Multi-skill Collaborative Teams based on Densest Subgraphs

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Abstract

We consider the problem of identifying a team of skilled individuals for collaboration, in the presence of a social network. Each node in the input social network may be an expert in one or more skills - such as theory, databases or data mining. The edge weights specify the affinity or collaborative compatibility between respective nodes. Given a project that requires a set of specified number of skilled individuals in each area of expertise, the goal is to identify a team that maximizes the collaborative compatibility. For example, the requirement may be to form a team that has at least three databases experts and at least two theory experts.

We explore team formation where the collaborative compatibility objective is measured as the density of the induced subgraph on selected nodes. The problem of maximizing density is NP-hard even when the team requires a certain number of individuals of only one specific skill. We present a 3-approximation algorithm that improves upon a naive extension of the previously known algorithm for densest at least $k$ subgraph problem. We further show how the same approximation can be extended to a special case of multiple skills as well. Our problem generalizes the formulation studied by Lappas et al. [KDD '09]. Further, they measured collaborative compatibility in terms of diameter and the spanning tree costs. Our density based objective also turns out to be more robust in certain aspects.

Experiments are performed on a crawl of the DBLP graph where individuals can be skilled in at most four areas - theory, databases, data mining, and artificial intelligence. In addition to our main algorithm, we also present heuristic extensions to trade off between the size of the solution and its induced density. These density-based algorithms outperform the diameter-based objective on several metrics for assessing the collaborative compatibility of teams. The solutions suggested are also intuitively meaningful and scale well with the increase in the number of skilled individuals required.

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1 Introduction

A team formation problem consists of forming a team from a large set of candidates such that the resulting team is best suited to perform the assignment. The main difficulty in providing an automated way to form a team from the solution space is the categorization of the desired attributes quantitatively. In spite of this, the problem has attracted many researchers and various interesting approaches have been suggested over the years, as we mention them in the related work section. In this spirit, we study this problem in the context of social network with a goal to identify the most collaborative team that satisfies the skill-set requirements of the project. Certainly, the naive approach would be to just find the candidates that match the requirements the best. However, considering the social network associated with the candidates add a value to the solution because, intuitively, such team is more likely to demonstrate better collaborative compatibility. This is also evident in practice, where many companies tend to promote employee referral program while hiring a candidate.

We model this team formation problem in the social network context by considering the network graph that connects the individuals, wherein each individual is represented by a node in the graph and an association between individuals is represented by an edge in the graph. In a more generic sense, each node can be assigned a set of desired attributes and an edge can be assigned a weight representing the collaborativeness between the individuals it is connecting. Note that, this model could further be extended in multiple dimensions and we believe that the work we present in this paper could be a good starting point with this regard. For example, one possible extension to this graph model would be a hypergraph model wherein we can accomodate many criteria - weight associated with hyperedge could define the collaborative compatibility between the set of nodes (instead of just two nodes), hyperedge could also be used to denote the set of nodes that represent a certain group, etc.

In this paper, as a starting point, we define the problem where each node is associated with a set of skills and a weight of the edge reflects the cohesiveness between two connecting nodes (users), and a goal is to form a collaborative team for a project that requires a specified number of people in each of a set of skills. In this setting, two users can collaborate better as a team if they have a high-weight edge (strong affinity for interaction) between them. Specifically, consider the following example where a social network of computer scientists is presented. Each user is skilled in a subset of areas between theory, databases and data mining. A company wants to hire people for a predeterminated project. The goal of the project requires that the team consists of at least three database researchers, at least two theory researchers, and at least one researcher with expertise in data mining. Presented with the social network where edges reflect collaborative interactions, how should the company go about hiring a team for the project?

A special case of this problem was studied in [12]. They consider team formation when the team requires at most one person each in a set of different skills. Our problem formulation generalizes this by allowing the team to require multiple skilled individuals in any skill. Clearly there are projects where multiple people with specific skills may be desired. It turns out that this generalization makes the problem significantly harder and more interesting. For example, the problem is no longer trivial even when the social network contains users that are either skilled or not skilled in just one specific area. Suppose a project requires eight database researchers, and the social network contains people who are either skilled in databases or not, how does one go about choosing the team? We shall mention the complexity as well as algorithmic results for this special case as well shortly.

A critical question in team formation based on a social network is to determine the collaborative quality of a team. The edges specify the collaborative compatibility of two nodes. However, given a subset of say $k$ nodes in the social network (let us even say these $k$ nodes are connected), how do we know how collaborative this team is? To tackle this, [12] suggested two objectives: one based on the diameter of the subgraph induced by these $k$ nodes, and another based on the spanning tree cost of these nodes; and demonstrated the potential of these ideas through experimental results. These objectives can certainly be applied to solve the problem we define in this paper. In fact, we provide the extention to their diameter-based algorithm, prove the 2-approximation bound and also complement with experimental results. Similarly, the minimum-spanning tree based approach could also be extended to the problem defined here. However, the main focus of our paper is a novel density based objective that we propose for this problem; therefore, the majority of this paper’s contributions are related to this density objective. Specifically, we define the collaborative affinity of a team of $k$ nodes to be proportional to the density of the induced subgraph. Using density as a measure of the quality of an induced subgraph of nodes has certain intuitive merits over using diameter or minimum
spanning tree costs; we describe these in section 3.

We briefly summarize the problem definition here: given a set of skills $1, 2, \ldots, t$, and requirements $k_1, k_2, \ldots, k_t$, and a social network of nodes connected by (weighted) edges, the goal is to pick a subset of nodes such that at least $k_i$ distinct nodes possess skill $i$, for $1 \leq i \leq t$. The same node, however, may contribute to two different skills. The objective value of the solution is the density of the induced subgraph on these nodes. The goal is to maximize this objective. Notice that the number of returned nodes may be as small as $k_{\text{max}} = \max_i k_i$ or be even larger than $\sum_i k_i$. We now summarize the contributions of this paper.

**Our Contributions.**

- We present a novel problem definition for team formation to maximize collaborative compatibility. The constraint of the problem requires the team to comprise of at least a specified number of skilled individuals in each of a set of skills. This generalizes previous work that required forming a team with at least one skilled individual in each of a set of skills.

- As a measure of collaborative compatibility, we suggest a density based objective. Density is a novel metric for this domain and we show that it has certain desirable properties for measuring compatibility. Our density based team formation problem also generalizes previous graph algorithms work on finding densest subgraphs with size constraints.

- We address the collaborative team formation problem when the team requires one or more skills. We show that optimizing even the special case of a single skill is NP-hard under our density-based metric, as well as the previously suggested diameter-based metric. The main theoretical result of the paper is to present a novel 3-approximation algorithm for the density based team formation problem for both single as well as a special case of multiple skills. This improves upon a naive extension of previous work on size constrained densest subgraph problems. We also show how previous work on a 2-approximation for the diameter-based objective can be extended to our generalized problem.

- We present several heuristic algorithms that build on our 3-approximation for density-based team formation. These algorithms trade-off between the size of the returned solution and the density, while respecting the constraints on the skill requirements.

- We perform experiments on all these algorithms on the DBLP graph. Experiments show that density-based algorithms perform well in practice, identifying tightly knit and highly skilled teams and also scale well with the size of the team and skill requirements.

- We measure qualitative evidence of the teams reported by both density-based and diameter-based algorithms and show that the density-based algorithms compare favorably to the diameter-based algorithms on a number of different metrics. Further analysis of the teams (by inspecting the members of the team) reported show that the density-based approach suggest the teams that are more intuitive and meaningful compared to diameter-based teams.

**Overview.** We mention related work in Section 2. The various problem definitions, notations and some properties are formalized in Section 3. Our theoretical contributions, including the main 3-approximation algorithm for our density based objective are described in Section 4. The theoretical work on a diameter based objective is presented in Section 5. Finally, some additional heuristic algorithms and experimental results are detailed in Section 6.

## 2 Related Work

Various interesting approaches for team formation have been studied over the years. In operations research, the problem is defined as finding an optimal match between people and demanded functional requirements. It is often solved using techniques such as simulated annealing, branch-and-cut or genetic algorithms. Another interesting problem formulation requires taking into consideration the psychological aspects of the individuals involved in order to form a team of efficient collaboration, e.g., the work by Fitzpatrick and Askin, and Chen and Lin. Although all these approaches are interesting, they do not use the possible presence of a social graph structure between the individuals. Therefore, these
approaches are complementary to ours. Further, Gaston et al. \[7\] provide an experimental study on the effects of a graph structure among individuals on the performance of a team.

Our problem formulation differs from these fundamentally by requiring a solution where the optimality is determined based on the properties associated with a social graph structure among the individuals. In particular, we aim to form a team that contains at least $k_i$ nodes of skill $i$ such that the density of the resulting solution subgraph is maximized. A similar problem has been addressed by Lappas et. al. \[12\]. They try to find a team that contains at least 1 node for each skill $i$, with the cost of a solution measured in terms of either a diameter or a minimum spanning tree. Our problem definition generalizes this requirement and suggests a new density based measure for solution’s objective.

The problem of finding size-bound densest subgraphs is well-studied. Finding a maximum density subgraph on an undirected graph can be solved in polynomial time \[8, 13\]. However, the problem becomes NP-hard when a size restriction is enforced. In particular, finding a maximum density subgraph of size exactly $k$ is NP-hard \[2, 5\] and no approximation scheme exists under a reasonable complexity assumption \[9\]. Khuller and Saha \[10\] considered the problem of finding densest subgraphs with size restrictions and showed that these are NP-hard. Khuller and Saha \[10\] and also Andersen and Chellapilla \[1\] gave constant factor approximation algorithms. Our problem definition varies from these because we not only require to find the maximum density subgraph of size at least $k$, but, we also require that this subgraph contain $k_i$ nodes of property (or skill) $i$ such that $k = k_1 + k_2 + ... + k_n$. Thus, we also generalize past work on finding size-bound maximum density subgraphs.

3 Problem Definition

**Notation.** Let $\mathcal{X} = \{1, \ldots , n\}$ denote a set of $n$ individuals and $\mathcal{A} = \{a_1, \ldots, a_m\}$ denote a set of $m$ skills. Each individual $i$ is associated with a set of skills $X_i \subseteq \mathcal{A}$. If $a_j \in X_i$, then an individual $i$ has skill $a_j$. For each skill $a$, we define its support set, $S(a)$, as the set of individuals in $\mathcal{X}$ with skill $a$. That is, $S(a) = \{i | i \in \mathcal{X}$ and $a \in X_i\}$. A task $\mathcal{T}$ is a set of pairs where each pair, $(a_j, k_j)$, specifies that at least $k_j$ individuals of skill $a_j$ are required to perform the task.

Let $G(\mathcal{X}, E)$ denote the undirected, weighted graph representing the social network associated with the set of individuals $\mathcal{X}$. We use the notations $E(G)$ and $V(G)$ to represent the edge set and vertex set associated with the graph $G$. If $\mathcal{X}' \subseteq V(G)$, we use $G[\mathcal{X}']$ to denote the subgraph of $G$ induced by the nodes in $\mathcal{X}'$. Further, $W(\mathcal{X}')$ denotes the sum of the edge-weights associated with all the edges in the subgraph induced by the nodes in $\mathcal{X}'$. We also define a distance function between any two nodes $i$ and $i'$ in a graph $G$ as the sum of the edge-weights along the shortest path between $i$ and $i'$ in $G$. Further, without loss of generality, we assume that the graph $G$ is connected; we can transform every disconnected subgraph to a connected one by simply adding an edge that denotes zero collaborative compatibility. Given a measure of collaborative compatibility $Cc()$, we now formalize the problems considered in this paper.

**Single Skill Team Formation (sTF).** Given a set of $n$ individuals $\mathcal{X} = \{1, \ldots , n\}$, a graph $G(\mathcal{X}, E)$, task $\mathcal{T} = \{< a, k >\}$, find $\mathcal{X}' \subseteq \mathcal{X}$, such that $|\mathcal{X}' \cap S(a)| \geq k$, and the collaborative compatibility $Cc(\mathcal{X}')$ is optimized.

**Multiple Skill Team Formation (mTF).** Given a set of $n$ individuals $\mathcal{X} = \{1, \ldots , n\}$, a graph $G(\mathcal{X}, E)$, task $\mathcal{T} = \{< a_1, k_1 >, < a_2, k_2 >, \ldots, < a_m, k_m >\}$, find $\mathcal{X}' \subseteq \mathcal{X}$, such that $|\mathcal{X}' \cap S(a_j)| \geq k_j$ for each $j \in \{1, \ldots, m\}$ and the collaborative compatibility $Cc(\mathcal{X}')$ is optimized.

The main metric that we consider for collaborative compatibility for $sTF$ and $mTF$ is the following density based objective. In addition to this, we consider a diameter based objective as well (suggested in \[12\]) for comparison.

**Maximum Density(D).** Given a graph $G(\mathcal{X}, E)$ and a set of individuals $\mathcal{X}' \subseteq \mathcal{X}$, we define the density collaborative compatibility of $\mathcal{X}'$, denoted by $Cc-D(\mathcal{X}')$ to be the density of the induced subgraph $G[\mathcal{X}']$. Recall that the density $d(G)$ of a graph $G$ is defined as $d(G) = \frac{W(G)}{|V(G)|}$. The higher the value of the density, the better is the collaborative compatibility. An optimal solution $\mathcal{X}' \subseteq \mathcal{X}$, is the team that can perform task $\mathcal{T}$ and has maximum density.

**Minimum Diameter(R).** Given a graph $G(\mathcal{X}, E)$ and a set of individuals $\mathcal{X}' \subseteq \mathcal{X}$, we define the diameter collaborative compatibility of $\mathcal{X}'$, denoted by $Cc-R(\mathcal{X}')$, to be the diameter of the subgraph $G[\mathcal{X}']$. Recall that the diameter of a graph is the largest shortest path between any two nodes in the graph. An optimal solution $\mathcal{X}' \subseteq \mathcal{X}$, is the team that can perform task $\mathcal{T}$ and has minimum diameter.
In the following sections, we refer to the Single Skill Team Formation (sTF) and Multiple Skill Team Formation (mTF) problems with collaborative compatibility Cc-R as Diameter-sTF and Diameter-mTF, respectively. Similarly, for the collaborative compatibility Cc-D we refer to the corresponding problems as Density-sTF and Density-mTF respectively.

Properties. We now describe some properties of the maximum density objective. Notice that neither of these properties hold on Diameter-sTF or Diameter-mTF. For brevity, we mention the intuition without a rigorous definition or proof.

Strict Monotonicity. If a communication edge (with positive weight) is added between two nodes in the solution set for the Density-sTF or Density-mTF problem, then the collaborative compatibility objective Cc-D for the solution necessarily increases. Similarly, if a communication edge already present is deleted, then the Cc-D objective value decreases. This seems intuitive as an added collaboration between two people in the team enhances the quality of the team. However, in the case of diameter, adding or deleting an edge may not affect the solution at all.

Sensitivity. The Cc-D value for Density-sTF or Density-mTF does not increase or decrease radically upon adding or deleting an edge. Specifically, it can only change to an extent depending on the weight of the added or deleted edge, compared to the total weight of edges in the solution. However, adding or deleting an edge can radicantly change the diameter (for example make it finite from infinite) for an induced subgraph; this implies that the diameter objective is highly sensitive to small change.

The properties for density based objectives fall out of the fact that adding or deleting edges only gradually alters the density of a solution subgraph. Diameter based objectives (or even the minimum spanning tree based objective suggested in [12] that we do not consider in this paper) are not smooth in this sense; altering the graph slightly can change the objective radically. These properties make density based objectives somewhat more suitable. One drawback, however, of density as an objective arises from the fact that the optimal solution may contain disconnected components. Notice that this is not the case for the diameter based objective, however, although the solution returned is connected it may be of large size including non-skilled (undesired) nodes that are required to ensure the connectivity. To ensure the connectivity property for the density-based solutions, in the experimental section we suggest several heuristic algorithms.

Eventually, the quality of teams produced by different definitions needs to be evaluated (potential for collaboration) based on the measures neutral to these definitions; we make such objective comparisons in the experimental section.

4 Density-based objective

In this section, we claim that Density-sTF and Density-mTF are NP-hard problems. We then present the algorithms s-DensestAlk (Algorithm 1) and m-DensestAlk (Algorithm 2) for Density-sTF and Density-mTF, respectively. Further, we prove that Density-sTF achieves 3-approximation factor.

Theorem 1 Density-sTF and Density-mTF problems are NP-complete.

Proof: We prove the claim by a reduction from the Densest at least k subgraph (DalkS) problem defined in [10]. An instance of DalkS consists of a graph \( G(\mathcal{X}, E) \), and a constant \( k \), and the solution is a maximum density subgraph with at least \( k \) nodes. We transform it into an instance of Density-sTF problem by defining a skill \( a \) for every node \( v \in V \) in which case a solution would be a maximum density subgraph with at least \( k \) nodes that have skill \( a \). And since skill \( a \) is defined for every node in \( G \), it is easy to see that \( \mathcal{X}' \subseteq \mathcal{X} \) is the solution to the problem Density-sTF iff it is a solution to the problem DalkS. The problem Density-sTF is a special case of Density-mTF which implies that Density-mTF is NP-hard.

4.1 3-approximation algorithm for Density-sTF

Intuition: To begin with, the algorithm s-DensestAlk (Algorithm 1) accepts the graph and the skill requirements as an input. It then finds the densest subgraph and removes it from the input graph and adds it to the solution subgraph (which is initially empty). It then checks if the solution subgraph satisfies the skill requirements. Until the solution subgraph constructed meets the skill requirements, the algorithm continues to iterate through the process of finding the densest subgraph from the remaining input graph and adding it
to the solution subgraph. Since in each iteration the algorithm adds the densest subgraph, it is ensured that the solution subgraph has sufficiently high density. Note that although we are able to prove that the algorithm guarantees a 3-approximation ratio in terms of density, no bound on the size is guaranteed. We overcome this drawback by applying various simple heuristic algorithms which are described later in the section 6.2.

**Details:** The algorithm s-DensestAlk(G, T) takes as input the social graph G and a task T = \{ja, k_i\} where at least k individuals/nodes of skill a are required to perform the task T. As explained intuitively, the algorithm then proceeds through multiple iterations. In each iteration, i, it finds the maximum density subgraph of G_i, say H_{i+1}, removes it from G_i using the routine \textit{shrink}(G_i, H_{i+1}) and constructs a new solution subgraph D_{i+1} using the routine \textit{union}(D_i, H_{i+1}). The routine \textit{shrink}(G, H) removes H from G such that for each v ∈ (G − H), if v has l edges to the vertices in H, then it adds l self-loops to v with the corresponding edge-weights. Inside the routine \textit{union}(D, H), then for each loop, we look at its corresponding edge, say e(u, v), in the original input graph, G, and if u ∈ D, v ∈ H (or vice-versa), we replace the loop by an edge e(u, v). Finally, once the loop-termination condition is satisfied, the algorithm then examines each of the intermediate solution subgraphs, D_i, constructed in previous iterations and adds sufficient number of skilled nodes to it so that each D_i satisfies the skill requirement. The algorithm then picks up the one with the highest density as the final solution subgraph.

Our algorithm is very similar to the DensestAtleastK algorithm in [10] that calculates the maximum density subgraph containing at least k vertices without any skill constraints imposed. The naive extension would be to just add k skilled nodes to the solution returned by algorithm DensestAtleastK. And since their algorithm guarantees an approximation factor of 2 for density, this naive extension would guarantee an approximation factor of 4 (proof omitted for brevity). But, since the additional k nodes are picked at random the solution may suffer from many disconnected components making it practically infeasible to be of any use. Therefore, we propose the algorithm s-DensestAlk that differs mainly in the loop-termination condition imposed. This condition ensures that the resulting solution satisfies the constraints of at least k skilled nodes, improves the approximation ratio to 3 from 4, and has good connectivity properties.

Although the proof for 4-approximation is simple, it turns out that proving a 3-approximation to Density-sTF is significantly harder. While the algorithm is simple, the analysis is fairly detailed. The key idea is to consider various cases about the returned subgraph and carefully examine the density of each component. The analysis is similar to [10] at the high level. However, due to the skill-set constraints, several sub-cases need to be considered.

**Algorithm 1** s-DensestAlk(G, T)

1: \( D_0 \leftarrow \emptyset \), \( G_0 \leftarrow G \), \( i \leftarrow 0 \)
2: while \( |D_i \cap S(a)| < k \) where \( T = \{ja, k_i\} \) do
3: \( H_{i+1} \leftarrow \text{maximum-density-subgraph}(G_i) \)
4: \( D_{i+1} \leftarrow \text{union}(D_i, H_{i+1}) \)
5: \( G_{i+1} \leftarrow \text{shrink}(G_i, H_{i+1}) \)
6: \( i \leftarrow i + 1 \)
7: end while
8: for each \( D_i \) do
9: \( n_a = \text{number of nodes of skill a in } D_i \)
10: Add \( \max(k - n_a, 0) \) nodes of skill a to \( D_i \) to form \( D'_i \)
11: end for
12: Return \( D'_i \) which has the maximum density

**Theorem 2** The algorithm s-DensestAlk achieves an approximation factor of 3 for the Density-sTF problem.

**Proof:** Let \( H^* \) denote an optimal solution and \( d^* = \frac{W(H^*)}{|V(H^*)|} \) denote density of the optimal solution.

If the number of iterations is 1, then \( H_1 \) is the maximum density subgraph that contains at least \( k \) nodes of skill \( a \). Therefore, \( H^* = H_1 \) and the algorithm returns it. Otherwise, say the algorithm iterates for \( l \geq 2 \) rounds. There can be two cases:

**Case 1:** There exists an \( l' < l \) such that \( W(D_{l'-1} \cap H^*) < \frac{W(H^*)}{2} \) and \( W(D_{l'} \cap H^*) \geq \frac{W(H^*)}{2} \).

**Case 2:** There exists no such \( l' < l \).
Before analyzing the two cases in detail, note that by construction \(\text{density}(H_i) \leq \text{density}(D_i) \leq \text{density}(D_{i-1})\). We now consider case 2 first and later case 1.

**Proof for Case 2.**

Since the algorithm terminates after \(l\) iterations, \(D_l\) contains at least \(k\) nodes of skill \(a\). Further, we know that for each \(j \leq l - 1\), \(W(D_j \cap H^*) < \frac{W(H^*)}{2}\)

\[
W(G_j \cap H^*) \geq \frac{W(H^*)}{2}
\]

\[
\frac{W(G_j \cap H^*)}{|V(G_j \cap H^*)|} \geq \frac{W(H^*)}{2|V(H^*)|} \geq \frac{d^*}{2}
\]

\(G_j\) contains a subgraph of density \(\geq \frac{d^*}{2}\)

\[
\text{density}(H_i) \geq \frac{d^*}{l}
\]

\[
\text{density}(D_i) \geq \frac{d^*}{2}
\]

Thus, \(D_l\) has density \(\geq \frac{d^*}{2}\) and contains at least \(k\) nodes of skill \(a\). Therefore, the algorithm indeed returns a subgraph of density at least \(\geq \frac{d^*}{2}\).

**Proof for Case 1.**

\[
W(D_{l-1} \cap H^*) < \frac{W(H^*)}{2} \quad \text{and} \quad W(D_l \cap H^*) \geq \frac{W(H^*)}{2}
\]

\[
W(G_{l} \cap H^*) \geq \frac{W(H^*)}{2}
\]

\[
\frac{W(G_{l} \cap H^*)}{|V(G_{l} \cap H^*)|} \geq \frac{W(H^*)}{2|V(H^*)|} = \frac{d^*}{2}
\]

\(G_l\) has a subgraph of density \(\geq \frac{d^*}{2}\)

\[
\text{density}(H_{l}) \geq \frac{d^*}{2} \quad (H_{l}\text{ is densest subgraph of } G)
\]

\[
\text{density}(D_{l}) \geq \frac{d^*}{2}
\]

Now, let us divide **Case 1** into following 4 parts

(a) \(|V(D_{l})| \leq \frac{k}{l}\)

According to step 10, algorithm adds at most \(k\) vertices to \(D_{l}\) to obtain the subgraph, say \(D\), with density \(d\)

\[
d \geq \frac{W(D_{l})}{|V(D_{l})|} \geq \frac{W(H^*)}{2 + k} \geq \frac{W(H^*)}{3|V(H^*)|} = \frac{d^*}{3}
\]

(b) \(|V(D_{l})| \geq 2k\)

According to step 10, algorithm adds at most \(k\) vertices to \(D_{l}\). Further, we know that \(\text{density}(D_{l}) \geq \frac{d^*}{2} \quad \text{therefore, the resulting subgraph, } D'_{l}\text{ has density}

\[
d = \frac{W(D_{l})}{|V(D_{l})|} \geq \frac{W(D_{l})}{2|V(D_{l})|} \geq \frac{d^*}{3}
\]

(c) \(\frac{k}{l} < |V(D_{l})| < 2k \quad \text{and} \quad |V(D_{l}) \cap V(H^*)| \geq \frac{|V(H^*)|}{2}\)

According to step 10, algorithm adds at most \(\frac{|V(H^*)|}{2}\) nodes to \(D_{l}\) to form \(D'_{l}\) with density, say \(d\).

i \(|V(D_{l})| \geq |V(H^*)|\)

\[
d \geq \frac{W(D_{l})}{|V(D_{l})|} \geq \frac{W(H^*)}{2} \geq \frac{d^*}{3}
\]

ii \(|V(D_{l})| < |V(H^*)|\)

\[
d \geq \frac{W(D_{l})}{|V(D_{l})| + \frac{|H^*|}{2}} \geq \frac{W(D_{l})}{|V(H^*)| + \frac{|H^*|}{2}} \geq \frac{W(H^*)}{3|V(H^*)|} \geq \frac{d^*}{3}
\]
Figure 2: $D_{i1} = D_i \cap G'$, $D_{i2} = shrink(D_i, D_{i1})$, $G'' = shrink(G', D_{i1})$

(d) $\frac{k}{2} < |V(D_{i'})| < 2k$ and $|V(D_{i'}) \cap V(H^*)| < \frac{|V(H^*)|}{2}$.

If $d_{i'} = \text{density}(D_{i'}) \geq d^*$, then adding at most $k$ vertices gives a subgraph $D'_{i'}$ with density, say $d$ such that

$$d = \frac{W(D_{i1})}{|V(D_{i1})| + k} \geq \frac{W(D_{i1})}{|V(D_{i1})| + 2|V(D_{i1})|} \geq \frac{W(D_{i1})}{3|V(D_{i1})|} \geq d^*.$$

Therefore, $D_{i'}$ is a subgraph that contains at least $k$ nodes of skill $a$ and has density $d \geq \frac{d^*}{3}$. We are done here.

Now, assume that $d_{i'} < d^*$.

In the rest of the proof, we divide $D_{i'}$ into subgraphs as explained below and shown in Figure 1.

Let $G' = D_{i'} \cap H^*$.

Claim 1 $W(G') \geq \frac{W(H^*)}{2}$ and density($G'$) $\geq d^*$.

Proof: $|V(G')| = |V(D_{i'}) \cap H^*)| < \frac{|V(H^*)|}{2}$ and $W(G') = W(D_{i'}) \cap H^*) \geq \frac{W(H^*)}{2}$.

$\Rightarrow$ density($G'$) $\geq \frac{W(H^*)}{2|V(H^*)|} \geq d^*$.

Define $i$ such that density($H_i$) $\geq d^*$ and density($H_{i+1}$) $< d^*$. Such an $i \leq l'$ exists due to Claim 1 and since $d_{i'} < d^*$.

$\Rightarrow$ density($D_i$) = $d_i \geq d^*$.

Let, $n_i = |V(D_i)|$. We now consider two sub-cases.

i $n_i \geq \frac{|V(H^*)|}{2}$: Add at most $k$ vertices to $D_i$ to get a subgraph $D'_i$ with density($D'_i$) = $d$, such that

$$d = \frac{W(D_{i1})}{|V(D_{i1})| + k} \geq \frac{W(D_{i1})}{|V(D_{i1})| + |V(H^*)|} \geq \frac{W(D_{i1})}{3|V(D_{i1})|} \geq d^*.$$

Thus, $D'_i$ is a subgraph containing at least $k$ nodes of skill $a$ and density $d \geq \frac{d^*}{3}$ and we are done here.

ii $n_i < \frac{|V(H^*)|}{2}$: We know that density($G'$) $\geq d^*$, density($H_i$) $\geq d^*$ and density($H_{i+1}$) $< d^*$. Therefore, $G' \cap D_i \neq \phi$. We now introduce a few definitions and prove claims about them.

Let, $D_{i1} = D_i \cap G'$, $D_{i2} = shrink(D_i, D_{i1})$, and $G'' = shrink(G', D_{i1})$ (Figure 2). Further, let $X = shrink(D_{i'}, D_i)$.

Claim 2 $W(D_{i1}) \geq \frac{|V(H^*)|d^*}{2} - W(G'').$

Proof: $W(G') = W(G'') + W(D_{i1})$ since $G'' = shrink(G', D_{i1})$; but $W(G') \geq \frac{W(H^*)}{2}$ (using Claim 1).

Claim 3 density($D_{i2}$) $> \frac{d^*}{7}$.
In this section, we present the algorithm \textit{m-DensestAlk} is an extension of the algorithm \textit{s-DensestAlk} that described for \textit{m-DensestAlk} in section 4.1. Since each node contributes to atmost one skill, an optimal solution, \( H^* \), has at least \( k \) vertices. The

\[ \text{Proof: } \] Recall that for each \( j \leq i, \text{density}(H_j) > d^* \). Further, \( H_j = \text{densest subgraph of shrink}(G, D_{j-1}) \). Therefore, for each \( v \in H_j \), the degree of \( v \) induced in \( H_j \) is at least \( d^* \). Therefore, for all \( v \in D_i \), degree(v) > \( d^* \) (here we abuse notation to denote \( v \)'s degree induced in \( D_i \) by degree(\( v \))).

For convenience, let \( n_x = |V(X)|, n_{x'} = |V(D_{x'})|, n_{i1} = |V(D_{i1})|, n_{i2} = |V(D_{i2})|, \) and \( n'' = |V(G'')| \).

**Claim 4** \( W(X) - W(G'') \geq \frac{d^*}{3} (n_x - n'') \).

**Proof:** Since \( H_t \) is the maximum density subgraph of \( \text{shrink}(G, D_{t-1}) \), \( \text{density}(H_t) \geq \text{density}(S) \) for any \( S \subseteq H_t \). Further, since \( X = \text{shrink}(D_t, D_i) \), and \( G'' = \text{shrink}(G', D_i \cap G') \), we have \( G'' \subseteq X \). Therefore, \( \text{density}(H_j) \geq \text{density}(H_j \cap G'') \) (for all \( i < j \leq t \)).

Therefore, \( W(X) - W(G'') = \sum_{j=i+1}^{t'} W(H_j) - \sum_{j=i+1}^{t'} W(H_j \cap G'') \geq \sum_{j=i+1}^{t'} \text{density}(H_j)(|H_j| - |H_j \cap G''|) \geq \frac{d^*}{3} (n_x - n'') \).

Notice that we have (lower) bounded the density or the weight of each of \( D_{i1}, D_{i2}, \) and \( X \), the three components that add up to \( D_t \). We are now ready to argue about the density of \( D_{i'} \) when \( k \) vertices are added to it. Before initiating this analysis, we briefly state a claim relating the sizes of these components.

**Claim 5** \( n_{i2} + n_x - n'' \geq n_{i'} - \frac{|V(H^*)|}{2} \).

**Proof:** This follows using \( |V(G')| \leq \frac{|V(H^*)|}{2} \) and the definition \( G'' = \text{shrink}(G', D_{i1}) \).

We now complete the analysis.

\[ d = \text{density}(D) \geq \frac{W(D)}{n_x + k} \]
\[ = \frac{W(D_{i1})+W(X)}{n_{i1} + k} = \frac{W(D_{i1})+W(D_{i2})+W(X)}{n_{i1} + k} \]
\[ \geq \frac{d^*|V(H^*)|+W(G'')+d^*n_{i2}}{n_{i1} + k} + W(X) \text{ (using Claim 2)} \]
\[ \geq \frac{d^*|V(H^*)|+d^*n_{i2}}{n_{i1} + k} + \frac{d^*(n_x - n'')}{3} \text{ (using Claim 1)} \]
\[ \geq \frac{d^*}{2} \left( \frac{2n_{i2}+k}{n_{i1} + k} \right) - \frac{W(H^*)}{2} \text{ (using Claim 5)} \]
\[ \geq d^* \frac{2n_{i2}+k}{n_{i1} + k} - d^* \frac{2}{3} \text{ (since } k \leq n_{i2} \text{).} \]

Remark: Cases (c) and (d) do not use the bound \( |V(D_{i'})| < 2k| \); so they together subsume case (b), but we have presented (b) for clarity.

### 4.2 Algorithm for \textit{Density-mTF}

In this section, we present the algorithm \textit{m-DensestAlk} (Algorithm 2) for the \textit{Density-mTF} problem. This is an extension of the algorithm \textit{s-DensestAlk} for the \textit{Density-sTF} problem described earlier. The algorithm \textit{m-DensestAlk} accepts input parameters: graph \( G \) and task \( T = \{ < a_1, k_1 >, < a_2, k_2 >, \ldots, < a_m, k_m > \} \) which requires at least \( k_i \) individuals of skill \( a_i \) to perform the task \( T \). Each iteration within the algorithm \textit{m-DensestAlk} is exactly similar to the \textit{s-DensestAlk} described earlier except that here the termination condition verifies that the solution subgraph contains at least \( k_i \) nodes with skill \( a_i \) for \( i \in \{1, \ldots, m \} \) and thus satisfying the multiple skill requirement instead of single skill requirement. The details of the algorithm are similar to that described for \textit{s-DensestAlk} in the section 3.4.

**Theorem 3** The algorithm \textit{m-DensestAlk} achieves an approximation factor of 3 for the special case of \textit{Density-mTF} problem where each node in the graph has at most one skill.

**Proof:** Let \( m = |T| \) and \( k = \sum_{j=1}^{m} k_j \) where \( k_j \) number of individuals are required of skill \( a_j \) s.t. \( < a_j, k_j > \in T \). Since each node contributes to atmost one skill, an optimal solution, \( H^* \), has at least \( k \) vertices. The
Algorithm 2 \textit{m-DensestAlk}(G, T)

1: \( D_0 \leftarrow \phi, \ G_0 \leftarrow G, \ i \leftarrow 0 \)
2: \textbf{while} \( |D_i \cap S(a_j)| < k_j \) for any \(<a_j, k_j> \in T\) \textbf{do}
3: \( H_{i+1} \leftarrow \text{maximum-density-subgraph}(G_i) \)
4: \( D_{i+1} \leftarrow \text{union}(D_i, H_{i+1}) \)
5: \( G_{i+1} \leftarrow \text{shrink}(G_i, H_{i+1}) \)
6: \( i \leftarrow i + 1 \)
7: \textbf{end while}
8: \textbf{for each} \( D_i \) \textbf{do}
9: \( D_i' \leftarrow D_i \)
10: \textbf{for each} \(<a_j, k_j> \in T\) \textbf{do}
11: \( n_{aj} = \text{number of nodes of skill } a_j \text{ in } D_i \)
12: \( \text{Add } \max(k_j - n_{aj}, 0) \text{ nodes of skill } a_j \text{ to } D_i' \)
13: \textbf{end for}
14: \textbf{end for}
15: \textbf{Return} \( D_i' \) which has the maximum density

proof for \textit{m-DensestAlk} is analogous to the proof for \textit{s-DensestAlk} with the only difference that instead of adding any \( k \) nodes of skill \( a \) to \( D_i \)'s, we add \( k_j \) nodes of skill \( a_j \) s.t. \(<a_j, k_j> \in T\).

We are unable to bound the performance of \textit{m-DensestAlk} for the general case of \textit{Density-mTF} problem. Further, the time complexity of \textit{m-DensestAlk} is \( O(kn^3) \) which can be inefficient for very large graphs but is manageable at the scale at which we run experiments. Directly using the linear time algorithm for the densest at least \( k \) subgraph problem in [4, 10] or \( O(n^3) \)-time algorithm from [4, 10] for \textit{Density-sTF} problem would result in a weaker bound i.e. 6 and 4-approximation respectively. In both cases, however, one may possibly get many disconnected components.

5 Diameter-based objective

In this section, we mention theoretical results for \textit{Diameter-sTF} and \textit{Diameter-mTF}. We show that these problems are NP-hard (note that the NP-hardness of \textit{Diameter-sTF} does not follow from any previous work). We further present an algorithm \textit{MinDiameter} (Algorithm 3) which is an extension of \textit{RarestFirst} in [12], and prove that it achieves a 2-approximation factor.

Algorithm 3 \textit{MinDiameter}(G, T)

1: \textbf{for each} \(<a, k> \in T\) \textbf{do}
2: \( S(a) = \{i \mid a \in X_i\} \)
3: \textbf{end for}
4: \( a_{\text{rare}} = \arg \min_{<a,k> \in T} |S(a)| \)
5: \textbf{for each} \( i \in S(a_{\text{rare}}) \) \textbf{do}
6: \textbf{for each} \(<a, k> \in T\) \textbf{do}
7: \( R_{ia} = d_k(i, S(a), k) \)
8: \textbf{end for}
9: \( R_i \leftarrow \max_a R_{ia} \)
10: \textbf{end for}
11: \( i^* \leftarrow \arg \min R_i \)
12: \( \mathcal{X}' = \{i^*\} \)
13: \textbf{for each} \(<a, k> \in T\) \textbf{do}
14: \( \mathcal{X}' = \mathcal{X}' \cup \{\text{Path}_k(i^*, S(a), k)\} \)
15: \textbf{end for}

Theorem 4 \textit{Diameter-sTF} and \textit{Diameter-mTF} problems are NP-complete.
Proof: The problems Diameter-$sTF$ and Diameter-$mTF$ are in NP because for a given candidate solution, in polynomial time, it can be verified that the skill-set requirement is satisfied. We prove that Diameter-$sTF$ is NP-hard by reduction from the 3-satisfiability problem. Consider a 3-SAT instance, say $\Psi = C_1 \land C_2 \land \cdots \land C_m$, where each clause, $C_j = (x \lor y \lor z)$, and $\{x, y, z\} \subseteq U = \{u_1, u_2, \ldots, u_n\}$. Let, $C = \{C_1, C_2, \ldots, C_m\}$. Let $N, M$ denote the number of variables and clauses, respectively. We construct an instance of Diameter-$sTF$ problem corresponding to the 3-SAT instance $\Psi$ using the following rules.

Rule 1 For each variable $x$, create two nodes $x, \neg x$ in $G$ and set $d(x, \neg x) = r'$.

Rule 2 For each clause $C_j$, create two nodes, $C_{j1}$ and $C_{j2}$ in $G$ and set $d(C_{j1}, C_{j2}) = r'$.

Rule 3 Pick any $r$ such that $r < r'$. For each pair of variables $(x, y)$, where $y \neq \neg x$, set $d(x, y) = r$. Similarly, for each pair of clauses $(C_f, C_g)$, where $d(C_f, C_g)$ is not set by rule 2, set $d(C_f, C_g) = r$.

Rule 4 For each clause, $C_j = (x \lor y \lor z)$, set $d(C_{j1}, x) = d(C_{j2}, x) = \frac{r}{2}$ and $d(C_{j1}, z) = d(C_{j2}, z) = \frac{r}{2}$ and $d(C_{j1}, u) = d(C_{j2}, u) = r$ for each $u \in U - \{x, y, z\}$.

Rule 5 For each $u_i, \neg u_i \in U$, associate a skill $a$ to node $u_i, \neg u_i$. And for each $C_j \in C$, associate a skill $a$ to the nodes $C_{j1}, C_{j2}$.

Claim 6 In $G$, $d(x, \neg x) > r$ where $x, \neg x \in U$.

Proof: In $G$, for each variable $y(\neq x \neq \neg x)$, $d(x, y) = d(y, \neg x) = r$ and $d(x, \neg x) = r' > r$ (rule 1, 3). Further, both $x$ and $\neg x$ cannot appear together in any clause $C_j \in C$ (pre-processing). Therefore, in $G$, $d(C_{j1}, x) = d(C_{j2}, x) = \frac{r}{2}$ and $d(C_{j1}, \neg x) = d(C_{j2}, \neg x) = r$ (rule 3, 4). 

Claim 7 Let $X$ be the subgraph of $G$ and $V(X)$ denote the nodes in $X$. Let $C_{j1}, C_{j2} \in V(X)$ where $C_j = (x \lor y \lor z)$. Then, in $X$, $d(C_{j1}, C_{j2}) = r$ iff $V(X) \cap \{x, y, z\} \neq \emptyset$.

Proof: Assume $V(X) \cap \{x, y, z\} = \emptyset$.

In $G$, for each clause $C_j (\neq C_{j1} \neq C_{j2})$, $d(C_{j1}, C_j) = d(C_{j2}, C_j) = r$ and $d(C_{j1}, C_{j2}) = r' > r$ (rule 2, 3). Further, for each $u \in U - \{x, y, z\}$, $d(C_{j1}, u) = d(C_{j2}, u) = r$ (rule 4). Therefore, in $X$, $d(C_{j1}, C_{j2}) = r$.

However, this is a contradiction because, in $X$, $d(C_{j1}, C_{j2}) > r$.

Therefore, $V(X) \cap \{x, y, z\} = \emptyset$.

Claim 8 Let $k = N + 2M$. If $\Psi$ has a satisfying assignment then $G$ has a sub-graph $X'$ with $|X' \cap S(a)| \geq k$ and $\text{diameter}(X') \leq r$.

Proof: If $\Psi$ has a satisfying assignment, then $G$ has a subgraph $X'$ such that $X'$ contains $C_{j1}, C_{j2}$ for each clause $C_j \in C$, and $u(\text{or } \neg u) \in U$ if $u(\text{or } \neg u)$ is set to 1 in the satisfying assignment for $\Psi$. Note that in the satisfying assignment for $\Psi$ either $u$ or $\neg u$ appears in the assignment. Thus, $X'$ contains exactly $N$ variables and twice the number of clauses. Thus, $|X' \cap S(a)| = N + 2M = k$ (rule 5).

Since $X'$ contains either a variable or its negation, for each pair of variables $(x, y) \in V(X) \cap U$, $d(x, y) = r$ (rule 3). Further, in the satisfying assignment for $\Psi$ each clause $C_j = x \lor y \lor z$, has at least one of the variables set to 1. So, for each pair of nodes $(p, q) \in V(X) \cap C$, $d(p, q) = r$ (Claim 7 and rule 3). Therefore, distance between any two nodes in $X'$ is at most $r$ (rule 4).

Thus, if $\Psi$ has a satisfying assignment then $G$ has a subgraph $X'$ with $|X' \cap S(a)| = k$ and $\text{diameter}(X') = r$.

Claim 9 Let $k = N + 2M$. If $G$ has a sub-graph $X'$ with $|X' \cap S(a)| \geq k$ and $\text{diameter}(X') \leq r$ then $\Psi$ has a satisfying assignment.

Proof: If $\text{diameter}(X') \leq r$ then it contains either $u$ or $\neg u$ but not both because $d(u, \neg u) > r$ (Claim 6). Since $k = N + 2M$, for each variable $u \in U$, $X'$ contains a node corresponding to either $u$ or $\neg u$ (not both) and for each clause $C_j \in C$, $X'$ contains nodes corresponding to $C_{j1}$ and $C_{j2}$ (rule 5). Now, since $\text{diameter}(X') \leq r$, it implies that $d(C_{j1}, C_{j2}) \leq r$. This implies that at least one of the nodes corresponding
to $x, y, z$ in $C_j$ is included in the sub-graph $\mathcal{X}'$ (Claim 7). Now, if each variable $u \in U \cap V(\mathcal{X}')$ is set to 1, then $\Psi$ has a satisfying assignment.

Claims 8 and 9 prove that Diameter-sTF is NP-hard. Since Diameter-sTF is the special case of Diameter-mTF, its NP-hardness proof follows.

**Theorem 5** For any graph distance function $d$ that satisfies the triangle inequality, the algorithm MinDiameter achieves an approximation factor of 2 for the Diameter-sTF and Diameter-mTF problems.

**Proof:** The analysis we present here is similar to the analysis of the RarestFirst algorithm presented in [12]. First, consider the solution $\mathcal{X}'$ output by the MinDiameter algorithm, and let $a_{\text{rare}} \in T$ be the skill possessed by the least number of individuals in $X$. Also, let $i^*$ be the individual picked from set $S(a_{\text{rare}})$ to be included in the solution $\mathcal{X}'$. Now, consider two other skills $a_1 \neq a_2 \neq a_{\text{rare}}$ and individuals $i, i' \in X$ such that $i \in S(a_1), i \not\in S(a_2) \text{ and } i' \not\in S(a_1), i' \in S(a_2)$. If $i, i'$ are part of the team reported by the MinDiameter algorithm, it means that $d(i^*, i) \leq d_k(i^*, S(a_1), k_1)$ and $d(i^*, i') \leq d_k(i^*, S(a_2), k_2)$. Due to the way the algorithm operates, we can lowerbound the $\text{Cc-R}$ cost of the optimal solution, $\mathcal{X}^*$, as follows:

$$d(i^*, i) \leq \text{Cc-R}(\mathcal{X}^*) \text{ and } d(i^*, i') \leq \text{Cc-R}(\mathcal{X}^*) \quad (1)$$

Since we have assumed that the distance function $d$ satisfies the triangle inequality, $d(i, i') \leq d(i, i^*) + d(i^*, i')$. By applying the bounds given in (1), we get the proposed approximation factor.

$$d(i, i') \leq \text{Cc-R}(\mathcal{X}^*) + \text{Cc-R}(\mathcal{X}^*) \leq 2\cdot \text{Cc-R}(\mathcal{X}^*)$$

Algorithm MinDiameter is as follows. For each individual, say $i_r \in S(a_{\text{rare}})$ where $a_{\text{rare}}$ is the rarest skill (the skill with the minimum size support set $S$), and for each skill $a_i \in T$, the algorithm finds the distance to all the nodes in the support set $S(a_i)$. Then, for each support set $S(a_i)$, it chooses the $k_r$-size subset of $S(a_i)$ such that the maximum shortest path distance between $i_r$ and the nodes in this subset is minimum among all $k_r$-size subsets of $S(a_i)$. We call this distance as $d_k(i_r, S(a_i), k_r)$. Further, we denote the set of $k_i$ shortest paths between $i_r$ and each of the nodes belonging to the corresponding $k_i$-size subset of $S(a_i)$ as $\text{Path}_k(i_r, S(a_i), k_i)$. Thus, for each $i_r \in S(a_{\text{rare}})$ the algorithm has identified $k_i$ nodes of skill $a_i$, thereby forming a possible solution team that satisfies the constraints. Finally, the algorithm then picks one of these solutions that has minimum diameter. The time complexity of the algorithm MinDiameter, assuming that all pairs shortest paths are pre-computed, is $O(n^2)$.

### 6 Experiments

In this section, we evaluate various team formation algorithms using the collaboration graph extracted from the DBLP bibliography server. We show that the density of the subgraph returned by our algorithms s-DensestAlk and m-DensestAlk perform favorably in comparison to the algorithm MinDiameter. We also show that our algorithm for density version provides high-quality results in terms of effective communication and collaboration (according to several metrics). In this section, we also present three simple heuristic extensions that can be used to process the solutions returned by s-DensestAlk and m-DensestAlk in order to further improve these solutions by reducing size and improving connectivity, while maintaining high density. Finally, examples of teams reported by our methods qualitatively corroborate the effectiveness of our framework.

#### 6.1 Experimental Setup

We use a snapshot of the DBLP data downloaded on May 17, 2010 to create a benchmark data set for our experiments. We only consider the papers published in the domains of Database (DB), Data Mining (DM), Artificial Intelligence (AI) and Theory (T) conferences. We select papers from a total of 21 conferences categorized as follows: $DB = \{\text{SIGMOD, VLDB, ICDE, ICDT, EDBT, PODS}\}$, $DM = \{\text{WWW, KDD, SDM, PKDD, ICDM}\}$, $AI = \{\text{ICML, ECML, COLT, UAI}\}$, and $T = \{\text{SODA, FOCS, STOC, STACS, ICALP, ESA}\}$. We define the skill set $T = \{T, AI, DB, DM\}$. The set of skilled individuals $X_{\text{dblp}}$ consists of the set of authors with at least three papers in these domains. Two authors $i_1,i_2$ are connected in the graph $G_{\text{dblp}}(X_{\text{dblp}}, E)$ if they appear as
co-authors in at least two papers in DBLP. The above procedure creates a set $X_{dblp}$ consisting of 6137 individuals. The maximum component size is 3869. We use this for all the experiments. The skill set $X_i$ of each such author $i$ is defined as $X_i = \{t \mid t \in T \text{ and } P_i(t) \neq \phi\}$ where $P_i(t)$ denotes the set of papers coauthored by $i$ that are published in the conferences in the domain $t$.

Maximum Density Team Formation. To evaluate the algorithms 11 and 2 for each edge $e(i_1, i_2)$, we set the edge weight $w(i_1, i_2) = |P_{i_1} \cap P_{i_2}|$, where $P_{i_1}$ and $P_{i_2}$ represent the set of papers published by $i_1$ and $i_2$ respectively. For the subgraph, say $G'(V', E')$ returned by these algorithms, we calculate the density, $d' = \frac{W(G')}{|V'(G')|}$.

Minimum Diameter Team Formation. Here, we set edge-weight $w(i_1, i_2) = 1 - \frac{|P_{i_1} \cap P_{i_2}|}{|P_{i_1} \cap P_{i_2}|}$ as suggested in the paper [12]. For comparison, when a subgraph $G'(V', E')$ is returned by the MinDiameter, we compute its density by considering the induced subgraph on vertices $V''$, say $G''$ (which could contain more edges that $E'$). The density calculated is $d'' = \frac{W(G'')}{|V''(G'')|}$ with edge weights $w(i_1, i_2) = |P_{i_1} \cap P_{i_2}|$.

6.2 Heuristic algorithms

The objectives for sTF-Density and mTF-Density are to find subgraphs with maximum density satisfying the skill requirements. However, this does not necessitate a connected graph: disconnectedness makes meaningful collaboration in real-life difficult. This is an artifact of the objective function, rather than the algorithm. While the solutions returned by our algorithms sTF-Density and mTF-Density never had more than three components, we would like solutions with only one component. This is the motivation for heuristic improvements. A dual benefit in our suggested heuristics is that we are able to reduce the number of nodes in the returned subgraph. The hope is that these can be achieved without compromising significantly on the density.

Algorithm 4 EnhanceComponent($G', T$)

1: \{(Note: $T = \{< a, k >\}$)
2: for each component $C_i \in G'$ do
3: \quad $C'_i \leftarrow C_i$, $N_i \leftarrow N(C_i) - C_i$
4: \quad (note: $N(C_i)$ denotes neighbors of nodes in $C_i$)
5: for each node $v \in N_i$ do
6: \quad if $|V(C'_i) \cap S(a)| \geq k$ then
7: \quad \quad $C' \leftarrow C' \cup C'_i$
8: \quad \quad break for loop
9: \quad end if
10: \quad if $v \in S(a)$ then
11: \quad \quad $C'_i \leftarrow C'_i \cup v$
12: \quad end if
13: end for
14: end for

Algorithm 5 EnhancedDense($G, T$)

1: $G' \leftarrow s$DensestAlk($G, T$)
2: $C' \leftarrow$ EnhanceComponent($G', T$)
3: Return arg min$_{C'_i \in C'} |C'_i|$
the non-skilled nodes one by one without making the component disconnected. The \textit{PartialTrimmedDense} algorithm allows at most $k$ non-skilled nodes in the component whereas \textit{CompleteTrimmedDense} attempts to remove as many non-skilled nodes as possible. The smallest resulting component with the required skilled nodes is then picked. This helps reduce the size of the solution, which is now a single component, and hopefully still sufficiently dense since the heuristic started with a 3-approximation to the density objective.

\begin{align*}
\text{Algorithm 6} \ & \text{PartialTrimmedDense}(G, T) \\
1: \ (\text{Note: } T = \{< a, k >\}) \\
2: \ G' \leftarrow \text{s-DensestAlk}(G, T) \\
3: \ C' \leftarrow \text{EnhanceComponent}(G', T) \\
4: \ & \text{for each component } C'_i \in C' \text{ do} \\
5: \ & \quad Q \leftarrow \{u \mid u \in C'_i \text{ and } u \notin S(a)\} \\
6: \ & \quad \text{while } Q \text{ not empty and } |V(C'_i) - S(a)| > k \text{ do} \\
7: \ & \quad \quad u_{\text{min}} \leftarrow \text{pop lowest degree node from } Q \\
8: \ & \quad \quad \text{if } (C'_i - u_{\text{min}}) \text{ is connected then} \\
9: \ & \quad \quad \quad C'_i \leftarrow C'_i - u_{\text{min}} \\
10: \ & \quad \quad \text{end if} \\
11: \ & \quad \text{end while} \\
12: \ & \quad \text{if } |V(C'_i) - S(a)| > k \text{ then} \\
13: \ & \quad \quad C' \leftarrow C' - C'_i \\
14: \ & \quad \text{end if} \\
15: \ & \text{end for} \\
16: \ & \text{Return } \arg \max_{C'_i \in C'} \text{density}(C'_i) 
\end{align*}

\begin{align*}
\text{Algorithm 7} \ & \text{CompleteTrimmedDense}(G, T) \\
1: \ (\text{Note: } T = \{< a, k >\}) \\
2: \ G' \leftarrow \text{s-DensestAlk}(G, T) \\
3: \ C' \leftarrow \text{EnhanceComponent}(G', T) \\
4: \ & \text{for each component } C'_i \in C' \text{ do} \\
5: \ & \quad Q \leftarrow V(C'_i) - S(a) \\
6: \ & \quad \text{while } Q \text{ is not empty do} \\
7: \ & \quad \quad u_{\text{min}} \leftarrow \text{pop lowest degree node from } Q \\
8: \ & \quad \quad \text{if } (C'_i - u_{\text{min}}) \text{ is connected then} \\
9: \ & \quad \quad \quad C'_i \leftarrow C'_i - u_{\text{min}} \\
10: \ & \quad \quad \text{end if} \\
11: \ & \quad \text{end while} \\
12: \ & \quad \text{end for} \\
13: \ & \text{Return } \arg \min_{C'_i \in C'} |V(C'_i)|
\end{align*}

\subsection{Single Skill Team Formation}

We run the single skill experiments for $k \in \{3, 5, 7, 9, 11, 13, 15\}$. For each value of $k$, we have a separate run for each skill $a \in \{T, A1, DB, DM\}$. We calculate statistics, such as density, size, and number of connected components for each solution and present the mean over these four runs as the final statistic.

Figures 6(a) and 6(b) show ($k$ vs. density) and ($k$ vs. size) plots, respectively. From these plots, we can see that the density obtained by $s$-\text{DensestAlk} significantly outperforms the density obtained by \text{MinDiameter} algorithm. This is of course expected. However, the downside is that the size of the solution to $s$-\text{DensestAlk} is also larger (and in some cases disconnected). The heuristic \text{EnhancedDense} essentially adds neighbors to each component in the solution so that the resulting component satisfies the required skill-set and then picks the one with the smallest size. Therefore connectivity is guaranteed. Further, the reduction in density is not much and even the cardinality has reduced compared to the original solution. This also means that the
solution returned by \( s\text{-DensestAlk} \) contained a good component to start with - by good component we mean a component that has most of the skills satisfied and has high density.

Now, notice that by applying heuristics PartialTrimmedDense and CompleteTrimmedDense, we attempt to remove the non-skilled nodes one by one from each of these enhanced components (while maintaining connectivity). As the plots show again, this serves the purpose of significantly reducing the cardinality of the solution and as a hard constraint the algorithm still satisfies the skill requirement. It can be observed from the plots that PartialTrimmedDense has density almost equal to the \( s\text{-DensestAlk} \) and the cardinality is reduced by more than fifty percent. Further, CompleteTrimmedDense gives a solution that has cardinality almost equal to \( k \) (which would be optimal), with very little reduction in density. Finally, we plot \((k \text{ vs. density per node})\) in Figure 3(c). While this figure can be deduced, we present it to highlight the observation that the heuristics reduce the cardinality without compromising on the density. Notice that in this plot, CompleteTrimmedDense has the highest value of density per node, for every value of \( k \).

Given that density is intuitively a better measure of team collaboration, these results show that we are completely able to eliminate connectivity issues inherent in this objective, and output small yet sufficient, and highly collaborative (dense) teams.

### 6.4 Multiple Skill Team Formation

We run the multiple skill experiments for \( k \in \{3, 8, 13, 18, 23, 28\} \) and for each run, we randomly choose \( k \) skills from \( A = \{T, A1, DB, DM\} \). For example, when \( k = 3 \), we may choose a skill (multi)set \( \{T, T, DM\} \) which means we want a subgraph that contains at least two authors of skill \( T \) and one author of skill \( DM \). Recall that a given author can have multiple skills and therefore the solution may consist of a subgraph whose size is less than the value of \( k \).

**Figure 3: single skill experiments**

**Figure 4: multiple skills experiments**

Figures 4(a) and 4(b) plots \((k \text{ vs. density})\) and \((k \text{ vs. size})\), respectively, for multiple skill team formation experiments. Note that the plots for multiple skill experiments fluctuate more than single skill experiments.
This is due to the randomness in picking the multiple skills requirements. Also, some solutions returned are of the same size even as \( k \) is increased. This is because sometimes the same solution satisfies different required skill sets.

In these figures, we again see that \( m\text{-DensestAlk} \) algorithm has the highest density. Note that the solution with density 0 and size 1 corresponds to an individual that has all the required skills. Further, similar to single skill experiments, we apply the heuristics mentioned earlier in order to get a connected subgraph without compromising on the density much. Figure 4(b) shows that the heuristics have been effective in reducing cardinality. In fact, the cardinality of the solution obtained by \( \text{CompleteTrimmedDense} \) is lesser than \( k \) because a single individual can satisfy more than one skills. Further, for the \( k \geq 13 \) tasks, the density achieved by the heuristics is also close to that of \( m\text{-DensestAlk} \). While sometimes certain heuristics have low density (e.g., \( k = 3 \) or \( k = 8 \)), all heuristics offer a nice trade-off between size and density (and return connected solutions by design). For each value of \( k \), there exists at least one solution with density close to maximum-density and small cardinality. We omit the density per node plot here due to lack of space, and because it can be deduced from Figures 4(a), (b).

6.5 Density Vs. Diameter Analysis

In the previous sections, we demonstrated the effectiveness of various heuristic algorithms in order to obtain a solution subgraph that is connected, small and dense. The intuition behind suggesting the density as a metric for team collaborative compatibility is that a denser graph has more edges between nodes, resulting in a greater possibility for collaboration. Small diameter does not necessarily guarantee this property. In this section, we consider three metrics for comparing Density and Diameter based approaches: \( \text{teamPubs}, \text{partialTeamPubs} \) and \( \text{teamPubRatio} \). The metric \( \text{teamPubs} \) defines the number of publications where all the authors of the publication belong to the solution subgraph. \( \text{partialTeamPubs} \) defines the number of publications where at least half of the authors of the publication belong to the solution subgraph. These two metrics give a good indication of the collaboration compatibility of reported teams. In addition, we propose another metric \( \text{teamPubRatio} \) which is essential for the comparative study because it is affected not only by the team-members’ collaboration compatibility but also on the size of the team. In this case, for each publication, say \( p' \), we compute the ratio of \( |X' \cap A'| \) where \( X' \) is the set of authors in the solution subgraph and \( A' \) is the set of authors of the publication \( p' \). That is, \( \text{teamPubRatio} \) measures the Jaccard similarity between a publication’s author set and a team’s author set. We then take the average of this quantity over all the publications.

We now describe the details of the evaluation strategy used to calculate these metrics. For both single skill team formation and multiple skill team formation problems, we consider the teams that were proposed as a solution in the previously described experiments. In particular, we consider the solutions reported by the algorithms \( \text{CompleteTrimmedDense} \) and \( \text{MinDiameter} \). We choose only \( \text{CompleteTrimmedDense} \) algorithm for density because it reports the smallest solutions. The goal is to establish that the small teams obtained by \( \text{CompleteTrimmedDense} \) also achieve superior results for the three metrics of collaboration compatibility mentioned above. The results of metric evaluation are shown in the plots 5 and 6 for single skill and multi skill experiments, respectively. In each plot, value of \( k \) is plotted along the \( x \)-axis and the value of the the metrics for the corresponding solution subgraphs are along the \( y \)-axis. In case of single skill experiments, for each \( k \), the metric value reported is the average of metric values for the solutions corresponding to each of the skills \( \{ \text{T, AI, DB, DM} \} \). Further, for the metrics \( \text{teamPubs} \) and \( \text{partialTeamPubs} \) the \( y \)-axis defines the resulting number of publications whereas for the metric \( \text{teamPubRatio} \), the \( y \)-axis defines the scaled \((100000 \times)\) metric value. From these plots it can be observed that in both single skill and multi skill team formation problems, the algorithm \( \text{CompleteTrimmedDense} \) consistently outperforms the algorithm \( \text{MinDiameter} \) for all the three metrics. In case of single skill, for each of the three metrics, and for most values of \( k \), the metric value for \( \text{CompleteTrimmedDense} \) is about twice that of \( \text{MinDiameter} \). In multi skill, the variation is somewhat larger, but \( \text{CompleteTrimmedDense} \) consistently displays superior metric values for all cases. Recall that the size of the solution teams by both these algorithms were very similar (and the metric \( \text{teamPubRatio} \) does not necessarily benefit with larger team size); therefore, these experiments suggest that density-based team formation leads to teams with better collaborative compatibility than the diameter-based team formation.
6.6 Qualitative evidence

To analyze the quality of teams that are returned by our algorithms for maximum density, we refer to the Most Cited Computer Science Authors list maintained by CiteSeerX (citeseerx.ist.psu.edu/stats/authors?all=true) which contains most cited 10000 authors. We also refer to the list Central Authors: Computer Science (all-time) published at (confsearch.org/confsearch/ca.jsp) [11]. This list contains 1000 researchers ranked on the basis of DBLP publications.

We examine the authors of teams returned by s-DensestAlk and m-DensestAlk algorithms in order to determine how many authors in the team are among top 500 and top 1000 most cited authors according to the list maintained by CiteSeerX. Due to space constraints, we present only some representative lists from single skill team formation in Table 1. The lists are for $k = 3$ for $T$ and $DB$, and for $k = 15$ for $DM$ and $AI$. Team members who appear among the top 500 and 1000 cited authors are indicated by bold and italic font, respectively. We can see from these results that in each team, we have many top cited and prolific/famous authors (who may not be in the top 1000 list). These results show that teams formed by choosing the objective of maximum density subgraph are intuitively meaningful.

Complementary results are seen on using the second list, i.e. a list of top 1000 ranked researchers [11]. Instead of presenting another table with author names corresponding to this list, we adopt a different approach.
Table 1: Teams reported by s-DensestAlk.

| Skills | Authors |
|--------|---------|
| T(3)   | Prabhakar Raghavan, Ravi Kumar, Philip S. Yu, D. Sivakumar, Sridhar Rajagopalan, Andrew Tomkins |
| DB(3)  | Philip S. Yu, Haixun Wang, Jiawei Han, Xifeng Yan, Wei Fan, Hong Cheng, Charu C. Aggarwal |
| DM(15) | Jiawei Han, Zheng Chen, Haixun Wang, Philip S. Yu, Amr El Abbadi, Benyu Zhang, Wei Fan, Jun Yan, Shuiyong Yan, Hong Cheng, Qiang Yang, Ning Liu, Jian Pei, Charu C. Aggarwal, Xifeng Yan, Divyakant Agrawal |
| AI(15) | Ravi Kumar, Ronald Fagin, Philip S. Yu, Christos Faloutsos, Zheng Chen, Wei-Ying Ma, Andrei Z. Broder, Jian-Tao Sun, Hongjun Lu, Dou Shen, Shuiyong Yan, Anthony K. H. Tung, Wei Fan, Sridhar Rajagopalan, Qiang Yang, Eli Upfal, Andrew Tomkins, Jure Leskovec |

Table 2: Team ranks based on top-ranked authors.

| Skills | {s/m} - CompleteTrimmed DensityAlk | Min Dense | Min Diameter |
|--------|-----------------------------------|-----------|--------------|
| T(3)   | 23.42                             | 8.11      | 0            |
| AI(3)  | 20.81                             | 17.34     | 0            |
| DB(3)  | 18.25                             | 18.25     | 0            |
| DM(3)  | 18.25                             | 18.25     | 0            |
| T(15)  | 14.95                             | 19.67     | 2.05         |
| AI(15) | 15.25                             | 14.48     | 1.86         |
| DB(15) | 10.54                             | 10.80     | 0.75         |
| DM(15) | 9.55                              | 9.93      | 1.05         |
| T(1), DB(1), DM(1) | 18.25 | 100 | 24.39 |
| T(8), AI(6), DB(8), DM(6) | 9.49 | 6.3 | 4.1 |

for measuring quality. We determine the overall rank of a team using the ranks of the individual authors within the team. To be specific, we compute the mean reciprocal rank of all the skilled individuals in the team and report the final rank of the team as \( r = 1000 \sum \frac{1}{r_i} \) where \( r_i \) denotes the rank of a skilled individual and \( n_s \) denotes the skilled individuals in the team. Similar findings are observed if this quantity includes non-skilled nodes as well. We report the ranks observed in Table 2. Our original algorithms for maximum density and the subsequent heuristics form a team of highly ranked authors and perform significantly better than the minimum-diameter algorithm. The validation of these algorithms over two different qualitative approaches provides further credence to this framework of team formation using a density based objective.

7 Conclusions and Future Work

We presented a novel approach for skilled collaborative team formation based on finding dense subgraphs. On the theoretical front, we showed constant factor approximation algorithms. On the practical side, we showed several heuristic improvements to our main provable algorithm, and compared it to the previous approach based on identifying small diameter subgraphs. Our experimental results show that the densest subgraph approach significantly outperforms the previous techniques on multiple different measures of collaborative compatibility.

The formulations in this paper as well as [12] assume that for any given skill, each node in the network is either skilled or not skilled. A nice generalization would be to consider a range of expertise for any skill, modeled as a value between 0 and 1. Another specific open question is to present more efficient algorithms for all objectives. Further, these definitions can be extended along many dimensions. In reality a team’s value depends on several complex assets such as cultural backgrounds, geographical location, personalities, ability...
to work in teams etc. Some of these characteristics cannot even be measured easily. Yet, while the current models are a good start, it would be nice to investigate these directions and move closer to the motivating realistic scenario.

References

[1] R. Andersen and K. Chellapilla. Finding dense subgraphs with size bounds. In WAW ’09: Proceedings of the 6th International Workshop on Algorithms and Models for the Web-Graph, pages 25–37, 2009.

[2] Y. Asahiro, R. Hassin, and K. Iwama. Complexity of finding dense subgraphs. Discrete Appl. Math., 121(1-3):15–26, 2002.

[3] A. Baykasoglu, T. Dereli, and S. Das. Project team selection using fuzzy optimization approach. Cybern. Syst., 38(2):155–185, 2007.

[4] S. J. Chen and L. Lin. Modeling team member characteristics for the formation of a multifunctional team in concurrent engineering. IEEE Transactions on Engineering Management, 51(2):111–124, 2004.

[5] U. Feige, G. Kortsarz, and D. Peleg. The dense k-subgraph problem. Algorithmica, 29, 1999.

[6] F. L. Fitzpatrick and R. G. Askin. Forming effective worker teams with multi-functional skill requirements. Comput. Ind. Eng., 48(3):593–608, 2005.

[7] M. Gaston, J. Simmons, and M. desJardins. Adapting network structures for efficient team formation. In Proceedings of the AAAI Fall Symposium on Artificial Multi-agent Learning, 2004.

[8] A. V. Goldberg. Finding a maximum density subgraph. Technical Report UCB/CSD-84-171, EECS Department, University of California, Berkeley, 1984.

[9] S. Khot. Ruling out ptas for graph min-bisection, dense k-subgraph, and bipartite clique. SIAM J Computing, 36(4):1025–1071, 2006.

[10] S. Khuller and B. Saha. On finding dense subgraphs. In ICALP (1), pages 597–608, 2009.

[11] M. Kuhn and R. Wattenhofer. The theoretic center of computer science. SIGACT News, 38(4):54–63, 2007.

[12] T. Lappas, K. Liu, and E. Terzi. Finding a team of experts in social networks. In KDD, pages 467–476, 2009.

[13] E. Lawler. Combinatorial optimization - networks and matroids. Holt, Rinehart, and Winston, 1976.

[14] H. Wi, S. Oh, J. Mun, and M. Jung. A team formation model based on knowledge and collaboration. Expert Syst. Appl., 36(5):9121–9134, 2009.

[15] A. Zzkarian and A. Kusiak. Forming teams: an analytical approach. IIE Transactions, 31(1):85–97, 2004.