Bounding Helly numbers via Betti numbers

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Abstract

We prove the following Helly-type theorem. For any non-negative integers \(b\) and \(d\) there exists an integer \(h(b,d)\) such that the following holds. If \(\mathcal{F}\) is a finite family of subsets of \(\mathbb{R}^d\) such that \(\tilde{\beta}_i(\cap \mathcal{G}) \leq b\) for any \(\mathcal{G} \subseteq \mathcal{F}\) and every \(0 \leq i \leq \lceil d/2 \rceil - 1\) then \(\mathcal{F}\) has Helly number at most \(h(b,d)\).

Here \(\tilde{\beta}_i\) denotes the reduced Betti numbers (with singular homology). The main strength of this result is that very weak assumptions on \(\mathcal{F}\) are sufficient to guarantee a bounded Helly number.

We obtain this result by combining homological non-embeddability results with a Ramsey-based approach to build, given an arbitrary simplicial complex \(K\), well-behaved chain map \(C_*(K) \to C_*(\mathbb{R}^d)\). Both techniques are of independent interest.

1 Introduction

Helly’s classical theorem [Hel23], a cornerstone of convex geometry, asserts that if a finite family of convex subsets of \(\mathbb{R}^d\) has the property that any \(d+1\) of the sets have a point in common then the whole family must have a point in common. Stated in the contrapositive, if \(\mathcal{F}\) is a finite family of convex subsets of \(\mathbb{R}^d\) with empty intersection then \(\mathcal{F}\) contains a sub-family \(\mathcal{G}\) of size at most \(d+1\) that already has empty intersection. This inspired the definition of the Helly number of a family \(\mathcal{F}\) of arbitrary sets. If \(\mathcal{F}\) has empty intersection then its Helly number is defined as the size of the largest sub-family \(\mathcal{G} \subseteq \mathcal{F}\) with the following properties: \(\mathcal{G}\) has empty intersection and any proper sub-family of \(\mathcal{G}\) has nonempty intersection; if \(\mathcal{F}\) has nonempty intersection then its Helly number is, by convention, 1. With this terminology, Helly’s theorem simply states that any finite family of convex sets in \(\mathbb{R}^d\) has Helly number at most \(d+1\). Such uniform bounds that are independent of the cardinality of the family are of particular interest.

In the spirit of Helly’s theorem, bounds on Helly numbers were given for a variety of situations in discrete geometry (such bounds are often referred to as Helly-type theorems); we refer to the surveys [Eck93, Wen04, Tan13] for an overview of the abundant literature on this topic. The classical questions are of two types, existential and quantitative: identify conditions under which Helly numbers can be bounded uniformly, and obtain sharp bounds. In this paper, we focus on the existential question and give a new homological condition sufficient to bound Helly numbers.

The study of topological conditions (as opposed to more geometric ones like convexity) ensuring bounded Helly numbers started with Helly’s topological theorem [Hel30] (see also [Deb70] for a modern
version of the proof), which states that a finite family of open subsets of $\mathbb{R}^d$ has Helly number at most $d + 1$ if the intersection of any sub-family of at most $d$ members of the family is either empty or a homology cell. This includes the case of finite open good cover in $\mathbb{R}^d$, where the same bound follows easily from the classical Nerve theorem [Bor48, Bj03]. This “good cover” condition was subsequently relaxed by Matoušek [Mat97] who showed that it is sufficient to control the low-dimensional homotopy of intersections: for any integers $b$ and $d$ there exists a constant $c(b,d)$ such that any finite family of subsets of $\mathbb{R}^d$ in which every sub-family intersects in at most $b$ connected components, each $([d/2] − 1)$-connected, has Helly number at most $c(b,d)$.

Our main result is a Helly-type theorem under a homological condition that relaxes Matoušek’s condition (as well as that of Helly’s topological theorem). Throughout this paper, we will work with homology with coefficients in the field $\mathbb{Z}_2$ and $\beta_i(X)$ will denote the $i$th reduced Betti number (over $\mathbb{Z}_2$) of a space $X$.

In what follows, we use the notation $\bigcap F := \bigcap_{U \in F} U$ to denote the intersection of a family of sets.

**Theorem 1.** For any non-negative integers $b$ and $d$ there exists an integer $h(b,d)$ such that the following holds. If $F$ is a finite family of subsets of $\mathbb{R}^d$ such that $\beta_i(\bigcap G) \leq b$ for any $G \subseteq F$ and every $0 \leq i \leq [d/2] − 1$ then $F$ has Helly number at most $h(b,d)$.

**Remarks 2.** (a) By Hurewicz’ Theorem and the Universal Coefficient Theorem [Hat02, Theorem 4.37 and Corollary 3A.6], a $k$-connected space $X$ satisfies $\beta_i(X) = 0$ for all $i \leq k$. Thus, our condition indeed relaxes Matoušek’s, in two ways: by using $\mathbb{Z}_2$-homology instead of the homotopy-theoretic assumptions of $k$-connectedness, and by allowing an arbitrary fixed bound $b$ instead of $b = 0$.

(b) Quantitatively, the bound on $h(b,d)$ that we obtain is very large as it follows from successive applications of Ramsey’s theorems. However, as far as only the existence of uniform bounds is concerned, Theorem 1 not only generalizes Matoušek’s result (which also uses Ramsey’s theorem), but also subsumes a series of Helly-type theorems due to Amenta [Ame96], Kalai and Meshulam [KM08], Colin de Verdière et al. [CGG12], and Montejean [Mon13]. Note that for results that hold in rather general spaces, e.g. [KM08, CGG12, Mon13], Theorem 1 only subsumes the case of $\mathbb{R}^d$.

(c) Our method also proves a bound of $d + 1$ on the Helly number of any family $F$ such that $\beta_i(\bigcap G) = 0$ for all $i \leq d$ and all $G \subseteq F$ (see Corollary 19), which generalizes Helly’s topological theorem as the sets of $F$ are, for instance, not assumed to be open. Under the weaker assumption that $\beta_i(\bigcap G) = 0$ for all subfamilies $G \subseteq F$ but only for $i \leq [d/2] − 1$, our method still yields a bound of $d + 2$ on the Helly number (see Corollary 18). In both cases the bounds are tight (see Remark 20).

(d) Theorem 1 is “qualitatively sharp”, in the sense that all (reduced) Betti numbers $\beta_i$ with $0 \leq i \leq [d/2] − 1$ need to be bounded to obtain a bounded Helly number. To see this, fix some $k$ with $0 \leq k \leq [d/2] − 1$. For $n$ arbitrarily large, consider a geometric realization in $\mathbb{R}^d$ of the $k$-skeleton of the $(n−1)$-dimensional simplex (see [Mat03, Section 1.6]); more specifically, let $V = \{v_1, \ldots, v_n\}$ be a set of points in general position in $\mathbb{R}^d$ (for instance, $n$ points on the moment curve) and consider all geometric simplices $\sigma_A := \text{conv}(A)$ spanned by subsets $A \subseteq V$ of cardinality $|A| \leq k + 1$. By general position, $\sigma_A \cap \sigma_B = \sigma_{A \cap B}$, so this yields indeed a geometric realization.

For $1 \leq j \leq n$, let $U_j$ be the union of all the simplices not containing the vertex $v_j$. We set $F = \{U_1, \ldots, U_n\}$. Then, $\bigcap F = \emptyset$, and for any proper sub-family $G \subseteq F$, the intersection $\bigcap G$ is either $\mathbb{R}^d$ (if $G = \emptyset$) or (homeomorphic to) the $k$-dimensional skeleton of a $(n−1 − |G|)$-dimensional homology group is trivial, as is the case if $X = \mathbb{R}^d$ or $X$ is a single point. Here and in what follows, we refer the reader to standard textbooks like [Hat02, Munk84] for further topological background and various topological notions that we leave undefined.

2An open good cover is a finite family of open subsets of $\mathbb{R}^d$ such that the intersection of any sub-family of at most $d$ members is either empty or is contractible (and hence, in particular, a homology cell).

3We recall that a topological space $X$ is $k$-connected, for some integer $k \geq 0$, if every continuous map $S^i \to X$ from the $i$-dimensional sphere to $X$, $0 \leq i \leq k$, can be extended to a map $D^{i+1} \to X$ from the $(i+1)$-dimensional disk to $X$.

4We also remark that our condition can be verified algorithmically since Betti numbers are easily computable, at least for sufficiently nice spaces that can be represented by finite simplicial complexes, say. By contrast, it is algorithmically undecidable whether a given 2-dimensional simplicial complex is 1-connected, see, e.g., the survey [Seq04].

5In the original proof, this assumption is crucial and used to ensure that the union of the sets must have trivial homology in dimensions larger than $d$; this may fail if the sets are not open.
simplex. Thus, the Helly number of $F$ equals $n$. Moreover, the $k$-skeleton $\Delta_{m-1}^{(k)}$ of an $(m-1)$-dimensional simplex has reduced Betti numbers $\beta_i = 0$ for $i \neq k$ and $\beta_k = \binom{m-1}{k}$. Thus, we can indeed obtain arbitrarily large Helly number as soon as at least one $\beta_k$ is unbounded. In particular, setting $k = 0$ yields the lower bound $h(b, d) \geq b + 1$.

Consequences for optimisation problems. Various optimization problems can be formulated as the minimization of some function $f : \mathbb{R}^d \to \mathbb{R}$ over some intersection $\bigcap_{i=1}^n C_i$ of subsets $C_1, C_2, \ldots, C_n$ of $\mathbb{R}^d$. If, for $t \in \mathbb{R}$, we let $L_t = f^{-1}((-\infty, t])$ and $F_t = \{ C_1, C_2, \ldots, C_n, L_t \}$ then

$$\min_{x \in \bigcap_{i=1}^n C_i} f(x) = \min \left\{ t \in \mathbb{R} : \bigcap F_t \neq \emptyset \right\}.$$ 

If the Helly number of the families $F_t$ can be bounded uniformly in $t$ by some constant $h$ then there exists a subset of $h - 1$ constraints $C_{i_1}, C_{i_2}, \ldots, C_{i_{h-1}}$ that suffice to define the minimum of $f$:

$$\min_{x \in \bigcap_{i=1}^n C_i} f(x) = \min_{x \in \bigcap_{i=1}^{n-1} C_{i_j}} f(x).$$

A consequence of this observation, noted by Amenta [Ame94], is that the minimum of $f$ over $C_1 \cap C_2 \cap \ldots \cap C_n$ can \footnote{This requires that $f$ be generic in the sense that its local minima attains pairwise distinct values.} be computed in randomized $O(n)$ time by generalized linear programming [SW92]. Together with Theorem 1, this implies that an optimization problem of the above form can be solved in randomized linear time if it has the property that every intersection of some subset of the constraints with a level set of the function has bounded “topological complexity” (measured in terms of the sum of the first $[d/2]$ Betti numbers). Let us emphasize that this linear-time bound holds in a real-RAM model of computation, where any constant-size subproblems can be solved in $O(1)$-time; it therefore concerns the combinatorial difficulty of the problem and says nothing about its numerical difficulty.

Identifying new Helly-type theorems. Let us illustrate how Theorem 1 helps identify concrete situations in which Helly numbers are bounded by giving an example which, to the best of our knowledge, is not covered by any other Helly-type theorem appearing in the literature.

By an affine $k$-sphere in $\mathbb{R}^d$ for $0 \leq k \leq d - 1$ we simply mean a geometric sphere of arbitrary center and radius inside some affine $(k + 1)$-space of $\mathbb{R}^d$. An affine sphere is an affine $k$-sphere for some $k \in \{0, \ldots, d - 1\}$. Theorem 1 implies that the Helly number of an arbitrary family of affine spheres in $\mathbb{R}^d$ is bounded since an arbitrary intersection of affine spheres is an empty set, singleton, or an affine sphere, all of them having bounded Betti numbers. A careful analysis can of course lead to a much better bound on the Helly number than the one given by Theorem 1. However, note that Theorem 1 immediately reveals that the Helly number is bounded.

Structure of the paper. We prove Theorem 1 in three steps. We first establish, in Section 2, an analogue in homology of the Van Kampen-Flores Theorem on non-embeddability of certain simplicial complexes in $\mathbb{R}^d$. We then present, in Section 3, a general principle, which we learned from Matoušek [Mat97], to derive Helly-type theorems from non-embeddability results. Finally, in Section 4, we refine this principle and combine it with our homological Van Kampen-Flores Theorem to prove Theorem 1.

2 Homological Representations

In this section, we define homological representations, an analogue of topological embeddings on the level of chain maps, and show that certain simplicial complexes do not admit homological representations in $\mathbb{R}^d$ in analogy to classical non-embeddability results due to Van Kampen and Flores. In fact, when this comes at no additional cost we phrase the auxiliary results in a slightly more general setting, replacing $\mathbb{R}^d$ by a general topological space $R$. Readers that focus on the proof of Theorem 1 can safely replace every occurrence of $R$ with $\mathbb{R}^d$. 

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We assume that the reader is familiar with basic topological notions and facts concerning simplicial complexes and singular and simplicial homology, as described in textbooks like [Hat02, Mun84]. As remarked above, throughout this paper we will work with homology with \( \mathbb{Z}_2 \)-coefficients unless explicitly stated otherwise. Moreover, while we will consider singular homology groups for topological spaces in general, for simplicial complexes we will work with simplicial homology groups. In particular, if \( X \) is a topological space then \( C_i(X) \) will denote the singular chain complex of \( X \), while if \( K \) is a simplicial complex, then \( C_i(K) \) will denote the simplicial chain complex of \( K \) (both with \( \mathbb{Z}_2 \)-coefficients).

**Notations.** Let \( K \) be a (finite, abstract) simplicial complex. The **underlying topological space** of \( K \) is denoted by \( |K| \). Moreover, we denote by \( K(i) \) the \( i \)-dimensional skeleton of \( K \), i.e., the set of simplices of \( K \) of dimension at most \( i \); in particular \( K(0) \) is the set of vertices of \( K \). For an integer \( n \geq 0 \), let \( \Delta_n \) denote the \( n \)-dimensional simplex.

### 2.1 Non-Embeddable Complexes

We recall that an embedding of a finite simplicial complex \( K \) into \( \mathbb{R}^d \) is simply an injective continuous map \( |K| \to \mathbb{R}^d \). The fact that the complete graph on five vertices cannot be embedded in the plane has the following generalization.

**Proposition 3** (Van Kampen [vK32], Flores [Flo33]). For \( k \geq 0 \), the complex \( \Delta_{2k+2}^{(k)} \), the \( k \)-dimensional skeleton of the \( (2k+2) \)-dimensional simplex, cannot be embedded in \( \mathbb{R}^{2k} \).

A basic tool for proving the non-embeddability of a simplicial complex is the so-called Van Kampen obstruction. To be more precise, we emphasize that in keeping with our general convention regarding coefficients, we work with the \( \mathbb{Z}_2 \)-coefficient version\(^7\) of the Van Kampen obstruction, which will be reviewed in some detail in Section 2.3 below. Here, for the benefit of readers who are willing to accept certain topological facts as given, we simply collect those statements necessary to motivate the definition of homological representations and to follow the logic of the proof of Theorem 1.

Given a simplicial complex \( K \), one can define, for each \( d \geq 0 \), a certain cohomology class \( \sigma^d(K) \) that resides in the cohomology group \( H^d(K) \) of a certain auxiliary complex \( K \) (the quotient of the combinatorial deleted product by the natural \( \mathbb{Z}_2 \)-action, see below); this cohomology class \( \sigma^d(K) \) is called the Van Kampen obstruction to embeddability into \( \mathbb{R}^d \) because of the following fact:

**Proposition 4.** Suppose that \( K \) is a finite simplicial complex with \( \sigma^d(K) \neq 0 \). Then \( K \) is not embeddable into \( \mathbb{R}^d \). In fact, a slightly stronger conclusion holds: there is no almost-embedding \( f: |K| \to \mathbb{R}^d \), i.e., no continuous map such that the images of disjoint simplices of \( K \) are disjoint.

Another basic fact is the following result (for a short proof see, for instance, [Mel09, Example 3.5]).

**Proposition 5** ([vK32, Flo33]). For every \( k \geq 0 \), \( \sigma^{2k} \left( \Delta_{2k+2}^{(k)} \right) \neq 0 \).

As a consequence, one obtains Proposition 3, and in fact the slightly stronger statement that \( \Delta_{2k+2}^{(k)} \) does not admit an almost-embedding into \( \mathbb{R}^{2k} \).

### 2.2 Homological Representations and a Van Kampen–Flores Result

For the proof of Theorem 1, we wish to replace homotopy-theoretic notions (like \( k \)-connectedness) by homological assumptions (bounded Betti numbers). The simple but useful observation that allows us to do this is that in the standard proof of Proposition 4, which is based on (co)homological arguments, maps can be replaced by suitable chain maps at every step.\(^8\) The appropriate analogue of an almost-embedding is the following.

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\(^7\)There is also a version of the Van Kampen obstruction with integer coefficients, which in general yields more precise information regarding embeddability than the \( \mathbb{Z}_2 \)-version, but we will not need this here. We refer to [Mel09] for further background.

\(^8\)This observation was already used in [Wag11] to study the (non-)embeddability of certain simplicial complexes. What we call a **homological representation** in the present paper corresponds to the notion of a **homological minor** used in [Wag11].
Definition 1. Let $R$ be a (nonempty) topological space, $K$ be a simplicial complex, and consider a chain map $\gamma: C_\ast(K) \to C_\ast(R)$ from the simplicial chains in $K$ to singular chains in $R$.

(i) The chain map $\gamma$ is called nontrivial$^9$ if the image of every vertex of $K$ is a finite set of points in $R$ (a 0-chain) of odd cardinality.

(ii) The chain map $\gamma$ is called a homological representation of a simplicial complex $K$ in $R$ if it is nontrivial and if, additionally, the following holds: whenever $\sigma$ and $\tau$ are disjoint simplices of $K$, their image chains $\gamma(\sigma)$ and $\gamma(\tau)$ have disjoint supports, where the support of a chain is the union of the images of the singular simplices with nonzero coefficient in that chain.

Remark 6. Suppose that $f: |K| \to \mathbb{R}^d$ is a continuous map.

(i) The induced chain map$^{11}$ $f_2: C_\ast(K) \to C_\ast(\mathbb{R}^d)$ is nontrivial.

(ii) If $f$ is an almost-embedding then the induced chain map is a homological representation.

Moreover, note that without the requirement of being nontrivial, we could simply take the constant zero chain map, for which the second requirement is trivially satisfied.

We have the following analogue of Proposition 4 for homological representations.

Proposition 7. Suppose that $K$ is a finite simplicial complex with $\sigma^d(K) \neq 0$. Then $K$ does not admit a homological representation in $\mathbb{R}^d$.

As a corollary, we get the following result, which underlies our proof of Theorem 1.

Corollary 8. For any $k \geq 0$, the $k$-skeleton $\Delta(k)^{2k+2}$ of the $(2k+2)$-dimensional simplex has no homological representation in $\mathbb{R}^{2k}$.

We conclude this subsection by two facts that are not needed for the proof of the main result but are useful for the presentation of our method in Section 3.

If the ambient dimension $d = 2k + 1$ is odd, we can immediately see that $\Delta^{(k+1)}_{2k+2}$ has no homological representation in $\mathbb{R}^{2k+1}$ since it has no homological representation in $\mathbb{R}^{2k+2}$; this result can be slightly improved:

Corollary 9. For any $d \geq 0$, the $[d/2]$-skeleton $\Delta^{(d/2+1)}_{d+2}$ of the $(d + 2)$-dimensional simplex has no homological representation in $\mathbb{R}^d$.

Proof. The statement for even $d$ is already covered by the case $k = d/2$ of Corollary 8, so assume that $d$ is odd and write $d = 2k + 1$. If $K$ is a finite simplicial complex with $\sigma^d(K) \neq 0$ and if $C K$ is the cone over $K$ then $\sigma^d(CK) \neq 0$ (for a proof, see, for instance, [BKK02, Lemma 8]). Since we know that $\sigma^{2k}(\Delta^{(k+1)}_{2k+2}) \neq 0$ it follows that $\sigma^{2k+1}(\Delta^{(k+1)}_{2k+2}) \neq 0$. Consequently, $\sigma^{2k+1}(\Delta^{(k+1)}_{2k+2}) \neq 0$ since $C \Delta^{(k+1)}_{2k+2}$ is a subcomplex of $\Delta^{(k+1)}_{2k+3}$. Proposition 7 then implies that $\Delta^{(k+1)}_{2k+3}$ admits no homological representation in $\mathbb{R}^{2k+1}$.

The next fact is the following analogue of Radon’s lemma, proved in the next subsection along the proof of Proposition 7.

Lemma 10 (Homological Radon’s lemma). For any $d \geq 0$, $\sigma^d(\partial \Delta_{d+1}) \neq 0$. Consequently, the boundary of $(d + 1)$-simplex $\partial \Delta_{d+1}$ admits no homological representation in $\mathbb{R}^d$.

$^9$We recall that a chain map $\gamma: C_\ast \to D_\ast$ between chain complexes is simply a sequence of homomorphisms $\gamma_n: C_n \to D_n$ that commute with the respective boundary operators, $\gamma_{n-1} \circ \partial_C = \partial_D \circ \gamma_n$.

$^{10}$If we consider augmented chain complexes with chain groups also in dimension $-1$, then being nontrivial is equivalent to requiring that the generator of $C_{-1}(K) \cong \mathbb{Z}_2$ (this generator corresponds to the empty simplex in $K$) is mapped to the generator of $C_{-1}(R) \cong \mathbb{Z}_2$.

$^{11}$The induced chain map is defined as follows: We assume that we have fixed a total ordering of the vertices of $K$. For a $p$-simplex $\sigma$ of $K$, the ordering of the vertices induces a homeomorphism $h_\sigma: [\Delta_p] \to |\sigma| \subseteq |K|$. The image $f_2(\sigma)$ is defined as the singular $p$-simplex $f \circ h_\sigma$. 

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2.3 Deleted Products and Obstructions

Here, we review the standard proof of Proposition 4 and explain how to adapt it to prove Proposition 7, which will follow from Lemma 14 and Lemma 15 (b) below. The reader unfamiliar with cohomology and willing to admit Proposition 7 can safely proceed to Section 3.

\[ \mathbb{Z}_2 \text{-spaces and equivariant maps.} \]  

We begin by recalling some basic notions of equivariant topology: An action of the group \( \mathbb{Z}_2 \) on a space \( X \) is given by an automorphism \( \nu: X \to X \) such that \( \nu \circ \nu = 1_X \); the action is free if \( \nu \) does not have any fixed points. If \( X \) is a simplicial complex (or a cell complex), then the action is called simplicial (or cellular) if it is given by a simplicial (or cellular) map. A space with a given (free) \( \mathbb{Z}_2 \)-action is also called a (free) \( \mathbb{Z}_2 \)-space.

A map \( f: X \to Y \) between \( \mathbb{Z}_2 \)-spaces \((X,\nu)\) and \((Y,\mu)\) is called equivariant if it commutes with the respective \( \mathbb{Z}_2 \)-actions, i.e., \( f \circ \nu = \mu \circ f \). Two equivariant maps \( f_0, f_1: X \to Y \) are equivariantly homotopic if there exists a homotopy \( F: X \times [0,1] \to Y \) such that all intermediate maps \( f_t := F(\cdot, t) \), \( 0 \leq t \leq 1 \), are equivariant.

A \( \mathbb{Z}_2 \)-action \( \nu \) on a space \( X \) also yields a \( \mathbb{Z}_2 \)-action on the chain complex \( C_*(X) \), given by the induced chain map \( \nu_2: C_*(X) \to C_*(X) \) (if \( \nu \) is simplicial or cellular, respectively, then this remains true if we consider the simplicial or cellular chain complex of \( X \) instead of the singular chain complex), and if \( f: X \to Y \) is an equivariant map between \( \mathbb{Z}_2 \)-spaces then the induced chain map is also equivariant (i.e., it commutes with the \( \mathbb{Z}_2 \)-actions on the chain complexes).

\[ \text{Spheres.} \]  

Important examples of free \( \mathbb{Z}_2 \)-spaces are the standard spheres \( S^d \), \( d \geq 0 \), with the action given by antipodality, \( x \mapsto -x \). There are natural inclusion maps \( S^{d-1} \hookrightarrow S^d \), which are equivariant. Antipodality also gives a free \( \mathbb{Z}_2 \)-action on the union \( S^\infty = \bigcup_{d \geq 0} S^d \), the infinite-dimensional sphere. Moreover, one can show that \( S^\infty \) is contractible, and from this it is not hard to deduce that \( S^\infty \) is a universal \( \mathbb{Z}_2 \)-space, in the following sense (see, for instance, [Koz08, Prop. 8.16 and Thm. 8.17]).

**Proposition 11.** If \( X \) is any cell complex with a free cellular \( \mathbb{Z}_2 \)-action, then there exists an equivariant map \( f: X \to S^\infty \). Moreover, any two equivariant maps \( f_0, f_1: X \to S^\infty \) are equivariantly homotopic.

Any equivariant map \( f: X \to S^\infty \) induces a nontrivial equivariant chain map \( f_\sharp: C_*(X) \to C_*(S^\infty) \). A simple fact that will be crucial in what follows is that Proposition 11 has an analogue on the level of chain maps.

We first recall the relevant notion of homotopy between chain maps: Let \( C_*(X) \) and \( C_*(Y) \) be (singular or simplicial, say) chain complexes, and let \( \varphi, \psi: C_*(X) \to C_*(Y) \) be chain maps. A chain homotopy \( \eta \) between \( \varphi \) and \( \psi \) is a family of homomorphisms \( \eta_t: C_j(X) \to C_{j+1}(Y) \) such that

\[ \varphi_j - \psi_j = \partial j+1 \circ \eta_j + \eta_j \circ \partial j \]

for all \( j \).\(^{12}\) If \( X \) and \( Y \) are \( \mathbb{Z}_2 \)-spaces then a chain homotopy is called equivariant if it commutes with the (chain maps induced by) the \( \mathbb{Z}_2 \)-actions.\(^{13}\)

**Lemma 12.** If \( X \) is a cell complex with a free cellular \( \mathbb{Z}_2 \)-action then any two nontrivial equivariant chain maps \( \varphi, \psi: C_*(X) \to C_*(S^\infty) \) are equivariantly chain homotopic.\(^{14}\)

**Proof of Lemma 12.** Let the \( \mathbb{Z}_2 \)-action on \( X \) be given by the automorphism \( \nu: X \to X \). For each dimension \( i \geq 0 \), the action partitions the \( i \)-dimensional cells of \( X \) (the basis elements of \( C_i(X) \)) into pairs \( \sigma, \nu(\sigma) \). For each such pair, we arbitrarily pick one of the cells and call it the representative of the pair.

We define the desired equivariant chain homotopy \( \eta \) between \( \varphi \) and \( \psi \) by induction on the dimension, using the fact that all reduced homology groups of \( S^\infty \) are zero.\(^{15}\)

We start the induction in dimension at \( j = -1 \) (and for convenience, we also use the convention that all chain groups, chain maps, and \( \eta_t \) are understood to be zero in dimensions \( i < -1 \)). Since we assume

\(^{12}\)Here, we use subscripts and superscripts on the boundary operators to emphasize which dimension and which chain complex they belong to; often, these indices are dropped and one simply writes \( \varphi - \psi = \partial \eta + \eta \partial \).

\(^{13}\)We also recall that if \( f, g: X \to Y \) are (equivariantly) homotopic then the induced chain maps are (equivariantly) chain homotopic. Moreover, chain homotopic maps induce identical maps in homology and cohomology.

\(^{14}\)We stress that we work with the cellular chain complex for \( X \).

\(^{15}\)This just mimics the argument for the existence of an equivariant homotopy, which uses the contractibility of \( S^\infty \).
that both \( \varphi \) and \( \psi \) are nontrivial, we have that \( \varphi_{-1}, \psi_{-1} : C_{-1}(X) \to C_{-1}(S^\infty) \) are identical, and we set 
\[
\eta_{-1} : C_{-1}(X) \to C_0(S^\infty)
\]
to be zero.

Next, assume inductively that equivariant homomorphisms \( \eta_i : C_i(X) \to C_i(S^\infty) \) have already been defined for \( i < j \) and satisfy 
\[
\varphi_i - \psi_i = \eta_{i-1} \circ \partial + \partial \circ \eta_i
\]
for all \( i < j \) (note that initially, this holds true for \( j = 0 \)).

Suppose that \( \sigma \) is a \( j \)-dimensional cell of \( X \) representing a pair \( \sigma, \nu(\sigma) \). Then \( \partial \sigma \in C_{j-1}(X) \), and so 
\[
\eta_{j-1}(\partial \sigma) \in C_j(S^\infty)
\]
is already defined. We are looking for a suitable chain \( c \in C_{j+1}(S^\infty) \) which we can take to be \( \eta_j(\sigma) \) in order to satisfy the chain homotopy relation (1) also for \( i = j \), such a chain \( c \) has to satisfy \( \partial c = b \), where
\[
b := \varphi_j(\sigma) - \psi_j(\sigma) - \eta_{j-1}(\partial(\sigma)).
\]
To see that we can find such a \( c \), we compute
\[
\partial b = \partial \varphi_j(\sigma) - \partial \psi_j(\sigma) - \partial \eta_{j-1}(\partial(\sigma))
\]
\[
= \varphi_{j-1}(\partial \sigma) - \psi_{j-1}(\partial \sigma) - \left( \varphi_j(\sigma) - \psi_j(\sigma) - \eta_{j-1}(\partial(\sigma)) \right) = 0
\]

Thus, \( b \) is a cycle, and since \( H_j(S^\infty) = 0 \), \( b \) is also a boundary. Pick an arbitrary chain \( c \in C_{j+1}(S^\infty) \) with \( \partial c = b \) and set \( \eta_j(\sigma) := c \) and \( \eta_j(\nu(\sigma)) := \nu_j(c) \). We do this for all representative \( j \)-cells \( \sigma \) and then extend \( \eta_j \) by linearity. By definition, \( \eta_j \) is equivariant and (1) is now satisfied also for \( i = j \). This completes the induction step and hence the proof.

### Deleted products and Gauss maps.

Let \( K \) be a simplicial complex. Then the Cartesian product \( K \times K \) is a cell complex whose cells are the Cartesian products of pairs of simplices of \( K \). The (combinatorial) deleted product \( K \) of \( K \) is defined as the polyhedral subcomplex of \( K \times K \) whose cells are the products of vertex-disjoint pairs of simplices of \( K \), i.e., \( K := \{ \sigma \times \tau : \sigma, \tau \in K, \sigma \cap \tau = \emptyset \} \). The deleted product is equipped with a natural free \( \mathbb{Z}_2 \)-action that simply exchanges coordinates, \((x,y) \mapsto (y,x)\). Note that this action is cellular since each cell \( \sigma \times \tau \) is mapped to \( \tau \times \sigma \).

**Lemma 13.** If \( f : |K| \to \mathbb{R}^d \) is an embedding (or, more generally, an almost-embedding) then\(^{16}\) there exists an equivariant map \( \tilde{f} : \tilde{K} \to S^{d-1} \).

**Proof.** Define \( \tilde{f}(x,y) := \frac{f(x) - f(y)}{\|f(x) - f(y)\|} \). This map, called the Gauss map, is clearly equivariant. \( \Box \)

For the proof of Proposition 7, we use the following analogue of Lemma 13.

**Lemma 14.** Let \( K \) be a finite simplicial complex. If \( \gamma : C_\ast(K) \to C_\ast(\mathbb{R}^d) \) is a homological representation then there is a nontrivial equivariant chain map (called the Gauss chain map) \( \tilde{\gamma} : C_\ast(\tilde{K}) \to C_\ast(S^{d-1}) \).

The proof of this lemma is not difficult but a bit technical, so we postpone it until the end of this section.

### Obstructions.

Here, we recall a standard method for proving the non-existence of equivariant maps between \( \mathbb{Z}_2 \)-spaces. The arguments are formulated in the language of cohomology, and, as we will see, what they actually establish is the non-existence of nontrivial equivariant chain maps.

Let \( K \) be a finite simplicial complex and let \( \tilde{K} \) be its (combinatorial) deleted product. By Proposition 11, there exists an equivariant map \( G_K : \tilde{K} \to S^\infty \), which is unique up to equivariant homotopy. By factoring out the action of \( \mathbb{Z}_2 \), this induces a map \( \overline{G}_K : \tilde{K} \to \mathbb{R}P^\infty \) between the quotient spaces \( \overline{K} = \tilde{K}/\mathbb{Z}_2 \) and \( \mathbb{R}P^\infty = S^\infty/\mathbb{Z}_2 \) (the infinite-dimensional real projective space), and the homotopy class of the map \( \overline{G}_K \) depends only\(^{17}\) on \( K \). Passing to cohomology, there is a uniquely defined induced homomorphism
\[
\overline{G}_K^* : H^\ast(\mathbb{R}P^\infty) \to H^\ast(\overline{K}).
\]

\(^{16}\)We remark that a classical result due to Haefliger and Weber [Hae63, Web67] asserts that if \( \dim K \leq (2d - 3)/3 \) (the so-called metastable range) then the existence of an equivariant map from \( K \) to \( S^{d-1} \) is also sufficient for the existence of an embedding \( K \to \mathbb{R}^d \) (outside the metastable range, this fails); see [Sko08] for further background.

\(^{17}\)We stress that this does not mean that there is only one homotopy class of continuous maps \( \overline{K} \to \mathbb{R}P^\infty \); indeed, there exist such maps that do not come from equivariant maps \( K \to S^\infty \), for instance the constant map that maps all of \( \overline{K} \) to a single point.
It is known that $H^d(\mathbb{R}P^\infty) \cong \mathbb{Z}_2$ for every $d \geq 0$. Letting $\xi^d$ denote the unique generator of $H^d(\mathbb{R}P^\infty)$, there is a uniquely defined cohomology class

$$\phi^d(K) := \overline{c}_K^*(\xi^d),$$

called the van Kampen obstruction (with $\mathbb{Z}_2$-coefficients) to embedding $K$ into $\mathbb{R}^d$. For more details and background regarding the van Kampen obstruction, we refer the reader to [Mel09].

The basic fact about the van Kampen obstruction (and the reason for its name) is that $K$ does not embed (not even almost-embed) into $\mathbb{R}^d$ if $\phi^d(K) \neq 0$ (Proposition 4). This follows from Lemma 13 and Part (a) of the following lemma:

**Lemma 15.** Let $K$ be a simplicial complex and suppose that $\phi^d(K) \neq 0$.

(a) Then there is no equivariant map $\tilde{K} \to S^{d-1}$.

(b) In fact, there is no nontrivial equivariant chain map $C_*(\tilde{K}) \to C_*(S^{d-1})$.

Together with Lemma 14, Part (b) of the lemma also implies Proposition 7, as desired. The simple observation underlying the proof of Lemma 15 is the following

**Observation 16.** Suppose $\varphi: C_*(\tilde{K}) \to C_*(S^\infty)$ is a nontrivial equivariant chain map (not necessarily induced by a continuous map). By factoring out the action of $\mathbb{Z}_2$, $\varphi$ induces a chain map $\overline{\varphi}: C_*(\overline{\tilde{K}}) \to C_*(\overline{S^\infty})$. The induced homomorphism in cohomology

$$\overline{\varphi}^*: H^*(\mathbb{R}P^\infty) \to H^*(\overline{K})$$

is equal to the homomorphism $\overline{c}_K^*$ used in the definition of the Van Kampen obstruction, hence in particular

$$\phi^d(K) = \overline{\varphi}^*(\xi^d).$$

**Proof.** By Lemma 12, $\varphi$ is equivariantly chain homotopic to the nontrivial equivariant chain map $(G_K)_\sharp$ induced by the map $G_K$. Thus, after factoring out the $\mathbb{Z}_2$-action, the chain maps $\varphi$ and $(G_K)_\sharp$ from $C_*(\overline{K})$ to $C_*(\overline{S^\infty})$ are chain homotopic, and so induce identical homomorphisms in cohomology.

**Proof of Lemma 15.** If there exists an equivariant map $f: \tilde{K} \to S^{d-1}$, then the induced chain map $f_*: C_*(\tilde{K}) \to C_*(S^{d-1})$ is equivariant and nontrivial, so (b) implies (a), and it suffices to prove the former.

Next, suppose for a contradiction that $\psi: C_*(\tilde{K}) \to C_*(S^{d-1})$ is a nontrivial equivariant chain map. Let $\iota: S^{d-1} \to S^\infty$ denote the inclusion map, and let $\iota_*: C_*(S^{d-1}) \to C_*(S^\infty)$ denote the induced equivariant, nontrivial chain inclusion. Then the composition $\varphi = (\iota_\sharp \circ \psi): C_*(\tilde{K}) \to C_*(S^\infty)$ is also nontrivial and equivariant, and so, by the preceding observation, for the induced homomorphism in cohomology, we get

$$\phi^d(K) = (\iota_\sharp \circ \psi)^*(\xi^d) = \iota^*(\overline{\varphi}^*(\xi^d)).$$

However, $\overline{\varphi}^*(\xi^d) \in H^d(\mathbb{R}P^{d-1}) = 0$ (for reasons of dimension), hence $\phi^d(K) = 0$, contradicting our assumption.

**Remark 17.** The same kind of reasoning also yields the well-known Borsuk–Ulam Theorem, which asserts that there is no equivariant map $S^d \to S^{d-1}$, using the fact that the inclusion $\iota: \mathbb{R}P^d \to \mathbb{R}P^\infty$ (induced by the equivariant inclusion $i: S^d \to S^\infty$) has the property that $\overline{\varphi}^*(\xi^d)$, the pullback of the generator $\xi^d \in H^d(\mathbb{R}P^\infty)$, is nonzero.\(^{18}\) In fact, once again one gets a homological version of the Borsuk–Ulam theorem for free: there is no nontrivial equivariant chain map $C_*(S^d) \to C_*(S^{d-1})$.

**Proof of Lemma 10.** It is not hard to see that the deleted product $\overline{\partial \Delta}_{d+1} = \overline{\Delta}_{d+1}$ of the boundary of $(d + 1)$-simplex is combinatorially isomorphic to the boundary of a certain convex polytope and hence homeomorphic to $S^d$ (respecting the antipodal action), see [Mat03, Exercise 5.4.3]. Thus, the assertion $\phi^d(\overline{\partial \Delta}_{d+1}) \neq 0$ follows immediately from the preceding remark (the homological proof of the Borsuk–Ulam theorem). Together with Proposition 7, this implies that there is no homological representation of $\partial \Delta_{d+1}$ in $\mathbb{R}^d$.\(^{18}\)

\(^{18}\)In fact, it is known that $H^*(\mathbb{R}P^\infty)$ is isomorphic to the polynomial ring $\mathbb{Z}_2[\xi]$, that $H^*(\mathbb{R}P^d) \cong \mathbb{Z}_2[\xi]/(\xi^{d+1})$, and that $\overline{\varphi}^*$ is just the quotient map.
The proof of Proposition 7 is complete, except for the following:

Proof of Lemma 14. Once again, we essentially mimic the definition of the Gauss map on the level of chains. There is one minor technical difficulty due to the fact that the cells of $\hat{K}$ are products of simplices, whereas the singular homology of spaces is based on maps whose domains are simplices, not products of simplices (this is the same issue that arises in the proof of K"unneth-type formulas in homology).

Assume that $\gamma: C_*(K) \to C_*(\mathbb{R}^d)$ is a homological representation. The desired nontrivial equivariant chain map $\tilde{\gamma}: C_*(\hat{K}) \to C_*(S^{d-1})$ will be defined as the composition of three intermediate nontrivial equivariant chain maps

$$C_*(\hat{K}) \xrightarrow{\alpha} D_* \xrightarrow{\beta} C_*(\mathbb{R}^d) \xrightarrow{p_s} C_*(S^{d-1}).$$

These maps and intermediate chain complexes will be defined presently.

We define $D_*$ as a chain subcomplex of the tensor product $C_*(\mathbb{R}^d) \otimes C_*(\mathbb{R}^d)$. The tensor product chain complex has a basis consisting of all elements of the form $s \otimes t$, where $s$ and $t$ range over the singular simplices of $\mathbb{R}^d$, and we take $D_*$ as the subcomplex spanned by all $s \otimes t$ for which $s$ and $t$ have disjoint supports (note that $D_*$ is indeed a chain subcomplex, i.e., closed under the boundary operator, since if $s$ and $t$ have disjoint supports, then so do any pair of simplices that appear in the boundary of $s$ and of $t$, respectively). The chain complex $C_*(K)$ has a canonical basis consisting of cells $\sigma \times \tau$, and the chain map $\alpha$ is defined on these basis elements by “tensoring” $\gamma$ with itself, i.e.,

$$\alpha(\sigma \times \tau) := \gamma(\sigma) \otimes \gamma(\tau).$$

Since $\gamma$ is nontrivial, so is $\alpha$, the disjointness properties of $\gamma$ ensure that the image of $\alpha$ does indeed lie in $D_*$, and $\alpha$ is clearly $\mathbb{Z}_2$-equivariant.

Next, consider the Cartesian product $\mathbb{R}^d \times \mathbb{R}^d$ with the natural $\mathbb{Z}_2$-action given by flipping coordinates. This action is not free since it has a nonempty set of fixed points, namely the “diagonal” $\Delta = \{(x, x) : x \in \mathbb{R}^d\}$. However, the action on $\mathbb{R}^d \times \mathbb{R}^d$ restricts to a free action on the subspace $\mathbb{R}^d := (\mathbb{R}^d \times \mathbb{R}^d) \setminus \Delta$ obtained by removing the diagonal (this subspace is sometimes called the topological deleted product of $\mathbb{R}^d$). Moreover, there exists an equivariant map $p: \mathbb{R}^d \to S^{d-1}$ defined as follows: we identify $S^{d-1}$ with the unit sphere in orthogonal complement $\Delta^\perp = \{(w, -w) : w \in \mathbb{R}^d\}$ and take $p: \mathbb{R}^d \to S^{d-1}$ to be the orthogonal projection onto $\Delta^\perp$ (which sends $(x, y)$ to $\frac{1}{2}(x - y, y - x)$), followed by renormalizing,

$$p(x, y) := \frac{1}{\|x - y, y - x\|}(x - y, y - x) \in S^{d-1} \subset \Delta^\perp.$$ 

The map $p$ is equivariant and so the induced chain map $p_*$ is equivariant and nontrivial.

It remains to define $\beta: D_* \to C_*(\mathbb{R}^d)$. For this, we use a standard chain map

$$\text{EML}: C_*(\mathbb{R}^d) \otimes C_*(\mathbb{R}^d) \to C_*(\mathbb{R}^d \times \mathbb{R}^d),$$

sometimes called the Eilenberg–Mac Lane chain map, and then take $\beta$ to be the restriction to $D_*$.

Given a basis element $s \otimes t$ of $C_*(\mathbb{R}^d) \otimes C_*(\mathbb{R}^d)$, where $s: \Delta_p \to \mathbb{R}^d$ and $t: \Delta_q \to \mathbb{R}^d$ are singular simplices, we can view $s \otimes t$ as the map $s \otimes t: \Delta_p \times \Delta_q \to \mathbb{R}^d \times \mathbb{R}^d$ with $(x, y) \mapsto (s(x), t(y))$. This is almost like a singular simplex in $\mathbb{R}^d \times \mathbb{R}^d$, except that the domain is not a simplex but a prism (product of simplices). The Eilenberg–Mac Lane chain map is defined by prescribing a systematic and coherent way of triangulating products of simplices $\Delta_p \times \Delta_q$ that is consistent with taking boundaries; then $\text{EML}(s \otimes t) \in C_{p+q}(\mathbb{R}^d \times \mathbb{R}^d)$ is defined as the singular chain whose summands are the restrictions of the map $\sigma \otimes \tau: \Delta_p \times \Delta_q$ to the $(p + q)$-simplices that appear in the triangulation of $\Delta_p \times \Delta_q$. We refer to [GDR05] for explicit formulas for the chain map EML. What is important for us is that the chain map EML is equivariant and nontrivial. Both properties follow more or less directly from the construction of the triangulation of the prisms $\Delta_p \times \Delta_q$, which can be explained as follows: Implicitly, we assume that the vertex sets $\{0, 1, \ldots, p\}$ and $\{0, 1, \ldots, q\}$ are totally ordered in the standard way. The vertex set of $\Delta_p \times \Delta_q$ is the grid $\{0, 1, \ldots, p\} \times \{0, 1, \ldots, q\}$, on which we consider the coordinatewise partial
order defined by \((x, y) \leq (x', y')\) if \(x \leq x'\) and \(y \leq y'\). Then the simplices of the triangulation are all totally ordered subsets of this partial order. Thus, if \(\sigma = \{(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)\}\) is a simplex that appears in the triangulation of \(\Delta_p \times \Delta_q\) then the simplex \(\sigma' = \{(y_0, x_0), (y_1, x_1), \ldots, (y_n, x_n)\}\) obtained by flipping all coordinates appears in the triangulation of \(\Delta_q \times \Delta_p\); see Figure 1. This implies equivariance of EML (and it is nontrivial since it maps a single vertex to a single vertex).

\[\square\]

3 Helly-type theorems from non-embeddability

We derive Theorem 1 from obstructions to embeddability using a technique we learned from the work of Matoušek [Mat97]. In this section, we illustrate this technique, which in fact already appears in the classical proof of Helly’s convex theorem from Radon’s lemma, on a few examples, then formalize its ingredients.

Notation. Given a set \(X\) we let \(2^X\) and \(\binom{X}{k}\) denote, respectively, the set of all subsets of \(X\) (including the empty set) and the set of all \(k\)-element subsets of \(X\). If \(f: X \to Y\) is an arbitrary map between sets then we abuse the notation by writing \(f(S)\) for \(\{f(s) \mid s \in S\}\) for any \(S \subseteq X\); that is, we implicitly extend \(f\) to a map from \(2^X\) to \(2^Y\) whenever convenient.

3.1 Homotopic assumptions

Let \(\mathcal{F} = \{U_1, U_2, \ldots, U_n\}\) denote a family of subsets of \(\mathbb{R}^d\). We assume that \(\mathcal{F}\) has empty intersection and that any proper subfamily of \(\mathcal{F}\) has nonempty intersection. Our goal is to show how various conditions on the topology of the intersections of the subfamilies of \(\mathcal{F}\) imply bounds on the cardinality of \(\mathcal{F}\). For any (possibly empty) proper subset \(I\) of \([n] = \{1, 2, \ldots, n\}\) we write \(U_I\) for \(\bigcap_{i \in [n] \setminus I} U_i\). We also put \(U_{[n]} = \mathbb{R}^d\).

Path-connected intersections in the plane. Consider the case where \(d = 2\) and the intersections \(\bigcap \mathcal{G}\) are path-connected for all subfamilies \(\mathcal{G} \subseteq \mathcal{F}\). Since every intersection of \(n - 1\) members of \(\mathcal{F}\) is nonempty, we can pick, for every \(i \in [n]\), a point \(p_i \in U_{[n]}\). Moreover, as every intersection of \(n - 2\) members of \(\mathcal{F}\) is connected, we can connect any pair of points \(p_i\) and \(p_j\) by an arc \(s_{i,j}\) inside \(U_{[n]}\). We thus obtain a drawing of the complete graph on \([n]\) in the plane in a way that the edge between \(i\) and \(j\) is contained in \(U_{[n]}\) (see Figure 2). If \(n \geq 5\) then the stronger form of non-planarity of \(K_5\) implies that there exist two edges \(\{i, j\}\) and \(\{k, \ell\}\) with no vertex in common and whose images intersect (see Proposition 4 and Lemma 5). Since \(U_{\{i, j\}} \cap U_{\{k, \ell\}} = \bigcap \mathcal{F} = \emptyset\), this cannot happen and \(\mathcal{F}\) has cardinality at most 4.

\([d/2]\)-connected intersections in \(\mathbb{R}^d\). The previous argument generalizes to higher dimension as follows. Assume that the intersections \(\bigcap \mathcal{G}\) are \([d/2]\)-connected\(^{19}\) for all subfamilies \(\mathcal{G} \subseteq \mathcal{F}\). Then we can build by induction a function \(f\) from the \([d/2]\)-skeleton of \(\Delta_{n-1}\) to \(\mathbb{R}^d\) in a way that for any simplex \(\sigma\), the image \(f(\sigma)\) is contained in \(U_\sigma\). The previous case shows how to build such a function.

\(^{19}\)Recall that a set is \(k\)-connected if it is connected and has vanishing homotopy in dimension 1 to \(k\).
from the 1-skeleton of $\Delta_{n-1}$. Assume that a function $f$ from the $\ell$-skeleton of $\Delta_{n-1}$ is built. For every $(\ell+1)$-simplex $\sigma$ of $\Delta_{n-1}$, for every facet $\tau$ of $\sigma$, we have $f(\tau) \subset U_\tau \subset U_\tau$. Thus, the set

$$\bigcup_{\tau \text{ facet of } \sigma} f(\tau)$$

is the image of an $\ell$-dimensional sphere contained in $U_\tau$, which is contractible. We can fill this sphere by a $(\ell+1)$-dimensional ball contained in $U_\tau$, thus extending $f$ to the $(\ell+1)$-skeleton of $\Delta_{n-1}$.

The Van Kampen-Flores theorem asserts that for any continuous function from $\Delta^k_{2k+2}$ to $\mathbb{R}^k$ there exist two disjoint faces of $\Delta^k_{2k+2}$ whose images intersect (see Proposition 4 and Lemma 5). So, if $n \geq 2[d/2] + 3$, then there exist two disjoint simplices $\sigma$ and $\tau$ of $\Delta^d_{[d/2]+2}$ such that $f(\sigma) \cap f(\tau)$ is nonempty. Since $f(\sigma) \cap f(\tau)$ is contained in $U_\sigma \cap U_\tau = \bigcap \mathcal{F} = \emptyset$, this is a contradiction and $\mathcal{F}$ has cardinality at most $2[d/2] + 2$.

By a more careful inspection of odd dimensions, the bound $2[d/2] + 2$ can be improved to $d + 2$. We skip this in homotopic setting, but we will do so in homological setting (which is stronger anyway); see Corollary 18 below.

**Contractible intersections.** Of course, the previous argument works with other non-embeddability results. For instance, if the intersections $\bigcap \mathcal{G}$ are contractible for all subfamilies then the induction yields a map $f$ from the $d$-skeleton of $\Delta_{n-1}$ to $\mathbb{R}^d$ with the property that for any simplex $\sigma$, the image $f(\sigma)$ is contained in $U_\sigma$. The topological Radon theorem [BB79] (see also [Mat03, Theorem 5.1.2]) states that for any continuous function from $\Delta^k_{d+1}$ to $\mathbb{R}^d$ there exist two disjoint faces of $\Delta^k_{d+1}$ whose images intersect. So, if $n \geq d + 2$ we again obtain a contradiction (the existence of two disjoint simplices $\sigma$ and $\tau$ such that $f(\sigma) \cap f(\tau) \neq \emptyset$ whereas $U_\sigma \cap U_\tau = \bigcap \mathcal{F} = \emptyset$), and the cardinality of $\mathcal{F}$ must be at most $d + 1$.

### 3.2 From homotopy to homology

The previous reasoning can be transposed in homology as follows. Assume that for $i = 0, 1, \ldots, k-1$ and all subfamilies $\mathcal{G} \subseteq \mathcal{F}$ we have $\hat{\beta}_i(\bigcap \mathcal{G}) = 0$. We construct a nontrivial\(^{20}\) chain map $f$ from the simplicial chains of $\Delta_{n-1}^{(k)}$ to the singular chains of $\mathbb{R}^d$ by increasing dimension:

- For every $\{i\} \subset [n]$ we let $p_i \in U_{\bigcup \mathcal{G}}$. This is possible since every intersection of $n-1$ members of $\mathcal{F}$ is nonempty. We then put $f(\{i\}) = p_i$ and extend it by linearity into a chain map from $\Delta_{n-1}^{(0)}$ to $\mathbb{R}^d$. Notice that $f$ is nontrivial and that for any 0-simplex $\sigma \subseteq [n]$, the support of $f(\sigma)$ is contained in $U_\sigma$.

- Now, assume, as an induction hypothesis, that there exists a nontrivial chain map $f$ from the simplicial chains of $\Delta_{n-1}^{(k)}$ to the singular chains of $\mathbb{R}^d$ with the property that for any $(\leq \ell)$-simplex

\(^{20}\)See Definition 1.
σ ⊆ [n], ℓ < k, the support of f(σ) is contained in Uγ. Let σ be a (ℓ + 1)-simplex in \( \Delta^{(\ell+1)}_{n-1} \). For every ℓ-dimensional face τ of σ, the support of f(τ) is contained in Uγ ⊆ Uγ. It follows that the support of f(∂σ) is contained in Uγ, which has trivial homology in dimension ℓ + 1. As a consequence, f(∂σ) is a boundary in Uγ. We can therefore extend f to every simplex of dimension ℓ + 1 and then, by linearity, to a chain map from the simplicial chains of \( \Delta^{(\ell+1)}_{n-1} \) to the singular chains of \( \mathbb{R}^d \). This chain map remains nontrivial and, by construction, for any (≤ ℓ + 1)-simplex σ ⊆ [n], the support of f(σ) is contained in Uγ.

If σ and τ are disjoint simplices of \( \Delta^{(k)}_{n-1} \) then the intersection of the supports of f(σ) and f(τ) is contained in \( U_γ \cap U_γ = \emptyset \) and these supports are disjoint. It follows that f is not only a nontrivial chain map, but also a homological representation in \( \mathbb{R}^d \). We can then use obstructions to the existence of homological representations to bound the cardinality of F. Specifically, since we assumed that F has empty intersection and any proper subfamily of F has nonempty intersection, Corollary 9 implies:

**Corollary 18.** Let F be a family of subsets of \( \mathbb{R}^d \) such that \( \tilde{β}_i(\bigcap G) = 0 \) for every \( G \subseteq F \) and \( i = 0, 1, \ldots, \lceil d/2 \rceil - 1 \). Then the Helly number of F is at most \( d + 2 \).

The homological Radon’s lemma (Lemma 10) yields (noting \( \partial \Delta_{d+1} = \Delta^{(d)}_{d+1} \)):

**Corollary 19.** Let F be a family of subsets of \( \mathbb{R}^d \) such that \( \tilde{β}_i(\bigcap G) = 0 \) for every \( G \subseteq F \) and \( i = 0, 1, \ldots, d - 1 \). Then the Helly number of F is at most \( d + 1 \).

**Remark 20.** The following modification of Remark 2(d) shows that the two previous statements are sharp in various ways. First assume that for some values \( k, n \) there exists some embedding \( f \) of \( \Delta^{(k)}_{n-1} \) into \( \mathbb{R}^d \). Let \( K_i \) be the simplicial complex obtained by deleting the ith vertex of \( \Delta^{(k)}_{n-1} \) (as well as all simplices using that vertex) and put \( U_i := f(K_i) \). The family \( F = \{U_1, \ldots, U_n\} \) has Helly number exactly \( n \), since it has empty intersection and all its proper subfamilies have nonempty intersection. Moreover, for every \( G \subseteq F \), \( \bigcap G \) is the image through \( f \) of the \( k \)-skeleton of a simplex on \( |F \setminus G| \) vertices, and therefore \( \tilde{β}_i(\bigcap G) = 0 \) for every \( G \subseteq F \) and \( i = 0, \ldots, k - 1 \). Now, such an embedding exists for:

- \( k = d \) and \( n = d + 1 \), as the d-dimensional simplex easily embeds into \( \mathbb{R}^d \). Consequently, the bound of \( d + 1 \) is best possible under the assumptions of Corollary 19.

- \( k = d - 1 \) and \( n = d + 2 \), as we can first embed the \((d - 1)\)-skeleton of the d-simplex linearly, then add an extra vertex at the barycenter of the vertices of that simplex and embed the remaining faces linearly. This implies that if we relax the condition of Corollary 19 by only controlling the first \( d - 2 \) Betti numbers then the bound of \( d + 1 \) becomes false. It also implies that the bound of \( d + 2 \) is best possible under (a strengthening of) the assumptions of Corollary 18.

(Recall that, as explained in Remark 2(d), the \( \lceil d/2 \rceil - 1 \) in the assumptions of Corollary 18 cannot be reduced without allowing unbounded Helly numbers.)

**Constrained chain map.** Let us formalize the technique illustrated by the previous example. We focus on the homological setting, as this is what we use to prove Theorem 1, but this can be easily transposed in homotopy.

Considering a slightly more general situation, we let \( F = \{U_1, U_2, \ldots, U_n\} \) denote a family of subsets of some topological space \( \mathbb{R} \). As before for any (possibly empty) proper subset \( I \) of \( [n] = \{1, 2, \ldots, n\} \) we write \( U_I \) for \( \bigcap_{i \in [n] \setminus I} U_i \) and we put \( U_{[n]} = \mathbb{R} \).

Let \( K \) be a simplicial complex and let \( γ : C_*(K) \to C_*(\mathbb{R}) \) be a chain map from the simplicial chains of \( K \) to the singular chains of \( \mathbb{R} \). We say that \( γ \) is constrained by \((F, Φ)\) if:

1. \( Φ \) is a map from \( K \) to \( 2^{[n]} \) such that \( Φ(σ ∩ τ) = Φ(σ) ∩ Φ(τ) \) for all \( σ, τ ∈ K \) and \( Φ(∅) = ∅ \).
2. For any simplex \( σ ∈ K \), the support of \( γ(σ) \) is contained in \( U_{\Phi(σ)} \).

See Figure 3. We also say that a chain map \( γ \) from \( K \) is constrained by \( F \) if there exists a map \( Φ \) such that \( γ \) is constrained by \((F, Φ)\). In the above constructions, we simply set \( Φ \) to be the identity. As we already saw, constrained chain maps relate Helly numbers to homological representations (see Definition 1) via the following observation:
**Lemma 21.** Let $\gamma : C_*(K) \to C_*(\mathbb{R})$ be a nontrivial chain map constrained by $\mathcal{F}$. If $\bigcap \mathcal{F} = \emptyset$ then $\gamma$ is a homological representation of $K$.

**Proof.** Let $\Phi : K \to 2^{[n]}$ be such that $\gamma$ is constrained by $(\mathcal{F}, \Phi)$. Since $\gamma$ is nontrivial, it remains to check that disjoint simplices are mapped to chains with disjoint support. Let $\sigma$ and $\tau$ be two disjoint simplices of $K$. The supports of $\gamma(\sigma)$ and $\gamma(\tau)$ are contained, respectively, in $U_{\Phi(\sigma)}$ and $U_{\Phi(\tau)}$, and

$$U_{\Phi(\sigma)} \cap U_{\Phi(\tau)} = U_{\Phi(\sigma) \cap \Phi(\tau)} = U_{\Phi(\sigma) \cap \Phi(\tau)} = U_{\Phi(\emptyset)} = U_0 = \bigcap \mathcal{F}.$$ 

Therefore, if $\bigcap \mathcal{F} = \emptyset$ then $\gamma$ is a homological representation of $K$. 

\[\square\]

### 3.3 Relaxing the connectivity assumption

In all the examples listed so far, the intersections $\bigcap G$ must be connected. Matoušek [Mat97] relaxed this condition into “having a bounded number of connected components”, the assumptions then being on the topology of the components, by using Ramsey’s theorem. The gist of our proof is to extend his idea to allow a bounded number of homology classes not only in the first dimension but in any dimension. Let us illustrate how Matoušek’s idea works in two dimension:

**Theorem 22 ([Mat97, Theorem 2 with $d = 2$]).** For every integer $b$ there is an integer $h(b)$ with the following property. If $\mathcal{F}$ is a finite family of subsets of $\mathbb{R}^2$ such that the intersection of any subfamily has at most $b$ path-connected components, then the Helly number of $\mathcal{F}$ is at most $h(b)$.

Let us fix $b$ above and assume that for any subfamily $G \subseteq \mathcal{F}$ the intersection $\bigcap G$ consists of at most $b$ path-connected components and that $\bigcap \mathcal{F} = \emptyset$. We start, as before, by picking for every $i \in [n]$, a point $p_i$ in $U_{\{i\}}^{(i)}$. This is possible as every intersection of $n - 1$ members of $\mathcal{F}$ is nonempty. Now, if we consider some pair of indices $i, j \in [n]$, the points $p_i$ and $p_j$ are still in $U_{\{i,j\}}^{(i,j)}$ but may lie in different connected components. It may thus not be possible to connect $p_i$ to $p_j$ inside $U_{\{i,j\}}^{(i,j)}$. If we, however, consider $b + 1$ indices $i_1, i_2, \ldots, i_{b+1}$ then all the points $p_{i_1}, p_{i_2}, \ldots, p_{i_{b+1}}$ are in $U_{\{i_1,i_2,\ldots,i_{b+1}\}}^{(i_1,i_2,\ldots,i_{b+1})}$ which has at most $b$ connected components, so at least one pair among of these points can be connected by a path inside $U_{\{i_1,i_2,\ldots,i_{b+1}\}}^{(i_1,i_2,\ldots,i_{b+1})}$. Thus, while we may not get a drawing of the complete graph on $n$ vertices we can still draw many edges.

To find many vertices among which every pair can be connected we will use the hypergraph version of the classical theorem of Ramsey:
Theorem 23 (Ramsey [Ram29]). For any $x$, $y$ and $z$ there is an integer $R_x(y, z)$ such that any $x$-uniform hypergraph on at least $R_x(y, z)$ vertices colored with at most $y$ colors contains a subset of $z$ vertices inducing a monochromatic sub-hypergraph.

From the discussion above, for any $b+1$ indices $i_1 < i_2 < \ldots < i_{b+1}$ there exists a pair $\{k, \ell\} \in \left(\begin{bmatrix} b+1 \\ 2 \end{bmatrix}\right)$ such that $p_{i_k}$ and $p_{i_\ell}$ can be connected inside $U_{\{i_1, i_2, \ldots, i_{b+1}\}}$. Let us consider the $(b+1)$-uniform hypergraph on $[n]$ and color every set of indices $i_1 < i_2 < \ldots < i_{b+1}$ by one of the pairs in $\left(\begin{bmatrix} b+1 \\ 2 \end{bmatrix}\right)$ that can be connected inside $U_{\{i_1, i_2, \ldots, i_{b+1}\}}$ (if more than one pair can be connected, we pick one arbitrarily). Let $t$ be some integer to be fixed later. By Ramsey’s theorem, if $n \geq R_{b+1} \left(\begin{bmatrix} b+1 \\ 2 \end{bmatrix}, t\right)$ then there exist a pair $\{k, \ell\} \in \left(\begin{bmatrix} b+1 \\ 2 \end{bmatrix}\right)$ and a subset $T \subseteq [n]$ of size $t$ with the following property: for any $(b+1)$-element subset $S \subset T$, the points whose indices are the $k$th and $\ell$th indices of $S$ can be connected inside $U_T$.

Now, let us set $t = 5 + \binom{\binom{2}{b+1}}{1} (b-1) = 10b - 5$. We claim that we can find five indices in $T$, denoted $i_1, i_2, \ldots, i_5$, and, for each pair $\{i_u, i_v\}$ among these five indices, some $(b+1)$-element subset $Q_{u,v} \subset T$ with the following properties:

(i) $i_u$ and $i_v$ are precisely in the $k$th and $\ell$th position in $Q_{u,v}$, and

(ii) for any $1 \leq u, v, u', v' \leq 5$, $Q_{u,v} \cap Q_{u',v'} = \{i_u, i_v\} \cap \{i_{u'}, i_{v'}\}$.

We first conclude the argument, assuming that we can obtain such indices and sets. Observe that from the construction of $T$, the $i_u$‘s and the $Q_{u,v}$‘s we have the following property: for any $u, v \in [5]$, we can connect $p_{i_u}$ and $p_{i_v}$ inside $U_{Q_{u,v}}$. This gives a drawing of $K_5$ in the plane. Since $K_5$ is not planar, there exist two edges with no vertex in common, say $\{u, v\}$ and $\{u', v'\}$, that cross. This intersection point must lie in $U_{Q_{u,v}} \cap U_{Q_{u,v'}} = U_{Q_{u,v} \cup Q_{u,v'}} = U_{\{i_u, i_v\} \cup \{i_{u'}, i_{v'}\}} = U_\emptyset = \bigcap \mathcal{F} = \emptyset$, a contradiction. It must then be that the assumption that $n \geq R_{b+1} \left(\begin{bmatrix} b+1 \\ 2 \end{bmatrix}, t\right)$ is false and $\mathcal{F}$ has cardinality at most $R_{b+1} \left(\begin{bmatrix} b+1 \\ 2 \end{bmatrix}, 10b - 5\right) - 1$, which is our $b(b)$.

The selection trick. It remains to derive the existence of the $i_u$‘s and the $Q_{u,v}$‘s. It is perhaps better to demonstrate the method on a simple example to develop some intuition before we formalize it.

Example. Let us fix $b = 4$ and $\{k, \ell\} = \{2, 3\} \in \left(\begin{bmatrix} 4+1 \\ 2 \end{bmatrix}\right)$. We first make a ‘blueprint’ for the construction inside the rational numbers. For any two indices $u, v \in [5]$ we form a set $Q_{u,v} \subseteq \mathbb{Q}$ of size $b + 1 = 5$ by adding three rational numbers (different from $1, \ldots, 5$) to the set $\{u, v\}$ in such a way that $\{u, v\}$ appear on the 2nd and 3rd position of $Q_{u,v}$. For example, we can set $Q_{1,4}$ to be $\{0.5; 1; 4; 4.7; 5.13\}$. Apart from this, we require that we add a different set of rational numbers for each $\{u, v\}$. Thus $Q_{u,v} \cap Q_{u',v'} = \{u, v\} \cap \{u', v'\}$. Our blueprint now appears inside the set $T' := \bigcup_{1 \leq u < v \leq 5} Q_{u,v}$; note that both this set $T'$ and the set $T$ inside which we search for the sets $Q_{u,v}$ have 35 elements. To obtain the required indices $i_u$ and sets $Q_{u,v}$ we must consider the unique strictly increasing bijection $\pi_0: T' \to T$ and set $i_u := \pi_0(u)$ and $Q_{u,v} := \pi_0(Q_{u,v})$.

The general case. Let us now formalize the generalization of this trick that we will use to prove Theorem 1. Let $Q$ be a subset of $[w]$. If $e_1 < e_2 < \ldots < e_w$ are the elements of a totally ordered set $W$ then we call $\{e_i : i \in Q\}$ the subset selected by $Q$ in $W$.

Lemma 24. Let $1 \leq q \leq w$ be integers and let $Q$ be a subset of $[w]$ of size $q$. Let $Y$ and $Z$ be two finite totally ordered sets and let $A_1, A_2, \ldots, A_r$ be $q$-element subsets of $Y$. If $|Z| \geq |Y| + r(w - q)$, then there exist an injection $\pi: Y \to Z$ and $r$ subsets $W_1, W_2, \ldots, W_r \in \binom{[w]}{w}$ such that for every $i \in [r]$, $Q$ selects $\pi(A_i)$ in $W_i$. We can further require that $W_i \cap W_j = \pi(A_i \cap A_j)$ for any two $i, j \in [r]$, $i \neq j$.

Proof. Let $\pi_0$ denote the monotone bijection between $Y$ and $[|Y|]$. For $i \in [r]$ we let $D_i$ denote a set of $w - q$ rationals, disjoint from $[|Y|]$, such that $Q$ selects $\pi_0(A_i)$ in $D_i \cup \pi_0(A_i)$. We further require that the $D_i$ are pairwise disjoint, and put $Z' = [|Y|] \cup \left(\bigcup_{i \in [r]} D_i\right)$. Since $|Z| \geq |Y| + r(w - q) = |Z'|$ there exists a strictly increasing map $\nu: Z' \to Z$. We set $\pi := \nu \circ \pi_0$ and $W_i := \nu(D_i \cup \pi_0(A_i)) \in \binom{[w]}{w}$. The desired condition is satisfied by this choice. See Figure 4. \qed
Figure 4: Illustration for the proof of Lemma 24. We assume that $w = 4$ and $Q = \{1, 3, 4\}$.

4 Constrained chain maps and Helly number

We now generalize the technique presented in Section 3 to obtain Helly-type theorems from non-embeddability results. We will construct constrained chain maps for arbitrary complexes. As above, $\mathcal{F} = \{U_1, U_2, \ldots, U_n\}$ denotes a family of subsets of some topological space $R$ and for $I \subseteq [n]$ we keep the notation $U_I$ as used in the previous section. Note that although so far we only used the reduced Betti numbers $\beta$, in this section it will be convenient to work with standard (non-reduced) Betti numbers. Let $\beta_1(K)$ be the number of vertices of $K$.

Proposition 25. For any finite simplicial complex $K$ and integer $b$ there exists a constant $h_K(b)$ such that the following holds. For any finite family $\mathcal{F}$ of at least $h_K(b)$ subsets of a topological space $R$ such that $\bigcap \mathcal{G} \neq \emptyset$ and $\beta_i(\bigcap \mathcal{G}) \leq b$ for any $\mathcal{G} \subseteq \mathcal{F}$ and any $0 \leq i < \dim K$, there exists a nontrivial chain map $\gamma : C_*(K) \to C_*(R)$ that is constrained by $\mathcal{F}$.

The case $K = \Delta^{(k)}_{2k+2}$, with $k = \lceil d/2 \rceil$ and $R = \mathbb{R}^d$, of Proposition 25 implies Theorem 1.

Proof of Theorem 1. Let $b$ and $d$ be fixed integers, let $k = \lceil d/2 \rceil$ and let $K = \Delta^{(k)}_{2k+2}$. Let $h_K(b+1)$ denote the constant from Proposition 25 (we plug in $b+1$ because we need to switch between reduced and non-reduced Betti numbers). Let $\mathcal{F}$ be a finite family of subsets of $\mathbb{R}^d$ such that $\beta_i(\bigcap \mathcal{G}) \leq b$ for any $\mathcal{G} \subseteq \mathcal{F}$ and every $0 \leq i \leq \dim K = \lceil d/2 \rceil - 1$, in particular $\beta_i(\bigcap \mathcal{G}) \leq b + 1$ for such $\mathcal{G}$. Let $\mathcal{F}^*$ denote an inclusion-minimal sub-family of $\mathcal{F}$ with empty intersection: $\bigcap \mathcal{F}^* = \emptyset$ and $\bigcap (\mathcal{F}^* \setminus \{U\}) \neq \emptyset$ for any $U \in \mathcal{F}^*$. If $\mathcal{F}^*$ has size at least $h_K(b+1)$, it satisfies the assumptions of Proposition 25 and there exists a nontrivial chain map from $K$ that is constrained by $\mathcal{F}^*$. Since $\mathcal{F}^*$ has empty intersection, this chain map is a homological representation by Lemma 21. However, no such homological representation exists by Corollary 8, so $\mathcal{F}^*$ must have size at most $h_K(b+1) - 1$. As a consequence, the Helly number of $\mathcal{F}$ is bounded and the statement of Theorem 1 holds with $h(b,d) = h_K(b+1) - 1$.

The rest of this section is devoted to prove Proposition 25. We proceed by induction on the dimension of $K$, Section 4.1 settling the case of 0-dimensional complexes and Section 4.3 showing that if Proposition 25 holds for all simplicial complexes of dimension $i$ then it also holds for all simplicial complexes of dimension $i + 1$. As the proof of the induction step is quite technical, as a warm-up, we provide the reader with a simplified argument for the induction step from $i = 0$ to $i = 1$ in Section 4.2. We let $V(K)$ and $v(K)$ denote, respectively, the set of vertices and the number of vertices of $K$. 

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holds with $5$ holds when dim.

As a warm-up, we now prove Proposition 4.2 Principle of the induction mechanism (4.1 Initialization (dim $K = 0$))

If $K$ is a 0-dimensional simplicial complex then Proposition 25 holds with $h_K(b) = v(K)$. Indeed, consider a family $F$ of at least $v(K)$ subsets of $R$ such that all proper subfamilies have nonempty intersection. We enumerate the vertices of $K$ as $\{v_1, v_2, \ldots, v_{v(K)}\}$ and define $\Phi(\{v_i\}) = \{i\}$; in plain English, $\Phi$ is a bijection between the set of vertices of $K$ and $\{1, 2, \ldots, v(K)\}$. We first define $\gamma$ on $K$ by mapping every vertex $v \in K$ to a point $p(v) \in U_{\Psi(v)}$ then extend it linearly into a chain map $\gamma : C_0(K) \to C_0(R)$. It is clear that $\gamma$ is nontrivial and constrained by $(F, \Phi)$, so Proposition 25 holds when dim $K = 0$.

4.2 Principle of the induction mechanism (dim $K = 1$)

As a warm-up, we now prove Proposition 25 for 1-dimensional simplicial complexes. While this merely amounts to reformulating Matousek’s proof for embeddings [Mat97] in the language of chain maps, it still introduces several key ingredients of the induction while avoiding some of its complications. To avoid further technicalities, we use the non-reduced version of Betti numbers here.

Let $K$ be a 1-dimensional simplicial complex with vertices $\{v_1, v_2, \ldots, v_{v(K)}\}$ and assume that $F$ is a finite family of subsets of a topological space $R$ such that for any $G \subseteq F$, $\bigcap G \neq \emptyset$ and $\beta_1(G) \leq b$. Let $s \in N$ denote some parameter, to be fixed later. We assume that the cardinality of $F$ is large enough (as a function of $s$) so that, as argued in Subsection 4.1, there exist a bijection $\Psi : \Delta_{s}^{(0)} \to [s + 1]$ and a nontrivial chain map $\gamma' : C_*(\Delta_{s}^{(0)}) \to C_*(R)$ constrained by $(F, \Psi)$. We extend $\Psi$ to $\Delta_{s}$ by putting $\Psi(\sigma) = \cup_{v \in \sigma} \Psi(v)$ for any $\sigma \in \Delta_{s}$ and $\Psi(\emptyset) = \emptyset$. Remark that for any $\sigma, \tau \in \Delta_{s}$ we have $\Psi(\sigma \cap \tau) = \Psi(\sigma) \cap \Psi(\tau)$.

We now look for an injection $f$ of $V(K)$ into $V(\Delta_{s})$ such that the chain map $\gamma' \circ f_2 : C_*(\Delta_{s}^{(0)}) \to C_*(R)$ can be extended into a chain map $\gamma : C_*(K) \to C_*(R)$ constrained by $F$. Let $e = \{u, v\}$ be an edge in $K$. If we could arrange that $\gamma'(f(u) + f(v))$ is a boundary in $U_{\Psi(f(v))}$ then we could simply define $\gamma(e)$ to be a chain in $U_{\Psi(f(v))}$ bounded by $\gamma'(f(u) + f(v))$ (see Figure 5). Unfortunately this is too much to ask but we can still follow the Ramsey-based approach of Subsection 3.3: we add “dummy” vertices to $\Psi([f(u), f(v)])$ to obtain a set $W_e$ such that $\gamma'(f(u) + f(v))$ is a boundary in $U_{\Psi(W_e)}$. If we use different dummy vertices for distinct edges then setting $\gamma(e)$ to be a chain in $U_{\Psi(W_e)}$ bounded by $\gamma'(f(u) + f(v))$ still yields a chain map constrained by $F$. We spell out the details in four steps.
Step 1. Any set $S$ of $2^b + 1$ vertices of $\Delta_s$ contains two vertices $u_S, v_S \in S$ such that $\gamma'(u_S + v_S)$ is a boundary in $U_{\Phi(S)}$. Indeed, first notice that for any $u \in S$, the support of $\gamma'(u)$ is contained in $U_{\Phi(S)}$. The assumption on $\mathcal{F}$ about bounded Betti numbers of intersections of subfamilies of $\mathcal{F}$ then ensures that there are at most $2^b$ distinct elements in $H_0(U_{\Phi(S)}).$ Thus, there are two vertices $u_S, v_S \in S$ such that $\gamma'(u_S)$ and $\gamma'(v_S)$ are in the same homology class in $H_0(U_{\Phi(S)}).$ Since we consider homology with coefficients over $\mathbb{Z}_2$, the sum of two chains that are in the same homology class is always a boundary. In particular, $\gamma'(u_S + v_S) = \gamma'(u_S) + \gamma'(v_S)$ is a boundary in $U_{\Phi(S)}$.

Step 2. We use Ramsey’s theorem (Theorem 23) to ensure a uniform “2-in-(2^b + 1)” selection. Let $t$ be some parameter to be fixed in Step 3 and let $H$ denote the $(2^b + 1)$-uniform hypergraph with vertex set $V(\Delta_s)$. For every hyperedge $S \in H$ there exists (by Step 1) a pair $Q_S \in \binom{(2^b + 1)}{2}$ that selects a pair whose sum is mapped by $\gamma'$ to a boundary in $U_{\Phi(S)}$. We color $H$ by assigning to every hyperedge $S$ the “color” $Q_S$. Ramsey’s theorem thus ensures that if $s \geq R_{2^b+1}\left(\binom{2^b+1}{2},t\right)$ then there exist a set $T$ of $t$ vertices of $\Delta_s$ and a pair $Q^* \in \binom{(2^b + 1)}{2}$ so that $Q^*$ selects in any $S \in \binom{T}{2^b+1}$ a pair $\{u_S, v_S\}$ such that $\gamma'(u_S + v_S)$ is a boundary in $U_{\Phi(S)}$.

Step 3. Now, let $r$ be the number of edges of $K$ and let $\sigma_1, \sigma_2, \ldots, \sigma_r$ denote the edges of $K$. We define

$$h_K(b) = R_{2^b+1}\left(\binom{2^b+1}{2}, r(2^b - 1) + v(K)\right) + 1$$

and we assume that $s \geq h_K(b) - 1$. We set the parameter $t$ introduced in Step 2 to $t = r(2^b - 1) + v(K)$. We can now apply Lemma 24 with $Y = V(K)$, $Z = T$, $q = 2$, $w = 2^b + 1$, and $A_i = \sigma_i$ for $i \in [r]$. As a consequence, there exist an injection $f : V(K) \to T$ and $W_1, W_2, \ldots, W_r \in \binom{T}{2^b+1}$ such that (i) for each $i$, $Q^*$ selects $f(\sigma_i)$ in $W_i$, and (ii) $W_i \cap W_j = f(\sigma_i \cap \sigma_j)$ for $i, j \in [r], i \neq j$.

Step 4. We define $\Phi$ by

$$\Phi(\emptyset) = \emptyset$$
$$\Phi(\{\sigma_i\}) = \Psi(f(\sigma_i)) \quad \text{for } i = 1, 2, \ldots, v(K)$$
$$\Phi(\sigma_i) = \Psi(W_i) \quad \text{for } i = 1, 2, \ldots, r$$

We define $\gamma$ over the vertices of $K$ by putting $\gamma(v) = \gamma'(f(v))$ for any $v \in V(K)$. Now remark that for any edge $\sigma_i = \{u, v\}$ of $K$, $\gamma'(f(u) + f(v))$ is a boundary in $U_{\Phi(W_i)}$, this follows from the definition of $T$ and the fact that $Q^*$ selects $\{f(u), f(v)\}$ in $W_i$. We can therefore define $\gamma(\{u, v\})$ to be some (arbitrary) chain in $U_{\Phi(W_i)}$ with boundary $\gamma'(f(u) + f(v))$. We then extend this map linearly into a chain map $\gamma : C_*(K) \to C_*(R)$.

To conclude the proof of Proposition 25 for 1-dimensional complexes it remains to check that the chain map $\gamma$ and the function $\Phi$ defined in Step 4 have the desired properties.

Observation 26. $\gamma$ is a nontrivial chain map constrained by $(\mathcal{F}, \Phi)$.

Proof. First, it is clear from the definition that $\gamma$ is a chain map. Moreover, the definition of $\gamma'$ ensures that for every vertex $v \in K$ the support of $\gamma(v)$ is a finite set of points with odd cardinality. So $\gamma$ is indeed a nontrivial chain map.

The map $\Phi$ is from $K$ to $2^{b+1}$ and $\Phi(\emptyset)$ is by definition the empty set. The next property to check is that the identity $\Phi(\sigma \cap \tau) = \Phi(\sigma) \cap \Phi(\tau)$ holds for all $\sigma, \tau \in K$. When $\sigma$ and $\tau$ are vertices this follows from the injectivity of $\Psi$ and $f$. When $\sigma$ and $\tau$ are edges this follows from the same identity for $\Psi$ and $f$. We could require that $\gamma'$ sends every vertex to a point in $U_{\Phi(W_i)}$, i.e. is a chain map induced by a map, and simply argue that since $U_{\Phi(W_i)}$ has at most $b$ connected components, any $b + 1$ vertices of $\Delta_s$ contains some pair that can be connected inside $U_{\Phi(W_i)}$. This argument does not, however, work in higher dimension. Since Section 4.2 is meant as an illustration of the general case, we choose to follow the general argument.

\footnote{We could require that $\gamma'$ sends every vertex to a point in $U_{\Phi(W_i)}$, i.e. is a chain map induced by a map, and simply argue that since $U_{\Phi(W_i)}$ has at most $b$ connected components, any $b + 1$ vertices of $\Delta_s$ contains some pair that can be connected inside $U_{\Phi(W_i)}$. This argument does not, however, work in higher dimension. Since Section 4.2 is meant as an illustration of the general case, we choose to follow the general argument.}

\footnote{\textit{Remark:} $H_0(U_{\Phi(W_i)}) \simeq \mathbb{Z}_2^m$ for some $m \leq b$.}
the fact that Step 4 guaranteed that \( W_i \cap W_j = f(\sigma_i \cap \sigma_j) \) for \( i, j \in [r], i \neq j \). The remaining case is when \( \sigma = \sigma_i \) is an edge and \( \tau \) is a vertex. Then, by construction, \( \tau \in \sigma \) if and only if \( f(\tau) \in W_i \), and

\[
\Phi(\sigma_i) \cap \Phi(\tau) = \Psi(W_i) \cap \Psi(f(\tau)) = \Psi(W_i \cap f(\tau)) = \left\{ \begin{array}{ll}
\Psi(\emptyset) & \text{if } f(\tau) \notin W_i \\
\Psi(f(\tau)) & \text{if } f(\tau) \in W_i
\end{array} \right\} = \Phi(\sigma_i \cap \tau).
\]

It remains to check that for any simplex \( \sigma \in K \), the support of \( \gamma(\sigma) \) is contained in \( U_{\Phi(\sigma)} \). When \( \sigma = \{v\} \) is a vertex then \( \gamma(\sigma) = \gamma'(f(v)) \). Since \( \gamma' \) is constrained by \( (F, \Psi) \), the support of \( \gamma'(f(v)) \) is contained in \( U_{\Phi(f(v))} = U_{\Phi(\sigma)} \), so the property holds. When \( \sigma = \sigma_i \) is an edge, \( \gamma(\sigma_i) \) is, by construction, a chain in \( U_{\Phi(W_i)} = U_{\Phi(\sigma)} \) and the property also holds.

4.3 The induction

Let \( k \geq 2 \), let \( K \) be a simplicial complex of dimension \( k \) and assume that Proposition 25 holds for all simplicial complexes of dimension \( k - 1 \) or less. Let \( F \) be a finite family of subsets of a topological space \( R \) such that for any \( G \subseteq F \) and any \( 0 \leq i \leq k - 1 \), \( \bigcap G \neq \emptyset \) and \( \beta_i(\bigcap G) \leq b \). We want to construct, assuming \( F \) contains sufficiently many sets, a nontrivial chain map \( \gamma : C_*(K) \to C_*(R) \) constrained by \( F \).

**Preliminary example.** When going from \( k = 0 \) to \( k = 1 \), the first step (as described in Section 4.2) is to start with a constrained chain map \( \gamma' : C_*(K^{(0)}) \to C_*(R) \) and observe that for some 1-simplices \( \{u, v\} \in K \) the chain \( \gamma'(\partial\{u, v\}) \) must already be a boundary. To see that this is not the case in general, consider the drawing of \( \Delta_i^{(1)} \) in an annulus depicted in the figure on the left. Observe that for every triangle \( \{i, j, k\} \in \Delta_i^{(2)} \) the image, in this drawing, \( \partial\{i, j, k\} \) is a cycle going around the hole of the annulus and is therefore not a boundary. So, if we start with a chain map \( \gamma' \) corresponding to that drawing, we will not be able to extend it by “filling” any triangle directly. This is not a peculiar example, and a similar construction can easily be done with arbitrarily many vertices. Observe, though, that the cycle going from 1 to 2, then 4, 3 and then back to 1 is a boundary; in other words, if we replace, in the triangle \( \partial\{1, 2, 3\} \), the edge from 2 to 3 by the concatenation of the edges from 2 to 4 and from 4 to 3, we build, using a chain map of \( \Delta_i^{(1)} \) where no 2-face can be filled, a chain map of \( \Delta_i^{(2)} \) where the 2-face can be filled. We systematize this observation using the barycentric subdivision of \( K \).

**Barycentric subdivision.** The idea behind the notion of barycentric subdivision is that the geometric realization of a simplicial complex \( K' \) can be subdivided by inserting a vertex at the barycentre of every face, resulting in a new, finer, simplicial complex, denoted \( \text{sd } K' \), that is still homeomorphic to \( K' \). Formally, the vertices of \( \text{sd } K' \) consist of the faces of \( K' \), except for the empty face, and the faces of \( \text{sd } K' \) are the collections \( \{\sigma_1, \ldots, \sigma_t\} \) of faces of \( K' \) such that

\[
\emptyset \neq \sigma_1 \subsetneq \sigma_2 \subsetneq \cdots \subsetneq \sigma_t.
\]

In other words, the set of vertices of \( \text{sd } K' \) is \( K' \setminus \{\emptyset\} \) and the faces of \( \text{sd } K' \) are the chains of \( K' \setminus \{\emptyset\} \). For \( \sigma \in K' \) we abuse the notation and let \( \text{sd } \sigma \) denote the subdivision of \( \sigma \) regarded as a subcomplex of \( \text{sd } K' \), that is,

\[
\text{sd } \sigma = \{\{\sigma_1, \ldots, \sigma_t\} \subseteq K' : \emptyset \neq \sigma_1 \subsetneq \sigma_2 \subsetneq \cdots \subsetneq \sigma_t \}. \subseteq \sigma \}
\]

We will mostly manipulate barycentric subdivision through the \( \text{sd } \sigma \). For further reading on barycentric subdivisions we refer the reader, for example, to Mat03, Section 1.7.

**Overview of the construction of \( \gamma \).** Let \( s \in \mathbb{N} \) be some parameter depending on \( K \) and to be determined later. To construct \( \gamma \) we will define three auxiliary chain maps

\[
C_*(K^{(k-1)}) \xrightarrow{\alpha} C_*(\text{sd } K^{(k-1)}) \xrightarrow{\beta_s} C_*(\Delta_i^{(k-1)}) \xrightarrow{\gamma'} C_*(R)
\]
As before, \( \gamma' \) is a chain map from \( C_s(\Delta_s^{(k-1)}) \) constrained by \( \mathcal{F} \) and is obtained by applying the induction hypothesis. Unlike in Section 4.2, we do not inject the vertices of \( K \) in those of \( \Delta_s \) directly but proceed through \( sd K \), the barycentric subdivision of \( K \). We “inject” \( K^{(k-1)} \) into \( sd K^{(k-1)} \) by means of a chain map \( \alpha \). We then construct an injection \( \beta \) of the vertices of \( sd K \) into the vertices of \( \Delta_s \) which we extend linearly into a chain map \( \beta_2 \). The key idea is the following:

The boundary of any \( k \)-simplex \( \sigma \) of \( K \) is mapped, under \( \alpha \), to a sum of \( k! \) boundaries of \( k \)-simplices of \( sd K \), all of which are mapped through \( \beta_2 \) to chains with the same homology in some appropriate \( U_{\mathcal{F}_{\sigma}} \).

Since \( k! \) is even and we consider homology with coefficients in \( \mathbb{Z}_2 \), it follows that \( \gamma' \circ \beta_2 \circ \alpha(\sigma) \) is a boundary in \( U_{\mathcal{F}_{\sigma}} \). We therefore construct \( \gamma \) as an extension of \( \gamma' \circ \beta_2 \circ \alpha \).

**Definition of \( \gamma' \).** Since \( \Delta_s^{(k-1)} \) has dimension \( k-1 \), the induction hypothesis ensures that if the cardinality of \( \mathcal{F} \) is large enough then there exists a nontrivial chain map \( \gamma' : C_s(\Delta_s^{(k-1)}) \to C_s(\mathbb{R}) \) constrained by \( \mathcal{F} \). We denote by \( \Psi \) a map such that \( \gamma' \) is constrained by \( (\mathcal{F}, \Psi) \). Remark that \( \Psi \) must be monotone over \( \Delta_s^{(k-1)} \) as for any \( \sigma \subseteq \tau \in \Delta_s^{(k-1)} \) we have \( \Psi(\sigma) = \Psi(\sigma \cap \tau) = \Psi(\sigma) \cap \Psi(\tau) \subseteq \Psi(\tau) \). It follows that for any \( \sigma \in \Delta_s^{(k-1)} \) we have

\[
\Psi(\sigma) = \bigcup_{\tau \in \Delta_s^{(k-1)}, \tau \subseteq \sigma} \Psi(\tau)
\]

We use this identity to extend \( \Psi \) to \( \Delta_s \), that is we define:

\[
\forall A \subseteq V(\Delta_s), \quad \Psi(A) = \bigcup_{\tau \in \Delta_s^{(k-1)}, \tau \subseteq A} \Psi(\tau).
\]

Remark that the extended map still commutes with the intersection:

**Lemma 27.** For any \( A, B \subseteq V(\Delta_s) \) we have \( \Psi(A) \cap \Psi(B) = \Psi(A \cap B) \).

**Proof.** For any \( A, B \subseteq V(\Delta_s) \) we have

\[
\Psi(A) \cap \Psi(B) = \left( \bigcup_{\sigma \in \Delta_s^{(k-1)}, \sigma \subseteq A} \Psi(\sigma) \right) \cap \left( \bigcup_{\tau \in \Delta_s^{(k-1)}, \tau \subseteq B} \Psi(\tau) \right)
\]

Distributing the union over the intersections we get

\[
\Psi(A) \cap \Psi(B) = \bigcup_{\sigma, \tau \in \Delta_s^{(k-1)}, \sigma \subseteq A, \tau \subseteq B} \Psi(\sigma) \cap \Psi(\tau)
\]

and as \( \Psi(\sigma \cap \tau) = \Psi(\sigma) \cap \Psi(\tau) \) if \( \sigma, \tau \) are simplices of \( \Delta_s^{(k-1)} \) this rewrites as

\[
\Psi(A) \cap \Psi(B) = \bigcup_{\sigma, \tau \in \Delta_s^{(k-1)}, \sigma \subseteq A, \tau \subseteq B} \Psi(\sigma \cap \tau).
\]

Finally, observing that

\[
\{ \sigma \cap \tau : \sigma, \tau \in \Delta_s^{(k-1)}, \sigma \subseteq A, \tau \subseteq B \} = \{ \vartheta : \vartheta \in \Delta_s^{(k-1)}, \vartheta \subseteq A \cap B \}
\]

we get

\[
\Psi(A) \cap \Psi(B) = \bigcup_{\vartheta \in \Delta_s^{(k-1)}, \vartheta \subseteq A \cap B} \Psi(\vartheta) = \Psi(A \cap B)
\]

which proves the desired identity. \( \square \)
Figure 6: The map $\alpha$ applied to a simplex $\sigma$ (left) and to $\partial \sigma$ (right). Significant parts of the boundaries $\partial \tau$ cancel out.

**Definition of $\alpha$.** Now we define a chain map $\alpha : C_* \left( K^{(k-1)} \right) \rightarrow C_* \left( \text{sd} K^{(k-1)} \right)$ by first putting

$$\alpha : \sigma \in K^{(k-1)} \mapsto \sum_{\tau \in \text{sd} \sigma \atop \dim \tau = \dim \sigma} \tau,$$

and then extending that map linearly to $C_* \left( K^{(k-1)} \right)$. See Figure 6. Remark that $\alpha$ behaves nice with respect to the differential:

$$\alpha(\partial \sigma) = \sum_{\tau \in \text{sd} \sigma \atop \dim \tau = \dim \sigma} \partial \tau.$$

Note that the formula above makes sense and is valid even if $\sigma$ is a $k$-simplex although we define $\alpha$ only up to dimension $k-1$.

**Definition of $\beta$.** We now construct the injection $\beta : V(\text{sd} K) \rightarrow V(\Delta_s)$ and, for constraining purposes, an auxiliary function $\kappa$ associating to every $k$-dimensional simplex of $K$ some simplex of $\Delta_s$. We want these functions to satisfy:

(P1) For any simplex $\sigma \in K$, $\kappa(\sigma) \cap \text{Im} \beta = \beta(V(\text{sd} \sigma))$.

(P2) For any $k$-simplices $\sigma, \tau \in K$, $\kappa(\sigma) \cap \kappa(\tau) = \beta(V(\text{sd} \sigma)) \cap \beta(V(\text{sd} \tau))$.

(P3) For any $k$-simplex $\sigma \in K$, when $\tau$ ranges over all $k$-simplices of $\text{sd} \sigma$, all chains $\gamma' \circ \beta_1(\partial \tau)$ have support in $U_{\Psi(\kappa(\sigma))}$ and are in the same homology class in $H_{k-1}(U_{\Psi(\kappa(\sigma))})$.

The intuition behind these properties is that $\kappa(\sigma)$ should augment $\beta(V(\text{sd} \sigma))$ by “dummy” vertices (P1) in a way that distinct simplices use disjoint sets of “dummy” vertices (P2). Property (P3), will allow building $\gamma$ over $k$-simplices as explained in the preceding overview.

We start the construction of $\beta$ and $\kappa$ with a combinatorial lemma. Let $\ell = 2^{k+1} - 1$ stand for the number of vertices of the barycentric subdivision of a $k$-dimensional simplex, and set $m = R_{k+1}(2^k, \ell)$.

**Claim 1.** For any integer $t$, if $s \geq R_m \left( \binom{m}{t} , t \right)$ then there exist a set $T$ of $t$ vertices of $\Delta_s$ and a set $Q^* \in \binom{\ell}{t}$ such that $Q^*$ selects in any $M \in \binom{t}{m}$ a subset $L_M$ with the following property: when $\sigma$ ranges over all $k$-simplices of $\Delta_s$ with $\sigma \subseteq L_M$, all chains $\gamma'(\partial \sigma)$ are in the same homology class in $H_{k-1} \left( U_{\Psi(M)} \right)$.

**Proof.** Let $M$ be a subset of $m$ vertices of $\Delta_s$. Since $\gamma'$ is constrained by $(F, \Psi)$, for every $k$-simplex $\sigma \subseteq M$ the support of $\gamma'(\partial \sigma)$ is contained in $U_{\Psi(\partial \sigma)} \subseteq U_{\Psi(\sigma)} \subseteq U_{\Psi(M)}$. We can therefore color the $(k+1)$-uniform hypergraph on $M$ by assigning to every hyperedge $\sigma$ the homology class of $\gamma'(\partial \sigma)$ in $U_{\Psi(M)}$. Since $\beta_{k-1} \left( U_{\Psi(M)} \right) \leq b$, there are at most $2^b$ colors in this coloring. As $m = R_{k+1}(2^k, \ell)$, Ramsey’s Theorem implies that there exists a subset $L \subset M$ of $\ell$ vertices inducing a monochromatic hypergraph. We let $Q_M$ denote an element of $\binom{\ell}{t}$ that selects such a subset $L$. It remains to find a subset $T$ of vertices of $\Delta_s$ so that all $m$-element subsets $M \subseteq T$ give rise to the same $Q_M$. This is done by another application of Ramsey’s theorem to the $m$-uniform hypergraph on the vertices of $\Delta_s$ where each hyperedge $M$ is colored by the $\ell$-element subset $Q_M$. The subset $T$ can have size $t$ as soon as $s \geq R_m \left( \binom{m}{t} , t \right)$, which proves the statement. □
Now, back to the construction of \( \beta \) and \( \kappa \). We first want a subset of \( V(\Delta_s) \) with a “uniform \( \ell \)-in-\( m \) selection” property of Claim 1 large enough that we can inject \( V(\text{sd } K) \) using Lemma 24. We set:

\[
t = v(\text{sd } K) + r(m - \ell) \quad \text{and} \quad s^* = R_m \left( \frac{m}{\ell} \right) t,
\]

and assume that \( s \geq s^* \); since \( s^* \) only depends on \( b \) and \( K \), this merely requires that \( \mathcal{F} \) is large enough, again as a function of \( b \) and \( K \), so that \( \gamma' \) still exists. We let \( T \) and \( q^* \) denote the subset of \( V(\Delta_s) \) and the element of \( \left( \frac{m}{\ell} \right) \) whose existence follows from applying Claim 1. Let \( \sigma_1, \sigma_2, \ldots, \sigma_r \) denote the \( k \)-dimensional simplices of \( K \). We apply Lemma 24 with

\[
Y = V(\text{sd } K), \quad Z = T, \quad A_i = V(\text{sd } \sigma_i), \quad q = \ell, \quad \text{and } w = m,
\]

and obtain an injection \( \pi : Y \to Z \) and \( W_1, W_2, \ldots, W_r \in (\mathbb{Z}/m) \) such that (i) for every \( i \leq r \), \( \pi^* \) selects \( \pi(A_i) \) in \( W_i \), and (ii) for any \( i \neq j \leq r \), \( W_i \cap W_j = \pi(A_i \cap A_j) \). This injection \( \pi \) is our map \( \beta \) and we may extend \( \kappa(\sigma_i) = W_i \). It is clear that Property (P1) holds, and since

\[
\kappa(\sigma_i) \cap \kappa(\sigma_j) = W_{ij} \quad \text{if } i \neq j,
\]

Property (P2) also holds. The set \( \pi^* \) selects \( \pi(A_i) \) in \( W_i \) (Lemma 24) so Claim 1 ensures that when \( \tau \) ranges over all \( k \)-simplices of \( \Delta_s \), all chains \( \gamma'(\partial \tau) \) have support in \( U_{\Psi(\partial \tau)} \) and are in the same homology class in \( H_{k+1} \left( U_{\Psi(\partial \tau)} \right) \). Substituting \( \pi(A_i) = \beta(V(\text{sd } \sigma_i)) \) and \( W_i = \kappa(\sigma_i) \), we see that (P3) holds.

**Construction of \( \gamma \).** Recall that we have the chain maps\(^{23}\):

\[
C_* \left( K^{(k-1)} \right) \xrightarrow{\alpha} C_* \left( \text{sd } K^{(k-1)} \right) \xrightarrow{\beta_k} C_* \left( \Delta_s^{(k-1)} \right) \xrightarrow{\gamma'} C_* (\mathbb{R}).
\]

We define \( \gamma = \gamma' \circ \beta_k \circ \alpha \) as a chain map from \( C_* \left( K^{(k-1)} \right) \) to \( C_* (\mathbb{R}) \). Let \( \sigma \) be a \( k \)-dimensional simplex of \( K \). From the definition of \( \alpha \) we have

\[
\gamma(\partial \sigma) = \sum_{\tau \in \text{sd } \sigma} \gamma'(\partial \tau) \circ \beta_k(\partial \tau).
\]

By property (P3), all summands in the above chain have support in \( U_{\Psi(\kappa(\sigma))} \) and belong to the same homology class in \( H_{k+1} \left( U_{\Psi(\kappa(\sigma))} \right) \). There is an even number of summands, namely \( k! \) and we are using homology over \( \mathbb{Z}/2 \), so \( \gamma' \circ \beta_k \circ \alpha(\partial \sigma) \) has support in \( U_{\Psi(\kappa(\sigma))} \) and is a boundary in \( U_{\Psi(\kappa(\sigma))} \). We can therefore extend \( \gamma \) into a chain map from \( C_* (K) \) to \( C_* (\mathbb{R}) \) in a way that for any \( k \)-simplex \( \sigma \) of \( K \), the support of \( \gamma(\sigma) \) is contained in \( U_{\Psi(\kappa(\sigma))} \).

**Properties of \( \gamma \).** If \( v \) is a vertex of \( K \) then \( \text{sd } v \) consists of a single simplex, also a vertex. The chain \( \alpha(v) \) thus consists of a single term, with support a vertex of \( \text{sd } K \), and the support of \( \beta_k \circ \alpha(v) \) is a single vertex, \( \beta(\text{sd } v) \). Since \( \gamma' \) is nontrivial, the support of \( \gamma(v) \) is an odd number of points and \( \gamma \) is also nontrivial. It remains to argue that \( \gamma \) is constrained by \((\mathcal{F}, \Phi)\) where:

\[
\Phi : \begin{cases} 
K \to 2^\mathcal{F} \\
\sigma \mapsto \begin{cases} 
\Psi(\beta(V(\text{sd } \sigma))) & \text{if } \dim \sigma \leq k - 1 \\
\Psi(\kappa(\sigma)) & \text{if } \dim \sigma = k
\end{cases}
\end{cases}
\]

It is clear that \( \Phi(\emptyset) = \Psi(\emptyset) = \emptyset \) by definition of \( \Psi \). Also, the construction of \( \gamma \) immediately ensures that for any \( \sigma \in K \) the support of \( \gamma(\sigma) \) is contained in \( U_{\Psi(\kappa(\sigma))} \). To conclude the proof that \( \gamma \) is constrained by \((\mathcal{F}, \Phi)\) and therefore the induction it only remains to check that \( \Phi \) commutes with the intersection:

**Claim 2.** For any \( \sigma, \tau \in K \), \( \Phi(\sigma \cap \tau) = \Phi(\sigma) \cap \Phi(\tau) \).

\(^{23}\beta_k \) is the chain map induced by \( \beta \) restricted to chains of dimension at most \( (k - 1) \).
Proof. The claim is obvious for \( \sigma = \tau \), so from now on assume that this is not the case. First assume that \( \sigma \) and \( \tau \) have dimension at most \( k-1 \). Then,

\[
\Phi(\sigma) \cap \Phi(\tau) = \Psi(\beta(V(\text{sd} \sigma))) \cap \Psi(\beta(V(\text{sd} \tau))) = \Psi(\beta(V(\text{sd} \sigma)) \cap \beta(V(\text{sd} \tau))),
\]

the last equality following from Lemma 27. Since the map \( \beta \) on subsets of \( V(\Delta_s) \) is induced by a map \( \beta \) on vertices of \( \Delta_s \), we have \( \beta(V(\text{sd} \sigma)) \cap \beta(V(\text{sd} \tau)) = \beta(V(\text{sd} \sigma) \cap V(\text{sd} \tau)) \). Moreover, by the definition of the barycentric subdivision we have \( V(\text{sd} \sigma) \cap V(\text{sd} \tau) = V(\text{sd}(\sigma \cap \tau)) \). Thus,

\[
\Psi(\beta(V(\text{sd} \sigma)) \cap \beta(V(\text{sd} \tau))) = \Phi(\sigma \cap \tau),
\]

and the statement holds for simplices of dimension at most \( k-1 \).

Now assume that \( \sigma \) and \( \tau \) are both \( k \)-dimensional so that

\[
\Phi(\sigma) \cap \Phi(\tau) = \Psi(\kappa(\sigma)) \cap \Psi(\kappa(\tau)) = \Psi(\kappa(\sigma) \cap \kappa(\tau)) = \Psi(\beta(V(\text{sd} \sigma)) \cap \beta(V(\text{sd} \tau))),
\]

the last identity following from Property (P2) of the map \( \kappa \). Again, from the definition of \( \beta \) and the barycentric subdivision we have

\[
\beta(V(\text{sd} \sigma)) \cap \beta(V(\text{sd} \tau)) = \beta(V(\text{sd} \sigma) \cap \text{sd} \tau).
\]

We thus obtain

\[
\Phi(\sigma) \cap \Phi(\tau) = \Psi \circ \beta \circ V(\text{sd} \sigma \cap \text{sd} \tau) = \Phi(\sigma \cap \tau),
\]

the last identity following from the definition of \( \Phi \) on simplices of dimension at most \( k-1 \). The statement also holds for simplices of dimension \( k \).

Finally assume that \( \sigma \) and \( \tau \) are of dimension \( k \) and at most \( k-1 \) respectively. Then, applying Lemma 27 we have:

\[
\Phi(\sigma) \cap \Phi(\tau) = \Psi(\kappa(\sigma)) \cap \Psi(\beta(V(\text{sd} \tau))) = \Psi(\kappa(\sigma) \cap \beta(V(\text{sd} \tau))).
\]

Note that \( \beta(V(\text{sd} \tau)) \subseteq \text{Im} \beta \) and that, by property (P1), \( \kappa(\sigma) \cap \text{Im} \beta = \beta(V(\text{sd} \sigma)) \). We thus have

\[
\kappa(\sigma) \cap \beta(V(\text{sd} \tau)) = \beta(V(\text{sd} \sigma)) \cap \beta(V(\text{sd} \tau)) = \beta(V(\text{sd}(\sigma \cap \tau))),
\]

the last equality following, again, from the definition of barycentric subdivision. As \( \sigma \cap \tau \) has dimension at most \( k-1 \) we have

\[
\Phi(\sigma) \cap \Phi(\tau) = \Psi(\beta(V(\text{sd}(\sigma \cap \tau)))) = \Phi(\sigma \cap \tau)
\]

and the statement holds for the last case.

\[ \square \]

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