The power-law expansion universe and dark energy evolution

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Abstract

In order to depict the transition from deceleration to acceleration expansion of the universe we use a power-law expansion scale factor, $a \sim t^{n_0 + bt^m}$, with $n_0$, $b$ and $m$ three parameters determined by $H_0$, $q_0$ and $z_T$. For the spatially flat, isotropic and homogeneous universe, such a scale factor leads to the results that the dark energy density is slowly changing currently, and predicts the equation of state $w_X$ changes from $w_X > -1$ to $w_X < -1$.

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I. INTRODUCTION

The type Ia supernova (SNe) observations \[1\] and the measurements on the cosmic microwave background (CMB) \[2\] indicate that the universe is acceleratedly expanding and is spatially flat. The faintness of high-redshift SNe has been interpreted as evidence that the expansion of the universe is accelerating, which suggests the universe transition from deceleration to acceleration has happened. The transition redshift of the universe from deceleration to acceleration provides us an important information of dark energy. Providing that the apparent faintness of SNe Ia provides the direct evidence for the accelerating expansion of the universe, then it happened at \(z_T \approx 0.5\) \[3\]. It should have happened at \(z > 0.3\), with the best fit at about \(z_T \approx 0.45\) \[4\], or \(z_T \approx 0.40\) \[5\]. Using a simple expansion model, \(q \simeq q_0 + z(dq/dz)_{z=0}\), and the statistical method, the transition redshift is determined to be \(z_T \approx 0.46\) \[6\]. From modified Friedmann equation, the constraint from the SNe Ia observation gives \(z_T = 0.52 \sim 0.73\) \[4\]. From a joint analysis of SNe+CMB, one can get \(z_T \approx 0.39\), but it can be \(z_T \approx 0.57\) for \(\Omega_{0m} \approx 0.27\), \(h \approx 0.71\) and \(w_\Lambda \approx -1\) \[7\]. Anyway, the results from astronomical observations show our universe could have undergone an expansion transition from deceleration to acceleration \[3, 4, 5, 6, 7, 8, 9\].

The problem of the dynamical origin of the dark energy has attracted a great many attentions \[10, 11, 12, 13, 14, 15, 16, 17, 18, 19\]. Current supernova, CMB, and LSS data already rule out dark energy models with dark energy densities varying fast with time \[20\]. From the viewpoint of geometry the scale factor of the universe is a fundamental quantity. It is possible to catch a glimpse of the nature of dark energy. Rather than attempting to obtain a whole description for the universe and the dark energy, the approach here aims to construct a scale factor by using the current observations: the Hubble parameter \(H_0\), the deceleration parameter \(q_0\) and the transition redshift. It is expected that such a scale factor provides an approximate description for the universe and the dark energy during a history of evolution including at least the transition of the universe.

The determination of parameters in a parametrized quantity depends on the observations. An appropriate scale factor should satisfy at least the two requirements from the observations: the past transition of the universe from acceleration to deceleration expansion phase and the currently slow variation of the dark energy density. Recently, motivated by describing the possible super-accelerated transition \[15\], we discuss the scalar factor of the
form $a \sim t^{n_0 + bt}$. In this paper, we will extend this form to $a \sim t^{n_0 + bt^m}$. This needs the three observation quantities to determine the three parameters, which will be chosen to be $H_0$, $q_0$ and $z_T$. In Sec. II, we determine the three parameters in the scale factor by using $H_0 \approx 0.7 \times 10^{-10}$ yr$^{-1}$, $q_0 = -0.5$ and $z_T = 0.5$, as well as $t_0 \approx 1.4 \times 10^{10}$ yr. In Sec. III, we show the results of dark energy produced from the above scale factor with three parameters, which illustrates that the dark energy density is slowly changing currently and the equation of state $w_X$ can change from $w_X > -1$ to $w_X < -1$. The latter result can give a strict constraint on the dark energy model.

II. APPROACH ON THE POWER-LAW EXPANSION UNIVERSE

For each epoch, the radiation, matter and dark energy epoch, the universe may be characterized by a power-law scale factor with a constant exponent depending on the equation of state of the dominant component in that epoch. Actually, the exponent change with time since the fractions of the different energy components in the universe have been varying. We begin by writing down the scale factor

$$a = a_c \left( \frac{t}{T} \right)^n,$$

with $n$ a function of time, where $a_c$ and $T$ are two constants. From (1), the Hubble parameter and its derivative are

$$H = \dot{n} \ln t + \frac{n}{t},$$

$$\dot{H} = \frac{\ddot{n}}{t} - \frac{n}{t^2},$$

where a dot denotes the derivative with respect to time.

Without observations, there is no priori way of determining the scale factor. We anticipate, from our knowledge of the evolution of the universe such as the existence of the transition, that the exponent should increase with time at least from the matter-dominated epoch to today. Assuming the scale factor (1), the simplest form of the exponent to satisfy the above requirement should be the linear approximation $n = n_0 + bt$, as in [15], which can always be a good approximation to the regular exponent $n$ in an enough short time.
However, we require a scale factor can well describe the universe in a long interval of time, here. For this, we will attempt to test the following form of the scale factor

\[ n = n_0 + bt^m, \]  

(4)

where \( n_0 \), \( m \) and \( b \) are three positive constants.

Putting Eq.(4) and \( \dot{n} = bmt^{m-1} \) in Eqs. (2) and (3) yields

\[ H = b(m \ln t + 1)t^{m-1} + n_0 t^{-1}, \]  

(5)

\[ \dot{H} = b[m(m-1) \ln t + 2m - 1]t^{m-2} - n_0 t^{-2}. \]  

(6)

Eqs. (5) and (6) shows the dependence of \( H \) and \( \dot{H} \) on \( m \). For \( m < 1 \), Eq. (6) leads to \( \dot{H} \) is always negative and (5) indicates the universe gets gradually flat for late time. For \( m \geq 1 \), Eqs. (5) and (6) show obviously the late-time phantom property of the dark energy [15].

From \( \ddot{a}/a = \dot{H} + H^2 \) and \( H = \dot{a}/a \), one has the deceleration parameter

\[ q = -\frac{\ddot{a}}{a^2} = -(1 + \frac{\dot{H}}{H^2}). \]  

(7)

Defining \( x = \ln t, A = 2n_0 + 2m - 1 \) and \( B = m(2n_0 + m - 1) \), putting Eqs. (2) and (3) in (7) then we obtain

\[ q = -H^{-2}[b^2(1 + mx)^2 t^{2m-2} + b(A + Bx)t^{m-2} + (n_0^2 - n_0 )t^{-2}]. \]  

(8)

Using \( H_0, q_0, \) and \( t_0 \), then from Eqs. (5) and (8) we obtain the following two equations

\[ H_0 = b(mx_0 + 1)t_0^{m-1} + n_0 t_0^{-1}, \]  

(9)

\[ -q_0 H_0^2 = b^2(1 + mx_0)^2 t_0^{2m-2} + b(A + Bx_0)t_0^{m-2} + (n_0^2 - n_0 )t_0^{-2}, \]  

(10)

with \( x_0 = \ln t_0 \). Solving Eqs. (9) and (10) gives rise to

\[ n_0 = H_0 t_0 - b(mx_0 + 1)t_0^m, \]  

\[ b = \frac{1 - (q_0 + 1)H_0 t_0}{m(mx_0 + 2)}H_0 t_0^{1-m}. \]  

(11)

According to [5], there can be the current deceleration parameter \( q_0 > -0.5 \). From [6, 8], there can be \( q_0 < -0.5 \). Let us adopt a modest value \( q_0 = -0.5 \), here. The current Hubble parameter and the cosmological age are taken as \( H_0 \approx 0.7 \times 10^{-10} yr^{-1} \) and \( t_0 \approx 1.4 \times 10^{10} yr \).
[21], here. The three parameters \( H_0 \approx 0.7 \times 10^{-10} yr^{-1}, q_0 = -0.5, z_T = 0.5 \), as well as, \( t_0 \approx 1.4 \times 10^{10} yr \) will be used to determine \( n_0, b \) and \( m \). For some special \( m \), the parameters \( b \) and \( n_0 \) are exhibited in Table I. Notice that the deceleration parameter \( q \) is given by (7).

For the five cases \( m = 2.9 \sim 3.3 \), the variations of the deceleration parameter \( q \) with time are shown in Fig. 1. From Fig. 1, one can see the expected converge of the curves characterized by \( m \) to \( t = t_0 \) and the unexpected converge to \( t \approx 0.6 \times 10^{10} yr \). The second convergence of the curves with different \( m \) is an intriguing feature, it means the validity of the power-law scale factor may be extended to the range, \( t < 0.6 \times 10^{10} yr \).

### Table I: The parameters \( b, n_0 \) are given for \( q_0 = -0.50 \) and some special values of \( m \).

| \( m \) | 2.90  | 3.00  | 3.10  | 3.20  | 3.30  |
|--------|-------|-------|-------|-------|-------|
| \( b \) | \( 9.32 \times 10^{-33} \) | \( 8.42 \times 10^{-34} \) | \( 7.63 \times 10^{-35} \) | \( 6.93 \times 10^{-36} \) | \( 6.31 \times 10^{-37} \) |
| \( n_0 \) | 0.810 | 0.816 | 0.821 | 0.826 | 0.830 |

FIG. 1: The figure shows the variations of \( q \) for \( m = 2.90, 3.00, 3.10, 3.20, 3.30 \), respectively.

The expression, \( 1 + z = a_0/a(t) \), holds valid for any spatially flat, isotropic and homogeneous universe. Table 2 exhibits the predicted transition time \( t_T = 8.75 \sim 9.06 Gyr \) for \( m = 2.9 \sim 3.3 \). From \( z_T = n_1/n_2 - 1 \) with \( n_1 = n_0 + b t_0^m \) and \( n_2 = n_0 + b t_T^m \), the corresponding transition redshift is \( z_T = 0.528 \sim 0.485 \). For \( z_T = 0.5 \), \( m \) is determined as \( m \approx 3.16 \), through (11) which gives \( n_0 \approx 0.8239 \) and \( b \approx 1.809 \times 10^{-35} yr^{-m} \).
TABLE II: The predicted transition time $t_T$ and the corresponding redshift $z_T$.

| $m$ | 2.90 | 3.00 | 3.10 | 3.20 | 3.30 |
|-----|------|------|------|------|------|
| $t_T(10^{10} \text{yr})$ | 0.875 | 0.883 | 0.891 | 0.8984 | 0.906 |
| $z_T$ | 0.528 | 0.517 | 0.506 | 0.496 | 0.485 |

TABLE III: This table shows $\Delta n = bt^m$ with $b \simeq 1.809 \times 10^{-35} \text{yr}^{-m}$ and $m = 3.16$.

| $t(10^{10} \text{yr})$ | 0.6 | 0.9 | 1.4 | 2.0 |
|-----------------------|-----|-----|-----|-----|
| $\Delta n$            | 0.0001 | 0.0005 | 0.0021 | 0.0064 |

TABLE III shows the variation of $n$ with time is very slow, which changes only about 0.0064 or $n \simeq 0.8239 \sim 0.8303$ in $t = 6\text{Gyr} \sim 20\text{Gyr}$. This illustrates that the scale factor discussed currently can be applicable from the late matter-dominated epoch. In this section, we have worked out the three parameters $n_0$, $b$ and $m$ by using the three observation quantities $H_0$, $q_0$ and $z_T$. In the next section, we will extract some information of the dark energy from the scale factor with the above three parameters.

III. RESULTING DARK ENERGY EVOLUTION FROM SCALE FACTOR WITH THREE PARAMETERS

Though the scale factor (11) with the exponent (11) is somewhat an ad hoc choice, it has taken into account the actual observations such as the transition of universe from acceleration to deceleration expansion. It is expected that the power-law expansion universe proposed here can yield the consistent result with the observations for the dark energy.

For the spatially flat, isotropic and homogeneous universe, the Einstein equations reads

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{3}{8\pi G}(\rho_X + \rho_M),$$

with $\rho_X$ and $\rho_M$ the energy densities of dark energy and matter. For the matter of the pressureless fluid, the energy density evolves in terms of

$$\rho_M = \rho_M(1 + z)^3 = \Omega_M \rho_0 \frac{t^3(\rho_0+bt^m)}{t^3(\rho_0+bt^m)};$$
where \( \rho_0 \) is the total energy density of the current universe, \( \rho_{M0} \) and \( \Omega_{M0} \) denote the today’s energy density and fraction of the matter. From Eqs. (12) and (13), it follows that the energy density of dark energy

\[
\rho_X = \rho - \rho_M = \left[ \frac{n_0 + b(m \ln t + 1)t^m}{H_0^2t^2} \right]^2 - \Omega_{M0} \frac{t_0^{3(n_0 + b_0^m)}}{t^{3(n_0 + b^m)}} \rho_0. \tag{14}
\]

Fig. 2 shows how \( \rho_X \) varies with time. From Fig. 2, one can see that from \( t = 12.6 \text{Gyr} \) (\( z \approx 0.1 \)) to \( t = 14 \text{Gyr} \) the dark energy density changes only about \( 6\% \sim 4\% \) for \( \Omega_M = 0.27 \sim 0.33 \), this illustrates that the dark energy is slowly changing currently \[20, 22\]. The conserved equation for dark energy reads

\[
\dot{\rho}_X + 3H(\rho_X + p_X) = 0. \tag{15}
\]

Noting that the equation of state \( p_X = w_X \rho_X \), Eq. (15) yields the approximate calculation formula of the equation of state, \( \bar{w}_X = -1 - \frac{\Delta \rho_X}{3H \rho_X \Delta t} \), where \( \dot{\rho}_X \), \( H \) and \( \rho_X \) have been replaced
by the mean variation rate $\frac{\Delta \rho_X}{\Delta t}$, the mean values $\bar{H}$ and $\bar{\rho}_X$ for $\Delta t \ll 1\text{Gyr}$, respectively. From the $\bar{w}_X$ formula, then there is

$$\bar{w}_{X0} = -1 - \frac{\Delta \rho_X}{3H_0 \rho_{X0} \Delta t}, \quad (16)$$

where $\bar{w}_{X0}$ denotes the average of $w_X$ in $\Delta t = t_0 - t$ with $t_0$ the age of the universe. Taking $t = 13.9\text{Gyr}$, i.e., $\Delta t = 0.1\text{Gyr}$, then $\frac{\Delta \rho_X}{\rho_{X0}} \approx -0.0031, -0.0025, -0.0018$, and (16) yields $w_{X0} = \bar{w}_{X0} \approx -0.85, -0.88, -0.92$ for $\Omega_{M0} = 0.27, 0.30, 0.33$, respectively. Clearly, the above values of $w_{X0}$ can agree with the most results of the dark energy equation of state.

In addition, Fig. 2 illustrates $\rho_X$ has the minimum, which implies that $w_X$ can change from $w_X > -1$ to $w_X < -1$ at a certain future time, which is quite similar to the result [20, 23] that $w_X$ can be across $-1$ at $z < 0.2$. Looking somewhat different, these two results don’t mean any disagreement since the most observations such as the equation of state of dark energy are still given only in a quite uncertainty range. Instead, it can provide an important information of the dark energy. If getting confirmed from the further observations it will provide a strong constrain on the dark energy models.

To sum up, the predicted results of the dark energy from the parametrized scale factor with three parameters show no contradiction with the known those from observations. Thus, it is believed that it can give a good approximation descriptions for the universe and dark energy during a long evolution course, from $6\text{Gyr}$ to a future time, saying $20\text{Gyr}$. We hope that the work in this paper can be the beginning of a meaningful approach on the universe evolution and the behavior of dark energy.

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