Can the Chaplygin gas be a plausible model for dark energy?

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Abstract

In this note two cosmological models representing the flat Friedmann Universe filled with a Chaplygin fluid, with or without dust, are analyzed in terms of the recently proposed "statefinder" parameters [1]. Trajectories of both models in the parameter plane are shown to be significantly different w.r.t. "quiescence" and "tracker" models. The generalized Chaplygin gas model with an equation of state of the form $p = -\frac{A}{\rho^\alpha}$ is also analyzed in terms of the statefinder parameters.

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In the search for cosmological models describing the observed cosmic acceleration [2, 3, 4], the inspiration coming from inflation has suggested mainly models making use of scalar fields [5, 6, 7, 8, 9]. There are of course alternatives; in particular, in [10, 11, 12] an elementary model has been presented describing a Friedmann universe filled with a perfect fluid obeying the Chaplygin equation of state

$$p = -\frac{A}{\rho},$$

where $A$ is a positive constant (for a thorough review see Ref. [13]). The interesting feature of this model is that it naturally provides a universe that undergoes a transition from a decelerating phase, driven by dust-like matter, to a cosmic acceleration at later stages of its evolution (see [10] for details). An interesting attempt to justify this model [14] makes use of an effective field theory for a three-brane universe [15].

In the flat case, the model can be equivalently described in terms of a homogeneous minimally coupled scalar field $\phi$, with potential [10]

$$V(\phi) = \frac{1}{2} \sqrt{A} \left( \cosh 3\phi + \frac{1}{\cosh 3\phi} \right).$$

However, since models trying to provide a description (if not an explanation) of the cosmic acceleration are proliferating, there exists the problem of discriminating between the various contenders. To this aim a new proposal introduced in [1] makes use of a pair of parameters $\{r, s\}$, called “statefinder”. The relevant definition is as follows:

$$r \equiv \frac{a}{aH^3}, \quad s \equiv \frac{r - 1}{3(q - 1/2)},$$

where $H \equiv \dot{a} / a$ is the Hubble constant and $q \equiv -\frac{\ddot{a}}{aH^2}$ is the deceleration parameter. The new feature of the statefinder is that it involves the third derivative of the cosmological radius.
Trajectories in the \(\{s, r\}\)-plane corresponding to different cosmological models exhibit qualitatively different behaviors. \(\Lambda\)CDM model diagrams correspond to the fixed points \(s = 0, r = 1\). The so-called "quiescence" models \([1]\) are described by vertical segments with \(r\) decreasing from \(r = 1\) down to some definite value. Tracker models \([18]\) have typical trajectories similar to arcs of parabola lying in the positive quadrant with positive second derivative.

The current location of the parameters \(s\) and \(r\) in these diagrams can be calculated in models (given the deceleration parameter); it may also be extracted from data coming from SNAP (SuperNovae Acceleration Probe)-type experiments \([3]\). Therefore, the statefinder diagnostic combined with future SNAP observations may possibly be used to discriminate between different dark energy models.

In this short paper we apply that diagnostic to Chaplygin cosmological models (the direct comparison of the already available supernova data with the Chaplygin gas model was also undertaken recently in \([16, 17]\)). We consider both the one-fluid pure Chaplygin gas model and a two-fluid model where dust is present as well. We show that these models are different from those considered in \([1]\) and they are worth of further study.

To begin with, let us rewrite the formulae for the statefinder parameters \([1]\) in a form convenient for our purposes. We shall need the Friedmann equation for the flat universe

\[
H^2 = \frac{\dot{a}^2}{a^2} = \rho \tag{4}
\]

and the energy conservation equation

\[
\dot{\rho} = -3H(\rho + p). \tag{5}
\]

Using these two equations it is easy to find that

\[
q = \frac{1}{2} + \frac{3p}{2\rho}, \tag{6}
\]

and then

\[
r = 1 - \frac{3\dot{\rho}}{2p\sqrt{\rho}}; \quad s = -\frac{\dot{\rho}}{3p\sqrt{\rho}}. \tag{7}
\]

For a one-component fluid\([2]\) these formulae become especially simple. Since

\[
\dot{\rho} = \frac{\partial p}{\partial \rho} \rho \dot{\rho} = -3\sqrt{\rho}(\rho + p)\frac{\partial p}{\partial \rho}, \tag{8}
\]

we easily get:

\[
r = 1 + \frac{9}{2} \left(1 + \frac{p}{\rho}\right) \frac{\partial p}{\partial \rho}; \quad s = \left(1 + \frac{p}{\rho}\right) \frac{\partial p}{\partial \rho}. \tag{9}
\]

For the Chaplygin gas one has simply that

\[
v_s^2 = \frac{\partial p}{\partial \rho} = \frac{A}{\rho^2} = -\frac{\rho}{p} = 1 + s \tag{10}
\]

\((v_s^2\) is the square of the velocity of sound) and therefore

\[
r = 1 - \frac{9}{2} s(1 + s). \tag{11}
\]

Thus, the curve of \(r(s)\) is an arc of parabola. To find the admissible values of \(s\), we note that Eqs. (1), (4) and (5) easily give the following dependence of the energy density on the cosmological scale factor \([10]\):

\[
\rho = \sqrt{A + \frac{B}{a^6}}. \tag{12}
\]

\(^1\)We confine ourselves to the case, which is fulfilled in our models, of a fluid for which the equation of state has the form \(p = p(\rho)\).
where $B$ is an integration constant; therefore

$$v_s^2 = 1 + s = \frac{A}{A + B}.$$  \hspace{1cm} (13)

When the cosmological scale factor $a$ changes from 0 to $\infty$ the velocity of sound varies from 0 to 1 and $s$ varies from $-1$ to 0. Thus in our model the statefinder $s$ takes negative values; this feature is not shared by quiessence and tracker models considered in [1].

As $s$ varies in the interval $[-1, 0]$, $r$ first increases from $r = 1$ to its maximum value and then decreases to the $\Lambda$CDM fixed point $s = 0, r = 1$ (see Fig. 1).

![Figure 1: s-r evolution diagram for the pure Chaplygin gas](image)

If $q \approx -0.5$ the current values of the statefinder (within our model) are $s \approx -0.3$, $r \approx 1.9$. In [1] it is reported an interesting numerical experiment based on 1000 realizations of a SNAP-type experiment, probing a fiducial $\Lambda$CDM model. Our values of the statefinder lie outside the three-sigma confidence region displayed in [1]. Based on this fact it can be expected that future SNAP experiments should be able to discriminate between the pure Chaplygin gas model and the standard $\Lambda$CDM model.

Let us consider now a more "realistic" cosmological model which, besides a Chaplygin’s component, contains also a dust component. For a two-component fluid Eqs. (13) take the following form:

$$r = 1 + \frac{9}{2(\rho + p_1)} \left[ \frac{\partial p}{\partial \rho} (\rho + p) + \frac{\partial p_1}{\partial \rho_1} (p_1 + p_1) \right],$$  \hspace{1cm} (14)

$$s = \frac{1}{\rho + p_1} \left[ \frac{\partial p}{\partial \rho} (\rho + p) + \frac{\partial p_1}{\partial \rho_1} (p_1 + p_1) \right].$$  \hspace{1cm} (15)

If one of the fluids is dust, i.e. $p_1 = p_d = 0$, the above formulae become

$$r = 1 + \frac{9(\rho + p)}{2(\rho + p_d)} \frac{\partial p}{\partial \rho}, \quad s = \frac{\rho + p}{p} \frac{\partial p}{\partial \rho}. \hspace{1cm} (16)$$
If the second fluid is the Chaplygin gas, proceeding exactly as before we obtain the following relation:

\[ r = 1 - \frac{9}{2} \frac{s(s+1)}{1 + \frac{\kappa}{s}}. \]  

(17)

To find the term \( \rho_d/\rho \) we write down the dependence of the dust density on the cosmological scale factor:

\[ \rho_d = \frac{C}{a^3}. \]  

(18)

where \( C \) is a positive constant. Eq. (13) gives that \( An^6 + B = -\frac{B}{s} \) and therefore

\[ \frac{\rho_d}{\rho} = \frac{C}{\sqrt{An^6 + B}} = \kappa \sqrt{-s}, \]  

(19)

where the constant \( \kappa = C/\sqrt{B} \) is the ratio between the energy densities of dust and of the Chaplygin gas at the beginning of the cosmological evolution (cf. Eq. (12)). Thus

\[ r = 1 - \frac{9}{2} \frac{s(s+1)}{1 + \kappa \sqrt{-s}}. \]  

(20)

Graphs of the function (20) for different choices of \( \kappa \) are plotted in Fig. 2.

In this case there are choices of the parameters so that the current values of the statefinder are close to the \( \Lambda \)CDM fixed point; indeed for \( \kappa = 1 \) we have \( s = -0.09 \) and \( r = 1.2835 \). By increasing \( \kappa \) we get closer and closer. Already for \( \kappa = 2 \) we get \( s = 0.035, r = 1.11 \) while for \( \kappa > 10 \) the statefinder essentially coincides with the \( \Lambda \)CDM fixed point (see Fig. 2).

Thus our two-fluid cosmological models (with \( \kappa \) say bigger than 2) cannot be discriminated from the \( \Lambda \)CDM model on the basis of the statefinder analysis. On the other hand this fact could also be interpreted as an argument in favor of the model because the \( \Lambda \)CDM model does not contradict observations up to now.

The attractive point of our two-fluid cosmological model with a parameter \( \kappa \) of order one is that it may suggest a solution to the cosmic coincidence conundrum: indeed here the initial values of the energies of dust and of the Chaplygin gas are of the same order of magnitude.

The comparison of the two-fluid (i.e. the Chaplygin gas plus dust) cosmological model with observational data has also been studied in [16, 17]. An analysis of the data obtained from 26 Supernovae leads to the conclusion [16] that the best fitting model seems to be the pure Chaplygin gas without dust. On the other hand [17] by combining the results of the observations of 92 Supernovae with the matter power spectrum a wide range of values of \( \kappa \) seems to be admissible.

Thus, for getting more precise constraints on the parameters of the Chaplygin models one needs both new observations and additional diagnostic techniques. From this point of view an application of the statefinder analysis could be useful.

Similar conclusions also hold for a one-fluid model of a generalized Chaplygin gas with a modified equation of state, as introduced in [11]:

\[ p = -\frac{A}{\rho^\alpha}, \]  

(21)

with \( 0 \leq \alpha \leq 1 \). This gives a cosmological evolution from an initial dust-like behavior to an asymptotic cosmological constant, with an intermediate epoch that can be seen as a mixture of a cosmological constant with a fluid obeying the state equation \( p = \alpha \rho \) \((\alpha = 0 \) corresponds to the \( \Lambda \)CDM model). This generalized model was studied in some detail in [14]. Equation (21) gives the following dependence of the energy density \( \rho \) on the scale factor \( a \):

\[ \rho = \left( A + \frac{B}{a^{3(\alpha+1)}} \right)^{\frac{1}{1-\alpha}}. \]  

(22)
In this case the squared velocity of sound

\[ v_s^2 = \frac{\partial p}{\partial \rho} = -\frac{\alpha p}{\rho} = \frac{A\alpha}{A + \frac{B}{s(\alpha + 1)}} \]  

(23)

varies from 0 to \( \alpha \). From Eq. (23) it follows that \( s = v_s^2 - \alpha \) and therefore the admissible values of \( s \) are now in the interval between \(-\alpha\) and 0. From Eq. (23) we get

\[ r = 1 - \frac{9 s(s + \alpha)}{2\alpha} \]  

(24)

For small values of \( \alpha \) the generalized Chaplygin gas model becomes indistinguishable from the standard \( \Lambda \)CDM cosmological model.

In conclusion. From the statefinder viewpoint both the pure Chaplygin model and the two-fluid mixture have different behaviors w.r.t. other commonly studied models. In particular, a future larger amount of data on high \( z \) type Ia supernovae may allow to distinguish between the pure Chaplygin gas and the \( \Lambda \)CDM models. Instead, the mixed two-fluid model becomes practically indistinguishable from \( \Lambda \)CDM for sufficiently large values of the parameter \( \kappa \) (\( \kappa > \kappa_0 \approx 5 \)). On the other hand a mixed

Figure 2: \( s \)-r evolution diagram for the Chaplygin gas mixed with dust. Dots locate the current value of the statefinder.
two-fluid model with $\kappa$ of order one is attractive from the point of view of a possible solution of the cosmic coincidence problem.

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