Limiting behavior of relative Rényi entropy
in a non-regular location shift family

Masahito Hayashi
Laboratory for Mathematical Neuroscience, Brain Science Institute, RIKEN

Abstract

We calculate the limiting behavior of relative Rényi entropy when the first probability distribution is close to the second one in a non-regular location-shift family which is generated by a probability distribution whose support is an interval or a half-line. This limit can be regarded as a generalization of Fisher information, and plays an important role in large deviation theory.

Keywords: relative Rényi entropy, α-divergence, non-regular location shift family, Weibull distribution, gamma distribution, beta distribution

1 Introduction

In a regular distribution family, Cramér-Rao inequality holds, and the maximum likelihood estimator (MLE) converges to a normal distribution whose variance is the inverse of the Fisher information because the Fisher information converges and is well-defined in this family. However, in a non-regular location shift family which is generated by a distribution of \( \mathbb{R} \) whose support is not \( \mathbb{R} \) (e.g., a Weibull distribution, gamma distribution, or beta distribution), the Fisher information diverges and cannot be defined. Thus, one might think that a substitute information quantity is necessary for a discussion of the asymptotic theory. Akahira and Takeuchi \([1]\) proposed the limit of the Hellinger affinity

\[
- \log \int p_{\theta}^{\frac{1}{2}}(\omega)p_{\theta+\epsilon}^{\frac{1}{2}}(\omega) \, d\omega
\]

as a substitute information quantity. This value is obtained by a transformation from the Hellinger distance. Moreover, Akahira \([2]\) proposed the relative Rényi entropy (Chernoff’s distance)

\[
I^s(p\|q) := -\log \int p^s(\omega)q^{1-s}(\omega) \, d\omega \quad (0 < s < 1)
\]

as a substitute information quantity for a non-regular location shift family. This quantity is linked with α-divergence

\[
D^\alpha(p\|q) := \frac{1}{1-\alpha^2} \left( 1 - \int \Omega p^{\frac{1}{2}\alpha}(\omega)q^{\frac{1}{2}\alpha}(\omega) \, d\omega \right)
\]

which was introduced by Amari-Nagaoka \([3]\) from an information geometrical viewpoint, by the monotone transformation \( x \mapsto -\log \left( 1 - \frac{1-\alpha^2}{4}x \right) \). Since α-divergence is a special case of f-divergence introduced by Csiszár \([4]\), which satisfies the information processing inequality, the relative Rényi entropy satisfies the information processing inequality

\[
I^s(p\|q) \geq I^s(p \circ f^{-1}\|q \circ f^{-1})
\]

for any map \( f \). Moreover, as is shown by Chernoff’s formula \([5]\) and Hoeffding’s formula \([6]\), the asymptotic error exponents in simple hypothesis testing are characterized by the relative Rényi entropy. Thus, it can be regarded as a suitable information quantity.

As was proven by Hayashi \([7]\), the upper bounds of large deviation type bounds are given by these limits of the relative Rényi entropies. These upper bounds are outlined in section \([8]\). Therefore, the calculation of these limits for the non-regular location shift family is an important topic. These limits can be regarded as suitable substitutes for the Fisher information because when the Kullback-Leibler divergence is finite, the relative Rényi entropies are connected with the Kullback-Leibler divergence by the relation

\[
D(p\|q) = \lim_{s \to 1} \frac{1}{s(1-s)} I^s(p\|q) = \lim_{s \to 0} \frac{1}{s(1-s)} I^s(q\|p).
\]

As is known, if a one-parameter distribution family \( \mathcal{S} := \{ p_\theta \mid \theta \in \Theta \subset \mathbb{R} \} \) satisfies suitable regularity conditions, Kullback-Leibler divergence is closely related to the Fisher information \( J_\theta \) defined by \([8]\) as

\[
\lim_{\epsilon \to 0} \frac{1}{\epsilon^2} D(p_{\theta+\epsilon}\|p_\theta) = \frac{1}{2} J_\theta
\]

\[
J_{\theta_0} := \int_\Omega \left( \frac{\partial \log p_{\theta_0}(\omega)}{\partial \theta} \right)^2 p_{\theta_0}(\omega) \, d\omega.
\]
However, when the support depends on the parameter $\theta$, the equation does not hold because the divergence is infinite. As was shown by Akahira [3], under suitable regularity conditions, the equation

$$\lim_{s \to 0} \frac{1}{c^2 s(1-s)} I^s(p_\theta) = \frac{1}{2} J_\theta$$

holds. As the examples in sections 2 and 3 show, there are cases where relation (4) holds, but equation (2) does not. The above facts indicate that the limit of the relative Rényi entropy is a suitable substitute for the Fisher information in a non-regular location shift family.

Moreover, in a regular family, since Fisher information is well-defined, the Riemann metric can be naturally defined on every tangent space. However, in a non-regular location shift family, as was pointed out by Amari [8], the natural metric on the tangent space is not a Riemann metric, but a general Minkowski metric. Such a manifold with a general Minkowski metric on every tangent space is called a Finsler space. Amari [8] proposed that to treat the asymptotic behavior of the MLE, we should regard a non-regular location shift family as a Finsler space with the Minkowski metric $F(\theta) := \lim_{\epsilon \to 0} \frac{1}{2} H(p_\theta || p_{\theta + \epsilon})^\frac{1}{2}$, where $H$ is the Hellinger distance. Unfortunately, the relation between the MLE and this Minkowski metric has not been adequately clarified, and the value of this Minkowski metric has not been calculated. Our result for the case $s = \frac{1}{2}$ gives the value of this Minkowski metric.

2 Interval support case

In this section, we discuss the location shift family generated by a $C^3$ continuous probability density function $f$ whose support is an open interval $(a, b) \subset \mathbb{R}$. We assume conditions (1) and (3) for $f$:

$$f_1(x) := f(a + x) \equiv A_1 x^{\kappa_1 - 1} \quad \text{as } x \to +0$$
$$f_2(x) := f(b - x) \equiv A_2 x^{\kappa_2 - 1} \quad \text{as } x \to +0,$$

where $\kappa_1, \kappa_2 > 0$. In addition, if $\kappa_i \neq 1$, we assume the following conditions:

$$f'_i(x) \equiv A_i (\kappa_i - 1) x^{\kappa_i - 2} \quad \text{as } x \to +0$$
$$f''_i(x) \equiv A_i (\kappa_i - 1)(\kappa_i - 2) x^{\kappa_i - 3} \quad \text{as } x \to +0 \text{ if } \kappa_i \neq 2$$
$$xf''_i(x) \to 0 \quad \text{as } x \to +0 \text{ if } \kappa_i = 2.$$  

If $\kappa_i = 1$, we assume the existence of the limits $\lim_{x \to +0} f'_i(x)$ and $\lim_{x \to +0} f''_i(x)$. If $\kappa_i > 2$, we assume that

$$J_f := \int_a^b f^{-1}(x)(f')^2(x) \, dx < \infty.$$  

For example, when $f$ is the beta distribution $f(x) = \frac{1}{B(\alpha, \beta)} x^{\alpha-1}(1-x)^{\beta-1}$ whose support is $(0, 1)$, the above conditions are satisfied and we have

$$\kappa_1 = \alpha, \quad \kappa_2 = \beta, \quad A_1 = A_2 = \frac{1}{B(\alpha, \beta)}.$$  

In this paper, we denote the beta function by $B(x, y)$. Then, we have the following theorem.

**Theorem 1** Assume that $\kappa := \kappa_1 = \kappa_2$.

$$\lim_{\epsilon \to +0} \frac{I^\epsilon(f_\theta || f_{\theta + \epsilon})}{\epsilon^\kappa} = \begin{cases} 
\frac{1 - \kappa}{\kappa} (A_1 s + A_2 (1-s)) & 0 < \kappa < 1 \\
A_1 s + A_2 (1-s) \quad & \kappa = 1 \\
A_1 s + A_2 (1-s) (s + s + s + s) & 1 < \kappa < 2 \\
A_1 s + A_2 (1-s) s (1-s) & \kappa = 2 \\
A_1 s + A_2 (1-s) & \kappa < 2
\end{cases}$$

$$\lim_{\epsilon \to +0} \frac{I^\epsilon(f_\theta || f_{\theta + \epsilon})}{\epsilon^2 \log \epsilon} = \begin{cases} 
\frac{1}{2} J_f & 0 < \kappa < 1 \\
A_1 s + A_2 (1-s) & \kappa = 1 \\
\frac{1}{2} J_f & 1 < \kappa < 2 \\
\frac{1}{2} J_f & \kappa = 2 \\
\frac{1}{2} J_f & \kappa < 2
\end{cases}$$  

where $f_\theta(x) := f(x - \theta)$. These convergences are uniform for $0 < s < 1$. If $\kappa_1 < \kappa_2$, substituting $\kappa := \kappa_1$, $A_2 := 0$, we obtain the above equations.
When is $f$ Half-line support case

These convergences are uniform for $0 < s < 1$ where relation (3) holds, but relation (2) does not. Note that when $0 < \kappa < 2$, in general, the equation $\lim_{\epsilon \to +0} \frac{I^*(f_{-\epsilon}||f_{+\epsilon})}{\epsilon^{\kappa}} = \lim_{\epsilon \to -0} \frac{I^*(f_{-\epsilon}||f_{+\epsilon})}{\epsilon^{\kappa}}$ does not hold.

Proof: Since $I^*(f_{0}||f_{\pm \epsilon}) = I^*(f_{-\epsilon}||f_{0})$, Lemma 2 yields equation (12).

Lemma 1 For any $c \in (a, b)$, we define

$$
I_-^s(c, f, \epsilon) := \int_a^c f^{1-s}(x) f^s(x + \epsilon) \, dx - \int_a^c f(x) \, dx - f(c) s \epsilon - \frac{s}{2} f'(c) \epsilon^2,
$$

$$
I_+^s(c, f, \epsilon) := \int_c^b f^{1-s}(x) f^s(x + \epsilon) \, dx - \int_c^b f(x) \, dx + f(c) s \epsilon + \frac{s}{2} f'(c) \epsilon^2.
$$

\[
\begin{align*}
\lim_{\epsilon \to +0} \frac{I_-^s(c, f, \epsilon)}{\epsilon^{\kappa_1}} & = \begin{cases} 
-\frac{1 - \kappa_1}{\kappa_1} A_1 s B(s + \kappa_1(1 - s), 1 - \kappa_1) & 0 < \kappa_1 < 1 \\
-\frac{\kappa_1}{\kappa_1} A_1 s B(s + \kappa_1(1 - s), 1 - \kappa_1) & 1 < \kappa_1 < 2 \\
A_1 s(1 - s) & \kappa_1 = 2
\end{cases} \\
\lim_{\epsilon \to +0} \frac{I_-^s(c, f, \epsilon)}{-\epsilon^2 \log \epsilon} & = -\frac{s(1 - s)}{2} J_{f,c}^- \\
\lim_{\epsilon \to +0} \frac{I_-^s(c, f, \epsilon)}{\epsilon^2} & = -\frac{s(1 - s)}{2} J_{f,c}^-
\end{align*}
\]

and

\[
\begin{align*}
\lim_{\epsilon \to +0} \frac{I_+^s(c, f, \epsilon)}{\epsilon^{\kappa_2}} & = \begin{cases} 
\frac{1 - \kappa_2}{\kappa_2} A_2 s B(1 - s + \kappa_2 s, 1 - \kappa_2) & 0 < \kappa_2 < 1 \\
\frac{\kappa_2}{\kappa_2} A_2 s B(1 - s + \kappa_2 s, 1 - \kappa_2) & 1 < \kappa_2 < 2 \\
A_2 s(1 - s) & \kappa_2 = 2
\end{cases} \\
\lim_{\epsilon \to +0} \frac{I_+^s(c, f, \epsilon)}{-\epsilon^2 \log \epsilon} & = -\frac{s(1 - s)}{2} J_{f,c}^+ \\
\lim_{\epsilon \to +0} \frac{I_+^s(c, f, \epsilon)}{\epsilon^2} & = -\frac{s(1 - s)}{2} J_{f,c}^+
\end{align*}
\]

where $J_{f,c}^-$ and $J_{f,c}^+$ are defined as

$$J_{f,c}^- := \int_a^c f^{-1}(x)(f'(x))^2 \, dx, \quad J_{f,c}^+ := \int_c^b f^{-1}(x)(f'(x))^2 \, dx.$$

These convergences are uniform for $0 < s < 1$.

3 Half-line support case

In this section, we discuss the case where the support is the half-line $(0, \infty)$ and the probability density function $f$ is $C^3$ continuous. Similarly to (2) and (3), we assume that

$$f(x) \equiv A_2 x^{\kappa-1} \text{ as } x \to 0. \quad (15)$$

When $\kappa \neq 1$, we assume the following conditions:

$$f'(x) \equiv A_1(\kappa - 1)x^{\kappa-2} \quad \text{as } x \to +0 \quad (16)$$

$$f''(x) \equiv A_1(\kappa - 1)(\kappa - 2)x^{\kappa-3} \quad \text{as } x \to +0 \text{ if } \kappa \neq 2 \quad (17)$$

$$xf'''(x) \to 0 \quad \text{as } x \to +0 \text{ if } \kappa = 2. \quad (18)$$
When \( \kappa = 1 \), we assume the existence of the limits \( \lim_{x \to +0} f'(x) \) and \( \lim_{x \to +0} f''(x) \). In addition, we assume that there exist real numbers \( c > 0 \) and \( \epsilon > 0 \) such that

\[
\int_{c}^{\infty} f^{-1}(x)(f'(x))^2 \, dx < \infty
\]  
(19)

\[
\int_{c}^{\infty} \sup_{0 \leq t_1 \leq \epsilon} f(x + t_1) \sup_{0 \leq t_2 \leq \epsilon} |f^{-3}(x + t_2)(f')^3(x + t_2)| \, dx < \infty
\]  
(20)

\[
\int_{c}^{\infty} \sup_{0 \leq t_1 \leq \epsilon} f(x + t_1) \sup_{0 \leq t_2 \leq \epsilon} |f^{-2}(x + t_2)f'(x + t_2)f''(x + t_2)| \, dx < \infty
\]  
(21)

\[
\int_{c}^{\infty} \sup_{0 \leq t_1 \leq \epsilon} f(x + t_1) \sup_{0 \leq t_2 \leq \epsilon} |f^{-1}(x + t_2)f'''(x + t_2)| \, dx < \infty.
\]  
(22)

For example, when \( f \) is Weibull distribution \( f(x) = \alpha \beta x^{\alpha-1} e^{\beta x^{\alpha}} \), the above conditions are satisfied and we have

\[
\kappa = \alpha, \quad A = \alpha \beta.
\]  
(23)

When \( f \) is gamma distribution \( f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{\beta x} \), the above conditions are satisfied and

\[
\kappa = \alpha, \quad A = \frac{\beta^\alpha}{\Gamma(\alpha)}.
\]  
(24)

Now, we obtain the following theorem.

**Theorem 2** We obtain

\[
\lim_{\epsilon \to +0} \frac{I^*(f_0 || f_{\theta + \epsilon})}{\epsilon^2} = \begin{cases} 
\frac{1 - \kappa}{\kappa} (\alpha \beta (s + \kappa(1-s)), 1 - \kappa) & 0 < \kappa < 1 \\
\alpha \beta (s + \kappa(1-s)) & \kappa = 1 \\
\alpha \beta (s + \kappa(1-s) - \kappa B(s + \kappa(1-s), 2 - \kappa)) & 1 < \kappa < 2 \\
\frac{\alpha \beta (s + \kappa(1-s))}{2} J_f & \kappa = 2
\end{cases}
\]  
(25)

where

\[
J_f := \int_{0}^{\infty} f^{-1}(x)(f')^2(x) \, dx.
\]  
(26)

These convergences are uniform for \( 0 < s < 1 \).

Similarly to Theorem 1, Theorem 2 is proven from Lemma 1 and Lemma 3.

**Lemma 2** For a real number \( c > 0 \) satisfying \((19)-(22)\), we define

\[
I^+_s(c, f, \epsilon) := \int_{c}^{\infty} f^{1-s}(x)f^s(x + \epsilon) \, dx - \int_{c}^{b} f(x) \, dx + f(c) \epsilon + \frac{s}{2} f'(c) \epsilon^2.
\]

We obtain

\[
\lim_{\epsilon \to +0} \frac{I^+_s(c, f, \epsilon)}{\epsilon^2} = -\frac{s(1-s)}{2} J^+_f(c)
\]  
(27)

where

\[
J^+_f(c) := \int_{c}^{\infty} f^{-1}(x)(f'(x))^2 \, dx
\]

and the convergence of \((27)\) is uniform for \( 0 < s < 1 \).
4 Relation between main results and large deviation theory

We will outline a relation between Theorems 1 and 2 and large deviation theory only for a location shift family \( \{f_\theta(x) := f(x-\theta) | \theta \in \mathbb{R} \} \), where \( f \) satisfies the conditions given in Section 2 or Section 3. This relation was discussed by Hayashi [7] more precisely. As generalizations of Bahadur’s large deviation type bound, we define the following quantities:

\[
\alpha_1(\theta) := \limsup_{\eps \to +0} \frac{1}{g(\eps)} \sup_{\theta - \eps \leq \theta' \leq \theta + \eps} \beta(T, \theta', \eps)
\]

\[
\alpha_2(\theta) := \sup_{T} \liminf_{\eps \to +0} \frac{1}{g(\eps)} \inf_{\theta - \eps \leq \theta' \leq \theta + \eps} \beta(T, \theta', \eps)
\]

\[
\beta(T, \theta, \eps) := \liminf \frac{1}{n} \log f_\theta^n \{ |T_n - \theta| > \eps \},
\]

where \( T = \{T_n\} \) is a sequence of estimators and \( g(\eps) \) is chosen by

\[
g(\eps) = \begin{cases} 
\frac{e^\kappa}{\eps} & 0 < \kappa < 2 \\
-\epsilon^2 \log \epsilon & \kappa = 2 \\
\epsilon^2 & \kappa > 2.
\end{cases}
\]

As Ibragimov and Has’minskii [9] pointed out, when KL-divergence is in finite, there exists a super efficient estimator \( T \) such that \( \beta(T, \theta, \eps) \) and \( \lim_{\eps \to +0} \frac{1}{g(\eps)} \beta(T, \theta, \eps) \) are infinite at one point \( \theta \). Therefore, we need to take the infimum \( \inf_{\theta - \eps \leq \theta' \leq \theta + \eps} \) into account. Of course, in a regular case, as was proven by Hayashi [7], the two bounds \( \alpha_1(\theta) \) and \( \alpha_2(\theta) \) coincide.

If the convergence \( \lim_{s \to 0} I_s^{(p_0 \to p_1[p_0 \to p_1])} / g(\epsilon) \) is uniform for \( s \in (0,1) \) and \( \theta \in K \) for any compact set \( K \subset \mathbb{R} \), these quantities are evaluated as

\[
\alpha_1(\theta) \leq \overline{\alpha}_1(\theta) := \begin{cases}
2^s \sup_{0 < s < 1} \frac{I_{g, \theta}^s}{s^{1-s}} & \text{if } 0 < \kappa < 2 \\
4 \sup_{0 < s < 1} \frac{I_{g, \theta}^s}{s^{1-s}} & \text{if } \kappa \geq 2
\end{cases}
\]

\[
\alpha_2(\theta) \leq \overline{\alpha}_2(\theta) := \begin{cases}
\sup_{0 < s < 1} \frac{I_{g, \theta}^s}{s^{1-s}} \left( s^{\frac{1}{1-s}} + (1-s)^{\frac{1}{1-s}} \right)^{\kappa-1} & \text{if } 0 < \kappa < 1 \\
2 \frac{I_{g, \theta}^1}{s^{1-s}} & \text{if } \kappa = 1 \\
\inf_{0 < s < 1} \frac{I_{g, \theta}^s}{s^{1-s}} \left( s^{\frac{1}{1-s}} + (1-s)^{\frac{1}{1-s}} \right)^{\kappa-1} & \text{if } 2 > \kappa > 1 \\
\inf_{0 < s < 1} \frac{I_{g, \theta}^s}{s^{1-s}} & \text{if } 2 \leq \kappa,
\end{cases}
\]

where \( I_{g, \theta}^s \) are defined by

\[
I_{g, \theta}^s := \lim_{\eps \to +0} \frac{I_s^{(p_0 \to p_1[p_0 \to p_1])}}{g(\epsilon)} \quad 1 \geq s \geq 0.
\]

Note that the uniformity of the convergence concerning \( 0 < s < 1 \) is necessary for deriving the above inequalities. In Hayashi [7], these inequalities were proven and the attainability of bounds \( \overline{\alpha}_1(\theta) \) and \( \overline{\alpha}_2(\theta) \) was discussed.

5 Conclusion

We have calculated the limit of the relative Rényi entropy. As mentioned in Section 1, this calculation plays an important role in large deviation type asymptotic theory. On the other hand, we conjecture that these limits characterize the asymptotic behavior of the MLE. This relation, though, still has to be clarified.

A Proof of Lemma 1

A.1 Asymptotic behavior of \( I_s^-(c, f, \epsilon) \)

In the following, when the limit \( \lim_{s \to +0} g(x + \epsilon) (\lim_{s \to +0} g(x - \epsilon)) \) exists for a function \( g \), we denote it by \( g(x + 0) (g(x - 0)) \), respectively. Our situation is divided into five cases: (i) \( 0 < \kappa_1 < 1 \), (ii) \( \kappa_1 = 1 \), (iii)
\[
1 < \kappa_1 < 2, \text{ (iv) } \kappa_1 = 2, \text{ and (v) } \kappa_1 > 2. \text{ First, we discuss cases (ii) and (v).}
\]
\[
\begin{align*}
\int_a^c f^{1-s}(x) f^s(x + \epsilon) \, dx \\
= \int_a^c f^{1-s}(x) \left| f^s(x + \epsilon) - \left( f^s(x) + (f^s)'(x) \epsilon + (f^s)''(x) \frac{\epsilon^2}{2} \right) \right| \, dx \\
+ \int_a^c \left( f(x) + f^{1-s}(x)(f^s)'(x) \epsilon + f^{1-s}(x)(f^s)''(x) \frac{\epsilon^2}{2} \right) \, dx \\
\end{align*}
\]
Equation (28)

The first term is calculated by
\[
\begin{align*}
\int_a^c \left( f(x) + f^{1-s}(x)(f^s)'(x) \epsilon + f^{1-s}(x)(f^s)''(x) \frac{\epsilon^2}{2} \right) \, dx \\
= \int_a^c \left( f(x) + f'(x) \epsilon + \left( \frac{s(s-1)}{2} f^{-1}(x) (f'(x))^2 + \frac{s}{2} \frac{f''(x)}{2} \epsilon^2 \right) \right) \, dx \\
= \int_a^c \left( f(x) dx + f(c) \epsilon + f'(c) \frac{s \epsilon^2}{2} - f(a+0) \epsilon \right) \\
+ \left( \frac{s(s-1)}{2} \int_a^c f^{-1}(x) (f'(x))^2 \, dx - \frac{s}{2} f'(a+0) \epsilon \right) \, dx \\
\end{align*}
\]
Equation (29)

The term
\[
\frac{1}{c^2} \int_a^c f^{1-s}(x) \left| f^s(x + \epsilon) - \left( f^s(x) + (f^s)'(x) \epsilon + (f^s)''(x) \frac{\epsilon^2}{2} \right) \right| \, dx
\]
goes to 0 uniformly for $0 < s < 1$ as $\epsilon \to +0$. Thus, in case (ii), since $f(a+0) = A_1$, we obtain (13) and the uniformity for $0 < s < 1$. In case (v), since $f(a+0) = f'(a+0) = 0$, we obtain (13) and the uniformity for $\kappa_1 > 2$.

Next, we discuss cases (i), (iii), and (iv). We can calculate $I^*_s(c, f, \epsilon)$ as
\[
\begin{align*}
\int_a^c f^{1-s}(x) f^s(x + \epsilon) \, dx \\
= \int_{a+\delta}^c f^{1-s}(x) \left| f^s(x + \epsilon) - \left( f^s(x) + (f^s)'(x) \epsilon + (f^s)''(x) \frac{\epsilon^2}{2} \right) \right| \, dx \\
+ \int_{a+\delta}^c \left( f(x) + f^{1-s}(x)(f^s)'(x) \epsilon + f^{1-s}(x)(f^s)''(x) \frac{\epsilon^2}{2} \right) \, dx + \int_a^{a+\delta} f^{1-s}(x) f^s(x + \epsilon) \, dx \\
\end{align*}
\]
Equation (30)

In the following, we discuss only case (i). Concerning the second term of (30), we have
\[
\begin{align*}
\int_{a+\delta}^c \left( f(x) + f^{1-s}(x)(f^s)'(x) \epsilon + f^{1-s}(x)(f^s)''(x) \frac{\epsilon^2}{2} \right) \, dx + \int_a^{a+\delta} f(x) \, dx \\
= \int_a^c f(x) \, dx + \left( \int_{a+\delta}^c f'(x) \, dx \right) \epsilon + \frac{s(s-1)}{2} \left( \int_{a+\delta}^c f^{-1}(x) (f'(x))^2 \, dx \right) \epsilon^2 + \frac{s}{2} \left( \int_{a+\delta}^c f''(x) \, dx \right) \\
= \int_a^c f(x) \, dx + \left( f(c) - f(a+\delta) \right) \epsilon + \left( f'(c) - f'(a+\delta) \right) \frac{s \epsilon^2}{2} + \frac{s(s-1)}{2} \int_{a+\delta}^c f^{-1}(x) (f'(x))^2 \, dx \epsilon^2 \\
= \int_a^c f(x) \, dx + f(c) \epsilon + \frac{s}{2} f'(c) \epsilon^2 \\
- f(a+\delta) \epsilon - f'(a+\delta) \frac{s \epsilon^2}{2} + \frac{s(s-1)}{2} \left( \int_{a+\delta}^c f^{-1}(x) (f'(x))^2 \, dx \right) \epsilon^2 \\
\end{align*}
\]
Equation (31)
Concerning the third term of \([30]\), we can calculate

\[
\int_a^{a+\delta} f^{1-s}(x)f^s(x + \epsilon)\,dx - \int_a^{a+\delta} f(x)\,dx
\]

\[= \int_a^{a+\delta} (f^{1-s}(x)f^s(x + \epsilon) - f(x))\,dx \]

\[= \int_a^{a+\delta} \int_0^c f^{1-s}(x)(f^s)'(x + y)\,dy\,dx \]

\[= \int_0^c \int_0^{\frac{\delta}{f_1'(y)}} s\frac{f_1^{1-s}(yz)}{f_1'(y)}\frac{f_1'(y(z + 1))}{f_1'(y)}\,dz\,dy \]

\[
(32)
\]

Since

\[
\int_0^{\infty} \frac{z^{(\kappa_1-1)(1-s)}}{(1+z)^{(\kappa_1-1)(1-s)+2-\kappa_1}}\,dz = B(\kappa_1 + s - \kappa_1 s, 1 - \kappa_1),
\]

using \([3]\) and \([3]\), we can prove that for any \(\epsilon' > 0\) real numbers \(\delta > 0\) and \(\epsilon > 0\) exist independently for \(s\) such that

\[
\left| \int_0^c \int_0^{\frac{\delta}{f_1'(y)}} s\frac{f_1^{1-s}(yz)}{f_1'(y)}\frac{f_1'(y(z + 1))}{f_1'(y)}\,dz\,dy - B(\kappa_1 + s - \kappa_1 s, 1 - \kappa_1) \right| < \epsilon'
\]

(33)

for \(\epsilon > \forall y > 0\). For any \(\epsilon' > 0\), there exists a real \(\epsilon > 0\) such that

\[
\left| \int_0^c y\frac{f_1'(y)}{f_1'(y)}\,dy - A_1\frac{\kappa_1-1}{\kappa_1} \right| < \epsilon'.
\]

(34)

Therefore,

\[
\left| \int_0^c \int_0^{\frac{\delta}{f_1'(y)}} s\frac{f_1^{1-s}(yz)}{f_1'(y)}\frac{f_1'(y(z + 1))}{f_1'(y)}\,dz\,dy \right| \leq A_1 B(\kappa_1 + s - \kappa_1 s, 1 - \kappa_1)\frac{s(1-\kappa_1)}{\kappa_1}
\]

\[
\left| \int_0^c y\frac{f_1'(y)}{f_1'(y)}\,dy \right| \leq A_1 \frac{1-\kappa_1}{\kappa_1} + \epsilon' \left( A_1 \frac{1-\kappa_1}{\kappa_1} + \epsilon' + \sup_{0<\epsilon<1} B(\kappa_1 + s - \kappa_1 s, 1 - \kappa_1) \right)
\]

(35)

From \((30), (31), (32),\) and \((33),\) for any \(\epsilon'' > 0\), there exist \(\epsilon > 0\) and \(\delta > 0\) such that

\[
\left| I_1^- (\epsilon, f, s) - B(\kappa_1 + s - \kappa_1 s, 1 - \kappa_1)\frac{s(1-\kappa_1)}{\kappa_1} \right|
\]

\[
\leq \frac{1}{\epsilon^{\kappa_1}} \left[ \int_0^c f^{1-s}(x) \left( f^s(x + \epsilon) - \left( f^s(x) + (f^s)'(x)\epsilon + (f^s)''(x)\frac{\epsilon^2}{2} \right) \right)\,dx \right.
\]

\[
+ \left| f(a + \delta)s\epsilon + f'(a + \delta)\frac{\epsilon^2}{2} - \frac{s(s-1)}{2} \left( \int_0^c f^{-1}(x)f'(x)\,dx \right) \right|^2 \right] + \epsilon''.
\]

(36)

The first term is less than any \(\epsilon'' > 0\) when we chose \(\epsilon > 0\) to be sufficiently small for \(\delta, \epsilon'' > 0\). The independence of \(\epsilon > 0\) for \(0 < s < 1\) is shown as follows. For any \(\epsilon > 0\), there exists \(0 \leq t(x, \epsilon) \leq 1\) such that

\[
f^s(x + \epsilon) - \left( f^s(x) + (f^s)'(x)\epsilon + (f^s)''(x)\frac{\epsilon^2}{2} \right) = (f^s)'''(x + t(x, \epsilon)\epsilon)\frac{\epsilon^3}{6}.
\]

(37)
Since
\[(f^s)'''(x) = s(s-1)(s-2)f^{s-3}(x)(f')^3(x) + 3s(s-1)f^{s-2}(x)f''(x)f'''(x) + sf^{s-1}(x)f'''(x),\]  
we can evaluate
\[
\int_{a+\delta}^{c} f^{1-s}(x) \left| f^s(x + \epsilon) - \left( f^s(x) + (f^s)'(x)\epsilon + (f^s)''(x)\epsilon^2 + \frac{(f^s)'''(x)}{2!}\epsilon^3 \right) \right| dx
\]
\[
= \int_{a+\delta}^{c} f^{1-s}(x) \left| (f^s)'(x + t(x, \epsilon))\epsilon + \frac{(f^s)''(x)}{2!}\epsilon^2 + \frac{(f^s)'''(x)}{3!}\epsilon^3 \right| dx
\]
\[
\leq \frac{c^3}{6} \int_{a+\delta}^{c} f^{1-s}(x) f^s(x + t(x, \epsilon)) \left| s(s-1)(s-2)f^{-3}(x + t(x, \epsilon))\epsilon(f')^3(x + t(x, \epsilon))
        + 3s(s-1)f^{-2}(x + t(x, \epsilon))\epsilon f''(x + t(x, \epsilon)) f'''(x + t(x, \epsilon)) + sf^{-1}(x + t(x, \epsilon))\epsilon f'''(x + t(x, \epsilon)) \right| dx
\]
\[
\leq \frac{c^3}{6} \int_{a+\delta}^{c} \sup_{0\leq t_1 \leq \epsilon} f(x + t_1) \left[ 2 \sup_{0\leq t_2 \leq \epsilon} |f^{-3}(x + t_2)(f')^3(x + t_2)|
        + 3 \sup_{0\leq t_3 \leq \epsilon} |f^{-2}(x + t_3)f''(x + t_3)| + \sup_{0\leq t_4 \leq \epsilon} |f^{-1}(x + t_4)f'''(x + t_4)| \right] dx.
\]
From the \(C^3\) continuity of \(f\), the coefficient of \(\epsilon^3\) at (38) is finite. Thus, we can show the independence of \(\epsilon > 0\).
We obtain (38) and the uniformity in case (i).

Next, we discuss cases (iii) and (iv). \(\kappa_1 = 2\). Concerning the second term of (38), we can calculate
\[
\int_{a+\delta}^{c} \left( f(x) + f^{1-s}(x)(f^s)'(x) + f^{1-s}(x)(f^s)''(x)\frac{\epsilon^2}{2} \right) dx + \int_{a}^{a+\delta} f(x) + f^{1-s}(x)(f^s)'(x) dx
\]
\[
= \int_{a}^{c} \left( f(x) + f'(x)\epsilon \right) dx + \frac{s(s-1)}{2} \left( \int_{a+\delta}^{c} f^{-1}(x)(f'(x))^2 dx \right) \epsilon^2 + \frac{s}{2} \left( \int_{a+\delta}^{c} f'''(x) dx \right) \epsilon^2
\]
\[
= \int_{a}^{c} f(x) dx + (f(c) - f(a + 0))\epsilon + (f'(c) - f'(a + \delta))\epsilon \frac{\epsilon^2}{2} + \frac{s(1-s)}{2} \left( \int_{a+\delta}^{c} f^{-1}(x)(f'(x))^2 dx \right) \epsilon^2.
\]
Concerning the last term of (38), we have
\[
\int_{a}^{a+\delta} f^{1-s}(x)f^s(x + \delta) dx - \int_{a}^{a+\delta} f(x) + f^{1-s}(x)(f^s)'(x) dx
\]
\[
= \int_{a}^{c} \int_{0}^{\epsilon} \frac{1}{y_2} \left[ \frac{f^{1-s}(y_2z)}{f^{1-s}(y_2(z + 1))} \frac{f''(y_2z)}{f''(y_2(z + 1))} \frac{(f')^2(y_2z)}{(f')^2(y_2)}
        + s(s-1) \frac{f^{1-s}(y_2z)}{f^{1-s}(y_2(z + 1))} \frac{f(y_2)}{f(y_2(z + 1))} \frac{(f')^2(y_2z)}{(f')^2(y_2)} \right] dy_2 dy_1
\]
\[
= \frac{1}{0} \int_{0}^{\epsilon} \frac{1}{y_2} \left[ \frac{f^{1-s}(y_2z)}{f^{1-s}(y_2(z + 1))} \frac{f''(y_2z)}{f''(y_2(z + 1))} \frac{(f')^2(y_2z)}{(f')^2(y_2)}
        + s(s-1) \frac{f^{1-s}(y_2z)}{f^{1-s}(y_2(z + 1))} \frac{f(y_2)}{f(y_2(z + 1))} \frac{(f')^2(y_2z)}{(f')^2(y_2)} \right] dy_2 dy_1.
\]
for $\epsilon > \forall y_2 > 0$. For any $\epsilon' > 0$, there exists a real number $\epsilon > 0$ such that

$$\left| \int_a^\infty \int_0^{y_1} \frac{f''(y_2)}{f(y_2)} y_2 dy_2 \ dy_1 - \frac{\epsilon - 1}{\epsilon_1} \right| < \epsilon'. \tag{44}$$

Similarly to (35), it follows from (43) and (44) that for any $\epsilon'' > 0$ there exist real numbers $\delta > 0$ and $\epsilon > 0$ such that

$$\left| \int_a^\infty \int_0^{y_1} f^{1-s}(x)(f^s)'(x + y_2) dy_2 \ dy_1 + A_1 B(1 + (1 - s)(\kappa_1 - 1), 2 - \kappa_1) \frac{s(2 - \kappa_1 + (1-s)(\kappa_1-1))}{\epsilon_1} \right| < \epsilon''. \tag{45}$$

From (30), (40), (41), and (44), we can evaluate

$$\left| I_s^- (c, f, \epsilon) + A_1 B(1 + (1 - s)(\kappa_1 - 1), 2 - \kappa_1) \frac{s(2 - \kappa_1 + (1-s)(\kappa_1-1))}{\epsilon_1} \right|$$

$$\leq \frac{1}{\epsilon_1} \left[ \int_0^c f^{1-s}(x) \left| f^s(x + \epsilon) - \left( f^s(x) + (f^s)'(x) \epsilon + (f^s)''(x) \frac{\epsilon^2}{2} \right) \right| \right] \ dx$$

$$+ \left| -f'(a + \delta) \frac{\epsilon^2}{2} + \frac{s(1 - s)}{2} \left( \int_0^c f^{-1}(x)(f')^2 \ dx \right) \frac{\epsilon^2}{2} \right| + \epsilon'.' \tag{46}$$

Note that $f(a + 0) = 0$. The first term is less than any $\epsilon'' > 0$ when we chose $\epsilon > 0$ to be sufficiently small for $\delta > 0$ and $\epsilon' > 0$. Similarly to (36), we can show that the choice of $\epsilon > 0$ does not depend on $0 < s < 1$. Thus, we obtain (13) and the uniformity in case (iii).

In the following, we discuss case (iv). Using the conditions (5), (7), and (9), we can prove that for any $\epsilon' > 0$, there exist real numbers $\delta > 0$ and $\epsilon > 0$ such that

$$\left| \int_0^{s/2} \left( \frac{f^{1-s}(y_2)}{f(y_2)} \right) \ f''(y_2) \ f^{s/(y_2+z+1)} + s(s-1) \ f^{1-s}(y_2) \ f^{s/(y_2+z+1)} \ f^{s/(y_2+z+1)} - A_1 \ f^{s/(y_2+z+1)} \ f^{s/(y_2+z+1)} \ dz + s(1-s)(-\log y_2) \right|$$

$$< \epsilon' \tag{47}$$

for $\epsilon > y_2 > 0$. For any $\epsilon' > 0$, there exists a real number $\epsilon > 0$ such that

$$\left| \int_0^c \int_0^{y_1} - \log y_2 \ \frac{f''(y_2)}{f(y_2)} y_2 dy_2 \ dy_1 - A_1 (1 - \frac{1}{2} \epsilon^2 \log \epsilon) \right| < \epsilon''. \tag{48}$$

Similarly to (33), for any $\epsilon'' > 0$ there exist $\delta > 0$ and $\epsilon > 0$ such that

$$\left| \int_a^\infty \int_0^{y_1} f^{1-s}(x)(f^s)'(x + y_2) dy_2 \ dy_1 + A_1 \frac{s(1-s)}{2} (1 - \log \epsilon) \right| < \epsilon''. \tag{49}$$

From (30), (40), (41), and (49), we can evaluate this as

$$\left| I_s^- (c, f, \epsilon) + A_1 \frac{s(1-s)}{2} (1 - \log \epsilon) \right|$$

$$< \frac{1}{\epsilon_1} \left[ \int_0^c f^{1-s}(x) \left| f^s(x + \epsilon) - \left( f^s(x) + (f^s)'(x) \epsilon + (f^s)''(x) \frac{\epsilon^2}{2} \right) \right| \right] \ dx$$

$$+ \left| -f'(a + \delta) \frac{\epsilon^2}{2} + \frac{s(1-s)}{2} \left( \int_0^c f^{-1}(x)(f')^2 \ dx \right) \frac{\epsilon^2}{2} \right| + \epsilon''. \tag{50}$$

Note that $f(a + 0) = 0$. The first term is less than any $\epsilon'' > 0$ when we chose $\epsilon > 0$ to be sufficiently small for $\delta > 0$ and $\epsilon' > 0$. Similarly to (36), we can show that the choice of $\epsilon > 0$ does not depend on $0 < s < 1$. Thus, we obtain (13) and the uniformity in case (iv).
A.2 Asymptotic behavior of $I^+_s(c, f, \epsilon)$

As in Section A.1, our situation is divided into five cases: (i) $0 < \kappa_2 < 1$, (ii) $\kappa_2 = 1$, (iii) $1 < \kappa_2 < 2$, (iv) $\kappa_2 = 2$, and (v) $\kappa_2 > 2$. First, we consider cases (ii) and (v).

\[
\int_c^{b-\epsilon} f^{1-s}(x) f^s(x + \epsilon) \, dx
\]

= \int_c^{b-\epsilon} f^{1-s}(x) \left| f^s(x + \epsilon) - \left( f^s(x) + (f^s)'(x)\epsilon + (f^s)''(x)\epsilon^2 \right) \right| \, dx

+ \int_c^{b-\epsilon} \left( f(x) + f'(x)s\epsilon + f^{-1}(x)(f')^2(x)\frac{s(s-1)}{2}\epsilon^2 + f''(x)\frac{s}{2}\epsilon^2 \right) \, dx

= \int_c^b \left( f(x) + f'(x)s\epsilon + f^{-1}(x)(f')^2(x)\frac{s(s-1)}{2}\epsilon^2 + f''(x)\frac{s}{2}\epsilon^2 \right) \, dx

- \int_{b-\epsilon}^b (f(b) + f'(b)(x-b) + f'(b)s\epsilon) \, dx

+ \int_c^{b-\epsilon} f^{1-s}(x) \left| f^s(x + \epsilon) - \left( f^s(x) + (f^s)'(x)\epsilon + (f^s)''(x)\epsilon^2 \right) \right| \, dx

- \int_{b-\epsilon}^b \left( f(x) + f'(x)s\epsilon + f^{-1}(x)(f')^2(x)\frac{s(s-1)}{2}\epsilon^2 + f''(x)\frac{s}{2}\epsilon^2 \right) - (f(b-0) + f'(b-0)(x-b) + f'(b-0)s\epsilon) \, dx.

The first and second terms are calculated as

\[
\int_c^b \left( f(x) + f'(x)s\epsilon + f^{-1}(x)(f')^2(x)\frac{s(s-1)}{2}\epsilon^2 + f''(x)\frac{s}{2}\epsilon^2 \right) \, dx

- \int_{b-\epsilon}^b (f(b-0) + f'(b)(x-b) + f'(b-0)s\epsilon) \, dx

= \int_c^b f(x) \, dx + (f(b-0) - f(c))s\epsilon + (f'(b-0) - f'(c))\frac{s}{2}\epsilon^2

+ \left( \int_c^b f^{-1}(x)(f')^2(x) \, dx \right) \frac{s(s-1)}{2}\epsilon^2 - \int_{b-\epsilon}^b (f(b-0) + f'(b-0)(x-b) + f'(b)s\epsilon) \, dx

= \int_c^b f(x) \, dx - f(c)s\epsilon - f'(c)\frac{s}{2}\epsilon^2 + f(b)s\epsilon + f'(b)\frac{s}{2}\epsilon^2

+ \left( \int_c^b f^{-1}(x)(f')^2(x) \, dx \right) \frac{s(s-1)}{2}\epsilon^2 - f(b-0)s\epsilon + f'(b-0)\epsilon^2 - f'(b-0)s\epsilon^2

= \int_c^b f(x) \, dx - f(c)s\epsilon - f'(c)\frac{s}{2}\epsilon^2

+ f(b-0)(s-1)\epsilon + f'(b-0)(1-s)\epsilon^2 + \left( \int_c^b f^{-1}(x)(f')^2(x) \, dx \right) \frac{s(s-1)}{2}\epsilon^2.

The term

\[
\frac{1}{\epsilon^2} \left[ \int_c^{b-\epsilon} f^{1-s}(x) \left| f^s(x + \epsilon) - \left( f^s(x) + (f^s)'(x)\epsilon + (f^s)''(x)\epsilon^2 \right) \right| \, dx

+ \int_{b-\epsilon}^b \left( f(x) + f'(x)s\epsilon + f^{-1}(x)(f')^2(x)\frac{s(s-1)}{2}\epsilon^2 + f''(x)\frac{s}{2}\epsilon^2 \right) - (f(b) + f'(b)(x-b) + f'(b)s\epsilon) \, dx \right]
\]
In the following, we discuss only case (i). Letting $\epsilon$ goes to 0 uniformly for $0 < s < 1$ as $\epsilon \to +0$. In case (ii), the $C^3$ continuity of $f$ and the existence of $f''_2(0)$ and $f''_2'(0)$ guarantee

$$\int_a^b f^{-1}(x)(f')^2(x) \, dx < \infty.$$  

Thus, from the existence of $f''_2(0)$ and the relation $f(b - 0) = A_2$, we obtain (14) and the uniformity in case (ii). From (10) and the relations $f(b - 0) = f'(b - 0) = 0$, we obtain (14) and the uniformity in case (v).

Next, we consider cases (i), (iii), and (iv).

$$\int_c^{b-\delta} f^{-1}(x)(f')(x + \epsilon) \, dx
\quad = \int_c^{b-\delta} \left| f'(x + \epsilon) - \left( f'(x) + (f')'(x)\epsilon + (f'')'\epsilon^2 \right) \epsilon^2 \right| \, dx
\quad + \int_c^{b-\delta} \left( f(x) + \epsilon f^{1-s}(x)(f')'(x) + \frac{\epsilon^2}{2} f^{1-s}(x)(f'')''(x) \right) \, dx
\quad + \int_{b-\delta}^{b-\delta+\epsilon} \epsilon f^{1-s}(x)(f')'(x) \, dx. \quad \text{(51)}$$

In the following, we discuss only case (i).

$$\int_c^{b-\delta} f(x) + \epsilon f^{1-s}(x)(f')'(x) + \frac{\epsilon^2}{2} f^{1-s}(x)(f'')''(x) \, dx + \int_{b-\delta+\epsilon}^b f(x) \, dx
\quad = \int_c^b f(x) \, dx - \int_{b-\delta}^{b-\delta+\epsilon} f(x) \, dx
\quad + \left( \int_c^{b-\delta} f^{1-s}(x)(f')'(x) \, dx \right) \epsilon + \left( \int_c^{b-\delta} f^{1-s}(x)(f'')''(x) \, dx \right) \frac{\epsilon^2}{2}
\quad = \int_c^b f(x) \, dx - \int_{b-\delta}^{b-\delta+\epsilon} f(x) \, dx + \left( \int_c^{b-\delta} f'(x) \, dx \right) s\epsilon
\quad + \left( \frac{s(s-1)}{2} \int_c^{b-\delta} (s-1)f^{-1}(x)(f')^2 \, dx + \frac{s}{2} \int_c^{b-\delta} f''(x) \, dx \right) \epsilon^2
\quad = \int_c^b f(x) \, dx - \int_{b-\delta}^{b-\delta+\epsilon} f(x) \, dx + (f(b - \delta) - f(c))s\epsilon + (f'(b - \delta) - f'(c)) \frac{s\epsilon^2}{2}
\quad + \frac{s(s-1)}{2} \left( \int_c^{b-\delta} f^{-1}(x)(f')^2 \, dx \right) \epsilon^2. \quad \text{(52)}$$

Letting $z := \frac{b-x}{y}$, we have

$$\int_{b-\delta}^{b-\delta+\epsilon} f^{1-s}(x)(f')(x + \epsilon) \, dx - \int_{b-\delta+\epsilon}^b f(x) \, dx
\quad = \int_{b-\delta+\epsilon}^b \left( f^{1-s}(x - \epsilon) - f'(x) \right) \, dx
\quad = \int_{b-\delta+\epsilon}^b f'(x) \int_{-\epsilon}^0 (f^{1-s})'(x + y) \, dy \, dx
\quad = \int_{-\epsilon}^0 \int_{0}^{\frac{s-1}{s}} \frac{f_2'(yz)}{f_2'(y(z + 1))} \frac{f_2'(y(z + 1))}{f_2'(y)} \, dy \, dx.
\quad = \int_{0}^{\frac{s-1}{s}} \frac{f_2'(yz)}{f_2'(y(z + 1))} \frac{f_2'(y(z + 1))}{f_2'(y)} \, dy \, dx. \quad \text{(53)}$$
Similarly to (33), we can prove that for any $\epsilon'' > 0$, there exist real numbers $\delta > 0$ and $\epsilon > 0$ such that

$$\left| \int_0^{b-\delta} \frac{f_x^z(y) - f_x^z(y+1)}{\epsilon^{\kappa_2}} dy + B(\kappa_2 + s - \kappa_2s, 1 - \kappa_2) \frac{s(1 - \kappa_2)}{\kappa_2} \right| < \epsilon''$$  \hspace{1cm} (54)

for $\epsilon > \forall y > 0$. Therefore, from (51), (52), (53), and (54), similarly to (36), we can prove that for any $\epsilon' > 0$, an real number $\epsilon > 0$ exists independently for $s$, such that

$$\left| I_s^c(c, f, \epsilon) + A_2B(\kappa_2 + (1 - s) - \kappa_2(1 - s), 1 - \kappa_2) \frac{(1-s)(1-\kappa_2)}{\kappa_2} \epsilon^{\kappa_2} \right| < \epsilon''.$$  \hspace{1cm} (55)

Thus, we obtain (14) and the uniformity in case (i).

Next, we consider cases (iii) and (iv). Concerning the second term of (51), we have

$$\int_c^{b-\delta} f(x) + f'(x)(f^s)'(x) \, dx + \frac{c^2}{2} f^{1-s}(x)(f^s)'(x) \, dx + \int_{b-\delta+\epsilon}^{b} f(x) - f^s(x)(f^{1-s})'(x) \epsilon \, dx$$

$$= \int_c^{b-\delta} \left( f(x) + f'(x)se \right) \, dx + \left( \int_c^{b-\delta} f^{1-s}(x)(f^s)'(x) \, dx \right) \frac{c^2}{2}$$

$$- \int_{b-\delta}^{b-\delta+\epsilon} f(x) + f'(x)se \, dx + \int_{b-\delta+\epsilon}^{b} (f(x) - f^s(x)(f^{1-s})'(x) \epsilon) \, dx$$

$$= \int_c^{b} f(x) \, dx + f(b-0)se - f(c)se + \left( \int_c^{b-\delta} f^{1-s}(x)(f^s)'(x) \, dx \right) \frac{c^2}{2}$$

$$- \int_{b-\delta}^{b-\delta+\epsilon} (f(x) + f'(x)se) \, dx + \int_{b-\delta+\epsilon}^{b} (f(x) - f^s(x)(f^{1-s})'(x) \epsilon) \, dx.$$  \hspace{1cm} (56)

We can evaluate this as

$$\left| \int_{b-\delta}^{b} (f(x) + f'(x)se) \, dx + \int_{b-\delta+\epsilon}^{b} (f(x) - f^s(x)(f^{1-s})'(x) \epsilon) \, dx \right|$$

$$= \left| \int_{b-\delta}^{b-\delta+\epsilon} f(x) \, dx - \int_{b-\delta}^{b-\delta+\epsilon} f'(x)se \, dx - \int_{b-\delta+\epsilon}^{b} f'(x) \epsilon \, dx \right|$$

$$= \int_{b-\delta}^{b-\delta+\epsilon} f(b-\delta+\epsilon) - f(x) - f'(x)se \, dx$$

$$\leq \int_{b-\delta}^{b-\delta+\epsilon} |f(b-\delta+\epsilon) - f(x)| + |f'(x)|se \, dx$$

$$\leq \max_{0 \leq t \leq 1} |f'(b-\delta+\epsilon)| \frac{3}{2} \epsilon^2.$$  \hspace{1cm} (57)
Thus, we obtain (14) and the uniformity in case (iii). Similarly to (49), in case (iv), we can prove that

$$\int_{b-\delta}^{b} f^{1-s}(x) f^{s}(x + \epsilon) \, dx + \int_{b-\delta+\epsilon}^{b} -f(x) + f^{s}(x)(f^{1-s})'(x) \epsilon \, dx$$

$$= \int_{b-\delta+\epsilon}^{b} f^{s}(x) \left( (f^{1-s}(x) - f^{1-s}(x - \epsilon) + (f^{1-s})'(x) \epsilon \right) \, dx$$

Similarly to (50), from (51), (56), (57), (58), and (61), we have

Concerning the third term of (51), we have

$$0 < \int_{y_1}^{y} \int_{0}^{\delta(x_2)} \left( f_{y_1}^{2}(y_2) \right) f_{y_2}^{2}(y_2) \left[ \frac{f_{y_2}^{2}(y_2(z + 1))}{f_{y_2}^{2}(y_2(z + 1))} f_{y_2}^{2}(y_2(z + 1)) \right] \, dy_2 \, dy_1$$

Similarly to (58), from (57), (58), and (60), we can prove that for any \( \epsilon'' > 0 \) there exists a real number \( \epsilon > 0 \) such that

$$\left| I_{s}^{+}(c, f, \epsilon) + A_2 B(1 + s(\kappa_2 - 1), 2 - \kappa_2) \frac{(1-s)(2-\kappa_2+s(\kappa_2-1)) \epsilon \kappa_2}{\kappa_2} \right| \, \epsilon'' \quad (59)$$

Similarly to (59), from (51), (52), (57), (58), and (59), we can prove that for any \( \epsilon'' > 0 \) there exists a real number \( \epsilon > 0 \) such that

Thus, we obtain (14) and the uniformity in case (iii). Similarly to (18), in case (iv), we can prove that

$$\left| f_{b-\delta+\epsilon}^{s}(x) \left( -\int_{\epsilon}^{y_1} \int_{0}^{\delta(x_2)} (f^{1-s})''(x + y_2) \, dy_2 \, dy_1 \right) \, dx + A_2 B(1 + s(\kappa_2 - 1), 2 - \kappa_2) \frac{(1-s)(2-\kappa_2+s(\kappa_2-1)) \epsilon \kappa_2}{\kappa_2} \right| \, \epsilon'' \quad (60)$$

$$\left| I_{s}^{+}(c, f, \epsilon) + A_2 B(1 + s(\kappa_2 - 1), 2 - \kappa_2) \frac{(1-s)(2-\kappa_2+s(\kappa_2-1)) \epsilon \kappa_2}{\kappa_2} \right| \, \epsilon'' \quad (61)$$

Similarly to (51), from (51), (52), (57), (58), and (59), we have

Thus, we obtain (14) and the uniformity in case (iv).

### B  Proof of Lemma \( 2 \)

We can calculate

$$\int_{c}^{\infty} f^{1-s}(x) f^{s}(x + \epsilon) \, dx$$

$$= \int_{c}^{\infty} f^{1-s}(x) \left( f^{s}(x + \epsilon) - \left( f^{s}(x) + (f^{s})'(x) \epsilon + (f^{s})''(x) \frac{\epsilon^2}{2} \right) \right) \, dx$$

$$+ \int_{c}^{\infty} f^{1-s}(x) \left( f^{s}(x) + (f^{s})'(x) \epsilon + (f^{s})''(x) \frac{\epsilon^2}{2} \right) \, dx.$$
The second term of (63) is calculated as

\[
\int_c^\infty f^{1-s}(x) \left( f^s(x) + (f^s)'(x)\epsilon + (f^s)''(x)\frac{\epsilon^2}{2} \right) \, dx \\
= \int_c^\infty f(x) + f'(x)s\epsilon + f^{-1}(x)(f')^2(x)\frac{s(s-1)}{2}\epsilon^2 + f''(x)\frac{s\epsilon^2}{2} \, dx \\
= \int_c^\infty f(x) \, dx + \int_c^\infty f^{-1}(x)(f')^2(x)\frac{s(s-1)}{2} \, dx - f(c)s\epsilon - f'(c)\frac{s\epsilon^2}{2} \, dx
\]

(64)

Similarly to (39), we can evaluate the first term of (63) by

\[
\int_c^\infty \left| f^s(x) - \left( f^s(x) + (f^s)'(x)\epsilon + (f^s)''(x)\frac{\epsilon^2}{2} \right) \right| \, dx \\
\leq \frac{\epsilon^3}{6} \int_c^\infty \sup_{0 \leq t_1 \leq \epsilon} f(x + t_1) \left[ 2 \sup_{0 \leq t_2 \leq \epsilon} |f^{-3}(x + t_2)(f')^3(x + t_2)| \\
+ 3 \sup_{0 \leq t_3 \leq \epsilon} |f^{-2}(x + t_3)f'(x + t_3)f''(x + t_3)| + \sup_{0 \leq t_4 \leq \epsilon} |f^{-1}(x + t_4)f'''(x + t_4)| \right] \, dx.
\]

(65)

Conditions (20) - (22) guarantee that the coefficient of (65) is finite. From (63), (64), and (65), we obtain (27) and the uniformity for 0 < s < 1.

References

[1] Akahira, M. and Takeuchi, K. (1995) Non-regular Statistical Estimation Lecture Notes in Statistics No 107, Springer.

[2] Akahira, M. (1996) Loss of information of a statistic for a family of non-regular distributions, Ann. Inst. Statist. Math. Vol. 48, No. 2, 349-364.

[3] Amari, S. and Nagaoka, H. (2000) Methods of Information Geometry, AMS & Oxford University Press.

[4] Csiszár, I. (1967) On topological properties of f-divergence. Studia Scientiarum Mathematicarum Hungarica, 2, 329-339.

[5] Chernoff, H. (1952) A measure of asymptotic efficiency for tests of a hypothesis based on the sum of observations. Ann. Math. Stat., 23, 493-507.

[6] Hoeffding, W. (1965) Ann. Math. Stat., 36, 369-400.

[7] Hayashi, M. (2002) Two non-regular extensions of the large deviation bound, BSIS Technical Reports No.02-4, http://www.bsis.brain.riken.go.jp/BSIS-TR.html eprint math.PR/0212076.

[8] Amari, S. (1984) Non-regular probability family and Finsler geometry, RIMS koukyuroku 6, 27, (In Japanese).

[9] Ibragimov, I. A. and Has’minskii, R. Z. (1981) Statistical Estimation, Springer.