PRE-LIE ALGEBRAS AND INCIDENCE CATEGORIES OF
COLORED ROOTED TREES

MATT SZCZESNY

Abstract. The incidence category $C_F$ of a family $F$ of colored posets closed under disjoint unions and the operation of taking convex sub-posets was introduced by the author in [12], where the Ringel-Hall algebra $H_F$ of $C_F$ was also defined. We show that if the Hasse diagrams underlying $F$ are rooted trees, then the subspace $n_F$ of primitive elements of $H_F$ carries a pre-Lie structure, defined over $\mathbb{Z}$, and with positive structure constants. We give several examples of $n_F$, including the nilpotent subalgebras of $\mathfrak{sl}_n$, $\mathfrak{gl}_n$, and several others.

1. Introduction

A left pre-Lie algebra is a $k$–vector space $A$ endowed with a binary bilinear operation $\triangleright$ satisfying the identity

$$((a \triangleright b) \triangleright c) - a \triangleright (b \triangleright c) = (b \triangleright a) \triangleright c - b \triangleright (a \triangleright c)$$

It follows easily from 1.1 that anti-symmetrizing $\triangleright$ yields a Lie bracket

$$[a, b] = a \triangleright b - b \triangleright a$$
on $A$. However, not every Lie algebra arises from a pre-Lie algebra. Pre-Lie algebras first appeared in the works of E.B. Vinberg [13] and M. Gerstenhaber [3], and have since found applications in several areas. One prominent example is perturbative quantum field theory [4], where insertion of Feynman graphs into each other equips them with a pre-Lie structure which controls the combinatorics of the renormalization procedure.

In this paper, we show that pre-Lie algebras arise naturally from incidence categories introduced by that author in [12]. An incidence category is built from a collection $F$ of colored posets, which is closed under the operations of disjoint union and convex subposet - we will denote it by $C_F$. The objects of $C_F$ are the posets in $F$, and for $P_1, P_2 \in F$

$$\text{Hom}(P_1, P_2) := \{(I_1, I_2, f)|I_j \text{ is an order ideal in } P_j, f : P_1 \setminus I_1 \to I_2 \text{ an isomorphism}\}$$

Here, the poset $I_1$ should be viewed as the kernel of the morphism, and $I_2$ as the image. All morphisms in $C_F$ have kernels and cokernels, and so the notion of exact sequence makes sense. In [12], the Ringel-Hall algebra $H_{C_F}$ of $C_F$ was defined.

The author is supported by an NSA grant.
$H_{C_F}$ is the $\mathbb{Q}$–vector space of finitely supported functions on isomorphism classes of $C_F$:

$$H_{C_F} := \{ f : \text{Iso}(C_F) \to \mathbb{Q} | \text{supp}(f) | < \infty \}$$

with product given by convolution:

$$f \ast g(M) = \sum_{A \subset M} f(A)g(M/A).$$

$H_{C_F}$ possesses a co-commutative co-product given by

$$\Delta(f)(M, N) = f(M \oplus N)$$

(where $M \oplus N$ denotes the disjoint union of $M$ and $N$) as well as an antipode, making it a Hopf algebra. $H_{C_F}$ is graded, connected, and co-commutative, and so by the Milnor-Moore theorem isomorphic to $U(n_F)$, where $n_F$ is the Lie algebra of its primitive elements. It follows from 1.3 that

$$n_F = \text{span}\{\delta_P | P \in F, P \text{ connected} \}$$

We show that if $F$ consists of posets whose Hasse diagrams are rooted trees, then $n_F$ carries a pre-Lie structure $\triangleright$, with

$$\delta_{P_1} \triangleright \delta_{P_2} := \delta_{P_1} \ast \delta_{P_2} - \delta_{P_1 \oplus P_2}.$$ 

A more concrete description of $\triangleright$ is the following: $\delta_{P_1} \triangleright \delta_{P_2}$ is a sum of delta-functions supported on connected posets $P \in F$ whose Hasse diagram is obtained by grafting the root of $P_1$ onto a vertex of $P_2$. It follows from the definition of $\triangleright$ that the structure constants are non-negative integers.

The paper is organized as follows. Section 2 recalls the definition of pre-Lie algebra and introduces the universal example, namely the pre-Lie algebra of colored rooted trees. In section 3 we recall the construction of the incidence category $C_F$ as well as its main properties. The Ringel-Hall algebra of $C_F$ is introduced in section 4. In section 5 we define the pre-Lie structure $\triangleright$ on $n_F$ and verify that it satisfies the identity 1.1. Finally, section 6 is devoted to examples - among these are pre-Lie structures on nilpotent Lie subalgebras of $\mathfrak{sl}_n$ and $L\mathfrak{gl}_n$.

Acknowledgements: The author is very grateful to Pavel Etingof for valuable discussions and suggestions.

2. PRE-LIE ALGEBRAS

In this section, we recall the definition and some examples of (left) pre-Lie algebras. Let $k$ be a field.

Definition 1. A left pre-Lie algebra is a $k$–vector space $A$ endowed with a binary bilinear operation $\triangleright$ satisfying the left pre-Lie identity

$$(a \triangleright b) \triangleright c - a \triangleright (b \triangleright c) = (b \triangleright a) \triangleright c - b \triangleright (a \triangleright c)$$

for $a, b, c \in A$. 

One checks easily that antisymmetrizing the operation
\[ [a, b] = a \triangleright b - b \triangleright a \]
gives \( A \) the structure of a Lie algebra.

**Example 1.** Any associative \( k \)-algebra \( A \) is a pre-Lie algebra with the pre-Lie structure given by
\[ a \triangleright b := ab, \]
where the right hand side refers to the associative multiplication in \( A \).

**Example 2.** One of the most important examples of pre-Lie algebras is given by colored rooted trees. Recall that a tree is a graph with no cycles. We denote by \( E(t), V(t) \) the edge and vertex sets of \( t \) respectively. Let \( S \) be a finite set. By a **rooted tree colored by** \( S \), we mean a tree with a distinguished vertex \( r(t) \in V(t) \) called the **root**, and an assignment of an element of \( S \) to each \( v \in V(t) \). We adopt the convention that rooted trees are always drawn with the root on top. For example, if \( S = \{a, b\} \), then the following are rooted trees colored by \( S \):

Let \( \mathbb{T}_S \) denote the set of rooted trees whose vertices are colored by \( S \). Given \( t \in \mathbb{T}_S \), and \( e \in E(t) \), removing \( e \) disconnects \( t \) into two colored rooted trees: \( R_e(t) \) containing \( r(t) \) and \( P_e(t) \), whose root is the end of \( e \). Let \( \mathcal{T}_S \) be the \( k \)-vector space spanned by \( \mathbb{T}_S \). We have
\[ \mathcal{T}_S = \bigoplus_{n=0}^{\infty} \mathcal{T}_S[n] \]
where \( \mathcal{T}_S[n] \) is the subspace of \( \mathcal{T}_S \) spanned by trees with \( n \) vertices. For colored rooted trees \( t_1, t_2 \in \mathbb{T}_S \), let
\[ t_1 \triangleright t_2 := \sum_{s \in \mathbb{T}_S} n(t_1, t_2, s)s \]
where
\[ n(t_1, t_2, s) = \#\{e \in E(s)|P_e(s) = t_1, R_e(s) = t_2\}. \]
For example, we have

\[ b \triangleright a = 2 \quad b \triangleright a = 2 \quad a \triangleright b = 1 \quad a \triangleright b = 1 \]

It is well-known (see for instance [2]) that \( \triangleright \) defines a pre-Lie structure on \( \mathcal{T}_S \). The following theorem is proven in [2]

**Theorem 1.** \( \mathcal{T}_S \) is the free pre-Lie algebra on \( |S| \) generators.

**Remark 1.** In what follows, unless stated otherwise, \( k = \mathbb{Q} \).
3. Incidence categories

3.1. Recollections on posets. We begin by recalling some basic notions and terminology pertaining to posets (partially ordered sets) following [10, 11].

(1) An interval is a poset having unique minimal and maximal elements. For $x, y$ in a poset $P$, we denote by $[x, y]$ the interval

$$[x, y] := \{z \in P : x \leq z \leq y\}$$

If $P$ is an interval, we will often denote by $0_P$ and $1_P$ the minimal and maximal elements.

(2) An order ideal in a poset $P$ is a subset $L \subseteq P$ such that whenever $y \in L$ and $x \leq y$ in $P$, then $x \in L$.

(3) A sub-poset $Q$ of $P$ is convex if, whenever $x \leq y$ in $Q$ and $z \in P$ satisfies $x \leq z \leq y$, then $z \in Q$. Equivalently, $Q$ is convex if $Q = L \setminus I$ for order ideals $I \subset L$ in $P$.

(4) Given two posets $P_1, P_2$, their disjoint union is naturally a poset, which we denote by $P_1 + P_2$. In $P_1 + P_2$, $x \leq y$ if both lie in either $P_1$ or $P_2$, and $x \leq y$ there.

(5) A poset which is not the union of two non-empty posets is said to be connected.

(6) The cartesian product $P_1 \times P_2$ is a poset where $(x, y) \leq (x', y')$ iff $x \leq x'$ and $y \leq y'$.

(7) A distributive lattice is a poset $P$ equipped with two operations $\land, \lor$ that satisfy the following properties:

   (a) $\land, \lor$ are commutative and associative
   (b) $\land, \lor$ are idempotent - i.e. $x \land x = x$, $x \lor x = x$
   (c) $x \land (x \lor y) = x = x \lor (x \land y)$
   (d) $x \land y = x \iff x \lor y = y \iff x \leq y$
   (e) $x \lor (y \land z) = (x \lor y) \land (x \lor z)$
   (f) $x \land (y \lor z) = (x \land y) \lor (x \land z)$

(8) For a poset $P$, denote by $J_P$ the poset of order ideals of $P$, ordered by inclusion. $J_P$ forms a distributive lattice with $I_1 \lor I_2 := I_1 \cup I_2$ and $I_1 \land I_2 := I_1 \cap I_2$ for $I_1, I_2 \in J_P$. If $P_1, P_2$ are posets, we have $J_{P_1 + P_2} = J_{P_1} \times J_{P_2}$, and if $I, L \in J_P$, and $I \subseteq L$, then $[I, L]$ is naturally isomorphic to the lattice of order ideals $J_{L \setminus I}$.

Remark 2. Suppose that the Hasse diagram of a poset $P$ is a rooted tree - that is, $P$ has a unique maximal element $r(P)$, and the Hasse diagram contains no cycles. It is then easy to see that order ideals $I \subseteq P$ correspond to admissible cuts of $P$, where the latter is a collection of edges $C \subseteq E(P)$, having the property that at most one edge of $C$ is encountered along any path from root to leaf. For
instance, the dotted edges of the poset $T$ below yield an admissible cut:

Each admissible cut $C \subset E(P)$ divides the tree into a rooted connected tree $R_C(P)$ containing $r(P)$, and a rooted forest (a disjoint union of rooted trees) $P_C(P)$. The notation is clearly an extension of that used in example 2. In the last example, we have

$$R_C(T) = \begin{array}{c} b \\ \bullet \\ b \end{array} \quad \quad P_C(T) = \begin{array}{c} b \\ \bullet \\ b \end{array}$$

3.2. From posets to categories. Let $\mathcal{F}$ be a family of colored posets which is closed under the formation of disjoint unions and the operation of taking convex subposets, and let $\mathcal{P}(\mathcal{F}) = \{J_P : P \in \mathcal{F}\}$ be the corresponding family of distributive lattices of order ideals. For each pair $P_1, P_2 \in \mathcal{F}$, let $M(P_1, P_2)$ denote the set of colored poset isomorphisms $P_1 \to P_2$. It follows that $M(P, P)$ forms a group, which we denote $\text{Aut}_M(P)$.

3.2.1. The category $\mathcal{C}_\mathcal{F}$. We proceed to define a category $\mathcal{C}_\mathcal{F}$, called the incidence category of $\mathcal{F}$ as follows. Let

$$\text{Ob}(\mathcal{C}_\mathcal{F}) := \mathcal{F} = \{P \in \mathcal{F}\}$$

and

$$\text{Hom}(P_1, P_2) := \{(I_1, I_2, f) : I_1 \in J_{P_1}, f \in M(P_1 \setminus I_1, I_2)\} \quad i = 1, 2$$

We need to define the composition of morphisms

$$\text{Hom}(P_1, P_2) \times \text{Hom}(P_2, P_3) \to \text{Hom}(P_1, P_3)$$

Suppose that $(I_1, I_2, f) \in \text{Hom}(P_1, P_2)$ and $(I_2', I_3', g) \in \text{Hom}(P_2, P_3)$. Their composition is the morphism $(K_1, K_3, h)$ defined as follows.

- We have $I_2' \subset I_2$, and since $f : P_1 \setminus I_1 \to I_2$ is an isomorphism, $f^{-1}(I_2 \setminus I_2')$ is an order ideal of $P_1 \setminus I_1$. Since in $J_{P_1}$, $[I_1, P] \simeq J_{P_1 \setminus I_1}$, we have that $f^{-1}(I_2 \setminus I_2')$ corresponds to an order ideal $K_1 \in J_{P_1}$ such that $I_1 \subset K_1$.
- We have $I_2' \subset I_2 \vee I_2'$, and since $[I_2', P_2] \simeq J_{P_2 \setminus I_2'}$, $I_2 \vee I_2'$ corresponds to an order ideal $L_2 \in J_{P_2 \setminus I_2'}$. Since $g : P_2 \setminus I_2' \to I_3'$ is an isomorphism, $g(L_2) \subset J_{P_3}$, and since $J_{P_3} \subset J_{P_3}$, $g(L_2)$ corresponds to an order ideal $K_3 \in J_{P_3}$ contained in $I_3'$.
- The isomorphism $f : P_1 \setminus I_1 \to I_2$ restricts to an isomorphism $\tilde{f} : P_1 \setminus K_1 \to I_2 \setminus I_2' = I_2 \setminus I_2'$, and the isomorphism $g : P_2 \setminus I_2'$ restricts to an isomorphism $\tilde{g} : I_2 \vee I_2' \setminus I_2' = I_2 \setminus I_2' \to K_3$. Thus, $g \circ f : P_1 \setminus K_1 \to K_3$ is an isomorphism and $g \circ f \in M(P_1 \setminus K_1, K_3)$ by the property (4) above.
As shown in [12], the composition of morphisms is associative.

**Remark 3.**
- We refer to $I_2$ as the *image* of the morphism $(I_1, I_2, f) : P_1 \to P_2$.
- We denote by $\text{Iso}(C_F)$ the collection of isomorphism classes of objects in $C_F$, and by $[P]$ the isomorphism class of $P \in C_F$.

### 3.3. Properties of the categories $C_F$.
We now enumerate some of the properties of the categories $C_F$.

1. The empty poset $\emptyset$ is an initial, terminal, and therefore null object. We will sometimes denote it by $\emptyset$.

2. We can equip $C_F$ with a symmetric monoidal structure by defining $P_1 \oplus P_2 := P_1 + P_2$.

3. The indecomposable objects of $C_F$ are the $P$ with $P$ a connected poset in $\mathcal{F}$.

4. The simple objects of $C_F$ are the $P$ where $P$ is a one-element poset.

5. Every morphism
   \[(I_1, I_2, f) : P_1 \to P_2\]
   has a kernel
   \[(\emptyset, I_1, \text{id}) : I_1 \to P_1\]

6. Similarly, every morphism 3.1 possesses a cokernel
   \[(I_2, P_2 \setminus I_2, \text{id}) : P_2 \to P_2 \setminus I_2\]
   We will use the notation $P_2/P_1$ for $\text{coker}((I_1, I_2, f))$.

**Note:** Properties 5 and 6 imply that the notion of exact sequence makes sense in $C_F$.

7. All monomorphisms are of the form
   \[(\emptyset, I, f) : Q \to P\]
   where $I \in J_P$, and $f : Q \to I \in M(Q, I)$. Monomorphisms $Q \to P$ with a fixed image $I$ form a torsor over $\text{Aut}_M(I)$. All epimorphisms are of the form
   \[(I, \emptyset, g) : P \to Q\]
where \( I \in J_P \) and \( g : P \setminus I \to Q \in M(P \setminus I, Q) \). Epimorphisms with fixed kernel \( I \) form a torsor over \( \text{Aut}_M(P \setminus I) \).

(8) Sequences of the form
\[
\emptyset \xrightarrow{(0,0,\text{id})} I \xrightarrow{(0,1,\text{id})} P \xrightarrow{(I,0,\text{id})} P \setminus I \xrightarrow{(P \setminus I,0,\text{id})} \emptyset
\]
with \( I \in J_P \) are short exact, and all other short exact sequences with \( P \) in the middle arise by composing with isomorphisms \( I \to I' \) and \( P \setminus I \to Q \) on the left and right.

(9) Given an object \( P \) and a subobject \( I, I \in J_P \), the isomorphism \( J_{P \setminus I} \simeq [I, P] \) translates into the statement that there is a bijection between subobjects of \( P/I \) and order ideals \( J \in J_P \) such that \( I \subset J \subset P \). The bijection is compatible with quotients, in the sense that \( (P/I)/(J/I) \simeq J/I \).

(10) Since the posets in \( \mathcal{F} \) are finite, \( \text{Hom}(P_1, P_2) \) is a finite set.

(11) We may define Yoneda \( \text{Ext}^n(P_1, P_2) \) as the equivalence class of \( n \)-step exact sequences with \( P_1, P_2 \) on the right and left respectively. \( \text{Ext}^n(P_1, P_2) \) is a finite set. Concatenation of exact sequences makes
\[
\mathbb{E}xt^* := \bigcup_{A,B \in I(\mathcal{C}_\mathcal{F}), n} \text{Ext}^n(A, B)
\]
into a monoid.

(12) We may define the Grothendieck group of \( \mathcal{C}_\mathcal{F} \), \( K_0(\mathcal{C}_\mathcal{F}) \), as
\[
K(\mathcal{C}_\mathcal{F}) = \bigoplus_{A \in \mathcal{C}_\mathcal{F}} \mathbb{Z}[A]/\sim
\]
where \( \sim \) is generated by \( A + B - C \) for short exact sequences
\[
\emptyset \to A \to C \to B \to \emptyset
\]
We denote by \( k(A) \) the class of an object in \( K_0(\mathcal{C}_\mathcal{F}) \).

4. Ringel-Hall algebras

For an introduction to Ringel-Hall algebras in the context of abelian categories, see [8]. We define the Ringel-Hall algebra of \( \mathcal{C}_\mathcal{F} \), denoted \( \mathcal{H}_{\mathcal{C}_\mathcal{F}} \), to be the \( \mathbb{Q} \)-vector space of finitely supported functions on isomorphism classes of \( \mathcal{C}_\mathcal{F} \). I.e.
\[
\mathcal{H}_{\mathcal{C}_\mathcal{F}} := \{ f : \text{Iso}(\mathcal{C}_\mathcal{F}) \to \mathbb{Q} || \text{supp}(f) || < \infty \}
\]
As a \( \mathbb{Q} \)-vector space it is spanned by the delta functions \( \delta_A, A \in \text{Iso}(\mathcal{C}_\mathcal{F}) \). The algebra structure on \( \mathcal{H}_{\mathcal{C}_\mathcal{F}} \) is given by the convolution product:
\[
f \ast g(M) = \sum_{A \subset M} f(A)g(M/A)
\]
for \( M \in \text{Iso}(\mathcal{C}_F) \). In what follows, it will be conceptually useful to choose a representative in each isomorphism class. For \( M, N, Q \in \text{Iso}(\mathcal{C}_F) \), let \( F^Q_{M,N} \) be the number of exact sequences

\[
\emptyset \to M \overset{i}{\to} Q \overset{\pi}{\to} N \to \emptyset
\]

where \((i, \pi)\) and \((i', \pi')\) are considered equivalent iff \( i = i' \) and \( \pi = \pi' \) (this makes sense, since we have fixed a representative in each isomorphism class). It follows from the definition 4.1 that

\[
\delta_M \times \delta_N = \sum_{Q \in \text{Iso}(\mathcal{C}_F)} \frac{F^Q_{M,N}}{|\text{Aut}(M)||\text{Aut}(N)|} \delta_Q,
\]

from which it is apparent that \( H_{\mathcal{C}_F} \) encodes the structure of extensions in \( \mathcal{C}_F \).

\( H_{\mathcal{C}_F} \) possesses a co-commutative co-product given by

\[
(4.2) \quad \Delta(f)(M, N) = f(M \oplus N)
\]

as well as a natural \( K_0^+(\mathcal{C}_F) \)-grading in which \( \delta_A \) has degree \( k(A) \in K_0^+(\mathcal{C}_F) \). If \( \mathcal{F} \) is colored by the set \( S \), it is easy to see that \( K_0^+(\mathcal{C}_F) \simeq \mathbb{N}^{[S]} \).

The subobjects of \( P \in \mathcal{C}_F \) are exactly \( I \in J_P \), and the product 4.1 becomes

\[
f \ast g([P]) = \sum_{I \in J_P} f([I])g([P \setminus I]).
\]

It is shown in [8] that the product is associative, the co-product co-associative and co-commutative, and that the two are compatible, making \( H_{\mathcal{C}_F} \) into a co-commutative bialgebra. Recall that a bialgebra \( A \) over a field \( k \) is connected if it possesses a \( \mathbb{Z}_{\geq 0} \)-grading such that \( A_0 = k \). In addition to the \( K_0^+(\mathcal{C}_F) \)-grading, \( H_{\mathcal{C}_F} \) possesses a grading by the order of the poset - i.e. we may assign \( \deg(\delta_P) = |P| \). This gives it the structure of graded connected bialgebra, and hence Hopf algebra. The Milnor-Moore theorem implies that \( H_{\mathcal{C}_F} \) is the enveloping algebra of the Lie algebra of its primitive elements, which we denote by \( n_{\mathcal{F}} \) - i.e. \( H_{\mathcal{C}_F} \simeq U(n_{\mathcal{F}}) \). It follows from 4.2 that \( f \in n_{\mathcal{F}} \) is primitive if it is supported on the isomorphism classes of connected posets. Thus, we have that

\[
n_{\mathcal{F}} = \text{span}\{\delta_P|P \in \mathcal{F}, P \text{ connected} \}
\]

We will use the notation \( \mathcal{F}^{\text{conn}} \subset \mathcal{F} \) to denote the sub-collection of \( \mathcal{F} \) consisting of connected posets. We have thus established the following:

**Theorem 2.** The Ringel-Hall algebra of the category \( \mathcal{C}_F \) is a co-commutative graded connected Hopf algebra, isomorphic to \( U(n_{\mathcal{F}}) \), where \( n_{\mathcal{F}} \) denotes the graded Lie algebra of its primitive elements. \( n_{\mathcal{F}} = \text{span}\{\delta_P|P \in \mathcal{F}^{\text{conn}}\} \).

**Remark 4.** \( H_{\mathcal{C}_F} \) is a special case of an incidence Hopf algebra introduced by Schmitt in [10, 9].
5. A Pre-Lie structure on $n_F$

We assume now that the collection $\mathcal{F}$ consists of colored posets whose underlying Hasse diagrams are rooted trees. Recall that $\mathcal{F}$ was assumed to be:
- closed under the operation of taking convex sub-posets
- closed under disjoint unions

It is immediate that to produce an $\mathcal{F}$ satisfying these two requirements, one may start with an arbitrary collection $\mathcal{F}'$ of colored posets, and close it with respect to each operation - i.e. adjoin to $\mathcal{F}'$ all convex sub-posets and all disjoint unions of these. If $\mathcal{F}$ arises in this way as the closure of $\mathcal{F}'$, we will write $\mathcal{F} = \overline{\mathcal{F}'}$.

**Example 3.** Suppose that $\mathcal{F}'$ consists of a single poset, whose Hasse diagram is an $n$-vertex ladder colored by the set $S = \{1, \ldots, n\}$.

```
  1
 / \  \
 2   3
 /   / \
\cdot   \cdot   \cdot
/  \  / \
/   / / \
/   / / /
\cdot   3  2
```

Let us adopt the notation $L(a_1, a_2, \ldots, a_k)$ for a $k$-vertex ladder Hasse diagram labeled by $a_1, a_2, \ldots, a_k$ root-to-leaf ($\mathcal{F}'$ thus consisting of $L(1, 2, \ldots, n)$). To close $\mathcal{F}'$ with respect to convex subsets, we must adjoin to it $L(r, r+1, r+2, \ldots, r+m)$, where $1 \leq r \leq r + m \leq n$.

```
  1 \cdots n
 /  /  /  /  \
2  n 2  n-1 2
 /  /  /  /  / \
3  n 3  n-1 3
 /  /  /  / \
\cdot  2  3  2
```

Finally, closing with respect to disjoint unions, we can identify elements of $\mathcal{F} = \overline{\mathcal{F}'}$ with Young diagrams having at most $n$ rows, each of whose columns is labeled by $k, k+1, \ldots, k+m$. For instance

```
2 1 3 4
3 2
4 3
5
```
is identified with the poset
\[ L(2, 3, 4, 5) + L(1, 2, 3) + L(3) + L(4). \]

We proceed to equip \( n_F \) with a pre-Lie structure. For \( a, b \in F_{\text{conn}} \), we define
\[ (5.1) \quad \delta_a \triangleright \delta_b = \delta_a \centerdot \delta_b - \delta_{a \oplus b} \]
and extend the product \( \triangleright \) to all of \( n_F \) by linearity. The subtraction of the term \( \delta_{a \oplus b} \) in 5.1 has the effect of removing the delta-function supported on the one split extension of \( b \) by \( a \), and so the right-hand side of 5.1 does indeed lie in \( n_F \).

It follows easily that we may re-write the definition 5.1 as:
\[ (5.2) \quad \delta_a \triangleright \delta_b = \sum_{t \in F} n(a, b, t) \delta_t \]
where \( n(a, b, t) \) is defined as in example 2.

**Theorem 3.** Let \( \mathcal{F} \) be a collection of colored posets closed with respect to taking convex sub-posets and disjoint unions. If the Hasse diagrams of posets in \( \mathcal{F} \) are rooted trees, then \( \triangleright \) equips \( n_{\mathcal{F}} \) with the structure of a pre-Lie algebra.

**Proof.** A two-sided pre-Lie ideal in a pre-Lie algebra \( A \) is a subspace \( I \subset A \) such that if \( x \in I \), then \( a \triangleright x \in I \) and \( x \triangleright a \in I \) \( \forall a \in A \). One checks easily that the quotient \( A/I \) inherits a pre-Lie structure. Let \( \mathcal{F} \) be a collection of colored rooted forests colored by \( S \), closed under the operations of disjoint union and convex sub-poset, and \( \mathcal{F}_{\text{conn}} \subset \mathcal{F} \) the connected ones (i.e. the rooted trees). \( \mathcal{F}_{\text{conn}} \) is closed under taking convex sub-posets. I claim that \( J = T_S \setminus \mathcal{F}_{\text{conn}} \) is a two-sided pre-Lie ideal in \( T_S \). Let \( u \in T_S \) and \( s \in J \). We have
\[ \delta_u \triangleright \delta_s = \sum_{t \in T_S} n(u, s, t) \delta_t \]
Suppose that \( n(u, s, t) \neq 0 \) and \( t \in T_S \setminus J = \mathcal{F}_{\text{conn}} \). \( t \) has an edge \( e \) such that \( P_e(t) = u \) and \( R_e(t) = s \), and since both are convex sub-posets of the poset \( t \in \mathcal{F}_{\text{conn}}, u, s \in \mathcal{F}_{\text{conn}}, \) contradicting the fact that \( s \in J \). It follows that \( \delta_u \triangleright \delta_s \in J \). The same argument shows that \( \delta_s \triangleright \delta_u \in J \). The quotient \( T_S/J \) is canonically identified with \( n_{\mathcal{F}} \) with the bracket 5.2. \( \square \)

We give a second proof, very close to the one for \( \mathcal{F} = T_S \) given in [2].

**Proof.** We need to verify the identity 2.1. It follows from 5.2 that for \( a, b, c \in \mathcal{F}_{\text{conn}} \),
Because $F$ is closed under taking convex sub-posets, $P_c(t) \in \mathcal{F}^{\text{conn}}$ and $R_s(t) \in \mathcal{F}^{\text{conn}}$, $\forall t \in \mathcal{F}^{\text{conn}}$. The sum $\sum_{t \in \mathcal{F}^{\text{conn}}} n(a, b, t)n(t, c, s)$ may be identified with the number of pairs of edges $\pi = \{e_1, e_2\} \subset E(s)$, such that the resulting cut is NOT admissible (i.e. both edges lie along a single path from root to leaf in $s$), and the three connected components when $\pi$ is removed, are, top-to-bottom, $c, b$ and $a$. Similarly, the sum $\sum_{t \in \mathcal{F}^{\text{conn}}} n(b, c, t)n(a, t, s)$ may identified with the number of pairs $\pi' = \{e_1, e_2\} \subset E(s)$ such that the corresponding cut of $s$ results in three components $a, b, c$, with $r(s) \in c$, and no element of $a$ greater than an element of $b$. The coefficient of $\delta_s$ in

$$\delta_a \triangleright (\delta_b \triangleright \delta_c) = \sum_{s, t \in \mathcal{F}^{\text{conn}}} a \triangleright (\sum_{t \in \mathcal{F}^{\text{conn}}} n(b, c, t)\triangleright n(a, t, s)$$

therefore counts the number of admissible two-edge cuts of $s$ such that the connected component containing $r(s)$ is isomorphic to $c$, and the remaining two to $a, b$ respectively.

Applying the same analysis to the right-hand-side of 2.1 proves the equality. \hfill \Box

**Remark 5.** It follows from 5.2 that $n_F$ is defined over $\mathbb{Z}$, and that the structure constants are non-negative.

### 6. Examples

In this section, we consider different examples of families $\mathcal{F}$, and the resulting pre-Lie algebras $n_F$. Recall that since $n_F$ is graded by $\mathbb{N}$, the Lie algebra $n_F$ is pro-nilpotent (nilpotent if $n_F$ is finite-dimensional).

**Example 4.** Let $S$ be a finite set, and $\mathcal{F} = \overline{T}_S$, the set of rooted forests colored by $S$. We then obtain the pre-Lie algebra structure on $S$-labeled rooted trees described in example 2.
Example 5. Suppose $S$ consists of a single element, and let $\mathcal{F} = \mathcal{F}'$, where $\mathcal{F}'$ is the collection of all ladders:

\begin{center}
\begin{tikzpicture}
\draw (0,0) -- (0,1);
\draw (0,1) -- (0,2);
\draw (0,2) -- (0,3);
\draw (0,3) -- (0,4);
\end{tikzpicture}
\end{center}

(since there is only one color, we suppress the labeling). Denote by $L_n$ the $n$-vertex ladder. We have

$$\delta_{L_n} \triangleright \delta_{L_m} = \delta_{L_{m+n}}.$$ 

so the Lie algebra $\mathfrak{n}_F$ is abelian. In the Ringel-Hall algebra $H_{\mathcal{C}_F}$ we have

$$\delta_{L_n} \triangleright \delta_{L_m} = \delta_{L_{m+n}} + \delta_{L_m \otimes L_n}$$

and

$$\Delta(L_m) = L_m \otimes 1 + 1 \otimes L_m.$$ 

It is well-known (see eg. [6]) that the Hopf algebra $H_{\mathcal{C}_F}$ is isomorphic to the Hopf algebra of symmetric functions, with $L_m$ corresponding to the $m$th power sum.

Example 6. Let $S = \{1, 2, \cdots, n\}$, and let $\mathcal{F} = \mathcal{F}'$, where $\mathcal{F}'$ consists of singleton vertices colored by $S$. $\mathcal{F}$ is thus the collection of all finite sets colored by $S$, with trivial partial order. Denote by $X(m_1, m_2, \cdots, m_n)$ the set of $m_1 + m_2 + \cdots + m_n$ elements, with $m_i$ colored $i$, $1 \leq i \leq n$. $\mathfrak{n}_F$ is therefore spanned by the $\delta_{X(0, \cdots, 1, \cdots, 0)}$. The operation $\triangleright$ is identically 0, so the Lie algebra $\mathfrak{n}_F$ is abelian. In $H_{\mathcal{C}_F}$ we have

$$\delta_{X(m_1, \cdots, m_n)} \triangleright \delta_{X(m_1', \cdots, m_n')} = \left(\prod_{i=1}^{n} \binom{m_i + m_i'}{m_i}\right) \delta_{(m_1 + m_1', \cdots, m_n + m_n')}.$$ 

Example 7. Let $S = \{1, 2, \cdots, n\}$, and let $\mathcal{F} = \mathcal{F}'$, where $\mathcal{F}'$ consists of all $S$–colored ladder trees

\begin{center}
\begin{tikzpicture}
\draw (0,0) -- (0,1);
\draw (0,1) -- (0,2);
\draw (0,2) -- (0,3);
\draw (0,3) -- (0,4);
\end{tikzpicture}
\end{center}
Denote by \( L(a_1, \cdots, a_k) \) the \( k \)-vertex ladder whose \( i \)th vertex counting from the leaf is colored \( a_i \). We have
\[
\delta_{L(a_1, \cdots, a_n)} \triangleright \delta_{L(b_1, \cdots, b_m)} = \delta_{L(a_1, \cdots, a_n, b_1, \cdots, b_m)}
\]
Let \( \mathbb{Q} < X_1, \cdots, X_s > \) denote the free associative algebra on \( S \) viewed as a Lie algebra. There is a linear isomorphism
\[
\rho : n_{\mathcal{F}} \to \mathbb{Q} < X_1, \cdots, X_s >
\]
\[
\rho(L(a_1, \cdots, a_k)) = X_{a_1} X_{a_2} \cdots X_{a_k}
\]
It follows from 6.1 that \( \rho \) is a Lie algebra isomorphism.

**Example 8.** Consider the collection \( \mathcal{F} \) from example 3, where \( \mathcal{F} = \overline{L(1, 2, \cdots, n)} \).
Here \( n_{\mathcal{F}} = \text{span}\{\delta_{L(k, \cdots, k+m)}\} \), \( 1 \leq k \leq k + m \leq n \). We have
\[
\delta_{L(p, \cdots, p+r)} \triangleright \delta_{L(k, \cdots, k+m)} = \begin{cases} 
\delta_{L(k, \cdots, p+r)} & \text{if } k + m + 1 = p \\
0 & \text{otherwise}
\end{cases}
\]
so that in the Lie algebra \( n_{\mathcal{F}} \),
\[
[\delta_{L(p, \cdots, p+r)}, \delta_{L(k, \cdots, k+m)}] = \begin{cases} 
\delta_{L(k, \cdots, p+r)} & \text{if } k + m + 1 = p \\
0 & \text{otherwise}
\end{cases}
\]
Let \( E_{i,j} \) denote the \((n+1) \times (n+1)\) matrix with a 1 in entry \((i,j)\) and zeros everywhere else. Then the commutation relations 6.2 imply that the map
\[
\phi : n_{\mathcal{F}} \to \text{Mat}_{n+1}
\]
\[
\phi(\delta_{L(k, \cdots, k+m)}) = -E_{k,k+m+1}
\]
is an isomorphism of \( n_{\mathcal{F}} \) onto the Lie algebra of upper-triangular \((n+1) \times (n+1)\) matrices.

**Example 9.** Let \( S = \{1, 2\} \), and let \( \mathcal{F} = \overline{\mathcal{F}'} \), where \( \mathcal{F}' \) consists of all \( S \)-colored ladders where the colors alternate. Let us denote by \( L(i, n) \), \( i \in S, n \geq 1 \) the alternating ladder with \( n \) vertices, whose root is colored \( i \). Then \( n_{\mathcal{F}} = \text{span}\{L(i, n)\}, i \in S, n \geq 1 \). We have
\[
\delta_{L(i, n)} \triangleright \delta_{L(i, m)} = \begin{cases} 
\delta_{L(i, n+m)} & \text{if } m \equiv 0 \ mod \ 2, \ i \in S \\
0 & \text{otherwise}
\end{cases}
\]
\[
\delta_{L(i, n)} \triangleright \delta_{L(j, m)} = \begin{cases} 
\delta_{L(j, n+m)} & \text{if } m \equiv 1 \ mod \ 2, \ i \neq j \in S \\
0 & \text{otherwise}
\end{cases}
\]
It follows that

\[
\delta_{L_i,2k}, \delta_{L_j,2l}] = 0
\]

\[
\delta_{L_i,2k}, \delta_{L_j,2l+1}] = \begin{cases} -\delta_{L_j,2(k+l+1)} & \text{if } i = j \\ \delta_{L_j,2(k+l+1)} & \text{if } i \neq j \end{cases}
\]

\[
\delta_{L_i,2k+1}, \delta_{L_j,2l+1}] = \delta_{L_j,2(k+l+1)} - \delta_{L_i,2(k+l+1)}
\]

Recall that \(gl_2 = Mat_2 = n_- \oplus h \oplus n_+\), where

\[
n_- = \text{span}\{f\}, \quad n_+ = \text{span}\{e\}, \quad h = \text{span}\{h_1, h_2\}
\]

and

\[
f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad h_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad h_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}
\]

Let \(Lgl_2 = gl_2 \otimes \mathbb{Q}[t, t^{-1}]\) be the loop algebra of \(gl_2\), with bracket

\[
[X \otimes t^m, Y \otimes t^n] = [X, Y] \otimes t^{m+n}
\]

\(Lgl_2\) also has a triangular decomposition \(Lgl_2 = Lgl_2^+ \oplus h \oplus Lgl_2^-\), where

\[
Lgl_2^+ = n_+ \oplus gl_2 \otimes t\mathbb{Q}[t] \quad Lgl_2^- = n_- \oplus gl_2 \otimes t^{-1}\mathbb{Q}[t^{-1}]
\]

Let

\[
\phi : n_F \rightarrow Lgl_2^+
\]

\[
\phi(\delta_{L_i,2k+1}) = e \otimes t^k
\]

\[
\phi(\delta_{L_2,2k+1}) = f \otimes t^{k+1}
\]

\[
\phi(\delta_{L_i,2k}) = -h_1 \otimes t^k
\]

\[
\phi(\delta_{L_2,2k}) = -h_2 \otimes t^k
\]

It follows from 6.3 that \(\phi\) is an isomorphism. It follows that \(U(Lgl_2^+)\) has an integral basis which may be identified with Young diagrams whose columns are colored by alternating strings of 1’s and 2’s.

**Example 10.** A straightforward generalization of the previous example, with \(S = \{1, \ldots, n\}\) and \(F'\) consisting of ladders periodically colored by 1, \ldots, \(n\) yields \(n_F \simeq Lgl_n^+\).
Example 11. Let $S = \{1, 2\}$, and let $\mathcal{F} = \mathcal{F}'$, where $\mathcal{F}'$ is the set of all ladders colored by a sequence of 1’s followed by a sequence of 2’s.

Denote by $L(i, j)$ the ladder with $i$ 1’s followed by $j$ 2’s. We have

\[
\begin{align*}
\delta_{L(i,j)} & \triangleright \delta_{L(m,n)} = 0 \text{ if } ij > 0 \text{ and } mn > 0 \\
\delta_{L(i,0)} & \triangleright \delta_{L(m,n)} = \begin{cases} 
\delta_{L(i+m,0)} & \text{if } n = 0 \\
0 & \text{otherwise}
\end{cases} \\
\delta_{L(0,j)} & \triangleright \delta_{L(m,n)} = \delta_{L(m,n+j)} \\
\delta_{L(i,j)} & \triangleright \delta_{L(m,0)} = \delta_{L(i+m,j)} \\
\delta_{L(i,j)} & \triangleright \delta_{L(0,n)} = \begin{cases} 
\delta_{L(0,j+n)} & \text{if } i = 0 \\
0 & \text{otherwise}
\end{cases}
\end{align*}
\]

so that we obtain the following non-zero commutation relations (i.e. all other commutators are 0):

\[
\begin{align*}
[\delta_{L(i,0)}, \delta_{L(0,n)}] &= -\delta_{L(i,n)} \\
[\delta_{L(i,0)}, \delta_{L(m,n)}] &= -\delta_{L(m+i,n)} \text{ if } n > 0 \\
[\delta_{L(0,j)}, \delta_{L(m,n)}] &= \delta_{L(m,n+j)} \text{ if } m > 0
\end{align*}
\]

Example 12. Let $S = \{1, 2, \ldots, n\}$, and let $\mathcal{F} = \mathcal{F}'$, where $\mathcal{F}'$ consists of all $S$–colored corollas (rooted trees where all leaves are connected directly to the root)

Closing $\mathcal{F}'$ with respect to convex sub-posets means adjoining singleton colored trees. Denote by $X(i)$ the singleton tree colored by $1 \leq i \leq n$, and by $Y(i, a_1, \ldots, a_n)$ the corolla whose root is colored $i$ and which has $a_1 + a_2 + \cdots + a_n$
leaves, with $a_1$ colored 1, $a_2$ colored 2 etc. In $\mathfrak{n}_F$ we have

\[
\delta X(i) \triangleright \delta X(j) = \delta Y(j,0,\ldots,1,\ldots,0)
\]

\[
\delta X(i) \triangleright \delta Y(j,a_1,\ldots,a_n) = \delta Y(j,a_1,\ldots,a_i+1,\ldots,a_n)
\]

\[
\delta Y(j,a_1,\ldots,a_n) \triangleright \delta X(i) = 0
\]

\[
\delta Y(j,a_1,\ldots,a_n) \triangleright \delta Y(j,b_1,\ldots,b_n) = 0
\]

which leads to the following commutation relations:

\[
[\delta X(i), \delta X(j)] = \delta Y(j,0,\ldots,1,\ldots,0) - \delta Y(i,0,\ldots,1,\ldots,0)
\]

\[
[\delta X(i), \delta Y(j,a_1,\ldots,a_n)] = \delta Y(j,a_1,\ldots,a_i+1,\ldots,a_n)
\]

\[
[\delta Y(j,a_1,\ldots,a_n), \delta Y(j,b_1,\ldots,b_n)] = 0
\]
References

[1] Cartier, P. A primer of Hopf algebras. Frontiers in number theory, physics, and geometry. II, 537–615, Springer, Berlin, 2007.
[2] Chapoton, F.; Livernet, M. Pre-Lie algebras and the rooted trees operad. Internat. Math. Res. Notices 2001, no. 8, 395–408.
[3] Gerstenhaber, M. The cohomology structure of an associative ring, Ann. of Math. 78 (1963), 267288.
[4] Kreimer, D. On the Hopf algebra structure of perturbative quantum field theory. Adv. Theor. Math. Phys. 2 303-334 (1998).
[5] Kremonizer K. and Szczesny M. Feynman graphs, rooted trees, and Ringel-Hall algebras. Comm. Math. Phys. 289 (2009), no. 2 561–577.
[6] Macdonald, I. G. Symmetric functions and Hall polynomials. Second edition. With contributions by A. Zelevinsky. Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1995. x+475 pp.
[7] Rota, G-C. On the Foundations of Combinatorial Theory I: Theory of Mbius Functions”, Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete 2: 340368, (1964).
[8] Schiffmann, O. Lectures on Hall algebras. Preprint math.RT/0611617.
[9] Schmitt, W. R. Antipodes and Incidence Coalgebras. Journal of Comb. Theory. A 46 (1987), 264-290.
[10] Schmitt, W. R. Incidence Hopf algebras. J. Pure Appl. Algebra 96 (1994), no. 3, 299–330.
[11] Stanley, R. Enumerative combinatorics Vol. 1. Cambridge Studies in Advanced Mathematics, 49. Cambridge University Press, Cambridge, 1997.
[12] Szczesny, M. Incidence categories. J. Pure and Appl. Algebra, in press.
[13] Vinberg, E.B. The theory of homogeneous convex cones, Transl. Moscow Math. Soc. 12 (1963), 340403.

Department of Mathematics and Statistics, Boston University, Boston MA, USA
E-mail address: szczesny@math.bu.edu