IMPULSIVE CONTROL OF CONSERVATIVE PERIODIC EQUATIONS AND SYSTEMS: VARIATIONAL APPROACH

Pavel Drábek*
Department of Mathematics and NTIS, University of West Bohemia
Univerzitní 8, 301 00 Plzeň, Czech Republic

Martina Langerová
NTIS, University of West Bohemia
Univerzitní 8, 301 00 Plzeň, Czech Republic

(Communicated by Manuel del Pino)

Abstract. Using the variational structure of the second order periodic problems we find an optimal impulsive control which forces the conservative system into a periodic motion. In particular, our main results concern the system of charged planar pendulums with external disturbances and neglected friction. Such a system might serve as a model for coupled micromechanical array.

1. Introduction. This paper is devoted to the study of the second order differential equations and systems with a variational structure. We formulate conditions on a system of control impulses applied at fixed times which force a solution of the impulsive problem to satisfy periodic boundary conditions. We discuss the optimality of such control as well as the energy and multiplicity of periodic solutions.

As a simple motivation we can consider an undamped pendulum equation

\[ x''(t) + \sin x(t) = f(t), \quad t \in \mathbb{R}. \]

For an arbitrary \( f \in L^1(0,T), \ T > 0, \) we find optimal conditions on a system of impulses \( u_j, \ j = 1, \ldots, p, \) applied at times \( 0 < t_1 < \ldots < t_p < T, \) which guarantee the existence of a \( T \)-periodic solution of impulsive equation

\[ x''(t) + \sin x(t) = f(t) + \sum_{j=1}^{p} u_j \delta(t-t_j). \]  \hspace{1cm} (1)

Here, \( \delta = \delta(t), \ t \in \mathbb{R}, \) denotes the Dirac delta impulse concentrated at zero. The impulses in (1) which force the system into a periodic motion result in a solution which exhibits finite jumps in the derivative of magnitude \( u_j. \) These jumps occur at fixed time instants \( t_j, \ j = 1, \ldots, p. \) Our first result states that control impulses satisfying

\[ \sum_{j=1}^{p} u_j = -\int_0^T f(t) \, dt \]

will force the pendulum into a periodic motion with sudden changes of velocity at times \( t_j, \ j = 1, \ldots, p. \) In order to minimize the Euclidean
norm of a vector of impulses \( u = (u_1, \ldots, u_p) \), we have to choose \( u_j = \frac{-1}{p} T \int_0^T f(t) \, dt \) for all \( j = 1, \ldots, p \), see Section 3. In Section 4, we discuss the multiplicity of the solution of the impulsive problem, as well as its energy. In Sections 5 and 6, we generalize these results for systems.

In particular, as a simple motivation we consider two charged pendulums with an attractive interaction,

\[
\begin{align*}
\begin{cases}
x''_1(t) - x_1(t) + x_2(t) + \sin x_1(t) = f_1(t) + \sum_{j=1}^{p_1} u_{1j} \delta(t - t_{1j}), \\
x''_2(t) - x_2(t) + x_1(t) + \sin x_2(t) = f_2(t) + \sum_{j=1}^{p_2} u_{2j} \delta(t - t_{2j}),
\end{cases}
\tag{2}
\end{align*}
\]

see Figure 1. Here, an external forcing is given by \( f_i \) and impulsive control is given by \( \{u_{ij}\} \), \( i = 1, 2 \), \( j = 1, \ldots, p_i \), \( 0 < t_{i1} < \ldots < t_{ip_i} < T \).

![Figure 1. A model of 2 coupled charged pendulums.](image)

Our result states that for arbitrary \( f_i \in L^1(0, T) \), \( i = 1, 2 \), the control impulses satisfying

\[
\sum_{j=1}^{p_1} u_{1j} + \sum_{j=1}^{p_2} u_{2j} = -\int_0^T (f_1(t) + f_2(t)) \, dt
\tag{3}
\]

will force this conservative system into a \( T \)-periodic motion described by a vector function \( \mathbf{x}(t) = (x_1(t), x_2(t)) \), \( x_i(0) = x_i(T), \ x'_i(0) = x'_i(T), \ i = 1, 2 \), which minimizes the energy functional associated with (2). An interesting observation following from (3) consists in the fact that for any external force \( \mathbf{f}(t) = (f_1(t), f_2(t)) \) just one impulse of magnitude \( u \) concentrated at \( \tau \in (0, T) \), and applied, for example, to the first pendulum, will force the system into a \( T \)-periodic motion provided that

\[
u = -\int_0^T (f_1(t) + f_2(t)) \, dt.
\]

In such a case, the first component of \( \mathbf{x} = \mathbf{x}(t) \) will exhibit the jump in the derivative of magnitude \( u \) at time \( t = \tau \),

\[
\Delta x'_1(\tau) := x'_1(\tau^+) - x'_1(\tau^-) = u,
\]
while the second component \( x_2 = x_2(t) \) will be a smooth \( C^1 \)-function.

Note that in Section 6 we consider general system of \( N \) pendulums for any \( N \geq 2 \). Such a system can be regarded as a mechanical analogue in order to model coupled micromechanical array, see [3] for details.

The practical importance of mathematical models the solutions of which possess instantaneous impulses that result in discontinuities of velocity but with no sudden change of position was stressed e.g. in papers [4, 5, 11].

The variational structure of our problems allows us to treat them in the Sobolev space of periodic functions which seems to be a “natural environment” for solutions having jumps in the derivative. In this connection we mention pioneering works [15] and [17], where the variational approach to impulsive problems had been invented systematically. The authors of this paper applied variational approach to both semilinear and quasilinear impulsive problems with nonlinearities and impulses satisfying the so called Landesman-Lazer type conditions, see [6] and [7]. We also refer to [2, 10, 16] for the general theory of impulsive differential equations and to [19] which deals with the impulsive control theory. However, in these books topological and analytical methods (based on fixed point theorems, degree theory or method of sub- and supersolutions) are extensively used, rather than the application of variational methods.

2. Preliminaries. Let \( H := \{ x \in H^1(0,T) : x(0) = x(T) \} \) be the Sobolev space of periodic functions equipped with the scalar product

\[
(x,y) = \int_0^T (x'(t)y'(t) + x(t)y(t)) \, dt
\]

and the induced norm \( \|x\| = \sqrt{(x,x)} \). We will also work with the norm \( \|x\|_{L^\infty} = \max_{t \in [0,T]} |x(t)| \) on the space \( C[0,T] \), \( \|x\|_{L^1} = \int_0^T |x(t)| \, dt \) on the space \( L^1(0,T) \) and the scalar product

\[
(x,y)_{L^2} = \int_0^T x(t)y(t) \, dt
\]

on the space \( L^2(0,T) \) with the induced norm \( \|x\|_{L^2} = \sqrt{(x,x)_{L^2}} \).

Let \( \delta = \delta(t) \) be the Dirac delta impulse concentrated at 0, i.e., \( \delta(t) = 0 \) for \( t \neq 0 \), \( \delta(0) = +\infty \), and \( \int_{-\infty}^{+\infty} \delta(t) \, dt = 1 \). Let \( g : \mathbb{R} \to \mathbb{R} \) be continuous, \( 2\pi \)-periodic function satisfying

\[
\int_0^{2\pi} g(s) \, ds = 0,
\]

and \( f \in L^1(0,T), T > 0 \). Moreover, let \( 0 < t_1 < \ldots < t_p < T \), \( p \in \mathbb{N} \), and \( u_j, j = 1, \ldots, p \), be given. In this section we consider the impulsive problem

\[
\begin{align*}
\begin{cases}
  x''(t) + g(x(t)) &= f(t) + \sum_{j=1}^p u_j \delta(t-t_j), & t \in [0,T], \\
  x(0) &= x(T), & x'(0) = x'(T).
\end{cases}
\end{align*}
\]
We say that $x \in H$ is a weak solution of (5) if the integral identity
\[
\int_0^T x'(t)y'(t) \, dt - \int_0^T g(x(t))y(t) \, dt + \int_0^T f(t)y(t) \, dt + \sum_{j=1}^p u_j y(t_j) = 0
\]
holds for any test function $y \in H$.

Let us introduce the energy functional $E : H \to \mathbb{R}$ associated with (5),
\[
E(x) := \frac{1}{2} \int_0^T |x'(t)|^2 \, dt - \int_0^T G(x(t)) \, dt + \int_0^T f(t)x(t) \, dt
\]
\[+ \sum_{j=1}^p u_j x(t_j), \quad x \in H,
\]
where $G(s) = \int_0^s g(\sigma) \, d\sigma$. Then, $E$ has continuous Fréchet derivative in $H$, and
\[
(E'(x), y) = 0 \quad \text{for any } y \in H
\]
is equivalent to (6). Therefore, $x = x(t)$ is a weak solution of (5) if and only if it is a critical point of $E$.

It is a matter of simple application of the integration by parts formula and the standard regularity argument for ordinary differential equations of the second order (cf. [9] or [15]) that any weak solution $x = x(t)$ satisfies the impulsive conditions
\[
\Delta x'(t_j) := x'(t_j^+) - x'(t_j^-) = u_j, \quad j = 1, 2, \ldots, p,
\]
the equation
\[
x''(t) + g(x(t)) = f(t)
\]
holds a.e. in $(0, T)$, and the periodic conditions $x(0) = x(T)$, $x'(0) = x'(T)$ hold.

For any element $f \in L^1(0, T)$ we write $f(t) = \bar{f} + \tilde{f}(t)$, where $\bar{f} = \frac{1}{T} \int_0^T f(t) \, dt$ and $\int_0^T \tilde{f}(t) \, dt = 0$. It is proved in [12, Section 4] that the equation (8) with $f = \bar{f}$ has at least two distinct solutions which do not differ by a multiple of $2\pi$ and, moreover, one of them minimizes associated energy functional
\[
\tilde{E}(x) := \frac{1}{2} \int_0^T |x'(t)|^2 \, dt - \int_0^T G(x(t)) \, dt + \int_0^T \tilde{f}(t)x(t) \, dt.
\]

3. Existence results. In this section we first prove the existence of an impulsive control and then we discuss its optimality.

**Theorem 3.1.** Let $f \in L^1(0, T)$. Then for any control impulses $u_j$, $j = 1, \ldots, p$, satisfying
\[
\sum_{j=1}^p u_j = -\int_0^T f(t) \, dt
\]
the problem (5) admits at least one solution $x_0 = x_0(t)$ which minimizes the energy functional $E$ given in (7).
Proof. The proof follows the lines of [12, Section 4] taking into account the influence of impulses.

We write $x \in H$ as $x(t) = \bar{x} + \hat{x}(t)$ and similarly $f \in L^1(0, T)$ as $f(t) = \bar{f} + \hat{f}(t)$. Then, due to (9) we get

$$
\mathcal{E}(x) = \frac{1}{2} \int_0^T |\dot{\bar{x}}(t)|^2 \, dt - \int_0^T G(\bar{x} + \hat{x}(t)) \, dt + \int_0^T f(t) \dot{\bar{x}}(t) \, dt
$$

$$
+ \sum_{j=1}^{p} u_j \bar{x} + \sum_{j=1}^{p} u_j \hat{x}(t_j)
$$

(10)

Recall the continuous embedding $H \hookrightarrow C[0, T]$ and the Sobolev inequality

$$
||\hat{x}||_{L^\infty} \leq \frac{\sqrt{T}}{2\sqrt{3}} \left( \int_0^T |\dot{x}'(t)|^2 \, dt \right)^{\frac{1}{2}},
$$

see [14, p. 9]. Applying the Hölder inequality we get from (10) that

$$
\mathcal{E}(x) \geq \frac{1}{2} \int_0^T |\dot{\bar{x}}(t)|^2 \, dt - T \max_{s \in [0, 2\pi]} |G(s)|
$$

$$
- \frac{\sqrt{T}}{2\sqrt{3}} \left( \int_0^T |\hat{f}(t)| \, dt \right) \left( \int_0^T |\dot{x}'(t)|^2 \, dt \right)^{\frac{1}{2}}
$$

$$
- \frac{\sqrt{T}}{2\sqrt{3}} \max_{i=1, \ldots, p} |u_i| \left( \int_0^T |\dot{x}'(t)|^2 \, dt \right)^{\frac{1}{2}}
$$

(11)

for all $x \in H$. Notice that $\max_{s \in [0, 2\pi]} |G(s)| < \infty$ due to (4). It follows from (11) that $\inf_{x \in H} \mathcal{E}(x) > -\infty$, i.e., $\mathcal{E}$ is bounded from below in $H$.

Let $\{x_n\}_{n=1}^{\infty} \subset H$ be a minimizing sequence, i.e., $\lim_{n \to \infty} \mathcal{E}(x_n) = \inf_{x \in H} \mathcal{E}(x)$. It follows from (4) that $G(s + 2\pi) = G(s)$ for all $s \in \mathbb{R}$. Therefore, without loss of generality, we assume that $\bar{x}_n \in [0, 2\pi)$. From here and from (11) we deduce that $\{x_n\}_{n=1}^{\infty}$ is a bounded sequence in $H$. Hence, we may assume that there exist $x_0 \in H$ and a subsequence $\{x_{n_k}\}_{k=1}^{\infty} \subset \{x_n\}_{n=1}^{\infty}$ such that $x_{n_k} \to x_0$ (weakly) in $H$. It follows from the compact embedding $H \hookrightarrow C[0, T]$ and the weak lower semicontinuity of $x \mapsto \frac{1}{2} \int_0^T |\dot{x}'(t)|^2 \, dt$ that $\mathcal{E}$ is weakly lower semicontinuous on $H$. Therefore,

$$
\mathcal{E}(x_0) \leq \lim_{n \to \infty} \inf \mathcal{E}(x_n) = \inf_{x \in H} \mathcal{E}(x),
$$

i.e., $\mathcal{E}(x_0) = \inf_{x \in H} \mathcal{E}(x)$ and so $x_0 = x_0(t)$ is a global minimizer of $\mathcal{E}$ on $H$. We then conclude that $(\mathcal{E}'(x_0), y) = 0$ for all $y \in H$ and hence $x_0 = x_0(t)$ is a solution of (5) which minimizes $\mathcal{E}$.

\qed
Let $u = (u_1, \ldots, u_p)$ be the vector of impulses at given points $t_j, j = 1, \ldots, p$, and
\[ C(u) = \sqrt{u_1^2 + \cdots + u_p^2}. \]
From Theorem 3.1 we get "optimal impulsive control" which guarantees the existence of a solution of (5).

Corollary 1. Let $f \in L^1(0, T)$. Then there exists a unique optimal control
\[ u_0 = \left( -\frac{1}{p} \int_0^T f(t) \, dt, \ldots, -\frac{1}{p} \int_0^T f(t) \, dt \right) \]
which minimizes $C$ and $C(u_0) = \frac{1}{\sqrt{p}} \left| \int_0^T f(t) \, dt \right|$.

The proof of this assertion follows from simple analysis of elementary optimization problem for a quadratic function of $p$ variables with a linear constraint.

4. Multiplicity results. Similarly to a nonimpulsive case (see [12, 13] or [18]) the problem (5) has another solution which is different from $x_0 = x_0(t)$ and which does not necessarily minimize the energy functional $E$ on $H$.

Theorem 4.1. Under the assumptions of Theorem 3.1 the problem (5) has another solution $x_1 = x_1(t)$ which does not differ from $x_0 = x_0(t)$ by a multiple of $2\pi$ and $E(x_1) \geq E(x_0)$.

Proof. The proof again follows the lines of nonimpulsive case [12, Section 4].

Let $x_0$ be a solution guaranteed by Theorem 3.1 and $x_0$ is not a strict local minimum of $E$. Then there is another $x_1 = x_1(t)$ arbitrarily close to $x_0$ which also minimizes $E$ and hence it is a critical point of $E$. The assertion then follows. Therefore, further we assume that $x_0$ is a strict local minimum of $E$, and we proceed to prove that $E$ has a Mountain Pass-type geometry locally around $x_0$. Indeed, there exists $R > 0$ such that $E(x) > E(x_0)$ for all $x \in H$, $0 < \|x - x_0\| \leq R$. Due to $2\pi$-periodicity of $g$ we have $R < 2\pi \sqrt{T}$. Therefore,
\[ b := \inf_{\|x - x_0\| = R} E(x) \geq E(x_0). \]

Next we show that $b > E(x_0)$. Assume by contradiction that $b = E(x_0)$. Then there is a sequence $\{x_n\}_{n=1}^\infty$, $\|x_n - x_0\| = R$ and $E(x_n) \to E(x_0)$ as $n \to \infty$. Without loss of generality we may assume that there are an element $x^* \in H$ and a subsequence $\{x_{n_k}\}_{k=1}^\infty \subset \{x_n\}_{n=1}^\infty$ such that $x_{n_k} \to x^*$ in $H$. By the weak lower semicontinuity of the norm in $H$ and by the compact embedding $H \hookrightarrow C[0,T]$ we infer
\[ \|x^* - x_0\| \leq R \text{ and } E(x^*) \leq \liminf_{k \to \infty} E(x_{n_k}) = E(x_0). \]
Since $E(x) > E(x_0)$ for all $0 < \|x - x_0\| \leq R$, we necessarily have $x^* = x_0$. Hence $x_{n_k} \to x_0$ in $H$ and $\lim_{k \to \infty} \|x_{n_k} - x_0\|_{L^\infty} = 0$ by the compact embedding $H \hookrightarrow C[0,T]$. Set
\[ F(x) := -\frac{1}{2} \int_0^T |x(t)|^2 \, dt - \int_0^T G(x(t)) \, dt + \int_0^T f(t)x(t) \, dt + \sum_{j=1}^p u_j x(t_j). \]
By above mentioned compact embedding

\[ \lim_{k \to \infty} F(x_{nk}) = F(x_0). \]

Then

\[ \frac{1}{2}\|x_{nk}\|^2 = \mathcal{E}(x_{nk}) - F(x_{nk}) \to \mathcal{E}(x_0) - F(x_0) = \frac{1}{2}\|x_0\|^2, \]

i.e., \( \|x_{nk}\| \to \|x_0\| \). This fact together with the weak convergence \( x_{nk} \to x_0 \) in \( H \)

imply \( \lim_{k \to \infty} \|x_{nk} - x_0\| = 0 \), a contradiction with \( \|x_{nk} - x_0\| = R \).

Now, we proceed to prove that \( \mathcal{E} \) satisfies the Palais-Smith condition, i.e., if \( \{x_n\}_{n=1}^\infty \subset H \) is such that \( \mathcal{E}(x_n) \) is bounded in \( \mathbb{R} \) and \( \mathcal{E}'(x_n) \to 0 \) in \( H^* \) (dual space of \( H \)) when \( n \to \infty \) (\( \{x_n\}_{n=1}^\infty \) is called a Palais-Smith sequence) then there exist \( x \in H \) and a subsequence \( \{x_{nk}\}_{k=1}^\infty \subset \{x_n\}_{n=1}^\infty \) such that \( x_{nk} \to x \) in \( H \).

Indeed, let \( \{x_n\}_{n=1}^\infty \) be a Palais-Smith sequence. Due to \( 2\pi \)-periodicity of \( g \) we may assume that \( x_n \in [0, 2\pi) \). The estimate (11) with \( x \) replaced by \( x_n \) and the boundedness of \( \mathcal{E}(x_n) \) yield the boundedness of \( \{\|\tilde{x}_n\|_{L^2}\}_{n=1}^\infty \). Therefore, \( \{x_n\}_{n=1}^\infty \) is bounded, too. Hence, up to a subsequence, we may assume that \( x_n \to x \) in \( H \) for a certain \( x \in H \), and by the compact embedding \( H \hookrightarrow C[0,T] \) also \( \lim_{n \to \infty} \|x_n - x\|_{L^\infty} = 0 \). It follows from \( \mathcal{E}'(x_n) \to 0 \) in \( H^* \) and \( x_n \to x \) in \( H \) that

\[
0 \leftarrow \langle \mathcal{E}'(x_n), x_n - x \rangle = \int_0^T \tilde{x}_n(t)(\tilde{\tilde{x}}_n(t) - \tilde{x}(t)) \, dt - \int_0^T g(x_n(t))(x_n(t) - x(t)) \, dt \quad \rightarrow 0 \quad \text{as} \quad n \to \infty \]

\[
+ \int_0^T f(t)(x_n(t) - x(t)) \, dt + \sum_{j=1}^p u_j(x_n(t_j) - x(t_j)).
\]

Here \( \langle \cdot, \cdot \rangle \) denotes the duality between \( H^* \) and \( H \). This implies

\[
\|\tilde{x}_n\|_{L^2}^2 - \langle \tilde{x}_n', \tilde{x}' \rangle_{L^2} \to 0 \quad \text{as} \quad n \to \infty.
\]

Since we also have \( \langle \tilde{x}_n', \tilde{x}' \rangle_{L^2} \to \|\tilde{x}'\|^2_{L^2} \) by the weak convergence \( x_n \to x \) in \( H \), we deduce \( \|\tilde{x}_n'\|_{L^2} \to \|\tilde{x}'\|_{L^2} \). Therefore, \( \|x_n\| \to \|x\| \) as \( n \to \infty \). We finally conclude \( x_n \to x \) in \( H \).

It follows from the Mountain Pass Theorem (see e.g. [8, Theorem 7.4.5]) that there exists \( x_1 \in H \) satisfying \( \mathcal{E}(x_1) \geq b \) and \( \mathcal{E}'(x_1) = 0 \). Since \( \mathcal{E}(x_1) \geq b > \mathcal{E}(x_0), \)

\( x_1 \) and \( x_0 \) do not differ by a multiple of \( 2\pi \). This completes the proof of Theorem 4.1.

5. **Conservative systems.** In this section we deal with the direct generalization of the results from Sections 3 and 4 to systems of \( N \) equations with \( N \geq 2 \). We will work with vector functions \( x = x(t), t \in [0,T], x(t) = (x_1(t), \ldots, x_N(t)) \) from the spaces \( H := \{ x \in H^1([0,T], \mathbb{R}^N) : x(0) = x(T) \}, C := C([0,T], \mathbb{R}^N), \)

\( L^1 := L^1([0,T], \mathbb{R}^N) \) and \( L^2 := L^2([0,T], \mathbb{R}^N) \). Then \( x \in H \) (resp. \( C, L^1, L^2 \)) is equivalent to \( x_i \in H \) (resp. \( C[0,T], L^1[0,T], L^2[0,T] \)) for any \( i = 1, \ldots, N \). We define the scalar product on \( H \) as follows

\[
(x,y) := \int_0^T [(x'(t), y'(t))_{\mathbb{R}^N} + (x(t), y(t))_{\mathbb{R}^N}] \, dt, \quad x, y \in H,
\]
where $(\cdot, \cdot)_{\mathbb{R}^N}$ is usual (Euclidean) scalar product in $\mathbb{R}^N$ and $|\cdot|_{\mathbb{R}^N} = \sqrt{(\cdot, \cdot)_{\mathbb{R}^N}}$ is Euclidean norm. We set $\|x\| = \sqrt{(x, x)}$. We also define

$$
\|x\|_{L^\infty} := \max_{i=1,\ldots,N} \left\{ \max_{t \in [0,T]} |x_i(t)| \right\} \quad \text{for } x \in C, \quad \|x\|_{L^1} := \max_{i=1,\ldots,N} \left\{ \int_0^T |x_i(t)| \, dt \right\}
$$

for $x \in L^1$ and $\|x\|_{L^2} := (\sqrt{(x, x)}_{L^2})$, $(x, y)_{L^2} = \int_0^T (x(t), y(t))_{\mathbb{R}^N} \, dt$ for $x, y \in L^2$.

We use the following convention: $x'(t) = (x'_1(t), \ldots, x'_{N}(t))$ and $\int_0^T x(t) \, dt = \left( \int_0^T x_1(t) \, dt, \ldots, \int_0^T x_N(t) \, dt \right)$. Let $e_i = (0, \ldots, 1, \ldots, 0) \in \mathbb{R}^N$ be basis unit elements, $i = 1, \ldots, N$. For $T_i > 0, i = 1, \ldots, N$, and $T > 0$ we consider a function $G : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ satisfying the Carathéodory conditions and

$$
G(t, x + T_i e_i) = G(t, x)
$$

for a.e. $t \in [0,T]$ and all $x \in \mathbb{R}^N$. We also assume that there exists $D_x G := \left( \frac{\partial G}{\partial x_1}, \ldots, \frac{\partial G}{\partial x_N} \right) : [0,T] \times \mathbb{R}^N \to \mathbb{R}^N$ and it satisfies the Carathéodory conditions.

Finally, let $0 < t_1 < \ldots < t_p < T_1, \ldots, 0 < t_{N1} < \ldots < t_{NP} < T, p_i \in \mathbb{N} \cup \{0\}$, $i = 1, \ldots, N$, be fixed time instants, and $u_{ij}, i = 1, \ldots, N, j = 1, \ldots, p_i$, be given real values. We consider the generalization of the scalar equation (5) in the following form

$$
\begin{cases}
\begin{aligned}
ex''(t) + D_x G(t, x(t)) &= f(t) + \sum_{i=1}^N \left( \sum_{j=1}^{p_i} u_{ij} \delta(t - t_{ij}) \right) e_i, \quad t \in [0,T], \\
x(0) &= x(T), \quad x'(0) = x'(T).
\end{aligned}
\end{cases}
$$

Notice that (13) reduces to (5) if $N = 1, T_1 = 2\pi, x = x$ and $D_x G = g$. The existence and multiplicity results easily carry over component-wise from the scalar case (5) to the system (13). Indeed, weak solutions $x \in H$ of (13) are critical points of $\mathcal{E} : H \to \mathbb{R}$, where

$$
\mathcal{E}(x) = \frac{1}{2} \int_0^T |x'(t)|_{\mathbb{R}^N}^2 \, dt - \int_0^T G(t, x(t)) \, dt + \int_0^T (f(t), x(t))_{\mathbb{R}^N} \, dt
$$

$$
+ \sum_{i=1}^N \sum_{j=1}^{p_i} u_{ij} x_i(t_{ij}),
$$

$x \in H$. Similarly to the scalar case, every weak solution $x = x(t)$ of (13) satisfies the impulsive conditions

$$
\Delta x'_i(t_{ij}) := x'_i(t_{ij}^+) - x'_i(t_{ij}^-) = u_{ij}, \quad i = 1, \ldots, N, j = 1, \ldots, p_i,
$$

the system of equations

$$
ex''(t) + D_x G(t, x(t)) = f(t)
$$

holds a.e. in $(0,T)$, and the periodic conditions $x(0) = x(T), x'(0) = x'(T)$ hold.

The existence result reads as follows.

**Theorem 5.1.** Let $f \in L^1$ be arbitrary. Then for any system of control impulses $u_{ij}, i = 1, \ldots, N, j = 1, \ldots, p_i$, satisfying

$$
\sum_{j=1}^{p_i} u_{ij} = - \int_0^T f_i(t) \, dt, \quad i = 1, \ldots, N,
$$

...
the problem (13) admits at least one solution \( x_0 = x_0(t) \) which minimizes the energy functional \( E \) given on \( H \).

Sketch of the proof. Taking the advantage of (12) there exists \( C > 0 \) such that
\[
\left| \int_0^T G(t, x(t)) \, dt \right| \leq C
\]
for all \( x \in H \). We split \( x = x(t) \) as \( x(t) = \bar{x} + \tilde{x}(t) \), where \( \bar{x} = (\bar{x}_1, \ldots, \bar{x}_N) \) and \( \tilde{x}(t) = (\tilde{x}_1(t), \ldots, \tilde{x}_N(t)) \). With the help of (15), (16) and the Sobolev inequality, we get
\[
E(x) \geq \frac{1}{2} \int_0^T |\tilde{x}'(t)|^2 \, dt - C - \|\tilde{x}\|_{L^\infty} \|f\|_{L^1} - \max_{j=1,\ldots,p_i} \|u_{i,j}\| \|\tilde{x}'\|_{L^2}.
\]

Hence \( \inf_{x \in H} E(x) > -\infty \). Thanks to (12) we may assume that \( \bar{x}_i \in [0, T_i] \) for all \( i = 1, \ldots, N \). This fact combined with (17) imply that a minimizing sequence for \( E \) is bounded in \( H \). The existence of a minimizer \( x_0 \in H \) then follows from the weak lower semicontinuity argument along the same lines as in the proof of Theorem 3.1.

Remark 1. Let \( u_i = (u_{i1}, \ldots, u_{ip_i}) \) be the vectors of impulses at times \( t_{ij} \) and
\[
C_i(u_i) = \sqrt{u_{i1}^2 + \ldots + u_{ip_i}^2}, \quad i = 1, \ldots, N.
\]
Let \( f \in L^1 \) be arbitrary. Then, for every \( i = 1, \ldots, N \), there exists a unique optimal control
\[
u_i = \left( -\frac{1}{p_i} \int_0^T f_i(t) \, dt, \ldots, -\frac{1}{p_i} \int_0^T f_i(t) \, dt \right)
\]
which minimizes \( C_i \), and we have
\[
C_i(u_i) = \frac{1}{\sqrt{p_i}} \left| \int_0^T f_i(t) \, dt \right|.
\]

Remark 2. As a counterpart of Theorem 4.1 we have the following nonuniqueness result: Under the assumptions of Theorem 5.1 the problem (13) has, besides of solution \( x_0 = x_0(t) \) which minimizes the energy functional \( E \), at least one other solution \( x_1 = x_1(t) \) such that \( E(x_1) \geq E(x_0) \) and
\[
\|x_0 - x_1\| < \frac{1}{\sqrt{T}} \min\{T_1, \ldots, T_N\}.
\]

The proof of this fact follows the lines of the proof of Theorem 4.1. We also refer the reader to [13] for the proof of nonimpulsive case.
6. System of mutually attracted pendulums. Similar system of equations studied in this section was used in [3] to model a coupled micromechanical array. Let us consider a system of \(N\) planar pendulums, \(N \geq 2\), hanged on fixed frame at mutual distance \(a > 0\), as in Figure 2. Application of voltage \(V\) gives rise to an attractive interaction between each pendulum and its nearest neighbors, \(\varphi(s) = -C(s)\frac{V^2}{s^2}\), where \(s\) is the instantaneous distance between the interacting pendulums and \(C\) is the capacitance.

![Figure 2. A model of \(N\) mutually attracted pendulums.](image)

The equations of motion of the conservative system with external disturbances given by \(f_i = f_i(t)\), \(i = 1, \ldots, N\), take the following form:

\[
\begin{align*}
mx''_1(t) &= -m \sin x_1(t) - \varphi''(a)(x_1(t) - x_2(t)) + f_1(t), \\
mx''_j(t) &= -m \sin x_j(t) - \varphi''(a)(2x_j(t) - x_{j-1}(t) - x_{j+1}(t)) + f_j(t), \\
mx''_N(t) &= -m \sin x_N(t) - \varphi''(a)(x_N(t) - x_{N-1}(t)) + f_N(t),
\end{align*}
\]

where \(j = 2, \ldots, N-1\), \(m\) is the mass of each pendulum and \(-\varphi''(a) > 0\) due to the attractive nature of the interaction between nearest neighbors (cf. [1, p. 430] and also [3, p. 804]). Normalizing \(m = -\varphi''(a) = 1\), the above system takes the form

\[
x''(t) - Ax(t) + D_x G(t, x(t)) = f(t),
\]

where

\[
A = \begin{pmatrix}
1 & -1 & & & \\
-1 & 2 & -1 & & \\
& -1 & 2 & -1 & \\
& & & \ddots & \\
0 & & & -1 & 2 & -1 \\
& & & & -1 & 1
\end{pmatrix}
\]

is the symmetric positive semi-definite tridiagonal \(N \times N\) matrix and

\[
D_x G(t, x(t)) = (\sin x_1(t), \ldots, \sin x_N(t)).
\]

Given \(f \in L^1\), our goal is to find an impulsive control \(\{u_{ij}\}, i = 1, \ldots, N, j = 1, \ldots, p_i\), such that the periodic impulsive system

\[
\begin{cases}
x''(t) - Ax(t) + D_x G(t, x(t)) = f(t) + \sum_{i=1}^{N} \left( \sum_{j=1}^{p_i} u_{ij} \delta(t - t_{ij}) \right) e_i, \ t \in [0, T], \\
x(0) = x(T), \ x'(0) = x'(T)
\end{cases}
\]

(19)
has a solution. Notice that a solution \( x = x(t) \) of (19) is a solution of (18) in the sense of Carathéodory which satisfies the impulsive conditions (14). Similarly as in Section 5, the set of solutions of (19) coincides with the set of critical points of the associated energy functional

\[
\mathcal{E}(x) := \frac{1}{2} \int_0^T \left[ |x'(t)|^2_{\mathbb{R}^N} + (Ax(t), x(t))_{\mathbb{R}^N} \right] \, dt - \int_0^T G(t, x(t)) \, dt \\
+ \int_0^T (f(t), x(t))_{\mathbb{R}^N} \, dt + \sum_{i=1}^N \sum_{j=1}^{p_i} u_{ij} x_i(t_{ij}),
\]

\( x \in H \). Clearly, there exists a constant \( C > 0 \) such that

\[
\left| \int_0^T G(t, x(t)) \, dt \right| \leq C
\]

(20) for all \( x \in H \). The matrix \( A \) satisfies \( \dim \ker A = 1 \) and \( \ker A \) is spanned by \( \phi = \left( \frac{1}{\sqrt{NT}}, \ldots, \frac{1}{\sqrt{NT}} \right) \). In this section we will split any element \( x \in L^1 \) as follows

\[
x(t) = \hat{x} + \ddot{x}(t) = (\hat{x}, \ldots, \hat{x}) + (\ddot{x}_1(t), \ldots, \ddot{x}_N(t)),
\]

where

\[
\hat{x} = \frac{1}{\sqrt{NT}} \langle \phi, x \rangle_{L^2} = \frac{1}{NT} \sum_{i=1}^N \int_0^T x_i(t) \, dt, \quad \ddot{x}_i(t) = x_i(t) - \hat{x}, \quad i = 1, \ldots, N.
\]

Observe, that for any \( x, y \in L^1 \) we have

\[
(\hat{x}, y)_L^2 = \int_0^T \sum_{i=1}^N \ddot{x}_i(t) y_i(t) \, dt = \hat{x} \int_0^T \sum_{i=1}^N y_i(t) - \frac{1}{NT} \sum_{l=1}^N \int_0^T y_l(\tau) \, d\tau \, dt
\]

\[
= \hat{x} \int_0^T \sum_{i=1}^N y_i(t) \, dt - \hat{x} \int_0^T \sum_{l=1}^N y_l(t) \, dt = 0.
\]

Similarly, we also derive

\[
(A\hat{x}, y)_L^2 = (A\hat{x}, A\dot{y})_{L^2} = 0.
\]

**Lemma 6.1.** There exists \( c > 0 \) such that for all \( \hat{x} \in H \) we have

\[
\int_0^T \left[ ||\ddot{x}'(t)||_{\mathbb{R}^N} + (Ax(t), x(t))_{\mathbb{R}^N} \right] \, dt \geq c\|\ddot{x}\|^2.
\]

**Proof.** Notice that \( \lambda_1 = 0 \) is the first and simple eigenvalue of the operator \( H \to H \) given by

\[
x \mapsto -x'' + Ax
\]

and the corresponding eigenspace is spanned by \( \phi \). The inequality (22) then follows from the standard Courant-Weinstein variational characterization of higher eigenvalues of (23) employing the positive semi-definiteness of \( A \) and the compact embedding \( H \hookrightarrow L^2 \). \( \square \)

We continue with the existence result for the impulsive problem.
Theorem 6.2. Let \( f \in L^1 \) be arbitrary. Then for any system of control impulses \( u_{ij}, i = 1, \ldots, N, j = 1, \ldots, p_i, \) satisfying
\[
\sum_{i=1}^{N} \sum_{j=1}^{p_i} u_{ij} = -\sum_{i=1}^{N} \int_0^T f_i(t) \, dt,
\]
the problem (19) admits at least one solution \( x_0 = x_0(t) \) which minimizes the energy functional \( E \) on \( H \).

Proof. We use (20) - (22) and (24) to estimate \( E \) from below,
\[
E(x) := \frac{1}{2} \int_0^T \left[ |\dot{x}'(t)|_H^2 + (A x(t), x(t))_{R^N} \right] \, dt - \int_0^T G(t, \dot{x} + \ddot{x}(t)) \, dt
+ N T f \dot{x} + \int_0^T (\ddot{f}(t), \ddot{x}(t))_{R^N} \, dt + \dot{x} \sum_{i=1}^{N} \sum_{j=1}^{p_i} u_{ij} + \sum_{i=1}^{N} \sum_{j=1}^{p_i} u_{ij} \ddot{x}(t_{ij})
\geq \frac{C}{2} \|\ddot{x}\|^2 - C - \|f\|_{L^1} \|\dot{x}\|_{L^\infty} - \max_{j=1,\ldots,p_i} \{ |u_{ij}| \} \|\ddot{x}\|_{L^\infty}
\geq \frac{C}{2} \|\ddot{x}\|^2 - C - d \left[ \|f\|_{L^1} + \max_{j=1,\ldots,p_i} \{ |u_{ij}| \} \right] \|\dot{x}\|_{L^\infty},
\]
where \( d > 0 \) is a constant of the embedding \( H \hookrightarrow C, \) i.e.,
\[
\|x\|_{L^\infty} \leq d \|x\| \quad \text{for all} \quad x \in H.
\]
It follows from (25) that \( \inf_{x \in H} E(x) > -\infty. \) Let \( \{x_n\}_{n=1}^\infty \subset H \) be a minimizing sequence, \( x_n(t) = \ddot{x}_n + \ddot{x}_n(t). \) Since \( D_x G(t, \cdot) \) is \( 2\pi \)-periodic, we may assume, without loss of generality, that \( \ddot{x}_n \in [0, 2\pi]^N. \) Then from (25) with \( x \) replaced by \( x_n \) we deduce that \( \{x_n\}_{n=1}^\infty \) is bounded in \( H. \) The existence of \( x_0 \in H \) such that \( E(x_0) = \inf_{x \in H} E(x) \) now follows from the weak lower semicontinuity argument as in the proof of Theorem 3.1. \( \Box \)

Let \( u = (u_{11}, \ldots, u_{1p_1}, u_{21}, \ldots, u_{Np_N}) \) be a vector of impulses applied at fixed time instants \( t_{ij}, i = 1, \ldots, N, j = 1, \ldots, p_i, \)
\[
\mathcal{C}(u) = \left( \sum_{i=1}^{N} \sum_{j=1}^{p_i} u_{ij}^2 \right)^{\frac{1}{2}}.
\]

Corollary 2. Let \( f \in L^1 \) and \( P = \sum_{i=1}^{N} p_i. \) Then there exists a unique optimal control
\[
u = \left( -\frac{1}{P} \sum_{i=1}^{N} \int_0^T f_i(t) \, dt, \ldots, -\frac{1}{P} \sum_{i=1}^{N} \int_0^T f_i(t) \, dt \right)
\]
which minimizes \( \mathcal{C} \), and \( \mathcal{C}(\nu) = \frac{1}{\sqrt{P}} \left| \sum_{i=1}^{N} \int_0^T f_i(t) \, dt \right|. \)
Remark 3. Note that only one impulse applied to one single pendulum forces the entire system into a periodic motion. The sudden change of velocity occurs only for the pendulum this impulse is applied to. The velocities of other pendulums are continuous functions.

Theorem 6.3. Under the assumptions of Theorem 6.2 the problem (19) has, besides of solution \( x_0 = x_0(t) \) which minimizes the energy functional \( E \), at least one other solution \( x_1 = x_1(t) \) such that \( E(x_1) \geq E(x_0) \), and \( x_0 \) and \( x_1 \) do not differ by a multiple of \( 2\pi \).

Proof. Similarly as in the proof of Theorem 4.1 we may assume that \( x_0 \) is a strict local minimum of the energy functional \( E \) on \( H \) and show that \( E \) has a Mountain Pass-type geometry locally around \( x_0 \). Indeed, there exists \( R < 2\pi \sqrt{T} \) such that \( E(x) > E(x_0) \) for \( 0 < \|x - x_0\| \leq R \), and therefore

\[
\text{b} := \inf_{\|x - x_0\| = R} E(x) \geq E(x_0).
\]

The proof of \( b > E(x_0) \) follows the same lines as that of Theorem 4.1 and it is based on the weak lower semicontinuity argument for \( x \mapsto \frac{1}{2} \|x'\|_2^2 \) and the compactness of

\[
x \mapsto \frac{1}{2} \int_0^T (Ax(t), x(t))_{\mathbb{R}^N} \, dt - \int_0^T G(t, x(t)) \, dt + \int_0^T (f(t), x(t))_{\mathbb{R}^N} \, dt + \sum_{i=1}^N \sum_{j=1}^{p_i} u_{ij}(x(t_{ij})).
\]

It remains to prove the Palais-Smale condition. Let \( \{x_n\}_{n=1}^\infty \) be a Palais-Smale sequence. Its boundedness follows, as in the proof of Theorem 4.1, using the fact that \( (Ax(t), x(t))_{\mathbb{R}^N} \geq 0 \), \( t \in [0,T] \), for all \( x \in H \). Therefore, passing to a subsequence if necessary, we may assume that there is \( x \in H \) such that \( x_n \rightharpoonup x \) in \( H^* \), \( x_n \to x \) in \( \mathbb{R}^N \) and \( x_n \to x \) in \( L^2 \) and in \( C \). The assumption \( E'(x_n) \to 0 \) in \( H^* \) then yields

\[
0 \leq \langle E'(x_n), x_n - x \rangle = \langle x_n', x_n' - x' \rangle_{L^2} + \langle Ax_n, x_n - x \rangle_{L^2} \to 0 \text{ as } n \to \infty \\
- \langle DxG(\cdot, x_n), x_n - x \rangle_{L^2} + \langle f; x_n - x \rangle_{L^2} \to 0 \text{ as } n \to \infty \\
+ \sum_{i=1}^N \sum_{j=1}^{p_i} u_{ij}(x(t_{ij}) - x(t_{ij}), e_i)_{\mathbb{R}^N} \to 0 \text{ as } n \to \infty
\]

Hence, \( \|x_n'\|_{L^2}^2 - (x_n', x_n')_{L^2} \to 0 \) as \( n \to \infty \). Since also \( (x_n', x_n')_{L^2} \to \|x'\|_{L^2}^2 \) by the weak convergence \( x_n \rightharpoonup x \) in \( H \), we conclude \( \|x_n'\|_{L^2} \to \|x'\|_{L^2} \). Therefore, \( \|x_n\| \to \|x\| \) and so we finally conclude \( x_n \to x \) in \( H \). This proves the Palais-Smale condition.

The existence of the second solution \( x_1 = x_1(t) \) now follows from the Mountain Pass Theorem.

Acknowledgments. This research was supported by the Grant 13-00863S of the Grant Agency of Czech Republic and by the project LO1506 of the Czech Ministry of Education, Youth and Sports.
REFERENCES

[1] N. W. Ashcroft and N. D. Mermin, Solid State Physics, Saunders College, Philadelphia, 1976.
[2] D. D. Bainov and P. S. Simeonov, Impulsive Differential Equations: Periodic Solutions and Applications, Harlow, Longman, 1993.
[3] E. Buks and M. L. Roukes, Electrically tunable collective response in a coupled micromechanical array, J. Microelectromech. Syst., 11 (2002), 802–807.
[4] T. E. Carter, Optimal impulsive space trajectories based on linear equations, J. Optim. Theory Appl., 70 (1991), 277–297.
[5] T. E. Carter, Necessary and sufficient conditions for optimal impulsive rendezvous with linear equations of motion, Dynam. Control, 10 (2000), 219–227.
[6] P. Drábek and M. Langerová, Quasilinear boundary value problem with impulses: Variational approach to resonance problem, Bound. Value Probl., 2014 (2014), 14pp.
[7] P. Drábek and M. Langerová, On the second order periodic problem at resonance with impulses, J. Math. Anal. Appl., 428 (2015), 1339–1353.
[8] P. Drábek and J. Milota, Methods of Nonlinear Analysis, Applications to Differential Equations, 2nd edition, Birkhäuser, Springer Basel, 2013.
[9] B. S. Kalinin, On oscillations of mathematical pendulum with striking impulse, Differ. Uravn., 7 (1969), 1267–1274. (in Russian)
[10] V. Lakshmikantham, D. D. Bainov and P. S. Simeonov, Theory of Impulsive Differential Equations, World Scientific, Cambridge, 1989.
[11] X. Liu and A. R. Willms, Impulsive controllability of linear dynamical systems with applications to maneuvers of spacecraft, Math. Probl. Eng., 2 (1996), 277–299.
[12] J. Mawhin and M. Willem, Multiple solutions of the periodic boundary value problem for some forced pendulum-type equations, J. Differential Equations, 52 (1984), 264–287.
[13] J. Mawhin and M. Willem, Variational methods and boundary value problems for vector second order differential equations and applications to the pendulum equation, in Nonlinear Analysis and Optimization, Lect. Notes Math., 1107, Springer, Berlin, (1984), 181–192.
[14] J. Mawhin and M. Willem, Critical Point Theory and Hamiltonian Systems, Springer-Verlag, New York, 1989.
[15] J. J. Nieto and D. O’Regan, Variational approach to impulsive differential equations, Nonlinear Anal. Real World Appl., 10 (2009), 680–690.
[16] A. M. Samoilenko and N. A. Perestyuk, Impulsive Differential Equations, World Scientific, Singapore, 1995.
[17] Y. Tian and W. Ge, Applications of variational methods to boundary-value problem for impulsive differential equations, Proc. Edinb. Math. Soc., 51 (2008), 509–527.
[18] M. Willem, Oscillations forcées de systèmes hamiltoniens, Publications Mathématiques de la Faculté des Sciences de Besançon, Besançon, 1981.
[19] T. Yang, Impulsive Control Theory, Lecture Notes in Control and Information Sciences, 272, Springer-Verlag Berlin Heidelberg, 2001.

Received July 2017; revised November 2017.

E-mail address: pdrabek@kma.zcu.cz
E-mail address: mlanger@ntis.zcu.cz