LOCAL HÖLDER REGULARITY FOR SET-INDEXED PROCESSES

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Abstract. In this paper, we study the Hölder regularity of set-indexed stochastic processes defined in the framework of Ivanoff-Merzbach. The first key result is a Kolmogorov-like Hölder-continuity Theorem derived from the approximation of the indexing collection by a nested sequence of finite subcollections. Increments for set-indexed processes are usually not simply written as $X_U - X_V$, hence we considered different notions of Hölder-continuity. Then, the localization of these properties leads to various definitions of Hölder exponents.

In the case of Gaussian processes, almost sure values are proved for these exponents, uniformly along the sample paths. As an application, the local regularity of the set-indexed fractional Brownian motion and the set-indexed Ornstein-Uhlenbeck process are proved to be constant, with probability one.

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1. Introduction

Sample path properties of stochastic processes have been deeply studied for a long time, starting with the works of Kolmogorov, Lévy and others on the modulus of continuity and laws of the iterated logarithm of the Brownian motion. Since the late 1960s, these results were extended to general Gaussian processes, while a finer study of the local properties of these sample paths was carried out (we refer to Berman [14, 15], Dudley [19, 20], Orey and Pruitt [38], Orey and Taylor [39] and Strassen [41], for the early study of Gaussian paths and their rare events). Among the large literature dealing with fine analysis of regularity, Hölder exponents continue to be widely used as a local measure of oscillations (see [10, 12, 35, 36, 43] for examples of recent works in this area). Two different definitions, called local and pointwise Hölder exponents, are usually considered for a stochastic process $\{X_t; t \in \mathbb{R}_+\}$, depending whether the increment $X_t - X_s$ is compared with a power $|t - s|^{\alpha}$ or $\rho^\alpha$ inside a ball $B(t_0, \rho)$ when $\rho \to 0$. As an example, with probability one, the local regularity of fractional Brownian motion $\{B^H; t \in \mathbb{R}_+\}$ is constant along the path: the pointwise and local Hölder exponents at any $t \in \mathbb{R}_+$ are equal to the self-similarity index $H \in (0, 1)$ (e.g. see [26]).

This field of research is also very active in the multiparameter context and a non-exhaustive list of authors and recent works in this area includes Ayache [8], Dalang [16], Khoshnevisan [16, 34], Lévy-Véhel [26], Xiao [37, 44, 45]. As an extension to the multiparameter one, the set-indexed context appeared to be the natural framework to describe invariance principles studying convergence of empirical processes (e.g. see
The understanding of set-indexed processes and particularly their regularity is a more complex issue than on points of \( \mathbb{R}^N \). The simple continuity property is closely related to the nature of the indexing collection. As an example, Brownian motion indexed by the lower layers of \([0,1]^2\) (i.e. the subsets \( A \subseteq [0,1]^2 \) such that \([0,t] \subseteq A \) for all \( t \in A \)) is discontinuous with probability one (we refer to [2] or [32] for the detailed proof). As a matter of fact, necessary and sufficient conditions for the sample path continuity property were investigated, starting with Dudley [20] who introduced a sufficient condition on the metric entropy of the indexing set, followed by Fernique [22] who gave a necessary conditions in the specific case of stationary processes on \( \mathbb{R}^N \). Talagrand gave a definitive answer in terms of majorizing measures [42] (see [5] or [33] for a complete survey and also [6, 7] for a LIL and Lévy’s continuity moduli for set-indexed Brownian motion). The question was left open so far concerning the exact Hölder regularity of set-indexed processes.

A formal set-indexed setting has been introduced by Ivanoff and Merzbach in order to study standard issues of stochastic processes, such as martingale and Markov properties (see [30, 32]). In this framework, an indexing collection \( A \) is a collection of subsets of a measure space \((T, m)\), which is assumed to satisfy certain properties such as stability by intersection of its elements. Section 2 of the present paper uses these properties, instead of conditions on the metric entropy, to derive a Kolmogorov-like criterion for Hölder-continuity of a set-indexed process. The collection of sets \( A \) is endowed with a metric \( d_A \) and a nested sequence \( \mathcal{A} = (A_n)_{n \in \mathbb{N}} \) of finite subcollections of \( A \) such that each element of \( A \) can be approximated as the decreasing limit (for the inclusion) of its projections on the \( A_n \)'s. We consider a supplementary Assumption \((\mathcal{H}_A)\) on \( A \) and \( d_A \) which impose that: 1) the distance from any \( U \in A \) to \( A_n \) can be related to the cardinal \( k_n = \#A_n \), roughly by \( d_A(U, A_n) = O(k_n^{-1/q_A}) \), where \( q_A \) is called the discretization exponent of \((A_n)_{n \in \mathbb{N}}\); and 2) a minimality condition on the class \((A_n)_{n \in \mathbb{N}}\) that is verified in most cases. This is discussed in Section 2, together with the links between our assumption and entropic conditions of previous works. We prove in Theorem 2.9: If \( \{X_U; U \in A\} \) is a set-indexed process and \( \alpha, \beta, K \) are positive constants such that \( \mathbb{E}[|X_U - X_V|^\gamma] \leq K d_A(U,V)^{q_A + \beta} \) for all \( U, V \in A \), then for all \( \gamma \in (0, \beta/\alpha) \), there exist a random variable \( h^* \) and a constant \( L > 0 \) such that almost surely

\[
\forall U, V \in A: \ d_A(U, V) < h^* \Rightarrow |X_U - X_V| \leq L d_A(U, V)^\gamma.
\]

Alternatively, Hölder-continuity can be based on the usual definition for increments of set-indexed processes. Instead of quantities \( X_U - X_V \), the increments of a set-indexed process \( \{X_U; U \in A\} \) are defined on the class \( C \) of sets \( C = U_0 \setminus \bigcup_{1 \leq k \leq n} U_k \) where \( U_0, U_1, \ldots, U_n \in A \) by the inclusion-exclusion formula

\[
\Delta X_C = X_{U_0} - \sum_{k=1}^n \sum_{j_1 < \cdots < j_k} (-1)^{k-1} X_{U_0 \cap U_{j_1} \cap \cdots \cap U_{j_k}}.
\]

This definition extends the notion of rectangular increments for multiparameter processes. For instance, quantities like \( \Delta_{[u,v]} B = B_v - B_{(u_1,v_2)} - B_{(v_1,u_2)} + B_u \), where \( u < v \in \mathbb{R}^2 \) and \( B \) is the Brownian sheet, were proved to be useful to derive geometric sample path properties of the process (see e.g. some of the works of Dalang and Walsh [17]). Let us notice that some processes can satisfy an increment stationarity property with respect to these increments while they do not for quantities \( X_U - X_V \). Moreover,
this inclusion-exclusion principle is very useful when it comes to martingale and Markov properties. According to this definition, another way to express the Hölder-continuity of \( X \) is \( |\Delta X_C| \leq L m(C)^\gamma \), for \( C \in \mathcal{C} \). This question is clarified in Section 2.2.

The purpose of Hölder exponents is the (optimal) localization of the Hölder-continuity concept. Following the previous discussion, the first definition for local and pointwise Hölder exponents is based on the comparison between \( |X_U - X_V| \) and a power \( d_A(U, V)^\alpha \) or \( \rho^\alpha \) in a ball \( B_{d_A(U_0, \rho)} \) around \( U_0 \in \mathcal{A} \) when \( \rho \to 0 \). Another definition compares \( |\Delta X_C| \) for \( C = U \setminus \bigcup_{1 \leq k \leq n} V_k \) in \( \mathcal{C} \) with \( d_A(U, U_0) < \rho \) and \( d_A(U_0, V_k) < \rho \) for each \( k \), to a power \( m(C)^\alpha \) when \( \rho \to 0 \). As in the real-parameter setting, these two kinds of exponents, precisely defined in Sections 3 and 3.1, provide a fine knowledge of the local behaviour of the sample paths. In Section 4, the different Hölder exponents are linked to the Hölder regularity of projections of the set-indexed process on increasing paths.

The pointwise continuity has been introduced in the multiparameter setting in [4] and in the set-indexed setting in [29] as a weak form of continuity. In this definition, the point mass jumps are the only kind of discontinuity considered. Without any supplementary condition on the indexing collection, the set-indexed Brownian motion satisfies this property, even on lower layers where it is not continuous. In Section 3.2, we define the pointwise continuity Hölder exponent of a pointwise continuous process \( X \) by a comparison between \( \Delta X_{C_n(t)} \) with a power \( m(C_n(t))^{\alpha} \) when \( n \to \infty \), where \( (C_n(t))_{n \in \mathbb{N}} \) is a decreasing sequence of elements in \( \mathcal{C} \) which converges to \( t \in \mathcal{T} \).

In the Gaussian case, we prove in Section 5 that the different aforementioned Hölder exponents admit almost sure values. Assumption \( (\mathcal{H}_A) \) is the key to extend this result from the multiparameter to the set-indexed setting. Moreover these almost sure values can be obtained uniformly on \( \mathcal{A} \) for the local exponent. However, this a.s. result cannot be obtained for the pointwise exponent (even for multiparameter processes). Nevertheless, we proved that it holds for the set-indexed fractional Brownian motion (defined in [27]) in Section 6, thus improving on a result in the multiparameter case [26]. As this requires some specific extra work, we believe that the uniform almost sure result might not be true for the pointwise exponent of any Gaussian process. Finally, we also applied our results to the set-indexed Ornstein-Uhlenbeck (SIOU) process [11], for which all exponents are almost surely equal to 1/2 at any set \( U \in \mathcal{A} \).

2. Hölder continuity of a set-indexed process

In the classical case of real-parameter (or multiparameter) stochastic processes, Kolmogorov’s continuity criterion is a useful tool to study sample path Hölder-continuity (e.g. see [33, 26]). In this section, we focus on the definition of a suitable assumption on the indexing collection, that allows to prove an extension of this result to the set-indexed (possibly non-Gaussian) setting.

2.1. Indexing collection for set-indexed processes. A general framework was introduced by Ivanoff and Merzbach to study martingale and Markov properties of set-indexed processes (we refer to [30, 32] for the details of the theory). The structure of these indexing collections allowed the study of the set-indexed extension of fractional Brownian motion [27], its increment stationarity property [28] and a complete characterization of the class of set-indexed Lévy processes [29].
Let $\mathcal{T}$ be a locally compact complete separable metric and measure space, with metric $d$ and Radon measure $m$ defined on the Borel sets of $\mathcal{T}$. All stochastic processes will be indexed by a class $\mathcal{A}$ of compact connected subsets of $\mathcal{T}$.

In the whole paper, the class of finite unions of sets in any collection $\mathcal{D}$ will be denoted by $\mathcal{D}(u)$. In the terminology of [32], we assume that the indexing collection $\mathcal{A}$ satisfies stability and separability conditions in the sense of Ivanoff and Merzbach:

**Definition 2.1** (adapted from [32]). A nonempty class $\mathcal{A}$ of compact, connected subsets of $\mathcal{T}$ is called an indexing collection if it satisfies the following:

1. $\emptyset \in \mathcal{A}$ and for all $A \in \mathcal{A}$, $A^c \neq A$ if $A \notin \{\emptyset, \mathcal{T}\}$.
2. $\mathcal{A}$ is closed under arbitrary intersections and if $A, B \in \mathcal{A}$ are nonempty, then $A \cap B$ is nonempty. If $(A_i)$ is an increasing sequence in $\mathcal{A}$ then $\bigcup_i A_i \in \mathcal{A}$.
3. The $\sigma$-algebra $\sigma(\mathcal{A})$ generated by $\mathcal{A}$ is the collection $\mathcal{B}$ of all Borel sets of $\mathcal{T}$.
4. Separability from above: There exists an increasing sequence of finite subclasses $\mathcal{A}_n = \{\emptyset, A_{1n}, \ldots, A_{kn}\}$ ($n \in \mathbb{N}, k_n \geq 1$) of $\mathcal{A}$ closed under intersections and a sequence of functions $g_n : \mathcal{A} \to \mathcal{A}_n$ defined by

$$\forall U \in \mathcal{A}, \quad g_n(U) = \bigcap_{V \in \mathcal{A}_n, V \supseteq U} V,$$

and such that for each $U \in \mathcal{A}$, $U = \bigcap_{n \in \mathbb{N}} g_n(U)$.

(Note: ‘($\cdot)^c$’ and ‘($\cdot)^{\circ}$’ denote respectively the closure and the interior of a set.)

Standard examples of indexing collections can be mentioned, such as rectangles $[0, t]$ of $\mathbb{R}^N$, arcs of the circle $\mathbb{S}^2$ or lower layers. Some of them are detailed in Examples 2.5 and 2.6 below.

**Distances on sets.** In order to study the Hölder-continuity of set-indexed processes, we consider a distance on the indexing collection. Along this paper, we may sometimes specify the distance on $\mathcal{A}$ that we are using. Among them, the following distances are of special interest:

- The classical Hausdorff metric $d_H$ defined on $\mathcal{K} \setminus \emptyset$, the nonempty compact subsets of $\mathcal{T}$, by

$$\forall U, V \in \mathcal{K} \setminus \emptyset; \quad d_H(U, V) = \inf \{\epsilon > 0 : U \subseteq V^{\epsilon} \text{ and } V \subseteq U^{\epsilon}\},$$

where $U^{\epsilon} = \{x \in \mathcal{T} : d(x, U) \leq \epsilon\}$;

- and the pseudo-distance $d_m$ defined by

$$\forall U, V \in \mathcal{A}; \quad d_m(U, V) = m(U \triangle V),$$

where $m$ is the measure on $\mathcal{T}$ and $\triangle$ denotes the symmetric difference of sets.
Remark 2.2. In the case of $\mathcal{A} = \{(0, t); t \in \mathbb{R}^N_+\}$, $(s, t) \mapsto d_m([0, s], [0, t])$ induces a distance on $\mathbb{R}^N_+$. This distance can be compared to the classical distances of $\mathbb{R}^N$,

\[
\begin{align*}
d_1 : (s, t) \mapsto ||t - s||_1 &= \sum_{i=1}^{N} |t_i - s_i|, \\
d_2 : (s, t) \mapsto ||t - s||_2 &= \sum_{i=1}^{N} (t_i - s_i)^2, \\
d_\infty : (s, t) \mapsto ||t - s||_\infty &= \max_{1 \leq i \leq N} |t_i - s_i|.
\end{align*}
\]

If $m$ is the Lebesgue measure of $\mathbb{R}^N$, the distance $d_m$ is equivalent to $d_1$, $d_2$ and $d_\infty$ on any compact of $\mathbb{R}^N_+ \setminus \{0\}$.

More precisely, for all $a < b$ in $\mathbb{R}^N_+ \setminus \{0\}$, there exist two positive constants $m_{a,b}$ and $M_{a,b}$ such that

$$\forall s, t \in [a, b]; \quad m_{a,b} d_1(s, t) \leq m([0, s] \triangle [0, t]) \leq M_{a,b} d_\infty(s, t).$$

We refer to [25] for a proof of these assertions.

**Total boundedness of indexing collections.** As discussed in the introduction, the study of continuity of stochastic processes is closely related to the control of the metric entropy of the indexing collection. Following the conditions of Definition 2.1, some additional assumptions on the collection $\mathcal{A}$ are required to guarantee that $(\mathcal{A}, d_\mathcal{A})$ is totally bounded (or at least locally totally bounded). We recall that a metric space $(\mathcal{T}, d)$ is totally bounded if for any $\epsilon > 0$, $\mathcal{T}$ can be covered by a finite number of balls of radius smaller than $\epsilon$. The minimal number of such balls is called the metric entropy and is denoted by $N(\mathcal{T}, d, \epsilon)$.

Before getting to the main assumption on the metric $d_\mathcal{A}$ and the finite subclasses $(\mathcal{A}_n)_{n \in \mathbb{N}}$ that approximate $\mathcal{A}$, we notice that the sequence $(k_n)_{n \in \mathbb{N}} = (#\mathcal{A}_n)_{n \in \mathbb{N}}$ is an increasing sequence that tends to $\infty$, as $n \to \infty$. This property comes from condition (4) in Definition 2.1. We will say that $(\mathcal{A}_n, k_n)_{n \in \mathbb{N}}$ is admissible if:

$$\forall \delta > 0, \quad \sum_{n=1}^{\infty} \frac{k_{n+1}}{k_n^{1+\delta}} < \infty. \quad (2.1)$$

This should not appear as a restriction anyhow, because: if $(k_n)_{n \in \mathbb{N}}$ was going to $\infty$ too slowly, it would suffice to extract a subsequence ; and in the opposite situation, the gap between one scale to the other is too large and can then be filled with additional subclasses.

**Assumption $(\mathcal{H}_\mathcal{A})$.** Let $d_\mathcal{A}$ be a (pseudo-)distance on $\mathcal{A}$. Let us suppose that for $\mathcal{A} = (\mathcal{A}_n)_{n \in \mathbb{N}}$, there exist positive real numbers $q_\mathcal{A}$ and $M_1$ such that:

1. For all $n \in \mathbb{N}$,

$$\sup_{U \in \mathcal{A}_n} d_\mathcal{A}(U, g_n(U)) \leq M_1 k_n^{-1/q_\mathcal{A}}, \quad (H1)$$

2. and the collection $(\mathcal{A}_n)_{n \in \mathbb{N}}$ is minimal in the sense that: setting for all $n \in \mathbb{N}$ and all $U \in \mathcal{A}_n$,

$$\mathcal{V}_n(U) = \{V \in \mathcal{A}_n : V \supseteq U, d_\mathcal{A}(U, V) \leq 3M_1 k_n^{-1/q_\mathcal{A}}\},$$

where $\mathcal{V}_n(U)$ is the family of all the finite subclasses of $\mathcal{A}_n$ that approximate $\mathcal{A}$.

$$\forall U \in \mathcal{A}_n, \quad \text{there exist } V_1, V_2 \in \mathcal{V}_n(U) \text{ such that } \mathcal{V}_n(U) = \mathcal{V}_n(V_1) \cup \mathcal{V}_n(V_2).$$

Let us notice that $(\mathcal{V}_n(U))_{U \in \mathcal{A}_n}$ is a nested family of finite subclasses of $\mathcal{A}_n$ such that:

$$\forall U \in \mathcal{A}_n, \quad \mathcal{V}_n(U) \text{ is admissible}.$$
the sequence \((N_n)_{n \geq 1}\) defined by \(N_n = \max_{U \in \mathcal{A}_n} \# \mathcal{V}_n(U)\) for all \(n \geq 1\) satisfies
\[
\forall \delta > 0, \quad \sum_{n=1}^{\infty} k_n^{-\delta} N_n < \infty. \tag{H2}
\]

The real \(q_A\) is not unique and it depends \textit{a priori} on the distance \(d_A\) and the sub-\semilattices \(\mathcal{A} = (\mathcal{A}_n)_{n \in \mathbb{N}}\). Such a real \(q_A\) is called \textit{discretization exponent} of \((\mathcal{A}_n)_{n \in \mathbb{N}}\). Note that if \(N_n\) can be bounded independently of \(n\), then the last assumption is satisfied by admissibility of \(k_n\).

\textbf{Remark 2.3.} Without loss of generality, the distance \(d_A\) can be normalized such that \(M_1 = 1\).

\textbf{Remark 2.4.} The summability condition \((H2)\) of Assumption \((\mathcal{H}_A)\) is close to the notion of entropy with inclusion developed by Dudley \cite{Dudley} in the context of empirical processes. On the contrary to the present work, \cite{Dudley} focused exclusively on the sample path boundedness and continuity in the Brownian case.

The following example shows that Assumption \((\mathcal{H}_A)\) is satisfied in simple situations.

\textbf{Example 2.5.} \begin{itemize}
\item In the case of \(\mathcal{A} = \{[0, t] ; t \in [0, 1] \subset \mathbb{R}_+\}\), the subclasses \(\mathcal{A}_n (n \in \mathbb{N})\) are commonly \([0, k.2^{-n}] ; k = 0, \ldots, 2^n\}. The two \textit{(pseudo-)distances} \(d_H\) and \(d_\lambda\) (where \(\lambda\) is the Lebesgue measure on \(\mathbb{R}\)) on \(\mathcal{A}\) are equal to \[d_A : ([0, s], [0, t]) \mapsto |t - s|,\]
and we have \[\forall k = 0, \ldots, 2^n - 1; \quad d_A([0, k.2^{-n}], [0, (k + 1).2^{-n}]) = 2^{-n}.\]
Then the two conditions of Assumption \((\mathcal{H}_A)\) both are satisfied for \(q_A = 1\).
\item In the case of \(\mathcal{A} = \{[0, t] ; t \in [0, 1] \subset \mathbb{R}_N^+\}\), the subclasses \(\mathcal{A}_n (n \in \mathbb{N})\) can be chosen as \[\{[0, 2^{-n}.(l_1, \ldots, l_N)]; 0 \leq l_1, \ldots, l_N \leq 2^n\}.\]
Let \(U\) be a set in \(\mathcal{A}\). The distance (induced by the Lebesgue measure \(\lambda\)) between \(U\) and \(g_n(U)\) is the volume difference between the two sets. It can be easily bounded from above by the sum of the volumes of the outer faces, minus a negligible residue \[\sup_{U \in \mathcal{A}} d_\lambda(U, g_n(U)) = \sup_{U \in \mathcal{A}} \lambda(g_n(U) \setminus U) = N.2^{-n} + o(2^{-n}).\]
Since \(k_n = (2^n + 1)^N\), this leads to \[d_\lambda(U, g_n(U)) = O(k_n^{-1/q_A}),\]
with \(q_A = N\) and the other condition of Assumption \((\mathcal{H}_A)\) are satisfied.
\end{itemize}

On the contrary to the rectangles case, the following example shows that the collection of \textit{lower layers} of \(\mathbb{R}^N\) does not satisfy Assumption \((\mathcal{H}_A)\). We will see later that this result is not surprising in the view of Theorem 2.9, since Brownian motion indexed by lower layers of \([0, 1]^2\) does not have a continuous modification, as can be seen for instance in \cite{2, 32}.
Example 2.6. Let $\mathcal{A}$ be the collection of lower layers of $[0,1]^2$, i.e. the subsets $A$ of $[0,1]^2$ such that $\forall t \in A$, $[0,t] \subseteq A$. For all $n \in \mathbb{N}$, let $\mathcal{A}_n$ be the collection of finite unions of sets in the dissecting collection of the diadic rectangles of $[0,1]^2$, i.e. $\mathcal{A}_n = \left\{ \bigcup_{\text{finite}} [0,x] : 2^n x \in \mathbb{Z}^2 \cap (0,2^n]^2 \right\} \cup \{0\} \cup \{\emptyset\}$.

Then, it can be shown that the cardinal $k_n$ of $\mathcal{A}_n$ satisfies $k_n \geq 2^{2n}$ for all $n \in \mathbb{N}$. For all $U \in \mathcal{A}_n$, we can see that $\inf_{V \in \mathcal{A}_n, V \supseteq U} d_{\mathcal{A}}(U,V) = 2^{-2n}$, hence there does not exist any $q_{\mathcal{A}}$ such that $2^{-2n}$ and $k_n^{-1/q_{\mathcal{A}}}$ are of the same order. Consequently the subclasses $(\mathcal{A}_n)_{n \in \mathbb{N}}$ cannot verify Assumption $(\mathcal{H}_{\mathcal{A}})$.

To conclude this section, we emphasize the fact that Assumption $(\mathcal{H}_{\mathcal{A}})$ implies the total boundedness of $(\mathcal{A}, d_{\mathcal{A}})$. Since $\forall n \in \mathbb{N}$, $d_{\mathcal{A}}(U,g_n(U)) \leq k_n^{-1/q_{\mathcal{A}}}$, $\mathcal{A}_n$ constitutes a $k_n^{-1/q_{\mathcal{A}}}$-net for all $n \in \mathbb{N}$, and thus $(\mathcal{A}, d_{\mathcal{A}})$ is totally bounded.

2.2. Kolmogorov’s criterion. As $(\mathcal{A}, d_{\mathcal{A}})$ is not generally totally bounded, for any deterministic function $f : \mathcal{A} \to \mathbb{R}$, we consider the modulus of continuity on any totally bounded $B \subseteq \mathcal{A}$

$$\omega_{f,B}(\delta) = \sup_{U,V \in B, d_{\mathcal{A}}(U,V) \leq \delta} |f(U) - f(V)|.$$ 

Recall that the function $f$ is said H"older continuous of order $\alpha > 0$ if for all totally bounded $B \subseteq \mathcal{A}$ one of the following equivalent conditions holds (e.g. see [33], Chapter 5)

(i) \[ \limsup_{\delta \to 0} \delta^{-\alpha} \omega_{f,B}(\delta) < \infty. \]

(ii) There exists $M > 0$ and $\delta_0 > 0$ such that for all $U,V \in B$ with $d_{\mathcal{A}}(U,V) < \delta_0$, $|f(U) - f(V)| \leq M d_{\mathcal{A}}(U,V)^\alpha$.

For any general set-indexed Gaussian process, Dudley’s Corollary 2.3 in [20] allows to compute a modulus of continuity (giving the same kind of result than following Corollary 2.10). This result holds under certain entropic conditions on the indexing collection, which are different from these of our setting. Assumption $(\mathcal{H}_{\mathcal{A}})$ and more precisely its second condition allows to prove a continuity criterion in the non-Gaussian case. Although Adler and Taylor [5] emphasize that the Gaussian property is only used through the exponential decay of the tail probability of the process in the proof of the previous results, they do not suggest any Kolmogorov criterion for non-Gaussian processes. The following Theorem 2.9 do so in the general set-indexed framework of Ivanoff and Merzbach, thanks to the discretization exponent.

Definition 2.7. A (pseudo-)distance $d_{\mathcal{A}}$ on $\mathcal{A}$ is said:

(i) Outer-continuous if for any non-increasing sequence $(U_n)_{n \in \mathbb{N}}$ in $\mathcal{A}$ converging to $U = \bigcap_{n \in \mathbb{N}} U_n \in \mathcal{A}$, $d_{\mathcal{A}}(U_n, U)$ tends to 0 as $n$ goes to $\infty$;
Proof. Let us fix \( n \) and for surely \( \gamma > 0 \) and \( d \), the most important metrics in the context of set-indexed processes, \( d_m \) and \( d_H \), are contractive.

Assumption \((\mathcal{H}_A)\) on the subcollections \((\mathcal{A}_n)_{n \in \mathbb{N}}\) and the contractivity of the metric \( d_A \) allow to state the first important result of the paper:

**Theorem 2.9.** Let \( d_A \) be a contractive (pseudo-)distance on the indexing collection \( \mathcal{A} \), whose subclasses \( \mathcal{A} = (\mathcal{A}_n)_{n \in \mathbb{N}} \) satisfy Assumption \((\mathcal{H}_A)\) with a discretization exponent \( q_A > 0 \). Let \( X = \{X_U; \; U \in \mathcal{A}\} \) be a set-indexed process such that

\[
\forall U, V \in \mathcal{A}, \quad E \left[ |X_U - X_V|^\alpha \right] \leq K \, d_A(U, V)^{q_A + \beta}\tag{2.2}
\]

where \( K, \alpha \) and \( \beta \) are positive constants.

Then, the sample paths of \( X \) are almost surely locally \( \gamma \)-Hölder continuous for all \( \gamma \in (0, \frac{\beta}{\alpha}) \), i.e. there exist a random variable \( h^* \) and a constant \( L > 0 \) such that almost surely

\[
\forall U, V \in \mathcal{A}, \quad d_A(U, V) < h^* \Rightarrow |X_U - X_V| \leq L \, d_A(U, V)^\gamma.
\]

**Proof.** Let us fix \( \gamma \in (0, \frac{\beta}{\alpha}) \) and denote \( \mathcal{D} = \bigcup_{n \in \mathbb{N}} \mathcal{A}_n \) a countable dense subset of \( \mathcal{A} \). First, let \((a_j)_{j \in \mathbb{N}}\) be any sequence of positive real numbers such that \( \sum_{j \in \mathbb{N}} a_j < +\infty \), and for \( n \in \mathbb{N} \) such that \( \sum_{j \geq n} a_j \leq 1 \), we have

\[
P \left( \sup_{U \in \mathcal{D}} |X_U - X_{g_n(U)}| \geq k^{-\gamma/q_A}_{n+1} \right)
\]

\[
\leq P \left( \exists U \in \mathcal{D}, \sum_{j=n}^{\infty} |X_{g_{j+1}(U)} - X_{g_j(U)}| \geq k^{-\gamma/q_A}_{n+1} \right)
\]

\[
\leq P \left( \exists U \in \mathcal{D}, \exists j \geq n, |X_{g_{j+1}(U)} - X_{g_j(U)}| \geq a_j k^{-\gamma/q_A}_{n+1} \right) \tag{2.3}
\]

\[
\leq P \left( \exists j \geq n, \exists V \in \mathcal{A}_{j+1}, |X_V - X_{g_j(V)}| \geq a_j k^{-\gamma/q_A}_{n+1} \right)
\]

\[
\leq \sum_{j=n}^{\infty} \sum_{V \in \mathcal{A}_{j+1}} P \left( |X_V - X_{g_j(V)}| \geq a_j k^{-\gamma/q_A}_{n+1} \right).
\]

Now applying successively Tchebyshev’s inequality, (2.2) and Equation (H1) of Assumption \((\mathcal{H}_A)\),

\[
P \left( \sup_{U \in \mathcal{D}} |X_U - X_{g_n(U)}| \geq k^{-\gamma/q_A}_{n+1} \right) \leq \sum_{j=n}^{\infty} k_{j+1}^{-\alpha} a_j^{-\alpha} k_{n+1}^{-\gamma/q_A} \sup_{V \in \mathcal{A}_{j+1}} E \left[ |X_V - X_{g_j(V)}|^\alpha \right]
\]

\[
\leq K \, k_{n+1}^{-\gamma/q_A} \sum_{j=n}^{\infty} a_j^{-\alpha} k_{j+1} \sup_{V \in \mathcal{A}_{j+1}} d_A(V, g_j(V))^{q_A + \beta}
\]

\[
\leq K \, k_{n+1}^{-\gamma/q_A} \sum_{j=n}^{\infty} a_j^{-\alpha} k_{j+1}^{-\beta/q_A} k_{j}^{-\beta/q_A}.
\]
The admissibility of \((k_n)_{n \in \mathbb{N}}\) implies that for \(\delta > 0\), and for \(n\) large enough (depending on \(\delta\)), \(k_n^{\alpha \gamma / q \Delta} \leq (k_n^{\alpha \gamma / q \Delta})^{1+\delta}\), so that:

\[
P\left( \sup_{U \in \mathcal{D}} |X_U - X_{g_n(U)}| \geq k_n^{-\gamma / q \Delta} \right) \leq K \ k_n^{\delta \alpha \gamma / q \Delta} \sum_{j=n}^{\infty} a_j^{-\alpha} k_{j+1}^{k_j - \beta / q \Delta} k_n^{\alpha / q \Delta} \]
\[
\leq K \ k_n^{\delta \alpha \gamma / q \Delta} \sum_{j=n}^{\infty} a_j^{-\alpha} k_{j+1}^{k_j - (\beta - \gamma \alpha) / q \Delta}.
\]

Since \(\beta - \alpha \gamma > 0\), \((a^*_j)_{j \in \mathbb{N}}\) can be chosen equal to \((k_j^{-(\beta - \alpha \gamma) / 3q \Delta})_{j \in \mathbb{N}}\) (which is indeed summable because \(k_n\) is admissible), and then:

\[
P\left( \sup_{U \in \mathcal{D}} |X_U - X_{g_n(U)}| \geq k_n^{-\gamma / q \Delta} \right) \leq K \ k_n^{\delta \alpha \gamma / q \Delta} \sum_{j=n}^{\infty} k_{j+1}^{k_j - 2(\beta - \gamma \alpha) / 3q \Delta},
\]

which finally leads to, for \(\delta = (\beta - \alpha \gamma)/(6\alpha \gamma)\),

\[
P\left( \sup_{U \in \mathcal{D}} |X_U - X_{g_n(U)}| \geq k_n^{-\gamma / q \Delta} \right) \leq K \ k_n^{-\delta \alpha \gamma / q \Delta} \sum_{j=n}^{\infty} k_{j+1}^{k_j - (\beta - \gamma \alpha) / 3q \Delta}.
\]

Thus, this probability is summable and Borel-Cantelli’s theorem implies the existence of \(\Omega^* \subseteq \Omega\) with \(P(\Omega^*) = 1\) such that \(\forall \omega \in \Omega^*\),

\[
\exists n^*(\omega) \in \mathbb{N}, \forall n \geq n^*, \forall U \in \mathcal{D}, \ |X_U - X_{g_n(U)}| < k_n^{-\gamma / q \Delta}.
\]

(2.4)

Now, we develop the same argument for the following probability:

\[
P\left( \sup_{U \in \mathcal{A}_n} \sup_{V \in \mathcal{V}_n(U)} |X_U - X_V| \geq k_n^{1/q \Delta} \right) \leq k_n \ N_n \ \sup_{U \in \mathcal{A}_n} \sup_{V \in \mathcal{V}_n(U)} P\left( |X_U - X_V| \geq k_n^{1/q \Delta} \right)
\]
\[
\leq K \ k_n \ k_n^{\alpha / q \Delta} k_n^{-\beta / q \Delta}
\]
\[
\leq K \ k_n \ k_n^{-(\beta - \gamma \alpha) / 2q \Delta},
\]

(2.5)

where we used \(\delta\) as in the previous paragraph. This is summable by (H2), hence there exists \(\Omega''\) a measurable subset of \(\Omega\) of probability 1 and \(n''\) a integer-valued finite random variable such that on \(\Omega''\):

\[
\forall n \geq n'', \ \sup_{U \in \mathcal{A}_n} \sup_{V \in \mathcal{V}_n(U)} |X_U - X_V| < k_n^{-\gamma / q \Delta}.
\]

(2.6)

For any \(U, V \in \mathcal{D}\), there is a unique \(n \in \mathbb{N}\) such that \(k_n^{-1/q \Delta} \leq d_A(U, V) < k_n^{-1/q \Delta}\). Let \(I_n = [k_n^{-1/q \Delta}, k_n^{-1/q \Delta}]\). Without any restriction, we assume that \(U \subseteq V\). Indeed, if this not the case, we shall consider \(X_U - X_V = X_U - X_{U \cap V} + X_{U \cap V} - X_V\), where \(d_A(U, U \cap V) \leq d_A(U, V)\) by contractivity. Since this implies that \(g_n(V) \in \mathcal{V}_n(g_n(U))\), we will write, on \(\Omega^* \cap \Omega''\), for any \(n \geq n'' \lor n''\):

\[
\sup_{U, V \in \mathcal{D}} |X_U - X_V| \leq \sup_{d_A(U, V) \in I_n} \left( |X_U - X_{g_n(U)}| + |X_{g_n(U)} - X_{g_n(V)}| + |X_{g_n(V)} - X_V| \right)
\]
\[
\leq 3 \ k_n^{-1/q \Delta}
\]
\[
\leq 3 \ d_A(U, V)^{\gamma},
\]

(2.7)
as a consequence of Equations (2.4) and (2.6). Since $\Omega^* \cap \Omega^{**}$ is of probability 1, we have proved that there exist a constant $L > 0$ and a random variable $h^*$ such that
\[ \forall U, V \in D; \quad d_A(U, V) < h^* \Rightarrow |X_U - X_V| \leq L d_A(U, V)^\gamma \text{ a.s.} \tag{2.8} \]

In the last part of the proof, we need to extend (2.8) to the whole class $A$. From the outer-continuity of $d_A$, we can claim:

On $\Omega^*$, for all $\varepsilon \in (0, h^*)$, for all $U$ and $V$ in $A$ with $d_A(U, V) < h^* - \varepsilon$, there exists $n_0 > n^*$ such that $d_A(g_n(U), g_m(V)) < h^*$ for all $n \geq n_0$ and $m \geq n_0$. Thus by (2.8),
\[ \forall n > n_0, \forall m > n_0; \quad |X_{g_n(U)} - X_{g_m(V)}| \leq L d_A(g_n(U), g_m(V))^\gamma. \tag{2.9} \]

We define the process $\tilde{X}$ by
- $\forall \omega \notin \Omega^*$, $\forall U \in A$, $\tilde{X}_U(\omega) = 0$,
- $\forall \omega \in \Omega^*$,
  - $\forall U \in D$, $\tilde{X}_U(\omega) = X_U(\omega)$
  - $\forall U \in A \setminus D$, $\tilde{X}_U(\omega) = \lim_{n \to \infty} X_{g_n(U)}(\omega)$.

Applying (2.9) with $V = U$, the outer-continuity property of $d_A$ implies that $(X_{g_n(U)}(\omega))_{n \in \mathbb{N}}$ is a Cauchy sequence and then converges in $\mathbb{R}$.

The process $\tilde{X}$ satisfies almost surely
\[ \forall U, V \in A; \quad d_A(U, V) < h^* \Rightarrow |\tilde{X}_U - \tilde{X}_V| \leq L. d_A(U, V)^\gamma. \]

Moreover,
- $\forall U \in D$, $\tilde{X}_U = X_U$ almost surely.
- $\forall U \in A \setminus D$, by construction, $X_{g_n(U)} \xrightarrow{a.s.} \tilde{X}_U$ as $n \to \infty$.

Since $E \left[ |X_{g_n(U)} - X_U|^\alpha \right]$ converges to 0 when $n \to \infty$, the sequence $(X_{g_n(U)})_{n \in \mathbb{N}}$ converges in probability to $X_U$. Then, there exists a subsequence converging almost surely.

From these two facts, we get $\tilde{X}_U = X_U$ a.s.

As in the multiparameter’s case, a simpler statement holds for Gaussian processes (see [33] for a detailed study of the Kolmogorov criterion in the multiparameter frame).

**Corollary 2.10.** Let $d_A$ be a (pseudo-)distance on the indexing collection $A$, whose subclasses $A = (A_n)_{n \in \mathbb{N}}$ satisfy Assumption (H$_A$). Let $X = \{X_U; U \in A\}$ be a centered Gaussian set-indexed process such that
\[ \forall U, V \in A, \quad E \left[ |X_U - X_V|^2 \right] \leq K d_A(U, V)^{2\beta} \]

where $K > 0$ and $\beta > 0$.

Then, the sample paths of $X$ are almost surely locally $\gamma$-Hölder continuous for all $\gamma \in (0, \beta)$.

**Proof.** For any $p \in \mathbb{N}^*$, there exists a constant $M_p > 0$ such that for all centered Gaussian random variable $Y$, we have $E \left[ Y^{2p} \right] = M_p (E \left[ Y^2 \right])^p$. Then,
\[ \forall U, V \in A; \quad E \left[ |X_U - X_V|^{2p} \right] \leq K M_p d_A(U, V)^{2p\beta}. \]

For all $\gamma \in (0, \beta)$, there exists $p \in \mathbb{N}^*$ such that $2p\beta > q_A$, where $q_A$ is the discretization exponent of $(A_n)_{n \in \mathbb{N}}$. By Theorem 2.9 the result follows. \qed
Remark 2.11. The proof of Theorem 2.9 shows that when Condition \((H2)\) is removed from Assumption \(\mathcal{H}_A\), the conclusion remains true when the hypothesis (2.2) is strengthened in
\[
\forall U, V \in \mathcal{A}; \quad E[|X_U - X_V|^\alpha] \leq K d_A(U, V)^{2q_A+\beta}.
\]
The result follows from the simple estimation \(N_n \leq k_n\) in Equation (2.5). In that case, the validity of Corollary 2.10 persists, since the integer \(p\) can be chosen such that \(2p\beta > 2q_A\) (instead of \(2p\beta > q_A\)).

As previously mentioned, the Brownian motion indexed by the lower layers of \([0, 1]^2\) is discontinuous with probability one (e.g. see Theorem 1.4.5 in [5] or [2, 32]). The previous Theorem 2.9 and Corollary 2.10 do not contradict this fact, since the collection of lower layers of \([0, 1]^2\) do not satisfy Assumption \(\mathcal{H}_A\) according to Example 2.6 in the specific case of the separating subclasses \(\{\mathcal{A}_n\}_{n \in \mathbb{N}}\) mentioned there. This latter result is improved by the following corollary of Theorem 2.9.

Corollary 2.12. Any subclasses \(\{\mathcal{A}_n\}_{n \in \mathbb{N}}\) satisfying Condition (4) of Definition 2.1 for the indexing collection of lower layers of \([0, 1]^2\) do not satisfy Assumption \(\mathcal{H}_A\).

Following the early work of Dudley, the restriction of the set-indexed Brownian motion to an indexing collection satisfying certain conditions can admit a continuous modification. We refer to [5] for a modern survey of these results. In particular, the set-indexed Brownian motion is a.s. continuous over any Vapnik-Červonenkis class of sets (see Corollary 1.4.10 in [5]), as the collection of rectangles of \(\mathbb{R}^N\) is an example.

Remark 2.13. According to Dudley’s Theorem (see Theorem 2.7.1 of Chapter 5 in [33] and also Theorems 1.3.5 and 1.5.4 in [5]), the existence of a continuous modification of a centered Gaussian \(\mathcal{A}\)-indexed process can be proved if \((\mathcal{A}, d_A)\) is totally bounded and \(\int_0^1 \sqrt{\log N(\mathcal{A}, \varepsilon)} \, d\varepsilon < +\infty\), where \(N(\mathcal{A}, \varepsilon)\) denotes the entropy function (relative to the distance \(d_A\)).

Following the continuity on processes indexed by Vapnik-Červonenkis classes of sets and the role of Assumption \(\mathcal{H}_A\) in Theorem 2.9, we emphasize the fact that upper bounds for the entropy function can be obtained in the two cases. Let us define
\[
\phi(n) = k_n^{-1/q_A}.
\]
Let us also define, for \(\varepsilon \in (0, 1/2]\), \(n(\varepsilon) = \inf\{k : \phi(k) < \varepsilon\}\).

From Condition \((H1)\) of Assumption \(\mathcal{H}_A\), for all \(U \in \mathcal{A}\),
\[
d_A(U, g_{n(\varepsilon)}(U)) \leq \phi(n(\varepsilon)) \leq \varepsilon,
\]
which implies \(N(\mathcal{A}, \varepsilon) \leq k_{n(\varepsilon)}\).

We can see easily that:
\[
0 < \varepsilon \leq k_{n(\varepsilon)}^{-1/q_A},
\]
which allows to get a bound for the entropy function (relative to the distance \(d_A\)),
\[
N(\mathcal{A}, \varepsilon) \leq k_{n(\varepsilon)} \leq \varepsilon^{-q_A}, \quad (2.10)
\]

In the case of a Vapnik-Červonenkis class \(\mathcal{D}\) of sets in a measure space \((E, \mathcal{E}, \nu)\), the entropy function (relative to the distance \(\nu(B \odot B)\)) is bounded as:
\[
\forall 0 < \varepsilon \leq 1/2, \quad N(\mathcal{D}, \varepsilon) \leq K \varepsilon^{-2\nu} |\ln \varepsilon|^\nu, \quad (2.11)
\]
where $K$ and $v$ are positive constants (e.g. see [5], Theorem 1.4.9).

2.3. C-increments. So far, we only considered simple increments of $X$ of the form $X_U - X_V$ for $U, V \in \mathcal{A}$ not necessarily ordered. However, these quantities do not constitute the natural extension of the one-parameter $X_t - X_s$ ($s, t \in \mathbb{R}_+$) to multiparameter (e.g. [33, 4, 24]) and set-indexed (e.g. [32, 28]) settings, particularly when increment stationarity property is concerned. This section is devoted to usual increments of set-indexed processes, which extend the rectangular increments of multiparameter processes. Let us define, for any given indexing collection $\mathcal{A}$, the collection $\mathcal{C}$ of subsets of $\mathcal{T}$, defined as

$$\mathcal{C} = \{U_0 \setminus \bigcup_{i=1}^{k} U_i; \ U_0, U_1, \ldots, U_k \in \mathcal{A}, k \in \mathbb{N}\}.$$  

This collection is used to index the process $\Delta X$, defined by $\Delta X_C = X_{U_0} - \sum_{i \geq 0} \Delta X_{U_0 \cap U_1} U_i$ for $C = U_0 \setminus \bigcup_{i=1}^{k} U_i$, where $\Delta X_{U_0 \cap U_1} U_i$ is given by the inclusion-exclusion formula

$$\Delta X_{U_0 \cap U_1} U_i = \sum_{i=1}^{k} \sum_{j_1 < \ldots < j_l} (-1)^{l-1} X_{U_0 \cap U_{j_1} \cap \ldots \cap U_{j_l}}. \tag{2.12}$$

The existence of the increment process $\Delta X$ indexed by $\mathcal{C}$ requires that for any $C \in \mathcal{C}$, the value $\Delta X_C$ does not depend on the representation of $C$.

**Corollary 2.14.** Under the hypotheses of Theorem 2.9 and if the distance $d_\mathcal{A}$ on the class $\mathcal{A}$ is assumed to be contractive, for each fixed integer $l \geq 1$, for all $\gamma \in (0, \beta/\alpha)$, there exist a random variable $h^{**}$ and a constant $L > 0$ such that, with probability one,

$$\forall C = U \setminus \bigcup_{i \leq l} V_i \text{ with } U, V_1, \ldots, V_l \in \mathcal{A}, \quad \max_{i \leq l} \{m(U \setminus V_i)\} < h^{**} \Rightarrow |\Delta X_C| \leq L m(C)^{\gamma}. \tag{2.13}$$

For a proof of this result, see Appendix A.

Corollary 2.14, as a result on the class $\mathcal{C}^l = \{U \setminus V; \ U \in \mathcal{A}, V \in \mathcal{B}^l\}$ where $\mathcal{B}^l = \{U_{i=1}^l \cap V; \ V_1, \ldots, V_l \in \mathcal{A}\}$, does not extend to the whole $\mathcal{C} = \bigcup_{i \geq 1} \mathcal{C}^l$, as the following example shows. The next result is an adaptation of an example in [5, 32] to the set-indexed setting. It states that the Brownian motion can be unbounded on $\mathcal{C}$ when $\mathcal{A}$ is the collection of rectangles of $[0, 1]^2$.

**Proposition 2.15.** Let $W$ be a Brownian motion indexed by the Borelian sets of $[0, 1]^2$, i.e. a centered Gaussian process with covariance structure

$$\mathbb{E}[W_{C}W_{C'}] = \lambda(C \cap C'), \quad \forall C, C' \in \mathcal{B}([0, 1]^2)$$

where $\lambda$ denotes the Lebesgue measure.

Let $\mathcal{A}$ be the collection of rectangles of $[0, 1]^2$. In the sequel, we consider the restriction on the class $\mathcal{C}$, related to $\mathcal{A}$, of the Brownian motion defined above.

Then for all $h > 0$, all $M > 0$, and for almost all $\omega \in \Omega$, there exist sequences of sets $(C_n(\omega))_{n \in \mathbb{N}}$, $(C'_n(\omega))_{n \in \mathbb{N}}$ in $\mathcal{C}$ such that $\lambda(C_n(\omega)) \lor \lambda(C'_n(\omega)) < h$ and for $n$ big enough,

$$\max\{|W_{C_n(\omega)}(\omega)|, |W_{C'_n(\omega)}(\omega)|\} > \frac{M}{8}.$$
Without any stronger condition than Assumption \((\mathcal{H}_A)\) on the sub-semilattices \((\mathcal{A}_n)_{n \in \mathbb{N}}\), the previous example of set-indexed Brownian motion dismisses a possible definition of the Hölder continuity for stochastic processes of the form:

\[ \exists M > 0, \exists \delta_0 > 0 : \forall C \in \mathcal{C} \text{ with } m(C) < \delta_0, \ |\Delta X_C| \leq M.m(C)^\alpha. \]

3. Hölder exponents for set-indexed processes

Back to the beginning of Section 2.2, localizing the two expressions (i) and (ii) for Hölder-continuity leads to two different notions. Indeed, for the distance \(d_A\) on \(A\), if \(B_{d_A}(U_0, \rho)\) (or simply \(B(0, \rho)\) if the context is clear) denotes the open ball centered in \(U_0 \in A\) and whose radius is \(\rho > 0\), we get

(i) \(\text{loc}\) lim sup \(\delta \to 0^+ \delta^{-q} \sup_{U,V \in B_{d_A}(U_0, \delta)} |f(U) - f(V)| < \infty.\)

(ii) \(\text{loc}\) There exist \(M > 0\) and \(\delta_0 > 0\) such that

\[ \forall U,V \in B_{d_A}(U_0, \delta_0), \ |f(U) - f(V)| \leq M \ d_A(U,V)^q. \]

Although the conditions (i) and (ii) are equivalent, localizing around \(U_0 \in A\) only gives \((\text{ii})_{\text{loc}} \Rightarrow (\text{i})_{\text{loc}}\). This leads usually to consider two kinds of Hölder exponent at \(U_0 \in A\):

- the pointwise Hölder exponent
  \[ \alpha_f(U_0) = \sup \left\{ \alpha : \limsup_{\rho \to 0} \sup_{U,V \in B(U_0, \rho)} \frac{|f(U) - f(V)|}{\rho^\alpha} < \infty \right\}, \quad (3.1) \]
  - and the local Hölder exponent
  \[ \tilde{\alpha}_f(U_0) = \sup \left\{ \alpha : \limsup_{\rho \to 0} \sup_{U,V \in B(U_0, \rho)} \frac{|f(U) - f(V)|}{d_A(U,V)^\alpha} < \infty \right\}. \quad (3.2) \]

Each one allows to measure the regularity of the function \(f\). In general, we have

\[ \tilde{\alpha}_f \leq \alpha_f, \quad (3.3) \]

but the inequality can be strict.

**Remark 3.1.** We can see that condition \((\text{i})_{\text{loc}}\) is equivalent to \(q < \alpha(U_0)\), and condition \((\text{ii})_{\text{loc}}\) is equivalent to \(q < \tilde{\alpha}(U_0)\). Then \(\tilde{\alpha}_f \leq \alpha_f\) is another statement for \((\text{ii})_{\text{loc}} \Rightarrow (\text{i})_{\text{loc}}\). Note that the discussion of this whole paragraph is not specific to indexing collection, but can be adapted to any totally bounded metric space.

**Example 3.2.** Consider the case of the metric space \((\mathbb{R}, |\cdot|)\). Fix \(\gamma > 0\) and \(\delta > 0\). Let \(f\) be a chirp function defined by \(t \mapsto |t|^\gamma \sin \frac{1}{1+t^\gamma}\). The two Hölder exponents at 0 can be computed and \(\tilde{\alpha}_f(0) = \frac{\gamma}{1+\delta} < \alpha_f(0) = \gamma\).

This example shows that the sole pointwise exponent is not sufficient to describe the irregularity of the function. The local exponent can see the oscillations around 0, while the pointwise exponent cannot. These two notions can be applied to study sample path regularity of a stochastic process.
In the case of Gaussian processes (see [26]), we define the deterministic pointwise Hölder exponent

$$\alpha_X(U_0) = \sup \left\{ \sigma; \limsup_{\rho \to 0} \sup_{U \in B(U_0, \rho)} \mathbb{E} \frac{|X_U - X_V|^2}{\rho^{2\sigma}} < \infty \right\}$$

(3.4)

and the deterministic local Hölder exponent

$$\tilde{\alpha}_X(U_0) = \sup \left\{ \sigma; \limsup_{\rho \to 0} \sup_{U \in B(U_0, \rho)} \mathbb{E} \frac{|X_U - X_V|^2}{d_A(U, V)^{2\sigma}} < \infty \right\}.$$  

(3.5)

On the space \((\mathbb{R}^N, \|\cdot\|)\), it is shown in [26] that for all \(t_0 \in \mathbb{R}^N \), the pointwise and local Hölder exponents of \(X\) at \(t_0\) satisfy almost surely

$$\alpha_X(t_0) = \alpha_X(t_0) \quad \text{and} \quad \tilde{\alpha}_X(t_0) = \tilde{\alpha}_X(t_0).$$

In the following sections, several other definitions are studied for Hölder regularity of set-indexed processes. They are connected to the various ways to study the local behaviour of the sample paths of \(X\) around a given set \(U_0 \in \mathcal{A}\).

### 3.1. Definition of Hölder exponents on \(C^l\).

Following expression (2.12) for the definition of the increments of a set-indexed process, we consider alternative definitions for Hölder exponents, where the quantities \(X_U - X_V\) are substituted with \(\Delta X_{U,V}\).

As stated in Section 2.3, it is not wise to consider \(\Delta X_{U,V}\) when \(U \in \mathcal{A}\) and \(V \in \mathcal{A}(u)\) are close to a given \(U_0 \in \mathcal{A}\). Indeed, Proposition 2.15 shows that the quantity \(|\Delta X_{U,V}|\) can stay far away from 0 when \(m(U \setminus V)\) is small, even in the simple case of a Brownian motion indexed by \([0, 1]^2\). However, when \(U \in \mathcal{A}\) and \(V\) is restricted to sets of the form \(V = \bigcup_{1 \leq i \leq l} V_i\) where \(l\) is fixed and \(V_1, \ldots, V_l \in \mathcal{A}\), the Hölder regularity can be defined from the study of \(\Delta X_{U,V}\).

Fix any integer \(l \geq 1\) and set for all \(U \in \mathcal{A}\) and \(\rho > 0\),

$$B^l(U, \rho) = \left\{ \bigcup_{1 \leq i \leq l} V_i; V_1, \ldots, V_l \in \mathcal{A}, \max_{1 \leq i \leq l} d_A(U, V_i) < \rho \right\}.$$

The pointwise and local Hölder \(C^l\)-exponents at \(U_0 \in \mathcal{A}\) are respectively defined as

$$\alpha_{X,C^l}(U_0) = \sup \left\{ \alpha; \limsup_{\rho \to 0} \sup_{U \in B^l(U_0, \rho), \ V \in B^l(U_0, \rho)} \frac{|\Delta X_{U,V}|}{\rho^\alpha} < \infty \right\},$$

and

$$\tilde{\alpha}_{X,C^l}(U_0) = \sup \left\{ \alpha; \limsup_{\rho \to 0} \sup_{U \in B^l(U_0, \rho), \ V \in B^l(U_0, \rho)} \frac{|\Delta X_{U,V}|}{d_A(U, V)^\alpha} < \infty \right\}.$$

The following result shows that the \(C^l\)-exponents do not depend on \(l\) and, consequently, they provide a definition of Hölder exponents on the class \(C\). Moreover, these exponents can be compared to the exponents defined by (3.1) and (3.2).
Proposition 3.3. If $d_A$ is a contractive distance, for any $U_0 \in \mathcal{A}$, the exponents $\alpha_{X,C}(U_0)$ and $\tilde{\alpha}_{X,C}(U_0)$ do not depend on the integer $l \geq 1$. They are denoted by $\alpha_{X,C}(U_0)$ and $\tilde{\alpha}_{X,C}(U_0)$ respectively. Moreover, for all $U_0 \in \mathcal{A}$ and all $\omega \in \Omega$,

$$\alpha_{X,C}(U_0)(\omega) \geq \alpha_X(U_0)(\omega) \quad \text{and} \quad \tilde{\alpha}_{X,C}(U_0)(\omega) \geq \tilde{\alpha}_X(U_0)(\omega).$$

Proof. We only detail the case of the pointwise exponent. The proof for the local exponent is totally similar.

From the definition of the $C^l$-exponents, since $l \geq l'$ implies $B^l(U_0, \rho) \subseteq B^l(U_0, \rho)$, it is clear that

$$\forall \omega \in \Omega, \forall l \geq l', \quad \alpha_{X,C^l}(U_0)(\omega) \leq \alpha_{X,C^l}(U_0)(\omega).$$

For the sake of readability, we prove the converse inequality for $l = 2, l' = 1$ (the other cases are very similar). For any $\rho > 0$, let $U \in B_{d_A}(U_0, \rho)$, and $V = V_1 \cup V_2 \in B^l(U_0, \rho)$ with $V_1, V_2 \in \mathcal{A}$. From the inclusion-exclusion formula,

$$|\Delta X_{U\setminus V}| = |X_U - X_{U \cap V_1} - X_{U \cap V_2} + X_{U \cap V_1 \cap V_2}| = |\Delta X_{U \cap V_1} + \Delta X_{U \cap V_2} - \Delta X_{U \cap (V_1 \cap V_2)}| \leq |\Delta X_{U \cap V_1}| + |\Delta X_{U \cap V_2}| + |\Delta X_{U \cap (V_1 \cap V_2)}|.$$

We have $d_A(U_0, V_1) \leq \rho$, $d_A(U_0, V_2) \leq \rho$ and

$$d_A(U_0, V_1 \cap V_2) \leq d_A(U_0, V_1) + d_A(V_1, V_1 \cap V_2) \leq d_A(U_0, V_1) + d_A(V_1, V_2) \leq 2d_A(U_0, V_1) + d_A(U_0, V_2) \leq 3\rho,$$

using $d_A(V_1, V_1 \cap V_2) \leq d_A(V_1, V_2)$ from the contracting property of $d_A$.

Then, for all $\alpha < \alpha_{X,C^l}(U_0)(\omega)$,

$$\limsup_{\rho \downarrow 0} \sup_{U \in B^l(U_0, \rho), \ V \in B^l(U_0, \rho)} \frac{|\Delta X_{U \setminus V}|}{\rho^\alpha} < \infty,$$

which says that $\alpha < \alpha_{X,C^l}(U_0)(\omega)$. Thus, $\alpha_{X,C^l}(U_0)(\omega) \leq \alpha_{X,C^l}(U_0)(\omega)$.

This inequality achieves to prove that $\alpha_{X,C^l}(U_0)(\omega)$ does not depend on the integer $l \geq 1$.

To prove the second part of the Proposition, it suffices then to prove the inequality for $l = 1$. This is straightforward, since for a fixed $U \in B_{d_A}(U_0, \rho)$,

$$\sup_{V \in B^l(U_0, \rho)} |\Delta X_{U \setminus V}| \leq \sup_{W \in B_{d_A}(U_0, \rho)} |X_U - X_W|.$$

Hence $\alpha_{X}(U_0) \leq \alpha_{X,C}(U_0)$. The inequality for the local exponent can be obtained identically, or one can notice that it is a direct consequence of Corollary 2.14.

The converse inequality does not hold in general since quantities $|X_U - X_V|$ cannot be obtained from the increment process $\Delta X$ when $U, V$ are not ordered.

Remark 3.4. The previous definition of the pointwise Hölder exponent on $C^l$ is not equivalent to the quantity

$$\sup \left\{ \alpha : \limsup_{\rho \downarrow 0} \sup_{U \in B_{d_A}(U_0, \rho), \ V \in B^l(U_0, V < \rho)} \frac{|\Delta X_{U \setminus V}|}{\rho^\alpha} < \infty \right\},$$

as the following example shows.

In the particular case of the indexing collection \( \mathcal{A} \) equal to the rectangles of \( \mathbb{R}^2 \) and the distance \( d_\lambda = \lambda(\bullet \Delta \bullet) \) induced by the Lebesgue measure \( \lambda \) of \( \mathbb{R}^2 \), we show that the assertion \( (V \in \mathcal{B}^l : d_\lambda(U_0, V) < \rho) \) is not equivalent to \( (V \in \mathcal{B}^l(U_0, \rho)) \).

Consider \( V_1 = [0; (n^2, n^2 + \frac{1}{n})] \), \( V_2 = [0; (n^2 + \frac{1}{n}, n^2)] \) and \( U = [0; (n^2 + \frac{1}{n}, n^2 + \frac{1}{n})] \). We have

\[
d_\lambda(U, V_1 \cup V_2) = \frac{1}{n^2} \quad \text{while} \quad d_\lambda(U, V_1) = d_\lambda(U, V_2) \approx n.
\]

Then, \( V_1 \cup V_2 \notin \mathcal{B}^2(U, \rho) \) for small \( \rho \) and it is not possible to control the quantity \( |X_U - \Delta X_{V_1 \cup V_2}| \) using \( |X_U - X_{V_1}|, |X_U - X_{V_1}| \) and \( |X_U - X_{V_1 \cap V_2}| \) as was done in the previous proofs.

The notation \( \alpha_{X,C} \) must be considered with care: Proposition 2.15 shows that the Hölder exponents cannot be defined directly by taking the supremum on \( \mathcal{A} \) and \( V \in \mathcal{A}(u) \) with \( d_\mathcal{A}(U_0, U) < \rho \) and \( d_\mathcal{A}(U_0, V) < \rho \) (and then, on the class \( \mathcal{C} \)). This is the reason why the set \( V \) is restricted to be in \( \mathcal{B}^2(U, \rho) \).

The arguments of the proof of Proposition 3.3 in the particular case of \( l = 1 \) leads to:

for all \( \omega \),

\[
\alpha_{X,C}(U_0)(\omega) \geq \sup \left\{ \alpha : \limsup_{\rho \to 0} \sup_{U \in B_{d_\mathcal{A}}(U_0, \rho)} \sup_{V \subseteq \mathcal{V}} \left| \frac{X_U(\omega) - X_V(\omega)}{\rho^\alpha} \right| < \infty \right\},
\]

and

\[
\tilde{\alpha}_{X,C}(U_0)(\omega) \geq \sup \left\{ \alpha : \limsup_{\rho \to 0} \sup_{U \in B_{d_\mathcal{A}}(U_0, \rho)} \sup_{V \subseteq \mathcal{V}} \left| \frac{X_U(\omega) - X_V(\omega)}{d_\mathcal{A}(U, V)^\alpha} \right| < \infty \right\}.
\]

The converse inequalities follow from the fact that the set of \( U, V \in B_{d_\mathcal{A}}(U_0, \rho) \) with \( U \subseteq V \) is included in the set of \( U \in B_{d_\mathcal{A}}(U_0, \rho) \) and \( V \in \mathcal{B}^1(U_0, \rho) \). Then, we can state:

**Corollary 3.5.** If \( d_\mathcal{A} \) is a contractive distance, the pointwise and local Hölder \( C \)-exponents at \( U_0 \in \mathcal{A} \) are respectively given by

\[
\alpha_{X,C}(U_0) = \sup \left\{ \alpha : \limsup_{\rho \to 0} \sup_{U \in B_{d_\mathcal{A}}(U_0, \rho)} \sup_{V \subseteq \mathcal{V}} \left| \frac{X_U - X_V}{\rho^\alpha} \right| < \infty \right\},
\]

and

\[
\tilde{\alpha}_{X,C}(U_0) = \sup \left\{ \alpha : \limsup_{\rho \to 0} \sup_{U \in B_{d_\mathcal{A}}(U_0, \rho)} \sup_{V \subseteq \mathcal{V}} \left| \frac{X_U - X_V}{d_\mathcal{A}(U, V)^\alpha} \right| < \infty \right\}.
\]

3.2. **Pointwise continuity.** As previously mentioned, the set-indexed Brownian motion can be not continuous, when the indexing collection is not a Vapnik-Cervonenkis class (see [5, 32] for the detailed study).

In [29], a weak form of continuity is considered in the study of set-indexed Poisson process, set-indexed Brownian motion and more generally set-indexed Lévy processes. In particular, the sample paths of the set-indexed Brownian motion are proved to be
pointwise continuous as a set-indexed Lévy process with Gaussian increments. Notice that such a property does not require Assumption $(\mathcal{H}_A)$ on $A$. We recall the following definitions:

**Definition 3.6** ([29]). The point mass jump of a set-indexed function $x : A \to \mathbb{R}$ at $t \in T$ is defined by

$$J_t(x) = \lim_{n \to \infty} \Delta x_{C_n(t)}, \quad \text{where} \quad C_n(t) = \bigcap_{C \in C_n} C$$

(3.6)

and for each $n \geq 1$, $C_n$ denotes the collection of subsets $U \setminus V$ with $U \in A_n$ and $V \in A_n(u)$.

**Definition 3.7** ([29]). A set-indexed function $x : A \to \mathbb{R}$ is said pointwise continuous at $t \in T$ if $J_t(x) = 0$.

Let us recall that a subset $A'$ of $A$ which is closed under arbitrary intersections is called a lower sub-semilattice of $A$. The ordering of a lower sub-semilattice $A' = \{A_1, A_2, \ldots\}$ is said to be consistent if $A_i \subset A_j \Rightarrow i \leq j$. Proceeding inductively, we can show that any lower sub-semilattice admits a consistent ordering, which is not unique in general (see [32]).

If $\{A_1, \ldots, A_n\}$ is a consistent ordering of a finite lower sub-semilattice $A'$, the set $C_i = A_i \setminus \bigcup_{j \leq i-1} A_j$ is called the left neighbourhood of $A_i$ in $A'$. Since $C_i = A_i \setminus \bigcup_{A \in A', A \not\in A_i} A$, the definition of the left neighbourhood does not depend on the ordering.

As in the classical Kolmogorov criterion of continuity, the pointwise continuity of a set-indexed process $X$ can be proved from the study of $\mathbb{E}[|\Delta X_{C_n(t)}|^p]$ when $n$ goes to infinity.

**Proposition 3.8.** Let $X = \{X_U; U \in A\}$ be a set-indexed process and let $U_{\text{max}}$ be a subset in $A$ such that $m(U_{\text{max}}) < +\infty$ and assume that there exist $p > 0$, $q > 1$, $N \geq 1$ and $K > 0$ such that for all $t \in U_{\text{max}}$ and all $n \geq N$,

$$\mathbb{E}[|\Delta X_{C_n(t)}|^p] \leq K m(C_n(t))^q.$$  

(3.7)

Then, for any $\gamma \in (0, (q-1)/p)$, there exists an increasing function $\varphi : \mathbb{N} \to \mathbb{N}$ and a random variable $n^* \geq 1$ satisfying, with probability one,

$$\forall t \in U_{\text{max}}, \forall n \geq n^*, \quad |\Delta X_{C_{\varphi(n)}(t)}| \leq m(C_{\varphi(n)}(t))^{\gamma}.$$  

**Proof.** Up to restricting the indexing collection to $\{U \cap U_{\text{max}}, U \in A\}$, we assume in this proof that the indexing collection $A$ is included in $U_{\text{max}}$.

For all $0 < \gamma < q - 1/p$, we consider $S_n = \sup \left\{ \frac{|\Delta x_{C_n(t)}|}{m(C_n(t))^{\gamma}}; t \in U_{\text{max}} \right\}$, where $C_n(t)$ is defined in (3.6). When $t$ ranges $U_{\text{max}}$, the subset $C_n(t)$ ranges $\mathcal{C}'(A_n)$, the collection of the disjoint left-neighbourhoods of $A_n$. Consequently we can write $S_n = \sup \left\{ \frac{|\Delta x_{C}|}{m(C)^{\gamma}}; C \in \mathcal{C}'(A_n) \right\}$. 


For any integer \( p \geq 1 \), we have
\[
P(S_n \geq 1) \leq \sum_{C \in C'(A_n)} P(|\Delta X_C| \geq m(C)\gamma) \leq \sum_{C \in C'(A_n)} \mathbb{E} \left( \frac{|\Delta X_C|^p}{m(C)^\gamma} \right) \leq K \sum_{C \in C'(A_n)} m(C)^{q-\gamma p}.
\]

Since \( q - \gamma p > 1 \), we have
\[
P(S_n \geq 1) \leq K \left( \sum_{C \in C'(A_n)} m(C) \right) \sup_{C \in C'(A_n)} \left\{ m(C)^{q-\gamma p-1} \right\}
\]
\[
\leq K m(U_{\text{max}}) \sup_{C \in C'(A_n)} \left\{ m(C)^{q-\gamma p-1} \right\}
\]
where the fact that the \( C \in C'(A_n) \) are disjoint is used. Up to choosing an extraction \( \varphi \) for the sequence \( u_n = \sup_{C \in C'(A_n)} \left\{ m(C)^{q-\gamma p-1} \right\} \), we can assume that \( u_n \) is summable. Hence the Borel-Cantelli Lemma implies that for \( 0 < \gamma < (q-1)/p \), \( \{S_{\varphi(n)} < 1\} \) happens infinitely often, which gives the result.

**Remark 3.9.** Proposition 3.8 does not require Assumption \((\mathcal{H}_A)\) for the collection \( \{A_n\}_{n \in \mathbb{N}} \) and the distance \( d_m \).

From Proposition 3.8, it is natural to define local Hölder regularity of a set-indexed process by a comparison of \( \Delta X_{C_n(t)} \) to quantities \( m(C_n(t))^{\alpha} \) with \( \alpha > 0 \), when \( n \) is large.

**Definition 3.10.** The pointwise continuity Hölder exponent at any \( t \in \mathcal{T} \) is defined by
\[
\alpha_{X}^{pc}(t) = \sup \left\{ \alpha : \limsup_{n \to \infty} \frac{|\Delta X_{C_n(t)}|}{m(C_n(t))^{\alpha}} < \infty \right\}.
\]

According to Proposition 3.8, if \( X \) is a \( A \)-indexed process satisfying hypothesis (3.7), then with probability one, \( \alpha_{X}^{pc}(t) \geq (q-1)/p \) for all \( t \in U_{\text{max}} \).

**Remark 3.11.** As in the continuity criterion (Theorem 2.9 and Corollary 2.10), the proof of Proposition 3.8 can be improved for \( \gamma \in (0, (kq - 1)/kp) \) for any \( k \in \mathbb{N} \), when the process is Gaussian. In that specific case, the upper bound for admissible values of \( \gamma \) is \( q/p \) (instead of \( (q-1)/p \)).

### 4. Connection with Hölder exponents of projections on flows

In this section, we consider the concept of flow, which is a useful tool to reduce characterization or convergence problems to a one-dimensional issue. Flows have been used to characterize: strong martingales [32], set-Markov processes [9], set-indexed fractional Brownian motion [28] and set-indexed Lévy processes [29].
Definition 4.1 ([32]). An elementary flow is defined to be a continuous increasing function \( f : [a, b] \subset \mathbb{R}_+ \to A \), i.e. such that
\[
\forall s, t \in [a, b]; \, s < t \Rightarrow f(s) \subseteq f(t)
\]
\[
\forall s \in [a, b); \, f(s) = \bigcap_{u > s} f(u)
\]
\[
\forall s \in (a, b); \, f(s) = \bigcup_{u < s} f(u).
\]
A simple flow is a continuous function \( f : [a, b] \to A(u) \) such that there exists a finite sequence \( (t_0, t_1, \ldots, t_n) \) with \( a = t_0 < t_1 < \cdots < t_n = b \) and elementary flows \( f_i : [t_{i-1}, t_i] \to A \) (\( i = 1, 2, \ldots, n \)) such that
\[
\forall s \in [t_{i-1}, t_i]; \, f(s) = f_i(s) \cup \bigcup_{j=1}^{i-1} f_j(t_j).
\]

The set of all simple (resp. elementary) flows is denoted \( S(A) \) (resp. \( S^e(A) \)).

According to [28], we use the parametrization of flows which allows to preserve the increment stationarity property under projection on flows (it avoids the appearance of a time-change).

Definition 4.2 ([28]). For any set-indexed process \( X = \{X_U; \, U \in A\} \) on the space \((\mathcal{T}, A, m)\) and any simple flow \( f : [a, b] \to A(u) \), the \( m \)-standard projection of \( X \) on \( f \) is defined as the process
\[
X^{f,m} = \left\{ X^f_t = \Delta X_{f \circ \theta^{-1}(t)}; \, t \in \theta([a, b]) \right\},
\]
where \( \theta \) is the function \( t \mapsto m[f(t)] \) and \( \theta^{-1} \) its right inverse.

The importance of flows in the study of set-indexed processes follows the fact that the finite dimensional distributions of an additive \( A \)-indexed process \( X \) determine and are determined by the finite dimensional distributions of the class \( \{X^{f,m}, \, f \in S(A)\} \) ([30], Lemma 6).

As the projection of a set-indexed process on any flow is a real-parameter process, its classical Hölder exponents can be considered and compared to the exponents of the set-indexed process. In the sequel, we study how regularity of flows connects the exponents \( \alpha_X(U_0) \) (resp. \( \alpha^e_X(U_0) \)) and \( \alpha_{X^{f,m}}(t_0) \) (resp. \( \alpha^e_{X^{f,m}}(t_0) \)), when \( U_0 \in A \) and \( f \circ \theta^{-1}(t_0) = U_0 \).

For any \( U_0 \in A \), let us denote by \( S(A, U_0) \) the subset of \( S(A) \) containing all the simple flows \( f : \theta^{-1}(I_f) \to A(u) \) such that there exists \( t_0 > 0 \) satisfying \( f \circ \theta^{-1}(t_0) = U_0 \), and where \( I_f \) is a closed interval of \( \mathbb{R}_+ \) containing a ball centered in \( t_0 \). Such a \( t_0 \) does not depend on the flow \( f \), since \( t_0 = m(U_0) \). In the same way, we define \( S^e(A, U_0) \) for elementary flows.

Lemma 4.3. Let \( f \in S(A, U_0) \) and \( \eta > 0 \) such that \( B(t_0, \eta) \subseteq I_f \). For all \( t \in B(t_0, \eta) \), \( f \circ \theta^{-1}(t) \in B^e_{dm}(U_0, \eta) = \{ A \in A(u) : m(A \triangle U_0) < \eta \} \).
Proof. \( \theta^{-1}(t) = \inf \{ x \in I_f : \theta(x) \geq t \} \). As \( \theta \) is increasing, \( \theta^{-1} \) is increasing as well. We assume without loss of generality that \( t \geq t_0 \). Then,

\[
d_m(f \circ \theta^{-1}(t), U_0) = m \left( f \circ \theta^{-1}(t) \triangle f \circ \theta^{-1}(t_0) \right) = m \left( f \circ \theta^{-1}(t) \setminus f \circ \theta^{-1}(t_0) \right) = m(f \circ \theta^{-1}(t)) - m(f \circ \theta^{-1}(t_0)) = t - t_0.
\]

□

Using Lemma 4.3, we can compare the Hölder regularity of \( X \) and the Hölder regularity of its projections on flows.

**Proposition 4.4.** Let \( X = \{ X_U; U \in \mathcal{A} \} \) be a set-indexed process on \((\mathcal{T}, \mathcal{A}, m)\), with finite Hölder exponents at \( U_0 \in \mathcal{A} \). Then,

\[
\inf_{f \in S^c(\mathcal{A}, U_0)} \alpha_{X,t,m}(t_0) = \alpha_{X,c}(U_0) \geq \alpha_{X}^{c}(U_0) \quad \text{a.s.}
\]

\[
\inf_{f \in S(\mathcal{A}, U_0)} \tilde{\alpha}_{X,t,m}(t_0) = \tilde{\alpha}_{X,c}(U_0) \geq \tilde{\alpha}_{X}^{c}(U_0) \quad \text{a.s.}
\]

where the metric considered on \( \mathcal{A} \) is \( d_m \).

**Proof.** The proof is only given for the pointwise Hölder exponent. The case of the local Hölder exponent is totally similar.

From Proposition 3.3, the inequality \( \alpha_{X,c}(U_0) \geq \alpha_{X}^{c}(U_0) \) for all \( \omega \in \Omega \) is already known.

The equality \( \inf_{f \in S^c(\mathcal{A}, U_0)} \alpha_{X,t,m}(t_0) = \alpha_{X,c}(U_0) \) follows from Corollary 3.5 and Lemma 4.3. □

The natural question is then to wonder if the previous inequality could be improved in an equality. The answer is generally no, as the following example shows.

**Example 4.5.** In this example, we only consider deterministic functions, instead of random processes. Let \( F \) be a set-indexed function on \( \mathcal{A} \), the usual collection of rectangles of \([0,1]^2\). Let \( U_0 \in \mathcal{A} \) and assume that \( F \) is \( \alpha \)-Hölder continuous in \( U_0 \), for some \( \alpha \in (0,1) \). We assume without loss of generality that \( F(U_0) = 0 \).

Let us divide \( \mathcal{A} \) into four quadrants around \( U_0 = [0,(x_0,y_0)] \) in the following manner:

\[
Q_1 = \{ [0,(x,y)] \in \mathcal{A} : x \leq x_0 \text{ and } y < y_0 \},
\]

\[
Q_2 = \{ [0,(x,y)] \in \mathcal{A} : x \leq x_0 \text{ and } y \geq y_0 \},
\]

\[
Q_3 = \{ [0,(x,y)] \in \mathcal{A} : x > x_0 \text{ and } y \geq y_0 \},
\]

\[
Q_4 = \{ [0,(x,y)] \in \mathcal{A} : x > x_0 \text{ and } y < y_0 \}.
\]
Let us fix $\epsilon > 0$. As $F$ is $\alpha$-Hölder continuous at $U_0$, for all $K > 0$, there exists a sequence of sets in $\mathcal{A}$ converging to $U_0$ and such that
\[ \forall n \geq 0, \quad |F(U_n)| = |F(U_n) - F(U_0)| > K \, d_\mathcal{A}(U_n, U_0)^{\alpha + \epsilon}. \]

There is at least one of the quadrants in which there are infinitely many sets $U_n$. Up to a rotation, assume $Q_4$ is this quadrant. We now assume (without restriction) that a subsequence of $(U_n)$ belongs to a closed subset $S \subset Q_4$ (see figure 1).

Let $G$ be a smooth function except maybe at $U_0$, taking its values in $[0, 1]$ and such that $G(U_0) = 0$ and $G(U) = 0$ for all $U \in Q_1 \cup Q_2 \cup Q_3$ (see figure 1 above), and $G(U) = 1$ for all $U \in S$. We denote by $H$ the product of $F$ and $G$.

Up to an extraction that we detailed previously, the sequence $(U_n)_{n \in \mathbb{N}}$ belongs to $S$. Then,
\[ \forall n \geq 0, \quad |H(U_n) - H(U_0)| = |H(U_n)| = |F(U_n)| > K \, d_\mathcal{A}(U_n, U_0)^{\alpha + \epsilon}. \]

Thus, if $H$ is $\beta$-Hölder continuous at $U_0$, then the inequality $\beta \leq \alpha$ holds necessarily.

For $\gamma < \alpha$, there exist $\rho > 0$ and $K > 0$ such that
\[ \forall U \in B_{d_\mathcal{A}}(U_0, \rho), \quad |F(U)| = |F(U) - F(U_0)| \leq K \, d_\mathcal{A}(U, U_0)^{\gamma}. \]

Thus,
\[ |H(U) - H(U_0)| = |H(U)| \leq G(U) \cdot |F(U)| \leq K \, d_\mathcal{A}(U, U_0)^{\gamma}. \]

We have built a function $H$ which is $\alpha$-Hölder continuous. On the other hand, the projection of $H$ on any elementary flow $f \in S^e(\mathcal{A}, U_0)$ is uniformly 0 and consequently, $\inf_{f \in S^e(\mathcal{A}, U_0)} \tilde{\alpha}_{H,f,m} = \infty > \alpha$.

5. Almost sure values for the Hölder exponents

5.1. Separability of stochastic processes. As in the real-parameter case, we prove that the random Hölder exponents of the sample paths have almost sure values when the process is Gaussian: these values are determined in Theorems 5.3 and 5.4.
Defining Hölder exponents by expressions (3.1) and (3.2) leads us to ask whether they are random variables, in order to consider measurable events related to these quantities. This question was first answered by Doob (see [18]) for linear parameter space, see [33] for a contemporary exposition.

**Definition 5.1** ([18]). A process \( \{X_U, U \in \mathcal{A}\} \) is said separable if there exist an at most countable collection \( S \subset \mathcal{A} \) and a null set \( \Lambda \) such that for all closed sets \( F \subset \mathbb{R} \) and all open set \( O \) for the topology induced by \( d_A \),

\[
\{ \omega : X_U(\omega) \in F \text{ for all } U \in O \cap S \} \setminus \{ \omega : X_U(\omega) \in F \text{ for all } U \in O \} \subset \Lambda
\]

This definition is well suited for set-indexed processes since we have the following:

**Theorem 5.2** (from [23, Theorem 2 p.153]). Any stochastic process from a separable metric space with values in a locally compact space admits a separable modification. Hence, if the sub-collections \( (\mathcal{A}_n)_{n \in \mathbb{N}} \) and the metric \( d_A \) satisfy Assumption \((\mathcal{H}_A)\), any \( \mathbb{R} \)-valued set-indexed stochastic process \( X = \{X_U; U \in \mathcal{A}\} \) has a separable modification.

We shall now consider that all our processes are separable. As a consequence, assuming without any restriction on the probability space, variables such as \( \sup_{U \in O} X_U \), for \( O \) an open set of \( \mathcal{A} \), are indeed measurable. Hence the random Hölder coefficients aforementioned are random variables.

5.2. Uniform results for Gaussian processes. Recall that according to Remark 2.11, Condition \((H2)\) can be removed from Assumption \((\mathcal{H}_A)\) when the process \( X \) is Gaussian and therefore in all this section.

**Theorem 5.3.** Let \( X = \{X_U; U \in \mathcal{A}\} \) a set-indexed centered Gaussian process, where \( (\mathcal{A}_n)_{n \in \mathbb{N}} \) and \( d_A \) satisfy Assumption \((\mathcal{H}_A)\). If the deterministic local Hölder exponent of \( X \) at \( U_0 \in \mathcal{A} \) is positive and finite, we have

\[
P(\tilde{\alpha}_X(U_0) = \tilde{\alpha}_X(U_0)) = 1,
\]

and

\[
P(\alpha_X(U_0) = \alpha_X(U_0)) = 1.
\]

In a similar way to Theorem 3.14 of [26], we can also obtain almost sure results on the exponents \( \alpha_X(U_0) \) and \( \tilde{\alpha}_X(U_0) \) uniformly in \( U_0 \in \mathcal{A} \).

**Theorem 5.4.** Let \( X = \{X_U; U \in \mathcal{A}\} \) be a set-indexed centered Gaussian process, where \( (\mathcal{A}_n)_{n \in \mathbb{N}} \) and \( d_A \) satisfy Assumption \((\mathcal{H}_A)\). Suppose that the functions \( U_0 \mapsto \liminf_{U \to U_0} \tilde{\alpha}_X(U) \) and \( U_0 \mapsto \liminf_{U \to U_0} \alpha_X(U) \) are positive over \( \mathcal{A} \). Then, with probability one,

\[
\forall U_0 \in \mathcal{A}, \quad \liminf_{U \to U_0} \tilde{\alpha}_X(U) \leq \tilde{\alpha}_X(U_0) \leq \limsup_{U \to U_0} \tilde{\alpha}_X(U)
\]

and

\[
\forall U_0 \in \mathcal{A}, \quad \liminf_{U \to U_0} \alpha_X(U) \leq \alpha_X(U_0).
\]
The proof of Theorem 5.3 is an adaptation of proofs in [24], but with a conceptual improvement due to the well-suited formulation of Assumption \((\mathcal{H}_A)\), and a technical improvement in Section B.1 that we obtained through Theorem 2.9. The proofs are detailed in Appendix B. The proof of Theorem 5.4 is given in Section B.3.

### 5.3. Corollaries for the \(C\)-Hölder exponents and the pointwise continuity exponent

Theorem 5.3 can be transposed to the \(C\)-Hölder exponent, and the pointwise continuity exponent. If \(X\) is a Gaussian set-indexed process, we define respectively the deterministic pointwise and local \(C\)-Hölder exponents on one hand, for all integer \(l \geq 1\),

\[
\varphi_{X,C}(U_0) = \sup \left\{ \alpha : \limsup_{\rho \to 0} \sup_{V \in \mathcal{B}^0(U_0,\rho)} \frac{\mathbb{E}[|\Delta X_{U \setminus V}|^2]}{\rho^{2\alpha}} < \infty \right\},
\]

\[
\tilde{\varphi}_{X,C}(U_0) = \sup \left\{ \alpha : \limsup_{\rho \to 0} \sup_{V \in \mathcal{B}^0(U_0,\rho)} \frac{\mathbb{E}[|\Delta X_{U \setminus V}|^2]}{d(U,V)^{2\alpha}} < \infty \right\}
\]

and the deterministic pointwise continuity exponent on the other hand,

\[
\varphi_X^{pc}(t_0) = \sup \left\{ \alpha : \limsup_{n \to \infty} \frac{\mathbb{E}[|\Delta X_{C_n(t_0)}|^2]}{m(C_n(t_0))^{2\alpha}} < \infty \right\}.
\]

Similarly to Proposition 3.3, the pointwise and local deterministic exponents do not depend on \(l\). Hence they are denoted respectively by \(\varphi_{X,C}(U_0)\) and \(\tilde{\varphi}_{X,C}(U_0)\).

**Corollary 5.5.** Let \(X = \{X_U, U \in \mathcal{A}\}\) be a centered Gaussian set-indexed process. If the subcollections \((\mathcal{A}_n)_{n \in \mathbb{N}}\) satisfy Assumption \((\mathcal{H}_A)\) and if the deterministic \(C\)-Hölder exponents are finite, then for \(U_0 \in \mathcal{A}\),

\[
\varphi_{X,C}(U_0) = \varphi_{X,C}(U_0) \text{ a.s. and } \tilde{\varphi}_{X,C}(U_0) = \tilde{\varphi}_{X,C}(U_0) \text{ a.s.}
\]

**Proof.** It suffices to prove the result for \(l = 1\), which corresponds to \(V \subseteq U\) in the definition of the standard Hölder exponent. Thus one can apply the previous proofs (Sections B.1 and B.2) which are still valid when restricted to \(V \subseteq U\). \(\square\)

**Corollary 5.6.** Let \(X = \{X_U, U \in \mathcal{A}\}\) be a centered Gaussian set-indexed process. If the deterministic exponent of pointwise continuity is finite, then for \(t_0 \in \mathcal{T}\),

\[
\varphi_X^{pc}(t_0) = \varphi_X^{pc}(t_0) \text{ a.s.}
\]

Moreover, for any \(U_{\max} \in \mathcal{A}\) such that \(m(U_{\max}) < \infty\),

\[
\mathbb{P} \left( \forall t \in U_{\max}, \varphi_X^{pc}(t) \geq \varphi_X^{pc}(t_0) \right) = 1.
\]

**Proof.** Fix \(t_0 \in \mathcal{T}\). Let \(\alpha < \varphi_X^{pc}(t_0)\). The inequality \(\alpha < \varphi_X^{pc}(t_0)\) a.s. is a direct consequence of Proposition 3.8. This gives \(\varphi_X^{pc}(t_0) \geq \varphi_X^{pc}(t_0)\) almost surely.

For the converse inequality, denote \(\mu = \varphi_X^{pc}(t_0)\). Then for all \(\epsilon > 0\), there exist a subsequence \((C_{\varphi(n)}(t_0))_{n \in \mathbb{N}}\) of \((C_n(t_0))_{n \in \mathbb{N}}\) and a constant \(c > 0\) such that

\[
\forall n \in \mathbb{N}^*, \mathbb{E}[|\Delta X_{C_{\varphi(n)}(t_0)}|^2] \geq c \cdot m(C_{\varphi(n)}(t_0))^{2\mu + \epsilon}.
\]
For all \( n \in \mathbb{N} \), the law of the random variable \( \frac{\Delta X_{\varphi(n)}(t_0)}{m(C_{\varphi(n)}(t_0))^{\alpha+\epsilon}} \) is \( \mathcal{N}(0, \sigma_n^2) \). The previous inequality implies that \( \sigma_n \to \infty \) as \( n \to \infty \). Then for all \( \lambda > 0 \), the same computation as in Lemmas B.1 and B.2 leads to

\[
P\left( \frac{m(C_{\varphi(n)}(t_0))^{\alpha+\epsilon}}{\Delta X_{\varphi(n)}(t_0)} < \lambda \right) = P\left( \frac{\Delta X_{\varphi(n)}(t_0)}{m(C_{\varphi(n)}(t_0))^{\alpha+\epsilon}} > \frac{1}{\lambda} \right) = \int_{|x| > \frac{1}{\lambda}} \frac{1}{\sqrt{2\pi\sigma_n}} \exp\left( -\frac{x^2}{2\sigma_n^2} \right) \, dx = \frac{1}{2\pi} \int_{|x| > \frac{1}{\lambda\sigma_n}} \exp\left( -\frac{x^2}{2} \right) \, dx \to 1 \text{ as } n \to \infty.
\]

Therefore the sequence \( \left( \frac{m(C_{\varphi(n)}(t_0))^{\alpha+\epsilon}}{\Delta X_{\varphi(n)}(t_0)} \right)_{n \in \mathbb{N}} \) converges to 0 in probability. As a consequence, there exists a subsequence which converges to 0 almost surely. Then for all \( \epsilon > 0 \), we have almost surely \( \alpha^{pc}_{X}(t_0) \leq \mu + \epsilon \). Taking \( \epsilon \in \mathbb{Q}^+ \), this yields \( \alpha^{pc}_{X}(t_0) \leq \alpha^{pc}_{X}(t_0) \) a.s.

The second equation is a direct consequence of Proposition 3.8.

\[\square\]

6. Application: Hölder regularity of the set-indexed fractional Brownian motion and the set-indexed Ornstein–Uhlenbeck process

The various general results proved in Section 5 allow to describe the local behaviour of recent set-indexed extensions of two well-known stochastic processes: fractional Brownian motion and Ornstein–Uhlenbeck process.

6.1. Hölder exponents of the SIfBm. The local regularity of fractional Brownian motion \( B^H = \{B^H_t; \ t \in \mathbb{R}^+\} \) is known to be constant a.s. and given by the self-similarity index \( H \in (0, 1) \). More precisely, the two classical Hölder exponents satisfy, with probability one,

\[\forall t \in \mathbb{R}^+, \ \alpha_{Bu}(t) = \tilde{\alpha}_{Bu}(t) = H.\]

In [27, 28], a set-indexed extension for fractional Brownian motion has been defined and studied. A mean-zero Gaussian process \( \mathbf{B}^H = \{\mathbf{B}^H_U; \ U \in \mathcal{A}\} \) is called a set-indexed fractional Brownian motion (SIfBm for short) on \((\mathcal{T}, \mathcal{A}, m)\) if

\[\forall U, V \in \mathcal{A}, \ \mathbf{E} \left[ \mathbf{B}^H_U \mathbf{B}^H_V \right] = \frac{1}{2} \left[ m(U)^{2H} + m(V)^{2H} - m(U \triangle V)^{2H} \right], \tag{6.1}\]

where \( H \in (0, 1/2] \) is the index of self-similarity of the process.

In [25], the deterministic local Hölder exponent and the almost sure value of the local Hölder exponent have been determined for the particular case of an SIfBm indexed by the collection \( \{[0, t]; \ t \in \mathbb{R}^+_N\} \cup \{\emptyset\} \), called the multiparameter fractional Brownian motion. If \( X \) denotes the \( \mathbb{R}^+_N \)-indexed process defined by \( X_t = \mathbf{B}^H_{[0,t]} \) for all \( t \in \mathbb{R}^+_N \), it is proved that for all \( t_0 \in \mathbb{R}^+_N \), \( \tilde{\alpha}_X(t_0) = H \) and with probability one, for all \( t_0 \in \mathbb{R}^+_N \), \( \tilde{\alpha}_X(t_0) = H \).
However, the local regularity has not been studied so far, in the general case of an indexing collection which is not reduced to the collection of rectangles of $\mathbb{R}^N_+$. Theorem 5.3, Theorem 5.4 and Corollary 5.6 provide new results for the sample paths of SIFBm.

In Section 5, Theorem 5.4 failed to provide a uniform almost sure upper bound for the pointwise Hölder exponent of a general Gaussian set-indexed process. In the specific case of the set-indexed fractional Brownian motion, this result can be improved under some additional requirement. We consider a supplementary condition on the collection $\mathcal{A}$ and the distance $d_m$: there exists $\eta > 0$ such that

$$
\inf_{\rho > 0} \sup \left\{ \frac{d_m(U, g_n(U))}{\rho} ; \ n \in \mathbb{N}, \ U, g_n(U) \in B_{d_m}(U_0, \rho) \right\} \geq \eta.
$$

(6.2)

Theorem 6.1. Let $B^H$ be a set-indexed fractional Brownian motion on $(\mathcal{T}, \mathcal{A}, m)$, $H \in (0, 1/2]$. Assume that the subclasses $(\mathcal{A}_n)_{n \in \mathbb{N}}$ satisfy Assumption ($\mathcal{H}_A$). Then, the local and pointwise Hölder exponents of $B^H$ at any $U_0 \in \mathcal{A}$, defined with respect to the distance $d_m$, or any equivalent distance, satisfy

$$
P \left( \forall U_0 \in \mathcal{A}, \ \tilde{\alpha}_{B^H}(U_0) = H \right) = 1
$$

and if the additional Condition (6.2) holds,

$$
P \left( \forall U_0 \in \mathcal{A}, \ \alpha_{B^H}(U_0) = H \right) = 1.
$$

Consequently, when $\mathcal{A}$ is the collection of rectangles of $\mathbb{R}^N_+$ and $\lambda$ is the Lebesgue measure, i.e. $B^H$ is a multiparameter fractional Brownian motion, we have

$$
P \left( \forall U_0 \in \mathcal{A}, \ \alpha_{B^H}(U_0) = \tilde{\alpha}_{B^H}(U_0) = H \right) = 1.
$$

Proof. From the definition of the set-indexed fractional Brownian motion, the following expression of the incremental variance,

$$
\forall U, V \in \mathcal{A}, \quad \mathbb{E} \left[ |B^H_U - B^H_V|^2 \right] = m(U \triangle V)^{2H},
$$

directly implies that the deterministic pointwise and local Hölder exponents are equal to $H$. By Theorem 5.3, the random exponents on an indexing collection satisfying Assumption ($\mathcal{H}_A$) are also equal to $H$.

For the uniform almost sure result on $\mathcal{A}$, according Theorem 5.4, it remains to prove that $P \left( \forall U_0 \in \mathcal{A}, \ \alpha_{B^H}(U_0) \leq H \right) = 1$. This fact is the object of the following Section 6.2.

For the particular case of the multiparameter fractional Brownian motion, it suffices to notice that the collection $\mathcal{A}$ of rectangles of $\mathbb{R}^N$ endowed with the Lebesgue measure $\lambda$ satisfies Condition (6.2).

Let us recall that for any $U_0 \in \mathcal{A}$, $d_{\lambda}(U_0, g_n(U_0)) = N.2^{-n} + o(2^{-n})$. Hence for a given $\rho > 0$, choosing the smallest integer $n$ such that $N.2^{-n} \leq \rho/2$ ensures that

$$
\frac{d_{\lambda}(U_0, g_n(U_0))}{\rho} \geq \frac{N.2^{-(n+1)}}{\rho} \geq \frac{1}{8},
$$

and that $g_n(U_0) \in B_{d_{\lambda}}(U_0, \rho)$.

$\square$
If the collection \( \mathcal{A} \) or the metric \( d_m \) do not satisfy the additional requirement (6.2), then the lower bound for the pointwise exponent remains true by Theorem 5.4: \( \mathbb{P} ( \forall U_0 \in \mathcal{A}. \ \alpha_{B^H}(U_0) \geq H ) = 1. \)

In [27], it is shown that for all \( U, V \in \mathcal{A} \), \( \mathbb{E}[|\Delta B^H_U|] = m(U \backslash V)^{2H} \). This implies that for all \( U_0 \in \mathcal{A} \), \( \bar{\alpha}_{B^H}(U_0) = \alpha_{B^H}(U_0) = H \), and so by Corollary 5.5:

\[
\bar{\alpha}_{B^H}(U_0) = \alpha_{B^H}(U_0) = H \quad \text{a.s.}
\]

The case of the exponent of pointwise continuity needs to determine the behaviour of \( \mathbb{E}[|\Delta B^H_C|^2] \) when \( C \in \mathcal{C} \) (and not only \( C = U \backslash V \in \mathcal{C}_0 \), with \( U, V \in \mathcal{A} \) as previously).

In the specific case of an SIfBm with \( H = 1/2 \), we can state:

**Proposition 6.2.** Let \( B \) be a Brownian motion on \( \mathcal{A} \). Then, for all \( t_0 \in T \),

\[
\alpha_{B^C}(t_0) = \alpha_{B^C}(t_0) = \frac{1}{2} \quad \text{a.s.}
\]

A uniform lower bound in any \( U_{\max} \in \mathcal{A} \) such that \( m(U_{\max}) < \infty \), is given by:

\[
\mathbb{P} \left( \forall t_0 \in U_{\max}, \ \alpha_{B^C}(t_0) \geq \alpha_{B^C}(t_0) = \frac{1}{2} \right) = 1.
\]

**Proof.** Since \( \mathbb{E}[|\Delta B_C|^2] = m(C) \), the result follows from Corollary 5.6.

This property cannot be extended directly to any SIfBm for which \( H < 1/2 \), since we do not have \( \mathbb{E}[|\Delta B^H_C|^2] = m(C)^{2H} \) for all \( C \in \mathcal{C} \) (see [27]). However, the results of Proposition 6.2 hold in the specific case of rectangles of \( \mathbb{R}^N \), i.e. for the multiparameter fractional Brownian motion (see Remark 6.9).

### 6.2. Proof of the uniform a.s. pointwise exponent of the SIfBm

In [1], the isotropic fractional Brownian field is proved to have a uniform pointwise exponent equal to \( H \) using techniques such as local times; and in [10], the same result holds for the regular multifractional Brownian motion (mBm), with a proof based on the integral representation of the mBm. This result relies on tools that are not available in the set-indexed framework, although some attempts have been made to introduce set-indexed local times ([31]).

In [10], the following technical lemma is proved for a multifractional Brownian motion. We restrict it to fBm’s case:

**Lemma 6.3.** Let \( B^H = \{B^H_t, t \in \mathbb{R}_+\} \) be a fractional Brownian motion of index \( H \in (0, 1) \). Let \( \epsilon > 0, \rho > 0, 0 \leq s < t, n \in \mathbb{N}^* \) and \( \delta u = \frac{\rho}{\epsilon} \). Then, let \( u_0 = s \) and for all \( k \in \{0, \ldots, n\} \), \( u_{k+1} = u_k + \delta u \). We have the following:

\[
\mathbb{P} \left( \bigcap_{k=1}^n \{|B_{u_k}^H - B_{u_{k-1}}^H| < \rho^{H+\epsilon}\} \right) \leq \left( \frac{2}{\sqrt{2\pi}} \right)^n \left( \frac{\rho^{H+\epsilon}}{C.(\delta u)^H} \right)^n,
\]

where \( C \) is a constant depending only on \( H \).

In the sequel, for \( U \subset V \in \mathcal{A} \), we denote by \( \mathcal{R}(f, U \to V) \), the range of the elementary flow \( f : [0, d] \to \mathcal{A} \) such that \( f(0) = U \) and \( f(d) = V \), where \( d = d_m(U, V) \) (the distance considered here is always \( d_m = m(\Delta \bullet \bullet) \)). Hence \( \mathcal{R}(f, U \to V) \) is a
totally ordered subset of \( \mathcal{A} \) which forms a continuum. We also denote by \( \mathcal{R}_n(f,U) \), the range \( \mathcal{R}(f,U \to g_n(U)) \). Since the choice of a particular \( f \) does not matter, these notations can be used without specifying \( f \), considering that a choice has been made.

**Lemma 6.4.** Let \( \mathbf{B}^H \) be a SIfBm on \( (\mathcal{A}, \mathcal{T}, m) \) of index \( H \in (0, \frac{1}{2}] \). Let \( U \in \mathcal{A} \), \( i \in \mathbb{N} \) and \( \rho_i = d_m(U, g_i(U)) \). Let \( \epsilon > 0 \), \( n \in \mathbb{N}^* \). In any \( \mathcal{R}_i(f,U) \), there exist an increasing sequence \( (U_j)_{0 \leq j \leq n} \) such that \( U_0 = U \), \( U_n = g_i(U) \), and \( \delta_j = d_m(U_{j-1}, U_j) = \frac{\rho_i}{n} \) for all \( j \in \{1, \ldots, n\} \). Then,

\[
\mathbb{P}\left( \bigcap_{k=1}^{n} \left\{ |\mathbf{B}_{U_k}^H - \mathbf{B}_{U_{k-1}}^H| < \rho_i^{H+\epsilon} \right\} \right) \leq \left( \frac{2}{\sqrt{2\pi}} \right)^n \left( \frac{\rho_i^{H+\epsilon}}{\sigma} \right)^n
\]

where \( \sigma = C. (\delta U)^H \) and \( C > 0 \) only depends on \( H \). Equivalently, there exists a constant \( \tilde{C} > 0 \), which only depends on \( H \), such that

\[
\mathbb{P}\left( \bigcap_{k=1}^{n} \left\{ |\mathbf{B}_{U_k}^H - \mathbf{B}_{U_{k-1}}^H| < \rho_i^{H+\epsilon} \right\} \right) \leq \left( \tilde{C} \ n^H \rho_i^\epsilon \right)^n . \tag{6.3}
\]

**Proof.** Let us consider the range \( \mathcal{R}_i(f,U) \) of a flow \( f \) connecting \( U \) to \( g_i(U) \). The standard projection of \( X = \mathbf{B}^H \) on \( f \) is a standard fractional Brownian motion that we denote \( X^{f,m} = \{ X^{f,m}_t, t \in [0, \rho_i] \} \). As usual, \( \theta = m \circ f \) and in the present situation, \( \theta : [0, \rho_i] \to [m(U), m(g_i(U))] \). For \( k \in \{0, \ldots, n\} \), let \( u_k = m(U) + k \frac{\rho_i}{n} \) and define \( U_k = f \circ \theta^{-1}(u_k) \). The \( U_k \)'s constitute the sequence of the statement and we remark that

\[
\mathbb{P}\left( \bigcap_{k=1}^{p} \left\{ |X_{U_k} - X_{U_{k-1}}| < \rho_i^{H+\epsilon} \right\} \right) = \mathbb{P}\left( \bigcap_{k=1}^{p} \left\{ |X_{u_k}^{f,m} - X_{u_{k-1}}^{f,m}| < \rho_i^{H+\epsilon} \right\} \right).
\]

The result follows from Lemma 6.3. \( \square \)

The following Proposition 6.5 is the key result to prove the uniform almost sure upper bound for the SIfBm.

**Proposition 6.5.** Let \( \mathbf{B}^H \) be a SIfBm on \( (\mathcal{A}, \mathcal{T}, m) \) of parameter \( H \in (0, 1/2] \). We assume that \( (\mathcal{A}_n)_{n \in \mathbb{N}} \) endowed with the distance \( d_m \) satisfies Assumption \( \mathcal{H}_A \) and that Condition \( (6.2) \) holds.

Then, with probability one, for all \( \epsilon > 0 \), there exists a random variable \( h > 0 \) such that for all \( \rho \leq h(\omega) \) and for all \( U_0 \in \mathcal{A} \),

\[
\sup_{U,V \in B_{d_m}(U_0, \rho)} |\mathbf{B}_U^H - \mathbf{B}_V^H| \geq \rho^{H+\epsilon}.
\]

**Proof.** Let us fix \( \epsilon > 0 \). For all \( U \in \mathcal{A} \), let \( \rho_{n,U} = d_m(U, g_n(U)) \) and \( p_{n,U} = \lfloor \rho_{n,U}^{-\epsilon} \rfloor \). For all \( N \in \mathbb{N}^* \), we consider the event

\[
A_N = \bigcup_{n \geq N} \bigcup_{U \in \mathcal{A}_n} \left\{ \forall V, W \in \mathcal{R}_n(f,U), |X_V - X_W| < \rho_{n,U}^{H+\epsilon} \right\}.
\]
We have
\[ P(A_N) \leq \sum_{n \geq N} \sum_{U \in A_n} P(\forall V, W \in \mathcal{R}_n(f, U), |X_V - X_W| < \rho_{n,U}^{H+\epsilon}) \]
\[ \leq \sum_{n \geq N} \sum_{U \in A_n} P\left( \bigcap_{k=1}^{p_n,U} \{|X_{U_k} - X_{U_{k-1}}| < \rho_{n,U}^{H+\epsilon}\} \right), \]
where \( U_0, \ldots, U_{p_n,U} \) are defined as in Lemma 6.4.

Following equation (6.3) and since \( \rho_{n,U} = d_A(U, g_n(U)) \leq k_n^{-1/q_A} \), there exist positive constants \( C_1 \) and \( C_2 \) such that
\[ P\left( \bigcap_{k=1}^{p_n,U} \{|X_{U_k} - X_{U_{k-1}}| < \rho_{n,U}^{H+\epsilon}\} \right) \leq \left( C_1 \rho_{n,U}^{(1-H)} \right)^{p_n,U-1} \]
\[ \leq \left( C_2 k_n^{-1/q_A} \right)^{(1-H)(k^{1/q_A}-1)}. \]

Going back to the previous equation, we obtain
\[ P(A_N) \leq \sum_{n \geq N} k_n \left( C_2 k_n^{-1/q_A} \right)^{\epsilon(1-H)(k^{1/q_A}-1)-1} = R_N. \]

Since \( k_n \) is admissible, we can easily show that \( \sum_{N \in \mathbb{N}} R_N < \infty \). Hence, Borel-Cantelli Lemma implies the existence of a random variable \( N(\omega) \) such that: with probability one, for all \( n \geq N(\omega) \) and for all \( U \in A_n \),
\[ \exists V, W \in \mathcal{R}_n(f, U); \quad |X_V - X_W| \geq \rho_{n,U}^{H+\epsilon}. \quad (6.4) \]

For \( U_0 \in \mathcal{A} \) and \( \rho > 0 \), Assumption (6.2) gives the existence of \( \mathcal{R}_n(f, U) \subset B_{d_A}(U_0, \rho) \), for some \( n \geq N(\omega) \) and \( U \in \mathcal{A} \) such that \( \rho_{n,U} \geq \eta \rho \). Then, there exist \( V, W \in \mathcal{A} \) (the same that in (6.4)), such that
\[ |X_V - X_W| \geq \rho_{n,U}^{H+\epsilon} \geq (\eta^{H+\epsilon}) \rho^{H+\epsilon} \]
which concludes the proof. \( \square \)

As a consequence of Proposition 6.5, with probability one, the random pointwise Hölder exponent of a SIBm is uniformly smaller than \( H \) (and thus, equal to \( H \), by Theorem 5.4), provided that Assumption (\( \mathcal{H}_A \)) and the additional requirement (6.2) hold.

6.3. Hölder exponents of the SIOU process. Theorems 5.3 and 5.4 can be also applied to derive Hölder exponents of the set-indexed Ornstein-Uhlenbeck (SIOU in short) process, studied in [11]. This process was introduced as an example of set-indexed process satisfying some stationarity and Markov properties.

A mean-zero Gaussian process \( Y = \{Y_U; U \in \mathcal{A}\} \), where \( \mathcal{A} \) is an indexing collection on the measure space \( (\mathcal{T}, m) \), is called a stationary set-indexed Ornstein-Uhlenbeck process if
\[ \forall U, V \in \mathcal{A}, \quad \mathbb{E}[Y_U Y_V] = \frac{\sigma^2}{2\gamma} \exp(-\gamma m(U \triangle V)), \]
for given positive constants \( \gamma \) and \( \sigma \).
Fixing $U_0 \in \mathcal{A}$, and for all $U,V$ close to $U_0$ for the metric $d_m$, $E[|Y_U - Y_V|^2] = \frac{\sigma^2}{\gamma^2} (1 - e^{-\gamma(m(U \Delta V))})$ implies that $E[|Y_U - Y_V|^2] = \sigma^2 [m(U \Delta V) + o(m(U \Delta V))]$. This leads to $\omega_Y(U_0) = \tilde{\omega}_Y(U_0) = 1/2$. Consequently, the following result follows directly from Theorem 5.4.

**Proposition 6.6.** Let $Y = \{Y_U; U \in \mathcal{A}\}$ be a stationary set-indexed Ornstein-Uhlenbeck process on $(\mathcal{T}, \mathcal{A}, m)$. Assume that the subclasses $(\mathcal{A}_n)_{n \in \mathbb{N}}$ of $\mathcal{A}$ satisfy Assumption $(\mathcal{H}_A)$.

Then, the pointwise and local Hölder exponents satisfy, with probability one,

$$\forall U_0 \in \mathcal{A}, \quad \tilde{\alpha}_Y(U_0) = \frac{1}{2} \quad \text{and} \quad \alpha_Y(U_0) \geq \frac{1}{2}$$

and $\forall U_0 \in \mathcal{A}, \quad \alpha_Y(U_0) = \frac{1}{2}$ a.s.

As another consequence of the previous remark, the equality holds for the $C$-Hölder exponents, for all $U_0 \in \mathcal{A}$, almost surely.

As mentioned in the case of the SfBm, the computation of the exponent of pointwise continuity requires a fine estimation of the variance of the process over $C$. When $\mathcal{A}$ is the collection of the rectangles of $\mathbb{R}^N_+$, the estimation of $E[|\Delta Y_{C_n(t)}|^2]$ is easier, as the example of the SIOU process shows.

**Lemma 6.7.** Let $\mathcal{A} = \{[0,t]: t \in [0,1]^N\}$ endowed with the usual dissecting class $(\mathcal{A}_n)$ made of the dyadics. Let $t \in (0,1)^N$, $t = (t_1, \ldots, t_N)$ and define:

$$t^n_j = \begin{cases} t_j & \text{if } 2^n t_j \in \mathbb{N} \\ 2^{-n}[2^n t_j + 1] & \text{otherwise,} \end{cases} \quad \text{and} \quad \tilde{t}^n_k = \begin{cases} 2^{-n}[2^n t_k - 1] & \text{if } 2^n t_k \in \mathbb{N} \\ 2^{-n}[2^n t_k] & \text{otherwise.} \end{cases}$$

Then,

$$C_n(t) = [0, (t^n_1, \ldots, t^n_N)] \setminus \bigcup_{k=1}^N [0, (t^n_1, \ldots, \tilde{t}^n_k, \ldots, t^n_N)].$$

**Proof.** We recall that $C_n(t)$, the left-neighbourhood of $A_t$ in $\mathcal{A}_n$, is defined as $\bigcap_{C \in \mathcal{C}} C$. In the particular case of the rectangles, it corresponds to the expression given in the lemma.

As usual, let $\lambda$ be the Lebesgue measure of $\mathbb{R}^N$. A direct consequence of this result is that any Gaussian process $X$ satisfying the assumptions of Corollary 2.10 satisfies, for all $t \in [0,1]^N$ and for all $\omega$,

$$\tilde{\alpha}_{X,C}(A_t) \leq \alpha_{X,C}(A_t) \leq \alpha_{X,C}^p(t),$$

with respect to the Lebesgue measure $\lambda$ and the distance $d_\lambda$.

More precise results are available for the SIOU process and the SfBm.

**Proposition 6.8.** Let $Y = \{Y_U; U \in \mathcal{A}\}$ be a SIOU process, where $\mathcal{A}$ refers to the rectangles of $[0,1]^N$ as in the Lemma 6.7. Then, the pointwise continuity of $Y$ with respect to the Lebesgue measure $\lambda$ of $\mathbb{R}^N$ satisfies

$$\forall t_0 \in [0,1]^N, \quad P\left(\alpha_{Y,C}^p(t_0) = \omega_{Y,C}^p(t_0) = \frac{1}{2}\right) = 1,$$
and,

$$P \left( \forall t_0 \in [0, 1]^N, \quad \alpha_{pc}^Y(t_0) \geq \alpha_{pc}^V(t_0) = \frac{1}{2} \right) = 1.$$ 

Proof. For the sake of readability, the proof is written for $N = 2$. Let $t = (t_1, t_2) \in [0, 1]^N$. To show there is no difference in the final result, we assume $t_1$ is dyadic and $t_2$ is not. Let $k, l \in \mathbb{N}, k < 2^l$ such that $t_1 = k.2^{-l}$. Let $n \in \mathbb{N}, n \geq l$.

First, we notice that, by Lemma 6.7,

$$C_n(t) = \left[ 0, (t_1, 2^{-n}[2^n t_2 + 1]) \right] \setminus \left\{ 0, (2^{-n}[2^n t_1 - 1], 2^{-n}[2^n t_2 + 1]) \right\} \cup [0, (t_1, 2^{-n}[2^n t_2])] .$$

Re-writing this for short $C_n(t) = A_n \setminus \{B_{1,n} \cup B_{2,n}\}$, the inclusion-exclusion formula gives

$$E[|\Delta Y_{C_n(t)}|] = E[Y_{A_n}^2 + Y_{B_{1,n}}^2 + Y_{B_{2,n}}^2 + Y_{B_{1,n} \cap B_{2,n}}^2 - 2Y_{A_n} Y_{B_{1,n}} - 2Y_{A_n} Y_{B_{2,n}}$$

$$+ 2Y_{A_n} Y_{B_{1,n} \cap B_{2,n}} + 2Y_{B_{1,n}} Y_{B_{2,n}} - 2Y_{B_{1,n}} Y_{B_{1,n} \cap B_{2,n}} - 2Y_{B_{2,n}} Y_{B_{1,n} \cap B_{2,n}}] .$$

Combined with the covariance of the SIOU, a second-order Taylor expansion gives:

$$E[|\Delta Y_{C_n(t)}|^2] = \frac{\sigma^2}{2\gamma} (8\gamma 2^{-2n} + 16\gamma^2 2^{-4n}[2^n t_1][2^n t_2] + o(2^{-2n})) .$$

Considering the fact that $\lambda(C_n(t)) = 2^{-2n}$, the previous expansion implies $\alpha_{pc}^V(t) = \frac{1}{2}$. Therefore, Corollary 5.6 gives the result. 

\[\square\]

Remark 6.9. With the notations of Proposition 6.8, we can consider the case of the SIfBm $B^H$ indexed by $A = \{0, t\}, t \in \mathbb{R}_+^N \cup \{0\}$,

$$E \left[ |\Delta B^H_{C_n(t)}|^2 \right] = m(A_n \setminus B_{1,n})^{2H} + m(A_n \setminus B_{2,n})^{2H} - m(A_n \setminus (B_{1,n} \cap B_{2,n})^{2H}$$

$$- m(B_{1,n} \triangle B_{2,n})^{2H} + m(B_{1,n} \setminus B_{2,n})^{2H} + m(B_{2,n} \setminus B_{1,n})^{2H} .$$

Then, the same development as the previous proof gives $\alpha_{pc}^B(t_0) = H$ for all $t_0 \in [0, 1]^N$. Consequently, we can state:

$$\forall t_0 \in [0, 1]^N, \quad P \left( \alpha_{pc}^B(t_0) = \alpha_{pc}^V(t_0) = H \right) = 1,$$

and,

$$P \left( \forall t_0 \in [0, 1]^N, \quad \alpha_{pc}^B(t_0) \geq \alpha_{pc}^V(t_0) = H \right) = 1 .$$

\section*{Appendices: technical results}

A. Proof of Corollary 2.14

In order to prove Corollary 2.14, we need the following lemma:

\textbf{Lemma A.1.} If the distance $d_A$ on the class $A$ is contracting, then for $U, V_1, V_2 \in A$,

$$d_A(U, V_1) \lor d_A(U, V_2) \leq \rho \Rightarrow d_A(U, V_1 \cap V_2) \leq 3\rho .$$

Moreover, for any integer $l \geq 1$ and for all $U, V_1, \ldots, V_l \in A$,

$$\max_{i \leq l} \{d_A(U, V_i)\} \leq \rho \Rightarrow d_A(U, V_1 \cap \cdots \cap V_l) \leq K(l) \rho ,$$

for some constant $K(l) > 0$ which only depends on $l$. 

Proof of Lemma A.1. The proof relies on the triangular inequality and the contracting property of $d_A$. \hfill \Box

Proof of Corollary 2.14. Assuming that $g_n$ can be extended to $A(u)$ in the following way:

$$\forall V_1, \ldots, V_p \in A, \quad g_n \left( \bigcup_{i=1}^{p} V_i \right) = \bigcup_{i=1}^{p} g_n(V_i),$$

the following inequality holds:

$$|X_U - \Delta X_{\cup V_i}| \leq |X_{g_n(U)} - \Delta X_{g_n(\cup V_i)}| + \sum_{j \geq n_0} |X_{g_{j+1}(U)} - X_{g_j(U)}|$$

$$+ \sum_{j \geq n_0} |\Delta X_{g_{j+1}(\cup V_i)} - \Delta X_{g_j(\cup V_i)}|. \quad (A.1)$$

Since for all $V_1, \ldots, V_p \in A$,

$$\Delta X_{\cup V_i} = \sum_{i=1}^{p} X_{V_i} + \cdots + (-1)^{k-1} \sum_{i_1 < \cdots < i_k} X_{\cap V_{i_1} \cap \cdots \cap V_{i_k}} + \cdots + (-1)^{p-1} X_{V_1 \cap \cdots \cap V_p},$$

one can express

$$|\Delta X_{g_{n+1}(U \cup V_i)} - \Delta X_{g_n(U \cup V_i)}| \leq \sum_{i=1}^{p} |X_{g_{n+1}(V_i)} - X_{g_n(V_i)}| + \cdots$$

$$+ \sum_{i_1 < \cdots < i_k} |X_{g_{n+1}(\cap V_{i_1} \cap \cdots \cap V_{i_k})} - X_{g_n(\cap V_{i_1} \cap \cdots \cap V_{i_k})}| + \cdots$$

$$+ |X_{g_{n+1}(\cap \cap V_i)} - X_{g_n(\cap \cap V_i)}|. \quad (A.2)$$

Now assume that $U, V_1, \ldots, V_p \in D$. When $p \leq l$, the number of terms in the right side of inequality (A.2) is bounded by a constant, independent of the set $V_1, \ldots, V_p \in A$. Thus, there exists a positive constant $K_2(l)$ such that

$$|\Delta X_{g_{n+1}(U \cup V_i)} - \Delta X_{g_n(U \cup V_i)}| \leq K_2(l) \sup_{W \in D} |X_{g_{n+1}(W)} - X_{g_n(W)}|. \quad (A.3)$$

Using the same sequence $(a_j)_{j \in \mathbb{N}}$ as in the proof of Theorem 2.9, and the above equation (A.3) in the third inequality below:

$$\mathbb{P} \left( \sup_{V_1, \ldots, V_p \in D} \sum_{j \geq n_0} |\Delta X_{g_{j+1}(U \cup V_i)} - \Delta X_{g_j(U \cup V_i)}| \geq K_2(l) k_{n_0+1}^{-\gamma/q_4} \right)$$

$$\leq \mathbb{P} \left( \exists V_1, \ldots, V_p \in D, \exists j \geq n_0, \ |\Delta X_{g_{j+1}(U \cup V_i)} - \Delta X_{g_j(U \cup V_i)}| \geq a_j K_2(l) k_{n_0+1}^{-\gamma/q_4} \right)$$

$$\leq \mathbb{P} \left( \exists W \in D, \exists j \geq n_0, \ |X_{g_{j+1}(W)} - X_{g_j(W)}| \geq a_j k_{n_0+1}^{-\gamma/q_4} \right).$$

We obtain the same expression (2.3) that we had in the proof of Theorem 2.9, thus the same conclusion holds: if $\max_{i \leq l} \{m(U \setminus V_i)\} \leq k_{n_0}^{-1/q_4}$, then almost surely, $k_{n_0}^{-1/q_4} \leq h^*$ implies that:

$$\sup_{V_1, \ldots, V_p \in D} \sum_{j \geq n_0} |\Delta X_{g_{j+1}(U \cup V_i)} - \Delta X_{g_j(U \cup V_i)}| \leq K_2(l) k_{n_0+1}^{-\gamma/q_4}.$$
In the same way, the second term of the upper bound (A.1) is proved to be bounded by $K_4(q, q_\Delta) m(C)\gamma$, where $K_4(q, q_\Delta) > 0$ only depends on $\gamma$ and $q_\Delta$.

The first term of (A.1) can be bounded by a finite sum (whose number of terms only depends on $l$) of the form $|X_{g_{\rho_0}}(U) - X_{g_{\rho_0}}(V_{i_1}, \ldots, i_k)|$, where $V_{i_1, \ldots, i_k} = V_{i_1} \cap \cdots \cap V_{i_k}$ for $i_1 < \cdots < i_k \leq l$:

$$|X_{g_{\rho_0}}(U) - \Delta X_{g_{\rho_0}}(U)| \leq \sum_{j=1}^{l} \sum_{i_1 < \cdots < i_j} |X_{g_{\rho_0}}(U) - X_{g_{\rho_0}}(V_{i_1}, \ldots, i_j)|. \quad (A.4)$$

Finally, if $\max_{i \leq l} \{m(U \setminus V_i)\} \leq k_{n_0}^{-1/q_\Delta}$, condition (H1) of Assumption $(H_4)$ and Lemma A.1 imply

$$d_m(g_{\rho_0}(U), g_{\rho_0}(V_{i_1}, \ldots, i_j)) \leq d_m(g_{\rho_0}(U), U) + d_m(U, V_{i_1}, \ldots, i_j) + d_m(V_{i_1}, \ldots, i_j, g_{\rho_0}(V_{i_1}, \ldots, i_j)) \leq K(l) \max_{i \leq l} \{m(U \setminus V_i)\} + 2k_{n_0}^{-1/q_\Delta} \leq (K(l) + 2)k_{n_0}^{-1/q_\Delta}.

Hence, Theorem 2.9 implies that when $k_{n_0}^{-1/q_\Delta} < (K(l) + 2)^{-1} h^*$, each term of equation (A.4) is bounded by a quantity proportional to $m(C)\gamma$. Then, the random variable $h^{**}$ of the statement can be chosen to be $(K(l) + 2)^{-1} h^*$ and the result follows. \(\square\)

**B. Proof of Theorems 5.3 and 5.4**

**B.1. Lower bound for the pointwise and local Hölder exponents.** A lower bound for the local Hölder exponent is directly given by Corollary 2.10. For all $U_0 \in \mathcal{A}$ and all $0 < \alpha < \tilde{\alpha}X(U_0)$, there exists $\rho_0 > 0$ and $K > 0$ such that

$$\forall U, V \in B_{d_{\Delta}}(U_0, \rho_0); \quad \mathbb{E}[|X_U - X_V|^2] \leq K\ d_{\Delta}(U, V)^{2\alpha}.$$

Therefore, the sample paths of $X$ are almost surely $\nu$-Hölder continuous in $B_{d_{\Delta}}(U_0, \rho_0)$ for all $\nu \in (0, \alpha)$, which leads to $\alpha \leq \tilde{\alpha}X(U_0)$ almost surely. Then we get

$$\mathbb{P}(\tilde{\alpha}X(U_0) \geq \tilde{\alpha}X(U_0)) = 1.$$

By inequality (3.3), any lower bound for the local Hölder exponent is also a lower bound for the pointwise exponent. Moreover it can be improved in the case of strict inequality $0 < \tilde{\alpha}X(U_0) < \alpha_X(U_0)$.

For any $\epsilon > 0$, there exist $0 < \rho_1 < \rho_0$ and $M > 0$ such that

$$\forall \rho < \rho_1, \forall U, V \in B(U_0, \rho); \quad \mathbb{E}\left[\frac{|X_U - X_V|}{\rho^{\rho_X(U_0) - \epsilon}}\right]^2 \leq M\ \rho^{\epsilon}.$$

Then setting $\gamma = \alpha_X(U_0) - \epsilon$, the exponential inequality for the centered Gaussian variable $X_U - X_V$ implies

$$\mathbb{P}(\ |X_U - X_V| \geq \rho^{\gamma}) \leq \exp\left(-\frac{1}{2} \mathbb{E}[|X_U - X_V|^2]\rho^{2\gamma}\right) \leq \exp\left(-\frac{1}{2} M\rho^{\epsilon}\right).$$
We consider the particular case \( \rho = k_n^{-1/q_A} < \rho_1 \) for \( n \in \mathbb{N} \) large enough. Using the above estimate in the proof of Theorem 2.9 still leads to equation (2.7), where we had that on \( \Omega^* \), for all \( n \geq n^* \):

\[
\sup_{U, V \in D} \frac{|X_U - X_V|}{d_A(U, V) \leq \rho} \leq 3 \rho^\gamma.
\]

Hence this inequality gives:

\[
\sup_{U, V \in B(U_0, k_n^{-1/q_A})} |X_U - X_V| \leq C k_n^{-\gamma/q_A} \text{ a.s.}
\]

and since the sequence \( \left( k_n^{-1/q_A} \right)_{n \in \mathbb{N}} \) is decreasing,

\[
\limsup_{\rho \to 0} \sup_{U, V \in B(U_0, \rho)} \frac{|X_U - X_V|}{\rho^\gamma} < \infty \text{ a.s.}
\]

Therefore, \( \forall \epsilon > 0, \alpha_X(U_0) \geq \omega_X(U_0) - \epsilon \) almost surely and \( \mathbb{P}(\alpha_X(U_0) \geq \omega_X(U_0)) = 1 \).

B.2. Upper bounds for the pointwise and local Hölder exponents. As in [24], upper bounds for the pointwise and local Hölder exponents are given by the following two lemmas. Their proof are totally identical to multiparameter setting.

**Lemma B.1.** Let \( X = \{X_U; U \in \mathcal{A}\} \) be a centered Gaussian process. Assume that for \( U_0 \in \mathcal{A} \), there exists \( \mu \in (0, 1) \) such that for all \( \epsilon > 0 \), there exist a sequence \( (U_n)_{n \in \mathbb{N}^*} \) of \( \mathcal{A} \) converging to \( U_0 \), and a constant \( c > 0 \) such that

\[
\forall n \in \mathbb{N}^*; \quad \mathbb{E} \left[ |X_{U_n} - X_{U_0}|^2 \right] \geq c \, d_A(U_n, U_0)^{2\mu + \epsilon}.
\]

Then, we have almost surely

\[
\alpha_X(U_0) \leq \mu.
\]

Since the process \( X \) has a finite deterministic Hölder exponent, for \( \mu = \omega_X(U_0) \), one can find a sequence \( (U_n) \) as in Lemma B.1. Hence \( \mathbb{P}(\alpha_X(U_0) \leq \omega_X(U_0)) = 1 \).

**Lemma B.2.** Let \( X = \{X_U; U \in \mathcal{A}\} \) be a centered Gaussian process. Assume that for \( U_0 \in \mathcal{A} \), there exists \( \mu \in (0, 1) \) such that for all \( \epsilon > 0 \), there exist two sequences \( (U_n)_{n \in \mathbb{N}^*} \) and \( (V_n)_{n \in \mathbb{N}^*} \) of \( \mathcal{A} \) converging to \( U_0 \), and a constant \( c > 0 \) such that

\[
\forall n \in \mathbb{N}^*; \quad \mathbb{E} \left[ |X_{U_n} - X_{V_n}|^2 \right] \geq c \, d_A(U_n, V_n)^{2\mu + \epsilon}.
\]

Then, we have almost surely

\[
\tilde{\alpha}_X(U_0) \leq \mu.
\]

As for the pointwise case, \( \mathbb{P}(\tilde{\alpha}_X(U_0) \leq \tilde{\omega}_X(U_0)) = 1 \) follows from Lemma B.2 with \( \mu = \tilde{\omega}_X(U_0) \).
B.3. **Proof of the uniform almost sure result.** This section is devoted to the proof of Theorem 5.4. We only consider the local Hölder exponent. The uniform almost sure lower bound for the pointwise exponent is proved in a similar way.

Starting with the lower bound, from Theorem 2.9, for all $U_0 \in \mathcal{A}$ and all $\epsilon > 0$, there is a modification $Y_{\tilde{U}_0}$ of $X$ which is $\alpha$-Hölder continuous for all $\alpha \in (0, \tilde{\alpha}_X(U_0) - \epsilon)$ on $B_{d_A}(U_0, \rho_0)$.

- In the first step, $\tilde{\alpha}_X$ is assumed to be constant over $\mathcal{A}$. Hence the local Hölder exponent of $Y_{\tilde{U}_0}$ satisfies almost surely
  \[ \forall U \in B_{d_A}(U_0, \rho_0), \quad \tilde{\alpha}_{Y_{\tilde{U}_0}}(U) \geq \tilde{\alpha}_X - \epsilon. \quad (B.1) \]
  The collection $\mathcal{A}$ is totally bounded, so it can be covered by a countable number of balls of radius at most $\eta$, for all $\eta > 0$. Let $B$ be one of these balls. For all $U_0 \in \mathcal{A}$, we consider $\rho_0 > 0$ such that $(B.1)$ holds. We have obviously
  \[ B \subseteq \bigcup_{U_0 \in B} B_{d_A}(U_0, \rho_0). \]
  For each open ball, there exists an integer $n$ such that $B_{d_A}(U_0, \rho_0) \cap \mathcal{A}_n \neq \emptyset$ so that for $V_0 \in B_{d_A}(U_0, \rho_0) \cap \mathcal{A}_n$, there exists an integer $m_0$ such that $U_0 \in B_{d_A}(V_0, 2^{-m_0}) \subseteq B_{d_A}(U_0, \rho_0)$. Thus
  \[ B \subseteq \bigcup_{n \geq m_0} B_{d_A}(V_0, 2^{-m_0}), \]
  where the union is countable. Each of these balls satisfies
  \[ \mathbb{P} \left( \forall U \in B_{d_A}(V_0, 2^{-m_0}), \quad \tilde{\alpha}_X(U) \geq \tilde{\alpha}_X - \epsilon \right) = 1, \]
  and since $\mathcal{A}$ is a countable union of balls $B_{d_A}(V_0, 2^{-m_0})$, we get
  \[ \mathbb{P} \left( \forall U \in \mathcal{A}, \quad \tilde{\alpha}_X(U) \geq \tilde{\alpha}_X - \epsilon \right) = 1. \]
  Taking $\epsilon \in \mathbb{Q}_+^*$, we conclude that
  \[ \mathbb{P} \left( \forall U \in \mathcal{A}, \quad \tilde{\alpha}_X(U) \geq \tilde{\alpha}_X \right) = 1. \quad (B.2) \]

- In the general case of a not constant exponent $\tilde{\alpha}_X$, for any ball $B$ of radius $\eta$ previously introduced, we set $\beta = \inf_{U \in B} \tilde{\alpha}_X(U) - \epsilon$, $\epsilon > 0$. Then, there exists a constant $C > 0$ such that
  \[ \forall U, V \in B, \quad \mathbb{E}[|X_U - X_V|^2] \leq C \, d_A(U, V)^{2\beta}. \]
  In a similar way as we proved (B.2), we deduce the existence of an event $\Omega^* \subseteq \Omega$ of probability one such that for all $\omega \in \Omega^*$:
  \[ \forall U \in \mathcal{A}, \forall n \geq 0, \forall \epsilon \in \mathbb{Q}_+^*, \quad \forall U_0 \in B_{d_A}(U, 2^{-n}), \quad \tilde{\alpha}_X(U_0) \geq \inf_{V \in B_{d_A}(U, 2^{-n})} \tilde{\alpha}_X(V) - \epsilon. \]
  By letting $n \to \infty$, the previous equation leads to
  \[ \mathbb{P} \left( \forall U_0 \in \mathcal{A}, \quad \tilde{\alpha}_X(U_0) \geq \liminf_{U \to U_0} \tilde{\alpha}_X(U) \right) = 1. \]
In order to prove the converse inequality (which holds only for the local exponent), we adapt a proof in [26]. We first assume that \( \tilde{\alpha}_X \) is constant on \( \mathcal{A} \).

Using the fact that \( D = \bigcup_{n \in \mathbb{N}} A_n \) is countable, Lemma B.2 gives

\[
P(\forall U \in D, \ \tilde{\alpha}_X(U) \leq \tilde{\alpha}_X) = 1.
\]

Let \( \Omega' \in \mathcal{F} \) be the set of \( \omega \), such that \( \tilde{\alpha}_X(U) \leq \tilde{\alpha}_X \) for all \( U \in D \). Let \( U_0 \in A \setminus D \). Let \( (U^{(i)}_i)_{i \in \mathbb{N}} \) be a sequence in \( D \) converging to \( U_0 \). On \( \Omega' \), \( \tilde{\alpha}_X(U^{(i)}_i) \leq \tilde{\alpha}_X \) for all \( i \in \mathbb{N} \). For each fixed \( i \in \mathbb{N} \), there exist two sequences \( (V^{(i)}_n)_{n \in \mathbb{N}} \) and \( (W^{(i)}_n)_{n \in \mathbb{N}} \) in \( A \) converging to \( U^{(i)}_i \) as \( n \to \infty \), and for all \( n \in \mathbb{N} \),

\[
\lim_{n \to +\infty} \frac{|X_{V^{(i)}_n} - X_{W^{(i)}_n}|}{d_A(V^{(i)}_n, W^{(i)}_n) \tilde{\alpha}_X + \varepsilon} = +\infty.
\]

As in [26], we build two other sequences \( (V_n)_{n \in \mathbb{N}} \) and \( (W_n)_{n \in \mathbb{N}} \) so that \( V_n \to U_0 \) and \( W_n \to U_0 \) and

\[
\lim_{n \to +\infty} \frac{|X_{V_n} - X_{W_n}|}{d_A(V_n, W_n) \tilde{\alpha}_X + \varepsilon} = +\infty.
\]

This implies the expected inequality for all \( U_0 \in A \).

The general case for \( \tilde{\alpha}_X \) not constant is proved in the same way as for the lower bound.

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