BRAIDED ANTISYMMETRIZER AS BIALGEBRA HOMOMORPHISM

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Abstract. For an Yang Baxter operator we show that a bialgebra homomorphism from a free braided tensor bialgebra to a cofree braided shuffle bialgebra is the Woronowicz braided antisymmetrizer [5]. A cofree braided shuffle bialgebra is a braided generalization of a cofree shuffle bialgebra introduced by Sweedler [4]. Its graded dual bialgebra is a free braided tensor bialgebra [3].

1. Introduction

Let $k$ be an associative ring with unit and $V$ be a $k$-bimodule. For an Yang Baxter operator $B \in \text{End}(V^{\otimes 2})$ Woronowicz [5] defined a braided antisymmetrizer $W(B) \in \text{End}(V^{\otimes})$, which is the generalization of the antisymmetrizer for the flip $B = -\tau$.

For the operator $W_n(B) = W(B)|_{V^{\otimes n}}$ exist two operators $\mu(B)$ and $\Delta(B)$ such that

$$W_{k+l}(B) = \mu_{k,l}(B) \circ \{W_k(B) \otimes W_l(B)\} = \{W_k(B) \otimes W_l(B)\} \circ \Delta_{k,l}(B).$$

We point out that two braided bialgebras are associated with those decompositions, the cofree braided shuffle bialgebra $bShV$ and its graded dual: the free braided tensor bialgebra $bTV$. We show that the cofree braided shuffle bialgebra is the generalization of the cofree shuffle bialgebra considered by Sweedler [4].

We prove that the kernel of the braided antisymmetrizer $W(B)$ is the biideal in the free braided bialgebra $bTV$.

Theorem.
The homomorphism $W$ from the braided tensor bialgebra $bTV$ to the braided shuffle bialgebra $bShV$, such that $W|_k = \text{id}$ and $W|_V = \text{id}$, is the braided antisymmetrizer $W(B)$.

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2. Notations

Let $B_n$ be a braid group with generators $\{\sigma_1, \sigma_2, \ldots, \sigma_{n-1}\}$ and relations
\[
\sigma_i \sigma_j = \sigma_j \sigma_i \quad |i - j| > 1,
\]
\[
\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad i = 1, 2, \ldots, n-2.
\]

The generators of a permutation group $P_n$: $\{t_1, t_2, \ldots, t_{n-1}\}$ satisfy relations (1) and the additional relation $\forall i = 1, \ldots, n-1 : t_i^2 = 1$. For a permutation $p \in P_n$ let $I(p)$ be the number of inversed pairs in the sequence $(p(1), p(2), \ldots, p(n))$, then
\[
p = t_{k_1} t_{k_2} \cdots t_{k_{I(p)}}.
\]

One can define [5] a map $\pi$ from $P_n$ to a braid group $B_n$. For the permutation (2) we define $\pi(t_i) = \sigma_i$ and
\[
b = \pi(p) = \sigma_{k_1} \sigma_{k_2} \cdots \sigma_{I(p)} \in B_n.
\]

A braid (3) is independent of the choice of the decomposition (2), see [5]. Denote by $\Xi$ the image of the map $\pi$,
\[
\Xi_n := \pi(P_n) \subset B_n.
\]

A $(k, l)$-shuffle for $k, l \geq 0$ is a permutation $p \in P_{k+l}$ satisfying
\[
p(1) < p(2) < \ldots < p(k) \text{ and } p(k+1) < p(k+2) < \ldots < p(l).
\]

A $(0, n)$ and $(n, 0)$ shuffle is an identity. The subset of $(k, l)$-shuffles will be denoted by $Sh_{k,l} \subset P_{k+l}$. For this set we have the corresponding subset of braided shuffles
\[
bSh_{k,l} := \pi(Sh_{k,l}) \subset B_n.
\]

For $k$-bimodule $V$ denote by $\mathcal{V}$ the $\mathbb{Z}$-graded $k$-bimodule
\[
\mathcal{V} = k \oplus V \oplus V \otimes 2 \oplus \ldots.
\]

Let $[v_1|v_2|, \ldots, |v_n]$ be an element of $\mathcal{V}^n \equiv \mathcal{V} \cap \mathcal{V}^{\otimes n}$ and $v_0 = 1 \in k$. The tensor algebra $TV$ means in this case [2] the pair $TV = \{\mathcal{V}, \otimes\}$, where $\otimes : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ is a free product.

Let $V^* = \text{Hom}(V, k)$ be the dual $k$-bimodule for $V$. For the evaluation $< V^*, V >$ are two possibilities to evaluate $< V^* \otimes V^*, V \otimes V >$. We use the following evaluation (note that transposition depends on this choice),
\[
< \alpha \otimes \beta, v \otimes w > = < \beta, v > < \alpha, w >, \quad \forall \alpha, \beta \in V^* \quad \text{and} \quad \forall v, w \in V.
\]

For the bimodule $\mathcal{V}$ the graded dual bimodule $\mathcal{V}^g$ means [4]
\[
\mathcal{V}^g = k \oplus V^* \oplus V^* \otimes 2 \oplus \ldots.
\]
Let $B$ be the Yang Baxter operator, i.e. an invertible endomorphism of the $k$-bimodule $V \otimes V$ which satisfies the braid equation
\begin{equation}
(id \otimes B)(B \otimes id)(id \otimes B) = (B \otimes id)(id \otimes B)(B \otimes id) \in \text{End}(V \otimes^3).
\end{equation}
Let $B_k = id_{k-1} \otimes B \otimes id_{n-k-1}$ of $V^n \subset V \in \text{End}(V \otimes^n)$ be bimodule endomorphisms. For the Yang Baxter operator $B$ we can define the representation $\rho_B$ of the braid group $B_n$ in the $\mathbb{Z}$-graded $k$-bimodule $V$,
\begin{equation}
\rho_B \in \text{group}\{B_n, \text{End}(V^n)\} : \rho_B(\sigma_k) = B_k.
\end{equation}

3. TWO BRAIDED BIALGEBRAS

For the Yang Baxter operator $B$ on the bimodule $V$ the subset $\Xi_n \subset B_n$ is acting on the $\mathbb{Z}$-graded $k$-bimodule through the representation $\rho_B$ (11). Let us introduce the notion of the cofree braided shuffle algebra. For the $k$-bimodule $V$ one can define \footnote{1} the cofree comultiplication $\Delta^\otimes$
\begin{equation}
\Delta^\otimes[v_1|v_2|\ldots|v_n] = \sum_{k=0}^{n} [v_1|v_2|\ldots|v_k] \otimes [v_{k+1}|\ldots|v_n].
\end{equation}

The braided shuffle multiplication $\mu(B) : V \otimes V \to V$ is defined for $\mu_{k,n-k}(B) = \mu(B)|V^k \otimes V^{n-k}$
\begin{equation}
\mu_{k,n-k}(B) = \sum_{b \in bSh_{k,n-k}} \rho_B(b) : V^k \otimes V^{n-k} \to V^n,
\end{equation}
\begin{equation}
\mu_{k,n-k}(B)([v_1|v_2|\ldots|v_k] \otimes [v_{k+1}|\ldots|v_n]) = \sum_{b \in bSh_{k,n-k}} \rho_B[v_1|v_2|\ldots|v_{n}].
\end{equation}

For example $\mu_{1,2}(B) = id_3 + B \otimes \text{id} + (B \otimes \text{id})(\text{id} \otimes B)$.

\textbf{Lemma 3.1.} The multiplication $\mu(B)$ (13) is associative and $B$-braided.

\textit{Proof.} Denote an element $[v_1|v_2|\ldots|v_{I}]$ by $v_{I}$. Then we have
\begin{equation}
\mu(B)(v_I \otimes v_J) = \sum_{K=I+J} v_K \in V^K.
\end{equation}
The associativity condition is proved by the following equation
\begin{equation}
[\mu(B) \circ (\mu(B) \otimes \text{id})](v_I \otimes v_J \otimes v_K)
= \sum_{L=I+J+K} v_L = [\mu(B) \circ (\text{id} \otimes \mu(B))](v_I \otimes v_J \otimes v_K).
\end{equation}
\hfill \square
Consider the $B$-braided monoidal category. A bialgebra is defined over this category if the multiplication $m$ and comultiplication $\triangle$ are morphisms and satisfy the following compatibility condition
\begin{equation}
\triangle \circ m = (m \otimes m) \circ (\text{id} \otimes B \otimes \text{id}) \circ (\triangle \otimes \triangle).
\end{equation}

**Lemma 3.2.** The triple $(V, \mu(B), \triangle^\otimes)$ is the braided bialgebra.

The proof by induction for the term $V^k \otimes V^l$ is omitted.

Consider the $Z$-graded dual $k$-bimodule $V^g$. The free braided tensor bialgebra $bT V$ is the graded dual to the cofree braided shuffle bialgebra. Then the free braided tensor bialgebra $bT V$ means the triple ${V^g, \otimes, \triangle^\mu(B)}$, where the multiplication $\otimes$ and the comultiplication $\triangle^\mu$ are graded duals in the following sense
\begin{equation}
\triangle^\mu(B) = \mu^g(B), \quad \text{and} \quad \otimes = \triangle^{\otimes g}.
\end{equation}

Dual version of the lemma 3.2 is the following braided bialgebra.

**Lemma 3.3.** The triple $(V, \otimes, \triangle^\mu(B))$ is the braided bialgebra.

### 4. Braided Antisymmetrizer

For an Yang Baxter operator $B$ the braided antisymmetrizer $W(B)$ was defined by Woronowicz [5] as
\begin{equation}
W(B) = \sum_{b \in \Xi_n} \rho_B(b).
\end{equation}

For $B = -\pi$ we get the Woronowicz form of the braided antisymmetrizer with the sign of the permutation.

**Lemma 4.4.** For the braided antisymmetrizer $W(B)$, the multiplication $\mu(B)$ and the comultiplication $\triangle^\mu(B)$ the following recurrent formula is holds
\begin{equation}
W_{n+1}(B) = \mu(B)_{n,1} \circ [W_n(B) \otimes \text{id}] = [\text{id} \otimes W_n(B)] \circ \triangle^\mu_{1,n}(B).
\end{equation}

For the proof see [5] for details.

From this lemma by induction we can prove
\begin{equation}
W_{k+l}(B) = \mu(B)_{k,l} \circ [W_k(B) \otimes W_l(B)].
\end{equation}

and dually
\begin{equation}
W_{k+l}(B) = [W_k(B) \otimes W_l(B)] \circ \triangle^\mu_{1,n}(B).
\end{equation}

Consider braided bialgebras $bT V$ and $bShV$. A map $W : bT V \to BShV$ is a homomorphism of these bialgebras $W \in bialg(bT V, bShV)$ if satisfies two conditions

- $W$ is the algebra map:
  \begin{align*}
  W \in \text{alg}(\otimes, \mu(B)), \quad W \circ \otimes &= \mu(B) \circ (W \otimes W).
  \end{align*}
• W is the coalgebra map:
\[ W \in \text{coalg}(\Delta^\mu(B), \Delta^\otimes), \quad \Delta^\otimes \circ W = (W \otimes W) \circ \Delta^\mu(B). \]

**Theorem 4.5.** The bialgebra homomorphism \( W \in \text{bialg}(bTV, bShV) \), such that \( W|k = \text{id} \) and \( W|V = \text{id} \), is the braided antisymmetrizer \( W(B) \).

**Proof.** From the assumption that \( W \) is the algebra map we obtain
\[ W_n = \mu_{n-1,1}(B) \circ (W_{n-1} \otimes \text{id}). \]
Then by induction we have
\[ W_n = \mu_{n-1,1}(B) \circ \mu_{n-2,1}(B) \circ \ldots \circ \mu_{1,1}(B). \]
From the assumption \( W|k = \text{id} \) and \( W|V = \text{id} \) this is the braided antisymmetrizer \( W = W(B) \).

\[ \square \]

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**References**

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