Nonexistence of Bigeodesics in Integrable Models of Last Passage Percolation

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Abstract

Bi-infinite geodesics are fundamental objects of interest in planar first passage percolation. A longstanding conjecture states that under mild conditions there are almost surely no bigeodesics, however the result has not been proved in any case. For the exactly solvable model of directed last passage percolation on \( \mathbb{Z}^2 \) with i.i.d. exponential passage times, we study the corresponding question and show that almost surely the only bigeodesics are the trivial ones, i.e., the horizontal and vertical lines. The proof makes use of estimates for last passage time available from the integrable probability literature to study coalescence structure of finite geodesics, thereby making rigorous a heuristic argument due to Newman [2].

1 Introduction

We consider the following directed last passage percolation (LPP) model on \( \mathbb{Z}^2 \). For each vertex \( v \in \mathbb{Z}^2 \) associate i.i.d. weight \( \xi_v \) distributed as \( \text{Exp}(1) \). Define \( u \preceq v \) if \( u \) is coordinate wise smaller than \( v \) in \( \mathbb{Z}^2 \). For any oriented path \( \gamma \) from \( u \) to \( v \) let the passage time of \( \gamma \) be defined by

\[
\ell(\gamma) := \sum_{v' \in \gamma \setminus \{v\}} \xi_{v'}.
\]

For \( u \preceq v \) define the last passage time from \( u \) to \( v \), denoted \( T_{u,v} \), by \( T_{u,v} := \max_{\gamma} \ell(\gamma) \) where the maximum is taken over all up/right oriented paths from \( u \) to \( v \). Observe that by continuity of the exponential distribution, almost surely there exists a unique path between every pair of (ordered) points \( u \) and \( v \) that attains this maximum. We shall denote by \( \Gamma_{u,v} \) the path between \( u \) and \( v \) that attains the last passage time \( T_{u,v} \) and call \( \Gamma_{u,v} \) the geodesic between \( u \) and \( v \).

Our object of interest is a bigeodesic, a bi-infinite up/right path \( \gamma = \{v_i\}_{i \in \mathbb{Z}} \) such that for each \( i < j \) the restriction of \( \gamma \) between \( v_i \) and \( v_j \) is the geodesic from \( v_i \) to \( v_j \). It is trivial to observe that the horizontal and vertical lines, that is the lines \( \{x = i\} \) and \( \{y = j\} \) for \( i, j \in \mathbb{Z} \), are bigeodesics. We call these bigeodesics the trivial bigeodesics and any other bigeodesic a non-trivial bigeodesic. Our main theorem in this paper is the following.

**Theorem 1.** For directed last passage percolation on \( \mathbb{Z}^2 \) with i.i.d. exponential passage times, almost surely there does not exist any non-trivial bigeodesic.

1.1 Background

Kardar, Parisi, and Zhang predicted in their seminal work [25] that a large class of randomly growing interfaces exhibit a universal behaviour that is now known as the KPZ universality, including the longitudinal and transversal fluctuation exponents of \( 1/3 \) and \( 2/3 \) respectively. Directed last
passage percolation and first passage percolation (where one puts i.i.d. weights on the edges of $\mathbb{Z}^2$ and studies the first passage time i.e. minimum weight path between two vertices) models on the plane are believed to belong to the KPZ universality class under very general conditions on the passage time distributions. However, the scaling exponents and the scaling limits have only been rigorously established for a handful of so-called integrable models (e.g. LPP with exponential or geometric weights) where exact distributional formulae for the passage times are available due to some remarkable bijections and connections with random matrix theory and orthogonal polynomials. Although some geometric consequences of the algebraic formulae was already studied by Johansson [24] who established the scaling exponent $2/3$ for transversal fluctuation in Poissonian LPP, sharper geometric estimates and interesting consequences thereof has only recently started being explored [7, 9, 5, 4].

In another related, but separate direction of works, a lot of progress has been made in studying planar first passage percolation, another model believed to be in the KPZ universality class that, however, is not exactly solvable. In the absence of exact formulae, the study of first passage percolation has relied mostly on a geometric understanding of the geodesics, the study of which was initiated by Newman and co-authors as summarized in his ICM paper [29] where certain coalescence results are established under curvature assumptions on the limit shape. Although much less is rigorously known, the connection between understanding properties of semi-infinite and bi-infinite geodesics, limit shapes and the KPZ predicted fluctuation exponents has been clear for some years. Much progress has been made in recent years in understanding the geodesics starting with the idea of Hoffman [22] of studying infinite geodesics using Busemann functions. These techniques have turned out to be extremely useful, providing a great deal of geometric information on the structure of geodesics in first passage percolation [12, 13, 1]. Some of these techniques have also recently been applied to last passage percolation models with our without integrable structure [17, 16, 30].

The question of the existence of bigeodesics in planar first passage percolation has been one of the most important longstanding problems in the field. Although Benjamini and Tessera recently showed that bigeodesics do exist for first passage percolation on certain hyperbolic graphs [9], it is believed that under some mild conditions on the passage time distribution almost surely bigeodesics do not exist (observe that there are no trivial bigeodesics in the first passage percolation setting) for the two dimensional Euclidean lattice. However, it is only rigorously known that under certain regularity assumptions on the boundary of limit shape bigeodesics along fixed directions do not exist [12, 13, 1]. In this paper, we prove the nonexistence of bigeodesics for the exactly solvable model of exponential LPP, where not only the exact limit shape is known, much finer information about the coalescence structure of finite geodesics can be obtained from the moderate deviation estimates available in the integrable probability literature.

1.2 An outline of the Argument

In an AIM workshop in 2015, Newman presented a heuristic argument for almost sure non-existence of bigeodesics in FPP predicated on the transversal fluctuation exponent $\xi > 1/2$; see [2]. Part of this paper follows the general outline of that argument, however, with some significant modifications and additional ingredients. To implement this program we establish new results about the coalescence structure of geodesics in exponential last passage percolation, which are of independent interest and useful in other contexts as well [6].

First observe the following: by translation invariance and ergodicity, we know that existence of a bigeodesic is a $0 - 1$ event and hence it follows that if almost surely bigeodesics exist, then with positive probability there must exist bigeodesics passing through the orgin, denoted $0$. We shall
prove Theorem 1 by showing that almost surely there does not exist any non-trivial bigeodesic passing through $0$. Let $\gamma = \{v_i\}_{i \in \mathbb{Z}}$ be a bi-infinite path passing through $0$. Without loss of generality assume $v_0 = 0$. Let us set $v_i := (x_i, y_i)$. Observe that if $\gamma$ is a bigeodesic then $\gamma^+ := \{v_0, v_1, \ldots\}$ and $\gamma^- := \{v_0, v_{-1}, \ldots\}$ are both semi-infinite geodesic rays

It is known [29] that almost surely every geodesic ray emanating from a fixed vertex has a direction, i.e., except on a set of zero probability $\lim_{n \to \infty} \frac{y_i}{x_i} := h(\gamma^+) \in [0, \infty]$ and $\lim_{n \to -\infty} \frac{y_i}{x_i} := h(\gamma^-) \in [0, \infty]$ exist. For a bigeodesic $\gamma$ passing through $0$ we shall call $h(\gamma^+)$ and $h(\gamma^-)$ the forward limiting direction and the backward limiting direction of $\gamma$ respectively. As already pointed out, the vertical and horizontal directions are somewhat special, we shall take care of them separately. For $h \in (0, 1)$, let $E_h$ denote the event that there exists a bigeodesic passing though $0$ such that either its forward limiting direction is in $(h, \frac{1}{h})$, or its backward limiting direction is in $(h, \frac{1}{h})$. Let $E_s$ denote the event that there exists a bigeodesic $\gamma$ passing through the origin which has either $h(\gamma^-) = h(\gamma^+) = 0$ (i.e., it is horizontally directed) or $h(\gamma^-) = h(\gamma^+) = \infty$ (i.e., it is vertically directed). It is immediate that Theorem 1 will follow from the next two propositions.

**Proposition 1.1.** For each $h \in (0, 1)$, we have $\mathbb{P}(E_h) = 0$.

**Proposition 1.2.** We have $\mathbb{P}(E_s) = 0$.

Observe that the situation in Proposition 1.2 does not occur in the FPP setting as that model is not directed. A different argument is required to establish Proposition 1.2 and rule out the vertical and horizontal non-trivial bigeodesics. Let us, for now, focus on the situation of Proposition 1.1, and describe how this proposition is established following Newman’s general heuristics. Clearly it suffices to prove Proposition 1.1 for $h$ sufficiently small. Let $S_n$ denote the square $[-n, n]^2 \cap \mathbb{Z}^2$. We shall denote the union of its left and bottom side by $\text{Ent}_n$ and the union of its top and right side by $\text{Exit}_n$. Observe that any bi-infinite path through $0$ must enter $S_n$ through a point on $\text{Ent}_n$, and exit $S_n$ via a point on $\text{Exit}_n$. Clearly if the path is a bigeodesic, then its restriction to $S_n$ must give a geodesic between a point on $\text{Ent}_n$ and a point on $\text{Exit}_n$. Moreover, on $E_h$, one must also have that for all $n$ sufficiently large, the line joining the endpoints of the putative bigeodesic restricted to $S_n$ must have slope in $(\frac{h}{2}, \frac{2}{h})$. Let $E_{n,h}$ denote the event that there exists points $u \in \text{Ent}_n$ and $v \in \text{Exit}_n$ such that slope$(u, v) \in (\frac{h}{2}, \frac{2}{h})$ and $0 \in \Gamma_{u,v}$. Clearly if $\mathbb{P}(E_h) > 0$ then $\lim_{n \to \infty} \mathbb{P}(E_{n,h}) > 0$. This is contradicted by the following proposition which, therefore, implies Proposition 1.1.

**Proposition 1.3.** Let $h \in (0, 1)$ be fixed. There there exists $C = C(h) > 0$ such that $\mathbb{P}(E_{n,h}) \leq Cn^{-1/3}$ for infinitely many $n$.

Newman’s heuristic for showing that $\mathbb{P}(E_{n,h}) = o(1)$ is the following. Divide the intervals $\text{Ent}_n$ and $\text{Exit}_n$ into disjoint subintervals of length $n^\chi$ where $\chi$ is the transversal fluctuation exponent (known to be equal to $2/3$ in our case). For most pairs of intervals $(I, J)$, the point $0$ is “far” (at the transversal fluctuation scale) from the straight lines joining points in $I$ to points in $J$, so the contribution for such pairs should be negligible and the main contribution should come from the “opposite pairs”. Also for each pair of “opposite” sub-intervals $I$ and $J$, $I \subseteq \text{Ent}_n$, and $J \subseteq \text{Exit}_n$, the geodesics from points in $I$ to points in $J$ “should coalesce” and hence the chance of there being any geodesic passing through the origin should be $\approx n^{-\chi}$. Taking a union bound over $(n^{1-\chi})$ many pairs of opposite intervals, we should get the required probability bound as long as $\chi > 1/2$.

There are a number of obvious issues with this heuristic, even if the transversal fluctuation exponent in known to be bigger than $1/2$, as was already pointed out in [2]. First, as was shown

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1Semi-infinite geodesics, or geodesic rays, are naturally defined as follows. A path $\gamma = \{v_i\}_{i \in \mathbb{Z}_{\geq 0}}$ is called a semi-infinite geodesic if $v_i \preceq v_{i+1}$ for all $i$ (or $v_{i+1} \preceq v_i$ for all $i$), and the restriction of $\gamma$ between $v_i$ and $v_j$ is a geodesic from $v_i$ to $v_j$ for all $i < j$ (resp. for all $i > j$).
recently in \[30, 6\] coalescence (of all geodesics) in an on-scale rectangle (i.e., an \(n \times n^{2/3}\) rectangle) happens with positive probability, but not with high probability. Second, one needs to deal with the correlated events of coalescence and the geodesic passing through the origin. To circumvent these issues we show that even though all paths might not coalesce, most of the paths do, a result of independent interest (see Theorem 2). The other issue is to deal with the contribution of the pairs of intervals that are not exactly opposite one another. This issue is circumvented by an averaging argument, where instead of looking at the probability of some geodesic passing through the origin we look at the average number of vertices near the origin that are on such geodesics.

**Inputs from Integrable Probability**

This paper continues the general program of understanding the geometry of geodesics in exactly solvable models of last passage percolation using inputs from integrable probability initiated in \[7\] and continued in \[6, 5, 4\]. As such, we use the same integrable inputs, and quote many of the consequences of the same derived in these papers, especially \[7\]. For the convenience of the reader, let us briefly recall here the type of estimates we shall be using, and also collect the precise statements at the end in Appendix B.

The two fundamental ingredients are the convergence of the rescaled passage time for the Exponential LPP \[23\] and a moderate deviation estimate for the same \[8, 26\]. Specifically we use that, for \(h \in (0, \infty)\), \(h^{-1/6} n^{1/3} (T_{0,(n,hn)} - n(1 + \sqrt{h})^2)\) convergence weakly to a scalar multiple of the GUE-Tracy Widom distribution \[2\] and the above statistic has exponential tails uniformly in high \(n\) and also in \(h\). In the situation of Proposition 1.1 we are in the case where \(h\) is bounded away from 0 and \(\infty\); we recall the precise statements of the relevant results in Theorem B.1 and Theorem B.2.

We would like to point out here that such moderate deviation estimates are known for a number of other exactly solvable model of last passage percolation models such as the Poissonian last passage percolation on \(\mathbb{R}^2\) \[27, 28\] and LPP on \(\mathbb{Z}^2\) with geometric passage times \[11\], and hence one might expect that variants of our results can be proved for these models as well.

Using Theorem 13.1 and Theorem 13.2 therein (Theorems B.1 and B.2), \[7\] established a number of useful consequences which we shall extensively use. In particular, for rectangles of dimension \(n \times n^{2/3}\), where the pair of longer sides have slope bounded away from 0 and \(\infty\), \[7\] established that the supremum (and infimum) of centered and scaled (by \(n^{1/3}\)) passage times over all pairs of points, one from the each shorter side of the rectangle, has uniform exponential tails. See Proposition B.3, Proposition B.4 and Proposition B.5.

One further consequence of the moderate deviation estimates established in \[7\] was a quantitative control of the transversal fluctuation of a geodesic. It was established in \[7\] that the maximum distance of the geodesic from \(0\) to \(n\) (for \(r \in \mathbb{Z}\) we shall denote the point \((r, r)\) by \(r\)) scales as \(n^{2/3}\) (as mentioned already the exponent was identified in \[24\]) and has stretched exponential tails at this scale. We shall need this result as well; see Proposition B.6 for a precise statement.

To deal with the axial directions we also need a moderate deviation estimate for \(h^{-1/6} n^{1/3} (T_{0,(n,hn)} - n(1 + \sqrt{h})^2)\) when \(h\) is allowed to become arbitrarily large or small (see Theorem 4.5). The integrable input is provided by \[26\], and a necessary analogue of Proposition B.4 using Theorem 4.5 (see Theorem 4.6).

\footnote{Strictly speaking, one usually proves such results in the model of Exponential LPP where the weight of the last vertex is also included in the definition of \(T\). However for large \(n\) this does not make any difference and we shall ignore this issue henceforth.}
The axial directions

Before wrapping up this section, let us present a brief outline of the argument for proving Proposition 1.2. Let us only consider the vertical direction. Simple translation invariance and ergodicity considerations show that there cannot exist a vertically directed bigeodesic which only moves finitely many steps in the horizontal direction. So it suffices to show that there cannot exist any semi-infinite geodesic started from origin directed vertically upwards that moves infinitely many steps to the right. We prove this by contradiction. If such a geodesic exists with positive probability, then with positive probability it will also take $M$ rightward steps before $L$ upward steps for some large $M$ and large $L$ depending on $M$.

To rule this out, we establish the following two results. First we show that for $\varepsilon$ arbitrarily small the transversal fluctuation of the geodesic from $0$ to $(\varepsilon n, n)$ is $O(\varepsilon^{2/3} n^{2/3})$ with high probability (see Proposition 4.7); this generalizes Johansson’s transversal fluctuation result [24] to steep geodesics. We further prove a local version of the above transversal fluctuation result showing that the local transversal fluctuation of the geodesic from $0$ to $(\varepsilon n, n)$ at height $L$ is $O(\varepsilon^{2/3} L^{2/3})$ (see Theorem 4.4). This generalizes Theorem 3 of [6], where such a result was proved for $\varepsilon$ bounded away from $0$ and $\infty$.

Once we have this result at our disposal we can simply take $\varepsilon$ sufficiently small depending on $L$, and argue that if the geodesic from $0$ to $(\varepsilon n, n)$ took $M$ rightward steps before $L$ upward steps then it would have atypically large transversal fluctuation at height $L$. Observing that any semi-infinite geodesic started at $0$ and directed vertically upward will be to the left of the geodesic $0$ to $(\varepsilon n, n)$ for all $n$ sufficiently large completes the proof.

Organization of the paper

The rest of the paper is organized as follows. In Section 2 we state and prove Theorem 2, a result about rarity of multiple disjoint geodesics across a rectangle of size $n \times n^{2/3}$, a result of independent interest. In Section 3 we complete the proof of Proposition 1.3 using Theorem 2.8, a consequence of Theorem 2 and a generalization of that, Lemma 3.1. In Section 4 we prove Proposition 1.2 and complete the proof of Theorem 1. In Appendix A we sketch how Lemma 3.1 can be proved following the same steps as in the proof of Theorem 2.8. Finally, in Appendix B we collect the integrable inputs and their consequences from [7] that we use throughout the paper.

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2 Rarity of Multiple Disjoint Geodesics

In this section we prove a result concerning number of disjoint geodesics across an on-scale rectangle, i.e., a rectangle of size $n \times n^{2/3}$. The result says essentially says that the maximum number of disjoint geodesics from one side of the rectangle to the other is tight at $O(1)$ scale, and has nice stretched
exponential tails. Rarity of disjoint geodesics is a question of independent interest, and has been investigated in [19] in the context of Brownian last passage percolation using the Brownian Gibbs property of [10]. Showing that a large number of disjoint geodesics is sufficiently rare, has a number of applications. In this paper, we shall use this result to prove Proposition 1.3. In [6], this is used to prove optimal tail estimates for distance to coalescence for semi-infinite geodesics started at distinct points. This can also be used to give an optimal solution to the midpoint problem for exponential last passage percolation; see Remark 2.11. For applications in studying the locally Brownian nature of Airy processes, see [18, 19, 21, 20].

Let us first state the result in the simplest possible setting, where the rectangle concerned has the pair of larger sides parallel to the diagonal line $x = y$. For $r \in \mathbb{Z}$, let $L_r$ denote the line $x + y = 2r$. Let $A_n$ (resp. $B_n$) denote the line segment on $L_0$ (resp. $L_n$) of length $2n^{2/3}$ with midpoint $0$ (resp. $n$). For points $u, v$ on $A_n$ (or on $B_n$) we say $u < v$ if $v = u + i(-1, 1)$ for some $i \in \mathbb{N}$. For $\ell \in \mathbb{N}$, let $E_\ell$ denote the event $^4$ that there exists $u_1 < u_2 < \cdots < u_\ell$ on $A_n$, and $v_1 < v_2 < \cdots < v_\ell$ on $B_n$, such that the geodesics $\Gamma_{u_i,v_i}$ are disjoint. The next theorem is our main result in this section.

**Theorem 2.** There exists constant $n_0, \ell_0 \in \mathbb{N}$ such that for all $n > n_0$ and for all $\ell_0 < \ell < n^{0.01}$ we have

$$\mathbb{P}(E_\ell) \leq e^{-c\ell^{1/4}}$$

for some absolute constant $c > 0$.

Observe that Theorem 2 immediately implies that if $N_n$ denote the maximum number of pairwise disjoint geodesics from points on $A_n$ to points on $B_n$, then we have $\mathbb{E}N_n \leq C$ for some absolute constant $C$. A variant of this can be used to solve the so-called midpoint problem with the optimal exponents; see Remark 2.11.

Before delving into the details of the proof of Theorem 2 let us briefly explain the idea. First we shall show that the length of the geodesic from any point on $A_n$ to any point in $B_n$ is unlikely to be too small, i.e., even the minimum geodesic length is typically $4n - \Theta(n^{1/3})$. Now the question is reduced to showing that it is unlikely to have a large number of disjoint paths from $A_n$ to $B_n$ that have length at least $4n - Cn^{1/3}$. To this end we observe that if there are a large number of disjoint paths from $A_n$ to $B_n$, one of them must be constrained to be contained in a thin region; such paths are known to typically be much smaller in length (see e.g. [5, 4, 14]). We shall then use the BK inequality to conclude that a number of such paths existing disjointly is unlikely enough to beat the entropy of the number of tuples of such thin regions. Let us now move towards making the above heuristic rigorous.

Let $n$ be sufficiently large and let $\ell < n^{0.01}$ be fixed and sufficiently large. Let $U_n$ and $V_n$ be the line segments on $L_0$ and $L_n$) of length $2\ell^{1/3}n^{2/3}$ with midpoint $0$ and $n$ respectively. Let $R$ denote the rectangle whose one pair of opposite sides are $U_n$ and $V_n$. The following lemma says that geodesics from $A_n$ to $B_n$ will typically be completely contained in $R$.

**Lemma 2.1.** Let $F_\ell$ denote the event that there exist $u \in A_n$ and $v \in B_n$ such that $\Gamma_{u,v}$ exits $R$. Then $\mathbb{P}(F_\ell) \leq e^{-c\ell^{1/4}}$ for some $c > 0$.

**Proof.** Let $u_0$ and $u'_0$ (resp. $v_0$ and $v'_0$) denote the smallest and the largest vertices of $A_n$ (resp. $B_n$) in the order defined above. It is easy to see that all $\Gamma_{u,v}$’s (for $u \in A_n, v \in B_n$) are sandwiched between $\Gamma_{u_0,v_0}$ and $\Gamma_{u'_0,v'_0}$, this fact is often refereed to as polymer ordering. So it suffices to show that it is unlikely that $\Gamma_{u_0,v_0}$ or $\Gamma_{u'_0,v'_0}$ will exit $R$. This follows from Proposition 2.6.

$^4$Notation in this section is independent of the rest of the paper, so this $E_\ell$ is not to be confused with the event $E_\ell$ defined earlier. Also, whenever we use $\ell(\cdot)$ for the weight of a path, we shall explicitly mention the argument, so there will be no scope for confusion with the two uses of $\ell$. 

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The next lemma shall show that none of the geodesic lengths from $A_n$ to $B_n$ can be too small.

**Lemma 2.2.** For each fixed constant $c_1 > 0$, there exists $c > 0$ such that for all $\ell$ and all $n$ sufficiently large we have

$$
\mathbb{P}\left( \inf_{u \in A_n, v \in B_n} T_{u,v} \leq 4n - c_1 \ell^{1/4} n^{1/3} \right) \leq e^{-c \ell^{1/4}}.
$$

**Proof.** It follows from Proposition B.5 that $\inf_{u \in A_n, v \in B_n} E T_{u,v} \geq 4n - C n^{1/3}$ for some absolute constant $C > 0$. The result now is immediate Proposition B.3.

Let $G^\ell$ denote the event that there exists $u_1 < u_2 < \cdots < u_\ell$ on $A_n$, and $v_1 < v_2 < \cdots < v_\ell$ on $B_n$, and disjoint paths $\gamma_i$ joining $u_i$ and $v_i$ contained in $R$ such that $\ell(\gamma_i) \geq 4n - c_1 \ell^{1/4} n^{1/3}$. In view of Lemma 2.1 and Lemma 2.2 the following proposition suffices to prove Theorem 2.

**Proposition 2.3.** In the above set-up, we have

$$
\mathbb{P}(G^\ell) \leq e^{-c \ell^{1/4}}.
$$

We shall need some preparation to prove Proposition 2.3. We shall divide the rectangle $R$ into an $h \times \ell^{1/8} m$ grid of sub-rectangles, see Figure 1. We shall choose a suitable $h$ and $m$ later. More precisely, consider lines $L_i$ with slope $-1$ equally spaced with internal spacing $\frac{2n}{h}$ such that $L_0 = L_0$ and $L_h = L_n$. Observe that each of these lines intersects $R$ in a line segment of length $2n^{2/3}$, abusing notation let $L_i$ denote those line segments. Divide the line segment $L_i$ into equally spaced line segment $L_{i,j}$ each of length $\frac{n^{2/3}}{m}$.

Figure 1: Dividing the $n \times 2\ell^{1/8} n^{2/3}$ rectangle $R$ into an $h \times 2\ell^{1/8} m$ grid as in the proof of Proposition 2.3. To show that having too many disjoint paths across this rectangle none of which is much smaller than typical, we fix a sequence $J = \{j_0, j_1, \ldots, j_h\}$ that encodes where the path crosses different lines of the grid. Let $\gamma_J$ denotes the best path with encoding $J$. Lemma 2.3 shows that for appropriate choices of $h$ and $m$, $\gamma_J$ is likely rather small, and a union bound together with the BK inequality finishes the proof of Theorem 2.

Our next objective is the following. For a fixed sequence $J := \{j_0, j_1, j_2, \ldots, j_{h-1}, j_h\}$ taking values in $[-\ell^{1/8} m, \ell^{1/8} m) \cap \mathbb{Z}$, we shall consider the best path $\gamma_J$ from $A_n$ to $B_n$ that passes through the line segment $L_{i,j_i}$ for each $i = 0, 1, \ldots, h$. We shall show that for suitable choice of parameters $\ell(\gamma_J)$ is typically much smaller than $4n$. Before stating this result we need the following auxiliary lemma to fix our parameters.
Lemma 2.4. Let \( A_\ast \) denote the line segment joining \((-\frac{c_0 n^{2/3}}{2}, \frac{c_0 n^{2/3}}{2})\) and \((\frac{c_0 n^{2/3}}{2}, -\frac{c_0 n^{2/3}}{2})\), and let \( B_\ast = A_\ast + (n, n)\). For \( c_0 \) sufficiently small there exists \( c_2, c_3 > 0 \) such that for all \( n \) sufficiently large we have
\[
\mathbb{E} \sup_{u \in A_\ast, v \in B_\ast} T_{u, v} \leq 4n - c_2 n^{1/3}; \quad \text{Var} \sup_{u \in A_\ast, v \in B_\ast} T_{u, v} \leq c_3 n^{2/3}.
\]

Proof. Observe first that Theorem B.2 implies that \( |\sup_{u \in A_\ast, v \in B_\ast} \mathbb{E} T_{u, v} - 4n|, |\inf_{u \in A_\ast, v \in B_\ast} \mathbb{E} T_{u, v} - 4n| \leq C n^{1/3} \) for some absolute constant \( C \). It follows now from Proposition B.5 that for all \( t \) large, we have
\[
\mathbb{P} \left( \sup_{u \in A_\ast, v \in B_\ast} T_{u, v} - 4n \leq t n^{1/3} \right) \leq e^{-ct}
\]
for some \( c > 0 \). This implies the second part of the statement.

For the first part consider the points \( u_0 = (-c_0^{3/2} n, c_0^{3/2} n) \) and \( v_0 = (n + c_0^{3/2} n, n + c_0^{3/2} n) \). Clearly
\[
\mathbb{E} \sup_{u \in A_\ast, v \in B_\ast} T_{u, v} \leq \mathbb{E} T_{u_0, v_0} - \mathbb{E} \inf_{u \in A_\ast} T_{u_0, u} - \mathbb{E} \inf_{v \in B_\ast} T_{v, v_0}.
\]
It follows from Lemma 2.2 that \( \mathbb{E} \inf_{u \in A} T_{u_0, u}, \mathbb{E} \inf_{v \in B} T_{v, v_0} \geq 4c_0^{3/2} n - C_1 c_0^{1/2} n^{1/3} \) for some constant \( C_1 > 0 \). From the Tracy-Widom convergence result of [23] (Theorem B.1), the uniform tail estimates (Theorem B.2), and the fact that the GUE Tracy-Widom distribution has negative mean, it follows that for \( n \) sufficiently large \( \mathbb{E} T_{u_0, v_0} \leq 4(n + 2c_0^{3/2} n) - C n^{1/3} \) for some \( C > 0 \). Putting these together and choosing \( c_0 \) sufficiently small completes the proof of the first part.

Fix \( c_0 \) such that the conclusion of Lemma 2.4 holds. Recall that we have fixed \( \ell < n^{0.01} \) sufficiently large and \( n \) sufficiently large. For the next lemma we shall fix \( h \leq \sqrt{\ell} \), and choose \( m = \frac{h^{2/3}}{c_0} \). Consider the line segments \( L_{i,j} \) as described above for this choice of \( h \) and \( m \). Let us also fix sequence \( J := \{ j_0, j_1, j_2, \ldots, j_h-1, j_h \} \) taking values in \([-\ell^{1/8} m, \ell^{1/8} m] \cap \mathbb{Z} \), and let \( \gamma_J \) denote the best path from \( A_n \) to \( B_n \) that passes through the line segment \( L_{i,j_i} \) for each \( i = 0, 1, \ldots, h \). We have the following lemma.

Lemma 2.5. There exists \( c_1, c > 0 \) such that for all \( h \) and \( m \) as above, we have for each \( J \)
\[
\mathbb{P}(\ell(\gamma_J) \geq 4n - c_1 h^{2/3} n^{1/3}) \leq e^{-ch^{1/2}}.
\]

We shall use this lemma for \( h = \sqrt{\ell} \) only, however we believe it is interesting to state this result for general \( h \), as Lemma 2.5 gives control on how long the paths passing through a thin region can be. Fixing \( A \) as in the statement of the lemma, observe that \( \ell(\gamma_J) \) is bounded above by
\[
\sum_{i=0}^{h-1} \sup_{u \in L_{i,j_i}, v \in L_{i+1,j_{i+1}}} T_{u,v}.
\]
This quantity is a sum of independent variables, whose mean and variance has been bounded in Lemma 2.4. We are now ready to prove Lemma 2.5.

Proof of Lemma 2.5. First fix a \( J \) as in the statement of the lemma. Observe that by our choice of \( \ell \) and \( h \) one has that the slope of the line joining the midpoints of \( L_{i,j_i} \) and \( L_{i+1,j_{i+1}} \) is between \( 1/2 \)
and 2, and hence the arguments in Lemma 2.4 will continue to hold for \( \sup_{u \in L_{i,j_i}, v \in L_{i+1,j_{i+1}}} T_{u,v} \). More precisely we shall have \( c_0 \) and \( c_2 \) as in Lemma 2.4 and for each \( i \)

\[
\mathbb{E} \sup_{u \in L_{i,j_i}, v \in L_{i+1,j_{i+1}}} T_{u,v} \leq 4 \frac{n}{h} - c_2(n/h)^{1/3};
\]

\[
\mathbb{P} \left( \sup_{u \in L_{i,j_i}, v \in L_{i+1,j_{i+1}}} T_{u,v} - 4 \frac{n}{h} \geq t(n/h)^{1/3} \right) \leq e^{-ct}.
\]

By a Bernstein type bound on sum of independent variables having exponential tails, it follows that for each \( t > 0 \) sufficiently large we have

\[
\mathbb{P} \left( \ell(\gamma_J) - (4n - c_2 h^{2/3} n^{1/3}) \geq th^{1/6} n^{1/3} \right) \leq e^{-ct}
\]

for some \( c > 0 \). We can now set \( c_1 = c_2/2 \) and \( t = c_1 h^{1/2} \) to recover the statement of the lemma. \( \square \)

To complete the proof of Proposition 2.3, we need to control the entropy of the sequence \( \ell \)-tuple \((J_1, J_2, \ldots, J_\ell)\) associated with \( \ell \) disjoint paths as is predicated to exist on the event \( \mathcal{G}_\ell \). Let \( \mathcal{J} \) denote the set of all sequences \( J \) as described above. We have the following lemma.

**Lemma 2.6.** There exists a deterministic set \( \mathcal{C} = \mathcal{C}_{\ell,m,h} \subseteq \mathcal{J}^\ell \) with $$|\mathcal{C}| \leq (\ell + 2\ell^{1/8} m)^{2\ell^{1/8} m (h + 1)},$$ such that on the event \( \mathcal{G}_\ell \), there exists \((J_1, J_2, \ldots, J_\ell) \in \mathcal{C} \) such that \( \ell(\gamma_i) \leq \ell(\gamma_{J_i}) \) for each \( i = 1, 2, \ldots, \ell \).

**Proof.** On \( \mathcal{G}_\ell \), let \( \gamma_1, \gamma_2, \ldots, \gamma_\ell \) be naturally ordered set of disjoint paths as given by the definition of the events. For each \( i \in [\ell] := \{1, 2, \ldots, \ell\} \), let \( J_i = (j_{i}^{(1)}, \ldots, j_{i}^{(k_i)}) \) be the element of \( \mathcal{J} \) such that \( \gamma_i \) intersects \( L_{k_i j_{i}^{(1)}} \) for each \( k_i \). Clearly \( \ell(\gamma_i) \leq \ell(\gamma_{J_i}) \). So now we need to bound the total number of all possible tuples \((J_1, J_2, \ldots, J_\ell)\). Observe that the ordering implies, if \( i_1 < i_2 \), we must have \( j_{i_1}^{(k_{i_1})} \leq j_{i_2}^{(k_{i_2})} \) for each \( k_i \). It follows that \( \mathcal{C} \) can be enumerated by picking \((h+1)\) many the non-decreasing sequences of length \( \ell \) where each co-ordinate takes values in \(-\ell^{1/8} m \leq y_1 \leq y_2 \leq \cdots \leq y_\ell \leq \ell^{1/8} m \). So our task is reduced to enumerating positive integer sequences \(-\ell^{1/8} m \leq y_1 \leq y_2 \leq \cdots \leq y_\ell \leq \ell^{1/8} m \). By looking at the difference sequence \( z_k = (y_k - y_{k-1}) \) this reduces to enumerating sequences with \( z_1 + z_2 + \cdots + z_\ell \leq 2\ell^{1/8} m \). It is a standard counting exercise to see that number of such sequences is bounded by \( \left( \frac{\ell + 2\ell^{1/8} m}{2\ell^{1/8} m} \right) \). The result follows. \( \square \)

We can now complete the proof of Proposition 2.3.

**Proof of Proposition 2.3** Clearly it suffices to prove the theorem for \( n \) sufficiently large and \( \ell \) sufficiently large. For a fixed sufficiently large \( \ell < n^{0.01} \), set \( h = \ell^{1/2} \) and let \( m = \frac{h^{2/3}}{c_0} \) as in the statement of Lemma 2.5. For \((J_1, J_2, \ldots, J_\ell) \in \mathcal{C} \) let \( A_{J_1, J_2, \ldots, J_\ell} \) denote the event that there exist disjoint paths \( \gamma_1, \gamma_2, \ldots, \gamma_\ell \) satisfying the condition in the definition of \( \mathcal{G}_\ell \) with \( J_i \) being the sequence corresponding to \( \gamma_i \) as constructed in the proof of Lemma 2.6 (in particular this implies \( \ell(\gamma_i) \leq \ell(\gamma_{J_i}) \)) and \( \ell(\gamma_i) \geq 4n - c_1 \ell^{1/8} n^{1/3} \). Using Lemma 2.6 it follows that \( \mathbb{P}(\mathcal{G}_\ell) \) is upper bounded by

\[
\sum_{(J_1, J_2, \ldots, J_\ell) \in \mathcal{C}} \mathbb{P}(A_{J_1, J_2, \ldots, J_\ell}).
\]
Now observe that for any path $\gamma$ the event that $\ell(\gamma) \geq 4n - c_1 \ell^{1/4} n^{1/3}$ is increasing in the vertex weights and hence by the BK inequality probability of a number of such events happening disjointly is upper bounded by the product of the marginal probabilities. It therefore follows using Lemma 2.5 that $P(A_{i_1, i_2, \ldots, i_4}) \leq e^{-c_2 \ell^{1/4}}$. By Lemma 2.6 it follows that for any $\epsilon > 0$ we have $|C| \leq e^{C\ell n^{4+\epsilon}}$ and hence the result follows.

We now discuss briefly a generalization of Theorem 2 which is useful in other contexts, even though we shall not need it. Observe that the proof of Proposition 2.3 does not use the fact that the endpoints of $\gamma_i$’s were specified to lie on $A_n$ and $B_n$ respectively. The same result will be true verbatim if we replace $A_n$ by $U_n$ and $B_n$ by $V_n$ in the definition of the event $G_\ell$. Consider now the line segment $A_n’$ incident on $A_n$ (resp. $B_n’$ incident on $B_n$) of length $2\ell^{1/4} n^{2/3}$ and midpoint at $0$ (resp. $n$). It follows from Theorem B.2 and elementary calculations that $\inf_{u \in A_n’, v \in B_n’} ET_{u, v} = 4n - \Theta(\ell^{1/8} n^{1/3})$. Proposition B.5 now implies the following strengthening of Lemma 2.2.

For each fixed constant $c_1 > 0$, there exists $c > 0$ such that for all $\ell$ and all $n$ sufficiently large we have

$$P(\inf_{u \in A_n’, v \in B_n’} T_{u, v} \leq 4n - c_1 \ell^{1/4} n^{1/3}) \leq e^{-c\ell^{1/4}}.$$ 

Observe further that Lemma 2.1 also continues to hold (with possibly a different constant $c$) if we change $A_n$ to $A_n’$ and $B_n$ to $B_n’$ in its definition; i.e., the probability that any geodesic from $A_n’$ to $B_n’$ exits $R$ is upper bounded by $e^{-c_1 \ell^{1/4}}$ for $\ell$ and $n$ sufficiently large. Let $E_{\ell}'$ denote the event that there exists $u_1 < u_2 < \cdots < u_4$ on $A_n’$, and $v_1 < v_2 < \cdots < v_4$ on $B_n’$, such that the geodesics $\Gamma_{u_1, v_1}$ are disjoint. Using the above discussion and the proof of Proposition 2.3 one can obtain the following generalization of Theorem 2 whose proof we shall omit.

**Corollary 2.7.** There exists $n_0, \ell_0 > 0$, such that for all $n > n_0, n^{0.01} > \ell > \ell_0$ and we have $P(E_{\ell}') \leq e^{-c_1 \ell^{1/4}}$.

This corollary is used in [6] to prove optimal tail estimates for coalescence of semi-infinite geodesics.

### 2.1 Most geodesics coalesce quickly

One can use Theorem 2 to deduce a stronger result: not only are disjoint geodesics unlikely, most pairs of geodesics between points on $A_n$ and $B_n$ actually merge together rather quickly. This is the result we shall need to rule out bigeodesics. To state the result formally, we introduce the following terminology. For $x, x’ \in A_n$ and $y, y’ \in B_n$ we say that $(x, y) \sim (x’, y’)$ if $\Gamma_{x, y}$ and $\Gamma_{x’, y’}$ are same between the lines $L_{n/3}$ and $L_{2n/3}$. It is easy to see that $\sim$ is an equivalence relation. Let $M_n$ denote the number of equivalence classes. We have the following theorem.

**Theorem 2.8.** There exists $c > 0$ such that for all $\ell < n^{0.01}, n \in \mathbb{N}$ we have

$$P(M_n \geq \ell) \leq e^{-c \ell^{1/128}}.$$ 

Observe that the only significant difference between Theorem 2 and Theorem 2.8 is that in the former case we were considering disjoint paths from $A_n$ to $B_n$ which were naturally ordered. In the set-up of Theorem 2.8 we have to consider also geodesics that can potentially cross each other. To circumvent this issue we show that if there exists a large number of different equivalence classes there must be some stretch of linear length between $L_0$ and $L_n$ such that the geodesics corresponding to the equivalence classes are disjoint in this stretch. To this end we have the following lemma.
Lemma 2.9. Suppose \( u_1 < u_2 < \cdots < u_k \) (resp. \( v_1, v_2, \ldots, v_k \) with \( v_i \neq v_j \)) be points on \( \mathbb{L} \) (resp. \( \mathbb{L}' \)) such that each of the pairs \((u_i, v_i)\) are from a different equivalence class, i.e., for each pair \((i, j)\) with \( i \neq j \), \( \Gamma_{u_i, v_i} \) and \( \Gamma_{u_j, v_j} \) do not coincide between the lines \( \mathbb{L}_{n/3} \) and \( \mathbb{L}_{2n/3} \). Then there exists a subset \( I \subset [k] \) with \( |I| \geq k^{1/8} \) that the restrictions of \( \{\Gamma_{u_i, v_i}\}_{i \in I} \) are disjoint either between the lines \( \mathbb{L}_0 \) and \( \mathbb{L}_{n/6} \) or \( \mathbb{L}_{n/6} \) and \( \mathbb{L}_{n/3} \) or \( \mathbb{L}_{n/3} \) and \( \mathbb{L}_{2n/3} \) or \( \mathbb{L}_{2n/3} \) and \( \mathbb{L}_n \).

Proof. By Erdős-Szekers theorem, there exists a subset \( I_1 = \{i_1, i_2, \ldots, i_{\sqrt{k}}\} \) of \([k]\) such that either \( v_{i_1} < v_{i_2} < \cdots < v_{i_{\sqrt{k}}} \) or \( v_{i_1} > v_{i_2} > \cdots > v_{i_{\sqrt{k}}} \). In the former case it is easy to see that there exists a subset of \( I_1 \) that satisfies the condition in the statement of the lemma. Suppose the contrary, and without loss of generality assume \( I_1 = [\sqrt{k}] \). Consider then the points \( w_1, w_2, \ldots, w_{k^{1/4}} \) to be the points where \( \Gamma_{u_i, v_i} \) intersects \( \mathbb{L}_{n/6} \) for \( i \in I_1 \). Clearly if any point among those is repeated more that \( k^{1/8} \) times we are done, so without loss of generality let us assume that \( w_1, w_2, \ldots, w_{k^{3/8}} \) are distinct. Now again by Erdős-Szekers theorem we assume without loss of generality that either \( w_1 < w_2 < \cdots < w_{k^{1/6}} \) or \( w_1 > w_2 > \cdots > w_{k^{1/6}} \). In the former case we must have that \( \Gamma_{u_i, v_i} \) for \( i \leq k^{1/6} \) is disjoint between \( \mathbb{L}_0 \) and \( \mathbb{L}_{n/6} \). In the latter case we are back to the first case of the proof when looking at geodesics between \( \mathbb{L}_{n/6} \) and \( \mathbb{L}_n \) with \( k^{3/16} \) replacing \( k^{1/2} \). By the same argument as before we are done.

We can now prove Theorem 2.8

Proof of Theorem 2.8. Fix \( \ell < n^{0.01} \) sufficiently large and \( n \) sufficiently large. Observe first that using Lemma 2.1 we can restrict ourselves on the event that none of the geodesics from \( A_n \) to \( B_n \) exit \( \mathbb{R} \). It is easy to see that either there exists \( k = \ell^{1/4} \) many geodesics satisfying the hypothesis of Lemma 2.9 or there exists at least \( \ell^{1/4} \) that are disjoint between \( \mathbb{L}_0 \) and \( \mathbb{L}_{n/3} \) or \( \mathbb{L}_{2n/3} \) and \( \mathbb{L}_n \). The probability of the latter case can be bounded using Theorem 2. In the former case we can apply Lemma 2.9 and get an interval of length \( n/6 \) on which at least \( \ell^{1/32} \) many paths are disjoint. Applying Theorem 2 again completes the proof.

We complete this section with the following immediate corollary of Theorem 2.8 whose proof we omit.

Corollary 2.10. In the set-up of Theorem 2.8. Let \( N_n \) denote the number of vertices \( v = (v_1, v_2) \) with \( 2n/3 \leq v_1 + v_2 \leq 4n/3 \) such that \( v \) lies on a geodesic \( \Gamma_{u,w} \) for some \( u \in A_n \) and \( w \in B_n \). Then there exists \( C > 0 \) such that \( \mathbb{E} N_n \leq C n \).

We finish this section with a remark on the midpoint problem which was alluded to before.

Remark 2.11. Consider the geodesic \( \Gamma_n \) from \( 0 \) to \( n \). Assuming \( n \in 2 \mathbb{N} \) what is the probability that \( \Gamma_n \) passes through the midpoint \( n/2 \)? This question was asked in [17] in the context of first passage percolation and became popular as the “midpoint problem”. It is natural to conjecture that the probability is \( \Theta(n^{-2/3}) \) for models in KPZ universality class where the transversal fluctuation exponent is believed to be 2/3. However, for non-integrable models, to show even that this probability is \( o(1) \) remained open for many years and was only recently settled in [17]. For the integrable model of exponential LPP, it is easy to see that one can use Theorem 2 to show that \( \mathbb{P}(\Gamma_n \text{ passes through } n/2) = O(n^{-2/3}) \). Indeed, Theorem 2.8 implies that the number of vertices on the line \( x + y = n \) contained in a geodesic from \( (t, -t) \) to \( (n + t, n - t) \) for some \( t \) with \( |t| \leq n^{2/3} \) has expectation uniformly bounded above by a constant. This, together with the translation invariance of the model gives the desired result. Let us also briefly describe how to get a matching lower bound of \( cn^{-2/3} \) for this probability. One can, for example, use the barrier construction of [4, 9], to conclude that
with probability bounded away from 0 the following event occurs: the geodesic from \((n^{2/3}, -n^{2/3})\) to \((n + n^{2/3}, n - n^{2/3})\) and the geodesic from \((-n^{2/3}, n^{2/3})\) to \((n - n^{2/3}, n + n^{2/3})\) intersect the line \(x + y = n\) at the same point \((n/2 + t, n/2 - t)\) for some \(t\) with \(|t| \leq n^{2/3}\). The desired lower bound will then follow from translation invariance as before; we omit the details.

3 A geodesic not directed axially hitting the origin is unlikely

We shall prove Proposition 1.3 in this section. The key to the proof will be a generalization of Corollary 2.10 that will work for more general rectangles and parallelograms, as long as the slope of the line joining the midpoints of the pair of shorter sides is not too small. We now move towards a precise statement to this effect. Fix \(h \in (0, 1)\) sufficiently small.

![Figure 2: The bottom and left side and the top and right side of \([-n, n]^2\) is divided into line segments of length \(n^{2/3}\). Fix a pair of such line segments \((I, J)\); \(I\) from the bottom and left side, and \(J\) from the top and right side. Arguing as in the proof of Theorem 2.8 one can show that paths from \(I\) to \(J\) coalesce into \(O(1)\) many highways before passing near the origin. Indeed this implies, as stated in Lemma 3.1 that the expected number of vertices on such geodesics in a linear size box around \(0\) is \(O(n)\). Taking a union bound over \(O(n^{2/3})\) many pairs of intervals, one concludes that the number of vertices in a linear size box around the origin that also line geodesics across the square is \(o(n^2)\), which suffices to prove Proposition 1.1.

Now divide \(\text{Ent}_n\) and \(\text{Exit}_n\) into intervals of length \(n^{2/3}\) each. More specifically for

\[
i = -hn^{1/3}, -hn^{1/3} + 1, \ldots, 0, 1, 2, \ldots, n^{1/3},
\]

let \(\mathcal{B}_i\) denote the line segment \([-in^{2/3}, (-i + 1)n^{2/3}] \times \{-n\}\) and let \(\mathcal{L}_i\) denote the line segment \(\{-n\} \times [-in^{2/3}, (-i + 1)n^{2/3}]\). Similarly, let \(\mathcal{T}_i\) denote the line segment \([i(1)n^{2/3}, in^{2/3}] \times \{n\}\) and let \(\mathcal{R}_i\) denote the line segment \(\{n\} \times [(i - 1)n^{2/3}, in^{2/3}]\). For \(I \in \cup \mathcal{B}_i \cup \mathcal{L}_i\) and \(J \in \cup \mathcal{T}_i \cup \mathcal{R}_i\), we say that the pair \((I, J)\) is \(h\)-compatible if the straight line joining the mid-point of \(I\) to the midpoint of \(J\) has slope in \((\frac{h}{10}, \frac{10}{h})\); see Figure 2. The following lemma generalizes Corollary 2.10.

**Lemma 3.1.** Let \(h \in (0, 1)\) be fixed and sufficiently small. There exists a constant \(C = C(h) > 0\) such that for each \(h\)-compatible pair of line segments \((I, J)\) we have the following. Let \(N = N_n(I, J)\) denote the number of vertices \(w\) in \([-nh^{100}, nh^{100}]^2\) such that there exists \(u \in I\) and \(v \in J\) with \(w \in \Gamma_{u,v}\). Then \(\mathbb{E}N_n(I, J) \leq Cn\).
We shall not provide a detailed proof of this lemma. It follows, similarly to Corollary \[2.10\] from the exact analogue of Theorem \[2.8\] for the quadrilateral whose one pair of opposite sides are \(I\) and \(J\). That a variant of Theorem \[2.8\] holds for this quadrilateral can be proved by essentially re-doing the same calculations as in the proof of Theorem \[2.8\] with some minor modifications.

Observe that essentially the only ingredients we used were Proposition \[B.3\] and Proposition \[B.4\] (or the adapted version Proposition \[B.5\]), both of which continue to hold for \(h\)-compatible pairs of intervals \(I\) and \(J\) as long as \(h\) remains bounded away from 0 and \(\infty\). This is because those results, in turn, depended on the weak convergence result of \[23\] and the moderate deviation estimates of \[3\], as explained in \[7\]. Further observe that, both these estimates (Theorem \[B.1\] and Theorem \[B.2\]) are uniform in \(m \in \left(\frac{h}{100}, \frac{100}{h}\right)\) where \(m\) denotes the slope of the line joining the pairs of points in question, and hence we can get a uniform constant \(C\) depending only on \(h\). We omit the details here, but shall provide a sketch of the argument in Appendix A.

Using Lemma \[3.1\] we can now complete the proof of Proposition \[1.3\]

**Proof of Proposition \[1.3\]** Let \(h\) be as in the statement of the proposition and fix \(n \in \mathbb{N}\) sufficiently large. Observe that taking a union bound over all \(h\)-compatible pairs \((I, J)\) it follows that that

\[
\mathbb{E}\left\{ v : \left[-\frac{nh}{100}, \frac{nh}{100}\right]^2 : \exists u \in \text{Ent}_n, w \in \text{Exit}_n, \text{slope}(u, w) \in \left(\frac{h}{4}, \frac{4}{h}\right) \text{ and } v \in \Gamma_{u,w} \right\} \leq Cn^{5/3}
\]

for some constant \(C = C(h) > 0\). This implies that there exists a deterministic \(v \in \left[-\frac{nh}{100}, \frac{nh}{100}\right]^2 \cap \mathbb{Z}^2\) such that

\[
\mathbb{P}\left( \exists u \in \text{Ent}_n, w \in \text{Exit}_n, \text{slope}(u, w) \in \left(\frac{h}{4}, \frac{4}{h}\right) \text{ and } v \in \Gamma_{u,w} \right) \leq 10^4 h^{-2} C n^{-1/3}.
\]

the probability that there exists \(\Gamma_{u,w}\) containing \(v\) where \(u \in \text{Ent}_n, w \in \text{Exit}_n, \text{slope}(u, w) \in \left(\frac{h}{4}, \frac{4}{h}\right)\) is at most \(10^4 h^{-2} C n^{-1/3}\). Observe that by translation invariance (consider the translation that takes \(v\) to \(0\)), and simple geometric considerations it follows that there exists \(n' \in [n, n + \frac{nh}{100}] \cap \mathbb{N}\) such that

\[
\mathbb{P}\left( \exists u \in \text{Ent}_{n'}, w \in \text{Exit}_{n'}, \text{slope}(u, w) \in \left(\frac{h}{2}, \frac{2}{h}\right) \text{ and } 0 \in \Gamma_{u,w} \right) \leq 10^4 h^{-2} C n^{-1/3}.
\]

It follows that for infinitely many \(n\), \(\mathbb{P}(\mathcal{E}_{n,h}) \leq C(h)n^{-1/3}\) for some constant \(C(h) > 0\) completing the proof of the proposition.

\[
\square
\]

## 4 Ruling out bigeodesics in axial directions

We prove Proposition \[1.2\] in this section. That is, we show that non-trivial bigeodesics passing through \(0\) directed in axial directions almost surely do not exist. By the obvious symmetry of the problem, it suffices only to rule out vertically directed bigeodesics. We first make the following definition. A bigeodesic \(\gamma = \{v_i\}_{i \in \mathbb{Z}}\) with \(v_0 = 0\) with forward and backward limiting direction both equal to \(\infty\) is called a **finite width bigeodesic** if \(\lim_{n \to \infty} x_n - x_{-n} < \infty\) where \(v_n = (x_n, y_n)\) for \(n \in \mathbb{Z}\). A vertically upward directed semi-infinite geodesic \(\gamma = \{v_i\}_{i \in \mathbb{Z}_+}\) started from \(0\) is called an **infinite width geodesic** if \(x_n \to \infty\) as \(n \to \infty\) where \(v_n = (x_n, y_n)\). By symmetry and translation invariance, Proposition \[1.2\] is a consequence of Lemma \[4.1\] and Lemma \[4.2\] below.
Lemma 4.1. Almost surely there does not exist any non-trivial vertically directed finite width bigeodesic.

Proof. For \( a < b \in \mathbb{Z} \), we call a vertically directed finite width bigeodesic \( \gamma = \{v_i\}_{i \in \mathbb{Z}} \) (not necessarily passing through 0) an \((a,b)\) bigeodesic if \( \lim_{n \to -\infty} x_n = a \) and \( \lim_{n \to \infty} x_n = b \). Clearly it suffices to show that, for each \((a,b)\) bigeodesic, Fix \( a < b \in \mathbb{Z} \). For \( i \in \mathbb{Z} \); let \( \mathcal{C}(a;b;i) \) denote the event that there exists an \((a,b)\) bigeodesic \( \gamma \) such that \((a,i) \in \gamma \) and \((a+1,i) \in \gamma \). Clearly, by translation invariance \( \mathbb{P}(\mathcal{C}(a;b;i)) \) does not depend on \( i \). Observe also that; for almost every given realization of vertex weights; there can be at most one \((a,b)\) bigeodesic and hence \( \mathcal{C}(a,b;i) \) can hold for at most one \( i \). This implies that \( \mathbb{P}(\mathcal{C}(a,b;i)) = 0 \) for all \( i \in \mathbb{Z} \) which, in turn, implies that almost surely there does not exist any \((a,b)\) bigeodesic. Taking an union bound over all pairs \((a,b)\), we get the result. \( \square \)

Lemma 4.2. Almost surely there does not exist any semi-infinite infinite width geodesic started from 0 directed vertically upwards.

We shall need the following lemma to prove Lemma 4.2.

Lemma 4.3. There exists \( \varepsilon_0, c_0 > 0 \) and \( C_0, L_0, N_0 \in \mathbb{N} \) with \( C_0 < \varepsilon_0 L_0 \) such that for all \( n > N_0 \), \( L \in (L_0, n) \) and \( C_0 L / \varepsilon_0 \leq \varepsilon < \varepsilon_0 \), and \( M \geq 2 \varepsilon L \) and sufficiently large, the following holds. Let \( \Gamma \) denote the geodesic from 0 to \((\varepsilon n, n)\). Let \( (X_L, L) \) denote the right-most point of \( \Gamma \) on the line \( y = L \). Then we have \( \mathbb{P}(X_L \geq M) \leq e^{-c_0 \varepsilon^{-2/3} ML^{-2/3}} \) for some positive constant \( c \) not depending on \( \varepsilon, L, \) and \( M \).

Lemma 4.3 is an immediate consequence of a more general result Theorem 4.4 that controls the local transversal fluctuations of almost vertically directed geodesics. We state and prove it separately in the next subsection as this result is of independent interest. First we complete the proof of Lemma 4.2 using Lemma 4.3.

Proof of Lemma 4.2. We shall prove Lemma 4.2 by contradiction. Suppose there exists a semi-infinite geodesic as in the statement of the lemma with probability \( \delta > 0 \). Fix \( M \in \mathbb{N} \) sufficiently large. There exists \( L = L(M) > 0 \) such that with probability at least \( \delta/2 \) there exists a semi-infinite geodesic \( \gamma \) started from 0 directed vertically upwards such that there exists a point \( v \) on \( \gamma \) to the right of the point \((M,L)\) on the line \( y = L \). Now let \( \varepsilon = \frac{C_0 L}{\varepsilon_0} \) where \( C_0 \) is as in Lemma 4.3 and observe that \( \gamma \) must pass through points to the left of \((\varepsilon n, n)\) for all sufficiently large \( n \). Polymer ordering, Lemma 4.3 and choosing \( M \geq 2 C_0 \) sufficiently large so that \( 2e^{-c_0 C_0^{-2/3} M} < \delta \) \( (c_0 \) and \( C_0 \) as in Lemma 4.3) leads to a contradiction and completes the proof. \( \square \)

4.1 Transversal Fluctuation of Steep Geodesics

For any path \( \gamma \) from 0 to \((\varepsilon n, n)\), let the local transversal fluctuation of \( \gamma \) at length scale \( L \) be

\[
TF_L(\gamma) := \sup\{(x-\varepsilon L)_+ : (x,L) \in \gamma\}.
\]

We have the following result.

Theorem 4.4. There exists \( \varepsilon_0, x_0, c > 0 \) and \( C_0, L_0, N_0 \in \mathbb{N} \) with \( C_0 < \varepsilon_0 L_0 \) such that for all \( n > N_0 \), \( L \in (L_0, n) \), \( x > x_0 \) and \( C_0 L / \varepsilon_0 \leq \varepsilon < \varepsilon_0 \), we have the following: if \( \Gamma \) is the geodesic from 0 to \((\varepsilon n, n)\), then

\[
\mathbb{P}(TF_L(\Gamma) \geq x \varepsilon^{2/3} L^{2/3}) \leq e^{-cx}.
\]
Observe that in the case where $\varepsilon$ is bounded away from 0 and $\infty$, the global transversal fluctuation (i.e., $\sup_L TF_L(\Gamma)$) exponent was derived by Johansson [24], a more quantitative result was obtained in [7] (see Proposition B.6). The local transversal fluctuation in that case was obtained in [6]. Indeed, Theorem 4.4 should be compared to Theorem 3 of [6]. The integrable input used in that proof is the uniform tail estimate from [3] (Theorem B.2). The proof of Theorem 4.4 is also similar to the proof of Theorem 3 in [6], except that we use Proposition 4.6 as the input. Fix $\varepsilon > 0$ sufficiently large. Clearly if $x > m\varepsilon - 2/3n^{1/3}$ there is nothing to prove, so let us assume that $x \leq m\varepsilon - 2/3n^{1/3}$. For $j \geq 0$, let $A_j$ denote the line segment joining $(mn/2 + (x + j)\varepsilon^{2/3}n^{2/3}, n/2)$ and $(mn/2 + (x + j + 1)\varepsilon^{2/3}n^{2/3}, n/2)$. Let $A_j$ denote the event that
\[
\sup_{v \in A_j} T_{0,v} + T_{v,(mn,n)} \geq n(1 + \sqrt{m})^2 - x\varepsilon^{-1/6}n^{1/3}.
\]
Let $B$ denote the event that
\[
T_{0,(mn,n)} \leq n(1 + \sqrt{m})^2 - x\varepsilon^{-1/6}n^{1/3}.
\]

**Theorem 4.5** ([26]). There exists $\varepsilon_0 > 0, x_0 > 0$ and $N_0 \in \mathbb{N}$ such that for all $\varepsilon \in (\varepsilon_0, \varepsilon_0)$, $x \leq x_0$, and $n > N_0$ we have
\[
\mathbb{P}(\left| T_{0,((en,n)} - n(1 + \varepsilon)^2 \right| \geq x\varepsilon^{-1/6}n^{1/3}) \leq e^{-cx}
\]
for some absolute constant $c > 0$.

Using Theorem 4.5 Proposition 4.4 can be proved as in the proof of Theorem 3 in [6]. We outline the steps below. First we need one auxiliary result that controls simultaneously the distance between all pairs of points in a parallelogram of size $n \times \varepsilon^{2/3}n^{2/3}$. We need some notation. For $m, \varepsilon > 0$, $n \in \mathbb{N}$, let $U = U_{m,\varepsilon}$ denote the parallelogram whose vertices are $0$, $(\varepsilon^{2/3}n^{2/3}, 0)$, $(mn, n)$, $(mn + \varepsilon^{2/3}n^{2/3}, n)$. Let $A_U$ and $B_U$ denote the bottom and top side of $U$ respectively. We have the following proposition.

**Proposition 4.6.** There exist constants $C_0, c, x_0, \varepsilon_0 > 0$ such that for each $\varepsilon \in (\varepsilon_0, \varepsilon_0)$, $m \in (\varepsilon_0, 100\varepsilon)$, $n$ sufficiently large and $x > x_0$, we have the following:
\[
\mathbb{P}\left( \sup_{u \in A_U, v \in B_U} T_{u,v} - \mathbb{E}T_{u,v} \geq x\varepsilon^{-1/6}n^{1/3} \right) \leq e^{-cx}.
\]

The proof of this proposition is identical to the proof of Proposition B.4 (Proposition 10.5 in [7]), adapted to the exponential LPP case, except that we use Theorem 4.5 instead of Theorem B.2 as the integrable input. We shall omit the proof.

Proposition 4.6 together with Theorem 4.5 can be used to control (global) transversal fluctuation for steep geodesics. We have the following result.

**Proposition 4.7.** There exist constants $C_0, c, x_0, \varepsilon_0 > 0$ such that for each $\varepsilon \in (\varepsilon_0, \varepsilon_0)$, $m \in (\varepsilon_0, 10\varepsilon)$, $n$ sufficiently large and $x > x_0$, we have the following. Let $\Gamma$ denote the geodesic from $0$ to $(mn, n)$. Then we have
\[
\mathbb{P}(TF_{2}(\Gamma) \geq x\varepsilon^{2/3}n^{2/3}) \leq e^{-cx}.
\]

**Proof.** This proof is similar to the proof of Proposition B.6 (see Lemma 11.3 of [7]), except that we use Proposition 4.6 as the input. Fix $x > 0$ sufficiently large. Clearly if $x > m\varepsilon - 2/3n^{1/3}$ there is nothing to prove, so let us assume that $x \leq m\varepsilon - 2/3n^{1/3}$. For $j \geq 0$, let $A_j$ denote the line segment joining $(mn/2 + (x + j)\varepsilon^{2/3}n^{2/3}, n/2)$ and $(mn/2 + (x + j + 1)\varepsilon^{2/3}n^{2/3}, n/2)$. Let $A_j$ denote the event that
\[
\sup_{v \in A_j} T_{0,v} + T_{v,(mn,n)} \geq n(1 + \sqrt{m})^2 - x\varepsilon^{-1/6}n^{1/3}.
\]
Let $B$ denote the event that
\[
T_{0,(mn,n)} \leq n(1 + \sqrt{m})^2 - x\varepsilon^{-1/6}n^{1/3}.
\]
Finally let $C$ denote the event that
\[ T_0((mn,n/2)) + T_{(0.9mn,n/2),(mn,n)} \geq n(1 + \sqrt{m})^2 - x_{n/3} - 1/6. \]
Clearly,
\[ \mathbb{P}(TF_2(\Gamma) \geq x_{n/3}^2/3) \leq \mathbb{P}(C) + \mathbb{P}(B) + \sum_{j=0}^{0.4m} \mathbb{P}(A_j). \]
It is easy to see from Theorem 4.5 that $\mathbb{P}(B) + \mathbb{P}(C) \leq e^{-cx}$ for some absolute constant $c > 0$. It also follows from Theorem 4.5 that $\sup_{v \in A_j} ET_{0,v} + ET_{v,(mn,n)} \leq n(1 + \sqrt{m})^2 - c' \varepsilon x_{n/3} - 1/6$ for some constant $c' > 0$. Using Proposition 4.6 it now follows that $\mathbb{P}(A_j) \leq e^{-c(x+j)}$. This completes the proof of the proposition.

Figure 3: Theorem 4.4 shows that it is unlikely that the geodesic from $0$ to $(\varepsilon n, n)$ has a large transversal fluctuation (at the scale $\varepsilon_n^2/3 L^{2/3}$) at height $L$. To prove this one shows that it is unlikely that the best path from $0$ to $(2\varepsilon L + x\varepsilon^2/3 (2L)^{2/3}, 2L)$ via $(\varepsilon L + x\varepsilon^2/3 (2L)^{2/3}, L)$ is competitive with the geodesic from $0$ to $(2\varepsilon L + x\varepsilon^2/3 (2L)^{2/3}, 2L)$. Doing this calculation at all dyadic scales and summing over scales gives the desired result.

We can now complete the proof of Theorem 4.4. This proof is similar to the proof of Theorem 3 in [2].

**Proof of Theorem 4.4.** Without loss of generality we shall assume that $n = 2^j L$ for some $j_0 \in \mathbb{N}$. Fix $x$ sufficiently large. Fix a real number $\alpha \in (1, \sqrt{2})$. For $j \leq j_0$ let $A_j$ denote the event that $TF_{2^j L}(\Gamma) \geq x(\alpha^j x^2 L)^{2/3}$. Clearly it suffices to show that $\sum_{j \geq 1} \mathbb{P}(A_j^c \cap A_{j-1}) \leq e^{-cx}$. Now observe that, by polymer ordering, one has on $A_j \cap A_{j-1}$, the geodesic $\Gamma^*$ from 0 to $v' := (2^j L + x(\alpha^{j-1} x^2 L)^{2/3}, 2^j L)$ passes through some point to the right of $(2^j L + x(\alpha^{j-1} x^2 L)^{2/3}, 2^j L)$ on the line $y = 2^{j-1} L$, and consequently, and choosing $\alpha \in (1, \sqrt{2})$ appropriately, we have $TF_{2^{j-1} L}(\Gamma^*) \geq 0.1 x(\alpha^{j-1} x^2 L)^{2/3}$. Now we need to consider two cases. If $x(\alpha^{j-1} x^2 L)^{2/3} \leq 10x^2 L$, then Proposition 4.7 applies and we get $\mathbb{P}(A_j \cap A_{j-1}) \leq e^{-c' x^{2/3} L}$ for some $c > 0$. Now in the other case, apply Proposition 4.7 with $x$ replaced by $\varepsilon^j := (2^j L)^{-1} x(\alpha^{j-1} x^2 L)^{2/3}$; notice that this now is the slope of the line joining 0 and $v'$, up to a factor of 10. Proposition 4.7 in this case implies, given, $\varepsilon^j L$ is sufficiently large, that
\[ \mathbb{P}(A_j \cap A_{j-1}) \leq e^{-c' x(\varepsilon^j L)^{1/3}}. \]
for some constant $c' > 0$. By definition of $\epsilon'$ and the assumption that $x\alpha^{2j/3}(\epsilon 2^j L)^{2/3} \geq 10\epsilon 2^j L$, it follows that $\mathbb{P}(A_j \cap A^{c}_{j-1}) \leq e^{-cx\alpha^{2j/3}}$ in this case also, for some $c > 0$. Summing this over all $j$ gives the result.

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In this appendix, we shall indicate how to prove Lemma 3.1 by following the same line of arguments as in the proof of Corollary 2.10. Let us only focus on the case where $I$ and $J$ are parallel, for notational convenience; the other case is similar. As already explained, we need an analogue of Theorem 2 for the quadrilateral whose one pair of opposite sides is $I$ and $J$. We first state the following general result.

A Proof of Lemma 3.1
For $m > 0$, let $U = U_m$ denote the parallelogram whose vertices are $a = (0, n^{2/3})$, $b = (0, -n^{2/3})$, $(n, mn) + a$ and $(n, mn) + b$. Let $A$ and $B$ denote the left and right edge of $U$ respectively. As in the set-up of Theorem 2, for points $u, v$ on $A$ (or on $B$) we say $u < v$ if $v = u + i(0,1)$ for some $i \in \mathbb{N}$. For $\ell \in \mathbb{N}$, let $E_\ell$ denote the event that there exists $u_1 < u_2 < \cdots < u_\ell$ on $A_n$, and $v_1 < v_2 < \cdots < v_\ell$ on $B_n$, such that the geodesics $\Gamma_{u_i,v_i}$ are disjoint. We have the following analogue of Theorem 2

**Theorem A.1.** Let $\psi \in (0,1)$ be fixed. There exist constants $n_0, \ell_0 \in \mathbb{N}$ such that for all $n > n_0$ and for all $\ell_0 < \ell < n^{0.01}$ and for all $m \in (\psi, \frac{1}{\psi})$, we have

$$\mathbb{P}(E_\ell) \leq e^{-c\ell^{1/4}}$$

for some absolute constant $c = c(\psi) > 0$.

The proof of Theorem A.1 follows the same outline as the proof of Theorem 2. Let $\ell < n^{0.01}$ be sufficiently large and fixed and let $n$ be sufficiently large. Fix $m \in (\psi, \frac{1}{\psi})$ and set $R$ to be the parallelogram with opposite sides $U := \ell^{1/8}A$ and $V = (n, mn) + U$. The analogue of Lemma 2.1 holds in this setting with tails estimates uniform in $m \in (\psi, \frac{1}{\psi})$, using Proposition B.6. While considering Lemma 2.2 we encounter the only significant difference between the setting here and the setting of Theorem 2. In the setting of Theorem 2 we had $|ET_{u,v} - ET'_{u',v'}| = O(n^{1/3})$ where $u, u'$ (resp. $(v, v')$) varied over points in $A_n$ (resp. $B_n$). This is no longer true in our case, and consequently we need to change the centering.

**Lemma A.2.** For each fixed constant $c_1 > 0$, there exists $c = c(\psi) > 0$ such that for all $\ell$ and all $n$ sufficiently large we have

$$\mathbb{P}(\inf_{u \in A, v \in B} T_{u,v} - ET_{u,v} \leq -c_1\ell^{1/4}n^{1/3}) \leq e^{-c\ell^{1/4}}.$$ 

The proof as before is using Proposition B.3 which included the case of the more general slope. With these results at our disposal Theorem A.1 follows from the following analogue of Proposition 2.3

Let $G_\ell$ denote the event that there exists $u_1 < u_2 < \cdots < u_\ell$ on $A_n$, and $v_1 < v_2 < \cdots < v_\ell$ on $B_n$, and disjoint paths $\gamma_i$ joining $u_i$ and $v_i$ contained in $R$ such that $\ell(\gamma_i) - ET_{u_i,v_i} \geq 4n - c_1\ell^{1/4}n^{1/3}$.

**Proposition A.3.** In the above set-up, we have $\mathbb{P}(G_\ell) \leq e^{-c\ell^{1/4}}$ for some constant $c = c(\psi) > 0$.

Again the proof is re-doing the steps of the proof of Proposition 2.3. The only changes come via having to re-centre by the expected weight of individual polymers rather than the common re-centering as before. As in the set-up of Lemma 2.5 we shall divide the parallelogram $R$ into an $h \times 2\ell^{1/8}m$ grid of sub-parallelograms with sides parallel to $R$, let the line segments $L_{i,j}$ for $i = 0, 1, 2, \ldots, h$ and $j \in [-\ell^{1/8}m, \ell^{1/8}m] \cap \mathbb{Z}$ be defined analogous to before. Fix, as before, a sequence $J := \{j_0, j_1, j_2, \ldots, j_{h-1}, j_h\}$ taking values in $[-\ell^{1/8}m, \ell^{1/8}m] \cap \mathbb{Z}$. Let us consider the path $\gamma_j$ from $A$ to $B$ that passes through the line segments $L_{i,j_i}$ for each $i = 0, 1, \ldots, h$, and maximizes (over all such paths) $\ell(\gamma_j) - ET_{u,v}$ where $u$ and $v$ are the end points of the paths. We have the following analogue of Lemma 2.5

**Lemma A.4.** There exists $c_0 > 0$ sufficiently small and $c_1 > 0$ such that for all $h \leq \sqrt{\ell}$ and $c_0m = h^{2/3}$, we have for each $J$, as above, $\gamma_J$ as above with endpoints $u$ and $v$

$$\mathbb{P}(\ell(\gamma_J) - ET_{u,v} \geq -c_1h^{2/3}n^{1/3}) \leq e^{-c\ell^{1/2}}$$

for some $c = c(\psi) > 0$. 

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Fixing a \( J \) as above, observe that,

\[
\ell(\gamma_J) - ET_{u,v} \leq \sum_{i=0}^{h-1} \sup_{u_i \in L_{i,j_i}, v_i \in L_{i+1,j_{i+1}}} T_{u_i,v_i} - ET_{u_i,v_i} + \mathcal{L}(J)
\]

where

\[
\mathcal{L}(J) := \sup_{u_i \in L_{i,j_i}} \sum_{i=0}^{h-1} ET_{u_i,u_{i+1}} - ET_{u_0,u_h}.
\]

One can complete the proof of Lemma A.4 along the lines of the proof of Lemma 2.5 provided we establish the following two facts:

(i) There exists \( C = C(\psi) > 0 \) such that for all \( m \in (\psi, \frac{1}{\psi}) \) and for all \( J \) as above we have

\[
\mathcal{L}(J) \leq -Ch^{2/3}n^{1/3}.
\]

(ii) In the same set-up, we have for each \( J \) and for each \( i \in \{0, 1, \ldots, h-1\} \)

\[
E \left[ \sup_{u_i \in L_{i,j_i}, v_i \in L_{i+1,j_{i+1}}} T_{u_i,v_i} - ET_{u_i,v_i} \right] \leq C'(n/h)^{1/3}.
\]

where \( C' \) can be made arbitrarily small by taking \( c_0 \) sufficiently small.

To prove (i) above we simply appeal to the weak convergence result of Johansson (Theorem B.1), which together with the moderate deviation estimates (Theorem B.2) and the fact the the GUE Tracy-Widom distribution has a negative mean to conclude

\[
ET_{u_i,u_{i+1}} \leq (\sqrt{x_{i+1} - x_i} + \sqrt{y_{i+1} - y_i})^2 - C(n/h)^{1/3}
\]

for some constant \( C > 0 \). The rest is easy algebra.

The proof of (ii) is along the same lines as that of Lemma 2.4 using Proposition B.3. We omit the details.

With these two results, we can as before use a concentration for sum of independent sub-exponential random variables to complete the proof of Lemma A.4. Observe that the two basic ingredients used here are Theorem B.1 and Theorem B.2 both of which are uniform in \( m \in (\psi, \frac{1}{\psi}) \), so the we can get the constants in the statement of the lemma depending only on \( \psi \).

Once we Lemma A.4 at our disposal, the proof of Theorem A.1 can be completed as that of Theorem 2, using an analogue of Lemma 2.6 to control the entropy of the sequence \( J \). This part of the proof is almost identical to the one before, and we shall omit the details.

Now we can use Theorem A.4 to obtain an analogue of Theorem 2.8. Let \( A \) and \( B \) be as above, the left and right boundaries of the parallelogram \( U \). For \( u, u' \in A \) and \( v, v' \in B \) we say that \( (u,v) \sim (u',v') \) if \( \Gamma_{u,v} \) and \( \Gamma_{u',v'} \) are same between the lines \( y = \frac{n}{3} \) and \( y = \frac{2n}{3} \). As before \( \sim \) is an equivalence relation. Let \( M_n \) denote the number of equivalence classes.

**Theorem A.5.** There exists \( c = c(\psi) > 0 \) such that for all \( \ell < n^{0.01}, n \in \mathbb{N} \) and for all \( m \in (\psi, \frac{1}{\psi}) \) we have

\[
\mathbb{P}(M_n \geq \ell) \leq e^{-c\ell^{1/128}}.
\]
The proof is analogous to that of Theorem 2.8 using Theorem A.1 instead of Theorem 2 and an analogue of Lemma 2.9. Again the proof is almost identical and we omit the details.

Coming back to Lemma 3.1, Theorem A.5 immediately implies this lemma in the case where $I$ and $J$ are parallel vertical line segments. The same proof works when $I$ and $J$ are both horizontal line segments. In the case where $I$ is vertical and $J$ is horizontal (or vice versa), one needs an analogue of Theorem A.1 in such cases. We have the following general result to this end.

Let $a = (0, n^{2/3})$, $b = (0, -n^{2/3})$ and $A$ be the line segment joining $a$ and $b$ as before. Let $C$ denote the line segment joining $(n, mn) + (n^{2/3}, 0)$ and $(n, mn) + (-n^{2/3}, 0)$. For $u, u' \in A$ and $v, v' \in C$ we say that $(u, v) \sim (u', v')$ if $\Gamma_{u,v}$ and $\Gamma_{u',v'}$ are same between the lines $y = \frac{n}{3}$ and $y = \frac{2n}{3}$. Let $M_n'$ denote the number of equivalence classes. We have the following analogue of Theorem A.5.

**Theorem A.6.** There exists $c = c(\psi) > 0$ such that for all $\ell < n^{0.01}, n \in \mathbb{N}$ and for all $m \in (\psi, \frac{1}{\psi})$ we have
\[
\mathbb{P}(M_n' \geq \ell) \leq e^{-c\ell^{1/128}}.
\]

Let $D$ denote the line segment joining the points $c := (n, mn) + (-n^{2/3}, 0)$ and $d := c - (4n^{2/3}, 0)$. By appealing to Theorem 3 of [6] we conclude that all geodesic from $A$ to $C$ intersects $D$ with failure probability at most $e^{-c\ell^{1/128}}$ by taking $n$ sufficiently large. One can then apply Theorem A.5 to the quadrilateral $Q$ with vertical opposite sides $A$ and $D$ to conclude Theorem A.6. We omit the details. This concludes the proof of Lemma 3.1 in the case where $I$ and $J$ are not parallel.

### B Integrable inputs and their useful consequences

We shall recall here the precise statements for the integrable inputs and their consequences that we have used throughout the paper. All these results use the following two basic ingredients. First we need the Tracy-Widom convergence result from [23].

**Theorem B.1 ([23]).** For each $h > 0$ we have
\[
\frac{T_{0,(n,h)}}{h^{-1/6}n^{1/3}} \to F_{TW}
\]
as $n \to \infty$ as $n \to \infty$ where $F_{TW}$ denotes the GUE Tracy-Widom distribution.

The argument in [23] shows that the convergence is uniform in $h \in [\psi, 1/\psi]$ for any fixed $\psi > 0$. The next ingredient is the moderate deviation estimate corresponding to the above weak convergence result which is implicit in [3], as explained in [7].

**Theorem B.2 ([7 Theorem 13.2]).** There exist constants $N_0, t_0, c > 0$ depending only on $\psi \in (0, 1)$ such that we have for all $n > N_0, t > t_0$ and all $h \in (\psi, 1/\psi)$
\[
\mathbb{P}(|T_{0,(n,h)} - n(1 + \sqrt{h})^2| \geq tn^{1/3}) \leq e^{-ct}.
\]

Observe that as $h \to 0$ the above result is complemented by Theorem 1.5 ([26 Theorem 2]) which gives the moderate deviation estimates at the fluctuation scale for $h$ arbitrarily small.

The next set of results will give us the control on the maximum and minimum fluctuation of the passage times for pairs of points in an on-scale parallelogram. For $m > 0$, and $n \in \mathbb{N}$, let us denote by $U_{m, n}$ the parallelogram whose end points are $(0, 0), (n, mn), (0, n^{2/3}), (n, mn + n^{2/3})$. Let $L_U$ and $R_U$ denote the left and right side of $U$. The following two results are quoted from [7] where it
was proved for Poissonian LPP. The proof for the exponential case follows almost verbatim except that one needs to use Theorem [B.2] as the integrable input in stead of the moderate deviations for Poissonian LPP that was used there.

**Proposition B.3.** There exist constants \( c_1 > 0, n_0 > 0 \) and \( t_0 > 0 \) depending on \( \psi \) such that we have for all \( n > n_0 \) and \( t > t_0 \) and all \( m \in (\psi, 1/\psi) \)

\[
\mathbb{P} \left( \inf_{u \in L_U, v \in R_U} T_{u,v} - \mathbb{E} T_{u,v} \leq -tn^{1/3} \right) \leq e^{-c_1 t}.
\]

This is contained in Proposition 10.1 of [7].

**Proposition B.4.** There exist constants \( c_1 > 0, n_0 > 0 \) and \( t_0 > 0 \) depending on \( \psi \) such that we have for all \( n > n_0 \) and \( t > t_0 \) and all \( m \in (\psi, 1/\psi) \)

\[
\mathbb{P} \left( \sup_{u \in L_U, v \in R_U} T_{u,v} - \mathbb{E} T_{u,v} \geq tn^{1/3} \right) \leq e^{-c_1 t}.
\]

This is contained in Proposition 10.5 of [7]. Observe that the case where \( m \) is not bounded away from 0 is treated in Proposition 4.6 the proof of which is identical to that of Proposition B.4 except that we use Theorem 4.5 instead of Theorem B.2.

Proposition B.3 and Proposition B.4 are robust in the following senses. First, the same result (with the same proof) continues to hold if the height of the parallelogram is changed from \( n^{2/3} \) to \( Cn^{2/3} \) as long as \( C \) is bounded away from \( \infty \). Further, the results work for all pairs of points in \( U \) which have slopes bounded away from 0 and \( \infty \) by some fixed functions of \( \psi \). Also, the result does not require the shorter pair of sides of the parallelogram be vertical. Indeed, perhaps the more natural way of phrasing such a result is to look at an \( n \times n^{2/3} \) rectangles whose pair of shorter sides are parallel to the line \( x + y = 0 \). We have the following more general result whose proof is a minor variant of the proofs of Propositions B.3 and B.4 and will therefore be omitted.

Let \( \tilde{U} \) denote the parallelogram whose pairs of shorter sides \( A \) and \( B \) of length \( n^{2/3} \) each are aligned with the lines \( x + y = 0 \) and \( x + y = n \) respectively, and let the slope joining the midpoints of \( A \) and \( B \) be \( m \). We have the following result.

**Proposition B.5.** There exist constants \( c_1 > 0, n_0 > 0 \) and \( t_0 > 0 \) depending on \( \psi \) such that we have for all \( n > n_0 \) and \( t > t_0 \) and all \( m \in (\psi, 1/\psi) \)

\[
\mathbb{P} \left( \sup_{u \in A, v \in B} |T_{u,v} - \mathbb{E} T_{u,v}| \geq tr^{1/3} \right) \leq e^{-c_1 t}.
\]

This result was used in Section 2 to show that a path constrained to be in a thin cylinder is unlikely to be competitive with the unconstrained best path between two points.

Finally we also need the following transversal fluctuation estimate. Let \( TF(s,n,m) \) denote the event that the geodesic between \((0,0)\) and \((n,mn)\) exits a strip of width \( sn^{2/3} \) around the straight line joining the end points. The following result was established in [7].

**Proposition B.6.** There exists \( c > 0, n_0 > 0 \) and \( s_0 > 0 \) depending on \( \psi \) such that we have for all \( n > n_0 \) and \( s > s_0 \) and all \( m \in (\psi, 1/\psi) \)

\[
\mathbb{P}(TF(s,n,m)) \leq e^{-cs^2}.
\]
For the case $m = 1$, this is Theorem 11.1 of [7], and a more local version was proved in Theorem 3 of [6]. The general case, with the same proof was stated in Corollary 11.7 of [7]. Although the statements of Theorem 11.1 and Corollary 11.7 of [7] has an upper bound of $e^{-cs}$, the proof gives an upper bound of $e^{-cs^2}$. As an aside, the optimal exponent here is $s^3$; see Remark 1.6 in [6]. Observe that the analogue of Proposition B.6 for the case when $m$ becomes arbitrarily small is established in Proposition 4.7.

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