ON POPA’S COCYCLE SUPERRIGIDITY THEOREM

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ABSTRACT. These notes contain an Ergodic-theoretic account of the Cocycle Superrigidity Theorem recently discovered by Sorin Popa. We state and prove a relative version of the result, discuss some applications to measurable equivalence relations, and point out that Gaussian actions (of “rigid” groups) satisfy the assumptions of Popa’s theorem.

1. INTRODUCTION AND STATEMENT OF THE RESULTS

A recent paper [22] of Sorin Popa contains a remarkable new Cocycle Superrigidity theorem. Here is a special case of this result (see [22] Theorem 3.1, and Theorem 1.3 below for the general case):

1.1. Theorem (Popa’s Cocycle Superrigidity, Special Case). Let $\Gamma$ be a discrete group with Kazhdan’s property (T), let $\Gamma_0 < \Gamma$ be an infinite index subgroup, $(X_0, \mu_0)$ an arbitrary probability space, and let $\Gamma \curvearrowright (X, \mu) = (X_0, \mu_0)^{\Gamma/\Gamma_0}$ be the corresponding generalized Bernoulli action. Then for any discrete countable group $\Lambda$ and any measurable cocycle $\alpha : \Gamma \times X \to \Lambda$ there exist a homomorphism $\varphi : \Gamma \to \Lambda$ and a measurable map $\phi : X \to \Lambda$ so that: $\alpha(g, x) =\varphi(g.x)\varphi(g)\varphi(x)^{-1}$.

Note the unprecedented strength of this theorem: there are no assumptions on the nature of the discrete target group $\Lambda$, no conditions are imposed on the cocycle $\alpha$, and the assumption on the acting group $\Gamma$ is of a general and non-specific nature. At the same time the conclusion is unusually strong: the cocycle $\alpha$ is “untwisted” within the given discrete group $\Lambda$. The result however is specific to a particular class of actions – Bernoulli actions, or more generally malleable actions.

This theorem continues a series of ground breaking results in the area of von Neumann Algebras recently obtained by Sorin Popa [27], [25], [23], [24], [26], and by Popa and collaborators: Popa-Sasyk [28], Ioana-Peterson-Popa [16], Popa-Vaes [29]. The reader is also referred to Vaes’ Seminaire Bourbaki [32] for an overview and more references. The aims of these notes are:

(1) To give a rather short, self contained, purely Ergodic-theoretic proof of Popa’s Cocycle Superrigidity Theorem. We use this opportunity to give a relative version of the result (Theorem 1.3 below). Our proof follows Popa’s general strategy, but implements some of the steps differently. The proof is essentially contained in sections 3 and 4 below.

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(2) To point out that the class of actions for which Popa proved his Cocycle Superrigidity property (weakly mixing malleable actions) extends from the class of (generalized) Bernoulli actions to a larger class of all weakly mixing Gaussian actions.

(3) To discuss the relationship between the Cocycle Superrigidity results of Zimmer and Popa, showing in particular that algebraic actions are not quotients of malleable actions.

(4) To discuss some applications of Popa’s Cocycle Superrigidity Theorem to Ergodic theory, in particular to II_1 equivalence relations.

In these notes we emphasize the Ergodic theoretic point of view, trying to complement the original Operator Algebra framework of Popa’s [22] (also Vaes’ [32]). As a result we do not discuss at all the results on von Neumann equivalence of group actions, and refer the reader to the above mentioned papers (see also [29]).

1.a. The Statement of Popa’s Cocycle Superrigidity Theorem. Theorem [14] is only a special case of Popa’s Cocycle Superrigidity theorem. Before formulating the more general statement [13] below it is necessary to recall/define some notions which we shall use hereafter:

w-normal and wq-normal subgroups: The following notions are generalizations of normality, and subnormality. A closed subgroup $H < G$ is called weakly normal (w-normal), if there exists an increasing family $H = H_0 < H_1 < \cdots H_\eta = G$ of closed subgroups $\{H_i\}_{0 \leq i \leq \eta}$, well ordered by inclusion, and such that $(\bigcup_{i<j} H_i) \triangleleft H_j$ for each ordinal $0 \leq j \leq \eta$. A further weakening of this notion is: $H < G$ is weakly quasi-normal (wq-normal), if there exists an increasing family $H = H_0 < H_1 < \cdots H_\eta = G$ of closed subgroups $\{H_i\}_{0 \leq i \leq \eta}$ well ordered by inclusion, such that denoting $H'_j \overset{\text{def}}{=} (\bigcup_{i<j} H_i)$ for each ordinal $0 \leq j \leq \eta$ the group $H_j$ is generated by the set $\{g \in G \mid g^{-1}H'_j g \cap H'_j \text{ is not precompact in } G\}$.

groups of finite type: A topological group is said to be of finite type, or to belong to class $\mathcal{U}_{\text{fin}}$, if it can be embedded as a closed subgroup of the unitary group of a finite von-Neumann algebra (i.e. a von Neumann algebra with a faithful normal final trace). Class $\mathcal{U}_{\text{fin}}$ contains the class $\mathcal{G}_{\text{disc}}$ of all discrete countable groups, and the class $\mathcal{G}_{\text{cmp}}$ of all second countable compact groups. $\mathcal{U}_{\text{fin}}$ also contains such groups as the inner automorphism group $\text{Inn}(\mathcal{R})$ (a.k.a. the full group) of a II_1 countable relation $\mathcal{R}$. All groups in $\mathcal{U}_{\text{fin}}$ admit a complete metric which is bi-invariant; hence semi-simple Lie groups are not in $\mathcal{U}_{\text{fin}}$ (see [2.4] for further discussion).

(relative) property (T): A locally compact second countable (l.c.s.c. ) group $G$ has Kazhdan’s property (T) if every unitary $G$-representation which almost has invariant vectors, has non-trivial invariant vectors. A closed subgroup $H < G$ in a l.c.s.c. group $G$ is said to have relative property (T) in $G$ if every unitary $G$-representation which almost has $G$-invariant vectors, has non-trivial $H$-invariant vectors.
L-Cocycle-Superrigid actions (extensions): Let $G \curvearrowright (X,\mu)$ be an ergodic probability measure preserving (p.m.p.) action of some l.c.s.c. group $G$, and $L$ be some Polish group. We shall say that the action $G \curvearrowright (X,\mu)$ is $L$-Cocycle-Superrigid if for every measurable cocycle $\alpha : G \times X \to L$ there exists a measurable map $\phi : X \to L$ and a homomorphism $\varrho : G \to L$ so that for each $g \in G$:

$$\alpha(g,x) = \phi(g,x)\varrho(g)\phi(x)^{-1}$$

for $\mu$-a.e. $x \in X$. More generally, if $p : (X,\mu) \to (Y,\nu)$ is a measurable equivariant quotient map of p.m.p. $G$-actions, we shall say that $G \curvearrowright (X,\mu)$ is $L$-Cocycle-Superrigid relatively to $(Y,\nu)$ if every measurable cocycle $\alpha : G \times X \to L$ can be written as

$$\alpha(g,x) = \phi(g,x)\varrho(g,p(x))\phi(x)^{-1}$$

where $\phi : X \to L$ is a measurable map and $\varrho : G \times Y \to L$ is a measurable cocycle. If an action $G \curvearrowright (X,\mu)$ is $L$-Cocycle-Superrigid for all groups $L$ in some class $\mathcal{C}$ (e.g. $\mathcal{U}_{\text{fin}}$) we shall say that $G \curvearrowright (X,\mu)$ is $\mathcal{C}$-Cocycle-Superrigid. Similarly one defines the relative notion.

Weakly mixing actions (extensions): A p.m.p action $G \curvearrowright (X,\mu)$ is said to be weakly mixing if the diagonal action of $G$ on $(X \times X,\mu \times \mu)$ is ergodic. If $G \curvearrowright (X,\mu)$ has a quotient p.m.p. action $G \curvearrowright (Y,\nu)$ one can form a fibered product space $X \times_Y X \overset{\text{def}}{=} \{(x_1,x_2) \in X \times X \mid p(x_1) = p(x_2)\}$ on which $G$ acts diagonally preserving a certain probability measure $\mu \times_\nu \mu$, associated to the disintegration of $p : \mu \mapsto \nu$. If the latter action is ergodic, the original action $G \curvearrowright (X,\mu)$ is said to be weakly mixing relative to the quotient $G \curvearrowright (Y,\nu)$; weak mixing is equivalent to weak mixing relative to the trivial action on a point (see [2,e] for more details).

Malleable actions (extensions): Given a probability space $(Z,\zeta)$ denote by $\text{Aut}(Z,\zeta)$ the group of all measure space automorphisms of $(Z,\zeta)$ considered modulo null sets and endowed with the weak topology which makes it into a Polish group (see [2,a]). A p.m.p action $G \curvearrowright (X,\mu)$ is malleable if the connected component of the identity of the centralizer $\text{Aut}(X \times X,\mu \times \mu)^G$ of the diagonal $G$-action on $(X \times X,\mu \times \mu)$ contains the flip $F : (x,y) \mapsto (y,x)$ or, an element of the form

$$F \circ (T \times S) : (x,y) \mapsto (S(y),T(x)) \quad \text{where} \quad T,S \in \text{Aut}(X,\mu)^G.$$  

More generally, if $(X,\mu) \overset{\text{pr}}{\rightarrow} (Y,\nu)$ is a morphism of p.m.p. $G$-actions we shall say that $G \curvearrowright (X,\mu)$ is malleable relative to $(Y,\nu)$ if the flip $F : (x_1,x_2) \mapsto (x_2,x_1)$, or a transformation of the form $F \circ (T \times S)$ with $T,S \in \text{Aut}(X,\mu)^G$, $p \circ T = p \circ S$, lies in the connected component of the identity in the centralizer of $G$ in $\text{Aut}(X \times Y,\mu \times \nu \mu)$. *

Generalized Bernoulli actions: Given a probability space $(X_0,\mu_0)$, an infinite countable set $I$, and a permutation action $\sigma : G \to \text{Sym}(I)$, there is a

*We use a variation of the original definitions (cf. [22, 4.3]), using connectivity rather than path connectivity and allowing for $S,T$. 
p.m.p. $G$-action on the product space $(X, \mu) = (X_0, \mu_0)^I = \prod_{i \in I}(X_0, \mu_0)_i$ by permutation of the coordinates: $(g.x)_i = (x)_{\sigma(g^{-1})(i)}$. This action $G \acts (X, \mu)$ is ergodic iff the $G$-action on the index set $I$ has no finite orbits. The classical (two-sided) Bernoulli shift corresponds to $G = \mathbb{Z}$ acting on itself.

**Gaussian actions:** are p.m.p. actions $G \acts (X, \mu)$ constructed out of unitary, or rather orthogonal, representations $\pi : G \to O(H)$ of the symmetric powers of $\pi$, which in particular contains $\pi$ itself as a subrepresentation. For Gaussian actions ergodicity is equivalent to weak mixing and occurs iff $\pi$ is weakly mixing (i.e., does not contain a finite dimensional subrepresentation). We note that if $G$ acts by permutations on a countable set $I$, then the Bernoulli $G$-action on $([0, 1], m)^I$ is isomorphic to the Gaussian action corresponding to the $G$-representation on $\ell^2(I)$; hence the class of Gaussian actions contains that of generalized Bernoulli actions with the non atomic base space.

The notions of $w$-normality and $wq$-normality, the class $\mathcal{U}_{\text{fin}}$ of groups, and the concept of malleability (and other variants of these notions) were introduced and studied by Sorin Popa. The notion of malleability is of particular importance in his work and in our discussion of it. Generalized Bernoulli actions (with a non-atomic base space) were the main example of malleable actions in [22]. Here we observe that the class of malleable actions contains all Gaussian actions.

1.2. **Theorem.** All Gaussian actions, including generalized Bernoulli actions with a non-atomic base space, of any l.c.s.c. group are malleable.

Note that the notion of malleability itself does not involve any ergodicity assumptions, and so in the above statement no assumptions are imposed on the Bernoulli or Gaussian actions. Yet in what follows we shall need to require (weak) mixing of the action of the group $G$, or even of a certain subgroup $H < G$. For a Bernoulli action $G \acts (X_0, \mu_0)^I$ the subgroup $H$ would be weakly mixing iff its action on the index set $I$ has no finite orbits (we always assume that the base space $(X_0, \mu_0)$ is not a point, but allow it to have atoms). For Gaussian actions arising from an orthogonal $G$-representation $\pi$, the action of a subgroup $H < G$ is weakly mixing iff the restriction $\pi |_H$ has no finite dimensional subrepresentations. We are ready to state the general form of Popa’s Cocycle Superrigidity Theorem (cf. [22, Theorem 3.1]).

1.3. **Theorem (Popa’s Cocycle Superrigidity (with the relative version)).** Let $G$ be a l.c.s.c. group with a closed subgroup $H < G$ which has relative property (T) in $G$, and let $G \acts (X, \mu)$ be an ergodic p.m.p. action (with a quotient $(Y, \nu)$). Suppose that:

(a) $G \acts (X, \mu)$ is malleable (resp. malleable relatively to $(Y, \nu)$), and either
(b) $H$ is $w$-normal in $G$ and $H \acts (X, \mu)$ is weakly mixing (resp. weakly mixing relatively to $(Y, \nu)$), or
then G of Popa - Vaes [29].

For a non-atomic probability space \((X_0, \mu_0)\) the combination of the above Theorem 1.3 and the malleability 1.2 imply \(\mathcal{U}_{\text{fin}}\)-Cocycle Superrigidity of Bernoulli actions \(\Gamma \curvearrowright (X_0, \mu_0)^{\Gamma/\Gamma_0}\) as in Theorem 1.1. To deduce \(\mathcal{U}_{\text{fin}}\)-Cocycle Superrigidity for Bernoulli actions where the base space \((X_0, \mu_0)\) has atoms, Popa uses the following argument (cf. [22, Lemma 2.11], see Lemma 3.2 below):

1.4. **Proposition (Weakly Mixing Extensions).** Let \(G\) be a l.c.s.c. group, \((X', \mu') \to (X, \mu) \to (Y, \nu)\) be p.m.p extensions with \((X', \mu') \to (X, \mu)\) being relatively weakly mixing. If \(G \curvearrowright (X', \mu')\) is \(\mathcal{U}_{\text{fin}}\)-Cocycle Superrigid (relatively to \((Y, \nu)\)), then the same applies to the intermediate action \(G \curvearrowright (X, \mu)\).

Indeed, any probability space \((X_0, \mu_0)\) is a quotient of the non-atomic probability space \(([0, 1], m)\) where \(m\) is the Lebesgue measure; hence \((X, \mu) = (X_0, \mu_0)^I\) is a \(G\)-quotient of \((X', \mu') = ([0, 1], m)^I\), which is easily seen to be relatively weakly mixing.

Skew-products provide examples of relative \(\mathcal{U}_{\text{fin}}\)-Cocycle Superrigid actions.

1.5. **Example (Skew products).** Let \(H < G\) be a w-normal subgroup with relative property (T) with a p.m.p. action \(G \curvearrowright (Y, \nu)\) and a measurable cocycle \(\sigma : G \times Y \to V\) with values in a l.c.s.c. group \(V\), and let \(V \curvearrowright (Z, \zeta)\) be a malleable p.m.p. action, e.g. generalized Bernoulli or Gaussian action. Let \(G\) act on \((X, \mu) = (Y \times Z, \nu \times \zeta)\) by \(g : (x, z) \mapsto (g.x, \sigma(g, x).z)\). If the \(H\)-action on \((X, \mu)\) is weakly mixing relative to \((Y, \nu)\), i.e., if \(H\) acts ergodically on \(Y \times Z \times Z\), then \(G \curvearrowright (X, \mu)\) is \(\mathcal{U}_{\text{fin}}\)-Cocycle Superrigid relatively to \((Y, \nu)\).

First we discuss some applications of the “absolute case” of the theorem to equivalence relations, presenting the Ergodic theoretic rather than Operator Algebra point of view. For very interesting further applications see the recent paper of Popa - Vaes [29].

1.b . **Applications to \(\Pi_1\) Equivalence Relations.** Let \(\Gamma \curvearrowright (X, \mu)\) be an ergodic action of a countable group \(\Gamma\). The “orbit structure” of this action is captured by the equivalence relation

\[ R_{X, \Gamma} = \{(x, x') \in X \times X \mid \Gamma . x = \Gamma . x'\}. \]

Two actions \(\Gamma \curvearrowright (X, \mu)\) and \(\Lambda \curvearrowright (Y, \nu)\) are said to be *Orbit Equivalent (OE)* if their orbit relations are isomorphic, where isomorphism of equivalence relations, is a measure space isomorphism \(\theta : X' \to Y'\) between conull subsets of \(X\) and \(Y\) identifying the restrictions of the corresponding equivalence relations to these subsets. Isomorphism of equivalence relations and OE of actions can be weakened by allowing \(X'\) and \(Y'\) to be arbitrary measurable subsets of *positive* (rather

\[ b') H \text{ is wq-normal in } G \text{ and } G \curvearrowright (X, \mu) \text{ is mixing (resp. mixing relatively to } (Y, \nu)^\dagger). \]

Then the \(G\)-action on \((X, \mu)\) is \(\mathcal{U}_{\text{fin}}\)-Cocycle-Superrigid (resp. relatively to \((Y, \nu)\)).
than full) measure. This is *weakly isomorphism* of relations, and *weakly Orbit Equivalent* (or *stably Orbit Equivalent*) of actions; if the original actions are of type $\Pi_1$ the ratio $\nu(Y')/\mu(X')$ is called the *compression constant* of $\theta$ (§). Zimmer’s Cocycle Superrigidity Theorem [34] paved the way to many very strong OE rigidity results in ergodic theory for higher rank lattices and other similar groups (see Zimmer’s monograph [37] and references therein, Furman [7], [8], [9], Monod - Shalom [19], Hjorth - Kechris [15]), also (Gaboriau [11], [12], and Shalom [30] for a survey).

Popa’s Cocycle Superrigidity Theorem [1,3] allows to study not only (weak) OE isomorphisms, but also (weak) OE “embeddings” and other “morphisms” between equivalence relations. By a *morphism* $\theta : (X, \mu, R) \to (Y, \nu, S)$ between ergodic countable equivalence relations we mean a measurable map $\theta : X' \to Y'$ between conull spaces $X' \subset X, Y' \subset Y$ with $\theta_*\mu|_{X'} \sim \nu|_{Y'}$, and $\theta \times \theta(R|_{X' \times X'}) \subset S|_{Y' \times Y'}$. The *kernel* of the morphism $\theta$ is the relation on $(X, \mu)$ given by

$$\text{Ker}(\theta) = \{(x_1, x_2) \in R \mid \theta(x_1) = \theta(x_2)\}$$

(note that Ker($\theta$) it is not ergodic if Y is a non-trivial space); the *image* $\text{Im}(\theta)$ of $\theta$ is the obvious subrelation of $S$ on $(Y, \nu)$.

1.6. **Definition (Relation Morphisms).** Let $\theta$ be a weak relation morphism from an ergodic equivalence relation $R$ on $(X, \mu)$ to $S$ on $(Y, \nu)$. We shall say that $\theta$ is an *injective relation morphism* if Ker($\theta$) is trivial, that $\theta$ is a *surjective relation morphism* if $\text{Im}(\theta) = S$, and that $\theta$ is a *bijective relation morphism* if Ker($\theta$) is trivial and $\text{Im}(\theta) = S$. The notion of *weakly injective/surjective/bijective relation morphism* corresponds to $X'$ and $Y'$ being positive measure subsets, rather than conull ones. In the $\Pi_1$ context weak morphisms come equipped with a compression constant $\dagger$.

1.7. **Remark.** Note that a bijective relation morphism is not necessarily an isomorphism, because $\theta : X \to Y$ need not be invertible. For example, any $\Gamma$-equivariant quotient map $\theta : (X, \mu) \to (Y, \nu)$ of free ergodic actions of some countable group $\Gamma$ defines a bijective relation morphism of the corresponding orbit relations $(X, \mu, R_{X, \Gamma}) \to (Y, \nu, R_{Y, \Gamma})$.

The following theorem summarizes some of the consequences of Theorem 1.3 to orbit relations; it is parallel to and somewhat more detailed than Popa’s [22, Theorems 0.3, 0.4, 5.6–5.8]. We denote by $G_{\text{disc}} \subset \mathcal{U}_{\text{fin}}$ the class of all discrete countable groups.

1.8. **Theorem (Superrigidity for Orbit Relations).** Let $\Gamma \acts (X, \mu)$ be a $G_{\text{disc}}$-Cocycle Superrigid p.m.p. ergodic action of a countable group $\Gamma$, and $\Lambda \acts (Y, \nu)$ be an ergodic measure preserving essentially free action of some countable group $\Lambda$ on a (possibly infinite) measure space $(Y, \nu)$ and let

$$X \supset X' \overset{\theta}{\to} Y' \subset Y$$

be a weak morphism of the orbit relations $R_{X, \Gamma}$ and $R_{Y, \Lambda}$. Then there exist:

$\dagger$In [22] a bijective relation morphism is called *local Orbit Equivalence*. 
The space of \( \Gamma \) and \( T \) is a composition \( T : (X, \mu) \longrightarrow (Y, \nu_1) \rightarrow (Y_1, \nu_1) \), where \((X, \mu_1)\) is the space of \( \Gamma_0 \)-ergodic components equipped with the natural action of \( \Gamma_1 \cong \Gamma/\Gamma_0 \), and \( T_1 : (X_1, \mu_1) \rightarrow (Y_1, \nu_1) \) is a quotient map of \( \Gamma_1 \)-actions.

Furthermore, with the notations above:

(i) If \( \Gamma \curvearrowright (X, \mu) \) is essentially free, then \( \theta \) is a weakly injective morphism iff \( \Gamma_0 = \operatorname{Ker}(\theta) \) is finite; in this case \( (X, \mu) \) is a finite extension of \((X_1, \mu_1)\).

(ii) If \( \nu(Y) < \infty \) and \( \theta \) is a weakly surjective morphism, then \( \Gamma_1 = \theta(\Gamma) \) is of finite index in \( \Lambda \) and, assuming \( \Gamma \curvearrowright (X, \mu) \) is aperiodic, \( \nu(Y)/\nu(\nu_1) \) is an integer dividing \([\Lambda : \Gamma_1]\).

(iii) If \( \nu(Y) = 1 \), \( \Gamma \curvearrowright (X, \mu) \) is essentially free and aperiodic, and \( \theta \) is a weakly bijective morphism (in particular, if \( \theta \) is a weak isomorphism) of \( \mathcal{R}_{X, \Gamma} \) to \( \mathcal{R}_{Y, \Lambda} \), then both \([\Gamma_0] \) and \([\Lambda : \Gamma_1] \) are finite, the compression constant is rational and is given by

\[
c(\theta) = \frac{\nu(\nu_1)}{\nu(Y)} \frac{[\Lambda : \Gamma_1]}{[\Gamma_0]}
\]

Such a weakly bijective morphism (or weak isomorphism) \( \theta \) extends to a bijective morphism (resp. an isomorphism) \( \tilde{\theta} : X \rightarrow Y \) iff \( c(\theta) = 1 \).

1.9. Remarks. (1) An action \( \Gamma \curvearrowright (X, \mu) \) is called aperiodic, if it has no non-trivial finite quotients; equivalently, if the restriction of the action \( \Gamma' \curvearrowright (X, \mu) \) to any finite index subgroup \( \Gamma' < \Gamma \) is ergodic. Weakly mixing actions are aperiodic.

(2) Note that often \( \Gamma_0 \) is forced to be finite, even in the context of general weak isomorphisms of orbit relations (coming from \( \mathcal{G}_{\text{disc}} \)-Cocycle Superrigid action to a free action of some countable group). Finiteness of \( \Gamma_0 \) is guaranteed if every infinite normal subgroup of \( \Gamma \) acts ergodically on \( (X, \mu) \); which is indeed the case if \( \Gamma \curvearrowright (X, \mu) \) is mixing, or if \( \Gamma \) has no proper infinite normal subgroups (e.g. \( \Gamma \) is an irreducible lattice in a higher rank Lie group, see [13, Ch VIII]).

(3) In (ii) one cannot reverse the implication. If \( \Gamma < \Lambda \) is a finite index inclusion of countable groups with property (T) so that \( \Gamma \) has no epimorphism onto \( \Lambda \), and \((X, \mu) = (X_0, \mu_0)^\Lambda \) with the Bernoulli action of the groups \( \Gamma < \Lambda \), then the identity map \( \theta(x) = x \) is a relation morphism from \( \mathcal{R}_{X, \Gamma} \) to \( \mathcal{R}_{X, \Lambda} \) which is not weakly surjective. In fact there does not exist weakly surjective morphisms from \( \mathcal{R}_{X, \Gamma} \) to \( \mathcal{R}_{X, \Lambda} \).

The rigidity with respect to weak self isomorphisms as in (iii) provides the basic tool for the computation of certain invariants such as the fundamental group,
first cohomology, the outer automorphism group of the orbit relations of \(\mathcal{U}_{\text{fin}}\) Cocycle Superrigid actions. It can also be used to produce equivalence relations which cannot be generated by an essentially free action of any group (see [28, 29], for actions of higher rank lattices see [8], [9]). It seems, however, that the full strength of Popa’s Cocycle Superrigidity Theorem, manifested in the first part of Theorem 1.8 should yield qualitatively new rigidity phenomena related to morphisms of relations.

1.c. Comparison with Zimmer’s Cocycle Superrigidity. It is natural to compare Popa’s Theorem with Zimmer’s Cocycle Superrigidity. Zimmer’s result (see [37, Theorem 5.2.5]), generalizing Margulis’ Superrigidity (see [18, Ch VII]), is a theorem about untwisting of measurable cocycles \(\alpha : G \times X \to H\) over ergodic (irreducible) actions of (semi-)simple algebraic groups \(G\) and taking values in semi-simple algebraic groups \(H\) (the result is subject to the assumption that \(\alpha\) is Zariski dense and is not “precompact”). Zimmer’s cocycle superrigidity has a wide variety of applications with two most prominent areas being:

(a) smooth volume preserving actions of higher rank groups/lattices on manifolds (here the cocycle is the derivative cocycle),

(b) orbit relations of groups actions in Ergodic theory (using the “rearrangement” cocycle).

The comparison with Popa’s result lies in the latter area. In this context Zimmer’s Cocycle Superrigidity Theorem is usually applied to cocycles \(\alpha : \Gamma \times X \to \Lambda\), where \(\Gamma < G\) is a lattice in a simple Lie group, and \(\Lambda\) is a lattice or a more general subgroup in a (semi)simple Lie group \(H\). Assuming the cocycle satisfies the assumptions of the theorem (which is the case if \(\alpha\) comes from an OE, or weak OE of free p.m.p. actions of \(\Gamma \actson (X,\mu)\) and \(\Lambda \actson (X',\mu')\)) the conclusion is that: there exists a homomorphism (local isomorphism) \(\varrho : G \to H\) and a measurable map \(\phi : X \to H\) so that \(\alpha(\gamma, x) = \phi(\gamma.x) \varrho(\gamma) \phi(x)^{-1}\). In comparison with Popa’s result note the following points:

(i) there is no claim that \(\varrho(\Gamma)\) is a subgroup of \(\Lambda < H\);

(ii) even if this is the case, say if \(G = H\), \(\Gamma = \Lambda\) and \(\varrho\) is the identity, one still cannot expect the “untwisting” map \(\phi(x)\) to take values in \(\Lambda < H\);

(iii) at the same time there is no specific assumptions on the action \(\Gamma \actson (X,\mu)\) beyond ergodicity (or irreducibility if \(G\) is semi-simple).

The point (ii) is extensively studied in [9]. The simplest example of a cocycle \(\Gamma \times X \to \Gamma < G\) which is cohomologous to the identity homomorphism as a cocycle into \(G\) but not as a cocycle into \(\Gamma\), appears in the left translation \(\Gamma\)-action on \(X = G/\Gamma\). The cocycle is \(\alpha = \varrho|_{\Gamma \times G/\Gamma}\) – the restriction of the canonical cocycle \(\varrho : G \times G/\Gamma \to \Gamma\) defined by \(\varrho(g,x) = f(g,x)\varrho f(x)^{-1}\), where \(f : G/\Gamma \to G\) is a Borel cross-section of the projection \(G \to G/\Gamma\). So Zimmer’s Cocycle Superrigidity does not, and in general cannot, untwist cocycles within a discrete group, even if the target group is \(\Gamma\) itself. This is, of course, in a sharp contrast to Popa’s result.

Another feature of Popa’s Cocycle Superrigidity is that it has no assumptions on the target group besides it being discrete, or more generally, in class \(\mathcal{U}_{\text{fin}}\). The main point of [7] and [8] was to prove a result of this type for cocycles \(\alpha : \Gamma \times X \to \Lambda\)
over a general ergodic action $\Gamma \curvearrowright (X, \mu)$ of a higher rank lattice $\Gamma$, with $\Lambda$ being an arbitrary discrete countable group. These results, however, require the following condition on the cocycle $\alpha$:

1.10. Definition. Let $\Gamma$ and $\Lambda$ be discrete countable groups, $\Gamma \curvearrowright (X, \mu)$ be a p.m.p. action and $\alpha : \Gamma \times X \to \Lambda$ be a measurable cocycle. We shall say that $\alpha$ is a Measure Equivalence cocycle (ME-cocycle), or finite covolume cocycle, if the $\Gamma$-action on the infinite Lebesgue space

$$(\tilde{X}, \tilde{\mu}) = (X \times \Lambda, \mu \times m_\Lambda) \quad g : (x, \ell) \mapsto (g.x, \alpha(g, x)\ell) \quad (g \in \Gamma)$$

is essentially free, and has a Borel fundamental domain of finite $\tilde{\mu}$-measure.

1.11. Example. Cocycles corresponding to (weak) Orbit Equivalence of essentially free actions have finite covolume. The same can be shown for cocycles corresponding to (weakly) bijective morphisms of orbit relations of essentially free p.m.p. actions. As the name suggests cocycles associated to any Measure Equivalence coupling between two countable groups are ME-cocycles (see [S]).

1.12. Theorem (ME-Cocycle Superrigidity). Let $k$ be a local field, $G$ be a $k$-connected simple algebraic group defined over $k$ with rank$_k(G) \geq 2$, $G = G(k)$ the l.c.s.c. group of the $k$-points of $G$. Denote $G = \text{Aut} (G/Z(G))$ and let $g \mapsto \tilde{g}$ be the homomorphism $G \to \tilde{G}$ with finite kernel $Z(G)$ and finite cokernel $\text{Out}(G/Z(G))$.

Let $\Gamma < G$ be a lattice and $\Gamma \curvearrowright (X, \mu)$ be an ergodic aperiodic p.m.p. action, and let $A = \{ \alpha_i \}$ be a family of measurable cocycles $\alpha_i : \Gamma \times (X, \mu) \to \Lambda_i$ of finite covolume taking values in arbitrary countable groups $\Lambda_i$.

Then there exists a measurable $\Gamma$-equivariant quotient $(X, \mu) \xrightarrow{p}(Y_A, \nu_A)$ with the following properties:

1. for each cocycle $\alpha_i : \Gamma \times (X, \mu) \to \Lambda_i$ in $A$ there exist: a short exact sequence $N_i \to \Lambda_i \to \tilde{\Lambda}_i$ with $N_i$ being finite, and $\tilde{\Lambda}_i$ being a lattice in $\tilde{G}$; a measurable map $\phi_i : X \to \tilde{\Lambda}_i$, and a measurable cocycle $g_i : \Gamma \times Y_A \to \Lambda_i$, so that, denoting $\tilde{\alpha}_i : \Gamma \times X \to \tilde{\Lambda}_i$, we have

$$\tilde{\alpha}_i(\gamma, x) = \phi_i(\gamma.x) g_i(\gamma, p(x)) \phi_i(x)^{-1}.$$ 

2. $(Y_A, \nu_A)$ is the minimal quotient with the above properties, i.e., if all $\alpha \in A$ reduce to a quotient $(Y', \nu')$ then $(Y', \nu')$ has $(Y_A, \nu_A)$ as a quotient.

In the case of a one point set $A = \{ \alpha \}$, where $\alpha : \Gamma \times X \to \Lambda$ is a finite covolume cocycle, the quotient $(Y_{\{\alpha\}}, \nu_{\{\alpha\}})$ is either

(i) trivial, in which case $\varrho_\alpha : \Gamma \to \tilde{\Lambda}$ is an inner homomorphism $\varrho_\alpha(\gamma) = \varrho_0 \gamma^{-1} \varrho_0$ by some $\varrho_0 \in \tilde{G}$ (in particular, $\varrho_0 \tilde{\Gamma} \varrho_0^{-1} < \tilde{\Lambda}$), or

(ii) is $(G/\Lambda, m_{G/\Lambda})$ and $\varrho_\alpha : \Gamma \times G/\Lambda \to \Lambda$ is the restriction of the standard cocycle.

In the general case $(Y_A, \nu_A)$ is the join $\bigvee_{x \in A}(Y_{\{\alpha\}}, \nu_{\{\alpha\}})$. For a finite set $A = \{ \alpha_1, \ldots, \alpha_n \}$ of cocycles the quotient $\Gamma \curvearrowright (Y_A, \nu_A)$ is isomorphic to the diagonal $\Gamma$-action on $(\prod_{j=1}^k \tilde{G})/\Delta$, where $\Delta$ is a lattice in $\prod_{j=1}^k \tilde{G}$ containing a product $\prod_{j=1}^k \Lambda_{ij}$ of lattices in the factors.
1.13. Remarks. (1) Assuming that the target group $\Lambda$ has only infinite conjugacy classes (ICC) one has $N_i = \{e\}$, $\bar{\Lambda}_i = \Lambda_i$, $\bar{\alpha}_i = \alpha_i$.

(2) We recall that many aperiodic actions $\Gamma \ltimes (X, \mu)$ do not have any measurable quotients of the form $G/\Lambda$ (see [8]); for such actions case (a) is ruled out, and so in this case we get an absolute ME-Cocycle Superrigidity result (for general countable ICC targets).

(3) The above Theorem is an extension of [8] Theorems B,C], the latter corresponds to the single ME-cocycle case $(Y_\{\alpha\}, \nu_{\{\alpha\}})$. We have also used this opportunity to extend the framework from higher rank real simple Lie groups to higher rank simple algebraic groups over general fields (this extension does not require any additional work, the original proofs in [7], [8] applied verbatim to simple algebraic groups over general local field).

(4) The definition 1.10 applies not only to discrete groups but to all unimodular l.c.s.c. groups $\Lambda$. Theorem 1.12 applies to this more general setting with the following adjustments: in (1) the kernel $N_i$ is compact, rather than just finite, and $\Lambda_i$ in addition to being a lattice in $G$ may also be $G/Z(G)$, $G$, or any intermediate closed group; statements (i) and (ii) are claimed only if $\bar{\Lambda}$ is discrete (i.e., is a lattice in $G$).

(5) The results in [6] suggest that in many situations it is possible to define a canonical Cocycle Superrigid quotient $(Y_A, \nu_A)$ for a given p.m.p. action $G \ltimes (X, \mu)$ and a set $A$ of measurable cocycles $\alpha : G \times X \to L_\alpha$ with target groups $L_\alpha \in \mathcal{G}_{\text{inv}}$.

1.d. Standard algebraic actions vs. Malleable actions. Continuing the line of comparison between Popa’s and Zimmer’s Cocycle Superrigidity Theorems it is natural to ask whether $\mathcal{U}_\text{fin}-$Cocycle Superrigid actions $\Gamma \ltimes (X, \mu)$ of a higher rank lattice $\Gamma < G$ admit an algebraic action $\Gamma \ltimes G/\Lambda$ as a measurable quotient. We answer this question (posed by Popa) in the negative. In fact, a much more general class of algebraic actions cannot appear as a quotient of a $\mathcal{U}_\text{fin}-$Cocycle Superrigid action, and furthermore the corresponding orbit relations are not compatible. More precisely:

1.14. Theorem. Let $H$ be a semi-simple Lie group, $\Delta < H$ an irreducible lattice, let $\Lambda < H$ be an unbounded countable subgroup acting by left translations on $(H/\Delta, m_{H/\Delta})$.

Then the algebraic ergodic p.m.p. action $\Delta \ltimes (H/\Delta, m_{H/\Delta})$, is not a quotient of any $\mathcal{U}_\text{fin}-$Cocycle Superrigid action. In fact, if $\Gamma \ltimes (X, \mu)$ is a p.m.p. $\Delta-$Cocycle Superrigid action (e.g. a $\mathcal{U}_\text{fin}-$Cocycle Superrigid action), of some countable group $\Gamma$, then the orbit relations $R_{X,\Gamma}$ has no weak relation morphisms to $R_{H/\Delta,\Lambda}$.

1.15. Remarks. Let $\Gamma < G$ be a lattice in a simple a algebraic group $G$ of higher rank, and $\Gamma \ltimes (X, \mu) = (X_0, \mu_0)^f$ be an ergodic generalized Bernoulli action. This action is $\mathcal{U}_\text{fin}-$Cocycle Superrigid by Popa’s Cocycle Superrigidity because such $\Gamma$ has property (T).
ON POPA’S COCYCLE SUPERRIGIDITY THEOREM

(1) It follows from Theorem 1.14 above that $\Gamma \actson (X, \mu)$ has no quotients of the form $\Gamma \actson (G/\Lambda, m_{G/\Lambda})$. Thus in Theorem 1.12 case (i) applies.

(2) The proof of Theorem 1.14 is based on Popa’s Cocycle Super rigidity. However, one can use a direct argument to show that in this context $\Gamma \actson (X_0, \mu_0)$ has no non-trivial quotient actions $\Gamma \actson (Y, \nu)$ where each element $g \in \Gamma$ has finite Kolmogorov-Sinai entropy: $h(Y, \nu, g) < \infty$ (see Section 5). Thus ME-Cocycle Superrigidity for Bernoulli actions of higher rank lattices could have been proved before/independently of Popa’s Cocycle Superrigidity result.

(3) Using Theorem 1.8 the above statement can be strengthened to say that the orbit relation $R_{X, \Gamma}$ of a Bernoulli action of a higher rank lattice $\Gamma$ has no weak relation morphisms to orbit relations $R_{Y, \Lambda}$ of any essentially free p.m.p. action $\Lambda \actson (Y, \nu)$ with element-wise finite entropy: $h(Y, \nu, \ell) < \infty$ for all $\ell \in \Lambda$.

1.e. Some more applications. Theorem 1.3 has some incidental applications to the structure of ergodic actions. The following is immediate from the definitions:

1.16. Proposition (Skew products). Any skew-product of an $L$-Cocycle Superrigid action $G \actson (X, \mu)$ by a p.m.p. action $L \actson (Z, \zeta)$ is isomorphic to a diagonal action on $(X \times Z, \mu \times \zeta)$ via some homomorphism $\varphi : G \to L$.

We denote by $\mathcal{G}_{\text{cmp}} \subset \mathcal{G}_{\text{fin}}$ the class of all separable compact groups.

1.17. Proposition (Extension of weakly mixing malleable actions). Let $G \actson (X, \mu)$ be a $\mathcal{G}_{\text{cmp}}$-Cocycle Superrigid action (e.g. a $\mathcal{G}_{\text{fin}}$-Cocycle Superrigid action), and $(X', \mu') \overset{p}{\to} (X, \mu)$ be a p.m.p. extension. If $G \actson (X', \mu')$ is a weakly mixing action then $p$ is relatively weakly mixing extension.

1.18. Examples. The following are examples of weakly mixing actions $\Gamma \actson (X, \mu)$ with a p.m.p. quotient $(X, \mu) \overset{p}{\to} (Y, \nu)$ which is not relatively weakly mixing; in fact, here $p$ is a compact extension.

(1) Let $\Gamma \actson (X, \mu) = (X_0, \mu_0)^{\Gamma}$ be an ergodic generalized Bernoulli action of some group $\Gamma$. (Recall that such actions are weakly mixing). Assuming that the base space $(X_0, \mu_0)$ is not an atomic probability space with distinct weights, its automorphism group $\text{Aut}(X_0, \mu_0)$ contains a non-trivial compact group $K$. The latter acts diagonally on $(X, \mu) = (X_0, \mu_0)^{\Gamma}$ commuting with $\Gamma$. Thus $\Gamma \actson (X, \mu)$ has a quotient action $\Gamma \actson (Y, \nu) = (X, \mu)/K$, and the natural projection $(X, \mu) \overset{p}{\to} (Y, \nu)$ is a compact extension of $\Gamma$-actions. (These actions exhibits very interesting rigidity phenomena extensively studied by Popa in [26], and by Popa and Vaes in [29].)

(2) Let $\Gamma < G$ be a lattice in a non-compact center free simple Lie group $G$, and $\Gamma \to K$ be a homomorphism into a compact group $K$ having a dense image (e.g. $\Gamma = \text{SL}_n(Z) \to K = \text{SL}_n(Z_p)$). Then the $\Gamma$-action on $(K, m_K)$ by translation is ergodic, and so is the induced $G$-action on $X = (G \times K)/\Gamma \cong G/\Gamma \times K$. By Howe-Moore theorem the action $G \actson
$X = G/\Gamma \times K$ is mixing, and hence weakly mixing. Yet it is a compact extension of the $G$-action on $Y = G/\Gamma$. The restriction of these actions to any unbounded subgroup $\Gamma' < G$ (e.g. $\Gamma' = \Gamma$) inherits the above properties (another use of Howe-Moore).

1.19. Remarks. (1) In the above examples $\Gamma \curvearrowright (X,\mu)$ is a compact extension of $\Gamma \curvearrowright (Y,\nu)$ defined via some cocycle $\alpha : \Gamma \times Y \to K$ where $K$ is a compact group with $X = Y \times K$. This cocycle $\alpha : \Gamma \times Y \to K$ cannot be untwisted in $K$, because being weakly mixing the action $\Gamma \curvearrowright (X,\mu)$ admits no non-trivial compact quotients.

(2) Taking $\Gamma \curvearrowright (X,\mu)$ to be $U_{\text{fin}}$-Cocycle Superrigid (e.g. Bernoulli action of a property (T) group $\Gamma$) one gets a quotient action which fails to be $U_{\text{fin}}$-Cocycle Superrigid; so relative weak mixing is a necessary assumption in Proposition 1.4.

(3) Note also that $\Gamma' \curvearrowright (G/\Gamma,m_{G/\Gamma})$ is not $U_{\text{cmp}}$-Cocycle Superrigid, even though it is ME-Cocycle Superrigid if $\Gamma$ and $\Gamma'$ are not commensurable.

Popa’s Cocycle Superrigidity implies triviality of certain skew-product constructions also in the context of equivalence relations:

1.20. Theorem. Let $\Gamma \curvearrowright (X,\mu)$ be a $U_{\text{fin}}$-Cocycle Superrigid action of a countable group $\Gamma$, denote $R = R_{X,\Gamma}$, let $\mathcal{S}$ some type $\Pi_1$ equivalence relation on some $(Y,\nu)$, and let $Q$ be an ergodic subrelation of $R \times \mathcal{S}$ on $(X \times Y,\mu \times \nu)$, such that the projection $X \times Y \to X$ is a relation reduction of $Q$ to $R$. Then there exists a p.m.p. $\Gamma$-action $\Gamma \curvearrowright (Y,\nu)$ with $R_{Y,\Gamma} \subset \mathcal{S}$, so that $Q$ differs from the orbit relation $R_{X \times Y,\Gamma}$ of the diagonal action $\Gamma$-action on $(X \times Y,\mu \times \nu)$ by an inner automorphism of $R \times \mathcal{S}$.

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2. Generalities and Preliminaries

2.a. $\text{Aut} (X,\mu)$ as a topological group. Let $(X,\mu)$ be a Lebesgue measure space (finite or infinite). We denote by $\text{Aut} (X,\mu)$ the group of all measure preserving measurable bijections between co-null sets of $X$, where two maps which agree $\mu$-a.e. are identified. This group has two natural topologies: which we call uniform, and weak (also known as the vague or coarse topology). The uniform topology can be defined by the metric

$$d^{(u)}(T,S) = \min(1, \mu \{ x \in X \mid T(x) \neq S(x) \}) \quad (T,S \in \text{Aut} (X,\mu)),$$

where the truncation by 1 is relevant only to the infinite measure case. Note that $d^{(u)}$ is a bi-invariant metric on $\text{Aut} (X,\mu)$, which is easily seen to be complete. However, $\text{Aut} (X,\mu)$ is not separable in this metric.

We shall be almost exclusively using another, weaker, metrizable topology on $\text{Aut} (X,\mu)$, with respect to which it becomes a Polish group, i.e., complete second countable (equivalently separable) topological group. This topology is defined by
a family of pseudo-metrics

$$d_E^{(w)}(T, S) = \mu(T E \triangle S E)$$

where \( E \subset X \) are measurable sets of finite measure. One easily checks that

$$|d_E^{(w)} - d_F^{(w)}| \leq 2\mu(E \triangle F)$$

so the above topology is determined by countably many pseudo-metrics \( d_E^{(w)} \) where \( \{E_i\}_{i=1}^\infty \) are dense in the \( \sigma \)-algebra of \( X \). Hence this topology is metrizable, the group is separable and is easily seen to be complete.

Both the uniform and the weak topologies are quite natural under the embedding \( \pi : \text{Aut}(X, \mu) \to U(L^2(X, \mu)) \) in the unitary group of \( L^2(X, \mu) \). Under this embedding the uniform topology corresponds to the topology of the operator norm, while the weak topology corresponds to the strong/weak operator topology (recall that the two coincide on the unitary group). The fact that the image of \( \text{Aut}(X, \mu) \) is closed in \( U(L^2(X, \mu)) \) both in the weak/strong operator topology and in the norm topology follows from the completeness of the uniform and the weak topologies on \( \text{Aut}(X, \mu) \).

2.b. Target groups – classes \( \mathcal{U}_{\text{fin}} \subset \mathcal{G}_{\text{inv}} \). In [22] Sorin Popa singles out a very interesting class of groups \( \mathcal{U}_{\text{fin}} \). By definition \( \mathcal{U}_{\text{fin}} \) consists of all Polish groups which can be embedded as closed subgroups of the unitary group of a finite von Neumann algebra. Groups of this class are well adapted to techniques involving unitary representations. Popa points out that the class \( \mathcal{U}_{\text{fin}} \) is contained in the class \( \mathcal{G}_{\text{inv}} \) of all Polish groups which admit a bi-invariant metric (see [22, 6.5]), and it seems to be unknown whether \( \mathcal{U}_{\text{fin}} \subset \mathcal{G}_{\text{inv}} \) is a proper inclusion. We shall use the defining property of the class \( \mathcal{G}_{\text{inv}} \), which turns out to be very convenient for manipulations with abstract measurable cocycles (see Section 3 below and forthcoming [6]). We shall consider the following subclasses as the main examples of groups in \( \mathcal{U}_{\text{fin}} \subset \mathcal{G}_{\text{inv}} \):

- \( \mathcal{G}_{\text{lsc}} \) – the class of all discrete countable groups. \( \mathcal{G}_{\text{lsc}} \subset \mathcal{U}_{\text{fin}} \) since a discrete group \( \Lambda \) embeds in the unitary group of its von Neumann algebra \( L(\Lambda) \) which has type \( \Pi_1 \). The discrete metric on \( \Lambda \) is clearly bi-invariant.
- \( \mathcal{G}_{\text{cmp}} \) – the class of all second countable compact groups. If \( K \) is a compact separable group then its von Neumann algebra \( L(K) \) is of type \( \Pi_1 \) (the trace is given by integration with respect to the Haar measure). To see directly that compact groups admit a bi-invariant metric, start from any metric \( d_0 \) defining the topology on \( K \) and minimize, or average over \( K \times K \)-orbit to obtain an equivalent bi-\( K \)-invariant metric:

$$d_{\text{inf}}(g, h) = \inf_{K \times K} d_{k,k'}(g, h), \quad d_{\text{ave}}(g, h) = \int_{K \times K} d_{k,k'}(g, h) \, dk \, dk'$$

where \( d_{k,k'}(g, h) = d_0( kgk', khk' ) \), \((k, k' \in K)\).
- \( \text{Inn}(\mathcal{R}) \) – the inner automorphism group (a.k.a. as the full group) of a \( \Pi_1 \) countable equivalence relation on a standard probability space \((X, \mu)\)

$$\text{Inn}(\mathcal{R}) = \{ T \in \text{Aut}(X, \mu) \mid (x, T(x)) \in \mathcal{R} \text{ a.e. } x \in X \}$$
Then $\text{Inn}(\mathcal{R})$ embeds in the unitary group of the Murray-von Neumann algebra $M(\mathcal{R})$ associated to $\mathcal{R}$; the type of the algebra is that of the the relation, i.e. $\text{II}_1$ - a finite type.

More explicitly, $\text{Inn}(\mathcal{R})$ embeds as a closed subgroup of $\text{Aut}(\mathcal{R}, \tilde{\mu})$, where $\tilde{\mu}$ denotes the infinite Lebesgue measure on $\mathcal{R}$ obtained by lifting $\mu$ on $X$ via $p_1: \mathcal{R} \to X$, using the counting measure on the fibers. Hence $\text{Inn}(\mathcal{R})$ inherits the weak topology from $\text{Aut}(\mathcal{R}, \tilde{\mu})$, coming from the weak/strong operator topology on the unitary group of $L^2(\mathcal{R}, \tilde{\mu})$. The same topology can also be obtained as the restriction of the uniform topology on $\text{Aut}(X, \mu)$ to the subgroup $\text{Aut}(\mathcal{R})$ (see (2.a)). Since $d^{(a)}$ is a bi-invariant metric on $\text{Aut}(X, \mu)$, in which $\text{Inn}(\mathcal{R})$ is easily seen to be closed, this gives another explanation to the inclusion $\text{Inn}(\mathcal{R}) \subseteq \mathcal{G}_{\text{inv}}$.

Both classes $\mathcal{G}_{\text{fin}} \subset \mathcal{G}_{\text{inv}}$ are closed under: passing to closed subgroups, taking countable direct sums, taking direct integrals (i.e., passing from $G$ to $[0,1]^G$).

2.c. Quotients and Skew products. A morphism between two p.m.p. actions of $G$ on $(X, \mu)$ and $(Y, \nu)$ is a measurable map $p: X \to Y$ with $p_*\mu = \nu$, such that for each $g \in G$: $p(g.x) = g.p(x)$ for $\mu$-a.e. $x \in X$. We say that $Y$ is a quotient of $X$, and/or that $X$ is an extension of $Y$. Given such a map one obtains a disintegration: a measurable map $Y \ni y \mapsto \mu_y \in \text{Prob}(X)$ such that

$$\mu = \int_Y \mu_y d\nu(y), \quad \mu_y(p^{-1}(\{y\})) = 1 \quad \text{for } \nu\text{-a.e. } y \in Y.$$ 

Such a disintegration is unique (modulo null sets), and so it follows that for each $g \in G$: $g_*\mu_y = \mu_{g.y}$ for $\nu$-a.e. $y \in Y$.

Let $G \curvearrowright (Y, \nu)$ be a p.m.p. action, let $\sigma: G \times Y \to H$ be a measurable cocycle taking values in some Polish group $H$ which has a p.m.p. action $H \curvearrowright (Z, \zeta)$. Then one can construct the skew-product $G$-action

$$(X, \mu) = (Y \times Z, \nu \times \zeta), \quad g: (x, z) \mapsto (g.x, \sigma(g, x).z)$$

For the skew-product action $G \curvearrowright (X, \mu)$ to be ergodic it is necessary, though not sufficient, that both $G \curvearrowright (Y, \nu)$ and $V \curvearrowright (Z, \zeta)$ are ergodic.

We say that a morphism $(X, \mu) \to (Y, \nu)$ of $G$-actions is an isometric extension if there exists a compact group $K$, a closed subgroup $K_0 < K$, and a measurable cocycle $\sigma: G \times Y \to K$, so that $G \curvearrowright (X, \mu)$ is measurably isomorphic to the skew product extension of $G \curvearrowright (Y, \nu)$ by $Z = K/K_0$ with the corresponding Haar measure.

The $G$-action on $(Y, \nu)$ is a quotient of $G \curvearrowright (X, \mu)$; in fact, for ergodic p.m.p. one can view any quotient $(X, \mu) \to (Y, \nu)$ of $G$-actions can be viewed as a skew-product defined by an action of $H < \text{Aut}(Z, \zeta)$. Here $(Z, \zeta)$ is a standard nonatomic Lebesgue space, unless $X \to Y$ is a finite extension in which case we can take $Z = \{1, \ldots, k\}$ and $H = S_k$. The latter case is an example of an isometric extension of $Y$ using $K = S_k$ and $K_0 \cong S_{k-1}$.
2.d. Fibered Products. Given two p.m.p. actions $G \curvearrowright (X_i, \mu_i), \ i = 1, 2,$ which have a common quotient action $G \curvearrowright (Y, \nu)$, via $p_i : X_i \rightarrow Y,$ one can define the corresponding fibered product space 
$$X_1 \times_Y X_2 = \{(x_1, x_2) \in X_1 \times X_2 \mid p_1(x_1) = p_2(x_2)\}$$
equipped with a fibered product probability measure 
$$\mu_1 \times_Y \mu_2 = \int_Y \mu_{1,y} \times \mu_{2,y} \, d\nu(y)$$
where $\mu_i = \int \mu_{i,y} \, d\nu(y)$ are the corresponding disintegrations. The measure $\mu_1 \times_Y \mu_2$ is invariant under the diagonal $G$-action $g : (x_1, x_2) \mapsto (g.x_1, g.x_2)$ on $X_1 \times_Y X_2.$ The definition of fibered products has an obvious extension to more than two factors. We shall denote by $(X^n_Y, \mu^n)$ the fibered product of $1 \leq n \leq \infty$ copies of a given $G$-action $G \curvearrowright (X, \mu)$ with the quotient $G$-action on $(Y, \nu).$ The usual product (which can be viewed as the fibered product over the trivial $Y$) will be denoted by $(X^n, \mu^n).$

2.e. Relative weak mixing. Recall that a p.m.p. action $G \curvearrowright (X, \mu)$ is weakly mixing if the diagonal $G$-action on $(X \times X, \mu \times \mu)$ is ergodic. If $G \curvearrowright (X, \mu)$ is weakly mixing then the diagonal $G$-action on $(X \times X', \mu \times \mu')$ is ergodic for any ergodic p.m.p. action $G \curvearrowright (X', \mu')$, and it follows that the diagonal $G$-action on the product $(X^n, \mu^n)$ is ergodic, and weakly mixing.

We shall be interested in the relative version of the notion of weak mixing, introduced by Furstenberg in [10], and Zimmer in [34, 35]. Glasner’s [13, Ch 9, 10] is a good recent reference.

2.1. Theorem (Relative weak mixing). Let $(X, \mu) \rightarrow (Y, \nu)$ is a morphism of ergodic G-actions. The following are equivalent:

1. The morphism is relatively weakly mixing, i.e., $G \curvearrowright (X^n_Y, \mu^n)$ is ergodic.
2. For any ergodic p.m.p. action $G \curvearrowright (X', \mu')$ which has $G \curvearrowright (Y, \nu)$ as a quotient, the $G$-action on $(X \times_Y X', \mu \times_Y \mu')$ is ergodic.
3. The diagonal $G$-action on $(X^n_Y, \mu^n)$ is ergodic for all $n \geq 2.$
4. There does not exist intermediate quotients $(X, \mu) \rightarrow (Y', \nu') \rightarrow (Y, \nu)$ of the $G$-actions, where $(Y', \nu') \rightarrow (Y, \nu)$ is a non-trivial isometric extension.

Of course $G \curvearrowright (X, \mu)$ is weakly mixing iff it is weakly mixing relative to the trivial action on a point.

Warning. If $(X, \mu) \rightarrow (Y, \nu) \rightarrow (Z, \zeta)$ are morphisms of ergodic $G$-actions and $(X, \mu) \rightarrow (Z, \zeta)$ is relatively weak mixing, then $(Y, \nu) \rightarrow (Z, \zeta)$ is also relatively weakly mixing, but $(X, \mu) \rightarrow (Y, \nu)$ need not be such. In particular, a weakly mixing action $G \curvearrowright (X, \mu)$ can fail to be weakly mixing relative to some quotients (recall Examples [1, 15]).

2.f. Relative Property (T). Let $\pi : G \rightarrow U(\mathcal{H}_\pi)$ be a unitary representation of some topological group (unitary representations are always assumed to be continuous with respect to the weak/strong topology on the unitary group). Given a
compact subset $K \subset G$ and $\epsilon > 0$ let

$$V_{K,\epsilon} = \left\{ v \in H \left| \|v\| = 1, \max_{g \in K} \|\pi(g)v - v\| < \epsilon \right\}$$

Vectors in $V(K,\epsilon)$ are called $(K,\epsilon)$-almost $\pi(G)$-invariant. A unitary $G$-representation $\pi$ is said to almost have invariant vectors if $V_{K,\epsilon}$ is non-empty for any compact $K \subset G$ and $\epsilon > 0$.

A l.c.s.c. group $G$ has Kazhdan’s property (T) if every unitary $G$-representation which almost has invariant vectors, actually has non-trivial invariant vectors. A closed subgroup $H < G$ is said to have relative property (T) in $G$, if every unitary $G$-representation $\pi$ which almost has $G$-invariant vectors, has $H$-invariant vectors.

The fact that a sequence of unitary representations can be organized into a single one, by taking their direct sum, can be used to show that the quantifiers in the definition of property (T) can be interchanged to say that: $G$ has property (T) (resp. $H < G$ has relative (T)) if there exist compact $K \subset G$ and $\epsilon > 0$, so that any unitary $G$-representation with non-empty $V_{K,\epsilon}$ has non-trivial $G$-invariant vectors (resp. $H$-invariant vectors). Furthermore, the invariant vectors can be found close to the almost invariant ones:

2.2. Theorem (Jolissaint [17]). If $H < G$ has relative property (T), then for any $\delta > 0$ there exists a compact subset $K \subset G$ and $\epsilon > 0$ such that: any unitary $G$-representation $\pi$ with non-empty $V_{K,\epsilon}$ has an $H$-invariant unit vector $u$ with $\text{dist}(u,V_{K,\epsilon}) < \delta$.

For property (T) groups (i.e. the case where $H = G$) or, more generally, in the case where $H$ is normal in $G$, the above result is easy (consider the $G$-representation on the orthogonal complement of the space of $H$-invariant vectors). For the general case of an arbitrary subgroup $H < G$ with relative property (T), a different argument is needed (see [17], or the appendix in [21]).

Many property (T) groups are known (see [3], [38]), but there are many many more groups with normal subgroups with relative property (T). Basic example of this type is $H = Z^2$ in $G = SL_2(Z) \rtimes Z^2$. Many arithmetic examples are constructed in Alain Valette’s [33]. A very general construction of relative property (T) inclusions of an Abelian $H$ into semi-direct products $G = \Gamma \rtimes H$ were recently obtained by Talia Fernós in [4].

2.g. Gaussian Actions. Let us briefly describe this basic construction of p.m.p. actions stemming from unitary (or rather orthogonal) representations. For references see [2], [13], [11] and references therein.

The standard Gaussian distribution on $R$ has density $(2\pi)^{-1/2}e^{-x^2/2} dx$; the direct product of $k$ copies of this measure, denoted $\mu_{R^k}$, is the standard Gaussian measure on $R^k$; it has density $(2\pi)^{-k/2}e^{-\|x\|^2/2} dx$ on $R^k$. The standard measure $\mu_{R^k}$ is invariant under orthogonal transformations, and orthogonal projections $R^k \rightarrow R^l$ map $\mu_{R^k}$ to $\mu_{R^l}$. A general (centered) Gaussian measure on a finite dimensional vector space $R^n$ is a pushforward $A_*\mu_{R^k}$ of the standard Gaussian measure $\mu_{R^k}$ by a linear map $A : R^k \rightarrow R^n$ (one can always take $k \leq n$ and
\( A = UD \) where \( D \) is diagonal and \( U \) is orthogonal). For \( \mu = A_\ast \mu \llcorner_k \) the coordinate projections have correlation matrix \( C = AA^T \); in fact a general (centered) Gaussian measure on \( \mathbb{R}^n \) is determined by its correlation matrix \( C \) and therefore will be denoted \( \mu_C \). If \( C \) is invertible, we say that \( \mu_C \) is non-degenerate; in this case \( d\mu_C(y) = (2\pi)^{-n/2} \det(C)^{-1} e^{-(y,C^{-1}y)/2} \, dy \).

Let \( I \) be a countable set, \( \{v_i\}_{i \in I} \) a sequence of unit vectors in some real Hilbert space \( \mathcal{H} \), and \( C : I \times I \to [-1,1] \) be given by \( C_{i,j} = \langle v_i, v_j \rangle \). Then for any finite subset \( J = \{i_1, \ldots, i_n\} \subset I \) the \( n \times n \) matrix \( C_J \) is a correlation matrix, and defines a centered Gaussian probability measure \( \mu_{C_J} \) on \( \mathbb{R}^J \). It is easy to see that for any two finite subsets \( K \subset J \) of \( I \) the projection of \( \mu_{C_J} \) under \( \mathbb{R}^J \to \mathbb{R}^K \) is \( \mu_{C_K} \). Hence by Kolmogorov’s extension theorem there exists a unique probability measure \( \mu_C \) on \( \mathbb{R}^I \) whose projections on \( \mathbb{R}^J \) are \( \mu_{C_J} \) for any finite subset \( J \subset I \).

Now let \( \pi : G \to O(\mathcal{H}) \) be a linear orthogonal representation of a topological group \( G \), and assume that \( G \) leaves invariant the set \( \{v_i\}_{i \in I} \). Then for any \( g \in G \):

\[
\langle \pi(g)v_i, \pi(g)v_j \rangle = \langle v_i, v_j \rangle,
\]

and therefore the map \( g : \mathbb{R}^J \to \mathbb{R}^gJ \) maps \( \mu_{C_J} \) onto \( \mu_{C_{gJ}} \). This defines a measure preserving \( G \)-action on \( (\mathbb{R}^J, \mu_C) \), which is called the Gaussian process/action associated to the unitary \( G \)-action on the set \( V = \{v_i\}_{i \in I} \). Assuming \( V \) spans a dense subspace of \( \mathcal{H} \), we get an isometric embedding \( \mathcal{H} \to L^2(\mathbb{R}^J, \mu_C) \) sending \( v_i \) to the coordinate projection \( f_i : \mathbb{R}^J \to \mathbb{R} \). In particular \( \pi \) becomes a subrepresentation of the \( G \)-representation on \( L^2(\mathbb{R}^J, \mu_C) \).

2.3. Remark. Observe that Bernoulli actions with a non-atomic base space is a particular case of a Gaussian process, corresponding to \( C \) being the identity matrix, i.e., \( \{v_i\}_{i \in I} \) being orthonormal, the situation arising in the \( G \)-representation on \( \mathcal{H} = \ell^2(I) \).

Gaussian actions can be defined not only for representations of discrete groups, but for continuous orthogonal representations \( \pi : G \to O(\mathcal{H}) \) of an arbitrary l.c.s.c. group \( G \). The fastest explanation is the following: given \( \pi : G \to O(\mathcal{H}) \) of such a group \( G \) choose a dense countable subgroup \( \Lambda < G \) and apply the above construction to a \( \pi(\Lambda) \)-invariant spanning set \( \{v_i\}_{i \in I} \subset \mathcal{H} \). Note that continuity of the representation \( \pi \) implies continuity of the \( \Lambda \)-action on \( (\mathbb{R}^J, \mu_C) \), i.e., if \( \lambda_n \to 1 \) in \( G \) then

\[
\mu_C(\lambda_n E \Delta E) \to 0, \quad (E \in \mathcal{B}(\mathbb{R}^J)).
\]

Indeed, it suffices to check this for sets \( E \) which appear as a preimage of a Borel subset of \( \mathbb{R}^J \) of the projection \( \mathbb{R}^J \to \mathbb{R}^J \), where \( J \subset I \) is finite. For such sets \( E \) follows from the fact that \( \lim_{n \to \infty} \max_{i \in J} \| \pi(g_n)v_j - v_j \| = 0 \). Hence, the \( \Lambda \)-action on \( (X, \mu) \) extends to \( G \to \text{Aut} (X, \mu) \) by continuity. Since \( G \) is l.c.s.c. this homomorphism is realized as a measurable action \( G \acts (X, \mu) \).

---

Footnote: Often Gaussian actions are described as the action on a probability space on which a generating family \( \{X_i\}_{i \in I} \) of standard Gaussian variables is defined with correlations \( E(X_i X_j) = \langle v_i, v_j \rangle \); the \( G \)-action corresponds to the permutation of \( X_i \)’s. In our description the underlying space is \( (\mathbb{R}^J, \mu_C) \) and \( X_i \)’s are coordinate projections.
3. Cohomology of Measurable Cocycles

This section contains some results about measurable cocycles $G \times X \to L$ which do not depend on property (T) of $G$ or on malleability of the action $G \leq (X, \mu)$ (results related to the latter notions are discussed in Section [1]). Hence the nature of the acting l.c.s.c. group $G$ is immaterial. However, it will be important to impose certain conditions on the target group $L$. Here we will assume that $L \in \mathcal{G}_{\text{inv}}$, namely that it admits a bi-invariant metric $d$ with respect to which it is a complete separable group. The importance of this condition and of the weak mixing assumption will become clear in the following Lemma 3.2, which is a basis for the main statements 3.0–3.5 of this section. These statements parallel Lemma 2.11, Proposition 3.5, Theorem 3.1, and Lemma 3.6 of Popa’s [22], where they are proved in the context of Operator Algebras. Here we work in the framework of the class $\mathcal{G}_{\text{inv}} \supset \mathcal{G}_{\text{fn}}$ and use only elementary Ergodic theoretic arguments. We remark that similar results hold for some other classes of target groups and, sometimes without weak mixing assumption. This is discussed in the forthcoming paper [6].

We start by an elementary observation:

3.1. Lemma (Separability argument). Let $(X, \mu)$ be a probability space, $(M, d)$ a separable metric space, $\Phi : X \to M$ a Borel map such that $d(\Phi(x_1), \Phi(x_2)) = d_0$ for $\mu \times \mu$-a.e. $(x_1, x_2) \in X \times X$. Then $d_0 = 0$ and $\Phi_\ast \mu = \delta_{m_0}$ for some $m_0 \in M$.

Proof. Assume $d_0 > 0$. By separability, $M$ can be covered by countably many open balls $M = \bigcup_{i=1}^\infty B_i$ with $\text{diam}(B_i) = \sup\{d(p, q) \mid p, q \in B_i\} < d_0$. Note that $E_i = \Phi^{-1}(B_i)$ has $\mu(E_i) = 0$, but this is impossible because $1 = \sum_{i=1}^\infty \mu_y(E_i)$. Hence $d_0 = 0$ and therefore $\Phi_\ast \mu$ is a Dirac measure $\delta_{m_0}$ at some $m_0 \in M$. □

The following should be compared to Popa’s original [22, Lemma 2.11].

3.2. Lemma (Basic). Let $G \leq (X, \mu)$ be an ergodic action, $(X, \mu) \overset{p}{\twoheadrightarrow} (Y, \nu)$ a $G$-equivariant measurable quotient which is relatively weakly mixing. Let $L$ be a group in class $\mathcal{G}_{\text{inv}}$, and let $\alpha, \beta : G \times Y \to L$ be two measurable cocycles. Let $\Phi : X \to L$ be a measurable function, so that for each $g \in G$ for $\mu$-a.e. $x \in X$:

$$\alpha(g, p(x)) = \Phi(g,x)\beta(g, p(x))\Phi(x)^{-1}.$$  

Then $\Phi$ descends to $Y$: there exists a measurable $\phi : Y \to L$ so that $\Phi = \phi \circ p$ a.e. on $X$, and

$$\alpha(g, y) = \phi(g, y)\beta(g, y)\phi(y)^{-1}.$$  

This Lemma describes the injectivity of the pull-back map

$$H^1(G \times X, L) \overset{p^\ast}{\twoheadleftarrow} H^1(G \times Y, L)$$

between the sets of equivalence classes of measurable cocycles, corresponding to weakly mixing morphisms $X \overset{p}{\twoheadrightarrow} Y$ and coefficients $L$ from $\mathcal{G}_{\text{inv}}$. This implies Proposition [1.4] in the introduction.

3.3. Remark. The assumption of relative weak mixing is essential (recall Remark [1.9]). For ergodic extensions $(X, \mu) \overset{p}{\twoheadrightarrow} (Y, \nu)$ which are not necessarily relatively weakly mixing one can prove that if cocycles $\alpha, \beta : G \times Y \to L$ (with
L ∈ ℓ^∞(G) are cohomologous when lifted to G × X → L, then they are already cohomologous when lifted to an intermediate isometric extension (Y', ν') → (Y, ν) (a finite extension, if L is countable). This is discussed in more detail in [3].

Proof. Consider the diagonal G-action on the fibered product (X × Y, µ × ν). It is ergodic by the assumption of relative weak mixing. The function f : X × Y → [0, ∞) defined by

\[ f(x_1, x_2) = d(\Phi(x_1), \Phi(x_2)) \]

is measurable and is G-invariant. Indeed for any g ∈ G for µ × ν-a.e. (x_1, x_2) we have, denoting by \( y = p(x_1) = p(x_2) \):

\[
\begin{align*}
f(g.x_1, g.x_2) &= d(\Phi(g.x_1), \Phi(g.x_2)) \\
&= d(\alpha(g,y)\Phi(x_1)\beta(g,y)^{-1}, \alpha(g,y)\Phi(x_2)\beta(g,y)^{-1}) \\
&= d(\Phi(x_1), \Phi(x_2)) = f(x_1, x_2).
\end{align*}
\]

Ergodicity of the G-action on the fibered product (i.e. the assumption of relative weak mixing) implies that f is essentially a constant \( d_0 \), i.e. \( f(x_1, x_2) = d_0 \) for µ × ν-a.e. (x_1, x_2). For ν-a.e. \( y \in Y \) we apply Lemma 3.1 to \((X, \mu_y)\) to deduce that \( \Phi_*\mu_y = \delta_{\phi(y)} \) for some \( \phi : Y \to L \). Measurability of \( \phi \) follows from that of \( \Phi \).

□

The following is a relative version of Popa’s [22, Theorem 3.1]. Our proof is based on the Basic Lemma above.

3.4. Theorem (Untwisting cocycles). Let G ↾ (X, µ) be p.m.p. action which is weakly mixing relative to a quotient action G ↾ (Y, ν); let L be a group from class ℓ^∞, and let \( \alpha, \beta : G × X \to L \) be two measurable cocycle. Assume that there exists a map \( F : X × Y × X → L \) so that for every \( g ∈ G \): for \( \mu^G_y \)-a.e. \( (x_1, x_2) ∈ X^2_Y \)

\[ \alpha(g, x_1) = F(g.x_1, g.x_2) \beta(g.x_2) F(x_1, x_2)^{-1}. \]

Then there exists a measurable cocycle \( \phi : G × Y → L \) and measurable maps \( \phi, \psi : X → L \) so that

\[
\begin{align*}
\alpha(g,x) &= \phi(g.x) \phi(g,p(x)) \phi(x)^{-1}, \\
\beta(g,x) &= \psi(g.x) \psi(g,p(x)) \psi(x)^{-1},
\end{align*}
\]

for all \( g ∈ G \) and µ-a.e. \( x ∈ X \).

Of course the “absolute case” corresponds to \( G ↾ (Y, ν) \) being the trivial action on a point; in this case \( \phi : G → L \) is a homomorphism. The reader, interested in this case only, should read the following short proof consistently replacing the phrase “for ν-a.e. \( y ∈ Y \) for \( \mu^G_y \)-a.e. ...” by just “for µ-a.e. ...”.

Proof. By the assumption for ν-a.e. \( y ∈ Y \) the relation (4) holds for \( \mu_y × \mu_y \)-a.e. \( (x_1, x_2) \). By Fubini for ν-a.e. \( y ∈ Y \) for \( \mu_y × \mu_y × \mu_y \)-a.e. \( (x_1, x_2, x_3) \) we have both relations:

\[
\begin{align*}
\alpha(g, x_1) &= F(g.x_1, g.x_2) \beta(g, x_2) F(x_1, x_2)^{-1} \quad \text{and} \\
\alpha(g, x_3) &= F(g.x_3, g.x_2) \beta(g, x_2) F(x_3, x_2)^{-1}.
\end{align*}
\]
Substituting the first in the second, we obtain the following identities which hold \( \mu^3 \)-almost everywhere on \( X_Y^3 \):

\[
\alpha(g, x_3) = F(g, x_3, g, x_2) \beta(g, x_2) F(x_3, x_2)^{-1} = F(g, x_3, g, x_2) F(g, x_1, g, x_2)^{-1} \alpha(g, x_1) F(x_1, x_2) F(x_3, x_2)^{-1}.
\]

Setting \( \Phi(x_1, x_2, x_3) \overset{\text{def}}{=} F(x_1, x_2) F(x_3, x_2)^{-1} \) the above takes the form

\[
\alpha(g, x_1) = \Phi(g, x_1, g, x_2, g, x_3) \alpha(g, x_3) \Phi(x_1, x_2, x_3)^{-1}
\]
or, equivalently:

\[
\Phi(g, x_1, g, x_2, g, x_3) = \alpha(g, x_1) \Phi(x_1, x_2, x_3) \alpha(g, x_3)^{-1}.
\]

Next, observe that the morphism of the diagonal \( G \)-actions

\[
q : (X_Y^3, \mu^3_Y) \to (X_Y^2, \mu^2_Y), \quad q(x_1, x_2, x_3) = (x_1, x_3)
\]
is relatively weakly mixing. Indeed, weak mixing of this morphism is equivalent to the ergodicity of the diagonal \( G \)-action on the fibered product

\[
X_Y^3 \times_{(X_Y^1)} X_Y^3 \cong X_Y^4.
\]

Ergodicity of the \( G \)-action on the latter follows from relative weak mixing (Theorem 2.4 (1) \( \implies \) (3)).

Hence, by the Basic Lemma 3.2 \( \Phi(x_1, x_2, x_3) = f(x_1, x_3) \) for some measurable map \( f : X_Y^2 \to L \). Therefore

\[
F(x_1, x_2) F(x_3, x_2)^{-1} = \Phi(x_1, x_2, x_3) = f(x_1, x_3)
\]
meaning that \( \mu^3_Y \)-a.e. on \( X_Y^3 \):

\[
F(x_1, x_2) = f(x_1, x_3) F(x_3, x_2).
\]

Using Fubini one can choose a measurable section \( s : Y \to X \) so that for \( \nu \)-a.e. \( y \in Y \) for \( \mu_y \times \mu_y \)-a.e. \( (x_1, x_2) \):

\[
F(x_1, x_2) = f(x_1, s(y)) F(s(y), x_2).
\]

Defining \( \phi, \psi : X \to L \) by

\[
\phi(x) \overset{\text{def}}{=} f(x, s \circ p(x)), \quad \psi(x) \overset{\text{def}}{=} F(s \circ p(x), x)^{-1}
\]
we get that \( F(x_1, x_2) \) splits as \( F(x_1, x_2) = \phi(x_1) \psi(x_2)^{-1} \). This allows to rewrite 4 as \( \mu^3_Y \)-a.e. identity:

\[
\phi(g, x_1)^{-1} \alpha(g, x_1) \phi(x_1) = \psi(g, x_2)^{-1} \beta(g, x_2) \psi(x_2).
\]

Hence for each \( g \in G \): for \( \nu \)-a.e. \( y \in Y \) the above holds for \( \mu_y \times \mu_y \)-a.e. pair \((x_1, x_2)\). Therefore the left and the right hand sides are each \( \mu_y \)-a.e. constant, which we denote by \( \varrho(g, y) \). This measurable function \( \varrho : G \times Y \to L \) inherits the cocycle equation from \( \alpha \) and/or \( \beta \). \( \square \)
In the proof of the Popa Cocycle Superrigidity theorem in the case of group inclusion \( H < G \) with relative property (T), one first shows that the restriction of a cocycle \( \alpha : G \times X \to L \) to \( H \times X \to L \) can be "untwisted" so to descend to a cocycle \( \varrho : H \times Y \to L \) (or to become a homomorphism \( \varrho : H \to L \) in the "absolute" case). One then needs to show that this is enough to untwist the whole cocycle \( \alpha : G \times X \to L \). The next Lemma, parallel to Popa's \[22\] Lemma 3.6 (going back to \[27\], Lemma 5.7), shows how the "untwisting" on \( H \) extends to a larger subgroup \( \hat{H} \subset G \).

3.5. Lemma (Promoting homomorphisms). Let \( G \acts (X,\mu) \) be a p.m.p. action with a quotient \((Y,\nu)\), \( H < G \) be subgroups, and \( \alpha : G \times X \to L, \varrho : H \times Y \to L \) be measurable cocycles taking values in a group \( L \) from class \( \mathcal{G}_{\text{inv}} \). Suppose that the restriction of \( \alpha \) to \( H \times X \) descends to \( \varrho \), namely:
\[
\alpha(h,x) = \varrho(h,p(x)) \quad (h \in H).
\]
and that \( H \acts (X,\mu) \) is relatively weakly mixing relative to \((Y,\nu)\).

Then, denoting \( \hat{H} = N_G(H) \), the cocycle \( \varrho \) extends to \( \hat{\varrho} : \hat{H} \times Y \to L \) and the restriction of \( \alpha \) to \( \hat{H} \times X \) descends to \( \hat{\varrho} \):
\[
\alpha(g,x) = \hat{\varrho}(g,p(x)) \quad (g \in \hat{H}).
\]
Moreover, if there exists a subgroup \( K < H \) so that \( K \acts (X,\mu) \) is weakly mixing relative to \((Y,\nu)\), then the same conclusion applies to the group \( \hat{H} \) generated by the set \( G_{K,H} \defeq \{ g \in G \mid gKg^{-1} < H \} \supseteq N_G(H) \).

Proof. Let \( K < H \) (possibly \( K = H \)) be a subgroup so that \( K \acts (X,\mu) \) is weakly mixing relative to \((Y,\nu)\). For \( g \in G \) we denote by \( k \mapsto k^g \defeq gkg^{-1} \) the conjugation isomorphism \( K^g \overset{\sim}{\to} K^g = \{ k^g \mid k \in K \} \).

Fix some \( g \in G_{K,H} \). Computing \( \alpha(gk,x) = \alpha(k^gg,x) \) using the cocycle identity gives (using that \( K^g < H \)):
\[
\begin{align*}
\alpha(gk,x) &= \alpha(g,k.x) \alpha(k,x) = \alpha(g,k,x) \varrho(k,y) \\
\alpha(k^gg,x) &= \alpha(k^g,g.x) \alpha(g,x) = \varrho(k^g,g.y) \alpha(g,x)
\end{align*}
\]
where \( y = p(x) \). This gives a \( \mu \)-a.e. identity
\[
\alpha(g,k.x) = \varrho(k^g,g.y) \alpha(g,x) \varrho(k,y)^{-1}.
\]
Observe that the map \( \varrho^g : K \times Y \to L \) defined by \( \varrho^g(k,y) \defeq \varrho(k^g,y) \) is a cocycle. Therefore using Basic Lemma \[32\] the map \( \Phi(x) = \alpha(g,x) \), satisfying \( \Phi(k.x) = \varrho^g(k,y) \Phi(x) \varrho(k,y)^{-1} \), descends to a function \( Y \to L \). The set
\[
G_1 = \{ g \in G \mid \alpha(g,x) \text{ depends only on } y = p(x) \}
\]
forms a subgroup of \( G \). The above arguments shows that \( G_1 \supseteq G_{K,H} \supseteq N_G(H) \). Hence \( G_1 > \hat{H} \) and \( \hat{\varrho}(g,y) \) denotes the value of \( \alpha(g,x) \), \( y = p(x) \), for \( g \in G_1 > \hat{H} \).

The following Lemma is parallel to Popa’s \[22\] Proposition 3.5. We shall refer to this Lemma only in Remark \[13\].
3.6. **Lemma (Tightness for untwisting).** Let $G \actson (X, \mu)$ be an ergodic action, $(X, \mu) \stackrel{\nu}{\to} (Y, \nu)$ a $G$-equivariant measurable quotient which is relatively weakly mixing, $L$ a group in class $F_{\text{inv}}$, $M < L$ a closed subgroup, $\alpha : G \times X \to M$ and $\varrho : G \times Y \to M$ be measurable cocycles. Suppose that there exists a measurable map $\Phi : X \to L$ so that for all $g \in G$ for $\mu$-a.e. $x$:

\[ \varrho(g, p(x)) = \Phi(g.x) \alpha(g, x) \Phi(x)^{-1}. \]

Then there exists a measurable map $\Phi' : X \to M$ and a cocycle $\varrho' : G \times Y \to M$ so that for all $g \in G$ for $\mu$-a.e. $x$:

\[ \varrho'(g, p(x)) = \Phi'(g.x) \alpha(g, x) \Phi'(x)^{-1}. \]

In the particular case of trivial $Y$, i.e., if $G \actson (X, \mu)$ is weakly mixing and the cocycle $\alpha : G \times X \to M$ can be untwisted to a homomorphism $\varrho : G \to M < L$ in $L$, then $\alpha$ can be untwisted to a homomorphism $\varrho' : G \to M$ within $M$.

**Proof.** The assumption that $L \in F_{\text{inv}}$ is used to define the following $L$-invariant metric $\tilde{d}$ on $L/M$:

\[ \tilde{d}(\ell_1 M, \ell_2 M) = \text{dist}(\ell_2^{-1} \ell_1, M) = \inf_{m \in M} d(\ell_2^{-1} \ell_1, m). \]

Let $f : X \times_Y X \to [0, \infty)$ be given by $f(x_1, x_2) = \tilde{d}(\Phi(x_1) M, \Phi(x_2) M)$. This measurable function is invariant for the diagonal $G$-action on $X \times_Y X$ because

\[
\begin{align*}
    f(g.x_1, g.x_2) &= \tilde{d}(\Phi(g.x_1) M, \Phi(g.x_2) M) \\
                     &= \tilde{d}(\varrho(g, y) \Phi(x_1) \alpha(g, x)^{-1} M, \varrho(g, y) \Phi(x_2) \alpha(g, x)^{-1} M) \\
                     &= \tilde{d}(\varrho(g, y) \Phi(x_1) M, \varrho(g, y) \Phi(x_2) M) \\
                     &= \tilde{d}(\Phi(x_1) M, \Phi(x_2) M) = f(x_1, x_2)
\end{align*}
\]

where $y = p(x) \in Y$. Ergodicity of the $G$-action on the fibered product (i.e. the assumption of relative weak mixing) implies that $f(x_1, x_2)$ is $\mu \times_{\nu} \mu$-a.e. constant $d_0$. Using Lemma 341 we deduce $d_0 = 0$, and so for some measurable $\phi : Y \to L$ we have

\[ \Phi(x) M = \phi(\pi(x)) M, \]

which implies that $\Phi'(x) = \phi(\pi(x))^{-1} \Phi(x)$ takes values in $M$. Let $\varrho'(g, y) = \phi(g, y)^{-1} \varrho(g, y) \phi(y)$. This is a cocycle on $G \times Y$ satisfying [341]. All the terms in the right hand side of this equation take values in $M$ and so does $\varrho' : G \times Y \to M$ as claimed. \qed

4. **Rigidity vs. Deformation**

**4.a. The topology on $Z^1(G \times X, L)$.** Let $G \actson (X, \mu)$ be a fixed p.m.p. action of a l.c.s.c. group, let $L$ be some Polish group (which will be assumed to be in class $F_{\text{inv}}$). Consider the space $Z^1(G \times X, L)$ of all measurable cocycles $G \times X \to L$ (identified modulo $\mu$-null sets) with the topology of convergence in measure, which can be defined as follows. Fix a left-invariant (below bi-invariant) metric $d$ on $L$. 
Given a compact subset $K \subset G$ and $\epsilon > 0$ let $V_{K,\epsilon}$ denote the set of pairs $(\alpha, \beta)$ of cocycles $\alpha, \beta : G \times X \to L$ such that

$$\mu \{ x \in X \mid d(\alpha(g, x), \beta(g, x)) < \epsilon \} > 1 - \epsilon \quad (g \in K).$$

Sets $V_{K,\epsilon}$ form the base of topology on the cocycles. This topology is complete and is metrizable, but we shall not use this fact.

4.1. **Proposition (Continuity).** Let $G \curvearrowright (X, \mu)$ be an ergodic p.m.p actions of a l.c.s.c. group $G$, and $L$ be a group of class $\mathcal{G}_{inv}$. Then the action of the centralizer $\text{Aut}(X, \mu)^G$ on the space $Z^1(G \times X, L)$ of measurable cocycles $G \times X \to L$ is continuous.

*Proof for the discrete case.* We assume that both $G$ and $L$ are discrete groups, and denote them by $\Gamma$, $\Lambda$ respectively. We should prove that given a finite set $F \subset \Gamma$ and an $\epsilon > 0$ there is a a neighborhood $U$ of the identity in $\text{Aut}(X, \mu)$ so that

$$\mu \{ x \in X \mid \alpha(g, x) = \alpha(g, T(x)) \} > 1 - \epsilon \quad (\forall g \in F)$$

for all $T \in U \cap \text{Aut}(X, \mu)^\Gamma$. Indeed, for each $g \in \Gamma$ there exists a finite set $\Lambda_g \subset \Lambda$ so that

$$\mu \{ x \in X \mid \alpha(g, x) \in \Lambda_g \} > 1 - \epsilon/2.$$

Let $E_{g,\ell} = \{ x \in X \mid \alpha(g, x) = \ell \}$. Then there exists a neighborhood $U$ of the identity in $\text{Aut}(X, \mu)$ so that for $T \in U$ for each $g \in F$ we have:

$$\sum_{\ell \in \Lambda_g} \mu(E_{g,\ell} \triangle TE_{g,\ell}) < \epsilon/2.$$

It now follows that (7) holds for all $T \in U$. Restricting $T$ to commute with $\Gamma$ guarantees that $\alpha(g, T(x))$ is a cocycle. \hfill $\Box$

The general argument is only slightly more involved, and is postponed to the Appendix (section 6). The following is a reformulation of Popa’s [22] Proposition 4.2.

4.2. **Theorem (Local Rigidity of Cocycles).** Let $H < G$ be a closed subgroup which has relative property (T) in a l.c.s.c. group $G$, and let $G \curvearrowright (X, \mu)$ be a p.m.p. action such that the restricted action $H \curvearrowright (X, \mu)$ is ergodic.

Then for any group of finite type $L \in \mathcal{G}_{\text{fin}}$ the map

$$Z^1(G \times X, L) \subset Z^1(H \times X, L) \rightarrow H^1(H \times X, L)$$

has a discrete image. More precisely, there exists a compact subset $K \subset G$ and $\epsilon > 0$ s.t. for any two cocycles $\alpha, \beta : G \times X \to L$ with

$$\mu \{ x \in X \mid d(\alpha(g, x), \beta(g, x)) < \epsilon \} > 1 - \epsilon \quad (g \in K)$$

there exists a measurable $\phi : X \to L$ so that as $H$-cocycles we have:

$$\alpha(h, x) = \phi(h, x) \beta(h, x) \phi(x)^{-1} \quad (h \in H).$$
Here we shall the proof for the special case of discrete target group, following Hjorth’s [14]. This argument can be adapted to the case of \( L \) being compact, or \( L = \text{Inn}(\mathcal{R}) \) for a \( \text{II}_1 \)-equivalence relation \( \mathcal{R} \). For the general case see Popa’s paper [22, Proposition 4.2].

**Proof - case of discrete target group.** Let \( H < G \) and \( G \curvearrowright (X,\mu) \) be as in the theorem, and let \( \alpha : G \times X \to \Lambda \) be a measurable cocycle taking values in a discrete countable group \( \Lambda \). Using Theorem 2.2 we formulate the assumption that \( H < G \) has relative property (T) in \( G \) as follows: there exists a compact subset \( K \subset G \) and \( \epsilon > 0 \) so that: for any unitary \( G \)-representation \( \pi : G \to U(\mathcal{H}_\pi) \) which has a unit vector \( v_0 \in \mathcal{H}_\pi \) with

\[
\inf_{g \in K} |\langle \pi(g)v_0, v_0 \rangle| \geq 1 - \epsilon
\]

there exists an \( H \)-invariant unit vector \( v \in \mathcal{H}_\pi \) with \( \|v - v_0\| < 1/10 \).

The claim is that any cocycle \( \beta : G \times X \to \Lambda \) with \( (\alpha, \beta) \in V_{K,\epsilon} \), i.e., such that

\[
\mu \{ x \in X \mid \alpha(g,x) = \beta(g,x) \} > 1 - \epsilon \quad (g \in K),
\]

relation [8] holds.

To see this, consider the product \( \tilde{X} = X \times \Lambda \) with \( \tilde{\mu} = \mu \times m_\Lambda \), where \( m_\Lambda \) is the counting measure on \( \Lambda \). With \( \alpha, \beta : G \times X \to \Lambda \) consider the measure preserving \( G \)-action on the infinite measure space \( (\tilde{X}, \tilde{\mu}) \) by

\[
g : (x, \ell) \mapsto (g.x, \alpha(g,x) \ell \beta(g,x)^{-1})
\]

and defines a unitary representation \( \pi \) on \( \mathcal{H} = L^2(\tilde{X}, \tilde{\mu}) \). Denoting by \( F_0 \in \mathcal{H} \) the characteristic function of the set \( X \times \{ e \} \subset X \times \Lambda \), we note that \( F_0 \) is a unit vector with

\[
\langle \pi(g)F_0, F_0 \rangle = \mu \{ x \in X \mid \alpha(g,x) = \beta(g,x) \} > 1 - \epsilon \quad (g \in K).
\]

Thus there exists an \( H \)-invariant unit vector \( F \in \mathcal{H} \) close to \( F_0 \). For \( x \in X \) consider \( |F(x,\cdot)|^2 \) as a function \( \Lambda \to [0,\infty) \). By Fubini for \( \mu \)-a.e. \( x \in X \) this function is summable; let \( w(x) \) denote its sum, \( p(x) \) denote its maximal value, and \( k(x) \) – the number of points on which this value is attained:

\[
p(x) = \max_{\ell \in L} |F(x,\ell)|^2, \quad k(x) = \text{card} \{ \ell \in \Lambda \mid |F(x,\ell)|^2 = p(x) \}.
\]

Then \( w, p \) and \( k \) are measurable \( H \)-invariant functions; hence by ergodicity these functions are \( \mu \)-a.e. constants: \( w(x) = \|F\|^2 = 1 \), \( p(x) = p \) and \( k(x) = k \). In fact we claim that \( k = 1 \). Indeed \( k(x) \cdot p(x) \leq w(x) \) gives \( p \leq 1/k \); in particular \( |F(x,e)|^2 \leq 1/k \). Thus

\[
\|F_0 - F\|^2 \geq 1 - 1/\sqrt{k}
\]

which would contradict \( \|F - F_0\| < 1/10 \) unless \( k = 1 \). Hence we can define a measurable function \( \phi : X \to \Lambda \) by \( |F(x,\phi(x))|^2 = p \). As \( F \) is \( H \)-invariant we get that for each \( h \in H \):

\[
\phi(h.x) = \alpha(h,x)\phi(x)\beta(h,x)^{-1}
\]

which is equivalent to [8]. \( \square \)
4.3. Remark. In the general case of \( L \in \mathcal{U}_{\text{fin}} \) (see [22] Proposition 4.2), where \( L \) is imbedded in the unitary group \( U \) of a finite von Neumann algebra \( M \), Popa applies the relative property (T) to the representation associated to \( M \), and then induced through \( X \) to a unitary representation of \( G \). After some work he shows that \( \alpha : G \times X \to L < U \) can be untwisted to a homomorphism \( \varrho : G \to L < U \) (or reduced to a cocycle \( \varrho : G \times Y \to L < U \)) by a conjugation in \( U \). It is here that Lemma 3.6 is used to untwist \( \alpha \) to some \( \varrho' : G \to L \) (or \( \varrho' : G \times Y \to L \)) within \( L \) itself.

4.b. Proof of Popa’s Cocycle Superrigidity Theorem. Consider the fibered product \( (Z, \zeta) = (X \times_Y X, \mu \times \nu, \mu) \) with the diagonal action of \( G \). Let \( C = \text{Aut}(Z, \zeta)^G \) denote the centralizer of \( G \) in \( \text{Aut}(Z, \zeta) \), and let \( C^0 \) denote the connected component of the identity in \( C \) considered with the weak topology. By the relative malleability assumption, \( C^0 \) contains a map of the form

\[
F : (x, y) \mapsto (T(y), S(x))
\]

where \( T, S \in \text{Aut}(X, \mu)^G \). The relative weak mixing assumption means that \( H \) acts ergodically on \( (Z, \zeta) \).

Given any measurable cocycle \( \alpha : G \times X \to L \) where \( L \in \mathcal{U}_{\text{fin}} \), denote by \( \alpha_i : G \times Z \to L \) the pull-back of \( \alpha \) via the projections \( p_i : Z \to X, p_i(x, y) = x_i \):

\[
\alpha_i(g, (x_1, x_2)) \overset{\text{def}}{=} \alpha(g, x_i) \quad (i = 1, 2).
\]

The group \( C^0 < C = \text{Aut}(Z, \zeta)^G \) acts continuously on the space \( Z^1(G \times Z, L) \) of all measurable cocycles (Proposition 4.1.1). It follows from the Local Rigidity Theorem 4.2 that the space \( H^1(H \times Z, L) \) inherits a discrete topology from \( Z^1(G \times Z, L) \), and therefore the action of the connected group \( C^0 \) on \( Z^1(G \times Z, L) \) gives rise to a trivial action on \( H^1(H \times Z, L) \). This means that any two cocycles in the \( C^0 \)-orbit on \( Z^1(G \times Z, L) \) are cohomologous as \( H \)-cocycles. In particular, there exists a measurable map \( F : Z = X \times_Y X \to L \) so that

\[
\alpha(h, x_1) = F(h, x_1, h, x_2) \alpha(h, S(x_2)) F(x_1, x_2)^{-1} \quad (h \in H < G).
\]

By Theorem 3.4 there exist a measurable map \( \phi : X \to L \) and a measurable cocycle \( \varrho_0 : H \times Y \to L \) so that:

\[
\alpha(h, x) = \phi(h, x) \varrho_0(h, p(x)) \phi(x)^{-1} \quad (h \in H).
\]

Define a cocycle \( \beta : G \times X \to L \) by \( \beta(g, x) \overset{\text{def}}{=} \phi(g, x)^{-1} \alpha(g, x) \phi(x) \) and let

\[
G_1 \overset{\text{def}}{=} \{g \in G \mid \beta(g, x) \text{ is a.e. a function of } y = p(x)\}.
\]

It follows from the cocycle identity that \( G_1 \) is a subgroup of \( G \), and we already know that \( H < G_1 \). It remains to show that \( G_1 = G \).

If condition (b) in the theorem is satisfied, \( H \) is w-normal in \( G \). Lemma 3.5 states that for any subgroup \( H < K < G_1 \) we have \( N_G(K) < G_1 \) (the assumption that \( H \lhd (X, \mu) \) is weakly mixing relative to \( (Y, \nu) \) passes to all the sup-groups \( K > H \)). The assumption that \( H \) is w-normal in \( G \) implies, using transfinite induction starting with \( H_0 = H \), that \( G_1 = G \).
If condition (b') in the theorem is satisfied, i.e., $H$ is only wq-normal, one makes use of the second part of Lemma 3.5. Let $H = H_0 < H_1 < \cdots < H_\eta = G$ be the well ordered chain of subgroups as in the definition of wq-normality. For each ordinal $j \leq \eta$ let $H'_j = \bigcup_{i<j} H_i$ and let $A_j$ be the set of elements $g \in G$ for which the group $K_{j,g} \overset{\text{def}}{=} H'_j \cap g^{-1}H'_jg$ is not compact, and hence acts on $(X,\mu)$ weakly mixing relative to $(Y,\nu)$. By the assumption of wq-normality $H_j$ is generated by $A_j$. Thus Lemma 3.5 shows that if $H_i < G_1$ for all $i < j$, then $H_j < G_1$, and $G = G_1$ follows by the transfinite induction.

With $G = G_1$ shown, it follows that $\beta(g,x) = \varphi(g,p(x))$ for some measurable function $\varphi : G \times Y \rightarrow L$, which is necessarily a cocycle, and we conclude that $\alpha(g,x) = \varphi(g,x) \varphi(g,p(x)) \varphi(x)^{-1}$ ($g \in G$) as claimed. If $Y$ is a point space with the trivial action $\varphi$ is just a homomorphism $G \rightarrow L$. This completes the proof of Theorem 1.3.

4.c. Malleable actions - proof of Theorem 1.2.

**Bernoulli Actions.** For reader’s convenience we start with the generalized Bernoulli actions (see Popa [22, Lemma 4.5]). Let $I$ be a countable set with an action $\sigma : G \rightarrow \text{Sym}(I)$ of a l.c.s.c. group, and let $(X,\mu) = (X_0,\mu_0)^I$ be the product space with the $G$-action by coordinate permutation. The diagonal $G$-action on $(X \times X,\mu \times \mu)$ is the Bernoulli action of $G$ on $(X_0 \times X_0,\mu_0 \times \mu_0)^I$. Hence malleability of $G \acts (X,\mu)$ follows from the following observations 4.4 and 4.5.

4.4. Lemma. Let $(X_0,\mu_0)$ be a non-atomic probability measure space. Then in the Polish group $\text{Aut}(X_0 \times X_0,\mu_0 \times \mu_0)$, considered with the weak topology, the flip $F_0(x,y) = (y,x)$ is in the (path) connected component of the identity.

**Proof.** Without loss of generality we may assume that $(X_0,\mu_0)$ is the unit interval $[0,1]$ with the Lebesgue measure $m$. For $0 \leq t \leq 1$ let $T_t \in \text{Aut}([0,1]^2,m^2)$ be defined by

$$T_t(x,y) \overset{\text{def}}{=} \begin{cases} (x,y) & \text{if } x,y \in [0,t] \\ (y,x) & \text{otherwise} \end{cases}.$$ 

Then $T_0 = F_0$ is the flip, and $T_1 = \text{Id}$ is the identity. The path $t \in [0,1] \mapsto T_t \in \text{Aut}([0,1]^2,m^2)$ is continuous in the strong, and hence also in the weak, topologies. \qed

Let $(Z_0,\zeta_0)$ be some probability space, and $(Z,\zeta) = (Z_0,\zeta_0)^I$ be the product space with the Bernoulli action of $G$. Let $\Delta : \text{Aut}(Z_0,\zeta_0) \rightarrow \text{Aut}(Z,\zeta)$ denote the diagonal embedding. The following fact is obvious:

4.5. Lemma. $\Delta$ is a continuous embedding, and its image commutes with $G$.

Since Bernoulli actions with a non-atomic base space are included in the family of Gaussian actions [28], the above also follows from the following case.
We now turn to Gaussian actions (described in 2.2) corresponding to an orthogonal representation $\pi : G \to O(H)$. First assume that $G$ is countable and the action is constructed from some $\pi(G)$-invariant set $V = \{ v_i \}_{i \in I} \subset H$; hence $G \hom \{(I, \mu_C)\}$.

The product measure $\mu_C \times \mu_C$ on $\mathbf{R}_1 \times \mathbf{R}_1 = (\mathbf{R}^2)^I$ is “built from” the standard Gaussian measures $\mu_{\mathbf{R}^2}$ on $\mathbf{R}^2$, and is therefore invariant under the diagonal action of the rotation group $SO(\mathbf{R}^2)$. More precisely, the claim is that the rotation $R_\theta$ by angle $\theta$ acting on $\mathbf{R}_1 \times \mathbf{R}_1$ by:

$$R_\theta : ((x_i)_{i \in I}, (y_j)_{j \in I}) \mapsto ((x_i \cos \theta - y_i \sin \theta)_{i \in I}, (x_j \sin \theta + y_j \cos \theta)_{j \in I})$$

preserves the product measure $\mu_C \times \mu_C$. Indeed the latter is the Gaussian measure $\mu_\mathbf{C}$, where $\mathbf{C}$ is a “block diagonal matrix” $C \otimes I_2$ corresponding to the set $V = V \otimes e_1 \cup V \otimes e_2 \subset H \otimes \mathbf{R}^2 \cong \mathbf{H} \otimes \mathbf{H}$. For any finite subset $J \subset I$ the measure $\mu_\mathbf{C}_J$ on $(\mathbf{R}^2)^J$ is invariant under the rotations $R_\theta$. Therefore $\mu_\mathbf{C} = \mu_{\mathbf{R}^2} \times \mu_C$ is also $R_\theta$-invariant. The continuous path $\{ R_\theta \mid 0 \leq \theta \leq \pi/2 \}$ in $\text{Aut}(\mathbf{R}_1 \times \mathbf{R}_1, \mu_C \times \mu_C)$ commuting with the diagonal $G$-action connects the identity with the map

$$R_{\pi/2} : (x,y) \mapsto (-y,x)$$

proving the malleability of the Gaussian $G$-action on $(\mathbf{R}_1, \mu_C)$. This argument extends to Gaussian actions of non-discrete groups using a dense countable subgroup $\Lambda < G$ and using the continuity argument as outlined in 2.2. This completes the proof of Theorem 1.8.

4.6. Remark. This argument is in fact similar to a rotation argument used by Popa in the non-commutative context in [23, 1.6.3].

5. Applications

Proof of Theorem 1.8. First we reduce the weak morphism $X \supset X' \xrightarrow{\theta} Y' \subset Y$ to a morphism of the form $X \xrightarrow{\theta'} Y' \subset Y$ (see 3 Section 2) for a more detailed discussion). First note that there exists a measurable “retraction” map $\pi : X \to X'$ with $(x, \pi(x)) \in \mathcal{R}_{X,\Gamma}$ for $\mu$-a.e. $x \in X$. Indeed, enumerating elements of $\Gamma$ as $\{ g_n \}_{n \in \mathbf{N}}$ (say, with $g_1 = e$) let $\pi(x) \overset{\text{def}}{=} g_{n(x)} x$ where $n(x) = \min \{ n \in \mathbf{N} \mid g_n.x \in X' \}$, which is finite for $\mu$-a.e. $x \in X$ by ergodicity. Consider the map

$$\sigma : X \xrightarrow{\pi} X' \xrightarrow{\theta'} Y' \subset Y.$$

Since $\pi_* \mu \sim \mu_{X'}$ we have $\sigma_* \mu \sim \nu_{Y'} \sim \nu$.

Observe that $\sigma(\Gamma.x) \subset \Lambda.\sigma(x)$ for $\mu$-a.e. $x \in X$, and since $\Lambda$ is assumed to act freely on $(Y, \nu)$ (after possibly disregarding a null set), this defines a measurable cocycle $\alpha : \Gamma \times X \to \Lambda$ by

$$\sigma(g.x) = \alpha(g, x).\sigma(x).$$

Applying Popa’s Cocycle Superrigidity Theorem 1.3 there is a homomorphism $\varphi : \Gamma \to \Lambda$ and a measurable $\phi : X \to \Lambda$ so that

$$\alpha(g, x) = \phi(g.x) \varphi(g) \phi(x)^{-1}.$$
Define a measurable map $T : X \rightarrow Y$ by $T(x) \overset{\text{def}}{=} \phi(x)^{-1} \cdot \sigma(x)$. More explicitly, we have a measurable countable partition $X = \bigcup X_\ell$ into sets $X_\ell = \phi^{-1}(\{\ell\})$, so that for $x \in X_\ell$: $T(x) = \ell \cdot \sigma(x)$. In particular, this shows that $T_\ast \mu \prec \nu$. Let

$$f(y) = \frac{dT \ast \mu}{d\nu}(y), \quad Y_1 = \{y \in Y \mid f(y) > 0\}, \quad F = f \circ T.$$ 

Note that $F : X \rightarrow \mathbb{R}$ is a $\Gamma$-equivariant measurable function, because

$$T(g.x) = \phi(g.x)^{-1} \cdot \sigma(g.x) = \phi(g.x)^{-1} \cdot \alpha(g,x) \cdot \sigma(x) = \phi(g) \phi(x)^{-1} \cdot \sigma(x) = \phi(g) \cdot T(x).$$

Hence, by ergodicity, $F(x)$ is $\mu$-a.e. a finite positive constant $c = \nu(Y_1)/\mu(X) = \nu(Y_1)$, and the morphism $T : X \rightarrow Y_1 \subset Y$ satisfies

$$T(g.x) = \phi(g) \cdot T(x) \quad (g \in \Gamma).$$

Since $T(g.x) = T(x)$ for every $g \in \Gamma_0 = \text{Ker}(g)$, the map $T$ factors through $(X_1, \mu_1)$ -- the space of $\Gamma_0$-ergodic components. The $\Gamma$-action on $(X_1, \mu_1)$ factors through the action of $\Gamma_1 = \phi(\Gamma) \cong \Gamma/\Gamma_0$, and the corresponding map $T_1 : (X_1, \nu_1) \rightarrow (Y_1, \nu_1)$ is $\Gamma_1$-equivariant. Being a quotient of an ergodic action $\Gamma \curvearrowright (Y_1, \nu_1)$ is ergodic. This proves (1)–(3).

For the proof of (i)–(iii) we need the following

**Claim.** There exist positive measure subsets $A \subset X$ and $B \subset Y_1$ and fixed elements $a \in \Gamma$, $b \in \Lambda$, so that $T(A) = B$ and $T(x) = b \cdot \theta(a,x)$ for a.e. $x \in A$.

**Proof.** This follows from the construction of the map $T : X \rightarrow Y_1$ as

$$T(x) = \ell_x \cdot \theta(\pi(x)) = \ell_x \cdot \theta(g_{n(x)}, x) \quad \text{where} \quad \ell_x = \phi(x)^{-1}.$$ 

Indeed, since both $\Gamma$ and $\Lambda$ are countable, there exists a positive measure set $A \subset X$ so that $\mu(n(x)) = \text{constant} n_0$ on $A$ (we denote $a = g_{n_0}$), while $\ell_x$ is constant on $A$. The set $B = b \cdot \theta(a,A) = T(A)$ has positive measure and $A = T^{-1}(B)$. 

(i). We assume the action $\Gamma \curvearrowright (X, \mu)$ to be free, and $\theta : X' \rightarrow Y'$ to be an injective morphism of the restrictions $\mathcal{R}_{X,\Gamma}|_{X' \times X'}$ into $\mathcal{R}_{Y,\Lambda}|_{Y' \times Y'}$. Using the above claim it follows that also $T : A \rightarrow B$ is an *injective* relation morphism of the restrictions $\mathcal{R}_{X,\Gamma}|_{A \times A} \rightarrow \mathcal{R}_{Y,\Lambda}|_{B \times B}$. Note that for any $g \in \Gamma$, if $x \in A \cap g^{-1}A$ then

$$\mathcal{R}_{X,\Gamma}|_{A \times A} \ni (x, g.x) \xrightarrow{T} (T(x), \phi(g).T(x)) \in \mathcal{R}_{Y,\Lambda}|_{B \times B}.$$ 

Thus the fact that $T : A \rightarrow B$ is an injective relation morphism, together with the freeness of the $\Gamma$-action, imply that for $g \in \text{Ker}(g) \setminus \{e\}$ we have $\mu(A \cap g^{-1}A) = 0$. Therefore

$$|\Gamma_0| \cdot \mu(A) = \sum_{g \in \Gamma_0} \mu(gA) = \mu\left( \bigcup_{g \in \Gamma_0} gA \right) \leq \mu(X) = 1,$$

implying the finiteness of $\Gamma_0 \triangleleft \Gamma$. 


On the other hand, if $\Gamma_0$ is finite, then $(X_1, \mu_1)$ is just the space of $\Gamma_0$-orbits in $(X, \mu)$, and there exists a Borel section $f : X_1 \to X$ of the finite extension $(X, \mu) \to (X_1, \mu_1) = (X, \mu)/\Gamma_0$. Observe that $T$ mapping $X' \overset{\text{def}}{=} f(X_1)$ to $Y_1$ as

$$X \ni x \overset{T}{\longrightarrow} X_1 \ni f(x) \in \tilde{Y}_1 \subset Y$$

is an injective relation morphism from $\mathcal{R}_{X,\Gamma}|_{X' \times X'}$ into $\mathcal{R}_{Y,\Lambda}|_{Y_1 \times Y_1} = \mathcal{R}_{Y_1,\Gamma_1}$.

(ii). We assume that $\nu(Y) < \infty$ and that $\theta : X' \to Y'$ is a surjective morphism of the restrictions $\mathcal{R}_{X,\Gamma}|_{X' \times X'}$ onto $\mathcal{R}_{Y,\Lambda}|_{Y' \times Y'}$. Using the above claim it follows that $T : A \to B$ is a surjective relation morphism of the restrictions $\mathcal{R}_{X,\Gamma}|_{A \times A}$ onto $\mathcal{R}_{Y,\Lambda}|_{B \times B}$.

Let $\ell \in \Lambda$ be such that $\nu(B \cap \ell^{-1}B) > 0$. Then for $\nu$-a.e. $y \in B \cap \ell^{-1}B$ we have $(y, \ell, y) \in \mathcal{R}_{Y,\Lambda}|_{B \times B}$ and, by surjectivity, we can find and $x \in A$ and $g \in \Gamma$ with $y = T(x)$ and $\ell, y = T(g, x) = g(y).T(x)$. The essential freeness of the action $\Lambda \curvearrowright (Y, \nu)$ implies that $\ell = g(y) \in \Gamma_1$.

Reversing this implication, we conclude that $\nu(B \cap \ell^{-1}B) = 0$ whenever $\ell \notin \Gamma_1$. Let $\{\ell_i\}_{i \in I}$ be representatives of distinct cosets $\Lambda/\Gamma_1$. Thus $\ell_i \cap \ell_j \notin \Gamma_1$ for $i \neq j$ and $\nu(\ell_i B \cap \ell_j B) = \nu(B \cap \ell_i^{-1} \ell_j B) = 0$. Hence

$$[\Lambda : \Gamma_1] \cdot \nu(B) = \sum_{i \in I} \nu(\ell_i B) = \nu(\bigcup_{i \in I} \ell_i B) \leq \nu(Y) = \infty$$

which yields $[\Lambda : \Gamma_1] < \infty$.

Let $\Lambda' = \bigcap_{\ell \in \Lambda} \ell \Gamma_1 \ell^{-1}$ – the maximal normal subgroup in $\Lambda$ contained in $\Gamma_1$. It has finite index in $\Lambda$. Thus $\Gamma' = g^{-1}(\Lambda')$ has finite index in $\Gamma$, and acts ergodically on $(X, \mu)$, by the aperiodicity assumption on $\Gamma \curvearrowright (X, \mu)$. Hence $\Lambda' \curvearrowright (Y_1, \nu_1)$, which is a quotient of $\Gamma' \curvearrowright (X, \mu)$, is an ergodic action. Since $\Lambda' \lhd \Lambda$ the decomposition of $(Y, \nu)$ into $\Lambda'$-ergodic components (one of which is $(Y_1, \nu_1)$) is acted upon by the finite group $\Lambda/\Lambda'$. This action is transitivity due to ergodicity of the $\Lambda$-action on $(Y, \nu)$, and each has the same $\nu$-measure. Thus $\nu(Y)/\nu(Y_1) = k$ is an integer dividing the index $[\Lambda : \Gamma_1]$.

(iii) follows from the combination of (i) and (ii). The last claim, that if $c(\theta) = 1$ then the weak relation morphism $\theta$ extends to a relation morphism bijection $\tilde{\theta} : X \to Y$ (up to null sets), follows from an easy adaptation of $[5]$, Proposition 2.7. The latter treats weak relation isomorphisms rather than weak bijective morphisms.

Sketch of the proof of Theorem $[5, 7]$. The case of a single cocycle $A = \{\alpha\}$ is basically $[5]$, Theorems B, C. For a set $A$ of cocycles we take $Y_A = \bigvee_{\alpha \in A} Y_{(\alpha)}$. In other words $L^\infty(Y_A, \nu_A)$ is the abelian sub-algebra of $L^\infty(X, \mu)$ generated by $L^\infty(Y_{(\alpha)}, \nu_{(\alpha)})$, $\alpha \in A$. The properties (1) and (2) of $(Y_A, \nu_A)$ are evident. The structure of $(Y_A, \nu_A)$ when $A = \{\alpha_1, \ldots, \alpha_n\}$ is a finite set follows from the fact (see $[2]$) that Ratner’s theorem allows to explicitly describe $\Gamma$-invariant ergodic probability measures on $\prod_{i=1}^n Y_{(\alpha_i)} = \prod_{i=1}^n \tilde{G}/\Lambda_i$ with marginals $m_{\tilde{G}/\Lambda_i}$ as homogeneous measures for some subdiagonal embedding of $\tilde{G}^k$ in $\tilde{G}^n$.

$\square$
Proof of Theorem 1.14. Starting from a weak relation morphism \( \theta \) from \( R_{X,1} \) to \( R_{H/\Delta,1} \). Theorem 1.8 gives a homomorphism \( g_1 : \Gamma \to \Lambda \) and the chain of \( \Gamma \)-equivariant measurable quotients

\[
T : (X, \mu) \xrightarrow{\operatorname{erg}} (X_1, \mu_1) \xrightarrow{T_1} (Y_1, \nu_1), \quad Y_1 \subset H/\Delta.
\]

The image \( \Gamma_1 = g_1(\Lambda) \) acts ergodically on a non-trivial space \( (Y_1, \nu_1) \). In fact \( \nu_1 \prec m_{H/\Delta} \) is absolutely continuous with respect to Haar measure. Thus \( \Gamma_1 < \Delta < H \) is infinite and hence unbounded, and therefore acts ergodically on \( (H/\Delta, m_{H/\Delta}) \) by Howe-Moore theorem. So \( Y_1 = Y = H/\Delta \) and \( \nu_1 = \nu = m_{H/\Delta} \). Let \( Z_1 = Z_H(\Gamma_1) \) denote the centralizer of \( \Gamma_1 \) in \( H \). For later use we observe that \( Z_1 \) is a proper algebraic subgroup in \( H \), and therefore is a null set with respect to the Haar measure \( m_H \).

Let \( f : H/\Delta \to H \) be a Borel cross-section of the projection \( H \to H/\Delta \), and let \( c : H \times H/\Delta \to \Delta \) be the corresponding measurable cocycle

\[
c(h, y) = f(h, y) h f(y)^{-1} \quad (h \in H, \ y \in H/\Delta).
\]

Denote by \( \alpha : \Gamma \times X \to \Delta \) the lift of \( c \) to \( X \), namely \( \alpha(g, x) \overset{\text{def}}{=} c(g_1(g), T(x)) \). We have

\[
\alpha(g, x) = \Phi_1(g, x) g_1(g) \Phi_1(x)^{-1}, \quad \text{where} \quad \Phi_1 : X \xrightarrow{T} H/\Delta \xrightarrow{f} H.
\]

The assumption that \( \Gamma \prec (X, \mu) \) is \( \Delta \)-Cocycle Superrigid means that there exists a homomorphism \( g_2 : \Gamma \to \Delta \) and a measurable map \( \Phi_2 : X \to \Delta \) so that

\[
\alpha(g, x) = \Phi_2(g, x) g_2(g) \Phi_2(x)^{-1}.
\]

We shall deduce a contradiction by comparing \( \Phi_1 \) to \( \Phi_2 \). To do so rewrite the identity

\[
\Phi_2(g, x) g_2(g) \Phi_2(x)^{-1} = \alpha(g, x) = \Phi_1(g, x) g_1(g) \Phi_1(x)^{-1}
\]

as \( \Phi(g, x) = g_1(g) \Phi(x) g_2(g)^{-1} \) with \( \Phi(x) \overset{\text{def}}{=} \Phi_1(x)^{-1} \Phi_2(x) \). Next define a measurable map \( \Psi : X \times X \to H \) by \( \Psi(x', x) \overset{\text{def}}{=} \Phi(x') \Phi(x)^{-1} \). We have

\[
\Psi(g, x', g, x) = g_1(g) \Phi(x', x) g_1(g)^{-1}.
\]

Therefore the push-forward probability measure \( \omega = \Psi_*(\mu \times \mu) \) on \( H \) is invariant under conjugations by \( \Gamma_1 = g_1(\Gamma) < \Delta < H \).

It is a general fact (see [5, p. 38]) that the only probability measure on a semi-simple center free group \( H \) which is invariant under conjugations by elements of a subgroup \( \Gamma_1 < H \) is supported on the centralizer \( Z_1 = Z_H(\Gamma_1) \).

Thus \( \Phi(x') \Phi(x)^{-1} \in Z_1 \) for \( \mu \times \mu \text{-a.e. } (x', x) \in X \times X \). By Fubini, there exists \( h_0 \in H \) (take \( h_0 = \Phi(x')^{-1} \) for \( \mu \text{-a.e. } x' \)) so that for \( \mu \text{-a.e. } x \in X \):

\[
\Phi_2(x)^{-1} \Phi_1(x) = \Phi(x)^{-1} \in h_0 Z_1.
\]

Since \( \Phi_2 : X \to \Delta \) it follows that for \( \mu \text{-a.e. } x \in X \):

\[
\Phi_1(x) \in \Phi_2(x) h_0 Z_1 \subset \bigcup_{h \in \Delta} hh_0 Z_1.
\]
The right hand side is a countable union of translates of $Z_1$, and therefore is a null set with respect to the Haar measure $m_H$ on $H$. At the same time $\Phi_1 \ast \mu < m_H$ because

$$\Phi_1 \ast \mu = f \ast m_{H/\Delta}$$

is the restriction of the Haar measure on $H$ to a fundamental domain for $\Delta < H$.

The contradiction shows that there could not exist a weak relation morphism from $R_{X,G}$ to $R_{H/\Delta,\Lambda}$. $\square$

**On Remark 1.15(2).** Let $\Gamma$ be a higher rank lattice, $(X, \mu) = (X_0, \mu_0)^\ell$ an ergodic generalized Bernoulli action, and $(X, \mu) \to (Y, \nu)$ a measurable quotient where each $g \in \Gamma$ acts with finite entropy. Then $(Y, \nu)$ is a point. This follows from the following facts:

1. By Stuck and Zimmer [31] an ergodic action of a higher rank lattice is essentially free, unless the action is transitive one on a finite set. Since $\Gamma \acts X$ is weakly mixing, it follows that $\Gamma \acts Y$ is essentially free.
2. Higher rank lattices $\Gamma$ always contain a copy of $\mathbb{Z}^2$; hence restricting to $\mathbb{Z}^2$ we get that $\mathbb{Z}^2 \acts Y$ is an essentially free action which is a quotient of a Bernoulli action of $\mathbb{Z}^2$ on $(X, \mu)$.
3. Hence $\mathbb{Z}^2 \acts Y$ is isomorphic to a non-trivial Bernoulli action $\mathbb{Z}^2 \acts (Z_0, \zeta_0)^\mathbb{Z}^2$, being an essentially free quotient of a Bernoulli action $\mathbb{Z}^2 \acts (X, \mu)$.
4. The generators $g_1, g_2$ of the free Abelian group $\mathbb{Z}^2$, must have infinite infinite Kolmogorov-Sinai entropy on $(Y, \nu) = (Y_0, \nu_0)^\mathbb{Z}^2$, because these are Bernoulli $\mathbb{Z}$-actions with a non-atomic base space $(Y_1, \nu_1) = (Y_0, \nu_0)^\mathbb{Z}$:

$$h(Y, \nu, g_1) = H(Y_1, \nu_1) = \infty \cdot H(Y_0, \nu_0) = \infty$$

since $(Y_0, \nu_0)$ is not a singleton. $\square$

**Proof of Proposition 1.17.** If $(X', \mu') \overset{p}{\to} (X, \mu)$ is not relatively weakly mixing, then there exists an intermediate extension $p : (X', \mu') \overset{r}{\to} (Z, \zeta) \overset{q}{\to} (X, \mu)$, where $q$ is a non-trivial relatively compact extension, i.e., $(Z, \zeta) = (X \times K/K_0, \mu \times m_{K/K_0})$ and the $G$-action on $Z$ is given by a measurable cocycle $\alpha : G \times X \to K$. Using $f_{\text{cmp}}$-Cocycle Superrigidity, $\alpha(g, x) = \phi(g, x) \varrho(g) \phi(x)^{-1}$ with $\varrho : G \to K$ homomorphism, and $\phi : X \to K$ is a measurable map. Then $f : Z \to Z$ given by $f(x, kK_0) = (x, \phi(x) kK_0)$ conjugates the given action $G \acts (Z, \zeta)$ to the diagonal action $G \acts X \times K/K_0$, $g : (x, kK_0) \mapsto (g \cdot x, g \cdot kK_0)$. The map

$$(X' \overset{r}{\to} Z \overset{f}{\to} X \times K/K_0 \overset{p}{\to} K/K_0)$$

shows that $G \acts (X', \mu')$ has the isometric action $G \acts K/K_0$ as a quotient, contrary to the assumption that $G \acts (X', \mu')$ is weakly mixing. $\square$

**Proof of Theorem 1.20.** Define a measurable cocycle $\Gamma \acts X \to \text{Inn}(S)$ as follows: for $g \in \Gamma$ and $\mu$-a.e. $x \in X$ let $a_{g,x} \in \text{Inn}(S)$ be defined by

$$((x, y), (g \cdot x, a_{g,x} \cdot y)) \in \mathcal{Q}.$$
It follows that \( \alpha \) is a cocycle. Since \( \text{Inn}(\mathcal{S}) \in \mathcal{W}_\text{fin} \) and \( \Gamma \curvearrowright (X, \mu) \) is assumed to be \( \mathcal{W}_\text{fin}-\text{Cocycle Superrigid} \) we have a homomorphism \( g : \Gamma \to \text{Inn}(\mathcal{S}) \) and a measurable map \( \phi : X \to \text{Inn}(\mathcal{S}) \) so that \( \alpha(g, x) = \phi(g, x) \phi(g) \phi(x)^{-1} \). Let \( f \in \text{Aut}(X \times Y, \mu \times \nu) \) be given by \( f(x, y) = (x, \phi(x), y) \). This is an inner relation automorphism of \( \mathcal{R} \times \mathcal{S} \).

Note that the diagonal \( \Gamma \)-action on \( (X \times Y, \mu \times \nu) \), \( g : (x, y) \mapsto (g, x, \phi(g, y)) \) is also inner to \( \mathcal{R} \times \mathcal{S} \), and \( \Phi \) identifies the orbit relation of this \( \Gamma \)-action with \( \mathcal{Q} \). \( \square \)

6. Appendix

6.a. Proof of Proposition 4.1 - the general case. We consider a fixed measurable cocycle \( \alpha : G \times X \to L \) where \( L \) is a Polish group with some bi-invariant metric \( d \). For \( T \in \text{Aut}(X, \mu)^G \) we shall measure the distance between \( \alpha \) and the shifted cocycle \( \alpha(g, T(x)) \) by the function:

\[
  f_T(g, x) = d(\alpha(g, x), \alpha(g, T(x)))
\]

and will use the following sets depending on the parameters \( \epsilon > 0, \delta > 0 \):

\[
  E_{\epsilon}(T, g) = \{ x \in X \mid f_T(g, x) > \epsilon \}, \\
  A_{\epsilon, \delta}(T) = \{ g \in G \mid \mu(E_{\epsilon}(T, g)) < \delta \}.
\]

6.1. Lemma. The above functions and sets satisfy the following relations:

\[
  (9) \quad f_T(gh, x) \leq f_T(g, h, x) + f_T(g, x), \quad (g, h \in G) \\
  (10) \quad f_{TS}(g, x) \leq f_T(g, S(x)) + f_S(g, x), \quad (T, S \in \text{Aut}(X, \mu)^G) \\
  (11) \quad \mu(E_{\epsilon_1 + \epsilon_2}(T, gh)) \leq \mu(E_{\epsilon_1}(T, h)) + \mu(E_{\epsilon_2}(T, g)) \\
  (12) \quad \mu(E_{\epsilon_1 + \epsilon_2}(TS, g)) \leq \mu(E_{\epsilon_1}(T, g)) + \mu(E_{\epsilon_2}(S, g)) \\
  (13) \quad A_{\epsilon_1, \delta_1}(T)^{-1} \cdot A_{\epsilon_2, \delta_2}(T) \subset A_{\epsilon_1 + \epsilon_2, \delta_1 + \delta_2}(T)
\]

For fixed \( g \in G, \epsilon > 0 \) and \( \delta > 0 \) there exist an open neighborhood \( U = U(g, \epsilon, \delta) \) of the identity in \( \text{Aut}(X, \mu) \) so that:

\[
  (14) \quad g \in \bigcap_{T \in U \cap \text{Aut}(X, \mu)^G} A_{\epsilon, \delta}(T).
\]

Proof. \( \square \). This inequality follows from the cocycle identity, combined with the following “parallelogram inequality” which takes advantage of the bi-invariance of the metric \( d \) on \( L \):

\[
  d(a_1a_2, b_1b_2) = d(b_1^{-1}a_1, b_2a_2^{-1}) \leq d(b_1^{-1}a_1, e) + d(e, b_2a_2^{-1}) \\
  = d(a_1, b_1) + d(a_2, b_2)
\]

for every \( a_1, a_2, b_1, b_2 \in L \).

---

\( \square \) is immediate from the triangle inequality

\[
  d(\alpha(g, x), \alpha(g, TSx)) \leq d(\alpha(g, x), \alpha(g, Sx)) + d(\alpha(g, Sx), \alpha(g, TSx)).
\]

\( \square \) follows from the inclusion

\[
  E_{\epsilon_1 + \epsilon_2}(T, gh) \subset E_{\epsilon_1}(T, h) + h^{-1}E_{\epsilon_2}(T, g)
\]

which is implied by \( \square \).
is immediate from (10).

(13) follows from (11) and the easy fact: $A_{\epsilon, \delta}(T) = A_{\epsilon, \delta}(T)^{-1}$.

(14) Choose an $\epsilon/2$-dense sequence $\{\ell_i\}_{i=1}^\infty$ in $L$, let

$$X_i = \{ x \in X \mid d(\alpha(g_0, x), \ell_i) < \epsilon/2 \},$$

and choose $N$ large enough to ensure $\mu(\bigcup_{i=1}^N X_i) > 1 - \delta/2$. Set

$$U(g, \epsilon, \delta) = \bigcap_{i=1}^N \left\{ T \in \text{Aut} (X, \mu) \mid \mu (TX_i \triangle X_i) < \frac{\delta}{2N} \right\}.$$

For $T \in U(g, \epsilon, \delta)$ we have $\mu(\bigcup_{i=1}^N (X_i \cap T^{-1}X_i)) > 1 - \delta$, while for $x \in X_i \cap T^{-1}X_i$:

$$d(\alpha(g, x), \alpha(g, T(x))) \leq d(\alpha(g, x), \ell_i) + d(\ell_i, \alpha(gT(x))) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

This completes the proof of the Lemma. □

Proof of Proposition 4.1. Fix a compact subset $K \subset G$ and $\epsilon > 0$ and $\delta > 0$.

It follows from (14) that there exists an open neighborhood $U_0$ of the identity in $\text{Aut} (X, \mu)$ so that there is a set $B = B_{\epsilon/3, \delta/3}$ of positive Haar measure of elements $g \in G$ with

$$B \subset A_{\epsilon/3, \delta/3}(T).$$

Then, by a standard Fubini argument, there exits an open neighborhood $V$ of the identity in thel. c.s.c. group $G$ s.t. $V \subset B^{-1}B$. Hence for every $T \in U_0 \cap \text{Aut} (X, \mu)^G$ we have, using (13):

$$V \subset B^{-1}B \subset A_{\epsilon/3, \delta/3}(T)^{-1}A_{\epsilon/3, \delta/3}(T) \subset A_{2\epsilon/3, 2\delta/3}(T).$$

Since $K$ is compact, there exists a finite set $\{g_1, \ldots, g_m\} \subset K$ so that $K \subset \bigcup_{i=1}^m g_iV$. For each $g_i$ there is a neighborhood $U_i$ of the identity in $\text{Aut} (X, \mu)$ so that $g_i \in A_{\epsilon/3, \delta/3}(T)$ for all $T \in U_i \cap \text{Aut} (X, \mu)^G$. Therefore taking $U = U_0 \cap U_1 \cap \cdots \cap U_m$, and using (13) again, we can deduce that

$$K \subset \bigcap_{T \in U \cap \text{Aut} (X, \mu)^G} A_{\epsilon, \delta}(T).$$

This proves the Proposition. □

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