Quantum- Classical Correspondence of Shortcuts to Adiabaticity

Manaka Okuyama and Kazutaka Takahashi

Department of Physics, Tokyo Institute of Technology, Tokyo 152-8551, Japan

(Dated: September 1, 2018)

We formulate the theory of shortcuts to adiabaticity in classical mechanics. For a reference Hamiltonian, the counterdiabatic term is constructed from the dispersionless Korteweg-de Vries (KdV) hierarchy. Then the adiabatic theorem holds exactly for an arbitrary choice of time-dependent parameters. We use the Hamilton-Jacobi theory to define the generalized action. The action is independent of the history of the parameters and is directly related to the adiabatic invariant.

Shortcuts to adiabaticity (STA) is a method controlling dynamical systems. The implementation of the method results in dynamics that are free from nonadiabatic transitions for an arbitrary choice of time-dependent parameters in a reference Hamiltonian. It was developed in quantum systems and its applications have been studied in various fields of physics and engineering. It is important to notice that this method, decomposing the Hamiltonian into the reference term and the counterdiabatic term, is applied to any dynamical systems and offers a novel insight into the systems.

In the quantum system, the adiabatic theorem is described by the adiabatic state constructed from the instantaneous eigenstate of a reference Hamiltonian. When the time-dependence of the parameters in the Hamiltonian is weak, the solution of the Schrödinger equation can be approximated by the adiabatic state. On the other hand, the adiabatic theorem in the classical system is described by the phase volume defined in periodic systems. The closed trajectory in phase space for a fixed parameter gives the adiabatic invariant

\[ J = \int dx dp \theta(E_0 - H_0), \tag{1} \]

where \( E_0 \) denotes the instantaneous energy. \( J \) is defined instantaneously and the adiabatic theorem states that \( J \) is approximately conserved when the parameter change is slow.

Thus the quantum and classical adiabatic theorems look very different and the relation between them is not obvious. The quantum STA is introduced so that the quantum adiabatic theorem holds exactly and we expect that the same holds for the classical case. These two formulations will allow us to make a link between two theorems. In this letter, we develop the theory of the classical STA. First, we formulate the classical STA in a general way so that the counterdiabatic term can be calculated, in principle, from the derived formula. Second, we show that the adiabatic theorem holds exactly in the classical STA and the adiabatic invariant is obtained directly from the nonperiodic trajectory. Third, we show that the quantum STA reduces to the classical STA by taking the limit \( \hbar \to 0 \), which suggests some relation between quantum and classical adiabatic dynamics.

We consider classical systems with one degree of freedom for simplicity. The system is characterized by the Hamiltonian \( H = H(x, p; \alpha(t)) = H_0 + H_{\text{CD}} \). The dynamical variables in phase space are denoted by \( x \) and \( p \).

To consider the counterdiabatic driving, we use a time-dependent parameter \( \alpha(t) \). It is a straightforward task to generalize the present formulation to systems including several parameters. In the adiabatic time evolution, \( \alpha(t) \) represents a slowly-varying function. Here, we do not impose any conditions on \( \alpha(t) \). In the quantum counterdiabatic driving, the state is determined instantaneously.

The same must be implemented for classical dynamics and we impose the condition that, when the solution of the equation of motion \( (x, p) = (x(t; \alpha(t)), p(t; \alpha(t))) \) is substituted, the reference Hamiltonian \( H_0(x, p, \alpha(t)) \) is equal to the instantaneous energy as

\[ H_0(x(t; \alpha(t)), p(t; \alpha(t)), \alpha(t)) = E_0(\alpha(t)). \tag{2} \]

Then, by considering the time derivative, we find that the counterdiabatic term \( H_{\text{CD}} = H - H_0 = \dot{\alpha}(t) \xi(x, p, \alpha) \), added to the Hamiltonian, satisfies

\[ \frac{\partial H_0(x, p, \alpha)}{\partial \alpha} = \{ \xi(x, p, \alpha), H_0(x, p, \alpha) \} + \frac{dE_0(\alpha)}{d\alpha}, \tag{3} \]

where \{ \cdot, \cdot \} denotes the Poisson bracket. This is obtained by using the equation of motion (Section A of Ref. [1]). If we find \( \xi \) that satisfies this equation, we can realize an ideal time evolution that is characterized by \( E_0(\alpha(t)) \) at each time \( t \). This equation corresponds to the equation derived in Ref. [6] and may be related to the equation for the dynamical invariant in the quantum STA [4, 11].
The commutator in the quantum system is replaced by the Poisson bracket in the classical limit. We note that $\xi$ is not uniquely determined from Eq. (3) [3]. This arbitrariness is discussed in the formulation discussed below.

It is a simple task to show that the adiabatic theorem holds in this counterdiabatic driving. Taking the derivative of $J$ in Eq. (11) with respect to $\alpha$ and using Eq. (3), we obtain

$$\frac{dJ}{d\alpha} = -\int dx dp \{\xi, H_0\}\delta (E_0 - H_0).$$  

(4)

This integral is evaluated by the surface contributions and goes to zero in systems with a smooth trajectory, as we see in the following examples (Section A of Ref. [10]). This result means that $J$ is determined by the initial condition and is independent of $t$. The proof clearly indicates that the counterdiabatic term is introduced so that the adiabatic theorem holds exactly. We note that the time variable $t$ does not appear in Eq. (3) explicitly, which allows us to handle the adiabatic invariant defined geometrically in phase space. This result is the same as that in Ref. [6].

The solution of Eq. (3) can be studied systematically as was done in Ref. [12] for the quantum system. As an example, we set the reference Hamiltonian in a standard form

$$H_0(x, p, \alpha(t)) = p^2 + U(x, \alpha(t)).$$  

(5)

Then we show that the solution of Eq. (3) is given by the dispersionless Korteweg–de Vries (KdV) hierarchy [13, 14] (Section B of Ref. [10]). The corresponding method in the quantum STA was developed in Ref. [12] and the KdV hierarchy was found. The reference Hamiltonian and the counterdiabatic term represent the Lax pair in the corresponding nonlinear integrable system [15]. The dispersionless KdV hierarchy is known as the “classical” limit of the KdV hierarchy [16].

When $\xi(x, p, \alpha)$ is linear in $p$, the potential is of the form

$$U(x, \alpha(t)) = \frac{1}{\gamma^2(t)} u \left( \frac{x - x_0(t)}{\gamma(t)} \right),$$  

(6)

where $u$ is an arbitrary function, and $\alpha$ represents both $x_0$ and $\gamma$, the former represents a translation, and the latter a dilation. The counterdiabatic term is given by

$$H_{CD} = \dot{x}_0 p + \frac{\dot{\gamma}}{\gamma} (x - x_0) p.$$  

(7)

This is known as the scale-invariant driving and was found in previous works [3, 7, 17]. A new result is obtained when we set that $\xi$ is third order in $p$. We find that the counterdiabatic term is given by

$$H_{CD} = \dot{\alpha} \xi = \dot{\alpha} \left( p U(x, \alpha) + \frac{2}{3} p^3 \right).$$  

(8)

and the potential satisfies the dispersionless KdV equation

$$\frac{\partial U(x, \alpha)}{\partial \alpha} + U(x, \alpha) \frac{\partial U(x, \alpha)}{\partial x} = 0.$$  

(9)

This equation can be obtained by removing the third derivative term, the dispersion term, in the KdV equation [18]. The form of the potential is different between the quantum and classical STA for the same form of the counterdiabatic term in Eq. (8). This property is contrasted with that in the scale-invariant system where the quantum and classical STA give the same result. In the same way, we can find the correspondence between the KdV and dispersionless KdV hierarchies at each odd order in $p$. Although it is a difficult problem to implement the higher order terms in an actual experiment, some deformation of the counterdiabatic term is possible to represent the term by a potential function [12].

Before studying the solutions of the dispersionless KdV equation, we reformulate the classical STA by using the Hamilton–Jacobi theory. The standard classical adiabatic theorem is described in periodic systems since the validity of the approximation is written in terms of the period $T$ as $T|\dot{\alpha}/\alpha| \ll 1$. Although the adiabatic invariant is treated in ergodic systems [12, 21], its generalization is a delicate and difficult problem. To find the quantum-classical correspondence of the adiabatic systems, we need to extend the formulation to general systems. This can be done by the Hamilton–Jacobi theory. The adiabatic invariant is related to the action $S = \int_0^t dt' L = \int_0^t dt' (\dot{x} p - H)$. This is a function of $x(t)$, $t$, and the whole history of $\alpha(t)$: $S = S(x(t), t, \{\alpha(t)\})$. The property that $S$ is independent of the history of $x(t)$ is shown by using the equation of motion. In the counterdiabatic driving, trajectories in phase space are determined from Eq. (2) and the Hamiltonian satisfies Eq. (9) that has no explicit time dependence. These properties imply that the dynamics is characterized at each $t$, irrespective of past history. By considering the variation $\alpha(t') \to \alpha(t') + \delta \alpha(t')$ of $S$ at an arbitrary $t'$ between 0 and $t$, and using the equation of motion, we obtain the deviation of the action as (Section C of Ref. [10])

$$\delta S = \int_0^t dt' \delta \alpha(t') \left( \frac{\partial H_0}{\partial \alpha} + \{\xi, H_0\} \right) - [\delta \alpha(t')\xi]_0.$$  

(10)

We use Eq. (3) to find that the function defined as

$$\Omega = S(x(t), t, \{\alpha(t)\}) + \int_0^t dt' E_0(\alpha(t'))$$  

(11)

is independent of the history of $\alpha(t)$. This function is a simple generalization of the Hamilton’s characteristic function, or the abbreviated action, which is usually defined for constant $E_0$ by the Legendre transformation. It satisfies

$$\frac{\partial \Omega}{\partial x} = p(x, \alpha), \quad \frac{\partial \Omega}{\partial \alpha} = -\xi(x, p(x, \alpha), \alpha).$$  

(12)
The momentum $p$ is represented as a function of $x$ and $\alpha$ as we see from Eq. (2). These derivatives have no explicit $t$ dependence. This implies that $\Omega$ is a function of $x$ and $\alpha$, and not of $t$, just like the property of the Legendre transformation. The explicit $t$ dependence of $\Omega$ can be removed by adding a time-dependent term, which does not change the trajectory, to the Hamiltonian. As a result we can set $\Omega = \Omega(t(x), \alpha(t))$.

The Hamilton–Jacobi equation is given by $\frac{\partial \Omega}{\partial t} + \frac{\partial \Omega}{\partial x} p + \frac{\partial \Omega}{\partial \alpha} = 0$ with $p = \frac{\partial \mathcal{H}}{\partial x}$. We substitute $\Omega$ to this equation. Noting that the time derivative is replaced in the present system with $\partial_t \rightarrow \partial_t + \dot{\alpha} \partial_\alpha$, we find that the Hamilton–Jacobi equation for the counterdiabatic driving is decomposed as

$$H_0 \left( x, p = \frac{\partial \Omega(x, \alpha)}{\partial x}, \alpha \right) = E_0(\alpha),$$

$$\xi \left( x, p = \frac{\partial \Omega(x, \alpha)}{\partial x}, \alpha \right) = - \frac{\partial \Omega(x, \alpha)}{\partial \alpha}. \quad (14)$$

These equations are solved as a function of $x$ and $\alpha$, and have no explicit time dependence. We note that the counterdiabatic term is given by $H_{CD} = \dot{\alpha} \xi(x, p, \alpha)$. Thus the counterdiabatic driving is characterized by $\Omega$.

The definition of the action shows that $\Omega$ is written as

$$\Omega(x(t), \alpha(t)) = \int_0^t dt' \left( \dot{\alpha} \dot{\xi}(x(t'), \alpha(t')) \right)$$

where we use the equation of motion in the second line. This expression shows that $\Omega$, as a function of $t$ and $\alpha(t)$, satisfies the following equations:

$$\left( \frac{\partial \Omega}{\partial t} \right)_\alpha = \frac{\partial H_0}{\partial p} p, \quad \left( \frac{\partial \Omega}{\partial \alpha} \right)_t = \frac{\partial \xi}{\partial p} p - \xi. \quad (16)$$

Second equation states that $\Omega$ is independent of $\alpha(t)$ when the counterdiabatic term is linear in $p$. This is the case of the scale-invariant driving where $H_0$ is of the form $\mathcal{H}$ with $\mathcal{F}$. The Hamilton–Jacobi equation reads

$$\left( \frac{\partial \Omega}{\partial x} \right)^2 + \frac{1}{\gamma^2} \frac{1}{\gamma} \left( \frac{x - x_0}{\gamma} \right) = E_0(\alpha),$$

and we find that $E_0$ and $\Omega$ take the form $E_0(\alpha(t)) = \epsilon_0/\gamma^2(t)$ and $\Omega = \Omega((x - x_0)/\gamma)$, respectively. By setting $\gamma(0) = 1$, we can regard $\epsilon_0$ as the initial energy at $t = 0$. The counterdiabatic term is calculated as

$$H_{CD} = -\dot{x}_0 \frac{\partial \Omega}{\partial x_0} - \gamma \frac{\partial \Omega}{\partial \gamma} = \dot{x}_0 p + \frac{\dot{\gamma}}{\gamma} (x - x_0) p. \quad (18)$$

where we use the property that the derivatives of $\Omega$ with respect to $x_0$ and $\gamma$ are translated to that with $x$ in the present system. $\Omega$ is also a function of $E_0(\alpha(0)) = \epsilon_0$ and its definition shows that the derivative of $\Omega$ with respect to $\epsilon_0$ gives the relation

$$\frac{\partial \Omega}{\partial \epsilon_0} = \int_0^t dt' \frac{\tau(t')}{\gamma^2(t')} = \tau(t), \quad (19)$$

where the last equality is the definition of the rescaled time $\tau(t)$. We conclude that $\Omega$ as a function of $t$ and $\alpha$ in the scale-invariant system satisfies the relation

$$\Omega(x(t; \alpha(t)), \alpha(t)) = \Omega(x(\tau(t); \alpha(0)), \alpha(0)). \quad (20)$$

The left-hand side represents $\Omega$ at $t$ obtained in the protocol $\alpha(t)$ and the right-hand side represents $\Omega$ at $\tau(t)$ in the fixed protocol $\alpha(0)$. When the latter system gives a closed trajectory, $\Omega$ at the period is equal to the adiabatic invariant in Eq. (11) and is written as $\Omega = \frac{\partial}{\partial p} \mathcal{H}$ (Section D of Ref. [10] for an example of the harmonic oscillator). This relation shows that the adiabatic invariant is directly obtained from the corresponding nonperiodic trajectory.

For nonscale-invariant systems, Eq. (20) is not satisfied. We treat the dispersionless KdV system as an example. $H_0$ is given by Eq. (5) and the potential $\mathcal{U}$ satisfies the dispersionless KdV equation (9). We can rederive Eq. (8) in the present formalism by assuming that $E_0$ is constant. Substituting $\mathcal{U} = E_0 - (\partial_x \Omega)^2$ to Eq. (9), we find (Section C of Ref. [11])

$$\frac{\partial \Omega}{\partial \alpha} + \mathcal{U} \frac{\partial \Omega}{\partial p} + \frac{2}{3} \left( \frac{\partial \Omega}{\partial x} \right)^3 = 0. \quad (21)$$

This equation shows that the counterdiabatic term obtained from Eq. (14) is given by Eq. (8).

Equation (9) is solved by the hodograph method as

$$\mathcal{U}(x, \alpha) = f(x - \alpha \mathcal{U}(x, \alpha)), \quad (22)$$

where $f$ is an arbitrary function [22, 23]. As a simple example we consider the case $f(x) = x^2$. Then, by solving the quadratic equation, we obtain

$$\mathcal{U}(x, \alpha) = \frac{2ax + 1 - \sqrt{4ax + 1}}{2a^2}. \quad (23)$$

We take the negative branch of the equation so that the trajectories are bound. This potential is well-defined for $x > -\frac{1}{4a}$ and we set the parameters so that this relation is satisfied throughout the time evolution. By taking the limit $\alpha \rightarrow 0$, we have the harmonic oscillator $U(x, 0) = x^2$.

The equation of motion is solved numerically and we show the trajectories in phase space and $\Omega(x(t; \alpha(t)), \alpha(t))$ in Figs. [11] and [2] respectively. We use the protocol $\alpha(t) = \alpha(0) (1 - \sin^2(st))$. We see from Fig. 2 that Eq. (20) is not satisfied since $\Omega$ is not necessarily a monotone increasing function and the time rescaling cannot give the result at $s = 0$. The counterdiabatic
where $T$ is x imagined $J$ is an integer trajectory which is defined for a fixed $\alpha$ with $s$. The solid line represents $s = 0.0$, the dashed line $s = 4.5$, and the dotted line $s = 9.5$.

This relation holds for an arbitrary choice of $\alpha(t)$ and can be proved by assuming that $T$ is independent of the initial energy. We use the theory of action-angle variables for the proof. See Sec. E of Ref. [10]. The period $T$ can also be calculated there and we find $T = \pi$ in the present case. Thus the adiabatic invariant can be calculated directly from the real nonperiodic trajectory.

The Hamilton–Jacobi theory of the classical STA makes a link between the classical and quantum systems. In scale-invariant Hamiltonian in Eq. [5] with [16], the counterdiabatic term in classical system becomes the same as that in quantum system if we use the symmetrization as $px \rightarrow (\hat{p}x + \hat{x}\hat{p})/2$. In the KdV systems, the form of the Hamiltonian is unchanged but the potential function $U$ satisfies an equation that is different from Eq. [4].

In the quantum system the state is described by the wavefunction which satisfies the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} \psi(x,t) = (H_0 + H_{\text{CD}})\psi(x,t).$$

(25)

For example, we consider the Hamiltonian (5) and (8) with the replacement $p\nu \rightarrow (\hat{p}\nu + \hat{\nu}\hat{p})/2$. This setting gives a counterdiabatic driving when the potential satisfies the KdV equation [18]

$$\frac{\partial U(x,\alpha)}{\partial \alpha} + U(x,\alpha) \frac{\partial U(x,\alpha)}{\partial x} + \frac{\partial^3 U(x,\alpha)}{\partial x^3} = 0.$$ (26)

By substituting the wavefunction

$$\psi(x,t) = e^{-iE_nt/\hbar} A(x,\alpha(t)) e^{i\Omega(x,\alpha(t))/\hbar},$$

(27)

where $A$ and $\Omega$ are real functions, to the Schrödinger equation and taking the limit $\hbar \rightarrow 0$, we can obtain Eq. (21). Thus the classical STA is obtained from the quantum version by taking the classical limit. To find this relation, it is crucial to develop the classical STA by the Hamilton–Jacobi theory as we discuss in this letter.

In the quantum STA, the wavefunction is given by the adiabatic state of $H_0(x,\nu,\alpha(t))$. By using the instantaneous eigenstate of $H_0$, $|\alpha(t)\rangle$, and the instantaneous energy of $H_0$, $E_n(\alpha(t))$, we can write the wave function as

$$|\psi_n^{(\text{ad})}(t)\rangle = \exp \left( -\frac{i}{\hbar} \int_0^t dt' E_n(\alpha(t')) \right) |\psi_n(\alpha(t))\rangle,$$

(28)

where

$$|\tilde{\psi}_n(\alpha(t))\rangle = \exp \left( -\int_{\alpha(0)}^{\alpha(t)} d\alpha' \langle n(\alpha')| \frac{\partial}{\partial \alpha'} |n(\alpha')\rangle \times |n(\alpha(t))\rangle \right).$$

(29)

The point is that $|\tilde{\psi}_n(\alpha(t))\rangle$, the adiabatic state without the dynamical phase, is written in terms of $\alpha$, not of $t$. We define the unitary operator $V$ as

$$|\tilde{\psi}_n(\alpha(t))\rangle = V(\alpha(t)) |\psi_n(\alpha(0))\rangle.$$
This operator was introduced to develop the path integral formulation of the adiabatic theorem [24]. Then we can show that the counterdiabatic term is written as

$$\hat{H}_{CD}(t) = i\hbar \alpha(t) \partial V(\alpha) / \partial \alpha V^\dagger(\alpha).$$

(31)

We note that, in this expression, the unitary operator $\hat{V}$ can be replaced by the total time evolution operator to find the formula by Demirplak and Rice [1]. By using $\hat{V}$, we can write the formula of $\hat{H}_{CD}$ in a more suggestive form. We write $\hat{V}$ as

$$\hat{V}(\alpha) = \exp \left( \frac{i}{\hbar} \hat{\Omega}(\alpha) \right),$$

(32)

and this operator $\hat{\Omega}$ is the quantum analogue of the characteristic function $\Omega(x, \alpha)$ in the classical system. For example, in the scale-invariant Hamiltonian [5] with [6], $\hat{\Omega}$ is given by

$$\hat{\Omega}(\alpha(t)) = -\hat{p}(x_0(t) - x_0(0)) - \frac{1}{2} \left[ \hat{\gamma}(\hat{\alpha}(t)) + \hat{\alpha}(t) \hat{\gamma}(0) \right].$$

(33)

The counterdiabatic term in this case is written as

$$\hat{H}_{CD}(t) = -\alpha(t) \partial \hat{\Omega}(\alpha) / \partial \alpha.$$

(34)

This expression formally coincides with Eq. (14).

To summarize, we have developed the classical STA by using the Hamilton–Jacobi theory. The system is characterized by the generalized characteristic function $\Omega(x, \alpha)$ and the counterdiabatic term is obtained from this function. The equation for the counterdiabatic term can be studied systematically and is solvable when the system falls in the dispersionless KdV hierarchy. Our formulation also gives a relation to the adiabatic theorem. We can also show that the classical STA is reduced from the quantum STA by taking the standard semiclassical approximation.

We acknowledge financial support from the ImPACT Program of the Council for Science, Technology, and Innovation, Cabinet Office, Government of Japan. K.T. was supported by JSPS KAKENHI Grant No. 26400385.

[1] M. Demirplak and S. A. Rice, Adiabatic population transfer with control fields, J. Phys. Chem. A 107, 9937 (2003).
[2] M. Demirplak and S. A. Rice, Assisted adiabatic passage revisited, J. Phys. Chem. B 109, 6838 (2005).
[3] M. V. Berry, Transitionless quantum driving, J. Phys. A 42, 365303 (2009).
[4] X. Chen, A. Ruschhaupt, S. Schmidt, A. del Campo, D. Guéry-Odelin, and J. G. Muga, Fast optimal frictionless atom cooling in harmonic traps: Shortcut to adiabaticity, Phys. Rev. Lett. 104, 063002 (2010).
[5] E. Torrontegui, S. Ibáñez, S. Martínez-Garaot, M. Modugno, A. del Campo, D. Guéry-Odelin, A. Ruschhaupt, X. Chen, and J. G. Muga, Shortcuts to adiabaticity, Adv. At. Mol. Opt. Phys. 62, 117 (2013).
[6] C. Jarzynski, Generating shortcuts to adiabaticity in quantum and classical dynamics, Phys. Rev. A 88, 040101(R) (2013).
[7] S. Deffner, C. Jarzynski, and A. del Campo, Classical and quantum shortcuts to adiabaticity for scale-invariant driving, Phys. Rev. X 4, 021013 (2014).
[8] A. Patra and C. Jarzynski, Classical and quantum shortcuts to adiabaticity in a tilted piston, J. Phys. Chem. B 10.1021/acs.jpca.6b08769 (2016).
[9] C. Jarzynski, S. Deffner, A. Patra, and Y. Subaşı, Fast forward to the classical adiabatic invariant, arXiv:1611.06437 (2016).
[10] See Supplemental Material for the detailed calculations.
[11] H. R. Lewis and W. B. Riesenfeld, An exact quantum theory of the time-dependent harmonic oscillator and of a charged particle in a time-dependent electromagnetic field, J. Math. Phys. 10, 1458 (1969).
[12] M. Okuyama and K. Takahashi, From classical nonlinear integrable systems to quantum shortcuts to adiabaticity, Phys. Rev. Lett. 117, 070401 (2016).
[13] D. R. Lebedev, Conservation laws and Lax representation of Benney’s long wave equations, Phys. Lett. A 74, 154 (1979).
[14] V. E. Zakharov, Benney equations and quasiclassical approximation in the method of the inverse problem, Functional Analysis and Its Applications 14, 89 (1980).
[15] P. D. Lax, Integrals of nonlinear equations of evolution and solitary waves, Commun. Pure Appl. Math. 21, 467 (1968).
[16] K. Takasaki and T. Takabe, Quasi-classical limit of KP hierarchy, W-symmetries and free fermions, Zap. Nauchn. Sem. POMI 235, 295 (1996) [J. Math. Sci. 94, 1635 (1999)].
[17] A. del Campo, Shortcuts to adiabaticity by counterdiabatic driving, Phys. Rev. Lett. 111, 100502 (2013).
[18] D. J. Korteweg and G. de Vries, On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves, Philos. Mag. 39, 422 (1895).
[19] P. Hertz, Über die mechanischen Grundlagen der Thermodynamik, Ann. Phys. (Leipzig) 338, 225 (1910).
[20] P. Hertz, Über die mechanischen Grundlagen der Thermodynamik, Ann. Phys. (Leipzig) 338, 537 (1910).
[21] E. Ott, Goodness of ergodic adiabatic invariants Phys. Rev. Lett. 42, 1628 (1979).
[22] Y. Kodama, A method for solving the dispersionless KP equation and its exact solutions, Phys. Lett. A 129, 223 (1988).
[23] Y. Kodama and J. Gibbons, A method for solving the dispersionless KP hierarchy and its exact solutions II, Phys. Lett. A 135, 167 (1989).
[24] T. Kashiwa, S. Nima, and S. Sakoda, Berry’s phase and euclidean path integral, Ann. Phys. 220, 248 (1992).
**A. CLASSICAL STA**

We start from Eq. (2) to derive the formula of the classical STA. Taking the time derivative, we have

$$\frac{dH_0(x, p, \alpha)}{dt} = \dot{\alpha}(t) \frac{dE_0(\alpha)}{d\alpha}. \tag{A.1}$$

The left-hand side is rewritten by using the equations of motion \( \dot{x} = \frac{\partial U}{\partial p} \) and \( \dot{p} = -\frac{\partial U}{\partial x} \) as

$$\dot{\alpha} \frac{\partial H_0}{\partial \alpha} = \alpha \frac{dE_0}{d\alpha}. \tag{A.2}$$

This gives Eq. (3).

The adiabatic theorem is proved as follows. We take the derivative with respect to \( \alpha \) of Eq. (1) to find

$$\frac{dJ}{d\alpha} = \int dxdp \left( \frac{dE_0}{d\alpha} - \frac{\partial H_0}{\partial \alpha} \right) \delta (E_0 - H_0). \tag{A.3}$$

Using Eq. (3), we can write

$$\frac{dJ}{d\alpha} = - \int dxdp \{ \xi, H_0 \} \delta (E_0 - H_0)$$

$$= \int dxdp \left( \frac{\partial \xi}{\partial \alpha} \frac{\partial \xi}{\partial p} - \frac{\partial \xi}{\partial \alpha} \frac{\partial \xi}{\partial x} \right) \theta (E_0 - H_0). \tag{A.4}$$

This integration is written by the surface contributions as

$$\frac{dJ}{d\alpha} = \int dx \left[ \frac{\partial \xi}{\partial \alpha} \theta (E_0 - H_0) \right]_{x_+}^{x_-}$$

$$- \int dp \left[ \frac{\partial \xi}{\partial \alpha} \theta (E_0 - H_0) \right]_{p_+}^{p_-}. \tag{A.5}$$

When \( x \) is equal to its maximum or minimum value, \( x_+ \) or \( x_- \), \( p \) takes a fixed value for a smooth trajectory. In this case, this derivative goes to zero and we can prove the adiabatic theorem.

**B. DISPERSIONLESS KDV HIERARCHY**

For the reference Hamiltonian \( H_0 \) in Eq. (5), we study possible forms of the potential \( U \) and the corresponding counterdiabatic term \( H_{CD} = \dot{\alpha} \xi(x, p, \alpha) \). \( \xi \) is obtained by solving Eq. (3). In the present case, it is written as

$$\left( -p \frac{\partial}{m \partial x} + \frac{\partial U}{\partial x} \frac{\partial}{\partial p} \right) \xi = \frac{dE_0}{d\alpha} - \frac{\partial U}{\partial \alpha}. \tag{B.1}$$

This equation shows that \( \xi \) is odd order in \( p \). We note that the additional relation \( p^2 = E_0(\alpha) - U(x, \alpha) \) is used to solve this equation.

As an example, we first assume that \( \xi \) is linear in \( p \):

$$\xi(x, p, \alpha) = p \xi_0(x, \alpha). \tag{B.2}$$

We also assume

$$E_0(\alpha) = \epsilon_0 \alpha^k, \tag{B.3}$$

where \( \epsilon_0 \) is constant. Equation (B.1) reads

$$\left[ -2(E_0 - U) \frac{\partial}{\partial x} + \frac{\partial U}{\partial x} \right] \xi_0 = -\frac{\partial U}{\partial \alpha} + \frac{k}{2} E_0. \tag{B.4}$$

This equation holds for an arbitrary choice of initial energy and we obtain

$$\tilde{\xi}_0(x, \alpha) = -\frac{k}{2 \alpha} x + c_0(\alpha), \tag{B.5}$$

where \( c_0 \) is an arbitrary function of \( \alpha \). The potential function satisfies

$$\alpha \frac{\partial U}{\partial \alpha} + \left( -\frac{k}{2} x + \alpha c_0(\alpha) \right) \frac{\partial U}{\partial x} = kU = 0. \tag{B.6}$$

Substituting \( U(x, \alpha) = \alpha^k \tilde{U}(x, \alpha) \) to this equation, we obtain

$$\alpha \frac{\partial \tilde{U}}{\partial \alpha} + \left( -\frac{k}{2} x + \alpha c_0(\alpha) \right) \frac{\partial \tilde{U}}{\partial x} = 0. \tag{B.7}$$

This equation shows that the potential has the scale-invariant form

$$U = \alpha^k \tilde{U} = \alpha^k U_0 \left( \frac{x - x_0(\alpha)}{\alpha^{k/2}} \right), \tag{B.8}$$

where \( U_0 \) is an arbitrary function and \( x_0 \) is determined from the equation

$$\alpha^k \frac{dx_0(\alpha)}{d\alpha} - \frac{k}{2} x_0(\alpha) = \alpha c_0(\alpha). \tag{B.9}$$

By using a proper reparametrization of the parameters, we obtain the potential in Eq. (6).

We next consider the case where \( \xi \) is third order in \( p \):

$$\xi(x, p, \alpha) = p \tilde{\xi}_0(x, \alpha) + p^2 \tilde{\xi}_2(x, \alpha). \tag{B.10}$$

We also assume that \( E_0 \) is constant:

$$\frac{dE_0(\alpha)}{d\alpha} = 0. \tag{B.11}$$
Then Eq. (3) is written as
\[
\left(-2p^2 \frac{\partial}{\partial x} + \frac{\partial U}{\partial x}\right) \xi_0 + \left(-2p^4 \frac{\partial}{\partial x} + 3p^2 \frac{\partial U}{\partial x}\right) \xi_2 = \frac{\partial U}{\partial \alpha},
\]
and we obtain the conditions
\[
\frac{\partial \xi_2}{\partial x} = 0, \quad \frac{1}{m} \frac{\partial \xi_0}{\partial x} + 3 \frac{\partial U}{\partial x} \xi_2 = 0, \quad \frac{\partial U}{\partial x} \xi_0 = -\frac{\partial U}{\partial \alpha}.
\]
The first equation shows that \(\tilde{\xi}_2\) is independent of \(x\):
\[
\tilde{\xi}_2(x, \alpha) = c_2(\alpha),
\]
the second equation shows that \(\tilde{\xi}_2\) is written in terms of \(U\):
\[
\tilde{\xi}_0 = 3mc_2(\alpha)U + c_0(\alpha),
\]
and the third equation gives
\[
\frac{1}{3mc_2(\alpha)} \frac{\partial U}{\partial \alpha} + \left( U + \frac{c_0(\alpha)}{3mc_2(\alpha)} \right) \frac{\partial U}{\partial x} = 0.
\]
This is essentially equivalent to the dispersionless KdV equation in Eq. (9).

We can go further to derive the higher order dispersionless KdV equations. They are fifth order in \(p\), seventh order, and so on.

**C. HAMILTON-JACOBI THEORY**

We briefly review the Hamilton-Jacobi theory to formulate the classical STA. The system is characterized by the action defined from the Lagrangian:
\[
S = \int_0^t dt' L(x(t'), \dot{x}(t'); \alpha(t)).
\]
It satisfies, by definition,
\[
\frac{dS}{dt} = L.
\]
On the other hand we can write
\[
\frac{dS}{dt} = \frac{\partial S}{\partial x} \dot{x} + \frac{\partial S}{\partial t} = \frac{\partial L}{\partial \dot{x}} \dot{x} + \frac{\partial S}{\partial t},
\]
where we use the conjugate momentum
\[
p = \frac{\partial L}{\partial \dot{x}} = \frac{\partial S}{\partial x}.
\]
The first equality is the definition of the momentum and the second is derived from the variation of the action, \(x(t') \to x(t') + \delta x(t')\) as
\[
\delta S = \int dt' \left( \delta x(t') \frac{\partial L}{\partial \dot{x}} + \delta \dot{x}(t') \frac{\partial L}{\partial \dot{x}} \right)
= \int dt' \left( \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial \dot{x}} \frac{\partial \dot{x}}{\partial \dot{x}} \right) + \delta \dot{x}(t') \frac{\partial L}{\partial \dot{x}} \bigg|_0.
\]
The first term goes to zero if we use the equation of motion. This means that the action is independent of the history of \(x\) and is determined as a function of \(x(t)\). The derivative of the action with respect to \(x(t)\) gives the second equality in Eq. (C.4).

In a similar way we can consider the variation \(\alpha(t') \to \alpha(t') + \delta \alpha(t')\):
\[
\delta S = \int_0^t dt' \delta \alpha \left( \frac{\partial p}{\partial \alpha} \left( \dot{x} - \frac{\partial H}{\partial p} \right) - \frac{\partial H_0}{\partial \alpha} \right.
+ \left. \frac{\partial \xi}{\partial x} + \frac{\partial \xi}{\partial p} + \dot{\alpha} \frac{\partial \xi}{\partial \alpha} - \delta \alpha \xi \right)_0.
\]
Using the equation of motion and the condition for the counterdiabatic term in Eq. (3), we obtain
\[
\delta S = \int_0^t dt' \delta \alpha \left( \frac{\partial H_0}{\partial \alpha} + \xi, H_0 \right) - |\delta \alpha \xi|_0
= -\int_0^t dt' \delta \alpha \frac{dE_0(\alpha)}{d\alpha} - |\delta \alpha \xi|^t_0.
\]
This shows that \(\Omega\) defined as Eq. (11) is independent of the history of \(\alpha(t)\).

The Hamilton-Jacobi equation is derived by equating Eqs. (C.2) and (C.3). Using the definition of the Hamiltonian \(H = \dot{x}p - L\), we obtain
\[
\frac{\partial S(x, t, \{\alpha(t)\})}{\partial t} + H \left( x, p = \frac{\partial S}{\partial x}; \alpha(t) \right) = 0.
\]
We note that the action is dependent on \(x, t,\) and \(\alpha(t)\), and the time derivative acts on \(\alpha\). By replacing \(\partial_t + \dot{\alpha} \partial_\alpha\), we write
\[
\frac{\partial S(x, t, \{\alpha(t)\})}{\partial t} + H_0 \left( x, p = \frac{\partial S}{\partial x}; \alpha(t) \right)
+ \dot{\alpha} \left( \frac{\partial S(x, t, \{\alpha(t)\})}{\partial \alpha} + \xi \left( x, p = \frac{\partial S}{\partial x}; \alpha(t) \right) \right) = 0.
\]
This equation and the definition of \(\Omega\) show that the Hamilton-Jacobi equation is decomposed as Eqs. (13) and (14).

The derivation of the counterdiabatic term of the scale-invariant system was done in Eq. (18). Here we consider the case of the dispersionless KdV equation. Substituting \(U = E_0 - (\partial_x \Omega)^2\) to Eq. (9) and assuming \(E_0\) is constant, we have
\[
-2 \frac{\partial \Omega}{\partial x} \left[ \frac{\partial}{\partial x} \left( \frac{\partial \Omega}{\partial x} + U \frac{\partial \Omega}{\partial x} \right) + 2 \frac{\partial^2 \Omega}{\partial x^2} \frac{\partial \Omega}{\partial x} \right] = 0.
\]
FIG. D.1. Trajectories in phase space for the harmonic oscillator in Eq. \((D.1)\). We use the protocol \(\omega(t) = 1 + \frac{1}{2}\sin(st)\), and take the initial condition as \(E_0(\omega(0)) = 1\) and \((x(0), p(0)) = (-2, 0)\).

This gives

\[
\frac{\partial}{\partial x} \left[ \frac{\partial \Omega}{\partial \omega} + U \frac{\partial \Omega}{\partial x} + \frac{2}{3} \left( \frac{\partial \Omega}{\partial x} \right)^3 \right] = 0. \tag{C.11}
\]

By gauging out an irrelevant constant term, we obtain Eq. \((21)\).

D. HARMONIC OSCILLATOR

We consider an example of the harmonic oscillator

\[
H_0 = p^2 + \frac{\omega^2(t)}{4} x^2, \tag{D.1}
\]

with \(\omega(t) = 1 + \frac{1}{2}\sin(st)\). This system is described as a scale-invariant system and the parameter \(\omega\) represents the dilation effect as \(\gamma = 1/\sqrt{\omega}\). At \(s = 0\), we have a static system and the trajectory in phase space becomes a closed curve. This is not the case for systems with \(s \neq 0\) as we see in Fig. \([D.1]\). \(\Omega(t)\) for several values of \(s\) are plotted in Fig. \([D.2]\). By changing the horizontal axis as \(\tau(t) = \int_0^t dt' \omega(t')\), we can confirm that all curves fall on the curve at \(s = 0\). In the present example, \(\Omega\) can be calculated analytically and is given by

\[
\Omega(x(t; \omega(t)), \omega(t)) = \frac{E_0(\omega(t))}{\omega(t)} (\tau(t) - \sin(\tau(t)) \cos(\tau(t))). \tag{D.2}
\]

E. ACTION-ANGLE VARIABLES

The adiabatic invariant \(J\) is independent of the parameter \(\alpha(t)\). It is a function of the initial energy \(E_0(\alpha(0)) = \epsilon_0\): \(J = J(\epsilon_0)\). Then the characteristic function is written as a function of \(x\), \(\alpha\), and \(J\): \(\Omega = \Omega(x, \alpha, J)\).

In the periodic systems, we know that the action-angle variables are useful to characterize the system. We define the angle variable

\[
w = \frac{\partial \Omega}{\partial J}. \tag{E.1}
\]

In the theory of canonical transformation, \(w\) and the action variable \(J\) are interpreted as the canonical variables. \(J\) represents the conjugate momentum of \(w\).

We assume, for simplicity, that the instantaneous energy is constant: \(E_0(\alpha(t)) = \epsilon_0\). In this case, the angle variable is represented as

\[
w = \frac{\partial \epsilon_0}{\partial J} \frac{\partial \Omega}{\partial \epsilon_0} = \frac{\partial \epsilon_0}{\partial J}. \tag{E.2}
\]

We note that \(J(\epsilon_0)\) is independent of \(\alpha(t)\). This means that \(w\) is proportional to \(t\): \(w = t/T(J)\). Taking the derivative of \(w\) with respect to \(x\), we obtain

\[
\frac{\partial w}{\partial x} = \frac{\partial^2 \Omega}{\partial x \partial J} = \frac{\partial p}{\partial J}. \tag{E.3}
\]

For a fixed \(\alpha\), we consider the integration of \(w\) over the closed trajectory. Then we obtain

\[
w = \oint dx \frac{\partial w}{\partial x} = \oint dx \frac{\partial p}{\partial J} = \oint \frac{\partial}{\partial J} p = 1. \tag{E.4}
\]

This result shows that \(T\) is equal to the period of the trajectory.

Integrating \(w\) with respect to \(J\), we write

\[
\Omega(x, \alpha, J) = \int_0^J dJ' \frac{t(x, \alpha, J')}{T(J')}. \tag{E.5}
\]
where we set the boundary condition $\Omega = 0$ at $J = 0$. We note that this expression does not mean that $\Omega$ is proportional to $t$. $t$ in the right hand side is defined as $t = \frac{\partial \Omega}{\partial \epsilon_0}$ and is a function of $x$, $\alpha$, and $J$. Thus Eq. (E.5) has no explicit time dependence and the real dynamics is not implemented in this expression.

Now we substitute $x$ and $\alpha$ at $t = T(J)$ to Eq. (E.5). Then

$$\Omega(x(T(J), J), \alpha(T(J)), J) = \int_0^J \frac{t(x(T(J), J), \alpha(T(J)), J')}{T(J')} \, dJ'. \quad (E.6)$$

We note that $x$ is a function of $t$ and $J$. If $T$ is independent of $J$, the function $t$ in the right-hand side of this equation must be equal to $T$. We obtain in that case

$$\Omega(x(T, J), \alpha(T), J) = J. \quad (E.7)$$

As an example, we consider the dispersionless KdV system with the potential in Eq. (23). The adiabatic invariant is represented as

$$J = 2 \int_{x_-}^{x_+} dx \sqrt{\epsilon_0 - U(x, \alpha)}, \quad (E.8)$$

where $x_\pm$ represent the end points of the trajectory. Since the potential has the form $U(x, \alpha) = u(\alpha x)/\alpha^2$, we can write

$$J = \frac{2}{\alpha^2} \int_{z_-}^{z_+} dz \sqrt{\epsilon_0 \alpha^2 - u(z)}, \quad (E.9)$$

where $z_\pm = \alpha x_\pm$. We know from the adiabatic theorem that this function is independent of $\alpha$. Then we can evaluate this integral by taking the limit $\alpha \to 0$. $u(z)$ is replaced with $z^2$ and we find

$$J = 2 \frac{\pi \epsilon_0 \alpha^2}{\alpha^2} = \pi \epsilon_0. \quad (E.10)$$

This shows that the period of the trajectory is given by

$$T = \frac{\partial J}{\partial \epsilon_0} = \pi. \quad (E.11)$$

This is independent of the initial energy $\epsilon_0$. 

