On physical meaning of Weyl vector

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Abstract
The present paper considers if the new proposed conformal geometrodynamics (CGD) can extend the Nature features compared with general theory of relativity (GTR). The answer for this question can be connected with unique phenomenon arising from Riemann space transition used in GTR, to Weyl space used in CGD. We have in mind the possibility to set up the perfect correspondence in certain spatial regions between the equations of different physical phenomena: (1) Phenomena associated with Weyl degrees of freedom in a plane space. (2) Phenomena described in terms of half-integer spin particles and observed quantities corresponding to the full set of bispinors. The said phenomenon is described in the present paper. Analyzed here are the new prospects in the problem of combination of quantum physics concepts and GTR, unification of physical interactions and understanding of many of the effects known from the experiments but not properly understood yet.

1 Introduction
The general theory of relativity (GTR) is used in modern science and technology from Global Positioning System to cosmological models to describe space-time relations between the events. This is not by accident because of number of direct experimental correctness proofs based on GTR (see reviews in [1]). It turned out that whenever there is a possibility to set up an experiment to check GTR, predictions of GTR met experimental results in the best way rather than alternative theories.

However, the most convincing GTR check is confined to a weak gravitational field region, or as they usually say, post-Newtonian approximation (more accurate in some cases). In regard to high gravitational fields, modern science encounters considerable difficulties. The physics of the processes in material and space-time in the region of event horizons and singularity is still
obscure. The attempts to develop energy-momentum tensor $T_{\alpha\beta}$ of the right side of GTR\(^1\) equation led to no result

$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = T_{\alpha\beta},$$

(1)

from considerations of geometry. The attempts combine the most successful theories of the 20\(^{th}\) century – GTR and quantum theory.

Given the state of space-time science, the attempts to fall outside the limits of GTR are not insensible. One of the variants of such a fall was suggested and developed in the series of papers [3]-[6] et al. By conformal geometrodynamics (CGD) we shall call the specified theory. CGD equations are given by (1), only $T_{\alpha\beta}$ tensor has been never used as energy-momentum tensor of the particular material. $T_{\alpha\beta}$ tensor is written as $A_\alpha$ vector and $\lambda$ scalar function in CGD as follows:

$$T_{\alpha\beta} = -2A_\alpha A_\beta - g_{\alpha\beta}A^2 - 2g_{\alpha\beta}A_\mu^\nu + A_{\alpha;\beta} + A_{\beta;\alpha} + g_{\alpha\beta}\lambda.$$  

(2)

The word “conformal” in reference to geometrodynamics is connected with the fact that equations (1) with right side of in the form of tensor (2) show invariance as regard to transformations

$$g_{\alpha\beta} \rightarrow g_{\alpha\beta} \cdot \exp (2\sigma), \quad A_\alpha \rightarrow A_\alpha - \sigma_\alpha, \quad \lambda \rightarrow \lambda \cdot \exp (-2\sigma).$$  

(3)

Here $\sigma(x)$ - is coordinate arbitrary scalar function. Metrics transformations by law (3) are called conformal. From the point of view of physicist, the transformations (3) are of considerable value as they maintain cause-and-effect relations between the events. Mathematicians put this property of transformations (3) in other words: transformations (3) maintain the variety of light cone congruence.

“Geometrodynamics” term used in CGD abbreviation is explained by the possibility to set up the equation (1) with right side in the form of tensor (2) only in terms of geometric objects of Weyl space:

$$g_{\alpha\beta} = -\frac{1}{\lambda} \cdot \mathcal{R}(\alpha\beta).$$  

(4)

Here $\mathcal{R}(\alpha\beta)$ - is symmetrical part of Ricci tensor for Weyl space. We are not going to specify Weyl space properties given in multiplicity of works

\(^1\) From now on we shall use notations from the known book in GTR [2].
beginning with [7]. Note that $A_\alpha$ vector has geometrical-only meaning as it is Weyl vector entering into connection structure

$$\Gamma^\mu_{\alpha\nu} = \left(\frac{\nu}{\alpha\nu}\right) + A_\alpha \delta^\mu_\nu + A_\nu \delta^\mu_\alpha - g_\alpha^\nu g^\mu_\epsilon A_\epsilon. \quad (5)$$

However, covariant derivatives of Weyl space object are found by the formulae

$$\nabla Y^\beta = Y^\beta_{\alpha} + \Gamma^\beta_{\alpha\epsilon} Y^\epsilon + n \cdot A_\alpha Y^\beta. \quad (6)$$

Here $n$ is Weyl object weight ($Y^\beta$ vector in that case).

The number of CGD equation nontrivial properties and solutions is described in [6]. Let us enumerate these properties:

1. Cauchy problem for equation (1) with right side in the form of tensor is set without Cauchy data.

2. Space-like hypersurface crack solutions are admitted.

3. There occurs preserving current vector along with a possibility to “process” $T_{\alpha\beta}$ tensor using phenomenological thermodynamics.

4. Geometrodynamical medium proved to be considered as relativistic simple viscous liquid. Equations of state for this medium, as well as viscosity result from CGD equations. There occurs a new state function analogous to entropy.

5. Gauge vector and lambda term can be interpreted in terms of the degree of freedom of $1/2$ spin particles.

Several solutions of the equations of CGD have been obtained in recent years, the analysis of these equations has verified the enumerated properties (See [6], [8]-[10]). It is proved, for example, that all known exact solutions of GTR equations (exterior and interior solutions of Schwarzschild, de Sitter, Friedmann etc.) can be arbitrary closely approximated in a certain space-time region by CGD solutions. Thus, it is reasonable to suppose that not only GTR but also CGD can be used to describe large-scale physical processes.

Either of the given above properties is interesting in its own way, and is likely to be analyzed. The most intriguing is the last property. For the first time geometrical objects proved to be connected with dynamics of operator of state of microscopic objects with half-integer spin. None of the attempts of
physical interaction unification gave such a result. Thus, it is no wonder that it is the 5th property that is thoroughly analyzed. The algorithm of tensor system mapping onto bispinor degrees of freedom developed in a number of works ([11], [12], etc.) was used. The results of [13] research exceeded all expectations. Outline of the results is the aim of this work.

The work includes auxiliary sections 2, 4, 5. The first one gives formulae of thermodynamic phenomenological analysis of geometrodynamic analysis of the medium described by energy-momentum tensor ([2]). Section 4 contains well-known information of the theory of Dirac matrix. Section 5 gives the variant of tensor system mapping onto bispinor degrees of freedom. The new results are presented in sections 3, 6. Section 7 gives one of the possible exact solutions of CGD equations – solution of Yukava potential type. The given solution is particular, but suffices to describe methods of application of the new results. The discussion of the results obtained is given at the end of the work.

2 Thermodynamic analysis of CGD equations

It follows from (28) that vector $j_\alpha = Const \cdot (\lambda_\alpha - 2\Lambda A_\alpha)$ generally satisfies the equation of continuity

\[ j^{\alpha}_{,\alpha} = 0. \]  

(7)

The timelike vector $j^\alpha$ can be represented in the space-time domain in the form of

\[ j^\alpha \equiv \rho u^\alpha, \]  

(8)

where $u^\alpha$ - is the unit timelike vector. In signature $(-+++)$

\[ u^2 = -1. \]  

(9)

The timelike vector $j^\alpha$ in the scheme satisfying the equation of continuity [7], means that the scheme contains certain strongly conserved substance. The density $\rho$ of this substance is defined by the formula $\rho = \sqrt{- (j^\alpha j_\alpha)}$ from [8], (9).

Let us take a conserved substance to imply some charge which is taken for strongly conserved in elementary particle theory. The specific volume $V$ is defined as a reciprocal of $\rho$,
\[ V = \frac{1}{\rho}. \]  

(10)

Two projection operators can be set up in the ordinary way with the help of \( u^\alpha \):

\[- u^\alpha u^\beta, \quad s^{\alpha\beta} \equiv g^{\alpha\beta} + u^\alpha u^\beta. \]  

(11)

The tensor \( T_{\alpha\beta} \) in (2) can be represented in the form of

\[ T_{\alpha\beta} = U \cdot u_\alpha u_\beta + (u_\alpha q_\beta + u_\beta q_\alpha) + W_{\alpha\beta}, \]  

(12)

where the values \( U, q_\alpha, W_{\alpha\beta} \) are defined by the formulae

\[ U \equiv (u^\mu T_{\mu\nu} u^\nu), \quad q_\alpha \equiv -s_\alpha^\mu T_{\mu\nu} u^\nu, \quad W_{\alpha\beta} \equiv s_\alpha^\mu s_\beta^\nu T_{\mu\nu}. \]  

(13)

Henceforth we shall follow (13), that is to say: \( U \) - is the energy density, \( q_\alpha \) - is the energy flux vector, \( W_{\alpha\beta} \) - is the strain tensor. \( W_{\alpha\beta} \) tensor is often represented as the sum of two summands,

\[ W_{\alpha\beta} = P \cdot s_{\alpha\beta} - \tau_{\alpha\beta}, \]  

(14)

where \( P \) - is the pressure, and \( \tau_{\alpha\beta} \) - is the tensor of viscous strain, satisfying the condition of

\[ \tau^\nu_\nu = 0. \]  

(15)

The condition (15) means that the tensor \( \tau_{\alpha\beta} \) does not contain second viscosity terms. The fulfillment of the condition (15) the representation of (14) is one-valued.

It is pertinent to note that \( U \) and \( P \) treatment as energy density and medium pressure is in agreement with the treatment of the similar values in the case of energy-impulse tensor of the perfect liquid, i.e. in the case when

\[ T_{\alpha\beta} = (U + P) \cdot u_\alpha u_\beta + P \cdot g_{\alpha\beta}. \]  

(16)

Here \( U \) is defined by \( U \equiv (u^\mu T_{\mu\nu} u^\nu) \), and \( P \) is found by

\[ P = \frac{1}{3} T_{\alpha\beta} (g^{\alpha\beta} + u^\alpha u^\beta) = \frac{1}{3} T_{\alpha\beta} s_{\alpha\beta}. \]  

(17)

That is in the case of the perfect liquid the formulae for \( U \) and \( P \) are the same as (13), (17) for these values in the case of CGD.
The explicit form of the values $U$, $q_\alpha$, $W_{\alpha\beta}$, $P$, $\tau_{\alpha\beta}$, introduced above depends on the gauge choice. When $\lambda$ constancy condition is used as gauge condition, Lorentz condition $A^\alpha;\alpha = 0$ will be satisfied, and the formulae for the values introduced can be written in covariant form. These formulae are given by:

$$U = -\frac{3}{4} \cdot \frac{\rho^2}{\lambda^2} + \frac{1}{\lambda} (u^\nu \rho_{,\nu}) - \lambda,$$

$$q_\alpha = s_\alpha^\beta \left( \frac{V_\beta}{2 \lambda V^2} + \frac{1}{2 \lambda V} w_\beta \right),$$

$$W_{\alpha\beta} = -\frac{\rho}{2 \lambda} (u_{\alpha;\beta} + u_{\beta;\alpha}) - \frac{\rho^2}{4 \lambda^2} \cdot s_{\alpha\beta} + \lambda \cdot s_{\alpha\beta} + \frac{\rho}{2 \lambda} [u_{\alpha} w_{\beta} + u_{\beta} w_{\alpha}],$$

$$P = \frac{\rho^2}{4 \lambda^2} + \frac{1}{3 \lambda} (u^\nu \rho_{,\nu}),$$

$$\tau_{\alpha\beta} = \frac{\rho}{2 \lambda} s_\alpha^\mu s_\beta^\nu \left( u_{\mu;\nu} + u_{\nu;\mu} - \frac{2}{3} s_{\mu\nu} (u^\sigma;\sigma) \right).$$

The vector $w_\alpha$ from (19), (20) is defined by the formula $w_\alpha = u^\sigma u_{\alpha;\sigma}$, i.e. it is a four-dimensional acceleration vector.

The equations (18), (21) imply that there is the following connection between $U, P, V$:

$$P = \frac{1}{3} U + \frac{4}{3} \lambda + \frac{1}{2 V^2 \lambda^2}.$$

These formulae is none other then the equation of state of geometrodynamic continuum.

The formula for isentropic sound speed $c_s$, defined as

$$c_s^2 = -V^2 \left( \frac{\partial P}{\partial V} \right),$$

is given by

$$c_s^2 = \frac{4}{3} V (P - \lambda) + \frac{1}{2 \lambda^2 V}.$$
3 Equations for vector and antisymmetric tensor in CGD in the case of flat space

Equation (1) with the right side in the form of tensor (2) implies that formulae

\[ T_{\alpha ;\beta} = 0 \]  

(26)

should be performed. When the antisymmetric tensor is introduced

\[ F_{\alpha \beta} = A_{\beta,\alpha} - A_{\alpha,\beta}, \]

(27)

the formulae

\[ F_{\alpha ;\beta} = \lambda_{\alpha} - 2\lambda A_{\alpha}. \]

(28)

follows from (26). If

\[ \lambda = \text{Const}, \]  

(29)

is taken as gauge condition, then Weyl vector will satisfy Lorentz condition

\[ A^{\nu ;\nu} = 0. \]  

(30)

There are problems in CGD and GTR in which the evolution of gravitational degrees of freedom as well as degrees of freedom connected with matter fields should be consistently considered. Among these problems is Schwarzschild problem. On the other hand there is a large number of problems in physics which are set up to consider the matter fields dynamics without regard of gravitational degree of freedom. Here the space is set up to be flat, while the dynamic equations for matter fields are derived from (26).

The flat space problem set up in the case of CGD equations implies that the dynamic equations should result from (26) for the complementary degree of freedom providing the energy-momentum tensor used for tensor (2). In case that \( T_{\alpha ;\beta} \) is obtained, the equation is changed with (1), and given the gauge (29), we get the following four equations:

\[ J_{\beta,\alpha} - J_{\alpha,\beta} = 4m \cdot H_{\alpha\beta}, \]  

(31)

\[ H_{\alpha ;\beta} = -m \cdot J_{\alpha}, \]  

(32)
\[ J_\mu^\nu = 0 \]  
(33)

\[ m = \text{Const.} \]  
(34)

Providing the initial CGD equations written in terms of Weyl vector \( A_\alpha \) and lambda term \( \lambda \), the equations (31)-(34) are written in new terms: \( J^\alpha, \, H^{\alpha\beta}, \, m \). The connection between the values depends on the gauge conditions. Here

\[ J_\alpha = \frac{2}{m} A_\alpha; \quad H_{\alpha\beta} = \frac{1}{2m^2} F_{\alpha\beta}, \]  
(35)

\[ \lambda = 2m^2. \]  
(36)

### 4 Dirac matrices. Dirac equation

For the coherency of the treatment and convenience of the references let us mention some properties of Dirac matrices and Dirac equation which will be used later.

Our concern will be the case when the space can be considered plane. In this case the metric tensor of Riemannian space \( g_{\alpha\beta} \) can be considered the same as metric tensor of Minkowsky space, which is given by \( g_{\alpha\beta} = \text{diag} [-1, 1, 1, 1] \) in Cartesian coordinates. Dirac matrices (DM) \( \gamma_\alpha \) are defined by

\[ \gamma_\alpha \gamma_\beta + \gamma_\beta \gamma_\alpha = 2g_{\alpha\beta} \cdot E. \]  
(37)

The symbol \( E \) means the identity matrix \( 4 \times 4 \) in (37). DM \( \gamma_\alpha \) are constant in the entire space. We use Majorana system of matrices

\[ \gamma_0 = -i\rho_2 \sigma_1, \quad \gamma_1 = \rho_1, \quad \gamma_2 = \rho_2 \sigma_2, \quad \gamma_3 = \rho_3, \]  
(38)

in case the explicit form of DM is required. The elements of this system are integral real numbers.\(^2\)

Suppose that field functional \( Z \) is general the \( 4 \times 4 \) matrix satisfying Dirac equation

\(^2\) There is a real system of DM among (37) solutions in case the signature \((- + + +)\) is used.
\[ \gamma^\nu (\nabla_\mu u Z) = m \cdot Z. \tag{39} \]

By \( Z \) we shall mean the bispinor matrix. The bispinor states are selected from \( Z \) by its right multiplication by the projections.

The equation

\[ \left( \nabla_\nu Z^+ \right) D \gamma^\nu = -m \cdot Z^+ D. \tag{40} \]

is combined with (39). \( D \) matrix in (40) is defined by

\[ D \gamma_\mu D^{-1} = -\gamma_\mu^+. \tag{41} \]

Covariant derivatives of bispinor matrix in (39), (40), are set down as:

\[ \nabla_\alpha Z = Z;_\alpha - Z \Gamma_\alpha \]
\[ \nabla_\alpha Z^+ = Z^+_\alpha + \Gamma_\alpha Z^+ \tag{42} \]

The value of \( \Gamma_\alpha \) in (42) will be referred to as bispinor connectivity.\(^3\) It is the whole complex of real anti-Hermitean matrices:

\[ \Gamma^*_\alpha = \Gamma_\alpha, \quad \Gamma^+_\alpha = -\Gamma_\alpha. \tag{43} \]

Let us mention some formulae resulting from Dirac equation. Equation (39) is multiplied by \( \gamma^\alpha \) on the left side.

\[ \gamma^\alpha \gamma^\nu (\nabla_\nu Z) = m \cdot \gamma^\alpha Z. \tag{44} \]

Let us write the product \( \gamma^\alpha \gamma^\nu \) as

\[ \gamma^\alpha \gamma^\nu = g^{\alpha\nu} + S^{\alpha\nu} \tag{45} \]

(here \( S^{\mu\nu} = \frac{1}{2} (\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu) \)) and (45) are inserted into (44).

\[ (\nabla_\alpha Z) = -S^{\alpha\nu} (\nabla_\nu Z) + m \cdot \gamma_\alpha Z. \tag{46} \]

After Hermitean conjugation (46) and \( D \) multiplication from the right side we get:

\[ \left( \nabla_\alpha Z^+ \right) D = \left( \nabla_\nu Z^+ \right) DS^{\alpha\nu} - m \cdot Z^+ D \gamma_\alpha. \tag{47} \]

\(^3\) The bispinor connectivity and gauge field agree within constant factor.
5 Tensor mapping onto bispinor matrix

Suppose that the set of five types of tensors enumerated in 1 is defined in 4D Riemannian space.

Let us construct the matrix $M$ according to the rule

$$M \equiv a \cdot i D^{-1} + b \cdot i \gamma_5 D^{-1} + J_\alpha \cdot \gamma^\alpha D^{-1} + s_\alpha \cdot i \gamma_5 \gamma^\alpha D^{-1} + H_{\alpha\beta} \cdot S^{\alpha\beta} D^{-1}. \quad (48)$$

Here $\gamma_5 = \gamma_0 \gamma_1 \gamma_2 \gamma_3$. Each of the summands in the right side of (48) is Hermitian matrix, so that $M$ matrix is Hermitian as well. The full number of tensor components enumerated in Table 1 is 16.

| Tensor type                  | Symbol |
|------------------------------|--------|
| 1 Scalar                     | $a$    |
| 2 Vector                     | $J^\alpha$ |
| 3 Antisymmetric tensor       | $H^{\alpha\beta}$ |
| 4 Pseudovector               | $s^\alpha$ |
| 5 Pseudoscalar               | $b$    |

Table 1: Five types of tensor

$M$ matrix is changed as $M \rightarrow M' = LML^+$ under Lorentz transformations of DM $\gamma_\alpha \rightarrow \gamma'_\alpha = L_{\gamma^\alpha} L^{-1}$, i.e. the matrix is the object of bispinor product type by conjugated Hermitian bispinor.

$$\text{If} \quad \gamma_\alpha \rightarrow \gamma'_\alpha = L_{\gamma^\alpha} L^{-1}, \quad \text{then} \quad M \rightarrow M' = LML^+. \quad (49)$$

Being Hermitian, the matrix (48) can have any rank up to four included. When the rank is given, the eigenvalue spectrum can include the real values (positive and negative), as well as complex values (complex conjugated values). Here we confine ourselves to the analysis of the space-time regions in which the rank of matrix (48) is 4 and all the eigenvalues are positive.

It is known from the general theory of matrices that “the square root” can be taken from $M$ i.e. $M$ matrix can be represented as

$$M = Z \cdot Z^+. \quad (50)$$

If $Z$ in (50) implies the arithmetical root, then the procedure of “square-root” generation is single-valued.
(48) and (50) imply that the original tensor are related to $Z$ with

$$
\begin{align*}
    a &= -\frac{i}{4} \cdot Sp (Z^+ DZ), \\
    b &= \frac{i}{4} \cdot Sp (Z^+ D\gamma_5 Z), \\
    s^\alpha &= -\frac{i}{4} \cdot Sp (Z^+ D\gamma_5 \gamma^\alpha Z), \\
    J^\alpha &= \frac{1}{4} \cdot Sp (Z^+ D\gamma^\alpha Z), \\
    H^{\alpha\beta} &= -\frac{1}{8} \cdot Sp (Z^+ D S^{\alpha\beta} Z).
\end{align*}
$$

(51)

The fulfillment of the condition of tractability (50) means that

$$
\text{det} (Z) \neq 0
$$

(52)
as well as that there is $Z^{-1}$ matrix along with $Z$ matrix.

In particular, the real numbers are quite sufficient to solve the problem of tensor mapping onto the bispinor matrix when only $J^\alpha$ and $H^{\alpha\beta}$ tensor are nonzero among all the tensors enumerated in the Table 1, and DM are used as in real representation as DM system. This case will be of the particular concern. The full number of $J^\alpha$ and $H^{\alpha\beta}$ components is 10. In this particular case Hermitian $M$ matrix is given by

$$
M = J^\alpha \cdot (\gamma_\alpha D^{-1}) + H^{\alpha\beta} \cdot (S_{\alpha\beta} D^{-1}).
$$

(53)

All the mapping results presented above have the algebraic character as they belong to tensor and bispinor matrices at one arbitrary point of Riemannian space. It stands to reason that if the fields $J^\alpha (x), H^{\alpha\beta} (x)$ are defined in certain region of space and if the condition of $M (x)$ matrix positivity is fulfilled, then $Z (x)$ matrix is set up at each point and hence we have the mapping onto the bispinor matrix of two tensor fields: $J^\alpha (x), H^{\alpha\beta} (x)$.

6 Dirac equation in terms of the observed values

Suppose that the fields $J^\alpha (x), H^{\alpha\beta} (x)$ are set in an arbitrary region of space and at each point $M (x)$ matrix can be mapped onto $M (x)$ bispinor matrix. Let us raise the question: from which law will $Z$ matrix vary if the vector and antisymmetric vector from (53) obey CGD equations, i.e. (31)-(34)?

The answer can be obtained from three theorems proof given below.
6.1 Theorem 1

Let us put Theorem 1 as

$$J^{\alpha} = 0.$$  (54)

Let us prove that first, it is fulfilled if the vector $J^{\alpha}$ is expressed in terms of bispinor matrix according to the formula (51),

$$J^{\alpha} = \frac{1}{4} \cdot Sp \left( Z^+ D\gamma^{\alpha} Z \right),$$

and second, $Z, Z^+$ matrices satisfy Dirac equations as (39) and (40). Let us differentiate the formula for $J^{\alpha}$.

$$J^{\alpha}_{\alpha} = \frac{1}{4} \cdot Sp \left( \left( \nabla^{\alpha} Z^+ \right) D\gamma^{\alpha} Z \right) + \frac{1}{4} \cdot Sp \left( Z^+ D\gamma^{\alpha} \left( \nabla_{\alpha} Z \right) \right).$$

We use Dirac equation.

$$J^{\alpha}_{\alpha} = \frac{1}{4} \cdot Sp \left( -m \cdot Z^+ DZ + m \cdot Z^+ DZ \right) = 0.$$

Hence, the theorem is proved.

6.2 Theorem 2

Theorem 2 states that if bispinor matrix obeys Dirac equation (39) and the values of $J_{\alpha}, H_{\alpha\beta}$ are related with (51), then the formula

$$(J_{\beta,\alpha} - J_{\alpha,\beta}) = 4m \cdot H_{\alpha\beta} + E_{\alpha\beta\mu \nu} \frac{1}{4} Sp \left\{ \left( \nabla_{\nu} Z^+ \right) D\gamma_{\mu} \gamma_{\nu} Z - Z^+ D\gamma_{\mu} \gamma_{\nu} \left( \nabla_{\nu} Z \right) \right\}$$

is true.

The theorem will be proved in several steps. First, we try to value $(J_{\beta,\alpha} - J_{\alpha,\beta})$, using (46), (47). (51) implies:

$$(J_{\beta,\alpha} - J_{\alpha,\beta}) = \frac{1}{4} Sp \left\{ \left( \nabla_{\alpha} Z^+ \right) D\gamma_{\beta} Z + Z^+ D\gamma_{\beta} \left( \nabla_{\alpha} Z \right) \right\} - \frac{1}{4} Sp \left\{ \left( \nabla_{\beta} Z^+ \right) D\gamma_{\alpha} Z + Z^+ D\gamma_{\alpha} \left( \nabla_{\beta} Z \right) \right\}.$$  (56)

Use (46), (47).
\[(J_{\beta,\alpha} - J_{\alpha,\beta}) = \]
\[= \frac{1}{4} \text{Sp} \{(\nabla_\nu Z^+)^\alpha \gamma_\beta S_{\alpha}^\nu\} - m \cdot Z^+ D\gamma_\alpha \gamma_\beta Z - Z^+ D\gamma_\beta S_{\gamma_\alpha}^\nu (\nabla_\nu Z) + \]
\[+ m \cdot Z^+ D\gamma_\beta \gamma_\alpha Z - \frac{1}{3} \text{Sp} \{(\nabla_\nu Z^+) D\gamma_\beta S_{\gamma_\alpha}^\nu (\nabla_\nu Z) - \]
\[\neg m \cdot Z^+ D\gamma_\beta \gamma_\alpha Z - Z^+ D\gamma_\alpha S_{\gamma_\beta}^\nu (\nabla_\nu Z + m \cdot Z^+ D\gamma_\alpha \gamma_\beta Z\}. \quad (57)\]

Let us combine the terms with and without the derivatives in the right side of (57) separately.

\[(J_{\beta,\alpha} - J_{\alpha,\beta}) = \frac{1}{4} m \cdot \text{Sp} \{-Z^+ D\gamma_\alpha \gamma_\beta Z + Z^+ D\gamma_\beta \gamma_\alpha Z\}
\[+ \frac{1}{4} \text{Sp} \{(\nabla_\nu Z^+) D\gamma_\beta S_{\gamma_\alpha}^\nu (\nabla_\nu Z) - (\nabla_\nu Z^+)^\alpha \gamma_\beta Z - Z^+ D\gamma_\beta S_{\gamma_\alpha}^\nu (\nabla_\nu Z) \} \quad (58)\]

Let us replace DM product such as \(S_{\alpha}^\nu \gamma_\beta, \gamma_\beta S_{\alpha}^\nu\), in (58) using the following formulae:

\[
S_{\alpha}^\nu \gamma_\beta = -\eta_\alpha \eta_\beta \gamma^\nu + \delta_\beta^\nu \gamma_\alpha + E_{\alpha}^{\nu \beta \mu} \gamma_5 \gamma^\mu
\]

\[
\gamma_\beta S_{\alpha}^\nu = \eta_\alpha \eta_\beta \gamma^\nu - \delta_\beta^\nu \gamma_\alpha + E_{\alpha}^{\nu \beta \mu} \gamma_5 \gamma^\mu
\]

We get:

\[(J_{\beta,\alpha} - J_{\alpha,\beta}) = -m \cdot \text{Sp} \{Z^+ D\gamma_\alpha \gamma_\beta Z\}
\[+ \frac{1}{4} \text{Sp} \{(\nabla_\nu Z^+) D\gamma_\beta S_{\gamma_\alpha}^\nu (\nabla_\nu Z)\} \quad (59)\]

The terms in (59), containing the metric tensor, are reduced. The other give:

\[(j_{\beta,\alpha} - j_{\alpha,\beta}) = +8 m \cdot h_{\alpha \beta}
\[+ \frac{1}{4} \text{Sp} \{(\nabla_\nu Z^+) D\gamma_\alpha Z\} + E_{\alpha}^{\nu \beta \mu} \frac{1}{2} \text{Sp} \{(\nabla_\nu Z^+) D\gamma_5 \gamma^\mu Z\}
\[+ \frac{1}{4} \text{Sp} \{Z^+ D\gamma_\alpha (\nabla_\nu Z)\} - E_{\alpha}^{\nu \beta \mu} \frac{1}{2} \text{Sp} \{Z^+ D\gamma_5 \gamma^\mu (\nabla_\nu Z)\}
\[+ \frac{1}{4} \text{Sp} \{- (\nabla_\alpha Z^+) D\gamma_\beta Z\} + E_{\alpha}^{\nu \beta \mu} \frac{1}{2} \text{Sp} \{(\nabla_\nu Z^+) D\gamma_5 \gamma^\mu Z\}
\[+ \frac{1}{4} \text{Sp} \{- Z^+ D\gamma_\beta (\nabla_\alpha Z)\} - E_{\alpha}^{\nu \beta \mu} \frac{1}{2} \text{Sp} \{+ Z^+ D\gamma_5 \gamma^\mu (\nabla_\nu Z)\}. \]

The terms containing \(D\gamma_\alpha\), are reduced to \(- (J_{\beta,\alpha} - J_{\alpha,\beta})\). The other are combined into \(E_{\alpha}^{\nu \beta \mu} \frac{1}{2} \text{Sp} \{(\nabla_\nu Z^+) D\gamma_5 \gamma^\mu Z - Z^+ D\gamma_5 \gamma^\mu (\nabla_\nu Z)\}\). As a result we get
\[(J_{\beta,\alpha} - J_{\alpha,\beta}) = 8m \cdot H_{\alpha\beta} - (J_{\beta,\alpha} - J_{\alpha,\beta}) + \mathcal{E}_{\alpha\beta\mu\nu} \mathcal{S} \{ (\nabla_{\nu} Z^+) D\gamma^\nu \gamma^\mu Z - Z^+ D\gamma^\nu \gamma^\mu (\nabla_{\nu} Z) \} \].

(60)

We get the formula which agrees with (55) after identity transformations of (60). Hence, the theorem 2 is proved.

6.3 Theorem 3

Theorem 3 states that the formula

\[ H_{\alpha\nu} = -\frac{1}{8} \mathcal{S} \{ (\nabla_{\alpha} Z^+) DZ - Z^+ D(\nabla_{\alpha} Z) \} - m \cdot J_{\alpha} \]

(61)

is true.

The proof involves the direct check of validity of (61). We have:

\[ H_{\alpha\nu} = -\frac{1}{8} \mathcal{S} \{ Z^+ D S_{\alpha\nu} Z \}_{\nu} = -\frac{1}{8} \mathcal{S} \{ (\nabla_{\nu} Z^+) D S_{\alpha\nu} Z \} - \frac{1}{8} \mathcal{S} \{ Z^+ D S_{\alpha\nu} (\nabla_{\nu} Z) \}. \]

(62)

In the first case we change the matrix \( S_{\alpha\nu} \) according to

\[ S_{\alpha\nu} = \delta^\nu_{\alpha} - \gamma^\nu \gamma_{\alpha}, \]

(63)

and in the second case according to

\[ S_{\alpha\nu} = -\delta^\nu_{\alpha} + \gamma_{\alpha} \gamma^\nu. \]

(64)

We get:

\[ H_{\alpha\nu} = -\frac{1}{8} \mathcal{S} \{ (\nabla_{\nu} Z^+) D (\delta^\nu_{\alpha} - \gamma^\nu \gamma_{\alpha}) Z \} \]

\[ = -\frac{1}{8} \mathcal{S} \{ Z^+ D (\delta^\nu_{\alpha} - \gamma^\nu \gamma_{\alpha}) (\nabla_{\nu} Z) \} = -\frac{1}{8} \mathcal{S} \{ Z^+ D (\delta^\nu_{\alpha} - \gamma^\nu \gamma_{\alpha}) Z \} = -\frac{1}{8} \mathcal{S} \{ Z^+ D (\delta^\nu_{\alpha} - \gamma^\nu \gamma_{\alpha}) (\nabla_{\nu} Z) \} = \]

\[ = -\frac{1}{8} \mathcal{S} \{ (\nabla_{\alpha} Z^+) DZ - Z^+ D(\nabla_{\alpha} Z) \} + \frac{1}{8} \mathcal{S} \{ (\nabla_{\nu} Z^+) D\gamma^\nu \gamma_{\alpha} Z \} - \frac{1}{8} \mathcal{S} \{ Z^+ D\gamma^\nu \gamma_{\alpha} (\nabla_{\nu} Z) \} \]

(65)

After using (39), (40) we get:

\[ H_{\alpha\nu} = \frac{1}{8} \mathcal{S} \{ (\nabla_{\alpha} Z^+) DZ - Z^+ D(\nabla_{\alpha} Z) \} - 2m \cdot \frac{1}{8} \mathcal{S} \{ Z^+ D\gamma_{\alpha} Z \}. \]

(66)
Theorem number | Theorem tells that
---|---
1 | $J^\alpha_{\cdot \alpha} = 0$
2 | $(J_{\beta, \alpha} - J_{\alpha, \beta}) = 4m \cdot H_{\alpha \beta} + E_{\alpha \beta \mu \nu} \frac{1}{4} \text{Sp} \{((\nabla_{\nu} Z^+) D_{\gamma_5} \gamma^\mu Z - Z^+ D_{\gamma_5} \gamma^\mu (\nabla_{\nu} Z)\}$
3 | $H_{\alpha \nu} = -\frac{1}{8} \text{Sp} \{(\nabla_\alpha Z^+) LZ - Z^+ D(\nabla_\alpha Z)\} - m \cdot J_\alpha$

Table 2: Relations resulting from Dirac equation

The obtained relation (66) agrees with (61). Hence, it is proved that the relation (61) follows from Dirac equation, i.e. the theorem 3 is proved.

The results of the proved theorems are summed in Table 2.

If all the spur terms in Table 2 are zero, the relation between the vector $J^\alpha$ and the tensor $H^{\alpha \beta}$ will take the form which agree completely with (31) - (34). Let us prove that it is necessary to set bispinor connectivity in Table 2 as

$$\Gamma_\alpha = \frac{1}{2} \left[ \left( Z^{-1} Z_\alpha \right) - \left( Z^+_\alpha Z^{-1}_\alpha \right) \right]$$

(67)
to make the spur terms zero. Actually, it is necessary to prove that on substituting (67) in spur terms, these terms go to zero, i.e. the following equalities are performed:

$$E_{\alpha \beta \mu \nu} \text{Sp} \{((\nabla_{\nu} Z^+) D_{\gamma_5} \gamma^\mu Z - Z^+ D_{\gamma_5} \gamma^\mu (\nabla_{\nu} Z)\} = 0.$$  

(68)

$$\text{Sp} \{(\nabla_\alpha Z^+) LZ - Z^+ D(\nabla_\alpha Z)\} = 0.$$  

(69)

Let us check it by the example of one of the equalities (68) - (69). For example, the equality (69). We use spur properties.

$$\text{Sp} \{(\nabla_\alpha Z^+) LZ - Z^+ D(\nabla_\alpha Z)\} =$$

$$= 2 \cdot \text{Sp} [\Gamma_\alpha (Z^+ DZ)] + \text{Sp} \{Z^+_{\alpha \cdot} DZ - Z^+ DZ_{\cdot \alpha}\}$$

$$\text{Sp} \{(\nabla_\alpha Z^+) LZ - Z^+ D(\nabla_\alpha Z)\} =$$

$$= 2 \cdot \text{Sp} [\Gamma_\alpha (Z^+ DZ)] + \text{Sp} \{Z^+_{\alpha \cdot} DZ - Z^+ DZ_{\cdot \alpha}\}$$

The identity transformations show that
\[ Sp \left[ (\nabla_\alpha Z^+) DZ - Z^+ D (\nabla_\alpha Z) \right] = 0, \]

i.e. the equality (69) is fulfilled when bispinor connectivity is defined by (67). The validity of (68) is proved in a similar way.

7 The example of the exact solution of CGD equations

7.1 The exact solution

The solution concerned takes the form:

\[ J_0 = u, \quad J_k = 0 \]
\[ H_{0k} = -\frac{1}{4m} u' \frac{x_k}{r}, \quad H_{mn} = 0. \]  \hspace{1cm} (70)

Here

\[ u = u(r), \]  \hspace{1cm} (71)

so the solution is steady-state and spherically symmetrical. The Anzats (70) provides automatic execution of the equations (31), (33), (34). The equations (32) should be satisfied in order that (70) is the solution of CGD equations. Substitute (70) in (32). It turns out that (32) is satisfied if u function is the solution of

\[ u'' + \frac{2}{r} u' - 4m^2 u = 0. \]  \hspace{1cm} (72)

The general solution of (72) consists of two summands:

\[ u = C_1 \cdot e^{-2mr \frac{2}{m}} + C_2 \cdot e^{2mr \frac{2}{m}}. \]  \hspace{1cm} (73)

Each summand is included in (73) with dimensionless integration \( C_1, C_2 \). We shall consider the case when \( C_1 = 0 \). If \( m \) constant is positive, the solution of (73) grows exponentially,

\[ u = -c \cdot \frac{e^{2mr}}{mr}. \]  \hspace{1cm} (74)
In (74) \( C_2 \) constant stands for \(-c \). Let us substitute (74) in (70) and get:

\[
J_0 = -c \cdot \frac{e^{2mr}}{mr}, \quad H_{0k} = -\frac{c}{4m^2} \cdot e^{2mr} \cdot \left[ \frac{12}{r^2} - \frac{2m}{r} \right] \cdot \frac{x_k}{r}.
\] (75)

Let us substitute (75) in (53) and use DM in the form of (38) to obtain \( M \) matrix. We get:

\[
M = \begin{pmatrix}
-u - \frac{1}{2m} u' \frac{y}{r} & 0 & -\frac{1}{2m} u' \frac{z}{r} & \frac{1}{2m} u' \frac{z}{r} \\
0 & -u - \frac{1}{2m} u' \frac{z}{r} & -\frac{1}{2m} u' \frac{z}{r} & -\frac{1}{2m} u' \frac{z}{r} \\
-\frac{1}{2m} u' \frac{y}{r} & -\frac{1}{2m} u' \frac{y}{r} & -u' + \frac{1}{2m} u' \frac{y}{r} & 0 \\
\frac{1}{2m} u' \frac{y}{r} & -\frac{1}{2m} u' \frac{y}{r} & 0 & -u' + \frac{1}{2m} u' \frac{z}{r}
\end{pmatrix}
\] (76)

The eigenvalues of \( M \) matrix:

\[
\mu_1 = -u - \frac{1}{2m} u'; \quad \mu_2 = -u - \frac{1}{2m} u'; \quad \mu_3 = -u + \frac{1}{2m} u'; \quad \mu_4 = -u + \frac{1}{2m} u'.
\] (77)

The substitution of (74) in (77) for \( u \) gives:

\[
\mu_1 = \mu_2 = -\frac{c \cdot e^{2mr}}{2m^2r^2} \cdot [1 - 4mr]; \quad \mu_3 = \mu_4 = -\frac{c \cdot e^{2mr}}{2m^2r^2}.
\] (78)

It follows from (78) that positivity condition of all eigenvalues first consists of constant \( c \) constraint

\[
c > 0,
\] (79)

second, of constraints of the range of radial variable values

\[
r > 1/4m.
\] (80)

Some of the eigenvalues are negative when the conditions (79), (80) are violated.

Let us denote the normalized eigenvectors corresponding to the eigenvalues (77) by \( \xi^I, \xi^{II}, \xi^{III}, \xi^{IV} \). These vectors satisfy the following formulae:

\[
M \xi^I = \xi^I \mu_1, \quad M \xi^{II} = \xi^{II} \mu_2, \quad M \xi^{III} = \xi^{III} \mu_3, \quad M \xi^{IV} = \xi^{IV} \mu_4.
\] (81)

Let us reduce the vectors \( \xi^I, \xi^{II}, \xi^{III}, \xi^{IV} \) in the explicit form:
\[ \xi^I = \begin{pmatrix} y & -2r(r-z) \\ x & 2r(r-z) \\ 0 & 0 \\ -\sqrt{r-z} & \sqrt{2r} \end{pmatrix} \quad \xi^{II} = \begin{pmatrix} y & -2r(r-z) \\ x & 2r(r-z) \\ 0 & 0 \\ -\sqrt{r-z} & \sqrt{2r} \end{pmatrix} \]

\[ \xi^{III} = \begin{pmatrix} y & -2r(r+z) \\ x & 2r(r+z) \\ 0 & 0 \\ -\sqrt{r+z} & \sqrt{2r} \end{pmatrix} \quad \xi^{IV} = \begin{pmatrix} y & -2r(r+z) \\ x & 2r(r+z) \\ 0 & 0 \\ -\sqrt{r+z} & \sqrt{2r} \end{pmatrix} \]

(82)

Let us compile \( \mu \) scalar matrix, where \( \mu_1, \mu_2, \mu_3, \mu_4 \) stand diagonally, along with \( \xi \) matrix, where the columns are composed of vector components \( \xi^I, \xi^{II}, \xi^{III}, \xi^{IV} \).

\[ \mu = \begin{pmatrix} \mu_1 & 0 & 0 & 0 \\ 0 & \mu_2 & 0 & 0 \\ 0 & 0 & \mu_3 & 0 \\ 0 & 0 & 0 & \mu_4 \end{pmatrix} \]  

(83)

\[ \xi = \begin{pmatrix} -y & x & y & -x \\ \sqrt{2r(r-z)} & \sqrt{2r(r-z)} & \sqrt{2r(r+z)} & \sqrt{2r(r+z)} \\ \sqrt{2r(r-z)} & \sqrt{2r(r-z)} & \sqrt{2r(r+z)} & \sqrt{2r(r+z)} \\ 0 & 0 & 0 & \sqrt{r+z} \end{pmatrix} \]  

(84)

\( \xi \) matrix of the form of (84) is orthogonal. In terms of the matrices introduced (83), (84), the relations (81) will be written in the form of matrix equality

\[ M \cdot \xi = \xi \cdot \mu. \]  

(85)

Multiply (85) on the right side by \( \xi^+ \).

\[ M = \xi \cdot \mu \cdot \xi^+. \]  

(86)

If all the eigenvalues \( \mu_1, \mu_2, \mu_3, \mu_4 \) are positive, the relation (86) can be written in the form of (51), where

\[ Z = \xi \cdot \sqrt{\mu}, \]  

(87)

and the matrix
is denoted by $\sqrt{\mu}$. Let us write out $Z$ bispinor matrix in the explicit form for the exact solution considered.

\[
Z = \begin{pmatrix}
-\frac{y\sqrt{\mu_1}}{\sqrt{2r(r-z)}} & \frac{x\sqrt{\mu_2}}{\sqrt{2r(r-z)}} & \frac{y\sqrt{\mu_3}}{\sqrt{2r(r+z)}} & -\frac{x\sqrt{\mu_4}}{\sqrt{2r(r+z)}} \\
\frac{x\sqrt{\mu_1}}{\sqrt{2r(r-z)}} & -\frac{y\sqrt{\mu_2}}{\sqrt{2r(r-z)}} & -\frac{y\sqrt{\mu_3}}{\sqrt{2r(r+z)}} & \frac{y\sqrt{\mu_4}}{\sqrt{2r(r+z)}} \\
0 & \frac{y\sqrt{r-z}}{\sqrt{2r}} & -\frac{x\sqrt{\mu_1}}{\sqrt{2r}} & 0 \\
\frac{x\sqrt{r-z}}{\sqrt{2r}} & 0 & 0 & \frac{x\sqrt{r-z}}{\sqrt{2r}} \\
\end{pmatrix}
\]  

\[\text{(89)}\]

For $\sqrt{\mu}$ and $\xi$ orthogonal matrix are parent matrices to find $\Gamma_\alpha$ bispinor connectivity from (67). The computing sequence $\Gamma_\alpha$ includes:

- Formula for $\xi^+$, according to (84) for $\xi$ matrix.
- Partial derivatives $\xi^+_0, \xi^+_1, \xi^+_2, \xi^+_3$.
- Finding $\langle \xi^+_0 \xi \rangle, \langle \xi^+_1 \xi \rangle, \langle \xi^+_2 \xi \rangle$.
- Computation of combinations $\sqrt{\mu} \cdot \langle \xi^+_0 \xi \rangle \cdot \frac{1}{\sqrt{\mu}}$, $\sqrt{\mu} \cdot \langle \xi^+_1 \xi \rangle \cdot \frac{1}{\sqrt{\mu}}$, $\sqrt{\mu} \cdot \langle \xi^+_2 \xi \rangle \cdot \frac{1}{\sqrt{\mu}}$ and substitution in (67) for $\Gamma_\alpha$.

Each of the operations enumerated is quite simple, though the treatment seems to be too lengthy. We give the final output introducing the auxiliary function

\[
\Omega = \left( \sqrt{\frac{\mu_1}{\mu_3}} + \sqrt{\frac{\mu_3}{\mu_1}} \right) = \left( \sqrt{\frac{\mu_2}{\mu_4}} + \sqrt{\frac{\mu_4}{\mu_2}} \right) = \left( \sqrt{\frac{\mu_5}{\mu_6}} + \sqrt{\frac{\mu_6}{\mu_5}} \right) = \frac{4m_u}{\sqrt{4m^2u^2 - u^2}}.
\]

\[\text{(90)}\]

The bispinor connectivity components can be written in the compact form as follows:

\[
\Gamma_0 = 0,
\]

\[\text{(91)}\]
\[
\Gamma_1 = -\frac{y}{2r(r - z)} \cdot i\sigma_2 - \Omega \cdot \frac{xz}{4r^2\sqrt{r^2 - z^2}} \cdot i\rho_2 + \Omega \cdot \frac{y}{4r\sqrt{r^2 - z^2}} \cdot i\rho_1\sigma_2,
\]

\[
\Gamma_2 = \frac{x}{2r(r - z)} \cdot i\sigma_2 - \Omega \cdot \frac{yz}{4r^2\sqrt{r^2 - z^2}} \cdot i\rho_2 - \Omega \cdot \frac{x}{4r\sqrt{r^2 - z^2}} \cdot i\rho_1\sigma_2,
\]

\[
\Gamma_3 = \Omega \cdot \frac{\sqrt{r^2 - z^2}}{4r^2} \cdot i\rho_2.
\]

As we might expect, the formulae (91)-(94) obtained correspond to the group of gauge generations \(SO(4)\).

### 7.2 Solution analysis

Let us write the formulae for energy-momentum tensor components corresponding to the solution of (75). Suppose that the values included in energy-momentum tensor are of the structure as follows:

\[
A_\alpha = (A_0(x, y, z), 0, 0, 0), \quad \lambda = \text{Const}.
\]

We have:

\[
\begin{align*}
T_{00} &= -3(A_0)^2 - \lambda \\
T_{0k} &= A_{0,k} \\
T_{mn} &= \delta_{mn} [(A_0)^2 + \lambda]
\end{align*}
\]

Change \(A_\alpha, \lambda\) in (96) to \(J_\alpha, m\) according to the formulae (35), (36). We get:

\[
\begin{align*}
T_{00} &= -3m^2(J_0)^2 - 2m^2 \\
T_{0k} &= mJ_{0,k} \\
T_{mn} &= \delta_{mn} [m^2(J_0)^2 + 2m^2]
\end{align*}
\]

(97) implies the formulae to express \(U\) energy density and \(P\) pressure:

\[
U = -3m^2(J_0)^2 - 2m^2,
\]

\[
P = m^2(J_0)^2 + 2m^2.
\]
It follows from (98), (99) that the energy density is the negative value, and the pressure is positive,

\[ U < 0, \quad P > 0. \]  

(100)

Here \( U \) and \( P \) are related as

\[ U = -3P + 4m^2. \]  

(101)

If we compare (101) and (99) we get the following relation

\[ P = \frac{1}{16V^2m^4} + 2m^2. \]  

(102)

The substitution of (102) for (101), gives:

\[ U = -\frac{3}{16V^2m^4} - 2m^2. \]  

(103)

The comparison of (103) and (98) gives:

\[ J^0 = \frac{1}{4Vm^3}. \]  

(104)

It follows from the general theory that the vectors \( J^\alpha \) and \( u^\alpha \) should be collinear, i.e.

\[ J^\alpha = \text{Const} \cdot \frac{u^\alpha}{V}. \]  

(105)

By taking the index \( \alpha = 0 \) in (105) and using the formulae (104) and (75) for \( J^0 \), we get:

\[ \rho = \frac{1}{V} = 4cm^3 \cdot \frac{e^{2mr}}{mr}. \]  

(106)

(106), (103), (102) give us radial coordinate dependence of energy density and pressure:

\[ U = -3e^2 \cdot \frac{e^{4mr}}{r^2} - 2m^2, \]  

(107)

\[ P = e^2 \cdot \frac{e^{4mr}}{r^2} + 2m^2. \]  

(108)
It follows from (107), (108) that when \( r \to \infty \), \( U \) and \( P \) are related as \( U = -P \).

Now let us find the formula for \( c_s \), isentropic velocity of sound. Substitute (108) and (106) for \( P \) and \( 1/V \) in (24). We get:

\[
\frac{c^2_s}{6} \cdot c \cdot \frac{e^{2m r}}{m^2 r}.
\] (109)

(109) states that isentropic velocity of sound goes to infinity when \( r \to \infty \). Such dependence implies that the velocity of sound should come up to the velocity of light at certain finishing radius \( \tau \). In order for \( \tau \) to be found, (109) should be set equal to \( 1/m \) - the unique constant in length dimension problem.

\[
(m \tau) e^{-2m \tau} = \frac{6}{11} \cdot c.
\] (110)

It is clear that if

\[
c \leq 11/12c,
\] (111)

the solution of (110) is always available. The perturbation velocity of geometrodynamic medium does not achieve the velocity of light.

Evidently, the accumulating perturbations reconstruct the solution within the framework of CGD solutions if the character of solution formation is evolutionary in the range of radii near \( \tau \). In other words, \( \tau \) - is the radius within which the solution can branch, i.e. one solution branch can be changed by the other. Note that the solution of (74) type can be the other solution branch with decreasing exponent not increasing.

Thus, the solution (70) is valid for all values of the variable \( r > 0 \). However, there are two values of the variable when the solution and/or its treatment undergo changes. The first value is \( 1/4m \), and the second value \( \tau \) is determined from (110). The full solution includes the description of \( J_\alpha \) vector and \( H_{\alpha \beta} \) tensor in three domains I, II, III. Domain II is

\[
(1/4m) < r < \tau,
\] (112)

it differs in the fact that if the integration constant satisfies the inequality (111), the condition of positivity of the polarization matrix of \( M \) density from (53) is fulfilled in the range (112). It means that the solution (70) can be interpreted in terms of \( Z \) bispinor matrix (89) in the range of (112). Because
this matrix is the direct sum of four bispinors, the solution can be described in terms of four particles with 1/2 spin in the domain $II$. As for the domains $I, III$, the polarization matrix of $M$ density from $[53]$ is positively definite in these domains. $M$ matrix requires additional analysis in these domains.

8 Result discussion

The work is one of the fundamental areas of the modern theoretical physics – determination of physical meaning of Weyl space-time degrees of freedom. The problem emerged in 1918 when H. Weyl [7] suggested considering general space in relativity theory not Riemannian space. The additional space properties were connected with the vector introduced (Weyl vector). Physicists and mathematicians have been clarifying the meaning of the vector for more than 90 years. Partial success attends the research confirming the assumption of the fundamental function of Weyl vector in physics. Thus, the conceptual models appeared in which Weyl degrees of freedom are connected with parameters of dark material and energy in the Universe, with cosmological red shift, with change of scale to measure the space-time interval ([8], [14]-[19], etc.). In some works ([20]-[22], etc.) Weyl degrees of freedom are considered as Weyl integrable space quality (i.e. the space in which Weyl vector is the gradient of scalar function) resulting in Schrödinger equation.

The present work suggests the solution to the problem of physical interpretation of Weyl space degrees of freedom. In terms of CGD, Weyl vector depends on the range of the phenomena considered. Weyl vector acts as current density vector of conserved charge on a large scale, i.e. in case the left side of the equation (1) with the right side in the form of tensor (2) cannot be omitted and the space curvature cannot be neglected. CGD does not pre-define the type of conserved charge, but shows that this charge does exist. The availability of such a strongly conserved current density vector allows introduction of specific volume notion, as well as get up phenomenological thermodynamics for geometrodynamic continuum.

Weyl vector is proportional to sum vector of probability density of all the particles with half-integer spin on microparticle scale in terms of which the geometrodynamic medium dynamics can be described. The gauge is always available when the divergence of Weyl vector goes to zero.

It is our opinion that the agreement of CGD equations with Dirac derived equations given in Table[2] is the trenchant argument confirming the viability
of interpretation suggested.

However, it should be understood that the dynamics of Weyl degrees of freedom is described by (31)-(34) at any polarization structure of $M$ matrix. However, quantum-field interpretation of these equations is not always possible, it is applicable when all the eigenvalues of $M$ matrix are positive. This condition fits the requirements imposed on polarization density matrices in quantum mechanics and quantum field theory. Evidently, the violation of positivity condition of $M$ matrix, as well as vanishing of matrix determinant do not interfere with the analogous interpretation – it is a matter that requires additional analysis. The results of the work assume the number of generalizations, for example: complexification, introduction of internal spaces, various cases of $M$ matrix eigenvalues, etc.

We note finally that Weyl vector interpretation described opens up new possibilities when Standard model of elementary particles, particularly confinement model, is theoretically proved.

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