Boson-Fermion pairing in a Boson-Fermion environment

A. Storozenko,1,2 P. Schuck,1,3 T. Suzuki,4 H. Yabu,4 J. Dukelsky5
1Institut de Physique Nucléaire, IPN-CNRS, Université Paris-Sud, F-91406 Orsay Cédex, France
2Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, 141980 Dubna, Russia
3Laboratoire de Physique et Modélisation des Milieux Condensés, CNRS & Université Joseph Fourier,
Maison des Magasins, B.P. 166, 38042 Grenoble Cedex 9, France
4Department of Physics, Tokyo Metropolitan University,
1-1 Minami-Ohsawa, Hachioji, Tokyo 192-0397, Japan
5Instituto de Estructura de la Materia, Serrano 123, Madrid 28006, Spain

(Dated: January 7, 2022)

Propagation of a Boson-Fermion (B-F) pair in a B-F environment is considered. The possibility of formation of stable strongly correlated B-F pairs, embedded in the continuum, is pointed out. The new Fermi gas of correlated B-F pairs shows a strongly modified Fermi surface. The interaction between like particles is neglected in this exploratory study. Various physical situations where our new pairing mechanism could be of importance are invoked.

PACS numbers: 03.75.Fi, 05.30.Fk

The physics of ultra cold atomic gases is making progress at a rapid pace, which has led to a realization of Boson-Fermion mixtures of atomic gases [1, 2, 3, 4]. Boson-Fermion mixtures may exhibit the richest variety of phenomena of all. They may show very different behaviour from pure Fermion or pure Bose gases [5, 6]. Especially interesting is a possible instability of the mixture when there is an attraction between Bosons and Fermions [5, 7, 8, 9], as the recent experiment in fact suggests a collapse of the mixture [10].

In the present work we propose and study quite a different scenario for an attractively interacting Boson-Fermion mixture. To simplify the problem in a first survey we shall consider the situation where there is no interaction between atoms of the same kind. As we will discuss at the end of the paper, this is not a severe approximation to cases where the interaction between like atoms is repulsive. More precisely we want to address the question what happens to a mixture of free Fermions and Bosons when a (tunable) attraction is switched on between Fermions and Bosons. We imagine that correlated B-F pairs will be created. These B-F pairs are composite Fermions and as such these B-F pairs should form a new Fermi gas of composites. Besides in ultracold atomic gases such a situation can exist in other branches of physics. For example in nuclear systems (e.g. neutron stars) of high density of nucleons in the so-called hadronisation transition. Further examples may be added to this list.

For a numerical example, we take a mixture of 40K - (fermion) and 41K - (boson) atoms throughout the paper. They are known as candidates for a realization of such kind of quantum systems. While their scattering lengths are not well fixed at present, different values have been reported experimentally [11] it is not crucial at the moment because our study will be mostly academic, elaborating on the basic phenomenon. Applications to realistic systems shall be left for the future.

Let us consider a single B-F pair propagating in the background of a homogeneous gas of free one component Fermions and spinless Bosons. We will formulate our approach for a situation at finite temperature T, though later on in our application we will concentrate on the T = 0 case. We have in mind an analogous study Cooper performed long time ago [12] for the propagation of two fermions (spin up/down) in the background of a homogeneous gas of two component free Fermions. In other words we consider a situation where in the original Cooper problem one Fermion type (let us say spin down) is replaced by spinless bosons. The B-F propagator at finite temperature T and finite centre of mass momentum P of the pair which is added to the system with momenta P/2 + p (Fermion) and P/2 − p (Boson) is

\[
G_{p,p'}^T(P) = -i \theta(t-t') \left\{ \left( b_{\frac{p}{2}+p}^c c_{\frac{p}{2}+p} \right)^t, \left( c_{\frac{p}{2}+p}^t b_{\frac{p}{2}+p}^c \right)^{t'} \right\}
\]

where \{,\} is the anticommutator and \( c^+ \) and \( b^+ \) are Fermion and Boson creation operators, respectively. In ladder approximation the integral equation for \( G_{p,p'}(P,E) \) reads [13]

\[
G_{p,p'}(P,E) = G_{p}^0(p,E) \delta(p-p') + \int \frac{dp_1}{(2\pi)^3} G_{p}^0(p,E) V(p,p_1) G_{p_1,p'}(P,E)
\]

(1)

In graphical form this equation is represented in Fig. 1. In (1) \( V(p,p_1) \) is the B-F interaction and \( G_{p}^0(p,E) \) is the free retarded B-F propagator in the B-F background:

\[
G_{p}^0(p,E) = \frac{1 - f_{\frac{p}{2}+p} g_{\frac{p}{2}-p}}{E - e_f \left( \frac{p}{2} + p \right) - e_b \left( \frac{p}{2} - p \right) + i\eta}
\]
Here \( f(p) \) and \( g(p) \) are the Fermi-Dirac and Bose-Einstein distributions with chemical potentials \( \mu_f \) and \( \mu_b \), respectively, and the term with the condensate fraction \( n_0 \) of Bosons only appears for \( T < T_c \) where \( T_c \) is the critical temperature for Bose condensation. We further have \( c_f(p) = c_b(p) = p^2/2m \) which are the kinetic energies of Fermions and Bosons which we suppose of equal mass: \( m_b = m_f = m \). For simplicity we disregard mass shifts from selfenergy corrections which may drive the masses of Fermions and Bosons apart, even if in free space they are equal. Had we considered F-F propagation in a two component Fermi gas (spin up/dowm), as Cooper did in his original work, than in (2) the bosonic distribution +\( g(\frac{E}{2} - p) \) would have to be replaced by \( -f(\frac{E}{2} - p) \) with, of course, \( n_0 = 0 \). As in Cooper’s work, equation (1) only treats the propagation of one pair and neglects the influence of the other pairs on the pair under consideration. We therefore only can study situations with a very low density of B-F pairs.

For the B-F case we will make the schematic ansatz of separability of the force:

\[
V(p, p') = -\lambda v(p)v(p'), \quad \lambda > 0
\]

with a Yukawa type of form factor

\[
v(p) = \frac{1}{\sqrt{m(p^2 + \beta^2)}}
\]

where, in principle, the two parameters \( \lambda \) and \( \beta \) may be related to the scattering length and the effective range parameters of the low energy B-F scattering in free space. However, in this exploratory study we will consider \( \lambda \) and \( \beta \) as free parameters especially in view of the fact that the interaction strength can be shifted using the Feshbach resonance phenomenon whose application to K-atoms has been discussed in [13, 17]. The integral equation can then easily be solved with only a quadrature to be done numerically. The result is

\[
G_{p,p'}(P, E) = G_{p}(P,E)\delta(p-p') - \frac{1}{(2\pi)^3} \frac{\lambda G_0^0 P, E) v(p)v(p')G_{p'}(P,E)}{1 + \lambda J_0(E, P)}
\]

where

\[
J_0(E, P) = \int \frac{dp'}{(2\pi)^3} G_0(p, E) v^2(p)
\]

Without loss of generality we can consider the simpler propagator integrated over relative momentum

\[
G(P, E) = \int \frac{dp'}{(2\pi)^3} \int dp v(p)v(p')G_{p,p'}(P, E)
\]

\[
= \frac{J_0(E, P)}{1 + \lambda J_0(E, P)}
\]

We will be interested in the \( T \)-matrix \[18\]

\[
T_E^B (q, q') = \frac{-\lambda v(q)v(q')}{1 + \lambda J_0(E, P)}
\]

and want to study the pole structure of this function, first at \( T = 0 \), as a function of \( P \). To this purpose we show in Fig.2 the imaginary part of \( J_0 \) as a function of energy \( E \) for different values of the total momenta. Solid line corresponds to the \( P^2/2m = 0 \); dotted line - \( P^2/2m = 0.01\mu_f \); dashed line - \( P^2/2m = 0.4\mu_f \) and dash-dotted line - \( P^2/2m = \mu_f \).
with \( k_F = \sqrt{\frac{2m}{\pi\rho_F}} \) the Fermi momentum. We also see a sharp peak at \( E = P^2/2m \) (the finite width is numerical). This peak corresponds to the motion of the free Fermions when the Bosons are at rest in the condensate. We can call this peak the one of the free B-F pairs. The corresponding free B-F propagator is

\[
C^0_{n_0}(P, E) = \frac{n_0}{E - P^2/2m + i\eta} = \mathcal{P} \frac{n_0}{E - P^2/2m - i\pi n_0 \delta(E - P^2/2m)}
\]

(8)

We see that this part of the propagator is equal to the pure single Fermion propagator multiplied with \( n_0 \) which is the free boson propagator at \( T = 0 \). The peak of (8) and the threshold of the continuum part of Im \( J_0 \) approach one another for increasing \( P \) and meet exactly at \( E = k_F^2/2m \) when the free BF pair moves with \( P = k_F \).

In this work we will restrict to \( T = 0 \) when the phase space factors in (9) reduce to \( g(p) = 0 \) and \( f(p) = \theta(k_F^2 - p^2) \).

\[
\frac{1}{E - P^2/2m} = -\frac{1}{\lambda n_0 \nu^2(P/2)}
\]

(10)

In Fig. 4 we show for \( \lambda' = 58 \) in comparison with dispersion of pure collective pole \( E^{\text{Coll}}_1 \) and the free B-F pair \( E^0_0 \) (thick lines).

In Fig. 4 we show for \( \lambda' = 58 \) the dispersion of the collective pole (9) and of the ordinary pole (10) which describes to very good approximation (with the parameters used here \( \lambda n_0 \nu^2(P/2) \ll 1 \)) the center of mass motion of a non interacting B-F pair i.e. \( E^0_0 = P^2/2m \) (thick lines). On the same figure we also show the true dispersion of the two roots (thin lines). We notice that at \( P \approx k_F/2 \) there is a level crossing between (9) and (10). However, in reality, due to the no crossing rule [10] and the level-level repulsion the two roots do not cross but, as well known [19], nevertheless exchange their character around crossing. For \( P \ll k_F/2 \) the collective pole is above the ordinary root whereas for \( P \gg k_F/2 \) it is the inverse. This interchange has dramatic consequences: all B-F pairs with center of mass momenta \( k_F/2 \ll P < k_F \) will populate the lower branch, i.e. the collective pole. Due to its strong collectivity the upper part of the Fermi sphere becomes strongly modified, as we will see later. Of course, this interpretation is qualitative, since we only considered a single B-F pair and, as in the case of ordinary Cooper pairing [20], pair-pair interaction may modify the scenario quantitatively. How much of the original free Fermi surface melts and turns into a new momentum space shell filled with a gas of B-F pairs depends, of course, on the interaction. For \( \lambda' < 58 \) the new shell will be thinner than the one of Fig. 4 and at \( \lambda' = \lambda'_{\text{cr}} \) the shell of B-F pairs disappears. For the parameters used here this happens for \( \lambda'_{\text{cr}} = 54.46 \). One may, however, also define another critical value \( \lambda' = \lambda'_{\text{tot}} \) which corresponds to the conversion of practically all original bosons into B-F pairs. For our case this occurs at \( \lambda'_{\text{tot}} \simeq 58.75 \). Increasing the interac-
faction further, part of the B-F pairs will be converted into bound B-F molecules with negative binding energy. The various scenarios are depicted in Fig. 5 where we show the dispersion of the collective pole (9) in comparison with the free gas B-F dispersion \( E_0' = P^2/2m \) for four cases \( \lambda' = \lambda'_{cr} = 54.46, \lambda' = 58, \lambda' = \lambda'_{tot} \approx 58.75 \) and \( \lambda' = 59 \). We see that for \( \lambda' < \lambda'_{cr} \) the new dispersion of the B-F pairs undershoots the free gas B-F dispersion everywhere and that for \( \lambda' = 59 \) molecules appear in the range \( 0 \leq P \lesssim 0.47k_F \).

For \( \lambda' < \lambda'_{cr} \) still a stable pole i.e with no imaginary part exists down to infinitesimally small attraction where \( P = 0 \) the collective pole hits the value \( 2\mu_f \). This is due to the logarithmic divergency seen in Fig. 3, which is of the same origin as in the original Cooper problem of Fermions [14], namely the sharp Fermi function in (9) at \( P = T = 0 \). In regions where the collective pole lies above the ordinary B-F pole one would call the collective pole a B-F pair vibration which for \( \lambda' \lesssim \lambda'_{cr} \) can become of considerable collectivity as seen in Fig. 5. Whether like in nuclear physics [19] such pair vibrations can be detected experimentally is an open question.

To evaluate the ratio of the fermions and bosons which participate in the composite BF pair let us consider the fermion (boson) occupation numbers

\[
n_f (\mu, T) = \frac{dE}{2\pi} f (E) A (p, E) \tag{11}
\]

which can be found by the help of the single particle spectral function \( A (p, E) \)

\[
A_{b,f} (p, E) = \frac{-2 \text{Im} \Sigma_{b,f}^{ret} (E, p)}{(E - e^{b,f}_p - \text{Re} \Sigma_{b,f}^{ret} (E, p))^2 + \text{Im} \Sigma_{b,f}^{ret} (E, p)^2} \tag{12}
\]

calculated via the retarded self-energy \( \Sigma_{b,f}^{ret} (E, p) \). The retarded self-energy can be defined through the Matsubara self energies \( \Sigma_{b,f} (z_n, p) \) (again we keep the formalism general and work at finite temperature; however, at the end we set \( T = 0 \)). To find \( \Sigma_{b,f} (z_n, p) \) we express them in terms of the \( T \) matrix calculated in ladder approximation

\[
\Sigma_{b,f} (z_n, p) = \pm T \sum_{z_n'} \int \frac{dp'}{(2\pi)^3} F_{z_n' + z_n} (q, q) G_{0}^{b,f} (z_n', p') \tag{13}
\]

where \( K = p + p', q = (p - p')/2 \) and \( G_{0}^{b,f} (z_n', p') \) are the free single particle Green’s functions

\[
G_{0}^{b} (z_n, p) = \frac{1}{iz_n - e^b_p} - (2\pi)^3 T^{-1} n_0 (p) \delta_{n,0} \tag{14}
\]

\[
G_{0}^{f} (z_n, p) = \frac{1}{iz_n - e^f_p} \tag{15}
\]

The sum over the Matsubara frequencies can be performed using the spectral representation of the \( T \)-matrix and the s.p. GF and transforming the sum into contour integral . The corresponding imaginary parts will be

\[
\text{Im} \Sigma_{b}^{ret} (E, p) = \int \frac{dp'}{(2\pi)^3} \text{Im} T_{e^b_p + E} \nu^f (q, q) \times \left[ f (e^b_{p'}) - f (e^f_{p'} + E) \right] \tag{15}
\]

\[
\text{Im} \Sigma_{f}^{ret} (E, p) = n_0 \text{Im} T_{E + i\delta} \nu^0 (p/2, p/2) + \int \frac{dp'}{(2\pi)^3} \text{Im} T_{e^b_p + E + i\delta} \nu^f (q, q) \times \left[ g (e^b_p) + f (e^b_p + E) \right] \tag{16}
\]

The real parts can be calculated from the imaginary ones by using the dispersion relation

\[
\text{Re} \Sigma (E, p) = \Sigma^0 (p) + p \int \frac{dE' \Gamma (E', p)}{E - E'} \tag{17}
\]

where \( \Gamma (E, p) = i (\Sigma^{ret} (E, p) - \Sigma^{adv} (E, p)) \).

For the energies below \( \text{E}^{thr} \) the quantity \( J_0^f (E, K) \) is zero and the \( T^- \) matrix exhibits poles at \( 1 + \lambda J_0^f (E, K) = 0 \). The corresponding value of \( \text{Im} T_{e^b_p + E + i\delta} \nu^0 (q, q) \) in this case is the following

\[
\text{Im} T_{E + i\delta} \nu^0 (q, q) = V (q, q) \sum_{\nu} A_{\nu}^K \delta (E - E_{\nu}^E) \tag{18}
\]

where

\[
A_{\nu}^K = \left[ \frac{dE_{\nu}^K}{dE} \right]^{-1} \tag{19}
\]

and \( E_{\nu}^K \) is the solution of the secular equation \( 1 + \lambda J_0^f (E_{\nu}^E, K) = 0 \).

For the calculation of the occupation numbers we use the quasiparticle approximation (we checked this to be very accurate).
where \( \xi_p^{b,f} \) is the solution of the following equation

\[
\xi_p^{b,f} - \xi_p^{b,f} - \text{Re}\Sigma^\text{ret}_p(\xi_p^{b,f},p) = 0 \tag{21}
\]

The fermionic distribution function \( n_p^{f}(\mu_f,T) \) was calculated at zero temperature in the approximation that \( \text{Im}\Sigma^\text{ret}(E,p) \) of eq. (16) contains only the term with the boson condensate. We estimated that this term gives the by far largest contribution in comparison with the second term. At weak interaction the redistribution of the fermions due to their interaction with bosons is small. An example, in Fig. 6a we show \( n_p^{f} \) calculated for \( \lambda' = 52.0 \). We can see that the usual Fermi step function \( \tilde{\mu}^{a} \) and the corresponding dispersion \( \tilde{\mu}^{b} \) of eq. (16) contains via the operator (16) contains only the term with the quasifermion energies, solutions of (21). Since the mass operator (16) contains via the \( T \)-matrix the two poles we have discussed in Figs. 3, 4, eq. (21) will have three roots which in the weak coupling limit considered here correspond approximately to the free solution \( \xi_1^{f} \sim \xi_p^{b,f} \), a second one \( \xi_2^{f} \) corresponding to (10) and almost degenerate with \( \xi_1^{f} \) up to the level crossing, and a third one which corresponds to the collective BF pair (9) corresponding to the highest root in Fig. 6b. Since our interaction strength is still sufficiently weak so that the original chemical potential is barely changed, the intersection at \( p^2/2m\mu_F \simeq 1.02 \) of the \( \xi_2^{f} \) with \( \mu_F \) gives the upper step in Fig. 6a whereas the solution \( \xi_1^{f} \) still intersects at \( p^2/2m \simeq \mu_F \) and gives raise to the intermediate step in Fig. 6a. When the interaction further decreases, the two lowest roots become completely degenerate and we have the usual Fermi step.

With the growth of \( \lambda \) the low lying BF roots become more and more collective and when \( \lambda' \) approaches the value equal to \( \sim 54.0 \) the BF roots corresponding to total momenta \( K \sim k_F \) become enough collective to change the Fermion distribution strongly. In Figs. 7a we thus show \( n_p^{f} \) for \( \lambda' = 54.0 \) and the corresponding dispersion is displayed in Fig 7b. After the level crossing, as already discussed above, (9) and (10) exchange their properties and therefore for \( p^2/2m\mu_F \gtrsim 1 \) the lowest root becomes the collective BF pair and the highest one becomes degenerate with the free solution.

At the interaction \( \lambda' \gtrsim 54.0 \) level crossing takes place below the Fermi momentum. Such a case is displayed in Figs. 8a and 8b for \( \lambda' = 56.0 \). We can see one small step at the momenta slightly below \( k_F \), which will disappear with an increase of the interaction, and a rather long tail in the Fermi distribution which corresponds to the strong collective BF pairs. With further increasing of the interaction this tail goes to infinity whereas the energy of the collective BF pair goes to zero. After that the BF pair converts into the molecular state and it is necessary to apply another kind of theory.

We prefer not to increase \( \lambda' \) further because the redistribution of \( n_p^{f} \) strongly varies with \( \lambda' \) and for strong \( \lambda' \) the one pair approximation becomes invalid. On the other hand the stronger values of \( \lambda' \) employed in Figs. 4

\[
A_{b,f}(p,E) = \frac{2\pi\delta(E - \xi_p^{b,f})}{1 - \frac{d}{dE}\text{Re}\Sigma^\text{ret}_{b,f}(E,p)|_{E=\xi_p^{b,f}}} \tag{20}
\]
and 5 have just been chosen for illustration purposes and for a qualitative discussion of the roots of $\text{Re} \ J_0 = -\lambda^{-1}$ this seems quite appropriate.

We therefore very nicely see that with the parameters chosen we have around $\mu_F$ a mixture of the gas of the old free fermions and the new composite ones formed out of a boson and a fermion. The interaction was chosen sufficiently weak so that the one pair description is approximately valid and yet sufficiently strong so that the coexistence of the two Fermi gases can clearly be seen.

In principle with (1), (12) and (15) one can also calculate the new boson occupation numbers, i.e. of those bosons which, due to the B-F correlations, are scattered out of the condensate. However, with our choice of parameters, the influence of the bosons on the fermions remains modest and the one pair approximation is justified. On the other hand the action of the fermions on the bosons even for relatively small interactions is enormous because of the much greater number of fermions than bosons invalidating the one pair approximation. One therefore would have to iterate the calculation for the occupation numbers, that is we insert the new occupation numbers into (1), solve it, calculate new occupation numbers and so on up to convergence. At the same time we will have to reajust the chemical potentials for bosons and fermions to preserve number of particles. This shall be done in future work and we do not show the Boson occupation here.

The reader may have noticed that our approach is equivalent to a particle-particle RPA formulation \[ \text{in the BF channel. Since our BF pairs are discrete states an eigenvalue variant of the present approach can give useful additional information. We shortly present this in the Appendix.} \]

In conclusion we considered Boson-Fermion propagation in a B-F environment and found that the original free gas converts for sufficiently strong attractive interaction into a completely new state of matter as a new Fermi gas of B-F Cooper pairs with a strongly modified Fermi surface. The most interesting and novel feature concerns the fact that, due to the Pauli exclusion principle, this transition can occur for interaction strength insufficient to form bound B-F molecules in free space. In other words the B-F pairs are at positive energies much analogous to Cooper pairs in a pure Fermi gas. On the other hand the collective B-F pairs are still Fermions building a new Fermi gas of composites and a new Fermi surface. Whether this transition has anything to do with the recently discovered collapse of a B-F mixture \[ \text{remains to be seen.} \]

The only system parameter we varied in our work is the strength of the interaction. Of course, a variety of other parameters could be changed: the densities of bosons and fermions can be varied in strong proportions, their masses could be strongly different, we worked strictly at zero temperature only, we consider a homogeneous system and not the geometry of traps, etc. Such investigations shall be performed in the future. It also should again be mentioned that in this pioneering work we considered only a very idealized situation, disregarding any interaction between like particles, i.e. between bosons or between fermions. We suspect that as long as the interactions between like particles are repulsive nothing qualitative will change: the Fermi surface will become slightly rounded and some depletion of the condensed bosons will occur. The constellation of moderate repulsion between particles of the same kind and attraction between different kinds is not unrealistic \[21, 22\].

In this respect we also mention that an attractive B-F interaction can induce, via e.g. second order processes, an effective attraction between Fermions \[23\]. In this paper we consider a weak coupling scenario where the interaction is weaker as the one needed to form a B-F molecule. Therefore we suppose that induced F-F attraction is weak and in any case weaker than the direct F-F repulsion which we implicitly can assume here. In any case, in this work we are only treating a one component Fermi gas where s-wave scattering is suppressed, unless the force is finite range. Then only dipole or higher odd multipole interactions could lead to F-F attraction. On the other hand, did one consider Fermions with spin together with bosons, F-F attraction is possible more easily and in that case our scenario still may change strongly. Indeed it is conceivable that in this case standard purely fermionic Cooper pairs form new Cooper pairs of triples in pairing up with bosons. A strong enhancement of our present effect could occur since now ‘bosons’ (the F-F Cooper pairs) pair with bosons (atoms). What exactly will happen under these conditions is unknown at this point. Of course with attraction among Fermions, it is
also conceivable that two B-F Cooper pairs form a quartet. Those quartets would be different from the purely fermionic quartets which may be possible when four different species of Fermions are trapped in a pure fermi gas, as recently discussed in the nuclear physics context \((\alpha\text{- particles})\). One sees that a great variety of quantum condensation phenomena may still be explored with ultra-cold atomic gases consisting out of Bosons and Fermions.

Note added in proof: when this work was completed we learned about the related work in \([21]\) where, however, the BF pairs are treated in the molecular state and not as collective B-F pairs embedded as a sharp states in the continuum as in this work.

Acknowledgments

Interest in this work together with discussions by G.F. Bertsch is gratefully acknowledged.

One of the authors (A. S.) gratefully acknowledges a post-doctoral fellowship from the French Ministry of Research.

APPENDIX

A further more formal but eventually useful aspect of our theory is to relate it to the language of a particle-particle RPA \([19]\) in the BF channel. We only consider this here at \(T = 0\) and present the extension to \(T \neq 0\) in a future publication. To this purpose we write the following RPA excitation operator

\[
Q^+_l = \sum_{pq} \frac{X^l_{pq} c^+_{pq}}{\sqrt{1 + \delta_{q,0} n_0}} + \sum_h \frac{Y^l_{h0} c^+_{h0}}{\sqrt{n_0}} \tag{22}
\]

where \(p(h)\) is a fermion momentum above (below) the Fermi sea (a 'particle' ('hole') state) and \(q\) is a boson momentum which can take on all values. With the definition of an excited state of the \(N + 2\) particle system (the addition mode)

\[
|l> = Q^+_l |
\]

we arrive with the usual condition

\[
|Q_l| \geq 0 \tag{24}
\]

at the following secular equation \([12]\)

\[
<| \{ \delta Q, [H, Q^+_l] \} |> = E_l <| \{ \delta Q, Q^+_l \} |>
\]

where \(\delta Q\) is a variation with respect to either \(X\) or \(Y\). The usual linearisation of (25) consists in evaluating the expectation values with the uncorrelated ground state that is a product of a Slater determinant and an ideal Bose condensate. Defining the Hamiltonian as

\[
H = \sum_q \left( \varepsilon^b_q b^+_q b_q + \varepsilon^f_q c^+_q c_q \right) + \sum_{q_1, q_2, q_4} V_{q_1, q_2, q_4} b^+_q c_{q_1} c_{q_2} b_{q_4}
\]

\[
+ \sum_{q_1, q_2, q_4} V_{q_1, q_2, q_4} b^+_q c_{q_1} c_{q_2} b_{q_4}
\]

where

\[
\varepsilon^f_q = \varepsilon_q^f - \mu_f
\]

eq. (25) then reads as

\[
\begin{pmatrix}
A_{p'q',pq} D_{p'q',h0} & \left( Y^l_{pq} \right) \\
B_{h',0,pq} & C_{h',0,h0}
\end{pmatrix}
\left( X^l_{pq} \right) = E_l \left( X^l_{pq} \right)
\]

where

\[
A_{p'q',pq} = \delta_{pp'} \delta_{qq'} \left( \varepsilon^f_q + \varepsilon^b_p \right)
\]

\[
+ \sqrt{1 + \delta_{q,0} n_0} V_{p'q',pq} \sqrt{1 + \delta_{q,0} n_0}
\]

\[
B_{h',0,pq} = V_{h',0,pq} \sqrt{n_0} \sqrt{1 + \delta_{q,0} n_0}
\]

\[
D_{h',0,pq} = V_{p'q',h0} \sqrt{n_0} \sqrt{1 + \delta_{q,0} n_0}
\]

\[
C_{h',0,h0} = \delta_{hh'} \left( \varepsilon^f_{h} + \varepsilon^b_{h'} \right) + V_{h',0,h0} n_0
\]

We immediately see that with \(V_{p'q',pq} = -\lambda^v \left( p' - q' \right) \left( p - q \right) \delta_{p+q,p'+q'}\) the eigen values are given as before by \(Re J_0 = -\lambda^v - 1\) and therefore the RPA description is completely equivalent to the Green’s function approach we used at the beginning.

The interesting aspect of this formulation is that eq. (27) contains a quasifermion approximation, i.e. the BF pairs in (22) are treated as ideal Fermions.

\[
F_{pq}^+ = \frac{c^+_{pq} b^+_{q}}{\sqrt{1 + \delta_{q,0} n_0}}
\]

\[
F_{h}^+ = \frac{c^+_{h0} b^+_{h}}{\sqrt{n_0}}
\]

with

\[
\{ F_{p'q',F_{pq}} \} = \delta_{pp'} \delta_{qq'}
\]

\[
\{ F_{h',F_{h}} \} = \delta_{hh'}
\]

This is quite in analogy to the standard RPA for a pure fermi system where a fermion pair \(c^+_{p} c^+_{p}\) is treated as a quasiboson \([12]\). This quasifermion approximation contained in (27) allows to write down the new approximate groundstate. It is given by a Slater determinant of the new BF pairs

\[
|uc> > \exp \left( \sum_{p'q'h'} Z_{p'q'h'} F_{p'q'}^+ F_{h'}^+ \right) |uc>
\]

where \(|uc>\) is the 'uncorrelated' vacuum. From the condition (24) we also can find the system of equations which defines the coefficients \(Z_{p'q'h'}\):

\[
Y^l_{pq} + \sum_{pq} X^l_{pq} Z_{pqh'} = 0
\]
The initial Hamiltonian (25) in the basis of the new quasifermions (22) will be the following

\[ H = \sum_{l \geq 0} E_l Q_l^+ Q_l \]  

where \( l < 0 \) corresponds to the negative roots of (27) or (which is the same) to the roots of the RPA for the \( N-2 \) system (the removal mode). And finally we can define the following correlation energy

\[ E_{\text{corr}} = \langle H \rangle - \langle uc \vert H \vert uc \rangle = \sum_{l < 0} E_l - \text{Tr}C \]  

which may be evaluated for realistic systems in the future. This correlation energy calculated in B-F pp-RPA is exactly the analog to the correlation energy calculated for an electron gas in ph-RPA via the summation of ring diagrams [18]. We therefore see that our B-F calculation also leads to an improved equation of state.

[1] A.G. Truscott, K.E. Strecker, W.I. McAlexander, G.B. Partridge and R.G. Hulet, Science 291, 2570 (2001).
[2] F. Schreck, L. Khaykovich, K.L. Corwin, G. Ferrari, T. Bourdel, J. Cubizolles and C. Salomon, Phys. Rev. Lett. 87, 080403 (2001).
[3] Z. Hadzibabic, C.A. Stan, K. Dieckmann, S. Gupta, M.W. Zwierlein, A. Gorlitz, W. Ketterle, Phys. Rev. Lett. 88, 160401 (2002).
[4] G. Roati, F. Riboli, G. Modugno, M. Inguscio, Phys. Rev. Lett. 89, 150403 (2002).
[5] K. Mølmer, Phys. Rev. Lett. 80, 1804 (1998).
[6] T. Miyakawa, T. Suzuki and H. Yabu, Phys. Rev. A62, 063613 (2000); A. Minguzzi and M.P. Tosi, Phys. Lett. A268, 142 (2000); T. Sogo, T. Suzuki and H. Yabu, Phys. Rev. A68, 063607 (2003).
[7] T. Miyakawa, T. Suzuki and H. Yabu, Phys. Rev. A64, 033611 (2001).
[8] R. Roth, Phys. Rev. A66, 013614 (2002).
[9] P. Capuzzi, A. Minguzzi and M.P. Tosi, J. Phys B37, S73 (2004).
[10] G. Modugno, G. Roati, F. Riboli, F. Ferlaino, R.J. Brecha and M. Inguscio, Science 297, 2240 (2002); M. Modugno, F. Ferlaino, F. Riboli, G. Roati, G. Modugno, M. Inguscio, Phys. Rev. A68, 043626 (2003).
[11] H.-J. Schulze, M. Baldo, U. Lombardo, J. Cugnon and A. Lejeune, Phys. Lett. B355, 21 (1995); Phys. Rev. C57, 704 (1998).
[12] K. Rajagopal, F. Wilczek, hep-ph/0011333 M. Alford, Ann. Rev. Nucl. Part. Sci. 51, 131 (2001).
[13] R. Cote, A. Dalgarno, H. Wang, and W.C. Stwalley, Phys. Rev. A57, R4118 (1998); John L. Bohn, James P. Burke, Jr., and Chris H. Greene, H. Wang, P.L. Gould, and W.C. Stwalley, Phys. Rev. A59, 3660 (1999).
[14] Leon N. Cooper, Phys. Rev. 104, 1189 (1956).
[15] H. Yabu, T. Takayama, T. Suzuki and P. Schuck, Nucl. Phys. A738, 273 (2004).
[16] Y. Yamaguchi, Phys. Rev. 95, 1628 (1954); Y. Yamaguchi and Y. Yamaguchi, Phys. Rev. 95, 1635 (1954).
[17] J.L. Bohn, Phys. Rev. A61, 053409 (2000).
[18] A.L. Fetter, J. D. Walecka, Quantum Theory of Many-Particle Systems (McGraw-Hill, New York, 1971).
[19] P. Ring, P. Schuck, The Nuclear Many Body Problem (Springer, New York, 1980).
[20] J.R. Schrieffer, Theory of Superconductivity (Benjamin, Reading, Mass., 1964).
[21] W. Hofstetter, J.I. Cirac, P. Zoller, E. Demler, and M.D. Lukin, Phys. Rev. Lett. 89, 220407 (2002).
[22] J.N. Milstein, S.J.J.M. Kokkelmans, and M.J. Holland, cond-mat/0204334.
[23] M.J. Bijlsma, B.A. Heringa and H.T.C. Stoof, Phys. Rev. A61, 053601 (2000); H. Heiselberg, C.J. Pethick, H. Smith and L. Viverit, Phys. Rev. Lett. 85, 2418 (2000); L. Viverit, C.J. Pethick and H. Smith, Phys. Rev. A61, 053605 (2000).
[24] G. Röpke, A. Schnell, P. Schuck, P. Nozières, Phys. Rev. Lett. 80, 3177 (1998).
[25] T. Yamada, P. Schuck, Phys. Rev C69, 024309 (2004).
[26] M.Yu. Kagan, I.V. Brodsky, D.V. Efremov, A.V. Klaptsov, Phys. Rev A70, 023607 (2004).