Hamiltonian dynamics of linear affine in acceleration theories

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Abstract. We study the constrained Ostrogradski-Hamilton framework for the equations of motion $M_{\mu \nu} \dddot{x}^\nu = F_\mu (x^\nu, \dot{x}^\nu)$ provided by mechanical systems described by second-order derivative actions with a linear dependence in the accelerations. We stress the peculiar features provided by the surface terms arising for this type of theories and we discuss some important properties for this kind of actions in order to pave the way for the construction of a well defined quantum counterpart by means of canonical methods. In particular, we analyse in detail the constraint structure for these theories, and its relation to the inherent conserved quantities, where the associated energies together with a Nöether charge may be identified. We also provide some examples where our approach is explicitly applied, and emphasize the way in which our original arrangement results propitious for the Hamiltonian formulation of covariant field theories.

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1. Introduction

Gauge theories with higher derivatives have a potential interest for the study of the existence of several symmetries given their intrinsic nature. The Hamiltonian formalism provides a convenient way to formulate and explore the symmetries of field theories as well as to explore the ghost problem and the unitary properties of quantum gauge field theories for systems with higher-order singular Lagrangians. A common strategy to develop the Hamiltonian formalism for this type of theories is by means of the introduction of auxiliary variables which is accompanied by certain constraints enforcing the definition of the new variables. However, this procedure sometimes may become hard to manipulate since for a generic system we must work out the constraint
Lagrangians affine in acceleration

algebra of an enlarged set of constraints. Nonetheless, not all higher-order gauge theories fall into this category. Some interesting theories have a linear dependence on acceleration, including General Relativity, brane models, and Electromagnetism with Chern-Simons corrections, for example, thus deserving special attention mainly when canonical quantization procedure is going to be introduced.

In this paper we review the Ostrogradski-Hamilton framework for the so-called *Lagrangians linear affine in acceleration*, that is, Lagrangians with a linear dependence on accelerations [1, 2, 3, 4, 5, 6, 7, 8, 9]. We have opted to keep all the variables in the enlarged phase space in order to construct a natural Dirac algorithm to determine the physical constrained surface where dynamics will take place. In our opinion, this is an issue that has been overlooked in many contributions. Additionally, we study the relation between this approach and Dirac’s one for first-order theories. We find that this constrained Ostrogradski-Hamilton structure may be inferred from the associated Euler-Lagrange equations of motion. This approach has been of particular relevance for the analysis of the canonical formalism for certain brane models [10, 11, 12, 13] (see also [14, 15] for a comparison with a different method), bringing back into life the old idea that considers General Relativity as a second-order field theory [16].

Lagrangians affine in acceleration have been discussed extensively in the regular case (see, for example, [1, 2, 3, 17, 18, 19, 20]). For the singular case, however, there are several points which seem to remain unclear and, therefore, this will be our purpose here: to analyse the singular nature of these Lagrangians by means of an Ostrogradski-Hamiltonian approach. In particular, we resolve the conditions under which our system allows surface terms and explore the role that this kind of terms play on the constraint structure of the theory. Further, we also emphasize the character of geometric invariants which provide well-defined energies and Nöether charges, and examine their relevance for covariant systems through the so-called Zermelo conditions [21, 22, 23, 24, 25].

This paper is organized as follows. In Section 2 we explain at some length why these Lagrangians are best understood straightforwardly as standard second-order Lagrangians instead of neglecting the surface term from the beginning. Also, this Section serves as a guide in the transition towards the Ostrogradski-Hamilton formalism for this type of theories. In Section 3 we describe the important role of the surface term, and we decompose our Lagrangian into two components, one related to the dynamic term, and the other related to the surface term. In Section 4 we complete the Ostrogradski-Hamiltonian approach within Dirac formulation for constrained systems. Section 5 is dedicated to the geometric analysis of the constraints. We provide a couple of examples of applications of our scheme in Section 6, related to the chiral oscillator and the geometric dynamics resulting from a second-order 2-form, respectively. In Section 7 we specialize our formulation to the analysis of covariant theories, and in particular, we develop our results for covariant brane theories. In addition, we present one further example given by the electrically charged bubble, in order to prove the results previously obtained within this section. In Section 8, we include some concluding remarks. Finally, we address some technical issues related to conserved quantities and
2. Lagrangians affine in acceleration

Let us consider the dynamical evolution of physical systems governed by the local action

\[ S[x^\mu] = \int_c d\tau L(x^\mu, \dot{x}^\mu, \ddot{x}^\mu), \]

where the integral is well-defined for a parametric curve \( c \), and the Lagrange function \( L : T^2M \rightarrow \mathbb{R} \) defined on a basis manifold \( M \) is of the explicit form

\[ L(x^\mu, \dot{x}^\mu, \ddot{x}^\mu) = K_\mu(x^\nu, \dot{x}^\nu) \ddot{x}^\mu + V(x^\mu, \dot{x}^\mu). \]

Here, \( K_\mu(x^\nu, \dot{x}^\nu) \) and \( V(x^\mu, \dot{x}^\mu) \) are two arbitrary \( C^2(M) \) functions and \( \mu, \nu = 0, 1, 2, \ldots, N - 1 \) label the local coordinates on the extended configuration space. An overdot stands for a derivative with respect to a convenient parameter \( \tau \). We will consider for simplicity systems with a finite number of degrees of freedom. The Lagrangian \((2)\) is said to be affine in acceleration due to the fact that the Hessian \( H_{\mu\nu} = (\partial^2 L/\partial \ddot{x}^\mu \partial \ddot{x}^\nu) \) vanishes identically \([5]\).

As it is expected \( L \) is determined up to a total time derivative of the form

\[ \tilde{L}(x^\mu, \dot{x}^\mu, \ddot{x}^\mu) = L(x^\mu, \dot{x}^\mu, \ddot{x}^\mu) + \frac{d}{d\tau} Y(x^\mu, \dot{x}^\mu), \]

that clearly do not affect the equations of motion as far as the function \( Y(x^\mu, \dot{x}^\mu) \) is an arbitrary function of the coordinates and the velocities, that is, \( \partial Y/\partial \ddot{x}^\mu = 0 \). This arbitrariness leads to the following transformations laws for the functions \( K_\mu \) and \( V \)

\[ \tilde{K}_\mu(x^\nu, \dot{x}^\nu) = K_\mu + \frac{\partial Y}{\partial \dot{x}^\mu}, \quad (4a) \]

\[ \tilde{V}(x^\mu, \dot{x}^\mu) = V + \frac{\partial Y}{\partial x^\mu} \ddot{x}^\mu. \quad (4b) \]

These transformation laws closely resemble a sort of gauge transformation. Indeed, equation \((4a)\) looks like an Abelian gauge transformation in the velocity coordinates sector whereas equation \((4b)\) seems like an Abelian gauge transformation for a scalar potential in the position coordinates sector. Analogous transformations for Lagrangians affine in velocity were developed in reference \([26]\).

Associated with a generic second-order Lagrange function, \( L \), the Euler-Lagrange equations of motion read

\[ E^{(0)}_\mu(L) = 0, \]

where the differential operator \( E^{(0)}_\mu \) is defined on the basis manifold as

\[ E^{(0)}_\mu := \frac{\partial}{\partial x^\mu} - \frac{d}{d\tau} \left( \frac{\partial}{\partial \dot{x}^\mu} \right) + \frac{d^2}{d\tau^2} \left( \frac{\partial}{\partial \ddot{x}^\mu} \right). \]

As usual, solutions to equations of motion \((5)\) correspond to extremal curves for the action. Note also that \( E^{(0)}_\mu \) is the first of the Craig-Synge covectors associated to a differentiable Lagrangian of second order, as we will discuss in Appendix A.
In our particular formulation, we have
\[ \frac{\partial L}{\partial x^\mu} = \frac{\partial K_\nu}{\partial x^\mu} \ddot{x}^\nu + \frac{\partial V}{\partial x^\mu}, \]
\[ \frac{\partial L}{\partial \dot{x}^\mu} = \frac{\partial K_\nu}{\partial \dot{x}^\mu} \ddot{x}^\nu + \frac{\partial V}{\partial \dot{x}^\mu}, \]
\[ \frac{\partial L}{\partial \ddot{x}^\mu} = K_\mu, \]
from which we obtain the equations of motion in the compact form
\[ M_{\mu\nu} \ddot{x}^\nu = F_\mu, \tag{7} \]
provided that
\[ \frac{\partial K_\mu}{\partial \ddot{x}^\nu} = \frac{\partial K_\nu}{\partial \ddot{x}^\mu}. \tag{8} \]
We will explain in detail this last condition below.

Equations of motion in the form (7) suggests that \( M_{\mu\nu} \) may be interpreted as a “mass-like matrix”, while the term \( F_\mu \) may be interpreted as a force vector. These quantities are defined as
\[ M_{\mu\nu} := \frac{\partial P_\nu}{\partial x^\mu} - \frac{\partial p_\mu}{\partial \dot{x}^\nu}, \tag{9} \]
\[ F_\mu := \frac{\partial p_\mu}{\partial x^\nu} \dot{x}^\nu - \frac{\partial V}{\partial x^\mu}, \tag{10} \]
respectively. Here, we have introduced the quantities
\[ P_\mu := \frac{\partial L}{\partial \ddot{x}^\mu} = K_\mu, \tag{11a} \]
\[ p_\mu := \frac{\partial L}{\partial \ddot{x}^\mu} - \frac{d}{d\tau} \left( \frac{\partial L}{\partial \dot{x}^\mu} \right) = \frac{\partial V}{\partial \dot{x}^\mu} - \frac{\partial K_\mu}{\partial x^\alpha} \dot{x}^\nu, \tag{11b} \]
which are nothing but the canonical momenta in the so-called Ostrogradski-Hamiltonian approach, as will be explained in Section 4. Conditions (8) ensure that no third-order derivatives appear in the equations of motion (7). A trivial example of this is obtained by considering \( K_\mu = K_\mu(x^\nu) \) as thus the equations of motion are second order and conditions (8) vanish identically. A different example is considered if \( K_\mu \) specializes to \( K_\mu = K_\mu(\dot{x}^\nu) \) for which the form (7) is preserved with \( M_{\mu\nu} = -\partial p_\mu/\partial \dot{x}^\nu \). We want to emphasize that, within our formulation, equations (7) are of second-order even if the action (1) involves second-order derivative terms. This is of a remarkable physical relevance since within this framework neither we have propagation of extra degrees of freedom nor we confront the instability issues associated to higher-order theories [27, 28, 29, 30]. This is so because condition (8) allows to split the original Lagrangian in two terms where one of them is a total time derivative which contains the acceleration dependence of the Lagrangian \( L \). According to the dependence of \( K_\mu \) and \( V \), we have that \( P_\mu = P_\mu(x^\nu, \dot{x}^\nu) \) and \( p_\mu = p_\mu(x^\nu, \dot{x}^\nu, P_\nu) \) and, in consequence \( M_{\mu\nu} = M_{\mu\nu}(x^\alpha, \dot{x}^\alpha) \) and \( F_\mu = F_\mu(x^\alpha, \dot{x}^\alpha) \).

For a regular matrix \( M_{\mu\nu} \), there is an inverse matrix \( M_{\mu\nu}^{-1} \) such that \( M_{\mu\nu}^{-1} M_{\alpha\nu} = \delta_{\mu\nu} \) or \( M_{\mu\alpha} M_{\alpha\nu}^{-1} = \delta_{\mu\nu} \). Hence, we can solve (7) for the accelerations \( \ddot{x}^\mu = M^{-1\mu\nu} F_\nu =: \)
Lagrangians affine in acceleration

$F^\mu(x^\nu, \dot{x}^\nu)$. This regular case is related to the so-called inverse problem in Lagrangian mechanics which has been studied at length in [1, 2, 3, 19]. Clearly, we emphasize that in order to study the singular nature of the Lagrangian (2) we must focus on the analysis of the matrix $M_{\mu\nu}$. The case of a singular $M_{\mu\nu}$ will be analyzed below.

In addition, we may note that condition (8) corresponds to an identically vanishing curl of $P_\mu$ in the velocity configuration space sector, namely

$$N_{\mu\nu} := \frac{\partial P_\nu}{\partial \dot{x}_\mu} - \frac{\partial P_\mu}{\partial \dot{x}_\nu} = 0. \quad (12)$$

It is straightforward to check that the quantities $M_{\mu\nu}, F_\mu$ and $N_{\mu\nu}$ are all invariant under the transformations (4a) and (4b). Hence, it is not surprising that all these quantities will play a fundamental role in the following sections.

To end this section, we describe the conserved quantities for our second order Lagrangian theory (see Appendix A, and references therein, for further details). In our notation, the conserved energies of the system are explicitly given by

$$(13a) \quad E^{(1)c}(L)_c := -\dot{x}^\mu K_\mu,$$

$$(13b) \quad E^{(2)c}(L)_c := \frac{\partial V}{\partial \dot{x}^\mu} \dot{x}^\mu - V + \left(\frac{\partial E^{(1)c}(L)_c}{\partial x^\mu}\right) \dot{x}^\mu \dot{x}^\nu.$$  

We will realize in the following sections that the energy $E^{(1)c}(L)_c$ corresponds to the term leading to a total time derivative in the Lagrangian (2), while the energy $E^{(2)c}(L)_c$ corresponds to the canonical Hamiltonian when written in terms of the momenta $1\alpha$ and $1\beta$. Also note that energies $E^{(1)c}(L)_c$ and $E^{(2)c}(L)_c$ are related through the identity $E^{(2)c}(L)_c = (\partial V/\partial \dot{x}^\mu) \dot{x}^\mu - V + (\partial E^{(1)c}(L)_c/\partial x^\mu) \dot{x}^\mu$ as may be deduced from general expressions (A.10) and (A.11). We must emphasize that these energies are only conserved along the solutions of the Craig-Synge covectors, as described in Appendix A, where we also show the way in which these energies are helpful in order to construct a Noether theorem for the Lagrangian we are working with, and which states that the function

$$G(L, \phi) := W^\mu \left(\frac{\partial K_\nu}{\partial \dot{x}^\mu} \ddot{x}^\nu + \frac{\partial V}{\partial \dot{x}^\mu}\right) - \frac{d}{d\tau} W^\mu K_\mu - \eta E^{(2)c}(L)_c + \frac{d\eta}{d\tau} E^{(1)c}(L)_c - \phi, \quad (14)$$

is conserved through an infinitesimal transformation of the form

$$x^\mu \mapsto x^\mu + \epsilon W^\mu(x, \tau),$$

$$\tau \mapsto \tau + \epsilon \eta(x, \tau), \quad (15)$$

along the solution curves of the Euler-Lagrange equations (5). In relations (14) and (15) we have introduced the differentiable vector field $W^\mu$ locally defined along a curve $c$ and such that it vanishes at the endpoints of that curve, the sufficiently small real number $\epsilon > 0$, and the smooth arbitrary (locally defined) functions $\eta := \eta(x, \tau)$ and $\phi := \phi(x, \dot{x})$. Also, it is assumed that points connected through the infinitesimal transformation (15) belong to the same domain of a local chart in the basis manifold. On physical grounds, the function $G(L, \phi)$ is nothing but the Noether charge associated to the action (11), which may be expressed as $G(L, \phi) = W^\mu p_\mu - \dot{W}^\mu P_\mu - \eta E^{(1)c}(L)_c + \dot{\eta} E^{(2)c}(L)_c - \phi,$
where equations (11a) and (11b) have been invoked. Further details may be found in Appendix A.

### 3. On the surface term and \( s \)-equivalent Lagrangians

We turn now to explore the conditions on \( K_\mu \) such that partial integrations of the linear term in the accelerations in (2) does not lead to total time derivatives. The condition \( d(\dot{x}^\mu K_\mu)/d\tau = 0 \) is equivalent to

\[
\left( K_\mu + \frac{\partial K_\nu}{\partial \dot{x}^\mu} \dot{x}^\nu \right) \ddot{x}^\mu = -\frac{\partial K_\mu}{\partial x^\nu} \dot{x}^\mu \dot{x}^\nu.
\]

(16)

Notice that the transformation (4a) implies the existence of an infinite class of equivalent Lagrangians (2). Taking advantage of this, given arbitrary functions \( K_\mu \), a transformation (4a) may lead to obtain divergence-free functions \( \tilde{K}_\mu \) provided \( Y \) satisfies the equation

\[
\frac{\partial Y}{\partial \dot{x}^\mu} \ddot{x}^\mu + \frac{d}{d\tau} \left( \frac{\partial Y}{\partial \dot{x}^\mu} \right) \dot{x}^\mu = -K_\mu \ddot{x}^\mu - \frac{dK_\mu}{d\tau} \dot{x}^\mu.
\]

(17)

Suppose now that we can identify a total time derivative. Without loss of generality we may define alternative quantities and rewrite the Lagrangian (2) as follows

\[
L(x^\mu, \dot{x}^\mu, \ddot{x}^\mu) = f(x^\mu, \dot{x}^\mu) + \frac{d}{d\tau} \left[ g(x^\mu) h(\dot{x}^\mu) \right],
\]

(18)

where \( f(x^\mu, \dot{x}^\mu) \), \( g(x^\mu) \) and \( h(\dot{x}^\mu) \) are smooth functions. In fact, the function \( h(\dot{x}^\mu) \) is defined up to an integration constant. This results as a consequence that equation (8) only implies the existence of a boundary function \( \Lambda(x^\mu, \dot{x}^\mu) \) such that \( K_\mu = \partial\Lambda/\partial \dot{x}^\mu \) (see also [1]).

By expanding the total time derivative and comparing to equation (2) yields

\[
K_\mu(x^\nu, \dot{x}^\nu) = g \frac{\partial h}{\partial \dot{x}^\mu},
\]

(19)

\[
V(x^\nu, \dot{x}^\nu) = f + \dot{x}^\mu \frac{\partial g}{\partial x^\mu} h.
\]

(20)

Expression (19) fulfils condition (8), indeed. Hence, we can rewrite the original Lagrangian (2) as the sum of two Lagrangian functions

\[
L_d := f = V - \frac{\partial g}{\partial x^\mu} h \dot{x}^\mu,
\]

(21)

\[
L_s := \frac{d(g h)}{d\tau} = g \frac{\partial h}{\partial \dot{x}^\mu} \ddot{x}^\mu + \frac{\partial g}{\partial x^\mu} h \dot{x}^\mu.
\]

(22)

Note that \( L_d = L_d(x^\mu, \dot{x}^\mu) \) while \( L_s = L_s(x^\mu, \dot{x}^\mu, \ddot{x}^\mu) \), thus the Lagrangian \( L_s \) contains all the information related to the second-order derivatives. We will refer to these Lagrangian functions as the “dynamic” Lagrangian and the “surface” Lagrangian, respectively.

\^ For simplicity, from now on, we consider separation of variables for the boundary term. From a more general viewpoint, analogous conclusions follow if we adopt the boundary function \( \Lambda(x^\mu, \dot{x}^\mu) \) instead of the functions \( g(x^\mu) \) and \( h(\dot{x}^\mu) \).
Lagrangians (2) and (21), belonging to the set known as s-equivalent Lagrangians, are characterized by equations of motion which are solved by the same orbits. The advantage of this separation is particularly conspicuous in certain cases of covariant theories. Indeed, sometimes \( f(x^\mu, \dot{x}^\mu) \), regarded as an ordinary first-order Lagrangian, does not allow for a consistent Hamiltonian treatment whenever \( M_{\mu \nu} \) is singular (see, for example, \( [10, 11, 31, 32] \)). In addition, this separation helps to understand deeply the role of \( M_{\mu \nu} \).

With respect to the surface Lagrangian, \( L_s \), and considering the definitions (11a) and (11b), we have the momenta

\[
P_\mu = g \frac{\partial h}{\partial \dot{x}^\mu},
\]

\[
p_\mu = \frac{\partial g}{\partial x^\mu} h.
\]

Analogously, with respect to the first-order dynamic Lagrangian, \( L_d \), we have the momenta

\[
p_\mu = \frac{\partial f}{\partial \dot{x}^\mu} = \frac{\partial V}{\partial \dot{x}^\mu} - \frac{\partial g}{\partial x^\nu} \frac{\partial h}{\partial \dot{x}^\mu} \dot{x}^\nu - \frac{\partial g}{\partial x^\mu} h.
\]

Obviously, we have that with respect to the original Lagrangian (2) the conjugate momenta to the position variables (11b) are given by

\[
p_\mu = p_\mu + p_\mu = \frac{\partial V}{\partial \dot{x}^\mu} - \frac{\partial g}{\partial x^\nu} \frac{\partial h}{\partial \dot{x}^\mu} \dot{x}^\nu,
\]

that is, the total momenta \( p_\mu \) associated to the Lagrangian (2) (or (18)) is obtained by adding a contribution from both the surface and the dynamic momenta. At this point it is clear what we must to do in order to express the original Lagrangian (2) as an ordinary first-order Lagrangian plus a total time derivative term. Starting from equation (19) we need to identify first \( g \) and \( \partial h / \partial \dot{x}^\mu \). Then we proceed to integrate the partial derivatives to obtain \( h \). We must insert this information in (20) in order to obtain \( f \) and, finally, we finish the process by inserting these quantities in (18).

We also note from equations (23a) and (23b) that we have the relation

\[
\frac{\partial P_\nu}{\partial x^\mu} = \frac{\partial p_\mu}{\partial \dot{x}^\nu}.
\]

This identity is the key point when we try to connect the two points of view for the Lagrangians (2) and (18). It also turns out that from equations (9), (25) and (26) we have

\[
M_{\mu \nu} = - \frac{\partial p_\mu}{\partial \dot{x}^\nu} = - \frac{\partial^2 L_d}{\partial \dot{x}^\mu \partial \dot{x}^\nu} = M_{\nu \mu}.
\]

We have thus established that the mass-like matrix (9) corresponds to the Hessian matrix for the s-equivalent Lagrangian \( L_d \). By virtue of the symmetry of \( M_{\mu \nu} \), we have that

\[
\frac{\partial p_\mu}{\partial \dot{x}^\nu} \frac{\partial p_\nu}{\partial \dot{x}^\mu} = \frac{\partial^2 L_d}{\partial \dot{x}^\mu \partial \dot{x}^\nu} = M_{\mu \nu}.
\]

\[
\frac{\partial M_{\mu \nu}}{\partial \dot{x}^\rho} = \frac{\partial M_{\rho \mu}}{\partial \dot{x}^\nu}.
\]
Moreover, the force vector becomes
\[ F_\mu = \frac{\partial p_\mu}{\partial x^\nu} \dot{x}^\nu + \frac{\partial^2 g}{\partial x^\mu \partial x^\nu} h \dot{x}^\nu - \frac{\partial V}{\partial x^\mu}, \] (29)
and, by considering (26) we have in addition the identity
\[ \Theta_{\mu\nu} := \frac{\partial P_\mu}{\partial x^\nu} - \frac{\partial P_\nu}{\partial x^\mu}, \] (30a)
\[ = \frac{\partial p_\mu}{\partial \dot{x}^\nu} - \frac{\partial p_\nu}{\partial \dot{x}^\mu}, \] (30b)
where the last relation establishes that the curl of \( P_\mu \) in the coordinate configuration space sector corresponds to the curl of the non-dynamical momenta \( p_\mu \) in the velocity configuration space sector.

4. Ostrogradski-Hamiltonian approach

Since the Lagrangian (2) depends on the accelerations, in accordance with higher-order derivative theories, it is quite natural that we try to develop a canonical analysis by means of the Ostrogradski-Hamiltonian approach [33, 34]. The configuration space is spanned by \( C = \{ x^\mu; \dot{x}^\mu \} \) and in consequence the ordinary phase space is enlarged and spanned by \( \Gamma = \{ x^\mu, p_\mu, \dot{x}^\mu, P_\mu \} \), where the conjugate momenta to \( \dot{x}^\mu \) and \( x^\mu \) are given by (11a) and (11b), respectively. The highest momenta \( P_\mu \) must satisfy the \( N \) primary constraints
\[ C_\mu = P_\mu - K_\mu(x^\nu, \dot{x}^\nu) \approx 0, \] (31)
which follows directly from the definition (11a). Squaring the primary constraints in order to obtain a resulting Hamiltonian quadratic in the momenta \( P_\mu \) does not prove to be always correct [35]. The canonical Hamiltonian is
\[ H_0 = p_\mu \dot{x}^\mu + P_\mu \ddot{x}^\mu - L(x^\mu, \dot{x}^\mu, \ddot{x}^\mu) = p_\mu \dot{x}^\mu - V(x^\mu, \dot{x}^\mu), \] (32)
so that the total Hamiltonian of the system is given by
\[ H = p_\mu \dot{x}^\mu - V + u^\mu C_\mu. \] (33)
Here, \( u^\mu \) are Lagrange multipliers enforcing the primary constraints (31). These are a priori functions of the phase space variables and possibly of time.

For two phase space functions \( F \) and \( G \), we introduce the generalized Poisson bracket
\[ \{ F, G \} := \frac{\partial F}{\partial x^\mu} \frac{\partial G}{\partial p_\mu} + \frac{\partial F}{\partial \dot{x}^\mu} \frac{\partial G}{\partial P_\mu} - (F \longleftrightarrow G). \] (34)
Under this definition we observe easily that the primary constraints are in involution in a strong sense
\[ \{ C_\mu, C_\nu \} = 0. \] (35)
Lagrangians affine in acceleration

Therefore, there must be secondary constraints. Persistence of the primary constraints under evolution of the parameter $\tau$ requires that

$$\{C_\mu, H\} \approx \{C_\mu, H_0\} + u^\nu \{C_\mu, C_\nu\},$$

$$= -\frac{\partial K_\mu}{\partial x^\nu} \dot{x}^\nu - p_\mu + \frac{\partial V}{\partial \dot{x}^\mu} \approx 0. \quad (36)$$

As a consequence we can identify the $N$ secondary constraints

$$C_\mu := p_\mu - \frac{\partial V}{\partial \dot{x}^\mu} + \frac{\partial K_\mu}{\partial x^\nu} \dot{x}^\nu \approx 0. \quad (37)$$

Now, a straightforward calculation shows that the Poisson brackets of primary and secondary constraints is directly related to the matrix $M_{\mu\nu}$

$$\{C_\mu, C_\nu\} = M_{\mu\nu}, \quad (38)$$

and thus the process of generation of more constraints is finished. Certainly, the stationary condition on the secondary constraints is

$$\{C_\mu, H\} \approx \{C_\mu, H_0\} + u^\nu \{C_\mu, C_\nu\},$$

$$= -F_\mu + u^\nu M_{\mu\nu} \approx 0, \quad (39)$$

which merely provides a restriction on the Lagrange multipliers $u^\mu$. In addition, the Poisson brackets among the secondary constraints is

$$\{C_\mu, C_\nu\} = \frac{\partial p_\nu}{\partial x^\mu} - \frac{\partial p_\mu}{\partial x^\nu} =: X_{\mu\nu}.$$

This expression stands for the curl of the momenta $p_\mu$ in the coordinate space sector and it will play a fundamental role in the subsequent discussion. Further, $X_{\mu\nu}$ is also invariant under the transformations (4a) and (4b). In terms of the momenta associated to the dynamic Lagrangian (21) we have that

$$X_{\mu\nu} = \frac{\partial p_\nu}{\partial x^\mu} - \frac{\partial p_\mu}{\partial x^\nu}. \quad (41)$$

As can easily be verified, the matrix (40) satisfies the Bianchi identity

$$\frac{\partial X_{\mu\nu}}{\partial x^\rho} + \frac{\partial X_{\rho\mu}}{\partial x^\nu} + \frac{\partial X_{\nu\rho}}{\partial x^\mu} = 0. \quad (42)$$

There are other relationships, known as Helmholtz conditions, which relate to matrices $M_{\mu\nu}$ and $X_{\mu\nu}$ with all the momenta. See Appendix B for details.

4.1. Hamiltonian equations

The Hamiltonian equations are obtained in the standard way. First,

$$\dot{x}^\mu \approx \{\dot{x}^\mu, H\} = u^\nu \frac{\partial C_\nu}{\partial P_\mu} = u^\mu, \quad (43)$$

i.e., $u^\mu$ are nothing but the accelerations which were not possible to solve out from (11a). Notice that Eq. (39) is reduced to the equations of motion (5) when $u^\mu$ is inserted. The second Hamilton equation for $\dot{x}^\mu$ is a mere identity

$$\dot{x}^\mu \approx \{x^\mu, H\} = \frac{\partial H_0}{\partial p_\mu} = \dot{x}^\mu, \quad (44)$$
Lagrangians affine in acceleration

since the only dependence on \( p_\mu \) in the canonical Hamiltonian is through the term \( p_\mu \dot{x}^\mu \).

With respect to the Hamilton equation for \( \dot{P}_\mu \) we have

\[
\dot{P}_\mu \approx \{ P_\mu, H \} = -\frac{\partial H}{\partial \dot{x}^\mu} - u^\nu \frac{\partial C_\nu}{\partial \dot{x}^\mu},
\]

\[
= -p_\mu + \frac{\partial V}{\partial x^\mu} + u^\nu \frac{\partial K_\nu}{\partial x^\mu}.
\]

This expression reproduces the form of the momenta (11) once we insert the Lagrange multiplier (43). Finally, the Hamilton equation for \( \dot{p}_\mu \) reads

\[
\dot{p}_\mu \approx \{ p_\mu, H \} = -\frac{\partial H}{\partial x^\mu} - u^\nu \frac{\partial C_\nu}{\partial x^\mu},
\]

\[
= \frac{\partial V}{\partial x^\mu} + u^\nu \frac{\partial K_\nu}{\partial x^\mu}.
\]

Here again, the Euler-Lagrange equations of motion (7) are recovered when we insert the Lagrange multiplier (43). Thus, the two descriptions are entirely equivalent.

5. Analysis of the constraints

Following the ordinary Dirac treatment for constrained systems we require to decompose the set of primary and secondary constraints into first- and second-class constraints [36, 37, 38]. According to equations (35), (38) and (40) we construct the matrix \( \Omega_{ij} \) whose elements are the Poisson brackets among all the constraints

\[
\Omega_{ij} = \begin{pmatrix}
\{ C_\mu, C_\nu \} & \{ C_\mu, C_\nu \} \\
\{ C_\mu, C_\nu \} & \{ C_\mu, C_\nu \}
\end{pmatrix} = \begin{pmatrix}
0_{N \times N} & -M_{\mu \nu} \\
M_{\mu \nu} & X_{\mu \nu}
\end{pmatrix},
\]

where \( M_{\mu \nu} \) and \( X_{\mu \nu} \) were defined in (9) and (40), respectively, and \( i, j = 1, 2, \ldots, N, N+1, \ldots, 2N \). The rank of this non-singular matrix is \( 2N \) which indicates the presence of \( 2N \) second-class constraints. Indeed, the primary and secondary constraints exhaust the whole set of constraints being second-class and we can use them as mere identities strongly equal to zero. This is useful in order to express some canonical variables in terms of others. The counting of degrees of freedom (dof) is readily obtained [37]:

\[
dof = \frac{[4N - 2N]}{2} = N.
\]

We may therefore construct the Dirac bracket as usual [37, 38]. For two phase space functions, \( F \) and \( G \), we define

\[
\{ F, G \}^* := \{ F, G \} \Omega^{-1}_{ij} \{ \chi_i, G \},
\]

where \( \chi_i \) denotes the second-class constraints \( \chi_i = (C_\mu, C_\mu) \) and \( \Omega^{-1}_{ij} \) is the inverse of the matrix \( \Omega_{ij} \) such that \( \Omega^{-1}_{ij} \Omega^j_k = \delta_{ik} \). As a result we have a reduced phase-space Hamiltonian description where we must replace the Poisson bracket (34) by the Dirac bracket (48) in order that the second-class constraints hold strongly.

It is worthwhile commenting on these results in connection with the inverse problem of Lagrangian mechanics. The representation (47) for the non-singular matrix \( \Omega_{ij} \) is consistent with the results found in [39, 40] where \( \Omega_{ij} \) takes the form of a non-singular matrix which makes the equations of motion (7) derivable from a variational principle.
Additionally, for a matrix $\Omega_{ij}$ of type (17), they gives rise to three conditions on $M_{\mu\nu}$ and $X_{\mu\nu}$, namely the Helmholtz integrability conditions given by equations (28b), (42) and (B.2), respectively.

5.1. The singular case

A wide range of interesting physical systems does not have a regular matrix $M_{\mu\nu}$. For a singular matrix $M_{\mu\nu}$ of rank $R_M$ there exist $n = N - R_M$ independent left (or right) zero-modes eigenvectors $\xi^\mu_{(n)}(x^\nu, \dot{x}^\nu)$, satisfying

$$\xi^\mu_{(n)} M_{\mu\nu} = 0 \quad \text{or} \quad M_{\mu\nu} \xi^\nu_{(n)} = 0.$$  

(49)

This condition clearly imposes certain restrictions on the previously introduced arbitrary function $V(x^\mu, \dot{x}^\mu)$ and the momenta $p_\mu$. Indeed, from equations of motion (7) we have the Lagrangian constraints

$$\varphi_{(n)}(x^\mu, \dot{x}^\mu) := \xi^\mu_{(n)} F_\mu = \xi^\mu_{(n)} \left( \frac{\partial p_\mu}{\partial x^\nu} \dot{x}^\nu - \frac{\partial V}{\partial x^\mu} \right) = 0,$$  

(50)

for each independent left zero-mode $\xi^\mu_{(n)}$. On the contrary, if these conditions are not satisfied then the action (1) does not provide a consistent theory. So, in the case of a singular matrix $M_{\mu\nu}$ and a consistent action (1), the general solutions to the equations of motion involve arbitrary time-dependent functions, as expected for gauge systems.

6. Examples

We turn now to consider some explicit examples of applications of the approach developed above.

6.1. Chiral oscillator

Consider the 2-dimensional non-relativistic oscillator with a Chern-Simons-like term described by the Lagrangian [41, 42, 43, 44]

$$L(x^\mu, \dot{x}^\mu, \ddot{x}^\mu) = -\frac{\lambda}{2} \epsilon_{\mu\nu} \ddot{x}^\mu \ddot{x}^\nu + \frac{m}{2} \ddot{x}^\mu \ddot{x}^\mu,$$  

(51)

where $\lambda$ and $m$ are non-vanishing constants, $x^\mu = (x, y)$ and $\epsilon_{\mu\nu}$ is the Levi-Civita symbol such that $\epsilon_{12} = 1$ ($\mu, \nu = 1, 2$). For this case we immediately identify

$$K_\mu = \frac{\lambda}{2} \epsilon_{\mu\nu} \dot{x}^\nu = \frac{\lambda}{2} (\dot{y}, -\dot{x}),$$  

$$V = \frac{m}{2} \dot{x}_\mu \dot{x}^\mu.$$  

(52)

(53)

Notice that both $K_\mu$ and $V$ only depend on the velocities. Condition (16) is satisfied by (52) and, in consequence it is not possible to identify a surface term. It is crucial to note that this is an effect of the fact that condition (8) is not followed by this system

$$\frac{\partial K_\mu}{\partial \ddot{x}^\nu} \neq \frac{\partial K_\nu}{\partial \ddot{x}^\mu},$$  

(54)
which is a signal that equations of motion will not be of second-order. In fact, the form of the equations of motion (7) is not valid anymore for this case. Indeed, from (5) we obtain
\[ \lambda \epsilon_{\mu
u} \ddot{x}^\nu - m \dddot{x}_\mu = 0. \] (55)
Despite the third-order of these equations, it is possible to reduce this system of equations of motion to a soluble second-order one.

From (11a) and (11b) the Ostrogradski-Hamilton momenta associated to (51) are
\[ P_\mu = \lambda \epsilon^{\mu\nu} \dot{x}_\nu, \] (56)
\[ p_\mu = m \ddot{x}_\mu - \lambda \epsilon_{\mu\nu} \dddot{x}_\nu. \] (57)
The canonical Hamiltonian reads
\[ H_0 = p_\mu \dot{x}_\mu - \frac{m}{2} \dot{x}_\mu \dot{x}_\mu. \] (58)
The two primary constraints are given by \( C_\mu = P_\mu - \frac{\lambda}{2} \epsilon_{\mu\nu} \dot{x}_\nu \approx 0 \) and, in consequence the total Hamiltonian is
\[ H = p_\mu \dot{x}_\mu - \frac{m}{2} \dot{x}_\mu \dot{x}_\mu + u^\mu \left( P_\mu - \frac{\lambda}{2} \epsilon_{\mu\nu} \dot{x}_\nu \right), \] (59)
where \( u^\mu \) are Lagrange multipliers enforcing the primary constraints. The Poisson brackets among the primary constraints results
\[ \{ C_\mu, C_\nu \} = \lambda \epsilon_{\nu\mu}. \] (60)
Evolution in time of \( C_\mu \) leads only to a restriction of \( u^\mu \)
\[ \dot{C}_\mu = \{ C_\mu, H \} \approx -p_\mu + m \ddot{x}_\mu + \lambda \epsilon_{\nu\mu} u^\nu = 0, \] (61)
and in consequence \( C_\mu \) are second-class constraints. The number of degrees of freedom is readily obtained: dof = \( [8 - 2]/2 = 3 \). See [41] for a detailed quantum description for this model.

6.2. Geometric dynamics inferred from a second-order form

Consider the so-called second-order general energy Lagrangian given by [4, 5]
\[ L(x^\mu, \dot{x}^\mu, \ddot{x}^\mu) = \omega_\mu(x^\alpha) \dot{x}^\mu + \omega_{\mu\nu}(x^\alpha) \dot{x}^\mu \dot{x}^\nu, \] (62)
where the terms \( \omega_\mu \) are considered as potentials on a Riemannian manifold \((\mathbb{R}^n, \delta_{\mu\nu})\) where \( \delta_{\mu\nu} \) is the metric and, the other terms satisfy the property \( \omega_{\mu\nu} = \omega_{\nu\mu} \). These coefficients \((\omega_\mu, \omega_{\mu\nu})\) completely characterize the second-order form \( \omega = \omega_\mu(x^\alpha) d^2x^\mu + \omega_{\mu\nu}(x^\alpha) dx^\mu \otimes dx^\nu \). The geometric properties of this Lagrangian have been extensively studied in a huge variety of contexts ranging from biomathematics, mathematical economics, to a non-standard version of electrodynamics [4, 5, 15]. From our point of view, this Lagrangian is interesting as the inherent equations of motion are straightforwardly related to the geodesic equation, as we will see below.
The tensor
\[ N_{\mu\nu} := \frac{1}{2}(\partial_\nu \omega_\mu + \partial_\mu \omega_\nu), \] (63)
which is known as the deformation rate tensor field, allows us to introduce a new metric
\[ g_{\mu\nu} := \omega_{\mu\nu} - N_{\mu\nu}, \] (64)
where we assume that \( \det(g_{\mu\nu}) \neq 0 \). We identify immediately that, in our notation, \( K_\mu = \omega_\mu \) and \( V = \omega_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \). Note further that (8) is trivially satisfied. From (11a) and (11b) we have
\[ P_\mu = \omega_\mu, \] (65)
\[ p_\mu = (2\omega_{\mu\nu} - \partial_\nu \omega_\mu) \dot{x}^\nu = (2g_{\mu\nu} + \partial_\mu \omega_\nu) \dot{x}^\nu. \] (66)

The mass-like matrix (9) becomes \( M_{\mu\nu} = -2g_{\mu\nu} \). Similarly, the force vector is
\[ F_\mu = 2(\omega_{\alpha\beta\mu} - \Omega_{\alpha\beta\mu}) \dot{x}^\alpha \dot{x}^\beta, \] (67)
where we have introduced the connection symbols associated to \( \omega_{\mu\nu} \)
\[ \omega_{\alpha\beta\mu} := \frac{1}{2}(\partial_\alpha \omega_{\beta\mu} + \partial_\beta \omega_{\alpha\mu} - \partial_\mu \omega_{\alpha\beta}), \] (68)
\[ \Omega_{\mu\alpha\beta} := \frac{1}{2}(\partial_\alpha \partial_\beta \omega_\mu + \partial_\mu \partial_\beta \omega_\alpha - \partial_\mu \partial_\alpha \omega_\beta). \] (69)

Thus, from (7), the equations of motion read
\[ g_{\mu\alpha} \ddot{x}^\alpha + \Gamma_{\alpha\beta\mu} \dot{x}^\alpha \dot{x}^\beta = 0, \] (70)
where \( \Gamma_{\alpha\beta\mu} := \omega_{\alpha\beta\mu} - \Omega_{\alpha\beta\mu} \). Technically, as discussed in [5], we have obtained the dynamics of the geodesics governed by an Otsuki type connection \( \Gamma_{\alpha\beta\mu} \) [46].

The canonical Hamiltonian is
\[ H_0 = p_\mu \dot{x}^\mu - \omega_{\mu\nu} \dot{x}^\mu \dot{x}^\nu, \] (71)
and we have \( N \) primary constraints
\[ C_\mu = P_\mu - \omega_\mu \approx 0, \] (72)
which are in involution in a strong sense, \( \{C_\mu, C_\nu\} = 0 \). The total Hamiltonian is \( H = H_0 + u^\mu C_\mu \). By requiring that the constraints \( C_\mu \approx 0 \) hold in time, we generate the \( N \) secondary constraints
\[ C_\mu = p_\mu - (2\omega_{\mu\nu} - \partial_\nu \omega_\mu) \dot{x}^\nu \approx 0. \] (73)

The Poisson brackets among the primary and secondary constraints read
\[ \{C_\mu, C_\nu\} = -2g_{\mu\nu}. \] (74)
As discussed before, no further constraints exist as evolution in time of the secondary constraints \( C_\mu \approx 0 \) leads to an expression that determines the Lagrange multipliers
\[ 2g_{\mu\nu}(\ddot{x}^\mu - u^\mu) \approx 0. \] (75)
Indeed, \( u^\mu = \ddot{x}^\mu \), in agreement with (43). In addition, we have
\[ \{C_\mu, C_\nu\} = X_{\mu\nu} = 2(\partial_\mu g_{\nu\rho} - \partial_\nu g_{\mu\rho}) \dot{x}^\rho. \] (76)
Lagrangians affine in acceleration

Hence, from Eq. (47) we have

\[
(\Omega_{ij}) = \begin{pmatrix} 0 & 2g_{ij} \\ -2g_{ij} & 2g_{ij} \end{pmatrix},
\]

and its inverse matrix

\[
(\Omega^{-1}_{ij}) = \frac{1}{4} \begin{pmatrix} X_{\alpha\beta}g^{\alpha\mu}g^{\beta\nu} - 2g^{\mu\nu} \\ 2g^{\mu\nu} \\ 0 \end{pmatrix},
\]

where \(g^{\mu\nu}\) stands for the inverse matrix associated to the metric \(g_{\mu\nu}\). It follows that the corresponding Dirac brackets become

\[
\{ F, G \}^* = \{ F, G \} - \frac{1}{4} X_{\alpha\beta}g^{\alpha\mu}g^{\beta\nu} \{ F, C_\mu \} \{ C_\nu, G \} + \frac{1}{2} g^{\mu\nu} (\{ F, C_\mu \} \{ C_\nu, G \} - \{ F, C_\mu \} \{ C_\nu, G \}).
\]

(79)

7. Covariant theories

As it is well known, gauge theories invariant under reparametrizations may be interpreted exceedingly well in a purely geometric fashion. One of the most important consequences is that the Hamiltonian vanishes identically, thus we confront a non-dynamical system [36, 37, 38]. From a geometric point of view, the vanishing of the Hamiltonian is a consequence of Zermelo conditions (see [21, 22, 23] and Appendix A) which state the invariance of the Lagrangian under the Lie derivative of the Liouville vector fields associated to a second-order Lagrangian [47], that is, \(\mathcal{L}_{\Gamma^{(i)}}(L) = 0\) for any Liouville vector field \(\Gamma^{(i)}\). In other words, Zermelo conditions are the necessary conditions for an action integral (in our case (1)) to be invariant under parametrization of the curve \(c\) (see Appendix A for further details). From a physical point of view, for our case, the action integral (1) together with (2), must be a Lorentz invariant as well as a homogeneous function in the generalized velocities. Indeed, from equation (2) the associated Legendre transformation (see also equation (32)) yields

\[
H_0 = \frac{\partial L}{\partial \dot{x}^\mu} \dot{x}^\mu + \left[ \frac{\partial L}{\partial \ddot{x}^\mu} - \frac{d}{d\tau} \left( \frac{\partial L}{\partial \dot{x}^\mu} \right) \right] \ddot{x}^\mu - L,
\]

\[
= \frac{\partial V}{\partial \dot{x}^\mu} \dot{x}^\mu - \frac{\partial K_\mu}{\partial x^\nu} \dot{x}^\nu \dot{x}^\mu - V,
\]

\[
= \dot{x}^\mu \frac{\partial V}{\partial x^\mu} - V - \dot{x}^\nu \frac{\partial}{\partial \dot{x}^\nu} (\dot{x}^\mu K_\mu).
\]

(80)

Note that the Hamiltonian \(H_0\) is equivalent to the energy \(\mathcal{E}_c^{(2)}\) introduced in relation (13b), which, together with the energy \(\mathcal{E}_c^{(1)}\), vanishes whenever Zermelo conditions are considered. Zermelo conditions (A.9) may be interpreted in our notation as the condition that the function \(V\) has to be a homogeneous function of first degree in the velocities, while the function \(K_\mu\) has to be orthogonal to the velocity vector. Under these Zermelo conditions, our canonical Hamiltonian vanishes accordingly as shown in (A.11).
A different but equivalent viewpoint shows also the vanishing of the canonical Hamiltonian $H_0$ in terms of the functions $f$ and $h$ introduced in (18). Indeed, from equation (21) we see that the function $f(x^\mu, \dot{x}^\mu)$ must be also homogeneous of first degree in the velocities, while it is required that the function $h(\dot{x}^\mu)$ be homogeneous of order zero, that is,

$$\dot{x}^\mu \frac{\partial h}{\partial \dot{x}^\mu} = 0. \quad (81)$$

By virtue of $\dot{x}^\mu \partial f / \partial \dot{x}^\mu = f$, we have that $\dot{x}^\nu (\partial^2 f / \partial \dot{x}^\mu \partial \dot{x}^\nu) = -\dot{x}^\nu M_{\mu\nu} = 0$ and in consequence the velocity vector results to be one of the zero-modes of $M_{\mu\nu}$. This last fact is important since if $M_{\mu\nu}$ is singular and if the rows (or columns) are proportional to each other, hence $M_{\mu\nu}$ can be written as the direct product of two vectors. For our case,

$$M_{\mu\nu} = l(x^\alpha, \dot{x}^\alpha) n_\mu n_\nu, \quad (82)$$

where $n_\mu$ is a unit spacelike vector such that $n_\mu \dot{x}^\mu = 0$ and $n_\mu n^\mu = 1$, and $l(x^\alpha, \dot{x}^\alpha)$ is a function of the configuration space.

In the light of this geometrical interpretation, we can see directly from equations (23a) and (25) that contraction of the momenta $P_\mu$ and $p_\mu$ with the velocity vector leads to the vanishing of $P_\mu \dot{x}^\mu$ as well as to the vanishing of the canonical Hamiltonian $H_0$, as given by (80) and (32). This, of course, is a direct result from Zermelo’s conditions (A.9). We also comment that these important relations are classified as first-class constraints in the Dirac approach for constrained systems (see for example, Theorem 1.3 in [37]). As a byproduct, from Eq. (81), it follows that $\partial h / \partial \dot{x}^\mu$ is proportional to a normal vector, $n^\mu$, satisfying also $n_\mu \dot{x}^\mu = 0$,

$$\frac{\partial h}{\partial \dot{x}^\mu} = m(\dot{x}^\nu) n_\mu, \quad (83)$$

where $m$ is function depending on the velocities. This fact is also useful in the search of more constraints. All these features comprise the hallmark of reparametrization invariant systems such as General Relativity and brane theories [10, 11, 16, 48].

By the way, the fact that $H_0$ appears as a secondary constraint is inferred from the beginning in equation (50) by considering $\xi^\mu = \dot{x}^\mu$. Clearly,

$$\dot{x}^\mu F_\mu = \dot{x}^\mu \frac{\partial}{\partial x^\mu} (p_\nu \dot{x}^\nu - V),$$

$$= \frac{d}{d\tau} (p_\mu \dot{x}^\mu - V) - \frac{\partial (p_\nu \dot{x}^\nu - V)}{\partial \dot{x}^\mu} \ddot{x}^\mu.$$ 

By using equations (9), (11a) and (11b) we may easily verify the identity $\partial (p_\nu \dot{x}^\nu - V) / \partial \dot{x}^\mu = -M_{\mu\nu} \dot{x}^\nu = 0$, which simplifies the last equation to

$$\dot{x}^\mu F_\mu = \frac{d}{d\tau} (p_\mu \dot{x}^\mu - V). \quad (84)$$

Thus, we have outlined the constrained scheme for the Lagrangian (2) which will be confirmed by means of Dirac formalism for constrained systems below.
7.1. Covariant brane theories

In the representation provided by (17), under certain conditions, the whole set of second-class constraints $\chi_i$ could involve a hidden sector of first-class constraints. Certainly, for covariant theories the matrix $M_{\mu\nu}$ is singular and we can uncover these constraints as follows. From equation (19) we infer that

$$f^1_{(n)} := \xi^\mu_{(n)} C_\mu,$$

are $n$ first-class constraints. Indeed, by contracting equations (35) and (38) on the right with $\xi^\mu_{(n)}$, we have $\{C_\mu, f^1_{(n)}\} \approx 0$ and $\{C_\mu, f^1_{(n)}\} \approx 0$. We note that these identities hold off-shell. Note further that still we are left with $R_M$ second-class constraints.

Likewise, we can uncover another set of first-class constraints. To find this, we first project (38) on the left by $\xi^\mu_{(n)}$ and we obtain that $\{f^2_{(n)}, C_\mu\} \approx 0$ where

$$f^2_{(n)} := \xi^\mu_{(n)} C_\mu.$$

We deduce similarly from (49), (38) and (35) that $\{f^1_{(n)}, f^2_{(n)}\} \approx 0$ where $n, n' = 1, 2, \ldots, N - R_M$.

With respect to the second-class constraints, for this specific case and adopting a geometric viewpoint, we need to find the whole set of orthogonal vectors to $\xi^\mu_{(n)}$, say $n^\mu_{(s)}$, and then contract $C_\mu$ and $C_\mu$ with $n^\mu_{(s)}$, where $s$ keeps track of the number of orthogonal vectors to $\xi^\mu_{(n)}$. To prove this, we rely on a geometrical identity that relates the complete orthogonal basis vector [8]. Subsequently, we rewrite the primary and secondary constraints by using the metric in order to expand these constraints in terms of the orthogonal basis, i.e., $C_\mu = g^{\mu\nu} C_\nu = H^{\mu\nu} C_\nu + \perp^{\mu\nu} C_\nu$. From here, by linear independence, we can identify an equivalent set of constraints to $C_\mu$ and $C_\mu$. We can check by straightforward computation that the resulting expressions $C_\mu n^\mu_{(s)} = 0$ and $C_\mu n^\mu_{(s)} = 0$ are second-class constraints. Hence, the vectors $\xi^\mu_{(n)}$ span a basis for the first-class constraint surfaces in the phase space whereas $n^\mu_{(s)}$ span a basis for the second-class constraint surfaces.

As we have noted for covariant theories, it is convenient to deal with the primary and secondary constraints transformed, instead of their original form, by projecting them along suitable independent vectors $Z^\mu_{(s)} = Z^\mu_{(s)} (x^\nu, \dot{x}^\nu)$ in order to have the constraints [34]

$$P_\mu Z^\mu_{(s)} = A_{(s)} (x^\mu, \dot{x}^\mu), \quad (88)$$

$$p_\mu Z^\mu_{(s)} = B_{(s)} (x^\mu, \dot{x}^\mu), \quad (89)$$

where $r$ denotes the dimension of the orthonormal basis.

§ In any $N$-dimensional manifold with a non-degenerate metric $g_{\mu\nu}$ that is used to lower and raise indexes, for covariant theories defined in an immersed $(p + 1)$-dimensional surface, at any point of the surface the following decomposition holds

$$g^{\mu\nu} = H^{\mu\nu} + \perp^{\mu\nu}, \quad (87)$$

where $H^{\mu\nu} = g^{\mu\nu} e^a_{\mu} e^\nu_{b}$ is the projection tensor of rank $(p + 1)$ on the surface, $g_{ab}$ is the induced metric, and $e^\mu_{a}$ denotes the tangent vectors to the surface. Further, $\perp^{\mu\nu} = n^\mu_{(s)} n^\nu_{(s)}$ is the complementary projector tensor of rank $(N - p - 1)$, orthogonal to the surface.
Thus, we have changed to an equivalent set of constraints
\[ C_\mu \rightarrow \phi_\mu = A_\mu^{\nu^\prime}C_\nu, \]  
\[ C_\mu \rightarrow \varphi_\mu = B_\mu^{\nu^\prime}C_\nu, \tag{90a} \]

with \( \phi_\mu = (f_1^{(n)}, x_1^{(s)}) \) and \( \varphi_\mu = (f_2^{(n)}, x_2^{(s)}) \).

### 7.2. Electrically charged bubble

As another illustration we consider now a relativistic bubble in the presence of an electromagnetic field with a total electric charge \( q \) on the shell \([11, 49]\). The Lagrangian is given by
\[ L(\dot{t}, \dot{r}, r, \ddot{t}, \ddot{r}) = -\alpha r^2 N^2 (\dot{r} \ddot{t} - \dot{t} \ddot{r}) - 2\alpha \dot{r} \dot{t} - \beta \frac{q^2 \dot{t}}{r}, \tag{91} \]

where \( \alpha \) and \( \beta \) are constants. Here, \( N = \sqrt{\dot{t}^2 - \dot{r}^2} \). For this case \( \mu, \nu = 1, 2 = t, r \). We recognize from (91)
\[ K_1 = K_t = \alpha \frac{r^2 \dot{r}}{N^2}, \tag{92} \]
\[ K_2 = K_r = -\alpha \frac{r^2 \dot{t}}{N^2}, \tag{93} \]
\[ V(t, r, \dot{r}) = -2\alpha \dot{r} - \beta \frac{q^2 \dot{r}}{r}. \tag{94} \]

As in the previous case, from condition (16) we obtain that \( d(\dot{r} \dot{t})/d\tau = 0 \) which shows an inconsistency and thus it is possible to identify a surface term \([11]\).

For this case, from equation (19) we observe that \( g(r) = \alpha r^2 \) and \( \partial h(\dot{t}, \dot{r})/\partial \dot{t} = \dot{r}/N^2 \) and \( \partial h(\dot{t}, \dot{r})/\partial \dot{r} = -\dot{t}/N^2 \). Thus, integrating we have up to a constant, \( h(\dot{t}, \dot{r}) = -\tanh^{-1}(\dot{r}/\dot{t}) \). In summary, we have
\[ g(r) = \alpha r^2 \quad \text{and} \quad \partial h/\partial \dot{x}^\mu = -\frac{1}{N}n_\mu, \tag{95} \]

(see notation below). From Eq. (20) we have now \( f(r, \dot{t}, \dot{r}) = 2\alpha \dot{r} \tanh^{-1}(\dot{r}/\dot{t}) - 2\alpha \dot{t} - \beta \frac{q^2 \dot{r}}{r} \). Finally, we can obtain the form of the Lagrangian associated to the total time derivative (22)
\[ L_s = \frac{d}{d\tau} \left[ -\alpha r^2 \tanh^{-1} \left( \frac{\dot{r}}{\dot{t}} \right) \right], \tag{96} \]

which is in agreement with the results found in \([11]\).

From Eqs. (11a) and (11b) we have the momenta associated to this theory
\[ P_1 = P_t = \alpha \frac{r^2 \dot{r}}{N^2}, \tag{97} \]
\[ P_2 = P_r = -\alpha \frac{r^2 \dot{t}}{N^2}, \tag{98} \]
\[ p_1 = p_t = -\frac{2\alpha \dot{r}^2}{N^2} - \beta \frac{q^2}{r} =: -\Omega, \tag{99} \]
\[ p_2 = p_r = \frac{2\alpha \dot{r}^2}{N^2}, \tag{100} \]
where $\Omega$ is the conserved bulk energy. Then, by considering these momenta in (9) and (40) it is found that

\[
(M_{\mu \nu}) = -\frac{4\alpha r \dot{t}}{N^4} \begin{pmatrix} \dot{r}^2 & -\dot{r} \dot{t} \\ -\dot{r} \dot{t} & \dot{t}^2 \end{pmatrix},
\]

\[
(X_{\mu \nu}) = \begin{pmatrix} 0 & \frac{2\alpha r^2}{N^2} - \frac{\beta q^2}{r^2} \\ -\frac{2\alpha r^2}{N^2} + \frac{\beta q^2}{r^2} & 0 \end{pmatrix}.
\] (101)

Note that $(M_{\mu \nu})$ is singular. In consequence, we have a left (right) zero-mode given by $\xi^\mu = (\dot{t}, \dot{r})$. Hence, from (31), (97) and (98) we have a first-class constraint

\[
f_1 = \xi^\mu C_\mu = P_\mu \dot{x}^\mu = P_t \dot{t} + P_r \dot{r} \approx 0.
\] (102)

Similarly, by contracting (37) and considering (99) and (100) we have another first-class constraint

\[
f_2 = \xi^\mu C_\mu = P_\mu \dot{x}^\mu = \left(2\alpha r + \beta q^2 \right) \dot{t} = p_t \dot{t} + p_r \dot{r} + \left(2\alpha r + \beta q^2 \right) \dot{t} = 0.
\] (103)

An orthogonal vector to $\xi^\mu = \dot{x}^\mu$, under a Minkowski metric, is provided by the unit spacelike vector $n^\mu = \frac{1}{N}(\dot{r}, \dot{t})$. In consequence, the matrix $M_{\mu \nu}$ in (101) may be expressed as

\[
M_{\mu \nu} = -\frac{4\alpha r \dot{t}}{N^2} n_\mu n_\nu,
\] (104)

which is in agreement with equation (82).

Hence, by contracting (31) and (37) along $n^\mu$, and considering equations (97) to (100), we find the second-class constraints

\[
s_1 = NP_\mu n^\mu + \alpha r^2 = P_t \dot{r} + P_r \dot{t} + \alpha r^2 \approx 0,
\] (105)

\[
s_2 = NP_\mu n^\mu + \beta \frac{q^2}{r} = p_t \dot{r} + p_r \dot{t} + \beta \frac{q^2}{r} \approx 0.
\] (106)

All these results are in complete agreement with the results found in [11, see equations (28), (29), (34) and (35) there]. We also note that if we calculate either the Poisson bracket or the Dirac bracket between the first-class constraints, $f_1$ and $f_2$, we explicitly get $\{f_1, f_2\} = -f_2$ which resembles a truncated Virasoro algebra of the form $\{L_m, L_n\} = (m-n)L_{m+n}, \ (m = 0, n = 1)$ by redefining the constraints as $L_0 := f_1$ and $L_1 := f_2$. We must emphasize here that this algebra was obtained straightforwardly in our formulation, without any appeal to auxiliary variables as in [14].

The canonical Hamiltonian is

\[
H_0 = p_t \dot{t} + p_r \dot{r} + \left(2\alpha r + \beta \frac{q^2}{r} \right) \dot{t},
\] (107)

and the total Hamiltonian is given by

\[
H = H_0 + u^t \left( P_t - \alpha \frac{r^2 \dot{r}}{N^2} \right) + u^r \left( P_r + \alpha \frac{r^2 \dot{t}}{N^2} \right).
\] (108)

where $u^t$ and $u^r$ are Lagrange multipliers enforcing the primary constraints (97) and (98), respectively.
8. Concluding remarks

We have presented the Hamiltonian formulation for Lagrangians affine in acceleration. Our presentation was strongly based on the geometric analysis of the quantities involved and, in particular, on the conditions under which our Lagrangian accepted a decomposition into a true dynamic term plus a surface term. We highlighted the role that the surface term played in our formulation, and the existence of surface equivalent Lagrangians. The main interest for the study of these systems has been motivated by certain brane models where the existing canonical approaches are not entirely suitable, although several other examples may be found in the literature. In this sense, we emphasize that most of the available examples enclose regular Lagrangians, for which the Hamiltonian is developed by the introduction of auxiliary variables resulting, from our point of view, in a cumbersome description hiding the true geometric interpretation available for these sort of systems. Our claim is that the description developed in this work, allowed us, in a very natural way, to obtain a Hamiltonian version for which the geometric invariants are explicitly written. Our geometric formulation allowed us to straightforwardly incorporate our results to the analysis of either non-regular or covariant Lagrangians. As stated in the Introduction, we argue that our formulation paves the way for the quantisation of this sort of systems, at least within canonical schemes. We will developed quantum aspects for theories with Lagrangians affine in acceleration somewhere else.

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Appendix A. Nöether theorem and energies of the system

This section follows, as close as possible, the notation in references [22, 23, 24]. We start this section by introducing a differentiable vector field $W^\mu$ along the curve $c$ at which the action (1) takes place. We impose that the vector field $W^\mu$ is (at least locally) regular and satisfies the condition that the vector and its first derivatives vanish at the end-points of the action integral.

Associated to the field vector $W^\mu$ we may define the linearly-independent operators

\begin{align}
\frac{dW}{d\tau} &:= W^\mu \frac{\partial}{\partial x^\mu} + dW^\mu \frac{\partial}{d\tau} \frac{\partial}{\partial \dot{x}^\mu} + \frac{d^2W^\mu}{d\tau^2} \frac{\partial}{\partial \ddot{x}^\mu}, \\
I^{(1)}_W &:= 2W^\mu \frac{\partial}{\partial \ddot{x}^\mu}, \\
I^{(2)}_W &:= W^\mu \frac{\partial}{\partial \ddot{x}^\mu} + 2\frac{dW^\mu}{d\tau} \frac{\partial}{\partial \ddot{x}^\mu}.
\end{align}
In particular, operator (A.1) is called the total derivative in the direction of the vector field $W^\mu$. The relevance of these operators mainly relies on the fact that for any differentiable Lagrangian, $L(x, \dot{x}, \ddot{x})$, $dW/d\tau$, $I^{(1)}_W(L)$ and $I^{(2)}_W(L)$ become real scalar fields, that is, they are invariant under local coordinate transformations on $T^2M$. It is straightforward to show that these operators are related to the Euler-Lagrange operator $E^{(0)}_\mu$ defined in (6), through the identity

$$
\frac{dW}{d\tau} - \frac{d}{d\tau} \left( I^{(2)}_W - \frac{1}{2} \frac{d}{d\tau} I^{(1)}_W \right) = W^\mu E^{(0)}_\mu.
$$

(A.4)

Indeed, this relation states the projection of the Euler-Lagrange operator in the direction of $W^\mu$. It is also crucial to note that, for the particular case $W^\mu = dx^\mu/d\tau$, the operator (A.1) becomes the total derivative with respect to the parameter $\tau$, while (A.2) and (A.3) become identical to the Lie derivatives along the flows of the Liouville vector fields

$$
\Gamma^{(1)} := 2i^\mu \frac{\partial}{\partial \dot{x}^\mu},
$$

(A.5)

$$
\Gamma^{(2)} := i^\mu \frac{\partial}{\partial x^\mu} + 2i^\mu \frac{\partial}{\partial \ddot{x}^\mu},
$$

(A.6)

that is, $I^{(1)}_\dot{x} = \mathcal{L}_{\Gamma^{(1)}}$ and $I^{(2)}_\dot{x} = \mathcal{L}_{\Gamma^{(2)}}$. For any differential Lagrangian $I^{(1)}_\dot{x}(L)$ and $I^{(2)}_\dot{x}(L)$ are called the main invariants and, for the specific choice of our Lagrangian (2), these become

$$
I^{(1)}_\dot{x}(L) = 2x^\mu K^\mu,
$$

(A.7)

$$
I^{(2)}_\dot{x}(L) = \left( \frac{\partial K^\mu}{\partial \dot{x}^\nu} \dot{x}^\nu + 2K^\mu \right) \ddot{x}^\mu + \frac{\partial V}{\partial \ddot{x}^\mu} \ddot{x}^\mu.
$$

(A.8)

Both invariants (A.7) and (A.8) serve to establish conservation theorems, as we will see below. Also, from these last relations we may deduce Zermelo conditions [21], which state the necessary conditions for an action integral to be independent of the parametrization of the curve $c$, namely,

$$
I^{(1)}_\dot{x}(L) = 0,
$$

$$
I^{(2)}_\dot{x}(L) = L.
$$

(A.9)

Indeed, the first of Zermelo conditions (A.9) stands for the invariance of the Lagrangian along the vector field $I^{(1)}_\dot{x}$, while the second sets $I^{(2)}_\dot{x}$ as a genuine Liouville vector field when applied to the Lagrangian function. In our notation, Zermelo conditions may be explicitly obtained by combining equations (A.7) to (A.9).

Now, the energies for a given second-order Lagrangian, $L$, are given in terms of the main invariants as

$$
\mathcal{E}^{(1)}_c(L) := -\frac{1}{2} I^{(1)}_\dot{x}(L),
$$

(A.10)

$$
\mathcal{E}^{(2)}_c(L) := I^{(2)}_\dot{x}(L) - \frac{1}{2} \frac{d}{d\tau} I^{(1)}_\dot{x}(L) - L,
$$

(A.11)
which for our Lagrangian \([2]\) correspond to equations \([13a]\) and \([13b]\), respectively. Conservation of these energies is dictated by the relations

\[
\frac{dE_c^{(1)}(L)}{d\tau} + \frac{1}{2} I_c^{(2)}(L) = -\frac{1}{2} \frac{dx^\mu}{d\tau} E_\mu^{(1)}(L),
\]

\[
\frac{dE_c^{(2)}(L)}{d\tau} = -\frac{dx^\mu}{d\tau} E_\mu^{(0)}(L).
\]

The first of these identities may be obtained straightforwardly, while the second is a consequence of identity \([A.4]\). Here, we used the \(\mathbb{R}\)-linear covector fields \(E_\mu^{(0)}\) (defined in \([6]\)), \(E_\mu^{(1)} := -\partial/\partial \dot{x}^\mu + 2d/d\tau(\partial/\partial \ddot{x}^\mu)\) which, together with \(E_\mu^{(2)} := \partial/\partial \ddot{x}^\mu\), are the so-called Craig-Synge covectors associated to a differentiable second-order Lagrangian \([50, 51]\). In this way, we see in particular that \(E_c^{(2)}\), which is related to the canonical Hamiltonian \(H_0\) (see \([32]\) or \([80]\)), is conserved only along the solution curve to Euler-Lagrange equations \(E_\mu^{(0)}(L) = 0\). Furthermore, we see from equations \([A.10]\) and \([A.11]\) that these energies are identically vanishing whenever Zermelo conditions \([A.9]\) are considered, that is, for covariant systems.

Finally, in order to study the behaviour of the function \([14]\), we choose two points, \((x, \tau)\) and \((x', \tau')\) belonging to the same domain of a local chart \(U \times (a, b) \subset M \times \mathbb{R}\). These points are connected through an infinitesimal transformation of the form \([15]\), where \(\epsilon \in \mathbb{R}\) is a sufficiently small positive number, and \(\eta := \eta(x, \tau)\) is an arbitrary smooth function locally defined at the point \((x, \tau)\). Then, it is direct to show that the infinitesimal transformation \([13]\) is a local symmetry of the Lagrangian \(L(x, \dot{x}, \ddot{x})\) if and only if for any \(C^\infty\)-function \(F(x, \dot{x})\) the following equation holds

\[
L \left( x', \frac{dx'}{d\tau}, \frac{d^2 x'}{d\tau^2} \right) d\tau' = L \left( x, \frac{dx}{d\tau}, \frac{d^2 x}{d\tau^2} \right) + \frac{d}{d\tau} \left( F \left( x, \frac{dx}{d\tau} \right) \right) d\tau.
\]

From this last relation, we may Taylor expand the left hand side around unprimed coordinates, and keeping first order terms in \(\epsilon\), we may find, after some calculus, the identity

\[
\frac{dG(L, \phi)}{d\tau} = \left( W^\mu - \eta \frac{dx^\mu}{d\tau} \right) E_\mu^{(0)}(L),
\]

where we have defined the function

\[
G(L, \phi) := I_W^{(2)}(L) - \frac{1}{2} \frac{d}{d\tau} I_W^{(1)}(L) - \eta E_c^{(2)}(L) + \frac{d\eta}{d\tau} E_c^{(1)}(L) - \phi,
\]

and \(\phi\) stands for the first-order term in the \(\epsilon\)-expansion of the function \(F(x, \dot{x})\), that is, \(F(x, \dot{x}) = \epsilon \phi(x, \dot{x})\). From this, we are ready to establish the Nöether theorem, which state that, along the solution curves of Euler-Lagrange equations of motion \(E_\mu^{(0)}(L) = 0\), the function \(G(L, \phi)\) is conserved under evolution of the parameter \(\tau\). Note that the conserved function \(G(L, \phi)\) depends solely on the invariants \(I_W^{(1)}, I_W^{(2)}\) and the energies \(E_c^{(1)}, E_c^{(2)}\). Also, we must note that whenever Zermelo conditions \([A.9]\) hold, the conserved function \(G(L, \phi)\) is reduced to

\[
G(L, \phi) = I_W^{(2)}(L) - \frac{1}{2} \frac{d}{d\tau} I_W^{(1)}(L) - \phi,
\]
thus, we expect (A.17) to be conserved for covariant theories. The explicit form of the function $G(L, \phi)$ in our notation is given by equation (14).

**Appendix B. On Helmholtz conditions**

Concerning the integrability conditions for $M_{\mu\nu}$ and $K_{\mu}$, our starting point will be the matrix introduced in (11). Making use of the partial derivatives with respect to the coordinates $x^\rho$ and considering the skew-symmetric part with respect to the indices $\mu$ and $\rho$, respectively, we get

$$\frac{\partial M_{\mu\nu}}{\partial x^\rho} - \frac{\partial M_{\rho\nu}}{\partial x^\mu} = \frac{\partial^2 P_\nu}{\partial x^\rho \partial x^\mu} - \frac{\partial^2 P_\mu}{\partial x^\rho \partial x^\nu} + \frac{\partial}{\partial \dot{x}^\mu} \left( \frac{\partial p_\mu}{\partial x^\rho} - \frac{\partial p_\rho}{\partial x^\mu} \right).$$  \hspace{1cm} (B.1)

From equation (40) we thus have

$$\frac{\partial X_{\mu\nu}}{\partial \dot{x}^\rho} = \frac{\partial M_{\mu\nu}}{\partial x^\rho} - \frac{\partial M_{\rho\nu}}{\partial x^\mu}. \hspace{1cm} (B.2)$$

Now, by taking partial derivatives of the force term (10) with respect to $\dot{x}^\mu$, and taking the symmetric part with respect to the indices $\mu$ and $\nu$, we get

$$\frac{\partial F_{\mu}}{\partial \dot{x}^\nu} + \frac{\partial F_{\nu}}{\partial \dot{x}^\mu} = \dot{x}^\alpha \frac{\partial}{\partial x^\alpha} \left( \frac{\partial p_{\mu}}{\partial \dot{x}^\nu} + \frac{\partial p_{\nu}}{\partial \dot{x}^\mu} \right) + \frac{\partial}{\partial x^\nu} \left( p_{\mu} - \frac{\partial V}{\partial \dot{x}^\mu} \right) + \frac{\partial}{\partial x^\mu} \left( p_{\nu} - \frac{\partial V}{\partial \dot{x}^\nu} \right).$$  \hspace{1cm} (B.3)

Definition (11b) can now be used to express the last two terms on the right hand side of this equation in terms of the highest momenta $P_{\mu}$ as

$$\frac{\partial F_{\mu}}{\partial \dot{x}^\nu} + \frac{\partial F_{\nu}}{\partial \dot{x}^\mu} = \dot{x}^\alpha \frac{\partial}{\partial x^\alpha} \left( \frac{\partial p_{\mu}}{\partial \dot{x}^\nu} + \frac{\partial p_{\nu}}{\partial \dot{x}^\mu} \right) - \dot{x}^\alpha \frac{\partial}{\partial x^\alpha} \left( \frac{\partial P_{\mu}}{\partial x^\nu} + \frac{\partial P_{\nu}}{\partial x^\mu} \right).$$  \hspace{1cm} (B.4)

Now, by splitting the momenta $p_{\mu}$ in terms of $p_{\mu}$ and $\dot{p}_{\mu}$, and taking into account the identity (26) we obtain

$$\frac{\partial F_{\mu}}{\partial \dot{x}^\nu} + \frac{\partial F_{\nu}}{\partial \dot{x}^\mu} = \dot{x}^\alpha \frac{\partial}{\partial x^\alpha} \left( \frac{\partial p_{\mu}}{\partial \dot{x}^\nu} + \frac{\partial \dot{p}_{\mu}}{\partial \dot{x}^\mu} \right),$$  \hspace{1cm} (B.5)

whereby we will have, from equation (27), the condition

$$2 \dot{x}^\alpha \frac{\partial M_{\mu\nu}}{\partial x^\alpha} = - \left( \frac{\partial F_{\mu}}{\partial \dot{x}^\nu} + \frac{\partial F_{\nu}}{\partial \dot{x}^\mu} \right).$$ \hspace{1cm} (B.6)

Finally, proceeding in a similar fashion as for (B.6), we consider now partial derivatives of the force term (10) with respect to $x^\mu$, and considering the symmetric part with respect to indices $\mu$ and $\nu$, we get

$$\dot{x}^\alpha \frac{\partial X_{\mu\nu}}{\partial x^\alpha} = \frac{\partial F_{\mu}}{\partial x^\nu} + \frac{\partial F_{\nu}}{\partial x^\mu}.$$ \hspace{1cm} (B.7)

Equations (27), (28b), (B.6) and (B.7) are usually referred to as the Helmholtz integrability conditions satisfied for a non-singular matrix $M_{\mu\nu}$ and the vector $K_{\mu}$ in order to obtain equations of motion (7) from a variational principle.
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