FUKAYA’S CONJECTURE ON \( S^1 \)-EQUIVARIANT DE RHAM COMPLEX

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ABSTRACT. Getzler-Jones-Petrack \cite{7} introduced \( A_\infty \) structures on the equivariant complex for manifold \( M \) with smooth \( S^1 \) action, motivated by geometry of loop spaces. Applying Witten’s deformation by Morse functions followed by homological perturbation we obtained a new set of \( A_\infty \) structures. We extend and prove Fukaya’s conjecture \cite{6} relating this Witten’s deformed equivariant de Rham complexes, to a new Morse theoretical \( A_\infty \) complexes defined by counting gradient trees with jumping which are closely related to the \( S^1 \) equivariant symplectic cohomology proposed by Siedel \cite{15}.

1. INTRODUCTION

In the influential paper \cite{17} by Witten, harmonic forms on a compact oriented Riemannian manifold \((M, g)\) are related to the Morse complex

\[
(CM_f^* := \bigoplus_{p \in \text{Crit}(f)} \mathbb{C} \cdot p) \quad \text{on} \quad M
\]

with a Morse function \( f \). More precisely, Witten introduced the twisted Laplacian

\[
\Delta_{f, \lambda} = d^*_{f, \lambda} \circ d + d \circ d^*_{f, \lambda}
\]

with a large real parameter \( \lambda \), and an isomorphism

\[
\phi : (CM_f^*, \delta) \to (\Omega_{f, <1}^*(M), d)
\]

where \( \Omega_{f, <1}^*(M) \) refers to the small eigensubspace of \( \Delta_{f, \lambda} \) (see Section 2.2). The detailed analysis of \( \phi \) is later carried out in \cite{9, 11, 10, 12} and readers may also see \cite{18} for this correspondence.

In \cite{6}, Fukaya conjectured that Witten’s isomorphism \( \phi \) can be enhanced to an isomorphism of \( A_\infty \) algebras (or categories), a generalization of differential graded algebras (abbrev. dga), encoding rational homotopy type by work of Quillen \cite{14} and Sullivan \cite{16}. The \( A_\infty \) structures \( m_k(\lambda) \)'s on \( \Omega_{f, <1}^*(M) \) are obtained by pulling back the structures of the de Rham dga \((\Omega^*(M), d, \wedge)\) using the homological perturbation lemma (see e.g. \cite{13}) with homotopy operator \( H_{f, \lambda} = d_{f, \lambda}^* G_{f, \lambda} \). The Morse \( A_\infty \) structures \( m_k^{\text{Morse}} \)'s are defined via counting gradient flow trees of Morse functions as in \cite{5}. Fukaya conjectured that they are related by

\[
\lim_{\lambda \to \infty} m_k(\lambda) = m_k^{\text{Morse}}
\]

via the Witten’s isomorphism \( \phi \). This conjectured is proven in \cite{3} by extending the analytic technique in \cite{12} to incorporate the homotopy operator \( H_{f, \lambda} \).

When \( M \) is equipped with a smooth \( S^1 \) action, motivated by the geometry of loop space \( S^1 \acts \mathcal{L}X \) for some \( X \), Getzler-Jones-Petrack \cite{7} introduced an enhancement of the equivariant de Rham complex on \( M \). They defined new \( A_\infty \) algebra structures consisting of

\[
\tilde{m}_k : (\Omega^*(M)[[u]]) \otimes^k \to \Omega^*(M)[[u]]
\]

by adding higher order (in \( u \)) operations \( uP_k \)'s (see Section 2.1) to ordinary de Rham dga structures. Witten’s deformed \( A_\infty \) structures \( m_k(\lambda) \)'s are constructed from \( \tilde{m}_k \)'s in \( 1.3 \) using the technique of homological perturbation as in original Fukaya’s conjecture.

\begin{itemize}
    \item[1] Here \( \text{Crit}(f) \) refers to set of critical points of \( f \), and the differential \( \delta \) is given by counting gradient flow lines.
    \item[2] We let \( d_{f, \lambda}^* \) to be the adjoint of \( d \), and \( G_{f, \lambda} \) to be Witten’s Green function of \( \Delta_{f, \lambda} \) w.r.t. volume form \( e^{-2\lambda f \cdot \text{vol}_M} \).
\end{itemize}
Inspired by Fukaya’s correspondence, we define new Morse theoretic type counting structures \( m_{k}^{\text{Morse}} \)'s (where \( m_{1}^{\text{Morse}} \) is known before in [2]) associated to \( S^1 \acts M \), counting of Morse flow trees with jumpings coming from the \( S^1 \) action (see the following Section 1.1). We prove the generalization of (1.2) for \( S^1 \acts M \) relating these two structures.

**Theorem 1.1** (=Theorem 2.11). We have
\[
\lim_{\lambda \to \infty} m_k(\lambda) = m_{k}^{\text{Morse}}.
\]

1.1. **The operation \( m_{k}^{\text{Morse}} \)'s.** To describe \( m_{k}^{\text{Morse}} \)'s, we fix a generic sequence (see Definition 2.8) of functions \((f_0, \ldots, f_k)\) such that their differences \( f_{ij} := f_j - f_i \) are assumed to be Morse-Smale as in Definition 2.5. The Morse theoretical \( A_\infty \) product \( m_{k}^{\text{Morse}} \)'s take the form
\[
m_{k}^{\text{Morse}} := \sum_{T} m_{k,T}^{\text{Morse}} : CM_{(k-1)k}^*[u] \otimes \cdots \otimes CM_{f_0}^*[u] \to CM_{f_0}^*[u]
\]
which is a summation over directed labeled ribbon \( k \)-tree \( T \) with \( k \)-incoming edges and 1 outgoing edge, where internal vertices are either labeled by 1 or by \( u \). For example (see Section 2.3 for details), if we take the tree \( T \) to be the one with two incoming edges \( e_{12} \) and \( e_{01} \) joining the vertex \( v_r \) connected to the outgoing edge \( e_{02} \), with \( v_r \) being labeled by \( u \). The gradient flow trees with type \( T \) will be consisting of gradient flow lines of \( f_{12}, f_{01} \) and \( f_{02} \) which ending at critical points \( q_{12}, q_{01} \) and \( q_{02} \) respectively, that can be joined together at a point \( x_{v_r} \in M \) with further help of the \( S^1 \) action \( \sigma_t : M \to M \) (for some \( t \)) as shown in the Figure 1. As a consequence of the above Theorem 1.1, the Morse (pre)-category (here pre-category means this operation only defined for generic sequence \((f_0, \ldots, f_k)\) on \( S^1 \acts M \) is an \( A_\infty \) (pre)-category.

![Gradient tree with jumping](image)

**Figure 1.** Gradient tree with jumping of type \( T \)

**Corollary 1.2.** The operations \( m_{k}^{\text{Morse}} \)'s satisfy the \( A_\infty \) relation for generic sequences of functions.

**Remark 1.3.** In [15, Section 8b], Seidel proposed the \( A_\infty \) operators \( m_{k}^{\text{Floer}} \) on the symplectic cochain complex for a Liouville domain \( X \), which corresponds to \( m_{k}^{\text{Morse}} \)'s if we think of \( M \) as a finite dimensional analogue of \( \mathcal{L}X \). The corresponding \( m_{1}^{\text{Floer}} \) operation is studied in details in [19]. The above Theorem 1.1 suggest how Witten deformation can provide a linkage between the Getzler-Jones-Petrack’s operation \( \tilde{m}_k \) on \( \mathcal{L}X \) and the Floer theoretical operations introduced by Seidel through the investigation of the corresponding finite dimensional situation.
This paper consists of three parts. In Section 2 we set up the Witten deformation of Getzler-Jones-Petrack’s $A_\infty$ operations $\tilde{m}_k$’s, the definition of counting gradient flow trees with jumping, and state our Main Theorem 2.11. In Section 3.1 we recall the necessary analytic result by following [3]. The rest of Section 3 will be a proof of Theorem 2.11 by figuring out the exact relations between the operations $m_{k,T}(\lambda)$ and counting of gradient trees.

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2. Witten’s deformation of $S^1$-equivariant de Rham complex

We always let $(M, g)$ to be an $n$-dimensional compact oriented Riemannian manifold, and denote it volume form by $\text{vol}_M$ (or simply $\text{vol}$). We assume there is an smooth $S^1$ action $\sigma : S^1 \times M \to M$ on $M$ preserving $(g, \text{vol})$. We should write $\sigma_t : M \to M$ to be the action for a fixed $t \in S^1$.

2.1. $S^1$-equivariant de Rham complex and category. We begin with recalling the Definition of $S^1$-equivariant de Rham $A_\infty$ algebra introduced in [7], which is reformulated to be $A_\infty$ category as follows for the convenient of presentation of this paper.

Definition 2.1. The $S^1$-equivariant de Rham $A_\infty$ category $dR(M)$ consisting of object being smooth functions $f : M \to \mathbb{R}$, with morphism $\text{Hom}(f,g) := \Omega^*(M)[[u]]$ where $u$ is a formal variable. The $A_\infty$ operations $\tilde{m}_k : \text{Hom}(f_{k-1}, f_k) \otimes \cdots \otimes \text{Hom}(f_0, f_1) \cong (\Omega^*(M)[[u]])^\otimes k \to \text{Hom}(f_0, f_k) \cong \Omega^*(M)[[u]]$ is defined by $\tilde{m}_1(\alpha_{01}) = d(\alpha_{01}) + uP_1(\alpha_{01})$, $\tilde{m}_2(\alpha_{12}, \alpha_{01}) = (-1)^{|\alpha_{12}|+|\alpha_{01}|} \alpha_{12} \wedge \alpha_{01} + uP_2(\alpha_{12}, \alpha_{01})$ and $\tilde{m}_k(\alpha_{(k-1)k}, \ldots, \alpha_{01}) = uP_k(\alpha_{(k-1)k}, \ldots, \alpha_{01})$ for $\alpha_{ij} \in \text{Hom}(f_i, f_j)$.

Here the operator $P_k$ is defined by the action $P_1(\alpha_{ij}) = \int_{S^1}(\tau \circ (\sigma^*(\alpha_{ij}))) dt$, and for $k \geq 2$ we use

$$P_k(\alpha_{(k-1)k}, \ldots, \alpha_{01}) := \int_{0 \leq t_k \leq \cdots \leq t_1 \leq 1} \left( (\tau \circ (\sigma^*(\alpha_{(k-1)k}))) \wedge \cdots \wedge (\tau \circ (\sigma^*(\alpha_{01}))) \right) dt_k \cdots dt_1.$$

The fact that the about operations $\tilde{m}_k$’s form an $A_\infty$ category is proven in [7] Theorem 1.7.

2.2. Homological perturbation via Witten’s deformation. We follow [3] Section 2.2. to introduce the Witten deformation with a real parameter $\lambda > 0$, which is originated from [17]. For each $f_i$ and $f_j$, we twist the volume form vol by $f_{ij} := f_j - f_i$ as $\text{vol}_{ij} := e^{-2\lambda f_i} \text{vol}$, and let $d_{ij} := e^{2\lambda f_{ij}} d e^{-2\lambda f_i} = d + 2\lambda \nabla f_i$ to be the adjoint of $d$ with respect to the volume form $\text{vol}_{ij}$. The Witten Laplacian is defined by $\Delta_{ij} := dd^*_{ij} + d^*_j d^*_{ij}$, acting on the complex $\Omega^*(M)[[u]]$ We denote the span of eigenspaces with eigenvalues contained in $[0, 1)$ by $\Omega^*_{ij, < 1}(M)[[u]]$, or simply $\Omega^*_{ij, < 1}[[u]]$. We use construction in [3] originated from [6] using homological perturbation lemma [13], which obtain a new $A_\infty$ structure from $m_k$’s as follows.

Definition 2.2. A (directed) $k$-tree labeled $T$ consists of a finite set of vertices $\bar{T}^{[0]}$ together with a decomposition $\bar{T}^{[0]} = T^{[0]}_\infty \sqcup T^{[0]}_o \sqcup \{v_o\}$, where $T^{[0]}_\infty$, called the set of incoming vertices, is a set of size $k$ and $v_o$ is called the outgoing vertex (we also write $T^{[0]}_{\infty} := T^{[0]}_\infty \cup \{v_o\}$ and $T^{[0]}_{o} := T^{[0]}_o \cup \{v_o\}$), a finite set of edges $\bar{T}^{[1]}$, two boundary maps $\partial_{\infty}, \partial_o : \bar{T}^{[1]} \to \bar{T}^{[0]}$ (here $\partial_{\infty}$ stands for incoming and$$3$$Strictly speaking, the differential forms here depend on the real parameter $\lambda$ while we prefer to suppress the dependence in our notation.
∂₀ stands for outgoing), and a labeling of every internal vertices $T^{[0]}$ by either 1 or $u$, satisfying the following conditions:

1. Every vertex $v \in T^{[0]}_{in}$ has valency one, and satisfies $\#\partial^{-1}_o(v) = 0$ and $\#\partial^{-1}_i(v) = 1$; we let $T^{[1]} := T^{[1]} \setminus \partial^{−1}_i(T^{[0]}_{in})$.
2. Every vertex $v \in T^{[0]}$ has an unique edge $e_{v,o} \in T^{[1]}$ such that $\partial^{−1}_i(e_{v,o}) = v$, and only trivalent vertices in $T^{[0]}$ can be labeled with 1.
3. For the outgoing vertex $v_o$, we have $\#\partial^{-1}_o(v_o) = 1$ and $\#\partial^{-1}_i(v_o) = 0$; we let $e_o := \partial^{-1}_o(v_o)$ be the outgoing edge and denote by $v_r \in T^{[0]}_{in} \cup T^{[0]}$ the unique vertex (which we call the root vertex) with $e_o = \partial^{-1}_i(v_r)$.
4. The topological realization $|\tilde{T}| := (\bigcup_{v \in T^{[1]}[0,1]} 1)/\sim$ of the tree $T$ is connected and simply connected; here $\sim$ is the equivalence relation defined by identifying boundary points of edges if their images in $T^{[0]}$ are the same.

By convention we also allow the unique labeled 1-tree with $T^{[0]} = \emptyset$. Two labeled $k$-trees $T_1$ and $T_2$ are isomorphic if there are bijections $T^{[0]}_1 \cong T^{[0]}_2$ and $T^{[1]}_1 \cong T^{[1]}_2$ preserving the decomposition $T^{[0]}_i = T^{[0]}_{in} \cup T^{[0]}_o \cup \{v_i,o\}$ and boundary maps $\partial^{−1}_i$ and $\partial^{−1}_o$ and the labeling of $T^{[0]}$. The set of isomorphism classes of labeled $k$-trees will be denoted by $\mathbb{T}_k$. For a labeled $k$-tree $T$, we will abuse notations and use $T$ (instead of $|T|$) to denote its isomorphism class.

A labeled ribbon $k$-tree is a $k$-tree $T$ with a cyclic ordering of $\partial^{−1}_i(v) \cup \partial^{−1}_o(v)$ for each trivalent vertex $v \in T^{[0]}$, and isomorphism of labeled ribbon $k$-trees are further required to preserve this ordering. A labeled ribbon $k$-tree can have its topological realization $|\tilde{T}|$ being embedded into the unit disc $D$, with $T^{[0]}_D$ lying on the boundary $\partial D$ such that the cyclic ordering of $\partial^{−1}_i(v) \cup \partial^{−1}_o(v)$ agree with the anti-clockwise orientation of $D$. The set of isomorphism classes of labeled ribbon $k$-trees will be denoted by $\mathbb{LT}_k$.

**Notations 2.3.** For each $T \in \mathbb{LT}_k$, we can associated to each edge $e \in T^{[1]}$ a numbering by pair of integer $ij$ using the embedding $|T| \to D$ by the rules: there are $k + 1$ connected components of $D \setminus |T|$, and we assign each component by integers $0, \ldots, k$; each (directed) edge $e \in T^{[1]}$ with region numbered by $i$ on its left and region numbered by $j$ on its right is numbered by $ij$; the incoming edges numbered by $e_{(k−1)k}, \ldots, e_{01}$ and the outgoing edge $e_{0k}$ are in clockwise ordering of $\partial D$.

A pair of $v \in T^{[0]} \cup \{v_o\}$ attached to an edge $e \in T^{[1]}$ is called a flag, and we will let $f(T)$ to be the set of all flags. For every flag $(e, v)$, we let $T_{e,v}$ to be the unique subtree with outgoing vertex being $v$ if $\partial_o(e) = v$, and we let $T_{e,v}$ to be the unique subtree with outgoing edge being $e$ if $\partial_i(e) = v$.

**Definition 2.4.** Given a labeled ribbon $k$-tree $T \in \mathbb{LT}_k$ with an embedding $|\tilde{T}| \to D$, we associate to it an operation $m_{k,T}(\lambda) : \Omega^{*}_{(k−1)k,\leq 1}[\{i\}] \otimes \cdots \otimes \Omega^{*}_{01,\leq 1}[\{i\}] \to \Omega^{*}_{0k,\leq 1}[\{i\}]$ by the following rules:

1. aligning the inputs $\varphi_{(k−1)k}, \ldots, \varphi_{01}$ at the incoming vertices $T^{[0]}_{in}$ according to the clockwise ordering induced from $D$;
2. if a vertex $v \in T^{[0]}$ has incoming edges $e_{v,1}, \ldots, e_{v,l}$ and outgoing edge $e_{v,o}$ attached to it such that $e_{v,1}, \ldots, e_{v,l}, e_{v,o}$ is in clockwise orientation, we apply the operation $\wedge$ if $v$ is labeled with 1 (and hence trivalent) and the operation $\vee$ if $v$ is labeled with $\bar{u}$;
3. for an edge $e \in T^{[0]}$ which is numbered by $ij$, we apply the homotopy operator $H_{ij} := d^*_ij G_{ij}$ where $G_{ij}$ is the Witten’s twisted Green operator associated to the Witten Laplacian $\Delta_{ij}$;
4. for the unique outgoing edge $e_o$, we apply the operator $P_{0k}$ which is the orthogonal projection $P_{0k} : \Omega^{*}_{\{i\}} \to \Omega^{*}_{0k,\leq 1}[\{i\}]$ with respect to the twisted $L_2$-norm obtained from the volume form $\text{vol}_{0k}$. 


Definition 2.7. Given a sequence $a_{ij}$ for the unique tree with $T^{[0]} = \emptyset$ to be the restriction of $d$ on $\Omega_{ij}^{*}$. For each labeled ribbon $k$-tree $T$, we assign $n_T$ to be the number of vertices in $T^{[0]}$ labeled with $u$, and we let $m_k(\lambda) := \sum_{T \in \mathcal{LT}_k} u_T m_{k,T}(\lambda)$ to be the homological perturbed $A_\infty$ structure.

It is well-known that (see e.g. [11, Chapter 8]) the perturbed $A_\infty$ structure $m_k(\lambda)$'s satisfy the $A_\infty$ relation. And we obtain a new category $dR_{<1}(M)$ via Witten deformation.

2.3. Relation with $S^1$-equivariant Morse flow trees. In [12, 17, 18], a relation between the Morse complex $CM_{f_{ij}}$ and $\Omega_{ij}^{*}$ is established when $f_{ij}$ is a Morse-Smale function in following Definition 2.5. Following [18], it is an isomorphism

$$\Phi_{ij} : \Omega_{ij}^{*} \rightarrow CM_{f_{ij}}; \quad \Phi_{ij}(\alpha) := \sum_{p \in \text{Crit}(f_{ij})} \int_{V_p^-} \alpha,$$

where $\text{Crit}(f_{ij})$ is the finite set of critical points of $f_{ij}$ (with Morse index of $p$ given by number of negative eigenvalues of $\nabla^2 f_{ij}(p)$), and $V_p^-$ (Notice that we further choose an orientation of $V_p^-$ by choosing a volume element of the normal bundle $\mathcal{N}V_p^+$) is the unstable submanifold associated to $p$ which is the union of all gradient flow lines $\gamma(s)$ of $\nabla f_{ij}$ which limit toward $p$ as $s \rightarrow \infty$. Furthermore, the de Rham differential is identified with the Morse differential $\delta_1$ defined via counting Morse flow lines.

Definition 2.5. A Morse function $f_{ij}$ is said to satisfy the Morse-Smale condition if $V_p^+$ and $V_q^-$ intersecting transversally for any two critical points $p \neq q$ of $f_{ij}$.

We illustrate how the technique in [3] can be used to establish a relation between $\lambda \rightarrow \infty$ limit of the operation $m_k^T(\lambda)$ with a new Morse-theoretical counting for $S^1 \rightarrow M$ defined as follows.

Notations 2.6. A metric labeled $k$-tree (ribbon) $T$ is a labeled (ribbon) $k$-tree together with a length function $l : T^{[1]} \setminus \{e_o\} \rightarrow (0, +\infty)$. For each $e \in T^{[1]}$, we let $\mathcal{I}_e = (-\infty, 0]$ if $e \in T^{[1]}_{\infty}$, $\mathcal{I}_e = [0, l(e)]$ for $e \in T^{[1]} \setminus \{e_o\}$ and $\mathcal{I}_{e_o} = [0, \infty)$. The space of metric structure on $T$, denoted by $S(T)$, is a copy of $(0, +\infty)^{|T^{[1]}|-1}$. The space $S(T)$ can be partially compactified to a manifold with corners $(0, +\infty)^{|T^{[1]}|-1}$, by allowing the length of internal edges going to be infinity. In particular, it has codimension-1 boundary $\partial S(T) = \bigsqcup_{T = T' \sqcup T''} S(T') \times S(T'')$.

For every vertex $v \in T$, we use $\nu(v) + 1$ to denote the valency of $v$. We write $\Delta_l := \{(t_l, \ldots, t_1) \in [0, 1]^l \mid 0 \leq t_l \leq \cdots \leq t_1 \leq 1\}$ for $l > 1$, and $\Delta_1 = S^1$ [4] and attach to each vertex $v$ labeled with $u$ a simplex $\Delta_{\nu(v)}$. Writing $LT^{[0]}$ to be the collection of all vertices with label $u$, we let $S(T) := \prod_{v \in LT^{[0]}} \Delta_{\nu(v)} \times S(T)$.

Definition 2.7. Given a sequence $\vec{f} = (f_0, \ldots, f_k)$ such that all the difference $f_{ij}$'s are Morse, with a sequence of points $\vec{q} = (q_{k-1}, \ldots, q_0, q_k)$ such that $q_{ij}$ is a critical point of $f_{ij}$, and a metric labeled ribbon $k$-tree $T$, a gradient flow tree (with jumping) $T$ (readers may see Figure [4] for an example) of type $(T, \vec{f}, \vec{q})$ consisting of a gradient flow line $\gamma_{ij} : \mathcal{I}_{e_{ij}} \rightarrow M$ of the Morse function $f_{ij}$ for each edge $e_{ij} \in T^{[1]}$ numbered by $ij$, and a point $t_v = (t_v, \nu(v), \ldots, t_{v,1}) \in \Delta_{\nu(v)}$ for every $v \in LT^{[0]}$ satisfying:

1. $\lim_{s \rightarrow -\infty} \gamma_{e_{i(i+1)}}(s) = q_{i(i+1)}$ for the incoming edges $e_{i(i+1)} \in T^{[1]}_{\infty}$, and $\lim_{s \rightarrow \infty} \gamma_{e_{ij}}(s) = q_{0k}$ for the outgoing edge $e_{0k}$;
2. for a trivalent vertex $v \in T^{[0]}$ labeled by $1$ with two incoming edges $e_{ij}, e_{ij}$ and outgoing edge $e_{il}$, we require that $\gamma_{ij}(l(e_{ij})) = \gamma_{ij}(l(e_{ij})) = \gamma_{il}(0)$;

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4This is not the 1-simplex, but we would like to unify our notation in this way.
(3) for a vertex \( v \in LT^{[0]} \) with incoming edges \( e_{i_1}u_{i_1}, \ldots, e_{i_{k-1}}u_{i_{k-1}} \) and outgoing edge \( e_{i_{k}}u_{i_{k}} \), we require that \( \sigma(-t_{i_1}, \gamma_{i_1}(l(e_{i_1})) = \cdots = \sigma(-t_{i_{k-1}}, \gamma_{i_{k-1}}(l(e_{i_{k-1}}))) = \gamma_{i_{k}}(0) \), where \( l = \nu(v) \) and \( \sigma \) is the \( S^1 \) action map in the beginning of Section 2.8.

We will let \( M_T(\bar{f}, \bar{q}) \) to denote the moduli space (as a set) of gradient flow lines of type \( T \). For the unique tree with \( T^{[0]} = \emptyset \), we let \( M_T(\bar{f}, \bar{q}) \) to be the moduli space of gradient flow lines quotient by the extra \( R \) symmetry by convention.

Similar to the moduli space of gradient flow trees without \( S^1 \) action (see e.g. [3] Section 2.1.), we can describe \( M_T(\bar{f}, \bar{q}) \) as intersection of stable and unstable manifolds.

**Definition 2.8.** Given the sequence \( f \) and \( q \) as in the above Definition 2.7, we define a smooth map \( f_{T,i(i+1)} : V^{+}_{q(i+1)} \times S(T) \to M \) for each \( i = 0, \ldots, k - 1 \) as follows. Given a incoming edge \( e_{i(i+1)} \), there is a unique sequence of edges \( e_{i_0}u_0 = e_{i(i+1)}, e_{i_1}u_1, \ldots, e_{i_{m+1}}u_{m+1} = e_0 \) with \( v_d := \partial_0(e_{i_{d+1}}) \) forming a path from the incoming vertex \( v_{i(i+1)} \) to the outgoing vertex \( v_0 \). Fixing a point \( x_0 \in V^{+}_{q(i+1)} \), and a point \( (\{t_v\}_{v \in LT^{[0]}}, \{l(e_i)\}_{e_i \in T^{[1]}}, \{e_{i_0}\}) \in S(T) \), we determine a point \( x_d \in M \) inductively for \( 0 \leq d \leq m + 1 \) by the rules:

1. if \( v_d \) is labeled with 1, we simply take \( x_{d+1} \) to be the image of \( x_d \) under \( l(e_{i_{d+1}}) = 1 \) time flow of \( \nabla f_{i_{d+1}u_{d+1}} \) for \( d < m \), and \( x_{d+1} = x_d \) for \( d = m \);
2. if \( v_d \) is labeled with \( u \), we take \( x_{d+1} \) to be the image of \( \sigma(-t_{v_d}, x_d) \) under the \( l(e_{i_{d+1}}) \) time flow of \( \nabla f_{i_{d+1}u_{d+1}} \) if \( d < m \), and \( x_{d+1} = \sigma(-t_{v_d}, x_d) \) for \( d = m \), where \( e_{i_{d+1}} \) is the \( l \)-th incoming edge attached to \( v_d \) in the anti-clockwise orientation.

These map can be put together as \( f_T : V^{+}_{q_0} \times V^{+}_{q_{(k-1)}k} \times \cdots \times V^{+}_{q_{01}} \times S(T) \to M^k \) using the natural embedding \( V^{+}_{q_0} \hookrightarrow M \) for the first component. Therefore we see that \( M_T(\bar{f}, \bar{q}) = f_T^{-1}(D) \) where \( D = M \leftrightarrow M^{k+1} \) is the diagonal.

We say a sequence of function \( f \) generic if for any sequence of critical points \( q \), any labeled tree \( T \) the associated intersection \( f_T \) with \( D \) is transversal with expected dimension (meaning that it is empty when expected negative dimensional intersection), and the same hold when restricting \( f_T \) on any boundary strata of \( V^{+}_{q_0} \times V^{+}_{q_{(k-1)}k} \times \cdots \times V^{+}_{q_{01}} \times S(T) \) (the stratification coming from that of \( T^{[1]} \)) and for any subsequence of \( f \).

Suppose we are given a generic sequence \( f \) with \( q \) and \( T \) as in the above Definition 2.8 then we can compute the dimension of the moduli space as

\[
\text{dim}(M_T(\bar{f}, \bar{q})) = \text{deg}(q_{(0)}) - \sum_{i=0}^{k-1} \text{deg}(q_{i(i+1)}) + \sum_{v \in LT^{[0]}} \nu(v) + |T^{[1]}| - 1.
\]

**Definition 2.9.** Given generic \( f, q \) and \( T \) as in the above Definition 2.8, such that \( \text{dim}(M_T(\bar{f}, \bar{q})) = 0 \), with a flow tree \( \Gamma \in M_T(\bar{f}, \bar{q}) \), we assign a sign \( (-1)^{\chi(\Gamma)} \) by assigning a differential form \( \text{vol}_v \in \wedge^n T^* M_{\gamma(v)} \) (Here we abuse the notation to use \( v \) to stand for the corresponding point in \( \mathcal{I}_\nu \)) for each flag \( (e, v) \in f(T) \) inductively along the tree \( T \) as follows:

1. for an incoming edge \( e_{i(i+1)} \) with \( v = \partial_0(e_{i(i+1)}) \), we let \( \text{vol}_{e_{i(i+1)}, v} \) to be the restriction of the volume form of the normal bundle \( NV^{+}_{q(i+1)} \) onto \( \gamma_{e_{i(i+1)}}(v) \);
2. for a vertex \( v \in T^{[0]} \) with incoming edges \( e_{i_{1}}u_{i_{1}}, \ldots, e_{i_{1}}u_{i_{1}} \) and outgoing edge \( e_{i_{0}}u_{i_{0}} \), we take \( (-1)^{\text{vol}_{e_{i_{1}}u_{i_{1}}, v} + 1} \text{vol}_{e_{i_{1}}u_{i_{1}}, v} \wedge \text{vol}_{e_{i_{0}}u_{i_{0}}, v} \) arranged in clockwise orientation with \( \text{vol}_{e_{i_{1}}, v} \) defined, we let \( \text{vol}_{e_{i_{1}}u_{i_{1}}, v} := (-1)^{|\text{vol}_{e_{i_{1}}u_{i_{1}}, v} + 1}} \text{vol}_{e_{i_{1}}u_{i_{1}}, v} \wedge \text{vol}_{e_{i_{0}}u_{i_{0}}, v} \).
Therefore, for the outgoing edge $e_v$ we obtain a differential form $\nu_{e_v}$ when $v$ is labeled with $a$.

(3) for an edge $e_{ij}$ with incoming vertex $v_0 = \partial_{in}(e_{ij})$ and outgoing vertex $v_1 = \partial_{out}(e_{ij})$, we let $\nu_{e_{ij},v_1} = (\tau(\partial_{in}(e_{ij})))^* (\nu_{e_{ij},v_0})$ where $\tau(\partial_{in}(e_{ij}))$ is the gradient flow of $\nabla f_{ij}$ for time $l(e_{ij})$.

Therefore, for the outgoing edge $e_0$ starting at the root vertex $v_r$ and ending at the outgoing vertex $v_o$, we obtain a differential form $\nu_{e_0,v_r}$ from the above construction, and we determine the sign $(-1)^{\chi(\Gamma)}$ by $(-1)^{\chi(\Gamma)} \nu_{e_0,v_r} \wedge \nu_{e_0,v_o} = \nu_M$ where $\nu_{e_0}$ is the chosen volume element in $NV^+$ for the critical point $q_0$. (For the case $T^{[0]} = \emptyset$, we define by convention that $(-1)^{\chi(\Gamma)} \Gamma' \wedge \nu_p \wedge^* \nu_q = \nu_M$ for a gradient flow line $\Gamma$ from $p$ to $q$.)

**Definition 2.10.** Given a generic sequence of functions $\tilde{f} = (f_0, \ldots, f_k)$, with a sequence of critical points $(q_{(k-1)k}, \ldots, q_{01})$ we define the operation $m_k^{\text{Morse}}(q_{(k-1)k}, \ldots, q_{01}) \in CM_{f_0}^*[[u]]$ by extending linearly the formula

$$m_k^{\text{Morse}}(q_{(k-1)k}, \ldots, q_{01}) := \begin{cases} \sum_{q_0 \in \text{Crit}(f_0)} \sum_{\Gamma \in \mathcal{M}_{\tilde{f}}(\tilde{q})} (-1)^{\chi(\Gamma)} q_{0} & \text{if } \dim(\mathcal{M}_{\tilde{f}}(\tilde{q})) = 0, \\ 0 & \text{otherwise,} \end{cases}$$

where $\tilde{q} = ((q_{(k-1)k}, \ldots, q_{01}, q_{0k})$). We further let $m_k^{\text{Morse}} = \sum_{T \in \mathcal{L}_k} u^{n_T} m_k^{\text{Morse}}$ where $n_T = |\mathcal{L}_k^{[0]}|$.

We have the following **Theorem 2.11** which is the main result for this paper.

**Theorem 2.11.** Given a generic sequence of functions $\tilde{f} = (f_0, \ldots, f_k)$, with a sequence of critical points $\tilde{q} = (q_{(k-1)k}, \ldots, q_{01}, q_{0k})$, then we have

$$\lim_{\lambda \to \infty} \Phi(m_k,T(\lambda)(\phi(q_{(k-1)k}), \ldots, \phi(q_{01}))) = m_k^{\text{Morse}}(q_{(k-1)k}, \ldots, q_{01}),$$

where $\phi := \Phi^{-1}$ is the inverse of the isomorphism in equation (2.1).

As a consequence, the Morse product $m_k^{\text{Morse}}$ satisfies the $A_\infty$-relation whenever we consider a generic sequence of functions such that every operation appearing in the formula is well-defined.

### 3. Proof of Theorem 2.11

#### 3.1. Analytic results.

For the proof of Theorem 2.11, we assume $T^{[0]} \neq \emptyset$ since this is exactly the case carried out by [12]. We begin with recalling the necessary analytic results from [12, 18, 3].

#### 3.1.1. Results for a single Morse function.

We will assume that the function $f_{ij}$ we are dealing with satisfy the Morse-Smale assumption 2.5. Due to difference in convention, $e^{-\lambda f_{ij}} \Delta_{ij} e^{\lambda f_{ij}}$ is called the Witten’s Laplacian in [3], and result stated in this Section is obtain by the corresponding statements in [3] by conjugating $e^{\lambda f_{ij}}$.

**Theorem 3.1 ([12, 18]).** For each $f_{ij}$, there is $\lambda_0 > 0$ and constants $c, C > 0$ such that we have $\text{Spec}(-\Delta_{ij}) \cap [ce^{-\lambda_0}, C\lambda^{1/2}] = \emptyset$, for $\lambda > \lambda_0$. The map $\Phi = \Phi_{ij} : \Omega^*_{ij,c} \to CM_{f_{ij}}^*$ in equation (2.1) is a chain isomorphism for $\lambda$ large enough. We will denote the inverse by $\phi = \phi_{ij}$.

We will the asymptotic behaviour of $\phi(q)$ for a critical point $q$ of $f_{ij}$, and we will need the following Agmon distance $d_{ij}$ for this purpose.

---

5Hence we have valency of $v$ being 3.

6We omit the numbering $ij$ from our notation here.
Definition 3.2. For a Morse function $f_{ij}$, the Agmon distance $d_{ij}$ or simply denoted by $d$, is the distance function with respect to the degenerated Riemannian metric $\langle \cdot, \cdot \rangle_{f_{ij}} = |df_{ij}|^2 \langle \cdot, \cdot \rangle$, where $\langle \cdot, \cdot \rangle$ is the background metric. We will also write $\rho_{ij}(x,y) := d_{ij}(x,y) - f_{ij}(y) + f_{ij}(x)$.

Lemma 3.3. We have $\rho_{ij}(x,y) \geq 0$ with equality holds if and only if $x$ is connected to $y$ via a generalized flow line $\gamma : [0,1] \to M$ with $\gamma(0) = x$ and $\gamma(1) = y$. Here a generalized flow line means that $\gamma$ is continuous, and there is a partition $0 = t_0 < t_1 < \cdots < t_i = 1$ such that $\gamma|_{(t_r,t_{r+1})}$ is a reparameterization of a gradient flow line of $f_{ij}$ and $\gamma(t_r) \in \text{Crit}(f_{ij})$ for $0 < r < i$.

Lemma 3.4. Let $\gamma \subset \mathbb{C}$ to be a subset whose distance from $\text{Spec}(\Delta_{ij})$ is bounded below by a constant. For any $j \in \mathbb{Z}_+$ and $\epsilon > 0$, there is $k_j \in \mathbb{Z}_+$ and $\lambda_0 = \lambda_0(\epsilon) > 0$ such that for any two points $x_0, y_0 \in M$, there exist neighborhoods $V$ and $U$ (depending on $\epsilon$) of $x_0$ and $y_0$ respectively, and $C_{j,\epsilon} > 0$ such that $\| \nabla^j((z - \Delta_{ij})^{-1}u) \|_{C^0(V)} \leq C_{j,\epsilon}e^{-\lambda(\rho_{ij}(x_0,y_0) - \epsilon)} \| u \|_{W^{k,j,2}(U)}$, for all $\lambda > \lambda_0$ and $u \in C^0_c(U)$, where $W^{k,p}$ refers to the Sobolev norm.

We will also need modified version of the resolvent estimate for $G_{ij}$, which can be obtained by applying the original resolvent estimate to the the formula

\begin{equation}
G_{ij}(u) = \int_s z^{-1}(z - \Delta_{ij})^{-1}u.
\end{equation}

Lemma 3.5. For any $j \in \mathbb{Z}_+$ and $\epsilon > 0$, there is $k_j \in \mathbb{Z}_+$ and $\lambda_0 = \lambda_0(\epsilon) > 0$ such that for any two points $x_0, y_0 \in M$, there exist neighborhoods $V$ and $U$ (depending on $\epsilon$) of $x_0$ and $y_0$ respectively, and $C_{j,\epsilon} > 0$ such that $\| \nabla^j(G_{ij}u) \|_{C^0(V)} \leq C_{j,\epsilon}e^{-\lambda(\rho_{ij}(x_0,y_0) - \epsilon)} \| u \|_{W^{k,j,2}(U)}$, for all $\lambda > \lambda_0$ and $u \in C^0_c(U)$, where $W^{k,p}$ refers to the Sobolev norm.

For a critical point $q$ of $f_{ij}$, $\phi(q)$, has certain exponential decay measured by the Agmon distance from the critical point $q$.

Lemma 3.6. For any $\epsilon$, there exists $\lambda_0 = \lambda_0(\epsilon) > 0$ such that for $\lambda > \lambda_0$, we have $\phi(q) = \mathcal{O}_\epsilon(e^{-\lambda(g^+_q(x) - \epsilon)})$, and same estimate holds for the derivatives of $\phi_{ij}(q)$ as well. Here $\mathcal{O}_\epsilon$ refers to the dependence of the constant on $\epsilon$ and $g^+_q(x) = \rho_{ij}(q,x) = d_{ij}(q,x) + f_{ij}(q) - f_{ij}(x)$.

Remark 3.7. We notice that $g^+_q$ is a nonnegative function with zero set $V^+_q$ that is smooth and Bott-Morse in a neighborhood $W$ of $V^+_q \cup V^-_q$. Similarly, if we write $g^-_q = d_{ij}(q,x) + f_{ij}(x) - f_{ij}(q)$ which is a nonnegative function with zero set $V^-_q$ and is smooth and Bott-Morse in $W$, and we have $\phi_{ij}(q)/\| \phi_{ij}(q)e^{-\lambda f_{ij}} \|^2 = \mathcal{O}_\epsilon(e^{-\lambda(g^-_q - \epsilon)})$ where $\ast_{ij} = \ast_{e^{-2\lambda f_{ij}}}$ comparing to the usual star operator $\ast$.

Lemma 3.8. The normalized basis $\phi(q)/\| \phi(q) \|$’s are almost orthonormal basis with respect to the twisted inner product $\langle \cdot, \cdot \rangle_{e^{-2\lambda f_{ij}}}$. More precisely, there is a $C, c > 0$ and $\lambda_0$ such that when $\lambda > \lambda_0$, we will have $\int_M \langle \frac{\phi(q)}{\| \phi(q) \|}, \frac{\phi(q)}{\| \phi(q) \|} \rangle \text{vol}_{ij} = \delta_{pq} + Ce^{-c\lambda}$.

Restricting our attention to a small enough neighborhood $W$ containing $V^+_q \cup V^-_q$, the above decay estimate of $\phi(q)$ from [24] can be improved from an error of order $\mathcal{O}_\epsilon(e^{e\lambda})$ to $\mathcal{O}(\lambda^{-\infty})$.

Lemma 3.9. There is a WKB approximation of the $\phi(q)$ as $\phi(q) \sim \lambda^{\deg(q)/2}e^{-\lambda g^+_q} (\omega_{q,0} + \omega_{q,1} \lambda^{-1/2} + \ldots)$ which is an approximation in any precompact open subset $K \subset W_q$ of the form

\begin{equation}
\| e^{\lambda g^+_q} \nabla^j (\lambda^{-\deg(q)/2} \phi(q) - e^{-\lambda g^+_q} \sum_{l=0}^N \omega_{q,j} \lambda^{-l/2}) \|_{L^\infty(K)} \leq C_{j,K,N} \lambda^{-N-1+2j}.
\end{equation}

\[\text{Readers may see [5] for its basic properties.} \]

\[\text{Notice that we indeed have } \omega_{q,j+1} = 0 \text{ in this case while we prefer to write it in this form to unify our notations.} \]
for any \( j, N \in \mathbb{Z}_+ \), where \( W_q \supset V^+_q \cup V^-_q \) is an open neighborhood of \( V^+_q \cup V^-_q \).

Furthermore, the integral of the leading order term \( \omega_{q,0} \) in the normal direction to the stable submanifold \( V^+_q \) is computed in [12].

**Lemma 3.10.** Fixing any point \( x \in V^+_q \) and \( \chi \equiv 1 \) around \( x \) compactly supported in \( W \), we take any closed submanifold (possibly with boundary) \( NV_{q,x}^+ \) of \( W \) intersecting transversally with \( V^+_q \) at \( x \). We have

\[
\lambda^{\deg(q)/2} \int_{NV_{q,x}^+} e^{-\lambda g^+_q} \chi \omega_{q,0} = 1 + \mathcal{O}(\lambda^{-1});
\]

\[
\frac{\lambda^{\deg(q)}}{||e^{-\lambda f_{ij}} \phi_{ij}(q)||^2} \int_{NV_{q,x}^+} e^{-\lambda g^-_q} \chi \cdot \omega_{q,0} = 1 + \mathcal{O}(\lambda^{-1}),
\]

for any point \( x \in V^-_q \), with \( NV_{q,x}^- \) intersecting transversally with \( V^-_q \).

### 3.1.2. WKB for homotopy operator

We recall the key estimate for the homotopy operator \( H_{ij} \) proven in [3 Section 4]. Let \( \gamma(t) \) be a flow line of \( \nabla f_{ij} / |\nabla f_{ij}|_{d_{ij}} \) starts at \( \gamma(0) = x_S \) and \( \gamma(T) = x_E \) for a fixed \( T > 0 \) as shown in the following figure [2]. We consider an input form \( \zeta_S \) defined in a neighborhood \( W_S \) of \( x_S \). Suppose we are given a WKB approximation of \( \zeta_S \) in \( W_S \), which is an approximation of \( \zeta_S \) according to order of \( \lambda \) of the form

\[
\zeta_S \sim e^{-\lambda g_S}(\omega_{S,0} + \omega_{S,1} \lambda^{-1/2} + \omega_{S,2} \lambda^{-1} + \ldots)
\]

which means we have \( \lambda_{j,0} > 0 \) such that when \( \lambda > \lambda_{j,N,0} \) we have

\[
\|e^{\lambda g_S} \nabla_j (\zeta_S - e^{-\lambda g_S}(\sum_{i=0}^N \omega_{S,i} \lambda^{-i/2}))\|_{L^\infty(W_S)} \leq C_{j,N} \lambda^{-N-1+2j},
\]

for any \( j, N \in \mathbb{Z}_+ \). We further assume that \( g_S \) is a nonnegative Bott-Morse function in \( W_S \) with zero set \( V_S \) such that \( \gamma \) is not tangent to \( V_S \) at \( x_S \). We consider the equation

\[
\Delta_{ij} \zeta_E = (I - P_{ij}) d^*_{ij}(\chi_S \zeta_S),
\]

where \( \chi_S \) is a cutoff function compactly supported in \( W_S \), \( P_{ij} : \Omega^*(M) \to \Omega^*_{ij,<1} \) is the projection. We want to have a WKB approximation of \( \zeta_E = H_{ij}(\chi_S \zeta_S) \)

**Lemma 3.11.** For \( \text{supp}(\chi_S) \) small enough (the size only depends on \( g_S \) and \( f_{ij} \)), there is a WKB approximation of \( \zeta_E \) in a small enough neighborhood \( W_E \) of \( x_E \), of the form \( \zeta_E \sim e^{-\lambda g_E} \lambda^{-1/2} (\omega_{E,0} + \omega_{E,1} \lambda^{-1/2} + \ldots) \) in the sense that we have \( \lambda_{j,0} > 0 \) such that when \( \lambda > \lambda_{j,N,0} \) we have

\[
\|e^{\lambda g_E} \nabla_j (\zeta_E - e^{-\lambda g_E}(\sum_{i=0}^N \omega_{E,i} \lambda^{-(i+1)/2}))\|_{L^\infty(W_E)} \leq C_{j,N} \lambda^{-N+2j}.
\]
Furthermore, the function $g_E$ (only depending on $g_S$ and $f_{ij}$) is a nonnegative function which is Bott-Morse in $W_E$ with zero set $V_E = \bigcup_{-\infty < t < +\infty} \{ t(V_S) \} \cap W_E$ which is a closed submanifold in $W_E$, where $c_t$ is the t-time $\nabla f_{ij}/|\nabla f_{ij}|^2$.

Finally, we have the following Lemma 3.12 from [3] relating the integrals of $\omega_{S0}$ and $\omega_{E0}$.

**Lemma 3.12.** Using same notations in lemma 3.11 and suppose $\chi_S$ and $\chi_E$ are cutoff functions supported in $W_S$ and $W_E$ respectively, then we have

$$\lambda^{-\frac{1}{2}} \int_{N_{x_B}} e^{-\lambda g_s} \chi_E \omega_{E0} = \left( \int_{N_{x_S}} e^{-\lambda g_s} \chi_S \omega_{S0} \right) (1 + O(\lambda^{-1})).$$

Furthermore, suppose $\omega_{S0}(x_S) \in \bigwedge^{top} N(V_S)_{x_S}$, we have $\omega_{E0}(x_E) \in \bigwedge^{top} N(V_E)_{x_E}$. Here $\bigwedge^{top} E$ refers to the $r$-rank vector bundle $E$. Here $N_{x_S}$ and $N_{x_E}$ are any closed submanifold of $W_S$ and $W_E$ intersecting $V_S$ and $V_E$ transversally at $x_S$ and $x_E$ respectively.

### 3.2. Apriori Estimate.

**Notations 3.13.** From now on, we will consider a fixed generic sequence $\hat{f} = (f_0, \ldots, f_k)$ with corresponding sequence of critical points $\hat{q} = (q_{(k-1)k}, \ldots, q_{01}, q_{0k})$ and a fixed labeled ribbon $k$-tree $T$ such that $\dim(M_T(\hat{f}, \hat{q})) = 0$ (the dimension is given by formula (2.2)). We use $q_{ij}$ to denote a fixed critical point of $f_{ij}$. $\phi(q_{ij})$ associated to $q_{ij}$ is abbreviated by $\phi_{ij}$.

**Notations 3.14.** For $T \in T_k$ or $LT_k$ with $\hat{q}$, we let $\Delta_T := \coprod_{v \in LT_k} \Delta_{\nu(v)}$ of dimension $\nu(T) := \sum_{v \in LT_k} \nu(v)$, and we also let $\deg(T) := \sum_{i=0}^{k-1} \deg(q_{(i+1)}) - [T[1]] - \nu(T)$. We inductively define a volume form $\nu_T$ on $\Delta_T$ for labeled ribbon tree $T \in LT_k$ by: letting $\nu_1 = dt_1 \wedge \ldots \wedge dt_1$ on the $\Delta_1$; and for $\nu_v$ labeled with 1 we split $T$ at $v_v$ into $T_2$ and $T_1$ such that $T_2, T_1, e_o$ is clockwisely oriented, then we take $\nu_T = \nu_{T_2} \wedge \nu_{T_1}$; and for $\nu_v$ labeled with $u$ we split $T$ at $v_v$ into $T_1, \ldots, T_1$ clockwisely, and we take $\nu_T = \nu_{T_1} \wedge \ldots \wedge \nu_{T_1} \wedge v_v$. We should also write $\nu_T$ to be the polyvector field dual to $\nu_T$.

**Definition 3.15.** Given a labeled ribbon $k$-tree $T$ with $\tilde{f}$ and $\tilde{q}$ as above, we associate to it a length function $\hat{\rho}_T$ on $\mathcal{M}(T) := \Delta_T \times \mathbb{R}_{[T_{ni}]}^+ \to \mathbb{R}_{[T_{ni}]}^+$ with coordinates $(\tilde{v}_T, \tilde{x}_T)$ (where $\tilde{v}_T = (v_v)_{v \in LT_k}$ and $\tilde{x}_T = (x_v)_{v \in LT_k}$) inductively along the tree by the rules:

1. for the unique tree with one edge $e$ numbered by $ij$, we take $\hat{\rho}_T(x_v) := \rho_{ij}(q_{ij}, x_v)$;
2. when $v_v$ is labeled with 1, we split $T$ at the root vertex $v_v$ into $T_2, T_1$. We notice that $\mathcal{M}(T) = \mathcal{M}(T_2) \times_M \mathcal{M}(T_1) \times M_{v_v}$ (with coordinates $\tilde{v}_T = (\tilde{v}_T, \tilde{x}_T)$, and $\tilde{x}_T = (\tilde{x}_T, \tilde{x}_T, \tilde{x}_v)$ such that $x_{T_2, v} = x_{T_1, v} = x_v$ in $M$) and we let

$$\hat{\rho}_T(\tilde{v}_T, \tilde{x}_T) = \hat{\rho}_{ij}(x_v, x_v) + \sum_{j=1}^2 \hat{\rho}_{ij}(\tilde{v}_T, \tilde{x}_j)$$

if the numbering on $e_o$ is $ij$;
3. when $v_v$ is labeled with $u$, we split $T$ at $v_v$ into $T_1, \ldots, T_1$ and we can write $\mathcal{M}(T) = \mathcal{M}(T_1) \times_M \ldots \times_M \mathcal{M}(T_1) \times M(\Delta_l \times M_{v_v})$ where $l = \nu(v_v)$. By writing coordinates $(\tilde{v}_{T_l}, \tilde{x}_{T_l})$ for $\mathcal{M}(T_l)$, $t_{v_v} = (t_{v_v,l}, t_{v_v,1})$ for $\Delta_l$, $x_{v_v}$ for $M_{v_v}$ and $x_v$ for $M_{v_v}$ satisfying $x_{T_1, v} = \sigma_{t_{v_v,l}}(x_v), \ldots, x_{T_1, v} = \sigma_{t_{v_v,1}}(x_v)$, we let

$$\hat{\rho}_T(\tilde{v}_T, \tilde{x}_T) := \hat{\rho}_{ij}(x_v, x_v) + \sum_{j=1}^l \hat{\rho}_{ij}(\tilde{v}_{T_l}, \tilde{x}_{T_l})$$

if the numbering on $e_o$ is $ij$.

---

9Here $T_{ni}$ is the set of all vertices besides incoming edges introduced in Definition 2.2.
Fixing the outgoing point \( x_{v_0} = q_0 \) giving coordinates \( \bar{x}_T = (x_v)_{v \in T[0]} \) for \( M[T[0]] \), we let \( \hat{\rho}_T(\bar{u}_T, \bar{x}_T) := \hat{\rho}_T(\bar{u}_T, \bar{x}_T, q_0) \).

**Example 3.16.** Suppose that \( T \) is the labeled ribbon 2-tree with two incoming vertices \( v_2 \) and \( v_1 \) joining to \( v \) labeled with \( u \) by \( e_{12} \) and \( e_0 \), and \( v \) is joining to the root vertex \( v_r \) labeled with \( u \) via \( e \). Then we have \( \Delta \times M[T[0]] \) \( = \Delta_2 \times S^1 \times M^3 \) and \( \hat{\rho}_T(t_{v,2}, t_{v,1}, t_{v_r}, x_{v_r}, x_{v_2}, x_{v_1}) = \rho_{02}(x_{v_r}, x_{v_2}) + \rho_{02}(x_{v}, \sigma_{t_{v,2}}(x_{v_2})) + \rho_{12}(q_{12}, \sigma_{t_{v,1}}(x_{v_1})) + \rho_{01}(q_{01}, \sigma_{t_{v,1}}(x_{v_1})) \). The following Figure 3 shows the tree \( T \) and its associated \( \hat{\rho}_T \).

![Image](image.png)

**Figure 3.** Distance function associated to \( T \)

From its construction and Lemma 3.3, we notice that \( \rho_T(\bar{u}_T, \bar{x}_T) \geq 0 \) and equality holds if and only if for each edge \( e \) numbered by \( ij \) with \( \partial_n(e) = v_1 \) and \( \partial_0(e) = v_2 \), there is a generalized flow line of \( \nabla f_{ij} \) joining \( x_{v_1} \) to \( x_{v_2} \), where \( x_{v_2} = x_{v_2}^0 \) when \( v_2 \) is labeled by 1; and \( x_{v_2} = \sigma_{t_{v_2,j}}(x_{v_2}) \) if \( v_2 \) is labeled by \( u \) with and \( e \) is the \( j \)th incoming edges of \( v_2 \) in the anti-clockwise orientation. Therefore, we have a generalized flow tree (with jumping) of type \((T, \bar{f}, \bar{q})\) (which is a generalization of flow tree in Definition 2.7) by allow broken flow lines as in Definition 3.3. With the condition that \( \dim(M(\bar{f}, \bar{q})) = 0 \) as mentioned in Notation 3.13, we notice that every such generalized flow line is an actual flow line from the generic assumption 2.8 for \( \bar{f} \), because the expected dimension for flow tree with broken flow line is negative.

**Notations 3.17.** We let \( \Gamma_1, \ldots, \Gamma_d \) be the gradient flow tree of type \((T, \bar{f}, \bar{q})\), such that each \( \Gamma_i \) is associated with a point \( t_{\Gamma_i,v} \in \Delta_{v(v)} \) (for \( v \in LT[0] \) and \( x_{\Gamma_i,v} \in M \) (for \( v \in T[0] \)) such that

1. \( x_{\Gamma_i,v} \) is the starting point of a gradient flow line \( \gamma_e \) associated to edge \( e \) if \( \partial_n(e) = v \), and we write \( x_{\Gamma_i,e,v} = x_{\Gamma_i,v} \) in this case;
2. \( x_{\Gamma_i,v} \) is the end point of the gradient flow line \( \gamma_e \) if \( v \) is labeled by 1 if \( \partial_0(e) = v \), and we write \( x_{\Gamma_i,e,v} = x_{\Gamma_i,v} \) in this case;
3. and \( \sigma_{\Gamma_i,e,j}(x_{\Gamma_i,v}) \) is the end point of a gradient flow line \( \gamma_e \) associated to \( j \)th-edge \( e \) clockwise if \( v \) is labeled by \( u \) and \( \partial_0(e) = v \), and we write \( x_{\Gamma_i,e,v} = \sigma_{\Gamma_i,e,j}(x_{\Gamma_i,v}) \) in this case.

We consider a sequence of cut off functions \( \tilde{\chi} := (\chi_v)_{v \in T[0]} \) such that \( \chi_v \) compactly supported in a ball \( U_v := B(x_v, r/2) \) of radius \( r \) centered at a fixed point \( x_v \in M \), and \( (\tilde{\chi}_v)_{v \in LT[0]} \) with \( \chi_v \).
Lemma 3.20. We fix a point \( \tilde{v}_T, \tilde{x}_T \) in \( \mathcal{M}(T) \) with the cut off functions \( \check{\chi} \) and \( \check{\nu} \) and \( m_{\check{\nu}, \check{\chi}}^T \) as before Definition 3.18, for any \( \epsilon > 0 \) we have \( \lambda_0(\epsilon) \) and small enough radius \( r = r(\epsilon) \) of cut off functions (which is described before Definition 3.18) such that when \( \lambda > \lambda_0 \) we have the norm estimate

\[
\| m_{\check{\nu}, \check{\chi}}^T \|_{\| e^{-\lambda} \phi_0 \|_2} \leq C_{\epsilon} e^{-\lambda (\rho_T(\tilde{v}_T, \tilde{x}_T) - r \epsilon)},
\]

\( \lambda \) as before Definition 3.18}
for any $j \in \mathbb{Z}_+$ (Here we fix an arbitrary metric on the simplices $\Delta_1$’s), where $b_T$ is a constant depending the combinatorics of $T$.

Proof. We prove by induction along the tree $T$ that for each flag $(e, v)$ with $\partial_0(e) = v \neq v_o$ we have

$$\|m_{\hat{v}, \hat{e}}^{(e,v)}\|_{C^j(\mathbf{A}(\hat{e}, v) \times U_v)} \leq C_{j, \epsilon, \bar{\chi}, \bar{\tau}} \exp \left(-\lambda(\hat{\rho}_{T_{T,v}}(\hat{v}_{T_{T,v}}, \hat{x}_{T_{T,v}}) - b_{T_{T,v}})\right),$$

where $U_v = B(x_v, r/2)$, for any points $\hat{v}_T \in \mathbf{A}_T, \hat{x}_T \in M^{[T_{T,v}]}_{\text{geo}}$ with the associated cut off functions $\hat{\tau}$ and $\bar{\chi}$ with small enough $r$. The initial case follows from the estimate in Lemma 3.6. For induction we consider an edge $e$ with $\partial_0(e) = v$ and $\partial_0(e) = \bar{v}$. We take subtrees (of $T$) $T_1, \ldots, T_l$ with edges $e_1, \ldots, e_l$ attached to $v$ such that $e_1, \ldots, e_l$ is clockwise oriented. There are two cases.

The first case is when $v$ is labeled with 1 and we have $l = 2$. In this case we have the estimate

$$\|m_{\hat{v}, \hat{e}}^{(e_2,v)} \wedge m_{\hat{v}, \hat{e}}^{(e_1,v)}\|_{C^j(\mathbf{A}(e_2, v) \times \mathbf{A}(e_1, v) \times U_v)} \leq C_{j, \epsilon, \bar{\chi}, \bar{\tau}} \exp \left(-\lambda(\hat{\rho}_{T_{T,v}}(\hat{v}_{T_{T,v}}, \hat{x}_{T_{T,v}}) - b_{T_{T,v}})\right),$$

by choosing $b_{T_{T,v}} = b_{T_1} + b_{T_2}$, where we require $x_{T_1,v} = x_{T_2,v} = x_v$ in the R.H.S. of the above equation. Assuming that $e$ is numbered by $ij$, and we apply the Lemma 3.5 to the term $m_{\hat{v}, \hat{e}}^{(e, v)} = d_{ij}^e G_{ij}(\nu_v m_{\hat{v}, \hat{e}}^{(e_2,v)} \wedge m_{\hat{v}, \hat{e}}^{(e_1,v)})$ (we choose smaller $r$ if necessary) we obtain the estimate

$$\|d_{ij}^e G_{ij}(\nu_v m_{\hat{v}, \hat{e}}^{(e_2,v)} \wedge m_{\hat{v}, \hat{e}}^{(e_1,v)})\|_{C^j(\mathbf{A}(e_2, v) \times U_v)} \leq C_{j, \epsilon, \bar{\chi}, \bar{\tau}} \exp \left(-\lambda(\hat{\rho}_{T_{T,v}}(\hat{v}_{T_{T,v}}, \hat{x}_{T_{T,v}}) - b_{T_{T,v}})\right),$$

by taking $b_{T_{T,v}} \geq b_{T_{T,v}} + 1$ which is the desired estimate.

The second case is when $v$ is labeled with $u$, and we have the estimate

$$\|\sigma_1^* (t_{w_{v,1}} \wedge \nu_v) m_{\hat{v}, \hat{e}}^{(e,v)} \wedge \cdots \wedge \sigma_l^* (t_{w_{v,1}} \wedge \nu_v) m_{\hat{v}, \hat{e}}^{(e,v)}\|_{C^j(\Pi_{j=1}^l \mathbf{A}(t_j) \times C_u \times U_i)} \leq C_{j, \epsilon, \bar{\chi}, \bar{\tau}} \exp \left(-\lambda\sum_{j=1}^l \hat{\rho}_{T_j}(\hat{v}_{T_j}, \hat{x}_{T_j}) - b_{T_{T,v}}\right),$$

using the induction hypothesis and by taking $b_{T_{T,v}} \geq l + \sum_{j=1}^l b_{T_j}$, for $(t_1, \ldots, t_l)$ varying in small enough neighborhood $C_u$ of $(t_{w_1,1}, \ldots, t_{w_l,1})$ ($C_u$ introduced in the paragraph before Definition 3.18), where we require that the identity $x_{T_{T,v}} = \sigma_{T_{T,v}}(x_v)$ on the R.H.S. as in the Definition 3.18. By applying $d_{ij}^e G_{ij}$ (if $e$ is numbered by $ij$) to the term $m_{\hat{v}, \hat{e}}^{(e,v)} = \nu_{T_{T,v}} \nu_v \sigma_1^* (t_{w_{v,1}} \wedge \nu_v) m_{\hat{v}, \hat{e}}^{(e,v)} \wedge \cdots \wedge \sigma_l^* (t_{w_{v,1}} \wedge \nu_v) m_{\hat{v}, \hat{e}}^{(e,v)}$ as in Definition 3.18, and using Lemma 3.5 again we have the desired estimate

$$\|m_{\hat{v}, \hat{e}}^{(e,v)}\|_{C^j(\mathbf{A}(T_{T,v}) \times U_v)} \leq C_{j, \epsilon, \bar{\chi}, \bar{\tau}} \exp \left(-\lambda(\hat{\rho}_{T_{T,v}}(\hat{v}_{T_{T,v}}, \hat{x}_{T_{T,v}}) - b_{T_{T,v}})\right),$$

where we take $b_{T_{T,v}} \geq b_{T_{T,v}} + 1$.

To obtain the statement of the Lemma, we observe that if $T_1, \ldots, T_l$ are the incoming trees joining to the root vertex we have

$$\|m_{\hat{v}, \hat{e}}^{(e,v)}\|_{C^j(\mathbf{A}(T) \times U_v)} \leq C_{j, \epsilon, \bar{\chi}, \bar{\tau}} \exp \left(-\lambda\sum_{j=1}^l \hat{\rho}_{T_j}(\hat{v}_{T_j}, \hat{x}_{T_j}) - b_{T_{T,v}}\right)$$

in a small enough neighborhood $U_v$ of $x_v$, where we have $l = 2$ and $x_{T_{T,v}} = x_{T_1,v} = x_{T_2,v}$ in R.H.S. as in the first case with $v$ labeled with 1, and $x_{T_{T,v}} = \sigma_{T_{T,v}}(x_v)$ in R.H.S. as in the second case that $v$ is labeled with $u$. The Lemma follows from the estimate for $m_{\hat{v}, \hat{e}}^{(e,v)}$ and that for $e^{-2\lambda(\phi + \phi_0)\|\epsilon\|^2}$ in Remark 3.7.
The above Lemma allows us to estimate the terms $m^T_{\tilde{\chi},\tilde{\zeta}}$ appearing in the R.H.S., and from the discussion after Example 3.16, we notice that it is closely related to gradient flow tree of type $T$. With the gradient flow trees $\Gamma_i$’s as in Notation 3.17, we assume there are open neighborhoods $D_{\Gamma_i,v}$ and $W_{\Gamma_i,v}$ of $x_{\Gamma_i,v}$ for $v \in T^{[0]}$ such that $D_{\Gamma_i,v} \subset W_{\Gamma_i,v}$ together with $\chi_{\Gamma_i,v} \equiv 1$ on $D_{\Gamma_i,v}$ which is compactly supported in $W_{\Gamma_i,v}$ giving $\tilde{\chi}_i = (\chi_{\Gamma_i,v})_{v \in T^{[0]}}$. Similarly, we also assume there are open neighborhoods $C_{\Gamma_i,v}$ and $E_{\Gamma_i,v}$ of $t_{\Gamma_i,v}$ in $\triangle_{u(v)}$ satisfying $C_{\Gamma_i,v} \subset E_{\Gamma_i,v}$ together with $\chi_{\Gamma_i,v} \equiv 1$ on $C_{\Gamma_i,v}$ which is compactly supported in $E_{\Gamma_i,v}$ giving $\tilde{\chi}_i = (\chi_{\Gamma_i,v})_{v \in LT^{[0]}}$. We should further prescribe the size of these neighborhood $W_{\Gamma_i,v}$’s and $E_{\Gamma_i,v}$ in the upcoming Section 3.3 which is defined along the gradient tree $\Gamma_i$’s together with the WKB approximation [11]. By writing $\overline{D}_{\Gamma_i} = \prod_{v \in T^{[0]}} D_{\Gamma_i,v}$ and $\overline{C}_{\Gamma_i} = \prod_{v \in LT^{[0]}} C_{\Gamma_i,v}$, we have $\rho_T \geq c > 0$ for some constant $c$ outside $\bigcup_{i=1}^d \overline{C}_{\Gamma_i} \times \overline{D}_{\Gamma_i}$ by continuity of $\rho_T$ and the discussion after Example 3.16. As a result, we can fix a small enough $\epsilon$ (and the associated $r(\epsilon)$) such that $b_T \epsilon < c/2$. The following Figure 4 show the situation for these open subsets $W_{\Gamma_i,v}$’s and $E_{\Gamma_i,v}$’s for the tree in Example 3.16.

![Figure 4: Open subsets near gradient tree $\Gamma_i$](image)

We can take a finite collection $\{\tilde{\chi}_i\}_{i \in I}$ and $\{\tilde{\zeta}_j\}_{j \in J}$ in the paragraph before Lemma 3.20 such that $\{\tilde{\chi}_i\}_{i \in I} \cup \{\tilde{\chi}_{\Gamma_1}, \ldots, \tilde{\chi}_{\Gamma_d}\}$ forms a partition of unity of $M|T^{[0]}|$ and finite collection $\{\tilde{\zeta}_j\}_{j \in J} \cup \{\tilde{\zeta}_{\Gamma_1}, \ldots, \tilde{\zeta}_{\Gamma_d}\}$ forms a partition of unity of $\triangle_T$ respectively, further satisfying $(\text{Supp}(\tilde{\chi}_i) \times \text{Supp}(\tilde{\zeta}_j)) \cap \overline{C}_{\Gamma_i} \times \overline{D}_{\Gamma_i} = \emptyset$ for each flow tree $\Gamma_i$ and any $i, j$. Therefore we have the estimate $\|m^T_{\tilde{\chi}_i,\tilde{\zeta}_j} \wedge e^{-2\lambda f_{\phi_{0k}} \phi_{0k}} \|_{C^0(\triangle_T \times M)} \leq C_{\epsilon,\tilde{\chi}_i,\tilde{\zeta}_j} \epsilon^{-\lambda c/2}$. As a conclusion of this Section 3.2, we have

$$\int_M m_{k,T}(\lambda) (\hat{\phi}_{(k-1)k}, \phi_{01}) \wedge e^{-2\lambda f_{\phi_{0k}} \phi_{0k}} = \sum_{i=1}^d \int_{\triangle_T \times M} m^T_{\tilde{\chi}_{\Gamma_i},\tilde{\zeta}_{\Gamma_i}} \wedge e^{-2\lambda f_{\phi_{0k}} \phi_{0k}} + O(\epsilon^{-\lambda c/2}),$$

ROUGHLY speaking, these are the open subsets that WKB approximation for $m^T_{\tilde{\chi},\tilde{\zeta}}$ can be constructed. These open subsets does not depend on $m^T_{\tilde{\chi},\tilde{\zeta}}$ but rather depend on the geometry of gradient flow tree $\Gamma_i$’s when applying Lemma 3.9 and Lemma 3.11 along $\Gamma_i$’s.
where $O(e^{-\lambda c/2})$ refers to function in $\lambda$ bounded by $Ce^{-\lambda c/2}$ for some $C$. This cut off the contribution to integral near the gradient flow trees $\Gamma_i$'s.

### 3.3. WKB approximation method.

#### 3.3.1. WKB expansion for $m_{(e,v)}^{(e,v)}$. We fix a particular gradient flow tree $\Gamma = \Gamma_i$ (we omit $i$ in our notations for the rest of this paper) and compute the contribution from the integral $\int_{\Delta \times M} m_{\Delta,0}^T \cdot \lambda_f \omega \frac{e^{-\lambda_f \phi_{0k}}}{\lambda_f^{-\phi_{0k}}} \quad \text{in the above equation 3.6 using techniques from \cite{3} Section 3].}$

We inductively define the open subset $\lambda f = \lambda f_{ij}$ for $(e,v)$ and $(e,v) = \lambda f_{ij}$ for the incoming edges $(1)$ for the incoming edges $(4)$ for an edge $\lambda f_{ij}$ for the outgoing edge $(2)$ for $(e,v)$ to $\lambda f_{ij}$ with $\tau^j_{ij} \subset \lambda f_{ij}$. In this case we have $\lambda f_{ij} = \lambda f_{ij} \subset \lambda f_{ij}$ with the product WKB expansion as $\lambda f_{ij} = \lambda f_{ij} \times \lambda f_{ij}$. Therefore we have the WKB expansion $m_{(e,v)}^{(e,v)} \sim \lambda^e_{e,v} \cdot \lambda^g_{e,v} \cdot (\omega(e,v),0 + \omega(e,v),1 \lambda^{-\frac{1}{2}} + \cdots)$ by taking $\lambda^e_{e,v} = \lambda^e_{e,v} \cdot \lambda^g_{e,v} = \lambda^e_{e,v} + \lambda^g_{e,v}$ and $\omega(e,v) = \sum_{i+j=0}^1 \omega(e,v),i \cdot \omega(e,v),j + \omega(e,v),j$ (Here $e$ is given (2) in Definition 3.18). In this case we have $g_{e,v}$ being a non-negative Bott-Morse function in $E_{e,v}$ together with the WKB expansion for $\tau^j_{ij} \subset \lambda f_{ij}$ with $\tau^j_{ij} \subset \lambda f_{ij}$.

(1) for the incoming edges $e_{ij}$ with $\partial \omega(e_{ij}) = v$, we define $W_{e_{ij},v}$ to be a open subset of $x_{\lambda f, e_{ij},v}$ (We use the notation as in Notation 3.17) together with the WKB expansion for $\phi_{ij}$ in $\lambda f_{ij}$ from Lemma 3.9 with $r_{e_{ij},v} = \deg(q_{ij})_v$ and $g_{e_{ij},v} = g_{e_{ij},v}$. In this case we have $W_{e_{ij},v} = V_{e_{ij},v} \cap W_{e_{ij},v}$ being the stable submanifold;

(2) for $(e,v)$ with $\partial \omega(e,v) = v$ with $v$ is labeled with 1, we let $T_2, T_1$ to be subtrees with outgoing edges $e_2, e_1$ ending at $v$ such that $e_2, e_1$ are clockwisely oriented, we let $E_{e_{ij},v} = E_{T_2} \times E_{T_1}$ and $W_{e,v} = W_{e_2,v} \cap W_{e_1,v}$ with the product WKB expansion as

$$\sum_{i+j=0}^1 \omega(e,v),i \cdot \omega(e,v),j \lambda^{\frac{1}{2}} + \cdots$$

by taking $\lambda^e_{e,v} = \lambda^e_{e,v} \cdot \lambda^g_{e,v} = \lambda^e_{e,v} + \lambda^g_{e,v}$ and $\omega(e,v) = \sum_{i+j=0}^1 \omega(e,v),i \cdot \omega(e,v),j + \omega(e,v),j$ (Here $e$ is given (2) in Definition 3.18). In this case we have $g_{e,v}$ being a non-negative Bott-Morse function in $E_{e,v}$ together with the WKB expansion for $\tau^j_{ij} \subset \lambda f_{ij}$ with $\tau^j_{ij} \subset \lambda f_{ij}$.

(3) when we have $v$ labeled with $u$, we let $\mathcal{L}_1, \mathcal{L}_2$ be subtrees with outgoing edges $e_1, \ldots, e_1$ ending at $v$ with $e_1, \ldots, e_1$ are clockwisely oriented, we let $E_{e,v} = E_{T_2} \times E_{T_1} \times E_{e,v}$ and $W_{e,v} = W_{e_2,v} \cap W_{e_1,v}$ with the product WKB expansion as

$$\sum_{i+j=0}^1 \omega(e,v),i \cdot \omega(e,v),j \lambda^{\frac{1}{2}} + \cdots$$

by taking $\lambda^e_{e,v} = \lambda^e_{e,v} \cdot \lambda^g_{e,v} = \lambda^e_{e,v} + \lambda^g_{e,v}$ and $\omega(e,v) = \sum_{i+j=0}^1 \omega(e,v),i \cdot \omega(e,v),j + \omega(e,v),j$ (Here $e$ is given (2) in Definition 3.18). In this case we have $g_{e,v}$ being a non-negative Bott-Morse function in $E_{e,v}$ together with the WKB expansion for $\tau^j_{ij} \subset \lambda f_{ij}$ with $\tau^j_{ij} \subset \lambda f_{ij}$.

(4) for an edge $e$ numbered by $ij$ with $\partial \omega(e) = v_0$ and $\partial \omega(e) = v_1$ with $v_1$ not being the outgoing vertex $v_0$, we apply the Lemma 3.11 by taking $C_S = m_{(e,v)}^{(e,v)}$ and shrinking $W_{e,v}$ if necessary together with its WKB approximation, therefore we obtain the WKB approximation for

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12Here $T_{e,v}$ is the combinatorial subtree of $T$ as in Notation 2.3.
\[\zeta_E = m_{\chi,\overline{z}}^{(e, v_1)}\] in a neighborhood of zero section in the normal bundle.

In this case we have \(V_{e,v_1} = \cup_{t \in \mathbb{R}} \sigma_t(V_{e,v_0} \cap (\mathcal{E}_{T,e,v_1} \times W_{e,v_1})\) where \(\sigma_t\) is the t-time flow of \(\nabla f_{ij}/\nabla f_{ij}\)'s extended to \(\mathcal{E}_{T,e,v_1} \times (\mathcal{M} \setminus \text{Crit}(f_j))\) by taking product with \(\mathcal{E}_{T,e,v_1}\).

For the outgoing edge \(e_o\) with outgoing vertex \(v_o\), we simply take the WKB expansion of \(m_{\chi,\overline{z}}^{(e_o, v_o)}\) to be that of \(m_{\chi,\overline{z}}^{(e, v_1)}\). In this case we have \(V_{e_o,v_0} = V_{e_o,v_1}\).

Having the WKB approximation of \(m_{\chi,\overline{z}}^{(e, v_0)}\), together with that for \(e - 2\lambda f_0 \phi_0 k \sim \frac{\lambda^{\deg(\phi_0)}/2}{\|e - \lambda f_0 \phi_0 k\|^2} \quad \star e - \lambda f_0 \phi_0 k \|e - \lambda f_0 \phi_0 k\|\), from Lemma 3.9 (here we abbreviated \(g_{\phi_0}\) and \(\omega_{\phi_0, j}\)’s by \(g_{\phi_0}\) and \(\omega_{\phi_0}\)’s respectively), we obtain

\[
\int_{\mathbf{T} \times M} m_{\chi,\overline{z}}^{\mathbf{r},\mathbf{r}} \wedge \frac{\lambda^{\deg(\phi_0)/2}}{\|e - \lambda f_0 \phi_0 k\|} \quad \int_{\mathbf{T} \times M} e^{-\lambda g_{\phi_0} + \omega_{\phi_0}} + \omega_{\phi_0,0} + \mathcal{O}(\lambda^{-1/2}).
\]

3.3.2. Explicit computation of the integral. From the generic assumption of \(f\) in Definition 2.8, we notice that all the points \(t_\mathbf{r}_\mathbf{v} \in \mathbf{r}_\mathbf{v}(\mathbf{T})\). In the above WKB construction, by shrinking \(\mathcal{E}_{T,e,v}\)’s if necessary, we may always assume that \(\pi_{e,v} : \mathcal{E}_{T,e,v} \times W_{e,v} \rightarrow W_{e,v}\) being identified with a neighborhood of zero section in the normal bundle \(N_{\mathcal{E}_{T,e,v}} \times W_{e,v}\). We notice that the element \(\nu_{T,e,v} \wedge \text{vol}_{e,v}\) (Here vol_{e,v} is introduced in Definition 2.9 as a top degree element in \(\wedge^* N_{e,v}\)) is a top degree element in \(\wedge^* N_{e,v}\), serves as an orientation in the normal direction (by extending to whole \(W_{e,v}\)).

We show inductively along gradient tree \(\Gamma\) that the integration along fiber

\[(\pi_{e,v})_*(\lambda^{\mathbf{r},e,v} e^{-\lambda g_{\mathbf{r},e,v} \omega_{\mathbf{r},e,v}}, 0) = 1 + \mathcal{O}(\lambda^{-1/2})\]

at the point \((\mathbf{r}_{e,v}, \pi_{e,v})\) (here \(\mathbf{r}_{e,v}\) is introduced in Notation 3.17) in \(V_{e,v}\) (Here \((\pi_{e,v})_*\) refers integration along fibers of \(\pi_{e,v}\) with respect to orientation \(\nu_{T,e,v} \wedge \text{vol}_{e,v}\) using techniques from \[3\] Section 3). Since \(g_{e,v}\) is non-negative Bott-Morse function with zero set \(V_{e,v}\), using the well-known stationary phase expansion (see e.g. [4] or [3, Lemma 58]) we notice the leading order in \(\lambda^{-1/2}\) is strictly positive. Below we only consider \(v\) in \(V_{e,v}\) modulo error \(\mathcal{O}(\lambda^{-1/2})\). (\(\mathbf{r}\) as in (2) Definition 3.18).

(1) for the incoming edges \(e_i\) with \(\partial_0(e_i) = v\), this is exactly Lemma 3.10

(2) for \((e, v)\) with \(\partial_1(e) = v\) with \(v\) is labeled with 1, with subtree \(T_2, T_1\) and outgoing edges \(e_2, e_1\) ending at \(v\), we have \(V_{e,v} = (V_{e_2,v} \times \mathcal{E}_{T_2}) \cap (V_{e_1,v} \times \mathcal{E}_{T_1})\) and we can compute

\[(\pi_{e,v})_*(\lambda^{\mathbf{r},e,v} e^{-\lambda g_{\mathbf{r},e,v} \omega_{\mathbf{r},e,v}}, 0) = (-1)\mathbf{\varepsilon}(\pi_{e_2,v})_*\lambda^{\mathbf{r},e_2,v} e^{-\lambda g_{\mathbf{r},e_2,v} \omega_{\mathbf{r},e_2,v}}(\pi_{e_1,v})_*\lambda^{\mathbf{r},e_1,v} e^{-\lambda g_{\mathbf{r},e_1,v} \omega_{\mathbf{r},e_1,v}}, 0) = 1\]

at the point \((\mathbf{r}_{e,v}, \pi_{e,v})\) in \(V_{e,v}\) modulo error \(\mathcal{O}(\lambda^{-1/2})\) (\(\varepsilon\) as in (2) Definition 3.18);

(3) when we have \(v\) labeled with \(u\), let \(T_1, \ldots, T_1\) be subtrees with outgoing edges \(e_1, \ldots, e_1\) ending at \(v\) with \(e_i, \ldots, e_1\) clockwise ordered, we notice that \(V_{e,v} = \bigcap_{j=1}^t V_{e_j,v}\) from WKB construction in previous Section 3.3. From the induction, we can compute the integral

\[(\pi_{e_j,v})_*(\lambda^{\mathbf{r},e_j,v} e^{-\lambda g_{\mathbf{r},e_j,v} \omega_{\mathbf{r},e_j,v}}, 0) = 1 + \mathcal{O}(\lambda^{-1/2})\]

as function on \(\tau_j^{-1}(\mathcal{E}_{T_1} \times W_{e,v})\) with \(\mathcal{O}(\lambda^{-1/2})\) if we identify a neighborhood \(\tau_j^{-1}(\mathcal{E}_{T_1} \times W_{e,v})\) with a neighborhood of zero section in the pull back normal bundle \(\tau_j^{-1}(N_{\mathcal{E}_{T_1}})\) as treat \(\pi_{e,v} : \tau_j^{-1}(N_{\mathcal{E}_{T_1}}) \rightarrow (V_{e,v})\) as
integration along fibers. We obtain the identity

\[(\pi_{e,v})_{\ast}(\lambda_{T_{e,v}} e^{\lambda g_{e,v}} \omega_{e,v}) = \prod_{j=1}^{l} (\pi_{e,j,v})_{\ast}(\lambda_{T_{e,j,v}} e^{-\lambda \tau_{j}}(g_{e,j,v}) \tau_{j}(\omega(e,j,v),0)) = 1,\]

at \((\tilde{t}_{G_{e,v}}, x_{G_{e,v}})\) modulo error \(O(\lambda^{-\frac{1}{2}})\);

(4) for an edge \(e\) numbered by \(ij\) with \(\partial_{ri}(e) = v_0\) and \(\partial_{i}(e) = v_1\) with \(v_1\) not being the outgoing vertex \(v_o\), we can compute \((\pi_{e,v_1})_{\ast}(\lambda_{T_{e,v_1}} e^{\lambda g_{e,v_1}} \omega_{e,v_1}) = 1 + O(\lambda^{-\frac{1}{2}})\) at the point \((\tilde{t}_{G_{e,v_1}}, x_{G_{e,v_1}})\) using the fact that \((\pi_{e,v_0})_{\ast}(\lambda_{T_{e,v_0}} e^{-\lambda g_{e,v_0}} \omega_{e,v_0}) = 1 + O(\lambda^{-\frac{1}{2}})\) at the point \((\tilde{t}_{G_{e,v_0}}, x_{G_{e,v_0}})\) by applying Lemma 3.12 with \(x_S = x_{G_{e,v_0}}\) an \(x_E = x_{G_{e,v_1}}\) (notice that \(\tilde{t}_{G_{e,v_0}} = t_{G_{e,v_1}}\));

(5) for the outgoing edge \(e_o\) with outgoing vertex \(v_o\), since we have \(V_{e_o,v_o}\) and \(E_T \times V_{0k}^\perp\) intersecting transversally at \((\tilde{t}_{G_{e,v}}, x_{G_{e,v}})\), we can compute

\[
\frac{\lambda_{T_{e_o,v_o}}^{\deg(q_{0k})/2}}{\|e^{\lambda f_{0k}} \phi_{0k}\|^2} \int_{\Delta_T \times M} e^{-\lambda(g_{e_o,v_o} + g_{0k})} \omega_{e_o,v_o} \wedge \omega_{0k,0} = \pm (\pi_{e_o,v_0})_{\ast}(\lambda_{T_{e_o,v_0}} e^{-\lambda g_{e_o,v_0}} \omega_{e_o,v_0})(\lambda^{\deg(q_{0k})/2} ||e^{\lambda f_{0k}} \phi_{0k}\|^2 \int_{NV_{T_{e_o,v_0}}} e^{-\lambda g_{0k} \omega_{0k,0}} + O(\lambda^{-\frac{1}{2}})) = \pm 1 + O(\lambda^{-\frac{1}{2}})
\]

where the \(\pm\) sign depending on whether the sign of gradient flow tree \(\Gamma\) obtained by comparing \(\text{vol}_{e_o,v_r} \wedge \text{vol}_{0k}\) with \(\text{vol}_M\) as described in Definition 2.9.

As a conclusion, we have proven that

\[
\int_M m_{k,T}(\lambda)(\phi_{(k-1)k}, \ldots, \phi_{01}) \wedge * e^{-2\lambda f_{0k}} \phi_{0k} = \sum_{i=1}^{d} (-1) \chi(\Gamma_i) + O(\lambda^{-\frac{1}{2}})
\]

and hence Theorem 2.11

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