Abstract. We study metric projections onto cones in the Wasserstein space of probability measures, defined by stochastic orders. Dualities for backward and forward projections are established under general conditions. Dual optimal solutions and their characterizations require study on a case-by-case basis. Particular attention is given to convex order and subharmonic order. While backward and forward cones possess distinct geometric properties, strong connections between backward and forward projections can be obtained in the convex order case. Compared with convex order, the study of subharmonic order is subtler. In all cases, Brenier–Strassen type polar factorization theorems are proved, thus providing a full picture of the decomposition of optimal couplings between probability measures given by deterministic contractions (resp. expansions) and stochastic couplings. Our results extend to the forward convex order case the decomposition obtained by Gozlan and Juillet, which builds a connection with Caffarelli’s contraction theorem. A further noteworthy addition to the early results is the decomposition in the subharmonic order case where the optimal mappings are characterized by volume distortion properties. To our knowledge, this is the first time in this occasion such results are available in the literature.

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1. Introduction

Stochastic ordering of distributions is ubiquitous in probability and statistics. Depending on its context of application, the order of distributions are determined according to their behavior under a given group $\mathcal{A}$ of test functions. Two probability measures $\mu, \nu$ are called increasing in a stochastic order defined by $\mathcal{A}$ if they satisfy

$$\int \varphi d\mu \leq \int \varphi d\nu \text{ for all } \varphi \in \mathcal{A}.$$ 

Convex order and subharmonic order are two frequently used stochastic orders which corresponds respectively to $\mathcal{A}$ being the set of convex functions and subharmonic functions. The larger the test set $\mathcal{A}$ is, the stronger the stochastic order becomes. So subharmonic order is more restrictive than convex order. There are also many other widely used stochastic orders, e.g. in one dimension, an increasing concave order is defined by increasing concave functions [28]. Stochastic ordering of high dimensional distributions can also be defined according to the orderings of their one dimensional projections [16]. Stochastic ordering of multiple distributions is defined similarly [17]. We refer to [35] for a full account of various stochastic orders and their applications in operations research and economics etc.

Stochastic order is a functional way to characterize the properties of couplings between a pair of probability measures. Strassen theorem shows that convex order relationship is equivalent to the existence of a martingale coupling [36]. This is generalized to subharmonic order which is proved to be necessary and sufficient for the existence of a Brownian martingale [20]. Generalizations to other stochastic orders are also considered [10]. Strassen theorem is the starting point of many recent studies on optimal martingale transport and its applications in mathematical finance [8,19], Skorokhod embedding and related topics [7,21,22].

Despite its wide applications, stochastic ordering is usually hard to implement in practice. For one thing, sampling probability measures via naive simulation is costly. For another, discretizing probabilities in a given stochastic order is tricky, since stochastic order relation is usually unstable, i.e. discretized measures could easily violate the original stochastic order. This has been observed on the level of convex order. A large amount of research has been devoted to the stability issues of optimal martingale transport. Under some conditions in one dimension, it is stable [6,26,29], but not in high dimensions [12]. These issues have become a great hindrance to the numerical pursuits of stochastic orders.

As a general tool of sampling probability measures in stochastic order, we propose to study Wasserstein projections onto the cones defined by a given stochastic order. One such projection for convex order was employed by Gozlan and Juillet [24] to obtain a martingale version of Brenier’s polar factorization [11]. Note that Brenier’s motivation for the investigation of polar factorization was to address the instability issues in the numerical study of perfect incompressible fluids. We intend to report numerical benefits offered by Wasserstein projections in a separate article. In the current article, we instead build the necessary mathematical framework required of downstream applications, and demonstrate its uses in exploring the properties of optimal mappings between probability measures.
For any probability measures $\mu$, $\nu$ and cost function $c$, we define the Wasserstein transport cost as

$$T_c(\mu, \nu) = \inf_{\pi \in \Pi(\mu, \nu)} \int_{X \times Y} c(x, y) d\pi(x, y).$$

We also write

$$T_k(\mu, \nu) = T_c(\mu, \nu) \text{ if } c(x, y) = |x - y|^k, \ k \geq 1.$$  

and denote

$$W_k(\mu, \nu) = (T_k(\mu, \nu))^{1/k}.$$  

Given a pair of probability measures $\mu$, $\nu$ and a stochastic order determined by a function class $A$, we study projections onto backward cone with vertex $\nu$ and forward cone with vertex $\mu$. Specifically, the backward cone $P^A_{\leq \nu}$ is the set of probability measures less than $\nu$ w.r.t. the stochastic order defined by $A$. The forward cone $P^A_{\mu \leq}$ is the set of probability measures greater than $\mu$ w.r.t. the stochastic order defined by $A$. The projection problems we have in mind is defined w.r.t. the Wasserstein transport cost $T_c$ for a given cost function $c(x, y)$:

(1.1) (backward projection) $\inf_{\bar{\mu} \in P^A_{\leq \nu}} T_c(\mu, \bar{\mu}),$

(1.2) (forward projection) $\inf_{\bar{\nu} \in P^A_{\mu \leq}} T_c(\bar{\nu}, \nu).$

The word \textit{backward} emphasizes the fact that the projection we look for locates in the direction "backward in time axis" relative to the vertex $\nu$ of the cone. Similarly, the word \textit{forward} emphasizes the fact that the projection we look for locates in the direction "forward in time axis" relative to the vertex $\mu$ of the cone. These are illustrated in Figure 1.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure1.png}
\caption{Illustration of backward and forward projections onto cones defined by stochastic order $A$. Dotted line indicates the direction of increasing stochastic order.}
\end{figure}

We first present the dual theorems for backward and forward projections.
Theorem 1. Let $X, Y$ be locally compact polish spaces. Given $\mu \in \mathcal{P}(X), \nu \in \mathcal{P}(Y)$, a cost function $c(x,y)$ and a defining function class $\mathcal{A}$ associated with a stochastic order, the following dualities hold under appropriate conditions.

(1) backward duality (Theorem 4.3):

\[
\inf_{\bar{\mu} \in \mathcal{P}_c A} \mathcal{T}_c(\mu, \bar{\mu}) = \sup_{(u, \varphi)} \left\{ \int_X ud\mu - \int_Y \varphi d\nu \right\},
\]

where $u(x) - \varphi(y) \leq c(x,y)$ with $u$ continuous and $\varphi \in \mathcal{A}$.

(2) forward duality (Theorem 4.4):

\[
\inf_{\bar{\nu} \in \mathcal{P}_c A} \mathcal{T}_c(\bar{\nu}, \nu) = \sup_{(\varphi, v)} \left\{ \int_X \varphi d\mu - \int_Y vd\nu \right\},
\]

where $\varphi(x) - v(y) \leq c(x,y)$ with $\varphi \in \mathcal{A}$ and $v$ continuous.

The duality theorems for Wasserstein projections given by Theorem 1 include as a special case the classical Kantorovich duality. This happens when the stochastic order becomes degenerate, see Remark 4.9. Another special case is the duality for backward convex order projection, this is previously proved by first establishing the equivalence between backward convex order projection and the weak optimal transport introduced in [25], and then using the duality theorem for the weak optimal transport. In one dimension, the equivalence is proved in [24], then generalized to higher dimensions [34] under the condition that $\mu$ has a density w.r.t. the Lebesgue measure. The general case is proved in [23] and [1]. The duality for weak optimal transport is proved in [25] and [4] via different approaches. In the compact case it is also proved in [2] without using measurable selection theorems.

It is worth noting that in proving the backward duality (1.3) in Theorem 1, we do not rely on weak optimal transport or other indirect reformulation via the method described in the last paragraph. The approach taken here is direct and general enough to handle all linear stochastic orders (Definition 2.1) at the same time. The way we handle stochastic orders is more in line with the perspective of Strassen theorem and has not been studied in the current context.

In appearance, the backward duality (1.3) and the forward duality (1.4) look so familiar that people might unconsciously fall under the illusion that backward and forward projections are conjugate or even equivalent to each other. But actually their relationship depends critically on the defining class $\mathcal{A}$. Except in some special situations, there is no immediate connection between backward and forward projection for general stochastic orders, this will be explained in detail in section 8. In fact, in the most promising case, i.e. the convex order case, the difference is already prominent in numerical computation [1] and it is observed that, compared with the forward convex order projection, the backward convex order projection is easier to manage due to its natural connection with the weak optimal transport. Such a connection is not available for forward convex order projection.

The defining class $\mathcal{A}$ determines the admissible functions of the dual, thus gives properties of the projections specific to $\mathcal{A}$. We now focus on the convex order and the subharmonic order where we obtain interesting geometric properties of the optimal mappings such as contraction and expansion. We summarize the main results about convex order projections as below, notations are defined later.
Theorem 2. Let $\mu, \nu \in P_2(\mathbb{R}^d)$, $c(x,y) = |x-y|^2$. Denote by $T_2(\mu, \mathcal{P}^{ce}_{\leq \nu}), D_2(\mu, \mathcal{P}^{ce}_{\leq \nu})$ the optimal primal, dual values of the backward convex order projection, and respectively $T_2\left(\mathcal{P}^{ce}_{\leq \mu}, \nu\right), D_2\left(\mathcal{P}^{ce}_{\leq \mu}, \nu\right)$ of the forward convex order projection. Then under appropriate conditions we have the following.

1) The duality for backward convex order projection,
$$T_2(\mu, \mathcal{P}^{ce}_{\leq \nu}) = D_2(\mu, \mathcal{P}^{ce}_{\leq \nu}),$$
and the duality for forward convex order projection,
$$T_2\left(\mathcal{P}^{ce}_{\leq \mu}, \nu\right) = D_2\left(\mathcal{P}^{ce}_{\leq \mu}, \nu\right).$$

2) The optimal dual value for backward convex order projection is attained (Theorem 6.1), and the optimal mapping from $\mu$ to the unique projection is characterized by convex contraction (Definition 7.1 and Theorem 7.4).

3) The optimal dual value for forward convex order projection is attained (Theorem 6.2), and the optimal mapping from $\nu$ to the unique projection is characterized by convex expansion (Definition 7.2 and Theorem 7.6).

4) The optimal mappings for backward and forward convex order projections are inverse to each other (Theorem 8.3 and Corollary 8.5).

Item (1) is an instance of Theorem 1 enriched with desirable traits for convex order projections (see Theorem 5.3). Dual attainment in item (2) and (3) are key results. Item (2) is originally given by Gozlan and Juillet [23]. This gives the backward decomposition: given two probability measures, there is a transport plan between them given by the gradient map of a convex contraction, followed by a martingale coupling. This establishes a link with the celebrated Caffarelli’s contraction theorem [15] (see also [30] and [18]): if $\nu$ is a log-concave perturbation of the Gaussian measure $\mu$, then the optimal transport map from $\mu$ to $\nu$ is given by the gradient of a convex function which is a contraction. In the language of item (2), the optimal map is a contraction when the projection $\bar{\mu}$ to the backward cone is equal to $\nu$. Item (3) is novel and it reinforces the link with Caffarelli’s contraction theorem by showing a forward decomposition: given two probability measures, there is a transport plan between them given by a martingale coupling followed by an expansion map which is the gradient of a convex function. Item (3) is a natural companion to item (2); however, it is not a straightforward result, since the backward and forward convex order cone have distinct geometric properties, for example, one is geodesically convex in the Wasserstein space $P_2(\mathbb{R}^d)$ while the other is not; see section 8. The properties of backward and forward mapping and their relation in item (4) are remarkable, however it seems to be unique to the convex order case. In one dimension, these properties are obtained by [1, 5] via methods specific to one dimension.

Theorem 2 immediately prompts one to ask whether similar results hold for other stochastic orders, whether the projections can be characterized in some special way, in particular giving a link to Caffarelli’s contraction type results. We are able to obtain a result similar to Theorem 2 for subharmonic order. Notice that subharmonic order is stronger than the convex order, therefore the corresponding cones become much smaller, for example, in $\mathbb{R}^d$ ($d \geq 2$), there is no subharmonic order between discrete measures. This makes us expect weaker properties for the projection mappings than in the convex order case. These weaker properties are
also natural in view of Theorem 1 since the class $\mathcal{A}$ consists of subharmonic functions that are less special than convex functions. Indeed, in the following theorem the characterizations of the projection mappings are given by what we call Laplacian contraction and expansion, which resembles a linearized version of the convex contraction and expansion.

**Theorem 3.** Given $\mu, \nu$ supported in a bounded smooth domain, $c(x,y) = |x - y|^2$. Denote by $T_2(\mu, \mathcal{P}_{sh}^{\leq \nu})$, $D_2(\mu, \mathcal{P}_{sh}^{\leq \nu})$ the optimal primal, dual values of the backward subharmonic order projection, and respectively $T_2(\mathcal{P}_{sh}^{\mu \leq \nu}, \nu)$, $D_2(\mathcal{P}_{sh}^{\mu \leq \nu}, \nu)$ of the forward subharmonic order projection. Then under appropriate conditions we have the following.

1. The duality for backward subharmonic order projection,
   \[ T_2(\mu, \mathcal{P}_{sh}^{\leq \nu}) = D_2(\mu, \mathcal{P}_{sh}^{\leq \nu}), \]
   and the forward subharmonic order projection,
   \[ T_2(\mathcal{P}_{sh}^{\mu \leq \nu}, \nu) = D_2(\mathcal{P}_{sh}^{\mu \leq \nu}, \nu). \]

2. The optimal dual value for backward subharmonic order projection is attained (Theorem 10.1), and the optimal mapping from $\mu$ to the unique projection is characterized by Laplacian contraction (Definition 11.1 and Theorem 11.5).

3. The optimal dual value for forward subharmonic order projection is attained (Theorem 10.5), and the optimal mapping from $\nu$ to the unique projection is characterized by Laplacian expansion (Definition 11.2 and Theorem 11.7). Moreover, the forward projection mapping, say $\nabla \bar{\psi}^*_0$, is a volume increasing map. In particular, if $\nu$ is absolutely continuous then the forward projection $\bar{\nu} = (\nabla \bar{\psi}^*_0)_# \nu$ is also absolutely continuous, and their densities (with the same notation) satisfy
   \[ \bar{\nu}(\nabla \bar{\psi}^*_0(x)) \leq \nu(x), \text{ a.e. } x. \]

This theorem seems to be the first occasion in the literature where a connection is made between contraction type properties of optimal mappings and the subharmonic order (thus with Brownian martingales). The volume increasing property of the forward projection mapping in item (3) of Theorem 3 is remarkable, and it gives a weaker counterpart to the convex order case. It seems a similar property is not available for the backward subharmonic order projection. Item (2) and (3) show that optimal couplings between two probability measures are composed by Laplacian contraction or expansion and a Brownian martingale transport. They also raise a natural question: when would the projection $\bar{\mu}$ of $\mu$ onto $\mathcal{P}_{sh}^{\mu \leq \nu}$ be equal to $\nu$, and respectively the projection $\bar{\nu}$ of $\nu$ onto $\mathcal{P}_{sh}^{\mu \leq \nu}$ be equal to $\mu$? Answering these questions may give Caffarelli’s contraction type results for a large class of measures beyond the known cases.

The dual attainment is usually nontrivial, there is no one-size-fits-all approach to it. The situation for subharmonic order is subtler than convex order. Convex functions enjoy many vital properties under $c$-transforms. But almost all these properties break in the case of subharmonic order, this has led to a series of difficulties both in the proof of attainment and characterization of optimal mappings. Moreover, the crucial double convexification trick is unfortunately unavailable for subharmonic order projections. We will discuss more in section 10.

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**Notation:**

$X$: polish space. With slight abuse of notation, metric on this space is written as $\| \cdot \|_X$, i.e., the distance from a reference point which we do not explicitly specify. If $X$ a (separable) Banach space, then $\| \cdot \|_X$ is the norm on $X$. When no confusion arises, we write $\| \cdot \|$ for simplicity.

$M(X)$, $M_+(X)$: (nonnegative) finite Radon measures on $X$.  
$M_k(X)$, $M_{k,+}(X)$: (nonnegative) measures with finite $k$-th moment. $M_0(X) = M(X)$, $M_{0,+}(X) = M_+(X)$.

$P(X)$, $P^{ac}(X)$: probability measures on $X$ (which are absolutely continuous w.r.t. the reference measure of $X$).

$P_k(X)$, $P^{ac}_k(X)$: probability measures in $P(X)$, $P^{ac}(X)$ with finite $k$-th moment.

$P_0(X) = P(X)$, $P^{ac}_0(X) = P^{ac}(X)$.

$\Pi(\mu, \nu)$: the set of probability couplings with marginals $\mu$ and $\nu$.

$A^*$: the defining class of a stochastic order, $A_{cx}$ stands for the defining class of convex order and $A_{sh}$ for subharmonic order.

$\mu \leq_A \nu$: $\mu$ is smaller than $\nu$ in the stochastic order defined by $A$.

$\mathcal{P}^{A}_{k, \leq \nu}$: the backward cone,

$$\mathcal{P}^{A}_{k, \leq \nu} = \{ \eta \in P_k(Y) : \eta \leq_A \nu \}.$$  

the subscripts $k$ are omitted and written $\mathcal{P}_{\leq \nu}$ if the underlying space is bounded.

$\mathcal{P}^{A}_{k, \mu}$: the forward cone,

$$\mathcal{P}^{A}_{k, \mu} = \{ \xi \in P_k(X) : \mu \leq_A \xi \}.$$  

the subscripts $k$ are omitted and written $\mathcal{P}^{A}_{\mu}$ if the underlying space is bounded.

$C^0_\infty(X)$: the set of smooth functions with compact support in $X$.

$C_0(X)$: continuous functions which go to zero at infinity.

$C_b(X)$: bounded continuous functions.

$\mathcal{P}_{b,k}(X)$: bounded measurable functions.

$C_{b,k}(X) : k \geq 0$, continuous functions with growth no more than $\|x\|^k$.

$C_{0,k}(X) : k \geq 0$, continuous functions with asymptotic order $o(\|x\|^k)$.

$\mathcal{P}_{b,k}(X) : k \geq 0$, measurable functions with growth no more than $\|x\|^k$.

$g^*$: the Legendre-Fenchel dual of a function $g$.

$g_e$: the subharmonic envelope of a function $g$.

2. **Linear stochastic order**

The stochastic orders we are interested in are those which are characterized by a class of admissible functions. We call them linear stochastic orders. The term linear is used to emphasize the sort of problems our method can be applied to.

**Definition 2.1.** Let $A$ be a nonempty class of measurable functions which form a convex cone:

1. if $f \in A$, then $af \in A$, $\forall a \geq 0$.
2. if $f, g \in A$, then $af + (1 - a)g \in A$, $\forall 0 \leq a \leq 1$.  


A measure $\mu \in M_+(X)$ is smaller than $\nu \in M_+(X)$ in the linear stochastic order defined by $\mathcal{A}$, denoted by $\mu \trianglelefteq_{\mathcal{A}} \nu$, if $\mu, \nu$ have equal mass and

$$\int f d\mu \leq \int f d\nu$$

for all $f \in \mathcal{A}$ such that both integrals exist in the extended sense. The class $\mathcal{A}$ is called the defining class of the associated stochastic order.

Note by definition $0 \in \mathcal{A}$. Hereafter, with slight abuse of notation, we will use a defining class $\mathcal{A}$ to mean both the function class itself and the stochastic order associated with it. We simply call $\mathcal{A}$ a (linear) stochastic order. Here are a few examples of linear stochastic orders.

**Example 2.2.** On the real line, the usual stochastic order is defined by the set of all increasing functions. (Increasing) convex order corresponds to the set of all (increasing) convex functions. These can also be generalized to consider the so-called $m$-Convex order with $m \geq 1$ being an integer, defined by the set of functions whose $m$-th derivative is nonnegative.

**Example 2.3.** In arbitrary dimension, two commonly encountered stochastic orders are convex order and subharmonic order, which are respectively defined by the set of all lower semicontinuous convex functions and the set of all subharmonic functions. The two notions coincide in one dimension.

**Example 2.4.** When the defining class $\mathcal{A}$ is the set of all bounded continuous functions, the associated stochastic order becomes degenerate. In this special case, we call it a trivial order. Measures in trivial order are identical.

In these examples, the set of admissible functions for which (2.1) is equality contains nontrivial elements. For $m$-Convex order, (2.1) is equality for all polynomials with degree no greater than $m - 1$. For convex order, (2.1) is equality for all for linear functions. For subhamarmonic order, (2.1) becomes equality for all for harmonic functions. These could be useful for performing normalizations on the admissible class.

The following representation of linear stochastic order is straightforward.

**Lemma 2.5.** Let $\mathcal{A}$ be a defining function class as defined in Definition 2.1. Assume that $\{-1, 1\} \subset \mathcal{A}$. Let $\mu, \nu \in M_+(X)$. Then

$$\sup_{f \in \mathcal{A}} \left\{ \int_X f d\mu - \int_X f d\nu \right\} = \begin{cases} 0, & \mu \trianglelefteq_{\mathcal{A}} \nu, \\ \infty, & \text{otherwise}. \end{cases}$$

**Remark 2.6.** The assumption $\{-1, 1\} \subset \mathcal{A}$ ensures that

$$\mu \text{ and } \nu \text{ are of equal mass.}$$

In some situations, the constraint (2.3) can be implied from other accompanying constraints. If this is the case, then we a priori know that the measures have identical mass, thus the assumption $\{-1, 1\} \subset \mathcal{A}$ in Lemma 2.5 can be omitted.

### 2.1. Backward and forward projection

We investigate two types of Wasserstein projections associated with a given linear stochastic order $\mathcal{A}$.

#### Backward projection

Given $\mu \in P(X), \nu \in P(Y)$. Define the backward convex cone

$$\mathcal{P}_{\trianglelefteq_{\mathcal{A}}} = \{ \eta \in P(Y) : \eta \trianglelefteq_{\mathcal{A}} \nu \}$$
consisting of measures in \( P(Y) \) which are smaller than \( \nu \) in the linear stochastic order \( \mathcal{A} \). The backward projection of \( \mu \) onto the cone \( \mathcal{P}_\mathcal{A}^\mu \) is defined as any \( \bar{\mu} \in \mathcal{P}_\mathcal{A}^\mu \) which attains
\[
\mathcal{T}_c(\mu, \mathcal{P}_\mathcal{A}^\mu) \triangleq \inf_{\bar{\mu} \in \mathcal{P}_\mathcal{A}^\mu} \mathcal{T}_c(\mu, \bar{\mu}),
\]
i.e., the transportation cost between \( \mu \) and \( \mathcal{P}_\mathcal{A}^\mu \).

**Forward projection**

The forward convex cone, denoted by
\[
\mathcal{P}_\mathcal{A}^\mu = \{ \xi \in P(X) : \mu \leq_\mathcal{A} \xi \},
\]
is the set of measures in \( P(X) \) which are greater than \( \mu \) in the linear stochastic order \( \mathcal{A} \). The forward projection of \( \nu \) onto \( \mathcal{P}_\mathcal{A}^\mu \) is any \( \bar{\nu} \in \mathcal{P}_\mathcal{A}^\mu \) which attains
\[
\mathcal{T}_c(\mathcal{P}_\mathcal{A}^\mu, \nu) \triangleq \inf_{\bar{\nu} \in \mathcal{P}_\mathcal{A}^\mu} \mathcal{T}_c(\bar{\nu}, \nu),
\]
i.e., the transportation cost between \( \mathcal{P}_\mathcal{A}^\mu \) and \( \nu \).

### 2.2. Compact vs non-compact case.

When we mention the projection problems, we use the term *compact case* to mean the underlying spaces \( X \) and \( Y \) are compact, not just that the measures \( \mu, \nu \) have compact supports. Correspondingly, the term *non-compact case* (or *general case*) means \( X \) and \( Y \) are not necessarily compact.

It is important to note that, the set \( \mathcal{P}_\mathcal{A}^\mu \) given by (2.5) is a subset of \( P(X) \). All measures in \( \mathcal{P}_\mathcal{A}^\mu \) live in \( X \). Therefore, forward projection where \( X \) is a proper subset of the underlying space and forward projection where \( X \) equal the underlying space are different. Take \( \mathbb{R}^d \), if \( X \) is only a proper subset of \( \mathbb{R}^d \), then \( \mathcal{T}_c(\mathcal{P}_\mathcal{A}^\mu, \nu) \) optimizes over all admissible measures sitting in \( X \). Measures not in \( X \) are not admissible to the optimization. If \( X \) equals \( \mathbb{R}^d \), then \( \mathcal{T}_c(\mathcal{P}_\mathcal{A}^\mu, \nu) \) optimizes over all admissible measures without restrictions on where they locate. So, it is preferable to be aware of this distinction when dealing with forward projection. This distinction for backward projection does not exist however, see e.g. Lemma 8.1.

### 3. Duality the compact case

The rigorous proof of the duality in the general case, where the underlying space is not necessarily compact, is delicate and requires additional preparations. Compared with the general case, the compact case is less restrictive in the assumptions and requires minimal preparations in the proof. So we first deal with the compact case in this section. The general case will be the topic of the next section. Starting with the compact case also helps the reader grasp the main idea more easily.

#### 3.1. Backward projection.

The following simple lemma is very useful.

**Lemma 3.1.** Let \( X, Y \) be Polish spaces and \( \mu \in P(X) \). Then, for any \( \xi \in M_+(Y) \), \( \pi \in M_+(X \times Y) \),
\[
\sup_{(u,v)} \left\{ \int u d\mu - \int v \xi - \int (u(x) - v(y)) d\pi(x,y) \right\} = \begin{cases} 0, & \xi \in P(Y), \pi \in \Pi(\mu, \xi), \\ \infty, & \text{otherwise,} \end{cases}
\]
where the supremum runs over \( (u,v) \in C_b(X) \times C_b(Y) \).
We define the class of bounded measurable functions on $X$,
\[ \mathcal{B}(X) = \{ u \text{ bounded, measurable} \} . \]

**Theorem 3.2.** Let $X, Y$ be compact Polish spaces, $\mu \in P(X)$, $\nu \in P(Y)$, $c : X \times Y \mapsto [0, \infty]$ be lower semicontinuous and $A$ be a defining function class as defined in Definition 2.1. Assume that $A$ and $A \cap \mathcal{B}$ define the same stochastic order.

(i) The backward duality holds
\[ T_c(\mu, \mathcal{P}_{\mu}^A) = \sup_{(u, \varphi) \in \mathcal{V}_c^\ast \cap \mathcal{C}_b} \left\{ \int_X u d\mu - \int_Y \varphi d\nu \right\}, \]
where $\mathcal{V}_c^\ast$ is the set of measurable functions $(u, \varphi)$ such that
\[ \varphi \in A \text{ and } u(x) - \varphi(y) \leq c(x, y), \forall x, y. \]

(ii) The alternative form of backward duality holds
\[ T_c(\mu, \mathcal{P}_{\mu}^A) = \sup_{\varphi \in A \cap \mathcal{C}_b} \left\{ \int_X Q_c(\varphi) d\mu - \int_Y \varphi d\nu \right\}, \]
where
\[ Q_c(\varphi)(x) = \inf_{y \in Y} \{ \varphi(y) + c(x, y) \}. \]

In both forms (i) and (ii), $\varphi \in A \cap \mathcal{C}_b$ can be relaxed to $\varphi \in A \cap \mathcal{B}$.

**Remark 3.3.** Here we loosely write $\mathcal{V}_c^\ast \cap \mathcal{C}_b$ to mean couples $(u, \varphi)$ such that
\[ (3.1) \quad \varphi \in A \text{ and } u(x) - \varphi(y) \leq c(x, y), \forall x, y. \]
and
\[ (3.2) \quad Q_c(\varphi)(x) = \inf_{y \in Y} \{ \varphi(y) + c(x, y) \}. \]

The duality remains true if (3.4) is replaced with
\[ (3.6) \quad u(x) - v(y) \leq c(x, y), \mu\text{-a.e. } x, \forall y. \]

Upon obtaining this duality, we then conclude the proof of the theorem in step 3.

1. To prove (3.3), we first show that
\[ \sup_{\varphi \in \mathcal{V}_c \cap (L^1 \times \mathcal{B})} \left\{ \int_X u d\mu - \int_Y \varphi d\nu \right\} \leq \sup_{\varphi \in (L^1 \times \mathcal{B})} \left\{ \int_X u d\mu - \int_Y \varphi d\nu \right\} \leq T_c(\mu, \mathcal{P}_{\mu}^A). \]
Only the second inequality needs explanation. For $\tilde{\mu} \in \mathcal{P}_{\mu}^A$, $\pi \in \Pi(\mu, \tilde{\mu})$, $(u, v, \varphi) \in \mathcal{V}_c \cap \{ L^1 \times \mathcal{B} \}$, we have $\tilde{\mu} \in P(Y)$ and $v, \varphi$ are bounded, hence $\int v d\tilde{\mu}$, $\int \varphi d\tilde{\mu}$
exist and are finite. Then we can write
\[
\int_X ud\mu - \int_Y \varphi dv = \int_X ud\mu - \int_Y vd\tilde{\mu} + \int_Y \varphi d\tilde{\mu} - \int_Y \varphi dv \\
\leq \int_X ud\mu - \int_Y vd\tilde{\mu} + \int_Y \varphi d\tilde{\mu} - \int_Y \varphi dv \\
\leq \int_X ud\mu - \int_Y vd\tilde{\mu} \\
\leq \int_{X \times Y} c(x, y)d\pi
\]
The first inequality uses (3.5). The second inequality uses the fact \(\varphi \in (A \cap S_\beta)\) and the stochastic order relationship. The third inequality uses (3.4). Now taking infimum over \(\tilde{\mu} \in \mathcal{P}_{A \leq \nu}\), \(\pi \in \Pi(\mu, \tilde{\mu})\) and supremum over \((u, v, \varphi) \in \mathcal{V}_c \cap (L^1 \times S_\beta \times S_\beta)\), we obtain
\[
\sup_{\mathcal{V}_c \cap (L^1 \times S_\beta \times S_\beta)} \left\{ \int_X ud\mu - \int_Y \varphi dv \right\} \leq T_c(\mu, \mathcal{P}_{A \leq \nu}).
\]
Clearly we get the same result if (3.4) is replaced with (3.6), since (3.6) implies
\[
u(x) - v(y) \leq c(x, y), \ \pi\text{-a.e. } (x, y),
\]
for any \(\tilde{\mu} \in \mathcal{P}_{A \leq \nu}, \ \pi \in \Pi(\mu, \tilde{\mu})\).

2. Next we prove the duality
\[
(3.7) \quad T_c(\mu, \mathcal{P}_{A \leq \nu}) = \sup_{(u, v, \varphi) \in \mathcal{V}_c \cap C_b} \left\{ \int_X ud\mu - \int_Y \varphi dv \right\}.
\]
For this, we introduce the functionals
\[
\mathcal{G} : p \in C_b(X \times Y) \mapsto \begin{cases} 0, & p(x, y) \geq -c(x, y), \\ \infty, & \text{otherwise.} \end{cases}
\]
\[
\mathcal{H} : q \in C_b(Y) \mapsto \begin{cases} 0, & q(y) \geq 0, \\ \infty, & \text{otherwise.} \end{cases}
\]
\[
\mathcal{I} : (p, q) \in C_b(X \times Y) \times C_b(Y) \mapsto \begin{cases} \int_Y qdv - \int_X pd\mu, & p(x, y) = v(y) - u(x) \text{ for some } (u, v) \in C_b, \\ \infty, & q(y) = \varphi(y) - v(y) \text{ for some } \varphi \in \mathcal{A} \cap C_b, \\ \infty, & \text{otherwise.} \end{cases}
\]
Note \(\mathcal{I}\) is convex in view of the definition of \(\mathcal{A}\). \(\mathcal{I}\) is well-defined, indeed, suppose
\[
p(x, y) = v_1(y) - u_1(x) = v_2(y) - u_2(x), \ \forall x, y,
\]
\[
q(y) = \varphi_1(y) - v_1(y) = \varphi_2(y) - v_2(y), \ \forall x, y,
\]
then, for some constant \(a \in \mathbb{R}\),
\[
u_1 = u_2 - a, \ v_1 = v_2 - a, \ \varphi_1 = \varphi_2 - a.
\]
Hence
\[
\int_Y \varphi_1(y)dv(y) - \int_X u_1(x)d\mu(x) = \int_Y \varphi_2(y)dv(y) - \int_X u_2(x)d\mu(x),
\]
showing that the definition of $I$ does not depend on the way $p(x, y)$ and $q(x)$ are split. Straightforward calculations yield the Legendre transforms of the above functionals,

$$
G^*(-\pi) = \sup_{p \in C_b(X \times Y)} \left\{ -\int_{X \times Y} p(x, y) d\pi(x, y) \right\}
= \left\{ \begin{array}{ll}
\int_{X \times Y} c(x, y) d\pi(x, y), & \pi \in M+(X \times Y), \\
\infty, & \text{otherwise}.
\end{array} \right.
\]

$$

$$
H^*(-\bar{\mu}) = \sup_{q \in C_b(Y), q \geq 0} \left\{ -\int_Y q(y) d\bar{\mu}(y) \right\}
= \left\{ \begin{array}{ll}
0, & \bar{\mu} \in M+(Y), \\
\infty, & \text{otherwise}.
\end{array} \right.
\]

$$

$$
I^*(\pi, \bar{\mu}) = \sup_{(u, v) \in C_b} \left\{ \int_{X \times Y} (v(y) - u(x)) d\pi(x, y) + \int_X u(x) d\mu(x) - \int_Y \varphi(y) d\nu(y) \right\}
+ \int_Y \varphi(y) d\bar{\mu}(y) - \int_Y v(y) d\bar{\mu}(y)
= \sup_{(u, v) \in C_b} \left\{ \int_{X \times Y} (v(y) - u(x)) d\pi(x, y) + \int_X u(x) d\mu(x) - \int_Y v(y) d\bar{\mu}(y) \right\}
+ \int_Y \varphi(y) d\bar{\mu}(y) - \int_Y \varphi(y) d\nu(y)
= \left\{ \begin{array}{ll}
0, & \bar{\mu} \in \mathcal{P}^A_{\leq \nu}, \pi \in \Pi(\mu, \bar{\mu}), \\
\infty, & \text{otherwise}.
\end{array} \right.
\]

In the calculation of $G^*(-\pi)$, the assumption on the cost function $c$ is used. The last equality in $I^*(\pi, \bar{\mu})$ uses Lemma 3.1, Lemma 2.5 and the assumption that $A$ and $A \cap C_b$ define the same stochastic order. Let

$$
\Theta(p, q) = G(p) + H(q), \quad \Xi(p, q) = I(p, q).
\]

Since $0 \in A \cap C_b$, we can set $\varphi(y) \equiv 0, v(y) \equiv -1, u(x) \equiv -2$ and define

$$
p_0 = v(y) - u(x) = 1, \quad q_0 = \varphi(y) - v(y) = 1.
\]

Then $(p_0, q_0)$ is in the effective domain of $\Theta$ and $\Xi$,

$$
\Theta(p_0, q_0) = 0 < \infty, \quad \Xi(p_0, q_0) = 2 < \infty.
\]

Moreover $\Theta$ is obviously continuous at $(p_0, q_0)$. We are then in a position to invoke Fenchel-Rockafellar theorem [38, Theorem 1.9],

$$
\inf_{(p, q)} \{\Theta(p, q) + \Xi(p, q)\} = \sup_{(\pi, \bar{\mu})} \{-\Theta^*(-\pi, -\bar{\mu}) - \Xi(\pi, \bar{\mu})\}.
\]

Plugging in early calculations, we obtain (3.7), which together with step 1 proves the duality (3.3).

3. Finally we show that the duality (3.3) leads to the dualities of the theorem. It is easy to see that, for any triple $(u, v, \varphi) \in V_c \cap C_b$, $(u, \varphi)$ is an admissible pair in $V^*_c \cap C_b$. On the other hand, for any $(u, \varphi) \in V^*_c \cap C_b$, the triple $(u, \varphi, \varphi)$ is in $V_c \cap C_b$. Hence

$$
\sup_{(u, v, \varphi) \in V_c \cap C_b} \left\{ \int_X u d\mu - \int_Y \varphi d\nu \right\} = \sup_{(u, \varphi) \in V^*_c \cap C_b} \left\{ \int_X u d\mu - \int_Y \varphi d\nu \right\}.
\]
The LHS of (3.8) equals $T_c(\mu, \mathcal{P}_{\leq} A)$ by step 2. Hence the duality in (i) follows. To see the alternative duality form in (ii), we first note that

$$\sup_{(u,v,\varphi) \in U_c \cap C_b \times L^1} \left\{ \int_X u \, d\mu - \int_Y v \, d\nu \right\} \leq \sup_{\varphi \in A \cap C_b} \left\{ \int_X Q_c(\varphi) \, d\mu - \int_Y \varphi \, d\nu \right\},$$

On the other hand, following the idea of step 1, we have that

$$\sup_{\varphi \in A \cap C_b} \left\{ \int_X Q_c(\varphi) \, d\mu - \int_Y \varphi \, d\nu \right\} \leq T_c(\mu, \mathcal{P}_{\leq} A).$$

Therefore, in view of step 2, the proof of (ii) is completed. □

3.2. **Forward projection.** Now we derive the duality formula for forward projection.

**Theorem 3.4.** Let $X, Y$ be compact, $\mu \in P(X)$, $\nu \in P(Y)$, $c : X \times Y \mapsto [0, \infty]$ be lower semicontinuous and $A$ be a defining function class as defined in Definition 2.1. Assume that $A$ and $A \cap C_b$ define the same stochastic order.

(i) The forward duality holds

$$T_c(\mathcal{P}_{\leq} A, \nu) = \sup_{(\varphi,u,v) \in U^* \cap C_b \times L^1} \left\{ \int_X \varphi \, d\mu - \int_Y u \, d\nu \right\},$$

where $U^*$ is the set of measurable functions $(\varphi,u)$ such that

$$\varphi \in A \text{ and } u(x) - v(y) \leq c(x,y), \quad \forall x,y.$$

(ii) The alternative form of forward duality holds

$$T_c(\mathcal{P}_{\leq} A, \nu) = \sup_{\varphi \in A \cap C_b} \left\{ \int_X \varphi \, d\mu - \int_Y Q_c(\varphi) \, d\nu \right\},$$

where

$$Q_c(\varphi)(y) = \sup_{x \in X} \{ \varphi(x) - c(x,y) \}. \quad (3.10)$$

In both forms (i) and (ii), $\varphi \in A \cap C_b$ can be relaxed to $\varphi \in A \cap \mathcal{H}_b$.

**Proof.** The proof is similar to the backward case, Theorem 3.2. We only indicate the difference. As before, the proof of the theorem boils down to the intermediate duality

$$T_c(\mathcal{P}_{\mu \leq \nu}) = \sup_{(\varphi,u,v) \in U \cap C_b \times L^1 \times L^1} \left\{ \int_X \varphi \, d\mu - \int_Y u \, d\nu \right\},$$

where $U$ is the collection of triples $(\varphi,u,v)$ of measurable functions such that

$$u(x) - v(y) \leq c(x,y), \quad \forall x,y, \quad (3.11)$$

and

$$\varphi \in A \text{ and } \varphi(x) \leq u(x), \quad \forall x. \quad (3.12)$$

The duality remains true if (3.11) is replaced with

$$u(x) - v(y) \leq c(x,y), \quad \forall x, \nu\text{-a.e. } y.$$

This is proved in 2 steps.

1. Similar to step 1 in the proof of Theorem 3.2, we have

$$\sup_{U \cap C_b} \left\{ \int \varphi \, d\mu - \int v \, d\nu \right\} \leq \sup_{U \cap C_b \times L^1} \left\{ \int \varphi \, d\mu - \int v \, d\nu \right\} \leq T_c(\mathcal{P}_{\mu \leq \nu}).$$
2. Next we have
\[ \mathcal{T}_c(\Phi_{\mu,\nu}) = \sup_{(\varphi,u,v) \in U_c \cap C_b} \left\{ \int_X \varphi d\mu - \int_Y v d\nu \right\}. \]
This is proved by following the same lines as in step 2 of Theorem 3.2, except that the functional \( \mathcal{T} \) is now defined as
\[ \mathcal{T} : (p,q) \in C_b(X \times Y) \times C_b(X) \rightarrow \left\{ \begin{array}{ll} \int_Y v d\nu - \int_X \varphi d\mu, & p(x,y) = v(y) - u(x) \text{ for some } (u,v) \in C_b, \\
\infty, & q(x) = u(x) - \varphi(x) \text{ for some } \varphi \in \mathcal{A} \cap C_b, \end{array} \right. \text{ otherwise.} \]

4. Duality the general case

In the previous section, we have proved the duality formulas for backward and forward projection in the case the underlying spaces \( X, Y \) (resp. \( X \times Y \)) are compact.

At first sight, one might think that the proof there can be carried out easily to the general case, e.g. \( \mathbb{R}^d \). The typical cutting and gluing technique would be the first to appear in our mind. A careful thinking, however, reveals the difficulty in such an approach. The reason lies in the special structure of measures in stochastic order, which makes it hard to localize. Another obstacle comes from the function spaces. If we use \( C_0 \) instead of \( C_b \) in this case, the function \( \mathcal{T} \) in Theorem 3.2 and Theorem 3.4 would be useless, since the decomposition of \( p(x,y) \) as the sum of functions of individual variables is possible only in a trivial way. The admissible set \( \mathcal{A} \) adds a further layer of difficulty, it renders both the spaces \( C_0 \) and \( C_b \) useless in the general case, since those spaces are not large enough to accommodate nontrivial test functions in \( \mathcal{A} \). Consider for instance the convex order defined by the set of all convex functions, there are no non-constant convex functions in \( C_b(\mathbb{R}^d) \).

We introduce appropriate function spaces to address these issues.

4.1. The space \( C_{b,k} \) and its dual \( (C_{b,k})^* \). Let \( X \) be a locally compact polish space. Let \( k \geq 0 \) be an integer. We introduce the function space \( C_{b,k}(X) \) defined by
\[ C_{b,k}(X) = \left\{ u \in C(X) : \frac{u}{1 + \|x\|^k} \in C_b(X) \right\}. \]
with the norm
\[ \|u\|_{b,k} = \sup_{x \in X} \frac{|u(x)|}{1 + \|x\|^k}. \]
Then
\[ C_{0,k}(X) = \left\{ u \in C(X) : \frac{u}{1 + \|x\|^k} \in C_0(X) \right\} \]
is a closed subspace of \( C_{b,k}(X) \). Its topological dual space is identified with the space of finite Borel measures with \( k \)-th moment, i.e.
\[ (C_{0,k}(X))^* \cong M_k \triangleq \left\{ \eta \in M(X) : \left( 1 + \|x\|^k \right) \eta \in M(X) \right\}. \]
We denote by \( M_{k,+} \) the set of nonnegative measures in \( M_k \).
We also introduce
\[ \mathcal{H}_{b,k}(X) = \left\{ u \text{ measurable} : \frac{u}{1 + \|x\|^k} \in \mathcal{H}(X) \right\}. \]

Clearly when \( k = 0 \), \( C_{b,0}(X) \) (resp. \( C_{0,0}(X) \), \( \mathcal{H}_{b,0}(X) \)) reduces to the usual space \( C_b(X) \) (resp. \( C_0(X) \), \( \mathcal{H}_b(X) \)). In addition, if \( X \) is bounded, then \( C_{0,k}(X) = C_0(X) \), \( C_{b,k}(X) = C_b(X) \), \( \mathcal{H}_{b,k}(X) = \mathcal{H}_b(X) \), for \( k \geq 0 \).

The following crucial lemmas provide decomposition and representation of continuous linear functionals on \( C_{b,k} \).

**Lemma 4.1.** Let \( k \geq 0 \) be an integer and \( X, Y \) be locally compact, \( \sigma \)-compact polish spaces. Let \( L \) be a nonnegative continuous functional on \( C_{b,k}(X \times Y) \). Then
\[ L = \pi + R, \]
where \( \pi \in M_{k,+}(X \times Y) \) and \( R \) a nonnegative continuous linear functional supported at infinity, i.e.,
\[ (R, u) = 0, \forall u \in C_{0,k}(X \times Y). \]

**Proof.** The proof is similar to [38, Lemma 1.24]. \( \square \)

**Lemma 4.2.** Let \( X, Y \) be locally compact, \( \sigma \)-compact polish spaces and \( k \geq 0 \). Let \( \mu \in M_+(X) \), \( \nu \in P_k(Y) \) be Borel probabilities. If \( L \in (C_{b,k}(X \times Y))^* \) is nonnegative such that, for all \( u \in C_{b,k}(X), v \in C_{b,k}(Y) \),
\[ \langle L, u + v \rangle = \int_X u(x)d\mu + \int_Y v(y)d\nu, \]
then \( \mu \in P_k(X), L \in P_k(X \times Y) \) and \( L \in \Pi(\mu, \nu) \).

Note the assumption is that \( \mu \) is a nonnegative measure. It is part of the conclusion that \( \mu \) is a probability of \( k \)-th moment.

**Proof.** 1. When \( k = 0 \) or both \( X \) and \( Y \) are bounded (thus \( C_{b,k} \) reduces to \( C_b \)), this is [38, Lemma 1.25].

2. Now consider \( k \geq 1 \) and either \( X \) or \( Y \) is unbounded, say \( X \) is unbounded. To simplify notation, we assume w.l.g. that the origin \( 0 \) is in \( X \) and \( Y \), so that we can use \( 0 \) as the reference point for the metrics on \( X \) and \( Y \). In view of Lemma 4.1, we can write
\[ L = \pi + R, \]
where \( \pi \in M_{k,+}(X \times Y) \) and \( R \) a nonnegative continuous linear functional supported at infinity in the sense of (4.1). To complete the proof, it suffices to show that \( R = 0 \). Let \( A_n \) be an increasing compact sets of \( X \) such that
\[ A_n \subset \text{int}(A_{n+1}), \quad \bigcup_n A_n = X. \]

By Urysohn’s Lemma, there are continuous functions \( a_n(x) \in C_0(X) \) satisfying
\[ 0 \leq a_n(x) \leq 1, \quad a_n(x) = 1 \text{ on } A_n, \quad 0 \text{ on } A_{n+1}. \]
Clearly \( \{a_n(x)\} \) is an increasing sequence of functions and \( a_n(x) \to 1 \). Since \( k \geq 1 \) and one of the underlying spaces is unbounded, \( C_b(X) \subset C_{0,k}(X \times Y) \). Therefore \( R \) vanishes on \( C_b(X) \) by (4.1). Taking \( u \in C_b(X), v = 0 \) as test functions, we have that
\[ \int_{X \times Y} u(x)d\pi = \langle L, u \rangle = \int_X u(x)d\mu, \forall u \in C_b(X). \]
If we substitute $u$ with
\[ u_n(x) = a_n(x) \|x\|_X^k \in C_b(X), \; n \geq 1, \]
we get
\[ \int_{X \times Y} u_n(x) d\pi = \int_X u_n(x) d\mu, \; \forall n \geq 1. \]
It follows, by monotone convergence theorem, that
\[ \int_{X \times Y} \|x\|_X^k d\pi = \int_X \|x\|_X^k d\mu. \tag{4.3} \]
Since $\pi \in M_{k,+}(X \times Y)$, this indicates that $\mu$ has $k$-th moment, i.e., $\mu \in M_k(X)$.
Analogously we have
\[ \int_{X \times Y} \|y\|_Y^k d\pi = \int_Y \|y\|_Y^k d\nu. \tag{4.4} \]
Now taking $u = \|x\|_X^k$, $v = \|y\|_Y^k$ as test functions in (4.2) we get
\[ \int_{X \times Y} (\|x\|_X^k + \|y\|_Y^k) d\pi + \langle R, \|x\|_X^k + \|y\|_Y^k \rangle = \int_X \|x\|_X^k d\mu + \int_Y \|y\|_Y^k d\nu. \]
This together with (4.3) and (4.4) implies
\[ \langle R, \|x\|_X^k + \|y\|_Y^k \rangle = 0. \]
From these we conclude that $R = 0$. Indeed, for any $u \in C_{b,k}(X \times Y)$, there exist constants $a > 0$, $b > 0$ such that
\[ -a \left( \|x\|_X^k + \|y\|_Y^k \right) - b \leq u(x,y) \leq a \left( \|x\|_X^k + \|y\|_Y^k \right) + b. \]
Note 1 $\in C_{0,k}(X \times Y)$, therefore, when $R$ acts on the first and last term of the above inequalities, they all vanish. Since $R$ is nonnegative, we see that
\[ \langle R, u - \left[ -a \left( \|x\|_X^k + \|y\|_Y^k \right) - b \right] \rangle \geq 0 \text{ which yields } \langle R, u \rangle \geq 0, \]
and
\[ \langle R, \left[ a \left( \|x\|_X^k + \|y\|_Y^k \right) + b \right] - u \rangle \geq 0 \text{ which yields } \langle R, u \rangle \leq 0. \]
Hence
\[ \langle R, u \rangle = 0, \; \forall u \in C_{b,k}(X \times Y). \]

3. Now we know $L \in M_{k,+}(X \times Y)$, the remaining proof is similar to Lemma 3.1. \hfill \Box

4.2. General duality for Wasserstein projections. There is one caveat before we prove the general duality. The stochastic cones we defined earlier in (2.4) and (2.5) do not explicitly require moments of measures, this will not cause problems in the compact case, since moments of the probabilities exist automatically. However, we have to make this requirement precise in the general case. So we retain similar notation of the cones, but make it more precise for the general duality that the cones are contained in the space of probability measures with $k$-th moment, i.e. for $\mu \in P_k(X)$,
\[ \mathcal{P}^A_{k,\leq \mu} = \{ \eta \in P_k(X) : \eta \leq A \nu \} \]
and for $\nu \in P_k(Y)$,
\[ \mathcal{P}^A_{k,\mu \leq} = \{ \xi \in P_k(X) : \mu \leq A \xi \}. \]
Whenever the underlying spaces in question are bounded, the moments are not relevant, the subscripts \( k \) will be omitted so that they are consistent with the notations introduced earlier.

**Theorem 4.3 (Backward duality).** Let \( X, Y \) be locally compact Polish spaces, \( j \geq k \geq 0 \) be integers, \( \mu \in P_j(X) \), \( \nu \in P_j(Y) \) and \( A \) be a defining function class as defined in Definition 2.1. Assume that

(a1) \( A \) and \( A \cap C_{b,k} \) define the same stochastic order,

(a2) the cost \( c(x, y) \) is nonnegative, lower semicontinuous and there exist \( \alpha > 0 \) and \( \beta \in \mathbb{R} \) such that

\[
c(x, y) + \frac{1}{\alpha} \|x\|_X^k - \alpha \|y\|_Y^k \geq \beta, \quad \forall x, y.
\]

Then we have the following.

(i) Let \( V_{c^*} \) be defined in (3.1), then

\[
T_c(\mu, \mathcal{P}_{A,\leq \nu}) = \sup_{(u, \varphi) \in V_{c^*} \cap C_{b,k}} \left\{ \int_X ud\mu - \int_Y \varphi d\nu \right\}.
\]

(ii) Let \( Q_c(\cdot) \) be defined in (3.2), then

\[
T_c(\mu, \mathcal{P}_{A,\leq \nu}) = \sup_{\varphi \in A \cap C_{b,k}} \left\{ \int_X Q_c(\varphi)d\mu - \int_Y \varphi d\nu \right\}.
\]

In both (i) and (ii), \( \varphi \in A \cap C_{b,k} \) can be relaxed to \( \varphi \in A \cap \mathcal{S}_{b,k} \).

**Proof.** Since the a probability measure on a Polish space has \( \sigma \)-compact support, we may assume in the following that \( X, Y \) are \( \sigma \)-compact. We would like to follow the steps as in Theorem 3.2. The conclusion of the theorem follows easily, once we can prove

\[
T_c(\mu, \mathcal{P}_{A,\leq \nu}) = \sup_{(u, v, \varphi) \in V_{c} \cap C_{b,k} \times \mathcal{S}_{b,k}} \left\{ \int_X ud\mu - \int_Y \varphi d\nu \right\}
\]

where \( V_c \) is defined through (3.4)(3.5). In contrast with Theorem 3.2, the correct domain for the functionals \( \Theta \) and \( \Xi \) is \( C_{b,k}(X \times Y) \times C_0(Y) \). However, if we take a look at the functionals \( G(\cdot), H(\cdot) \) defined in Theorem 3.2, we would soon realize that they are never continuous in \( C_0(Y) \) with the supremum norm induced by \( C_b(Y) \). To fix this problem, we employ a usual perturbation trick to circumvent this difficulty. Given \( \epsilon_1, \epsilon_2 > 0 \), we define

\[
G_{\epsilon_1} : p \in C_{b,k}(X \times Y) \mapsto \begin{cases} 0, & p(x, y) \geq -c(x, y) - \epsilon_1, \\ \infty, & \text{otherwise}. \end{cases}
\]

\[
H_{\epsilon_2} : q \in C_0(Y) \mapsto \begin{cases} 0, & q(y) \geq -\epsilon_2, \\ \infty, & \text{otherwise}. \end{cases}
\]

\[
I : (p, q) \in C_{b,k}(X \times Y) \times C_0(Y) \mapsto \begin{cases} \int_Y \varphi d\nu - \int_X ud\mu, & p(x, y) = v(y) - u(x) \text{ for some } (u, v) \in C_{b,k}, \\ \infty, & q(y) = \varphi(y) - v(y) \text{ for some } \varphi \in A \cap C_{b,k}, \text{ otherwise}. \end{cases}
\]
Note $\mathcal{I}$ is convex, well-defined and nontrivial. We have the following,
\[
G^*_e(-L) = \sup_{p \in C_{b,k}(X \times Y), \, p \geq -e_x} \{-(L, p)\} = \sup_{p \in C_{b,k}(X \times Y), \, p \geq c} \{-(L, p)\} + \epsilon_1,
\]
which is $\infty$ unless $L \in (C_{b,k}(X \times Y))^\ast$ is nonnegative,
\[
\mathcal{H}^*(\bar{\mu}) = \sup_{q \in C_0(Y), \, q \geq -\epsilon_2} \left\{ -\int_Y q(y)d\bar{\mu} \right\} = \left\{ \epsilon_2, \quad \bar{\mu} \in M_+(Y), \right\}.
\]
\[
\mathcal{I}^*(L, \bar{\mu}) = \sup_{(u,v) \in C_{b,k} \varphi \in \mathcal{A} \cap C_{b,k}} \left\{ (L, v(y) - u(x)) + \int_X u(x)d\mu(x) - \int_Y \varphi(y)d\nu(y) + \int_Y (\varphi(y) - v(y))d\bar{\mu} \right\}
\]
\[
= \sup_{(u,v) \in C_{b,k} \varphi \in \mathcal{A} \cap C_{b,k}} \left\{ (L, v(y) - u(x)) - \int_Y v(y)d\bar{\mu} + \int_X u(x)d\mu(x) + \int_Y \varphi(y)d\bar{\mu} - \int_Y \varphi(y)d\nu(y) \right\}.
\]
By virtue of Lemma 4.2, $\mathcal{I}^*(L, \bar{\mu}) = 0$ if and only if $\bar{\mu} \leq_\mathcal{A} \nu$ and $L \in \Pi(\mu, \bar{\mu})$. When this happens, we have $\bar{\mu} \in P_k(Y)$ and $L \in P_k(X \times Y)$. Therefore
\[
\mathcal{I}^*(L, \bar{\mu}) = \begin{cases} 
0, & \bar{\mu} \in \mathcal{P}_\leq_{\mathcal{A}, \nu}, \, L \in \Pi(\mu, \bar{\mu}), \\
\infty, & \text{otherwise.}
\end{cases}
\]
Let
\[
\Theta_x(p, q) = G_x(p) + \mathcal{H}_x(q), \quad \Xi(p, q) = \mathcal{I}(p, q).
\]
To complete the proof, it remains to show that there exists $(p_0, q_0) \in C_{b,k}(X \times Y) \times C_0(Y)$ such that
\[
\Xi(p_0, q_0) < \infty, \quad \Theta_x(p_0, q_0) < \infty \text{ and } \Theta_x \text{ is continuous at } (p_0, q_0).
\]
Define
\[
\varphi_0(y) = 0, \quad v_0(y) = 0, \quad u_0(x) = -\frac{1}{a}\|x\|_X^k - b,
\]
where $a, b$ are constants to be determined later. Clearly $\varphi_0 \in \mathcal{A} \cap C_{b,k}, u_0, v_0 \in C_{b,k}$. Set
\[
p_0(x, y) = v_0(y) - u_0(x), \quad q_0(y) = \varphi_0(y) - v_0(y), \quad \forall x, y.
\]
Then
\[
p_0 = \frac{1}{a}\|x\|_X^k + b \in C_{b,k}(X \times Y), \quad q_0 \equiv 0 \in C_0(Y).
\]
A $\delta$-neighbourhood $U_\delta(p_0, q_0)$ of $(p_0, q_0)$ in $C_{b,k}(X \times Y) \times C_0(Y)$ is given by all functions $(p, q) \in C_{b,k}(X \times Y) \times C_0(Y)$ satisfying
\[
\|p - p_0\|_{b,k} + \|q - q_0\|_b = \sup_{(x,y) \in X \times Y} \frac{|p(x, y) - p_0(x, y)|}{1 + (\|x\|_X + \|y\|_Y)^k} + \sup_{y \in Y} |q(y) - q_0(y)| < \delta.
\]
It follows that, for any $(p, q) \in U_\delta(p_0, q_0)$,
\[
p(x, y) \geq p_0 - \delta - \delta 2^{k-1} (\|x\|_X^k + \|y\|_Y^k)
\]
\[
= \left(1 - \delta 2^{k-1}\right)\|x\|_X^k - \delta 2^{k-1}\|y\|_Y^k + b - \delta, \quad \forall x, y,
\]
and
\[
q(x, y) \geq q_0 - \delta = -\delta, \quad \forall x, y.
\]
Now choose $\epsilon_1 > 0$, $\epsilon_2 > 0$, $b > 0$ large, $a > 0$, $\delta > 0$ small such that

$$-\delta > -\epsilon_2, \quad \frac{1}{a} - \delta 2^{k-1} \geq \frac{1}{\alpha}, \quad \alpha \geq \delta 2^{k-1}$$

and $b - \delta > -\beta - \epsilon_1$.

With these constants and the assumption on the cost function $c$, we see that

$$p \geq -c - \epsilon_1, \quad q \geq -\epsilon_2, \quad \forall (p, q) \in U_\delta(p_0, q_0).$$

Hence

$$\Theta_c(p, q) = 0, \quad \forall (p, q) \in U_\delta(p_0, q_0).$$

Therefore the duality is proved upon cancelling $\epsilon_1 + \epsilon_2$ from both sides.

The duality for forward projection can be proved as Theorem 4.3. We leave its proof to the reader.

**Theorem 4.4 (Forward duality).** Let $X$, $Y$ be locally compact Polish spaces, $j \geq k \geq 0$ be integers, $\mu \in P_j(X)$, $\nu \in P_j(Y)$ and $A$ be a defining function class as defined in Definition 2.1. Assume that

(a1) $A$ and $A \cap C_{b,k}$ define the same stochastic order,

(a2) the cost $c(x, y)$ is nonnegative, lower semicontinuous and there exist $\alpha > 0$ and $\beta \in \mathbb{R}$ such that

$$c(x, y) + \frac{1}{\alpha} ||y||_X^k - \alpha ||x||_Y^k \geq \beta, \quad \forall x, y.$$

Then we have the following.

(i) Let $U^*_c$ be defined in (3.9), then

$$\mathcal{T}_c(\mathcal{P}^A_{k, \mu \leq \nu}) = \sup_{(\varphi, \psi) \in U^* \cap C_{b,k}} \left\{ \int_X \varphi d\mu - \int_Y \psi d\nu \right\}.$$

(ii) Let $Q_c(\cdot)$ be defined in (3.10), then

$$\mathcal{T}_c(\mathcal{P}^A_{k, \mu \leq \nu}) = \sup_{\varphi \in A \cap C_{b,k}} \left\{ \int_X \varphi d\mu - \int_Y Q_c(\varphi) d\nu \right\}.$$

In both (i) and (ii), $\varphi \in A \cap C_{b,k}$ can be relaxed $\varphi \in A \cap \mathcal{J}_{b,k}$. 


The assumption (a2) on the cost function of the duality theorems (Theorem 4.3 and Theorem 4.4) might not be the most general one, but it already includes important examples encountered in applications. If the underlying spaces are bounded, then (a2) is automatic. In the general case, the following example shows that all power functions satisfy this assumption.

**Example 4.5.** Let \( k \geq 1 \) be an integer, \( h(s) : [0, \infty) \mapsto [0, \infty) \) be a continuous function such that

\[
    m_k \triangleq \inf_{s \geq a} \frac{h(s)}{s^k} > 0 \text{ for some } a > 0.
\]

Then the cost function \( c(x, y) = h(|x - y|) \) on \( \mathbb{R}^d \times \mathbb{R}^d \) satisfies assumption (a2) of Theorem 4.3 (resp. Theorem 4.4). In particular, the quadratic cost \( c(x, y) = |x - y|^2 \) satisfies assumption (a2) with \( k = 1 \) or \( 2 \).

**Proof.** We show that assumption (a2) of Theorem 4.3 is satisfied by the cost function \( c(x, y) = h(|x - y|) \), the proof for Theorem 4.4 is similar. Let \( 0 < \epsilon < 1 \).

Consider

\[
    f(x, y) = h(|x - y|) + \frac{1}{\epsilon} |x|^k - \epsilon |y|^k.
\]

With a change of variable \( z = x - y \), we can rewrite the function as

\[
    f(z, y) = h(|z|) + \frac{1}{\epsilon} |z + y|^k - \epsilon |y|^k.
\]

Since we only need a lower bound of \( f \) and it is bounded from below when \( y = 0 \) or \( z = 0 \), we may assume \( y \neq 0 \) and \( z \neq 0 \) in the following. Using triangle inequality we have

\[
    f(z, y) \geq h(|z|) + \frac{1}{\epsilon} |z| - |y|^k - \epsilon |y|^k = |y|^k \left[ \frac{h(|z|)}{|z|^k} |z|^k + \frac{1}{\epsilon} \frac{|z|}{|y|} - 1 \right] - \epsilon.
\]

We distinguish two scenarios:

\[
    (s1) \quad \frac{|z|}{|y|} - 1 > \frac{1}{2} \quad (s2) \quad \frac{|z|}{|y|} - 1 \leq \frac{1}{2} \text{ or } \frac{1}{2} \leq \frac{|z|}{|y|} \leq \frac{3}{2}.
\]

In scenario (s1), we have

\[
    f(z, y) \geq |y|^k \left[ \frac{1}{2\epsilon} - \epsilon \right].
\]

In scenario (s2), if \( |z| \geq a \), then

\[
    f(z, y) \geq |y|^k \left[ \frac{mh}{2k} - \epsilon \right].
\]

If \( |z| < a \), then \( |y| \leq 2 |z| < 2a \), thus

\[
    f(z, y) \geq -\epsilon |y|^k \geq -\epsilon (2a)^k.
\]

Therefore we can choose \( \epsilon > 0 \) small such that

\[
    \frac{mh}{2k} - \epsilon > 0 \text{ and } \frac{1}{2k\epsilon} - \epsilon > 0,
\]

then in all cases the function \( f \) is bounded from below by \( -\epsilon (2a)^k \).

We make a few remarks concerning backward projection Theorem 4.3 and forward projection Theorem 4.4.
Remark 4.6. Theorem 4.3 and Theorem 4.4 do not assume the transportation costs $T_c(\mu, A_{k,\mu}^A)$, $T_c(\mathcal{P}_{k,\mu}^A, \nu)$ are finite, therefore, the optimal projections and optimal couplings are generally nonnegative linear functionals on $C_b$ or $C_{b,k}$, as can be seen from the proof. However, once the transportation costs are finite, then they become true probabilities in $P_k$. Under mild conditions of the cost function, the transportation costs are finite. Consider for example $k \geq 1$, $\mu \in P_k(\mathbb{R}^d)$, $\nu \in P_k(\mathbb{R}^d)$ and there is $A > 0$ such that

$$0 \leq c(x, y) \leq A(|x|^k + |y|^k).$$

Since $\nu \in \mathcal{P}_{k,\mu}^A$,

$$0 \leq T_c(\mu, \mathcal{P}_{k,\mu}^A) \leq T_c(\mu, \nu) = \inf_{\pi \in \Pi(\mu, \nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} c(x, y) d\pi(x, y)$$

$$\leq A \int_{\mathbb{R}^d \times \mathbb{R}^d} (|x|^k + |y|^k) d\pi(x, y) \quad (\forall \pi \in \Pi(\mu, \nu))$$

$$= A \left( \int_{\mathbb{R}^d} |x|^k d\mu + \int_{\mathbb{R}^d} |y|^k d\nu \right) < \infty.$$

The same estimate holds for forward projection $T_c(\mathcal{P}_{k,\mu}^A, \nu)$.

Remark 4.7. It is part of Fenchel-Rockafellar theorem that, once the optimal transportation costs are finite, then the optimal values $T_c(\mu, \mathcal{P}_{k,\mu}^A)$, $T_c(\mathcal{P}_{k,\mu}^A, \nu)$, $T_c(\mu, \mathcal{P}_{k,\mu}^A)$, $T_c(\mathcal{P}_{k,\mu}^A, \nu)$ etc. are attained.

Remark 4.8. In the duality theorems, the requirement $\varphi \in \mathcal{A} \cap \mathcal{F}_{b,k}$ cannot be further relaxed to $\varphi \in \mathcal{A} \cap \mathcal{L}^1$, because at some point of the proof, we will need to integrate $\varphi$ w.r.t. the free marginal (i.e. the projection). If $\varphi$ is only known to be integrable w.r.t. the fixed marginal, then there is not enough information to ensure the integration of $\varphi$ w.r.t. the free marginal exists. However, for backward projection, it is possible to further relax $\mathcal{A} \cap \mathcal{F}_{b,k}$ to:

$$\left\{ \varphi : \varphi \in \mathcal{A} \cap \mathcal{L}^1(X, d\nu) \text{ and } \varphi \geq f_\varphi \text{ for some } f_\varphi \in \mathcal{F}_{b,k} \right\}.$$

When this relaxed condition holds, then, for any $\tilde{\mu} \in \mathcal{P}_{k,\mu}^A$, $\tilde{\mu} \leq \mathcal{A} \nu$, we have

$$\int f_\varphi d\tilde{\mu} \leq \int \varphi d\tilde{\mu} \leq \int \varphi d\nu < \infty,$$

which implies $\int \varphi d\tilde{\mu}$ is finite. Similarly, for forward projection, it is possible to further relax $\mathcal{A} \cap \mathcal{F}_{b,k}$ to:

$$\left\{ \varphi : \varphi \in \mathcal{A} \cap \mathcal{L}^1(Y, d\mu) \text{ and } \varphi \leq g_\varphi \text{ for some } g_\varphi \in \mathcal{F}_{b,k} \right\}.$$

Remark 4.9. If $\mathcal{A}$ defines a trivial order (see Example 2.4) and $k = 0$, then the dualities of Wasserstein projections reduce to the classical Kantorovich duality.

Remark 4.10. If the defining classes $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_1 \cap \mathcal{F}_{b,k}$ $\mathcal{A}_2 \cap \mathcal{F}_{b,k}$ define the same stochastic order, then they give equal optimal dual value. Take backward projection for example,

$$\sup_{\varphi \in \mathcal{A}_1 \cap \mathcal{F}_{b,k}} \left\{ \int_X Q_c(\varphi) d\mu - \int_Y \varphi d\nu \right\} = \sup_{\varphi \in \mathcal{A}_2 \cap \mathcal{F}_{b,k}} \left\{ \int_X Q_c(\varphi) d\mu - \int_Y \varphi d\nu \right\}.$$
This follows from the proof of the duality formula and the fact that these defining classes produce the same stochastic order cones, i.e. $\mathcal{P}^{A_1}_{k, \leq \nu} = \mathcal{P}^{A_2}_{k, \leq \nu}$. The same is true for forward projections. It is worth noting that it is not immediately obvious that running the supremum over different sets should result in equal optimal dual value.

The following theorem gives the relationship between optimal primal solutions and optimal dual solutions of Wasserstein projections.

**Theorem 4.11.** Let $X, Y$ be locally compact Polish spaces, $k \geq 0$ be an integer, $\mu \in P_k(X)$, $\nu \in P_k(Y)$ and $A$ be a defining function class as defined in Definition 2.1.

(i) Assume that the conditions of Theorem 4.3 are satisfied. Let $\bar{\mu}$ be the optimizer for $T_c(\mu, P^{A}_{k, \leq \nu})$ and suppose $\varphi \in A \cap L^1(d\nu)$ is an optimal dual solution for backward projection of Theorem 4.3 which is bounded from below by some function in $S_{b,k}$, then
$$\int \varphi d\bar{\mu} = \int \varphi d\nu.$$ 
In particular $\varphi$ is an optimal potential for $T_c(\mu, \bar{\mu})$.

(ii) Assume that the conditions of Theorem 4.4 are satisfied. Let $\bar{\nu}$ be the optimizer for $T_c(P^{A}_{k, \mu \leq \nu}, \nu)$ and suppose $\varphi \in A \cap L^1(d\mu)$ is an optimal dual solution for forward projection of Theorem 4.4 which is bounded from above by some function in $S_{b,k}$, then
$$\int \varphi d\mu = \int \varphi d\bar{\nu}.$$ 
In particular $\varphi$ is an optimal potential for $T_c(\bar{\nu}, \nu)$.

**Proof.** We only prove (i), the proof of (ii) is similar. In view of Remark 4.8, the integral $\int \varphi d\bar{\mu}$ is finite. Then using the optimality of $\bar{\mu}$ and $\varphi$, we obtain
$$T_c(\mu, \bar{\mu}) = T_c(\mu, P^{A}_{k, \leq \nu}) = \int Q_c(\varphi) d\mu - \int \varphi d\nu$$
$$= \int Q_c(\varphi) d\mu - \int \varphi d\bar{\mu} + \int \varphi d\bar{\mu} - \int \varphi d\nu$$
$$\leq T_c(\mu, \bar{\mu}) + \left( \int \varphi d\bar{\mu} - \int \varphi d\nu \right) \leq T_c(\mu, \bar{\mu}).$$

The first inequality is due to the fact that $(Q_c(\varphi), \varphi)$ is an admissible pair for the transportation $T_c(\mu, \bar{\mu})$. The second inequality follows from the fact that $\bar{\nu} \leq_A \nu$. \qed

4.3. **Uniqueness.** Now we turn to the uniqueness of the projections. If the cones $\mathcal{P}^{A}_{k, \mu \leq \nu}, \mathcal{P}^{A}_{k, \leq \nu}$ are convex along Wasserstein geodesics, then the uniqueness would be immediate. However, except in a few special situations, these cones are generally not convex along Wasserstein geodesics. That being said, the uniqueness of the projection can still be obtained using the convexity of these cones under linear interpolation. The case of convex order projection is proved in [1]. The proof follows the classical strict convexity argument.

**Theorem 4.12 (Uniqueness).** Let $k \geq 0$, $X, Y$ be convex subsets of $\mathbb{R}^d$, $\mu \in P_k(X)$, $\nu \in P_k(Y)$ and $A$ be a defining function class as defined in Definition 2.1. The cost
function $c(x, y) = h(x - y)$ for some strictly convex function $h : \mathbb{R}^d \mapsto [0, \infty)$. If $\mu \in P_k(X)$ and the transport cost of the backward projection is finite, then the projection exists and is unique. Similarly, if $\nu \in P_k(Y)$, then forward projection exists and is unique.

Proof. We consider forward projection and $X = Y = \mathbb{R}^d$, other cases are proved similarly. Assume that $\nu \in P_k(\mathbb{R}^d)$ and the transport cost of the forward projection is finite. In view of Remark 4.7, the forward projection exists. Let $\bar{\nu}_0$, $\bar{\nu}_1$ be forward projections of $\nu$ onto $\mathcal{P}_{k, \mu}^\Delta$, i.e.

$$\mathcal{T}_c(\bar{\nu}_0, \nu) = \mathcal{T}_c(\bar{\nu}_1, \nu) = \mathcal{T}_c(\mathcal{P}_{k, \mu}^\Delta, \nu).$$

Denote by $\pi_s$ (i.e. $\bar{\nu}_s$) the optimal coupling between $\bar{\nu}_i$ and $\nu$. Then, for $s \in (0, 1)$ fixed,

$$\pi_s \triangleq (1 - s)\pi_0 + s\pi_1,$$

is a coupling between $\bar{\nu}_s \triangleq (1 - s)\bar{\nu}_0 + s\bar{\nu}_1$ and $\nu$. Since $\mathcal{P}_{k, \mu}^\Delta$ is convex under linear interpolation, $\bar{\nu}_s \in \mathcal{P}_{k, \mu}^\Delta$. Hence

$$\begin{align*}
\mathcal{T}_c(\mathcal{P}_{k, \mu}^\Delta, \nu) &\leq \mathcal{T}_c(\bar{\nu}_s, \nu) \\
&\leq \int_{\mathbb{R}^d \times \mathbb{R}^d} c(x, y) d\pi_s \\
&= (1 - s) \int_{\mathbb{R}^d \times \mathbb{R}^d} c(x, y) d\pi_0 + s \int_{\mathbb{R}^d \times \mathbb{R}^d} c(x, y) d\pi_1 \\
&= (1 - s) \mathcal{T}_c(\bar{\nu}_0, \nu) + s \mathcal{T}_c(\bar{\nu}_1, \nu) = \mathcal{T}_c(\mathcal{P}_{k, \mu}^\Delta, \nu).
\end{align*}$$

It follows that

$$\mathcal{T}_c(\mathcal{P}_{k, \mu}^\Delta, \nu) = \mathcal{T}_c(\bar{\nu}_s, \nu) = \int_{\mathbb{R}^d \times \mathbb{R}^d} c(x, y) d\pi_s,$$

i.e. $\bar{\nu}_s$ is a forward projection of $\nu$ onto $\mathcal{P}_{k, \mu}^\Delta$ and $\pi_s$ is the optimal coupling between $\bar{\nu}_s$ and $\nu$. Since $\nu$ is absolutely continuous w.r.t. the Lebesgue measure, for each $i = 0, 1$, there exists a $\nu$-unique optimal mapping $T_i$ from $\nu$ to $\bar{\nu}_i$, i.e.,

$$d\pi_i(x, y) = \delta_{T_i(y)}(x) d\nu(y), \ i = 0, 1.$$

In view of (4.5),

$$T_s(y) = (1 - s)T_0(y) + sT_1(y), \ \nu-a.e.y.$$

We claim that $T_0(y) = T_s(y) = T_1(y)$ for $\nu$-a.e. $y$. Otherwise we would have, due to the strict convexity of the cost, that

$$\begin{align*}
\mathcal{T}_c(\mathcal{P}_{k, \mu}^\Delta, \nu) &= \mathcal{T}_c(\bar{\nu}_s, \nu) = \int_{\mathbb{R}^d} c(T_s(y), y) d\nu = \int_{\mathbb{R}^d} c(1 - s)T_0(y) + sT_1(y), y) d\nu \\
&< (1 - s) \int_{\mathbb{R}^d} c(T_0(y), y) d\nu + s \int_{\mathbb{R}^d} c(T_1(y), y) d\nu \\
&= (1 - s) \mathcal{T}_c(\bar{\nu}_0, \nu) + s \mathcal{T}_c(\bar{\nu}_1, \nu) = \mathcal{T}_c(\mathcal{P}_{k, \mu}^\Delta, \nu),
\end{align*}$$

which is a contradiction. Therefore

$$\delta_{T_0(y)}(x) d\nu(y) = \delta_{T_s(y)}(x) d\nu(y) = \delta_{T_1(y)}(x) d\nu(y).$$

It follows $\pi_0, \pi_s, \pi_1$ are equal, thus have equal first marginals, i.e. $\bar{\nu}_0 = \bar{\nu}_s = \bar{\nu}_1$. □
5. Convex Order Projections

This section is devoted to convex order projections and their duality theorems.

**Definition 5.1 (Convex order).** Given two probabilities \( \mu, \nu \in P_1(\mathbb{R}^d) \), we call \( \mu \) smaller than \( \nu \) in convex order, denoted by \( \mu \leq_{cx} \nu \), if the inequality

\[
\int \varphi \, d\mu \leq \int \varphi \, d\nu
\]

holds for all real-valued convex function \( \varphi \) such that both integrals of \( \varphi \) w.r.t. \( \mu, \nu \) exist in the extended sense.

The following lemma gives several equivalent definitions of convex order.

**Lemma 5.2.** Let \( \mu, \nu \in P_1(\mathbb{R}^d) \). Regarding the inequality (5.1) the following statements are equivalent.

(i) (5.1) holds for any lower semicontinuous proper convex function \( \varphi \) such that both integrals exist in the extended sense.

(ii) (5.1) holds for any lower semicontinuous proper convex function \( \varphi \) which is bounded from below.

(iii) (5.1) holds for any convex function \( \varphi \) which is Lipschitz continuous.

**Proof.** 1. First prove (i) and (ii) are equivalent. It suffices to show (ii) implies (i). Let \( \varphi \) be any lower semicontinuous proper convex function such that both integrals w.r.t. \( \mu, \nu \) exist in the extended sense. Then, for \( \forall K \leq 0 \), the function \( \max\{ \varphi, K \} \) is convex and bounded from below. Hence

\[
\int \varphi^+ \, d\mu + \int \{ \varphi < 0 \} \max\{ \varphi, K \} \, d\mu = \int \max\{ \varphi, K \} \, d\mu
\]

\[
\leq \int \max\{ \varphi, K \} \, d\nu
\]

\[
= \int \varphi^+ \, d\nu + \int \{ \varphi < 0 \} \max\{ \varphi, K \} \, d\nu.
\]

Note \( \int \{ \varphi < 0 \} \max\{ \varphi, K \} \, d\mu, \int \{ \varphi < 0 \} \max\{ \varphi, K \} \, d\nu \) are finite. If \( \int \varphi^+ \, d\mu = \infty \), then from the above inequality we infer that \( \int \varphi^+ \, d\nu = \infty \). Since both integrals of \( \varphi \) w.r.t. \( \mu, \nu \) exist in the extended sense, it follows that \( \int \varphi \, d\mu = \int \varphi \, d\nu = \infty \). In this case (5.1) holds trivially with both sides equal to \( \infty \). In the other extreme case where \( \int \varphi^+ \, d\mu < \infty \) and \( \int \varphi^+ \, d\nu = \infty \), (5.1) holds trivially too. It remains to consider the case where \( \int \varphi^+ \, d\mu < \infty \), \( \int \varphi^+ \, d\nu < \infty \). In this case, we can let \( K \to -\infty \) and use monotone convergence theorem to obtain

\[
\int \max\{ \varphi, K \} \, d\mu \to \int \{ \varphi < 0 \} \varphi \, d\mu, \quad \int \max\{ \varphi, K \} \, d\nu \to \int \{ \varphi < 0 \} \varphi \, d\nu.
\]

It follows that

\[
\int \varphi \, d\mu \leq \int \varphi \, d\nu.
\]

Therefore, (i) holds in all cases.

2. For the equivalence between (ii) and (iii), it suffices to show (iii) implies (ii). Consider any lower semicontinuous proper convex function \( \varphi \) on \( \mathbb{R}^d \). Since \( \varphi \) is lower semicontinuous, there is a sequence of \( n \)-Lipschitz convex functions \( \varphi_n \) which increase to \( \varphi \) in a pointwise manner as \( n \to \infty \). Note \( \varphi_n \in L^1(d\mu) \cap L^1(d\nu), \forall n.\)
Then using monotone convergence theorem, we obtain that \( \varphi \) satisfies (5.1). Thus (ii) is proved. \( \square \)

Define
\[
A_{cx} = \{ \varphi : \varphi \text{ proper convex, lower semicontinuous} \}.
\]

Note that for any \( \mu \in P_1(\mathbb{R}^d) \) and any proper convex function \( \varphi \), the integral \( \int \varphi d\mu \) always exists in the extended sense. This results from the fact that \( \varphi \) is supported by a linear function, thus \( \int \min\{\varphi, 0\} d\mu > -\infty \). This together with Lemma 5.2 indicates that the three defining classes \( A_{cx}, A_{cx} \cap \mathcal{J}_{b,1}, A_{cx} \cap \mathcal{J}_{b,2} \) produce the same convex order relation. Instead of \( A_{cx} \), we can also use the following class to get the same convex order,
\[
A_{cx,lb} = \{ \varphi \in A_{cx} : \varphi \text{ bounded from below} \}.
\]

Note we have
\[
(5.2) \quad A_{cx} \cap C_{b,1} = A_{cx} \cap \mathcal{J}_{b,1} = \{ \varphi : \varphi \text{ convex, Lipschitz} \}.
\]

Let \( k \geq 1 \). The backward and forward convex order cones are denoted by
\[
\mathcal{P}_{k,\leq} = \{ \eta \in P_k(\mathbb{R}^d) : \eta \leq_{cx} \nu \}.
\]

and
\[
\mathcal{P}_{k,\mu \leq} = \{ \xi \in P_k(\mathbb{R}^d) : \mu \leq_{cx} \xi \}.
\]

In view of Lemma 5.2, the defining classes \( A_{cx}, A_{cx} \cap \mathcal{J}_{b,1}, A_{cx,lb} \) etc. give the same convex order cones.

5.1. The duality theorems. Now we state the dualities for backward and forward convex order projections.

**Theorem 5.3.** Let \( \mu \in P_2(\mathbb{R}^d), \nu \in P_2(\mathbb{R}^d) \).

(i) For \( k = 1, 2 \), the duality for backward convex order projection holds
\[
T_2(\mathcal{P}_{1,\leq}, \mu) = T_2(\mathcal{P}_{2,\leq}, \mu) = \sup_{\varphi \in A_{cx} \cap C_{b,k}} \left\{ \int_{\mathbb{R}^d} Q_2(\varphi) d\mu - \int_{\mathbb{R}^d} \varphi d\nu \right\}
\]
\[
= \sup_{\varphi \in A_{cx} \cap L^1(d\nu)} \left\{ \int_{\mathbb{R}^d} Q_2(\varphi) d\mu - \int_{\mathbb{R}^d} \varphi d\nu \right\},
\]

where
\[
(5.3) \quad Q_2(\varphi)(x) = \inf_{y \in \mathbb{R}^d} \left\{ \varphi(y) + |x - y|^2 \right\}.
\]

(ii) For \( k = 1, 2 \), the duality for forward convex order projection holds
\[
T_2(\mathcal{P}_{1,\mu \leq}, \nu) = T_2(\mathcal{P}_{2,\mu \leq}, \nu) = \sup_{\varphi \in A_{cx} \cap C_{b,k}} \left\{ \int_{\mathbb{R}^d} \varphi d\mu - \int_{\mathbb{R}^d} Q_2(\varphi) d\nu \right\}
\]
\[
= \sup_{x \in \mathbb{R}^d} \left\{ \varphi(x) - |x - y|^2 \right\}.
\]

Note in both (i) and (ii), the optimal values are equal regardless of \( k \), moreover \( \varphi \in A_{cx} \cap C_{b,k} \) can be relaxed to \( \varphi \in A \cap \mathcal{J}_{b,k} \).
Proof. The dualities for individual \( k \) are obtained by invoking Theorem 4.3 and Theorem 4.4. To see that the optimal values are equal regardless of \( k = 1, 2 \), we proceed as below.

1. We prove that 
\[
\mathcal{P}^{cx}_{1, \leq \nu} = \mathcal{P}^{cx}_{2, \leq \nu},
\]
whence 
\[
\mathcal{T}_2(\mu, \mathcal{P}^{cx}_{1, \leq \nu}) = \mathcal{T}_2(\mu, \mathcal{P}^{cx}_{2, \leq \nu}).
\]
Since \( \mathcal{P}^{cx}_{2, \leq \nu} \subset \mathcal{P}^{cx}_{1, \leq \nu} \), so it suffices to prove the opposite inclusion. For any \( \eta \in P_1(\mathbb{R}^d) \) and \( \eta \leq_{cx} \nu \), we have 
\[
\int_{\mathbb{R}^d} \varphi d\eta \leq \int_{\mathbb{R}^d} \varphi d\nu \text{ for all } \varphi \in A_{cx} \cap \mathcal{J}_{b, 2}.
\]
Since \( \nu \in P_2(\mathbb{R}^d) \), we can take as \( \varphi \) the convex function \(|y|^2| \) to obtain 
\[
\int_{\mathbb{R}^d} |y|^2 d\eta \leq \int_{\mathbb{R}^d} |y|^2 d\nu < \infty,
\]
which shows that \( \eta \in P_2(\mathbb{R}^d) \). Therefore \( \mathcal{P}^{cx}_{1, \leq \nu} \subset \mathcal{P}^{cx}_{2, \leq \nu} \).

2. That \( \varphi \in A_{cx} \cap \mathcal{C}_{b, k} \) in the backward duality can be relaxed to \( \varphi \in A_{cx} \cap L^1(d\nu) \) follows from Remark 4.8 and the fact that real-valued convex functions are supported by linear functions. This completes the proof of item (i).

3. We prove that  
\[
(5.5) \quad \mathcal{T}_2(\mathcal{P}^{cx}_{1, \mu \leq}, \nu) = \mathcal{T}_2(\mathcal{P}^{cx}_{2, \mu \leq}, \nu).
\]
The idea of step 1 does not apply here. In general we only have \( \mathcal{P}^{cx}_{2, \mu \leq} \subset \mathcal{P}^{cx}_{1, \mu \leq} \), whence 
\[
\mathcal{T}_2(\mathcal{P}^{cx}_{1, \mu \leq}, \nu) \subseteq \mathcal{T}_2(\mathcal{P}^{cx}_{2, \mu \leq}, \nu).
\]
To show the reverse inequality, we prove that there exists \( \tilde{\nu} \in \mathcal{P}^{cx}_{2, \mu \leq} \) such that 
\[
(5.6) \quad \mathcal{T}_2(\tilde{\nu}, \nu) = \mathcal{T}_2(\mathcal{P}^{cx}_{1, \mu \leq}, \nu),
\]
which indicates 
\[
\mathcal{T}_2(\mathcal{P}^{cx}_{2, \mu \leq}, \nu) \subseteq \mathcal{T}_2(\tilde{\nu}, \nu) = \mathcal{T}_2(\mathcal{P}^{cx}_{1, \mu \leq}, \nu).
\]
Therefore, once (5.6) is established, the proof of (5.5) would be completed. Note this actually yields a direct proof of the existence for \( \mathcal{T}_2(\mathcal{P}^{cx}_{1, \mu \leq}, \nu), \mathcal{T}_2(\mathcal{P}^{cx}_{2, \mu \leq}, \nu) \). To prove (5.6), first note, in view of Remark 4.6, \( \mathcal{T}_2(\mathcal{P}^{cx}_{1, \mu \leq}, \nu) \) is finite. Let \( \bar{\nu}_n \in \mathcal{P}^{cx}_{1, \mu \leq} \) be a minimizing sequence for \( \mathcal{T}_2(\mathcal{P}^{cx}_{1, \mu \leq}, \nu) \). Then the sequence \( \mathcal{T}_2(\bar{\nu}_n, \nu) \) is bounded by some constant \( M > 0 \). By \( \mathcal{W}_2 \)-triangle inequality, 
\[
\int_{\mathbb{R}^d} |x|^2 d\bar{\nu}_n = \mathcal{T}_2(\bar{\nu}_n, \delta_0) \leq (\mathcal{T}_2(\bar{\nu}_n, \nu) + \mathcal{T}_2(\nu, \delta_0))^2 \\
\leq 2(\mathcal{T}_2(\bar{\nu}_n, \nu) + \mathcal{T}_2(\nu, \delta_0)) \leq 2 \left( M + \int_{\mathbb{R}^d} |x|^2 d\nu \right) < \infty, \quad \forall n.
\]
Here \( \delta_0 \) is the Dirac measure concentrated at the origin. By virtue of Markov inequality the sequence \( \bar{\nu}_n \) is tight (see e.g. [9]). Therefore there exists a subsequence \( \bar{\nu}_{n_i} \) converging weakly to some probability \( \bar{\nu} \). Meanwhile 
\[
\sup_i \int_{|x| \geq R} |x|^2 d\bar{\nu}_{n_i} \leq \sup_i \left( \frac{1}{R} \int_{|x| \geq R} |x|^2 d\bar{\nu}_{n_i} \right) \leq \frac{1}{R} \sup_n \int_{|x| \geq R} |x|^2 d\bar{\nu}_n \to 0 \text{ as } R \to \infty.
\]
By [38, Theorem 7.12 (ii)],
\[ W_1(\bar{\nu}_n, \bar{\nu}) \rightarrow 0 \text{ as } i \rightarrow \infty. \]

Hence for any \( g \in \mathcal{S}_{b,1} \),
\[ \int_{\mathbb{R}^d} gd\bar{\nu}_n \rightarrow \int_{\mathbb{R}^d} gd\bar{\nu} \text{ as } n \rightarrow \infty. \]

Using Lemma 5.2 and \( \mu \leq \bar{\nu}_n \), we infer that \( \mu \leq_{cx} \bar{\nu} \). Moreover \( \bar{\nu} \) has second-order moment. Indeed, by virtue of Layer cake theorem [31, p26, 1.13],
\[ \int_{\mathbb{R}^d} |x|^2 d\bar{\nu}_n = 2 \int_0^\infty t\bar{\nu}_n(|x| > t)dt. \]

Since \( \bar{\nu}_n \) converges weakly to \( \bar{\nu} \),
\[ \lim_{i \rightarrow \infty} \bar{\nu}_n(|x| > t) = \bar{\nu}(|x| > t), \text{ for a.e.-t.} \]

Now using Fatou lemma
\[ \int_{\mathbb{R}^d} |x|^2 d\bar{\nu} = 2 \int_0^\infty t\bar{\nu}(|x| > t)dt \leq \liminf_{i \rightarrow \infty} 2 \int_0^\infty t\bar{\nu}_n(|x| > t)dt = \liminf_{i \rightarrow \infty} \int_{\mathbb{R}^d} |x|^2 d\bar{\nu}_n \leq \sup_n \int_{\mathbb{R}^d} |x|^2 d\bar{\nu}_n < \infty. \]

Hence \( \bar{\nu} \in P_2(\mathbb{R}^d) \). Therefore \( \bar{\nu} \in \mathcal{P}_{1,\mu}^{cx} \subset \mathcal{P}_{1,\mu} \). Finally, we show that \( \bar{\nu} \) is optimal for \( T_2(\mathcal{P}_{1,\mu}^{cx}, \nu) \). Let \( \bar{\pi}_n \in \Pi(\bar{\nu}_n, \nu) \) be the optimal couplings for \( T_2(\bar{\nu}_n, \nu) \).

Since \( \bar{\nu}_n \in \mathcal{P}_{1,\mu}^{cx} \) converges weakly to \( \bar{\nu} \), we may assume that \( \bar{\pi}_n \) converges weakly to some \( \bar{\pi} \in \Pi(\bar{\nu}, \nu) \). For any \( R > 0 \),
\[ \int_{\mathbb{R}^d \times \mathbb{R}^d} (|x-y|^2 \wedge R) d\bar{\pi}(x,y) = \lim_{i \rightarrow \infty} \int_{\mathbb{R}^d \times \mathbb{R}^d} (|x-y|^2 \wedge R) d\bar{\pi}_n(x,y) \leq \lim_{i \rightarrow \infty} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x-y|^2 d\bar{\pi}_n(x,y) = T_2(\mathcal{P}_{1,\mu}^{cx}, \nu). \]

Sending \( R \rightarrow \infty \) and using monotone convergence theorem,
\[ \int_{\mathbb{R}^d \times \mathbb{R}^d} |x-y|^2 d\bar{\pi}(x,y) \leq T_2(\mathcal{P}_{1,\mu}^{cx}, \nu). \]

Since \( \bar{\nu} \in \mathcal{P}_{1,\mu}^{cx} \), this shows \( \bar{\pi} \) is an optimal coupling which achieves \( T_2(\mathcal{P}_{1,\mu}^{cx}, \nu) \), thus (5.6) is satisfied. \( \square \)

**Remark 5.4.** From the proof of Theorem 5.3, any projection onto \( \mathcal{P}_{1,\mu}^{cx} \) is also a projection onto \( \mathcal{P}_{1,\mu}^{cx} \), and vice versa. The same is true for \( \mathcal{P}_{1,\mu}^{cx} \) and \( \mathcal{P}_{1,\mu}^{cx} \).

### 6. Dual attainment for convex order projections

In this section we will prove the attainment of the optimal dual value associated with backward convex order projection and forward convex order projection. Characterization of these optimal dual potentials will be given in the following section. These are tackled by combining the treatments in the classical optimal transportation with duality formulas proved in the earlier sections.
6.1. Backward projection. 

**Theorem 6.1.** Let \( \mu \in P_2(\mathbb{R}^d) \), \( \nu \in P_2(\mathbb{R}^d) \). Then, for each \( k = 1 \) or \( 2 \), there is a convex function \( \varphi_0 \in \mathcal{A}_{cx} \cap L^1(d\nu) \) with values in \( \mathbb{R} \cup \{ \infty \} \) which achieves the optimal dual value

\[
D_2(\mu, \mathcal{P}_{k, \leq \nu}) \Deltaq \sup_{\varphi \in \mathcal{A}_{cx} \cap \mathcal{A}_{b,k}} \left\{ \int Q_2(\varphi)d\mu - \int \varphi d\nu \right\},
\]

where \( Q_2(\cdot) \) is defined in (5.3). If both \( \mu \) and \( \nu \) have compact supports, then there exists a Lipschitz convex optimal dual solution. Moreover

\[
D_2(\mu, \mathcal{P}_{k, \leq \nu}) = D_2(\mu, \mathcal{P}_{2, \leq \nu}).
\]

**Proof.** Using the equivalence between backward convex order projection and weak optimal transport, it is proved in [23, Proposition 1.1] that

\[
\mathcal{T}_2(\mu, \mathcal{P}_{k, \leq \nu}) = D_{2, \text{lb}}(\mu, \mathcal{P}_{k, \leq \nu}) \Deltaq \sup_{\varphi \in \mathcal{A}_{cx} \cap \mathcal{A}_{b,k}} \left\{ \int_{\mathbb{R}^d} Q_2(\varphi)d\mu - \int_{\mathbb{R}^d} \varphi d\nu \right\}.
\]

The optimal dual value \( D_{2, \text{lb}}(\mu, \mathcal{P}_{k, \leq \nu}) \) is attained at some \( \varphi_0 \in \mathcal{A}_{cx} \cap L^1(d\nu) \) with values in \( \mathbb{R} \cup \{ \infty \} \). Note in general, \( \varphi_0 \) might not be a member of \( \mathcal{A}_{cx} \cap \mathcal{A}_{b,1} \). It is also proved that, if both \( \mu \) and \( \nu \) have compact supports, then \( \varphi_0 \) can be chosen to be Lipschitz convex. In view of Theorem 5.3,

\[
\mathcal{T}_2(\mu, \mathcal{P}_{k, \leq \nu}) = D_2(\mu, \mathcal{P}_{k, \leq \nu}) = D_2(\mu, \mathcal{P}_{2, \leq \nu}),
\]

hence the same function \( \varphi_0 \) which attains \( D_{2, \text{lb}}(\mu, \mathcal{P}_{k, \leq \nu}) \) also achieves \( D_2(\mu, \mathcal{P}_{k, \leq \nu}) \) and \( D_2(\mu, \mathcal{P}_{2, \leq \nu}) \).

Therefore \( \varphi_0 \) is the desired function we look for. \( \square \)

6.2. Forward projection. The next result is about the attainment of forward convex order projection. The proof combines a preliminary trick with the strategy used in the proof of [23, Theorem 1.2].

**Theorem 6.2.** Let \( \mu \in P_2(\mathbb{R}^d) \), \( \nu \in P_2(\mathbb{R}^d) \). Then, for each \( k = 1 \) or \( 2 \), there exists a convex function \( \bar{\varphi}_0 \in \mathcal{A}_{cx} \cap C_{b,2} \) which satisfies \( \bar{\varphi}_0 \leq |x|^2 \) and achieves the optimal dual value

\[
D_2(\mathcal{P}_{k, \leq \nu}) \Deltaq \sup_{\varphi \in \mathcal{A}_{cx} \cap \mathcal{A}_{b,k}} \left\{ \int \varphi d\mu - \int Q_2(\varphi)d\nu \right\},
\]

where \( Q_2(\cdot) \) is defined in (3.10). Moreover

\[
D_2(\mathcal{P}_{1, \mu, \leq \nu}) = D_2(\mathcal{P}_{2, \mu, \leq \nu}).
\]

**Proof.** That the two optimal dual values are equal is a result of Theorem 5.3. The proofs of attainment for \( k = 1 \) and \( 2 \) are similar, but note that the function which achieves \( D_2(\mathcal{P}_{1, \mu, \leq \nu}) \) might not be in \( \mathcal{A}_{b,1} \). Here we only give proof for \( k = 2 \).

1. We first show that the optimization can be restricted to \( \mathcal{A}_{bx}^0 \), where

\[
\mathcal{A}_{bx}^0 = \{ \bar{\varphi} \in \mathcal{A}_{cx} : \bar{\varphi} = Q_2(\bar{\varphi}), \ Q_2(\bar{\varphi})(0) = 0 \}.
\]

Indeed, first recall Remark 4.6, the optimal dual value is finite under the conditions of the theorem. If \( \varphi \in \mathcal{A}_{cx} \cap \mathcal{A}_{b,2} \) is such that \( Q_2(\varphi) \) is identically \( \infty \), then the dual value is \(-\infty\), since the integral \( \int \varphi d\mu \) is finite. So we can avoid these functions in
the supremum in $D_2(\mathcal{P}_{x,\mu,\nu}^{\mathbb{R}})$. For any $\varphi \in A_{\mathcal{C}} \cap \mathcal{J}_{b,2}$ such that $Q_2(\varphi)$ is not identically $\infty$, there is $y_0 \in \mathbb{R}^d$ (depending on $\varphi$) for which

$$Q_2(\varphi)(y_0) < \infty.$$ 

Let

$$\tilde{\varphi}(x) \triangleq \inf_{y \in \mathbb{R}^d} \left\{ Q_2(\varphi)(y) + |x - y|^2 \right\} = Q_2(\varphi)(x).$$

Then

$$\tilde{\varphi}(x) \leq Q_2(\varphi)(y_0) + |x - y_0|^2 < \infty, \ \forall x.$$ 

Since $\tilde{\varphi}$ is convex and $\varphi \leq \tilde{\varphi}$, we see that $\tilde{\varphi} \in A_{\mathcal{C}} \cap C_{b,2}$. By (A.3), we have

$$Q_2(\tilde{\varphi})(y) = Q_2(\varphi)(y_0) + |x - y_0|^2,$$

whence

$$\tilde{\varphi}(x) = Q_2(\varphi)(y_0) + |x - y_0|^2 = Q_2(\tilde{\varphi})(y),$$

whence

$$\tilde{\varphi}(x) = Q_2(Q_2(\varphi))(x) = Q_2(\tilde{\varphi})(x).$$

So

$$\int \varphi d\mu - \int Q_2(\varphi) d\nu \leq \int \tilde{\varphi} d\mu - \int Q_2(\tilde{\varphi}) d\nu \leq D_2(\mathcal{P}_{x,\mu,\nu}^{\mathbb{R}}).$$

The last inequality is due to $\tilde{\varphi} \in A_{\mathcal{C}} \cap \mathcal{J}_{b,2}$, so it is admissible to the supremum in $D_2(\mathcal{P}_{x,\mu,\nu}^{\mathbb{R}})$. Moreover adding a constant to $\tilde{\varphi}$ will not change the difference of the integrals, we may assume that $Q_2(\tilde{\varphi})(0) = 0$. Therefore we have proved

$$D_2(\mathcal{P}_{x,\mu,\nu}^{\mathbb{R}}) = \sup_{\tilde{\varphi} \in A_{\mathcal{C}} \cap \mathcal{J}_{b,2}} \left\{ \int \tilde{\varphi} d\mu - \int Q_2(\tilde{\varphi}) d\nu \right\}.$$ 

2. Let $\varphi_n \in A_{\mathcal{C}} \cap \mathcal{J}_{b,2}$ be a maximization sequence for $D_2(\mathcal{P}_{x,\mu,\nu}^{\mathbb{R}})$. Write

$$\varphi_n = Q_2(\varphi_n).$$

Then $\varphi_n = Q_2(\varphi_n)$ with $\varphi_n(0) = 0$. We assume, without loss of generality, that $\int y d\nu = 0$. First we have an upper bound (uniform in $n$) on $\varphi_n$,

$$(6.1) \quad \varphi_n(x) = Q_2(\varphi_n)(x) = \inf_y \left\{ \varphi_n(y) + |x - y|^2 \right\} \leq |x|^2.$$ 

Next we will obtain a pointwise lower bound (uniform in $n$) on $\varphi_n$. Since

$$|x|^2 - \varphi_n(x) = 2 \sup_y \left\{ x \cdot y - \frac{1}{2} \left( |y|^2 - \varphi_n(y) \right) \right\}$$

is convex, we get by Jensen inequality

$$|\bar{x}|^2 - \varphi_n(\bar{x}) \leq \int \left( |x|^2 - \varphi_n(x) \right) d\mu,$$

where

$$\bar{x} = \int x d\mu.$$ 

It follows that

$$\varphi_n(\bar{x}) \geq |\bar{x}|^2 - \int \left( |x|^2 - \varphi_n(x) \right) d\mu$$

$$= |\bar{x}|^2 + \int \varphi_n(x) d\mu - \int \varphi_n(y) d\nu - \int |x|^2 d\nu + \int \varphi_n(y) d\nu.$$ 

Note

$$\int \varphi_n(x) d\mu - \int \varphi_n(y) d\nu$$

is bounded.
Since \( \varrho_n \) is convex,
\[
\int \varrho_n(y) \, d\nu \geq \varrho_n \left( \int y \, d\nu \right) = \varrho_n(0) = 0.
\]
Therefore \( \bar{\varphi}_n(\bar{x}) \) is bounded from below. Using the convexity of \( \bar{\varphi}_n \) and (6.1), we get
\[
\bar{\varphi}_n(\bar{x}) \leq \frac{1}{2} \bar{\varphi}_n(2\bar{x} - x) + \frac{1}{2} \bar{\varphi}_n(x) \leq \frac{1}{2} |2\bar{x} - x| + \frac{1}{2} \bar{\varphi}_n(x).
\]
It follows that
\[
\bar{\varphi}_n(x) \leq \frac{1}{2} \bar{\varphi}_n(\bar{x}) - \frac{1}{2} |2\bar{x} - x|.
\]
In view of (6.1) and the fact that \( \bar{\varphi}_n(\bar{x}) \) is bounded from below, we have \( \bar{\varphi}_n(x) \) is bounded for each \( x \). Therefore we may assume, up to a subsequence, that
\[
\bar{\varphi}_n(x) \to \bar{\varphi}_0(x), \quad \forall x,
\]
where \( \bar{\varphi}_0(x) \) is a convex function. Now we show \( \bar{\varphi}_0 \) is the optimal dual solution we look for. Clearly \( \bar{\varphi}_0 \in A_{cx} \cap C_{b,2} \). Note
\[
\varrho_n(x) \geq \bar{\varphi}_0(x) - |x - y|^2, \quad \forall x, y.
\]
Hence
\[
\liminf_{n \to \infty} \varrho_n(y) \geq \bar{\varphi}_0(x) - |x - y|^2, \quad \forall x, y.
\]
So
\[
\liminf_{n \to \infty} \varrho_n(y) \geq \sup_x \{ \bar{\varphi}_0(x) - |x - y|^2 \} = Q_2(\bar{\varphi}_0)(y).
\]
Note \( \bar{\varphi}_n \) is bounded from above by a \( \mu \)-integrable function (ref. (6.1)) and \( \varrho_n \) is bounded from below by a common \( \nu \)-integrable quadratic function (ref. (6.2) and (6.3)). Therefore, using Fatou lemma,
\[
D_2(\mathcal{P}^{cx}_{2,\mu \leq, \nu}) = \lim_{n \to \infty} \left\{ \int \bar{\varphi}_n(x) \, d\mu - \int \varrho_n(y) \, d\nu \right\} = \limsup_{n \to \infty} \int \bar{\varphi}_n(x) \, d\mu - \liminf_{n \to \infty} \int \varrho_n(y) \, d\nu \leq \int \bar{\varphi}_0(x) \, d\mu - \int Q_2(\bar{\varphi}_0)(y) \, d\nu \leq D_2(\mathcal{P}^{cx}_{2,\mu \leq, \nu}).
\]
So \( \bar{\varphi}_0 \) is the optimal solution we look for. \( \square \)

Unlike the backward case in Theorem 6.1, requiring \( \mu \) and \( \nu \) to have compact supports does not ease the way to find a more regular optimal dual solution. This results from the major difference between backward and forward projection, which we will discuss in Section 8.

### 7. Characterization of convex order projections

In this section, we present results on the characterization of optimal mappings for backward and forward convex order projections. The optimal mappings possess special properties which we call convex contraction and convex expansion. These are consequences of the duality and attainment obtained in previous sections. Recall that the Legendre transform of a function \( \phi \) is denoted by \( \phi^* \).
Definition 7.1 (Convex contraction). Let $\varphi$ be a proper lower semicontinuous convex function. We call $\varphi$ a convex contraction if $\varphi = \phi^*$ for some proper function $\phi$ such that $D^2 \phi \geq \text{Id}$ in the sense of distribution.

Definition 7.2 (Convex expansion). Let $\varphi$ be a proper lower semicontinuous convex function. We call $\varphi$ a convex expansion if $\varphi = \phi^*$ for some proper function $\phi$ such that $D^2 \phi \leq \text{Id}$ in the sense of distribution.

Remark 7.3. That $\varphi$ is a convex contraction is equivalent to $D^2 \varphi \leq \text{Id}$. Indeed, in view of Lemma A.2, if $\varphi$ is a convex contraction then $D^2 \varphi \leq \text{Id}$. Conversely, if $\varphi$ is a proper lower semicontinuous convex function such that $D^2 \varphi \leq \text{Id}$, then using again Lemma A.2, $D^2 \varphi^* \geq \text{Id}$.

7.1. Backward projection. The following characterization of optimal mapping for backward convex order projection is proved in [23, Theorem 1.2, Theorem 2.1], we include them and restated in our terms for readers’ convenience.

Theorem 7.4. Suppose that $\mu \in P_2(\mathbb{R}^d)$, $\nu \in P_2(\mathbb{R}^d)$ and $\varphi \in A_{\text{cx}} \cap L^1(d\nu)$ is the optimizer of the dual $D_2(\mu, \mathcal{P}_{1,\leq,\nu}^{\text{cx}})$ obtained in Theorem 6.1. Let

$$\varphi_0(y) = \frac{1}{2} |y|^2 + \varphi(y).$$

Then

(i) $\varphi_0^* \in C^1(\mathbb{R}^d)$ is a convex contraction.

(ii) $(\nabla \varphi_0^*)_b \mu$ is the unique projection of $\mu$ onto $\mathcal{P}_{1,\leq,\nu}^{\text{cx}}$.

Theorem 7.5. Let $\mu \in P_2(\mathbb{R}^d)$, $\nu \in P_2(\mathbb{R}^d)$. Then the following are equivalent.

(i) The projection of $\mu$ onto $\mathcal{P}_{1,\leq,\nu}^{\text{cx}}$ coincides with $\nu$.

(ii) There exists a convex contraction $\phi$ such that $(\nabla \phi)_b \mu = \nu$.

Note in view of the step 1 in the proof of Theorem 5.3, $\mathcal{P}_{1,\leq,\nu}^{\text{cx}} = \mathcal{P}_{2,\leq,\nu}^{\text{cx}}$. So we can also replace $\mathcal{P}_{1,\leq,\nu}^{\text{cx}}$ with $\mathcal{P}_{2,\leq,\nu}^{\text{cx}}$ in the above theorems.

7.2. Forward projection. A key feature of our duality formulation of the Wasserstein projection in stochastic order is that the property of the optimal dual solution is inherited from the defining class of the stochastic order (e.g. the convexity in the convex order case). The optimal dual solution, if exists, not only gives rise to the optimal mapping to the projection (Theorem 4.11), but also produces special properties of the optimal mapping. Moreover, our duality formula allows us to handle the backward and forward case in a unified manner. The backward convex order projection in the previous theorem yields a special optimal mapping: a convex contraction. The next theorem will give us the exact opposite of the backward case: a convex expansion. Note that we even have a very precise relation between the optimal mappings of the backward and forward convex order projections: they are actually inverse to each other, which is somewhat surprising (see Section 8).

Theorem 7.6. Suppose that $\mu \in P_2(\mathbb{R}^d)$, $\nu \in P_2(\mathbb{R}^d)$ and $\varphi \in A_{\text{cx}} \cap L^1(d\nu)$ is the optimizer of the dual $D_2(\mathcal{P}_{2,\mu \leq,\nu}^{\text{cx}})$ obtained in Theorem 6.2. Let

$$\tilde{\varphi}_0(x) = \frac{1}{2} |x|^2 - \frac{1}{2} \varphi(x).$$
Then
(i) $\tilde{\varphi}_0^*$ is a convex expansion. In addition, $\tilde{\varphi}_0^*$ is uniformly convex.
(ii) $(\nabla \tilde{\varphi}_0^*)_\# \nu$ is the unique projection of $\nu$ onto $\mathcal{P}^\infty_{2,\mu \leq}$.

Proof. By Theorem 4.12, the forward projection is unique. Let $\tilde{\nu}$ be the unique forward convex order projection and $\pi \in \Pi(\tilde{\nu}, \nu)$ such that
\[
T_2(\mathcal{P}^\infty_{2,\mu \leq}, \nu) = T_2(\tilde{\nu}, \nu) = \int |x-y|^2 d\pi(x,y).
\]
By Theorem 5.3, the optimality of $\varphi$ and Theorem 4.11,
\[
\int |x-y|^2 d\pi(x,y) = \int \varphi(x) d\mu - \int Q_2(\varphi)(y) d\nu = \int \varphi(x) d\tilde{\nu} - \int Q_2(\varphi)(y) d\nu = \int (\varphi(x) - Q_2(\varphi)(y)) d\pi(x,y).
\]
Hence
\[
\int \left[ |x-y|^2 - (\varphi(x) - Q_2(\varphi)(y)) \right] d\pi(x,y) = 0.
\]
Since the integrand is nonnegative, we have
\[(7.1) \quad |x-y|^2 = \varphi(x) - Q_2(\varphi)(y), \text{ $\pi$-a.e. (}x,y\text{).}\]
Using Lemma A.1,
\[(7.2) \quad Q_2(\varphi)(y) = 2\tilde{\varphi}_0^*(y) - |y|^2, \text{ where } \tilde{\varphi}_0(x) = \frac{1}{2} |x|^2 - \frac{1}{2} \varphi(x).\]
Since $Q_2(\varphi)$ is convex for convex $\varphi$, we get from (7.2) that $\tilde{\varphi}_0^*$ is uniformly convex and $D^2 \tilde{\varphi}_0^* \geq Id$. Thus $\tilde{\varphi}_0^*$ is a convex expansion (Remark 7.3). Finally, combining (7.1) and (7.2), we have
\[
|x-y|^2 = \varphi(x) - \left(2\tilde{\varphi}_0^*(y) - |y|^2\right), \text{ $\pi$-a.e. (}x,y\text{),}
\]
which can be rewrite as
\[
x \cdot y = \tilde{\varphi}_0(x) + \tilde{\varphi}_0^*(y), \text{ $\pi$-a.e. (}x,y\text{).}
\]
Since $Q_2(\varphi) \in L^1(Y, d\nu)$, it is finite for $\nu$-a.e. $y$. Hence $\tilde{\varphi}_0^*$ is finite for $\nu$-a.e. $y$. This together with the absolute continuity of $\nu$ implies that $\tilde{\varphi}_0^*$ is $\nu$-a.e. differentiable and
\[
x = \nabla \tilde{\varphi}_0^*(y), \text{ $\nu$-a.e. } y.
\]
Therefore
\[
x = \nabla \tilde{\varphi}_0^*(y), \text{ $\pi$-a.e. (}x,y\text{).}
\]
This shows that $\tilde{\nu} = (\nabla \tilde{\varphi}_0^*)_\# \nu$ is the unique projection of $\nu$ onto $\mathcal{P}^\infty_{2,\mu \leq}$. \qed

In contrast to Theorem 7.6, we need to assume the absolute continuity of $\nu$ to ensure the uniqueness of the forward projection in Theorem 7.6, this results from the difference in the geometric properties of backward and forward convex order cones, see Section 4.3 and Section 8.

It turns out the properties of the potential given in Theorem 7.6 are also optimal. The follow complements the corresponding result of backward convex order projection given in Theorem 7.5.
Theorem 7.7. Let $\mu \in P_2(\mathbb{R}^d)$, $\nu \in P_{2,\mu}^{ac}(\mathbb{R}^d)$. Then the following are equivalent.

(i) The projection of $\nu$ onto $\mathcal{P}_{2,\mu}^{ac}$ coincides with $\mu$.

(ii) There exists a convex expansion $\phi$ such that $(\nabla \phi)_{\#}\nu = \mu$.

Proof. 1. That (i) implies (ii) follows from Theorem 7.6 and Lemma A.2.

2. Suppose that there exists a convex expansion $\phi$ such that $(\nabla \phi)_{\#}\nu = \mu$. Let

$$\varphi(x) = |x|^2 - 2\phi^*(x).$$

Since $\phi$ is a convex expansion, hence $\varphi$ is convex. In addition, by Lemma A.2 $\phi^* \in C^1$, $\nabla \phi^*$ is 1-Lipschitz, thus $\phi^*$ has at most quadratic growth. Therefore $\varphi \in \mathcal{A}_{ex} \cap C_{b,2}$. Since $\phi$ is convex,

$$\nabla \phi(y) \cdot y = \phi^*(\nabla \phi(y)) + \phi(y), \ a.e. \ y.$$  

Noting $\nu$ is absolutely continuous w.r.t. the Lebesgue measure, we get

$$\nabla \phi(y) \cdot y = \phi^*(\nabla \phi(y)) + \phi(y), \nu-a.e. \ y.$$  

Therefore

$$\mathcal{T}_2(\mu, \nu) \leq \int |\nabla \phi(y) - y|^2 d\nu = \int |\nabla \phi(y)|^2 d\nu - 2 \int (\nabla \phi(y))_\cdot y \, d\nu + \int |y|^2 d\nu$$

$$= \int |x|^2 d\mu - 2 \int (\phi(y) + \phi^*(\nabla \phi(y))) d\nu + \int |y|^2 d\nu$$

$$= \int \left( |x|^2 - 2\phi^*(x) \right) d\mu - \int \left( 2\phi(y) - |y|^2 \right) d\nu.$$  

Using the definition of $\varphi$ and applying Lemma A.2 to the term, we continue writing

$$\mathcal{T}_2(\mu, \nu) \leq \int \varphi(x) d\mu - \int Q_2(\varphi)(y) d\nu$$

$$\leq \sup_{g \in \mathcal{A}_{ex} \cap C_{b,2}} \left\{ \int g(x) d\mu - \int Q_2(g)(y) d\nu \right\} \left( = \mathcal{D}_2(\mathcal{P}_{2,\mu}^{ac}, \nu) \right)$$

$$\leq \sup_{g \in L^1(\mu)} \left\{ \int g(x) d\mu - \int Q_2(g)(y) d\nu \right\} = \mathcal{T}_2(\mu, \nu).$$

This shows that $\varphi$ is an optimizer of the dual, i.e.,

$$\mathcal{D}_2(\mathcal{P}_{2,\mu}^{ac}, \nu) = \int \varphi(x) d\mu - \int Q_2(\varphi)(y) d\nu.$$  

Hence by Theorem 7.6, the image $(\nabla \phi)_{\#}\nu$, which is $\mu$ by assumption, is the projection of $\nu$ onto $\mathcal{P}_{2,\mu}^{ac}$. Therefore we have proved that the projection of $\nu$ onto $\mathcal{P}_{2,\mu}^{ac}$ is $\mu$. 

\[\square\]

8. Backward projection versus forward projection

In appearance, the backward and forward duality formulas correspond to the two forms of classical Kantorovich duality where one can go from one form to the other through a $c$-transform. People might thus be tempted to think that the backward duality and forward duality for Wasserstein projections can be bridged by a $c$-transform as in the classical case, hence are equivalent in some sense. This, however, is not true in general, simply because the defining class $\mathcal{A}$ might not be invariant under the $c$-transforms $Q_c$ and $Q_{\tilde{c}}$. That being said, the case of convex order is an exception, the backward and forward convex order projections are indeed equivalent in an appropriate sense. This result is somewhat surprising, since even
in the convex order case, although the backward cone is geodesically convex, the forward cone is not, see Example 8.2.

8.1. **Supports of measures in convex order.** For any two measures in convex order, their supports, informally speaking, are increasing with convex order. This implies that for backward projection problem (1.1), the support of the backward projection \( \bar{\mu} \) is implicitly known to be contained in the support of \( \nu \). Forward projection does not enjoy this property. This is made precise in the following simple result, for which we provide a proof for completeness.

**Lemma 8.1.** Let \( \mu, \nu \in P(\mathbb{R}^d) \) and \( \mu \preceq_{cx} \nu \). Then
\[ \text{clconv}(\text{supp}(\mu)) \subset \text{clconv}(\text{supp}(\nu)), \]
where \( \text{clconv}(\cdot) \) denotes the closure of the convex hull of a given set.

**Proof.** Suppose to the contrary that the conclusion is not true. Then there exists a nonempty open ball \( B(x_0) \) such that \( \mu(B(x_0)) > 0 \) and \( \bar{B}(x_0) \cap \text{clconv}(\text{supp}(\nu)) = \emptyset \).

Then there is a linear function \( l(x) \) which strictly separates \( \text{clconv}(\text{supp}(\nu)) \) and \( \bar{B}(x_0) \):
\[ l(x) > 0, \, x \in \bar{B}(x_0); \quad l(x) < 0, \, x \in \text{clconv}(\text{supp}(\nu)). \]

Consider the convex function
\[ \varphi(x) = \max(l(x), 0), \quad x \in \mathbb{R}^d. \]

It is easy to see that
\[ \int \varphi(x) d\mu > 0 = \int \varphi(x) d\nu. \]

This contradicts the assumption that \( \mu \preceq_{cx} \nu \). \( \square \)

8.2. **Convexity of the cones.** Given \( \mu, \nu \in P(\mathbb{R}^d) \). We discuss the geodesic convexity of the backward cone \( \mathcal{P}^{cx}_{2, \preceq_{cx} \nu} \) and forward cone \( \mathcal{P}^{cx}_{2, \preceq \mu} \). In view of [3, Proposition 9.3.2], \( \mathcal{P}^{cx}_{2, \preceq_{cx} \nu} \) is convex along generalized geodesics, hence also convex along geodesics. The forward cone \( \mathcal{P}^{cx}_{2, \preceq \mu} \), on the other hand, is generally not geodesically convex as illustrated by the example below which is inspired by [12].

![Figure 2. Probabilities (solid red and blue) in convex order](image)

**Example 8.2.** Let
\[ \mu = \text{probability supported on solid red balls in Figure 2}, \]
and
\[ \nu = \text{probability supported on solid blue balls in Figure 2}. \]
Figure 3. The optimal transport from the probability on red balls to the one on blue balls. Circles indicate displacement interpolation.

Consider the cone $\mathcal{P}_{\{\mu, \nu\}}^{c, 2}$. Clearly $\nu \in \mathcal{P}_{\{\mu, \nu\}}^{c, 2}$. In Figure 2, the indicated distance $b$ is larger than $a$. When the Wasserstein distance between $\mu$ and $\nu$ is computed, it is easy to see that the optimal path to transport $\mu$ to $\nu$ is along the dotted red lines as shown in Figure 3, where an intermediate displacement interpolation, say $[\mu, \nu]_s$ with $s \in (0, 1)$, is given and supported on the red circles in Figure 3. The closure of the convex hull of $\text{supp}([\mu, \nu]_s)$ is indicated by shaded area, which does not include $\text{clconv}(\text{supp}(\mu))$. In view of Lemma 8.1, the relation $\mu \leq_{c, 2} [\mu, \nu]_s$ does not hold, in other words, $[\mu, \nu]_s$ is not in the cone $\mathcal{P}_{\{\mu, \nu\}}^{c, 2}$. Therefore $\mathcal{P}_{\{\mu, \nu\}}^{c, 2}$ is not geodesically convex.

8.3. Relation between backward and forward solution. In the following we show that the backward and forward convex order projection costs are equal and the optimal backward and forward mappings are inverse to each other. Given the fact that the backward and forward convex order cones possess distinct geometric properties as we have already seen, the equality and inverse relations are surprising.

Theorem 8.3. Let $\mu \in P_2(\mathbb{R}^d)$, $\nu \in P_2(\mathbb{R}^d)$. Then

$$T_2(\mathcal{P}_{\{\mu, \nu\}}^{c, 2}) = D_2(\mathcal{P}_{\{\mu, \nu\}}^{c, 2}) = D_2(\mu, \mathcal{P}_{\{\mu, \nu\}}^{c, 2}) = T_2(\mu, \mathcal{P}_{\{\mu, \nu\}}^{c, 2}).$$

Moreover the following are true.

(i) Let $\varphi^f \in \mathcal{A}_{c, 2} \cap C_{\{\mu, \nu\}}$ be the optimal dual solution for forward projection obtained in Theorem 6.2, then

$$Q_2(\varphi^f) \in \mathcal{A}_{c, 2} \cap L^1(\nu)$$

is an optimal dual solution for backward projection.

(ii) Let $\varphi^b \in \mathcal{A}_{c, 2} \cap L^1(\nu)$ be the optimal dual solution for backward projection obtained in Theorem 6.1, then

$$Q_2(\varphi^b) \in \mathcal{A}_{c, 2} \cap J_{b, 2}$$

is an optimal dual solution for forward projection.

Proof. It suffices to verify that

$$D_2(\mu, \mathcal{P}_{\{\mu, \nu\}}^{c, 2}) = D_2(\mu, \mathcal{P}_{\{\mu, \nu\}}^{c, 2}).$$

Recall the dualities in Theorem 5.3,

$$D_2(\mu, \mathcal{P}_{\{\mu, \nu\}}^{c, 2}) = \sup_{\varphi \in \mathcal{A}_{c, 2} \cap L^1(\nu)} \left\{ \int Q_2(\varphi) d\mu - \int \varphi d\nu \right\},$$

and

$$D_2(\mu, \mathcal{P}_{\{\mu, \nu\}}^{c, 2}) = \sup_{\varphi \in \mathcal{A}_{c, 2} \cap J_{b, 2}} \left\{ \int \varphi d\mu - \int Q_2(\varphi) d\nu \right\}.$$


Since for \( \varphi \in \mathcal{A}_{cx} \cap \mathcal{J}_{b,2} \), \( Q_2(\varphi) > -\infty \) is convex, thus bounded from below by its supporting plane. Hence \( \int Q_2(\varphi) d\nu \) is bounded from below. Since \( \int Q_2(\varphi) d\nu = \infty \) would not contribute to the supremum in (8.3), we can restrict to those \( \varphi \in \mathcal{A}_{cx} \cap \mathcal{J}_{b,2} \) such that \( \int Q_2(\varphi) d\nu \) is finite, i.e. \( Q_2(\varphi) \in L^1(d\nu) \). Therefore we can write

(8.4) \[
D_2(\mathcal{P}_{2,\mu}^{\text{cx}}, \nu) = \sup_{\varphi \in \mathcal{A}_{cx} \cap \mathcal{J}_{b,2}} \left\{ \int \varphi d\mu - \int Q_2(\varphi) d\nu \right\}.
\]

Then, for any \( \varphi \) admissible to the supremum of (8.4), it is legitimate to substitute \( \varphi \) with \( Q_2(\varphi) \) in the supremum of (8.2) and write

(8.5) \[
D_2(\mu, \mathcal{P}_{2,\mu}^{\text{cx}}) \geq \int Q_2(Q_2(\varphi)) d\mu - \int Q_2(\varphi) d\nu \geq \int \varphi d\mu - \int Q_2(\varphi) d\nu.
\]

Since \( \varphi \) runs over all functions admissible to the supremum of (8.4), we get

\[
D_2(\mu, \mathcal{P}_{2,\mu}^{\text{cx}}) \geq D_2(\mathcal{P}_{2,\mu}^{\text{cx}}, \nu).
\]

On the other hand, for any \( \varphi \in \mathcal{A}_{cx} \cap L^1(d\nu) \), \( Q_2(\varphi) \) is convex and \( \varphi \) is \( \nu \)-a.e. finite, thus \( Q_2(\varphi) \in \mathcal{A}_{cx} \cap \mathcal{J}_{b,2} \). So it is legitimate to substitute \( \varphi \) with \( Q_2(\varphi) \) in the supremum of (8.3) and write

(8.6) \[
\int Q_2(\varphi) d\mu - \int \varphi d\nu \leq \int Q_2(Q_2(\varphi)) d\mu - \int Q_2(\varphi) d\nu \leq D_2(\mathcal{P}_{2,\mu}^{\text{cx}}, \nu).
\]

Since this holds for any \( \varphi \in \mathcal{A}_{cx} \cap L^1(d\nu) \), we have

\[
D_2(\mu, \mathcal{P}_{2,\mu}^{\text{cx}}) \leq D_2(\mathcal{P}_{2,\mu}^{\text{cx}}, \nu).
\]

Thus (8.1) is proved. To see (i) and (ii), we note that, by the optimality of \( \varphi^f \),

\[
D_2(\mathcal{P}_{2,\mu}^{\text{cx}}, \nu) = \int \varphi^f d\mu - \int Q_2(\varphi^f) d\nu.
\]

Since the dual value is finite and \( \varphi^f \in \mathcal{A}_{cx} \cap C_{b,2} \), \( Q_2(\varphi^f) \in \mathcal{A}_{cx} \cap L^1(d\nu) \). So we can insert \( \varphi^f \) into (8.5) to saturate the inequalities. Therefore \( Q_2(\varphi^f) \) is an optimal solution for \( D_2(\mu, \mathcal{P}_{2,\mu}^{\text{cx}}) \). Similarly \( \varphi^b \in \mathcal{A}_{cx} \cap L^1(d\nu) \) implies \( Q_2(\varphi^b) \in \mathcal{A}_{cx} \cap \mathcal{J}_{b,2} \). Therefore we can substitute \( \varphi^b \) into (8.6) to see that \( Q_2(\varphi^b) \) is optimal for \( D_2(\mathcal{P}_{2,\mu}^{\text{cx}}, \nu) \).

\[ \square \]

**Remark 8.4.** Either following the same argument as above, or using the above result together with Theorem 5.3, we have \( D_2(\mathcal{P}_{1,\mu}^{\text{cx}}, \nu) = D_2(\mu, \mathcal{P}_{1,\mu}^{\text{cx}}) \). Although Theorem 8.3 is proved for quadratic cost \( c = |x - y|^2 \), it is easy to generalize to other convex cost functions of the form \( c = h(x - y) \).

The equality \( T_k(\mu, \mathcal{P}_{k,\mu}^{\text{cx}}) = T_k(\mathcal{P}_{k,\mu}^{\text{cx}}, \nu) \) with \( k > 1 \) is proved in [1] using a different method based on the primal formulations of the projection problems. In one dimension, it is shown there that the optimal mappings for backward and forward convex order projections are inverse to each other. The following result confirms these properties on the optimal mappings in general dimensions and sheds more lights on the deep connection between backward and forward convex order projection.

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Corollary 8.5. Let $\mu \in P_2(\mathbb{R}^d)$, $\nu \in P_{ac}^2(\mathbb{R}^d)$, $\tilde{\mu}$ the optimizer of $\mathcal{T}_2(\mu, \mathcal{P}_{2,\leq}^{\infty})$, and $\tilde{\nu}$ the optimizer of $\mathcal{T}_2(\mathcal{P}_{2,\leq}^{\infty}, \mu \leq \nu)$. There exists a real-valued convex function $\Phi \in C^1$ such that
\[ (\nabla \Phi)_{#} \mu = \tilde{\mu}, \quad (\nabla \Phi^*)_{#} \nu = \tilde{\nu}. \]
Moreover $\tilde{\nu} \in P_{ac}^2(\mathbb{R}^d)$.

Proof. Let $\varphi^f$ be the optimal dual solution of the forward convex order projection obtained in Theorem 6.2. Recall that $\varphi^f$ is convex continuous and $\varphi^f \leq |x|^2$. By Theorem 8.3, $Q_2(\varphi^f)$ is an optimal dual solution for backward convex order projection. By virtue of Theorem 7.6, the unique projection of $\nu$ onto $\mathcal{P}_{2,\leq}^{\infty}$ is given by $(\nabla \tilde{\varphi}_0^*)_{#} \nu$, where
\[ (8.7) \quad \tilde{\varphi}_0(x) \equiv \frac{1}{2} |x|^2 - \frac{1}{2} \varphi^f(x). \]
In particular, $D^2 \tilde{\varphi}_0^* \geq I_d$. Now according to Theorem 7.4, the unique projection of $\mu$ onto $\mathcal{P}_{2,\leq}^{\infty}$ is given by $(\nabla \tilde{\varphi}_0^*)_{#} \mu$, where
\[ (8.8) \quad \varphi_0(y) \equiv \frac{1}{2} |y|^2 + \frac{1}{2} Q_2(\varphi^f)(y). \]
Substituting (8.7) into (8.8), we obtain
\[
\varphi_0(y) = \frac{1}{2} |y|^2 + \frac{1}{2} Q_2 \left( \left[ |x|^2 - 2 \tilde{\varphi}_0(x) \right] \right)(y) \\
= \frac{1}{2} |y|^2 + \frac{1}{2} \sup_{x \in \mathbb{R}^d} \left\{ |x|^2 - 2 \tilde{\varphi}_0(x) - |x - y|^2 \right\} \\
= \tilde{\varphi}_0^*(y).
\]
Let $\Phi = \tilde{\varphi}_0^{**}$. Then $\Phi$ is the desired real-valued convex function. Furthermore, $\Phi^* = \tilde{\varphi}_0^*$ is uniformly convex, so the forward projection $\tilde{\nu}$ under the mapping $\nabla \Phi^*$ is absolutely continuous and $\Phi = \Phi^{**} \in C^1$. □

Remark 8.6. Convexity is crucial in the proof of Theorem 8.3 and Corollary 8.5. We do not expect other stochastic orders (e.g. the subharmonic order) to have the above equality and the inverse relation between backward and forward projection.

9. Subharmonic order projections

In this section we consider another important instance of stochastic order, i.e. subharmonic order.

Definition 9.1. An upper semicontinuous function $\psi$ with values in $[-\infty, \infty)$ is subharmonic on an open set $X$, if the sub-mean value inequality
\[ \psi(x_0) \leq \frac{1}{\omega_d(r)} \int_{\partial B_r(x_0)} \psi d\sigma_r \]
holds for any ball $B_r(x_0)$ contained in $X$, where $\omega_d(r)$ is the surface area of the ball $B_r(x_0)$.

If a function $\psi$ satisfies the above sub-mean value properties, but not necessarily upper semicontinuous, then it is called almost subharmonic. This terminology is justified by the fact that every almost subharmonic function equals an (upper semicontinuous) subharmonic function almost everywhere [37].
Definition 9.2 (Subharmonic order). Let $X \subset \mathbb{R}^d$ be a convex bounded open set and $\mu, \nu \in P(X)$, we call $\mu$ smaller than $\nu$ in subharmonic order, denoted by $\mu \leq_{sh} \nu$, if the inequality
\begin{equation}
\int_X \psi d\mu \leq \int_X \psi d\nu,
\end{equation}
holds for all subharmonic $\psi \in C_b(X)$.

Subharmonic order is sufficient to induce a Brownian transport between $\mu$ and $\nu$ (see e.g. [20, Proposition 3.4]). By performing convolutions with smooth radial kernels, each subharmonic function can be approximated in any compact subsets of $X$ by decreasing sequence of smooth subharmonic functions. Similar to Lemma 5.2, one can also restrict the integrands of (9.1) to those which are bounded from below.

Define
\[ A_{sh} = \{ \psi : \psi \text{ subharmonic, bounded from below} \}. \]

For notational simplicity, the domain of definition for functions in $A_{sh}$ is not explicitly specified and will be clear from the context.

Let $\mu \in P(X), \nu \in P(Y)$. Define the backward and forward subharmonic order cone,
\[ \mathcal{P}^\sh_{\leq \nu} = \{ \eta \in P(Y) : \eta \leq_{sh} \nu \}, \]
and
\[ \mathcal{P}^\sh_{\mu \leq} = \{ \xi \in P(X) : \mu \leq_{sh} \xi \}. \]

9.1. The duality theorems.

Theorem 9.3. Let $X, Y \subset \mathbb{R}^d$ be bounded convex open subsets, $\mu \in P(X), \nu \in P(Y)$. Then
(i) for backward subharmonic order projection it holds that
\[ T_2(\mu, \mathcal{P}^\sh_{\leq \nu}) = \sup_{\psi \in A_{sh} \cap C_b(Y)} \left\{ \int_X Q_2(\psi) d\mu - \int_Y \psi d\nu \right\}. \]
(ii) for forward subharmonic order projection it holds that
\[ T_2(\mathcal{P}^\sh_{\mu \leq}, \nu) = \sup_{\psi \in A_{sh} \cap C_b(X)} \left\{ \int_X \psi d\mu - \int_Y Q_2(\psi) d\nu \right\}. \]

Here $Q_2(\cdot), Q_2(\cdot)$ are defined in Theorem 5.3. In both (i) and (ii), $\psi \in A_{sh} \cap C_b$ can be relaxed to $\psi \in A_{sh} \cap C_b$.

Proof. The proof is completed by applying Theorem 4.3 and Theorem 4.4 to the defining class $A_{sh} \cap C_b$. \qed

10. Dual attainment for subharmonic order projections

Obtaining a solution in the required subharmonic function class with appropriate regularity is challenging. One difficulty lies in the definition of subharmonic function. To determine whether or not a function is subharmonic, one usually needs to test sub-mean value property over spheres or balls. This implicitly uses Lebesgue measure as a reference. In the dualities of subharmonic order projections, subharmonic functions have to interact with arbitrary probability measures.
To be able to traverse from an arbitrary probability measure to the Lebesgue measure, assumptions such as absolute continuity, bounds away from zero are usually unavoidable.

Another property missing from subharmonic functions involves the transforms $Q_2(\cdot), Q_\bar{2}(\cdot)$. These transforms preserve convexity, which plays a key role in the convex order case. However, $Q_2(\cdot), Q_\bar{2}(\cdot)$ not always preserve subharmonicity for general subharmonic functions. Subharmonicity is only preserved by $Q_\bar{2}(\cdot)$ in subdomain, this is proved by [32] in Alexandrov space, which includes Euclidean space as a special case [13, Example 4.2.1]. Therefore, the double convexification trick, which we use in the attainment for convex order projections, no longer work in the subharmonic order case.

Despite these difficulties, we are able to obtain optimal dual solutions for subharmonic order projections and a weak parallel of the convex contraction and expansion, which we call Laplacian contraction and Laplacian expansion, see Definition 11.1 and Definition 11.2.

We will first prove attainment of the duality. The method used here seems to have the weakest assumptions on the measures.

10.1. Backward projection.

**Theorem 10.1.** Let $X, Y$ be bounded convex open subsets of $\mathbb{R}^d$, $\mu \in P(X)$ and $\nu \in P_{ac}(Y)$. Assume that the density of $\nu$ is bounded away from zero in $Y$. Then there exists $\psi_0 \in A_{sh} \cap L^1(Y, d\nu)$ which is bounded from below such that

$$D_2(\mu, \mathcal{P}_{\leq \nu}^{sh}) \leq \sup_{\psi \in A_{sh} \cap C_b(Y)} \left\{ \int_X Q_2(\psi) d\mu - \int_Y \psi d\nu \right\} \leq \int_X Q_2(\psi_0)(x) d\mu - \int_Y \psi_0(y) d\nu.$$  

Denote by $\bar{\mu}$ the backward subharmonic order projection of $\mu$ onto $\mathcal{P}_{\leq \nu}^{sh}$. If in addition

$$(10.1) \quad \int_Y \psi_0(y) d\bar{\mu} \leq \int_Y \psi_0(y) d\nu,$$

then $\psi_0$ achieves the optimal dual value.

**Proof.** Since $T_2(\mu, \mathcal{P}_{\leq \nu}^{sh})$ is finite (ref. Remark 4.6), the optimal dual $D_2(\mu, \mathcal{P}_{\leq \nu}^{sh})$ is finite. We can also write

$$(10.2) \quad D_2(\mu, \mathcal{P}_{\leq \nu}^{sh}) = \sup_{\psi \in A_{sh} \cap C_b(Y)} \left\{ \int_X Q_2(Q_2(\psi))(x) d\mu - \int_Y \psi(y) d\nu \right\}.$$  

Let $\psi_n \in A_{sh} \cap C_b(Y)$ be a maximizing sequence and denote $\bar{\psi}_n = Q_2(Q_2(\psi_n))$. Since the measures sit in a compact set containing both $X$ and $Y$, the cost function is Lipschitz continuous there. Thus $\bar{\psi}_n$ and $Q_2(\bar{\psi}_n)$ are Lipschitz continuous with a common Lipschitz constant inherited from the cost function. When necessary, we can regard these functions as defined on $\bar{Y}$ and $\bar{X}$ by extending them to the boundaries. By adding constant(s), we may assume without loss of generality that

$$\min_{y \in Y} \bar{\psi}_n(y) = 0, \forall n.$$  

Now it is readily seen that $\bar{\psi}_n$ is uniformly bounded in $Y$. It follows that $Q_2(\bar{\psi}_n)$ is also uniformly bounded in $X$. Therefore the sequence $\bar{\psi}_n$, and so $Q_2(\bar{\psi}_n)$, are
uniformly bounded and equicontinuous. Hence, by Arzela-Ascoli theorem, we may suppose that it holds uniformly that

\[ \hat{\psi}_n \to \hat{\psi}_0 \text{ and } Q_2(\hat{\psi}_n) \to Q_2(\hat{\psi}_0), \quad n \to \infty. \]

where \( \hat{\psi}_0 \) is Lipschitz continuous on \( Y \).

Hence

\[ \int_X Q_2(\hat{\psi}_n)(x) d\mu \to \int_X Q_2(\hat{\psi}_0)(x) d\mu, \quad n \to \infty. \]

It follows that \( \int Q_2(\hat{\psi}_n) d\mu \) is bounded, so we obtain from (10.2) and the boundedness of the optimal dual value that \( \int \psi_n d\nu \) is bounded. Note

(10.3) \[ 0 \leq \hat{\psi}_n(y) \leq \psi_n(y), \quad \forall y. \]

By assumption, there exists \( a_0 > 0 \) such that

\[ a_0 \int_Y \psi_n(y) dy \leq \int_Y \psi_n(y) d\nu, \]

which implies \( \int \psi_n dy = \int |\psi_n| dy \) is bounded. Then by Lemma B.1, there exists a subharmonic function \( \psi_0 \in L^1(Y, dy) \) such that, up to a subsequence,

(10.4) \[ \psi_n(y) \to \psi_0(y), \text{ a.e. } y. \]

It follows that

\[ \psi_n(y) \to \psi_0(y), \text{ } \nu\text{-a.e. } y. \]

By Fatou lemma

\[ \liminf_{n \to \infty} \int_Y \psi_n(y) d\nu \geq \int_Y \psi_0(y) d\nu. \]

Therefore \( \psi_0 \in L^1(Y, d\nu) \). Using (10.3)(10.4),

\[ 0 \leq \hat{\psi}_0(y) \leq \psi_0(y), \text{ a.e. } y. \]

Since \( \hat{\psi}_0 \) is continuous and \( \psi_0 \) is upper semicontinuous, we obtain,

\[ 0 \leq \hat{\psi}_0(y) \leq \psi_0(y), \quad \forall y. \]

Thus the subharmonic function \( \psi_0 \) is bounded from below and we obtain

\[ D_2(\mu, \mathcal{SH}) = \lim_{n \to \infty} \left\{ \int_X Q_2(\hat{\psi}_n)(x) d\mu - \int_Y \psi_n(y) d\nu \right\} \]

\[ \leq \limsup_{n \to \infty} \int_X Q_2(\hat{\psi}_n)(x) d\mu - \liminf_{n \to \infty} \int_Y \psi_n(y) d\nu \]

\[ = \int_X Q_2(\hat{\psi}_0)(x) d\mu - \int_Y \psi_0(y) d\nu \]

\[ \leq \int_X Q_2(\psi_0)(x) d\mu - \int_Y \psi_0(y) d\nu. \]

If \( \psi_0 \) satisfies (10.1), then the above inequality continues as

\[ \int_X Q_2(\psi_0)(x) d\mu - \int_Y \psi_0(y) d\nu \leq D_2(\mu, \mathcal{SH}). \]

Therefore the optimal dual value is attained at \( \psi_0 \). \( \square \)
Remark 10.2. The assumption (10.1) is added because a priori the subharmonic function \( \psi_0 \in A_{sh} \cap L^1(Y, dv) \) may not satisfy the submartingale inequality, unless further informations are available. In our context, \( \nu \) is generated by a stopped Brownian motion which has initial distribution \( \bar{\mu} \) and stays in \( Y \), hence one might think that the assumption (10.1) is redundant. However, there are examples showing the submartingale inequality might fail even though the Brownian motion is stopped before it exits \( Y \). Here is an example due to Zhen-Qing Chen. Let \( Y \) be the 2-dimensional unit ball. Up to a conformal transform, we can reduce \( Y \) to the upper half space \( H = \{ z = (x, y) \in \mathbb{R}^2 : y > 0 \} \) and consider the Poisson kernel for \( H \),

\[
\psi(z) = \psi(x, y) = \frac{1}{\pi} \frac{y}{x^2 + y^2}.
\]

Denote by \( \tau_0 \) the hitting time of the parabolic curve \( \Gamma : y = ax^2 \ (a > 0) \) by the Brownian motion \( W_{z_0}(t) \) emanating from \( z_0 = (0, 1) \in \mathbb{R}^2 \). Since the origin \((0, 0)\) is polar for \( W_{z_0}(t) \), it is almost surely never reached. Therefore \( \tau_0 \) is strictly less than the exit time \( \tau_H \) of \( H \). Note \( \psi(z) \leq \frac{a}{\pi} \) on \( \Gamma \). Hence

\[
E(\psi(W_{z_0}(\tau_0))) \leq \frac{a}{\pi} \leq \frac{\pi}{a} = E(\psi(W_{z_0}(0))) \text{ for a small,}
\]

i.e., the submartingale inequality fails for the harmonic function \( \psi \). One can also modify this example to make a similar counterexample where the distribution of the stopped Brownian motion has a density with a positive lower bound.

10.2. Forward projection. This section deals with the dual attainment of the forward projection. In contrast to Theorem 10.1, we will get a Lipschitz continuous optimal dual solution for the forward projection.

Recall that, for convex cost \( c \), the \( c \)-transforms \( Q_c(\cdot) \) and \( Q_c(\cdot) \) defined by (3.2)(3.10) preserves convexity. This has played a crucial role in our proof of the dual attainment in the convex order case. However, analogous preservation property does not hold for subharmonic functions. To address this issue we introduce a composite of the \( c \)-transforms with the operation of taking subharmonic envelopes.

Given \( g \in C(X) \) on a connected open set \( X \) of \( \mathbb{R}^d \), its subharmonic envelope is defined as

\[
(10.5) \quad g_c \triangleq \sup\{ \psi : \psi \text{ subharmonic, } \psi \leq g \}.
\]

It is well-known that \( g_c \in C(X) \). If \( X \) is convex and \( g \in C(\bar{X}) \), then \( g_c \in C(\bar{X}) \).

Definition 10.3. For a function \( g \), we write \( Q_{cc}(g) \) as the subharmonic envelope of \( Q_c(g) \).

Lemma 10.4. Let \( a \in \mathbb{R} \), the cost \( c(x, y) \) is bounded from below and admit a modulus of continuity. Assume that all functions as a result of the \( c \)-transforms are defined on bounded domains. Then for any subharmonic function \( \psi \) on a bounded open set \( X \subset \mathbb{R}^d \) with values in \([a, \infty)\),

\[
Q_c(\psi) = Q_c(Q_{cc}(Q_c(\psi))), \quad Q_{cc}(\psi) = Q_{cc}(Q_c(Q_{cc}(\psi))).
\]

Proof. Assume without loss of generality that the cost \( c \) is nonnegative. For any function \( g \geq a \) on \( X \), it is easy to see that \( Q_c(g) \geq a \) and \( Q_c(g) \) is bounded from below, so it makes sense to consider subharmonic envelopes of these functions as a result of \( Q_c(\cdot) \) and \( Q_c(\cdot) \) transforms. Recalling (A.1)(A.2), we have

\[
Q_c(Q_c(\psi)) \leq \psi, \quad Q_c(Q_c(\psi)) \geq \psi.
\]
It follows that
(i) \( Q_\varepsilon(Q_{ce}(\psi)) \leq \psi \), (ii) \( Q_{ce}(Q_\varepsilon(\psi)) \geq \psi \).

Note in obtaining (ii) the fact that \( \psi \) is subharmonic is used, while in obtaining (i) the subharmonicity of \( \psi \) is not used. Now applying \( Q_\varepsilon(\cdot) \) on both sides of (ii), we have
\[ Q_\varepsilon(Q_{ce}(Q_\varepsilon(\psi))) \geq Q_\varepsilon(\psi). \]
Since \( Q_\varepsilon(\psi) \) is bounded from below, we may replace \( \psi \) with \( Q_\varepsilon(\psi) \) in (i) and obtain
\[ Q_\varepsilon(Q_{ce}(Q_\varepsilon(\psi))) \leq Q_\varepsilon(\psi). \]
Therefore
\[ Q_\varepsilon(\psi) = Q_\varepsilon(Q_{ce}(Q_\varepsilon(\psi))). \]

Similarly applying \( Q_{ce}(\cdot) \) on both sides of (i),
\[ Q_{ce}Q_{ce}(Q_{ce}(\psi)) \leq Q_{ce}(\psi), \]
Since \( Q_{ce}(\psi) \) is subharmonic, \( Q_{ce}(\psi) \) cannot take \( \infty \), we may then replace \( \psi \) with \( Q_{ce}(\psi) \) in (ii) to get
\[ Q_{ce}Q_{ce}(Q_{ce}(\psi)) \geq Q_{ce}(\psi). \]
Therefore
\[ Q_{ce}(\psi) = Q_{ce}Q_{ce}(Q_{ce}(\psi)). \]

\[ \square \]

Lemma 10.4 is crucial for the dual attainment for forward subharmonic order projection. It enables us to use a version of double convexification trick. Note that the optimal solution in this case is Lipschitz and subharmonic.

**Theorem 10.5.** Let \( X, Y \) be bounded convex open subsets of \( \mathbb{R}^d \), \( \mu \in \mathcal{P}^{ac}(X) \) and \( \nu \in \mathcal{P}(Y) \). Then there exists \( \bar{\psi}_0 \in A_{sh} \cap \text{Lip}(X) \) which achieves the supremum of the dual value,
\[
D_2(\mathcal{P}^{sh}_{\mu \leq \varepsilon}, \nu) \triangleq \sup_{\psi \in A_{sh} \cap \text{Lip}(X)} \left\{ \int_X \psi d\mu - \int_Y Q_2(\psi) d\nu \right\}.
\]

**Proof.** In view of Lemma 10.4 and Theorem 9.3,
\[
D_2(\mathcal{P}^{sh}_{\mu \leq \varepsilon}, \nu) = \sup_{\psi \in A_{sh} \cap \text{Lip}(X)} \left\{ \int_X \psi d\mu - \int_Y Q_2(Q_{2e}(\psi)) d\nu \right\}
\]
\[
\leq \sup_{\psi \in A_{sh} \cap \text{Lip}(X)} \left\{ \int_X Q_2(Q_{2e}(\psi)) d\mu - \int_Y Q_2(Q_{2e}(\psi)) d\nu \right\}
\]
\[
= \sup_{\psi \in A_{sh}^1 \cap \text{Lip}(X)} \left\{ \int_X \psi d\mu - \int_Y Q_2(\psi) d\nu \right\} \leq D_2(\mathcal{P}^{sh}_{\mu \leq \varepsilon}, \nu),
\]
where
\[ A_{sh}^1 = \{ \bar{\psi} : \bar{\psi} = Q_{2e}(Q_2(\psi)) \text{ for some } \psi \in A_{sh} \} \subset A_{sh}. \]
Let \( \bar{\psi}_n \in A_{sh}^1 \cap \mathcal{C} \) be a maximizing sequence, i.e. \( \bar{\psi}_n = Q_{2e}(Q_2(\psi_n)) \) for some \( \psi_n \in A_{sh} \). Note that, similar to Theorem 10.1, \( Q_2(\bar{\psi}_n) \) and \( Q_2(Q_2(\psi_n)) \) are Lipschitz continuous with a common Lipschitz constant inherited from the cost function. In view of [14, Theorem 2] or [33], \( \bar{\psi}_n \) and \( Q_2(\bar{\psi}_n) \) are Lipschitz continuous with a common Lipschitz constant. Now we can assume without loss of generality that
\[ \max_x \bar{\psi}_n(x) = 0, \forall n. \]
Therefore the sequence $\bar{\psi}_n$, and thus $Q_2(\bar{\psi}_n)$, are uniformly bounded and equicontinuous. By Arzela-Ascoli theorem, we may suppose that it holds uniformly that
\[
\bar{\psi}_n \to \bar{\psi}_0, \quad Q_2(\bar{\psi}_n) \to Q_2(\bar{\psi}_0),
\]
where $\bar{\psi}_0$ is a Lipschitz continuous subharmonic function. We call $\bar{\psi}_n$ a Laplacian expansion if $Q_2(\bar{\psi}_n)$ satisfies $\Delta \bar{\psi}_n = \bar{\psi}_0$.

\[\int_Y Q_2(\bar{\psi}_n)(y)d\nu \to \int_Y Q_2(\bar{\psi}_0)(y)d\nu.\]

Therefore
\[
D_2(\mathcal{A}_{sh}^\mu, \nu) = \lim_{n \to \infty} \left\{ \int_X \bar{\psi}_n(x)d\mu - \int_Y Q_2(\bar{\psi}_n)(y)d\nu \right\}
\]
\[= \int_X \bar{\psi}_0(x)d\mu - \int_Y Q_2(\bar{\psi}_0)(y)d\nu \leq D_2(\mathcal{A}_{sh}^\mu, \nu).\]

The last inequality uses the fact that $\bar{\psi}_0 \in \mathcal{A}_{sh} \cap \mathcal{C}_b$. Thus $\bar{\psi}_0$ is an optimal dual solution, and, if we repeat the inequalities at the beginning of the proof, then we can see that $\bar{\psi}_0$ satisfies $\bar{\psi}_0 = Q_{2e}(Q_2(\bar{\psi}_0)).$ \qed

It seems a similar proof cannot be applied to the backward dual attainment, using the operator $Q_{2e}(\cdot)$ defined similarly to $Q_{2s}(\cdot)$ (Definition 10.3) To make the trick work, we will need the identity $Q_2(\cdot) = Q_2(Q_{2e}(Q_2(\cdot)))$. However, in general, we only have $Q_{2e}(\cdot) = Q_2(Q_2(Q_2(\cdot)))$.

11. Characterization of subharmonic order projections

In parallel with section 7, we show that the optimal mappings for backward and forward subharmonic order projectitons are characterized by Laplacian contraction and Laplacian expansion.

**Definition 11.1 (Laplacian contraction).** Let $\psi$ be a proper lower semicontinuous convex function. We call $\psi$ a Laplacian contraction if $\psi = \phi^*$ for some proper function $\phi$ such that $\phi$ is bounded from below and $\Delta \phi \geq d$ in the sense of distribution.

**Definition 11.2 (Laplacian expansion).** Let $\psi$ be a proper lower semicontinuous convex function. We call $\psi$ a Laplacian expansion if $\psi = \phi^*$ for some proper function $\phi$ such that $\phi$ is bounded from below and $\Delta \phi \leq d$ in the sense of distribution.

**Remark 11.3.** Note the definition of Laplacian contraction and Laplacian expansion are consistent with that of the convex contraction and convex expansion defined in section 7, and the use of Legendre transform instead of the function itself is essential for subharmonic order case. While $D^2 \phi \geq Id$ implies $D^2 \phi^* \leq Id$ (Lemma A.2), it is not generally true that $\Delta \phi \geq d$ implies $\Delta \phi^* \leq d$.

Laplacian expansion has a simple but significant geometric consequence.

**Lemma 11.4.** Let $\psi$ be a Laplacian expansion. Then the map $\nabla \psi$ satisfies
\[
\det D^2 \psi(x) \geq 1, \text{ a.e. } x.
\]

This explains the term expansion, since the mapping $\nabla \psi$ increases volumes. Moreover, if $\nu$ absolutely continuous w.r.t. the Lebesgue measure, then so is $\tilde{\nu} = (\nabla \psi)_\# \nu$, in addition, (still using $\nu$, $\tilde{\nu}$ to denote their respective densities),
\[
\tilde{\nu}(\nabla \psi(x)) \leq \nu(x), \text{ a.e. } x.
\]
Proof. Let \( \phi \) be a function such that \( \phi \) is bounded from below, \( \psi = \phi^* \) and \( \Delta \phi \leq d \).

1. First consider the case where \( \phi \) is convex. An important observation is that from the arithmetic-geometric inequality, we have for convex \( \phi \),

\[
(\det D^2 \phi(x))^{1/d} \leq \frac{1}{d} \Delta \phi(x), \text{ a.e. } x.
\]

Here we used the almost second-order differentiability of convex functions. Then \( \Delta \phi \leq d \) implies

\[
(11.1) \quad (\det D^2 \phi(x))^{1/d} \leq 1, \text{ a.e. } x.
\]

Recall that \( (\nabla \psi)_{\#} \nu = \bar{\nu} \) is equivalent to

\[
\bar{\nu}(E) = \int_{(\nabla \psi)^{-1}E} d\nu(x) \text{ for any measurable } E,
\]

where \( (\nabla \psi)^{-1}(E) \) denotes the inverse image of \( E \). We infer from (11.1) that \( \bar{\nu}(E) \geq \nu(E) \). If \( \bar{\nu} \) is absolutely continuous, then so is \( \nu \). Still using \( \nu, \bar{\nu} \) to denote their respective density, we then have

\[
\bar{\nu}(\nabla \psi(x)) \leq \nu(x), \text{ a.e. } x,
\]

which means the target density gets smaller than the source under the mapping \( \nabla \psi \).

2. Now consider the case where the convexity of \( \phi \) is not assumed. Note, the function \( \phi^{**} = \psi^* \) is convex and we have

\[
\Delta \phi^{**}(x) \leq \Delta \phi(x) \text{ on } S \triangleq \{ x : \phi(x) = \phi^{**}(x) \}.
\]

Moreover, \( \nabla \phi^{**} \) (which is nothing but \( \nabla \psi^* \)) is the inverse map (almost surely) to \( \nabla \psi \). If a point \( x \) is mapped by \( \nabla \psi \) (the optimal map from \( \nu \) to \( \bar{\nu} \)) outside of the contact set \( S \), then the point \( x \) belongs to the set where \( \psi \) is not differentiable, and this set has zero mass, so zero mass under \( \nu \) by absolute continuity. Therefore, what happens outside the contact set \( S \) does not affect the densities. As a result, we can replace \( \phi \) with \( \phi^{**} \) in step 1 and follow the same argument there to conclude the same result for \( \psi \).

\( \square \)

A similar argument does not work in the Laplacian contraction case, since we cannot use the arithmetic-geometric inequality as above.

11.1. Backward projection.

**Theorem 11.5.** Let \( X, Y \) be bounded convex open subsets of \( \mathbb{R}^d \), \( \mu \in P(X) \) and \( \nu \in P^{ac}(Y) \). Assume that the conditions of Theorem 10.1 are satisfied and \( \psi \in \mathcal{A}_{sh} \cap L^1(Y, d\nu) \) is the optimal dual solution for \( D_{2}(\mu, \mathcal{P}_{\leq \nu}^{sh}) \) obtained there. Then the unique projection \( \bar{\mu} \) of \( \mu \) onto \( \mathcal{P}_{\leq \nu}^{sh} \) is given by \( (\nabla \psi_0^*)_{\#} \mu \), where

\[
\psi_0(y) = \frac{1}{2} |y|^2 + \frac{1}{2} \psi(y).
\]

**Proof.** We skip the proof, since it is similar to Theorem 7.6. Only note that since \( Q_2(\psi) \in L^1(X, d\mu) \), it is finite for \( \mu \)-a.e. \( x \). Hence \( \psi_0^* \) is finite for \( \mu \)-a.e. \( x \). Then we can use the absolute continuity of \( \mu \) to infer the desired conclusion. \( \square \)
Theorem 11.6. Let $X, Y$ be bounded convex open subsets of $\mathbb{R}^d$, $\mu \in P(X)$ and $\nu \in P^{ac}(Y)$. Assume that the conditions of Theorem 10.1 are satisfied. Consider the following statements.

(i) The projection of $\mu$ onto $\mathcal{P}^{sh}_{\subseteq \nu}$ is $\nu$.

(ii) There is a Laplacian contraction $\phi$ such that $(\nabla \phi)_{\#} \mu = \nu$.

We always have that (i) implies (ii). If $\phi$ in (ii) is such that $\phi^* \in C_b$ and $\Delta \phi^* \geq d$, then (ii) implies (i).

Proof. That (i) implies (ii) follows from 11.5. To see the reverse implication, we note by definition of Laplacian contraction, there is a function $\xi$ such that $\xi$ is bounded from below, $\Delta \xi \geq d$ and $\phi = \xi^*$. Define

$\psi(y) \triangleq 2\phi^*(y) - |y|^2 = 2\xi^{**}(y) - |y|^2$.

Then $\psi$ is continuous and subharmonic by assumption. Using the convexity of $\phi$ we can write

$\nabla \phi(x) \cdot x = \phi^*(\nabla \phi(x)) + \phi(x), \text{ a.e. } x$.

Therefore, similar to Theorem 7.7, we have

$T_2(\mu, \nu) \leq \int \left( |x|^2 - 2\phi(x) \right) d\mu - \int \left( 2\phi^*(y) - |y|^2 \right) d\nu$

$= \int Q_2(\psi)(x) d\mu - \int \psi(y) d\nu \quad (\psi \text{ is subharmonic by (ii)})$

$\leq \sup_{g \in A_{sh} \cap C_b} \left\{ \int Q_2(g)(x) d\mu - \int g(y) d\nu \right\} \leq T_2(\mu, \nu)$,

which shows that $\psi$ is an optimizer of $D_2(\mu, \mathcal{P}^{sh}_{\subseteq \nu})$. Then the desired conclusion follows. \qed

11.2. Forward projection.

Theorem 11.7. Let $X, Y$ be bounded convex open subsets of $\mathbb{R}^d$, $\mu \in P^{ac}(X)$ and $\nu \in P(Y)$. Assume that $\psi \in A_{sh} \cap \text{Lip}(X)$ is the optimal dual solution for $D_2(\mathcal{P}^{sh}_{\mu \subseteq \nu}, \nu)$ obtained in Theorem 10.5. Then the unique projection of $\nu$ onto $\mathcal{P}^{sh}_{2, \mu \subseteq \nu}$ is given by $(\nabla \tilde{\psi}_0)_{\#} \nu$, where

$\tilde{\psi}_0(x) = \frac{1}{2} |x|^2 - \frac{1}{2} \psi(x)$.

We omit the proof, since it is similar to Theorem 7.6.

Theorem 11.8. Let $X, Y$ be bounded convex open subsets of $\mathbb{R}^d$, $\mu \in P^{ac}(X)$ and $\nu \in P(Y)$. Consider the following statements.

(i) The projection of $\nu$ onto $\mathcal{P}^{sh}_{\mu \subseteq \nu}$ is $\mu$.

(ii) There is a Laplacian expansion $\phi$ such that $(\nabla \phi)_{\#} \mu = \nu$.

We always have that (i) implies (ii). If $\phi$ in (ii) is such that $\phi^* \in C_b$ and $\Delta \phi^* \leq d$, then (ii) implies (i).

Proof. That (i) implies (ii) follows from 11.7. To see the reverse implication, we note by definition of Laplacian expansion, there is a function $\eta$ such that $\eta$ is bounded from below, $\Delta \eta \leq d$ and $\phi = \eta^*$. Define

$\psi(x) \triangleq |x|^2 - 2\phi^*(x) = |x|^2 - 2\eta^{**}(x)$.

Then the rest of the proof follows as in Theorem 11.6. \qed
APPENDIX A. $c$-TRANSFORMS

We recall a few properties of the $c$-transforms $Q_c(\cdot)$ and $Q_{\bar{c}}(\cdot)$:

$$
Q_c(g)(x) = \inf_{y \in Y} \{ g(y) + c(x, y) \}, \quad Q_{\bar{c}}(g)(y) = \sup_{x \in X} \{ g(x) - c(x, y) \}
$$

It is well-known that $Q_c(\cdot)$ preserves Lipschitz continuity, convexity and concavity. Since

$$Q_c(g) = -Q_{-c}(g),$$

$Q_{\bar{c}}$ has the same properties as $Q_c$. Moreover, for any function $g : Y \to \mathbb{R} \cup \{\infty\}$,

$Q_c(Q_c(g)) \leq g,$

and for any function $g : X \to \mathbb{R} \cup \{-\infty\}$,

$Q_c(Q_c(g)) \geq g.$

In particular,

$Q_c(g) = Q_c(Q_c(g)), \quad Q_{\bar{c}}(g) = Q_{\bar{c}}(Q_{\bar{c}}(g)).$

When $c(x, y) = |x - y|^2$, the $c$-transforms are written $Q_2(\cdot)$, $Q_2(\cdot)$.

**Lemma A.1.** Let $g$ be a function defined in a subset $\Omega$ of $\mathbb{R}^d$. The following identities hold.

(i) For $x \in \mathbb{R}^d$,

$$Q_2(g)(x) = |x|^2 - 2g_0^*(x), \text{ where } g_0(y) = \frac{1}{2}|y|^2 + \frac{1}{2}g(y).$$

(ii) For $y \in \mathbb{R}^d$,

$$Q_2(g)(y) = 2g_0^*(y) - |y|^2, \text{ where } g_0(x) = \frac{1}{2}|x|^2 - \frac{1}{2}g(x).$$

**Proof.** Straightforward calculations yield

$$Q_2(g)(x) = \inf_{y \in \Omega} \{ g(y) + |x - y|^2 \} = \inf_{y \in \Omega} \{ g(y) + |y|^2 - 2x \cdot y \} + |x|^2$$

$$= |x|^2 - 2\sup_{y \in \Omega} \left\{ x \cdot y - \left( \frac{1}{2}|y|^2 + \frac{1}{2}g(y) \right) \right\}$$

$$= |x|^2 - 2g_0^*(x).$$

Similarly

$$Q_2(g)(y) = \sup_{x \in \Omega} \{ g(x) - |x - y|^2 \} = \sup_{x \in \Omega} \{ g(x) - |x|^2 + 2x \cdot y \} - |y|^2$$

$$= 2\sup_{x \in \Omega} \left\{ x \cdot y - \left( \frac{1}{2}|x|^2 - \frac{1}{2}g(x) \right) \right\} - |y|^2$$

$$= 2g_0^*(y) - |y|^2.$$ 

\[\square\]

**Lemma A.2.** Let $g : \mathbb{R}^d \to \mathbb{R}$ be a function.

(i) If $\frac{1}{2}|x|^2 - g(x)$ is convex, then $g^*(y) - \frac{1}{2}|y|^2$ is convex.

(ii) If $g(y) - \frac{1}{2}|y|^2$ is convex, then $\frac{1}{2}|x|^2 - g^*(x)$ is convex.

(iii) If $g$ is a lower semicontinuous proper convex function, then

$$\frac{1}{2}|x|^2 - g(x) \text{ is convex if and only if } g^*(y) - \frac{1}{2}|y|^2 \text{ is convex.}$$
In either case of (iii), we have $g \in C^1$. In matrix form, (iii) can be rewritten as: $D^2g \leq Id$ if and only if $D^2g^{**} \geq Id$. Second order derivatives of convex functions are understood in distributional sense and $Id$ is the $d \times d$ identity matrix.

Proof. (i) and (ii) follow from the straightforward calculations

$$Q_2\left([|x|^2 - 2g]\right)(y) = \sup_x \left\{|x|^2 - 2g(x) - |x-y|^2\right\} = 2g^*(y) - |y|^2,$$

and

$$Q_2\left([2g - |y|^2]\right)(x) = \inf_y \left\{2g(y) - |y|^2 + |x-y|^2\right\} = |x|^2 - 2g^*(x).$$

If $g$ is a lower semicontinuous proper convex function, then $g = g^{**}$. So (iii) follows from (ii). If either case of (iii) is true, then

$$g^*(y) = \left(g^*(y) - \frac{1}{2}|y|^2\right) + \frac{1}{2}|y|^2$$

is uniformly convex, hence $g = g^{**} \in C^1$. \hfill \Box

**Appendix B. Convergence of Subharmonic Functions**

The following result is adapted from [27, Theorem 4.1.9] to serve its purpose in our setting.

**Lemma B.1.** Let $X$ be a connected open subset of $\mathbb{R}^d$ and $\psi_n$ a sequence of subharmonic functions such that $\int_X |\psi_n|d\mathcal{L}$ is bounded by some constant $A > 0$. Then up to a subsequence $\psi_n$ converges almost surely to a subharmonic function $\psi \in L^1(X, d\mathcal{L})$ in the usual sense of Definition 9.1.

**Proof.** 1. By identifying $L^1(X, d\mathcal{L})$ with a subset of $M(X)$, we may assume that $\psi_n$ converges weakly to some $\xi \in M(X)$. Since $\psi_n$ is subharmonic, $\Delta \psi_n$ is a non-negative distribution, thus a nonnegative measure by [27, Theorem 2.1.7]. Hence we may also assume that $\Delta \psi_n$ converges weakly to some nonnegative distribution $\eta \geq 0$. Then for any $\phi \in C^0_0(X)$,

$$\int_X \Delta \phi \xi(x) = \lim_{n \to \infty} \int_X \Delta \phi \psi_n(x) = \lim_{n \to \infty} \int_X \phi \Delta \psi_n dx = \int_X \phi d\eta(x).$$

Therefore $\Delta \xi = \eta \geq 0$ in the sense of distribution. By [27, Theorem 4.1.8], $\xi$ is a subharmonic function $\psi \in L^1_{loc}(X, d\mathcal{L})$ in the usual sense of Definition 9.1.

2. Let $B_r = \{|x| \leq r\}$, $\delta > 0$ and $0 \leq \rho \in C^\infty_0(\mathbb{R}^d)$ be a radially symmetric function supported in $B_1$ such that $\int \rho = 1$. Given a compact set $K \subset X$, we claim that, as $n \to \infty$,

$$\psi_n * \rho_\delta(x) \to \psi * \rho_\delta(x) \quad \text{uniformly for } x \in K.$$

By assumption

(B.1) \[ |\int \psi_n \varphi dx| \leq A \sup_x |\varphi|, \forall \varphi \in C^\infty_0(K_\delta), \]

where $K_\delta$ is the compact set given by $K + B_\delta$. $\delta$ is small so that $K_\delta \subset X$. Since $\psi_n$ converges to the function $\psi$ in distribution, it follows that

(B.3) \[ |\int \psi \varphi dx| \leq A \sup_x |\varphi|, \forall \varphi \in C^\infty_0(K_\delta). \]
Let $\epsilon > 0$. Since $K$ is compact, there is a finite net $K_{\text{net}} = \{x_j : j = 1, ..., N\} \subset K$ satisfying: for each $x \in K$ there is a $x_{j_0} \in K_{\text{net}}$ such that

$$2A \sup_{z \in \mathbb{R}^d} \left| \rho \left( \frac{x - z}{\delta} \right) - \rho \left( \frac{x_{j_0} - z}{\delta} \right) \right| \leq \frac{\epsilon}{2}.$$  

This is feasible since $\rho$ is uniformly continuous on $\mathbb{R}^d$. For each $x_j \in K_{\text{net}}$, we have

$$\psi_n * \rho_\delta(x_j) = \int \psi_n(z) \rho \left( \frac{x_j - z}{\delta} \right) dz \to \int \psi(z) \rho \left( \frac{x_j - z}{\delta} \right) dz.$$  

Hence, there is $n_0$ such that

$$|\psi_n * \rho_\delta(x_j) - \psi * \rho_\delta(x_j)| \leq \frac{\epsilon}{2} \text{ for } j = 1, ..., N, \ n \geq n_0.$$  

Therefore, for $x \in K$, $n \geq n_0$ and some $x_{j_0} \in K_{\text{net}}$ depending on $x$,

$$|\psi_n * \rho_\delta(x) - \psi * \rho_\delta(x)|$$

$$\leq \left| \psi_n * \rho_\delta(x) - \psi * \rho_\delta(x) - (\psi_n * \rho_\delta(x_{j_0}) - \psi * \rho_\delta(x_{j_0})) \right| + \frac{\epsilon}{2}$$

$$= \left| \left( \int (\psi_n(z) - \psi(z)) \rho \left( \frac{x - z}{\delta} \right) - \rho \left( \frac{x_{j_0} - z}{\delta} \right) dz \right) + \frac{\epsilon}{2} \right|$$

$$\leq 2A \sup_{z \in \mathbb{R}^d} \left| \rho \left( \frac{x - z}{\delta} \right) - \rho \left( \frac{x_{j_0} - z}{\delta} \right) \right| + \frac{\epsilon}{2} \leq \epsilon.$$  

Note the second inequality is due to the fact that

$$z \mapsto \rho \left( \frac{x - z}{\delta} \right) - \rho \left( \frac{x_{j_0} - z}{\delta} \right)$$

is a function in $C_0^\infty(K_\delta)$ so that (B.2)-(B.3) apply.

3. Let $g_K \in C_0^\infty(X)$ such that $0 \leq g_K \leq 1$ and $g_K = 1$ on $K$. Let $\epsilon > 0$ and $\delta > 0$ be smaller than the distance between $\partial X$ and $\text{supp}(g_K)$. By the definition of $\rho_\delta$ (in step 2) and the subharmonicity,

$$\psi_n * \rho_\delta(x) \geq \psi_n(x), \ \psi * \rho_\delta(x) \geq \psi(x), \ \forall x \in K.$$  

Together with (B.1), this implies that for large $n$,

$$\psi * \rho_\delta(x) + \epsilon - \psi_n(x) > 0, \ \psi * \rho_\delta(x) + \epsilon - \psi(x) > 0, \ \forall x \in K.$$  

Then, for $x \in K$,

$$|\psi_n(x) - \psi(x)| \leq |\psi_n(x) - \psi(x)| g_K(x)$$

$$\leq |\psi * \rho_\delta(x) + \epsilon - \psi_n(x)| g_K(x) + |\psi * \rho_\delta(x) + \epsilon - \psi(x)| g_K(x)$$

$$= (\psi * \rho_\delta(x) + \epsilon - \psi_n(x)) g_K(x) + (\psi * \rho_\delta(x) + \epsilon - \psi(x)) g_K(x).$$  

Therefore

$$\limsup_{n \to \infty} \int_K |\psi_n - \psi| dx \leq 2 \int_X (\psi * \rho_\delta + \epsilon - \psi) g_K dx.$$  

Sending $\delta \to 0$, $\epsilon \to 0$, we obtain $\psi_n \to \psi$ in $L^1(K, dx)$. Since $K$ is an arbitrary compact set, it follows that $\psi_n \to \psi$ in $L^1_{\text{loc}}(X, dx)$.

4. Since the open set $X$ can be covered by a countable number of closed balls, on each of these balls we may extract a subsequence of $\psi_n$ which converges almost
surely to $\psi$. Utilizing a diagonal procedure, we obtain that up to a subsequence $\psi_n$ converges to $\psi$ almost surely. By Fatou lemma,
\[
\liminf_{n \to \infty} \int_X |\psi_n| \, dx \geq \int_X |\psi| \, dx,
\]
hence $\psi \in L^1(X, dx)$. □

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