A non-relativistic theory of quantum mechanics and gravity with local modulus symmetry

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Abstract

Inspired by the similarities between quantum field theory and general relativity, in that each theory encompasses two universal constants and a local symmetry, we set out to construct a non-relativistic theory of quantum mechanics and gravity based upon two assumptions: quantum system remains invariant to local modulus transformation, and physical laws reduce to those of conventional quantum mechanics in small enough region, i.e. a modified version of the equivalence principle. Imposing local modulus symmetry brings a number of changes, including the replacement of wave function’s complex conjugate by a new scalar function with the same phase but different modulus, and particle momentum operator built on covariant derivative, with a connection of purely imaginary velocity field that is identified as the gravitational escape velocity. Three quantum metric functions are defined to signify the kinematic change of quantum state brought by gravity. The modified equivalence principle enables us to relate the escape velocity field with the quantum metric functions. Equation of motion and field equation that are covariant to local modulus transformation are constructed. New features in these equations offer potential mechanisms to account for the dark energy, the mass discrepancies in the universe, and the quantum state reduction of macroscopic objects.

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I. Introduction

A. Universal constants

The Planck constant $\hbar$, the speed of light in vacuum $c$, and the gravitational constant $G$, are considered the most fundamental constants in physics. Together they form the natural unit system, where physical quantities in the Planck scale, e.g. Planck length and Planck time, take the numerical value of 1. One can also examine the fundamental theories of physics from the perspective of these three constants, as shown in Fig. 1. In the standard unit SI system based on human scale, $G$, $\hbar$, and $c^{-1}$ are all quite small numerically, thus the crudest approximation in SI system is to equate all three to 0. This results into, crudely speaking, the “0th order” theory, i.e. classical mechanics, with the kinematics of Newtonian absolute space and time, but without the effect of gravity, as shown on the lowest level in Fig. 1. The next level, “1st order” approximation is to make $c^{-1}$, $G$ and $\hbar$ finite separately. This then results into three theories, namely special relativity (SR), Newtonian gravity, and non-relativistic quantum mechanics (QM). The next level “2nd order” approximation is to make $c^{-1}$, $G$, and $\hbar$ finite two at a time. Thus, as shown at the highest level in Fig. 1, the theory with finite $c^{-1}$ and $G$ but with $\hbar$ set to 0 is general relativity (GR), the theory with finite $c^{-1}$ and $\hbar$ but with $G$ set to 0 is quantum field theory (QFT). Currently these two theories are generally considered the most important and fundamental theories in physics. Based on the perspective presented in Fig. 1, however, there should be a third “2nd order” theory, with finite $G$ and $\hbar$ but with $c^{-1}$ set to 0. Sitting at the same level as GR and QFT, this theory is then conceivably of fundamental importance as well. Yet curiously, such a theory, which is called general quantum mechanics here, in reference to its symmetric status to general relativity in Fig. 1, has received little attention.

So far the best effort in this direction, to the author’s knowledge, is through the approach of the so-called Schrödinger-Newton equations, a term first proposed by Penrose. The Schrödinger-Newton equations, consisting of the Schrödinger equation as the equation of motion, and the gravitational Poisson equation as the field equation, essentially describe quantum particles moving in a Newtonian gravitational potential.
created by the particles’ own mass density distribution. Thus, the potential energy in the Schrödinger equation is determined by the gravitational Poisson equation, with the source matter density proportional to the probability density of the particles. This probability density in turn is determined by the wave function of the particles, i.e., these two non-relativistic equations are coupled. In the case of the simplest system, i.e., a single particle with mass $m$, these equations are shown below as\(^2\):

\[
\tag{1}
\imath \hbar \frac{\partial \psi(\mathbf{r})}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi(\mathbf{r}) + U(\mathbf{r}) \psi(\mathbf{r})
\]

\[
\nabla^2 U(\mathbf{r}) = 4\pi G m^2 |\psi(\mathbf{r})|^2 \tag{2}
\]

where $\psi(\mathbf{r})$ is the wave function of the particle at position $\mathbf{r}$, $U(\mathbf{r})$ is the gravitational self-interacting potential energy, and $\nabla^2$ is the Laplacian operator. If one excludes the self-interacting gravitational energy of the particle under consideration, and $U(\mathbf{r})$ is redefined as the potential energy caused by external gravitational source, then the term $m |\psi(\mathbf{r})|^2$ in Eq. (2) needs to be replaced by an external mass density $\rho_{\text{ext}}(\mathbf{r})$. Eq. (1) and (2), obviously uncoupled now, are just the conventional Schrödinger equation with an external gravitational field and the conventional gravitational Poisson equation, respectively.

The main motivation of Penrose et al. studying these equations is to tackle the quantum measurement problem by exploring the possibility of gravitation-induced quantum state reduction.\(^2\) It should be noted that similar schemes have been explored before.\(^3,4\) Starting from the generally covariant Newton-Cartan theory of gravity, Christian has used a variational approach to combine it with non-relativistic quantum mechanics, and he also reached the same Schrödinger-Newton equations.\(^5\) Instead of quantizing GR or “general relativizing QFT”, he then argued that one can try to “special relativize” these Schrödinger-Newton equations, and this approach then constitutes a third way to explore quantum gravity and related Planck scale physics.

Regardless of the motivations, from the perspective of Fig. 1, the Schrödinger-Newton equations remain unsatisfying, mainly in two aspects. First, one of the signatures in any “2nd order” theory is term or terms containing two universal constants. In the case of GR, they are terms containing both $G$ and $c^{-1}$; in QFT, terms containing both $\hbar$ and $c^{-1}$. Yet in Eq. (1) and (2), there are no terms that explicitly have both $G$ and $\hbar$. Secondly, and more importantly, both GR and QFT are local theories with global symmetries (Lorentz and gauge symmetries, respectively) elevated to local symmetries. The approach of Schrödinger-Newton equations does not have this key element. The main purpose of this paper is to construct a non-relativistic theory that couples gravity with quantum mechanics without the need of self-interaction, at the same time remedies these two deficiencies.

### B. Local symmetries

To combine quantum mechanics with gravity, it is helpful to learn from GR, the last successful theory that combines a “1st order” fundamental theory, i.e., special relativity, with gravity. In the process of introducing the new kinematics of SR, Einstein stressed the importance of symmetry and universal constant (in this case $c$). This is manifested by the two famous postulates in SR.\(^6\) In the process of developing GR, Einstein recognized the importance of the weak equivalence principle of Newtonian gravity on the gravitational side, and the importance of Lorentz symmetry on the SR side. His way of combining them
is to elevate the Lorentz symmetry from a global one in SR to a local one in GR, in the form of Einstein equivalence principle; at the same time, he introduced a new element, the varying metric tensor, into the kinematics to express the effect of gravity.\footnote{7}

It is worth emphasizing here the difference between the weak equivalence principle and Einstein equivalence principle.\footnote{8} The weak equivalence principle is essentially the equality between the inertial mass and the gravitational mass for any massive object. It is a firm feature of Newtonian mechanics and gravity, strongly supported by experimental evidences going back to Galileo’s time.\footnote{9} The Einstein equivalence principle states that in small enough region of spacetime, the law of non-gravitational physics reduces to those of SR. It is a generalization of weak equivalence principle in the context of relativistic spacetime.\footnote{8} We note that the geometric nature of GR is obviously closely related to the Einstein equivalence principle, but the root of geometrization is not in weak equivalence principle. Instead, it should be traced back to Minkowski’s geometric formulation of SR\footnote{10}.

In a broader context not specifically tied to relativistic theory, one can argue, that the essence of the equivalence principle is the expression of gravity through kinematics instead of dynamics, as gravity is not a normal force associated with conventional dynamics, and it can be transformed away locally by certain kinematic transformation.

It is also beneficial here to learn from QFT, the other “2nd order” fundamental theory. Similar to general covariance in GR, the most important symmetry in QFT is the gauge symmetry. One starts from the observation that in QM the phase of wave function has a global symmetry. In contrast to classical waves where the absolute value of phase can be measured, one can add an arbitrary uniform phase to the quantum wave function, and that does not alter anything in the quantum system. This global symmetry is then elevated into a local gauge symmetry, with phase transformation varying from point to point over the whole spacetime. Similar to GR, this also leads to covariant derivatives, and the associated connections are related to gauge fields that correspond to electromagnetic, weak and strong forces. Interestingly, wave function in QM has another global symmetry, in its modulus. Again different from classical waves, one can multiply a constant real and positive number to the unnormalized quantum wave function, and that does not alter anything in the quantum system.

With these insights from the two existing “2nd level” theories, we can start to outline the framework of the theory that combines gravity with QM, but without relativity. On the gravitational side, it is natural to take the equivalence principle as the foundation. The equivalence principle here is not the Einstein equivalence principle, but it should encompass the weak equivalence principle. Additionally, some new element needs to be introduced into the kinematics to express gravitational effect, similar to the metric tensor in GR. One should also be able to transform away this new element locally, so that in small enough region, the laws of physics reduce to those of conventional QM, just as in GR they reduce to SR locally. This can serve as the new definition of the equivalence principle in the theory here. On the QM side, we will show that the relevant symmetry is related to the modulus of the wave function. Similar to the global phase symmetry being elevated into a local phase symmetry in QFT, the global symmetry of the modulus of wave function in QM mentioned above will be elevated into a local symmetry in the theory here. This will also lead to a covariant derivative, with the associated connection related to gravitational field.
II. Theory

A. Quantum state

In this section we will discuss the relation between state vector and wave function in conventional non-relativistic quantum mechanics. Similarity between wave function in QM and vector field in SR will be highlighted.

Mathematically QM is built on the notion that the state of a quantum system is described by a single vector in the Hilbert space. This state vector can be denoted by a ket vector \(|\psi\rangle\), or equally well by its Hermitian conjugate, the bra vector \langle \psi |. For reasons that will become clear later, spatial position representation will be used exclusively in the discussion below. Suppose the position basis ket is denoted as \(|r\rangle\), and \langle r | is the position basis bra, with \( r \) representing any point in space, and suppose \( \psi(r) = \langle r | \psi \rangle \), then inserting the identity operator \( \int d^3 r \langle r |(r) \) into \(|\psi\rangle\), one has \(|\psi\rangle = \int d^3 r \langle r |(r)|\psi\rangle = \int d^3 r \psi(r)|r\rangle\). Here \( \psi(r) \) is understood as the component of the state vector \(|\psi\rangle\) along the \(|r\rangle\) basis in the Hilbert space. Similarly the bra vector \( \langle \psi |\) can be decomposed as \( \langle \psi | = \int d^3 r \psi^*(r)|r\rangle \), where \( \psi^*(r) \) is the complex conjugate of \( \psi(r) \). The common practice then is to treat \( \psi(r) \) or \( \psi^*(r) \), like \(|\psi\rangle \) or \( \langle \psi |\), as containing all the information of the quantum state. Since \( \psi(r) \), or the time dependent form \( \psi(r, t) \), is a function of real space and time, mathematically it can be taken as a complex scalar field, with the Schrödinger equation in Eq. (1) offering solution for such a field. \( \psi(r, t) \) is commonly known as the wave function of the quantum system. Similar statement can be made for \( \psi^*(r, t) \).

The equivalence of \( \psi(r) \) to \(|\psi\rangle\), and \( \psi^*(r) \) to \langle \psi |\), are based on \( \int d^3 r \langle r |(r) \) being an identity operator. This in turn is based on the orthonormal condition:

\[
\langle r' |(r) = \delta(r' - r) = \delta(x' - x)\delta(y' - y)\delta(z' - z)
\]

Here and in the discussion below, for convenience we adopt the Cartesian coordinates as the default coordinate system, with the basis ket \(|r\rangle = |x, y, z\rangle = |x\rangle|y\rangle|z\rangle\), and the basis bra \langle r | = \langle z, y, x | = \langle z|\langle y|\langle x|\). \( \delta(r' - r) \) is the 3 dimensional Dirac delta function.

As already mentioned, development of this new theory can be advanced by drawing analogies with GR and QFT. In this context, it is worth mentioning a similarity between QM and SR. In SR, any vector \( V \), its component denoted as \( V^\mu \), has a corresponding dual vector (or covector) \( \tilde{V} \), with component \( \tilde{V}_\nu \), and the relation \( V^\mu = \eta^{\mu\nu} \tilde{V}_\nu \), where \( \eta^{\mu\nu} \) is the Minkowski metric tensor. Since \( \eta^{\mu\nu} \) is constant over the whole spacetime, \( \tilde{V}_\nu \) contains the same information as \( V^\mu \). In QM, the bra vector \( \langle \psi |\) was initially introduced by Dirac as the dual vector of the ket vector \(|\psi\rangle\) in the Hilbert space.\(^{11}\) Since the bra vector is the Hermitian conjugate of ket vector, \( i.e. \langle \psi | = |\psi\rangle^\dagger \), \(|\psi\rangle \) and \( \langle \psi |\) essentially contain the same information, similar to the vector and its dual vector in SR.

To extend the analogy further, we note that in SR, the vector component \( V^\mu \) and its dual vector component \( \tilde{V}_\nu \), in general are fields over real spacetime. They can thus be more adequately denoted as \( V^\mu(x) \) and \( \tilde{V}_\nu(x) \), with \( x \) representing any point in spacetime in the context of SR and GR. (In the context of QM, \( x \) is used to represent one of the three dimensions of Cartesian coordinates, but since the discussions here are clearly delineated, there should be no confusions.) In QM, as mentioned above, the wave function and its complex conjugate are functions of real space and time, thus are more mathematically analogous to the vector and dual vector field in SR than the single vector \(|\psi\rangle\) and \( \langle \psi |\) in the Hilbert space. To make the analogy even closer, assume that in SR \( \hat{e}_\mu \) and \( \hat{e}^\nu \) represent
the sets of basis vectors and basis dual vectors for the vector field \( V(x) \) and dual vector field \( \tilde{V}(x) \), respectively, so \( V(x) = V^\mu(x) \hat{e}_\mu \) and \( \tilde{V}(x) = V_\nu(x) \hat{e}^\nu \cdot \psi(r, t) \) and \( \psi^*(r, t) \) in QM are thus similar to \( V^\mu(x) \) and \( V_\nu(x) \) in SR, while the orthonormal \( |r| \) basis ket and \( \langle r | \) basis bra in QM play roles similar to the Cartesian-like orthonormal bases of \( \hat{e}_\mu \) and \( \hat{e}^\nu \) in SR, \emph{i.e.} constant over the whole spacetime. When gravity is present and spacetime is curved, global Cartesian-like coordinate system is not possible anymore. Coordinate basis vectors, with varying magnitudes from point to point, are commonly used in GR instead of the orthonormal basis. As shown below, similar effects will be found for the position basis kets and bras when one combines gravity with QM.

**B. Modulus transformation**

We now introduce the local modulus transformation. Similar to the simplest gauge transformation in QFT, \emph{i.e.} the local phase transformation with \( U(1) \) symmetry in quantum electrodynamics (QED), one can change the modulus of non-relativistic wave function of any quantum system by multiplying an addition factor \( e^{\lambda(r)} \),

\[
\psi(r, t) \rightarrow e^{\lambda(r)} \psi(r, t) \tag{4}
\]

Here \( \lambda(r) \) is an arbitrary, real, smooth, and differentiable scalar function of spatial position \( r \). The exponential form is chosen here to guarantee the factor positive definite.

Note that \( \lambda(r) \) is only a function of space, and time is not explicitly present. This does not mean one performs this operation continuously over time. Rather, in the context of non-relativistic theory, this operation should be understood as a one-off event. It can be performed at any given time, thus no explicit time in \( \lambda(r) \), but once the wave function’s modulus is changed by the additional factor of \( e^{\lambda(r)} \), the system is allowed to evolve without further interference. This is different from the QED case, because of the non-relativistic nature of the theory here.

We now proceed to introduce one of the most important postulates in our theory here. Similar to the general covariance in GR and the gauge symmetry in QFT, we postulate that quantum system remains invariant when its wave function undergoes any local modulus transformation.

Before we discuss the consequences of this postulate, it is worthwhile to note the lessons in GR again. In GR, one may take the vector field \( V(x) \) and covector field \( \tilde{V}(x) \) as abstract geometric objects over spacetime, and express them as \( V(x) = V^\mu(x) \hat{e}_\mu(x) \) and \( \tilde{V}(x) = V_\nu(x) \hat{e}^\nu(x) \), with \( \hat{e}_\mu(x) \) and \( \hat{e}^\nu(x) \) now representing the coordinate basis vectors and dual vectors that vary over spacetime. The components \( V^\mu(x) \) and \( V_\nu(x) \) transform inversely as their respective basis vectors and basis dual vectors \( \hat{e}_\mu(x) \) and \( \hat{e}^\nu(x) \) do, indicated by their respective super and subscript indices. This guarantees \( V(x) \) and \( \tilde{V}(x) \) as geometric invariant objects under local coordinate transformation.

In previous section we draw the analogy between \( V^\mu(x) \hat{e}_\mu \) in SR and \( \psi(r, t) |r\rangle \) in QM. Let us formalize it with the definition of \( \hat{\psi}(r, t) \) as

\[
\hat{\psi}(r, t) = \psi(r, t) |r\rangle \tag{5}
\]

and we can extend the analogy between \( V(x) \) in GR and \( \hat{\psi}(r, t) \) in the theory here by identifying \( \hat{\psi}(r, t) \) as an object that stays invariant under local modulus transformation.
Since \( \psi(\mathbf{r}, t) \) transforms according to Eq. (4), to keep \( \tilde{\psi}(\mathbf{r}, t) \) invariant, similar to the coordinate basis in GR, we demand \( |\mathbf{r}| \) to transform inversely as:
\[
|\mathbf{r}| \rightarrow e^{-\lambda(\mathbf{r})}|\mathbf{r}|
\]
(6)
On the other hand, since the Dirac delta function is obviously an invariant, the orthonormal condition between \( |\mathbf{r}| \) and \( \langle \mathbf{r} \rangle \) in Eq. (3) means \( \langle \mathbf{r} \rangle \) should transform as:
\[
\langle \mathbf{r} \rangle \rightarrow e^{\lambda(\mathbf{r})}\langle \mathbf{r} \rangle
\]
(7)
To complete the analogy between vector and covector here and in GR, we define another invariant object \( \tilde{\psi}(\mathbf{r}, t) \) as:
\[
\tilde{\psi}(\mathbf{r}, t) = \tilde{\psi}(\mathbf{r}, t)|\mathbf{r}|
\]
(8)
\( \tilde{\psi}(\mathbf{r}, t) \) is a new complex scalar function of space and time. Since \( \langle \mathbf{r} \rangle \) transforms the same way as \( \psi(\mathbf{r}, t) \), to keep \( \tilde{\psi}(\mathbf{r}, t) \) invariant, \( \psi(\mathbf{r}, t) \) should transform inversely as \( \psi(\mathbf{r}, t) \):
\[
\psi(\mathbf{r}, t) \rightarrow e^{-\lambda(\mathbf{r})}\psi(\mathbf{r}, t)
\]
(9)
We note that the covector of conventional QM is identified as \( \psi^*(\mathbf{r}, t)\langle \mathbf{r} \rangle \) in previous discussion. \( \psi^*(\mathbf{r}, t) \), being the complex conjugate of \( \psi(\mathbf{r}, t) \), shares with \( \psi(\mathbf{r}, t) \) the same modulus. Thus \( \psi^*(\mathbf{r}, t) \) transforms the same way as \( \psi(\mathbf{r}, t) \) under local modulus transformation, and cannot form an invariant object with \( |\mathbf{r}| \). This is why we need a new complex scalar function \( \tilde{\psi}(\mathbf{r}, t) \) in Eq. (8). However, since the phase of any complex function is not altered by the modulus transformation, \( \tilde{\psi}(\mathbf{r}, t) \) is defined to share the same phase as \( \psi^*(\mathbf{r}, t) \). Alternatively, we can define a new real positive dimensionless smooth function \( \gamma(\mathbf{r}) \) as the ratio between the modulus of \( \tilde{\psi}(\mathbf{r}, t) \) and that of \( \psi^*(\mathbf{r}, t) \) so that
\[
\psi(\mathbf{r}, t) = \gamma(\mathbf{r})\psi^*(\mathbf{r}, t)
\]
(10)
with a transformation rule of
\[
\gamma(\mathbf{r}) \rightarrow e^{-2\lambda(\mathbf{r})}\gamma(\mathbf{r})
\]
(11)
to guarantee the transformation rule of \( \tilde{\psi}(\mathbf{r}, t) \) as Eq. (9). Since \( \lambda(\mathbf{r}) \) is an arbitrary real function, for any given pair of \( \psi(\mathbf{r}, t) \) and \( \psi^*(\mathbf{r}, t) \), one can always set \( \lambda(\mathbf{r}) = \frac{1}{2}\ln \gamma(\mathbf{r}) \) to transform them into a new pair with the same modulus, i.e. to make \( \tilde{\psi}(\mathbf{r}, t) = \psi^*(\mathbf{r}, t) \), and the new \( \gamma(\mathbf{r}) \) as a constant of 1. \( \gamma(\mathbf{r}) \) is not a function of time, because it supposedly can be transformed away by \( \lambda(\mathbf{r}) \). We call \( \tilde{\psi}(\mathbf{r}, t) \) the co-wave function.

With the introduction of \( \tilde{\psi}(\mathbf{r}, t) \), \( \psi(\mathbf{r}, t) \) \( \tilde{\psi}(\mathbf{r}, t) \) and \( \gamma(\mathbf{r}) \), as well as the transformation rules for \( \psi(\mathbf{r}, t) \), \( \tilde{\psi}(\mathbf{r}, t) \), \( |\mathbf{r}| \) and \( \langle \mathbf{r} \rangle \), we can now investigate the consequences of this postulate, first on the kinematics of QM in the next section, then on the dynamics of QM in the section after.

**C. Quantum kinematics**

In conventional QM, all observables are real numbers that result from some combination of \( \psi(\mathbf{r}, t) \) and \( \psi^*(\mathbf{r}, t) \). To keep these observables invariant to local modulus transformation, one needs to replace \( \psi^*(\mathbf{r}, t) \) by \( \tilde{\psi}(\mathbf{r}, t) \). For example, the probability density \( \rho \) is defined now as \( \rho(\mathbf{r}, t) = \tilde{\psi}(\mathbf{r}, t)\psi(\mathbf{r}, t) \), which is obviously invariant due to the inversely related transformation rules of \( \psi(\mathbf{r}, t) \) and \( \tilde{\psi}(\mathbf{r}, t) \). To make the transition from QM to the theory here, the first thing to do is then to systematically go over everything and replace \( \psi^*(\mathbf{r}, t) \) by \( \tilde{\psi}(\mathbf{r}, t) \), or equivalently by \( \gamma(\mathbf{r})\psi^*(\mathbf{r}, t) \).
The next object whose transformation property needs to be considered is the spatial derivative of wave function $\nabla \psi(r, t)$. To make it behave under local modulus transformation, the mathematical treatment is very similar to local phase transformation in QED. We first define the covariant spatial derivative $D_r$ that will be used to replace the ordinary derivative $\nabla$ as $D_r = \nabla + Y(r)$, with $Y(r)$ being the vector connection. Thus

$$D_r \psi(r, t) = \nabla \psi(r, t) + Y(r) \psi(r, t) \quad (12)$$

Under local modulus transformation $Y(r)$ should transform as:

$$Y(r) \rightarrow Y(r) - \nabla \lambda(r) \quad (13)$$

so that we have

$$D_r \psi(r) \rightarrow [\nabla + Y(r) - \nabla \lambda(r)] [e^{\lambda(r)} \psi(r)] = e^{\lambda(r)} [D_r \psi(r)] \quad (14)$$

Thus the covariant spatial derivative of $\psi(r)$ transforms exactly like $\psi(r)$ itself. To conform with local modulus symmetry, the second thing to do then is to systematically replace ordinary derivative $\nabla$ by covariant derivative $D_r$.

Different from conventional QM, the position basis ket $|r\rangle$ and bra $\langle r|$ can change from place to place, implied from the transformation rules in Eq. (6) and (7). Thus one expects the spatial derivatives of $|r\rangle$ and $\langle r|$ will play important roles in our theory. Yet we will not discuss them in this and next section. This is due to the similarity between the basis ket and bra we use here and the coordinate basis in GR. Because of the convenient transformation rules for tensors and their bases in coordinate basis system, calculations in GR can be performed almost exclusively on tensor components and their covariant derivatives with its associated connection, without worrying about the varying basis vectors and their derivatives. In discussions on the kinematics and dynamics of quantum particles, a similar strategy adopted for $|r\rangle$ and $\langle r|$ here will allow us to focus only on $\psi(r, t)$ and $\check{\psi}(r, t)$, with gravitational effect confined in the connection $Y(r)$ alone. Later on when we discuss how to relate $Y(r)$ with $|r\rangle$ and $\langle r|$, we then need to discuss the details of $|r\rangle$, $\langle r|$, and their derivatives.

We now move on to quantum operators. As we discussed before, with position representation, the operator $\int d^3r |r\rangle \langle r|$ is an identity operator due to the orthonormal condition in Eq. (3). Since this condition remains the same even when $|r\rangle$ and $\langle r|$ change from place to place, $\int d^3r |r\rangle \langle r|$ stays as an identity operator, and the projector operator $P_r = |r\rangle \langle r|$ also remains invariant. Essentially both the inner and outer product of $|r\rangle$ and $\langle r|$ remain invariant, as their transformation rules in Eq. (6) and (7) dictate the same results in both cases. Consequently, we do not expect any change in the form of quantum operators in position representation here. This is also one of the reasons why position representation is used exclusively in this paper.

We can rewrite the kinematics of quantum particles based on the 3 principles laid out above, namely replacing $\psi^*(r, t)$ by $\check{\psi}(r, t)$, replacing $\nabla$ by $D_r$, and using the same form of quantum operators. For example, the expectation value of particle position should be $\langle \hat{r} \rangle = \int d^3r \check{\psi}(r, t) r \psi(r, t)$, where $\hat{r}$ is the position operator.

Let us construct the particle momentum operator $\hat{p}$ based on covariant derivative:

$$\hat{p} = \frac{\hbar}{i} D_r = \frac{\hbar}{i} \nabla + \frac{\hbar}{i} Y(r) = \frac{\hbar}{i} \nabla - imv^s(r) \quad (15)$$

Here we have replaced the connection $Y(r)$ by a velocity field $v^s(r)$ with

$$v^s(r) = \frac{\hbar}{m} Y(r) \quad (16)$$
and \( m \) is the mass of the particle.

The Hermitian conjugate of \( \hat{p} \), denoted as \( \hat{p}^\dagger \), is easily shown as:
\[
\hat{p}^\dagger = \hat{p}^T = \frac{\hbar}{i} \nabla - \frac{\hbar}{i} Y(r) = \frac{\hbar}{i} \nabla + im\nu^e(r)
\]  
(17)

thus \( \hat{p}^\dagger \neq \hat{p} \). This brings up the question of whether momentum related physical quantities in our theory have real eigenvalues. Even more seriously, since Hermitian operators are closely related to the unitarity of quantum system, this raises the question whether quantum state can evolve unitarily in our theory. We will address the first question now, and the second question in the next section, with positive answers in both cases.

We first note that the definition of covariant derivative for \( \psi(r,t) \) in Eq. (12) together with its transformation rule in Eq. (14) cannot be applied to \( \bar{\psi}(r,t) \). Instead, we need to define a different covariant derivative \( \mathcal{D}_r \) for \( \bar{\psi}(r,t) \) as:
\[
\mathcal{D}_r \bar{\psi}(r,t) = \nabla \bar{\psi}(r,t) - Y(r)\bar{\psi}(r,t)
\]  
(18)

with the same connection \( Y(r) \). Inserting its transformation rule in Eq. (13) one has:
\[
\mathcal{D}_r \bar{\psi}(r) \rightarrow [\nabla - Y(r) + \nu \lambda(r)]\left[e^{-\lambda(r)}\bar{\psi}(r)\right] = e^{-\lambda(r)}[\mathcal{D}_r \bar{\psi}(r)]
\]  
(19)

and \( \hat{p}^\dagger = \frac{\hbar}{i} \mathcal{D}_r \) compared to \( \hat{p} = \frac{\hbar}{i} D_r \). We note that in QED, there is only one definition of covariant derivative, but in GR the covariant derivatives for vector field and covector field are very similar to those in Eq. (12) and (18). It is also obvious to see here, that once \( \bar{\psi}(r,t) \) is used instead of \( \psi^*(r,t) \), it is inevitable that \( \hat{p}^\dagger \neq \hat{p} \).

In conventional QM, the expectation value of \( \hat{p} \) is:
\[
\bar{p} = \langle \psi | \hat{p} \psi \rangle = \int \psi^*(r,t) \frac{\hbar}{i} \nabla \psi(r,t) d^3r
\]  
(20)
The probability density flow \( j(r,t) \), on the other hand, is defined locally as:
\[
j(r,t) = \frac{1}{2m} [\psi^*(r,t) \hat{p} \psi(r,t) - \psi(r,t) \hat{p} \psi^*(r,t)] = \frac{1}{m} \text{Re}[\psi^* \hat{p} \psi]
\]  
(21)

where Re stands for real part. If the modulus of \( \psi(r,t) \) is \( R(r,t) \) and its phase is \( S(r,t)/\hbar \) so that we have \( \psi(r,t) = R(r,t)e^{iS(r,t)/\hbar} \), and keep in mind that the particle probability density \( \rho(r,t) = R^2(r,t) \) in conventional QM, one has:
\[
\bar{p} = \int \rho(r,t) \nabla S(r,t) d^3r
\]  
(22)
\[
j(r,t) = \frac{1}{m} \rho(r,t) \nabla S(r,t)
\]  
(23)

This means \( \bar{p} \), which is real by its definition, and \( j(r,t) \), which is taken as a real part of a complex quantity \( \psi^* \hat{p} \psi \), are essentially built from the same local quantity \( \rho(r,t) \nabla S(r,t) \) in QM, i.e. the gradient of the phase weighted by the probability density.

We stick to the same principle in our theory here. When we make the redefinition of any physical observable that is related to momentum, this expectation value has to be real, and locally it should be built solely from \( \rho(r,t) \nabla S(r,t) \), with \( \rho(r,t) \) redefined as:
\[
\rho(r,t) = \bar{\psi}(r,t) \psi(r,t) = \gamma(r) R^2(r,t)
\]  
(24)

We note that local modulus transformation does not affect \( S(r,t) \), thus \( \rho(r,t) \nabla S(r,t) \) is an invariant to this transformation. For \( j(r,t) \), we can just replace \( \psi^*(r,t) \) by \( \bar{\psi}(r,t) \), use Eq. (15) for the momentum operator \( \hat{p} \), and otherwise use the same definition in Eq. (21) for \( j(r,t) \). One can then easily show that this results into the same Eq. (23) for \( j(r,t) \). For the expectation value of momentum, we redefine it as:
\[ \bar{p} = \int \text{Re}[\bar{\psi}(r, t) \bar{\partial} \psi(r, t)] d^3r = \int \text{Re}[\bar{\psi}(r, t) \frac{\hbar}{i} \nabla \psi(r, t)] d^3r \]  
(25)

Again one can easily show that this leads to the same Eq. (22) for \( \bar{p} \).

In QED, the canonical momentum operator, defined as \( \frac{\hbar}{i} \partial_\mu \), corresponds to the total momentum that includes both parts from particle and the electromagnetic field. One needs to subtract the 4-potential from it to get the particle momentum. Our theory here has a purely imaginary momentum field \( im\mathbf{v}^\mu(r) \), and it does not play any role in the particle momentum and probability density flow, following the definitions in Eq. (23) and (25). Physically this is consistent with the notion that relativistic field has its own momentum, while non-relativistic field acts instantaneously, and does not have its own momentum.

In summary, we have outlined here a self-consistent method that allows us to systematically reformulate quantum kinematics under local modulus symmetry requirement. After replacing \( \psi^*(r, t) \) by \( \bar{\psi}(r, t) \), and replacing \( \nabla \) by \( \nabla_r \), one can use the same form of quantum operators as in conventional QM, and all expectation values are defined either as before (if they are already real), or as the real part of the original form (if they are complex). This procedure guarantees that all expectation values are real, invariant to local modulus transformation, and revert to their QM forms when \( Y(r) = 0 \).

**D. Quantum dynamics**

So far we have focused on local modulus symmetry’s impact on kinematics, and all the discussions are in the context of occurring at the same given time. Now we turn our attention to their effects on time evolution of the wave function and the dynamic equation of motion. It is important to stress, that since our theory is non-relativistic, the local modulus transformation here is a one-off event. In other words, the transformation can be done at any given moment, but once it is done, quantum system is allowed to evolve by itself without any further interference.

Suppose there are two adjacent points, P and Q, in a one dimensional space. A wave packet, described by the wave function \( \psi(x, t) \), passes through space. Suppose at time \( t \), the peak of the wave packet is at point P, and at time \( t + \delta t \), this peak reaches point Q. In the case where the wave packet is long with the energy centered around \( E = p^2 / 2m \), the group velocity is \( p/m \), so the displacement from P to Q is \( \delta x = \delta t \cdot p/m \). In the case of a general wave packet made of superposition of plane waves with a wide spread of different wavelengths, one needs to replace \( p \) by the canonical momentum operator \( \hat{P} = \frac{\hbar}{i} \partial_x \). Now suppose one makes a local modulus transformation at time \( t \), which results into an additional factor of \( e^{\lambda(x_P)} \) at point P and a different factor \( e^{\lambda(x_Q)} \) at Q. Since \( f(r, t) \) is an invariant, the group velocity remains the same, so at time \( t + \delta t \), the peak of the wave packet reaches point Q again.

To find out the time derivative \( \partial \psi(x, t) / \partial t \) at point Q, one needs to compare the wave function \( \psi(x_Q, t) \) with \( \psi(x_Q, t + \delta t) \), with the latter originating from the wave function at P and time \( t \), i.e. \( \psi(x_P, t) \). However, at time \( t \), the wave function is modulated differently at P and Q by the local modulus transformation. Thus \( \partial \psi(x, t) / \partial t \) at Q, after the transformation, not only depends on the factor \( e^{\lambda(x_Q)} \), but also on \( e^{\lambda(x_P)} \). As a result, \( \partial \psi(x, t) / \partial t \) does not transform covariantly as \( \psi(x, t) \). The same argument can obviously be generalized into the 3 dimensional case.
To make both sides of the Schrödinger equation covariant to local modulus transformation, one has to introduce a new form of time derivative, named here as the covariant time derivative $D_t$, with the property that $D_t \psi(r, t)$ transforms the same way as $\psi(r, t)$. To obtain the exact form of $D_t$, we use the example above but now generalized to 3 dimensions. Since one knows that the wave function $\psi(r_Q, t + \delta t)$ originates from $\psi(r_p, t)$ and thus transforms the same way as $\psi(r_p, t)$, one can use the connection $Y(r)$ to transport $\psi(r_Q, t + \delta t)$, so that the resulted wave function, denoted as $\psi(P \to Q, t + \delta t)$, is assumed to transform the same as $\psi(r_Q, t)$. Based on the same relation of parallel transport and covariant derivative of vectors in GR, Eq. (12) then implies that $\psi(P \to Q, t + \delta t) = \psi(r_Q, t + \delta t) + Y(r_Q) \cdot (r_Q - r_p) \psi(r_Q, t + \delta t)$ . Let us define $\delta r = r_p - r_Q$, as everything is based on point Q, so $r_Q$ should serve as the origin. To the first order approximation, one can also replace $\psi(r_Q, t + \delta t)$ in the last term by $\psi(r_Q, t)$ as $r_Q - r_p = -\delta r$ is already a first order infinitesimal. Putting everything together, one can now define the temporal covariant derivative at point Q as:

$$D_t \psi(r_Q, t) = \frac{\psi(P \to Q, t + \delta t) - \psi(r_Q, t)}{\delta t} = \frac{\psi(r_Q, t + \delta t) - \psi(r_Q, t) - Y(r_Q) \cdot \delta r \psi(r_Q, t)}{\delta t}$$

The terms in this definition are now all based on the point Q, and since Q can be any point, we will omit this index below, and Eq. (26) can be written as:

$$D_t \psi(r, t) = \frac{\partial \psi(r, t)}{\partial t} - Y(r) \cdot \frac{\delta r}{\delta t} \psi(r, t)$$

As mentioned above, $\delta r = \delta t \cdot \hat{P}/m = \delta t \cdot \frac{\hbar}{2im} \nabla$, and $Y(r) = \frac{m}{\hbar} v^e(r)$, thus one can rewrite the above equation as:

$$D_t \psi(r, t) = \frac{\partial \psi(r, t)}{\partial t} + i v^e(r) \cdot \nabla \psi(r, t)$$

Having obtained both the spatial and temporal covariant derivatives that transform the same way as $\psi(r, t)$, we can now rewrite the Schrödinger equation to make it covariant to local modulus transformation as

$$i \hbar D_t \psi(r, t) = H \psi(r, t) = \frac{\hat{p}^2}{2m} \psi(r, t) = \frac{-\hbar^2}{2m} D_r^2 \psi(r, t)$$

Here we assume that the system is not subject to any force other than that related to gravity. One can certainly insert any other real potential energy $U(r)$ into the Hamiltonian so $H = \frac{-\hbar^2}{2m} D_r^2 + U(r)$, and it is obvious that the equation is still covariant to local modulus transformation. However, since this paper is focused on gravity and quantum mechanics, in the following discussion we will ignore any other type of potential energy’s effect.

One can now insert Eq. (15) and (28) into Eq. (29), and after expanding the quadratic $\hat{p}^2$ term one has:

$$\left\{i \hbar \frac{\partial}{\partial t} - \hbar v^e(r) \cdot \nabla\right\} \psi(r, t) = \left\{\frac{-\hbar^2}{2m} \nabla^2 - \hbar v^e(r) \cdot \nabla - \frac{1}{2} m v^e(r)^2 - \frac{\hbar}{2} \nabla \cdot v^e(r)\right\} \psi(r, t)$$

$$= \left\{\frac{-\hbar^2}{2m} \nabla^2 - \hbar v^e(r) \cdot \nabla - \frac{1}{2} m v^e(r)^2 - \frac{\hbar}{2} \nabla \cdot v^e(r)\right\} \psi(r, t)$$
The second terms on both sides of the equation are the same, cancelling it out yields:

$$i\hbar \frac{\partial}{\partial t} \psi(r, t) = \left\{ -\frac{\hbar^2}{2m} \nabla^2 - \frac{1}{2} m \mathbf{v}^e(r)^2 - \frac{\hbar}{2} \nabla \cdot \mathbf{v}^e(r) \right\} \psi(r, t)$$  \hspace{1cm} (31)

In Newtonian gravity, if $U_g(r)$ is the gravitational potential energy of a particle with a mass $m$ at a given point $r$, one can define the escape velocity $\mathbf{v}^e$ at this point as $U_g(r) + \frac{1}{2} m \mathbf{v}^e(r)^2 = 0$, or

$$U_g(r) = -\frac{1}{2} m \mathbf{v}^e(r)^2$$  \hspace{1cm} (32)

This is exactly the second term on the right side of Eq. (31). We thus identify $\mathbf{v}^e(r)$, defined in Eq. (16), as the escape velocity of gravitational field. Eq. (31) is therefore the quantum equation of motion of a particle in an external gravitational field. There is one new term, $-\frac{\hbar}{2} \nabla \cdot \mathbf{v}^e(r)$, in Eq. (31). Since escape velocity always contains gravitational constant $G$, this term thus has both $\hbar$ and $G$ inside it. Adding the fact that Eq. (31) is covariant to local modulus transformation, we have thus fulfilled the goal of remedying the two deficiencies in the first equation of the Schrödinger-Newton equations, as mentioned in the introduction. With both $\hbar$ and $G$ inside it, this last term in Eq. (31) is expected to be quite small compared to the second term. However, it is interesting that this term, as some form of particle potential energy, is independent of particle mass. We will discuss it in detail in the discussion part.

It is worth noting here that in Newtonian gravity, escape velocity, as defined in Eq. (32), is not an accurate name, since only the magnitude of $\mathbf{v}^e(r)$ is well defined. So it should be called escape speed. The escape velocity $\mathbf{v}^e(r)$, as defined in Eq. (16), is a vector field linearly proportional to the connection $\mathbf{Y}(r)$. So it not only has direction, but also transforms, similar to $\mathbf{Y}(r)$, as

$$\mathbf{v}^e(r) \rightarrow \mathbf{v}^e(r) - \hbar \nabla \lambda(r)/m$$  \hspace{1cm} (33)

The exact relation between the gravitational source and $\mathbf{v}^e(r)$ will be developed in the next two sections.

The definition of both spatial and temporal covariant derivatives above are based on an velocity field $\mathbf{v}^e(r)$. It thus bears some similarities to fluid dynamics. Heuristically one can imagine that the space is filled by a flow of purely imaginary velocity $i\mathbf{v}^e(r)$, and gravitational information is encoded in $\mathbf{v}^e(r)$. If quantum particle’s momentum relative to space is described by the canonical momentum operator $\hat{P} = \frac{\hbar}{i} \nabla$, then this particle’s momentum relative to the flow should be expressed as $\frac{\hbar}{i} \nabla - im\mathbf{v}^e(r)$. This is the same as the definition in Eq. (15), based on the spatial covariant derivative. On the other hand, a particle in the flow of velocity field $i\mathbf{v}^e(r)$ has a Lagrangian derivative of $\partial_t + i\mathbf{v}^e(r) \cdot \nabla$, thus it matches the temporal covariant derivative that we obtain Eq. (28).

Consistent with the non-relativistic theory here, the purely imaginary nature of this velocity field does not contribute anything to the observed particle momentum, as we already discussed before. However, the square of the same momentum contributes to the Hamiltonian a real and negative definite term of $-\frac{1}{2} m \mathbf{v}^e(r)^2$. This matches the nature of gravity, which is always attractive, thus always produces a negative potential energy.

In quantum mechanics, if the potential energy in Eq. (1) is of real value, then the Schrödinger equation guarantees that the dynamic evolution is unitary, as the continuity
equation can be derived from Eq. (1). In our current theory, with the new definition of probability density in Eq. (24), and its associated density flow defined in Eq. (23), can the dynamic equation of motion, as shown in Eq. (31), still guarantee the unitary evolution of the system? We will present the proof below that it is indeed the case.

Let us define a reduced Hamiltonian $H_r$ as:

$$H_r = \frac{-\hbar^2}{2m} \nabla^2 - \frac{1}{2} m v^e(r)^2 - \frac{\hbar}{2} \nabla \cdot v^e(r)$$  \hspace{1cm} (34)

So now we can rewrite the Schrödinger equation in Eq. (31) as:

$$i\hbar \frac{\partial}{\partial t} \psi(r, t) = H_r \psi(r, t)$$  \hspace{1cm} (35)

Even though the full Hamiltonian of the system is not Hermitian, $H_r$ in Eq. (34) is obviously Hermitian. Based on the same derivation as in conventional QM, we can have the following continuity equation

$$\frac{\partial \rho_0}{\partial t} + \nabla \cdot j_0 = 0$$  \hspace{1cm} (36)

Here $\rho_0 = \psi^*(r, t)\psi(r, t)$, while $j_0 = \frac{1}{m} \text{Re} [\psi^* \hat{p} \psi]$, as defined in conventional QM. The particle probability density $\rho$ and density flow $j$ in our theory here, on the other hand, are dependent on $\tilde{\psi}(r, t)$ instead of $\psi^*(r, t)$. Thus $\rho(r, t) = \gamma(r) \rho_0(r, t)$, and $j(r, t) = \gamma(r) j_0(r, t)$. Note that $\gamma(r)$ is not a function of time, so $\partial_t \rho(r, t) = \gamma(r) \partial_t \rho_0(r, t)$. On the other hand, $\nabla \cdot j(r, t) = \gamma(r) \nabla \cdot j_0(r, t) + \nabla \gamma(r) \cdot j_0(r, t)$. Since we have the freedom of attaching an arbitrary factor of $e^{-2\lambda(r)}$ to $\gamma(r)$, as shown in Eq. (11), we can set $\nabla \gamma(r) = 0$ at any given point, and this is the case even when we do not set $\gamma(r)$ as a constant over space. Thus $\nabla \cdot j(r, t) = \gamma(r) \nabla \cdot j_0(r, t)$ under such condition, (or take the QFT language, in such a gauge). Multiply a factor $\gamma(r)$ onto Eq. (36), we find

$$\gamma(r) \frac{\partial \rho_0}{\partial t} + \gamma(r) \nabla \cdot j_0 = \frac{\partial \rho}{\partial t} + \nabla \cdot j = 0$$  \hspace{1cm} (37)

Since both $\rho$ and $j$ are invariant to local modulus transformation, if the continuity equation above is valid in one modulus condition, it should be valid in all conditions. Thus we prove that quantum system always evolves unitarily in our theory.

E. Metric functions

Having found the equation of motion in the last section to replace the Schrödinger equation in Eq. (1) of conventional QM, we can begin the search of the field equation to replace the Poisson equation in Eq. (2) of Newtonian gravity. Again we look for guidance from GR. In GR the connection in the equation of motion is the Christoffel symbol, denoted here as $\Gamma^\lambda_{\mu\nu}$, while the field equation is about finding solution for the metric tensor $g_{\mu\nu}$. The mathematical relation between $g_{\mu\nu}$ and $\Gamma^\lambda_{\mu\nu}$ is facilitated by the Einstein equivalence principle. In this section we will first define entities called quantum metric functions, which will play a similar role like $g_{\mu\nu}$ in GR. We then proceed to find out its relation with the connection $Y(r)$. In the next section we will discuss the field equation that can determine the quantum metric functions.

In previous sections of this theory part, we are focused on the wave function, quantum operators and dynamic equations, but defer the discussion on the basis ket $|r\rangle$ and basis bra $\langle r|$. To study their relation with $Y(r)$, we can analyze the gradient of $\tilde{\psi}(r, t)$. 


Since $\psi(r, t)$ is defined as an object that is invariant to local modulus transformation, $\nabla \psi(r, t)$ should also be an invariant. On the other hand, based on Leibnitz rule one has:

$$\nabla \psi(r, t) = \nabla [\psi(r, t)|r]| = [\nabla \psi(r, t)]|r| + \psi(r, t)[\nabla |r|]$$

(38)

Since $\psi(r, t)$ and $|r|$ transform inversely, to keep $\nabla \psi(r, t)$ invariant, we need to set

$$\nabla |r| = Y(r)|r|$$

(39)

Using the same procedure with $\nabla \psi(r, t)$, we have

$$\nabla |r| = -Y(r)|r|$$

(40)

The above two equations are symbolically quite simple, but not very useful for obtaining the exact function of $Y(r)$. The reason is, while the gradient $\nabla$ is an operator in real space, $|r|$ and $\langle r|$ are basis vector and covector in the Hilbert space. This again is similar to the situation in GR, where the basis vectors of adjacent points in principle reside in different tangent space. Consequently taking derivative on them directly is not very meaningful. The way around this in GR is to define the metric tensor $g_{\mu\nu}(x)$ as $g_{\mu\nu}(x) = \hat{e}_{\mu}(x) \cdot \hat{e}_{\nu}(x)$, where $\hat{e}_{\mu}(x)$ and $\hat{e}_{\nu}(x)$ are the coordinate basis vectors. The gravitational information encoded in $\hat{e}_{\mu}(x)$ and $\hat{e}_{\nu}(x)$ are thus transferred to $g_{\mu\nu}(x)$, a rank 2 tensorial function of spacetime position $x$. By relating $g_{\mu\nu}(x)$ to the connection $\Gamma^{\lambda}_{\mu\nu}(x)$, and solving the field equation for $g_{\mu\nu}(x)$, one can then find the exact form for $\Gamma^{\lambda}_{\mu\nu}(x)$.

We note that in GR, $g_{\mu\nu}(x)$ is a $(0, 2)$ type tensor. In QM, other than the ket vectors and bra covectors, the physical entities we usually encounter are either scalars resulted from the inner product of kets and bras, or quantum operators as the outer product of kets and bras. In GR language, if the ket and bra are $(1, 0)$ and $(0, 1)$ type tensors, respectively, then these quantum operators are $(1, 1)$ mixed rank tensors. Due to the orthonormal conditions between $|r|$ and $\langle r|$, even when $|r|$ and $\langle r|$ are not constant over space, these operators stay invariant to local modulus transformation. To extract gravitational information out of $|r|$ and $\langle r|$, we thus need to construct $g_{\mu\nu}(x)$ type $(0, 2)$ tensor instead.

Let us first define 3 new operators and 3 related scalar functions as:

$$\hat{\gamma}_1 [x] = \gamma_1(r) \langle x|$$

$$\hat{\gamma}_2 [y] = \gamma_2(r) \langle y|$$

$$\hat{\gamma}_3 [z] = \gamma_3(r) \langle z|$$

(41)

Due to the orthonormal condition in Eq. (3), Eq. (41) above implies that:

$$\hat{\gamma}_1 = \gamma_1(r) \langle x|$$

$$\hat{\gamma}_2 = \gamma_2(r) \langle y|$$

$$\hat{\gamma}_3 = \gamma_3(r) \langle z|$$

(42)

Note here $\langle bra \rangle_i$, with $i = 1, 2, 3$, corresponds to the basis bra $\langle x|$, $\langle y|$, and $\langle z|$, respectively. $\hat{\gamma}_i$ in Eq. (42) above can be understood as three $(0, 2)$ tensors in their respective dimensions, with $\langle x|$, $\langle y|$, and $\langle z|$ serving as the tensor bases, while $\gamma_1(r)$, $\gamma_2(r)$, $\gamma_3(r)$ as the tensor components. Note that these basis tensors look quite similar to the basis bra of composite system in conventional QM, but of course they represent very different things. $\hat{\gamma}_i$ are $(0, 2)$ type operators in the Hilbert space, while $\gamma_i(r)$ are assumed to be smooth, dimensionless, real, and positive definite functions of $r$, so $\gamma_i(r)$ are scalar functions in real space.
We further define the complex conjugate operator $T_0$ as:

$$T_0 \psi(r, t) = \psi^*(r, t)$$

so $T_0$ turns wave function into its complex conjugate. Since $T_0^2 \psi(r, t) = \psi(r, t)$, the inverse operator of $T_0$, denoted as $T_0^{-1}$, is the same as $T_0$, i.e. $T_0 = T_0^{-1}$.

Finally we define the operator $\hat{T}$ as:

$$\hat{T} = T_0 \hat{T}_1 \hat{T}_2 \hat{T}_3$$

with $\hat{T}$ having the following property:

$$\hat{T}\psi(r, t) = \psi(r, t)$$

By combining Eq. (42), (43), (44), (45), and (10) we find that

$$\gamma(r) = \gamma_1(r)\gamma_2(r)\gamma_3(r)$$

Similar to the role the metric tensor plays in GR of turning vector into their covector, the combined effect of $\hat{T}_{1-3}$ with $T_0$ is to convert the vector $\psi(r, t)$ into its covector $\bar{\psi}(r, t)$ in our theory here, as shown in Eq. (45). We thus call $\gamma_1(r)$, $\gamma_2(r)$ and $\gamma_3(r)$ the quantum metric functions.

When the local modulus transformation is performed, how do $\gamma_i(r)$ transform? To find that, we need to make an important assumption about how $|x|$, $|y|$, and $|z|$ transform. We postulate that the extra factor $e^{\lambda(r)}$ is always distributed evenly between them, i.e.

$$|x| \to e^{\lambda(r)/3}|x|, \quad |y| \to e^{\lambda(r)/3}|y|, \quad |z| \to e^{\lambda(r)/3}|z|$$

Since we assume $\hat{T}_i$ are tensor-like objects that are invariant to local modulus transformation, from Eq. (47) we know the transformation properties of $\gamma_i(r)$ must be:

$$\gamma_i(r) \to e^{-2\lambda(r)/3} \gamma_i(r)$$

Combining them leads to $\gamma(r) \to e^{-2\lambda(r)}\gamma(r)$ in Eq. (11).

We denote the inverse operators of $\hat{T}_i$ as $\hat{T}_i^{-1}$, so $\hat{T}_i = \hat{T}_i^{-1}$. They are (2, 0) type tensors in the Hilbert space. Apply them on Eq. (42), one can get their expressions as:

$$\hat{T}_1 = \gamma_1(r)^{-1} |x\rangle \langle x|$$

$$\hat{T}_2 = \gamma_2(r)^{-1} |y\rangle \langle y|$$

$$\hat{T}_3 = \gamma_3(r)^{-1} |z\rangle \langle z|$$

Note that all operators defined above commute with each other. Finally we find the inverse of $\hat{T}$ as:

$$\hat{T}^{-1} = (T_0 \hat{T}_1 \hat{T}_2 \hat{T}_3)^{-1} = T_0 \hat{T}_1 \hat{T}_2 \hat{T}_3$$

so

$$\hat{T}\psi(r, t) = \psi(r, t)$$

As mentioned above, we have defined the basis ket and bra like the coordinate basis in GR, which allows us to focus on the components of tensors without concerns about their bases. That is the case in previous sections, that is also the case here, where we do not really need to deal with operators like $\hat{T}_i$ or $\hat{T}_i$. Instead, from now on we can focus our attention only on $\gamma_i(r)$, and study how they are related to $Y(r)$ and $\nu(r)$, and how they are decided by the gravitational source.
Like $g_{\mu\nu}$, $\gamma_i(\mathbf{r})$ is expected to be completely determined by gravitational effect. Thus, when gravity can be neglected, $\gamma_1(\mathbf{r})$, $\gamma_2(\mathbf{r})$, and $\gamma_3(\mathbf{r})$ should all be constants over space. If one uses the values of $\gamma_1$, $\gamma_2$, and $\gamma_3$ at any given position as the lengths of 3 sides to construct a rectangular prism at that position, then in the absence of gravity, one would expect an orthorhombic crystalline lattice formed over space, as these unit cells have the same dimension at different positions. In the presence of gravity, each side’s length changes from place to place, and changes differently from the other 2 sides, as $\gamma_1(\mathbf{r})$, $\gamma_2(\mathbf{r})$, and $\gamma_3(\mathbf{r})$ are all different functions. Local modulus transformation, in such a picture, is equivalent to changing the volume of these unit cells, as $\gamma(\mathbf{r})$, based on Eq. (46), is the product of $\gamma_1(\mathbf{r})$, $\gamma_2(\mathbf{r})$, and $\gamma_3(\mathbf{r})$, and it can be changed by an arbitrary factor of $e^{-2A(\mathbf{r})}$. However, the ratios between $\gamma_1(\mathbf{r})$, $\gamma_2(\mathbf{r})$, and $\gamma_3(\mathbf{r})$ cannot be changed by local modulus transformation, according to their transformation rules in Eq. (48). Conventional QM, in such a picture, is equivalent to having a cubic crystalline lattice over space. Even though one cannot change an orthorhombic lattice to a cubic one through local modulus transformation, these two scenarios have really no difference in substance, as all these constant $\gamma_i$ can be normalized away. Both of them represent the no gravity case. Since the volume of each unit cell can be changed arbitrarily, what gravity really decides is the change of anisotropy of these unit cells over space. If the quantum system is in one dimension, thus can be described by just one ket or one bra, and only one metric function instead of three different $\gamma_i(\mathbf{r})$, then the only metric function of the system can be easily transformed into a constant over space by local modulus transformation. The three different $\gamma_i(\mathbf{r})$ is thus a crucial feature of our theory here. They cannot in principle be transformed into constants globally at the same time by local modulus transformation.

Having found out the transformation rule for $\gamma_{1-3}(\mathbf{r})$ in Eq. (48), we still need to find out the form of covariant derivative for $\gamma_{1-3}(\mathbf{r})$, as this will be the next step toward associating $\gamma_i(\mathbf{r})$ with the connection $\mathbf{Y}(\mathbf{r})$. Note that the vector connection $\mathbf{Y}(\mathbf{r})$ is first proposed as a result of establishing a covariant derivative for the scalar wave function $\psi(\mathbf{r}, t)$ under local modulus transformation. Now there are 3 kets of $|x\rangle$, $|y\rangle$, and $|z\rangle$ multiplied together that represent the basis ket $|\mathbf{r}\rangle$ at any position, not the single value $R$ representing the modulus of the scalar wave function. Connecting such states for adjacent points in 3 spatial dimensions therefore requires a $3 \times 3$ matrix connection, denoted here as $Y_{ij}(\mathbf{r})$, with $i = 1, 2, 3$ corresponding to the 3 basis kets, and $j = x, y, z$ corresponding to the 3 spatial directions. We note that the three spatial derivatives $(\partial_x, \partial_y, \partial_z)$ of the three $\gamma_i$ also form a $3 \times 3$ matrix $\partial_j \gamma_i$, so are their covariant derivatives, denoted here as $D_j \gamma_i$. While a vector connection $\mathbf{Y}(\mathbf{r})$ is enough for the kinematics and dynamics of quantum particle, as shown in previous sections, one now needs the rank 2 tensor field $Y_{ij}(\mathbf{r})$ to construct the covariant derivatives of $\gamma_i(\mathbf{r})$, i.e. to relate $D_j \gamma_i$, with $\partial_j \gamma_i$.

If we insert $|\mathbf{r}\rangle = |x\rangle|y\rangle|z\rangle$ into Eq. (39), together with the definition of

$$
\partial_j|x\rangle = Y_{1j}(\mathbf{r})|x\rangle; \quad \partial_j|y\rangle = Y_{2j}(\mathbf{r})|y\rangle; \quad \partial_j|z\rangle = Y_{3j}(\mathbf{r})|z\rangle
$$

(52)

we can easily prove that $Y_j(\mathbf{r})$, the $j$th component of $\mathbf{Y}(\mathbf{r})$, is linked to $Y_{ij}(\mathbf{r})$ as:

$$
Y_j(\mathbf{r}) = Y_{1j}(\mathbf{r}) + Y_{2j}(\mathbf{r}) + Y_{3j}(\mathbf{r})
$$

(53)

so $\mathbf{Y}(\mathbf{r})$ and $Y_{ij}(\mathbf{r})$ are closely linked. However, the fact that we have now two types of connections is a new feature that has no parallels in GR. The origin of this is the different number of components vs. bases in $\dot{\psi}(\mathbf{r}, t)$ and $\dot{\psi}(\mathbf{r}, t)$. In GR, the number of vector

16
components and the number of basis vectors are always the same. Here for the vector field \(\dot{\psi}(r,t)\) for example, we have one scalar component \(\psi(r,t)\), yet three ket bases \(|x\rangle, |y\rangle,\) and \(|z\rangle\) that change differently over space. Furthermore, the product of \(|x\rangle, |y\rangle,\) and \(|z\rangle\) has physical meaning, but not the summation of them. In GR the opposite is true. These differences make the covariant derivative of \(g_{\mu\nu}\) in GR less a guidance for the construction of the covariant derivative of \(\gamma_i(r)\) here. Instead, we need to find solutions by examining \(\gamma_i(r)\)’s own characteristics.

Let us first imagine that we can move the quantum \((0,2)\) tensor \(\hat{T}\) from one point to its neighboring point in a way similar to the parallel transport in GR. This can help us to find out how the covariant derivatives of the components of \(\hat{T}\), namely \(D_j\gamma_i\), is different from their ordinary derivatives \(\partial_j\gamma_i\). We first note that as long as the Cartesian coordinate system is maintained here, any process, including the parallel transport process, cannot change \(\gamma_1\) into a mixture of \(\gamma_2\) and/or \(\gamma_3\). This kind of mixing is as meaningless as mixing \(x\) with \(y\) or \(z\) coordinate in a single Cartesian coordinate system. This means that the covariant derivative of a specific \(\gamma_i\) should not be related to the other two \(\gamma_i\). Assume the parallel transport process here, like that in GR, only involves linear change, then we can conclude that \(D_j\gamma_i - \partial_j\gamma_i\) should be proportional to the specific \(\gamma_i\) only, and unrelated to the other two \(\gamma_i\). This obviously is different from the situation in GR.

Since \(\gamma_i\) make up the components of tensor \(\hat{T}\), the difference between the covariant and ordinary derivative of \(\gamma_i\) should entirely stem from the change of the bases of \(\hat{T}\). Eq. (52) shows that the changes of these bases result into \(Y_{ij}(r)\). We can therefore expect that other than \(\gamma_i\), the matrix \(D_j\gamma_i - \partial_j\gamma_i\) should be entirely dependent on the matrix \(Y_{ij}\). We further note that when \(\hat{T}\) is parallel transported, the basis tensor involved is \(|r\rangle\langle r|\). Since \(\hat{T}\) is a product of three \(\hat{T}_i\) that commute with each other, inside \(\hat{T}\) the associative law of multiplication means one can choose to lump \(\gamma_i\) with \(|y\rangle\langle y|\) as legitimately as to lump \(\gamma_1\) with \(|x\rangle\langle x|\). Consequently, one would expect the covariant derivative of \(\gamma_1\), for example, is not only related to \(Y_{1j}(r)\), but also related to \(Y_{2j}(r)\) and \(Y_{3j}(r)\) as well. Therefore, each component of the matrix \(D_j\gamma_i - \partial_j\gamma_i\) is not only related to the \(Y_{ij}(r)\) component with the same \(i\) and \(j\), but also related to other components inside the connection matrix.

How is the matrix \(D_j\gamma_i - \partial_j\gamma_i\) related to the matrix \(Y_{ij}(r)\) then? We note that the deviation of basis bras \(|x\rangle, |y\rangle,\) and \(|z\rangle\) from those in conventional QM can be divided into two categories. One is the difference among \(|x\rangle, |y\rangle,\) and \(|z\rangle\) at a given position \(r\), the other is the change of these basis bras along the 3 different directions in real space, \(i.e.\) the real space anisotropy of the bras. We postulate that the first type can be accounted for by the \(Y_{ij}(r)\) components with the same \(i\) index (representing the \(i\)th bra) averaged over three different \(j\) (representing 3 different directions), while the second type by the \(Y_{ij}(r)\) components with the same \(j\) averaged over three different \(i\). Putting these discussions together, we postulate that the covariant derivative of \(\gamma_i(r)\) can be expressed as:

\[
D_j\gamma_i = \partial_j\gamma_i - \frac{1}{3}(Y_{ix} + Y_{iy} + Y_{iz})\gamma_i - \frac{1}{3}(Y_{1j} + Y_{2j} + Y_{3j})\gamma_i
\]  (54)

Note that the last two terms on the right side of Eq. (54) have negative signs. This is because \(\gamma_i\) are components of \((0, 2)\) type tensor \(\hat{T}\).
Earlier we have postulated that the extra factor $e^{\lambda(r)}$ is always distributed evenly among $(x|, (y|, and (z|$. This leads to the transformation rule for $\gamma_i$ in Eq. (48). Applying the local modulus transformation to Eq. (54), together with the transformation rule for $Y(r)$ in Eq. (13), and $Y(r)$ as the sum of its component $Y_{ij}$ shown in Eq. (53), we find that the matrix $Y_{ij}$ needs to be symmetric, i.e., $Y_{ij} = Y_{ji}$, to ensure that $\gamma_i(r)$ and $Y(r)$ transform consistently. Conversely, we can start with the assumption of a symmetric $Y_{ij}$, and conclude that the local modulus transformation always distributes the extra factor $e^{\lambda(r)}$ equally among $(x|, (y|, and (z|$. Note that the connection in GR, the Christoffel symbol $\Gamma_{\mu\nu}^{\lambda}$, is assumed to be torsion free, therefore also symmetric in its $\mu$ and $\nu$ indices.

In the introduction part we have postulated a new equivalence principle. It states that in small enough region, the laws of physics reduce to those of conventional QM. Since in conventional QM $\gamma_i$ are constant over space, it is natural to demand the connection here being metric compatible, i.e., $D_j \gamma_i = 0$. Again, this is very similar to the situation in GR. Consequently we have:

$$\partial_j \gamma_i = \frac{1}{3} (Y_{ix} + Y_{iy} + Y_{iz}) \gamma_i + \frac{1}{3} (Y_{1j} + Y_{2j} + Y_{3j}) \gamma_i$$  \hspace{1em} (55)$$

Since $Y_{ij}$ is symmetric, this equation means:

$$\partial_j \ln \gamma_i(r) = \partial_i \ln \gamma_j(r)$$  \hspace{1em} (56)$$

Let us now define $v_{ij}^e(r) = \frac{\hbar}{m} Y_{ij}(r)$, and a new real valued function $\tau_i(r)$ as:

$$\tau_i(r) = \frac{\hbar}{m} \ln \gamma_i(r)$$  \hspace{1em} (57)$$

For reasons that will become clear later, we call $\tau_i(r)$ the regularized metric functions. We also define the sum of them as $\tau(r)$ so

$$\tau(r) = \tau_1(r) + \tau_2(r) + \tau_3(r) = \frac{\hbar}{m} \ln \gamma(r)$$  \hspace{1em} (58)$$

Based on Eq. (11) we find that the transformation rule for $\tau(r)$ should be

$$\tau(r) \rightarrow \tau(r) - 2h\lambda(r)/m$$  \hspace{1em} (59)$$

Eq. (55) can then be written as:

$$\partial_j \tau_i = \frac{1}{3} (v_{ij}^e + v_{iy}^e + v_{iz}^e) + \frac{1}{3} (v_{ij}^e + v_{2j}^e + v_{3j}^e)$$  \hspace{1em} (60)$$

Note that based on Eq. (53) we can write the $j$th component of the escape velocity $v^e(r)$ as $v_j^e(r)$ with $v_j^e(r)$ as:

$$v_j^e(r) = v_{1j}^e(r) + v_{2j}^e(r) + v_{3j}^e(r)$$  \hspace{1em} (61)$$

Eq. (60) is essentially a system of 9 linear equations, and in principle we can solve them to find the 9 components of $v_{ij}^e(r)$ as a function of $\tau_i(r)$. The determinant formed by the numerical coefficients of $v_{ij}^e$, however, turns out to be zero, so there is no unique solution for each $v_{ij}^e$. Fortunately, we just need to know $v_j^e(r)$ as a function of $\tau_i(r)$ in order to connect $\gamma_i(r)$ with $Y(r)$ and $v^e(r)$, and $v_j^e(r)$ as a function of $\tau_i(r)$ can be uniquely decided. Take the example of $v_x^e$, we find that:

$$\partial_x \tau_1 + \partial_x \tau_2 + \partial_x \tau_3 + \partial_y \tau_1 - \partial_y \tau_3 + \partial_z \tau_1 - \partial_z \tau_2 = (v_{1x}^e + v_{1y}^e + v_{1z}^e) + (v_{1x}^e + v_{2x}^e + v_{3x}^e)$$  \hspace{1em} (62)$$

18
Since \( \nu_0^e \) is symmetric, and \( \nu^e = \nu^e_1 + \nu^e_2 + \nu^e_3 \), we can immediately solve \( \nu^e_1 \) as a function of \( \tau_1 \). Using similar procedures we can also solve \( \nu^e_2 \) and \( \nu^e_3 \), and we express all three of them in the following important equation:

\[
\begin{align*}
\nu^e_1 &= \frac{1}{2} \left[ \partial_1 (\tau_1 + \tau_2 + \tau_3) + \partial_2 (\tau_1 - \tau_3) + \partial_3 (\tau_1 - \tau_2) \right] \\
\nu^e_2 &= \frac{1}{2} \left[ \partial_2 (\tau_1 + \tau_2 + \tau_3) + \partial_3 (\tau_2 - \tau_1) + \partial_1 (\tau_2 - \tau_3) \right] \\
\nu^e_3 &= \frac{1}{2} \left[ \partial_3 (\tau_1 + \tau_2 + \tau_3) + \partial_1 (\tau_3 - \tau_2) + \partial_2 (\tau_3 - \tau_1) \right]
\end{align*}
\]

(63)

We thus have established the escape velocity field \( \nu^e (r) \) as a function of the regularized metric functions \( \tau_1 (r) \), \( \tau_2 (r) \) and \( \tau_3 (r) \); or equivalently the connection \( \gamma_i (r) \) as a function of the quantum metric functions \( \gamma_i (r) \).

Put the three components of \( \nu^e \) together, we find that:

\[
\begin{align*}
\nu^e (r) &= \frac{1}{2} \nu[\tau_1 (r) + \tau_2 (r) + \tau_3 (r)] + \frac{1}{2} \nu \tau (r) \\
&= \frac{1}{2} \nu [\tau_1 (r) + \tau_2 (r) + \tau_3 (r)] \\
&+ \frac{1}{2} [\partial_1 (\tau_1 - \tau_3) + \partial_2 (\tau_1 - \tau_2)] \mathbf{i} + \frac{1}{2} [\partial_2 (\tau_2 - \tau_1) + \partial_3 (\tau_3 - \tau_2)] \mathbf{j} + \frac{1}{2} [\partial_3 (\tau_3 - \tau_1) + \partial_1 (\tau_3 - \tau_2)] \mathbf{k}
\end{align*}
\]

(64)

with \( \mathbf{i}, \mathbf{j}, \) and \( \mathbf{k} \) representing the unit vectors along the \( x, y, z \) directions, respectively. We denote the first term on the right side of Eq. (64) as \( \nu^e_\mu (r) \) so

\[
\nu^e_\mu (r) = \frac{1}{2} \nu \tau (r)
\]

Under local modulus transformation, we find that \( \nu^e_\mu (r) \) is responsible for the entire change of \( \nu^e (r) \) in Eq. (33), while each of the other three terms on the right side of Eq. (64) stays invariant. We denote the sum of these 3 terms as \( \nu^e_\mu (r) \):

\[
\begin{align*}
\nu_\mu (r) &= \frac{1}{2} \partial_1 (\tau_1 - \tau_3) + \partial_2 (\tau_1 - \tau_2)] \mathbf{i} + \frac{1}{2} \partial_2 (\tau_2 - \tau_1) + \partial_3 (\tau_3 - \tau_2)] \mathbf{j} + \frac{1}{2} \partial_3 (\tau_3 - \tau_1) + \partial_1 (\tau_3 - \tau_2)] \mathbf{k}
\end{align*}
\]

(66)

so

\[
\nu^e (r) = \nu^e_\mu (r) + \nu^e_\mu (r)
\]

(67)

If \( \nu^e (r) \) is independent of particle mass \( m \), then Eq. (63) implies that all 3 \( \tau_i \) are also independent of \( m \). This can sometimes bring much clarity to our discussion, particularly when gravity, not quantum particle, is the only thing under investigation. In these cases \( \tau_i \) will be the preferred metric function to use. On the other hand, when quantum particle is the focus of study, then \( \gamma_i \) is often the preferred choice.

**F. Field equation**

We now proceed to construct the field equation that relates external gravitational source’s mass density \( \rho_{ext} (r) \) with the regularized metric functions \( \tau_i (r) \). Once \( \tau_i (r) \) are solved, we can find \( \nu^e (r) \) based on Eq. (63), and consequently put \( \nu^e (r) \) into the equation of motion in Eq. (31) to find solution of the particle wave function \( \psi (r, t) \). In our theory \( \psi (r, t) \) is not the only thing that decides the quantum state, we still need \( \gamma (r) \) to find \( \hat{\psi} (r, t) \). Fortunately \( \gamma (r) \) is quite simply related to \( \tau (r) \) through Eq. (58). So \( \tau_i (r) \) is the key, and the field equation is really the last major missing piece of our theory.

We have two basic requirements for the field equation. One is that the Newtonian Poisson equation should be recovered when \( h \to 0 \), based on corresponding principle. The other is that the field equation should be covariant to local modulus transformation. Let us
first use Eq. (32) to replace the gravitational potential energy by its corresponding escape velocity, and rewrite the Poisson equation of Newtonian gravity as:

\[ \nabla^2 [v^e(r)^2] = -8\pi G \rho_{ext}(r) \]  

(68)

We note that mass density is always invariant to local modulus transformation in our theory, so the right side of Eq. (68) is an invariant. However, the left side is only dependent on \( v^e(r) \), and \( v^e(r) \) is not invariant to local modulus transformation. As a result, we need to modify the left side.

As we already discussed above, \( v^e(r) \) has two parts, \( v^e_v(r) \) and \( v^e_u(r) \), with \( v^e_u(r) \) invariant to local modulus transformation. On the other hand, Eq. (65) shows that \( v^e_v(r) \) is irrotational as it is half of the gradient of the scalar function \( \tau(r) \). Since \( v^e_v(r) \) transforms the same way as \( v^e(r) \), from which an arbitrary irrotational field \( \hbar \nabla \lambda(r)/m \) can be subtracted through local modulus transformation, we can set \( v^e_v(r) \) to zero by using this modulus freedom. In such a setting \( v^e(r) = v^e_u(r) \). A natural solution to fix the left side of Eq. (68) is then to replace \( v^e(r) \) by \( v^e_u(r) \), and we propose the new field equation as:

\[ \nabla^2 [v^e_u(r)^2] = -8\pi G \rho_{ext}(r) \]  

(69)

In this new equation both sides are invariant to local modulus transformation. When we choose our modulus setting so that \( v^e_u(r) = 0 \), Eq. (69) becomes the Poisson equation in Eq. (68). Thus, the two basic requirements we put up in the beginning are satisfied.

Let us now use Eq. (66) to replace \( v^e_u(r) \) by \( \tau_i(r) \), and we can express the field equation as:

\[
\nabla^2 \left[ \left( \partial_y (\tau_1 - \tau_3) + \partial_z (\tau_1 - \tau_2) \right)^2 + \left( \partial_z (\tau_2 - \tau_1) + \partial_x (\tau_2 - \tau_3) \right)^2 + \left[ \partial_x (\tau_3 - \tau_2) + \partial_y (\tau_3 - \tau_1) \right]^2 \right] = -32\pi G \rho_{ext}(r)
\]

(70)

We know from Eq. (56) that

\[ \partial_j \tau_i(r) = \partial_i \tau_j(r) \]

(71)

Together with an arbitrary scalar function \( \lambda(r) \) that we can use to set \( \tau(r) \) to any scalar function we want, Eq. (70) and Eq. (71) then enable us to solve \( \tau_1(r), \tau_2(r) \) and \( \tau_3(r) \), given any distribution of \( \rho_{ext}(r) \).

The particle mass \( m \) does not appear in Eq. (70), which means \( \tau_i(r) \) are not dependent on \( m \). Eq. (63) in turn shows that \( v^e(r) \) is not dependent on \( m \) either. Thus we have corroborated the weak equivalence principle in our theory. Furthermore, \( \hbar \) does not appear in Eq. (63) and (70), \( v^e(r) \) is thus completely gravitational, and not quantum mechanical. There is actually no need for the corresponding principle here.

III. Discussions

A. Practical applications

Having reached the main goal we set at the beginning of this paper, namely to develop a self-consistent non-relativistic theory of quantum mechanics and gravity with local symmetry, it is natural to inquire what kind of new phenomena can be expected from this theory. To make quantitative predictions, one first needs to solve the field equation for \( \tau_i(r) \) in some fashion, yet the field equation in Eq. (70) is a non-linear third order partial differential equation, and the task of solving it, even in highly symmetric case, looks formidable. However, even without solving the field equation, we may still examine it as well as the new equation of motion to see whether they predict any qualitatively new
physics, and if there are, whether they are related to any experiments and observations that cannot be explained by current theories. This is exactly what we set out to do in this section, but we want to emphasize, that until quantitative solutions to the field equation are found, all the predictions we make here and their link to experiments and observations are tentative.

We briefly mentioned before that the term $-\frac{\hbar}{2} \nabla \cdot \mathbf{v}^e(r)$ inside the equation of motion of Eq. (31) looks quite peculiar, let us now examine it more closely. First of all, since $\mathbf{v}^e(r)$ is proportional to $G^{1/2}$, this term, among all the terms in both the equation of motion and the field equation, is the only one that explicitly has both $G$ and $\hbar$. It is thus expected to contain new physics that does not exist either in QM or in Newtonian gravity. Secondly, as some form of a potential energy of the particle appearing in the particle’s equation of motion, it is not related to the particle’s properties, particularly its mass. Indeed we can express this term’s unit mass potential energy as $-\frac{\hbar}{2m} \nabla \cdot \mathbf{v}^e(r)$. It is now explicitly dependent on the particle mass. In this sense it even violates the weak equivalence principle, unless we count it as some form of vacuum energy that exists in the background, and without much relation to the particle itself. This naturally brings up the possibility of its association with dark energy and cosmological constant.

As we know, a cosmological constant term in the Einstein equation is thought necessary to explain the accelerating expansion of the universe. General relativity does not produce such a term naturally on its own. Instead, its physical origin is thought very likely to lie somewhere else. The vacuum energy of QFT, on the other hand, is far greater than the observed value to account as its origin. In our theory, $-\frac{\hbar}{2m} \nabla \cdot \mathbf{v}^e(r)$ appears naturally out of putting the spatial covariant derivative into the Hamiltonian. It seems to be associated only with vacuum, not particles. Furthermore, being proportional to $G^{1/2} \hbar$, it is expected to be of quite small value, likely too small to be directly detected within current experimental or observational limit. Yet it can be ubiquitous in space as long as $\mathbf{v}^e(r)$ has an irrotational component. Its nature is gravitational, yet as long as the divergence of $\mathbf{v}^e(r)$ is negative, this energy term can be positive. All these features make this term a plausible candidate for dark energy.

However, in order to compare it directly with cosmological observations, we have some obstacles to overcome. The first is that this term is an energy in the quantum particle’s equation of motion. It is not an energy density. Furthermore, the quantum particle in our theory is not restricted to elementary particles, it can indeed be any system with 3 spatial degree of freedoms. Thus it is not immediately clear how to translate this term into an energy density in the universe. Secondly, the universe is known as isotropic and homogenous at the cosmological scale. If we set $\rho_{\text{ext}}(r)$ as a constant over space with a value of the average density of the universe, the field equation in Eq. (70) does not have a solution, the same problem people run into when Newtonian Poisson equation is used to model the universe. (On the other hand, the Einstein equation of GR, not the field equation here, is supposed to be the fundamental equation for cosmology, so the solution of $\mathbf{v}^e(r)$ at the cosmological scale might come out of GR.) Despite these difficulties, we think overall it is worthwhile to explore this term further, and investigate whether it is really associated with dark energy.

Next we come to the field equation and examine its difference with the Poisson equation in Newtonian gravity. We note that in our field equation of Eq. (69), the unit mass
potential energy is \( u_g(r) = -\frac{1}{2} v_u^e(r)^2 \), while in the equation of motion of Eq. (31), this term is \(-\frac{1}{2} v^e(r)^2 = -\frac{1}{2} v^e(r)^2 - \frac{1}{2} v_u^e(r)^2 - v_u^e \cdot v_u^e \). In the Poisson equation and equation of motion for Newtonian gravity, on the other hand, they are both \(-\frac{1}{2} v^e(r)^2 \). As we noted above, if we use the local modulus transformation to set \( v^e = 0 \), so \( v_u^e = v^e \), then as far as \( u_g(r) \) is concerned, it is the same as in Newtonian gravity. However, when \( v^e \) cannot be set to zero, then we should expect that the gravitational force on particles will deviate from that given by Newtonian theory. This deviation is caused by the extra potential energy of \(-\frac{1}{2} v^e(r)^2 - v^e \cdot v_u^e \) that is not accounted for by the Poisson like equation of Eq. (69). Note that even though our theory is a theory of quantum mechanics and gravity, this modification on Newtonian gravity is entirely classical. \( v^e \), \( v_u^e \), and \( v_u^e \) are all related to \( G \) but not to \( h \). This is different from our above discussion on \(-\frac{h}{2} \nabla \cdot v^e(r) \).

In the standard model of elementary particles, the masses of elementary particles arise from the Higgs mechanism, with the vacuum state having a spontaneously broken gauge symmetry. Similar process can happen here. Our theory, including both the field equation and the equation of motion, are manifestly covariant to local modulus transformation. Yet the state itself, characterized by the 3 \( \gamma \), or alternatively the 3 \( \tau \), can have spontaneously broken modulus symmetry. This can result in a \( \tau(r) \) that cannot be set as constant over space, which leads to a finite \( v^e \) and deviations from Newtonian gravity.

In the solar system where all experiments are conducted so far, we expect the modification from a finite \( v^e \) on Newtonian gravity to be quite small. If the gravitational acceleration originating from \( v^e \) is denoted as \( a_c \), and that from \( v_u^e \) denoted as \( a_u \), (neglecting the effect of their coupling term \( v^e \cdot v^e \) for the moment), then one would expect \( a_u \gg a_c \) in the solar system, otherwise \( a_c \) should already be detected. On the other hand, in systems where \( a_u \) itself is small, \( a_c \) can be comparable or even much greater in magnitude than \( a_u \). In such cases the effect of a finite \( v^e \), and the deviation from Newtonian gravity, can be readily subject to observation.

Gravitational pull on galaxies and galaxy clusters is generally weak in magnitude, much weaker than the acceleration found in the solar system. Starting from the observation by Zwicky in the 1930s,\(^\text{13}\) it is noticed that the dynamics of galaxies and galaxy clusters often cannot be adequately described by Newtonian gravity if one uses the observed baryonic mass of these galaxies and galaxy clusters as the sole gravitational source. \( \Lambda \)CDM, the current standard model of cosmology, attributes such deviation from Newtonian gravity to the existence of non-baryonic dark matter inside galaxies and galaxy clusters.\(^\text{14}\) The process of big bang nucleosynthesis and large scale structure formation evolved from the initial condition written on the cosmic microwave background also provide strong argument for dark matter.\(^\text{14}\) Yet despite many efforts, no direct detection of dark matter has ever been confirmed. On the other hand, there is also an alternative theory for the mass discrepancy problem. It is based on modified Newtonian dynamics (MOND), and was first proposed by Milgrom.\(^\text{15}\) MOND is a phenomenological theory where a characteristic acceleration \( a_0 \) is introduced first, with a magnitude of about \( 10^{-10}\text{ms}^{-2} \), roughly 11 orders of magnitude weaker than the acceleration on the surface of the earth. MOND states that if the acceleration \( a \) of the system is far great than \( a_0 \), the gravitational law is essentially Newtonian; if \( a \ll a_0 \), then the gravity in MOND will deviate from Newtonian gravity significantly. MOND can explain many features of galaxy dynamics.
without resorting to dark matter. However, it is less successful in the behavior of galaxy clusters. It is also less convincing than the $\Lambda$CDM model with regard to the formation of large structures in the evolution of the universe.\textsuperscript{16}

If we assume that $a_c$ in our theory takes the value close to $a_0$ when the object under investigation is on the scale of galaxy, then we do expect a significant deviation from Newtonian gravity when $a_u$ drops below $a_0$ at the galaxy scale. Moreover, since $a_c$’s origin is in spontaneous breaking of local modulus symmetry, its value can potentially evolve over time. Given the relation between the two pairs of $a_c$ and $a_u$ and $v_c^e$, $v_u^e$, particularly the effect from the coupling term $v_c^e \cdot v_u^e$, it is possible that $a_c$’s value can also change over different length scale. So potentially the evolutionary formation as well as current dynamic behaviors of galaxies and galaxy clusters can find good fittings in our theory here without the addition of dark matter. In the end, $\Lambda$CDM is built on the theoretical foundation of GR and QFT. If our theory is correct, it seems plausible that it should be taken into consideration together with GR and QFT at the beginning. This seems particularly true where gravitation is weak.\textsuperscript{16} There are certainly many unknowns in our theory without solutions of the field equation. Moreover, the details of the spontaneous modulus symmetry breaking are not clear yet. Nevertheless, the abundant observational data and greatly advanced simulation methods currently available make us to believe, that this can be a fertile research area in the near future.

Having touched upon the potential impacts of our theory on gravity, we now discuss its possible new effects on quantum mechanics. Besides the covariant derivative introduced to account for gravity, the other fundamentally new feature in our theory is the co-wave function $\tilde{\psi}(r,t)$. It replaces $\psi^*(r,t)$, the complex conjugate of wave function in conventional QM, and their difference is the factor $\gamma(r)$. Up to now we did not comment much on $\gamma(r)$ except to mention that it can be set as a constant by local modulus transformation. We want to study the effect of $\gamma(r)$ on quantum states in more details now. Let us first express it in terms of $\tau(r)$. Based on Eq. (58) we have:

$$\gamma(r) = e^{\frac{m}{\hbar} \tau(r)} \quad (72)$$

Note that $\tau(r)$ only depends on gravity, not on quantum mechanics, and it is independent of particle properties. Furthermore, spontaneous modulus symmetry breaking can set it to non-zero and varying values at different positions.

To get a rough estimate of $\gamma(r)$ we make the simplest assumption of a constant $a_c$ near the surface of our earth system, with $a_c$ taking the value of $a_0 = 10^{-10} \text{ms}^{-2}$. We also attribute the origin of $a_c$ entirely to a modulus symmetry breaking related gravitational potential energy of $- \frac{1}{2} v_c^e(r)^2$, with the direction of both $a_c$ and $v_c^e(r)$ set along the radial direction of the earth sphere. This reduces the degree of freedom in the problem to one.

With $r$ as the distance to the center of the earth, we find $v_c^e(r) \sim \sqrt{2 a_0} \frac{1}{r^\frac{3}{2}}$. Note that this is essentially the same formula for the conventional definition of escape velocity at earth surface if one set $a_0$ to 9.8 $\text{ms}^{-2}$ instead of $10^{-10} \text{ms}^{-2}$. From Eq. (65) we have $v_c^e(r) = \frac{1}{2} \nabla \tau(r)$, so up to a constant, as $\tau(r) = \frac{4\sqrt{2}}{3} a_0^\frac{1}{2} r^\frac{3}{2}$. This can be put into Eq. (72) to find out $\gamma$ as a function of $r$. What we are interested is not the absolute value of $\gamma(r)$, since any constant can be normalized away. Rather, we want to estimate the change of $\gamma(r)$ over a certain length, as this determines how rapidly quantum particle’s probability density
changes over space due to the presence of a varying $\gamma(r)$. Let us examine two cases, one is the electron, the lightest microscopic particle in our everyday life, the other a macroscopic particle with one gram of mass. In both cases, we set the distance between two adjacent points at 1 Å, a typical length for the resolution of powerful transmission electron microscopes. We find for the electron, the change of $\gamma(r)$ over 1 Å is about $e^{10^{-16}}$, while for the 1g object, it is about $e^{10^{11}}$. Over a distance of 1 micron, the change of $\gamma(r)$ is about $e^{10^{-10}}$ for the electron, and $e^{10^{17}}$ for the 1g object. In both length scales, it is quite clear that for electron, the change of $\gamma(r)$ is too small to be detected; for the 1g object, the change of $\gamma(r)$ is so steep that the probability density of the object can be regarded as concentrated on a point in all practical sense. On scale even larger, for micro object it becomes hard to maintain quantum coherence, so experimentally it will be difficult to single out the influence of $\gamma(r)$. For macro object, the change of $\gamma(r)$ will increase further.

This kind of binary results between micro and macro objects can offer the mechanism for the solution of the quantum measurement problem. The drastic change of $\gamma(r)$ for any macro object over any measurable length renders it to a quantum mechanical pointer state for all practical purposes. This is independent of how its wave function $\psi(r,t)$ evolves in the Schrödinger equation, and it means we do not need to rely on dynamics to solve the problem. Instead, this new feature $\gamma(r)$ in quantum kinematics can prevent the spread of probability density of macro objects over space in the first place. For microscopic objects, the change of $\gamma(r)$ over any microscopic length scale is too small to be detected. Thus for microscopic objects, our theory produces practically the same results as conventional QM. The reason behind such binary results is the expression of $\gamma(r)$ in Eq. (72), where the exponent is proportional to the mass of the object. With a change of more than 20 orders of magnitude for the exponent, it is easy to see why micro and macro objects are located at the two extreme ends, and the in-between scenario is hard to realize. It is also for this reason that we call $\tau_i(r)$ the regularized metric functions, as the drastic change of $\gamma(r)$ caused by mass $m$ is stripped away from $\tau_i(r)$, besides being a logarithmic function.

Of course we have performed a very simple calculation using very simplified assumptions. There are many complications in the real world. For example, space is always 3 dimensional and $\tau(r)$ can change differently along different directions. The value that $a_r$ takes can be different from what we use. The coupling term $\mathbf{p}^c \cdot \mathbf{r}^e$ can be too big to be neglected, etc. Nevertheless, the change of the exponent in Eq. (72) is so much greater than unity for macro objects, and so much less than unity for micro objects, that it seems plausible, that one can still expect a binary outcome outlined above even after all the necessary changes to account for real world situations. In other words, quantitative change without qualitative change. However, in order to make quantitative prediction, one still needs to perform detailed calculations. A clear understanding of modulus symmetry breaking is also essential.

A related topic is the tension between the asymmetry of time in the second law of thermodynamics with the time reversal symmetry of dynamic law in classical mechanics first, and quantum mechanics later. As we know, the fundamental dynamic law of QM has time reversal symmetry. If we make the transformation of $t \rightarrow -t$, then $\psi^*(r,-t)$ is still the solution of the complex conjugate of the Schrödinger equation, provided that the potential energy of the system is real. Based on Eq. (31), this is still the case here, so $\psi^*(r,-t)$ is still the solution. However, the co-wave function of the system now, after time
reversal transformation, is \( \tilde{\psi}(\mathbf{r}, -t) \), not \( \psi^*(\mathbf{r}, -t) \). As long as \( \gamma(\mathbf{r}) \) is not constant, the two are not the same. In principle \( \psi^*(\mathbf{r}, -t) \) does not have any physical meaning for the quantum system here, while \( \tilde{\psi}(\mathbf{r}, -t) \) is not a solution of the complex conjugate of the dynamic equation. Thus time reversal symmetry is broken at the molecular level, at least in principle, even though the spatial change of \( \gamma(\mathbf{r}) \) for microscopic object is very small, as we have shown above. Is this time reversal symmetry breaking at the molecular level responsible for the asymmetry of time in the second law of thermodynamics? This is an open and important question that demands more investigation.

If our theory is right, then there is a unifying rule that is applicable to both macro and micro objects. There is no need to assign the rule of classical mechanics to macro objects. Micro or macro, they are all quantum objects. If our estimates are right, then it is gravity that ultimately causes the nearly total disappearance of quantum spatial fluctuation for macro objects, as Penrose has long advocated.\(^\text{17}\) He also envisioned the link between the asymmetry of time in the second law of thermodynamics and the reduction of quantum fluctuation on the macroscopic level.\(^\text{17}\) The argument distinctive here is that one does not need relativistic theory to realize it, and the way macro objects achieve this reduction of quantum fluctuation in our theory is through kinematics instead of dynamics. On the other hand, the Copenhagen interpretation divides the world into micro and macro worlds with different rules for each.\(^\text{18}\) In light of the binary outcome for macro and micro objects implied in Eq. (72), this division is probably a necessary arrangement in the early development of quantum mechanics. Nevertheless, it is a temporary solution. To justify a temporary solution by insisting on the impossibility of a unifying law for both macro and micro objects in principle, from the perspective of our theory outlined above, is confusing the means with the ends.

**B. Theoretical framework**

Our theory bears a few resemblances to quantum field theory. The local modulus symmetry here is quite similar to the \( U(1) \) gauge symmetry in quantum electrodynamics, at least mathematically. Consequently it is not surprising that the resulted covariant derivative in our theory is also quite similar mathematically to that in quantum electrodynamics. In addition, we have proposed that the local modulus symmetry is spontaneously broken. This broken symmetry can result into a gravitational law that deviates from Newtonian gravity. It can also potentially lead to pointer state for macroscopic objects. In QFT the mechanism of broken gauge symmetry is also very important, since only through it can the massive gauge Bosons acquire their masses.

Yet it is quite clear that our theory is closer to GR in its essence. There are two fundamental factors that contribute to this closeness. As we point out in the beginning of the theory part, quantum mechanics and special relativity has structural similarities, particularly in their shared use of vector and covector in linear space to describe states. When they are fused with gravity, many of these features remain intact. The second factor is the equivalence principle. Even though the exact statements are not the same, as they refers to the reduction of laws to SR and QM in small enough region, respectively, the essence is the same for both cases. It is the expression of gravity through kinematics instead of dynamics, as gravity is not a normal force associated with conventional dynamic law, and it can be transformed away locally by certain kinematic transformation. This is the fundamental feature of gravity that other forces do not share. As Fig. 1 illustrates, the
The finiteness of the 3 fundamental constants have made fundamental changes on the Newtonian kinematics of absolute space and time. So another angle that shows the uniqueness of gravity is that while its strength is proportional to $G$, one of the 3 fundamental constants, that is not the case with other forces. This is also the key reason why our theory in essence is closer to GR than to QFT. If the four dimensional spacetime is curved by gravity in GR, in some sense one can similarly state that the Hilbert space of quantum system in position representation is curved by gravity in our theory.

In the end, the purpose of our theory is to unite quantum mechanics with gravity in a non-relativistic context, just as the purpose of GR is to unite SR with gravity in a non-QM context, while QFT unites SR with QM without gravity. As we discussed in the introduction, in such a framework, it is not surprising that they share the same characteristics of encompassing two fundamental constants, as well as remaining invariant to certain local transformations. While all three first level theories in Fig. 1, SR, QM and Newtonian gravity, do not have local symmetry, the requirement of local symmetry seems necessary to fuse two of them together at a time.

While QFT is the theory for the smallest scale, and GR is the theory for the largest, it seems the theory we propose here occupies a scale in between. For small particles and molecules, the suggested deviation from QM is too small to be detected currently. If our estimate is close, it will take a particle with a mass about as large as $10^{-11}$ gram to have a probability density significantly different from conventional QM predictions. Consequently our theory is unlikely to have much implication for elementary particle physics, which is well described by QFT. On the other hand, the kinematics of human scale objects can be greatly modified by our theory, as discussed above. Thus, from elementary particles to macroscopic objects, the adequate theories may follow the sequence of QFT $\rightarrow$ QM $\rightarrow$ this theory.

On the gravitational side, the suggested modification on Newtonian law becomes evident only when gravity is very weak. This is indeed opposite to GR, where one expects more deviations from Newtonian law with stronger gravity. Thus on a scale of gravity from the weakest to the strongest, the adequate theories seems to follow the sequence of this theory $\rightarrow$ Newtonian gravity $\rightarrow$ GR. The separation between our theory and GR in the sequence also makes it possible to apply them separately in different situations, for example GR for black holes, and this theory for the motions of galaxies and galaxy clusters. On the largest scale, the universe itself, it is clear that GR is the basic foundational theory that underpins cosmology, yet potentially with input from the $-\frac{1}{2}m\mathbf{v}(\mathbf{r})^2$ term here for the cosmological constant in the Einstein equation. On the other hand, it seems that $\mathbf{v}(\mathbf{r})$ at the scale of the universe cannot be solved by the field equation here. Instead, Einstein equation may be needed for its solution. In this sense, the two theories may be coupled at the cosmological scale.

The in-between status of the theory proposed here is not limited to the scale of its applications, but also features in its theoretical structure. In QED, the connection in the covariant derivative is the 4-potential, while in GR, the connection is the gravitational field. Here we have a connection $\mathbf{Y}(\mathbf{r})$, or equivalently the velocity field $\mathbf{v}(\mathbf{r})$, that is neither a potential nor a field. Instead, the potential is $-\frac{1}{2}m\mathbf{v}(\mathbf{r})^2$. Taking the gradient of this potential yields a field that is a product of the connection $\mathbf{v}(\mathbf{r})$ and its spatial derivatives. So the field is neither like the connection in GR, nor like the connection’s exterior.
derivative in QED, but somewhere in between as a combination of both. Whether there is any deeper meaning of such a structure awaits further exploration.

At the end of this section, it is natural to ask about the relativistic parent of this theory. By extending the perspective presented in Fig. 1 to one level above, it seems clear that such a relativistic parent theory should also be the gravitational parent theory of QFT, as well as the quantum mechanical parent theory of GR. In other words, it is the theory of ultimate precision, with $G$, $\hbar$, and $c^{-1}$ all set at their true values. Such a theory obviously is essential for the description of Planck scale physics. However, for the vast majority of situations in the current universe, a combination of GR, QFT and potentially the theory proposed here may be sufficient, with different situations calling for different theories or the right combinations of them. This may even be true for the vast majority of the evolutionary history of the universe, with the notable exception of the very early stage of the big bang where Planck scale physics is clearly needed. On the other hand, non-relativistic characteristics are embedded in our theory here. For example, the imaginary momentum field $i m \nu^e(r)$ is automatically removed from the particle momentum expectation value based on the rules in our theory. This fits well with the fact that non-relativistic field has no momentum of its own. Changing this to a relativistic version does not look compatible with the structure of our current theory. Thus, the road to a relativistic parent theory does not look like to be an easy extension. Again this is not very surprising, considering the difficulties people encounter in the attempt of quantizing the gravitational field or fusing QFT and GR together.

IV. Conclusions
Quantum field theory and general relativity, the two most fundamental theories in physics, share the common features of possessing local symmetry and encompassing two universal constants ($\hbar$, $c$ and $G$, $c$, respectively). Inspired by these commonalities, we set out to build a non-relativistic theory that encompasses $G$ and $\hbar$ with local symmetry. Starting with the observation that the modulus of wave function can be changed by an arbitrary constant factor in conventional quantum mechanics, we elevate this global symmetry into a local one. The requirement of local modulus symmetry results into a number of important changes from conventional quantum mechanics. Foremost is the replacement of the complex conjugate of wave function by a new entity called co-wave function. They differ by a real and positive factor of $\gamma(r)$ that may change from position to position. Another important change is the replacement of ordinary spatial derivative by a covariant spatial derivative, with a purely imaginary connection $i m \nu^e(r)$ attached to the canonical momentum operator to form particle momentum operator. The ordinary temporal derivative also needs to be replaced by a covariant temporal derivative that bears resemblance to the Lagrangian derivative in fluid dynamics, with a purely imaginary velocity field of $i \nu^e(r)$. With these changes we are able to make the modified Schrödinger equation covariant to local modulus transformation. We find the potential energy term now is $-\frac{1}{2} m \nu^e(r)^2$. This can be identified as gravitational potential energy if $\nu^e(r)$ is identified as the escape velocity.

To account for gravity, we define 3 quantum metric functions $\gamma_i(r)$, with their product equal to $\gamma(r)$, and their corresponding regularized metric functions $\tau_i(r)$ are defined too. We postulate that in small enough region, physical laws reduce to conventional quantum mechanics in our theory. Based on this new equivalence principle, we are able to
find out $v^e(r)$ as a function of $\tau_i(r)$. It turns out that $v^e(r)$ can be divided into two parts, with one part $v^e_u(r)$ that is invariant to local modulus transformation, and another part $v^e_c(r)$ that takes up all the changes required for $v^e(r)$ under local modulus transformation. Replacing $v^e(r)$ by $v^e_u(r)$ in the potential energy term of $-\frac{1}{2}mv^e(r)^2$, we find a new gravitational Poisson equation that is invariant to local modulus transformation. It can also revert to the Newtonian Poisson equation when $\gamma(r)$ is set as a constant.

With the theory established, we find out a number of new features from this theory that have potential applications. Foremost is the additional term of $-\frac{h}{2}\nabla \cdot v^e(r)$ in the equation of motion that may be linked to the cosmological constant. In the case of a spontaneously broken modulus symmetry that does not allow us to set $\gamma(r)$ as a constant, we find the field equation differs from the Newtonian field equation. This can potentially provide the mechanism to account for the mass discrepancy widely observed in galaxies and galaxy clusters. On the other hand, we find $\gamma(r)$ scales exponentially with particle mass. With the same symmetry breaking mechanism, we find that $\gamma(r)$ barely registers any change for microscopic objects. For macroscopic objects, $\gamma(r)$ changes so rapidly that their quantum states become pointer states in any practical sense. It thus provides a potential solution to the quantum measurement problem. A related consequence brought by the changing $\gamma(r)$ is the breaking of time reversal symmetry by the quantum system’s co-wave function. This may have implications for the asymmetry of time in the second law of thermodynamics.
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