Family size decomposition of genealogical trees

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Abstract

We study the path of family size decompositions of varying depth of genealogical trees. We prove that this decomposition as a function on (equivalence classes of) ultra-metric measure spaces to the Skorohod space describing the family sizes at different depths is perfect onto its image, i.e. there is a suitable topology such that this map is continuous closed surjective and pre-images of compact sets are compact. We also specify a (dense) subset so that the restriction of the function to this subspace is a homeomorphism. This property allows us to argue that the whole genealogy of a Fleming-Viot process with mutation and selection as well as the genealogy in a Feller branching population can be reconstructed by the genealogical distance of two randomly chosen individuals.

Keywords: Genealogical distance, (ultra-)metric measure spaces, mass coalescent, family size decomposition.

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1 Introduction

There are several approaches to study genealogical properties of a Wright-Fisher population. For example, one can use the Kingman coalescent \cite{Kin82} which naturally generates the genealogical tree of a neutral Moran model or a neutral Wright-Fisher population at a fixed time. When selection is present, things get harder. This is because in contrast to the neutral model, the tree of the current population now depends on the whole type evolution until the present time (see Theorem 2 in \cite{DGP12}). Nevertheless, one can use, for example, the ancestral selection graph introduced by Krone and Neuhauser (see \cite{KN97} and \cite{NK97}) or the lookdown construction introduced by Donelly and Kurtz (see \cite{DK96} and \cite{DK99}), to construct or read off genealogical properties. But still, it is quite hard to get explicit results on the genealogy.

Depperschmidt, Greven, Pfaffelhuber, Winter \cite{GPW09}, (compare also \cite{GPW13}; and \cite{DGP12} for a survey) followed a different approach. They constructed the genealogy dynamically as a tree-valued process (we explain the notion “tree” in more details below). This has the advantage, that one can use the generator of this Markov process to get, for example, recurrence relations of the genealogical distance of two randomly chosen individuals in equilibrium.

Here we try to connect the “classical” approach of coalescent models and the setting in Depperschmidt, Greven, Pfaffelhuber and Winter. Namely we ask for a quantity that both, (exchangeable) coalescents and (genealogical) trees, have in common.

Recall, that one can use the Kingman coalescent to construct the genealogy in the neutral case and that the law of the Kingman coalescent is determined by the law of its block frequencies via Kingman’s paint box construction (see for example \cite{Ber06}). The process associated with the evolution of the block frequencies is usually called mass coalescent. We will show that a tree naturally contains the concept of “mass coalescent”, which we call path of family size decompositions (or family size decomposition for simplicity) in this context. Our goal in this paper is to study properties of this decomposition and how to apply these results to (large) Wright-Fisher populations and Feller branching populations to gain information about genealogy.

To get a bit more precise (see Section 2 and Section 3 for all details), we consider the space $U$ of (equivalence classes of) ultra-metric measure spaces.
$(X, r, \mu)$, where we interpret $X$ as a set of individuals, $r$ as a genealogical distance and $\mu$ as a sampling measure. We assume that $(X, r)$ is complete and separable and note that one can map $(X, r)$ isometrically to the leaves of a rooted $\mathbb{R}$-tree (see Remark 2.7 in [DGP12] and Remark 2.2 in [DGP11]), which justifies the name tree for an ultra-metric measure space.

Now, we decompose $X$ into balls $\bar{B}_h(x) := \{ y \in X : r(x, y) \leq h \}$ for some $h > 0$ and note that $\bar{B}_h(x) \cap \bar{B}_h(y) = \emptyset$ for $r(x, y) > h$, since $r$ is an ultra-metric, and the number of balls needed to cover $X$ is countable, since $X$ separable. We can interpret $\bar{B}_h(x)$ as a family descending from an ancestor who lived at the time $h$ (measured backwards) (see Figure 1). When we now calculate the sizes, $\mu(\bar{B}_h(x))$, of the different families and denote the size ordered vector by $\mathbf{a}(h) := (a_1(h), a_2(h), \ldots)$, i.e. $a_k(h) \geq a_{k+1}(h)$ for all $k$, then we finally get the notion of family size decomposition of trees:

We call the function $\mathfrak{F} : U \to D((0, \infty), S^\downarrow)$ that maps an ultra-metric measure space to the (cadlag) function $h \mapsto \mathbf{a}(h)$ family size decomposition.

Here,

$$S^\downarrow := \left\{ (x_1, x_2, \ldots) \in [0, \infty)^\mathbb{N} : \sum_{i \in \mathbb{N}} x_i < \infty, \; x_1 \geq x_2 \geq \ldots \right\} \quad (1.1)$$

Figure 1: On the left side we draw an ultra-metric measure space $(X, r, \mu)$, where $|X| = 7$ and $\mu(\{x\}) = 1$ for all $x \in X$. We can decompose this tree into four disjoint (closed) balls of radius $h > 0$ (drawn on the right side).

Our first goal in this paper is to study properties of this map and we can summarize our results as follows:

*The function $\mathfrak{F}$ is continuous, closed and preimages of compact sets are compact, i.e. $\mathfrak{F}$ is perfect.*
Of course, we have to define suitable topologies on the respective spaces $U$ and $D((0, \infty), S^2)$, so that the result is valid. While we equip the space of cadlag functions $h \mapsto a(h)$ with the Skorohod topology, we need to specify the topology on $U$. Typically, one would equip $U$ with the so called Gromov-weak topology. Convergence in this topology is equivalent to convergence of the corresponding distance matrix distributions $\nu^m, (X, r, \mu)$, where

$$\nu^m(X, r, \mu)(\cdot) := \mu^{\otimes m}(\{x_1, \ldots, x_m : (r(x_i, x_j))_{1 \leq i < j \leq m} \in \cdot\})$$  \hspace{1cm} (1.2)

(see Section 2 for details) but we point out that the function $\Phi$ is not continuous in this topology. This is the reason why we need to introduce a finer topology which we call Gromov-weak atomic topology.

Convergence of a sequence $u_n \in U, n \in \mathbb{N}$ to a limit object $u \in U$ in this new topology is equivalent to convergence of the distance matrix distribution, i.e. convergence in the Gromov-weak topology, and convergence of the following quantities

(a) $\nu^{2, u_n} := \sum_{h \geq 0} \nu^{2, u_n}(\{h\})^2 \delta_h \Rightarrow (\nu^{2, u})^*$ as $n \to \infty$, where “$\Rightarrow$” denotes the convergence in the weak topology on finite measures.

(b) $\nu^{2, u_n}(\{0\}) \to \nu^{2, u}(\{0\})$ as $n \to \infty$.

The following example shows the differences between the convergence in the Gromov-weak and Gromov-weak atomic topology (see Section 2 for all details).

**Example 1.1.** (Convergence in the Gromov-weak atomic topology) We consider the sequence $(\{x_1, x_2, x_3\}, r_n, \delta_{x_1} + \delta_{x_2} + \delta_{x_3})$, with

$$r_n(x_1, x_2) = 1,$$

$$r_n(x_1, x_3) = r_n(x_2, x_3) = 1 + \frac{1}{n},$$

$$r_n(x_i, x_i) = 0, \ i = 1, 2, 3 \hspace{1cm} (1.3)$$

for $n \geq 1$ and the ultra-metric measure space $(\{x_1, x_2, x_3\}, r, \delta_{x_1} + \delta_{x_2} + \delta_{x_3})$, where

$$r(x_i, x_j) = 1, \ i \neq j, \quad r(x_i, x_j) = 0, \ i = j. \hspace{1cm} (1.4)$$

then $u_n \to u$ in the Gromov-weak topology (see Figure 2) but $u_n \not\rightarrow u$ not in the Gromov-weak atomic topology.
In view of the above example we can interpret convergence in the Gromov-weak atomic topology as convergence in the Gromov-weak topology plus some additional conditions on the convergence of the “branching points of the tree”.

Even though an ultra-metric measure space \((X, r, \mu)\) cannot be reconstructed by the value \(\mathfrak{F}(X, r, \mu)\) in general (since \(\mathfrak{F}\) is not injective), we may hope that, as in the case of the Kingman coalescent, we can find a “nice” subspace on which a reconstruction is possible. Indeed we prove that

there is a dense \(G_\delta\) subset of \(U\) such that \(\mathfrak{F}\) restricted to this subspace is a homeomorphism (onto its image)

and call elements \(u\) of this subspace identifiable by family sizes.

Next we want to apply our results on the tree-valued Feller diffusion and the tree-valued Fleming-Viot process and note that even though both processes are different, from a genealogical perspective they look quite similar: One can show that conditioned on the total mass, the genealogy in a Feller diffusion is a time inhomogeneous tree-valued Fleming-Viot process. We do not want to go into detail but refer to \cite{GGR16, G13}, Chapter 5 or \cite{GD18}. The important observation is, that it is enough to consider the tree-valued Fleming-Viot process to gain genealogical information for the tree-valued Feller diffusion and vice versa. Having that in mind, we get the following result:

\textit{The genealogy, denoted by} \(U_t\) \textit{as an} \(U\)-\textit{valued random variable}, \textit{in a (large) Wright-Fisher population (with or without selection) at time} \(t > 0\) \textit{is completely determined by the distribution of the distance of two randomly chosen individuals, i.e.} \(\mathcal{L}(\nu_2^{2,kt})\).
Note that due to the above observation, this result stays valid in the situation of a tree-valued Feller diffusion.

In fact, we think that in the “most” infinite population models the genealogy at a given time $t$ is identifiable by family sizes, where we have processes in mind that arise as large population limits of graphically constructed finite population models equipped with the uniform distribution on the set of individuals. The reason is Proposition 3.16 that mainly says that, whenever the vector of family sizes at some depth is absolutely continuous to the (product-)Lebesgue measure, we can reconstruct the whole genealogy by the path of family size decompositions of varying depth.

We also point out that the above result is not a probabilistic result in the sense that we use probabilistic arguments to reconstruct the law of the genealogy, but rather a result about the “states” of the genealogy, i.e. the genealogy realizes its values (with probability one) in a subspace of $U$ on which the function $\nu^2$ or $\mathcal{F}$ is a homeomorphism (onto its image).

Although we think that the function $\mathcal{F}$ (and the concept of family size decompositions) is an interesting object itself it can also be used to construct compact subsets of $U$ and therefore gives a tool to prove compact containment conditions for evolving genealogies (see Corollary 3.12). To be a bit more precise we have the following application in mind: (1) Construct a tree-valued process $U^N$ via a graphical construction. (2) Show that it has a large population limit, i.e. $U^N \Rightarrow U$, when $U$ is equipped with the Gromov-weak atomic topology. (3) Use the continuous mapping theorem to deduce that $\mathcal{F}(U^N) \Rightarrow \mathcal{F}(U)$ (see also Corollary 3.12). (4) Assume that the process $U$ is again indexed, i.e. $U = U^M$, for some $M = 1, 2, \ldots$ and we want to study the behavior of $U^M$ for $M \to \infty$. The key idea in doing that is to observe that for an evolving genealogy $U = (U_t)_{t \geq 0}$ we can not only consider $\mathcal{F}(U_t) = (f(U_t, h))_{h \geq 0}$ for fixed $t$ (which corresponds to a backward in time picture), but also $(f(U_t, t + h))_{h \geq 0}$ for fixed $h$ (which corresponds to a forward in time picture). It will turn out that a combination of both a backward as well as a forward in time picture can be used to get tightness of the evolving genealogies in terms of simpler processes. Roughly speaking, it is enough to prove convergence of $(f(U^M_t, t + h_i))_{i \geq 0}$ for fixed $M \in \mathbb{N}$ to get tightness of $U^M$. The above is not quite rigorous (and will be part of an upcoming paper) but should give a hint that it is important to understand the function $\mathcal{F}$ in order to get results on $U$-valued processes.

At this point we should note that the results presented here are generalizations and extensions of results in [Gri17], but for the sake of completeness we include all proofs needed for the results in this paper.
2 Metric measure spaces and the Gromov weak atomic topology

Here we give the definition and basic properties of (ultra-)metric measure spaces, the subspaces we are interested in and the Gromov-weak (see [DGP11] and [GPW09]) and Gromov-weak atomic topology.

Recall that the support of a finite Borel measure $\mu$, denoted by $\text{supp}(\mu)$, on some separable metric space $(X,d)$ is defined as the smallest closed set $C$ with $\mu(X\setminus C) = 0$. Note that $\text{supp}(\mu)$ is also given as

$$ \text{supp}(\mu) = \{ x \in X \mid \forall \varepsilon > 0 : \mu(B_\varepsilon(x)) > 0 \}, \quad (2.1) $$

where $B_\varepsilon(x)$ is the open ball of radius $\varepsilon$ around $x$.

**Definition 2.1.** (Metric measure spaces) We call the triple $(X,r,\mu)$

1. a metric measure space, short mm-space, if
   
   (a) $(X,r)$ is a complete separable metric space, where we assume that $X \subset \mathbb{R}$ (one needs this to avoid set theoretic pathologies).
   
   (b) $\mu \in \mathcal{M}_f(X)$, i.e. $\mu$ is a finite measure on the Borel sets generated by $r$.

2. ultra-metric, if $r(x_1,x_2) \leq r(x_1,x_3) \lor r(x_3,x_2)$ for $\mu$-almost all $x_1,x_2,x_3$,

3. compact, if $\text{supp}(\mu)$ is compact,

4. purely atomic, if $\sum_{x \in X} \mu(\{x\}) = \mu(X)$, and non atomic if $\sum_{x \in X} \tilde{\mu}(\{x\}) = 0$.

5. identifiable (by family sizes), if it is ultra-metric and

$$ \sum_{x \in A_h^1} \mu(\bar{B}_h(x)) \neq \sum_{x \in A_h^2} \mu(\bar{B}_h(x)), \quad (2.2) $$

for all $h > 0$ and all measurable subsets $A_h^1, A_h^2 \subset \text{supp}(\mu)$ with

$$(A_h^1)^h := \{ y \in \text{supp}(\mu) : \exists x \in A_h^1, \ r(x,y) \leq h \} \neq (A_h^2)^h \quad (2.3)$$

and $x,y \in A_h^i$, $x \neq y$ implies $r(x,y) > h$, $i = 1,2$.

6. (non simultaneous) binary if $r(x_1,x_2) = r(x_3,x_4)$ implies either $x_1 = x_3$ and $x_2 = x_4$ or $x_1 = x_4$ and $x_2 = x_3$ for $\mu$ almost all $x_1,x_2,x_3,x_4$. 


We say that two mm-spaces \((X, r_X, \mu_X)\) and \((Y, r_Y, \mu_Y)\) are equivalent if there is a measure-preserving isometry between this spaces, i.e. a map \(\varphi : \text{supp}(\mu_X) \to \text{supp}(\mu_Y)\) with \(r_X(x, y) = r_Y(\varphi(x), \varphi(y)), x, y \in \text{supp}(\mu_X)\) and \(\mu_Y = \mu_X \circ \varphi^{-1}\). This property defines an equivalence relation, and we denote by \([X, r, \mu]\) the equivalence class of a mm-space \((X, r, \mu)\).

We define the following sets:

\[
\mathbb{M} := \{[X, r, \mu] : (X, r, \mu) \text{ is a metric measure space}\},
\]
\[
\mathbb{U} := \{[X, r, \mu] \in \mathbb{M} : (X, r, \mu) \text{ is an ultra-metric measure space}\},
\]
\[
\mathbb{U}^a := \{[X, r, \mu] \in \mathbb{U} : (X, r, \mu) \text{ is purely atomic}\},
\]
\[
\mathbb{U}^c := \{[X, r, \mu] \in \mathbb{U} : (X, r, \mu) \text{ is non-atomic}\},
\]
\[
\mathbb{I} := \{[X, r, \mu] \in \mathbb{U} : (X, r, \mu) \text{ is identifiable}\},
\]
\[
\mathbb{B} := \{[X, r, \mu] \in \mathbb{U} : (X, r, \mu) \text{ is binary}\}.
\]

We also use combinations of the above spaces such as \(\mathbb{U}^a \cap \mathbb{U}^c\) etc.

We will typically use \(m\) and \(u\) for elements of \(\mathbb{M}\) and \(\mathbb{U}\).

**Remark 2.2.** Clearly, the property of being measure preserving isometric is reflexive and transitive. To see that it is symmetric one can first show that the image \(\varphi(\text{supp}(\mu_X))\) is dense in \(\text{supp}(\mu_Y)\) and then extend the inverse to a measure-preserving isometry \(\text{supp}(\mu_Y) \to \text{supp}(\mu_X)\). We may therefore assume w.l.o.g. that the measure-preserving isometries are surjective.

**Example 2.3.** (Identifiable elements) Let \((U_i)_{i \in \mathbb{N}}\) be independent \(\mathbb{R}_+\)-valued random variables, which are all absolutely continuous to the Lebesgue measure and satisfy \(\sum_i U_i < \infty\) almost surely. Let \((\mathbb{N}, r)\) be a complete ultra-metric space, then

\[
\left[\mathbb{N}, r, \sum_{i \in \mathbb{N}} U_i \delta_i\right] \in \mathbb{I}, \quad \text{almost surely.}
\]

**Definition 2.4.** (Distance matrix distribution) Let \(k \in \mathbb{N}_{\geq 2}, m = [X, r, \mu] \in \mathbb{M}\) and set

\[
R^{k,x,r} : \left\{ X^k \to \mathbb{R}_{+}^{\binom{k}{2}}, \right.
\]
\[
\left. (x_i)_{1 \leq i \leq k} \mapsto (r(x_i, x_j))_{1 \leq i < j \leq k}. \right\}
\]
We define the distance matrix distribution of order $k$ by:

$$
\nu^{k,m} := (R^k(X, r)) \otimes \mu \in \mathcal{M}_f \left( \mathbb{R}_+^{(k)} \right),
$$

where $\mathbb{R}_+^{(k)}$ is equipped with the product topology. For $k = 1$ we define

$$
\nu^{1,m} := \overline{m} := \mu(X).
$$

\[\square\]

**Remark 2.5.** Note that $\nu^{k,m}$ in the above definition does not depend on the representative $(X, r, \mu)$ of $m$. In particular $\nu^{k,m}$ is well defined for all $k \in \mathbb{N}$.

\[\clubsuit\]

**Definition 2.6.** (Gromov-weak topology) Let $m_m, m_{m_1}, m_{m_2}, \ldots \in M$. We say $m_n \to m$ for $n \to \infty$ in the Gromov-weak topology, if

$$
\nu^{k,m_n} \overset{n \to \infty}{\to} \nu^{k,m}
$$

in the weak topology on $\mathcal{M}_f \left( \mathbb{R}_+^{(k)} \right)$ for all $k \in \mathbb{N}$.

\[\diamond\]

**Remark 2.7.** Since $\overline{m} = \sqrt{\nu^2(m) \mathbb{R}_+}$, we have $m \mapsto \overline{m}$ is continuous in the Gromov-weak topology.

\[\spadesuit\]

For our results it will be necessary to introduce a finer topology:

**Definition 2.8.** (Gromov-weak atomic topology) Let $u, u_1, u_2, \ldots \in U$. We say $u_n \to u$ for $n \to \infty$ in the Gromov-weak atomic topology, if $u_n \to u$ for $n \to \infty$ in the Gromov-weak topology and

a) $(\nu^{2,u_n})^* \to (\nu^{2,u})^*$, where $(\nu^{2,u})^* = \sum_{h \geq 0} \nu^{2,u}(\{h\})^2 \delta_h$.

b) $\nu^{2,u_n}(\{0\}) \to \nu^{2,u}(\{0\})$.

\[\Diamond\]
Remark 2.9. This topology is related to the so called weak atomic topology on finite measures, introduced by [EK94], where one says that a sequence $\mu_n \in \mathcal{M}_f(X)$, $n \in \mathbb{N}$ of finite Borel-measures converges to a finite Borel-measure $\mu \in \mathcal{M}_f(X)$ in the weak atomic topology, when $\mu_n \Rightarrow \mu$ (i.e. convergence in the weak topology) and $\mu_*^n := \sum_{x \in X} \mu_n(\{x\})^2 \delta_x \Rightarrow \mu^*$. This explains the origin of the name “Gromov-weak atomic”.

Example 2.10. (Convergence in the Gromov-weak atomic topology - Example 1.1 continued) Assume we are in the situation of Example 1.1 then $u_n \rightarrow u$ in the Gromov-weak topology. Note that
\[
(\nu^{2, u_n})^* = 3^2 \delta_0 + 2^2 \delta_1 + 4^2 \delta_{1+\frac{1}{n}},
\]
\[
(\nu^{2, u})^* = 3^2 \delta_0 + 6^2 \delta_1.
\]
Hence
\[
(\nu^{2, u_n})^* \Rightarrow 3^2 \delta_0 + (2^2 + 4^2) \delta_1 \neq (\nu^{2, u})^*.
\]
This means $u_n \nRightarrow u$ in the Gromov-weak atomic topology.

Now recall the definition of the Prohorov distance of two finite measures $\mu_1$ and $\mu_2$ on a metric space $(E, r)$ with Borel $\sigma$-field $\mathcal{B}(E)$
\[
d_{Pr}(\mu_1, \mu_2) := \inf \left\{ \varepsilon > 0 : \mu_1(A) \leq \mu_2(A^\varepsilon) + \varepsilon, \right. \]
\[
\left. \mu_2(A) \leq \mu_1(A^\varepsilon) + \varepsilon \text{ for all } A \text{ closed} \right\},
\]
where
\[
A^\varepsilon := \left\{ x \in E : r(x, x') < \varepsilon, \text{ for some } x' \in A \right\}.
\]
The next proposition summarizes some important facts about the Gromov-weak topology (see [DGP11] and [LVW15] section 2.1; compare also [GPW09]).

Proposition 2.11. (Properties of the Gromov-weak topology) (a) $\mathcal{M}$ equipped with the Gromov-weak topology is Polish and the subspace $U \subset \mathcal{M}$ is closed.

(b) An example for a complete metric on $\mathcal{M}$ (respectively $U$) is the Gromov-Prohorov metric $d_{GPr}$, where for two mm-spaces $[X, r_X, \mu_X]$ and $[Y, r_Y, \mu_Y]$
\[
d_{GPr}([X, r_X, \mu_X], [Y, r_Y, \mu_Y]) := \inf_{(\varphi_X, \varphi_Y, Z)} d_{Pr}^{Z,r_Z}(\mu_X \circ \varphi_X^{-1}, \mu_Y \circ \varphi_Y^{-1}),
\]
where
where the infimum is taken over all isometric embeddings \( \varphi_X \) and \( \varphi_Y \) from \( \text{supp}(\mu_X) \) and \( \text{supp}(\mu_Y) \) into some complete separable metric space \((Z, \rho_Z)\) and \( d_{Pr}^{(Z,\rho_Z)} \) denotes the Prohorov distance on \( M_f(Z) \).

We close this section with some properties of the Gromov-weak atomic topology:

**Theorem 2.12.** *(Properties of the Gromov-weak atomic topology)* If we equip \( U \) with the Gromov-weak atomic topology then the following holds (recall (2.9)):

(a) \( U \) is a Polish space,

(b) \( I \subset U \) is dense.

(c) Let \( U_c \) be equipped with the subspace topology, then \( B \subset U_c \) is closed.

**Remark 2.13.** (1) As in [EK94] (see the discussion after (2.3)), the Borel sets generated by the Gromov-weak topology coincide with the Borel-sets generated by the Gromov-weak atomic topology.

(2) \( U_c \subset U \) is measurable in the Gromov-weak topology and therefore measurable in the Gromov-weak atomic topology (see Remark 2.8 and Corollary 3.6 in [ALW16]).

3 Family size decomposition of ultra-metric measure spaces

We will now introduce the function \( \mathfrak{F} \) that gives the size of the different families of an ultra-metric measure space \( u \).

3.1 Definitions

We start with the following Lemma, that gives us the existence of an “almost surely” disjoint decomposition of an ultra-metric measure space into closed balls.

**Lemma 3.1.** Let \( 0 < h, u = [X, r, \mu] \in U \) and \( B_h(x) \) be the closed ball of radius \( \leq h \) around \( x \in X \). Then there is a \( n(h) \in \mathbb{N} \cup \{\infty\} \) and a family \( \{r^h_i : i \in \{1, 2, \ldots, n(h)\}\} \) of elements of \( \text{supp}(\mu) \) with

\[
\mu(\overline{B}(r^h_i, h) \cap \overline{B}(r^h_j, h)) = 0, \quad (3.1)
\]
for $i \neq j$ and
\[
\mu(X) = \sum_{i=1}^{n(h)} \mu(\bar{B}(v_i^h, h)).
\]  
(3.2)

Moreover, if $0 < \delta \leq h$, then there is a partition \( \{I_i\}_{i \in \{1, \ldots, n(h)\}} \) of \( \{1, \ldots, n(\delta)\} \) such that
\[
\mu(\bar{B}(v_i^h, h)) = \sum_{j \in I_i} \mu(\bar{B}(v_j^\delta, \delta)), \quad \forall i = 1, \ldots, n(h).
\]  
(3.3)

**Remark 3.2.** (i) By the definition of the support we get \( \mu(\bar{B}(v_i^h, h)) > 0 \) for all \( i \in \{1, \ldots, n(h)\} \).

(ii) The analogue of Lemma 3.1 holds if we replace \( \leq \) by \( < \).

♣

**Remark 3.3.** Another important observation is the following: Given a finite ultra-metric space \( (\{1, \ldots, n(0)\}, r) = (X, r) \), then the path of partitions \( h \mapsto (\{I_i^h\}_{i = 1, \ldots, n(h)}) \) of \( \{1, \ldots, n(0)\} \) contains all information of the metric \( r \), i.e. the function that maps an ultra-metric \( r \) on \( X \) to the path of partitions is an injection and given an element \( (h \mapsto \pi^h) \) contained in the range of this map, the corresponding metric \( r \) can be reconstructed by
\[
r(k, l) := \inf \{h > 0 : k, l \in \pi_i^h \text{ for some } i = 1, \ldots, n(h)\},
\]  
(3.4)

for \( k, l \in \{1, \ldots, n(0)\} \).

♣

Let \( C \geq 0 \) and set
\[
S_C^\downarrow := \left\{ (x_1, x_2, \ldots) \in \mathbb{R}^\infty : \sum_{i \in \mathbb{N}} x_i \leq C, \ x_1 \geq x_2 \geq \ldots \right\},
\]  
(3.5)

\[
S^\downarrow := \left\{ (x_1, x_2, \ldots) \in \mathbb{R}^\infty : \sum_{i \in \mathbb{N}} x_i < \infty, \ x_1 \geq x_2 \geq \ldots \right\}.
\]  
(3.6)

We consider the following two distances on \( S_C^\downarrow \) and \( S^\downarrow \):
\[
d^1(x, y) = \sum_{i=1}^\infty |x_i - y_i| = \|x - y\|_1
\]  
(3.7)

and
\[
d^\infty(x, y) = \max_{i \in \mathbb{N}} |x_i - y_i|.
\]  
(3.8)

We note that \( S_C^\downarrow \) and \( S^\downarrow \) are typically equipped with the \( \ell^1 \)-distance, \( d^1 \).
Definition 3.4. (Definition of $f$) Let $u \in \mathbb{U}$. We define the map $f(u, \cdot) : (0, \infty) \to \mathcal{S}^\downarrow$, 
$$f(u, h) = (a_1(h), a_2(h), \ldots),$$
(3.9)
where the $a_k(h)$ are given by
$$a_k(h) = \max \left\{ c \geq 0 : \sum_{i=1}^{n(h)} \mathbb{1}(\mu(\overline{B}(r^i_{h}, h)) \geq c) \geq k \right\}, \quad k = 1, 2, \ldots, n(h),$$
$$a_k(h) = 0, \quad \text{for } k > n(h).$$
(3.10)

Note that $a_k(h) \geq a_{k+1}(h)$ is the non-increasing reordering of the sequence $(\mu(\overline{B}(r^i_{h}, h)))_{i=1,\ldots,n(h)}$. ♦

Remark 3.5.

(i) Let $(X, r_X, \mu_X)$ and $(Y, r_Y, \mu_Y)$ be two equivalent ultra-metric measure spaces and let $\varphi : \text{supp}(\mu_X) \to \text{supp}(\mu_Y)$ be a measure preserving isometry. Then \{${r^i_{h} : i \in \{1, 2, \ldots, n(h)\}}$\} $\subset \text{supp}(\mu_X)$ satisfies the conditions in Lemma 3.1 if and only if \{${\varphi(r^i_{h}) : i \in \{1, 2, \ldots, n(h)\}}$\} $\subset \text{supp}(\mu_X)$ satisfies the conditions.

(ii) If $x \in \text{supp}(\mu_X)$ and $h > 0$, then there is exactly one $i \in \{1, \ldots, n(h)\}$ with
$$\mu_X(\overline{B}(r^i_{h}, h)) = \mu_X(\overline{B}(r^i_{h}, h) \cap \overline{B}(x, h)) = \mu_X(\overline{B}(x, h)).$$
(3.11)
As a consequence, the definition of $f$ does not depend on the representatives. ♣

Note that the domain of $f(u, \cdot)$ is $(0, \infty)$. In some cases it is also possible to add 0 to the domain and we close this section with the following remark:

Remark 3.6. In the case, where $u \in \mathbb{U}^a$ is purely atomic we can extend the function $f(u, \cdot)$ to a function $\hat{f}(u, \cdot) : [0, \infty) \to \mathcal{S}^\downarrow$. ♣

3.2 Results

We start with the following definition:
Definition 3.7. (Definition of $\mathfrak{F}$) We define
\[ \mathfrak{F} : U \to (S^↓)^{(0, \infty)}, \quad u \mapsto f(u, \cdot). \tag{3.12} \]

The first observation is, that $\mathfrak{F}$ maps ultra-metric measure spaces to cadlag (i.e. right continuous with left limits) functions:

Lemma 3.8. $F := \mathfrak{F}(U) \subset D((0, \infty), S^↓)$, where $S^↓$ is equipped with $d^1$.

In the following
\[ D((0, \infty), S^↓) \] is always equipped with the Skorohod topology, given in Appendix A.

Now the question is whether $\mathfrak{F}$ is continuous when $U$ is equipped with the Gromov-weak topology.

Example 3.9. Assume we are in the situation of Example 1.1. Observe that if we take for example $t_n = 1 + \frac{1}{n} \to 1$ then
\[ f(u_n, t_n) \equiv (2, 1, 0, \ldots) \notin \{(1, 1, 1, 0, \ldots), (3, 0, 0, \ldots)\} = \{f(u, 1), f(u, 1-\}) \],
\[ f(u_n, \cdot) \not\to f(u, \cdot) \text{ in the Skorohod topology (see Proposition 3.6.5 in [EK86]).} \tag{3.13} \]

In other words we can not expect $\mathfrak{F}$ to be continuous, when $U$ is equipped with the Gromov-weak topology. But as we have seen in Example 1.1 the sequence $u_n$ does not converge in the Gromov-weak atomic topology and in fact, this is the reason why we introduced this new topology.

Recall that a function $f : X \to Y$ between two topological spaces is called perfect, if it is continuous, surjective, closed (i.e. maps closed sets to closed sets) and $f^{-1}(\{y\})$ is compact in $X$ for all $y \in Y$. We remark the following:

Remark 3.10. If $X$ is a topological space and $Y$ is a compactly generated Hausdorff space (for example a metric space) and $f : X \to Y$ is surjective, then the following is equivalent (see for example [Pal70]):

(i) $f$ is perfect,
(ii) \( f \) is continuous and proper, i.e. \( f^{-1}(K) \) is compact in \( X \) for all compact sets \( K \subset Y \).

Note that a perfect map is also a quotient map, i.e. surjective and \( f^{-1}(U) \) is open in \( X \) iff \( U \) is open in \( Y \).

\[ \clubsuit \]

**Theorem 3.11.** (Properties of \( F \)) Let \( U \) be equipped with the Gromov-weak atomic topology, then \( F : U \to \mathbb{F} \) has the following properties:

i) \( F \) is perfect.

ii) The restriction \( F|_{\mathbb{I}} \) of \( F \) to \( \mathbb{I} \) (see Definition 2.1) is a homeomorphism onto its image.

Recall that a collection of cadlag process \( \{X^n : n \in \mathbb{N}\} \) with values in some Polish space \( E \) satisfies a compact containment condition if for all \( \varepsilon > 0 \) and \( T > 0 \) there is a compact set \( K \subset E \) such that

\[
\inf_{n \in \mathbb{N}} P(X^n(t) \in K \forall t \in [0,T]) \geq 1 - \varepsilon. \quad (3.14)
\]

**Corollary 3.12.** Let \( U_n \) be a sequence in \( U \) and \( \mathbb{U} \) be equipped with the Gromov-weak atomic topology. Then

(i) \( (\mathcal{L}(U_n))_{n \in \mathbb{N}} \) is tight if and only if \( (\mathcal{L}(\mathcal{F}(U_n)))_{n \in \mathbb{N}} \) is tight.

(ii) \( U_n \Rightarrow U \) for some \( U \)-valued random variable \( U \) implies \( \mathcal{F}(U_n) \Rightarrow \mathcal{F}(U) \).

Moreover, the map \( D([0,\infty)) \times \mathcal{F}(\mathbb{U}) \to D([0,\infty),\mathcal{F}(\mathbb{U})) \) is continuous and a collection of \( U \)-valued cadlag processes \( \{(U^n_t)_{t \geq 0} : n \in \mathbb{N}\} \) satisfies a compact containment condition if and only if \( \{(\mathcal{F}(U^n_t))_{t \geq 0} : n \in \mathbb{N}\} \) satisfies a compact containment condition.

**Proof.** This is a direct consequence of Theorem 3.11, the continuous mapping theorem, Prohorov’s theorem and the fact that continuous images of compact sets are compact. See Problem 3.13 in [EK86]. \( \square \)

Another interesting observation is, that even though, the above result is related to the Gromov-weak atomic topology, we also get a result for the Gromov-weak topology:

**Proposition 3.13.** Let \( U^n, n = 1,2,\ldots \) be a sequence of \( U \)-valued random variables and let \( \mathbb{U} \) be equipped with the Gromov-weak topology. Assume that for all \( \delta > 0 \) and all \( \varepsilon > 0 \)
(i) there is a compact set $\Gamma \subset S^\downarrow$ such that
\[
\limsup_{n \to \infty} P\left( f(U_n, \delta) \in \Gamma^c \right) \leq \varepsilon,
\]
(3.15)

(ii) there is an $H \geq 0$ such that
\[
\limsup_{n \to \infty} P\left( \left( \sum_{i=1}^{\infty} f(U_n, H)^2 \right) - \sum_{i=1}^{\infty} f(U_n, H)^2 \geq \varepsilon \right) \leq \varepsilon
\]
(3.16)

and that the total mass $\nu^{1_M^n}$ is tight. Then, $(U^n)_{n \in \mathbb{N}}$ is tight.

Remark 3.14. Note that $S^\downarrow_C$ equipped with $d^\infty$ is a compact space (this follows analogue to Proposition 2.1. in [Ber06]). It is not hard to see that $\Gamma \subset S^\downarrow_C$, equipped with $d^1$, is compact, if for all $\varepsilon > 0$ there is a $M \in \mathbb{N}$ such that
\[
\sup_{f \in \Gamma} \sum_{i \geq M} f_i \leq \varepsilon.
\]
(3.17)

We will discuss this property in more detail in Section 8.2.

Even though the function $\mathcal{F}$ is not injective on the whole space it is at least injective on a dense subset (see Remark 2.12).

Remark 3.15. By Lavrentiev’s Theorem (see Section 35.II in [Kur14]) there are two $G_\delta$ sets $I \subset I^* \subset U$ and $\mathcal{F}(I) \subset I^*_\delta \subset \mathcal{F}(U)$ and a homeomorphism $\mathcal{F}^*: I^* \to I^*_\delta$ extending $\mathcal{F}$. In addition $I^*$ is dense in $U$ since $I$ is dense in $U$.

Next, we give a criterion when a $U$-valued random variable takes values in $I$:

Proposition 3.16. Assume that $U$ is an $U$-valued random variable. Let $N_h \in \mathbb{N} \cup \{\infty\}$ be the number of non-zero entries of $f(U, h)$. If $L(f(U, h)) \ll \lambda^{\otimes N_h}$ conditioned on $N_h$ for all $h > 0$, where $\lambda$ denotes the Lebesgue measure, then $U \in I$ almost surely.

We close this section with a result that is even stronger than the above result, when one considers the subspace $\mathcal{B} \cap I$ (see Definition 2.1):

Theorem 3.17. (Properties of $\mathcal{B} \cap I$) Let $\mathcal{B} \cap I$ be equipped with the Gromov-weak atomic topology, then we have $\mathcal{V} : \mathcal{B} \cap I \to \mathcal{V}(\mathcal{B} \cap I) \subset \mathcal{M}_f(\mathbb{R}_+)$, $u \mapsto \nu^{2,u}$ is a homeomorphism.
Remark 3.18. As in Remark 3.15 we can extend the homeomorphism to $G_\delta$ subsets.

4 Application to the tree-valued Fleming-Viot process

In this section we give a short introduction to tree-valued Fleming-Viot processes and show that these processes live in the subspace $\mathbb{B} \cap \tilde{\mathbb{I}}$ (see Theorem 3.17). For simplicity, we will only introduce the neutral model and refer to [DGP12] for the general case.

In section 4.1 we define the neutral tree-valued Moran model of a given size $N$ (the population size). This model was defined by [GPW13] and extended by [DGP12] to include selection and mutation. In section 4.2 we consider the large population limit (i.e. $N \to \infty$) of the tree-valued Moran models, the so called tree-valued Fleming-Viot process, and give our main result for this process.

4.1 Definition of the neutral model

We want to describe the genealogy of a population, consisting of $N \in \mathbb{N}$ individuals, that evolves according to the following dynamic:

**Resampling:** Every pair $i \neq j$ of individuals is replaced with rate one. If such an event occurs, $i$ is replaced by an offspring of $j$ with probability $\frac{1}{2}$, or $j$ is replaced by an offspring of $i$ with probability $\frac{1}{2}$.

In order to describe the evolution of this process formally, let $I_N := [N] := \{1, \ldots, N\}$, $N \in \mathbb{N}$ and

$$\{\eta^{ij} : i, j \in I_N, i \neq j\}$$

be a realization of a family of independent rate 1 Poisson point processes.

For $i, i' \in I_N$, $0 \leq h < t < \infty$ we say that there is a path from $(i, h)$ to $(i', t)$ if there is an $n \in \mathbb{N}$, $h \leq t_1 < t_2 < \cdots < t_n \leq t$ and $j_1, \ldots, j_n \in I_N$ such that for all $k \in \{1, \ldots, n+1\}$ ($j_0 := i, j_{n+1} := i'$)

$$\eta^{j_{k-1}j_k}\{t_k\} = 1, \eta^{x,j_{k-1}}((t_{k-1}, t_k)) = 0$$

for all $x \in I_N$.

Note that for all $i \in I_N$ and $0 \leq h \leq t$ there exists an unique element

$$A_h(i, t) \in I_N$$

(4.2)
Figure 3: On the left side we see the graphical construction of the Moran model; \( \rightarrow \) indicates a resampling event. On the right side we see the genealogical tree of the population at time \( t \). In this case the ancestor of all individuals at time \( h \) would be individual 4, i.e. \( A_h(i, t) = 4 \) for all \( i = 1, \ldots, 4 \) with the property that there is a path from \( (A_h(i, t), h) \) to \( (i, t) \). We call \( A_h(i, t) \) the \emph{ancestor of} \( (i, t) \) \emph{at time} \( h \) (see Figure 3).

Let \( r_0 \) be an ultra-metric on \( I_N \) and \( i, j \in I_N \). Then we define the following (pseudo) ultra-metric on \( I_N \):

\[
    r_t(i, j) := \begin{cases} 
    t - \sup\{h \in [0, t] : A_h(i, t) = A_h(j, t)\}, & \text{if } A_0(i, t) = A_0(j, t), \\
    t + r_0(A_0(i, t), A_0(j, t)), & \text{if } A_0(i, t) \neq A_0(j, t).
    \end{cases}
\]

Since \( r_t \) is only a pseudo-metric, we consider the following equivalence relation \( \approx_t \) on \( I_N \): \( x \approx_t y \iff r_t(x, y) = 0 \). We denote by \( \bar{I}_N \) the set of equivalence classes and note that we can find a set of representatives \( \bar{I}_N \) such that \( \bar{I}_N \rightarrow \bar{I}_N \), \( x \rightarrow [x]_{\approx_t} \) is a bijection.

Let \( \mu^N \in M_1(I_N) \) be the uniform distribution on \( I_N \), i.e.

\[
    \mu^N = \frac{1}{N} \sum_{k \in I_N} \delta_k
\]

and define

\[
    \tilde{r}_t(\bar{i}, \bar{j}) = r_t(\bar{i}, \bar{j}), \quad \tilde{\mu}^N_t(\{\bar{i}\} \times \cdot) = \mu^N_t([\bar{i}]_{\approx_t} \times \cdot), \quad \bar{i}, \bar{j} \in \bar{I}_N.
\]
Then the \textit{tree-valued Moran model} of size $N$ is defined as

$$U_t^N := [I_{Nt}^N, r_t, \mu_t^N].$$

### 4.2 Results for the tree-valued Fleming-Viot process

Assume that $\mathcal{L}(U_0^N) \Rightarrow \mu \in \mathcal{M}_1(U)$, where $U$ is equipped with the Gromov-weak topology. Then

$$(U_t^N)_{t \geq 0} \overset{N \to \infty}{\Rightarrow} (U_t)_{t \geq 0}$$

weakly in the Skorohod topology on $D([0, \infty), U)$, where $\mathcal{L}(U_0) = \mu$ and $(U_t)_{t \geq 0}$ is the solution of a well-posed martingale problem (see Theorem 2 in \cite{GPW13}). We call the process $U = (U_t)_{t \geq 0}$ \textit{tree-valued Fleming-Viot process}.

**Proposition 4.1. (Convergence of the tree-valued Moran models)** Let $U$ be equipped with the Gromov-weak atomic topology and let $(V_i)_{i \in \mathbb{N}}$ be a sequence of independent $[0, 1]$-uniformly distributed random variables. We assume that $U_0^N = [[0, 1], r_0, \mu_0^N]$, where $([0, 1], r_0)$ is a compact binary ultra-metric space, and $\mu_0^N = \frac{1}{N} \sum_{i=1}^{N} \delta_{V_i}$. Then

$$U_t^N \Rightarrow U_t \quad \text{for all } t \geq 0.$$  

**Remark 4.2.** Even though we choose a special initial condition, the proof for general initial conditions should be similar but more technical (one needs to use Lemma 5.8 in \cite{GPW09} for example). We also note that the initial condition does not really matter when one wants to study genealogical properties that are generated by an evolving population.

We are now ready for our main result:

**Theorem 4.3. (State space of tree-valued FV-processes)** Recall Remark \ref{rem:measurable} and assume that $P(U_0 \in \mathbb{B} \cap \mathbb{U}_c) = 1$, then

$$P(U_t \in \mathbb{B} \cap \mathbb{I} \cap \mathbb{U}_c) = 1, \quad \forall t > 0.$$  

**Remark 4.4.** Even though, it is not hard to see that $\mathbb{B} \cap \mathbb{I}$ is measurable we can also apply Remark \ref{rem:Gdelta} and replace $\mathbb{B} \cap \mathbb{I}$ in the above theorem by a suitable $G_\delta$-set.
Since we did not define the model with selection we need to refer all interested readers to [DGP12]. But, as a direct consequence of the Girsanov transform - Theorem 2 in this paper, one can prove the following.

**Corollary 4.5.** If we denote by $\mathcal{U}^\alpha$ the tree-valued Fleming-Viot process with mutation and selection parameter $\alpha \geq 0$, defined in [DGP12], with $P(\mathcal{U}_0^\alpha \in \mathbb{B} \cap \mathbb{U}_c) = 1$, then

$$P(\mathcal{U}_t^\alpha \in \mathbb{B} \cap \mathbb{I} \cap \mathbb{U}_c) = 1, \quad \forall t > 0. \quad (4.10)$$

## 5 Preparations for the proofs

We start with some preparations needed for the proofs of our results. In section 5.1 we prove some bounds for the Gromov-Prohorov metric and in section 5.2 we introduce the notion of concatenation of trees, which will be useful in order to prove the continuity of $\mathfrak{F}$.

### 5.1 Bounds for the Gromov-Prohorov metric and the function $\Phi$

We start with the following observation.

**Remark 5.1.** Let $(X, r, \mu)$ and $(\tilde{X}, \tilde{r}, \tilde{\mu})$ be two equivalent ultra-metric measure spaces. If we denote by $\{r^h_i : i = 1, \ldots, n(h)\}$ and $\{\tilde{r}^h_i : i = 1, \ldots, \tilde{n}(h)\}$ two families of representatives in the sense of Lemma 3.1, then it is not hard to see (see also Remark 3.5) that

$$\begin{align*}
\left[ \left\{ r^h_i : i \in \{1, \ldots, n(h)\} \right\}, r, \sum_{i \in \{1, \ldots, n(h)\}} \mu(\bar{B}^r(r^h_i, h))\delta_{r^h_i} \right] &= \\
&= \left[ \left\{ \tilde{r}^h_i : i \in \{1, \ldots, \tilde{n}(h)\} \right\}, \tilde{r}, \sum_{i \in \{1, \ldots, \tilde{n}(h)\}} \tilde{\mu}(\bar{B}^{\tilde{r}}(\tilde{r}^h_i, h))\delta_{\tilde{r}^h_i} \right],
\end{align*}$$

and it is possible to define for $h > 0$

$$\Phi_h(u) = \left[ \left\{ r^h_i : i \in \{1, \ldots, n(h)\} \right\}, r, \mu_h \right] \quad (5.2)$$

and

$$\Phi_h(u) = \left[ \left\{ r^h_i : i \in \{1, \ldots, n(h)\} \right\}, r - h \cdot 1(r^h_i \neq r^h_j), \mu_h \right], \quad (5.3)$$
where
\[ \mu_h := \sum_{i \in \{1, \ldots, n(h)\}} \mu\left(\tilde{B}(x_i^h, h)\right) \delta_{x_i^h}. \] (5.4)

These functions will appear in several proofs. The reason is the following Lemma:

**Lemma 5.2.** Let \( 0 < h \) and \( u = [X, r, \mu] \in \mathbb{U} \).

(i) If \( A \subset X \) is measurable, and \( \mu_A(\cdot) := \mu(\cdot \cap A) \) then
\[ d_{GP}(\mu_A, [X, r, \mu]) \leq \mu(X \setminus A). \] (5.5)

(ii) If \( u' = [X, r, \mu'] \in \mathbb{U} \), then
\[ d_{GP}(u, u') \leq d_{Pr}(\mu, \mu'), \] (5.6)
where the Prohorov distance is taken on the set of Borel-measures on \( X \) (see (2.18)).

(iii) Let \( \Phi_h \) and \( \tilde{\Phi}_h \) be the functions from Remark 5.1. Then
\[ d_{GP}(u, \Phi_h(u)) \leq h, \quad d_{GP}(\Phi_h(u), \tilde{\Phi}_h(u)) \leq h. \] (5.7)

(iv) The functions \( h \mapsto \Phi_h(u) \) and \( h \mapsto \tilde{\Phi}_h(u) \) as functions from \( (0, \infty) \rightarrow \mathbb{U} \) are both cadlag.

**Proof.** (i) Note that the identity \( id : X \rightarrow X \) is an isometric embedding from \( A \) to \( X \). Using the definition of the Gromov-Prohorov metric from Proposition 2.11, it is enough to bound (note that \( \mu_A \leq \mu \)):
\[ d_{Pr}(\mu_A, \mu) = \inf \{ \epsilon > 0 : \mu(B) \leq \mu_A(B^\epsilon) + \epsilon, \ \forall B \subset X \text{ Borel-measurable} \}, \] (5.8)
where
\[ B^\epsilon = \{ x \in X : \exists x' \in B, \ r(x, x') < \epsilon \}. \] (5.9)
Note that if \( \mu(X \setminus A) = 0 \) then \( d_{Pr}(\mu_A, \mu) = 0 \) and if \( \mu(X \setminus A) > 0 \) we can take \( \epsilon = \mu(X \setminus A) \) and the result follows.

(ii) As in (i) one can use the identity as isometric embedding.
(iii) We use the notation of Remark 5.1 and note that \( id \) is an isometric embedding of \( \{v_i^h, i \in \mathbb{N}\} \) in \( X \). Define the measure \( \bar{\mu} \) on \( X \times X \) by

\[
\bar{\mu}(A_1 \times A_2) := \sum_{i \in \mathbb{N}} \mu(A_1 \cap \bar{B}(v_i^h, h))\delta_{v_i^h}(A_2). \tag{5.10}
\]

for all measurable sets \( A_1, A_2 \subset X \) and observe that \( \bar{\mu} \) is a coupling of \( \mu \) and \( \mu_h \). Since \( \mu_h(\{v_i^h, i \in \mathbb{N}\}) = \mu(X) \) and by the definition of the Gromov-Prohorov metric from Proposition 2.11 together with Theorem 3.1.2 in [EK86] (with the obvious extension to couplings of finite measures with the same mass), we get

\[
d_{GP}(u, \hat{\Phi}_h(u)) \leq \inf \inf \left\{ \varepsilon > 0 : \nu(\{(x, x') \in X \times X : r(x, x') \geq \varepsilon\}) \leq \varepsilon \right\}, \tag{5.11}
\]

where the infimum is taken over all couplings \( \nu \) of \( \mu \) and \( \mu_h \). It follows that

\[
d_{GP}(u, \hat{\Phi}_h(u)) \leq \inf \{\varepsilon > 0 : \bar{\mu}(\{(x, x') \in X \times X : r(x, x') \geq \varepsilon\}) \leq \varepsilon\} \tag{5.12}
\]

and if we choose \( \varepsilon > h \) then

\[
\bar{\mu}(\{(x, x') \in X \times X : r(x, x') \geq \varepsilon\}) \leq \sum_{i,j \in \mathbb{N}, i \neq j} \mu(B(v_i^h, h) \cap \bar{B}(v_i^h, h))\delta_{v_i^h}(B(v_j^h, h)) = 0. \tag{5.13}
\]

For the second part, we use the same argument as in section 3 in [Loe13]: Let \( Y := \{v_i^h : i \in \{1, \ldots, n(h)\}\} \), \( r^1 = r \), \( r^2 = r - h1(x \neq y) \) and \( \mu^1 = \mu^2 = \mu_h \). We denote by \( Y \uplus Y \) the disjoint union of \( Y \) and \( Y \) and let \( \varphi_i : Y \to Y \uplus Y \) be the canonical embeddings, \( i = 1, 2 \). Define the metric \( d \) on \( Y \uplus Y \) by

\[
d(\varphi_1(x), \varphi_1(y)) = r^1(x, y), \tag{5.14}
\]

\[
d(\varphi_2(x), \varphi_2(y)) = r^2(x, y), \tag{5.15}
\]

\[
d(\varphi_1(x), \varphi_2(y)) = \inf_{z \in Y} (r^1(x, z) + r^2(y, z)) + h, \tag{5.16}
\]

where \( x, y \in Y \). Then, as in [Loe13] it is easy to see that this is a metric on \( Y \uplus Y \) that extends the metrics \( r^1 \) and \( r^2 \) (i.e. \( \varphi_i \) is an isometry for \( i = 1, 2 \)) and we have

\[
\varphi_2(\varphi_1^{-1}(F)) \subset F^{h_0} := \{x \in Y \uplus Y : \exists x' \in F \text{ s.t. } d(x, x') < h_0\}, \tag{5.17}
\]
for all $h_0 > h$. Since $\mu^1 = \mu^2$ this gives:
\[
\mu^1 \circ \varphi_1^{-1}(F) = \mu^2 \circ \varphi_1^{-1}(F) \leq \mu^2 \circ \varphi_2^{-1}(\varphi_1^{-1}(F)) \leq \mu^2 \circ \varphi_2^{-1}(F^{h_0}) + h_0,
\]
for all $h_0 > h$ and the result follows.

(iv) A similar argument as in (iii) shows that $\hat{\Phi}_{h'}(u) \to \hat{\Phi}_h(u)$ for $h' \downarrow h$ and by definition we have $\Phi_{h+\delta}(u) = \Phi_{\delta}(\Phi_h(u))$ and hence by (iii) $\Phi_{h'}(u) \to \Phi_h(u)$ for $h' \downarrow h$. This shows the right continuity. For the existence of the left limits set
\[
u_h := \left\{ \{ r_i^h : i \in \{1, \ldots, n(h)\} \}, \; r - h \cdot 1(\tau_i \neq \tau_j), \; \mu_h^2 \right\},
\]
(5.19)
\[
u_h := \left\{ \{ r_i^h : i \in \{1, \ldots, n(h)\} \}, \; r, \; \mu_h^2 \right\},
\]
(5.20)
where
\[
\mu_h^2(A) := \sum_{i \in \{1, \ldots, n(h)\}} \mu \left( B(r_i^h, h) \right) \delta_{r_i^h}(A)
\]
(5.21)
is given in terms of open balls with radius $< h$ (instead of $\leq h$ - see Remark 3.2 (ii)). By a similar argument as in (iii) together with the fact that
\[
\lim_{h' \uparrow h} \mu \left( B(r_i^h, h) \setminus \bar{B}(r_i^h, h') \right) = 0,
\]
(5.22)
we get
\[
d_{GP}(\Phi_{h'}(u), u_h) \lor d_{GP}(\hat{\Phi}_{h'}(u), \hat{u}_h) \to 0, \quad h' \uparrow h.
\]
(5.23)

5.2 Concatenation of trees

We summarize some properties of the concatenation of trees given in [GGR16] (see also [EM17]).

**Definition 5.3.** (Concatenation of trees) Let $h > 0$ and $u_i = [X_i, r_i, \mu_i]$, $i \in I$ ($I \subset \mathbb{N} \cup \{\infty\}$) be a sequence in $U$ with $\sum_{i \in I} \mu_i(X_i) \leq C < \infty$,
\[
\overline{u}_i := \mu_i(X_i) = \sqrt{\nu^{2,u_i}[0, \infty])} > 0
\]
(5.24)
and
\[
\nu^{2,u_i}(h, \infty) = 0.
\]
(5.25)
We define the concatenation:
\[
\left( \bigcup_{i \in I} u_i := u_{i_1} \sqcup u_{i_2} \sqcup \ldots := \bigcup_{i \in I} X_i, r^h, \sum_{i \in I} \mu_i \right),
\]
where \( \bigcup_{i \in I} X_i \) is the disjoint union of the \( X_i \) and
\[
r^h(x, y) = \begin{cases} 
  r_i(x, y), & \text{for } x, y \in X_i, \\
  h, & \text{for } x \in X_i, \ y \in X_j, \ i \neq j.
\end{cases}
\]

**Definition 5.4.** (\( h \)-top) In the sense of Lemma 3.1 we define for \( 0 < h \) and \( u = [X, r, \mu] \in \mathbb{U} \) the \( h \)-top \( \Psi_h(u) \in \mathbb{U} \) as
\[
\Psi_h(u) := \left( \bigcup_{i \in \{1, \ldots, n(h)\}} B(\tau^h_i, h), r, \mu|_{B(\tau^h_i, h)} \right)
\]
where \( \mu|_{B(\tau^h_i, h)}(\cdot) = \mu(\cdot \cap B(\tau^h_i, h)) \). By Remark 3.5 this definition is independent of the representative \( (X, r, \mu) \).

**Remark 5.5.** Note that \( \Psi_h([X, r, \mu]) = [X, r^h, \mu] \) with
\[
r^h(x, y) = \begin{cases} 
  r(x, y), & \text{if } r(x, y) \leq h, \\
  h, & \text{otherwise}.
\end{cases}
\]

**Remark 5.6.** (i) Let \( h > 0 \) and \( u_i := [\bar{B}(\tau^h_i, h), r, \mu|_{\bar{B}(\tau^h_i, h)}] \in \mathbb{U} \), then
\[
\nu^{2, u}((h, \infty)) = 0 \text{ for all } i = 1, 2, \ldots, n(h).
\]
(ii) Let \( u \in \mathbb{U} \) and \( u_i \) given as in (i) for some \( h > 0 \). If \( x \in \text{supp}(\mu) \), then there is an \( i, j \in \{1, \ldots, n(h)\} \) such that
\[
\mu(\bar{B}(x, h)) = \bar{u}_i = f(h, u_i) = f(h, u)_j.
\]
(iii) As in (ii) we get for \( h > 0 \):
\[
\bar{u} = \mu(X) = \sum_{i=1}^{n(h)} f(h, u)_i = \sum_{i=1}^{n(h)} \bar{u}_i.
\]
Note that \( u \mapsto \bar{u} \) is continuous, since \( \bar{u} = \sqrt{\nu^{2, u}([0, \infty))} \).
Definition 5.7. (Concatenation as partial order) Define for \(0 < h\) the relation \(\leq_h\) on \(U\) by saying \(u \leq_h v\) if there is a \(u' \in U\) with \(\nu^{2,u'}(h, \infty) = 0\) such that \(\Psi_h(v) = \Psi_h(u) \sqcup_h u'\).

Lemma 5.8. Let \(0 < h, u, v \in U\), \((u_n), (v_n)\) be two sequences in \(U\) and \(U\) be equipped with the Gromov-weak topology.

(i) Suppose that \(\Psi_h(u_n) \to u\) and \(\chi_n := \Psi_h(u_n) \sqcup_h \Psi_h(v_n) \to \chi\), for \(n \to \infty\). Then \(u \leq_h \chi\).

(ii) If \(d_{GPr}(u_n, u) \to 0\), then \(\Psi_h(u_n) \to \Psi_h(u)\) for all \(h > 0\), i.e. \(u \mapsto \Psi_h(u)\) is continuous. Moreover, if \(h_n \to h\), then \(\Psi_{h_n}(u) \to \Psi_h(u)\).

(iii) If \(u_n \to u\), for \(n \to \infty\), and \(v_n \leq_h u_n\), for all \(n \in \mathbb{N}\), then \(\{\Psi_h(v_n) : n \in \mathbb{N}\}\) is compact.

(iv) Assume we are in the situation of Remark 5.6 for some \(h > 0\), then \(\nu^{2,u_i} \leq \nu^{2,u}\) for all \(i = 1, \ldots, n(h)\).

(v) Let \(u_n, v_n \in U\) such that \(u_n \to u \in U\) and \(v_n \to v \in U\) and assume that \(u_n, u, v_n, v\) satisfy \((5.25)\), then \(u_n \sqcup_h v_n \to u \sqcup_h v\).

Proof. This is a summary of Proposition 2.17, Lemma 3.3 and Lemma 3.5 in [GGR16].

6 A short note on the weak atomic topology

Here we give a short introduction to the weak atomic topology (see [EK94]) and prove a Proposition that gives a characterization of convergence in this topology in terms of cumulative distribution functions.

Definition 6.1. (Weak atomic topology) Let \((E, r)\) be a complete separable metric space and \(\mu_1, \mu_2, \ldots \in M_f(E)\) (space of finite Borel-measures on \(E\)). We say that \(\mu_n \to \mu\) in the weak-atomic topology if

- \(\mu_n \Rightarrow \mu\) in the weak topology and

- \(\mu_n^* \Rightarrow \mu^*\) in the weak topology, where \(\mu^* := \sum_{x \in E} \mu(\{x\})^2 \delta_x\).

\[\Box\]
Proposition 6.2. Assume that $E = \mathbb{R}$, and let $\mu, \mu_1, \mu_2, \ldots \in \mathcal{M}_F(E)$ be finite measures with $\mu_n(E) \to \mu(E)$. Then $\mu_n \to \mu$ in the weak atomic topology if and only if $F_n \to F$ in the Skorohod topology on $D(\mathbb{R}, \mathbb{R}_+)$, where $F(t) := \mu((\infty, t])$, $F_1(t) := \mu_1((\infty, t])$, $F_2(t) := \mu_2((\infty, t])$, $\ldots$, $t \geq 0$.

Proof. First observe that a classical result says that $\mu_n \to \mu$ is equivalent to $F_n(t) \to F(t)$ and $\mu_n(E) \to \mu(E)$ for all continuity points $t$ of $F$.

“$\Rightarrow$” If $\mu\{t\} > 0$ for some $t \in \mathbb{R}$, then, according to Lemma 2.5 in [EK94], there is an unique sequence $(t_n)_{n \in \mathbb{N}}$ with $(\mu\{t\}, t_n) \to (\mu\{t\}, t)$ and all other sequences $s_n \to t$ with $s_n \neq t_n$ satisfy $\mu\{s_n\} \to 0$. Moreover, a simple application of the Portmanteau Theorem gives: For all $\varepsilon > 0$ there is a $\delta > 0$ such that $\mu((t - \delta, t_n) \cup (t_n, t + \delta)) < \varepsilon$ for all $n$ large enough and $\mu((t - \delta, t) \cup (t, t + \delta)) < \varepsilon$ for all $\delta < \delta$. If we now choose $\delta > 0$ in such a way that $t - \delta$ is a continuity point of $F$, then

$$
\lim_{n \to \infty} |F_n(t_n) - F(t)| = \lim_{n \to \infty} |\mu_n((\infty, t_n]) - \mu((\infty, t])|
\leq \lim_{n \to \infty} |F_n(t - \delta) - F(t - \delta)| + \lim_{n \to \infty} |\mu_n\{t_n\} - \mu\{t\}|
+ \limsup_{n \to \infty} |\mu_n((t - \delta, t_n]) - \mu((t - \delta, t])|
(6.1)
$$

and hence $F_n(t_n) \to F(t)$. A similar argument shows that the conditions of Proposition 3.6.5 in [EK86] are satisfied and therefore $F_n \to F$ in the Skorohod topology.

“$\Leftarrow$” Now let $F_n \to F$ in the Skorohod topology. Then for all discontinuity points $t$ of $F$ there is one sequence $(t_n)_{n \in \mathbb{N}}$ such that $F(t_n) \to F(t)$ and $F(t_n -) \to F(t-)$ (see (6.20) in the proof of Proposition 3.6.5 in [EK86]). Since $\mu\{t\} = F(t) - F(t-)$ this gives

$$
\lim_{n \to \infty} \mu_n\{t_n\} = \lim_{n \to \infty} (F_n(t_n) - F_n(t_n-)) = F(t) - F(t-) = \mu\{t\} > 0.
(6.2)
$$

Moreover, all other sequences $(s_n)_{n \in \mathbb{N}}$ with $s_n < t_n$ and $s_n \to t$ satisfy $|F_n(s_n) - F(t-)| \to 0$ and hence

$$
\lim_{n \to \infty} \mu_n\{s_n\} = \lim_{n \to \infty} (F_n(s_n) - F_n(s_n-)) = 0
(6.3)
$$

and the analogue holds for sequences $s_n > t_n$ and $s_n \to t$. Hence we can apply Lemma 2.5 in [EK94] and get the result. 

\[\square\]
7 Proof of Theorem 2.12 (a), (b)

(a) First of all observe that

\[ d_{GP_a}(u,u') := d_{GP_2}(u,u') + \left| \nu^2(u(\{0\}) - \nu^2(u'(\{0\}) \right| + \rho_a(\nu^2,\nu^2) \]  

(7.1)

is a metric on \( U \), where \( \rho_a \) is given in [EK94]. Now, the properties follow analogue to Lemma 2.3 (combined with Lemma 2.5) in [EK94] and Proposition 5.6 in [GPW09].

(b) Recall the notation in Remark 5.1 and note that for \( u := [X,r,\mu] \in U \), \( \mu_h \) is purely atomic, \( h > 0 \). Let \( A_h \) be a finite subset of \( \{ x \in X : \mu_h(\{x\}) > 0 \} \) with the property, that

\[ \mu_h(X \setminus A_h) < h \]  

(7.2)

and let \( \bar{\mu}_h(\cdot) := \mu_h(\cdot \cap A_h) \), then, by Lemma 5.2

\[ [X,r,\bar{\mu}_h] \to u. \]  

(7.3)

In addition, note that

\[ \nu^2(u(\{h'\}) = \nu^2,\hat{\Phi}_h(u(\{h'\}) \]  

for all \( 0 < h < h' \) (7.4)

and

\[ \left| \nu^2[X,r,\mu_h](\{h'\}) - \nu^2[X,r,\mu_h](\{h'\}) \right| \leq 2\mu_h(X) \cdot \mu_h(X \setminus A_h). \]  

(7.5)

This shows \( [X,r,\bar{\mu}_h] \to u \) in the Gromov-weak atomic topology (see again Section 6 or [EK94]).

Let \( n \in \mathbb{N} \) and \( a \in \mathbb{R}_+^n \). Using an induction argument and the fact that \( \bigcup_{k \in \mathbb{N}} \{ \sum_{i \in I} a_i^k : I \subset \{1, \ldots, n\} \} \) is countable, where \( a^k \in \mathbb{R}_+^n \), \( k \in \mathbb{N} \), it is straightforward to see that one can approximate \( a \) (pointwise) by a sequences \( a^k \) with

\[ \forall I, J \subset \{1, \ldots, n\}, I \cap J = \emptyset : \sum_{i \in I} a_i^k \neq \sum_{j \in J} a_j^k. \]  

(7.6)

When we now take \( [X,r,\mu] \in U \) with \( \mu = \sum_{i=1}^n a_i \delta_{x_i} \) for \( x_i \in X \), \( i = 1, \ldots, n \), this shows, that the sequence of measures \( \mu^k = \sum_{i=1}^n a_i^k \delta_{x_i} \) satisfy \( \mu^k \Rightarrow \mu \). Using a similar argument as above together with Lemma 7.2 below, finally gives the result (compare also Proposition 5.6 in [GPW09]).
Assume \( A_h = \{x_1, \ldots, x_n\} \). Then, another another way of proving this result is to disturb the measure a bit, i.e. to add a realization of independent, positive, variables \( U_1, \ldots, U_n \) with \( \sum_i U_i = 1/n \) to \( \mu_h(\{x_1\}, \ldots, \mu(\{x_n\}) \) (compare Example 2.3).

In order to prove (c) of this theorem, we need some more results on the function \( \mathcal{F} \). Therefore, we skip the proof at this point and refer to Section 9.

**Remark 7.1.** Note that the above argument can be modified to prove that \( \mathcal{F}(\mathbb{I}) \) is dense in \( \mathcal{F}(\mathbb{U}) \).

**Lemma 7.2.** Let \( u = [\{x_1, \ldots, x_n\}, r, \mu] \in \mathbb{U} \), \( n \in \mathbb{N} \cup \{\infty\} \) and \( a_i = \mu(\{x_i\}) \), \( i \in \mathbb{N} \). Then \( u \in \mathbb{I} \) if and only if

\[
\sum_{i \in I} a_i \neq \sum_{i \in J} a_i, \quad \forall \ I, J \subset \{1, \ldots, n\}, \ I \neq J. \tag{7.7}
\]

**Proof.** This follows directly by definition, since for all \( h \geq 0 \) and \( x \in \{x_1, \ldots, x_n\} \) there is a set \( I \) such that \( \mu(\bar{B}_h(x)) = \sum_{i \in I} \mu(\{x_i\}) \).

\[
8 \text{ Proofs for Section 3}
\]

Here we give the proofs of our results on the function \( \mathcal{F} \).

**8.1 Proof of Lemma 3.1**

For the first part we observe that since \((\text{supp}(\mu), r)\) is separable there is a countable set \( J \subset \text{supp}(\mu) \), such that

\[
\text{supp}(\mu) \subset \bigcup_{x \in J} \bar{B}(x, h). \tag{8.1}
\]

We define the set

\[
\mathcal{I} := \{ I \subset J : \mu(\bar{B}(x, h) \cap \bar{B}(y, h)) = 0, \ \forall x, y \in I, \ x \neq y \}. \tag{8.2}
\]

Note that \( \subset \) defines a partial order on \( \mathcal{I} \). If we take a totally ordered subset \( \mathcal{T} \subset \mathcal{I} \), then \( \bigcup_{A \in \mathcal{T}} A \in \mathcal{I} \) (for two different elements \( x, y \in \bigcup_{A \in \mathcal{T}} A \), there is a set \( A' \in \mathcal{T} \), since \( \mathcal{T} \) is totally ordered, such that \( x, y \in A' \) is an upper bound for \( \mathcal{T} \). By Zorn’s lemma, we can find a maximal set \( I \in \mathcal{I} \).
It remains to proof that
\[ \mu(X) = \mu(\text{supp}(\mu)) = \sum_{x \in I} \mu(\bar{B}(x, h)). \quad (8.3) \]

Note that for \( x, y \in \text{supp}(\mu) \), since \( r \) is an ultra-metric \( \mu \) almost surely, we either have
\[ \mu(\bar{B}(x, h) \cap \bar{B}(y, h)) = 0, \quad (8.4) \]
or
\[ \mu(\bar{B}(x, h) \cap \bar{B}(y, h)) = \tilde{\mu}(\bar{B}(y, h)). \quad (8.5) \]

By (8.1), we have
\[ \mu(X) = \mu\left( \bigcup_{x \in J} \bar{B}(x, h) \right). \quad (8.6) \]

If we would assume that \( \mu(X) > \sum_{x \in I} \mu(\bar{B}(x, h)) \), then, since \( I \subset J \), we would find a \( \tilde{x} \in J \) such that
\[ \mu(\bar{B}(\tilde{x}, h) \cap \bar{B}(x, h)) = 0, \quad \forall x \in I. \quad (8.7) \]

This is a contradiction, since \( I \) is a maximal element of \( \mathcal{I} \).

For the second part we set
\[ I_i := \left\{ j \in \{1, \ldots, n(\delta)\} : \mu(\bar{B}(\tau^h_j, \delta) \cap \bar{B}(\tau^h_i, h)) > 0 \right\}. \quad (8.8) \]

Since \( r \) is an ultra-metric \( \mu \)-almost surely, we get \( \mu(\bar{B}(\tau^h_j, \delta) \cap \bar{B}(\tau^h_i, h)) = \mu(\bar{B}(\tau^h_j, \delta)) \) for all \( j \in I_i \). This together with the first part implies \( \geq \).

Let \( A := \bar{B}(\tau^h_i, h) \setminus \bigcup_{j \in I_i} \bar{B}(\tau^h_j, \delta) \). If we assume that \( \mu(A) > 0 \), then we can take a \( x \in A \cap \text{supp}(\mu) \) and, by Remark 3.5 we find a \( j \) such that
\[ \mu(\bar{B}(x, \delta) \cap \bar{B}(\tau^h_j, \delta)) = \mu(\bar{B}(x, \delta)). \]

It follows that \( \mu(\bar{B}(\tau^h_i, h) \cap \bar{B}(\tau^h_j, \delta)) = \mu(\bar{B}(\tau^h_j, \delta)) > 0 \) and hence \( j \in I_i \). A contradiction and therefore \( \mu(A) = 0 \). To see that \( \{I_i\}_{i=1}^{n(h)} \) forms a partition follows by similar arguments.

### 8.2 A result on relative compactness and proof of Lemma 3.8 and Proposition 3.13

We have the following result on relative compactness:

**Proposition 8.1.** (Relative compactness and further properties) Let \( (u_n)_{n \in \mathbb{N}} \) be a sequence in \( U \) and \( u \in U \).


(i) If $u_n \to u$ in the Gromov-weak topology and $(h_n)_{n \in \mathbb{N}}$ is a sequence in $(0, \infty)$ with $h_n \to h \in (0, \infty)$, then $\{f(u_n, h_n) : n \in \mathbb{N}\}$ is relatively compact in $S^\downarrow$, equipped with $d^1$.

(ii) $\mathfrak{F}(\mathbb{U}) \subset D((0, \infty), S^\downarrow)$ and
\[
\lim_{h \downarrow 0} \max_{i} |f(u, h)_i - f(u, 0)_i| = \lim_{h \downarrow 0} d^\infty(f(u, h), f(u, 0)) = 0, \quad (8.9)
\]
where $f([X, r, \mu], 0) := (\mu\{x\}_{x \in X})^\downarrow$, i.e. the decreasing rearrangement of the atoms of $\mu$.

(iii) If $d^1(f(u_n, h), f(u, h)) \to 0$ for all continuity points $h$ of $f(u, \cdot)$, then $\{u_n : n \in \mathbb{N}\}$ is relatively compact with respect to the Gromov-weak topology.

Remark 8.2. Note that (ii) is Lemma 3.8.

Before we start we need the following result on monotonicity, which is a direct consequence of Lemma 3.1:

Lemma 8.3. Let $0 < \delta \leq h$, $u \in \mathbb{U}$ and assume we are in the situation of Lemma 3.7. Then $n(h) \leq n(\delta)$. Moreover, for $M \leq n(h)$:
\[
\sum_{i=1}^{M} f(u, h)_i \geq \sum_{i=1}^{M} f(u, \delta)_i. \quad (8.10)
\]

We are now able to prove Proposition 8.1 (i).

Proof. Proposition 8.1 - (i) Note that if we equip $S^\downarrow_C$, $C > 0$ with
\[
d^\infty(x, y) := \max\{|x_i - y_i| : i \in \mathbb{N}\}, \quad x, y \in S^\downarrow_C, \quad (8.11)
\]
then $(S^\downarrow_C, d^\infty)$ is a compact space (this follows analogue to Proposition 2.1 in [Ber06]).

Note that $u_n \to u$ implies $\overline{u}_n \to \overline{u}$, where $\overline{u} = \sqrt{\nu^2([0, \infty))}$ and hence we find a constant $C > 0$ such that
\[
\sup_{n \in \mathbb{N}} \overline{u}_n = \sup_{n \in \mathbb{N}} \sum_{i \in \mathbb{N}} f(u_n, h_n)_i \leq C. \quad (8.12)
\]

It follows that $\{f(u_n, h_n) : n \in \mathbb{N}\}$ is relatively compact in $(S^\downarrow_C, d^\infty)$. Hence, there is a $x \in S^\downarrow_C$ such that $d^\infty(f(u_{n_k}, h_{n_k}), x) \to 0$ along some
subsequence and we have to show that \( d^1(f(u_{n_k}, h_{n_k}), x) \to 0 \). We suppress the dependence on the subsequence and set

\[
\delta := \inf_{n \in \mathbb{N}} h_n > 0. \tag{8.13}
\]

Next we prove that for all \( 0 < \varepsilon \leq \delta \) there is a \( M \in \mathbb{N} \) such that

\[
\sup_{n \in \mathbb{N}} \sum_{i=M+1}^{\infty} f(u_n, h_n)_i < \varepsilon. \tag{8.14}
\]

We assume the converse, i.e. assume there is an \( \varepsilon > 0 \) with \( \varepsilon \leq \delta \) such that for all \( M \in \mathbb{N} \) there is a \( n \in \mathbb{N} \) with

\[
\sum_{i=M+1}^{\infty} f(u_n, h_n)_i \geq \varepsilon. \tag{8.15}
\]

Note that \( f(u_n, \bar{\varepsilon})_i \leq \frac{C}{M} \) for all \( i \geq M, M \in \mathbb{N}, \bar{\varepsilon} > 0 \). Moreover, note that when \( \bar{\varepsilon} \leq \delta \), Lemma 8.3 implies

\[
\sum_{i=M+1}^{\infty} f(u_n, \bar{\varepsilon})_i \geq \varepsilon. \tag{8.16}
\]

Since \([X_n, r_n, \mu_n] = u_n \to u\), we have (see Proposition 7.1 in [GPW09] and Proposition B.2 in [GGR16]):

\[
0 = \lim_{M \to \infty} \sup_{n \in \mathbb{N}} \nu_C^M (u_n)
= \lim_{M \to \infty} \sup_{n \in \mathbb{N}} \inf \left\{ \bar{\varepsilon} > 0 : \mu_n \left( \left\{ x \in X_n : \mu_n(B^{\bar{r}_n}(x, \bar{\varepsilon})) \leq \frac{C}{M} \right\} \right) \leq \bar{\varepsilon} \right\}
\geq \lim_{M \to \infty} \sup_{n \in \mathbb{N}} \inf \left\{ \bar{\varepsilon} > 0 : \mu_n \left( \left\{ x \in X_n : \mu_n(B^{\bar{r}_n}(x, \bar{\varepsilon})) \leq \frac{C}{M} \right\} \right) \leq \bar{\varepsilon} \right\}
= \lim_{M \to \infty} \sup_{n \in \mathbb{N}} \inf \left\{ \bar{\varepsilon} > 0 : \sum_{i=1}^{\infty} f(u_n, \bar{\varepsilon})_i 1 \left( f(u_n, \bar{\varepsilon})_i \leq \frac{C}{M} \right) \leq \bar{\varepsilon} \right\}
\geq \lim_{M \to \infty} \sup_{n \in \mathbb{N}} \inf \left\{ \bar{\varepsilon} > 0 : \sum_{i=M+1}^{\infty} f(u_n, \bar{\varepsilon})_i \leq \bar{\varepsilon} \right\}
\geq \varepsilon, \tag{8.17}
\]

a contradiction and (8.14) follows.
If we now define for $\varepsilon > 0$

$$M^x := \min \left\{ K : \sum_{i=K+1}^{\infty} x_i < \varepsilon \right\},$$

(8.18)

then (8.14) implies for all $0 < \varepsilon \leq \delta$, there is a $M \in \mathbb{N}$ such that

$$\sum_{i=1}^{\infty} |f(u_n, h_n)_i - x_i| \leq (M \lor M^x) \cdot \max_{i \in \mathbb{N}} |f(u_n, h_n)_i - x_i| + 2\varepsilon, \quad \forall n \in \mathbb{N}.$$

(8.19)

Therefore

$$\sum_{i=1}^{\infty} |f(u_n, h_n)_i - x_i| \nrightarrow 0.$$

(8.20)

\[\square\]

**Remark 8.4.** The above proof also shows: If $x \in S^1_C$, $u_n \rightarrow u$ in the Gromov-weak topology and $0 < h_n \rightarrow h > 0$, then the following is equivalent:

(i) $d^1(f(u_n, h_n), x) \rightarrow 0$,

(ii) $d^\infty(f(u_n, h_n), x) \rightarrow 0$.

Moreover, by Proposition 2.1. in [Ber06], the above is equivalent to

(iii) $f(u_n, h_n)_i \rightarrow x_i$ for all $i \in \mathbb{N}$.

\[\bullet\]

Next we show that $\mathcal{F}$ takes values in the space of cadlag functions. Before we do that we need the following Lemma:

**Lemma 8.5.** Let $u = [X, r, \mu], u_1 = [X_1, r_1, \mu_1], u_2 = [X_2, r_2, \mu_2], \ldots \in \mathbb{U}$ and assume that $u_n \rightarrow u$ in the Gromov-weak topology and $\nu^{2,u_n}(\{0\}) \rightarrow \nu^{2,u}(\{0\})$. Then there is a complete separable metric space $(Z, r_Z)$ and isometric embeddings $\varphi_n : X_n \rightarrow Z, n \in \mathbb{N}$ and $\varphi : X \rightarrow Z$ such that $\mu_n \circ \varphi_n^{-1} \rightarrow \mu \circ \varphi^{-1}$ in the weak atomic topology.

**Proof.** Observe that

$$\mu^*_n := \sum_{x \in X_n} \mu_n(\{x\})^2 = \nu^{2,u_n}(\{0\}) \rightarrow \nu^{2,u}(\{0\}) = \sum_{x \in X} \mu(\{x\})^2.$$  

(8.21)

When we now apply Lemma 5.8 in [GPW09] combined with Lemma 2.1 and 2.2 in [EK94], the result follows. \[\square\]
Proof. Proposition 8.1 - (ii) Let \( h \in (0, \infty) \) and \((h_n)_{n \in \mathbb{N}}\) be a sequence in \((0, \infty)\) with \( h_n \downarrow h \) and \( u = [X, r, \mu] \). Let \( \{v^h_i : i \in \{1, \ldots, n(h)\}\} \) be as in Lemma 3.1. Then we have
\[
\bar{B}(v^h_i, h) = \bigcap_{n \in \mathbb{N}} \bar{B}(v^h_i, h_n), \quad \forall i = 1, \ldots, n(h) \tag{8.22}
\]
and the \( \sigma \)-continuity of the measure \( \mu \) gives \( f(u, h_n)_i \rightarrow f(u, h)_i \) for all \( i \in \{1, \ldots, n(h)\} \). By Remark 8.4, this is enough to get the right continuity.

In order to prove that the limits from the left exist, assume \( h_n \uparrow h \).
Note that by Proposition 8.1 (i), \( \{f(u, h_n) : n \in \mathbb{N}\} \) is relatively compact in \((S^1, d^1)\). Let \( \tilde{x} \in S^1 \) be a limit point along some subsequence \( (n_k)_{k \in \mathbb{N}}, \varepsilon > 0 \) and \( M \in \{1, \ldots, n(h)\} \). By Lemma 8.3 we have for \( m \) large enough
\[
\sum_{i=1}^M f(u, h)_i \geq \sum_{i=1}^M f(u, h_n)_i \geq \sum_{i=1}^M f(u, h_m)_i, \quad \forall n \geq m. \tag{8.23}
\]
Hence \( 0 \leq \sum_{i=1}^M f(u, h_n)_i \) is a monotonically increasing sequence and therefore it converges to some \( S_M \geq 0 \). It follows that \( f(u, h_n)_1 \rightarrow S_1 \) and for \( M > 1 \)
\[
f(u, h_n)_M = \sum_{i=1}^M f(u, h)_i - \sum_{i=1}^{M-1} f(u, h_n)_i \xrightarrow{n \to \infty} S_M - S_{M-1}, \tag{8.24}
\]
i.e. \( \tilde{x}_i = S_i - S_{i-1} (S_0 := 0) \) for all \( i \in \{1, \ldots, n(h)\} \) independent of the subsequence and the existence of the left limit follows.

For the second part, set \( u_n := \Phi_{h_n}(u) = [X_n, r_n, \mu_n] \), for \( h_n \downarrow 0 \). Then, by Lemma 5.2 \( u_n \rightarrow u \), in the Gromov weak topology. Moreover, by the definition of \( \mu_n \) (see Remark 5.1), we have
\[
\nu^2(u_n) = \nu^2(u_n([0, t_n])) \rightarrow \nu^2(u([0, t])) \tag{8.25}
\]
We can now apply Lemma 2.5 in [EK94] combined with Lemma 8.5 to get the result. \( \square \)

Remark 8.6. Let \( \mu_h \) be the measure given in Remark 5.1. As we have seen in the above proof, \( \mu_h \rightarrow \mu \) in the weak atomic topology when \( h \downarrow 0 \) (on \((X, r))\). In particular, by Definition 6.1,
\[
\sum_{i \in \mathbb{N}} f(u, h)_i^2 \rightarrow \sum_{i \in \mathbb{N}} f(u, 0)_i^2, \quad \text{for } h \downarrow 0. \tag{8.26}
\]
Note also that
\[ \nu^{2,u}(\{0\}) = \sum_{i=1}^{\infty} f(u,0)_i^2. \quad \text{(8.27)} \]

Before we prove Proposition 8.1 (iii), we need the following connection of \( \mathcal{F}(u) \) for a given \( u \in U \) and the pairwise distance matrix distribution \( \nu^{2,u} \):

**Lemma 8.7.** Let \( u = [X, r, \mu] \in U, h > 0 \). Then
\[ \nu^{2,u}[0, h] = \sum_{i \in \mathbb{N}} (f(u,h)_i)^2 \quad \text{(8.28)} \]
and \( \nu^{2,u}\{h\} = 0 \) if and only if \( d^1(f(u,h), f(u,h)) = 0 \), i.e. the continuity points of \( \nu^{2,u} \) are exactly the ones of \( f(u,\cdot) \).

**Remark 8.8.** Since \( f(u,\cdot) \) is cadlag and constant between its jump points, the measure \( \nu^{2,u} \) is purely atomic.

**Proof.** Recall Lemma 3.1. By definition we have
\[ \nu^{2,u}[0, h] = \mu \otimes \mu(\{(x,y) \in X \times X : r(x,y) \leq h\}) \]
\[ = \sum_{i \in \{1, \ldots, n(h)\}} \mu(B(e_i,h))^2 \]
\[ = \sum_{i \in \mathbb{N}} (f(u,h)_i)^2. \quad \text{(8.29)} \]

Moreover, since \( \mathcal{F}(u) \) is cadlag, where \( S^1 \) is equipped with \( d^1 \), we get
\[ \nu^{2,u}[0, h] = \lim_{\varepsilon \downarrow 0} \nu^{2,u}[0, h - \varepsilon] = \sum_{i \in \mathbb{N}} (f(u,h)_i)^2. \quad \text{(8.30)} \]

It follows that
\[ \nu^{2,u}\{h\} = \sum_{i \in \mathbb{N}} ( (f(u,h)_i)^2 - (f(u,h-)_i)^2 ). \quad \text{(8.31)} \]

As a consequence we get (recall \( \overline{u} := \sqrt{\nu^{2,u}([0,\infty))} \) is the total mass.)
\[ \nu^{2,u}\{h\} = \sum_{i \in \mathbb{N}} ( (f(u,h)_i)^2 - (f(u,h-)_i)^2 ) \]
\[ \leq \sum_{i \in \mathbb{N}} |f(u,h)_i - f(u,h-)_i| (f(u,h)_i + f(u,h-)_i) \]
\[ \leq 2 \overline{u} \cdot d^1(f(u,h), f(u,h-)). \quad \text{(8.32)} \]
and “⇐” follows. Now let \( \{ q^h_i : i \in \{1, 2, \ldots, n'(h)\} \} \) be as in Lemma 3.1 (for \((X, r, \mu)\)) where we replaced the closed balls by open balls (see Remark 3.2). Following the proof of Lemma 3.1 we find for each \( i \in \{1, \ldots, n(h)\} \) a set \( I_i \subset \{1, 2, \ldots, n'(h)\} \) such that

\[
\mu(\bar{B}(r^h_i, h)) = \sum_{j \in I_i} \mu(B(q^h_j, h))
\]

(8.33)

and \( \{ I_i \}_{i=1}^{n(h)} \) is a partition of \( \{1, 2, \ldots, n'(h)\} \). Now, using (8.29) and (8.30), we get

\[
\sum_{i \in \mathbb{N}} f(u, h−)^2_i = \sum_{i=1}^{n'(h)} \mu(B(q^h_i, h))^2
\]

(8.34)

and \( \nu^2_{\mu}({\{h\}}) = 0 \) implies that

\[
\sum_{i=1}^{n'(h)} \mu(B(q^h_i, h))^2 = \sum_{i=1}^{n(h)} \mu(\bar{B}(c^h_i, h))^2 = \sum_{i=1}^{n(h)} \left( \sum_{j \in I_i} \mu(B(q^h_j, h)) \right)^2.
\]

(8.35)

But this is equivalent to

\[
\sum_{i=1}^{n(h)} \left( \sum_{j \in I_i} \mu(B(q^h_j, h))^2 - \left( \sum_{j \in I_i} \mu(B(q^h_j, h)) \right)^2 \right) = 0.
\]

(8.36)

Since

\[
\sum_{j \in I_i} \mu(B(q^h_j, h))^2 - \left( \sum_{j \in I_i} \mu(B(q^h_j, h)) \right)^2 \leq 0,
\]

(8.37)

we get

\[
\sum_{j \in I_i} \mu(B(q^h_j, h))^2 = \left( \sum_{j \in I_i} \mu(B(q^h_j, h)) \right)^2,
\]

(8.38)

for all \( i \in \{1, \ldots, n(h)\} \) and hence \( |I_i| = 1 \) (note that \( \mu(B(q^h_j, h)) > 0 \) for all \( j \) - see Remark 3.2). But this shows that \( \mu(\bar{B}(r^h_i, h)) = \mu(B(q^h_i, h)) \) for all \( i \) and by definition of \( f \) (as the reordering of such masses) the result follows.

We can now finish the proof of Proposition 8.1.
Proof. Proposition 8.1 - (iii) To prove relative compactness recall that
\[ \nu_2^u[0, h] = \sum_{i \in N} (f(u, h)_i)^2, \]  
(8.39)
\[ \nu_2^u([0, \infty)) = \sqrt{\sum_{i \in N} f(u, h)_i}, \]  
(8.40)
for all \( h > 0 \) and that \( h > 0 \) is a continuity point of \( f(u, \cdot) \) iff \( \nu_2^u[0, h] := \nu_2^u[0, h], \) by Lemma 8.7.

Since \( d_1(f(u_n, h), f(u, h)) \to 0 \) for all continuity points \( h \) of \( f(u, \cdot) \) and
\[ 0 \leq \limsup_{n \to \infty} \nu_2^{u_n} (\{0\}) \leq \nu_2^u([0, \delta]) \]  
(8.41)
for all continuity points \( \delta > 0 \), we get
\[ \lim_{n \to \infty} \nu_2^{u_n} (\{0\}) = \limsup_{n \to \infty} \nu_2^{u_n} (\{0\}) = 0, \]  
(8.42)
if \( \nu_2^u\{0\} = 0 \). This gives
\[ \nu_2^{u_n} \xrightarrow{n \to \infty} \nu_2^u. \]  
(8.43)

For the relative compactness of \( \{u_n : n \in \mathbb{N}\} \) it remains to show, that (see Theorem 2 and Remark 2.11 in [GPW09]; see also Proposition B.2 in [GGR16]):
\[ \lim_{\delta \to 0} \limsup_{n \to \infty} \nu_\delta(u_n) = 0, \]  
(8.44)
where \( \nu_\delta(\cdot) \) is the modulus of mass distribution:
\[ \nu_\delta([X, r, \mu]) = \inf \left\{ \tilde{\varepsilon} > 0 : \mu(\{x \in X : \mu(B(x, \tilde{\varepsilon})) \leq \delta\}) \leq \varepsilon \right\}. \]  
(8.45)

Note that if \( u_n = [X_n, r_n, \mu_n] \), then (see Lemma 8.7 and its proof)
\[ \mu_n(\{x \in X_n : \mu_n(B^{r_n}(x, 2\varepsilon)) \leq \delta\}) \leq \mu_n(\{x \in X_n : \mu_n(B^{r_n}(x, \varepsilon)) \leq \delta\}) \]
\[ = \sum_{i=1}^{\infty} f(u_n, \varepsilon)_i \cdot I(f(u_n, \varepsilon)_i \leq \delta), \]  
(8.46)
for all \( \varepsilon > 0 \). Define
\[ M(\delta; \varepsilon) := \min\{i \in \mathbb{N} : f(u, \varepsilon)_i < \delta\}. \]  
(8.47)
If \( \varepsilon \) is a continuity point of \( f(u, \cdot) \) we get \( d^1(f(u_n, \varepsilon), f(u, \varepsilon)) \to 0 \) and hence

\[
\limsup_{n \to \infty} \sum_{i=1}^{\infty} f(u_n, \varepsilon)_i \cdot 1(f(u_n, \varepsilon)_i < \delta) = \limsup_{n \to \infty} \sum_{i=M(\delta; \varepsilon)}^{\infty} f(u_n, \varepsilon)_i \\
= \sum_{i=M(\delta; \varepsilon)}^{\infty} f(u, \varepsilon)_i.
\]  

(8.48)

Since the right hand side converges to 0 when \( \delta \downarrow 0 \), the result follows. \( \Box \)

Finally we give the proof of Proposition 3.13:

\textbf{Proof.} (Proposition 3.13) First observe that, by assumption,

\[
\limsup_{n \to \infty} P(\nu^2 L^n([H, \infty)) \geq \varepsilon) \\
= \limsup_{n \to \infty} P \left( \left( \sum_{i=1}^{\infty} f(U^n_i, H) \right)^2 - \sum_{i=1}^{\infty} f(U^n_i, \delta)^2 \geq \varepsilon \right) \\
\leq \varepsilon.
\]  

(8.49)

Hence, \( (L(\nu^2 L^n))_{n \in \mathbb{N}} \) is tight. On the other hand, recall the definition of \( \nu_\beta \) (see (8.45)) and note that by condition (i), for all \( \delta > 0 \) and \( \varepsilon > 0 \) there is a \( M \) such that

\[
\limsup_{n \to \infty} P(\sum_{i=M}^{\infty} f(U^n_i, \delta) \geq \varepsilon) \leq \varepsilon.
\]  

(8.50)

Therefore (compare the proof of Proposition 8.1 - (iii)), for all \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that

\[
\limsup_{n \to \infty} P(\nu_\delta(U^n) \geq \varepsilon) \\
\leq \limsup_{n \to \infty} P \left( \sum_{i=1}^{\infty} f(u_n, \varepsilon)_i \cdot 1(f(u_n, \varepsilon)_i \leq \delta) \geq \varepsilon \right) \\
= \limsup_{n \to \infty} P \left( \sum_{i=M}^{\infty} f(u, \varepsilon)_i \geq \varepsilon \right) \\
\leq \varepsilon.
\]  

(8.51)

Hence, the result follows by Theorem 3 in \[GPW09\] (compare also Remark 3.2 and Remark 7.2 (ii) in \[GPW09\]). \( \Box \)
8.3 Proof of Theorem 3.11 and Theorem 3.17

The main ingredient for the proof of continuity in Theorem 3.11 is the following lemma.

Lemma 8.9. Let \((u_n)_{n \in \mathbb{N}}\) be a sequence in \(U\) and \(u \in U\). If \(d_{GPr}(u_n, u) \to 0\) then \(d^1(f(u_n, h), f(u, h)) \to 0\) for all continuity points \(h\) of \(f(u, \cdot)\).

Before we can prove this Lemma we need the following:

Lemma 8.10. Recall the notation of Section 5.2 and let \(U\) be equipped with the Gromov-weak topology. Let \((u_n)_{n \in \mathbb{N}}\) be a sequence in \(U\), \(u \in U\) and \(u^n_\iota, u_\iota\) be as in Remark 5.6. Moreover, let \(\hat{u}_n := \bigsqcup_{i \in J_n} u^n_\iota\) and assume that \(|J_n| \equiv C \in \mathbb{N}\). If \(h > 0\) is a continuity point of \(f(u, \cdot)\), \(\hat{u}_n \to \hat{u}\) and \(u_n \to u\) in the Gromov-weak topology, then \(\hat{u} \leq h \Psi_h(u)\) and \(\hat{u} = \bigsqcup_{i \in J} u_i \in U\) with \(|J| \leq C\).

Proof. Note that \(\hat{u}_n \leq h \Psi_h(u_n)\) and hence Lemma 5.8 (i) and (ii) implies \(\hat{u} \leq h \Psi_h(u)\). Moreover, by Lemma 8.7 and Lemma 5.8 (iv) we get

\[\nu^{2, u_i}([h, \infty)) = 0, \quad \forall i = 1, \ldots, n(h)\]  

and hence Theorem 2.13 in [GGR16] implies the existence of a set \(J\) such that

\[\hat{u} = \bigsqcup_{i \in J} u_i.\]  

If we take a sequence \(i_n \in J_n\), then Lemma 5.8 (iii) and (i) gives the existence of a subsequence such that \(u^n_{i_n} \to \hat{u} \leq h \hat{u}\) and hence, there is a subset \(\tilde{J} \subset J\) such that

\[\tilde{u} = \bigsqcup_{i \in \tilde{J}} u_i.\]  

We assume that

\[|\tilde{J}| \geq 2.\]  

Then there are \(j_1, j_2 \in \tilde{J}\) such that (see Lemma 8.7, Remark 5.6 and Lemma 3.1):

\[\nu^{2, \tilde{u}}(\{j\}) \geq \overline{u}_{j_1} \cdot \overline{u}_{j_2} > 0\]  

(8.56)
and for every $\delta > 0$, by the Portmanteau Theorem:

$$0 < \nu^{2,\tilde{u}}(\{h\}) \leq \nu^{2,\tilde{u}}(h - \delta, h + \delta) \leq \liminf_{k \to \infty} \nu^{2,u_{nk}}(h - \delta, h + \delta) \leq \limsup_{k \to \infty} \nu^{2,u_{nk}}[h - \delta, h + \delta].$$

(8.57)

Since $\bigcap_{\delta > 0}[h - \delta, h + \delta] = \{h\}$, this is a contradiction to Lemma 8.7. Hence $|\tilde{J}| \leq 1$ and therefore there is either one $i \in J$ such that

$$u_{nk} \to u_i,$$  

(8.58)

or

$$u_{nk} \to [0, 0, 0] =: e,$$  

(8.59)

the neutral element with respect to $\sqcup$, i.e. $u \sqcup e = u = e \sqcup u$. Combining this with Lemma 5.8 (iii) and (v) implies $|J| \leq C$.

\begin{remark}
In fact, the above argument shows that the number of balls map, i.e. $n(h) : U \to \mathbb{N} \cup \{\infty\}$, for $h > 0$ (see Lemma 3.1), is lower semi-continuous, when $h$ is a continuity point.
\end{remark}

Now we can prove Lemma 8.9

\textbf{Proof.} (Lemma 8.9) Let $(n_k)_{k \in \mathbb{N}}$ such that $d^1(f(u_{nk}, h), x) \to 0 \in S^1$ (see Proposition 8.1 (i)). We will now prove that $x = f(u, h)$ and therefore we may assume w.l.o.g. that $f(u_n, h) \to x$ for the following. Define

$$\tilde{x}_1 := x_1,$$
$$\tilde{x}_l := \max\{x_i : i \in \mathbb{N}\} \setminus \{\tilde{x}_1, \ldots, \tilde{x}_{l-1}\}, \quad l \geq 1.$$  

(8.60)

Note that $d^1(f(u_n, h), x) \to 0$ implies: For all $L \in \mathbb{N}$ there is a $\tilde{\varepsilon} > 0$ and $N \in \mathbb{N}$ such that

$$\{i \in \mathbb{N} : |f(u_n, h)_i - \tilde{x}_l| < \varepsilon\} = \{i \in \mathbb{N} : x_i = \tilde{x}_l\} =: C_l,$$  

(8.61)

for all $n \geq N$, $0 < \varepsilon < \tilde{\varepsilon}$ and $l \leq L$. Let $(u_{nk}^n)_{i \in \mathbb{N}}$ and $(u_i)_{i \in \mathbb{N}}$ be as in Remark 5.6 $\delta > 0$ and $L \in \mathbb{N}$ large enough such that

$$\limsup_{n \to \infty} \left| \sum_{l \leq L} [C_l |\tilde{x}_l - \sum_{i \in \mathbb{N}} f(u_n, h)_i| \right| \leq \delta.$$  

(8.62)
and define
\[ U_n^l := \{ i \in \mathbb{N} : |\bar{u}_n^l - \bar{x}_l| < \varepsilon \}. \] (8.63)

Then, by Remark 5.6 and for all \( n \) large enough,
\[ |U_n^l| = |\{ i \in \mathbb{N} : |f(u_n, h)_i - \bar{x}_l| < \varepsilon \}| = |C_l| \in \mathbb{N} \] (8.64)

and by Lemma 8.10 and Lemma 5.8 (iii) and (v), we can find a subsequence \( (n_k)_{k \in \mathbb{N}} \) and a set \( \hat{U}^l \subset \mathbb{N} \) with \( |\hat{U}^l| \leq |C_l| \) such that
\[ \{ u_{n_k}^l : i \in U_{n_k}^l \} \to \{ u_i : i \in \hat{U}^l \} \] (8.65)
(where we allow duplications in the above sets) and
\[ \bigsqcup_{i \in U_{n_k}^l} u_{n_k}^l \to \bigsqcup_{i \in \hat{U}^l} u_i, \] (8.66)
for all \( l \leq L \). Let
\[ U_l := \{ i \in \mathbb{N} : \bar{u}_i = \bar{x}_l \}. \] (8.67)

Since \( \bar{u}_n^l \to \bar{x}_l \) for every sequence \( i_n \in U_n^l \) we get \( \hat{U}^l \subset U^l \) (independent of the choice of \( (n_k) \)) for all \( l \leq L \) and, by the observation in (8.59), we have \( |\hat{U}^l| = |C_l| \). On the other hand, if there is a \( l^* \leq L \) such that \( |U_{n^*}^l| > |C_{l^*}| \) then
\[ \bar{u} = \lim_{n \to \infty} \bar{u}_n = \lim_{n \to \infty} \sum_{l \in \mathbb{N}} f(u_n, h)_l \]
\[ \leq \lim_{n \to \infty} \sum_{l \leq L} |C_l|\bar{x}_l + \delta = \sum_{l \leq L} \sum_{i \in U_{n_k}^l} \bar{u}_i^l + \delta \]
\[ = \sum_{l \leq L} \sum_{i \in \hat{U}^l} \bar{u}_i + \delta \leq \sum_{l \leq L} \sum_{i \in U^l} \bar{u}_i - \hat{x}_l^* + \delta \]
\[ \leq \bar{u} - \hat{x}_l^* + \delta, \] (8.68)
which implies \( |U_l| = |C_l| \) for all \( l \leq L \) with \( \hat{x}_l \geq \delta \). Hence, \( \hat{U}^l = U^l \) for all those \( l \) and therefore, letting \( \delta \downarrow 0, x = f(u, h) \). \( \square \)

In order to prove Theorem 3.11 we need the following Lemma.

**Lemma 8.12.** Let \( (x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}} \) be a sequence in \( (0, \infty) \) with \( x_n \to x > 0, y_n \to y > 0 \) for \( n \to \infty \) and \( x_n < y_n \) for all \( n \in \mathbb{N} \). Moreover, let \( (u_n)_{n \in \mathbb{N}} \) be a sequence in \( U \) with \( u_n \to u \in U \), for \( n \to \infty \) (with respect to the Gromov-weak topology). Then
(i) $v^{2,u_n}(x_n, y_n) \to 0$ iff $d^1 \left( f(u_n, x_n), f(u_n, y_n) \right) \to 0$, for $n \to \infty$.

(ii) $v^{2,u_n}[x_n, y_n] \to 0$ iff $d^1 \left( f(u_n, x_n - ), f(u_n, y_n - ) \right) \to 0$, for $n \to \infty$.

(iii) $v^{2,u_n}(x_n, y_n) \to 0$ iff $d^1 \left( f(u_n, x_n), f(u_n, y_n - ) \right) \to 0$, for $n \to \infty$.

Proof. Similar to (8.32) we get “$\Rightarrow$” and it remains to prove “$\Leftarrow$”. By Lemma 8.7 we have

$$v^{2,u_n} (x_n, y_n) = \sum_{i \in \mathbb{N}} \left( (f(u_n, x_n)_i)^2 - (f(u_n, y_n)_i)^2 \right)$$

and by Lemma 3.1 we find for all $n \in \mathbb{N}$ a partition $\{I^n_i\}_{i=1,\ldots,N(y_n)}$ of $\{1, \ldots, N(x_n)\}$ (where we write $N(h)$ instead of $n(h)$ to avoid confusion with the index of the sequence $u_n$) such that

$$\sum_{i \in \mathbb{N}} \left( (f(u_n, y_n)_i)^2 - (f(u_n, x_n)_i)^2 \right)$$

$$= \sum_{i=1}^{N(y_n)} \left( \sum_{j \in I^n_i} f(u_n, x_n)_j \right)^2 - \sum_{i=1}^{N(x_n)} (f(u_n, x_n)_i)^2$$

$$= \sum_{i=1}^{N(y_n)} \sum_{k,l \in I^n_i, k \neq l} f(u_n, x_n)_k \cdot f(u_n, x_n)_l$$

$$= 2 \sum_{i=1}^{N(y_n)} \mathbb{1}(|I^n_i| \geq 2) \sum_{k,l \in I^n_i, k < l} f(u_n, x_n)_k \cdot f(u_n, x_n)_l$$

$$\xrightarrow{n \to \infty} 0.$$ 

We apply Proposition 8.1 (i) twice and find $a, b \in \mathcal{S}^↓$ such that

$$d^1 \left( f(u_{n_k}, x_{n_k}), a \right) \to 0, \quad d^1 \left( f(u_{n_k}, y_{n_k}), b \right) \to 0, \quad (8.71)$$

along some subsequence $(n_k)_{k \in \mathbb{N}}$. Using an induction argument, we will now prove that $a = b$ holds.

We suppress the dependence on the subsequence, set $m^↓_1 := \min(I^n_i)$ and
observe that by (8.70)

$$b_1 = \lim_{n \to \infty} f(u_n, y_n)_1 = \lim_{n \to \infty} \sum_{j \in I^n} f(u_n, x_n)_j$$

$$= \lim_{n \to \infty} f(u_n, x_n)_{m^n_h} + \lim_{n \to \infty} \frac{1}{f(u_n, x_n)_{m^n_h}} \sum_{j \in I^n, j > m^n_h} f(u_n, x_n)_j$$

$$= \lim_{n \to \infty} f(u_n, x_n)_{m^n_h} \leq a_1. \tag{8.72}$$

Since \( \{I^n\}_{i=1,\ldots,N(y_n)} \) is a partition of \( \{1, \ldots, N(x_n)\} \) we find an \( \tilde{n} \in \{1, \ldots, N(y_n)\} \), such that \( 1 \in I^n_{\tilde{n}}. \) Again, by (8.70), it follows that

$$b_1 = \lim_{n \to \infty} f(u_n, y_n)_1 \geq \lim_{n \to \infty} f(u_n, y_n)_{\tilde{n}}$$

$$= \lim_{n \to \infty} \sum_{j \in I^n_{\tilde{n}}} f(u_n, x_n)_j = \lim_{n \to \infty} f(u_n, x_n)_1 = a_1 \tag{8.73}$$

and hence \( a_1 = b_1. \) Let \( k \in \mathbb{N} \) and assume that \( a_l = b_l \) for all \( l = 1, 2, \ldots, k, \) i.e.

$$\lim_{n \to \infty} |f(u_n, y_n)_l - f(u_n, x_n)_l| = 0. \tag{8.74}$$

Then, after a suitable reordering, we can assume that \( l \in I^n \) for all \( l = 1, 2, \ldots, k \) and all \( n \) large enough. We can now apply the argument in (8.72) and in (8.73) to get \( b_{k+1} \leq a_{k+1} \) and \( b_{k+1} \geq a_{k+1}. \)

Similar arguments show that the other cases hold (see again Lemma 3.1 and its proof).

Now, we are ready to prove the main result:

Proof. (Theorem 3.11) (i) - Continuity

We start by proving \( u_n \to u \) in the Gromov-weak atomic topology implies \( \tilde{\mathcal{F}}(u_n) \to \tilde{\mathcal{F}}(u) \) in the Skorohod topology, i.e. we prove that \( \tilde{\mathcal{F}} \) is continuous.

We apply Remark 2.7, Proposition 6.2 and Corollary 3.3.2 in [EK86] and get that \( F_n \to F \) in the Skorohod topology, where \( F_n(h) := \nu^2 u_n([0, h])1(h \geq 0) \) and \( F(h) := \nu^2 u([0, h])1(h \geq 0). \) Take a sequence \( h_n \to h > 0 \) such that \( h_n \to h > 0 \) and assume that \( F_n(h_n) \to F(h) \). Let \( \bar{\varepsilon} > 0 \) and take an \( \varepsilon > 0 \) such that \( |F(h + \varepsilon) - F(h)| < \bar{\varepsilon} \) and \( h + \varepsilon \) is a continuity point of \( F \). It follows that \( |F_n(h_n) - F_n(h + \varepsilon)| < \bar{\varepsilon} \) for all \( n \) large enough and hence, by Lemma 8.12 for all \( \bar{\varepsilon} > 0 \) there is an \( \varepsilon > 0 \) such that

$$d^1\left( f(u_n, h_n), f(u_n, h + \varepsilon) \right) < \bar{\varepsilon} \tag{8.75}$$
for all $n$ large enough. Since we can choose $\varepsilon$ such that $h + \varepsilon$ is a continuity point of $F$ and hence of $f(\cdot, u)$ (see Lemma 8.7), Lemma 8.9 implies that for all $n$ large enough
\[
 d^1\left(f(u_n, h_n), f(u, h + \varepsilon)\right) < \varepsilon. \tag{8.76}
\]
Since $f(u, h + \varepsilon) \to f(u, h)$ for $\varepsilon \downarrow 0$, this gives
\[
 d^1\left(f(u_n, h_n), f(u, h)\right) \to 0. \tag{8.77}
\]
If $F_n(h_n) \to F(h^-)$ we can use a similar argument to show that
\[
 d^1\left(f(u_n, h_n), f(u, h^-)\right) \to 0. \tag{8.78}
\]
We can now apply Proposition 3.6.5 in [EK86] (see also Section A) and get $\mathfrak{F}(u_n) \to \mathfrak{F}(u)$ in the Skorohod topology, when we can show that $0 \leq t_n \to 0$ implies $d^\infty(f(u_n, t_n), f(u, 0)) \to 0$.

By Lemma 8.5, we may assume w.l.o.g. that $u_n = [X, r, \mu_n], u = [X, r, \mu]$ and $\mu_n \to \mu$ in the weak atomic topology and applying Lemma 2.5 in [EK94] gives $d^\infty(f(u_n, 0), f(u, 0)) \to 0$. Since $\nu^{2,u}$ is purely atomic (see Remark 8.8) and again, by Lemma 2.5 in [EK94], it is not hard to see that $\nu^{2,u_n}(0, t_n) \to 0$ (otherwise two atoms would merge in the limit which is prohibited in the weak atomic topology). Moreover, by Lemma 5.2
\[
 d_{GPr}(\Phi_{t_n}(u_n), u) \leq d_{GPr}(\Phi_{t_n}(u_n), u_n) + d_{GPr}(u_n, u) \to 0 \tag{8.79}
\]
and
\[
 \nu^{2,\Phi_{t_n}(u_n)}(\{0\}) = \nu^{2,u_n}(0, t_n) = \nu^{2,u_n}(\{0\}) + \nu^{2,u_n}((0, t_n]) \to \nu^{2,u}(\{0\}). \tag{8.80}
\]
Applying again Lemma 8.5 and the argument from above gives
\[
 d^\infty(f(u_n, t_n), f(u, 0)) \to 0 \tag{8.81}
\]
and the result follows.

(i) - Perfectness
Let $K \subset \mathfrak{F}(U)$ be compact, then, because of the continuity of $\mathfrak{F}$, $\mathfrak{F}^{-1}(K)$ is closed and, as a direct consequence of Proposition 8.1, $\mathfrak{F}^{-1}(K)$ is relatively compact in the Gromov-weak topology.
If we now take \( u_n \in \mathcal{F}^{-1}(K) \) this gives \( u_{n_k} \to u \in \mathcal{F}^{-1}(K) \) (where the closure is taken with respect to the Gromov-weak topology) along some subsequence (where we suppress this dependence in the following) in the Gromov-weak topology. Since \( \mathcal{F}(u_n) \in K, \}\{ \mathcal{F}(u_n) : n \in \mathbb{N} \} \) is relatively compact in the Skorohod topology. Moreover, since 

\[
|\nu^{u_n}(0, h]) - \nu^u([0, h])| \leq d^1(\bar{f}(u_n, h), \bar{f}(u, h)), \quad \text{for all } h > 0, \quad (8.82)
\]

we can apply Theorem 3.6.3. in [EK86] together with Proposition 6.2 (recall that continuous images of compact sets are compact) to get that \( \nu^{u_n} |_{[\delta, \infty)} : n \in \mathbb{N} \) is relatively compact in the weak atomic topology for all continuity points \( \delta > 0 \) of \( \nu^u([0, \cdot]) \). It follows that \( \nu^{u_n} |_{[\delta, \infty)} \to \nu^u |_{[\delta, \infty)} \) in the weak atomic topology and therefore \( \Phi_\delta(u_n) \to \Phi_\delta(u) \) in the Gromov-weak atomic topology, for all such \( \delta > 0 \). Note that this also gives \( \mathcal{F}(\Phi_\delta(u_n)) = (\bar{f}(u_n, h))_{h \geq \delta} \to (\bar{f}(u, h))_{h \geq \delta} \) and hence, since \( \mathcal{F}(u_n) \) is relatively compact, \( \mathcal{F}(u_n) \to \mathcal{F}(u) \) in the Skorohod topology.

It remains to prove that

(i) \( \lim_{\delta \downarrow 0} \limsup_n \nu^{u_n}([0, \delta]) = 0 \),

(ii) \( \nu^{u_n}(\emptyset) \to \nu^u(\emptyset) \).

Note that the Portmanteau Theorem gives "(ii) \( \Rightarrow \) (i)".

In terms of Lemma 8.5 we may assume w.l.o.g. that \( u_n = [X, r, \mu_n], \) \( u = [X, r, \mu] \) with \( \mu_n \Rightarrow \mu \) and need to prove \( \mu_n \to \mu \) in the weak atomic topology. But by Lemma 2.5 in [EK94], this is implied by the convergence \( d^\infty(\bar{f}(u_n, 0), \bar{f}(u, 0)) \to 0 \).

(ii) - Results for the subspace

We start by proving that \( \mathcal{F} \) restricted to \( \mathbb{I} \) is injective. Assume \( \mathcal{F}(u) = \mathcal{F}(u') \) for \( u = [X, r, \mu], u' = [X', r', \mu'] \in \mathbb{I} \) but \( u \neq u' \). Since \( u \in \mathbb{I} \) iff \( \Phi_h(u) \in \mathbb{I} \) and \( u = u' \) iff \( \Phi_h(u) \in \Phi_h(u') \) for all \( h > 0 \), we assume w.l.o.g. that \( u \in \mathbb{I} \cap \mathbb{U}^a \).

Using the idea in the proof of Theorem 2.12 (ii) we may further assume that \( \mu \) and \( \mu' \) have only finitely many atoms and we denote the atoms by \( (a_i)_{i=1,\ldots,m} \) and \( (a'_i)_{i=1,\ldots,m'} \), \( m, m' \in \mathbb{N} \).

Let \( 0 = t_0 < t_1 < \ldots < t_n \) be the discontinuity points of \( \bar{f}(u, \cdot) = \bar{f}(u', \cdot) \) and note that these points correspond exactly to the points \( \{ r(x, y) : x, y \in \text{supp}(\mu), x \neq y \} \). Moreover, since \( \bar{f}(u, 0) = \bar{f}(u', 0) \) we get (up to reordering) \( (a_i)_{i=1,\ldots,m} = (a'_i)_{i=1,\ldots,m'} \). When we now apply Lemma 7.2 we find for all \( t_i \) pairwise disjoint sets \( (A^k_i)_{k=1,\ldots,K} \), where \( K \) is the number of the non zero
elements of \( f(u, t_i) \) such that
\[
f(u, t_i)_k = \sum_{j \in A^k_i} a_j, \quad k = 1, \ldots, K
\] (8.83)
and the choice of \((A^k_i)_{k=1,\ldots,K}\) is unique. This uniqueness allows us to define the ultra-metric \( \tilde{r} \) on \( \{1, \ldots, m\} \) by
\[
\tilde{r}(x, y) = t_i \iff \min \left\{ j \in \{1, \ldots, n\}, \exists k \text{ s.t. } x, y \in A^k_j \right\} = i \quad (8.84)
\]
and we get
\[
\left[ \{1, \ldots, m\}, \tilde{r}, \sum_{i=1}^{m} a_i \delta_i \right] = u \quad (8.85)
\]
(compare also Remark 3.3). Since all the quantities that were necessary for the above construction are completely determined by the values of \( f(u, \cdot) \), we have \( u = u' \) and get the injectivity of \( F\big|_{I} \).

Since \( F\big|_{I} \) is continuous, it remains to prove that \( \mathcal{F}(\mathcal{I}) \ni f_n \to f \in \mathcal{F}(\mathcal{I}) \) implies \( u_n := \mathcal{F}^{-1}(f_n) \to \mathcal{F}^{-1}(f) =: u \).

Since \( \{f_n : n \in \mathbb{N}\} \) is relatively compact, we can apply part (i) of this theorem and get \( \{u_n : n \in \mathbb{N}\} \) is relatively compact and hence satisfies \( u_{n_k} \to \bar{u} \in U \) along some subsequence. It follows that
\[
\mathcal{F}(u_{n_k}) = f_{n_k} \to f = \mathcal{F}(\bar{u}) \in \mathcal{F}(\mathcal{I}).
\] (8.86)
Since \( u \in \mathcal{I} \) if and only if \( \mathcal{F}(u) \in \mathcal{F}(\mathcal{I}) \) (this is exactly the construction from above), we get \( \bar{u} = \mathcal{F}^{-1}(f) \).

**Remark 8.13.** As we have seen in the proof of part (ii), the following holds:
\( u \in \mathcal{I} \) if and only if \( \mathcal{F}(u) \in \mathcal{F}(\mathcal{I}) \).

**Proof.** (Proof of Theorem 3.17) We use the ideas of Theorem 3.11(ii), take \( u = [X, r, \mu] \in \mathcal{B} \cap \mathcal{I} \) and assume \( \mu \) has only finitely many atoms \((a_i)_{i=1,\ldots,m}\). Denote the discontinuity points of \( \nu^2u[0, \cdot] \) by \( t_0 := 0 < t_1 < \ldots < t_n \) and observe that, by Lemma 8.7, these points correspond to \( \{r(x, y) : x, y \in \text{supp}(\mu), x \neq y\} \). In particular \( m = n + 1 \).

Take \( u' = [X', r', \mu'] \in \mathcal{B} \cap \mathcal{I} \) and assume \( \nu^2u = \nu^2u' \), then, as in the proof of Theorem 3.11, the set of discontinuity points of \( \nu^2u'[0, \cdot] \) coincides with \( t_0 := 0 < t_1 < \ldots < t_n \). Moreover, since \( \nu^2u([0, t_n]) = \nu^2u'([0, t_n]) \) we get \( \tilde{f}(u, t_n) = \tilde{f}(u', t_n) \).
Assume now, that $f(u, t_k) = f(u', t_k)$ for all $k = n, n - 1, \ldots, K + 1$ and some $K \leq n - 1$ and define $a^K_i := f(u, t_K)$. If $f(u, t_K) \neq f(u', t_k)$, then, by definition of $B \cap I$, there is a $l^*, i_1, i_2$ and $j^*, j_1, j_2$ such that

$$a^{K+1}_l = \begin{cases} a^K_l, & l < n^*, \\ a^K_{i_1} + a^K_{i_2}, & l = n^*, \\ a^K_{i_1}, & l \in \{n^* + 1, \ldots, i_1\}, \\ a^K_{i_1}, & l \in \{i_1 + 1, \ldots, i_2 - 1\}, \\ a^K_{i_1}, & l > i_2. \end{cases}$$

(8.87)

and

$$\tilde{a}^{K+1}_l = \begin{cases} \tilde{a}^K_l, & l < n^*, \\ \tilde{a}^K_{j_1} + \tilde{a}^K_{j_2}, & l = n^*, \\ \tilde{a}^K_{j_1}, & l \in \{n^* + 1, \ldots, j_1\}, \\ \tilde{a}^K_{j_1}, & l \in \{j_1 + 1, \ldots, j_2 - 1\}, \\ \tilde{a}^K_{j_1}, & l > j_2. \end{cases}$$

(8.88)

and for a given $u \in B \cap I$, $i^*, i_1, i_2$ are unique. By assumption, we therefore need to prove that $\{a^K_i, a^K_{i_2}\} = \{\tilde{a}^K_j, \tilde{a}^K_{j_2}\}$.

If $i^* = j^*$, then, since $\nu^{2,u} = \nu^{2,u'}$,

$$\begin{align*}
(1) & \ a^K_{i_1} + a^K_{i_2} = \tilde{a}^K_{j_1} + \tilde{a}^K_{j_2} \iff a^{K+1}_{i^*} = a^{K+1}_{j^*}, \\
(II) & \ (a^K_{i_1})^2 + (a^K_{i_2})^2 + (a^{K+1}_{j^*})^2 = (\tilde{a}^K_{j_1})^2 + (\tilde{a}^K_{j_2})^2 + (a^{K+1}_{j^*})^2
\end{align*}$$

(8.89)

and the result follows. Hence, we assume $i^* \neq j^*$, (which implies $a^{K+1}_{i^*} \neq a^{K+1}_{j^*}$). Note that by Theorem 3.11(ii), we have

$$\Phi_{t_{K+1}}(u) = \Phi_{t_{K+1}}(u') = \left\{1, \ldots, n - K\right\}, r, \sum_{l=1}^{n-K} a^K_{i+l} \delta_l \right\}.$$  

(8.90)

We assume w.l.o.g. that $i_1 = 1, i_2 = 2, j^* = 3$ and define the ultra-metric $\tilde{r}^1$ on \{1, 2, 3\} by

$$\tilde{r}^1(1, 2) = t_{K+1} - t_k =: s^1, \quad \tilde{r}^1(1, 3) = r(i^*, j^*) + t_{K+1} - t_k =: s^2.$$  

(8.91)

On the other hand we assume, w.l.o.g. $j_1 = 4, j_2 = 5, i^* = 6$ and define the ultra-metric $\tilde{r}^2$ on \{4, 5, 6\} by

$$\tilde{r}^2(4, 5) = t_{K+1} - t_k = s^1, \quad \tilde{r}^2(4, 6) = r(i^*, j^*) + t_{K+1} - t_k = s^2.$$  

(8.92)

By assumption $\nu^{2, u} = \nu^{2, u'}$, where

$$u^1 = \left\{1, 2, 3\right\}, \tilde{r}^1, a^K_{i_1} \delta_1 + a^K_{i_2} \delta_2 + a^{K+1}_{j_2} \delta_3$$  

(8.93)
and
\[ u^2 = \{4, 5, 6\}, \tau^2, a_{ij}^K \delta_1 + a_{ij}^K \delta_2 + a_{ij}^{K+1} \delta_3 \].  \tag{8.94}

For simplicity, set \( x := a_{i1}^K, y := a_{i2}^K, z := a_{j}^{K+1} \), then the following holds:
\[
\begin{align*}
    b_0 &= \nu^2 u([0]) = x^2 + y^2 + z^2, \\
    b_1 &= \nu^2 u([0, t_1]) = (x + y)^2 + z^2, \\
    b_2 &= \nu^2 u([0, t_2]) = (x + y + z)^2.
\end{align*}
\tag{8.95}
\]

Now, up to reordering and with respect to the constraints on the \( a_i \), there is only one possible solution to the above system of equations, given by
\[
\begin{align*}
x &= \frac{\sqrt{2\sqrt{b_2} \cdot \sqrt{2b_1 - b_2 + 2b_1} + \sqrt{2\sqrt{b_2} \cdot \sqrt{2b_1 - b_2 - 2b_1} + 8b_0 - 6b_1}}}{4}, \\
y &= \frac{\sqrt{2\sqrt{b_2} \cdot \sqrt{2b_1 - b_2 + 2b_1} - \sqrt{2\sqrt{b_2} \cdot \sqrt{2b_1 - b_2 - 2b_1} + 8b_0 - 6b_1}}}{4}, \\
z &= \frac{\sqrt{b_2 - \sqrt{2b_1 - b_2}}}{2},
\end{align*}
\tag{8.96}
\]
and the result follows. \(\square\)

### 8.4 Proof of Proposition 3.16

Note that by definition of \( I \), \( u \in I \) if and only if \( \Phi_h(u) \in I \) for all \( h > 0 \), \( \Phi_h(u) \in \bar{I} \) implies \( \Phi_{h'}(u) \in \bar{I} \) whenever \( h' > h \).

Fix an \( h > 0 \) and let \((a_i^h)_{i=1,...,N_h} = f(U, h)\). Since \((a_i^h)_{i=1,...,N_h}\) are, conditioned on \( N_h \), independent and \( L(a_i^h) \) are absolutely continuous with respect to the Lebesgue measure for all \( i \), we get
\[
P \left( \exists I \subset \{1, \ldots, N_h\} : \sum_{i \in I} a_i = x \bigg| N_h \right) = P \left( \exists I \subset \{1, \ldots, N_h\} : a_{\min(I)} = x - S_I \bigg| N_h \right) = 0,
\tag{8.97}
\]
for all \( x \in \mathbb{R} \), where \( S_I = \sum_{i \in I \backslash \{\min(I)\}} a_i \). We can now apply Lemma 7.2 and get
\[ P (U_t \in I) = P (\Phi_h (U_t) \in I, \forall h > 0) \]
\[ = \lim_{h \downarrow 0} P (\Phi_h (U_t) \in I) \]
\[ = 1 - \lim_{h \downarrow 0} P (\Phi_h (U_t) \not\in I) \]
\[ \geq 1 - \lim_{h \downarrow 0} E \left[ E \left[ 1 \left( \mathcal{L} \left( (a_i^h)_{i=1, \ldots, N_h} \right) \not\ll \lambda^\otimes N_h \right) \bigg| N_h \right] \right] \]
\[ = 1. \]  

(8.98)

9 Proof of Theorem 2.12 (c)

As we have seen in Remark 8.11, the number of balls map has some nice properties. In fact these properties allow us to proof our result and we start by formalizing this remark.

**Definition 9.1.** Define the number of balls map \( \mathcal{N} : U_c \to D((0, \infty), \mathbb{N}) \), \( u \mapsto (h \mapsto \min \{ i \in \mathbb{N} : f(u, h)_{i+1} = 0 \} ) \) and \( n(f(u, h)) := \mathcal{N}(u)_h. \)

**Remark 9.2.** (1) In terms of Lemma 3.1, \( \mathcal{N}(u)_h = n(f(u, h)) = n(h). \)

(2) Using Lemma 3.8, it is not hard to see that the range of the map is a subset of the Skorohod space.

(3) Since \( u \in U_c, \mathcal{N}(u)_h < \infty \) for all \( h > 0. \)

**Lemma 9.3.** \( u \in B \cap U_c \) if and only if \( u \in U_c \) and \( n(f(u, h-)) - n(f(u, h)) \leq 1 \) for all \( h > 0. \)

**Proof.** Assume that \( u = [X, r, \mu] \) and that there is an \( h > 0 \) such that \( n(f(u, h-)) - n(f(u, h)) \geq 2. \) Then, by the argument in the second part of Lemma 3.1, there are at least three points \( x, y, z \in X \) such that
\[ \min (r(x, y), r(x, z), r(y, z)) \geq h \]  
(9.1)

and a \( \tau \in X \) such that
\[ \mu(B(\tau, h) \cap B(x, h)) = \mu(B(x, h)), \]  
(9.2)
\[ \mu(B(\tau, h) \cap B(y, h)) = \mu(B(y, h)), \]  
(9.3)
\[ \mu(B(\tau, h) \cap B(z, h)) = \mu(B(z, h)), \]  
(9.4)

which contradicts the definition of \( B. \)
Proof. (Theorem 2.12(c)) Let \(u_n \in B \cap U_c\) with \(u_n \rightarrow u \in U_c\) in the Gromov-weak atomic topology. Then \(\mathfrak{F}(u_n) \rightarrow \mathfrak{F}(u)\), by Theorem 3.11, and hence \(f(u_n, \delta) \rightarrow f(u, \delta)\) with respect to \(d_1\) for all continuity points \(\delta > 0\) of \(f(u, \cdot)\). Therefore, we can find for all \(\varepsilon > 0\) a \(K \in \mathbb{N}\) such that

\[
\sup_{n \in \mathbb{N}} \sum_{i \geq K} f(u_n, \delta)_i \leq \varepsilon. \tag{9.5}
\]

Now, as we have seen in the proof of Lemma 8.9 for all \(\varepsilon > 0\) small enough and all continuity points \(h > 0\) of \(f(u, \cdot)\), there is a \(N \in \mathbb{N}\) such that \(n(f(u_n, h)\varepsilon) \equiv n(f(u, h))\) for all \(n \geq N\), where

\[
f(u_n, \delta)_i^\varepsilon := f(u_n, \delta)_i \mathbb{1}(f(u_n, \delta)_i > \varepsilon), \quad i \in \mathbb{N}. \tag{9.6}
\]

Finally observe that by the convergence in the Skorohod topology and since \((f(u, h))_{h \geq \delta}\) has only finitely many jumps for all \(u \in U_c\) and \(\delta > 0\), two jumps of size larger than \(\varepsilon\) (with respect to \(d_1\)) are uniformly separated, say by \(\eta\) (see Theorem 3.6.3 in [EK86]), and therefore,

\[
n(f(u, h - )) - n(f(u, h)) = n(f(u, h - \eta/3)) - n(f(u, h + \eta/3))
\]

\[
= n(f(u_n, h - \eta/3)\varepsilon) - n(f(u_n, h + \eta/3)\varepsilon) \leq 1.
\]

for all \(n\) large enough and all \(h > 0\), where we assumed that \(\varepsilon\) is small enough and that \(h \pm \eta/3\) is a continuity point of \(f(u, \cdot)\).

10 Proofs for Section 4.2

We start by proving that the convergence of the marginals of the tree-valued Moran model to the tree-valued Fleming-Viot process also holds, when the space \(U\) is equipped with the Gromov-weak atomic topology. Then we apply this result to prove our main result.

10.1 Proof of Proposition 4.1

In order to see that \(U^N_t \Rightarrow U_t\), when \(U\) is equipped with the Gromov-weak atomic topology, we will prove:

Lemma 10.1. Under the assumptions of Proposition 4.1 one has weak convergence of \(\mathfrak{F}(U^N_t)\) for all \(t \geq 0\).
Once we have shown this Lemma, we can apply Theorem 3.11 to get relative compactness and hence convergence of $U^N_t$ to $U_t$, that is Proposition 4.1.

**Remark 10.2.** Note that the above implies $F(U^N_t) \Rightarrow F(U_t)$. ♣

In order to prove Lemma 10.1 we need the following.

### 10.1.1 Connection to the Kingman-coalescent

Here we give the connection of the tree-valued Moran model and the Kingman-coalescent (see the proof of Proposition 6.15 in [GR16]). For the definition and properties of the Kingman-coalescent we refer to [Ber06] and [Ber09].

For fixed $t \geq 0$, we set $A_h(i) := A_{t-h}(i, t)$ (see (4.2)), $0 \leq h \leq t$ and $[N] := \{1, \ldots, N\}$. Then $\{A_h(i) : i \in [N]\}$ can be described as a family of processes in $[N]^N$ that starts in $A_0(i) = i$ and has the following dynamic: Whenever $\eta^{i,j}(\{t-h\}) = 1$ for some $i, j \in [N] = I_N$ we have the following transition:

$$A_{h-}(k) \rightarrow A_h(k) = i, \quad \forall k \in \{l \in [n] : A_{h-}(l) = j\}. \quad (10.1)$$

It is now straightforward to see that the time it takes to decrease the number of different labels, $|\{A_h(i) : i \in [N]\}|$, by 1, given there are $k$ different labels, is exponential distributed with parameter $\binom{k}{2}$ and that the two labels (the one that replaces and the one that is replaced) are sampled uniformly without replacement under all existing labels. If we define

$$\kappa_i(h) = \{j \in [N] : A_h(j) = A_h(i)\}, \quad (10.2)$$

this implies $\kappa^N = (\{\kappa_1(h), \ldots, \kappa_N(h)\})_{0 \leq h \leq t}$ is a Kingman $N$-coalescent (up to time $t$). If we now set

$$\nu^N_t = \left[1, \ldots, N, r^N_t, \frac{1}{N} \sum_{k=1}^N \delta_k\right], \quad (10.3)$$

where

$$r^N_t(i, j) = \begin{cases} \inf\{h \geq 0 \mid \exists k : i, j \in \kappa^N_k(h)\}, & \text{if } \exists k : i, j \in \kappa^N_k(t-), \\ r_0(i, j), & \text{otherwise}, \end{cases} \quad (10.4)$$

and $r_0$ is the ultra-metric given in the definition of $U^N_t$ (see (4.3)), then the above implies
Lemma 10.3. $\mathcal{L}(\mathcal{V}_t^N) = \mathcal{L}(\mathcal{U}_t^N)$.

Now, note that the Kingman-coalescent $\kappa = (\kappa_t)_{t \geq 0}$ satisfies the consistency relation:

(C) $\kappa|_N := (\kappa(t)|_N)_{t \geq 0}$ is a Kingman $N$-coalescent started from $\kappa(0)|_N$, where for $\pi = \{\pi_1, \pi_2, \ldots\} \in \mathcal{P}(\mathbb{N})$ (set of partitions of $\mathbb{N}$) we define $\pi|_N \in \mathcal{P}([N])$ (set of partitions of $[N]$) as the element induced by $\pi_1 \cap [N], \pi_2 \cap [N], \ldots$.

Note further, that the law of the Kingman-coalescent is determined by this property and the initial configuration. As a consequence, we may assume:

Assumption 10.4. Our processes are defined on a probability space, where the Kingman $N$-coalescents are coupled (for different $N$) in such a way that they are restrictions of an underlying Kingman coalescent $\kappa = (\kappa_t)_{t \geq 0}$, i.e. we assume that $(\kappa^N)_{N \in \mathbb{N}} := (\kappa|_N)_{N \in \mathbb{N}}$.

10.1.2 Proof of Lemma 10.1

First note that by the construction in 10.1.1 we have that $f(\mathcal{U}_t^N, h)_{0 \leq h \leq t}$ is given by the decreasing reordering of

$$\left(\frac{1}{N} |\kappa_i^N(h)|\right)_{i \in [N]} = \left(\frac{1}{N} |\kappa_i(h) \cap [N]|\right)_{i \in \mathbb{N}},$$

where $\kappa^N(0) = \{\{1\}, \ldots, \{N\}\}$ and $|\cdot|$ denotes the number of elements in $\cdot$. Next, we define for $A \subset \mathbb{N}$

$$|A|_f := \lim_{N \to \infty} \frac{|A \cap [N]|}{N},$$

if it exists, and call $|A|_f$ in this case the asymptotic frequency of $A$.

Proof. (Lemma 10.1) We first note that the Kingman-coalescent at a time $t > 0$ forms an exchangeable random partition and therefore possesses asymptotic frequency almost surely (see [Ber06]). That is, when we denote by $(\tau_k)_{k \in \mathbb{N}}$ the first times of the Kingman-coalescent to have $k$ blocks, i.e. $\tau_k := \inf\{t > 0 : |\kappa(t)| = k\}$, then $|K_i(k)|_f := |\kappa_i(\tau_k)|_f$ exists for all $i \in \mathbb{N}$ almost surely. We denote by $K^i(k)$ the decreasing reordering of the block frequencies $(|K_i(k)|_f)_{i=1,\ldots,k}$ and define $f : (0, \infty) \to S^i$ by

$$f(t) := \begin{cases} K^i(1), & \text{when } t \geq \tau_1, \\ K^i(k), & \text{when } t \in [\tau_k, \tau_{k-1}), \ k \geq 2. \end{cases}$$

(10.7)
Note that \( f(U^N_t, t) = f(U^N_t, t^-) \) almost surely, hence, by construction, \( f \in D((0, \infty), S^\downarrow) \) and

\[
(f(U^N_t, h))_{\delta \leq h \leq t} \to (f(h))_{\delta \leq h \leq t}
\]  

(10.8)
in the Skorohod topology almost surely for all \( \delta > 0 \), where \( S^\downarrow \) is equipped with \( d^1 \) (recall that the Kingman-coalescent comes down from infinity, i.e. there are only finitely many elements of \( f(h) \) that are non-zero).

Moreover, by definition, \( \Phi_t(U^N_t) = [0, 1], r_0, \mu^N_t] \) and \( \Phi_t(U_t) = [0, 1], r_0, \mu_t] \), where

\[
\mu^N_t = \sum_{i \in \mathbb{N}} f(U^N_t, t)_i \delta_{V_i}
\]  

(10.9)
and

\[
\mu_t = \sum_{i \in \mathbb{N}} f(t)_i \delta_{V_i}
\]  

(10.10)

and \( \mu^N_t \Rightarrow \mu_t \). By the coupling of the Kingman-coalescent and the Kingman-\(N\)-coalescent, we know further, that for \( N \) large enough, the number of non-zero entries of \( f(U^N_t, t) \) (which corresponds to the number of blocks in a Kingman-\(N\)-coalescent) equals the number of non-zero entries of \( f(t) \). Combining this observation with the fact that \( \sum_{i \in I} f(U^N_t, t)_i \to \sum_{i \in I} f(t)_i \), for all \( I \subset \mathbb{N}, |I| < \infty \), and Lemma 3.1, we get

\[
f(U^N_t, h)_{h \geq t} \Rightarrow f(h)_{h \geq t}.
\]  

(10.11)

Finally observe that for \( \varepsilon > 0 \) and \( \delta > 0 \),

\[
\limsup_{N \to \infty} P \left( \sup_{h \in [0, \delta]} d^\infty(f(U^N_t, h), f(U^N_t, 0)) \geq \varepsilon \right) 
\]

\[
= \limsup_{N \to \infty} P \left( \left| f(U^N_t, \delta)_1 - \frac{1}{N} \right| \geq \varepsilon \right) 
\]

\[
\leq \limsup_{N \to \infty} P \left( f(U^N_t, \delta)_1 \geq \varepsilon \right) 
\]

\[
\leq P \left( f(\delta)_1 \geq \varepsilon \right).
\]  

(10.12)

Note that \( K^+(n) \) is the decreasing rearrangement of a random variable that is uniform distributed on the simplex (see again [Ber06]), i.e. \( K^+(n) = \max(X_1, \ldots, X_n)/\sum_{i=1}^n X_i \), where \( X_1, \ldots, X_n \) are independent exponential-1-distributed. We can now apply the strong law of large numbers together with a simple calculation to show that for \( \delta > 0 \) small enough

\[
\limsup_{N \to \infty} P \left( \sup_{h \in [0, \delta]} d^\infty(f(U^N_t, h), f(U^N_t, 0)) \geq \varepsilon \right) \leq \varepsilon
\]  

(10.13)
(in fact one could prove that $E \left[ f(\delta) \right] = \frac{1}{k} \sum_{l=1}^{k} \frac{1}{l}$ - see for example [OW11]).

10.2 Proof of Theorem [4.3]

Note that the tree-valued Fleming-Viot process $(\mathcal{U}_t)_{t \geq 0}$ takes values in the space of compact ultra-metric measure spaces $\mathbb{U}_c$ for all $t > 0$ (see Proposition 2.11 in [GPW13]). Fix a $t > 0$. It follows that for all $h > 0$ there is an almost surely finite random variable $N_h \in \mathbb{N}$ such that $\Phi_h(\mathcal{U}_t) = \{X, r, \mu_h\}$ satisfies $|\{x \in X | \mu_h(\{x\}) > 0\}| = N_h$ almost surely. Since the reordering of atoms of $\mu_h$ is distributed as the decreasing rearrangement of a random variable that is uniformly distributed on the simplex (see [Ber06]; compare also the proof of Lemma 10.1), the result follows by Proposition 3.16 once we have shown that $\mathcal{U}_t \in \mathcal{B}$ almost surely. But, since $\mathcal{U}_t \in \mathcal{B}$ almost surely (recall Section 10.1.2), $\mathcal{B}$ is closed in the Gromov-weak atomic topology (see Theorem 2.12), and $\mathcal{U}_t \Rightarrow \mathcal{U}_t$, the result follows by the Portmanteau theorem.

A Skorohod topology

The space $D((0, \infty), \mathcal{S}^1)$ is equipped with the Skorohod topology, which is induced by the following metric

$$d^{SK}(f, g) := d^{SK,1}(f, g) + d^{SK,\infty}(f, g)$$

where

$$d^{SK,1}(f, g) := \inf_{\lambda \in \Lambda} \left( \gamma(\lambda) \vee \int_0^\infty e^{-u} \rho_1(f, g, \lambda, u) du \right)$$

$$d^{SK,\infty}(f, g) := \inf_{\lambda \in \Lambda} \left( \gamma(\lambda) \vee \int_0^\infty e^{-u} \rho_2(f, g, \lambda, u) du \right),$$

with

$$\rho_1(f, g, \lambda, u) := \sup_{t \geq 0} d^1(f(t \land u), g(\lambda(t) \land u)) \land 1,$$

$$\rho_2(f, g, \lambda, u) := \sup_{t \geq 0} d^\infty(f(t \land u), g(\lambda(t) \land u)) \land 1$$

and $\Lambda$ is the set of strictly increasing surjective Lipschitz continuous functions $\lambda : [0, \infty) \rightarrow [0, \infty)$ such that

$$\gamma(\lambda) := \text{ess sup}_{s > t \geq 0} \left| \log \left( \frac{\lambda(s) - \lambda(t)}{s - t} \right) \right| < \infty.$$
Here

\[
d^1(x, y) := \sum_{i=1}^{\infty} |x_i - y_i|, \quad (A.7)
\]

\[
d^{\infty}(x, y) := \max_{i \in \mathbb{N}} |x_i - y_i|. \quad (A.8)
\]

We note that the idea for the definition of the first term of \(d^{SK}\) follows the same idea as in the case where one wants to include cadlag functions defined on \(\mathbb{R}_+\) (and not only on compact intervals) and one can use the same techniques as for example in Section 3 of [EK86] to prove that

**Proposition A.1.**

(i) The space \(D((0, \infty), S^1)\) equipped with \(d^{SK}\) is a complete separable metric space.

(ii) The characterization of convergence in this topology is analogue to the characterization given in Proposition 3.6.5 in [EK86].

Let \(f, f_1, f_2, \ldots \in D((0, \infty), S^1)\). Then \(\lim_{n \to \infty} d^{SK}(f_n, f) = 0\) if and only if whenever \(t, t_1, t_2, \ldots \in [0, \infty)\), and \(\lim_{n \to \infty} t_n = t\), the following conditions holds.

(a) If \(t > 0\) then \(\lim_{n \to \infty} d^1(f_n(t_n), f(t)) \wedge d^1(f_n(t_n), f(t-)) = 0\) and if \(t = 0\), then \(\lim_{n \to \infty} d^{\infty}(f_n(t_n), f(t)) = 0\).

(b) If \(t > 0\), \(\lim_{n \to \infty} d^1(f_n(t_n), f(t)) = 0\) and \(s_n \geq t_n\) for each \(n\), and \(\lim_{n \to \infty} s_n = t\), then \(\lim_{n \to \infty} d^1(f_n(s_n), f(t)) = 0\).

(c) If \(t > 0\), \(\lim_{n \to \infty} d^1(f_n(t_n), f(t-)) = 0\) and \(0 \leq s_n \leq t_n\) for each \(n\), and \(\lim_{n \to \infty} s_n = t\), then \(\lim_{n \to \infty} d^1(f_n(s_n), f(t-)) = 0\).
References

[ALW16] Siva Athreya, Wolfgang Löhr, and Anita Winter. The gap between Gromov-vague and Gromov-Hausdorff-vague topology. *Stochastic Process. Appl.*, 126(9):2527–2553, 2016.

[Ber06] Jean Bertoin. *Random fragmentation and coagulation processes*, volume 102 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2006.

[Ber09] Nathanaël Berestycki. *Recent progress in coalescent theory*, volume 16 of *Ensaios Matemáticos [Mathematical Surveys]*. Sociedade Brasileira de Matemática, Rio de Janeiro, 2009.

[DGP11] Andrej Depperschmidt, Andreas Greven, and Peter Pfaffelhuber. Marked metric measure spaces. *Electron. Commun. Probab.*, 16:174–188, 2011.

[DGP12] Andrej Depperschmidt, Andreas Greven, and Peter Pfaffelhuber. Tree-valued Fleming-Viot dynamics with mutation and selection. *Ann. Appl. Probab.*, 22(6):2560–2615, 2012.

[DK96] Peter Donnelly and Thomas G. Kurtz. A countable representation of the Fleming-Viot measure-valued diffusion. *Ann. Probab.*, 24(2):698–742, 1996.

[DK99] Peter Donnelly and Thomas G. Kurtz. Genealogical processes for Fleming-Viot models with selection and recombination. *Ann. Appl. Probab.*, 9(4):1091–1148, 1999.

[EK86] Stewart N. Ethier and Thomas G. Kurtz. *Markov processes: Characterization and convergence*. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. John Wiley & Sons, Inc., New York, 1986.

[EK94] S. N. Ethier and Thomas G. Kurtz. Convergence to Fleming-Viot processes in the weak atomic topology. *Stochastic Process. Appl.*, 54(1):1–27, 1994.

[EM17] Steven N. Evans and Ilya Molchanov. The semigroup of metric measure spaces and its infinitely divisible probability measures. *Trans. Amer. Math. Soc.*, 369(3):1797–1834, 2017.
[GD18] Andreas Greven and Andrej Depperschmidt. Tree-valued Feller diffusion. in preparation, 2018.

[GGR16] P. Gloede, A. Greven, and T. Rippl. Branching trees I: Concatenation and infinite divisibility. ArXiv e-prints, December 2016.

[Glö13] Patric Karl Glöde. Dynamics of Genealogical Trees for Autocatalytic Branching Processes. doctoralthesis, Friedrich-Alexander-Universität Erlangen-Nürnberg (FAU), 2013.

[GPW09] Andreas Greven, Peter Pfaffelhuber, and Anita Winter. Convergence in distribution of random metric measure spaces (Λ-coalescent measure trees). Probab. Theory Related Fields, 145(1-2):285–322, 2009.

[GPW13] Andreas Greven, Peter Pfaffelhuber, and Anita Winter. Tree-valued resampling dynamics martingale problems and applications. Probab. Theory Related Fields, 155(3-4):789–838, 2013.

[GR16] M. Grieshammer and T. Rippl. Partial orders on metric measure spaces. ArXiv e-prints, May 2016.

[Gri17] Max Grieshammer. Measure representations of genealogical processes and applications to Fleming-Viot models. doctoralthesis, Friedrich-Alexander-Universität Erlangen-Nürnberg (FAU), 2017. https://opus4.kobv.de/opus4-fau/frontdoor/index/docId/8565.

[Kin82] J. F. C. Kingman. The coalescent. Stochastic Process. Appl., 13(3):235–248, 1982.

[KN97] Stephen M. Krone and Claudia Neuhauser. Ancestral processes with selection. Theoretical population biology, 51(3):210–237, 1997.

[Kur14] Kazimierz Kuratowski. Topology, volume 1. Elsevier, 2014.

[Loe13] Wolfgang Loehr. Equivalence of Gromov-Prohorov- and Gromov’s □1-metric on the space of metric measure spaces. Electron. Commun. Probab., 18:no. 17, 10, 2013.

[LVW15] Wolfgang Löh, Guillaume Voisin, and Anita Winter. Convergence of bi-measure R-trees and the pruning process. Ann. Inst. Henri Poincaré Probab. Stat., 51(4):1342–1368, 2015.
[NK97] Claudia Neuhauser and Stephen M. Krone. The genealogy of samples in models with selection. *Genetics*, 145(2):519–534, 1997.

[OW11] Shmuel Onn and Ishay Weissman. Generating uniform random vectors over a simplex with implications to the volume of a certain polytope and to multivariate extremes. *Ann. Oper. Res.*, 189:331–342, 2011.

[Pal70] Richard S Palais. When proper maps are closed. *Proceedings of the American Mathematical Society*, 24(4):835–836, 1970.