in the original paper by Varopoulos, and subsequently improved by Hebey, [27, 20]. Namely, they proved that if \( \text{Ric} \geq -(m-1)D \) for some positive constant \( D \), then

\[
\left( \int_M \varphi |x|^p \, d\text{vol}_g \right)^{\frac{m-p}{mp}} \leq A \left( \int_M |\nabla \varphi|^p \, d\text{vol}_g \right)^{\frac{1}{p}} + A \left( \int_M \varphi^p v^\beta \, d\text{vol}_g \right)^{\frac{1}{p}},
\]

where \( 1 \leq p \leq m \), \( \alpha \) and \( \beta \) are real constants satisfying \( \beta/p - \alpha(m-p)/(mp) \geq 1/m \), and \( v(x) = (\text{vol}_g(B_1(x)))^{-1} \). With respect to (3), on the one hand Theorem 1.6 permits non-constant lower bounds on the Ricci curvature. On the other hand a further assumption on the injectivity radius is required, and the sole case \( p = 2 \) is covered. Moreover the two inequalities seem different in nature, since in (3) the same function \( v^\beta \) weights both the energy and the potential terms at RHS. Note that more general disturbed Sobolev-type inequalities recovering both (1) and Varopoulos-Hebey’s inequality (for \( p = 2 \)) could be obtained combining (3) with the conformal method used in the proof of Theorem 1.6.

The organization of this paper is as follows. In Section 2 we recap some known results about the existence of special sequences of cut-off functions and see how these can be used to obtain density results for second order Sobolev spaces. In Section 3 we see explicitly how, on a suitable ball centered at each point of a manifold with a non-constant Ricci curvature bound and suitably decaying injectivity radii, we can control in harmonic coordinates the metric. In Section 4 we construct good exhaustion function in this generality, first dealing with sub-quadratic Ricci curvature growth and suitably decaying injectivity radii and then with the situation in which we have sub-quadratic sectional curvature growth (and no assumptions on the injectivity radii). Starting from these exhaustion functions, in Section 5 we construct the cut-off functions needed for the proof of our first density result (Theorem 1.3). In Section 6 we focus on the case \( p = 2 \) and give the proof of Theorem 1.6. In this case we are assuming only a quadratic negative lower bound on the Ricci curvature. Making use of the weak Laplacian cut-off functions constructed in [6], under these assumptions, we are able to prove the density result by applying the divergence theorem to a suitable compactly supported vector field together with the Bochner formula. Finally in Section 7 we prove Theorem 1.6.

2. **Sequences of Cut-off functions and applications to density problems**

Sequences of Laplacian and Hessian cut-off functions where defined in [14] and [17]. Here we will need to introduce also the slightly different notions of weak Laplacian and weak Hessian cut-off functions. Namely

**Definition 2.1.** A complete Riemannian manifold \((M, g)\) is said to admit a sequence \( \{\chi_n\} \subset C^\infty_c(M) \) of Laplacian cut-off functions, if \( \{\chi_n\} \) has the following properties:

- (C1) \( 0 \leq \chi_n(x) \leq 1 \) for all \( n \in \mathbb{N} \), \( x \in M \);
- (C2) for all compact \( K \subset M \), there is a \( n_0(K) \in \mathbb{N} \) such that for all \( n \geq n_0(K) \), one has \( \chi_n|_K = 1 \);
- (C3) \( \|\nabla \chi_n\|_\infty \to 0 \) as \( n \to \infty \);
- (C4) \( \|\Delta \chi_n\|_\infty \to 0 \) ad \( n \to \infty \).

Furthermore, \((M, g)\) is said to admit a sequence \( \{\chi_n\} \subset C^\infty_c(M) \) of weak Laplacian cut-off functions, if \( \{\chi_n\} \) satisfies (C1), (C2), and there exist constants \( A_1, A_2 \) such that, for all \( n \in \mathbb{N} \),

- (C3’) \( \|\nabla \chi_n\|_\infty \leq \frac{A_1}{n} \);
- (C4’) \( \|\Delta \chi_n\|_\infty \leq \frac{A_2}{n} \).

**Definition 2.2.** \((M, g)\) is said to admit a sequence \( \{\chi_n\} \subset C^\infty_c(M) \) of Hessian cut-off functions, if \( \{\chi_n\} \) satisfies (C1), (C2), (C3), and

- (C4”) \( \|\text{Hess}(\chi_n)\|_\infty \to 0 \) as \( n \to \infty \).
Furthermore, \((M, g)\) is said to admit a sequence \(\{\chi_n\} \subset C^\infty_c(M)\) of weak Hessian cut-off functions, if \(\{\chi_n\}\) satisfies (C1), (C2), and there exist constants \(A_1, A_2\) such that, for all \(n \in \mathbb{N}\),

\[
\begin{align*}
(C3') \quad & \|\nabla \chi_n\|_\infty \leq A_1; \\
(C4'') \quad & \|\text{Hess} \chi_n\|_\infty \leq A_2.
\end{align*}
\]

The following proposition (partially taken from \cite{15}) should give the state of the art on the existence of such sequences of cut-off functions: point (i) follows by \cite{25} (see also \cite{14} for an alternative proof in the case \(C = 0\)), point (ii) was proved in \cite{6}; point (iii) is a consequence of \cite{8}, point (iv) was proven in \cite{24} sharpening a construction given in \cite{26}, (v) is a consequence of \cite{22} Theorem 1.3.

**Proposition 2.3.** Let \((M^m, g)\) be a complete Riemannian manifold.

(i) If \((M, g)\) has Ric\(g \geq -C\) for some constant \(C \geq 0\), then \(M\) admits a sequence of Laplacian cut-off functions.

(ii) More generally, fix a reference point \(o\) in \((M, g)\), and denote by \(r(x) = \text{dist}_g(x, o)\). If

\[
\text{Ric}_g \geq -(m - 1)\frac{C^2}{(1 + r^2)^{\alpha/2}},
\]

with \(\alpha \in [-2, 2]\) then, for every \(R \geq 1\) when \(\alpha \in [-2, 2)\) and for every \(R > 0\) when \(\alpha = 2\), and for every \(\gamma > \Gamma(\alpha, k, d)\) there exists a sequence \(\{\phi_R\} \subset C^\infty_c(M)\) of cut-off functions such that

1. \(\phi_R \equiv 1\) on \(B_R(o)\);
2. \(\text{supp}(\phi_R) \subset B_{\gamma R}(o)\);
3. \(|\nabla \phi_R| \leq \frac{C_1}{R}\);
4. \(|\Delta \phi_R| \leq \frac{C_2}{R^{1+\alpha/2}}\).

In particular, for \(\alpha \in (-2, 2)\) this is a sequence of Laplacian cut-off functions.

(iii) If \(\|\text{Ric}_g\|_\infty < \infty\) then there exists a sequence of Hessian cut-off functions.

(iv) If \(\|\text{Ric}_g\|_\infty < \infty\) and \(\text{inj}(M) > 0\) then there exists a sequence of Hessian cut-off functions

(v) There exist \(\varepsilon(m)\) and \(\lambda'(m)\) such that if \(-\Lambda' \leq \text{Ric}_g \leq \Lambda\) and \(\text{vol}_g(B_1(x)) \geq (1 - \varepsilon)\omega_m\), for all \(x \in M\) and for some \(\Lambda \geq 0\), then there exists a sequence of Hessian cut-off functions.

Following the terminology introduced in \cite{17}, we recall that a \(L^p\)-Calderón-Zygmund inequality (CZ\((p)\)) is said to hold on \((M, g)\) for some \(1 < p < \infty\) if there are constants \(C_1 \geq 0\) and \(C_2 > 0\), such that for all \(u \in C^\infty_c(M)\) one has

\[
\text{CZ}(p) \quad \|\text{Hess}(u)\|_{L^p} \leq C_1 \|u\|_{L^p} + C_2 \|\Delta u\|_{L^p}.
\]

Note that, as in \cite{17}, here we have left out the case \(p = 1\), since such an inequality indeed fails for the Euclidean Laplace operator in \(\mathbb{R}^m\).

The following result was proven in \cite{14}; see also Proposition 3.6 in \cite{17}.

**Proposition 2.4** (Theorem 2.6 in \cite{14} and Proposition 3.6 in \cite{17}). (a) Assume that (CZ\((p)\)) holds for some \(1 < p < \infty\) and that \(M\) admits a sequence of Laplacian cut-off functions. Then one has that \(W^2_{0, p}(M) = W^{2, p}(M)\).

(b) If \(M\) admits a sequence of weak Hessian cut-off functions, then one has \(W^2_{0, p}(M) = W^{2, p}(M)\) for all \(1 < p < \infty\).

**Remark 2.5.** Actually, what is asked in point (b) of Proposition 3.6 in \cite{17} is the existence of a sequence of genuine Hessian cut-off functions. Here we observe that what is really needed for the density result is that the gradient and the Hessian of the cut-offs are uniformly bounded. Indeed, first note that \(C^\infty_c(M) \cap W^{2, p}(M)\) is dense in \(W^{2, p}(M)\) (see for instance \cite{13}, Theorem 2). Then, given a smooth \(f \in W^{2, p}(M)\), pick a sequence \(\{\chi_n\}\) of weak Hessian cut-off functions and define \(f_n = \chi_n f\). Proceeding as in \cite{17}, we get that

\[
\begin{align*}
(4) \quad & \|f_n - f\|_{L^p} = \|(1 - \chi_n)f\|_{L^p} \\
(5) \quad & \|\nabla (f_n - f)\|_{L^p} \leq \|f \nabla \chi_n\|_{L^p} + \|(1 - \chi_n)\nabla f\|_{L^p} \\
(6) \quad & \|\text{Hess} (f_n - f)\|_{L^p} \leq \|f \text{Hess} (\chi_n)\|_{L^p} + \|\nabla \chi_n\|_{L^p} + \|(1 - \chi_n)\text{Hess}(f)\|_{L^p}
\end{align*}
\]
Both \((1 - \chi_n), \nabla \chi_n\) and \(\text{Hess}(\chi_n)\) are uniformly bounded and supported in \(\text{supp}(1 - \chi_n)\). Moreover by property (C2), given any compact set \(K \subset M\), we have that \(\text{supp}(1 - \chi_n) \subset M \setminus K\) for \(n\) large enough. Since \(f \in W^{2,p}(M)\) this permits to conclude that all the terms at the RHS of \((4), (5)\) and \((6)\) tend to 0 as \(n \to \infty\).

3. Harmonic Coordinates and Rescalings

Recall that a local coordinate system \(\{x^i\}\) is said to be harmonic if for any \(i\), \(\Delta_g x^i = 0\). The harmonic radius is then defined as follows.

**Definition 3.1.** Let \((M^m, g)\) be a smooth Riemannian manifold and let \(x \in M\). Given \(Q > 1\), \(k \in \mathbb{N}\), and \(\alpha \in (0, 1)\), we define the \(C^{k,\alpha}\) harmonic radius at \(x\) as the largest number \(r_H = r_H(Q, k, \alpha)(x)\) such that on the geodesic ball \(B_{r_H}(x)\) of center \(x\) and radius \(r_H\), there is a harmonic coordinate chart such that the metric tensor is \(C^{k,\alpha}\) controlled in these coordinates. Namely, if \(g_{ij}, i, j = 1, \ldots, m\), are the components of \(g\) in these coordinates, then

\[
\begin{align*}
(1) & \quad Q^{-1}\delta_{ij} \leq g_{ij} \leq Q\delta_{ij} \quad \text{as bilinear forms}; \\
(2) & \quad \sum_{1 \leq |\beta| \leq k} r_H^{|\beta|} \sup |\partial_\beta g_{ij}(y)| + \sum_{|\beta| = k} r_H^{k+\alpha} \sup_{y \neq z} \frac{|\partial_\beta g_{ij}(z) - \partial_\beta g_{ij}(y)|}{d_{g}(y, z)^\alpha} \leq Q - 1.
\end{align*}
\]

We then define the (global) harmonic radius \(r_H(Q, k, \alpha)(M)\) of \((M, g)\) by

\[
r_H(Q, k, \alpha)(M) = \inf_{x \in M} r_H(Q, k, \alpha)(x)
\]

where \(r_H(Q, k, \alpha)(x)\) is as above.

As a consequence of \([2]\) we have the validity of the following

**Proposition 3.2.** Let \(\alpha \in (0, 1), Q > 1, \delta > 0\). Let \((M^m, g)\) be a smooth Riemannian manifold, and \(\Omega\) an open subset of \(M\). Set

\[
\Omega(\delta) = \{x \in M \mid d_g(x, \Omega) < \delta\}.
\]

Suppose that

\[
|\text{Ric}_g(x)| \leq 1 \quad \text{and} \quad \text{inj}_g(x) \geq i \quad \text{for all} \quad x \in \Omega(\delta),
\]

then, there exists a positive constant \(C_{HR} = C_{HR}(m, Q, \kappa, \alpha, \delta, i)\), such that for any \(x \in \Omega\)

\[
r_H(Q, 1, \alpha)(x) \geq C_{HR}.
\]

Since

\[
\partial_\beta g^i_j = -\partial_\beta g_{lk} g^l_i g^{k}_j,
\]

note that under these assumptions, for every \(x \in \Omega\), on \(B_{r_H}(x)\) we have also that:

\[
\begin{align*}
(1') & \quad Q^{-1}\delta_{ij} \leq g_{ij} \leq Q\delta_{ij}; \\
(2') & \quad \sum_\beta r_H |\partial_\beta g_{ij}(y)| + \sum_\beta r_H^{1+\alpha} \sup_{y \neq z} \frac{|\partial_\beta g_{ij}(z) - \partial_\beta g_{ij}(y)|}{d_{g}(y, z)^\alpha} \leq C(Q),
\end{align*}
\]

for some constant \(C(Q)\), depending only on \(Q\).

Fix \(o \in M\), denote by \(r(x) = d_g(x, o)\) and assume that \(|\text{Ric}_g(x)| \leq \lambda^2(r(x))\), for some non-decreasing \(\lambda : \mathbb{R} \to \mathbb{R}^+\), and that \(\text{inj}_g(x) \geq \frac{i_0}{\lambda(r(x))}\), for some uniform constant \(i_0 > 0\). Given \(x \in M \setminus \overline{B_2^2(o)}\), for any \(y \in B_1^2(x)\) we have that \(|\text{Ric}_g(y)| \leq \lambda^2(r(x) + 1)\) and \(\text{inj}_g(y) \geq \frac{i_0}{\lambda(r(x) + 1)}\). Denoting by \(\lambda_1 = \lambda(r(x) + 1)\), we set

\[
g_\lambda = \lambda_1^2 g.
\]

Then

\[
\forall y \in B_{\lambda_1}^2(x), \quad |\text{Ric}_{g_\lambda}(y)| \leq 1, \quad \text{inj}_{g_\lambda}(y) \geq \lambda_1 \text{inj}_g(y) \geq \lambda_1 \frac{i_0}{\lambda_1} = i_0.
\]

By Proposition 4.2 we have that there exists a \(C_{HR}(m, Q, \alpha, \delta, i_0)\) such that \(\forall y \in B_{\lambda_1}^{2\alpha}(x)\), on \(B_{\lambda_1}^{2\alpha}(y)\), there exist harmonic coordinates for which the metric \(g_\lambda\) satisfies the analogous relations to \((1), (1'), (2)\) and \((2')\) with \(r_H = C_{HR}\) (and \(k = 1\)).

Hence for every \(y \in B_{\lambda_1}^2(x)\) we can find on \(B_{\lambda_1}^2\) harmonic coordinates with respect to which

\[
(\text{i}) \quad Q^{-1}\lambda_1^{-2}\delta_{ij} \leq g_{ij} \leq Q\lambda_1^{-2}\delta_{ij};
\]
(ii) \( \sum_{\beta} \lambda_{1}^{2} C_{HR} \sup |\partial_{\beta} g_{ij}(y)| + \sum_{\beta} C_{HR}^{1+\alpha} \lambda_{1}^{2-\alpha} \sup_{y \neq z} \frac{|\partial_{\beta} g_{ij}(z) - \partial_{\beta} g_{ij}(y)|}{d_{g}(y,z)^{\alpha}} \leq Q - 1; \)

and thus also

(i') \( Q^{-1} \lambda_{1}^{2} \delta_{ij} \leq g_{ij} \leq Q \lambda_{1}^{2} \delta_{ij}; \)

(ii') \( \sum_{\beta} \lambda_{1}^{-2} C_{HR} \sup |\partial_{\beta} g_{ij}(y)| + \sum_{\beta} C_{HR}^{1+\alpha} \lambda_{1}^{-2-\alpha} \sup_{y \neq z} \frac{|\partial_{\beta} g_{ij}(z) - \partial_{\beta} g_{ij}(y)|}{d_{g}(y,z)^{\alpha}} \leq C(Q), \)

for some constant \( C(Q). \)

4. Construction of controlled exhaustion functions

The key step in our construction of sequences of (weak) Hessian cut-off functions is to exhibit suitable smooth exhaustion functions with a good explicit control on the gradient and the Hessian in terms of the distance function \( r \) to a fixed reference point. It is already known that this is possible when \( \| \text{Riem} \|_{\infty} < \infty, \) or when \( \| \text{Ric} \|_{\infty} < \infty \) and \( \text{inj}_{g}(M) > 0, \) [24]. For a further recent result see also [22]. Here, we will deal with the situation in which the curvature is controlled by a sub-quadratic function of the distance from a fixed reference point.

4.1. Sub-quadratic Ricci growth. Let \( o \in M, ~ r(x) \doteq \text{dist}_{g}(x, o) \) and suppose that we are in the assumption (i) of Theorem 1.5. Up to change the values of the constants \( D \) and \( i_{0} \), this is equivalent to assume that for some \( 0 < \eta \leq 1, ~ D > 0 \) and some \( i_{0} > 0, \)

\[
\text{(7)} \quad |\text{Ric}_{g}(x)| \leq D^{2}(1 + r(x)^{2})^{\eta} = \lambda^{2}(r(x)), \quad \text{inj}_{g}(x) \geq \frac{i_{0}}{\lambda(r(x))}, \quad \text{on } M.
\]

All over this section, \( C \) will denote real constants greater than 1, all independent of \( x \in M, \) whose explicit values can possibly change from line to line.

**Step 0:** Exhaustion functions with controlled Laplacian.

Let \( h \in C^{\infty}(M) \) be the exhaustion function given in [6, Theorem 2.1]. Then

(i) \( C^{-1} r(x)^{1+\eta} \leq h(x) \leq C \max \{ r(x)^{1+\eta}, 1 \} \) on \( M; \)

(ii) \( |\nabla h| \leq C r^{\eta} \) on \( M \setminus \bar{B}_{1}(o); \)

(iii) \( |\Delta h| \leq C r^{2\eta} \) on \( M \setminus \bar{B}_{1}(o). \)

Moreover, by the construction in the proof of [6, Theorem 2.1], \( h \) is a solution of

\[
\text{(8)} \quad \Delta h = |\nabla h|^{2} - \theta r^{2\eta} \doteq f
\]

on \( M \setminus \bar{B}_{1}(o), \) where \( \theta \) is a positive fixed constant and \( \bar{r} \in C^{\infty}(M) \) is a smooth 1st order approximation of the distance function, which satisfies in particular

- \( C^{-1} r(x) \leq \bar{r}(x) \leq C \max \{ r(x), 1 \} \)
- \( |\nabla \bar{r}| \leq C \)

on \( M. \)

**Step 1:** using harmonic coordinates.

Given \( x \in M \setminus \bar{B}_{2}(0), \) we define \( h_{x} : B_{2}(x) \to \mathbb{R} \) by

\[
h_{x}(y) = h(y) - h(x).
\]

Then \( h_{x}(x) = 0, ~ h_{x} \) satisfies \([5], \) and

- \( |\nabla h_{x}| \leq C r^{\eta}; \)
- \( |\Delta h_{x}| \leq C r^{2\eta}; \)
- \( |\text{Hess } h_{x}| = |\text{Hess } h| \),

Fix now \( \alpha \in (0, 1), ~ Q > 1 \) and a sufficiently small \( \delta > 0. \) By \([7], \) and Section \([3], \) we know that there exists a constant \( C_{HR}(m, Q, \alpha, \delta, i_{0}) \) such that we can find on \( B_{C_{HR}/\lambda_{1}(x)} \) a harmonic chart

\[
\varphi_{H} = (y^{1}, \ldots, y^{m}) : B_{C_{HR}/\lambda_{1}(x)} \to U \subset \mathbb{R}^{m},
\]

such that \( \varphi_{H}(x) = 0, \) and with respect to which

(i) \( Q^{-1} \lambda_{1}^{-2} \delta_{ij} \leq g_{ij} \leq Q \lambda_{1}^{-2} \delta_{ij}; \)
(ii) \[ \sum_{\beta} \lambda_{1}^{2} C_{HR} \sup_{y \neq z} \left| \partial_{\beta} g_{ij}(y) \right| + \sum_{\beta} C_{HR}^{1+\alpha} \lambda_{1}^{2-\alpha} \sup_{y \neq z} \left| \frac{\partial_{\beta} g_{ij}(y) - \partial_{\beta} g_{ij}(y)}{d_{y}(y,z)^{\alpha}} \right| \leq Q - 1; \]

(i') \[ Q^{-1} \lambda_{1}^{2} \delta^{ij} \leq g^{ij} \leq Q \lambda_{1}^{2} \delta^{ij}; \]

(ii') \[ \sum_{\beta} \lambda_{1}^{2-2} C_{HR} \sup_{y \neq z} \left| \partial_{\beta} g^{ij}(y) \right| + \sum_{\beta} C_{HR}^{1+\alpha} \lambda_{1}^{2-\alpha} \sup_{y \neq z} \left| \frac{\partial_{\beta} g^{ij}(y) - \partial_{\beta} g^{ij}(y)}{d_{y}(y,z)^{\alpha}} \right| \leq C_{Q}, \]

for some constant \( C_{Q} \).

**STEP 2:** pointwise Schauder estimate. For any \( \beta > 0 \), denote by \( \mathbb{B}_{\beta}(0) \) the Euclidean ball of radius \( \beta \) centered at the origin. Note that

\[ \mathbb{B}_{C_{HR}/\sqrt{Q}}(0) \subset U = \varphi_{H}(B_{C_{HR}/\lambda_{1}}(x)). \]

Define \( \hat{h}_{x} \equiv h_{x} \circ \varphi_{H}^{-1}, \hat{f} \equiv f \circ \varphi_{H}^{-1}, \) and \( \hat{g}^{ij} \equiv g^{ij} \circ \varphi_{H}^{-1} \). Letting \( \beta \equiv C_{HR}/(2\sqrt{Q}) \), define \( \hat{h}_{\beta,x} : \mathbb{B}_{2}(0) \to \mathbb{R} \) by \( \hat{h}_{\beta,x}(v) = \hat{h}_{x}(\beta v) \). Then

\[ \partial_{ij}^{2} \hat{h}_{\beta,x}(v) = \beta^{2} \partial_{ij}^{2} \hat{h}_{x}(\beta v). \]

By \( (\text{S}) \), we hence get that in these coordinates, on \( \mathbb{B}_{2}(0) \),

\[ \hat{g}^{ij}(\beta v) \partial_{ij}^{2} \hat{h}_{\beta,x}(v) = \beta^{2} \hat{g}^{ij}(\beta v) \partial_{ij}^{2} \hat{h}_{x}(\beta v) = \beta^{2} \hat{f}(\beta v). \]

Hence

\[ \lambda_{1}^{-2} \hat{g}^{ij}(\beta v) \partial_{ij}^{2} \hat{h}_{\beta,x}(v) = \lambda_{1}^{-2} \beta^{2} \hat{f}(\beta v) = \hat{f}_{\beta}(v). \]

Note that

\[ \lambda_{1}^{-2} \hat{g}^{ij}(\beta \cdot ) \geq \lambda_{1}^{-2} Q^{-1} \lambda_{1}^{2} \delta^{ij} = Q^{-1} \delta^{ij} \quad \text{on} \ \mathbb{B}_{2}(0), \]

\[ |\lambda_{1}^{-2} \hat{g}^{ij}(\beta \cdot )| \leq Q \lambda_{1}^{-2} \lambda_{1}^{2} = Q \quad \text{on} \ \mathbb{B}_{2}(0), \]

\[ [\lambda_{1}^{-2} \hat{g}^{ij}(\beta \cdot )]_{C_{L_{\infty}}^{\phi_{0},0}}(0) \leq \frac{\lambda_{1}^{-2} \beta}{C_{HR}} C_{Q} \lambda_{1}^{2} = C_{Q}/2\sqrt{Q}. \]

Applying classical pointwise Schauder estimates for second order elliptic operators (see in particular [18 Theorem 1.1] or [19 Theorem 5.20]) to equation \( (\text{S}) \) we hence get that

\[ \left[ \hat{h}_{\beta,x} \right]_{C_{L_{\infty}}^{2,\alpha}}(0) \leq C \left\{ \left[ \hat{h}_{\beta,x} \right]_{L_{\infty}(\mathbb{B}_{1}(0))} + \left[ \hat{f}_{\beta} \right]_{L_{\infty}(\mathbb{B}_{1}(0))} + \left[ \hat{f} \right]_{C_{L_{\infty}}^{\phi_{0},0}(0)} \right\}. \]

From this it follows that

\[ \beta^{2} \left[ \partial_{ij}^{2} \hat{h}_{x}(0) \right] \leq C \left\{ \left[ \hat{h}_{x} \right]_{L_{\infty}(\mathbb{B}_{2}(0))} + \frac{\beta^{2}}{\lambda_{1}^{2}} \left[ \hat{f} \right]_{L_{\infty}(\mathbb{B}_{2}(0))} + \frac{\beta^{2+\alpha}}{\lambda_{1}^{2}} \left[ \hat{f} \right]_{C_{L_{\infty}}^{\phi_{0},0}(\mathbb{B}_{2}(0))} \right\}, \]

i.e.

\[ \left[ \partial_{ij}^{2} \hat{h}_{x}(0) \right] \leq C \left\{ \frac{1}{\beta^{2}} \left[ \hat{h}_{x} \right]_{L_{\infty}(\mathbb{B}_{2}(0))} + \frac{1}{\lambda_{1}^{2}} \left[ \hat{f} \right]_{L_{\infty}(\mathbb{B}_{2}(0))} + \frac{\beta^{2+\alpha}}{\lambda_{1}^{2}} \left[ \hat{f} \right]_{C_{L_{\infty}}^{\phi_{0},0}(\mathbb{B}_{2}(0))} \right\}. \]

From now on we will denote \( \hat{h}_{x} \) simply by \( \hat{h} \), being understood the fact that it depends on the point \( x \) we have fixed on \( M \). Moreover the Euclidean balls will be always centered at the origin, unless otherwise stated.

About the first term on the RHS of \( (\text{I}) \), we note that for any \( v \in \mathbb{B}_{\beta} \), letting \( y = \varphi_{H}^{-1}(v) \), we have that

\[ \left| \hat{h}_{x}(v) \right| = \left| h_{x}(y) \right| = \left| h(y) - h(x) \right| \leq C d_{y}(x,y) \sup_{\zeta \in B_{d_{y}(x,y)}(x)} \left| r^{\zeta}(\zeta) - \zeta \right| \leq C \left( r(x) + \frac{C_{HR}}{\lambda_{1}} \right)^{\eta} \frac{C_{HR}}{\lambda_{1}} \leq C. \]

Here the last inequality comes from the definition of \( \lambda_{1} \).
About the second term, note that
\[ |\hat{f}(v)| = |f(y)| = \left| |\nabla h_x|^2(y) - \theta \tilde{r}(y)^{2\eta}\right| \]
\[ \leq Cr(y)^{2\eta} \]
\[ \leq C \left( r(x) + \frac{CHR}{\lambda_1} \right)^{2\eta} \]
\[ \leq C \rho^{2\eta} r(x)^{2\eta}, \]
for some positive constant \( \rho > 1 \). In particular,

(13)
\[ \frac{1}{\lambda_1^2} \| \hat{f} \|_{L^\infty(\mathbb{B}_\beta)} \leq C \frac{r(x)^{2\eta}}{\lambda_1^2} \leq C. \]

It remains to estimate \( \hat{f} \) on \( C_{L^\infty(\mathbb{B}_\beta)}^{0,\alpha} \). Letting \( v, w \in \mathbb{B}_\beta \), note that

(14)
\[ \left| \frac{\hat{f}(v) - \hat{f}(w)}{|v - w|^\alpha} \right| \leq \frac{C}{\lambda_1^2} \frac{|\nabla h_x|^2 \circ \varphi_H^{-1} |(v)| - |\nabla h_x|^2 \circ \varphi_H^{-1} |(w)|}{|v - w|^\alpha} \]
\[ \leq \frac{C}{\lambda_1^2} \frac{|\nabla h_x|^2 \circ \varphi_H^{-1} |(v)| - |\nabla h_x|^2 \circ \varphi_H^{-1} |(w)|}{|v - w|^\alpha} + C \frac{|\tilde{r}^{2\eta}(v) - \tilde{r}^{2\eta}(w)|}{|v - w|^\alpha}. \]

The first term will be estimated in Step 3 and Step 4 below. About the second term, letting \( y = \varphi_H^{-1} (v) \) and \( z = \varphi_H^{-1} (w) \), we have that

(15)
\[ \left| \tilde{r}^{2\eta}(\varphi_H^{-1} (v)) - \tilde{r}^{2\eta}(\varphi_H^{-1} (w)) \right| \leq \sup \left\{ 2\eta \tilde{r}(\tilde{r}(\tilde{r})^{2\eta-1} |\nabla \tilde{r}|(\zeta) : \zeta \in B_{d_y(x,y)}(x) \right\} \frac{d_y(y,z)}{|v - w|^\alpha} \]
\[ \leq 2\eta C \sup \left\{ \tilde{r}(\tilde{r})^{2\eta-1} : \zeta \in B_{d_y(x,y)}(x) \right\} \frac{\sqrt{Q}}{\lambda_1} |v - w|^{1-\alpha} \]
\[ \leq 2\eta C \frac{\sqrt{Q}}{\lambda_1} (2\beta)^{1-\alpha} \rho^{2\eta-1} r(x)^{2\eta-1}. \]

**Step 3:** estimate of \( |\nabla h_x|^2 \circ \varphi_H^{-1} |_{C_{L^\infty(\mathbb{B}_\beta)}^{0,\alpha}} \) using Sobolev embeddings.

By the Euclidean Sobolev embeddings (see e.g. [11, p. 109]), we have

\[ |\nabla h_x|^2 \circ \varphi_H^{-1} |_{C_{L^\infty(\mathbb{B}_\beta)}^{0,\alpha}} \leq K_3 \left\{ \| |\nabla h_x|^2 \circ \varphi_H^{-1} |_{L^p(\mathbb{B}_\beta)} \| + \sum_j \| \partial_j (|\nabla h_x|^2 \circ \varphi_H^{-1} |_{L^p(\mathbb{B}_\beta)}) \| \right\}^{1/p}, \]
with \( K_3 \) a positive constant depending only on \( p, m \) and \( \alpha \). Concerning the first term in the RHS, since \( |\nabla h_x| \leq Cr^\eta \) we get

(16)
\[ |\nabla h_x|^2 \circ \varphi_H^{-1} |_{L^p(\mathbb{B}_\beta)} \leq C \tilde{r}^{2\eta} \circ \varphi_H^{-1} |_{L^p(\mathbb{B}_\beta)} \leq \omega_m C \beta^m \| r \|^{2np}_{L^\infty(BC_{K_H}(x))} \]
\[ \leq C \left( r(x) + \frac{CHR}{2\lambda_1} \right)^{2np} \]

where \( \omega_m \) is the volume of the \( m \)-dimensional unit sphere. Concerning the second term, let us compute
\[ \partial_j (|\nabla h_x|^2 \circ \varphi_H^{-1} |) = \partial_j (\partial_k h_x \partial_i h_x g^{ki}) \]
\[ = 2\delta_k \partial_i h_x \partial_j h_x g^{ki} + \delta_i \partial_k h_x \partial_j g^{ki}, \]
so that
\[
\|\partial_j((|\nabla h_x|^2 \circ \varphi_H^{-1})|_{L^p(\mathbb{B}_\beta)}) \leq 2\|\partial_j \partial_k \hat{h} \partial_l \hat{g}^{kl}\|_{L^p(\mathbb{B}_\beta)} + \|\partial_k \hat{h} \partial_l \hat{g}^{kl}\|_{L^p(\mathbb{B}_\beta)}
\]
\[
\leq 2\sum_{k,l} \|\partial_j \partial_k \hat{h}\|_{L^p(\mathbb{B}_\beta)} \|\partial_l \hat{h}\|_{L^\infty(\mathbb{B}_\beta)} Q \lambda_1^2 + \sum_{k,l} \|\partial_k \hat{h}\|_{L^p(\mathbb{B}_\beta)} \|\partial_l \hat{h}\|_{L^\infty(\mathbb{B}_\beta)} C(Q) C_{HR} \lambda_1^2
\]
\[
\leq 2m^2 \|D^2 \hat{h}\|_{L^p(\mathbb{B}_\beta)} \|D \hat{h}\|_{L^\infty(\mathbb{B}_\beta)} Q \lambda_1^2 + m^2 \|D \hat{h}\|_{L^p(\mathbb{B}_\beta)} \|D \hat{h}\|_{L^\infty(\mathbb{B}_\beta)} C(Q) C_{HR} \lambda_1^2
\]
\[
\leq 2m^2 \|D^2 \hat{h}\|_{L^p(\mathbb{B}_\beta)} \|\nabla h_x| \circ \varphi_H^{-1}\|_{L^\infty(\mathbb{B}_\beta)} Q^{3/2} \lambda_1
\]
\[
+ m^2 \|\nabla h_x| \circ \varphi_H^{-1}\|_{L^p(\mathbb{B}_\beta)} \|\nabla h_x| \circ \varphi_H^{-1}\|_{L^\infty(\mathbb{B}_\beta)} Q C(Q) C_{HR} \lambda_1^2.
\]
where we used the fact that
\[
(17) \|D \hat{h}\| = \partial_k \partial_l \hat{g}^{kl} \leq \lambda_1^{-2} Q \partial_k \partial_l \hat{g}^{kl} = \lambda_1^{-2} Q |\nabla h_x|^2 \circ \varphi_H^{-1}.
\]
Reasoning as in (16) we get
\[
\|\partial_j((|\nabla h_x|^2 \circ \varphi_H^{-1})|_{L^p(\mathbb{B}_\beta)}) \leq 2m^2 C \|D^2 \hat{h}\|_{L^p(\mathbb{B}_\beta)} \|r^n \circ \varphi_H^{-1}\|_{L^\infty(\mathbb{B}_\beta)} Q^{3/2} \lambda_1
\]
\[
+ m^2 C \|r^n \circ \varphi_H^{-1}\|_{L^p(\mathbb{B}_\beta)} \|r^n \circ \varphi_H^{-1}\|_{L^\infty(\mathbb{B}_\beta)} Q C(Q) C_{HR} \lambda_1^2
\]
\[
\leq C \lambda_1 \left( r(x) + \frac{C_{HR}}{2 \lambda_1} \right)^{\eta} \|D^2 \hat{h}\|_{L^p(\mathbb{B}_\beta)} + C \left( r(x) + \frac{C_{HR}}{2 \lambda_1} \right)^{2\eta}.
\]

**Step 4: estimate of \( \|D^2 \hat{h}\|_{L^p(\mathbb{B}_\beta)} \) by a Calderón-Zygmund inequality.**

Let \( \phi \in C_c^\infty(\mathbb{B}_{2\beta}) \) be such that \( 0 \leq \phi \leq 1 \) and \( \phi \equiv 1 \) on \( \mathbb{B}_\beta \) and \( \max\{\|\nabla \phi\|_\infty; \|\Delta \phi\|_\infty\} < C_1 \) for some \( C_1 = C_1(\beta, m) \in \mathbb{R} \). According to the Calderón-Zygmund inequality, [11 Corollary 9.10], there exists a constant \( C_2 = C_2(m, p) > 0 \) such that
\[
(18) \|D^2 \hat{h}\|_{L^p(\mathbb{B}_\beta)} \leq \|D^2 (\phi \hat{h})\|_{L^p(\mathbb{B}_{2\beta})}
\]
\[
\leq C_2 \|\Delta_0 (\phi \hat{h})\|_{L^p(\mathbb{B}_{2\beta})}
\]
\[
\leq C_2 \left( \|\phi \Delta_0 \hat{h}\|_{L^p(\mathbb{B}_{2\beta})} + \|\hat{h} \Delta_0 \phi\|_{L^p(\mathbb{B}_{2\beta})} + 2 \|D \hat{h} \cdot D \phi\|_{L^p(\mathbb{B}_{2\beta})} \right)
\]
\[
\leq C_2 \left( \|\Delta_0 \hat{h}\|_{L^p(\mathbb{B}_{2\beta})} + C_1 \|\hat{h}\|_{L^p(\mathbb{B}_{2\beta})} + 2C_1 \|D \hat{h}\|_{L^p(\mathbb{B}_{2\beta})} \right),
\]
where \( \Delta_0 = \sum_i \partial_i \partial_i \) is the Euclidean Laplacian.

Now,
\[
\|\Delta_0 \hat{h}\|_{L^p(\mathbb{B}_{2\beta})} \leq C \lambda_1^2 \left( r(x) + \frac{C_{HR}}{\lambda_1} \right)^{2\eta}.
\]
Accordingly,
\[
\|\Delta_0 \hat{h}\|_{L^p(\mathbb{B}_{2\beta})} \leq C \lambda_1^2 \left( r(x) + \frac{C_{HR}}{\lambda_1} \right)^{2\eta}.
\]

As in (12) we have that \( \|\hat{h}\|_{L^\infty(\mathbb{B}_{2\beta})} \leq C \), so that \( \|\hat{h}\|_{L^p(\mathbb{B}_{2\beta})} \leq C \). Moreover, using (17),
\[
\|D \hat{h}\|_{L^p(\mathbb{B}_{2\beta})} \leq \lambda_1^{-1} \sqrt{Q} \|\nabla h_x| \circ \varphi_H^{-1}\|_{L^p(\mathbb{B}_{2\beta})} \leq \lambda_1^{-1} C \left( r(x) + \frac{C_{HR}}{\lambda_1} \right)^{\eta}.
\]
Inserting these estimates in (18) gives
\[
\|D^2 \hat{h}\|_{L^p(\mathbb{B}_g)} \leq C_2 \left[ C \lambda_1^{-2} \left( r(x) + \frac{C_{HR}}{\lambda_1} \right)^{2\eta} + C \left( 1 + \lambda_1^{-1} \left( r(x) + \frac{C_{HR}}{\lambda_1} \right)^{\eta} \right) \right] \leq C.
\]

Coming back to Step 3,
\[
\|\partial_j (|\nabla h_x|^2 \circ \varphi_H^{-1})\|_{L^p(\mathbb{B}_g)} \leq C r^{2\eta}
\]
and
\[
||\nabla h_x|^2 \circ \varphi_H^{-1}|^\alpha_{C_{L^\infty}(\mathbb{B}_g)} \leq C r^{2\eta}.
\]

**Step 5: estimate of \(|\mathrm{Hess} h|\).**

Using (19) and (15), we get by (14) that
\[
\frac{\beta^\alpha}{\lambda_1^4} \left[ \hat{f} \right]_{C^{0,\alpha}_{L^\infty}(\mathbb{B}_g)} \leq C,
\]
and hence, by (11), (12) and (13),
\[
|\partial_{ij}^2 h_x(x)| = |\partial_{ij}^2 \hat{h}_x(0)| \leq C.
\]

Recalling now that
\[
\nabla_i \nabla_j h_x = \partial_{ij}^2 h_x - \Gamma_{ij}^k \partial_k h_x,
\]
we can compute that
\[
|\mathrm{Hess} h_x| (x) = \left[ g^{ik} g^{jl} \left( \partial_{ij}^2 h_x - \Gamma_{ij}^s \partial_s h_x \right) \left( \partial_{kl}^2 h_x - \Gamma_{kl}^t \partial_t h_x \right) \right]^{\frac{1}{2}} (x)
\leq \left[ g^{ik} g^{jl} \partial_{ij}^2 h_x \partial_{kl}^2 h_x + g^{ik} g^{jl} \Gamma_{ij}^s \Gamma_{kl}^t \partial_s h_x \partial_t h_x
- 2 g^{ik} g^{jl} \partial_{ij}^2 h_x \Gamma_{kl}^t \partial_t h_x \right]^{\frac{1}{2}} (x).
\]

Since
\[
|g^{ik} g^{jl} \partial_{ij}^2 h_x \partial_{kl}^2 h_x| (x) \leq Q^2 \lambda_1^4 \left| \partial_{ij}^2 h_x \partial_{kl}^2 h_x \right| (x) = Q^2 \lambda_1^4 |\partial_{ij} h_x|^2 (x) \leq C \lambda_1^4 \leq C r^{4\eta}(x),
\]
\[
|g^{ik} g^{jl} \Gamma_{ij}^s \Gamma_{kl}^t \partial_s h_x \partial_t h_x| (x) \leq \frac{9}{4} C Q^5 \frac{(Q - 1)^2}{C_{HR}} \lambda_1^2 \lambda_1^{-r\eta}(x)
\leq C r^{4\eta}(x),
\]
\[
|2 g^{ik} g^{jl} \partial_{ij}^2 h_x \Gamma_{kl}^t \partial_t h_x| (x) \leq 2 C^2 Q^6 \sqrt{Q} \frac{(Q - 1)}{C_{HR}} \lambda_1^4 \lambda_1^{-1} r^\eta(x)
\leq C r^{4\eta}(x),
\]
we eventually obtain that
\[
(20) \quad |\mathrm{Hess} h| (x) = |\mathrm{Hess} h_x| (x) \leq C r^{2\eta}(x).
\]

We have thus proved the following

**Theorem 4.1.** Let \((M, g)\) be a complete Riemannian manifold and \(o \in M\) a fixed reference point, \(r(x) \equiv \text{dist}_g(x, o)\). Suppose that for some \(0 < \eta \leq 1\), some \(D > 0\) and some \(i_0 > 0\),
\[
|\text{Ric}_g| (x) \leq D^2 (1 + r(x)^2)^\eta, \quad \text{inj}_g(x) \geq \frac{i_0}{D (1 + r(x)^2)^\eta} > 0 \quad \text{on } M.
\]

Then there exists an exhaustion function \(h \in C^\infty(M)\) such that, for some positive constant \(C\) independent of \(x\) and \(o\), we have that
(i) \( C^{-1}r(x)^{1+\eta} \leq h(x) \leq C \max \{r(x)^{1+\eta}, 1\} \) on \( M \);

(ii) \(|\nabla h| \leq Cr^\eta \) on \( M \setminus B_2(o) \);

(iii) \(|\text{Hess } h|(x) \leq Cr(x)^{2\eta} \) on \( M \setminus B_2(o) \).

Finally note that the first part of Theorem 1.5 is equivalent to Theorem 4.1 up to introduce the new function \( H \in C^\infty(M) \) defined by \( H = h^{1/\eta} \).

4.2. Sub-quadratic sectional curvature growth (no assumptions on injectivity radius). Let \( o \in M \), \( r(x) \doteq \text{dist}_g(x,o) \) and let us assume that, for some \( 0 < \eta \leq 1 \) and \( D > 0 \),

\[
|\text{Sect}_g(x)| \leq D^2(1 + r(x)^2)^\eta. \tag{21}
\]

As in Step 0 of Subsection 4.1 we start with the exhaustion function \( h \) given in [6].

Given \( R_0 \in \mathbb{R}^+ \), to be chosen later, and \( x \in M \) such that \( r(x) > 1 + R_0 \), we have that on \( B_{R_0}(x) \)

\[
|\text{Sect}_g| \leq D^2(1 + (R_0 + r(x))^2)^\eta \doteq K_{x,R_0}.
\]

By a localized version of the Cartan-Hadamard theorem (see e.g. [16] Lemma 2.7) we have that for every \( 0 < R < \min \{\pi/\sqrt{K_{x,R_0}}, R_0\} \), there exists a smooth complete Riemannian manifold \( (\bar{M}, \bar{g}) \), \( \bar{x} \in M \), and a smooth surjective local isometry

\[
F \doteq F_{g,x,R} : B^\bar{g}_{\bar{R}}(\bar{x}) \to B^g_R(x),
\]

such that

- \( F(\bar{x}) = x \);
- \( \text{inj}_{\bar{g}}(\bar{x}) \geq \bar{R} \);
- \( |\text{Sect}_{\bar{g}}(\bar{y})| \leq K_{x,R_0} \) on \( B^\bar{g}_{\bar{R}}(\bar{x}) \);
- \( F(B^\bar{g}_{\bar{R}}(\bar{x})) = B^g_R(x) \), for all \( 0 < \bar{R} < R \).

In particular, for every \( \bar{y} \in B^\bar{g}_R(\bar{x}) \), we have that

\[
|\text{Sect}_{\bar{g}}(\bar{y})| \leq K_{x,R_0}, \quad \text{inj}_{\bar{g}}(\bar{y}) \geq d_{\bar{g}}(\bar{y}, \partial B^\bar{g}_{\bar{R}}(\bar{x})) \geq \frac{R}{2}.
\]

We define \( \bar{h}_x : B^\bar{g}_R(\bar{x}) \to \mathbb{R} \) by

\[
\bar{h}_x(\bar{y}) = h(F(\bar{y})) - h(F(\bar{x})).
\]

Then \( \bar{h}_x(\bar{x}) = 0 \), and

- \( |\nabla \bar{h}_x| \leq C r^\eta \);
- \( |\Delta \bar{h}_x| \leq C r^{2\eta} \);
- \( |\text{Hess}^\bar{g}_x \bar{h}_x(\bar{y})| = |\text{Hess}^g h|_g(y) \) on \( B^g_R(x) \),

with \( \bar{r} = r \circ F \). Moreover, by (3),

\[
\Delta \bar{h}_x = |\nabla \bar{h}_x|^2 - \theta(\bar{r} \circ F)^{2\eta} \doteq \bar{f}.
\]

Letting \( \lambda^2_{R_0} = (m - 1)K_{x,R_0} \), we set

\[
\bar{g}_\lambda = \lambda^2_{R_0} g.
\]

Then, by (22)

\[
|\text{Ric}_{\bar{g}_\lambda}(\bar{y})| \leq 1, \quad \text{inj}_{\bar{g}_\lambda}(\bar{y}) \geq \lambda R_0 \frac{R}{2}.
\]

Assuming that \( R_0 \geq \left(\frac{\pi}{2}\right)^{\frac{1}{1-\eta}} \), we can take \( R = \frac{\pi}{2\sqrt{K_{x,R_0}}} \), getting that

\[
\text{inj}_{\bar{g}_\lambda}(\bar{y}) \geq \frac{\sqrt{m - 1} \pi}{4} \doteq i_0.
\]

Given \( \alpha \in (0, 1) \), \( Q > 1 \) and \( \delta > 0 \), Proposition 3.2 hence yields that there exists a constant \( C_{HR}(m, Q, \alpha, \delta, i_0) \) such that for every \( \bar{y} \in B^\bar{g}_{\bar{R}}(\bar{x}) \) we can find on \( B^{g}_{C_{HR}(g)}(\bar{y}) \) harmonic coordinates with respect to which

(i) \( Q^{-1}\lambda^2_{R_0} \delta_{ij} \leq \bar{g}_{ij} \leq Q \lambda^2_{R_0} \delta_{ij} \);

(ii) \( \sum_\beta \lambda^2_{R_0} C_{HR} \sup_{\bar{y} \neq \bar{z}} \left| \frac{\partial \bar{g}_{ij}(\bar{y})}{\partial \bar{y}_i} - \frac{\partial \bar{g}_{ij}(\bar{y})}{\partial \bar{y}_j} \right| \leq Q - 1;
\]
and thus also

(i') \( Q^{-1} \lambda_{R_0}^2 \delta^{ij} \leq \tilde{g}^{ij} \leq Q \lambda_{R_0}^2 \delta^{ij} \);

(ii') \( \sum_{\beta} \lambda_{R_0}^{-2} C_{HR} \sup |\partial_\beta \tilde{g}^{ij}(y)| + \sum_{\beta} C_{HR}^{1+\alpha} \sup_{y \neq z} \lambda_{R_0}^{-2-\alpha} \frac{|\partial_\delta \tilde{g}^{ij}(z)-\partial_\delta \tilde{g}^{ij}(y)|}{d(y,z)^\alpha} \leq C(Q) \),

for some constant \( C(Q) \). Traveling through again Step 2, 3, 4 and 5 of Subsection 4.1 we eventually get that

\[ |\text{Hess}_g|_g(x) = |\text{Hess}_\tilde{g}|_\tilde{g}(x) \leq C r^{2\eta}(\tilde{x}) = C r^{2\eta}(x). \]

We have thus proved the following

**Theorem 4.2.** Let \((M, g)\) be a complete Riemannian manifold and \( o \in M \) a fixed reference point, \( r(x) = \text{dist}_g(x, o) \). Suppose that for some \( \eta < 1 \) and \( D > 0 \),

\[ |\text{Sect}_g|(x) \leq D^2(1 + r(x)^2)^\eta. \]

Then there exists an exhaustion function \( h \in C^\infty(M) \) such that, for some positive constants \( C > 1 \) independent of \( x \) and \( o \) and for some radius \( R_0 \), we have that

(i) \( C^{-1} r(x)^{1+\eta} \leq h(x) \leq C \max \{ r(x)^{1+\eta}, 1 \} \) on \( M \);

(ii) \( |\nabla h|(x) \leq C r^{\eta} \) on \( M \setminus B_{1+R_0}(o) \);

(iii) \( |\text{Hess}_h|(x) \leq C r^{2\eta} \) on \( M \setminus B_{1+R_0}(o) \).

Once again, defining \( H \in C^\infty(M) \) by \( H = h^{1+\eta} \), we get that the second part of Theorem 1.3 is equivalent to Theorem 4.2.

5. **Hessian cut-off functions**

In this section we construct (weak) Hessian cut-off functions starting from the distance-like functions obtained in the previous sections. This will permit to conclude the proof of Theorem 1.3.

**Lemma 5.1.** Let \((M, g)\) be a complete Riemannian manifold and \( o \in M \) a fixed reference point. If there exists an exhaustion function \( h \in C^\infty(M) \) such that for some positive constants \( D_2, \rho > 0 \), \( \beta > \varepsilon_i \geq 0 \), we have that

(i) \( D_1 r(x)^\beta \leq h(x) \leq D_2 \max \{ 1, r(x)^\beta \} \) for every \( x \in M \);

(ii) \( |\nabla h|(x) \leq D_3 r(x)^{\beta-\varepsilon_i} \), for every \( x \in M \setminus B_{\rho}(o) \);

(iii) \( |\text{Hess}(h)|(x) \leq D_4 r(x)^{\beta-\varepsilon_i} \), for every \( x \in M \setminus B_{\rho}(o) \).

Then given a \( \gamma > (D_2/D_1)^{1/\beta} \), there exists a family of cut-off functions \( \{ \chi_R \} \) such that

1. \( \chi_R = 1 \) on \( B_{\rho}(o) \) and \( \chi_R = 0 \) on \( M \setminus B_{\gamma R}(o) \);

2. \( |\nabla \chi_R| \leq \frac{C_1}{R^\gamma} \);

3. \( |\text{Hess}(\chi_R)| \leq \frac{C_2}{R^{\min(2\gamma, \gamma^2)}} \).

In particular \( \{ \chi_R \} \) is a family of weak Hessian cut-off functions for every \( \varepsilon_1, \varepsilon_2 > 0 \) and of genuine Hessian cut-off functions whenever \( \varepsilon_1 \varepsilon_2 > 0 \).

**Proof.** Let \( \Gamma = \frac{D_2}{D_1} \geq 1 \), and \( \gamma > \Gamma^{1/\beta} \) a real number. Let \( \phi \in C^\infty([0, 1]) \) be such that

\( \phi|_{(-\infty, 1]} = 1, \phi|_{[\gamma^a, \gamma^a]} = 0, |\phi'| + |\phi''| \leq a, \)

for some \( a > 0 \). For any \( R > 0 \), let \( \phi_R \in C^\infty([0, +\infty)) \) be defined by

\[ \phi_R(t) = \phi \left( \frac{t}{D_1 R^\beta} \right). \]

Then

\[ |\phi'_R| \leq \frac{a}{D_1 R^{\beta}}, \quad |\phi''_R| \leq \frac{a}{D_1^2 R^{2\beta}}. \]

For each radius \( R \gg 1 \), define \( \chi_R := \phi_R \circ h \). Then it is immediate to verify that \( \{ \chi_R \} \) meets the required properties. \( \square \)

We are finally in the position to give the
Proof (of Theorem 1.3). Under the assumptions (a) or (b) of the theorem, applying respectively Theorem 4.1 or Theorem 4.2 we get the existence of a distance-like function \( h \) with suitably controlled growth of the derivatives up to the 2nd order. Hence Lemma 5.1 applies and guarantees the existence of a sequence of (weak)-Hessian cut-off functions. Then Theorem 1.3 is a direct consequence of Proposition 2.4 (b). \( \square \)

6. THE SPECIAL CASE \( p = 2 \)

Proof of Theorem 1.4. Let \( \lambda^2(r(x)) = -D^2(1 + r(x)^2) \). By Corollary 2.3 we know that there exist a large constant \( \gamma > 1 \) and a sequence of weak Laplacian cut-off functions \( \{\chi_n\} \subset C^\infty_c(M) \) such that

- (1) \( \chi_n \equiv 1 \) on \( B_n(o) \);
- (2) \( \text{supp}(\chi_n) \subset B_{\gamma n}(o) \);
- (3) \( |\nabla \chi_n| \leq \frac{C_1}{n} \);
- (4) \( |\Delta \chi_n| \leq \frac{C_2}{n} \).

with \( C_1 \) and \( C_2 \) independent of \( n \).

As noticed in Remark 2.5, it is sufficient to consider \( f \in C^\infty(M) \cap W^{1,2}(M) \). We want to prove that \( \chi_n f \) converges to \( f \) in \( W^{2,2}(M) \). By properties (1) and (2) and the dominated convergence theorem it follows that

\[
\int_M |f - \chi_n f|^2 d\text{vol}_g \to 0,
\]

as \( n \to \infty \). Furthermore, by properties (1), (2), (3), and the dominated convergence theorem, we have that

\[
\int_M |\nabla f - \nabla(\chi_n f)|^2 d\text{vol}_g = \int_M |\nabla f - (\chi_n \nabla f + f \nabla \chi_n)|^2 d\text{vol}_g
\]

\[
= C \int_M f^2 |\nabla \chi_n|^2 d\text{vol}_g + C \int_M (1 - \chi_n)^2 |\nabla f|^2 d\text{vol}_g \to 0,
\]

as \( n \to \infty \). We now note that

\[
\int_M (\text{Hess}(\chi_n f) - \text{Hess} f)^2 d\text{vol}_g = \int_M (1 - \chi_n)^2 |\text{Hess} f|^2 d\text{vol}_g + 2 \int_M |\nabla \chi_n|^2 |\nabla f|^2 d\text{vol}_g + \int_M |f|^2 |\text{Hess} \chi_n|^2 d\text{vol}_g.
\]

Reasoning as above we get that the first two term on the RHS converge to 0. About the last term, using Bochner formula and our curvature assumption, we have that

\[
\text{div} \left( f^2 \frac{\nabla |\nabla \chi_n|^2}{2} \right) = f^2 \frac{\Delta |\nabla \chi_n|^2}{2} + f \langle \nabla f, \nabla |\nabla \chi_n|^2 \rangle
\]

\[
= f^2 |\text{Hess} \chi_n|^2 + \text{Ric}_g(\nabla \chi_n, \nabla \chi_n) + \langle \nabla \chi_n, \nabla \Delta \chi_n \rangle + 2 f |\nabla \chi_n| \langle \nabla f, \nabla |\nabla \chi_n| \rangle
\]

\[
\geq f^2 |\text{Hess} \chi_n|^2 + 2 f \text{Ric}_g(\nabla \chi_n, \nabla \chi_n) + f^2 |\nabla \chi_n, \nabla \Delta \chi_n \rangle - \frac{f^2}{2} |\nabla |\nabla \chi_n|^2| - 2 |\nabla \chi_n|^2 |\nabla f|^2
\]

\[
\geq \frac{f^2}{2} |\text{Hess} \chi_n|^2 + 2 |\nabla \chi_n|^2 |\nabla f|^2 - 2 |\nabla \chi_n|^2 |\nabla \chi_n|^2 + \text{div} (f^2 \Delta \chi_n \nabla \chi_n)
\]

\[
- 2 f \Delta \chi_n \langle \nabla f, \nabla \chi_n \rangle - f^2 |\Delta \chi_n|^2
\]

\[
\geq \frac{f^2}{2} |\text{Hess} \chi_n|^2 - 2 |\nabla \chi_n|^2 |\nabla f|^2 - \lambda^2 f^2 |\nabla \chi_n|^2 + \text{div} (f^2 \Delta \chi_n \nabla \chi_n) - 2 f^2 |\Delta \chi_n|^2
\]

Integrating, we get that

\[
1 \int_M f^2 |\text{Hess} \chi_n|^2 d\text{vol}_g \leq \int_M \lambda^2 f^2 |\nabla \chi_n|^2 d\text{vol}_g + 2 \int_M f^2 (\Delta \chi_n)^2 d\text{vol}_g + 3 \int_M |\nabla f|^2 |\nabla \chi_n|^2 d\text{vol}_g.
\]

By property (3), and the dominated convergence theorem the last term on the RHS of (23) converges to 0 as \( n \to \infty \). Moreover, by properties (1) and (2), we have that \( \nabla \chi_n \) and \( \Delta \chi_n \) are supported in \( B_{\gamma n}(o) \setminus B_n(o) \).
Hence, using property (3), the definition of $\lambda$, and the fact that $f \in L^2(M)$, we have that
\[
\int_M \chi^2 |\nabla \chi|^2 f^2 d\vol_g = \int_{B_{2n}(o) \setminus B_n(o)} \chi^2 |\nabla \chi|^2 f^2 d\vol_g
\leq \int_{M \setminus B_n(o)} C \frac{1 + \gamma^2 n^2}{n^2} f^2 d\vol_g \leq \tilde{C} \int_{M \setminus B_n(o)} f^2 d\vol_g \to 0,
\]
as $n \to \infty$. Similarly, by property (4),
\[
\int_M f^2 (\Delta \chi)^2 d\vol_g \leq C^2 \int_{M \setminus B_n(o)} f^2 d\vol_g \to 0,
\]
as $n \to \infty$. This finally gives our claim.

\[\square\]

7. A disturbed Sobolev inequality

In this last section we prove Theorem 1.6.

**Proof.** Let $\phi \in C^\infty(M)$ be a positive function to be choosen later, and define a new conformal metric $\tilde{g} = e^{2\phi} g$. Then, for any $x \in TM$, we have that

\[
\text{Ric}_{\tilde{g}}(X, X) = \text{Ric}_g(X, X) - (m - 2) \left[ \text{Hess}(\phi)(X, X) - g(X, \nabla \phi)^2 \right] + g(X, X) \left( \Delta \phi - (m - 2)|\nabla \phi|^2 \right)
\]

Accordingly
\[
|\text{Ric}_{\tilde{g}}| \leq C(m) e^{-2\phi} \left\{ |\text{Ric}_g| + ||\text{Hess}(\phi)| + |\nabla \phi|^2 \right\}
\]

Now, let $H \in C^\infty(M)$ be the exhaustion function given by Theorem 1.5. Without loss of generality we can suppose that $H > 1$ on $M$. Choose $\phi = \eta \ln(H)$. In particular since $e^\phi = H^\eta > 1$, $(M, \tilde{g})$ is complete. Moreover,
\[
|\nabla \phi|(x) = \eta \left| \frac{\nabla H}{H} \right| \leq \eta C
\]
and
\[
||\text{Hess}\phi|(x) = \eta \left| \frac{\text{Hess} H}{H} - \frac{dH \otimes dH}{H^2} \right| \leq 2\eta^3 C^2 r(x)^\eta
\]

Thus there exists a constant $\tilde{C} > 0$ depending on $m, \eta$ and $C$ such that

\[
(24) \quad |\text{Ric}_{\tilde{g}}|(x) \leq \frac{C(m)}{H^{2\eta}} \left\{ \lambda(r(x)) + 2\eta^3 C^3 r(x)^\eta + \eta^2 C^2 \right\} \leq \tilde{C}.
\]

We want now to compare volumes of geodesic balls in the metrics $g$ and $\tilde{g}$. First note that it exists $R > 0$ such that for all $x \in M \setminus B_{2R}(o)$, $\rho > 0$, one has

\[
(25) \quad B_{\rho r^{-\eta}(x)}^g(x) \subset B_{2\rho}^\tilde{g}(x)
\]

In fact, consider a point $y \in B_{\rho r^{-\eta}(x)}^g(x)$. Let $\sigma : [0, a] \to B_{\rho r^{-\eta}(x)}^g(x)$ be a minimizing geodesic of $(M, g)$ joining $x$ to $y$. Then
\[
\rho r^{-\eta}(x) > L_g(\sigma)
= \int_0^a e^{-\phi(\gamma(s))} \tilde{g}(\dot{\gamma}, \dot{\gamma})^{1/2}(s) ds
\geq d_{\tilde{g}}(x, y) \inf \{ e^{-\phi(z)} : z \in B_{\rho r^{-\eta}(x)}^g(x) \},
\]
from which we deduce that
\[
d_{\tilde{g}}(x, y) \leq \rho r^{-\eta}(x) \sup \{ e^{\phi(z)} : z \in B_{\rho r^{-\eta}(x)}^g(x) \} \leq \rho r^{-\eta}(x)(r(x) + \rho)^\eta \leq 2\rho
\]
for $r(x)$ large enough and (25) is proved. Then

\begin{equation}
\text{vol}_{\tilde{g}}(B_{2\rho}^g(x)) \geq \text{vol}_{\tilde{g}}(B_{\rho r^{-\eta}(x)}^g(x)) \\
\geq \inf \{e^{m\phi(x)} : z \in B_{\rho r^{-\eta}(x)}^g(x)\} \text{vol}_{\tilde{g}}(B_{\rho r^{-\eta}(x)}^g(x)) \\
\geq \left( r(x) - \rho r^{-\eta}(x) \right)^{mn} \text{vol}_{\tilde{g}}(B_{\rho r^{-\eta}(x)}^g(x)) \\
\geq \frac{r^{mn}(x)}{2} \text{vol}_{\tilde{g}}(B_{\rho r^{-\eta}(x)}^g(x))
\end{equation}

for $r(x)$ large enough. Recall the following result by Croke.

**Lemma 7.1** (Proposition 14 in [10]). There exists a dimensional constant $C_m > 0$ such that for any $x \in M$ and $i > 0$, if

\[ \forall y \in B_{\frac{\rho}{2}}^g(x), \quad \text{inj}_g(y) > i, \]

then

\[ \text{vol}_{\tilde{g}}(B_{\frac{\rho}{2}}^g(x)) \geq C_m i^m. \]

Let $E = i_0/D$ and choose $i = i(x) = E(1 + r(x))^{-\eta}/2$. There exists a positive radius $R_{\eta}$ large enough depending on $\eta$ such that for all $x \in M \setminus B_{R_{\eta}}^g(o)$ and for all $y \in B_{\frac{\rho}{4}}^g(x)$ we have

\[ \text{inj}_g(y) \geq \frac{E}{(1 + r(y))^{\eta}} \geq \frac{1}{2} \frac{E}{(1 + r(x))^{\eta}} = i. \]

Then Lemma 7.1 applies and we get that for all $x \in M \setminus B_{R_{\eta}}^g(o)$

\[ \text{vol}_{\tilde{g}} \left( B_{E(1 + r(x))^{-\eta}/4}^g(x) \right) \geq C_m \frac{E^m}{2^m(1 + r(x))^{mn}}. \]

Choosing $\rho = E/4$ in (26), we finally obtain that

\begin{equation}
\text{vol}_{\tilde{g}} \left( B_{E/2}^g(x) \right) \geq \frac{r^{mn}(x)}{2} \text{vol}_{\tilde{g}}(B_{E^{r^{-\eta}(x)/2}}^g(x)) \\
\geq C_m \frac{r^{mn}(x)}{2} \frac{E^m}{2^m(1 + r(x))^{mn}} \\
\geq \frac{C_mE^m}{2}
\end{equation}

if $r(x)$ is large enough. In particular $\text{vol}_{\tilde{g}}(B_{E/4}^g(x))$ is uniformly lower bounded on $M$.

According to the well known result by Varopoulos alluded to above, [27, 9], the bounds on the Ricci curvature and on the volumes of small balls given in [24] and (27) guarantee the continuity of the embedding $W^{1,2} \subset L^{2m/(m-2)}$ on $(M, \tilde{g})$. Namely $(M, \tilde{g})$ supports the Sobolev inequality

\[ \left( \int_M u^{\frac{2m}{m-2}} d\text{vol}_{\tilde{g}} \right)^{\frac{m-2}{m}} \leq A \int_M \tilde{g}(\nabla u, \nabla u)^2 d\text{vol}_{\tilde{g}} + B \int_M u^2 d\text{vol}_{\tilde{g}} \]

for all $u \in C^\infty(M)$, and for some positive constants $A$ and $B$ independent of $u$. Since the scalar curvature $\text{Scal}_{\tilde{g}}$ is bounded on $(M, \tilde{g})$, by an integration by parts we can write the Sobolev inequality in the form

\begin{equation}
\left( \int_M u^{\frac{2m}{m-2}} d\text{vol}_{\tilde{g}} \right)^{\frac{m-2}{m}} \leq A \int_M -u\text{L}_{\tilde{g}} u d\text{vol}_{\tilde{g}} + B' \int_M u^2 d\text{vol}_{\tilde{g}}
\end{equation}

where $B' = B + \frac{m-2}{4(m-1)} A \|\text{Scal}_{\tilde{g}}\|_\infty$ and

\[ L_{\tilde{g}} u = \Delta_{\tilde{g}} u + \frac{m-2}{4(m-1)} \text{Scal}_{\tilde{g}}(x) u \]

is the conformal Laplacian of $(M, \tilde{g})$. Using standard ideas introduced by Schoen and Yau, [25], we set $v = e^{\frac{m-2}{2} \phi} = H^{\frac{m-2}{2} \eta}$ and $\phi = uv$, and note that

\[ uL_{\tilde{g}} u d\text{vol}_{\tilde{g}} = \phi L_{\tilde{g}} \phi d\text{vol}_{\tilde{g}} \]
Acknowledgement. The first author is partially supported by INdAM-GNSAGA. The second and third authors are partially supported by INdAM-GNAMPA.

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