ON A CLASS OF DIFFUSION-AGGREGATION EQUATIONS

YUMING PAUL ZHANG*

Department of Mathematics, University of California, Los Angeles
Los Angeles, CA 90025, USA

(Communicated by José A. Carrillo)

Abstract. We investigate the diffusion-aggregation equations with degenerate diffusion $\Delta u^m$ and singular interaction kernel $K_s = (-\Delta)^{-s}$ with $s \in (0, \frac{d}{2})$. The equation is related to biological aggregation models. We analyze the regime where the diffusive force is stronger than the aggregation effect. In such regime, we show the existence and uniform boundedness of solutions in the case either $s > \frac{1}{2}$ or $m < 2$. Hölder regularity is obtained when $d \geq 3$, $s > \frac{1}{2}$ and uniqueness is proved when $s \geq 1$.

1. Introduction. We consider the following nonlinear, degenerate drift-diffusion equation

$$u_t = \Delta u^m - \nabla \cdot (u \nabla K_s u) \quad \text{in} \quad \mathbb{R}^d \times [0, \infty),$$

with nonnegative initial data $u(x, 0) = u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, where the degeneracy arises due to the range of $m$, $m > 1$. The nonlocal drift is given by the Riesz kernel

$$K_s u = cK_s * u \quad \text{where} \quad s \in \left(0, \frac{d}{2}\right), \quad K_s(z) = |z|^{-d+2s}, c > 0.$$  

For a suitable choice of $c = c(d, s) > 0$, the convolution is governed by a fractional diffusion process: $K_s u = (-\Delta)^{-s} u$ ([26]).

The model arises from the macroscopic description of cell motility due to cell adhesion and chemotaxis phenomena, see [11, 6, 29]. In the context of biological aggregation, $u$ describes the population density and the degenerate diffusion models the local repulsion taking over-crowding effects into consideration. This effect can also be found in many physical applications, including fluids in porous medium ([30, 19]). The homogeneous singular kernel models long-range attractive interactions between cells, with smaller $s$ representing stronger aggregation at near-distances and therefore more singular. For larger $s$, we consider stronger force at long-distances. The competition between the diffusion and the nonlocal aggregation is one of the core subjects in the study of aggregation models.

To find the balance of the two competing effects, we use a scaling argument, also see [12, 5]. Define

$$u_r(x, t) := r^d u(rx, r^{d(m-1)+2}t),$$

and then formally $(-\Delta)^{-s} u_r = r^{d-2s} (-\Delta)^{-s} u$. It is straightforward to check

$$\partial_t u_r = \Delta u_r^m - r^{2d-dm-2s} \nabla \cdot (u_r \nabla \cdot K_s u_r).$$

2010 Mathematics Subject Classification. Primary: 35K55, 45G05; Secondary: 35K65, 35B65.

Key words and phrases. Aggregation equation, degenerate diffusion, a priori estimates, uniform boundedness, singular integral.

* Corresponding author: Yuming Paul Zhang.
So $m = 2 - 2s/d$ leads to a compensation between the diffusion and the aggregation. The range $m > 2 - 2s/d$ where the diffusion dominates over the aggregation is often referred to as the subcritical regime. The range $m < 2 - 2s/d$ is called supercritical.

When $s = 1$, $\mathcal{K}_1$ represents the Newtonian potential and (1) is the well-known degenerate Patlak-Keller-Segel equation. In the corresponding subcritical regime, the well-posedness, boundedness and continuity regularity properties of solutions have been established in [4, 2, 14]. When $m = 2 - 2/d$, it has been shown in [15, 5] that the mass of the initial data plays an important role. More precisely, if the initial mass is larger than one critical value, solutions can blow up in finite time and otherwise they always stay regular. In the supercritical regime, finite time blow up is again possible, see [27, 2].

In this paper, we consider the natural extension of the Newtonian potential: $\mathcal{K}_s = (-\Delta)^{-s}$ with $s \in (0, \frac{d}{2})$ (see (6) for details). For this kernel, to the best of our knowledge, only stationary solutions have been analyzed before. [9] studied the existence of stationary solutions in the fair competition regime: $m = 2 - 2s/d$. It was shown in [11] that stationary solutions are radially symmetric decreasing with compact supports and have certain regularity properties in most of the subcritical regime. If putting a negative sign in front of the nonlocal drift, the term is repulsive which is studied by [8, 23]. Recently in [13, 10], the equations with repulsive-attractive kernel, of the form $K(z) = |z|^a - |z|^b$ with $2 \geq a, b > -d$, are studied. [10] analyzed the asymptotic and the singular limits as $m \to \infty$, and [13] proved the boundedness of solutions when the repulsive potential has a stronger singularity.

Our goal is to initiate investigating the dynamic equation (1) in the subcritical regime, starting with its well-posedness and regularity properties. Many questions stay open as we discuss below.

**Summary of our result.**

Throughout the paper, we assume

1. $s \in (0, \frac{d}{2})$, $m > 2 - \frac{2s}{d}$, \hspace{1cm} (4)
2. $u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and $u_0$ is nonnegative. \hspace{1cm} (5)

**Theorem 1.1.** [Existence and Boundedness] Consider general dimensions $d \geq 1$. Suppose (4)-(5), and either $s > \frac{1}{2}$ or $m < 2$ hold. Then there exists a nonnegative weak solution $u$ to (1) with mass preserved and $u$ is uniformly bounded for all $t \in [0, \infty)$. The bound only depends on $s, m, d$, and $\|u_0\|_1 + \|u_0\|_\infty$.

The proof is given in Theorems 4.2-4.3. This theorem can be seen as a variate result as compared to [18, 28, 2] where Keller-Segel systems or equations are considered. We approach the problem by two approximations: regularization of the gradient of the kernel and elimination of the degeneracy, see (14). The existence of smooth solutions for the approximated problems is standard [3].

With the approximations in mind, the first goal is to obtain a priori uniform in time $L^\infty$-bound of solutions. We use an iteration method which can be found in [21], and show a sequence of differential inequalities. The key is to bound the attracting term by the degenerate diffusion. By scaling, the condition $m > 2 - 2s/d$ is critical, the use of which will be highlighted.

If $1/2 < s \leq 1$, the uniform bound is obtained separately when $m < 2$ and $m \geq 2$, and only for the former range of $m$ if $s \leq 1/2$. Here $s = \frac{1}{2}$ is a borderline, because $|\nabla \mathcal{K}_s|$ is only locally integrable when $s > \frac{1}{2}$. In the proof, we will apply Sobolev inequalities and properties of fractional Laplacian. It is essential that each
estimate needs to be consistent with the scaling and this turns out to be a useful hint for us, for example the choice of exponents in inequality (31). When \( s > 1 \), interpolation inequalities for fractional differentiation are no longer helpful. To go around the technical difficulty, we adopt a different argument by studying the singular convolution integrals in three different ways according to the steps of the iteration, see the proof of Theorem 3.5.

As for the remaining range \( m \geq 2 \) and \( s \leq 1/2 \), we conjecture the same a priori bound. While likely a technical issue, challenges for \( s > \) due to the loss of the integrability of \( \nabla K_s \), \( \nabla K_s * u \) is not well-defined for \( u \in L^1 \cap L^\infty \). To overcome this difficulty, we observe the following a priori estimate under the condition \( m < 2 \),

\[
\nabla u \in L^2_{loc}([0, \infty), L^2(\mathbb{R}^d)).
\]

Using this, we can make sense of \( \nabla (-\Delta)^{-s} u \) in the space \( L^2_{loc}([0, \infty), L^2(\mathbb{R}^d)) \), see Lemma 4.1.

Next let us state the uniqueness result.

**Theorem 1.2.** [Uniqueness] When \( s \geq 1, d \geq 3 \), there is a unique weak solution to (1) with initial data \( u_0 \).

In [3, 2], the uniqueness problem was solved when \( s = 1, d \geq 3 \). We will take their approach to study (for each \( t \)) the solution \( u(\cdot, t) \) in the \( \dot{H}^{-1}(\mathbb{R}^d) \) and prove for \( s \in (1, \frac{d}{2}) \). For \( d \leq 2 \), the solution \( u(\cdot, t) \in L^1 \cap L^\infty \) may lie outside of \( H^{-1}(\mathbb{R}^d) \).

As for the regularity property, with the help of [20], we have the following theorem.

**Theorem 1.3.** [Hölder Regularity] Suppose \( s \in (\frac{1}{2}, \frac{d}{2}) \) with \( d \geq 2 \). Let \( u(\cdot, \cdot) \) be a uniformly bounded weak solution to (1) with initial data \( u_0 \). Then for any \( \tau > 0 \), \( u \) is uniformly Hölder continuous in \( \mathbb{R}^d \times (\tau, \infty) \).

A lot of open questions remain to be investigated in the subcritical regime: existence result for \( s < 1/2 \) and \( m > 2 \), uniqueness for \( s < 1 \) and Hölder regularity for \( s \leq 1/2 \).

Let us comment that our results and proofs adapt to more general kernels \( K \) such that \( |K_s(x, y, t)|, |\nabla_x K_s(x, y, t)|, |D^2_x K_s(x, y, t)| \) share the same singularity as \( |x - y|^{-d+2s}, |x - y|^{-d-1+2s}, |x - y|^{-d-2+2s} \) respectively near \( x = y \). Some modifications are needed if we only assume \( |K_s(x, y, t)|, |\nabla_x K_s(x, y, t)|, |D^2_x K_s(x, y, t)| \) to be bounded away from \( x = y \).

**Outline of the paper.**

Section 2 contains preliminary definitions and notations. Section 3 proves a priori estimate that solutions are uniformly bounded for all time in the case either \( s > 1/2 \) or \( m < 2 \). In section 4 we show the existence of solutions. In section 5, we demonstrate the uniqueness result when \( d > 3 \) and \( s \geq 1 \) and the Hölder regularity result when \( s > 1/2 \).

2. **Preliminaries and notations.** We use the notation \( -(-\Delta)^r \) with \( r \in (0, 1] \) for fractional Laplacian operator which is defined on the Schwartz class of functions on \( \mathbb{R}^d \) by Fourier multiplier with symbol \( -|\xi|^{2r} \), see chapter V [26]. Alternatively,
Further assume that the operator can be extended naturally to the Sobolev space $\text{W}^{2r,2}(\mathbb{R}^d)$. We denote the constant in front of the singular integral as $c_{d,r}$. The domain of the operator can be extended naturally to the Sobolev space $\text{W}^{2r,2}(\mathbb{R}^d)$.

We define the following bilinear form associated to the space $\text{W}^{r,2}(\mathbb{R}^d)$:

$$B_r(v, w) = c_{d,r} \int \frac{(v(x) - v(y))(w(x) - w(y))}{|x - y|^{d+2r}} dxdy$$

for $v, w \in \text{W}^{r,2}(\mathbb{R}^d)$. Then

$$B_r(v, w) = \langle (-\Delta)^r v, w \rangle_{L^2} := \int_{\mathbb{R}^d} w(-\Delta)^r v dx.$$  

Using Parseval’s identity and definitions, we have for $0 < r_1 < r$

$$\langle (-\Delta)^{r_1} v, w \rangle_{L^2} = \langle |\nabla|^{r-r_1} v, |\nabla|^{r_1} w \rangle_{L^2}.$$  

For details, we refer readers to [22] and Section 3 [7].

**Proposition 1** (Proposition 3.2 [7]). For every $v, w \in \text{W}^{1,2}(\mathbb{R}^d)$, we have

$$B_r(v, w) = C \int \frac{\nabla v(x) \cdot \nabla w(y)}{|x - y|^{d-2s+2r}} dxdy.$$  

The inverse operator of fractional Laplacian is denoted by $(-\Delta)^{-s}$ which can be realized as the Riesz potential

$$(-\Delta)^{-s} u(x) := \int_{\mathbb{R}^d} K_s(x, y)u(y)dy;$$  

and

$$K_s(x, y) := \frac{2^{2s}\Gamma\left(\frac{d-2s}{2}\right)}{\pi^{d/2}\Gamma(s)} |x - y|^{-d+2s}.$$  

Here $s \in (0, \frac{d}{2})$ and $u$ is a function integrable enough for (6) to make sense. We refer readers to [22, 8, 25] for more details.

When $s > \frac{d}{2}$, $\nabla K_s u$ is well defined for $u \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. When $s < \frac{d}{2}$, if we further assume that $u$ is $\gamma$-Hölder continuous with $\gamma > 1 - 2s$, $\nabla K_s u$ can be defined via a Cauchy principal value

$$\nabla K_s u(x) := \int_{\mathbb{R}^d} \nabla_x K_s(x, y)(u(y) - u(x))dy.$$  

Next we give the notion of weak solutions to (1) which is similar to those in [3, 8].

**Definition 2.1.** Let $u_0(x) \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ be nonnegative and $T \in (0, \infty]$. We say that a nonnegative function $u : \mathbb{R}^d \times [0, T] \rightarrow [0, \infty)$ is a weak solution to (1) in time $[0, T]$ with initial data $u_0$ if

$$u \in C([0, T], L^1(\mathbb{R}^d)) \cap L^\infty(\mathbb{R}^d \times [0, T]), \quad u^m \in L^2(0, T, \dot{H}^1(\mathbb{R}^d)),$$

and

$$u\nabla K_s u \in L^2(\mathbb{R}^d \times [0, T])$$  

and for all test function $\phi \in C_c^\infty(\mathbb{R}^d \times (0, T))$,

$$\int_0^T \int_{\mathbb{R}^d} u \phi_t dx dt = \int_0^T \int_{\mathbb{R}^d} (\nabla u^m - u\nabla K_s u) \nabla \phi dx dt.$$  

(8)
Now we collect some known inequalities which will be used later.

**Lemma 2.2.** [Young’s convolution inequality] For all $p, q, r \in [1, \infty]$ satisfying $1 + 1/q = 1/p + 1/r$, we have for all functions $f \in L^p(\mathbb{R}^d)$, $g \in L^r(\mathbb{R}^d)$

$$
\|f * g\|_{L^q} \leq \|f\|_{L^p} \|g\|_{L^r}.
$$

**Lemma 2.3.** [Gagliardo-Nirenberg Interpolation Inequality] Let $\alpha, r, q, s$ be nonnegative constants such that the following holds

$$
0 \leq s \leq \alpha < 1, 1 < r, p, q < +\infty
$$
and

$$
\frac{1}{p} = \frac{s}{d} + \left(\frac{1}{r} - \frac{1}{\alpha}\right) \alpha + \frac{1 - \alpha}{q}.
$$

Then there exists a constant $C$ depending only on $\alpha, r, q, s$ such that for any function $u : \mathbb{R}^d \to \mathbb{R}$ satisfying $u \in L^q(\mathbb{R}^d)$ and $\nabla u \in L^r(\mathbb{R}^d)$, we have

$$
\|\nabla^s u\|_p \leq C \|\nabla u\|_\alpha \|u\|_{q}^{1-\alpha}.
$$

Condition (9) can be replaced by

$$
0 < s < \alpha < 1, 1 < r, p < +\infty, 1 \leq q < +\infty.
$$

If $s = 0$, the inequality is classical and (9) can be replaced by

$$
0 \leq \alpha < 1, 1 < r, p < +\infty, 1 \leq q < +\infty.
$$

This lemma is not given in the most general form, which is unnecessary for our purpose. We refer readers to [24] for the classical Gagliardo-Nirenberg inequality. The validity of the inequality with fractional derivatives can be found in Corollary 1.5 [17]. But they did not cover the case when (10) and (11) hold with $q = 1$. We postpone the completion of the proof to the appendix.

The following lemma is useful which can be viewed as an application of Theorem 2.2 [26] or Theorem 4.3.3 [16]. We will only sketch the proof of the lemma below.

**Lemma 2.4.** There exists a constant $C_d > 0$ such that for all $1 < p < \infty$ and $u \in W^{1,p}(\mathbb{R}^d)$

$$
\|\nabla u\|_p \leq C_d \max\{p, (p-1)^{-1}\} \|\nabla u\|_p.
$$

**Proof.** Recall $\langle|\nabla u|^\ast \rangle = 2\pi|\xi| \hat{u}(\xi)$. We can write

$$
2\pi|\xi| = 2\pi \sum_{j=1}^{d} \frac{\xi_j^2}{|\xi|} = \sum_{j=1}^{d} (-i\frac{\xi_j}{|\xi|}) \times 2\pi i \xi_j. =: \sum_{j=1}^{d} m_j(\xi)2\pi i \xi_j.
$$

We only need to show that the Riesz multiplier $m_j$ are bounded in $L^p$ i.e. the operator $R_j$ defined by $R_j u := \tilde{m}_j * u$ is bounded in $L^p$. If it is bounded, then

$$
\|\nabla u\|_p \leq \sum_{j=1}^{\infty} \|R_j(\partial_j u)\|_p \leq \sum_{j=1}^{\infty} \|\partial_j u\|_p \leq \|\nabla u\|_p.
$$

Now we want to apply Theorem 2.2 [26] or Theorem 4.3.3 [16]. The kernel of the Riesz transform is a Calderón-Zygmund convolution kernel: if $d \geq 2$

$$
K_j(x) := \tilde{m}_j = (-\frac{i}{|\xi|}) \varphi = \frac{-1}{2\pi} \partial_{x_j} \left(\frac{\pi^{d/2+1} \Gamma(d+1)}{\pi^{-d/2} \Gamma(d)} \frac{1}{|x|^{d-1}}\right) = \frac{1}{2} \pi^{-\frac{d}{2}} \frac{\Gamma(d+1)}{\Gamma(d)} (d-1) \frac{x_j}{|x|^{d+1}}.
$$
When \( d = 1 \), we have \( K(x) = (\pi^{\frac{d}{2}} \Gamma(\frac{1}{2}))^{-\frac{1}{2}} \). Clearly \( |K_j(x)| \lesssim \frac{1}{|x|^d} \) and \( |\nabla K_j(x)| \lesssim \frac{1}{|x|^d} \). Therefore by the theorems,
\[
\|R_j(\partial_j u)\|_p = \|K_j * (\partial_j u)\|_p \leq C \max\{p, (p - 1)^{-1}\}\|\partial_j u\|_p
\]
and (*) is justified. We finished the proof. \( \Box \)

Recall the homogeneous Sobolev space \( \dot{H}^s(\mathbb{R}^d) \):

**Definition 2.5.** Let \( \dot{H}^s \) be a measurable set in \( \mathbb{R}^d \). For \( |s| < \frac{d}{2} \), \( \dot{H}^s \) can be considered as the dual space of \( \dot{H}^{-s} \) through the following bilinear functional: for any \( f \in \dot{H}^s, g \in \dot{H}^{-s} \), \( (f, g) = \int_{\mathbb{R}^d} f(x)g(x)dx \). Furthermore if \( s = 1 \), \( \dot{H}^1 \) is the subset of tempered distributions with locally integrable Fourier transforms and such that \( |\nabla f| \in L^2(\mathbb{R}^d) \).

The proof is given in [1].

**Notations.** We write \( \mathbb{N} \) as all natural numbers and \( \mathbb{N}^+ \) as all positive natural numbers. For \( p \geq 1 \), for simplicity, we denote \( \| \cdot \|_p := \| \cdot \|_{L^p(\mathbb{R}^d)} \). We write \( B_R(x) \) as a ball in \( \mathbb{R}^d \) centered at \( x \) with radius \( R \). Denote \( B_R := B_R(0) \).

Throughout this paper, the constants \( \{C\} \) represent universal constants, by which we mean various constants that only depend on \( m, d, s \) and the \( L^1, L^\infty \) norms of the initial data \( u_0 \). We may write \( C(A) \) or \( C_A \) to emphasize the dependence of \( C \) on \( A \).

We write \( A \lesssim B \) if \( A \leq CB \) for some universal constant \( C \). By writing \( A \lesssim_B B \), we mean \( A \leq CB \) where \( C \) depends on universal constants and \( D \) (with particular emphasis on the dependence of \( D \)). By \( A \sim B \), we mean both \( A \lesssim B \) and \( B \lesssim A \) are fulfilled.

Let \( S \) be a measurable set in \( \mathbb{R}^d \). The indicator function \( \chi_S(x) \) equals 1 if \( x \in S \) and it equals 0 otherwise.

3. A priori estimates. In this section several a priori estimates (\( L^\infty L^p_t \) and \( L^\infty_t L^\infty_x \) bounds) are obtained. We start with the following lemma. We refer readers to Lemma 3.1 [20] for the proof.

**Lemma 3.1.** Suppose \( n_0 = 1 \) and \( n_{k+1} := 2n_k + a \) for all \( k \geq 0 \) with \( a > -1 \). Let \( \{A_k(t), k \in \mathbb{N}^+\} \) be a sequence of differentiable, positive functions on \( [0, \infty) \) satisfying
\[
\frac{d}{dt}A_k + C_0 A_k \leq C_1 n_k + C_1^k (A_{k-1})^{2+C_1 n_k^{-1}},
\]
for some positive constants \( C_0, C_1 \). Let \( B_k(t) := A_k^{(n_k^{-1})}(t) \) and suppose \( \{B_0(t), B_k(0)\} \) are uniformly bounded with respect to \( k \in \mathbb{N}, t > 0 \). Then \( \{B_k(t)\} \) are uniformly bounded for all \( t > 0 \) and \( k \in \mathbb{N}^+ \).

Now let us regularize the equation (1). Instead of modifying \( K_s \), we consider the following approximation
\[
V_{s,\epsilon}(x) := \zeta_{\epsilon}(x) \nabla_x K_s(x, 0)
\]
where $\epsilon > 0$ is a small parameter and $\zeta_\epsilon$ is a smooth, radially symmetric, nonnegative function that
\[
\zeta_\epsilon = 0 \text{ for } |x| \leq \epsilon \text{ and } |x| \geq 2/\epsilon, \quad \zeta_\epsilon = 1 \text{ for } |x| \in [2\epsilon, 1/\epsilon],
\]
\[
|\nabla \zeta_\epsilon| \lesssim 1/\epsilon \text{ for } |x| \leq 2\epsilon, \quad |\nabla \zeta_\epsilon| \lesssim \epsilon \text{ for } |x| \geq 1/\epsilon.
\]
(13)

It is not hard to see
\[
(1) \ V_{s,\epsilon} \text{ is a smooth vector field and } V_{s,\epsilon}(x) = c(-d + 2s)|x|^{-d-2+2s} \text{ for } |x| \in [2\epsilon, 1/\epsilon];
\]
\[
(2) \ |\nabla \cdot V_{s,\epsilon}(x)| \leq C|x|^{-d-2+2s} \text{ holds for some } C > 0 \text{ only depending on } d, s \text{ and } \epsilon.
\]

Consider the following uniformly parabolic problem with no-flux boundary condition:
\[
\begin{aligned}
\frac{\partial}{\partial t} u_\epsilon &= \epsilon \Delta u_\epsilon + \Delta u_\epsilon^m - \nabla \cdot (u_\epsilon V_{s,\epsilon} * u_\epsilon) = 0 \quad \text{in } B^\frac{d}{2} \times [0, \infty), \\
(\epsilon \nabla u_\epsilon + \nabla u_\epsilon^m - (u_\epsilon V_{s,\epsilon} * u_\epsilon)) \cdot \nu &= 0 \quad \text{on } \partial B^\frac{d}{2} \times [0, \infty), \\
u_\epsilon(x, 0) &= u_0 \ast \rho_\epsilon(x) \quad \text{on } B^\frac{d}{2}.
\end{aligned}
\]

where $\rho_\epsilon$ is a smooth modifier which converges to the delta mass and the interaction kernel $V_{s,\epsilon}$ is smooth compactly supported. By Theorem 4.2 [3], there exists a unique solution $u_\epsilon$ to (14) which is nonnegative and smooth. It turns out that considering the following approximation is simpler for the discussion and the proofs remain the same. Later we will assume $u_\epsilon$ solves
\[
\begin{aligned}
\frac{\partial}{\partial t} u_\epsilon &= \epsilon \Delta u_\epsilon + \Delta u_\epsilon^m - \nabla \cdot (u_\epsilon V_{s,\epsilon} * u_\epsilon) = 0 \quad \text{in } \mathbb{R}^d \times [0, \infty), \\
u_\epsilon(x, 0) &= u_0(x) \quad \text{on } \mathbb{R}^d.
\end{aligned}
\]

In the following subsequent theorems, we are going to prove that $u_\epsilon$ are uniformly bounded for all time independent of $\epsilon$. As mentioned before, we will treat the following five cases separately: $\{m < 2, 1/2 < s \leq 1\}$, $\{m \geq 2, 1/2 < s \leq 1\}$, $\{m < 2, s \leq 1/2\}$, $\{m < 2, 1 < s < d/2\}$ and $\{m \geq 2, 1 < s < d/2\}$.

**Theorem 3.2.** Suppose $(s, \epsilon) \in (\frac{1}{2}, 1]$ and $m \in (2 - \frac{2s}{d}, 2)$. Let $u := u_\epsilon$ be a solution to (14). Then $u$ is uniformly bounded for all time and the bound only depends on $d, s, m$ and $\|u_0\|_{L^1} + \|u_0\|_{L^\infty}$.

**Proof.** Define a sequence $\{n_k, k \in \mathbb{N}^+\} \subset \mathbb{R}^+$ by
\[
n_0 = 1, \quad n_{k+1} := 2n_k + 1 - m \quad \text{for all } k \geq 0.
\]
(15)

We find $n_k = 2^k(2 - m) - 1 + m$. It follows from $m < 2$ that $n_k \to \infty$ as $k \to \infty$.

Fix any $k \geq 1$ and we know $n_k \geq n_1 = 3 - m$. For simplicity of notation, let us write
\[
n = n_k, \quad l = n_{k-1} = \frac{m + n - 1}{2} < n.
\]

Now, without loss of generality, suppose that the total mass of $u_0$ is 1 and so is the total mass of $u(\cdot, t)$ by the equation. Since $u$ is smooth, we multiply $u^{n-1}$ on both sides of (14) and find
\[
\begin{aligned}
\partial_t \int_{\mathbb{R}^d} u^n dx &\leq -n \int_{\mathbb{R}^d} \nabla u^m \nabla u^{n-1} dx + n \int_{\mathbb{R}^d} (u V_{s,\epsilon} * u) \cdot \nabla u^{n-1} dx \\
&\leq -C_m \int_{\mathbb{R}^d} |\nabla u_\epsilon|^2 dx + (n - 1) \int_{\mathbb{R}^d} V_{s,\epsilon} * u \nabla u^n dx.
\end{aligned}
\]
(16)
By property (2) of \( V_{s,c} \), we obtain
\[
X := \int_{\mathbb{R}^d} V_{s,c} \ast u\nabla u^\alpha dx = \int_{\mathbb{R}^d} (-\nabla \cdot V_{s,c}) \ast u\ n^\alpha dx \\
\leq C \int_{\mathbb{R}^{2d}} \frac{(u(x) - u(y))(u^\alpha(x) - u^\alpha(y))}{|x-y|^{d+2-2s}} dxdy
\]
Since \( u \) is nonnegative
\[
(u(x) - u(y))(u^\alpha(x) - u^\alpha(y)) \leq (u^l(x) - u^l(y))(u^{n+1-l}(x) - u^{n+1-l}(y)),
\]
and thus
\[
X \leq C \int_{\mathbb{R}^{2d}} \frac{(u^l(x) - u^l(y))(u^{n+1-l}(x) - u^{n+1-l}(y))}{|x-y|^{d+2-2s}} dxdy
\]
\[
= C \int_{\mathbb{R}^d} (-\Delta)^{-s} u^l \nabla u^{n+1-l} dx \quad (\text{by Proposition 1})
\]
\[
\leq C \int_{\mathbb{R}^d} (\Delta)^{-s} u^l \ n^{n+1-l} dx \\
\leq C \left\| \nabla^2 2^{-2s} u^l \right\|_2 \left\| u^{n+1-l} \right\|_2 \quad (\text{by Hölder’s inequality})
\]
\[
= C \left\| \nabla^2 2^{-2s} u^l \right\|_2 \left\| u^l \right\|^{\frac{n+1-l}{2}}_2^{\frac{1}{\frac{n+1-l}{2}}}.
\]
By Gagliardo-Nirenberg interpolation inequality (Lemma 2.3)
\[
\left\| \nabla^2 2^{-2s} u^l \right\|_2 \lesssim \left\| \nabla u^l \right\|^\alpha_2 \left\| u^l \right\|^{1-\alpha}_1,
\]
\[
\left\| u^l \right\|_2^{\frac{n+1-l}{2}} \lesssim \left\| \nabla u^l \right\|^\beta_2 \left\| u^l \right\|^{1-\beta}_1
\]
Now by Hölder’s inequality, for any small $\delta > 0$
\[ X \leq \frac{\delta}{n} \| \nabla u_1^\prime \|_2^2 + C_\delta n c_n \| u_1^\prime \|_1^{\theta'} \]  
(19)
where
\[ \theta' = \theta'(n) := 2 + \frac{2(2 - m)}{l(2 - \theta(n))} \leq 2 + Cn^{-1} \quad \text{and} \quad c_n := \frac{\theta(n)}{2 - \theta(n)}. \]

According to (18), $\{c_n\}$ are uniformly bounded for all $n \geq 3 - m$. By Gagliardo-Nirenberg inequality
\[ \| u_1^\prime \|_\varphi \lesssim \| \nabla u_1^\prime \|_2 \| u_1^\prime \|_1^{1-\gamma} \]
where
\[ \gamma = \left( \frac{n-1}{n} \right) / \left( \frac{1}{2} + \frac{1}{d} \right). \]

Direct calculation shows $\frac{2n}{\varphi} < 2$. Next by Young’s inequality
\[ \int_{\mathbb{R}^d} u^n dx \lesssim \| \nabla u_1^\prime \|_2 \| u_1^\prime \|_1^{(1-\gamma)n} \leq \delta \| \nabla u_1^\prime \|_2^2 + C_\delta \| u_1^\prime \|_1^{\gamma'} \]
where
\[ \gamma' = \gamma'(n) := 2 - \frac{2(m-1)}{2l - \gamma n} \leq 2 - Cn^{-1} \quad \text{and} \quad \text{(not hard to check)} \quad \gamma' > 0. \]

Finally by (16), (19) and (20), we obtain for all $n \geq 3 - m$
\[ \partial_t \int_{\mathbb{R}^d} u^n dx + c \int_{\mathbb{R}^d} u^n dx \leq Cn \left( \int_{\mathbb{R}^d} u^n dx \right)^{\gamma'} + Cn^{-1} \left( \int_{\mathbb{R}^d} u^n dx \right)^{\theta'} \]
where $c, C$ are independent of $n$.

Recall (15). Since for some universal $C$
\[ n_k \sim_m 2^k, \quad \theta', \gamma' \leq 2 + Cn^{-1}, \quad c_n \leq C, \]
writing $A_k = \int_{\mathbb{R}^d} u^{nk} dx$ yields
\[ \frac{d}{dt} A_{k+1} + cA_{k+1} \leq C^k + C^k A_k^{2 + C2^{-n}} \quad \text{for all} \quad k \geq 0. \]

Finally applying Lemma 3.1 concludes the theorem. \(\square\)

**Theorem 3.3.** Theorem 3.2 holds in the regime $m \geq 2$ and $s \in (\frac{1}{2}, 1]$.

**Proof.** Denote $u_1 = \max\{u - 1, 0\}$, $\tilde{u} = u - u_1 \leq 1$. For some $n \geq 2$, let us multiply $u_1^{n-1}$ on both sides of (14). We get
\[ \partial_t \int_{\mathbb{R}^d} u_1^n dx = n \int_{\mathbb{R}^d} u_1^{n-1} u dx \]
\[ \leq -mn \int_{\mathbb{R}^d} u^{m-1} \nabla u \nabla u_1^{n-1} dx + n \int_{\mathbb{R}^d} \left( V_{s,c} + u \right) u \nabla u_1^{n-1} dx. \]

Because
\[ \int_{\mathbb{R}^d} u^{m-1} \nabla u \nabla u_1^{n-1} dx = \int_{\mathbb{R}^d} \left( u_1 + 1 \right)^{m-1} \nabla u_1 \nabla u_1^{n-1} dx \]
\[ \geq C_m \int_{\mathbb{R}^d} \left( \left| \nabla u_1^{\frac{n+1}{2}} \right|^2 + \left| \nabla u_1^{\frac{n}{2}} \right|^2 \right) dx \]
for some $C_m > 0$ bounded from below for all $m \geq 2$, we obtain
\[ \partial_t \int_{\mathbb{R}^d} u_1^n dx \leq -C_m \int_{\mathbb{R}^d} \left( \left| \nabla u_1^{\frac{n+1}{2}} \right|^2 + \left| \nabla u_1^{\frac{n}{2}} \right|^2 \right) dx + nX. \]
(21)
Let us now estimate $X$:

$$X = \int_{\mathbb{R}^d} V_{s,\epsilon} \ast u_1 u \nabla u_1^{n-1} dx + \int_{\mathbb{R}^d} V_{s,\epsilon} \ast \tilde{u} u \nabla u_1^{n-1} dx$$

$$\lesssim \int_{\mathbb{R}^d} V_{s,\epsilon} \ast u_1 \nabla u_1^n dx + \int_{\mathbb{R}^d} V_{s,\epsilon} \ast u_1 \nabla u_1^{n-1} dx + \int_{\mathbb{R}^d} V_{s,\epsilon} \ast \tilde{u} u \nabla u_1^{n-1} dx$$

$$=: Y_n + Y_{n-1} + X_1. \quad (22)$$

We will first consider $X_1$. By the fact that

$$s > \frac{1}{2}, \quad \tilde{u} \leq 1, \quad \tilde{u} \in L^1 \text{ and } |V_{s,\epsilon}(x)| \lesssim |x|^{-d-1+2s},$$

we have

$$|V_{s,\epsilon} \ast \tilde{u}|(x) \lesssim \int_{\mathbb{R}^d} |x-y|^{-d-1+2s} \tilde{u}(y) dy$$

$$\leq C \int_{\mathbb{R}^d} \tilde{u} dy + \int_{|x-y| \leq 1} |x-y|^{-d-1+2s} dy \leq C.$$

Hence for any small $\delta > 0$

$$X_1 = C \int_{\mathbb{R}^d} u \left| \nabla u_1^{n-1} \right| dx \lesssim \int_{\mathbb{R}^d} u \left| \nabla u_1^{n} \right| dx$$

$$\leq C_\delta n \int_{\mathbb{R}^d} \left( u_1^n + u_1^{n-2} \right) dx + \frac{\delta}{n} \| \nabla u_1^n \|_2^2$$

$$\leq C_\delta n \int_{\mathbb{R}^d} u_1^n dx + C_\delta n + \frac{\delta}{n} \| \nabla u_1^n \|_2^2. \quad (23)$$

In the last inequality (23), we applied

$$\int_{\mathbb{R}^d} u_1^{n-2} dx \leq \int_{u_1 \geq 1} u_1^n dx + \int_{1 \leq u_1 \leq 2} 1 dx \leq \left\| u_1^{\frac{n}{2}} \right\|_2^2 + 1.$$

Next by Gagliardo-Nirenberg and Young’s inequalities

$$C_\delta \left\| u_1^{\frac{n}{2}} \right\|_2^2 \leq C_\delta C_\alpha \left\| \nabla u_1^n \right\|_2^{2\alpha} \left\| u_1^{\frac{n}{2}} \right\|_1^{2(1-\alpha)} \leq C_\delta n^{d+1} \left\| u_1^{\frac{n}{2}} \right\|_1 + \frac{\delta}{n} \left\| \nabla u_1^n \right\|_2^2 \quad (24)$$

where we picked

$$\alpha = \frac{1}{2} / \left( \frac{1}{2} + \frac{1}{d} \right).$$

So by (23), for some universal small $\delta > 0$

$$X_1 \leq C_\delta n + C_\delta n^{d+1} \left\| u_1^{\frac{n}{2}} \right\|_1 + \frac{\delta}{n} \left\| \nabla u_1^n \right\|_2^2. \quad (25)$$

For $Y_i$ with $l = n - 1, n$, as proved before (in Theorem 3.2)

$$Y_i \lesssim \int_{\mathbb{R}^d} \frac{\left( u_1^{\frac{n+1}{2}}(x) - u_1^{\frac{n+1}{2}}(y) \right)^2}{|x-y|^{d+2-2s}} dxdy \lesssim \int_{\mathbb{R}^d} \nabla (-\Delta)^{-s} u_1^{\frac{n+1}{2}} \nabla u_1^{\frac{n+1}{2}} dx.$$
By Fourier transformation and Hölder’s inequality,
\[
Y_l \lesssim \left( \int_{\mathbb{R}^d} |\xi|^{2-2s} |u_1^{\frac{d+1}{2}}|^2 d\xi \right)^{1-s} \left( \int_{\mathbb{R}^d} \left| \left( \int_{\mathbb{R}^d} |\xi|^{2} |u_1^{\frac{d+1}{2}}|^2 d\xi \right)^{1-s} \int_{\mathbb{R}^d} u_1^{\frac{d+1}{2}} d\xi \right) \right)^s
\]
\[
\lesssim \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} \left| \nabla u_1^{\frac{d+1}{2}} \right|^2 dx \right)^{1-s} \int_{\mathbb{R}^d} u_1^{\frac{d+1}{2}} dx \right)^s
\]
\[
\leq C_\delta n^{-\frac{s}{d+1}} \left\| u_1^{\frac{d+1}{2}} \right\|_2^2 + \frac{\delta}{n} \left\| \nabla u_1^{\frac{d+1}{2}} \right\|^2_2
\]
\[
\leq C_\delta n^{nX} \left\| u_1^{\frac{d+1}{2}} \right\|_2^2 + \frac{\delta}{n} \left\| \nabla u_1^{\frac{d+1}{2}} \right\|^2_2 \quad (\text{since } s > \frac{1}{2}).
\]
(26)

When \( l = n - 1 \), by (26) and (24), we get for some \( C \) only depending on \( \delta \)
\[
Y_{n-1} \leq C n^{d+1} \left\| u_1^2 \right\|_1^2 + \frac{\delta}{n} \left\| \nabla u_1^2 \right\|_2^2.
\]

When \( l = n \), as done previously
\[
Y_n \leq C n^{d+1} \left\| u_1^{\frac{d+1}{2}} \right\|_1^2 + \frac{\delta}{n} \left\| \nabla u_1^{\frac{d+1}{2}} \right\|_2^2.
\]

By Gagliardo-Nirenberg,
\[
\left\| u_1^{\frac{d+1}{2}} \right\|_1^2 = \left\| u_1^{\frac{d+1}{2}} \right\|_{\frac{n+1}{2}}^{2\beta_1} \leq C \left\| \nabla u_1^{\frac{d+1}{2}} \right\|_2^{2\beta_1} \left\| u_1^{\frac{d+1}{2}} \right\|_1^{2\beta_2}
\]
where
\[
\beta_1 = \beta_1(n) = \frac{1}{n} \left( \frac{1}{2} + \frac{1}{d} \right), \quad \beta_2 = \beta_2(n) = \frac{n+1}{n} - \beta_1.
\]

By Young’s inequalities
\[
C \left\| \nabla u_1^{\frac{d+1}{2}} \right\|_2^{2\beta_1} \left\| u_1^{\frac{d+1}{2}} \right\|_1^{2\beta_2} \leq C \left( \frac{\epsilon^p \left\| \nabla u_1^{\frac{d+1}{2}} \right\|_2^{2\beta_1 p}}{p} + \left\| u_1^{\frac{d+1}{2}} \right\|_1^{2\beta_2 q} \right)
\]
\[
\leq C_\delta n^{c_n} \left\| u_1^{\frac{d+1}{2}} \right\|_1^{\gamma_n} + \frac{\delta}{n^{d+2}} \left\| \nabla u_1^{\frac{d+1}{2}} \right\|^2_2
\]
where we pick
\[
p = \frac{1}{\beta_1} \sim n, \quad q = \frac{p}{p-1}, \quad Ce^p/p = \frac{\delta}{n^{d+2}}.
\]

Thus \( \epsilon^p \sim \frac{\delta}{n^{d+2}} \). Now since \( \frac{q}{p} = \frac{1}{p-1} = \frac{\beta_2}{1-\beta_1} \), we find \( e^{-q} = C_\delta n^{-\frac{\beta_2(d+1)}{1-\beta_1}} \),
\[
\gamma_n = \frac{2\beta_2}{1 - \beta_1} = 2 + \frac{2}{n(1-\beta_1)} \quad \text{and} \quad c_n = \frac{\beta_1(d+1)}{1-\beta_1}.
\]

It is not hard to check that for all \( n \geq 2 \), \( \beta_1(n) \leq \beta_1(2) < 1 \). And so \( c_n \) is uniformly bounded for all \( n \geq 2 \). Thus we proved that for any small \( \delta > 0 \)
\[
Y_n \leq C_\delta n^{c_n} \left\| u_1^{\frac{d+1}{2}} \right\|_{\gamma_n}^2 + \frac{\delta}{n} \left\| \nabla u_1^{\frac{d+1}{2}} \right\|^2_2 + \frac{\delta}{n} \left\| \nabla u_1^{\frac{d+1}{2}} \right\|^2_2.
\]

Combining with (22) and (25), for some \( c = c_n + d + 1 > 0 \) uniformly bounded for all \( n \geq 2 \)
\[
nX \leq C_\delta n^2 + C_\delta n^{c} \left( \left\| u_1^{\frac{d+1}{2}} \right\|_1^2 + \left\| u_1^{\frac{d+1}{2}} \right\|_{\gamma_n}^2 + \delta \left\| \nabla u_1^{\frac{d+1}{2}} \right\|^2_2 + \delta \left\| \nabla u_1^{\frac{d+1}{2}} \right\|^2_2.\]
Multiplying Proof.

Recall (15), and for any $\gamma$

Recall here

By (27), (28), we have

Theorem 3.4. Theorem 3.2 holds in the regime:

Therefore by (21)

Again by Galiardo-Nirenberg inequality and Young’s inequality

where $\theta = \frac{1}{2}/\left(\frac{1}{2} + \frac{1}{n}\right)$. So for some universal $C, c > 0$

By (27), (28), we have

Recall here $\gamma_n \leq 2 + \frac{c}{n}$.

Now letting $n = 2^k$ with $k \in \mathbb{N}^+$ and $A_k = \int_{\mathbb{R}^d} u_1^{2^k} \, dx$ gives

By Lemma 3.1 and (29), $u_1(x, t)$ is uniformly bounded for all $t \geq 0$ and so is $u(x, t)$.

Theorem 3.4. Theorem 3.2 holds in the regime: $m \in (2 - 2s/d, 2)$ and $s \in (0, \frac{1}{2}]$.

Proof. Recall (15), and for any $k \geq 1$ let $n = n_k \geq 3 - m$, $l = n_{k-1} = \frac{m+n-1}{2}$.

Multiplying $u^{n-1}$ on both sides of (14), we obtain

For the interaction term:

By Young’s inequality, for any $p > 1, q > 1$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$, the above

To choose the $p, q$, we use the scaling by considering $u_r = r^d u(rx, t)$. Then

$$
\left\| \nabla |^{1-2s} u_r^l \right\|_p^p = \left( r^{1-2sd_l} \right)^p \left\| \nabla |^{1-2s} u^l \right\|_p^p;
$$

$$
\left\| \nabla | u_r^{n+1-l} \right\|_q^q = \left( r^{rd(n+1-l)} \right)^q \left\| \nabla | u^{n+1-l} \right\|_q^q.
$$
To match the scaling, we want \((1 - 2s + dl)p = (1 + d(n + 1 - l))q\) in the case when \(m = 2 - \frac{2s}{d}\). Using \(\frac{1}{p} + \frac{1}{q} = 1\), we find

\[
p(n) = \frac{2 + d(m + n - 1)}{1 + d(m + l - 2)}, \quad q(n) = \frac{2 + d(m + n - 1)}{1 + d(n + 1 - l)}.
\] (31)

These are the values we pick for \(p, q\). When \(n = 3 - m, l = 1\), we obtain

\[
p(3 - m) = \frac{2 + 2d}{1 + d(3 - m)}, \quad q(3 - m) = \frac{2 + 2d}{1 + d(3 - m)}.
\] (32)

While as \(n \to \infty\), \(p(n)\) is monotonically decreasing, \(q(n)\) is monotonically increasing and

\[
p(n) \to 2, \quad q(n) \to 2.
\]

Also it is not hard to see that

\[
p(n) > 1, \quad q(n) > 1, \quad 1 \leq \frac{p(n)}{q(n)} = \frac{1 + d(3 - m)}{1 + d(m - 1)} =: c_1(d, m),
\]

\[
p(n) - 2 = \frac{2d(2 - m)}{1 + d(m + l - 2)} \sim \frac{1}{n},
\] (33)

\[
2 - q(n) = \frac{2d(2 - m)}{1 + d(n + 1 - l)} \sim \frac{1}{n}.
\] (34)

By Lemma 2.4 and Young’s inequality, for any \(\delta \in (0, 1)\)

\[
X_2 \leq C \|\nabla u^{n+1-l}\|_q^q = C' \int_{\mathbb{R}^d} u^{(n+1-2l)} |\nabla u|^q \, dx
\]

\[
\leq C_\delta n \left\| u^{(n+1-2l)} \right\|_{\frac{q}{1-\delta}}^{\frac{q}{1-\delta}} + \delta \left\| |\nabla u|^q \right\|_{\frac{q}{1-\delta}}^{\frac{q}{1-\delta}}
\]

\[
= C_\delta n \left\| u^l \right\|_2^2 + \delta \left\| \nabla u^l \right\|_2^2.
\]

In the last inequality, we used (34). Now by Gagliardo-Nirenberg interpolation inequality

\[
C_\delta \left\| u^l \right\|_2^2 \leq C_\delta \left\| \nabla u^l \right\|_2^{2\beta} \left\| u^l \right\|_1^{2(1-\beta)} \leq \frac{\delta}{n} \left\| \nabla u^l \right\|_2^2 + C_\delta n^{c_0} \left\| u^l \right\|_1^2
\]

where

\[
\beta = \frac{1}{2} \left( \frac{1}{2} + \frac{1}{d} \right), \quad c_0 = \frac{\beta}{1-\beta} = \frac{d}{2}.
\]

Therefore

\[
X_2 \leq C_\delta n^{c_0+1} \left\| u^l \right\|_1^2 + 2\delta \left\| \nabla u^l \right\|_2^2.
\] (35)

As for \(X_1\), again by Gagliardo-Nirenberg interpolation inequality

\[
X_1 = \left\| |\nabla|^{1-2s} u^l \right\|_p \leq \left\| \nabla u^l \right\|_2^{\alpha} \left\| u^l \right\|_1^{1-\alpha}
\]

where we need to put

\[
\alpha = \alpha(n) = \left( \frac{1-2s}{d} - \frac{1}{p} + 1 \right) / \left( \frac{1}{d} + \frac{1}{2} \right).
\]

It is not hard to check that \(s < \alpha(n) < 1\) uniformly for all \(n \geq 3 - m\). Moreover, we claim that

\[
\sup_{n \geq 3-m} \alpha(n)p(n) < 2.
\]
By monotonicity of $\alpha(n)p(n)$ in $n$, we only need to check when $n = (3 - m)$. By (32) and direct calculations,

$$\alpha(3 - m) p(3 - m) < 2 \iff m > 2 - \frac{2s}{d}.$$  

With this, we obtain

$$X_1^p \lesssim \frac{\delta}{n^{c_1 + 1}} \|\nabla u\|_2^n + C_{\delta} n^{c_2} \|u^l\|_1^{2+\gamma}$$  

(36)

where $c_2$ is a constant that

$$c_2 \geq (c_1 + 1) \frac{\alpha(1 - \alpha)p^2}{2 - \alpha p}$$

and by (33)

$$\gamma = \gamma(n) = \frac{2(p - 2)}{2 - \alpha p} \sim \frac{1}{n}.$$  

Putting together (35) and (36) shows

$$n \int V_{s,\epsilon} * u \nabla u^n dx \leq n^{c_1 + 1} X_1^p + X_2$$

$$\leq C_{\delta} n^{c_2 + c_1 + 1} \|u^l\|_1^{2+\gamma} + C_{\delta} n^{c_1 + 1} \|u^l\|_2^n + 3\delta \|\nabla u^l\|_2^n.$$  

Picking $\delta$ small enough, (30) shows for $c^* = \max\{c_2 + c_1 + 1, c_0 + 1\}$

$$\partial_t \int_{R^d} u^n dx + \int_{R^d} |\nabla u|^2 dx \leq C_{\delta} n^{c^*} \|u^l\|_1^{2+\gamma}.$$  

(38)

As done in (20), for some $\gamma' \in (0, 2)$

$$\|u^n\|_1 \lesssim \|\nabla u^l\|_2 + \|u^l\|_1^{\gamma'} \lesssim \|\nabla u^l\|_2^n + \|u^l\|_1^{2+\gamma} + 1.$$  

To conclude, we find out that

$$\frac{d}{dt} \|u^n\|_1 + \|u^n\|_1 \leq C + Cn^{c^*} \|u^l\|_1^{2+\gamma}$$

where $C, c^* > 0$ only depends on $s, d, m$.

Finally as in Theorem 3.2, since $n = n_k, l = n_{k-1}$ in (15), we proved the desired differential inequalities for all $k$. By considering $A_k = \int_{R^d} u^n dx$, we conclude the proof after applying Lemma 3.1. 

□

**Theorem 3.5.** Theorem 3.2 holds in the regime: $m \in (2 - 2s/d, 2)$ and $s \in (1, \frac{d}{2})$.

**Proof.** For $n \geq 3 - m$, denote $l = \frac{n+m-1}{2} \geq 1$. We multiply $u^{n-1}$ on both sides of (14) and obtain

$$\partial_t \int_{R^d} u^n dx \leq -mn \int_{R^d} u^{m-1} \nabla u \nabla u^{n-1} dx + n \int_{R^d} (V_{s,\epsilon} * u) \nabla u \nabla u^{n-1} dx$$

$$\leq -C_m \int_{R^d} |\nabla u|^2 dx - (n - 1) \int_{R^d} (\nabla \cdot V_{s,\epsilon} * u) u^n dx. \quad \text{(39)}$$  

Let $\chi(x) = \chi_{|x| \leq 1}(x)$ be an indicator function. Taking $A_1 := \chi \nabla \cdot V_{s,\epsilon}$ and $A_2 := (1 - \chi) \nabla \cdot V_{s,\epsilon}$ yield

1. $A_1$ is compactly supported and $|A_1|(z) \leq |z|^{-d-2+2s}$,
2. $|A_1|$ bounded in $L^{\frac{d}{d+2-2s}}(R^d)$ for all $1 < s' < s$,
3. $A_2$ is bounded.
Fix one $s'$ such that 
\[ s' \in (1, s) \quad \text{and} \quad m > 2 - \frac{2s'}{d}. \]
We have
\[ X \leq \int_{R^d} |A_1| * u u^a dx + C \int_{R^{2d}} |A_2| u(y) u^a(x) dx dy =: X_1 + X_2. \]
By Young’s convolution inequality
\[ X_1 \leq \|u^n\|_p \|u\|_q \] (40)
with
\[ p, q \geq 1 \quad \text{satisfying} \quad \frac{1}{p} + \frac{1}{q} = 1 + \frac{2s'}{d}. \] (41)

**Claim.** for any $\delta > 0$, there exists $C(\delta), c(\delta) > 0$ such that for all $n \geq 3 - m, l = \frac{n+m-1}{2}$
\[ X_1 \leq C n^\epsilon + C n^\epsilon \|u^l\|_1^{2+\frac{\epsilon}{2}} + \frac{\delta}{n} \|\nabla u^l\|_2^2. \] (42)
We will discuss the proof below according to different values of $n$.

(i) Suppose $n \leq \frac{d}{2s' - 2}$. We can write
\[ \|u^n\|_p = \|u^l\|_\frac{2p}{p+2}, \quad \|u\|_q = \|u^l\|_\frac{q}{q+2}. \] (43)
By Gagliardo-Nirenberg inequality and Young’s inequality,
\[ \|u^l\|_2 \leq C \|\nabla u^l\|_2 \|u^l\|_1^{1-\alpha}, \quad \|u^l\|_4 \leq C \|\nabla u^l\|_2 \|u^l\|_1^{1-\beta}. \] (44)
where $\alpha, \beta$ are given by
\[ \frac{l}{np} + \left(\frac{1}{2} + \frac{1}{d}\right)\alpha = 1, \quad \frac{l}{q} + \left(\frac{1}{2} + \frac{1}{d}\right)\beta = 1. \] (45)
Now we take
\[ p = \frac{n+1}{n} \left(1 + \frac{2s' - 2}{d}\right), \quad q = \frac{n+1}{n} \left(1 + \frac{2s' - 2}{d}\right). \]
Since $n \leq \frac{d}{2s' - 2}$, (41) is fulfilled. With this choice of $p, q$, by (45) we have
\[ \alpha = \beta = \left(\frac{1}{2} + \frac{1}{d}\right)^{-1} \left(1 - \frac{l}{n+1} \left(1 + \frac{2s' - 2}{d}\right)\right) > 0 \]
uniformly in $n \geq 3 - m$ by $n \leq \frac{d}{2s' - 2}$. By definitions of $n, l$, we compute
\[ 2 - \alpha \frac{n+1}{l} = \left(\frac{1}{2} + \frac{1}{d}\right)^{-1} \left(2 - \frac{n+1}{l} + \frac{2s'}{d}\right) \geq \left(\frac{1}{2} + \frac{1}{d}\right)^{-1} (m - 2 - \frac{2s'}{d}) \]
with equality holds when $n = 3 - m$. Thus we get
\[ \sup_{n \geq 3-m} \alpha \frac{n+1}{l} < 2 \] (46)
and this is exactly equivalent to $m > 2 - \frac{2s'}{d}$. In particular due to $n+1 \geq 2l$, we have $\sup_{n \geq 3-m} \alpha < 1$.
From (40), (43) and (44), it follows that
\[ X_1 \leq C \|\nabla u^l\|_2^{\frac{2p}{p+2}} \|u^l\|_1^{(1-\alpha) \frac{p}{q+2}}. \]
By Young’s inequality,
\[ X_1 \leq \frac{\epsilon^{p_1} \|\nabla u\|_2^{\frac{\alpha}{2} + \frac{\beta}{2} + 1}}{p_1} + \frac{C^{q_1} \|u\|_1^{(1-\alpha)\frac{\beta}{2} + (1-\beta)\frac{\alpha}{2}}}{\epsilon^{q_1}} \text{ for any } \epsilon > 0, \]
where we select
\[ p_1(n) = \frac{2}{\alpha n + 1}, \quad q_1(n) = \frac{p_1(n)}{p_1 - 1}. \]

By (46), we obtain \( \inf_{n \geq 3-m} p_1(n) > 1 \) and thus \( q_1 \) is uniformly bounded. Pick \( \epsilon = (\delta/n)^{1/p_1} \). Finally because \( q_1 \) is bounded and \( \frac{\alpha + 1}{\alpha} \leq 2 + \frac{c}{n} \) for some universal \( c \), we conclude with (42).

(ii) If \( l \leq \frac{d}{2 \nu - 2} \leq n \), use \( q = \frac{d}{2 \nu - 2} \), \( p = 1 \) in (40) and calculate \( \alpha, \beta \) accordingly by (45). In this situation, we get
\[ 1 > \frac{l}{np} \quad \Rightarrow \quad \frac{l}{q} > \frac{1}{2}, \]
and thus it is immediately to check that \( \alpha, \beta \in (0, 1) \). (40) and (44) yield
\[ X_1 \leq C \|\nabla u\|_2^{\alpha \frac{1}{2} + \beta \frac{1}{2}} \|u\|_1^{(1-\alpha)\frac{1}{2} + (1-\beta)\frac{1}{2}}. \]  
(47)

Direction computation shows
\[ \sup_{n \geq 3-m} \frac{n}{l(n)} + \frac{B}{l(n)} < 2 \]
and this is equivalent to \( m > 2 - \frac{2 \nu - 2}{\nu} \). Because
\[ \alpha \frac{n}{l} + \beta \frac{1}{l} + (1-\alpha) \frac{n}{l} + (1-\beta) \frac{1}{l} = \frac{n+1}{l} \sim 2 + \frac{c}{n} \]
holds for some universal \( c > 0 \), we obtain (42) by (47) and Hölder’s inequality.

(iii) Lastly suppose \( n > l(n) \geq \frac{d}{2 \nu - 2} \). We take \( p = 1, q = \frac{d}{2 \nu - 2} \) in (40). Since \( \|u\|_1 \) is bounded, the set \( \{u > 1\} \) is of finite measure. By the assumption, we get \( q \leq l \). By Jensen’s inequality
\[ \int_{\mathbb{R}^d} u^q \, dx \leq \int_{u < 1} u^q \, dx + \int_{u > 1} u^q \, dx \leq C \|u\|_1^q. \]
Thus
\[ X_1 = \|u^n\|_1 \|u\|_q \leq C \|u^n\|_1 (1 + \|u\|_1^\frac{1}{q}). \]  
(48)

By Gagliardo-Nirenberg,
\[ \|u^n\|_1^\frac{\alpha}{2} = \|u\|_q \leq C \|\nabla u\|_2^{\alpha} \|u\|_1^{1-\alpha} \]
where \( \alpha \) is given by \( \frac{l}{n} + (\frac{1}{2} + \frac{1}{\nu}) \alpha = 1 \). From this we get
\[ X_1 \leq C \|\nabla u\|_2^{\alpha \frac{1}{2}} \left( \|u\|_1^{(1-\alpha)\frac{1}{2}} + \|u\|_1^{(1-\alpha)\left(\frac{1}{2} + \frac{1}{\nu}\right)} \right) \]
\[ \leq C \|\nabla u\|_2^{\alpha \frac{1}{2}} \left( 1 + \|u\|_1^{(1-\alpha)\left(\frac{1}{2} + \frac{1}{\nu}\right)} \right) \]  
(49)
\[ \leq C n^\frac{\nu}{n} \|\nabla u\|_2^{\frac{1}{2}} + C \|\nabla u\|_2^{\alpha \frac{1}{2}} \|u\|_1^{(1-\alpha)\left(\frac{1}{2} + \frac{1}{\nu}\right)}. \]

In the last line we used the mean inequality and
\[ \alpha \frac{n}{l} = \frac{n - l}{l} \left( \frac{1}{2} + \frac{1}{\nu} \right) < \frac{d}{d+2} < 2. \]
Next compute
\[ 2 - \frac{n}{T} \alpha = (\frac{1}{2} + \frac{1}{d})^{-1}(2 + \frac{2}{d} - \frac{n}{T}) \geq (\frac{1}{2} + \frac{1}{d})^{-1} \frac{2}{d}, \]
and we find
\[ \sup_{n \geq 3-m} \frac{n}{T} \alpha < 2. \]

As before after applying Hölder’s inequality in (49), we can prove (42). (50) shows that the constants in (42) can be chosen independent of \( n \).

In all we finished the proof of the claim and proved (42). As for \( X_2 \), note that \( X_2 \leq C \|u^n\|_1 \), so it can be handled similarly as the estimate (48).

Putting together the estimates (39), (42) and taking \( \delta \) to be small, we obtain for all \( n \geq 3 - m \)
\[ \partial_t \|u^n\|_1 + c \|\nabla u^n\|^2 \leq C_3 n^c + C_3 n^c \|u^n\|_1 \leq \frac{\alpha}{2} \]
for some \( C, c > 0 \) independent of \( n \). Using (20) and (51), we get
\[ \frac{dt}{dt} A_{k+1} + c A_{k+1} \leq C^k + C^k A_{k+1}^{2+C^2-k} \]
with \( A_k = \|u^{n_k}\|_1 \) and \( n_k = 2^k(2 - m) - 1 + m \). Finally the proof follows as before.

**Theorem 3.6.** *Theorem 3.2 holds in the regime: \( m \geq 2 \) and \( s \in (1, \frac{d}{2}) \).*

**Proof.** For any \( n > 1 \), we multiply \( u_1^{n-1} \) on both sides of (14) where \( u_1 = (u-1)_+ \). We have
\[ \partial_t \int_{\mathbb{R}^d} u_1^n dx = n \int_{\mathbb{R}^d} u_1^{n-1} u_t dx \]
\[ \leq -mn \int_{\mathbb{R}^d} u_1^{n-1} \nabla u_1 \nabla u_1^{n-1} dx + n \int_{\mathbb{R}^d} (V_{s,\epsilon} * u) u_1 \nabla u_1^{n-1} dx \]
Since \( m \geq 2 \),
\[ n \int_{\mathbb{R}^d} u_1^{n-1} \nabla u_1 \nabla u_1^{n-1} dx \geq n \int_{\mathbb{R}^d} (1 + u_1) \nabla u_1 \nabla u_1^{n-1} dx \]
\[ \geq C \int_{\mathbb{R}^d} \left| \nabla u_1^n \right|^2 + \left| \nabla u_1^{n+1} \right|^2 dx, \]
we obtain
\[ \partial_t \int_{\mathbb{R}^d} u_1^n dx \leq -C_m \int_{\mathbb{R}^d} \left| \nabla u_1^n \right|^2 + \left| \nabla u_1^{n+1} \right|^2 dx + nY. \]

Recall the notation \( \bar{u} = u - u_1 \). For \( Y \), we have
\[ Y = \frac{n-1}{n} \int_{\mathbb{R}^d} V_{s,\epsilon} * u \nabla u_1^n dx + \int_{\mathbb{R}^d} V_{s,\epsilon} * u \nabla u_1^{n-1} dx \]
\[ \leq \int_{\mathbb{R}^d} |\nabla \cdot V_{s,\epsilon}| * u \nabla u_1^n dx + \int_{\mathbb{R}^d} |\nabla \cdot V_{s,\epsilon}| * u \nabla u_1^{n-1} dx \]
\[ \leq \int_{\mathbb{R}^d} |\nabla \cdot V_{s,\epsilon}| * (u_1 + \bar{u}) \nabla u_1^n dx + \int_{\mathbb{R}^d} |\nabla \cdot V_{s,\epsilon}| * u (u_1^n + 1_{u_1 < 1}) dx \]
\[ \leq \int_{\mathbb{R}^d} |\nabla \cdot V_{s,\epsilon}| * u_1 \nabla u_1^n dx + \int_{\mathbb{R}^d} |\nabla \cdot V_{s,\epsilon}| * \bar{u} \nabla u_1^n dx + \int_{\mathbb{R}^d} |\nabla \cdot V_{s,\epsilon}| * 1_{u_1 < 1} u dx. \]
Because $|\nabla \cdot V_{s,\epsilon}| \lesssim |x|^{-d-2+2s}$, $s > 1$ and
\[ |\tilde{u}|, 1_{\{u_1 < 1\}} \leq 1; \quad \tilde{u}(\cdot, t), 1_{\{u_1 < 1\}}(\cdot, t) \in L^1(\mathbb{R}^d), \]
we have for some universal constant $C$
\[ |\nabla \cdot V_{s,\epsilon}| * \tilde{u} + |\nabla \cdot V_{s,\epsilon}| * 1_{\{u_1 < 1\}} \leq C. \]
Also due to (53) and $u(\cdot, t) \in L^1(\mathbb{R}^d)$, we deduce
\[ Y \lesssim \int_{\mathbb{R}^d} |\nabla \cdot V_{s,\epsilon}| * u_1 \nabla u_1^3 dx + \int_{\mathbb{R}^d} u_1^3 dx + 1. \]
Next fix one $\tilde{s} \in (1, s)$. By Young’s convolution inequality,
\[ Y \lesssim \|u_1^p\|_p\|u_1\|_q + \|u_1^q\|_q + 1 \]
where $p, q$ satisfy
\[ p, q \geq 1, \quad \frac{1}{p} + \frac{1}{q} = 1 + \frac{2\tilde{s} - 2}{d}. \]
Now fix one $\tilde{m} \in \left(2 - \frac{2\tilde{s}}{d}, 2\right)$ and set $l = \frac{n + \tilde{m} - 1}{2}$. Note that
\[ \|\nabla u_1^l\|_2^2 \lesssim \|\nabla u_1^1\|_2^2 + \|\nabla u_1^{\tilde{m}}\|_2^2. \]
It then follows from (52) and (53) that
\[ \partial_t \|u^n\|_1 + c \|\nabla u^n\|_2^2 \leq C_n\|u_1^n\|_p\|u_1\|_q + C_n\|u_1^n\|_1 + C_n. \]
Then we only need to show (51) with $m, s$ replaced by $\tilde{m}, \tilde{s}$, and after that the proof follows the same as before.

To show (51), actually $\|u_1^n\|_1$ can be treated the same as in (20). For $\|u_1^n\|_p\|u_1\|_q$, we go to (40) and follow the proof of the Claim below it.

4. **Existence of solutions.** In this section, we show the existence of weak solutions to (1) in the subcritical regime with $s \in (0, \frac{d}{2})$. Let $u_\epsilon$ be a solution to (14). By Theorems 3.2-3.6,
\[ \sup_{\epsilon \in (0, 1), t \geq 0} \|u_\epsilon(\cdot, t)\|_1 + \|u_\epsilon(\cdot, t)\|_\infty < \infty. \] (54)
First let us consider the situation when $s \leq \frac{1}{2}$. We need the following a priori estimate.

**Lemma 4.1.** Assume (5), $s \in (0, \frac{1}{2}]$, $m \in \left(2 - \frac{2s}{d}, 2\right)$ and let $u_\epsilon$ be the solution to (14). Then for any $T > 0$, there exists a constant $C_T$ independent of $\epsilon$ such that
\[ \|V_{s,\epsilon} \ast u_\epsilon\|_{L^2(\mathbb{R}^d \times [0, T])} \leq C_T, \] (55)
\[ \|\nabla u_\epsilon\|_{L^2(\mathbb{R}^d \times [0, T])} \leq C_T. \] (56)

**Proof.** Recall (38) and we take $n = 3 - m$. After integrating in time, we find
\[ \int_{\mathbb{R}^d} u_\epsilon^{3-m} dx(T) + \int_{\mathbb{R}^d \times [0, T]} |\nabla u_\epsilon|^2 dx dt \leq \int_{\mathbb{R}^d} u_\epsilon^{3-m} dx(0) + C |3-m| c \int_0^T \|u_\epsilon\|_1^{2+\gamma} dt. \]
Since $\|u_\epsilon\|_3-m(t), \|u_\epsilon\|_1(t)$ are uniformly bounded in time,
\[ \|\nabla u_\epsilon\|^2_{L^2(\mathbb{R}^d \times [0, T])} \leq C + CT. \] (57)

Denote $\rho = |x|$ and recall (12), we have
\[ V_{s,\epsilon}(x) := c \zeta(\rho) \nabla \rho^{-d+2s}. \]
We can find \( g(\rho) : (0, \infty) \to \mathbb{R} \) such that
\[
g'(\rho) = c(-d + 2s)c_\rho(\rho)\rho^{-d-1+2s} \quad \text{and} \quad g(1) = c.
\] (58)

By (13), we have
\[
|g(\rho)| \leq C\rho^{-d+2s}, \quad |g'(\rho)| \leq C\rho^{-d-1+2s}
\]
for some \( C > 0 \) only depend on \( d, s \). Actually for \( \rho \in [2\epsilon, 1/\epsilon] \), we know \( g(\rho) = C\rho^{-d+2s} \).

Let \( \varphi : [0, \infty) \to [0, 1] \) be a smooth bump function that \( \varphi(\rho) = 1 \) for \( \rho \leq 1 \) and \( \varphi(\rho) = 0 \) for \( \rho \geq 2 \). Let us decompose \( g \) into two parts and write
\[
g = g_s + g_b := \varphi g + (1 - \varphi)g.
\]

Seeing from (58), for some universal constant \( C = C(d, s) > 0 \)
\[
\|g_s\|_1 \leq c \int^2 \rho^{-d+2s}\rho^{-d-1}d\rho \leq C,
\] (59)
\[
\|\nabla g_b\|_2^2 \leq C \int_{|x| \geq 1} |(\nabla \varphi) g|^2(x) + |\nabla g|^2(x)dx
\]
\[
\leq C \int^2 (\rho^{-d+2s})^2\rho^{-d-1}d\rho + C \int^\infty (\rho^{-d-1+2s})^2\rho^{-d-1}d\rho
\]
\[
\leq C + C \int^\infty \rho^{-d-3+4s}d\rho
\]
\[
\leq C(s,d) \quad (\text{since } 2s \leq 1 \text{ and } 2s < d).
\]

It is not hard to see
\[
\|V_{s,\epsilon} * u_{\epsilon} \|^2_{L^2(\mathbb{R}^d \times [0,T])}
\]
\[
\leq 2 \|g_s(|x|) \ast \nabla u_{\epsilon}\|^2_{L^2(\mathbb{R}^d \times [0,T])} + 2 \|\nabla g_b(|x|) \ast u_{\epsilon}\|^2_{L^2(\mathbb{R}^d \times [0,T])}
\]
\[
=: 2X_1 + 2X_2.
\]

Using Lemma 2.2, (57) and (59) give
\[
X_1 = \int_0^T \|g_s \ast \nabla u_{\epsilon}\|^2_{L^2} dt \leq \int_0^T \|g_s\|^2_1 \|\nabla u_{\epsilon}\|^2_{L^2} dt
\]
\[
= \|g_s\|^2_1 \|\nabla u_{\epsilon}\|^2_{L^2(\mathbb{R}^d \times [0,T])} < \infty.
\]

For \( X_2 \), by (60) we obtain
\[
X_2 \leq \int_{\mathbb{R}^d \times [0,T]} \int_{\mathbb{R}} |\nabla g_b|^2(x-y)u_{\epsilon}^2(y)dydxdt \leq C \int_{\mathbb{R} \times [0,T]} u_{\epsilon}^2(y)dydt < \infty.
\]

In all, we have proved
\[
\|V_{s,\epsilon} * u_{\epsilon}\|_{L^2(\mathbb{R}^d \times [0,T])} \leq C
\]
where \( C \) only depends on \( d, s, T \) and \( \|u_{\epsilon}\|_1 + \|u_{\epsilon}\|_{\infty} \).

\( \square \)

**Theorem 4.2.** Assume (5) and \( s \in (0, \frac{1}{2}) \), \( m \in (2 - \frac{2s}{d}, 2) \). Then there exists a nonnegative weak solution \( u \) to (1) with initial data \( u_0 \). Furthermore, the mass of \( u \) is preserved and \( u \) is uniformly bounded for all \( t \in [0, \infty) \). And the bound only depends on \( s, m, d, \) and \( \|u_0\|_1 + \|u_0\|_{\infty} \).
Proof. For any small $\epsilon > 0$, let $u_\epsilon$ be a solution to (14). Let us show the following tightness of $\{u_(\cdot,t)\}_\epsilon$ in $L^1(\mathbb{R}^d)$: for any $T > 0$

$$\lim_{R \to \infty} \int_{B_1^R} u_\epsilon(x,t)dx \to 0 \text{ uniformly in } \epsilon \in (0,1) \text{ and } t \in [0,T].$$

(61)

Take a function $\varphi = \varphi_{N,R} \in C^\infty_0(\mathbb{R}^d)$ such that for some $N >> 1$

$$\varphi = 0 \quad \text{in } |x| \leq R,$$

$$\varphi = 1 \quad \text{in } 2R \leq |x| \leq NR,$$

$$0 \leq \varphi \leq 1, \quad |\nabla \varphi| \leq R^{-1}, \quad |\Delta \varphi| \lesssim R^{-1} \quad \text{in } \mathbb{R}^d.$$

By the equation (14), for any $t \in (0,T]$,

$$\int_{\mathbb{R}^d} u_\epsilon \varphi dx(t) = \int_{\mathbb{R}^d} u_\epsilon \varphi dx(0) + \int_0^t \int_{\mathbb{R}^d} (\epsilon u_\epsilon + u_\epsilon^m) \Delta \varphi dxdt + \int_0^t \int_{\mathbb{R}^d} (u_\epsilon V_{s,\epsilon} * u_\epsilon) \nabla \varphi dxdt.$$

By (54) and the condition $|\Delta \varphi| \lesssim R^{-1}$, $Y_1$ converges to 0 as $R \to \infty$ uniformly in $\epsilon$ and $t \leq T$. Next by Hölder’s inequality and Lemma 4.1,

$$Y_2 \leq \|V_{s,\epsilon} * u_\epsilon\|_2 \|u_\epsilon \nabla \varphi\|_2 \leq C_T R^{-1}.$$

Combining the assumption that $u_0 \in L^1(\mathbb{R}^d)$, we have

$$\int_{\mathbb{R}^d} u_\epsilon(x,t)\varphi(x)dx \to 0 \text{ as } R \to 0$$

uniformly in $\epsilon, N$. Finally letting $N \to \infty$, we proved (61).

Now we want to show precompactness of $u_\epsilon$ in $L^1$. As done in the proof of Theorem 2.13 [3] we show that the family $\{u_\epsilon\}_\epsilon$ satisfy the Riesz-Frechet-Komogorov compactness criterion:

1. For all $\theta > 0$ there exists $0 < h_0 \leq \theta$ such that for all $\epsilon > 0$ and $h \leq h_0$

$$\int_0^{T-\theta} \int_{\mathbb{R}^d} |u_\epsilon(x,t+h) - u_\epsilon(x,t)|dxdt \leq \theta.$$

2. For all $\theta \in (0,T)$ there exists $0 < h_0 \leq \theta$ such that for all $\epsilon > 0$, $h \leq h_0$ and $i = 1,...,d$

$$\int_0^T \int_{\mathbb{R}^d} |u_\epsilon(x,t+he_i) - u_\epsilon(x,t)|dxdt \leq \theta.$$

By (61), we only need to verify the above two in bounded spatial domains i.e. we are allowed to replace $\mathbb{R}^d$ by $B_R$. While for bounded domains, the criterion is validated in Theorem 2.13 and Lemma 2.8 [3] if $u_\epsilon$ are uniformly bounded and $\|\nabla u_\epsilon^m\|_{L^2(\mathbb{R}^d \times [0,T])} \leq C_T$. Therefore our (56) in Lemma 4.1 and (54) finish the proof of the criterion.

Thus we conclude that $\{u_\epsilon\}_\epsilon$ is precompact in $L^1(0,T; L^1(\mathbb{R}^d))$. By passing $\epsilon \to 0$ along subsequences, we have $u_\epsilon \to u$ in $L^1(0,T; L^1(\mathbb{R}^d))$. Since $u_\epsilon$ are uniformly bounded independent of $x,t$ and $\epsilon$, the limit $u$ is uniformly bounded a.e. in $x,t$.

Due to (56), we can have

$$u \in L^2(0,T; \dot{H}^1(\mathbb{R}^d)).$$

(62)
Therefore $\nabla (-\Delta)^{-s}u$ is well-defined which is a bounded function in $L^2(\mathbb{R}^d \times [0, T])$. Now we need to show the weak convergence of $V_{s, \epsilon} * u_\epsilon$ to $\nabla (-\Delta)^{-s}u$.

Let $\xi \in C_0^\infty(\mathbb{R}^d \times [0, T], \mathbb{R}^d)$ be a test function. We have

\[
\iint_{\mathbb{R}^d \times [0, T]} (V_{s, \epsilon} * u_\epsilon - \nabla (-\Delta)^{-s}u) \xi \, dx dt
= \iint_{\mathbb{R}^d \times [0, T]} V_{s, \epsilon} * \xi u_\epsilon - \nabla \cdot (-\Delta)^{-s}\xi u \, dx dt
\leq C \iint_{\mathbb{R}^d \times [0, T]} |V_{s, \epsilon} * \xi - \nabla \cdot (-\Delta)^{-s}\xi| \, dx dt
\leq C \iint_{\mathbb{R}^d \times [0, T]} |\nabla \cdot (-\Delta)^{-s}\xi| |u_\epsilon - u| \, dx dt.
\]

(63)

Here we used that $u_\epsilon, u$ are uniformly bounded. Note that $u_\epsilon \to u$ in $L^1(\mathbb{R}^d \times [0, T])$, and hence to show the above integral converges to 0 as $\epsilon \to 0$, we only need to show $X \to 0$. Suppose $\xi = 0$ in $B_{R_\xi}^c \times [0, T]$ for some $R_\xi \in (0, 4/\epsilon)$ and hence by (12),

\[
X \leq C \iint_{\mathbb{R}^d \times [0, T]} |(\xi(x - y) - 1)\nabla_x K_s(x, y)| |\xi(y, t) - \xi(x, t)| \, dy dx dt
\leq CLip(\xi) \iint_{|x - y| \leq 2\epsilon} |x - y|^{-d+1+2s} |x - y| \left(\chi_{|x| \leq R_\epsilon} + \chi_{|y| \leq R_\epsilon}\right) \, dx dy dt
\leq 2CLip(\xi) T \iint_{|z| \leq R_\epsilon, |z| \leq 2\epsilon} |z|^{-d+2s} \, dz dx
\leq C(d, s) Lip(\xi) T R_\xi^d \epsilon^{2s}
\]

which converges to 0 as $\epsilon \to 0$. Thus $V_{s, \epsilon} * u_\epsilon \to \nabla (-\Delta)^{-s}u$ weakly in distribution.

Again by (55)-(56) and interpolation,

\[
\|\nabla (-\Delta)^{-s}u_\epsilon\|_{L^2(\mathbb{R}^d \times [0, T])} \leq C_T
\]

So actually we have

\[
V_{s, \epsilon} * u_\epsilon \to \nabla (-\Delta)^{-s}u \text{ weakly in } L^2(\mathbb{R}^d \times [0, T])
\]

which gives

\[
u_\epsilon V_{s, \epsilon} * u_\epsilon \to u \nabla (-\Delta)^{-s}u \text{ weakly in } L^2(\mathbb{R}^d \times [0, T]).
\]

Then we proved that $u$ satisfies the equation (1). From the equation and (61), we deduce the mass preservation of $u$ for all $t > 0$.

Finally we discuss the continuity property of $u$. For any fixed domain $B_R$, from the equation, $u \in H^1(0, T, H^{-1}(B_R))$ where $H^{-1}(B_R)$ denotes the dual of $H^1(B_R)$. And therefore $u(t)$ is uniformly continuous (depending on $R$) in $H^{-1}(B_R)$ for all $t \in [0, T]$. It follows from Lemma 2.6 [3] and (62) that we can improve the continuity of $u$ in the weak topology $H^{-1}(B_R)$ to the continuity of $u^m$ in $L^2(B_R)$ topology. Then by Lemma 2.9 [3] using convexity of $u^m$, it can be shown that $u(t)$ is continuous with respect to $L^2(B_R)$. By interpolation and boundedness, $u(t)$ is continuous with respect to $L^1(B_R)$. In the end by (61), we find $u \in C([0, T], L^1(\mathbb{R}^d))$. \qed
Next consider the case when \( s > \frac{1}{2} \). This case is simpler since \( |V_{s,e}| \) is locally integrable near the origin. And thus

\[
|V_{s,e} * u_\epsilon(x,t)| \leq C \int_{|x-y| \leq 1} |V_{s,e}(x-y,t)| dy + C \int_{|x-y| > 1} u_\epsilon(y,t) dy < \infty \quad (64)
\]

uniformly in \( \epsilon, x, t \). We have the following theorem.

**Theorem 4.3.** Assume (5) and \( s \in (\frac{1}{2}, \frac{d}{2}) \), \( m > 2 - \frac{2s}{d} \). Then there exists a weak solution \( u \) to (1) with initial data \( u_0 \). Furthermore, the mass of \( u \) is preserved and \( u \) is uniformly bounded for all \( t \in [0, \infty) \). And the bound only depends on \( s, m, d \), and \( \|u_0\|_1 + \|u_0\|_\infty \).

**Proof.** Let us only sketch the proof below. Multiply \((m + 1)u_\epsilon^m \) on both sides of (14) and do integration in \( \mathbb{R}^d \times [0, T] \) to get

\[
\int_{\mathbb{R}^d} u_\epsilon^{m+1} dx(T) - \int_{\mathbb{R}^d} u_\epsilon^{m+1} dx(0) + (m + 1) \int_{\mathbb{R}^d \times [0,T]} \nabla u_\epsilon^m dx dt = (m + 1) \int_{\mathbb{R}^d \times [0,T]} (u_\epsilon V_{s,e} * u_\epsilon) \cdot \nabla u_\epsilon^m dx dt.
\]

By the uniform bound (54) on \( u_\epsilon \), we have

\[
\int_{\mathbb{R}^d \times [0,T]} |\nabla u_\epsilon^m|^2 dx dt \leq C + \int_{\mathbb{R}^d \times [0,T]} (u_\epsilon V_{s,e} * u_\epsilon) \cdot \nabla u_\epsilon^m dx dt \\
\leq C + C \int_{\mathbb{R}^d \times [0,T]} |u_\epsilon \nabla u_\epsilon^m| dx dt \quad \text{(by (64))}
\]

\[
\leq C + C \int_{\mathbb{R}^d \times [0,T]} |u_\epsilon|^2 dx dt + \frac{1}{2} \int_{\mathbb{R}^d \times [0,T]} |\nabla u_\epsilon^m|^2 dx dt.
\]

Here we used Hölder’s inequality in the last inequality. Since \( \int_{\mathbb{R}^d \times [0,T]} |u_\epsilon|^2 dx dt \) is bounded, we obtain

\[
\int_{\mathbb{R}^d \times [0,T]} |\nabla u_\epsilon^m|^2 dx dt \leq C_T
\]

for some \( C_T \) uniformly for all \( \epsilon \in (0, 1) \).

With the help of (64), we can reprove the tightness property (61) for \( u_\epsilon \). Similarly as before \( \{u_\epsilon\}_\epsilon \) is precompact in \( L^1(0, T, L^1(\mathbb{R}^d)) \). Passing \( \epsilon \to 0 \) along subsequences, we have \( u_\epsilon \to u \) in \( L^1(0, T, L^1(\mathbb{R}^d)) \) and the limit \( u \) is uniformly bounded a.e. in \( x, t \). Due to (64) and (65), it is not hard to see that along subsequences

\[
\begin{align*}
&u_\epsilon V_{s,e} * u_\epsilon \to u \nabla (-\Delta)^{-s} u \\
&\nabla u_\epsilon^m \to \nabla u^m
\end{align*}
\]

weakly in \( L^2(\mathbb{R}^d \times [0, T]) \) and therefore \( u \) satisfies the equation. The desired regularity of \( u \) can be achieved as before.

\[\square\]

5. **Uniqueness and Hölder regularity.** This section is concerned with the uniqueness and the continuity properties of (1) in the subcritical regime for some \( s \).

**Theorem 5.1.** Assume (5) and \( s \in (1, \frac{d}{2}) \), \( m \geq 1, d \geq 3 \). Then there is a unique weak solution to (1) with initial data \( u_0 \).

**Proof.** Fix any \( T > 0 \), let \( u_1, u_2 \) be two weak solutions in \( \mathbb{R}^d \times [0, T] \) with the same initial data. By definition they satisfy (7). We will follow the approach of [3, 2] and estimate the difference of \( u_1, u_2 \) in \( H^{-1} \).
For each $t > 0$, define $\phi(\cdot, t)$ through

$$\Delta \phi(x, t) = u_1(x, t) - u_2(x, t) \quad \text{and} \quad \lim_{|x| \to \infty} \phi(x, t) = 0.$$ 

Since $u(\cdot, t) \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, this problem is well-posed when $d \geq 3$.

By the equation

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^d} |\nabla \phi|^2 dx = \int_{\mathbb{R}^d} (\nabla u_1^m - \nabla u_2^m) \nabla \phi dx - \int_{\mathbb{R}^d} (u_1 - u_2)(\nabla K_s u_1) \nabla \phi dx$$

$$- \int_{\mathbb{R}^d} u_2(\nabla K_s (u_1 - u_2)) \nabla \phi dx =: X_1 + X_2 + X_3.$$ 

Direct computations yields

$$X_1 = - \int_{\mathbb{R}^d} (u_1^m - u_2^m)(u_1 - u_2) \leq 0.$$ 

Note that $|D^2K_s(z)| \sim |z|^{-d-2+2s}$ and $d + 2 - 2s \in (2, d)$. Therefore, denoting

$$A_1(z) := \chi_{|z| \geq 1} D^2 K_s(z), \quad A_2(z) := \chi_{|z| < 1} D^2 K_s(z),$$

we have $A_1(z)$ is bounded and $A_2(z) \in L^1$. Since $u_1$ is uniformly bounded for $t \in [0, T]$,

$$|D^2 K_s u_1| (x) \leq C \int_{\mathbb{R}^d} |A_1(x-y)||u_1(y)|dy + C \int_{\mathbb{R}^d} |A_2(x-y)||u_1(y)|dy,$$

$$\lesssim \int_{\mathbb{R}^d} u_1(y)dy + \int_{\mathbb{R}^d} |A_2(x-y)|dy \leq C.$$ 

We get

$$X_2 = - \int_{\mathbb{R}^d} \Delta \phi \nabla K_s u_1 \nabla \phi dx = \int_{\mathbb{R}^d} \nabla \phi D^2 K_s u_1 \nabla \phi dx,$$

$$\leq C \int_{\mathbb{R}^d} |D^2 K_s u_1||\nabla \phi|^2 dx \leq C \|\nabla \phi\|_2^2.$$ 

As for $X_3$, by Young’s convolution inequality,

$$X_3 = \int_{\mathbb{R}^d} u_2(D^2 K_s * \nabla \phi)\nabla \phi dx,$$

$$= \int_{\mathbb{R}^d} u_2(A_1(z) * \nabla \phi)\nabla \phi dx + \int_{\mathbb{R}^d} u_2(A_2(z) * \nabla \phi)\nabla \phi dx,$$

$$\leq C \|A_1 * \nabla \phi\|_2 \|\nabla \phi\|_2 + C \|A_2\|_1 \|\nabla \phi\|_2^2 \leq C \|\nabla \phi\|_2^2 \quad (\text{since } A_1 \text{ is bounded}).$$ 

Set $\eta(t) = \|\nabla \phi\|_2^2$ and we find

$$\frac{d}{dt} \eta(t) \leq C \eta(t).$$ 

Also we have $\eta(0) = 0$ due to $u_1(x, 0) - u_2(x, 0) = 0$. By Gronwall’s inequality $\eta(t) = 0$ and we find $u_1(\cdot, t) = u_2(\cdot, t)$ in $\dot{H}^{-1}$ for all $t \in [0, T]$. Since $T$ is arbitrary, we conclude the proof of the theorem.  

Now consider the regularity problems with $s > 1/2$.  

Proof of Theorem 1.2. Let \( u \) be a solution to (1) and denote

\[
V(x, t) := -\nabla K_u(x, t).
\]

Then we can rewrite the equation as

\[
u_t = \Delta u^m + \nabla \cdot (Vu).
\]  \hspace{1cm} (66)

By Theorems 3.2- 3.6, in the subcritical regime, \( u \) is uniformly bounded in \( L^\infty(\mathbb{R}^d \times [0, \infty)) \) and \( \|u(\cdot, t)\|_1 = \|u_0\|_1 < \infty \). Thus

\[
|V(x, t)| = |\nabla K_u(x, t)| \lesssim \int_{\mathbb{R}^d} |x - y|^{-d+2s}u(y, t)dy
\]

\[
\lesssim \int_{|x-y| \leq 1} |y|^{-d+2s}dy + \int_{|x-y| \geq 1} u(y, t)dy
\]

\[
\leq C
\]

with \( C > 0 \) only depending on \( d, s, \|u_0\|_1 + \|u\|_\infty \).

For any fixed \( \tau > 0 \), take any \((x_0, t_0) \in \mathbb{R}^d \times (2\tau, \infty) \). Consider

\[
\tilde{u}(x, t) := (2\tau)^{-\frac{d}{2s}} u(x_0 + 2\tau x, t_0 + 2\tau t)
\]

which then solves (66) in \( Q_1 := B_1 \times (-1,0) \) with \( V \) replaced by

\[
\tilde{V}(x, t) := V(x_0 + 2\tau x, t_0 + 2\tau t).
\]

It follows from the definition and (67) that

\[
\|\tilde{u}\|_{L^\infty(Q_1)} \leq (2\tau)^{-\frac{d}{2s}} \|u\|_{L^\infty_{x,t}} \|\tilde{V}\|_{L^\infty(Q_1)} \leq \|V\|_{L^\infty_{x,t}} \leq C.
\]

Hence, according to Theorem 4.1 [20], \( \tilde{u} \) is Hölder continuous in both space and time in \( B_{\frac{1}{2}} \times (-\frac{1}{2}, 0) \) and the continuity norm only depends on \( m, d \) and the upper bounds of \( \|\tilde{u}\|_{L^\infty(Q_1)} \) and \( \|\tilde{V}\|_{L^\infty(Q_1)} \). Therefore, after rescaling back, our solution \( u(x, t) \) is Hölder continuous in \((x_0 + B_\tau \times (t_0 - \tau, t_0) \), with the Hölder norm only depending on \( \tau, d, s, m \) and \( \|u_0\|_1 + \|u\|_\infty \). Since \((x_0, t_0) \) is arbitrarily picked in \( \mathbb{R}^d \times (2\tau, \infty) \), we conclude that \( u \) is uniformly Hölder continuous with respect to both space and time in \( \mathbb{R}^d \times (\tau, \infty) \). \( \square \)

Appendix A. Proof of Lemma 2.3. For \( q > 1 \), the result is covered by Corollary 1.5 [17]. We only need to consider the case when \( q = 1 \) and \( s \neq 0 \). Fix \( s, \alpha, r, p \) that \( \alpha > s \) and (10), (11) are satisfied. From (10) it follows that

\[
\frac{1}{p} = \frac{s}{d} + \left( \frac{1}{r} - \frac{1}{d} \right) \alpha + 1 - \alpha = \frac{s}{d} + \left( 1 + \frac{1}{d} - \frac{1}{r} \right) \alpha + 1.
\]

Since \( \alpha > s, r > 1 \), we can take \( \alpha', q' \) such that

\[
s \leq \alpha' < \alpha, \quad 1 < q' < r,
\]

\[
\frac{1}{p} = \frac{s}{d} + \left( \frac{1}{r} - \frac{1}{d} \right) \alpha' + \frac{1 - \alpha'}{q'}.
\]  \hspace{1cm} (68)

By Corollary 1.5 [17]

\[
\|\nabla u\|_p \leq C \|\nabla u\|_{\alpha'} \|u\|_1^{1-\alpha'}.
\]  \hspace{1cm} (69)

By the classical Gagliardo-Nirenberg inequality

\[
\|u\|_{q'} \leq C \|\nabla u\|_{\beta'} \|u\|_1^{1-\beta'}
\]  \hspace{1cm} (70)
where $\beta$ satisfies
\[
\frac{1}{q'} = \left( \frac{1}{r} - \frac{1}{d} \right) \beta + 1 - \beta.
\]
Since $q' < r$,
\[
\beta = \left( 1 - \frac{1}{q'} \right) / \left( 1 + \frac{1}{d} - \frac{1}{r} \right) \in (0, 1).
\]
By (68) and simple calculations
\[
\frac{1}{p} = \frac{s}{d} + \left( \frac{1}{r} - \frac{1}{d} \right) \alpha' + \left( \frac{1}{r} - \frac{1}{d} \right) \beta + 1 - \beta
\]
Comparing this with (10), we obtain
\[
\alpha = \alpha' + \beta - \alpha' \beta.
\]
Finally plugging (70) into (69) gives
\[
\|\nabla|^s u\|_p \leq C \|\nabla u\|_{\alpha} \|u\|_{1}^{1-\alpha}.
\]

Acknowledgments. The author would like to thank his advisor Inwon Kim for her guidance and stimulating discussions. The author would also like to thank José A. Carrillo, Franca Hoffmann, Kyungkeun Kang and Monica Visan for helpful discussions and suggestions.

REFERENCES

[1] H. Bahouri, J.-Y. Chemin and R. Danchin, *Fourier Analysis and Nonlinear Partial Differential Equations*, volume 343, Springer Science & Business Media, 2011.
[2] J. Bedrossian, N. Rodríguez and A. L Bertozzi, Local and global well-posedness for aggregation equations and patlak–keller–segel models with degenerate diffusion, *Nonlinearity*, 24 (2011), 1683–1714.
[3] A. L Bertozzi and D. Slepcev, Existence and uniqueness of solutions to an aggregation equation with degenerate diffusion, *Communications on Pure and Applied Analysis*, 9 (2010), 1617–1637.
[4] A. Blanchet, V. Calvez and J. A Carrillo, Convergence of the mass-transport steepest descent scheme for the subcritical patlak–keller–segel model, *SIAM Journal on Numerical Analysis*, 46 (2008), 691–721.
[5] A. Blanchet, J. A Carrillo and P. Laurençot, Critical mass for a patlak–keller–segel model with degenerate diffusion in higher dimensions, *Calculus of Variations and Partial Differential Equations*, 35 (2009), 133–168.
[6] S. Boi, V. Capasso and D. Morale, Modeling the aggregative behavior of ants of the species polyergus rufescens, *Nonlinear Analysis: Real World Applications*, 1 (2000), 163–176.
[7] L. Caffarelli, F. Soria and J. L. Vázquez, Regularity of solutions of the fractional porous medium flow, *Journal of the European Mathematical Society*, 15 (2013), 1701–1746.
[8] L. Caffarelli and J. L. Vazquez, Nonlinear porous medium flow with fractional potential pressure, *Archive for Rational Mechanics and Analysis*, 202 (2011), 537–565.
[9] V. Calvez, J. A. Carrillo and F. Hoffmann, Equilibria of homogeneous functionals in the fair-competition regime, *Nonlinear Analysis*, 159 (2017), 85–128.
[10] J. A. Carrillo, K. Craig and Y. Yao, Aggregation-diffusion equations: Dynamics, asymptotics, and singular limits, In *Active Particles*, 2 (2019), 65–108.
[11] J. A. Carrillo, F. Hoffmann, E. Mainini and B. Volzone, Ground states in the diffusion-dominated regime, *Calculus of Variations and Partial Differential Equations*, 57 (2018), Art. 127, 28 pp.
[12] J. A. Carrillo and G. Toscani, Asymptotic $l^1$-decay of solutions of the porous medium equation to self-similarity, *Indiana University Mathematics Journal*, 49 (2000), 113–142.
J. A. Carrillo and J. Wang, Uniform in time $l^\infty$-estimates for nonlinear aggregation-diffusion equations, *Acta Applicandae Mathematicae*, (2018), 1–19.

L. Chayes, I. Kim and Y. Yao, An aggregation equation with degenerate diffusion: Qualitative property of solutions, *SIAM Journal on Mathematical Analysis*, 45 (2013), 2995–3018.

J. Dolbeault and B. Perthame, Optimal critical mass in the two dimensional keller–segel model in $\mathbb{R}^2$, *Comptes Rendus Mathematique*, 339 (2014), 611–616.

L. Grafakos, *Classical Fourier Analysis*, volume 2, Springer, 2008.

H. Hajaiej, L. Molinet, T. Ozawa and B. Wang, Necessary and sufficient conditions for the fractional gagliardo-nirenberg inequalities and applications to navier-stokes and generalized boson equations (harmonic analysis and nonlinear partial differential equations), *RIMS Kokyuroku Bessatsu*, 26 (2011), 159–175.

D. Horstmann and M. Winkler, Boundedness vs. blow-up in a chemotaxis system, *Journal of Differential Equations*, 215 (2005), 52–107.

H. E. Huppert and A. W. Woods, Gravity-driven flows in porous layers, *Journal of Fluid Mechanics*, 292 (1995), 55–69.

I. Kim and Y. P. Zhang, Regularity properties of degenerate diffusion equations with drifts, *SIAM Journal on Mathematical Analysis*, 50 (2018), 4371–4406.

R. Kowalczyk, Preventing blow-up in a chemotaxis model, *Journal of Mathematical Analysis and Applications*, 305 (2005), 566–588.

M. Kwaśnicki, Ten equivalent definitions of the fractional laplace operator, *Fractional Calculus and Applied Analysis*, 20 (2017), 7–51.

Q.-H. Nguyen and J. L. Vázquez, Porous medium equation with nonlocal pressure in a bounded domain, *Communications in Partial Differential Equations*, 43 (2018), 1502–1539.

L. Nirenberg, On elliptic partial differential equations, In *Il Principio di Minimo e sue Applicazioni Alle Equazioni Funzionali*, Springer, 2011, 1–48.

M. Riesz, L’intégrale de riemann-liouville et le problème de cauchy, *Acta mathematica*, 81 (1949), 1–222.

E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, Princeton, N.J., 1970.

Y. Sugiyama et al, Time global existence and asymptotic behavior of solutions to degenerate quasi-linear parabolic systems of chemotaxis, *Differential and Integral Equations*, 20 (2007), 133–180.

Y. Tao and M. Winkler, Boundedness in a quasilinear parabolic–parabolic keller–segel system with subcritical sensitivity, *Journal of Differential Equations*, 252 (2012), 692–715.

C. M. Topaz, A. L. Bertozzi and M. A. Lewis, A nonlocal continuum model for biological aggregation, *Bulletin of Mathematical Biology*, 68 (2006), 1601–1623.

J. L. Vázquez, *The Porous Medium Equation: Mathematical Theory*, Oxford University Press, 2007.

Received March 2019; 1st revision June 2019; final revision August 2019.

E-mail address: yzhangpaul@math.ucla.edu