Charging advantages of Lipkin-Meshkov-Glick quantum battery

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We investigate the performance of the Lipkin-Meshkov-Glick quantum battery based on shortcuts to adiabaticity (STA). We mainly consider the situation where the coupling strength of any two sites in the quantum battery is a sinusoidal function with respect to time. The charging efficiency of the quantum battery can be greatly enhanced via STA. We also analyze the influences of parameters, including particle number, anisotropic parameter, the amplitude and frequency of the driving fields. It is found that an efficient charging process and thus high charging advantages can be achieved by adjusting these parameters properly. Moreover, we calculate the energy fluctuation, von Neumann entropy and energy cost during charging. The STA can make the stored energy and the von Neumann entropy change periodically during the charging process and reduce the energy fluctuation, and the minimal energy fluctuation always occurs in the proximity of minima of the von Neumann entropy.

I. INTRODUCTION

Conventional batteries store energy in the form of chemical energy and transform it into electrical energy while working, which have significant limitations in microscopic electronic equipments and renewable energy. The trend towards miniaturization of electronic devices has led to the demand for small energy storage devices to improve flexibility [1], especially at the atomic and molecular scale [2], where conventional batteries are not well suited. On the other hand, the reduction of fossil fuels promotes the development of renewable energy and has higher requirements for energy storage equipment. With the development of quantum technologies, these problems are expected to be solved by so-called quantum batteries. Quantum batteries are new energy storage media that utilize quantum resources to achieve higher performance than their classical counterparts [2–4]. They are a fundamental concept in quantum thermodynamics [5, 6] and are able to improve performance through quantum mechanical effects since the proposed in 2013 [7]. The maximal amount of energy extractable from the quantum battery is dubbed as ergotropy [8–16].

There have been a large number of researches on quantum batteries, especially many-body quantum batteries, such as Dicke quantum battery [17–24], Sachdev-Ye-Kitaev (SYK) quantum battery [25, 26], spin-chain quantum battery [27–48]. The Dicke quantum battery can be constructed by embedding $N$ two-level systems in a single cavity and interacting with a single mode of cavity photons for charging [18]. The charging power in the collective charging protocol of Dicke quantum battery is $\sqrt{N}$ times of that in the parallel charging protocol that assigns a cavity to each two-level system under the same charging condition [18, 19]. Furthermore, the performance of a Dicke quantum battery can be further improved by adjusting the charging process to be dominated by the two-photon interaction [20] and the interatomic interactions or driving field [24]. The SYK quantum battery involves a strongly interacting system [49] with non-local correlations and the presence of non-local correlations strongly suppress the fluctuations of its average energy stored [25]. The numerical calculations of the SYK quantum battery show that entanglement can accelerate its quantum charging dynamics, resulting in a quantum advantage [25, 26]. In a spin-chain quantum battery, the spin-spin interactions can provide an enhancement in charging power [27–32]. The transverse-field Ising spin chain quantum battery can lead to a higher transfer of energy and better stability of the stored energy by periodically modulation [48] and the battery cells are charged by a finite number of spins through a general Heisenberg XY interaction [30]. For the quantum battery based on a disordered Heisenberg spin chain model, the interaction can suppress temporal energy fluctuations, and the work extraction capability is much better in low entanglement [34] and can enhance self-discharge process [45]. In terms of the quantum battery constructed by the XYZ spin chain, defects or impurities have a positive effect on the generation of quenched averaged power from the battery [35].

The Lipkin-Meshkov-Glick (LMG) model analyzes the infinite-range interaction between a set of spin–1/2 particles in the presence of an external magnetic field [50, 51], and is suitable for constructing a many-body quantum battery. Over the years, researches on the LMG model involve many aspects [51–61], such as the residual energy in adiabatic quantum dynamics close to its quantum critical point [53], the critical signatures subject to dissipative environments [54–56], the spectrum in the thermodynamical limit [57], the orthogonality catastrophe in quantum many-body systems [52] and multipartite nonlocality [58]. Simulations of the LMG model have been implemented by ultracold atoms or atoms near nanostructures [61], trapped ions [59] or circuit quantum electrodynamics [60]. Recently, the charging process and the bound on the stored or extractable energy of LMG quantum batteries with a constant coupling strength have been discussed [61]. It is also shown that quantum entanglement enhances the charging power.
of LMG quantum batteries. We consider whether the charging process can be accelerated to improve the LMG quantum battery performance through shortcuts to adiabaticity (STA).

The STA is a set of techniques that is broader than counterdiabatic driving (CD), also known as transitionless quantum driving [62–68]. It is applicable to speed up the adiabatic evolution of quantum systems from the initial state to the final state and can be realized by adding a counterdiabatic driving field [69–71]. It has been widely used in adiabatic quantum computation [62, 63], detecting and separating chiral molecules [65], quantum heat engines [72–74], many-body spin systems [75, 76] and so on. The costs assisting driving field (including energetic and thermodynamic costs) of STA in different system are also reviewed [62, 74, 77–81]. The STA can provide a fast approach to population control in two-level or three-level atoms [66]. Recently, the application of STA to the quantum battery can achieve an efficient and stable charging or discharging process [82–84]. Importantly, the STA has also been applied to the LMG model for studying quantum annealing [67], adiabatic generation of cat states [68], and quantum speed limit [77, 78, 85].

In this paper, we investigate the LMG quantum battery based on STA. Firstly, we adjust the coupling strength between any two particles in the LMG quantum battery from the original constant to a sinusoidal function of time, and study the charging dynamics in the non-STA charging protocol. Then, we modify the Hamiltonian by adding an auxiliary field to maintain the system in the instantaneous ground state of the Hamiltonian during the charging process, and study the charging dynamics in the STA charging protocol. We explore the stored energy and the charging power in these two charging protocols, compared with the charging protocol where the coupling strength is constant, to study the role of the STA technology in the charging process of the LMG quantum battery. Afterwards, we discuss the influences of parameters such as particle number, anisotropic parameter, amplitude and frequency of the driving field on the maximum stored energy and maximum power, and also focus on the von Neumann entropy [7, 61, 86], the energy fluctuation [5, 87–92] and energy cost [77, 93, 94] during charging.

The paper is organized as follows. In Sec. II, we introduce the LMG quantum battery model and charging protocols. Then, we analyze the charging characteristics of the LMG quantum battery, including the charging energy and power in Sec. III. In Sec. IV, we analyze the von Neumann entropy, the energy fluctuation and energy cost. Finally, we give a discussion and summary in Sec. V.

II. MODEL

The LMG quantum battery consists of an array of spin-1/2 particles, which are charged by exposure to an external magnetic field [50, 51]. Before accelerating the evolution of the quantum state with the application of STA, the charging Hamiltonian is described by [58, 61, 95]

\[ H_0 = \frac{\lambda}{N} \sum_{i<j} (\sigma_i^z \sigma_j^z + \gamma \sigma_i^y \sigma_j^y) + \frac{1}{2} \sum_{j=0}^{N-1} \sigma_j^z, \]

where \( N \) is particle number, \( \sigma_i^z, \sigma_i^y, \sigma_i^x \) are Pauli matrices, \( \gamma \) is the anisotropy parameter, \( \lambda \) is the coupling strength between any two sites \( i \) and \( j \), and \( H_B = \frac{1}{2} \sum_{j=0}^{N-1} \sigma_j^z \) defines the battery Hamiltonian. It is noted that the model is equivalent to the long-range Ising chain when \( \gamma = 0 \) [96, 97]. We consider that the spin system is initially in the ground state of the battery Hamiltonian \( H_B \).

By considering the collective spin operators

\[ S_\alpha = \sum_{j=0}^{N-1} \frac{\sigma_j^\alpha}{2} \quad (\alpha = x, y, z), \]

the charging Hamiltonian can be rewritten as

\[
H_0 = \frac{\lambda}{2N} [(1 + \gamma) (S_x S_- + S_- S_x - N) + (1 - \gamma) (S_y^2 + S_z^2)] + S_z.
\]

Here, we have attached the constant energy shift. The ladder operators \( S_\pm \) are derived from the total spin operators by \( S_+ = S_x + iS_y \) and \( S_- = S_x - iS_y \). Thus, in the total spin notation, the battery Hamiltonian reads \( H_B = S_z \).

We consider Dicke states to describe our quantum system, which is the Hilbert subspace characterized by the maximal total angular momentum \( S = N/2 \) [98]. In the basis \( |S, S_z\rangle \) (\( S_z = m, m = -N/2, \ldots, N/2 \)), we can diagonalize Eq. (3) to find the complete spectrum [53, 99]. The matrix representation of the Hamiltonian Eq. (3) is a \((N + 1) \times (N + 1)\) symmetric matrix, where the ladder operators are calculated as

\[
\langle S, m \pm 1 | S_\pm | S, m \rangle = \sqrt{(S(S + 1) - m(m \pm 1))}.
\]

Then, to guarantee adiabaticity and drive the eigenstates of \( H_0 \) exactly, the correction term is calculated from the spectrum of \( H_0 \) [70, 100, 101], which reads

\[ H_{cd}(t) = i \sum_\psi \partial_t \psi_n(t) \langle \psi_n(t) \rangle, \]

or

\[ H_{cd}(t) = i \sum_{m \neq n} \frac{|\psi_m(t)\rangle \langle \psi_m(t) | \partial_t H_0 | \psi_n(t) \rangle \langle \psi_n(t) |}{E_n - E_m}, \]

where \( |\psi_m(t)\rangle \) and \( |\psi_n(t)\rangle \) are the instantaneous eigenstates of \( H_0 \). And the total Hamiltonian is

\[ H = H_0 + H_{cd}. \]
At the initial moment, we adjust the coupling strength $\lambda$ to zero and prepare the system in the instantaneous ground state of $H_0$, equivalently,

$$|\psi(0)\rangle = |S, -N/2\rangle, \quad (8)$$

representing an discharged quantum battery. At any time, the evolved state is derived from

$$|\psi(t)\rangle = e^{-iHt} |\psi(0)\rangle. \quad (9)$$

The stored energy of the quantum battery can be expressed as

$$C(t) = E(t) - E(0), \quad (10)$$

with

$$E(t) = \langle \psi(t) | S_z | \psi(t) \rangle. \quad (11)$$

Simultaneously, the charging power is defined by

$$P(t) = \frac{C(t)}{t}. \quad (12)$$

During charging, the von Neumann entropy is calculated by the formula [102]

$$S = -Tr(\rho \log_2 \rho), \quad (13)$$

which provides a convenient measure of entanglement. Here $\rho$ is the reduced density matrices of system.

In addition, the energy fluctuation is obtained by

$$\Delta W = \sqrt{\langle \psi(t) | S_z^2 | \psi(t) \rangle - (\langle \psi(t) | S_z | \psi(t) \rangle)^2}. \quad (14)$$

The energy cost assisting the process with the STA protocol is defined as follows (see Ref. [34] for a more detailed discussion),

$$C_0 = \int_0^\tau \left\| \frac{\partial H_0(t)}{\partial \tau}, \rho(t) \right\| \, dt. \quad (15)$$

Accordingly, the instantaneous cost is $\partial_t C_0 = \partial C_0 / \partial t$.

III. STORED ENERGY AND CHARGING POWER

To probe the performance of quantum battery, we consider a driven version of the LMG model with a time-dependent coupling, which has been studied extensively [98, 103–105]. For example, a set of nonequilibrium quantum phase transitions in a periodically driven LMG model has been established and the external driving induces a rich phase diagram that characterizes the multistability in the system [98]. As a common and easy to implement example, here we assume a time-dependent coupling strength $\lambda(t) = A \sin(\omega t)$, with amplitude $A$ and frequency $\omega$ [27, 98]. This modulation is different from the original LMG quantum battery where the coupling strength is a constant [61], and also different from the previous form of coupling strength for studying quantum annealing [67], adiabatic generation of cat states [68] and quantum speed limit [85] through the STA method.

The time evolution of the stored energy and power during charging is shown in Fig. 1. Here, we have numerically simulated the STA charging protocol and the non-STA charging protocol, and the results are expressed by blue solid lines and red dash-dotted lines respectively. To show the advantages of STA charging protocol more intuitively, we also simulate the stored energy and power when the coupling strength is constant, and exhibit them with a black dash line. Under this set of parameters, we find that the non-STA charging protocol with the coupling strength varying sinusoidally with time can improve the performance of the LMG quantum battery, while the performance can be further improved by the STA. In our numerical simulation, the non-STA protocol increases the...
maximum stored energy by about one time and the maximum power by about 20%, while the STA protocol increases the maximum stored energy by about two times and the maximum power by nearly 140%.

Figs. 2(a) and 2(c) show the maximum stored energy and charging power as functions of $N$ in different charging protocols. In these simulations, we start our calculations from $N = 3$, and find that the maximum stored energy and charging power increase as the number of particles $N$. The difference is that the maximum stored energy and charging power of the STA charging protocol increase linearly with the increase of $N$, while they tend to gradually flatten when the coupling strength is constant, following the scaling laws $C_{\text{MAX}} \propto N^{0.52}$ and $P_{\text{MAX}} \propto N^{0.36}$ (we assume the maximum charging power with the form $P_{\text{MAX}} \propto N^\alpha$ and use linear fitting to obtain the scaling exponent $\alpha$ by taking the logarithm, i.e., $\log(P_{\text{MAX}}) \propto \alpha \log(N)$. Here the scaling exponent $\alpha$ essentially reflects the nature of the battery in charging performance). The performance of the quantum battery with STA satisfies $C_{\text{MAX}} \propto N^{1.02}$ and $P_{\text{MAX}} \propto N^{0.97}$. Further calculation shows that high charging advantage can always be achieved in STA battery, that is, the scaling exponent arrives to even beyond $\alpha = 1.2$ by adjusting the external field parameters (see Table I). The dependence of the maximum stored energy and the maximum charging power on the anisotropy parameter $\gamma$ is indicated in Figs. 2(b) and 2(d) for different charging protocols. Within the calculation range we consider, the advantage of the STA charging protocol on the maximum charging power decreases sharply with the increase of parameter $\gamma$ until $\gamma = -0.16$, and then slightly recovers. On the other hand, this charging protocol shows stored energy stability, especially after $\gamma > 0$, the change of the maximum stored energy is negligible, while the maximum stored energy of the other two charging protocols tends to zero.

| $\gamma$ | $\lambda = A = 5$ | $\lambda = A \sin(\omega t)$ | $\lambda = A \sin(\omega t), \text{STA}$ |
|---------|-----------------|-----------------|-----------------|
|         | $\omega = 5$    | $\omega = 10$   | $\omega = 10$   |
| $0.1$   | 0.03            | 0.31            | 1.00            |
|         | 0.01            | 0.51            | 1.00            |
|         | 0.01            | 0.001           | 1.00            |
| $-0.1$  | 0.36            | 0.63            | 0.97            |
|         | 0.33            | 0.11            | 1.00            |
|         | 0.61            | 0.81            | 1.20            |
|         | 0.61            | 0.27            | 1.20            |
| $1.0$   | 0.82            | 0.83            | 1.20            |
|         | 0.82            | 0.47            | 1.20            |
|         | 0.83            | 0.90            | 1.18            |
|         | 0.84            | 0.82            | 1.13            |

Figs. 3(a) and 3(c) illustrate the dependencies of the maximum stored energy and charging power of the LMG quantum battery on the amplitude $A$ in different charging protocols. For the case of weak coupling, the STA charging protocol enables the LMG quantum battery to store more stored energy and charge faster. However, it is noted that the amplitude of the coupling strength...
and that entropy is constantly oscillating during the charging process. A comparison of these Figs. 1(a) and 4(a) shows higher entanglement to correspond to higher energy, supporting the idea that entanglement drives the high performance of the battery [39]. The entropy value changes periodically in the STA charging protocol, while in the other two charging protocols it oscillates irregularly. In Fig. 4(b), we find that the non-STA charging protocol increases the energy fluctuation, while the STA technology decreases the fluctuation. Compared with the case where the coupling strength is constant, the non-STA charging protocol increases the maximum fluctuation of the stored energy by about 74%. The maximum fluctuation in the STA charging protocol is the lowest of the three, which is reduced by about 53%. In other words, the STA charging protocol can effectively suppress the fluctuation in the stored energy. In addition, we show that a critical behavior will appear. The minimal energy fluctuation always occurs in the proximity of minima of the von Neumann entropy (also corresponding to the lowest values of energy and power) and for particular choices of parameters it can exactly coincide with them. The minima of the von Neumann entropy tend to zero, the state is very close to pure state. Therefore the system can extract more work with entanglement than without. Beyond that threshold, the von Neumann entropy immediately increases to its maximum value, meaning that the optimal energy transport is achieved in this case.

An interesting and key topic in the STA is to assess the cost of implementing the STA protocol. Following the method in Ref. [93], we calculate the instantaneous energy cost \( \partial_t C_0 \), that is

\[
\partial_t C_0 = \| \partial_t H_0, \rho(t) \| = \sqrt{2} \lambda \sqrt{\text{Var}(H_1)}, \tag{16}
\]

where

\[
H_1 = \left[ (1 + \gamma)(S_+ S_- + S_- S_+) - N \right] + (1 - \gamma) \left( S_+^2 + S_-^2 \right) \frac{2N}{\lambda^2},
\]

and \( \text{Var}(H_1) \) denotes the variance of the \( H_1 \). To further show the energy cost at different particle numbers, we further define the maximum instantaneous energy cost as follows

\[
(\partial_t C_0)_{\text{max}} = \max(\partial_t C_0). \tag{17}
\]

These results are shown in Fig. 5. Here the \( \tau \) is the rescaling time of the evolution and we take the duration of the STA as the time to reach the first maximum energy value during the evolution process. The instantaneous energy cost will increase as the number of particles increases. More interestingly, 1) the cost becomes 0 at the maximum energy point and 2) when the number of particles increases to a certain extent, the maximum instantaneous energy cost gradually becomes stable, which means that the STA works better for large numbers of particles (see also Fig. 5(b), for small particle number \( N \), the slope of the maximum instantaneous energy cost is large, while in the case of large particle number, it becomes smaller.).

FIG. 4. The dependence of (a) the von Neumann entropy \( S \) and (b) the energy fluctuation \( \Delta W \) on \( t \) during charging. Relative parameters, color coding and labeling are the same as in Fig. 4.

IV. ENTROPY, ENERGY FLUCTUATION AND ENERGY COST

Finally, we investigate the time evolution of the von Neumann entropy and the fluctuation in the stored energy, and the results are shown in Fig. 4. Similarly, we compare three charging protocols. It can be found from Fig. 4(a) that entropy is constantly oscillating during the charging process. A comparison of these Figs. 1(a) and
FIG. 5. The dependence of (a) the instantaneous energy cost $\partial_t C_0$ on the rescaling evolution time $\tau$ (the duration takes $0.3\pi$) for different $N$ during charging and (b) the maximum instantaneous energy cost $(\partial_t C_0)_{\text{max}}$ on particle number $N$ during charging. Other parameters are the same as in Fig. 3.

V. DISCUSSIONS AND CONCLUSIONS

It is important to point out that the analytic assessment of $H_{cd}$ term for many-body systems is quite challenging. For small $N$, the correction term can be analytically calculated and is always related to the collective spin operators $S_x S_y + S_y S_x$. For $N > 3$, similar to Ref. [51], one can build the corrections by a hybrid strategy combining a STA and optimal control. Another challenge in the context of the proposal put forward here is the physical implementation of the driving term, which is common to STA-based protocols in quantum many-body systems [51]. A seemingly potential candidate system could be the one put forward in Ref. [106], where Hamiltonian terms of the form can be engineered [51]. The nonlocal terms of the type in the auxiliary Hamiltonian can also be implemented using the stroboscopic techniques demonstrated in the laboratory [75].

Here we have adjusted the coupling strength of the LMG quantum battery to a sinusoidal function of time. Afterwards, we have applied the STA protocol based on counterdiabatic driving to the charging process of the quantum battery and realized fast charging. The STA can greatly improve the maximum stored energy and charging power of the LMG quantum battery. Simultaneously, by comparing the performance of the LMG quantum battery with different parameters, we found that when the particle number $N$ or the frequency $\omega$ of the coupling strength is large enough, the quantum battery charged by the STA has an absolute advantage in the charging power, and this advantage increases by adjusting $N$, $\omega$ and $\gamma$. Within the range of our calculation, the advantage of the charging power brought by the STA requires that the amplitude $A$ of coupling strength should not be too large, and the charging power will decrease sharply with the increase of the anisotropy parameter $\gamma$ before the critical point. In addition, we have considered the energy fluctuation, von Neumann entropy and energy cost. It is shown that the STA can make the von Neumann entropy change periodically and reduce the energy fluctuation. The minimal energy fluctuation always occurs in the proximity of minima of the von Neumann entropy. This means that the charging behavior of the quantum battery is correlated with entanglement, supporting the notion that the entanglement enhances the charging capacity of the LMG quantum battery. The maximum instantaneous energy cost will be stabilized as the number of particles increases. Besides the LMG quantum battery, we believe that the STA technology can also improve the performance of other many-body quantum batteries such as the Dicke quantum battery.

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