ON \textit{n}-TRANSLATION ALGEBRAS

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Abstract. Motivated by Iyama’s higher representation theory, we introduce \textit{n}-translation quivers and \textit{n}-translation algebras. The classical \(\mathbb{Z}\mathbb{Q}\) construction of the translation quiver is generalized to construct an \((n+1)\)-translation quiver from an \(n\)-translation quiver, using trivial extension and smash product. We prove that the quadratic dual of \(n\)-translation algebras have \((n-1)\)-almost splitting sequences in the category of its projective modules. We also present a non-Koszul \(1\)-translation algebra whose trivial extension is \(2\)-translation algebra, thus also provides a class of examples of \((3, m-1)\)-Koszul algebras (and also a class of \((m-1, 3)\)-Koszul algebras) for all \(m \geq 2\).

1. Introduction

Auslander Reiten quiver plays a very important role in representation theory of algebras\[^{5, 28, 1}\] and it is widely used in the study of Cohen-Macaulay modules, cluster algebras and Calabi-Yau algebras and categories\[^{2, 3, 23, 27, 10}\]. There are many algebras, such as Auslander algebras, preprojective algebras, related to Auslander Reiten quivers\[^{3, 5, 6}\]. One important feature of Auslander Reiten quiver is that it is translation quiver in the sense that there is a translation, related to the almost splitting sequences, equipped on it. Recently, Iyama develops the higher representation theory\[^{19, 20, 21}\], where a class of higher translation quiver also plays an important role.

In\[^{13}\], we introduce translation algebras as algebras with translation quivers as their quivers and the translation corresponds to an operation related to the Nakayama functor. They are algebras close to self-injective algebras with vanishing radical cube. The quadratic dual of Auslander algebras and of preprojective algebras are translation algebras. In this paper we introduce \(n\)-translation quivers and \(n\)-translation algebras as a generalization of translation quivers and translation algebras. Besides the \(n\)-translation quiver, we also need \((p, q)\)-Koszulity which unifying Koszulity in\[^{21}\] and almost Koszulity in\[^{5}\] in the definition of an \(n\)-translation algebra.

Another motivation of this paper is Iyama’s construction of an \((n+1)\)-complete algebra from the cone of an \(n\)-complete algebra, the bound quivers of these algebras are higher translation quivers\[^{21}\]. It is observed that the quiver of an \((n+1)\)-complete algebra is certain truncations of copies of the quiver of the \(n\)-complete algebra connected by new arrows\[^{17}\]. It is well known that the preinjective and preprojective components of the Auslander Reiten quiver is also obtained by connecting copies of the quiver (or opposite quiver) of the original algebra by certain new arrows\[^{28}\]. Iyama’s construction of absolute \(n\)-complete algebras is realized as

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truncations of the McKay quivers of Abelian groups in [16], where the new arrows is obtained by inserting ‘returning arrows’ in the original quiver.

Returning arrows also appear in recent research on cluster algebras [10]. In [17], such returning arrows are obtained by constructing trivial extension for a graded self-injective algebra. Trivial extension is also used to construct \((n + 1)\)-translation quiver from certain \(n\)-translation quiver. For an \(n\)-translation algebra whose the trivial extension is \((n + 1)\)-translation algebra, which we called extendible, we define its \(v\)-extension as the smash product of its trivial extension with \(\mathbb{Z}_v\), for a non-negative integer \(v\). The quiver of a \(v\)-extension of an \(n\)-translation algebra with quiver \(Q\) is \(\mathbb{Z}_v|_n Q\), a construction generalize the stable translation quiver \(\mathbb{Z}Q\) in the classical cases. We proves that in stable Koszul case, the trivial extension of an \(n\)-translation algebra is an \((n + 1)\)-translation algebra.

Analog to what happens in the representation finite path case, we point out that the Auslander Reiten quiver of the cone of Iyama’s absolute 2-complete algebra \(T^2_m(k)\), which is the quadratic dual of a 1-translation algebra, is a truncation of \(\mathbb{Z}_v|_1 Q\). We also discuss connection of our construction with the \(n\)-representation-infinite algebras introduces in [18]. From the viewpoint of algebras, our \(n\)-translation algebras look like a quadratic dual version of algebras in \(n\)-representation theory [19, 20, 21, 22, 18].

Almost Koszul rings are introduced by Brenner, Butler and King in [8]. The almost Koszulity builds a bridge between \(n\)-translation algebras and \(n\)-almost splitting sequences. We show that the category of finitely generated projective modules of the quadratic dual of an \(n\)-translation algebra has \(n\)-almost splitting sequences. We also characterize the \(n\)-almost splitting sequences using \(\tau\)-hammocks of the \(n\)-translation quiver. In fact, we use quadratic duality to establish the correspondence between radical layers of the indecomposable projective module of the \(n\)-translation algebra and terms in the \(n\)-almost splitting sequences starting from the correspond indecomposable projective module of its quadratic dual, which we call a partial Artin-Schelter \(n\)-regular algebra.

One of the main result of [8] is prove that the preprojective algebras and the trivial extensions of a representation finite hereditary algebras are periodic. For the Auslander algebra of a hereditary algebra of type \(A_m\) with linear order, which is one of Iyama’s absolute 2-complete algebra, we prove that the trivial extension of its quadratic dual, which is a 1-translation algebra, is \((3, m - 1)\)-Koszul, hence is also 2-translation algebra. We prove this by developing a method of using cuboid to computer the first \(m - 1\) terms of the minimal projective resolutions of the simples. This give an example of extendible \(n\)-translation algebra which is not Koszul stable, it also serves as a new examples of \((3, m - 1)\)-Koszul algebras (and also \((m - 1, 3)\)-Koszul algebras by taking their quadratic dual) and as a new examples of algebra with periodic projective resolution.

In Section 2, we recall concepts and results on algebras, quivers, and higher representation needed. Notions and results on almost Koszul algebra are recalled and modified in the last part of this section. \(n\)-translation quivers and \(n\)-translation algebras are introduced in Section 3. In Section 4, we prove that the trivial extension of a stable Koszul \(n\)-translation algebra is \((n + 1)\)-translation algebra. The \((n + 1)\)-translation quiver of a \(v\)-extension of an extendible \(n\)-translation algebra is described in Section 5. In Section 6, we introduce partial Artin-Schelter \(n\)-regular algebra and verify the Koszul duality between \(n\)-translation algebras and partial
Artin-Schelter $n$-regular algebras, we also discuss the connection with the algebras in the higher representation in [21, 18] as examples. In the last section, we study an example of non-Koszul extendible 2-translation algebra, by constructing it main part of the minimal projective resolution of simples, and obtain new examples of $(3, m - 1)$-Koszul algebras.

2. Preliminaries

Algebras and quivers. Let $k$ be a field and let $\Lambda$ be a graded algebra, that is, $\Lambda = \Lambda_0 + \Lambda_1 + \cdots$, with $\Lambda_0$ semisimple and $\Lambda$ is generated by $\Lambda_1$ over $\Lambda_0$. We assume that $\Lambda$ is \emph{locally finite} in the sense that $\Lambda_0$ is a direct sum (possibly, infinite copies) of $k$ and for any element $x \in \Lambda$, there is an idempotent $e \in \Lambda$ such that $x \in e\Lambda e$, and $e\Lambda_1 e$ is finite dimensional. If $e\Lambda e$ is finite-dimensional for each idempotent, $\Lambda$ is called \emph{locally finite-dimensional}. Write $J = \Lambda_1 + \Lambda_2 + \cdots$, call it the Jacobson radical of $\Lambda$.

Let $\Lambda_0 = \bigoplus_{i \in Q_0} k_i$, with $k_i \simeq k$ as algebras. Write $\otimes = \otimes_{\Lambda_0}$, the tensor product over $\Lambda_0$, set $M_{\otimes 1} = M$ and $M_{\otimes t+1} = M_{\otimes t} \otimes M$ for a $\Lambda_0$-$\Lambda_0$-bimodule $M$. Let $T_{\Lambda_0}(\Lambda_1) = \bigoplus_{t=0}^{\infty} \Lambda_{\otimes t}$ be the tensor algebra of $\Lambda_1$ over $\Lambda_0$, with multiplication given by the tensor product. Then $\Lambda_1$ are $\Lambda_0$-$\Lambda_0$-bimodules and we have compatible bimodule epimorphisms $\mu_t : \Lambda_{\otimes t} \longrightarrow \Lambda_t$. This induces and epimorphism of algebras $\mu : T_{\Lambda_0}(\Lambda_1) \longrightarrow \Lambda$

with $\text{Ker} \mu \subseteq \bigoplus_{t=2}^{\infty} \Lambda_{\otimes t}$.

Such algebras are presented by quivers with relations. Let $e_i$ be the image of the identity of $k$ under the canonical embedding of the $k_i$ in to $\Lambda_0$. Then $\{e_i | i \in Q_0\}$ is a complete set of orthogonal primitive idempotents in $\Lambda_0$ and

$$\Lambda_1 = \bigoplus_{i,j \in Q_0} e_j \Lambda_1 e_i$$

as $\Lambda_0$-$\Lambda_0$-bimodules. Fix a basis $Q_1^{ij}$ of $e_j \Lambda_1 e_i$ for any pair $i,j \in Q_0$, the elements of $Q_1^{ij}$ are called arrows from $i$ to $j$. Take

$$Q_1 = \bigcup_{(i,j) \in Q_0 \times Q_0} Q_1^{ij}.$$  

Thus we get a quiver $Q$ with vertex set $Q_0$ and arrow set $Q_1$ which is \emph{locally finite} in the sense that there are only finitely many arrows starting or ending at each vertex. The path algebra $kQ$ of $Q$ is defined as usual, and we have that $kQ \simeq T_{\Lambda_0}(\Lambda_1)$.

For an arrow $\alpha : i \rightarrow j$ in the quiver, denote by $s(\alpha) = i$ its starting vertex and by $t(\alpha) = j$ its ending vertex. For a path $p = \alpha_r \cdots \alpha_1$, denote by $s(p) = s(\alpha_1)$ its starting vertex and by $t(p) = t(\alpha_r)$ its ending vertex. For a path $p$, call the number of arrows in $p$ the length of $p$ and write it as $l(p)$.

Thus $\Lambda \simeq kQ/I$ for some ideal $I = \text{Ker} \mu \subseteq \bigoplus_{t \geq 2} \Lambda_{\otimes t}$, the ideal generated by the paths of length larger than or equal to 2 in $kQ$. Let $\rho$ be a set of generators of $I$ and call it a relation set. The elements in $\rho$ are linear combinations of paths of length larger or equal to two in $kQ$. From now on, by a quiver we usually mean a\emph{ bound quiver} $Q = (Q_0, Q_1, \rho)$, that is, a quiver with the vertex set $Q_0$, the arrow set $Q_1$ and the relation set $\rho$. We may assume that $\rho$ is a set of linear combinations of paths of the same length, since $\Lambda$ is graded. $\Lambda$ is called \emph{quadratic} if $I$ is generated by $I \cap (\Lambda_1 \otimes \Lambda_1)$. In this case, $\rho$ can be chosen as a subset of $\Lambda_1 \otimes \Lambda_1$. The grading
of $\Lambda$ is induced by the lengths of the paths, \{\(e_i \mid i \in Q_0\)\} is a complete set of paths of length zero which forms a basis of $\Lambda_0$.

We assume that the algebras are connected in the sense its quiver is connected throughout this paper, when not specialized. We use the same notation $Q$ for both the quiver and the bound quiver, denote by $k(Q) = kQ/(\rho)$ the algebra given by the bound quiver $Q$, that is, the quotient algebra of the path algebra $kQ$ of $Q$ modulo the ideal generated by the relations. A path is called a \textit{bound path} if its image in $k(Q)$ is nonzero.

For a locally finite graded algebra $\Lambda$ over a field $k$, denote by $\text{Mod}\Lambda$ the category of left $\Lambda$-modules, morphisms and extensions in this categories will be denoted by $\text{Hom}_\Lambda$ and $\text{Ext}_\Lambda$. Denote by $\text{Gr}\Lambda$ the category of graded left $\Lambda$-modules, morphisms and extensions in this categories will be denoted by $\text{hom}_\Lambda$ and $\text{ext}_\Lambda$. A module $M$ in $\text{Gr}\Lambda$ has a decomposition $M = \sum_{t \in \mathbb{Z}} M_t$ as a direct sum of vector spaces satisfying $\Lambda_t M_s \subseteq M_{t+s}$ for $t, s \in \mathbb{Z}$. For $M, N$ in $\text{Gr}\Lambda$, $\text{hom}_\Lambda$ is the set of homomorphisms $f$ satisfying $f(M_t) \subseteq N_t$ for each $t \in \mathbb{Z}$. Write $[\ ]$ for the shifting functor, that is $M[t] = M_{t+r}$ for any graded $\Lambda$-module $M$. We have that $\text{Hom}_\Lambda(M, N) = \sum_{t \in \mathbb{Z}} \text{hom}_\Lambda(M, N[t])$ for graded $\Lambda$-module $M$ and $N$. In this case the $\text{Hom}_\Lambda(M, N)$ and $\text{Ext}_\Lambda(M, N)$ are graded naturally.

A graded $\Lambda$-module $M = \sum_t M_t$ is called \textit{locally finitely generated} (resp. \textit{locally finite-dimensional}) if for any idempotent $e \in \Lambda$, the submodule $\Lambda eM$ is finitely generated (resp. finite-dimensional). If $\Lambda$ is locally finitely generated graded algebra and $M, N$ are locally finitely generated, then $\text{Hom}_\Lambda(M, N)$ and $\text{Ext}_\Lambda(M, N)$ are locally finitely generated.

Denote by $\text{gr}\Lambda$ the category of locally finitely generated graded left $\Lambda$-modules. This category contains finitely generated graded $\Lambda$-modules. If $\Lambda$ is locally finite dimensional, $\text{Gr}\Lambda$ is an Abelian category closed under extensions. Left graded $\Lambda$-modules are also left graded $\Lambda_0$-modules. Denote by $\text{mod}\Lambda$ the category of locally finitely generated left $\Lambda$-modules.

Let $M$ be a locally finitely generated $\Lambda$-module. Clearly $M = \sum_{t \in Q_0} e_t M_t$, each $\Lambda e_t M$ is finite generated and we have that

$$M = \bigoplus_{t \in Q_0} \bigoplus_{i \in Q_0} e_i M_t$$

as directed sum of vector spaces and each $e_i M_t$ is finite dimensional. Define the local dual

$$D_* M = \bigoplus_{t \in Q_0} \bigoplus_{i \in Q_0} \text{Hom}_k(e_i M_t, k) \subseteq \text{Hom}_k(M, k) = DM,$$

as vector space. This defined a locally finitely generated right $\Lambda$-module via the action

$$(\sum_{t \in Q_0} f_t) \alpha(\sum_{j \in Q_0} x_j) = \sum_{t \in Q_0} f_t(\alpha x_t)$$

for any $\alpha \in \Lambda$, $f = \sum_{t \in Q_0} f_t \in D_* M$ and $x = \sum_{j \in Q_0} x_j \in M$. In fact, it is a duality between the categories of locally finitely generated left $\Lambda$-modules and the categories of locally finitely generated right $\Lambda$-modules, which restricts to the usual dual functor on the finite dimensional modules.

Write $P(i) = \Lambda e_i$ for the indecomposable projective module corresponding to the vertex $i$. Then $S(i) \simeq P(i)/JP(i)$ is the top of $P(i)$. Let $I(i)$ be the injective envelope of $S(i)$.
For a \( \Lambda \)-module \( M \), write \( P(M) = P^0(M) \) for its projective cover, and
\[
\cdots \rightarrow P^r(M) \xrightarrow{f_r} \cdots \xrightarrow{f_2} P^1(M) \xrightarrow{f_1} P^0(M) \xrightarrow{f_0} M \rightarrow 0
\]
for a minimal projective resolution of \( M \). Recall for a \( t \geq 0 \), the syzygy \( \Omega M \) of \( M \) is defined as \( \Omega M = \text{Im} f_1 \), and \( \Omega^t M = \text{Im} f_t \).

**Higher representation theory.** Iyama recently introduces higher Auslander-Reiten theory, we recall some basic notions \cite{21}. For \( M \in \text{mod} \Lambda \), denote by \( \text{add} M \) the subcategory of \( \text{mod} \Lambda \) consisting of direct summands of finite direct sums of copies of \( M \).

A subcategory \( \mathcal{C} \) of \( \text{mod} \Lambda \) is called **contravariantly finite** if for any \( X \in \text{mod} \Lambda \), there exists a morphism \( f \in \text{Hom}_\Lambda(C, X) \) with \( C \in \mathcal{C} \) such that \( \text{Hom}_\Lambda(\ - , X) \rightarrow 0 \) is exact on \( \mathcal{C} \). Dually a **covariantly finite subcategory** is defined. A contravariantly and covariantly finite subcategory is called **functorially finite**.

Let \( \Lambda \) be a locally finite-dimensional algebra and let \( n \geq 1 \). Let \( \mathcal{C} \) be a subcategory of \( \text{mod} \Lambda \). We call \( \mathcal{C} \) **\( n \)-cluster tilting** if it is functorially finite and
\[
\mathcal{C} = \{ X \in \text{mod} \Lambda \mid \text{Ext}^i_\Lambda(X, C) = 0, 0 < i < n \}
\]
We call an object \( T \in \text{mod} \Lambda \) **\( n \)-cluster tilting** if so is \( \text{add} T \).

Let \( \mathcal{C} \) be a Krull-Schmidt category.

(a) For an object \( M \in \mathcal{C} \), a morphism \( f_0 \in J_\mathcal{C}(M, C_1) \) is called **left almost split** if \( C_1 \in \mathcal{C} \) and
\[
\text{Hom}_\mathcal{C}(C_1, -) \xrightarrow{f_0} J_\mathcal{C}(M, -) \rightarrow 0
\]
is exact on \( \mathcal{C} \). A left minimal and left almost split morphism is called a **source morphism**.

(b) We call a complex
\[
M \xrightarrow{f_0} C_1 \xrightarrow{f_1} C_2 \xrightarrow{f_2} \cdots
\]
(1)
a **source sequence** of \( M \) if the following conditions are satisfied.
(i) \( C_i \in \mathcal{C} \) and \( f_i \in J_\mathcal{C} \) for any \( i \),
(ii) we have the following exact sequence on \( \mathcal{C} \).
\[
\cdots \xrightarrow{f_i} \text{Hom}_\mathcal{C}(C_2, -) \xrightarrow{f_1} \text{Hom}_\mathcal{C}(C_1, -) \xrightarrow{f_0} J_\mathcal{C}(M, -) \rightarrow 0
\]
(2)
A **sink morphism** and a **sink sequence** are defined dually.

(c) We call a complex
\[
0 \rightarrow M \rightarrow C_1 \rightarrow C_2 \rightarrow \cdots \rightarrow C_n \rightarrow N \rightarrow 0
\]
(3)
an **\( n \)almost split sequence** if this is a source sequence of \( M \in \mathcal{C} \) and a sink sequence of \( N \in \mathcal{C} \).

A source sequence (1) corresponds to a minimal projective resolution (2) of a functor \( J_\mathcal{C}(M, -) \) on \( \mathcal{C} \). Thus any indecomposable object \( M \in \mathcal{C} \) has a unique source sequence up to isomorphisms of complexes if it exists.
(p,q)-Koszulity. Let $\Lambda$ be a locally finite graded algebra. Assume that $\{S(i) = \Lambda_0 e_i | i \in Q_0\}$ is a complete set of non-isomorphic graded $\Lambda_0$-modules concentrated at degree 0, which is also a complete set of left graded simple $\Lambda$-modules. In this section, we discuss almost Koszul (or $(p,q)$-)Koszul algebras, following [8] and [7].

With the convention that $\Lambda_\infty = 0$ and $M_\infty = 0$ for any left graded $\Lambda$-module $M$. We have the following version of left almost Koszul ring in [8].

For $p, q \in \mathbb{N} \cup \{\infty\}$, a locally finitely generated graded algebra $\Lambda = k(Q)$ is called left almost Koszul of type $(p,q)$ if it satisfies the following conditions:

1. $\Lambda_t = 0$ for $t > p$, and for each $i \in Q_0$.
2. there is a graded complex

$$P^*(i) : 0 \longrightarrow P^t(i) \longrightarrow \cdots \longrightarrow P^1(i) \longrightarrow P^0(i) \longrightarrow 0$$

of projective left $\Lambda$-modules such that each $P^t(i)$ is generated by its component of degree $t$ with non-zero homology $\Lambda_0 e_i$ in degree 0, the possible only another non-zero homology in degree $p + q$, and in this case, it is $W(i) = \Lambda_0 \otimes P^q_0(i)$.

A simple module $S(i)$ satisfying (2) will be called a $(p,q)$-Koszul simple module. Right almost Koszul algebra is defined similarly using right modules. In this paper, such algebra is abbreviated as left (right) $(p,q)$-Koszul algebra.

Obviously, a Koszul algebra of Loewy length $p$ is a $(p,q)$-Koszul for $q > \text{gl.dim} \Lambda$, and for $p$ larger than the Loewy lengths of all the indecomposable projective models. A $(p,q)$-Koszul algebra $\Lambda$ is Koszul if one of $p, q$ is $\infty$.

Similar to [8], we also have the following results on the left (right) $(p,q)$-Koszulity for locally finitely generated graded algebras, and we omit the proofs since they can be easily done by modifying those in [8].

**Proposition 2.1.** Assume $q \geq 1$, then $\Lambda$ is a left $(p,q)$-Koszul if and only if

1. $\text{Ext}_\Lambda^t(S(i), S(j))_s = 0$ for $i,j \in Q_0$ and $s \neq t \leq q$, and
2. $\text{Ext}_\Lambda^{q+1}(S(i), S(j))_s = 0$ for $s \neq p + q$.

**Proposition 2.2.** A left $(p,q)$-Koszul algebra is also a right $(p,q)$-Koszul algebra.

Thus we do not need to distinguish left and right for $(p,q)$-Koszul algebra, we just call it a $(p,q)$-Koszul algebra.

**Proposition 2.3.** A $(p,q)$-Koszul algebra $\Lambda$ is generated by $\Lambda_0$ and $\Lambda_1$, and, if $q \geq 2$, then $\Lambda$ is quadratic.

Now we consider quadratic algebra $\Lambda$. Let $S = \Lambda_0$ and let $V = \Lambda_1 = \bigoplus_{i,j} e_j V e_i$ be the $S$-$S$-bimodule, with the arrows from $i$ to $j$ in $Q_1$ as a basis of $e_j V e_i$. Let $R$ be the $S$-$S$-sub-bimodule of $V \otimes V$ generated by the elements in $\rho$.

Recall that for two graded $S$-$S$-modules $M = \bigoplus M_t$ and $N = \bigoplus N_t$, the gradation of the tensor product is defined by $(M \otimes N)_t = \bigoplus_{t+e = t} M_e \otimes N_{e'}$. As in [8, 7], we define $S$-$S$-sub-bimodules $K^t$ of $V^{\otimes t}$ by the formulae $K^0 = S$, $K^1 = V$, $K^2 = R$ and $K^{t+1} = V^{K^t} \cap K^t V$ for all $t \geq 2$. $K^t$ is a $S$-$S$-bimodule concentrated in degree $t$. The left Koszul complex of $\Lambda$ is

$$\cdots \longrightarrow \Lambda \otimes K^t \longrightarrow \cdots \longrightarrow \Lambda \otimes K^1 \longrightarrow \Lambda \otimes K^0 \longrightarrow 0.$$  

The differential is the restriction of the map $\Lambda \otimes V^{\otimes t} \longrightarrow \Lambda \otimes V^{\otimes t-1}$,

$$a \otimes v_1 \otimes \cdots \otimes v_t \longrightarrow \alpha v_1 \otimes \cdots \otimes v_t,$$

where $\alpha$ is a suitable element.
which is the composite of the following sequence of natural maps
\[ \Lambda \otimes K^t \hookrightarrow \Lambda \otimes V K^{t-1} \longrightarrow \Lambda \Lambda_1 \otimes V K^{t-1} \hookrightarrow \Lambda \otimes K^{t-1}. \]

Similar to Theorem 2.6.1 in [7], \( \Lambda \) is a Koszul if and only if its left Koszul complex is a projective resolution of \( \Lambda_0 \). For each \( i \in Q_0 \), (4) restrict to a left Koszul complex for \( \Lambda_0 e_i \):

\[ \cdot \longrightarrow \Lambda \otimes K^t e_i \longrightarrow \cdot \longrightarrow \Lambda \otimes K^i e_i \longrightarrow \Lambda \otimes K^0 e_i \longrightarrow 0. \]

We can also decompose each term as a direct sum of indecomposable projective modules: \( \Lambda \otimes K^t e_i = \bigoplus_{j \in Q_0} \Lambda e_j \otimes e_j K^t e_i = \bigoplus_{j \in Q_0} \Lambda e_j \otimes e_j D_s(\Lambda_j)e_i \). The following proposition is an adaption of Proposition 3.9 of [5].

**Proposition 2.4.** Assume \( p, q \geq 2 \), let \( \Lambda \) be a quadratic algebra. Then \( \Lambda \) is \( (p,q) \)-Koszul if and only if

1. \( \Lambda_t = 0 \) for all \( t > q \),
2. \( K^s = 0 \) for all \( s > p \), and
3. for each \( i \in Q_0 \), the only non-zero homology modules of \( \Lambda \otimes K^i e_i \) in degree 0, and 0 or \( \Lambda_p \otimes K^q e_i \) in degree \( p + q \).

In this case the left Koszul complex for \( \Lambda \) provides the first \( q + 1 \) terms of the minimal projective resolution of \( \Lambda_0 \). There is another description of the left and right Koszul complexes associated with the quadratic algebra \( \Lambda \) in 2.8 of [7], which makes use of the quadratic dual of \( \Lambda \).

There is a decomposition of \( V = \bigoplus_{i,j} e_i V e_j \) as \( S \)-bimodule, and we have that \( D e_j V e_i \simeq e_i V e_j \) as \( S \)-bimodule. Thus \( D_s V \) defines the opposite quiver \( Q^{op} \) of \( Q \), with \( Q_0^{op} = Q_0 \) and \( Q_1^{op} = \cup_{(i,j) \in Q_0 \times Q_0} Q_1^{ij} \), where \( \cup \{ \alpha^* \alpha \in Q_1^i \} \) whose elements form a dual basis of the arrows in \( Q_1^i \). Note that \((D_s V \otimes \cdots \otimes D_s V)\) is identified with \( D_s(V \otimes \cdots \otimes V) \), via

\[ f_t \otimes \cdots \otimes f_1(x_1 \otimes \cdots \otimes x_t) = f_t(\cdots f_2(f_1(x_1)x_2) \cdots x_t) = f_1(x_1)f_2(x_2) \cdots f_t(x_t), \]

for \( f_1, \ldots, f_t \in D_s V \) and \( x_1, \ldots, x_t \in V \). The quadratic dual algebra of \( \Lambda \), denoted \( \Lambda' \), is defined by the formula

\[ \Lambda' = T_S(D_s V)/(R^1), \]

where \( R^1 = \{ f \in D_s(V^2) | f(x) = 0, \text{ for any x \in R} \} \subset D_s(V^2) = (D_s V)^2 \) is the annihilator of \( R \subset V^2 \). The following proposition follows directly from the definition.

**Proposition 2.5.** If \( \Lambda \) is locally finite, so is \( \Lambda' \) and

\[ (\Lambda')^1 = \Lambda. \]

Thus the bound quiver of \( \Lambda' \) is the opposite quiver \( Q'^{op} = (Q_0, Q_1^{op}, \rho^*) \) of the bound quiver \( Q \) of \( \Lambda \) with the dual basis \( \rho^* \) of a chosen base \( \rho \) of \( R \) in \((D_s V)^2 \) as the relations. Its connection with the Koszul complexes of \( \Lambda \) is made via the formulae

\[ D_s(\Lambda'_1) = K^t. \]

Let \( \mu : \Lambda'_{i-1} \otimes \Lambda'_1 \longrightarrow \Lambda'_1 \) be the multiplication map, \( \partial \) be its dual map, then

\[ \partial : D_s \Lambda'_1 \longrightarrow D_s(\Lambda'_{i-1} \otimes \Lambda'_1) = D_s \Lambda'_1 \otimes D_s \Lambda'_{i-1} = V \otimes K^{t-1}, \]

by identifying \( D_s \Lambda'_1 \) with \( \Lambda_1 = V \).
Thus the left Koszul complex of $\Lambda$ is isomorphic to

$$\cdots \rightarrow \Lambda \otimes D_* (\Lambda'_0^{t+1}) \rightarrow \Lambda \otimes D_* (\Lambda'_1^{t}) \rightarrow \Lambda \otimes D_* (\Lambda'_0^{t}) = \Lambda \rightarrow 0$$

where the differential $\partial : \Lambda \otimes D_* (\Lambda'_1^{t}) \rightarrow \Lambda \otimes D_* (\Lambda'_0^{t})$ is induced by the dual of the multiplication map $\Lambda'_1^{t+1} \otimes \Lambda'_1^{t} \rightarrow \Lambda'_1^{t}$, and the identification of $D_* (\Lambda'_1^{t})$ with $\Lambda'_1$. Note that $\Lambda \otimes D_* (\Lambda'_1^{t})$ is a right $\Lambda \otimes \Lambda^{op}$-module with action defined by $(x \otimes y)(a \otimes a') = xa \otimes a'y$. Write $u = \sum_{\alpha \in Q_1} \alpha \otimes \alpha^*$ formally, write $(e \otimes e)u = \sum_{\alpha \in Q_1, e\alpha \neq 0} \alpha \otimes \alpha^* \in \Lambda \otimes \Lambda$ for any idempotent $e$ in $A_0 = D_0 A_0$; $(e \otimes e)u \in \Lambda \otimes D_* (\Lambda'_1^{t})$. Define $(\sum x_i \otimes y_i)u = (\sum x_i \otimes y_i)((e \otimes e)u)$ for $x_i \in eA_c$. Then $(\Lambda \otimes D_* (\Lambda'_1^{t}))u \subset \Lambda \otimes D_* (\Lambda'_0^{t})$ and right multiplication by $u$ defines exactly the differential of the Koszul complex.

If $\Lambda$ is $(p,q)$-Koszul, we have that $\Lambda'_1^{t} = 0$ for $t > q$. The only non-zero homology of the above complex is $Se_i$ in degree 0 and $\bigoplus_{j \in Q_0} \Lambda_p \otimes D_* (\Lambda'_1^{j}) e_i$ in degree $p + q$.

The above complex decomposes into $p + q + 1$ homogeneous sub-complexes as (see [5])

$$\begin{eqnarray*}
\Lambda_0 \otimes D_* (\Lambda'_0^{p})e_i & \cdots & \Lambda_0 \otimes D_* (\Lambda'_2^{p})e_i & \cdots & \Lambda_0 \otimes D_* (\Lambda'_1^{p})e_i & \cdots & \Lambda_0 \otimes D_* (\Lambda'_0^{p})e_i = Se_i \\
\Lambda_1 \otimes D_* (\Lambda'_0^{p})e_i & \cdots & \Lambda_1 \otimes D_* (\Lambda'_2^{p})e_i & \cdots & \Lambda_1 \otimes D_* (\Lambda'_1^{p})e_i & \cdots & \Lambda_1 \otimes D_* (\Lambda'_0^{p})e_i \\
\Lambda_2 \otimes D_* (\Lambda'_0^{p})e_i & \cdots & \Lambda_2 \otimes D_* (\Lambda'_2^{p})e_i & \cdots & \Lambda_2 \otimes D_* (\Lambda'_1^{p})e_i & \cdots & \Lambda_2 \otimes D_* (\Lambda'_0^{p})e_i \\
\vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots \\
\Lambda_q \otimes D_* (\Lambda'_0^{p})e_i & \cdots & \Lambda_q \otimes D_* (\Lambda'_2^{p})e_i & \cdots & \Lambda_q \otimes D_* (\Lambda'_1^{p})e_i & \cdots & \Lambda_q \otimes D_* (\Lambda'_0^{p})e_i.
\end{eqnarray*}$$

The total space of this complex can be viewed as a bigraded $\Lambda_1^{\Lambda}$-bimodule with a differential of bidegree $(1,-1)$ which has non-zero homology only in bidegrees $(0,0)$ and $(p,q)$. On applying to it the exact functor $\bigoplus_{i \in Q_0} \text{Hom}_S (\text{Hom}_S (e_i S, \, \_), S)$, we obtain a bigraded $\Lambda_1^{\Lambda}$-$\Lambda_1$-bimodule, with a differential of bidegree $(1,-1)$ which has non-zero homology only in bidegrees $(0,0)$ and $(q,p)$, and which is the total space of the complex

$$0 \rightarrow \Lambda_1^{\Lambda} \otimes D_* \Lambda_p e_i \rightarrow \cdots \rightarrow \Lambda_1^{\Lambda} \otimes D_* \Lambda_1 e_i \rightarrow \Lambda_1^{\Lambda} \otimes D_* \Lambda_0 e_i \rightarrow 0.$$

This complex has non-zero homology only in degree 0, where it is $\Lambda_1^{\Lambda} e_i$, and possibly in degree $p + q$, which is $\Lambda_1^{\Lambda} \otimes D_* \Lambda_p e_i$. Together with the fact that $\Lambda_1^{\Lambda} = 0$ for all $t > q$, this proves the following result from [3].

**Proposition 2.6.** If $\Lambda$ is a $(p,q)$-Koszul algebra with $p,q \geq 2$, then $\Lambda_1^{\Lambda}$ is a $(q,p)$-Koszul algebra.

The facts that the left Koszul complex provides the first $q + 1$ terms of the minimal projective resolution of $S$ as a left $\Lambda$-module and the above isomorphism of complexes mean that

$$\Lambda_1^{\Lambda} \cong \bigoplus_{i,j \in Q_0} \text{Ext}^1_\Lambda (Se_i, Se_j),$$

for $0 \leq t \leq q$. Write

$$\mathcal{E}(\Lambda) = \bigoplus_{t=0}^{q} \bigoplus_{i,j \in Q_0} \text{Ext}^1_\Lambda (Se_i, Se_j),$$

Then we have that $\mathcal{E}(\Lambda) = \Lambda^{op}$, we also call $\mathcal{E}(\Lambda)$ quadratic dual of $\Lambda$. 
Recall that the Yoneda-Ext algebra $E(\Lambda)$ is defined as the vector space

$$E(\Lambda) = \bigoplus_{t=0}^{\infty} \bigoplus_{i,j \in Q_0} \text{Ext}^t_{\Lambda}(S(i), S(j)),$$

with the multiplication defined by the Yodeda product. It follows from [8] that the following hold:

**Proposition 2.7.** If $\Lambda$ is a left $(p, q)$-Koszul algebra with $p > 1$, then

$$E(E(\Lambda)) = \Lambda$$

and $E(\Lambda)$ is a subalgebra of $E(\Lambda)$ generated by $E_0(\Lambda)$ and $E_1(\Lambda)$.

Furthermore, $E(\Lambda)$ is generated by $E(\Lambda)$ and $\bigoplus_{i,j \in Q_0} \text{Ext}^{t+1}_{\Lambda}(S_i, S_j)$.

Let $\Lambda$ be a $(p, q)$-Koszul algebra. Define

$$E(M) = \bigoplus_{t=0}^{q} \bigoplus_{i,j \in Q_0} \text{Ext}^t_{\Lambda}(M, S_i),$$

for $M \in \text{gr} \Lambda$.

Thus for each $i$,

$$E(\Lambda)_t e_i = \bigoplus_{j \in Q_0} \text{Ext}^t_{\Lambda}(S_i, S_j),$$

for $0 \leq t \leq q + 1$.

The following theorem follows easily.

**Theorem 2.8.** Let $\Lambda$ be a $(p, q)$-Koszul algebra. Then

1. if $S$ is a simple $\Lambda$-module, then $E(S)$ is an indecomposable projective $E(\Lambda)$-module and

$$E_E(\Lambda)(E(\Lambda)(S)) \simeq S,$$

2. if $P$ is an indecomposable projective $\Lambda$-module, then $E(P)$ is a simple $E(\Lambda)$-module and

$$E_E(\Lambda)(E(\Lambda)(P)) \simeq P.$$

In fact, if $S = \Lambda_0 e_i$, then $E(S) = E(\Lambda)_t e_i$, if $P = \Lambda e_i$ then $E(P) = E(\Lambda)_0 e_i$.

Let $P^\bullet(M)$ be a minimal projective resolution of the module $M$, which is a complex of projective modules with only nonzero homology $M$ in degree 0. Denote by $M^a$ the direct sum of $a$ copies of the module $M$. Then we have the following proposition (see also [11]).

**Proposition 2.9.** Let $\Lambda$ be a $(p, q)$-Koszul algebra. Then

1. Let $S$ be a simple $\Lambda$-module with $P^t(S) = \bigoplus_{i \in Q_0} (\Lambda e_i)^{a_{t,i}}$, then

$$\text{rad}^t E(\Lambda)(S) / \text{rad}^{t+1} E(\Lambda)(S) \simeq \bigoplus_{i \in Q_0} (E(\Lambda)_0 e_i)^{a_{t,i}},$$

for $0 \leq t \leq q$.

2. Let $P$ be an indecomposable projective $\Lambda$-module with $\text{rad}^t P / \text{rad}^{t+1} P \simeq \bigoplus_{i \in Q_0} (\Lambda_0 e_i)^{a_{t,i}}$, then

$$P^t(E(P)) = \bigoplus_{i \in Q_0} (E(\Lambda)_0 e_i)^{a_{t,i}},$$

for $0 \leq t \leq p$. 
3. \textit{n}-\textit{TRANSLATION QUIVERS AND n-TRANSLATION ALGEBRAS}

Let \( n \) be a non-negative integer, and let \( Q = (Q_0, Q_1, \rho) \) be a bound quiver with \textit{homogeneous relations}, that is, elements in \( \rho \) are linear combinations of paths of the same length. Assume that a bijective map \( \tau : Q_0 \setminus \mathcal{P} \longrightarrow Q_0 \setminus \mathcal{I} \) is defined for two subsets \( \mathcal{P}, \mathcal{I} \subset Q_0 \). \( Q \) is called an \textit{n-translation quiver} if it satisfies the following conditions:

(1) Any maximal bound path is of length \( n + 1 \) from \( \tau v \) in \( Q_0 \setminus \mathcal{P} \) to \( v \), for some vertex \( v \) in \( Q_0 \setminus \mathcal{I} \).
(2) Two bound paths of length \( n + 1 \) from \( \tau v \) to \( v \) are linearly dependent, for any \( v \in Q_0 \setminus \mathcal{P} \).

In this case \( \tau \) is called the \textit{n-translation} of \( Q \), we also write it as \( \tau_{[n]} \) when we need to specify \( n \). The vertices in \( \mathcal{P} \) are called \textit{projective vertices} and vertices in \( \mathcal{I} \) are called \textit{injective vertices}.

\textbf{Example 3.1.} (1) A \( 0 \)-translation quiver is a bound quiver whose connected component is either a subquiver of the quiver of type \( A_{\infty} \), or the quiver of type \( \tilde{A}_l \), with linear orientation and all the paths of length 2 as relations.

(2) Bound quiver with each bound path of length \( \leq n \) is an \( n \)-translation quiver with \( \mathcal{P} = \mathcal{I} = Q_0 \), we call such \( n \)-translation quiver \textit{null translation quiver}.

An \( m \)-translation quiver for \( m < n \) is a \( n \)-translation quiver.

(3) \( 1 \)-translation quiver is the usual translation quiver.

\textbf{Proposition 3.2.} Let \( Q \) be a finite \( n \)-translation quiver, then \( \mathcal{P} = \emptyset \) if and only if \( \mathcal{I} = \emptyset \).

In this case, the \( n \)-translation quiver \( Q \) is just a \textit{stable bound quiver} of Loewy length \( n + 2 \) defined in \([15]\). We call a \( n \)-translation quiver a \textit{stable \( n \)-translation quiver} if the \( n \)-translation is defined everywhere, or equivalently \( \mathcal{P} = \emptyset = \mathcal{I} \).

A graded algebra \( \Lambda \) is called an \textit{n-pretranslation algebra} if its bound quiver is an \( n \)-translation quiver. \( 0 \)-pretranslation algebras are uniserial algebras with radical squared zero. \( 1 \)-pretranslation algebras are the translation algebras defined in \([13]\).

A stable bound quiver of Loewy length \( n + 2 \) is an \( n \)-translation quiver and a graded self-injective algebra of Loewy length \( n + 2 \) is a \( n \)-pretranslation algebra. We have the following proposition (see \([15]\)).

\textbf{Proposition 3.3.} An \( n \)-pretranslation algebra \( \Lambda \) is self-injective if and only if its quiver is stable.

An \( n \)-pretranslation algebra \( \Lambda \) is called an \textit{n-translation algebra} if there is an \( l \in \mathbb{N} \cup \{\infty\} \) such that \( \Lambda \) is \((n + 1, l)\)-Koszul. We call \( l \) the \textit{generalized Coxeter number} of \( \Lambda \), as is indicated in the next proposition. The bound quiver of an \( n \)-translation algebra is called a \textit{proper \( n \)-translation quiver}.

Since \( 0 \)-pretranslation algebras are Koszul, they are \( 0 \)-translation algebras.

\( 1 \)-pretranslation algebras are algebras with vanishing radical cube and so they are also \( 1 \)-translation algebra. The self-injective \( 1 \)-translation algebras are just self-injective algebras with vanishing radical cube. Their relationship with Koszulity is described in \([24]\), see also corollary 4.3 of \([8]\), we rewrite these properties in the language of \( 1 \)-translation algebra as follows.

\textbf{Proposition 3.4.} Let \( \Lambda \) be a connected stable \( 1 \)-translation algebra, and let \( l \) be its generalized Coxeter number. Then either
(1) \( \Lambda \) is representation infinite and \( l = \infty \), or
(2) \( \Lambda \) is representation finite and \( l = h - 1 \), where \( h \) is the Coxeter number of its Dynkin diagram.

4. Trivial extensions of \( n \)-translation algebras

Recall that the trivial extension \( \Lambda \ltimes M \) of an algebra \( \Lambda \) by a \( \Lambda \)-bimodule \( M \) is the algebra defined on the vector spaces \( \Lambda \oplus M \) with the multiplication defined by

\[
(a, x)(b, y) = (ab, ay + xb)
\]

for \( a, b \in \Lambda \) and \( x, y \in M \). The trivial extension \( \tilde{\Lambda} = \Lambda \ltimes D_\Lambda \) of the \( \Lambda \) by the bimodule \( D_\Lambda \) is called the trivial extension of \( \Lambda \).

Let \( Q \) be an \( n \)-translation quiver with the translation \( \tau \). For each \( i \in Q_0 \setminus P \), fix a path \( p_i \) of length \( n + 1 \) from \( \tau i \) to \( i \). Let \( \Lambda = \Lambda_Q \) be the \( n \)-pretranslation algebra defined by \( Q \). Then we have the following result.

**Lemma 4.1.** Write \( e_j kQ_1 e_i \) for the space spanned by the arrows from \( i \) to \( j \). Then for any \( i, j \in Q_0 \setminus P \), there is an isomorphism of vector spaces \( \tau(i, j) : e_j kQ_1 e_i \to e_j kQ_1 e_{\tau i} \) such that for any path \( p \) of length \( n \), and arrow \( \alpha \in e_j kQ_1 e_i \), we have in \( \Lambda \)

\[
\alpha p = p_j \quad \text{and} \quad p \tau(i, j)(\alpha) = p_i.
\]

**Proof.** Identify \( kQ_0 + kQ_1 \) with its image in \( \Lambda \). Since paths of length \( n + 1 \) from \( \tau j \) to \( j \) are linear dependent in \( \Lambda \), for each pair \( i, j \in Q_0 \setminus P \), let \( \alpha_1, \cdots, \alpha_s \) be the \( \alpha \) such that \( t \in \tau j \) to \( i \) such that for \( 1 \leq t \leq s \), \( \alpha_t q(t) = p_j \) and \( \alpha_t q(t') = 0 \) for \( t \neq t' \) in \( \Lambda \). Similarly, since \( q(1), \cdots, q(s) \) are linear combinations of paths of length \( n \) from \( \tau j \) to \( i \) and they are linearly independent, there are uniquely determined linear combinations \( z_1, \cdots, z_s \) of arrows from \( \tau i \) to \( j \), such that for \( 1 \leq t \leq s \), \( q(t)z_t = p_i \) and \( q(t')z_t = 0 \) for \( t \neq t' \) in \( \Lambda \). Define \( \tau(i, j)\alpha_t = z_t \) for \( t = 1, \cdots, s \), it is easy to see that \( \tau(i, j) \) is the desired map. \( \square \)

Clearly, \( \tau(i, j) \) is in fact an isomorphism of the vector spaces, for any pair \( i, j \in Q_0 \setminus P \). Set \( \tau(i, j) = 0 \) for \( (i, j) \notin (Q_0 \setminus P) \times (Q_0 \setminus P) \), and let

\[
\tau = \bigoplus_{i,j\in Q_0} \tau(i, j) : \bigoplus_{i,j\in Q_0} e_j kQ_1 e_i \to \bigoplus_{i,j\in Q_0} e_j kQ_1 e_i.
\]

This extends \( \tau \) to \( kQ_1 \).

An \( n \)-translation quiver \( Q \) is called admissible if the following are satisfied:

1. For each bound path \( p \), either there is a path \( q \) such that \( pq \) is bound path of length \( n + 1 \), or there is a path \( q' \) such that \( q'p \) is bound path of length \( n + 1 \);
2. For any \( i, j \in Q_0 \setminus P \), if \( p \) is a bound path from \( i \) to \( \tau j \), then all the vertices of \( p \) are non-projective.
3. If \( i \) is a projective vertex, \( j \) is not projective, and there is an arrow from \( i \) to \( j \), then there is no injective vertex \( j' \) on any bound path from \( \tau j \) to \( i \).

Clearly, a stable \( n \)-translation quiver is admissible.

The following proposition is a generalization of Theorem 2.4 of [17].

**Proposition 4.2.** Let \( \Lambda \) be a \( n \)-pretranslation algebra given by the admissible \( n \)-translation quiver \( Q \) with translation \( \tau \). Let \( \Lambda \) be its trivial extension and \( \tilde{\Lambda} = (Q_0, \tilde{Q}, \tilde{\rho}) \) be its bound quiver. Then
(1) \( \hat{Q}_0 = Q_0 \) and \( \hat{Q}_1 = Q_1 \cup \{ \beta_i : i \to \tau i | i \in Q_0 \setminus \mathcal{P} \} \)

(2) \( \hat{p} = p \cup \{ \beta_i, \beta_i, i \in Q_0 \setminus \mathcal{P} \} \cup \{ \tau (\alpha) \beta_i - \beta_j \alpha : i \to j \in Q_1, i, j \in Q_0 \setminus \mathcal{P} \} \)

(3) \( \Lambda \) is a stable \((n+1)\)-pretranslation algebra with trivial translation.

Proof. Note that \((D_\Lambda, \Lambda)^2 = 0\) in \( \Lambda \), thus it is a nilpotent ideal of \( \Lambda \) and is contained in the radical of \( \Lambda \). Take a basis \( B \) of \( \Lambda \) consisting of paths and containing \( p_i \) for all \( i \in Q_0 \setminus \mathcal{P} \). Let \( B^* \) be the dual basis of \( D_\Lambda, \Lambda \), and let \( p_i^* \) be the basic element dual to the path \( p_i \) of maximal length from \( \tau i \) to \( i \). For any \( i, j \in Q_0 \) and \( 0 \leq t \leq n+1 \), let \( q_1, \ldots, q_r \) be the bound paths of length \( t \) from \( j \) to \( i \) in \( B \), either there are linear combinations \( q'_1, \ldots, q'_r \) of paths from \( \tau i \) to \( j \) such that \( q'_i q_t = p_i \) and \( q'_i q_t = 0 \) for \( t' \neq t \), or there are linear combinations \( q''_1, \ldots, q''_r \) of paths from \( i \) to \( \tau^{-1} j \) such that \( q''_i q_t = p_{r-i} \) and \( q''_i q_t = 0 \) for \( t' \neq t \). Thus \( q'_i = \alpha q''_i p_i^* \) in the first case and \( q''_i = p_{i-r} q'_i \) in the second, and hence \( \{ q'_i p_{i-r} q''_i | t(q''_i) \in Q_0 \setminus \mathcal{P}, l(q''_i) \leq n+1 \} \) spans \( D_\Lambda \) as \( k \)-space. Especially, \( D_\Lambda \) is generated by \( \{ p_i^* | i \in Q_0 \setminus \mathcal{P} \} \) as a \( \Lambda \)-bimodule.

So \( \Lambda \) is generated by \( \Lambda_1 \) and \( \{ p_i^* | i \in Q_0 \setminus \mathcal{P} \} \) over \( \Lambda_0 = \Lambda_0 \). Clearly, we have that \( e_j p_i^* e_i = 0 \) if and only if \( i \not\sim i \) and \( j' = \tau i \), and it is linearly independent of the arrows in \( Q \), so it can be regarded as an arrows from \( i \) to \( \tau i \). Write \( \beta_i \) for the arrow represented by \( p_i^* \) and regard it as an element of degree 1. This proves that the quiver of \( \Lambda \) has the same set of vertices as \( Q \) and has the arrow set \( Q_1 = Q_1 \cup \{ \beta_i : i \to \tau i | i \in Q_0 \} \).

We have \((D_\Lambda, \Lambda)^2 = 0\), so \( p_i^* p_j^* = 0 \), and by Lemma 1, we have that \( \tau (\alpha) p_i^* - p_j^* \alpha \) for any arrow \( \alpha : i \to j \) with \( i,j \in Q_0 \setminus \mathcal{P} \). Thus we have an epimorphism \( \psi \) from \( k\hat{Q}/\langle \hat{p} \rangle \) to \( \Lambda \) sending \( k\hat{Q}/\langle \hat{p} \rangle \) to \( \Lambda \) and \( \beta_i \) to \( p_i^* \).

For each path \( p \in B \) with \( t(p) = j \) is not projective, there is a linear combination \( p' \) of paths such that \( pp' = p_j \), and \( qp' = 0 \) for any \( p \neq q \in B \), and \( p' p_j^* = p^* \). Fix \( p' \) for such \( p \). If \( t(p) \) is projective, then by admissibility, \( i = s(p) \) is not injective, there is a linear combination \( p'' \) of paths such that \( p'' p = p_i \) and \( p'' q = 0 \) for any \( p \neq q \in B \), and \( p_i^* p'' = p^* \). Fix \( p'' \) for such \( p \). It is easy to see that for \( i, j \in Q_0 \) and \( 0 \leq h \leq n+1 \), the set \( \{ p' | p \in B \cap e_j \Lambda_n e_i \} \) is a basis of \( e_i \Lambda_{n+1-h} e_j \) if \( j \) is not projective, and the set \( \{ p'' | p \in B \cap e_j \Lambda_n e_i \} \) is a basis of \( e_i \Lambda_{n+1-h} e_j \) if \( i \) is not injective.

Now let

\[
C = \{ p' \beta_i(p) | p \in B, t(p) = s(p') \in Q_0 \setminus \mathcal{P}, l(p') \leq n+1 \} \cup \{ \beta_i s(p') p'' | p \in B, t(p) = s(p'') \in Q_0 \setminus \mathcal{P}, l(p'') \leq n+1 \}
\]

If \( i, j \not\in \mathcal{P} \) and there is a bound path \( p \in \Lambda \) from \( i \) to \( \tau j \), then \( \beta_i p \beta_j = 0 \). Since all the vertices of \( p \) are non-projective and we may reduce the length of \( p \) by applying the relations of the form \( \tau (\alpha) \beta_j - \beta_j \alpha \) if \( p \) is not trivial. When \( p \) is reduced to a trivial path and then apply the relations in of the form \( \beta_i \beta_i \) to get that \( \beta_i p \beta_j = 0 \) in \( k\hat{Q}/\langle \hat{p} \rangle \).

Thus each element in \( k\hat{Q}/\langle \hat{p} \rangle \) can be written as a linear combination of the form \( p \beta_i q \) with \( p \) a bound path starting from \( \tau i \) and \( q \) a bound path ending at \( i \). If there is no projective vertex in \( q \), then \( p \beta_i q \) is written as a linear combination of the paths of the form \( p' \beta_j \), due to the relations of the form \( \tau (\alpha) \beta_j - \beta_j \alpha \) for each arrow \( \alpha \) with non-projective \( i' \), \( j' \). If there is a projective vertex \( i_0 \) in \( q \), then by the admissibility, all the vertices in \( p \) are non-injective. It follows from the relations of the form \( \tau (\alpha) \beta_j - \beta_j \alpha \) for each arrow \( \alpha \) with non-projective \( s(\alpha) = i', t(\alpha) = j' \), we get that \( \alpha' \beta_j - \beta_j \alpha \) for each arrow \( \alpha' \) with non-injective vertices \( s(\alpha') =
This shows that $k\bar{Q}/(\bar{\rho})$ is spanned by $B$ and $C$ and thus is isomorphic to $\bar{\Lambda}$.

Set $\{p_i^*|i \in Q_0 \setminus P\}$ as elements of degree 1, then $\Lambda_1 = \Lambda_0 + \sum_{i \in Q_0} k p_i^*$, and we have that $\Lambda = \Lambda_0 + \Lambda_1 + \cdots + \Lambda_{n+2}$, with $\Lambda_i = \Lambda_i^0$ for $1 \leq i \leq n+2$.

It is clearly that each maximal path in $k\bar{Q}/(\bar{\rho})$ contains an arrow $\beta_j$ for some $j \in Q_0$, has length $n+2$ and has the same vertex as its starting and ending vertex. Thus $\Lambda \simeq k\bar{Q}/(\bar{\rho})$ is a stable $(n+1)$-pretranslation algebra with trivial translation.

If $\Lambda$ is a Koszul self-injective algebra, then all its projective modules have the same Loewy length, say $n+2$, in this case $\Lambda$ is a stable $n$-translation algebra with an admissible quiver. So we have the following theorem in this case.

**Theorem 4.3.** Let $\Lambda$ be a Koszul self-injective algebra. If $\Lambda$ is an $n$-translation algebra, then $\Lambda$ is an $(n+1)$-translation algebra.

**Proof.** By Lemma 3.2 of [17], the trivial extension of a Koszul self-injective algebra of Loewy length $n+2$ is Koszul self-injective of Loewy length $n+3$. The Theorem follows directly from the definition. \qed

If the trivial extension of an $n$-translation algebra $\Lambda$ with admissible quiver is an $(n+1)$-translation algebra, then $\Lambda$ is called extendible.

5. **Smash products with cyclic groups and $(n+1)$-translation quiver $Z_v|_nQ$**

Let $Z_v = \mathbb{Z}/v\mathbb{Z}$ be a cyclic group of $v$ elements for a nonnegative integer $v$, and we write $Z_0 = \mathbb{Z}$, conventionally. Let $\Lambda = \Lambda_0 + \Lambda_1 + \cdots$ be a graded algebra. We assume that $\Lambda$ has a $Z_v$ grading which is determined by a homogeneous subspace $U \subset \Lambda_1$ of $\Lambda$ with a complement $U'$ in $\Lambda_1$. Let $W_0$ be the subalgebra generated by $\Lambda_0$ and $U'$. Define $W_r$ inductively by

$$W_r = \sum_{s=0}^{r-1} W_s U W_{r-1-s}$$

for $r = 1, 2, \ldots$. Let $\Lambda'_r = \sum_{t \in r+vZ} W_t$. Then $\Lambda = \Lambda'_0 + \Lambda'_1 + \cdots$ is a $Z_v$-graded algebra, we call this a $Z_v$-grading determined by the subspace $U$ of $\Lambda_1$.

Recall that the smash product $\Lambda \# kZ_v^*$ of $\Lambda$ with $Z_v$, defined on the $Z_v$-grading of $\Lambda$ determined by $U$, is the free $\Lambda$ module $\Lambda \otimes kZ_v^*$ with basis $Z_v^* = \{p_g|g \in Z_v\}$, and the multiplication is defined by

$$(x \otimes p_h)(y \otimes p_g) = xy_{h-g} \otimes p_g$$

for $x, y \in \Lambda, y = \sum_{h \in Z_v} y_h$ with $y_h \in \Lambda_h$. Write $ap_g = a \otimes p_g$ for the element of $\Lambda \# kZ_v^*$. For any $\Lambda$-module $M$ which has a $Z_v$-gradation $M = \sum_{h \in Z_v} M_h$, the $\Lambda \# kZ_v^*$-module $M \# kZ_v^*$ is defined as $M \otimes kZ_v^*$ with

$$(a \otimes p_g)(m \otimes p_h) = am_{h-g} \otimes p_h$$

for all $a \in \Lambda, m = \sum_{h \in Z_v} m_h \in M_h$ and $g, h \in Z_v$. Similarly, we write its elements simply as $mp_h$ for $m \in M$ and $h \in Z_v$.

The original gradation of $\Lambda$ induces a grading on $\Lambda \# kZ_v$:

$$\Lambda \# kZ_v = \Lambda_0 \otimes kZ_v + \Lambda_1 \otimes kZ_v + \cdots$$
with \(\{e_ip\}_{i \in D_0, h \in \mathbb{Z}_n}\) a complete set of orthogonal primitive idempotents, which forms a \(k\)-basis of \(\Lambda_0 \otimes_k k\mathbb{Z}_v\). Let \(\Lambda_1 = \Lambda'_1 + \Lambda''_1\) as a direct sum of vector spaces. Note that \(\Lambda\) is generated by \(\Lambda_0\) and \(\Lambda_1\), so \(\Lambda_t\) is spanned by the elements of the form \(x_1 \cdots x_r\), with \(x_t \in \Lambda'_1 \cup \Lambda''_1\). Thus \(\Lambda_t \otimes_k k\mathbb{Z}_v^t\) is spanned by the elements of the form \(x_1 \cdots x_rp_h\) with \(x_t \in \Lambda'_1 \cup \Lambda''_1\) and \(h \in \mathbb{Z}_v\). But we have that

\[
x_1 \cdots x_lp_h = x_1p_h, \ldots, x_lp_h
\]

with \(h_v = h\) and \(h_{t-1} = h_t\) if \(x_t \in \Lambda'_1\) and \(h_{t-1} = h_t + 1\) if \(x_t \in \Lambda''_1\). This shows that \(\Lambda^*\) is generated by \(\Lambda_0 \otimes_k k\mathbb{Z}_v\) and \(\Lambda_1 \otimes k\mathbb{Z}_v^t\).

Since \(\{p_h\}_{h \in \mathbb{Z}_v^*}\) forms a basis of \(\Lambda^*\). If a relation subspace for \(\Lambda\) is \(R\), then \(R \otimes_k k\mathbb{Z}_v^t\) is a relation subspace for \(\Lambda^*\).

Assume that \(\Lambda\) is a quadratic algebra, then \(R\) can be chosen as a subspace of \(\Lambda_1 \otimes_{\Lambda_0} \Lambda_1\). Thus \(R \otimes_k k\mathbb{Z}_v^* \subset \Lambda_1 \otimes_{\Lambda_0 \otimes_k k\mathbb{Z}_v} \Lambda_1 \otimes k\mathbb{Z}_v^*\) and \(\Lambda^* \otimes k\mathbb{Z}_v^*\) is a quadratic algebra. Recall that the \(\Lambda_0\)-\(\Lambda_0\)-sub-bimodules \(K^t\) of \(V^t\) is defined by \(K^0 = \Lambda_0, K^1 = V, K^2 = R\) and \(K^{t+1} = V K^t \cap K^t V\) for all \(t \geq 2\). Define \(\Lambda_0 \otimes k\mathbb{Z}_v^*\)-\(\Lambda_0 \otimes k\mathbb{Z}_v^*\)-sub-bimodules \(K(t)\) of \((\Lambda_0 \otimes k\mathbb{Z}_v^*) \otimes_{\Lambda_0 \otimes k\mathbb{Z}_v^*} \Lambda_1 \otimes k\mathbb{Z}_v^*\) and \(\Lambda^* \otimes k\mathbb{Z}_v^*\) for all \(t \geq 2\). Similarly, we have the following

**Lemma 5.1.**

\[
\tilde{K}^t = K^t \otimes k\mathbb{Z}_v^t,
\]

and

\[
(\Lambda^* \otimes k\mathbb{Z}_v^* \tilde{K}^t) \tilde{K}^t \approx \Lambda \otimes K^t \otimes k\mathbb{Z}_v^t.
\]

\(K^t\) is a \(\Lambda_0\)-\(\Lambda_0\)-bimodule concentrated in degree \(t\), thus \(\tilde{K}^t\) is a \(\Lambda_0 \otimes_k k\mathbb{Z}_v^*\)-\(\Lambda_0 \otimes_k k\mathbb{Z}_v^*\)-bimodule concentrated in degree \(t\). Apply \(- \otimes_k k\mathbb{Z}_v^*\) on the Koszul complex \(\mathcal{K}\) of \(\Lambda\), we get

**Proposition 5.2.**

\[
\cdots \to \Lambda \otimes K^t \otimes k\mathbb{Z}_v^t \to \cdots \to \Lambda \otimes K^2 \otimes k\mathbb{Z}_v^t \to \Lambda \otimes K^0 \otimes k\mathbb{Z}_v^t \to 0 \quad (6)
\]

is the left Koszul complex of \(\Lambda^* \otimes k\mathbb{Z}_v^*\).

If \(\Lambda\) is \((p,q)\)-Koszul with \(p, q \geq 2\), then we easily get the following properties from Proposition 2.3

1. \(\Lambda_t \otimes k\mathbb{Z}_v^* = 0\) for all \(t > q\),
2. \(K^s \otimes k\mathbb{Z}_v^* = 0\) for all \(s > p\), and
3. the only non-zero homology modules of \((6)\) are \(\Lambda_0\) in degree 0, and \(\Lambda_p \otimes K^q \otimes k\mathbb{Z}_v^*\) in degree \(p + q\).

The following theorem follows easily.

**Theorem 5.3.** Assume that \(\Lambda\) is a quadratic algebra and \(p, q \geq 2\). Then \(\Lambda\) is a \((p,q)\)-Koszul algebra if and only if \(\Lambda^* \otimes k\mathbb{Z}_v^*\) is a \((p, q)\)-Koszul algebra.

If \(\Lambda\) is an \(n\)-pretranslation algebra with an admissible \(n\)-translation quiver, the \(v\)-extension of \(\Lambda\) is defined as the smash product \(\Lambda^* \otimes \mathbb{Z}_v^*\) of the trivial extension \(\Lambda\) of \(\Lambda\) with \(\mathbb{Z}_v^*\), defined on the \(\mathbb{Z}_v^*\)-grading determined by the vector spaces spanned by the new arrows of the trivial extension.

As a corollary of Theorem 5.3 we have the following proposition.

**Proposition 5.4.** Let \(\Lambda\) be an extendible \(n\)-translation algebra with an admissible \(n\)-translation quiver, then its \(v\)-extension is an \((n + 1)\)-translation algebra.
Let $Q$ be an admissible $n$-translation quiver with $n$-translation $\tau = \tau_0$, and projective vertex set $P$ and injective vertex set $I$. We define the $n + 1$ translation quiver $Z_{v,n}Q$ as follows.

The vertex set
\[ Z_{v,n}Q_0 = Q_0 \times Z_v = \{(i, t) | i \in Q_0, t \in Z_v\}, \]
and the arrow set
\[
Z_{v,n}Q_1 = Z_v \times Q_1 \cup Z_v \times (Q_0 \setminus P) \cup \{(i, t) : (i, t) \rightarrow (j, t) | \alpha : i \rightarrow j \in Q_1, t \in Z_v, \}
\]
Let $t \in Z_v$. Denote by $p(t)$ the path
\[ p(t) = (\alpha_1, t) \cdots (\alpha_1, t) \]
for each path $p = \alpha_1 \cdots \alpha_1$ in the quiver $Q$ and denote by $z(t)$ the linear combination
\[ z(t) = \sum a_sp_s(t) \]
for each linear combination of paths $z = \sum a_sp_s$ in the quiver $Q$.

Let $Q(t)$ be the subquiver with vertex set $Q(t)_0 = \{(i, t) | i \in Q_0\}$ and arrow set $Q(t)_1 = \{(\alpha, t) | \alpha \in Q_1\}$ and relation set $\rho(t) = \{z(t) | z \in \rho\}$, for each $t \in Z$. Then $Q(t) = (Q(t)_0, Q(t)_1, \rho(t))$ is a subquiver of $Z_{v,n}$ isomorphic to $Q$ as $n$-translation quivers.

Let $\rho(t)^* = \{((\tau(t), t + 1)(\beta), t + 1)(\alpha)(t), i, j \in Q_0 \setminus P, \alpha : i \rightarrow j \in Q_1\} \cup \{(\beta)(\beta, t + 1)|i, t \in Q_0 \setminus P\}$, take
\[ \rho_{Z_{v,n}Q} = \cup_{t \in Z_v}(\rho(t) \cup \rho(t)^*) \]
as the relation set for the bound quiver $Z_{v,n}Q$.

Define $\tau_{n+1} : (i, t) \rightarrow (i, t - 1)$ for $(i, t) \in Z_{v,n}Q_0$.

By Proposition 4.2, we know that the trivial extension $\Lambda$ is an $(n + 1)$-pretranslation algebra with stable $(n + 1)$-translation quiver $\hat{Q} = (Q_0, Q_1, \hat{\rho})$ with trivial translation. This quiver has the same vertex set as $Q$, arrow set $\hat{Q}_1 = Q_1 \cup \{\beta_i : i \rightarrow \tau_{n+1}i | i \in Q_0 \setminus P\}$ and a relation set $\hat{\rho} = \rho \cup \{\beta_{i}i | i \in Q_0 \setminus P\} \cup \{\tau_{n+1}(\alpha) \beta_i - \beta_j \alpha : i \rightarrow j \in Q_1, i, j \in Q_0 \setminus P\}$. Here $\beta_i = p_i^*$ for some fixed bound path $p_i$ of maximal length in $\Lambda$ for each $i \in Q_0 \setminus P$. Let $\Lambda_1$ be the space spanned by $\Lambda_1$ and $\{\beta_i : i \rightarrow \tau_{n+1}i | i \in Q_0 \setminus P\}$, we have that $\Lambda = \Lambda_0 + \Lambda_1 + \cdots + \Lambda_{n+1}$, with $\Lambda_t = \Lambda_j^t$ for $1 \leq t \leq n + 1$.

Let $U$ be the subspace of $\Lambda_1$ spanned by the set $\{\beta_i : i \rightarrow \tau_{n+1}i | i \in Q_0 \setminus P\}$ and let $\Lambda \# Z_v$ be the smash product defined on the $Z_v$-grading determined by $U$.

**Proposition 5.5.** If $Q$ is an admissible $n$-translation quiver, and let $\Lambda$ be the $n$-pretranslation algebra defined by $Q$. Then $Z_{v,n}Q$ is the bound quiver for $\Lambda \# Z_v$.

Especially $Z_{v,n}Q$ is a stable $(n + 1)$-translation quiver with translation $\tau_{n+1}$.

**Proof.** Let $\overline{Q} = (Q_0, Q_1, \overline{\rho})$ be the bound quiver of $\Lambda \# Z_v$. Clearly, we have that $\Lambda \# Z_v = \Lambda_0 \otimes_v Z_v + \Lambda_1 \otimes_v Z_v + \cdots + \Lambda_{n+1} \otimes_v Z_v$. Note that $\{e_i \delta_m | i \in Q_0, n \in Z_v\}$ is a basis of $\Lambda_0 \otimes_v Z_v$, and we have that
\[
e_i \delta_m \cdot e_j \delta_{m'} = \begin{cases} e_i \delta_m & \text{if } i = j, m = m' \\ 0 & \text{otherwise} \end{cases},\]
\[ \rho \]

the path 
\[ i \]
and 
\[ \nu \]
\( \{ \nu \} \) is a complete set of orthogonal primitive idempotents. Write 
\( (i, n) = e_i \delta_n \), thus we have 
\( \overline{Q}_0 = \mathbb{Z}_e |_0 Q_0 \).

Since 
\( Q_1 \cup \{ \beta_i : i \to \tau_{[n]} i | i \in Q_0 \ \setminus \mathcal{P} \} \) forms a basis of \( \Lambda_1 \), 
\( \Lambda_0 \otimes_k \mathbb{Z}_e^* \) and 
\( (Q_1 \times \mathbb{Z}_e^*) \cup \{ (\beta_i : i \to \tau_{[n]} i | i \in Q_0 \ \setminus \mathcal{P} \} \times \mathbb{Z}_e^* \) is a basis of \( \Lambda \# \mathbb{Z}_e^* \), and one see easily that 
\( e_i \rho_{mn} e_i = \alpha \delta_m e_i e_{m''} = \alpha \delta_m \) for any arrow 
\( \alpha = i \to j \) in \( Q_1 \) if and only if 
\( i' = i, j' = j \) and 
\( m' = m'' = m, e_i \rho_{mn} e_i e_{m''} = \beta \delta_m \) if and only if 
\( i = i', j = j, m = m' = m + 1 \) and 
\( m'' = m \). Set 
\( (a, m) = \alpha \delta_m \) be an arrow from 
\( (i, m) \) to 
\( (j, m) \) for an arrow 
\( \alpha = i \to j \) and set 
\( (\beta_i, m) = \beta \delta_m \) be the arrow from 
\( (i, m) \) to 
\( (\tau_{[n]} i, m + 1) \), for each 
\( m \in \mathbb{Z} \), then we have that 
\( \overline{Q}_1 = \mathbb{Z}_e |_n Q_1 \).

By Proposition 1.2 the relation set for \( \Lambda \) is 
\( \rho = \rho \cup \{ \beta_{[n]} | \beta | i, i \in Q_0 \ \setminus \mathcal{P} \} \cup \{ \tau_{[n]} (\alpha) \beta - \beta_j \alpha : j \to i \in Q_1, j, i \in Q_0 \ \setminus \mathcal{P} \} \). Note that for a path 
\( p = \alpha_{\tau_i} \cdots \alpha_1 \) and 
\( m \in \mathbb{Z} \), we have a unique path 
\( p \delta_m = \alpha_{\tau_i} \delta_m \cdots \alpha_1 \delta_m \), and we also have 
\( \alpha \beta \delta_m = \alpha \delta_m e_i e_{m''} = \alpha \delta_m \) for any arrow 
\( \alpha = i \to j \) in \( Q_1 \) if and only if 
\( i' = i, j' = j \) and 
\( m' = m'' = m, e_i \rho_{mn} e_i e_{m''} = \beta \delta_m \) if and only if 
\( i = i', j = j, m = m' = m + 1 \) and 
\( m'' = m \). Set 
\( (a, m) = \alpha \delta_m \) be an arrow from 
\( (i, m) \) to 
\( (j, m) \) for an arrow 
\( \alpha = i \to j \) and set 
\( (\beta_i, m) = \beta \delta_m \) be the arrow from 
\( (i, m) \) to 
\( (\tau_{[n]} i, m + 1) \), for each 
\( m \in \mathbb{Z} \), then we have that 
\( \overline{Q}_1 = \mathbb{Z}_e |_n Q_1 \).

For each 
\( i \in Q_0 \), there is a maximal bound path in \( \Lambda \) of the form 
\( p \beta_i q \) with paths 
\( p, q \) of 
\( Q \) such that 
\( l(p) + l(q) = n \) and 
\( l(p) = s(q) = i \). So for each 
\( (i, m) \in \mathbb{Z}_e |_n \), we have a maximal bound path in 
\( \Lambda \# \mathbb{Z}_e^* \) has the form 
\( p \beta_i q \delta_m \) from 
\( (i, m) \) to 
\( (i, m + 1) \). Thus 
\( \mathbb{Z}_e |_n Q \) is a stable 
\( (n + 1) \)-translation quiver with translation 
\( \tau_{[n+1]} : (i, m+1) \to (i, m) \).

It is well known that for a path algebra 
\( kQ \) of a quiver 
\( Q \) without oriented cycle, the preprojective and preinjective components of its Auslander Reiten quiver are truncations of the translation quiver 
\( \mathbb{Z} Q \).

Take 
\( Q \) as the quiver with bipartite orientation (when it exists), then 
\( kQ \) is an 
\( (1, 2) \)-Koszul algebra with admissible 
1-translation quiver and is extendible. Let 
\( kQ_0 \) be the 0-extension of 
\( kQ \), then by Proposition 5.4, 
\( kQ_0 \) is a 2-translation algebra with 2-translation quiver 
\( \mathbb{Z}_e |_2 Q \) exactly the translation quiver 
\( \mathbb{Z} Q \).

So the preprojective and preinjective components of the Auslander Reiten quiver of the 1-translation algebras 
\( kQ \) are truncations of the bound quiver of its 0-extension.

If 
\( \Lambda \) is an extendible 
\( n \)-translation algebra with 
\( n \)-translation quiver 
\( Q \), which is directed in the sense that there is no oriented cycle in the quiver. Let 
\( \Lambda \) be its 
\( \tau \)-trivial extension, then 
\( \Lambda \) is a graded self-injective algebra with Loewy length 
\( n + 3 \). It follows from [13] that 
\( \Lambda \) is a \( \tau \)-slice algebra of 
\( \Lambda \), and we have that the following results

\textbf{Theorem 5.6.} \( \Lambda \# \mathbb{Z}_e^* \) is the repetitive algebra of 
\( \Lambda \) and the bounded derived category 
\( \mathcal{D}(\Lambda) \) of the category 
\( \text{mod} \ \Lambda \) of finite generated \( \Lambda \)-modules is equivalent to 
the stable category 
\( \text{mod} \ \Lambda \# k \mathbb{Z}_e^* \) of finitely generated 
\( \Lambda \# k \mathbb{Z}_e^* \)-modules.

\textbf{6. Koszul Duality and \( n \)-Almost Splitting Sequences}

Let 
\( \Gamma = \Gamma_0 + \Gamma_1 + \cdots \) be a basic locally finite algebra, and let 
\( \{ e_i | i \in Q_0 \} \) be a complete set of orthogonal primitive idempotents of 
\( \Gamma \). \( \Gamma \) is called a \textit{partial Artin-Schelter \( n \)-regular algebra} if there are subsets 
\( \mathcal{P}, \mathcal{I} \) of 
\( Q_0 \) and bijection map 
\( \nu : Q_0 \ \setminus \mathcal{P} \to Q_0 \ \setminus \mathcal{I} \) such that the following conditions are satisfied for positive integers 
\( n, l \), and for each vertex 
\( i \in Q_0 \ \setminus \mathcal{P} \): (i) 
\( \text{Ext}^l_\Gamma (\Gamma_0 e_i, \Gamma) = 0 \) for 
\( 0 < l < n + 1 \); (ii) 
\( \text{Ext}^{l+1}_\Gamma (\Gamma_0 e_i, \Gamma)[l] \simeq e_i \Gamma_0 \) as 
\( \Gamma^op \)-modules.
\( l \) is called \textit{Gorenstein parameter} of \( \Gamma \). Conventionally, if \( \Lambda \) is a \((p,q)\)-Koszul algebra with \( p \) finite, we set \( q = \infty \) when \( \Lambda \) is Koszul. Let \( l(q) = q \) if it is finite and 0 if \( q = \infty \). Using an analogue argument as in \cite{29}, we have

**Theorem 6.1.** Assume that \( p = n + 1 \) is finite. Let \( \Lambda \) be a \((p,q)\)-Koszul algebra, then \( \Lambda \) is an \( n \)-translation algebra with generalized Coxeter number \( q \) if and only if \( \Gamma = \Lambda' \) is a partial Artin-Schelter \( n \)-regular algebra with Gorenstein parameter \( l(q) \).

**Proof.** Since \( \Lambda \) is a \((p,q)\)-Koszul algebra, \( \Gamma = \Lambda' \) is a \((q,p)\)-Koszul algebra.

If \( \Lambda \) is \( n \)-translation algebra with null \( n \)-translation quiver, then \( \Lambda_p = 0 \) for \( p = n + 1 \) and \( \Lambda \) is Koszul of projective dimension \( q \), and this holds if and only if \( \Gamma = \Lambda' \) is a Koszul of projective dimension \( \leq p \), and \( Q_0 = \mathbb{I} = \mathcal{P} \). Thus, if and only if \( Q_0 \ \setminus \mathcal{P} = 0 \), and \( \Gamma \) is a partial Artin-Schelter \( n \)-regular algebra.

If the bound quiver of \( \Lambda \) is not null \( n \)-translation quiver, then we have \( \Lambda_p \neq 0 \). Thus \( \Gamma = \Lambda' \) is \((q,p)\)-Koszul, and for each \( i \in Q_0 \), we have a Koszul complex

\[
0 \rightarrow \bigoplus_{i} (\nu_i \Delta_i (\Gamma_{0}^i)) e_i \rightarrow \cdots \rightarrow \bigoplus_{i} (\nu_i \Delta_i (\Gamma_{0}^i)) e_i \rightarrow \bigoplus_{i} (\nu_i \Delta_i (\Gamma_{0}^i)) e_i = \bigoplus_{i} e_i \rightarrow 0,
\]

with non-zero homology only in degree 0, where it is \( \Gamma_0 e_i \), and in degree \( p + q \), where it is \( \Gamma_0 \otimes (\nu_i \Delta_i (\Gamma_{0}^i)) e_i \). This complex is indecomposable since it is in a minimal projective resolution of \( \bigoplus_{i} e_i \). Applying \( \text{Hom}_{\mathcal{P}}(\bigoplus_{i} e_i, \Gamma) \) on this complex one gets

\[
0 \rightarrow (\bigoplus_{i} e_i \Gamma_1 \otimes \Gamma \oplus \cdots \oplus (\bigoplus_{i} e_i \Gamma_{p-1} \otimes \Gamma \oplus \bigoplus_{i} e_i \Gamma_p \otimes \Gamma \rightarrow 0. \quad (7)
\]

\( \Gamma \) is partial Artin-Schelter \( n \)-regular if and only if for each \( i \in Q_0 \ \setminus \mathcal{P} \), the only non-zero homologies in \( \mathbb{I} \) are at two ends, and the one at the finial position is \( e_i \Gamma_0 \). On the other hand, \( \Gamma \) is right \((q,p)\)-Koszul, so there is a Koszul complex for \( e_i \Gamma_0 \):

\[
0 \rightarrow (\bigoplus_{i} e_i \nu_i \Delta_i (\Gamma_{0}^i) \otimes \Gamma \oplus \cdots \oplus (\bigoplus_{i} e_i \nu_i \Delta_i (\Gamma_{0}^i) \otimes \Gamma \rightarrow (\bigoplus_{i} e_i \nu_i \Delta_i (\Gamma_{0}^i) \otimes \Gamma = e_i \Gamma \rightarrow 0. \quad (8)
\]

Since \( \Gamma \) is \((q,p)\)-Koszul and both complexes \((7)\) and \((8)\) are the first \( p + 1 \) term in a minimal projective resolution of \( e_i \Gamma_0 \), so there is an isomorphism of complexes

\[
e_i \Gamma_1 \otimes \Gamma \oplus \cdots \oplus e_i \Gamma_{p-1} \otimes \Gamma \oplus e_i \Gamma_p \otimes \Gamma \
\]

\[
e_i \nu_i \Delta_i (\Gamma_{0}^i) \otimes \Gamma \oplus \cdots \oplus e_i \nu_i \Delta_i (\Gamma_{0}^i) \otimes \Gamma \
\]

Here \( d \) and \( \partial \) are left actions by \( u = \sum_{a} \alpha^* \otimes \alpha \) and \( \phi = \Phi \otimes 1 \) for some isomorphism \( \Phi : e_i \Gamma \rightarrow e_i \nu_i \Delta_i (\Gamma) = \Delta_i (\Gamma e_i) \) of graded vector spaces. Thus \( \partial d = d \phi \), that is, if \( x \otimes y \in \Gamma \otimes \Gamma \) for bound paths \( x \) and \( y \), then

\[
\Phi(\sum_{a \in Q_1} \alpha^* x \otimes \alpha y) = (\Phi \otimes 1)(\alpha^* x \otimes \alpha y) = (\Phi \otimes 1)(d(x \otimes y)) = d(\Phi \otimes 1)(x \otimes y) = d(\Phi(x) \otimes y) = \sum_{a \in Q_1} (\alpha^* \Phi(x) \otimes \alpha y)
\]

So \( \Phi(x) = \alpha^* \Phi(x) \) and \( \Phi : e_i \Gamma_1 \rightarrow \Delta_i (\Gamma e_i) \) is an isomorphism of right \( \Gamma' \)-modules. This is equivalent to that \( \Delta_i (\Gamma e_i) \simeq \Gamma e_i \nu_i \Delta_i (\Gamma e_i) \) is projective-injective as left \( \Gamma' \)-module, and the injective hull of \( S(i) \) and the projective cover of \( S(v_i) \) are the same. Now take \( v \) as Nakayama translation \( \tau \), then \( \tau i \in Q_0 \ \setminus \mathcal{I} \), and we have that paths from \( \tau i \) to \( i \) are maximal ones with length \( p \) and any two such paths are linear dependent, and there is no bound path of length \( p \) from \( \tau i \) to \( j \) for any \( j \neq i \).

This equivalent to that the bound quiver of \( \Lambda = \Gamma' \) is an \( n \)-translation quiver and \( \Lambda \) is a \( n \)-translation algebra, since \( n = p - 1 \). This proves our theorem. \( \square \)
Let $\Lambda = kQ$ be an $n$-translation algebra with bound quiver $Q = (Q_0, Q_1, \rho)$. Then $\Gamma = \Lambda^!$ is a partial Artin-Schelter $n$-regular algebra, and we call it the dual $n$-translation algebra of the proper $n$-translation quiver $Q$. Combine Theorem 6.1 with Proposition 3.3, we get first part of the following theorem.

**Theorem 6.2.** An $n$-translation algebra $\Lambda$ is self-injective if and only if its quiver is stable, if and only if the dual $n$-translation algebra $\Gamma$ is a partial Artin-Schelter $n$-regular algebra with $P = 0 = I$.

If $\Lambda$ is Koszul, then $\Gamma$ is an Artin-Schelter regular algebra of dimension $n + 1$.

**Proof.** It follows from Proposition 2.9 that $\Gamma = \Lambda^{op}$ is an $n$-translation algebra with $\text{dim} \text{dim} \Gamma = \Lambda^{op}$ be the dual $n$-translation algebra. Let $\calQ$ be the category of finite generated projective $\Gamma$-modules. We have the following result.

**Proposition 6.3.** If $i \in Q_0 \setminus P$ is not projective, then there is an $(n - 1)$-almost sequence in $\calE(S(i))$ ending at $\calE(S(i))$

$$\calE(r^{n+1}P(S(i))) \rightarrow \cdots \rightarrow \calE(rP(S(i))/r^2P(S(i))) \rightarrow \calE(S(i))$$

If $i \in Q_0 \setminus I$, then there is an $(n - 1)$-almost sequence in $\calQ$ starting at $\calE(S(i))$

$$\calE(S(i)) \rightarrow \calE(\text{soc}^2I(S(i))/\text{soc}I(S(i))) \rightarrow \cdots \rightarrow \calE(\text{soc}^{-1}I(S(i))/\text{soc}^{-1}I(S(i))).$$

**Proof.** It follows from Proposition 6.3 that $\calE$ sends the radical series of the projective cover of a simple $\Lambda$-module $\Lambda_0e_i$

$$J^n\Lambda e_i, \cdots, J^2\Lambda e_i, \Lambda e_i/J^2\Lambda e_i$$

to the linear part

$$\calE(J^n\Lambda e_i) \rightarrow \cdots \rightarrow \calE(\Lambda e_i/J\Lambda e_i) = \calE(S(i))$$

(9)

of a minimal projective resolution of the corresponding simple $\Gamma$-module $\Gamma_0e_i$, which is also the linear part of a minimal injective resolution of $\Gamma_0e_i$. So all maps in this sequence are in the radical of $\calQ$. Clear, the above sequences is exactly the sequence

$$\text{Hom}_\calQ(-, \calE(J^p\Lambda)) \rightarrow \cdots \rightarrow \text{Hom}_\calQ(-, \calE(J^p\Lambda e_i))/\calE(S(i)) \xrightarrow{J^p} \text{Hom}_\calQ(-, J(\calE(S(i)))) \rightarrow 0.$$

This shows that (9) is a sink sequence.

Since $\text{Ext}_\calQ^l(\Gamma_0e_i, \Gamma) = 0$ for $0 < l < n + 1$ and for each vertex $i \in Q_0 \setminus P$ and $\text{Ext}_\calQ^{n+1}(\Gamma_0e_i, \Gamma)[l] \simeq e_i\Gamma_0$ as $\Gamma^{op}$-modules for each vertex $i \in Q_0 \setminus P$, we have that

$$\text{Hom}_\calQ(\calE(\Lambda/J\Lambda e_i), -) \rightarrow \cdots \rightarrow \text{Hom}_\calQ(\calE(J^n\Lambda e_i)/J^p\Lambda e_i), -) \rightarrow 0$$

is exact. Hence the sequence (9) is also a source sequence, and hence is also an $(n - 1)$-almost splitting sequence.

This proves the first assertion, the second one is proven dually.

By Proposition 6.3, for each indecomposable projective injective $\Lambda$-module $M = P(S(\tau_i)) = I(S(i))$, there is an $(n - 1)$-almost sequence

$$\calE(J^{n+1}M) \rightarrow \calE(J^nM/J^{n+1}M) \rightarrow \cdots \rightarrow \calE(JM/J^2M) \rightarrow \calE(M/JM)$$

in $\calQ$.

Define

$$\tau : \calE(S(i)) \rightarrow \calE(S(\tau_i))$$
and call it the \((n - 1)\)-Auslander-Reiten translation of \(Q\). Our Proposition [6,3] can be restated as follows.

**Theorem 6.4.** \(Q\) has \((n - 1)\)-almost splitting sequences. That is, for each \(i \in Q_0 \setminus P\), there is an \((n - 1)\)-almost sequence
\[
\Gamma e_i \rightarrow U_n \rightarrow \cdots \rightarrow U_1 \rightarrow \Gamma e_i
\]
in \(Q\).

The \((n - 1)\)-almost splitting sequences in \(Q\) can be characterized using \(\tau\)-hammocks [15]. Let \(i, j \in Q_0\), write \(Q^t_{ij}\) for the set of bound paths of length \(t\) in \(Q\). For each \(i \in Q_0\) define the \(\tau\)-hammock starting at \(i\) as the quiver \(H^i\) with the vertex set
\[
H^i_0 = \{(j, b) | j \in Q_0, Q^b_{ij} \neq \emptyset\},
\]
and the arrow set
\[
H^i_1 = \{(\alpha, b) : (j, b) \rightarrow (j', b + 1) | \alpha \in Q^t_{ij'}, and there is p \in Q^t_{ij'} such that op \in Q^b_{ij'}^{t+1}\}.
\]
Dually, we define the \(\tau\)-hammock ending at \(i\) as the quiver \(H_i\) with the vertex set
\[
H_{i,0} = \{(j, -b) | j \in Q_0, Q_{ji}^{-b} \neq \emptyset\},
\]
and the arrow set
\[
H_{i,1} = \{(\alpha, b) : (j, b - 1) \rightarrow (j', b) | \alpha \in Q^t_{ij'}, and there is p \in Q^t_{ij'} such that po \in Q_{ji}^{b+1}\}.
\]
If \(i \in Q_0 \setminus P\), then \(\tau i \in Q_0 \setminus I\) and we have
\[
H_i[-n-1] = H_{i}^{-i},
\]
where \(H_i[-n-1]\) is obtained from \(H_i\) by decreasing the second indices of the vertices and arrows by \(n + 1\).

Define the hammock functions \(\mu^i : H^i_0 \rightarrow \mathbb{Z}\) and \(\mu_i : H_{i,0} \rightarrow \mathbb{Z}\) as the integral maps on the vertices of \(H^i\) and \(H_i\), respectively, as follow. For each \((j, b) \in H^i_0\), define \(\mu^i(j, b)\) to be the number of bound paths of length \(b\) from \(i\) to \(j\), and for each \((j, -b) \in H_{i,0}\), define \(\mu_i(j, -b)\) to be the number of bound paths of length \(b\) from \(j\) to \(i\).

We clearly have the following proposition describing the radical series of indecomposable projective modules and the socle series of indecomposable injective \(\Lambda\)-modules.

**Proposition 6.5.**
\[
\Lambda e_i \simeq J^bP(i)/J^{b+1}P(i) = \bigoplus S(j)^{\mu^i(j,b)}
\]
and
\[
\text{soc}^{b+1}I(i)/\text{soc}^bI(i) = \bigoplus S(j)^{\mu_i(j,-b)}
\]

Thus we have the following theorem describing the \((n - 1)\)-almost splitting sequences in \(Q\), by Propositions [6,3] and [6,5].

**Theorem 6.6.** Let
\[
M^n \rightarrow M^{n-1} \rightarrow \cdots \rightarrow M^0
\]
be an \((n-1)\)-almost splitting sequence in \(Q\).

If \(M^0 = \Gamma e_i\), then \(M^i = \bigoplus_{(j,t) \in H^i_0}(\Gamma e_j)^{\mu^i(j,t)}\).

If \(M^n = \Gamma e_i\), then \(M^i = \bigoplus_{(j,t) \in H_{i,0}}(\Gamma e_j)^{\mu_i(j,t)}\).

As an corollary, we have
Corollary 6.7. Let $\Lambda$ be a Koszul self-injective algebra of Loewy length $n+2$ and $\Gamma = E(\Lambda)$ be its Yoneda algebra. Let $\tau$ be the Nakayama translation defined on the category $S$ of the simple $\Lambda$-modules. Let $Q = Q(\Gamma)$ be the category of finite generated graded projective $\Gamma$-modules. Then $Q$ has $(n-1)$-almost splitting sequences and 
$$\tau : E(S) \longrightarrow E(\tau S)$$
is the $(n-1)$-Auslander-Reiten translation.

The following examples show how $n$-translation algebra and higher representation theory \cite{21} \cite{18} are related.

Example 6.8. In \cite{21}, Iyama characterized absolute $n$-complete algebra $T_n^m(k)$ with dominant dimension no less than $n$ and global dimension no more than $n$. We show that such algebras can be obtained as truncations from the Skew group algebras of cyclic groups over the polynomial algebras \cite{10}.

Using results in this paper, we point out that the Auslander Reiten quiver of the cone of Iyama’s absolute $n$-complete algebra $T_n^m(k)$ can also be obtained as a truncation from the quiver of the 0-extension of an $n$-translation algebra for $n = 1, 2$ (see next section for the algebras concerned).

$T_1^m(k)$ has the same quiver \cite{10} as the 0-translation algebra $\Lambda(1)$, in fact, $T_1^m(k)$ is just the quadratic dual $\Gamma(1)$ of $\Lambda(1)$.

It follows from \cite{8}, that the trivial extension $\tilde{\Lambda}(1)$ of $\Lambda(1)$ is $(2, m-1)$-Koszul, hence $\Lambda(1)$ is extendible. Now we have a 1-translation algebra $\overline{\Lambda}(1) = \tilde{\Lambda}(1) \# \mathbb{Z}$ with 1-translation quiver $\mathbb{Z}/1Q(1) \simeq \mathbb{Z}Q(1)$ \cite{12}. In this quiver $Q(1) \simeq Q(1) \times \{1\}$ is a $\tau_{[1]}$-slice. Now for each vertex $\tilde{i} = (i, 1)$, consider the forward hammock $H_{\tilde{i}}$ initiated from $\tilde{i}$, which is used in \cite{13} to describe the linear part of the projective resolution of $S(i)$. Let $Q(2)$ be the truncation of the forward hammocks initiated from the vertices on $Q(1) \times \{1\}$ (see \cite{13} in next section). We have a 1-translation algebra $\Lambda(2)$ defined on $Q(2)$, and we have that the category of the finite generated projective modules of its quadratic dual $\Gamma(2)$ have $n$-almost splitting sequence, by Theorem 6.4 and $Q(2)$ is just its $n$-Auslander Reiten quiver. In fact, this category is equivalent to the cone of $T_1^1(k)$, and we have that $\Gamma(2) \simeq T_2^m(k)$, and its quiver is a truncation of $\mathbb{Z}Q(1) = \mathbb{Z}_{[0]}Q(1)$.

It is proved in the next section that the algebra $\Lambda(2)$ is extendible.

Now $\overline{\Lambda}(2)$ has bound quiver $\mathbb{Z}/2Q(2)$, which is a 2-translation quiver, and $Q(2) \times \{0\}$ is a $\tau_{[2]}$-slice. For each vertex $(w, 0) \in Q(2) \times \{0\}$, the cuboid $C(w)$ of the vertex $w$ embedded in $\mathbb{Z}/1Q(2)$ with $w$ as the initial vertex, we use the same notation $C(w)$ for its image under this embedding. Now let $Q(3)$ be the full subquiver with vertices in $\cup_{w \in Q(2)} C(w)$. Let $\Lambda(3)$ be the 2-translation algebra with the 2-translation quiver $Q(3)$. Let $\Gamma(3)$ quadratic dual of $\Lambda(3)$. Let $Q(3)$ be the category of finite generated projective $\Gamma(3)$-modules, the $Q(3)$ has 2-almost splitting sequences. Clear $\Gamma(3)$ is isomorphic to $T_3^m(k)$, and thus $Q(3)$ is equivalent to the cone of $T_3^m(k)$.

This shows that the Auslander Reiten quiver of the cone of $T_n^m(k)$ is a (finite) truncation of $\mathbb{Z}/1Q(2)$.

This can be regard as a version of the classical result that the Auslander Reiten quiver of the path algebra $kQ$ for a Dynkin quiver $Q$ is a truncation of $\mathbb{Z}Q$.

The representation theory of an hereditary algebra with radical square zero are closely related to that of a selfinjective algebra with radical cube zero \cite{30} \cite{13}. For an hereditary algebra with radical square zero, we know that it is representation...
finite if and only if its trivial extension is \((2, h)\)-Koszul. So we may define an extendible \(n\)-translation algebra weakly representation finite if its trivial extension is \((n + 1, h)\)-Koszul for some finite \(h\), and call \(h\) the Coxeter number of \(\Lambda\). Define an extendible \(n\)-translation algebra weakly representation-infinite if its trivial extension is \((n + 1, \infty)\)-Koszul. Call an extendible \(n\)-translation algebra weakly representation-tame if its trivial extension is \((n + 1, \infty)\)-Koszul with finite complexity [12, 13].

It is interesting to known if a weakly representation finite \(n\)-translation algebra is a quadratic dual \(n\)-representation finite in the sense of [21, 22].

Recall that an algebra with quiver \(Q\) is called directed if there is no oriented cycle in \(Q\).

**Example 6.9.** Let \(\Lambda\) be a directed extendible \(n\)-translation algebra with \(n\)-translation quiver \(Q\). Let \(\tilde{\Lambda}\) be the trivial extension of \(\Lambda\), let \(\tilde{\Lambda} = \tilde{\Lambda} \# \mathbb{Z}^+\) be the \(0\)-extension, and let \(\tilde{Q}\) and \(\tilde{Q}^+\) be their quivers, respectively. Both \(\tilde{Q}\) and \(\tilde{Q}^+\) are \((n + 1)\)-translation quivers.

Both \(\tilde{\Lambda}\) and \(\tilde{\Lambda}^\ast\) are stable \((n + 1)\)-translation algebras. \(\tilde{Q}^+ = \tilde{Q}|_{\tilde{\Lambda}}\) is isomorphic to a connected component of the separated directed quiver of \(\tilde{Q}\). The full subquiver with vertex set \(\{(i, 0) | i \in Q^0\}\) is a \(\tau_{\tilde{\Lambda}}\)-slice and \(\Lambda\) a \(\tau_{\tilde{\Lambda}}\)-slice algebra (see [15]).

Let \(\Gamma, \tilde{\Gamma}\) and \(\Gamma^\ast\) be the quadratic dual of \(\Lambda, \tilde{\Lambda}\) and \(\tilde{\Lambda}^\ast\), respectively. They share same quiver as \(\Lambda, \tilde{\Lambda}\) and \(\tilde{\Lambda}^\ast\), but with relations \(\rho^\ast, \tilde{\rho}^\ast\) and \(\rho^\ast\) which span the orthogonal spaces of the subspaces spanned by the relations \(\rho, \tilde{\rho}\) and \(\rho^\ast\) in the quadratic space. Denote their bound quivers by \(Q^\ast, \tilde{Q}^\ast\) and \(\tilde{Q}^\ast\), respectively.

If \(\tilde{\Lambda}\) is Koszul \((n + 1)\)-translation, then \(\text{gl.dim} \Gamma = n\) and \(\tilde{\Gamma}\) is also Koszul \((n + 1)\)-translation algebra. Recall that for \(n \geq 1\), the \(n\)-Auslander-Reiten translations are defined by \(\tau_n = \text{DExt}^n(D, \Gamma): \text{mod} \Gamma \rightarrow \text{mod} \Gamma^\ast, \text{and} \quad \tau_n^\ast = \text{Ext}^n(\text{D}, \Gamma^\ast): \text{mod} \Gamma^\ast \rightarrow \text{mod} \Gamma\) [20]. If \(\Gamma\) is \(n\)-representation-infinite, then both \(\mathcal{P} = \mathcal{P}(\Gamma) = \text{add}\{\tau_n^\ast \Gamma | t \geq 0\}\) and \(\mathcal{I} = \mathcal{I}(\Gamma) = \text{add}\{\tau_n \Gamma | t \geq 0\}\) have \(n\)-almost splitting sequences with the ending terms connected by \(n\)-Auslander-Reiten translations.

Let \(\tilde{Q}^+ = \tilde{\mathbb{N}}|_{\tilde{\Lambda}}\) and \(\tilde{T}^- = \tilde{\mathbb{N}}|_{\tilde{\Lambda}}\) be the full subquivers with vertex sets the vertices with nonnegative second indices and with nonpositive second indices, respectively. Let \(\Gamma^\ast N\) and \(\Gamma_N^\ast\) be the algebras given by the bound quivers \(\tilde{Q}^+\) and \(\tilde{T}^-\), respectively. Then their quadratic dual, \(\tilde{\Gamma}^\ast\) and \(\tilde{\Gamma}^\ast\) are \((n + 1)\)-translation algebras with \((n + 1)\)-translation quivers \(\tilde{Q}^\ast = \tilde{\mathbb{N}}|_{\tilde{\Lambda}}\) and \(\tilde{Q}^\ast = \tilde{\mathbb{N}}|_{\tilde{\Lambda}}\), and \((n + 1)\)-translation \(\tau_{\tilde{\Lambda}}\)-slice in \(\tilde{Q}\). The vertices on \(Q\) \(\ast\{0\}\) are exactly the projective vertices in \(\tilde{Q}^\ast\) and injective vertices in \(\tilde{T}^-\).

It follows from Theorem 6.3 that there exist \(n\)-almost splitting sequences in \(\mathcal{Q}(\tilde{T}^-)\) and \(\mathcal{Q}(\tilde{T}^+)\), the categories of finite generated graded projective \(\tilde{T}^-\)-modules and \(\tilde{T}^+\)-modules, respectively. The ending terms of these \(n\)-almost splitting sequences are connected by the translation of these quivers, and from Theorem 6.6 these \(n\)-almost splitting sequences are characterized by the \(\tau\)-hammocks.

On the other hand, it follows from induction that the \(n\)-almost splitting sequences in \(\mathcal{P}\) and \(\mathcal{Q}(\tilde{T}^+)\) are obtained in the same way and those in \(\mathcal{I}\) and \(\mathcal{Q}(\tilde{T}^-)\) are obtained in the same way, since \(\Gamma\) is directed. So \(\mathcal{Q}(\tilde{T}^+) = \tilde{\mathbb{N}}Q^\ast\) and \(\mathcal{Q}(\tilde{T}^-) = -\tilde{\mathbb{N}}Q^\ast\) are Auslander-Reiten quivers of \(\mathcal{P}\) and of \(\mathcal{I}\), respectively.

7. Non Koszul case: 2-translation algebras of type A

It is proved in [8] that trivial extension of the path algebra of an bipartite Dynkin quiver is almost Koszul. Now we take the case \(A_n\), view it from a different view
point and try to prove similar results in this case for a 1-translation algebra. In this way, we also construct a family of \((3, m - 1)\)-Koszul algebras, which are 2-translation algebras, and a family of \((m - 1, 3)\)-Koszul algebras for any \(m > 2\).

Let’s start with a quiver \(Q\) of Dynkin diagram \(A_m\) with linear orientation, so the vertex set can be taken as \(Q_\circ = \{1, 2, \cdots, m\}\), with arrow set \(Q_\times = \{\alpha_i : i \to i+1| i = 1, 2, \ldots, m - 1\}\). So we have the quiver

\[
Q(1) = Q : \circ \overset{\alpha_1}{\longrightarrow} \circ \cdots \circ \overset{\alpha_m}{\longrightarrow} \circ.
\]

Let \(\Lambda(1)\) be the algebra given by \(Q\) with the relation \(\rho = \{\alpha_{i+1}\alpha_i | i = 1, \ldots, m-2\}\), it is a 0-translation algebra, and it is extendible. Let \(\tilde{\Lambda}(1) = T(\Lambda(1))\) be the trivial extension of \(\Lambda(1)\), this is a symmetric algebra with double quiver of a Dynkin diagram \(A_m\) as its bound quiver and is isomorphic to the trivial extension of the path algebra of the bipartite quiver of Dynkin diagram \(A_m\). This is given by quiver \(\tilde{Q}(1)\)

\[
\tilde{Q}(1) : \circ \overset{\alpha_1}{\longrightarrow} \circ \circ \circ \cdots \overset{\alpha_{m-1}}{\longrightarrow} \circ \overset{\alpha_m}{\longrightarrow} \circ
\]

with relations

\[
\tilde{\rho}(1) = \{\alpha_{i+1}\alpha_i, \alpha_i^*\alpha_{i+1}^*, \alpha_i\alpha_i^* - \alpha_{i+1}^*\alpha_i| i \leq m - 2\}.
\]

This algebra \(\tilde{\Lambda}(1)\) is isomorphic to the trivial extension obtained from the path algebra of quiver the bipartite \(A_m\), \(\tilde{\Lambda}(1)\) is \((2, m - 1)\)-Koszul, by \([3]\). \(\Lambda(1)\) is not extendible, that is, the trivial extension \(T(\Lambda(1))\) of \(\Lambda(1)\) is not almost Koszul. Let \(\overline{\Lambda}(1) = \tilde{\Lambda} \# \mathbb{Z}\) be the 0-extension of \(\Lambda(1)\), this is a 1-translation algebra since \(\overline{\Lambda}(1)\) is.

\(\Lambda(1)\) is given by the quiver \(\overline{Q}(1)\)

\[
\overline{Q}(1) = \overline{Q(1)}
\]

with relations

\[
\overline{\rho}(1) = \{\alpha_{(i+1),t}\alpha_{(i,t)} \alpha^*_{(i+1,t)}, \alpha_{(i,t)}^*\alpha_{(i+1,t)}^*, \alpha_{(i,t)}\alpha_{(i+1,t)} - \alpha^*_{(i+1,t)}\alpha_{(i,t)} | i \leq m - 2, t \in \mathbb{Z}\}.
\]

\(\overline{Q} = \overline{Q(1)}\) is a translation quiver with translation \(\tau(i, t) = (i, t - 1)\) for all the vertices \((i, t)\). Now we truncate an extensible 1-translation algebra from \(\overline{\Lambda}(1)\).

By \([13]\), the linear part of the projective resolution of a simple \(S(v)\) can be described with the additive function defined by the vertex \(v\) and the corresponding forward hammock. Take a copy of \(Q(1)\) in \(\overline{Q(1)}\), say \(Q(1) \times \{1\}\), let \(Q(2)\) be the full subquiver with vertex set the union of the vertices in the forward hammocks
initiated with the vertex in $Q(1) \times \{1\}$:

Let $\Lambda(2)$ be the algebra given by the quiver $Q(2)$ with relation set

$$
\rho(2) = \{\alpha_{i+1,t} \alpha_{i,t+1} | 1 \leq i \leq m-1, 1 \leq t \leq m-i-1\} \\
\cup \{\alpha_{i,t}^* \alpha_{i+1,t+1}^* | 3 \leq i \leq m, (i, t) \in Q(2)\} \\
\cup \{\alpha_{i,t}^* \alpha_{i,t+1} - \alpha_{i+1,t} \alpha_{i+1,t+1} | 2 \leq i \leq m-1, 1 \leq t \leq m-i\}.
$$

This is the Auslander Reiten quiver of the quadratic dual of $\Lambda(1)$, which is the path algebra $kQ(1)$ of quiver of type $A_m$ with linear orientation. It is easy to see that $\Lambda(2)$ is an 1-translation algebra with a admissible 1-translation quiver. We remark that the algebra $\Lambda(2)$ is the quadratic dual of the Auslander algebra of the path algebra $kQ(1)$, and is also the quadratic dual of Iyama’s absolute 2-complete algebra $T^{(2)}_m(k)$ [21].

Let $\Lambda(2) = T(\Lambda(2))$ be the trivial extension of $\Lambda(2)$, then $\Lambda(2)$ is a symmetry algebra of Loewy length 4, and its bound quiver is stable 2-translation quiver:

$$
\tilde{Q}:
$$

with relation set

$$
\tilde{\rho} = \rho(2) \cup \{\gamma_{i,t} \gamma_{i,t+1} | 1 \leq i \leq m-2, 2 \leq m+i \} \\
\cup \{\alpha_{i,t}^* \gamma_{i,t+1} - \gamma_{i-1,t+1} \alpha_{i,t}^* | 2 \leq i \leq m-1, 1 \leq t \leq m+i \} \\
\cup \{\alpha_{i,t} \gamma_{i,t+1} - \gamma_{i+1,t+1} \alpha_{i,t+1} | 1 \leq i \leq m-2, 1 \leq t \leq m+i \}.
$$

Note that $\tilde{\Lambda}(2)$ is the quadratic dual of the 3-preprojective algebra of $T^{(2)}_m(k)$ [22, 23].

Write $\tilde{\Lambda} = \tilde{\Lambda}(2)$ in the remaining of this section, when no confusion will appear.

Let $S(i, t)$ be the simple $\tilde{\Lambda}$-module corresponding to the vertex $(i, t)$, and let $\tilde{P}(i, t)$ be its projective cover as $\tilde{\Lambda}$-module. Then as graded module, $\tilde{P}(i, t) = \tilde{P}(i, t)_0 + \tilde{P}(i, t)_1 + \tilde{P}(i, t)_2 + \tilde{P}(i, t)_3$ with $\tilde{P}(i, t)_0 \simeq S(i, t) \simeq \tilde{P}(i, t)_3$.

We have following proposition describing the indecomposable $\tilde{\Lambda}$-modules.
Proposition 7.1.  
(1) For \( v = (1,1), (m,1), (1,m), \) \( \dim_k \bar{P}(v) = 4 \) and we have
\[
\bar{P}(1,1) \simeq S(2,1) \quad \bar{P}(1,1) \simeq S(1,2) \\
\bar{P}(m,1) \simeq S(m-1,2) \quad \bar{P}(m,1) \simeq S(m-1,1) \\
\bar{P}(1,m) \simeq S(1,m-1) \quad \bar{P}(1,m) \simeq S(2,m-2) \\
\]
(2) For \( v = (1,t), 2 \leq t \leq m, \) or \( v = (i,1), \) or \( (i,m+1-i) \) for \( 2 \leq i \leq m-1, \) \( \dim_k \bar{P}(v) = 6 \) and we have
\[
\bar{P}(1,t) \simeq S(2,t) \oplus S(1,t-1) \\
\bar{P}(i,1) \simeq S(i+1,1) \oplus S(i-1,2) \\
\bar{P}(i,m+1-i) \simeq S(i,m-i) \oplus S(i-1,m-i) \\
\bar{P}(1,t) \simeq S(2,t-1) \oplus S(i+1,t-1) \\
\bar{P}(i,1) \simeq S(i-1,1) \oplus S(i,2) \\
\bar{P}(i,m+1-i) \simeq S(i+1,m-i) \oplus S(i-1,m-i-1) \\
\]
(3) For \( v = (i,t), 2 \leq i \leq m-1, 2 \leq t \leq m-i, \) \( \dim_k \bar{P}(v) = 8 \) and we have
\[
\bar{P}(i,t) \simeq S(i+1,t) \oplus S(i-1,t+1) \oplus S(i,t-1) \\
\bar{P}(i,t) \simeq S(i+1,t) \oplus S(i+1,t-1) \oplus S(i-1,t) \\
\]
Now we study the projective resolution of a simple \( \bar{A} \)-module \( S(i,t) \).
For any \( 1 \leq i \leq m, 1 \leq t \leq m+1-i, \) set \( b_1(i,t) = i-1, \) \( b_2(i,t) = m+1-i-t \)
and \( b_3(i,t) = t-1. \) Let \( b = b(i,t) = (b_1(i,t), b_2(i,t), b_3(i,t)) \), and let
\[
C(i,t) = \{(a_1, a_2, a_3)|0 \leq a_1 \leq b_1(i,t), 0 \leq a_2 \leq b_2(i,t), 0 \leq a_3 \leq b_3(i,t)\} \subset \mathbb{Z}^3, 
\]
is the set of the integral vertices on a cuboid with sides of length \( b_1(i,t), b_2(i,t), \)
and \( b_3(i,t) \), with the initial vertex \( C_0(i,t) = \{(0,0,0)\} \). Let \( \mathbb{Z}^3_i = \{a = (a_1, a_2, a_3)|a_1 + a_2 + a_3 = l\}, \) and let
\[
C_l(i,t) = \{a = (a_1, a_2, a_3) \in C(i,t)|a_1 + a_2 + a_3 = l\} = \mathbb{Z}^3_i \cap C(i,t), 
\]
for \( l = 0, \ldots , m-1 \), we call \( l = a_1 + a_2 + a_3 \) the length of \( a \). With the edges the line segments parallel to the coordinate axes, \( C(i,t) \) forms a net inside the cuboid consisting of vertices which can be reached from the initial vertex in \( l \) steps.
Let \( a = (a_1, a_2, a_3) \) be a vertex in \( C(i,t) \). If \( a_j < b_j(i,t) \) define
\[
\zeta_j^+ a = (a_1', a_2', a_3') \text{ with } a_j' = \begin{cases} a_j' = a_j + 1 & j' \neq j \\ a_j' = j & j' = j \end{cases} 
\]
If \( a_j > 0 \) define
\[
\zeta_j^- a = (a_1', a_2', a_3') \text{ with } a_j' = \begin{cases} a_j' = a_j - 1 & j' \neq j \\ a_j' = j & j' = j \end{cases} 
\]
\( \zeta_j^+ \) is a map from a subset of \( C_l(i,t) \) to \( C_{l+1}(i,t) \) and \( \zeta_j^- \) is a map from a subset of \( C_{l+1}(i,t) \) to \( C_l(i,t) \) for \( l = 1, \ldots , m-2 \). They extend to well defined maps on \( \mathbb{Z}^3 \), and their operations are commutative.

For \( a = (a_1, a_2, a_3) \in \mathbb{Z}^3 \) define \( (u(a), v(a)) = (i - a_1 + a_2, t + a_1 - a_3) \), if \( (u(a), v(a)) \) is a vertex of \( \bar{Q} \) in this case, we say that \( a \) is a quiver vertex. If \( a \in C(i,t) \), then \( (u(a), v(a)) \) is always a quiver vertex. The following lemma follows easily.

Lemma 7.2. If \( a, a' \in \mathbb{Z}^3 \), then \( (u(a), v(a)) = (u(a'), v(a')) \) if and only if \( a = a' \).
If $a$ is a quiver vertex, we use $a$ instead of $(u(a), v(a))$ as indices. So
$$e_a = e_{(u(a), v(a))}, \tilde{P}(a) = \tilde{P}(u(a), v(a)) \text{ and } S(a) = S(u(a), v(a))$$
if $a$ is a quiver vertex. We set conventionally $e_a$, $\tilde{P}(a)$ and $S(a)$ to be zero if $a$ is not a quiver vertex.

For the sake of simplifying our discuss, we introduce new notations for the arrows. Write $\gamma_a^{(1)} = a^+_a$, $\gamma_a^{(2)} = a^+_a$, $\gamma_a^{(3)} = \gamma_a$. Obviously, we have
$$e_{(i', t')}(a) \neq 0 \text{ if and only if } (i', t') = (u(\psi_j^+ a), v(\psi_j^+ a)) \text{ and it is a vertex of } \tilde{Q}.$$

For each $a \in \mathbb{Z}^3$, let $T^+(a)$ be the cube consisting of the vertices between $a$ and $c_3^+ c_2^+ c_1^+ a$. So the vertex set of this cube is
$$\{a, c_1^+ a, c_2^+ a, c_3^+ a, c_2^+ c_1^+ a, c_3^+ c_2^+ a, c_3^+ c_2^+ c_1^+ a\}.$$ This cube is the support of $\tilde{P}(a)$. More precisely, if $a$ is a quiver vertex, it follows from Proposition 8 that the support of $\tilde{P}(a)$ are just the quiver vertices in $T^+(a)$ and the degree $t$ component $\tilde{P}(a)_t$ is supported by those vertices which can be reached in $t$ steps from $a$.

Similarly, let $T^-(a)$ be the cube consisting of the vertices between $c_3^+ c_2^+ c_1^+ a$ and $a$. If $a$ is a quiver vertex, then the injective cover of the simple module $S(a)$ is supported by the quiver vertices in $T^-(a)$. And if $S(a)$ is a composition factor of $\tilde{P}(d)_t$ if and only if $d \in T^-(a)$, or more precisely,
$$d = c_1^+ a \quad l = 1$$
$$d = c_2^+ c_1^+ a \quad l = 2$$
$$d = c_3^+ c_2^+ c_1^+ a \quad l = 3.$$ for pairwise different $j$, $j'$ and $j''$.

For each $j \in \mathbb{Z}/3\mathbb{Z}$, the set
$$F_j = \{a \in C(i, t)|a_j = 0\}$$
is called a lower face and
$$F_j' = \{a \in C(i, t)|a_j = b_j(i, t)\}$$
is called an upper face. Clearly, we have that if $a = (a_1, a_2, a_3) \in C(i, t)$, then $T^+(a)$ is in $C(i, t)$ if and only if $a_j < b_j(i, t)$ for $j = 1, 2, 3$. This is equivalent to that $a \not\in \cup_{j \in \mathbb{Z}/3\mathbb{Z}} F_j'$, that is, it is not in any upper faces.

For each $j \in \mathbb{Z}/3\mathbb{Z}$, we have a lower edge
$$E_j = \{a = (a_1, a_2, a_3)|0 \leq a_j \leq b_j(i, t), a_{j'} = 0 \text{ if } j' \neq j\}$$
starting from the initial vertex $(0, 0, 0)$ and an upper edge
$$E_j' = \{a = (a_1, a_2, a_3)|0 \leq a_j \leq b_j(i, t), a_{j'} = b_{j'}(i, t) \text{ if } j' \neq j\}$$
leading to the terminal vertex $(b_1(i, t), b_2(i, t), b_3(i, t))$. For each pair $j, j' \in \mathbb{Z}/3\mathbb{Z}$ with $j \neq j'$, we have a middle edge
$$E_{j, j'} = \{a = (a_1, a_2, a_3)|a_j = b_j(i, t), a_{j'} = 0, 0 \leq a_{j''} \leq b_{j''}(i, t) \text{ for } j'' \neq j, j'\}$$
connecting $E_j$ and $E_{j'}$.

If $a \in E_j$ and $a \neq (0, 0, 0)$, then $c_j^- a \in E_j$, and $c_j^- a$ is not in $C(i, t)$ if $j' \neq j$.
If $a \in E_j'$, then $c_j^- a \in C(i, t)$, and $c_j^- a \in E_{j'}$ if $a_{j''} \neq 0$, $c_j^- a$ is not in $C(i, t)$. If $a \in E_j$, then $c_j^- a \in E_j'$ if $a_j > 0$, and $c_j^- a \in C(i, t)$ for $j' \neq j$. 
Lemma 7.3. 

\begin{equation}
\cdots \rightarrow \tilde{P}^2(S(i,t)) \xrightarrow{f_3} \tilde{P}^1(S(i,t)) \xrightarrow{f_2} \tilde{P}^0(S(i,t)) \xrightarrow{f_1} S(i,t) \rightarrow 0
\end{equation}

be a minimal projective resolution of the simple \( \Lambda \)-modules \( S(i,t) \) of corresponding to the vertex \((i,t)\).

Consider the linear part of this projective resolution. We assert that for \( l = 0, 1, \ldots , m - 1 \), we have

\begin{equation}
\tilde{P}^l(S(i,t)) \simeq \bigoplus_{a \in C_l(i,t)} \tilde{P}(a).
\end{equation}

Let \( \{ \epsilon^{(l)}_a \mid a \in C_l(i,t) \} \) be the standard basis of this direct sum. Then \( e_a \epsilon^{(l)}_a = \epsilon^{(l)}_a \) and \( e_{(i',t')} \epsilon^{(l)}_a = 0 \) if \((i',t') \neq (u(a), v(a))\). We assume that \( \epsilon^{(l)}_a = 0 \) when \( a \notin C_l(i,t) \), for \( l = 1, \ldots, m - 1 \), conventionally.

With the convention that \( \zeta^{(l)}_{i,j} a_j = 0 \) if \( \zeta^{(l)}_{i,j} a \notin C_l(i,t) \). For each \( a \in C_{l+1}(i,t) \), there are elements \( \zeta^{(l)}_{i,j} a_{1,1} \zeta^{(l)}_{i,j} a_{2,2} \zeta^{(l)}_{i,j} a_{3,3} \in k \), \( \zeta^{(l)}_{i,j} a_j \neq 0 \) if \( \zeta^{(l)}_{i,j} a \in C_l(i,t) \), such that \( \text{Ker} f_l \) is generated by the set

\begin{equation}
K_l = \{ \zeta^{(l)}_{i,j} a_{1,1} \zeta^{(l)}_{i,j} a_{2,2} \zeta^{(l)}_{i,j} a_{3,3} a \mid a \in C_{l+1}(i,t) \}.
\end{equation}

Write \( \theta l_a = \zeta^{(l)}_{i,j} a_{1,1} \zeta^{(l)}_{i,j} a_{2,2} \zeta^{(l)}_{i,j} a_{3,3} \zeta^{(l)}_{i,j} a \). For each \( a \in C_{l+1}(i,t) \), we have

\begin{equation}
e_a \theta l_a = \theta l_a.
\end{equation}

The map \( f_l \) is defined by

\begin{equation}
f_l(\epsilon^{(l)}_a) = \theta l_a = \zeta^{(l-1)}_{i,j} a_{1,1} \epsilon^{(l-1)}_{i,j} a + \zeta^{(l-1)}_{i,j} a_{2,2} \epsilon^{(l-1)}_{i,j} a + \zeta^{(l-1)}_{i,j} a_{3,3} \epsilon^{(l-1)}_{i,j} a,
\end{equation}

for each \( a \in C_l(i,j) \).

Lemma 7.3. In the projective resolution \( \{ \tilde{P}(S(i,t)) \} \), we have

1. \( \{ \tilde{P}(S(i,t)) \} \) hold for \( 0 \leq l \leq m - 1 \);
2. \( \{ \tilde{P}(S(i,t)) \} \) hold for \( 1 \leq l \leq m - 1 \);
3. \( \{ \tilde{P}(S(i,t)) \} \) generates \( \text{Ker} f_l \) for \( 0 \leq l \leq m - 2 \);
4. \( \text{C}_{m-1}(i,j) = \{ b \}, \zeta^{(l)}_{i,j} \zeta^{(l)}_{i,j} b \) is a quiver vertex and \( \text{Ker} f_{m-1} \simeq S(\zeta^{(l)}_{i,j} \zeta^{(l)}_{i,j} b) \).

Proof. We now prove \( \{ \tilde{P}(S(i,t)) \} \), \( \{ \tilde{P}(S(i,t)) \} \) and \( \{ \tilde{P}(S(i,t)) \} \), using induction. The assertions for \( \tilde{P}^0(S(i,t)) \), \( \text{Ker} f_0 \) and \( f_1 \) are obvious.

Assume that \( 1 \leq l \leq m - 1 \) and the assertions hold for \( \tilde{P}^h(i,t) \), \( \text{Ker} f_h \) and \( f_h \) for \( h < l \). Write \( L(v_1, \ldots , v_r) \) for the vector space spanned by the vectors \( v_1, \ldots , v_r \). By the inductive assumption, \( \text{Ker} f_{l-1} \) is a graded module generated in degree \( l \) and there are nonzero \( \hat{c}^{(l-1)}_{a'_{i,j}} \in k \) for each \( a' \in C_{l-1}(i,t) \), and conventionally \( \hat{c}^{(l-1)}_{a_{i,j}} = 0 \) if \( \zeta^{(l-1)}_{i,j} a \notin C_{l-1}(i,t) \) for \( a \in C_l(i,t) \), we have

\( \text{Ker} f_{l-1} ) = \text{top} ( \text{Ker} f_{l-1} ) = L( e_a \hat{c}^{(l-1)}_a | a \in C_l(i,t) ) = \sum_{a \in C_l(i,t)} k \theta^{(l-1)}_a \).

By \( \{ \tilde{P}(S(i,t)) \} \), as \( \Lambda_0 \)-modules, we have

\begin{equation}
k \theta^{(l-1)}_a \simeq \tilde{\Lambda}_0 e_a
\end{equation}

Since for each \( a \in C_l(i,t) \), we have exactly one nonzero element \( \hat{c}^{(l-1)}_a \) in \( K_{l-1} \). Thus

\( \text{top} ( \text{Ker} f_{l-1} ) \simeq \bigoplus_{a \in C_l(i,t)} \tilde{\Lambda}_0 e_a \).
Thus $\hat{P}^l(S(i,t)) \simeq \bigoplus_{a \in C(i,t)} \hat{P}(a)$.

This proves (14) for $l$.

Define the homomorphism $f_l$ from $\hat{P}^l(S(i,t))$ to $\hat{P}^{l-1}(S(i,t))$ with

$$f_l(a^{(l)}) = \theta_a^{(l-1)}$$

Clearly $f_l$ is an epimorphism from $\hat{P}^l(S(i,t))$ to $\ker f_{l-1}$, and $f_l(\hat{P}^l(S(i,t)))_p \subseteq (\hat{P}^{l-1}(S(i,t)))_{p+1}$ for $p \in \mathbb{Z}$. This proves (20) for $1 \leq l \leq m - 1$.

Now we compute generators of $\ker f_l$ for $l \leq m - 2$. Clearly $(\ker f_l)_l = 0$ and $(\ker f_l)_{l-1} = \text{soc} \hat{P}^l(S(i,t))$.

Assume $0 \neq x_p$ is a homogeneous element in $(\ker f_l)_p$, then we have $x_{l+1} = \sum_{a \in C(i,t)} (z_a, \gamma_a^{(1)}) + z_a, \gamma_a^{(2)} + z_a, \gamma_a^{(3)} e_a^{(l)}$. So

$$0 = f_l(x_{l+1}) = \sum_{a \in C(i,t)} (z_a, \gamma_a^{(1)}) + \sum_{a, \gamma_a^{(2)} + z_a, \gamma_a^{(3)} e_a^{(l)}) f_l(e_a^{(l)})$$

$$= \sum_{a \in C(i,t)} (z_a, \gamma_a^{(1)}) + \sum_{a, \gamma_a^{(2)} + z_a, \gamma_a^{(3)} e_a^{(l)}) f_l(e_a^{(l)})$$

$$= \sum_{a \in C(i,t)} (z_a, \gamma_a^{(1)}) + \sum_{a, \gamma_a^{(2)} + z_a, \gamma_a^{(3)} e_a^{(l)}) f_l(e_a^{(l)})$$

Let $a^{(l)} = (a'_1, a'_2, a'_3) \in \mathbb{Z}^3$ be a quiver vertex with $a'_1 + a'_2 + a'_3 = l + 1$ such that $a'_i \in C(i,t)$ for some $j \in \mathbb{Z}/3\mathbb{Z}$. Assume that $\mathbb{Z}/3\mathbb{Z} = \{j, j', j''\}$.

If $a'_j \in C(i,t)$ for all $j \in \mathbb{Z}/3\mathbb{Z}$, then $a' \in C(i+1,t)$. In this case, $a'_j \in C(i,t)$ are all in $C(i,t)$ for pairwise different $j, j', j'' \in \mathbb{Z}/3\mathbb{Z}$. Left multiply (22) with $e_{a^{(l)}}$, then by Lemma (7,2)

$$0 = f_l(e_{a^{(l)}} x_{l+1}) = \sum_{a \in C(i,t)} (z_a, \gamma_a^{(1)}) + \sum_{a, \gamma_a^{(2)} + z_a, \gamma_a^{(3)} e_a^{(l)}) f_l(e_{a^{(l)}})$$

This is equivalent to the homogeneous linear equations

$$z_{a^{(1)}}, \gamma_{a^{(1)}}^{(1)} + z_{a^{(2)}}, \gamma_{a^{(2)}}^{(1)} + z_{a^{(3)}}, \gamma_{a^{(3)}}^{(1)} = 0$$

and all the coefficients are nonzero, by inductive assumption.

By Lemma (7,2) we get an exact sequence for the composition factors isomorphic to $S(a')$ in the degree $l + 1$ part of the first $l + 1$ terms of the minimal projective resolution (16) of $S(i,t)$:

$$0 \rightarrow e_{a^{(l)}}(\ker f_{l+1}) \rightarrow \bigoplus_{j \in \mathbb{Z}/3\mathbb{Z}} e_{a^{(l)}} \lambda_1 e_{a^{(j)}} \rightarrow \bigoplus_{j \in \mathbb{Z}/3\mathbb{Z}} e_{a^{(l)}} \lambda_2 e_{a^{(j-1)}} e_{a^{(j-2)}} \rightarrow e_{a^{(l)}} \lambda_3 e_{a^{(1)}} e_{a^{(0)}} \rightarrow 0.$$
then each $\zeta^j_{\gamma^j a^j, j} \neq 0$ and

$$\theta^l_{\gamma^l a^l} = \zeta^l_{\gamma^l a^l, 1} j^{-1} \varepsilon^l_{\gamma^l a^l, 2} \varepsilon_{\gamma^l a^l, 3} \varepsilon_{\gamma^l a^l}$$

is a nonzero element of $e_{a^l}(\text{Ker}f_l)_{l+1}$.

If $\zeta^j_{\gamma^j a^j} \notin C_l(i,t)$ and $\zeta^j_{\gamma^j a^j} \notin C_l(i,t)$, then $\zeta^j_{\gamma^j a^j} \notin F_j$, $a^j \in F_j \setminus E_j$ and $\zeta^j_{\gamma^j a^j} \notin C_l(i,t)$. Left multiplying (22) with $e_{a^j}$, one gets

$$0 = f_l(e_{a^j}x_{l+1}) = z_{\zeta^j a^j, j} \varsigma_{\zeta^j, j} \zeta_{\gamma^j a^j, j} e_{a^j} \gamma_{\gamma^j a^j} \varsigma_{\gamma^j a^j, j} \varsigma_{\gamma^j, j} = z_{\zeta^j a^j, j} \varsigma_{\zeta^j, j} \zeta_{\gamma^j a^j, j} e_{a^j} \gamma_{\gamma^j a^j} \varsigma_{\gamma^j a^j, j} \varsigma_{\gamma^j, j} a^j.$$

So

$$z_{\zeta^j a^j, j} \varsigma_{\zeta^j, j} \zeta_{\gamma^j a^j, j} + z_{\zeta^j a^j, j} \varsigma_{\zeta^j, j} \zeta_{\gamma^j a^j, j} = 0,$$

with $\zeta_{\zeta^j a^j, j} \varsigma_{\zeta^j, j} \zeta_{\gamma^j a^j, j}$ nonzero, by the inductive assumption.

In the first $l + 1$ term of the minimal projective resolution (10) of $S(i, t)$, the composition factors isomorphic to $S(a^j)$ in degree $l + 1$ form an exact sequence

$$0 \rightarrow e_{a^j}(\text{Ker}f_l)_{l+1} \rightarrow e_{a^j} \tilde{A}_1 e_{\zeta^j a^j} \oplus e_{a^j} \tilde{A}_2 e_{\zeta^j a^j} \rightarrow 0.$$

We have that $\dim_k(e_{a^j} \tilde{A}_1 e_{\zeta^j a^j} \oplus e_{a^j} \tilde{A}_2 e_{\zeta^j a^j}) = 2$ and $\dim e_{a^j} \tilde{A}_2 e_{\zeta^j a^j} = 1$, this means that the solution of (20) is one-dimensional. Let $z_{\zeta^j a^j, j} = \zeta_{\zeta^j, j} z_{\zeta^j a^j, j}$ be a nonzero solution then $z_{\zeta^j, j} \neq 0 \neq z_{\zeta^j, j}$. Take $\zeta_{\zeta^j a^j, j} = \zeta_{\zeta^j, j} \zeta_{\gamma^j a^j, j}$ and $\zeta_{\zeta^j a^j, j} = 0$ since $\zeta_{\gamma^j a^j}, \notin C_l(i, t)$, then $\theta^l_{\gamma^j a^j} = \sum_{j \in \mathbb{Z}/2\mathbb{Z}} \zeta^j_{\gamma^j a^j} \gamma^j_{\gamma^j a^j} \varsigma_{\gamma^j a^j, j} \varsigma_{\gamma^j, j} \in (\text{Ker}f_l)_{l+1} e_{\zeta^j a^j}$.

Assume that $\zeta^j_{\gamma^j a^j}, \zeta^j_{\gamma^j a^j} \notin C_l(i, t)$.

If $a^j \in C_l(i, t)$ then $\zeta^j_{\gamma^j a^j} \in E_j$ and $a^j \in E_j$. This implies that $f_l(e_{\zeta^j a^j}) = \zeta^l_{\gamma^l a^l, 1} \gamma^l_{\gamma^l a^l} \varsigma^l_{\gamma^l a^l, 2} \varsigma^l_{\gamma^l a^l, 3} \varsigma^l_{\gamma^l a^l} e_{\zeta^j a^j}$, and $f_l(\gamma^j_{\gamma^j a^j} e_{\zeta^j a^j}) = 0$. We also have that $\zeta^j_{\gamma^j a^j}, \notin C_l(i, t)$ and $\zeta^j_{\gamma^j a^j} \notin C_l(i, t)$. Thus $S(a^j)$ appears only once in the degree $l + 1$ part of the first $l + 1$ term of the projective resolution (10). Let $\gamma^j_{\gamma^j a^j, j} = 1$ and $\zeta^j_{\gamma^j a^j, j} = 0 = \zeta^j_{\gamma^j a^j, j} \gamma^j_{\gamma^j a^j} \varsigma^j_{\gamma^j a^j, j} = \sum_{j \in \mathbb{Z}/2\mathbb{Z}} \zeta^j_{\gamma^j a^j} \gamma^j_{\gamma^j a^j} \varsigma^j_{\gamma^j a^j, j} \gamma^j_{\gamma^j a^j} = \theta^j_{\gamma^j a^j} \in (\text{Ker}f_l)_{l+1} e_{\zeta^j a^j}$.

If $a^j \notin C_l(i, t)$ then $\zeta^j_{\gamma^j a^j} \notin F_j$ and $\zeta^j_{\gamma^j a^j, j} \notin C_l(i, t)$, and we have that $l \geq b_j(i, t)$.

Now we prove by induction on $l - b_j(i, t)$ that $S(a^j)$ is not a composition factor of $(\text{Ker}f_l)_{l+1}$.

If $l = b_j(i, t)$, then $a^j$ is not a quiver vertex.

Assume that $l - b_j(i, t) > 0$ and assume that $a^j$ is a quiver vertex.

If $\zeta^j_{\gamma^j a^j} \in E_j$, the only quiver vertices of $T^- (a^j)$ are $\zeta^j_{\gamma^j a^j}$ and $\zeta^j_{\gamma^j a^j}$. So we have that in the first $l + 1$ term of the minimal projective resolution (10)

$$0 \rightarrow e_{a^j} \tilde{A}_1 e_{\zeta^j a^j} \rightarrow e_{a^j} \tilde{A}_2 e_{\zeta^j a^j} \rightarrow 0,$$

and $e_{a^j}(\text{Ker}f_l)_{l+1} = 0$.

If $\zeta^j_{\gamma^j a^j} \notin E_j$, then $\zeta^j_{\gamma^j a^j}$ and $\zeta^j_{\gamma^j a^j} a^j$ are both in $F_j \cup C_l(i, t)$ and we have that $\zeta^j_{\gamma^j a^j} \zeta^j_{\gamma^j a^j} a^j = \zeta^j_{\gamma^j a^j} \zeta^j a^j \in F_j \cap C_l(i, t)$. The quiver vertices of $T^- (a^j)$
are $\zeta_j a'$, $\zeta_j^{-1} a'$, $\zeta_j^{-2} a'$ and $\zeta_j^{-n} \zeta_j^{-1} a'$. In the first $l + 1$ term of the minimal projective resolution \([9]\), we have exact sequence

$$0 \to e_a^* \Lambda e_{\zeta_j^{-1}} a^* \to e_a^* \Lambda_2 e_{\zeta_j^{-1}} a^* \oplus e_a^* \Lambda_2 e_{\zeta_j^{-1}} a^* \to e_a^* \Lambda_3 e_{\zeta_j^{-1}} a^* \to 0,$$

hence $e_a^* (\text{Ker } f_{l+1}) = 0$.

Thus $e_a^* (\text{Ker } f_{l+1}) \neq 0$ if and only if $a' \in C^{l+1} (i, t)$, and in this case, we have that $\dim e_a^* (\text{Ker } f_{l+1}) = 1$. Thus $\text{Ker } f_{l+1}$ is spanned by $K_i$ in \([8]\).

Assume that $a'' \in \mathbb{Z}_n^{l+2}$ is a quiver vertex with $\zeta_j^{-n} \zeta_j a'' \in C_l(i, t)$ for some $j, j'' \in \mathbb{Z}/3\mathbb{Z}$.

If $\zeta_j a'' \in C_l(i, t)$ for all $j \in \mathbb{Z}/3\mathbb{Z}$, then $\zeta_j a'' \in C_l(i, t)$ and all the vertices of $T_{a''}$ are quiver vertices. Since $K_i$ spans $(\text{Ker } f_{l+1})$, so for each $j \in \mathbb{Z}/3\mathbb{Z}$, there are nonzero $e_{\zeta_j^{-1}} a'' \in \mathbb{Z}/3\mathbb{Z}$, such that

$$\theta_{\zeta_j^{-1}} a'' \equiv \sum_{j'' \in \mathbb{Z}/3\mathbb{Z}} e_{\zeta_j^{-1}} a'' \mu_{j''} e_{\zeta_j^{-1}} a'' \equiv e_{\zeta_j^{-1}} a'' \sigma, \quad \text{is a nonzero element of } e_{\zeta_j^{-1}} a'' (\text{Ker } f_{l+1}).$$

Thus we have

$$\zeta_{j''} e_{\zeta_j^{-1}} (\zeta_j)^{(j)}_{j''} a'' \zeta_j^{-1} a'' = (j-1)_{j''} \zeta_j^{-1} a'' + \zeta_{j''} e_{\zeta_j^{-1}} (\zeta_j)^{(j)}_{j''} a'' (j-1)_{j''} \zeta_j^{-1} a''$$

are three nonzero elements of $e_a^* (\text{Ker } f_{l+2})$ for $j \in \mathbb{Z}/3\mathbb{Z}$. Any two of them are linearly independent and they span a subspace of dimension 2 in $e_a^* (\text{Ker } f_{l+2})$.

Now consider the $l + 1$ component of the first $l + 1$ term of the minimal projective resolution of the simple $S(i, t)$:

$$0 \to e_a^* (\text{Ker } f_{l+2}) \to \bigoplus_{j \in \mathbb{Z}/3\mathbb{Z}} e_a^* \Lambda_3 e_{\zeta_j^{-1}} a'' \to e_a^* \Lambda_3 e_{\zeta_j^{-1}} a'' \to 0.$$
So \( \dim \ker(a^v \tilde{\Lambda}_1 \theta^{(l)}_{\epsilon_{i,t} + n^v}) \) is an one dimensional subspace of \( \ker(a^v \ker(f_t)_{i+2}) \) as above. This proves that \( \ker(a^v \ker(f_t)_{i+2}) \) is in the submodule of \( \tilde{\mathcal{P}}^d(S(i, t)) \) generated by \( K_t \).

This proves that \( \ker(f_t)_{i+2} \) is in the submodule of \( \tilde{\mathcal{P}}^d(S(i, t)) \) generated by \( K_t \), thus \( K_t \) is generated by \( K_t \) for \( l < m - 2 \).

By definition, we have that \( \mathcal{C}_{m-1}(i, t) = \{ b \} = \{ (B_1(i, t), B_2(i, t), B_3(i, t)) \} \). Now we consider \( \ker(f_m - 1) \). If \( b(j, t) > 0 \) for all \( j \), the vertices in \( T^+(b) \) are all quiver vertices. So the nonzero components of degrees \( m, m+1, m+2 \) parts of the projective resolution \( \mathcal{T} \) of \( S(i, t) \) are

\[
0 \to e_{\epsilon_{j_t}^+, \epsilon_{j_t}^{-}}(\ker(f_m - 1)) \to 0 \\
0 \to e_{\epsilon_{j_t}^+, \epsilon_{j_t}^{-}}(\ker(f_m - 1)) \to 0 \\
0 \to e_{\epsilon_{j_t}^+, \epsilon_{j_t}^{-}}(\ker(f_m - 1)) \to 0 \\
0 \to e_{\epsilon_{j_t}^+, \epsilon_{j_t}^{-}}(\ker(f_m - 1)) \to 0 \\
0 \to e_{\epsilon_{j_t}^+, \epsilon_{j_t}^{-}}(\ker(f_m - 1)) \to 0 
\]

for all \( j' \in \mathbb{Z}/3\mathbb{Z} \). But for any \( j' \in \mathbb{Z}/3\mathbb{Z} \), we have that \( \dim e_{\epsilon_{j_t}^+, \epsilon_{j_t}^{-}}(\ker(f_m - 1)) = 1 \), \( \dim e_{\epsilon_{j_t}^+, \epsilon_{j_t}^{-}}(\ker(f_m - 1)) = 2 \) and \( \dim e_{\epsilon_{j_t}^+, \epsilon_{j_t}^{-}}(\ker(f_m - 1)) = 1 \), this implies that \( e_{\epsilon_{j_t}^+, \epsilon_{j_t}^{-}}(\ker(f_m - 1)) = 0 \). Similarly, for any \( j', j'' \in \mathbb{Z}/3\mathbb{Z} \), if \( \epsilon_{j_t}^+, \epsilon_{j_t}^{-} \) is a quiver vertex, then \( \dim e_{\epsilon_{j_t}^+, \epsilon_{j_t}^{-}}(\ker(f_m - 1)) = 1 \) and \( \dim e_{\epsilon_{j_t}^+, \epsilon_{j_t}^{-}}(\ker(f_m - 1)) = 1 \), so we get that \( e_{\epsilon_{j_t}^+, \epsilon_{j_t}^{-}}(\ker(f_m - 1)) = 0 \). Thus

\[
\ker(f_m - 1) = e_{\epsilon_{j_t}^+, \epsilon_{j_t}^{-}}(\ker(f_m - 1)) = \text{soc} \tilde{\Lambda}_3 \epsilon_b^{(m-1)} \cong S(\epsilon_{j_t}^+, \epsilon_{j_t}^{-}).
\]

If \( b(j, t) = 0 \) and \( b(j, t) \neq 0 \) for all \( j' \neq j \), then the quiver vertices of \( T^+(b) \) are \( \epsilon_{j_t}^+, \epsilon_{j_t}^{-}, \epsilon_{j_t}^+, \epsilon_{j_t}^{-}, \epsilon_{j_t}^+, \epsilon_{j_t}^{-}, \epsilon_{j_t}^+, \epsilon_{j_t}^{-} \). So the nonzero components of degrees \( m, m+1, m+2 \) parts of the projective resolution \( \mathcal{T} \) of \( S(i, t) \) are

\[
0 \to e_{\epsilon_{j_t}^+, \epsilon_{j_t}^{-}}(\ker(f_m - 1)) \to e_{\epsilon_{j_t}^+, \epsilon_{j_t}^{-}}(\ker(f_m - 1)) \to 0 \\
0 \to e_{\epsilon_{j_t}^+, \epsilon_{j_t}^{-}}(\ker(f_m - 1)) \to e_{\epsilon_{j_t}^+, \epsilon_{j_t}^{-}}(\ker(f_m - 1)) \to 0 \\
0 \to e_{\epsilon_{j_t}^+, \epsilon_{j_t}^{-}}(\ker(f_m - 1)) \to e_{\epsilon_{j_t}^+, \epsilon_{j_t}^{-}}(\ker(f_m - 1)) \to 0 \\
0 \to e_{\epsilon_{j_t}^+, \epsilon_{j_t}^{-}}(\ker(f_m - 1)) \to e_{\epsilon_{j_t}^+, \epsilon_{j_t}^{-}}(\ker(f_m - 1)) \to 0 \\
0 \to e_{\epsilon_{j_t}^+, \epsilon_{j_t}^{-}}(\ker(f_m - 1)) \to e_{\epsilon_{j_t}^+, \epsilon_{j_t}^{-}}(\ker(f_m - 1)) \to 0 
\]

Similarly, we can prove that

\[
\ker(f_m - 1) = \text{soc} \tilde{\Lambda}_3 \epsilon_b^{(m-1)} \cong S(\epsilon_{j_t}^+, \epsilon_{j_t}^{-}).
\]

If \( b_{j-t}(i, t) = 0 = b_{j-t}(i, t) \neq 0 \), then the quiver vertices of \( T^+(b) \) are \( \epsilon_{j_t}^+, \epsilon_{j_t}^+, \epsilon_{j_t}^+, \epsilon_{j_t}^+, \epsilon_{j_t}^+, \epsilon_{j_t}^+, \epsilon_{j_t}^+, \epsilon_{j_t}^{-} \), \( \epsilon_{j_t}^+, \epsilon_{j_t}^+, \epsilon_{j_t}^{-} \). Thus the nonzero components of degrees \( m, m+1, m+2 \) parts of the projective resolution \( \mathcal{T} \) of \( S(i, t) \) are

\[
0 \to e_{\epsilon_{j_t}^+, \epsilon_{j_t}^{-}}(\ker(f_m - 1)) \to e_{\epsilon_{j_t}^+, \epsilon_{j_t}^{-}}(\ker(f_m - 1)) \to 0 \\
0 \to e_{\epsilon_{j_t}^+, \epsilon_{j_t}^{-}}(\ker(f_m - 1)) \to e_{\epsilon_{j_t}^+, \epsilon_{j_t}^{-}}(\ker(f_m - 1)) \to 0 \\
0 \to e_{\epsilon_{j_t}^+, \epsilon_{j_t}^{-}}(\ker(f_m - 1)) \to e_{\epsilon_{j_t}^+, \epsilon_{j_t}^{-}}(\ker(f_m - 1)) \to 0 \\
0 \to e_{\epsilon_{j_t}^+, \epsilon_{j_t}^{-}}(\ker(f_m - 1)) \to e_{\epsilon_{j_t}^+, \epsilon_{j_t}^{-}}(\ker(f_m - 1)) \to 0 \\
0 \to e_{\epsilon_{j_t}^+, \epsilon_{j_t}^{-}}(\ker(f_m - 1)) \to e_{\epsilon_{j_t}^+, \epsilon_{j_t}^{-}}(\ker(f_m - 1)) \to 0 
\]

Thus similarly

\[
\ker(f_m - 1) = \text{soc} \tilde{\Lambda}_3 \epsilon_b^{(m-1)} \cong S(\epsilon_{j_t}^+, \epsilon_{j_t}^{-} \epsilon_{j_t}^+, \epsilon_{j_t}^{-} \epsilon_{j_t}^+, \epsilon_{j_t}^{-} \epsilon_{j_t}^+, \epsilon_{j_t}^{-}).
\]

This proves our lemma. \( \square \)
Remark 7.4. (1) In our construction, the trivial extensions $\hat{\Lambda}(1)$ and $\hat{\Lambda}$ can be replaced by twisted trivial extensions.

(2) Since the first $m$ term of the minimal projective resolution of $S(i, t)$ is also the first $m$ term of the minimal projective resolution of the simple module $S(u(c_1^+ c_2^+ c_3^+ (b(i, t))), v(c_1^+ c_2^+ c_3^+ (b(i, t))))$, our lemma has a dual statements for minimal injective resolutions using dual cuboid $\hat{C}(i, t)$.

As a corollary, we have the following theorem

Theorem 7.5. $\hat{\Lambda}$ is almost Koszul algebra of type $(3, m-1)$, and $\hat{\Lambda}'$ is almost Koszul algebra of type $(m-1, 3)$.

$\hat{\Lambda}$ is a stable 2-translation algebra and $\hat{\Lambda}'$ is a stable $(m-1)$-translation algebra.

$\Lambda$ is a periodic algebra with minimal periodicity 3$m$ and $\hat{\Lambda}'$ is a periodic algebra with minimal periodicity 12.

Proof. The first two assertion follows directly from the lemma.

For the last assertion, note that for any vertex $(i, t) \in \hat{Q}_0$, $b(i, t) = (i-1, m+1-i-t, t-1)$ and

$$c_1^+ c_2^+ c_3^+ (b(i, t)) = (i, m+2-i-t, t),$$

thus $u(c_1^+ c_2^+ c_3^+ (b(i, t))) = (m+2-i-t, i)$, $\Omega^m S(i, t) = S(m+2-i-t, i)$, $\Omega^m S(i, t) = \Omega^m S(m+2-i-t, i) = S(t, m+2-i-t, i)$, $\Omega^m S(i, t) = \Omega^m S(t, m+2-i-t, i) = S(i, t)$.

This shows that the minimal periodicity of $\hat{\Lambda}$ is 3$m$. It follows from the Koszul duality that the minimal periodicity of $\hat{\Lambda}'$ is $3 \cdot 4 = 12$.

□

It seems very natural to general our results in this section to higher cases.

Take our example, it follows from [15] that $\Lambda(2)$ is a $\tau_{[2]}$-slice algebra. By taking $\tau$-mutations of $\Lambda(2)$ in $\hat{\Lambda} \# \mathbb{Z}^{*}$, we get isomorphic trivial extension. Theorems [15] provide a family of examples of weakly representation finite 2-translation algebras.

In fact, the $\tau_{[2]}$-mutations of $\Lambda(2)$ are the quadratic duals of the $n$-APR tilts of $T_n(k)$ [22].

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