Stochastic Consensus over Time-Varying Networks of Linear Symmetric Agents

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Multi-agent systems over noisy networks with linear symmetric agents are considered. The topology of the network which defines the availability of communication among the agents is assumed to be a directed, weakly connected and balanced graph or an undirected and connected one. Furthermore, the communication graph is allowed to be time varying. The aim of this study is to establish a stochastic averaging consensus algorithm under a noisy environment for each time-varying network. The convergence analysis of the proposed algorithms reveals an explicit relation between the number of iterations and the closeness of the agreement, which gives a stopping rule for the consensus algorithm. The results are illustrated through numerical examples.

1. Introduction

Multi-agent consensus is a fundamental tool in several application fields such as distributed communication, distributed optimization, opinion formation, and so on [1–4]. An objective of the consensus problems is to achieve an agreement of the states. The states represent conditions of the machines, environment, or agents themselves. For example, the states correspond to the positions in vehicle formation problems [2], local time in clock synchronization problems [3], or agents' opinions [4]. Since the multi-agent consensus is often used over noisy networks in practical situations, its convergence analysis and stopping rule should be established in stochastic settings [5]. It is also significant to consider a time varying topology case. For example, there is a work [6] which gives consensus conditions under noisy and time-varying networks. Here agents are assumed as first order dynamical systems, while they can be higher order dynamical systems in a certain case. Symmetric systems [7] are examples of such a higher order dynamics.

In this paper, we consider multi-agent systems over noisy and time-varying networks, where each agent is a linear symmetric system. A topology of the network which represents the availability of communication among the agents is defined as a graph. We assume that the graph is directed, weakly connected, and balanced or undirected and connected. We propose a communication gain for each network to achieve a stochastic consensus of the multi-agent system under an appropriate assumption for the symmetric system. Moreover, we reveal a relation between the closeness of the agreement and the number of iterations explicitly with a probabilistic guarantee. It gives a rigorous stopping rule for the consensus algorithm. These results include the existing ones [5] as a special case, that is, if the agents have a first order dynamics and form a time-invariant network, the results of the present study coincide with those of [5].

This paper is organized as follows. We define multi-agent systems to be considered in Section 2. We analyze the convergence of the consensus algorithm which gives the stopping rule for directed and undirected topology cases in Section 3. We show numerical examples in Section 4. Finally, we conclude this paper in Section 5. The preliminary versions of this study were presented at conferences [8,9].

2. Multi-Agent Systems

Let us consider $N$ linear symmetric agents having the same dynamics

\[ x_i(k+1) = Ax_i(k) + bu_i(k), \]
\[ y_i(k) = c^T x_i(k), \quad i \in \mathcal{V}, \]

(1)
where $x_i(k) \in \mathbb{R}^n$ is the state, $u_i(k) \in \mathbb{R}$ is the input, $y_i(k) \in \mathbb{R}$ is the output, and $\mathcal{V} = \{1, 2, \ldots, N\}$, $N \in \mathbb{N}$, and $k \in \mathbb{N}$. The conditions on the coefficient matrices

$$A = A^T, \quad b = c,$$

represent symmetry of the agents [7]. We assume that

$$\det A \neq 0, \quad 0 \leq A - \kappa b c^T < I_n, \quad \forall \kappa \in (0, 1],$$

where $I_n \in \mathbb{R}^{n \times n}$ is the identity matrix. That is, each agent can be stabilized by an arbitrarily low gain output feedback. This assumption is equivalent to

$$0 < A \leq I_n, \quad 0 \leq A - b c^T \leq (1 - \eta) I_n, \quad (2)$$

with some $\eta \in (0, 1]$. This $\eta$ will be used later. We remark that the linear symmetric agent can be the standard first order agent considered in [5]. In fact, the standard one with $n = 1$ satisfies this assumption with $\eta = 1$, where $A = 1$, $b = 1$, and $c = 1$.

For the set of symmetric agents, we introduce interactions

$$u_i(k) = r(k) \sum_{j \in \mathcal{N}_i(k)} (z_{ij}(k) - y_i(k)),$$

$$z_{ij}(k) = y_j(k) + w_{ij}(k), \quad i \in \mathcal{V}, j \in \mathcal{N}_i(k), \quad (3)$$

where $r(k) \in \mathbb{R}$ is the communication gain to be determined later, and $z_{ij}(k) \in \mathbb{R}$ is the information which the agent $i$ receives from the agent $j$ ($i \neq j$). The communication noise $w_{ij}(k) \in \mathbb{R}$ follows an independent and identically distributed noise with respect to $i$, $j$, and $k$ which satisfies

$$\mathbb{E}[w_{ij}(k)] = 0, \quad \text{Var}[w_{ij}(k)] = \sigma^2 < \infty,$$

where $\mathbb{E}[a]$ and $\text{Var}[a]$ denote the expectation and the variance of a random variable $a$. We assume that $\sigma^2$ is known to all of the agents. Furthermore, $\mathcal{N}_i(k)$ is the set of agents that can communicate the agent $i$ at time $k$, which introduces a time-varying graph. Let $G(k) = (\mathcal{V}, \mathcal{E}(k))$ be the time-varying graph, where $G(k) \in \{G_1, G_2, \ldots, G_p\}$ and $G_t = (\mathcal{V}, \mathcal{E}_t)$. Here $\mathcal{V}$ denotes the set of agents and $\mathcal{E}_t \subseteq \mathcal{V} \times \mathcal{V}$ denotes the set of edges. In this paper, we deal with undirected and undirected graphs. We assume that $G_t$ is weakly connected and balanced in the directed graph and connected in the undirected one. With these representations, $\mathcal{N}_i(k) = \{j \in \mathcal{V} | (i, j) \in \mathcal{E}(k)\}$. Note that the edge $(i, j) \in \mathcal{E}_t$ is ordered pair in the directed graph, i.e., $i$ and $j$ are the tail and the head of the edge. For $j \notin \mathcal{N}_i(k)$, it is defined as $w_{ij}(k) = 0$ because of no communication.

The stochastic consensus problem of the multi-agent system (1) with agent interactions (3) which contains the communication noise is to achieve

$$\lim_{k \to \infty} \mathbb{P} \left( \exists k \in \mathcal{V} \text{ s.t. } \|x_i(k) - \frac{1}{N} \sum_{i=1}^{N} x_i(k) \| \geq \epsilon \right) = 0 \quad (4)$$

for all $\epsilon > 0$, where $\mathbb{P}$ is the probability measure on the noise sequence and $\| \cdot \|$ is the Euclidean norm.

In order to represent (1) and (3) as a compact form, we introduce the graph Laplacian $L_t$ of $G_t$ [3] as

$$[L_t]_{ij} = \begin{cases} -1 & \text{if } (j, i) \in \mathcal{E}_t, \\ d_t^{(i)} & \text{if } j = i, \\ 0 & \text{otherwise}, \end{cases}$$

where $d_t^{(i)}$ is the number of the neighbors at the agent $i$ with $G_t$. Note that, since the graph is assumed to be weakly connected and balanced in the directed graph and connected in the undirected one, the graph Laplacian $L_t$ satisfies

$$1_N^T L_t = 0, \quad L_t 1_N = 0, \quad \text{rank } L_t = N - 1,$$

for any $t \in \{1, 2, \ldots, p\}$, where $1_N$ is the $N$-dimensional vector whose elements are all $1$ [3]. By using the graph Laplacian $L(k) \in \{L_t | t = 1, 2, \ldots, p\}$, the agents (1) and the interaction (3) can be rewritten as

$$x(k+1) = (I_N \otimes A) x(k) + (I_N \otimes b) u(k),$$

$$y(k) = (I_N \otimes c^T) x(k), \quad (5)$$

and

$$u(k) = -r(k) L(k) y(k) + r(k) W(k) 1_N, \quad (6)$$

where

$$x(k) = [x_1^T(k) \ x_2^T(k) \ \cdots \ \ x_N^T(k)]^T \in \mathbb{R}^{Nn},$$

$$y(k) = [y_1(k) \ y_2(k) \ \cdots \ \ y_N(k)]^T \in \mathbb{R}^n,$$

$$u(k) = [u_1(k) \ u_2(k) \ \cdots \ \ u_N(k)]^T \in \mathbb{R}^n,$$

$$W(k) = \begin{bmatrix} w_{11}(k) & \cdots & w_{1N}(k) \\ \vdots & \ddots & \vdots \\ w_{N1}(k) & \cdots & w_{NN}(k) \end{bmatrix} \in \mathbb{R}^{N \times N}. \quad (7)$$

Here $\otimes$ is the operator of the Kronecker product. Substituting eq. (6) into eq. (5), we have

$$x(k+1) = \left( (I_N \otimes A) - r(k) (I_N \otimes b) L(k) (I_N \otimes c^T) \right) x(k)$$

$$+ r(k) (I_N \otimes b) W(k) 1_N. \quad (7)$$

We then employ a state coordinate transformation

$$[\xi_1(k)] = \begin{bmatrix} S^T \otimes I_n \\ \frac{1}{\sqrt{N}} \end{bmatrix} x(k),$$

$$[\xi_2(k)] = \begin{bmatrix} S \otimes I_n \\ \frac{1}{\sqrt{N}} \end{bmatrix} x(k), \quad (8)$$

where $S \in \mathbb{R}^{N \times (N-1)}$ is the orthonormal complement of $1_N^T / \sqrt{N}$. That is, $S$ is a matrix which satisfies

$$\begin{bmatrix} S^T \\ \frac{1}{\sqrt{N}} \end{bmatrix} \begin{bmatrix} S \frac{1}{\sqrt{N}} \\ \frac{1}{\sqrt{N}} \end{bmatrix} = I_N.$$ 

Applying eq. (8) to the overall system (7), we have
\[ \xi_1(k+1) = ((I_{N-1} \otimes A) - r(k) (S^T L_t(k) S \otimes b \sigma^T)) \xi_1(k) + r(k) (I_{N-1} \otimes b) \bar{w}_1(k), \]
\[ \xi_2(k+1) = A \xi_2(k) + r(k) b w_2(k), \]
where
\[ \bar{w}_1(k) = S^T W(k) 1_N \in \mathbb{R}^{(N-1)}, \]
\[ \bar{w}_2(k) = \frac{1}{\sqrt{N}} W(k) 1_N \in \mathbb{R}. \]

Then we see that \( \bar{w}_1(k) \) and \( \bar{w}_2(k) \) satisfy
\[ \mathbb{E}[\bar{w}_1(k)] = 0, \quad \mathbb{Cov}[ar{w}_1(k)] \leq N \bar{v}^2 I_{N-1}, \]
\[ \mathbb{E}[\bar{w}_2(k)] = 0, \quad \mathbb{Var}[\bar{w}_2(k)] \leq N \bar{v}^2 \]

independently of given graph structure, where \( \mathbb{Cov}[a] \) is the covariance of a random vector \( a \).

We now define the average and the deviation of the states of all agents at \( k \) as
\[ \bar{x}(k) = \left( \frac{1}{\sqrt{N}} \otimes I_n \right) x(k), \]
\[ \bar{x}(k) = x(k) - (1_N \otimes I_n) \bar{x}(k) = \left( I_{Nn} - \frac{1}{\sqrt{N}} 1_N \otimes I_n \right) x(k). \]

The state coordinate transformation says that these can be rewritten as
\[ \bar{x}(k) = \frac{1}{\sqrt{N}} \xi_2(k), \quad \bar{x}(k) = (S \otimes I_n) \xi_1(k), \]
which implies that \( \|\bar{x}(k)\| = \|\xi_1(k)\| \). That is, we can consider the convergence of \( \bar{x}(k) \) as that of \( \xi_1(k) \). We therefore see that the condition (4) of the stochastic consensus can be rewritten as
\[ \lim_{k \to \infty} P(\|\xi_1(k)\| \geq \epsilon) = 0, \]
for all \( \epsilon > 0 \).

### 3. Stochastic Consensus

#### 3.1 Directed Graph Topology Case

Let \( \lambda_k^{(i)} \) be the \( i \)-th largest eigenvalues of \( (L_t + L_t^T)/2 \). Since \( G_t \) is weakly connected and balanced for any \( t \in \{1,2,\ldots,p\} \), the eigenvalues satisfy \( 0 = \lambda_k^{(1)} < \lambda_k^{(2)} \leq \cdots \leq \lambda_k^{(N)} \) [3]. We define \( \sigma_{\lambda_t} \) as the largest singular value of \( L_t \).

The following lemma is the key tool which will be used in the proof of the main result.

**Lemma 1** The graph Laplacian \( L_t \) satisfies
\[ \left\| (I_{N-1} \otimes A) - \frac{\eta \lambda_t^{(2)}}{(\bar{\sigma}_t)^2} (S^T L_t S \otimes b \sigma^T) \right\|^2 \leq 1 - \frac{\eta^2 \lambda_t^{(2)^2}}{2(\bar{\sigma}_t)^2}, \]
where
\[ a = \arg \min_{t=1,2,\ldots,p} \lambda_t^{(2)}, \quad b = \arg \max_{t=1,2,\ldots,p} \sigma_t, \]
and \( \|\cdot\| \) denotes the spectral norm.

**Proof** For simplicity, we introduce \( L_t^s = S^T L_t S \) and \( \alpha = \eta \lambda_t^{(2)} / (\bar{\sigma}_t)^2 \). We will prove that
\[ \left\| (I_{N-1} \otimes A) - \alpha \left( L_t^s \otimes b \sigma^T \right) \right\|^2 \leq 1 - \frac{\alpha \eta \lambda_t^{(2)}}{2} \] is satisfied.

From the assumption (2), we see that
\[ \left\| (I_{N-1} \otimes A) - \alpha \left( L_t^s \otimes b \sigma^T \right) \right\|^2 = \left\| (I_{N-1} \otimes A^1/2) \right\|^2 + \left\| (I_{N-1} \otimes A^1/2) - \alpha \left( L_t^s \otimes A^{-1/2} b \sigma^T \right) \right\|^2 \leq \left\| (I_{N-1} \otimes A^1/2) \right\|^2 + \left\| (I_{N-1} \otimes A^1/2) - \alpha \left( L_t^s \otimes A^{-1/2} b \sigma^T \right) \right\|^2 = \lambda_{\text{max}} \left( (I_{N-1} \otimes A) - \alpha \left( (L_t^s + (L_t^s)^T) \otimes b \sigma^T \right) \right) + \alpha^2 \left( (L_t^s)^2 L_t^s \otimes b \sigma^T A^{-1} b \sigma^T \right) \]
is satisfied, where \( \lambda_{\text{max}}(A) \) is the largest eigenvalue of \( A \). Since \( (L_t^s + (L_t^s)^T)/2 \) is positive definite, we obtain a lower bound of the second term in eq. (10) as
\[ \alpha \left( (L_t^s + (L_t^s)^T) \otimes b \sigma^T \right) \geq 1\tau N - 2\alpha \lambda_t^{(2)} b \sigma^T. \]
From the assumption (2), \( 0 \leq b \sigma^T = c b^T \leq A \leq I_n \). Thus we have
\[ 0 \leq \left[ \begin{array}{c} b \\ b \end{array} \right] \left[ \begin{array}{c} c^T \\ c^T \end{array} \right] \leq \left[ \begin{array}{c} I_n \\ b c^T \end{array} \right] \]
that is, \( b c^T A^{-1} b \sigma^T \leq I_n \). Using this fact with \( (L_t^s)^2 L_t^s \leq (\bar{\sigma}_t)^2 I_{N-1} \), we see that an upper bound of the third term in eq. (10) is given by
\[ \alpha^2 \left( (L_t^s)^2 L_t^s \otimes b \sigma^T A^{-1} b \sigma^T \right) \leq \alpha^2 (\bar{\sigma}_t)^2 \left( (I_{N-1} \otimes I_n) \right). \]
Applying these bounds to eq. (10), we see that
\[ \left\| (I_{N-1} \otimes A) - \alpha \left( L_t^s \otimes b \sigma^T \right) \right\|^2 \leq \lambda_{\text{max}} \left( (I_{N-1} \otimes A) - \alpha \left( (L_t^s + 2\alpha b \lambda_t^{(2)} \sigma^T) \right) \right) + \alpha^2 (\bar{\sigma}_t)^2 \left( (I_{N-1} \otimes I_n) \right). \]
From the assumption (2), we obtain
\[ A - 2\alpha b \lambda_t^{(2)} \sigma^T = (1 - 2\alpha b \lambda_t^{(2)}) A + 2\alpha b \lambda_t^{(2)} (A - b \sigma^T) \]
\[ \leq (1 - 2\alpha b \lambda_t^{(2)}) I_n + 2\alpha b \lambda_t^{(2)} (1 - \eta) I_n \]
\[ = (1 - 2\alpha \eta \lambda_t^{(2)}) I_n. \]
We therefore see that
\[ \left\| (I_{N-1} \otimes A) - \alpha \left( L_t^s \otimes b \sigma^T \right) \right\|^2 \leq 1 - 2\alpha \eta \lambda_t^{(2)} + \alpha^2 (\bar{\sigma}_t)^2 \]
Then we have the first main result. 

\[
\text{(Proof)} \quad \text{For given constants } \alpha \in (0, \infty), \beta \in (0, \infty), \text{ and } \gamma \in (0, 1), \text{ select } k_f \in \mathbb{N} \text{ which satisfies }
\]
\[
k_f \geq \max\{\tau_1, \tau_2\},
\]
\[
\tau_1 = \left(\frac{1}{\alpha} - 1\right) k_0 + 1, \quad \tau_2 = \frac{4N(N-1)\sigma^2}{\beta^2 \gamma^2 (\lambda_a^{(2)})^2} - k_0 + 1.
\]

Then the \(k_f\)-th deviation \(\bar{x}(k_f)\) satisfies
\[
P\left(\|\bar{x}(k_f)\| \leq \alpha \|\bar{x}(1)\| + \beta\right) \geq 1 - \gamma
\]
(11)

for any initial state \(x(1)\). Furthermore, the average \(\bar{x}(k)\) satisfies
\[
E[\bar{x}(k)] = A^{k-1}x(1), \quad \text{Cov}[\bar{x}(k)] \leq \frac{2\pi^2 \nu^2}{3\eta^2 (\lambda_a^{(2)})^2} I_n
\]
for any initial state \(x(1)\) and \(k \in \mathbb{N}\).

(Proof) For \(k \in \mathbb{N}\),
\[
\xi_1(k) = \Phi(k, 1) \xi_1(1) + \sum_{m=1}^{k-1} r(m) \Phi(k, m+1) (I_{N-1} \otimes b) \bar{w}_1(m),
\]
where
\[
\Phi(k, m) =
\begin{cases}
((I_{N-1} \otimes A) - r(k-1) (L^*(k-1) \otimes bc^T)) & \text{if } m = 1,

-(I_{N-1} \otimes A) - r(k-2) (L^*(k-2) \otimes bc^T) & \text{if } m = 2,

\vdots

-(I_{N-1} \otimes A) - r(m) (L^*(m) \otimes bc^T) & \text{if } m > 0,

I_{(N-1)n} & \text{otherwise},
\end{cases}
\]
and the summation with no summands is zero. Here \(L^*(k) = S^T L(k) S \in \{I_s^* | s = 1, 2, \ldots, p\}\). Then, the expectation of \(\xi_1(k)\) is
\[
E[\xi_1(k)] = \Phi(k, 1) \xi_1(1),
\]
and the covariance of \(\xi_1(k)\) is
\[
\text{Cov}[\xi_1(k)] = E\left[\sum_{m=1}^{k-1} r^2(m) \Phi(k, m+1) (I_{N-1} \otimes b) \bar{w}_1(m)\right]
\]
which establishes Lemma 1.

By Markov's inequality, we have
\[
P\left(\|\xi_1(k) - E[\xi_1(k)]\| \leq \frac{\text{Tr}(\text{Cov}[\xi_1(k)])}{\gamma}\right) \geq 1 - \gamma,
\]
for any initial state \(x(1)\). Furthermore, the average \(\bar{x}(k)\) satisfies
\[
E[\bar{x}(k)] = A^{k-1}x(1), \quad \text{Cov}[\bar{x}(k)] \leq \frac{2\pi^2 \nu^2}{3\eta^2 (\lambda_a^{(2)})^2} I_n
\]
for any initial state \(x(1)\) and \(k \in \mathbb{N}\).

Then, by the definition of \(\Phi(k, m)\),
\[
\|\Phi(k, m+1)\| \leq \prod_{\ell=m+1}^{k-1} \|I_{(N-1)n}\| - \frac{\nu}{\eta (\lambda_a^{(2)})^2} (L^*(k) \otimes bc^T)
\]
\[
\leq \prod_{\ell=m+1}^{k-1} \left(1 - \frac{1}{k_0 + \ell}\right)
\]
\[
= \prod_{\ell=m+1}^{k-1} \frac{k_0 + \ell - 1}{k_0 + \ell} = \frac{k_0 + m}{k_0 + k - 1}
\]
(14)

is derived. Thus, if \(k \geq \tau_1\), we have
\[
\|E[\xi_1(k)]\| \leq \|\Phi(k, 1)\| \|\xi_1(1)\|
\]
\[ \leq \frac{k_0}{k_0 + k - 1} \| \xi_1(1) \| \]
\[ \leq \frac{k_0}{k_0 + \tau_1 - 1} \| \xi_1(1) \| = \alpha \| \xi_1(1) \|. \quad (15) \]

Using eq. (2) and eq. (14), for \( k \geq \tau_2 \) we have
\[
\text{Tr}(\text{Cov}[\xi_1(k)]) 
\leq \text{Tr}(Nv^2I_{N-1}) \sum_{m=1}^{k-1} r^2(m) \| \Phi(k, m + 1) \|^2 
\leq \frac{4N(N-1)v^2}{\nu^2(\lambda_a^2)^2} \sum_{m=1}^{k-1} \frac{1}{(k_0 + m)^2} \left( \frac{k_0 + m}{k_0 + k - 1} \right)^2 
\leq \frac{4N(N-1)v^2}{\nu^2(\lambda_a^2)^2} \frac{1}{k_0 + k - 1} 
\leq \frac{4N(N-1)v^2}{\nu^2(\lambda_a^2)^2} = \beta^2 \gamma. \quad (16) \]

Applying the relations eq. (15) and eq. (16) to eq. (13), we see that eq. (11) of Theorem 1 is satisfied if \( k_f \) is chosen as \( k_f \geq \tau_1 \) and \( k_f \geq \tau_2 \).

Next, we prove the second part of the theorem.

For \( k \in \mathbb{N} \), we have
\[
\bar{x}(k) = \frac{1}{\sqrt{N}} \bar{\xi}_2(k) 
= \frac{1}{\sqrt{N}} \left\{ A^{k-1} \xi_2(1) + \sum_{m=1}^{k-1} r(m)A^{k-1-m}b\bar{w}_2(m) \right\}. 
\]

Since \( \mathbb{E}[\bar{w}_2(k)] = 0 \), we obtain
\[
\mathbb{E}[\bar{x}(k)] = \frac{1}{\sqrt{N}} A^{k-1} \xi_2(1) = A^{k-1} \bar{x}(1). 
\]

The covariance of \( \bar{x}(k) \) then satisfies
\[
\text{Cov}[\bar{x}(k)] = \frac{1}{N} \mathbb{E} \left[ \left( \sum_{m=1}^{k-1} r^2(m)A^{k-1-m}b\bar{w}_2(m) \right. \right. 
\left. \left. \bar{w}_2^\top(m) b^\top \left( A^{k-1-m} \right)^\top \right) \right]. 
\]

By using Var[\( \bar{w}_2(k) \)] \( \leq Nv^2 \) and (2) with \( r(k) \), we see that
\[
\text{Cov}[\bar{x}(k)] \leq v^2 \sum_{m=1}^{k-1} r^2(m)A^{k-1-m}bc^\top \left( A^{k-1-m} \right)^\top 
\leq v^2 \left( \frac{4}{\nu^2(\lambda_a^2)^2} \frac{1}{k_0 + k - 1} \right) I_n 
\leq v^2 \left( \frac{4}{\nu^2(\lambda_a^2)^2} \frac{1}{6} \right) I_n = \frac{2v^2}{3\nu^2(\lambda_a^2)^2} I_n, 
\]
which concludes the second part of Theorem 1. \( \square \)

### 3.2 Undirected Graph Topology Case

In this section, we redefine \( \lambda^{(i)}_t \) as the \( i \)-th largest eigenvalues of the graph Laplacian \( L_t \). Since \( G_t \) is connected for any \( t = 1, 2, \ldots, p \), the eigenvalues satisfy
\[ 0 = \lambda^{(1)}_t < \lambda^{(2)}_t < \cdots < \lambda^{(N)}_t \quad [3]. \]

The following lemma is the key tool which will be used in the proof of Theorem 2.

**Lemma 2.** The graph Laplacian \( L_t \) satisfies
\[
\left\| (I_{N-1} \otimes A) - \frac{1}{\lambda_b^{(N)}} \left( S^\top L_t S \otimes b c^\top \right) \right\| \leq 1 - \frac{\eta \lambda_a^{(2)}}{\lambda_b^{(N)}} , 
\]

where
\[ a = \arg \min_{t=1,2,\ldots,p} \lambda_t^{(2)}, \quad b = \arg \max_{t=1,2,\ldots,p} \lambda_t^{(N)}. \]

**Proof.** Since \( L_t^* = S^\top L_t S \) is positive definite,
\[
I_{N-1} \otimes \lambda_b^{(2)} bc^\top \leq L_t^* \otimes bc^\top \leq I_{N-1} \otimes \lambda_b^{(N)} bc^\top 
\]
is satisfied for any \( t = 1, 2, \ldots, p \). Thus, we have
\[
I_{N-1} \otimes (A - bc^\top) \leq (I_{N-1} \otimes A) - \frac{1}{\lambda_b^{(N)}} \left( L_t^* \otimes bc^\top \right) 
\leq I_{N-1} \otimes \left( A - \frac{\lambda_b^{(2)}}{\lambda_b^{(N)}} bc^\top \right). \quad (17) 
\]

From the assumption (2), we have
\[
A - \frac{\lambda_b^{(2)}}{\lambda_b^{(N)}} bc^\top = \left( 1 - \frac{\lambda_a^{(2)}}{\lambda_b^{(N)}} \right) A + \frac{\lambda_a^{(2)}}{\lambda_b^{(N)}} (A - bc^\top) 
\leq \left( 1 - \frac{\lambda_a^{(2)}}{\lambda_b^{(N)}} \right) I_n + \frac{\lambda_a^{(2)}}{\lambda_b^{(N)}} (1 - \eta) I_n. 
\]

Substituting the assumption (2) and the above inequality into eq. (17), we obtain
\[
0 \leq (I_{N-1} \otimes A) - \frac{1}{\lambda_b^{(N)}} \left( L_t^* \otimes bc^\top \right) 
\leq I_{N-1} \otimes \left( 1 - \frac{\lambda_a^{(2)}}{\lambda_b^{(N)}} \right) I_n + \frac{\lambda_a^{(2)}}{\lambda_b^{(N)}} (1 - \eta) I_n 
= I_{N-1} \otimes \left( 1 - \frac{\eta \lambda_a^{(2)}}{\lambda_b^{(N)}} \right) I_n. 
\]

Thus we have Lemma 2. \( \square \)

Now, let us select the communication gain as
\[
r(k) = \frac{1}{\eta \lambda_a^{(2)}(k_0 + k)}, \quad k_0 \geq \frac{\lambda_b^{(N)}}{\eta \lambda_a^{(2)}} - 1, 
\]
where \( k_0 \in \mathbb{N} \). Then we have the second main result.

**Theorem 2.** For given constants \( \alpha \in (0, \infty) \), \( \beta \in (0, \infty) \), and \( \gamma \in (0, 1) \), select \( k_f \in \mathbb{N} \) which satisfies
\[
k_f \geq \max\{\tau_1, \tau_2\}, \quad \tau_1 = \left( \frac{1}{\alpha} - 1 \right) k_0 + 1, \quad \tau_2 = \frac{N(N-1)v^2}{\beta^2 \gamma^2 \lambda_a^{(2)}}, \quad k_0 + 1. \quad (18) 
\]

Then the \( k_f \)-th deviation \( \bar{x}(k_f) \) satisfies
\[
\mathbb{P}(\| \bar{x}(k_f) \| \leq \alpha \| \bar{x}(1) \| + \beta) \geq 1 - \gamma \quad (19) 
\]
for any initial state \( x(1) \). Furthermore, the average \( \bar{x}(k) \) satisfies
\[ E[\bar{x}(k)] = A^{k-1} \bar{x}(1), \quad \text{Cov}[\bar{x}(k)] \leq \frac{\pi^2 \nu^2}{6\eta \lambda^{(2)}_n} I_n \]

for any initial state \( x(1) \) and \( k \in \mathbb{N} \).

(Proof) The first part of the theorem can be proved in a similar manner by the proof of Lemma 2 in the previous section with Lemma 2.

From Lemma 2, we obtain
\[
\| (I_{N-1} \otimes A) - \frac{1}{\eta \lambda^{(2)}_n (k_0 + k)} \left( L^*_n (k) \otimes b c^T \right) \| \leq 1 - \frac{1}{k_0 + k}.
\]

Then, by the definition of \( \Phi(k,m) \),
\[
\| \Phi(k,m+1) \| \leq \prod_{\ell = m+1} (1 - \frac{1}{k_0 + \eta \lambda^{(2)}_n (k_0 + \ell)}) \leq k_0 + m \leq k_0 + k - 1
\]

is derived. Thus, we have
\[
\| E[\xi_1(k)] \| \leq \frac{k_0}{k_0 + k - 1} \| \xi_1(1) \| \leq \frac{k_0}{k_0 + k - 1} \| \xi_1(1) \| = \alpha \| \xi_1(1) \|. \quad (21)
\]

Using eq. (2) and eq. (20), for \( k \geq \tau_2 \), we have
\[
\text{Tr}(\text{Cov}[\xi_1(k)]) \leq \text{Tr}(N \nu^2 I_{N-1}) \sum_{m=1}^{k-1} r^2(m) \| \Phi(k,m+1) \|^2 \leq N(N - 1) \nu^2 \sum_{m=1}^{k-1} \frac{1}{(k_0 + m)^2} \leq \frac{N(N - 1) \nu^2}{\lambda^{(2)}_n} \leq \frac{N(N - 1) \nu^2}{\lambda^{(2)}_n} \leq \frac{N(N - 1) \nu^2}{\lambda^{(2)}_n} = \beta^2 \gamma. \quad (22)
\]

Applying the relations eq. (21) and eq. (22) to eq. (13), we see that eq. (19) of Theorem 2 is satisfied if \( k_f \) is chosen as \( k_f \geq \tau_1 \) and \( k_f \geq \tau_2 \).

We now prove the second part of the theorem.

Since the expectation of the average can be proved by the same way in Theorem 1, we will prove only the covariance. By using (2) with \( r(k) \), we see that
\[
\text{Cov}[\bar{x}(k)] \leq \nu^2 \sum_{m=1}^{k-1} \frac{1}{\eta \lambda^{(2)}_n} \frac{1}{(k_0 + m)^2} I_n \leq \nu^2 \frac{1}{\eta \lambda^{(2)}_n} \frac{\pi^2 \nu^2}{6 \eta \lambda^{(2)}_n} I_n = \frac{\pi^2 \nu^2}{6 \eta \lambda^{(2)}_n} I_n,
\]

which concludes the second part of Theorem 2. □

Here we remark that, when the graphs are time-invariant and \( \eta = 1 \), Theorems 1 and 2 become identical to the result derived in [5]. Since the previous work [5] deals with the standard agent case with \( n = 1 \), we can see Theorems 1 and 2 as a natural extension of the existing result [5] to the higher order agent case, where the symmetry (2) plays a crucial role.

The theorems show an explicit relation between the closeness of the agreement and the number of iterations with a probabilistic guarantee, which gives a stopping rule for the consensus algorithm. The stopping rules are characterized by \( \lambda^{(2)}_n \) and \( \bar{\sigma}_n \) for directed graphs and \( \lambda^{(2)}_n \) and \( \lambda^{(2)}_n \) for undirected graphs. In general, \( a \) and \( b \) are chosen from different candidate graphs \( G_t \) for time-variant cases.

4. Numerical Examples

The system consists of six agents \( (N = 6) \). We used \( n = 3 \) and the following settings:
\[
A = \begin{bmatrix} 0.50 & 0.25 & 0.00 \\ 0.25 & 0.75 & 0.25 \\ 0.00 & 0.25 & 0.50 \end{bmatrix}, \quad b = c = \begin{bmatrix} 0.50 \\ 0.50 \\ 0.50 \end{bmatrix}.
\]

These satisfy the assumption (2) when \( \eta \) is smaller than or equal to 0.5. The initial state was set as
\[
x_1(1) = \begin{bmatrix} 1.0 \\ 2.0 \\ 3.0 \end{bmatrix}, \quad x_2(1) = \begin{bmatrix} 4.0 \\ 5.0 \\ 6.0 \end{bmatrix}, \quad x_3(1) = \begin{bmatrix} 7.0 \\ 8.0 \\ 9.0 \end{bmatrix}, \quad x_4(1) = \begin{bmatrix} 7.7 \\ 8.7 \\ 9.7 \end{bmatrix}, \quad x_5(1) = \begin{bmatrix} 4.5 \\ 2.5 \end{bmatrix}, \quad x_6(1) = \begin{bmatrix} 1.5 \end{bmatrix}.
\]

The graphs \( G_1, G_2, G_3 \) in the directed graph case were defined by their graph Laplacians
\[
L_1 = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & -1 \\ 0 & -1 & 2 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ -1 & 0 & -1 & 0 & 0 \end{bmatrix}, \quad L_2 = \begin{bmatrix} -1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 2 & -1 \\ -1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad L_3 = \begin{bmatrix} -1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & -1 \\ 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},
\]

The maximum value \( \bar{\sigma}_n \) of the largest singular values is 2.9745 and the minimum value \( \lambda^{(2)}_n \) of the second smallest eigenvalues of \( (L + L^T) / 2 \) is 0.7192. In our numerical examples, the graphs were varied according
Theorem 1 and the minimum value $\lambda^{(2)}_1$ of the second smallest eigenvalue is $1.4384$. Fig. 2 shows the topology of the graphs and the time-varying sequence. The probabilistic parameters which guarantee the closeness of the agreement were chosen as the same numbers of the directed graph case. Based on Theorem 2, the number of iterations $k_f$ is $3,474$. Since $\|\bar{x}(1)\|$ is $14.1185$, the theorem gives the inequality
\[
P(\|\bar{x}(3,474)\| \leq 1.9119) \geq 0.8.
\]
We tried 10,000 times with different noise sequences whose variance $v^2$ is $1$. Figs. 5, 6 show that the behavior of the average $\bar{x}(k)$ and the deviation $\tilde{x}(k)$ at a certain trial. Then, the worst value of $\|\bar{x}(3,474)\|$ in our trials is $0.1341$, i.e., the inequality $\|\bar{x}(3,474)\| \leq 1.9119$ is always satisfied for all trials. Thus, this result is consistent with the statement of Theorem 2.

5. Conclusions

We have analyzed stochastic consensus for the multi-agent systems over noisy time-varying directed and undirected networks with linear symmetric agents. The theorems establish the relation between the closeness of the agreement and the number of iterations.
explicitly with a probabilistic guarantee, which gives a stopping rule for the consensus algorithms.

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