Quantitative Timed Simulation Functions and Refinement Metrics for Timed Systems *

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Abstract. We introduce quantitative timed refinement and timed simulation (directed) metrics, incorporating zenoness checks, for timed systems. These metrics assign positive real numbers between zero and infinity which quantify the timing mismatches between two timed systems, amongst non-zeno runs. We quantify timing mismatches in three ways: (1) the maximal timing mismatch that can arise, (2) the “steady-state” maximal timing mismatches, where initial transient timing mismatches are ignored; and (3) the (long-run) average timing mismatches amongst two systems. These three kinds of mismatches constitute three important types of timing differences. Our event times are the global times, measured from the start of the system execution, not just the time durations of individual steps. We present algorithms over timed automata for computing the three quantitative simulation distances to within any desired degree of accuracy. In order to compute the values of the quantitative simulation distances, we use a game theoretic formulation. We introduce two new kinds of objectives for two player games on finite-state game graphs: (1) eventual debit-sum level objectives, and (2) average debit-sum level objectives. We present algorithms for computing the optimal values for these objectives in graph games, and then use these algorithms to compute the values of the timed simulation distances over timed automata.

1 Introduction

Theories of system approximation for continuous systems are used for analyzing systems that differ to a small extent, as opposed to the traditional boolean yes/no view of system refinement for discrete systems. These theories are necessary as formal models are only approximations of the real world, and are subject to estimation and modelling errors. Approximation theories have been traditionally developed for continuous control systems [ASG01] and more recently for linear and non-linear systems [GJP08; GPT10; Pol+10], timed systems [HMP05], labeled Markov Processes [Des+04], probabilistic automata [Bre+03], quantitative transition systems [AFS09], games [Cha+10a], and software systems [CGL12].

Timed and hybrid systems model the evolution of system outputs as well as the timing aspects related to the system evolution. In this work we develop a theory of system approximation for timed systems by quantifying the timing differences between corresponding system events. We first generalize timed refinement relations to metrics on timed systems that quantitatively estimate the closeness of two systems. Given a timed model $T_s$ denoting the abstract specification model, and a model $T_r$ denoting the concrete refined implementation of $T_s$, we assign a positive real number between zero and infinity to the pair $(T_r, T_s)$ which denotes the quantitative refinement distance.

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between \( T_r \) and \( T_s \). Given a trace \( tr_r \) of \( T_r \), and a trace \( tr_s \) of \( T_s \), we define various distances between the two traces, e.g., the distance being \( \infty \) if the untimed trace sequences differ, and being the supremum of the differences of the matching timepoints for matching events otherwise. Our event times are the \textit{global times}, measured from the start of the system execution, not just the time durations of individual steps. The distance between the systems \( T_r \) and \( T_s \) is taken to be the supremum of closest matching trace differences from the initial states.

Timed trace inclusion is undecidable on timed automata [AD94], thus timed refinement is conservatively estimated using \textit{timed simulation relations} [Cer92]. Simulation relations take a branching time view, unlike the linear view of refinement relations, and can be defined using two player \textit{games}.

We generalize timed simulation relations to quantitative timed simulation functions, and define the values of quantitative timed simulation functions as the real-valued outcome of games played on the corresponding timed graphs.

\textit{Zeno} runs where time converges is an artifact present in models of timed systems due to model imperfections; such runs are obviously absent in the physical systems which our timed models are meant to represent. We thus exclude Zeno runs in our computation of quantitative timed refinement and quantitative timed simulation relations.

We define three illustrative quantitative timed simulation directed distances which measure three important system differences. The \textit{maximal time difference} quantitative simulation distance denotes the maximal time discrepancy that can arise amongst matching transitions. The \textit{eventual maximal time difference} quantitative simulation distance denotes the eventual maximal time discrepancy that arises (ignoring finite time trace prefix discrepancies) amongst matching transitions. This corresponds to the “steady-state” difference between systems, ignoring transient behavior. The \textit{(long-run) average time difference} quantitative simulation distance denotes the average time discrepancy amongst matching transitions. This distance measures the long-run average time discrepancies, per transition, amongst two timed systems. Ideally, we want all three simulation distances to be as small as possible between the specification and the implementation systems, but minimizing one may lead to increase in values for others. Thus, all three simulation distances give important information about systems. We illustrate the various quantitative timed simulation distances via examples.

\textit{Example 1 (Maximal Time Difference).} Consider the two timed automata \( \mathcal{T}_1, \mathcal{T}_2 \) in Figure 1. The locations are labelled with the observations. The starting location of each automaton is the one labelled with the observation \( a \), and the starting value of the clock \( x \) is 0. Let us first look at the value of the \textit{maximal time difference} quantitative timed simulation distance \( S_{\text{MaxDiff}} \) for the state pair \( (a, x = 0)^{\mathcal{T}_1}, (a, x = 0)^{\mathcal{T}_2} \). The value is (1) infinity if the state of \( \mathcal{T}_1 \) does not time-abstract simulate (combined with time-divergence encoded as a fairness constraint [Cer92; HKR02], to allow only time-divergent runs in \( \mathcal{T}_1 \)) the state of \( \mathcal{T}_2 \); (2) the maximal time difference
between matching transitions of $T_1$ and $T_2$ otherwise, amongst time-divergent runs. For the two timed automata in Figure 1, it can be checked that $\langle a, x = 0 \rangle^{T_1}$ time-abstract simulates $\langle a, x = 0 \rangle^{T_2}$, and that the maximal time difference between matching transitions is 9 time units, (e.g., between the paths $\langle a, x = 0 \rangle^{T_1} \xrightarrow{10} \langle b, x = 0 \rangle^{T_1} \xrightarrow{0} \langle c, x = 0 \rangle^{T_1} \xrightarrow{5} \langle c, x = 0 \rangle^{T_1} \xrightarrow{5} \cdots$ and $\langle a, x = 0 \rangle^{T_2} \xrightarrow{1} \langle b, x = 0 \rangle^{T_2} \xrightarrow{9} \langle c, x = 0 \rangle^{T_2} \xrightarrow{5} \langle c, x = 0 \rangle^{T_2} \xrightarrow{5} \cdots$).

Example 2 (Global Event Times). Consider the two timed automata in Figure 2. The value of $x = 1$ reset $x$ $T_3$ $x = 1$ reset $x$ $T_4$

the maximal time difference quantitative timed simulation distance $S_{\text{MaxDiff}}$ for the state pair $(\langle a, x = 0 \rangle^{T_3}, \langle a, x = 0 \rangle^{T_4})$ is $\infty$, since timing mismatch corresponding to the $n$-th transition is $n$ (the $n$-th transition in $T_3$ occurs at global time $n$, the $n$-th transition in $T_4$ occurs at global time $2 \cdot n$). We depict the timelines in Figure 3.

Example 3 (Eventual Maximal Time Difference). Consider the two timed automata $T_1$ and $T_2$ in Figure 1. Let us look at the value of the eventual maximal time difference quantitative timed simulation distance $S_{\text{LimMaxDiff}}$ for the state pair $\left(\langle a, x = 0 \rangle^{T_1}, \langle a, x = 0 \rangle^{T_2}\right)$. The value is (1) infinity if the state of $T_1$ does not time-abstract simulate (combined with time-divergence encoded as a fairness constraint to allow only time-divergent runs in $T_1$) the state of $T_2$; (2) the eventual maximal time difference between matching transitions of $T_1$ and $T_2$ otherwise (ignoring the time differences amongst finite timed trace prefixes), amongst time-divergent runs. In the automata $T_1, T_2$, there is a time mismatch only at the transitions from $a$, and this transition can only occur before time 10. Once the executions reach the location $c$, the automaton $T_2$ is able to match the transitions of $T_1$ at the exact times, with zero time discrepancy. Thus, $S_{\text{LimMaxDiff}}$ denotes the “steady-state” time discrepancy between $T_1, T_2$, and this value is zero for the state pair $\left(\langle a, x = 0 \rangle^{T_1}, \langle a, x = 0 \rangle^{T_2}\right)$, in contrast to the value of 9 for $S_{\text{MaxDiff}}$ for the state pair. Note that we ignore time-discrepancies for
finite time (by discarding Zeno runs), not just finite trace prefixes. If we ignore only finite trace prefixes, then we would have obtained a value of 9, as $T_1$ can loop on the location $b$ by preventing time from progressing (note that the clock $x$ is not reset on the $b$ loop transition).

Example 4 (Eventual Maximal Time Difference). Consider the two timed automata $T_5$ and $T_6$ in Figure 4. Let us look at the value of the eventual maximal time difference quantitative timed simulation distance $S_{\text{LimMaxDiff}}$ for the state pair $\left(\langle a,x=0 \rangle^{T_5}, \langle a,x=0 \rangle^{T_6}\right)$. In this case, a time difference of 9 occurs infinitely often in time-divergent runs, (e.g. between the paths $\langle a,x=0 \rangle^{T_5} \xrightarrow{10} \langle b,x=0 \rangle^{T_5} \xrightarrow{0} \langle c,x=0 \rangle^{T_5} \xrightarrow{5} \langle a,x=0 \rangle^{T_5} \xrightarrow{10} \cdots$ and $\langle a,x=0 \rangle^{T_6} \xrightarrow{1} \langle b,x=0 \rangle^{T_6} \xrightarrow{9} \langle c,x=0 \rangle^{T_6} \xrightarrow{5} \langle a,x=0 \rangle^{T_6} \xrightarrow{1} \cdots$. The maximal time difference of 9 time units arises when taking the transitions from the $a$-labelled states. Thus, the value of $S_{\text{LimMaxDiff}}$ for the state pair $\left(\langle a,x=0 \rangle^{T_5}, \langle a,x=0 \rangle^{T_6}\right)$ is 9. It can be checked that in this case, the value of $S_{\text{MaxDiff}}$ for the state pair is also 9.

Example 5 (Average Time Difference). Consider the two timed automata $T_5$ and $T_6$ in Figure 4. Let us look at the value of the (long-run) average time difference quantitative timed simulation distance $S_{\text{AvgDiff}}$ for the state pair $\left(\langle a,x=0 \rangle^{T_5}, \langle a,x=0 \rangle^{T_6}\right)$. As usual, for the value to be finite, we require time-abstract simulation (with time-divergence). If time-abstract simulation holds, we take the average with respect to the number of transitions (over non-Zeno runs). For the state pair, a time difference of 9 occurs infinitely often, but this difference occurs in only one-third of the transitions (the transitions from $a$ locations). For the transitions from $b$ and $c$, the time discrepancy is zero. Thus, the value for $S_{\text{AvgDiff}}$ is $\frac{9+0+0}{3} = 3$.

To compute the values of the three simulation functions, we use the framework of turn-based games on finite-state game graphs. We introduce two new game theoretic objectives (these objectives are required for computing two of the quantitative simulation functions) on these game graphs, namely, eventual debit-sum level and average debit-sum level objectives, and present novel solutions for both. We need to consider the sums of the weights encountered as in our quantitative simulation functions, the global time is the sum of the time durations of all the preceding transitions.

Eventual debit-sum level and average debit-sum level games are also interesting on their own. We next illustrate average debit-sum level games. These games are played on two-player finite-state turn-based game graphs. Each transition in the game graph incurs a cost (denoted by a negative weight), or a reward (denoted by a positive weight). These costs can be viewed as monetary losses, or monetary gains. The debit-sum level at a stage in the game denotes the absolute value of the
monetary balance if the balance is negative (the balance is the sum of all the positive and negative costs and rewards). The objective of player 1 is to have the lowest possible average debit-sum level. As a financial application, consider the case when banks have to take overnight loans from the Central Bank loan windows in case of need (these loans need to be renewed each day the loan is not repaid). It is in the banks interests to minimize the average of the loan amount per day.

Example 6 (Debit Sum-Level Turn Based Games). Consider the turn-based game depicted in Figure 5. All locations are player-1 locations. The numbers on the edges denote the costs or rewards that player-1 gets when that transition is taken. Positive weights denotes rewards, and negative weights denotes costs. Viewing the weights as monetary transactions, and starting with a monetary balance of zero at a, if player 1 loops around the left loop, then the trace, together with the monetary balances during the run of the play. The average negative balance, i.e, the average debit-sum level (per unit location visit), is \( \frac{0+9+5+1}{4} = \frac{15}{4} \). If player 1 loops around the right loop, then the trace, together with the balances is: \((a, 0) (b_4, -5) (b_5, -3) (b_6, -1) \)\( \omega \). The average negative balance is \( \frac{0+5+3+1}{4} = \frac{9}{4} \). Thus the optimum average debit sum-level value for player 1 is \( \frac{9}{4} \), and the optimum strategy is to loop around the right-hand side, where it needs to borrow less, on average.

Our Contributions. Our main contributions in the present work are as follows.

* We define three quantitative refinement metrics incorporating Zenoness conditions semantically, that is our refinement metrics ignore artificial Zeno runs present in systems due to modelling artifacts. We also show that these quantitative functions are actually (directed) metrics.

* We define quantitative timed simulation functions corresponding to the refinement metrics using a game theoretic formulation. These quantitative simulation functions also incorporate Zenoness conditions for obtaining physically meaningful system differences. As far we know, this is is the first work which handles Zeno runs when computing simulation functions.

* We present algorithms for computing all the defined quantitative timed simulation functions to within any desired degree of accuracy for any given timed automaton.

* We introduce new game theoretic objectives on finite-state turn-based game graphs, namely, eventual debit-sum level objectives and average debit-sum level objectives, and present novel solutions for both. These new objectives are required in the computation of the defined quantitative simulation functions.

We have considered the (more challenging) framework of global event times in our quantitative simulation functions. Our solution framework is also applicable where the mismatches are only with respect to transition durations (simple algorithms are applicable in this case). Our algorithms can easily be generalized to consider quantitative simulation functions in which an observation \( \sigma \) is allowed to match a different observation \( \sigma' \), but with some matching penalty in case \( \sigma \neq \sigma' \) (the penalty being in addition to the timing mismatch of \( \sigma, \sigma' \)). Thus, our algorithms apply to the
computation of quantitative simulation functions which consider the Skorokhod metric [JS03] over mismatches.

**Related Work.** There has been a recent body of work on the theory of approximate bisimulation for continuous and switched systems (e.g. [Gir12; Gir13; GJP08; GP11; GPT10; Tab08]). The focus of the approach in [GJP08; GPT10] is on systems with real-valued outputs and the approximations are targeted towards output values which change in a continuous fashion. The focus is not on the timing aspect. The simulation relations are constrained to match output values at exactly the same sample points, thus there is no mechanism to incorporate the time discrepancies. The work in [Gir13] uses quantized system values in the bisimulation relation, and shows how this can be used to synthesize controllers of lower complexity. Approximate bisimilar models have also been used in symbolic frameworks to design controllers for various classes of systems and desired properties, see, e.g. [Gir12; Tab08]. The work in [QFD11] presents a similarity relation for hybrid systems (which are more general than the timed automata in our work) where the approximation is with respect to the maximal timing mismatch over runs, as well as output values. Computation of similarity relations is reduced to solving a class of derived hybrid games, however, these games are not decidable. The work gives a sufficient condition which ensures decidability. For timed systems, the work in [HMP05] presented maximal time difference quantitative timed simulation functions, however, Zeno issues were ignored. Our solutions for the new objectives on finite-state game graphs builds on previous work on mean payoff parity games, multi-dimensional mean payoff, and energy games [Bou+11; CD10; Cha+10b; Cha10; CHJ05]. The new game objectives presented in our work, that are required for the quantitative timed simulation functions, were previously unstudied, and require new ideas in their solutions.

## 2 Quantitative Timed Trace Difference and Refinement Metrics

In this section we define quantitative refinement functions on timed systems which quantify timing mismatches. These functions allow approximate matching of timed traces and generalize timed and untimed refinement relations.

**Timed Transition System (TTS).** A timed transition system (TTS) is a tuple $A = \langle S, \Sigma, \rightarrow, \mu, S_0 \rangle$ where

- $S$ is the set of states.
- $\Sigma$ is a set of atomic propositions (the observations).
- $\rightarrow \subseteq S \times \mathbb{R}^+ \times S$ is the transition relation such that for all $s \in S$ there exists at least one $s' \in S$ such that for some $\Delta$, we have that $(s, \Delta, s')$ belongs to $\rightarrow$.
- $\mu : S \mapsto 2^\Sigma$ is the observation map which assigns a truth value to atomic propositions in each state.
- $S_0 \subseteq S$ is the set of initial states.

We write $s \xrightarrow{t} s'$ if $(s, t, s')$ belongs to $\rightarrow$. A state trajectory is an infinite sequence $s_0 \xrightarrow{t_0} s_1 \xrightarrow{t_1} \cdots$, where for each $j \geq 0$, we have $s_j \xrightarrow{t_j} s_{j+1}$. The state trajectory is initialized if $s_0 \in S_0$ is an initial state. A state trajectory $s_0 \xrightarrow{t_0} s_1 \cdots$ induces a trace given by the observation sequence $\mu(s_0) \xrightarrow{t_0} \mu(s_1) \xrightarrow{t_1} \cdots$. To emphasize the initial state, we say $s_0$-trace for a trace induced by a state trajectory starting from $s_0$. A trace is initialized if it is induced by an initialized state trajectory. Given a trace $tr$ induced by a state trajectory $s_0 \xrightarrow{t_0} s_1 \xrightarrow{t_1} \cdots$, let $\text{time}_{tr}[i]$ denote $\sum_{j=0}^{i} t_j$, i.e. the time of the $i$-th transition. The trace $tr$ is time-convergent or Zeno if $\lim_{i \to \infty} \text{time}_{tr}[i]$ is finite;
otherwise it is \textit{time-divergent} or \textit{non-Zeno}. We denote the sets all of time-divergent initialized trajectories, and of time-divergent initialized traces of a timed transition system $A$ by $\text{Timediv}(A)$ and $\mu(\text{Timediv}(A))$ respectively, and the sets of initialized trajectories, and of all initialized traces of $A$ by $\text{Trajs}(A)$ and $\mu(\text{Trajs}(A))$ respectively. A TTS is \textit{well-formed} if from every $s_0 \in S_0$, there exists a $s_0$-trace in $\text{Timediv}(A)$. We consider only well-formed TTSs in the sequel. The TTS $A_f$ \textit{refines} or \textit{implements} the TTS $A_s$ (the specification) if every initialized trace of $A_f$ is also an initialized trace of $A_s$. We first define various quantitative notions of refinement that quantify if the behavior of an implementation TTS is “close enough” to a specification TTS. We begin by defining several metrics on trace differences and refinements.

**Maximal Timing Mismatch Trace Difference Distance.** Given two traces $tr = \sigma_0 \xrightarrow{t_0} \sigma_1 \xrightarrow{t_1} \sigma_2 \ldots$ and $tr' = \sigma'_0 \xrightarrow{t'_0} \sigma'_1 \xrightarrow{t'_1} \sigma'_2 \ldots$, the maximal timing mismatch trace difference distance $D_{\text{MaxDiff}}(tr, tr')$ is defined by

$$D_{\text{MaxDiff}}(tr, tr') = \begin{cases} \infty & \text{if } \sigma_n \neq \sigma'_n \text{ for some } n \\ \sup_n \{|\text{time}_{tr}[n] - \text{time}_{tr'}[n]|\} & \text{otherwise} \end{cases}$$

The distance $D_{\text{MaxDiff}}(tr, tr')$ indicates the maximal time discrepancy between matching observations in the two traces $tr$ and $tr'$.

**Proposition 1.** The function $D_{\text{MaxDiff}}()$ is a metric on timed traces.

**Refinement Distance Induced by $D_{\text{MaxDiff}}$.** The trace difference metric $D_{\text{MaxDiff}}$ induces a \textit{refinement distance} between two TTSs. Given two TTSs $A_f$ (the refined system) and $A_s$ (the specification), with initial state sets $S_f, S_s$ respectively, the \textit{refinement distance} of $A_f$ with respect to $A_s$ induced by $D_{\text{MaxDiff}}$ is given by

$$R_{\text{MaxDiff}}(A_f, A_s) = \sup_{tr_f \in \mu(\text{Timediv}(A_f))} \inf_{tr_s \in \mu(\text{Trajs}(A_s))} D_{\text{MaxDiff}}(tr_f, tr_s)$$

where $tr_f$ (respectively, $tr_s$) is a $q_f$-trace (respectively, $q_s$-trace) for some $q_f \in S_f$ (respectively, $q_s \in S_s$). We quantify over time-divergent traces of the refinement as Zeno traces are physically unrealizable\(^1\). Notice that this refinement distance is asymmetric: it is a \textit{directed distance} [AFS09]. The refinement distance $R_{\text{MaxDiff}}(A_f, A_s)$ indicates quantitatively how well initialized time-divergent traces of $A_f$ match corresponding initialized traces of $A_s$ with respect to the $D_{\text{MaxDiff}}$ trace difference metric.

**Proposition 2.** The function $R_{\text{MaxDiff}}()$ is a directed metric on timed transition systems.

**Proof.** The proof is similar to the proof of Proposition 4.

We next define several other trace difference metrics, which in turn induce their own refinement distances on TTSs.

\(^1\) Note that we do not need to put any time-divergence requirement on the traces from $A_s$; the “inf” operator ensures that only time-divergent traces are considered from the well-formed TTS $A_s$. The distance $D_{\text{MaxDiff}}(tr_f, tr_s)$ is infinite if one trace is time-divergent and the other Zeno.
Let \( \sigma\rightarrow_1^{t_1} \sigma_2\rightarrow_2^{t_2} \ldots \) and \( \sigma'_0\rightarrow_1^{t'_1} \sigma'_1\rightarrow_1^{t'_2} \sigma'_2\rightarrow_2^{t'_3} \ldots \), the limit-maximal timing mismatch trace difference distance \( D_{\text{LimMaxDiff}}(\sigma, \sigma') \) is defined by

\[
D_{\text{LimMaxDiff}}(\sigma, \sigma') = \begin{cases} 
\lim_{M \to \infty} \sup_{n \geq M} \{ |\text{time}_\sigma[n] - \text{time}_{\sigma'}[n]| \} & \text{if } \sigma \neq \sigma' \text{ for some } n \\
0 & \text{otherwise}
\end{cases}
\]

The distance \( D_{\text{LimMaxDiff}}(\sigma, \sigma') \) indicates the limit-maximal time discrepancy between matching observations in the two traces \( \sigma \) and \( \sigma' \). That is, it indicates the eventual “steady state” maximal time discrepancy, ignoring any initial spikes in the time discrepancy between the two traces (we still require all observations to be matched).

In the sequel, we view limits as having values on the extended real line (i.e. in \( \mathbb{R} \cup \{-\infty, \infty\} \)).

**Lemma 1.** Let \( a_n \) and \( b_n \) both be non-decreasing or both be non-increasing sequences of real numbers for \( n \geq 0 \). Then \( \lim_{n \to \infty}(a_n) \) and \( \lim_{n \to \infty}(b_n) \) both exist and \( \lim_{n \to \infty}(a_n) + \lim_{n \to \infty}(b_n) = \lim_{n \to \infty}(a_n + b_n) \).

**Lemma 2.** Let \( a_n \) and \( b_n \) be real numbers for \( n \geq 0 \) and let \( M \geq 0 \). Then \( \sup_{n \geq M}(a_n) + \sup_{n \geq M}(b_n) \geq \sup_{n \geq M}(a_n + b_n) \).

**Proposition 3.** The function \( D_{\text{LimMaxDiff}}() \) is a metric on timed traces.

**Proof.** We prove \( D_{\text{LimMaxDiff}}(\sigma_1, \sigma_2) + D_{\text{LimMaxDiff}}(\sigma_2, \sigma_3) \geq D_{\text{LimMaxDiff}}(\sigma_1, \sigma_3) \).

If all the observation sequences of \( \sigma_1, \sigma_2, \sigma_3 \) are not the same, or if \( D_{\text{LimMaxDiff}}(\sigma_1, \sigma_2) \) or \( D_{\text{LimMaxDiff}}(\sigma_2, \sigma_3) \) is infinite, then the claim is straightforward. So consider that the observation sequences of the three traces are the same and that \( D_{\text{LimMaxDiff}}(\sigma_1, \sigma_2) \) and \( D_{\text{LimMaxDiff}}(\sigma_2, \sigma_3) \) are both finite. We have

\[
D_{\text{LimMaxDiff}}(\sigma_1, \sigma_2) + D_{\text{LimMaxDiff}}(\sigma_2, \sigma_3) = \lim_{M \to \infty} \sup_{n \geq M} \{ |\text{time}_{\sigma_1}[n] - \text{time}_{\sigma_2}[n]| \} + \lim_{M \to \infty} \sup_{n \geq M} \{ |\text{time}_{\sigma_2}[n] - \text{time}_{\sigma_3}[n]| \}
\]

\[
= \lim_{M \to \infty} \left( \sup_{n \geq M} \{ |\text{time}_{\sigma_1}[n] - \text{time}_{\sigma_2}[n]| \} \right) + \lim_{M \to \infty} \left( \sup_{n \geq M} \{ |\text{time}_{\sigma_2}[n] - \text{time}_{\sigma_3}[n]| \} \right) \text{ by Lemma 1.}
\]

\[
\geq \lim_{M \to \infty} \left( \sup_{n \geq M} \{ |\text{time}_{\sigma_1}[n] - \text{time}_{\sigma_2}[n]| \} \right) + \lim_{M \to \infty} \left( \sup_{n \geq M} \{ |\text{time}_{\sigma_2}[n] - \text{time}_{\sigma_3}[n]| \} \right) \text{ by Lemma 2.}
\]

\[
\geq \lim_{M \to \infty} \sup_{n \geq M} \{ |\text{time}_{\sigma_1}[n] - \text{time}_{\sigma_3}[n]| \}
\]

\[
= D_{\text{LimMaxDiff}}(\sigma_1, \sigma_3).
\]

The desired result follows.

**Refinement Distance Induced by \( D_{\text{LimMaxDiff}} \).** The trace difference metric \( D_{\text{LimMaxDiff}}(A_t, A_s) \) induces the refinement distance \( R_{\text{LimMaxDiff}}(A_t, A_s) \). Formally, given two timed transition systems \( A_t, A_s \) with initial state sets \( S_t, S_s \) respectively, the refinement distance of \( A_t \) with respect to \( A_s \) induced by \( D_{\text{LimMaxDiff}} \) is given by

\[
R_{\text{LimMaxDiff}}(A_t, A_s) = \sup_{\tau_{q_t} \in \mu(\text{TimeDiv}(A_t))} \inf_{\tau_{q_s} \in \mu(\text{Trajsi}(A_s))} D_{\text{LimMaxDiff}}(\tau_{q_t}, \tau_{q_s})
\]

where \( \tau_{q_t} \) (respectively, \( \tau_{q_s} \)) is a \( q_t \)-trace (respectively, \( q_s \)-trace) for some \( q_t \in S_t \) (respectively, \( q_s \in S_s \)).
Proposition 4. The function \( R_{\text{LimMaxDiff}}() \) is a directed metric on timed transition systems.

Proof. We prove \( R_{\text{LimMaxDiff}}(A_1, A_2) + R_{\text{LimMaxDiff}}(A_2, A_3) \geq R_{\text{LimMaxDiff}}(A_1, A_3) \).

The interesting case is when both \( R_{\text{LimMaxDiff}}(A_1, A_2) \) and \( R_{\text{LimMaxDiff}}(A_2, A_3) \) are finite. Let \( R_{\text{LimMaxDiff}}(A_1, A_2) = K_{1,2} \) and let \( R_{\text{LimMaxDiff}}(A_2, A_3) = K_{2,3} \). Consider any \( tr_1 \in \mu(\text{Timediv}(A_1)) \). Since \( K_{1,2} = \sup_{tr_{q_1} \in \mu(\text{Timediv}(A_1))} \inf_{tr_{q_2} \in \mu(\text{Trajs}(A_3))} \{ D_{\text{LimMaxDiff}}(tr_{q_1}, tr_{q_2}) \} \), we have that \( K_{1,2} \geq \inf_{tr_{q_2}} \{ D_{\text{LimMaxDiff}}(tr_{1}, tr_{q_2}) \} \). Hence we have that for any given \( \epsilon > 0 \), there exists \( tr_2 \in \mu(\text{Trajs}(A_2)) \) such that \( D_{\text{LimMaxDiff}}(tr_1, tr_2) < K_{1,2} + \epsilon \). Now, \( tr_2 \) must be time divergent (i.e. \( tr_2 \in \mu(\text{Timediv}(A_2)) \)), otherwise \( D_{\text{LimMaxDiff}}(tr_1, tr_2) \) is not finite. Using a similar argument, we have that there exists a trace \( tr_3 \in \mu(\text{Trajs}(A_3)) \) such that \( D_{\text{LimMaxDiff}}(tr_2, tr_3) < K_{2,3} + \epsilon \).

Since \( D_{\text{LimMaxDiff}}(tr_1, tr_2) + D_{\text{LimMaxDiff}}(tr_2, tr_3) \geq D_{\text{LimMaxDiff}}(tr_1, tr_3) \), we have that

\[
D_{\text{LimMaxDiff}}(tr_1, tr_3) < K_{1,2} + K_{2,3} + 2 \cdot \epsilon.
\]

Since there exists a \( tr_3 \) such that the above inequality holds for any \( \epsilon > 0 \), we have that

\[
\inf_{tr_{q_3} \in \mu(\text{Trajs}(A_3))} D_{\text{LimMaxDiff}}(tr_1, tr_{q_3}) \leq K_{1,2} + K_{2,3}.
\]

And since this inequality holds for any \( tr_1 \in \mu(\text{Timediv}(A_1)) \), we have

\[
\sup_{tr_{q_1} \in \mu(\text{Timediv}(A_1))} \inf_{tr_{q_3} \in \mu(\text{Trajs}(A_3))} D_{\text{LimMaxDiff}}(tr_{q_1}, tr_{q_3}) \leq K_{1,2} + K_{2,3}.
\]

The desired result follows. \( \square \)

Limit-Average Trace Difference Distance. Given two traces \( tr = \sigma_0 \xrightarrow{t_0} \sigma_1 \xrightarrow{t_1} \sigma_2 \ldots \) and \( tr' = \sigma'_0 \xrightarrow{t'_0} \sigma'_1 \xrightarrow{t'_1} \sigma'_2 \ldots \), the limit-average trace difference distance \( D_{\text{AvgDiff}}(tr, tr') \) is defined by

\[
D_{\text{AvgDiff}}(tr, tr') = \left\{ \begin{array}{ll}
\lim_{M \to \infty} \left( \sup_{n \geq M} \left\{ \frac{\sum_{i=0}^{n} |time_{\sigma[i]} - time_{\sigma'[i]}|}{n} \right\} \right) & \text{if } \sigma_j \neq \sigma'_j \text{ for some } j \\
0 & \text{otherwise}
\end{array} \right.
\]

The distance \( D_{\text{AvgDiff}}(tr, tr') \) indicates the long-run average of the time discrepancies between the two traces.

Proposition 5. The function \( D_{\text{AvgDiff}}() \) is a metric on timed traces.

Proof. We prove \( D_{\text{AvgDiff}}(tr_1, tr_2) + D_{\text{AvgDiff}}(tr_2, tr_3) \geq D_{\text{AvgDiff}}(tr_1, tr_3) \).

If all the observation sequences of \( tr_1, tr_2, tr_3 \) are not the same, or if \( D_{\text{AvgDiff}}(tr_1, tr_2) \) or \( D_{\text{AvgDiff}}(tr_2, tr_3) \) is infinite, then the claim is straightforward. So consider that the observation sequences of the three traces are the same and that \( D_{\text{AvgDiff}}(tr_1, tr_2) \) and \( D_{\text{AvgDiff}}(tr_2, tr_3) \) are both...
finite. We have $D_{\text{AvgDiff}}(tr_1, tr_2) + D_{\text{AvgDiff}}(tr_2, tr_3)\]

$$\leq \lim_{M \to \infty} \left( \sup_{n \geq M} \left\{ \frac{\sum_{i=0}^{n} (|time_{tr_1}[i] - time_{tr_2}[i]|)}{n} \right\} \right) + \lim_{M \to \infty} \left( \sup_{n \geq M} \left\{ \frac{\sum_{i=0}^{n} (|time_{tr_2}[i] - time_{tr_3}[i]|)}{n} \right\} \right)$$

$$= \lim_{M \to \infty} \left( \sup_{n \geq M} \left\{ \frac{\sum_{i=0}^{n} (|time_{tr_1}[i] - time_{tr_2}[i]|)}{n} \right\} \right)$$

$$\geq \lim_{M \to \infty} \left( \sup_{n \geq M} \left\{ \frac{\sum_{i=0}^{n} (|time_{tr_1}[i] - time_{tr_2}[i]|)}{n} \right\} \right)$$

$$= D_{\text{AvgDiff}}(tr_1, tr_3).$$

The desired result follows. ☐

**Refinement Distance Induced by $D_{\text{AvgDiff}}$.** The trace difference metric $D_{\text{AvgDiff}}$ induces the refinement distance $R_{\text{AvgDiff}}(A_t, A_s)$. The formal definition is as that for $R_{\text{LimMaxDiff}}(A_t, A_s)$, replacing $D_{\text{LimMaxDiff}}$ with $D_{\text{AvgDiff}}$.

**Proposition 6.** The function $R_{\text{AvgDiff}}()$ is a directed metric on timed transition systems.

*Proof.* The proof is similar to the proof of Proposition 4. ☐

**A Note on Zeno-Asymmetry in Refinement Metrics.** There is an asymmetry in the definitions for refinement metrics with respect to Zenoness as only Zeno behaviors of $A_t$ are given special treatment. This is because in case of Zeno behavior by the specification, our definitions automatically give a value of $\infty$, which is the correct notion. That is, for $\Psi \in \{D_{\text{MaxDiff}}, D_{\text{LimMaxDiff}}, D_{\text{AvgDiff}}\}$, we have $\Psi(tr_{q_t}, tr_{q_s}) = \infty$ if $tr_{q_t}$ is time divergent, and $tr_{q_s}$ is time convergent.

### 3 Timed Simulation Relations

The general trace inclusion problem for timed systems is undecidable [AD94]; and simulation relations allow us to restrict our attention to a computable relation. In this section we recall the definitions of timed and untimed simulation relations. We also present timed and untimed simulation games which give an alternative way of defining the simulation relations. This will motivate the game theoretic definitions of quantitative timed simulation functions in the sequel.

**Timed Simulation Relations.** Let $A_t$ and $A_s$ be two TTSs. A binary relation $\preceq \subseteq S_t \times S_s$ is a **timed simulation** if $s_t \preceq s_s$ implies the following conditions: (1) $\mu(s_t) = \mu(s_s)$; and (2) If $s_t \overset{t}{\to} s'_t$, then there exists $s'_s$ such that $s_s \overset{t}{\to} s'_s$, and $s'_t \preceq s'_s$. The state $s_t$ is timed simulated by the state $s_s$ if there exists a timed simulation $\preceq$ such that $s_t \preceq s_s$. A binary relation $\equiv$ is a **timed bisimulation** if it is a symmetric timed simulation. Two states $s_t$ and $s_s$ are timed bisimilar if there exists a timed bisimulation $\equiv$ with $s_t \equiv s_s$. Timed bisimulation is stronger than timed simulation which in turn is stronger than trace inclusion. If state $s_t$ is timed simulated by state $s_s$, then every $s_t$-trace is also a $s_s$-trace.
Untimed Simulation Relations. Untimed simulation and bisimulation relations are defined analogously to timed simulation and bisimulation relations by ignoring the duration of time steps. Formally, a binary relation \( \preceq_u \subseteq S_T \times S_u \) is an (untimed) simulation if \( s_t \preceq_u s_u \) implies the following conditions: (1) \( \mu(s_t) = \mu(s_u) \). (2) If \( s_t \xrightarrow{t} s'_t \), then there exists \( s'_u \in S_u \) such that \( s_u \xrightarrow{t'} s'_u \), and \( s'_t \preceq s'_u \). A symmetric untimed simulation relation is called an untimed bisimulation.

Timed simulation and bisimulation require that times be matched exactly. This is often too strict a requirement, especially since timed models are approximations of the real world. On the other hand, untimed simulation and bisimulation relations ignore the times on moves altogether.

Analogous to the notions of quantitative refinement presented in Section 2, we will define quantitative notions of simulation functions which lie in between these extremes in Section 5. We will define quantitative simulation functions in a game theoretic framework. The motivation for the game theoretic framework for simulation relations is presented next.

Timed and Untimed Simulation Games. There exists an alternative equivalent game theoretic view of timed simulation (a similar view exists for untimed simulation). Given two timed transition systems \( A_T \) and \( A_u \), consider a two player turn-based bipartite timed transition game structure \( \mathcal{G}_T(A_T, A_u) \) with state space \( (S_T \times S_u) \cup (S_T \times S_u) \) (the full formal definitions of game structures will be presented in Section 4). The states of player 2 (the antagonist) are \( S_T \times S_u \times \{1\} \) and the states of player 1 (the protagonist) are \( S_T \times S_u \times \{2\} \). The transitions are:

**Player-2 transitions.** \( \langle s_t, s_u, 2 \rangle \xrightarrow{A} \langle s'_t, s_u, 1 \rangle \) such that \( s_t \xrightarrow{A} s'_t \) is a valid transition in \( A_T \).

**Player-1 transitions.** \( \langle s_t, s_u, 1 \rangle \xrightarrow{A} \langle s'_t, s_u, 2 \rangle \) such that \( s_u \xrightarrow{A} s'_u \) is a valid transition in \( A_u \).

To decide if \( s_u \) time-simulates \( s_t \), we play the following game. Let \( \langle s_t, s_u, 2 \rangle \) be the initial state such that \( \mu(s_t) = \mu(s_u) \). Player-2 picks a transition of some duration \( \Delta_t \) from this state and moves to some state \( \langle s'_t, s_u, 1 \rangle \). From \( \langle s'_t, s_u, 1 \rangle \), player 1 then picks a transition of duration \( \Delta_u \) such that \( \Delta_u = \Delta_t \) and moves to \( \langle s'_t, s'_u, 2 \rangle \) such that \( \mu(s'_u) = \mu(s'_u) \). If no such transition exists, then player 1 loses. If the game can proceed forever without player-1 losing, then player 2 loses and player 1 wins. If player 1 has a winning strategy from \( \langle s_t, s_u, 2 \rangle \), then \( s_u \) time-simulates \( s_t \). For untimed simulation, we ignore the time durations of the moves (player 1 can pick transitions of any duration from \( A_u \)). We denote the two player turn-based bipartite untimed transition game as \( \mathcal{G}_u(A_T, A_u) \).

4 Finite-state Game Graphs

We will define the values of quantitative timed simulation functions in Section 5 through game theoretic formulations of problems for finite-state turn based game graphs. In this section, we first present the basic background on finite-state game graphs, and the relevant known results; then introduce new game theoretic objectives (that were not studied before but are required for quantitative timed simulation functions) and present solutions for the new objectives.

4.1 Basic Definitions and Known Results

In this section we present definitions of finite game graphs, plays, strategies, objectives, notion of winning, and the decision problems.

**Game Graphs.** A game graph \( G = \langle Q, E \rangle \) consists of a finite set \( Q \) of states partitioned into player-1 states \( Q_1 \) and player-2 states \( Q_2 \) (i.e., \( Q = Q_1 \cup Q_2 \) and \( Q_1 \cap Q_2 = \emptyset \)), and a set \( E \subseteq Q \times Q \) of directed edges such that for all \( q \in Q \), there exists (at least one) \( q' \in Q \) such that \( (q, q') \in E \). A
player-1 game is a game graph where $Q_1 = Q$ and $Q_2 = \emptyset$. The subgraph of $G$ induced by $S \subseteq Q$ is the graph $(S, E \cap (S \times S))$ (which is not a game graph in general); the subgraph induced by $S$ is a game graph if for all $s \in S$ there exist $s' \in S$ such that $(s, s') \in E$.

**Plays and Strategies.** A game on $G$ starting from a state $q_0 \in Q$ is played in rounds as follows. If the game is in a player-1 state, then player 1 chooses the successor state from the set of outgoing edges; otherwise the game is in a player-2 state, and player 2 chooses the successor state from the set of outgoing edges. The game results in a play from $q_0$, i.e., an infinite path $\rho = q_0q_1 \ldots$ such that $(q_i, q_{i+1}) \in E$ for all $i \geq 0$. The prefix of length $n$ of $\rho$ is denoted by $\rho(n) = q_0 \ldots q_n$. A strategy for player 1 is a function $\pi_1 : Q^*Q_1 \to Q$ such that $(q, \pi_1(\rho \cdot q)) \in E$ for all $\rho \in Q^*$ and $q \in Q_1$. An outcome of $\pi_1$ from $q_0$ is a play $q_0q_1 \ldots$ such that $\pi_1(q_0 \ldots q_i) = q_{i+1}$ for all $i \geq 0$ such that $q_i \in Q_1$.

An outcome of player 2 is defined analogously. A player-1 strategy is memoryless if it is independent of the history and depends only on the current state, and hence can be described as a function $\pi_1 : Q_1 \to Q$. Memoryless strategies for player 2 are defined analogously. We denote by $\Pi_1$ and $\Pi_2$ the set of strategies for player 1 and player 2, respectively. Given a starting state $q$, a strategy $\pi_1$ for player 1 and a strategy $\pi_2$ for player 2, we have a unique play $q_0q_1q_2\ldots$, such that $q_0 = q$ and for all $i \geq 0$ we have that (i) if $q_i$ is a player-1 state, then $q_{i+1} = \pi_1(q_0, q_1, \ldots, q_i)$; and (ii) if $q_i$ is a player-2 state, then $q_{i+1} = \pi_2(q_0, q_1, \ldots, q_i)$. We denote the unique play as $\rho(\pi_1, \pi_2, q)$.

**Objectives.** In this work we consider both qualitative and quantitative objectives. We first introduce qualitative objectives that we use in our work. A **qualitative objective** for $G$ is a set $\phi \subseteq Q^\omega$ of winning plays. For a play $\rho$, we denote by $\inf(\rho)$ the set of states that occur infinitely often in $\rho$. We consider Büchi objectives, and its dual coBüchi objectives which are defined as follows. A Büchi objective consists of a set of Büchi states, and requires that the set $B$ is visited infinitely often. Formally, the Büchi objective defines the following set of winning plays: $\text{Büchi}(B) = \{ \rho \mid \inf(\rho) \cap B \neq \emptyset \}$. Dually the coBüchi objective consists of a set $C$ of coBüchi states and requires that states outside $C$ be visited only finitely often, and defines the set $\text{coBüchi}(C) = \{ \rho \mid \inf(\rho) \subseteq C \}$ of winning plays.

When we will consider qualitative objectives, the objective of player 1 will be disjunction of two Büchi objectives, and the objective of player 2 will be the complement (conjunction of two Büchi objectives). The qualitative objectives will be used to model Zeno runs. We now introduce several quantitative objectives.

**Quantitative Objectives.** A **quantitative objective** for $G$ is a function $f : Q^\omega \to \mathbb{R}$ that maps every play to a real-valued number (in contrast a qualitative objective can be interpreted as a function $\phi : Q^\omega \to \{0,1\}$ that maps plays to Boolean rewards, with 1 for winning plays). Let $w : E \to \mathbb{Z}$ be a weight function and let us denote by $W$ the largest weight (in absolute value) according to $w$. For a finite prefix $\rho(n) = q_0q_1 \ldots q_n$ of a play we denote by $\text{Sum}(w)(\rho(n)) = \sum_{i=0}^{n-1} w(q_i, q_{i+1})$ the sum of the weights of the prefix. The **debit level** at the end of the prefix $\rho(n)$ is defined by

$$\text{Deb}(w)(\rho(n)) = \max(0, -\sum_{i=0}^{n-1} w(q_i, q_{i+1})).$$

Note the negative sign in the definition. The debit level denotes the amount by which the accumulated sum of the weights has dipped below 0 at the end of $\rho(n)$ (if the sum of the weights is positive, *i.e.* there is a credit, then the debit level is defined to be 0). We will consider the following quantitative objective functions.

**Maximum debit level.** For a play $\rho$, the maximum debit level is the maximal debit level that occurs in it. Formally, for a play $\rho$ and the weight function $w$ we have $\text{MaxDeb}(w)(\rho) = \sup_n \text{Deb}(w)(\rho(n)) = \inf\{v_0 \mid \forall n \geq 0, v_0 + \text{Sum}(w)(\rho(n)) \geq 0\}$. 

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**Eventual maximal debit level.** For a play $\rho$, the eventual maximum debit level is the maximal debit level that occurs after some point in the play (i.e., it is the maximal debit level that occurs infinitely often in the play). Formally, for a play $\rho$ and the weight function $w$ we have $\text{EvMaxDeb}(w)(\rho) = \limsup_{n \to \infty} \text{Deb}(w)(\rho(n)) = \lim_{M \to \infty} \sup_{n \geq M} \text{Deb}(w)(\rho(n)) = \inf \{ v_0 \mid \exists n_0 \geq 0. \forall n \geq n_0, v_0 + \sum(w)(\rho(n)) \geq 0 \}$.

**Average weight.** The mean-payoff (or limit-average weight) objective function on a play $\rho = q_0q_1 \ldots$ is the long-run average of the weights of the play, i.e., $\text{Avg}(w)(\rho) = \limsup_{n \to \infty} \frac{1}{n} \cdot \sum(w)(\rho(n))$.

**Average debit-sum.** Along with the previous objective, we introduce a new objective function, which we call the average debit level that assigns to every play the long-run average of the debit levels. Formally, $\text{AvgDeb}(w)(\rho) = \limsup_{n \to \infty} \frac{\sum_{i=0}^{n} \text{Deb}(w)(\rho(i))}{n}$. Note that since the debit level is defined to be 0 if the accumulated sum is positive (i.e., a positive credit level), a positive credit cannot cancel out a positive debit-sum in the averaging process in $\text{AvgDeb}(w)(\rho)$. Observe that in contrast to mean-payoff objective that is the average of the weights, the average debit level has the flavor of the average of the partial sums of the weights.

In the sequel, when the weight function $w$ is clear from context we will omit it and simply write $\text{Sum}(\rho(n))$ and $\text{Avg}(\rho)$, and so on. For each of the above quantitative objectives, we will consider a version of the quantitative objective that is a disjunction with a coBüchi objective. Formally for a quantitative objective $f$ and coBüchi objective $\text{coBüchi}(C)$, the quantitative objective that is the disjunction of the two objectives is defined as follows for a play $\rho$: if $\rho \in \text{coBüchi}(C)$, then the objective function assigns value 0 to $\rho$, otherwise it assigns value $f(\rho)^2$. We will refer to the corresponding version of the quantitative objectives with disjunction with coBüchi objective as $\text{MaxDebCB}$, $\text{EvMaxDebCB}$, $\text{AvgCB}$, and $\text{AvgDebCB}$, respectively (and when the weight function $w$ and the coBüchi set $C$ is clear from the context we drop them for simplicity).

**Winning Strategies, Optimal Value, and Optimal Strategies.** A player-1 strategy $\pi_1$ is winning in a state $q$ (we also say that player 1 is winning, or that $q$ is a winning state) for a qualitative objective $\phi$ if $\rho \in \phi$ for all outcomes $\rho$ of $\pi_1$ from $q$. The optimal value for a quantitative objective is the minimal value that player 1 can guarantee against all strategies of player 2. Formally, for a quantitative objective $f$ that maps plays to real-valued rewards, the optimal value $\text{Opt}(f)(q)$ at state $q$ is defined as $\text{Opt}(f)(q) = \inf_{\pi_1 \in \Pi_1} \sup_{\pi_2 \in \Pi_2} f(\rho(\pi_1, \pi_2, q))$. A strategy for player 1 is optimal if it achieves the optimal value against all strategies of player 2, i.e., a strategy $\pi_1^*$ is optimal if we have $\text{Opt}(f)(q) = \sup_{\pi_2 \in \Pi_2} f(\rho(\pi_1^*, \pi_2, q))$. Similarly, a player-2 strategy $\pi_2^*$ is optimal if we have $\text{Opt}(f)(q) = \inf_{\pi_1 \in \Pi_1} f(\rho(\pi_1, \pi_2^*, q))$.

We now present a theorem that summarizes known results about Büchi and coBüchi games, maximum debit level (also known as minimal initial credit for energy games), and mean-payoff games. The results of Büchi and coBüchi objectives follow from [EJ91], the results for maximum debit level games follows from the results on energy games of [CD10], and the result for mean-payoff games follows from [Bou+11; CHJ05] (also note that in [Bou+11; CD10; CHJ05] player 1 has a conjunction of energy (or mean-payoff) with parity objectives (parity objectives generalize coBüchi objectives), whereas in our setting player 1 has the disjunction of energy (or mean-payoff) with coBüchi, and thus the roles of player 1 and player 2 in this work are exchanged as compared to [Bou+11; CD10; CHJ05]).

**Theorem 1.** The following assertions hold for finite-state game graphs.

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2 We focus on objectives involving debits, and 0 is the best possible debit value for player 1.
1. The set of winning states in games with disjunction of two coBüchi objectives can be computed in time $O(|Q| \cdot |E|)$, and memoryless winning strategies exist for player 1 and winning strategies of player 2 require one-bit memory (from their respective winning states).

2. The optimal value for maximum debit level functions with coBüchi disjunctions can be computed in time $O(|Q|^2 \cdot |E| \cdot W)$, and memoryless optimal strategies exist for player 1 and optimal strategies for player 2 require finite memory. If the optimal value is not $+\infty$, then the optimal value is at most $|Q| \cdot |W|$.

3. The optimal value for limit-average functions with coBüchi disjunctions can be computed in time $O(|Q|^2 \cdot |E| \cdot W)$, and memoryless optimal strategies exist for player 1 and the optimal strategies of player 2 may require infinite memory.

4.2 New Results and Algorithms – Eventual Maximal Debit Level Objectives

In this section we present a solution for games with eventual maximal debit level objectives (i.e. for minimal initial credit for eventual survival). We will solve the problem by a reduction to a coBüchi game. We start with a lemma that is required for the reduction.

**Lemma 3.** For all game graphs with a weight function $w$, the following assertions hold:

1. The optimal value of the eventual maximal debit level objective is at most the optimal value of the maximum debit level objective i.e., for all states $q$ we have $\text{Opt}(\text{EvMaxDeb})(q) \leq \text{Opt}(\text{MaxDeb})(q)$.

2. The optimal value of the maximum debit level objective is $+\infty$ iff the optimal value of the eventual maximal debit level is $+\infty$.

**Proof.** The first item follows from definition. The proof of the second item is as follows: if we have a sequence $\{x_n\}_{n \geq 0}$ of integers, then $\sup x_n = +\infty$ iff $\limsup x_n = +\infty$. Considering $\{x_n\}_{n \geq 0}$ to be the sequence $\{\text{Sum}(\rho(n))\}_{n \geq 0}$, we obtain the result for all plays. Hence the result follows.

**Reduction of EvMaxDeb objective games to coBüchi Games.** The solution for the optimal value for the EvMaxDeb objective is obtained as follows: (1) We compute $\text{Opt}(\text{MaxDeb})(q)$ using algorithms of Theorem 1, and if $\text{Opt}(\text{MaxDeb})(q)$ is infinite, then $\text{Opt}(\text{EvMaxDeb})(q)$ is infinite (by Lemma 3); (2) if $\text{Opt}(\text{MaxDeb})(q)$ is finite, then by Lemma 3 and by Theorem 1 we have $\text{Opt}(\text{EvMaxDeb})(q)$ to be finite and that $\text{Opt}(\text{EvMaxDeb})(q) \leq |Q| \cdot W$. If $\text{Opt}(\text{EvMaxDeb})(q)$ is finite, the procedure to check whether $\text{Opt}(\text{EvMaxDeb})(q) \leq D$ for $0 \leq D \leq |Q| \cdot W$ is as follows: we construct a coBüchi game where we keep track of the current sum of weights; the coBüchi states are those where the tracked sum is not below $-D$. We restrict the set that the tracked sums belong to as follows. First, we divide $Q$ into two disjoint subsets: $Q = Q_\infty \uplus Q_\infty^\infty$, based on the value of $\text{Opt}(\text{MaxDeb})$ at states in these subsets. States in $Q_\infty$ have $\text{Opt}(\text{MaxDeb})$ to be $+\infty$; states in $Q_\infty^\infty$ have $\text{Opt}(\text{MaxDeb})$ to be finite (observe that, by Theorem 1, all states in $Q_\infty^\infty$ have $\text{Opt}(\text{MaxDeb})$ to be at most $|Q| \cdot W$). Consider an optimal strategy $\pi_1$ for player 1 for the EvMaxDeb objective starting from a state in $Q_\infty$. This strategy must ensure that a state in $Q_\infty$ (with any tracked sum) is never visited, as from these states $\text{Opt}(\text{MaxDeb})$ (and hence $\text{Opt}(\text{EvMaxDeb})$) is $+\infty$, thus if $\pi_1$ were to visit a state in $Q_\infty$ (with any tracked sum) starting from $q$, then player 2 could make $\text{EvMaxDeb}$ to be $+\infty$ from $q$ for that player-1 strategy. Thus, we define $Q_{\infty}^\infty$ (with any tracked sum) to be losing sink states in the coBüchi game (i.e. these sink states are defined to not belong to the coBüchi set).
Bounds on Tracked Weight-Sums. Starting from the initial state $q$, an optimal player-1 strategy for the $\text{EvMaxDeb}$ objective must ensure that the game always stays in $Q_{\infty}$ states by above. Observe that if player 1 cannot avoid staying inside a negative weight-sum cycle, then all states in that cycle have $\text{Opt}(\text{MaxDeb}) = \infty$ (and hence $\text{Opt}(\text{EvMaxDeb}) = \infty$). Thus all states in that cycle will be outside $Q_{\infty}$. Moreover, it is in the interest of player 1 to never complete a negative weight-sum cycle. Thus, we have that optimal player-1 strategies for the $\text{EvMaxDeb}$ objective ensure that the game always stays in $Q_{\infty}$ states, and that a negative weight-sum cycle is never formed. Since the sum of the negative weights in a cycle is at most $-|Q| \cdot W$, we thus only need to keep track of weight-sums that are at least $-|Q| \cdot W$. If the tracked sum of the weights ever falls below $-|Q| \cdot W$, we transition to a losing sink state in the coBüchi game (i.e. a sink state which is defined to not belong to the coBüchi set).

We now show that we only need to track weight sums that are below $|Q| \cdot W$ as follows. Consider the states $Q_{\infty}$. From these states, starting with an initial weight sum of 0, player 1 has a strategy to ensure that the sum of weights never goes below $-|Q| \cdot W$ by definition. This means that from these states, starting with an initial weight sum of $|Q| \cdot W$, player 1 has a strategy to ensure that the sum of weights never goes below 0. Thus, in the game for $\text{EvMaxDeb}$, if the tracked sum of the weights ever exceeds $|Q| \cdot W$ at a $Q_{\infty}$ state, we transition to a winning sink state in the coBüchi game (i.e. a sink state which is defined to belong to the coBüchi set).

From the above two cases it follows that we only need to keep track of the sum of weights that lie between $-|Q| \cdot W$ and $|Q| \cdot W$. If the sum of the weights is greater than or equal to $-D$ (and it is a $Q_{\infty}$ state), then we call the state a coBüchi state, otherwise it is a bad state for the coBüchi objective. The goal of player 1 is the coBüchi objective, which is equivalently the objective to ensure that from some point on the sum of the weights is always greater than or equal to $-D$. Using a binary search for $D$ for values between 0 and $|Q| \cdot W$ we obtain the optimal value. Also observe that the games we construct for the binary searches have at most $O(|Q|^2 \cdot W)$ states and $O(|E| \cdot |Q| \cdot W)$ edges.

For disjunction with a coBüchi objective, we have the same reduction as above, but in the end we obtain a game with disjunction of two coBüchi objectives.

**Theorem 2.** The optimal player-1 strategy, and the optimal value $\text{Opt}(\text{EvMaxDebCB})(q)$ for the eventual maximal debit level objective with coBüchi disjunction can be computed in time $O(|Q|^3 \cdot |E| + W^2 \cdot \log(|Q| \cdot W))$. 

The next example illustrates the difference between maximum debit level and eventual maximal debit level objectives.

**Example 7 (Maximum debit level vs eventual maximal debit level).** Consider the game graph $G_0$ in Figure 6. The game $G_0$ has only one play from $q_0$, namely, $q_0 \rightarrow q_1 \rightarrow (q_2 \rightarrow q_3 \rightarrow)^\omega$. It can be seen that $\text{Opt}(\text{MaxDeb})(q_0)$ is 10 as a debit level of 10 is seen on the transition from $q_0$ to $q_1$. However,
Opt(\text{EvMaxDeb})(q_0) \text{ is only 2, as the debit level 10 occurs only once in the play. The debit level 2 however occurs infinitely often in the play. Thus, Opt(\text{EvMaxDeb})(q_0) \text{ is 2.} \quad \square 

4.3 New Results and Algorithms – Average Debit Level Objectives

In this section we present a solution for games with average debit level objectives. We start with an example that illustrates average debit level objectives.

Example 8. Consider the game graph $G_1$ in Figure 7. The game $G_1$ has only one play from $q_0$, namely, $(q_0 \rightarrow q_1 \rightarrow q_2)\omega$ (and similarly only one play from any state). For this play we compute the debit and credit levels: let $(q, d, c)$ denote the state $q$, and $d, c$ the debit and credit levels at that point in the play (note that only either the debit, or credit level can be non-zero, by definition). The play together with debit and credit levels is: $(q_0, 0, 0) \rightarrow ((q_1, 1, 0) \rightarrow (q_2, 0, 1) \rightarrow (q_0, 0, 0) \rightarrow)\omega$.

Thus the average debit level $\text{AvgDeb}(w)(q_0) = 1/3$. Now consider the only play from $q_2$. The play annotated with debit and credit levels is: $(q_2, 0, 0) \rightarrow ((q_0, 1, 0) \rightarrow (q_1, 2, 0) \rightarrow (q_2, 0, 0) \rightarrow)\omega$. Note that credit levels never rise above 0 in this play. The average debit level $\text{AvgDeb}(w)(q_2)$ for this play is 1. Thus, where we “enter” in a cycle affects the value of the average debit level. $\square$

The next lemma is a technical lemma on integer sequences.

Lemma 4. Let $x_0, x_1, \ldots$ be a sequence of integers. The following assertions hold.

1. If $x_i$ is positive for every $i$, and there exist $i_0 \geq 0$ and $N > 0$ such that for all $i \geq i_0$, there exists $1 \leq m_i \leq N$ such that $x_{i+m_i} > x_i$. Then, $\lim_{M \to \infty} \left( \sup_{k \geq M} \left\{ \frac{\sum_{i=0}^{k-1} x_i}{k} \right\} \right) = \infty$.

2. Suppose (i) there exists $W < \infty$ such that for all $i \geq 0$, we have $|x_{i+1} - x_i| \leq W$; and (ii) there exist $i_0 \geq 0$ and $N > 0$ such that for all $i \geq i_0$, there exists $1 \leq m_i \leq N$ such that $x_{i+m_i} < x_i$. Then, there exists $M \geq 0$ such that $x_i < 0$ for all $i \geq M$.

Proof. We present both items of the proof.

1. Consider $\sum_{i=i_0}^{i_0+\alpha \cdot N + j} x_i$ for $\alpha \geq 0$ and $0 \leq j < N$. Consider the set

$$X_\alpha = \{ x_j \mid i_0 + \alpha \cdot N \leq j < i_0 + \alpha \cdot (N + 1) \}$$

It follows by induction that for every $\alpha \geq 0$, we have: (i) there exists $x_i \in X_\alpha$, such that $x_i \geq \alpha$ (informally, the claims hold because there is an increment of at least one, starting from $x_{i_0}$, in every $N$ steps); and hence, (ii) $\sum_{i=i_0}^{i_0+\alpha \cdot N + j} x_i \geq 0 + 1 + \cdots + \alpha$ (since we can pick $x_i \in X_\alpha$ such that $x_i \geq \alpha$). Thus,

$$\frac{\sum_{i=i_0}^{i_0+\alpha \cdot N + j} x_i}{i_0 + \alpha \cdot N + j} \geq \frac{\alpha \cdot (\alpha + 1)}{2 \cdot (i_0 + \alpha \cdot N + j)} \geq \frac{\alpha \cdot (\alpha + 1)}{2 \cdot (i_0 + (\alpha + 1) \cdot N)}$$

Fig. 7. Game Graph $G_1$
for every \( \alpha \geq 0 \) and \( 0 \leq j < N \). Thus,

\[
\frac{\sum_{i=i_0}^{i_0+\alpha\cdot N+j} x_i}{i_0 + \alpha \cdot N + j} \geq \frac{\alpha}{2 \cdot (\frac{3n}{\alpha + 1} + N)}
\]

Therefore, for every \( \alpha \geq 0 \), we have

\[
\sup_{k > N} \left\{ \frac{\sum_{i=0}^{k-1} x_i}{k} \right\} \geq \frac{\sum_{i=0}^{i_0+\alpha\cdot N} x_i}{i_0 + \alpha \cdot N} \geq \frac{\alpha}{2 \cdot (\frac{3n}{\alpha + 1} + N)}
\]

Letting \( \alpha \to \infty \), we have the desired result.

2. It follows from induction that for every \( \alpha \geq 0 \), there exists \( x_{i_0} \in \{x_j \mid i_0 + \alpha \cdot N \leq j < i_0 + \alpha \cdot (N + 1)\} \), such that \( x_{i_0} + \alpha \leq x_{i_0} \) (that is, \( x_{i_0} \) is at least \( \alpha \) less than \( x_{i_0} \)). Informally, the claims hold because there is a decrement of at least one, starting from \( x_{i_0} \), in every \( N \) steps.

Consider any \( \alpha > 1 + N \cdot W + x_{i_0} \). Consider the set

\[
X_\alpha = \{x_j \mid i_0 + \alpha \cdot N \leq j < i_0 + \alpha \cdot (N + 1)\}
\]

Since, \( |x_{i+1} - x_i| \leq W \) for \( i \) in the given sequence, for any \( x, x' \in X_\alpha \), we must have \( |x - x'| \leq N \cdot W \). Also, there exists \( x_\alpha \in X_\alpha \) such that \( x_{i_0} + \alpha \leq x_{i_0} \). Thus, for all \( x \in X_\alpha \), we have

\[
x + \alpha \leq x_{i_0} + N \cdot W.
\]

Since \( \alpha > 1 + N \cdot W + x_{i_0} \), we have,

\[
x + 1 + N \cdot W + x_{i_0} \leq x_{i_0} + N \cdot W.
\]

Rearranging, we get \( x \leq -1 \). Thus, for all \( i > (2 + N \cdot W + x_{i_0}) \cdot N \), we have \( x_i \leq -1 \).

\( \square \)

**Corollary 1.** Consider a play \( \rho = q_0 q_1 \ldots \) of a finite-state game graph \( G \). The following assertions hold.

1. Suppose there exist \( i_0 \geq 0 \) and \( N > 0 \) such that for all \( i \geq i_0 \), there exists \( 1 \leq m_i \leq N \) such that

   \( \text{Sum}(\rho(i)) > \text{Sum}(\rho(i + m_i)) \). Then, \( \text{AvgDeb}(\rho) = \infty \).

2. Suppose there exist \( i_0 \geq 0 \) and \( N > 0 \) such that for all \( i \geq i_0 \), there exists \( 1 \leq m_i \leq N \) such that

   \( \text{Sum}(\rho(i)) < \text{Sum}(\rho(i + m_i)) \). Then, \( \text{AvgDeb}(\rho) = 0 \).

\( \square \)

**Mean-Payoff Supremal Games for Solving Games with Average Debit Level Objectives.**

We define a dual objective and game to \( \text{AvgCB} \) in which player 1 is trying to maximize the value; it will be used in the solution for average debit level objectives. Let the quantitative objective function \( \text{AvgCBSup} \) on a play \( \rho \) be defined as \( +\infty \) (the best payoff for player 1) if the coBüchi objective is satisfied, else it is \( \liminf_{n \to \infty} \frac{1}{n} \cdot \text{Sum}(w)(\rho(n)) \). Let \( \text{OptSup}(\text{AvgCBSup}) \) be the value of the game when player 1 is trying to maximize the value of \( \text{AvgCBSup} \), defined as \( \text{OptSup}(\text{AvgCBSup}) = \)
sup_{\pi_1 \in \Pi^1} \inf_{\pi_2 \in \Pi^2} \text{AvgCBSup}(\rho(\pi_1, \pi_2, q)). We call these games mean-payoff \textit{supremal} games. The algorithm for the solution of OptSup(AvgCBSup) is similar to the algorithm for the solution of Opt(AvgCB); and results for mean-payoff games in Theorem 1 apply also to mean-payoff supremal games.

Lemma 5. The following assertions hold: consider a weight function w, and coBüchi objective coBüchi(C), and then we have

1. If Opt(MaxDebCB)(q) = +\infty, then Opt(AvgDebCB)(q) = +\infty.
2. If OptSup(AvgCBSup)(q) > 0, then Opt(AvgDebCB)(q) = 0.

Proof. We present proof of both the items.
1. If Opt(MaxDebCB)(q) = \infty, then consider a finite-memory optimal strategy \pi_2^* for player 2 (such a strategy exists by Theorem 1). Once the strategy \pi_2^* is fixed we obtain a graph where only player 1 makes choices. Since Opt(MaxDebCB)(q) = \infty, it follows that for every cycle U in the graph the sum of the weights in U is negative, and there is at least one state in U that is not a coBüchi state (i.e., U \cap (Q \setminus C) \neq \emptyset). Since all cycles are negative the first condition of Corollary 1 is satisfied for all paths with N as the size of the graph. Moreover the coBüchi objective is also falsified. This concludes the proof of the first item.

2. Suppose OptSup(AvgCBSup)(q) > 0. Consider a memoryless optimal strategy \pi_1 for player 1 for the mean-payoff supremal objective with coBüchi disjunction (such a strategy exists by Theorem 1). Since OptSup(AvgCBSup)(q) > 0, it follows that, in the graph obtained by fixing the strategy \pi_1, for every cycle U, either the sum of the weights is positive or U \subseteq C. Consider a play \rho which is an outcome of \pi_1. For the play, either (i) the coBüchi objective is satisfied; or (ii) AvgCBSup(\rho) > 0, and then the second condition of Corollary 1 is satisfied. In either case the desired result of the second item follows.

Reduction of Average Debit coBüchi Games to Mean-Payoff coBüchi Games. We now use Lemma 5 to solve the average debit problem. By Lemma 5, we have that (i) if Opt(MaxDebCB) is +\infty, then Opt(AvgDebCB)(q) = +\infty; and (ii) if OptSup(AvgCBSup) > 0, then Opt(AvgDebCB)(q) = 0. We now consider the remaining case when (i) Opt(MaxDebCB)(q) is not +\infty, and (ii) OptSup(AvgCBSup)(q) \leq 0. We reduce the average debit level with coBüchi disjunction problem to solving a larger mean-payoff with coBüchi disjunction supremal game as follows: the new weights in the larger mean-payoff supremal game correspond to tracked negative \textit{sums} of original weights in the average debit level game (i.e. we track debit levels). For the mean-payoff supremal game, we construct a new weight function according to the current sum of original weights, i.e., if the current sum of original weights is \ell, then the new weight function assigns value max(−\ell, 0) to \textit{states} (note that games with weights on states can be easily transformed to games with weights on edges by assigning all outgoing edges from a state q the state-weight of q). The optimal value of this constructed supremal game with the new weight function with mean-payoff with coBüchi disjunction supremal objective is equal to the optimal value for the average debit level with coBüchi disjunction objective in the original game for the original weight function.

Bounds on Tracked Weight-Sums. Consider the case when for a state q, we have (i) Opt(MaxDebCB)(q) to be not +\infty, and (ii) OptSup(AvgCBSup)(q) \leq 0.

\footnotesize
1 To avoid confusion, and as a memory aid, we use the term “supremal” exclusively in games where player 1 is the maximizer. The absence of “supremal” denotes games where player 1 is the minimizer.

\normalsize
1. Since $\text{Opt}((\text{MaxDebCB}(q))$ is not $+:\infty$, the following fact follows from Theorem 1.

**Fact 1:** There exists a memoryless player-1 strategy $\pi_1$ such that in all plays that arise from $\pi_1$, all cycles $U$ formed during the play either (a) have a weight-sum that is not negative or, (b) consist of only coBüchi states.

2. Since $\text{OptSup}(\text{AvgCBSup}(q)) \leq 0$, the following fact follows from Theorem 1.

**Fact 2:** Player 2 has a strategy $\pi_2$ such that in all plays that arise from $\pi_2$, all cycles $U$ formed during the play (a) have a weight-sum, that is not strictly positive; and (b) contain at least one non-coBüchi state.

Now consider average debit level games.

1. Due to **Fact 1**, player 1 can ensure that all cycles $U$ formed during the play have a weight-sum of at least 0, or consist of only coBüchi states. Observe that such strategies are good for player 1, since a negative cycle is only favorable for player 2, i.e., if a negative cycle is executed then the average debit level increases.

2. Due to **Fact 2**, player 2 can ensure that no matter the strategy of player 1, all the cycles formed will have a weight-sum of at most 0, and will contain at least one non-coBüchi state. Observe that such strategies are good for player 2, since a positive cycle is only favorable for player 1 for the average debit level objective, i.e., if a positive cycle is executed the average debit level decreases.

Combining the above two statements, we have that there exist optimal plays (plays when both players are playing optimally), for the average debit level objective, where for all cycles formed along the play, the weight-sum will be exactly be 0, and there will be at least one non-coBüchi state. Thus, if both players are playing optimally in the average debit coBüchi game, then the tracked original weight-sums will stay in the range $-|Q| \cdot W$ to $|Q| \cdot W$, and the coBüchi objective will not be satisfied. We use this fact to restrict the set that these tracked original weight-sums belong to in the constructed mean-payoff supremal game. We assign a new weight function (and add two new non-coBüchi sink states) in the constructed mean-payoff supremal game (recall that in this game, player 1 is the maximizer) to ensure that both players play such that the resulting plays have tracked original weight-sums in the range $-|Q| \cdot W$ to $|Q| \cdot W$ as follows.

First, if the tracked original weight-sum ever goes below $-|Q| \cdot W$, we transition to a special sink non-coBüchi state $q_-$ with new state weight $-1$. This state weight assignment ensures that player 1 plays such that the tracked original-weight sums never go below $-|Q| \cdot W$, since all states with tracked original-weight sums not smaller than $-|Q| \cdot W$ have new state weights that are at least 0, and thus since player 1 is the maximizer in this new game, it is in its interest to not fall into the sink non-coBüchi state with the lower state weight. This first modification gives us a mean-payoff supremal game with tracked original weight-sums in the range $-|Q| \cdot W$ to $\infty$.

We apply a second modification to the weight function: if the tracked original weight-sum ever goes above $|Q| \cdot W$, we transition to a special sink non-coBüchi state $q_+$ with new state weight $|Q| \cdot W + 1$. The second state weight assignment above ensures that player 2 plays such that the tracked original weight-sums never go above $|Q| \cdot W$, since all states with tracked original-weight-sums at least $-|Q| \cdot W$ (by the first modification above), and at most $|Q| \cdot W$ (by the second modification), have new state weights at most $|Q| \cdot W$. Since player 2 is the minimizer in this new game, it is in its interest to not fall into the sink non-coBüchi state with the higher weight $|Q| \cdot W + 1$. Thus, these two modifications ensure that we only need to keep track of original-weight sums from $-|Q| \cdot W$ to $|Q| \cdot W$ in the larger mean-payoff supremal game. The constructed game has $O(|Q|^2 \cdot W)$ states,
\[O(|E| \cdot |Q| \cdot W)\] edges, and the supremal absolute value of the new weight function is \(O(|Q| \cdot W)\).
Thus our reduction and Theorem 1 yield the following result for average debit level objectives.

**Theorem 3.** The optimal player-1 strategy, and the optimal value \(\text{Opt}(\text{AvgDeb})(q)\) for average debit level objective with coBüchi disjunction can be computed in time \(O(|Q|^6 \cdot |E| \cdot W^4)\).

### 4.4 From Debit Level to Difference Level Objectives

An easy extension of the debit level objectives is difference level objectives — instead of the debit levels that arise in plays, we consider the absolute values of the sum of the weights (i.e., we consider \(|\text{Sum}(\rho(n))|\) values). We call the corresponding versions as Diff instead of Deb. These games can be solved using two weight functions (the original weight function and its negation), and then applying results for two-dimensional energy and mean-payoff games with disjunction with coBüchi objectives.

Applying our techniques to solve eventual maximal debit level, and average debit level, along with the results of [Cha10b; Cha10; VR11] we obtain the following result.

**Theorem 4.** The optimal player-1 strategy, and the optimal value for difference-sum function with coBüchi disjunction, \(\text{Opt}(\text{MaxDiffCB})(q)\), the optimal value \(\text{Opt}(\text{EvMaxDiffCB})(q)\) for the eventual maximal difference level objective with coBüchi disjunction, and the optimal value \(\text{Opt}(\text{AvgDiffCB})(q)\) for average difference level objective with coBüchi disjunction, can all be computed in \(O(\text{poly}(Q,E,W))\) time, where poly is a polynomial function.

### 5 Quantitative Timed Simulation Functions

In this section, define quantitative timed simulation functions (QTSFs) for timed transition systems in a game theoretic framework.

**Timed Transition Game Structures.** A timed transition game structure is a tuple \(\mathcal{G} = (S, \rightarrow)\) where
- \(S\) is the set of states, consisting of player-1 states \(S_1\) and player-2 states \(S_2\) (i.e., \(S = S_1 \cup S_2\) and \(S_1 \cap S_2 = \emptyset\)),
- \(\rightarrow \subseteq S \times \mathbb{R}^+ \times S\) is the transition relation such that for all \(s \in S\) there exists at least one \(s' \in S\) such that for some \(\Delta\), we have \(s \xrightarrow{\Delta} s'\).

Plays, objectives, strategies, outcomes etc. are as in finite games (Section 4).

**Quantitative Timed Simulation Functions (QTSFs).** Analogous to the game theoretic presentation of timed simulation games, we now present a game theoretic definition of QTSFs. Recall the two player turn-based bipartite timed transition game structure \(\mathcal{G}(A_1, A_2)\) defined in Section 3.

Consider a play \(\rho\) in \(\mathcal{G}(A_1, A_2)\): \(\langle s_0^0, s_0^1, \Delta^0 \rangle \xrightarrow{\Delta^1} \langle s_1^0, s_1^1, 1 \rangle \xrightarrow{\Delta^0} \langle s_1^1, s_1^2, 2 \rangle \xrightarrow{\Delta^1} \cdots\). Let \(\rho(\tau)\) be the projection on \(A_1\), thus \(\rho(\tau)\) is the \(A_1\) trajectory \(s_1^0 \xrightarrow{\Delta^0} s_1^1 \xrightarrow{\Delta^1} \cdots\). Note that \(\rho(\tau)\) is a valid trajectory in \(A_1\). We define \(\rho(\hat{s})\) similarly.

**Definition 1 (Quantitative Timed Simulation Objectives Over Game Plays).** Recall the \(D_{\text{MaxDiff}}, D_{\text{LimMaxDiff}}, D_{\text{AvgDiff}}\) trajectory difference metrics defined in Section 2. For \(\Psi \in \{D_{\text{MaxDiff}}, D_{\text{LimMaxDiff}}, D_{\text{AvgDiff}}\}\), we define \(\Psi^\text{Timediv}()\) as follows for a play \(\rho\) in \(\mathcal{G}(A_1, A_2)\):

\[
\Psi^\text{Timediv}(\rho) = \begin{cases} 
0 & \text{if } \rho(\tau) \notin \text{Timediv}(A_1) \\
\Psi(\rho(\tau), \rho(\hat{s})) & \text{otherwise}
\end{cases}
\]

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Definition 2 (QTSFs). Let $A_t, A_s$ be timed transition systems, and let $G_t(A_t, A_s)$ be the two player turn-based bipartite timed transition game structure defined in Section 3. The value of the QTSF, denoted $S_{\Psi \text{Timediv}}(t,s)$, for states $s_t$ and $s_s$ of $A_t$ and $A_s$ respectively, and for $\Psi \text{Timediv} \in \{\text{MaxDiff, LimMaxDiff, D Timediv}\}$, is defined as follows:

$$S_{\Psi \text{Timediv}}(t,s_t) = \inf_{\pi_s \in I_s} \sup_{\pi_t \in I_t} \Psi_{\text{Timediv}}(\rho(\pi_t, \pi_s, (t,s,t), 2))$$

where $\rho(\pi_t, \pi_s, (t,s,t), 2)$ is the trajectory which results given the player-1 strategy $\pi_t \in I_t$ and the player-2 strategy $\pi_s \in I_s$. Equivalently, $S_{\Psi \text{Timediv}}(t,s_t) = \text{Opt}(\Psi_{\text{Timediv}})((t,s,t), 2))$.

The next proposition states that the refinement distance between two systems $A_t, A_s$ with the initial states $q_t, q_s$ respectively is at most the value of the corresponding simulation function.

Proposition 7 (QTSFs Over-Approximate Refinement Distances). Let $A_t$ and $A_s$ be two TTSs with the initial states $q_t$ and $q_s$ respectively. For $(A, \Psi_{\text{Timediv}}) \in \{(\text{MaxDiff, D Timediv}), (\text{LimMaxDiff, D Timediv}), (\text{AvgDiff, D Timediv})\}$, we have

$$\mathcal{R}_A(A_t, A_s) \leq S_{\Psi_{\text{Timediv}}}(t,s_t).$$

Proof. Consider any $\vartheta > S_{\Psi_{\text{Timediv}}}(t,s_t)$, and any time-divergent $s$-trajectory $\vartraj_t$ of $A_s$. The trajectory $\vartraj_t$ corresponds naturally to a player-2 strategy $\pi_t$ in the game $G_t(A_t, A_s)$, where player 2 picks transitions which lead to the $A_t$-projected trajectory $\vartraj_s$ no matter what player 1 does. By the definition of the simulation function, there must exist a player-1 strategy such that the $A_s$-projected trajectory is at most $\vartheta$ away from $\vartraj_t$ according to the metric $D_A$. The desired result follows.

We next show that the simulation functions are actually directed metrics.

Proposition 8 (QTSFs are Directed Metrics). For $\Psi_{\text{Timediv}} \in \{D_{\text{MaxDiff}}, D_{\text{LimMaxDiff}}, D_{\text{AvgDiff}}\}$, the function $S_{\Psi_{\text{Timediv}}}$ is a directed metric over states of TTSs.

Proof. It is clear that for any TTS and for any state $s$, we have $S_{\Psi_{\text{Timediv}}}(s,s) = 0$. We now show the triangle inequality. Let $A_a, A_b, A_c$ be timed transition systems, with initial states $s_a, s_b, s_c$ respectively. We show $S_{\Psi_{\text{Timediv}}}(s_a, s_b) + S_{\Psi_{\text{Timediv}}}(s_b, s_c) \geq S_{\Psi_{\text{Timediv}}}(s_a, s_c)$.

Given two states $s_a \in A_a$ and $s_b \in A_b$ for any two systems $A_a$ and $A_b$, consider the game $G_t(A_a, A_b)$. We recall that player 1 is the player which is trying to simulate the other player. We say a player-1 strategy $\pi_a$ is $\varepsilon$-optimal for the objective $\Psi_{\text{Timediv}}$ from the state $(s_a, s_b)$ if $S_{\Psi_{\text{Timediv}}}(s_a, s_b) + \varepsilon \geq \sup_{\pi_a \in I_a} \psi_{\text{Timediv}}(\rho(\pi_a, \pi_a, (s_a, s_b), 2))$. Let $K_{a,b,c}, K_{a,c}, K_{a,c}$ denote $S_{\Psi_{\text{Timediv}}}(s_a, s_b), S_{\Psi_{\text{Timediv}}}(s_b, s_c), S_{\Psi_{\text{Timediv}}}(s_a, s_c)$ respectively. It suffices to show that for every $\varepsilon > 0$ there exists a player-1 strategy $\pi_{a,c}, 1$ such that $K_{a,b} + K_{b,c} + \varepsilon \geq \sup_{\pi_{a,c} \in I_{a,c}} \psi_{\text{Timediv}}(\rho(\pi_{a,c}, 1, \pi_{a,c}, 2, (s_a, s_b), 2))$ for the game $G_t(A_a, A_c)$ from the state $(s_a, s_c)$ for the objective $\Psi_{\text{Timediv}}$. We construct such a player-1 strategy as follows.

Consider the case when $K_{a,b} < \infty$ and $K_{b,c} < \infty$ (otherwise the claim is trivially proved). Consider an $\varepsilon/2$-optimal player-1 strategy $\pi_{a,b,1}$ for the game $G_t(A_a, A_b)$ from the state $(s_a, s_b, 2)$ for the objective $\Psi_{\text{Timediv}}$. The player-1 strategy $\pi_{a,b,1}$ can be used to map any finite or infinite $s_a$-trajectory $\vartraj_a$ of $A_a$ to a unique $s_b$-trajectory $\vartraj_b$ of $A_b$: consider a player-2 strategy which "looks" only at the $A_a$ component and blindly generates $\vartraj_a$; the trajectory $\vartraj_b$ is the corresponding $A_b$.
trajectory as a result of $\pi_{a,b}^{\varepsilon/2}$ playing against this player-2 strategy. Let this map from trajectories of $A_a$ to those of $A_b$ be denoted as $\mathcal{M}_{a,b,\varepsilon/2}$. Observe that for an infinite time-divergent trajectory $\text{traj}_a$, we have $\Psi(\text{traj}_a, \mathcal{M}_{a,b,\varepsilon/2}(\text{traj}_a)) < K_{ab} + \varepsilon/2$ by the definition of the player-1 strategy $\pi_{a,b}^{\varepsilon/2}$ being $\varepsilon/2$ optimal. Similarly, there exists an $\varepsilon/2$-optimal player-1 strategy $\pi_{b,c}^{\varepsilon/2}$ for the game $\mathcal{G}_t(A_b, A_c)$ from the state $\langle s_b, s_c, 2 \rangle$ for the objective $\Psi_{\text{Timediv}}$.

We now define a player-1 strategy $\pi_{a,c,1}^*$ for the game $\mathcal{G}_t(A_a, A_c)$ from the state $\langle s_a, s_c, 2 \rangle$ as follows (it essentially picks the same moves as $\pi_{b,c}^{\varepsilon/2}$). Consider a $\mathcal{G}_t(A_a, A_c)$ play $\rho_{a,c}^f = \langle s_a, 0, 0, 2 \rangle \xrightarrow{\Delta_0^a} \langle s_a, 1, 0, 1 \rangle \xrightarrow{\Delta_0^a} \langle s_a, 1, 1, 2 \rangle \xrightarrow{\Delta_1^a} \cdots \xrightarrow{\Delta_n^a} \langle s_a, n, n, 1 \rangle$ Let $\mathcal{M}_{a,b,\varepsilon/2}(\langle \rho_{a,c}^f \rangle) = s_a^0 \xrightarrow{\Delta_0^a} s_a^1 \cdots \xrightarrow{\Delta_n^a} s_a^n$. Using $\mathcal{M}_{a,b,\varepsilon/2}()$, we map a play $\rho_{a,c}^f$ of $\mathcal{G}_t(A_a, A_c)$ to a play $\mathcal{G}_t(b,c,\varepsilon/2)(\rho_{a,c}^f)$ of $\mathcal{G}_t(A_b, A_c)$ using $\mathcal{M}_{a,b,\varepsilon/2}(\rho_{a,c}^f)$ on the $A_a$ components, leaving the $A_c$ components unchanged, as follows.

The play $\mathcal{G}_t(b,c,\varepsilon/2)(\rho_{a,c}^f)$ of $\mathcal{G}_t(A_b, A_c)$ is defined to be $\langle s_b, 0, 0, 2 \rangle \xrightarrow{\Delta_0^b} \langle s_b, 1, 0, 1 \rangle \xrightarrow{\Delta_0^b} \langle s_b, 1, 1, 2 \rangle \xrightarrow{\Delta_1^b} \cdots \xrightarrow{\Delta_n^b} \langle s_b, n, n, 1 \rangle$ (note that the components with the subscript $c$ remain unchanged). We note that $\mathcal{G}_t(b,c,\varepsilon/2)(\rho_{a,c}^f)$ is a valid play of $\mathcal{G}_t(A_b, A_c)$. Finally, $\pi_{a,c,1}(\rho_{a,c}^f)$ is defined to be $\pi_{b,c,1}(\mathcal{G}_t(b,c,\varepsilon/2)(\rho_{a,c}^f))$; that is, $\pi_{a,c,1}(\rho_{a,c}^f)$ is defined as the $A_c$-state $s_c^k$ where $s_c^k$ is the state prescribed by the player-1 strategy $\pi_{b,c,1}^*$ on the $\mathcal{G}_t(A_b, A_c)$-run obtained from $\rho_{a,c}^f$ by changing all the $A_b$ components to those in $\mathcal{M}_{a,b,\varepsilon/2}(\rho_{a,b,c}^f(a))$. Intuitively, given a finite play $\rho_{a,c}^f$ of $\mathcal{G}_t(A_a, A_c)$, player 1 (i) first obtains a finite trajectory $\xi_b$ in $A_b$ by mapping $\rho_{a,c}^f(a)$ (the $\rho_{a,c}^f$ trajectory projected onto $A_a$) to $\xi_b$ using the game $\mathcal{G}_t(A_a, A_b)$ and the player-1 strategy $\pi_{a,b,1}^{\varepsilon/2}$; (ii) utilizes the fact that the finite trajectories $\rho_{a,c}^f(c)$ and $\xi_b$ correspond to a play $\rho_{b,c}^f$ of $\mathcal{G}_t(A_b, A_c)$, and (iii) uses $\pi_{b,c,1}^{\varepsilon/2}(\rho_{b,c}^f)$ to prescribe the next $A_c$-state in the game $\mathcal{G}_t(A_a, A_c)$ from $\rho_{a,c}^f$.

We claim the player-1 strategy $\pi_{a,c,1}^*$ is such that $K_{a,b} + K_{b,c} + \varepsilon \geq \sup_{\pi_{a,c,2} \in \Pi_{a,c,2}} \Psi_{\text{Timediv}}(\rho_{a,c,1}, \pi_{a,c,2}, (s_a, s_c, 2)))$. Consider any player-2 strategy $\pi_{a,c,2}$ in the game $\mathcal{G}_t(A_a, A_c)$, and the resultant play $\rho_{a,c}^*$, when player 1 plays with the strategy $\pi_{a,c,1}^*$. If $\rho_{a,c}^*(a)$ is time-convergent, we are done. Assume $\rho_{a,c}^*(a)$ is time-divergent. The function $\mathcal{G}_t(b,c,\varepsilon/2)()$ can be seen as also mapping infinite plays $\rho_{a,c}$ of $\mathcal{G}_t(A_a, A_c)$ to plays $\rho_{b,c}$ of $\mathcal{G}_t(A_b, A_c)$ (the infinite mapping being the limit of the finite mappings, and noting that our games have infinite plays). We have the following facts (denoting $\mathcal{G}_t(b,c,\varepsilon/2)(\rho_{a,c}^f) = \rho_{b,c}^f$):

1. $\Psi(\rho_{a,c}^*(a), \rho_{b,c}^*(b)) < K_{a,b} + \varepsilon/2$ (due to the properties of $\mathcal{M}_{a,b,\varepsilon/2}(\rho_{a,c}^f)$).
2. Recall that $\rho_{b,c}^*$ is defined to the play obtained from $\rho_{a,c}^f$ by substituting the $A_a$ projected run with $\mathcal{M}_{a,b,\varepsilon/2}(\rho_{a,c}^f(a))$. We claim the following fact: $\rho_{b,c}^*$ is also equal to the play which arises in the game $\mathcal{G}_t(A_b, A_c)$ when player 1 plays with the strategy $\pi_{b,c,1}^{\varepsilon/2}$ against a player-2 strategy which simply picks $A_b$-transitions leading to the projected $A_b$ trajectory $\mathcal{M}_{a,b,\varepsilon/2}(\rho_{a,c}^*(a))$. That is, the $A_c$ run which arises due to the player-1 strategy $\pi_{a,c,1}^{\varepsilon/2}$ in the play $\rho_{a,c}^*$ is the same as the $A_c$ run which arises due to the player-1 strategy $\pi_{b,c,1}^{\varepsilon/2}$ in the play $\rho_{b,c}^*$. Thus $\rho_{b,c}^*(c) = \rho_{a,c,1}^*(c)$. The reason for this fact is that $\rho_{a,c}^*$ is the play when player 1 plays with the strategy $\pi_{a,c,1}^*$. The strategy $\pi_{a,c,1}^*$ is such that, by definition, the $A_c$-state proposed by $\pi_{b,c,1}^{\varepsilon/2}$ on $\rho_{b,c}^*[0..n]$, and the $A_c$-state proposed by $\pi_{a,c,1}^*$ on $\rho_{a,c}^*[0..n]$ are exactly the same.
From the second item above, we have that \( \Psi \left( \rho^*_b,c(b), \rho^*_b,c(c) \right) < K_{b,c} + \varepsilon/2 \) as \( \pi_{b,c,1}^\varepsilon \) is an \( \varepsilon/2 \) optimal player-1 strategy in the game \( \mathcal{G}_t(A_b, A_c) \). Combining with the first item in the enumeration above, we get by Propositions 1, 3 and 5 that \( \Psi \left( \rho^*_a,c(a), \rho^*_a,c(c) \right) < K_{a,b} + K_{b,c} + \varepsilon. \) Noting that \( \rho^*_a,c(a) \) is time-divergent, we get that \( \Psi^{\text{Timediv}}(\rho^*_a,c) < K_{a,b} + K_{b,c} + \varepsilon. \) The triangle inequalities for the simulation functions follow. \( \square \)

# 6 Computation of Quantitative Simulation Functions on Timed Automata

In this section we obtain algorithms for computing QTSFs on timed automata [AD94] (which suggest a finite syntax for specifying infinite-state timed structures), by reducing the problem to games on finite-state graphs. The solution involves the following steps. We first enlarge the TTS corresponding to the given timed automaton \( \mathcal{T} \), in Subsection 6.2, in order to measure elapsed time, and to measure the integer “time-ticks” (or integer time boundaries) crossed during executions of \( \mathcal{T} \) (if the current real-valued time is \( \Delta \), then \( \lfloor \Delta \rfloor \) integer time-ticks have elapsed). In Subsection 6.3, we define integer-time QTSFs which depend only on the elapsed integer time-ticks and show that these integer-time simulation functions are close to the original (real-valued) simulation functions. Next, we show that these integer-time QTSFs can be computed by a reduction to finite state game graphs in Subsection 6.4. Finally, we present the algorithm which ties all the steps together, and show that we can compute the quantitative simulation functions to within any desired degree of accuracy on timed automata.

## 6.1 Timed Automata

In this section, we briefly recall the model (for a detailed treatment, see [AD94]). A timed automaton \( \mathcal{T} \) is a tuple \( \langle L, \Sigma, C, \mu, \rightarrow, \gamma, S_0 \rangle \), where

- \( L \) is the set of locations; and \( \Sigma \) is the set of atomic propositions.
- \( C \) is a finite set of clocks. A clock valuation \( \nu : C \mapsto \mathbb{R}^+ \) for a set of clocks \( C \) assigns a real value to each clock in \( C \).
- \( \mu : L \mapsto 2^\Sigma \) is the observation map (it does not depend on clock values).
- \( \rightarrow \subseteq L \times L \times 2^C \times \Phi(C) \) gives the set of transitions, where \( \Phi(C) \) is the set of clock constraints generated by \( \psi : = x \leq d \mid d \leq x \mid \neg \psi \mid \psi_1 \land \psi_2 \).
- \( \gamma : L \mapsto \text{Constr}(C) \) is a function that assigns to every location an invariant on clock valuations.

All clocks increase uniformly at the same rate. When at location \( l \), a valid execution must move out of \( l \) before the invariant \( \gamma(l) \) expires. Thus, the timed automaton can stay at a location only as long as the invariant is satisfied by the clock values.

- \( S_0 \subseteq L \times \mathbb{R}^+[C] \) is the set of initial states.

Each clock increases at rate 1 inside a location. A clock valuation is a function \( \kappa : C \mapsto \mathbb{R}_{\geq 0} \) that maps every clock to a nonnegative real. The set of all clock valuations for \( C \) is denoted by \( K(C) \). Given a clock valuation \( \kappa \in K(C) \) and a time delay \( \Delta \in \mathbb{R}_{\geq 0} \), we write \( \kappa + \Delta \) for the clock valuation in \( K(C) \) defined by \( (\kappa + \Delta)(x) = \kappa(x) + \Delta \) for all clocks \( x \in C \). For a subset \( \lambda \subseteq C \) of the clocks, we write \( \kappa[\lambda := 0] \) for the clock valuation in \( K(C) \) defined by \( (\kappa[\lambda := 0])(x) = 0 \) if \( x \in \lambda \), and \( (\kappa[\lambda := 0])(x) = \kappa(x) \) if \( x \notin \lambda \). A clock valuation \( \kappa \in K(C) \) satisfies the clock constraint \( \theta \), written \( \kappa \models \theta \), if the condition \( \theta \) holds when all clocks in \( C \) take on the values specified by \( \kappa \). A state \( s = \langle l, \kappa \rangle \) of the timed automaton \( \mathcal{T} \) is a location \( l \in L \) together with a clock valuation \( \kappa \in K(C) \) such that the invariant at the location is satisfied, that is, \( \kappa \models \gamma(l) \). We let \( S \) be the set of all states
of \( T \). An edge \( \langle l, l', \lambda, g \rangle \) represents a transition from location \( l \) to location \( l' \) when the clock values at \( l \) satisfy the constraint \( g \). The set \( \lambda \subseteq C \) gives the clocks to be reset to 0 with this transition. The semantics of timed automata are given as timed transition systems. This is standard (see e.g. [AD94]), and omitted here.

Clock Region Equivalence. Clock region equivalence, denoted as \( \equiv \) is an equivalence relation on states of timed automata. The equivalence classes of the relation are called regions, and induce a time abstract bisimulation on the corresponding timed transition system. There are finitely many clock regions; more precisely, the number of clock regions is bounded by \( |L| \cdot |\prod_{x \in C} (c_x + 1) \cdot |C|! \cdot |C|^2| \).

For a real \( t \geq 0 \), let \( \text{frac}(t) = t - \lfloor t \rfloor \) denote the fractional part of \( t \). Given a timed automaton game \( T \), for each clock \( x \in C \), let \( c_x \) denote the largest integer constant that appears in any clock constraint involving \( x \) in \( T \) (let \( c_x = 1 \) if there is no clock constraint involving \( x \)). Two states \( \langle l_1, \kappa_1 \rangle \) and \( \langle l_2, \kappa_2 \rangle \) are said to be region equivalent if all the following conditions are satisfied: (a) \( l_1 = l_2 \), (b) for all clocks \( x \), we have \( \kappa_1(x) \leq c_x \) iff \( \kappa_2(x) \leq c_x \), (c) for all clocks \( x \) with \( \kappa_1(x) \leq c_x \), we have \( \lfloor \kappa_1(x) \rfloor = \lfloor \kappa_2(x) \rfloor \), (d) for all clocks \( x, y \) with \( \kappa_1(x) \leq c_x \) and \( \kappa_1(y) \leq c_y \), \( \text{frac}(\kappa_1(x)) \leq \text{frac}(\kappa_1(y)) \) iff \( \text{frac}(\kappa_2(x)) \leq \text{frac}(\kappa_2(y)) \), and (e) for all clocks \( x \) with \( \kappa_1(x) \leq c_x \), we have \( \text{frac}(\kappa_1(x)) = 0 \) iff \( \text{frac}(\kappa_2(x)) = 0 \). Given a state \( \langle l, \kappa \rangle \) of \( T \), we denote the region containing \( \langle l, \kappa \rangle \) as \( \text{Reg}(\langle l, \kappa \rangle) \).

Region Graph. The region graph \( \text{Reg}(T) \) corresponding to \( T \) is the time-abstract bisimulation quotient graph induced by the region equivalence relation (see [AD94] for details). The states of \( \text{Reg}(T) \) are the regions of \( T \). There is a transition \( R \rightarrow R' \) in the region graph iff there exists \( s \in R \) and \( s' \in R' \) such that there is a transition from \( s \) to \( s' \) according to the semantics of \( T \).

### 6.2 Enlarging the Timed Automaton TTS

For ease of presentation we assume that all clocks are bounded, \( i.e., \) that the invariants of each location can be conjuncted with the clause \( \wedge_{x \in C} (x \leq c_{\text{max}}) \) for some constant \( c_{\text{max}} \). The general case where clocks may be unbounded can be solved using similar algorithms, with some additional bookkeeping.

Given a timed automata \( T \) where all the clocks are bounded by \( c_{\text{max}} \), let \( \lceil T \rceil \) denote the timed transition system obtained by adding to \( T \) an extra clock \( z \), which cycles between 0 and 1, for measuring elapsed time, and an integer valued variable \( \text{ticks} \) which takes on values in \( \mathbb{N}_{\leq c_{\text{max}}} \), where \( \mathbb{N}_{\leq c_{\text{max}}} \) denotes the set \( \{0, 1, \ldots, c_{\text{max}}\} \). Formally, the set of states of \( \lceil T \rceil \) is \( S_{\lceil T \rceil} = S \times \mathbb{R}_{[0,1)} \times \mathbb{N}_{\leq c_{\text{max}}} \), where \( S \) is the set of states of \( T \). The state \( \langle s, \bar{z}, d \rangle \) of \( \lceil T \rceil \) has the following components:

- \( s \) is the state of the original timed automaton \( T \);
- \( \bar{z} \) is the value of the added clock \( z \) which gets reset to 0 every time it crosses 1 (i.e., if \( \kappa' \) is the clock valuation resulting from letting time \( \Delta \) elapse from an initial clock valuation \( \kappa \), then \( \bar{z} = \kappa'(z) = (\kappa(z) + \Delta) \mod 1 \)); and
- \( d \) denotes the value of the integer variable \( \text{ticks} \), and is equal to the number of integer boundaries crossed by the added clock \( z \) since the last transition: if the clock valuation in the previous state was \( \kappa \), and the transition time duration is \( \Delta \), then \( d = |\kappa(z) + \Delta| \) in the current state, where \( | \cdot \) denotes the integer floor function. Note that since all the clocks in \( T \) are bounded by \( c_{\text{max}} \), we have \( d \leq c_{\text{max}} \), as the maximum duration of a transition is \( c_{\text{max}} \), and \( \kappa(z) < 1 \) in the previous state. Note that \( d \) can have a value of 1 as the result of a transition of duration \( \Delta < 1 \), \( e.g., \) if the clock \( z \) had a value of 0.9 in the previous state, and \( \Delta = 0.2 \), then \( d = 1 \).

The region equivalence relation can be expanded to \( \lceil T \rceil \) states. Two states \( \langle \langle l_1, \kappa_1 \rangle, \bar{z}_1, d_1 \rangle \) and \( \langle \langle l_2, \kappa_2 \rangle, \bar{z}_2, d_2 \rangle \) of \( \lceil T \rceil \) are defined to be region equivalent if \( \langle l_1, d_1 \rangle = \langle l_2, d_2 \rangle \), and \( \kappa_2^{\bar{z}_2 \mod 2} = \kappa_2^{\bar{z}_1 \mod 2} \), where
where $\kappa^z_{s_i}$ denotes the clock valuation $\kappa_i$ on $C$ expanded to a clock valuation to $C \cup \{z\}$ by mapping $z$ to $\bar{z}_i$ (we denote the enlarged clock valuation be denoted as $\bar{\kappa}$). Similar to the region graph $\text{Reg}(\mathcal{T})$, we define an untimed finite state bisimulation quotient graph $\text{Reg}(\langle \mathcal{T} \rangle)$ for $\langle \mathcal{T} \rangle$.

Given a state $s$ of $\mathcal{T}$, we denote by $[s]$ the state $\langle s, 0, 0 \rangle$ of $\langle \mathcal{T} \rangle$. For a state trajectory $\text{traj} = s_0 \xrightarrow{t_1} s_1 \xrightarrow{t_1} \ldots$, we let $\text{traj}[i]$ denote the state $s_i$. Given a state trajectory $\text{traj}$ of the timed automaton $\mathcal{T}$, we denote by $[\text{traj}]$ the $\langle \mathcal{T} \rangle$ trajectory $[\text{traj}[0]] \xrightarrow{t_1} \hat{s}_1 \xrightarrow{t_1} \hat{s}_2 \ldots$, where $\hat{s}_i = \langle s_i, \bar{z}_i, d_i \rangle$, and $\bar{z}_i, d_i$ values are according to the times of the transitions (letting $[\text{traj}[0]] = \hat{s}_0$). That is, $[\text{traj}]$ denotes the trajectory obtained by adding the clock $z$, and the integer variable $\text{ticks}$, where the values for both the new variables are set to 0 in the starting state $[\text{traj}[0]]$. The new variables just observe the time, and the integer boundaries crossed for each transition according to the semantics for $\langle \mathcal{T} \rangle$ described previously. The first component of $[\text{traj}[i]]$ is the same as the state $\text{traj}[i]$ for all $i$.

The next lemma shows that a trajectory is time-divergent iff it satisfies a Büchi constraint.

**Lemma 6.** Let $\text{traj}$ be a trajectory of a timed automaton $\mathcal{T}$ in which all clocks are bounded by $c_{\max}$. The trajectory $\text{traj}$ is time-divergent iff $[\text{traj}]$ satisfies the Büchi condition $\bigvee_{i=1}^{c_{\max}} \text{ticks} = i$.

**Proof.** The proof follows from the fact that trajectory $\text{traj}$ is not time-divergent iff global time does not progress beyond some integer $U$. This happens iff time crosses only finitely many integer boundaries. Now, global time crosses an integer boundary at step $n$ iff $\bigvee_{i=1}^{c_{\max}} \text{ticks} = i$ is true at step $n$. Thus trajectory $\text{traj}$ is not time-divergent iff $\bigvee_{i=1}^{c_{\max}} \text{ticks} = i$ is true only finitely often. Equivalently, trajectory $\text{traj}$ is time-divergent iff $\bigvee_{i=1}^{c_{\max}} \text{ticks} = i$ is true infinitely often.

### 6.3 Integer-Time Quantitative Timed Simulation Functions

In this subsection we define quantitative simulation functions where only the integer “time-ticks” encountered are of relevance (as opposed to the exact real-valued times for the original QTSFs). The utility of these integer-time simulation functions is that they can be computed over timed automata by reductions to finite state games. These simulation functions are also close in value to the real-valued QTSFs. First, we define a notion of integer time which measures the number of integer time-ticks crossed up to the current time point.

**Definition 3 (Integer Time).** For the trajectory $[\text{traj}]$, let $\text{time}_{[\text{traj}]}^{\text{int}} [i]$ denote the number of integer boundaries crossed up to the $i$-th transition: $\text{time}_{[\text{traj}]}^{\text{int}} [i] = \lfloor \text{time}_{[\text{traj}]}[i] \rfloor$.

We have the following lemma which expresses $\text{time}_{[\text{traj}]}^{\text{int}} [i]$ in terms of the of the values of the $\text{ticks}$ variable in trajectories. Note that the value of the $\text{ticks}$ variable is zero in the first state of a valid trajectory $[\text{traj}]$ of $\langle \mathcal{T} \rangle$.

**Lemma 7.** Let $\text{traj}$ be a trajectory of a timed automaton $\mathcal{T}$ in which all clocks are bounded. We have: $\text{time}_{[\text{traj}]}^{\text{int}} [i] = \sum_{j=0}^{i} d_j$, where $d_i$ is the value of the $\text{ticks}$ variable at the state $[\text{traj}][i]$.

**Proof.** The proof follows from the definition of the $\text{ticks}$ variable updates: the updates count the integer boundaries crossed by the clock $z$ which measures elapsed time.

Using the notion of integer-time, we next define integer-time trajectory difference metrics.
Definition 4 (The Integer-Time Trajectory Difference Metrics $D_{\text{MaxDiff}}^{\text{int}}$, $D_{\text{LimMaxDiff}}^{\text{int}}$, and $D_{\text{AvgDiff}}^{\text{int}}$). Corresponding to the trajectory difference metric $D_\varphi()$, for $\varphi = \text{MaxDiff, LimMaxDiff, AvgDiff}$, we define the integer-time trace difference metric $D_\varphi^{\text{int}}()$, by substituting $\text{time}^{\text{int}}()$ for $\text{time}()$ in the definition of $D_\varphi()$. E.g., letting $[\text{traj}]_n = (l_n, \kappa_n, \delta_n, \mu_n)$ and $[\text{traj}']_n = (l'_n, \kappa'_n, \delta'_n, \mu'_n)$, we have:

$$D_{\text{MaxDiff}}^{\text{int}}([\text{traj}], [\text{traj}']) = \begin{cases} \infty & \text{if } \mu(l_n) \neq \mu(l'_n) \text{ for some } n \\ \sup_n \{|\text{time}^{\text{int}}_{[\text{traj}]}(n) - \text{time}^{\text{int}}_{[\text{traj}']} (n)| \} & \text{otherwise} \end{cases}$$

Proposition 9. The functions $D_{\text{MaxDiff}}^{\text{int}}$, $D_{\text{LimMaxDiff}}^{\text{int}}$, and $D_{\text{AvgDiff}}^{\text{int}}$ are metrics over timed trajectories.

Proof. The proofs are along similar lines to the corresponding claims for $D_{\text{MaxDiff}}$, $D_{\text{LimMaxDiff}}$, and $D_{\text{AvgDiff}}$. \(\square\)

The following lemma shows that $D_{\varphi}^{\text{int}}()$ closely approximates $D_\varphi()$.

Lemma 8. Let $\text{traj}_1$ and $\text{traj}_2$ be two trajectories of a timed automaton $\mathcal{A}$. The following assertions are true for $\varphi \in \{\text{MaxDiff, LimMaxDiff, AvgDiff}\}$.

1. $D_\varphi([\text{traj}_1], [\text{traj}_2]) = \infty$ iff $D_{\varphi}^{\text{int}}([\text{traj}_1], [\text{traj}_2]) = \infty$.

2. If both $D_\varphi([\text{traj}_1], [\text{traj}_2])$ and $D_{\varphi}^{\text{int}}([\text{traj}_1], [\text{traj}_2])$ are less than $\infty$, then

$$D_\varphi([\text{traj}_1], [\text{traj}_2]) + 1 \geq D_{\varphi}^{\text{int}}([\text{traj}_1], [\text{traj}_2]) \geq D_\varphi([\text{traj}_1], [\text{traj}_2]) - 1.$$

Proof. The first claim is obvious. We prove the second claim. Let us denote the sequence $\text{time}^{\text{int}}_{[\text{traj}]}(n)$ as $x(n)$, the sequence $\text{time}^{\text{int}}_{[\text{traj}]}(n)$ as $x'(n)$, the sequence $\text{time}^{\text{int}}_{[\text{traj}]}(n)$ as $y(n)$ and the sequence $\text{time}^{\text{int}}_{[\text{traj}]}(n)$ as $y'(n)$. We have for all $n \geq 0$,

$$x(n) - 1 < y(n) \leq x(n)$$

$$x'(n) - 1 < y'(n) \leq x'(n).$$

Thus, we have

$$x(n) - x'(n) - 1 < y(n) - y'(n) < x(n) - x'(n) + 1$$

Hence

$$|x(n) - x'(n)| - 1 < |y(n) - y'(n)| < |x(n) - x'(n)| + 1$$

It follows that

$$\sup_n |x(n) - x'(n)| - 1 < \sup_n |y(n) - y'(n)| < \sup_n |x(n) - x'(n)| + 1$$

Thus, we have the results for $\varphi = \text{MaxDiff}$.

We also have the following two relationships:

$$\lim_{U \to \infty} \sup_{n > U} |x(n) - x'(n)| - 1 < \lim_{U \to \infty} \sup_{n > U} |y(n) - y'(n)|$$

$$\lim_{U \to \infty} \sup_{n > U} |y(n) - y'(n)| < \lim_{U \to \infty} \sup_{n > U} |x(n) - x'(n)| + 1.$$
This gives the results for $\varphi = \lim \max \text{Diff}$.

Next, we note that for every $n$, the following two relationships hold.

$$\frac{\sum_{j=0}^{n} x(j) - n}{n+1} < \frac{\sum_{j=0}^{n} y(j)}{n+1} \leq \frac{\sum_{j=0}^{n} x'(j)}{n+1} \leq \frac{\sum_{j=0}^{n} y'(j)}{n+1} < \frac{\sum_{j=0}^{n} x'(j) - n}{n+1}$$

And thus,

$$\frac{\sum_{j=0}^{n} x(j)}{n+1} - 1 < \frac{\sum_{j=0}^{n} y(j)}{n+1} \leq \frac{\sum_{j=0}^{n} x'(j)}{n+1} \leq \frac{\sum_{j=0}^{n} y'(j)}{n+1} < \frac{\sum_{j=0}^{n} x'(j) - 1}{n+1}$$

Then, applying similar reasoning as in $\varphi = \lim \max \text{Diff}$, we get the results for $\varphi = \text{AvgDiff}$. □

Using $D_{\text{MaxDiff}}^\text{int}$, $D_{\text{LimMaxDiff}}^\text{int}$, and $D_{\text{AvgDiff}}^\text{int}$, we can define integer-time quantitative simulation functions $S_{\Psi_{\text{int}}}^{\text{Timediv}}$ which approximate $S_{\Psi}^{\text{Timediv}}$ for $\Psi^{\text{Timediv}} \in \{D_{\text{MaxDiff}}^\text{int}, D_{\text{LimMaxDiff}}^\text{int}, D_{\text{AvgDiff}}^\text{int}\}$. The definitions follow along similar lines to the definitions for $S_{\Psi}^{\text{Timediv}}$. We present them formally below.

First, we present integer-time quantitative objectives which map simulation game plays to integer valued numbers.

**Definition 5 (Integer-Time Quantitative Objectives for Timed Simulation Games).** For the trajectory difference metrics $\Psi_{\text{int}} \in \{D_{\text{MaxDiff}}^\text{int}, D_{\text{LimMaxDiff}}^\text{int}, D_{\text{AvgDiff}}^\text{int}\}$, we define the integer valued quantitative objective $\Psi_{\text{int}}^{\text{Timediv}}$ as follows for a play $\rho$ in the timed simulation game $\Sigma_t([T_r], [T_s])$:

$$\Psi_{\text{int}}^{\text{Timediv}}(\rho) = \begin{cases} 
0 & \text{if } \rho(\tau) \notin \text{Timediv}([T_r]) \\
\Psi_{\text{int}}^{\text{Timediv}}(\rho(\tau), \rho(s)) & \text{otherwise}
\end{cases}$$

The integer-time quantitative simulation functions $S_{\Psi_{\text{int}}}^{\text{Timediv}}([s_r], [s_s])$, can now be defined exactly as in Definition 2, using $\Psi_{\text{int}}^{\text{Timediv}}$ instead of $\Psi^{\text{Timediv}}$. The formal definition is given below in Definition 6.

**Definition 6 (Integer-Time QTSF).** Let $T_r, T_s$ be timed automata, with the corresponding enlarged timed transition systems $[T_r], [T_s]$ respectively, and let $\Sigma_t([A_r], [A_s])$ be the two player turn-based bipartite timed simulation game structure. The value of the integer-time QTSF $S_{\Psi_{\text{int}}}^{\text{Timediv}}([s_r], [s_s])$, for $[s_r]$ and $[s_s]$ states of $[T_r]$ and $[T_s]$ respectively, and for $\Psi_{\text{int}} \in \{D_{\text{MaxDiff}}^\text{int}, D_{\text{LimMaxDiff}}^\text{int}, D_{\text{AvgDiff}}^\text{int}\}$, is defined as follows.

$$S_{\Psi_{\text{int}}}^{\text{Timediv}}([s_r], [s_s]) = \inf_{\pi_s \in \Pi_s} \sup_{\pi_t \in \Pi_t} \Psi_{\text{int}}^{\text{Timediv}}(\rho(\pi_t, \pi_s, ([s_r], [s_s], 2)))$$

where $\rho(\pi_t, \pi_s, ([s_r], [s_s], 2))$ is the trajectory which results given the player-1 strategy $\pi_s \in \Pi_s$ and the player-2 strategy $\pi_t \in \Pi_t$. □
Lemma 9. Let $T_r, T_s$ be timed automata, with the corresponding enlarged timed transition systems $[T_r], [T_s]$ respectively, and let $G_t([T_r], [T_s])$ be the two player turn-based bipartite timed simulation game structure. The following assertions are true for $\langle \Psi_{int}, \Psi \rangle \in \{\langle D_{\text{MaxDiff}}, D_{\text{MaxDiff}}\rangle, \langle D_{\text{LimMaxDiff}}, D_{\text{LimMaxDiff}}\rangle, \langle D_{\text{AvgDiff}}, D_{\text{AvgDiff}}\rangle\}$, for any play $\rho$ of $G_t([A_r], [A_s])$.

1. $\Psi_{int}^{\text{Timediv}}(\rho) = \infty \iff \Psi^{\text{Timediv}}(\rho) = \infty$.
2. If both $\Psi_{int}^{\text{Timediv}}(\rho)$ and $\Psi^{\text{Timediv}}(\rho)$ are less than $\infty$, then $|\Psi_{int}^{\text{Timediv}}(\rho) - \Psi^{\text{Timediv}}(\rho)| \leq 1$

Proof. The results follow from Lemma 8 and by the definitions of $\Psi_{int}^{\text{Timediv}}(\rho)$ and $\Psi^{\text{Timediv}}(\rho)$.

We next present a result concerning number sets. This result will be used to show that the integer simulation functions are close in value to the real-valued simulation functions.

Lemma 10. Let $\{x_{r,s}, y_{r,s} \mid r \in R, s \in S\}$ be a set of tuples of numbers for some given sets $R, S$ such that $x_{r,s} \in \mathbb{R}_+^\infty$ and $y_{r,s} \in \mathbb{R}_+^\infty$, where $\mathbb{R}_+^\infty = \mathbb{R}_+ \cup \{\infty\}$. Let both the following conditions hold:

1. For all $r, s$ we have $x_{r,s} = \infty \iff y_{r,s} = \infty$.
2. There exists some $\alpha \in \mathbb{R}_+^\infty$ such that for all $r, s$, if $x_{r,s}$ and $y_{r,s}$ are both finite, then $|x_{r,s} - y_{r,s}| \leq \alpha$.

Then, the following assertion are true.

1. If both $\inf_{s \in S} \sup_{r \in R} x_{r,s} = \infty$ and $\inf_{s \in S} \sup_{r \in R} y_{r,s} = \infty$ then $\inf_{s \in S} \sup_{r \in R} x_{r,s} - \inf_{s \in S} \sup_{r \in R} y_{r,s} \leq \alpha$.

Proof. We prove both the assertions.

1. Suppose $\inf_{s \in S} \sup_{r \in R} x_{r,s} = \infty$ (the other direction is symmetric). We must have that for every $s \in S$ the entity $\sup_{r \in R} x_{r,s} = \infty$ (otherwise the inf would have been smaller). We show that:

   **Fact-1:** For every $s \in S$, if $\sup_{r \in R} x_{r,s} = \infty$, then the entity $\sup_{r \in R} y_{r,s} = \infty$.

   Fix some $s \in S$.

   - If there exists some $r \in R$ such that $x_{r,s} = \infty$, then by the conditions of the lemma, $y_{r,s} = \infty$.
   - Thus $\sup_{r \in R} y_{r,s} = \infty$.

   - Suppose for all $r \in R$ we have $x_{r,s} < \infty$. By the conditions of the lemma, for all $r \in R$ we have $|x_{r,s} - y_{r,s}| \leq \alpha$. Thus, if $\sup_{r \in R} x_{r,s} = \infty$, then $\sup_{r \in R} y_{r,s} = \infty$.

   This **Fact-1** is true. Hence, $\sup_{r \in R} y_{r,s} = \infty$ for every $s \in S$. Thus, $\inf_{s \in S} \sup_{r \in R} y_{r,s} = \infty$.

2. Suppose we have both $\inf_{s \in S} \sup_{r \in R} x_{r,s} < \infty$ and $\inf_{s \in S} \sup_{r \in R} y_{r,s} < \infty$.

   Fix some $s \in S$.

   - Suppose $\sup_{r \in R} x_{r,s} = \infty$. We have that $\sup_{r \in R} y_{r,s} = \infty$ by **Fact-1** above.
   - Suppose $\sup_{r \in R} x_{r,s} < \infty$ (note that there must exist at least one such $s$ otherwise $\inf_{s \in S} \sup_{r \in R} x_{r,s} = \infty$). Thus, for this $s$, we have that for all $r \in R$, the quantity $x_{r,s} < \infty$.

   By the conditions of the lemma, we have that for this $s$, for all $r \in R$, the quantity $y_{r,s} < \infty$, and that $|x_{r,s} - y_{r,s}| \leq \alpha$. This implies that $\sup_{r \in R} y_{r,s} < \infty$, and that $\left|\sup_{r \in R} x_{r,s} - \sup_{r \in R} y_{r,s}\right| \leq \alpha$. 

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Let \( p_s = \sup_{r \in R} x_{r,s} \), and \( q_s = \sup_{r \in R} y_{r,s} \). From above, we have that for all \( s \), it holds that either
- \( p_s = q_s = \infty \), or
- \( |p_s - q_s| \leq \alpha \).

Also, it holds that for at least one \( s \), we have \( p_s < \infty \). Thus, can throw away the \( p_s \) numbers such that \( p_s = \infty \) in the computation of \( \inf_s p_s \). For the rest, since \( |p_s - q_s| \leq \alpha \), and since \( p_s \geq 0 \) and \( q_s \geq 0 \), we have that \( |\inf_s p_s - \inf_s q_s| \leq \alpha \). Thus, the second part of the assertion is true. \( \square \)

The following proposition states that the integer simulation functions closely approximate the original Qtsfs.

**Proposition 10 (Integer-Time Qtsfs Approximate Exact Qtsfs).** Let \( T_r, T_s \) be timed automata, with the corresponding enlarged timed transition systems \([T_r], [T_s]\) respectively, and let \( \mathcal{S}_t([A_t], [A_s]) \) be the two player turn-based bipartite timed simulation game structure. For \( (\Psi_{\text{int}}, \Psi) \in \{\langle D_{\text{MaxDiff}}, D_{\text{MaxDiff}}\rangle, \langle D_{\text{LimMaxDiff}}, D_{\text{LimMaxDiff}}\rangle, \langle D_{\text{AvgDiff}}, D_{\text{AvgDiff}}\rangle\} \), we have the following assertions to be true.

1. \( S_{\Psi_{\text{int}}}^{\text{Timediv}}(\langle [s_t], [s_s] \rangle) = \infty \) iff \( S_{\Psi}^{\text{Timediv}}(\langle [s_t], [s_s] \rangle) = \infty \).
2. If \( S_{\Psi_{\text{int}}}^{\text{Timediv}}(\langle [s_t], [s_s] \rangle) < \infty \) and \( S_{\Psi}^{\text{Timediv}}(\langle [s_t], [s_s] \rangle) < \infty \), then
   \[
   0 \leq S_{\Psi_{\text{int}}}^{\text{Timediv}}(\langle [s_t], [s_s] \rangle) - S_{\Psi}^{\text{Timediv}}(\langle [s_t], [s_s] \rangle) \leq 1
   \]

**Proof.** The proof follows from Lemma 10 and Lemma 9. \( \square \)

### 6.4 Reduction to Games on Finite Weighted Game Graphs

In this section we show how to compute the values of the integer-time Qtsfs by reducing the problem to finite state games. First, we show that the values of the integer-time Qtsfs are exactly the same on discrete time region graphs as on timed automata.

**The Integer Trace Difference Metrics and Simulation Functions on Region Graphs.** We first lift the integer trace difference metrics \( \Psi_{\text{int}}^{\text{Timediv}} \) for \( \Psi_{\text{int}} \in \{D_{\text{MaxDiff}}, D_{\text{LimMaxDiff}}, D_{\text{AvgDiff}}\} \) to region graphs. Let \( \text{Reg}([T]) \) be the region graph corresponding to the enlarged timed automaton structure \([T] \) as defined in Subsection 6.2. We note that the \( \text{ticks} \) variable in \([T] \) counts the elapsed integer time boundaries crossed by the global clock \( z \) since the last transition in \( T \). Thus \( \text{Reg}([T]) \) can be viewed as a discrete time transition structure. For a trajectory \( [\text{tra}_i] \) of \( \text{Reg}([T]) \), we use Lemma 7 as defining \( \text{time}_{\text{int}}^{\text{Reg}}([\text{tra}_i]).i \) in terms of the \( \text{ticks} \) variable. For TTS corresponding to region graph \( \text{Reg}([T]) \) we define \( \text{Timediv}(\text{Reg}([T])) \) as the set of runs satisfying the Büchi condition Büchi \( (\bigvee_{i=1}^{\text{max}} \text{ticks} = i) \). By Lemma 6, this has the intended meaning of encoding time divergence. Let the Consider the (discrete) timed simulation game \( \mathcal{S}_t(\text{Reg}([T_r]), \text{Reg}([T_s])) \), and let us use the observation function \( \mu_{\Theta} \) defined as \( \mu_{\Theta}(l, \kappa, z, d) = \mu(l) \). We define \( \Psi_{\text{int}}^{\text{Timediv}} \) for \( \Psi_{\text{int}} \in \{D_{\text{MaxDiff}}, D_{\text{LimMaxDiff}}, D_{\text{AvgDiff}}\} \) on plays of \( \mathcal{S}_t(\text{Reg}([T_r]), \text{Reg}([T_s])) \) as usual using \( \text{time}_{\text{int}}^{\text{Reg}} \). The next lemma states that the values of the integer simulation functions on the region graphs are the same as that on timed automata.

**Lemma 11.** Let \( T_r, T_s \) be timed automata, and let \( \text{Reg}([T_r]), \text{Reg}([T_s]) \) be region graphs of the corresponding enlarged timed game structures \([T_r], [T_s]\) respectively. For any states \( [s_t], [s_s] \) of \([T_r]\) and \([T_s]\) of \([T_s]\), we have

\[
S_{\Psi_{\text{int}}}^{\text{Timediv}}([s_t], [s_s]) = S_{\Psi_{\text{int}}(\text{Reg}([T_r]), \text{Reg}([T_s])]}(\text{Reg}([s_t]), \text{Reg}([s_s]))
\]
where $\Psi_{int} \in \{ D_{\text{MaxDiff}}^{\text{int}}, D_{\text{LimMaxDiff}}^{\text{int}}, D_{\text{AvgDiff}}^{\text{int}} \}$.

Proof. For any timed automata $T$, we have that $\text{Reg}([T])$ is a bisimulation quotient of $[T]$ for the enlarged region equivalence relation (as defined in Subsection 6.2). Thus, given any play $\rho$ of $\mathcal{G}_X([T_1], [T_2])$, there exists a play $\rho_{\text{Reg}}$ of $\mathcal{G}_X(\text{Reg}([T_1]), \text{Reg}([T_2]))$ such that $\rho_{\text{Reg}}(t)$ and $\rho_{\text{Reg}}(s)$ have the same integer time observation trace sequences as $\rho(t)$ and $\rho(s)$ (note that the enlarged region equivalence relation ensures that the values of the ticks variables match at each step). The dual fact for any play $\rho_{\text{Reg}}$ of $\mathcal{G}_X(\text{Reg}([T_1]), \text{Reg}([T_2]))$ also holds due to the bisimulation. Since $\Psi_{int}$ depends only on the integer time plays of the game structures, we have the desired result. □

The weighted finite untimed game graph $\mathcal{F}(\text{Reg}([T_1]), \text{Reg}([T_2]))$. Now we construct a finite weighted game graph $\mathcal{F}(\text{Reg}([T_1]), (T_2))$, on which we can use the algorithms of Section 4, to compute the values of the integer-time QTSFs for the game $\mathcal{G}_X(\text{Reg}([T_1]), \text{Reg}([T_2]))$. The game structure $\mathcal{F}$ is essentially the simulation game $\mathcal{G}_X$ over the region graphs, where weights are assigned to transitions based on the tick values of the region states. Formally, $\mathcal{F}(\text{Reg}([T_1]), \text{Reg}([T_2]))$ (denoted $\mathcal{F}$ in short) is the tuple $(S^\delta, \rightarrow^\delta, w^\delta)$, where

$- S^\delta = S_1^\delta \cup S_2^\delta$, and

$- \ast$ The set of player-2 states is $S_2^\delta = S^\text{Reg}([T_1]) \times S^\text{Reg}([T_2]) \times \{2\}$, where $S^\text{Reg}([T_1])$ is the set of states of $\text{Reg}([T_1])$, and $S^\text{Reg}([T_2])$ is the set of states of $\text{Reg}([T_2])$.

$- \ast$ The set of player-1 states is $S_1^\delta = S^\text{Reg}([T_1]) \times S^\text{Reg}([T_2]) \times \{1\}$.

$- \rightarrow^\delta$ is the set of edges where

$\ast$ The player-2 transitions are:

$\langle \text{Reg}((l_t, \hat{\kappa}_t, d_t)), \text{Reg}((s_2)) \rangle, \{2\} \rightarrow \langle \text{Reg}((l'_t, \hat{\kappa}'_t, d'_t)), \text{Reg}(s_2) \rangle, \{1\}$, such that $\text{Reg}((l_t, \hat{\kappa}_t, d_t)) \rightarrow \text{Reg}((l'_t, \hat{\kappa}'_t, d'_t))$ is a valid transition in $\text{Reg}([T_2])$.

$\ast$ The player-1 transitions are:

$\langle \text{Reg}((s_1)), \text{Reg}((l_s, \hat{\kappa}_s, d_s)) \rangle, \{1\} \rightarrow \langle \text{Reg}((s_1)), \text{Reg}((l'_s, \hat{\kappa}'_s, d'_s)) \rangle, \{2\}$, such that

1. $\text{Reg}((l_s, \hat{\kappa}_s, d_s)) \rightarrow \text{Reg}((l'_s, \hat{\kappa}'_s, d'_s))$ is a valid transition in $\text{Reg}([T_1])$; and

2. $\mu(\text{Reg}((s_1))) = \mu(\text{Reg}((s'_1)))$, that is, the observation on the (timed automaton) location of $\text{Reg}(s'_1)$ is the same as the observation on the location of $\text{Reg}(s_1)$.

If there is no outgoing transition from a player-1 state according to the above rules, we add a dummy transition to a sink state $s_{\text{sink}}$ which we define to be such that the $\text{Opt}$ value for player 1 is $\infty$ for all objectives from $s_{\text{sink}}$.

$- w^\delta(e_2) = 0$ for any edge $e_2$ originating from a player-2 state.

$- w^\delta\left(\langle \text{Reg}((l_t, \hat{\kappa}_t, d_t)), \text{Reg}((l_s, \hat{\kappa}_s, d_s)) \rangle, \{1\} \rightarrow \langle \text{Reg}((l_t, \hat{\kappa}_t, d_t)), \text{Reg}((l'_t, \hat{\kappa}'_t, d'_t)) \rangle, \{2\}\right)$ is the value $d_t - d'_t$.

We note that $d'_t$ is the number of integer boundaries crossed by the clock $z$ in a transition to go from any state in $\text{Reg}((l_s, \hat{\kappa}_s, d_s))$ to any state in $\text{Reg}((l'_s, \hat{\kappa}'_s, d'_s))$, and similarly for $d_t$. Thus, the quantity $d_t - d'_t$ encodes the difference of the integer boundaries crossed by the clock $z$ in the region graphs $\text{Reg}([T_1])$ and $\text{Reg}([T_2])$ during the last step in the simulation game.

Intuitive explanation of $\mathcal{F}$: The simulation game $\mathcal{G}_X$ can be viewed as proceeding in a sequence of rounds – in each round first player 2 picks a transition in $T_1$, and then player 1 picks a transition in $T_2$, trying to the match the move of player 2. The weighted game $\mathcal{F}$ can similarly be viewed as proceeding in a sequence of rounds. First player 2 takes a transition from a state $\langle \text{Reg}((l_t, \hat{\kappa}_t, d_t)), \text{Reg}(s_2) \rangle$ to $\langle \text{Reg}((l'_t, \hat{\kappa}'_t, d'_t)) \rangle, \text{Reg}(s_2), \{1\}$ corresponding to the transition in the timed automaton $T$. The integer boundaries crossed by the global clock are recorded in
the update \( \Delta'_t \) (but the weight of the transition is taken as 0). Denoting \( \langle \rho'_t, \delta_t, \Delta'_t \rangle \) as \([s'_t]\), and letting \([s_\tau]\) = \(\langle \rho_\tau, \delta_\tau, \Delta_\tau \rangle\), the next transition is from \(\langle \\text{Reg}([s'_t]), \\text{Reg}([\langle \rho_\tau, \delta_\tau, \Delta_\tau \rangle, 1) \rangle \to \\text{Reg}([s'_t]), \\text{Reg}([\langle \rho'_t, \delta'_t, \Delta'_t \rangle, 2)\rangle, \) corresponding to a player-1 transition in the timed automaton \(\mathcal{T}_s\) in the simulation game \(\mathcal{G}_t\). The duration of the player-1 transition in \(\mathcal{T}_s\) corresponds to \(\Delta'_t\) integer boundaries being crossed by the clock \(z\) of \(\mathcal{T}_s\). Thus, the difference in the integer boundaries crossed in the trajectories of \(\mathcal{T}_t\) and \(\mathcal{T}_s\) for this round is \(\Delta'_t - \Delta'_t\), and this is the weight of the second transition of \(\mathcal{F}\).

The next lemma states that to compute the values of the integer-time QTSFs on the region graphs, we can use the objectives \(\text{MaxDiffCB}\), \(\text{EvMaxDiffCB}\), \(\text{AvgDiffCB}\) on the weighted finite game \(\mathcal{F}(\\text{Reg}([\mathcal{T}_t]), \\text{Reg}([\mathcal{T}_s]))\).}

**Lemma 12.** Let \(\mathcal{T}_t\) and \(\mathcal{T}_s\) be well-formed timed automata such that all clocks are bounded by \(c_{\text{max}}\), and let \(\mathcal{F}(\\text{Reg}([\mathcal{T}_t]), \\text{Reg}([\mathcal{T}_s]))\) be the weighted game structure corresponding to \(\mathcal{G}_t(\\text{Reg}([\mathcal{T}_t]), \\text{Reg}([\mathcal{T}_s]))\), as described above. Fix the coBüchi objective \(\text{coBuchi}\) in the following. For \(\psi_{\text{int}}\) equal to \(\langle \mathcal{D}^\text{int}_{\text{MaxDiff}}, \text{MaxDiffCB} \rangle\), or \(\langle \mathcal{D}^\text{int}_{\text{AvgDiff}}, \text{AvgDiffCB} \rangle\), or \(\langle \mathcal{D}^\text{int}_{\text{AvgDiff}}, \text{AvgDiffCB} \rangle\), we have

\[
\mathcal{F}(\psi_{\text{int}}, \text{Timediv}) \left( \langle \text{Reg}([s_t]), \text{Reg}([s_s]) \rangle \right) = \left( \text{Opt}^\text{int}_\text{AvgDiff}(\langle \text{Reg}([s_t]), \text{Reg}([s_s]) \rangle) \right) \left( \langle \text{Reg}([s_t]), \text{Reg}([s_s]), 2 \rangle \right).
\]

**Proof.** Note that every finite play \(\rho^{\mathcal{G}_t}\) of \(\mathcal{G}_t(\\text{Reg}([\mathcal{T}_t]), \\text{Reg}([\mathcal{T}_s]))\) in which player 1 has not lost (in the simulation game) corresponds to a finite play \(\rho^\mathcal{F}\) in \(\mathcal{F}\) in which the sink location \(s_{\text{sink}}\) has not been visited, and similarly for the other direction (for starting states \(\langle \text{Reg}([s_t]), \text{Reg}([s_s]), 2 \rangle\)). The move choices for both players are the same, apart from \(s_{\text{sink}}\) transitions.

Observe that any two states \(\text{Reg}([s_t]), \text{Reg}([s_s])\) are untimed similar in \(\mathcal{G}_t(\\text{Reg}([\mathcal{T}_t]), \\text{Reg}([\mathcal{T}_s]))\) if and only if, for every player-1 strategy, player 2 has a strategy which forces the play into the sink location and thus leads to an \(\infty\) value for all the quantitative objectives. Thus, consider the case where \(\text{Reg}([s_t]), \text{Reg}([s_s])\) are untimed similar. Now, \(\text{Timediv}\) has been shown to be equivalent to \(\text{Buchi}(\sum_{i=1}^{c_{\text{max}}} \text{ticks} = i)\) earlier, on the region graphs. This Büchi condition is equivalent to \(\neg \text{coBuchi}(\text{ticks} = 0)\). Thus, the condition \(\rho^{\mathcal{G}_t}(\tau) \notin \text{Timediv}\) holds if \(\rho^\mathcal{F} \notin \text{coBuchi}(\text{ticks} = 0)\) holds. Finally, we note that for any play \(\rho^{\mathcal{G}_t}(\pi_t, \pi_s, \langle \text{Reg}([s_t]), \text{Reg}([s_s]), 2 \rangle)\), the corresponding play \(\rho^\mathcal{F}(\pi_t, \pi_s, \langle \text{Reg}([s_t]), \text{Reg}([s_s]), 2 \rangle)\) is such that

1. For all \(i > 0\), we have \(\text{time}^\mathcal{F}(\rho^{\mathcal{G}_t}(\tau)) = \sum_{j=1}^i w^\mathcal{F}(\rho^\mathcal{F}[2j - 1] \rightarrow \rho^\mathcal{F}[2j])\).
2. For every \(i \geq 0\), we have \(w^\mathcal{F}(\rho^\mathcal{F}[2i] \rightarrow \rho^\mathcal{F}[2i + 1]) = 0\)

The desired results follow.

**6.5 Integer-Time Simulation Functions Approximate Real-Valued Simulation Functions**

**Precision of the Integer-Time QTSFs.** Given a positive integer \(\alpha \geq 1\), and a timed automaton \(\mathcal{T}\), let \(\alpha \cdot \mathcal{T}\) denote the timed automaton obtained from \(\mathcal{T}\) by multiplying every constant by \(\alpha\). Note that if clocks are bounded by \(c_{\text{max}}\) in \(\mathcal{T}\), then clocks are bounded by \(\alpha \cdot c_{\text{max}}\) in \(\alpha \cdot \mathcal{T}\). The automaton \(\alpha \cdot \mathcal{T}\) is just \(\mathcal{T}\) with a blown up timescale. One time unit in \(\mathcal{T}\) corresponds to \(\alpha\) time units in \(\alpha \cdot \mathcal{T}\).

We let \(\alpha \cdot \mathcal{I} = \langle \alpha \cdot \mathcal{I} \rangle\), and \(\alpha \cdot \langle l, \kappa, \delta \rangle = \langle l, \alpha \cdot \kappa, \text{frac}(\alpha \cdot \delta), \alpha \cdot \delta \rangle\) where \(\text{frac}(\beta)\) denotes the fractional part of \(\beta\), i.e., \(\beta - \lfloor \beta \rfloor\) for \(\beta \geq 0\). Note that in \(\alpha \cdot \mathcal{I}\), the clock \(z\) still cycles from 0
Lemma 13. Let $\mathcal{T}_r, \mathcal{T}_s$ be timed automata, with the corresponding enlarged timed transition systems $[\mathcal{T}_r], [\mathcal{T}_s]$ respectively, and let $\mathcal{G}_r([A_r], [A_s])$ be the two player turn-based bipartite timed simulation game structure. Let $\Psi \in \{\mathcal{D}_{\text{MaxDiff}}, \mathcal{D}_{\text{LimMaxDiff}}, \mathcal{D}_{\text{AvgDiff}}\}$, for any $\alpha$ a positive integer, and for any states $[s_r]$ and $[s_s]$ of $[\mathcal{T}_r]$ and $[\mathcal{T}_s]$ respectively, we have

$$\alpha \cdot S_{\mathcal{G}_r}^\text{Timediv}([s_r], [s_s]) = S_{\mathcal{G}_r}^\text{Timediv}([\alpha \cdot [s_r], \alpha \cdot [s_s]])$$

Proof. The proof follows from observing that the times in $\alpha \cdot \mathcal{T}$ are just the times in $\mathcal{T}$ multiplied by $\alpha$.

The following lemma states that integer-time QTSFs can approximate the exact QTSFs to within any desired degree of accuracy.

Proposition 11 (Integer-Time QTSFs Approximate Exact QTSFs to Any Desired Degree). Let $\mathcal{T}_r, \mathcal{T}_s$ be timed automata, with the corresponding enlarged timed transition systems $[\mathcal{T}_r], [\mathcal{T}_s]$ respectively, and let $\mathcal{G}_r([A_r], [A_s])$ be the two player turn-based bipartite timed simulation game structure. For $\Psi, \alpha > 0$, we have the following assertions to be true.

1. $S_{\Psi}^\text{Timediv}([\alpha \cdot [s_r], \alpha \cdot [s_s]]) = \infty$ iff $S_{\Psi}^\text{Timediv}([\alpha \cdot [s_r], \alpha \cdot [s_s]]) = \infty$.
2. If $S_{\Psi}^\text{Timediv}([\alpha \cdot [s_r], \alpha \cdot [s_s]]) < \infty$ and $S_{\Psi}^\text{Timediv}([\alpha \cdot [s_r], \alpha \cdot [s_s]]) < \infty$, then

$$|\alpha^{-1} \cdot S_{\Psi}^\text{Timediv}([\alpha \cdot [s_r], \alpha \cdot [s_s]]) - S_{\Psi}^\text{Timediv}([\alpha \cdot [s_r], \alpha \cdot [s_s]])| \leq \frac{1}{\alpha}$$

Proof. The proof follows from Lemma 13 and Proposition 10 applied to $\alpha \cdot \mathcal{T}_r$ and $\alpha \cdot \mathcal{T}_s$.

6.6 Final Algorithms and Results

We now present the final algorithm for computing the values for the QTSFs $S_{\Psi}^\text{Timediv}$ for $\Psi \in \{\mathcal{D}_{\text{MaxDiff}}, \mathcal{D}_{\text{LimMaxDiff}}, \mathcal{D}_{\text{AvgDiff}}\}$, to within any desired degree of accuracy. The algorithm is listed in the function $h_{\Psi, \alpha}(s_r, s_s)$. The proof of the correctness of the algorithm uses Proposition 11, and Lemma 12, and the results of the previous section on games on finite state game graphs.

Theorem 5. Let $\mathcal{T}_r$ and $\mathcal{T}_s$ be well-formed timed automata such that all clocks are bounded by $c_{\text{max}}$, and let $\alpha \geq 1$ be a positive integer. Let $S_{\Psi}^\text{Timediv}$ denote the QTSF for $\Psi \in \{\mathcal{D}_{\text{MaxDiff}}, \mathcal{D}_{\text{LimMaxDiff}}, \mathcal{D}_{\text{AvgDiff}}\}$. The function $h_{\Psi, \alpha}(s_r, s_s)$ is such that for any states $s_r$ of $\mathcal{T}_r$ and $s_s$ of $\mathcal{T}_s$, either

1. $S_{\Psi}^\text{Timediv}(s_r, s_s) = h_{\Psi, \alpha}(s_r, s_s) = \infty$; or
2. Both values are finite and $|S_{\Psi}^\text{Timediv}(s_r, s_s) - h_{\Psi, \alpha}(s_r, s_s)| \leq \frac{1}{\alpha}$

Proof. The proof follows from Proposition 11 and Lemma 12. Since $\exists (\mathcal{R}(\alpha \cdot \mathcal{T}_r), \mathcal{R}(\alpha \cdot \mathcal{T}_s))$ is a finite weighted game graph, the value of $h_{\Psi, \alpha}(s_r, s_s)$ can be computed using the algorithms of Section 4.
Input: States $s_r, s_s$ from $T_r, T_s$ respectively; $\alpha$ a positive integer; 
$\Psi \in \{D_{\text{MaxDiff}}, D_{\text{LimMaxDiff}}, D_{\text{AvgDiff}}\}$

Output: A number approximating $S_{\Psi} \text{Timediv}(\langle s_r, s_s \rangle)$ (maximum error difference: $1/\alpha$)

1. $\text{Reg}([\alpha \cdot T_r])$, $\text{Reg}([\alpha \cdot T_s]) :=$ Region graphs of the expanded timed game structures $[\alpha \cdot T_r]$ and $[\alpha \cdot T_s]$;
2. $\mathcal{F} := \mathcal{F}(\text{Reg}([\alpha \cdot T_r]), \text{Reg}([\alpha \cdot T_s])); /*$ Finite weighted game constructed from the region graphs */
3. switch $\Psi$ do
   case $D_{\text{MaxDiff}}$
      $\Xi := \text{MaxDiffCB}_{\text{coB"uchi}}(\text{ticks}_r=0);
   $ case $D_{\text{LimMaxDiff}}$
      $\Xi := \text{EvMaxDiffCB}_{\text{coB"uchi}}(\text{ticks}_r=0);
   $ case $D_{\text{AvgDiff}}$
      $\Xi := \text{AvgDiffCB}_{\text{coB"uchi}}(\text{ticks}_1=0);
   $
4. return $\alpha^{-1} \cdot \text{Opt}(\mathcal{F}((\text{Reg}([\alpha \cdot s_r]), \text{Reg}([\alpha \cdot s_s]), 2));$

Function $h_{\Psi,\alpha}(s_r, s_s)$

7 Conclusion Remarks

We have defined three ways of quantifying timing mismatches, and have presented algorithms for computing the values of three kinds of quantitative timed simulation functions which quantify corresponding timing mismatches between two timed automata to within any desired degree of accuracy. We note that the optimal player-1 strategies in the weighted game $\mathcal{F}$ used in Function $h_{\Psi,\alpha}()$ are also computable, and are witnesses to the quantitative simulation function values (similar to simulation relations witnessing the simulation decision problem). We expect that the algorithms presented in this paper will contribute to the further development of approximation theories for continuous, switched and hybrid dynamical systems for the automatic synthesis of more powerful controllers.

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