CERTAIN NIL CLEAN CONDITIONS ON ZERO-DIVISORS

Huanyin Chen
Department of Mathematics, Hangzhou Normal University
Hangzhou 310036, People’s Republic of China

Abstract: An element of a ring is (very) nil clean if it is the sum of an (a very) idempotent and a nilpotent element. In this paper we investigate the uniqueness of (very) nil cleanness, especially, on zero-divisors. A ring $R$ is very (D-very) nil clean if every element (zero-divisor) can be uniquely written as the sum of a (very) idempotent and a nilpotent element. The structure of these rings are determined. For instance, we prove that a ring $R$ is very nil clean if and only if $R$ is abelian; $J(R)$ is nil and $R/J(R)$ is isomorphic to $Z_3$, a Boolean ring, or $Z_3 \bigoplus B$ where $B$ is Boolean. A periodic ring $R$ is D-very nil clean if and only if $R$ is abelian and $R/J(R)$ is isomorphic to a field $F$, $Z_3 \bigoplus Z_3$, $Z_3 \bigoplus B$ where $B$ is Boolean, or a Boolean ring. In particular, the structure of uniquely (D-uniquely) nil clean rings are also studied.

Key words: Zero-divisor; Very Nil Clean Ring; D-Very Nil Clean Ring; Uniquely Nil Clean Ring; D-Uniquely Nil Clean Ring.

MR(2010) Subject Classification: 16E50, 16U99.

1. INTRODUCTION

Let $R$ be a ring with an identity. We say that $a \in R$ is a left (right) zero-divisor if there exists a nonzero $b \in R$ such that $ab = 0 (ba = 0)$. An element that is a left and a right zero-divisor is simply called a zero-divisor. Zero-divisors occur in many classes of rings. An element $a$ in a ring $R$ is (uniquely) nil clean if it is the sum of (unique) an idempotent $e \in R$ and a nilpotent. A ring $R$ is (uniquely) nil clean provided that every element in $R$ is (unique) nil clean. An element $a \in R$ is a very idempotent provided that $a$ or $-a$ is an idempotent. An element $a \in R$ is called very nil clean provided that there exists a very idempotent $e \in R$ such that $a - e \in N(R)$, and that $a - f \in N(R)$ with a very idempotent $f \in R$ implies that $e^2 = f^2$. A ring $R$ is very nil clean if every element in $R$ is very nil clean. The motivation of this paper is to explore structure of rings with these nil clean conditions on zero-divisors.

A ring $R$ is called $D$-very nil clean provided that every zero-divisor in $R$ is very nil clean. A ring $R$ is called $D$-uniquely nil clean provided that every zero-divisor in $R$ is uniquely nil clean. In Section 2, we prove that a ring $R$ is very nil clean if and only if $R$ is abelian; $J(R)$
is nil and \( R/J(R) \) is isomorphic to \( \mathbb{Z}_3 \), a Boolean ring, or \( \mathbb{Z}_3 \oplus B \) where \( B \) is a Boolean. A ring \( R \) is called a periodic ring if for any \( a \in R \) there exist distinct \( m, n \in \mathbb{N} \) such that \( a^m = a^n \). In Section 3, we shall determine the structure of \( D \)-very nil clean rings in periodic case. We show that a periodic ring \( R \) is \( D \)-very nil clean if and only if \( R \) is abelian and \( R/J(R) \) is isomorphic to a field \( F \), \( \mathbb{Z}_3 \oplus \mathbb{Z}_3 \), \( \mathbb{Z}_3 \oplus B \) where \( B \) is Boolean, or a Boolean ring. Furthermore, in Section 4, we are concern several special cases. We prove that a ring \( R \) is an abelian ring in which every zero-divisor in \( R \) is a very idempotent or a nilpotent element if and only if \( R \) is isomorphic to one of the following: a \( D \)-ring, a Boolean ring, \( \mathbb{Z}_3 \oplus \mathbb{Z}_3 \), and \( \mathbb{Z}_3 \oplus B \) where \( B \) is a Boolean. In Section 5, we characterize uniquely nil clean rings. We prove that a ring \( R \) is uniquely nil clean if and only if \( 2 \in R \) is nilpotent and \( R \) is very nil clean. From this, we show that \( RG \) is uniquely nil clean if and only if \( R \) is uniquely nil clean and \( I(R, G) \) is nil. Finally, in the last section, we explore the structure of \( D \)-uniquely nil clean rings. We show that a ring \( R \) is a \( D \)-uniquely nil clean ring if and only if \( R \) is a \( D \)-ring or \( R \) is uniquely nil clean.

Throughout, all rings are associative with an identity. We use \( Id(R) \), \( N(R) \) and \( J(R) \) to denote the sets of all idempotents, all nilpotent elements and the Jacobson radical of a ring \( R \). \( Z(R) \) and \( NZ(R) \) stand for the sets of all zero-divisors and non zero-divisors of a ring \( R \).

### 2. VERY NIL CLEAN RINGS

The aim of this is to investigate the structure of very nil clean rings. The necessary and sufficient conditions under which a group ring is very nil clean are also obtained.

**Lemma 2.1** [2, Theorem 2.28]. Let \( R \) be a ring. Then every element in \( R \) is a very idempotent if and only if \( R \) is isomorphic to one of the following:

- (a) \( \mathbb{Z}_3 \),
- (b) a Boolean ring, or
- (c) \( \mathbb{Z}_3 \oplus B \) where \( B \) is a Boolean.

**Theorem 2.2.** Let \( R \) be a ring. Then \( R \) is very nil clean if and only if

1. \( R \) is abelian;
2. \( J(R) \) is nil;
3. \( R/J(R) \) is isomorphic to one of the following:
   - (a) \( \mathbb{Z}_3 \),
   - (b) a Boolean ring, or
   - (c) \( \mathbb{Z}_3 \oplus B \) where \( B \) is a Boolean.

**Proof.** Suppose that \( R \) is very nil clean. For any idempotent \( e \in R \) and any \( a \in R \), \( e + ea(1-e) \in R \) is an idempotent. Since \( (e + ea(1-e)) + 0 = e + ea(1-e) \), by the uniqueness, \( ea(1-e) = 0 \); hence, \( ea = eae \). Likewise, \( ae = eae \), and so \( ea = ae \). Thus, \( R \) is abelian. Let \( x \in J(R) \). Then we have a very idempotent \( e \in R \) such that \( w := x - e \in N(R) \). Clearly, \( e \in R \) is central; hence, \( wx = w(w + e) = (w + e)w = wx \). This implies that
1 - e = (1 - x) + w = (1 - x)(1 + (1 - x)^{-1}w) \in U(R), and so 1 - e = 1. Thus, e = 0, and then x = w \in N(R). That is, J(R) is nil. Let a \in R. Then there exists a central very idempotent e \in R such that w := a - e \in N(R). If e^2 = e, then \( a - a^2 = w - 2ew-w^2 \in N(R). If e^2 = -e, then a + a^2 = w + 2ew + w^2 \in N(R). \) In any case, we can find some \( n \in \mathbb{N} \) such that \( a^n = a^{n+1}f(a) \) where \( f(t) \in R[t]. \) In view of Herstein’s Theorem, R is periodic, and then \( N(R) \) forms an ideal of \( R. \) Therefore, \( J(R) = N(R), \) and so every element in \( R/J(R) \) is a very idempotent. In light of Lemma 2.1, (3) is satisfied.

Conversely, assume that (1) – (3) hold. Let \( a \in R. \) Then \( \overline{a} \) is a very idempotent, in terms of Lemma 2.1. As \( J(R) \) is nil, every idempotent lifts modulo \( J(R), \) and so every very idempotent lifts modulo \( J(R). \) Thus, we can find a very idempotent \( e \in R \) such that \( \overline{e} = \overline{e}. \) Hence, \( v := a - e \in J(R) \subseteq N(R). \) If there exists a very idempotent \( f \in R \) such that \( w := a - f \in N(R), \) then \( e^2 - f^2 = (a - v)^2 - (a - w)^2 = (-av - va + v^2) + (aw + wa - w^2). \) As \( v \in J(R), \) we see that \( -av - va + v^2 \in J(R). \) Furthermore, \( aw + wa - w^2 \in N(R) \) since \( aw = wa. \) This implies that \( 1 - (e^2 - f^2) = -(-av - va + v^2) + (1 - (aw + wa - w^2)) \in U(R). \) As \( e^2, f^2 \in R \) are idempotents, we have \( (e^2 - f^2)^3 = e^2 - f^2, \) and so \( (e^2 - f^2)(1 - (e^2 - f^2)) = 0. \) Accordingly, \( e^2 = f^2, \) as asserted. \( \Box \)

**Corollary 2.3.** Let \( R \) be a local ring. Then \( R \) is very nil clean if and only if

1. \( J(R) \) is nil;
2. \( R/J(R) \) is isomorphic to \( \mathbb{Z}_2 \) or \( \mathbb{Z}_3. \)

**Proof.** Suppose that \( R \) is very nil clean. Then \( J(R) \) is nil. Since every idempotent in \( R \) is 0 or 1, \( R/J(R) \) is isomorphic to \( \mathbb{Z}_2, \) or a Boolean ring. If \( R/J(R) \) is isomorphic to a Boolean ring, then \( R/J(R) \cong \mathbb{Z}_2, \) as desired.

The converse is clear from Theorem 2.2, as \( R \) is abelian. \( \Box \)

**Corollary 2.4.** Let \( R \) be a commutative ring. Then every element in \( R \) is the sum of a very idempotent and a nilpotent element if and only if

1. \( J(R) \) is nil;
2. \( R/J(R) \) is isomorphic to one of the following:
   
   a. \( \mathbb{Z}_3, \)
   b. a Boolean ring, or
   c. \( \mathbb{Z}_3 \oplus B \) where \( B \) is a Boolean.

**Proof.** This is obvious by Theorem 2.2 \( \Box \)

Let \( R = \mathbb{Z}_{2^m} \times \mathbb{Z}_{3^n} (m, n \in \mathbb{N}). \) Then \( R \cong \mathbb{Z}_{2^m} \oplus \mathbb{Z}_{3^n}. \) But \( J(\mathbb{Z}_{2^m} \oplus \mathbb{Z}_{3^n}) = 2\mathbb{Z}_{2^m} \oplus 3\mathbb{Z}_{3^n} \) is nil, and so \( R/J(R) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_3. \) Therefore every element in \( R \) is the sum of a very idempotent and a nilpotent element, in terms of Corollary 2.4.

**Corollary 2.5.** Let \( R \) be a ring, and let \( \sigma : R \rightarrow R \) be an endomorphism. Then \( R[[x, \sigma]] \) is very nil clean if and only if

1. \( R \) is very nil clean;
2. \( \sigma(e) = e \) for all idempotents \( e \in R. \)
Suppose that $R[[x,\sigma]]$ is very nil clean. Then $R[[x,\sigma]]$ is abelian, by Theorem 2.2. For any idempotent $e \in R$, we see that $ex = xe = \sigma(e)x$, and so $e = \sigma(e)$. Let $a \in R$ be a zero-divisor. Then $a \in R[[x,\sigma]]$ is a zero-divisor. By hypothesis, there exists a unique idempotent $e(x) \in R[[x,\sigma]]$ such that $a - e(x) \in N(R[[x,\sigma]])$. Hence, we have a unique idempotent $e(0) \in R$ such that $a - e(0) \in N(R)$. Therefore, $R$ is very nil clean.

Conversely, assume that (1) and (2) hold. Let $e(x) = \sum_{i=0}^{\infty} e_i x^i \in R[[x,\sigma]]$ be an idempotent. Then

$$
e_0^2 = e_0;$$
$$e_0 e_1 + e_1 \sigma(e_0) = e_1;$$
$$e_0 e_2 + e_1 \sigma(e_1) + e_2 \sigma^2(e_0) = e_1;$$

In view of Theorem 2.2, $R$ is abelian. Hence, $(2e_0 - 1)e_1 = 0$. As $(2e_0 - 1)^2 = 1$, we get $e_1 = 0$. Likewise, $e_2 = \cdots = 0$. Hence, $e(x) = e_0 \in R$ is an idempotent. This implies that $R[[x,\sigma]]$ is abelian. By using Theorem 2.2 again, $R/J(R)$ is one of the following:

(a) $\mathbb{Z}_3$,
(b) a Boolean ring, or
(c) $\mathbb{Z}_3 \oplus B$ where $B$ is a Boolean.

Obviously,

$$R[[x,\sigma]]/J(R[[x,\sigma]]) \cong R/J(R).$$

Therefore $R[[x,\sigma]]/J(R[[x,\sigma]])$ is one of the preceding. Accordingly, we complete the proof, in terms of Theorem 2.2.

\section*{Proposition 2.6}

Let $R$ be a ring, suppose that

(1) $N(R)$ is commutative;

(2) Every element in $R$ is the sum of a very idempotent and a nilpotent element.

Then $J(R)$ is nil and $R/J(R)$ is isomorphic to one of the following:

(a) $\mathbb{Z}_3$,
(b) a Boolean ring, or
(c) $\mathbb{Z}_3 \oplus B$ where $B$ is a Boolean.

\textbf{Proof.} Let $a \in J(R)$. Then there exists an idempotent $e \in R$ such that $a + e \in N(R)$ or $a - e \in N(R)$. If $v := a + e \in N(R)$, then $1 - e = (1 + a) - v \in U(R)$, and so $e = 0$. If $w := a - e \in N(R)$, then $1 - e = (1 - a) + w \in U(R)$, and so $e = 0$. In any case, we have $a \in N(R)$. Therefore, $J(R)$ is nil.

Let $e \in R$ be an idempotent, and let $x \in N(R)$. Then $(1 - e)xe \in N(R)$, and so $x(1 - e)xe = (1 - e)xex$. Hence, $x^2 e - xexe = xex - xex$, and so $ex^2 e - xexe = 0$. Thus, $ex^2 e = (ex)^2$. Write $x^m = 0 (m \geq 1)$. By induction, we get $(exe)^2 = ex^2 e = 0$, and then $exe \in N(R)$. This implies that $xe = exe + (1 - e)xe \in N(R)$. Likewise, $e x \in N(R)$. Let $r \in R$. Then we have an idempotent $e \in R$ such that $r + e \in N(R)$ or $r - e \in N(R)$. Since $N(R)$ is commutative, we see that $rx, x r \in N(R)$. We infer that $N(R)$ is an ideal of $R$. Hence, $N(R) = J(R)$. It follows that every element in $R/J(R)$ a very idempotent. Therefore the result follows, by Lemma 2.1. \hfill $\square$
Let $P(R)$ be the intersection of all prime ideals of $R$, i.e., $P(R)$ is the prime radical of $R$. As is well known, $P(R)$ is the intersection of all minimal prime ideals of $R$.

**Theorem 2.7.** Let $R$ be a ring. Then $R$ is very nil clean if and only if

1. $R$ is abelian;
2. $R/P(R)$ is very nil clean.

**Proof.** Suppose that $R$ is very nil clean. Then $R$ is abelian. In view of Theorem 2.2, $R$ is clean, and so it is an exchange ring. Thus, $R/P(R)$ is abelian. Obviously, $J(R/P(R)) = J(R)/P(R)$ is nil. Further, $(R/P(R))/J(R/P(R)) \cong R/J(R)$. By Theorem 2.2 again, $R/P(R)$ is very nil clean.

Conversely, assume that (1) and (2) hold. For any $x \in J(R)$, we see that $\omega \in J(R/P(R))$ is nilpotent. Since $P(R)$ is nil, we see that $x \in R$ is an nilpotent; hence that $J(R)$ is nil. As $R/J(R) \cong (R/P(R))/J(R/P(R))$, it follows from Theorem 2.2 that $R$ is very clean, as asserted.  \hfill $\square$

Let $R$ be a ring, and let $G$ be a group. The augmentation ideal $I(R,G)$ of the group ring $RG$ is the kernel of the homomorphism from $RG$ to $R$ induced by collapsing $G$ to 1. That is, $I(R,G) = \ker(\omega)$, where $\omega = \{ \sum_{g \in G} r_g g \mid \sum_{g \in G} r_g = 0 \}$.

**Lemma 2.8.** Let $R$ be a ring, and let $G$ be a group. If $RG$ is very nil clean, then so is $R$.

**Proof.** Let $a \in R$. Then we have a very idempotent $e \in RG$ such that $a - e \in N(RG)$ and that such representation is unique. Hence, $a - \omega(e) = \omega(a - e) \in N(R)$. Obviously, $\omega(e) \in R$ is a very idempotent. If $a - f \in N(R)$ for a very idempotent $f \in R$, then $e^2 = f^2$, as desired. \hfill $\square$

**Theorem 2.9.** Let $R$ be a ring, and let $G$ be a group. If $I(R,G)$ is nil, then $RG$ is very nil clean if and only if so is $R$.

**Proof.** One direction is obvious by Lemma 2.8. Conversely, assume that $R$ is very nil clean. Let $x \in RG$. Then $x = \omega(x) + (x - \omega(x))$. By hypothesis, there exists a very idempotent $e \in R$ such that $w := \omega(x) - e \in N(R)$. Hence, $x = e + (w + (x - \omega(x)))$. Since $\ker(\omega)$ is nil, we see that $v := w + (x - \omega(x)) \in N(R)$. Assume that $x = f + w$ where $f \in RG$ is an very idempotent and $w \in N(RG)$. Then $f - \omega(f) \in \ker(\omega)$ is nil. As $R$ is very nil clean, $R$ is abelian. Hence, $(f - \omega(f))(1 - (f - \omega(f))^2) = 0$, and so $f = \omega(f) \in R$. It is easy to verify that $vw = (x - e)(x - f) = (x - f)(x - e) = vw$, and then $e - f = w - v \in N(R)$. It follows from $(e - f)^2 = 0$ that $e = f$, as needed. \hfill $\square$

**Corollary 2.10.** Let $R$ be a ring with a prime $p \in J(R)$, and let $G$ be a locally finite $p$-group. Then $RG$ is very nil clean if and only if $R$ is very nil clean.

**Proof.** One direction is obvious. Conversely, assume that $R$ is very nil clean. Then $J(R)$ is nil by Theorem 2.2. We first suppose $G$ is finite and prove the claim by induction on $|G|$. As the center of a nontrivial finite $p$-group contains more than one element, we may take $x \in G$ be an element in the center with the order $p$. Let $(x)$ be the subgroup of $G$ generated by $x$. Then $G = G/(x)$ has smaller order. By induction hypothesis, $\ker(\varphi(x))$ is nil, where $\varphi: R\rightarrow G$. Let $\varphi: R\rightarrow R\bar{G}, \sum_{g \in \bar{G}} r_g g \rightarrow \sum_{g \in \bar{G}} r_g \bar{g}$. Then $\ker(\varphi) = (1 - x)RG$. Since $x^p = 1$, we see that $(1 - x)^p \in p\bar{G}$; hence, $1 - x \in RG$ is nilpotent. But $\varphi(\ker(\omega)) = \ker(\varphi(x))$.
is nil. For any \( z \in \text{ker}(\omega) \), we have some \( m \in \mathbb{N} \) such that \( z^m \in \text{ker}(\varphi) \) is nilpotent. Thus, \( z \in RG \) is nilpotent. We conclude that \( \text{ker}(\omega) \) is nil, and therefore \( RG \) is very nil clean, in terms of Theorem 2.2.

\[ \square \]

3. FACTORIZATION OF ZERO-DIVISORS

A ring \( R \) is \( D \)-very nil clean provided that every zero-divisor of \( R \) is very nil clean. This section is concern on such rings. We begin with the relation of these rings with very nil clean ones.

**Theorem 3.1.** Let \( R \) be a ring. Then \( R \) is very nil clean if and only if

1. \( R \) is periodic;
2. \( R \) is \( D \)-very nil clean;
3. \( U(R) = \{x \pm 1 \mid x \in N(R)\} \).

**Proof.** Suppose that \( R \) is very nil clean. As in the proof of Theorem 2.2, \( R \) is periodic. (2) is obvious. Let \( x \in U(R) \). Then we have a very idempotent \( e \in R \) such that \( w := x - e \in N(R) \). As \( R \) is abelian, we see that \( e = x - w \) and \( ew = we \), and so \( e = \pm 1 \). Therefore \( x = w \pm 1 \), as desired.

Conversely, assume that (1) – (3) hold. Let \( a \in R \). Then we have distinct \( m, n \in \mathbb{N}(m > n) \) such that \( a^m = a^n \). If \( a \) is a zero-divisor, then \( a \) is very nil clean. If \( a \) is a non zero-divisor, \( a^m - n = 1 \). By (3), we see that \( a \) is very nil clean. This completes the proof.

\[ \square \]

**Lemma 3.2.** Every \( D \)-very nil clean ring is abelian.

**Proof.** Let \( e \in R \) be an idempotent, and let \( x \in R \). Then \( e + ex(1 - e) \in R \) is an idempotent. If \( e = 1 \), then \( ex = exe \). If \( 1 - e = ex(1 - e) \), then \( ax = exe \). If \( e \neq 1 \) and \( 1 - e \neq ex(1 - e) \), then \( e + ex(1 - e) \in R \) is a zero-divisor, as

\[ (1 - e)(e + ex(1 - e)) = 0 = (e + ex(1 - e))(1 - e - ex(1 - e)). \]

Since \( e + ex(1 - e) = e + ex(1 - e) + 0 \), by hypothesis, \( e^2 = (e + ex(1 - e))^2 \), and then \( ex(1 - e) = 0 \). That is, \( ex = exe \). Likewise, \( xe = exe \). Thus, \( exe \). This completes the proof.

\[ \square \]

We say that a ring \( R \) is a \( D \)-ring if every zero-divisor in \( R \) is nilpotent. For instance, \( \mathbb{Z}_{p^k} \) (\( p \) is prime, \( k \geq 1 \)). This concept coincides with that introduced by Abu-Khuzam and Yaqub [1] for a commutative ring.

**Theorem 3.3.** Every \( D \)-very nil clean ring is a \( D \)-ring or the product of two \( D \)-very nil clean rings.

**Proof.** Let \( R \) be a \( D \)-very nil clean ring. In view of Lemma 3.2, \( R \) is abelian.

Case I. \( R \) is indecomposable. Then every zero-divisor is nilpotent or invertible. The later is imposable, and so \( R \) is a \( D \)-ring.

Case II. \( R \) is decomposable. Write \( R = A \oplus B \). Let \( a \in A \). Then \( (a, 0) \in R \) is a zero-divisor. By hypothesis, there exists a very idempotent \( (e, e') \in R \) such that \( (a, 0) - (e, e') \in N(R) \), and that \( (a, 0) - (f, f') \in N(R) \) with a very idempotent \( (f, f') \in R \) implies that
(e, e′)² = (f, f′)². Thus, a − e ∈ N(R). If there exists a very idempotent g ∈ A such that a − g ∈ N(A). Then (a, 0) − (g, 0) ∈ N(R). This implies that (g, 0)² = (e, e′)², and so g² = e². Therefore A is very nil clean. Similarly, B is very nil clean, as asserted. □

Example 3.4. Z₃ is very nil clean, but Z₃ ⊕ Z₃ is not very nil clean.
Proof. Clearly, Z₃ is very nil clean. One easily checks that (1, 2) ∈ Z₃ ⊕ Z₃ is not very nil clean, and we are through. □

Lemma 3.5. Let R be a ring. Then every zero-divisor in R is a very idempotent if and only if R is isomorphic to one of the following:

1. a domain,
2. Z₃ ⊕ Z₃,
3. Z₃ ⊕ B where B is a Boolean, or
4. a Boolean ring.

Proof. Suppose that every zero-divisor in R is a very idempotent. By Lemma 3.2, R is abelian.

Case I. R is indecomposable. Then Id(R) = {0, 1} and −Id(R) = {0, −1}. Thus, the only zero-divisor is zero. Hence, R is a domain.

Case II. R is decomposable. Then we have S, T ≠ 0 such that R = S ⊕ T. For any t ∈ T, (0, t) ∈ R is a zero-divisor. By hypothesis, (0, t) or −(0, t) is an idempotent; hence that t or −t is an idempotent in T. Therefore every element in T is a very idempotent. In light of Lemma 2.1, T is isomorphic to one of the following:

(i) Z₃,
(ii) a Boolean ring, or
(iii) Z₃ ⊕ B where B is a Boolean.

Likewise, S is isomorphic to one of the preceding. Thus, R is isomorphic to one of the following: R is isomorphic to one of the following:

(i) Z₃ ⊕ Z₃,
(ii) a Boolean ring, or
(iii) Z₃ ⊕ B where B is a Boolean.

(iv) Z₃ ⊕ Z₃ ⊕ B where B is a Boolean.

But in Case (iv), (1, 2, 0) ∈ Z₃ ⊕ Z₃ ⊕ B is a zero-divisor, while it is not a very idempotent. Therefore Case (iv) will not appear, as desired.

Conversely, if R is a domain, then every zero-divisor is zero. If R = Z₃ ⊕ Z₃, then NZ(R) = {(1, 1), (2, 1), (2, 1)}, Id(R) = {(0, 0), (0, 1), (0, 1), (1, 1)} and −Id(R) = {(0, 0), (0, 2), (2, 0), (2, 2)}. Therefore R = NZ(R) ∪ Id(R) ∪ −Id(R). If R = Z₃ ⊕ B where B is a Boolean, then Id(R) = {(0, b), (1, b) | b ∈ B} and −Id(R) = {(0, b), (2, b) | b ∈ B}. Therefore R = Id(R) ∪ −Id(R). If R is a Boolean ring, then every element in R is an idempotent. In any case, every element in R is a very idempotent, and we are done. □

We come now to the main result of the section.

Theorem 3.6. Let R be a periodic ring. Then R is D-very nil clean if and only if
(1) $R$ is abelian;

(2) $R/J(R)$ is isomorphic to one of the following:

(a) a field $F$,

(b) $\mathbb{Z}_3 \oplus \mathbb{Z}_3$,

(c) $\mathbb{Z}_3 \oplus B$ where $B$ is Boolean, or

(d) a Boolean ring.

Proof. Suppose that $R$ is $D$-very nil clean. Then $R$ is abelian. In view of [4, Theorem ], $N(R)$ is an ideal of $R$. As $R$ is periodic, $J(R)$ is nil; hence, $J(R) = N(R)$. As every idempotent lifts modulo $N(R)$, we see that $R/J(R)$ is abelian. Let $\overline{a} \in R/N(R)$ be a zero-divisor. If $a \in R$ is not a zero-divisor, then $a \in U(R)$, and so $\overline{a} \in U(R/N(R))$, a contradiction. Thus, $a \in R$ is a zero-divisor. By hypothesis, $a$ is the sum of a very idempotent and a nilpotent. Hence, $a$ is a very idempotent. That is, every zero-divisor in $R/J(R)$ is a very idempotent.

In light of Lemma 3.5, $R/J(R)$ is isomorphic to one of the following:

(i) a domain $F$,

(ii) $\mathbb{Z}_3 \oplus \mathbb{Z}_3$,

(iii) $\mathbb{Z}_3 \oplus B$ where $B$ is Boolean, or

(iv) a Boolean ring.

If $R = F$ is a domain, then for any $a \in R$, $a = 0$ or $a^m = 1$ for some $m \in \mathbb{N}$. This shows that $R$ is a field, as required.

Conversely, assume that (1) and (2) hold. In view of [4, Theorem ], $N(R)$ forms an ideal of $R$. Let $a \in R$ be a zero divisor. Then $\overline{a} \in R/J(R)$ is a zero-divisor; otherwise, $\overline{a} \in R/J(R)$ is invertible, and so $a \in R$ is invertible, a contradiction. According to Lemma 3.5, $\overline{a}$ is a very idempotent in $R/J(R)$. As $R$ is periodic, $J(R)$ is nilpotent, and so every idempotent modulo $J(R)$. This implies that $v := a - e \in N(R)$ for some very idempotent $e \in R$. Let $f \in R$ be a very idempotent such that $w := a - f \in N(R)$. Then $e^2 - f^2 = (a - v)^2 - (a - w)^2 = -av - va + v^2 + aw + wa - w^2 \in N(R)$. As $e, f \in R$ are very clean, we see that $e^2, f^2 \in R$ are idempotents. It is easy to verify that $(e^2 - f^2)(1 - (e^2 - f^2)^2) = 0$, and so $e^2 = f^2$. Therefore we complete the proof. \[ \square \]

Let $R$ be a ring, and let $\sigma : R \to R$ be an endomorphism. $T(R, \sigma) = \{ \begin{pmatrix} a & b \\ a \\ \end{pmatrix} \mid a, b \in R \}$. Here,

$$\begin{pmatrix} a & b \\ a \\ \end{pmatrix} + \begin{pmatrix} a' & b' \\ a' \\ \end{pmatrix} = \begin{pmatrix} a + a' & b + b' \\ a + a' \\ \end{pmatrix};$$

$$\begin{pmatrix} a & b \\ a \\ \end{pmatrix} \begin{pmatrix} a' & b' \\ a' \\ \end{pmatrix} = \begin{pmatrix} aa' + ab' + b\sigma(a') \\ aa' \\ \end{pmatrix}.$$

Corollary 3.7. Let $R$ be a periodic ring, and let $\sigma : R \to R$ be an endomorphism. Then $T(R, \sigma)$ is $D$-very nil clean if and only if

(1) $R$ is $D$-very nil clean;

(2) $\sigma(e) = e$ for all idempotents $e \in R$.
Proof. Obviously, $T(R, \sigma)$ is periodic. Suppose that $T(R, \sigma)$ is $D$-very nil clean. Then $T(R, \sigma)$ is abelian, by Theorem 3.6. Let $e \in R$ be an idempotent, and let $x \in R$. Then

$$
\begin{pmatrix}
e & e \\e & e
\end{pmatrix}
\begin{pmatrix}0 & 1 \\0 & 0
\end{pmatrix}
= 
\begin{pmatrix}0 & 1 \\0 & 0
\end{pmatrix}
\begin{pmatrix}e & e \\e & e
\end{pmatrix},
$$

and so $e = \sigma(e)$. Let $a \in R$ be a zero divisor. Then $\begin{pmatrix}a & a \\a & a
\end{pmatrix} \in T(R, \sigma)$ is a zero-divisor.

By hypothesis, there exists a very idempotent $\begin{pmatrix}e & g \\e & e
\end{pmatrix} \in T(R, \sigma)$ such that

$$
\begin{pmatrix}a & a \\a & a
\end{pmatrix} - \begin{pmatrix}e & g \\e & e
\end{pmatrix} \in N(T(R, \sigma)).
$$

It follows that we have a very idempotent $e \in R$ such that $a - e \in N(R)$. If $a - f \in N(R)$ with a very idempotent $f \in R$, then we have a very idempotent $\begin{pmatrix}f & 0 \\f & f
\end{pmatrix} \in T(R, \sigma)$ such that

$$
\begin{pmatrix}a & a \\a & a
\end{pmatrix} - \begin{pmatrix}f & 0 \\f & f
\end{pmatrix} \in N(T(R, \sigma)).
$$

Thus, we get $\begin{pmatrix}e & g \\e & e
\end{pmatrix}^2 = \begin{pmatrix}f & 0 \\f & f
\end{pmatrix}^2$. It follows that $e^2 = f^2$, and therefore $R$ is $D$-very nil clean.

Conversely, assume that (1) and (2) hold. Then $T(R, \sigma)$ is abelian, as in the preceding discussion. By using Theorem 3.6, $R/J(R)$ is isomorphic to one of the following:

(a) a field $F$,
(b) $\mathbb{Z}_3 \oplus \mathbb{Z}_3$,
(c) $\mathbb{Z}_3 \oplus B$ where $B$ is Boolean, or
(d) a Boolean ring.

But $T(R, \sigma)/J(T(R, \sigma)) \cong R/J(R)$, we see that $T(R, \sigma)/J(T(R, \sigma))$ is isomorphic to one of the preceding. Consequently, $T(R, \sigma)$ is $D$-very nil clean, in terms of Theorem 3.6. □

4. SPECIAL CASES

The purpose of this section is to explore the structure of rings in which every zero-divisor is a very idempotent or a nilpotent element. These form a subset of all $D$-very nil clean rings.

**Lemma 4.1.** Every ring in which every element is a very idempotent or a nilpotent element is abelian.

**Proof.** Let $e \in R$ be an idempotent, and let $x \in R$. Then $1 - ex(1 - e) \in U(R)$. If $(1 - ex(1 - e))^2 = 1 - ex(1 - e)$, then $ex(1 - e) = 0$, and so $ex = exe$. If $(1 - ex(1 - e))^2 = -(1 - ex(1 - e))$, then $ex(1 - e) = 2$, and so $ex(1 - e) = 2e(1 - e) = 0$. Hence, $ex = exe$. 
Huanyin Chen

If \(1 - ex(1 - e) \in N(R)\), this will be a contradiction. Thus, \(ex = exe\). Likewise, \(xe = exe\). Therefore \(ex = xe\), hence the result. \(\square\)

**Lemma 4.2.** Let \(R\) be a ring. Then \(R = N(R) \cup Id(R) \cup -Id(R)\) if and only if \(R\) is isomorphic to one of the following:

(a) \(\mathbb{Z}_3, \mathbb{Z}_4\),

(b) a Boolean ring, or

(c) \(\mathbb{Z}_3 \oplus B\) where \(B\) is a Boolean.

**Proof.** Suppose that \(R = N(R) \cup Id(R) \cup -Id(R)\). In view of Lemma 4.1, \(R\) is abelian.

Case I. \(R\) is indecomposable. Then \(R = N(R) \cup \{0, 1, -1\}\). Let \(a \in N(R)\). Then \(1 - a \in U(R)\). If \((1 - a)^2 = 1 - a\), then \(a = 0\). If \((1 - a)^2 = -(1 - a)\), then \(a = 2\). If \(1 - a \in N(R)\), then we get a contradiction. Therefore \(R = \{0, 1, -1, 2\}\), and so \(R \cong \mathbb{Z}_2, \mathbb{Z}_3\) or \(\mathbb{Z}_4\).

Case II. \(R\) is decomposable. Write \(R = S \oplus T\). For any \(t \in T\), \((1, t) \in R\) is not nilpotent. Then \((1, t) \in R\) is a very idempotent. This implies that \(t \in T\) is a very idempotent. Hence, every element in \(T\) is a very idempotent. Likewise, every element in \(S\) is a very idempotent. In view of Lemma 2.1, \(S\) and \(T\) are isomorphic to one of the following:

(i) \(\mathbb{Z}_3\),

(ii) a Boolean ring, or

(iii) \(\mathbb{Z}_3 \oplus B\) where \(B\) is a Boolean.

One easily checks that \((1, 2) \in \mathbb{Z}_3 \oplus \mathbb{Z}_3\) and \((1, 2, 0) \in \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus B\) where \(B\) is Boolean are neither a very idempotent nor a nilpotent. Therefore \(R\) is isomorphic to one of \((a) - (c)\). The converse is obvious. \(\square\)

**Theorem 4.3.** Let \(R\) be a ring. Then \(R\) is an abelian ring in which every zero-divisor in \(R\) is a very idempotent or a nilpotent element if and only if \(R\) is isomorphic to one of the following:

(a) a \(D\)-ring,

(b) a Boolean ring,

(c) \(\mathbb{Z}_3 \oplus \mathbb{Z}_4\),

(d) \(\mathbb{Z}_3 \oplus B\) where \(B\) is a Boolean.

**Proof.** Suppose that \(R\) is an abelian ring in which every zero-divisor in \(R\) is a very idempotent or a nilpotent element.

Case I. \(R\) is indecomposable. Then every very idempotent is 0, 1 or \(-1\). Hence, every zero-divisor in \(R\) is nilpotent. Hence, \(R\) is a \(D\)-ring.

Case II. \(R\) is decomposable. Write \(R = S \oplus T\). For any \(t \in T\), \((0, t)\) is a very idempotent or a nilpotent element. We infer that every element in \(T\) is a very idempotent or a nilpotent element. Similarly, every element in \(S\) is a very idempotent or a nilpotent element. By virtue of Lemma 4.2, \(S\) and \(T\) are both isomorphic to one of the following:

(a) \(\mathbb{Z}_3, \mathbb{Z}_4\),

(b) a Boolean ring, or
(c) \( \mathbb{Z}_3 \oplus B \) where \( B \) is a Boolean.

But one easily checks that \( Z(R) \neq \text{Id}(R) \cup -\text{Id}(R) \cup N(R) \) for any of those types

(1) \( \mathbb{Z}_3 \oplus \mathbb{Z}_4 \),
(2) \( \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus B \) where \( B \) is a Boolean ring,
(3) \( \mathbb{Z}_4 \oplus \mathbb{Z}_4 \),
(4) \( \mathbb{Z}_4 \oplus B \) where \( B \) is a Boolean ring, and
(5) \( \mathbb{Z}_3 \oplus \mathbb{Z}_4 \oplus B \) where \( B \) is a Boolean ring.

Therefore \( R \) is isomorphic to one of (a) – (d).

Conversely, \( R \) is abelian, as every \( D \)-ring is connected. One easily checks that any of these four types of rings satisfy the desired condition.

\[ \blacksquare \]

**Corollary 4.4.** Let \( R \) be a ring. Then the following are equivalent:

(1) \( R \) is an abelian ring in which every zero-divisor in \( R \) is an idempotent or a nilpotent element.
(2) \( R \) is a \( D \)-ring or a Boolean ring.

**Proof.** (1) \( \Rightarrow \) (2) In view of Theorem 4.3, \( R \) is isomorphic to one of the following:

(a) a \( D \)-ring,
(b) a Boolean ring,
(c) \( \mathbb{Z}_3 \oplus \mathbb{Z}_4 \),
(d) \( \mathbb{Z}_3 \oplus B \) where \( B \) is a Boolean.

But in the case \( \mathbb{Z}_3 \oplus \mathbb{Z}_3 \), \( (0, 2) \notin \text{Id}(R) \cup N(R) \). In the case \( \mathbb{Z}_3 \oplus B \), \( (2, 0) \notin \text{Id}(R) \cup N(R) \). Therefore proving (2).

(2) \( \Rightarrow \) (1) This is obvious. \[ \blacksquare \]

The following example shows that the abelian condition in Corollary 4.4 is necessary.

**Example 4.5.** Every zero-divisor in \( T_2(\mathbb{Z}_2) \) is an idempotent or a nilpotent element. But \( T_2(\mathbb{Z}_2) \) is neither a Boolean ring nor a \( D \)-ring. In this case, \( R \) is not abelian.

**Proposition 4.6.** Let \( R \) be a ring in which every element in \( R \) is a very idempotent or a nilpotent element. Then every zero-divisor in \( R \) which is not 2 is a very idempotent.

**Proof.** Let \( 2 \neq a \in R \). Assume that \( a \in R \) is nilpotent. Then \( 1-a \in U(R) \). If \( (1-a)^2 = 1-a \), then \( a = 0 \). If \( (1-a)^2 = -(1-a) \), then \( a = 2 \), a contradiction. If \( 1-a \in N(R) \), then \( 1-a = 0 \), an absurd. Therefore \( a \in R \) is a very idempotent, as asserted. \[ \blacksquare \]

5. **UNIQUELY NIL CLEAN RINGS**
A ring $R$ is said to be uniquely nil clean provided that for any $a \in R$ there exists a unique idempotent $e \in R$ such that $a - e \in R$ is nilpotent. Recently, this type of nil clean rings was studied in [9]. We shall give the connection of uniquely nil clean rings and very nil clean rings, and then obtain the conditions under which a group ring is uniquely nil clean. We add several new characterizations of such rings.

**Theorem 5.1.** Let $R$ be a ring. Then $R$ is uniquely nil clean if and only if

1. $R$ is abelian.

2. $R/J(R)$ is Boolean and $J(R)$ is nil.

**Proof.** Assume that $R$ is uniquely nil clean. As in the proof of Theorem 2.2, $R$ is abelian, and so $R$ is strongly nil clean. For any $a \in R$, it follows that $a - a^2 \in N(R)$. Write $(a - a^2)^m = 0$ for some $m \in \mathbb{N}$. Then $a^m = a^{m+1}b$ for a $b \in R$, and so $R$ is strongly $\pi$-regular. If $x \in J(R)$, we have some $n \in \mathbb{N}$ such that $x^n = x^{n+1}y$ for a $y \in R$; hence, $x^n(1 - xy) = 0$. Clearly, $1 - xy \in U(R)$, and so $x^n = 0$. This implies that $J(R)$ is nil. As $R$ is abelian, it follows by [4, Theorem] that $N(R)$ forms an ideal of $R$. Thus, $N(R) \subseteq J(R)$, and therefore $R/J(R)$ is Boolean.

Assume that (1) and (2) hold. For any $a \in R$, $a - a^2 \in J(R)$, and so we have an idempotent $e \in R$ such that $a - e \in J(R)$, as $J(R)$ is nil. Write $a = e + v$. Then $v \in J(R) \subseteq N(R)$. If there exists an idempotent $f \in R$ and a $w \in N(R)$ such that $a = f + w$, then $e - f = (a - w) - (a - w) = w - v$. Clearly, $wv = (a - f)(a - e) = (a - e)(a - f) = vw$, and so $e - f \in N(R)$. Since $(e - f)^3 = e - f$, we see that $e - f = 0$, and then $e = f$. Therefore $R$ is uniquely nil clean. □

**Corollary 5.2.** Let $R$ be a ring. Then $R$ is uniquely nil clean if and only if $R$ is uniquely clean, and $J(R)$ is nil.

**Proof.** This is obvious from Theorem 5.1. □

**Corollary 5.3.** A ring $R$ is uniquely nil clean if and only if

1. $R$ is $\pi$-regular;

2. Every idempotent in $R$ is central;

3. $R/J(R)$ is Boolean.

**Proof.** Suppose that $R$ is uniquely nil clean. Then $R$ is strongly nil clean, and so $R$ is $\pi$-regular. Furthermore, proving (2) and (3) by Theorem 5.1.

Conversely, assume that (1), (2) and (3) hold. Clearly, $R$ is an exchange ring, and so every idempotent lifts modulo $J(R)$. Let $x \in J(R)$. Since $R$ is $\pi$-regular, we have $n \in \mathbb{N}$ and $y \in R$ such that $x^n = x^n y x^n$; hence, $x^n(1 - y x^n) = 0$. This implies that $x^n = 0$. That is, $J(R)$ is nil. Therefore we complete the proof, thanks to Theorem 5.1. □

We note that $\mathbb{Z}/4\mathbb{Z}$ is uniquely nil clean and $\mathbb{Z}/6\mathbb{Z}$ is not uniquely nil clean, though they are both $\pi$-regular rings with all idempotents central.

**Theorem 5.4.** Let $R$ be a ring. Then $R$ is uniquely nil clean if and only if

1. $R$ is abelian.

2. $R$ is strongly nil clean.
Proof. One direction is obvious by Theorem 5.1.

Conversely, assume that (1) and (2) hold. For any \( a \in R \), there exists an idempotent \( e \in R \) and a \( w \in N(R) \) such that \( a = e + w \). Write \( a = f + v, f = f^2 \in R, v \in N(R) \). In light of Theorem 5.1, \( N(R) \) forms an ideal of \( R \), and so \( e - f = (a - w) - (a - v) = v - w \in N(R) \). As \( R \) is abelian, \((e - f)^2 = e - f\); hence, \( e = f \), as desired. \( \square \)

Corollary 5.5. Every homomorphic image of a uniquely nil clean ring is uniquely nil clean.

Proof. Let \( R \) be uniquely nil clean, and let \( I \) be an ideal of \( R \). In light of Theorem 5.4, \( R \) is abelian strongly nil clean. This shows that \( R/I \) is strongly nil clean. Clearly, \( R \) is strongly clean. We infer that every idempotent lifts modulo \( I \), and then \( R/I \) is abelian. By Theorem 5.2 again, \( R/I \) is uniquely nil clean, as required. \( \square \)

Corollary 5.6. Let \( R \) be a ring. Then the following are equivalent:

1. \( R \) is uniquely nil clean.
2. \( T = \{ (a_{ij}) \in T_n(R) \mid a_{11} = \cdots = a_{nn} \} \) is uniquely nil clean.
3. \( R[x]/(x^n) \) is uniquely nil clean for all \( n \geq 2 \).

Proof. This is obvious by Theorem 5.4. \( \square \)

Theorem 5.7. Let \( R \) be a ring. Then \( R \) is uniquely nil clean if and only if

1. \( 2 \in R \) is nilpotent;
2. \( R \) is very nil clean.

Proof. Suppose that \( R \) is uniquely nil clean. In view of Theorem 5.1, \( 2^2 = 2 \) in \( R/J(R) \), and so \( 2 \in J(R) \) is nilpotent. By Theorem 5.1 and Theorem 2.2, we observe that every uniquely nil clean ring is very nil clean.

Conversely, assume that (1) and (2) hold. As \( 2 \in \mathbb{Z}_3 \) is not nilpotent. In view of Theorem 2.2, we see that \( R \) is abelian, \( J(R) \) is nil, and that \( R/J(R) \) is Boolean. The result follows by Theorem 5.1. \( \square \)

Corollary 5.8. Let \( R \) be a ring. Then \( R \) is uniquely nil clean if and only if

1. \( R \) is abelian;
2. \( R/P(R) \) is uniquely nil clean.

Proof. One direction is obvious, by Theorem 2.7 and Corollary 5.7.

Conversely, assume that (1) and (2) hold. By virtue of Theorem 5.7, \( 2 \in R/P(R) \) is nilpotent. We infer that \( 2 \in R \) is nilpotent. Furthermore, \( R/P(R) \) is very nil clean. According to Theorem 2.7, \( R \) is very nil clean. By using Theorem 5.7 again, \( R \) is uniquely nil clean. \( \square \)

Corollary 5.9. Let \( R \) be a ring, and \( G \) be a group. Then \( RG \) is uniquely nil clean if and only if \( R \) is uniquely nil clean and \( I(R, G) \) is nil.

Proof. Suppose \( RG \) is uniquely nil clean. Then \( RG \) is very nil clean and \( 2 \in N(RG) \), by Theorem 5.7. Hence, \( R \) is very nil clean and \( 2 \in N(R) \). By using Theorem 5.7 again, \( R \) is uniquely nil clean. On the other hand, \( RG \) is uniquely clean, thanks to Corollary 5.2. Hence, \( RG/J(RG) \) is Boolean. For any \( g \in G \), we see that \( (1 - g) - (1 - g)^2 \in J(RG) \); hence, \( 1 - g \in J(RG) \). This implies that \( ker(\omega) \subseteq J(RG) \) is nil, as desired.
Conversely, assume that $R$ is uniquely nil clean and $\ker(\omega)$ is nil. In light of Theorem 5.7 and Theorem 2.9, $2 \in N(R)$ and $RG$ is very nil clean. By using Theorem 5.7 again, $RG$ is uniquely nil clean, as asserted. □

**Example 5.10.** Let $G$ be a 3-group. Then $\mathbb{Z}_3G$ is is not uniquely nil clean, while it is very nil clean.

### 6. D-UNIQUELY NIL RINGS

A ring $R$ is said to be $D$-uniquely nil clean provided that for any zero-divisor $a \in R$ there exists a unique idempotent $e \in R$ such that $a - e \in R$ is nilpotent. The aim of this is to give the connection of $D$-uniquely nil clean rings and uniquely nil clean rings, and then characterize the structure of $D$-uniquely nil clean periodic rings.

**Lemma 6.1.** Every $D$-uniquely nil clean ring is abelian.

*Proof.* This is similar to that in Lemma 3.2. □

**Theorem 6.2.** Let $R$ be a ring. Then $R$ is a $D$-uniquely nil clean ring if and only if $R$ is a $D$-ring or $R$ is uniquely nil clean.

*Proof.* Suppose $R$ is a $D$-uniquely nil clean ring. Then $R$ is abelian by Lemma 6.1.

Case I. $R$ is indecomposable. Let $a \in R$ be a zero-divisor. Then $a \in R$ is nilpotent or $a \in U(R)$. This shows that every zero-divisor is nilpotent, i.e., $R$ is a $D$-ring.

Case II. $R$ is decomposable. Write $R = A \oplus B$. For any $x \in A$, $(x, 0) \in R$ is a zero-divisor. Hence, we can find a unique idempotent $(e, f) \in R$ such that $(x, 0) - (e, f) \in N(R)$. Thus, $x - e \in N(R)$ for an idempotent $e \in R$. If there exists an idempotent $g \in R$ such that $x - g \in N(R)$, then $(x, 0) - (g, f) \in N(R)$. By the uniqueness, we get $g = e$, and therefore $A$ is uniquely nil clean. Similarly, $B$ is uniquely nil clean, and then $R$ is uniquely nil clean.

Conversely, if $R$ is a $D$-ring, then $R$ is a $D$-uniquely nil clean ring. So we assume that $R$ is uniquely nil clean, and therefore $R$ is a $D$-uniquely nil clean ring. □

**Corollary 6.3.** Let $R$ be $D$-uniquely nil clean. Then the ring $T(R, R) = \{ \left( \begin{array}{cc} a & b \\ a & \end{array} \right) | a, b \in R \}$ is $D$-uniquely nil clean.

*Proof.* By virtue of Theorem 6.2, $R$ is a $D$-ring or $R$ is uniquely nil clean.

Case I. $R$ is a $D$-ring. Let $\left( \begin{array}{cc} a & x \\ a & \end{array} \right) \in Z(T(R, R))$. Then $0 \neq \left( \begin{array}{cc} b & y \\ b & \end{array} \right), \left( \begin{array}{cc} c & z \\ c & \end{array} \right) \in T(R, R)$ such that

$$\left( \begin{array}{cc} a & x \\ a & \end{array} \right) \left( \begin{array}{cc} b & y \\ b & \end{array} \right) = \left( \begin{array}{cc} c & z \\ c & \end{array} \right) \left( \begin{array}{cc} a & x \\ a & \end{array} \right) = 0.$$ 

If $b, c \neq 0$, it follows from $ab = 0 = ca$ that $a \in R$ is a zero-divisor. If $b = c = 0$, then $y, z \neq 0$. It follows from $ay = 0 = za$ that $a \in R$ is a zero-divisor. If $b = 0, c \neq 0$ or $b \neq 0, c = 0$, then $ca = 0 = ay$ or $ab = 0 = za$. Then $a \in R$ is a zero-divisor. Hence, $a \in R$ is nilpotent. Therefore we conclude that $\left( \begin{array}{cc} a & x \\ a & \end{array} \right)$ is nilpotent, and so $T(R, R)$ is a $D$-ring.

Case II. $R$ is uniquely nil clean. In light of Corollary 5.6, $T(R, R)$ is uniquely nil clean.
Therefore $T(R, R)$ is $D$-uniquely nil clean, in terms of Theorem 6.2.

Proposition 6.4. A ring $R$ is $D$-uniquely nil clean if and only if for any zero-divisor $a \in R$ there exists a central idempotent $e \in R$ such that $a - e \in N(R)$.

Proof. One direction is obvious from Lemma 6.1. Conversely, letting $e \in R$ be an idempotent, we have a central idempotent $f \in R$ such that $w := e - f \in N(R)$. Thus, $(e - f)^3 = e - f$, and so $(e - f)(1 - (e - f)^2) = 0$. This implies that $e = f$, and then $R$ is abelian. Let $a \in R$ be a zero-divisor. Then there exists a central $e \in R$ such that $a - e \in N(R)$. If there exists an idempotent $f \in R$ such that $a - f \in N(R)$, then $e - f = (a - f) - (a - e) \in N(R)$. It follows from $(e - f)^3 = e - f$ that $e = f$, which completes the proof.

Theorem 6.5. Let $R$ be a periodic ring. Then $R$ is $D$-uniquely nil clean if and only if

1. $R$ is abelian;
2. $R$ is local or $R/J(R)$ is Boolean.

Proof. Suppose that $R$ is $D$-uniquely nil clean. Then $R$ is abelian by Lemma 6.1.

In view of [4, Theorem ], $N(R)$ forms an ideal of $R$. Hence, $J(R) = N(R)$. Let $f \in R/J(R)$ is a zero-divisor. Then $a \in R$ is a divisor; otherwise, $a \in U(R)$ as $R$ is periodic, a contradiction. Hence, $a$ is the sum of an idempotent and a nilpotent element. This shows that $a$ is an idempotent. Therefore, every zero-divisor in $R/J(R)$ is an idempotent.

Set $S = R/J(R)$. Suppose that $S$ has a nonzero zero-divisor. Then we have some $x, y \in R$ such that $xy = 0, x, y \neq 0$. Hence, $(yx)^2 = 0$. If $yx \neq 0$, then $yx \in R$ is a zero-divisor. So $yx \in R$ is an idempotent. Thus, $yx = (yx)^2 = 0$. This implies that $x \in R$ is a zero-divisor, and so $x = x^2$. It follows that $1 - x \in R$ is a zero-divisor; hence that $1 - x = (1 - x)^2$. Therefore $x^2 = x$.

Let $a \in R$. Then $(xa(1 - x))^2 = 0$. Hence, $xa(1 - x) = 0$; otherwise, $xa(1 - x) \in R$ is an idempotent, and so $xa(1 - x) = 0$, a contradiction. Thus, $xa(1 - x) = 0$, hence, $xa = xax$. Likewise, $ax = xax$. Thus, $xa = ax$. If $xa = 0$, then $a \in R$ is a zero-divisor, and so it is an idempotent. If $xa \neq 0$, it follows from $xa(1 - x) = 0$ that $xa \in R$ is a zero-divisor, and so $xa = (xa)^2$. Hence, $xa(1 - a) = 0$. This implies that $1 - a \in R$ is a zero-divisor, and then $1 - a = (1 - a)^2$. Thus, $a = a^2$. Therefore $a \in R$ is an idempotent. Consequently, $R/J(R)$ is Boolean or $R/J(R)$ is a domain. If $R/J(R)$ is a domain, the periodic property implies that $R$ is a field. Thus, $R$ is local or $R/J(R)$ is Boolean.

Conversely, assume that (1) and (2) hold. Let $a \in R$ be a zero-divisor. If $R$ is local, then $a \in J(R)$. As $R$ is local, we see that $J(R)$ is nil; hence, $a = 0 + a$ is the sum of an idempotent and a nilpotent. If $a = e + w$ with an idempotent $e \in R$ and a nilpotent $w \in R$, then $e = a - w$ with $aw = (e + w)w = w(e + w) = wa$, as $R$ is abelian. This shows that $e \in R$ is nilpotent. Hence, $e = 0$. Thus, there exists a unique idempotent $e \in R$ such that $a - e \in N(R)$. If $R/J(R)$ is Boolean, we can find an idempotent $e \in R$ such that $a - e \in N(R)$, as $J(R)$ is nil. Since $R$ is abelian, we see that such idempotent $e$ is unique. Therefore $R$ is $D$-uniquely nil clean.

Corollary 6.6. Let $R$ be a periodic ring, and let $\sigma : R \to R$ be an endomorphism. Then $T(R, \sigma)$ is $D$-uniquely nil clean if and only if

1. $R$ is $D$-uniquely nil clean;
2. $\sigma(e) = e$ for all idempotents $e \in R$. 

Proof. As $R$ is a periodic ring, then so is $T(R, \sigma)$. Suppose that $T(R, \sigma)$ is $D$-uniquely nil clean. Then $T(R, \sigma)$ is abelian, by Lemma 6.1. Thus, $e = \sigma(e)$ for any idempotent $e \in R$, as in the proof of Corollary 3.7. Let $a \in R$ be a zero divisor. Then $\begin{pmatrix} a & e \\ e & a \end{pmatrix} \in T(R, \sigma)$ is a zero-divisor. By hypothesis, there exists a unique idempotent $\begin{pmatrix} e & f \\ e & f \end{pmatrix} \in T(R, \sigma)$ such that $\begin{pmatrix} a & e \\ e & a \end{pmatrix} = \begin{pmatrix} e & f \\ e & f \end{pmatrix} \in N(T(R, \sigma)).$

It follows that we have a unique idempotent $e \in R$ such that $a - e \in N(R)$. Therefore, $R$ is $D$-uniquely nil clean.

Conversely, assume that (1) and (2) hold. Let $\begin{pmatrix} e & f \\ e & f \end{pmatrix} \in T(R, \sigma)$ be an idempotent.

Then $e = e^2$ and $ef + f\sigma(e) = f$. As $R$ is $D$-uniquely nil clean, we see that $R$ is abelian. Hence, $(2e - 1)f = 0$. It follows from $(2e - 1)^2 = 1$ that $f = 0$. This implies that $T(R, \sigma)$ is abelian. In view of Theorem 6.5, $R/J(R)$ is a division ring or a Boolean ring. Since $T(R, \sigma)/J(T(R, \sigma)) \cong R/J(R)$, we see that $T(R, \sigma)$ is local, or $T(R, \sigma)/J(T(R, \sigma))$ is Boolean. Therefore $T(R, \sigma)$ is $D$-uniquely nil clean, in terms of Theorem 6.5. \hfill \Box

REFERENCES

[1] H. Abu-Khuzam and A. Yaqub, Structure of rings with certain conditions on zero-divisors, *Int. J. Math. Math. Sci.*, **15**(2006), 1–6.
[2] M.S. Ahn, Weakly Clean Rings and Almost Clean rings, Ph.D. Thesis, The University of Iowa, Iowa, 2003.
[3] D. Andrica and G. Călugăreanu, A nil-clean $2 \times 2$ matrix over the integers which is not clean, *J. Algebra Appl.*, **13**(2014), 1450009 [9 pages] DOI: 10.1142/S0219498814500091.
[4] A. Badawi, On abelian $\pi$–regular rings, *Comm. Algebra*, **25**(1997), 1009–1021.
[5] H. Chen, On uniquely clean rings, *Comm. Algebra*, **39**(2011), 189–198.
[6] H. Chen, Strongly nil clean matrices over $R[x]/(x^2 - 1)$, *Bull. Korean Math. Soc.*, **49**(2012), 589–599.
[7] H. Chen, On strongly nil clean matrices, *Comm. Algebra*, **41**(2013), 1074–1086.
[8] A.J. Diesl, Nil clean rings, *J. Algebra*, **383**(2013), 197–211.
[9] V.A. Hiremath and S. Hegde, Using ideals to provide a unified approach to uniquely clean rings, *J. Aust. Math. Soc.*, **96**(2014), 258–274.
[10] M.T. Kosan; T.K. Lee and Y. Zhou, When is every matrix over a division ring a sum of an idempotent and a nilpotent? *Linear Algebra Appl.*, **450**(2014), 7–12.
[11] A. Stancu, On some constrictions of nil-clean, clean, and exchange rings, arXiv: 1404.2662v1 [math.RA], 10 Apr 2014.