Differentiating Majorana from Andreev Bound States in a Superconducting Circuit

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We investigate the low-energy theory of a topological insulator nanowire threaded with magnetic flux and coupled in proximity to a finite capacitance Josephson junction. Charge fluctuations across the junction couple to resonant microwave fields and can be used to probe microscopic excitations such as Majorana and Andreev bound states. We show that the hybridization between the localized excitations and the charge degree of freedom gives rise to key features in the spectroscopic patterns, which reveal the presence of Majorana fermions.

**Introduction.**—Quantized supercurrent oscillations in Josephson junctions strongly coupled to cavity photons, within the framework of circuit quantum electrodynamics (cQED), have become a prominent source, not only for the development of quantum processors [1–9], but also in the study of mesoscopic solid-state phenomena [10–14]. Their high-Q superconducting resonator environment and the nonlinearity of the junction, allow precise control and high resolution microwave probing while maintaining strong coherence throughout the system. New generation hybrid devices combine additional solid-state components in order to enhance their tunability, control their responsiveness to external fields and develop a framework that can support unique quantum states, which may be difficult to probe and control in other systems [15–23].

A prime candidate for such a hybrid device is based on a topological insulator nanowire such as Bi$_2$Se$_3$ and Bi$_2$Te$_3$ embedded in a Josephson junction. The proximity effect gives rise to an unconventional, topological superconducting pairing [24–31], whose properties may prove useful for quantum computation. These materials are known for their large Dirac surface states bandwidth and are therefore excellent candidates for the realization of Majorana zero-energy states [32–38]. However, due to their normal surface states, they can also host Andreev bound states in the junction’s weak-link [39–42]. Differentiating between the two types of bound states is an undertaking which is at the heart of current experimental exploration.

In this paper we consider a setting which consists of a single cylindrical topological insulator nanowire embedded inside the superconducting islands of a transmon [2] with an applied magnetic flux $\Phi$ (see Fig. 1). The anharmonic spectrum of the transmon can be controlled by external shunted capacitors, and displays strong dipole transitions around the Josephson plasma frequency. The presence of bound states in the junction generates a fine structure around the plasma frequency, allowing us to target processes related to the Andreev bound states and their interaction with the Majorana fermions. We develop an effective model for the dynamics of the weak-link, revealing the interplay of microscopic and macroscopic degrees of freedom. We use this model to obtain the spectroscopic signatures of the device, which manifest the fermionic parity constraints in the superconducting islands and the hybridization between the different degrees of freedom. We show that these patterns contain unambiguous signatures of the two types of bound states in the weak link, revealing their unique properties.

**Description of the Model.**—Let us consider the full system consisting of a single topological insulator nanowire in proximity to a transmon, with the two sides of the nanowire embedded in the superconducting plates. Due to quantum confinement in the radial direction of the nanowire, multiple bands exist separated by $\sim v/R$, with $R$ the radius of the nanowire and $v$ the fermi velocity. This separation between the bands allows us to reduce the nanowire to a one-dimensional system. We tune the magnetic flux close to $\Phi = \frac{hc}{2e}$, noting that any discrepancy from this flux value will open a finite magnetic gap $B$ in the Dirac spectrum (see Supplemental Material), and set the chemical potential to $|\mu| < \frac{\hbar}{R} - B$. This con-
dition ensures that only the lowest non-degenerate band is occupied, thus creating effectively a one-dimensional system. The system is modeled by the action $S = S_{\text{sc}} + S_{\text{w}} + S_{\text{nm}} + S_{\text{J}}$. The first term reads $S_{\text{sc}} = \sum_j \int \text{d}t \text{d}z \Psi_j^\dagger G_{j}^{-1} \Psi_j$ and describes the proximity induced superconductivity in the left ($j = 1$) and right ($j = 2$) islands, given by the inverse Green function

$$G_j^{-1} = i \partial_t - \left( iv \partial_z \sigma_y + B \sigma_z + \frac{\partial_t \phi_j}{2} \right) \tau_z + \Delta \sigma_y \tau_y.$$  (1)

Here $\sigma_i$ and $\tau_i$ are Pauli matrices in spin and Nambu space respectively, operating on the spinor $\Psi = \frac{1}{\sqrt{2}}(\psi_\uparrow, \bar{\psi}_\downarrow, \psi_\downarrow)^T$. The superconducting phase in the pairing term $\Delta \phi_j(t)$ is treated beyond the mean-field approximation. A crucial matter, as it allows us to take into account effects of charge fluctuations. We use the gauge transformation $\Psi \rightarrow e^{-i \frac{\phi_j(t)}{\gamma} \tau_z} \Psi$ which removes the phase from the pairing term and adds $\partial_t \phi(t)$ to the kinetic term. In addition, for convenience and without affecting the main results, we have set the chemical potential to $\mu = 0$ throughout. The weak-link is modeled as a two-state system, given by

$$S_{\text{w}} = \int \text{d}t \text{d}z \left(C^\dagger (i \partial_t - \varepsilon \tau_z - B \tau_z) C - U c_1^\dagger c_1 c_2^\dagger c_2 \right),$$

where $C = \frac{1}{\sqrt{2}}(c_1, c_2, -c_1^\dagger, -c_2^\dagger)^T$ and the two operators $c_1$, $c_2$ represent low momentum modes with spin orientation along the nanowire. This is justified due to the weak-link’s finite size and the resulting level quantization. We have also included $\varepsilon$ which serves as a local gate operating on the weak-link, and a repulsive Coulomb interaction $U$. The coupling of the weak-link to the islands is given by the tunneling term $S_{\text{nm}} = \sum_j (\lambda \int \text{d}t \Psi_j^\dagger(0)e^{\frac{i \phi_j(t)}{\gamma} \tau_z} C + \text{h.c.})$ and the action of the transmon, which also serves as the superconductivity inducing system, is $S_{\text{J}} = \int \text{d}t \left( \frac{(\dot{\phi}_1 - \dot{\phi}_2)^2}{16 E_\text{C}} + \frac{(\dot{\phi}_1 + \dot{\phi}_2)^2}{16 E'_\text{C}} + E_J \cos(\phi_1 - \phi_2) \right)$. Here $E_\text{C}$ and $E'_\text{C}$ define the scale of the charging effect, originating from the finite capacitance of the mesoscopic device, with the ratio $E'_\text{C}/E_\text{C} \leq 1$ controlling the strength of the mutual capacitance. Throughout we will assume that the system operates in the transmon regime $E'_\text{C}/E_\text{C} \ll E_J$[2], where $E_J$ is the Josephson energy.

The effects of charge transfer between the islands and the weak-link, and the related dipole transitions, are dominated mainly by low-lying excitations. To this end, we derive a low-energy theory which focuses on the interaction of Cooper pairs with the bound states in the weak-link, by systematically integrating all highly fluctuating degrees of freedom (for a full derivation see the Supplemental Material). This results in an effective Hamiltonian of the form $H_{\text{eff}} = H_{\text{C}} + H_{\gamma} + H_{\text{J}}$ which we use for our main analysis of the system. The effective dynamics of the junction are given by a modified transmon Hamiltonian

$$H_J = 4 E_C (\hat{n} - n_g)^2 - E_J \cos(\phi) + E'_C \left( \hat{N}^2 + 2a C^1 \tau_z C \hat{N} \right).$$  (2)

We have presented the results using the phase difference $\phi = \phi_1 - \phi_2$ and the average phase $\delta = (\phi_1 + \phi_2)/2$, conjugate to $\hat{n} = \frac{1}{2} (\hat{n}_1 - \hat{n}_2)$ and $\hat{N} = \hat{n}_1 + \hat{n}_2$ respectively. The operator $\hat{n}$ denotes the relative number of Cooper pairs between the islands, and $\hat{N}$ is the total number of excess Cooper pairs in the islands. The operator $e^{-i q \phi}$ transfers $q$ amount of charge from the left to the right island, and $e^{-i q \delta}$ describes the transfer of charge $q$ from the weak-link to the superconducting islands. The values $E_C$ and $E'_C$ were redefined to include both the capacitance of the nanowire and the parent superconductors. In order to account for the gate voltage we added $n_w$ which represents the offset charge, measured in units of the Cooper pair charge. The parameter $\alpha \equiv \alpha_c + \frac{\lambda^2}{2} \left( 1 - \frac{B^2}{\Delta^2} \right)^{-1}$, controls the interaction between the weak-link and the islands, and is comprised of two contributions: one is capacitive, given by a phenomenological parameter $\alpha_c$, and the other is a consequence of the induced superconductivity in the weak-link. The weak-link is governed by

$$H_C = (\bar{\xi} + B)c_1^\dagger c_1 + (\xi - B)c_1 c_1^\dagger + U c_1^\dagger c_1 c_2 c_2^\dagger$$

$$+ 2 \Gamma \cos(\phi/2) \left( e^{i \delta} c_1^\dagger c_1 + e^{-i \delta} c_2^\dagger c_2 \right) + \text{h.c.}$$  (3)

with an induced pairing of strength $\Gamma = \lambda^2/v$. Since this term emerges from integration of high-energy degrees of freedom the pairing should satisfy $\Gamma \ll \Delta$, and for our purposes we further assume $\Gamma \ll E_J$. The repulsive interaction and the local gate values were modified to $U \sim U + 2 E'_C \alpha^2$ and $\bar{\xi} = \xi + E'_C \alpha^2$. The weak-link fermions are coupled to the Majorana fermions via the Hamiltonian

$$H_{\gamma} = w e^{i \delta/2} \left[ -i e^{i \varphi} \gamma_1 (c_1^\dagger + c_1) - e^{-i \varphi} \gamma_2 (c_1 - c_1^\dagger) \right] + \text{h.c.}$$  (4)

where $\gamma_1$, $\gamma_2$ are hermitian operators denoting the Majorana fermions localized near the weak-link and $w \sim \sqrt{\Gamma B}$. Assuming negligible hybridization energy between each pair of Majorana fermions, we exclude their remote counter-parts $\gamma_1$, $\gamma_2$ (see Supplemental Material).

Note that in Eq. (4) the process of charge transfer is also accompanied by a change of fermionic parity by means of $\gamma_1$ and $\gamma_2$. As each side of the junction can be characterized by an addition of non-local fermion zero modes $f_1$ and $f_2$, defined by $\gamma_1 = i (f_1^\dagger - f_1)$ and $\gamma_2 = f_2^\dagger + f_2$. Their occupation defines the fermionic parity in each island. Since the system is only capacitively shunted the total number of particles is conserved and can be fixed by a neutrality condition $2 \hat{N} + \hat{n}_w = 0$, where $\hat{n}_w = c_1^\dagger c_1 + c_2^\dagger c_2$. With this constraint and the different parity
The plasma frequency corresponding to two shifted harmonic oscillators with three different coupling strengths: (a) $\Gamma/E_0 = 0.5$, $\kappa/2\pi = 3$ MHz, (b) $\Gamma/E_0 = 0.12$, $\kappa/2\pi = 0.4$ MHz, (c) $\Gamma/E_0 = 0.04$, $\kappa/2\pi = 0.3$ MHz. Both lines have the same periodicity $n_g \rightarrow n_g + 1$, which indicates the tunneling of solely Cooper pairs. In (d) we have increased the magnetic gap to $B/E_0 = 11.3$ and set $\Gamma/E_0 = 0.01$, which resulted in additional transition lines. Owing to the mechanisms of single particle tunneling, all transition lines exhibit an apparent form. The central lines show spectral holes, and are enveloped by a form of type (1) with an increased gap $\Delta f = 0.95$ MHz. In all of the magnified insets we used $\kappa/2\pi = 0.1$ MHz.

combinations, we end up with eight different subspaces which we denote by $\{p_1, p_2, \sigma_w\}$, where $p_j = 0, 1$ indicates the occupation of $f_j$ and $\sigma_w = 0, \uparrow, \downarrow, \uparrow\downarrow$ correspond to the four possible spin configurations in the weak-link.

**Spectroscopic signatures of bound states.**—To study the effect of the bound states on the spectroscopic signatures, we take for a long wire $U, \alpha_c \ll E_C$. We consider first the case where the Majorana fermions are absent, by setting $B = 0$. This results in a Hamiltonian which conserves the fermionic parity and thus by projecting $H_{\text{eff}}$ onto the $\{0, 0, 0, 0, 0\}$ subspace we get

$$H = \begin{pmatrix} H_f[n_w = 0] & 2\Gamma \cos(\varphi/2) & \frac{2\Gamma}{\cos(\varphi/2)} \ H_f[n_w = 2] + 2 \xi \end{pmatrix},$$

where $n_w = \langle p_1, p_2, \sigma_w | \hat{n}_w | p_1, p_2, \sigma_w \rangle$. A qualitative picture can be attained by focusing on solutions localized near $\varphi \sim 0$, as is characteristic to the transmon. If we ignore the effect of the offset charge, a straightforward diagonalization of Eq. (5) gives us two independent sectors $H_{\pm} \sim 4E_C\hat{n}_w^2 + E_J\hat{\varphi}^2/2 \pm \sqrt{(E_C^2 + 2\xi)^2 + 16\Gamma^2}/2$, corresponding to two shifted harmonic oscillators with the plasma frequency $\omega_p = \sqrt{8E_CE_J}$. In addition we have used $\Gamma/\Delta \ll 1$ to simplify the expression. Note that the exact system also has an anharmonicity of order $E_C$, which can be accounted for by including higher orders in $\varphi$. The split spectrum is a result of the Andreev bound states inducing charge fluctuations in the weak-link, added to those which appear in the traditional transmon. One can appreciate this by looking at the charge distribution given by $\langle \hat{n}_w \rangle_{\pm} \sim 1 \pm \tanh \left( \frac{E_C^2 + 2\xi}{4\Gamma} \right)$ in each sector of $H_{\pm}$. To obtain a quantitative description of the model we construct the Hilbert space using the eigenstates of $\hat{n}$ and the values of $n_w$. Since only Cooper pairs are allowed to tunnel in this regime, the $n_w = 0$ sector imposes $n \in Z$, while in the $n_w = 2$ sector it is $n \in Z + 1/2$, due to the absence of a Cooper pair in one of the islands. This division between the sectors is illustrated in the dependence of the energy spectrum on $n_g$ - the charge dispersion $|2, 4\rangle$, and can be seen in the system’s spectroscopy (see Fig. 2).

The charge fluctuations between the two sides of the junction result in a coupling of the system to an electromagnetic field via the dipole operator $\hat{n}$. By calculating its matrix elements for varying values of $n_w$, we can find experimentally accessible patterns (see Fig. 2). As in the traditional transmon, the dominant dipole transitions are between neighboring levels separated by $\sim \omega_p$. 

![Figure 2](image-url)
Here however, each of the two sectors of $\mathcal{H}_\pm$ contributes a transition line shifted with respect to its partner by $n_g \rightarrow n_g + 1/2$, which results in a doublet-like pattern. For $\Gamma \rightarrow 0$ the transition lines cross at the degeneracy points $n_g = 1/4 + \mathbb{Z}/2$ in a manner which is seen in experimental measurements of the transmon [11] and is usually a result of quasi-particle poisoning. Here however, the effect is due to a coherent occupation of the weak-link.

The dependence of the charge distribution in the weak-link on the local gate $\varepsilon$, suggests a special symmetry point at $\varepsilon = -E_C/2$. By tuning the system to this point, each sector of $\mathcal{H}_\pm$ contributes a single fermion to the weak-link which occupies the Andreev bound state and results in an added pair of transition lines to Fig. 2(a-c). This behavior is superficially similar to the Majorana-transmon [8, 10], where neighboring Majorana fermions hybridize in the weak-link. By shifting the gate away from this finely tuned symmetry point, one can easily distinguish between the role performed by Andreev bound states and that of neutral Majorana fermions, as the latter is not affected by the gate.

We now increase the flux beyond the half-flux quanta and open a magnetic gap $B \neq 0$. This uncovers the inherent differences between Andreev bound states and Majorana fermions, as observed in their distinct sensitivity to $n_g$. Combining the two types of bound states results in a rich array of parity configurations due to the addition of single fermion transfer. The $n_g = 0, 2$ subspaces, now have variants of even and odd fermionic parity on each side while in total keeping a symmetric combination, i.e. $p_1 = p_2$. In both variants the coupling between the islands is mediated by a transfer of a spin-singlet state without any direct response to $B$. The $n_g = 1$ subspaces, on the other hand, which have an asymmetric parity combination, accommodate spin-polarized Andreev bound states which hybridize with the Majorana fermions and result in a Zeeman splitting around the magnetic gap. Here we show four dipole transition lines as a function of $n_g$, following the pattern in inset (2) of Fig. 2 for varying values of the magnetic gap: (a) $B = 11E_C$, (b) $B = 11.3E_C$, (c) $B = 11.8E_C$, (d) $B = 12.5E_C$. All patterns are centered around a frequency which increases approximately linearly with $B$ in this range, from $f = 5.1713$ GHz to $f = 5.4455$ GHz. The patterns have a similar dipole size and are plotted with a resolution given by $\kappa/2\pi = 0.1$ MHz.

Conclusion.— In this work we investigated the physics of coupled low-energy bound states in a topological insulator nanowire embedded in a Josephson junction, where the charging term plays an important role. We have shown that by applying an external flux we can tune the junction between two remarkably different behaviors. The first is characterized by the absence of Majorana fermions, with Andreev bound states generating dipole transitions similar to those found in the traditional transmon. The second, in contrast, marked by the nucleation of Majorana fermions, displays a striking difference in the vicinity of the $n_g = 1/4$ point, where some of the transition lines develop a vanishing intensity. This signature emerges despite the presence of Andreev bound states and is distinct from their behaviour. While zeros in the intensities at $n_g = 1/4$ might occur accidentally also in the absence of Majorana fermions, the application of a local gate reveals the persistent neutrality of the Majorana fermions by maintaining a sharp zero at this value of the offset charge.

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**SUPPLEMENTARY MATERIAL**

**TI Nanowire**

Let us consider a cylindrical TI with radius \( R \) threaded with magnetic flux along its symmetry axis. Such systems with curved surfaces were studied previously \[34,37,38,44\] and we outline here only the details needed for our setup. This includes the reduction of the TI to a one-dimensional system and a derivation of the Majorana zero modes when superconductivity is added. The surface Hamiltonian in cylindrical coordinates is given by

\[
H_{\text{surf}} = i v \partial_z (\sigma_y \cos \theta - \sigma_x \sin \theta) - \frac{v \sigma_z}{R} (i \partial_\theta + \Phi),
\]

where Fermi velocity \( v \) and a dimensionless flux \( \Phi \) which includes both orbital and Zeeman contributions. The reduction to a one-dimensional system can be understood more easily by rotating the Hamiltonian with

\[
e^{-i \frac{v \sigma_z}{R} H_{\text{surf}}} e^{i \frac{v \sigma_z}{R} H_{\text{surf}}} = i v \partial_z \sigma_y - \frac{v \sigma_z}{R} (i \partial_\theta + \Phi).
\]

This in turn changes the angular boundary condition to \( 4\pi \)-periodicity, and results in a half-integer quantization \( -i \partial_\theta \to \ell \in \mathbb{Z} + \frac{1}{2} \). For \( \Phi = 0 \) the rotational symmetry \( \ell \leftrightarrow -\ell \) results in double degenerate branches. However, by increasing the flux we obtain a single low energy branch (\( \ell = \frac{1}{2} \)) where the system can be treated as one-dimensional, as long as higher values of \( \ell \) are not excited. In the case \( \Phi = \frac{1}{2} \), the gap formed due to the nanowire’s finite radius is closed and a linear Dirac spectrum is formed. Introducing \( B \) as the magnetic gap that the Dirac spectrum acquires, we can define the flux in general as \( \Phi = \frac{1}{2} - \frac{BR}{2\pi} \). Focusing on \( \ell = \frac{1}{2} \) we obtain the one-dimensional Hamiltonian

\[
H_{\text{surf}}' = i v \partial_z \sigma_y + B \sigma_z.
\]

It is known \[37\] that when a spatially varying pairing term \( \Delta(z) \psi_\uparrow^\dagger \psi_\downarrow + \text{h.c.} \) is included, the nanowire supports Majorana fermions. Here we derive the wave functions of the two Majorana fermions in a single wire with edges at \( z = 0 \) and \( z = -L \). For a finite \( L \) a small energy splitting occurs due to the hybridization of edge modes, but we will ignore this effect and assume exact zero-modes. To find the zero-modes we solve \( H f(z) = 0 \) with

\[
H = (i v \partial_z \sigma_y + B \sigma_z) \tau_z - \Delta(z) \sigma_y \tau_y,
\]

where \( \Delta(z) \) has a step-like profile localized around the edges. At the right edge \( \Delta(z \gg 0) \rightarrow 0 \) and \( \Delta(z \ll 0) \rightarrow \Delta \), and the mirror image at the left edge. The magnetic gap \( B < \Delta \) is kept constant throughout. We will denote the right edge solution by \( f(z) \) and the left edge solution by \( \tilde{f}(z) \). Using the transformation \( \tilde{H} = U^\dagger H U \) with \( U = e^{-i \frac{B}{2} \tau_z} e^{i \frac{\Delta(z)}{2} \tau_y} \) we obtain

\[
\tilde{H} = (B \sigma_x - i v \partial_z \sigma_y) \tau_z - \Delta(z) \sigma_y \tau_y.
\]

Giving us four independent equations of the form \( (\partial_z - W_j(z)) u_j(z) = 0 \), where \( u_j \) are elements of the spinor solving \( \tilde{H} g(z) = 0 \). In general the solution has the form \( u_j(z) = N_\ell e^{f_\ell \int_0^z W_j(z') dz'} \) with a normalization factor \( N_\ell \). Given the \( \Delta(z) \) profile outlined above only one finite solution exists, and all others, depending on which edge, diverge. The appropriate solution at the right edge \( (z = 0) \) is \( u(z) = N_\ell e^{f_\ell \int_0^z \Delta(z') dz'} \), while at the left edge \( (z = -L) \) it is \( \tilde{u}(z) = N_\ell e^{-f_{-L} \int_{-L}^z \Delta(z') dz'} \).

In spinor form the wave functions are

\[
g(z) = u(z)(0, i, 0, 0)^T
\]

\[
\tilde{g}(z) = \tilde{u}(z)(0, 0, 1, 0)^T.
\]

By choosing a sharp step-function profile for \( \Delta(z) \) we obtain the normalization factor \( N_\ell = \sqrt{\frac{2B(\Delta-B)}{v \Delta}} \). Reverting to the original basis we get the spinors

\[
f(z) = \frac{1}{2} u(z)(i, i, -i, -i)^T
\]

\[
\tilde{f}(z) = \frac{1}{2} \tilde{u}(z)(1, -1, 1, -1)^T,
\]

which are used to define the two types of hermitian Majorana operators \( \gamma, \gamma^\dagger \). This can be generalized trivially to a Josephson junction by combining two such wires, each hosting a pair: \( \gamma_1, \gamma_1^\dagger \) at left side and \( \gamma_2, \gamma_2^\dagger \) at the right.

**Effective Field Theory**

Here we show a full derivation of the effective theory by focusing on the low-energy excitations. We first consider the integration of high-energy bulk quasi-particles, and afterwards include the Majorana fermions. This will give us a full description of the bound states in conjunction with the mesoscopic Josephson junction.

**High Energy Quasi-particles**

We concentrate on the action describing the superconducting regions of the nanowire given by Eq. (1) in the main text, and their coupling to the weak-link. Throughout we use \( \Delta \) as the largest energy scale to make controlled approximations. In particular we assume that the phase dynamics are slow compared to the superconducting quasi-particles, with the exception of the Majorana fermions. It will be sufficient to examine only one of the junction sides which is governed by the action

\[
S = \int dt dz \left[ \Psi^\dagger G^{-1} \Psi + \Psi^\dagger \eta + \eta^\dagger \Psi \right].
\]

Here we defined the auxiliary source field \( \eta(z,t) = \lambda \delta(z) e^{-i \frac{\Delta}{2} \tau_y} C \) and for simplicity omitted the index \( j \). Integrating the fermionic fields \( \Psi, \Psi^\dagger \) results in the form
$S' = S_0 + S_1$, where
\[ S_0 = -\text{tr} \ln(G^{-1}), \quad (S9) \]
and
\[ S_1 = \int dt dz \int dt' dz' \eta^\dagger(z', t') G(z - z', t, t') \eta(z, t). \quad (S10) \]

The notation “tr” designates the trace over the orbital and the Nambu-spin subspaces. We start by evaluating the term $S_0$ which describes the slow phase dynamics. The inverse Green function can be written as $G^{-1} = g^{-1} - \chi(t)$, where $g = \sum_{\omega k} e^{i(kz - \omega t)} g(\omega, k)$ is the bare Green function valid in the mean-field regime, and $\chi(t) = \frac{1}{2} \phi \tau_z$ is the deviation from the mean-field. This allows us to recast Eq. (S9) as $S_0 = \text{tr} \ln(1 - g\chi)$, where we ignored the excess term $\text{tr} (g^{-1})$ since it is independent of the phase. An explicit form for $g$ is given by
\[
g(\omega, k) = \frac{1}{W} \begin{pmatrix} A(B) & -iP & iU & -D(B) \\ iP & A(-B) & D(-B) & -iU \\ -iU & D(-B) & A(-B) & iP \\ -D(B) & iU & -iP & A(B) \end{pmatrix}, \quad (S11)\]

with the matrix elements
\[
W = (B^2 + \Delta^2 + \nu^2 k^2 - \omega^2)^2 - 4B^2 \Delta^2 \\
A(B) = B(\Delta^2 - \nu^2 k^2 + \omega^2) - \omega B^2 - B^3 \\
- \omega(\Delta^2 + \nu^2 k^2 - \omega^2) \\
D(B) = \Delta(\Delta^2 + \nu^2 k^2 - (B + \omega)^2) \\
P = \nu k(B^2 + \Delta^2 + \nu^2 k^2 - \omega^2) \\
U = 2\nu k B \Delta. \quad (S12)\]

Since the phase fluctuations are small compared to the energy of the quasi-particles $\sim \Delta$, we can perform a 2nd order expansion in $g\chi$, which gives us
\[
S_0 \simeq \text{tr}(g\chi) + \frac{1}{2} \text{tr}(g\chi g\chi). \quad (S13)\]

Evaluating the 1st term we get
\[
\text{tr}(g\chi) = \text{Tr} \sum_{\omega' \omega} g(\omega, k) \chi(\omega - \omega') \delta_{\omega \omega'}, \quad (S14)\]

where the trace is divided into an orbital sum given by the Fourier components and a Nambu-spin summation denoted by Tr. The Fourier components of the phase fluctuations are given by $\chi(\omega - \omega') = \frac{1}{2} (\omega - \omega') \phi \delta_{\omega - \omega'} \tau_z$ and with the included delta function $\delta_{\omega \omega'}$ reduce the entire term to zero. The 2nd term in Eq. (S13) has the leading contribution
\[
\text{tr} (g\chi g\chi) = \frac{1}{4} \text{Tr} \sum_{\omega \Omega k} \Omega^2 |\phi\Omega|^2 g(\omega, k) \tau_z g(\omega - \Omega, k) \tau_z. \quad (S15)\]

Note that since $\chi \sim \omega$ and $g \sim \frac{1}{\omega}$, the expansion performed in Eq. (S13) is analogous to a 2nd order expansion in $(\omega/\Delta) \ll 1$. In this framework the bare Green function in (S15) is approximated using $g(\omega - \Omega) \simeq g(\omega) - g'(\omega)\Omega$ which results in
\[
\text{tr}(g\chi g\chi) = \frac{1}{4E_{C}^{(w)}} \left( \sum_{\Omega} \Omega^2 |\phi\Omega|^2 \right) \quad (S16)\]

Here we neglected terms proportional to $\Omega^3$, and defined the parameter $E_{C}^{(w)}$ via
\[
E_{C}^{(w)} = \left( \text{Tr} \sum_{\omega k} g(\omega, k) \tau_z g(\omega, k) \tau_z \right)^{-1}. \quad (S17)\]

Transforming Eq. (S16) back to the time-domain we obtain the free part of the phase dynamics resulting from the bulk fermions $S_0 = \int dt dz \int dt' dz' g(\omega, k) \phi \tau_z$ and as a result the action can be split into two terms $S_1 = S_1^{(0)} + S_1^{(\chi)}$. First we consider the case $\chi = 0$, in which the Green function can be replaced by the bare diagonal Green function $G(\omega, \omega'; k, k') = g(\omega, k) \delta_{\omega \omega'} \delta_{kk'}$, resulting in
\[
S_1^{(0)} = \chi^2 \int dt dz dz' e^{-i\omega(t-t')} \times \left( \sum_{\omega k} g(\omega, k) e^{-i\omega(t-t')} \right) e^{i\omega(t-t')} C(t). \quad (S18)\]

Seeing that we focus on the low energy theory for the $C$ fermions we can expand the Green function (S11) to
\[
\sum_{k} g(\omega, k) \simeq \sum_{k} \left( g(0, k) + \omega \frac{\partial g(\omega, k)}{\partial \omega} \right) \bigg|_{\omega=0}. \quad (S19)\]

To calculate the sum over $k$ we replace it with an integral $\sum_{k} \rightarrow \frac{1}{\nu} \int d\epsilon$, where $\epsilon = vk$, which results in
\[
g_0 \equiv \sum_{k} g(0, k) = \frac{1}{\nu} \sigma_y \tau_y, \quad (S20)\]

and
\[
g_1 = \frac{1}{\nu} \sum_{k} \frac{\partial g(\omega, k)}{\partial \omega} \bigg|_{\omega=0} = \frac{(\Delta + 2^{3/2} B \sigma_x \tau_y)}{\nu (B^2 - \Delta^2)}. \quad (S21)\]

Using the identities $\sum_{\omega} e^{-i\omega(t-t')} = \delta(t-t')$ and $\sum_{\omega} \omega e^{-i\omega(t-t')} = \delta(t-t')i \partial_t$, Eq. (S18) evaluates to
\[ S_1^{(0)} = \lambda^2 \int dt \left[ C^\dagger(t) e^{-i \frac{\omega(t)}{2} \tau_2} \left( g_0 - \frac{1}{2} (\partial_t \phi) g_1 \tau_z \right) e^{i \frac{\omega(t)}{2} \tau_2} C(t) \right. \\
+ \left. C^\dagger(t) \left( e^{-i \frac{\omega(t)}{2} \tau_2} g_1 e^{i \frac{\omega(t)}{2} \tau_2} \right) i \partial_t C(t) \right]. \]  

(S22)

We continue and examine the case \( \chi \neq 0 \). Contrary to the previous case, the Green function’s contribution to \( S^{(\chi)}_1 \) is of the form \( g \chi g \) and therefore not diagonal in \( \omega \) and \( k \). Explicitly:

\[ S_1^{(\chi)} = \lambda^2 \int dt dt' C^\dagger(t') e^{-i \frac{\omega(t')}{2} \tau_2} K(t, t') e^{i \frac{\omega(t')}{2} \tau_2} C(t), \]

where

\[ K(t, t') = \sum_{\omega, \omega'} \chi(\omega - \omega') g(\omega', \omega) e^{-i(\omega t - \omega' t')}. \]

(S24)

Since \( \chi \sim (\omega - \omega') \), it will be sufficient to take the zeroth order of \( g(\omega, k) \) which results in

\[ K(t, t') \approx \sum_k g(0, k) \tau_2 g(0, k) \sum_{\omega, \omega'} \chi(\omega - \omega') e^{-i(\omega t - \omega' t')} \].

By performing the sum over \( k \) and using Eq. (S11) we can show that \( K(t, t') = 0 \). Thus the \( S_1^{(0)} \) term given by Eq. (S22) is the only contribution to \( S_1 \).

Finally, combining both sides of the junction and adding \( S_w \), we get the full action for the weak-link fermions:

\[ S_C = \int dt C^\dagger \left[ J i \partial_t - (\varepsilon + B \sigma_z) \tau_z \right. \\
+ \sum_{j=1,2} \left( \Gamma_j e^{-i \phi_j(t) \tau_x} \sigma_y e^{i \phi_j(t) \tau_x} - \alpha_j \phi_j \tau_z \right) \right] C \]

(S26)

\[ - U C_i^\dagger C_i^\dagger C_i \],

where

\[ J = 1 + \sum_{j=1,2} \lambda_j^2 e^{-i \phi_j(t) \tau_x} g_1 e^{i \phi_j(t) \tau_x} \].

(S27)

We introduced \( \Gamma_j = \lambda_j^2 / v \) which serves as the induced pairing strength. Since the integration systematically removes high energy excitations it should satisfy \( \Gamma_j \ll \Delta \). Due to the integration process a fermion-phase interaction term appears with coupling constant \( \alpha_j = \lambda_j^2 \Delta / 2 v (\Delta^2 - B^2) \). This term is meaningful as long \( E_C \neq 0 \), otherwise it can be gauged out, similar to a vector potential of a massless particle.

**Majorana Fermions**

A general quasi-particle \( \Psi(z) \) in the superconductor can be defined as a combination of the Bogoliubov operators, where two are given by the Majorana fermions and the rest are high energy bulk states

\[ \Psi(z) = \frac{i u(z)}{\sqrt{2}} \tau_z \gamma + \frac{\bar{u}(z)}{\sqrt{2}} \sigma_z \bar{\gamma} + \sum_n U_n(z) \Gamma_n. \]  

(S28)

We have used the results and notation of Eq. (S7) to define the contribution of the Majorana fermions. \( \Gamma_n \) are spinors of Bogoliubov quasi-particles with their appropriate amplitudes in matrix form \( U_n(z) \). In the previous section we used momentum states as eigenstates of a one-dimensional superconducting wire. This approach can be justified here as well with the approximation \( \sum_n U_n(z) \Gamma_n \approx \sum_k e^{i k z} \Psi_k \), as the Majorana fermions are zero modes energetically isolated from the gapped quasi-particles. We can project Eq. (1) in the main text to the Majorana sector by the replacement \( \Psi_1(z) \rightarrow \frac{i u(z)}{\sqrt{2}} \tau_z \gamma \) and \( \Psi_2(z) \rightarrow \frac{\bar{u}(z)}{\sqrt{2}} \sigma_z \bar{\gamma} \), while the rest of the quasi-particles integrated. With this we obtain

\[ S_{\gamma} = \int dt \left( \gamma_1 i \partial_t \gamma_1 + \bar{\gamma}_2 i \partial_t \bar{\gamma}_2 \right. \]

\[ + \frac{i w_1}{\sqrt{2}} \gamma_1 e^{i \phi_1(t)} \tau_z C \]

\[ \left. + \frac{w_2}{\sqrt{2}} \bar{\gamma}_2 e^{i \phi_2(t)} \sigma_z C + h.c. \right) \],

(S29)

where \( w_1 = u(0) \lambda_1 \) and \( w_2 = \bar{u}(0) \lambda_2 \). Note that we neglected the Majorana fermions on the outermost edges of the junction since their hybridization is exponentially small in the length of the nanowire.

All the low-energy contributions are combined into the effective action \( S_{\text{eff}} = S_J + S_C + S_{\gamma} \), which is used in the next section to derive the effective Hamiltonian.

**Derivation of the Hamiltonian**

To derive the Hamiltonian we find the canonical variables from the Lagrangian in \( S_{\text{eff}} = \int dt L_{\text{eff}} \), written in terms of the phase difference \( \varphi = \phi_1 - \phi_2 \) and the average phase \( \delta = (\phi_1 + \phi_2) / 2 \). We define \( \mathcal{P}_X = \frac{\partial \mathcal{L}_{\text{eff}}}{\partial \dot{X}} \) as the conjugate momentum to \( X = \{ \varphi, \delta, \gamma_1, \gamma_2, C \} \) and by employing Legendre’s transformation we obtain the Hamiltonian in the form

\[ H_{\text{eff}} = \sum_X \mathcal{P}_X (\partial_t X) - L_{\text{eff}} = H_J + H_C + H_{\gamma}. \]  

(S30)

Since the phase dynamics \( \partial_t \phi_j \) dictate the charge fluctuations, we identify \( \mathcal{P}_{\varphi} \) with the relative number of cooper pairs between the two superconducting islands, and \( \mathcal{P}_{\delta} \) with the total number of excess cooper pairs residing in the islands. Explicitly these are

\[ \mathcal{P}_{\varphi} = \frac{\partial \varphi(t)}{8E_C} - \alpha - C^\dagger \tau_z C \]

\[ \mathcal{P}_{\delta} = \frac{\partial \delta(t)}{2E_C} - \alpha + C^\dagger \tau_z C, \]  

(S31)
where \( \alpha_- = \frac{1}{2}(\alpha_1 - \alpha_2) \) and \( \alpha_+ = \alpha_1 + \alpha_2 \). With these definitions we extract \( H_J \) from \( S_{\text{eff}} \) as

\[
H_J = 4E_C \left( \hat{n} + \alpha_- \tau_z \hat{C} \right)^2 + E'_C \left( \hat{N} + \alpha_+ \tau_z \hat{C} \right)^2 - E_J \cos(\varphi). \tag{S32}
\]

We used the quantized versions of the conjugate momenta: \( \mathcal{P}_\varphi \rightarrow \hat{n} \) and \( \mathcal{P}_\delta \rightarrow \hat{N} \). Note that in the symmetric regime (\( \lambda_1 = \lambda_2 \)), the one used in the main text, the coupling constant \( \alpha_- \) vanishes. Similarly to Eq. (S32), by using Legendre’s transformation, we get the Hamiltonian for the weak-link fermions

\[
H_C = \mathcal{C}^\dagger \left[ (\varepsilon + B \sigma_z) \tau_z + e^{-i \frac{\delta}{4} \tau_z} \left( \Gamma_1 + \Gamma_2 \right) \cos \left( \frac{\varphi}{2} \right) \tau_y 
- \left( \Gamma_1 - \Gamma_2 \right) \sin \left( \frac{\varphi}{2} \right) \tau_x \right] \mathcal{C} + U \hat{c}_{\uparrow}^\dagger \hat{c}_{\downarrow}^\dagger \hat{c}_{\downarrow} \hat{c}_{\uparrow}. \tag{S33}
\]

The coupling of the weak-link fermions to the Majorana fermions is given by

\[
H_\gamma = \frac{1}{\sqrt{2}} \left[ -iw_1 \gamma_1 e^{i \frac{(2\delta + \varphi)}{4} \tau_z} \tau_x - w_2 \bar{\gamma}_2 e^{i \frac{(2\delta - \varphi)}{4} \tau_y} \sigma_x \right] \mathcal{C} + \text{h.c.} \tag{S34}
\]