IDEMPOTENT MODULES, LOCUS OF COMPACTNESS AND LOCAL SUPPORTS

JON F. CARLSON

To the memory of Brian Parshall and Georgia Benkart, good friends and wonderful colleagues

Abstract. Let $kG$ be the group algebra of a finite group scheme defined over a field $k$ of characteristic $p > 0$. Associated to any closed subset $V$ of the projectivized prime ideal spectrum $\text{Proj} H^*(G, k)$ is a thick tensor ideal subcategory of the stable category of finitely generated $kG$-module, whose closure under arbitrary direct sums is a localizing tensor ideal in the stable category of all $kG$-modules. The colocalizing functor from the big stable category to this localizing subcategory is given by tensoring with an idempotent module $E$. A property of the idempotent module is that its restriction along any flat map $\alpha : \mathbb{k}[t]/(t^p) \to kG$ is a compact object. For a $kG$-module $M$, we define its locus of compactness in terms of such restrictions. With some added hypothesis, in the case that $V$ is a closed point, for a $kG$-module $M$, we show that in the stable category $\text{Hom}(E, M)$ is finitely generated over the endomorphism ring of $E$, provided the restriction along an associated flat map is a compact object. This leads to a notion of local supports. We prove some of its properties, give a realization theorem and display the images of the $L_\xi$ modules under the colocalizing functor.

1. Introduction and notation

Suppose that $k$ is a field of characteristic $p > 0$ and that $G$ is a finite group scheme defined over $k$. Let $kG$ denote its group algebra, which is a finite dimensional cocommutative Hopf algebra. Examples of such algebras include the group algebras of finite groups, restricted enveloping algebras of restricted $p$-Lie algebras and infinitesimal subgroups of algebraic groups defined over $k$. For any such algebra the stable categories $\text{stmod}(kG)$ of finitely generated $kG$-modules and $\text{StMod}(kG)$ of all $kG$-modules are tensor triangulated categories. There is a notion of support varieties, and the thick tensor ideals in $\text{stmod}(kG)$ have been classified using this construction and the spectrum $V_G(k) = \text{Proj} H^*(G, k)$ of the cohomology ring $[5, 12]$.  

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Associated to any thick tensor ideal in \textit{stmod}(kG) is a triangle consisting of the trivial module and a pair of idempotent modules \( \mathcal{E} \) and \( \mathcal{F} \). Tensoring with these modules induce colocalizing and localizing functors on the stable category. Except in trivial cases, these module are infinitely generated. None the less, they have the unusual property that their restrictions along any flat map (called a \( \pi \)-point) \( A = k[t]/(t^p) \to kG \) is a compact object in \textit{StMod}(A). So it would seem that the property of having compact restrictions is related to the structure of the category \textit{StMod}(kG). In this paper we begin an investigation of this issue.

We introduce a locus of compactness for a \( kG \)-module \( M \). It is the collection of all points in \( V_G(k) \) such that the restriction of \( M \) along any associated \( \pi \)-point is compact. The locus has some properties similar to support varieties, and in particular, it identifies some thick subcategories of \textit{StMod}(kG). In the case that \( kG \) is the restricted enveloping algebra of a Lie algebra, the properties are particularly nice, and the subcategories closed under tensor products. In other cases there is some dependence on the Hopf structure on \( kG \). Given a module \( M \), the inclusion of a point of \( V_G(k) \) in the locus of \( M \) depends on the image of \( M \) in the colocalized category generated by the idempotent \( \mathcal{E} \) module.

The idempotent \( \mathcal{E} \) module acts by tensor product as the identity on the colocalized subcategory that it generates. As a consequence, the group of homomorphisms between two modules in the subcategory is module over the endomorphism ring of the idempotent module. Using earlier work \[9\], we know precisely the structure of the endomorphism ring of the \( \mathcal{E} \) module in the case that \( kG \) is the group algebra of an elementary abelian \( p \)-group and the variety of the \( \mathcal{E} \) module is a single point \( V \) in \( V_G(k) \). While it is true in this case that the endomorphism ring is infinitely generated, modules in the subcategory can still be distinguished by the annihilator of their endomorphisms and homomorphisms. We show that, given any noninveritble element \( \zeta \) of the stable endomorphism ring \( \text{End}_{kG}(\mathcal{E}) \), there is a \( kG \)-module \( M_\zeta \) such that the annihilator of \( \text{Hom}_{kG}(\mathcal{E}, M) \) in \( \text{End}_{kG}(\mathcal{E}) \) is close to being the ideal generated by \( \zeta \). That is, the radicals of the ideals coincide. In addition, if the point \( V \) is in the locus of compactness of a module \( M \) in the subcategory, then the homomorphism group in the stable category \( \text{Hom}_{kG}(\mathcal{E}, M) \) is finitely generated over the endomorphism ring \( \text{End}_{kG}(\mathcal{E}) \).

After an early version of this paper was posted, it came to light that the group consisting of Dave Benson, Srikanth Iyengar, Henning Krause and Julia Pevtsova have results that overlap significantly with those in this paper. The BIKP collaboration approaches the subject from an abstract categorical perspective that contrasts sharply with the more elementary hands-on development presented here. In several cases, they prove stronger theorems, but without as much of the detailed information on the modules and maps. I am grateful to the group for letting me see their unfinished manuscript \[8\] and letting me borrow at least one of their ideas. I want
to also thank the Hausdorff Research Institute for Mathematics at the University of Bonn for the support and stimulating program during which some of work on this paper was completed.

For references on group representation and cohomology see the texts [3] or [10]. For triangulated categories, see [17] or the early sections of [2].

2. Background

Throughout the paper \( kG \) is the group algebra of a finite group scheme \( G \) that is defined over the field \( k \) of characteristic \( p > 0 \). Let \( \text{mod}(kG) \) denote the category of finitely generated \( kG \)-modules and \( kG \)-module homomorphisms, and let \( \text{Mod}(kG) \) be the category of all \( kG \)-modules and homomorphisms. The stable category \( \text{stmod}(kG) \) has the same objects as \( \text{mod}(kG) \), but in the stable category the set of morphisms from object \( M \) to object \( N \) is given by

\[
\text{Hom}_{kG}(M, N) = \frac{\text{Hom}_{kG}(M, N)}{\text{PHom}_{kG}(M, N)}
\]

where \( \text{PHom}_{kG}(M, N) \) is the set of all homomorphisms that factor through a projective module. The category \( \text{StMod}(kG) \) of all \( kG \)-modules is constructed the same way, by factoring out any map that factors through a projective module. Note that \( \text{stmod}(kG) \) is the subcategory of compact objects in \( \text{StMod}(kG) \).

For a \( kG \)-module \( M \), the modules \( \Omega(M) \) and \( \Omega^{-1}(M) \) are defined to be the kernel of a projective cover \( P \rightarrow M \) and the cokernel of an injective hull \( M \hookrightarrow I \), respectively. Recall that \( kG \) is a self-injective rings, so that the injective module \( I \) is projective and the projective module \( P \) is injective. Consequently, \( \Omega(\Omega^{-1}(M)t) \cong \Omega^{-1}(\Omega(M)) \) is the largest direct summand of \( M \) that has no projective summands.

The stable categories \( \text{stmod}(kG) \) and \( \text{StMod}(kG) \) are tensor triangulated categories. The tensor product of modules is over the base field \( k \) with the action of \( kG \) defined using the coalgebra structure on \( kG \). The triangles correspond roughly to short exact sequences in the module category. That is, a triangle has the form

\[
\begin{array}{c}
\longrightarrow X \overset{\alpha}{\longrightarrow} Y \overset{\beta}{\longrightarrow} Z \overset{\gamma}{\longrightarrow} \Omega^{-1}(X) \longrightarrow
\end{array}
\]

where for some projective module \( P \) and maps \( \alpha' \), \( \beta' \) representing the classes \( \alpha \) and \( \beta \), there is an exact sequence

\[
0 \longrightarrow X \overset{\alpha'}{\longrightarrow} Y \oplus P \overset{\beta'}{\longrightarrow} Z \longrightarrow 0 .
\]

The operator \( \Omega^{-1} \) is the translation or shift functor on the stable categories.

A \( \pi \)-point [12] is a flat map \( \alpha : K[t]/(t^p) \rightarrow KG_K \), where \( K \) is some extension of \( k \), and the map factors by flat maps through the group algebra \( KE \) of some unipotent abelian subgroup scheme \( E \) of \( G_K \). Two \( \pi \)-points, \( \alpha_K : K[t]/(t^p) \rightarrow KG_K \) and
the element \( \left[ 2 \right] \), \( \pi \)-classes of having the form both projective or both not projective.

In the case that \( G \) is a finite group, a unipotent subgroup scheme would be a subalgebra \( kE \) where \( E = \langle g_1, \ldots, g_r \rangle \) is an elementary abelian subgroup of order \( p^r \) for some \( r \). Then a \( \pi \)-point that is defined over \( k \) would have the form \( \alpha : k[t]/(t^p) \to kE \subseteq kG \) given by \( \alpha(1) = \sum_{i=1}^{r} \alpha_i (g_i - 1) + w \) where \( w \) is some element in \( \text{Rad}^2(kE) \) and for some \( i, \alpha_i \neq 0 \). The equivalence class of \( \alpha \) depends only on the element \( [\alpha_1] \ldots [\alpha_r] \in \mathcal{P}_k^{-1} \). Let \( V^*_G(k) \) denote the collection of all equivalence classes of \( \pi \)-points.

The cohomology ring \( H^*(G, k) \) is a finitely generated graded-commutative \( k \)-algebra. This was proved by Evens, Golod and Venkov for finite groups and by Friedlander and Suslin \([14]\) for general finite group schemes. The projectivized prime ideal spectrum \( V_G(k) = \text{Proj} H^*(G, k) \) is a finite dimensional projective variety. For \( M \) a finitely generated \( kG \)-module, let \( V_G(M) \subseteq V_G(k) \) be the collection of all homogeneous prime ideals that contain the annihilator of \( \text{Ext}_G^*(M, M) \).

Given a closed subvariety \( V \subseteq V_G(k) \), the collection \( \mathcal{M}_V \) of all finitely generated \( kG \)-modules \( M \) with \( V_G(M) \subseteq V \) is a thick tensor ideal in \( \text{stmod}(kG) \), meaning that it is a triangulated subcategory that is both closed under taking direct summands and closed under tensor products with arbitrary objects in \( \text{stmod}(kG) \).

For any such \( V \) and \( \mathcal{M}_V \), there is a distinguished triangle \([18]\) in \( \text{StMod}(kG) \) having the form

\[
\mathcal{S}_V : \quad \mathcal{E}_V \xrightarrow{\sigma_V} k \xrightarrow{\tau_V} \mathcal{F}_V \xrightarrow{\Omega(\mathcal{E}_V)}...
\]

where \( \mathcal{E}_V \) and \( \mathcal{F}_V \) are idempotent modules in that \( \mathcal{E}_V \otimes \mathcal{E}_V \cong \mathcal{E}_V \), and \( \mathcal{F}_V \otimes \mathcal{F}_V \cong \mathcal{F}_V \) in the stable category. In addition \( \mathcal{E}_V \otimes \mathcal{F}_V \cong 0 \), meaning that \( \mathcal{E}_V \otimes \mathcal{F}_V \) is a projective module. The triangle has a couple of universal properties. Let \( M \) be a finitely generated \( kG \)-module. The first property is that any map \( \gamma : N \to M \), with \( N \) in \( \mathcal{M}_V \), factors through \( \sigma_V \otimes \text{Id}_M : \mathcal{E}_V \otimes M \to k \otimes M \cong M \). The other property is that any map \( \gamma : M \to N \), such that the third object in the triangle of \( \gamma \) is in \( \mathcal{M}_V \), factors through \( \tau_V \otimes \text{Id}_M : M \cong k \otimes M \to \mathcal{F}_V \otimes M \).

As to support varieties, for any \( kG \)-module \( M \), \( \mathcal{V}_G(M) \) is the collection of all points in \( V_G(k) \) with the property that the pull back along a corresponding \( \pi \)-point \( \alpha_K : K[t]/(t^p) \to KG_K \) of the extended module \( K \otimes M \) is not projective. In the case of the idempotent modules, with \( V \) closed, \( \mathcal{V}_G(\mathcal{E}_V) \) is the set of all points in \( V \), while \( \mathcal{V}_G(\mathcal{F}_V) = \mathcal{V}_G(k) \setminus \mathcal{V}_G(\mathcal{E}_V) \), its complement.
3. The locus of compactness

First we make a series of observations that lead to the definition of many thick subcategories of \( \text{StMod}(kG) \) associated to subsets the projectived prime ideal spectrum of the cohomology ring \( \text{H}^*(G,k) \cong \text{Ext}_{kG}^*(k,k) \). We show that under certain circumstance, there exist some different sorts of support varieties based on cardinalities of dimensions of restrictions of the objects along \( \pi \)-points.

By an idempotent module, we always mean a \( kG \)-module \( M \) such that \( M \otimes M \cong M \) in the stable category. Or said another way, \( M \) is an idempotent module if \( M \otimes M \cong M \oplus P \) where \( P \) is a projective module. We recall that the only finitely generated idempotent module is the trivial module \( k \). We refer to the modules \( \mathcal{E}_V \) and \( \mathcal{F}_V \), for \( V \) closed in \( V_G(k) \) as the Rickard idempotent modules. In addition, there are many other idempotent modules, including those defined for more general collections of points in \( V_G(k) \). For example, a countable direct sum of copies of the trivial module is an idempotent module. If we let \( M = \sum_{n \geq 0} \Omega^n (k) \), then a countable direct sum of copies of \( M \) is an idempotent module.

For all of this, the Rickard idempotent modules have a very special property. To be precise we have the following.

**Lemma 3.1.** Suppose that \( M \) is one of \( \mathcal{E}_V \) or \( \mathcal{F}_V \) for \( V \) a closed subvariety of \( V_G(k) \). Then for any \( \pi \)-point \( \alpha_K : K[t]/(t^p) \rightarrow KG_K \) we have that the restriction \( \alpha_K(K \otimes M) \) is either the zero module or is isomorphic to \( K \) in \( \text{StMod}(K[t]/(t^p)) \). The same is true of the tensor \( \mathcal{E}_V \otimes \mathcal{F}_{V'} \) for \( V \) and \( V' \) closed subvarieties.

**Proof.** Consider the extended triangle \( K \otimes_k S_V \). If the \( \pi \)-point \( \alpha_K \) corresponds to a point in \( V \), then it does not correspond to any point in \( \mathcal{V}_k(\mathcal{F}_V) \). Consequently, \( \alpha_K^*(K \otimes \mathcal{F}_V) \) is a projective module, and by the triangle, \( \alpha_K^*(K \otimes \mathcal{E}_V) \cong K \) in the stable category. On the other hand, if \( \alpha_K \) does not correspond to a point in \( V \), then \( \alpha_K^*(K \otimes \mathcal{E}_V) \) is projective and \( \alpha_K^*(K \otimes \mathcal{F}_V) \cong K \). \( \square \)

The above lemma implies that all of the Rickard idempotent modules are contained in the full subcategory \( \mathcal{SF}(kG) \) of \( \text{StMod}(kG) \) consisting of all \( kG \)-modules \( M \) with the property that \( \alpha_K^*(K \otimes M) \) is finite dimensional for all \( \pi \)-points

\[
\alpha_K : K[t]/(t^p) \rightarrow KG_K.
\]

In other words, the restriction along any \( \pi \)-point of a module in \( \mathcal{SF}(kG) \) is a compact object in the stable category of \( K[t]/(t^p) \). The category \( \mathcal{SF}(kG) \) is triangulated because the restriction maps preserve triangles and if two objects in a triangle are compact then so it the third. It is easy to see that it is also closed under finite direct sums and taking direct summands. In addition, by an arguments that is very similar to that of the proof of Lemma 3.1, it contains the tensor product of any finite dimensional module with a Rickard idempotent module.
An observation of Benson, Iyengar, Krause and Pevtsova helps us to make sense of this.

**Proposition 3.2.** Suppose that $\alpha, \beta : k[t]/(t^p) \to kG$ are equivalent $\pi$-points that are defined over $k$. Let $M$ be a $kG$-module such that $\alpha^\ast(M)$ is a compact object in $\text{StMod}(k[t]/(t^p))$. Then $\beta^\ast(M)$ is also compact.

**Proof.** Let $M^* = \text{Hom}_k(M, k)$ be the $k$-dual of $M$. There is a natural map $M \to M^{**}$ from $M$ to its double dual. This is a $kG$-homomorphism. That is, while the action of $kG$ on $M^*$ is defined using the antipode of the Hopf structure, the action on $M^{**}$ involves applying the antipode twice, which is the identity. Assume $\alpha^\ast(M)$ is compact. Then the map to the double dual is an isomorphism in the stable category. In particular, the cokernel $U$ of the natural map $M \to M^{**}$, which is an injective map, is a projective module on restriction along $\alpha$. But then, $\beta^\ast(U)$ is also projective since $\beta$ is equivalent to $\alpha$. It follows, that $\beta^\ast(M)$ is a compact object. \qed

Suppose that $V$ is any subset of points in $V_G(k)$. Let $N_V$ be the full subcategory of all $kG$-modules $M$ with the property that for any $\pi$-point $\alpha_K : K[t]/(t^p) \to KG$, the restriction $\alpha_K^\ast(K \otimes M)$ is a compact object, meaning isomorphic to a finite dimensional object in the stable category. Then $N_V$ is a thick triangulated subcategory of $\text{StMod}(kG)$. All of this suggests a variant support variety. We thank Paul Balmer for suggesting the name.

**Definition 3.3.** For a $kG$-module $M$, let $U_G(M)$ be the collection of points in $V_G(k)$ with the property that for any $\pi$-point associated to this point,

$$\alpha_K : A = K[t]/(t^p) \longrightarrow KG,$$

the restriction $\alpha_K^\ast(K \otimes M)$ is isomorphic to a finite dimensional module in the stable category $\text{StMod}(A)$. We call $U_G(M)$ the locus of compactness or compact locus of the module $M$.

Note that $U_G(M)$ is not a closed subvariety of $V_G(k)$, but rather only a subset of points. Indeed, we see from item (2) in the next proposition, that $U_G(M)$ is likely to contain an open set of $V_G(M)$. We list some of the properties of $U_G$.

**Proposition 3.4.** Suppose that $M$ and $N$ are $kG$-modules.

1. If $M$ is a compact object or if $M$ is in $\mathcal{SF}(kG)$, then $U_G(M) = V_G(k)$.
2. If $V$ is a point that is not in $V_G(M)$, then $V \in U_G(M)$.
3. As noted above, any subset $V \subset V_G(k)$ defines a thick subcategory $N_V$ in $\text{StMod}(kG)$.
4. Suppose that $L \to M \to N$ is a triangle in $\text{StMod}(kG)$. Then

$$U_G(L) \cap U_G(N) \subset U_G(M).$$
(5) Suppose that $V$ is a closed subvariety of $V_G(k)$. Then

$$U_G(M \otimes \mathcal{E}_V) = (V \cap U_G(M)) \cup c(V)$$

and

$$U_G(M \otimes \mathcal{F}_V) = V \cap (U_G(M) \cap c(V)).$$

where $c(V)$ is the complement of $V$ in $V_G(k)$. In particular, if $M$ is a compact object, then $U_G(M \otimes \mathcal{E}_V) = V_G(k) = U_G(M \otimes \mathcal{F}_V)$.

Proof. All but the last of the items are obvious. For the last item, suppose that $\alpha_K : K[t]/(t^p) \to KG_K$ is a $\pi$-point for some extension $K$ of $k$. If the equivalence class of $\alpha_K$ is in $V$, then $\alpha_K^*(K \otimes \mathcal{E}_V)$ is projective. It follows that in this case,

$$\alpha_K^*(K \otimes (\mathcal{E}_V \otimes M)) = \alpha_K^*(K \otimes M).$$

Hence, $V \cap U_G(M) = V \cap U_G(\mathcal{E}_V \otimes M)$.

On the other hand, suppose that the class of $\alpha_K$ is not in $V$. Then $\alpha_K^*(K \otimes \mathcal{E}_V)$ is projective and so is $\alpha_K^*(K \otimes (\mathcal{E}_V \otimes M))$. Hence, the class of $\alpha_K$ is in $U_G(M)$. This proves the first statement. The proof of the decomposition of $U_G(M \otimes \mathcal{F}_V)$ is very similar. The last statement follows from the previous ones, noting that $U_G(M) = V_G(k)$. $\square$

We remark that statement (4) of the proposition turns the usual condition for support varieties on its head. That is, for $L \to M \to N$ a triangle in $\text{StMod}(kG)$ we have that

$$V_G(M) \subseteq V_G(L) \cup V_G(N)$$

which is something like a reverse of the condition in (4).

There is one case where we can go even further with known results.

**Example 3.5.** Suppose that $kG$ is the restricted enveloping algebra of a restricted $p$-Lie algebra. In this case, $V_G(k)$ is identified with the restricted null cone $N_1$ (see [13, 15]). That is, we have an embedding of the Lie algebra into its restricted enveloping algebra, which is the free tensor algebra on the Lie algebra modulo the relations given by the Lie bracket and the $p^{th}$-power operation. The restricted null cone $N$ is the set of all elements $x$ of the Lie algebra such that $x^{[p]} = 0$, where $x \mapsto x^{[p]}$ is the $p^{th}$-power operation. In this case, every element of $V_G(k) \cong N$ has a unique distinguished $\pi$-point that is a map of Hopf algebras. Consequently, we can define $U_G(M)$ to be the collection of all points in $V_G(k)$ such that the restriction of $M$ along the distinguished $\pi$-point associated to this point is compact. That is, instead of letting $U_G(M)$, for a module $M$, be a set of equivalence classes of $\pi$-points with the finite dimensionality property for all elements of the equivalence class, we measure $U_G(M)$ only on the distinguished $\pi$-point in each class.

The main fact is that if $\alpha_K : K[t]/(t^p) \to KG_K$ is such a distinguished $\pi$-point, then $\alpha_K$ is a Hopf algebra map. Most importantly, restriction along $\alpha_K$ commutes
with the tensor product operation. This means that there is a tensor product theorem, which says for modules $M$ and $N$ that

$$
\mathcal{U}_G(M \otimes N) = (\mathcal{U}_G(M) \cap \mathcal{U}_G(N)) \cup c(\mathcal{V}_G(M)) \cup c(\mathcal{V}_G(N)),
$$

where $c(\mathcal{V}_G(M))$ is the complement of $\mathcal{V}_G(M)$ in $\mathcal{V}_G(k)$. That is, for a distinguished $\pi$-point $\alpha_K$, $\alpha^*_K(K \otimes (M \otimes N))$ is finite dimensional in the stable category if the restrictions of both $M$ and $N$ are finite dimensional or if one of the two is projective. In particular, if $V$ is closed subvariety of $\mathcal{V}_G(k)$ we have that $\mathcal{N}_V$ is closed under tensor products in $\text{StMod}(kG)$.

4. Dependence on Hopf structure

We notice that in some cases a given algebra over $k$ may have many possible Hopf algebra structures. In particular, if $B$ is the truncated polynomial ring $B = k[t_1, \ldots, t_r]/(t_1^{p^r}, \ldots, t_r^{p^r})$, then we can make $B$ into a Hopf algebra by choosing any collection $X_1, \ldots, X_r \in \text{Rad}(kG)$ that generate the radical of $B$. This means that the elements should be linearly independent modulo $\text{Rad}^2(B)$. Then the elements $1 + X_1, \ldots, 1 + X_r$ generate an elementary abelian subgroup $H$ order $p^r$ in the group of units of $B$, and $B$ can be taken to be the group algebra $kH$. We can then impose the Hopf algebra structure of $kH$ on $B$. Similarly we can take the subspace of $B$ generated by $X_1, \ldots, X_r$ to be a commutative restricted Lie algebra $L$ with trivial $p^r$-power operation. Then $B$ is isomorphic to the restricted enveloping algebra of $L$, and it can be equipt with the induced Hopf structure.

One implication of the above is the following, a result that we find useful.

**Lemma 4.1.** Let $kG = k[t_1, \ldots, t_r]/(t_1^{p^r}, \ldots, t_r^{p^r})$. Assume that $A = k[t]/(t^p)$ is made into a Hopf algebra by regarding it either as a group algebra (the group being generated by $g = 1 + t$) or as a restricted enveloping algebra of the one dimensional Lie algebra $< t >$. Then for any $\pi$-point defined over $k$, $\alpha : A \to kG$, there is a Hopf algebra structure on $kG$ such that $\alpha$ is a homomorphism of Hopf algebras.

The ability to choose the Hopf algebra structure is important when considering restrictions on the categories. In particular, if we are given a subalgebra $A$ of $kG$ that is a flat embedding, then the restriction map to the subalgebra induces a functor on stable categories. However, it does not preserve the tensor structure unless $A$ is a Hopf subalgebra.

On the other hand, there are some cases where it does not matter. The following is one such case.

**Proposition 4.2.** Suppose that $kG$ is a finite group scheme and $\alpha : A = k[t]/(t^p) \to kG$ is a $\pi$-point defined over $k$. Let $V$ be the closed point of $\mathcal{V}_G(k)$ associated to $\alpha$ and let $\mathcal{E}_V$ be the associated idempotent module. For any $kG$-module $M$, the restriction
\( \alpha^*(M) \) is isomorphic in \( \text{StMod}(A) \) to a finite dimensional module if and only if the restriction \( \alpha^*(E_V \otimes M) \) is also isomorphic to a finite dimensional module.

**Proof.** The point is that the subcategory \( \mathcal{M}_V \) of all \( kG \)-modules whose variety is contained in \( V \) is the same regardless of the tensor product. Likewise, the triangle

\[
\mathcal{S}_V : \quad M \otimes \mathcal{E}_V \xrightarrow{1 \otimes \sigma_V} M \otimes k \xrightarrow{1 \otimes \tau_V} M \otimes \mathcal{F}_V
\]

is distinguished by universal properties that don’t depend on the tensor product (see Theorem 2.6 of [2]). In particular, \( \alpha^*(M \otimes \mathcal{F}_V) = \{0\} \) in the stable category \( \text{StMod}(A) \), since the variety of \( \mathcal{F}_V \) does not contain \( V \). Hence, we have an isomorphism \( \alpha^*(M \otimes \mathcal{E}_V) \cong \alpha^*(M) \). \( \square \)

### 5. Endomorphism rings

In this section we investigate the endomorphism rings of the module \( E_V \) and associated modules. Earlier work by Daugulis [11], showed that under reasonable hypotheses on \( V \), \( \text{End}_{kG}(E_V) \) is local and locally nilpotent. Krause [16] derived some properties of \( E_V \) using the observation that \( E_V \) is endofinite. He also showed that \( \text{End}_{kG}(E_V) \) is commutative, since it is the center of the localizing subcategory of all \( kG \)-modules with variety contained in \( V \). That is, it is the algebra of natural transformation on the identity functor on that category. In this section, we give an explicit calculation of \( \text{End}_{kG}(E_V) \) in a specific case.

We assume throughout that \( kG = k[t_1, \ldots, t_r]/(t_1^p, \ldots, t_r^p) \) is the group algebra of an elementary abelian \( p \)-group or the restricted enveloping algebra of a commutative Lie algebra with vanishing \( p^{th} \)-power operation. We assume a coalgebra structure, but in the end, it does not matter what that structure is.

Assume that we have a \( \pi \)-point \( \alpha : k[t]/(t^p) \to kG \) representing a closed point \( V \) in \( V_G(k) \). The structure of the idempotent module \( E_V \) is presented in [9]. It is described as follows. For notation, let \( kC \) be the image of \( \alpha \) and let \( Z = \alpha(t) \). Choose elements \( X_1, \ldots, X_{r-1} \) in \( \text{Rad}(kG) \) such \( X_1, \ldots, X_{r-1}, Z \) generate \( \text{Rad}(kG) \). Let \( kH \) be the subalgebra generated by 1 and \( X_1, \ldots, X_{r-1} \), so that \( kG \cong kH \otimes kC \).

**Theorem 5.1.** [9, Proposition 5.4] Suppose that

\[
(P_*, \varepsilon) : \quad P_2 \xrightarrow{\partial} P_1 \xrightarrow{\partial} P_0 \xrightarrow{\varepsilon} k \xrightarrow{} 0
\]

is a minimal projective \( kH \)-resolution of \( kG \). Let \( U \) be the inflation to \( kG \) of the indecomposable \( kC \)-module of dimension \( p-1 \), so that \( U_{\otimes C} \cong \Omega_{kC}(k) \), has basis \( 1, Z, \ldots, Z^{p-2} \) and has trivial action by \( H \). Note that if \( p = 2 \) then \( U \cong k \). Then the restriction of \( E_V \) to \( kH \) is the direct sum

\[
(E_V)_{\mid H} \cong P_0 \oplus (P_1 \otimes U) \oplus P_2 \oplus (P_3 \otimes U) \oplus \ldots .
\]
The action of $Z$ on $E$ is given by the following formula. Assume that $x \in P_n$. For $n$ odd let

$$Z(x \otimes Z^j) = \begin{cases} x \otimes Z^{j+1} \in P_n \otimes U & \text{if } 0 \leq j < p-2, \\ \partial(x) \in P_{n-1} & \text{if } j = p-2, \end{cases}$$

while for $n$ even

$$Z(x) = \begin{cases} \partial(x) \otimes 1 \in P_{n-1} \otimes U & \text{if } n > 0, \\ 0 & \text{if } n = 0. \end{cases}$$

Next we consider the endomorphism ring of $E$. The fact that $(P_*, \varepsilon)$ is a minimal resolution implies that $\partial(P_i) \subseteq \operatorname{Rad}_{kH}(P_{i-1})$ for all $i \geq 1$. As a consequence, $\operatorname{Hom}_{kG}(E, k) \cong \prod_{i=0}^\infty \operatorname{Hom}_{kH}(P, k)$. That is, the minimality of the resolution also means that $\operatorname{Hom}_{kH}(P, k) \cong H^i(H, k)$.

In addition, applying $\operatorname{Hom}_{kG}(E, -)$ to the distinguished triangle $S_V$, we get that $\operatorname{Hom}_{kG}(E, E)(\sigma_V) \to \operatorname{Hom}_{kG}(E, k)$ is an isomorphism. Putting these facts together, we have established the following.

**Proposition 5.2.** The endomorphism ring $\operatorname{End}_{kG}(E) \cong \prod_{i=0}^\infty H^i(H, k)$, as an additive group.

Any element $\gamma \in H^d(H, k)$ is represented by a unique cocycle $\gamma: P_d \to k$, which in turn, extends to a chain map $\{\gamma_i\}: P_\ast \to P_\ast$ that is unique up to homotopy. Here $\gamma_i$ maps $P_{d+i}$ to $P_i$. The product on the cohomology ring $H^*(H, k)$ can be taken to be the homotopy class of the composition of these chain maps. Moreover, as in the proof of [9, Theorem 7.1] such a chain map induces an endomorphism $\hat{\gamma}: E \to E$.

In the case that the degree $d$ of $\gamma$ is even or that $p = 2$, the map $\hat{\gamma}$ is given on the $kH$-summands as follows. For $n$ odd, and $x \in P_{n+d}$,

$$\hat{\gamma}_n(x \otimes Z^j) = \gamma_n(x) \otimes Z^j,$$

while for $n$ even, $\hat{\gamma}_n: P_{n+d} \to P_n$ by

$$\hat{\gamma}_n(x) = \gamma_n(x).$$

The formula is somewhat more complicated when $d$ is an odd integer and $p > 2$. In that case, for $n$ odd, $\hat{\gamma}_n: P_{n+d} \to P_n \otimes U$ is given by $\hat{\gamma}_n(x) = \gamma_n(x) \otimes Z^{p-2}$. For $n$ even, define $\hat{\gamma}_n: P_{n+d} \to P_n \oplus (P_{n-1} \otimes U)$ (assuming that $P_{-1} = \{0\}$) by

$$\hat{\gamma}_n(x \otimes Z^j) = \begin{cases} (\gamma_n(x), 0) & \text{if } j = 0 \\ (0, \partial \gamma_n(x) \otimes Z^{j-1}) & \text{if } j > 0 \end{cases}$$
Notice that in all cases, $\hat{\gamma}$ is a $kH$-homomorphism since $E_V$ is free as a $kH$-module. To prove that it is a $kG$-homomorphism, it is only necessary to show that $\hat{\gamma}$ commutes with the action of $Z$. We leave this as an exercise for the reader.

Taking compositions, we get the following.

**Lemma 5.3.** Suppose that the characteristic $p$ is odd. Let $\hat{\colon} H^*(H, k) \to \text{End}_{kG}(E_V)$ be the map that sends the cohomology element to the endomorphism as described above. Then we have for $\gamma \in H^n(G, k)$ and $\mu \in H^m(G, k)$, the product is given as follows.

1. If either $n$ or $m$ is even, the $\hat{\gamma} \hat{\mu} = \hat{\gamma \mu}$.
2. If both $n$ and $m$ are odd, then $\hat{\gamma} \hat{\mu} = 0$.

**Proof.** The proof of the first item is a matter of choosing chain maps representing the two elements and using the formula for the composition as above. For the second, we note that the image of the composition of the two chain maps is contained in $Z_{p^2}E_V$. But this is in the kernel of the map $\sigma_V : E_V \to k$. Because $\sigma_V : \text{Hom}_{kG}(E_V, E_V) \to \text{Hom}_{kG}(E_V, k)$ is an isomorphism we conclude that $\hat{\gamma} \hat{\mu} = 0$. \(\square\)

In this way we can characterize $\text{End}_{kG}(E_V)$. We must first define a variant of the cohomology ring. Assume that $p > 2$. Recall that

$$H^*(H, k) \cong k[\zeta_1, \ldots, \zeta_{r-1}] \otimes \Lambda(\eta_1, \ldots, \eta_{r-1})$$

where $\Lambda$ is the exterior algebra generated by the degree one elements $\eta_i$ and the elements $\zeta_i$ have degree 2. Let $R \subset \Lambda$ be the subalgebra spanned by the elements of even degree. Let $\Theta = R[\eta'_1, \ldots, \eta'_{r-1}] / J$ where $J$ is the ideal generated by all products $\eta_i' \eta_j'$ and all monomials $\eta_i' x$ such that $x \in R$ and $\eta_i x = 0$ in $\Lambda$. Notice that $\Theta$ is a graded $k$-algebra and as $k$-vector spaces, $\Theta \cong \Lambda$. Let $\Gamma = k[\zeta_1, \ldots, \zeta_{r-1}] \otimes \Theta$. Note that $\Gamma$ is a commutative $k$-algebra, and that we have an isomorphism of $k$-vector spaces $\varphi : H^*(G, k) \to \Gamma$ that is the identity on $k[\zeta_1, \ldots, \zeta_{r-1}]$ and on $R$ and sends $\eta_i$ to $\eta'_i$.

In the case that $p = 2$, let $\Gamma = H^*(G, k)$ and let $\varphi$ be the identity. Then we have the following.

**Proposition 5.4.** If $\alpha = (\alpha_1, \alpha_2, \ldots)$ and $\beta = (\beta_1, \beta_2, \ldots)$ are elements of $\text{End}_{kG}(E_V)$ as in Proposition 5.3 with $\alpha_i, \beta_i \in H^i(H, k)$, then the product $\alpha \beta = \gamma = (\gamma_1, \gamma_2, \ldots)$ is given by the relation

$$\gamma_n = \sum_{i=0}^{n} \varphi^{-1}(\varphi(\alpha_i) \varphi(\beta_{n-i})).$$

That is, $\text{End}_{kG}(E_V) \cong \prod_{i=0}^{\infty} \Gamma_i$.

**Remark 5.5.** One of the features that makes the construction of $E_V$ and its endomorphism ring (as above) possible is that $kC$ acts trivially on the trivial module
k. A similar thing can be done for any $kG$ module $M$ that is inflated from a $kH$-module, so that $Z$ annihilates $M$. That is, we can construct $\mathcal{E}_V(M) \cong \mathcal{E}_V \otimes M$ from a minimal projective $kH$-resolution of $M_{\downarrow H}$ with the action of $Z$ given by formulas as above. The final result is that $\operatorname{End}_{kG}(\mathcal{E}_V(M))$ is, as a $k$-vector space, the direct product $\prod_{i=0}^{\infty} \operatorname{Ext}^i_{kH}(M,M)$. The computation of the product poses similar problems as above, though products of even degree elements coincide with the cohomology products.

6. Finite generation

Assume that $kG$ is a unipotent commutative group scheme. In this section, we show that if the restriction of a module $M$ along a $\pi$-point is a compact object in the stable category, then $\operatorname{Hom}_{kG}(\mathcal{E}_V, M)$ is finitely generated as a module over the endomorphism ring of the unit object $\mathcal{E}_V$ in the colocalized category. This leads to definition of a local support variety for such a module. In the proof, we assume that the $\pi$-point is a map of Hopf algebras. This can be done as in Lemma 4.1. However, in the end, the consequences of the analysis do not depend on the Hopf algebra structure. Hence the consequences hold without the assumption. Note that this is not an issue in the case that $kG$ is the restricted enveloping algebra of a commutative Lie algebra with trivial $p^{th}$-power operation, as we saw in Example 3.5. The proof also uses the detailed knowledge of the structure of the idempotent module $\mathcal{E}_V$.

We first note a general principle.

Lemma 6.1. Suppose that $V$ is a closed subvariety of $V_G(k)$. For any $kG$-module $M$, $\operatorname{Hom}_{kG}(\mathcal{E}_V, M) \cong \operatorname{Hom}_{kG}(\mathcal{E}_V, \mathcal{E}_V \otimes M)$ as modules over $\operatorname{End}_{kG}(\mathcal{E}_V)$.

Proof. This follows from the distinguished triangle for $M$ associated to the tensor ideal $\mathcal{M}_V$:

$$\cdots \rightarrow \Omega(\mathcal{F}_V) \otimes M \rightarrow \mathcal{E}_V \otimes M \rightarrow k \otimes M \rightarrow \mathcal{F}_V \otimes M \rightarrow \cdots$$

in that $\operatorname{Hom}_{kG}(\mathcal{E}_V, \Omega(\mathcal{F}_V) \otimes M) = 0 = \operatorname{Hom}_{kG}(\mathcal{E}_V, \mathcal{F}_V \otimes M)$. □

Let $kG = k[t_1, \ldots, t_r]/(t_1^p, \ldots, t_r^p)$. We assume the notation of Theorem 5.1. In particular, we let $\alpha : A = k[t]/(t^p) \rightarrow kG$ be a $\pi$-point that is a map of Hopf algebras. Let $kC$ be the image of $\alpha$ with $Z = \alpha(t)$. We choose a complementary subalgebra $kH$ so that $kG \cong kH \otimes kC$. Let $V$ be the variety consisting of the closed point represented by $\alpha$.

The module $\mathcal{E}_V$ has a sequence of finitely generated submodules

$$\mathcal{E}_0 \subseteq \mathcal{E}_1 \subseteq \mathcal{E}_2 \subseteq \cdots$$

where

$$(\mathcal{E}_n)_{\downarrow H} = P_0 \oplus (P_1 \otimes U) \oplus P_2 \cdots \oplus (P_n \otimes W)$$
for \( W = U \) if \( n \) is odd and \( W = k \) if \( n \) is even, exactly as in Theorem 5.1. Let \( \iota_n : \mathcal{E}_n \to \mathcal{E}_V \) be the inclusion. Suppose that \( M \) is a \( kG \)-module such that \( \alpha^*(M) \) is compact in the stable category \( \text{StMod}(A) \).

**Definition 6.2.** For an element \( \zeta \in \text{Hom}_{kG}(\mathcal{E}_V, M) \), the degree of \( \zeta \) is the largest integer \( d \) such that there exist a representative \( \zeta' \in \text{Hom}_{kG}(\mathcal{E}_V, M) \) of the class of \( \zeta \) such that \( \zeta' \iota_{d-1} = 0 \). The leading term of \( \zeta \) is the class in \( \text{Hom}_{kG}((\mathcal{E}_d/\mathcal{E}_{d-1}), M) \) of the map induced by \( \zeta' \iota_d \), where \( \zeta' \) is a representative of the degree-\( d \) element \( \zeta \) with \( \zeta' \iota_{d-1} = 0 \).

**Lemma 6.3.** Suppose that \( \zeta : \mathcal{E}_V \to M \) is a \( kG \)-homomorphism such that, for some \( n \), \( \zeta(\mathcal{E}_n) = \{0\} \) and the induced map \( \hat{\zeta} : \mathcal{E}_{n+1}/\mathcal{E}_n \to M \) factors through a \( kG \)-projective module. Then there exist \( \gamma : \mathcal{E}_V \to M \) such that \( (\zeta - \gamma)(\mathcal{E}_{n+1}) = \{0\} \) and \( \gamma \) factors through a projective \( kG \)-module. In particular, the degree of \( \zeta \) is at least \( n+2 \).

**Proof.** First notice that since \( \hat{\zeta} \) factors through a projective module, then so does \( \zeta \iota_{n+1} \). So, we have a diagram

\[
\begin{array}{ccc}
\mathcal{E}_{n+1} & \xrightarrow{\zeta_{n+1}} & M \\
\downarrow{\iota_{n+1}} & & \\
\mathcal{E}_V & \xrightarrow{\beta} & P \end{array}
\]

where \( P \) is projective and \( \alpha \beta = \zeta \iota_{n+1} \). The point is that \( P \) is also injective, so that there exists \( \mu \) with \( \mu \iota_{n+1} = \beta \). Then let \( \gamma = \alpha \mu \). \qed

Using the last lemma it is an easy exercise to prove the following. We leave this exercise to the reader.

**Proposition 6.4.** Every element \( \zeta \) in \( \text{Hom}_{kG}(\mathcal{E}_V, M) \) has a finite degree and well defined leading term. In particular, if \( \zeta(\mathcal{E}_{d-1}) = \{0\} \), then the degree of \( \zeta \) is \( d \) if and only if \( \zeta \iota_d \) does not factor through a projective module.

**Proposition 6.5.** Suppose that \( \gamma : \mathcal{E}_n/\mathcal{E}_{n-1} \to M \) is a \( kG \)-homomorphism. Then \( \gamma \) factors through a projective module if and only if either \( n \) is odd and \( \gamma(\mathcal{E}_n/\mathcal{E}_{n-1}) \subseteq ZM \) or \( n \) is even and \( \gamma(\mathcal{E}_n/\mathcal{E}_{n-1}) \subseteq Z^{p-1}M \).

**Proof.** We prove the case for \( n \) odd. The case for \( n \) even is even easier. The point is that for \( n \) odd, \( \mathcal{E}_n/\mathcal{E}_{n-1} \cong P_n \otimes U \). Because \( P_n \) is a free \( kH \)-module, it is a direct sum of copies of \( kH \) and \( Z \) acts trivially on it. Thus, \( \mathcal{E}_n/\mathcal{E}_{n-1} \) is a direct sum of copies of \( kH \otimes U \). If a map from \( kH \otimes U \) factors through a projective module, then it factors through the injective hull \( \psi : kH \otimes U \to kG \), where \( \psi(1 \otimes 1) = Z \). Thus, if \( \gamma \) factors through a projective, then its image is in \( ZM \). On the other hand, if we have, for
some summand of $\mathcal{E}_n/\mathcal{E}_{n-1}$ that is isomorphic to $kH \otimes U$, that $\gamma(1 \otimes 1) = Zm$, for $m \in M$, then we can define $\theta : kG \to M$ by $\theta(1) = m$, and $\theta \psi$ coincides with $\gamma$ on that summand. \qed

We recall that $\text{End}_{kG}(\mathcal{E}_V) \cong \prod_{n \geq 0} H^*(H, k)$. The action of a nonzero element $\gamma \in H^n(G, k) \subseteq \text{End}_{kG}(\mathcal{E}_V)$, on $\mathcal{E}_V$ is determined by the chain map $\gamma_s : P_s \to P_s$ of degree $n$. Hence, its degree as an element of $\text{End}_{kG}(\mathcal{E}_V)$ is $n$ and its leading term is the map $\nu : \mathcal{E}_n/\mathcal{E}_{n-1} \to P_0 \cong \mathcal{E}_0 \subseteq \mathcal{E}_V$, defined by $\gamma_0$. (See the discussion following Proposition 5.2)

For any $i \geq 0$, there is a shift $\Omega^i(\nu)$ which is the induced map

$$\Omega^i(\nu) = \mathcal{E}_{n+i}/\mathcal{E}_{n+i-1} \longrightarrow \mathcal{E}_i/\mathcal{E}_{i-1}.$$ 

Note that, $\gamma(\mathcal{E}_{n+i-1}) \subseteq \mathcal{E}_{i-1}$, so there is such a map.

**Proposition 6.6.** Suppose that an element $\zeta \in \text{Hom}_{kG}(\mathcal{E}_V, M)$ has degree $d$ and leading term $\mu$. Let $\gamma \in \text{Hom}_{kG}(\mathcal{E}_V, \mathcal{E}_V)$ be an element with degree $n$ and leading term $\nu$. If $\mu \Omega^d(\nu) \neq 0$ (meaning a representative does not factor through a projective module), then $\gamma \zeta$ has degree $d + n$ and $\mu \Omega^d(\nu)$ is its leading term. Otherwise, it has degree greater than $d + n$.

**Proof.** We have that $\gamma$ has a representative (call it $\gamma$) in $\text{Hom}_{kG}(\mathcal{E}_V, \mathcal{E}_V)$ such that $\gamma(\mathcal{E}_{n-1}) = 0$. Likewise, $\zeta$ has a representative such that $\zeta(\mathcal{E}_{d-1}) = 0$. As a result, $\gamma \zeta(\mathcal{E}_{n+d-1}) = 0$ and the composition $\gamma \zeta$ induces a map that is a composition

$$\mathcal{E}_{n+d}/\mathcal{E}_{n+d-1} \longrightarrow \mathcal{E}_n/\mathcal{E}_{n-1} \longrightarrow M$$

that is easily seen to be $\mu \Omega^d(\nu)$. \qed

To explain the products of leading elements, we should note the following. For notation, let $\Gamma(M) = M^C/Z^{p-1}M$, where $M^C$ is the submodule of $kC$-fixed points.

With the assumption that $\alpha^s(M)$ is finite dimensional, so also is $\Gamma(M)$.

**Lemma 6.7.** For any $n$ we have that

$$\text{Hom}_{kG}(\mathcal{E}_n/\mathcal{E}_{n-1}, M) \cong \begin{cases} \Gamma(M) \otimes H^n(H, k) & \text{if } n \text{ is even}, \\ \Gamma(M \otimes U) \otimes H^n(H, k) & \text{if } n \text{ is odd}. \end{cases}$$

**Proof.** We note that, if $n$ is even, then $Z$ annihilates $\mathcal{E}_n/\mathcal{E}_{n-1}$. Consequently, the image of any homomorphism $\vartheta : \mathcal{E}_n/\mathcal{E}_{n-1} \to M$ lies in $M^C$. On the other hand, if this image lies in $Z^{p-1}M$, then $\vartheta$ factors through a projective module. This proves the first case. For the case that $n$ is odd, we notice that

$$\text{Hom}_{kG}(\mathcal{E}_n/\mathcal{E}_{n-1}, M) = \text{Hom}_{kG}(P_n \otimes U, M) \cong \text{Hom}_{kG}(P_n, M \otimes U)$$

since $U$ is self-dual. \qed
Putting the last two results together we easily get the following.

**Lemma 6.8.** In the notation of Proposition 6.6, if $\zeta$ has even degree $n$ and leading term $\mu = \sum_{i=1}^{t} m_i \otimes \vartheta_i$ for $m_i \in \Gamma(M)$ and $\vartheta_i \in H^p(H,k)$ as in Lemma 6.7, then $\mu \Omega^d(\nu) = \sum_{i=1}^{t} m_i \otimes \vartheta_i \nu$ where by $\vartheta_i \nu$ we mean the product in $H^p(H,k)$.

We can now prove the main theorem of the section.

**Theorem 6.9.** Let $kG = k[t_1, \ldots, t_r]/(t_1^p, \ldots, t_r^p)$. We let $\alpha : A = k[t]/(t^p) \to kG$ be a $\pi$-point defined over $k$. Let $V$ be the variety consisting of the closed point represented by $\alpha$. If $M$ is a $kG$-module whose support variety consists of the single point $V$, and if the restriction $\alpha^*(M)$ is a compact object in $\text{StMod}(A)$, then $\text{Hom}_{kG}(\mathcal{E}_V, M)$ is finitely generated as a module over $\text{End}_{kG}(\mathcal{E}_V)$.

**Proof.** As before, we write $kG \cong kH \otimes kC$ where $kC$ is the image of $\alpha$ and $kH$ is a complementary subalgebra. By Lemma 4.11 there is a Hopf algebra structure on $kG$ such that $\alpha$ is a map of Hopf algebras. We assume this structure so that the previous results of this section hold. Let $H^*_{ev} = \sum_{n \geq 0} H^{2n}(H,k)$. Notice that $H^*(H,k)$ is finitely generated over $H^*_{ev}$.

Let $\mathcal{L}_n$ be the collection of all leading terms of elements in $\text{Hom}_{kG}(\mathcal{E}_V, M)$ having degree at most $n$. Thus,

$$\mathcal{L}_n \subseteq \sum_{j=0}^{n} (\Gamma(M) \oplus \Gamma(M \otimes U)) \otimes H^j(H,k).$$

Let $\mathcal{H} = \sum_{j \geq 0} (\Gamma(M) \oplus \Gamma(M \otimes U)) \otimes H^j(H,k)$. Let $I_n \subset \mathcal{H}$ be the $H^*_{ev}$-submodule of $\mathcal{H}$ generated by $\mathcal{L}_n$. Then we have an increasing sequence of submodules $I_0 \subseteq I_1 \subseteq I_2 \ldots$. Because, $\mathcal{H}$ is finitely generated over $H^*_{ev}$, which is noetherian, the sequence terminates. That is, $I = \cup_n I_n$ is a finitely generated $H^*_{ev}$-submodule of $\mathcal{H}$ and $I = I_N$ for some $N$.

For each $j = 0, \ldots, N$, choose a finite set of elements $U_j \subseteq \text{Hom}_{kG}(\mathcal{E}_V, M)$ such that the set of leading terms of the elements of $U_j$ generated $I_j/I_{j-1}$. Let $U = \{u_1, u_2, \ldots, u_t\} = \cup_{j=0}^{N} U_j$. We claim that $U$ is a set of generators for $\text{Hom}_{kG}(\mathcal{E}_V, M)$ as a module over $H^*_{ev}$.

For each $j$, let $v_j$ be the leading term of $u_j$. Suppose that $\gamma \in \text{Hom}_{kG}(\mathcal{E}_V, M)$. Let $s$ be the degree of $\gamma$ and let $\mu$ be its leading term. Then we can write $\mu = \sum_{i=1}^{t} u_i \zeta_i$, where each $\zeta_i$ is an element of $H^s(H,k)$. Let $\zeta_i$ denote also an element in $\text{End}_{kG}(\mathcal{E}_V) = \prod_i H^1(H,k)$, that has leading term $\zeta_i$. Let $\beta_1 = \sum_{i=0}^{t} v_i \zeta_i$. Note that $\beta_1$ has degree $s$ and has the same leading term as $\gamma_1 = \gamma$. Then $\gamma_2 = \gamma_1 - \beta_1$ has larger degree than $s$, the degree of $\gamma$. Now repeat this process with $\gamma_2$ in place of $\gamma$. We get $\beta_2$ with the same degree and leading term as $\gamma_2$, so that $\gamma_3 = \gamma_2 - \beta_2$ has higher degree than $\gamma_2$. 

Taking a limit of this process we obtain a sequence $\beta_1, \beta_2, \ldots$ such that $\gamma = \beta_1 + \beta_2 + \ldots$. Note that we can add the elements of this infinite sequence because they have different degrees. For each $j$, we have that $\beta_j = \sum_{i=1}^{t} v_i \zeta_{i,j}$ for some elements $\zeta_{i,j} \in \text{End}_{kG}(E_V)$. Thus,

$$\alpha = \sum_{j \geq 1} \beta_j = \sum_{i=0}^{t} v_i (\sum_{j \geq 1} \zeta_{i,j})$$

It follows that the $\{v_i\}_{1 \leq i \leq t}$ is a set of generators for $\text{Hom}_{kG}(E_V, M)$ as asserted.

Finally, we point out that the conclusion of the theorem is independent of the Hopf algebra structure on $kG$. Consequently, the theorem is true even without the assumption on the Hopf nature of the $\pi$-point at the beginning of the proof. □

**Remark 6.10.** In [8] it is proved that Theorem 6.9 has a strong converse. That is, the authors of [8] show that the condition on the module $M$ in the hypothesis is equivalent to the condition that the module by dualizable, which in turn is equivalent to the finite generation condition on $\text{Hom}_{kG}(E_V, M)$.

7. (Co)Local supports

If a module is finitely generated over its base ring, then its annihilator in that ring is an ideal, and we can define the support variety of the module. Rings constructed as $\text{End}_{kG}(E_V)$, as in Section 5, are not often finitely generated as algebras over the base field $k$. We expect that the prime ideal spectrum of such a ring is chaotic. None the less, the annihilator of a module such as $\text{Hom}_{kG}(E_V, M)$ for a $M$ a $kG$-module, is an invariant and can be used to distinguish modules. In this section we take a brief look at some of the possibilities. In particular, we prove a form of a realization theorem.

To this end, let $\mathfrak{A}_V(M) = \text{Ann}_C(\text{Hom}_{kG}(E_V, M))$ where $C = \text{End}_{kG}(E_V)$ and $V$ is a closed subvariety of $V_G(k)$. From Lemma 6.1 we see that if $M$ is a finite dimensional $kG$-module, then $\mathfrak{A}_V(M) = \mathfrak{A}_V(M \otimes E_V)$.

We begin with a couple of easy results.

**Proposition 7.1.** Suppose that the variety of $M$ is in $V$ for $V$ as above, a close subvariety of $V_G(k)$. Then, $\text{Hom}_{kG}(E_V, M) \cong \text{Hom}_{kG}(k, M)$.

**Proof.** In the stable category $M \otimes F_V = 0$. Hence, $\text{Hom}_{kG}(E_V, M \otimes F_V) = 0$. Thus the result follows from the distinguished triangle $\mathcal{E}_V$ involving $E_V$. □

The next result follows directly from Remark 5.5.

**Proposition 7.2.** Suppose that $kG \cong kH \otimes kC$ as in Theorem 5.7. That is, $kC \cong k[t]/(t^p)$ is the image of a $\pi$-point $\alpha$ whose class $V$ in $V_G(k)$ is a closed point, and $kH$ is the group algebra of an elementary abelian $p$-group. Suppose that $M$ is the
inflation of a finitely generated $kH$-module. Then, $\mathfrak{A}_V(M)$ consists of all tuples $(\gamma_1, \gamma_2, \ldots)$ such that $\gamma_i$ annihilates $H^*(G, M)$ for all $i$.

Before further exploration, we need a technical result.

**Lemma 7.3.** Let $V$ be a closed subvariety of $V_G(k)$. Suppose that

$$
\xymatrix{ L \ar[r]^-{\alpha} & M \ar[r]^-{\beta} & N \ar[r] & \Omega^{-1}(L)}
$$

is a triangle of $kG$-modules and that $\zeta \in \mathfrak{A}_V(N)$, $\gamma \in \mathfrak{A}_V(L)$. Then $\zeta \gamma \in \mathfrak{A}_V(M)$.

**Proof.** For $\mu \in \text{Hom}_{kG}(E_V, M)$, we have that $\beta(\mu \zeta) = (\beta \mu) \zeta = 0$ in the stable category. Consequently, there exists $\nu \in \text{Hom}_{kG}(E_V, L)$ such that $\mu \zeta = \alpha \nu$. Thus, $\mu \zeta \gamma = \alpha(\nu \gamma) = 0$. □

Let $\ell = \ell(kG)$ denote the Loewy length of $kG$, i.e. the least integer $n$ such that $\text{Rad}^n(kG) = \{0\}$. Recall that for any $kG$-module $M$, $\text{Rad}^\ell(M) = \text{Rad}^\ell(kG)M = \{0\}$.

**Proposition 7.4.** Let $V$ be a closed subvariety of $V_G(k)$ and $M$ a $kG$-module. Suppose that $\zeta \in \mathfrak{A}_V(M)$. Then, for any finite dimensional $kG$-module $N$, $\zeta^{(kG)} \in \mathfrak{A}_V(M \otimes N)$.

**Proof.** First notice that it is sufficient to prove the proposition in the case that $G$ is a $p$-group. This is because whenever a $kG$-map has the property that its restriction to the Sylow $p$-subgroup $P$ of $G$ factors through a $kP$-projective module, then the map factors through a $kG$-projective module.

Let $R_j = \text{Rad}^j(N)$, so that we have a radical filtration

$$
\{0\} = R_n \subset R_{n-1} \subset \cdots \subset R_1 \subset R_0 = N,
$$

where every one of the quotients $R_i/R_{i+1}$ is a direct sum of copies of the trivial module $k$. Let $M_i = M \otimes R_i$. Thus, for $0 \leq i < n$ we have a triangle

$$
\xymatrix{ \text{Hom}_{kG}(E_V, M_{i+1}) \ar[r] & \text{Hom}_{kG}(E_V, M_i) \ar[r] & \text{Hom}_{kG}(E_V, M_i/M_{i+1}) }.
$$

Fix the index $i$. The quotient $M_i/M_{i+1} \cong M \otimes (R_i/R_{i+1}) \cong \oplus_{j \in J} M \otimes k \cong \oplus_{j \in J} M.$ for some finite indexing set $J$.

Let $\hat{\zeta} : E_V \to E_V$ be a representative of the class of $\zeta$. Choose any homomorphism $\mu : E_V \to M \otimes (R_i/R_{i+1})$. The composition $\mu \hat{\zeta}$ factors through a projective module since $R_i/R_{i+1}$ has finite dimension. Thus, we have shown that $\zeta$ annihilates $\text{Hom}_{kG}(E_V, (M \otimes R_i)/(M \otimes R_{i+1}))$.

Using this fact we prove the proposition by applying Lemma 7.3 repeatedly, beginning with $i = n - 1$. □
Lemma 7.5. Suppose that $V$ is a closed subvariety of $V_G(k)$. Suppose that $\zeta \in \text{End}_{kG}(E_V)$ and that $\gamma \in \text{Hom}_{kG}(E_V, \Omega(E_V))$. Then $\gamma \zeta = \Omega(\zeta)\gamma$ in $\text{StMod}(kG)$.

Proof. The lemma is a consequence of general principles. That is, the $kG$-module $E_V$ is the unit object in the subcategory $\mathcal{M}_V$ and its graded endomorphism ring $\text{End}_{\mathcal{M}_V}(E_V) = \sum_{n \in \mathbb{Z}} \text{Hom}_{kG}(E_V, \Omega^{-n}(E_V))$ is graded commutative. \hfill \QED

Theorem 7.6. Suppose that $V$ is a closed subvariety of $V_G(k)$. Choose an element $\zeta \in \text{End}_{kG}(E_V)$ and let $M_\zeta$ be the third object in the triangle of $\zeta$:

\[
\begin{array}{c}
\Omega(E_V) \xrightarrow{\gamma} M_\zeta \xrightarrow{\beta} E_V \xrightarrow{\zeta} E_V \xrightarrow{\zeta} \Omega^{-1}(M_\zeta)
\end{array}
\]

Then the radical of the ideal $\mathfrak{A}_V(M_\zeta)$ is the same as that of the ideal generated by $\zeta$.

Proof. We use the diagram below to show that if $\mu \in \mathfrak{A}_V(M_\zeta)$, then $\mu^\ell$ is a multiple of $\zeta$. To start, notice that $\Omega^{-1}(M_\zeta) \cong \Omega^{-1}(k) \otimes M_\zeta$ in the stable category and by Proposition 7.4 $\mu^\ell$ is in $\mathfrak{A}_V(\Omega^{-1}(M_\zeta))$. In the diagram the vertical maps are right multiplication by $\mu^\ell$.

\[
\begin{array}{ccc}
\text{Hom}_{kG}(E_V, E_V) & \xrightarrow{\zeta} & \text{Hom}_{kG}(E_V, E_V) \\
\mu^\ell \downarrow & & \uparrow \mu^\ell \\
\text{Hom}_{kG}(E_V, E_V) & \xrightarrow{\zeta} & \text{Hom}_{kG}(E_V, E_V) \\
\beta & & \beta
\end{array}
\]

Because the composition $\beta \mu^\ell = \mu^\ell \beta$ is zero on the middle term $\text{Hom}_{kG}(E_V, E_V)$, we have that there is a map $\sigma : \text{Hom}_{kG}(E_V, E_V) \rightarrow \text{Hom}_{kG}(E_V, E_V)$ such that $\zeta \sigma = \mu^\ell \beta$. Thus we have that $\sigma(\text{Id}_M) \circ \zeta = \mu^\ell \sigma(\text{Id}_M) = \text{Id}_M \circ \mu^\ell = \mu^\ell$.

Next we show that $\zeta^2 \in \mathfrak{A}_V(M_\zeta)$. First, we see from the exact sequence

\[
\begin{array}{c}
\text{Hom}_{kG}(E_V, M_\zeta) \xrightarrow{\beta^*} \text{Hom}_{kG}(E_V, E_V) \xrightarrow{\zeta} \text{Hom}_{kG}(E_V, E_V)
\end{array}
\]

that $\zeta \beta = 0$ where here $\zeta$ means left composition with $\zeta$. Hence, in the diagram

\[
\begin{array}{ccc}
\text{Hom}_{kG}(E_V, \Omega(E_V)) & \xrightarrow{\gamma^*} & \text{Hom}_{kG}(E_V, M_\zeta) \\
\zeta^* \downarrow & & \zeta^* \\
\text{Hom}_{kG}(E_V, \Omega(E_V)) & \xrightarrow{\gamma^*} & \text{Hom}_{kG}(E_V, M_\zeta)
\end{array}
\]

the composition $\beta \zeta^*$ is also zero. Note here, the down arrows marked $\zeta^*$ denote right multiplication by $\zeta$. However, there is no problem since $\text{Hom}_{kG}(E_V, E_V)$ is commutative. As a result, the map $\tau$ exist with $\gamma \tau = \zeta$. 
Finally, we consider the diagram

\[
\begin{array}{ccccccccc}
\text{Hom}_{kG}(\mathcal{E}_V, \Omega(\mathcal{E}_V)) & \xrightarrow{\Omega^*(\gamma)} & \text{Hom}_{kG}(\mathcal{E}_V, \Omega(\mathcal{E}_V)) & \xrightarrow{\gamma^*} & \text{Hom}_{kG}(\mathcal{E}_V, M_\zeta) \\
\downarrow{\zeta^*} & & \downarrow{\zeta^*} & & \downarrow{\zeta^*} \\
\text{Hom}_{kG}(\mathcal{E}_V, \Omega(\mathcal{E}_V)) & \xrightarrow{\Omega^*(\gamma)} & \text{Hom}_{kG}(\mathcal{E}_V, \Omega(\mathcal{E}_V)) & \xrightarrow{\gamma^*} & \text{Hom}_{kG}(\mathcal{E}_V, M_\zeta) \\
\end{array}
\]

For \(\mu \in \text{Hom}_{kG}(\mathcal{E}_V, M_\zeta)\) we have that \(\mu \zeta = \gamma \circ (\tau(\mu))\). Here we regard \(\tau(\mu)\) as an element of the object in the middle of the upper row of the diagram. Recall that \(\gamma^*\) is left composition with the map \(\gamma : \Omega(\mathcal{E}_V) \to M_\zeta\) in the triangle that contains \(\zeta\). Thus we have that

\[
\mu \zeta^2 = (\gamma(\tau(\mu))) \zeta = (\gamma \circ \tau(\mu)) \circ \zeta = \gamma \circ (\Omega(\zeta) \circ \tau(\mu)) = \gamma(\Omega(\zeta)(\tau(\mu))) = 0,
\]

by Lemma 7.5 and the triangle of \(\gamma\).

Suppose that \(0 \neq \zeta \in H^n(G, k)\). Assume that \(n\) is even if \(p\) is odd. Associated to \(\zeta\) is a module that generates the thick tensor ideal in \(\text{stmod}(kG)\) of all modules whose support variety is contained in the closed set of all prime ideals that contain \(\zeta\). The module \(L_\zeta\) is defined to be the kernel the map \(\hat{\zeta}: \Omega^n(k) \to k \to 0\) where \(\hat{\zeta}\) is a cocycle representing \(\zeta\). Thus we have that \(\hat{\zeta}\) is a cocycle representing \(\zeta\). Suppose that \(V\) is a single point as in the last theorem. Tensoring the above sequence with \(\mathcal{E}_V\) yields an exact sequence

\[
0 \rightarrow L_\zeta \otimes \mathcal{E}_V \rightarrow \Omega^n(k) \otimes \mathcal{E}_V \xrightarrow{1 \otimes \hat{\zeta}} \mathcal{E}_V \rightarrow 0.
\]

But recall that \(\mathcal{E}_V\) is periodic of period 1 if \(p = 2\) and period 2 otherwise. That is, for example, if \(p > 2\), then the tensor of \(\mathcal{E}_V\) with the sequence

\[
0 \rightarrow k \rightarrow k(G/H) \xrightarrow{Z} k(G/H) \rightarrow k \rightarrow 0
\]

where \(k(G/H)\) is the permutation module on which \(kH\)-acts trivially and the middle map is multiplication by \(Z\), yields a sequence with \(\mathcal{E}_V\) at the ends and middle terms projective. Thus we have that \(\Omega^n(k) \otimes \mathcal{E}_V \cong \mathcal{E}_V\) in the stable category. The consequence of this is that \(L_\zeta \otimes \mathcal{E}_V\) is isomorphic to some \(M_\gamma\). So what is \(\gamma\)? We sketch a proof of an answer to the question.

For notation, we recall that \(H^*(C, k) \cong k[\mu]\) if \(p = 2\) and, otherwise, \(H^*(C, k) \cong k[\mu, \nu]/(\mu^2)\) where \(\mu\) and \(\nu\) are in degrees one and two. Thus for all \(n\), \(H^n(C, k)\) has dimension one and has a basis element which we denote \(\mu_n\). So in the case \(p > 2\), \(\mu_{2n+1} = \mu \nu^n\) while \(\mu_{2n} = \nu^n\). Recall also that \(H^*(G, k) = H^*(H, k) \otimes H^*(C, k)\).
Proposition 7.7. Assume the hypothesis of the Theorem 7.6. Let $\zeta$ be an element in $H^n(G, k)$. Write $\zeta = \sum_{i=0}^{n} \mu_i \gamma_{n-i}$ for $\gamma_{n-i} \in H^{n-i}(H, k)$. Then in the stable category, $L_\zeta \otimes E_V \cong M_\gamma$ where

$$\gamma = (\gamma_0, \ldots, \gamma_n, 0, 0, 0, 0, \ldots) \in \prod_{i \geq 0} H^i(H, k) \cong \text{End}_{kG}(E_V).$$

Proof. The secret to the proof is finding a splitting of the middle vertical map in the diagram:

$$
\begin{array}{cccccc}
0 & \longrightarrow & L_\zeta \otimes E_V & \longrightarrow & \Omega^n(k) \otimes E_V & \stackrel{1 \otimes \zeta}{\longrightarrow} & E_V & \longrightarrow & 0 \\
0 & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & 0 \\
0 & \longrightarrow & M_\zeta & \longrightarrow & E_V & \longrightarrow & E_V & \longrightarrow & 0.
\end{array}
$$

The map itself can be taken to be $(\mu_n \otimes 1) \otimes 1$, since the kernel $L_{\mu_n}$ of the cocycle $\mu_n \otimes 1 : \Omega^n(k) \rightarrow k$ is free on restriction to $kC$, and $L_{\mu_n} \otimes E_V$ is a projective module.

We construct the splitting $\theta$ as follows. Let $k\hat{G} = kC \otimes kH \otimes kH$ and construct the idempotent $E$-module $\hat{E}$ as in Theorem 5.1. If $p = 2$, then on restriction to $kH \otimes kH$, $\hat{E}$ is a sum of the terms of the projective resolution $((P \otimes P)_*, \varepsilon \otimes \varepsilon)$ where $(P_*, \varepsilon)$ is the minimal projective $kH$-resolution of $k$. For $p$ odd, the restriction is a sum of terms of the form $(P \otimes P)_n \otimes X$ where $X$ is either $U$ or $k$ depending on the parity of $n$. In either case we have a chain map $\hat{\psi} : P_* \rightarrow (P \otimes P)_*$ that defines cup product on cohomology and splits both $1 \otimes \varepsilon$ and $\varepsilon \otimes 1$. This gives us a map $\psi : E_V \rightarrow \hat{E}$. At the same time, $\hat{E}$ when viewed as a $kG$-module by restriction (taking the diagonal embedding of $kH$ into $kH \otimes kH$) is very naturally isomorphic to $E_V \otimes E_V$.

Next we notice that the module $V$ whose restriction to $kH$ is the direct sum

$$V_{kH} = P_0 \oplus (P_1 \otimes U) \oplus P_2 \oplus (P_3 \otimes U) \oplus \cdots \oplus (P_{n-1} \otimes U) \oplus P_n / \partial(P_{n+1})$$

where $U = k$, if $p = 2$. Here, $Z$ acts as in the formula in Theorem 5.1 where defined, and by the induce map $Z : P_n \rightarrow P_{n+1}$ on the last summand. But now, we see from the formula for minimal projective $kG$-resolution of $k$ as the tensor product of $kC$ and $kH$ resolutions, that $V \cong \Omega^n(k)$. Letting $\sigma : E_V \rightarrow V$ be the obvious quotient, we get the splitting

$$E_V \stackrel{\psi}{\longrightarrow} \hat{E} \cong E_V \otimes E_V \stackrel{\sigma \otimes 1}{\longrightarrow} \Omega^n(k) \otimes E_V.$$

Now the theorem follows by composition of maps, using the splittings. \quad \square

In the case that $\gamma_0 \neq 0$, then the $\gamma$ is a unit in $\text{End}_{kG}(E_V)$, implying that $L_\zeta \otimes E_V$ is the zero module. The situation is consistent with what we know in that $\gamma_0 \neq 0$ implies that $\text{res}_{G,C}(\zeta) \neq 0$, and hence $L_\zeta$ is projective on restriction to $kC$ and $L_\zeta \otimes E_V$ is a projective module.
We end by pointing out another curiosity. Again, suppose that $G, H, C, V$ are as in Proposition 7.2. Let $\zeta$ be an element $\text{End}_{kG}(\mathcal{E}_V)$ with $\zeta$ not invertible, let $\mathcal{N}_\zeta$ be the subcategory of all $kG$-modules $M$ whose variety is in $V$ and such that such that $\zeta$ annihilates $\text{Hom}_{kG}(\mathcal{E}_V, M)$. Then $\mathcal{N}_\zeta$ is a thick subcategory of $\text{StMod}(kG)$. It is not however a localizing subcategory. We know this because the localizing tensor ideal have been classified [6, 7]. But also, we can see by Theorem 7.6, that the direct sum $M = \sum_{n=1}^{\infty} M_{\zeta}^n$ is not in $\mathcal{N}_\zeta$. Moreover, under some mild assumptions, the module $\mathcal{E}_V$ can be recovered as the third object in a triangle involving an endomorphism of the module $M$, above. This is a sort of homotopy colimit construction and will be explored in subsequent work.

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Department of Mathematics, University of Georgia, Athens, GA 30602, USA
Email address: jfc@math.uga.edu