Gravity Duals of Fractional Branes
and Logarithmic RG Flow

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Abstract

We study fractional branes in $\mathcal{N} = 2$ orbifold and $\mathcal{N} = 1$ conifold theories. Placing a large number $N$ of regular D3-branes at the singularity produces the dual $\text{AdS}_5 \times X^5$ geometry, and we describe the fractional branes as small perturbations to this background. For the orbifolds, $X^5 = S^5/\Gamma$ and fractional D3-branes excite complex scalars from the twisted sector which are localized on the fixed circle of $X^5$. The resulting solutions are given by holomorphic functions and the field-theoretic beta-function is simply reproduced. For $N$ regular and $M$ fractional D3-branes at the conifold singularity we find a non-conformal $\mathcal{N} = 1$ supersymmetric $SU(N + M) \times SU(N)$ gauge theory. The dual Type IIB background is $\text{AdS}_5 \times T^{1,1}$ with NS-NS and R-R 2-form fields turned on. This dual description reproduces the logarithmic flow of couplings found in the field theory.
1. Introduction

By now there exists an impressive body of evidence that Type IIB strings on $\text{AdS}_5 \times X^5$ are dual to large $N$ strongly coupled 4-d conformal gauge theories, in the sense proposed in [1,2,3]. Here $X^5$ are positively curved 5-d Einstein spaces whose simplest example $S^5$ corresponds to the $\mathcal{N} = 4$ supersymmetric $SU(N)$ gauge theory. One may also quotient this duality by a discrete subgroup $\Gamma$ of the $SU(4)$ R-symmetry [4,5]. The resulting backgrounds with $X^5 = S^5/\Gamma$ are dual to “quiver gauge theories” with gauge group $S(U(N)^n)$ and bifundamental matter [6], which describe D3-branes near orbifold singularities. In such orbifold theories, in addition to regular D-branes which can reside on or off the orbifold fixed plane there are also “fractional” D-branes pinned to the fixed plane [7,8]. Our goal in this paper is to consider the effect of such fractional branes on the dual supergravity background. There is good motivation for studying this problem because, as we discuss below, introduction of fractional branes breaks the conformal invariance and introduces RG flow.

It has also been possible to construct dual gauge theories for $X^5$ which are not locally $S^5$. The simplest example is $X^5 = T^{1,1} = (SU(2) \times SU(2))/U(1)$ which turns out to be dual to an $\mathcal{N} = 1$ superconformal $SU(N) \times SU(N)$ gauge theory with a quartic superpotential for bifundamental fields [8,10]. In this theory, which arises on D3-branes at the conifold singularity, it is also possible to introduce fractional D-branes [11,12], and we study their effects in this paper.

Having constructed the gravity duals of fixed-point theories, the next logical step is to study the dual picture of the RG flow. A natural setup for this problem is provided by the supersymmetric flows connecting orbifold theories and the (generalized) conifold theories. It is clear, though, that in order to have a consistent picture one cannot restrict oneself to the $\Gamma$ invariant supergravity fields only, and needs to add the massless fields coming from the twisted sectors. This program was initiated in [13,14], and in this paper we focus on the role of the twisted sector fields in creating RG flows. Considerable progress on supersymmetric RG flows in other situations has also been made recently [15] (for a review see [16]). For important work done in studies of non-supersymmetric RG flows see [17,18].

The basic picture common to all RG problems is that the radial coordinate $r$ of $\text{AdS}_5$ defines the RG scale of the field theory, hence the scale dependence of couplings may be read off from the radial dependence of corresponding supergravity fields.

Since the RG flows of couplings in physically relevant gauge theories are logarithmic, an important problem is to find gravity duals of logarithmic flows. Attempts in this direction have been made in the context of Type 0B string theory [18,19,20]. This is an NSR
string with the non-chiral GSO projection \((-1)^{F_+F} = 1\) which breaks all spacetime supersymmetry \([21]\) (it is also a \((-1)^{F_s}\) orbifold of the Type II B theory). Type 0B theory has two basic types of D3-branes, electric and magnetic, and we will see that it is appropriate to call them the fractional branes. If equal numbers of the electric and magnetic branes are stacked parallel to each other, then we find on their world volume a \(U(N) \times U(N)\) gauge theory coupled to six adjoint scalars of the first \(U(N)\), six adjoint scalars of the second \(U(N)\), and Weyl fermions in bifundamental representations. This theory is a “regular” \(\mathbb{Z}_2\) orbifold of the \(\mathcal{N} = 4\) \(U(2N)\) gauge theory and hence is conformal in the planar limit \([22,23]\).

In general, in orbifold theories there are as many types of fractional branes as there are nodes of the quiver diagram (i.e. the number of gauge groups in the product), or the number of the irreducible representations of the orbifold group \(\Gamma\). If one takes a collection of the fractional branes of each type and the branes corresponding to the irreducible representation \(\mathcal{R}_i\) are taken \(n_i = \dim \mathcal{R}_i\) times then one gets a single regular D-brane which can depart to the bulk. More specifically, the charge of the fractional brane of \(\mathcal{R}_i\) type is

\[
q_i = \frac{n_i}{|\Gamma|} \tag{1.1}
\]

where \(|\Gamma|\) is the order of the orbifold group. It is well known that \(\sum_i n_i q_i = 1\).

The fractional branes act as sources for the twisted closed string states of the orbifold theory. In the Type 0B example discussed above the two \(U(N)\) groups correspond to the electric and the magnetic D3-branes, hence these are the two types of fractional branes for this particular \(\mathbb{Z}_2\) orbifold. Indeed, such branes have tadpoles for the twisted RR 4-form and for the tachyon \([18,24]\). If we stack \(N\) parallel electric branes only, then we find an “irregular” orbifold theory where the \(\mathbb{Z}_2\) action does not act on the gauge indices. This gives \(SU(N)\) gauge theory coupled to six adjoint scalars, which is not a CFT \([18]\).

Equations satisfied by the gravity dual of this theory were derived in \([18]\), and solved after some assumptions in \([19,20]\). The RG flow of the dilaton, which is related to the gauge coupling, comes from the equation

\[
\nabla^2 \phi = -\frac{1}{4\alpha'} T^2 e^{\phi/2}. \tag{1.2}
\]

\(^1\) The \(\mathbb{Z}_2\) is generated by \((-1)^{F_s}\), where \(F_s\) is the fermion number, together with conjugation by \(\gamma = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\) where \(I\) is the \(N \times N\) identity matrix. This orbifold is called regular because \(\gamma\) is traceless.
Since the tachyon field $T$ has a source $F_5^2$, it departs from zero and causes the dilaton to depend on $r$. Assuming that $T$ approaches a constant for large $r$ it was found that the RG flow is logarithmic in the UV \cite{13,20}. While this scenario has a number of uncertainties (due to the lack of detailed knowledge of the $T$-dependence in the effective action) it suggests a mechanism for RG flow of couplings in the dual gravity picture of fractional branes. In particular, the presence of “twisted” fields sourced by the fractional branes plays the crucial role.

The stack of electric D3-branes defines gauge theory coupled to six scalars fields, but it is obviously of more interest to remove the scalars and study the pure glue theory \cite{25}. One way of embedding it into string theory is to consider a $\mathbb{Z}_2$ orbifold of Type 0B by reflection of six coordinates \cite{26}. The regular orbifold theory has gauge group $U(N)^4$ coupled to a chiral field content:

- four quadruples of bi-fundamental fermions transforming in $(N_i, N_{i+1})$, $i = 0, 1, 2, 3$; $4 \equiv 0$,

- four sextets of bi-fundamental scalars in $(N_i, N_{i+2})$.

Now there are 4 gauge groups in the product, hence there should be 4 different types of fractional branes. If we stack $N$ fractional D3-branes of the same type then we find pure glue $U(N)$ gauge theory on their world volume \cite{26}. This suggests that fractional branes may provide a link to string duals of realistic gauge theories.

With this eventual goal in mind, in this paper we study gravity duals of fractional branes in supersymmetric conifold and orbifold theories: the SUSY removes some of the effective action uncertainties present in the Type 0B case. To simplify matters further we consider theories where $\beta$–functions for ‘t Hooft couplings $g_{YM}^2 N$ are of order $1/N$ rather than of order 1. Such gauge theories occur on $M$ fractional D3-branes parallel to $N$ regular D3-branes, with $M$ held fixed in the large $N$ limit. In the simplest examples we consider, the gauge group is then $SU(N + M) \times SU(N)$.

The large number $N$ of regular D3-branes produces the dual $\text{AdS}_5 \times X^5$ background, and for our purposes we may ignore the back-reaction of the $M$ fractional D3-branes on it. However, the fractional branes act as sources for an extra set of fields, namely the 2-form potentials $B^{NSNS}$ and $B^{RR}$. The flux of these 2-forms through a certain 2-cycle of $X^5$ (more precisely its deviation from the value at the orbifold point) defines the difference between $g_{YM}^{-2}$ for the gauge groups factors. In the $\mathcal{N} = 2$ supersymmetric orbifold cases the 2-cycles are collapsed, so that the twisted sector fields corresponding to the 2-form fluxes are confined to $\text{AdS}_5 \times S^1$ where $S^1$ is the fixed circle of $S^5/\Gamma$. By studying the dependence

It seems unclear, however, whether the fractional D3-branes in the $\mathbb{Z}_4$ theory which are stuck to the fixed fourplane exist, for their flux has nowhere to escape to.
of these twisted sector fields on the $\text{AdS}_5$ radial coordinate $r$ we find the logarithmic flow of the gauge couplings consistent with field theory expectations. In fact, the twisted sector fields are given by holomorphic functions of the complex variable $z = re^{i\phi}$ where $\phi$ is the $S^1$ coordinate (previously known solutions of $\mathcal{N} = 2$ gauge theories are also characterized by holomorphic functions $[27,28,29]$). Remarkably, in the limit we are studying all string-scale corrections are small, so that the use of effective supergravity equations is justified.

2. Fractional Branes on the Conifold.

Let us start by reviewing what is known about regular D3-branes at conical singularities. If a large number $N$ of D3-branes is placed at the apex of a 6-d cone $Y^6$ with metric

$$ds^2_{\text{cone}} = dr^2 + r^2 ds^2_5,$$

then the near-horizon region of the resulting 10-d geometry has the metric

$$ds^2 = R^2 \left[ \frac{r^2}{R^4} (-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + \frac{dr^2}{r^2} + ds^2_5 \right], \quad R^4 \sim g_s N (\alpha')^2. \quad (2.2)$$

This geometry is $\text{AdS}_5 \times X^5$ where $X^5$ is the base of the cone (if $Y_6$ is Ricci-flat then $X^5$ is a positively curved Einstein space $[31,9]$). Type IIB theory on this background is conjectured to be dual to the conformal limit of the gauge theory on $N$ D3-branes placed at the apex $[9,10]$.

An example where such a duality has been tested extensively is when $Y^6$ is the conifold, which is a singular Calabi-Yau manifold described in terms of complex variables $w_1, \ldots, w_4$ by the equation $[32]$

$$\sum_{a=1}^4 w_a^2 = 0.$$

The base of this cone is $T^{1,1} = (SU(2) \times SU(2))/U(1)$ whose Einstein metric may be written down explicitly $[32],$

$$ds^2_{X_5} = \frac{1}{9} \left( d\psi + \cos \theta_1 d\phi_1 + \cos \theta_2 d\phi_2 \right)^2 + \frac{1}{6} \sum_{a=1}^2 \left( d\theta_a^2 + \sin^2 \theta_a d\phi_a^2 \right). \quad (2.3)$$

The $\mathcal{N} = 1$ superconformal field theory on $N$ regular D3-branes placed at the singularity of the conifold has gauge group $SU(N) \times SU(N)$ and global symmetry $SU(2) \times SU(2) \times U(1)$

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3 Supergravity duals of logarithmic RG flows in $\mathcal{N} = 2$ gauge theories are being independently studied in $[30]$. 

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The chiral superfields $A_1, A_2$ transform as $(\mathbf{N}, \mathbf{\bar{N}})$ and are a doublet of the first $SU(2)$; the chiral superfields $B_1, B_2$ transform as $(\mathbf{\bar{N}}, \mathbf{N})$ and are a doublet of the second $SU(2)$. The R-charge of all four chiral superfields is $1/2$ and the theory has an exactly marginal superpotential $W = \epsilon^{ij} \epsilon^{kl} \text{Tr} A_i B_k A_j B_l$.

IIB supergravity modes on $\text{AdS}_5 \times \mathbf{T}^{1,1}$ have been matched in some detail with operators in this gauge theory whose dimensions are of order 1 in the large $N$ limit \[33\]-\[34\]. In addition, string theory has heavy supersymmetric states obtained by wrapping D3-branes over 3-cycles of $\mathbf{T}^{1,1}$. They have been shown \[11\] to correspond to “dibaryon” operators whose dimensions grow as $3N/4$ (schematically, these operators have the form $\text{Det} A$ or $\text{Det} B$). Further, one may consider a domain wall in $\text{AdS}_5$ obtained by wrapping a D5-brane over the 2-cycle of $\mathbf{T}^{1,1}$ (topologically, $\mathbf{T}^{1,1}$ is $\mathbf{S}^2 \times \mathbf{S}^3$). If this domain wall is located at $r = r_*$ then, by studying the behavior of wrapped D3-branes upon crossing it, it was shown in \[11\] that for $r > r_*$ the gauge group changes to $SU(N + 1) \times SU(N)$. Note that this is precisely the gauge theory expected on $N$ regular and one fractional D3-branes!

Thus, a D5-brane wrapped over the 2-cycle is nothing but a fractional D3-brane placed at a definite $r$. The identification of a fractional D3-brane with a wrapped D5-brane is consistent with the results of \[33\]-\[36\]-\[12\].

As shown in \[11\], this suggests a construction of the Type IIB dual for the $N = 1$ $SU(N + M) \times SU(N)$ gauge theory. In particular, the background has to contain $M$ units of R-R 3-form flux through the 3-cycle of $\mathbf{T}^{1,1}$:

$$\int_{C^3} H^{RR} = M. \tag{2.4}$$

If $M$ is fixed as $N \to \infty$ then the back-reaction of $H^{RR}$ on the metric and the $F_5$ background may be ignored to leading order in $N$. However, as we will show, the background must also include the NS-NS 2-form potential

$$B^{NSNS} = e^{\phi} f(r) \omega_2, \tag{2.5}$$

where $\omega_2$ is the closed 2-form corresponding to the 2-cycle which is dual to $C^3$,

$$\int_{C^2} \omega_2 = 1. \tag{2.6}$$

The desired connection with the RG flow is due to the fact that \[33\]-\[34\]-\[10\]

$$\frac{1}{g_1^2} - \frac{1}{g_2^2} \sim e^{-\phi} \left( \int_{C^2} B^{NSNS} - \frac{1}{2} \right). \tag{2.7}$$

\[4\] Similarly, a regular D3-brane serves as a domain wall between $SU(N) \times SU(N)$ and $SU(N + 1) \times SU(N + 1)$ gauge theory. This has a simple interpretation in terms of Higgsing the theory.
where \( g_1 \) and \( g_2 \) are the gauge couplings for \( SU(N+M) \) and \( SU(N) \) respectively. Therefore, the \( f(r) \) in (2.3) gives the dual supergravity definition of the scale dependence of \( \frac{1}{g_1^2} - \frac{1}{g_2^2} \).

Before solving for \( f(r) \) let us recall the \( \beta \)-function calculation in field theory. There we have

\[
\frac{d}{d \log(\Lambda/\mu)} \frac{1}{g_1^2} \sim 3(N + M) - 2N(1 - \gamma_A - \gamma_B) ,
\]

\[
\frac{d}{d \log(\Lambda/\mu)} \frac{1}{g_2^2} \sim 3N - 2(N + M)(1 - \gamma_A - \gamma_B) ,
\]

where \( \gamma \) are the anomalous dimensions of the fields \( A_i \) and \( B_j \). For \( M = 0 \) we find a fixed point with \( \gamma_A = \gamma_B = -1/4 \) which corresponds to R-charge 1/2. This is the superconformal gauge theory dual to \( AdS_5 \times T^{1,1} \) with vanishing 2-form potentials. For \( M \neq 0 \) it is impossible to make both beta functions vanish (even if we allow the anomalous dimensions of \( A \) and \( B \) to be different) and the theory undergoes logarithmic RG flow:

\[
\frac{1}{g_1^2} - \frac{1}{g_2^2} \sim M \log(\Lambda/\mu)[3 + 2(1 - \gamma_A - \gamma_B)] .
\]

Near the fixed point we expect \( \gamma_A + \gamma_B = -1/2 \) plus small corrections, hence the RHS gives \( \sim M \log(\Lambda/\mu) \).

Let us reproduce this result in supergravity. We need the Type IIB SUGRA equations of motion involving the 2-form gauge potentials. We will write these equations in the \( AdS_5 \times T^{1,1} \) background with constant \( \tau = C_0 + ie^{-\phi} \) (this is the \( SL_2(\mathbb{Z}) \) covariant combination of the dilaton and the R-R scalar of the Type IIB theory):

\[
d^*G = iF_5 \wedge G \ .
\]

\( G \) is the complex 3-form field strength,

\[
G = H^{RR} + \tau H^{NSNS} ,
\]

which satisfies the Bianchi identity \( dG = 0 \). Note that the RHS of (2.10) originates from the Chern-Simons term

\[
\int C_4 \wedge H^{RR} \wedge H^{NSNS} .
\]

Since the fractional D3-brane (the wrapped D5-brane) creates R-R 3-form flux through \( T^{1,1} \), \( H^{RR} \) should be proportional to the closed 3-form which was constructed in [11],

\[
H^{RR} \sim Me^\psi \wedge (e^{\theta_1} \wedge e^{\phi_1} - e^{\theta_2} \wedge e^{\phi_2}) .
\]

\footnote{We will check later that this background has no corrections of order \( M/N \) which is the order at which we are working.}
Here we are using the basis 1-forms

\[ e^\psi = \frac{1}{3} (d\psi + \sum_{i=1}^{2} \cos \theta_i d\phi_i), \quad e^{\theta_i} = \frac{1}{\sqrt{6}} d\theta_i, \quad e^{\phi_i} = \frac{1}{\sqrt{6}} \sin \theta_i d\phi_i. \quad (2.14) \]

In these coordinates, the closed form \(\omega_2\) which enters \((2.5)\) is given by

\[ \omega_2 \sim e^{\theta_1} \wedge e^{\phi_1} - e^{\theta_2} \wedge e^{\phi_2}, \]

so that

\[ e^{-\phi} H^{NSNS} \sim df(r) \wedge (e^{\theta_1} \wedge e^{\phi_1} - e^{\theta_2} \wedge e^{\phi_2}). \]

Since \(F_5 = \text{vol(AdS}_5) + \text{vol(T}^{1,1})\), \(F_5 \wedge H^{NSNS} = 0\). Let us set the R-R scalar \(C_0 = 0\). Then we see that the real part of \((2.14)\) is satisfied for all \(f(r)\). From the imaginary part we have

\[ \frac{1}{r^3} \frac{d}{dr} \left( r^5 \frac{d}{dr} f(r) \right) \sim M, \]

which implies

\[ f(r) \sim M \log r. \quad (2.15) \]

Quite remarkably, our solution of Type IIb SUGRA equations has reproduced the field theoretic beta function for \(\frac{1}{g_1^2 N} - \frac{1}{g_2^2 N}\) to order \(M/N\). This establishes the gravity dual of the logarithmic RG flow in the \(\mathcal{N} = 1\) supersymmetric \(SU(N+M) \times SU(N)\) gauge theory on \(N\) regular and \(M\) fractional D3-branes placed at the conifold singularity.

It is not hard to see that there are no other effects of order \(M/N\): the back-reaction of the 3-form field strengths on the metric, dilaton and the \(F_5\) comes in at order \((M/N)^2\). Consider, for instance, the dilaton equation of motion:

\[ \nabla^2 \phi = \frac{1}{12} \left( e^{\phi} H_{RR}^2 - e^{-\phi} H_{NSNS}^2 \right). \]

Even without checking the relative normalizations of \(H_{RR}^2\) and \(H_{NSNS}^2\), we can immediately see that the variation of \(e^{-\phi}/N\) is at most of order \((M/N)^2\). In the field theory this quantity translates into \(\frac{1}{g_1^2 N} + \frac{1}{g_2^2 N}\). The fact that there is no \(\beta\)-function of order \(M/N\) for this quantity agrees with the field theory RG equations provided that the sum of the anomalous dimensions, \(\gamma_A + \gamma_B\), has no corrections of order \(M/N\). This is a simple gravitational prediction about the gauge theory. It is of further interest to study the order \((M/N)^2\) effects on the background and compare them with field theory, but we postpone these calculations for future work.
3. Fractional Branes in $\mathcal{N} = 2$ Orbifold Theories.

In this section we will be concerned with the $\mathcal{N} = 2$ supersymmetric orbifolds of Type IIB strings. Before introducing D-branes these are backgrounds of the form $\mathbb{R}^{5,1} \times \mathbb{R}^4 / \Gamma$. To write down gravity duals of fractional branes we follow the strategy used in the last section: we first stack a large number $N$ of regular D3-branes creating the dual $\text{AdS}_5 \times S^5 / \Gamma$ background and then introduce the fractional branes as small perturbations on this background. We find an important difference from the conifold case, however, because $S^5 / \Gamma$ does not have any finite volume 2-cycles. A blowup of this space produces 2-cycles but breaks the $\mathcal{N} = 2$ supersymmetry. For this reason we work with the singular space where the 2-cycles are collapsed (we will see that this turns out to be simpler than the non-singular case discussed in the previous section).

Thus we consider the Type IIB superstring in the background $\text{AdS}_5 \times S^5 / \Gamma$ where $\Gamma$ is one of the A,D,E subgroups of $SU(2)$ whose action on $S^5$ is induced from the standard action on $\mathbb{R}^4 \approx \mathbb{C}^2$ times the trivial action on $\mathbb{R}^2$. Therefore, we get the action of $\Gamma$ on $\mathbb{R}^6$ having a fixed two-plane $0 \times \mathbb{R}^2$. This action descends to $S^5$ and the fixed plane becomes a fixed circle $S^1$.

The metric on $S^5 / \Gamma$ reads:

$$ds^2_5 = d\theta^2 + \cos^2 \theta d\varphi^2 + \sin^2 \theta ds^2_{S^1 / \Gamma},$$

and the fixed circle is at $\theta = 0$.

String theory in this background has extra massless fields compared to the $\Gamma$-invariant fields of the ten dimensional IIB supergravity. These fields are localized at the fixed circle $S^1$. In the paper [13] the multiplets of the five dimensional gauge supergravity these fields fall in were identified.

The simple meaning of these fields is the following: if one were to blow up the fixed circle to obtain a smooth Einstein metric then the topology of the resulting fivefold is such that the non-contractable two-spheres $C_i$ are supported. These spheres intersect each other according to the Dynkin diagram of the corresponding A,D,E Lie group:

$$C_i \cap C_j = a_{ij}$$

Now, reducing the ten dimensional supergravity fields along these cycles leads to new fields in six dimensions spanned by $S^1 \times \text{AdS}_5$. Of particular importance for us are the NSNS and RR two-forms $B_{NSNS}$ and $B_{RR}$. They give rise to the scalars:

$$\beta^i_{NSNS} = \int_{C_i} B_{NSNS}, \quad \beta^i_{RR} = \int_{C_i} B_{RR} \quad (3.2)$$
These fields are present even if one did not perform the blowup: they come from the twisted sector of the string theory.

We now proceed with writing the effective action for these fields (here we have set the R-R scalar to zero):

$$\int_{S^1 \times AdS_5} \sqrt{g} g^{mn} a^{ij} \left( e^{-\phi} \partial_m \beta_i^{NSNS} \partial_n \beta_j^{NSNS} + e^\phi \partial_m \beta_i^{RR} \partial_n \beta_j^{RR} \right) + a^{ij} F_5 \wedge \beta_i^{NSNS} \wedge d\beta_j^{RR}$$

(3.3)

The last term comes from the 10-d Chern-Simons coupling of the form

$$\int F_5 \wedge B_{NSNS} \wedge dB_{RR} .$$

Introduce the complex fields:

$$\gamma_i = \tau \beta_i^{NSNS} + \beta_i^{RR} .$$

These fields transform nicely under the SL$_2(\mathbb{Z})$ group:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} \gamma_i \\ \tau \end{pmatrix} = \frac{1}{c\tau + d} \begin{pmatrix} \gamma_i \\ a\tau + b \end{pmatrix}$$

The metric on the $S^1 \times \text{AdS}_5$ space is:

$$ds^2 = R^2 \left( \frac{dr^2}{r^2} + d\varphi^2 \right) + \frac{r^2}{R^2} (-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) , \quad R^4 = 4\pi g_s N\ell_s^4 ,$$

where $\varphi$ is the coordinate on the circle $S^1$.

Let us introduce the complex coordinate on the space transverse to $\mathbb{R}^{1,3}$: $z = re^{i\phi}$. We are interested in fields which have no $\mathbb{R}^{1,3}$ dependence. For these fields the action (3.3) takes on the following form (assuming that $\tau$ is constant, otherwise we get covariant derivatives instead of ordinary ones):

$$S = \int r^5 a^{ij} \frac{1}{\tau_2} \frac{\partial \gamma_i}{\partial z} \frac{\partial \gamma_j}{\partial \bar{z}} dr d\varphi$$

(3.4)

where $\tau_2 = e^{-\phi} = \text{Im}\tau$. This action is SL$_2(\mathbb{Z})$ invariant.

**Gauge couplings.** The theory on the boundary of the AdS$_5$ space is the superconformal quiver gauge theory with the gauge group $SU(Nn_0) \times SU(Nn_1) \times \ldots \times SU(Nn_r)$ where $n_i$ are the Dynkin indices – the dimensions of the irreps of $\Gamma$. $n_0 = 1$ is the dimension of the trivial representation $\mathcal{R}_0$. 
The relation between the boundary values of $\gamma_i$ and the couplings of these gauge factors was shown in [5] to be:

$$\begin{align*}
\tau_i &= \gamma_i, \quad i = 1, \ldots, r \\
\tau_0 &= |\Gamma|\tau - \sum_i n_i \tau_i
\end{align*}$$

(3.5)

where

$$\tau_j = \frac{\theta_j}{2\pi} + \frac{4\pi i}{g_j^2}.$$

Running of the dilaton. Now let us discuss the validity of our assumption that $\tau$ is constant. The Lagrangian for the $\vec{x}$ independent dilaton reads as follows:

$$L = \int r^3 t^3 dr dtd\phi \frac{1}{\tau_2} \left( t^2 \partial_r \tau \partial_r \bar{\tau} + (1 - t^2) \partial_t \tau \partial_t \bar{\tau} + \frac{1}{1 - t^2} \partial_\phi \tau \partial_\phi \bar{\tau} \right)$$

(3.6)

where we introduced the notation $t = \sin \theta$. The fixed circle is at $t = 0$. The fields $\gamma$ acts as sources for the $\tau$ equations of motion. Irrespectively of the precise form of (3.6) the source term in the dilaton equation of motion is:

$$\sim \delta(t) r^5 e^\phi a^{ij} (\partial \gamma_i \bar{\partial} \gamma_j + \partial \bar{\gamma}_i \bar{\partial} \bar{\gamma}_j),$$

(3.7)

(this expression is valid for $C_0 = 0$, $\phi = \text{const}$). Thus, the source term vanishes for holomorphic $\gamma$ and the dilaton is allowed to remain constant. This argument may suffer from some subtleties in case of badly singular $\gamma_i$.

Fractional branes. In string theory $B_{NSNS}, B_{RR}$ do not have to be globally well-defined two-forms - they behave like gauge fields. The same applies to the scalars $\beta_i$ obtained by the reduction of the $B$-fields. In particular, if we add the $k$-th fractional threebrane (D5-brane wrapped over $C_k$) at some point $z_*$ then the scalars $\beta_j^{RR}$ pick up a shift when circled around its location:

$$\beta_j^{RR} \rightarrow \beta_j^{RR} + a_{kj}$$

The brane being BPS does not spoil the holomorphicity of the $\gamma$’s, hence we conclude that the solution must have a logarithmic monodromy:

$$\gamma_j \sim \frac{a^{kj}_{ij}}{2\pi i} \log(z - z_*) + \ldots$$

(3.8)

One may be concerned about the appearance of the logarithm because $\gamma_j$ is defined on a torus with modular parameter $\tau$. Luckily, $\tau_2 = \frac{1}{g_s}$ is of order $N$ in the ‘t Hooft limit and
the periodicity in this direction may be ignored for large \( N \) (this is because the \( \beta \)-functions for \( \gamma_j \) are of order 1 for a finite number of fractional branes, so that the evolution of the couplings is relatively slow). In any event, the theory with a fractional brane may be regulated far in the UV by adding a fractional anti-brane at large \( r \):

\[
\gamma_j \sim \frac{a^{kj}}{2\pi i} \left[ \log(z - z_\ast) - \log(z - z_{\text{reg}}) \right] + \ldots
\]

This way the theory becomes conformal again for \( r \gg |z_{\text{reg}}| \). On the other hand, the singularities of \( \gamma_i \) at the locations of fractional branes are presumably removed by the instanton corrections which will be discussed below.

**Absence of the dilaton running from the field theory expectations.** The fractional brane of the \( R_i \) type affects the gauge theory in a simple way: by changing the \( SU(Nn_i) \) factor into \( SU(Nn_i + 1) \) and not changing the rest.

Clearly this induces a non-trivial beta functions for all couplings \( \tau_j \), such that \( j = i \) or \( a_{ij} \neq 0 \). In fact:

\[
\frac{d}{d\log(\Lambda/\mu)} \tau_j = 2Nn_j - \sum_{k \neq i} a_{jk}Nn_k + (Nn_i + 1)a_{ij} = -a_{ij}
\]

\[
\frac{d}{d\log(\Lambda/\mu)} \tau_i = 2(Nn_i + 1) - \sum_j a_{ij}Nn_j = 2
\]

Clearly:

\[
\frac{d}{d\log(\Lambda/\mu)} \sum_{k=0}^{r} n_k \tau_k = \left( -\sum_j a_{ij}n_j + 2n_i \right) = 0
\]

On the other hand, from (3.5) we see that

\[
\tau = \frac{1}{|\Gamma|} \sum_k n_k \tau_k
\]

and we come to the complete agreement with the space-time picture of the dilaton being constant.

**Instanton corrections.** As the space-time dilaton is constant for the solutions we discuss we may hope that the solutions we find are valid, at least far away from the locations of the fractional branes, but not too far to make some of the gauge factors extremely strongly coupled. When we approach the branes, the \( \log(z) \) behaviour \( \gamma \) makes the fractional D-instantons favorable. The creation amplitude of the fractional D-instanton of the \( R_j \) type is clearly of the order of

\[
\sim \exp 2\pi i \gamma_j \to \infty \quad (3.9)
\]
near the fractional D3-brane. The fractional \((p, q)\) D-instantons are the euclidean \((p, q)\)-string world sheets wrapped around the collapsed two cycles \(C_j\). The field theory counterpart of the picture which we are advocating is the possibility of having instanton corrections coming from the instantons of each group factor. Also, the appearance of the \((p, q)\) types of the instantons is the consequence of the S-duality non-trivially realized in the field theories under consideration.

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References

[1] J. Maldacena, “The Large N limit of superconformal field theories and supergravity,” Adv. Theor. Math. Phys. 2 (1998) 231, hep-th/9711200.

[2] S.S. Gubser, I.R. Klebanov, and A.M. Polyakov, “Gauge theory correlators from non-critical string theory,” Phys. Lett. 428B (1998) 105, hep-th/9802109.

[3] E. Witten, “Anti-de Sitter space and holography,” Adv. Theor. Math. Phys. 2 (1998) 253, hep-th/9802150.

[4] S. Kachru and E. Silverstein, “4d conformal field theories and strings on orbifolds,” Phys. Rev. Lett. 80 (1998) 4855, hep-th/9802183.

[5] A. Lawrence, N. Nekrasov and C. Vafa, “On conformal field theories in four dimensions,” Nucl. Phys. B533 (1998) 199, hep-th/9803015.

[6] M. Douglas and G. Moore, “D-branes, quivers, and ALE instantons,” hep-th/9603167.

[7] E.G. Gimon and J. Polchinski, “Consistency Conditions for Orientifolds and D Manifolds”, Phys. Rev. D54 (1996) 1667, hep-th/9601038.

[8] M.R. Douglas, “Enhanced Gauge Symmetry in M(atrix) theory”, JHEP 007(1997) 004, hep-th/9612126.

[9] I.R. Klebanov and E. Witten, “Superconformal Field Theory on Threebranes at a Calabi-Yau Singularity,” Nucl. Phys. B536 (1998) 199, hep-th/9807080.

[10] D.R. Morrison and M.R. Plesser, “Non-Spherical Horizons, I,” hep-th/9810201.

[11] S.S. Gubser and I.R. Klebanov, “Baryons and Domain Walls in an N=1 Superconformal Gauge Theory,” Phys. Rev. D58 (1998) 125025, hep-th/9808075.

[12] K. Dasgupta and S. Mukhi, “Brane Constructions, Fractional Branes and Anti-de Sitter Domain Walls,” hep-th/9904131.

[13] S. Gubser, N. Nekrasov, S. Shatashvili, “Generalized Conifolds and 4d N=1 SCFT,” hep-th/9811230.

[14] D.Z. Freedman, S.S. Gubser, K. Pilch, N.P. Warner, “Renormalization Group Flows from Holography–Supersymmetry and a c-Theorem,” hep-th/9904017.

[15] S.S. Gubser, “Non-conformal examples of AdS/CFT,” hep-th/9910117.

[16] L. Girardello, M. Petrini, M. Porrati, A. Zaffaroni, “Novel Local CFT and Exact Results on Perturbations of N=4 Super Yang Mills from AdS Dynamics,” hep-th/9810126.

[17] J. Distler and F. Zamora, “Nonsupersymmetric conformal field theories from stable anti-de Sitter spaces,” hep-th/9810206; “Chiral Symmetry Breaking in the AdS/CFT Correspondence,” hep-th/9911040.

[18] I. R. Klebanov and A. A. Tseytlin, “D-Branes and Dual Gauge Theories in Type 0 Strings,” Nucl. Phys. B546 (1999) 155, hep-th/9811035.

[19] J. Minahan, “Glueball Mass Spectra and Other Issues for Supergravity Duals of QCD Models,” hep-th/9811156.
[20] I.R. Klebanov and A.A. Tseytlin, “Asymptotic Freedom and Infrared Behavior in the Type 0 String Approach to Gauge Theory,” Nucl. Phys. B547 (1999) 143, hep-th/9812089

[21] L. Dixon and J. Harvey, “String theories in ten dimensions without space-time supersymmetry”, Nucl. Phys. B274 (1986) 93;
N. Seiberg and E. Witten, “Spin structures in string theory”, Nucl. Phys. B276 (1986) 272;
C. Thorn, remarks at the workshop “Superstring Theories and the Mathematical Structure of Infinite-Dimensional Lie Algebras”, Santa Fe Institute, November 1985

[22] I. R. Klebanov and A. A. Tseytlin, “Non-supersymmetric CFT from Type 0 String Theory,” JHEP 9903(1999) 015, hep-th/9901101

[23] N. Nekrasov and S. Shatashvili, “On non-supersymmetric CFT in four dimensions,” hep-th/9902110, L. Okun Festschrift, North-Holland, in press

[24] O. Bergman and M. Gaberdiel, “A Non-supersymmetric Open String Theory and S-Duality,” Nucl. Phys. B499 (1997) 183, hep-th/9701137

[25] A.M. Polyakov, “The Wall of the Cave,” hep-th/9809057

[26] I.R. Klebanov, N. Nekrasov and S. Shatashvili, “An Orbifold of Type 0B Strings and Non-supersymmetric Gauge Theories,” hep-th/9909109

[27] N. Seiberg and E. Witten, “Monopole Condensation and Confinement In N=2 Supersymmetric Yang-Mills Theory,” Nucl. Phys. B426 (1994) 19

[28] E. Witten, “Solutions of Four-dimensional Field Theories via M-theory”, Nucl. Phys. B500 (1997) 3, hep-th/9703166

[29] S. Katz, P. Mayr, C. Vafa, “Mirror symmetry and Exact Solution of 4D \( \mathcal{N} = 2 \) gauge theories I”, hep-th/9706110, Adv.Theor.Math.Phys. 1 (1998) 53-114

[30] C. Johnson, A. Peet and J. Polchinski, in preparation; reported by C. Johnson, talk at IAS.

[31] A. Kehagias, “New Type IIB Vacua and Their F-Theory Interpretation,” hep-th/9805131

[32] P. Candelas and X. de la Ossa, “Comments on Conifolds,” Nucl. Phys. B342 (1990) 246.

[33] S.S. Gubser, “Einstein Manifolds and Conformal Field Theories,” Phys. Rev. D59 (1999) 025006, hep-th/9807164.

[34] A. Ceresole, G. Dall’Agata, R. D’Auria, and S. Ferrara, “Spectrum of Type IIB Supergravity on \( \text{AdS}_5 \times T^{1,1} \): Predictions On \( \mathcal{N} = 1 \) SCFT’s,” hep-th/9905226.

[35] C.V. Johnson and R.C. Myers, “Aspects of Type IIB Theory on ALE Spaces”, Phys. Rev. D55 (1997) 6382, hep-th/9610140

[36] D.-E. Diaconescu, M. Douglas and J. Gomis, “Fractional Branes and Wrapped Branes,” JHEP 02 (1998) 013.