Combinatoric explosion of renormalization tamed by Hopf algebra: 30-loop Padé-Borel resummation

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Abstract It is easy to sum chain-free self-energy rainbows, to obtain contributions to anomalous dimensions. It is also easy to resum rainbow-free self-energy chains. Taming the combinatoric explosion of all possible nestings and chainings of a primitive self-energy divergence is a much more demanding problem. We solve it in terms of the coproduct $\Delta$, antipode $S$, and grading operator $Y$ of the Hopf algebra of undecorated rooted trees. The vital operator is $S \ast Y$, with a star product effected by $\Delta$. We perform 30-loop Padé-Borel resummation of 463 020 146 037 416 130 934 BPHZ subtractions in Yukawa theory, at spacetime dimension $d = 4$, and in a trivalent scalar theory, at $d = 6$, encountering residues of $S \ast Y$ that involve primes with up to 60 digits. Even with a very large Yukawa coupling, $g = 30$, the precision of resummation is remarkable; a 31-loop calculation suggests that it is of order $10^{-8}$.

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1 Introduction

In this work we develop the Hopf algebra of renormalization \([1, 2, 3, 4, 5]\) to progress beyond the rainbow \([6, 7]\) and chain \([8, 9]\) approximations for anomalous dimensions.

**Summing rainbows:** In \(d\) dimensions, the massless scalar one-loop integral with propagators to the powers \(\alpha, \beta\) is

\[
G(\alpha, \beta; d) := g(\alpha)g(\beta)g(d - \alpha - \beta) = \frac{\Gamma(d/2 - \alpha)}{\Gamma(\alpha)} \cdot \frac{\Gamma(d/2 - \beta)}{\Gamma(\beta)} \cdot \frac{\Gamma(d/2 - \alpha - \beta)}{\Gamma(d/2 - \alpha - \beta)}
\]

(1)

Now consider the interaction \(g\phi^\dagger \sigma \phi\), with a neutral scalar particle \(\sigma\) coupled to a charged scalar \(\phi\), in the critical dimension, \(d_c = 6\). To find the anomalous field dimension \(\gamma\) of \(\phi\), in the rainbow approximation of \([7]\), one solves the consistency condition

\[
1 = aG(1, 1 + \gamma; 6) = \frac{a}{\gamma(\gamma - 1)(\gamma - 2)(\gamma - 3)}
\]

(2)

which ensures that the coupling \(a := g^2/(4\pi)^d_{c/2}\) cancels the insertion of the anomalous self energy. The perturbative solution of the resulting quartic is easily found:

\[
\gamma_{\text{rainbow}} = 3 - \frac{\sqrt{5} + 4\sqrt{1 + a}}{2} = -\frac{a}{6} + 11\frac{a^2}{6^3} - 206\frac{a^3}{6^5} + \cdots
\]

(3)

**Resumming chains:** At the other extreme, one may easily perform the Borel resummation of chains of self-energy insertions, within a single rainbow. Suppose that the self energy \(p^2\Sigma(a, p^2/\mu^2)\) is renormalized in the momentum scheme, and hence vanishes at \(p^2 = \mu^2\). The renormalized massless propagator is \(\overline{D} = 1/(p^2 - p^2\Sigma)\). Then (3) is the rainbow approximation for \(\partial\Sigma/\partial \log(\mu^2)\) at \(p^2 = \mu^2\). Following the methods of \([8]\), one finds that the corresponding asymptotic series for chains is Borel resummable:

\[
\gamma_{\text{chain}} = -6 \int_0^\infty \frac{\exp(-6x/a)dx}{(x + 1)(x + 2)(x + 3)} \simeq -\frac{a}{6} + 11\frac{a^2}{6^3} - 170\frac{a^3}{6^5} + \cdots
\]

(4)

which differs from the rainbow approximation at 3 loops, with 206 in (3) coming from the triple rainbow, while 170 in (4) comes from a chain of two self energies inside a third.

**Hopf algebra:** We shall progress beyond the rainbow and chain approximations by including all possible nestings and chainings of the one-loop self-energy divergence. In other words, we consider the full Hopf algebra of undecorated rooted trees, established in \([1]\) and implemented in \([4]\). Two figures suffice to exhibit the class of diagrams considered, and their divergence structure. The first exhibits a 12-loop example, the second exhibits its divergence structure. Due to the fact that we combine chains and rainbows, we have a full tree structure \([1]\): the depth of the tree is larger than one, and there can be more than one edge attached to a vertex.

There are 4 notable features of this analysis.

1. We use the coproduct \(\Delta\) to combine the antipode \(S\) and grading operator \(Y\) in a star product \(S \star Y\) whose residue delivers the contribution of each rooted tree.
2. We show that the rationality of rainbows [10] extends to the contribution of every undecorated rooted tree, as had been inferred from examples in [11].

3. We confirm that a recent analysis [12] of dimensional regularization applies at both $d_c = 4$ and $d_c = 6$, detecting poles of $\Gamma$ functions that occur in even dimensions.

4. We obtain, to 30 loops, highly non-trivial alternating asymptotic series, which we resum, to high precision, by combining Padé [13] and Borel [8, 9] methods.

2 Hopf-algebra method

Let $t$ be an undecorated rooted tree, denoting the divergence structure of a Feynman diagram. Then its coproduct is defined, recursively, by

$$\Delta(t) = t \otimes e + id \otimes B_+(\Delta(B_-(t)))$$

(5)

where $e$ is the empty tree, evaluating to unity, $id$ is the identity map, $B_-$ removes the root, giving a product of trees in general, and $B_+$ is the inverse of $B_-$, combining products by restoring a common root. The recursion terminates with $\Delta(e) = e \otimes e$ and develops a highly non-trivial structure by the operation of the coproduct on products of trees

$$\Delta(\prod_k t_k) = \prod_k \Delta(t_k)$$

(6)
between each removal and restoration of a root. In Sweedler notation, it takes the form

$$\Delta(t) = \sum_k a_k^{(1)} \otimes a_k^{(2)} = t \otimes e + e \otimes t + \sum'_k a_k^{(1)} \otimes a_k^{(2)}$$  \hspace{1cm} (7)

with single trees on the right and, in general, products on the left. The prime in the second summation indicates the absence of the empty tree. The field-theoretic role of the coproduct is clear: on the left products of subdivergences are identified; on the right these shrink to points. Subtractions are effected by the antipode, defined by the recursion

$$S(t) = -t - \sum'_k S(a_k^{(1)}) a_k^{(2)}$$  \hspace{1cm} (8)

for a non-empty tree, with $$S(\prod_k t_k) = \prod_k S(t_k)$$ for products and $$S(e) = e$$.

Renormalization involves a twisted antipode, $$S_R$$. Let $$\phi$$ denote the Feynman map that assigns a dimensionally regularized bare value $$\phi(t)$$ to the diagram whose divergence structure is labelled by the tree $$t$$. Then we apply the recursive definition [4]

$$S_R(t) = -R \left( \phi(t) + \sum_k S_R(a_k^{(1)}) \phi(a_k^{(2)}) \right)$$  \hspace{1cm} (9)

with a renormalization operator $$R$$ that sets $$p^2 = \mu^2$$, in both the momentum and MS schemes, and in the MS scheme selects only the poles in $$\varepsilon := (d_c - d)/2$$.

We can use the coproduct to combine operators. Suppose that $$O_1$$ and $$O_2$$ operate on trees and their products. Then we define the star product $$O_1 \star O_2$$ by

$$O_1 \star O_2(t) = \sum_k O_1(a_k^{(1)}) O_2(a_k^{(2)})$$  \hspace{1cm} (10)

with ordinary multiplication performed after $$O_1$$ operates on the left and $$O_2$$ on the right of each term in the coproduct. By construction, $$S \star id$$ annihilates everything except the empty tree, $$e$$. The presence of $$R$$ makes $$S_R \star \phi$$ finite and non-trivial. In particular, the renormalized Green function is simply

$$\Gamma_R(t) = \lim_{\varepsilon \to 0} S_R \star \phi(t)$$  \hspace{1cm} (11)

whose evaluation was efficiently encoded in [4], using a few lines of computer algebra.

Here we present a new – and vital – formula for efficiently computing the contribution of an undecorated tree to the anomalous dimension. It is simply

$$\gamma(t) = \lim_{\varepsilon \to 0} \varepsilon \phi(S \star Y(t))$$  \hspace{1cm} (12)

where $$Y$$ is the grading operator, with $$Y(t) = nt$$, for a tree with $$n$$ nodes. In general, $$Y$$ multiplies a product of trees by its total number of nodes. To see that this works, consider the terms in [4], in the momentum scheme, before taking the limit $$\varepsilon \to 0$$. Each term has a momentum dependence $$(p^2)^{n(d-d_c)/2}$$, where $$n$$ is the number of loops (and hence nodes) of the tree on the right of the term in the Sweedler sum. If we multiply by $$n\varepsilon$$, and then let $$\varepsilon \to 0$$, we clearly obtain the derivative w.r.t. $$\log(\mu^2/p^2)$$. Setting $$p^2 = \mu^2$$ we obtain the contribution to the anomalous dimension. Thus $$R$$ plays no role and we may replace $$S_R(t)$$ by $$\lim_{R \to id} S_R(t) = \phi(S(t))$$, where $$S$$ is the canonical antipode. Multiplication by $$n\varepsilon$$ is achieved by $$\varepsilon \phi(Y(t)) = n\varepsilon \phi(t)$$ on the right of the coproduct, where $$Y$$ acts only on single trees. Hence the abstract operator $$S \star Y$$ delivers the precise combination of products of trees whose bare evaluation as Feynman diagrams is guaranteed to have merely a $$1/\varepsilon$$ singularity, with residue equal to the contribution to the anomalous dimension. Thus we entirely separate the combinatorics from the analysis.
3 Example

By way of example, we show how the 3-loop expansions of (3,4) result from (12). The combinatorics are now clear. The analysis, at first sight, seems to entail the detailed properties of $\Gamma$ functions. However, appearances can be misleading.

In general, a dimensionally regularized bare value for a $n$-loop diagram, corresponding to the undecorated rooted tree $t$, is evaluated by the recursion

$$\phi(t) = \frac{L(\varepsilon, n\varepsilon)}{n\varepsilon} \prod_k \phi(b_k)$$

(13)

where $b_k$ are the branches originating from the root of $t$. It terminates with $\phi(\varepsilon) = 1$. For the scalar theory with $d_c = 6$, the master function is

$$L(\varepsilon, \delta) = \frac{a\delta}{(p^2)^2} G(1,1+\delta-\varepsilon;6-2\varepsilon) = -\frac{a}{(p^2)^2} \frac{\Gamma(1-\delta)\Gamma(1+\delta)\Gamma(2-\varepsilon)}{\Gamma(4-\delta-\varepsilon)\Gamma(1+\delta-\varepsilon)}$$

(14)

Now the wonderful feature of (12) is that it depends only on the derivatives of $L(\varepsilon, \delta)$ w.r.t. $\delta$ at $\varepsilon = 0$. This reflects the fact that the anomalous dimension, unlike the Green function, is insensitive to the details of the regularization method. Thus we may, with huge savings in computation time, replace the master function by

$$L(0, \delta) = \frac{a}{(\delta - 1)(\delta - 2)(\delta - 3)} = \sum_{n \geq 0} g_n \delta^n = -\frac{a}{6} + 11\frac{a\delta}{6^2} - 85\frac{a\delta^2}{6^3} + O(\delta^3)$$

(15)

which establishes that the contribution of each rooted tree is rational. The residue of the anomalous dimension operator $S \ast Y$ feels only the rational residues of $\Gamma$ functions; it is blind to the zeta-valued derivatives that contribute to the renormalized Green function.

Now that the analysis has been drastically simplified, we return to the combinatorics. The double rainbow, $t_2$, has coproduct $\Delta(t_2) = t_2 \otimes e + e \otimes t_2 + t_1 \otimes t_1$, where $t_1$ is the single rainbow, with $\Delta(t_1) = t_1 \otimes e + e \otimes t_1$. The antipodes are $S(t_1) = -t_1$ and $S(t_2) = -t_2 + t_1^2$. The star products are $S \ast Y(t_1) = t_1$ and $S \ast Y(t_2) = 2t_2 - t_1^2$. Hence the contributions to the anomalous dimensions are the residues of $L(0, \varepsilon)/\varepsilon$ and $(L(0, 2\varepsilon) - L(0, \varepsilon))L(0, \varepsilon)/\varepsilon^2$, namely $g_0 = -a/6$ and $g_1 g_0 = 11a^2/6^3$.

Following this simple example, the reader should find it easy to determine the anomalous dimension contributions of the two rooted trees at 3 loops. For $t_3$, the triple rainbow graph, $S \ast Y$ delivers $3t_3 - 3t_1 t_2 + t_1^3$, with residue $g_2 g_0^2 + g_0^3 = -(85 + 11^2)a^3/6^5$, in agreement with (3). For the other diagram, $t'_3$, with a double chain in a single rainbow, it delivers $3t'_3 - 4t_1 t_2 + t_1^3$ with residue $2g_2 g_0^2 = -2 \times 85a^3/6^5$, in agreement with (3). The Borel resummation (3) of chains corresponds to the result $n! g_n g_0^n$ for a chain of $n$ self energies, inside a single rainbow. Writing the anomalous dimension contribution of the full Hopf algebra as the asymptotic series

$$\gamma_{\text{hopf}} \simeq \sum_{n > 0} G_n \frac{(-a)^n}{6^{2n-1}}$$

(16)
we find that \( G_3 = 3 \times 85 + 11^2 = 376 \).

In this paper, we undertake Padé-Borel resummation of the full Hopf series (16), to 30 loops. We also resum

\[
\tilde{\gamma}_{\text{hopf}} \simeq \sum_{n>0} \tilde{G}_n \frac{(-a)^n}{2^{2n-1}}
\]

for the anomalous dimension of a fermion field with a Yukawa interaction \( g\psi\sigma\psi \), at \( d_c = 4 \), whose rainbow approximation

\[
\tilde{\gamma}_{\text{rainbow}} = 1 - \sqrt{1 + a}
\]

was obtained in [6]. At the other extreme, the Borel-resummed chain approximation

\[
\tilde{\gamma}_{\text{chain}} = -2 \int_0^\infty \exp(-2x/a)dx \quad \frac{dx}{x + 2}
\]

is easily obtained from the Yukawa generating function, \( \tilde{L}(0, \delta) = a/(\delta - 2) \).

4 Results to 30 loops

At 4 loops, there are 5 undecorated Wick contractions, corresponding to 4 rooted trees, one of which has weight 2. For the scalar theory, at \( d_c = 6 \), the tally is

\[
G_4 = 4890 + 4711 + 3595 + 3595 + 3450 = 20241 = 3^2 \times 13 \times 173
\]

Already this becomes tedious to compute by hand. Fortunately, the recursions (5, 8) of the coproduct and antipode make it sublimely easy to automate the procedure (12).

At \( n \) loops, the number of relevant Wick contractions is the Catalan number \( C_{n-1} \), where \( C_n := \frac{1}{n+1} \binom{2n}{n} \). At 30 loops, there are \( C_{29} = 1002242216651368 \) contractions. Symmetries reduce these to rooted trees, with weights determined recursively by \( W(t) = w(t) \prod_k W(b_k) \) where \( b_k \) are the branches obtained by removing the root of \( t \). The symmetry factor of the root is \( w(t) = (\sum_j n_j)!/\prod_j n_j! \) where \( n_j \) is the number of branches of type \( j \). The generating formula for \( R_n \), the number of rooted trees with \( n \) nodes, is \( \sum_{n>0} R_n x^n = x \prod_{n>0} (1 - x^n)^{-R_n} \) which expresses the fact that removal of roots from all trees with \( n \) nodes produces all products of trees with a total of \( n - 1 \) nodes. This gives \( R_{30} = 354426847597 \). The number of terms produced by applying the BPHZ procedure [15] to a single tree with \( n \) nodes is \( 2^n \).

From these enumerations, one finds – with some trepidation – that computation to 30 loops entails \( \sum_{n \leq 30} 2^n R_n = 463020146037416130934 \) subtractions, each requiring 30 terms in its Laurent expansion, with coefficients involving integers of \( O(10^{60}) \). Brute force would require processing of \( O(10^{24}) \) bits of data, which is far beyond anything contemplated by current computer science. The remedy is clear: recursion of coproduct and antipode, to compute the residues of the anomalous dimension operator \( S \star Y \).

Each new coproduct or antipode refers to others with fewer loops. By storing these we easily progressed to 13 loops, extending the sequence \( G_n \) to
For $\tilde{G}_n$, in the Yukawa case, we obtained the 13-loop sequence

$$
1, 1, 4, 27, 248, 2830, 38232, 593859, 10401712, 202601898, 4342263000, 101551822350, 2573779506192
$$

At this point, recursion of individual trees hit a ceiling imposed by memory limitations. Beyond 13 loops, we stored only the unique combination of terms that is needed at higher loops, namely the momentum-scheme renormalized self energy. Allocating 750 megabytes of main memory to Reduce 3.7 [16], the time to reach 30 loops was 8 hours. Of these, more than 2 hours were spent on garbage collection, indicating the combinatoric complexity. Results for the scalar and Yukawa theories are in Tables 1 and 2. They are highly non-trivial. Factorization of $G_{27} = 2^6 \times 5 \times 103 \times 184892457645048836717 \times 699431048506216812683294624581$ needed significant use of Richard Crandall’s elliptic curve routine [17], while $G_{29}/240$ is a 60-digit integer that is most probably prime.

### 5 Padé-Borel resummation

We combine Padé-approximant [13] and Borel-transformation [8, 9] methods. From (4) we obtain the pure chain contribution $G_{n+1}^{\text{chain}} = (2^n + (2^n - 1)3^{n+1})n!$ with, for example, $G_4^{\text{chain}} = (8 + 7 \times 81) \times 6 = 3450$ appearing in (20) as the smallest contribution of the 5 Wick contractions at 4 loops, while the pure rainbow contribution, 4711, is next to largest. This is far removed from the situation at large $n$, where the pure rainbow term is factorially smaller than the pure chain term. At large $n$, we combine $C_{n-1} \approx 4^{n-1}/\sqrt{n^3\pi}$ Wick contractions, some of which are of order $G_n^{\text{chain}}$, while some are far smaller. It is thus difficult to anticipate the large-$n$ behaviour of $G_n$. We adopted an empirical approach, finding that $S_n := 12^{1-n}G_n/\Gamma(n+2)$ varies little for $n \in [14, 30]$, as shown in the final column of Table 1. In the Yukawa case of Table 2, we found little variation in $\tilde{S}_n := 2^{1-n}\tilde{G}_n/\Gamma(n+1/2)$.

In the scalar case, at $d_c = 6$, Padé-Borel resummation may be achieved by the Ansatz

$$
\gamma_{\text{hopf}} \approx -\frac{a}{12} \int_0^\infty P(ax/3)e^{-x^2} dx
$$

(21)

where $P(y) = 1 + O(y)$ is a $[M\backslash N]$ Padé approximant, with numerator $1 + \sum_{m=1}^M c_m y^m$ and denominator $1 + \sum_{n=1}^N d_n y^n$, chosen so as to reproduce the first $M+N+1$ terms in the asymptotic series (14). We expect $P(y)$ to have singularities only in the left half-plane. In particular, a pole near $y = -1$ is expected, corresponding to the approximate constancy of $S_n$ in Table 1. We fitted the first 29 values of $G_n$ with a $[14\backslash 14]$ Padé approximant $P(y)$, finding a pole at $y \approx -0.994$. The other 13 poles have $\Re y < -1$. Moreover there is no zero with $\Re y > 0$. The test-value $G_{30}$ is reproduced to a precision of $5 \times 10^{-16}$.
In the Yukawa case, at $d_c = 4$, we made the Ansatz
\[ \tilde{\gamma}_{\text{hopf}} \approx -\frac{a}{\sqrt{\pi}} \int_0^\infty Q(ax/2)e^{-x}x^{1/2} \, dx; \quad Q(y) := \frac{\tilde{P}(y)}{1 + y} \] (22)
suggested by Table 2. Here we put in by hand the suspected pole at $y = -1$. The $[14/14]$ approximant to $\tilde{P}(y) = 1 + O(y)$ then has all its 14 poles at $\Re y < -1$ and no zero with $\Re y > 0$. The test-value $\tilde{G}_{30}$ is reproduced to a reassuring precision of $4 \times 10^{-17}$.

Table 3 compares resummation of the full Hopf results (16,17) with those from the far more restrictive chain and rainbow subsets. To test the precision of resummations (21,22), we used the star product (12) to perform the $2.6 \times 10^{21}$ BPHZ subtractions that yield the exact 31-loop coefficients
\[ G_{31} = 2^6 \times 3^3 \times 5 \times 139 \times 2957 \times 22279 \times 69318820356301 \times 9602299922477621 \times 144927172127490232568467 \] (23)
\[ \tilde{G}_{31} = 2^5 \times 3^4 \times 5 \times 71 \times 109 \times 13224049649 \times 473202021103152647613521 \] (24)
No change in the final digits of Table 3 results from using these. At the prodigious Yukawa coupling $g = 30$, corresponding to $a = (30/4\pi)^2 \approx 5.7$, a $[15/15]$ Padé approximant gives $\tilde{\gamma}_{\text{hopf}} \approx -1.85202761$, differing by less than 1 part in $10^8$ from the $[14/14]$ result $\tilde{\gamma}_{\text{hopf}} \approx -1.85202762$. It appears that resummation of undecorated rooted trees is under very good control, notwithstanding the combinatoric explosion apparent in (23,24).

6 Conclusions

As stated in the introduction, we achieved 4 goals. First, we found the Hopf-algebra construct (12) that delivers undecorated contributions to anomalous dimensions. Then we found that these are rational, with the $\Gamma$ functions of (14) contributing only their residues, via (15). Next, we exemplified the analysis of dimensional regularization in (12), at two different critical dimensions, $d_c = 6$ and $d_c = 4$. The residues of a common set (1) of $\Gamma$ functions determine both results. Finally, we obtained highly non-trivial results, from all combinations of rainbows and chains, to 30 loops. A priori, we had no idea how these would compare with the easily determined pure chain contributions. Tables 1 and 2 suggest that at large $n$ the full Hopf-algebra results exceed pure chains by factors that scale like $n^2 2^n$ and $n^{1/2} 2^n$, respectively. Padé approximation gave 15-digit agreement with exact 30-loop results. In Table 3, we compare the Borel resumptions (21,22) of the full Hopf algebra with the vastly simpler rainbow approximations (3,18) and the still rather trivial chain approximations (4,19). Even at the very large Yukawa coupling $g = 30$ we claim 8-digit precision. Apart from large-$N_f$ approximations (13), we know of no other large-coupling analysis of anomalous dimension contributions, at spacetime dimensions $d \geq 4$, that progresses beyond pure rainbows (3,18) or pure chains (3,8).

In conclusion: Hopf algebra tames the combinatorics of renormalization, by disentangling the iterative subtraction of primitive subdivergences from the analytical challenge of evaluating dimensionally regularized bare values for Feynman diagrams. Progress with
the analytic challenge shall require the expansion of skeleton graphs in the regularization parameter $D - 4$. After that, the Hopf algebra of decorated rooted trees provides the tool to take care of the combinatorial challenge of renormalization in general. Generalizations of the methods here to cases where decorations are different, but still analytically trivial, are conceivable. The results in [15] are of this form. In the present case, where the combinatoric explosion is ferocious, while the analysis is routine, the automation of renormalization by Hopf algebra is a joy. How else might one resum $2.6 \times 10^{21}$ BPHZ subtractions at 31 loops and achieve 8-digit precision at very strong coupling?

Acknowledgements: This work was undertaken during the workshop Number Theory and Physics at the ESI in November 1999, where we enjoyed discussions with Pierre Cartier, Werner Nahm, Ivan Todorov and Jean-Bernard Zuber. Past work with Alain Connes, Bob Delbourgo, John Gracey and Andrey Grozin supports the present paper.

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Table 1: Scalar coefficients in (16), with $S_n := 12^{1-n}G_n/\Gamma(n+2)$

| $n$ | $G_n$ | $S_n$ |
|-----|-------|-------|
| 14  | 16301356287284530869810308 | 0.1165 |
| 15  | 3161258841758986060906197536 | 0.1177 |
| 16  | 650090787950164885954804021185 | 0.1186 |
| 17  | 141326399508139818539694443090940 | 0.1194 |
| 18  | 32389192708918228594003667471390750 | 0.1200 |
| 19  | 7805642594117634874205145727265669184 | 0.1205 |
| 20  | 1973552096478662083584247237907087008846 | 0.1209 |
| 21  | 522399387732959889862436502331522596697560 | 0.1212 |
| 22  | 144486332652501966354908665093390779463113660 | 0.1215 |
| 23  | 41681362292986022786933211385817840822702468640 | 0.1217 |
| 24  | 12520661507532542738174037622803485508817145773050 | 0.1218 |
| 25  | 391033892820248668787314743084879349561179264255736 | 0.1220 |
| 26  | 12678911588003558444565289086726128521948839015617187260 | 0.1221 |
| 27  | 426237156086127437403654947366849019736474802601497417920 | 0.1221 |
| 28  | 148382376919675149120919349602375065827367635238832722748020 | 0.1222 |
| 29  | 5342813346724318054633091126922424219952183999220340144106320 | 0.1222 |
| 30  | 19876558632009586773182109989526780486481329823560105761256963720 | 0.1222 |
Table 2: Yukawa coefficients in (17), with $\tilde{S}_n := 2^{1-n} \tilde{G}_n / \Gamma(n + 1/2)$

| $n$ | $\tilde{G}_n$ | $\tilde{S}_n$ |
|-----|---------------|--------------|
| 14  | 70282204726396 | 0.3715       |
| 15  | 2057490936366320| 0.3750       |
| 16  | 64291032462761955| 0.3780       |
| 17  | 2136017303903513184| 0.3806  |
| 18  | 75197869250518812754| 0.3828 |
| 19  | 279647587260570979512| 0.3848 |
| 20  | 109549714522464120960474| 0.3865 |
| 21  | 450930291078349663256400| 0.3880 |
| 22  | 194584224274515194731540740| 0.3894 |
| 23  | 8784041120771057847338352720| 0.3906 |
| 24  | 41403213339839749469857933710| 0.3917 |
| 25  | 2034034274654424143487152873888| 0.3928 |
| 26  | 1039819967521866936447997028508900| 0.3937 |
| 27  | 55230362672853506023203822058592752| 0.3946 |
| 28  | 3043750896574866226650924152479935036| 0.3953 |
| 29  | 1738144768449358374050720641310171808| 0.3961 |
| 30  | 102726115862063537442587021757211879288| 0.3968 |

Table 3: Comparison of chain, rainbow and full Hopf contributions

| $a$  | $-\gamma_{\text{chain}}$ | $-\gamma_{\text{rainbow}}$ | $-\gamma_{\text{hopf}}$ | $-\tilde{\gamma}_{\text{chain}}$ | $-\tilde{\gamma}_{\text{rainbow}}$ | $-\tilde{\gamma}_{\text{hopf}}$ |
|------|-----------------------------|-----------------------------|-----------------------------|----------------------------------|----------------------------------|----------------------------------|
| 0.5  | 0.0727579                   | 0.0731322                   | 0.0742476                   | 0.2245593                         | 0.2247449                         | 0.2278233                         |
| 1.0  | 0.1301409                   | 0.1322419                   | 0.1373080                   | 0.4126913                         | 0.4142136                         | 0.4281423                         |
| 1.5  | 0.1773375                   | 0.1825988                   | 0.1937609                   | 0.5765641                         | 0.5811388                         | 0.6118625                         |
| 2.0  | 0.2172313                   | 0.2268615                   | 0.2455916                   | 0.7226572                         | 0.7320508                         | 0.7837372                         |
| 2.5  | 0.2516214                   | 0.2665867                   | 0.2939133                   | 0.8549759                         | 0.8708287                         | 0.9464649                         |
| 3.0  | 0.2817148                   | 0.3027756                   | 0.3394353                   | 0.9762193                         | 1.0000000                         | 1.1017856                         |
| 3.5  | 0.3083635                   | 0.3361156                   | 0.3826462                   | 1.0883141                         | 1.1213203                         | 1.2509126                         |
| 4.0  | 0.3321923                   | 0.3671015                   | 0.4239016                   | 1.1926947                         | 1.2360680                         | 1.3947383                         |
| 4.5  | 0.3536734                   | 0.3961033                   | 0.4634712                   | 1.2904639                         | 1.3452079                         | 1.5339452                         |
| 5.0  | 0.3731724                   | 0.4234058                   | 0.5015652                   | 1.3824908                         | 1.4494897                         | 1.6690711                         |
| 5.5  | 0.3909778                   | 0.4492331                   | 0.5383523                   | 1.4694751                         | 1.5495098                         | 1.8005504                         |
| 6.0  | 0.4073216                   | 0.4737658                   | 0.5739698                   | 1.5519895                         | 1.6457513                         | 1.9287404                         |