Group analysis of hydrodynamic-type systems

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Abstract

We study point and higher symmetries for the hydrodynamic-type systems with two independent variables $t$ and $x$ with and without explicit dependence of the equations on $t, x$. We consider those systems which possess an infinite group of the hydrodynamic symmetries, establish existence conditions for this property and, using it, derive linearizing transformations for these systems. The recursion operators for symmetries are obtained and used for constructing infinite series of exact solutions of the studied equations. Higher symmetries, i.e. the Lie-Bäcklund transformation groups, are also studied and the interrelation between the existence conditions for higher symmetries and recursion operators is established. More complete results are obtained for two-component systems, though $n$-component systems are also studied. In particular, we consider Hamiltonian and semi-Hamiltonian systems.

1 Introduction

In this paper we review our results of application of the group analysis of differential equations to systems of the hydrodynamic type. This concept includes a systematic study of point and higher symmetries, recursion operators, Hamiltonian structures and the usage of these results in order to obtain exact analytic solutions of these equations. Originally B.A. Dubrovin and S.P. Novikov meant by hydrodynamic-type systems quasi-linear systems of first order partial differential equations which possessed Hamiltonian structure [1,2]. Here we consider a more general class of hydrodynamic-type equations which specifically includes semi-Hamiltonian equations of S.P. Tsarev [3,4] and explicitly dependent on $t$ (or $x$) equations which are rich in symmetries. As a consequence of the last property they admit linearization and as soon as existence conditions for recursion operators are satisfied infinite series of their exact solutions can be obtained. Thus, they have as good integrability properties as the Hamiltonian equations.

Hydrodynamic-type systems describe various physical phenomena: gas dynamics and hydrodynamics, magnetic hydrodynamics [3], nonlinear-elasticity and phase-transition.
models, chromatography and electrophoresis equations from physical chemistry and biology.

Applications of another kind are obtained by a representation of physically interesting higher order equations as integrability conditions of hydrodynamic-type systems: the Euler and Poisson equations of nonlinear acoustics, the Born-Infeld equation of nonlinear electrodynamics, systems of relativistic-string equations. Modern applications of hydrodynamic-type systems arise in the theory of averaging nonlinear soliton equations.

An attractive feature of hydrodynamic-type systems is the possibility of a geometric formulation of their properties by close analogy with the Hamilton-Jacobi and eikonal equations of mechanics and optics.

Symmetry group analysis of these systems leads naturally and algorithmically to associated differential-geometric structures: metric, connection, curvature, curvilinear orthogonal coordinate systems and their transformations. If the Hamiltonian structure exists, then it turns out to be merely an aspect of this geometrical theory. Thus, here we can see clearly a relation of transformation groups to geometry which S. Lie and F. Klein meant by their Erlangen program.

A purely differential-geometric theory does not solve the problem of integrating hydrodynamic-type equations. We present here another approach to this problem based on a systematic study of higher symmetries and recursion operators. The essence of this approach is the following. We pick out a class of those hydrodynamic-type systems which possess an infinite set of hydrodynamic symmetries depending on arbitrary solutions of a linear system of partial differential equations. Formulae for the corresponding invariant solutions determine a linearizing transformation, which reduces the problem of obtaining solutions of a nonlinear hydrodynamic-type system to a problem of solving a linear system. Then we may construct recursion operators which map symmetries again into symmetries if we assume in addition their existence conditions. They give rise to recursions of hydrodynamic symmetries and, as a consequence, lead to recursions for solutions of the corresponding linear system. Thus, we obtain the recursion formulae which allow us to multiply solutions of the original nonlinear hydrodynamic-type system. The action of recursion operators on the hydrodynamic symmetries explicitly dependent on \( x, t \) gives rise to infinite series of higher symmetries. It turns out to be possible to find the corresponding invariant solutions explicitly. Hence we obtain infinite series of exact solutions of the hydrodynamic-type system. They are analogous to the similarity solutions which are well-known in gas and fluid mechanics and therefore we may expect that they correspond to a physically interesting behavior.

Symmetries, recursions, Hamiltonian structures and exact solutions of two-component spatially one-dimensional hydrodynamic-type systems were studied by the author mainly before S.P. Tsarev’s publications. Symmetries, linearizing transformations and geometric theory of the multi-component hydrodynamic-type systems explicitly dependent on \( x, t \) have been constructed by the author in 1989. Higher symmetries, higher conservation laws, their interrelations and the theory of recursion operators for multi-component hydrodynamic-type systems have been constructed by the author in 1993, 1994. Finally in the recent article we further developed the theory of integrability of the diagonal hydrodynamic-type systems with explicit \( t \) or \( x \).
dependence and presented a non-trivial example of such integrable system. There we also clarified and formulated precisely the concept of the hydrodynamic symmetries which are first order symmetries though being neither point, nor contact ones.

In section 2 we study two-component diagonal hydrodynamic-type systems with explicit dependence on \( t \) or \( x \). We find an infinite group of their hydrodynamic symmetries and establish their existence conditions. By considering the corresponding invariant solutions we derive linearizing transformations for these systems. Second order symmetries and first and second order recursion operators are obtained together with their existence conditions which gives rise to infinite series of exact solutions of the studied systems.

In section 3 the generalized gas-dynamics equations are studied which include in particular the equations of the one-dimensional isoentropic gas dynamics. We obtain all their hydrodynamic and higher symmetries up to the third order inclusively. We construct first order recursion operators which lead naturally to the Lax representation of these equations. The latter is used for obtaining the explicit formulae for infinite series of their invariant solutions.

In section 4 a particular class of the two-component systems which may depend explicitly on \( t \) and possess the Hamiltonian structure, the so-called separable Hamiltonian systems, is studied. We obtain their hydrodynamic and higher symmetries, linearizing transformation, second order recursion operator, the Lax representation and explicit formulas for the infinite series of exact (invariant) solutions.

In section 5 we consider the \( n \)-component diagonal hydrodynamic-type systems with no explicit dependence on \( t \) and \( x \) which possess an infinite-dimensional group of the hydrodynamic symmetries, the so-called semi-Hamiltonian systems. We obtain their hydrodynamic symmetries together with their existence conditions and the associated differential-geometric structure. A linearizing transformation, first and second order recursion operators, higher symmetries and infinite series of exact solutions are constructed.

Finally, in section 6 the \( n \)-component diagonal hydrodynamic systems with explicit dependence on \( t \) or \( x \) are studied. The hydrodynamic symmetries and linearizing transformations are obtained.

## 2 Symmetries, recursions and invariant solutions for two-component hydrodynamic-type systems

Here we study a two-component diagonal hydrodynamic-type system which may depend explicitly on time or space variable \( t \) or \( x \). It is linear homogeneous in derivatives of the unknowns \( s(x,t), r(x,t) \)

\[
s_t = \phi(s,r,t)s_x, \quad r_t = \psi(s,r,t)r_x
\]

where the subscripts denote partial derivatives with respect to \( t \) and \( x \). Functions \( \phi \) and \( \psi \) are real-valued and smooth and satisfy the following nondegeneracy conditions

\[
\phi \neq \psi, \quad \phi_r(s,r,t)\psi_s(s,r,t) \neq 0
\]
Symmetries of system (2.1) are generated by the Lie equations for one-parameter Lie-Bäcklund groups in a canonical representation \[23, 24\]

\[
s_r = f(x, t, s, r, s_x, r_x, \ldots, s_x^{(N)}, r_x^{(N)}),
\]

\[
r_r = g(x, t, s, r, s_x, r_x, \ldots, s_x^{(N)}, r_x^{(N)}),
\]

\[
x_r = t_r = 0,
\]

which are compatible with system (2.1). Here \(s_x^{(N)} = \partial^N s/\partial x^N\), \(r_x^{(N)} = \partial^N r/\partial x^N\) and \(\tau\) is a parameter of a symmetry group: \(s = s(x, t, \tau), r = r(x, t, \tau)\).

Compatibility conditions for systems (2.1) and (2.3) \(s_{\tau} = s_{tr}, r_{\tau} = r_{tr}\) take the form of the determining equations for characteristics \((f, g)\) of symmetries

\[
D_t[f]|_{(2.1)} - \phi D_x[f] - s_x(\phi_s f + \phi_r g) = 0,
\]

\[
D_t[g]|_{(2.3)} - \psi D_x[g] - r_x(\psi_s f + \psi_r g) = 0,
\]

where \(D_t\) and \(D_x\) are operators of the total derivatives with respect to \(t\) and \(x\)

\[
D_x = \frac{\partial}{\partial x} + s_x \frac{\partial}{\partial s} + r_x \frac{\partial}{\partial r} + \sum_{k=1}^{\infty} \left( s_x^{(k+1)} \frac{\partial}{\partial s_x^{(k)}} + r_x^{(k+1)} \frac{\partial}{\partial r_x^{(k)}} \right),
\]

\[
D_t|_{(2.1)} = \frac{\partial}{\partial t} + \phi s_x \frac{\partial}{\partial s} + \psi r_x \frac{\partial}{\partial r} + \sum_{k=1}^{\infty} \left( D_x^k[\phi s_x] \frac{\partial}{\partial s_x^{(k)}} + D_x^k[\psi r_x] \frac{\partial}{\partial r_x^{(k)}} \right)
\]

and \(D_t\) is calculated with the use of system (2.1).

2.1 Hydrodynamic symmetries of diagonal systems with explicit temporal dependence

For hydrodynamic symmetries put \(N = 1\) in the Lie equations (2.3)

\[
s_r = f(x, t, s, r, s_x, r_x), \quad r_r = g(x, t, s, r, s_x, r_x),
\]

and in determining equations (2.3). Analysis of equations (2.5) produces the following results. Define the functions

\[
\Phi_r(s, r, t) = \phi_r(s, r, t)/(\phi - \psi), \quad \Theta_s(s, r, t) = \psi_r(s, r, t)/(\psi - \phi),
\]

\[
\hat{\Phi}(s, r, t) = b(s)\Phi_s(s, r, t) + d(r)\Phi_r(s, r, t) + \Phi_0(s),
\]

\[
\hat{\Theta}(s, r, t) = b(s)\Theta_s(s, r, t) + d(r)\Theta_r(s, r, t) + \Theta_0(r),
\]

\[
\hat{\phi}(s, r, t) = b(s)\phi_s(s, r, t) + d(r)\phi_r(s, r, t),
\]

\[
\hat{\psi}(s, r, t) = b(s)\psi_s(s, r, t) + d(r)\psi_r(s, r, t).
\]

**Definition 2.1.** We call system (2.1) a generic system (with respect to hydrodynamic symmetries) if its coefficients \(\phi, \psi\) do not satisfy the constraints

\[
\phi_t = \beta(t)\phi^2 + \epsilon(t)\phi + \lambda(t), \quad \psi_t = \beta(t)\psi^2 + \epsilon(t)\psi + \lambda(t)
\]
with arbitrary smooth functions $\beta(t), \varepsilon(t), \lambda(t)$.

Results of the group analysis are formulated as the following basic theorem [13].

**Theorem 2.1.** A diagonal two-component generic hydrodynamic-type system (2.1) which may explicitly depend on $t$ possesses an infinite set of hydrodynamic symmetries with a functional arbitrariness iff the following two conditions are satisfied:

1. Coefficients $\phi, \psi$ of system (2.1) satisfy the equalities
   \[
   \Phi_{rt} = \beta \phi_r, \quad \Theta_{st} = \beta \psi_s, \tag{2.12}
   \]
   where $\beta$ is arbitrary (real) constant, the functions $\Phi(s, r, t), \Theta(s, r, t)$ are defined by equations (2.8) and partial derivatives with respect to $t$ are taken at constant values of $s$ and $r$;

2. There exist four functions of one variable $b(s), d(r), \Phi_0(s), \Theta_0(r)$ which satisfy the equalities
   \[
   \hat{\Phi}_r = \Phi_r(\hat{\Phi} - \hat{\Theta}), \quad \hat{\Theta}_s = \Theta_s(\hat{\Theta} - \hat{\Phi}) \tag{2.13}
   \]
   where the functions $\hat{\Phi}(s, r, t), \hat{\Theta}(s, r, t)$ are defined by the formulae (2.9).

These symmetries are generated by the Lie equations

\[
\begin{align*}
s_r &= f = \hat{\phi}(x, t, s, r)s_x + b(s), \\
r_r &= g = \hat{\psi}(x, t, s, r)r_x + d(r),
\end{align*} \tag{2.14}
\]

where functions $\hat{\phi}, \hat{\psi}$ are defined by the following formulae

if $\beta \neq 0$

\[
\begin{align*}
\hat{\phi}(x, t, s, r) &= a(s, r) \exp \left\{ \beta \left[ x + \int_0^t \phi(s, r, t)dt \right] \right\} + \frac{1}{\beta} \hat{\Phi}(s, r, t), \\
\hat{\psi}(x, t, s, r) &= c(s, r) \exp \left\{ \beta \left[ x + \int_0^t \psi(s, r, t)dt \right] \right\} + \frac{1}{\beta} \hat{\Theta}(s, r, t),
\end{align*} \tag{2.15}
\]

and if $\beta = 0$

\[
\begin{align*}
\hat{\phi}(x, t, s, r) &= a(s, r) + \int_0^t \hat{\phi}(s, r, t)dt - \\
&\quad - \hat{\Phi}(s, r) \left[ x + \int_0^t \phi(s, r, t)dt \right], \\
\hat{\psi}(x, t, s, r) &= c(s, r) + \int_0^t \hat{\psi}(s, r, t)dt - \\
&\quad - \hat{\Theta}(s, r) \left[ x + \int_0^t \psi(s, r, t)dt \right].
\end{align*} \tag{2.16}
\]

Here the integrals with respect to $t$ are taken at constant values of $s$ and $r$ and the functions $\hat{\phi}, \hat{\psi}$ are defined by the formulae (2.10). The functions $a(s, r), c(s, r)$ constitute arbitrary smooth solution of the linear system of equations

\[
\begin{align*}
a_r(s, r) &= \Phi_r(s, r, 0)(a - c), \\
c_s(s, r) &= \Theta_s(s, r, 0)(c - a). \tag{2.17}
\end{align*}
\]

**Remark 2.1.** We can use the freedom in the definition (2.8) of the functions $\Phi, \Theta$ to transform equations (2.12) in the condition 1 of Theorem 2.1 to a more simple form

\[
\Phi_t(s, r, t) = \beta \phi(s, r, t), \Theta_t(s, r, t) = \beta \psi(s, r, t). \tag{2.18}
\]
Remark 2.2. A solution manifold of linear system (2.17) is locally parametrized by two arbitrary functions \( c_1(s), c_2(r) \) of one variable. They determine a functional arbitrariness in the definitions (2.15), (2.16) of hydrodynamic symmetries (2.14) for system (2.1).
System (2.17) always has a trivial solution \( a(s, r) = c(s, r) = c_0 = \text{const} \).
The condition 2 of Theorem 2.1 always has a trivial solution 
\[ b(s) = d(r) = 0, \quad \Phi_0(s) = \Theta_0(r) = c_0 = \text{const}, \]
\[ \hat{\Phi} = \hat{\Theta} = c_0, \quad \hat{\phi} = \hat{\psi} = 0. \] (2.19)

Corollary 2.1. The condition 1 of Theorem 2.1 is necessary and sufficient for system (2.1) to possess an infinite set of hydrodynamic symmetries, generated by the Lie equations 
\[ s_\tau = \tilde{\phi}(x, t, s, r)s_x, \quad r_\tau = \tilde{\psi}(x, t, s, r)r_x, \] (2.20)
which are linear homogeneous in derivatives. The coefficients \( \tilde{\phi}, \tilde{\psi} \) of these equations are determined by the following formulae
if \( \beta \neq 0 \)
\[ \tilde{\phi}(x, t, s, r) = a(s, r) \exp \left\{ \beta \left[ x + \int_0^t \phi(s, r, t) dt \right] \right\} + c_0, \]
\[ \tilde{\psi}(x, t, s, r) = c(s, r) \exp \left\{ \beta \left[ x + \int_0^t \psi(s, r, t) dt \right] \right\} + c_0, \] (2.21)

and if \( \beta = 0 \)
\[ \tilde{\phi}(x, t, s, r) = a(s, r) + c_0 \left[ x + \int_0^t \phi(s, r, t) dt \right], \]
\[ \tilde{\psi}(x, t, s, r) = c(s, r) + c_0 \left[ x + \int_0^t \psi(s, r, t) dt \right]. \] (2.22)

Remark 2.3. The condition 1 of Theorem 2.1 with \( \beta = 0 \) is met in particular for system 2.1 with the coefficients \( \phi(s, r), \psi(s, r) \) explicitly independent of \( t \). Such a system always has an infinite set of hydrodynamic symmetries with a functional arbitrariness. In this case coefficients 2.22 of Lie equations have the form
\[ \tilde{\phi}(x, t, s, r) = a(s, r) \left[ x + t \phi(s, r) \right], \]
\[ \tilde{\psi}(x, t, s, r) = c(s, r) \left[ x + t \psi(s, r) \right]. \] (2.23)

Remark 2.4. The condition 2 of Theorem 2.1 is that additional constraint which provides the existence of symmetries with the Lie equations (2.14) linear inhomogeneous in derivatives. Every nontrivial solution of equations (2.13) generates such symmetries.

2.2 Infinite-dimensional Lie algebra of hydrodynamic symmetries and recursions of symmetries

Let the system (2.1) possess two one-parameter symmetry groups generated by Lie equations (2.14) with a parameter \( \tau \) and by Lie equations of the same form with a parameter \( \bar{\tau} \)
\[ s_\tau = \bar{f} = \tilde{\phi}(x, t, s, r)s_x + \bar{b}(s), \]
\[ r_\tau = \bar{g} = \tilde{\psi}(x, t, s, r)r_x + \bar{d}(r). \] (2.24)
Here the coefficients $\bar{\phi}, \bar{\psi}$ are determined by analogy with the formulae (2.15) or (2.16) if $\beta \neq 0$

$$\bar{\phi}(x, t, s, r) = \bar{a}(s, r) \exp \left\{ \beta \left[ x + \int_0^t \phi(s, r, t) dt \right] \right\} + \frac{1}{\beta} \dot{\Phi}(s, r, t),$$

$$\bar{\psi}(x, t, s, r) = \bar{c}(s, r) \exp \left\{ \beta \left[ x + \int_0^t \psi(s, r, t) dt \right] \right\} + \frac{1}{\beta} \dot{\Theta}(s, r, t),$$

(2.25)

and if $\beta = 0$

$$\bar{\phi}(x, t, s, r) = \bar{a}(s, r) + \int_0^t \dot{\phi}(s, r, t) dt - \dot{\Phi}(s, r) \left[ x + \int_0^t \phi(s, r, t) dt \right],$$

$$\bar{\psi}(x, t, s, r) = \bar{c}(s, r) + \int_0^t \dot{\psi}(s, r, t) dt - \dot{\Theta}(s, r) \left[ x + \int_0^t \psi(s, r, t) dt \right].$$

(2.26)

The functions $\dot{\Phi}, \dot{\Theta}$ are defined by analogy with the formulae (2.9)

$$\dot{\Phi} = \bar{b}(s)\Phi_s + \bar{d}(r)\Phi_r + \Phi_0(s),$$

$$\dot{\Theta} = \bar{b}(s)\Theta_s + \bar{d}(r)\Theta_r + \Theta_0(r),$$

(2.27)

and the functions $\dot{\phi}, \dot{\psi}$ are defined by analogy with the formulae (2.10)

$$\dot{\phi} = \bar{b}(s)\phi_s + \bar{d}(r)\phi_r, \quad \dot{\psi} = \bar{b}(s)\psi_s + \bar{d}(r)\psi_r.$$  

(2.28)

The functions $\dot{\Phi}, \dot{\Theta}$ must satisfy the condition 2 of the theorem 2.1

$$\dot{\Phi}_r = \Phi_r(\dot{\Phi} - \dot{\Theta}), \quad \dot{\Theta}_s = \Theta_s(\dot{\Theta} - \dot{\Phi}),$$

(2.29)

The functions $\bar{a}(s, r), \bar{c}(s, r)$ constitute arbitrary smooth solution of the same linear system (2.17)

$$\bar{a}_r(s, r) = \Phi_r(s, r, 0)(\bar{a} - \bar{c}),$$

$$\bar{c}_s(s, r) = \Theta_s(s, r, 0)(\bar{c} - \bar{a}).$$

(2.30)

Let $\sigma = (f, g)$ and $\bar{\sigma} = (\bar{f}, \bar{g})$ be characteristics of the symmetries (2.14) and (2.24). A canonical Lie-Bäcklund symmetry operator with the characteristic $\sigma$ is defined as follows [24]

$$\hat{X}_{\sigma} = f \frac{\partial}{\partial s} + g \frac{\partial}{\partial r} + \left( D_t[f] \right) \frac{\partial}{\partial s_t} + \left( D_t[g] \right) \frac{\partial}{\partial r_t} + \left( D_x[f] \right) \frac{\partial}{\partial s_x} +$$

$$+ \left( D_x[g] \right) \frac{\partial}{\partial r_x} + \left( D^2_{xx}[f] \right) \frac{\partial}{\partial s_{xx}} + \left( D^2_{xx}[g] \right) \frac{\partial}{\partial r_{xx}} + \ldots$$

$$\ldots + \left( D^N_{xx}[f] \right) \frac{\partial}{\partial s^{(N)}_x} + \left( D^N_{xx}[g] \right) \frac{\partial}{\partial r^{(N)}_x} + \ldots$$

(2.31)

Here operator $D_t$ of the total derivative with respect to $t$ is calculated with the use of equations (2.1). The formula for $\hat{X}_{\sigma}$ is obtained by a substitution of $(\bar{f}, \bar{g})$ for $(f, g)$ to the formula (2.31).
The usual Lie commutator

\[ [\hat{X}_\sigma, \hat{X}_{\bar{\sigma}}] = \hat{X}_{[\sigma, \bar{\sigma}]} \equiv \hat{X}_{\bar{\sigma}} \]  

(2.32)

generates the commutator of symmetry characteristics

\[ \bar{\sigma} = [\sigma, \bar{\sigma}] = \sigma' [\sigma] - \sigma' [\sigma]. \]  

(2.33)

Here \( \sigma' \) denotes the operator of the Frechét derivative \[ (\sigma')_{\alpha\beta} = \sum_{j=0}^{\infty} \frac{\partial \sigma_{\alpha}}{\partial u^{\beta(x)}(x)} D_x^j (\alpha, \beta = 1, 2), \]  

(2.34)

where \( \sigma_1 = f, \sigma_2 = g, u^1 = s, u^2 = r \). The commutator \( [\sigma, \bar{\sigma}] \) of symmetry characteristics for system \( (2.1) \) is again a characteristic \( \tilde{\sigma} = (\tilde{f}, \tilde{g}) \) of some symmetry for this system generated by the Lie equations with the parameter \( \tilde{\tau}: s_{\tilde{\tau}} = \tilde{f}, r_{\tilde{\tau}} = \tilde{g} \) where

\[ \tilde{f} = [\sigma, \bar{\sigma}]_1 = f \frac{\partial f}{\partial s} - f \frac{\partial \tilde{f}}{\partial s} + g \frac{\partial f}{\partial r} - g \frac{\partial \tilde{f}}{\partial r} + D_x[f] \frac{\partial \tilde{f}}{\partial s_x} - D_x[f] \frac{\partial \tilde{f}}{\partial r_x}, \]  

(2.35)

\[ \tilde{g} = [\sigma, \bar{\sigma}]_2 = f \frac{\partial g}{\partial s} - f \frac{\partial \tilde{g}}{\partial s} + g \frac{\partial g}{\partial r} - g \frac{\partial \tilde{g}}{\partial r} + D_x[g] \frac{\partial \tilde{g}}{\partial s_x} - D_x[g] \frac{\partial \tilde{g}}{\partial r_x}, \]  

(2.36)

**Theorem 2.2.** (see [13]).

A commutator of hydrodynamic symmetries \( (2.24) \) and \( (2.14) \) with the characteristics \( \bar{\sigma} \) and \( \sigma \) is again a hydrodynamic symmetry of system \( (2.1) \) with the characteristic \( \tilde{\sigma} = [\sigma, \bar{\sigma}] = (\tilde{f}, \tilde{g}) \), which is generated by Lie equations

\[ s_{\tilde{\tau}} = \tilde{f} = \tilde{\phi}(x, t, s, r)s_x + \tilde{b}(s), \]

\[ r_{\tilde{\tau}} = \tilde{g} = \tilde{\psi}(x, t, s, r)r_x + \tilde{d}(r) \]  

(2.37)

where the coefficients are determined by the following formulae

if \( \beta \neq 0 \)

\[ \tilde{\phi}(x, t, s, r) = \tilde{a}(s, r) \exp \left\{ \beta \left[ x + \int_0^t \phi(s, r, t)dt \right] \right\} + \frac{1}{\beta} \tilde{\phi}(s, r, t), \]

\[ \tilde{\psi}(x, t, s, r) = \tilde{c}(s, r) \exp \left\{ \beta \left[ x + \int_0^t \psi(s, r, t)dt \right] \right\} + \frac{1}{\beta} \tilde{\psi}(s, r, t), \]  

(2.38)
and if $\beta = 0$
\[
\tilde{\phi}(x,t,s,r) = \tilde{a}(s,r) + \int_0^t \tilde{\phi}(s,r,t)dt - \\
-\tilde{\Phi}(s,r) \left[ x + \int_0^t \phi(s,r,t)dt \right],
\]
\[
\tilde{\psi}(x,t,s,r) = \tilde{c}(s,r) + \int_0^t \tilde{\psi}(s,r,t)dt - \\
-\tilde{\Theta}(s,r) \left[ x + \int_0^t \psi(s,r,t)dt \right]
\]
and for any value of $\beta$
\[
\tilde{b}(s) = \begin{vmatrix} \tilde{b}(s) & b(s) \\ \tilde{b}'(s) & b'(s) \end{vmatrix}, \quad \tilde{d}(r) = \begin{vmatrix} \tilde{d}(r) & d(r) \\ d'(r) & d'(r) \end{vmatrix}.
\]

Here the following notation is used
\[
\tilde{\phi}(s,r,t) = \tilde{b}(s)\phi_s(s,r,t) + \tilde{d}(r)\phi_r(s,r,t),
\]
\[
\tilde{\psi}(s,r,t) = \tilde{b}(s)\psi_s(s,r,t) + \tilde{d}(r)\psi_r(s,r,t).
\]
\[
\tilde{\Phi}(s,r,t) = \tilde{b}(s)\Phi_s + \tilde{d}(r)\Phi_r - b(s)\hat{\Phi}_s - d(r)\hat{\Phi}_r,
\]
\[
\tilde{\Theta}(s,r,t) = \tilde{b}(s)\Theta_s + \tilde{d}(r)\Theta_r - b(s)\hat{\Theta}_s - d(r)\hat{\Theta}_r,
\]
\[
\tilde{a}(s,r) = \tilde{b}(s)[a_s(s,r) - \Phi_s a(s,r)] - b(s)[\tilde{a}_s(s,r) - \Phi_s \tilde{a}(s,r)] + \\
+\Phi_r[d(r)c_s(s,r) - \tilde{d}(r)c(s,r)] + \Phi_0(s)a(s,r) - \tilde{\Phi}_0(s)\tilde{a}(s,r),
\]
\[
\tilde{c}(s,r) = \tilde{d}(r)[c_r(s,r) - \Theta_r c(s,r)] - d(r)[\tilde{c}_r(s,r) - \Theta_r \tilde{c}(s,r)] + \\
+\Theta_s[b(s)\tilde{a}(s,r) - \tilde{b}(s)a(s,r)] + \Theta_0(r)\tilde{c}(s,r) - \tilde{\Theta}_0(r)c(s,r),
\]

$\Phi = \Phi(s,r,0), \Theta = \Theta(s,r,0)$ if $\beta \neq 0$ and $\Phi = \Phi(s,r), \Theta = \Theta(s,r)$ if $\beta = 0$.

**Corollary 2.2.** The formulae (2.42) and (2.43) determine recursions of solutions for the linear systems (2.13) and (2.17) respectively. They map any pair of solutions of the corresponding system again into its solution.

Consider now a special case
\[
b(s) = d(r) = 0, \quad \hat{\Phi} = \hat{\Theta} = c_0 = 0, \quad (\tilde{b}(s), \tilde{d}(r)) \neq (0,0).
\]

**Corollary 2.3.** Let the condition 1 of the theorem 2.1 be satisfied for the system (2.1) and inhomogeneous in derivatives hydrodynamic symmetry of the form (2.24) exist. Then the Lie commutator of the symmetry (2.24) and a homogeneous in derivatives hydrodynamic symmetry of the form (2.20)–(2.22) with $c_0 = 0$ is again a homogeneous symmetry of the same form
\[
s_\tau = \tilde{f} = \tilde{\phi}(x,t,s,r)s_x,
\]
\[
r_\tau = \tilde{g} = \tilde{\psi}(x,t,s,r)r_x
\]
where for any real value of \( \beta \)
\[
\tilde{\varphi}(x, t, s, r) = \tilde{a}(s, r) \exp \left\{ \beta \left[ x + \int_0^t \varphi(s, r, t) \, dt \right] \right\}, \\
\tilde{\psi}(x, t, s, r) = \tilde{c}(s, r) \exp \left\{ \beta \left[ x + \int_0^t \psi(s, r, t) \, dt \right] \right\},
\]
(2.46)
and
\[
\tilde{a}(s, r) = \bar{b}(s) a(s, r) - \left[ \bar{b}(s) \Phi_s(s, r, 0) + \Phi_0(s) \right] a(s, r) - \\
\quad - d(r) [\Phi_x(s, r, 0) c(s, r) - d(r) \Theta_s(s, r, 0) c(s, r)] - \\
\quad - \bar{b}(s) [\Theta_x(s, r, 0) a(s, r) - d(r) \Theta_x(s, r, 0) a(s, r)],
\]
(2.47)

Thus, the Lie commutator with the inhomogeneous in derivatives symmetry (2.24) is a linear operator acting on the space of homogeneous hydrodynamic symmetries with \( c_0 = 0 \). It generates the recursion (2.47) of solutions for the linear system (2.12).

**Remark 2.5.** Inhomogeneous in derivatives hydrodynamic symmetry generates via Lie commutator a recursion operator for homogeneous hydrodynamic symmetries.

**Corollary 2.4.** If the system (2.1) satisfies the condition 1 of the theorem 2.1, then it possesses an infinite set of mutually commuting hydrodynamic symmetries with a functional arbitrariness which are homogeneous in derivatives with \( c_0 = 0 \) and have the form (2.45), (2.46).

In particular, if for \( \beta = 0 \) the coefficients \( \varphi(s, r), \psi(s, r) \) of the system (2.1) do not depend explicitly upon \( t \), then we reproduce Tsarev’s result [3] about the commutability of hydrodynamic-type flows without explicit dependence on \( t, x \).

### 2.3 Hydrodynamic symmetries of diagonal systems with explicit spatial dependence

Consider a diagonal two-component hydrodynamic-type system with explicit dependence on \( x \)
\[ s_t = \phi^*(s, r, x) s_x, \quad r_t = \psi^*(s, r, x) r_x \]
(2.48)

Define functions \( \Phi(s, r, x), \Theta(s, r, x) \) by the following equations
\[
\Phi_x(s, r, x) = \left[ \varphi^{*-1}(s, r, x) \right]_r / \left[ \varphi^{*-1}(s, r, x) - \psi^{*-1}(s, r, x) \right], \\
\Theta_x(s, r, x) = \left[ \psi^{*-1}(s, r, x) \right]_s / \left[ \psi^{*-1}(s, r, x) - \phi^{*-1}(s, r, x) \right]
\]
(2.49)

**Theorem 2.3.** A diagonal generic hydrodynamic-type system (2.48) which may explicitly depend on a space coordinate \( x \) possesses an infinite set of hydrodynamic symmetries with a functional arbitrariness locally parametrized by two arbitrary functions \( c_1(s), c_2(r) \) of one variable iff the two conditions are satisfied
1. coefficients \( \phi^*, \psi^* \) of the system (2.48) satisfy the equalities
\[
\Phi_{sx}(s, r, x) = \beta \left( \phi^{*-1}(s, r, x) \right)_r, \\
\Theta_{sx}(s, r, x) = \beta \left( \psi^{*-1}(s, r, x) \right)_s
\]
(2.50)

with arbitrary real constant \( \beta \);
2. there exist such four functions of one variable \( b(s), d(r), \Phi_0(s), \Theta_0(r) \) which satisfy the equations
\[
\begin{align*}
\hat{\Phi}_r(s, r, x) &= \Phi_r(s, r, x)(\hat{\Phi} - \hat{\Theta}), \\
\hat{\Theta}_s(s, r, x) &= \Theta_s(s, r, x)(\hat{\Theta} - \hat{\Phi})
\end{align*}
\] (2.51)
where the functions \( \hat{\Phi}(s, r, x), \hat{\Theta}(s, r, x) \) are defined by the formulae
\[
\begin{align*}
\hat{\Phi}(s, r, x) &= b(s)\Phi_s(s, r, x) + d(r)\Phi_r(s, r, x) + \Phi_0(s), \\
\hat{\Theta}(s, r, x) &= b(s)\Theta_s(s, r, x) + d(r)\Theta_r(s, r, x) + \Theta_0(r).
\end{align*}
\] (2.52)
These symmetries are generated by the Lie equations
\[
\begin{align*}
s_r &= \tilde{\phi}^*(x, t, s, r)s_x + b(s), \\
r_r &= \tilde{\psi}^*(x, t, s, r)r_x + d(r)
\end{align*}
\] (2.53)
where the functions \( \tilde{\phi}^*, \tilde{\psi}^* \) are defined as follows
if \( \beta \neq 0 \)
\[
\begin{align*}
\tilde{\phi}^*(x, t, s, r) &= \phi^*(s, r, x) \left\{ a(s, r) \exp \left[ \beta(t + \int_0^x \phi^*(s, r, x)dx) \right] + \frac{1}{\beta} \hat{\Phi}(s, r, x) \right\}, \\
\tilde{\psi}^*(x, t, s, r) &= \psi^*(s, r, x) \left\{ c(s, r) \exp \left[ \beta(t + \int_0^x \psi^*(s, r, x)dx) \right] + \frac{1}{\beta} \hat{\Theta}(s, r, x) \right\}
\end{align*}
\] (2.54)
and if \( \beta = 0 \)
\[
\begin{align*}
\tilde{\phi}^*(x, t, s, r) &= \phi^*(s, r, x) \left\{ a(s, r) + \int_0^x \hat{\phi}(s, r, x)dx - \hat{\Phi}(s, r) \left[ t + \int_0^x \phi^*(s, r, x)dx \right] \right\}, \\
\tilde{\psi}^*(x, t, s, r) &= \psi^*(s, r, x) \left\{ c(s, r) + \int_0^x \hat{\psi}(s, r, x)dx - \hat{\Theta}(s, r) \left[ t + \int_0^x \psi^*(s, r, x)dx \right] \right\}
\end{align*}
\] (2.55)
where the functions \( \hat{\phi}, \hat{\psi} \) are defined as follows
\[
\begin{align*}
\hat{\phi}(s, r, x) &= b(s)[\phi^*(s, r, x)]_s + d(r)[\phi^*(s, r, x)]_r, \\
\hat{\psi}(s, r, x) &= b(s)[\psi^*(s, r, x)]_s + d(r)[\psi^*(s, r, x)]_r.
\end{align*}
\] (2.56)
Here the functions \( a(s, r), c(s, r) \) form an arbitrary smooth solution of the linear system (2.17) with the coefficients \( \Phi_r(s, r, 0), \Theta_s(s, r, 0) \) which are obtained from the formulae (2.49) at \( x = 0 \).

2.4 Invariant solutions and linearizing transformations for systems with an explicit dependence on \( t \) or \( x \)

Let the system (2.1) which may explicitly depend on \( t \) satisfy the condition 1 of the theorem 2.1 and hence possess an infinite set of homogeneous in derivatives hydrodynamic
symmetries (2.20). For the corresponding invariant solutions we have $s_r = r_r = 0$ and taking into account the condition $s_x r_x \neq 0$ we obtain $\tilde{\phi}(x, t, s, r) = 0, \tilde{\psi}(x, t, s, r) = 0$. Using here the formulae (2.21), (2.22) for $\tilde{\phi}, \tilde{\psi}$ with $c_0 = 1$ we obtain the following conditions for invariant solutions

if $\beta \neq 0$

\[
\begin{align*}
    a(s, r) + \exp \left\{ -\beta \left[ x + \int_0^t \phi(s, r, t) dt \right] \right\} &= 0, \\
    c(s, r) + \exp \left\{ -\beta \left[ x + \int_0^t \psi(s, r, t) dt \right] \right\} &= 0,
\end{align*}
\]

(2.57)

and if $\beta = 0$

\[
\begin{align*}
    a(s, r) + x + \int_0^t \phi(s, r, t) dt &= 0, \\
    c(s, r) + x + \int_0^t \psi(s, r, t) dt &= 0.
\end{align*}
\]

(2.58)

**Theorem 2.4** (see [13]). Let the coefficients $\phi, \psi$ of diagonal system (2.1) satisfy the condition 1 of the theorem 2.1. Then any solution of the system (2.57) for $\beta \neq 0$ or of the system (2.58) for $\beta = 0$ is also a solution of the system (2.1) if the following conditions are satisfied for $\beta \neq 0$

\[
\left\{ \left[ \ln |a(s, r)| + \beta \int_0^t \phi(s, r, t) dt \right]_s + \left[ \ln |c(s, r)| + \beta \int_0^t \psi(s, r, t) dt \right]_r \right\}_{(2.57)} \neq 0, \tag{2.59}
\]

and for $\beta = 0$

\[
\left\{ \left[ a(s, r) + \int_0^t \phi(s, r, t) dt \right]_s + \left[ c(s, r) + \int_0^t \psi(s, r, t) dt \right]_r \right\}_{(2.58)} \neq 0. \tag{2.60}
\]

And vice versa any smooth solution $s(x, t), r(x, t)$ of the system (2.1) can be obtained from the systems (2.57) or (2.58) if the condition (2.59) or (2.60) is met in the vicinity of any point $(x_0, t_0)$ where the condition $s_x(x_0, t_0) \cdot r_x(x_0, t_0) \neq 0$ is satisfied.

**Remark 2.6** (see [13]). Equations (2.57) for $\beta \neq 0$ and (2.58) for $\beta = 0$ determine in an implicit form the linearizing point transformation of the nonlinear system (2.1) with explicit $t$-dependence which satisfies the condition 1 of the theorem 2.1. Indeed, a search for solutions $s(x, t), r(x, t)$ of the system (2.1) reduces to a search for solutions $a(s, r), c(s, r)$ of the linear system (2.17) with variable coefficients. In particular, if the coefficients $\phi(s, r), \psi(s, r)$ of the system (2.1) are not explicitly $t$-dependent then the condition 1 of the theorem 2.1 is satisfied with $\beta = 0$ and the equations (2.58) coincide with the classical hodograph transformation

\[
\begin{align*}
    a(s, r) + x + t \phi(s, r) &= 0, \\
    c(s, r) + x + t \psi(s, r) &= 0
\end{align*}
\]

(2.61)

Consider now the system (2.48) with explicit $x$-dependence. Let it satisfy the condition (2.50) of the theorem 2.3 for possessing an infinite set of the hydrodynamic symmetries
(2.53) with \( b(s) = d(r) = 0 \). Then for invariant solutions determined by the equations \( s_\tau = r_\tau = 0 \) with the condition \( s_x r_x \neq 0 \) we obtain

\[
\tilde{\phi}^*(x, t, s, r) = 0, \quad \tilde{\psi}^*(x, t, s, r) = 0.
\](2.62)

Here the functions \( \tilde{\phi}^*, \tilde{\psi}^* \) are determined by the formulae (2.54) for \( \beta \neq 0 \) and (2.53) for \( \beta = 0 \) with \( \tilde{\Phi} = \Phi_0(s), \tilde{\Theta} = \Theta_0(r), \tilde{\phi} = \tilde{\psi} = 0 \). Let the existence conditions for the implicit vector-function determined by the equations (2.62) be met. They are similar to the conditions (2.59), (2.60).

Then the equations (2.62) determine a linearizing transformation for the system (2.48) with explicit dependence on \( x \) and a theorem analogous to the theorem 2.4 is obviously valid.

### 2.5 Higher symmetries of diagonal two-component systems

Higher symmetries of the second order are generated by the Lie equations (2.3) with \( N = 2 \)

\[
s_\tau = f(x, t, s, r, s_x, s_{xx}, r_x, r_{xx}), \quad r_\tau = g(x, t, s, r, s_x, s_{xx}, r_x, r_{xx})
\](2.63)

which are compatible with the system (2.1).

**Definition 2.2.** We call the system (2.1) generic system with respect to second order symmetries if its coefficients \( \phi, \psi \) do not satisfy the constraints

\[
\Phi(s, r, t) = \ln \frac{\phi_s}{c(s,t)\phi + d(s,t)}, \quad \Theta(s, r, t) = \ln \frac{\psi_s}{G(r,t)\psi + H(r,t)},
\]

\[
\Phi_t(s, r, t) = A(s, t)\phi + B(s, t), \quad \Theta_t(s, r, t) = E(r, t)\psi + F(r, t)
\]

(2.64)

with arbitrary smooth functions \( A, B, c, d, E, F, G, H \) and the functions \( \Phi, \Theta \) defined by the equations (2.8).

The analysis of the symmetries (2.63) for the generic system (2.1) produces the following results.

**Theorem 2.5.** A necessary existence condition for second order symmetries of the generic system (2.1) coincides with the necessary and sufficient condition for the system (2.1) to possess an infinite set of homogeneous in derivatives hydrodynamic symmetries, i.e. with the condition 1 of the theorem 2.1.

Define the function \( \Lambda(s, r) \) by the equation

\[
\Lambda_s(s, r) = -\Phi_r(s, r, t)\Theta_s(s, r, t).
\](2.66)

The fact that the function \( \Lambda \) has no explicit independence of \( t \) is a consequence of the equations (2.8), (2.12) and (2.66).

Let \( A(s), C(r), b(s), d(r), \Phi_0(s), \Theta_0(r) \) be arbitrary smooth functions of one variable. Define the functions \( \tilde{\Phi}, \tilde{\Theta} \)

\[
\tilde{\Phi}(s, r, t) = A(s)(\Phi_s^2 - \Phi_{ss} + 2\Lambda_{ss}) + A'(s)\Lambda_s + C(r)(2\Phi_r\Theta_r + \Phi_{rr} - \Phi_{r}^2) + C'(r)\Phi_r + b(s)\Phi_s + d(r)\Phi_r + \Phi_0(s).
\]

\[
\tilde{\Theta}(s, r, t) = A'(s)\Theta_s + A(s)(2\Theta_s\Phi_s + \Theta_{ss} - \Theta_s^2) + C(r)(\Theta_r^2 - \Theta_{rr} + 2\Lambda_{r}r) + C'(r)\Lambda_r(s, r) + b(s)\Theta_s + d(r)\Theta_r + \Theta_0(r).
\](2.67)
If $\beta = 0$ in the equations (2.12), then in virtue of the condition (2.18) the functions $\Phi(s, r), \Theta(s, r)$ do not dependent explicitly on $t$. Then define the functions $\hat{\phi}, \hat{\psi}$

$$
\hat{\phi}(s, r, t) = A(s)(2\Phi_s \phi_s - \phi_{ss}) + C(r)[2\Theta_r \phi_r + \phi_{rr} - 2\Phi_r (\phi_r - \psi_r)] + 
C'(r)\phi_r + b(s)\phi_s + d(r)\phi_r,
$$

$$
\hat{\psi}(s, r, t) = A(s)[2\Phi_s \psi_s + \psi_{ss} - 2\Theta_s (\psi_s - \phi_s)] + A'(s)\psi_s + 
C(r)(2\Theta_r (\psi_r - \psi_r)) + b(s)\psi_s + d(r)\psi_r.
$$

(2.68)

**Theorem 2.6.** (see [15]). A diagonal two-component generic hydrodynamic-type system (2.4) which may explicitly depend on time possesses an infinite set of second order higher symmetries with a functional arbitrariness which is locally parametrized by two arbitrary functions $c_1(s), c_2(r)$ of one variable iff the following two conditions are satisfied:

1. the coefficients $\hat{\phi}, \hat{\psi}$ of the system (2.4) satisfy the equations (2.12) with an arbitrary real constant $\beta$ where the functions $\Phi, \Theta$ are defined by equations (2.8) and partial derivatives with respect to $t$ are taken at constant values of $s$ and $r$;

2. there exist six functions $A(s), C(r), b(s), d(r), \Phi_0(s), \Theta_0(r)$ of one variable which satisfy the equalities

$$
\hat{\Phi}_r = \Phi_r \cdot (\hat{\Phi} - \hat{\Theta}), \quad \hat{\Theta}_s = \Theta_s \cdot (\hat{\Theta} - \hat{\Phi})
$$

(2.69)

with the functions $\hat{\Phi}(s, r, t), \hat{\Theta}(s, r, t)$ defined by the formulae (2.67).

These symmetries are generated by the Lie equations

$$
s_r = f = A(s)\frac{s_{xx}}{s_x^2} + \Phi_r \left[ A(s)\frac{r_x}{s_x} + C(r)\frac{s_x}{r_x} \right] + 
\beta A(s) + s_x \nu(x, t, s, r) + 2A(s)\Phi_s + b(s),
$$

$$
r_r = g = C(r)\frac{r_{xx}}{r_x^2} + \Theta_s \left[ A(s)\frac{r_x}{s_x} + C(r)\frac{s_x}{r_x} \right] + 
\beta C(r) + r_x \rho(x, t, s, r) + 2C(r)\Theta_x + d(r)
$$

(2.70)

with the coefficients $\nu, \rho$ defined by the formulae

if $\beta \neq 0$

$$
\nu(x, t, s, r) = a(s, r) \exp \left\{ \beta \left[ x + \int_0^t \phi(s, r, t) dt \right] \right\} + \frac{1}{\beta} \hat{\Phi}(s, r, t),
$$

$$
\rho(x, t, s, r) = c(s, r) \exp \left\{ \beta \left[ x + \int_0^t \psi(s, r, t) dt \right] \right\} + \frac{1}{\beta} \hat{\Theta}(s, r, t),
$$

(2.71)

and if $\beta = 0$

$$
\nu(x, t, s, r) = a(s, r) + \int_0^t \hat{\phi}(s, r, t) dt - 
\frac{1}{\beta} \hat{\Phi}(s, r) \left[ x + \int_0^t \phi(s, r, t) dt \right],
$$

$$
\rho(x, t, s, r) = c(s, r) + \int_0^t \hat{\psi}(s, r, t) dt - 
\frac{1}{\beta} \hat{\Theta}(s, r) \left[ x + \int_0^t \psi(s, r, t) dt \right].
$$

(2.72)
where the functions $\hat{\phi}, \hat{\psi}$ are defined by the formulae (2.68). Here the integrals with respect to $t$ are taken at constant values of $s, r$. The functions $a(s, r), c(s, r)$ form an arbitrary smooth solution of the linear system (2.17).

Remark 2.7. For $A(s) = C(r) = 0$ the higher symmetries (2.70) reduce to the hydrodynamic symmetries (2.14) and the coefficients $\nu, \rho$ coincide with $\hat{\phi}, \hat{\psi}$.

2.6 First order recursion operators

An effective description of an infinite set of symmetries of any order is obtained by means of recursion operators which by their definition map any symmetry again into a symmetry. Here we consider recursion operators which belong to the class of matrix-differential operators

$$R = A_N D_x^N + A_{N-1} D_x^{N-1} + \ldots + A_1 D_x + A_0, \quad (2.73)$$

where $A_i$ are $n \times n$ matrices for $n$-component system and here $n = 2$. The matrices $A_i$ may depend on $s, r$ and their derivatives of a finite order with respect to $x$.

Definition 2.3. If $A_N \neq 0$, then the integer $N$ is called order of the recursion operator (2.73).

Here we consider the case $N = 1$.

Define the functions

$$S(s, r) = \Phi(s, r, 0) = A(s) \Phi_s(s, r, 0) + C(r) \Phi_r(s, r, 0) + \Phi_0(s),$$
$$T(s, r) = \Theta(s, r, 0) = A(s) \Theta_s(s, r, 0) + C(r) \Theta_r(s, r, 0) + \Theta_0(r) \quad (2.74)$$

with the functions $S(s, r), T(s, r)$ defined by the formulae (2.74) with $\Phi_0(s) = \Theta_0(r) = 0$.

This recursion operator is given by the following formula

$$R = \left\{ \begin{pmatrix} A(s) & 0 \\ 0 & C(r) \end{pmatrix} \cdot (D_x - \beta) - \begin{pmatrix} A(s) D_x [\Phi_s(s, r, t)], & -\Phi_r(s, r, t) [A(s) r_x - C(r) s_x] \\ \Theta_s(s, r, t) [A(s) r_x - C(r) s_x], & C(r) D_x [\Theta(s, r, t)] \end{pmatrix} \right\} \cdot \begin{pmatrix} 1/s_x & 0 \\ 0 & 1/r_x \end{pmatrix}. \quad (2.76)$$

A first example of first order recursion operator for a class of two-component systems was given in the author’s paper [17].

Theorem 2.7 generalizes Teshukov’s results [26] for explicitly $t$-dependent systems.

Theorem 2.8. For the system (2.1) which meets the condition 1 of the theorem 2.1 there exists a first order recursion operator $R$ iff there exists a recursion operator $\hat{R}$ which

\footnote{It is equivalent to the redefinition $\Phi_s(s, r, 0), \Theta_r(s, r, 0)$.}
acts on a subspace of homogeneous in derivatives hydrodynamic symmetries of the form (2.20)–(2.22) with \( C_0 = 0 \). An action of the operator \( \hat{R} \) is defined by the appropriate restriction of \( R \)

\[
\begin{pmatrix}
    a(s, r) \exp \{ \beta [x + \int_0^t \phi(s, r, t) dt] \} s_x \\
    c(s, r) \exp \{ \beta [x + \int_0^t \psi(s, r, t) dt] \} r_x
\end{pmatrix} =
\begin{pmatrix}
    \hat{a}(s, r) \exp \{ \beta [x + \int_0^t \phi(s, r, t) dt] \} s_x \\
    \hat{c}(s, r) \exp \{ \beta [x + \int_0^t \psi(s, r, t) dt] \} r_x
\end{pmatrix} \tag{2.77}
\]

with functions \( \hat{a}(s, r), \hat{c}(s, r) \) defined by the formulae

\[
\begin{align*}
\hat{a}(s, r) &= A(s) [a_s(s, r) - \Phi_s(s, r, 0) a(s, r)] - \Phi_0(s) a(s, r) - C(r) \Phi_r(s, r, 0) c(s, r), \\
\hat{c}(s, r) &= C(r) [c_r(s, r) - \Theta_r(s, r, 0) c(s, r)] - \Theta_0(r) c(s, r) - A(s) \Theta_s(s, r, 0) a(s, r). \tag{2.78}
\end{align*}
\]

**Corollary 2.5.** For any smooth solution \( a(s, r), c(s, r) \) of the linear system (2.17) the functions \( \hat{a}(s, r), \hat{c}(s, r) \) defined by the formulae (2.78) are also a solution of this system iff the condition (2.73) is met. Formulae (2.78) determine a recursion of solutions of the system (2.17).

**Remark 2.8.** In virtue of the linearizing transformations (2.57) or (2.58) a search for any “nonsingular” solutions of the nonlinear system (2.1) reduces to a search for solutions \( a(s, r), c(s, r) \) of the linear system (2.17) with variable coefficients. However, the integration of the equations (2.17) is also a problem. The existence of a recursion operator and hence of the recursion (2.78) which allows to multiply solutions of the linear equations (2.17) is the important property of the system (2.1). It allows us to go over from linearization of the system (2.1) to its integration. For this to be true its coefficients \( \phi, \psi \) have to meet the condition 1 of the theorem 2.1. and the conditions (2.73).

**Remark 2.9.** Linear system (2.17) has two trivial solutions \( (a, c) = (-1, -1) \) and \( (a(s, r), c(s, r)) = (\phi(s, r, 0), \psi(s, r, 0)) \) (see the equations (2.18)). The recursion formulae (2.78) map them to nontrivial solutions \( \hat{a}(s, r), \hat{c}(s, r) \) of the equations (2.17).

In particular, the first solution is mapped to the solution (2.74): \( \hat{a} = S(s, r), \hat{c} = T(s, r) \). Substituting these new solutions for \( a, c \) to the equations (2.18) we obtain new nontrivial solutions of the system (2.17). Thus, we construct two infinite series of its solutions. Then the linearizing transformations (2.57) and (2.58) generate two infinite series of exact solutions of the equations (2.1) in explicit form.

**Corollary 2.6.** The transformation (2.73) maps any smooth solution \( (a, c) \) of the system (2.17) again into a solution \( (\hat{a}, \hat{c}) \) of this system iff it maps the trivial solution \( (a = 1, c = 1) \) into some solution of the equations (2.17).

**Corollary 2.7.** The recursion (2.47) for solutions of the linear system (2.17) generated by the Lie commutator with the inhomogeneous in derivatives hydrodynamic symmetry coincides with the recursion (2.18) generated by the first order recursion operator (2.76). The existence conditions for the recursion operator and for the inhomogeneous
in derivatives hydrodynamic symmetry of the system (2.1) coincide. If there exist several such symmetries of the form (2.24), then there exist several corresponding recursion operators of the form (2.76) with \( A(s) = b(s), C(r) = d(r) \).

Recursion operator \( R \) of the form (2.76) generates an infinite set of higher symmetries (2.3) of any order \( N \). Second order symmetries are generated by a twofold action of the operator \( R \) on homogeneous in derivatives hydrodynamic symmetries (2.20)–(2.22) with \( c_0 \neq 0 \) where we can put \( c_0 = 1 \), \( a(s, r) = c(s, r) = 0 \). For instance, in the case \( \beta = 0 \) the formulae (2.20) and (2.22) give the following form of this initial symmetry

\[
s_r = s_x \left[ x + \int_0^t \phi(s, r, t)dt \right], \quad r_r = r_x \left[ x + \int_0^t \psi(s, r, t)dt \right] \tag{2.79}
\]

### 2.7 Second order recursion operators and higher symmetries

**Definition 2.4.** We call the system (2.1) generic with respect to second order recursion operators if its coefficients do not satisfy any of the following constraints

\[
\frac{\Phi_{rs}}{\Phi_r} + \Theta_s = c_1(s)e^\Phi, \quad \frac{\Theta_{sr}}{\Theta_s} + \Phi_r = c_2(r)e^\Theta \tag{2.80}
\]

with arbitrary smooth functions \( c_1(s), c_2(r) \).

Define the functions

\[
S(s, r) = A(s)(\Phi_r^2 - \Phi_{ss} + 2\Lambda_{ss}) + A'(s)\Lambda_s + C(r)(2\Phi_r\Theta_r + \Phi_{rr} - \Phi_r^2) + C'(r)\Phi_r + b(s)\Phi_s + d(r)\Phi_r + \Phi_0(s),
\]

\[
T(s, r) = A'(s)\Theta_s + A(s)(2\Theta_s\Phi_s + \Theta_{ss} - \Theta_s^2) + C(r)(\Theta_r^2 - \Theta_{rr} + 2\Lambda_{rr}) + C'(r)\Lambda_r + b(s)\Theta_s + d(r)\Theta_r + \Theta_0(r). \tag{2.81}
\]

with arbitrary smooth functions \( A(s), C(r), b(s), d(r), \Phi_0(s), \Theta_0(r) \) and the functions \( \Phi(s, r, t), \Theta(s, r, t), \Lambda(s, r) \) defined by the formulae (2.8) and (2.66).

**Theorem 2.9.** (see [13]). Let the generic system (2.1) satisfy the condition 1 of the theorem 2.1. Then a second order recursion operator of the form (2.73) with \( N = 2 \) exists for the system (2.1) if there exist six functions \( A(s), C(r), b(s), d(r), \Phi_0(s), \Theta_0(r) \) of one variable which satisfy the conditions

\[
S_r(s, r) = \Phi_r(s, r, 0)(S - T), \quad T_r(s, r) = \Theta_s(s, r, 0)(T - S). \tag{2.82}
\]

This recursion operator is determined by the formula

\[
R = (AD_x^2 + BD_x + F) \begin{pmatrix}
\frac{1}{s_x} & 0 \\
0 & \frac{1}{r_x}
\end{pmatrix} \tag{2.83}
\]

with \( 2 \times 2 \) matrices \( A, B \) defined as follows

\[
A = \begin{pmatrix}
\frac{A(s)}{s_x} & 0 \\
0 & \frac{C(r)}{r_x}
\end{pmatrix} \tag{2.84}
\]

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is defined as an appropriate restriction of dynamic symmetries of the form (2.20)–(2.22) with c \neq 0.

There exists a second order recursion operator formulae (2.67). Here \( \Phi = \Phi(s, r, t) \) and the elements \( f_{ij} \) of the matrix \( F \) are defined by the equations

\[
B = \begin{pmatrix}
- A(s) \frac{s_{xx}}{s_x^2} + 2A(s) \left( \Phi_s + \Phi_r \frac{r_x}{s_x} + \frac{\beta}{s_x} \right) + b(s) \\
- \Theta_s(s, r, t) \left[ A(s) \frac{r_x}{s_x} - C(r) \frac{s_x}{r_x} \right] \\
\Phi_r(s, r, t) \left[ A(s) \frac{r_x}{s_x} - C(r) \frac{s_x}{r_x} \right] \\
- \left[ C(r) \frac{r_{xx}}{r_x^2} + 2C(r) \left( \Theta_s \frac{s_x}{r_x} + \Theta_r \frac{r_x}{s_x} + \frac{\beta}{s_x} \right) + d(r) \right]
\end{pmatrix} (2.85)
\]

and the elements \( f_{ij} \) of the matrix \( F \) are defined by the equations

\[
f_{12} = A(s) \left[ - \Phi_r \frac{r_x}{s_x} \left( \frac{s_{xx}}{s_x} - \frac{r_{xx}}{r_x} \right) + (\Phi_{rr} - \Phi_r^2) \frac{r_x^2}{s_x^2} \right] - C(r) \Lambda_{sr} \frac{s_x^2}{r_x^2} + \{ A(s)[2(\Phi_{rs} - \Phi_r \Phi_s) - \Lambda_{sr}] - b(s) \Phi_r \} r_x +
\]

\[
+ \{ C(r)(2 \Phi_r \Theta_r + \Phi_{rr} - \Phi_r^2) + [C'(r) + d(r)] \Phi_r \} s_x -
\]

\[- \beta \Phi_r \left[ A(s) \frac{r_x}{s_x} - C(r) \frac{s_x}{r_x} \right],
\]

\[
f_{21} = C(r) \left[ \Theta_s \frac{s_x}{r_x} \left( \frac{s_{xx}}{s_x} - \frac{r_{xx}}{r_x} \right) + (\Theta_{ss} - \Theta_s^2) \frac{s_x^2}{r_x^2} \right] -
\]

\[- A(s) \Lambda_{sr} \frac{s_x^2}{r_x^2} + \{ C(r)[2(\Theta_{sr} - \Theta_s \Theta_r) - \Lambda_{sr}] - d(r) \Theta_s \} s_x +
\]

\[+ \{ A(s)[2 \Theta_s \Phi_s + \Theta_{ss} - \Theta_s^2] + [A'(s) + b(s)] \Theta_s \} r_x +
\]

\[+ \beta \Theta_s \left[ A(s) \frac{r_x}{s_x} - C(r) \frac{s_x}{r_x} \right],
\]

\[
f_{11} + f_{12} = \hat{\Phi}(s, r, t)s_x + \beta \{ A(s) \left[ \frac{s_{xx}}{s_x^2} + 2 \Phi_s(s, r, t) + \frac{\beta}{s_x} \right] +
\]

\[+ \Phi_r(s, r, t) \left[ A(s) \frac{r_x}{s_x} + C(r) \frac{s_x}{r_x} \right] + b(s) \},
\]

\[
f_{21} + f_{22} = \hat{\Theta}(s, r, t)r_x + \beta \{ C(r) \left[ \frac{r_{xx}}{r_x^2} + 2 \Theta_r(s, r, t) + \frac{\beta}{r_x} \right] +
\]

\[+ \Theta_s(s, r, t) \left[ A(s) \frac{r_x}{s_x} + C(r) \frac{s_x}{r_x} \right] + d(r) \}.
\]

Here \( \Phi = \Phi(s, r, t) \), \( \Theta = \Theta(s, r, t) \) and the functions \( \hat{\Phi}(s, r, t) \), \( \hat{\Theta}(s, r, t) \) are defined by the formulæ (2.67).

**Theorem 2.10.** For the system (2.1) which satisfies the condition 1 of the theorem 2.1 there exists a second order recursion operator \( \hat{R} \) of the form (2.83) iff there exists the recursion operator \( R \) which acts on a subspace of homogeneous in derivatives hydrodynamic symmetries of the form (2.20)–(2.22) with \( c_0 = 0 \). The action of the operator \( \hat{R} \) is defined as an appropriate restriction of \( R \) by the formula (2.67) where the functions
\( \hat{a}(s, r), \hat{c}(s, r) \) are defined as follows

\[
\begin{align*}
\hat{a} &= A(s)a_{ss} - [2A(s)\Phi_s + b(s)]a_s - C(r)\Phi_r c_r + \\
&+ [A'(s)\Lambda_s + A(s)(\Phi^2_s - \Phi_{ss} + 2\Lambda_{ss}) + b(s)\Phi_s + \Phi_0(s)]a + \\
&+ [C'(r)\Phi_r + C(r)(2\Phi_r\Theta_r + \Phi_{rr} - \Phi^2_r) + d(r)\Phi_r]c,
\end{align*}
\]

(2.88)

Corollary 2.8. For any smooth solution \( a(s, r), c(s, r) \) of the linear system (2.17) the functions \( \hat{a}(s, r), \hat{c}(s, r) \) defined by the formulae (2.88) form also a solution of this system iff the condition (2.82) for the functions (2.81) is satisfied. The formulae (2.88) determine a recursion of solutions for the system (2.17).

The Remarks 2.8 and 2.9 and the Corollary 2.6 are still valid for the recursion (2.88). Thus, again we can construct two infinite series of solutions of the system (2.1) starting from a trivial solution of the system (2.17).

Theorem 2.11. Existence conditions (2.82) with the notation (2.81) for the second order recursion of solutions of the system (2.17) are less restrictive than the existence conditions (2.74) with the notation (2.73) for the first order recursion (2.78).

Higher symmetries of the system (2.1) are generated by the action of the recursion operator (2.83) on the homogeneous in derivatives hydrodynamic symmetries (2.21)–(2.22) with \( c_0 \neq 0 \).

Theorem 2.12. (see [15]). Second order symmetries (2.70) of the system (2.1) obtained as a general solution of determining equations coincide with the second order symmetries obtained by the action of the second order recursion operator (2.83) on the hydrodynamic symmetries (2.21)–(2.22) with \( c_0 \neq 0 \). Existence conditions for the second order higher symmetries and for the second order recursion operator also coincide.

Corollary 2.9. All the second order symmetries may be obtained by an action of the second order recursion operator on the homogeneous in derivatives hydrodynamic symmetries.

Remark 2.10. The method of calculation of higher order recursion operators developed in [14] is much more simple than a straightforward calculation of higher symmetries from determining equations. Thus, with the suitable extension of the corollary 2.9 we see that this easier way of calculation of symmetries by means of the \( N \)th-order recursion operators gives all higher symmetries of the same order. In particular, a squared first order recursion operator does not produce a general form of second order symmetries.

3 Generalized gas dynamics equations
3.1 Symmetries of one-dimensional isoentropic gas dynamics equations

We consider one-dimensional gas dynamics equations for the isoentropic plane-parallel gas flow

\[ u_t + uu_x + \alpha^2(\rho)\rho x = 0, \]
\[ \rho_t + \rho u_x + u\rho = 0. \]  

(3.1)

Here \( u(x, t), \rho(x, t) \) are gas velocity and density at the point \( x \) at the moment \( t \), \( c = \rho \alpha(\rho) \) is the speed of sound, \( \alpha(\rho) \) is an arbitrary smooth function. In practice \( \alpha(\rho) \) is determined by the gas state equation \( p = P(\rho) \), where \( p \) is a gas pressure: \( \alpha(\rho) = \frac{1}{\rho} \sqrt{P'(\rho)}. \)

**Theorem 3.1.** (see [3].) The system (3.1) can be reduced to diagonal form (2.1)

\[ s_t = \phi(s, r)s_x, \quad r_t = \psi(s, r) r_x, \]
\[ \phi(s, r) = -\left[ \frac{s + r}{2} - \rho \alpha(\rho) \right], \quad \psi(s, r) = -\left[ \frac{s + r}{2} + \rho \alpha(\rho) \right] \]  

by the transformation to Riemann invariants \( s, r \)

\[ s = u - \int_{\rho_0}^\rho \alpha(\rho)d\rho, \quad r = u + \int_{\rho_0}^\rho \alpha(\rho)d\rho \]  

(3.3)

where \( \rho_0 \) is an arbitrary fixed constant. The inverse transformation is given by the formulae

\[ u = \frac{r + s}{2}, \quad \int_{\rho_0}^\rho \alpha(\rho)d\rho = \frac{r - s}{2} \]  

(3.4)

where the last equality determines \( \rho \) as an implicit function \( \rho = \rho(r - s) \) for any fixed choice of the function \( \alpha(\rho) \).

The determining equation for symmetries of system (3.2) has the form

\[ (ID_t + A)(f, g)^T = (0, 0)^T \]  

(3.5)

where \( I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \), \( T \) means a transposed matrix, the operator of the total derivative \( D_t \) is calculated with the use of the equations (3.2) and the operator \( A \) has the form

\[ A = \begin{pmatrix} \frac{s + r}{2} - \rho \alpha(\rho), & 0 \\ 0 & \frac{s + r}{2} + \rho \alpha(\rho) \end{pmatrix} D_x + \]
\[ + \begin{pmatrix} s_x & 0 \\ 0 & r_x \end{pmatrix} \left( I + \frac{\rho \alpha'(\rho)}{2\alpha(\rho)} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \right). \]

(3.6)

Canonical Lie-Bäcklund symmetries (2.3), (2.4) of \( N \)th order for \( N = 1, 2, 3 \) were obtained as general solutions of the determining equation (3.3) for a generic \( \alpha(\rho) \) (see [16, 17]) explicitly independent of \( x, t \).
$N = 1$:
\[ f_1 = a(s, r)s_x, \quad g_1 = c(s, r)r_x \]  
(3.7)
where $a(s, r), c(s, r)$ is an arbitrary smooth solution of the linear system
\[ a_s(s, r) = \frac{\alpha'(\rho)}{4\alpha^2}(a - c), \quad c_s(s, r) = \frac{\alpha'(\rho)}{4\alpha^2}(a - c), \]  
(3.8)

$N = 2$:
\[ f_2 = -\frac{s_{sx}}{s_x^2} - \frac{\alpha'(\rho)}{4\alpha^2} \left( \frac{s_x - r_x}{s_x r_x} \right), \]  
\[ g_2 = -\frac{r_{sx}}{r_x^2} + \frac{\alpha'(\rho)}{4\alpha^2} \left( \frac{s_x - r_x}{s_x r_x} \right), \]  
(3.9)

$N = 3$:
\[ f_3 = -\left( \frac{s_{xxx}}{s_x^3} - \frac{3s_{sx}^2}{s_x^4} \right) - \frac{\alpha'(\rho)}{4\alpha^2} \left( \frac{1}{s_x^3} - \frac{1}{r_x^3} \right) s_x r_{xx} - \frac{3\alpha'(\rho)}{4\alpha^2} \frac{s_{xx}}{s_x^3} (s_x - r_x) - \left[ \frac{1}{2} \left( \frac{\alpha'}{\alpha^2} \right)^2 \left( \frac{1}{s_x} + \frac{1}{r_x} \right) - \left( \frac{\alpha'}{\alpha^2} \right)' \frac{1}{\alpha s_x} \right] \frac{(s_x - r_x)^3}{8s_x r_x}, \]  
\[ g_3 = -\left( \frac{r_{xxx}}{r_x^3} - \frac{3r_{sx}^2}{r_x^4} \right) - \frac{\alpha'(\rho)}{4\alpha^2} \left( \frac{1}{s_x^3} - \frac{1}{r_x^3} \right) r_x s_{xx} - \frac{3\alpha'(\rho)}{4\alpha^2} \frac{r_{xx}}{r_x^3} (s_x - r_x) + \left[ \frac{1}{2} \left( \frac{\alpha'}{\alpha^2} \right)^2 \left( \frac{1}{s_x} + \frac{1}{r_x} \right) - \left( \frac{\alpha'}{\alpha^2} \right)' \frac{1}{\alpha r_x} \right] \frac{(s_x - r_x)^3}{8s_x r_x}. \]  
(3.10)

Here every $N$th-order symmetry is presented up to addition of lower-order symmetries. Hence all the higher symmetries have a functional arbitrariness determined by the linear system (3.8) since we can add to them the terms (3.7).

Analysis of the determining equation (3.3) for second order ($N = 2$) symmetries produces the following special choices of $\alpha(\rho)$ which lead to extensions of the set of symmetries.

1. Function $\alpha(\rho)$ satisfies the differential equation
\[ \left[ (c \rho + b)\alpha^2(\rho)/\alpha'(\rho) \right]' = -\frac{A}{2} \alpha(\rho) \]  
(3.11)
with arbitrary constants $A, b, c$. The second order symmetries are determined by the formula
\[ f_2 = (As + \bar{A}) \frac{s_{xx}}{s_x^2} + \frac{\alpha'(\rho)}{4\alpha^2} \left[ (As + \bar{A}) \left( \frac{r_x}{s_x} - 2 \right) + (Ar + \bar{C}) \frac{s_x}{r_x} \right] + b \]  
\[ + a(s, r)s_x, \]  
\[ g_2 = (Ar + \bar{C}) \frac{r_{xx}}{r_x^2} - \frac{\alpha'(\rho)}{4\alpha^2} \left[ (Ar + \bar{C}) \left( \frac{s_x}{r_x} - 2 \right) + (As + \bar{A}) \frac{r_x}{s_x} \right] + b \]  
\[ + c(s, r)r_x, \]  
(3.12)
with arbitrary constants $A, \bar{A}, \bar{C}, b$ and functions $a(s, r), c(s, r)$ satisfying the linear system (3.5).
In particular, the equation (3.11) is satisfied for \( c = 0 \) by the physically interesting state equation of a polytropic gas

\[
P(\rho) = a^2 \rho^\gamma, \quad \alpha(\rho) = a\sqrt{\gamma}\rho^{(\gamma-3)/2}
\]

(3.13)

where \( a, \gamma \) are constants.

2. Function \( \alpha(\rho) \) satisfies the condition \( \alpha'(\rho) = 0 \). This is a polytropic gas with \( \gamma = 3 \). Then the second order symmetries

\[
f_2 = s_x \psi_1 \left( \frac{s_{xx}}{s_x^3}, s \right), \quad g_2 = r_x \psi_2 \left( \frac{r_{xx}}{r_x^3}, r \right)
\]

(3.14)

depend on two arbitrary smooth functions \( \psi_1, \psi_2 \).

The gas dynamics equations (3.2) have the form

\[
s_t = -ss_x, \quad r_t = -rr_x.
\]

(3.15)

Their general solution

\[
x - st = F(s), \quad x - rt = G(r)
\]

(3.16)

depends on two arbitrary smooth functions \( F \) and \( G \). Hence \( s_\tau = r_\tau = 0 \) and the solution manifold consists solely of invariant solutions.

3. The function \( \alpha(\rho) \) satisfies the condition \( \alpha'(\rho) = -(2/\rho)\alpha(\rho) \). This is the Chaplygin gas \([27]\) with the state equation

\[
P(\rho) = P_0 - \frac{a^2}{\rho}, \quad \alpha(\rho) = \frac{a}{\rho^2} \quad (P_0 > 0)
\]

(3.17)

with the constants \( P_0, a \). Then the set of second order symmetries depends again on two arbitrary functions \( \psi_1(s, s_q), \psi_2(r, r_q) \) where \( q \) is the Lagrangian coordinate defined by the equation \[ dq = \rho dx - p dt. \]

(3.18)

The diagonal form (3.2) of gas dynamics equations after the transformation to the Lagrangian coordinates \( q, t \) becomes

\[
s_t = as_q, \quad r_t = -ar_q.
\]

(3.19)

Its general solution

\[
s = F(q + at), \quad r = G(q - at)
\]

(3.20)

depends on two arbitrary smooth functions \( F \) and \( G \). Since \( x_\tau = t_\tau = 0 \) and hence \( q_\tau = 0 \), we have \( s_\tau = r_\tau = 0 \) as well and the solution manifold again consists solely of invariant solutions.

In the last two cases the reason for the gas dynamics equations to be integrable in explicit form is that the extent of arbitrariness of the set of invariant solutions and of the general solution manifold turns out to be the same, i.e. two arbitrary functions of one variable. Hence all the (nonsingular) solutions are invariant.
3.2 Recursion operators for gas dynamics equations

Theorem 3.2. A first order recursion operator for symmetries of the gas dynamics equations (3.2) is given by the formula

\[ R = \left( ID_x - \frac{\alpha_x(\rho)}{2\alpha} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \right) \begin{pmatrix} \frac{1}{s_x} & 0 \\ 0 & \frac{1}{r_x} \end{pmatrix}. \]  

(3.21)

It commutes with the operator of the determining equation (3.5) on the solution manifold of the equation (3.5)

\[ [ID_t + A, R] = 0 \]  

(3.22)

where the operator \( D_t \) is calculated with the use of the gas dynamics equations.

Corollary 3.1. Operator (3.21) raises the order of higher symmetries by one unit according to the recursion formula

\[ R(f_N, g_N)^T = (f_{N+1}, g_{N+1})^T, \quad (N = 2, 3, \ldots) \]  

(3.23)

and thus, generates an infinite countable set containing Lie-Bäcklund symmetries of any order.

We note the equalities

\[ \begin{pmatrix} f_2 \\ g_2 \end{pmatrix} = R \begin{pmatrix} 1 \\ 1 \end{pmatrix} = R^2 x \begin{pmatrix} s \\ r \end{pmatrix}_x. \]  

(3.24)

Corollary 3.2. A solution of the recursion relation (3.23) has the form

\[ \begin{pmatrix} f_N \\ g_N \end{pmatrix} = R^{N-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = R^N x \begin{pmatrix} s \\ r \end{pmatrix}_x \quad (N = 2, 3, \ldots). \]  

(3.25)

Formula (3.21) gives us a general form of the first order recursion operator for a generic function \( \alpha(\rho) \). If the function \( \alpha(\rho) \) satisfies the equation

\[ \left( \frac{\alpha'}{\alpha^2} \right)'[A(r - s) + \bar{c} - \tilde{a}] = -2A \frac{\alpha'}{\alpha} \]  

with arbitrary constant \( A, \bar{a}, \tilde{c} \), then we obtain a more general form of a recursion operator.

In particular, the equation (3.26) is satisfied by the state equation (3.13) of the polytropic gas if the constants satisfy the equations

\[ A = -\frac{\gamma - 1}{4a\sqrt{\gamma}\rho_0^{(\gamma - 1)/2}}, \quad (\rho_0 \neq 0), \]  

(3.27)

\[ \bar{a} = \tilde{c} \quad (\rho_0 = 0). \]

Theorem 3.3. For equations of the polytropic gas there exists a first order recursion operator which depends upon one essential arbitrary constant \( \bar{a}/\tilde{c} \) \( (\rho_0 \neq 0) \) or
\( \bar{a}/A \ (\rho_0 = 0) \) and has the form

\[
R = \begin{pmatrix}
As + \bar{a} & 0 \\
0 & Ar + \bar{c}
\end{pmatrix} D_x - \frac{\alpha'(\rho)}{4\alpha^2} \begin{pmatrix}
(As + \bar{a})(r_x - s_x) & -[(As + \bar{a})r_x - (Ar + \bar{c})s_x] \\
-(As + \bar{a})r_x - (Ar + \bar{c})s_x & (Ar + \bar{c})(r_x - s_x)
\end{pmatrix} \cdot 
\begin{pmatrix}
\frac{1}{s_x} & 0 \\
0 & \frac{1}{r_x}
\end{pmatrix}
\]

(3.28)

where \( \frac{\alpha'(\rho)}{4\alpha^2} = (\gamma - 3)/[8a\sqrt{\gamma\rho(\gamma - 1)/2}] \) and the constants \( A, \bar{a}, \bar{c} \) satisfy the equations (3.27).

Now consider the action of the recursion operator (3.21) on the subspace of hydrodynamic symmetries (3.7) subject to conditions (3.8)

\[
R \begin{pmatrix}
a(s, r)s_x \\
c(s, r)r_x
\end{pmatrix} = \begin{pmatrix}
a_1(s, r)s_x \\
c_1(s, r)r_x
\end{pmatrix}.
\]

(3.29)

Here the functions \( a_1, c_1 \) are generated from \( a, c \) by the transformation

\[
a_1(s, r) = a_s(s, r) + [\alpha'(\rho)/(4\alpha^2)](a - c),
\]

\[
c_1(s, r) = c_r(s, r) + [\alpha'(\rho)/(4\alpha^2)](a - c).
\]

(3.30)

**Corollary 3.3.** The transformation (3.30) determines a recursion for solutions of the linear system (3.8), i.e. if \( a(s, r), c(s, r) \) is its solution, then \( a_1(s, r), c_1(s, r) \) is also the solution.

From the equation (3.29) we obtain

\[
R^N \begin{pmatrix}
a(s, r)s_x \\
c(s, r)r_x
\end{pmatrix} = \begin{pmatrix}
a_N(s, r)s_x \\
c_N(s, r)r_x
\end{pmatrix} \quad (N = 1, 2, \ldots)
\]

(3.31)

where \( a_N(s, r), c_N(s, r) \) is a result of \( N \)-fold application of the transformation (3.30) to the solution \( a(s, r), c(s, r) \) of the equations (3.8). Then \( a_N(s, r), c_N(s, r) \) is also a solution of the equations (3.8).

### 3.3 Generalized gas dynamics equations, their symmetries and recursion operators

In the Lie equations for hydrodynamic symmetries (3.7) of gas dynamics equations (3.2)

\[
s_r = a(s, r)s_x, \quad r_r = c(s, r)r_x
\]

(3.32)

subject to the conditions (3.8) we consider the group parameter \( \tau \) as a new time variable. Then we change the notation \( a(s, r), c(s, r) \) to \( \phi(s, r), \psi(s, r) \). We obtain systems of the form (2.1) subject to the additional constraints

\[
s_t = \phi(s, r)s_x, \quad r_t = \psi(s, r)r_x,
\]

\[
\phi_r(s, r) = \psi_s(s, r) = [\alpha'(\rho)/4\alpha^2](\phi - \psi)
\]

(3.33)
i.e. \((\phi, \psi)\) is an arbitrary smooth solution of the linear system (3.8).

Equations (3.33) appeared for the first time in the papers of the author \([7, 18]\) and later Olver and Nutku had called them generalized gas dynamics (GGD) equations \([1]\). They also have pointed out many interesting applications of these equations in physics.

The determining equation for symmetries of any of GGD systems (3.33) has the form similar to the equation (3.5)

\[
(ID_t + A(\phi, \psi))(f, g)^T = (0, 0)^T
\]

where the operator \(A(\phi, \psi)\) is defined by the formula

\[
A(\phi, \psi) = -\left( \begin{array}{cc}
\phi(s, r) & 0 \\
0 & \psi(s, r)
\end{array} \right) D_x + \\
+ \left( \begin{array}{cc}
s_x & 0 \\
0 & r_x
\end{array} \right) \left\{ -\phi_1(s, r) & 0 \\
0 & \psi_1(s, r) + \frac{\alpha'(\rho)}{4\alpha^2}(\phi - \psi)
\right\}.
\]

(3.35)

The functions \(\phi_1, \psi_1\) are generated from \(\phi, \psi\) by the transformation (3.30)

\[
\phi_1(s, r) = \phi_s(s, r) + \frac{\alpha'(\rho)}{4\alpha^2}(\phi - \psi), \\
\psi_1(s, r) = \psi_r(s, r) + \frac{\alpha'(\rho)}{4\alpha^2}(\phi - \psi)
\]

(3.36)

and hence the functions \(\phi_1, \psi_1\) satisfy the equations (3.33).

**Theorem 3.4.** For the GGD equations (3.33) all the hydrodynamic symmetries homogeneous in derivatives and explicitly independent of \(x, t\) are generated by the Lie equations (3.32) with the coefficients \(a(s, r), c(s, r)\) satisfying the linear system (3.8) and hence coincide with the hydrodynamic symmetries of gas dynamics equations.

Thus for any GGD system hydrodynamic symmetries are generated by the Lie equations which belong to the same GGD hierarchy but have a different time variable \(\tau\).

**Theorem 3.5.** Operator \(R\) defined by the formula (3.21) is a recursion operator for symmetries of the whole infinite GGD hierarchy (3.33). It commutes with the operator of determining equation (3.34) on the solution manifold of the equation (3.34)

\[
[ID_t + A(\phi, \psi), R] = 0
\]

(3.37)

where the operator \(D_t\) is calculated with the use of GGD equations.

**Theorem 3.6.** For generalized gas dynamics equations (3.33) with the coefficients \(\phi(s, r), \psi(s, r)\) all the hydrodynamic symmetries are generated by the Lie equations

\[
s_{\tau} = a(s, r)s_x - \lambda[x + t\phi(s, r)]s_x + c_0[1 + t\phi_1(s, r)s_x], \\
r_{\tau} = c(s, r)r_x - \lambda[x + t\psi(s, r)]r_x + c_0[1 + t\psi_1(s, r)r_x]
\]

(3.38)

where the functions \(a(s, r), c(s, r)\) satisfy the equations (3.8) and \(\lambda, c_0\) are arbitrary constants.

**Theorem 3.7.** Generalized gas dynamics equations with the coefficients \(\phi(s, r), \psi(s, r)\) have an infinite series of higher symmetries of any order \(N = 2, 3, \ldots\) with a functional
By the Lie equations and an explicit $t$-dependence. Symmetries of the order $N = 2$ are generated by the Lie equations

$$
\begin{align*}
    s_r &= a_1(s, r)s_x - \lambda[1 + t\phi_1(s, r)s_x] + c_0[t\phi_2(s, r)s_x + f_2], \\
    r_r &= c_1(s, r)r_x - \lambda[1 + t\psi_1(s, r)r_x] + c_0[t\psi_2(s, r)r_x + g_2].
\end{align*}
$$

(3.39)

Here the functions $a_1, c_1$ and $\phi_1, \psi_1$ are obtained by the transformations (3.30) and (3.36) from the functions $a, c$ and $\phi, \psi$ respectively. The functions $\phi_2, \psi_2$ are obtained by twofold application of the transformation (3.36) to the functions $\phi, \psi$ and the functions $a_1(s, r), c_1(s, r)$ form an arbitrary smooth solution of the equations (3.8). The functions $f_2, g_2$ are defined by the equations (3.9). $(N + 1)$th-order symmetries $(N + 1 \geq 3)$ are generated by the Lie equations

$$
\begin{align*}
    \left( \begin{array}{c} s \\ r \end{array} \right) &= \left( \begin{array}{c} a_N(s, r)s_x \\ c_N(s, r)r_x \end{array} \right) - \lambda \left\{ \left( \begin{array}{c} f_N \\ g_N \end{array} \right) + t \left( \begin{array}{c} \phi_N(s, r)s_x \\ \psi_N(s, r)r_x \end{array} \right) \right\} \\
    &+ c_0 \left\{ \left( \begin{array}{c} f_{N+1} \\ g_{N+1} \end{array} \right) + t \left( \begin{array}{c} \phi_{N+1}(s, r)s_x \\ \psi_{N+1}(s, r)r_x \end{array} \right) \right\} \\
    &+ t^2 \left\{ \left( \begin{array}{c} \phi_{N+2}(s, r)s_x \\ \psi_{N+2}(s, r)r_x \end{array} \right) \right\} \\
    &+ \cdots
\end{align*}
$$

(3.40)

Here the subscript $N$ denotes $N$-fold application of the transformations (3.30) and (3.36), the functions $f_N, g_N$ are defined by the formula (3.23) with their explicit form for $N = 2, 3$ given by the formulae (3.9), (3.10) and $\lambda, c_0$ are arbitrary constants.

### 3.4 Noncommutative Lie-Bäcklund algebra associated with gas dynamics equations

Denote by $\hat{X}_{(a,c)}$ canonical Lie-Bäcklund operators of hydrodynamic symmetries generated by the Lie equations (3.32) subject to the condition (3.8). Let $\hat{X}_N$ denote canonical Lie-Bäcklund operators of the $N$th-order higher symmetries (3.27) $(N = 2, 3, \ldots)$ for gas dynamics equations.

**Theorem 3.8.** Hydrodynamic and higher symmetries of gas dynamics equations generate infinite-dimensional noncommutative Lie-Bäcklund algebra in which the hydrodynamic symmetries form an infinite-dimensional commutative ideal

$$
\begin{align*}
    [\hat{X}_{(a,c)}, \hat{X}_{(\tilde{a},\tilde{c})}] &= 0, \\
    \hat{X}_N, \hat{X}_{(a,c)} &= \hat{X}_{(a_{N,\epsilon},c)}, \\
    [\hat{X}_M, \hat{X}_N] &= 0 \\
    (M, N = 2, 3, \ldots).
\end{align*}
$$

(3.41)

Here the functions $a(s, r), c(s, r)$ and $\tilde{a}(s, r), \tilde{c}(s, r)$ satisfy the equations (3.8).

**Theorem 3.9.** Let $N = 2, 3, \ldots$. The generalized gas dynamics equations have a common infinite series of higher symmetries (3.40) of $(N + 1)$th or larger order iff the right-hand sides of these equations differ only by a term belonging to a kernel of the operator $R^N$, i.e. $\phi_N, \psi_N$ coincide for all these equations. If the right-hand sides $\phi s_x, \psi r_x$ of GGD equations (3.33) belong themselves to a kernel of the operator $R^N$, then for all such equations the operators $\hat{X}_{(\phi,\psi)}$ commute with all the operators $\hat{X}_n$ of higher symmetries (3.23) of the order $n = N, N + 1, \ldots$.

**Corollary 3.4.** Commutative symmetry subalgebras for gas dynamics equations are generated by the operators $\hat{X}_{(a,c)}$ of those hydrodynamic symmetries whose characteristics
\((as_x, cr_x)\) belong to a kernel of some integer degree \(R^N\) of the recursion operator \((N \geq 2)\), and by the operators \(\hat{X}_n\) of higher symmetries \((3.23)\) of the order \(n\) larger or equal to \(N\).

Thus, a problem of constructive description of the kernel of operator \(R^N\) naturally arises. It is solved by means of the inverse recursion operator \(R^{-1}\)

\[
R^{-1} = \begin{pmatrix} s_x & 0 \\ 0 & r_x \end{pmatrix} \left\{ I + \frac{\alpha}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} D^{-1} \frac{\alpha_x}{\alpha^2} \right\} D^{-1}
\]  

(3.42)

where \(D^{-1} = \int dx\) is the operator of integration with respect to \(x\) at a constant value of \(t\) with the integration ”constant” \(c(t)\) depending upon \(t\). Hence a kernel of the operator \(R\) has the form

\[
R^{-1} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = c_1(t) \begin{pmatrix} s_x \\ r_x \end{pmatrix} + c_2(t) \alpha(\rho) \begin{pmatrix} -s_x \\ r_x \end{pmatrix}.
\]  

(3.43)

A kernel of the operator \(R^2\) is given by the formula

\[
R^{-2} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = c_1(t) \left( \frac{(u - \rho \alpha(\rho))s_x}{(u + \rho \alpha(\rho))r_x} \right) + c_2(t) \left( \frac{[\int \alpha^2(\rho) d\rho - u \alpha(\rho)]s_x}{[\int \alpha^2(\rho) d\rho + u \alpha(\rho)]r_x} \right) + c_3(t) \begin{pmatrix} s_x \\ r_x \end{pmatrix} + c_4(t) \alpha(\rho) \begin{pmatrix} -s_x \\ r_x \end{pmatrix}.
\]  

(3.44)

Here \(c_i(t)\) are arbitrary smooth functions.

3.5 Lax representation and invariant solutions of generalized gas dynamics equations

Formula \((3.22)\) gives the Lax representation \((28)\) of the gas dynamics equations

\[
\frac{\partial R}{\partial t} = [R, A].
\]  

(3.45)

Here the recursion operator \((3.21)\) and the ”stationary part” \((3.6)\) of the operator of the determining equation form the Lax pair of the Ibragimov-Shabat type \((29–31)\). Equation \((3.37)\) also gives Lax representation for any generalized gas dynamics equations \((3.33)\)

\[
\frac{\partial R}{\partial t} = [R, A(\phi, \psi)]
\]  

(3.46)

with the recursion operator \((3.21)\) as L-operator and the operator \((3.35)\) as A-operator.

In 1982 L.D. Faddeev and P.P. Kulish in a private communication pointed out to the author that these Lax pairs were degenerate, \(i.e.\) the mapping of a potential to scattering data was singular, so that the method of inverse scattering transform could not be applied. However, it turned out that it is not a deficiency of GGD equations but the indication to a possibility of linearizing these equations in a classical sense by the hodograph transformation and not by the inverse scattering transform.

Nevertheless, the Lax representation for these equations turns out to be useful for construction of their invariant solutions by means of the inverse recursion operator \((3.42)\) \([17]\).
At first consider the solutions of the gas dynamics equations (3.1) which are invariant with respect to the higher symmetries given by the formula (3.25)

\[(f_N, g_N)^T = R^{N-1}(1, 1)^T = 0 \quad (N = 2, 3, \ldots)\]  

Define the matrix \(U\)

\[
U = \begin{pmatrix}
u \\ \rho \\ \int_0^\rho \alpha^2(\rho) d\rho \\ u
\end{pmatrix}
\]

and the matrix-integral operator \(K\)

\[
K = D_x^{-1}U_x = D_x^{-1} \begin{pmatrix}
u_x \\ \rho_x \\ \alpha^2(\rho)\rho_x \\ u_x
\end{pmatrix}.
\]

Define the operator \(K\) and its degrees \(K^n\) with an integer \(n\) so that all the constants \(c_i\) of integrations \(D_x^{-1} = \int dx\) do not depend on \(t\).

**Theorem 3.10.** For any \(N = 2, 3, \ldots\) the equalities

\[
K^{N-1} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = c_{2N-1} \begin{pmatrix} c_{2N-1} \\ c_{2N} \end{pmatrix} + c_{2N-2} \begin{pmatrix} \int \alpha^2(\rho) d\rho \\ u \end{pmatrix} + c_{2N-3} \begin{pmatrix} u \\ \rho \end{pmatrix} + 
\]

\[
c_{2N-4} \begin{pmatrix} u^2/2 + \int d\rho \int \alpha^2(\rho) d\rho \\ \int \alpha^2(\rho) d\rho \\ \int \alpha^2(\rho) d\rho \end{pmatrix} + \ldots =
\]

\[
= \begin{pmatrix} x - ut \\ -\rho t \end{pmatrix}
\]

determine an infinite series of exact solutions \(u = u(x, t), \rho = \rho(x, t)\) of gas dynamics equations (3.1) which are invariant with respect to higher symmetries (3.25). They are given in the form of an implicit function \(t = t(u, \rho), x = x(u, \rho)\).

For \(N = 1\) the formula (3.50) have the form

\[
\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} x - ut \\ -\rho t \end{pmatrix}
\]

and gives a trivial solution for which \(u_t + uu_x = 0, \rho_t + \rho u_x = 0\), i.e. the velocity and density \(u\) and \(\rho\) are constant in the Lagrangian frame which moves together with a gas particle.

For \(N = 2\) the formula (3.50) can be taken in the form

\[
u - \nu_0 = \bar{\rho}(t - t_0),
\]

\[
\bar{\rho}(t - t_0)^2 - \int_{\bar{\rho}_0}^\bar{\rho} P'(\lambda \bar{\rho}) \frac{1}{\bar{\rho}^2} d\bar{\rho} = x - x_0 - u_0(t - t_0),
\]

where \(\bar{\rho} = \rho/\lambda\) and \(\lambda, \nu_0, x_0, t_0, \bar{\rho}_0\) are constants. It corresponds to the motion of a piston in a gas flow after the explosion.

Now we consider invariant solutions of generalized gas dynamics equations expressed through variables \(u, \rho\)

\[
u_t = w(u, \rho)u_x + v(u, \rho)\alpha^2(\rho)\rho_x,
\]

\[
\rho_t = v(u, \rho)u_x + w(u, \rho)\rho_x
\]

(3.53)
where the coefficients \( v, w \) satisfy the linear system
\[
w_u = v_\rho, \quad w_\rho = \alpha^2(\rho)v_u. \tag{3.54}
\]

The corollary 3.4 implies that the higher symmetries (3.25) of gas dynamics equations of the orders \( N_0, N_0 + 1, \ldots \) are also higher symmetries of those GGD equations (3.33) whose right-hand sides belong to a kernel of the recursion operator \( R_{N_0} \), i.e. \( \phi_{N_0} = \psi_{N_0} = 0 \). We give now their explicit form allowing them to depend explicitly on \( t \)

\[
\begin{align*}
D_t \left( \begin{array}{c} u \\ \rho \end{array} \right) &= \left( \begin{array}{cc} u_x & \alpha^2(\rho) \rho_x \\ \rho_x & u_x \end{array} \right) K^{N_0-1} \left( \begin{array}{c} a_1(t) \\ a_2(t) \end{array} \right) \\
&= \left( \begin{array}{c} u_x \alpha^2(\rho) \rho_x \\ \rho_x u_x \end{array} \right),
\end{align*}
\]

\[
\begin{align*}
&\cdot \left\{ \left( \begin{array}{cc} a_{2N_0-1}(t) \\ a_{2N_0}(t) \end{array} \right) + a_{2N_0-2}(t) \left( \int \alpha^2(\rho) d\rho \right) + a_{2N_0-3}(t) \left( \frac{u}{\rho} \right) + a_{2N_0-4}(t). \right. \\
&\cdot \left( \frac{u^2}{2} + \int d\rho \int \alpha^2(\rho) d\rho \right) + a_{2N_0-5}(t) \left( \frac{\int u^2}{2} + \int \alpha^2(\rho) d\rho \right) + \ldots \right\}.
\end{align*}
\tag{3.55}
\]

Here in the definition (3.41) of operator \( K \) the ”constants” \( a_i(t) \) of the integration with respect to \( x \) may depend on \( t \) and are assumed to be arbitrary smooth functions.

We consider the solutions of GGD equations (3.55) which are invariant with respect to the higher symmetries (3.25).

**Theorem 3.11.** For any \( N = N_0, N_0 + 1, \ldots (N \geq 2) \) the equations
\[
\begin{align*}
K^{N-1} \left( \begin{array}{c} c_1 \\ c_2 \end{array} \right) &\equiv \left( \begin{array}{c} c_{2N-1} \\ c_{2N} \end{array} \right) + c_{2N-2} \left( \int \alpha^2(\rho) d\rho \right) + c_{2N-3} \left( \frac{u}{\rho} \right) + \\
&+ c_{2N-4} \left( \frac{u^2}{2} + \int d\rho \int \alpha^2(\rho) d\rho \right) + c_{2N-5} \left( \frac{\int u^2}{2} + \int \alpha^2(\rho) d\rho \right) + \ldots = \\
&= \left( \begin{array}{c} \frac{d}{dt} \end{array} \right) + \left( \begin{array}{c} a_{2N_0-1}(t) \\ a_{2N_0}(t) \end{array} \right) dt + \left( \begin{array}{c} \int \alpha^2(\rho) d\rho \\ \int \alpha^2(\rho) d\rho \end{array} \right) \int_0^t a_{2N_0-2}(t) dt + \\
&+ \left( \begin{array}{c} \frac{u}{\rho} \end{array} \right) \int_0^t a_{2N_0-3}(t) dt + \left( \begin{array}{c} \frac{u^2}{2} \end{array} \right) \int_0^t a_{2N_0-4}(t) dt + \\
&+ \left( \begin{array}{c} \frac{\int u^2}{2} \end{array} \right) \int_0^t a_{2N_0-5}(t) dt + \ldots
\end{align*}
\tag{3.56}
\]

determine an infinite series of exact solutions \( u = u(x, t), \rho = \rho(x, t) \) of any generalized gas dynamics equations of the form (3.56) (with arbitrarily fixed integer \( N_0 \)) which may be explicitly \( t \)-dependent. These solutions are invariant with respect to the \( N \)th-order higher symmetries (3.25) of these equations. Here the functions \( a_i(t) \) and the constants \( c_i \) with \( i \leq 0 \) must be put equal to zero. The definition of the operator \( K \) is the same as in the formula (3.55).

For the gas dynamics equations (3.1) we have \( a_{2N_0-3}(t) = -1, a_i(t) = 0 \) for \( i \neq 2N_0-3 \).
4 Separable two-component Hamiltonian systems

4.1 Hamiltonian structure of generalized gas dynamics equations

We consider two-component hydrodynamic-type Hamiltonian systems of the form

$$D_t \begin{pmatrix} u \\ \rho \end{pmatrix} = \sigma_1 D_x \begin{pmatrix} H_u(u, \rho) \\ H_\rho(u, \rho) \end{pmatrix}$$ (4.1)

where $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is the Pauli matrix. Here $H(u, \rho)$ is Hamiltonian density of the hydrodynamic-type which corresponds to the Hamiltonian $\mathcal{H} = \int_{-\infty}^{\infty} H(u, \rho) dx$. For short we call $H(u, \rho)$ also a Hamiltonian.

Equations (4.1) take the form of the Hamilton equations

$$D_t \begin{pmatrix} u \\ \rho \end{pmatrix} = \left\{ \begin{pmatrix} u \\ \rho \end{pmatrix}, H \right\}$$ (4.2)

with the Poisson bracket of the hydrodynamic-type

$$\{H, h\} = (h_u, h_\rho) D_x \sigma_1 (H_u, H_\rho)^T.$$ (4.3)

Define the Hamiltonian matrix

$$\hat{H} = \begin{pmatrix} H_{uu} & H_{u\rho} \\ H_{u\rho} & H_{\rho\rho} \end{pmatrix}.$$ (4.4)

Then the equations (4.1) take the form

$$\begin{pmatrix} u \\ \rho \end{pmatrix}_t = \hat{H} \begin{pmatrix} u \\ \rho \end{pmatrix}_x \iff (ID_t - \hat{H} D_x) \begin{pmatrix} u \\ \rho \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$ (4.5)

The gas dynamics equations (3.1) have the Hamiltonian form (4.1), (4.5) with the Hamiltonian

$$H(u, \rho) = -\left[ \rho u^2/2 + \int_0^\rho \int_0^\rho \alpha^2(\rho) \rho d\rho \right].$$ (4.6)

The same is valid for the generalized gas dynamics equations (3.55) whose Hamiltonians $H(u, \rho, t)$ are defined by the equality

$$\begin{pmatrix} H_\rho \\ H_u \end{pmatrix} = K^{N_0} \begin{pmatrix} a_1(t) \\ a_2(t) \end{pmatrix}.$$ (4.7)
with the operator $K$ defined by the formula (3.49) and they may depend explicitly upon $t$. For example the equation (4.7) for $N_0 = 2$ becomes

$$H(u, \rho, t) = a_1(t) \left[ \rho u^2 / 2 + \int_0^\rho \alpha^2(\rho) \rho d\rho \right] +$$

$$+ a_2(t) \left[ u^3 / 3! + u \int_0^\rho \alpha^2(\rho) d\rho \right] +$$

$$+ a_3(t) \rho u + a_4(t) \left[ u^2 / 2 + \int_0^\rho \rho \int_0^\rho \alpha^2(\rho) d\rho \right] +$$

$$+ a_5(t) \rho + a_6(t) u. \tag{4.8}$$

4.2 Separable Hamiltonian systems

**Definition 4.1.** We say that Hamiltonians $\int_{-\infty}^\infty H \, dx$ and $\int_{-\infty}^\infty h \, dx$ commute if the Poisson bracket (4.3) of their densities is the exact derivative with respect to $x$ of some function

$$\{H(u, \rho, t), h(u, \rho, t)\} = D_x[Q(u, \rho, t)]. \tag{4.9}$$

Then we also say that Hamiltonians $H$ and $h$ commute and the Hamiltonian matrices $\hat{H}$ and $\hat{h}$ defined by the equation (4.4) commute also: $[\hat{H}, \hat{h}] = 0$.

**Theorem 4.1.** For any hydrodynamic-type Hamiltonian $H(u, \rho)$ there exists an infinite set of Hamiltonians $h(u, \rho)$ which commute with it and with each other. They are arbitrary smooth solutions of the wave equation

$$H_{\rho\rho} h_{uu} - h_{\rho\rho} H_{uu} = 0 \tag{4.10}$$

or for $H_{\rho\rho} \neq 0$

$$h_{uu} - V(u, \rho) h_{\rho\rho} = 0 \tag{4.11}$$

with $V(u, \rho) = H_{uu} / H_{\rho\rho}$.

**Definition 4.2.** Those systems (4.3) for which the wave equation (4.11) admits a separation of variables $u, \rho$, i.e. $V(u, \rho) = \beta^2(u)/\alpha^2(\rho)$, are called separable Hamiltonian systems [6, 18]. Their Hamiltonians $H$ satisfy the equation ($H_{uu} H_{\rho\rho} \neq 0$)

$$(1/\beta^2(u)) H_{uu} = (1/\alpha^2(\rho)) H_{\rho\rho} \tag{4.12}$$

and just the same for all the Hamiltonians $h$ commuting with $H$

$$(1/\beta^2(u)) h_{uu} = (1/\alpha^2(\rho)) h_{\rho\rho}. \tag{4.13}$$

Gas dynamics equations (3.1) and GGD equations (3.33) with the Hamiltonians (4.6) and (4.7) are examples of separable systems with $\beta^2(u) = 1$.

**Definition 4.3.** If $H(u, \rho)$ is a Hamiltonian and the ratio $H_{uu} / H_{\rho\rho} = 1/\alpha^2(\rho)$ is a function of $\rho$ only, then $H$ is called Hamiltonian of generalized gas dynamics. In this case $\beta(u) = 1$.

Many interesting physical applications of the separable Hamiltonian systems are given in the paper [6].
4.3 Second order recursion operator and Lax representation for separable Hamiltonian systems

We show that separable Hamiltonian systems obtained by the Manin’s construction possess higher symmetries, a recursion operator, the Lax representation of the Ibragimov-Shabat type and good integrability properties \([18, 32]\).

We introduce the notation

\[
\partial_u = \frac{\partial}{\partial u}, \quad \partial_\rho = \frac{\partial}{\partial \rho}, \quad \partial_u^{-1} = \int_0^u du, \quad \partial_\rho^{-1} = \int_0^\rho d\rho \tag{4.14}
\]

where the integration with respect to one variable is performed at a constant value of another variable. The Manin’s construction gives rise to two fundamental series of mutually commuting Hamiltonians \([18]\)

\[
H^{(2m)}(2m)_{(1,0)} = \sum_{n=0}^{m} (\partial_u^{-2} \beta^2(u))^{m-n} (\partial_\rho^{-2} \alpha^2(\rho))^n \rho, \tag{4.15}
\]

\[
H^{(2m-1)}(2m-1)_{(1,0)} = \sum_{n=0}^{m} (\partial_u^{-2} \beta^2(u))^{m-n-1} (\partial_\rho^{-2} \alpha^2(\rho))^n u \rho, \tag{4.16}
\]

with \(m = 0, 1, 2, \ldots\) and \(H^{(-1)}(1,0) = 0, H^{(-1)}(0,1) = 1\). Here every operator \(\partial_u^{-1}, \partial_\rho^{-1}\) acts on all the factors standing to the right of it, e.g.

\[
(\partial_u^{-2} \beta^2(u))^2 = \int_0^u du \int_0^u \left[ \beta^2(u) \int_0^u \beta^2(u) du \right] du. \tag{4.17}
\]

For arbitrary constants \(c_1, c_2\) we define

\[
H^{(N)}(c_1, c_2) = c_1 H^{(N)}(1,0) + c_2 H^{(N)}(0,1). \tag{4.18}
\]

The basic Hamiltonians for the GGD equations \((3.55)\) are obtained at \(\beta(u) = 1\)

\[
H^{(N)}(1,0) = \sum_{n=0}^{\left[ \frac{N}{2} \right]} \frac{u^{N-2n}}{(N-2n)!} (\partial_\rho^{-2} \alpha^2(\rho))^n \rho, \tag{4.19}
\]

\[
H^{(N)}(0,1) = \sum_{n=0}^{\left[ \frac{N+1}{2} \right]} \frac{u^{N+1-2n}}{(N+1-2n)!} (\partial_\rho^{-2} \alpha^2(\rho))^n. \tag{4.20}
\]

We list explicitly several Hamiltonians from the series \((4.15), (4.16)\)

\[
H^{(0)}(1,0) = \rho, \quad H^{(1)}(1,0) = u \rho, \quad H^{(2)}(1,0) = (\partial_u^{-2} \beta^2(u) + \partial_\rho^{-2} \alpha^2(\rho)) \rho, \tag{4.21}
\]

\[
H^{(3)}(1,0) = (\partial_u^{-2} \beta^2(u) + \partial_\rho^{-2} \alpha^2(\rho)) u \rho;
\]

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\[ H^{(0)}(0, 1) = u, \quad H^{(1)}(0, 1) = \partial_u^{-2} \beta^2(u) + \partial_\rho^{-2} \alpha^2(\rho), \]
\[ H^{(2)}(0, 1) = (\partial_u^{-2} \beta^2(u) + \partial_\rho^{-2} \alpha^2(\rho))u, \quad \]
\[ H^{(3)}(0, 1) = (\partial_u^{-2} \beta^2(u))^2 + \partial_u^{-2} \beta^2(u) \partial_\rho^{-2} \alpha^2(\rho) + (\partial_\rho^{-2} \alpha^2(\rho))^2. \] (4.22)

In particular, for the Hamiltonians (4.19), (4.20) of the generalized gas dynamics we obtain
\[ H^{(2)}(1, 0) = \rho u^2/2 + \tilde{G}_1(\rho), \quad H^{(3)}(1, 0) = \rho u^3/6 + u\tilde{G}_1(\rho) \] (4.23)

where
\[ \tilde{G}_1(\rho) = \int_0^\rho (\rho - \sigma) \sigma \alpha^2(\sigma) d\sigma \] (4.24)
and
\[ H^{(1)}(0, 1) = u^2/2 + G_1(\rho), \quad H^{(2)}(0, 1) = u^3/6 + uG_1(\rho) \] (4.25)
where
\[ G_1(\rho) = \int_0^\rho (\rho - \sigma) \alpha^2(\sigma) d\sigma. \] (4.26)

Consider at first the special case of the generalized gas dynamics Hamiltonians, \( \beta(u) = 1 \). Define the integral-matrix operator \( K_0 \) using the matrix \( U \) defined by the formula (3.43)
\[ K_0 = \partial_x^{-1} U_x = \int_0^x dx \begin{pmatrix} u_x & \alpha^2(\rho) \rho_x \\ \rho_x & u_x \end{pmatrix} = \int_{(0,0)}^{(u,\rho)} U_u du + U_\rho d\rho = \partial_x^{-1} I + \partial_\rho^{-1} \begin{pmatrix} 0 & \alpha^2(\rho) \\ 1 & 0 \end{pmatrix}. \] (4.27)

Here the integral with respect to \( x \) is taken at a constant value of \( t \) and the integral operator in the second line is the curvilinear integral not depending on the integration path in the \((u, \rho)\) plane.

**Theorem 4.2.** For \( \beta(u) = 1 \) the generalized gas dynamics Hamiltonians of the series (4.19), (4.20) and their combinations \( H^{(N)}(c_1, c_2) \) defined by the formula (4.18) satisfy the recursion relation
\[ \left( H^{(N+1)}_\rho, H^{(N+1)}_u \right)^T = K_0 \left( H^{(N)}_\rho, H^{(N)}_u \right)^T \] (4.28)
for \( N = 0, 1, 2, \ldots \) with the recursion operator \( K_0 \) defined by the formula (4.27).

The solution of the equation (4.28) is given by the formula
\[ \left( H^{(N)}_\rho, H^{(N)}_u \right)^T = K_0^N \left( H^{(0)}_\rho, H^{(0)}_u \right)^T = K_0^N (c_1, c_2)^T \] (4.29)
where \( H^{(N)} = H^{(N)}(c_1, c_2) \).

For arbitrary functions \( \beta(u), \alpha(\rho) \) we define the matrices
\[ U_1 = \begin{pmatrix} u & \partial_\rho^{-1} \alpha^2(\rho) \\ \rho & \partial_u^{-1} \beta^2(u) \end{pmatrix}, \quad U_2 = \begin{pmatrix} \partial_\rho^{-1} \beta^2(u) & \partial_\rho^{-1} \alpha^2(\rho) \\ \rho & u \end{pmatrix} \] (4.30)
and the matrix-integral operators

\[ K_1 = \partial_x^{-1}U_{1x} = \int_0^x dx \begin{pmatrix} u_x & \alpha^2(\rho) \rho_x \\ \rho_x & \beta^2(u) u_x \end{pmatrix} = \int_{(0,0)} U_{1u} du + U_{1\rho} d\rho = \partial_u^{-1} \begin{pmatrix} 1 & 0 \\ 0 & \beta^2(u) \end{pmatrix} + \partial_{\rho}^{-1} \begin{pmatrix} 0 & \alpha^2(\rho) \\ 1 & 0 \end{pmatrix}, \]  

(4.31)

\[ K_2 = \partial_x^{-1}U_{2x} = \int_0^x dx \begin{pmatrix} \beta^2(u) u_x & \alpha^2(\rho) \rho_x \\ \rho_x & u_x \end{pmatrix} = \int_{(0,0)} U_{2u} du + U_{2\rho} d\rho = \partial_u^{-1} \begin{pmatrix} \beta^2(u) & 0 \\ 0 & 1 \end{pmatrix} + \partial_{\rho}^{-1} \begin{pmatrix} 0 & \alpha^2(\rho) \\ 1 & 0 \end{pmatrix} \]  

(4.32)

with the operator \( \partial_x^{-1} \) defined as in the equation (4.27). Here again in the second line we have curvilinear integrals which do not depend on the integration path.

Define also the operator

\[ K = K_1K_2 = \partial_x^{-1}U_{1x}\partial_x^{-1}U_{2x}. \]  

(4.33)

**Theorem 4.3.** The separable Hamiltonians (4.15), (4.16) and their combinations \( H^{(N)}(c_1, c_2) \) satisfy the recursion relation

\[ (H^{(N+2)}_\rho, H^{(N+2)}_u)^T = K(H^{(N)}_\rho, H^{(N)}_u)^T \]  

(4.34)

for \( N = 0, 1, 2, \ldots \) with the recursion operator \( K \) defined by the equation (4.33).

This equation has the solutions

\[ (H^{(2m)}_\rho, H^{(2m)}_u)^T = K^m(H^{(0)}_\rho, H^{(0)}_u)^T = K^m(c_1, c_2)^T \]  

(4.35)

for \( m = 0, 1, 2, \ldots \) and

\[ (H^{(2m-1)}_\rho, H^{(2m-1)}_u)^T = K^{m-1}(H^{(1)}_\rho, H^{(1)}_u)^T = K^{m-1}U_1(c_1, c_2)^T \]  

(4.36)

for \( m = 1, 2, \ldots \) where \( H^{(N)} = H^{(N)}(c_1, c_2) \).

Thus, all the Hamiltonians of the Manin’s series (4.15), (4.16) are generated by the recursion operator (4.33) (see [18]).

Consider now the inverse recursion operator for these Hamiltonians

\[ K^{-1} = K_2^{-1}K_1^{-1} = (U_{2x})^{-1}D_x(U_{1x})^{-1}D_x \]  

(4.37)

with the matrices

\[ (U_{1x})^{-1} = \frac{1}{\beta^2 u_x^2 - \alpha^2 \rho_x^2} \begin{pmatrix} \beta^2 u_x & -\alpha^2 \rho_x \\ -\rho_x & u_x \end{pmatrix}, \]  

(4.38)

\[ (U_{2x})^{-1} = \frac{1}{\beta^2 u_x^2 - \alpha^2 \rho_x^2} \begin{pmatrix} u_x & -\alpha^2 \rho_x \\ -\rho_x & \beta^2 u_x \end{pmatrix}. \]  

(4.39)

Define the second order matrix-differential operator

\[ L = D_x K^{-1}\partial_x^{-1} = D_x(U_{2x})^{-1}D_x(U_{1x})^{-1}. \]  

(4.40)
Consider the Lie equations for Lie-Bäcklund symmetries of the Hamiltonian system (4.5) with the Hamiltonian $H$ and the Hamiltonian matrix $\hat{H}$,

$$\begin{pmatrix} u \\ \rho \end{pmatrix}_\tau = \begin{pmatrix} f(x, t, u, \rho, u_x, \rho_x, \ldots, u_x^{(N)}, \rho_x^{(N)}) \\ g(x, t, u, \rho, u_x, \rho_x, \ldots, u_x^{(N)}, \rho_x^{(N)}) \end{pmatrix} \equiv \begin{pmatrix} f_n \\ g_n \end{pmatrix}$$

(4.41)

assuming that $x_\tau = t_\tau = 0$. These symmetries satisfy the determining equation

$$(ID_t - D_x \hat{H}) \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$  

(4.42)

**Theorem 4.4.** Any separable Hamiltonian system (4.5) possesses the second order matrix-differential recursion operator of the form (4.40) which satisfies the recursion relation

$$L(f_n, g_n)^T = (f_{n+2}, g_{n+2})^T \quad (n = 2, 3, \ldots).$$

(4.43)

**Corollary 4.1.** On the solution manifolds of the systems (4.5) the following equality is satisfied

$$[ID_t - D_x \hat{H}, L] = 0.$$  

(4.44)

Thus, we obtain the Lax representation for these systems

$$\frac{\partial L}{\partial t} = [L, A]$$

(4.45)

with the Lax pair of the Ibragimov-Shabat type [29, 30] where $A = -D_x \hat{H}$ and the operator $L$ is defined by the equation (4.40).

**Corollary 4.2.** For generalized gas dynamics equations we must put $\beta(u) = 1$, $U_1 = U_2 = U$ and the second order recursion operator (4.40) is equal to the squared first order recursion operator $L_0$

$$L = L_0^2, \quad L_0 = D_x(U_x)^{-1}. $$

(4.46)

The operator $L_0$ coincides with the recursion operator [3.21] transformed from the Riemann invariants $s, r$ to the separable variables $u, \rho$ and satisfies the relation

$$L_0(f_n, g_n)^T = (f_{n+1}, g_{n+1})^T \quad (n = 2, 3, \ldots).$$

(4.47)

**Theorem 4.5.** For the separable Hamiltonian systems (4.5) with the generic functions $\beta(u), \alpha(\rho)$ the first order recursion operator does not exist and the second order recursion operator (4.40) is not reduced to a square of a first order recursion operator or to a product of different first order operators.

### 4.4 Hydrodynamic symmetries and higher symmetries

**Theorem 4.6** Separable systems (4.5) with the Hamiltonian $H(u, \rho)$ which do not depend explicitly on $t$ possess an infinite set of homogeneous in derivatives hydrodynamic symmetries. All such symmetries with or without an explicit dependence upon $x, t$ are generated by the Lie equations

$$\begin{pmatrix} u \\ \rho \end{pmatrix}_\tau = (xI + t\hat{H} + \hat{h}) \begin{pmatrix} u \\ \rho \end{pmatrix}_x$$

(4.48)
or

\[
\begin{pmatrix}
  u \\
  \rho
\end{pmatrix}_x = \hat{h} \begin{pmatrix}
  u \\
  \rho
\end{pmatrix}_x \equiv \sigma_1 D_x \begin{pmatrix}
  h_u(u, \rho) \\
  h_\rho(u, \rho)
\end{pmatrix}
\]  

(4.49)

respectively. Here \( h(u, \rho) \) is an arbitrary smooth solution of the equation (4.13), \( \hat{H} \) and \( \hat{h} \) are Hamiltonian matrices of the form (4.4).

Corollary 4.3. All the hydrodynamic symmetries (4.49) without an explicit dependence on \( x, t \) are separable Hamiltonian systems with the Hamiltonians \( h(u, \rho) \) which mutually commute and also commute with the Hamiltonian \( H(u, \rho) \) of the system (4.5).

Thus, any separable Hamiltonian system (4.5) is included into the infinite hierarchy of commuting separable Hamiltonian flows.

Theorem 4.7. Hydrodynamic symmetries (4.49) of the system (4.5) without an explicit dependence on \( x, t \) form an invariant subspace for the second order recursion operator \( L \). This operator generates the recursion relation for the hydrodynamic symmetries

\[
L \hat{h} \begin{pmatrix}
  u \\
  \rho
\end{pmatrix}_x = \sigma_1 D_x \begin{pmatrix}
  h_{1u}(u, \rho) \\
  h_{1\rho}(u, \rho)
\end{pmatrix} \equiv \hat{h}_1 \begin{pmatrix}
  u \\
  \rho
\end{pmatrix}_x
\]  

(4.50)

preserving the Hamiltonian structure of the Lie equations (4.49) and generates the recursion of their Hamiltonians

\[
h_1(u, \rho) = \frac{1}{\beta^2(u)} h_{uu}(u, \rho) = \frac{1}{\alpha^2(\rho)} h_{\rho\rho}(u, \rho). 
\]  

(4.51)

Corollary 4.4. The hydrodynamic symmetries (4.49) of the separable Hamiltonian system (4.5) which do not depend explicitly on \( x, t \) form an infinite-dimensional commutative Lie-Bäcklund algebra. Its elements have a functional arbitrariness, i.e. depend on an arbitrary smooth solution \( h(u, \rho) \) of the wave equation (1.13).

Denote by \( h_m(u, \rho) \) the result of \( m \)-fold application of the transformation (4.51) to \( h(u, \rho) \)

\[
h_m(u, \rho) = (\beta^{-2}(u) \partial^2/\partial u^2)^m h(u, \rho) = (\alpha^{-2}(\rho) \partial^2/\partial \rho^2)^m h(u, \rho). 
\]  

(4.52)

Then the formula (4.50) gives the result

\[
L^m \hat{h} \begin{pmatrix}
  u \\
  \rho
\end{pmatrix}_x = h_m \begin{pmatrix}
  u \\
  \rho
\end{pmatrix}_x. 
\]  

(4.53)

Theorem 4.8. For any separable Hamiltonian system (4.5) all its higher symmetries of the even order \( n = 2m \) which are generated by the action of the recursion operators \( L^m \) upon the hydrodynamic symmetries (4.48) are given by the formula

\[
\begin{pmatrix}
  f_{2m} \\
  g_{2m}
\end{pmatrix} = L^m \begin{pmatrix}
  u \\
  \rho
\end{pmatrix}_x + (t \hat{H}_m + \hat{h}_m) \begin{pmatrix}
  u \\
  \rho
\end{pmatrix}_x
\]  

(4.54)

for \( m = 1, 2, \ldots \). Here \( H_m(u, \rho) \) is obtained by the transformation (4.52) out of \( H(u, \rho) \). For \( m = 1 \) this formula gives all second order symmetries of the equations (4.3).
Theorem 4.9. The separable system (4.3) with the Hamiltonian \( H(u, \rho) \) possesses an infinite series of higher symmetries of even orders 2\( m \) without the explicit dependence on \( t, x \) if for some integer \( N \) the Hamiltonian \( H \) meets the condition
\[
L^N \hat{H} \left( \begin{array}{c} u \\ \rho \end{array} \right)_x = \left( \begin{array}{c} 0 \\ 0 \end{array} \right) \iff \hat{H}_N = 0
\] (4.55)
i. e. the vector \( \hat{H}(u_x, \rho_x)^T \) belongs to the kernel of the operator \( L^N \), and then \( m \) is larger or equal to \( N \). These symmetries are determined by the formula (4.54) with \( \hat{H}_m = 0 \) and all of them are common symmetries of all the systems (4.3) with the right-hand side belonging to the kernel of the operator \( L^N \).

For \( h_m = 0 \) we obtain the special form of higher symmetries subject to the condition (4.55) with \( m \geq N \)
\[
\left( \begin{array}{c} u \\ \rho \end{array} \right)_x = L^m x \left( \begin{array}{c} u \\ \rho \end{array} \right)_x = \left( \begin{array}{c} f_{2m}(u, \rho, u_x, \rho_x, \ldots, u_x^{(2m)}, \rho_x^{(2m)}) \\ g_{2m}(u, \rho, u_x, \rho_x, \ldots, u_x^{(2m)}, \rho_x^{(2m)}) \end{array} \right)
\] (4.56)

For \( \beta(u) = 1 \) we have \( L = L_0^2 \) where \( L_0 \) is the first order recursion operator (4.46). Then the equations (4.3) form the GGD system of the section 4.2 and have the higher symmetries of any order \( n \geq 2 \) larger or equal to \( N \)
\[
\left( \begin{array}{c} u \\ \rho \end{array} \right)_x = L_0^n x \left( \begin{array}{c} u \\ \rho \end{array} \right)_x = \left( \begin{array}{c} f_n \\ g_n \end{array} \right)
\] (4.57)
if the condition
\[
L^n_0 \hat{H} \left( \begin{array}{c} u \\ \rho \end{array} \right)_x = \left( \begin{array}{c} 0 \\ 0 \end{array} \right)
\] (4.58)
is met.

For the generic function \( \beta(u) \) we shall denote by \( \hat{X}_h \) and \( \hat{X}_{2m} \) the canonical Lie-Bäcklund operators of the hydrodynamic symmetries (4.49) and of the higher symmetries (4.56) respectively.

Theorem 4.10. Hydrodynamic and higher symmetries of the separable Hamiltonian systems (4.3) which do not depend explicitly on \( x \) and \( t \) form the infinite-dimensional noncommutative Lie-Bäcklund algebra
\[
[\hat{X}_h, \hat{X}_h] = 0, \quad [\hat{X}_{2m}, \hat{X}_h] = \hat{X}_{h_m}, \quad [\hat{X}_{2m}, \hat{X}_{2n}] = 0
\] (4.59)
in which the hydrodynamic symmetries \( \hat{X}_h \) form an infinite-dimensional commutative ideal.

Corollary 4.5. Higher and hydrodynamic symmetries of the theorem 4.10 commute iff \( h_m = 0 \) where \( h(u, \rho) \) are the Hamiltonians of the hydrodynamic symmetries (4.49).

The theorem 4.9 poses a problem of obtaining the kernel of the operator \( L^N \) to solve the equation (4.53). In order to obtain its general solution we have to allow an explicit dependence of the Hamiltonians on \( t \) and use the Manin’s series (4.15), (4.16) of Hamiltonians \( H^{(N)}(1 - \nu, \nu) \) with \( \nu = 0, 1 \).
We denote
\[ H^{[N]}(u, \rho, t) = \sum_{k=0}^{N} \left[ a_{4(N-k)+1}(t)H^{(2k)}(1,0) + a_{4(N-k)+2}(t)H^{(2k)}(0,1) + a_{4(N-k)+3}(t)H^{(2k-1)}(1,0) + a_{4(N-k)+4}(t)H^{(2k-1)}(0,1) \right] \quad (N = 1, 2, \ldots) \] (4.60)
with arbitrary smooth functions \( a_i(t) \).

Consider the Hamiltonian system (4.5) with the Hamiltonians \( H^{[N]} \) \((N = 1, 2, \ldots)\) explicitly dependent on \( t \).

**Theorem 4.11.** The kernel of the operator \( L^N \) coincides with the right-hand side of the Hamiltonian system (4.61)
\[ L^{-N} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \hat{H}^{[N]}(u, \rho, t) \begin{pmatrix} u \\ \rho \end{pmatrix}_x \] (4.62)
where
\[ L^{-1} = U_1D_x^{-1}U_2D_x^{-1}. \] (4.63)

**Corollary 4.6.** Let \( N \) be some integer. Any separable Hamiltonian system which has higher symmetries (4.54) with \( \hat{H}_m = 0 \) (with no explicit dependence on \( t, x \)) of the order \( 2m \geq 2N \) has the form (4.61). In particular, such a system has the higher symmetries (4.56) which meet the condition \( m \geq N \).

**Theorem 4.12.** Any separable Hamiltonian system (4.5) with the explicitly \( t \)-dependent Hamiltonian \( \hat{H}(u, \rho, t) \) possesses an infinite set of the explicitly dependent on \( t, x \) hydrodynamic symmetries
\[ \left( \begin{array}{c} u \\ \rho \end{array} \right)_\tau = \left( \begin{array}{c} xI + \int_0^t \hat{H}(u, \rho, t)dt + \hat{h}(u, \rho) \end{array} \right) \left( \begin{array}{c} u \\ \rho \end{array} \right)_x \] (4.64)
with the functional arbitrariness determined by an arbitrary smooth solution \( \hat{h}(u, \rho) \) of the wave equation (4.13). Infinite series of higher symmetries of any even order \( 2m \) for such a system is generated by the action of recursion operators \( L^m \) on the symmetries (4.64)
\[ \left( \begin{array}{c} u \\ \rho \end{array} \right)_\tau = L^m \left( \begin{array}{c} u \\ \rho \end{array} \right)_x + \left[ \int_0^t \hat{H}_m(u, \rho, t)dt + \hat{h}(u, \rho) \right] \left( \begin{array}{c} u \\ \rho \end{array} \right)_x \] (4.65)
and have the same extent of arbitrariness as the symmetries (4.64). In the formulae (4.64) and (4.65) the integration with respect to \( t \) is performed at constant values of \( u, \rho \).
4.5 Invariant solutions and linearization

We are looking for series of exact solutions of any separable Hamiltonian system (4.61) that are invariant with respect to higher symmetries (4.65) which have no explicit dependence on \( t, x \) due to the condition \( \hat{H}^{[N]}_m(u, \rho, t) = 0 \) for \( m \geq N \).

Invariance conditions for these solutions have the form

\[
L^m_x \left( \begin{array}{c} u \\ \rho \end{array} \right)_x + \hat{h}(u, \rho) \left( \begin{array}{c} u \\ \rho \end{array} \right)_x = \left( \begin{array}{c} 0 \\ 0 \end{array} \right)
\]

(4.66)

for \( m = N, N + 1, \ldots \) with \( h(u, \rho) \) satisfying the equation (4.13). For obtaining these solutions we suggest the method which uses the Lax representation (4.44) and the inverse recursion operator (4.63) [17, 18].

Define the Hamiltonian \( H^{[m]}(u, \rho) \) by the formula (4.60) with \( N = m \) and arbitrary constants \( c_i \) substituted for functions \( a_i(t) \)

\[
H^{[m]}(u, \rho) = \sum_{k=0}^{m} \left[ c_{4(m-k)+1} H^{(2k)}(1, 0) + c_{4(m-k)+2} H^{(2k)}(0, 1) + c_{4(m-k)+3} H^{(2k-1)}(1, 0) + c_{4(m-k)+4} H^{(2k-1)}(0, 1) \right].
\]

(4.67)

Let \( H(u, \rho) \) denote an arbitrary solution of the wave equation (4.12).

**Theorem 4.13.** Invariant solutions of the system (4.61) with the Hamiltonian \( H^{[N]}(u, \rho, t) \) which meet the invariance condition (4.66) are given by the formula

\[
\left( \begin{array}{c} H_{pu}(u, \rho) \\ H_{uu}(u, \rho) \end{array} \right) = \left( \begin{array}{c} x \\ 0 \end{array} \right) + \int_0^t \left( \begin{array}{c} H^{[N]}_{pu}(u, \rho, t) \\ H^{[N]}_{uu}(u, \rho, t) \end{array} \right) dt
\]

(4.68)

if the existence conditions are met for the implicit function \((u(x, t), \rho(x, t))\) determined by the equation (4.68).

This equation defines the linearizing transformation which reduces the solution of the nonlinear separable Hamiltonian system (4.61), which is explicitly \( t \)-dependent, to the solution of the linear wave equation (4.12) for \( H(u, \rho) \).

**Corollary 4.7.** The formula

\[
\left( \begin{array}{c} H^{[m]}_{pu}(u, \rho) \\ H^{[m]}_{uu}(u, \rho) \end{array} \right) = \left( \begin{array}{c} x \\ 0 \end{array} \right) + \int_0^t \left( \begin{array}{c} H^{[N]}_{pu}(u, \rho, t) \\ H^{[N]}_{uu}(u, \rho, t) \end{array} \right) dt
\]

(4.69)

for \( m = N, N + 1, \ldots \) gives the infinite series of exact invariant solutions of the equations (4.61) subject to the condition (4.66) with \( h(u, \rho) = 0 \), i.e. invariant with respect to the higher symmetries (4.56). To obtain an explicit form of these solutions, the expression (4.67) for \( H^{[m]}(u, \rho) \) must be used.

In the formulae (4.68) and (4.69) integrations with respect to \( t \) are performed at constant values of \( u \) and \( \rho \).

**Corollary 4.8.** For the system (4.61) with the Hamiltonian \( H^{[N]}(u, \rho) \) without an explicit dependence on \( t \) the formula (4.68) gives the hodograph transformation

\[
\left( \begin{array}{c} H_{pu}(u, \rho) \\ H_{uu}(u, \rho) \end{array} \right) = \left( \begin{array}{c} x \\ 0 \end{array} \right) + t \left( \begin{array}{c} H^{[N]}_{pu}(u, \rho) \\ H^{[N]}_{uu}(u, \rho) \end{array} \right)
\]

(4.70)
which interchanges the roles of independent and dependent variables \((x, t)\) and \((u, \rho)\).

**Remark 4.1.** Explicitly \(t\)-dependent Hamiltonian system \([4.61]\) is not linearizable by the common hodograph transformation \([4.70]\). The formula \([4.68]\) linearizes all systems of the form \([4.61]\) and presents a generalization of the hodograph transformation for explicitly time-dependent systems.

**Theorem 4.14.** Consider any separable system of the form \([4.5]\) with explicitly \(t\)-dependent Hamiltonian \(H(u, \rho, t)\) satisfying the equation \([4.12]\). Its invariant solutions with respect to the hydrodynamic symmetries \([4.64]\) are given by the equalities

\[
x + \int_{0}^{t} H_{\rho u}(u, \rho, t) dt = h_{\rho u}(u, \rho),
\]

\[
\int_{0}^{t} H_{uu}(u, \rho, t) dt = h_{uu}(u, \rho) \iff \int_{0}^{t} H_{\rho \rho}(u, \rho, t) dt = h_{\rho \rho}(u, \rho)
\]

where \(h(u, \rho)\) is an arbitrary smooth solution of the linear equation \([4.13]\). These formulae reduce the solution of any separable time-dependent system \([4.5]\) to the solution of the linear equation \([4.13]\) for \(h(u, \rho)\). This is the linearizing transformation which generalizes the hodograph transformation for explicitly time-dependent separable Hamiltonian systems \([4.3]\).

## 5 Semi-Hamiltonian equations

### 5.1 Geometry of semi-Hamiltonian systems and hydrodynamic symmetries

Consider a system of quasi-linear first order equations homogeneous in derivatives

\[
u^{i} = v_{i}(u)u^{i}_{x} \quad (i = 1, 2, \ldots, n)
\]

with the diagonal \(n \times n\) matrix \(V(u) = \text{diag}(v_{i}(u))\). In the rest of the article no summation on repeated indices will be assumed. Here \(u = (u^{1}, u^{2}, \ldots, u^{n})\) is \(n\)-vector, \(u^{i}\) are the Riemann invariants \([4]\).

Lie-Bäcklund symmetries of the \(N\)th order of the system \((5.1)\) are generated by the Lie equations

\[
u^{i}_{\tau} = \eta_{i}(x, t, u, u_{x}, \ldots, u^{(N)}) \quad (i = 1, 2, \ldots, n)
\]

where \(\tau\) is the group parameter and we assume that \(x_{\tau} = t_{\tau} = 0\). Denote \(\eta = (\eta_{1}, \ldots, \eta_{n})^{T}\).

Consider the operators \(D_{x}, D_{t}\) of the total derivatives with respect to \(x, t\) and for the calculation of \(D_{t}\) use the system \((5.1)\)

\[
D_{x} = \frac{\partial}{\partial x} + \sum_{j=1}^{n} \left( v_{j}^{i} \frac{\partial}{\partial u^{j}} + \sum_{k=1}^{\infty} u^{(k)} \frac{\partial}{\partial u^{(k)}} \right),
\]

\[
D_{t} = \frac{\partial}{\partial t} + \sum_{j=1}^{n} \left( v_{j}^{i} \frac{\partial}{\partial u^{j}} + \sum_{k=1}^{\infty} D_{x}^{k} v_{j}^{i} (u) u_{x}^{(k)} \frac{\partial}{\partial u^{j(k)}} \right)
\]

where \(\partial/\partial x\) and \(\partial/\partial t\) are calculated at constant values of \(u, u_{x}^{(k)}\).
The compatibility conditions $u^i_{\tau_t} = u^i_{t\tau}$ of the systems (5.1) and (5.2) take the form of the determining equation for the symmetry characteristic $\eta$

$$\left( ID_t - VD_x - U_x \left( \frac{\partial V}{\partial U} \right) \right) [\eta] = 0. \quad (5.5)$$

Here $I$ is the unit matrix, $U_x = \text{diag}(u^i_x)$, $\partial V/\partial U = (v_{i,u_j})$ is the Jacobian matrix of the mapping $v = v(u)$.

Hydrodynamic symmetries are obtained if we choose $N = 1$ in the Lie equations (5.2). Here we consider only hydrodynamic symmetries linear homogeneous in derivatives

$$u^i_t = W^i_i(u,x,t)u^i_x \quad (i = 1, 2, \ldots, n). \quad (5.6)$$

Impose the condition

$$v_i(u) \neq v_j(u) \quad (i \neq j). \quad (5.7)$$

Define the symmetrical connection coefficients associated with the system (5.1) [3]

$$\Gamma_{ij}^i(u) = \Gamma_{ji}^j(u) = v_{i,u_j}/(v_j - v_i) \quad (i \neq j), \quad \Gamma_{ij}^j = -(g_{ii}/g_{jj}) \Gamma_{ij}^i \quad (i \neq j \neq k \neq i). \quad (5.8)$$

This connection is compatible with the non-degenerate diagonal metric

$$g_{ii}(u) = H_i^2(u) = e^{2\Phi_i(u)}, \quad g_{ij} = 0 \quad (i \neq j), \quad \det(g^{ij}) \neq 0. \quad (5.10)$$

Here $H_i(u)$ are Lamé coefficients.

The connection is determined by the metric

$$\Gamma_{ij}^i = (\ln \sqrt{g_{ii}})_{u^j} = (\ln H_i)_{u^j} = \Phi_{i,u^j}(u). \quad (5.11)$$

Integrability conditions of the system (5.11) $\Gamma_{ij,k}^i = \Gamma_{ik,u^j}$ with the use of the expressions (5.8) take the form of the Tsarev’s conditions [3]

$$[v_{i,u^j}/(v_j - v_i)]_{u^k} = [v_{i,u^k}/(v_k - v_i)]_{u^j} \quad (i \neq j \neq k \neq i). \quad (5.12)$$

**Definition 5.1** (see [4]). A diagonal system (5.1) is called semi-Hamiltonian if its coefficients $v_i(u)$ meet the conditions (5.7) and (5.12).

**Theorem 5.1** (see [4]). The diagonal system (5.1) which meets the condition (5.7) possesses an infinite set of the hydrodynamic symmetries without explicit dependence on $x,t$

$$u^i_t = W^i_i(u,x)u^i_x \quad (i = 1, 2, \ldots, n) \quad (5.13)$$

with a functional arbitrariness iff the system (5.1) is semi-Hamiltonian, i.e. the condition (5.12) is satisfied. These symmetries are generated by the Lie equations (5.13) with the coefficients $w_i(u)$ which form an arbitrary smooth solution of the linear system

$$w_{i,u^j} = \Gamma_{ij}^i(w_j - w_i) \quad (i \neq j). \quad (5.14)$$
The coefficients $\Gamma^i_{ij}(u)$ are defined by the formula (5.8). All these symmetries mutually commute. The set of these symmetries is locally parametrized by $n$ arbitrary functions $c_i(u)$ of one variable.

**Theorem 5.2** (see [12]). Any semi-Hamiltonian system of the form (5.1) possesses an infinite set of explicitly dependent on $x, t$ hydrodynamic symmetries with the same functional arbitrariness as in the theorem 5.1. These symmetries are generated by the Lie equations

$$u^i_x = [w_i(u) + c(x + tv_i(u))]u^i_x \quad (i = 1, 2, \ldots, n)$$

(5.15)

with the coefficients $w_i(u)$ which form an arbitrary smooth solution of the linear system (5.14). Here $c$ is an arbitrary constant.

**Theorem 5.3** (see [4]). Any semi-Hamiltonian system of the form (5.1) with the non-degenerate metric (5.10) is a Hamiltonian system iff the following components of the Riemann curvature tensor vanish:

$$R^i_{jjii} = 0 \quad (i \neq j).$$

Then the curvature tensor vanishes identically and the variables $u^i$ form an orthogonal curvilinear coordinate system in a flat (pseudo-Euclidean) space.

**Corollary 5.1.** For the semi-Hamiltonian system (5.1) the following components of the curvature tensor vanish

$$R^i_{jkl} = 0 \quad (i \neq j \neq k \neq l), \quad R^i_{ikj} \equiv \Gamma^i_{ij,u^k} - \Gamma^i_{ik,u^j} = 0,$$

(5.16)

$$R^i_{jki} \equiv \Gamma^i_{ij,u^k} - [\Gamma^j_{ik,\Gamma^j_{kj}} + \Gamma^i_{ij}(\Gamma^j_{jk} - \Gamma^i_{ik})] = 0.$$

(5.17)

**Remark 5.1.** The equalities (5.17) give compatibility conditions for the linear system (5.14) and they are equivalent to the Tsarev’s conditions (5.12).

### 5.2 Invariant solutions and linearization of semi-Hamiltonian systems

Consider explicitly dependent on $x, t$ hydrodynamic symmetries (5.15) with $c = -1$. Invariance conditions for solutions of the semi-Hamiltonian system (5.1) with respect to these symmetries with $u_x \neq 0$ have the form

$$w_i(u) = tv_i(u) + x \quad (i = 1, 2, \ldots, n).$$

(5.18)

**Theorem 5.4** (see [4]). Let the functions $w_i(u)$ in the equations (5.18) form an arbitrary smooth solution of the linear system (5.14). Then any smooth solution $u^i(x, t)$ of the system (5.18) is a solution of the semi-Hamiltonian system (5.1). Vice versa, any solution $u^i(x, t)$ of the system (5.1) may be locally represented as a solution of the system (5.18) in the vicinity of such a point $(x_0, t_0)$ where the condition $u^i_x(x_0, t_0) \neq 0$ is met for each value of $i = 1, 2, \ldots, n$.

**Remark 5.2.** The equalities (5.18) determine a linearizing transformation for the semi-Hamiltonian system (5.1), i.e. the solution of the nonlinear system (5.1) is reduced to the solution of the linear system (5.14) for the functions $w_i(u)$. It generalizes the classical hodograph transformation for the case of multi-component systems and hence is called the generalized hodograph transformation [4]. We have shown here the group-theoretical
origin of the linearizing transformation: every nonsingular solution of the system (5.1) is its invariant solution with respect to the hydrodynamic symmetries (5.15) and the extent of arbitrariness of the set of these symmetries and of the set of the corresponding invariant solutions is the same as for the general solution manifold of the system (5.1), i.e. $n$ arbitrary functions $c_i(u^i)$ of one variable.

Hence to obtain explicit formulae for the invariant solutions of the system (5.1) one must solve the linear system (5.14) with variable coefficients.

5.3 First order recursion operators

By definition, a recursion operator $R$ maps any symmetry of the semi-Hamiltonian system (5.1) again into its symmetry, i.e. any solution $\eta = (\eta_1, \ldots, \eta_n)^T$ of the determining equation (5.5) is mapped again into the solution $R[\eta]$. For this to be true it is sufficient for the operator $R$ to commute with the operator of the determining equation

$$\left[ ID_t - VD_x - U_x \left( \frac{\partial V}{\partial U} \right), R \right] = 0 \quad (5.19)$$

on solution manifolds of the equations (5.1) and (5.3). If $N$ is any integer, then $N$th-order recursion operator has the form (2.73).

For the systems (5.1) it is convenient to search for recursion operators in a slightly different form.

In particular, for the first order recursion operators we assume

$$R = (AD_x + B)U_x^{-1} \quad (5.20)$$

where $A = A(u), B = B(u, u_x)$ are $n \times n$ matrices.

Define the functions $(i = 1, 2, \ldots, n)$

$$S_i(u) = \sum_{k=1}^{n} \Gamma_{ik}^i(u) c_k(u^k) + d_i(u^i) \quad (5.21)$$

which depend on $2n$ functions $c_i(u^i), d_i(u^i)$ of one variable.

Theorem 5.5 (see [26]). For the semi-Hamiltonian system (5.1) there exists a first order recursion operator $R$ of the form (5.20) iff there exist $2n$ functions $c_i(u^i), d_i(u^i)$ of one variable which meet the conditions

$$S_{i,u}(u) = \Gamma_{ij}^i(S_j - S_i) \quad (i \neq j) \quad (5.22)$$

with the functions $S_i(u)$ defined by the formula (5.21). Matrix elements of this recursion operator are given by the formula

$$R_{ij} = \left[ \delta_{ij} c_j(u^i) \left( D_x + \sum_{k=1}^{n} \Gamma_{ik}^i(u) u_x^k \right) 
+ \Gamma_{ij}^i(u) \left( c_j(u^j) u_x^j - c_i(u^i) u_x^i \right) \right] \left( \frac{1}{u_x^i} \right) + d_j(u^j) \delta_{ij} \quad (5.23)$$

where $\delta_{ij}$ is the Kroneker symbol.
Theorem 5.6. For the semi-Hamiltonian system (5.1) homogeneous in derivatives hydrodynamic symmetries (5.13) with no explicit dependence on \(x, t\) subject to the condition (5.14) form the invariant subspace for the recursion operator (5.23). A restriction of the operator \(R\) to this subspace is determined as follows

\[
R \begin{pmatrix} \hat{w}_1(u)u_1^1 \\ \vdots \\ \hat{w}_n(u)u_1^n \end{pmatrix} = \begin{pmatrix} \hat{w}_1(u)u_1^1 \\ \vdots \\ \hat{w}_n(u)u_1^n \end{pmatrix}
\]

where the functions \(\hat{w}_i(u)\) are determined by the formula

\[
\hat{w}_i(u) = c_i(u)w_i(u) + d_i(u)w_i(u) + \sum_{k=1}^n \Gamma_{ik}^j(u)c_k(u)w_k(u).
\]

Corollary 5.2. For any solution \(\{w_i(u)\}\) of the linear system (5.14) the functions \(\hat{w}_i(u)\) form also a solution of this system, i.e. the formula (5.23) is a recursion for solutions of the system (5.14), iff the conditions (5.22) are met.

The first order recursion operator (5.23) for the multi-component system (5.1) was constructed at first by a straightforward solution of the equation (5.19) [26]. A new, more simple method for constructing the recursion operators, based on the study of symmetries of the set of hydrodynamic symmetries of the system (5.1), was developed by the author [14]. Thus, the group-theoretical origin of recursion operators was discovered. A simple geometrical meaning of the existence conditions (5.22) for the first order recursion operators was also clarified in this paper.

To present the last statement explicitly, we consider another orthogonal curvilinear coordinates \(\{r^i\}\), which will be specified later, with the Lamé coefficients

\[
H_i(r) = \sqrt{g_{ii}(r)} = e^{\Phi_i(r)}.
\]

Consider the rotation coefficients \(\beta_{ji}(r)\) of this coordinate system defined by the equations [4, 33]

\[
H_{ir^j} = \beta_{ji}H_j \quad (i \neq j).
\]

Let \(\rho = \{(r^k - r^l)\}\) denote the set of coordinate differences.

**Theorem 5.7** (see [14]). First order recursion operator for the semi-Hamiltonian system (5.1) exists iff the rotation coefficients \(\beta_{ji}(r)\) of some curvilinear orthogonal coordinate system \(r^i\) depend only on the coordinate differences \(\rho\): \(\beta_{ji} = \beta_{ji}(\rho)\).

### 5.4 Second order recursion operators

For second order recursion operators we assume the form

\[
R = (AD_x^2 + BD_x + F)U_x^{-1}
\]

where \(A = A(u, u_x), B = B(u, u_x, u_{xx}), F = F(u, u_x, u_{xx}, u_{xxx})\) are \(n \times n\) matrices.

Define the "connection potential" \(V(u)\) by a completely integrable (in the Frobenius sense) system

\[
V_{u^iu^j}(u) = \Gamma_{ij}^k \Gamma_{kji}(i \neq j).
\]
Its solution $V(u)$ depends upon $n$ arbitrary functions of one variable. The integrability conditions for the system (5.29) are met as a consequence of the semi-Hamiltonian property (5.12).

Define the functions

$$b_{ik}(u) = f_k(u^k) \left[ \Gamma^i_{ik}(2P^k_{kk} - \Gamma^i_{ik}) - \Gamma^i_{ik,u^k} \right] +$$

$$+ [c_k(u^k) - f'_k(u^k)] \Gamma^i_{ik} \quad (i \neq k), \tag{5.30}$$

$$b_{ii}(u) = f_i(u^i) \left[ \Gamma^i_{ii,u^i} + (\Gamma^i_{ii})^2 - 2V_{\mu u^\mu}(u) \right] -$$

$$- f'_i(u^i)V_{\mu}(u) + c_i(u^i)\Gamma^i_{ii} + d_i(u^i), \tag{5.31}$$

$$B_i(u) = \sum_{k=1}^{n} b_{ik}(u) \tag{5.32}$$

which depend upon $3n$ functions $f_i(u^i), c_i(u^i), d_i(u^i)$ of one variable.

Let $R_{ik}, A_{ik}, B_{ik}, F_{ik}$ denote matrix elements of the operator $R$ and the matrices $A, B, F$ respectively.

**Definition 5.2.** Let $H_i(u)$ and $\beta_{ij}(u)$ denote Lamé coefficients and rotation coefficients of the curvilinear orthogonal system \{\$u^i\$\} and $G_i(\bar{u}^i)$ denote a function of $u$ independent of $u^i$. Semi-Hamiltonian system (5.1) is called a generic system (with respect to the second order recursion operators) if none of the following special cases is met for $i, j, k = 1, 2, \ldots, n$

$$v_{i,u^i}(u) = 0, \tag{5.33}$$

$$v_{i,u^i}(u) = \left( F_i(u^i)G_i(\bar{u}^i)/g_{ii} \right) e^{V(u)}, \tag{5.34}$$

$$[\beta_{ji}/(v_j - v_i)][(\beta_{ki}/H_i)_{u^i}/v_{i,u^i}]_{u^i} =$$

$$= [\beta_{ki}/(v_k - v_i)][(\beta_{ji}/H_i)_{u^i}/v_{i,u^i}]_{u^i} \quad (i \neq j \neq k \neq i). \tag{5.35}$$

**Theorem 5.8** (see [14]).

For the semi-Hamiltonian generic system (5.1) a second order recursion operator $R$ of the form (5.28) exists iff there exist $3n$ functions $f_i(u^i), c_i(u^i), d_i(u^i)$ of one variable which meet the conditions

$$B_{i,u^i}(u) = \Gamma^i_{ij}(B_j - B_i) \quad (i \neq j) \tag{5.36}$$

with the functions $B_i(u)$ defined by the formulae (5.30)–(5.32). Its matrix elements are given by the formulae

$$R_{ik} = (A_{ik}D_x^2 + B_{ik}D_x + F_{ik})(1/u_x^k), \tag{5.37}$$

$$A_{ik} = \delta_{ik} f_i(u^i)/u_x^i, \tag{5.38}$$

$$B_{ik} = \Gamma^i_{ik}[f_k(u^k)(u_x^k/u_x^i) - f_i(u^i)(u_x^i/u_x^i)] \quad (i \neq k), \tag{5.39}$$

$$B_{ii} = -f_i(u^i)u_{xx}^i/[u_x^i]^2 + 2[f_i(u^i)/u_x^i] \sum_{j \neq i} \Gamma^i_{ij}u_x^j +$$

$$+ 2f_i(u^i)\Gamma^i_{ii} + c_i(u^i), \tag{5.40}$$

45
\[ F_{ik} = f_i(u^i)\Gamma^i_{ik}(u^k_x/u^i_x)((u^k_{xx}/u^i_x) - (u^k_{xx}/u^i_x)) + \]
\[ + f_k(u^k)((u^i_x)^2/(u^i_x^2))\Gamma^i_{ik} - f_i(u^i)((u^k_x)^2/(u^i_x^2))\Gamma^i_{ik} u^k + \]
\[ + \{ f_i(u^i)(\Gamma^i_{ik} - 2\Gamma^{i}_{i,ik}) - [2f_i(u^i)\Gamma^i_{ik} + c_i(u^i)\Gamma^i_{ik}] u^k + \]
\[ + b_{ik}(u)u^i_x - f_i(u^i)(u^k_x/u^i_x) \sum_{j \neq i, k} u^j_x(\Gamma^i_{ik} - \Gamma^i_{ij}) - \Gamma^i_{ik} + \}
\[ F_{ii} = B_i(u)u^i_x - \sum_{k \neq i} F_{ik}. \] (5.42)

**Theorem 5.9** (see [14]). For the semi-Hamiltonian system (5.1) the homogeneous in derivatives hydrodynamic symmetries (5.13) with no explicit dependence on \(x, t\), subject to the condition (5.14), form the invariant subspace for the second order recursion operator (5.37). Its action on this subspace is determined by the same formula (5.24) from the section 5.3 but with the different definition of functions (5.37). (5.42)

**Corollary 5.3.** For any solution \(\{w_i(u)\}\) of the linear system (5.14) the formula (5.43) gives also a solution \(\{\hat{w}_i(u)\}\) of this system iff the conditions (5.36) are met. Thus, the formula (5.43) is a second order recursion for solutions of the system (5.14).

**Theorem 5.10** (see [14]). If the first order recursion operator exists then there also exists the second order recursion operator equal to the squared first order recursion operator. The inverse is not true, *i.e.* the existence conditions (5.22) for the first order recursion operator do not follow from the existence conditions (5.36) for the second order recursion operators.

This means that the existence conditions for the second order recursion operator are less restrictive than for the first order operator.

### 5.5 Generation of infinite series of exact solutions

To obtain explicit formulae for invariant solutions of the system (5.1) one must search for solutions of the linear system (5.14) and substitute these solutions for the set of functions \(\{w_i(u)\}\) in the linearizing transformation (5.18). Existence of a recursion operator for the semi-Hamiltonian system (5.1) is the additional constraint which makes it possible to obtain particular solutions of the system (5.14).

The linear system (5.14) has two trivial solutions
\[ w_i = 1, \quad w_i = v_i(u) \quad (i = 1, 2, \ldots, n). \] (5.44)

They serve as initial elements for the generation of infinite series of nontrivial solutions by recursion operators.

In particular, assume that the first order recursion operator (5.23) exists for the system (5.1), *i.e.* the conditions (5.22) are met. It generates the recursion (5.23) for solutions of
the system (5.14)

\[
\hat{w}_i(u) = (\hat{R}_1[w])_i = \sum_{k=1}^{n}(\hat{R}_1)_{ik}[w_k]
\] (5.45)

with the first order recursion operator

\[
(\hat{R}_1)_{ik} = \delta_{ik} \left[ c_i(u^i) \frac{\partial}{\partial u^i} + d_i(u^i) \right] + \Gamma^i_{ik}(u)c_k(u^k).
\] (5.46)

Two trivial solutions (5.44) are mapped by the operator \(\hat{R}_1\) to the nontrivial solutions of the system (5.14)

\[
\hat{w}_i(u) = (\hat{R}_1[1])_i = \sum_{k=1}^{n} \Gamma^i_{ik}(u)c_k(u^k) + d_i(u^i) \equiv S_i(u),
\] (5.47)

\[
\hat{w}_i(u) = (\hat{R}_1[v])_i = c_i(u^i)v_{i,w}(u) + d_i(u^i)v_i(u) + \sum_{k=1}^{n} \Gamma^i_{ik}(u)c_k(u^k)v_k(u).
\] (5.48)

Substituting these expressions for \(w_i(u)\) to the equations (5.18) we obtain the explicit formulae for nontrivial solutions of the system (5.1)

\[
\sum_{k=1}^{n} \Gamma^i_{ik}(u)c_k(u^k) + d_i(u^i) = tv_i(u) + x,
\] (5.49)

\[
c_i(u^i)v_{i,w}(u) + d_i(u^i)v_i(u) + \sum_{k=1}^{n} \Gamma^i_{ik}(u)c_k(u^k)v_k(u) = tv_i(u) + x \quad (i = 1, 2, \ldots, n).
\] (5.50)

These equalities determine the exact solutions \(u^i = u^i(x, t)\) of the system (5.1) as implicit functions.

The action by powers of the operator \(\hat{R}_1\) on the trivial solutions (5.44) generates the explicit formulae for two infinite series of exact invariant solutions with \(N = 1, 2, \ldots\)

\[
(\hat{R}_1^N[1])_i = tv_i(u) + x,
\] (5.51)

\[
(\hat{R}_1^N[v])_i = tv_i(u) + x.
\] (5.52)

Assume now that the less restrictive existence conditions (5.36) for the second order recursion operator are met. Then there exists the recursion (5.43) for solutions of the system (5.14)

\[
\hat{w}_i(u) = (\hat{R}_2[w])_i = \sum_{k=1}^{n}(\hat{R}_2)_{ik}[w_k]
\] (5.53)

with the second order recursion operator

\[
(\hat{R}_2)_{ik} = \delta_{ik} \left\{ f_i(u^i) \frac{\partial^2}{(\partial u^i)^2} + [f_i(u^i)\Gamma^i_{ii}(u) + c_i(u^i)] \frac{\partial}{\partial u^i} \right\} + 
\quad + f_k(u^k)\Gamma^i_{ik}(u) \frac{\partial}{\partial u^k} + b_{ik}(u).
\] (5.54)
Here the functions \( b_{ik}(u) \) are defined by the formulae (5.30), (5.31).

The trivial solutions (5.44) are mapped by the operator \( \hat{R}_2 \) to the nontrivial solutions of the system (5.14) and via the equations (5.18) to the corresponding solutions of the system (5.1)

\[
(\hat{R}_2[1])_i \equiv \sum_{k=1}^{n} b_{ik}(u) \equiv B_i(u) = tv_i(u) + x,
\]

\[
(\hat{R}_2[v])_i \equiv f_i(u')v_{i,u'}(u) + [2f_i(u')\Gamma_{iu}^i(u) + c_i(u')]v_{i,u'}(u) + 
+ \sum_{k \neq i} f_k(u^k)\Gamma_{ik}^i(u) + \sum_{k=1}^{n} b_{ik}(u)v_k(u) = tv_i(u) + x.
\]

(5.55)

The powers \( \hat{R}_2^N \) generate two infinite series of exact invariant solutions out of initial solutions (5.44) with \( N = 1, 2, \ldots \)

\[
(\hat{R}_2^N[1])_i = tv_i(u) + x,
\]

\[
(\hat{R}_2^N[v])_i = tv_i(u) + x.
\]

(5.56)

(5.57)

We can also use linear combinations of solutions of the two series for the functions \( w_i(u) \) in the equations (5.18)

\[
c_1(\hat{R}_{1,2}^N[1])_i + c_2(\hat{R}_{1,2}^M[v])_i = tv_i(u) + x
\]

(5.58)

with any integers \( N, M \), the operator \( \hat{R}_1 \) or \( \hat{R}_2 \) and arbitrary constants \( c_1, c_2 \).

Further generalization is obtained if we substitute for the characteristic \( \eta^d = x + tv_i(u) \) of the dilatation symmetry group in the formulae (5.15), (5.18) the results of the action on \( \eta^d \) of the operators \( \hat{R}_1^L \) or \( \hat{R}_2^L \) where \( L \) is any integer. Then we obtain the formula

\[
c_1(\hat{R}_{1,2}^N[1])_i + c_2(\hat{R}_{1,2}^M[v])_i = t(\hat{R}_{1,2}^L[v])_i + x(\hat{R}_{1,2}^L[1])_i.
\]

(5.59)

In the cases when \( L > N \) and (or) \( L > M \) this formula is equivalent to using negative powers of the operators \( \hat{R}_1 \) or \( \hat{R}_2 \) in the equations (5.58).

**Remark 5.3.** The operators \( \hat{R}_1 \) and \( \hat{R}_2 \) coincide with the first and second order symmetry operators for the linear system (5.14) essential for the separation of variables in these equations (see [34,35]). Solution of the system (5.14) by means of the separation of variables would mean solving completely the original nonlinear system (5.1). Therefore the linearizing transformation (5.18) may be considered as the extension of the method of separation of variables to nonlinear systems "rich in symmetries".

### 5.6 Higher symmetries of semi-Hamiltonian systems

**Theorem 5.11** (see [12]). All second order symmetries of the semi-Hamiltonian system (5.1) are generated by the second order recursion operator (5.37) out of the hydrodynamic symmetries (5.15) (with \( c = 1 \)). The corresponding Lie equations have the form

\[
u'_x = \eta_{2,i} \equiv \sum_{j=1}^{n} R_{ij}[w_j^i(x + tv_j(u) + w_j(u))] \equiv
\]

\[
- f_i(u')w_{ixx}/(u_x^2) + (f_i(u')/u_x^2) \sum_{j \neq i} \Gamma_{ij}^i u_x +
+ u_x^i \sum_{j \neq i} \Gamma_{ij}^i f_j(u')/u_x^2 + 2f_i(u')\Gamma_{ii}^i + c_i(u') + u_x^i xB_i(u) +
+ t(\hat{R}_2[v])_i + \hat{w}_i(u) \quad (i = 1, 2, \ldots, n)
\]

(5.60)
where the set of functions \( \{ \hat{w}_i(u) \} \) is an arbitrary solution of the linear system (5.14). The existence conditions (5.36) for the second order symmetries and for the second order recursion operators coincide and must be met by a choice of functions \( f_i(u^i), c_i(u^i), d_i(u^i) \). In the case when \( B_i(u) \neq 0 \) and \( (\hat{R}_2[v])_i \neq 0 \) these symmetries explicitly depend on \( x, t \).

The action of the powers \( R^N \) of the recursion operator with \( N = 1, 2, \ldots \) on the same hydrodynamic symmetries generates the infinite series of higher symmetries of the system (5.1)

\[
 u_i^\tau = \sum_{j=1}^{n}(R^N)_{ij}[u_j^\tau(x + tv_j(u) + w_j(u))].
\] (5.61)

If \( R \) is a second order recursion operator, then all these symmetries are of the even order \( 2N \).

### 6 Multi-component diagonal systems explicitly dependent on \( t \) or \( x \)

#### 6.1 Hydrodynamic symmetries of \( t \)-dependent systems

We consider the first order quasi-linear diagonal system of explicitly \( t \)-dependent equations with \( n \geq 3 \)

\[
 u_i^t = v_i(u, t)u_i^x, \quad i = 1, 2, \ldots, n
\] (6.1)

subject to the condition \( v_i \neq v_j \) for \( i \neq j \). We search for hydrodynamic symmetries of this system with the Lie equations

\[
 u_i^\tau = \sum_{j=1}^{n}A_{ij}^\tau(u, t, x)u_j^x, \quad i = 1, 2, \ldots, n
\] (6.2)

and \( x^\tau = t^\tau = 0 \).

Define the functions

\[
 c_{ij}(u, t) = v_{i,u}(u, t)/(v_j - v_i) \quad (i \neq j),
\] (6.3)

\[
 \Gamma_{ij}^\tau(u) = c_{ij}(u, 0).
\] (6.4)

**Theorem 6.1** (see [19, 21, 22]). Diagonal \( n \)-component system (6.1) of the hydrodynamic-type with an explicit time dependence possesses an infinite set of hydrodynamic symmetries of the form (6.2) with a functional arbitrariness iff its coefficients meet the condition (5.12) and the condition

\[
 [v_{i,u}(u, t)/(v_i(u, t) - v_j(u, t))]_t = \beta v_{i,u}(u, t) \quad (i \neq j)
\] (6.5)

with an arbitrary real constant \( \beta \). These symmetries are generated by the Lie equations

\[
 u_i^\tau = A_i^\tau(u, t, x)u_i^x, \quad i = 1, 2, \ldots, n
\] (6.6)

with the coefficients \( A_i \) defined by the formulae
for $\beta \neq 0$

$$A_i(u, t, x) = a_i(u) \exp \left\{ \beta \left[ x + \int_0^t v_i(u, t) dt \right] \right\} + C$$  \hfill (6.7)

and for $\beta = 0$

$$A_i(u, t, x) = a_i(u) + C \left[ x + \int_0^t v_i(u, t) dt \right]$$  \hfill (6.8)

where $C$ is an arbitrary constant and the integrations with respect to $t$ are performed at a constant value of $u$. Here the set of functions $\{a_i(u)\}$ is an arbitrary smooth solution of the linear system

$$a_{i,u}(u) = \Gamma^i_{ij}(u)(a_j - a_i) \quad (i \neq j)$$  \hfill (6.9)

with the coefficients $\Gamma^i_{ij}$ defined by the formulae (6.3), (6.4). The solution manifold of the system (6.9) depends upon $n$ arbitrary functions $c_i(u')$ of one variable which locally parametrize the set of hydrodynamic symmetries.

### 6.2 Hydrodynamic symmetries of $x$-dependent systems

Let coefficients of the diagonal system of ($n \geq 3$) equations explicitly depend on the coordinate $x$

$$u^i_t = \tilde{v}_i(u, x) u^i_x, \quad i = 1, 2, \ldots, n$$  \hfill (6.10)

and meet the condition $\tilde{v}_i \neq \tilde{v}_j$ ($i \neq j$).

Define the functions

$$\tilde{c}_{ij}(u, x) = \tilde{v}_{i,uj}(u, x) / (\tilde{v}_j - \tilde{v}_i) \quad (i \neq j),$$

$$\Gamma^i_{ij}(u) = \tilde{c}_{ij}(u, 0) / \tilde{v}_i(u, 0).$$  \hfill (6.11)

**Theorem 6.2** (see [19, 20, 22]). Diagonal $n$-component system (6.10) of the hydrodynamic type with an explicit $x$-dependence possesses an infinite set of hydrodynamic symmetries of the form (6.2) with a functional arbitrariness iff its coefficients meet the condition (5.12) with a substitution of $\tilde{v}_i$ for $v_i$ and the condition

$$\left[ (\tilde{v}_i^{-1}(u, x))_{uj} / (\tilde{v}_i^{-1}(u, x) - \tilde{v}_j^{-1}(u, x)) \right]_x = \beta (\tilde{v}_i^{-1}(u, x))_{uj} \quad (i \neq j)$$  \hfill (6.13)

with an arbitrary real constant $\beta$. These symmetries are generated by the Lie equations

$$u^i_{\tau} = \tilde{A}_i(u, t, x) u^i_x, \quad i = 1, 2, \ldots, n$$  \hfill (6.14)

with the coefficients $\tilde{A}_i$ defined by the formulae

for $\beta \neq 0$

$$\tilde{A}_i(u, t, x) = \tilde{v}_i(u, x) \left\{ a_i(u) \exp \left[ \beta \left( t + \int_0^x \tilde{v}_i^{-1}(u, x) dx \right) \right] \right\} + C$$  \hfill (6.15)
and for $\beta = 0$

$$\tilde{A}_i (u, t, x) = \tilde{v}_i (u, x) \left\{ a_i (u) + C \left[ t + \int_0^x \tilde{v}^{-1}_i (u, x) dx \right] \right\}$$  \hspace{1cm} (6.16)

where $C$ is an arbitrary constant and the integrations with respect to $x$ are performed at a constant value of $u$. The set of functions $\{a_i (u)\}$ is an arbitrary smooth solution of the linear system (6.9) with the coefficients $\Gamma^i_{ij}$ defined by the formulae (6.11), (6.12). The extent of arbitrariness is the same as in the theorem 6.1.

### 6.3 Invariant solutions and linearization

Equations determining the invariant solutions of the systems (6.1) and (6.10) subject to the constraint $u^i_x \neq 0$ for $i = 1, 2, \ldots, n$ are obtained from the Lie equations (6.6) and (6.14) with the invariance condition $u^i_x = 0$. They have the form $A_i (u, t, x) = 0$ and $\bar{A}_i (u, t, x) = 0$ with the functions $A_i$ and $\bar{A}_i$ defined by the formulae (6.7), (6.8) and (6.15), (6.16) respectively where we put $c = 1$ without the loss of generality. Hence we obtain

for the system (6.1)

$$a_i (u) + \exp \left\{ -\beta \left[ x + \int_0^t v_i (u, t) dt \right] \right\} = 0 \hspace{1cm} (\beta \neq 0),$$  \hspace{1cm} (6.17)

$$a_i (u) + x + \int_0^t v_i (u, t) dt = 0 \hspace{1cm} (\beta = 0),$$  \hspace{1cm} (6.18)

and for the system (6.10)

$$a_i (u) + \exp \left[ -\beta \left( t + \int_0^x \tilde{v}^{-1}_i (u, x) dx \right) \right] = 0 \hspace{1cm} (\beta \neq 0),$$  \hspace{1cm} (6.19)

$$a_i (u) + t + \int_0^x \tilde{v}^{-1}_i (u, x) dx = 0 \hspace{1cm} (\beta = 0)$$  \hspace{1cm} (6.20)

where $i = 1, 2, \ldots, n$.

Here the set of functions $a_i (u)$ is an arbitrary smooth solution of the linear system (6.9). Thus, the equations (6.17)–(6.20) determine a linearizing transformation for the systems (6.1), (6.10), i.e. the solution of these systems reduces to the solution of the linear system (6.9). These equations determine the solutions $u^i = u^i (x, t)$ of the original nonlinear system if the conditions of the implicit-function theorem are met.

More complete results on the diagonal systems with explicit $t$ or $x$ dependence and the example of a new integrable system of this class can be found in the recent paper [22].
7 Conclusions

The existence of an infinite-dimensional group of the hydrodynamic symmetries for the equations of the hydrodynamic type is an important property which provides the existence of linearizing transformations. The reason for this is that the degree of generality of the set of symmetries coincides with the degree of generality of the general solution set for these equations. Therefore, almost all solutions are the invariant solutions and they are obtained by standard formulae provided the symmetries are already determined. Such a formula gives a linearizing transformation reducing the original non-linear problem to the linear problem of determining the symmetries. The additional property is the existence of the recursion operator which makes it possible to solve partially the linear problem by constructing infinite series of its solutions and hence solutions of the original non-linear equations. The existence of such an operator also has a group-theoretical basis, since the recursion operator is completely determined by the symmetries of the determining equations for the hydrodynamic symmetries, i.e., by the 'symmetries of symmetries'.

This shows a group-theoretical origin of linearizing transformations and of the integrability property by which we mean a possibility to construct infinitely many exact solutions. The Hamiltonian structure, if it exists, does not improve the integrability properties of the equations of the hydrodynamic type. We have to conclude that the symmetry is the major necessary property that insures the integrability which was the original idea of S. Lie.

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