A new $\mathbb{Z}_3$-graded quantum group

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Abstract

We introduce a $\mathbb{Z}_3$-graded version of exterior (Grassmann) algebra with two generators and using this object we obtain a new $\mathbb{Z}_3$-graded quantum group denoted by $O(\tilde{GL}_q(2))$. We also discuss some properties of $O(\tilde{GL}_q(2))$.

1 Introduction

Quantum plane [1] is a well known example in quantum group theory. One specific approach to represent a quantum group is to introduce quantum plane (and its dual). When there exists an appropriate set of noncommuting variables spanning linearly a representation space, the endomorphisms on that space preserving the noncommutative structure allows to set up a quantum group. The natural extension to $\mathbb{Z}_2$-graded space was introduced in [2]. The present work starts a $\mathbb{Z}_3$-graded version of the exterior plane, denoted by $\mathbb{R}^{0|2}_q$, where $q$ is a cubic root of unity. In this case, of course, it will not go back to the original objects. The term ”plane” is used as a formal title based upon its construction. Following the approach of the Manin’s to quantum group $GL(2)$ we see that there exists a $\mathbb{Z}_3$-graded (quantum) group acting on the $\mathbb{Z}_3$-graded exterior plane. A detailed discussion of this group are given in Sect. 3. In [3] Chung finds commutation relations between the elements of a $\mathbb{Z}_3$-graded quantum 2x2 matrix using the differential schema established on quantum (1+1)-superplane. With a similar idea, in [4] the author obtains similar (but not all the same) relations. However, all structures introduced in the present study are completely different from both [3] and [4] except for matrix $T$.

2 $\mathbb{Z}_3$-graded planes

The aim of this section is to introduce the $\mathbb{Z}_3$-graded version of the exterior algebra and its dual. It is known that the Manin’s quantum plane is introduced as a $q$-deformation of commutative plane in the sense that it becomes the classical plane when $q$ is equal to 1. In our case, the parameter $q$ is a cubic root of unity and there is no return. To understand what this means, let’s begin with recalling some facts about the exterior algebra.

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2.1 \textit{Z}_3\text{-gradation}

A \textit{Z}_3\text{-graded} vector space is a vector space \( V \) together with a decomposition \( V = V_0 \oplus V_1 \oplus V_2 \). Members of \( V_0 \oplus V_1 \oplus V_2 \) are called homogeneous elements. The grade (or degree) of a homogenous element \( v \in V_i \) is denoted by \( \tau(v) = i \), \( i \in Z_3 \). An element in \( V_0 \) (resp. \( V_1 \) and \( V_2 \)) is of degree 0 (resp. 1 and 2).

A \textit{Z}_3\text{-graded} algebra \( A \) is a \textit{Z}_3\text{-graded} vector space \( A = A_0 \oplus A_1 \oplus A_2 \) which is also an associative algebra such that \( A_i \cdot A_j \subset A_{i+j} \) or, equivalently, \( \tau(\xi_1 \cdot \xi_2) = \tau(\xi_1) + \tau(\xi_2) \) for all homogeneous elements \( \xi_1, \xi_2 \in A \).

2.2 The algebra of functions on the \textit{Z}_3\text{-graded} exterior plane

A possible way to generalize the \textit{Z}_3\text{-graded} exterior plane is to increase the power of nilpotency of its generators and to impose a \textit{Z}_3\text{-graded} commutation relation on the generators. We will assume that \( q \) is a cubic root of unity.

It is needed to put the wedge product between the coordinates of exterior plane, but it does not matter in the \textit{Z}_3\text{-graded} case.

**Definition 2.1** Let \( O(\tilde{\mathbb{R}}_0^0|2q) \) be the algebra with the generators \( \theta \) and \( \varphi \) obeying the relations

\[
\theta \cdot \varphi = q^2 \varphi \cdot \theta, \quad \theta^3 = 0 = \varphi^3
\]

where the coordinates \( \theta \) and \( \varphi \) are of grade 1 and 2, respectively. We call \( O(\tilde{\mathbb{R}}_0^0|2q) \) the algebra of functions on the \textit{Z}_3\text{-graded} exterior plane \( \tilde{\mathbb{R}}_0^0|2q \).

**Definition 2.2** The \textit{Z}_3\text{-graded} plane \( \tilde{\mathbb{R}}_{0}^{0|2} \) with the function algebra

\[
O(\tilde{\mathbb{R}}_{0}^{0|2}) = K\{\xi, x\}/\langle \xi x - x \xi \rangle
\]

is called \textit{Z}_3\text{-graded} dual exterior plane where the generators \( \xi, x \) are of degree 2, 0, respectively.

Hence, in accordance with Definition 2.2, we have

\[
\tilde{\mathbb{R}}_{0}^{0|2} \ni \begin{pmatrix} \xi \\ x \end{pmatrix} \iff \xi x = x \xi. \tag{2}
\]

3 The \textit{Z}_3\text{-graded} (quantum) group

The algebraic group \( SL(2, \mathbb{C}) \) has coordinate algebra \( O(SL(2, \mathbb{C})) \). This algebra is the quotient of the commutative polynomial algebra \( \mathbb{C}[a, b, c, d] \) by the two-sided ideal generated by the element \( ad - bc - 1 \) where the indeterminates \( a, b, c, d \) are the coordinate functions on \( SL(2, \mathbb{C}) \). Using the group structure in \( SL(2, \mathbb{C}) \), we can encode it in terms of maps \( m \) (multiplication), \( \eta \) (identity) and \( S \) (inversion). Dualizing these maps to \( O(SL(2, \mathbb{C})) \), we get the corresponding co-maps called comultiplication \( \Delta \), counit \( \epsilon \), and antipode \( S \), respectively. The
axioms for the group structure of $SL(2, \mathbb{C})$, in terms of the maps, are then reversed giving us relations among the co-maps. The natural axioms satisfied in $O(SL(2, \mathbb{C}))$ by the maps $m$, $\eta$, $\Delta$, $\epsilon$ and $S$, it makes a Hopf algebra. The quantum group $O_q(SL(2, \mathbb{C}))$ is a noncommutative deformation of $O(SL(2, \mathbb{C}))$.

General concepts related to quantum groups (Hopf algebras) can be found in the books of Klimyk and Schm"{u}dgen [5] or Majid [6].

In this section, we will consider the 2x2 matrices acting on the $\mathbb{Z}_3$-graded exterior plane and will discuss the properties of such matrices. So, let $a$, $\beta$, $\gamma$, $d$ be elements of an algebra $A$ where the generators $a$ and $d$ are of degree 0, the generators $\gamma$ and $\beta$ are of degree 1 and 2, respectively. Let $\tilde{M}(2)$ be defined as the polynomial algebra $k[a, \beta, \gamma, d]$. It will sometimes be convenient and more illustrative to write a point $(a, \beta, \gamma, d)$ of $\tilde{M}(2)$ in the matrix form

$$T = \begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix} = \begin{pmatrix} t_{11} & \cdots & t_{1n} \\ \vdots & \ddots & \vdots \\ t_{m1} & \cdots & t_{mn} \end{pmatrix}. \quad (3)$$

We constitute the $\mathbb{Z}_3$-graded matrix algebra $\tilde{M}(2)$ as follows: We divided the algebra $\tilde{M}(2)$ into three parts in form $\tilde{M}(2) = A_\bar{0} \oplus A_\bar{1} \oplus A_\bar{2}$. In this case, if a matrix has the form of

$$T_0 = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}, \quad \text{(resp. } T_1 = \begin{pmatrix} 0 & 0 \\ \gamma & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}),$$

then it is an element of $A_\bar{0}$ (resp. $A_\bar{1}$, $A_\bar{2}$) and is of grade 0 (resp. 1, 2). This gives a $\mathbb{Z}_3$-graded structure to the algebra of matrices, in the sense that $\tau(T_i T_j) = \tau(T_i) + \tau(T_j) \pmod{3}$. It is easy to check that the product of two $\mathbb{Z}_3$-graded matrices is also a $\mathbb{Z}_3$-graded matrix. As it can easily be shown, matrices of the form (3) form a group provided that $ad - \beta \gamma \neq 0$. We denote this group by $\tilde{GL}(2)$.

### 3.1 The algebra $O(\tilde{M}_q(2))$

To determine a $q$-analogue of the algebra $O(\tilde{M}(2))$, we will first obtain the commutation relations between the matrix elements of the matrix $T$.

If $A$ and $B$ are $\mathbb{Z}_3$-graded algebras, then their tensor product $A \otimes B$ is the $\mathbb{Z}_3$-graded algebra whose underlying space is $\mathbb{Z}_3$-graded tensor product of $A$ and $B$. The following definition [7] gives the product rule for tensor product of algebras.

**Definition 3.1** If $A$ is a $\mathbb{Z}_3$-graded algebra, then the product rule in the $\mathbb{Z}_3$-graded algebra $A \otimes A$ is defined by

$$(a_1 \otimes a_2)(a_3 \otimes a_4) = q^{\tau(a_2)\tau(a_3)}a_1a_3 \otimes a_2a_4 \quad (4)$$

where $a_i$’s are homogeneous elements in the algebra $A$.

**Remark 1.** It is well known that, the matrix $T$ given in (3) defines the linear transformation $T : \mathbb{R}_q^{0|2} \rightarrow \mathbb{R}_q^{0|2}$ and $T : \mathbb{R}_q^{a|0|2} \rightarrow \mathbb{R}_q^{a|0|2}$. As a result of
these, we have $T \Theta = \Theta' \in \tilde{R}^{0}_{q}$ and $T \Phi = \Phi' \in \tilde{R}^{0}_{q}$, where $\Theta = (\theta, \phi)^{t}$ and $\Phi = (\xi, x)^{t}$. However, the relation $\alpha_{1} \alpha_{2} = q^{\tau(\alpha_{1})} \tau(\alpha_{2}) \alpha_{1}$ for all elements $\alpha_{1}$ and $\alpha_{2}$ in the $Z_{3}$-graded algebra is inconsistent. Therefore, we will use the following transform while getting the commutation relations between the matrix elements of $T$.

Let $a, \beta, \gamma, d$ be elements of the algebra $O(\tilde{M}(2))$. We also assume that the generators $a$ and $d$ are of degree 0, the generators $\gamma$ and $\beta$ are of degree 1 and 2, respectively. Then we can change the coordinates of a vector in $\tilde{R}^{0}_{q}$ as follows

$$\Theta' = \left( \begin{array}{c} \theta' \\ \phi' \end{array} \right) := \left( \begin{array}{c} a \\ \beta \\ \gamma \\ d \end{array} \right) \otimes \left( \begin{array}{c} \theta \\ \phi \end{array} \right), \quad \Theta'' = \left( \begin{array}{c} \theta'' \\ \phi'' \end{array} \right) := \left( \begin{array}{c} \theta \\ \phi \end{array} \right) \otimes \left( \begin{array}{c} a \\ \beta \\ \gamma \\ d \end{array} \right).$$

So, we can give the following proposition that can be proved with straightforward computations.

**Proposition 3.2** The coordinates of $\Theta'$ and $\Theta''$ satisfy (1) if and only if the generators $a, \beta, \gamma, d$ fulfill the relations

$$a \beta = \beta a, \quad \beta \gamma = \gamma \beta, \quad d \beta = \beta d,$$

$$a \gamma = q \gamma a, \quad d \gamma = q^{2} \gamma d,$$

$$ad = da + (q - 1) \beta \gamma,$$

where $q$ is a cubic root of unity.

**Remark 2.** Unlike the usual quantum group [1], one interesting feature is that the element $\beta$ belongs to the center of the algebra.

**Definition 3.3** The $Z_{3}$-graded algebra $O(\tilde{M}_{q}(2))$ is the quotient of the free algebra $k\{a, \beta, \gamma, d\}$ by the two-sided ideal $J_{q}$ generated by the six relations (6)-(8) of Proposition 3.2.

By relation (8), we have

$$D_{q} := ad - q \beta \gamma = da - \beta \gamma.$$ (9)

This element of $O(\tilde{M}_{q}(2))$ is called the $Z_{3}$-graded determinant.

The proof of the following assertion is given by direct computation using the relations (6)-(8).

**Remark 3.** The $Z_{3}$-graded quantum determinant defined in (9) commutes with $a, \beta, \gamma$ and $d$, so that the requirement $D_{q} = 1$ is consistent.

**Proposition 3.4** Let $T$ and $T'$ be two matrices such that their matrix elements satisfy the relations (6)-(8). If all elements of $T$ commute according to the rule (4) with all elements of $T'$, then the elements of the matrix (tensor) product $TT'$ obey the relations (6)-(8). We also have

$$D_{q}(T \otimes T') = D_{q}(T) \otimes D_{q}(T').$$
**Proof.** Let the matrix (tensor) product of $T$ with $T'$ be
\[
T \otimes T' = \begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix} \otimes \begin{pmatrix} a' & \beta' \\ \gamma' & d' \end{pmatrix} = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}.
\]
Then using the relations (6)-(8) with (4) we get
\[
(3)
\]
With these maps, the algebra $\mathcal{O}$ is uniquely determined by $XZ$. It can be similarly shown that relations $XZ = qZX$, $YZ = ZY$, etc., are provided. Proof of the latter as follows:
\[
XW = a\gamma \otimes a'\beta' + ad \otimes a'd' + q\beta\gamma \otimes \gamma'\beta' + \beta d \otimes \gamma'd'
\]
\[
YZ = q^2a\gamma \otimes a'\beta' + ad \otimes \beta'\gamma' + \beta \gamma \otimes d'a' + \beta d \otimes d'\gamma'
\]
\[
XW - qYZ = ad \otimes (a'd' - q\beta'\gamma') - q\beta\gamma \otimes (d'a' - \gamma'\beta')
\]
and so $D_q(T \otimes T')$ reduces to $D_q(T) \otimes D_q(T')$.

\[\Box\]

### 3.2 Bialgebra structure on $\tilde{M}_q(2)$

We now supply the algebra $\mathcal{O}(\tilde{M}_q(2))$ with a bialgebra structure. The comultiplication and the counit will be the same as the usual quantum groups.

**Proposition 3.5** (1) There exist $\mathbb{Z}_3$-graded algebra homomorphisms
\[
\Delta : O(\tilde{M}_q(2)) \longrightarrow O(\tilde{M}_q(2)) \otimes O(\tilde{M}_q(2)), \quad \epsilon : O(\tilde{M}_q(2)) \longrightarrow \mathbb{C}
\]
uniquely determined by
\[
\Delta(a) = a \otimes a + \beta \otimes \gamma, \quad \Delta(\beta) = a \otimes \beta + \beta \otimes d, \tag{10}
\]
\[
\Delta(\gamma) = \gamma \otimes a + d \otimes \gamma, \quad \Delta(d) = \gamma \otimes \beta + d \otimes d, \tag{11}
\]
\[
\epsilon(a) = 1 = \epsilon(d), \quad \epsilon(\beta) = 0 = \epsilon(\gamma). \tag{12}
\]

(2) With these maps, the algebra $O(\tilde{M}_q(2))$ is a bialgebra which is neither commutative nor cocommutative.

(3) The quantum determinant $D_q$ is group-like element of $O(\tilde{M}_q(2))$.

**Proof.** (1) In order to prove that $\Delta$ and $\epsilon$ are algebra homomorphisms, it is enough to show that the relations (13)-(18) remain invariant under $\Delta$ and $\epsilon$. As an example let us show that $\Delta(a\beta) = \Delta(\beta a)$:
\[
\Delta(a\beta) = \Delta(a)\Delta(\beta) = (a \otimes a + \beta \otimes \gamma)(a \otimes \beta + \beta \otimes d)
\]
\[
= a^2 \otimes a\beta + a\beta \otimes ad + \beta a \otimes \gamma \beta + q^2 \beta^2 \otimes \gamma d
\]
\[
= a^2 \otimes \beta a + \beta a \otimes da + qa\beta \otimes \beta \gamma + \beta^2 \otimes \gamma d
\]
\[
\Delta(\beta a) = \Delta(\beta)\Delta(a) = (a \otimes \beta + \beta \otimes d)(a \otimes a + \beta \otimes \gamma)
\]
\[
= a^2 \otimes \beta a + qa\beta \otimes \beta \gamma + \beta a \otimes da + \beta^2 \otimes \gamma d.
\]
(2) It is not difficult to check that the comultiplication $\Delta$ is coassociative in the sense that
\[(\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta \quad (13)\]
and the counit $\epsilon$ has the property
\[m \circ (\epsilon \otimes \text{id}) \circ \Delta = \text{id} = m \circ (\text{id} \otimes \epsilon) \circ \Delta. \quad (14)\]

It follows that $O(\tilde{M}_q(2))$ is indeed a bialgebra.

(3) To prove that the $Z_3$-graded determinant $D_q$ is group-like, it is enough to show that
\[\Delta(D_q) = D_q \otimes D_q \quad \text{and} \quad \epsilon(D_q) = 1. \quad (15)\]

Indeed, some computations give
\[\Delta(D_q) = \Delta(a)\Delta(d) - q\Delta(\beta)\Delta(\gamma) \]
\[= ad \otimes ad + q\beta \gamma \otimes \beta \gamma - qad \otimes \beta \gamma - q\beta \gamma \otimes da \]
\[= ad \otimes (ad - q\beta \gamma) + q\beta \gamma \otimes (\beta \gamma - da) \]
\[= (ad - q\beta \gamma) \otimes (da - \beta \gamma) \]

and $\epsilon(ad - q\beta \gamma) = \epsilon(a)\epsilon(d) - q\epsilon(\beta)\epsilon(\gamma) = 1. \quad \Box$

The bialgebra $O(M_q(2))$ is called the coordinate algebra of the $Z_3$-graded (quantum) matrix space $\tilde{M}_q(2)$.

### 3.3 The $Z_3$-graded Hopf algebra $O(\widetilde{GL}_q(2))$

Using the quantum determinant $D_q$ belonging to the algebra $O(M_q(2))$, we can define a new Hopf algebra adding an inverse $t^{-1}$ to $O(M_q(2))$. Let $O(\widetilde{GL}_q(2))$ be the quotient of the algebra $O(M_q(2))$ by the two-sided ideal generated by the element $tD_q - 1$. For short we write
\[O(\widetilde{GL}_q(2)) := O(M_q(2))[t]/(tD_q - 1).\]

Then the algebra $O(\widetilde{GL}_q(2))$ is again a bialgebra.

**Lemma 3.6** The elements of the matrix
\[\tilde{T} = \begin{pmatrix} \tilde{a} & \tilde{\beta} \\ \tilde{\gamma} & \tilde{d} \end{pmatrix} = \begin{pmatrix} d & -\beta \\ -q\gamma & a \end{pmatrix} \quad (16)\]
satisfy the defining relations of the algebra $O(\widetilde{GL}_q^2(2))$ and thus $O(\widetilde{GL}_q^2(2))$ is the opposite algebra of $O(\widetilde{GL}_q(2)).$

**Proof.** The use of relations (10)-(13) imply
\[\tilde{a}\tilde{\beta} = \tilde{\beta}\tilde{a}, \quad \tilde{\beta}\tilde{\gamma} = \tilde{\gamma}\tilde{\beta}, \quad \tilde{\beta}\tilde{d} = \tilde{d}\tilde{\beta},
\[\tilde{a}\tilde{\gamma} = q^2\tilde{\gamma}\tilde{a}, \quad \tilde{\gamma}\tilde{d} = q^2\tilde{d}\tilde{\gamma},
\[\tilde{a}\tilde{d} = \tilde{d}\tilde{a} + (1 - q^2)\tilde{\beta}\tilde{\gamma},\]
which are the defining relations of the algebra $O(\widetilde{GL}_{q^2}(2))$. The second claim follows from the fact that $q^3 = 1$. □

**Proposition 3.7** The bialgebra $O(\widetilde{GL}_{q}(2))$ is a $Z_3$-graded Hopf algebra. The antipode $S$ of $O(\widetilde{GL}_{q}(2))$ is given by

$$S(a) = d D_q^{-1}, \quad S(\beta) = -\beta D_q^{-1}, \quad S(\gamma) = -q\gamma D_q^{-1}, \quad S(d) = a D_q^{-1}. \quad (17)$$

**Proof.** By Lemma 3.6, there exists an algebra anti-homomorphism $S$ from $O(\widetilde{GL}_{q}(2))$ to $O(\widetilde{GL}_{q^2}(2))$ such that $S(a) = \tilde{a}$, etc. To prove that $S$ is an antipode for $O(\widetilde{GL}_{q}(2))$, we have to check the antipode axiom

$$m \circ (S \otimes id) \circ \Delta = \epsilon = m \circ (id \otimes S) \circ \Delta \quad (18)$$

for the generators. To check the axiom (18) for the generators is equivalent to verify the following matrix equality

$$TT \tilde{D}_q = \epsilon(T) = \tilde{T}TD_q$$

which follows from $D_q = ad - q\beta\gamma$ in $O(\widetilde{GL}_{q}(2))$ with $S(T) = D_q^{-1} \tilde{T} = T^{-1}$. The details can be checked easily. □

**Definition 3.8** The $Z_3$-graded Hopf algebra $O(\widetilde{GL}_{q}(2))$ is called the coordinate algebra of the $Z_3$-graded (quantum) group $\widetilde{GL}_{q}(2)$.

### 3.4 Coactions on the $Z_3$-graded exterior plane

In bialgebra terminology, the second suggestion of Proposition 3.2 yields the following.

**Proposition 3.9** The algebra $O(\mathbb{R}_q^{0|2})$ is a left and right comodule algebra of the bialgebra $O(M_q(2))$ with left coaction $\delta_L$ and right coaction $\delta_R$ such that

$$\delta_L(\theta) = a \otimes \theta + \beta \otimes \varphi, \quad \delta_L(\varphi) = \gamma \otimes \theta + d \otimes \varphi, \quad (19)$$

$$\delta_R(\theta) = \theta \otimes a + \varphi \otimes \gamma, \quad \delta_R(\varphi) = \theta \otimes \beta + \varphi \otimes d. \quad (20)$$

**Proof.** It is not difficult to verify that (19) and (20) define algebra homomorphisms $\delta_L$ from $O(\mathbb{R}_q^{0|2})$ to $O(M_q(2)) \otimes O(\mathbb{R}_q^{0|2})$ and $\delta_R$ from $O(\mathbb{R}_q^{0|2})$ to $O(\mathbb{R}_q^{0|2}) \otimes O(M_q(2))$, respectively. It remains to be checked that $\delta_L$ and $\delta_R$ are coactions, i.e., the conditions

$$(\Delta \otimes id) \circ \delta_L = (id \otimes \delta_L) \circ \delta_L, \quad m \circ (\epsilon \otimes id) \circ \delta_L = id \quad (21)$$

and

$$(id \otimes \Delta) \circ \delta_R = (\delta_R \otimes id) \circ \delta_R, \quad m \circ (id \otimes \epsilon) \circ \delta_R = id \quad (22)$$
are satisfied. For examples,
\[
(\Delta \otimes \text{id})\delta_L(\theta) = (\Delta \otimes \text{id})(a \otimes \theta + \beta \otimes \varphi)
\]
\[
= (a \otimes a + \beta \otimes \gamma) \otimes \theta + (a \otimes \beta + \beta \otimes d) \otimes \varphi
\]
\[
= a \otimes (a \otimes \theta + \beta \otimes \varphi) + \beta \otimes (\gamma \otimes \theta + d \otimes \varphi)
\]
\[
= a \otimes \delta_L(\theta) + \beta \otimes \delta_L(\varphi)
\]
\[
= (\text{id} \otimes \delta_L)\delta_L(\theta)
\]
and
\[
m \circ (\epsilon \otimes \text{id})\delta_L(\theta) = m(\epsilon \otimes \text{id})(a \otimes \theta + \beta \otimes \varphi)
\]
\[
= m(1 \otimes \theta + 0 \otimes \varphi)
\]
\[
= \theta
\]
as expected. □

**Remark 4.** In fact, there exists a left coaction of \(O(\tilde{\mathbb{R}}_q^{*0}\mid^2)\) on the plane \(\tilde{\mathbb{R}}_q^{*0}\mid^2\), called a left comodule-\(O(\tilde{\mathbb{R}}_q^{*0}\mid^2)\) satisfying the conditions [24].

**Remark 5.** An easy computation shows that the ideal \((\vartheta := \theta\varphi - q^2\theta\varphi)\) of \(\tilde{\mathbb{R}}_q^{*0}\mid^2\) is a subcomodule of \(\tilde{\mathbb{R}}_q^{*0}\mid^2\). The proof is immediate: Indeed, since \(\delta_L\) is an algebra map, it is only necessary to show that \(\delta_L(\vartheta) = D_q \otimes \vartheta\). Using relations [19]-[23] with [23] we get
\[
\delta_L(\vartheta) = \delta_L(\theta)\delta_L(\varphi) - q^2\delta_L(\varphi)\delta_L(\theta)
\]
\[
= qa\gamma \otimes \theta^2 + ad \otimes \theta\varphi + q^2\beta\gamma \otimes \varphi\theta + \beta d \otimes \varphi^2 - q^2\gamma a \otimes \theta^2
\]
\[
- q\gamma\beta \otimes \theta\varphi - q^2 da \otimes \varphi\theta - d\beta \otimes \varphi^2
\]
\[
= (ad - q\beta\gamma) \otimes \theta\varphi - q^2(da - \beta\gamma) \otimes \varphi\theta = D_q \otimes \vartheta
\]
as expected. □

### 3.5 The Hopf algebra \(O(\tilde{SL}_q(2))\)

We know that, since the determinant \(D_q\) is group-like, the two-sided ideal \(\langle D_q - 1 \rangle\) generated by the element \(D_q - 1\) is a biideal of \(O(\tilde{\mathbb{M}}_q(2))\). So the quotient \(O(\tilde{SL}_q(2)) := O(\tilde{\mathbb{M}}_q(2))/\langle D_q - 1 \rangle\) is a bialgebra.

**Proposition 3.10** There exists a Hopf \(*\)-algebra structures on the Hopf algebra \(O(\tilde{SL}_q(2))\) such that
\[
a^* = a, \quad \beta^* = \beta, \quad \gamma^* = q\gamma, \quad d^* = d.
\]

### 4 \(Z_3\)-graded quantum algebra of \(\tilde{GL}_q(2)\)

In this section, using the method of [24], we give an \(R\)-matrix formulation for the \(Z_3\)-graded quantum group \(\tilde{GL}_q(2)\) and obtain a \(Z_3\)-graded universal enveloping algebra \(U_q(gl(2))\).
4.1 The FRT construction for $\widetilde{GL}_q(2)$

The $R$-matrix formulation (the FRT-relation $\hat{R}T_1T_2 = T_1T_2\hat{R}$) for the quantum matrix groups $[8]$ can be considered as a compact matrix form of the commutation relations between the generators of an associative algebra.

The formulation for the $\mathbb{Z}_3$-graded quantum group $\widetilde{GL}_q(2)$ has the same form, but matrix tensor product includes additional $q$-factors related to $\mathbb{Z}_3$-grading. Two matrices $A, B (\tau(A_{ij}) = \tau(i) + \tau(j))$ are multiplied according to the rule
\[
(A \otimes B)_{ij,kl} = q^{\tau(j)(\tau(i)+\tau(k))} A_{ik}B_{jl}.
\] (24)

Due to this prescription, $T_2 = I \otimes T$ has the same block-diagonal form as in the standard (ungraded) case while $T_1 = T \otimes I$ includes the additional factors $q$ for graded elements standing at some of odd rows of blocks. For the $\mathbb{Z}_3$-graded quantum group $\widetilde{GL}_q(2)$ the $R$-matrix satisfying the $\mathbb{Z}_3$-graded Yang-Baxter equation has in the form
\[
\hat{R} = \begin{pmatrix}
q & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & q^{-2} & 0 \\
0 & 0 & 0 & q
\end{pmatrix} = (\hat{R}_{ij})
\] (25)

where $\hat{R} = PR$ and $P$ denotes the $\mathbb{Z}_3$-graded permutation operator defined by $P(a \otimes b) = q^{\tau(a)\tau(b)}b \otimes a$ on homogeneous elements. A simple calculation shows that this operator represents the 3rd-root of the permutation operator $P$ with action $P(a \otimes b) = b \otimes a$.

The condition for the matrices to belong to the $\mathbb{Z}_3$-graded quantum group $\widetilde{GL}_q(2)$ is given below, but it will not be proved here.

Proposition 4.1 A 2x2-matrix $T$ is a $\mathbb{Z}_3$-graded quantum matrix if and only if
\[
\hat{RT}_1T_2 = T_1T_2\hat{R}
\] (26)

where matrix elements of $T$ are $\mathbb{Z}_3$-graded.

4.2 A $\mathbb{Z}_3$-graded universal enveloping algebra $U_q(\widetilde{gl}(2))$

The $\mathbb{Z}_3$-graded quantum algebra of $\widetilde{GL}_q(2)$ can be analogous construction to approach of the Leningrad school. The $\mathbb{Z}_3$-graded quantum algebra of $\widetilde{GL}_q(2)$ has four generators: $U$ and $V$ are of degree 0, $X_-$ and $X_+$ are of degrees 1 and 2, respectively.

Proposition 4.2 The generators of the $\mathbb{Z}_3$-graded quantum algebra satisfy the following relations
\[
UV = VU, \quad UX_\pm = q^{\pm 2}X_\pm U, \quad VX_\pm = q^{\mp 2}X_\pm V, \\
X_+X_- - X_-X_+ = \frac{UV^{-1} - VU^{-1}}{q^2 - q}
\] (27, 28)
proof. The generators $U, V, X_\pm$ can be written in two 2x2 matrix as follows

$$L^+ = \begin{pmatrix} U & \lambda X_+ \\ 0 & V \end{pmatrix}, \quad L^- = \begin{pmatrix} U^{-1} & 0 \\ \lambda X_- & V^{-1} \end{pmatrix}$$  \quad (29)$$

where $\lambda = q - q^2$. The matrices $L^\pm$ satisfy the following relations

$$R^+ L_1^+ L_2^\pm = L_2^\pm L_1^+ R^+,$$  \quad (30)$$

where the matrix $R^+$ is defined by $R^+ = PRP$. The relations follow from the relations \textcolor{red}{(30)}. To obtain the relation \textcolor{red}{(28)} we use the relation

$$R^+ L_1^- L_2^\pm = L_2^\pm L_1^- R^+.$$  \quad (31)$$

Proposition 4.3 The coproduct of the generators is given by

$$\Delta(L^\pm) = L^\pm \otimes L^\pm.$$  \quad (32)$$

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