THE $\theta = \infty$ CONJECTURE IMPLIES THE RIEMANN HYPOTHESIS

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Abstract. We show that the $\theta = \infty$ conjecture implies the Riemann hypothesis.

1. Introduction

Since the work of Levinson [4], it has been known that one can obtain lower bounds for the proportion of zeros of the Riemann zeta-function on the critical line by computing upper bounds for the mollified second moment

$$I_N(T_1, T_2) := \int_{T_1}^{T_2} |M_N(\frac{1}{2} + it)|^2 |\zeta(\frac{1}{2} + it)|^2 dt,$$

where $M_N(s)$ is a mollifier roughly of the form

$$M_N(s) := \sum_{n \leq N} \mu(n) \left( \frac{n}{s} \right) \left( 1 - \frac{\log n}{\log N} \right),$$

with $N \geq 2$ an integer. Levinson [4] computed the asymptotic formula

$$\lim_{T \to \infty} \frac{I_T(0, T)}{T} = 1 + \frac{1}{\theta}$$

for $0 < \theta < \frac{1}{2}$, and used this result to deduce that $\kappa > \frac{1}{4}$, where

$$\kappa := \frac{\# \{ \rho \mid \zeta(\rho) = 0, \ 0 < \Im \rho < T, \ \Re \rho = \frac{1}{2} \}}{\# \{ \rho \mid \zeta(\rho) = 0, \ 0 < \Im \rho < T \}}$$

is the proportion of the non-trivial zeros of $\zeta(s)$ that lie on the critical line. Conrey [1] later proved that (1.2) (with a slightly different mollifier) remains valid for $\theta < \frac{1}{4}$, and thereby deduced that $\kappa > \frac{2}{3}$.

Initially it was believed (see [2]) that (1.2) does not hold when $\theta > 1$. However, Farmer [2] produced a heuristic argument suggesting that it holds for every $\theta > 0$, and called this the “$\theta = \infty$ conjecture”. Moreover, he proved that this conjecture implies that $\kappa = 1$, in other words, that 100% of the non-trivial zeros of $\zeta(s)$ lie on the critical line. He also argued that a slightly stronger form of the conjecture implies Montgomery’s pair correlation conjecture. More recently, Radziwill [6] showed that, as $\theta \to \infty$, $M_{T^\theta}(t)$ is essentially the best possible mollifier of length $T^\theta$ for $\zeta(s)$. In particular, his work implies that Levinson’s method can give $\kappa = 1$ only if it is used with mollifiers of length $T^\theta$, where $\theta$ is arbitrarily large.

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The purpose of this note is to show that the \( \theta = \infty \) conjecture actually implies the Riemann hypothesis. Indeed, we show that even an upper bound of the form \( I_N(0,T) \ll T^{1+\delta} \) for some \( \theta > 1 \) and all \( N \) in the range \( 2 \leq N \leq T^{\theta} \) implies a zero-free region for the zeta-function of the form \( \Re s > 1 - \delta \) for some \( \delta > 0 \) depending on \( \theta \); in other words, a quasi-Riemann hypothesis.

**Theorem 1.** Let \( \theta > 0 \) and assume that for every \( \varepsilon > 0 \) we have \( I_N(0,T) \ll \varepsilon T^{1+\varepsilon} \) for \( N \) in the range \( 2 \leq N \leq T^{\theta} \). Then \( \zeta(s) \) has no zeros in the half-plane \( \Re s > \frac{1}{2} + \frac{1}{2\theta} \). In particular, if \( I_N(0,T) \ll \varepsilon T^{1+\varepsilon} \) for \( 2 \leq N \leq T^{\theta} \) with \( \theta \) arbitrarily large, then the Riemann hypothesis is true.

In a number of recent works on mean values of \( L \)-functions in the \( t \)-aspect, the integral is taken over \([T,2T]\) rather than over \([0,T]\). Thus, it is natural to ask whether one can obtain a version of Theorem 1 for the interval \([T,2T]\). Usually there is no difficulty in passing from one interval to the other. In our case, however, the problem for \([T,2T]\) is more subtle because one needs an \( \Omega \)-result for \( M_N(t) \) that is uniform in \( t \). Using ideas from [5] and [3], we prove the following.

**Theorem 2.** Let \( \theta > 0 \) and assume that for every \( \varepsilon > 0 \) we have \( I_N(T,2T) \ll \varepsilon T^{1+\varepsilon} \) for \( N \) in the range \( 2 \leq N \leq T^{\theta} \). Then \( \zeta(s) \) has no zeros in the half-plane \( \Re s > \frac{1}{2} + \frac{2}{\theta} \). In particular, if \( I_N(T,2T) \ll \varepsilon T^{1+\varepsilon} \) for \( 2 \leq N \leq T^{\theta} \) with \( \theta \) arbitrarily large, then the Riemann hypothesis is true.

Notice that Theorem 2 only implies a quasi-Riemann hypothesis when \( \theta > 4 \), so in this respect it is weaker than Theorem 1. However, Theorem 2 whose proof is more difficult than that of Theorem 1 is in a certain sense best possible. If, for example, one assumes that \( \zeta(s) \) has a unique simple zero \( \rho_0 = \beta_0 + i\gamma_0 \) such that \( \gamma_0 > 0 \) and \( \beta_0 > \frac{1}{2} \), one can show that

\[
I_N(T,2T) = c_1 \frac{N^{2\beta_0-1}}{T^3} \frac{\log T}{\log^2 N} \left( 1 + \Re \left( N^{2i\gamma_0} \frac{\zeta'(\rho_0)^2}{\zeta(\rho_0)^2} \right) + o(1) \right) + O\left( T^{1+\varepsilon} + \frac{N^{\beta_0-\frac{1}{2}+\varepsilon}}{T^\theta} \right)
\]

for some constant \( c_1 > 0 \), as \( T \to \infty \), and this is consistent with the assumption \( I_{\theta^2}(T,2T) \ll T^{1+\varepsilon} \) if \( \theta < 4 \). For the sake of comparison, we note that with the same zero configuration one has

\[
I_N(0,T) = \frac{N^{2\beta_0-1}}{\log^2 N} (C(N) + o(1)) + O(T^{1+\varepsilon} + N^{\beta_0-\frac{1}{2}+\varepsilon}T^\varepsilon)
\]

for some positive function \( C(N) \) bounded away from 0, so that \( I_{\theta^2}(0,T) \ll T^{1+\varepsilon} \) implies \( \beta_0 \leq \frac{1}{2} + \frac{1}{2\theta} \), which is consistent with Theorem 1.

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### 2. Proof of the Theorems

We will prove Theorem 1 and Theorem 2 at the same time. It should be pointed out, however, that an easier argument would suffice for the former.
We begin by extending our earlier definition of $M_N(s)$ slightly by writing
\begin{equation}
M_x(s) \log x = \sum_{n \leq x} \frac{\mu(n)}{n^s} \log(x/n)
\end{equation}
for $x > 0$ (with $M_1(s) := 0$). Notice that the right-hand side is zero when $0 < x \leq 1$ and that this also allows us to extend the definition of $I_N(T_1, T_2)$ in \[(1.1)\] to $I_x(T_1, T_2)$. Now, for $t \in \mathbb{R}$ we have
\[M_x(\frac{1}{2} + it) \log x = \frac{1}{2\pi i} \int_{1 - i\infty}^{1 + i\infty} \frac{x^z}{\zeta(\frac{1}{2} + it + z)} \frac{dz}{z^2}.
\]
Thus, by Mellin inversion we see that
\[H_t(w) := \int_1^\infty M_x(\frac{1}{2} + it)(\log x)x^{-w} dx = \frac{1}{(w - 1)^2 \zeta(w - \frac{1}{2} + it)}
\]
for $\Re w > \frac{3}{2}$. Next, assuming that $\rho_0 = \beta_0 + i\gamma_0$ is a fixed zero of $\zeta(w)$ with $\beta_0 \geq 1/2$, we define
\[G_t(w) := \frac{(w - 1)^2 (w - \frac{3}{2} + it) \zeta(w - \frac{1}{2} + it)}{(w + 1)^2 (w - \frac{1}{2} + it - \rho_0)(w + it + 1)^4}.
\]
In the half-plane $\Re w \geq 0$, $G_t(w)$ is holomorphic and satisfies $G_t(w) \ll (1 + |w + it|)^{-\frac{5}{2}}$. Thus, setting
\[g_t(u) = \frac{1}{2\pi i} \int_{3 - i\infty}^{3 + i\infty} G_t(w)u^{-w} dw
\]
for $u > 0$, we have
\begin{equation}
g_t(u) = \begin{cases} 
0 & \text{if } u > 1, \\
O(1) & \text{if } 0 \leq u \leq 1,
\end{cases}
\end{equation}
as can be seen by moving the line of integration to $\Re w = +\infty$ when $u > 1$, and to $\Re w = 0$ when $0 \leq u \leq 1$.

Now consider the integral
\begin{equation}
J_t(x) := \frac{1}{2\pi i} \int_{3 - i\infty}^{3 + i\infty} G_t(w)H_t(w)x^w dw = \frac{1}{2\pi i} \int_{3 - i\infty}^{3 + i\infty} \frac{(w - \frac{3}{2} + it)x^w}{(w + 1)^2 (w - \frac{1}{2} + it - \rho_0)(w + it + 1)^4} dw,
\end{equation}
where, from this point on, we assume that $x \geq 2$. On the one hand, by the convolution formula for products of Mellin transforms, and since $M_y(\frac{1}{2} + it)$ log $y = 0$ when $0 < y \leq 1$,
\[J_t(x) = \int_1^\infty M_y(\frac{1}{2} + it)(\log y)g_t(y/x) dy.
\]
Thus, by \[(2.2)\],
\begin{equation}
J_t(x) \ll \int_1^x |M_y(\frac{1}{2} + it)| \log y dy
\end{equation}
for $x \geq 2$. On the other hand, moving the line of integration in \[(2.3)\] to $\Re w = 0$, we see that
\begin{equation}
J_t(x) = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} G_t(w)H_t(w)x^w dw + \frac{x^{\rho_0 + \frac{1}{2} - it}(\rho_0 - 1)}{(\frac{1}{2} + \rho_0 - it)^2(\rho_0 + \frac{1}{2})^4}.
\end{equation}
The integral on the right is $O(1)$ since $H_t(w)G_t(w) \ll (1 + |w|)^{-2}$ for $\Re w = 0$. Thus, from (2.4) and (2.5) we deduce that

$$x^{\beta_0 + \frac{1}{2}} + 1 \ll \int_1^x |M_y(\frac{1}{2} + it)| \log y \, dy.$$ 

It follows from the Cauchy-Schwarz inequality that

$$x^{2\beta_0} + 1 \ll \int_1^x |M_y(\frac{1}{2} + it)|^2 log^2 y \, dy$$

for $x \geq 2$. Multiplying both sides by $|\zeta(\frac{1}{2} + it)|^2$ and integrating with respect to $t$ over the interval $[T_1, T_2]$, where $0 \leq T_1 \leq T_2/2$, we obtain

$$\int_{T_1}^{T_2} |\zeta(\frac{1}{2} + it)|^2 \left( \frac{x^{2\beta_0}}{(1 + t)^4} + \frac{1}{x} \right) dt \ll \int_1^x \log^2 y \int_{T_1}^{T_2} |M_y(\frac{1}{2} + it)\zeta(\frac{1}{2} + it)|^2 dt \, dy$$

$$\leq \log^2 x \int_1^x I_y(T_1, T_2) \, dy.$$ 

Now $\int_{T_1}^{T_2} |\zeta(\frac{1}{2} + it)|^2 dt \gg T_2 \log(T_2 + 2)$ for $0 \leq T_1 \leq T_2/2$, so

$$\frac{x^{2\beta_0} \log(T_1 + 2)}{|1 + T_1|^3} + \frac{T_2 \log(T_2 + 2)}{x} \ll \log^2 x \int_1^x I_y(T_1, T_2) \, dy.$$ 

Thus, if $I_N(0, T) \ll T^{1+\varepsilon}$ holds for $2 \leq N \leq T^\theta$ and for every $\varepsilon > 0$, then taking $T_1 = 0$, $T_2 = T$, and $x = T^\theta$, we obtain

$$T^{2\beta_0} \ll T^{1+\varepsilon+\theta}.$$ 

Letting $T \to \infty$ and letting $\varepsilon > 0$ be sufficiently small, we obtain $\beta_0 \leq \frac{1}{2} + \frac{1}{2\theta}$, as claimed in Theorem 1. Theorem 2 follows in the same way on taking $T_1 = T$ and $T_2 = 2T$.

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