TOWARD RESOLUTION OF SINGULARITIES
OVER A FIELD OF
POSITIVE CHARACTERISTIC

(THE IDEALISTIC FILTRATION PROGRAM)

Dedicated to Professor Heisuke Hironaka

Part II.
Basic invariants associated to the idealistic filtration and their properties

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CHAPTER 0

Introduction to Part II

§0.1. Overview of the series.

This is the second of the series of papers under the title
“Toward resolution of singularities over a field of positive characteristic
(the Idealistic Filtration Program)"

- Part I. Foundation; the language of the idealistic filtration
- Part II. Basic invariants associated to the idealistic filtration
  and their properties
- Part III. Transformations and modifications of the idealistic filtration
- Part IV. Algorithm in the framework of the idealistic filtration.

For a brief summary of the entire series, including its goal and the overview of the Idealistic Filtration Program, we refer the reader to the introduction in Part I.

Here we will concentrate ourselves on the outline of Part II, which is presented in the next section.

§0.2. Outline of Part II.

As described in the overview of the Idealistic Filtration Program (cf. §0.2 in Part I), we construct a strand of invariants, whose maximum locus determines each center of blowup of our algorithm for resolution of singularities. The strand of invariants consists of the units (cf. 0.2.3.2.2 in Part I), each of which is a triplet of numbers \((\sigma, \tilde{\mu}, s)\) associated to a certain idealistic filtration (cf. Chapter 2 in Part I) and a simple normal crossing divisor \(E\) called a boundary. (To be precise, the invariant \(\sigma\) is a sequence of numbers indexed by \(\mathbb{Z}_{\geq 0}\) as described in Definition 3.2.1.1 in Part I.) The purpose of Part II is to establish the fundamental properties of the invariants \(\sigma\) and \(\tilde{\mu}\). They are the main constituents of the unit, while the remaining factor \(s\) can easily be computed as the number of (certain specified) components in the boundary passing through a given point, and needs no further mathematical discussion. Our goal is to study the intrinsic nature of these invariants associated to a given idealistic filtration. The discussion in Part II does not involve the analysis regarding the exceptional divisors created by blowups, and hence could only be directly applied to the situation \(at\ year\ 0\) of our algorithm. The systematic discussion on how some subtle adjustments should be made in the presence of the exceptional divisors \(after\ year\ 0\) and on how the strand of invariants functions in the algorithm, built upon the analysis in Part III of the modifications and transformations of an idealistic filtration, will have to wait for Part IV.

In the appendix, we report a new development, unexpected at the time of writing Part I, which suggests a possibility of constructing an algorithm using only the \(D\)-saturation (or \(D_E\)-saturation) and without using the \(\mathfrak{m}\)-saturation, still within the framework of the Idealistic Filtration Program. This would avoid the problem of termination, which we specified in the introduction to Part I as the only missing piece toward completing our algorithm in positive characteristic. (See §0.3. Current status of the Idealistic Filtration Program at the end of the introduction here in Part II for further developments and “evolution” of IFP up to date.)
The following is a rough description of the content of each chapter and the appendix in Part II.

Throughout the description, let $R$ be the coordinate ring of an affine open subset of a nonsingular variety $W$ of dimension $d = \dim W$ over an algebraically closed field $k$ of positive characteristic $\text{char}(k) = p > 0$ or of characteristic zero $\text{char}(k) = 0$, where in the latter case we set $p = \infty$ formally (cf. 0.2.3.2.1 and Definition 3.1.1.1 (2) in Part I).

0.2.1. Invariant $\sigma$. Chapter 1 is devoted to the discussion of the invariant $\sigma$, which is defined for a $\sh{T}$-saturated idealistic filtration $\sh{I}$ over $R$ (cf. 2.1.2 in Part I). The subtle adjustment of the invariant $\sigma$, in the presence of the exceptional divisor $E$, which is defined for a $\sh{T}_E$-saturated idealistic filtration (cf. 1.2.2. Logarithmic differential operators in Part I), will be postponed until Part III and Part IV.

0.2.1.1 Leading algebra and its structure. We fix a closed point $P \in \text{Spec } R \subset W$, with $m_P$ denoting the maximal ideal for the local ring $R_P$. The leading algebra $L(\sh{I}_P)$ of the localization $\sh{I}_P$ of the idealistic filtration $\sh{I}$ at $P$ is defined to be the graded $k$-subalgebra of $G_P = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} (G_P)_n = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} m_P^n/m_P^{n+1}$ (cf. 3.1.1 in Part I)

$$L(\sh{I}_P) = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} L(\sh{I}_P)_n \subset G_P,$$

where

$$L(\sh{I}_P)_n = \{ f \mod m_P^{n+1}; (f, n) \in \sh{I}_P, f \in m_P^n \}.$$

For $e \in \mathbb{Z}_{\geq 0}$ with $p^e \in \mathbb{Z}_{> 0}$, we define the pure part $L(\sh{I}_P)^{\text{pure}}_{p^e}$ of $L(\sh{I}_P)^{p^e}$ by the formula

$$L(\sh{I}_P)^{\text{pure}}_{p^e} = L(\sh{I}_P)^{p^e} \cap F^e((G_P)_1) \subset L(\sh{I}_P)^{p^e},$$

where $F^e$ is the $e$-th power of the Frobenius map of $G_P$.

The most remarkable structure of the leading algebra $L(\sh{I}_P)$ is that it is generated by its pure part (cf. Lemma 3.1.2.1 in Part I), i.e.,

$$L(\sh{I}_P) = k[L(\sh{I}_P)^{\text{pure}}] \quad \text{where} \quad L(\sh{I}_P)^{\text{pure}} = \bigsqcup_{e \in \mathbb{Z}_{\geq 0}} L(\sh{I}_P)^{\text{pure}}_{p^e}.$$ 

This follows from the fact that $\sh{I}_P$ is $\sh{T}$-saturated, since so is $\sh{I}$ (cf. compatibility of $\sh{T}$-saturation with localization, discussed in §2.4 in Part I).

0.2.1.2 Definition of the invariant $\sigma$ and its computation. We define the invariant $\sigma(P)$ by the formula

$$\sigma(P) = (d - \ell^{\text{pure}}_{p^e}(P))_{e \in \mathbb{Z}_{\geq 0}} \quad \text{where} \quad \ell^{\text{pure}}_{p^e}(P) = \dim_k L(\sh{I}_P)^{\text{pure}}_{p^e},$$

which reflects the behavior of the pure part of the leading algebra $L(\sh{I}_P)$. Varying $P$ among all the closed points $m$-Spec $R$ (i.e., all the maximal ideals of $R$), we obtain the invariant

$$\sigma : m\text{-Spec } R \to \bigsqcup_{e \in \mathbb{Z}_{\geq 0}} \mathbb{Z}_{\geq 0}.$$

Recall that Lemma 3.1.2.1 in Part I gives the description of a specific set of generators for the leading algebra $L(\sh{I}_P)$ taken from its pure part. Using this lemma, we can compute the dimension of the pure part $\ell^{\text{pure}}_{p^e}(P) = \dim_k L(\sh{I}_P)^{\text{pure}}_{p^e}$ in terms of the dimension of the entire degree $p^e$ component $\ell^{p^e}(P) = \dim_k L(\sh{I}_P)^{p^e}$ and in terms of the dimensions of the pure parts $\ell^{\text{pure}}_{p^e}(P)$ for $\alpha = 0, \ldots, e-1$. That is to say, $\ell^{\text{pure}}_{p^e}(P)$ can be computed inductively from $\ell^{p^e}(P)$ and the dimensions of the pure parts of lower degree.

0.2.1.3 Upper semi-continuity of the invariant $\sigma$. We observe that $\ell^{p^e}(P)$ can be computed as the rank of a certain “Jacobian-like” matrix, and hence is easily seen to be lower semi-continuous as a function of $P$. The upper semi-continuity of the invariant
The upper semi-continuity of the invariant $\sigma$ as a function over $m$-Spec $R$ also allows us to extend its domain to Spec $R$. That is to say, we have the invariant $\sigma$ defined over the extended domain

$$\sigma : \text{Spec } R \rightarrow \prod_{e \in \mathbb{Z}_{\geq 0}} \mathbb{Z}_{\geq 0},$$

which is automatically upper semi-continuous as a function over Spec $R$.

### 0.2.1.4 Clarification of the meaning of the upper semi-continuity

We say by definition that a function $f : X \rightarrow T$, from a topological space $X$ to a totally ordered set $T$, is upper semi-continuous if the set $X_t = \{x \in X ; f(x) \geq t\}$ is closed for any $t \in T$. When the target space $T$ is not well-ordered, however, we have to be extra-careful as we try to see the equivalence of this definition to the other “well-known” conditions for the upper semi-continuity. The target space of the invariant $\sigma : m$-Spec $R \rightarrow \prod_{e \in \mathbb{Z}_{\geq 0}} \mathbb{Z}_{\geq 0}$ is a priori not well-ordered. Nevertheless, using the fact that $f^{\text{pure}}_{\ell'}(P)$ is non-decreasing as a function of $e \in \mathbb{Z}_{\geq 0}$ for a fixed $P \in m$-Spec $R$, we observe that the target space for the invariant $\sigma$ can be replaced by some well-ordered subset. It can be seen easily then that the upper semi-continuity of the invariant $\sigma$ in the above sense is actually equivalent to the condition that, given a point $P \in m$-Spec $R$, there exists a neighborhood $U_P$ of $P$ such that $\sigma(P) \geq \sigma(Q)$ for any point $Q \in U_P$.

From this upper semi-continuity, interpreted in the equivalent condition, it follows that the domain of the invariant $\sigma$ can be extended from $m$-Spec $R$ to Spec $R$, as mentioned at the end of §0.2.1.3. We summarize the basic facts surrounding the definition of the upper semi-continuity in Chapter I for the sake of clarification.

### 0.2.1.5 Local behavior of a leading generator system and its modification

We use the abbreviation “LGS” for the word “Leading Generator System”. Prof. Cossart kindly suggested to us that “LGS” could be read “Leading Giraud System” in honor of J. Giraud, whose contribution (cf. [Gir74], [Gir75]) is profound in search of the right notion of “a hypersurface of maximal contact” in positive characteristic.

**Theorem:**

Let $\mathbb{H} = \{(h_i, m_i)\}_{i=1}^N \subset \mathbb{Z}$ with associated nonnegative integers $0 \leq e_1 \leq \cdots \leq e_N$ be a leading generator system of $L(I_P)$, if the leading terms of its elements provide a specific set of generators $L(I_P)^{\text{pure}}_{\ell'}$ for any $e \in \mathbb{Z}_{\geq 0}$.

Since the leading algebra $L(I_P)$ is generated by its pure part

$$L(I_P)^{\text{pure}} = \bigcup_{e \in \mathbb{Z}_{\geq 0}} L(I_P)^{\text{pure}}_{\ell'},$$

we conclude from condition (ii) that the leading terms $\{h_i = (h_i \mod m_i^{e_i+1})\}_{i=1}^N$ of $\mathbb{H}$ provide a set of generators for $X(I_P)$, i.e., $L(I_P) = k[\{h_i\}_{i=1}^N]$.

A basic question then about the local behavior of a leading generator system is:

Does $\mathbb{H}$ remain being a leading generator system of $I_Q$ for any closed point $Q$ in a neighborhood $U_P$ of $P$ (if we take $U_P$ small enough)?

Even though the answer is no in general, we show that we can modify a given leading generator system $\mathbb{H}$ into a new one $\mathbb{H}'$ such that $\mathbb{H}'$ is a leading generator system of $I_Q$ for any closed point $Q$ in a neighborhood $U_P$, as long as $Q$ is on the local maximum locus of the invariant $\sigma$ (and $Q$ is on the support of $I$). The last extra condition is equivalent...
to saying $\sigma(Q) = \sigma(P)$ by the upper semi-continuity of the invariant $\sigma$. We say then $\mathbb{H}'$ is “uniformly pure”. We will use this modification as the main tool to derive the upper semi-continuity of the invariant $\tilde{\mu}$ in Chapter 3.

0.2.2. Power series expansion. Chapter 2 is devoted to the discussion of the power series expansion with respect to a leading generator system and its (weakly-)associated regular system of parameters.

0.2.2.1 Similarities between a regular system of parameters and a leading generator system. If we have a leading generator system $\mathbb{H} = \{(h_l, p^e)^{n_l}_{l=1}\}$ in characteristic zero (for a $\mathcal{D}$-saturated idealistic filtration $\mathbb{D}_P$ over $R_P$ at a closed point $P \in \text{Spec} ~ R \subseteq W$), then the elements in the leading generator system are all concentrated at level 1, i.e., $e_l = 0$ and $p^e_l = 1$ for $l = 1, \ldots, N$ (cf. Chapter 3 in Part I). This implies by definition of a leading generator system that the set of the elements $H = (h_1, \ldots, h_N)$ forms (a part of) a regular system of parameters $(x_1, \ldots, x_d)$. (Say, $h_1 = x_l$ for $l = 1, \ldots, N$.) In positive characteristic, this is no longer the case. However, we can still regard the notion of a leading generator system as a generalization of the notion of a regular system of parameters, and we may expect some similarities between the two notions. One of such expected similarities is the power series expansion, which we discuss next.

0.2.2.2 Power series expansion with respect to a leading generator system. In characteristic zero, any element $f \in R_P$ has a power series expansion (with respect to the regular system of parameters $X = (x_1, \ldots, x_d)$, where $h_l = x_l$ for $l = 1, \ldots, N$, with $H = (h_1, \ldots, h_N)$ consisting of the elements of a leading generator system as described in 0.2.2.1)

$$f = \sum_{c \in \mathbb{G}_{\mathbb{O}}/p^e} c_l X^l = \sum_{b \in \mathbb{G}_{\mathbb{O}}} a_b \mathcal{H}^b$$

where $c_l \in k$ and $a_b$ is a power series in terms of the remainder $(x_{N+1}, \ldots, x_d)$ of the regular system of parameters.

In positive characteristic, we expect to have a power series expansion with respect to a leading generator system. More specifically and more generally, the setting for Chapter 2 is given as follows. We have a subset $H = (h_1, \ldots, h_N) \subseteq R_P$ consisting of $N$ elements, and nonnegative integers $0 \leq e_1 \leq \cdots \leq e_N$ attached to these elements, satisfying the following conditions (cf. 4.1.1 in Part I):

(i) $h_l \in m_P^{e_l}$ and $h_l = (h_l, m_P^{e_l+1}) = v_l^{\alpha_l}$ with $v_l \in m_P/m_P^{e_l}$ for $l = 1, \ldots, N$,
(ii) $\{v_l ; l = 1, \ldots, N\} \subseteq m_P/m_P^{e_N}$ consists of $N$-distinct and $k$-linearly independent elements in the $k$-vector space $m_P/m_P^{e_N}$.

We also take a regular system of parameters $(x_1, \ldots, x_d)$ such that

$$v_l = x_l = (x_l \text{ mod } m_P^{e_l})$$

for $l = 1, \ldots, N$.

(We say that a regular system of parameters $(x_1, \ldots, x_d)$ is associated to $H = (h_1, \ldots, h_N)$ if the above condition (asc) is satisfied. For the description of the condition of $(x_1, \ldots, x_d)$ being weakly-associated to $H$, we refer the reader to Chapter 2.)

Now we claim that any element $f \in R_P$ has a power series expansion of the form

$$f = \sum_{b \in \mathbb{G}_{\mathbb{O}}} a_b \mathcal{H}^b$$

where $a_b = \sum_{K \in \mathbb{G}_{\mathbb{O}}} b_{b,K} X^K$,

with $b_{b,K}$ being a power series in terms of the remainder $(x_{N+1}, \ldots, x_d)$ of the regular system of parameters, and with $K = (k_1, \ldots, k_d)$ varying in the range satisfying the condition

$$0 \leq k_l \leq p^{e_l} - 1 \text{ for } l = 1, \ldots, N \text{ and } k_l = 0 \text{ for } l = N + 1, \ldots, d.$$
(x_1, \ldots, x_d) are the results stated independent of the notion of an idealistic filtration.

0.2.2.3 Formal coefficient lemma. In the general setting as described in 0.2.2.2, the discussion on the power series expansion does not involve the notion of an idealistic filtration. The most interesting and important result regarding the power series expansion, however, is obtained when we introduce and require the following condition for \( H \) to satisfy, involving a \( \mathcal{D} \)-saturated idealistic filtration \( I_p \) over \( R_p \):

\[
\text{(iii) } (h_1, p^{r_i}) \in I_p \text{ for } l = 1, \ldots, N.
\]

Now the formal coefficient lemma claims

\[
(f, a) \in I_p, f = \sum_{B \in \mathbb{Z}_{\geq 0}^N} a_B H_B \Rightarrow (a_B, a - \|B\|) \in I_p \text{ for any } B \in \mathbb{Z}_{\geq 0}^N.
\]

(We recall that, for \( B = (b_1, \ldots, b_N) \in \mathbb{Z}_{\geq 0}^N \), we denote \((p^{r_1} b_1, \ldots, p^{r_N} b_N)\) by \([B]\) and \(\sum_{i=1}^N p^{r_i-b_i} \|B\|\). For the definition of the completion \( I_p \) of the idealistic filtration \( I_p \), we refer the reader to §2.4 in Part I.) The statement of the formal coefficient lemma turns out to be quite useful and powerful. In fact, Lemma 4.1.4.1 (Coefficient Lemma) in Part I can be obtained as a corollary to this formal version in Part II. We will see some applications of the formal coefficient lemma not only in Chapter 3 when we study the invariant \( \tilde{\mu} \), but also in Part III when we analyze the modifications and transformations of an idealistic filtration and in Part IV when we give the description of our algorithm.

0.2.3. Invariant \( \tilde{\mu} \). Chapter 3 is devoted to the discussion of the invariant \( \tilde{\mu} \), which is a counterpart in the new setting of the Idealistic Filtration Program to the notion of the “weak order” in the classical setting, whose definition involves the exceptional divisors. Naturally, when we carry out our algorithm, the definition of the invariant \( \tilde{\mu} \) in the middle of its process involves the exceptional divisors created by blowups. It also involves the subtle adjustments we have to make to the notion of a leading generator system for a \( \mathcal{D}_E \)-saturated idealistic filtration in the presence of the exceptional divisor \( E \) (cf. 0.2.1). However, we restrict the discussion of the invariant \( \tilde{\mu} \) in Part II to the one with no exceptional divisors taken into consideration, and hence to the discussion which could only be directly applied to the situation at year 0 of the algorithm. The discussion with the exceptional divisors taken into consideration, i.e., the discussion which can then be applied to the situation after year 0 of the algorithm, will be postponed until it finds an appropriate place in Part III or Part IV, where we will show how we should adjust the arguments in Part II in the presence of the exceptional divisors.

0.2.3.1 Definition of \( \tilde{\mu} \). Let \( I \) be a \( \mathcal{D} \)-saturated idealistic filtration over \( R \) as before. Let \( P \in \text{Spec } R \subset W \) be a closed point. Take a leading generator system \( \mathcal{H} \) for \( I_p \), and let \( \mathcal{H} \) be the set consisting of its elements. Recall that in 3.2.2 in Part I we set

\[
\mu_\mathcal{H}(I_p) = \inf \left\{ \mu_\mathcal{H}(f, a) := \frac{\text{ord}_\mathcal{H}(f)}{a} : (f, a) \in I_p, a > 0 \right\}
\]

where

\[
\text{ord}_\mathcal{H}(f) = \sup \{ n \in \mathbb{Z}_{\geq 0} : f \in m_p^n + (\mathcal{H}) \},
\]

and that we define the invariant \( \tilde{\mu}(P) \) by the formula

\[
\tilde{\mu}(P) = \mu_\mathcal{H}(I_p).
\]

There are two main issues concerning the invariant \( \tilde{\mu}(P) \).

Issue 1: Is \( \tilde{\mu}(P) \) independent of the choice of \( \mathcal{H} \) and hence of \( \mathcal{H} \) ?

Issue 2: Is \( \tilde{\mu} \) upper semi-continuous as a function of the (closed) point \( P \in \text{Spec } R \subset W \) ?
0.2.3.2 \( \bar{\mu}(P) \) is independent of the choice of \( \mathcal{H} \). We settled Issue 1 affirmatively via Coefficient Lemma in Part I. We would like to emphasize, on one hand, that we carried out the entire argument in Part I at the algebraic level of a local ring. This argument, showing that the invariant \( \bar{\mu}(P) \) is determined independent of the choice of a leading generator system, seems to be in contrast to the argument by Włodarczyk, where he uses some (analytic) automorphism of the completion of the local ring, showing that certain invariants are determined independent of the choice of a hypersurface of maximal contact via the notion of homogenization. Note that the notion of a leading generator system is a collective substitute for the notion of a hypersurface of maximal contact. (cf. 0.2.3.2.1 in Part I).

We remark, on the other hand, that we can give an analytic interpretation of the invariant \( \bar{\mu}(P) \) using the power series expansion discussed in Chapter 2. In fact, we see that \( \text{ord}_H(f) = \text{ord}(a_0) \) where \( a_0 \) with \( 0 = (0, \ldots, 0) \in (\mathbb{Z}_{>0})^N \) is the “constant term” of the power series expansion of the form (\( \ast \)). This explicit interpretation leads to an alternative way to settle Issue 1, though quite similar in spirit to the proof at the algebraic level, via the formal coefficient lemma. Note that \( \bar{\mu}(P) \) is rational, i.e., \( \bar{\mu}(P) \in \mathbb{Q} \), if we assume that \( I \) is of r.f.g. type (and hence that so is \( \mathbb{I}_P \)).

0.2.3.3 Upper semi-continuity of \((\sigma, \bar{\mu})\). Regarding Issue 2, we have to emphasize first that the question asking the upper semi-continuity of the invariant \( \bar{\mu} \) by itself is ill-posed, and its answer is no when literally taken. The precise and correct question to ask is the upper semi-continuity of the pair \((\sigma, \bar{\mu})\) with respect to the lexicographical order. Since the invariant \( \sigma \) is upper semi-continuous, this is equivalent to asking if the invariant \( \bar{\mu} \) is upper semi-continuous along the local maximum locus of the invariant \( \sigma \). We settle Issue 2 affirmatively in this precise form.

The difficulty in studying the behavior of the invariant \( \bar{\mu}(P) = \mu_H(\mathbb{I}_P) \), as we let \( P \) vary along the local maximum locus of the invariant \( \sigma \), lies in the fact that we also have to change the leading generator system \( \mathbb{H} \) and hence \( \mathcal{H} \) simultaneously. This is caused by the fact that our definition of a leading generator system is a priori “pointwise” in nature and hence that we do not know, even if \( \mathbb{H} \) is a leading generator system for \( \mathbb{I}_P \) at a point \( P \), \( \mathbb{H} \) stays being a leading generator system for \( \mathbb{I}_Q \) at a point \( Q \) in a neighborhood of \( P \). In general, it does not. There arises the need to modify a given leading generator system into one which is uniformly pure as discussed in 0.2.1.5. With the modified and uniformly pure leading generator system, the upper semi-continuity at issue is reduced to that of the multiplicity of a function in the usual setting. The upper semi-continuity can also be verified if we look at the power series expansion with respect to a uniformly pure leading generator system, and study the behavior of its coefficients.

0.2.4. Appendix. In the appendix, we report a new development, which establishes the nonsingularity principle using only the \( \mathfrak{D} \)-saturation and without using the \( \mathfrak{R} \)-saturation. Recall that in Part I we established the nonsingularity principle using both the \( \mathfrak{D} \)-saturation and \( \mathfrak{R} \)-saturation (cf. 0.2.3.2.4 and Chapter 4 in Part I). This opens up a possibility of constructing an algorithm, still in the frame work of the Idealistic Filtration Program, using only the \( \mathfrak{D} \)-saturation and without using the \( \mathfrak{R} \)-saturation. Note that the \( \mathfrak{R} \)-saturation invites the problem of termination, which we specified in the introduction to Part I as the only missing piece toward completing our algorithm in positive characteristic. Therefore, we believe that this new development is a substantial step forward in our quest for establishing an algorithm for resolution of singularities in positive characteristic.

This finishes the description of the outline of Part II.
0.3. Current status of the Idealistic Filtration Program.

It has been more than a year since we posted the original version of Part II on the electronic archive in August, 2007. We would like to report on the current status of the IFP, and make a note to Part I.

0.3.1. Current status. Since the advent of the new nonsingularity principle as described in 0.2.4, we have been pursuing the scheme of constructing an algorithm using only the $D$-saturation (or $D_E$-saturation in the presence of an exceptional divisor $E$). In fact, in characteristic zero, the scheme works almost perfectly providing an algorithm for local uniformization, with the triplet $(\sigma, \tilde{\mu}, s)$ being the unit to constitute the strand of invariants. (In order to obtain the global resolution of singularities, one has to work a little bit more to fill in the gap between the maximum locus of the strand and the support of the modification of an idealistic filtration. The gap is an anomaly observed when $\tilde{\mu} = 1$.) In positive characteristic, as we do not use the $\mathfrak{R}$-saturation any more, the denominators of the invariant $\tilde{\mu}$ are well-controlled, being no obstruction to showing the termination of the algorithm. Recently, however, some “bad” examples surfaced; if we try to naively follow the analogy to the case in characteristic zero, the blowup of a “$(\sigma, \tilde{\mu}, s)$-permissible” center would lead to the strict increase of the invariant $\tilde{\mu}$, violating the principle that the strand of invariants we construct should never increase after blowup. A few of these examples also indicate that the so-called monomial case needs a more careful treatment in positive characteristic than in characteristic zero. In order to overcome these pathologies observed in the “bad” examples, we introduce and insert a new invariant $\tilde{\nu}$, making the quadruple $(\sigma, \tilde{\mu}, \tilde{\nu}, s)$ the new unit to constitute the strand of invariants. The invariant $\tilde{\nu}$ is closely related to the invariant “$\nu$” used in [CP08] and [CP07]. We are now testing if our algorithm, taking the “$(\sigma, \tilde{\mu}, \tilde{\nu}, s)$-permissible” center in a quite explicit way, will provide a solution to the problem of local uniformization (and global resolution) in positive characteristic. We want to emphasize that we consider these new developments as the events in the process of “evolution” of the IFP, rather than mutation, since the basic strategy of the IFP, as envisioned in Part I, remains intact throughout our project. We reported the current status of the evolution of the IFP at the workshop held at RIMS in December of 2008, and we refer the reader to [RIMS08] for the precise content of the report. More details will be published in our subsequent papers in the near future.

0.3.2. Roles of $\sigma$ and $\tilde{\mu}$. Despite all the changes in the evolution process of the IFP discussed above, the fundamental roles of the invariants $\sigma$ and $\tilde{\mu}$, as the first two factors of the unit constituting the strand of invariants, remain unchanged. Therefore, the main portion of Part II, discussing these fundamental roles, remain unchanged.

0.3.3. Note to Part I. After Part I was published from Publications of RIMS, we learned that the result stated as Proposition 2.3.2.4 in Part I has already appeared in [LT74]. The arguments both in Part I and [LT74] are closely related to the classical results of Nagata [Nag57]. Due to our negligence, this fact was never mentioned in Part I, even though [LT74] was included in the references for Part I.
CHAPTER 1

Invariant $\sigma$

The purpose of this chapter is to investigate the basic properties of the invariant $\sigma$.

In this chapter, $R$ represents the coordinate ring of an affine open subset $\text{Spec} R$ of a nonsingular variety $W$ of $\dim W = d$ over an algebraically closed field $k$ of positive characteristic $\text{char}(k) = p$ or of characteristic zero $\text{char}(k) = 0$, where in the latter case we formally set $p = \infty$ (cf. 0.2.3.2.1 and Definition 3.1.1.1 (2) in Part I).

Let $I$ be a $\Sigma$-saturated idealistic filtration over $R$, and $I_P$ its localization at a closed point $P \in \text{Spec} R \subset W$.

§ 1.1. Definition and its computation.

1.1. Definition of $\sigma$. First we recall the definition, given in § 3.2 in Part I, of the invariant $\sigma$ at a closed point $P \in \text{Spec} R \subset W$.

Definition 1.1.1.1. The invariant $\sigma$ at $P$, which we denote by $\sigma(P)$, is defined to be the following infinite sequence indexed by $e \in \mathbb{Z}_{\geq 0}$

$$\sigma(P) = \left( d - l^\text{pure}_{pr}(P), d - l^\text{pure}_{pr}(P), \ldots, d - l^\text{pure}_{pr}(P), \ldots \right) = \left( d - l^\text{pure}_{pr}(P) \right)_{e \in \mathbb{Z}_{\geq 0}}$$

where

$$d = \dim W, \quad l^\text{pure}_{pr}(P) = \dim_k L(I_P)^\text{pure}.$$ 

(We refer the reader to Chapter 3 in Part I or 0.2.1.1 in the introduction to Part II for the definitions of the leading algebra $L(I_P)$ of the idealistic filtration $I_P$, its degree $p^e$ component $L(I_P)^{p^e}$, and its pure part $L(I_P)^{\text{pure}}$.)

Remark 1.1.1.2.

(1) The reason why we take the infinite sequence $\left( d - l^\text{pure}_{pr}(P) \right)_{e \in \mathbb{Z}_{\geq 0}}$ instead of the infinite sequence $\left( l^\text{pure}_{pr}(P) \right)_{e \in \mathbb{Z}_{\geq 0}}$ is two-fold:

(i) If we consider the infinite sequence $\left( l^\text{pure}_{pr}(P) \right)_{e \in \mathbb{Z}_{\geq 0}}$, it is lower semi-continuous as a function of $P$. Taking the negative of each factor ($+d$) of the sequence, we have our invariant upper semi-continuous, as we will see below. (We consider that the bigger $l^\text{pure}_{pr}(P)$ is, the better the singularity is. Therefore, as the measure of how bad the singularity is, it is also natural to define our invariant as its negative.)

(ii) We reduce the problem of resolution of singularities of an abstract variety $X$ to that of embedded resolution. Therefore, it would be desirable or even necessary to come up with an algorithm which would induce the “same” process of resolution of singularities, no matter what ambient variety $W$ we choose for an embedding $X \hookrightarrow W$ (locally).

While the infinite sequence $\left( l^\text{pure}_{pr}(P) \right)_{e \in \mathbb{Z}_{\geq 0}}$ (or its negative $-l^\text{pure}_{pr}(P)$) is dependent of the choice of $W$, the infinite sequence $\left( \dim W - l^\text{pure}_{pr}(P) \right)_{e \in \mathbb{Z}_{\geq 0}}$ is not. Therefore, the latter is more appropriate as an invariant toward constructing such an algorithm.
(2) The dimension of the pure part is non-decreasing as a function of \( e \in \mathbb{Z}_{\geq 0} \), and is uniformly bounded from above by \( d = \dim W \), i.e.,
\[
0 \leq l_{\text{pure}}^\alpha(P) \leq l_{\text{pure}}^{\alpha+1}(P) \leq \cdots \leq l_{\text{pure}}^0(P) \leq \cdots \leq d = \dim W
\]
and hence stabilizes after some point, i.e., there exists \( e_M \in \mathbb{Z}_{\geq 0} \) such that
\[
l_{\text{pure}}^e(P) = l_{\text{pure}}^{e_M}(P) \quad \text{for } e \geq e_M.
\]

Therefore, although \( \sigma(P) \) is an infinite sequence by definition, essentially we are only looking at some finite part of it.

(3) In characteristic zero, the invariant \( \sigma(P) \) consists of only one term \( d - l_{\text{pure}}^0 \), while the remaining terms \( d - l_{\text{pure}}^e \) are not defined for \( e > 0 \), as we set \( p = \infty \) in characteristic zero. (However, we may still say \( \sigma(P) \) is an infinite sequence and write \( \sigma(P) \in \prod_{e \in \mathbb{Z}_{\geq 0}} \mathbb{Z}_{\geq 0} \), for the sake of simplicity of presentation, intentionally ignoring the particular situation in characteristic zero.) Note that \( l_{\text{pure}} = l_{e} = \dim L(\mathcal{I}_p) \) can be regarded as the number indicating “how many linearly independent hypersurfaces of maximal contact we can take” for \( I_p \) (cf. Chapter 3 in Part I).

### 1.1.2. Computation of \( \sigma \).

The next lemma computes \( l_{\text{pure}}^\alpha(P) \) in terms of \( l_{\text{pure}}^n(P) \) and in terms of \( l_{\text{pure}}^n(P) \) for \( \alpha = 0, \ldots, e-1 \), which we can assume inductively have already been computed. We also see that \( l_{\text{pure}}^n(P) \) can be computed as the rank of a certain “Jacobian-like” matrix, and hence that it is lower semi-continuous as a function of \( P \). This immediately leads to the lower semi-continuity of the sequence \( \{ l_{\text{pure}}^n(P) \}_{n \in \mathbb{Z}_{\geq 0}} \) and hence to the upper semi-continuity of \( \sigma(P) = (d - l_{\text{pure}}^n(P)) \) as a function of \( P \). We will discuss the upper semi-continuity of \( \sigma \) in detail in the next section.

#### Lemma 1.1.2.1. Case : \( P \notin \text{Supp}(\mathcal{I}) \).

In this case, since we assume \( \mathcal{I} \) is \( \mathcal{D} \)-saturated and since so is \( I_p \), we observe that
\[
I_p = R_p \times \mathbb{R}.
\]

Accordingly, the invariant \( \sigma(P) \) takes the absolute minimum \( 0 \) in the value set of the invariant \( \sigma \), i.e.,
\[
\sigma(P) = (\sigma(P)(e))_{e \in \mathbb{Z}_{\geq 0}} = 0 \quad \text{with } \sigma(P)(e) = 0 \quad \forall e \in \mathbb{Z}_{\geq 0}.
\]

**Case : \( P \in \text{Supp}(\mathcal{I}) \).**

In this case, fixing \( e \in \mathbb{Z}_{\geq 0} \), we compute \( l_{\text{pure}}^n(P) \) in the following manner:

Suppose we have already computed \( l_{\text{pure}}^n(P) \) for \( \alpha = 0, \ldots, e-1 \).

Let \( 0 \leq e_1 < \cdots < e_K \leq e - 1 \) be the integers indicating the places where \( l_{\text{pure}}^n(P) \) jumps, i.e.,
\[
0 = l_{\text{pure}}^{e_1}(P) = \cdots = l_{\text{pure}}^{e_1}(P) < l_{\text{pure}}^{e_2}(P) = \cdots = l_{\text{pure}}^{e_2}(P) < \cdots < l_{\text{pure}}^{e_K}(P) = \cdots = l_{\text{pure}}^{e_K}(P).
\]

Introduce variables \( \{ v_{ij} \}_{j=1}^{K} \), where the second subscript \( j \) ranges from 1 to \( l_{\text{pure}}^{e_{j-1}}(P) \), i.e., \( j = 1, \ldots, l_{\text{pure}}^{e_{j-1}}(P) - l_{\text{pure}}^{e_{j}}(P) \).

Then we compute \( l_{\text{pure}}^\alpha(P) \) as follows:
\[
l_{\text{pure}}^\alpha(P) = l_{\text{pure}}^\alpha(P) - l_{\text{mixed}}^\alpha(P)
\]
where the number $l^{\text{mixed}}_{p^r}(P)$ is by definition given by the formula below

$$
l^{\text{mixed}}_{p^r}(P) = \# \{ \text{monomials of the form } \prod_{i=1}^{K} x_{ij}^{p^{i}r_i} b_{ij}; \sum_{i,j} p^{i}r_i b_{ij} = p^r, \quad p^{i}r_i b_{ij} \neq p^s \forall i,j \}.
$$

Moreover, take a set of generators $\{s_1, \ldots, s_r\}$ of the ideal $I$ at level $p^s$, i.e., $(s_1, \ldots, s_r) = I_{p^s} \subset R$. Let $(x_1, \ldots, x_d)$ be a regular system of parameters at $P$. Then

$$
l_{p^r}(P) = \text{rank} \left[ \partial_{x^i}(s_{p^r}) \right]_{i=1}^{d}.
$$

Proof. Case: $P \not\in \text{Supp}(I)$. In this case, by definition, there exists an element $(f, a) \in I_{p^s}$ with $a > 0$ such that $\text{ord}_p(f) < a$. There also exists an appropriate differential operator $d$ of degree $\text{ord}_p(f)$ such that $d(f) = u$ is a unit of $R_p$. Then we have

$$
(d(f), a - \text{ord}_p(f)) = (u, a - \text{ord}_p(f)) \in I_p
$$

and hence by condition (differential) in Definition 2.1.2.1 in Part I

$$(I_p)_{a - \text{ord}_p(f)} = R_p.
$$

This implies by condition (ii) in Definition 2.1.1.1 in Part I that

$$(I_p)_{n(a - \text{ord}_p(f))} = R_p \quad \forall n \in \mathbb{Z}_{>0}.
$$

We conclude then by condition (iii) in Definition 2.1.1.1 in Part I that

$$
I_p = R_p \times \mathbb{R}.
$$

From this the assertion on $\sigma(P)$ easily follows, since we have $L(I_p) = G_p$.

Case: $P \in \text{Supp}(I)$. Let

$$
L(I_p) = \bigoplus_{n \geq 0} L(I_p)_n \subset G_p = \bigoplus_{n \geq 0} m_p^n / m_p^{n+1}
$$

be the leading algebra of $I_p$.

By Lemma 3.1.2.1 in Part I, we can choose $\{e_1 < \cdots < e_M\} \subset \mathbb{Z}_{>0}$ and $V_1 \cup \cdots \cup V_M \subset G_1$ with $V_i = \{v_{ij}\}_j$ satisfying the following conditions

(i) \quad $F^e(V_i) \subset L(I_p)_{p^r}$ for $1 \leq i \leq M$,

(ii) \quad $\bigcap_{e \leq e'} F^e(V_i)$ is a $k$-basis of $L(I_p)_{p^r}$ for any $e \in \mathbb{Z}_{\geq 0}$.

Since $L(I_p)_{p^r}$ generates $L(I_p)$, we have $L(I_p) = k[\bigcup_{i=1}^{M} F^e(V_i)]$.

From this it follows that

$$
l_{p^r}(P) = \# \{ \text{monomials of the form } \prod_{i=1}^{K} x_{ij}^{p^{i}r_i} b_{ij}; \sum_{i,j} p^{i}r_i b_{ij} = p^r \}
$$

and hence that

$$
l_{p^r}^{\text{pure}} = l_{p^r}(P) - l_{p^r}^{\text{mixed}}
$$

where

$$
l_{p^r}^{\text{mixed}}(P) = \# \{ \text{monomials of the form } \prod_{i=1}^{K} x_{ij}^{p^{i}r_i} b_{ij}; \sum_{i,j} p^{i}r_i b_{ij} = p^r, \quad p^{i}r_i b_{ij} \neq p^s \forall i,j \}.
$$

The assertion in “Moreover” part follows from the fact that $L(I_p)_{p^r}$ is generated as a $k$-vector space by the degree $p^s$ terms of the power series expansions of $\{s_{p^s}\}_{p=1}^{r}$ with respect to a regular system of parameters $(x_1, \ldots, x_d)$, i.e.,

$$
L(I_p)_{p^r} = \left\{ s_{p^s} \text{ mod } m_p^{p^{s+1}}; \beta = 1, \ldots, r \right\} = \left\{ s_{p^s} \text{ mod } (x_1, \ldots, x_d)^{p^{s+1}}; \beta = 1, \ldots, r \right\}
$$
§1.2. Upper semi-continuity.

and that their coefficients appear as the entries of the matrix given in the statement, i.e.,

$$s_\beta = \sum_{i=1}^{p} \partial x^i(s_\beta) X^i \mod (x_1, \ldots, x_p)^{p+1}.$$

This completes the proof of Lemma 1.1.2.1.

Remark 1.1.2.2. Let us consider \(\tau(P) = \left( t_P(P) \right)_{e \in \mathbb{Z}_{\geq 0}}\). Then noting \(t_P(P) = t_{\text{pure}}(P)\), we conclude by Lemma 1.1.2.1(1) that \(\sigma(P)\) determines \(\tau(P)\) and vice versa.

In particular, for \(P, Q \in m\cdot \text{Spec} \ R\), we have

\[
\sigma(P) = \sigma(Q) \iff \tau(P) = \tau(Q)
\]

\[
\sigma(P) \geq \sigma(Q) \iff \tau(P) \leq \tau(Q).
\]

Therefore, the upper semi-continuity of the invariant \(\sigma\), which we will show in the next section, is equivalent to the lower semi-continuity of the invariant \(\tau\).

§1.2. Upper semi-continuity.

1.2.1. Basic facts surrounding the definition of the upper semi-continuity. In this subsection, we clarify some basic facts surrounding the definition of the upper semi-continuity. We denote by \(f : X \to T\) a function from a topological space \(X\) to a totally-ordered set \(T\).

Definition 1.2.1.1. We say \(f\) is upper semi-continuous if the set

\[
X_f := \{ x \in X ; f(x) \geq t \}
\]

is closed for any \(t \in T\).

Lemma 1.2.1.2. Consider the conditions below:

(i) For any \(x \in X\), there exists an open neighborhood \(U_x\) such that \(f(x) \geq f(y)\) for any \(y \in U_x\).

(ii) The set \(X_f = \{ x \in X ; f(x) > t \}\) is closed for any \(t \in T\).

(iii) \(f\) is upper semi-continuous.

Then we have the following implications:

\((i) \iff (ii) \implies (iii)\).

Moreover, if \(T\) is well-ordered (in the sense that every non-empty subset has a least element), then conditions (ii) and (iii) are equivalent.

Proof. The proof is elementary, and left to the reader as an exercise.

Corollary 1.2.1.3. For the invariant \(\sigma : m\cdot \text{Spec} \ R \to \prod_{e \in \mathbb{Z}_{\geq 0}} \mathbb{Z}_{\geq 0}\), where the target space \(\prod_{e \in \mathbb{Z}_{\geq 0}} \mathbb{Z}_{\geq 0}\) is totally ordered with respect to the lexicographical order, conditions (i), (ii), (iii) in Lemma 1.2.1.2 are all equivalent.

Proof. As mentioned in Remark 1.1.2.2(2), the dimension of the pure part \(t_P^{\text{pure}}(P)\) is non-decreasing as a function of \(e \in \mathbb{Z}_{\geq 0}\). Accordingly, \(\sigma(P)(e) = d - t_P^{\text{pure}}(P)\) is non-increasing as a function of \(e \in \mathbb{Z}_{\geq 0}\). Therefore, instead of \(\prod_{e \in \mathbb{Z}_{\geq 0}} \mathbb{Z}_{\geq 0}\), we may take the subset \(T = \{ (t_e)_{e \in \mathbb{Z}_{\geq 0}} \in \prod_{e \in \mathbb{Z}_{\geq 0}} \mathbb{Z}_{\geq 0} ; t_{e_1} \geq t_{e_2} \text{ if } e_1 > e_2 \} \subset \prod_{e \in \mathbb{Z}_{\geq 0}} \mathbb{Z}_{\geq 0}\) as the target space for \(\sigma\). Observe that \(T\) is well-ordered (with respect to the total order induced by the one on \(\prod_{e \in \mathbb{Z}_{\geq 0}} \mathbb{Z}_{\geq 0}\)). In fact, for a non-empty subset \(S \subset T\), we can construct its least element \(s_{\min} = (s_{\min,e})_{e \in \mathbb{Z}_{\geq 0}}\) inductively by the following formula:

\[
s_{\min,e} = \min \left\{ s_e \in \mathbb{Z}_{\geq 0} ; \ s = (s_i)_{i \in \mathbb{Z}_{\geq 0}} \in S \text{ s.t. } s_i = s_{\min,i} \text{ for } i < e \right\}.
\]

Now the statement of the corollary follows from Lemma 1.2.1.2.

The following basic description of the stratification into the level sets can be seen easily, and its proof is left to the reader.
Corollary 1.2.1.4. Let \( f : X \to T \) be an upper semi-continuous function. Suppose that \( X \) is noetherian, and that \( T \) is well-ordered. Then \( f \) takes only finitely many values over \( X \), i.e.,
\[
\{ f(x) : x \in X \} = \{ t_1 < \cdots < t_n \} \subset T.
\]
Accordingly, we have a strictly decreasing finite sequence of closed subsets
\[
X = X_{\geq n} \supseteq \cdots \supseteq X_{\geq 0} \supseteq \emptyset,
\]
which provides the stratification of \( X \) into the level sets
\[
\{ x \in X ; f(x) = t_i \} = X_{\geq i} \setminus X_{\geq i+1} \quad \text{for} \quad i = 1, \ldots, n.
\]

1.2.2. Upper semi-continuity of the invariant \( \sigma \).

Proposition 1.2.2.1. The invariant
\[
\sigma : \text{m-Spec } R \to \prod_{n \in \mathbb{Z}_{\geq 0}} \mathbb{Z}_{\leq 0}
\]
is upper semi-continuous.

Proof. Set \( X = \text{m-Spec } R \) for notational simplicity.

Given \( t = (t_i)_{i \in \mathbb{Z}_{\geq 0}} \in \prod_{i \in \mathbb{Z}_{\geq 0}} \mathbb{Z}_{\leq 0} \), and \( n \in \mathbb{Z}_{\geq 0} \), denote by \( t_{\leq n} \) the truncation of \( t \) up to the \( n \)-th term, i.e.,
\[
t_{\leq n} = (t_i)_{i \leq n} \in \prod_{i \leq n} \mathbb{Z}_{\geq 0}.
\]

We define \( \sigma_{\leq n} : X \to \prod_{i > 0} \mathbb{Z}_{\leq 0} \) by \( \sigma_{\leq n}(x) = (\sigma(x))_{\leq n} \) for \( x \in X \).

We also set \( X(t_{\leq n}) = \{ x \in X ; \sigma_{\leq n}(x) \geq t_{\leq n} \} \). Then
\[
X_{\geq n} = \bigcap_{n = 0}^{\infty} X(t_{\leq n}).
\]

In order to show the upper semi-continuity of \( \sigma \), we have to show \( X_{\geq n} \) is closed for any \( n \in \mathbb{Z}_{\geq 0} \). The above equality implies that it suffices to show \( X(t_{\leq n}) \) is closed for any \( n \in \mathbb{Z}_{\geq 0} \). This follows if the function \( \sigma_{\leq n} \) is upper semi-continuous, which we will show by induction on \( n \).

The function \( \sigma_{\leq 0} = d - \rho_{pure} \) is upper semi-continuous, since \( \rho_{pure} \) is lower semi-continuous (cf. Lemma 1.2.1).

Assume we have shown \( \sigma_{\leq n-1} \) is upper semi-continuous. We show then that the function \( \sigma_{\leq n} \) satisfies condition (i) in Lemma 1.2.1.2 and hence that it is upper semi-continuous.

Suppose we are given \( x \in X \). Since \( \sigma_{\leq n-1} \) is upper semi-continuous (and since the target space \( \prod_{i > 0} \mathbb{Z}_{\leq 0} \) is well ordered), there exists an open neighborhood \( U_x \) such that \( \sigma_{\leq n-1}(x) \geq \sigma_{\leq n-1}(y) \) for any \( y \in U_x \) (cf. Lemma 1.2.1.2). Since the function \( \rho_{pure} \) is lower semi-continuous (cf. Lemma 1.2.1.1), by shrinking \( U_x \) if necessary, we may assume that \( \rho_{pure}(x) \leq \rho_{pure}(y) \) for any \( y \in U_x \).

Take \( y \in U_x \).

If \( \sigma_{\leq n-1}(x) > \sigma_{\leq n-1}(y) \), then we obviously have \( \sigma_{\leq n}(x) > \sigma_{\leq n}(y) \).

If \( \sigma_{\leq n-1}(x) = \sigma_{\leq n-1}(y) \), then from the definition it follows that \( \rho_{pure}^{\alpha}(x) = \rho_{pure}^{\alpha}(y) \) for \( \alpha = 0, \ldots, n - 1 \). This implies by Lemma 1.2.1.2(1) that \( \rho_{pure}\alpha(x) = \rho_{pure}\alpha(y) \). Therefore, we conclude that
\[
\rho_{pure}\alpha(x) = \rho_{pure}(x) - \rho_{mixed}\alpha(x) \leq \rho_{pure}(y) - \rho_{mixed}\alpha(y) = \rho_{pure}\alpha(y),
\]
and hence that
\[
d - \rho_{pure}(x) \geq d - \rho_{pure}(y).
\]
Thus we have \( \sigma_{\leq n}(x) \geq \sigma_{\leq n}(y) \).

This shows that \( \sigma_{\leq n} \) satisfies condition (i) in Lemma 1.2.1.2 and hence that it is upper semi-continuous, and completes the induction.

Therefore, we conclude \( \sigma : \text{m-Spec } R \to \prod_{n \in \mathbb{Z}_{\geq 0}} \mathbb{Z}_{\leq 0} \) is upper semi-continuous.

This completes the proof of Proposition 1.2.2.1.
Corollary 1.2.2.2. We can extend the domain from $\mathfrak{m}$-Spec $R$ to Spec $R$ to have the invariant $\sigma : \text{Spec } R \to \prod_{e \in \mathbb{Z}_{\geq 0}} \mathbb{Z}_{\geq 0}$, by defining
$$\sigma(Q) = \min \{ \sigma(P) : P \in \mathfrak{m} \text{-Spec } R, P \in Q \} \text{ for } Q \in \text{Spec } R.$$ The formula is equivalent to saying that $\sigma(Q)$ is equal to $\sigma(P)$ with $P$ being a general closed point on $Q$. The invariant $\sigma$ with the extended domain is also upper semi-continuous.

Moreover, since Spec $R$ is noetherian and since $\prod_{e \in \mathbb{Z}_{\geq 0}} \mathbb{Z}_{\geq 0}$ can be replaced with the well-ordered set $T$ as described in the proof of Corollary 1.2.1.3, conditions (i) and (ii) in Lemma 1.2.1.2, as well as the assertions of Corollary 1.2.1.4, hold for the upper semi-continuous function $\sigma : \text{Spec } R \to T$.

Proof. Observe that, given $Q \in \text{Spec } R$, the formula for $\sigma(Q)$ is well-defined, since the existence of the minimum (i.e., the least element) on the right hand side is guaranteed by the fact that the value set of the invariant $\sigma$ is well-ordered (cf. the proof of Corollary 1.2.1.3). Note that there exists a non-empty dense open subset $U$ of $\mathfrak{Q} \cap \mathfrak{m}$-Spec $R$ such that $\sigma(Q) = \sigma(P)$ for $P \in U$, a fact implied by condition (i) of the upper semi-continuity of the invariant $\sigma$. The upper semi-continuity of the invariant $\sigma$ with the extended domain Spec $R$ is immediate from the upper semi-continuity of the invariant $\sigma$ with the original domain $\mathfrak{m}$-Spec $R$.

The “Moreover” part follows immediately from the statements of Lemma 1.2.1.2, Corollary 1.2.1.3 and Corollary 1.2.1.4.

This completes the proof of Corollary 1.2.2.2.

§1.3. Local behavior of a leading generator system.

1.3.1. Definition of a leading generator system and a remark about the subscripts. We say that a subset $\mathbb{H} = \{(h_i, p^e_i)\}_{i=1}^N \subset \mathbb{I}_p$ with nonnegative integers $0 \leq e_i \leq \cdots \leq e_N$ attached is a leading generator system (of the localization $\mathbb{I}_p$ of the $\mathfrak{T}$-saturated idealistic filtration $\mathfrak{T}$ over $R$ at a closed point $P \in \mathfrak{m}$-Spec $R$), if the leading terms of its elements provide a specific set of generators for the leading algebra $L(\mathbb{I}_p)$. More precisely, it satisfies the following conditions (cf. Definition 3.1.3.1 in Part I):

(i) $h_i \in m_p^{e_i}$ and $\overline{h_i} = (h_i \mod m_p^{e_i + 1}) \in L(\mathbb{I}_p)_{m_p^{e_i}}$ for $i = 1, \ldots, N$,

(ii) $\overline{h_i^{m_p^{e_i}}}$; $e_i \leq e$ consists of $\#\{u^i_j; e_i \leq e\}$-distinct elements, and forms a $k$-basis of $L(\mathbb{I}_p)_{m_p^{e_i}}$ for any $e \in \mathbb{Z}_{\geq 0}$.

Since the leading algebra $L(\mathbb{I}_p)$ is generated by its pure part $L(\mathbb{I}_p)_{m_p^{e_i}} = \bigcup_{e \in \mathbb{Z}_{\geq 0}} L(\mathbb{I}_p)_{m_p^{e_i}}$ (cf. 0.2.1.1), we conclude from condition (ii) that the leading terms of $\mathbb{H}$
$$\overline{h_i} = (h_i \mod m_p^{e_i + 1})_{i=1}^N$$ provide a set of generators for $L(\mathbb{I}_p)$, i.e., $L(\mathbb{I}_p) = k[\overline{h_i}]^N_{i=1}$.

We remark that, for the subscripts of the leading generator system $\mathbb{H}$, we sometimes use the letter “$I$” as above, writing $\mathbb{H} = \{(h_{ij}, p^{e_{ij}})\}_{i=1}^M$ with nonnegative integers $0 \leq e_1 \leq \cdots \leq e_M$ attached, and that some other times we use the letters $i$ and $j$, writing $\mathbb{H} = \{(h_{ij}, p^{e_{ij}})\}_{i=1, \ldots, M}$ with nonnegative integers $0 \leq e_1 < \cdots < e_M$ attached. In the latter use of the subscripts, conditions (i) and (ii) are written as in 1.3.1 of Part I:

(i) $h_{ij} \in m_p^{e_{ij}}$ and $\overline{h_{ij}} = (h_{ij} \mod m_p^{e_{ij} + 1}) \in L(\mathbb{I}_p)_{m_p^{e_{ij}}}$ for any $ij$,

(ii) $\overline{h_{ij}^{m_p^{e_{ij}}}}$; $e_i \leq e$ consists of $\#\{ij; e_i \leq e\}$-distinct elements, and forms a $k$-basis of $L(\mathbb{I}_p)_{m_p^{e_{ij}}}$ for any $e \in \mathbb{Z}_{\geq 0}$.

In the future, we use the subscripts in both ways, while choosing one at a time, depending upon the situation and its convenience.
1. INVARIANT $\sigma$

1.3.2. A basic question. Let $\mathbb{H}$ be a leading generator system of $I_P$. If we take a neighborhood $U_P$ of $P$ small enough, then $\mathbb{H}$ is a subset of $I_Q$ for any closed point $Q \in U_P \cap m\cdot \text{Spec } R$. We may then ask the following question regarding the local behavior of the leading generator system:

Is $\mathbb{H}$ a leading generator system of $I_Q$?

A moment of thought reveals that the answer to this question in general is no. In fact, due to the upper semi-continuity of the invariant $\sigma$, by shrinking $U_P$ if necessary, we may assume $\sigma(P) \geq \sigma(Q)$ for any closed point $Q \in U_P \cap m\cdot \text{Spec } R$. If $\sigma(P) > \sigma(Q)$, then there is no way that $\mathbb{H}$ could be a leading generator system of $I_Q$. (Note that the invariant $\sigma$ is completely determined by the leading generator system.)

We refine our question to avoid the obvious calamity as above:

Is $\mathbb{H}$ a leading generator system of $I_Q$ for any closed point $Q \in C \cap m\cdot \text{Spec } R \subset U_P \cap m\cdot \text{Spec } R$ where $C = \{Q \in U_P; \sigma(P) = \sigma(Q)\}$?

The answer to this question, for an arbitrary leading generator system $\mathbb{H}$ of $I_P$, is still no. One of the conditions for $\mathbb{H}$ to be a leading generator system of $I_P$ requires an element $(h_{ij}, p^r) \in \mathbb{H}$ to be pure at $P$, i.e., $(h_{ij}, \text{mod } m_P^{r+1}) \in L(P)_{\text{pure}}^{r+1}$. However, even when a closed point $Q \in U_P \cap m\cdot \text{Spec } R$ satisfies the condition $Q \in C \cap m\cdot \text{Spec } R$, some element $(h_{ij}, p^r)$ may fail to be pure at $Q$, i.e., $(h_{ij}, \text{mod } m_Q^{r+1}) \notin L(Q)_{\text{pure}}^{r+1}$, and hence $\mathbb{H}$ fails to be a leading generator system at $Q$.

Now we refine our question further:

Can we modify a given generator system $\mathbb{H}$ of $I_P$ into $\mathbb{H}'$ so that $\mathbb{H}'$ stays being a leading generator system of $I_Q$ for any closed point $Q \in C \cap m\cdot \text{Spec } R \subset U_P \cap m\cdot \text{Spec } R$ where $C = \{Q \in U_P; \sigma(P) = \sigma(Q)\}$?

The main goal of the next subsection is to give an affirmative answer to this last question (adding one extra condition of the point $Q$ being on the support $\text{Supp}(I)$ of the idealistic filtration), and also to give an explicit description of how we make the modification. We say we modify the given leading generator system into one which is "uniformly pure" (along $C$ intersected with $\text{Supp}(I)$).

1.3.3. Modification of a given leading generator system into one which is uniformly pure.

Definition 1.3.3.1. Let $\mathbb{H}$ be a leading generator system of the localization $I_P$ of the $\mathcal{T}$-saturated idealistic filtration $I$ over $R$ at a closed point $P \in m\cdot \text{Spec } R$. We say $\mathbb{H}$ is uniformly pure (in a neighborhood $U_P$ of $P$ along the local maximum locus $C$ of the invariant $\sigma$ intersected with the support $\text{Supp}(I)$ of the idealistic filtration) if there exists an open neighborhood $U_P$ of $P$ such that the following conditions are satisfied:

1. $\mathbb{H} \subset I_Q \forall Q \in U_P$.
2. $\sigma(P)$ is the maximum of the invariant $\sigma$ over $U_P$, i.e., $\sigma(P) \geq \sigma(Q) \forall Q \in U_P$.
3. $C = \{Q \in U_P; \sigma(P) = \sigma(Q)\}$ is a closed subset of $U_P$, and
4. $\mathbb{H}$ is a leading generator system of $I_Q$ for any $Q \in C \cap \text{Supp}(I) \cap m\cdot \text{Spec } R$.

(For the definition of the support $\text{Supp}(I)$ of the idealistic filtration $I$, we refer the reader to Definition 2.1.1.1 in Part I.)

Remark 1.3.3.2. We remark that in condition (4) of Definition 1.3.3.1, in order for $\mathbb{H}$ to be uniformly pure, we require $\mathbb{H}$ is a leading generator system of $I_Q$ for any closed point $Q \in C \cap \text{Supp}(I) \cap m\cdot \text{Spec } R$ (i.e., we only consider those closed points on the support $\text{Supp}(I)$ of the idealistic filtration $I$), where in the last form of the basic question in 1.3.2, we merely wrote "$Q \in C \cap m\cdot \text{Spec } R$". The reason to add this extra condition on $Q$ (as mentioned in the last paragraph of 1.3.2) is as follows:

Consider the case when $\sigma(P) = \emptyset$. (Recall that the symbol $\emptyset = (0, \cdots, 0, \cdots)$ represents the absolute minimum in the value set of the invariant $\sigma$.)

By the upper semi-continuity of the invariant $\sigma$, for a sufficiently small open neighborhood $U_P$ of $P$, we have $\sigma(Q) = \sigma(P) = \emptyset$ for any closed point $Q \in U_P \cap m\cdot \text{Spec } R$. 


and hence we have \( C \cap m \cdot \text{Spec } R = U_P \cap m \cdot \text{Spec } R \). On the other hand, the condition \( \sigma(P) = \emptyset \) implies that, given any leading generator system \( \mathbb{H} \) of \( I_P \), the elements \( \{h_{ij}\} \) are generators of the maximal ideal \( m_P \) with \( \#(ij) = d \). (Note that, in this case, all the elements of a leading generator system are concentrated at level 1, i.e., \( 1 = i = M \) and \( 0 = e_1 = e_i = e_M \).) Therefore, \( \mathbb{H} \) cannot be a leading generator system of \( I_P \) for a closed point \( Q \in U_P \cap m \cdot \text{Spec } R \) if \( Q \neq P \). That is to say, it would not satisfy the condition described in the last form of the basic question. However, in this case, we have either \( U_P \cap \text{Supp}(I) = \emptyset \) or \( U_P \cap \text{Supp}(I) = \{P\} \) (if we take \( U_P \) sufficiently small). Therefore, condition (4) in Definition 1.3.3.1 is automatically satisfied.

Consider the case when \( \sigma(P) \neq \emptyset \).

In this case, we have \( C \cap m \cdot \text{Spec } R = C \cap \text{Supp}(I) \cap m \cdot \text{Spec } R \), since any closed point \( Q \in C \cap m \cdot \text{Spec } R \) (i.e., we have \( \sigma(Q) = \sigma(P) \neq \emptyset \) is necessarily in the support \( \text{Supp}(I) \) of the idealistic filtration (cf. Lemma 1.1.2.1). Therefore, there is no difference between the condition in the last form of the basic question and condition (4) in Definition 1.3.3.1.

In other words, the extra condition for \( Q \) to be in the support \( \text{Supp}(I) \) is introduced so that we can avoid the “obvious” counter example to an affirmative answer to the last form of the basic question in the special case \( \sigma(P) = \emptyset \).

**Proposition 1.3.3.3.** Let \( \mathbb{H} = \{(h_{ij}, p^{e_i})\}_{i=1,\ldots,M} \) be a leading generator system of the localization \( I_P \) of the \( \mathcal{I} \)-saturated idealistic filtration \( I \) over \( R \) at a closed point \( P \in m \cdot \text{Spec } R \), with nonnegative integers \( 0 \leq e_1 < \cdots < e_M \) attached. Then \( \mathbb{H} \) can be modified into another leading generator system \( \mathbb{H}' \), which is uniformly pure.

More precisely, there exists \( \{g_{ij}\} \subseteq m_P \), where the subscript \( B \) ranges over the set

\[
\text{Mix}_{\mathbb{H},I} = \{B = (b_{i\alpha}) \in (\mathbb{Z}_{\geq 0})^{\#(I)} \mid [B] = p^{e_i}, b_{i\alpha} = 0 \text{ if } \alpha \geq i, \text{ and } p^{\alpha} b_{i\alpha} \neq p^{\gamma} \forall \alpha \beta\},
\]

such that, setting \( h'_{ij} = h_{ij} - \sum g_{ij}B^{e_i} \), the modified set \( \mathbb{H}' = \{(h'_{ij}, p^{e_i})\}_{i=1,\ldots,M} \) is a leading generator system of \( I_P \) which is uniformly pure.

**Proof.** It suffices to prove that there exists an affine open neighborhood \( U_P = \text{Spec } R_f \) of \( P \), where \( R_f \) is the localization of \( R \) by an element \( f \in R \), such that the following conditions are satisfied:

1. \( \mathbb{H} \subseteq I_f \) (and hence \( \mathbb{H}' \subseteq I_f \) where \( \mathbb{H}' \) is described in condition (4)),
2. \( \sigma(P) \) is the maximum of the invariant \( \sigma \) over \( U_P \), i.e., \( \sigma(P) \geq \sigma(Q) \) \( \forall Q \in U_P \),
3. \( C = \{Q \in U_P \mid \sigma(P) = \sigma(Q)\} \) is a closed subset of \( U_P \), and
4. there exists \( \{g_{ij}\} \subseteq R_f \), where the subscript \( B \) ranges over the set \( \text{Mix}_{\mathbb{H},I} \), such that \( \{g_{ij}\} \subseteq m_P \) and that, setting \( h'_{ij} = h_{ij} - \sum g_{ij}B^{e_i} \), the modified set

\[
\mathbb{H}' = \{(h'_{ij}, p^{e_i})\}_{i=1,\ldots,M}
\]

is a leading generator system of \( I_P \) for any \( Q \in C \cap \text{Supp}(I) \cap m \cdot \text{Spec } R \).

**Step 1. Check conditions (1), (2) and (3).**

It is easy to choose an affine open neighborhood \( U_P = \text{Spec } R_f \) of \( P \) satisfying condition (1). By the upper semi-continuity of the invariant \( \sigma \), we may also assume condition (2) is satisfied (cf. condition (i) in Lemma 1.2.1.2). Then condition (3) automatically follows, since \( C = U_P \cap (\text{Spec } R_{f \cdot \sigma(P)}) \) is closed (cf. Definition 1.2.1.1).

We remark that in terms of the invariant \( \tau \) (cf. Remark 1.1.2.2) conditions (2) and (3) are equivalent to the following

(2) \( \tau(P) \) is the minimum of the invariant \( \tau \) over \( U_P \), i.e., \( \tau(P) \leq \tau(Q) \) \( \forall Q \in U_P \), and
(3) \( C = \{Q \in U_P \mid \tau(P) = \tau(Q)\} \).
Now we have only to check, by shrinking $U_p$ if necessary, that condition (4) is also satisfied.

**Step 2.** Preliminary analysis to check condition (4).

First consider the idealistic filtration $\mathfrak{J} = G_{ij}(\mathbb{H})$ generated by $\mathbb{H}$ over $R_f$. Note that $\mathfrak{J} \subset \mathfrak{J}_f$ but that $\mathfrak{J}$ may not be $\mathfrak{T}$-saturated. In order to distinguish the invariant $\tau$ for $\mathfrak{I}$ (or equivalently for $\mathfrak{J}_f$ over $U_p$) from the invariant $\tau$ for $\mathfrak{J}$, we denote them by $\tau_1$ and $\tau_2$, respectively.

Since $\tau_1$ is lower semi-continuous, by shrinking $U_p$ if necessary, we may assume

$\tau(P)$ is the minimum of the invariant $\tau_1$ over $U_p$, i.e., $\tau_1(P) \leq \tau_2(Q) \forall Q \in U_p$.

For any closed point $Q \in C \cap \text{Supp}(\mathfrak{I}) \cap \text{m- Spec } R$, we compute

$$\tau_1(P) = \tau_2(P) \leq \tau_2(Q) \leq \tau_1(Q) = \tau_1(P).$$

We remark that the first equality is a consequence of the fact that the set $\{h_{ij} \mod m_p^{\ell_j + 1}\}_{j=1, \ldots, M}$ generates $L(\mathfrak{I}_Q)$ as a $k$-algebra,

Moreover $\{h_{ij} \mod m_p^{\ell_j + 1}\}_{j=1, \ldots, M}$ generates both $L(\mathfrak{I}_Q)$ and $L(\mathfrak{I}_P)$ as $k$-algebras, the second inequality is a consequence of (2), the third inequality is a consequence of the inclusion $J \subset I$, and that the last equality follows from the definition of the closed subset $C$.

Therefore, we see that

$$\tau_1(P) = \tau_2(P) = \tau_2(Q) = \tau_1(Q) \forall Q \in C \cap \text{Supp}(\mathfrak{I}) \cap \text{m- Spec } R.$$  

**Step 3.** Some conclusions of the equality $\tau_1(P) = \tau_2(P) = \tau_2(Q) = \tau_1(Q)$ for any $Q \in C \cap \text{Supp}(\mathfrak{I}) \cap \text{m- Spec } R$.

The equality obtained at the end of Step 2 leads to a few conclusions that we list below:

(a) The set $\{h_{ij} \mod m_p^{\ell_j + 1}\}_{j=1, \ldots, M}$ generates $L(\mathfrak{I}_Q)$ as a $k$-algebra. Moreover $\{h_{ij} \mod m_p^{\ell_j + 1}\}_{j=1, \ldots, M}$ generates both $L(\mathfrak{I}_Q)$ and $L(\mathfrak{I}_P)$ as $k$-algebras, the second inequality is a consequence of (2), the third inequality is a consequence of the inclusion $J \subset I$, and that the last equality follows from the definition of the closed subset $C$.

Therefore, we see that

$$\tau_1(P) = \tau_2(P) = \tau_2(Q) = \tau_1(Q) \forall Q \in C \cap \text{Supp}(\mathfrak{I}) \cap \text{m- Spec } R.$$  

(b) There exist nonnegative integers $0 \leq e_1 < \cdots < e_M$, independent of $Q \in C \cap \text{Supp}(\mathfrak{I}) \cap \text{m- Spec } R$, such that the jumping of the dimension of the pure part only occurs at these numbers, i.e.,

$$0 = l_{p^0}^{\text{pure}}(Q) = \cdots = l_{p_{M-1}^{\text{pure}}}^{\text{pure}}(Q) < l_{p_{M-1}}^{\text{pure}}(Q) = \cdots = l_{p_{M-1}^{\text{pure}}}^{\text{pure}}(Q) < \cdots$$

as $l_{p_{M-1}^{\text{pure}}}(Q) = l_{p_{M-1}^{\text{pure}}}(Q)$ for any $Q \in C \cap \text{Supp}(\mathfrak{I}) \cap \text{m- Spec } R$ and $e \in \mathbb{Z}_{>0}$ (cf. Remark 1.1.2.2).

Applying Lemma 3.1.2.1 in Part I to $L(\mathfrak{I}_Q)$ for $Q \in C \cap \text{Supp}(\mathfrak{I}) \cap \text{m- Spec } R$, we see that we can take $V_1 \cup \cdots \cup V_{M-1} \subset G_{i, Q} = m_Q/m_Q^2$ with $V_{i, Q} = \{v_{ij, Q}\}_j$, where $1 \leq j \leq l_{p_j}^{\text{pure}}(Q) - l_{p_{j-1}}^{\text{pure}}(Q)$, satisfying the following conditions.
(i) \( F^e(V_{i,0}) \subset L(\mathcal{I}_{Q})_{p'}^{\text{pure}} \) for \( 1 \leq i \leq M \).
(ii) \( \bigcup_{e \leq \alpha} F^e(V_{i,0}) \) is a k-basis of \( L(\mathcal{I}_{Q})_{p'}^{\text{pure}} \) for any \( e \in \mathbb{Z}_{\geq 0} \).

Since \( L(\mathcal{I}_{Q})_{p'}^{\text{pure}} \) generates \( L(\mathcal{I}_{Q}) \), we have \( L(\mathcal{I}_{Q}) = k[\bigcup_{i=1}^{M} F^e(V_{i,0})] \).

Using this information, we also conclude the following.

(c) The \( \overline{h}_{i,j,0} \) are all pure when \( i = 1 \), i.e., \( \overline{h}_{1,j,0} \in L(\mathcal{I}_{Q})_{p_1}^{\text{pure}} \), and we take \( v'_{j,0} \in G_{i,0} \) so that \( F^e(v'_{j,0}) = \overline{h}_{i,j,0} \) for \( j = 1, \ldots, k_{i,0} \).

As can be seen by induction on \( i = 1, \ldots, M \), for each \( j \), there exists uniquely a set \( \{c_{i,j,0}\}_{i,j} \subset k \) such that

\[
\overline{h}_{i,j,0} = \sum_{B \in \text{Mix}_{i,j}} c_{i,j,0} \overline{h}_{0,i,j} \in L(\mathcal{I}_{Q})_{p_1}^{\text{pure}}.
\]

We set \( V'_{i,0} = \{v'_{j,0}\}_{j} \), we see that we can replace \( V_{i,0} \cup \cdots \cup V_{M,0} \) with \( V'_{i,0} \cup \cdots \cup V'_{M,0} \) in the assertions of \( \text{Lemma 3.1.2.1 in Part I} \).

(Existence) By induction hypothesis, we may replace \( V_{i,0} \cup \cdots \cup V_{M,0} \) in the assertions of \( \text{Lemma 3.1.2.1 in Part I} \). Expressing \( h_{i,j,0} \) as a degree \( p^e \) homogeneous polynomial in terms of \( F^e(V_{i,0}) \cup \cdots \cup F^e(V_{M,0}) \cup F^e(V_{i-1,0}) \cup F^e(V_{i,0}) \), we see that there exists \( \{c_{i,j,0}\}_{i,j} \subset k \) such that

\[
\overline{h}_{i,j,0} - \sum_{B \in \text{Mix}_{i,j}} c_{i,j,0} \overline{h}_{0,i,j} \in L(\mathcal{I}_{Q})_{p_1}^{\text{pure}}.
\]

We also have \( L(\mathcal{I}_{Q}) = k[\bigcup_{i=1}^{M} F^e(V'_{i,0})] \).

In fact, we prove below conclusion (c), claiming the existence and uniqueness of such \( \{c_{i,j,0}\} \subset k \) as described above, showing simultaneously by induction on \( i \) that we can replace \( V_{i,0} \cup \cdots \cup V_{M,0} \) with \( V'_{i,0} \cup \cdots \cup V'_{M,0} \) in the assertions of \( \text{Lemma 3.1.2.1 in Part I} \).

We remark that the set

\[
\left\{ F^{e_\alpha} \left( \overline{h}_{0,i,j} - \sum_{B \in \text{Mix}_{i,j}} c_{i,j,0} \overline{h}_{0,i,j} \right) \right\}_{\alpha=1, \ldots, i, \beta=1, \ldots, p^\alpha_{\text{pure}(Q)}, i, j, p^\beta_{\text{pure}(Q)}} \subset L(\mathcal{I}_{Q})_{p_1}^{\text{pure}}.
\]

is linearly independent, since

\[
\left\{ \overline{h}_{0,i,j} : B = (b_{ij}) \text{, } \|B\| = p^\alpha, \text{ and } b_{ij} = 0 \text{ if } e_i > e_j \right\}
\]

is linearly independent (cf. conclusion (a) above), and that its cardinality \( \sum_{\alpha=1}^{i} \left( p^\alpha_{\text{pure}(Q)} - 1 \right) \) is equal to \( p^\alpha_{\text{pure}(Q)} \). Therefore, we conclude that the above set forms a basis of \( L(\mathcal{I}_{Q})_{p_1}^{\text{pure}} \).
(Uniqueness) Suppose there exists another set \( \{ c'_{ijB,O} \}_{B \in \text{Mix}_{kx}} \subset k \) such that
\[
\sum_{B \in \text{Mix}_{kx}} c_{ijB,O} H^g_Q = 0
\]
Then
\[
\sum_{B \in \text{Mix}_{kx}} c_{ijB,O} H^g_Q - \sum_{B \in \text{Mix}_{kx}} c'_{ijB,O} H^g_Q = \sum_{B \in \text{Mix}_{kx}} (c_{ijB,O} - c'_{ijB,O}) H^g_Q \in L(I_Q)^{\text{pure}}.
\]
From the conclusion at the end of the argument for (Existence) it follows that there exists
\[
\left\{ \gamma_{ab} \right\}_{a=1, \ldots, i, b=1, \ldots, j} \subset k
\]
such that
\[
\sum_{a=1, \ldots, i, b=1, \ldots, j} \gamma_{ab} F^{e_{i,b}} \left( h_{a,b,0} - \sum_{B \in \text{Mix}_{kx}} c_{a,b,0} H^g_Q \right).
\]
Again since \( \left( H^g_Q : B = (b_{ij}), ||B|| = p^z \right) \) is linearly independent, we conclude that \( \gamma_{ab} = 0 \) for all \( B \in \text{Mix}_{kx} \).

This finishes the proof of the uniqueness.

Now take \( v_{i,j,Q}' \in G_{i,j} \) such that \( F^{e_{i,j}}(v_{i,j,Q}') = h_{i,j,Q} - \sum_{B \in \text{Mix}_{kx}} c_{ijB,O} H^g_Q \).

Setting \( \{ v_{i,j,Q}' \} = \{ v_{i,j,Q} \} \), we see that we can replace \( V_{1,0} \sqcup \cdots \sqcup V_{M,Q} \) with \( V_{1,0} \sqcup \cdots \sqcup V_{M,Q} \) in the assertions of Lemma 3.1.2.1 in Part I.

This completes the proof for conclusion (c) by induction on \( i \).

**Step 4.** *Finishing argument to check condition (4).*

In order to check condition (4), it suffices to show that there exists
\[
\left\{ g_{ijB} \right\}_{B \in \text{Mix}_{kx}} \subset R_f
\]
such that
\[
g_{ijB}(Q) = c_{ijB,Q} \quad \forall Q \in C \cap \text{Supp}(I) \cap \text{m-Spec } R.
\]

Fix a regular system of parameters \((x_1, \ldots, x_d)\) at \( P \). By shrinking \( U_P \) if necessary, we may assume that \((x_1, \ldots, x_d)\) is a regular system of parameters over \( U_P \), i.e., \((x_1 - x_1(Q), \ldots, x_d - x_d(Q))\) is a regular system of parameters at \( Q \) for any \( Q \in U_P \cap \text{m-Spec } R \).

Now we analyze the condition of \( h_{i,j,Q} - \sum_{B \in \text{Mix}_{kx}} c_{ijB,Q} H^g_Q \) being pure, i.e.,
\[
(\forall) \quad h_{i,j,Q} - \sum_{B \in \text{Mix}_{kx}} c_{ijB,Q} H^g_Q \in L(I_Q)^{\text{pure}}.
\]

This happens if and only if, when we compute the power series expansions of \( h_{i,j} \) and \( \sum_{B \in \text{Mix}_{kx}} c_{ijB,Q} H^g_Q \) with respect to the regular system of parameters \((x_1 - x_1(Q), \ldots, x_d - x_d(Q))\) and when we compare the degree \( p^z \) terms, their mixed parts coincide (even though their pure parts may well not coincide). Since the coefficients of (the mixed parts of) the power series can be computed using the partial derivatives with respect to \( X = (x_1, \ldots, x_d) \), we conclude that condition (\( \forall \)) is equivalent to the following linear equation
\[
(\forall') \quad \left[ \partial_X H^g_Q \right]_{B \in \text{Mix}_{kx}} \left[ c_{ijB,Q} \right]_{B \in \text{Mix}_{kx}} = \left[ \partial_X h_{i,j}(Q) \right]_{c \in \text{Mix}_{kx}},
\]
where
\[
\text{Mix}_{kx} = \{ I = (i_1, \ldots, i_d) : |I| = p^z, i_1 \neq p^z \forall i = 1, \ldots, d \}.
\]
and where
\[
\begin{align*}
\left[ \partial_X H^R(Q) \right]_{1 \leq j \leq \# \text{Mix}_{X^I}} & \quad \text{is a matrix of size } (\# \text{Mix}_{X^I}) \times (\# \text{Mix}_{X^I}), \\
\left[ c_{ij} R, Q \right]_{1 \leq j \leq \# \text{Mix}_{X^I}} & \quad \text{is a matrix of size } (\# \text{Mix}_{X^I}) \times 1, \text{ and} \\
\left[ \partial_X h_{ij}(Q) \right]_{1 \leq j \leq \# \text{Mix}_{X^I}} & \quad \text{is a matrix of size } (\# \text{Mix}_{X^I}) \times 1.
\end{align*}
\]

Therefore, there exists a subset \( S \subseteq \text{Mix}_{X^I} \) (Note that actually the matrix \( \text{det} \left( \partial_X h_{ij}(P) \right) \in k^{\times} \)). By shrinking \( U_F \) if necessary, we may assume
\[
\text{rank} \left( \partial_X H^R(P) \right)_{1 \leq j \leq \# \text{Mix}_{X^I}} = \# \text{Mix}_{X^I}.
\]

Then the solution \( \left[ c_{ij} R, P \right]_{1 \leq j \leq \# \text{Mix}_{X^I}} \) uniquely exists (cf. conclusion (c)), we conclude that the coefficient matrix of the linear equation has full rank, i.e.,
\[
\text{rank} \left( \partial_X H^R(P) \right)_{1 \leq j \leq \# \text{Mix}_{X^I}} = \# \text{Mix}_{X^I}.
\]

Therefore, there exists a subset \( S \subseteq \text{Mix}_{X^I} \) with \( \# S = \# \text{Mix}_{X^I} \) such that the corresponding minor has a nonzero determinant, i.e.,
\[
\text{det} \left( \partial_X H^R(P) \right)_{1 \leq j \leq \# \text{Mix}_{X^I}} \in k^{\times}.
\]

Then the solution \( \left[ c_{ij} R, P \right]_{1 \leq j \leq \# \text{Mix}_{X^I}} \) can be expressed as follows
\[
\left[ c_{ij} R, P \right]_{1 \leq j \leq \# \text{Mix}_{X^I}} = \left( \left[ \partial_X H^R(P) \right]_{1 \leq j \leq \# \text{Mix}_{X^I}} \right)^{-1} \left[ \partial_X h_{ij}(P) \right]_{1 \leq j \leq \# \text{Mix}_{X^I}}.
\]

(Note that actually the matrix \( \left[ c_{ij} R, P \right]_{1 \leq j \leq \# \text{Mix}_{X^I}} \) as well as the matrix \( \left[ \partial_X h_{ij}(P) \right]_{1 \leq j \leq \# \text{Mix}_{X^I}} \) is a zero matrix.) By shrinking \( U_F \) if necessary, we may assume
\[
\text{det} \left( \partial_X H^R \right)_{1 \leq j \leq \# \text{Mix}_{X^I}} \in (R_F)^{\times}
\]

and hence that
\[
\text{det} \left( \partial_X H^R(Q) \right)_{1 \leq j \leq \# \text{Mix}_{X^I}} \in k^{\times} \quad \forall Q \in C \cap \text{Supp}(f) \cap \text{m- Spec } R.
\]

Then the solution \( \left[ c_{ij} R, Q \right]_{1 \leq j \leq \# \text{Mix}_{X^I}} \) for (\( \triangle \)) can be expressed as follows
\[
\left[ c_{ij} R, Q \right]_{1 \leq j \leq \# \text{Mix}_{X^I}} = \left( \left[ \partial_X H^R(Q) \right]_{1 \leq j \leq \# \text{Mix}_{X^I}} \right)^{-1} \left[ \partial_X h_{ij}(Q) \right]_{1 \leq j \leq \# \text{Mix}_{X^I}}.
\]

It follows immediately from this that, if we define the set \( \{ g_{ij} R \}_{1 \leq j \leq \# \text{Mix}_{X^I}} \) by the formula
\[
\left[ g_{ij} R \right]_{1 \leq j \leq \# \text{Mix}_{X^I}} = \left( \left[ \partial_X H^R \right]_{1 \leq j \leq \# \text{Mix}_{X^I}} \right)^{-1} \left[ \partial_X h_{ij} \right]_{1 \leq j \leq \# \text{Mix}_{X^I}},
\]

then it satisfies the desired condition
\[
g_{ij} R(Q) = c_{ij} R Q \quad \forall Q \in C \cap \text{Supp}(f) \cap \text{m- Spec } R.
\]

Finally, by shrinking \( U_F \) if necessary so that the above argument is valid for any element \( h_{ij} \), taken from the given leading generator system \( \Xi \), we see that condition (4) is satisfied.

This completes the proof of Proposition 1.3.3.3.
CHAPTER 2

Power series expansion

As in Chapter 1, we denote by $R$ the coordinate ring of an affine open subset Spec $R$ of a nonsingular variety $W$ of dim $W = d$ over an algebraically closed field $k$ of positive characteristic $\text{char}(k) = p$ or of characteristic zero $\text{char}(k) = 0$, where in the latter case we formally set $p = \infty$ (cf. 0.2.3.2.1 and Definition 3.1.1.1 (2) in Part I).

We fix a closed point $P \in W$.

Let $\mathcal{I}_P$ be a $\mathcal{D}$-saturated idealistic filtration over $R_P = O_{W, P}$, the local ring at the closed point, with $m_P$ being its maximal ideal.

Let $\mathcal{H} = \{(h_l, p^e_l)\}_{l=1}^N$ be a leading generator system of $I_P$.

In characteristic zero, the elements in the leading generator system are all concentrated at level 1, i.e., $e_1 = 0$ and $p^e_l = 1$ for $l = 1, \ldots, N$ (cf. Chapter 3 in Part I). This implies by definition of a leading generator system that the set of the elements $H = (h_l; l = 1, \ldots, N)$ forms (a part of) a regular system of parameters $(x_1, \ldots, x_d)$. (Say $h_l = x_l$ for $l = 1, \ldots, N$.) In positive characteristic, this is no longer the case. However, we can still regard the notion of a leading generator system as a generalization of the notion of a regular system of parameters, and we may expect some similar properties between the two notions.

Now any element $f \in R_P$ (or more generally any element $f \in \hat{R}_P$) can be expressed as a power series with respect to the regular system of parameters and hence with respect to the leading generator system as above in characteristic zero. That is to say, we can write

$$f = \sum_{I \in \mathbb{Z}_{\geq 0}^d} c_I X^I = \sum_{B \in \mathbb{Z}_{\geq 0}^N} a_B H^B$$

where $c_I \in k$ and where $a_B$ is a power series in terms of the remainder $(x_{N+1}, \ldots, x_d)$ of the regular system of parameters.

Chapter 2 is devoted to the study of the power series expansion with respect to the elements in a leading generator system (and its (weakly-)associated regular system of parameters), one of the expected similar properties mentioned above, which is valid both in characteristic zero and in positive characteristic.

§2.1. Existence and uniqueness.

2.1.1. Setting for the power series expansion. First we describe the setting for Chapter 2, which is slightly more general than just dealing with a leading generator system. Actually, until we reach §2.2, our argument does not involve the notion of an idealistic filtration.

Let $\mathcal{H} = \{h_1, \ldots, h_N\} \subset R_P$ be a subset consisting of $N$ elements, and $0 \leq e_1 \leq \cdots \leq e_N$ nonnegative integers attached to these elements, satisfying the following conditions (cf. 4.1.1 in Part I):

(i) $h_I \in m_P^{v'_I}$ and $\overline{h_I} = (h_I \mod m_P^{v'_I}) = v'_I$ with $v_I \in m_P/m_P^2$ for $I = 1, \ldots, N$,

(ii) $\{v_I; l = 1, \ldots, N\} \subset m_P/m_P^2$ consists of $N$-distinct and $k$-linearly independent elements in the $k$-vector space $m_P/m_P^2$.

We also take a regular system of parameters $(x_1, \ldots, x_d)$ such that

$$v_I = \overline{x_I} = (x_I \mod m_P^2)$$

for $I = 1, \ldots, N$. 24
We say \((x_1, \ldots, x_d)\) is associated to \(H = (h_1, \ldots, h_N)\) if the above condition (asc) is satisfied.

2.1.2. Existence and uniqueness of the power series expansion.

Lemma 2.1.2.1. Let the setting be as described in 2.1.1. Then any element \(f \in \mathcal{R}_P\) has a power series expansion, with respect to \(H = (h_1, \ldots, h_N)\) and its associated regular system of parameters \((x_1, \ldots, x_d)\), of the form

\[
(\star) \quad f = \sum_{B \in \mathbb{Z}_{d,0}} a_B H^B \\
\text{where} \ a_B = \sum_{K \in \mathbb{Z}_{d,0}} b_{B,K} X^K,
\]

with \(b_{B,K}\) being a power series in terms of the remainder \((x_{N+1}, \ldots, x_d)\) of the regular system of parameters, and with \(K = (k_1, \ldots, k_d)\) varying in the range satisfying the condition

\[
0 \leq k_i \leq p^n - 1 \quad \text{for} \quad i = 1, \ldots, N \quad \text{and} \quad k_i = 0 \quad \text{for} \quad i = N + 1, \ldots, d.
\]

Moreover, the power series expansion of the form \((\star)\) is unique.

Proof. (Existence) We construct a sequence \(\{f_i\}_{i \in \mathbb{Z}_{d,0}} \subseteq \mathcal{R}_P\) in the following manner.

Case 1. Construction of \(f_0\).

In this case, choose \(f_0 = a_{0,0} \in k\) such that

(i) \(f - f_0 \in \mathfrak{m}_P^{r+1}\),

(ii) \(f_0 = \sum_{|B| = 0} a_{B,0} H^B\).

Case 2. Construction of \(f_{r+1}\) assuming that of \(f_r\).

Suppose inductively that we have constructed \(f_r\) satisfying the following conditions:

(i) \(f - f_r \in \mathfrak{m}_P^{r+1}\),

(ii) \(f_r = \sum_{|B| \leq r} a_{B,r} H^B\) where \(a_{B,r} = \sum b_{B,K,r} X^K\)

with \(b_{B,K,r}\) being a polynomial in \((x_{N+1}, \ldots, x_d)\), i.e.,

\[
b_{B,K,r} = \sum_{|L| = |K| - |r|} c_{B,K,L} X^L
\]

where \(c_{B,K,L} \in k\) and where \(J = (j_1, \ldots, j_d)\) with \(j_i = 0\) for \(i = 1, \ldots, N\), and \(K\) varying in the range specified above, and satisfying the condition \(||B|| + |K| + |J| \leq r\).

Now express \(f - f_r = \sum c_{i,r} X^i\) with \(c_{i,r} \in k\) as a power series expansion in terms of the regular system of parameters \(X = (x_1, \ldots, x_d)\).

Given \(I = (i_1, \ldots, i_d)\) with \(|I| = r + 1\), determine

\[
\begin{align*}
B &= (b_1, \ldots, b_N), \\
K &= (k_1, \ldots, k_N, 0, \ldots, 0) \in (\mathbb{Z}_{d,0})^d, \\
J &= (0, \ldots, 0, j_{N+1}, \ldots, j_d) \in (\mathbb{Z}_{d,0})^d
\end{align*}
\]

by the formulas below

\[
\begin{align*}
i_l &= b_l p^n + k_l \quad \text{with} \quad b_l \in \mathbb{Z}_{d,0} \quad \text{and} \quad 0 \leq k_l \leq p^n - 1 \quad \text{for} \quad l = 1, \ldots, N, \\
i_l &= j_l \quad \text{for} \quad l = N + 1, \ldots, d.
\end{align*}
\]

Then it is straightforward to see, after renaming \(c_{i,r}\) as \(c_{B,K,J}\), that the following equality holds

\[
\sum_{|I| = r+1} c_{i,r} X^i = \sum_{|B| = |K| + |J| = r+1} c_{B,K,J} X^J X^K H^B \mod \mathfrak{m}_P^{r+2}.
\]

Set

\[
\begin{align*}
b_{B,K,r+1} &= \sum_{|B| = |K| + |J| = r+1} c_{B,K,J} X^J \\
a_{B,r+1} &= \sum_{|B| = r+1} b_{B,K,r} X^K \\
f_{r+1} &= \sum_{|B| = r+1} a_{B,r+1} H^B
\end{align*}
\]
Then \( f_{r+1} \) clearly satisfies conditions (i) and (ii). This finishes the inductive construction of the sequence \( \{f_r\}_{r \in \mathbb{Z}_0} \subset R_P \).

Now set

\[
\begin{align*}
  b_{R,K} &= \lim_{r \to \infty} b_{R,K,r} = \sum_{J} c_{R,K,J}X^J, \\
a_{R} &= \lim_{r \to \infty} a_{R,r} = \sum b_{R,K}X^K,
\end{align*}
\]

where each of the above limits exists by condition (ii).

Then condition (i) implies

\[
f = \lim_{r \to \infty} f_r = \lim_{r \to \infty} \sum_{|B| \leq r} a_B H^B = \sum a_B H^B,
\]

proving the existence of a power series expansion of the form (\( \ast \)).

(Uniqueness) In order to show the uniqueness of the power series expansion of the form (\( \ast \)), we have only to verify

\[
0 = \sum_{B \in (\mathbb{Z}_0)^d} a_B H^B \text{ of the form (} \ast \text{) } \iff a_B = 0 \quad \forall B \in (\mathbb{Z}_0)^N.
\]

As the implication (\( \iff \)) is obvious, we show the opposite implication (\( \Leftarrow \)) in what follows.

Suppose \( 0 = \sum_{B \in (\mathbb{Z}_0)^d} a_B H^B \).

Assume that there exists \( B \in (\mathbb{Z}_0)^N \) such that \( a_B \neq 0 \).

Set \( s = \min \{\text{ord}(a_B H^B) : a_B \neq 0\} \).

Write

\[
a_B = \sum_{K \in (\mathbb{Z}_0)^d} b_{R,K} X^K \text{ and } b_{R,K} = \sum_{J} c_{R,K,J}X^J \text{ with } c_{R,K,J} \in \mathbb{C},
\]

where \( K = (k_1, \ldots, k_d) \) varies in the range satisfying the condition

\[
0 \leq k_l \leq \rho^s - 1 \text{ for } l = 1, \ldots, N \text{ and } k_l = 0 \text{ for } l = N + 1, \ldots, d,
\]

and where \( J = (j_1, \ldots, j_d) \) varies in the range satisfying the condition

\[
j_l = 0 \quad \text{for } l = 1, \ldots, N.
\]

Then we have

\[
0 = \sum_B a_B H^B = \sum_B \sum_K \left( \sum_{J} c_{R,K,J} X^J \right) H^B
= \sum_{|B|+|J|=s} c_{R,K,J} X^J \left( \prod_{1 \leq i \leq N} \rho^{j_i/b_i} \right) \mod \mathbb{m}_{P,s}.
\]

On the other hand, we observe that the set of all the monomials of degree \( s \) \( \{X^J X^K \left( \prod_{1 \leq i \leq N} \rho^{j_i/b_i} \right) \}_{|B|+|K|+|J|=s} = \{X^J \}_{|J|=s} \) obviously forms a basis of the vector space \( \mathbb{m}_{P,s} / \mathbb{m}_{P,s}^+ \), and that \( c_{R,K,J} \neq 0 \) for some \( B, K, J \) with \(|B| + |K| + |J| = s\) by the assumption and by the choice of \( s \).

This is a contradiction!

Therefore, we conclude that \( a_B = 0 \quad \forall B \in (\mathbb{Z}_0)^N \).

This finishes the proof of the implication (\( \Leftarrow \)), and hence the proof of the uniqueness of the power series expansion of the form (\( \ast \)).

This completes the proof of Lemma 2.1.2.1

Remark 2.1.2.2. (1) It follows immediately from the argument to show the existence and uniqueness of the power series expansion \( f = \sum a_B H^B \) of the form (\( \ast \)) that

\[
\text{ord}(f) = \min \{ \text{ord}(a_B H^B) \} = \min \{ \text{ord}(a_B) + |B| \}
\]
and hence that
\[ \text{ord}(a_B) \geq \text{ord}(f) - \|B\| \quad \forall B \in (\mathbb{Z}_{\geq 0})^N. \]

(2) In the setting \ref{setting:2.1.1} we defined the notion of a regular system associated to \( H = (h_1, \ldots, h_N) \). We say that a regular system of parameters \((x_1, \ldots, x_d)\) is weakly-associated to \( H = (h_1, \ldots, h_N) \), if the following condition holds:
\[
\det \left[ a_{i} (h^{e_i}_j - a_j) \right]_{i=1, \ldots, L} \in R_p^e \quad \text{for} \quad e = e_1, \ldots, e_N \quad \text{where} \quad L_e = \# \{ l ; e_l \leq e \}.
\]

All the assertions of Lemma \ref{lemma:2.1.2.1} hold, even if we only require a regular system of parameters \((x_1, \ldots, x_d)\) to be weakly-associated to \( H \), instead of associated to \( H \).

\section{2.2. Formal coefficient lemma.}

\subsection{2.2.1. Setting for the formal coefficient lemma.}

As we can see from the description of the setting \ref{setting:2.1.1} our discussion on the power series expansion of the form \((*)\) (cf. Lemma \ref{lemma:2.1.2.1}) so far does not involve the notion of an idealistic filtration. However, the most interesting and important result of Chapter 2 is obtained as Lemma \ref{lemma:2.2.2.1} below, which we call the formal coefficient lemma, when we get the notion of an idealistic filtration involved and impose an extra condition related to it as follows:

Let \( \mathcal{H} = \{ h_1, \ldots, h_N \} \subset R_p \) be a subset consisting of \( N \) elements, and \( 0 \leq e_1 \leq \cdots \leq e_N \) nonnegative integers attached to these elements, satisfying conditions (i) and (ii), as described in the setting \ref{setting:2.1.1} Let \( X = (x_1, \ldots, x_d) \) be a regular system of parameters associated to \( H = (h_1, \ldots, h_N) \) with \( h_l = x_l^{e_l} \mod m_{p^{l+1}} \) for \( l = 1, \ldots, N \).

Let \( \mathcal{I}_p \) be a \( \mathcal{I} \)-saturated idealistic filtration over \( R_p \).

We impose the following extra condition

(iii) \( (h_l, p^{e_l}) \in \mathcal{I}_p \) for \( l = 1, \ldots, N \).

\subsection{2.2.2. Statement of the formal coefficient lemma and its proof.}

Now our assertion is that, under the setting of \ref{setting:2.2.1} and given an element in the idealistic filtration, the coefficients of the power series expansion of the form \((*)\), with “appropriate” levels attached, belongs to (the completion of) the idealistic filtration. We formulate this assertion as the following formal coefficient lemma.

\begin{lemma}

Let the setting be as described in \ref{setting:2.2.1}. Let \( \mathcal{I}_p \) be the completion of the idealistic filtration \( \mathcal{I}_p \) \( \text{(cf. \S 2.4 in Part I)} \).

Take an element \((f, a) \in \mathcal{I}_p \).

Let \( f = \sum_{B \in (\mathbb{Z}_{\geq 0})^N} a_B H^B \) be the power series expansion of the form \((*)\) \( \text{(cf. Lemma} \ref{lemma:2.1.2.7} \text{)} \).

Then we have
\[
(a_B, a - \|B\|) \in \overline{\mathcal{I}_p} \quad \forall B \in (\mathbb{Z}_{\geq 0})^N.
\]

\end{lemma}

\begin{proof}

We will derive a contradiction assuming
\[
(a_B, a - \|B\|) \notin \overline{\mathcal{I}_p} \quad \text{for some} \quad B \in (\mathbb{Z}_{\geq 0})^N.
\]

Note that, under the assumption, there should exist \( B \in (\mathbb{Z}_{\geq 0})^N \) with \( B \neq \emptyset \) such that \( (a_B, a - \|B\|) \notin \overline{\mathcal{I}_p} \). (In fact, suppose \( (a_B, a - \|B\|) \notin \overline{\mathcal{I}_p} \quad \forall B \neq \emptyset \). Then the equality \( a_B = f - \sum_{B \neq 0} a_B H^B \) and the inclusions \((f, a) \in \overline{\mathcal{I}_p} \) and \((a_B H^B, a) \in \overline{\mathcal{I}_p} \quad \forall B \neq \emptyset \), would imply \((a_B, a) = (a_B, a - \|[B]\|) \notin \overline{\mathcal{I}_p} \), which is against the assumption.)

We introduce the following notations:
\begin{align*}
I_B &= \|B\| + \sup \{ n \in \mathbb{Z}_{\geq 0} : a_B \in (\mathcal{I}_p)_{n-\|B\|} + \overline{m}_p^{\|B\|} \} \quad \text{for } B \in (\mathbb{Z}_{\geq 0})^N \setminus \{ \emptyset \}, \\
L &= \min_{B \in (\mathbb{Z}_{\geq 0})^N, B \neq \emptyset} \{ I_B \}, \\
\Gamma_B &= (\mathcal{I}_p)_{n-\|B\|} + \overline{m}_p^{\|B\|} \quad \text{for } B \in (\mathbb{Z}_{\geq 0})^N, \\
L_B &= \max \{ B + K : a_B \in \Gamma_B + \sum_{k=1}^{N} \overline{m}_p^{\|B+K\|} H^M \} \quad \text{for } B \in (\mathbb{Z}_{\geq 0})^N \setminus \{ \emptyset \}, I_B = l, \\
B_o &= \max_{B \in (\mathbb{Z}_{\geq 0})^N, B \neq \emptyset, l_B = l} \{ B \}, \\
\Lambda_B &= \Gamma_B + \sum_{L \leq B + M} \overline{m}_p^{\|B+M\|} H^M \quad \text{for } B \in (\mathbb{Z}_{\geq 0})^N.
\end{align*}

Note that \( l < \infty \) by the assumption \( a_B \notin (\mathcal{I}_p)_{n-\|B\|} \) for some \( B \neq \emptyset \). Note that the maximum of \( B + K \), the minimum of \( l_B \), and the maximum of \( B \) are all taken with respect to the lexicographical order on \((\mathbb{Z}_{\geq 0})^N\). Note that, if \( l_B = l \), then \( |B| < a \). This guarantees the existence of the maximum of \( B \in (\mathbb{Z}_{\geq 0})^N \) with \( B \neq \emptyset, l_B = l, L_B = L \). We remark that, when \( r \leq 0 \), we understand by convention \( \overline{m}_p^r \) represents \( R_p \).

We claim, for \( B, K \in (\mathbb{Z}_{\geq 0})^N \),

\begin{enumerate}
  \item \( H^K \Lambda_{B+K} \subset \Lambda_B \).
  \item \( \partial_K(\Lambda_B) \subset \Lambda_{B+K} \).
\end{enumerate}

(We remark that we identify \([K]\), for \( K = (k_1, \ldots, k_N) \in (\mathbb{Z}_{\geq 0})^N \), with \((p^{k_1}, k_1, \ldots, p^{k_N}, k_N, 0, \ldots, 0) \in (\mathbb{Z}_{\geq 0})^{2N}\), and hence that we understand \( \partial_K \) denotes \( \partial_{X^k} = \partial_{X^{k_1} \ldots X^{k_N}} \) in claim (ii).)

In fact, since \((H^K, [[K]]) \in \mathcal{I}_p\) and since \( H^K \in \overline{m}_p^{\|K\|} \), we see

\begin{align*}
H^K \Lambda_{B+K} &= H^K \left( \Gamma_{B+K} + \sum_{L \leq B+K+M} \overline{m}_p^{\|B+K+M\|} H^M \right) \\
&= H^K \left( \mathcal{I}_p)_{n-\|B+K\|} + \overline{m}_p^{\|B+K\|+1} + \sum_{L \leq B+K+M} \overline{m}_p^{\|B+K+M\|} H^M \right) \\
&\subseteq (\mathcal{I}_p)_{n-\|B\|} + \overline{m}_p^{\|B\|+1} + \sum_{L \leq B+M} \overline{m}_p^{\|B+M\|} H^M \quad \text{(by replacing old } M + K \text{ with new } M) \\
&= \Gamma_B + \sum_{L \leq B+M} \overline{m}_p^{\|B+M\|} H^M = \Lambda_B,
\end{align*}

checking claim (i).
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In order to see claim (ii), observe

- \( \partial_{[K]} \left( \tilde{p}_{[\pi_{\Lambda, K}]} \right) \subset \tilde{p}_{[\pi_{\Lambda, K}]} H^M \), since \( \tilde{p}_p \) is \( \Xi \)-saturated,
- \( \partial_{[K]} \left( \tilde{m}_{p}^{l-|[B]|+1} \right) \subset \tilde{m}_{p}^{l-|[B]|+1} H^M \), and
  \[ \partial_{[K]}^{-1} \left( \tilde{m}_{p}^{l-|[B]|+1} H^M \right) \subset \tilde{m}_{p}^{l-|[B]|+1} H^M \] for \( I \leq [K] \),
- \( \partial_{J}(H^M) \subset \left( \binom{M+1}{J} \right) H^{M-J} + \tilde{m}_{p}^{l-|[B]|+1} \), and
  \[ \left( \binom{M+1}{J} \right) = 0 \] unless \( J = [J] \) for some \( J \in (\mathbb{Z}_{\geq 0})^N \).

Using these observations, we compute

\[
\partial_{[K]}(\Lambda_B) = \partial_{[K]} \left( \Gamma_B + \sum_{L < M} \tilde{m}_{p}^{l-|[B]+M]} H^M \right)
\]
\[
= \partial_{[K]} \left( \tilde{p}_{[\pi_{\Lambda, K}]} \tilde{m}_{p}^{l-|[B]|+1} + \sum_{L < M} \tilde{m}_{p}^{l-|[B]|+1} H^M \right)
\]
\[
= \partial_{[K]} \left( \tilde{p}_{[\pi_{\Lambda, K}]} + \partial_{[K]} \left( \tilde{m}_{p}^{l-|[B]|+1} H^M \right) \right)
\]
\[
= \partial_{[K]} \left( \tilde{p}_{[\pi_{\Lambda, K}]} \right) + \partial_{[K]} \left( \tilde{m}_{p}^{l-|[B]|+1} H^M \right)
\]
\[
+ \sum_{L < M} \left[ \sum_{I \leq [K]} \partial_{[K]-I} \left( \tilde{m}_{p}^{l-|[B]+M]} \right) \partial_{I}(H^M) \right]
\]

(by the generalized product rule (cf. Lemma 1.2.1.2 (3) in Part I))
\[
= \partial_{[K]} \left( \tilde{p}_{[\pi_{\Lambda, K}]} \right) + \partial_{[K]} \left( \tilde{m}_{p}^{l-|[B]|+1} H^M \right)
\]
\[
+ \sum_{L < M} \left[ \sum_{J \leq [K]} \sum_{I \leq [K]} \partial_{[K]-I} \left( \tilde{m}_{p}^{l-|[B]+M]} \right) \partial_{I}(H^M) \right]
\]
\[
\subset \tilde{p}_{[\pi_{\Lambda, K}]} + \tilde{m}_{p}^{l-|[B]+K]} + \sum_{L < M} \left[ \sum_{J \leq [K]} \sum_{J \leq [K]} \tilde{m}_{p}^{l-|[B]+M+K]} H^{M-J} \right]
\]
\[
= \Gamma_{B+K} + \sum_{L < M} \tilde{m}_{p}^{l-|[B]+K]} H^{M-J}
\]
\[
\subset \Gamma_{B+K} + \sum_{L < M} \tilde{m}_{p}^{l-|[B]+K]} H^{M-J} = \Lambda_{B+K}
\]

(by replacing old \( M - J \) with new \( M \),

checking claim (ii).

Now by definition, for each \( B \in (\mathbb{Z}_{\geq 0})^N \) with \( B \neq \emptyset, l_B = l, L_B = L \), we can choose \( b_B \in \tilde{m}_{p}^{l-|[L]} \) such that \( a_B - b_B H^{l-B} \in \Lambda_B \). For each \( B \in (\mathbb{Z}_{\geq 0})^N \) with \( B \neq \emptyset \) but \( l_B \neq l \) or \( L_B \neq L \), we set \( b_B = 0 \) and have \( a_B - b_B H^{l-B} \in \Lambda_B \).

Therefore, we have, for each \( B \in (\mathbb{Z}_{\geq 0})^N \) with \( B \neq \emptyset \),

\[
a_B - b_B H^{l-B} \in \Lambda_B
\]

and hence by claim (i) (with \( B, K \in (\mathbb{Z}_{\geq 0})^N \) there being equal to \( \emptyset, B \) below, respectively)

\[
(a_B - b_B H^{l-B}) H^B \in \Lambda_{\emptyset}.
\]
Now we compute (with the symbol “≡” denoting the equality modulo $\Lambda_{B_o}$):

$$\partial_{[B_o]}f = \partial_{[B_o]} \left( \sum_{B \neq \emptyset} a_B H^B \right) = \partial_{[B_o]} \left( \sum_{B \neq \emptyset} a_B H^B \right) = \partial_{[B_o]} \left( \sum_{B \neq \emptyset} b_B H^L \right)$$

(since $\sum_{B \neq \emptyset} a_B H^B - \sum_{B \neq \emptyset} b_B H^L \in \Lambda_\emptyset$ and by claim (ii))

$$= \sum_{B \neq \emptyset, j_B = L} \partial_{[B_o]} \left( b_B H^{L-B} H^B \right)$$

$$= \sum_{B \neq \emptyset, j_B = L} \left[ \sum_{I \subseteq [B_o]} \partial_I \left( b_B H^{L-B} \right) \partial_I \left( H^B \right) \right]$$

(by the generalized product rule)

$$\equiv \sum_{B \neq \emptyset, j_B = L} b_B H^{L-B} \partial_{[B_o]} \left( H^B \right)$$

(since for $K \neq \emptyset$ we have $\partial_K \left( b_B H^{L-B} \right) = \partial_K \left( \left( a_B - b_B H^{L-B} \right) \right) \in \Lambda_{B+K}$ and hence $\partial_K \left( b_B H^{L-B} \right) \partial_{[B_o]^{-1}} \left( H^B \right) \in \Lambda_{B+K} \partial_{[B_o]^{-1}} \left( H^B \right) \subset \Lambda_{B_o}$)

$$= b_B H^{L-B} \quad \text{(by the maximality of $B_o$).}$$

Note that the inclusion $\Lambda_{B+K} \partial_{[B_o]^{-1}} \left( H^B \right) \subset \Lambda_{B_o}$ used above is verified as follows:

$$\Lambda_{B+K} \partial_{[B_o]^{-1}} \left( H^B \right)$$

$$= \left( \prod_{L < B+K} + \sum_{L < B+K+M} \hat{m}_p^{-\lfloor L-(B+K+M) \rfloor} H^M \right) \partial_{[B_o]^{-1}} \left( H^B \right)$$

$$= \left( \prod_{L < B+K} + \sum_{L < B+K+M} \hat{m}_p^{-\lfloor L-(B+K+M) \rfloor} H^M \right) \partial_{[B_o]^{-1}} \left( H^B \right)$$

$$\subset \left( \prod_{L < B+K} + \sum_{L < B+K+M} \hat{m}_p^{-\lfloor L-(B+K+M) \rfloor} H^M \right) \left( \left[ B \right] \right) \left( \left[ B_o - K \right] \right) H^{B-(B_o-K)} + \hat{m}_p^{-\lfloor B-(B_o-K) \rfloor}$$

(since $\partial_{[B_o]^{-1}} \left( H^B \right) \in \left( \prod_{L < B+K+M} \right)$ and since $\partial_{[B_o]^{-1}} \left( H^B \right) \in \hat{m}_p^{-\lfloor B-(B_o-K) \rfloor}$

(refer also to the last observation used to see claim (ii))

$$\subset \left( \prod_{L < B+K} + \sum_{L < B+K+M} \hat{m}_p^{-\lfloor L-(B+K+M) \rfloor} H^M \right) \left( \left[ B \right] \right) \left( \left[ B_o - K \right] \right) H^{B-(B_o-K)} + \hat{m}_p^{-\lfloor B-(B_o-K) \rfloor}$$

$$\subset \left( \prod_{L < B+K} + \sum_{L < B+K+M} \hat{m}_p^{-\lfloor L-(B+K+M) \rfloor} H^M \right) \left( \left[ B \right] \right) \left( \left[ B_o - K \right] \right) H^{B-(B_o-K)} + \hat{m}_p^{-\lfloor B-(B_o-K) \rfloor}$$


which contradicts the choice of
ally leads to an explicit construction of the coe
conditions:

\[ \min \{ \eta(0), \eta(1) \} \in \mathbb{Z} \]

which contradicts the choice of \( B \), with \( L_B = L \).

As for the dependence on \( \nu \), we would like to remark that the same idea of the proof by contradiction above actually leads to an explicit construction of the coefficients through differential operators and taking limits. We present such a construction, which is of interest on its own and which

\[ \partial \{ \nu \} \in \mathcal{I}_\nu \subset \mathcal{A}_\nu, \]

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We would like to remark that the same idea of the proof by contradiction above actually leads to an explicit construction of the coefficients through differential operators and taking limits. We present such a construction, which is of interest on its own and which

\[ \partial \{ \nu \} \in \mathcal{I}_\nu \subset \mathcal{A}_\nu, \]
### Case 1. Construction of $g_0$.

Set

$$g_0 = f - a_{O}.$$  

Then we check

1. $(0)_0 \quad g_0 = f - a_{O} = \sum_{B \in \mathbb{Z}^d} a_B H^B \in \mathcal{H},$
2. $(1)_0 \quad (a_{O} + g_0, a) = (f, a) \in \overline{I}_p,$
3. $(2)_0 \quad$ the condition $(2)_0$ is void when $n = 0.$

### Case 2. Construction of $g_n$ assuming that of $g_{n-1}$.

We look at the power series expansion of the form $(\star)$

$$g_{n-1} = \sum_{B \in \mathbb{Z}^d} a_{B,g_{n-1}} H^B.$$  

Set

- $\nu = \text{ord}(g_{n-1})$
- $B_0 = \min \left\{ B ; \text{ord}(a_{B,g_{n-1}}, H^B) = \nu \right\}$
- $\partial_{[B_0]} = \partial_{(x_1^1, x_2^2, \ldots, x_N^N)}$
- $B_o = (b_{o1}, b_{o2}, \ldots, b_{oN})$

Note that $B_o \neq \emptyset,$ which follows from condition $(0)_{n-1}.$

We set

$$g_n = \left(1 - H^{B_o} \partial_{[B_o]} \right) g_{n-1}.$$  

We check conditions $(0)_n, (1)_n$ and $(2)_n$ in the following.

#### Condition $(0)_n$

We compute

$$g_n = \left(1 - H^{B_o} \partial_{[B_o]} \right) g_{n-1} = g_{n-1} - H^{B_o} \partial_{[B_o]} g_{n-1},$$

where

- $g_{n-1} \in \mathcal{H}$ by condition $(0)_{n-1},$ and
- $H^{B_o} \partial_{[B_o]} g_{n-1} \in \mathcal{H}$ since $B_o \neq \emptyset.$

Therefore, we conclude

$$g_n \in \mathcal{H},$$

checking condition $(0)_n.$

#### Condition $(1)_n$

By inducational hypothesis, condition $(1)_{n-1}$ holds, i.e., we have the first inclusion

$$(a_{O} + g_{n-1}, a) \in \overline{I}_p.$$  

Since $\overline{I}_p$ is $\mathfrak{D}$-saturated, so is $\overline{I}_p$ (cf. compatibility of completion with $\mathfrak{D}$-satisfaction, Proposition 2.4.2.1 (2) in Part I). Therefore, the first inclusion implies the second inclusion

$$(\partial_{[B_o]} (a_{O} + g_{n-1}), a - \|B_o\|) \in \overline{I}_p.$$  

The second inclusion combined with the third inclusion below

$$(H^{B_o}, \|B_o\|) \in \overline{I}_p$$  

implies the fourth inclusion

$$(H^{B_o} \partial_{[B_o]} (a_{O} + g_{n-1}), a) \in \overline{I}_p.$$
Therefore, we conclude
\[
(a_0 + g_n, a) = (a_0 + (1 - H^B \partial_{B_0})g_{n-1}, a)
\]
\[
= \left(1 - H^B \partial_{B_0}\right)(a_0 + g_{n-1}) \quad \text{(since } H^B \partial_{B_0} \text{ acts on } a_0 \text{)}
\]
\[
= (a_0 + g_{n-1}) - H^B \partial_{B_0}(a_0 + g_{n-1}) \in \widehat{\mathbb{F}}_p
\]

(11)_n

That is to say, we have
\[
(a_0 + g_n, a) \in \widehat{\mathbb{F}}_p,
\]
checking condition (1)_n.

**Condition (2)_n**

Observe that, for any \( B \in (\mathbb{Z}_{\geq 0})^N \) with ord \( (a_{B,g_0}, H^B) = \nu \), we have
\[
(1 - H^B \partial_{B_0}) \left( a_{B,g_0}, H^B \right) = a_{B,g_0 - 1} H^B - H^B \left( B_{B_0} \right) a_{B,g_0 - 1} H^B + s_B
\]
\[
= \left(1 - B_{B_0}\right) a_{B,g_0 - 1} H^B + r_B
\]

where \( s_B \) and \( r_B \) are elements in \( \widehat{\mathbb{F}}_p \) with \( \text{ord}(s_B) > \nu - ||B_o|| \) and \( \text{ord}(r_B) > \nu \), respectively.

Therefore, we compute
\[
g_n = (1 - H^B \partial_{B_0})g_{n-1} = \left(1 - H^B \partial_{B_0}\right) \left( \sum a_{B,g_0 - 1} H^B \right)
\]
\[
= \left(1 - H^B \partial_{B_0}\right) \left( \sum_{B : \text{ord}(a_{B,g_0 - 1}) = \nu} a_{B,g_0 - 1} H^B + \sum_{B : \text{ord}(a_{B,g_0 - 1}) > \nu} a_{B,g_0 - 1} H^B \right)
\]
\[
= \sum_{B : \text{ord}(a_{B,g_0 - 1}) = \nu} \left(1 - B_{B_0}\right) a_{B,g_0 - 1} H^B + r
\]
\[
= \sum_{B : \text{ord}(a_{B,g_0 - 1}) > \nu, B \succ B_o} \left(1 - B_{B_0}\right) a_{B,g_0 - 1} H^B + r
\]

where \( r \) is an element in \( \widehat{\mathbb{F}}_p \) with \( \text{ord}(r) > \nu \).

From this computation it immediately follows that
\[
\text{ord}(g_n) \geq \nu = \text{ord}(g_{n-1})
\]
and that, if \( \text{ord}(g_n) = \nu = \text{ord}(g_{n-1}) \), then

\[
\text{the 2nd factor in } \eta(g_n) = \min \{B : \text{ord}(a_{B,g_0}, H^B) = \nu\}
\]
\[
> B_0
\]
\[
= \text{the 2nd factor in } \eta(g_{n-1}).
\]

Thus we conclude
\[
\eta(g_{n-1}) < \eta(g_n),
\]
checking condition (2)_n.

This completes the inductive construction of the sequence \( \{g_n\}_{n \in \mathbb{Z}_{\geq 0}} \subset \widehat{\mathbb{F}}_p \) satisfying conditions (0)_n, (1)_n and (2)_n.

This completes the argument in Step 1, showing \( (a_0, a) \in \widehat{\mathbb{F}}_p \).

**Step 2.** We show the statement in the general case, i.e., we show \( (a_B, a - ||B||) \in \widehat{\mathbb{F}}_p \) for any \( B \in (\mathbb{Z}_{\geq 0})^N \).
We will construct a sequence \( \{g_n\}_{n \in \mathbb{Z}_{\geq 0}} \subset \hat{R}_p \) inductively, satisfying the following conditions:

\[
\begin{align*}
(0)_n & \quad (g_n, a) \in \hat{I}_P, \\
(1)_n & \quad (a_{B,f^{-g_n}} - [B]) \in \hat{I}_P \text{ for any } B \in (\mathbb{Z}_{\geq 0})^N, \\
(2)_n & \quad \eta(g_{n-1}) < \eta(g_n).
\end{align*}
\]

The construction of such a sequence is sufficient to prove the statement

\[
(a_B, a - [B]) \in \hat{I}_P \text{ for any } B \in (\mathbb{Z}_{\geq 0})^N.
\]

In fact, since there are only finitely many \( B \)'s with \( \text{ord}(H^B) \leq \nu \) for a fixed \( \nu \in \mathbb{Z}_{\geq 0} \), and since

\[
\{ B : \text{ord}(H^B) \leq \nu \} \supset \{ B : \text{ord}(a_B H^B) = \text{ord}(g) \}
\]

for any \( g \in \hat{R}_p \) with \( \text{ord}(g) = \nu \), we conclude by condition \( (2)_n \) that

\[
\lim_{n \to \infty} \text{ord}(g_n) = \infty \quad \text{and hence} \quad \lim_{n \to \infty} g_n = 0.
\]

This implies by condition \( (1)_n \) that, for any \( B \in (\mathbb{Z}_{\geq 0})^N \),

\[
(a_B, a - [B]) = (a_{B,f^{-g_n}} - [B]) = (\lim_{n \to \infty} (a_{B,f^{-g_n}}), a - [B]) = \lim_{n \to \infty} (a_{B,f^{-g_n}}, a) \in \hat{I}_P,
\]

since \( (\hat{I}_P)_a = (\hat{I}_P)_a \) is complete.

**Case 1. Construction of \( g_0 \).**

Set

\[
g_0 = f.
\]

Then we check

\[
\begin{align*}
(0)_0 & \quad (g_0, a) = (f, a) \in \hat{I}_P \subset \hat{I}_P, \\
(1)_0 & \quad (a_{B,f^{-g_0}} - [B]) = (0, a - [B]) \in \hat{I}_P \text{ for any } B \in (\mathbb{Z}_{\geq 0})^N, \\
(2)_0 & \quad \text{the condition } (2)_n \text{ is void when } n = 0.
\end{align*}
\]
§2.2. FORMAL COEFFICIENT LEMMA.

We look at the power series expansion of the form (\( \star \))

\[ g_{n-1} = \sum_{B \in (\mathbb{Z}_{\geq 0})^N} a_{B, g_{n-1}} H^B. \]

Set

\[
\begin{align*}
\nu &= \text{ord}(g_{n-1}) \\
B_{\nu} &= \min \left\{ B : \text{ord}(a_{B, g_{n-1}} H^B) = \nu \right\} \\
I_n &= \{ b_{n1}, b_{n2}, \ldots, b_{nN} \} \\
\partial_{\{B_{\nu}\}} &= \partial_{X^{b_{n1}}} \partial_{X^{b_{n2}}} \cdots \partial_{X^{b_{nN}}},
\end{align*}
\]

We set

\[ g_n = (1 - H^B a_{\partial_{\{B_{\nu}\}}} H^B) g_{n-1} = g_{n-1} - a_{\partial_{\{B_{\nu}\}}, \partial_{\{B_{\nu}\}}} H^B, \]

where \( a_{\partial_{\{B_{\nu}\}}, \partial_{\{B_{\nu}\}}} \) denotes the operator such that \( a_{\partial_{\{B_{\nu}\}}, \partial_{\{B_{\nu}\}}} g = a_{\partial_{\{B_{\nu}\}}, \partial_{\{B_{\nu}\}}} \) for any \( g \in \hat{\mathbb{R}}_p. \)

We check conditions (0)\(_n\), (1)\(_n\) and (2)\(_n\) in the following.

**Condition (0)\(_n\)**

By inductive hypothesis, condition (0)\(_{n-1}\) holds, i.e., we have the first inclusion

\[ (g_{n-1}, a) \in \hat{\mathbb{I}}_p. \]

Since \( \mathbb{I}_p \) is \( \mathcal{D} \)-saturated, so is \( \hat{\mathbb{I}}_p \) (cf. compatibility of completion with \( \mathcal{D} \)-saturation, Proposition 2.4.2.1 (2) in Part I). Therefore, the first inclusion implies the second inclusion

\[ (\partial_{\{B_{\nu}\}} g_{n-1}, a - [B_{\nu}]) \in \hat{\mathbb{I}}_p. \]

By Step 1 the second inclusion implies the third

\[ (a_{\partial_{\{B_{\nu}\}}, \partial_{\{B_{\nu}\}}} g_{n-1}, a - [B_{\nu}]) \in \hat{\mathbb{I}}_p. \]

The third inclusion combined with the fourth inclusion below

\[ (H^B, [B_{\nu}]) \in \hat{\mathbb{I}}_p \]

implies the fifth inclusion

\[ (a_{\partial_{\{B_{\nu}\}}, \partial_{\{B_{\nu}\}}} H^B, a) \in \hat{\mathbb{I}}_p. \]

Therefore, we conclude

\[ (g_n, a) = (g_{n-1} - a_{\partial_{\{B_{\nu}\}}, \partial_{\{B_{\nu}\}}} H^B, a) \in \hat{\mathbb{I}}_p, \]

checking condition (0)\(_n\).

**Condition (1)\(_n\)**

When \( B \neq B_{\nu} \), we have

\[ a_{B, f - g_{n}} = a_{B, f - g_{n-1}} a_{\partial_{\{B_{\nu}\}}, \partial_{\{B_{\nu}\}}} H^B = a_{B, f - g_{n-1}}. \]

Therefore, by condition (1)\(_{n-1}\), we conclude

\[ (a_{B, f - g_{n}}, a - [B]) \in \hat{\mathbb{I}}_p. \]

When \( B = B_{\nu} \), we have

\[ a_{B_{\nu}, f - g_{n}} = a_{B_{\nu}, f - g_{n-1}} a_{\partial_{\{B_{\nu}\}}, \partial_{\{B_{\nu}\}}} H^B = a_{B_{\nu}, f - g_{n-1}} - a_{\partial_{\{B_{\nu}\}}, \partial_{\{B_{\nu}\}}} H^B. \]

Therefore, by condition (1)\(_{n-1}\) and Step 1, we conclude

\[ (a_{B_{\nu}, f - g_{n}}, a - [B_{\nu}]) \in \hat{\mathbb{I}}_p. \]

This checks condition (1)\(_n\).
Condition (2)$_n$. 

Observe that, for any $B \in (\mathbb{Z}_{\geq 0})^N$ with ord$(a_{B,B_0}H^B) = \nu$, we have

\[
(1 - H^B a_{B_0} \partial_{[B]}) (a_{B,B_0} H^B) = a_{B,B_0} H^B - H^B a_{B_0} \partial_{[B_0]} (a_{B,B_0} H^B - B_0 + s_B) = (1 - \delta_{B,B_0}) a_{B,B_0} H^B + r'_B
\]

where $s_B$ and $r'_B$ are elements in $\mathbb{R}$ with ord$(s_B) > \nu - ||B_0||$ and ord$(r'_B) > \nu$, respectively, and where $\delta_{B,B_0}$ denotes the Kronecker delta.

Therefore, we compute

\[
g_n = (1 - H^B a_{B_0} \partial_{[B_0]}) g_{n-1} = (1 - H^B a_{B_0} \partial_{[B_0]}) \left( \sum a_{B,B_0} H^B \right) = \left( 1 - H^B a_{B_0} \partial_{[B_0]} \right) \left( \sum a_{B,B_0} H^B \right)
\]

where $r'$ is an element in $\mathbb{R}$ with ord$(r') > \nu$.

From this computation it immediately follows that

\[\text{ord}(g_n) \geq \nu = \text{ord}(g_{n-1})\]

and that, if ord$(g_n) = \nu = \text{ord}(g_{n-1})$, then

\[\eta(g_n) = \min \left\{ B ; \text{ord}(a_{B,B_0} H^B) = \nu \right\}\]

> $B_0$

= the 2nd factor in $\eta(g_{n-1})$.

Thus we conclude

\[\eta(g_{n-1}) < \eta(g_n),\]

checking condition (2)$_n$.

This completes the inductive construction of the sequence $\{g_n\}_{n \in \mathbb{Z}_{\geq 0}} \subset \mathbb{R}$ satisfying conditions $(0)$_n, $(1)$_n and (2)$_n$.

This completes the argument in Step 2, showing $(a_B, a - ||B||) \in \mathbb{I}_P$.

This completes the alternative proof of Lemma 2.2.2.1.

Remark 2.2.2.2. (1) The basic strategy of the above proof can be seen in a more transparent way, if we consider the following special case: Suppose that we can take the associated regular system of parameters $(x_1, \ldots, x_N)$ in such a way that $h_l = x_l$ for $l = 1, \ldots, N$.

Given $(f, a) \in \mathbb{I}_P$, with $f = \sum a_B H^B$ being the power series expansion of the form $(\ast)$, we proceed as follows.

Step 1. We compute

\[
\prod_{0 \leq ||B|| < a} (1 - H^B \partial_{[B]}) f = a_0 - \sum_{||B|| \geq a} c_B H^B
\]

for some $c_B \in \mathbb{R}$. (Note that, since the operators $\{1 - H^B \partial_{[B]}\}$ do not commute, the product symbol $\prod_{0 \leq ||B|| < a}$ is understood to align the factors from right to left according to the lexicographical order among $(||B||, B)$'s.)
§2.2. FORMAL COEFFICIENT LEMMA.

Since $\mathbb{I}_P$ is $\mathcal{D}$-saturated, we see $(\prod_{B \in B_{\geq a}} (1 - H^B \partial H^B) f, a) \in \mathbb{I}_P$, while we have obviously $(\sum_{B \in B_{\geq a}} c_B H^B, a) \in \mathbb{I}_P$. Therefore, we conclude

$$(a_0, a) \in \mathbb{I}_P.$$ 

Step 2. For $B \in (\mathbb{Z}_{\geq 0})^N$, set $g = \partial H^B f$. Let $g = \sum_a a_B a H^B$ be the power series expansion of the form $\star$. Observe $a_{0, g} = a_B$. Since $\mathbb{I}_P$ is $\mathcal{D}$-saturated, we see $(g, a - [B]) \in \mathbb{I}_P$. By the previous step, we conclude

$$(a_B, a - [B]) = (a_{0, g}, a - [B]) \in \mathbb{I}_P.$$ 

In the general case, since the set $\mathcal{H}$ does not coincide with a part of the associated regular system of parameters, we can not follow the steps of the special case above literally. However, by substituting $\partial X_B$ for $\partial H^B$ and by filling the gap of the substitution through the process of taking the limits, we can try to follow the steps of the special case in spirit. That is the basic strategy of the above proof.

(2) In Chapter 3, we derive Lemma 4.1.4.1 (Coefficient Lemma) in Part I as a corollary to the formal coefficient lemma above.
CHAPTER 3

Invariant $\tilde{\mu}$

The purpose of this chapter is to study the basic properties of the invariant $\tilde{\mu}$. As the unit for the strand of invariants in our algorithm is a triplet of numbers $(\sigma, \tilde{\mu}, s)$ (or a quadruplet $(\sigma, \tilde{\mu}, \nu, s)$ (cf. [3.3.1]), we also study the behavior of the pair $(\sigma, \tilde{\mu})$ endowed with the lexicographical order. The discussion of the invariant $\tilde{\mu}$ in this chapter is restricted to and concentrated on the case where there are no exceptional divisors involved, and hence can only be applied directly to the process at year 0 of our algorithm. We will postpone the general discussion, involving the exceptional divisors and hence applicable to the process after year 0 of our algorithm, to Part III or Part IV (cf. 0.2.3).

The setting for this chapter is identical to that of Chapter 1. Namely, $R$ represents the coordinate ring of an affine open subset $\text{Spec } R$ of a nonsingular variety $W$ of dim $W = d$ over an algebraically closed field $k$ of positive characteristic $\text{char}(k) = p$ or of characteristic zero $\text{char}(k) = 0$, where in the latter case we formally set $p = \infty$ (cf. 0.2.3.2.1 and Definition 3.1.1.1 (2) in Part I).

Let $I$ be an idealistic filtration over $R$. We assume that $I$ is $\mathcal{D}$-saturated. We remark that then, by compatibility of localization with $\mathcal{D}$-saturation (cf. Proposition 2.4.2.1 (2) in Part I), the localization $I_P$ is also $\mathcal{D}$-saturated for any closed point $P \in \text{Spec } R$.

§3.1. Definition of $\tilde{\mu}$

3.1.1. Definition of $\tilde{\mu}$ as $\mu_H$. We fix a closed point $P \in \text{Spec } R \subset W$. Take a leading generator system $H = \{(h_l, p^e_l)\}_{l=1,...,N}$ with associated nonnegative integers $0 \leq e_1 \leq \cdots \leq e_N$ for the $\mathcal{D}$-saturated idealistic filtration $I_P$. Let $H = \{h_l\}_{l=1,...,N}$ be the set of its elements in $R_P$, and $(H) \subset R_P$ the ideal generated by $H$.

Definition 3.1.1.1. First we recall a few definitions given in §3.2 in Part I. For $f \in R_P$ (or more generally for $f \in \hat{R}_P$), we define its multiplicity (or order) modulo $(H)$, denoted by $\text{ord}_H(f)$, to be

$$\text{ord}_H(f) = \sup \left\{ n \in \mathbb{Z}_{\geq 0} : f \in m_P^n + (H) \right\}.$$ 

Note that we set $\text{ord}_H(0) = \infty$ by definition. We also define

$$\mu_H(I_P) := \inf \left\{ \frac{\text{ord}_H(f)}{a} : (f, a) \in I_P, a > 0 \right\}.$$ 

(We remark that $\mu_H(I_P)$ is defined in a similar manner.)

Finally the invariant $\tilde{\mu}$ at $P$, which we denote by $\tilde{\mu}(P)$, is defined by the formula

$$\tilde{\mu}(P) = \mu_H(I_P).$$

In order to justify the definition, we should show that $\mu_H(I_P)$ is independent of the choice of $H$, i.e., independent of the choice of a leading generator system $H$ for $I_P$. We will show this independence in the next subsection.

Remark 3.1.1.2. (1) The usual order is multiplicative, i.e., we have an equality

$$\text{ord}(fg) = \text{ord}(f) + \text{ord}(g) \quad \forall f, g \in R_P.$$
The order modulo \( (\mathcal{H}) \) is also multiplicative if \( e_1 = \cdots = e_N = 0 \). However, in general, we can only expect that the order modulo \( (\mathcal{H}) \) is only weakly multiplicative, i.e., we have only an inequality

\[
\text{ord}(fg) \geq \text{ord}(f) + \text{ord}(g) \quad \forall f, g \in R_p.
\]

In fact, if \( e_l > 0 \) for some \( l = 1, \ldots, N \), then it is easy to see (cf. Remark 3.2.1.2(1)) that we indeed have a strict inequality for some \( f, g \in R_p \), i.e.,

\[
\text{ord}(fg) > \text{ord}(f) + \text{ord}(g) \quad \text{for some } f, g \in R_p.
\]

(2) Assume further that the idealistic filtration \( I \) is of r.f.g. type (cf. Definition 2.1.1.1 (4) and §2.3 in Part I). Then the invariant \( \bar{\mu} \) takes the rational values with some bounded denominator \( \delta \) (independent of \( P \)).

In fact, take a finite set of generators \( T \) for \( \bar{I} = G_H(T) \) of the form

\[
T = \{(f_a, a_1)\}_{a \in \Lambda} \subseteq R \times \mathbb{Q}_{>0}, \text{ with } a_1 = \frac{p_1}{q_1} \text{ where } p_1, q_1 \in \mathbb{Z}_{>0}.
\]

Set \( \delta = \prod_{a \in \Lambda} p_1 a_1 \).

Then\[
\bar{\mu}(P) = \mu_H(I) = \inf \left\{ \frac{\text{ord}_H(f_a)}{a}; (f, a) \in I, a > 0 \right\} = \min \left\{ \frac{\text{ord}_H(f_{a_1})}{a_1}; (f_{a_1}) = \frac{\text{ord}_H(f_{a_1}) \cdot p_1}{p_1} \right\}
\]

(cf. Lemma 2.2.1.2 (1) in Part I and Remark 3.1.1.2 (1) above)

\[
\epsilon \in \frac{1}{\delta} \mathbb{Z}_{>0} \cup \{\infty\}.
\]

### 3.1.2. Invariant \( \mu_H \) is independent of \( \mathcal{H} \).

We show that \( \mu_H(I) \) is independent of the choice of \( \mathcal{H} \).

**Proposition 3.1.2.1.** Let the setting be as described in [§7.7].

Then \( \mu_H(I) \) is independent of the choice of \( \mathcal{H} \), i.e., independent of the choice of a leading generator system \( \mathbb{H} \) for \( I_p \).

**Proof.** Suppose

\[
\mu_H(\bar{I}) = \inf \left\{ \frac{\text{ord}_H(f_a)}{a}; (f, a) \in I, a > 0 \right\} < 1.
\]

Then, since \( \bar{I} \) is \( \mathcal{D} \)-saturated, we have \( \bar{I} = R_p \times \mathbb{R} \) (cf. Lemma 1.1.2.1 Case: \( P \in \text{Supp}(\bar{I}) \)). We conclude that the set of elements \( \mathcal{H} \) in any leading generator system \( \mathbb{H} \) for \( I_p \) is a regular system of parameters \( \{x_1, \ldots, x_d\} \) for \( R_p \), where \( d = \dim W \). Accordingly, we have

\[
\mu_H(I_p) = 0,
\]

independent of the choice of \( \mathcal{H} \).

Therefore, in the following, we may assume \( 1 \leq \mu_H(\bar{I}) \) and hence that \( 1 \leq \mu_H(I) \leq \mu_H(I_p) \) for any choice of \( \mathcal{H} \).

**Case 1.** \( \mu_H(I_p) = 1 \) for any choice of \( \mathcal{H} \).

In this case, \( \mu_H(I_p) = 1 \) is obviously independent of the choice of \( \mathcal{H} \) by the case assumption.

**Case 2.** \( \mu_H(I_p) > 1 \) for some choice of \( \mathcal{H} \).
In this case, fixing the set of elements $\mathcal{H}$ of a leading generator system $\mathbb{H}$ for $\mathbb{I}_p$ with $\mu_{\mathbb{H}}(\mathbb{I}_p) > 1$, we show

\[ (*) \quad \mu_{\mathbb{H}}(\mathbb{I}_p) \geq \mu_{\mathbb{H}}(\mathbb{I}_p) \quad (> 1) \]

where $\mathcal{H}'$ is the set of elements of another leading generator system $\mathbb{H}'$ for $\mathbb{I}_p$.

This is actually sufficient to show the required independence, since by switching the roles of $\mathcal{H}$ and $\mathcal{H}'$, we conclude $\mu_{\mathbb{H}}(\mathbb{I}_p) \geq \mu_{\mathbb{H}}(\mathbb{I}_p)$ and hence $\mu_{\mathbb{H}}(\mathbb{I}_p) = \mu_{\mathbb{H}}(\mathbb{I}_p)$.

First we make the following two easy observations:

1. Let $\mathcal{H}'' = \{h''_i\}_{i=1,\ldots,N}$ be another set of elements in $\mathbb{R}_p$ obtained from $\mathcal{H}'$ by a linear transformation, i.e., for each $e \in \mathbb{Z}_{\geq 0}$ we have
   \[ [h''_i]^{P^e} : e_i \leq e = [h'_i]^{P^e} : e_i \leq e \]  
   for some $g_e \in \text{GL}(\# \{e_i : e_i \leq e\}, k)$.

Then $\mathcal{H}''$ is the set of elements of a leading generator system $\mathbb{H}'' = \{(h''_i, p^e)\}_{i=1}^N$ for $\mathbb{I}_p$, and we have

\[ \mu_{\mathcal{H}''}(\mathbb{I}_p) = \mu_{\mathcal{H}'}(\mathbb{I}_p). \]

Going back to our situation, we see that there is $\mathcal{H}''$, obtained from $\mathcal{H}'$ by a linear transformation, such that $\mathcal{H}''$ and $\mathcal{H}$ share the same leading terms.

Therefore, in order to show the inequality $(*)$, by replacing $\mathcal{H}'$ with $\mathcal{H}''$ we may assume that $\mathcal{H}$ and $\mathcal{H}'$ share the same leading terms, i.e.,

\[ h_i \equiv h'_i \mod m_p^{p^e+1} \quad \text{for} \quad l = 1, \ldots, N. \]

2. Assume that $\mathcal{H}$ and $\mathcal{H}'$ share the same leading terms. Then we have a sequence of the sets of elements of leading generator systems for $\mathbb{I}_p$

\[ \mathcal{H} = \mathcal{H}_0, \mathcal{H}_1, \ldots, \mathcal{H}_N = \mathcal{H}' \]

where the adjacent sets share all but one elements in common. We have only to show

\[ \mu_{\mathcal{H}_l}(\mathbb{I}_p) \geq \mu_{\mathcal{H}_l}(\mathbb{I}_p) \quad \text{for} \quad l = 1, \ldots, N \]

in order to verify the inequality $(*)$.

According to the observations above, therefore, we have only to show the inequality $(*)$ under the following extra assumptions:

1. $\mathcal{H}$ and $\mathcal{H}'$ share the same leading terms, i.e.,
   \[ h_i \equiv h'_i \mod m_p^{p^e+1} \quad \text{for} \quad l = 1, \ldots, N. \]

2. $\mathcal{H}$ and $\mathcal{H}'$ share all but one element in common, i.e.,
   \[ h_l = h'_l \quad \text{for} \quad l = 1, \ldots, N \quad \text{except} \quad l = l_0. \]

In order to ease the notation, we set

\[ h = h_{l_0}, h' = h'_{l_0}, G = \mathcal{H} \setminus \{h_{l_0}\} = \mathcal{H}' \setminus \{h'_{l_0}\}. \]

Let $v$ be any positive number such that $1 < v < \mu_{\mathcal{H}'}(\mathbb{I}_p)$.

Since $(h, p^{\nu v}), (h', p^{\nu v}) \in \mathbb{I}_p$, we have $(h - h', p^{\nu v}) \in \mathbb{I}_p$. Therefore, by definition of $\mu_{\mathcal{H}'}(\mathbb{I}_p)$ and by the inequality $1 < v < \mu_{\mathcal{H}'}(\mathbb{I}_p)$, we have

\[ h - h' \in m_p^{[\nu p^{\nu v}]} + (\mathcal{H}) \quad \text{i.e.,} \quad h - h' = f_1 + f_2 \quad \text{with} \quad f_1 \in m_p^{[\nu p^{\nu v}]} \quad \text{and} \quad f_2 \in (\mathcal{H}). \]

On the other hand, by extra assumption (1), we have

\[ h - h' \in m_p^{p^{e+1}}. \]

Observing $[\nu p^{\nu v}] \geq p^{e+1} + 1$ (recall $v > 1$), we thus conclude that

\[ f_2 = (h - h') - f_1 \in (\mathcal{H}) \cap m_p^{p^{e+1}} \]
and hence that
\[ h - h' = f_1 + f_2 \in m_{P}^{[\nu p^e]} + (H) \cap m_{P}^{\nu + 1} \]
\[ \subset m_{P}^{[\nu p^e]} + h m_{P} + (G) \cap m_{P}^{\nu + 1}, \]
where the second inclusion follows from Lemma 4.1.2.3 in Part I.
That is to say, we have
\[ h - h' = g_1 + h r + g_2 \in m_{P}^{[\nu p^e]}, r \in m_{P}, g_2 \in (G) \cap m_{P}^{\nu + 1}. \]
Therefore, we have
\[ (1 - r) h = g_1 + h' + g_2. \]
Since \( u = 1 - r \) is a unit in \( R_{P} \), we conclude
\[ h = u^{-1} g_1 + u^{-1} h' + u^{-1} g_2 \in m_{P}^{[\nu p^e]} + (H'). \]
Given an element \((f, a) \in I_{P} (a > 0)\), we hence have
\[ f = \sum_{b} (I_{P})_{a = ||b||} b^{H_{P}} \] (by Coefficient Lemma in Part I, where \((I_{P})' = (I_{P}) \cap m_{P}^{[\nu r]}\))
\[ = \sum_{b = b_{0}, C = b_{1}, \ldots, b_{s} = 1, \ldots} (I_{P})'_{a = ||c|| - bp^e} h^{b} h^{C} \subset \sum_{b} (I_{P})'_{a = bp^e} h^{b} + (G) \]
\[ \subset \sum_{b} (I_{P})'_{a = bp^e} m_{P}^{[\nu p^e]} + (H') \quad (\text{since } h \in m_{P}^{[\nu p^e]} + (H')). \]
Therefore, we compute
\[ \text{ord}_{H'}(f) \geq \min_{b} \left\{ \text{ord}_{P} \left( (I_{P})'_{a = bp^e} m_{P}^{b[\nu p^e]} \right) \right\} \]
\[ \geq \min_{b} \left\{ ||v(a - bp^e)|| + b[\nu p^e] \right\} \geq va. \]
This implies
\[ \mu_{H'}(f, a) = \frac{\text{ord}_{H'}(f)}{a} \geq v. \]
Since this inequality holds for any positive number with \( 1 < v < \mu_{H}(I_{P}) \), we conclude
\[ \mu_{H'}(f, a) \geq \mu_{H}(I_{P}). \]
Since \((f, a) \in I_{P} (a > 0)\) is arbitrary, we finally conclude
\[ \mu_{H'}(I_{P}) \geq \mu_{H}(I_{P}). \]
This completes the proof of the inequality (\( \ast \)), and hence the proof of Proposition 3.1.2.1.

\section*{3.2. Interpretation of \( \tilde{\mu} \) in terms of the power series expansion.}

The purpose of this section is to give an interpretation of the invariant \( \tilde{\mu} = \mu_{H} \) in terms of the power series expansion of the form (\( \ast \)) discussed in Chapter 2.
3.2.1. The order \( \text{ord}_H(f) \) of \( f \) modulo \((H)\) is equal to the order \( \text{ord}(a_\varnothing) \) of the constant term of the power series expansion for \( f \).

**Lemma 3.2.1.1.** Let the setting be as described in 3.1.1.
Then we have
\[
\text{ord}_H(f) = \text{ord}(a_\varnothing),
\]
where \( a_\varnothing \) is the “constant” term of the power series expansion \( f = \sum a_B H^B \) of the form (\( \ast \)) as described in Lemma 2.1.2.1.

**Proof.** Since \( f \equiv a_\varnothing \mod (H) \), we obviously have
\[
\text{ord}_H(f) = \text{ord}_H(a_\varnothing) \geq \text{ord}(a_\varnothing).
\]
Suppose \( \text{ord}_H(f) > \text{ord}(a_\varnothing) = r \). Then by definition we can write
\[
f = f_1 + f_2 \quad \text{with} \quad f_1 \in m_p^{r+1}, \ f_2 \in (H).
\]
Therefore, we have
\[
f_1 = f - f_2 = \sum a_B H^B - f_2.
\]
Since \( f_2 \in (H) \), we conclude by the uniqueness of the power series expansion (for \( f_1 \)) of the form (\( \ast \)) that the constant term \( a_\varnothing = a_{\varnothing, f} \) for \( f \) is also the constant term \( a_{\varnothing, f_1} \) for \( f_1 \), i.e.,
\[
a_\varnothing = a_{\varnothing, f_1}.
\]
On the other hand, we have by Remark 2.1.2.2 (1)
\[
r = \text{ord}(a_\varnothing) = \text{ord}(a_{\varnothing, f_1}) \geq \text{ord}(f_1) \geq r + 1,
\]
a contradiction!
Therefore, we have
\[
\text{ord}_H(f) = \text{ord}(a_\varnothing).
\]
This completes the proof of Lemma 3.2.1.1.

**Remark 3.2.1.2.** (1) We give the justification of Remark 3.1.1.2 (1), using Lemma 3.2.1.1.
Suppose \( e_l > 0 \) for some \( l = 1, \ldots, N \).
Take \( a, b \in \mathbb{Z}_{>0} \) such that \( a + b = p^{e_l} \). Set \( f = x_l^a \) and \( g = x_l^b \).
Then \( a_{\varnothing, f} = x_l^a \) and \( a_{\varnothing, g} = x_l^b \). Therefore, we have by Lemma 3.2.1.1
\[
\text{ord}_H(f) + \text{ord}_H(g) = \text{ord}(a_{\varnothing, f}) + \text{ord}(a_{\varnothing, g}) = a + b.
\]
On the other hand, we observe
\[
f g = x_l^a x_l^b = x_l^{a+b} \in m_p^{e_l+1} + (H) \subset m_p^{e_l+1} + (H),
\]
which implies
\[
\text{ord}_H(f g) \geq p^{e_l+1} > p^{e_l} = a + b = \text{ord}_H(f) + \text{ord}_H(g).
\]

(2) We remark that the above interpretation of \( \text{ord}_H(f) \) is still valid, even if we consider the power series expansion of the form (\( \ast \)) with respect to \( H = (h_1, \ldots, h_N) \) and a regular system of parameters only weakly-associated to \( H \) (cf. Remark 2.1.2.2 (2), instead of the power series expansion of the form (\( \ast \)) with respect to \( H \) and a regular system of parameters associated to \( H \) as described in Lemma 2.1.2.1.
3.2.2. Alternative proof to Coefficient Lemma. The interpretation given in \[3.2.1\] allows us to derive Coefficient Lemma (Lemma 4.1.4.1 in Part I) as a corollary to the formal coefficient lemma (Lemma \[3.2.1\] in Part II).

**Corollary 3.2.2.1.** (= Coefficient Lemma) Let \(\nu \in \mathbb{R}_{\geq 0}\) be a nonnegative number such that \(\nu < \mu H(I_p)\). Set

\[
(I_p)_\nu = (I_p)_\nu \cap m_p^{[\nu]},
\]

where we use the convention that \(m_p^n = R_p\) for \(n \leq 0\). Then for any \(a \in \mathbb{R}\), we have

\[
(I_p)_a = \sum_B (I_p)_a' \subseteq \langle B \rangle H^B.
\]

Proof. Note that we already gave a proof to Coefficient Lemma in Part I. Here we present a different proof based upon the formal coefficient lemma, although both proofs share some common spirit.

Since \(H^B \in \langle I_p \rangle \subseteq \mathbb{R}\), we clearly have the inclusion

\[
(I_p)_a \supset \sum_B (I_p)_a' \subseteq \langle B \rangle H^B.
\]

Therefore, we have only to show the opposite inclusion

\[
(I_p)_a = \sum_B (I_p)_a' \subseteq \langle B \rangle H^B.
\]

Now, as observed in Remark 4.1.4.2 (2) in Part I, we have

\[
\sum_B (I_p)_a' \subseteq \langle B \rangle H^B = \sum_{\|B\| < a + \nu B} (I_p)_a' \subseteq \langle B \rangle H^B.
\]

Therefore, actually we have only to show

\[
(I_p)_a = \sum_{\|B\| < a + \nu B} (I_p)_a'.
\]

Since \(\bar{R}_p\) is faithfully flat over \(R_p\), we have only to prove this inclusion at the level of completion. That is to say, we have only to show

\[
(I_p)_a = \sum_{\|B\| < a + \nu B} (I_p)_a'.
\]

noting

\[
\left\{ \begin{array}{l}
(I_p)_a \otimes R_p \bar{R}_p = (I_p)_a, \\\n(I_p)_a' \otimes R_p \bar{R}_p = (I_p)_a' \cap m_p^{[\nu]} \otimes R_p \bar{R}_p = (I_p)_a' \cap m_p^{[\nu]} = (I_p)_a'.
\end{array} \right.
\]

Take \(f \in (I_p)_a\).

Let \(f = \sum_B a_B H^B\) be the power series expansion of the form (\(\ast\)) as described in Lemma \[2.1.2.2\].

Observe that, for each \(C \in (\mathbb{Z}_{\geq 0})^N\) with \(\|C\| \geq a + \nu B\), there exists \(B_C \in (\mathbb{Z}_{\geq 0})^N\) with \(a \leq B_C \leq a + \nu B\) such that \(B_C < C\) (cf. Remark 4.1.4.2 (2) in Part I). We choose one such \(B_C\) and call it \(\phi(C)\).

For each \(B \in (\mathbb{Z}_{\geq 0})^N\) with \(a \leq \|B\| \leq a + \nu B\), we set

\[
a'_B = a_B + \sum_{C \text{ with } \phi(C)=B} a_C H_C - B.
\]

We see then

\[
a'_B \in \bar{R}_p = (I_p)_a' \subseteq \langle B \rangle H^B.
\]

since \(a - \|B\| \leq 0\).

On the other hand, for each \(B \in (\mathbb{Z}_{\geq 0})^N\) with \(\|B\| < a\), we have by the formal coefficient lemma

\[
a_B \in (I_p)_a' \subseteq \langle B \rangle H^B.
\]
We also have by Lemma 3.2.1.1
\[ \text{ord}(ab) = \text{ord}_A(ab) \geq \lceil \mu_H(1) (a - |B|) \rceil \geq \lceil \nu(a - |B|) \rceil. \]

Therefore, we see
\[ a_B \in (\mathcal{I}_P)_{a - |B|} \cap \mathcal{I}^*_{\nu(a - |B|)} = (\mathcal{I}_P)'_{a - |B|}. \]

We conclude
\[ f = \sum_B a_B H^B = \sum_{|B| < a} a_B H^B + \sum_{|B| = a} a_B H^B. \]

We observe that
\[ \text{ord}(h) \text{ cri} \text{ (Lemma 2.2.2.1)}. \]

leading generator system \( H \). Let \( H' \) be the set of elements of another leading generator system \( H' \). We want to show \( \mu_{H'}(\mathcal{I}_P) = \mu_{H'}(\mathcal{I}_P) \). By the same argument as in the proof of Proposition 3.1.2.1 we may assume that \( H \) and \( H' \) share the same leading terms, i.e.,
\[ h_l \equiv h'_l \mod w_p^{p+1} \quad \text{for} \quad l = 1, \ldots, N. \]

Since \( H \) and \( H' \) share the same leading terms, we can take a regular system parameters \((x_1, \ldots, x_d)\) associated both to \( H \) and to \( H' \) simultaneously. In the following, when we consider the power series expansion of the form \((\ast)\), we understand that it is with respect to \( H \) and \((x_1, \ldots, x_d)\) with respect to \( H' \) and \((x_1, \ldots, x_d)\).

Now since \( \mu_{H'}(\mathcal{I}_P) = \mu_{H'}(\mathcal{I}_P) \) and since \( \mu_{H'}(\mathcal{I}_P) = \mu_{H'}(\mathcal{I}_P) \), we have only to show
\[ \mu_{H'}(\mathcal{I}_P) = \mu_{H'}(\mathcal{I}_P). \]

We observe that
\[ \mu_H(\mathcal{I}_P) = \inf \left\{ \mu_H(f, a) = \frac{\text{ord}_A(f)}{a}; (f, a) \in \mathcal{I}_P, a > 0 \right\} \]
\[ = \inf \left\{ \frac{\text{ord}_A(f)}{a}; (f, a) \in \mathcal{I}_P, a > 0, f = \sum a_{B, f} H^B \right\} \]
\[ = \inf \left\{ \frac{\text{ord}_A(f)}{a}; (f, a) \in \mathcal{I}_P, a > 0, f = a_{0, f} \right\} \]  
(by the interpretation given in 3.2.1)

and similarly that
\[ \mu_H(\mathcal{I}_P) = \inf \left\{ \mu_H(f, a) = \frac{\text{ord}_A(f)}{a}; (f, a) \in \mathcal{I}_P, a > 0 \right\} \]
\[ = \inf \left\{ \frac{\text{ord}_A(f)}{a}; (f, a) \in \mathcal{I}_P, a > 0, f = \sum a_{B, f} H^B \right\} \]
\[ = \inf \left\{ \frac{\text{ord}_A(f)}{a}; (f, a) \in \mathcal{I}_P, a > 0, f = a_{0, f} \right\} \]  
(by the formal coefficient lemma).
§3.3. UPPER SEMI-CONTINUITY OF \((\sigma, \tilde{\mu})\).

On the other hand, the condition \(f = a_{\sigma,f}'\) is equivalent to saying that \(f\), as a power series in terms of \((x_1, \ldots, x_N, x_{N+1}, \ldots, x_d)\) is of the form \(f = \sum b_k x^k\), with \(b_k\) being a power series in terms of the remainder \((x_{N+1}, \ldots, x_d)\) of the regular system of parameters, and with \(K = (k_1, \ldots, k_d)\) varying in the range satisfying the condition

\[0 \leq k_l \leq p^0 - 1\] for \(l = 1, \ldots, N\) and \(k_l = 0\) for \(l = N + 1, \ldots, d\).

Since the regular system of parameters \((x_1, \ldots, x_N, x_{N+1}, \ldots, x_d)\) is associated both to \(H\) and \(H'\) simultaneously, this condition is no different from the condition \(f = a_{\sigma,f}\). That is to say, we have

\[f = a_{\sigma,f}' \iff f = a_{\sigma,f}.\]

Therefore, by looking at the last expressions for \(\mu_H(I_P)\) and \(\mu_{H'}(I_P)\) above, we conclude

\[\mu_H(I_P) = \mu_{H'}(I_P).\]

This completes the presentation of the alternative proof.

§3.3. Upper semi-continuity of \((\sigma, \tilde{\mu})\).

The purpose of this section is to establish the upper semi-continuity of \((\sigma, \tilde{\mu})\), where the pair is endowed with the lexicographical order.

Recall that we have a \(\mathcal{T}\)-saturated idealistic filtration \(\mathcal{I}\) over \(R\).

3.3.1. Statement of the upper semi-continuity of \((\sigma, \tilde{\mu})\) and its proof.

Proposition 3.3.1.1. The function

\[(\sigma, \tilde{\mu}) : X = \text{m- Spec } R \to \left( \prod_{e \in \mathbb{Z}_{\geq 0}} \mathbb{Z}_{\geq 0} \right) \times (\mathbb{R}_{\geq 0} \cup \{\infty\})\]

is upper semi-continuous with respect to the lexicographical order on \(\left( \prod_{e \in \mathbb{Z}_{\geq 0}} \mathbb{Z}_{\geq 0} \right) \times (\mathbb{R}_{\geq 0} \cup \{\infty\})\). That is to say, for any \((\alpha, \beta) \in \left( \prod_{e \in \mathbb{Z}_{\geq 0}} \mathbb{Z}_{\geq 0} \right) \times (\mathbb{R}_{\geq 0} \cup \{\infty\})\), the locus \(X_{(\alpha, \beta)}\) is closed (cf. Definition 1.2.1.2).

Assume further that the idealistic filtration \(\mathcal{I}\) is of r.f.g. type (cf. Definition 2.1.1.1 (4) and §2.3 in Part I). Then the invariant \(\tilde{\mu}\) takes the rational values with some bounded denominator \(\delta\), and this upper semi-continuity allows us to extend the domain to define the function

\[(\sigma, \tilde{\mu}) : \text{Spec } R \to \left( \prod_{e \in \mathbb{Z}_{\geq 0}} \mathbb{Z}_{\geq 0} \right) \times (\mathbb{R}_{\geq 0} \cup \{\infty\}),\]

where for \(Q \in \text{Spec } R\) we have by definition

\[(\sigma, \tilde{\mu})(Q) = \min \left\{ (\sigma, \tilde{\mu})(P) = (\sigma(P), \tilde{\mu}(P)) : P \in \text{m- Spec } R, P \in \overline{Q} \right\},\]

or equivalently \((\sigma, \tilde{\mu})(Q)\) is equal to \((\sigma, \tilde{\mu})(P)\) with \(P\) being a general closed point on \(\overline{Q}\). The function \((\sigma, \tilde{\mu})\) with the extended domain is upper semi-continuous.

Moreover, since \(\text{Spec } R\) is noetherian and since \(\left( \prod_{e \in \mathbb{Z}_{\geq 0}} \mathbb{Z}_{\geq 0} \right) \times (\mathbb{R}_{\geq 0} \cup \{\infty\})\) can be replaced with a well-ordered set \(T\) (e.g., \(T\) can be obtained by replacing the first factor \(\prod_{e \in \mathbb{Z}_{\geq 0}} \mathbb{Z}_{\geq 0}\) with the well-ordered set as described in the proof of Corollary 1.2.1.3 and the second factor \(\mathbb{R}_{\geq 0} \cup \{\infty\}\) with \(\mathbb{Z}_{\geq 0} \cup \{\infty\}\)), conditions (i) and (ii) in Lemma 1.2.1.2, as well as the assertions in Corollary 1.2.1.3 hold for the upper semi-continuous function \((\sigma, \tilde{\mu}) : \text{Spec } R \to T\).

Proof. First we show the upper semi-continuity of the function

\[(\sigma, \tilde{\mu}) : X = \text{m- Spec } R \to \left( \prod_{e \in \mathbb{Z}_{\geq 0}} \mathbb{Z}_{\geq 0} \right) \times (\mathbb{R}_{\geq 0} \cup \{\infty\}).\]
We have only to show that, for any \((\alpha, \beta) \in \left(\prod_{e \in \mathbb{Z}^0} \mathbb{Z}_{\geq 0}\right) \times (\mathbb{R}_{>0} \cup \{\infty\})\), the locus \(X_{\xi(\alpha, \beta)}\) is closed.

**Step 1. Reduction to the (local) situation where** \(X = \text{m-}\text{Spec } R\) is an affine open neighborhood of a fixed point \(P\), \(\alpha = \sigma(P) \neq \emptyset\) is the maximum of the invariant \(\sigma\), and where a leading generator system \(\mathbb{H}\) of \(\text{L}_P\) is uniformly pure along the (local) maximum locus \(C\) of the invariant \(\sigma\).

Observe \(X_{\xi(\alpha, \beta)} = X_{\xi(\alpha)} \cup (X_{\xi(\alpha)} \cap X_{\xi(\alpha, \beta)})\). Since \(X_{\xi(\alpha)}\) is a closed subset by the upper semi-continuity of the invariant \(\sigma\) (cf. Corollary \[1.2.1.3\] and Proposition \[1.2.2.1\]), we have only to show \(X_{\xi(\alpha, \beta)}\) is closed inside of the open subset \(X_{\xi(\alpha)}\), or equivalently inside of any affine open subset \(U\) contained in \(X_{\xi(\alpha)}\). By replacing \(X\) with \(U\), we may assume that the invariant \(\sigma\) never exceeds \(\alpha\) in \(X\). Then again by the upper semi-continuity of the invariant \(\sigma\), the maximum locus \(C = \{Q \in X; \sigma(Q) = \alpha\} = X_{\xi(\alpha)}\) of the invariant \(\sigma\) is a closed subset. Since \(X_{\xi(\alpha, \beta)} \subset C\), we have only to show that, for any point \(P \in C\), there exists an affine open neighborhood \(U_P\) of \(P\) such that \(U_P \cap X_{\xi(\alpha, \beta)}\) is closed.

Suppose \(\alpha = \sigma(P) = \emptyset\). Then, taking \(U_P\) sufficiently small, we have \(U_P \cap \text{Supp}(\mathcal{I}) = \emptyset\) or \(|P|\). Therefore, we conclude that \(U_P \cap X_{\xi(\alpha, \beta)} = U_P \cap \{P\} = \emptyset\), and hence is closed. (Note that, for a point \(Q \in U_P\), we have \((\sigma, \overline{\mu})(Q) = (\emptyset, 0)\) if \(Q \notin \text{Supp}(\mathcal{I})\), and \((\sigma, \overline{\mu})(Q) = (\emptyset, \infty)\) if \(Q \in \text{Supp}(\mathcal{I})\).)

Therefore, in the following, we may concentrate on the case where \(\alpha = \sigma(P) \neq \emptyset\). We take a leading generator system \(\mathbb{H}\) of \(\text{L}_P\). By Proposition \[1.3.3.3\] and by shrinking \(U_P\) if necessary, we may assume that \(\mathbb{H}\) is uniformly pure along \(C\). Note that \(C = C \cap \text{Supp}(\mathcal{I})\), due to the condition \(\sigma(P) \neq \emptyset\) (cf. Remark \[1.3.3.2\]).

Finally by replacing \(X\) with \(U_P\), we are reduced to the (local) situation as described in Step 1.

We may also assume by shrinking \(U_P\) if necessary, after taking a regular system of parameters \((x_1, \ldots, x_N)\) associated to \(H = (h_1, \ldots, h_N)\) at \(P\), that we have a regular system of parameters \((x_1, \ldots, x_N)\) over \(\text{Spec } R\) such that the matrices

\[
\begin{bmatrix}
\partial_{x_i}(h_j^{e - e_j}) \\
\hline
j = 1, \ldots, L_e
\end{bmatrix}
\]

are all invertible, and hence that the conditions described in the setting 4.1.1 of Part I for the supporting lemmas to hold are satisfied (at any point in \(C\)).

(We would like to bring the attention of the reader to the difference in notation between here in Part II and there in 4.1.1 of Part I. The symbol “\(R\)” here denotes the coordinate ring of an affine open subset \(\text{Spec } R\) in \(W\) (cf. the beginning of Chapter 3), while the symbol “\(R\)” there denotes the local ring at a closed point.)

**Step 2. Reduction to statement (●), which is further reduced to statement (∇).**

We observe that, in order to provide an argument for the upper semi-continuity, it suffices to prove the following slightly more general statement (●) (which does not involve any idealistic filtration):

(●) Let \(C \subset \text{m-}\text{Spec } R\) be a closed subset.

Let \(\mathcal{H} = \{h_1, \ldots, h_N\} \subset R\) be a subset consisting of \(N\) elements, and \(0 \leq e_1 \leq \cdots \leq e_N\) nonnegative integers attached to these elements, satisfying the following conditions at each point \(P \in C\) (cf. 4.1.1 in Part I):

(i) \(h_l \equiv m_p^{\ell_i} \text{ and } \overline{h_l} = (h_l \mod m_p^{\ell_i + 1}) \equiv v_l^{p_i} \) with \(v_l \equiv m_p/m_p^2\) for \(l = 1, \ldots, N\),

(ii) \(\{v_l; l = 1, \ldots, N\} \subset m_p/m_p^2\) consists of \(N\)-distinct and \(k\)-linearly independent elements in the \(k\)-vector space \(m_p/m_p^2\).
We also have a regular system of parameters \((x_1, \ldots, x_d)\) over \(\text{Spec} \ R\) such that the matrices
\[
\left[ \partial_{x_i} \rho(x_i \partial_{x_j}) \right]_{i=1}^{d-1} \quad \text{for} \quad e = e_1, \ldots, e_N \quad \text{where} \quad L_e = \# \{ l : e_l \leq e \}
\]
are all invertible.

Then for any \(f \in R\) and \(r \in \mathbb{Z}_{\geq 0}\) the locus
\[
V_r(f, \mathcal{H}) := \{ P \in C : f \in m_P^r R_P + (\mathcal{H}) R_P \} = \{ P \in C : \text{ord}_P(f)(P) \geq r \}
\]
is a closed subset.

In fact, if we prove statement (\textbullet\textcircled{1}), then
\[
X_\alpha = \bigcap_{(f, \mathcal{H})} V_{l_\alpha}(f, \mathcal{H})
\]
is closed for any \((\alpha, \beta) \in \left( \prod_{e \in Z_{\geq 0}} Z_{\geq 0} \right) \times (\mathbb{R}_{\geq 0} \cup \{ \infty \})\), and hence we have the required upper semi-continuity of the function \((\alpha, \beta)\).

Furthermore, in order to prove statement (\textbullet\textcircled{2}) for any \(f \in R\) and \(r \in \mathbb{Z}_{\geq 0}\):

(\textcircled{2}) There exist \(\omega_l \in R (l = 1, \ldots, N)\) such that
\[
V_r(f, \mathcal{H}) = \left\{ P \in C : f - \sum_{l=1}^{N} \omega_l h_l \in m_P^r R_P \right\}.
\]

In fact, if we show statement (\textcircled{2}), then \(V_r(f, \mathcal{H})\) being a closed set follows from the usual upper semi-continuity of the order function for \(f - \sum_{l=1}^{N} \omega_l h_l\), and hence we have statement (\textbullet\textcircled{1}).

**Step 3.** Show statement (\textcircled{2}) by induction on \(r\).

Step 3 is dedicated to showing statement (\textcircled{2}) by induction on \(r\).

We set
\[
\begin{align*}
e & := e_1 = \min \{ e_l : l = 1, \ldots, N \}, \\
L & := \max \{ l : l = 1, \ldots, N, e_l = e \} = \# \{ l : 1, \ldots, N, e_l = e \}, \\
e' & = e_{L+1} \quad \text{(if} \quad L = N, \text{then we set} \quad e' = \infty), \\
\chi & = \# \{ e_1, \ldots, e_N \}.
\end{align*}
\]

**Case 1.** \(r \leq p^e\)

In this case, we have only to set \(\omega_l = 0 \quad (l = 1, \ldots, l)\) in order to see statement (\textcircled{2}).

**Case 2.** \(r > p^e\)

Observing \(V_r(f, \mathcal{H}) \subset V_{r-1}(f, \mathcal{H})\) and replacing \(f\) with \(f - \sum_{l=1}^{N} \omega_l h_l\) via application of statement (\textcircled{2}) for \(r - 1\) by induction, we may assume
\[
f \in m_P^{r-1} R_P \quad \forall P \in V_r(f, \mathcal{H}).
\]

We also observe then, by Supporting Lemma 3 in Part I (cf. Lemma 4.1.2.3 in Part I), that, at each point \(P \in V_r(f, \mathcal{H})\), there exist \(\beta_i, P \in m_P^{r-1-p^e} R_P\) such that
\[
f - \sum_{l=1}^{N} \beta_i, P h_l \in m_P^r R_P.
\]

Now we use the induction on the pair \((\chi, L)\).

**Case:** \(\chi = 1 \quad (L = N, e' = \infty)\)
In this case, by applying Supporting Lemma 2 in Part I (cf. Lemma 4.1.2.2 in Part I) with \( v = r, s = r - 1 \) and \( \alpha = -f \), we see

\[
(*) \beta_L \in F_v(-f) + \sum_{1 \leq i \leq N \neq L} (F_i \beta_L) h_i \pm (h_i') + m_p^{-\alpha} R_p.
\]

\[
\subset F_v(-f) + \sum_{1 \leq i \leq N \neq L} h_i R_p + (h_i') + m_p^{-\alpha} R_p.
\]

See Supporting Lemmas 1 and 2 in Part I (cf. Lemma 4.1.2.1 and Lemma 4.1.2.2 in Part I) for the definition of the differential operator \( F_v \). We would like to emphasize that, even though Supporting Lemma 3 is a local statement at \( P \), the differential operator \( F_v \) is defined globally over \( \text{Spec } R \) and hence that \( F_v(-f) \in R \).

From (\( * \)), we conclude the following.

When \( N = 1 \), we have only to set \( \omega_1 = F_v(-f) \) in order to see statement (\( \triangledown \)).

When \( N > 1 \), we observe

\[
V_v(f, \mathcal{H}) = V_v(f, \{ h_1, \ldots, h_{N-1}, h_N \}) = V_v(f - F_v(-f) h_N, \{ h_1, \ldots, h_{N-1} \}).
\]

Now statement (\( \triangledown \)) for \( f \) and \( r \) with respect to \( \mathcal{H} = \{ h_1, \ldots, h_{N-1}, h_N \} \) follows from statement (\( \triangledown \)) for \( f - F_v(-f) h_N \) and \( r \) with respect to \( \{ h_1, \ldots, h_{N-1} \} \), which holds by induction on \( (\chi, L) = (1, N - 1) \).

**Case: \( \chi > 1 \)**

In this case, by applying Supporting Lemma 2 in Part I (cf. Lemma 4.1.2.2 in Part I) with \( v = p^{\chi-1} - 1, s = r - 1 \) and \( \alpha = -f \), we see

\[
(*) \beta_L \in F_v(-f) + \sum_{1 \leq i \leq N \neq L} (F_i \beta_L) h_i \pm (h_i') + m_p^{-\alpha} R_p.
\]

\[
\subset F_v(-f) + \sum_{1 \leq i \leq N \neq L} h_i R_p + (h_i') + m_p^{-\alpha} R_p.
\]

From (\( * \)), we conclude that

\[
f - F_v(-f) h_L \in \sum_{1 \leq i \leq N \neq L} h_i R_p + h_i'^{-\alpha} R_p + m_p R_p
\]

and hence that

\[
V_v(f, \mathcal{H}) = V_v(f, \{ h_1, \ldots, h_{L-1}, h_L, h_{L+1}, \ldots, h_N \}) = V_v(f - F_v(-f), \{ h_1, \ldots, h_{L-1}, h_L'^{-\alpha}, h_{L+1}, \ldots, h_N \}).
\]

Now statement (\( \triangledown \)) for \( f \) and \( r \) with respect to \( \mathcal{H} = \{ h_1, \ldots, h_{L-1}, h_L, h_{L+1}, \ldots, h_N \} \) follows from statement (\( \triangledown \)) for \( f - F_v(-f) h_N \) and \( r \) with respect to \( \{ h_1, \ldots, h_{L-1}, h_L'^{-\alpha}, h_{L+1}, \ldots, h_N \} \), which holds by induction on \( (\chi, L) \). (In fact, if originally \( L = 1 \), then the invariant \( \chi \) drops by 1, and if originally \( L > 1 \), then the invariant \( \chi \) remains the same but the invariant \( L \) drops by 1.)

This completes the proof of statement (\( \triangledown \)).
This completes the proof of the upper semi-continuity of the function

\[(\sigma, \tilde{\mu}) : X = m\cdot \text{Spec } R \rightarrow \left( \prod_{e \in \mathbb{Z}_{\geq 0}} \mathbb{Z}_{\geq 0} \right) \times (\mathbb{R}_{\geq 0} \cup \{\infty\}).\]

If we assume further that the idealistic filtration is of r.f.g. type, then, as shown in Remark 5.1.1.2(2), the invariant \(\tilde{\mu}\) takes the rational values with some bounded denominator \(\delta\). Then we may replace the target space for the function

\[(\sigma, \tilde{\mu}) : m\cdot \text{Spec } R \rightarrow \left( \prod_{e \in \mathbb{Z}_{\geq 0}} \mathbb{Z}_{\geq 0} \right) \times (\mathbb{R}_{\geq 0} \cup \{\infty\})\]

with a well-ordered set (e.g., \(T\) can be obtained by replacing the first factor \(\prod_{e \in \mathbb{Z}_{\geq 0}} \mathbb{Z}_{\geq 0}\) with the well-ordered set as described in the proof of Corollary 1.2.1.3 and the second factor \(\mathbb{R}_{\geq 0} \cup \{\infty\}\) with \(\mathbb{Z}_{\geq 0} \cup \{\infty\}\). Now the assertion regarding the extension of the domain of the function \((\sigma, \tilde{\mu})\) from \(m\cdot \text{Spec } R\) to \(\text{Spec } R\) and the rest of the assertions in Proposition 3.3.1.1 follow from the same argument as in Corollary 1.2.2.2 where we discussed the extension of the domain of the invariant \(\sigma\) from \(m\cdot \text{Spec } R\) to \(\text{Spec } R\).

This completes the proof of Proposition 3.3.1.1.

### 3.3.2. Alternative proof to the upper semi-continuity of \((\sigma, \tilde{\mu})\)

We can give an alternative proof to the upper semi-continuity of \((\sigma, \tilde{\mu})\) using the interpretation of \(\tilde{\mu}\) in terms of the power series expansion of the form (\(\ast\)) as presented in 3.3.2.

**Alternative Proof to the Upper Semi-continuity of \((\sigma, \tilde{\mu})\).** By the same argument as before, we are reduced to the (local) situation as described in Step 1 of the original proof.

Take a regular system of parameters \(X_P = (x_1, \ldots, x_d) = (x_1, p, \ldots, x_d, p)\) at \(P\), which is associated to \(H = (h_1, \ldots, h_N)\). By shrinking \(\text{Spec } R\) if necessary, we may assume that \(X_Q = (x_1, Q, \ldots, x_d, Q) = (x_1 - x_1(Q), \ldots, x_d - x_d(Q))\) with \(x_i(Q) \in k\) is a regular system of parameters, which is weakly-associated to \(H\) at any point \(Q \in C\).

By the same argument as before, we have only to show that, given \(f \in R\) and \(r \in \mathbb{Z}_{\geq 0}\), the locus \(V(f, \mathcal{H}) = \{Q \in C; \text{ord}_H(f)(Q) \geq r\}\) is a closed subset as in Step 2 of the original proof.

This is where the alternative argument using the interpretation of \(\tilde{\mu}\) presented in 3.3.2 begins: Let \(f = \sum a_{0,Q}H^0\) be the power series expansion of \(f\) at \(Q \in C\) with respect to \(H\) and the regular system of parameters \(X_Q\), which is weakly-associated to \(H\) at \(Q\). By Lemma 3.2.1.1 and Remark 3.2.1.2(2), we have

\[\text{ord}_H(f)(Q) = \text{ord}(a_{0,Q}).\]

Let

\[a_{0,Q} = \sum_{l_0} \gamma_{l_0} X_{Q}^{l_0}, \quad \gamma_{l_0} \in k\]

be the power series expansion of \(a_{0,Q}\) with respect to \(X_Q\). Then we have

\[\text{ord}(a_{0,Q}) \geq r \iff \gamma_{l_0} = 0 \quad \forall l_0 \text{ with } |l_0| < r.\]

On the other hand, since the coefficients \(\gamma_{l_0}\) can be computed from the coefficients of the power series expansions

\[
\begin{align*}
f &= \sum c_{l,f,Q} X_{Q}^{l} \\
h_l &= \sum c_{l,h,Q} X_{Q}^{l} \quad (l = 1, \ldots, N)
\end{align*}
\]

with \(c_{l,f,Q} \in k\) and \(c_{l,h,Q} \in k\), where

\[
\begin{align*}
c_{l,f,Q} &= \partial_{X_{Q}^{l}}(f)(Q) = \partial_{X_{Q}^{l}}(f)(Q) \\
c_{l,h,Q} &= \partial_{X_{Q}^{l}}(h_l)(Q) = \partial_{X_{Q}^{l}}(h_l)(Q)
\end{align*}
\]

for \(l = 1, \ldots, N\).
via the invertible matrices appearing in the condition of $X_Q$ being weakly-associated to $H$

$$
\left[ \partial_{x_{i_e}}(h_{i_e}^{e_i}) \right]_{i=1-L_e}^{j=1-L_e} (Q) = \left[ \partial_{x_{i_e}}(h_{i_e}^{e_i}) \right]_{i=1-L_e}^{j=1-L_e} (Q) \quad \text{for } e = e_1, \ldots, e_N
$$

where $L_e = \# \{ \ell; e_{\ell} \leq e \}$, we conclude that, for each $I$, there exists $\gamma_I \in R$ such that

$$
\gamma_I(Q) = \gamma_{I,0} \quad \forall Q \in C.
$$

Finally then we conclude that

$$
V_r(f, H) = \{ Q \in C; \text{ord}_{H}(f)(Q) \geq r \} = \{ Q \in C; \gamma_I(Q) = 0 \quad \forall I \text{ with } |I| < r \}
$$

is a closed subset.

This completes the alternative proof for the upper semi-continuity of $(\sigma, \tilde{\mu})$. 

Appendix

The purpose of this appendix is to present the new nonsingularity principle using only the $\mathcal{D}$-saturation, as opposed to the old nonsingularity principle using both the $\mathcal{D}$-saturation and $\mathcal{R}$-saturation. (The combination of the $\mathcal{D}$-saturation and $\mathcal{R}$-saturation was called the $\mathcal{B}$-saturation in Part I (cf. 2.1.5 and 2.2.3 in Part I).)

In Part I, we emphasized the importance of the $\mathcal{R}$-saturation (and of the $\mathcal{B}$-saturation) in carrying out the IFP. In fact, the $\mathcal{R}$-saturation was crucial in establishing the nonsingularity principle, as formulated in Chapter 4 of Part I, which sits at the heart of constructing an algorithm. However, the $\mathcal{R}$-saturation has also been the main culprit in our quest to complete the algorithm, causing the following problems:

- By taking the $\mathcal{R}$-saturation, we may increase the denominator of the invariant $\tilde{\mu}$ indefinitely, and hence may not have the descending chain condition on the value set of the strand of invariants consisting of the units of the form $(\sigma, \tilde{\mu}, s)$. This invites the problem of termination, as we mentioned in the introduction to Part I.
- If we take the $\mathcal{R}$-saturation, the value of the invariant $\tilde{\mu}$ may strictly increase under blowup, even when the value of the invariant $\sigma$ stays the same. This violates the principle that our strand of invariants, consisting of the units of the form $(\sigma, \tilde{\mu}, s)$, should never increase under blowup.

While writing Part II, we came to realize that we can establish the nonsingularity principle, as formulated below, with only the $\mathcal{D}$-saturation and without the $\mathcal{R}$-saturation. This indicates that we may construct an algorithm, still in the framework of the IFP, without using the $\mathcal{R}$-saturation, and hence that we may avoid the problem of termination, as well as the other technical problems, that the use of the $\mathcal{R}$-saturation invites.

Even though we are still in the evolution process of the IFP program (See §0.3.1 for the current status of the IFP), we consider this new nonsingularity principle a substantial step forward in our quest to construct an algorithm for local and global resolution of singularities in positive characteristic.

In this appendix, $R$ represents the coordinate ring of an affine open subset $\text{Spec } R$ of a nonsingular variety $W$ of dimension $d$ over an algebraically closed field of positive characteristic $\text{char}(k) = p$ or of characteristic zero $\text{char}(k) = 0$, where in the latter case we formally set $p = \infty$ (cf. 0.2.3.2.1 and Definition 3.1.1.1 (2) in Part I).

§A.1. Nonsingularity principle with only $\mathcal{D}$-saturation and without $\mathcal{R}$-saturation.

A.1.1. Statement.

**Theorem A.1.1.1.** Let $\mathcal{I}$ be an idealistic filtration over $R$.

Let $P \in \text{Spec } R \subset W$ be a closed point.

(1) Assume that $\mathcal{I}$ is $\mathcal{D}$-saturated and that $\tilde{\mu}(P) = \infty$.

Then there exists a regular system of parameters $(x_1, \ldots, x_N, y_{N+1}, \ldots, y_d)$ at $P$ such that

\[
\mathcal{H} = \left\{ (x_1^{e_1}, \ldots, x_N^{e_N}) \right\}_{e_1 \leq \cdots \leq e_N}^N \text{ is an LGS of } \mathcal{I}_P
\]

\[
\mathcal{I}_P = G_{\mathcal{R}_p}(\mathcal{H}).
\]

(See the footnote to §0.2.1.5.)
Assume further that $\mathcal{I}$ is of r.f.g. type.

Then there exists an affine open neighborhood $P \in U_P = \text{Spec } R$, of $P$ (Note that $R$, represents the localization of $R$ by $r \in R$.) such that $(x_1, \ldots, x_N, y_{N+1}, \ldots, y_d)$ is a regular system of parameters over $U_P$, and that

$$
\begin{align*}
\mathcal{H} &= \left\{ (x_i^p, p^s) \right\}_{i=1}^N \subseteq \mathbb{R}, \\
\mathcal{I}_r &= G_R(\mathcal{H}).
\end{align*}
$$

In particular, we have

- $\text{Supp}(\mathcal{I}) \cap U_P = V(x_1, \ldots, x_N)$, which is hence nonsingular, and
- $(\sigma(Q), \tilde{\mu}(Q)) = (\sigma(P), \infty)$ for any closed point $Q \in \text{Supp}(\mathcal{I}) \cap U_P$.

**Remark A.1.1.2.** (1) It is straightforward to see that assertion (1) actually gives the following characterization: An idealistic filtration $\mathcal{I}_p$ over $R_P$ is $\mathcal{E}$-saturated and $\tilde{\mu}(P) = \infty$ if and only if there exist a regular system of parameters $(x_1, \ldots, x_N, y_{N+1}, \ldots, y_d)$ and a subset of the form $\mathcal{H} = \left\{ (x_i^p, p^s) \right\}_{i=1}^N \subseteq \mathcal{I}_p$ such that $\mathcal{I}_p = G_R(\mathcal{H})$. (The subset $\mathcal{H}$ is then automatically an LGS of $\mathcal{I}_P$.)

(2) We construct the strand of invariants in our algorithm (cf. 0.2.3.2.2 in the introduction to Part I), and at year 0 it takes the following form

$$\text{inv}_{new}(P) = (\sigma^0_0, \tilde{\mu}_0, s^0_0)(\sigma^0_0, \tilde{\mu}_0, s^0_0) \cdots (\sigma^{n-1}_0, \tilde{\mu}_0, s^{n-1}_0)(\sigma^{n}_0, \tilde{\mu}_0, s^{n}_0),$$

with the last $n$-th unit $(\sigma^n_0, \tilde{\mu}_0, s^n_0)$ being equal to $(\sigma^n_0, \infty, 0)$. The subscript “0” refers to year “0”, while the superscript “$i$” refers to the stage “$i$”. (Note that, if we insert the new invariant $\tilde{\nu}$ so that the unit changes from the triplet $(\sigma, \tilde{\mu}, s)$ to the quadruplet $(\sigma, \tilde{\mu}, \tilde{\nu}, s)$, then the strand of invariants also changes accordingly (cf. [1.3.1]).)

The (local) maximum locus of the strand of invariants coincides with the support $\text{Supp}(\mathcal{I}_0^n)$ of the last $n$-th modification $\mathcal{I}_0^n$. (Note that in year 0 we always have $\tilde{\mu} > 1$ and hence that there is no gap between the (local) maximum locus and the support of the modification, an anomaly observed when $\tilde{\mu} = 1$.) The idealistic filtration $\mathcal{I}_0^n$ is $\mathcal{E}$-saturated with $\tilde{\mu}(P) = \infty$. Therefore, applying Theorem [A.1.1.1] we conclude that $\text{Supp}(\mathcal{I}_0^n)$ is nonsingular (in a neighborhood of $P$). (Note that all the idealistic filtrations we deal with in our algorithm are of r.f.g. type.) Therefore, we conclude that the center of blowup, which is chosen to be the maximum locus of the strand, is nonsingular. This is why Theorem [A.1.1.1] is called the (new) nonsingularity principle of the center. (After year 0, we have to make several technical adjustments, including an adjustment to overcome the gap between the (local) maximum locus and the support of the last modification and another adjustment to introduce the $\mathcal{E}_E$-saturation in the presence of the exceptional divisor $E$ instead of the usual $\mathcal{E}$-saturation. The basic tool for us to guarantee the nonsingularity of the center, however, is still Theorem [A.1.1.1].)

(3) If we assume further that $\mathcal{I}_P$ is $\mathcal{E}$-saturated, then after having assertion (1), we immediately come to the conclusion that $e_1 = \cdots = e_N = 0$, i.e., all the elements in the LGS (and hence of any LGS) are concentrated at level 1. That is to say, we obtain the old nonsingularity principle Theorem 4.2.1.1 in Part I as a corollary to the new nonsingularity principle Theorem [A.1.1.1] of this appendix.
A.1.2. Proof.

Proof for assertion (1). Step 1. Show $\mathcal{I}_P = G_{R_{\mathcal{I}}}(\mathbb{H})$ for any LGS $\mathbb{H}$ of $\mathcal{I}_P$.

First, note that, if $P \notin \text{Supp}(\mathcal{I})$, then we would have $\mathcal{I}_P = R_p \times \mathbb{R}$ (since $\mathcal{I}_P$ is $\mathcal{T}$-saturated cf. Case $P \notin \text{Supp}(\mathcal{I})$ of Lemma 1.1.2.1) and hence $\mu(P) = 0$. Thus our assumption $\mu(P) = \infty$ implies $P \in \text{Supp}(\mathcal{I})$. Second, we claim $\mathcal{I}_P = G_{R_{\mathcal{I}}}(\mathbb{H})$ for any LGS $\mathbb{H}$ of $\mathcal{I}_P$. In order to prove this claim, we can use the same argument as presented in the proof of the nonsingularity principle formulated in Chapter 4 of Part I. Note that this part of the proof did not use the assumption that $\mathcal{I}_P$ is $\mathfrak{R}$-saturated.

Alternatively, we can give a proof of the claim using the formal coefficient lemma (cf. Lemma 1.2.2.1) discussed here in Part II, without referring to the arguments in Part I.

Take an element $f \in (\mathcal{I}_P)_a < (\mathcal{I}_P)_u$, and let $f = \sum_{B \subseteq (\mathcal{Z}_{\mathbb{Z}_{\mathbb{Z}}})^N} a_B H^B$ be the power series expansion of the form ($\dagger$) as described in Lemma 1.2.2.1. From the formal coefficient lemma it follows that

$$(a_B, a - \|B\|) \in \mathcal{I}_P \quad \forall B \in (\mathbb{Z}_{\mathbb{Z}})^N.$$ 

Suppose there exists $B \in (\mathbb{Z}_{\mathbb{Z}})^N$ with $\|B\| < a$ such that $a_B \neq 0$. Then we would have

$$\mu(P) = \mu_H(\mathcal{I}_P) = \mu_H(\mathcal{I}_P) \leq \frac{\text{ord}_H(a_B)}{a - \|B\|} = \frac{\text{ord}(a_B)}{a - \|B\|} < \infty, \quad (\text{cf. Lemma 1.2.2.1})$$

a contradiction! Therefore, we conclude

$$a_B = 0 \quad \forall B \in (\mathbb{Z}_{\mathbb{Z}})^N \text{ with } \|B\| < a.$$ 

This implies

$$f \in (H^B; \|B\| \geq a).$$

Since $f \in (\mathcal{I}_P)_a$ is arbitrary, we conclude

$$(\mathcal{I}_P)_a \subset (H^B; \|B\| \geq a),$$

while the opposite inclusion $$(\mathcal{I}_P)_a \supset (H^B; \|B\| \geq a)$$ is obvious. Therefore, we finally conclude

$$(\mathcal{I}_P)_a = (H^B; \|B\| \geq a) \quad \forall a \in \mathbb{R},$$

which is equivalent to saying

$$\mathcal{I}_P = G_{R_{\mathcal{I}}}(\mathbb{H}).$$

Step 2. Inductive construction of an LGS and a regular system of parameters of the desired form via claim ($\dagger$).

Now, by induction, we assume that we have found an LGS $\mathbb{H} = \{ (h_{ij}, p^e) \}$ of $\mathcal{I}_P$ and a regular system of parameters $\{ (x_{ij}, y_{N+1}, \ldots, y_d) \}$ at $P$ such that

$$
\begin{cases}
  h_{ij} = x_{ij}^{p^e} & \text{if } e_i < e_u, \\
  h_{ij} = x_{ij}^{p^e} \mod m_p^{p^e+1} & \text{if } e_i \geq e_u.
\end{cases}
$$

Note that we use the double subscripts $h_{ij}$ for the elements in the LGS, where the first subscript indicates the level $p^e$ with $e_1 < \cdots < e_M$. So we have the total of $N$ elements at $M$ distinct levels in the LGS. (See 1.3.1.) The inductive assumption means that we have found an LGS and a regular system of parameters of the desired form up to the level $e_i = e_{u-1}$.

We want to show, by replacing $h_{ij}$ and $x_{ij}$ with $e_i = e_a$ via the use of claim ($\dagger$) which we state next, that we can also have

$$
\begin{cases}
  h_{ij} = x_{ij}^{p^e} & \text{if } e_i < e_{u+1}, \\
  h_{ij} = x_{ij}^{p^e} \mod m_p^{p^e+1} & \text{if } e_i \geq e_{u+1}.
\end{cases}
$$

This completes the proof.
Step 3. Statement and the proof of claim ($\diamond$)

This step is devoted to proving the following claim:

($\diamond$) Set, for $l \in \mathbb{Z}_{\geq 0}$,

$$J_l = F^{p^l}(m_P) + \left\{ \prod_{e_i < e_u} x_{ij}^{p^l \alpha_{ij}} ; C = (c_{ij} ; e_i < e_u), ||C|| \geq p^{e_u} \right\} + (I_P)_{p^l} \cap mp^{p^l+1} + mp^l.$$ 

(Recall that the symbol “$F$” represents the Frobenius map.)

Then we have the inclusion

$$(\mathbb{I}_P)_{p^l} \subset J_l \quad \forall l \in \mathbb{Z}_{\geq 0}. $$

Observe

$$(\mathbb{I}_P)_{p^l} = (H^B ; ||B|| \geq p^{e_u}) \quad \text{(since } \mathbb{I}_P = G_{R^B(H)} \text{)}$$

$$= \left\{ X^{C} ; C = (c_{ij} ; e_i < e_u), ||C|| \geq p^{e_u} \right\} + (h_{ij} ; e_i = e_u) + (h_{ij} ; e_i > e_u)$$

$$ \subset \left\{ X^{C} ; C = (c_{ij} ; e_i < e_u), ||C|| \geq p^{e_u} \right\} + F^{p^l}(m_P) + mp^{p^l+1}$$

$$ \subset J_{p^{l+1}}.$$ 

Therefore, the required inclusion holds for $l \leq p^{e_u} + 1$.

Now assume, by induction, that the required inclusion

$$\mathbb{I}_P)_{p^l} \subset J_l$$

holds for a fixed $l \geq p^{e_u} + 1$. We want to show

$$\mathbb{I}_P)_{p^{l+1}} \subset J_{l+1}.$$

Take an arbitrary element $f \in (\mathbb{I}_P)_{p^{l+1}} \subset J_l$.

We may choose $\{\alpha_{ST} ; S, T \} \subset k$ such that

$$f = \sum_{|S,T|=l} \alpha_{ST} X^S Y^T \in J_{l+1}.$$ 

Note that then there exists $w \in m_P$ such that

$$\sum_{|S,T|=l} \alpha_{ST} X^S Y^T \in (\mathbb{I}_P)_{p^{l+1}} - mp^{l+1},$$

Set

$$\left\{ \begin{array}{l}
    s_{ij} = p^{e_u} s_{ij} + s_{ij} \text{ with } 0 \leq s_{ij} < p^{e_u} \\
    S_{ij} = (s_{ij})_h \\
    S_T = (s_{ij}).
\end{array} \right.$$ 

Then we have $S = [S_{ij}] + S_r$ and $X^S Y^T = X^{[S_{ij}]} X^S Y^T$.

We analyze the terms in

$$\sum_{|S,T|=l} \alpha_{ST} X^S Y^T \in \sum_{|S,T|=l} \alpha_{ST} X^{[S_{ij}]} X^S Y^T.$$
Case 1. \( S_y = 0 \).

In this case, we write for simplicity

\[ X^S Y^T = X^S Y^T = Z^V \]

by setting

\[
\begin{aligned}
(X, Y) &= (\{x_1, y_{N+1}, \cdots, y_d\} = (z_1, \cdots, z_d) = Z \\
(S, T) &= (s_i, t_{N+1}, \cdots, t_j) = (v_1, \cdots, v_d). \\
\end{aligned}
\]

Subcase 1.1 : \( p^{\alpha} \mid V \).

In this subcase, we conclude

\[ \alpha ST X^S Y^T = \alpha ST Z^V \in F^e(\mathfrak{m}_P) \subset J_{i+1}. \]

Subcase 1.2 : \( p^{\alpha} \not\mid V \).

In this subcase, let \( v_{\omega} \) be a factor, not divisible by \( p^{\alpha} \), of \( V \). Set

\[ v_{\omega} = p^{\tilde{\alpha}} v_{\omega, \tilde{\alpha}} + v_{\omega, r} \text{ with } 0 < v_{\omega, r} < p^{\tilde{\alpha}}. \]

Apply \( \partial_{\omega} \) to \((\bigtriangledown)\) and obtain

\[
\begin{aligned}
\partial_{\omega} \left( v_{\omega} + \sum_{\emptyset \neq S, T \subseteq \emptyset \neq \emptyset} \alpha ST X^S Y^T \right) &= \partial_{\omega} \left( \sum_{\emptyset \neq S, T \subseteq \emptyset \neq \emptyset} \alpha ST X^S Y^T \right) \\
&= \alpha ST Z^{V - v_{\omega, \tilde{\alpha}}} + (\text{other monomials of degree } (l - v_{\omega, r})) \\
&\in (\mathcal{I}_P)_{p^{\alpha} - v_{\omega, \tilde{\alpha}}} + m_{p^{l - v_{\omega, r} + 1}} \\
&= (\mathcal{I}_P)_{p^{\alpha} - v_{\omega, \tilde{\alpha}}} \cap m_{p^{l - v_{\omega, r} + 1}}.
\end{aligned}
\]

On the other hand, we observe

\[ (\mathcal{I}_P)_{p^{\alpha} - v_{\omega, \tilde{\alpha}}} \cap m_{p^{l - v_{\omega, r} + 1}} \subset \sum_{1 \leq j \leq M} h_{ij} m_{p^{l - v_{\omega, r} - p^{\tilde{\alpha}}}}. \]

(We use the convention that \( m_{p^{l - v_{\omega, r} - p^{\tilde{\alpha}}}} = R_P \) if \( l - v_{\omega, r} - p^{\tilde{\alpha}} \leq 0 \).)

In fact, let \( g \in (\mathcal{I}_P)_{p^{\alpha} - v_{\omega, \tilde{\alpha}}} \cap m_{p^{l - v_{\omega, r}}} \) be an arbitrary element, and \( g = \sum a_B \mathcal{H}^B \) the power series expansion of the form \((\star)\) as described in Lemma 2.1.2.1. Then it follows from the condition \( \tilde{\mu}(\mathcal{P}) = \infty \) and \( 0 < p^{\alpha} - v_{\omega, \tilde{\alpha}} \) that \( a_B = 0 \) (cf. Lemma 2.1.1.1), and from the construction that \( \text{ord}_p(a_B) \geq (l - v_{\omega, r}) - ||B|| \) for any \( B \in (\mathbb{Z}_{\geq 0})^H \) (cf. Remark 2.1.2.1(1)). Therefore, we conclude \( f \in \sum_{1 \leq j \leq M} h_{ij} m_{p^{l - v_{\omega, r} - p^{\tilde{\alpha}}}} \). This proves the inclusion above. (Note that the inclusion above can also be derived using Lemma 4.1.2.3 in Part I via the fact that \( \mathfrak{I}_P = \mathcal{G}_{R_p}(\mathfrak{I}_P) \).)

However, this inclusion implies that any monomial of degree \( l - v_{\omega, r} \) in the power series expansion of an element in \( (\mathcal{I}_P)_{p^{\alpha} - v_{\omega, \tilde{\alpha}}} \cap m_{p^{l - v_{\omega, r}}} \), with respect to the regular system of parameters \( (x_1, \cdots, x_N, y_{N+1}, \cdots, y_d) \), should be divisible by some element in the set \( \{ x_i^p : 1 \leq i \leq M \} \), and hence that the monomial \( Z^{V - v_{\omega, \tilde{\alpha}}} \) can not appear as \( S_y = 0 \).

Therefore, in this subcase, we conclude

\[ \alpha ST = 0. \]
Case 2. \( S_q \neq 0 \)

Subcase 2.1 : \( s_{ijq} > 0 \) for some \( i \geq u \).

In this subcase, we compute

\[
X^S Y^T \in x_{ij}^{ρ_i} \left( h_{ij} + \sum_{ρ_i > ρ_j} m_{ρ_j} \right) \subset h_{ij} + \sum_{ρ_i > ρ_j} m_{ρ_j} \subset (I_P)_{ρ_j} \cap m_{ρ_j} \subset J_{l+1}.
\]

(Note that, in order to obtain the second last inclusion above, we use the fact that \( h_{ij} \in m_{ρ_j} \) if \( i > u \), and use the condition \( l \geq ρ_j + 1 \) if \( i = u \)).

Therefore, we conclude

\[
α_{ST} X^S Y^T \in J_{l+1}.
\]

Subcase 2.2 : \( s_{ijq} = 0 \) for any \( i \geq u \) and \( ||S_q|| \geq p^{ρ_j} \).

In this subcase, we conclude

\[
α_{ST} = α_{ST} X^S Y^T \in \left( X^C \right)_{C = (c_{ij}; e_i < e_u), ||C|| \geq p^{ρ_j}} \subset J_{l+1}.
\]

Subcase 2.3 : \( s_{ijq} = 0 \) for any \( i \geq u \) and \( ||S_q|| < p^{ρ_j} \).

Note that \( 0 < ||S_q|| \) by the case assumption.

In this subcase, apply \( \partial_{X^S Y^T} \) to (Ⅷ) and obtain

\[
\partial_{X^S Y^T} \left( w^{ρ_i} + \sum_{ρ_i > ρ_j} α_{ST} X^S Y^T \right) = \partial_{X^S Y^T} \left( \sum_{ρ_i > ρ_j} α_{ST} X^S Y^T \right) = α_{ST} X^S Y^T + (\text{other monomials of degree } (l - ||S_q||)) \in (I_P)_{ρ_j - ||S_q||} \cap m_{ρ_j - ||S_q||} = (I_P)_{ρ_j - ||S_q||} \cap m_{ρ_j - ||S_q||} + m_{ρ_j - ||S_q||}.
\]

On the other hand, we observe

\[
(I_P)_{ρ_j - ||S_q||} \cap m_{ρ_j - ||S_q||} \subset \sum_{1 ≤ j ≤ M} h_{ij} m_{ρ_j - ||S_q||}.
\]

(We use the convention that \( m_{ρ_j - ||S_q||} = R_P \) if \( l - ||S_q|| - p^{ρ_j} \leq 0 \). The inclusion follows from the same argument as in Subcase 1.2.)

However, this inclusion implies that any monomial of degree \( l - ||S_q|| \) in the power series expansion of an element in \( (I_P)_{ρ_j - ||S_q||} \cap m_{ρ_j - ||S_q||} \), with respect to the regular system of parameters \( (x_1, \ldots, x_N, y_{N+1}, \ldots, y_d) \) should be divisible by some element in the set \( \{ x_{ij}^{p_{ρ_j}} ; 1 ≤ i ≤ M \} \), and hence that the monomial \( X^S Y^T \) can not appear.

Therefore, in this subcase, we conclude

\[
α_{ST} = 0.
\]

From the above analysis of the terms in \( \sum_{1 ≤ j ≤ M} α_{ST} X^S Y^T \), it follows that

\[
f = \sum_{1 ≤ j ≤ M} α_{ST} X^S Y^T + J_{l+1} = J_{l+1}.
\]

Since \( f \in (I_P)_{ρ_j} \) is arbitrary, we conclude \( (I_P)_{ρ_j} \cap J_{l+1} \), completing the inductive argument for claim (Ⅷ).

Step 4. Finishing argument for the inductive construction.
Claim (\(\bigcirc\)) states
\[
(I_p)_{p^n} \subset J_l = F^{e_0}(m_p) + \{x^{\{C\}} : C = (c_{ij}; e_i < e_u), \|C\| \geq p^{e_0} \}
+ (I_p)_{p^n} \cap m_p^{p^{e_0}+1} + m_p^l \quad \forall l \in \mathbb{Z}_{\geq 0}.
\]
This implies
\[
(I_p)_{p^n} \subset F^{e_0}(m_p) + \{x^{\{C\}} : C = (c_{ij}; e_i < e_u), \|C\| \geq p^{e_0} \}
+ (I_p)_{p^n} \cap m_p^{p^{e_0}+1} + F^{e_0}(m_p^l)R_p \quad \forall l \in \mathbb{Z}_{\geq 0}.
\]
Since \(R_p\) is a finite \(F^{e_0}(R_p)\)-module, including
\[
F^{e_0}(m_p) + \{x^{\{C\}} : C = (c_{ij}; e_i < e_u), \|C\| \geq p^{e_0} \} + (I_p)_{p^n} \cap m_p^{p^{e_0}+1}
\]
as an \(F^{e_0}(R_p)\)-submodule, we conclude (cf. [Mat86], page 62 last line) that
\[
(I_p)_{p^n} \subset \bigcap_{l \in \mathbb{Z}_{\geq 0}} \left[ F^{e_0}(m_p) + \{x^{\{C\}} : C = (c_{ij}; e_i < e_u), \|C\| \geq p^{e_0} \}
+ (I_p)_{p^n} \cap m_p^{p^{e_0}+1} + F^{e_0}(m_p^l)R_p \right]
= F^{e_0}(m_p) + \{x^{\{C\}} : C = (c_{ij}; e_i < e_u), \|C\| \geq p^{e_0} \} + (I_p)_{p^n} \cap m_p^{p^{e_0}+1}.
\]
Since
\[
\{x^{\{C\}} : C = (c_{ij}; e_i < e_u), \|C\| \geq p^{e_0} \} \subset (I_p)_{p^n},
\]
we also conclude
\[
(I_p)_{p^n} \subset F^{e_0}(m_p) \cap (I_p)_{p^n} + \{x^{\{C\}} : C = (c_{ij}; e_i < e_u), \|C\| \geq p^{e_0} \} + (I_p)_{p^n} \cap m_p^{p^{e_0}+1}.
\]
Now choose
\[
\{h_{ij}' = x_{ij}^{p^{e_0}} \} \subset F^{e_0}(m_p) \cap (I_p)_{p^n}
\]
such that
\[
\{h_{ij}' \mod m_p^{p^{e_0}+1} \} \cup \{x_{ij}^{\{C\}} \mod m_p^{p^{e_0}+1} : C = (c_{ij}; e_i < e_u), \|C\| = p^{e_0} \}
\]
form a \(k\)-basis of \(L(I_p)_{p^n}\).

In order to finish the inductive argument (cf. Step 2) to complete the proof for assertion (1), we have only to replace \(h_{ij}\) and \(x_{ij}\) with \(h_{ij}'\) and \(x_{ij}'\).

**Proof for assertion (2).** Take a regular system of parameters \((x_1, \ldots, x_N, y_{N+1}, \ldots, y_d)\) and an LGS \(H\) of \(I_p\) as described in assertion (1).

By choosing an affine neighborhood \(P \in U_p = \text{Spec } R_p\) of \(P\) sufficiently small, we may assume that \((x_1, \ldots, x_N, y_{N+1}, \ldots, y_d)\) is a regular system of parameters over \(U_p\) and that \(H = \{(x_i^{p^r}, x_j^{p^s})\}_{i=1}^N \subset L_p\).

Now we know by assumption that \(I\) is of r.f.g. type, i.e., \(I = G_R((f_{ij}, a_{ij}))_{ij\in A}\) for some \((f_{ij}, a_{ij}))_{ij\in A} \subset R \times \mathbb{Q}_{\geq 0}\) with \#\(A < \infty\).

Since \(I_p = G_R(\mathbb{H})\), we can write each \(f_{ij}\) as a finite sum of the form \(\sum g_{B,\lambda}H^B\) with \(g_{B,\lambda} \in R_p\). By shrinking \(U_p = \text{Spec } R_p\) if necessary, we may assume that the coefficients \(g_{B,\lambda}\) are in \(R_p\) for all \(B\) and \(\lambda \in \Lambda\). Then we have
\[
I_p = G_R((f_{ij}, a_{ij}))_{ij\in A} \subset G_R(\mathbb{H}).
\]
Since the opposite inclusion \(I_p \supset G_R(\mathbb{H})\) is obvious, we conclude
\[
I_p = G_R(\mathbb{H}) = \text{Supp}(\mathbb{H}) \subset U_p = \text{Supp}(I_p) = \text{Supp}(G_R(\mathbb{H})).
\]

It follows immediately from the above conclusions that
\[
\text{Supp}(I) \cap U_p = \text{Supp}(I_p) = \text{Supp}(G_R(\mathbb{H}))
= \left\{ Q \in U_p : \mu_Q(x^{p^l}, p^{e_0}) \geq 1 \text{ for } l = 1, \ldots, N \right\}
= V(x_1, \ldots, x_N)
\]
which is nonsingular.
Given any closed point $Q \in \text{Supp}(I) \cap U_P = V(x_1, \cdots, x_N)$, it also follows from the above conclusions that $(x_1, \cdots, x_N)$ is a part of a regular system of parameters at $Q$ with a subset $H = \{(x_i^{p^i}, p^i)\}_{i=1}^N \subset \mathbb{Q}$ such that $I_Q = G_{R_Q}(H)$. This implies that $H$ is an LGS of $I_Q$ and that $\tilde{\mu}(Q) = \infty$. Therefore, we conclude

$$(\sigma(Q), \tilde{\mu}(Q)) = (\sigma(P), \infty).$$

This concludes the proof of Theorem A.1.1.1.
The list of references for Part II is largely identical to the one for Part I [Kaw07], which we reproduce below for the convenience of the reader, with a few more papers and Part I added.

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