Dispersive blow-up and persistence properties for the Schrödinger–Korteweg–de Vries system

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Abstract
We study the dispersive blow-up phenomena for the Schrödinger–Korteweg–de Vries (S-KdV) system. Roughly, dispersive blow-up has been called to the development of point singularities due to the focussing of short or long waves. In mathematical terms, we show that the existence of this kind of singularities is provided by the linear dispersive solution by proving that the Duhamel term is smoother. It seems that this result is the first regarding systems of nonlinear dispersive equations. To obtain our results we use, in addition to smoothing properties, persistence properties for solutions of the IVP in fractional weighted Sobolev spaces which we establish here.

Keywords: dispersive blow-up, persistence property, Schrödinger–Korteweg–de Vries system
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1. Introduction and main results

1.1. The model
This paper is concerned with properties of solutions of the initial value problem (IVP) associated to the Schrödinger–Korteweg–de Vries (S-KdV) system,
where $u = u(t,x)$ is a complex-valued function and $v(t,x)$ is a real-valued function. This system governs the interactions between shortwaves $u = u(t,x)$ and longwaves $v = v(t,x)$ and has been studied in several fields of physics and fluid dynamics (see \cite{1, 11, 15, 24, 26}).

The system (1.1) appears as a particular case (under appropriate transformations) of the more general system

\[
\begin{aligned}
  &i\partial_t S + is\partial_x S + \partial_x^2 S = \alpha \partial_x + \gamma |S|^2 S, \\
  &\partial_t L + c_L \partial_x L + \nu P(D_x)L + \lambda \partial_x L^2 = \beta \partial_x |S|^2, \\
  &u(x,0) = u_0(x), \quad v(x,0) = v_0(x),
\end{aligned}
\]

where $S$ is a complex-valued function representing the short wave, $L$ is a real-valued function representing the long wave and $P(D_x)$ is a differential operator with constant coefficients. This system has received considerable attention because of the vast variety of physical settings in which it arises. For instance, the internal gravity-wave packet and the capillary-gravity interaction wave when (see \cite{9, 12, 15})

\[\beta < 0, \quad c_S = c_L = \gamma = \lambda = \nu = 0.\]

Furthermore, when $\gamma = 0$ the previous system has been derived as a model for the resonant ion-sound/Langmuir wave interaction in plasma physics under the assumption that the ion-sound wave is unidirectional (see \cite{22, 24}). Moreover, setting $\nu = 0$ and $\lambda \in \{0, 1\}$ this system appears in the general theory of water wave interaction in a nonlinear medium. Finally, as a model for the motion of two fluids under capillary-gravity waves in a deep water flow (see \cite{11})

\[P(D_x) = D_x \partial_x, \quad \nu = 1, \quad c_S = c_L = \gamma = \lambda = 0, \quad \alpha, \beta > 0,\]

or the motion of two fluids under a shallow water flow ($c_S = c_L = \gamma = \lambda = 0, \alpha, \beta > 0$, see \cite{11}). From now on we shall only focus on the S-KdV system (1.1).

The Schrödinger–Korteweg–de Vries system (1.1) has been shown not to be a completely integrable system (see \cite{3}). Therefore the solvability of (1.1) is dependent upon the method of evolution equations.

The IVP (1.1) has been extensively studied from the viewpoint of local and global well-posedness. Inspired in the results obtained for the famous Korteweg–de Vries (KdV) \cite{17} and the cubic Schrödinger equation \cite{27} several authors have studied the IVP (1.1). In general, a coupled system like (1.1) is more difficult to handle in the same spaces as in the space the single equation is solved. In the case of the system (1.1) this is due to the antisymmetric nature of the characteristics of each linear part. In [2] Bekiranov, Ogawa and Ponce showed that the coupled system (1.1) is locally well-posed in $H^s(\mathbb{R}) \times H^{s-\frac{1}{2}}(\mathbb{R})$ with $s > 0$. In \cite{8} Corcho and Linares extended this result for weak initial data $(u_0, v_0) \in H^s(\mathbb{R}) \times H^{s-\frac{1}{2}}(\mathbb{R})$ for various values of $k$ and $s$, where the lowest admissible values are $k = 0$ and $s = -\frac{1}{2} + \delta$ with $0 < \delta \leq \frac{1}{4}$. The end-point $(k,s) = (0, -\frac{1}{2})$ was treated in \cite{13} by Guo and Wang. We observe that no local/global well-posedness results in weighted Sobolev spaces have been registered in the literature as far as we know.

1.2. Dispersive blow-up

The aim of this work is to study the dispersive blow-up for solutions of the S-KdV system. The notion of a dispersive singularity first appeared in \cite{4}, where the authors studied the linearized Korteweg–de Vries equation.
The main idea of the analysis outlined there was that a smooth and bounded initial data $u_0(x)$ having finite energy could result in a solution developing pointwise singularities in finite time. Roughly, dispersive blow up of wave equations is a focusing phenomenon due to both the unbounded domain and the unboundedness of the linear dispersion (so that energy can travel arbitrarily fast). These two aspects allow arranging initial data in such a way that at a certain prescribed time, infinitely many short-wave component come together, coalescing in a single point and consequently resulting in a loss of spatial smoothness of the solution at that time.

The rigorous mathematical analysis of dispersive blow-up started in [5] where Bona and Saut proved the existence of dispersive blow-up for solutions of the generalized KdV equation. More precisely, they proved the following

**Theorem 1 ([5])**. Let $T > 0$ be given and let $\{(x_n,t_n)\}_{n=1}^{\infty}$ be a sequence of points in $\mathbb{R} \times (0,T)$ without finite limit points and such that $\{t_n\}_{n=1}^{\infty}$ is bounded below by a positive constant. Let either $s = 0$ and $k = 1$ or $s \geq 2$ and $k \geq 1$ an arbitrary integer. Then there exists $\psi \in H^s(\mathbb{R}) \cap C^\infty(\mathbb{R})$ such that the solution of the IVP

$$
\begin{cases}
  u_t + uu_x + u_{xxx} = 0, \\
  u(x,0) = \psi(x)
\end{cases}
$$

satisfies

1. $u$ lies in $L^\infty([0,T] : H^s(\mathbb{R})) \cap L^2([0,T] : H^{s+k+1}_{loc}(\mathbb{R}))$, or in $C([0,T] : H^s(\mathbb{R})) \cap L^2([0,T] : H^{s+1}_{loc}(\mathbb{R}))$, if $s \geq 2$.
2. $\partial^k_t u$ is continuous on $\mathbb{R} \times (0,T) \setminus \{U_{n=1}^{\infty} \{(x_n,t_n)\}\}$, and
3. $\lim_{(x,t) \to (x_n,t_n)} \partial^k_t u(x,t) = +\infty$ for $n = 1, 2, \ldots$.

The main idea behind the proof is to show that the Duhamel term associated to the solution of the IVP is smoother than the linear term of the solution. In [20] Linares and Scialom proved for $k \geq 2$ by means of the smoothing effects established for the linear KdV equation without using weighted Sobolev spaces (see also [21]). Recently, Linares, Ponce and Smith [19] using fractional weighted spaces improved the previous result in the case $k = 1$, i.e. for the KdV equation.

The analogous phenomena also appears in other linear dispersive equations, such as the linear Schrödinger equation and the free surface water waves system linearized around the rest state [7]. In [7] Bona and Saut constructed smooth initial data with point singularities for solutions of the linear Schrödinger equation. Bona, Ponce, Saut and Sparber [6] established the dispersive blow-up for the semilinear Schrödinger equation in dimension $n$ and other Schrödinger type equations. The main tools employed to show these results were the intrinsic smoothing effects of these dispersive equations. We shall remark that the only $n$-dimensional result regarding dispersive blow-up is this one just above refereed for the nonlinear Schrödinger equation.

### 1.3. Main results

Inspired in the dispersive blow-up results for the KdV and Schrödinger equations it was natural to ask what was the situation for solutions for the Schrödinger–Korteweg–de Vries system concerning this property. In our study we got the following answer.

**Theorem 1.1.** There exist initial data

$$
u_0 \in C^\infty(\mathbb{R}) \cap H^{3/2-}(\mathbb{R}), \quad v_0 \in C^\infty(\mathbb{R}) \cap H^{3/2-}(\mathbb{R}),$$

$$u_t + uu_x + u_{xxx} = 0, \quad x \in \mathbb{R}, \quad t \in (0,\infty).$$
for which the corresponding solution \((u,v)(\cdot,\cdot)\) of the IVP \((1.1)\) provided by theorem 1.2 (below):

\[
u \in C([0,T] : H^{-1/2}(\mathbb{R})), \quad v \in C([0,T] : H^{3/2}(\mathbb{R})),
\]
satisfies that there exists \(T^* \in [0,T]\) such that

\[
u(\cdot,t^*) \notin C^{1,1+}(\mathbb{R}), \quad v(\cdot,t^*) \notin C^1(\mathbb{R}).
\]

To prove this result, we construct first initial data borrowing some ideas in [7] and [19]. To treat the nonlinear problem is not straightforward, as we shall see, in our case the NLS-KdV system presents several new difficulties because of its coupling terms. In addition to the smoothing effects, the new key ingredient in our arguments is the persistence property of solutions of the IVP \((1.1)\) on weighted spaces, which allow us to close some nonlinear estimates for the solution.

We have constructed initial data which yield the formation of point singularities for solutions of the linear problem. Physically it shows a focusing phenomena at a linear level. Moreover, our result shows that this phenomena is purely linear since the presence of that singularity does not depend on the nonlinear structure of the problem.

1.3.1. Persistence properties. Due to the presence of the KdV structure in the system we need to use weighted spaces in order to show that the Duhamel term is smoother than the linear part of the equation.

As we commented above even in the usual Sobolev spaces the coupling of the Schrödinger equation and KdV equation introduces some difficulties because of the structure of the ‘symbols’ of the linear equations. In our analysis we need the following result which includes local well-posedness of the IVP \((1.1)\) in fractional Sobolev spaces and a persistence property of these solutions in these spaces. More precisely,

**Theorem 1.2.** Let \(s, r_1, r_2\) be positive numbers such that \(s > 3/4, s + 1/2 \geq r_1 \) and \(s \geq 2r_2\) and consider initial data

\[
(u_0, v_0) \in H^{s+\frac{1}{2}}(\mathbb{R}) \cap L^2(|x|^{2r_1} dx) \times H^s(\mathbb{R}) \cap L^2(|x|^{2r_2} dx).
\]

Then there exist \(T = T(\|u_0\|_{s+\frac{1}{2}} + \|v_0\|_s) > 0\) and a unique solution \((u(t), v(t))\) of the IVP \((1.1)\) satisfying

\[
u \in C([0,T]; H^{s+\frac{1}{2}}(\mathbb{R}) \cap L^2(|x|^{2r_1} dx) ) , \quad v \in C([0,T]; H^s(\mathbb{R}) \cap L^2(|x|^{2r_2} dx)), \quad \forall x \in \mathbb{R},
\]

\[
\|D^s_x \partial_x u\|_{L^{\infty}} + \|D^s_x \partial_x v\|_{L^{\infty}} < \infty, \quad \forall x \in \mathbb{R},
\]

\[
\|u\|_{L^2} + \|v\|_{L^2} < \infty,
\]

\[
\|\partial_x u\|_{L^{\infty}} + \|\partial_x v\|_{L^{\infty}} < \infty,
\]

with the local existence time satisfying:

\[
T = T(\|u_0\|_{s+\frac{1}{2}} + \|v_0\|_s) \to +\infty \quad \text{as} \quad \|u_0\|_{s+\frac{1}{2}} + \|v_0\|_s \to 0.
\]

Moreover, given \(T' \in (0,T)\), the map data solution \((u_0, v_0) \mapsto (u,v)(t)\) from \(H^{s+\frac{1}{2}}(\mathbb{R}) \cap L^2(|x|^{2r_1} dx) \times H^s(\mathbb{R}) \cap L^2(|x|^{2r_2} dx)\) to the class defined by \((1.3)-(1.4)\) is Lipschitz continuous.
The proof of theorem 1.2 uses the contraction mapping principle which is combined with smoothing properties of solutions of the associated linear problems for the Schrödinger and KdV equations. The key ingredient in our analysis to prove the persistence property is a new pointwise formula that allows to commute the fractional weights $|x|^s$ with the Schrödinger group $e^{it\Delta}$ and the Airy group $e^{-t\partial_x^3}$. This pointwise formula was deduced by Fonseca, Linares and Ponce in [10].

**Remark 1.1.** The result in theorem 1.2 is not available in the literature and as the case for the dispersive blow-up it seems the first one for systems.

Next, we introduce some notation we will employ along this work.

### 1.4. Notation

Let $1 \leq p, q \leq \infty$ and $f : \mathbb{R} \times [0,T] \to \mathbb{R}$. We define the norm

$$
\|f\|_{L^p_x L^q_T} := \left( \int_{\mathbb{R}} \left( \int_0^T |f|^q \, dt \right)^{p/q} \, dx \right)^{1/p},
$$

with the usual modifications when $p = \infty$ or $q = \infty$. Similarly for $\|f\|_{L^p_T L^q_x}$.

We will denote $\hat{f}$ the Fourier transform of a tempered distribution $f$. For any $s \in \mathbb{R}$, we define the homogeneous derivatives $D^s f$ by Fourier transform

$$
\hat{D^s f}(\xi) := |\xi|^s \hat{f}(\xi).
$$

As usual for $s \in \mathbb{R}$ we shall denote by $H^s(\mathbb{R})$ the standard $L^2$-based Sobolev space:

$$
H^s(\mathbb{R}) := \{ f \in S^\prime(\mathbb{R}) : \|f\|_{s,2} = \|J^s f\|_2 < \infty \},
$$

where $J^s$ denotes the Bessel operator of order $-s$. To simplify the notation we shall also use $\| \cdot \|_s$ instead of $\| \cdot \|_{s,2}$.

Finally, in the reminder of this work we will adopt the following notation, for $s \geq 0$

$$
H^s_+ (\mathbb{R}) = \bigcup_{s' > s} H^{s'} (\mathbb{R}), \quad H^s_- (\mathbb{R}) = \bigcup_{0 \leq s' < s} H^{s'} (\mathbb{R}).
$$

### 1.5. Organization of this paper

This paper is organized as follows. In section 2 we state a series of results needed in our analysis. In section 3 we construct the initial data which develop dispersive blow-up. In section 4 we establish theorem 1.2. Finally, in section 5 we show our main result theorem 1.1.

### 2. Preliminaries

#### 2.1. Smoothing properties

In this subsection some technical results on the smoothing properties of the free Schrödinger group $S(t) := e^{it\Delta}$ and the KdV group $V(t) := e^{-t\partial_x^3}$ are reviewed. They will find use in sections 3–5.

Next lemma provides the smoothing effects of Kato type [14] for solutions of the linear KdV equation.
Lemma 2.1 ([16]).
\[
\sup_x \| \partial_t V(t)v_0 \|_{L^2_t} \leq c \| v_0 \|_{L^2}
\] (2.1)
and
\[
\| \partial_x \int_0^t V(t-t')F(\cdot,t') \, dt' \|_{L^2_x} \leq c \| F \|_{L^2_t L^2_x}.
\] (2.2)

The following lemma gives us the smoothing effects of Kato type for solutions of the linear Schrödinger equation in dimension \( n = 1 \).

Lemma 2.2 ([16]).
\[
\sup_x \| D^{1/2} S(t) u_0 \|_{L^2} \leq c \| u_0 \|_{L^2},
\] (2.3)
\[
\| D^{1/2} \int_0^t S(t-t') F(\cdot, t') \, dt' \|_{L^2_x} \leq c \| F \|_{L^1_t L^2_x T}.
\] (2.4)
and
\[
\sup_x \| \partial_x \int_0^t S(t-t') F(\cdot, t') \, dt' \|_{L^2_x} \leq c \| F \|_{L^1_t L^2_x T}.
\] (2.5)

Next, we present Strichartz estimates for both groups \( S(t) \) and \( V(t) \).

**Lemma 2.3.** Let \( 2 \leq p, q \leq \infty \) such that \( \frac{2}{q} = \frac{1}{2} - \frac{1}{p} \). Then the following holds
\[
\| S(t) f \|_{L^q_t L^p} \leq c \| f \|_{L^2_t L^2}.
\] (2.6)

**Lemma 2.4.** For \( (\alpha, \theta) = [0, 1/2] \times [0, 1] \) it holds that
\[
\| D^{\alpha \theta/2} V(t) f \|_{L^q_t L^p} \leq c \| f \|_{L^2_t L^2}.
\] (2.7)
where \((q,p) = (6/\theta(\alpha + 1), 2/(1 - \theta))\).

For a proof of these results see for instance [18]. Finally, we introduce maximal function estimates for the linear solutions.

**Lemma 2.5.** For \( s > 1/2 \) and \( \rho_1 > 1/4 \), it holds that
\[
\| S(t) f \|_{L^\infty_t L^\rho_1} \leq c (1 + T)^{\rho_1} \| f \|_{s,2}.
\] (2.8)

For \( s \geq 1/4 \), it holds
\[
\| S(t) f \|_{L^\infty_t L^\rho_1} \leq c \| f \|_{s,2}.
\] (2.9)

For \( s > 3/4 \) and \( \rho_2 > 3/4 \) it holds that
\[
\| V(t) f \|_{L^\infty_t L^\rho_2} \leq c (1 + T)^{\rho_2} \| f \|_{s,2}.
\] (2.10)

It holds
\[
\| V(t) f \|_{L^\infty_t L^\rho_2} \leq c \| D^{1/4}_t f \|_{L^2_t}.
\] (2.11)
Proof. See [25, 29] for a proof of (2.8) and (2.9). For a proof of (2.10) see [28]. For a proof of (2.11) see [16].

To end up this subsection we list the following interpolated estimates.

**Lemma 2.6.**

\[
\|V(t)f\|_{L^6_tL^{10}x} \leq c\|f\|_{L^2_tL^2x}
\]

(2.12)

and

\[
\|D_1^{1/2}V(t)f\|_{L^{20/3}_tL^5x} \leq c\|D_1^{1/4}f\|_{L^2_tL^2x}.
\]

(2.13)

**Proof.** The estimates (2.12) and (2.13) follow by interpolating (2.1) and (2.11). See [16].

### 2.2. Weighted estimates

Since we are going to deal with weighted spaces, the next interpolation estimates will be very helpful.

**Lemma 2.7 ([23]).** Let \( a, b > 0 \). Assume that \( J^a f = (1 - \Delta)^{a/2} f \in L^2(\mathbb{R}^n) \) and \( \langle x \rangle^b f = (1 + |x|^2)^{b/2} f \in L^2(\mathbb{R}^n) \). Then, for any \( \theta \in (0, 1) \)

\[
\|J^{\theta a}(\langle x \rangle^{(1-\theta)b}f)\|_{L^2_x} \leq c\|\langle x \rangle^{b}f\|_{L^2_x}^{\theta}\|f\|_{L^2_x}^{1-\theta}
\]

(2.14)

and

\[
\|(\langle x \rangle^{(1-\theta)b}D_1^{\theta a}f)\|_{L^2_x} \leq c\|\langle x \rangle^{b}f\|_{L^2_x}^{1-\theta}\|D^a f\|_{L^2_x}^{\theta}.
\]

(2.15)

Now, we recall a very useful formula derived in [10] for the Airy group: for \( \beta \in (0, 1) \) and \( t \in \mathbb{R} \), the following formula holds:

\[
|x|^\beta V(t)u_0(x) = e^{it\Delta}(|x|^\beta u_0) + e^{it\Delta}\left\{ \Lambda_{\beta, \theta}(\hat{u}_0)(\xi) \right\}_v,
\]

(2.16)

with

\[
\|\Phi_{t,\beta}(\hat{f})\|_{L^2}\leq c(1 + |t|)^{1/2}\|f\|_{L^2_x}.
\]

(2.17)

Using the same arguments as in [10], we can also deduce a formula for solutions of the linear Schrödinger equation.

**Lemma 2.8.**

1. Let \( \beta \in (0, 1) \) and \( t \in \mathbb{R} \). Then the following pointwise formula holds

\[
|x|^\beta S(t)u_0(x) = e^{it\Delta}(|x|^\beta u_0) + e^{it\Delta}\left\{ \Lambda_{\beta, \theta}(\hat{u}_0)(\xi) \right\}_v,
\]

(2.18)

where

\[
\|\Lambda_{\beta, \theta}(\hat{u}_0)(\xi)\|_{L^2} \leq c(1 + |t|)\left(\|u_0\|_{L^2} + \|D_1^\beta u_0\|_{L^2}\right).
\]

(2.19)
\(\beta\) \in (0, 1), \ t \in \mathbb{R} \) and \((p, q)\) such that \(2q = 1 - \frac{1}{p}\). Then,
\[
\|x|^{\beta} S(t) u_0\|_{L^q_t L^p_x} \leq c \|x|^{\beta} u_0\|_{L^q_x} + c(1 + |t|) \left(\|u_0\|_{L^2_x} + \|D_x^0 u_0\|_{L^2_x}\right). \tag{2.20}
\]

**Proof.** The first part of the lemma follows the same arguments used to prove (2.16) in [10]. For the second part we use Strichartz estimate (2.6) and the pointwise formula (2.18) to obtain
\[
\|x|^{\beta} S(t) u_0\|_{L^q_t L^p_x} \leq c \|S(t)(x|^{\beta} u_0)\|_{L^q_t L^p_x} + \|S(t)\{\Lambda_x^{\beta}(\hat{u}_0)(\xi)\}^{\vee}\|_{L^q_t L^p_x}
\leq c \|x|^{\beta} u_0\|_{L^q_x} + c(1 + |t|) \left(\|u_0\|_{L^2_x} + \|D_x^0 u_0\|_{L^2_x}\right),
\]
which concludes the proof. \(\square\)

The estimates (2.17)–(2.19) will be crucial in the proof of theorem 1.2.

### 2.3. Leibniz rule

To end up this section we present the fractional Leibniz rule which will be employed to deal with nonlinear terms.

**Theorem 2.9 ([16]).**

1. For \(s > 0\) and \(1 < p < \infty\), it holds
\[
\|D_x^s (fg) - fD_x^s g - gD_x^s f\|_{L^p} \leq c \|f\|_{L^\infty} \|D_x^s g\|_{L^p}. \tag{2.21}
\]

2. Let \(\alpha \in (0, 1)\), \(\alpha_1, \alpha_2 \in [0, \alpha]\) with \(\alpha = \alpha_1 + \alpha_2\). Let \(p_1, p_2, q_1, q_2 \in (1, \infty)\) with
\[
1 = \frac{1}{p_1} + \frac{1}{p_2} \quad \text{and} \quad \frac{1}{2} = \frac{1}{q_1} + \frac{1}{q_2}. \]
Then,
\[
\|D_x^\alpha (fg) - fD_x^\alpha g - gD_x^\alpha f\|_{L^1 L^{p_1}_x L^{q_1}_T} \leq c \|D_x^\alpha f\|_{L^1 L^{p_1}_x L^{q_1}_T} \|D_x^\alpha g\|_{L^1 L^{p_2}_x L^{q_2}_T}. \tag{2.22}
\]

### 3. Construction of the initial data

In this section, attention is turned to understand dispersive blow-up for each linear equation.

Let us divide the analysis in two cases, the linear case of the Schrödinger equation and the linear case of the KdV equation:

#### 3.1. Linear case: Schrödinger equation

We will follow the argument employed in [6] and [7] with some modifications.

Consider the IVP associated to the linear Schrödinger equation:
\[
\begin{align*}
\{ & i\partial_t u + \partial_x^2 u = 0, \\
& u(x, 0) = u_0(x),
\end{align*}
\tag{3.1}
\]

Now, recall that for any \(u_0 \in L^2(\mathbb{R})\), the unique solution \(u\) of (3.1) has the representation:
\[
u(x, t) = \frac{1}{(4\pi it)^{1/2}} \int_{\mathbb{R}} e^{\frac{i|x-y|^2}{4it}} u_0(y) \, dy,
\]

The estimates (2.17)–(2.19) will be crucial in the proof of theorem 1.2.

---

**Theorem 2.9 ([16]).**

1. For \(s > 0\) and \(1 < p < \infty\), it holds
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\|D_x^s (fg) - fD_x^s g - gD_x^s f\|_{L^p} \leq c \|f\|_{L^\infty} \|D_x^s g\|_{L^p}. \tag{2.21}
\]

2. Let \(\alpha \in (0, 1)\), \(\alpha_1, \alpha_2 \in [0, \alpha]\) with \(\alpha = \alpha_1 + \alpha_2\). Let \(p_1, p_2, q_1, q_2 \in (1, \infty)\) with
\[
1 = \frac{1}{p_1} + \frac{1}{p_2} \quad \text{and} \quad \frac{1}{2} = \frac{1}{q_1} + \frac{1}{q_2}. \]
Then,
\[
\|D_x^\alpha (fg) - fD_x^\alpha g - gD_x^\alpha f\|_{L^1 L^{p_1}_x L^{q_1}_T} \leq c \|D_x^\alpha f\|_{L^1 L^{p_1}_x L^{q_1}_T} \|D_x^\alpha g\|_{L^1 L^{p_2}_x L^{q_2}_T}. \tag{2.22}
\]

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\tag{3.1}
\]

Now, recall that for any \(u_0 \in L^2(\mathbb{R})\), the unique solution \(u\) of (3.1) has the representation:
\[
u(x, t) = \frac{1}{(4\pi it)^{1/2}} \int_{\mathbb{R}} e^{\frac{i|x-y|^2}{4it}} u_0(y) \, dy,
\]
where the integral is taken in the improper Riemann sense.

Let $u_0$ be defined as

$$u_0(x) := \frac{e^{-i\alpha(x-x_0)^2}}{(1+x^2)^{3/4}}, \quad \alpha > 0.$$  \hspace{1cm} (3.2)

It is not difficult to show that $u_0 \in H^s(\mathbb{R})$ for any $s \in [0, 2)$, but $u_0 \not\in H^2(\mathbb{R})$. Moreover, $u_0$ satisfies

$$\langle x \rangle^2 u_0(x) \in L^2(\mathbb{R}).$$

(See [6, 7] for the details).

In the next lemma we present a precise statement of the dispersive blow-up for the linear Schrödinger equation.

**Lemma 3.1.** Let $\alpha \in \mathbb{R}$ and $x_0 \in \mathbb{R}$ fixed and let $\varepsilon \in (0, \frac{1}{2})$. Consider the point $(x^*, t^*) = (x_0, \frac{1}{4\alpha})$ and the initial data

$$u_0(x) := \frac{e^{-i\alpha(x-x_0)^2}}{(1+x^2)^{3/4}}.$$  

Then, the initial data satisfies:

$$u_0 \in C^\infty(\mathbb{R}) \cap H^2(\mathbb{R}) \cap L^\infty(\mathbb{R}),$$

and the associated global in-time solution $u \in C(\mathbb{R}; H^2(\mathbb{R}))$ of (3.1) has the following properties:

1. For any time $t \in \mathbb{R}$ with $t \neq t^*$, the solution $u(\cdot, t) \in C^{1, \frac{1}{2}+\varepsilon}(\mathbb{R})$.
2. At time $t^*$, the solution $u(\cdot, t^*) \in C^{1, \frac{1}{2}+\varepsilon}(\mathbb{R} \setminus \{x^*\})$.
3. At time $t^*$, the solution $u(\cdot, t^*) \not\in C^{1, \frac{1}{2}+\varepsilon}(\mathbb{R})$.

**Proof.** See [6] or [7] for a detailed proof. \hfill $\square$

This concludes the case of the free Schrödinger equation. Now the attention is turned to construct the initial data for the linear Korteweg–de Vries equation.

### 3.2. Linear case: Korteweg–de Vries equation

For this case we recall the data constructed in the proof of dispersive blow-up for the KdV equation given in [19] (see section 3).

Consider the linear IVP associated to the linear Korteweg–de Vries equation:

$$\begin{cases}
\partial_t v + \partial_x^3 v = 0, & x \in \mathbb{R}, \ t > 0, \\
v(x, 0) = v_0(x),
\end{cases}$$  \hspace{1cm} (3.3)

whose solution is given by

$$v(x, t) = V(t)v_0(x) = e^{-i\partial_x^3}v_0 = I_t * v_0(x),$$

where,

$$I_t(x) := \frac{1}{3\sqrt{3t}} A_t \left( \frac{x}{3\sqrt{3t}} \right),$$
and $A_i(\cdot)$ denotes the Airy function. The following lemma gives us the detailed statement for the dispersive blow-up for the initial value problem associated to the linear KdV equation (3.3).

**Lemma 3.2 ([19]).** Let $\alpha \in \mathbb{R}$ fixed and consider the initial data

$$\varphi_0(x) := \sum_{j=1}^{\infty} \lambda_j V(-\alpha j)\phi(x), \quad \lambda_j > 0,$$

where $\lambda_j = ce^{-j^2}$ with $c > 0$ small enough and $\phi(x) := e^{-2|x|}$. Then,

$$\varphi_0 \in C^\infty(\mathbb{R}) \cap H^{1/2}(-\mathbb{R}) \cap L^2((x)^{3/2} \, dx) \cap L^\infty(\mathbb{R}),$$

and the associated global in-time solution $\varphi \in C(\mathbb{R}; H^{3/2}(-\mathbb{R}))$ of (3.3) has the following properties:

1. For any $t > 0$ with $t \notin \alpha \mathbb{Z}$, we have $\varphi(\cdot, t) \in C^1(\mathbb{R})$.
2. For any $t \in \alpha \mathbb{N}$ we have $\varphi(\cdot, t) \notin C^1(\mathbb{R})$.

**Proof.** For a detailed proof of this statement see [19] section 3.

\[ \square \]

### 4. Proof of theorem 1.2

In this section, we show the persistence property in weighted spaces of solutions of the IVP (1.1).

**Proof of theorem 1.2.** The idea of the proof is to apply the contraction principle to the system of integral equations equivalent to (1.1), that is,

$$\begin{align*}
\Phi(u) &= S(t)u_0 + \int_0^t S(t-t')(u\varphi)(t') \, dt' + \int_0^t S(t-t')(|u|^2 u)(t') \, dt', \\
\Psi(v) &= V(t)v_0 - \int_0^t V(t-t')(v\partial_t v)(t') \, dt' + \int_0^t V(t-t')\partial_x(|u|^2)(t') \, dt',
\end{align*}

(4.1)

where $\{S(t)\}$ and $\{V(t)\}$ are the unitary groups associated to the linear Schrödinger and the linear KdV equation respectively. We will give a sketch of the proof.

For $u : \mathbb{R} \times [0, T] \rightarrow \mathbb{C}$ and $v : \mathbb{R} \times [0, T] \rightarrow \mathbb{R}$ with $T$ fixed and $s > 3/4$, define

$$\begin{align*}
\mu_1^s(u) := \|u\|_{L^s_T H^{s+1/2}} + \|D_x^s D_x^{1/2}u\|_{L_x^2 L_t^s} + \|u\|_{L_x^2 L_t^\infty} + \|\partial_x u\|_{L_x^2 L_t^\infty}, \\
\mu_2^s(v) := \|v\|_{L^s_T H^s} + \|D_x^s \partial_x v\|_{L_x^2 L_t^s} + \|D_x^{s-1/2} \partial_x v\|_{L_x^2 L_t^s}
\end{align*}

(4.2)

\[ \begin{align*}
\mu_2^s(v) := \|v\|_{L^s_T H^s} + \|D_x^s \partial_x v\|_{L_x^2 L_t^s} + \|D_x^{s-1/2} \partial_x v\|_{L_x^2 L_t^s} + \|\partial_x v\|_{L_x^2 L_t^\infty}.
\end{align*} \]

(4.3)

By using the definition, group properties, Minkowski’s inequality, and Sobolev spaces properties we have
\[ \|D_t^{\alpha+\frac{1}{2}} \Phi(u)\|_{L^2} \leq c \|D_t^{\alpha+\frac{1}{2}} u_0\|_{L^2} + \int_0^T \|D_t^{\alpha+\frac{1}{2}} (u\partial_t \varphi)\|_{L^2} \, dt' + \int_0^T \|D_t^{\alpha+\frac{1}{2}} (|u|^2 u)\|_{L^2} \, dt' \]
\[ \leq c \|u_0\|_{\dot{H}^{\alpha+\frac{1}{2}}} + \int_0^T \|D_t^{\alpha+\frac{1}{2}} (u\partial_t \varphi)\|_{L^2} \, dt' + cT \sup_{[0,T]} \|u(t)\|_{\dot{H}^{\alpha+\frac{1}{2}}}. \tag{4.4} \]

To complete the estimate we use the commutator estimate (2.21), Sobolev spaces properties, the Cauchy–Schwarz inequality, and Hölder’s inequality in time to led to
\[ \int_0^T \|D_t^{\alpha+\frac{1}{2}} \varphi\|_{L^2} \, dt \]
\[ \leq c \int_0^T \|D_t^{\alpha-\frac{1}{2}} (u\partial_t \varphi)\|_{L^2} \, dt + c \int_0^T \|D_t^{\alpha-\frac{1}{2}} (\partial_t u\varphi)\|_{L^2} \, dt \]
\[ \leq c \int_0^T \|\partial_t \varphi\|_{L^2} \|D_t^{\alpha-\frac{1}{2}} u\|_{L^2} \, dt + c \int_0^T \|u D_t^{\alpha-\frac{1}{2}} \partial_t \varphi\|_{L^2} \, dt + c \int_0^T \|\partial_t u\|_{L^2} \|D_t^{\alpha-\frac{1}{2}} \partial_t \varphi\|_{L^2} \, dt \]
\[ \leq cT^{3/4} \|\partial_t \varphi\|_{L^{2\infty}} \sup_{[0,T]} \|u(t)\|_{\dot{H}^{\alpha+\frac{1}{2}}} + cT^{3/2} \|\varphi\|_{I^2} \|D_t^{\alpha-\frac{1}{2}} \partial_t \varphi\|_{L^{2\infty}} \]
\[ + cT^{3/2} \|\partial_t u\|_{L^2} \sup_{[0,T]} \|\varphi(t)\|_{\dot{H}^{\alpha+\frac{1}{2}}} + cT^{3/2} \|\varphi\|_{I^2} \|D_t^{\alpha-\frac{1}{2}} \partial_t u\|_{L^{2\infty}} \tag{4.5} \]

Combining (4.4) and (4.5) it follows that
\[ \|D_t^{\alpha+\frac{1}{2}} \Phi(u)\|_{L^2} \leq c \|u_0\|_{\dot{H}^{\alpha+\frac{1}{2}}} + cT^{1/2} \left(1 + T^{3/4} \mu_1^{\alpha} (\mu_2^\alpha (v) + T^{1/2} (\mu_2^\alpha (u))) \right). \tag{4.6} \]

Next, we estimate the \(H^\alpha\)-norm of \(\Psi(v)\). It is enough to estimate \(\|D_t^\alpha \Psi(v)\|_{L^2}\). To do so, we use group properties and Minkowski’s inequality to obtain
\[ \|D_t^\alpha \Psi(v)\|_{L^2} \leq c \|D_t^\alpha v_0\|_{L^2} + \int_0^T \|D_t^\alpha (v\partial_t \varphi)\|_{L^2} \, dt' + \int_0^T \|D_t^\alpha \partial_t (uv)\|_{L^2} \, dt' \]
\[ \leq c \|v_0\|_{\dot{H}^\alpha} + \int_0^T \|D_t^\alpha (v\partial_t \varphi)\|_{L^2} \, dt' + \int_0^T \|D_t^\alpha (uv\partial_t)\|_{L^2} \, dt' \]
\[ + \int_0^T \|D_t^\alpha (u\partial_t v)\|_{L^2} \, dt'. \tag{4.7} \]

The commutator estimates (2.21) and Holder’s inequality yield
\[ \int_0^T \|D_t^\alpha (v\partial_t \varphi)\|_{L^2} \, dt \leq cT^{3/4} \|\partial_t \varphi\|_{L^{2\infty}} \sup_{[0,T]} \|v(t)\|_{\dot{H}^\alpha} \]
\[ + cT^{1/2} \|\varphi\|_{I^2} \|D_t^\alpha \partial_t \varphi\|_{L^{2\infty}}. \tag{4.8} \]
Similarly, we get
\[ \int_0^T \| D_x^s \partial_t \Phi \|_{L^2_t L^\infty_x} \, dt' + \int_0^T \| D_x^s (u \partial_t \tau) \|_{L^2_t L^\infty_x} \, dt' \leq c T^{3/4} \| \partial_t \tau \|_{L^2_t L^\infty_x} \sup_{[0,T]} \| u(t) \|_s + c T^{1/2} \| u \|_{L^\infty_t L^2_x} \| D_x^s \partial_t \tau \|_{L^2_t L^\infty_x}. \tag{4.9} \]

Using the definition (4.3) and the inequalities (4.7)–(4.9) we deduce that
\[ \| D_x^s \Psi(v) \|_{L^2_t L^\infty_x} \leq c \| \Phi(v) \|_s + c T^{1/2} (1 + T^{1/4}) (\| \mu_2^s(v) \|_2 + (\mu_1^s(u))^2). \tag{4.10} \]

On the other hand, use of Kato’s smoothing effect (2.3) and the analysis in (4.4) and (4.5) yield
\[ \| D_x^s \partial_x \Phi(u) \|_{L^\infty_t L^2_x} \leq c \| u_0 \|_{s+1/2} + c T^{1/2} (1 + T^{1/4}) (\| \mu_2^s(u) \|_2 + (\mu_1^s(u))^2). \tag{4.11} \]

Same argument as above, now applying Kato’s smoothing effect (2.1) and the arguments in (4.7), (4.8) and (4.9) lead to
\[ \| D_x^s \partial_x \Psi(v) \|_{L^\infty_t L^2_x} \leq c \| v_0 \|_s + c T^{1/2} (1 + T^{1/4}) (\| \mu_2^s(v) \|_2 + (\mu_1^s(u))^2). \tag{4.12} \]

From the estimates (2.8) and (2.10) combined with the arguments in (4.4), (4.5) and (4.7)–(4.9) it follows that
\[ \| \Phi(u) \|_{L^\infty_t L^2_x} + \| \Psi(v) \|_{L^\infty_t L^2_x} \leq c \| u_0 \|_{s+1/2} + c (1 + T)^{s/2} \| v_0 \|_s + c T^{1/2} (1 + T)^{s/2} (\| \mu_2^s(u) \|_2 + (\mu_1^s(u))^2) + c T^{1/2} (1 + T)^{s/2} (\| \mu_2^s(v) \|_2 + (\mu_1^s(u))^2). \tag{4.13} \]

The Strichartz estimates (2.6) and (2.7) together with the analysis in (4.4), (4.5) and (4.7)–(4.9) lead to
\[ \| \partial_t \Phi(u) \|_{L^\infty_t L^2_x} + \| \partial_t \Psi(v) \|_{L^\infty_t L^2_x} \leq c \| u_0 \|_{s+1/2} + \| v_0 \|_s + c T^{1/2} (1 + T)^{s/2} (\| \mu_2^s(u) \|_2 + (\mu_1^s(u))^2) + c T^{1/2} (1 + T)^{s/2} (\| \mu_2^s(v) \|_2 + (\mu_1^s(u))^2). \tag{4.14} \]

Combining (4.5), (4.10)–(4.14), and the definitions (4.2) and (4.3), we have
\[ \| (\Phi(u), \Psi(v)) \| = \mu_1^s(\Phi(u)) + \mu_2^s(\Psi(v)) \leq c (\| u_0 \|_{s+1/2} + \| v_0 \|_s) + c T^{1/2} (\| \mu_1^s(u) \|_2 + (\mu_1^s(u))^2) + c T^{1/2} (\| \mu_2^s(v) \|_2 + (\mu_1^s(u))^2) + c T^{1/2} (\| (u,v) \|_2 + \| (u,v) \|^2). \tag{4.15} \]

for \( 0 < T < 1 \). Choosing \( a \leq 2c (\| u_0 \|_{s+1/2} + \| v_0 \|_s) \) and \( T \) such that
\[ c T^{1/2} (\| (u,v) \| + \| (u,v) \|^2) < \frac{1}{2} \tag{4.16} \]

we can show that the map \((\Phi(u), \Psi(v))\) applies the ball.
\[ X_3^T = \{ (u, v) \in C([0, T_0]; H^{\rho+1/2}) \times C([0, T]; H^\rho) : \| (\Phi(u), \Psi(v)) \| \leq a \} \]

into itself.

The same argument described above show that \((\Phi(u), \Psi(v))\) is a contraction in \(X_3^T\) and so there is a unique solution of the IVP (1.1)

\[
(u, v) \in C([0, T]; H^{\rho+1/2}(\mathbb{R})) \times C([0, T]; H^\rho(\mathbb{R})).
\]  

(4.17)

By uniqueness the previous argument gives us a solution \((u(t), v(t))\) defined by the class (4.2) and (4.3) of the integral equations.

\[
u(t) = V(t)v_0 - \int_0^t V(t - t')(\nu \partial_x \nu)(t') \, dt' + \int_0^t V(t - t')\partial_x(|u|^2)(t') \, dt'.
\]  

(4.18)

Next, we prove the persistence property in weighted spaces. For simplicity we will take \(s = \frac{1}{2} + \) in the following.

We consider

\[
u_0 \in H^{\frac{1}{2}+}(\mathbb{R}) \cup L^2(|x|^\sigma \, dx) \quad \text{and} \quad v_0 \in H^{\frac{1}{2}+}(\mathbb{R}) \cup L^2(|x|^\sigma \, dx)
\]  

(4.19)

and introduce the notation

\[
\mu_1^T(u) = \mu_1^T(u) = \mu_1^T(u) + \sup_{[0,T_0]} ||| |x|^{\frac{1}{2}+} u(t)|||_{L^2}
\]

\[
\mu_2^T(v) = \mu_2^T(v) = \mu_2^T(v) + \sup_{[0,T_0]} ||| |x|^{\frac{1}{2}+} v(t)|||_{L^2}
\]  

(4.20)

for some \(T_0 \in (0, T)\) to be determined below.

Thus, applying formula (2.18) to \(u\) in (4.18), we have

\[
\| |x|^{\frac{1}{2}+} (u(t)) \|_{L^2} \leq c \| |x|^{\frac{1}{2}+} S(t)u_0 \|_{L^2} + \| |x|^{\frac{1}{2}+} S(t) \int_0^t S(t')(|u| + |u|^2)\, dt' \|_{L^2}
\]

\[
\leq \| |x|^{\frac{1}{2}+} u_0 \|_{L^2} + c(1 + |t|) (\| u_0 \|_{L^2} + \| D^{\frac{1}{2}+} u_0 \|_{L^2})
\]

\[
+ c \int_0^{T_0} \| |x|^{\frac{1}{2}+} (|u| + |u|^2)\|_{L^2} \, dt
\]

\[
+ c (1 + T_0) \int_0^{T_0} \| (|u| + |u|^2)\|_{L^2} \, dt
\]

\[
+ c (1 + T_0) \int_0^{T_0} \| D^{\frac{1}{2}+} (|u| + |u|^2)\|_{L^2} \, dt
\]

\[
\leq \| |x|^{\frac{1}{2}+} u_0 \|_{L^2} + c(1 + T_0) \| u_0 \|_{L^2} + A_1 + A_2 + A_3.
\]  

(4.21)
Next we estimate $A_i$, $i = 1, 2, 3$. Holder’s inequality and Sobolev lemma lead to

$$A_1 \leq \int_0^{T_0} \left( ||x|^\frac{3}{2} u(t)||_{L^2} + ||x|^\frac{3}{2} u^2(t)||_{L^2} \right) dt$$

$$\leq c T_0 \left( ||x|^\frac{3}{2} u||_{L^\infty_{0,T_0} L^2} + ||x|^\frac{3}{2} u(t)||_{L^\infty_{0,T_0} L^2} \sup_{[0,T_0]} ||u(t)||_4^2 \right)$$

$$\leq c T_0 \left( \mu_3^3(u) \mu_2^2(v) + \mu_3^3(u) \right). \tag{4.22}$$

Holder’s inequality and Sobolev lemma yield

$$A_2 \leq c (1 + T_0) T_0 \left( \sup_{[0,T_0]} ||u(t)||_4 + \sup_{[0,T_0]} ||v(t)||_4 + \sup_{[0,T_0]} ||u(t)||_4^3 \right)$$

$$\leq c (1 + T_0) T_0 \left( \mu_1^3(u) \mu_2^2(v) + \mu_1^3(u) \right). \tag{4.23}$$

Applying Sobolev spaces properties we obtain

$$A_3 \leq c (1 + T_0) \int_0^{T_0} \left( ||x|^\frac{3}{2} (u w)(t)||_{L^2} + ||x|^\frac{3}{2} (u u^2)(t)||_{L^2} \right) dt$$

$$\leq c (1 + T_0) T_0 \left( \sup_{[0,T_0]} ||u(t)||_4 + \sup_{[0,T_0]} ||v(t)||_4 + \sup_{[0,T_0]} ||u(t)||_4^3 \right)$$

$$\leq c (1 + T_0) T_0 \left( \mu_1^3(u) \mu_2^2(v) + \mu_1^3(u) \right). \tag{4.24}$$

Now we estimate $||x|^\frac{3}{2} v(t)||_{L^2}$. Applying formula (2.16) to $v$ in (4.18) we get

$$||x|^\frac{3}{2} v(t)||_{L^2} \leq ||x|^\frac{3}{2} v_0||_{L^2} + c(1 + T_0) \left( ||v_0||_{L^2} + ||D_1^\frac{3}{2} v_0||_{L^2} \right)$$

$$+ \int_0^t \int_0^t V(t - t') ||x|^\frac{3}{2} v(t')||_{L^2} dt'$$

$$+ \int_0^t \int_0^t V(t - t') ||x|^\frac{3}{2} \partial_t ||_{L^2} dt'$$

$$+ c (1 + T_0) \int_0^T \left( ||v \partial_t v||_{L^2} + ||\partial_t u^2||_{L^2} \right) dt$$

$$+ c (1 + T_0) \int_0^T \left( ||v \partial_t v||_{L^2} + ||\partial_t u^2||_{L^2} \right) dt$$

$$+ c (1 + T_0) \int_0^T \left( ||v \partial_t v||_{L^2} + ||\partial_t u^2||_{L^2} \right) dt. \tag{4.25}$$

The last three terms above were previously estimated. We only need to bound the third and fourth term on the right hand side of (4.25).

Minkowski’s inequality, group properties and Hölder’s inequality yield

$$\left\| \int_0^T V(t - t') ||x|^\frac{3}{2} v(t')||_{L^2} dt' \right\|_{L^2} \leq \int_0^T \left( ||x|^\frac{3}{2} v(t)||_{L^2} \right) dt$$

$$\leq c T_0^{3/4} \sup_{[0,T_0]} ||x|^\frac{3}{2} v(t)||_{L^2} ||\partial_t v||_{L^2_{0,T_0}}$$

$$\leq c T_0^{3/4} \mu_1^3(v) \mu_2^2(v). \tag{4.26}$$
Similarly, we obtain
\[
\left\| \int_0^t V(t-t') |x|^{1+} \partial_x (|u|^2)(t') \, dt' \right\|_{L_x^2} \\
\leq \int_0^{T_0} \left( \left\| |x|^{1+} \bar{u} \right\|_{L_x^2} \left\| \partial_x u \right\|_{L_x^\infty} + \left\| |x|^{1+} \bar{u} \right\|_{L_x^2} \left\| \partial_x \bar{u} \right\|_{L_x^\infty} \right) \, dt \\
\leq cT_0^{3/4} \sup_{[0,T_0]} \left\| |x|^{1+} \bar{u}(t) \right\|_{L_x^2} \left\| \partial_x u \right\|_{L_x^\infty} \\
\leq cT_0^{3/4} \mu_1^0(u) \mu_1^0(u). \tag{4.27}
\]

Gathering the information in (4.21)–(4.24) we get that
\[
\left\| |x|^{1+} u(t) \right\|_{L_x^2} \leq c \left\| |x|^{1+} u_0 \right\|_{L_x^2} + c(1 + T_0) \left\| u_0 \right\|_{\frac{1}{4}+} \\
+ cT_0 (\mu_1^0(u) \mu_2^0(v) + \mu_2^0(u) (\mu_1^0(u))^2) \\
+ c(1 + T_0) T_0 (\mu_1^0(u) \mu_2^0(v) + (\mu_1^0(u))^3). \tag{4.28}
\]

On the other hand, from (4.25)–(4.27) we deduce that
\[
\left\| |x|^{1+} v(t) \right\|_{L_x^2} \leq c \left\| |x|^{1+} v_0 \right\|_{L_x^2} + c(1 + T_0) \left\| v_0 \right\|_{3/4+} \\
+ cT_0^{3/4} (\mu_2^0(v) \mu_2^0(v) + \mu_2^0(u) (\mu_2^0(u))^2) \\
+ cT_0^{1/2} ((\mu_2^0(v))^2 + (\mu_1^0(u))^2). \tag{4.29}
\]

Taking \( T_0 \in (0, T) \) such that (4.16) holds we obtain
\[
\left\| |x|^{1+} u(t) \right\|_{L_x^2} + \left\| |x|^{1+} v(t) \right\|_{L_x^2} \leq c \left\| |x|^{1+} u_0 \right\|_{L_x^2} + c(1 + T_0) \left\| u_0 \right\|_{\frac{1}{4}+} \\
+ c \left\| |x|^{1+} v_0 \right\|_{L_x^2} + c(1 + T_0) \left\| v_0 \right\|_{\frac{3}{4}+} \\
+ 2c(1 + T_0) T_0^{1/2} (\left\| u_0 \right\|_{3/4+} + \left\| v_0 \right\|_{3/4}) \tag{4.30}
\]

This basically completes the proof. \( \square \)

5. Proof of the main theorem 1.1

The following proof is built upon the linear analysis appearing in section 3.

Consider the IVP (1.1) associated to the Schrödinger–Korteweg–de Vries system, with initial data
\[
\begin{align*}
\psi_0 &\in C^\infty(\mathbb{R}) \cap H^2(\mathbb{R}) \cap L^2(\langle x \rangle^4 \, dx) \cap L^\infty(\mathbb{R}), \\
\psi_0 &\in C^\infty(\mathbb{R}) \cap H^{1/2}(\mathbb{R}) \cap L^2(\langle x \rangle^{3/2} \, dx) \cap L^\infty(\mathbb{R}),
\end{align*}
\]

and

\[
\begin{align*}
\psi_0 &\in C^\infty(\mathbb{R}) \cap H^{3/2}(\mathbb{R}) \cap L^2(\langle x \rangle^3 \, dx) \cap L^\infty(\mathbb{R}),
\end{align*}
\]
constructed in lemmas 3.1 and 3.2 respectively, choosing both parameters such that they develop dispersive blow-up for the linear equations at the same time $t'$ small enough.

As we proved in the previous section we have a solution of the IVP (1.1) given by

$$u(x, t) = S(t)u_0 + \int_0^t S(t - t')(uv)(t')\,dt' + \int_0^t S(t - t')(|u|^2 u)(t')\,dt', \quad (5.1)$$

and

$$v(x, t) = V(t)v_0 + \int_0^t V(t - t')\partial_x(v^2)(t')\,dt' + \int_0^t V(t - t')(|\partial_x v|^2)(t')\,dt'. \quad (5.2)$$

If the integral terms in (5.1) and (5.2) are $C^{1/2+\varepsilon}_s(\mathbb{R})$ and $C^1_s(\mathbb{R})$ functions for all $t \in [0, T]$ respectively, then the desired result will follow from what we already known about $S(t)u_0$ and $V(t)v_0$ in section 3. To do this we divide the analysis in two cases, the inhomogeneous terms at Schrödinger equation level and the other ones at KdV equation level.

The following two lemmas are sufficient to complete the proof of theorem 1.1.

**Lemma 5.1.** Let $s > 3/4$ and consider an initial data

$$(u_0, v_0) \in H^{s+\frac{1}{4}}(\mathbb{R}) \cap L^2(|x|^{2s+1}\,dx) \times H^s(\mathbb{R}) \cap L^2(|x|^s\,dx).$$

Let $(u(t), v(t))$ be the corresponding solution for the IVP (1.1) given by theorem 1.2,

$$u(x, t) = S(t)u_0 + \int_0^t S(t - t')(uv)(t')\,dt' + \int_0^t S(t - t')(|u|^2 u)(t')\,dt'$$

$$= S(t)u_0 + \mathcal{I}(x, t),$$

then $\mathcal{I} \in C([0, T]; H^{s+\frac{1}{4}}(\mathbb{R}))$.

In other words, the integral term $\mathcal{I}$ is smoother than the free propagator $e^{it\Delta}u_0$ by a quarter of derivative. In particular, this implies that for initial data as at the beginning of this section, the integral term $\mathcal{I} \in C([0, T]; C^{1+\frac{1}{2}}(\mathbb{R}))$.

**Lemma 5.2.** Let $s > 7/6$ and consider an initial data

$$(u_0, v_0) \in H^{s+\frac{1}{4}}(\mathbb{R}) \cap L^2(|x|^{2s+1}\,dx) \times H^s(\mathbb{R}) \cap L^2(|x|^s\,dx).$$

Let $(u(t), v(t))$ be the corresponding solution for the IVP (1.1) given by theorem 1.2,

$$v(x, t) = V(t)v_0 + \int_0^t V(t - t')\partial_x(v^2)(t')\,dt' + \int_0^t V(t - t')(|\partial_x v|^2)(t')\,dt'$$

$$= V(t)v_0 + \mathcal{II}(x, t),$$

then $\mathcal{II} \in C([0, T]; H^{s+\frac{1}{4}}(\mathbb{R}))$.

The lemma affirms that the integral term $\mathcal{II}$ is smoother than the free propagator $V(t)v_0$ by one sixth derivative. In particular, this implies that for initial data $v_0$ as above, the integral term $\mathcal{II} \in C([0, T]; C^1(\mathbb{R}))$.

**Proof of lemma 5.1.** First of all, recall that the local well-posedness theorem 1.2 guarantees the existence of the solution
Now, let us divide the analysis in two steps. First, define
\[ u_1(t) := \int_0^t S(t - t')(uv)(t')dt'. \]

We shall show that \( u_1(t) \in H^{s+\frac{1}{2}}(\mathbb{R}) \) for all \( t \in [0, T] \). In fact, by (2.4) we have
\[
\|D_s^{s+\frac{1}{2}} u_1\|_{L_t^2} \leqslant \|D_s^{s+\frac{1}{2}} \int_0^t S(t - t')D_s^{s+\frac{1}{2}}(uv)(t')dt'\|_{L_t^2} \\
\leqslant \|D_s^{s+\frac{1}{2}}(uv)\|_{L_t^2} \\
\leqslant \|\partial_x^{s+\frac{1}{2}} u\|_{L_t^2} + \|\partial_x^{s+\frac{1}{2}} v\|_{L_t^2} + E_1.
\]

Let us estimate each of these terms. Using Hölder’s inequality we can bound the first term of (5.3) by
\[
\|\partial_x^{s+\frac{1}{2}} u\|_{L_t^2} \leqslant \|v\|_{L_t^{\infty}} \|D_s^{s+\frac{1}{2}} u\|_{L_t^2} \\
\leqslant cT^{1/2}\|v\|_{L_t^{\infty}} \|D_s^{s+\frac{1}{2}} u\|_{L_t^2} < \infty.
\]

On the other hand, the second term of (5.3) can be bound by
\[
\|\partial_x^{s+\frac{1}{2}} v\|_{L_t^2} \leqslant c\|\partial_x^{s+\frac{1}{2}} v\|_{L_t^{\infty}} \|D_s^{s+\frac{1}{2}} v\|_{L_t^2} \\
< \infty.
\]

Now, to estimate \( E_1 \) we shall employ commutator estimates and interpolated norms of the previous terms. For the sake of completeness we sketch the proof.

\[
E_1 = \|D_s^{s+\frac{1}{2}}(u\partial_x v - uD_s^a\partial_x v - \partial_x vD_s^a u)\|_{L_t^2} \\
+ \|D_s^{s+\frac{1}{2}}(\partial_x^a u - \partial_x^a u - \partial_x vD_s^a u)\|_{L_t^2} \\
+ \|\partial_x^a u\|_{L_t^2} + \|\partial_x^a u\|_{L_t^2}
\]

where \( a \in (0, 1) \) is such that \( s + \frac{1}{3} = 1 + a \).

Applying the Leibniz rule (2.22) and Hölder’s inequality it follows that
\[
E_1 \leqslant c\|\partial_x^{1/4} \partial_x^a u\|_{L_t^4} \|\partial_x^{1/4} D_s^a v\|_{L_t^4} + c\|\partial_x^a v\|_{L_t^{20/3}} \|D_s^a u\|_{L_t^{20/3}}
= E_{1,1} + E_{2,2}.
\]

Let us first estimate the term \( E_{1,1} \). For \( \|\partial_x^{1/4} \partial_x^a u\|_{L_t^4} \) we will use Strichartz estimate (2.20) combined with (4.21). Indeed, Hölder inequality and interpolation inequalities in lemma 2.7 give us
\[
\| (\chi)^{1/4} \partial_t u \|_{L^2_t L^4_x} \leq c T^{1/8} \| (\chi)^{1/4} \partial_t u \|_{L^2_t L^4_x} + c T^{1/8} (1 + T) \| u \|_{L^\infty_T H^{s + 1/2}_x}
\]
\[
\leq c T^{1/8} \| u \|^{2/(2s + 1)}_{L^\infty_T H^{s + 1/2}_x} \| (\chi)^{\frac{3s - 1}{4}} u \|_{L^\infty_T L^2_x}^{(2s - 1)/(2s + 1)} + c T^{1/8} (1 + T) \| u \|_{L^\infty_T H^{s + 1/2}_x},
\]
which is finite since \(\frac{3s - 1}{4} < s + \frac{1}{2}\).

Now we estimate \(\| (\chi)^{1/4} D_x^s v \|_{L^2_t L^4_x}\). Using Sobolev’s embedding and the interpolation inequalities in lemma 2.7 we obtain
\[
\| (\chi)^{1/4} D_x^s v \|_{L^2_t L^4_x} \leq c T^{1/4} \| (\chi)^{1/4} D_x^s v \|_{L^\infty_T L^2_x} \leq c T^{1/4} \| v \|^{1/3}_{L^\infty_T H^s} \| (\chi)^{3/8} D_x^s v \|^{2/3}_{L^\infty_T L^2_x} \leq c T^{1/4} \| v \|^{(2s - 1)/2s}_{L^\infty_T H^s} \| (\chi)^{1/2} v \|^{1/2}_{L^\infty_T L^2_x}
\]
which is finite.

To estimate \(\| \partial_t v \|_{L^{20/9} T^{20/9}}\), we employ the linear estimate (2.13) and a similar argument to show that the solution \(v\) is in \(H^s\), \(s > 3/4\). To bound \(\| D_x^s u \|_{L^{20/9} T^{20/9}}\), the same ideas as in the previous estimate yield
\[
\| D_x^s u \|_{L^{20/9} T^{20/9}} \leq c (\chi)^{11/20} D_x^s u \|_{L^{20/9} T^{20/9}} \leq c T^{1/5} (\chi)^{11/20} D_x^s u \|_{L^\infty_T L^{20/9}} \leq c T^{1/5} (\chi)^{11/20} D_x^s u \|_{L^\infty_T L^2_x} + c T^{1/5} (1 + T) \| u \|_{L^\infty_T H^{s + 1/2}_x} \leq c T^{1/5} \| u \|^{(4s - 3)/(2s + 2)}_{L^\infty_T H^{s + 1/2}_x} (\chi)^{11(2s + 1)/10} u \|^{1/(4s + 2)}_{L^\infty_T L^2_x} + c T^{1/5} (1 + T) \| u \|_{L^\infty_T H^{s + 1/2}_x}
\]
which is finite because \(\frac{11(2s + 1)}{10} < s + \frac{1}{2}\). Therefore we have
\[u_1(t) \in H^{s + 1/2}(\mathbb{R}) \quad \text{for all } t \in [0, T].\]

Now, let us consider the second integral term of the solution \(u(x, t)\):
\[
u_2(t) := \int_0^t \int S(t - t') |u|^2 u(t') \, dt' \, dt.
\]
We shall show that \(u_2(t) \in H^{s + 1/2}(\mathbb{R})\) for all \(t \in [0, T]\). In fact, by the dual version of Kato’s smoothing effect (2.2) we have
\[
\| D_x^{-1/2} \nu_2 \|_{L^2_t L^2_x} \leq c \| D_x^{-1/2} (|u|^2 u) \|_{L^2_t L^2_x} \leq c \| u \|_{L^\infty_T L^\infty_x} \| D_x^{-1/2} u \|_{L^\infty_T L^2_x} + E_2 < \infty \tag{5.4}
\]
where, again, the terms in $E_2$ are easy to control by considering the commutator estimates (see [16]) and the interpolated norms of the previous terms, so we omit the details. Thus, we conclude that $I \in C([0, T]; H^{s+\frac{1}{2}}(\mathbb{R}))$.

\section*{Proof of lemma 5.2.} First of all, recall that the local well-posedness theorem 1.2 guarantees the existence of the solution $u \in C([0, T]; H^{s+\frac{1}{2}}(\mathbb{R}) \cap L^2(|x|^{2s+1} \, dx))$, $\varphi \in C([0, T]; H^s(\mathbb{R}) \cap L^2(|x|^s \, dx))$.

Now, let us divide the analysis in two steps. First, define

$$v_1(t) := \int_0^t \! \! V(t - t') \partial_x(v^2)(t') \, dt' \in C^1(\mathbb{R}).$$

We shall show that $v_1(t) \in H^{s+\frac{1}{2}}(\mathbb{R})$ for all $t \in [0, T]$. In fact, using the smoothing property (2.2), we obtain

$$\sup_{0 \leq t \leq T} \| D^{s+1/6}_t \varphi \|_{L^2_t L^6_x} \leq c \| \varphi \|_{L^2_t L^6_x},$$

where $E_1$ is easy to control by considering the commutator estimates (see [16]) and interpolated norms of the previous terms to be considered below, so we omit this proof. Now, from Strichartz estimates (2.7) with $p = q = 6$, $\theta = \frac{3}{5}$ and $\omega = \frac{1}{2}$ we obtain:

$$\| D^{s+1/6}_t \varphi \|_{L^2_t L^6_x} < \infty.$$

On the other hand, using (2.14) in lemma 2.7 we deduce:

$$\| v \|_{L^6_t L^6_x} \leq c \| \varphi \|_{L^6_t L^6_x} \leq c T^{1/3} \| \varphi \|_{L^6_t L^6_x} \leq c T^{1/3} \| J^{1/6} (\varphi) \|_{L^6_t L^6_x} \leq c T^{1/3} \| J^\lambda \varphi \|_{L^6_t L^6_x} \leq c T^{1/3} \| J^\lambda \varphi \|_{L^6_t L^6_x} \leq c T^{1/3} \| J^\lambda \varphi \|_{L^6_t L^6_x},$$

with $\gamma$ such that $\frac{\gamma}{T} = \frac{1}{2}^+$, i.e. $\gamma > \frac{1}{2}$, and such that $(1 - \gamma) s > 1/6$. Note that the last inequality imposes the restriction $s > \frac{7}{8}$. Thus we have $v_1(t) \in H^{s+\frac{1}{2}}(\mathbb{R})$ for all $t \in [0, T]$, which concludes the demonstration of the first step.

Now, let us consider the second integral term of the solution $v(t, x)$:

$$v_2(t) := \int_0^t \! \! V(t - t') (\partial_x u)^2(t') \, dt' \in C^1(\mathbb{R})$$

We shall show that $v_2(t) \in H^{s+\frac{1}{2}}(\mathbb{R})$ for all $t \in [0, T]$. For this, we use the inhomogeneous smoothing Kato effect (2.2), thus we obtain:

$$\sup_{0 \leq t \leq T} \| D^{s+1/2}_t \int_0^t \! \! V(t - t') (\partial_x u)^2(t') \, dt' \|_{L^2_t L^2_x} \leq c \| u \|_{L^2_t L^2_x} \| D^{s+1/2}_t u \|_{L^2_t L^2_x} + E_2$$

$$\leq c T^{1/2} \| u \|_{L^2_t L^2_x} \| D^{s+1/2}_t u \|_{L^2_t L^2_x} + E_2,$$
where \( \|u\|_{L^2_t L^\infty_x} < \infty \) due to theorem 1.2 and the terms in \( E_2 \) are easy to control by considering the commutator estimates and the interpolated norms of the previous terms.

This concludes the estimates for the solution \( v \).

Therefore, we have shown that the Duhamel terms associated to our solutions are smoother than the corresponding linear associated solutions. In consequence, if there is a point singularity it has to be provided by the linear solution.

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