Categories of modules and their deformations

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Abstract

We develop an obstruction theory for lifting compact objects to the stable \( \infty \) category of quasi-coherent modules over a derived geometric stack \( X \) from the category of modules over its underlying classical stack \( X^{cl} \). The obstructions live in Andre-Quillen cohomology.

1 Introduction

The derived category of quasi-coherent modules over a scheme or an algebraic stack is usually very badly behaved in the sense that it is not controlled by a small data. In certain cases it is possible to find a set of compact generators of the derived category of modules in question. For example, if \( R \) is a commutative ring then the triangulated category \( D(R) \) of chain complexes of \( R \)-modules modulo weak equivalences of chain cohomology is compactly generated. Similar thing is true of the unbounded derived category of quasi coherent modules over a quasi-compact separated scheme [4]. In general not all algebraic stacks have this property. Stable homotopy theory gives rises to more sources of interesting triangulated categories. For any \( E_{\infty} \)-ring spectrum \( A \) the derived category of \( A \)-modules is compactly generated. For any derived scheme, formed by gluing derived affine schemes \( \text{Spec}(A) \) along Zariski maps of \( E_{\infty} \)-rings, the derived category of quasi-coherent modules form a compactly generated triangulated category [1]. We are interested in the triangulated category of the derived category of quasi-coherent modules over any general derived algebraic stack. Throughout this paper we think of derived algebraic stacks, once rigidified, as being equivalent to cosimplicial connective \( E_{\infty} \)-rings.

Given a derived \( \infty \)-stack \( X \), we want to study the stable \( \infty \)-category of quasi-coherent modules over \( X \). If \( X \) is an algebraic stack, i.e. \( X \) admits an atlas by simplicial derived affine scheme \( U_{\bullet} \), we get a cosimplicial stable \( \infty \)-category \( \text{Mod}(U_{\bullet}) \). The stable \( \infty \)-category modules over the stack \( X \) is the totalization \( \text{Tot}(\text{Mod}(U_{\bullet})) \).

Let \( \mathcal{C}_{\infty} \) be the category of connective \( E_{\infty} \) rings. Another way to approach this is to consider the stack \( QC \) considered as a moduli functor

\[ QC : \mathcal{C}_{\infty} \to \text{Pr}_{s_f-\infty} \]

where the right side is the \( \infty \)-category of presentable stable \( \infty \)-categories, so that \( QC(A) = \text{Mod}(A) \) and \( QC \) takes a map of modules \( f : A \to B \) to the functor \( - \otimes_A B \). This naturally extends to a functor between \( \infty \)-categories.
The desired object, i.e. the $\infty$-category of quasi-coherent modules over any $\infty$-stack $X : \mathcal{C}_\infty \to SSet$ is $\text{Hom}_{\infty\text{-stacks}}(X, QC)$, the hom space in the $\infty$-topos of $\infty$-stacks.

If $A$ is a connective $E_\infty$ ring which admits a postnikov tower decomposition and $\mathcal{M}$ an $\infty$-stack which admits a cotangent complex and is infinitesimally cohesive [6], lifting a family of objects classified by $\mathcal{M}$ on the ordinary affine scheme $\text{Spec } \pi_0 A$ to the derived affine scheme $\text{Spec } A$ is a problem in deformation theory. It is controlled by the cotangent complex of the stack $\mathcal{M}$. Associated to any derived algebraic $\infty$-stack $X$ there is an ordinary algebraic $\infty$-stack $X^{cl}$ which admits an atlas of a cosimplicial ordinary ring obtained by taking sectionwise $\pi_0$ of the the atlas of $X$. We can think of $X$ as an infinitesimal extension of the underlying $X^{cl}$.

Let $X$ be a derived $\infty$-stack. Let $X^{cl}$ be it’s associated classical (non-derived stack). There is a natural map $i : X^{cl} \to X$. The induced map on derived categories:

$$D_{qc}(X) \to D_{qc}(X^{cl})$$

Given arbitrary $x$ in $D_{qc}(X)$ and a perfect $u$ in $D_{qc}(X^{cl})$, with a map $u \to x$ we would like to find cohomological obstructions for lifting $u$ to a perfect module $\bar{u}$ over $X$ and a map $\bar{u} \to x$ over $X$ which restricts to $u \to x$ over $X^{cl}$

The main result is

**Theorem 1.1.** Let $X$ be a perfect derived algebraic $n$-stack for some $n$ and let $\bar{X}$ be a square-zero extension of $X$. Let $x : \bar{X} \to QC$ be a complex of quasi-coherent modules over $\bar{X}$ and let $u : X \to QC^{\omega}$ be a complex of perfect modules over $X$, along with a map $u \to x$ in $QC(X)$.

- Then there exists an obstruction theory for deforming $u$ to a $\bar{u} : X \to QC^{\omega}$. The space of deformations is isomorphic to $\Omega \text{Hom}_{\mathcal{O}_X}(\alpha^* LQC^{\omega}, N)$ with loops based at the trivial derivation.

- If this space in non-empty and $\bar{u}$ is a deformation of $u$, then there exists a perfect module $y_\beta : X \to QC^{\omega}$ along with maps $\beta : u \to y_\beta$ and $y_\beta \to x$ in $QC(X)$ such that the triangle commutes in $QC(X)$

$$\begin{array}{ccc}
\bar{u} & \to & x \\
\downarrow \beta & & \downarrow \\
\bar{y}_\beta & \to & x
\end{array}$$

There is an obstruction theory for lifting $\beta$ to $\bar{\beta} : \bar{u} \to \bar{y}_\beta$ such that $\bar{u} \to \bar{y}_\beta \to x$ is a deformation of $\alpha : u \to x$.

More precisely, there exists a moduli functor $G : \Omega_{u,y_\beta} QC_{X \times X}$ and an cocycle in the Andre-Quillen cohomology

$$\alpha(u, y_\beta) \in \text{Hom}_{\mathcal{O}_X}(\beta^* L_G, N)$$

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such that, if $\alpha(u, y_\beta) = 0$ there exists a lift $\tilde{\beta}$. The space of all such deformations is isomorphic to

$$\Omega \text{Hom}_{\mathcal{O}_X}(\beta^* L_G, N)$$

where the loops are based at the trivial derivation.

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## 2 Derived $\infty$-stacks: Overview

In this section we give a brief introduction to geometric $\infty$-stacks. The reader may find the necessary details on $\infty$-categories an $\infty$-topoi in [3].

Let $\mathcal{C}^{\text{op}}$ denote a presentable $\infty$-category (connective $E_\infty$ rings, or simplicial commutative rings) with an topology $\tau$ on $\mathcal{C}$. A derived $\infty$-prestack is a functor $F : \mathcal{C}^{\text{op}} \to S$.

$$\text{Spec } A = \text{Hom}_{\mathcal{C}^{\text{op}}}(A, -)$$

$F$ is an $\infty$-stack if it satisfies Cech descent with respect to $\tau$; $X \in \text{Fun}^L(\mathcal{C}^{\text{op}}, S)$ and $F$ takes the Cech nerve of any $\tau$-cover $U \to X$ to a limit diagram.

$F$ is an algebraic $\infty$-stack if there is a cosimplicial object $A_\bullet \in (\mathcal{C}^{\text{op}})^N(\Delta)$ and $F(B) = |\text{Hom}_{\mathcal{C}^{\text{op}}}(A_\bullet, B)|$,

$$F = \text{colim}_{\Delta^{\text{op}}} \text{Spec } A_\bullet$$

in the $\infty$-category of $\infty$-stacks.
2.1 The quasi-coherent $\infty$-stack

A quasi-coherent sheaf on a scheme $X$ is a morphism of stacks $X \to Mod$ from $X$, considered as a stack, into the canonical stack

$$Mod : \text{Spec} A \mapsto \text{Mod}_A$$

of modules which corresponds to the bifibration

$$T_{CRings} \simeq Mod \to CRings$$

from the tangent category of the category of commutative rings to commutative rings.

This definition of quasi-coherent sheaves generalizes to any $(\infty, 1)$-topos, and over arbitrary $\infty$-sites. Let $\mathcal{C}$ be symmetric monoidal $\infty$-category equipped with Grothendieck $\infty$ topology such that $\mathcal{C}^{op}$ is presentable. The tangent $\infty$ category $T(\mathcal{C}^{op}) \to \mathcal{C}^{op}$ is the bifibration whose fibers over an object $A \in \mathcal{C}$ plays the role of the $\infty$-groupoid of modules over $A$, see [2].

Under the $\infty$-Grothendieck construction this corresponds to a $(\infty, 1)$ presheaf

$$\text{Mod}_\infty : \mathcal{C}^{op} \to \hat{\text{Cat}}_{\infty}$$

where $\text{Spec} R$ for $R \in \mathcal{C}^{op}$ is the affine object in the geometry defined over $\mathcal{C}$, or directly in terms of test spaces

$$\text{Mod}_\infty : U \mapsto \text{Stab}(\mathcal{U}/)$$

This makes $\text{Mod}_\mathcal{C} \in \text{Shv}_{(\infty,1)}(\mathcal{C}) = [\mathcal{C}^{op}, \hat{\text{Cat}}_{\infty}]$.

Let $H = \text{Shv}_{\infty}(\mathcal{C})$ be the $\infty$-topos of $\infty$-stacks on $\mathcal{C}$ and $X \in H$ be an $\infty$-stack. The stable $\infty$ category of quasi coherent modules over $X$ is the Hom space in the $\infty$-topos $H$;

**Definition 2.1.**

$$QC(X) = \text{Hom}_H(X, \text{Mod}_\infty) \tag{1}$$

Notice that $H \subset [\mathcal{C}^{op}, \hat{\text{Cat}}_{\infty}]$ as any $\infty$-groupoid is in $\hat{\text{Cat}}_{\infty}$. $QC(X)$ is computed using the Yoneda-Kan extension.

By definition $\text{Kan}(F)(Y) = \lim_{j(U) \to_Y F(U)}$, where $Y = \colim_{j(U) \to Y} j(U)$ in $P(\mathcal{C})$. The above adjunction is an equivalence of $\infty$ categories; it follows
from the standard adjunction \(\text{Fun}(A, B) \xrightarrow{\text{Kan}} \text{Fun}^L(P(A), B)\) being an equivalence of \(\infty\) categories.

For a prestack \(X \in P(C) = \text{Fun}(C^{op}, SSets)\), suppose \(X = \text{colim}_\alpha j(\text{Spec}R_\alpha)\) the
\[
\widehat{QC}(X) = \text{Kan}_j(Mod)(X) = \text{lim}_\alpha Mod(\text{Spec}R_\alpha) = \text{lim}_\alpha \text{Stab}(C^{op}/R_\alpha).
\]

If \(X\) is an \(\infty\)-stack, \(QC(X)\) can be expressed as a limit similarly,
\[
QC : \infty\text{-stacks} \xrightarrow{i^{op}} P(C)^{op}\widehat{QC} \xrightarrow{\text{Cat}} \text{Cat}.
\]

however since \(i^{op}\) doesn’t preserve limits, it is not straightforward to show.

If \(X\) is a geometric \(\infty\)-stack (i.e. atlas by a simplicial object in \(C\)), we want to compute \(QC(X)\). \(QC\) is the composition
\[
QC : \text{geometric-}\infty\text{-stacks}^{op} \cong C^{\Delta} \xrightarrow{i^{op}} P(C)^{op}\widehat{QC} \xrightarrow{\text{Cat}_\infty} \text{Cat}.
\]

If \(A_\bullet \in C^{\Delta}\) is the cosimplicial object such that simplicial object in \(C\), \(\text{Spec}(A_\bullet)\) (or simply, the simplicial affine scheme) is an atlas for \(X\), then
\[
i(A_\bullet) = \text{Hom}_{C^{op}}(A_\bullet, -)
\]
That is, as an object in the prestack category \(i(A_\bullet)\) evaluates on objects in \(C^{op}\) as the geometric realization
\[
i(A_\bullet)(R) = |\text{Hom}_{C^{op}}(A_\bullet, R)|.
\]
or, \(i(A_\bullet) = \text{colim}_{\Delta^{op}} \text{Spec}(A_\bullet)\) in the \(\infty\) category of affine \(C\)-schemes. Therefore,
\[
QC(A_\bullet) = \widehat{QC}(i^{op}(A_\bullet)) = \widehat{QC}(\text{lim}_{\Delta} \text{Spec}(A_\bullet)) = \text{lim}_{\Delta} \widehat{QC}(\text{Spec}A_\bullet) = \text{TotMod}(A_\bullet)
\]
where the limit/Tot is taken in the category of the stable presentable \(\infty\) categories.

3 Deformation Theory

In this section we describe the basic setup for doing deformation theory of geometric \(\infty\)-stacks. We will closely follow Lurie’s DAG IV [2].

Let \(D\) be a presentable \(\infty\) category, then the tangent category \(T_D\) is the fiberwise stabilization of the projection map
\[
\text{Fun}(\Delta^1, D) \rightarrow \text{Fun}(\{1\}, D) \simeq D
\]
Roughly speaking, an object of the tangent bundle \(T_D\) consists of a pair \((A, M)\), where \(A \in D\) and \(M \in \text{Stab}(D/A)\); here Stab is the stabilization construction applied to an \(\infty\) category. If \(D\) is the ordinary category of commutative
rings (replace stabilization with abelianization) then the associated tangent category is equivalent to the category of modules; the objects are pairs \((A, M)\), where \(A\) is a commutative ring and \(M\) is a \(A\)-module. If \(\mathcal{D}\) is the ∞-category of \(E_\infty\)-rings or simplicial commutative rings then the tangent category recovers the categories of modules over such objects. Using this analogy, we can define a module over an object \(A\) to be an object of the fiber of the tangent category \(T_D\) over \(\mathcal{D}\), i.e. the stable ∞-category \(T_D \times _D A \simeq \text{Stab}(\mathcal{D}/A)\).

The cotangent complex functor \(L : \mathcal{D} \to T_D\) is the left adjoint to the forgetful functor

\[ T_D \to \text{Fun}(\Delta^1, \mathcal{D}) \to \text{Fun}(\{0\}, \mathcal{D}) \simeq \mathcal{D} \]

such that the cotangent complex \(L_A\) of an object \(A\) is in \(\text{Stab}(\mathcal{D}/A)\). In other words, the composition

\[ \mathcal{C} \to L T_D \to \text{Fun}(\Delta^1, \mathcal{D}) \to \text{Fun}(1, \mathcal{D}) \simeq \mathcal{D} \]

is the identity functor.

The absolute cotangent complex functor \(L : \mathcal{D} \to T_D\) is defined to be the composition

\[ \mathcal{D} \to \text{Fun}(\Delta^1, \mathcal{D}) \to T_D \]

where the first map is the given by the the diagonal embedding and the second map is the left adjoint to the forgetful functor \(G : T_D \to \text{Fun}(\Delta^1, \mathcal{D})\)

Since the diagonal embedding is left adjoint to the evaluation map \(\text{Fun}(\Delta^1, \mathcal{D}) \to \text{hbox(\{0\}, \mathcal{D})}\) the absolute cotangent complex functor is left adjoint to the composition \(T_D \to \text{Fun}(\Delta^1, \mathcal{D}) \to \text{Fun}(\{0\}, \mathcal{D})\).

The fiber of the tangent bundle \(T_D\) over \(A \in \mathcal{D}\) can be identified with the stable envelope \(\text{Stab}(\mathcal{D}/A)\). Under this identification the cotangent complex \(L_A \in \text{Stab}(\mathcal{D}/A)\) corresponds to the image of \(\text{id}_A \in \mathcal{D}/A\) under the suspension functor

\[ \Sigma^\infty : \mathcal{D}/A \to \text{Stab}(\mathcal{D}/A). \]

The trivial square zero extension of \(A \in \mathcal{D}\) along a \(A\)-module \(M\), denoted by \(A \oplus M\) is the image of the \(M\) under the functor

\[ \Omega^\infty : \text{Stab}(\mathcal{D}/A) \to \mathcal{D}/A \to \mathcal{D} \]

Given an object \(A \in \mathcal{D}\) and a \(A\)-module \(M \in T_D \times _D \{A\}\), a derivation of \(A\) into \(M\) is a map \(\eta : L_A \to M\) in the ∞-category \(T_D \times _D \{A\}\). The derivation \(\eta\) equivalently gives a map from \(A\) to the trivial square-zero extension of \(A\) defined by \(M\) in the category \(\mathcal{D}\),

\[ d_\eta : A \to A \oplus M \]

The derivation classified by the zero map \(L_A \to M\) (this is a stable category) corresponds a canonical section \(d_0 : A \to A \oplus M\) in \(\mathcal{D}\). The square-zero extension
of $A$ defined by the derivation $\eta : L_A \to M$ is the pullback in the $\infty$ category $\mathcal{D}$.

![Diagram](attachment:image.png)

Let $f : \tilde{A} \to A$ be a morphism in $\mathcal{D}$. Then $f$ is a square-zero extension if there exists a derivation $\eta : L_A \to M$ and an equivalence $\tilde{A} \simeq A^\eta$ in the $\infty$-category $\mathcal{D}/A$.

The square-zero extension $\tilde{A}$ will also be alternatively denoted by $A \oplus_\eta \Omega M$, so that $A \oplus_0 \Omega M \simeq A \oplus M$.

### 3.1 Infinitesimal Extensions of $\infty$-stacks

Suppose $A_* \in \mathcal{C}^{op}$ and $X = \colim_{A_*} \Spec A_*$ the associated algebraic stack. Then

\[ \Mod_{\mathcal{O}_X} = \Hom_H(X, \Mod) \]

We have seen have how to compute this

\[ \Mod_{A_*} = \Tot_{[n] \in \Delta} \Stab(\mathcal{C}^{op}_{/A_n}) \simeq \Tot_{[n] \in \Delta} \Mod_{A_n} \]

Therefore a module over $\mathcal{O}_X$ is an object in the totalization of a cosimplicial stable $\infty$-category. The 0-simplices of the Tot stable $\infty$-category are exactly $A_0$-modules + descent data, i.e. $\mathcal{O}_X$-modules. A $\mathcal{O}_X$-module $N$ is a cosimplicial diagram of modules, $N_n \in \Stab(\mathcal{C}^{op}_{/A_n})$, and descent data. The trivial square zero extension defined by each $A_n$-module $N_n$ is the image of $N_n$ under the map $\text{ev}_0 \circ \Omega^\infty : \Stab(\mathcal{C}^{op}_{/A_n}) \to \mathcal{C}^{op}$.

Let $\Stab(\mathcal{C}^{op}_{/A_*})$ denote the cosimplicial stable $\infty$-category induced by the cosimplicial diagram $A_*$: given a map $A \to B$ in $\mathcal{C}^{op}$ there is a naturally induced map of stable $\infty$-categories

\[ \Stab(\mathcal{C}^{op}_{/A}) \to \Stab(\mathcal{C}^{op}_{/B}). \]

The limit of this diagram in the $\infty$-category of stable $\infty$-categories is the category whose objects consists of an object in each category and descent data required to glue them. In the general situation we will use the notation $\mathcal{D}/A \to B$ for the $\infty$-category $\lim (\mathcal{C}^{op}_{/A} \to \mathcal{C}^{op}_{/B})$. Therefore in our case of interest, the $\infty$-category $\Stab(\mathcal{C}^{op}_{/A_*})$ (where $\Stab$ is taken level-wise) is the limit category $\Tot_{[n] \in \Delta} \Stab(\mathcal{C}^{op}_{/A_n})$.

We can apply the functor $\Omega^\infty$ to the cosimplicial stable presentable $\infty$-category and compose with evaluation at $\{0\} \in \Delta^1$.

\[ \Stab(\mathcal{C}^{op}_{/A_*})^{\Omega^\infty} \xrightarrow{\text{ev}_0} \mathcal{C}^{op}_{A_*} \xrightarrow{\text{ev}_0} \mathcal{C}^{\Delta^1} \]
Let $A_\bullet \in (C^{op})^\Delta$ and $N$ a $A$-module, that is an object in the totalization of the cosimplicial category $\text{Stab}(C^{op})$. The the **trivial square-zero extension** of $A_\bullet$ defined by $N$ is the image of $N$ under the map $\Omega^\infty \circ ev_0$. Denote this cosimplicial object in $C^{op}$ by $A_\bullet \oplus N$.

If $X$ is the geometric $\infty$-stack whose atlas is the simplicial affine $C$-scheme $\text{Spec}A_\bullet$, we'll denote the trivial square zero extension by $O_X \oplus N$.

The absolute cotangent complex of a cosimplicial ring $A_\bullet$ is the absolute cotangent complex of the associated geometric stack $X = \text{colim}_{\Delta^{op}} \text{Spec}A_\bullet$, $L_X$ (defined in the next section).

$L_X \in \text{Stab}(C^{op}_{/A_\bullet})$. For any $O_X$-module $N$, a _derivation_ of $X$ into $N$ is a map on the stable $\infty$-category $\text{Stab}(C^{op}_{/A_\bullet})$

$$\eta : L_X \to N$$

By adjunction, this is equivalent to giving a map $A_\bullet \to A_\bullet \oplus N$ in $C^{\Delta^{op}}$.

The **square-zero extension of $A_\bullet$ defined by $\eta$** is the pullback in the $\infty$-category $C^{\Delta^{op}}$

$$\begin{array}{ccc}
A_\bullet^0 & \longrightarrow & A_\bullet \\
\downarrow & & \downarrow d_0 \\
\downarrow d_\eta & & \downarrow \\
A_\bullet & \longrightarrow & A_\bullet \oplus N
\end{array}$$

Denote the geometric $\infty$-stack defined by the atlas $\text{Spec}A_\bullet^0$ by $X \oplus_{\eta}[\Omega N]$.

### 3.2 Cotangent complexes of $\infty$-stacks

The **cotangent complex of an $\infty$-stack**. Let $F$ be an $\infty$-C-stack, i.e. an object in $\text{Fun}(C^{op},\text{SSet})$. For $A \in C^{op}$ and $M \in \text{Stab}(C^{op})$. Let $A \oplus M$ be the trivial square-zero extension of $A$ by $M$. Let

$$x : \text{Spec}A \to F$$

be a $A$-point. Fix the following notation

$$X := \text{Spec}A$$

$$X[M] := \text{Spec}(A \oplus M)$$

The natural augmentation $A \to A \oplus M$ gives a natural map of stacks $X \to X[M]$.

The _space of derivarions_ from $F$ to $M$ at $x$ is defined by

$$\text{Def}_F(x,M) := \text{Hom}_{X/\text{Aff}_C}(X[M],F)$$

As $M \to X[M]$ is functorial in $M$ is functorial in $M$, there is a well defined functor

$$\text{Def}_F(x,-) : \text{Mod}_A \to \text{SSets}$$

defined to be the homotopy fiber in the $\infty$-category of simplicial sets

$$\begin{array}{ccc}
\text{Def}_F(x,M) & \longrightarrow & F(X[M]) \\
\downarrow & & \downarrow \\
x & \longrightarrow & F(X)
\end{array}$$

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The map $X \to X[M]$ has a canonical section (the zero derivation $d_0 : A \to A \oplus M$). Therefore, $\text{Def}_F(x, M)$ is a pointed space.

$F$ has a cotangent complex at $x$ if the functor $\text{Def}_F(x, M)$ is corepresented by an $A$-module $L_{F,x}$. The module $L_{F,x} \in \text{Stab}(C^{op}_A)$ is the cotangent complex of $F$ at $x$.

The $\infty$ stack $F$ has an absolute cotangent complex if for any $A \in C^{op}$ and any $x \in F(A)$, $F$ has a cotangent complex $L_{F,x}$ at $x$ and for any commutative diagram in $\infty$-stacks $F$

\[
\begin{array}{ccc}
\text{Spec}A & \xrightarrow{u} & \text{Spec}B \\
\downarrow x & & \downarrow x' \\
F & \xleftarrow{F} & F'
\end{array}
\]

the natural morphism $u^*L_{F,x'} \to L_{F,x}$ is an equivalence in $\text{Stab}(C^{op}_A)$. In such a case denote the absolute cotangent complex of $F$ by $L_F$. This is a $O_F$-module.

Suppose there is a map of $\infty$-prestacks $F \to F'$. Since $A \oplus N \to A$ has a canonical section, given by the zero derivative, $\text{Def}_F(x, N)$ is a pointed set. Denote by

\[\text{Def}_{F/F'}(x,-) : \text{Mod}_A \to \text{SSets}\]

the homotopy fiber of the map

\[df : \text{Def}_F(x,-) \to \text{Def}_{F'}(x,-)\]

There is an alternate description of $\text{Def}_{F/F'}(x,-) : \text{Mod}_A \to \text{SSets}$. Consider the functor $G : C\text{-stacks}_{/F'} \to \text{SSets}$ which is the restriction of $F$ to along the natural map $C\text{-stacks}_{/F'} \to C\text{-stacks}$. Then for a point $x : \text{Spec}A \to F'$, there is a point $x : \text{Spec}A \to G$ where $\text{Spec}A$ is considered an object in the over-category $C\text{-stacks}$ via the map $\text{Spec}A \to F \to F'$. The relative deformation functor at $x$, $\text{Def}_{F/F'}(x,-)$ is then equivalent to the absolute deformation functor $\text{Def}_G(x,-)$.

$F \to F'$ has a relative cotangent complex at $x$ if $\text{Def}_{F/F'}(x,-)$ is corepresentable by an $n$-connective $A$-module $L_{F/F',x}$ for some integer $n$.

$F \to F'$ has a relative cotangent complex if $F \to F'$ has a relative cotangent complex at $x$ for all points $x$ and given a commutative diagram in $\infty$-stacks

\[
\begin{array}{ccc}
\text{Spec}A & \xrightarrow{u} & \text{Spec}B \\
\downarrow x & & \downarrow x' \\
F & \xleftarrow{F} & F'
\end{array}
\]

the natural morphism $u^*L_{F,F',x'} \to L_{F,F',x}$ is an equivalence in $\text{Mod}_A$.

Suppose there is a sequence of maps of $\infty$-prestacks

$F \to F' \to F''$

and suppose the relative cotangent complex $F'/F''$ exists, then there is an exact triangle in the stable $\infty$-category of $F$-modules

\[L_{F'/F''}F \to L_{F/F''} \to L_{F'/F'}\]
in the sense that if either of the second or the third term exist then so does the other and the triangle.

4 Obstruction Theory

In this section we extend the Toën-Vessozi [6] obstruction theory formalism for derived affine schemes to algebraic ∞-stacks.

Suppose $d_{\eta} : X[M] \to X$ is a derivation, induced by a map $\eta : L_A \to M$ in $\text{Stab}(\mathcal{C}^{\text{op}}_{/A})$. Define $X_{\eta}[\Omega M] := \text{Spec}(A \oplus_{\eta} \Omega M)$. Then the pullback square

\[
\begin{array}{ccc}
A \oplus_{\eta} \Omega M & \to & A \\
\downarrow & & \downarrow d_0 \\
A & \to & A \oplus M
\end{array}
\]

means $X_{\eta}[\Omega M]$ is the homotopy pushout $X \coprod^{h}_{X[\Omega M]} X$ in the ∞-category of affine $\mathcal{C}$-schemes.

**Definition 4.1.** (\[6\]) An ∞-prestack $F$ has an obstruction theory if

(i) $F$ is infinitesimally cohesive

(ii) $F$ has a cotangent complex

Geometric ∞-stacks always have an obstruction theory.

Suppose $F$ has an obstruction theory then there exists a natural obstruction $\alpha(x) \in \text{Hom}_{\text{Mod}}(L_{F,x}, M)$ for a $A$-point $x : X \to F$ and $X_{\eta}[\Omega M]$ as defined above. This cohomological (Andre-Quillen) obstruction vanishes iff the dotted arrow exists in the diagram

\[
\begin{array}{ccc}
X_{\eta}[\Omega M] & \to & F \\
\downarrow & & \downarrow x' \\
X & \to & X_{\eta}[\Omega M]
\end{array}
\]

If $\alpha(x) = 0$, the space of lifts of $x$, $\text{Hom}_{X/\text{Aff}_{\mathcal{C}}}(X_{\eta}[\Omega M], F)$, is isomorphic to $\text{Hom}_{\text{Mod}}(L_{F,x}, \Omega M) \simeq \Omega \text{Hom}_{\text{Mod}}(L_{F,x}, M)$.

Is there a similar obstruction theory for lifting a family of object over an algebraic ∞-stack classified by a moduli stack $F$ which has an obstruction theory? Suppose $X = \text{colimSpec} A_\bullet$ (colimit in the ∞-category $\mathcal{C}$, i.e. the category of affine $\mathcal{C}$-schemes) where $A_\bullet$ is cosimplicial $\mathcal{C}^{\text{op}}$-object. Let $N \in \text{Stab}(\mathcal{C}^{\text{op}}_{/A_\bullet})$ be a $A_\bullet$-module and let $A_{\bullet}^\eta$ be the square-zero extension of $A_\bullet$ along a derivation $\eta : L_X \to N$. We want to find an obstruction for existence of the dotted arrow in
where $x$ is a $X$-point of $F$. It is clear from definitions that $\text{Spec}(A^\bullet) \simeq \text{Spec}A^\bullet \coprod_{\text{Spec}A^\bullet \oplus N} \text{Spec}A^\bullet$.

We need to verify that the following is an equivalence of simplicial sets when $F$ is infinitesimally cohesive

$$F(A^\bullet) \simeq F(A^\bullet) \times_{F(A^\bullet \oplus N)} F(A^\bullet).$$

Here for any cosimplicial $C$-object $B^\bullet$, $F(B^\bullet)$ is defined to be $F(\text{colim}_{\Delta^{op}} \text{Spec}B^\bullet)$ using the Kan extension along the Yoneda map $C \to P(C)$.

The following sequence of equivalences gives our desired equivalence.

$$F(A^\bullet) \simeq \text{Tot}_{[n] \in \Delta} F(A^0_{[n]}),$$
$$\simeq \text{Tot}_{[n] \in \Delta} (F(A^0_{[n]}) \times_{F(A^0_{[n]} \oplus N_{[n]})} F(A^0_{[n]})),$$
$$\simeq \text{Tot} F(A^\bullet) \times_{\text{Tot} F(A^\bullet \oplus N)} \text{Tot} F(A^\bullet),$$
$$\simeq F(\text{colimSpec}A^\bullet) \times_{F(\text{colimSpec}(A^\bullet \oplus N))} F(\text{colimSpec}A^\bullet).$$

### 5 Moduli of compact objects of $QC(X)$

**Definition 5.1.** A object $x$ in an $\infty$-category $D$ is compact if the functor $\text{Hom}_D(x, -) : D \to \infty$-groupoids commutes with small colimits;

$$\text{Hom}_D(x, \text{colim}_\alpha y_\alpha) \simeq \text{colim}_\alpha \text{Hom}_D(x, y_\alpha).$$

$D$ is compactly generated if there exists a family of compact objects $\{x_\alpha\}_\alpha$ such that, any map $X \to Y$ in $D$ is an equivalence if and only if $\text{Hom}_D(x_\alpha, X) \to \text{Hom}_D(x_\alpha, Y)$ is an weak equivalence of simplicial sets for all $\alpha$.

A stable $\infty$-category $D$ is compactly generated if there is a family of compact objects such that $y \in D$ is the zero object iff $\text{Hom}_D(x_\alpha, y)$ is a contractible simplicial set for all $\alpha$. In other words, for any arbitrary $y$ which is not the zero object, there is a non-zero map $c \to y$ from some compact object $c$.

The $\infty$-stack of perfect quasi-coherent modules $QC^{perf}$. Consider the $\infty$-functor considered as an object in $P(C)$

$$\text{Mod} : C^{op} \to \hat{\text{Cat}}_{\infty, st}$$
$$A \mapsto \text{Stab}(C^{op}_{/A})^\omega.$$
to an \( \infty \)-functor. \( QC^{\text{perf}} \) is the \( \infty \)-stack (fppf topology over connective \( E_{\infty} \) rings)

\[
QC^{\text{perf}} : \infty - \text{stacks} \to \hat{\text{Cat}}_{\infty, \text{st}}
\]

obtained by Kan extension along the Yoneda embedding.

The objects of \( \text{Stab}(\text{con} \frac{E}{A})^{\omega} \) will be called \textit{perfect complexes} of modules over \( A \).

The stack \( QC^{\text{perf}} \) is key to understanding the question of compact generation of the stable \( \infty \)-category \( QC(X) \). We need that \( QC^{\text{perf}} \) has an obstruction theory. In order for this we need to establish two things about \( QC^{\text{perf}} \)

- \( QC^{\text{perf}} \) is infinitesimally cohesive
- \( QC^{\text{perf}} \) has a cotangent complex

It follows from a result of Toën-Vessozi \cite{Toën-Vessozi} that it is enough to show that

- \( QC^{\text{perf}} \) is infinitesimally cohesive
- The diagonal \( \Delta : QC^{\text{perf}} \to QC^{\text{perf}} \times QC^{\text{perf}} \) is \( n \)-geometric for some \( n \).

The first follows from the fact that \( QC \) is infinitesimally cohesive. For the second part, let \( A \in \text{con} \frac{E}{A} \) and let \( x, y \) be objects in \( QC^{\text{perf}}(\text{Spec} A) \). In other words \( x \) and \( y \) are perfect modules over \( A \). Let \( \Omega_{x,y} QC^{\text{perf}} \) be the pullback in the \( \infty \)-category of \( \infty \)-stacks.

\[
\begin{array}{ccc}
\Omega_{x,y} QC^{\omega} & \longrightarrow & QC^{\omega} \\
\downarrow & & \downarrow \Delta \\
\text{Spec} A & \longrightarrow & QC^{\omega} \times QC^{\omega}
\end{array}
\]

We’ll show that \( \Omega_{x,y} QC^{\omega} \) is an algebraic \( n \)-stack (\( n \)-truncated) for some \( n \) depending on \( A \), \( x \) and \( y \). The proof is based on the Artin-Lurie criterion.

**Theorem 5.1.** (Lurie) A functor \( F : \text{conn} \frac{E_{\infty}}{\text{rings}} \to \text{SSets} \) is a derived algebraic \( n \)-stack (in Lurie’s sense, \( n \)-truncated) iff the following are satisfied

(i) \( F \) is a sheaf in the etale topology

(ii) \( F \) is \( \omega \)-accessible, it preserves \( \omega \)-filtered colimits

(iii) \( F \) is nilcomplete, carries Postnikov towers to limits

(iv) \( F \) is infinitesimally cohesive

(v) \( F \) has a cotangent complex

(vi) \( F \) is formally effective

(vii) The restriction of \( F \) to discrete commutative rings factors through \( \text{SSets}^{\leq n} \).
We’ll show the existence of the cotangent complex for $\Omega_{x,y}QC^\omega$. Checking the other hypotheses in the Artin-Lurie criterion are easy.

Let $B$ be an object under $A$ in $C^{op}$. Then the restriction of the functor $\Omega_{x,y}QC^\omega$ to $C^{op}_A$ can be described as

$$\Omega_{x,y}QC^\omega = \text{Map}_{\text{Mod}_B}(x \otimes_A B, y \otimes_A B).$$

Use the notation $\mathcal{F} = \Omega_{x,y}QC^\omega_{/\text{Spec}A}$ for the restriction of the functor $\Omega_{x,y}QC^\omega: C^{op} \rightarrow \text{SSet}$ along the natural functor $C^{op}_A \rightarrow C^{op}$. The structure morphism $\text{Spec}A \rightarrow \text{Spec}S$ has a cotangent complex. Therefore in order to show that $\Omega_{x,y}QC^\omega$ has an absolute cotangent complex it is sufficient to show that $\Omega_{x,y}QC^\omega \rightarrow \text{Spec}A$ has a relative cotangent complex, which is simply the cotangent complex of $\mathcal{F}$.

Let $B \in C^{op}_A$, an object in $C^{\text{stacks}}_/\text{Spec}A$. Let $z: \text{Spec}B \rightarrow \mathcal{F}$ a map in $C^{\text{stacks}}_/\text{Spec}A$. We want to show that the functor $\text{Def}_{\Omega_{x,y}QC^\omega /\text{A}}(x, -) : \text{Mod}_B \rightarrow \text{SSet}$ is corepresentable. Recall this is equivalent to the functor $\text{Def}_F(x, -)$. Let $B \oplus M$ be the trivial square-zero extension of $B$ along $M \in \text{Mod}_B$. We have

$$\mathcal{F}(\text{Spec}B) = \text{Map}_{\text{Mod}_B}(x \otimes_A B, y \otimes_A B) \simeq \text{Hom}_{\text{Mod}_A}(x, y \otimes_A B)$$

$$\mathcal{F}(\text{Spec}(B \oplus M)) = \text{Map}_{\text{Mod}_B(B \oplus M)}(x \otimes_A (B \oplus M), y \otimes_A (B \oplus M)) \simeq \text{Hom}_{\text{Mod}_A}(x, y \otimes_A (B \oplus M))$$

$$\simeq \text{Hom}_{\text{Mod}_A}(x, y \otimes_A B) \times \text{Hom}_{\text{Mod}_A}(x, y \otimes_A M)$$

All these equivalences commute with the natural map

$$\mathcal{F}(\text{Spec}(B \oplus M)) \rightarrow \mathcal{F}(\text{Spec}B).$$

Therefore the deformation space $\text{Def}_F(x, M)$ which is the homotopy fiber of this map at $x$ is equivalent to $\text{Hom}_{\text{Map}_A}(x, y \otimes_A M)$. There is a chain of equivalences

$$\text{Def}_F(x, M) \simeq \text{Hom}_{\text{Mod}_A}(x, y \otimes_A M)$$

$$\simeq \Omega^\infty((\text{Mor}_A(x,y) \otimes_A M))$$

$$\simeq \Omega^\infty(((\text{Mor}_A(x,y) \otimes_A B) \otimes_B M))$$

$$\simeq \Omega^\infty((\text{Mor}_B((\text{Mor}_A(x,y) \otimes_A B)^v, M))$$

$$\simeq \text{Hom}_{\text{Mod}_B}((\text{Mor}_A(x,y) \otimes_A B)^v, M)$$

The notation $\text{Mor}_A(x,y)$ is used for $\text{Hom}_{\text{Mod}_A}(x,y)$ when considered as an object of the stable $\infty$-category $\text{Mod}_A$.

The equivalences follow from the facts that $\text{Mor}_A(x,y)$ is a compact object when $x$ and $y$ are compact, $\text{Mod}_A$ is compactly generated under filtered colimits by $A$ and compact objects are dualizable in $\text{Mod}_B$. 

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Therefore $\text{Def}_F(x, -)$ is corepresentable by the $B$-module $L_{F,x} := (\text{Mor}_A(x, y) \otimes_A B)^v$.

Suppose given a commutative diagram in $\mathcal{C}$-stacks, $\text{Spec}_A$:

$$
\begin{array}{ccc}
\text{Spec}C & \xrightarrow{u} & \text{Spec}B \\
\downarrow{w} & \searrow{z} & \\
\mathcal{F} & \xrightarrow{f} & \mathcal{F}
\end{array}
$$

we have the equivalences

$$
L_{F,w} \simeq (\text{Mor}_A(x, y) \otimes_A C)^v \simeq \text{Mor}_A(\text{Mor}_A(x, y), C)
$$

$$
u^* L_{F,z} \simeq (\text{Mor}_A(x, y) \otimes_A B)^v \otimes C \simeq \text{Mor}_A(\text{Mor}_A(x, y), B) \otimes_C B
$$

The equivalences follow simply from adjunction are compatible with the natural map $u^* L_{F,z} \to L_{F,w}$ making it an equivalence in $\text{Mod}_C$.

This completes the proof that $\Omega_{x,y}^{QCw}$ has a cotangent complex. We need to verify the rest of the Artin-Lurie conditions to show that it is an algebraic stack. Then applying the proposition of [6] it follows that $QCw$ has a cotangent complex.

### 6 Proof of the Main Theorem

**Definition 6.1.** ([P]) A derived $\infty$-$\mathcal{C}$-stack $X$ is perfect if

(i) $X$ has affine diagonal,

(ii) $QC(X)$ is a presentable stable $\infty$-category, or equivalently the triangulated category $\text{ho}(QC(X))$ is compactly generated.

Suppose $A_\bullet$ is a cosimplicial object in $\mathcal{C}^{op}$ which is level-wise truncated as objects in the $\infty$-category $\mathcal{C}^{op}$. Then the derived algebraic $\mathcal{C}$-stack $X = \text{colim}_{\Delta^{op}} \text{Spec}(A_\bullet)$ can be obtained as finitely many square-zero extensions of the (non-derived) classical algebraic $\infty$-$\mathcal{C}$-stack

$$X^{cl} = \text{colim}_{\Delta^{op}} (\text{Spec}(\pi_0 A_\bullet))$$

There is a natural map $i : X^{cl} \to X$. Suppose we know that $X^{cl}$ is perfect, what can be said about the perfectness of derived counterpart $X$? Since $X^{cl} \to X$ is an infinitesimal extension of stacks, we shall consider the following question: suppose $i : X \to \bar{X}$ is a square-zero extension of an $\infty$-algebraic stack $X$ and suppose $QC(X)$ is compactly generated. What can be said about the presentability of the stable category $QC(\bar{X})$?

(I) We’ve seen in the previous section that $QC^w$ has an obstruction theory. Therefore we can use $L_{QC^w}$ to lift the compact objects in $QC(X)$ to compact objects in $QC(\bar{X})$. The space of all such lifts is a deformation space
There is an obstruction in the Andre-Quillen cohomology group

$$\alpha(u) \in \Hom_{\Mod_{\O_{X}}}(u^{\ast}L_{QC^\omega}, N)$$

(where $N \in \text{Stab}(\C^\omega)$ is an $\O_{X}$-module, so that $\tilde{X} = \text{colimSpec}(\A_{X}^\omega)$ for some derivation $\eta : L_{X} \to N$). If $\alpha(u) = 0$ let $\tilde{u}$ be a deformation of $u$.

(II) Given $x \in QC(\tilde{X})$. Then $i^{\ast}(x) \in QC(X)$. Since $QC(X)$ is compactly generated, there exists $u \in QC(X)^\omega$ and a non-zero map $u \to i^{\ast}(x)$ in $QC(X)$. We want to know if there is a lift of the map $f : u \to i^{\ast}(x)$ in $QC(X)$ to an map $\tilde{u} \to x$ in $QC(\tilde{X})$ under the map of stable $\infty$-categories

$$QC(X) \to QC(\tilde{X})$$

induced by the natural map $i : X \to \tilde{X}$.

The space of all possible lifts is the space of deformations of the map $u \to i^{\ast}(x)$ and is controlled by the cotangent complex of the $\infty$-stack $\Omega_{u, i^{\ast}x}QC$.

We’ll give an description of the space of lifts of the map $f : u \to i^{\ast}(x)$ to $\tilde{f} : \tilde{u} \to x$ in $QC(\tilde{X})$.

That $\Omega_{u, i^{\ast}x}QC \cong X \times_{QC} X$ in the category of $\infty$-stacks means that for any affine scheme $\Omega_{u, i^{\ast}x}QC(\text{Spec} A)$ is the $\infty$ category $\text{Hom}_{\infty-stacks}($Spec$ A, \Omega_{u, i^{\ast}x}QC$) in which the 0-simplices are triplets $(f, g, \phi)$ where

$$f, g : \text{Spec} A \to X$$

and

$$\phi : f^{\ast}u \to g^{\ast}i^{\ast}(x)$$

is a map in $\text{Mod}_{A}$. The 1-cells are morphisms between such triplets defined in the natural way.

In particular, the if we take the test space to be $X$ itself and a square zero-extension $\tilde{X}$ of $X$, then the mapping spaces are

$$\Omega_{u, i^{\ast}x}QC(X) = \text{Hom}_{\infty-St}(X, X \times_{QC} X)$$

$$\Omega_{u, i^{\ast}x}QC(\tilde{X}) = \text{Hom}_{\infty-St}(\tilde{X}, X \times_{QC} X)$$

The first space is the $\infty$-category whose objects are triplets $(f, g : X \to X, \phi : f^{\ast}x \to g^{\ast}y \in \text{Mod}_{A})$. The second space is the $\infty$-category whose objects are triplets $(f', g' : \tilde{X} \to X, \phi' : f'^{\ast}x \to g'^{\ast}y \in \text{Mod}_{A})$. Here $f^{\ast}x$, $g^{\ast}y$ and $\phi'$ are not deformations of $f'^{\ast}x$, $g'^{\ast}y$ and $\phi$ respectively.
However if we consider the point in $\Omega_{u,i^*x}QC(X)$ represented by the object $(1,1,f)$ corresponds to the triplet $(x,y,f : u \to i^*x)$ in $X \times_{QC} X$, then the fiber of $\Omega_{u,i^*x}QC(\tilde{X}) \to \Omega_{u,i^*x}QC(X)$ over this point

![Diagram](image)

is the $\infty$-category of objects $(u',x',\tilde{f})$ which are respectively deformations of $x$, $u$ and $f : u \to i^*(x)$ to $QC(\tilde{X})$. This deformation space is larger than the one we need. We want the space of deformations of the map $f$ that keeps a fixed choice of deformations of the source $u$ and target $i^*x$.

Consider the moduli functor $\mathcal{F} : \infty\text{-stacks}_{/X \times X} \to \text{Cat}_\infty$ obtained by restricting $\Omega_{u,i^*x}QC$ along the natural functor $\infty\text{-stacks}_{/X \times X} \to \infty\text{-stacks}$. Let $z : \text{Spec} A \to \mathcal{F}$ be a map in $\infty\text{-St}_{/X \times X}$. Then the mapping space

$\mathcal{F}(\text{Spec}A)$

is the $\infty$-category whose objects are maps $\phi : f^*x \to g^*y$ in $\text{Mod}_{\Omega_X}$. Here $f,g : \text{Spec} A \to X \times X$ is the test space in $\infty\text{-St}_{/X \times X}$. Denote this test space by $\text{Spec} A^{f,g}$.

$X$ is naturally an object in $\infty\text{-St}_{/X \times X}$ via the identity maps. We will denote this version of $X \in \infty\text{-St}_{/X \times X}$ by $X^{1,1}$.

Let $\tilde{X}$ be considered an object in $\infty\text{-St}_{/X \times X}$ via the derivations $d_1,d_2 : \tilde{X} \to X \times X$ so that $d_1^*u = \tilde{u}$ and $d_2^*(i^*x) = x$. Denote this object of the over category by $\tilde{X}^{d_1,d_2}$.

Now consider the point in $\mathcal{F}(X^{1,1})$ corresponding to the map $f : u \to i^*x$. The fiber of the natural map $\mathcal{F}(\tilde{X}^{d_1,d_2}) \to \mathcal{F}(X^{1,1})$ over this point

![Diagram](image)

is the $\infty$-category whose objects are exactly the deformations of the map $f : u \to i^*x$ in $QC(X)$ to $\tilde{f} : \tilde{u} \to x$ in $QC(\tilde{X})$.

Therefore if $\mathcal{F}$ has an obstruction theory, this deformation problem of lifting the map $f$

![Diagram](image)
is controlled by the cotangent complex $L_F$. Recall that this equivalent to the relative cotangent complex $L_{Ω_{u,i}^*QC/X}$ with respect to the natural map $Ω_{u,i}^*QC = X × QC X \to X × X$ of $\infty$-stacks.

More precisely, there is a cohomological obstruction

$$β(f) ∈ Hom_{O_X}(f^*L_F, N)$$

Alternately this obstruction lives in

$$Hom_{O_X}((1, 1, f)^*L_{Ω_{u,i}^*QC}, N).$$

If $β(f) = 0$ there exists deformations of $f$. The space of all possible deformations $f : u \to x$ is

$$ΩHom_{O_X}(f^*L_{Ω_{u,i}^*QC}, N).$$

For these two steps to work we need the two moduli stacks $QCω$ and $Ω_{u,i}^*QC$ have deformation theory. In other words that they are infinitesimally cohesive and have cotangent complexes. This has already established for $QCω$. Checking that the second space is infinitesimally cohesive is formal. Now we come to the existence of the cotangent complex for $Ω_{u,i}^*QC$.

Since $i^*x$ need not be compact, $Ω_{u,i}^*QC$ does not have a cotangent complex in general.

However $i^*x ∈ ModO_X$ and $X$ is perfect. Therefore $i^*x$ is a filtered colimit of perfect modules over $O_X$. Let us suppose that $i^*x = colim y_β$, for $β : X → QCω$.

Then the natural map

$$Hom_{O_X}(u, colim y_β) → colim Hom_{O_X}(u, y_β)$$

is an equivalence since $u$ is compact. Therefore any map $f : u → colim y_β = i^*x$ factors through $β : u → y_β$ for some $β$.

Since $d_2^*$ is a left adjoint, it preserves colimits,

$$x = d_2^*(colim y_β) ≃ colim(d_2^*y_β)$$

It is clear that $d_2^*y_β$ need not be compact.

Replace the moduli stacks $Ω_{u,i}^*QC$ in the second step with $Ω_{u,y_β} QC$. This one does indeed have an obstruction theory. This means that the functor

$$G : C/X \to \hat{Cat}_∞$$

is infinitesimally cohesive and has a cotangent complex.

There exists a natural obstruction in the Andre-Quillen cohomology

$$α(u, y_β) ∈ Hom_{O_X}(β^*L_G, N)$$

for lifting the map $β : u → y_β$ to $\bar{β} : \bar{u} → d_2^*(y_β)$. The space of all such deformations is equivalent to the space

$$ΩHom_{O_X}(β^*L_G, N)$$

with loops based at the trivial derivation.

$i = colim d_2^*(y_β)$ implies there is a unique map $d_2^*(y_β) → x$. Compose this with $\bar{β}$ to obtain the desired lift of $u → i^*x$ to $\widehat{X}$. 17
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