Relation between Scattering and Production Amplitudes
—concerning $\sigma$ Particle in $\pi\pi$ System—

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In a series of previous papers the present authors and collaborators have shown strong
evidence for existence of the long-sought $\sigma$-particle by analyzing the $\pi\pi$-scattering and $\pi\pi$-
production processes, applying, respectively, the IA method and the VMW method. In
this paper we examine the relation between the scattering amplitude $T$ and the production
amplitude $F$ from the viewpoint of the unitarity and the final state interaction (FSI) theorem
by using a simple field theoretical model. As a result it is shown that the amplitudes in the
physical state representation are directly represented through the Breit-Wigner amplitudes
of the relevant resonances, and the respective forms of $T$ and $F$ coincide with those in the
IA and VMW methods, justifying our methods of analyses.

§1. Introduction

The light iso-singlet scalar $\sigma$ meson appears as a chiral partner of the $\pi$ meson
in the linear representation of chiral symmetry. In the Nambu-Jona-Lasinio (NJL)
model and its extended version (ENJL) adapted to the quark model, which simply realizes the physical situation of $D\chi$SB of QCD, the existence of the $\sigma$-meson (or scalar meson nonet) is predicted with mass $\approx 2m_q$ ($m_q$ being the constituent quark mass). This $\sigma$ gives quarks constituent masses, and in this sense it partly plays a role of the Higgs particle of QCD. The existence of the $\sigma$-meson has been long-sought from various viewpoints, both theoretically and phenomenologically. However, its existence as a resonant particle has not yet been generally accepted. A major reason for this is due to the negative results of conventional analyses of the $\pi\pi$ phase-shift obtained from the high-statistics data of a CERN-Munich experiment in 1974.

In the recent $pp$-central collision experiment, a huge event concentration in the
$I = 0$ $S$-wave $\pi\pi$ channel is seen in the region of $m_{\pi\pi}$ around 500 - 600 MeV,
which is too large to be explained as a simple “background” and strongly suggests
the existence of a resonant particle, which can be identified with $\sigma$. Actually, it has
been shown that the characteristic shape of the $\sigma$ effective mass spectra below

* The ENJL model predicts the existence of the NG pseudo-scalar boson, scalar meson, and, moreover, vector and axial-vector meson nonets.
** However, see the analyses which suggest the existence of $\sigma$. 
1 GeV can be explained by a coherent sum of the two (σ and \( f_0 \)) Breit-Wigner resonant amplitudes. This parametrization method of the ππ-production amplitude is called the VMW method.\(^{31}\)

A similar event concentration around the region of \( m_{\pi\pi} = 500 \sim 600 \) MeV is also observed in the \( \pi\pi \) system obtained in the \( J/\Psi \rightarrow \omega\pi\pi \) decay, suggesting the existence of an iso-singlet \( S \)-wave state, which also seems to be identified with \( \sigma \).

However, the claim of \( \sigma \)-existence has been criticized from the so-called “universality of \( \pi\pi \)-scattering amplitude” argument as follows: “unitarity requires a resonance that decays to \( \pi\pi \), for example, has to couple in the same way to this final state whether produced in \( \pi\pi \) scattering or centrally in \( pp \rightarrow pp(\pi\pi) \)...Thus claims of a narrow \( \sigma(500) \) in the GAMS results cannot be correct as no such state is seen in \( \pi\pi \) scattering.”

On the other hand being inspired by these experiments, we and our collaborators have recently made a re-analysis of the \( \pi\pi \) phase shift through a new \( S \)-matrix parametrization method, the interfering amplitude (IA) method, and found strong evidence for the existence of the \( \sigma \) particle. The reason we obtained a different result from that in the conventional analyses is due to the introduction of the repulsive background phase \( \delta_{BG} \), which cancels a main part of the attractive phase due to \( \sigma \) production. This cancellation mechanism is guaranteed by chiral symmetry.\(^{33}\) However, it has been overlooked in the conventional analyses,\(^{33}\) leading to the wrong conclusion against the \( \sigma \) existence. Several other groups have independently performed re-analyses also leading to a positive conclusion for \( \sigma \) existence.\(^3\) Thus, not only the above “universalilty” argument for the \( \pi\pi \)-production processes but also the contents of the “universalilty” itself must be re-considered.

In this paper we examine the validity of the methods, the IA method (VMW method) for scattering (production) amplitude applied in the above phenomenological analyses leading to \( \sigma \) existence, from the general viewpoint of the unitarity and the applicability of FSI-theorem, especially noting the relation to the universality argument.

§2. General problem

In treating the \( \pi\pi \)-scattering and production amplitudes, there are two general problems to be taken into account.

Unitarity The scattering amplitude \( \mathcal{T} \) (and its hermitian conjugate \( \mathcal{T}^\dagger \)) must satisfy the condition

\[
\mathcal{T} - \mathcal{T}^\dagger = 2i\mathcal{T}\rho\mathcal{T}^\dagger, \tag{2.1}
\]

where \( \rho \) is the \( \pi\pi \)-state density.

Final State Interaction (FSI) theorem The production amplitude \( \mathcal{F} \) must

\(^{\ast}\) Reflecting these results the \( \sigma \)-particle has been revived in the list of the latest edition of PDG,\(^{39}\) after missing for twenty years, with the somewhat tentative label, “\( f_0(400 \sim 1200) \) or \( \sigma \)”.
have the same phase\(^{47}\) as \(T\),

\[
T \propto e^{i\delta} \rightarrow F \propto e^{i\delta},
\]

(“Universality” of scattering amplitudes) Furthermore, a more restrictive relation between \(F\) and \(T\) is conventionally required on the basis of the “Universality of \(T\),”\(^{33}\) as mentioned in Introduction: That is,

\[
F = \alpha(s)T,
\]

with a smooth real function \(\alpha(s)\) of \(s\). Following this, most analyses of experimental data on \(F\) in the region \(m_{\pi\pi} \lesssim 1.5\) GeV, obtained in any type of production processes, are made as follows. First, \(T\), with comparatively more rigorous data, is analyzed. Then, using this result as input, \(F\) is analyzed by parametrizing the function \(\alpha(s)\). Thus, in this procedure, physical information can be obtained only through scattering experiments, while any production experiment loses its value in seeking new resonances.

Our results obtained by phenomenological analysis, using the VMW-method, on the production process was criticized\(^{33}\) along the above line of thought that the claims for \(\sigma\) in \(F\) cannot be correct, as no such resonant poles exist in \(T\) due to the conventional phase shift analyses. However, as a result of the re-analysis of the \(\pi\pi\) scattering with the IA method, which satisfies unitarity, there seems to indeed exist a \(\sigma\) pole in \(T\). Accordingly, the main reason for the above criticism has been lost. However, in the VMW method there still remains a problem; whether or not it is consistent with the FSI theorem.

In the following we re-examine the relation between \(F\) and \(T\) concretely, by using a simple model.\(^{48},^{49}\)

(Simple field theoretical model) In the NJL-type model as a low energy effective theory of QCD, (and in the linear \(\sigma\) model, \(L\sigma\)M, obtained as its local limit), or in the constituent quark model, the pion \(\pi\) and the resonant particles such as \(\sigma(600)\) or \(f_0(980)\) are the color-singlet \(q\bar{q}\)-bound states and are treated equivalently. These “intrinsic quark dynamics states,” denoted as \(\pi, \sigma\) and \(f\) are stable particles with zero widths and appear from the beginning. Actually, these particles have structures

\(^{a)}\) See detailed discussion in §3.4.
and interact with one another (and a production channel “P”) through the residual strong interaction
\[ \mathcal{L}^{\text{int}} = g_\sigma \bar{\sigma} \pi \pi + \tilde{g}_f \tilde{f} \pi \pi + \bar{g}_{2\pi} (\pi \pi)^2 (\mathcal{L}^{\text{prod}} = \tilde{\xi}_\sigma \tilde{\sigma} \text{“P”} + \tilde{\xi}_f \tilde{f} \text{“P”} + \tilde{\xi}_{2\pi} (\pi \pi) \text{“P”}). \quad (2.4) \]
Due to this, these bare states change into physical states, denoted by \( \pi = \bar{\pi} \), \( \sigma \) and \( f \) with finite widths, as shown in Fig. 1. In the following we consider only the virtual two \( \pi \)-meson effects for the resonant \( \sigma \) and \( f \) particles.

§3. Three Different ways of description of scattering amplitudes

There are the following three ways to represent scattering amplitudes, corresponding to the three types of basic states for describing the resonant particles, as is depicted in Fig. 2.

1. Intrinsic quark-dynamics states (bare states) representation

In the bases of zero-width bare states, denoted as \( |\bar{\alpha}\rangle \), the \( \pi \pi \)-scattering amplitude is represented in terms of the \( \pi \pi \)-coupling constants \( \bar{g}_\alpha \) and the propagator matrix \( \bar{\Delta} \) as
\[ T = \bar{g}_\alpha \bar{\Delta}_{\alpha\bar{\beta}} \bar{g}_{\bar{\beta}}. \quad (3.1) \]
By taking into account the effects of repetition of the \( \pi \pi \)-loop, the bare-state propagator acquires a finite width. The corresponding inverse propagator is represented by
\[ \bar{\Delta}^{-1}_{\alpha\bar{\beta}} = (\bar{M}^2 - s - i\bar{G})_{\alpha\bar{\beta}}. \quad (3.2) \]
The real and imaginary parts of the squared mass matrix take non-diagonal forms, which implies that the bare states have indefinite masses and lifetimes. The imaginary part of the inverse propagator is
\[ \bar{G}_{\alpha\bar{\beta}} = \bar{g}_\alpha \rho \bar{g}_{\bar{\beta}}; \quad (\rho = \sqrt{1 - 4m_\pi^2/s/16\pi}), \quad (3.3) \]
where \( \rho \) is the \( \pi \pi \)-state density. Then our \( T \) is easily shown to satisfy the unitarity, Eq. (2.1).

2. “K-matrix” states representation

The real part of \( \Delta^{-1} \) is symmetric and can be diagonalized by an orthogonal transformation: It transforms the bare states \( |\bar{\alpha}\rangle \) into the “K-matrix” states \( |\bar{\alpha}\rangle \), as
\[ |\bar{\alpha}\rangle \equiv |\bar{\alpha}\rangle o_{\alpha\bar{\alpha}}. \quad (3.4) \]
Correspondingly, \( T \) is represented by
\[ T = \bar{g}_\alpha \bar{\Delta}_{\alpha\bar{\beta}} \bar{g}_{\bar{\beta}}. \quad (3.5) \]
with the inverse propagator
\[ \bar{\Delta}^{-1}_{\alpha\bar{\beta}} = (\Delta^{-1}_K - i\bar{G})_{\alpha\bar{\beta}}; \quad \Delta^{-1}_K_{\alpha\bar{\beta}} = (\bar{m}_\alpha^2 - s)\delta_{\alpha\beta}, \quad \bar{G}_{\alpha\bar{\beta}} = \bar{g}_\alpha \rho \bar{g}_{\bar{\beta}}. \quad (3.6) \]
Intrinsic quark-dynamics states

\[ \{\tilde{\alpha}; \tilde{\sigma}, \tilde{f}\} \]

bare states

\[ T = \tilde{g}_{\tilde{\alpha}} \tilde{\Delta}_{\tilde{\alpha} \tilde{\beta}} \tilde{g}_{\tilde{\beta}} \]

(\tilde{\eta}: real const.)

stable intrinsic qq states

\[ \tilde{\Delta}_{\tilde{\alpha} \tilde{\beta}} = (\tilde{\eta}^2 - s - iG)_{\tilde{\alpha} \tilde{\beta}} \]

Inverse Prop.

\[ \tilde{G}_{\tilde{\alpha} \tilde{\beta}} = \frac{\tilde{g}_{\tilde{\alpha}} \rho_{\tilde{\beta}}}{\tilde{g}_{\tilde{\beta}}} \]

Unitarity

"K-matrix" states

\[ \{\tilde{\alpha}; \tilde{\sigma}, \tilde{f}\} \]

\[ T = \tilde{g}_{\tilde{\alpha}} \tilde{\Delta}_{\tilde{\alpha} \tilde{\beta}} \tilde{g}_{\tilde{\beta}} \]

\[ \tilde{\Delta}_{\tilde{\alpha} \tilde{\beta}} = (\tilde{\eta}^2 - s - iG)_{\tilde{\alpha} \tilde{\beta}} \]

(\tilde{\alpha}, \tilde{\beta}: complex orthogonal)

physical state

\[ |\tilde{\alpha}\rangle \equiv |\tilde{\alpha}\rangle \tilde{u}_{\tilde{\alpha}} \]

\[ \langle \tilde{\alpha}| \equiv \langle \tilde{\alpha}| \tilde{\alpha}\tilde{\alpha} \tilde{\alpha} \]

Fig. 2. Three different representations of scattering amplitudes.

where the coupling constant \( \tilde{g}_{\tilde{\alpha}} (= \tilde{g}_{\tilde{\alpha}} \tilde{g}_{\tilde{\alpha}}) \) is real. These states have definite masses but indefinite lifetimes. The propagator \( \tilde{\Delta} \) is, owing to Eq. (3.6), expressed in the form representing concretely the repetition of the \( \pi\pi \) loop, as

\[ \tilde{\Delta} = (1 - i\Delta K \tilde{G})^{-1} \Delta K = \Delta K + i\Delta \tilde{G} \Delta K. \]  

Then \( \mathcal{T} \), similarly as the \( \mathcal{K} \) matrix representation in potential theory, takes the form

\[ \mathcal{T} = \mathcal{K} + i\mathcal{P} \rho \mathcal{K} = \mathcal{K}(1 - i\rho K)^{-1}; \quad \mathcal{K} = \tilde{g}_{\tilde{\alpha}} \Delta K_{\tilde{\alpha} \tilde{\beta}} \tilde{g}_{\tilde{\beta}} = \tilde{g}_{\tilde{\alpha}} (\tilde{m}_{\tilde{\alpha}}^2 - s)^{-1} \tilde{g}_{\tilde{\alpha}}. \]
From the viewpoint of the present field-theoretical model, this "\(\mathcal{K}\) matrix," Eq. (3.8), has the physical meaning of the propagators of bare particles with infinitesimal imaginary widths, \(\tilde{m}_\alpha^2 \rightarrow \tilde{m}_\alpha^2 - i\epsilon\), while the original \(\mathcal{K}\) matrix in potential theory is purely real and has no direct meaning.

3. Physical resonant states representation,

The imaginary part of \(\tilde{\Delta}^{-1}\) in the \(\mathcal{K}\)-matrix state representation was remained in a non-diagonal form. \(\tilde{\Delta}^{-1}\) can be diagonalized by a complex orthogonal matrix \(u\), satisfying \(\tilde{t} uu = 1\). It transforms \(|\tilde{\alpha}\rangle\) into the unstable physical states \(|\alpha\rangle\) as

\[
|\alpha\rangle \equiv |\tilde{\alpha}\rangle u_{\tilde{\alpha}\alpha}, \quad \langle \alpha| \equiv \tilde{t} u_{\tilde{\alpha}\alpha} |\tilde{\alpha}\rangle.
\]

(3.9)

It is to be noted that the transformation is not unitary and \(|\alpha| \neq (|\alpha\rangle)^\dagger\). Correspondingly, the \(\mathcal{T}\) matrix is represented by

\[
\mathcal{T} = F_\alpha \Delta_{\alpha\beta} F_\beta = \sum_\alpha F_\alpha (\lambda_\alpha - s)^{-1} F_\alpha; \quad F_\alpha = \tilde{g}_\beta u_{\tilde{\beta}\alpha}
\]

(3.10)

where \(\lambda_\alpha\) is the physical squared mass of the \(|\alpha\rangle\) state, and the \(F_\alpha\) are the physical coupling constants, which are generally complex. The physical state has a definite mass and lifetime, and is observed as a resonant particle directly in experiments.

§4. Scattering and production amplitudes consistent to FSI-condition

In the following we show how the formulas in the IA and VMW methods satisfying the FSI theorem are derived effectively in the physical state representation. We start from the "\(\mathcal{K}\) matrix" states, which can be identified with the bare states \(|\tilde{\alpha}\rangle (\equiv |\tilde{\alpha}\rangle)\) without loss of essential points, since the reality of the coupling constant is unchanged through the orthogonal transformation Eq. (3.4). The real part of the mass correction generally does not have a sharp \(s\) dependence, and, accordingly, \(\tilde{g}\) is almost \(s\) independent, except for in the threshold region.

4.1. Derivation of IA-method and VMW-method satisfying FSI-theorem

First we consider the two \((\tilde{\sigma}, \tilde{f})\) resonance-dominating case, assuming \(\tilde{g}_{2\pi} = \xi_{2\pi} = 0\). The scattering amplitude \(\mathcal{T}\) in the bare state representation is given by Eq. (3.8) as

\[
\mathcal{T}^{\text{Res}} = \mathcal{K}^{\text{Res}}/(1 - i\rho \mathcal{K}^{\text{Res}}); \quad \mathcal{K}^{\text{Res}} = \tilde{g}_\sigma (\tilde{m}_\sigma^2 - s)^{-1} \tilde{g}_\sigma + \tilde{g}_f (\tilde{m}_f^2 - s)^{-1} \tilde{g}_f.
\]

(4.1)

The production amplitude \(\mathcal{P}\) is obtained by replacing \(\tilde{g}^2\), appearing in the numerator \(\mathcal{K}\) of \(\mathcal{T}\), by \(\tilde{\xi}\) as

\[
\mathcal{P}^{\text{Res}} = \mathcal{P}^{\text{Res}}/(1 - i\rho \mathcal{K}^{\text{Res}}); \quad \mathcal{P}^{\text{Res}} = \tilde{\xi}_\sigma (\tilde{m}_\sigma^2 - s)^{-1} \tilde{g}_\sigma + \tilde{\xi}_f (\tilde{m}_f^2 - s)^{-1} \tilde{g}_f,
\]

(4.2)

where \(\mathcal{P}^{\text{Res}}\) is the production "\(\mathcal{K}\) matrix". The FSI theorem is automatically satisfied, since both \(\mathcal{K}^{\text{Res}}\) and \(\mathcal{P}^{\text{Res}}\) can be treated as real and the phases of \(\mathcal{T}^{\text{Res}}\) and \(\mathcal{P}^{\text{Res}}\) come from the common factor \((1 - i\rho \mathcal{K}^{\text{Res}})^{-1}\).
The quantity \( \lambda M \) where the physical squared mass \( M \) is given by

\[
\lambda = (1/2)[\bar{m}_\sigma^2 + \bar{m}_f^2 - i\rho(\bar{g}_\sigma^2 + \bar{g}_f^2) \pm \sqrt{(\bar{m}_\sigma^2 - \bar{m}_f^2 - i\rho(\bar{g}_\sigma^2 - \bar{g}_f^2))^2 - 4\rho^2\bar{g}_\sigma^2\bar{g}_f^2}];
\]

\[
\equiv M_\alpha^2 - i\rho g_\alpha^2. \quad \alpha = f, \sigma.
\]

The quantity \( \lambda_\alpha \) in Eq. (4-4) is identified with \( M_\alpha^2 - i\rho g_\alpha^2 \) appearing in the usual Breit-Wigner formula. Thus we define the physical mass \( M_\alpha \) and the real physical coupling factor \( g_\alpha \) \((g_\alpha^2 \equiv -\text{Im} \lambda_\alpha/\rho)\). By a simple manipulation, the \( T \) amplitude obtained in this way automatically satisfies the FSI theorem.

In the physical state representation, \( T \) is given by Eq. (3-10) as

\[
T^\text{Res} = F_\sigma(\lambda_\sigma - s)^{-1}F_\sigma + F_f(\lambda_f - s)^{-1}F_f,
\]

where the physical squared mass \( \lambda_\alpha \) is given by

\[
\lambda_\alpha = \frac{g_\sigma^2}{\lambda_\sigma - s} + \frac{g_f^2}{\lambda_f - s} + \frac{2i\rho g_\sigma^2}{\lambda_\sigma - s\lambda_f - s};
\]

\[
(4.5)
\]

where the \( \lambda_\alpha \) and \( g_\alpha \) are represented by \( \bar{m}_\alpha, \bar{g}_\alpha \) and \( \rho(s) \) (given in Eq. (3-3)), and, accordingly, are almost \( s \) independent, except for the threshold region. Thus, Eq. (4-5) is understood to be just the form of the scattering amplitude applied in IA method.

Similarly, \( F^\text{Res} \) in the physical state representation is given by

\[
F^\text{Res} = \frac{r_\sigma e^{i\theta_\sigma}}{\lambda_\sigma - s} + \frac{r_f e^{i\theta_f}}{\lambda_f - s};
\]

\[
(4.6)
\]

where \( r_\sigma e^{i\theta_\sigma} \equiv \sum_\sigma F_\sigma \) and \( r_f e^{i\theta_f} \equiv \sum_f F_f \) \((\Sigma_\sigma(\equiv \bar{g}_\beta u_\beta) \) is the production coupling factor in the physical state representation, which is generally complex). By using the equation

\[
F^\text{Res} = \frac{\bar{r}_\sigma(\bar{m}_\sigma^2 - s) + \bar{r}_f(\bar{m}_f^2 - s)}{(\lambda_\sigma - s)(\lambda_f - s)}; \quad \bar{r}_\sigma \equiv \bar{g}_\sigma \bar{\xi}_\sigma, \quad \bar{r}_f \equiv \bar{g}_f \bar{\xi}_f,
\]

\[
(4.7)
\]

Equation (4-1) can be rewritten into the form \( \rho T^\text{Res} = -\text{Im} D/D \) (where \( D = (\bar{m}_\sigma^2 - s)(\bar{m}_f^2 - s) - i\rho(\bar{g}_\sigma^2(\bar{m}_\sigma^2 - s) + \bar{g}_f^2(\bar{m}_f^2 - s)) \). D is factorized as \( D = D_1 D_2 \), where \( D_1 = \lambda_\sigma - s \) and \( D_2 = \lambda_f - s \), and \( \rho T^\text{Res} \) is represented by \( \rho T^\text{Res} = -\text{Im}(D_1 D_2)/D_1 D_2 = (-D_2 \text{Im} D_1 - D_1 \text{Im} D_2 + 2i\text{Im} D_1 \text{Im} D_2)/D_1 D_2 \). This is equivalent to Eq.(4-5).
which is obtained from Eq. (4.2), \( r_\sigma, r_f, \theta_\sigma \) and \( \theta_f \) are given by

\[
 r_\alpha e^{i\theta_\alpha} = \frac{[\bar{r}_\alpha(m^2_\beta - \lambda_\alpha) + \bar{R}(m^2_\sigma - \lambda_\sigma)]}{(\lambda_\beta - \lambda_\alpha)}, \tag{4.8}
\]

where \((\alpha, \beta) = (\sigma, f)\) or \((f, \sigma)\). As can be seen from Eq. (4.8), \( r_\alpha \) and \( \theta_\alpha \) are almost \( s \) independent, except for the threshold region. Thus it is understood that Eq. (4.6) is the same formula as that applied in VMW method.

In the VMW method, essentially the three new parameters, \( r_\sigma \), \( r_f \) and the relative phase \( \theta(\equiv \theta_f - \theta_\sigma) \), independent of the scattering process, characterize the relevant production processes. Presently they are represented by the two production coupling constants, \( \xi_\sigma \) and \( \xi_f \). Thus, among the three parameters in the VMW method there is one constraint due to the FSI theorem. \( r_\sigma \) and \( r_f \), corresponding respectively to \( \bar{r}_\sigma \) and \( \bar{r}_f \), are regarded as free parameters. On the other hand, \( \theta(\equiv \theta_f - \theta_\sigma) \) is constrained by

\[
 Re^{i\theta} = -\frac{(\bar{m}_\sigma^2 - \lambda_f) + \bar{R}(m_\sigma^2 - \lambda_\sigma)}{(m_\sigma^2 - \lambda_\sigma) + R(m_\sigma^2 - \lambda_\sigma)}, \quad (R \equiv r_f/r_\sigma, \quad \bar{R} \equiv \bar{r}_f/\bar{r}_\sigma), \tag{4.9}
\]

which is obtained from Eq. (4.8).

Next we consider the effect of the non-resonant background. It can be introduced consistently with the FSI theorem. In the IA method the scattering S matrix takes a multiplicative form of the resonant and background parts as

\[
 S = S^{\text{Res}}S^{\text{BG}} = \frac{1 + i\rho K^{\text{Res}}}{1 - i\rho K^{\text{Res}}}, \quad \frac{1 + i\rho K^{\text{BG}}}{1 - i\rho K^{\text{BG}}}. \tag{4.10}
\]

Correspondingly, the \( T \) matrix is represented by the respective \( K \) matrices, \( K^{\text{Res}} \) and \( K^{\text{BG}} \). \( F \) is obtained in a manner similar to Eq. (4.2) as

\[
 F = K^{\text{Res}} + K^{\text{BG}} \rightarrow F = \frac{P^{\text{Res}} + P^{\text{BG}}}{1 - i\rho K^{\text{Res}}}, \quad (1 - i\rho K^{\text{BG}}), \tag{4.11}
\]

where \( P^{\text{Res}} \) and \( P^{\text{BG}} \) are the resonant and background production “\( K \) matrix”, respectively. \( K^{\text{BG}}(P^{\text{BG}}) \) is equal to the background coupling factor \( g_{2\pi}(s)(\xi_{2\pi}(s)) \) in Eq. (2.4) \( ] \) This \( F \) automatically satisfies the FSI theorem. It can be rewritten as

\[
 F = F^{\text{Res}}(1 + i\rho T^{\text{BG}}) + F^{\text{BG}}(1 + i\rho T^{\text{Res}}), \tag{4.12}
 F^{\text{BG}} = P^{\text{BG}}/(1 - i\rho K^{\text{BG}}) \equiv f_{\text{BG}}(s)e^{i\delta_{\text{BG}}}. \tag{4.13}
\]

Both the first and the second terms in Eq. (4.12) have \( \sigma \) and \( f \) poles, and Eq. (4.12) can be rewritten as

\[
 F = \frac{\bar{r}_\sigma(s)(\bar{m}_f - s) + \bar{r}_f(s)(\bar{m}_\sigma - s)}{(\lambda_\sigma - s)(\lambda_f - s)e^{-i\delta_{\text{BG}}}}. \tag{4.14}
\]

\[^{\ast})\text{ The background coupling factor is generally } s \text{ dependent in the “} K \text{ matrix” representation. For example, in the case with the background of hard core type, } g_{2\pi}(s) = -\frac{1}{p(\pi)}\tan p_{1}r_{c}.\]
This has a similar form to Eq. (4.7), except for the $s$ dependence of the production couplings, given by

$$
\tilde{r}_\sigma(s) = \tilde{r}_\sigma \cos \delta_{BG} + f_{BG}(s)(m_{BG}^2 - s), \quad \tilde{r}_f(s) = \tilde{r}_f \cos \delta_{BG}, \quad (4.15)
$$

and the phase factor $e^{-i\delta_{BG}}$. In the case that the production coupling $\tilde{\xi}_{2\pi}$ is so small that $\tilde{r}_\sigma \gg f_{BG}m_{BG}^2$, $\tilde{r}_\sigma(s)$ and $\tilde{r}_f(s)$ have weak $s$ dependences, and they are approximated with constants, $\tilde{r}_\sigma(m_{BG}^2)$ and $\tilde{r}_f(m_{BG}^2)$, respectively. Then, Eq. (4.14) effectively reduces to Eq. (4.7), and the VMW method with a constrained phase parameter is reproduced in this case with a non-resonant background phase.

4.2. Physical Meaning of the “Universality of $T_{\pi\pi}$” and the VMW method

The methods of analyses we have used in studying scattering and production processes, respectively, the IA and VMW methods, are compared with those of the conventional analyses based on the “universality” of $\pi\pi$ scattering pictorially in Fig. 4.

The $\pi\pi$ scattering is largely affected by the effect of the non-resonant repulsive background, and $T$ cannot be described by the usual Breit-Wigner amplitudes alone with a non-derivative coupling. The spectrum of $T$ shows a very wide peak around $\sqrt{s} \approx 850$ MeV, at which value the phase $\delta_{BG}$ passes through 90 degrees, and then falls off rapidly, as shown in Fig. 4. In contrast, the spectra of $F$ in the pp central collision and the $J/\Psi \to \omega\pi\pi$-decay have peaks at around $\sqrt{s} = m_{\sigma}$ (500 ~ 600 MeV).

In the conventional approach, with the universality relation $F = \alpha T$, $T$ is first analyzed and the phase shift $\delta$ around $\sqrt{s} = m_{\sigma}$ is interpreted as due to the background, instead of $\sigma$ contribution. Then $F$ is analyzed with $\alpha(s)$ arbitrarily chosen in the polynomial form

$$
\alpha(s) = \sum_{n=0}^{\lambda} \alpha_n s^n. \quad (4.16)
$$

In the most simple case with $\alpha =$-const, the universality relation implies that $\tilde{\xi}_\sigma = \alpha \tilde{g}_{\sigma}$, $\tilde{\xi}_f = \alpha \tilde{g}_f$ and $\tilde{\xi}_{2\pi} = \alpha \tilde{g}_{2\pi}$; that is, all the production couplings are proportional

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* The overall phase factor $e^{-i\delta_{BG}}$ appears only in the angular analysis given in §2 through the scalar-tensor interfering term. This factor has a weak $s$ dependence, and its effect may be regarded as being included in the phase parameters, the $\theta_{\alpha}$, of the VMW method.

** According to the relation $F = \alpha T$, any production amplitude $F$ vanishes at the same position as the zero position, $s = s_0^T$, of $T$, and this is clearly incorrect. To avoid this problem, modified forms, $\alpha(s) = \frac{1}{s-s_0^{BG}} \sum_{n=0}^{\lambda} \alpha_n s^n$ or $\alpha(s) = \sum_{n=0}^{\lambda} \alpha_n s^n + \frac{1}{s-s_0^{BG}}$, are used in the actual analyses. However, this operation is quite artificial and arbitrary, since we are free to choose any function which is zero at $s = s_0^T$, instead of $s - s_0^T$, to remove the zero of $T(K)$, as is seen from the above two different forms given by the original authors. In our scheme the zero position of $F$, $s = s_0^T$, is dependent on both of the scattering and the production couplings of the relevant resonances, and is different from $s_0^F$. In this sense the above prescription for the “common zeros” problem is taken into account automatically. (See further details in the criticism of our description and our reply to it.)
Comparison of present method with conventional one

Conventional method

(Au, Morgan and Pennington '87)

| Scattering | K-matrix method |
|------------|----------------|
| δ          | $|I_{\text{Tr}}|^2$ |

Present method

| Scattering | IA- method |
|------------|------------|
| δ          | $|I_{\text{Tr}}|^2$ |

| Production | "Universality" $F= \alpha(s)T$ |
|------------|--------------------------------|
| Simple case $\alpha = \text{const.}$ | same spectra for any production processes |

Actual analysis

$\frac{\xi_f}{\bar{\xi}_f} - \frac{\xi_f}{\bar{\xi}_f} - \frac{\xi_f}{\bar{\xi}_f} \sim 0$

$\xi_f \sim \frac{\xi_f}{\bar{\xi}_f} \sim \frac{\xi_f}{\bar{\xi}_f} \sim 0$

Production experiments lose values as means of seeking for resonances!

VMW- method

Fig. 4. Analyses by IA and VMW methods compared with the conventional analyses based on the “Universality” of $\pi\pi$ scattering.

to the corresponding $\pi\pi$ couplings, and the spectra of $F$ and $T$ become the same. Actually, they are different, and the difference is fitted by $\alpha_n$. The masses and widths of the resonances are determined only from the $\pi\pi$ scattering, and the analyses of $F$ on any production process become nothing but the determination of the $\alpha_n$ for respective processes, which have no direct physical meaning. Thus all the production experiments lose their values in seeking new resonances.

On the other hand, in the VMW method, only the physically meaningful parameters are introduced. The $\xi_f$, $\xi_f$ and $\bar{\xi}_f$ are independent parameters of the $\pi\pi$ scattering, and the difference between the spectra of $F$ and $T$ is explained intuitively
by supposing the relations among the coupling constants such as

\[
\frac{\tilde{\xi}_\sigma}{g_\sigma} \gg \frac{\tilde{\xi}_{2\pi}}{g_{2\pi}},
\]

that is, the ratio of background effects to the \(\sigma\)-effects are weaker in the production processes than in the scattering process. Thus in this case the large low-energy peak structure in \(|F|^2\) shows directly the \(\sigma\) existence. In this situation the properties of \(\sigma\) can be obtained more precisely in the production processes than in the scattering processes. It seems to us that this difference between the two methods reflects their basic standpoints: In the “universality” argument, only the stable (pion) state forms the complete set of meson states, while \(\bar{\sigma}\) and \(\bar{f}\), in addition to the pion, are necessary as bases of the complete set in the VMW method.

§5. Presence of the initial state phase and the VMW method

In the previous section, we showed that the VMW method is an effective method to determine the resonance properties from production processes, although the parameters for \(F\) have some constraint due to the FSI theorem. Here it should be noted that the FSI theorem is only applicable to the case in which the initial state has no strong phase. We must carefully examine whether the initial state has a phase or not in actual cases. We take the \(pp\) central collision process, \(pp \rightarrow pp\pi\pi\), and the \(J/\Psi \rightarrow \omega\pi\pi\)-decay as examples. The situations are schematically shown in Fig. 5.

First, in the \(pp\) central collision, the main contribution to the relevant process is usually considered to be due to the double pomeron exchange process, which is supposed to have no strong phase. However, it is known that this process is largely affected by the \(\Delta\)-resonance diagram (described by the Breit-Wigner formula with finite width), which causes the initial strong phase, and the FSI condition may be violated.

A similar situation occurs also in the \(J/\Psi \rightarrow \omega\pi\pi\)-decay: this process is largely affected by the effect of \(\omega\pi\)-resonances, such as \(b_1(1235)\) and its excited states, which supply the initial strong phase.

This type of initial strong phase generally exists in all processes under the effect of strong interactions, which can effectively be introduced in the VMW method by substitution of

\[
\tilde{r}_\alpha \rightarrow \tilde{r}_\alpha e^{i\tilde{\rho}_{\alpha\tau\tau}}.
\]

We have little knowledge of these initial phases, and we are forced to treat the parameters in the VMW method as being effectively free.

\*\*\* However, in weak decays, such as \(K \rightarrow 2\pi\) and \(K_{l4}\) decays, the FSI condition is exactly satisfied, and the analysis using the VMW method with free parameters is not applicable to these processes.
§6. Concluding remarks

In this paper the relation between scattering and production amplitudes was investigated from the general viewpoint of the unitarity and the applicability of the FSI theorem, by using a simple field-theoretical model. The methods used in our phenomenological analyses of the $\pi\pi$-phase shift and the production processes, the IA method and the VMW method, respectively, were derived directly in the physical state representation of scattering and production amplitudes. The relative phase parameters $\theta$ in the VMW method are constrained by the FSI theorem in the case it is applicable. However, in general production processes under the effect of strong
interactions, the initial states have unknown strong phases, and correspondingly the $\theta$ in VMW method are treated as being effectively free.

Furthermore, we have checked carefully the physical meaning of the “universality” argument, and have argued that the conventional analyses following it seem to be only parameter-fitting and meaningless in seeking new resonances, while the VMW method is an effective method applicable to general production processes under the effect of strong interactions, in determining the existence and properties of new resonances.

For many years, some experimental facts suggesting $\sigma$ existence, other than mentioned in the Introduction, obtained in various production processes, had been persuasively to be interpreted without the $\sigma$ meson by invoking the “universality” of $\pi\pi$-scattering amplitude. However, as is shown here, we may conclude that the conventional argument of the “universality” is incorrect. Accordingly, these production experiments should be re-analyzed through the VMW method by taking into account the possible effects of $\sigma$ existence. Especially in this connection the negative estimation on the results obtained from the GAMS experiment (, which was mentioned in the Introduction,) should be corrected.

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