Bandit Learning with General Function Classes: Heteroscedastic Noise and Variance-dependent Regret Bounds

Heyang Zhao∗ and Dongruo Zhou† and Jiafan He‡ and Quanquan Gu§

Abstract

We consider learning a stochastic bandit model, where the reward function belongs to a general class of uniformly bounded functions, and the additive noise can be heteroscedastic. Our model captures contextual linear bandits and generalized linear bandits as special cases. While previous works (Kirschner and Krause, 2018; Zhou et al., 2021) based on weighted ridge regression can deal with linear bandits with heteroscedastic noise, they are not directly applicable to our general model due to the curse of nonlinearity. In order to tackle this problem, we propose a multi-level learning framework for the general bandit model. The core idea of our framework is to partition the observed data into different levels according to the variance of their respective reward and perform online learning at each level collaboratively. Under our framework, we first design an algorithm that constructs the variance-aware confidence set based on empirical risk minimization and prove a variance-dependent regret bound. For generalized linear bandits, we further propose an algorithm based on follow-the-regularized-leader (FTRL) subroutine and online-to-confidence-set conversion, which can achieve a tighter variance-dependent regret under certain conditions.

1 Introduction

Over the past decade, stochastic bandit algorithms have found a wide variety of applications in online advertising, website optimization, recommendation system and many other tasks (Li et al., 2010; McInerney et al., 2018). In the model of stochastic bandits, at each round, an agent selects an action and observes a noisy evaluation of the reward function for the chosen action, aiming to maximize the sum of the received rewards. A general reward function governs the reward of each action from the eligible action set.

A common assumption used in stochastic bandit problems is that the observation noise is conditionally independent and satisfies a uniform tail bound. In real-world applications, however, the variance of observation noise is likely to be dependent on the evaluation point (chosen action) (Kirschner and Krause, 2018). Moreover, due to the dynamic environment in reality, the variance of each action may also be different at each round. This motivates the studies of bandit problems with heteroscedastic noise. For example, Kirschner and Krause (2018) introduce the heteroscedastic noise

∗IIIS, Tsinghua University, Beijing, CN; e-mail: zhaohy2817@gmail.com
†Department of Computer Science, University of California, Los Angeles, CA 90095, USA; e-mail: drzhou@cs.ucla.edu
‡Department of Computer Science, University of California, Los Angeles, CA 90095, USA; e-mail: jiafanhe19@ucla.edu
§Department of Computer Science, University of California, Los Angeles, CA 90095, USA; e-mail: qgu@cs.ucla.edu
setting where the noise distribution is allowed to depend on the evaluation point. They propose a weighted least squares to estimate the unknown reward function more accurately in the setting where the underlying reward function is linear or lies in a separable Hilbert space (Section 5, Kirschner and Krause 2018).

In this paper, we consider a general setting, where the unknown reward function belongs to a known general function class $\mathcal{F}$ with bounded eluder dimension (Russo and Van Roy, 2013). This captures multi-armed bandits, linear contextual bandits (Abbasi-Yadkori et al., 2011) and generalized linear bandits (Filippi et al., 2010) simultaneously. Since weighted least squares highly depends on the linearity of the function class, we propose a multi-level learning framework for our general setting. The underlying idea of the framework is to partition the observed data into various levels according to the variance of the noise. The agent then estimates the reward function at each level independently and then exploit all the levels when selecting an action at each round.

While previous work by Kirschner and Krause (2018) considers sub-Gaussian noise with nonuniform variance proxies, we only assume nonuniform variances of noise (Zhou et al., 2021; Zhang et al., 2021), which brings a new challenge of exploiting the variance information of the noise to obtain tighter variance-aware confidence sets.

Under our multi-level learning framework, we first design an algorithm based on empirical risk minimization and Optimism-in-the-Face-of-Uncertainty (OFU) principle, and prove a variance-dependent regret bound. For a special class of bandits namely generalized linear bandits with heteroscedastic noise, we further propose an algorithm using follow-the-regularized-leader (FTRL) as an online regression subroutine and adopting the technique of online-to-confidence-set conversion (Abbasi-Yadkori et al., 2012; Jun et al., 2017). This algorithm achieves a provably tighter regret bound when the range of the reward function is relatively wide compared to the range of noise.

Our main contributions are summarized as follows:

- We develop a new framework multi-level regression, which can be applied to bandits with heteroscedastic noise, even when the reward function class does not lie in a separable Hilbert space.

- Under our framework, we design tighter variance-aware upper confidence bounds for bandits with general reward functions, and propose an bandit learning algorithm based on empirical risk minimization. We show that our algorithm enjoys variance-dependent regret upper bounds.

- For generalized linear bandits, which is a special case of our model class, we further propose an algorithm based on online-to-confidence-set conversion. More specifically, we first derive a variance-dependent regret bound for follow-the-regularized-leader (FTRL) for the online regression problem derived from generalized linear function class, and then convert the online learning regret bound to the bandit learning confidence set. We show that our algorithm can achieve a tighter regret bound for generalized linear bandits.

- As a by-product, our regret bound for FTRL shows a remarkable improvement against the state-of-the-art result in stochastic online linear regression Maillard et al. (2021), and enjoys appealing properties.

Notation. We use lower case letters to denote scalars, and use lower and upper case bold face letters to denote vectors and matrices respectively. We denote by $[n]$ the set \{1, \ldots, n\}. For a vector $x \in \mathbb{R}^d$ and matrix $\Sigma \in \mathbb{R}^{d \times d}$, a positive semi-definite matrix, we denote by $\|x\|_2$ the vector’s
Euclidean norm and define \( \| x \|_\Sigma = \sqrt{x^\top \Sigma x} \). For two positive sequences \( \{a_n\} \) and \( \{b_n\} \) with \( n = 1, 2, \ldots \), we write \( a_n = O(b_n) \) if there exists an absolute constant \( C > 0 \) such that \( a_n \leq C b_n \) holds for all \( n \geq 1 \) and write \( a_n = \Omega(b_n) \) if there exists an absolute constant \( C > 0 \) such that \( a_n \geq C b_n \) holds for all \( n \geq 1 \). Let \( \mathcal{N}(F, \alpha, \| \cdot \|_\infty) \) denote the \( \alpha \)-covering number of \( F \) in the sup-norm \( \| \cdot \|_\infty \). If there is no ambiguity, we may write \( \mathcal{N}(F, \alpha, \| \cdot \|_\infty) \) as \( N_\alpha \) for short. We use \( \tilde{O}() \) to further hide the polylogarithmic factors other than log-covering numbers.

Table 1: A summary of our regret results under different settings with different function classes and noise assumptions.

| Function Class          | Assumption on Noise | Regret                                      |
|-------------------------|---------------------|---------------------------------------------|
| General function class  | Assumptions 3.5 + 4.1 | \( \tilde{O} \left( \sqrt{\text{dim}_E \log N_\alpha \sum_{t=1}^T \sigma_t^2} \right) \) |
| General function class  | Assumption 3.5       | \( \tilde{O} \left( \sqrt{\text{dim}_E \log N_\alpha J + \text{dim}_E \log N_\alpha RT} \right) \) |
| Generalized linear class| Assumption 3.5       | \( \tilde{O} \left( \frac{K}{\kappa} d \sqrt{\sum_{t=1}^T \sigma_t^2} + \frac{K}{\kappa} (KAB + R) \sqrt{dT} \right) \) |

Refer to Section 3 for the definitions of \( \text{dim}_E, \sigma_t, J \) and \( R \), Section 6 for the definitions of \( \kappa, K, A, B \).

2 Related Work

Learning with Heteroscedastic Noise. Heteroscedastic noise problem has been studied in many different settings other than bandit problems such as active learning (Antos et al., 2010), regression problems (Aitken, 1936; Chaudhuri et al., 2017; Goldberg et al., 1997; Kersting et al., 2007), principle component analysis (Hong et al., 2016, 2018) and Bayesian optimization (Assael et al., 2014).

In contrast, only a few works have considered heteroscedastic noise in bandit settings. Kirschner and Krause (2018) is the first to formally introduce the concept of stochastic bandits with heteroscedastic noise. In their model, the variance of the noise at each round \( t \) is a function of the evaluation point \( x_t, \rho_t = \rho(x_t) \), and they further assume that the noise is \( \rho_t \)-sub-Gaussian. Prior to their work, Cowan et al. (2015) considers a variant of multi-armed bandits where the noise at each round is a Gaussian random variable with unknown variance. Zhou et al. (2021) considers linear bandits with heteroscedastic noise and generalized the heteroscedastic noise setting proposed by Kirschner and Krause (2018) in the sense that they no longer assume the noise to be \( \rho_t \)-sub-Gaussian, but only requires the variance of noise to be upper bounded by \( \rho_t^2 \) and the variances are arbitrarily decided by the environment, which is not necessarily a function of the evaluation point. Our work basically considers the noise setting proposed by Zhou et al. (2021) and further generalizes their setting to bandits with general function classes.

Bandits with Known Function Classes. Moving beyond multi-armed bandits, there have been significant theoretical advances on stochastic bandits with function approximation. Among them, there is a huge body of literature on linear bandit problems where the reward function is assumed to be a linear function of the feature vectors attached to the actions (Dani et al., 2008; Abbasi-Yadkori et al., 2011; Chu et al., 2011; Li et al., 2019, 2021). Generalizing the restrictive linear rewards, there has also been a flurry of studies on generalized linear bandit problems (Filippi et al., 2010; Jun et al., 2017; Li et al., 2017; Kveton et al., 2020).
As for stochastic bandits with general function classes, the seminal work by Russo and Van Roy (2013) introduced the notion of eluder dimension to measure the complexity of the function class and provided a general UCB-like algorithm that works for any given class of reward functions with bounded eluder dimension. They further proved a regret upper bound of order $\tilde{O}(\sqrt{\dim_E \log N \cdot T})$ for their proposed algorithm where $\dim_E$ is the eluder dimension and $\log N$ stands for the log-covering number of the function class. Linear bandits and generalized linear bandit problems can be seen as special cases as their proposed general model.

**Online-to-confidence-set Conversion.** Abbasi-Yadkori et al. (2012) may be the first one to introduce the technique that takes in an online learning subroutine and turns the output of it into a confidence set at each round. While Abbasi-Yadkori et al. (2012) considers applying this technique in linear bandits, Jun et al. (2017) generalized and introduced the previous approach to Generalized Linear Online-to-confidence-set Conversion (GLOC) and applied it to generalized linear bandits.

**Online Regression For Linear Functions.** Online linear regression has long been studied in the setting where the response variables (or labels) are bounded and chosen by an adversary (Bartlett et al., 2015; Cesa-Bianchi et al., 1996; Kivinen and Warmuth, 1997; Littlestone et al., 1991; Malek and Bartlett, 2018), to mention a few. A recent work (Maillard et al., 2021) considers the stochastic setting where the response variables are unbounded and revealed by the environment with additional random noise on the true labels. Maillard et al. (2021) discusses the limitations of online learning algorithms in the adversarial setting and further advocates for the need of complementary analyses for existing algorithms under stochastic unbounded setting.

### 3 Multi-level Learning Framework

We will introduce our model, some basic concepts and our proposed framework in this section.

#### 3.1 Preliminaries

**General function class** We introduce the following notion, eluder dimension (Russo and Van Roy, 2013), to measure the complexity of the general function class $\mathcal{F}$.

**Definition 3.1 ($\epsilon$-dependence).** An action $a \in \mathcal{A}$ is $\epsilon$-dependent on actions $\{a_1, a_2, \ldots, a_n\} \in \mathcal{A}$ with respect to $\mathcal{F}$ if any pair of functions $f, \tilde{f} \in \mathcal{F}$ satisfying

$$\sqrt{\sum_{i=1}^{n} (f(a_i) - \tilde{f}(a_i))^2} \leq \epsilon$$

also satisfies $f(a) - \tilde{f}(a) \leq \epsilon$. Further, $a$ is $\epsilon$-independent of $\{a_1, \ldots, a_n\}$ with respect to $\mathcal{F}$ if $a$ is not $\epsilon$-dependent on $\{a_1, \ldots, a_n\}$.

**Definition 3.2 (eluder dimension, Russo and Van Roy 2013).** The $\epsilon$-eluder dimension $\dim_E(\mathcal{F}, \epsilon)$ is the length $d$ of the longest sequence of elements in $\mathcal{A}$ such that, for some $\epsilon' \geq \epsilon$, every element is $\epsilon'$-independent of its predecessors.

**Definition 3.3 (width).** Define a width of a subset $\tilde{\mathcal{F}} \subset \mathcal{F}$ at an action $a \in \mathcal{A}$ by

$$w_{\tilde{\mathcal{F}}}(a) = \sup_{f, \tilde{f} \in \tilde{\mathcal{F}}} (\tilde{f}(a) - f(a)).$$

(3.2)
To make the problem tractable when the action set is infinite or continuous, we require the following assumption that the reward function class is known in advance by the agent. Notice that when the action set is finite, $F$ can be set as the set including all the eligible functions so that multi-armed bandit is still captured by our model.

**Assumption 3.4 (Known Reward Function Class).** The unknown reward function $f^*$ belongs to a function class $F = \{ f_\theta : A \to \mathbb{R} | \theta \in \Theta \}$ which is available to the learning agent.

**Bandit models** We consider a heteroscedastic variant of the classic stochastic bandit problem with general function classes. At each round $t \in [T]$ ($T \in \mathbb{N}$), the agent observes a decision set $D_t \subseteq A$ which is chosen by the environment. The agent then select an action $a_t \in D_t$ and observes reward $r_t$ together with a corresponding variance upper bound $\sigma_t^2$. We assume that $r_t = f^*(a_t) + \varepsilon_t$ where $f^* : A \to \mathbb{R}$ is an underlying real-valued reward function which is unknown to the learner and $\varepsilon_t$ is a random noise. We make the following assumption on $\varepsilon_t$.

**Assumption 3.5.** For each $t$, $\varepsilon_t$ satisfies that $\varepsilon_t | a_{1:t}, \varepsilon_{1:t-1}$ is a $R$-sub-Gaussian random variable ($R > \sigma_t$) and

$$E[\varepsilon_t | a_{1:t}, \varepsilon_{1:t-1}] = 0, \ E[\varepsilon_t^2 | a_{1:t}, \varepsilon_{1:t-1}] \leq \sigma_t^2.$$  

For simplicity, let $J = \sum_{t=1}^{T} \sigma_t^2$. This assumption on $\varepsilon_t$ is a slightly generalized version of that in Zhou et al. (2021) in the sense that it is no longer bounded by $R$ but is assumed to be a $R$-sub-Gaussian random variable. The goal of the agent is to minimize the following cumulative regret:

$$\text{Regret}(T) := \sum_{t=1}^{T} [f^*(a_t^*) - f^*(a_t)], \quad (3.3)$$

where the optimal action $a_t^*$ at round $t \in [T]$ is defined as

$$a_t^* := \arg\max_{a \in D_t} f^*(a). \quad (3.4)$$

Denote $\Delta_t$ as the smallest gap between the reward of an optimal action and the reward of a sub-optimal action:

$$\Delta_t := \min_{a \in D_t, a \notin D_t^*} [f^*(a_t^*) - f^*(a)], \quad (3.5)$$

where $D_t^* := \arg\max_{a \in D_t} f^*(a)$. Let $\Delta$ be the smallest gap in all the rounds: $\Delta := \min_{t \in [T]} \Delta_t$.

### 3.2 The Proposed Multi-level Learning Framework

**Existing approach.** To tackle the heteroscedastic bandit problem, for the case where the $F$ is the linear function class (i.e., $f(a) = \langle \theta^*, a \rangle$ for some $\theta^* \in \mathbb{R}^d$), a weighted linear regression framework (Kirschner and Krause, 2018; Zhou et al., 2021) has been proposed. Generally speaking, at each round $t \in [T]$, weighted linear regression constructs a confidence set $C_t$ based on the empirical risk minimization (ERM) for all previous observed actions $a_s$ and rewards $r_s$ as follows:

$$\theta_t \leftarrow \arg\min_{\theta \in \mathbb{R}^d} \lambda \| \theta \|_2^2 + \sum_{s \in [t]} w_s(\langle \theta, a_s \rangle - r_s)^2,$$
Algorithm 1 ML\(^2\) with OFU principle

1: **Input:** \( T, \mathcal{A}, \mathcal{F}, R, \sigma > 0 \).

2: **Initialize:** Set

\[
L \leftarrow \lceil \log_2 \frac{R}{\sigma} \rceil \quad \text{and} \quad C_{1,l} \leftarrow \mathcal{F}, \quad \Psi_{1,l} \leftarrow \emptyset \quad \text{for all } l \in [L].
\]

3: **for** \( t = 1 \cdots T \) **do**

4: Observe \( D_t \).

5: Choose action

\[
a_t = \arg\max_{a \in D_t} \min_{l \in [L]} \max_{f \in C_{t,l}} f(a).
\]

6: Observe stochastic reward \( r_t \) and \( \sigma_t^2 \).

7: Find \( l_t \) such that

\[
2^{l_t + 1} \sigma \geq \max(\sigma, \sigma_t) \geq 2^{l_t} \sigma.
\]

8: Update \( \Psi_{t+1,l_t} \leftarrow \Psi_t \cup \{t\} \) and \( \Psi_{t+1,l} \leftarrow \Psi_t \) for all \( l \in [L] \setminus \{l_t\} \).

9: Update \( C_{t+1,l} \) according to \( \Psi_{t+1,l} \) through a regression subroutine (e.g., Algorithm 2).

10: **end for**

\[
C_t \leftarrow \left\{ \theta \in \mathbb{R}^d \, | \, \sum_{s=1}^t w_s (\langle \theta, a_s \rangle - (\theta_t, a_s))^2 \leq \beta_t \right\},
\]

where \( w_s \) is the weight, and \( \beta_t, \lambda \) are some parameters to be specified. \( w_s \) is selected in the order of the inverse of the variance \( \sigma_s^2 \) at round \( s \) to let the variance of the rescaled reward \( \sqrt{w_s r_s} \) upper bounded by 1. Therefore, after the weighting step, one can regard the heteroscedastic bandits problem as a homoscedastic bandits problem and apply existing theoretical results to it. To deal with the general function case, a direct attempt is to replace the \( \langle \theta, a \rangle \) appearing in above construction rules with \( f(a) \). However, such an approach requires that \( \mathcal{F} \) is close under the linear mapping, which does not hold for general function class \( \mathcal{F} \).

**Multi-level Learning framework (ML\(^2\)).** To deal with the nonlinearity issue, we propose a novel framework \( \text{ML}\(^2\)\) in Algorithm 1. At the core of our design is the idea of partitioning the observed data into several levels and ‘packing’ data with similar variance upper bounds into the same level as shown in line 7-8 of Algorithm 1. Note that we use a small real number \( \sigma \) to ensure that the number of levels is bounded. Specifically, for any two data belong to the same level with variances larger than \( \sigma \), their variance will be at most twice larger than the other. Next in line 9, our framework calls a subroutine to estimate \( f^* \) according to the data points in \( \Psi_{t+1,l} \). Since the variances of the data in the same level are nearly the same, we can let algorithms that work for homoscedastic bandit problem run on the data in the same level without any problems. Particularly, we use the Empirical risk minimization (ERM) algorithm as described in Algorithm 2 for Sections 4 and 5. In Section 6, we show the power of using Algorithm 4 as the regression subroutine. Then in line 5, the agent makes use of \( L \) confidence sets simultaneously to select an action based on the **optimism-in-the-face-of-uncertainty** (OFU) principle over all \( L \) levels. In the following sections, we will consider several different settings to show the power of \( \text{ML}\(^2\)\).

4 **Warmup: Noise with Additional Sub-Gaussian Assumption**

In this section, we first consider a simplified variant of our problem described in the last section.

**Assumption 4.1** (Sub-Gaussianity of Noise). \( \epsilon_t \) is conditionally \( \sigma_t \)-sub-Gaussian on \( a_{1:t}, \epsilon_{1:t-1} \).
Algorithm 2: Empirical risk minimization (ERM) for partitioned data

1: **Input:** Level \( l \), time \( t \) and set of data points \( \Psi_{t+1, l} \).
2: Compute \( \hat{f}_{t+1, l} \leftarrow \min_{f \in \mathcal{F}} \sum_{s \in \Psi_{t+1, l}} (f(a_s) - r_s)^2 \).
3: Return \( C_{t+1, l} \leftarrow \left\{ f \in \mathcal{F} \mid \sum_{s \in \Psi_{t+1, l}} (f(a_s) - \hat{f}_{t+1, l}(a_s))^2 \leq \beta_{t+1, l} \right\} \).

Such a sub-Gaussian assumption on noise has been considered by Kirschner and Krause (2018).

Next we show the regret upper bound for ML\(^2\) with ERM. For simplicity, in the following results, let \( \dim_E \) denote \( \dim_E(\mathcal{F}, 1/T^2) \).

**Theorem 4.2** (Gap-independent regret bound for bandits with heteroscedastic sub-Gaussian noise). Suppose Assumption 3.4 and 4.1 hold and \( |f^*(a)| \leq C \) for all \( a \in \mathcal{A} \). For all \( t \in [T], l \in [L] \) and \( \delta \in (0, 1), \alpha > 0, \sigma > 0 \), if we apply Algorithm 2 as a subroutine of Algorithm 1 (in line 9) and set \( \beta_{t, l} \) as the square root of

\[
8(2l^2 + 1 \cdot \sigma)^2 \log(2N_{\alpha}L/\delta) + 4\alpha \left( C + \sqrt{(2l^2 + 1 \cdot \sigma)^2 \log(4t(t + 1)L/\delta)} \right),
\]

where \( N_{\alpha} = \mathcal{N}(\mathcal{F}, \alpha, \| \cdot \|_\infty) \) and \( L = \lceil \log_2 R/\sigma \rceil \) (recall the definition of \( L \) in Algorithm 1), then with probability at least \( 1 - \delta \), the regret for the first \( T \) rounds is bounded as follows:

\[
\text{Regret}(T) \leq L + 2C \dim_E L + 8\sqrt{2L \dim_E (J + \sigma^2 T) \log(2N_{\alpha}L/\delta)} + 4\sqrt{L \dim_E \alpha \sqrt{C + 2R \sqrt{\log(4T(T + 1)L/\delta)}T}}.
\]

**Corollary 4.3.** Let the same conditions as in Theorem 4.2 hold. Set

\[
\alpha = T^{-2}, \sigma = \dim_E^{-1}(\log(2N_{\alpha}L/\delta)\sqrt{T})^{-1}.
\]

Then with probability at least \( 1 - \delta \), when \( T \) is large enough, the regret for the first \( T \) rounds is bounded as follows:

\[
\text{Regret}(T) = \tilde{O} \left( \sqrt{\dim_E \log(\mathcal{N}(\mathcal{F}, T^{-2}, \| \cdot \|_\infty)) J} \right).
\]

**Remark 4.4.** Our result strictly improves the regret bound

\[
\tilde{O} \left( R \sqrt{\dim_E \log(\mathcal{N}(\mathcal{F}, T^{-2}, \| \cdot \|_\infty)) T} \right)
\]

achieved by Russo and Van Roy (2013) since \( J = \sum_{t=1}^{T} \sigma_t^2 \leq R^2 T \). In the worst case, when \( \sigma_1 = \cdots = \sigma_T = R \), our result degrades to their result.

**Remark 4.5.** When restricted to linear contextual bandits with dimension \( d \), since \( \log \mathcal{N}(\mathcal{F}, T^{-2}, \| \cdot \|_\infty) = \tilde{O}(d), \dim_E = \tilde{O}(d) \) (Russo and Van Roy, 2013), our result can be written as \( \tilde{O}(d \sqrt{T}) \), which matches the result of using weighted linear ridge regression for heteroscedastic linear bandit under our assumptions on noise (Kirschner and Krause, 2018; Zhou et al., 2021).
Next we also provide a gap-dependent regret bound for general function class setting, generalizing the previous gap-dependent regret bound in linear bandits derived by Abbasi-Yadkori et al. (2011).

**Theorem 4.6 (Gap-dependent regret bound for bandits with heteroscedastic sub-Gaussian noise).** Suppose Assumption 3.4 and 4.1 hold and $|f^*(a)| \leq C$ for all $a \in A$. Let $\sigma_{\max} = \max_{t \in [T]} \sigma_t$ and suppose $\sigma_{\max} > \sigma$. If we apply Algorithm 2 as a subroutine of Algorithm 1 (in line 9) and set $\beta_{t,l}$ as the same value in Theorem 4.2, then with probability at least $1 - \delta$, the regret of Algorithm 1 for the first $T$ rounds is bounded as follows:

$$\text{Regret}(T) \leq \frac{L}{\Delta} \left( 4 \dim_E C^2 + 1/T \right) + 16 \frac{L T \alpha C}{\Delta} \dim_E (\log T + 1) + 128 \frac{L}{\Delta} \sigma_{\max}^2 \log(2N \alpha L/\delta) \dim_E (\log T + 1).$$

**Corollary 4.7.** Let the same conditions as in Theorem 4.6 hold. Set $\alpha = T^{-2}$. Then with probability at least $1 - \delta$, when $T$ is large enough, the regret for the first $T$ rounds is bounded as follows:

$$\text{Regret}(T) = \tilde{O} \left( \frac{\sigma_{\max}^2}{\Delta} \dim_E \log (N(F, T^{-2}, \| \cdot \|_\infty)) \right).$$

**Remark 4.8.** Corollary 4.7 immediately suggests an $\tilde{O}(R^2 \dim_E \log (N(F, T^{-2}, \| \cdot \|_\infty))/\Delta) \text{ gap-dependent regret}$ by the fact $\sigma_{\max} = O(R)$, which provides a novel instance-dependent bound for the original problem considered by Russo and Van Roy (2013). To our knowledge, this is the first result of its kind for the general bandit model.

**Remark 4.9.** When restricted to linear contextual bandits with dimension $d$, our result reduces to $\tilde{O}(\sigma_{\max}^2 d^2/\Delta)$, which matches the previous result derived in Abbasi-Yadkori et al. (2011).

### 5 General Results for Bandits with Heteroscedastic Noise

In this section, we consider the original setting introduced in Section 3 without Assumption 4.1. In the following subsections, we will show that Algorithm 2 still works with refined value of $\beta$.

#### 5.1 Variance-aware Confidence Set

In this general setting, directly applying the confidence set used in the previous work (Russo and Van Roy, 2013) gives no improvement since we can only upper bounded the variance proxies of the noise with $R^2$ now and $\beta_{t,l}$ becomes $\tilde{O}(R^2 \log N(F, \alpha, \| \cdot \|_\infty))$. As a result, it is necessary to design new confidence set which makes use of the variance information.

We show in the following theorem how to set appropriate value of $\beta$, to ensure that the confidence set is large enough to contain $f^*$ with high probability, and exploit the variance information at the same time.

**Theorem 5.1 (Variance-dependent confidence sets).** Suppose that $|f^*(a)| \leq C$ for all $a \in A$. For any $\alpha > 0$ and $\delta \in (0, 1/2)$, if we set $\beta_{t,l}$ as the square root of

$$12C \alpha t + 4 \alpha R t + \frac{8}{3} C R \log(2R t^2/\delta) + 16 \cdot (2^{l+1} \sigma)^2 \log^2(2R t^2/\delta),$$

where $R = R \sqrt{2 \log(4 t^2/\delta)}$, $N = N(F, \alpha, \| \cdot \|_\infty)$, then $f^* \in C_{t,l}$ with probability at least $1 - 2\delta$ for any fixed $t, l$. 

8
Remark 5.2. When we set $\alpha$ to a small enough value, the order of $\beta_{t,l}$ is $\tilde{O}(2^{2t}\sigma^2 \log N_{\alpha} + CR \log N_{\alpha})$. Compared with the corresponding previous result $\tilde{O}(R^2 \log N_{\alpha})$ (Russo and Van Roy, 2013; Ayoub et al., 2020), our confidence set is tighter when $C$ is relatively small compared to $R$.

5.2 Regret Upper Bounds for ML² with ERM

Following the similar proof techniques used in Section 4, we can derive our general results with the variance-aware confidences sets described in the last subsection. In this part, we write $\dim_E(F, T^{-1})$ as $\dim_E$ for short.

Theorem 5.3 (Gap-independent regret bound for bandits with heteroscedastic noise). Suppose Assumption 3.4 holds and $|f^*(a)| \leq 1$ for all $a \in A$. For all $t \in [T], l \in [L]$ and $\delta \in (0, 1), \alpha > 0, \sigma > 0$, if we apply Algorithm 2 as a subroutine of Algorithm 1 (in line 9) and set $\beta_{t,l}$ as the square root of

$$12\alpha t + 4aR\tilde{t} + \frac{8}{3}R \log(2N_{\alpha}L/\delta) + 16 \cdot (2^{t+1}\sigma)^2 \log(2N_{\alpha}L/\delta)$$

where $L = \lceil \log_2 R/\sigma \rceil, N_{\alpha} = N(F, \alpha, \| \cdot \|_{\infty})$ and $\tilde{R} = R \sqrt{2 \log(4L^2/\delta)}$ (with a slight abuse of notation), then with probability at least $1 - 2\delta$, the regret for the first $T$ rounds is bounded as follows:

$$\text{Regret}(T) \leq \sqrt{L} \left( 2\sqrt{\dim_E T + 1} + 4/\sqrt{\dim_E \log T + 1} \alpha \sqrt{3 + 4R\tilde{t}} \right)$$

$$+ 2\sqrt{\frac{8}{3}L \dim_E \log T + 1} \tilde{R} \log(2N_{\alpha}L/\delta) + 16 \sqrt{L \dim_E \log T + 1} \log(2N_{\alpha}L^2/\delta) \sqrt{J + T\sigma^2}.$$

Corollary 5.4. Suppose $R = \Omega(1)$. Let the same conditions as in Theorem 5.3 hold. Set $\alpha = T^{-2}, \sigma = 1$. Then with probability at least $1 - \delta$, when $T$ is large enough, the regret for the first $T$ rounds is bounded as follows:

$$\text{Regret}(T) = \tilde{O} \left( \sqrt{\dim_E \log N_{\alpha}J} + \sqrt{\dim_E \log N_{\alpha}RT} \right).$$

Remark 5.5. Compared with the result shown in Corollary 4.3, the additional term of order $\tilde{O}(\sqrt{\dim_E \log N_{\alpha}RT})$ is incurred by a larger confidence set due to the absence of Assumption 4.1.

Remark 5.6. When restricted to heteroscedastic linear contextual bandits of dimension $d$, our regret bound can be written as $\tilde{O}(d\sqrt{J} + \sqrt{R}d\sqrt{T})$. With a slightly more restricted assumption on noise, Zhou et al. (2021) achieves a result of order $\tilde{O}(d\sqrt{J} + Rd\sqrt{T})$. Our result is appealing when the sub-Gaussian parameter of noise $R$ is much larger than 1 (or the range of the reward function, equivalently). When $R$ is small, our result becomes sub-optimal due to the property of our variance-aware confidence set.

We also provide a gap-dependent regret bound in the following theorem.

Theorem 5.7 (Gap-dependent regret bound for bandits with heteroscedastic noise, informal). Suppose Assumption 3.4 holds and $|f^*(a)| \leq 1$ for all $a \in A$. Let $\sigma_{\max} = \max_{t \in [T]} \sigma_t$ (suppose $\sigma_{\max} > \sigma$) and $d = \dim_E(F, 1/T)$. If we apply Algorithm 2 as a subroutine of Algorithm 1 (in line 9) and set $\beta_{t,l}$ as the same value in Theorem 5.3, then with probability at least $1 - 2\delta$, the regret of Algorithm 1 for the first $T$ rounds is bounded as follows:

$$\text{Regret}(T) = \tilde{O} \left( \frac{\sigma_{\max}^2}{\Delta} d \log N_{\alpha} + \frac{R}{\Delta} d \log N_{\alpha} \right)$$

if $T$ is large enough and we set $\alpha = T^{-2}$. 

Remark 5.8. Similar to Remark 5.5, compared with the regret in Corollary 4.7, the regret in Theorem 5.7 has an additional term that depends on $R$.

6 Tighter Bounds for Generalized Linear Bandits

Our general result shown in Theorem 5.3 has an additional term of order $\tilde{O}(\sqrt{\dim \Theta} \log N \alpha R T)$ which makes the result sub-optimal when $R$ is close to the range of the reward function, as discussed in the above section.

In this section, we consider a special case, generalized linear bandits with heteroscedastic noise and show how to get rid of the $\tilde{O}(\sqrt{\dim \Theta} \log N \alpha R T)$ term in regret upper bound and achieve better result when value of $R$ is relatively small or close to the bound of the reward function.

6.1 Generalized Linear Bandits

Following the definition of Filippi et al. (2010) and Jun et al. (2017), we consider the following generalized linear function class for reward functions.

Assumption 6.1 (Generalized linear function class). Action set $\mathcal{A}$ and $\Theta$ in Assumption 3.4 are subsets of $\mathbb{R}^d$. There exists a known link function $h$, such that

$$\forall a \in \mathcal{A} \text{ and } f_\theta \in \mathcal{F}, f_\theta(a) = h(\theta^\top a).$$

Let $f^* = f_{\theta^*}$. Assume that

$$\|\theta^*\|_2 \leq B, \sup_{a \in \mathcal{A}} \|a\|_2 \leq A.$$

To make the problem tractable, we need the following assumption on $h$.

Assumption 6.2 (Assumption 1, Jun et al. 2017). $h$ is $K$-Lipschitz on $[-A \cdot B, A \cdot B]$ and continuously differentiable on $(-A \cdot B, A \cdot B)$. Furthermore, $\inf_{z \in (-A \cdot B, A \cdot B)} h'(z) = \kappa$ for some $\kappa > 0$.

Next we propose the follow-the-regularized-leader (FTRL) framework (Shalev-Shwartz and Singer, 2007; Xiao, 2010; Hazan, 2019) in Algorithm 3, which is the key component of our final algorithm. Note that when dealing with bandit setting, we do not feed all the data points into a FTRL online learner. Here we number the data points with $t = 1, 2, \cdots$ for simplicity with a slight abuse of notation. In Algorithm 3 we will use a loss function $\ell$ and a regularized function $\phi$. To align with our Assumption 6.1, we select the loss function as follows, following Jun et al. (2017):

Assumption 6.3 (Loss function, Jun et al. 2017). The loss function $\ell$ in Algorithm 3 is selected as

$$\ell(z, r) = -rz + m(z), \ell_t(\theta) = \ell(\theta^\top a_t, r_t)$$

where $m(z)$ satisfies $m'(z) = h(z)$.
Algorithm 3 Follow The Regularized Leader (FTRL)

1: Input: \( \mathcal{F}, R \).
2: for \( t \geq 1 \) do
3: Observe \( a_t \).
4: Set \( \theta_t \leftarrow \arg\min_{\theta \in \mathbb{R}^d} \phi(\theta) + \sum_{s=1}^{t-1} \ell(\theta^T a_s, r_s) \).
5: Output the prediction \( \theta_t \).
6: Observe \( r_t \).
7: end for

6.2 Variance-dependent Regret for FTRL

Before proposing our final algorithm for generalized linear bandits, we first propose a variance-dependent complexity result for FTRL, since it is already nontrivial and reveals some interesting properties about our setting. We define a notion of regret of online regression, named by \( \text{reg}_t \), as follows. The concept of regret of online regression has been introduced in the previous work (Abbasi-Yadkori et al., 2012; Jun et al., 2017). In detail, it is used to characterize the complexity for FTRL to learn the generalized linear function.

Definition 6.4. The regret of online regression as follows:

\[
\text{reg}_t = \sum_{s=1}^{t} \ell(a_s^T \theta_s, r_s) - \sum_{s=1}^{t} \ell(a_s^T \theta^*, r_s).
\]

Our definition of regret of online regression is slightly different with the typical definition. The online regression regret in the previous work (Abbasi-Yadkori et al., 2012; Jun et al., 2017) is defined as

\[
\text{reg}_t' = \sup_{\theta \in \Theta} \sum_{s=1}^{t} \ell_s(\theta_s) - \ell_s(\theta),
\]

which follows the typical definition in online learning theory. By setting \( \theta = \theta^* \), any bound for \( \text{reg}_t' \) is also a valid bound for \( \text{reg}_t \). From the perspective of online learning, the algorithms and the corresponding analyses are usually introduced for either realizable setting where there exists an underlying \( \theta^* \) that incurs zero loss or adversarial setting where the bounded label \( r_s \) in each round \( s \) can be arbitrarily chosen by the adversary. As a result, the previous approaches by Abbasi-Yadkori et al. (2012); Jun et al. (2017) do not exploit the ‘stochastic’ property of the labels. Provided that the labels are sequentially generated with additional stochastic noise, our definition is more reasonable and natural. A recent work focusing on stochastic online linear regression also discussed the limitation of adversarial setting (Section 2.2, Maillard et al. 2021).

We propose a bound for \( \text{reg}_t \) here, which is one of the key results in this paper. Addressing the middle ground between adversarial setting and realizable setting, our novel result for stochastic setting shows highly non-trivial improvement compared with the result considered in adversarial setting.

Theorem 6.5 (Regret of FTRL). If we set

\[
\phi(\theta) = \frac{2A^2 K^2}{\kappa} \|\theta\|_2^2,
\]
and assume that all the data points fed into the algorithm are of noise variance bounded by $\sigma^2_{\text{max}}$, then with probability at least $1 - 3\delta$, $\forall t \geq 1$, the regret of Algorithm 3 for the first $t$ rounds is bounded as follows:

$$\text{reg}_t \leq \frac{8A^2K^2B^2}{\kappa} + \frac{9}{2\kappa}R^2 \log^2(4t^2/\delta) + \frac{3\sigma^2_{\text{max}}}{\kappa} \lambda d \log \left(1 + \frac{tA\kappa^2}{4dK^2}\right).$$

**Remark 6.6.** Jun et al. (2017) analyzed the online learning regret for the same function class and loss function with our setting. Their result yields a $\text{reg}_t$ in the order of $\tilde{O}(K^2A^2B^2R^2d)$. Our result improves their result in two aspects:

- $R$ is strictly larger than $\sigma_{\text{max}}$ since a $R$-sub-Gaussian random variable is definitely of variance lower than $R^2$.
- When we consider cases where the bound of reward functions (i.e., $KAB$) is extremely large compared to $R$, their result becomes $\tilde{O}(K^2A^2B^2d/\kappa)$, which has an additional linear dependence on $d$.

**Remark 6.7.** Consider a special case where $\kappa = K = 1$. Our result degrades to $\tilde{O}(A^2B^2 + R + \sigma^2_{\text{max}}d)$. This is essentially a regret upper bound for stochastic online linear regression with square loss. Recent work by Maillard et al. (2021) studied this stochastic setting and managed to get rid of the $\tilde{O}(A^2B^2d)$ term in classic result for online linear regression considering adversarial setting. Maillard et al. (2021) derived a high probability regret bound of $\tilde{O}(R^2d^2)$ after omitting the $o(\log(T)^2)$ terms (Theorem 3.3, Maillard et al. 2021). Unlike their result, our result does not suffer from the quadratic dependence on $d$ and $\sigma_{\text{max}}$ is only the upper bound on the standard variance. As for the $\tilde{O}(R + A^2B^2)$ term in our result, it is not hard to see that this part of loss is inevitable, since at the first round, the algorithm has no prior knowledge of $\theta^*$.

### 6.3 Regret Bound of Algorithm 1 with GLOC

With our new technical tool presented in the last subsection, we are now at the point to show our final algorithm for the generalized linear bandit setting. We propose our algorithm in Algorithm 4. Generally speaking, Algorithm 4 is a multi-level version of the generalized linear online-to-confidence-set conversion (GLOC) algorithm proposed by Jun et al. (2017), equipped with FTRL.

As shown in Algorithm 4, we maintain $L$ FTRL online learners in parallel. Under our framework, a single learner only receives data with similar variances of noise. As a result, we can make use of the variance-dependent result shown in Theorem 6.5 to derive a tighter regret bound for generalized linear bandits with heteroscedastic noise.

**Theorem 6.8** (Regret bound for generalized linear bandits, informal). Suppose that Assumption 6.1 and 6.2 hold for the known reward function class $\mathcal{F}$. If we apply Algorithm 4 as a subroutine of Algorithm 1 (in line 9) and set $\beta_{t,l}$ to

$$1 + \frac{32A^2K^2B^2}{\kappa^2} + \frac{26}{\kappa^2}R^2 \log^2(4t^2L/\delta) + 12\frac{2^{2(t+1)}\sigma^2}{\kappa^2} \lambda d \log \left(1 + \frac{tA\kappa^2}{4dK^2}\right) + \lambda B^2$$


Algorithm 4 GLOC with multi-level FTRL learners

1: Initialize: $\overline{V}_{0,l} \leftarrow \lambda I$ for all $l \in [L]$. \\
2: while input $t, l_t, a_t, r_t$ do \\
3: Set $\theta_{t,l_t}$ as Algorithm 3 suggests \\
4: $\theta_{t,l_t} \leftarrow \argmin_{\theta \in \mathbb{R}^d} \phi(\theta) + \sum_{s \in \Psi_{t+1,l_t}} \ell(\theta^\top a_s, r_s)$. \\
5: Find $t' = \max\{\Psi_{t,l_t}\}$. \\
6: Update $\nabla_{t,l_t} \leftarrow \nabla_{t',l_t} + a_t a_t^\top$. \\
7: Update $z_{t,l_t} \leftarrow a_t^\top \theta_{t,l_t}$. \\
8: Compute $\hat{\theta}_{t,l_t} \leftarrow \nabla_{t,l_t}^{-1} \left(\sum_{s \in \Psi_{t+1,l_t}} z_{s,l_t} \cdot a_s\right)$. \\
9: Define $C_{t,l_t} \leftarrow \{\theta \in \mathbb{R}^d : \|\theta - \hat{\theta}_{t,l_t}\|_2 \leq \beta_{t,l_t}\}$. \\
10: Define $C_{t,l_t} \leftarrow C_{t-1,l_t}$ for all $l \in [L] \setminus \{l_t\}$. \\
11: Return $\mathcal{C}_{t,l_t} \leftarrow \{f_\theta \in \mathcal{F} | \theta \in C_{t,l_t}\}$ for all $l \in [L]$. \\
12: end while

for all $t \in [T], l \in [L]$, where $L = \lceil \log_2 R/\sigma \rceil, \sigma = R/\sqrt{d}$, then with probability $1 - 4\delta$, the regret of Algorithm 1 for the first $T$ rounds is bounded as follows:

$$\text{Regret}(T) = \tilde{O}\left(\frac{K}{\kappa} d\sqrt{J} + \frac{K}{\kappa} (K \cdot AB + R) \sqrt{dT}\right).$$

**Remark 6.9.** In the worst case, i.e. $\sigma_1 = \cdots = \sigma_T = R$, our result degraded to $\tilde{O}(KRD\sqrt{T}/\kappa + K^2 AB\sqrt{dT}/\kappa)$, which still slightly improve the $\tilde{O}\left(K(KAB + R)d\sqrt{T}/\kappa\right)$ result provided by Jun et al. (2017).

**Remark 6.10.** If we apply the regret bound in Corollary 5.4 in generalized linear bandits, we automatically obtain a regret bound of $\tilde{O}\left(K(d\sqrt{J} + d\sqrt{RT})/\kappa\right)$ for the case where $K \cdot A \cdot B = 1$ and $R = \Omega(1)$. Our bound here improved the general result when $R = o(d)$. 

**Remark 6.11.** When restricted to heteroscedastic linear bandits by setting $\kappa = K = 1$, our result becomes $\tilde{O}\left((R + A \cdot B)d\sqrt{dT} + d\sqrt{J}\right)$, which only has an additional term of order $\tilde{O}(A \cdot B\sqrt{dT})$ compared with the result derived by Zhou et al. (2021).

7 Conclusion and Future Work

In this work we study heteroscedastic stochastic bandits problem for a general reward function class. We propose a multi-level regression framework $\text{ML}^2$ to deal with the heteroscedastic noises. Under three different settings with additional assumptions on the noise and the function class, we study the performance of $\text{ML}^2$ and propose corresponding variance-dependent regret bounds, which strictly improves previous algorithms for homoscedastic bandit setting. We leave to study the optimal regret bound of heteroscedastic stochastic bandits for a general reward function class for future work.
A Proofs from Section 4

Lemma A.1 (Proposition 3, Russo and Van Roy 2013). If \((\beta \geq 0|t \in \mathbb{N})\) is a nondecreasing sequence and \(\mathcal{F}_t := \left\{ f \in \mathcal{F} : \sum_{s=1}^{t-1} \left( \hat{f}_t(a_s) - f(a_s) \right)^2 \leq \beta^2_t \right\}\), then

\[
\sum_{t=1}^{T} 1(w_{\mathcal{F}_t}(D_t) > \epsilon) \leq \left(\frac{4\beta^2_T}{\epsilon^2} + 1\right) \dim_E(\mathcal{F}, \epsilon)
\]

for all \(T \in \mathbb{N}\) and \(\epsilon > 0\).

Lemma A.2 (Lemma 2, Russo and Van Roy 2013). If \((\beta \geq 0|t \in \mathbb{N})\) is a nondecreasing sequence and \(\mathcal{F}_t := \left\{ f \in \mathcal{F} : \sum_{s=1}^{t-1} \left( \hat{f}_t(a_s) - f(a_s) \right)^2 \leq \beta^2_t \right\}\), then

\[
\sum_{t=1}^{T} w_{\mathcal{F}_t}(D_t) \leq \frac{1}{T} + w_{\mathcal{F}}(A) \cdot \dim_E(\mathcal{F}, T^{-2}) + 4\beta^2_T \sqrt{\dim_E(\mathcal{F}, T^{-2})T}
\]

for all \(T \in \mathbb{N}\).

Instead of using the previous approach that bounds the sum of widths, we take another approach bounding the sum of squares, which can further provide a novel gap-dependent result later.

Lemma A.3 (Bounding the sum of the square of widths). If \((\beta \geq 0|t \in \mathbb{N})\) is a nondecreasing sequence and \(\mathcal{F}_t := \left\{ f \in \mathcal{F} : \sum_{s=1}^{t-1} \left( \hat{f}_t(a_s) - f(a_s) \right)^2 \leq \beta^2_t \right\}\), then

\[
\sum_{t=1}^{T} w^2_{\mathcal{F}_t}(D_t) \leq \dim_E(\mathcal{F}, 1/\sqrt{T})w^2_{\mathcal{F}}(A) + 1 + 4\beta^2_T \dim_E(\mathcal{F}, 1/\sqrt{T})(\log T + 1)
\]

for all \(T \in \mathbb{N}\).

Proof. Following a similar approach as Russo and Van Roy (2013), we reorder the set \(\{w_{\mathcal{F}_t}(D_t)\}_{t \in [T]}\) to \(\{w_t\}_{t \in [T]}\), such that \(w_1 \geq w_2 \geq \cdots \geq w_T\).

Let \(T' = \max\{t \in [T], w_t \geq \frac{1}{T}\}\).

From Lemma A.1,

\[
t \leq \left(\frac{4\beta^2_T}{(w_t - \delta)^2} + 1\right) \dim_E(\mathcal{F}, w_t - \delta)
\]

for any \(\delta \in (0, \epsilon)\). Taking \(\delta \to 0\), we have

\[
w^2_t \leq \frac{4\beta^2_T \dim_E(\mathcal{F}, w_t)}{t - \dim_E(\mathcal{F}, w_t)}.
\]

Hence,

\[
\sum_{t=1}^{T} w^2_{\mathcal{F}_t}(D_t) = \sum_{t=1}^{T} w^2_t
\]
\[
\leq \dim_E(\mathcal{F}, 1/T)w_T^2(\mathcal{A}) + 1/T + \sum_{t = \dim_E(\mathcal{F}, 1/T) + 1}^{T'} w_t^2
\]
\[
\leq \dim_E(\mathcal{F}, 1/T)w_T^2(\mathcal{A}) + 1/T + \sum_{t = \dim_E(\mathcal{F}, 1/T) + 1}^{T'} \frac{4\beta_t^2 \dim_E(\mathcal{F}, 1/T)}{t - \dim_E(\mathcal{F}, 1/T)}
\]
\[
\leq \dim_E(\mathcal{F}, 1/T)w_T^2(\mathcal{A}) + 1/T + 4\beta_T^2 \dim_E(\mathcal{F}, 1/T)(\log T + 1),
\]

where the first inequality holds due to \(\sum_{t = T' + 1}^{T} w_t^2 \leq 1/T\) under our definition of \(T'\), the second inequality follows from (A.2), the third inequality is derived by taking the integral. \(\square\)

With Assumption 4.1, we can directly apply the previous result on confidence set by replacing the sub-Gaussianity \(\eta\) parameter by \(2^{l+1}\sigma\). Previous result by Russo and Van Roy (2013) achieved a confidence set of radius \(\tilde{O}(\sqrt{\eta^2 \log N_\alpha + \alpha t(C + \eta)})\). Ayoub et al. (2020) later provides a result of the same order with improvement in terms of smaller constants.

**Lemma A.4** (Theorem 5, Ayoub et al. 2020). Suppose that \(|f^*(a)| \leq C\) for all \(a \in \mathcal{A}\). For any \(\alpha > 0\), if we set
\[
\beta_{t,l} = \left[8(2^{l+1} \cdot \sigma)^2 \log(2N_\alpha L/\delta) + 4t\alpha(C + \sqrt{(2^{l+1} \cdot \sigma)^2 \log(4t(t+1)L/\delta)})\right]^{1/2},
\]
then with probability at least \(1 - \delta\), for all \(t \geq 1, l \in [L], f^* \in \mathcal{C}_{t,l}\).

**Theorem A.5** (Restatement of Theorem 4.2). Suppose Assumption 3.4 and 4.1 hold and \(|f^*(a)| \leq C\) for all \(a \in \mathcal{A}\). For all \(t \in [T], l \in [L]\) and \(\delta \in (0, 1), \alpha > 0, \sigma > 0\), if we apply Algorithm 2 as a subroutine of Algorithm 1 (in line 9) and set \(\beta_{t,l}\) as the square root of
\[
8(2^{l+1} \cdot \sigma)^2 \log(2N_\alpha L/\delta) + 4t\alpha(C + \sqrt{(2^{l+1} \cdot \sigma)^2 \log(4t(t+1)L/\delta)}),
\]
where \(N_\alpha = \mathcal{N}(\mathcal{F}, \alpha, \| \cdot \|_{\infty})\) and \(L = \lceil \log_2 R/\sigma \rceil\) (recall the definition of \(L\) in Algorithm 1), then with probability at least \(1 - \delta\), the regret for the first \(T\) rounds is bounded as follows:
\[
\text{Regret}(T) \leq L + 2C \dim_E L + 8\sqrt{2L \dim_E J + \sigma^2 T \log(2N_\alpha L/\delta)}
\]
\[+ 4\sqrt{L \dim_E \alpha \sqrt{C + 2R \sqrt{\log(4T(T+1)L/\delta)}} T}.\]

**Proof.** For simplicity, let \(d = \dim_E(\mathcal{F}, 1/T^2)\).

With probability at least \(1 - \delta\), we have
\[
\text{Regret}(T) = \sum_{t=1}^{T} (f^*(a_t^*) - f^*(a_t))
\]
\[
= \sum_{l \in [L]} \sum_{t \in \Psi_{T+1,l}} (f^*(a_t^*) - f^*(a_t))
\]
\[
\leq \sum_{l \in [L]} \sum_{t \in \Psi_{T+1,l}} \left(\max_{f \in \mathcal{C}_{t,l}} f(a_t) - f^*(a_t)\right)
\]
\[= \sum_{l \in [L]} \sum_{t \in \Psi_{T+1,l}} \left(\max_{f \in \mathcal{C}_{t,l}} f(a_t) - f^*(a_t)\right).
\]

15
\[
\sum_{l \in [L]} \left( 1 + 2C \cdot d + 4\beta_{T,l} \sqrt{d|\Psi_{T+1,l}|} \right) \\
\leq L + 2CdL + 2 \sqrt{L \sum_{l \in [L]} \beta_{T,l}^2 d|\Psi_{T+1,l}|},
\]  
(A.6)

where the first equality holds by the definition in (3.3), the second equality holds since \(\Psi_{T+1,l}\) forms a partition of \([T]\), the first inequality holds due to Lemma A.4, the second inequality follows from Lemma A.3, the third inequality is obtained by applying Cauchy-Schwarz inequality.

Then we continue to bound \(I_0\),

\[
I_0 = \sqrt{L \sum_{l \in [L]} \sum_{t \in \Psi_{T+1,l}} \beta_{T,l}^2 d} \\
\leq \sqrt{Ld} \sum_{l \in [L]} \sum_{t \in \Psi_{T+1,l}} 8(2^{t+1} \sigma)^2 \log(2N_\alpha L/\delta) \\
+ \sqrt{Ld} \sum_{l \in [L]} \sum_{t \in \Psi_{T+1,l}} 4T \alpha (C + \sqrt{(4R^2 \log(4T(T+1)L/\delta))}) \\
\leq \sqrt{Ld} \left( \sum_{l \in [L]} \sum_{t \in \Psi_{T+1,l}} 8(2^{t+1} \sigma)^2 \log(2N_\alpha L/\delta) + 2\sqrt{\alpha T} \sqrt{C + 2R \sqrt{\log(4T(T+1)L/\delta)}} \right) \\
\leq \sqrt{Ld} \left( \sum_{t=1}^{T} 32(\sigma_t^2 + \bar{\sigma}^2) \log(2N_\alpha L/\delta) + 2\sqrt{\alpha T} \sqrt{C + 2R \sqrt{\log(4T(T+1)L/\delta)}} \right) \\
\leq \sqrt{32Ld(J + \bar{\sigma}^2T) \log(2N_\alpha L/\delta) + 2\sqrt{Ld\alpha} \sqrt{C + 2R \sqrt{\log(4T(T+1)L/\delta)}}}, \tag{A.7}
\]

where the first inequality follows from the definition of \(\beta_{T,l}\), the third inequality holds due to the fact that

\[
\forall l \in [L], t \in \Psi_{T+1,l}, \quad 2^{t+1} \sigma = 2 \cdot 2^t \bar{\sigma} \leq 2 \max\{\sigma, \sigma_t\} = \sqrt{4 \max\{\bar{\sigma}^2, \sigma_t^2\}} \leq \sqrt{4(\bar{\sigma}^2 + \sigma_t^2)},
\]

the fourth inequality follows from the definition of \(J\).

Substituting (A.7) into (A.6), we obtain

\[
\text{Regret}(T) \leq L + 2CdL + 8\sqrt{2Ld(J + \bar{\sigma}^2T) \log(2N_\alpha L/\delta)} \\
+ 4\sqrt{Ld\alpha} \sqrt{C + 2R \sqrt{\log(4T(T+1)L/\delta)}}T,
\]

which completes the proof. \(\Box\)

**Theorem A.6** (Restatement of Theorem 4.6). Suppose Assumption 3.4 and 4.1 hold and \(|f^*(a)| \leq C\) for all \(a \in A\). Let \(\sigma_{\text{max}} = \max_{t \in [T]} \sigma_t\). If we apply Algorithm 2 as a subroutine of Algorithm 1 (in line 9) and set \(\beta_{T,l}\) as the same value in Theorem 4.2, then with probability at least 1 \(-\delta\), the regret of Algorithm 1 for the first \(T\) rounds is bounded as follows:

\[
\text{Regret}(T) \leq \frac{L}{\Delta} \left( 4 \dim_E C^2 + 1/T \right) + 16 \frac{LT\alpha C}{\Delta} \dim_E (\log T + 1)
\]

16
\begin{align*}
+ 128 \frac{L}{\Delta} \sigma_{\text{max}}^2 \log(2N_\alpha L/\delta) \dim E(\log T + 1) \\
+ 32 \frac{L}{\Delta} T \alpha \sigma_{\text{max}} \sqrt{\log(8T^2 L/\delta)} \dim E(\log T + 1).
\end{align*}

Proof. For simplicity, let \( d = \dim E(\mathcal{F}, 1/T^2) \).

Suppose the event described in Lemma A.4 holds. With probability at least 1 \(-\delta,
\begin{align*}
\text{Regret} (T) \\
= \sum_{t=1}^{T} (f^*(a^*_t) - f^*(a_t)) \\
= \sum_{l \in [L]} \sum_{t \in \Psi_{T+1,l}} (f^*(a^*_t) - f^*(a_t)) \\
\leq \sum_{l \in [L]} \sum_{t \in \Psi_{T+1,l}} (f^*(a^*_t) - f^*(a_t))^2 / \Delta \\
\leq \frac{1}{\Delta} \sum_{l \in [L]} \sum_{t \in \Psi_{T+1,l}} w_{C_{l,t}}^2 (D_t) \\
\leq \frac{1}{\Delta} \sum_{l \in [L]} (4dC^2 + 1/T + 4\beta_{T,l}^2 d(\log T + 1)) \\
\leq \frac{L}{\Delta} \cdot (4dC^2 + 1/T) \\
+ 4 \frac{L}{\Delta} d(\log T + 1) \left( 32\sigma_{\text{max}}^2 \log(2N_\alpha L/\delta) + 4T\alpha (C + 2\sqrt{\sigma_{\text{max}}^2 \log(4T(T+1)L/\delta)}) \right)
\end{align*}

where the first equality follows from the definition in (3.3), the second equality holds by the fact that \( \Psi_{T+1,l} \ (l \in [L]) \) forms a partition of \([T] \), the first inequality holds due to the definition of \( \Delta \) in Subsection 3.1, the second inequality follows from Lemma A.4, the fourth inequality holds due to Lemma A.3 and the last inequality is derived by directly substituting the value of \( \beta_{T,l} \).

\[ \square \]

B Proofs from Section 5

Lemma B.1 (Freedman 1975). Let \( M, v > 0 \) be fixed constants. Let \( \{x_i\}_{i=1}^n \) be a stochastic process, \( \{G_i\}_i \) be a filtration so that for all \( i \in [n] \), \( x_i \) is \( G_i \)-measurable, while most surely \( \mathbb{E}[x_i|G_{i-1}] = 0 \), \(|x_i| \leq M \) and

\[ \sum_{i=1}^n \mathbb{E}(x_i^2|G_i) \leq v. \]

Then, for any \( \delta > 0 \), with probability \( 1 - \delta \), for all \( t \in [n] \),

\[ \sum_{i=1}^{t} x_i \leq \sqrt{2v \log(2t^2/\delta)} + 2/3 \cdot M \log(2t^2/\delta). \]
**Lemma B.2.** Suppose \( a, b \geq 0 \). If \( x^2 \leq a + b \cdot x \), then \( x^2 \leq 2b^2 + 2a \).

*Proof.* By solving the root of quadratic polynomial \( q(x) := x^2 - b \cdot x - a \), we obtain \( \max\{x_1, x_2\} = (b + \sqrt{b^2 + 4a})/2 \). Hence, we have \( x \leq (b + \sqrt{b^2 + 4a})/2 \) provided that \( q(x) \leq 0 \). Then we further have

\[
x^2 \leq \frac{1}{4} \left( b + \sqrt{b^2 + 4a} \right)^2 \leq \frac{1}{4} \cdot 2 \left( b^2 + b^2 + 4a \right) \leq 2b^2 + 2a. \tag{B.1}
\]

\[\square\]

**Theorem B.3** (Restatement of Theorem 5.1). Suppose that \( |f^*(a)| \leq C \) for all \( a \in A \). For any \( \alpha > 0 \) and \( \delta \in (0, 1/2) \), if we set \( \beta_{t,l} \) as the square root of

\[
12C\alpha t + 4\alpha R t + \frac{8}{3} C R \log (2N\alpha t^2/\delta) + 16 \cdot (2^{l+1}\sigma)^2 \log (2N\alpha t^2/\delta)
\]

where

\[
\overline{R} = R \sqrt{2 \log (4t^2/\delta)},
\]

then \( f^* \in C_{t,l} \) for all \( t \) with probability at least \( 1 - 2\delta \) for any fixed \( l \).

*Proof.* By simple calculation, for all \( f \in F \) we have

\[
\sum_{s \in \Psi_{t,l}} (f(a_s) - f^*(a_s))^2 + 2 \sum_{s \in \Psi_{t,l}} \epsilon_t[f^*(a_s) - f(a_s)] = \sum_{s \in \Psi_{t,l}} (r_s - f(a_s))^2 - \sum_{s \in \Psi_{t,l}} (r_s - f^*(a_s))^2.
\]

(B.2)

By sub-Gaussianity of \( \epsilon_t \) we have

\[
\mathbb{P} \left( \exists t \geq 1, \max_{1 \leq s \leq t} |\epsilon_s| \geq R \sqrt{2 \log (4t^2/\delta)} \right) \leq \sum_{s \geq 1} \mathbb{P}(|\epsilon_s| \geq R \sqrt{2 \log (4s^2/\delta)}) \leq \sum_{s \geq 1} \delta/(2s^2) \leq \delta. \tag{B.3}
\]

For simplicity, let event \( \mathcal{E}_{\text{subG}} := \{ \forall t \geq 1, \max_{1 \leq s \leq t} |\epsilon_s| \leq R \sqrt{4 \log (4t^2/\delta)} \} \).

Let \( \mathcal{G}(\alpha) \subset F \) be an \( \alpha \)-cover of \( F \) in \( \| \cdot \|_\infty \).

From the definition of \( \hat{f}_{t,l} \), we have

\[
\sum_{s \in \Psi_{t,l}} (\hat{f}_{t,l}(a_s) - f^*(a_s))^2 + 2 \sum_{s \in \Psi_{t,l}} \epsilon_t[f^*(a_s) - \hat{f}_{t,l}(a_s)] = I(\hat{f}_{t,l}) \leq 0. \tag{B.4}
\]

Let \( g = \arg\min_{\mathcal{G}(\alpha)} \| \hat{f}_{t,l} - g \|_\infty \).

We then bound the gap \( I(g) - I(\hat{f}_{t,l}) \) under event \( \mathcal{E}_{\text{subG}}, \)

\[
I(g) - I(\hat{f}_{t,l}) = \sum_{s \in \Psi_{t,l}} \left[ (g(a_s) - f^*(a_s))^2 - (\hat{f}_{t,l}(a_s) - f^*(a_s))^2 \right] + 2 \sum_{s \in \Psi_{t,l}} \epsilon_t[\hat{f}_{t,l}(a_s) - g(a_s)] \\
\leq \sum_{s \in \Psi_{t,l}} (g(a_s) - \hat{f}_{t,l}(a_s))(g(a_s) + \hat{f}_{t,l}(a_s) - 2f^*(a_s)) + 2 \sum_{s \in \Psi_{t,l}} \alpha R \sqrt{2 \log (4t^2/\delta)}
\]

18
\[ \leq 4C_{\alpha}t + 2\alpha R \sqrt{2 \log(4t^2/\delta)} t. \] (B.5)

Fix an \( f \in \mathcal{F} \). Applying Freedman’s inequality (Lemma B.1), with probability at least 1 - \( \delta \), we have
\[
\sum_{s \in \Psi_{t,l}} \epsilon_t[f^*(a_s) - f(a_s)] \geq -2/3 \left( \max_{1 \leq s \leq t} \epsilon_s \right) C_\alpha \log(2t^2/\delta) - \sqrt{2 \cdot (2^{l+1} \sigma)^2 \sum_{s \in \Psi_{t,l}} (f(a_s) - f^*(a_s))^2 \log(2t^2/\delta)}. \] (B.6)

for all \( t \geq 1 \).

Using a union bound on all the \( f \in \mathcal{G}(\alpha) \) and \( \mathcal{E}_{\text{sub}G} \), we further obtain that
\[
\sum_{s \in \Psi_{t,l}} \epsilon_t[f^*(a_s) - f(a_s)] \geq -2/3 \max_{1 \leq s \leq t} \epsilon_s \left( C_\alpha \log(2N_{\alpha}t^2/\delta) - \sqrt{8 \cdot (2^{l+1} \sigma)^2 \log(2N_{\alpha}t^2/\delta)} \sum_{s \in \Psi_{t,l}} (f(a_s) - f^*(a_s))^2 \right) \] (B.7)

for all \( f \in \mathcal{G}(\alpha) \) with probability at least 1 - 2\( \delta \).

Substituting (B.7) into the definition of \( I(f) \), we have that for \( g \), it holds for probability at least 1 - 2\( \delta \) that
\[
4C_{\alpha}t + 2\alpha R \sqrt{2 \log(4t^2/\delta)} t \geq I(g) \geq -\frac{4}{3} R \sqrt{2 \log(4t^2/\delta)} C_\alpha \log(2N_{\alpha}t^2/\delta) - \sqrt{8 \cdot (2^{l+1} \sigma)^2 \log(2N_{\alpha}t^2/\delta)} \sum_{s \in \Psi_{t,l}} (f(a_s) - f^*(a_s))^2 \] (B.9)
\[
+ \sum_{s \in \Psi_{t,l}} (g(a_s) - f^*(a_s))^2 \] (B.10)

where the first inequality is obtained by substituting (B.5) and (B.4) into the inequality below
\[
I(g) \leq I(g) - I(\hat{f}_{t,l}) + I(\hat{f}_{t,l}), \]
the second inequality follows from the definition of \( I(f) \) and (B.7).

Using Lemma B.2, we can deduce that
\[
\sum_{s \in \Psi_{t,l}} (g(a_s) - f^*(a_s))^2 \leq 8C_{\alpha}t + 4\alpha R \sqrt{2 \log(4t^2/\delta)} t + \frac{8}{3} R C_\alpha \sqrt{2 \log(4t^2/\delta)} \log(2N_{\alpha}t^2/\delta)
+ 16 \cdot (2^{l+1} \sigma)^2 \log(2N_{\alpha}t^2/\delta).
\]

Then we can complete the proof by bounding the gap between \( \sum_{s \in \Psi_{t,l}} (g(a_s) - f^*(a_s))^2 \) and \( \sum_{s \in \Psi_{t,l}} (\hat{f}_{t,l}(a_s) - f^*(a_s))^2 \):
\[
\sum_{s \in \Psi_{t,l}} (\hat{f}_{t,l}(a_s) - f^*(a_s))^2 \leq \sum_{s \in \Psi_{t,l}} (g(a_s) - f^*(a_s))^2
\]
\[ + \left| \sum_{a \in \Psi_{t,l}} (g(a_s) - f^*(a_s))^2 - \sum_{a \in \Psi_{t,l}} (\hat{f}_{t,l}(a) - f^*(a))^2 \right| \]

\[ \leq 12C\alpha t + 4\alpha R \sqrt{2 \log(4t^2/\delta)t} + \frac{8}{3} RC \sqrt{2 \log(4t^2/\delta)} \log(2N_a t^2/\delta) \]

\[ + 16 \cdot (2^{l+1}\sigma)^2 \log(2N_a t^2/\delta). \]

\[ \square \]

**Theorem B.4** (Restatement of Theorem 5.3). Suppose Assumption 3.4 holds and \(|f^*(a)| \leq 1\) for all \(a \in A\). For all \(t \in [T], l \in [L]\) and \(\delta \in (0, 1), \alpha > 0, \sigma > 0\), if we apply Algorithm 2 as a subroutine of Algorithm 1 (in line 9) and set \(\beta_{t,l}\) as the square root of

\[ 12\alpha t + 4\alpha R t + \frac{8}{3} R \log(2N_a t^2 L/\delta) + 16 \cdot (2^{l+1}\sigma)^2 \log(2N_a t^2 L/\delta) \]

where \(N_a = \mathcal{N}(\alpha, \|\cdot\|_\infty)\) and \(R = R \sqrt{2 \log(4t^2 L/\delta)}\) (with a slight abuse of notation), then with probability at least \(1 - 2\delta\), the regret for the first \(T\) rounds is bounded as follows:

\[
\text{Regret}(T) \leq 4 \sqrt{L \dim_E (\log T + 1) \alpha \sqrt{3 + R T} + 2 \frac{8}{3} L \dim_E (\log T + 1) R \log(2N_a t^2 L/\delta) T}
\]

\[ + 16 \sqrt{L \dim_E (\log T + 1) \log(2N_a t^2 L/\delta) \sqrt{J + T \sigma^2} + \sqrt{L} (2 \sqrt{\dim_E T + 1})}. \]

**Proof.** For simplicity, let \(d = \dim_E (\mathcal{F}, 1/T)\).

Based on Theorem 5.1, for any fixed \(l\), we have \(f^* \in \mathcal{C}_{t,l}\) with probability \(1 - 2\delta/L\). Applying a union bound on all \(l \in [L]\), we have \(f^* \in \mathcal{C}_{t,l}\) for all \(t, l\) with probability at least \(1 - 2\delta\).

Then we further obtain, with probability at least \(1 - 2\delta\),

\[
\text{Regret}(T) = \sum_{t=1}^{T} (f^*(a^*_t) - f^*(a_t))
\]

\[ = \sum_{l \in [L]} \sum_{t \in \Psi_{T+1,l}} (f^*(a^*_t) - f^*(a_t)) \]

\[ \leq \sum_{l \in [L]} \sum_{t \in \Psi_{T+1,l}} \left( \max_{f \in \mathcal{C}_{t,l}} f(a_t) - f^*(a_t) \right) \]

\[ = \sum_{l \in [L]} \sum_{t \in \Psi_{T+1,l}} \left( \max_{f \in \mathcal{C}_{t,l}} f(a_t) - f^*(a_t) \right) \sqrt{\frac{1}{|\Psi_{T+1,l}|} \cdot \frac{1}{|\Psi_{T+1,l}|}} \]

\[ \leq \sqrt{L} \cdot \sum_{l \in [L]} \frac{|\Psi_{T+1,l}|}{\sum_{t \in \Psi_{T+1,l}} w^2_{C_{t,l}}(D_t)} \]

\[ \leq \sqrt{L} \cdot \sum_{l \in [L]} \frac{|\Psi_{T+1,l}|}{\left( 4d + 1/T + 4\beta^2_{T,l}d \log T + 1 \right)} \]

where the first equality follows from the definition in (3.3), the second equality holds by the fact that \(\Psi_{T+1,l} (l \in [L])\) forms a partition of \([T]\), the second inequality follows from Cauchy-Schwarz inequality and the definition of width in (3.2), the third inequality follows from Lemma A.3.
Following the definition of $\beta_{t,l}$, we further calculate

$$\text{Regret}(T) \leq \sqrt{L} \left( 2\sqrt{dT} + 1 \right) + \sqrt{L} \cdot \sqrt{\sum_{t \in [L]} \sum_{t \in [T+1]} 64(2^{l+1}\sigma)^2 \log(2N_{\alpha}T^2/L/\delta)d(\log T + 1)$$

$$+ 2\sqrt{Ld(\log T + 1)^2 \sqrt{12\alpha T + 4\alpha R T + \frac{8}{3}\bar{R} \log(2N_{\alpha}T^2L/\delta)}}$$

$$\leq \sqrt{L} \left( 2\sqrt{dT} + 1 \right) + 4\sqrt{Ld(\log T + 1)\alpha \sqrt{3 + \bar{R} T}}$$

$$+ 2\sqrt{\frac{8}{3}Ld(\log T + 1)\bar{R} \log(2N_{\alpha}L^2/\delta)T}$$

$$+ \sqrt{Ld(\log T + 1) \log(2N_{\alpha}T^2L/\delta)} \sqrt{\sum_{t=1}^{T} 256(\sigma_t^2 + \bar{\sigma}^2)}$$

$$\leq \sqrt{L} \left( 2\sqrt{dT} + 1 \right) + 4\sqrt{Ld(\log T + 1)\alpha \sqrt{3 + \bar{R} T}}$$

$$+ 2\sqrt{\frac{8}{3}Ld(\log T + 1)\bar{R} \log(2N_{\alpha}L^2/\delta)T}$$

$$+ 16\sqrt{Ld(\log T + 1) \log(2N_{\alpha}T^2L/\delta)} \sqrt{J + T\bar{\sigma}^2},$$

where the first inequality holds by the definition of $\beta_{t,l}$ and the fact that $\sqrt{a + b} \leq \sqrt{a} + \sqrt{b}$ for $a, b > 0$, the second inequality follows from the definition of $l_t$, the third inequality follows from the definition of $J$. □

**Theorem B.5** (Formal version of Theorem 5.7). Suppose Assumption 3.4 holds and $|f^*(a)| \leq 1$ for all $a \in A$. Let $\sigma_{\text{max}} = \max_{t \in [T]} \sigma_t$ and $d = \dim \mathcal{E}(\mathcal{F}, 1/T)$. If we apply Algorithm 2 as a subroutine of Algorithm 1 (in line 9) and set $\beta_{t,l}$ as the same value in Theorem 5.3, then with probability at least $1 - 2\delta$, the regret of Algorithm 1 for the first $T$ rounds is bounded as follows:

$$\text{Regret}(T) \leq \frac{L}{\Delta} (4d + 1/T) + \frac{L\alpha T}{\Delta} d(\log T + 1)(12 + 4\bar{R}) + \frac{32}{\Delta} \frac{Ld(\log T + 1) \log(2N_{\alpha}T^2L/\delta)}{\log(2N_{\alpha}T^2L/\delta)}$$

$$+ 256\frac{L}{\Delta} \sigma_{\text{max}}^2 d(\log T + 1) \log(2N_{\alpha}T^2L/\delta).$$

**Proof.** For simplicity, let $d = \dim \mathcal{E}(\mathcal{F}, 1/T)$.

Basically following the previous approach in Theorem 4.6, with probability $1 - 2\delta$, we have

$$\text{Regret}(T) = \sum_{t=1}^{T} (f^*(a_t^*) - f^*(a_t))$$

$$= \sum_{t \in [L]} \sum_{t \in [T+1]} (f^*(a_t^*) - f^*(a_t))$$

$$\leq \sum_{t \in [L]} \sum_{t \in [T+1]} (f^*(a_t^*) - f^*(a_t))^2 / \Delta$$

$$\leq \sum_{t \in [L]} \sum_{t \in [T+1]} \left( \max_{f \in C_{t,l}} f(a_t) - f^*(a_t) \right)^2 / \Delta$$

$$= \sum_{t \in [L]} \sum_{t \in [T+1]} \left( \max_{f \in C_{t,l}} f(a_t) - f^*(a_t) \right)^2 / \Delta$$

21
\[ \leq \frac{1}{\Delta} \sum_{l \in [L]} \sum_{t \in \Psi_{T+1,l}} w_c^2(t \mathcal{D}_t) \]
\[ \leq \frac{1}{\Delta} \sum_{l \in [L]} (4 + 1/T + 4\beta_{T,l}^2d(\log T + 1)) \]
\[ \leq \frac{L}{\Delta} \cdot (4d + 1/T) \]
\[ + 4\frac{L}{\Delta}d(\log T + 1) \left( 12\alpha T + 4\alpha RT + \frac{8T}{3} \log(2N\alpha T^2 L/\delta) + 64\sigma_{\text{max}}^2 \log(2N\alpha T^2 L/\delta) \right) \]
\[ = \frac{L}{\Delta} (4d + 1/T) + \frac{L\alpha T}{\Delta}d(\log T + 1)(12 + 4R) + \frac{32}{3} \frac{L}{\Delta}d(\log T + 1) \log(2N\alpha T^2 L/\delta) \]
\[ + 256\frac{L}{\Delta}\sigma_{\text{max}}^2d(\log T + 1) \log(2N\alpha T^2 L/\delta), \]
where the first equality holds by the definition in (3.3), the second equality holds by the fact that \( \Psi_{T+1,l} (l \in [L]) \) forms a partition of \([T]\), the first inequality follows from the definition of \( \Delta \) in (3.5), the third inequality follows from the definition in (3.2), the fourth inequality holds by Lemma A.3, the fifth inequality follows from the definition of \( \beta_{T,l} \).

\section*{C Proofs from Section 6}

\textbf{Lemma C.1.} Let \( reg_t := \sum_{s=1}^{t} \ell(a_s^\top \theta, r_s) - \sum_{s=1}^{t} \ell(a_s^\top \theta^*, r_s) \). Following Algorithm 3, with probability at least \( 1 - \delta \),

\[ \sum_{s=1}^{t} (a_s^\top (\theta_s - \theta^*))^2 \leq \frac{4}{\kappa} reg_t + \frac{8R^2}{\kappa^2} \log(4t^2/\delta) \quad \text{(C.1)} \]

for all \( t \geq 1 \). We denote the corresponding event by \( \mathcal{E}_1 \).

\textit{Proof.}

\[ reg_t = \sum_{s=1}^{t} \ell(a_s^\top \theta, r_s) - \sum_{s=1}^{t} \ell(a_s^\top \theta^*, r_s) \]
\[ = \sum_{s=1}^{t} \ell(a_s^\top \theta^*, r_s)a_s^\top (\theta_s - \theta^*) + \frac{\ell''(\xi_s, r_s)}{2}(a_s^\top \theta_s - a_s^\top \theta^*)^2 \]
\[ \geq -\sum_{s=1}^{t} \epsilon_s a_s^\top (\theta_s - \theta^*) + \frac{\kappa}{2}(a_s^\top \theta_s - a_s^\top \theta^*)^2 \]

where the first equality follows from Definition 6.4, the second equality holds by Taylor series expansion and \( \xi_s \) is a point between \( a_s^\top \theta \) and \( a_s^\top \theta^* \), the first inequality follows from Assumption 6.2 and Assumption 6.3.

Further we obtain

\[ \sum_{s=1}^{t} (a_s^\top (\theta_s - \theta^*))^2 \leq \frac{2}{\kappa} \sum_{s=1}^{t} \epsilon_s a_s^\top (\theta_s - \theta^*) + \frac{2}{\kappa} reg_t \]
\[ \leq \frac{2}{\kappa} R \sqrt{2 \sum_{s=1}^{t} (a_s^\top (\theta_s - \theta^*))^2 \log(2/\delta) + \frac{2}{\kappa} \text{reg}_t} \]

with probability \( 1 - \delta \), where the second inequality follows from the sub-Gaussianity of \( \epsilon_s \).

Applying Lemma B.2, we obtain that with probability at least \( 1 - \delta \),

\[ \sum_{s=1}^{t} (a_s^\top (\theta_s - \theta^*))^2 \leq \frac{4}{\kappa} \text{reg}_t + \frac{8}{\kappa^2} R^2 \log(2/\delta). \]

Applying union bound on all \( t \geq 1 \), we have with probability at least \( 1 - \delta \),

\[ \sum_{s=1}^{t} (a_s^\top (\theta_s - \theta^*))^2 \leq \frac{4}{\kappa} \text{reg}_t + \frac{8}{\kappa^2} R^2 \log(4t^2/\delta) \tag{C.2} \]

for all \( t \geq 1 \).

\[ \square \]

**Lemma C.2** (Lemma 11, Abbasi-Yadkori et al. 2011). For any \( \lambda > 0 \) and sequence \( \{x_t\}_{t=1}^{T} \subset \mathbb{R}^d \) for \( t \in \{0, 1, \cdots, T\} \), define \( Z_t = \lambda I + \sum_{i=1}^{t} x_i x_i^\top \). Then, provided that \( \|x_t\|_2 \leq M \) for all \( t \in [T] \), we have

\[ \sum_{t=1}^{T} \min\{1, \|x_t\|_2^2 \} \leq 2d \log \frac{d \lambda + TM^2}{d \lambda}. \]

**Lemma C.3.** For any \( \lambda > 0 \) and sequence \( \{x_t\}_{t=1}^{T} \subset \mathbb{R}^d \) for \( t \in \{0, 1, \cdots, T\} \), define \( Z_t = \lambda I + \sum_{i=1}^{t} x_i x_i^\top \). Then, provided that \( \|x_t\|_2 \leq M \) for all \( t \in [T] \), we have

\[ \sum_{t=1}^{T} \|x_t\|_2^2 \|Z_t^{-1}\] \leq 2d \log \frac{d \lambda + TM^2}{d \lambda}. \]

**Proof.** Applying matrix inversion lemma,

\[ \sum_{t=1}^{T} \|x_t\|_2^2 \|Z_t^{-1}\] \leq \sum_{t=1}^{T} x_t^\top x_t \]

\[ \leq \sum_{t=1}^{T} \|x_t\|_2^2 \|Z_t^{-1}\] \leq \sum_{t=1}^{T} \min\{1, \|x_t\|_2^2 \} \] \leq 2d \log \frac{d \lambda + TM^2}{d \lambda},

where the second equality follows from matrix inversion lemma, the second inequality holds by Lemma C.2. \[ \square \]
Lemma C.4. Let
\[
\Sigma_t := \frac{4A^2K^2}{\kappa}I + \kappa \sum_{i=1}^t a_i \cdot a_i^\top,
\]
\[
\bar{R}_t := \max_{1 \leq s \leq t} |\epsilon_s|,
\]
\[
\sigma_{\text{max}} := \max_{t \geq 1} \sigma_t.
\]

Then with probability at least 1 - \delta, it holds simultaneously for all \( t \geq 1 \) that
\[
\sum_{s=1}^t (\epsilon_s^2 - \mathbb{E}[\epsilon_s^2]) \|a_s\|_{\Sigma_{s-1}}^2 \leq \frac{\sigma_{\text{max}} \bar{R}_t}{K} \sqrt{d \log \left( 1 + \frac{tA\kappa^2}{4dK^2} \log(2t^2/\delta) \right)} + \frac{2}{3\kappa} \left( \sigma_{\text{max}}^2 + \bar{R}_t^2 \right) \log(2t^2/\delta).
\]

Proof. To bound the sum of variance of each term, we calculate
\[
\sum_{s=1}^t \text{Var} \left( \epsilon_s^2 - \mathbb{E}[\epsilon_s^2] \right) \|a_s\|_{\Sigma_{s-1}}^2 \leq \sum_{s=1}^t \mathbb{E}[\epsilon_s^2] \|a_s\|_{\Sigma_{s-1}}^4
\]
\[
\leq \sum_{s=1}^t \frac{\kappa}{4K^2} \mathbb{E}[\epsilon_s^2] \|a_s\|_{\Sigma_{s-1}}^2 \|a_s\|_{\Sigma_{s-1}}^2
\]
\[
\leq \sigma_{\text{max}}^2 \frac{\bar{R}_t^2}{2K^2} \sqrt{d \log \left( 1 + \frac{tA\kappa^2}{4dK^2} \right)},
\]
where the second inequality follows from the definition of \( \Sigma_t \), the third inequality holds by Lemma C.2.

Also note that
\[
\max_{1 \leq s \leq t} (\epsilon_s^2 - \mathbb{E}[\epsilon_s^2]) \|a_s\|_{\Sigma_{s-1}}^2 \leq \left( \sigma_{\text{max}}^2 + \bar{R}_t^2 \right) \frac{1}{K}
\]
since \( \Sigma_t \geq \frac{4A^2K^2}{\kappa}I + \kappa a_s \cdot a_s^\top \).

Then we apply Freedman’s inequality, which gives for arbitrary \( t \geq 1 \),
\[
\sum_{s=1}^t (\epsilon_s^2 - \mathbb{E}[\epsilon_s^2]) \|a_s\|_{\Sigma_{s-1}}^2 \leq \frac{\sigma_{\text{max}} \bar{R}_t}{K} \sqrt{d \log \left( 1 + \frac{tA\kappa^2}{4dK^2} \right) \log(1/\delta)} + 2/3 \cdot \frac{1}{\kappa} \left( \sigma_{\text{max}}^2 + \bar{R}_t^2 \right) \log(1/\delta)
\]
with probability at least 1 - \delta.

Applying a union bound on all \( t \geq 1 \), we have with probability at least 1 - \delta,
\[
\sum_{s=1}^t (\epsilon_s^2 - \mathbb{E}[\epsilon_s^2]) \|a_s\|_{\Sigma_{s-1}}^2 \leq \frac{\sigma_{\text{max}} \bar{R}_t}{K} \sqrt{d \log \left( 1 + \frac{tA\kappa^2}{4dK^2} \right) \log(2t^2/\delta)} + \frac{2}{3\kappa} \left( \sigma_{\text{max}}^2 + \bar{R}_t^2 \right) \log(2t^2/\delta)
\]
(C.3)
since \( \sum_{t \geq 1} \frac{\delta}{2t^2} \leq \delta \).
Theorem C.5 (Restatement of Theorem 6.5). If we set $\phi(\theta) = \frac{2A^2K^2}{\kappa} \|\theta\|^2$ and assume that all the data points fed into the algorithm are of noise variance bounded by $\frac{\sigma_{\text{max}}^2}{d}$, then with probability at least $1 - 3\delta$, for all $t \geq 1$, the regret of Algorithm 3 for the first $t$ rounds is bounded as follows:

$$r_{\text{reg}} \leq \frac{8A^2K^2B^2}{\kappa} + \frac{9}{2\kappa} R^2 \log^2(4t^2/\delta) + \frac{3\sigma_{\text{max}}^2}{\kappa} d \log \left(1 + \frac{t\kappa R^2}{4dK^2}\right).$$

Proof. For simplicity, let $L_t(\theta) = \sum_{s=1}^{t-1} \ell(\theta_a, s) + \phi(\theta)$ and $\text{loss}_t(\theta) = L_t(\theta) - \phi(\theta)$.

Suppose event $E_1$, event $E_{\text{subG}} := \left\{ \forall t \geq 1, \max_{1 \leq s \leq t} |\epsilon_s| \leq R \sqrt{2\log(4t^2/\delta)} \right\}$ and the event described in Lemma C.4 (denoted by $E_2$) simultaneously hold in the following proof.

By sub-Gaussianity of $\epsilon_t$ we have

$$P \left( \exists t \geq 1, \max_{1 \leq s \leq t} |\epsilon_s| \geq R \sqrt{2\log(4t^2/\delta)} \right) \leq \sum_{s \geq 1} P(|\epsilon_s| \geq R \sqrt{2\log(4s^2/\delta)}) \leq \sum_{s \geq 1} \delta/(2s^2) \leq \delta. \quad (C.4)$$

Hence, $P(E_{\text{subG}}) \geq 1 - \delta$. Applying Lemma C.1 and Lemma C.4, we have that $P(E_{\text{subG}} \cap E_1 \cap E_2) \geq 1 - 3\delta$ by union bound.

From the update rule of Algorithm 3, we calculate

$$\sum_{s=1}^{t} \ell(a_s^\top \theta_s, r_s) - \sum_{s=1}^{t} \ell(a_s^\top \theta_s^*, r_s)$$

$$= \sum_{s=1}^{t} [\text{loss}_s(\theta_s) - \text{loss}_{s+1}(\theta_{s+1}) + \ell(a_s^\top \theta_s, r_s)] + \text{loss}_{t+1}(\theta_{t+1}) - \text{loss}_{t+1}(\theta^*)$$

$$= \sum_{s=1}^{t} [L_{s+1}(\theta_s) - L_{s+1}(\theta_{s+1})] - \phi(\theta_1) + \phi(\theta^*) + L_{t+1}(\theta_{t+1}) - L_{t+1}(\theta^*)$$

$$\leq 2 \max_{\theta \in \Theta} |\phi(\theta)| + \sum_{s=1}^{t} \left[ \frac{L_{s+1}(\theta_s) - L_{s+1}(\theta_{s+1})}{I_1} \right]. \quad (C.5)$$

where the first equality follows from the definition of $\text{loss}$, the second equality holds by the definition of $L$, the first inequality holds since $L_{t+1}(\theta_{t+1}) = \min_{\theta \in \Theta} L_{t+1}(\theta)$.

Then we continue to bound $I_1$.

$$I_1 = \sum_{s=1}^{t} [L_{s+1}(\theta_s) - L_{s+1}(\theta_{s+1})]$$

$$= \sum_{s=1}^{t} \left[ - \left( \frac{\partial L_{s+1}}{\partial \theta} (\theta_s, \theta_{s+1} - \theta_s) - (\theta_{s+1} - \theta_s)^\top H_{s+1}(\theta'_s)(\theta_{s+1} - \theta_s) \right) \right]$$

$$= \sum_{s=1}^{t} \left[ -(h(a_s^\top \theta_s) - r_s)^\top (a_s, \theta_{s+1} - \theta_s) - (\theta_{s+1} - \theta_s)^\top H_{s+1}(\theta'_s)(\theta_{s+1} - \theta_s) \right]$$

$$\leq \frac{1}{4} \sum_{s=1}^{t} \left( h(a_s^\top \theta_s) - r_s \right)^2 \|a_s\|^2 \frac{1}{H_{s+1}(\theta'_s)}.$$
We bound
\[
I_2 \leq \left[ \frac{4}{\kappa^2} r_{\text{reg}} + \frac{8R^2}{\kappa^2} \log(4t^2/\delta) \right] \frac{\kappa}{4K^2}
\]
\[
\leq \frac{1}{K^2} r_{\text{reg}} + \frac{2R^2}{\kappa K^2} \log(4t^2/\delta).
\]

We bound \( I_3 \) by decomposing it into its expected value and a zero-mean term.
\[
I_3 \leq \sum_{s=1}^{t} \sigma^2_s \|a_s\|_{H_{s+1}^{-1}(\theta_s')}^2 + \sum_{s=1}^{t} \left( \epsilon^2_s - \mathbb{E}[\epsilon^2_s] \right) \|a_s\|_{H_{s+1}^{-1}(\theta_s')}^2
\]
\[
\leq \sum_{s=1}^{t} \sigma^2_s \|a_s\|_{\Sigma_s^{-1}}^2 + \sum_{s=1}^{t} \left( \epsilon^2_s - \mathbb{E}[\epsilon^2_s] \right) \|a_s\|_{\Sigma_s^{-1}}^2
\]
\[
\leq 2 \frac{\sigma^2_{\text{max}}}{\kappa} d \log \left( 1 + \frac{t\kappa^2}{4dK^2} \right) + \sum_{s=1}^{t} \left( \epsilon^2_s - \mathbb{E}[\epsilon^2_s] \right) \|a_s\|_{\Sigma_s^{-1}}^2, \tag{C.11}
\]

where the second inequality follows from\( C.9 \), the third inequality holds by Lemma C.3.

Substituting (C.11) and (C.10) into (C.6), we have
\[
I_1 \leq \frac{1}{2} r_{\text{reg}} + \frac{R^2}{\kappa} \log(4t^2/\delta) + \frac{\sigma^2_{\text{max}}}{\kappa} d \log \left( 1 + \frac{t\kappa^2}{4dK^2} \right) + \frac{1}{2} I_4. \tag{C.12}
\]

Applying Lemma C.4 to bound \( I_4 \), we calculate
\[
I_4 \leq \frac{\sigma_{\text{max}} R_t}{K} \sqrt{d \log \left( 1 + \frac{t\kappa^2}{4dK^2} \right) \log(2t^2/\delta) + \frac{2}{3K} \left( \sigma^2_{\text{max}} + R_t^2 \right) \log(2t^2/\delta)}
\]

26
where the second inequality holds due to event $E_{\text{subG}}$, the third inequality follows from $\sigma_{\text{max}} \leq R$.

Substituting (C.13) into (C.12), we have

$$ I_1 \leq \frac{1}{2} \text{reg}_t + \frac{R^2}{\kappa} \log(4t^2/\delta) + \frac{\sigma_{\text{max}}^2}{\kappa} d \log \left(1 + \frac{tA^2}{4dK^2}\right) $$

$$ + \frac{\sigma_{\text{max}} R}{2K} \sqrt{2d \log \left(1 + \frac{tA^2}{4dK^2}\right) \log(4t^2/\delta)} $$

$$ \leq \frac{1}{2} \text{reg}_t + \frac{2R^2}{\kappa} \log^2(4t^2/\delta) + \frac{\sigma_{\text{max}}^2}{\kappa} d \log \left(1 + \frac{tA^2}{4dK^2}\right) $$

$$ + \frac{1}{2\kappa} \frac{\sigma_{\text{max}}^2}{\kappa} d \log \left(1 + \frac{tA^2}{4dK^2}\right) + \frac{1}{4\kappa} R^2 \log^2(4t^2/\delta) $$

$$ = \frac{1}{2} \text{reg}_t + \frac{9}{4\kappa} R^2 \log^2(4t^2/\delta) + \frac{3}{2}\sigma_{\text{max}}^2 \frac{d}{\kappa} \log \left(1 + \frac{tA^2}{4dK^2}\right). \quad (C.14) $$

Substituting the upper bound of $I_1$ above to (C.5),

$$ \text{reg}_t \leq 2 \max_{\theta \in \Theta} |\phi(\theta)| + \frac{1}{2} \text{reg}_t + \frac{9}{4\kappa} R^2 \log^2(4t^2/\delta) + \frac{3}{2}\sigma_{\text{max}}^2 \frac{d}{\kappa} \log \left(1 + \frac{tA^2}{4dK^2}\right) $$

$$ \leq 8A^2K^2B^2 \frac{1}{\kappa} + \frac{9}{2\kappa} R^2 \log^2(4t^2/\delta) + \frac{3}{2}\sigma_{\text{max}}^2 \frac{d}{\kappa} \log \left(1 + \frac{tA^2}{4dK^2}\right), \quad (C.15) $$

from which we can further complete the proof by the arbitrariness of $t$. 

In the following lemma, we formally introduce the conversion from online learning regret to confidence set in our setting. For simplicity in analysis, we omit the level subscript and suppose all the data is fed into the same level.

**Lemma C.6.** Suppose we feed loss function $\{\ell_s(\theta)\}_{s=1}^t$ into a single online learner $B$. Assume that $B$ has an online learning (OL) regret bound $\overline{\text{reg}}_t$: $\forall t \geq 1$,

$$ \sum_{s=1}^t \ell_s(\theta) - \ell_s(\theta^*) \leq \overline{\text{reg}}_t. \quad (C.16) $$

Define $X_t$ as the design matrix consisting of $a_1, \cdots, a_t, z_t = [z_1, \cdots, z_t]$. Then, with probability at least $1 - 4\delta$,

$$ \forall t \geq 1, \|\theta^* - \hat{\theta}_t\|_{\overline{\text{reg}}_t}^2 \leq 1 + \frac{4}{\kappa} \overline{\text{reg}}_t + \frac{8R^2}{\kappa^2} \log \left(4t^2/\delta\right) + \lambda B^2 - \left(\|z_t\|^2 - \hat{\theta}_t^\top X_t^\top z_t\right). \quad (C.17) $$
Proof. With Lemma C.1, we can prove this lemma by following nearly the same proof for Theorem 1 in Jun et al. (2017). (We can set \(\beta_t\) in their proof to be \(1 + \frac{2 R^2}{\kappa^2} \log \left( \frac{4 t^2}{\delta} \right)\) according to Lemma C.1. )

Lemma C.7. For all \(t\), with \(z_t\) and \(X_t\) defined as in Lemma C.8, we have

\[
\|z_t\|_2^2 - \hat{\theta}_t^\top X_t^\top z_t \geq 0.
\]

Proof. After ridge regression, \(\hat{\theta}_t = \nabla_t^{-1} X_t^\top z_t\) where \(\nabla_t := \lambda I + X_t^\top X_t\). Then we have

\[
\|z_t\|_2^2 - \hat{\theta}_t^\top X_t^\top z_t = \|z_t\|_2^2 - \left( \nabla_t^{-1} X_t^\top z_t \right)^\top X_t^\top z_t \\
= \|z_t\|_2^2 - z_t^\top X_t \nabla_t^{-1} X_t^\top z_t \tag{C.18}
\]

We consider

\[
\begin{bmatrix} \lambda I + X_t^\top X_t & X_t^\top \\ X_t & I \end{bmatrix} \succeq [X_t \ I]^\top [X_t \ I] \succeq 0. \tag{C.19}
\]

From Schur complement theorem, we have

\[
I \succeq (I + X_t^\top X_t)^{-1} X_t^\top = X_t \nabla_t^{-1} X_t^\top. \tag{C.20}
\]

Then we can complete the proof by substituting (C.20) into (C.18).

Lemma C.8 (Variance-dependent confidence set for generalized linear bandits). Suppose that Assumption 6.1, 6.2, 6.3 hold. For any \(\delta \in (0,1/4)\), if we set

\[
\beta_{t,l} := 1 + \frac{32 A^2 K^2 B^2}{\kappa^2} + \frac{26}{\kappa^2} R^2 \log^2(4 t^2 L/\delta) + 12 \frac{2^{2(l+1)} \sigma^2}{\kappa^2} d \log \left( 1 + \frac{t A K^2}{4 d K^2} \right) + \lambda B^2, \tag{C.21}
\]

then with probability at least \(1 - 4\delta\), we have \(\theta^* \in C_{t,l}\) for all \(t \geq 1, l \in [L]\).

Proof. For any \(l \in [L]\), with probability at least \(1 - \frac{\delta}{T}\),

\[
\|\theta^* - \hat{\theta}_{t,l}\|_{\nabla_{t,l}}^2 \leq 1 + \frac{4 R^2}{\kappa^2} \log \left( \frac{4 t^2 L/\delta}{\kappa^2} \right) + \lambda B^2,
\]

making use of Lemma C.8 and Lemma C.7.

From Theorem 6.5, with probability at least \(1 - 3\frac{\delta}{T}\), we can set

\[
\overline{\text{reg}_{t,l}} := \frac{32 A^2 K^2 B^2}{\kappa^2} + \frac{9}{2 \kappa^2} R^2 \log^2(4 t^2 L/\delta) + 3 \frac{2^{2(l+1)} \sigma^2}{\kappa^2} d \log \left( 1 + \frac{t A K^2}{4 d K^2} \right). \tag{C.22}
\]

By Union Bound, with probability \(1 - 4\delta\), for all \(l \in [L]\),

\[
\|\theta^* - \hat{\theta}_{t,l}\|_{\nabla_{t,l}}^2 \leq 1 + \frac{32 A^2 K^2 B^2}{\kappa^2} + \frac{18}{\kappa^2} R^2 \log^2(4 t^2 L/\delta) + 12 \frac{2^{2(l+1)} \sigma^2}{\kappa^2} d \log \left( 1 + \frac{t A K^2}{4 d K^2} \right)
\]

\[
+ \frac{8 R^2}{\kappa^2} \log \left( \frac{4 t^2 L/\delta}{\kappa^2} \right) + \lambda B^2 \\
= 1 + \frac{32 A^2 K^2 B^2}{\kappa^2} + \frac{26}{\kappa^2} R^2 \log^2(4 t^2 L/\delta) + 12 \frac{2^{2(l+1)} \sigma^2}{\kappa^2} d \log \left( 1 + \frac{t A K^2}{4 d K^2} \right) + \lambda B^2.
\]
Theorem C.9. Suppose that Assumptions 6.1 and 6.2 hold for the known reward function class $\mathcal{F}$. If we apply Algorithm 4 as a subroutine of Algorithm 1 (in line 9) and set $\beta_{t,l}$ to

$$1 + \frac{32A^2K^2B^2}{\kappa^2} + \frac{26}{\kappa^2} R^2 \log^2(4t^2L/d) + 12\frac{2^{(l+1)}\bar{\sigma}^2}{\kappa^2} d \log \left(1 + \frac{tA\kappa^2}{4dK^2}\right) + \lambda B^2$$

for all $t \in [T], l \in [L], \bar{\sigma} = R/\sqrt{d}$, then with probability $1 - 4\delta$, the regret of Algorithm 1 for the first $T$ rounds is bounded as follows:

$$\text{Regret}(T) = \tilde{O} \left( \frac{K}{\kappa} d\sqrt{T} + \frac{K}{\kappa} (K \cdot AB + R) \sqrt{dT} \right).$$

Proof.

$$\text{Regret}(T) = \sum_{t \in [T]} h(x_{t,*^t}\theta^*) - h(x_{t^*} \theta^*)$$

$$\leq K \sum_{t \in [T]} (x_{t,*^t}\theta^* - x_{t^*} \theta^*)$$

$$\leq K \sum_{t \in [T]} \left( \max_{\theta \in \mathcal{F}_{1,l}} x_i^\top \theta - x_i^\top \theta^* \right)$$

$$\leq K \sum_{t \in [T]} \left( \max_{\theta \in \mathcal{F}_{1,l}} x_i^\top \theta - x_i^\top \theta^* \right)$$

$$\leq K \sum_{t \in [T]} \left( \max_{\theta \in \mathcal{F}_{1,l}} x_i^\top \theta - x_i^\top \theta^* \right)$$

$$\leq 2K \sum_{t \in [T]} \min \left( \beta_{t,l}^{1/2} \|x_t\|_{\bar{\mathcal{V}}_{t-1,l}}^{-1}, AB \right), \quad \text{(C.23)}$$

where the first equality holds by the definition in (3.3), the first inequality follows from Assumption 6.2, the second inequality holds by Lemma C.8.

For an arbitrary $l \in [L]$, let $\mathcal{I}_1(l) := \left\{ t \in \mathcal{F}_{t+1,l} \big\| \sum_{i \in [I]} x_i \|_{\bar{\mathcal{V}}_{t-1,l}}^1 \leq 1 \right\}$ and $\mathcal{I}_2(l) := \mathcal{F}_{t+1,l} \setminus \mathcal{I}_1(l)$.

$$\sum_{t \in \mathcal{I}_1} \min \left( \beta_{t,l}^{1/2} \|x_t\|_{\bar{\mathcal{V}}_{t-1,l}}^{-1}, AB \right) \leq \beta_{t,l}^{1/2} \sum_{t \in \mathcal{I}_1} \min(1, \|x_t\|_{\bar{\mathcal{V}}_{t-1,l}}^1)$$

$$\leq \sqrt{\beta_{t,l} |\mathcal{F}_{t+1,l}|} \sum_{t \in \mathcal{I}_1} \left[ \min(1, \|x_t\|_{\bar{\mathcal{V}}_{t-1,l}}^1) \right]^2$$

$$\leq \sqrt{2\beta_{t,l} |\mathcal{F}_{t+1,l}| d \log \frac{d \lambda + TA^2}{d \lambda}}, \quad \text{(C.24)}$$

where the second inequality holds by Cauchy-Schwartz inequality, the third inequality follows from Lemma C.2.

Similarly, we calculate

$$\sum_{t \in \mathcal{I}_2} \min \left( \beta_{t,l}^{1/2} \|x_t\|_{\bar{\mathcal{V}}_{t-1,l}}^{-1}, AB \right) \leq AB \sum_{t \in \mathcal{I}_2} \min(1, \|x_t\|_{\bar{\mathcal{V}}_{t-1,l}}^1)$$

29
Substituting (C.24) and (C.25) into (C.23), we obtain

\[
\text{Regret} \leq 2K \sum_{\ell \in [L]} (AB + \sqrt{\beta T, l}) \sqrt{2|\Psi_{T+1,l}|d\log \frac{d\lambda + TA^2}{d\lambda}}
\]

\[
\leq 2K \sqrt{L \sum_{\ell \in [L]} 2(AB + \sqrt{\beta T, l})^2 d|\Psi_{T+1,l}| \log \frac{d\lambda + TA^2}{d\lambda}}
\]

\[
\leq 4K \sqrt{L \left( 1 + \frac{4\sqrt{2AKB}}{\kappa} + \frac{6}{\kappa} R \log(4T^2L/\delta) + \sqrt{\lambda B + AB} \right) \sqrt{d\log \frac{d\lambda + TA^2}{d\lambda}}}
\]

\[
+ 4K \sqrt{L \sum_{\ell \in [L]} 12 \frac{d^2}{\kappa^2 (2l+1)^2} |\Psi_{T+1,l}| \log \frac{d\lambda + TA^2}{d\lambda} \log \left( 1 + \frac{tA\kappa^2}{4dK^2} \right)}
\]

\[
\leq 4K \sqrt{L \left( 1 + \frac{4\sqrt{2AKB}}{\kappa} + \frac{6}{\kappa} R \log(4T^2L/\delta) + \sqrt{\lambda B + AB} \right) \sqrt{d\log \frac{d\lambda + TA^2}{d\lambda}}}
\]

\[
+ 4K \sqrt{L \log \frac{d\lambda + TA^2}{d\lambda} \log \left( 1 + \frac{tA\kappa^2}{4dK^2} \right) \sum_{\ell \in [L]} \sum_{t \in \Psi_{T+1,l}} 12 \frac{d^2}{\kappa^2 (2l+1)^2} (\sigma_t^2 + \sigma^2)}
\]

\[
\leq 4K \sqrt{L \left( 1 + \frac{4\sqrt{2AKB}}{\kappa} + \frac{6}{\kappa} R \log(4T^2L/\delta) + \sqrt{\lambda B + AB} \right) \sqrt{d\log \frac{d\lambda + TA^2}{d\lambda}}}
\]

\[
+ 4K \sqrt{L \log \frac{d\lambda + TA^2}{d\lambda} \log \left( 1 + \frac{tA\kappa^2}{4dK^2} \right) \sum_{\ell \in [T]} \frac{48 d^2}{\kappa^2} (\sigma_t^2 + \sigma^2)}
\]

\[
\leq 4K \sqrt{L \left( 1 + \frac{4\sqrt{2AKB}}{\kappa} + \frac{6}{\kappa} R \log(4T^2L/\delta) + \sqrt{\lambda B + AB} \right) \sqrt{d\log \frac{d\lambda + TA^2}{d\lambda}}}
\]

\[
+ 24 \frac{K}{\kappa} d \sqrt{L \log \frac{d\lambda + TA^2}{d\lambda} \log \left( 1 + \frac{tA\kappa^2}{4dK^2} \right) \sqrt{J + \sigma^2 T}}
\]

where the second inequality holds due to Cauchy-Schwarz inequality, the fourth inequality follows from the definition of \(\beta_{T, l}\), the sixth inequality holds since \(l_t\) in Algorithm 1 satisfies \(2^l \sigma_t \leq \max\{\sigma_t, \sigma\}\), the last inequality follows from the definition of \(J\).
References

Abbasi-Yadkori, Y., Pál, D. and Szepesvári, C. (2011). Improved algorithms for linear stochastic bandits. *Advances in neural information processing systems* 24 2312–2320.

Abbasi-Yadkori, Y., Pál, D. and Szepesvári, C. (2012). Online-to-confidence-set conversions and application to sparse stochastic bandits. In *Artificial Intelligence and Statistics*. PMLR.

Aitken, A. C. (1936). Iv.—on least squares and linear combination of observations. *Proceedings of the Royal Society of Edinburgh* 55 42–48.

Anthony, A., Grover, V. and Szepesvári, C. (2010). Active learning in heteroscedastic noise. *Theor. Comput. Sci.* 411 2712–2728.

Assael, J.-A. M., Wang, Z., Shahriari, B. and de Freitas, N. (2014). Heteroscedastic treed bayesian optimisation. *arXiv preprint arXiv:1410.7172*.

Ayoub, A., Jia, Z., Szepesvári, C., Wang, M. and Yang, L. (2020). Model-based reinforcement learning with value-targeted regression. In *International Conference on Machine Learning*. PMLR.

Bartlett, P. L., Koolen, W. M., Malek, A., Takimoto, E. and Warmuth, M. K. (2015). Minimax fixed-design linear regression. In *Conference on Learning Theory*. PMLR.

Cesa-Bianchi, N., Long, P. M. and Warmuth, M. K. (1996). Worst-case quadratic loss bounds for prediction using linear functions and gradient descent. *IEEE Transactions on Neural Networks* 7 604–619.

Chaudhuri, K., Jain, P. and Natarajan, N. (2017). Active heteroscedastic regression. In *International Conference on Machine Learning*.

Chu, W., Li, L., Reyzin, L. and Schapire, R. (2011). Contextual bandits with linear payoff functions. In *Proceedings of the Fourteenth International Conference on Artificial Intelligence and Statistics*. JMLR Workshop and Conference Proceedings.

Cowan, W., Honda, J. and Katehakis, M. N. (2015). Normal bandits of unknown means and variances: Asymptotic optimality, finite horizon regret bounds, and a solution to an open problem. *arXiv preprint arXiv:1504.05823*.

Dani, V., Hayes, T. P. and Kakade, S. M. (2008). Stochastic linear optimization under bandit feedback .

Filippi, S., Cappe, O., Garivier, A. and Szepesvári, C. (2010). Parametric bandits: The generalized linear case. In *NIPS*, vol. 23.

Freedman, D. A. (1975). On tail probabilities for martingales. *the Annals of Probability* 100–118.

Goldberg, P. W., Williams, C. K. and Bishop, C. M. (1997). Regression with input-dependent noise: A gaussian process treatment. *Advances in neural information processing systems* 10 493–499.

Hazan, E. (2019). Introduction to online convex optimization. *arXiv preprint arXiv:1909.05207*.
Hong, D., Balzano, L. and Fessler, J. A. (2016). Towards a theoretical analysis of pca for heteroscedastic data. In 2016 54th Annual Allerton Conference on Communication, Control, and Computing (Allerton). IEEE.

Hong, D., Fessler, J. A. and Balzano, L. (2018). Optimally weighted pca for high-dimensional heteroscedastic data. arXiv preprint arXiv:1810.12862.

Jun, K.-S., Bhargava, A., Nowak, R. and Willett, R. (2017). Scalable generalized linear bandits: online computation and hashing. In Proceedings of the 31st International Conference on Neural Information Processing Systems.

Kersting, K., Plagemann, C., Pfaff, P. and Burgard, W. (2007). Most likely heteroscedastic gaussian process regression. In Proceedings of the 24th international conference on Machine learning.

Kirschner, J. and Krause, A. (2018). Information directed sampling and bandits with heteroscedastic noise. In Conference On Learning Theory. PMLR.

Kivinen, J. and Warmuth, M. K. (1997). Exponentiated gradient versus gradient descent for linear predictors. Information and computation 132 1–63.

Kveton, B., Zaheer, M., Szepesvari, C., Li, L., Ghavamzadeh, M. and Boutilier, C. (2020). Randomized exploration in generalized linear bandits. In International Conference on Artificial Intelligence and Statistics. PMLR.

Li, L., Chu, W., Langford, J. and Schapire, R. E. (2010). A contextual-bandit approach to personalized news article recommendation. In Proceedings of the 19th international conference on World wide web.

Li, L., Lu, Y. and Zhou, D. (2017). Provably optimal algorithms for generalized linear contextual bandits. In International Conference on Machine Learning. PMLR.

Li, Y., Wang, Y., Chen, X. and Zhou, Y. (2021). Tight regret bounds for infinite-armed linear contextual bandits. In International Conference on Artificial Intelligence and Statistics. PMLR.

Li, Y., Wang, Y. and Zhou, Y. (2019). Nearly minimax-optimal regret for linearly parameterized bandits. In Conference on Learning Theory. PMLR.

Littlestone, N., Long, P. M. and Warmuth, M. K. (1991). On-line learning of linear functions. In Proceedings of the twenty-third annual ACM symposium on Theory of computing.

Mallard, O.-A., Perchet, V. et al. (2021). Stochastic online linear regression: the forward algorithm to replace ridge. Advances in Neural Information Processing Systems 34.

Malek, A. and Bartlett, P. L. (2018). Horizon-independent minimax linear regression. Advances in Neural Information Processing Systems 31 5259–5268.

McInerney, J., Lacker, B., Hansen, S., Higley, K., Bouchard, H., Gruson, A. and Mehrotra, R. (2018). Explore, exploit, and explain: personalizing explainable recommendations with bandits. Proceedings of the 12th ACM Conference on Recommender Systems.
Russo, D. and Van Roy, B. (2013). Eluder dimension and the sample complexity of optimistic exploration. In *Advances in Neural Information Processing Systems*. Citeseer.

Shalev-Shwartz, S. and Singer, Y. (2007). A primal-dual perspective of online learning algorithms. *Machine Learning* **69** 115–142.

Xiao, L. (2010). Dual averaging methods for regularized stochastic learning and online optimization.

Zhang, Z., Yang, J., Ji, X. and Du, S. S. (2021). Variance-aware confidence set: Variance-dependent bound for linear bandits and horizon-free bound for linear mixture mdp. *arXiv preprint arXiv:2101.12745*.

Zhou, D., Gu, Q. and Szepesvari, C. (2021). Nearly minimax optimal reinforcement learning for linear mixture markov decision processes. In *Conference on Learning Theory*. PMLR.