HOMOLOGICAL ASPECTS OF GORENSTEIN FLAT MODULES
RELATIVE TO DUALITY PAIRS

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ABSTRACT. We study homological aspects of Gorenstein flat modules over a ring with respect to a duality pair \((\mathcal{L}, \mathcal{A})\). These modules are defined as cycles of exact chain complexes with components in \(\mathcal{L}\) which remain exact after tensoring by objects in \(\mathcal{A}\) which are left Ext-orthogonal to \(\mathcal{A}\). In the case where \((\mathcal{L}, \mathcal{A})\) is bi-complete (meaning in addition that \(\mathcal{L}\) is closed under extensions, products and coproducts, \(R \in \mathcal{L}, (\mathcal{A}, \mathcal{L})\) is also a duality pair, and \(\mathcal{A}\) is the right half of a hereditary complete cotorsion pair) we prove that these relative Gorenstein flat modules are closed under extensions, and that the corresponding Gorenstein flat dimension is well behaved in the sense that it recovers many of the properties and characterizations of its (absolute) Gorenstein flat counterpart (for instance, it can be described in terms of torsion functors). The latter in turn is a consequence of a Pontryagin duality relation that we show between these relative Gorenstein flat modules and certain Gorenstein injective modules relative to \(\mathcal{A}\).

1. INTRODUCTION

Let \(R\) be an associative ring with identity. Gorenstein flat \(R\)-modules were introduced by Enochs, Jenda and Torrecillas in \([11]\). These are defined as cycles of exact chain complexes of flat (left) \(R\)-modules, which remains exact after tensoring by injective (right) \(R\)-modules. These modules and some of its properties were studied by Holm in \([17]\). In particular, he proved that over a right coherent ring, Gorenstein flat left \(R\)-modules are closed under extensions. The validity of this closure property over an arbitrary ring was an open problem until 2020, when it was settled by Šaroch and Šťovíček in \([21]\).

Gorenstein flat modules have turned out to be key in Gorenstein homological algebra, due to being a good analog for flat modules, but also for their rich interactions with Gorenstein projective and Gorenstein injective modules. In the last two decades there has been an increasing interest for generalizations of Gorenstein flat modules, such as the Gorenstein AC-flat modules presented by Bravo, Estrada and Iacob in \([8]\). The latter, as one should expect, have appealing interactions with the Gorenstein AC-flat projective modules, defined by Bravo, Gillespie and Hovey in \([9]\). Further generalizations have also been defined and studied in \([12]\] and \([23]\). In the former reference, Estrada, Iacob and the second named author of the present article study the Gorenstein \(B\)-flat modules, that is, cycles of complexes of flat left \(R\)-modules which remain exact after tensoring by modules in a class \(B\) of right \(R\)-modules. Under certain condition for \(B\) (namely, that \(B\) is closed under products and contains an elementary cogenerator of its definable closure),

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these modules are closed under extensions, and also have many interesting ho-

mological properties such as the construction of Gorenstein $B$-flat covers. In the 

latter reference, Wang, Yang and Zhu investigate Gorenstein flat modules relative 
to a duality pair $(\mathcal{L}, A)$. The required conditions that $(\mathcal{L}, A)$ needs to fulfill so that 
these relative Gorenstein flat modules are closed under extensions are that $(\mathcal{L}, A)$ is 
a perfect duality, $\mathcal{L}$ is closed under epikernels, $A$ is closed under products, and 
$\text{Tor}^R_i(A, L) = 0$ for every $A \in A$, $L \in \mathcal{L}$ and $i \in \mathbb{Z}_{>0}$.

In the present article, we aim to continue with the study of Gorenstein flat mod-
ules relative to duality pairs $(\mathcal{L}, A)$. We shall consider duality pairs $(\mathcal{L}, A)$ such 
that $\mathcal{L}$ is closed under extensions, products and coproducts, $R \in \mathcal{L}$, $(A, \mathcal{L})$ is also a 
duality pair, and $A$ is the right half of a hereditary complete cotorsion pair. We 
shall refer to such duality pairs as \textit{bicomplete}. On the other hand, our relative ver-

cion of Gorenstein flat modules is defined as follows: cycles of exact chain com-
plexes with components in $\mathcal{L}$ which remain exact after tensoring by objects in $A$, 
which in addition are left Ext-orthogonal to $A$. Under these assumptions, we prove 
that these relative Gorenstein flat modules are closed under extensions, and that 
the corresponding Gorenstein flat dimension is well behaved in the sense that it 
recovers many of the properties and characterizations of its (absolute) Gorenstein 
flat counterpart (for instance, it can be described in terms of torsion functors). 
The previous will be a consequence of a Pontryagin duality relation that we show 
between these relative Gorenstein flat modules and certain Gorenstein injective 
modules relative to $A$ (in the sense of [5]). The condition $\text{Tor}^R_i(A, L) = 0$ for ev-
ery $A \in A$, $L \in \mathcal{L}$ and $i \in \mathbb{Z}_{>0}$, will not be required in our approach. Hence, in 
particular, our results are valid for the duality pair (level, absolutely clean).

\textbf{Organization.} In Section 2 we recall the necessary background on relative ho-
mological algebra of modules, such as the definitions and notations for approxima-
tions, homological dimensions, cotorsion pairs, and duality pairs. Regarding 
duality pairs, we present in Proposition 2.5 a way to induce new duality pairs, 
namely, if $(\mathcal{L}, A)$ is a duality pair satisfying certain conditions, then $(\mathcal{L}^n, A^\nu_n)$ will 
be also a duality pair for which the same conditions hold as well. Here, $\mathcal{L}^n$ and $A^\nu_n$ 
denote the classes of modules with $\mathcal{L}$-resolution and $A$-coresolution dimension at 
most $n$, respectively.

Section 3 is devoted to investigate the class of cycles of exact chain complexes 
with components in a class $\mathcal{L}$ of left $R$-modules, which remain exact after tensor-
ning by objects in a class $A$ of right $R$-modules which are also left orthogonal to 
$A$. The latter class will be denoted by $\nu$, and these cycles of such complexes will 
be called Gorenstein $(\mathcal{L}, \nu)$-flat $R$-modules, which form a class that we denote by 
$\mathcal{GF}_{(\mathcal{L}, \nu)}(R)$. Many interesting properties are obtained in the case where $(\mathcal{L}, A)$ is 
a bicomplete duality pair. The first important result about $\mathcal{GF}_{(\mathcal{L}, \nu)}(R)$ is its Pon-
tryagin duality relation with the class $\mathcal{GI}_{(\nu, A)}(R^o)$ of $(\nu, A)$-Gorenstein injective 
right $R$-modules (defined as cycles of exact chain complexes with components in 
$A$ which remain exact after applying the functor $\text{Hom}_{R^o}(\nu, -)$). Specifically, we 
show in Theorem 3.9 that $M \in \mathcal{GF}_{(\mathcal{L}, \nu)}(R)$ if, and only if, its Pontryagin dual 
$M^+ := \text{Hom}_{R}(M, \mathbb{Q}/\mathbb{Z})$ belongs to $\mathcal{GI}_{(\nu, A)}(R^o)$. Several remarkable outcomes of 
this result are comprised in Corollary 3.11, where we show that $\mathcal{GF}_{(\mathcal{L}, \nu)}(R)$ is a 
resolving class closed under direct summands, and that $(\mathcal{GF}_{(\mathcal{L}, \nu)}(R), \mathcal{GI}_{(\nu, A)}(R^o))$ 
is a perfect duality pair. In particular, $\mathcal{GF}_{(\mathcal{L}, \nu)}(R)$ is closed under extensions, and 
every left $R$-module has a Gorenstein $(\mathcal{L}, \nu)$-flat cover.
In Section 4 we define and study homological dimensions relative to \( \mathcal{GF}_{(L, \nu)}(R) \). Theorem 3.9 also implies, in the case where \((L, A)\) is a bicomplete duality pair, a duality relation between the Gorenstein \((L, \nu)\)-flat and the \((\nu, A)\)-Gorenstein injective dimensions, namely, that \( \text{Gid}_{(L, \nu)}(M) = \text{Gid}_{(\nu, A)}(M^+) \) for every left \( R \)-module \( M \), as proved in Proposition 4.3. On the other hand, we characterize in Theorem 4.5 the Gorenstein \((L, \nu)\)-flat dimension of \( M \) in terms of the vanishing of the torsion functors \( \text{Tor}_i^R(-, M) \) at the class \( \nu \). Moreover, we also study global relative Gorenstein flat dimensions of the ground ring \( R \). Specifically, we introduce the left weak Gorenstein \((L, \nu)\)-flat global dimension and the left Gorenstein \((L, \nu)\)-flat finitistic dimension of \( R \), denoted by \( l.\text{wGdim}_{(L, \nu)}(R) \) and \( l.\text{GF}_{(L, \nu)}\)-findim\( (R) \), respectively. In the case where \((L, A)\) is a bicomplete duality pair, we characterize in Proposition 4.9 the finiteness of \( l.\text{wGdim}_{(L, \nu)}(R) \) in terms of the flat dimension of modules in \( \nu \). Furthermore, under the same conditions we show in Proposition 4.13 that \( l.\text{GF}_{(L, \nu)}\)-findim\( (R) \) coincides with the global \( L \)-resolution dimension of \( R \).

We close our article with Appendix A, where we show that the notions of Gorenstein \((L, \nu)\)-flat and \((\nu, A)\)-Gorenstein injective modules carry over to the category of chain complexes. In particular, we characterize these complexes in terms of \( \mathcal{GF}_{(L, \nu)}(R) \) and \( \mathcal{GI}_{(\nu, A)}(R^\alpha) \), so that the duality relation in the context of modules in inherited by the chain complex counterparts of \( \mathcal{GF}_{(L, \nu)}(R) \) and \( \mathcal{GI}_{(\nu, A)}(R^\alpha) \). As a consequence, the closure properties, the existence of relative approximations and the construction of homological dimensions relative to \( \mathcal{GF}_{(L, \nu)}(R) \) are also valid for Gorenstein \((L, \nu)\)-flat complexes.

2. Preliminaries

Notations. In what follows, we shall work with categories of modules and chain complexes over an associative ring \( R \) with identity. We denote by \( \text{Mod}(R) \) and \( \text{Mod}(R^\alpha) \) the categories of left and right \( R \)-modules. For simplicity, we shall refer to these two classes of modules as \( R \)-modules and \( R^\alpha \)-modules, respectively. Unless otherwise specified, the definitions and notations below will be presented for \( R \)-modules.

The categories of complexes of \( R \)-modules and \( R^\alpha \)-modules will be denoted by \( \text{Ch}(R) \) and \( \text{Ch}(R^\alpha) \). Objects in \( \text{Ch}(R) \) are sequences

\[
X_\bullet = \cdots \to X_{m+1} \xrightarrow{\partial_{m+1}^X} X_m \xrightarrow{\partial_m^X} X_{m-1} \to \cdots
\]

such that \( \partial_m^X \circ \partial_{m+1}^X = 0 \) for every integer \( m \in \mathbb{Z} \). The cycles of \( X_\bullet \), denoted \( Z_m(X_\bullet) \), are defined as the kernel of \( \partial_m^X \). If \( \mathcal{X} \) is a class of \( R \)-modules, denoted \( \mathcal{X} \subseteq \text{Mod}(R) \), then \( \text{Ch}(\mathcal{X}) \) denotes the class of chain complexes of modules in \( \mathcal{X} \), that is, \( X_\bullet \in \text{Ch}(\mathcal{X}) \) if \( X_m \in \mathcal{X} \) for every \( m \in \mathbb{Z} \).

Among the most important subcategories of \( \text{Mod}(R) \), we mainly consider the classes of projective, injective and flat \( R \)-modules, which will be denoted by \( \mathcal{P}(R) \), \( \mathcal{I}(R) \) and \( \mathcal{F}(R) \), respectively.

Concerning functors defined on modules, we let

\[
\text{Ext}^i_R(-,-) : \text{Mod}(R) \times \text{Mod}(R) \to \text{Mod}(\mathbb{Z})
\]

denote the right \( i \)-th derived functor of

\[
\text{Hom}_R(-,-) : \text{Mod}(R) \times \text{Mod}(R) \to \text{Mod}(\mathbb{Z}),
\]
where \( \text{Mod}(Z) \) denotes the category of abelian groups. If \( M \in \text{Mod}(R^o) \) and \( N \in \text{Mod}(R) \), then \( M \otimes_R N \) denotes the tensor product of \( M \) and \( N \). Recall the construction of this tensor products defines a bifunctor

\[
- \otimes_R - : \text{Mod}(R^o) \times \text{Mod}(R) \rightarrow \text{Mod}(Z).
\]

The left derived functors of \( - \otimes_R - \) are denoted by

\[
\text{Tor}^R_i(-,-) : \text{Mod}(R^o) \times \text{Mod}(R) \rightarrow \text{Mod}(Z).
\]

**Orthogonality.** In what follows, we let \( \mathbb{Z}_{>0} \) and \( \mathbb{Z}_{\geq 0} \) denote the sets of positive and nonnegative integers, respectively. For \( \mathcal{X}, \mathcal{Y} \subseteq \text{Mod}(R) \), the notation

\[
\text{Ext}^1_R(\mathcal{X}, \mathcal{Y}) = 0
\]

means that \( \text{Ext}^1_R(X, Y) = 0 \) for every \( X \in \mathcal{X} \) and \( Y \in \mathcal{Y} \). In the case where \( \mathcal{Y} \) is the singleton \( \{Y\} \) for some \( Y \in \text{Mod}(R) \), we simply write \( \text{Ext}^1_R(\mathcal{X}, Y) = 0 \). The notation \( \text{Ext}^1_R(X, \mathcal{Y}) = 0 \) for \( X \in \text{Mod}(R) \) has a similar meaning. Moreover, by

\[
\text{Ext}^{\geq 1}_R(\mathcal{X}, \mathcal{Y}) = 0
\]

we shall mean that \( \text{Ext}^i_R(\mathcal{X}, \mathcal{Y}) = 0 \) for every \( i \in \mathbb{Z}_{>0} \). One also has similar meanings for \( \text{Ext}^i_R(\mathcal{X}, N) = 0 \), \( \text{Ext}^{\geq 1}_R(N, \mathcal{Y}) = 0 \) and \( \text{Ext}^{\geq 1}_R(X, Y) = 0 \). We can also replace \( \text{Ext} \) by \( \text{Tor} \) in order to obtain similar notations for \( \text{Tor} \)-orthogonality.

The right \( \text{Ext} \)-orthogonal complements of \( \mathcal{X} \) will be denoted by

\[
\mathcal{X}^{\perp^i} = \{ M \in \text{Mod}(R) : \text{Ext}^i_R(\mathcal{X}, M) = 0 \},
\]

\[
\mathcal{X}^{\perp} = \bigcap_{i \geq 1} \mathcal{X}^{\perp^i}.
\]

The left orthogonal complements, on the other hand, are defined similarly. If \( \mathcal{X} \subseteq \text{Mod}(R^o) \), the right \( \text{Tor} \)-orthogonal complements of \( \mathcal{X} \) will be denoted by

\[
\mathcal{X}^{\perp t^i} = \{ M \in \text{Mod}(R) : \text{Tor}_i^R(\mathcal{X}, M) = 0 \},
\]

\[
\mathcal{X}^{\perp t} = \bigcap_{i \geq 1} \mathcal{X}^{\perp t^i}.
\]

Left \( \text{Tor} \)-orthogonal complements \( ^t \mathcal{X} \) and \( ^t \mathcal{Y} \) are defined for classes \( \mathcal{Y} \subseteq \text{Mod}(R) \) of \( R \)-modules.

**Relative homological dimensions.** There are homological dimensions defined in terms of extension functors. Let \( M \in \text{Mod}(R) \) and \( \mathcal{X}, \mathcal{Y} \subseteq \text{Mod}(R) \). The *injective dimensions of \( M \) and \( \mathcal{Y} \) relative to \( \mathcal{X} \) are defined by*

\[
\text{id}_{\mathcal{X}}(M) := \inf \{ m \in \mathbb{Z}_{\geq 0} : \text{Ext}^{\geq m+1}_R(\mathcal{X}, M) = 0 \},
\]

\[
\text{id}_{\mathcal{X}}(\mathcal{Y}) := \sup \{ \text{id}_{\mathcal{X}}(Y) : Y \in \mathcal{Y} \}.
\]

In the case where \( \mathcal{X} = \text{Mod}(R) \), we write

\[
\text{id}_{\text{Mod}(R)}(M) = \text{id}(M) \quad \text{and} \quad \text{id}_{\text{Mod}(R)}(\mathcal{Y}) = \text{id}(\mathcal{Y})
\]

for the (absolute) injective dimensions of \( M \) and \( \mathcal{Y} \).

By an \( \mathcal{X} \)-resolution of \( M \) we mean an exact complex

\[
\cdots \rightarrow X_m \rightarrow X_{m-1} \rightarrow \cdots \rightarrow X_1 \rightarrow X_0 \rightarrow M
\]

with \( X_k \in \mathcal{X} \) for every \( k \in \mathbb{Z}_{\geq 0} \). For simplicity, any \( k \)-th syzygy in a \( \mathcal{X} \)-resolution of \( M \) will be denote by \( \Omega^\mathcal{X}_k(M) \). If \( X_k = 0 \) for \( k > m \), we say that the previous
resolution has length \( m \). The resolution dimension relative to \( \mathcal{X} \) (or the \( \mathcal{X} \)-resolution dimension) of \( M \) is defined as the value
\[
\text{resdim}_{\mathcal{X}}(M) := \min\{m \in \mathbb{Z}_{\geq 0} : \text{there exists an} \ \mathcal{X} \text{-resolution of} \ C \ \text{of length} \ m \}.
\]
Moreover, if \( \mathcal{Y} \subseteq \text{Mod}(R) \) then
\[
\text{resdim}_{\mathcal{X}}(\mathcal{Y}) := \sup\{\text{resdim}_{\mathcal{X}}(Y) : Y \in \mathcal{Y}\}
\]
defines the resolution dimension of \( \mathcal{Y} \) relative to \( \mathcal{X} \). The classes of objects with bounded (by some \( n \geq 0 \)) and finite \( \mathcal{X} \)-resolution dimensions will be denoted by
\[
\mathcal{X}^n_n := \{M \in \text{Mod}(R) : \text{resdim}_{\mathcal{X}}(M) \leq n\} \quad \text{and} \quad \mathcal{X}^\wedge := \bigcup_{n \geq 0} \mathcal{X}^n_n.
\]
Dually, we can define \( \mathcal{X} \)-coresolutions and the coresolution dimension of \( M \) and \( \mathcal{Y} \) relative to \( \mathcal{X} \) (denoted \( \text{coresdim}_{\mathcal{X}}(M) \) and \( \text{coresdim}_{\mathcal{X}}(\mathcal{Y}) \)). We also have the dual notations \( \mathcal{X}^\wedge_n \) and \( \mathcal{X}^\vee \) for the classes of \( R \)-modules with bounded and finite \( \mathcal{X} \)-coresolution dimension.

The resolution and coresolution dimensions define functions
\[
\text{resdim}_{\mathcal{X}}(-) : \text{Mod}(R) \to \mathbb{Z}_{\geq 0} \cup \{\infty\} \quad \text{and} \quad \text{coresdim}_{\mathcal{X}}(-) : \text{Mod}(R) \to \mathbb{Z}_{\geq 0} \cup \{\infty\},
\]
for which one can consider the following notion of stability.

**Definition 2.1.** We say that \( \text{resdim}_{\mathcal{X}}(-) \) is stable if for every \( M \in \text{Mod}(R) \) and \( n \in \mathbb{Z}_{\geq 0} \), the following two assertions are equivalent:

(a) \( \text{resdim}_{\mathcal{X}}(M) \leq n \).

(b) Any \((n-1)\)-th \( \mathcal{X} \)-syzygy of \( M \) belongs to \( \mathcal{X} \).

The stable description for \( \text{coresdim}_{\mathcal{X}}(-) \) is defined dually.

We can note the following property concerning stability.

**Proposition 2.2.** Let \( \mathcal{X} \subseteq \text{Mod}(R) \) such that \( \text{resdim}_{\mathcal{X}}(-) \) is stable. For every short exact sequence \( A \rightarrow X \rightarrow C \) with \( X \in \mathcal{X} \) and \( A, C \in \mathcal{X}^\wedge \) and every \( n \in \mathbb{Z}_{\geq 0} \), one has that \( \text{resdim}_{\mathcal{X}}(C) = n \) if, and only if, \( \text{resdim}_{\mathcal{X}}(A) = n - 1 \).

Some of the sufficient conditions that a class \( \mathcal{X} \subseteq \text{Mod}(R) \) needs to fulfill so that \( \text{resdim}_{\mathcal{X}}(-) \) is stable are comprised in the concept of left Frobenius pair. If one lets \( \omega \subseteq \mathcal{X} \), one says that \( (\mathcal{X}, \omega) \) is a left Frobenius pair if:

1. \( \mathcal{X} \) is left thick, that is, it is closed under extensions, epikernels and direct summands.
2. \( \text{id}_{\mathcal{X}}(\omega) = 0 \).
3. \( \omega \) is a relative cogenerator in \( \mathcal{X} \), that is, for every object \( X \in \mathcal{X} \) there is an embedding into an object of \( \omega \) with cokernel in \( \mathcal{X} \).

The following result can be found in [5, Prop. 2.14], and was originally proved in [2, Prop. 3.3].

**Proposition 2.3.** If \( (\mathcal{X}, \omega) \) is a left Frobenius, then \( \text{resdim}_{\mathcal{X}}(-) \) is stable.

**Approximations.** Given \( M \in \text{Mod}(R) \) and \( \mathcal{X} \subseteq \text{Mod}(R) \), recall that a morphism \( \varphi : X \rightarrow M \) with \( X \in \mathcal{X} \) is an \( \mathcal{X} \)-precover of \( M \) if for every morphism \( \varphi' : X' \rightarrow M \) with \( X' \in \mathcal{X} \), there exists a morphism \( h : X' \rightarrow X \) such that \( \varphi' = \varphi \circ h \). If in addition, in the case where \( X' = X \) and \( \varphi' = \varphi \), the equality \( \varphi' = \varphi \circ h \) can only be completed by automorphisms \( h \) of \( X \), then one says that \( \varphi \) is an \( \mathcal{X} \)-cover of \( M \). An \( \mathcal{X} \)-precover is special if it is epic and its kernel belongs to \( \mathcal{X}^{-1} \). We shall say that \( \mathcal{X} \)
is precovering if every $R$-module has an $\mathcal{X}$-precover. Dually, one has the notions of (special) (pre)envelopes and special precovering, covering, (special) preenveloping and enveloping classes.

**Cotorsion pairs.** Two classes $\mathcal{X}, \mathcal{Y} \subseteq \text{Mod}(R)$ form cotorsion pair $(\mathcal{X}, \mathcal{Y})$ if $\mathcal{X} = \perp \mathcal{Y}$ and $\mathcal{Y} = \perp \mathcal{X}$. The cotorsion pair $(\mathcal{X}, \mathcal{Y})$ is said to be complete if $\mathcal{X}$ is special precovering (or equivalently, if $\mathcal{Y}$ is special preenveloping).

A class $\mathcal{X} \subseteq \text{Mod}(R)$ is resolting if $\mathcal{P}(R) \subseteq \mathcal{X}$ and $\mathcal{X}$ is closed under extensions and epikernels (that is, given a short exact sequence $A \to B \to C$ with $C \in \mathcal{X}$, then $A \in \mathcal{X}$ if and only if $B \in \mathcal{X}$). Coresolving classes are defined dually. A cotorsion pair $(\mathcal{X}, \mathcal{Y})$ is hereditary if $\mathcal{X}$ is resolting (or equivalently, $\mathcal{Y}$ is coresolving). The latter two conditions are in turn equivalent to saying that $\text{Ext}_{R}^{\geq 1}(\mathcal{X}, \mathcal{Y}) = 0$. Note that $\mathcal{X} = \perp \mathcal{Y}$ and $\mathcal{Y} = \perp \mathcal{X}$ if $(\mathcal{X}, \mathcal{Y})$ is a hereditary cotorsion pair.

**Gorenstein injective modules relative to admissible pairs.** The following concepts will be presented for modules, although they carry over to abelian categories (see [5, Defs. 3.6 & 3.7]).

Given two classes $\mathcal{X}, \mathcal{Y} \subseteq \text{Mod}(R^{o})$ of $R^{o}$-modules, an $R^{o}$-module $C \in \text{Mod}(R^{o})$ is $(\mathcal{X}, \mathcal{Y})$-Gorenstein injective if there exists an exact and $\text{Hom}_{R^{o}}(\mathcal{X}, -$)-acyclic complex $Y_{\bullet} \in \text{Ch}(\mathcal{Y})$ such that $M \simeq Z_{0}(Y_{\bullet})$. By $\text{Hom}_{R^{o}}(\mathcal{X}, -$)-acyclic we mean that

\[
\text{Hom}_{R^{o}}(\mathcal{X}, Y_{\bullet}) := \cdots \to \text{Hom}_{R^{o}}(X, Y_{m-1}) \xrightarrow{\text{Hom}_{R^{o}}(X, Y_{m}^{\bullet})} \text{Hom}_{R^{o}}(X, Y_{m}) \to \cdots
\]

is an exact complex of abelian groups for every $X \in \mathcal{X}$.

The class of $(\mathcal{X}, \mathcal{Y})$-Gorenstein injective $R^{o}$-modules is denoted by $\mathcal{GI}(\mathcal{X}, \mathcal{Y})(R^{o})$. In order to have nice homological properties for $\mathcal{GI}(\mathcal{X}, \mathcal{Y})(R^{o})$ (for example, to be a coresolving class closed under direct summands) one needs a minimal set of conditions for $\mathcal{X}$ and $\mathcal{Y}$. These conditions are comprised in the concept of $\mathcal{GI}$-admissible pair, that is, pairs $(\mathcal{X}, \mathcal{Y})$ such that:

1. $\text{Ext}_{R}^{\geq 1}(\mathcal{X}, \mathcal{Y}) = 0$.
2. $\mathcal{X}$ and $\mathcal{Y}$ are closed under finite coproducts.
3. $\mathcal{Y}$ is closed under extensions.
4. $\mathcal{X} \cap \mathcal{Y}$ is a relative generator in $\mathcal{Y}$.
5. $\mathcal{Y}$ is a relative cogenerator in $\text{Mod}(R)$.

**Duality pairs.** Consider the Pontryagin duality functor

\[
(-)^{+} := \text{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z}) : \text{Mod}(R) \to \text{Mod}(R^{o}),
\]

which is exact since $\mathbb{Q}/\mathbb{Z}$ is an injective $\mathbb{Z}$-module. The notion of duality pair was introduced by Holm and Jørgensen in [18], in the following way: two classes $\mathcal{L} \subseteq \text{Mod}(R)$ and $\mathcal{A} \subseteq \text{Mod}(R^{o})$ form a duality pair $(\mathcal{L}, \mathcal{A})$ if:

1. $L \in \mathcal{L}$ if, and only if, $L^{+} \in \mathcal{A}$.
2. $\mathcal{A}$ is closed under direct summands and finite direct sums.

A duality pair $(\mathcal{L}, \mathcal{A})$ is called:

- (co)product-closed if $\mathcal{L}$ is closed under (co)products.
- perfect if it is coproduct closed, $\mathcal{L}$ is closed under extensions and contains $R$ (regarded as an $R$-module).

**Remark 2.4.** One can also consider the Pontryagin duality functor

\[
(-)^{+} := \text{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z}) : \text{Mod}(R^{o}) \to \text{Mod}(R),
\]
mapping $R^e$-modules to $R$-modules, and thus one gets a similar notion of duality pair $(\mathcal{L}, \mathcal{A})$ in the case where $\mathcal{L} \subseteq \text{Mod}(R^e)$ and $\mathcal{A} \subseteq \text{Mod}(R)$.

The previous two types of duality pairs are important since they induce approximations by the classes $\mathcal{L}$ and $\mathcal{A}$. Indeed, it is proved in [18, Thm. 3.1] that:

- If $(\mathcal{L}, \mathcal{A})$ is coproduct closed, then $\mathcal{L}$ is covering.
- If $(\mathcal{L}, \mathcal{A})$ is product closed, then $\mathcal{L}$ is preenveloping.
- If $(\mathcal{L}, \mathcal{A})$ is perfect, then $(\mathcal{L}, \mathcal{L}^{\perp})$ is a perfect cotorsion pair (that is, a cotorsion pair such that $\mathcal{L}$ is covering and $\mathcal{L}^{\perp}$ is enveloping).

Duality pairs were also studied by Bravo, Gillespie and Hovey in [14, Prop. 2.3] it is proved that if $\mathcal{L}$ is a (co)resolution dimension. Suppose we are given $\mathcal{L}$ and $\mathcal{A}$. Moreover, $\mathcal{L}$ is closed under direct limits and so $\mathcal{F}(R) \subseteq \mathcal{L}$ by Lazard-Govorov’s Theorem (see for instance [19, Thm. II.4.34]).

In [9, Appx. A] and [15, Def. 2.4], a duality pair $(\mathcal{L}, \mathcal{A})$ is called symmetric if $(\mathcal{A}, \mathcal{L})$ is also a duality pair (in the sense of Remark 2.4). A complete duality pair is a symmetric duality pair $(\mathcal{L}, \mathcal{A})$ such that $(\mathcal{L}, \mathcal{A})$ is a perfect duality pair.

For the rest of this section, we shall see how duality pairs induce new duality pairs via (co)resolution dimensions. Suppose we are given $\mathcal{L} \subseteq \text{Mod}(R)$ and $\mathcal{A} \subseteq \text{Mod}(R^e)$. We can show the following result.

**Proposition 2.5.** Let $\mathcal{L} \subseteq \text{Mod}(R)$ and $\mathcal{A} \subseteq \text{Mod}(R^e)$ be classes of modules such that $\text{resdim}_\mathcal{L}(-) : \text{Mod}(R) \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ and $\text{coresdim}_\mathcal{A}(-) : \text{Mod}(R^e) \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ are stable. Suppose $(\mathcal{L}, \mathcal{A})$ is a symmetric duality pair, then the following assertions hold for every $n \in \mathbb{Z}_{\geq 0}$:

1. If $\mathcal{L}$ is a relative generator in $\text{Mod}(R)$ and $\mathcal{A}$ is a relative cogenerator in $\text{Mod}(R^e)$, then $(\mathcal{L}_n, \mathcal{A}_n)$ is also a symmetric duality pair.
2. If $(\mathcal{L}, \mathcal{A})$ is (co)product closed, then so is $(\mathcal{L}_n, \mathcal{A}_n)$.
3. If $(\mathcal{L}, \mathcal{A})$ is perfect, then so is $(\mathcal{L}_n, \mathcal{A}_n)$.

**Proof.**

1. We need to split the proof of this part as follows:
   - (i) $M \in \mathcal{L}_n$ if, and only if, $M^+ \in \mathcal{A}_n$: First, the “only if” part follows by the facts that $\mathcal{L}^\perp \subseteq \mathcal{A}$ and that $(-)^\perp$ is an exact functor. For the “if” part, suppose $M^+ \in \mathcal{A}_n$. Since $\mathcal{L}$ is a generator in $\text{Mod}(R)$, we can construct an exact sequence
     
     $$\Omega_{n-1}^\mathcal{L}(M) \to L_{n-1} \to \cdots \to L_1 \to L_0 \to M$$
     
     with $L_k \in \mathcal{L}$ for every $0 \leq k \leq n - 1$. It follows that
     
     $$M^+ \to L^+_0 \to L^+_1 \to \cdots \to L^+_n \to (\Omega_{n-1}^\mathcal{L}(M))^+$$
     
     is an exact sequence with $L^+_k \in \mathcal{A}$ for every $0 \leq k \leq n - 1$. Using also the facts that $\text{coresdim}_\mathcal{A}(-)$ is stable and that $\text{coresdim}_\mathcal{A}(M^+) \leq n$, we have that $(\Omega_{n-1}^\mathcal{L}(M))^+ \in \mathcal{A}$, and so $\Omega_{n-1}^\mathcal{L}(M) \in \mathcal{L}$. Hence, $\text{resdim}_\mathcal{L}(M) \leq n$.
   - (ii) $N \in \mathcal{A}_n$ if, and only if, $N^+ \in \mathcal{L}_n$: This is dual to (i).
   - (iii) $\mathcal{A}_n$ is closed under direct summands and finite direct sums: The latter is clear since the same property holds for $\mathcal{A}$, and every finite direct sum of exact complexes is also exact. Now suppose that we are given
Suppose \( (i) \) this part is immediate. Then, there is an exact sequence

\[
N_1 \oplus N_2 \rightarrow A^0 \rightarrow A^1 \rightarrow \cdots \rightarrow A^{n-1} \rightarrow A^n
\]

with \( A^k \in \mathcal{A} \) for every \( 0 \leq k \leq n \). We shall see that \( N_1, N_2 \in \mathcal{A}_n \) by using induction on \( n \). The case \( n = 0 \) is immediate. For \( n > 0 \), we can form short exact sequences

\[
N_1 \rightarrow A^0 \rightarrow K_1 \quad \text{and} \quad N_2 \rightarrow A^0 \rightarrow K_2.
\]

Taking their direct sum yields

\[
N_1 \oplus N_2 \rightarrow A^0 \oplus A^0 \rightarrow K_1 \oplus K_2,
\]

which is exact with \( A^0 \oplus A^0 \in \mathcal{A} \). After applying the functor \((-)^+\), we obtain the short exact sequence

\[
K_1^+ \oplus K_2^+ \rightarrow (A^0 \oplus A^0)^+ \rightarrow N_1^+ \oplus N_2^+
\]

where \((A^0 \oplus A^0)^+ \in \mathcal{L}\) and \(N_1^+ \oplus N_2^+ \in \mathcal{L}_n^\perp\). Now since \( \mathcal{L} \) is a generator in \( \text{Mod}(R) \), we can consider an exact complex

\[
\Omega_{n-2}(K_1^+ \oplus K_2^+) \rightarrow L_{n-1} \rightarrow \cdots \rightarrow L_2 \rightarrow L_1 \rightarrow K_1^+ \oplus K_2^+
\]

where \( L_k \in \mathcal{L} \) for every \( 0 \leq k \leq n-2 \). Glueing this complex and the latter short exact sequence at \( K_1^+ \oplus K_2^+ \) yields the complex

\[
\Omega_{n-2}(K_1^+ \oplus K_2^+) \rightarrow L_{n-1} \rightarrow \cdots \rightarrow L_2 \rightarrow L_1 \rightarrow (A^0 \oplus A^0)^+ \rightarrow N_1^+ \oplus N_2^+,
\]

and using the fact that \( N_1^+ \oplus N_2^+ \in \mathcal{L}_n^\perp\) and the stability of \( \text{resdim}_{\mathcal{L}}(-) \), we have that \( \Omega_{n-2}(K_1^+ \oplus K_2^+) \in \mathcal{L}\). It follows that \( K_1^+ \oplus K_2^+ \in \mathcal{L}_{n-1}^\perp\), and by part (ii) this in turn implies that \( K_1 \oplus K_2 \in \mathcal{A}_{n-1}^\perp\). By induction hypothesis, \( K_1, K_2 \in \mathcal{A}_n^\perp\), and hence the result follows by the stability of \( \text{coresdim}_{\mathcal{A}}(-) \).

(iv) \( \mathcal{L}_n^\perp\) is closed under direct summands and finite direct sums: This is dual to (iii).

(2) This part is immediate.

(3) Suppose \( (\mathcal{L}, \mathcal{A}) \) is perfect, that is, \( \mathcal{L} \) is closed under coproducts, extensions and contains \( R \) (regarded as an \( R\)-module). It is clear that \( \mathcal{L}_n^\perp\) is also closed under coproducts and contains \( R\).

It is only left to show that \( \mathcal{L}_n^\perp\) is closed under extensions. Again, let us use induction on \( n \). The case \( n = 0 \) is immediate. Suppose that \( \mathcal{L}_{n-1}^\perp\) is closed under extensions, and that we are given an exact sequence

\[
A \rightarrow B \rightarrow C
\]

with \( A, C \in \mathcal{L}_n^\perp\). Since \( (\mathcal{L}, \mathcal{L}^\perp) \) is a complete cotorsion pair, there is a short exact sequence \( K \rightarrow L \rightarrow B \) with \( L \in \mathcal{L}\) and \( K \in \mathcal{L}^\perp\). The pullback of
A \rightarrow B \leftarrow L \text{ yields the following commutative exact diagram:}

\[
\begin{array}{ccc}
A & \rightarrow & B \\
\downarrow & & \downarrow \\
Q & \rightarrow & L & \rightarrow & C \\
\downarrow pb & & \downarrow & & \downarrow \\
A & \rightarrow & B & \rightarrow & C
\end{array}
\]

Since \( C \in \mathcal{L}_n^\wedge, L \in \mathcal{L} \) and \( \text{resdim}_\mathcal{L}(-) \) is stable, we have that \( Q \in \mathcal{L}_{n-1}^\wedge \) by Proposition 2.2. Now for \( A \) consider a short exact sequence

\[
\Omega_1^\wedge(A) \rightarrow L' \rightarrow A
\]

with \( L' \in \mathcal{L} \) and \( \Omega_1^\wedge(A) \in \mathcal{L}_{n-1}^\wedge \). The pullback of \( Q \rightarrow A \leftarrow L' \) yields the following solid diagram:

\[
\begin{array}{ccc}
\Omega_1^\wedge(A) & = & \Omega_1^\wedge(A) \\
\downarrow & & \downarrow \\
K & \rightarrow & P & \rightarrow & L' \\
\downarrow & & \downarrow pb & & \downarrow \\
K & \rightarrow & Q & \rightarrow & A
\end{array}
\]

By induction hypothesis, we can note that \( P \in \mathcal{L}_{n-1}^\wedge \). On the other hand, \( \text{Ext}_R^1(L', K) = 0 \), and so \( K \) is a direct summand of \( P \). This in turn implies that \( K \in \mathcal{L}_{n-1}^\wedge \), and so \( B \in \mathcal{L}_n^\wedge \) (using the central column of the first diagram).

\[\square\]

**Corollary 2.6.** If \((\mathcal{L}, A)\) is a complete duality pair, then so is \((\mathcal{L}_n^\wedge, A_n^\wedge)\), provided that \( \text{resdim}_\mathcal{L}(-) \) and \( \text{coresdim}_A(-) \) are stable.

**Proof.** It suffices to note that \( \mathcal{L} \) is a relative generator in \( \text{Mod}(R) \) and that \( A \) is a relative cogenerator in \( \text{Mod}(R^\circ) \), since \((\mathcal{L}, A)\) is a perfect duality pair, and so \( \mathcal{P}(R) \subseteq \mathcal{L} \) and \( \mathcal{I}(R^\circ) \subseteq A \).

\[\square\]

### 3. Relative Gorenstein Flat Modules

In what follows, we shall consider classes

\[
\mathcal{L} \subseteq \text{Mod}(R) \quad \text{and} \quad A \subseteq \text{Mod}(R^\circ).
\]

The following of Gorenstein flat \( R \)-modules relative to \((\mathcal{L}, A)\) is due to Wang, Yang and Zhu [23, Def. 2.1].

**Definition 3.1.** An \( R \)-module \( M \) is **Gorenstein \((\mathcal{L}, A)\)-flat** if there exists an exact and \((A \otimes_R -)\)-acyclic complex \( L_\bullet \in \text{Ch}(\mathcal{L}) \) such that \( M \simeq Z_0(L_\bullet) \). By \((A \otimes_R -)\)-acyclic we mean that \( A \otimes_R L_\bullet \) is an exact complex of abelian groups for every \( A \in \mathcal{A} \).

The class of Gorenstein \((\mathcal{L}, A)\)-flat \( R \)-modules will be denoted by \( GF_{(\mathcal{L}, A)}(R) \).
Remark 3.2.
(1) The containment $\mathcal{L} \subseteq \mathcal{GF}(\mathcal{L}, \nu)(R)$ is clear.
(2) The Gorenstein $(\mathcal{F}(R), \mathcal{I}(R^\nu))$-flat modules are precisely the Gorenstein flat modules.
(3) If $\mathcal{F}(R) \subseteq \mathcal{L}$ and $\mathcal{A} \subseteq \mathcal{I}(R^\nu)$, then every Gorenstein flat $R$-module is Gorenstein $(\mathcal{L}, \mathcal{A})$-flat.
(4) Every cycle of an exact and $(\mathcal{A} \otimes_R -)$-acyclic complex in $\text{Ch}(\mathcal{L})$ is Gorenstein $(\mathcal{L}, \mathcal{A})$-flat.

In the main results of the present article, we shall consider certain duality pairs $(\mathcal{L}, \mathcal{A})$ with some additional conditions, and Gorenstein flat modules relative to the pair $(\mathcal{L}, \perp \mathcal{A} \cap \mathcal{A})$. So for the rest of this article, let us set the notation $\nu := \perp \mathcal{A} \cap \mathcal{A}$.

Let us show some characterizations and general properties of the class $\mathcal{GF}(\mathcal{L}, \nu)(R)$.

Lemma 3.3. If $\mathcal{L}^+ \subseteq \mathcal{A}$, then $\text{Tor}_{\geq 1}^R(\perp \mathcal{A}, \mathcal{L}) = 0$.

Proof. For every $M \in \text{Mod}(R^\nu)$ and $L \in \text{Mod}(R)$, it is well known that one a natural isomorphism
$$\text{Tor}_i^R(M, L^+) \cong \text{Ext}_R^i(M, L^+)$$
for every $i \geq 1$ (see for instance Göbel and Trlifaj’s [16, Lem. 1.2.11 (b)]). Now in the case where $M \in \perp \mathcal{A}$ and $L \in \mathcal{L}$, we have that $L^+ \in \mathcal{A}$, and so $\text{Ext}_R^i(M, L^+) = 0$. It follows that $\text{Tor}_i^R(M, L)^{++} = 0$. Since $\text{Tor}_i^R(M, L)$ is a pure subgroup of $\text{Tor}_i^R(M, L)^{++}$, we can conclude that $\text{Tor}_i^R(M, L) = 0$. \qed

The next result follows the spirit of Bennis’ [6, Lem. 2.4].

Proposition 3.4. Consider the following assertions for $M \in \text{Mod}(R)$:

(a) $M \in \mathcal{GF}(\mathcal{L}, \nu)(R)$.
(b) $M \in \nu^\perp$ and admits a $(\nu \otimes_R -)$-acyclic $\mathcal{L}$-coresolution.
(c) There is an exact sequence $M \rightarrow L \rightarrow G$ with $L \in \mathcal{L}$ and $G \in \mathcal{GF}(\mathcal{L}, \nu)(R)$.

Then, the following assertions hold:

(1) $(a) \Rightarrow (c)$.
(2) Suppose in addition that $\mathcal{L}^+ \subseteq \mathcal{A}$:
(i) $(a) \Rightarrow (b) \Leftarrow (c)$. Moreover, $(c) \Rightarrow (a)$ if in addition $\mathcal{L}$ is a relative generator in $\text{Mod}(R)$.
(ii) If $\mathcal{L}$ is a relative generator in $\text{Mod}(R)$, then $(a) \Leftarrow (b) \Rightarrow (c)$.

Proof.

(1) The implication $(a) \Rightarrow (c)$ follows directly from the definition of $\mathcal{GF}(\mathcal{L}, \nu)(R)$.

(2) (i) $(a) \Rightarrow (b)$: We only show that $M \in \nu^\perp$, as the rest of the assertion is part of the definition of Gorenstein $(\mathcal{L}, \nu)$-flat $R$-modules. Since $M \in \mathcal{GF}(\mathcal{L}, \nu)(R)$, we can consider an exact and $(\nu \otimes_R -)$-acyclic exact sequence $\Omega_1^\nu(M) \rightarrow L_0 \rightarrow M$. Then for every $A \in \nu$, we have the homology exact sequence
$$\text{Tor}_1^R(A, L_0) \rightarrow \text{Tor}_1^R(A, M) \rightarrow A \otimes_R \Omega_1^\nu(M) \rightarrow A \otimes_R L_0,$$

Note that if $\mathcal{P}(R) \subseteq \mathcal{L}$, then $\mathcal{L}$ is trivially a relative generator in $\text{Mod}(R)$. Moreover, the implications $(a) \Leftarrow (b) \Rightarrow (c)$ are also valid in the case where $\mathcal{P}(R) \subseteq \mathcal{L}$, and without assuming $\mathcal{L}^+ \subseteq \mathcal{A}$. \footnote{Note that if $\mathcal{P}(R) \subseteq \mathcal{L}$, then $\mathcal{L}$ is trivially a relative generator in $\text{Mod}(R)$. Moreover, the the implications $(a) \Leftarrow (b) \Rightarrow (c)$ are also valid in the case where $\mathcal{P}(R) \subseteq \mathcal{L}$, and without assuming $\mathcal{L}^+ \subseteq \mathcal{A}.$}
we have \( A \in \mathcal{F}(\mathcal{L}, \nu)(R) \). For \( A \in \nu \) consider the homology exact sequence
\[
\text{Tor}_i^R(A, G) \to \text{Tor}_i^R(A, M) \to \text{Tor}_i^R(A, L).
\]
From the implication (a) \( \Rightarrow \) (b) we have that \( \text{Tor}_i^R(A, G) = 0 \). Also, by Lemma 3.3 we have \( \text{Tor}_i^R(A, L) = 0 \). It follows that the sequence \( M \to L \to G \) is \((\nu \otimes_R -)\)-acyclic and that \( M \in \nu^T \). Now consider for \( G \) a \((\nu \otimes_R -)\)-acyclic \( \mathcal{L} \)-coresolution \( G \to L^0 \to L^1 \to \cdots \). Splicing at \( G \) this complex and the previous short exact sequence yields a \((\nu \otimes_R -)\)-acyclic \( \mathcal{L} \)-coresolution \( M \to L \to L^0 \to L^1 \to \cdots \).

- (c) \( \Rightarrow \) (a): Follows applying part of the arguments appearing in (2)-(ii) below.

(2)-(ii)

- (b) \( \Rightarrow \) (a): On the one hand, we have from the hypothesis a \((\nu \otimes_R -)\)-acyclic \( \mathcal{L} \)-coresolution \( M \to L^0 \to L^1 \to \cdots \). On the other hand, there exists an exact sequence \( K \to L_0 \to M \) with \( L_0 \in \mathcal{L} \). Applying \( A \otimes_R - \) with \( A \in \nu \) to this sequence, we have the homology exact sequence
\[
\text{Tor}_{i+1}^R(A, M) \to \text{Tor}_i^R(A, K) \to \text{Tor}_i^R(A, L_0),
\]
where \( \text{Tor}_{i+1}^R(A, L_0) = 0 \) by Lemma 3.3, and \( \text{Tor}_i^R(A, M) = 0 \) from the assumption. Then, \( \text{Tor}_{i+1}^R(A, K) = 0 \). Furthermore, the exact sequence \( K \to L_0 \to M \) is also \((\nu \otimes_R -)\)-acyclic since \( M \in \nu^T \). Hence, glueing at \( M \) the previous sequence and the \( \mathcal{L} \)-coresolution of \( M \) yields a \((\nu \otimes_R -)\)-acyclic \( \mathcal{L} \)-coresolution \( K \to L_0 \to L^0 \to L^1 \to \cdots \) with \( K \in \nu^T \).

We can repeat this procedure for \( K \) in order to obtain a \((\nu \otimes_R -)\)-acyclic \( \mathcal{L} \)-resolution of \( M \). Therefore, \( M \in \mathcal{F}(\mathcal{L}, \nu)(R) \).

- (b) \( \Rightarrow \) (c): Follows by (b) \( \Rightarrow \) (a) \( \Rightarrow \) (c).

\[\Box\]

In order to have interesting homological and homotopical properties from relative Gorenstein flat modules, one needs \( \mathcal{F}(\mathcal{L}, \nu) \) to be closed under extensions. We are not aware if this closure property holds in general. This problem has been

\[
\begin{align*}
\mathcal{L} & \to \text{Mod}(R) + \mathcal{L}^+ \subseteq A \\
\mathcal{L} & \to \text{Mod}(R) + \mathcal{L}^+ \subseteq A \\
\mathcal{L} & \to \text{Mod}(R)
\end{align*}
\]

**Figure 1.** Implications in Proposition 3.4.
tackled previously. For example, in [12, Thm. 2.14] Estrada, Iacob and the second author show that Gorenstein \((\mathcal{F}(R), B)\)-flat modules are closed under extensions, provided that \(B\) is closed under products and contains an elementary cogenerator of its definable closure. Another approach is the one studied by Wang, Yang and Zhu in [23, Coroll. 2.19], where they show that if \((\mathcal{L}, A)\) is a complete duality pair with \(\mathcal{L}\) closed under epikernels and \(\text{Tor}_{\geq 1}^R(A, \mathcal{L}) = 0\), then \(\mathcal{G}_\mathcal{F}(\mathcal{L}, A)\) is closed under extensions.

We realize that the Tor-orthogonality relation \(\text{Tor}_{\geq 1}^R(A, \mathcal{L}) = 0\) is very restrictive, as it seems that duality pairs satisfying it are scarce in the literature. Of course this is trivial for the classical duality pair \((\mathcal{F}(R), \mathcal{I}(R^\infty))\), but in general duality pairs obtained from flat and injective modules relative to modules of finite type, for instance, are not Tor-orthogonal. The following example shows this in a precise way.

**Example 3.5.** Let \(\mathcal{FP}_\infty(R^o)\) denote the class of \(R^o\)-modules of type \(\mathcal{FP}_\infty\), defined as those \(M \in \text{Mod}(R^o)\) for which there is a finitely generated projective resolution. The classes of level \(R\)-modules and absolutely clean \(R^o\)-modules are defined as the orthogonal complements

\[
\mathcal{LV}(R) := (\mathcal{FP}_\infty(R^o))^{T_1} \quad \text{and} \quad \mathcal{AC}(R^o) = (\mathcal{FP}_\infty(R^o))^{\perp_1},
\]

respectively. By [9, Thms. 2.12 & 2.14], it is known that \((\mathcal{L}(R), \mathcal{AC}(R^o))\) is a complete duality pair. Moreover, by [10, Coroll. 4.2] this duality pair is bicomplete (in the sense of Definition 3.7 below). However, it is not true in general that \(\text{Tor}_{\geq 1}^R(\mathcal{AC}(R^o), \mathcal{L}(R)) = 0\).

Consider for instance the commutative quotient ring \(R = \mathbb{k}[x_1, x_2, \ldots]/(x_i^2)\), where \(i \geq 1\) and \(\mathbb{k}\) is a field. This is an example of a (non coherent) \(2\)-coherent ring, where \(\mathcal{FP}_\infty(R^o)\) coincides with the class of finitely generated projective \(R^o\)-modules. This is turn implies that \(\text{Mod}(R) = \mathcal{LV}(R) = \mathcal{AC}(R^o)\) (see [9, Prop. 2.5] and [10, Ex. 1.4] for details). One can see that there are modules \(M\) and \(N\) over this ring such that \(\text{Tor}^R_{\geq 1}(M, N) \neq 0\). Indeed, suppose the converse, that is, \(\text{Tor}^R_{\geq 1}(\text{Mod}(R), N) = 0\) for every \(N \in \text{Mod}(R)\). Then, every \(R\)-module is flat, and so \(\mathcal{LV}(R) \subseteq \mathcal{F}(R)\). This in turn implies that \(R\) is a coherent ring by [9, Coroll. 2.11], getting thus a contradiction.

**Example 3.6.** Some particular interesting families of Gorenstein \((\mathcal{L}, A)\)-flat \(R\)-modules are obtained without requiring that \((\mathcal{L}, A)\) is a complete duality pair. This is the case of:

- Gorenstein flat \(R\)-modules, which are relative to \((\mathcal{F}(R), \mathcal{I}(R^o))\);
- Šaroch and Šťovíček’s [21, §4]: projectively coresolved Gorenstein flat \(R\)-modules, which are relative to \((\mathcal{P}(R), \mathcal{I}(R^o))\); and
- [12, Defs. 2.1 & 2.6]: Gorenstein \(B\)-flat \(R\)-modules and projectively coresolved Gorenstein \(B\)-flat \(R\)-modules, which are relative to \((\mathcal{F}(R), B)\) and \((\mathcal{P}(R), B)\), respectively, with \(B\) a class of \(R^o\)-modules closed under products and containing an elementary cogenerator of its definable closure.

For all of the previous choices of \((\mathcal{L}, A)\), the class \(\mathcal{G}_\mathcal{F}(\mathcal{L}, A)\) is closed under extensions and \(\text{Tor}_{\geq 1}^R(A, \mathcal{L}) = 0\). If the latter holds, we shall say that the pair \((\mathcal{L}, A)\) is Tor-orthogonal.

In view of Example 3.5, Lemma 3.3 and Proposition 3.4, it is also worth studying Gorenstein flat \(R\)-modules relative to \((\mathcal{L}, \nu)\). Indeed, one of the main results proved below is that \(\mathcal{G}_\mathcal{F}(\mathcal{L}, \nu)\) is closed under extensions whenever \((\mathcal{L}, A)\) is a complete duality pair satisfying certain approximation properties, but dropping
the Tor-orthogonality condition $\text{Tor}^R_{\leq 1}(\mathcal{A}, \mathcal{L}) = 0$. This particular family of duality pairs is specified below.

**Definition 3.7.** We shall say that $(\mathcal{L}, \mathcal{A})$ is a **bicomplete duality pair** if the following conditions are satisfied:

1. $(\mathcal{L}, \mathcal{A})$ is a complete duality pair.
2. $(\perp \mathcal{A}, \mathcal{A})$ is a hereditary complete cotorsion pair in $\text{Mod}(R^o)$.
3. $\mathcal{L}$ is closed under products.

The closure under extensions of $GF(\mathcal{L}, \nu)(R)$ will be possible thanks to a Pontryagin duality relation between $GF(\mathcal{L}, \nu)(R)$ and $GI(\nu, \mathcal{A})(R^o)$. We shall need the following result.

**Proposition 3.8.** Let $(\mathcal{X}, \mathcal{Y})$ be a GI-admissible pair in $\text{Mod}(R^o)$. If we are given a short exact sequence $A \twoheadrightarrow B \rightarrow C$ with $A \in (\mathcal{X} \cap \mathcal{Y})^{\perp 1}$ and $B, C \in GI(\mathcal{X}, \mathcal{Y})(R^o)$, then $A \in GI(\mathcal{X}, \mathcal{Y})(R^o)$.

**Proof.** Since $(\mathcal{X}, \mathcal{Y})$ is a GI-admissible pair, we have by the dual of [5, Coroll. 3.25] that $\mathcal{X} \cap \mathcal{Y}$ is a relative generator in $GI(\mathcal{X}, \mathcal{Y})(R^o)$. So we can consider a short exact sequence $C' \rightarrow W \rightarrow C$ with $W \in \mathcal{X} \cap \mathcal{Y}$ and $C' \in GI(\mathcal{X}, \mathcal{Y})(R^o)$. On the other hand, the pullback of $B \rightarrow C \twoheadrightarrow W$ yields the following commutative exact diagram:

```
C' -----> C'  
|         |         |
A ------> N ----> W  
|         |         |
|         |         |  pb |
A ------> B ----> C
```

By the dual of [5, Coroll. 3.3] we know that $GI(\mathcal{X}, \mathcal{Y})(R^o)$ is closed by extensions and direct summands, and thus we have $N \in GI(\mathcal{X}, \mathcal{Y})(R^o)$. We also know that $A \in (\mathcal{X} \cap \mathcal{Y})^{\perp 1}$, which in turn implies that $A$ is a direct summand of $N \in GI(\mathcal{X}, \mathcal{Y})(R^o)$, and hence $A \in GI(\mathcal{X}, \mathcal{Y})(R^o)$.

**Theorem 3.9.** Consider the following assertions for $M \in \text{Mod}(R)$ and a pair $(\mathcal{L}, \mathcal{A})$ such that $\mathcal{L}^+ \subseteq \mathcal{A}$:

1. $M \in GF(\mathcal{L}, \nu)(R)$.
2. $M^{+} \in GI(\nu, \mathcal{A})(R^o)$.
3. $M \in \nu^\perp$ and admits a $\text{Hom}_R(-, \mathcal{L})$-acyclic $\mathcal{L}$-coresolution.

The following statements hold true:

1. (a) $\Rightarrow$ (b).
2. Suppose in addition that $A^{+} \subseteq \mathcal{L}$:
   - (i) If $\mathcal{L}$ is a relative generator in $\text{Mod}(R)$, then (c) $\Rightarrow$ (a).
   - (ii) If $\mathcal{L}$ is preenveloping and $(\nu, \mathcal{A})$ is a GI-admissible pair, then (b) $\Rightarrow$ (c).

---

3 Note that our definition includes this additional condition compared to the namesake concept proposed by Wang and Di in [22, Def. 3.2].

4 Although the result is stated for $R^o$-modules, the proof presented here carries over to abelian categories.

5 This implication also holds if we assume $\mathcal{P}(R) \subseteq \mathcal{L}$ instead.
In particular, if \((\mathcal{L}, \mathcal{A})\) is a bicomplete duality pair, then (a), (b) and (c) are equivalent.

\[\text{\texttt{L} is preenveloping + (\nu, \mathcal{A}) is \GI\text{-admissible}}\]

**Figure 2.** At each implication, the containment \(\mathcal{L}^+ \subseteq \mathcal{A}\) is assumed.

**Proof.**

1. It is straightforward from the assumption and the fact that \((-)^+\) is an exact functor and the natural isomorphism \((A \otimes R_L)_+^+ \cong \Hom_{R^e}(A, L^+_+)\).

2. (i) Let \(M \in \text{Mod}(R)\) satisfying condition (c). Thus, there is a \(\Hom_R(-, \mathcal{L})\)-acyclic \(\mathcal{L}\)-coresolution \(\eta: M \to L^0 \to L^1 \to \cdots\). Now for any \(A \in \nu\) we have that \((A \otimes R \eta)^+ \cong \Hom_R(\eta, A^+)\) is exact since \(A^+ \subseteq \mathcal{L}\), and so \(A \otimes R \eta\) is exact (being a pure subcomplex of the exact complex \((A \otimes R \eta)^{++}\)). In other words, we have that \(M \in \nu^\perp\) admits a \((\nu \otimes R -)\)-acyclic \(\mathcal{L}\)-coresolution. The result then follows by Proposition 3.8 (2)-(ii).

(ii) Suppose \(M^+ \in \GI_{(\nu, \mathcal{A})}(R^e)\). Firstly, since \(\GI_{(\nu, \mathcal{A})}(R^e) \subseteq \nu^\perp\), we have that \(\Tor^R_i(\nu, M)^+ \cong \Ext^i_{R^e}(\nu, M^+) = 0\) for every \(i \geq 1\), and so \(M \in \nu^\perp\). On the other hand, there exists a short exact sequence \(N \to A \to M^+\) with \(A \in \mathcal{A}\), and \(N \in \GI_{(\nu, \mathcal{A})}(R^e)\), which yields a short exact sequence \(M^{++} \to A^+ \to N^+\) with \(A^+ \in \mathcal{L}\). Now since \(\mathcal{L}\) is preenveloping, there is an \(\mathcal{L}\)-preenvelope \(\phi^1: M \to L^0\), and so there is an arrow \(L^0 \to A^+\) making the following triangle commute:

\[
\begin{array}{ccc}
M & \longrightarrow & M^{++} \\
\phi^1 & \downarrow & \downarrow \\
L^0 & \rightarrow & A^+
\end{array}
\]

It follows that \(\phi^1\) is monic, and thus we have obtained the short exact and \(\Hom_R(-, \mathcal{L})\)-acyclic sequence \(M \to L^0 \to C_1\) with \(L^0 \in \mathcal{L}\). Then, we have the exact sequence of the form \(C^+_1 \to (L^0)^+ \to M^+\) where \((L^0)^+ \in \mathcal{A}\) and \(M^+ \in \GI_{(\nu, \mathcal{A})}(R^e)\). We shall prove that \(\Ext^R_1(\nu, C^+_1) = 0\) in order to apply Proposition 3.8. So let us take \(A' \in \nu\) and consider the following commutative diagram with exact rows:

\[
\begin{array}{ccc}
\Hom_{R^e}(A', (L^0)^+) & \longrightarrow & \Hom_{R^e}(A', M^+) \\
\downarrow & & \downarrow \\
\Hom_{R^e}(L^0, (A')^+) & \longrightarrow & \Hom_{R^e}(M, (A')^+)
\end{array}
\]

The arrow \(\Hom_R(\phi^1, A^+): \Hom_R(L^0, (A')^+) \to \Hom_R(M, (A')^+)\) is epic since \((A')^+ \in \mathcal{L}\), which in turn implies that \(\Hom_{R^e}(A', (L^0)^+) \to \)
Hom\textsubscript{R}(A', M^+) is epic. On the other hand, \((L^0)^+ \in \mathcal{A}\) together with \(A' \in \nu\) implies that Ext\textsubscript{R}(A', (L^0)^+) = 0, and hence Ext\textsubscript{R}(A', C_1^+) = 0.

It then follows by Proposition 3.8 that \(C_1^+ \in \mathcal{G}\mathcal{I}(\nu, \mathcal{A})(R^o)\). By repeating this process we can construct a Hom\textsubscript{R}(\_, L)-acyclic L-coresolution of \(M\).

For the last assertion, it suffices to note that if \((\perp \mathcal{A}, \mathcal{A})\) is a hereditary complete cotorsion pair, then \((\nu, \mathcal{A})\) is a GI-admissible pair. □

One important fact about the previous theorem is that it implies that Gorenstein flat \(R\)-modules relative to a bicomplete duality pair \((L, \mathcal{A})\) are closed under extensions. Before showing this, let us prove a relation between this closure property and a characterization of the \(R\)-modules in \(\mathcal{G}\mathcal{F}(\mathcal{L}, \nu)(R)\) which is dual to Proposition 3.8. The following result is a generalization of [6, Thm. 2.3]. For pedagogical reasons, we provide a proof using arguments similar to those appearing in [6], in order to understand the minimal assumptions required for the relative case.

**Proposition 3.10.** Consider the following assertions:

(a) Given a short exact sequence \(G_1 \to G_0 \to M\) with \(G_0, G_1 \in \mathcal{G}\mathcal{F}(\mathcal{L}, \nu)(R)\) and \(M \in \nu^\perp\), then \(M \in \mathcal{G}\mathcal{F}(\mathcal{L}, \nu)(R)\).

(b) \(\mathcal{G}\mathcal{F}(\mathcal{L}, \nu)(R)\) is closed under extensions.

(c) \(\mathcal{G}\mathcal{F}(\mathcal{L}, \nu)(R)\) is preresolving.

If \(\mathcal{L}\) is a relative generator in \(\text{Mod}(R)^{\perp}\), then the following hold:

1. If \(\mathcal{L}\) is closed under extensions with \(\mathcal{L}^+ \subseteq \mathcal{A}\), then (a) \(\Rightarrow\) (b) \(\Rightarrow\) (c).
2. If \((\mathcal{L}, \mathcal{A})\) is a duality pair such that \(\mathcal{A}\) is closed under extensions and \(\nu\) is a relative generator in \(\mathcal{A}\), then (c) \(\Rightarrow\) (a).

In particular, if \((\mathcal{L}, \mathcal{A})\) is a bicomplete duality pair, then (a), (b) and (c) are equivalent.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{diagram.png}
\caption{At each implication, it is assumed that \(\mathcal{L}\) is a relative generator in \(\text{Mod}(R)\).}
\end{figure}

**Proof.**

1. (a) \(\Rightarrow\) (b): Consider a short exact sequence \(M_1 \to M_2 \to M_3\) with \(M_1, M_3 \in \mathcal{G}\mathcal{F}(\mathcal{L}, \nu)(R)\). We show that \(M_2 \in \mathcal{G}\mathcal{F}(\mathcal{L}, \nu)(R)\) from (a). By Proposition 3.4 (2)-(i), we have that \(M_1, M_3 \in \nu^\perp\), and so \(M_2 \in \nu^\perp\) since \(\nu^\perp\) is closed under extensions. On the other hand, there is an exact sequence \(M'_2 \to L \to M_3\) with \(L \in \mathcal{L}\) and \(M'_2 \in \mathcal{G}\mathcal{F}(\mathcal{L}, \nu)(R)\).

\[\text{Note: One can assume that } \mathcal{P}(R) \subseteq \mathcal{L} \text{ instead.}\]
Taking the pullback of $M_2 \to M_3 \leftarrow L$ yields the following commutative exact diagram:

\[
\begin{array}{c}
M'_3 \longrightarrow M'_3 \\
\downarrow \\
M_1 \longrightarrow N \longrightarrow L \\
\downarrow \text{pb} \\
M_1 \longrightarrow M_2 \longrightarrow M_3
\end{array}
\]

Now for $M_1$ there is also an exact sequence $M_1 \to L' \to M'_1$ with $L \in \mathcal{L}$ and $M'_1 \in \mathcal{G}\mathcal{F}(\mathcal{L},\nu)(R)$. Taking the pushout of $L' \leftarrow M_1 \to N$ yields the following commutative exact diagram:

\[
\begin{array}{c}
M_1 \longrightarrow N \longrightarrow L \\
\downarrow \text{po} \\
L' \longrightarrow N' \longrightarrow L \\
\downarrow \\
M'_1 \longrightarrow M'_1
\end{array}
\]

Since $\mathcal{L}$ is closed under extensions, one has that $N' \in \mathcal{L}$. Then from the central column and Proposition 3.4 (2)-(i) we have that $N \in \mathcal{G}\mathcal{F}(\mathcal{L},\nu)(R)$. Finally, from the central column of the previous pullback diagram and the assumption (a), we have that $M_2 \in \mathcal{G}\mathcal{F}(\mathcal{L},\nu)(R)$.

(b) ⇒ (c): It follows in the same way as the proof of (1) ⇒ (2) in [6, Thm. 2.3], and by using Proposition 3.4 (2).

(2) Suppose we are given a short exact sequence $G_1 \to G_0 \to M$, where $G_0, G_1 \in \mathcal{G}\mathcal{F}(\mathcal{L},\nu)(R)$ and $M \in \nu^{\top}$. First, there is a short exact sequence $G_1 \to L_1 \to G'_1$ where $L_1 \in \mathcal{L}$ and $G'_1 \in \mathcal{G}\mathcal{F}(\mathcal{L},\nu)(R)$. Taking the pushout of $L_1 \leftarrow G_1 \to G_0$ yields the following commutative exact diagram:

\[
\begin{array}{c}
G_1 \longrightarrow G_0 \longrightarrow M \\
\downarrow \text{po} \\
L_1 \longrightarrow G'_0 \longrightarrow M \\
\downarrow \\
G'_1 \longrightarrow G'_1
\end{array}
\]

Since $\mathcal{G}\mathcal{F}(\mathcal{L},\nu)(R)$ is closed under extensions by (c), we have that $G'_0 \in \mathcal{G}\mathcal{F}(\mathcal{L},\nu)(R)$. So there is a short exact sequence $G'_0 \to L_0 \to G''_0$ with $L_0 \in \mathcal{L}$ and $G''_0 \in \mathcal{G}\mathcal{F}(\mathcal{L},\nu)(R)$. Now taking the pushout of $L_0 \leftarrow G'_0 \to M$ yields
the following commutative exact diagram:

\[
\begin{array}{cccccc}
L_1 & \longrightarrow & G'_0 & \longrightarrow & M \\
\downarrow & & \downarrow & & \downarrow \text{po} \\
L_1 & \longrightarrow & L_0 & \longrightarrow & M' \\
\downarrow & & \downarrow & & \downarrow \\
G''_0 & \longrightarrow & G''_0
\end{array}
\]

From this point, the proof differs from the one appearing in [6]. Consider the induced short exact sequence of right \( R \)-modules \((M')^+ \rightarrow L_0^+ \rightarrow L_1^+\), where \( L_0^+, L_1^+ \in \mathcal{A} \). Since \( \nu \) is a relative generator in \( \mathcal{A} \), there is an exact sequence \( \mathcal{A} \rightarrow N \rightarrow L_1^+ \) with \( \mathcal{A} \in \mathcal{A} \) and \( N \in \nu \). Now take the pullback of \( L_0^+ \rightarrow L_1^+ \) to obtain the following commutative exact diagram:

\[
\begin{array}{cccccc}
A & \longrightarrow & A \\
\downarrow & & \downarrow & & \downarrow \text{pb} \\
(M')^+ & \longrightarrow & A' & \longrightarrow & N \\
\downarrow & & \downarrow & & \downarrow \\
(M')^+ & \longrightarrow & L_0^+ & \longrightarrow & L_1^-
\end{array}
\]

Since \( \mathcal{A} \) is closed under extensions, we have that \( A' \in \mathcal{A} \). Moreover, \( \text{Ext}^1_R(N,(M')^+) \cong \text{Tor}^R_1(N, M')^+ \), where \( \text{Tor}^R_1(N, M') = 0 \) since \( M \in \nu^{+1} \) from the assumption, \( G''_0 \in \nu^+ \) by Proposition 3.4 (2)-(i), and \( \nu^{+1} \) is closed under extensions. It follows that the central row in the previous diagram splits, and so \((M')^+ \in \mathcal{A} \). Hence, \( M' \in \mathcal{L} \) since \( (\mathcal{L}, \mathcal{A}) \) is a duality pair, and thus \( M \in \mathcal{G}_\mathcal{F}(\mathcal{L}, \nu)(R) \) by Proposition 3.4 (2)-(i).

\[\square\]

Let us point out some consequences of Theorem 3.9.

**Corollary 3.11.** Let \((\mathcal{L}, \mathcal{A})\) be a bicomplete duality pair. The following assertions hold:

1. \( \mathcal{G}_\mathcal{F}(\mathcal{L}, \nu)(R) \) is a resolving class closed under direct summands.
2. \( (\mathcal{G}_\mathcal{F}(\mathcal{L}, \nu)(R), \mathcal{G}_\mathcal{I}(\nu, \mathcal{A})(R^\mathcal{A})) \) is a perfect duality pair. Furthermore,
   \[ (\mathcal{G}_\mathcal{F}(\mathcal{L}, \nu)(R), (\mathcal{G}_\mathcal{F}(\mathcal{L}, \nu)(R))^\perp) \]
   is a perfect cotorsion pair in \( \text{Mod}(R) \). In particular, every \( R \)-module has a Gorenstein \((\mathcal{L}, \nu)\)-flat cover. If in addition, \( \mathcal{G}_\mathcal{F}(\mathcal{L}, \nu)(R) \) is closed under arbitrary direct products, then every \( R \)-module has a Gorenstein \((\mathcal{L}, \nu)\)-flat preenvelope.
3. If \( G_1 \rightarrow G_0 \rightarrow M \) is a short exact sequence with \( G_0, G_1 \in \mathcal{G}_\mathcal{F}(\mathcal{L}, \nu)(R) \) and \( M \in \nu^{+1} \), then \( M \in \mathcal{G}_\mathcal{F}(\mathcal{L}, \nu)(R) \).
4. \( \mathcal{G}_\mathcal{F}(\mathcal{G}_\mathcal{F}(\mathcal{L}, \nu)(R), \nu)(R) = \mathcal{G}_\mathcal{F}(\mathcal{L}, \nu)(R) \).

**Proof.**
(1) By Proposition 3.10 and the fact that \( \mathcal{P}(R) \subseteq \mathcal{L} \subseteq \mathcal{GF}_{(L, \nu)}(R) \), it suffices to show that \( \mathcal{GF}_{(L, \nu)}(R) \) is closed under direct summands and extensions. Both closure properties will follow from the duality relation between \( \mathcal{GF}_{(L, \nu)}(R) \) and \( \mathcal{GI}_{(\nu, A)}(R^n) \) proved in Theorem 3.9. Indeed, suppose that \( N \in \text{Mod}(R) \) is a direct summand of \( M \in \mathcal{GF}_{(L, \nu)}(R) \), that is, there is a split monomorphism \( N \rightarrowtail M \). Applying the (contravariant) functor \((-)^+\) yields a split epimorphism \( M^+ \twoheadrightarrow N^+ \), and so \( N^+ \) is a direct summand of \( M^+ \in \mathcal{GI}_{(\nu, A)}(R^n) \) (by Theorem 3.9). Being \((\nu, A)\) a GI-admissible pair, we know from the dual of [5, Coroll. 3.33] that \( \mathcal{GI}_{(\nu, A)}(R^n) \) is closed under direct summands, and so \( N^+ \in \mathcal{GI}_{(\nu, A)}(R^n) \). Again by Theorem 3.9, we conclude that \( N \in \mathcal{GF}_{(L, \nu)}(R) \). The remaining closure property follows similarly, using the fact that \((-)^+\) is an exact functor along with Theorem 3.9 and the dual of [5, Coroll. 3.33].

(2) First, the equivalence between \( M \in \mathcal{GF}_{(L, \nu)}(R) \) and \( M^+ \in \mathcal{GI}_{(\nu, A)}(R^n) \) follows by Theorem 3.9. On the other hand, \( \mathcal{GI}_{(\nu, A)}(R^n) \) is closed under direct summands and finite direct sums by the dual of [5, Coroll. 3.33], since \((\nu, A)\) is a GI-admissible pair. Hence, \( (\mathcal{GF}_{(L, \nu)}(R), \mathcal{GI}_{(\nu, A)}(R^n)) \) is a duality pair. Clearly, \( \mathcal{GF}_{(L, \nu)}(R) \) contains \( R \) and is closed under extensions by part (1). Finally, consider a family \( \{M_i : i \in I\} \) of \( R \)-modules in \( \mathcal{GF}_{(L, \nu)}(R) \).

For the coproduct \( \bigoplus_{i \in I} M_i \), one can easily note that \( \bigoplus_{i \in I} M_i \in \nu^+ \), since \( M_i \in \nu^+ \) by Proposition 3.4. On the other hand, each \( M_i \) admits a \((\nu \otimes_R -)\)-acyclic \( \mathcal{L} \)-coresolution, say

\[
\eta_i : M_i \rightarrowtail L^0_i \rightarrowtail L^1_i \rightarrowtail \cdots
\]

Since \( \mathcal{L} \) is closed under coproducts, and the coproduct of exact complexes of \( R \)-modules is again exact, we have that

\[
\bigoplus_{i \in I} \eta_i : \bigoplus_{i \in I} M_i \rightarrowtail \bigoplus_{i \in I} L^0_i \rightarrowtail \bigoplus_{i \in I} L^1_i \rightarrowtail \cdots
\]

is an \( \mathcal{L} \)-coresolution of \( \bigoplus_{i \in I} M_i \). Moreover, since tensor products preserve coproducts, the complexes \( A \otimes_R (\bigoplus \eta_i) \) and \( (A \otimes_R \bigoplus \eta_i) \) are isomorphic, where each complex \( A \otimes_R \eta_i \) is exact for every \( A \in \nu \). It then follows that \( A \otimes_R (\bigoplus \eta_i) \) is exact. Hence, the \( \mathcal{L} \)-coresolution \( \bigoplus \eta_i \) is \((\nu \otimes_R -)\)-acyclic. Again by Proposition 3.4, we obtain \( \bigoplus_{i \in I} M_i \in \mathcal{GF}_{(L, \nu)}(R) \).

(3) Follows by Proposition 3.10.

(4) Follows by [22, Thm. 2.7] since \( \mathcal{GF}_{(L, \nu)}(R) \) is closed by extensions.

4. Gorenstein flat dimensions relative to duality pairs

We are now interested in studying homological dimensions constructed from Gorenstein \((L, A)\)-flat modules.

**Definition 4.1.** Given \( M \in \text{Mod}(R) \), the Gorenstein \((L, A)\)-flat dimension of \( M \) is defined as

\[
\text{Gfd}_{(L, A)}(M) := \text{resdim}_{\mathcal{GF}_{(L, A)}(R)}(M).
\]

In this section, we focus on Gorenstein flat dimensions relative to \((L, \nu)\).

**Proposition 4.2.** If \( \mathcal{GF}_{(L, \nu)}(R) \) is closed under extensions, then for every \( M \in \text{Mod}(R) \) with finite Gorenstein \((L, \nu)\)-flat dimension, the following statements are true:

...
(1) There is an exact sequence

\[ \begin{array}{c}
K \rightarrow G \rightarrow M \\
\end{array} \]

with \( \text{resdim}_\mathcal{L}(K) = \text{Gfd}_\mathcal{L}(M) - 1 \) and \( G \in \mathcal{G}\mathcal{F}_{(\mathcal{L},\nu)}(R) \)

(2) Suppose in addition that \( \mathcal{L} \) is a relative generator in \( \text{Mod}(R) \) closed under extensions. If either \( \mathcal{L} \) is closed under epikernels or \( \mathcal{L}^+ \subseteq \mathcal{A} \), then there is an exact sequence

\[ \begin{array}{c}
M \rightarrow H \rightarrow G' \\
\end{array} \]

with \( \text{resdim}_\mathcal{L}(H) = n \) and \( G' \in \mathcal{G}\mathcal{F}_{(\mathcal{L},\nu)}(R) \). In particular, the result holds if \( \mathcal{L} \) is resolving.

Proof.

(1) Follows from [5, Thm. 2.8 (a)].

(2) • In the case where \( \mathcal{L} \) is a preresorlving relative generator in \( \text{Mod}(R) \), note that the proof given in [7, Lem. 2.2] can be adapted replacing \( \text{resdim}_\mathcal{L}(-) \) by the flat dimension \( \text{fd}(-) \), and using the fact that for every exact sequence \( K' \rightarrow F \rightarrow K \) with \( F \in \mathcal{L} \) and \( \text{resdim}_\mathcal{L}(K') < \infty \), one has that \( \text{resdim}_\mathcal{L}(K) = \text{resdim}_\mathcal{L}(K') + 1 \).

• Now suppose that \( \mathcal{L} \) is a relative generator in \( \text{Mod}(R) \) closed under extensions with \( \mathcal{L}^+ \subseteq \mathcal{A} \). By [5, Thm. 2.8 (a)], there is a short exact sequence \( M \rightarrow H \rightarrow G' \) with \( G' \in \mathcal{G}\mathcal{F}_{(\mathcal{L},\nu)}(R) \) and \( \text{resdim}_\mathcal{L}(H) \leq \text{Gfd}_{\mathcal{L},\nu}(M) \). So it is only left to show that \( \text{resdim}_\mathcal{L}(H) = \text{Gfd}_{\mathcal{L},\nu}(M) \).

Let us suppose the converse, that is, \( \text{resdim}_\mathcal{L}(H) < \text{Gfd}_{\mathcal{L},\nu}(M) \). Say that \( \text{Gfd}_{\mathcal{L},\nu}(M) = n \). Then, there is an exact sequence

\[ L_{n-1} \rightarrow L_{n-2} \rightarrow \cdots \rightarrow L_1 \rightarrow L_0 \rightarrow H. \]

Now taking the pullback of \( M \rightarrow H \rightarrow L_0 \) yields the following commutative exact diagram:

\[
\begin{array}{cccccccccccc}
& & & & & L_{n-1} & & & & & & L_{n-1} \\
& & & & & \downarrow & & & & & & \downarrow \\
L_{n-2} & & \cdots & & \cdots & & \cdots & & \cdots & & \cdots & \cdots & \cdots \\
& & & & & \downarrow & & & & & & \downarrow \\
& & & & & L_1 & & & & & & L_1 \\
& & & & & \downarrow & & & & & & \downarrow \\
L_0' & \rightarrow & L_0 & \rightarrow & G' & \rightarrow & G' & \rightarrow & \cdots \\
& & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
M & \rightarrow & H & \rightarrow & G' & \rightarrow & G' & \rightarrow & \cdots
\end{array}
\]

By Proposition 3.10, we have that \( \mathcal{G}\mathcal{F}_{(\mathcal{L},\nu)}(R) \) is closed under epikernels, since \( \mathcal{G}\mathcal{F}_{(\mathcal{L},\nu)}(R) \) is closed under extensions. Then \( L_0 \in \mathcal{L} \subseteq \mathcal{A} \).
The following result extends the duality between $\mathcal{GF}(\mathcal{L}, \nu)(R)$ and $\mathcal{GI}(\mathcal{L}, A)(R)$ (Theorem 3.9) to their corresponding dimensions. Recall that if $(X, Y)$ is a GI-admissible pair, the $(X, Y)$-Gorenstein injective dimension of $N \in \text{Mod}(R^n)$ is defined as
\[
\text{Gid}_{(X, Y)}(N) := \text{coresdim}_{\mathcal{GI}(X, Y)(R^n)}(N).
\]

**Proposition 4.3.** For every $M \in \text{Mod}(R)$ and every bicomplete duality pair $(\mathcal{L}, A)$, the following equality holds true:
\[
\text{Gid}_{(\nu, A)}(M^+) = \text{Gfd}_{(\mathcal{L}, \nu)}(M).
\]

**Proof.** Let us first analyze the case where $M \in \mathcal{GF}(\mathcal{L}, \nu)(R)^\wedge$, and say $\text{Gfd}_{(\mathcal{L}, \nu)}(M) = n$. Then, $M$ admits a finite Gorenstein $(\mathcal{L}, \nu)$-flat resolution of length $n$. By Theorem 3.9 and the exactness of $(-)^+$, we can note that $M^+$ admits a finite $(\nu, A)$-Gorenstein injective coreresolution of length $n$. Thus, $\text{Gid}_{(\nu, A)}(M^+) \leq n$, and we can set $\text{Gid}_{(\nu, A)}(M^+) = m$. Since $(\mathcal{GF}(\mathcal{L}, \nu)(R), (\mathcal{GF}(\mathcal{L}, \nu)(R))^\perp)$ is a perfect cotorsion pair by Corollary 3.11, we can construct a partial Gorenstein $(\mathcal{L}, \nu)$-flat resolution
\[
K \rightarrow G_{m-1} \rightarrow \cdots \rightarrow G_0 \rightarrow M
\]
with $G_i \in \mathcal{GF}(\mathcal{L}, \nu)(R)$ for every $1 \leq i \leq n-1$. Again by Theorem 3.9 and the exactness of $(-)^+$, we obtain the exact sequence
\[
M^+ \rightarrow G_0^+ \rightarrow \cdots \rightarrow G_{m-1}^+ \rightarrow K^+
\]
with $G_i^+ \in \mathcal{GI}(\nu, A)(R^n)$ for every $1 \leq i \leq m-1$. Moreover, since $\nu$ is closed under direct summands, by the dual of [5, Coroll. 4.10] we have that $K^+ \in \mathcal{GI}(\nu, A)(R^n)$. This in turn implies that $K \in \mathcal{GF}(\mathcal{L}, \nu)(R)$, and hence $\text{Gfd}_{(\mathcal{L}, \nu)}(M) \leq m$.

Finally, from the previous reasoning, it is clear that $\text{Gid}_{(\mathcal{L}, \nu)}(M) = \infty$ and $\text{Gid}_{(\nu, A)}(M^+) < \infty$ (or $\text{Gid}_{(\nu, A)}(M^+) = \infty$ and $\text{Gfd}_{(\mathcal{L}, \nu)}(M) < \infty$) combined imply a contradiction. \qed

Theorem 4.5 below gives a functorial description of the Gorenstein $(\mathcal{L}, \nu)$-flat dimension and shows its stability (recall Definition 2.1). In order to prove it, it will be useful to recall the following functorial characterization of relative Gorenstein injective functions, which follows from the dual of [5, Corolls. 4.10 & 4.11 (a)].

**Lemma 4.4.** Let $(X, Y)$ be a GI-admissible pair in $\text{Mod}(R^n)$ such that $X \cap Y$ is closed under direct summands. For every $N \in \text{Mod}(R^n)$ with $\text{Gid}_{(X, Y)}(N) < \infty$, the following statements are equivalent:

(a) $\text{Gid}_{(X, Y)}(N) \leq n$.

(b) $\text{Ext}^n_R((X \cap Y)^\vee, N) = 0$.

(c) $\text{Ext}^{n+1}_R(X \cap Y, N) = 0$.

(d) If $N \rightarrow G^0 \rightarrow G^1 \rightarrow \cdots \rightarrow G^{n-1} \rightarrow K^n$ is an exact sequence with $G^i \in \mathcal{GI}_{(X, Y)(R^n)}$ for every $1 \leq i \leq n-1$, then $K^n \in \mathcal{GI}_{(X, Y)(R^n)}$.

In particular, the $(X, Y)$-Gorenstein injective dimension is stable.

7 Although the result is stated for $R^n$-modules, the proof presented here carries over to abelian categories.
Theorem 4.5. The following assertions are equivalent for every bicomplete duality pair $(\mathcal{L}, \mathcal{A})$ and every $M \in \text{Mod}(R)$ with $\text{Gfd}_{(\mathcal{L}, \nu)}(M) < \infty$:

(a) $\text{Gfd}_{(\mathcal{L}, \nu)}(M) \leq n$.
(b) $\text{Tor}_{2n+1}(\nu^+, M) = 0$.
(c) $\text{Tor}_{2n+1}(\nu, M) = 0$.
(d) If $K_n \to G_{n-1} \to \cdots \to G_0 \to M$ is an exact sequence with $G_i \in \mathcal{GF}_{(\mathcal{L}, \nu)}(R)$ for every $1 \leq i \leq n - 1$, then $K_n \in \mathcal{GF}_{(\mathcal{L}, \nu)}(R)$.

In particular, the Gorenstein $(\mathcal{L}, \nu)$-flat dimension is stable.

Proof.

- (a) $\Rightarrow$ (b): On the one hand, assuming (a), we obtain from Proposition 4.3 that $\text{Gfd}_{(\nu, \mathcal{A})}(M^+) = \text{Gfd}_{(\mathcal{L}, \nu)}(M) \leq n$. Then by Lemma 4.4 we have that $\text{Ext}_{2n+1}^R(\nu^+, M^+) = 0$, and the natural isomorphism $\text{Ext}_R^i(-, M^+) \cong \text{Tor}_i^R(-, M)^+$ yields that $\text{Tor}_{2n+1}^R(\nu^+, M) = 0$. This in turn implies (b).
- (b) $\Rightarrow$ (c): Immediate.
- (c) $\Rightarrow$ (d): Proceeding as in (a) $\Rightarrow$ (b), we can obtain from (c) that (a) holds. The latter in turn implies (d) by Theorem 3.9, Proposition 4.3 and Lemma 4.4.
- (d) $\Rightarrow$ (a): By Corollary 3.11, we can construct an exact sequence

$$K_n \to G_{n-1} \to \cdots \to G_0 \to M$$

with $G_i \in \mathcal{GF}_{(\mathcal{L}, \nu)}(R)$ for every $1 \leq i \leq n - 1$. Since we are assuming (d), we get $K_n \in \mathcal{GF}_{(\mathcal{L}, \nu)}(R)$. Hence, by definition of $\text{Gfd}_{(\mathcal{L}, \nu)}(-)$ we have that $\text{Gfd}_{(\mathcal{L}, \nu)}(M) \leq n$.

We have here a different proof of the previous result, we state such a result in two independent parts and with different hypotheses.

Remark 4.6. Under weaker assumptions, we can still get an equivalence between conditions (a) and (d) in Theorem 4.5 (i.e., that $\text{Gfd}_{(\mathcal{L}, \nu)}(-)$ is stable), namely:

(i) $\mathcal{L}$ is a relative generator in $\text{Mod}(R)$ closed under extensions, coproducts and direct summands, which satisfies $\mathcal{L}^+ \subseteq \mathcal{A}$; and

(ii) $\mathcal{GF}_{(\mathcal{L}, \nu)}(R)$ is closed under extensions.

Indeed, by Proposition 3.10 we have that $\mathcal{GF}_{(\mathcal{L}, \nu)}(R)$ is a preresolving class. Now let $P$ be a projective $R$-module. Since $\mathcal{L}$ is a relative generator in $\text{Mod}(R)$, we have a split epimorphism $L \twoheadrightarrow P$. So $P$ is a direct summand of $L$, which in turn implies that $P \in \mathcal{L}$. Hence, $\mathcal{P}(R) \subseteq \mathcal{GF}_{(\mathcal{L}, \nu)}(R)$ and $\mathcal{GF}_{(\mathcal{L}, \nu)}(R)$ is resolving. On the other hand, the closure under coproducts of $\mathcal{L}$ implies the same property for $\mathcal{GF}_{(\mathcal{L}, \nu)}(R)$.

We have here a slight variant of the Theorem 4.5.

Theorem 4.7. Consider the statements below for $M \in \text{Mod}(R)$ with $\text{Gfd}_{(\mathcal{L}, \nu)}(M) < \infty$:
The following assertions hold true:

1. If \( A^+ \subseteq \mathcal{L}, \mathcal{L}^+ \subseteq A \), \( \mathcal{L} \) is preenveloping and \((\nu, A)\) is GI-admissible pair, then 
\((a) \Rightarrow (b) \Leftrightarrow (c)\).

2. If \((\mathcal{L}, A)\) is a duality pair such that \( \mathcal{L} \) is a relative generator in \( \text{Mod}(R) \) closed under extensions, coproducts and direct summands, and \( \mathcal{G}\mathcal{F}_{(\mathcal{L}, \nu)}(R) \) is closed under extensions, then 
\((b) \Rightarrow (a)\).

**Proof.**

1. (a) \( \Rightarrow \) (c): It will follow by induction on \( n \). The case \( n = 0 \) is a consequence of Theorem 3.9 (1) and (2)-(ii). Now suppose the implication \( (a) \Rightarrow (b) \) holds for every \( R \)-module with Gorenstein \((\mathcal{L}, \nu)\)-flat dimension at most \( n - 1 \), and let \( M \in \text{Mod}(R) \) with \( \text{Gfd}_{(\mathcal{L}, \nu)}(M) \leq n \). Then, there is a short exact sequence \( K \rightarrow G \rightarrow M \), with \( G \in \mathcal{G}\mathcal{F}_{(\mathcal{L}, \nu)}(R) \) and \( \text{Gfd}_{(\mathcal{L}, \nu)}(K) \leq n - 1 \). We know again by Theorem 3.9 that \( \text{Tor}^R_{\geq 1}(\nu, G) = 0 \) and \( \text{Tor}^R_{\geq n}(\nu, K) = 0 \). It then follows from the Tor homology sequence that \( \text{Tor}^R_{\geq n+1}(\nu, M) = 0 \).

2. (b) \( \Rightarrow \) (c) is trivial, and (b) \( \Leftrightarrow \) (c) follows by a dimension shifting argument.

We show \( (b) \Rightarrow (a) \). Suppose \( \text{Gfd}_{(\mathcal{L}, \nu)}(M) < \infty \) and \( \text{Tor}^R_{\geq n+1}(\nu^\vee, M) = 0 \). By Remark 4.6, we can construct for some \( m > n \) a partial projective resolution
\[
\Omega_{m-1}^L(M) \rightarrow P_{m-1} \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow M,
\]
that is, with \( P_k \in \mathcal{P}(R) \) for every \( 0 \leq k \leq m - 1 \), and guarantee that \( \Omega_{m-1}^L(M) \in \mathcal{G}\mathcal{F}_{(\mathcal{L}, \nu)}(R) \). Consider \( \Omega_{n-1}^L(M) \) in the previous partial projective resolution. The aim to prove that \( \Omega_{n-1}^L(M) \in \mathcal{G}\mathcal{F}_{(\mathcal{L}, \nu)}(R) \). Consider

**Figure 4.** A display of the implications studied in Remark 4.6 and Theorem 4.7 for pairs \((\mathcal{L}, A)\) with \( \mathcal{L}^+ \subseteq A \).
the short exact sequence
\[ \Omega^\mathcal{L}_{m-1}(M) \to P_{m-1} \to \Omega^\mathcal{L}_{m-2}(M). \]
By dimension shifting,
\[ \text{Tor}^R_i(A, \Omega^\mathcal{L}_{m-2}(M)) \cong \text{Tor}^R_{m-1+i}(A, M) = 0 \]
for every \( A \in \nu \) and \( i \in \mathbb{Z}_{>0} \), that is, \( \Omega^\mathcal{L}_{m-2}(M) \in \nu^\perp \). Then, since \( \Omega^\mathcal{L}_{m-1}(M), P_{m-1} \in \mathcal{G}F(\mathcal{L}, \nu)(R) \), we get from Proposition 3.10 and the hypothesis that \( \Omega^\mathcal{L}_{m-2}(M) \in \mathcal{G}F(\mathcal{L}, \nu)(R) \). Therefore, continuing this process finitely many times, we conclude that \( \Omega^\mathcal{L}_{m-1}(M) \in \mathcal{G}F(\mathcal{L}, \nu)(R) \).
\[ \square \]

We conclude this section defining and pointing out some properties of the weak Gorenstein \((\mathcal{L}, \mathcal{A})\)-flat global dimension of \( R \).

**Definition 4.8.** The left weak Gorenstein \((\mathcal{L}, \mathcal{A})\)-flat global dimension of \( R \) is defined as the value
\[ l.w\text{Ggdim}_{(\mathcal{L}, \mathcal{A})}(R) := \sup\{ \text{Gfd}_{(\mathcal{L}, \mathcal{A})}(M) : M \in \text{Mod}(R) \} \].

We have the following result regarding the finiteness of \( l.w\text{Ggdim}_{(\mathcal{L}, \nu)}(R) \), as a consequence of Theorem 4.5.

**Proposition 4.9.** Let \((\mathcal{L}, \mathcal{A})\) be a bicomplete duality pair in \( \text{Mod}(R) \) and suppose that \( l.w\text{Ggdim}_{(\mathcal{L}, \nu)}(R) < \infty \). Then, the following statements are equivalent:
\begin{enumerate}[(a)]
  \item \( l.w\text{Ggdim}_{(\mathcal{L}, \nu)}(R) \leq n < \infty \).
  \item \( \text{fd}(A) \leq n \) for every \( A \in \nu \).
  \item \( \text{fd}(N) \leq n \) for every \( N \in \nu^\perp \).
\end{enumerate}

We also compare the following two finitistic dimensions.

**Definition 4.10.** Given a class \( \mathcal{X} \subseteq \text{Mod}(R) \), the \( \mathcal{X} \)-finitistic dimension of \( R \) is defined as the value
\[ l.\text{\mathcal{X}-findim}(R) := \sup\{ \text{resdim}_\mathcal{X}(M) : M \in \mathcal{X}^\perp \} \].
In particular, the left Gorenstein \((\mathcal{L}, \mathcal{A})\)-flat finitistic dimension of \( R \) is defined as
\[ l.\text{GF}_{(\mathcal{L}, \mathcal{A})}\text{-findim}(R) := \sup\{ \text{Gfd}_{(\mathcal{L}, \mathcal{A})}(M) : M \in \mathcal{G}F(\mathcal{L}, \mathcal{A})(R)^\perp \} \].

**Lemma 4.11.** If \((\nu, \mathcal{A})\) is a GI-admissible pair with \( \mathcal{A} \) and \( \nu \) closed under direct summands, then the equality
\[ \mathcal{G}I_{(\nu, \mathcal{A})}(R^{\nu}) \cap \mathcal{A}^\nu = \mathcal{A} \]
holds for every \( m \in \mathbb{Z}_{>0} \).

**Proof.** We first show the case \( m = 0 \). So let \( N \) be a \((\nu, \mathcal{A})\)-Gorenstein injective \( R^{\nu} \)-module with \( \text{coresdim}_\mathcal{A}(N) < \infty \). By the dual of [5, Thm. 2.8], there exists a short exact sequence \( N \to A \to N' \) with \( A \in \mathcal{A} \) and \( N' \in \nu^\perp \). This sequence splits by Lemma 4.4, and hence \( N \) is a direct summand of \( A \). It follows that \( N \in \mathcal{A} \).

Now if \( N \in \mathcal{G}I_{(\nu, \mathcal{A})}(R^{\nu}) \cap \mathcal{A}^\nu \) with \( m \in \mathbb{Z}_{>0} \), since \((\nu, \mathcal{G}I_{(\nu, \mathcal{A})}(R^{\nu}))\) is a right Frobenius pair by the dual of [5, Thm. 2.8 & Coroll. 4.10], we can find a short...
exact sequence \( N \rightarrow G \rightarrow N' \) where \( G \in \mathcal{G}T_{(\nu, A)}(R^\nu) \) and \( N' \in \nu_{m-1}^\nu \). Now by the dual of [4, Thm. 2.1 (a)], we have that \( \mathcal{A}^\nu \) is closed under extensions, and thus \( G \in \mathcal{G}T_{(\nu, A)}(R^\nu) \cap \mathcal{A}^\nu = \mathcal{A} \) by the case \( m = 0 \). Hence, \( N \in \mathcal{A}_m^\nu \). \( \square \)

The following is the dual version of the previous lemma. It is a consequence of the lemma itself and Propositions 2.5 and 4.3.

**Lemma 4.12.** If \((\mathcal{L}, \mathcal{A})\) is a bicomplete duality pair, then

\[
\mathcal{G}F_{(\mathcal{L}, \nu)}(R)^\wedge_m \cap \mathcal{L}^\wedge = \mathcal{L}_m^\wedge
\]

for every \( m \in \mathbb{Z}_{\geq 0} \).

The following result generalizes [17, Thm. 3.24] to our setting.

**Proposition 4.13.** Suppose \( \mathcal{L} \) is a relative generator in \( \text{Mod}(R) \) closed under extensions, and that either \( \mathcal{L} \) is closed under epikernels or \( \mathcal{L}^+ \subseteq \mathcal{A} \). If \( \mathcal{G}F_{(\mathcal{L}, \nu)}(R) \) is closed under extensions, then the inequality

\[
l.\mathcal{G}F_{(\mathcal{L}, \nu)}\text{-findim}(R) \leq l.\mathcal{L}\text{-findim}(R)
\]

holds true. Furthermore, the equality holds in the case where \((\mathcal{L}, \mathcal{A})\) is a bicomplete duality pair.

**Proof.** We prove that

\[
l.\mathcal{G}F_{(\mathcal{L}, \nu)}\text{-findim}(R) \leq l.\mathcal{L}\text{-findim}(R). \tag{i}
\]

We may assume that \( n := l.\mathcal{L}\text{-findim}(R) < \infty \). For any \( M \in \mathcal{G}F_{(\mathcal{L}, \nu)}(R)^\wedge \), we know by Proposition 4.2 (2), that there is a short exact sequence \( M \rightarrow H \rightarrow G \) with \( G \in \mathcal{G}F_{(\mathcal{L}, \nu)}(R) \) and \( \text{Gfd}_{(\mathcal{L}, \nu)}(M) = \text{resdim}_\mathcal{L}(H) \leq n \). Hence, (i) holds.

In order to show the remaining inequality assuming that \((\mathcal{L}, \mathcal{A})\) is a bicomplete duality pair, it suffices to note from Lemma 4.12 that \( \text{Gfd}_{(\mathcal{L}, \nu)}(N) = \text{resdim}_\mathcal{L}(N) \) for every \( N \in \mathcal{L}^\wedge \). \( \square \)

**Appendix A. Relative Gorenstein injective and flat complexes**

We study the chain complex version of Gorenstein \((\mathcal{L}, \nu)\)-flat \( R \)-modules and dimensions. Before that, let us first recall how tensor products in \( \text{Ch}(R) \) are defined. Given two chain complexes \( X_\bullet \in \text{Ch}(R^\nu) \) and \( Y_\bullet \in \text{Ch}(R) \), the **standard tensor product** between \( X_\bullet \) and \( Y_\bullet \) is defined as the chain complex \( X_\bullet \otimes Y_\bullet \) of abelian groups given componentwise by

\[
(X_\bullet \otimes Y_\bullet)_m := \bigoplus_{k \in \mathbb{Z}} X_k \otimes_R Y_{m-k}
\]

and whose differentials \( \partial^{X_\bullet \otimes Y_\bullet}_m : (X_\bullet \otimes Y_\bullet)_m \rightarrow (X_\bullet \otimes Y_\bullet)_{m-1} \) are given at generators by

\[
X_k \otimes_R Y_{m-k} \ni x \otimes y \mapsto \partial^{X_\bullet}_k(x) \otimes y + (-1)^k x \otimes \partial^{Y_\bullet}_{m-k}(y).
\]

One can construct from \( X_\bullet \otimes Y_\bullet \) another variant of the tensor product of complexes, called the **modified tensor product**. We denote it as \( X_\bullet \overline{\otimes} Y_\bullet \) and it is defined at degree \( m \) by

\[
(X_\bullet \overline{\otimes} Y_\bullet)_m := \frac{(X_\bullet \otimes Y_\bullet)_m}{\text{Im}(\partial^{X_\bullet \otimes Y_\bullet}_{m+1})}
\]
and with differentials $\partial_m^X \otimes^\mathbb{R} Y^\bullet : (X^\bullet \otimes^\mathbb{R} Y^\bullet)_m \to (X^\bullet \otimes^\mathbb{R} Y^\bullet)_{m-1}$ given by

$$x \otimes y + \text{Im}(\partial^{X^\bullet \otimes^\mathbb{R} Y^\bullet}_{m+1}) \to \partial^X_k(x) \otimes y + \text{Im}(\partial^{X^\bullet \otimes^\mathbb{R} Y^\bullet}_k)$$

where $x \otimes y \in X_k \otimes_R Y_{m-k}$. The left derived functors of $(- \otimes -)$ will be denoted by $\text{Tor}^R_i(-, -)$ (see [20, § 4.4]).

Hom functors in $\text{Ch}(R)$ will be denoted by $\text{Hom}_{\text{Ch}(R)}(-, -)$, and its right derived functors by $\text{Ext}^i_{\text{Ch}(R)}(-, -)$.

In what follows, given a class $\mathcal{X} \subseteq \text{Mod}(R)$, we shall consider the following induced classes in $\text{Ch}(R)$, with the terminology and notation borrowed from [14]:

- $\mathcal{X}$-complexes: exact complexes $X^\bullet \in \text{Ch}(R)$ such that $Z_m(X^\bullet) \in \mathcal{X}$ for every $m \in \mathbb{Z}$. The class of $\mathcal{X}$-complexes is denoted by $\mathcal{X}^\bullet$.

We also need to consider the following special complexes:

- Disk complexes: Given an $R$-module $M \in \text{Mod}(R)$, the $m$-th disk centered at $M$ is defined as the complex $D^m(M)$ with $M$ at degrees $m$ and $m-1$, and zero elsewhere, such that the only nonzero differential $(D^m(M))_m \to (D^m(M))_{m-1}$ is the identity.

- Shifted complexes: Given an $R$-complex $X^\bullet \in \text{Ch}(R)$, its $n$-th suspension or shift is defined as the complex $X^\bullet[n]$ with components

$$\begin{align*}
(X^\bullet[n])_m := X_{n-m}
\end{align*}$$

for every $m \in \mathbb{Z}$, and whose differentials are given by

$$\partial^X_{m}[n] := (-1)^n \partial^X_m.$$ 

Now we define the chain complex analogs for $\mathcal{F}_\nu(R)$ and $\mathcal{I}_\nu(R^\circ)$.

**Definition A.1.** We say that an $R$-complex $X^\bullet \in \text{Ch}(R)$ is $\mathcal{F}_\nu(R)$-flat if there is an exact and $(\mathcal{F}_\nu(R^\circ)-)$acyclic complex in $\text{Ch}(\mathcal{F}_\nu(R))$. Dually, an $R^\circ$-complex $Y^\bullet \in \text{Ch}(R^\circ)$ is $\mathcal{I}_\nu(R^\circ)$-Gorenstein injective if there is an exact and $\text{Hom}_{\text{Ch}(R^\circ)}(\mathcal{I}_\nu(-)-, -)$-acyclic complex in $\text{Ch}(\mathcal{I}_\nu(R^\circ))$.

The classes of $\mathcal{F}_\nu(R)$-flat and $\mathcal{I}_\nu(R^\circ)$-Gorenstein injective $R^\circ$-complexes will be denoted by $\mathcal{F}_\nu(R)$ and $\mathcal{I}_\nu(R^\circ)$, respectively.

**Proposition A.2.** The following assertions hold true for every $R^\circ$-complex $Y^\bullet$:

1. If $Y^\bullet \in \mathcal{I}_\nu(R^\circ)$, then $Y_m \in \mathcal{I}_\nu(R^\circ)$ for every $m \in \mathbb{Z}$.
2. In addition, if $(\nu, A)$ is a GL-admissible pair with $A$ closed under direct summands and monokernels, and containing the injective $R^\circ$-modules, then $Y_m \in \mathcal{I}_\nu(R^\circ)$ for every $m \in \mathbb{Z}$ implies that $Y^\bullet \in \mathcal{I}_\nu(R^\circ)$. In other words,

$$\mathcal{I}_\nu(R^\circ) = \text{Ch}((\mathcal{I}_\nu(R^\circ)))$$

**Proof.**

1. By definition there is an exact and $\text{Hom}_{\text{Ch}(R^\circ)}(\mathcal{I}_\nu(-)-, -)$-acyclic sequence of complexes in $\mathcal{I}_\nu(R^\circ)$, say

$$A_0 \to \cdots \to A_1^1 \to A_0^1 \to A_1^0 \to A_0^0 \to \cdots$$

such that $Y^\bullet \simeq \text{Ker}(A_0^\bullet \to A_1^\bullet)$. In order to show that $Y^\bullet \in \mathcal{I}_\nu(R^\circ)$, let us note that for $A \in \nu$, one has $D^m(A) \in \mathcal{I}_\nu$. Moreover, by [14, Lem. 3.1]

Note that with this notation we mean a complex of complexes, in this case a complex whose components are $\mathcal{L}$-complexes.
(1) we have that the complexes $\text{Hom}_{\mathcal{Ch}(R^o)}(D^m(A), \mathbb{A}_m)$ and $\text{Hom}_{R^o}(A, \mathbb{A}_m)$ are naturally isomorphic, where

$$\mathbb{A}_m = \cdots \to A_m^1 \to A_m^0 \to A_m^{-1} \to A_m^{-2} \to \cdots$$

is exact in $\text{Ch}(A)$ (since $A$ is closed under extensions and for each $i \in \mathbb{Z}$ we have a short exact sequence $Z_m(A_i) \to A_m^i \to Z_m(A_{i+1})$). Moreover, the exactness of $\text{Hom}_{\text{Ch}(R^o)}(D^m(A), \mathbb{A}_m)$ implies the exactness of $\text{Hom}_{R^o}(A, \mathbb{A}_m)$. It follows that $\mathbb{A}_m$ is an exact and $\text{Hom}_{R^o}(-, -)$-acyclic complex in $\text{Ch}(A)$ such that $Z_0(\mathbb{A}_m) \cong Y_m$. In other words, $Y_m \in \mathcal{G} \mathcal{I}_{(\nu, A)}(R^o)$.

(2) First, since $\nu$ is a relative generator for $\mathcal{G} \mathcal{I}_{(\nu, A)}(R^o)$ with $\text{pd}_{\mathcal{G} \mathcal{I}_{(\nu, A)}(R^o)}(\nu) = 0$ and $A$ closed under direct summands, we have by the dual of [3, Prop. 4.3 (2)] that $\nu$ is a relative generator in $\text{Ch}(\mathcal{G} \mathcal{I}_{(\nu, A)}(R^o))$ satisfying

$$\text{pd}_{\text{Ch}(\mathcal{G} \mathcal{I}_{(\nu, A)}(R^o))}(\nu) = 0.$$

Then, for $Y_\bullet \in \text{Ch}(\mathcal{G} \mathcal{I}_{(\nu, A)}(R^o))$ there is a short exact sequence

$$K_0^\bullet \to A_0^\bullet \to Y_\bullet$$

with $A_0^\bullet \in \nu \subseteq \mathbb{A}$ and $K_0^\bullet \in \text{Ch}(\mathcal{G} \mathcal{I}_{(\nu, A)}(R^o))$. By (i), we can note that the previous sequence is $\text{Hom}_{\text{Ch}(R^o)}(\nu, -)$-acyclic. By repeating this procedure, we can obtain a $\text{Hom}_{\text{Ch}(R^o)}(\nu, -)$-acyclic $A$-resolution of $Y_\bullet$. Finally, since $\mathcal{I}(R^o)$ is precisely the class of injective $R^o$-complexes, $\mathcal{I}(R^o) \subseteq A$, we have that $A$ contains the class of injective $R^o$-complexes. Thus, we can complete the previous resolution to a $\text{Hom}_{\text{Ch}(R^o)}(\nu, -)$-acyclic complex in $\text{Ch}(A)$ whose 0-th cycle is isomorphic to $Y_\bullet$.

Theorem A.3 characterizes the Gorenstein $(\mathcal{L}, \nu)$-flat complexes and extends the duality relation proved in Theorem 3.9 to the chain complex level. Before proving this result, let us first recall how Pontryagin duality is defined for complexes, for which the notion of internal $\text{Hom}$ is needed.

Given two $R$-complexes $X_\bullet, Y_\bullet \in \text{Ch}(R)$, let $\text{Hom}(X_\bullet, Y_\bullet)$ denote the chain complex of abelian groups defined componentwise by

$$\text{Hom}(X_\bullet, Y_\bullet)_m := \prod_{k \in \mathbb{Z}} \text{Hom}_R(X_k, Y_{m+k}),$$

with differentials $\partial^\text{Hom}(X_\bullet, Y_\bullet)_m : \text{Hom}(X_\bullet, Y_\bullet)_m \to \text{Hom}(X_\bullet, Y_\bullet)_{m-1}$ given by

$$\text{Hom}(X_\bullet, Y_\bullet)_m \ni (f_k : X_k \to Y_{m+k})_{k \in \mathbb{K}} \mapsto (\partial^Y_{m+k} \circ f_k - (-1)^m f_k - (-1)^{m+k} \circ \partial^X_k)f_k_{k \in \mathbb{Z}}.$$

The modified internal $\text{Hom}$ is defined as the complex $\overline{\text{Hom}}(X_\bullet, Y_\bullet)$ with $\overline{\text{Hom}}(X_\bullet, Y_\bullet)_m := Z_m(\text{Hom}(X_\bullet, Y_\bullet))$ and differentials $\partial^\overline{\text{Hom}}(X_\bullet, Y_\bullet)_m : \overline{\text{Hom}}(X_\bullet, Y_\bullet)_m \to \overline{\text{Hom}}(X_\bullet, Y_\bullet)_{m-1}$ given by

$$\overline{\text{Hom}}(X_\bullet, Y_\bullet)_m \ni (f_k : X_k \to Y_{m+k})_{k \in \mathbb{K}} \mapsto (\partial^Y_{m+k} \circ f_k - (-1)^{m+k} \circ \partial^X_k)f_k_{k \in \mathbb{Z}}.$$

The right derived functors of $\overline{\text{Hom}}(-, -)$ will be denoted by $\overline{\mathcal{E}xt}(-, -)$ (see [20, § 4.4]).
Recall from [20, Def. 4.4.9 & Prop. 4.4.10] that the Pontryagin dual of $X_\bullet$ is the $R^\circ$-complex defined by

$$X^+ := \text{Hom}(X_\bullet, D^0(Q/Z)).$$

It can also be defined as the complex

$$\cdots \to X^+_{-m-1} \to \cdots,$$

**Theorem A.3.** Let $(\mathcal{L}, \mathcal{A})$ be a bicomplete duality pair in $\text{Mod}(R)$. The following statements are equivalent:

(a) $X_\bullet \in \mathcal{GF}^{\text{Ch}}_{(\mathcal{L}, \nu)}(R)$.

(b) $X_m \in \mathcal{GF}_{(\mathcal{L}, \nu)}(R)$ for every $m \in \mathbb{Z}$.

(c) $X^+_m \in \mathcal{GF}^{\text{Ch}}_{(\nu, \mathcal{A})}(R^\circ)$.

(d) $X_\bullet$ admits a $\text{Hom}_{\text{Ch}}(\mathcal{L}, \nu)$-acyclic $\mathcal{L}$-coresolution.

In particular,

$$\mathcal{GF}^{\text{Ch}}_{(\mathcal{L}, \nu)}(R) = \text{Ch}(\mathcal{GF}_{(\mathcal{L}, \nu)}(R)).$$

**Proof.**

- (a) $\Rightarrow$ (b): Let $X_\bullet \in \mathcal{GF}^{\text{Ch}}_{(\mathcal{L}, \nu)}(R)$, that is, $X_\bullet$ is the 0-th cycle of $(\nu \otimes -)$-acyclic and exact complex in $\text{Ch}(\mathcal{L})$, say

$$L_{\square} = \cdots \to L^1_{\bullet} \to L^0_{\bullet} \to L^{-1}_{\bullet} \to L^{-2}_{\bullet} \to \cdots.$$

Then for each $m \in \mathbb{Z}$ we have the exact complex

$$L_m : \cdots L^1_m \to L^0_m \to L^{-1}_m \to L^{-2}_m \to \cdots$$

in $\text{Ch}(\mathcal{L})$ with $X_m \cong Z_n(L_m)$. By [13, Prop. 4.2.1 (4)], for each $A \in \nu$ and $L^i_{\bullet}$ there is a natural isomorphism

$$A \otimes_R L^i_{\bullet} \cong D^1(A) \otimes L^i_{\bullet},$$

which yields

$$A \otimes_R L_{\square} \cong D^1(A) \otimes L_{\square},$$

where $D^1(A) \otimes L_{\square}$ is exact since $D^1(A) \in \bar{\nu}$. Thus $A \otimes_R L_{\square}$ is exact, which implies that $A \otimes_R L_m$ is exact. Therefore, $X_m \in \mathcal{GF}_{(\mathcal{L}, \nu)}(R)$ for every $m \in \mathbb{Z}$.

- (b) $\Rightarrow$ (c): We know that $X_m \in \mathcal{GF}_{(\mathcal{L}, \nu)}(R)$ for every $m \in \mathbb{Z}$, and then by Theorem 3.9 we get $X^+_m \in \mathcal{GF}_{(\nu, \mathcal{A})}(R^\circ)$. The result then follows by Proposition A.2.

- (c) $\Rightarrow$ (d): The idea is to replicate the arguments from (b) $\Rightarrow$ (c) in Theorem 3.9 within the ambient of chain complexes. First of all, note that it is clear that $X^+_\bullet[n] \in \mathcal{GF}^{\text{Ch}}_{(\nu, \mathcal{A})}(R^\circ)$ for every $n \in \mathbb{Z}$. Now for every $A_\bullet \in \bar{\nu}$, we have that $\text{Ext}_{\mathcal{Ch}(R^\circ)}^i(A_\bullet, X^+_\bullet[n]) = 0$, and thus by [20, Props. 4.4.7 & 4.4.13 (2)] this yields

$$\text{Tor}_{\mathcal{L}}^i(A_\bullet, X_\bullet)^+ \cong \text{Ext}_{\mathcal{Ch}(R^\circ)}^i(A_\bullet, X^+_\bullet) = 0$$

for every $i \in \mathbb{Z}_{>0}$. In other words, $X_\bullet \in \bar{\nu}^T$.

Another consequence of having $X^+_\bullet \in \mathcal{GF}^{\text{Ch}}_{(\nu, \mathcal{A})}(R^\circ)$ is the existence of an epimorphism $A^0_\bullet \to X^+_\bullet$, with $A^0_\bullet \in \mathcal{A}$. This induces a monomorphism $X^\bullet \to (A^0_\bullet)^+$, where $(A^0_\bullet)^+ \in \bar{\mathcal{L}}$ since $\mathcal{A}^+ \subseteq \mathcal{L}$ and the functor $(-)^+$ is
exact. On the other hand, since \((\mathcal{L}, \mathcal{A})\) is a product closed duality pair, we have by [25, § 4.2 & Thm. 4.2.1] that \((\mathcal{L}, \mathcal{A})\) is a product closed duality pair of complexes. By [24, Thm. 3.2], the class \(\mathcal{L}\) is preenveloping. We can then consider the following diagram

\[
\begin{array}{ccc}
X_\bullet & \xrightarrow{\phi^0} & X_\bullet^+ \xrightarrow{\mathrm{Ext}^1} (A_\bullet^0)^+ \\
& & \downarrow \scriptstyle \mathrm{Tor}_1 \\
& & L_\bullet^0
\end{array}
\]

where \(\phi^0\) is a \(\mathcal{L}\)-preenvelope of \(X_\bullet\). The dotted arrow exists since \((A_\bullet^0)^+ \in \mathcal{L}\). It follows that \(\phi^0\) is a monomorphism, and so we have an exact sequence

\[
\eta_\bullet : X_\bullet \rightarrow L_\bullet^0 \rightarrow X_1^\bullet
\]

with \(X_1^\bullet := \mathrm{CoKer}(\phi^0)\), is \(\mathrm{Hom}_{\mathrm{Ch}(R)}(-, \mathcal{L})\)-acyclic.

Now we show that \((X_1^\bullet)^+ \in \mathcal{G}\mathcal{T}_{(\nu, \mathcal{A})}^\mathcal{L}(R^\omega)\). Let \(A_\bullet \in \mathcal{P}\). Again by [25, Thm. 4.2.1], we have that \(A_\bullet^+ \in \mathcal{L}\), and so \(A_\bullet^+[n] \in \mathcal{L}\) for every \(n \in \mathbb{Z}\). Now applying the functor \(\mathrm{Hom}_{\mathrm{Ch}(R)}(-, A_\bullet^+[n])\) to \(\eta_\bullet\) yields the exact sequence \(\mathrm{Hom}_{\mathrm{Ch}(R)}(\eta_\bullet, A_\bullet^+[n])\). By [20, Props. 4.4.7 & 4.4.11], the complex \(\overline{\mathrm{Hom}}(\eta_\bullet, A_\bullet^+)\) is exact and naturally isomorphic to \((A_\bullet^0, \eta_\bullet)^+\). It follows that \(A_\bullet^0, \eta_\bullet\) is exact for every \(A_\bullet \in \mathcal{P}\). In particular, we have the exact sequence

\[
(A_\bullet^0 \otimes L_\bullet^0)^+ \rightarrow (A_\bullet^0, X_\bullet^1)^+ \rightarrow \overline{\mathrm{Tor}_1}(A_\bullet, X_\bullet^1)^+ \rightarrow \overline{\mathrm{Tor}_1}(A_\bullet, L_\bullet^0)^+
\]

is exact, where \(\overline{\mathrm{Tor}_1}(A_\bullet, L_\bullet^0)^+ \cong \overline{\mathrm{Ext}^1}(A_\bullet, (L_\bullet^0)^+) = 0\) since \(A_\bullet \in \mathcal{P}\) and \((L_\bullet^0)^+ \in \mathcal{A} \subseteq \mathcal{G}\mathcal{T}_{(\nu, \mathcal{A})}^\mathcal{L}(R^\omega)\). By the exactness of the previous sequence we get \(\overline{\mathrm{Tor}_1}(A_\bullet, X_\bullet^1)^+ = 0\), and hence \(\overline{\mathrm{Ext}^1}(A_\bullet, (X_\bullet^1)^+) = 0\). The latter implies \(\overline{\mathrm{Ext}^1}(A_\bullet, (X_\bullet^1)^+) = 0\) by [20, Prop. 4.4.7]. Degreewise, we then obtain \(\overline{\mathrm{Ext}^m}(\nu, (X_\bullet^1)^+) = 0\) for every \(m \in \mathbb{Z}\). We also have the short exact sequence

\[
(X_m^1)^+ \rightarrow (L_m^0)^+ \rightarrow X_m^1,
\]

where \((L_m^0)^+, X_m^1 \in \mathcal{G}\mathcal{T}_{(\nu, \mathcal{A})}^\mathcal{L}(R^\omega)\) by Proposition A.2. Therefore, from Proposition 3.8 we obtain that \((X_m^1)^+ \in \mathcal{G}\mathcal{T}_{(\nu, \mathcal{A})}^\mathcal{L}(R^\omega)\), and again Proposition A.2 allows us to conclude that \((X_m^1)^+ \in \mathcal{G}\mathcal{T}_{(\nu, \mathcal{A})}^\mathcal{L}(R^\omega)\).

Repeating the previous procedure infinitely many times gives rise to a \(\mathrm{Hom}_{\mathrm{Ch}(R)}(-, \mathcal{L})\)-acyclic \(\mathcal{L}\)-coresolution of \(X_\bullet\).

\(\bullet\) (d) \(\Rightarrow\) (a): It suffices to note that \(\mathcal{P}(R) \subseteq \mathcal{L}\), and thus that \(\mathcal{L}\) contains the class \(\mathcal{P}(R)\) of projective \(R\)-complexes.

\[\square\]

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