ADÈLE RESIDUE SYMBOL AND TATE’S CENTRAL EXTENSION FOR MULTILOOP LIE ALGEBRAS

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Abstract. We generalize the linear algebra setting of Tate’s central extension to arbitrary dimension. In general, one obtains a Lie \((n+1)\)-cocycle. We compute it to some extent. The construction is based on a Lie algebra variant of Beilinson’s adelic multidimensional residue symbol, generalizing Tate’s approach to the local residue symbol for 1-forms on curves.

Firstly, recall that to every Lie algebra \(g\) one can associate its loop Lie algebra \(g[t^\pm]\). Iterating this construction, we obtain so-called multiloop Lie algebras, \(g[t^\pm_1, \ldots, t^\pm_n]\).

To begin with, we show that various classes of interesting multiloop Lie algebras can all be embedded into a large (infinite-dimensional) Lie algebra:

**Theorem 1.** Let \(k\) be a field and \(n \geq 1\). There is a universal Lie algebra \(\mathfrak{G}\) naturally containing the following:

1. the abelian Lie algebra \(k[t^\pm_1, \ldots, t^\pm_n]\),
2. Lie algebras of derivations, e.g. spanned by \(t^{s_1}_1 \cdots t^{s_n}_n \partial_i\), (acting on \(k[t^\pm_1, \ldots, t^\pm_n]\))
3. for any finite-dimensional simple Lie algebra \(g\) the multiloop algebra \(g[t^\pm_1, \ldots, t^\pm_n]\).

The universal Lie algebra \(\mathfrak{G}\) has a canonical Lie \((n+1)\)-cocycle \(\phi \in H^{n+1}(\mathfrak{G}, k)\). For \(n = 1\) this cocycle determines a central extension (known as Tate’s central extension)

\[ 0 \rightarrow k \rightarrow \hat{\mathfrak{G}} \rightarrow \mathfrak{G} \rightarrow 0 \]

and the pullback of it to one of the above types of subalgebras yields (respectively)

1. the Heisenberg algebra,
2. the Virasoro algebra,
3. the affine Lie algebra \(\hat{g}\) associated to \(g\).

This will be stated in more detail and proven in [10]. It is not at all surprising that some Lie algebras can be embedded into larger ones. The interesting fact is that there is such a Lie algebra which carries a canonical cocycle, inducing the ones defining all these classical central extensions. For \(n = 1\) the above is well-known, see for example [3] \(\S 2.1\). For \(n = 1, 2\) see [7]. In the language of the latter, \(\mathfrak{G}\) is an example of a “master Lie algebra”.

We are interested in the nature of \(\phi\) for \(n > 1\) – even if such cocycles cannot be
interpreted as a central extension anymore (we get crossed modules, etc.). Indeed, they are meaningful, as we shall see.

A key point of this text is the actual computation of \( \phi \) (with a slight limitation):

**Theorem 2.** The cocycle \( \phi \in H^{n+1}(\mathfrak{g}, k) \) is given explicitly by
\[
\phi(f_0 \wedge f_1 \wedge \ldots \wedge f_n) = \text{tr} \sum_{\pi \in \mathfrak{S}_n} \text{sgn}(\pi) \sum_{\gamma_1, \ldots, \gamma_n \in \{\pm\}} (-1)^{\gamma_1 + \ldots + \gamma_n} (P_1^{-\gamma_1} \text{ad}(f_{\pi(1)}) P_1^{\gamma_1}) \ldots (P_n^{-\gamma_n} \text{ad}(f_{\pi(n)}) P_n^{\gamma_n}) f_0,
\]

whenever \( f_0 \otimes f_1 \wedge \ldots \wedge f_n \) is already a \( \mathfrak{g} \)-valued Lie cycle. The \( P_i^+ \), \ldots, \( P_i^- \) refer to certain commuting idempotents (see \([4]\) for details).

The proof and details regarding the \( P_i^\pm \) can be found in \([3]\). Effectively, we compute the composition
\[
(0.1) \quad H_n(\mathfrak{g}, \mathfrak{g}) \xrightarrow{I} H_{n+1}(\mathfrak{g}, k) \to k,
\]

with \( I \) a natural map to be explained in \([2]\) By the Universal Coefficient Theorem for Lie algebras, \( H^{n+1}(\mathfrak{g}, k) \cong H_{n+1}(\mathfrak{g}, k)^* \), referring to the dual space. As such, although \( \phi \) is well-defined, the formula only applies to those cycles admitting a lift under \( I \) (as soon as it exists, the choice does not matter). The formula is rather complicated. However, the pullback to particular subalgebras of \( \mathfrak{g} \) can be much nicer, for example for multiloop Lie algebras of simple Lie algebras, we get the following:

**Theorem 3.** Suppose \( \mathfrak{g}/k \) is a finite-dimensional centreless Lie algebra (e.g. simple). For \( Y_0, \ldots, Y_n \in \mathfrak{g} \) we call
\[
B(Y_0, \ldots, Y_n) := \text{tr}_{\text{End}_k(\mathfrak{g})}(\text{ad}(Y_0) \text{ad}(Y_1) \ldots \text{ad}(Y_n))
\]

the ‘generalized Killing form’. Then on all Lie cycles admitting a lift under \( I \) as in eq. \([7]\) the pullback of \( \phi \) to \( \mathfrak{g}[t_1^\pm, \ldots, t_n^\pm] \) is explicitly given by
\[
\phi(Y_0 t_1^{i_0, 1} \ldots t_n^{i_n, n} \wedge \ldots \wedge Y_n t_1^{c_n, 1} \ldots t_n^{c_n, n}) = (-1)^n \sum_{\pi \in \mathfrak{S}_n} \text{sgn}(\pi) B(Y_{\pi(1)}, \ldots, Y_{\pi(n)}, Y_0) \prod_{i=1}^n c_{\pi(i), i}
\]

whenever \( \forall i \in \{1, \ldots, n\} : \sum_{p=0}^n c_{p,i} = 0 \) and zero otherwise. Here \( c_{i,p} \in \mathbb{Z} \) for all \( i = 0, \ldots, n \) and \( p = 1, \ldots, n \).

If \( \mathfrak{g} \) is finite-dimensional simple and \( n = 1 \), then the class \( \phi \) yields the universal central extension of the loop Lie algebra \( \mathfrak{g}[t_1, t_1^{-1}] \), the associated affine Lie algebra \( \widehat{\mathfrak{g}} \) (without extending by a derivation),
\[
0 \to k \to \widehat{\mathfrak{g}} \to \mathfrak{g}[t_1, t_1^{-1}] \to 0.
\]

In this case \( B \) is obviously just the ordinary Killing form of \( \mathfrak{g} \). The above theorem will be proven in \([8]\).

Additionally, we should say that these computations have an application outside the theory of Lie algebras. For this we need to return to the roots of the subject. In 1967 J. Tate \([17]\) showed that the residue of a rational 1-form \( fdg \) at a closed point \( x \) on an algebraic
curve $X/k$ can be expressed as a certain operator-theoretic trace on an infinite-dimensional space. Arbarello, de Concini and Kac \cite{1} eq. (2.7) reformulated this as

\[ (0.2) \text{res}_x f dg = \text{tr}([\pi, g] f). \]

On the right-hand side the functions $f, g$ are to be read as multiplication operators acting on the local field $\text{Frac} \mathcal{O}_{X,x} \simeq \kappa((t))$, seen as a $\kappa$-vector space, and $\pi$ denotes some projector on the non-principal part, e.g. “we cut off the principal part of the Laurent series”. It is natural to ask whether there exists a generalization of this formula to higher residues. We can give such a formula; it will be proven in §7:

**Theorem 4.** For a multiple Laurent polynomial ring with residue field $k$, say

$R := k[t_{1}^{\pm}, \ldots, t_{n}^{\pm}]$,

and $f_0, \ldots, f_n \in R$ we have

\[
\text{res}_t f_0 df_1 \ldots df_n = (-1)^n \text{tr} \sum_{\pi \in S_n} \text{sgn}(\pi) \left( -1 \right)^{\gamma_1 + \cdots + \gamma_n} (P_1^{-\gamma_1} \text{ad}(f_\pi(1))P_1^{\gamma_1}) \cdots (P_n^{-\gamma_n} \text{ad}(f_\pi(n))P_n^{\gamma_n}) f_0,
\]

where $P_1^{\pm}, \ldots, P_n^{\pm}$ are suitable projectors (explained in \cite{1} eq. 7.3).

1. For $n = 1$ and $\pi := P_1^{+}$ the formula reduces to the familiar eq. 0.2, as in \cite{1}.

2. If we have $f_i = t_{1}^{c_{1,i}} \cdots t_{n}^{c_{i,n}}$ for $i = 0, \ldots, n$, the formula reduces to

\[
\text{res} f_0 df_1 \ldots df_n = \det \begin{pmatrix} c_{1,1} & \cdots & c_{n,1} \\ \vdots & \ddots & \vdots \\ c_{1,n} & \cdots & c_{n,n} \end{pmatrix} \text{ if } \forall i : \sum_{p=0}^{n} c_{p,i} = 0
\]

and the residue is zero if the condition on the right-hand side is not satisfied.

3. For $n = 1$ and $f_1 = t_1$ this reduces by linearity to the classical definition

\[
\text{res} \alpha t_1 dt_1 = \begin{cases} 
\alpha & \text{if } c_1 = -1 \\
0 & \text{if } c_1 \neq -1. 
\end{cases}
\]

How to construct the cocycle $\phi$?

There are various ways to approach this construction. Frenkel and Zhu \cite{7} use distinguished generators of the cohomology ring of infinite matrix algebras, based on computations of Feigin and Tsygan \cite{4}. This is a very natural approach. However, in this text we use a different approach based on Beilinson’s multidimensional adelic residue \cite{2}. Originally, this approach was only used to generalize Tate’s approach to the residue symbol to several variables, but it readily generalizes to the problem we are discussing here. This might be interesting also since \cite{2} does not give an explicit formula – and it is not totally trivial to extrapolate a formula from the definition:

**Theorem 5.** The formula in Thm. 4 arises from the construction of Beilinson in the paper \cite{2} Lemma 1], i.e. it is the composition

\[
\Omega^n_{R/k} (\begin{array}{c} \rightarrow \\ (-1)^{n} \end{array}) H^\text{Lie}_{n+1}(\mathfrak{g}, k) \overset{p_2}{\rightarrow} E_{n+1}^{n+1} \overset{(d_{n+1})^{-1}}{\rightarrow} E_{n+1,n}^{n+1} \overset{p_1}{\rightarrow} H^\text{Lie}_{0}(\mathfrak{g}, N^{n+1}) \overset{\text{tr}}{\rightarrow} k,
\]

\[
\text{tr} = k.
\]
where

- \( \kappa : f_0 df_1 \wedge \cdots \wedge df_n \mapsto f_0 \wedge \cdots \wedge f_n \),
- \( N^{n+1} \) is a certain \( \Theta \)-module (see [4] for the definition, or \( T_{*N} \) in [2]), and
- \( \rho_1, \rho_2 \) are edge maps and \( d_{n+1} \) a differential on the \((n+1)\)th page of a certain spectral sequence \( ^{\wedge}E_{\bullet \bullet}^\bullet \) (constructed in Lemma 5 or see [2, Lemma 1]).

This result is only meaningful to readers familiar with the paper [2].

The above theorem actually lies at the heart of our approach. We formulate a contracting homotopy for a mild variation of the relevant complexes in [2] and then, in a slightly tedious computation, make the spectral sequence differential \( d_{n+1} \) explicit on the basis of this.

Finally, for applications in algebraic geometry, e.g. the interpretation as a local residue, it is unfortunate to interpret the word “loop Lie algebra” as \( g[t, t^{-1}] \). It is better to work with Laurent series, i.e. \( g((t)) \), or even local components of adèles. Tate’s original work uses the language of adèles for example. For this reason, we shall axiomatize all these variations through the notion of a “cubically decomposed algebra” (essentially taken from [2], where it’s not given a name).

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0.2. What is not here. In the present text I only discuss the ‘linear algebra setting’ of Tate’s central extension ([3 §1] for the case \( n = 1 \)). There is also a ‘differential operator setting’ ([2 §2]), which I will treat in a future text. Roughly speaking, \( \Theta \) will be replaced by much smaller algebras of differential operators on a vector bundle.

Moreover, I do not treat the true multiloop analogue of an affine Kac-Moody algebra in the present text. Already for \( n = 1 \) I only consider the ‘plain’ affine Lie algebras without extending by a derivation. From the perspective of a triangular decomposition, this is a rather horrible omission: the root spaces are infinite-dimensional! However, as the reader can probably imagine from the computations in [7] [8] the calculation gets a lot more complicated in the presence of derivations. Thus, this aspect will also be deferred to a future text. The same applies to the analogue of the plain Virasoro algebra. There should also be a nonlinear analogue, distinguished cohomology classes for multiloop groups. The cases \( n = 1, 2 \) (along with a higher representation theory in categories) are treated in detail by Frenkel and Zhu in [7].

One should also mention that there are completely orthogonal generalizations of Kac-Moody/Virasoro cocycles to multiloop Lie algebras, see for example [8 §9], [16].

1. Basic framework

For an associative algebra \( A \) we shall write \( A_{\text{Lie}} \) to denote the associated Lie algebra.
Definition 1 ([2]). An \((n\text{-fold})\) cubically decomposed algebra (over a field \(k\)) is the datum \((A,(I^\pm_i),\tau)\):

- an associative unital (not necessarily commutative) \(k\)-algebra \(A\);
- two-sided ideals \(I^+_i, I^-_i\) such that \(I^+_i + I^-_i = A\) for \(i = 1, \ldots, n\);
- writing \(I^0_i := I^+_i \cap I^-_i\) and \(I_{tr} := I^0_1 \cap \cdots \cap I^0_n\), a \(k\)-linear map

\[
\tau : I_{tr,Lie}/[I_{tr,Lie}, A_{Lie}] \to k.
\]

For any finite-dimensional \(k\)-vector space \(V\) certain infinite matrix algebras act naturally on the \(k\)-vector space of multiple Laurent polynomials \(V[t^{\pm 1}_1, \ldots, t^{\pm 1}_n]\). This yields an example of this structure, see [1,1] below. There is also an analogue in loc. cit. the ideals \(I^+_i, I^-_i\) are called \(X^i, Y^i\). The latter gives the multidimensional generalization of the adèle formulation of Tate’s central extension. See [5, 9, 10, 15] for more background on higher-dimensional adèles and their uses.

1.1. Infinite matrix algebras. Fix a field \(k\). Let \(R\) be an associative \(k\)-algebra, not necessarily unital or commutative. Define an algebra of infinite matrices

\[
E(R) := \{ \phi = (\phi_{ij})_{i,j \in \mathbb{Z}, \phi_{ij} \in R} \mid \exists K_\phi : |i - j| > K_\phi \Rightarrow \phi_{ij} = 0 \}.
\]

Define a product by \((\phi \cdot \phi')_{ij} := \sum_{k \in \mathbb{Z}} \phi_{ij} \phi'_{jk}\); the usual matrix multiplication formula; this sum only has finitely many non-zero terms and one can choose \(K_{\phi \phi'} := K_\phi + K_{\phi'}\). Then \(E(R)\) becomes an associative \(k\)-algebra. If \(R\) is unital, \(E(R)\) is also unital. \(E\) is a functor from associative algebras to associative algebras; for a morphism \(\varphi : R \to S\) there is an induced morphism \(E(\varphi) : E(R) \to E(S)\) by using \(\varphi\) entry-by-entry, i.e. \((E(\varphi)\phi)_{ij} := \varphi(\phi_{ij})\). If \(I \subseteq R\) is an ideal (which is in particular a non-unital associative ring), \(E(I) \subseteq E(R)\) is an ideal. Moreover, for ideals \(I_1, I_2\) one has \(E(I_1 \cap I_2) = E(I_1) \cap E(I_2)\) and \(E(I_1 + I_2) = E(I_1) + E(I_2)\), as a sum of ideals. Next, define

\[
I^+(R) := \{ \phi \in E(R) \mid \exists B_\phi : i < B_\phi \Rightarrow \phi_{ij} = 0 \}
\]
\[
I^-(R) := \{ \phi \in E(R) \mid \exists B_\phi : j > B_\phi \Rightarrow \phi_{ij} = 0 \}
\]

and one checks easily that \(I^+(R), I^-(R)\) are two-sided ideals in \(E(R)\). The following figure attempts to visualize the shape of the matrices in \(E(R), I^+(R)\) and \(I^-(R)\) respectively:

\[
\begin{bmatrix}
\ast & \ast & \ast \\
\ast & \ast & \ast \\
\ast & \ast & \ast
\end{bmatrix}
\begin{bmatrix}
\ast & \ast \\
\ast & \ast \\
\ast & \ast
\end{bmatrix}
\begin{bmatrix}
\ast & \ast \\
\ast & \ast \\
\ast & \ast
\end{bmatrix}
\]

Define \(I^0(R) := I^+(R) \cap I^-(R)\) and one checks that

\[
I^0(R) := \{ \phi \in E(R) \mid \phi_{ij} = 0 \text{ for all but finitely many } (i,j) \}.
\]

There is a trace morphism

\[
\text{tr} : I^0(R) \to R; \quad \text{tr} \phi := \sum_{i,j \in \mathbb{Z}} \phi_{ij},
\]

\[
(1.2)
\]
the sum is obviously finite. One easily verifies that \( \text{tr}[\phi, \phi'] = \sum_{i,j \in \mathbb{Z}} [\phi_{ij}, \phi'_{ij}] \) and thus \( \text{tr}[I^0(R), E(R)] \subseteq [R, R] \). More generally, if \( R' \subseteq R \) is a subalgebra,

\[
\text{tr}[I^0(R'), E(R)] \subseteq [R', R].
\]

We note that this trace does not necessarily vanish on commutators. Moreover, every \( \phi \in E(R) \) can be written as \( \phi = \phi^+ + \phi^- \) with \( \phi^+_i := \delta_{i,0} \phi_{ij} \) (for this \( R \) need not be unital, use \( \phi_{ij} \) for \( i \geq 0 \) and 0 otherwise) and \( \phi^- = \phi - \phi^+ \). One checks that \( \phi^\pm \in I^\pm(R) \). It follows that \( I^+(R) + I^-(R) = E(R) \).

Finally, let \( M \) be an \( R \)-bimodule (over \( k \), i.e. a left-\((A \otimes_k A^{\text{op}})\)-module; \( R \)-bimodules form an abelian category). Analogously to \( E(R) \), define

\[
(1.3) \quad E(M) := \{ \phi = (\phi_{ij})_{i,j \in \mathbb{Z}} | \exists K_\phi : |i-j| > K_\phi \Rightarrow \phi_{ij} = 0 \}.
\]

Again using the matrix multiplication formula, \( E(M) \) is an \( E(R) \)-bimodule. If \( 0 \to M' \to M \to M'' \to 0 \) is an exact sequence of \( R \)-bimodules, \( 0 \to E(M') \to E(M) \to E(M'') \to 0 \) is an exact sequence of \( E(R) \)-bimodules. Note that for an ideal \( I \subseteq R \) the object \( E(I) \) is well-defined, regardless whether we regard \( I \) as an associative ring as in eq. (1.1) or an \( R \)-bimodule as in eq. (1.3).

Now let \( V \) be a finite-dimensional \( k \)-vector space and \( R_0 \) an arbitrary unital subalgebra of \( \text{End}_k(V) \). Define \( R_i := E(R_{i-1}) \) for \( i = 1, \ldots, n \). Note that via \( k \to R_0, \alpha \mapsto \alpha \cdot 1_{\text{End}_k(V)} \), \( k \) is embedded into the center of \( R_i \). Then \( R_n = (E \circ \cdots \circ E)(R_0) \) is a unital associative \( k \)-algebra. Its elements may be indexed \( \phi = (\phi_{(i, j)})_{i,j \in \mathbb{Z}^n} \in R_0 \). By the properties discussed above,

\[
I_i^\pm := \langle E \cdot \cdots \cdot E \circ I^\pm \circ E \cdot \cdots \cdot E \rangle(R_0) \quad (I^\pm \text{ in the } i\text{-th place}),
\]

is an ideal in \( R_n \) (we use centered subscripts only to emphasize the numbering). Moreover,

\[
I_i^+ + I_i^- = (E \cdot \cdots \cdot E \circ I^+ \circ E \cdot \cdots \cdot E)(R_0) + (E \cdot \cdots \cdot E \circ I^- \circ E \cdot \cdots \cdot E)(R_0)
= (E \cdot \cdots \cdot E \circ E \cdot \cdots \cdot E)(R_0) = R_n.
\]

By composing the traces of eq. (1.2) we arrive at a \( k \)-linear map \( \tau \),

\[
\tau : I_{tr} = I^0_0 \cap \cdots \cap I^0_n = (I^0 \circ \cdots \circ I^0)(R_0)
\]

\[
\xrightarrow{\text{tr}} \cdots \xrightarrow{\text{tr}} I^0(I^0(R_0)) \xrightarrow{\text{tr}} I^0(R_0) \xrightarrow{\text{tr}} R_0 \xrightarrow{\text{tr}} k,
\]

where “\( \text{Tr} \)” (as opposed to “\( \text{tr} \)” denotes the ordinary matrix trace of \( \text{End}_k(V) \) (\( \supseteq R_0 \)). Here we have used that \( V \) is finite-dimensional over \( k \). Using \( \text{tr}[I^0(R'), E(R)] \subseteq [R', R] \) (for subalgebras \( R' \subseteq R \)) inductively, one sees that

\[
\tau[I_{tr}, R_n] = \text{Tr}(\text{tr} \circ \cdots \circ \text{tr} \circ \text{tr})[I^0(I^0(\cdots)), E(E(\cdots))] \subseteq \text{Tr}(\text{tr} \circ \cdots \circ \text{tr})[I^0(\cdots), E(\cdots)] \subseteq \text{Tr}[R_0, E_0] = 0
\]

since the ordinary trace \( \text{Tr} \) vanishes on commutators. Hence, \( \tau \) factors to a morphism \( \tau : I_{tr, \text{Lie}}/[I_{tr, \text{Lie}}, R_{\text{Lie}}] \to k \). Summarizing, for every \( n \geq 1 \), every finite-dimensional \( k \)-vector space \( V \) and every unital subalgebra \( R_0 \subseteq \text{End}_k(V) \), \( (R_n, (I_i^\pm), \tau) \) is a cubically decomposed algebra.

Finally, note that for any associative algebra \( R, E(R) \) is a right-\( R \)-submodule of \( \text{right-} R \)-module endomorphisms \( \text{End}_R(R[t, t^{-1}]) \) of \( R[t, t^{-1}] \). Write elements as \( a = \sum_{i \in \mathbb{Z}} a_i t^i \), also denoted \( a = (a_i) \), with \( a_i \in R \), and let \( \phi = (\phi_{ij}) \) act by
(\phi \cdot a)_i := \sum_k \phi_{ik} a_k$. Moreover, each $a \in R[t, t^{-1}]$ determines a right-$R$-module endomorphism via the multiplication operator $x \mapsto a \cdot x$. We find

$$R[t, t^{-1}] \to E(R) \to \text{End}_R(R[t, t^{-1}]).$$

Multiplication with $t^i$ is represented by a matrix with a diagonal $\ldots, 1, 1, 1, \ldots$, shifted by $i$ off the principal diagonal. Inductively,

$$(1.4) \quad R_0[t_1^{\pm 1}, \ldots, t_n^{\pm 1}] \to R_n \to \text{End}_{R_0}(R_0[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]).$$

See for example [11 §1], [13 Lec. 4] for more information regarding the case $n = 1$ and [7 §3] for a similar procedure when $n = 2$.

2. Modified Chevalley-Eilenberg complexes

Suppose $k$ is a field and $\mathfrak{g}$ a Lie algebra over $k$. We recall that for any $\mathfrak{g}$-module the conventional Chevalley-Eilenberg complex is given by $C(M)_r := M \otimes \bigwedge^r \mathfrak{g}$ along with the differential

$$(2.1) \quad \delta := \delta^{[1]} + \delta^{[2]} : C(M)_r \to C(M)_{r-1} \delta^{[1]}(f_0 \otimes f_1 \wedge \ldots \wedge f_r) := \sum_{i=1}^r (-1)^i [f_0, f_i] \otimes f_1 \wedge \ldots \wedge \widehat{f_i} \wedge \ldots \wedge f_r$$

$$\delta^{[2]}(f_0 \otimes f_1 \wedge \ldots \wedge f_r) := \sum_{1 \leq i < j \leq r} (-1)^{i+j+1} f_0 \otimes [f_i, f_j] \wedge f_1 \wedge \ldots \wedge \widehat{f_i} \wedge \ldots \wedge \widehat{f_j} \wedge \ldots \wedge f_r$$

for $f_0 \in M$ and $f_1, \ldots, f_r \in \mathfrak{g}$. Its homology is (if one wants by definition) Lie homology with coefficients in $M$. There is also a cohomological analogue; we refer the reader to the literature for details, e.g. [14 Ch. 10]. We may view $k$ itself as a $\mathfrak{g}$-module with the trivial structure. There is an obvious isomorphism

$$(2.2) \quad I : C(\mathfrak{g})_r \to C(k)_{r+1} \quad f_0 \otimes f_1 \wedge \ldots \wedge f_r \mapsto (-1)^r 1_k \otimes f_0 \otimes f_1 \wedge \ldots \wedge f_r$$

and one checks easily that this commutes with the respective differentials and thus induces morphisms $H_r(\mathfrak{g}, \mathfrak{g}) \to H_{r+1}(\mathfrak{g}, k)$. The linear dual $\mathfrak{g}^* := \text{Hom}_k(\mathfrak{g}, k)$ is canonically a $\mathfrak{g}$-module via $(f \cdot \varphi)(g) := \varphi([g, f])$ for $\varphi \in \mathfrak{g}^*$ and $f, g \in \mathfrak{g}$. The cohomological analogue of eq. (2.2) is the morphism $I : H^{r+1}(\mathfrak{g}, k) \to H^r(\mathfrak{g}, \mathfrak{g}^*)$ given by

$$(I \phi)(f_1 \wedge \ldots \wedge f_r)(f_0) := (-1)^r \phi(f_0 \otimes f_1 \wedge \ldots \wedge f_r).$$

Remark 1. These maps could be viewed as a Lie-theoretic analogue of map $I$ in Connes’ periodicity sequence, see [14 §2.2]. We may view $H_{r-1}(\mathfrak{g}, \mathfrak{g})$ as a partial “uncyclic” counterpart of Lie homology. The true Hochschild analogue would be Leibniz homology, cf. [14 §10.6]. For the present purposes we have however no use for this analogue.

Let $j \subseteq \mathfrak{g}$ be a Lie ideal. As such, it is a $\mathfrak{g}$-module and we may consider $C(j)_\bullet$. Following [2] we may work with a ‘cyclically symmetrized’ counterpart: We write $j \wedge \bigwedge^{r-1} \mathfrak{g}$ to denote the $\mathfrak{g}$-submodule of $\mathfrak{g} \wedge \bigwedge^{r-1} \mathfrak{g} = \bigwedge^r \mathfrak{g}$ generated by elements $j \wedge f_1 \wedge \ldots \wedge f_{r-1}$ such that $j \in j$ and $f_1, \ldots, f_{r-1} \in \mathfrak{g}$. If $j_i$, $i = 1, 2, \ldots$, are Lie ideals, we denote by $(\bigoplus (j_i)) \wedge \bigwedge^{r-1} \mathfrak{g}$ the module $(\bigoplus (j_i) \wedge \bigwedge^{r-1} \mathfrak{g})$.

Example 1. If $k \langle s, t, u \rangle$ and $k \langle s \rangle$ denote a 3-dimensional abelian Lie algebra along with a 1-dimensional Lie ideal, then $\bigwedge^2 k \langle s, t, u \rangle$ is 3-dimensional with basis $s \wedge t$, $s \wedge u$ and $t \wedge u$. Then $k \langle s \rangle \wedge k \langle s, t, u \rangle$ is 2-dimensional with basis $s \wedge t$, $s \wedge u$. 
The $k$-vector spaces $CE(i)_r := j \wedge i^r \mathfrak{g}$ (for $r \geq 1$) and $CE(i)_0 := k$ define a subcomplex of $C(k)_\bullet$. In particular, the differential is given by

$$
\delta(f_0 \wedge f_1 \wedge \ldots \wedge f_r) := \sum_{0 \leq i < j \leq r} (-1)^{i+j} [f_i, f_j] \wedge f_0 \wedge \ldots \hat{f}_i \ldots \hat{f}_j \ldots \wedge f_r.
$$

It is well-defined since $j$ is a Lie ideal. We get morphisms generalizing $I$, notably $H_r(\mathfrak{g}, i) \to H_{r+1}(CE(i))$ via $j \otimes \bigwedge^r \mathfrak{g} \to j \wedge \bigwedge^r \mathfrak{g}$ and analogously $H^{r+1}(CE(i)) \to H^r(\mathfrak{g}, i^r)$. We have resisted the temptation to re-index $CE(-)_\bullet$, despite the unpleasant ($+1$)-shift in eq. 2.2 in order to remain compatible with standard usage in the following sense:

**Lemma 1** ([2] Lemma 1(a)). $CE(\mathfrak{g})_\bullet$ is a complex of $k$-vector spaces and is quasi-isomorphic to $k \otimes_{\mathcal{U}_\mathfrak{g}} k$. In particular

$$
H_i(\mathfrak{g}, k) = H_i(CE(\mathfrak{g})_\bullet) \text{ and } H^i(\mathfrak{g}, k) = H^i(\text{Hom}_k(CE(\mathfrak{g})_\bullet, k)).
$$

**Proof.** As we have explained above, $CE(\mathfrak{g})_\bullet$ agrees with the standard Chevalley-Eilenberg complex and the latter is well-known to represent $k \otimes_{\mathcal{U}_\mathfrak{g}} k$. □

We easily compute

$$
H_0(\mathfrak{g}, i) \xrightarrow{\cong} H_1(CE(\mathfrak{g})) \cong j/[\mathfrak{g}, i]
$$

$$
H^1(CE(\mathfrak{g})) \xrightarrow{\cong} H^0(\mathfrak{g}, i^* \mathfrak{g}) \cong (j/[\mathfrak{g}, i]^*)^*.
$$

In higher degrees the map $I$ ceases to be an isomorphism.

Nonetheless, this computation hints at the principle of computation which we shall use below. Beilinson uses $CE(-)_\bullet$, in his paper [2], whereas we will only be able to do manageable computations with $C(-)_\bullet$. The map $I$ will serve to deduce facts about $CE(-)_\bullet$ while working with $C(-)_\bullet$.

### 3. Cubically decomposed algebras

Let $(A, (I_i^+), \tau)$ be an $n$-fold cubically decomposed algebra over a field $k$, see Def. 1 i.e. we are given the following datum:

- an associative unital (not necessarily commutative) $k$-algebra $A$;
- two-sided ideals $I_i^+, I_i^-$ such that $I_i^+ + I_i^- = A$ for $i = 1, \ldots, n$;
- writing $I_i^0 := I_i^+ \cap I_i^-$ and $I_{tr} := I_1^0 \cap \cdots \cap I_n^0$, a $k$-linear map
  $$
  \tau : I_{tr, Lie}/[I_{tr, Lie}, A_{Lie}] \to k.
  $$

See [1] to see how this type of structure arises. As a shorthand, define $\mathfrak{g} := A_{Lie}$. For any elements $s_1, \ldots, s_n \in \{+, -, 0\}$ we define the degree $\deg(s_1, \ldots, s_n) := 1 + \#\{i \mid s_i = 0\}$. Next, following [2] we shall construct complexes of $\mathfrak{g}$-modules:

**Definition 2** ([2]). For every $1 \leq p \leq n + 1$ define

$$
\wedge^p \mathfrak{g} := \bigcap_{\deg(s_1, \ldots, s_n) = p} CE(I_s^+, \bullet),
$$

and $\wedge^p \mathfrak{g} := CE(\mathfrak{g})_\bullet$.

Each $CE(I_s^+, \bullet)$ is a complex and all their differentials are defined by the same formula, eq. 2.3, as such the intersection of these complexes has a well-defined differential and is a complex itself. Same for the coproduct. The complex $\wedge^p \mathfrak{g}$ is inspired by a cubical object used by Beilinson [2].
We shall denote the components $N$ an exact complex (by definition!), so by the functoriality and flatness we define a terms of $N$ and the same for $N$, e.g. $I_1^+ \wedge I_1^-$ is a subspace in degree two of the left-hand side, but not of the right-hand side.

Diverging from [2] we shall primarily use the following slightly different auxiliary construction (which we will later relate to the above one):

**Definition 3.** For $1 \leq p \leq n + 1$ let
\[
\otimes T^p_\bullet := \prod_{s_1, \ldots, s_n \in \{\pm, 0\}} C(I_1^{s_1} \cap I_2^{s_2} \cap \cdots \cap I_n^{s_n})_\bullet
\]
and $\otimes T^p_\bullet := C(\mathfrak{g})_\bullet$.

So, instead of the modified Chevalley-Eilenberg complex of [2] we just use the standard complexes for Lie homology with suitable coefficients. Clearly the morphism $I : C(\mathfrak{g})_r \to C(k)_{r+1}$ descends to morphisms
\[
C(\mathfrak{g})_r \supseteq C(I_1^{s_1})_r \to CE(I_1^{s_1})_{r+1} \subseteq C(k)_{r+1}
\]
\[
f_0 \otimes f_1 \wedge \ldots \wedge f_r \mapsto (-1)^{\sum_i (s_i - 0)} f_0 \wedge f_1 \wedge \ldots \wedge f_r
\]
As we take intersections of Lie ideals on the left $C(I_1^{s_1} \cap \ldots)$, as in eq. (3.2) the image lies in the intersection of the individual images, i.e. $CE(I_1^{s_1})_\bullet \cap \ldots$, as in eq. (4.1) As a result, we obtain morphisms
\[
\otimes T^p_\bullet \to \wedge T^p_{\bullet+1} \quad (\text{for all } p)
\]
and since they are a restriction of the map $I$ to subcomplexes, this is a morphism of complexes, and thus induces maps on homology.

4. **The cube complex**

Next, we shall define maps $\cdots \to \otimes T^2_\bullet \to \otimes T^1_\bullet \to \otimes T^0_\bullet \to 0$, so that $(\otimes T^p_\bullet)_\bullet$ becomes an exact superscript-indexed complex of (subscript-indexed complexes); and the same for $\wedge T^p_\bullet$. We begin by discussing $\otimes T^0_\bullet$.

We define a $\mathfrak{g}$-module $N^0 := \mathfrak{g}$ and for $p \geq 1$
\[
(4.1) \quad N^p := \prod_{s_1, \ldots, s_n \in \{+,-,0\}} I_1^{s_1} \cap I_2^{s_2} \cap \cdots \cap I_n^{s_n} \quad (\text{with } \deg(s_1, \ldots, s_n) = p).
\]
We shall denote the components $f = (f_{s_1, \ldots, s_n})$ of elements in $N^p$ with indices in terms of $s_1, \ldots, s_n \in \{+,-,0\}$. Clearly $N^p = 0$ for $p > n + 1$. We shall treat all $N^p$ as $\mathfrak{g}$-modules and observe that
\[
\otimes T^p_\bullet := C(N^p)_\bullet
\]
(by definition!), so by the functoriality and flatness of $C_\bullet$ it suffices to construct an exact complex $N^p_\bullet$ out of the $N^p$ and then $\otimes T^p_\bullet$ will be an exact complex in $p$.

\[\footnote{We just tensor $N^p$ with the vector spaces $\wedge^i \mathfrak{g}$. Being over a field, this preserves exact sequences.}\]
Example 3. For \( n = 1 \) we have
\[
N^2 = I_1^0, \quad N^1 = I_1^+ \oplus I_1^-
\]
and elements would be denoted \( f = (f_0) \in N^2 \) and \( g = (g_+ , g_-) \in N^1 \). For \( n = 2 \) we have
\[
N^3 = I_1^0 \cap I_2^0, \quad N^2 = \left( \prod_{s_1 \in \{ +, - \}} I_1^1 \cap I_2^0 \right) \oplus \left( \prod_{s_2 \in \{ +, - \}} I_1^0 \cap I_2^2 \right)
\]
\[
N^1 = \prod_{s_1 , s_2 \in \{ +, - \}} I_1^1 \cap I_2^2.
\]

We shall use the shorthand \( s_1 \ldots \pm \ldots s_n \) (resp. 0 instead of \( \pm \)) to indicate that \( s_i \in \{ +, - \} \) (resp. \( s_i = 0 \)) sits in the \( i \)-th place. Define \( g \)-module homomorphisms
\[
\begin{align*}
(\partial_i f)_{s_1 \ldots \pm \ldots s_n} &= (-1)^{\# \{ j | j > i \text{ and } s_j = 0 \}} f_{s_1 \ldots 0 \ldots s_n} \\
(\partial f)_{s_1 \ldots 0 \ldots s_n} &= \sum_{i=1}^n (\partial_i f)_{s_1 \ldots s_n}
\end{align*}
\]
(4.2)

One checks easily that \( \partial_i^2 = 0 \) and \( \partial_i \partial_j + \partial_j \partial_i = 0 \) for all \( i, j = 1, \ldots, n \). As a consequence, \( \partial^2 = 0 \). The components are given explicitly by
\[
(\partial f)_{s_1 \ldots s_n} = \sum_{i=1}^n (\partial_i f)_{s_1 \ldots s_n}
\]
(4.3)

Definition 4. Let \( (A, (I_i^\pm), \tau) \) be an \( n \)-fold cubically decomposed algebra over a field \( k \). A system of good idempotents are pairwise commuting elements \( P_i^+ \in A \) for \( i = 1, \ldots, n \) such that for all \( i \):
\begin{enumerate}
\item \( P_i^{+2} = P_i^+ \).
\item \( P_i^+ A \subseteq I_i^+ \).
\item \( P_i^- A \subseteq I_i^- \) \quad (where we define \( P_i^- := 1_A - P_i^+ \)).
\end{enumerate}

We note that the \( P_i^- \) are also pairwise commuting idempotents and \( P_i^+ + P_i^- = 1_A \). Next, for \( s_i \in \{ +, - \} \) define \( k \)-vector space homomorphisms
\[
\begin{align*}
(\varepsilon_i f)_{s_1 \ldots s_n} &= (-1)^{s_i} \sum_{\gamma_i \in \{ \pm \}} ( -1 )^{\gamma_i} f_{s_1 \ldots s_n} \\
(\varepsilon_i f)_{s_1 \ldots 0 \ldots s_n} &= 0,
\end{align*}
\]
where \( (-1)^{\pm} = \pm 1 \). By direct calculation one verifies the identities \( \varepsilon_i^2 = \varepsilon_i \) and \( \varepsilon_i \varepsilon_j = \varepsilon_j \varepsilon_i \) for all \( i, j = 1, \ldots, n \). Finally, define
\[
\begin{align*}
(H_i f)_{s_1 \ldots 0 \ldots s_n} &= (-1)^{\# \{ j | j > i \text{ and } s_j = 0 \}} \sum_{\gamma_i \in \{ \pm \}} ( -1 )^{\gamma_i} f_{s_1 \ldots s_n} \\
(H_i f)_{s_1 \ldots \pm \ldots s_n} &= 0.
\end{align*}
\]
The expression \( P_i^{-\gamma_i} \) means \( P_i^- \) for \( \gamma_i = + \) and \( P_i^+ \) for \( \gamma_i = - \). One checks that
\[
\begin{align*}
H_i^2 &= 0 \quad \text{and} \quad H_i H_j + H_j H_i = 0 \\
\partial_i \varepsilon_j &= \varepsilon_j \partial_i \quad \text{and} \quad H_i \varepsilon_j = \varepsilon_j H_i
\end{align*}
\]
for all \( i, j = 1, \ldots, n \). Moreover, \( \partial_i H_j + H_j \partial_i = 0 \) whenever \( i \neq j \). In the special case \( i = j \) one finds instead that
\[
\partial_i H_i + H_i \partial_i = 1 - \varepsilon_i.
\]
Define $H := H_1 + \epsilon_1 H_2 + \cdots + \epsilon_1 \epsilon_2 \cdots \epsilon_{n-1} H_n$. Using the identities established above, one finds very easily

\begin{equation}
H^2 = 0 \quad \text{and} \quad \partial H + H \partial = 1 - \epsilon_1 \cdots \epsilon_n.
\end{equation}

The fact $H^2 = 0$ was observed by the anonymous referee; it explains a certain cancellation in the proof of Prop. which had been rather mysterious in an earlier version of this text.

**Lemma 2.** An explicit formula for $H$ is given by

\begin{equation}
(Hf)_{s_1 \ldots s_n} = (-1)^{\deg(s_1 \ldots s_n)} (-1)^{s_1 + \cdots + s_n} P_{s_1}^1 \cdots P_{s_n}^n \sum_{\gamma_1 \cdots \gamma_{b+1} \in \{\pm\}} (-1)^{\gamma_1 + \cdots + \gamma_{b+1}} P_{b+1}^{\gamma_b+1} f_{\gamma_1 \cdots \gamma_{b+1} s_{b+2} \cdots s_n},
\end{equation}

where $b$ denotes the largest index such that $s_1, \ldots, s_b \in \{\pm\}$ or $b = 0$ if none (and so $s_{b+1} = 0$ if $b < n$; $b + 1$ is the index of the “leftmost zero”).

**Proof.** One shows that

\begin{equation}
(\epsilon_1 \cdots \epsilon_i f)_{s_1 \ldots s_n} = (-1)^{s_1 + \cdots + s_i} P_{s_1}^i \cdots P_{s_n}^n \sum_{\gamma_1 \cdots \gamma_i \in \{\pm\}} (-1)^{\gamma_1 + \cdots + \gamma_i} f_{\gamma_1 \cdots \gamma_i s_{i+1} \cdots s_n}
\end{equation}

(for $s_1, \ldots, s_i \in \{\pm\}$) by evaluating $(\epsilon_j \cdots \epsilon_i f)$ inductively along $j = i, i-1, \ldots, 1$. Plug in $H_{i+1} f$ for $f$ to obtain

\begin{equation}
(\epsilon_1 \cdots \epsilon_i H_{i+1} f)_{s_1 \ldots s_n} = (-1)^{\# \{j \mid j > i+1 \text{ and } s_j = 0\}} (-1)^{s_1 + \cdots + s_i} P_{s_1}^i \cdots P_{s_n}^n \sum_{\gamma_1 \cdots \gamma_{i+1} \in \{\pm\}} (-1)^{\gamma_1 + \cdots + \gamma_{i+1}} P_{i+1}^{\gamma_{i+1}} f_{\gamma_1 \cdots \gamma_{i+1} s_{i+2} \cdots s_n}
\end{equation}

for $s_1, \ldots, s_i \in \{\pm\}$ and $s_{i+1} = 0$. Otherwise, i.e. for $0 \in \{s_1, \ldots, s_i\}$ or $s_{i+1} \in \{\pm\}$, the respective component is zero. Thus,

\begin{equation}
H_{s_1 \cdots s_n} = \sum_{i=1}^n (\epsilon_1 \cdots \epsilon_i H_{i+1} f)_{s_1 \cdots s_n}.
\end{equation}

The summands with $i > b$ vanish since for them $0 \in \{s_1, \ldots, s_i\}$. The summands with $i < b$ vanish since for them $s_{i+1} \in \{\pm\}$. Thus,

\begin{equation}
H_{s_1 \cdots s_n} = (\epsilon_1 \cdots \epsilon_b H_{b+1} f)_{s_1 \cdots s_n}
\end{equation}

and we use the above explicit formula. Note that $\# \{j \mid j > b+1 \text{ and } s_j = 0\}$ is just one below the total number of slots with value 0 since $s_1, \ldots, s_b \in \{\pm\}$ and $s_{b+1} = 0$. Thus, $(-1)^{\# \{j \mid j > i+1 \text{ and } s_j = 0\}} = (-1)^{\deg(s_1 \ldots s_n)}$. \hfill \Box

The above maps are defined for $N^0$ in degrees $\geq 1$. We extend them to degree zero by

\begin{equation}
\hat{\partial} : N^1 \to N^0 \quad \text{and} \quad \hat{H} : N^0 \to N^1
\end{equation}

\begin{equation}
\hat{\partial} f := \sum_{s_1 \cdots s_n \in \{+, -\}} (-1)^{s_1 + \cdots + s_n} f_{s_1 \cdots s_n}
\end{equation}

\begin{equation}
(\hat{H} f)_{s_1 \cdots s_n} := (-1)^{s_1 + \cdots + s_n} P_{s_1}^1 \cdots P_{s_n}^n f.
\end{equation}

Along with these, we obtain the following crucial fact:
Lemma 3. Equipped with these morphisms

\[(4.8) \quad N^\bullet = [N^{n+1} \xrightarrow{\partial} N^n \xrightarrow{\partial} \cdots \xrightarrow{\partial} N^1 \xrightarrow{\partial} N^0]_{n+1,0} \]

is a complex of \(g\)-modules with differentials \(\partial_\bullet\) (resp. \(\hat{\partial}\)) and contracting homotopies \(H_\bullet\) (resp \(\hat{H}\)) in the category of \(k\)-vector spaces.

\[\text{Proof.}\] The identities \(\partial^2 = 0\) and \(\hat{\partial} \circ \partial = 0 : N^2 \to N^0\) are easy to check. Next, we confirm the contracting homotopy. We find \(\partial H + H \partial = 1 - \varepsilon_1 \cdots \varepsilon_n\) by a telescope cancellation. For \(f \in N^i\) with \(i \geq 2\) for each component \(f_{s_1 \cdots s_n}\) there must be at least one \(i\) with \(s_i = 0\) and thus \(\varepsilon_1 \cdots \varepsilon_n |_{N^i} = 0\) for \(i \geq 2\). It remains to treat \(i = 0, 1\). For \(i = 1\) we compute

\[\hat{H} \hat{\partial} f = (1)^{s_1+\cdots+s_n} P_{s_1} \cdots P_{s_n} \sum_{s_1 \cdots s_n \in \{+,-\}} (1)^{s_1+\cdots+s_n} f_{s_1 \cdots s_n} = \varepsilon_1 \cdots \varepsilon_n f\]

(as in eq. 4.9). Thus, \(\partial H + \hat{H} \hat{\partial} = 1\) on \(N^1\). Finally, for \(i = 0\) we compute \(\hat{\partial} H f = f\). \(\square\)

Corollary 1. \(0 \to \otimes T^{n+1} \to \otimes T^n \to \cdots \to \otimes T^0 \to 0\) with differential (and a contracting homotopy) induced by \(\partial \otimes \text{id}_A \otimes g\) (and \(\hat{\partial} \otimes \text{id}_A \otimes g\)) is an exact complex of (complexes of \(k\)-vector spaces).

For the corollary just use that tensoring with \(\wedge^r g\) is exact.

5. THE CUBE COMPLEX II

Next, it would be nice to give a discussion of the \(\wedge T^\bullet\) parallel to the one for \(\otimes T^\bullet\) in the previous section. We can only do this to a limited extent, however.

Lemma 4. The definition

\[(5.1) \quad (\partial f)_{s_1 \cdots s_n} = \sum_{\{i | s_i = +, -\}} (-1)^{\#(j | j > i \text{ and } s_j = 0)} f_{s_1 \cdots 0 \cdots s_n} \]

turns \(\wedge T^\bullet\) into a complex of (complexes of \(k\)-vector spaces) with respect to the superscript index. The morphisms \(\otimes T^\bullet \xrightarrow{I} \wedge T^\bullet\) yield a morphism of complexes.

\[\text{Proof.}\] Easy. Just check that the map \(\partial\) is well-defined and satisfies \(\partial^2 = 0\); in fact exactly the same computation as in eqs. 4.2 applies. For the second claim, we just need to show that the map \(I\) commutes with the differential of either complex, but this is clear since the differentials are given by the same formula, compare eq. 4.3 with eq. 5.4 \(\square\)

The complex \(\wedge T^\bullet\) is the central object in Beilinson’s construction \([2]\). We will use its analogue \(\otimes T^\bullet\) as an auxiliary computational device. Firstly, let us explain Beilinson’s construction. We need the following entirely homological tool:

Lemma 5. Suppose we are given an exact sequence

\[S^\bullet = [S^{n+1} \to S^n \to \cdots \to S^0]_{n+1,0}\]

with entries in \(\text{Ch}^+ \text{Mod}_k\), i.e. each \(S^i = S_i^\bullet\) is a bounded below complex of \(k\)-vector space\(]3)\.

\[\text{One may alternatively view this as a bicomplex supported horizontally in degrees } [0, n + 1], \text{ bounded from below, and whose rows are exact.}\]
(1) There is a second quadrant homological spectral sequence \((E^r_{p,q}, d_r)\) converging to zero such that
\[ E^1_{p,q} = H_q(S^p, \bullet). \quad (d_r : E^r_{p,q} \to E^r_{p-r, q+r-1}) \]

(2) There is a first quadrant cohomological spectral sequence \((E^r_p,q, d^r)\) converging to zero such that
\[ E^1_{p,q} = H^q(Hom_k(S^p, \bullet)). \quad (d^r : E^r_{p,q} \to E^{r+p, q-r+1}) \]

(3) The following differentials are isomorphisms:
\[ d_{n+1} : E^{n+1}_{n+1,1} \to E^{n+1}_{0,n+1} \quad \text{and} \quad d^{n+1} : E^{0,n+1}_{n+1} \to E^{n+1,1}_{n+1}. \]

(4) Suppose \(H_p : S^p \to S^{p+1}\) is a contracting homotopy for \(S^\bullet\). Then
\[ (d_{n+1})^{-1} = H_n \delta_1 H_{n-1} \cdots \delta_{n-1} H_1 \delta_n H_0 = H_n \prod_{i=1, \ldots, n} (\delta_i H_{n-i}) \]
(where the last product depends on the ordering and refers to composition), and
\[ (d^{n+1})^{-1} = H_0^* \delta^*_n H_1^* \cdots \delta^*_1 H^*_n = H_0^* \prod_{i=n, \ldots, 1} (\delta^*_i H^*_{n+1-i}), \]
where we write \(f^* = \text{Hom}_k(f, k)\) as a shorthand.

The construction is functorial in \(S^\bullet\), i.e. if \(S^\bullet \to S'^\bullet\) is a morphism of complexes as in our assumptions, then there are induced morphisms between their spectral sequences.

**Proof.** Parts (1)-(3) are Lemma 1(a)]. More precisely, for (1) use the bicomplex spectral sequence for
\[ E^0_{p,q} = S^p_q \quad \text{and} \quad E^0_{p,q} = \text{Hom}_k(S^p_q, k). \]
If we take differentials ‘\(-\to\)’ for forming the \(E^0\)-page, the \(E^1\)-page vanishes since \(S^\bullet\) is exact (as a complex of complexes) and so the individual sequences of \(k\)-vector spaces \(S^p_q\) for constant \(i\) are exact, so \(E^\infty = E^1 = 0\). Then use the bicomplex spectral sequences with differential ‘\(-\to\)’ on the \(E^0\)-page for our claim. It also converges to zero then: (2) is analogous. (3) The bicomplex is horizontally supported in \([0, n+1]\).

(4) Diagram chase. \(\square\)

We combine Lemma 4 with Lemma 5. Apply the latter to \(S^p_q := T^p_q\); we denote the resulting spectral sequence by \(^\wedge E^\bullet_{p,q}\). The fact that the \((b)\)complex of Lemma 5 is supported horizontally in \([n+1, 0]\) (homologically, i.e. for \(^\wedge E^r_{p,q}\)) and \([0, n+1]\) respectively (cohomologically, i.e. for \(^\wedge E^r_{p,q}\)) implies that we have edge morphisms
\[ \rho_1 : ^\wedge E^{n+1}_{n+1,1} \to ^\wedge E^{1}_{n+1,1} \quad \text{and} \quad \rho_2 : ^\wedge E^{1}_{0,n+1} \to ^\wedge E^{n+1}_{0,n+1} \]
\[ \varphi_1 : ^\wedge E^{0,n+1}_{n+1} \to ^\wedge E^{1,0+n+1}_{n+1} \quad \text{and} \quad \varphi_2 : ^\wedge E^{n+1,1}_{n+1} \to ^\wedge E^{0,n+1}_{n+1}. \]

Next, we identify the involved objects: Using Lemma 4 we compute
\[ ^\wedge E^{1}_{0,n+1} = H_{n+1}(C E(I^*_t) \bullet) \cong H_{n+1}(g, k) \]
\[ ^\wedge E^{1}_{n+1,1} = H_I(\cap_{i=1, \ldots, n} \cap_{s, t} C E(I^*_t) \bullet) = I_I/[I_{II}, g] \]
\[ ^\wedge E^{n+1,1}_{1} = \text{Hom}_k(I_I)/[I_{II}, g, k] \quad \text{and} \quad ^\wedge E^{0,n+1}_{1} = H^{n+1}(g, k). \]
Definition 5 (\cite{2}). Let \((A,(I^\pm),\tau)\) be an \(n\)-fold cubically decomposed algebra over a field \(k\) and \(g := A_{\text{Lie}}\) its Lie algebra. Define
\[
\operatorname{res}_*: H_{n+1}(g,k) \to k \quad \text{and} \quad \operatorname{res} := \tau \circ \rho_1 \circ (d_{n+1})^{-1} \circ \rho_2
\]
and
\[
\operatorname{res}^*: k \to H^{n+1}(g,k) \quad \text{where for} \quad \operatorname{res}^*(1) := (\varphi_1 \circ (d^{n+1})^{-1} \circ \varphi_2) \tau,
\]
where for \(\operatorname{res}^*\) we read \(\tau\) as an element of \(E_1^{n+1,1}\). We will call \(\phi := \operatorname{res}^*(1)\) the Tate extension class.

In the case \(n = 1\) it would also be justified to name this cohomology class after Kac-Petersen \cite{12}; it also appears in the works of the Japanese school, e.g. \cite{11}.

Remark 2. It follows from the construction of \(\operatorname{res}_*\), \(\operatorname{res}^*\) that
\[
(5.2) \quad \operatorname{res}^*(\alpha)(X_0 \wedge \ldots \wedge X_n) = \alpha \operatorname{res}_* X_0 \wedge \ldots \wedge X_n.
\]

Now we would like to compute these maps explicitly. Clearly, the most elusive map in the construction is the differential \(d_{n+1}\) (resp. \(d^{n+1}\)). We can render it explicit using Lemma 5.4 as soon as we have an explicit contracting homotopy for \(\otimes T^*\) by Lemma 3 and its corollary. Luckily for us, these complexes are closely connected. We may apply Lemma 5 also to \(S^\bullet_g := \otimes T^p_{q-1}\); this time denote the resulting spectral sequence by \(\otimes E^\bullet_\bullet\). We easily compute
\[
\otimes E^{0,1}_{0,n+1} = H_{n+1}(\otimes T^p_{q-1}, g) = H_n(C(I^\bullet)) \cong H_n(g,g),
\]
\[
\otimes E^{1,1}_{1,n+1} = H_0(C(I^\bullet)) \cong H_0(I^\bullet/[I^\bullet,g]) = I_{\text{tr}}/[I_{\text{tr}},g],
\]
\[
\otimes E^{0,n+1}_{1,1} = H^n(g,g) \quad \text{and} \quad \otimes E^{1,n+1}_{1,1} = \text{Hom}_k(I_{\text{tr}}/[I_{\text{tr}},g],k).
\]

We note that some groups even agree with their \(^\wedge T^p_q\)-counterpart; as we had already observed in eq. 2.4

Definition 6. Write \(\otimes \operatorname{res}_* : H_n(g,g) \to k\) and \(\otimes \operatorname{res}^*(1) \in H^n(g,g)\) for the counterparts of \(\operatorname{res}_*, \operatorname{res}^*\) in Def. 5 using \(\otimes E\) instead of \(^\wedge E\).

Lemma 6 (Compatibility). The morphism of bicomplexes \(\otimes T^\bullet \to \wedge T^\bullet\) induces a commutative diagram

\[
\begin{array}{ccc}
H_n(g,g) & \overset{\otimes E^{n+1}_{0,n+1}}{\longrightarrow} & H_0(g,g) \\
\downarrow & & \downarrow \\
H_{n+1}(g,k) & \overset{\wedge E^{n+1}_{0,n+1}}{\longrightarrow} & H_1(g,k).
\end{array}
\]

Proof. We had already observed in Lemma 3 that the morphisms \(I\) induce a morphism of bicomplexes. The spectral sequences \(\otimes E^\bullet_\bullet\) and \(\wedge E^\bullet_\bullet\) both arise from...
Lemma 5, so by the functoriality of the construction we get an induced morphism of spectral sequences. In particular, all squares

\[
\begin{array}{ccc}
\otimes E^r_{p,q} & \xrightarrow{d_r} & \otimes E^r_{p-r,q+r-1} \\
\downarrow & & \downarrow \\
\wedge E^r_{p,q} & \xrightarrow{d_r} & \wedge E^r_{p-r,q+r-1}
\end{array}
\]

commute, giving the middle square in our claim. The same applies to the edge maps, giving the outer squares. □

Absolutely analogously we obtain a cohomological counterpart,

\[
\begin{array}{ccc}
H^1(g, k) & \xrightarrow{\cong} & H^{n+1}(g, k) \\
\downarrow & & \downarrow \\
H^0(g, g^*) & \xrightarrow{\cong} & H^n(g, g^*),
\end{array}
\]

where we have a contracting homotopy for the lower row. We leave the details of this formulation to the reader.

6. Concrete Formalism

Let \((A, (I_i^\pm), \tau)\) be an \(n\)-fold cubically decomposed algebra over a field \(k\). In \(\S\) 5 we have constructed a canonical morphism

\[
\text{res}_* : H_{n+1}(g, k) \rightarrow k
\]

where \(g := A_{\text{Lie}}\) is the Lie algebra associated to \(A\). By Lemma 6 its values on the image of \(H_n(g, g) \rightarrow H_{n+1}(g, k)\) can be computed via \(\otimes \text{res}_*\). In this section we will obtain an explicit formula for the latter morphism.

Given the definition of \(\otimes \text{res}_*\), Lemma 5 tells us that it can be given explicitly in terms of differentials of the ordinary Chevalley-Eilenberg complexes \(C(-)_*\) (cf. \(\S\) 2) and contracting homotopies of the cube complex \(N^*\) (cf. Lemma 3 and its corollary), namely

\[
(6.1) \quad \otimes \text{res}_* = \tau \circ \rho_1 \circ (\otimes d_{n+1})^{-1} \circ \rho_2 = \tau \circ \rho_1 H \prod_{i=1, \ldots, n}(\delta, H)\rho_2
\]

via the spectral sequence \(\otimes E^*_{\cdot, \cdot}\). The contracting homotopy \(H\) depends on the choice of a good system of idempotents, see Def. 4 Different choices will yield formulas that may look different, but as \(\otimes \text{res}_*\) (just like \(\text{res}_*\) itself) was defined entirely independently of the choice of any idempotents, all such formulas actually must agree.

Suppose a representative \(\theta := f_0 \otimes f_1 \ldots \wedge f_n\) with \(f_0, \ldots, f_n \in N^0\) is given (note that \(N^0\) equals \(g\) as a left-\(Ug\)-module by definition, so it is valid to treat all \(f_i\) on equal footing). We shall compute \(\otimes \text{res}_* \theta\) in several steps, starting with
\( \theta_{0,n} := \rho_2 \theta \), then following

\[
\begin{array}{c|c|c|c|c}
\theta_{n+1,0} & H & \theta_{n,0} & \theta_{n,1} & H \theta_{n-1,1} \\
\downarrow & & \uparrow & & \downarrow \\
q & & p & & + \\
\end{array}
\]

\[\begin{array}{c|c|c|c|c}
0 & 0 & \cdots & 0 & 0 \\
\theta_{1,n} & H & \theta_{0,n} & n & q \\
\vdots & & \vdots & & \vdots \\
n & & n & & n-1 \\
\end{array}\]

as prescribed by eq. (6.1). This graphical arrangement elucidates the position of the term of each step in the computation in the spectral sequence from which eq. (6.1) originates, see Lemma 5. However, for us each \( \theta_{*,*} \) will be an \( E_0 \)-page representative of the respective \( E_1 \)-page term. Finally \( \otimes \) res. \( \theta = \tau \rho_1 \theta_{n+1,0} \). We note that \( \rho_1, \rho_2 \) are just edge maps, i.e. an inclusion of a subobject and a quotient surjection. Hence, as we work with explicit representatives anyway, the operation of these maps is essentially invisible (e.g. in the quotient case it just means that our representative generates a larger equivalence class).

We will need a convenient notation for elements of this complex.

**(Notation A)** We will write \( \theta_{p,q}^{w_1 \ldots w_p} \) for the summands in any expression of the shape

\[
\theta_{p,q}^{w_1 \ldots w_p} = \sum_{w_1 \ldots w_p} \sum_{s_1 \ldots s_n} \theta_{p,q-s_1 \ldots s_n}^{w_1 \ldots w_p} \otimes f_1 \wedge \ldots \wedge f_{w_1} \wedge \ldots \wedge f_{w_p} \wedge \ldots \wedge f_n,
\]

where

- \( p, q - p \) denotes the location of the element in the bicomplex as in fig. 6.2
- \( s_1, \ldots, s_n \in \{0, +, -\} \) denotes the component (direct summand) of \( N^p \) as in eq. (4.1), \( f_1, \ldots, f_n \in g \),
- the additional superscripts \( w_1, \ldots, w_p \in \{1, \ldots, n\} \) are used to indicate the omission of wedge factors.

Note that the values \( \theta_{p,q-s_1 \ldots s_n}^{w_1 \ldots w_p} \) are not necessarily uniquely determined since the individual wedge tails need not be linearly independent.

**(Notation B)** We also need a shorthand for the summands in any expression of the shape

\[
\theta_{p,q-p-1} = \sum_{w_1 \ldots w_p} \sum_{s_1 \ldots s_n} \theta_{p,q-s_1 \ldots s_n}^{w_1 \ldots w_p} \otimes f_1 \wedge \ldots \wedge f_{w_1} \wedge \ldots \wedge f_{w_p} \wedge \ldots \wedge f_n.
\]

Again \( s_1, \ldots, s_n \) denotes the component in \( N^p, w_1, \ldots, w_p \) omitted wedge factors. Moreover, \( w_a \) and \( w_b \) denote two additional omitted wedge factors and simultaneously indicate that \( [f_{w_a}, f_{w_b}] \) appears as an additional wedge factor. As for the previous notation, the elements \( \theta_{p,q-s_1 \ldots s_n}^{w_1 \ldots w_p} \) are not uniquely determined. We will explain how these expressions arise soon.
Combinatorial Preparation: We define for arbitrary \( 1 \leq p \leq n \) and \( w_1, \ldots, w_p \in \{1, \ldots, n\} \) the ‘sign function’ (a generalization of the signum of a permutation)

\[
\rho(w_1, \ldots, w_p) := (-1)^{\sum_{k=1}^n \sum_{j<k} \delta_{w_j < w_k}}.
\]

By abuse of language we do not carry the value \( p \) in the notation for \( \rho \) as it will always be clear from the number of arguments which variant is used. It is easy to see that \( \rho(w_1) = +1 \) and \( \rho(w_1, w_2) = (-1)^{\delta_{w_1 < w_2}} \). For \( p = n \) we have

\[
\rho(w_1, \ldots, w_n) = \text{sgn}
\begin{pmatrix}
1 & \cdots & n \\
w_1 & \cdots & w_n
\end{pmatrix}.
\]

We shall need the inductive formula (which is easy to check by induction)

\[
\rho(w_1, \ldots, w_{p+1}) = (-1)^{\# \{ w_i | 1 \leq i \leq p \text{ s.t. } w_i < w_{p+1} \}} \rho(w_1, \ldots, w_p).
\]

\section{Proposition 1.}

Suppose \( \theta := f_0 \otimes f_1 \wedge \ldots \wedge f_n \) with \( f_i \in N_0 = g \). Moreover, suppose \( P_1^+, \ldots, P_n^+ \) is a good system of idempotents as in Def. \ref{def:1} Then for every \( p \geq 0 \) the element \( \theta_{p+1,q} \) is of the shape as in eq. \ref{eq:6.3} and for \( \gamma_1 \ldots \gamma_n \in \{+, -\} \) we have

\[
\theta_{p+1,q}^{\gamma_1 \ldots \gamma_n-p, 0 \ldots 0} = (-1)^{\sum_{i=1}^{p-1}(u+1)} (-1)^{w_1 + \ldots + w_p} \rho(w_1, \ldots, w_p)
\]

\[
(-1)^{\gamma_1 + \ldots + \gamma_n-p} P_1^{\gamma_1} \ldots P_n^{\gamma_n-p} P_1^{\gamma_1} \ldots P_n^{\gamma_n-p}
\]

\[
\sum_{\gamma_n-p+1 \ldots \gamma_n \in \{\pm\}} (-1)^{\gamma_n-p+1 + \ldots + \gamma_n} P_n^{-\gamma_n-p+1} \text{ad}(f_{w_p}) P_n^{-\gamma_n-p+1}
\]

\[
\cdots \left(P_n^{(-\gamma_n-p)} \text{ad}(f_{w_p}) P_n^{\gamma_n-p} \right) f_0.
\]

Here \( \rho(w_1, \ldots, w_p) \) is the sign function defined in eq. \ref{eq:6.4}. For \( p = 0 \) the expression \( \rho(w_1, \ldots, w_p) \) and the whole sum (\( \Sigma_{\{\pm\}}(\cdots) \)) in (\( \Sigma_{\{\pm\}}(\cdots) f_0 \)) should be read as +1 (giving the right-hand side of eq. \ref{eq:6.3} below).

- Note that no terms of the shape as in eq. \ref{eq:6.4} appear. This is not entirely obvious in view of the definition of \( \delta^2 \), see eq. \ref{eq:2.1}
- The formula does not compute \( \theta_{p+1,q}^{s_1 \ldots s_n} \) for arbitrary \( s_1 \ldots s_n \) of degree \( p + 1 \). This is due to the fact that we only have further use for the ones treated.
- For \( p \leq 1 \) read \( \sum_{u=1}^{p-1}(u+1) \) as zero.

\section{Proof.}

We prove this by induction. For \( p = 0 \) the claim reads

\[
\theta_{1,q}^{\gamma_1 \ldots \gamma_n} = (-1)^{\gamma_1 + \ldots + \gamma_n} P_1^{\gamma_1} \ldots P_1^{\gamma_n} f_0
\]

and in view of eq. \ref{eq:4.7} this proves the claim in this case. Now we proceed by induction. Assume the case \( p \) is settled, i.e. in the notation of eq. \ref{eq:6.3} \( \theta_{p+1,q}^{\gamma_1 \ldots \gamma_n, 0 \ldots 0} \) is exactly as in our claim. Next, we need to apply the differential \( \delta_q = \delta_q^{[1]} + \delta_q^{[2]} \) of the Chevalley-Eilenberg resolution, see eq. \ref{eq:2.1} The contribution of \( \delta_q^{[1]} \) will be relevant, but for \( \delta_q^{[2]} \) we shall see that (after applying the next contracting homotopy) the contribution vanishes. We treat each \( \delta^{[i]}, i = 1, 2 \) separately:

\section{(1) Consider \( \delta_q^{[1]} \) in eq. \ref{eq:2.1}}

The sum \( \Sigma_i \) loc. cit. maps components indexed
by $w_1, \ldots, w_p$ to components of $\delta[^1]\theta_{p,q}$, indexed by $w_1, \ldots, w_p$ and an additional $w_{p+1} \in \{1, \ldots, n\} \setminus \{w_1, \ldots, w_p\}$—they correspond to the summands of $\delta[^1]\theta_{p,q}$ and to the additional omitted wedge factor respectively. Moreover, the formula imposes signs $(-1)^{i+1}$, but here $i$ depends on the numbering of the wedges $(\ldots \wedge \ldots \wedge \ldots)$. In the notation of eq. 6.1, the subscript $j$ of $f_j$ does not necessarily indicate the $f_j$ sits in the $j$-th wedge, due to the possible omission of wedge factors $f_{w_1}, \ldots, f_{w_p}$ on the left-hand side of it. To compensate for that in the following computation the term $(-1)^{\#\{w_i \mid 1 \leq i \leq p \text{ s.t. } w_i < w_{p+1}\}}$ appears, sign-counting the omission on the left of the new-to-be-omitted $w_{p+1} \in$ the component of $\delta[^1]\theta_{p+1,q}$. As $p$ remains constant, the indexing $\gamma_1 \ldots \gamma_n - p \ldots 0 \ldots 0$ remains unaffected. We get for $(\delta[^1]\theta_{p+1,q})^w_{p+1,q-1|\gamma_1 \ldots \gamma_n - p \ldots 0 \ldots 0}$ the expression

$$
= (-1)^{\sum_{i=1}^{p+1}(u+1)}(-1)^{\#\{w_i \mid 1 \leq i \leq p \text{ s.t. } w_i < w_{p+1}\}} \text{ad}(f_{w_{p+1}})

= (-1)^{u_1 + \cdots + w_p} \rho(w_1, \ldots, w_p)

= (-1)^{\gamma_1^{\uparrow} + \cdots + \gamma_{n-p}^{\uparrow} P_1^{\gamma_1} \cdots P_{n-p}^{\gamma_{n-p}}}

\sum_{\gamma_{n-p+1}^{\uparrow} \cdots \gamma_n^{\uparrow} \in \{\pm\}} (-1)^{\gamma_{n-p+1}^{\uparrow} + \cdots + \gamma_n^{\uparrow}}

(P_{n-p+1}^{-\gamma_{n-p+1}^{\uparrow}} \text{ad}(f_{w_p}) P_{n-p+1}^{-\gamma_{n-p+1}^{\uparrow}}) \cdots (P_n^{-\gamma_n^{\uparrow}} \text{ad}(f_{w_1}) P_n^{-\gamma_n^{\uparrow}}) f_0.
$$

Next, we need to apply the contracting homotopy $H : N^{p+1} \to N^{p+2}$. Note that we have $p + 1 \geq 1$, so eq. 4.1 applies. Note that for indices $\gamma_1^{\uparrow} \ldots \gamma_{n-p-1}^{\uparrow} 0 \ldots 0$ with $\gamma_1^{\uparrow} \ldots \gamma_{n-p-1}^{\uparrow} \in \{\pm\}$ (i.e. indices of degree $p + 2$, cf. eq. 4.1) the index $\gamma_1^{\uparrow} \ldots \gamma_{n-p-1}^{\uparrow} 0 \ldots 0$ has degree $p + 1$. The latter have been computed above. We obtain for

$$(H \delta[^1]\theta_{p+1,q})^w_{p+2,q-1|\gamma_1^{\uparrow} \ldots \gamma_{n-p-1}^{\uparrow} 0 \ldots 0}$$

the expression

$$
= (-1)^{p}( -1)^{\gamma_1^{\uparrow} + \cdots + \gamma_{n-p-1}^{\uparrow} P_1^{\gamma_1} \cdots P_{n-p-1}^{\gamma_{n-p-1}}}

\sum_{\gamma_{n-p}^{\uparrow} \gamma_{n-p+1}^{\uparrow} \cdots \gamma_n^{\uparrow} \in \{\pm\}} (-1)^{\gamma_1^{\uparrow} + \cdots + \gamma_{n-p-1}^{\uparrow} P_{n-p-1}^{\gamma_{n-p-1}} + 1}

(\delta \theta_{p+1,q})^w_{p+1,q-1|\gamma_1^{\uparrow} \ldots \gamma_{n-p-1}^{\uparrow} 0 \ldots 0}.
$$

In principle the first factor is $(-1)^{\text{deg}(\ldots)} = (-1)^{p+2}$, but switching to $p$ preserves the correct sign. Next, we expand this using our previous computation and obtain
(by noting that many signs are squares and thus +1)

\[ = (-1)^{\sum_{i=1}^{n-1}(u_i+1)} (-1)^{p+1} \]

\[ (\gamma_1^i \cdots \gamma_{n-1}^i) (-1)^{\# \{ w_i | 1 \leq i \leq p \text{ s.t. } w_i < w_{p+1} \}} \]

\[ (\gamma_1^i \cdots \gamma_{n-1}^i) \rho(w_1, \ldots, w_p) P_{n-p-1}^1 \cdots P_{n-p-1}^1 \sum_{\gamma_n \in \{ \pm \}} (-1)^{\gamma_n} \]

\[ \left( \sum_{\gamma_1, \ldots, \gamma_{n-1} \in \{ \pm \}} P_{n-p-1}^1 \cdots P_{n-p-1}^1 \right) P_{n-p}^{-\gamma_n} \text{ ad}(f_{w_{p+1}}) P_{n-p}^{-\gamma_n} \]

\[ \sum_{\gamma_{n-p+1} \cdots \gamma_n \in \{ \pm \}} (-1)^{\gamma_{n-p+1} + \cdots + \gamma_n} \]

\[ \left( P_{n-p+1}^{-\gamma_{n-p+1}} \right) \text{ ad}(f_{w_p}) P_{n-p+1}^{-\gamma_n} \cdots \left( P_n^{-\gamma_n} \right) \text{ ad}(f_{w_1}) P_n^{-\gamma_n} \right) f_0. \]

The sum in parantheses is the identity since for all \( i \) we have \( P_i^+ + P_i^- = 1 \) by Def. 4.

Up to the naming of the indices, and after using eq. 6.7 this is exactly our claim in the case \( p + 1 \) (and this is true despite the fact that we have only considered \( \delta^1 \) so far – because we shall next show that the contribution from \( H \circ \delta^2 \) vanishes).

(2) Consider \( \delta^2 \) in eq. 2.3. Using the notation of eq. 6.3, we may write

\[ \theta_{p+1,q} = \bigoplus_{\text{deg}(s_1, \ldots, s_n) = p+1} \sum_{\text{pairw. diff.}} w_1 \cdots w_p \theta_{p+1,q}^{w_1 \cdots w_p} \otimes f_1 \wedge \cdots \wedge f_n. \]

Therefore \( \delta^2 \theta_{p+1,q} \) equals

\[ = \bigoplus_{\text{deg}(s_1, \ldots, s_n) = p+1} \sum_{\text{pairw. diff.}} w_1 \cdots w_p \theta_{p+1,q}^{w_1 \cdots w_p} \otimes f_1 \wedge \cdots \wedge f_n. \]

\[ (-1)^{\# \{ w_i | 1 \leq i \leq p \text{ s.t. } w_i < w_{p+1} \}} (-1)^{\# \{ w_i | 1 \leq i \leq p \text{ s.t. } w_i < w_{p+1} \}} \]

\[ \theta_{p+1,q}^{w_1 \cdots w_p} \otimes \left[ f_{w_{p+1}}, f_{w_{p+2}} \right] \wedge f_1 \wedge \cdots \wedge f_{w_{p+1}} \wedge \cdots \wedge f_{w_{p+2}} \wedge f_n. \]

The two terms \((-1)^{\# \{ w_i | 1 \leq i \leq p \text{ s.t. } w_i < w_{p+1} \}}\) (and with \( w_i < w_{p+2} \) respectively) appear since the original summand in \( \delta^2 \) carries the sign \((-1)^{i+j}\), so we need to compute the number of the wedge slot correctly, respecting the omitted wedge factors; compare with the discussion in the first part of this proof. We observe that the first wedge factor remains unchanged under \( \delta^2 \). Hence, when we apply the contracting homotopy \( H \) in this induction step and in the next again, the summand will vanish thanks to \( H^2 = 0 \), cf. eq. 4.3. It will not do harm to verify this explicitly: We use the notation of eq. 6.3 and write the above in terms of

\[ (\delta^2 \theta_{p+1,q})_{p+1,q}^{w_1 \cdots w_p} = (-1)^{w_{p+1} + w_{p+2}} (-1)^{\# \{ w_i | 1 \leq i \leq p \text{ s.t. } w_i < w_{p+1} \}} \]

\[ (-1)^{\# \{ w_i | 1 \leq i \leq p \text{ s.t. } w_i < w_{p+2} \}} \theta_{p+1,q}^{w_1 \cdots w_p}. \]

Next, we apply \( H : N^{p+1} \rightarrow N^{p+2} \) (see eq. 4.5): Then for indices \( s_1, \ldots, s_n = \gamma_1^1 \cdots \gamma_{n-p-1}^1 \cdots \gamma_n^1 \) (which is of degree \( p + 2 \)) we obtain the expression

\[ (H \delta^2 \theta_{p+1,q})_{p+2,q-1}^{w_1 \cdots w_p} = P_1^{\gamma_1^1} \cdots P_n^{\gamma_n^1} \sum_{\gamma_1, \ldots, \gamma_{n-p-1} \in \{ \pm \}} (-1)^{\# \{ \gamma_1 \cdots \gamma_{n-p-1} \}} \theta_{p+1,q}^{w_1 \cdots w_p}. \]
where we have plugged in our previous computation and started to disregard the precise sign. We know the last term of this expression by our induction hypothesis and therefore obtain

\[ P_1^{\gamma_1} \cdots P_n^{\gamma_n-1} \]

\[ \sum_{\gamma_1, \ldots, \gamma_n \in \{\pm\}} (-1)^{\sum_{i=1}^{n} \gamma_i} (1) P_{n-p}^{\gamma_n-p} P_1^{\gamma_1} \cdots P_n^{\gamma_n-p} \]

\[ \left( P_{n-p+1}^{\gamma_n-p+1} \right) \text{ad}(f_{w_1}) P_{n-p+1}^{\gamma_n} \cdots \left( P_1^{\gamma_1} \right) \text{ad}(f_{w_1}) P_n^{\gamma_n} \right) f_0. \]

As the \( P_1^{\pm}, \ldots, P_n^{\pm} \) commute pairwise, the same holds for \( P_1^{\pm}, \ldots, P_n^{\pm} \) (by Def. 4). Thus, the underlined expression can be rearranged to \( P_{n-p}^{\gamma_n-p} P_{n-p}^{\gamma_n-p} \ldots \), but \( P_i^{\gamma_i} P_i^{\gamma_i} = P_i^{\gamma_i} (1 - P_i^{\gamma_i}) = 0 \) as \( P_i^{\pm} \) is an idempotent. The same for \( P_i^{\pm} P_i^{\pm} \). Hence, in all the indices \( s_1 \ldots s_n \) relevant for our claim \( H \delta^{[2]} \theta_{p+1, q} \) is zero. \( \square \)

This readily implies the following key computation:

**Theorem 6 (Main Theorem).** Let \( (A, (f_i^\pm), \tau) \) be an \( n \)-fold cubically decomposed algebra over a field \( k \). Then

\[ \otimes \text{res}^* (f_0 \otimes f_1 \wedge \ldots \wedge f_n) = (-1)^{\frac{(n-1)n}{2}} \]

\[ \tau \sum_{\pi \in \mathfrak{S}_n} \text{sgn}(\pi) \sum_{\gamma_1 \ldots \gamma_n \in \{\pm\}} (-1)^{\sum_{i=1}^{n} \gamma_i} (P_1^{\gamma_1} \text{ad}(f_{\pi(1)}) P_1^{\gamma_1}) \cdots (P_n^{\gamma_n} \text{ad}(f_{\pi(n)}) P_n^{\gamma_n}) f_0, \]

where \( P_1^{\pm}, \ldots, P_n^{\pm} \) is any system of pairwise commuting good idempotents in the sense of Def. 4 (the value does not depend on the choice of the latter). Analogously,

\[ (\otimes \text{res}^* \varphi)(f_1 \wedge \ldots \wedge f_n)(f_0)^{\otimes} \text{res}^* (f_0 \otimes f_1 \wedge \ldots \wedge f_n) \]

for every \( \varphi \in k \).

We remark that one can also write the above formula as

\[ \otimes \text{res}^* (f_0 \otimes f_1 \wedge \ldots \wedge f_n) = (-1)^{\frac{(n-1)n}{2}} \]

\[ \tau \sum_{\pi \in \mathfrak{S}_n} \text{sgn}(\pi) \sum_{\gamma_1 \ldots \gamma_n \in \{\pm\}} (-1)^{\sum_{i=1}^{n} \gamma_i} (P_1^{\gamma_1} \text{ad}(f_{\pi(1)}) P_1^{\gamma_1}) \cdots (P_n^{\gamma_n} \text{ad}(f_{\pi(n)}) P_n^{\gamma_n}) f_0 \]

since for any expression \( g \) we have

\[ P_i^{\gamma_i} \text{ad}(f_{w_i}) P_i^{\gamma_i} g = P_i^{\gamma_i} [f_{w_i}, P_i^{\gamma_i} g] = P_i^{\gamma_i} f_{w_i} P_i^{\gamma_i} g - P_i^{\gamma_i} f_{w_i} P_i^{\gamma_i} g f_w = P_i^{\gamma_i} f_{w_i} P_i^{\gamma_i} g \]

since \( P_i^{\gamma_i} P_i^{\gamma_i} = (1 - P_i^{\gamma_i}) P_i^{\gamma_i} = 0 \) and \( P_i^{\gamma_i} \) is an idempotent.

**Proof.** Use Prop. 4 with \( p = n \). Plugging these components into the shorthand notation of eq. 6.3 we unwind for \( \otimes \text{res}^* (f_0 \otimes f_1 \wedge \ldots \wedge f_n) \) the formula

\[ = -\tau (-1)^{\frac{n^2-n}{2}} \sum_{\substack{w_1, \ldots, w_n \in \{1, \ldots, n\} \atop w_1 + \cdots + w_n = 1}} \rho(w_1, \ldots, w_n) (-1)^{w_1 + \cdots + w_n} \]

\[ \sum_{\gamma_1 \ldots \gamma_n \in \{\pm\}} (-1)^{\sum_{i=1}^{n} \gamma_i} (P_1^{\gamma_1} \text{ad}(f_{w_1}) P_1^{\gamma_1}) \cdots (P_n^{\gamma_n} \text{ad}(f_{w_1}) P_n^{\gamma_n}) f_0. \]
We can clearly replace \( w_1, \ldots, w_n \) by a sum over all permutations of \( \{1, \ldots, n\} \). In order to obtain a nice formula (in the above formula the \( P_i \) appear in ascending order, while the \( w_i \) appear in descending order), we prefer to compose each permutation with the order-reversing permutation \( w_i := \pi(n - i + 1) \): Hence,

\[
= -\tau(-1)^{\frac{n^2 - n}{2}} \sum_{\pi \in \mathcal{S}_n} \rho(\pi(n), \ldots, \pi(1))(-1)^{1+\cdots+n}
\sum_{\gamma_1, \ldots, \gamma_n \in \{\pm\}} (-1)^{\gamma_1 + \cdots + \gamma_n} (P_1^{-\gamma_1} \text{ad}(f_{\pi(1)})P_1^{\gamma_1}) \cdots (P_n^{-\gamma_n} \text{ad}(f_{\pi(n)})P_n^{\gamma_n})f_0.
\]

To conclude, use eq. (6.6) and the (easy) fact that the order-reversing permutation has signum \((-1)^{\frac{n^2 - n}{2}}\), giving the sign of our claim. \(\square\)

**Proof of Thms. 1 & 2.** We define \( \mathcal{G} := E^n(k) \), where \( E \) is the functor defined in §1.1. As already discussed in §1.1 this contains \( k[t_1^\pm, \ldots, t_n^\pm] \) as a Lie subalgebra, acting as multiplication operators \( x \mapsto f \cdot x \). It is also easily checked that the differential operators \( t_1^\pm \cdots t_n^\pm \partial_i \), can be written as infinite matrices. If \( \mathfrak{g} \) is a finite-dimensional Lie algebra, observe that \( \mathcal{G} = E^n(k) \) and \( E^n(\text{End}_k(\mathfrak{g})) \) are actually isomorphic. If \( \mathfrak{g} \) is simple, it is centreless, so the adjoint representation gives an embedding \( \mathfrak{g} \hookrightarrow \text{End}_k(\mathfrak{g}) \), and thus

\[
\mathfrak{g}(t_1^\pm, \ldots, t_n^\pm) \hookrightarrow E^n(\text{End}_k(\mathfrak{g})) \simeq E^n(k) = \mathcal{G}.
\]

This shows that all Lie algebras in the claim are subalgebras of \( \mathcal{G} \). As shown in §1.1 \( \mathcal{G} \) is a cubically decomposed algebra, so we define \( \phi \) as in Def. §1.1 \( \phi := \text{res}^*(1) \).

Since we work with field coefficients, the Universal Coefficient Theorem for Lie algebras tells us that

\[
H^{n+1}(\mathfrak{g}, k) \simeq H_{n+1}(\mathfrak{g}, k)^*,
\]

i.e. knowing the values of a cocycle only on Lie cycles (instead of all of \( \wedge \mathfrak{g} \)) determines the cocycle uniquely, \( \text{res}^*(1)(\alpha) = \text{res}_* \alpha \). However, by Lemma 6 we may evaluate the cocycle on the image of \( I \) by using \( \otimes \text{res}_* \) instead. Using Thm. 9 we get an explicit formula for \( \otimes \text{res}^*(1) \), proving Thm. 2. Using the explicit formula, it is a direct computation to check that for \( n = 1 \) the cocycle agrees with the ones mentioned in the claim of Thm. 1. \(\square\)

### 7. Application to the Multidimensional Residue

In this section we will show that the Lie cohomology class of Def. §1 naturally gives the multidimensional (Parshin) residue.

We work in the framework of multivariate Laurent polynomial rings over a field \( k \), see §1.1. In other words, as our cubically decomposed algebra we take an infinite matrix algebra \( A = E^n(k) \) and \( \mathfrak{g} = A_{\text{Lie}} \). Via eq. 1.3 it acts on the \( k \)-vector space \( k[t_1^\pm, \ldots, t_n^\pm] \). The latter, now interpreted as a ring, also embeds as a commutative subalgebra into \( A \). In order to distinguish very clearly between the subalgebra of \( A \) and the vector space it acts on, we shall from now on write \( k[t_1^\pm, \ldots, t_n^\pm] \) for the \( k \)-vector space. Thus, when we write \( t_i \) we always refer to the associated multiplication operator \( x \mapsto t_i \cdot x \) in \( A \), e.g. \( t_i^m \cdot t_i^l = t_i^{m+l} \).
Following \[2\] Lemma 1(b) we may introduce a (not quite well-defined\(^3\)) ‘map’
\[
(7.1) \quad \varpi : \Omega_{k[t_1^\pm, \ldots, t_n^\pm]/k}^n \to H_{n+1}(g, k) \quad \text{ for } f_0 \wedge f_1 \wedge \ldots \wedge f_n \mapsto f_0 \wedge f_1 \wedge \ldots \wedge f_n.
\]
As \(k[t_1^\pm, \ldots, t_n^\pm]\) is commutative, the \(f_i\) commute pairwise and thus \(f_0 \wedge \ldots \wedge f_n\) is indeed a Lie homology cycle.

**Theorem 7.** The morphism
\[
\text{res}_\ast \circ \varpi : \Omega_{k[t_1^\pm, \ldots, t_n^\pm]/k}^n \to k
\]
(with \(\varpi\) as in eq. (7.1) and \(\text{res}_\ast\) as in Def. \[3\]) for \(c_{i,j} \in \mathbb{Z}\) is explicitly given by
\[
t_1^{c_{1,0}} \ldots t_n^{c_{n,0}} \rho(t_1^{c_{1,1}} \ldots t_n^{c_{n,1}}) \wedge \ldots \wedge (t_1^{c_{1,n}} \ldots t_n^{c_{n,n}}) \mapsto -(1)^{\sum_{i=0}^n c_{i,i}} \det \begin{pmatrix} c_{1,1} & \cdots & c_{n,1} \\ \vdots & \ddots & \vdots \\ c_{1,n} & \cdots & c_{n,n} \end{pmatrix}
\]
whenever \(\sum_{p=0}^n c_{p,p} = 0\) and is zero otherwise. In particular \(-1)^{\sum_{i=0}^n c_{i,i}} (\text{res}_\ast \circ \varpi)\) is the conventional multidimensional (Parshin) residue.

The complicated sign \(-1)^{\sum_{i=0}^n c_{i,i}}\) should not concern us too much; it is an artifact of homological algebra. Just by changing our sign conventions for bicomplexes, we could easily switch to an overall opposite sign. Letting \(c_{i,j} = \delta_{i,j}\) for \(i, j \in \{1, \ldots, n\}\) gives the familiar
\[
-(1)^{\sum_{i=0}^n c_{i,i}} \text{res}\ast(at_1^{c_{1,0}} \ldots t_n^{c_{n,0}} \wedge t_1 \wedge \ldots \wedge t_n) = \delta_{c_{0,1}} = -1 \cdots \delta_{c_{0,n}} = -1 a
\]
for \(a \in k\). In particular this assures us that the map \(\text{res}_\ast\) gives the correct notion of residue: it is the \((-1, \ldots, -1)-\)coefficient of the Laurent expansion.

**Proof.** After unwinding \(\varpi\) it remains to evaluate \(\text{res}_\ast(f_0 \wedge f_1 \wedge \ldots \wedge f_n)\) for \(f_i := t_1^{c_{i,0}} \cdots t_n^{c_{i,n}}\) \((i = 0, \ldots, n)\). Clearly \(f_0 \otimes f_1 \wedge \ldots \wedge f_n\) is a cycle in \(H_n(g, g)\), and so by Lemma \[6\] we may use \(\otimes \text{res}_\ast\) instead of \(\text{res}_\ast\). Then Thm. \[6\] reduces this to the matrix trace
\[
(7.2) \quad \text{res}_\ast(f_0 \wedge f_1 \wedge \ldots \wedge f_n) = -(1)^{\sum_{i=0}^n c_{i,i}} \sum_{\pi \in S_n} \text{sgn}(\pi) \tau M_\pi, \quad \text{where}
\]
\[
M_\pi := \sum_{\gamma_1, \gamma_2 \in \{\pm\}} (-1)^{\gamma_1 + \gamma_2 \gamma_3} (P_1^{\gamma_1} f_{\pi(1)} P_1^{\gamma_2}) \cdots (P_n^{\gamma_1} f_{\pi(n)} P_n^{\gamma_2}) f_0.
\]
For the evaluation of \(\tau M_\pi\) fix a permutation \(\tau\) and pick the (pairwise commuting) system of idempotents 
\[
\text{(7.3)} \quad P_j^+ t_1^{\lambda_1} \cdots t_n^{\lambda_n} = \delta_{\lambda_j, 0} t_1^{\lambda_1} \cdots t_n^{\lambda_n} \quad \text{(with } \lambda_1, \ldots, \lambda_n \in \mathbb{Z})
\]
Next, observe that the Laurent polynomial ring \(W := k[t_1^\pm, \ldots, t_n^\pm]\) is stable (i.e. \(\phi W \subseteq W\)) under the endomorphisms \(f_0, \ldots, f_n\) and the idempotents \(P_j^\pm\), and therefore under \(M_\pi\). Hence, it follows that it suffices to evaluate the trace of \(M_\pi\) on the \(k\)-vector subspace \(k[t_1^\pm, \ldots, t_n^\pm]\). We compute successively
\[
f_k P_j^+ t_1^{\lambda_1} \cdots t_n^{\lambda_n} = \delta_{\lambda_j, 0} t_1^{\lambda_1 + c_k} \cdots t_n^{\lambda_n + c_k}.
\]
\[
P_j f_k P_j^+ t_1^{\lambda_1} \cdots t_n^{\lambda_n} = \delta_{0, \lambda_j} t_1^{\lambda_1 + c_k} \cdots t_n^{\lambda_n + c_k}.
\]
\(^3\)It does not respect the relation \(d(ab) = bda + adb\); this artifact already occurs in Beilinson’s paper \[2\]. However, this ambiguity dissolves after composing with the residue (as in the theorem) and it is very convenient to treat this as some sort of a map for the moment.
and analogously for $P^+_j f_k P^-_j$. We find
\begin{equation}
\sum_{\gamma_j \in \{\pm\}} (-1)^{\gamma_j} \left( P^+_j f_k P^-_j \right) t^\lambda_1 \cdots t^\lambda_n
= (\delta_0 \leq \lambda_i < c_{w,j} - \delta - c_{w,j} \leq \lambda_i < 0) t^{\lambda_1 + c_{k,j} \cdots t^{\lambda_n + c_{k,n}}.}
\end{equation}

Now we claim:

- Subclaim: Writing $w_i := \pi(i)$ we have
\begin{equation}
M_\pi t^\lambda_1 \cdots t^\lambda_n = \prod_{i=1}^{n} (\delta_0 \leq \lambda_i + c_{w,i} + \sum_{p=1}^{n} c_{w,p,i} < -c_{u,i})
- \delta - c_{w,i} \leq \lambda_i + c_{w,i} + \sum_{p=1}^{n} c_{w,p,i} < 0)
\end{equation}

\begin{equation}
\sum_{\gamma_i \in \{\pm\}} (-1)^{\gamma_i} \left( \prod_{i=1}^{n} (-1)^{\gamma_i} \left( P_i f_w P_{i}^\gamma \right) \prod_{i=1}^{n} (-1)^{\gamma_i} \left( P_i f_w P_{i}^\gamma \right) \right) f_0
\end{equation}

so that
\begin{equation}
M_\pi^{(1)} = M_\pi
\end{equation}

\begin{equation}
M_\pi^{(n+1)} = f_0.
\end{equation}

We claim that
\begin{equation}
M_\pi^{(i)} t^\lambda_1 \cdots t^\lambda_n = \alpha t^\lambda_1 + c_{w,1} + \sum_{p=1}^{n} c_{w,p,1} \cdots t^\lambda_n + c_{w,n} + \sum_{p=1}^{n} c_{w,p,n}
\end{equation}

for some factor $\alpha \in \{\pm 1, 0\}$. For $i = n + 1$ this is clear since $f_0 = t_{1}^{c_{0,1}} \cdots t_{n}^{c_{0,n}}$, in particular $\alpha = 1$. Assuming this holds for $i + 1$, for $i$ we get by using eq. \ref{eq:7.7} (with the appropriate values plugged in: $j := i$ and $k := w_i$, and $\lambda_i$ as in eq. \ref{eq:7.6})
\begin{equation}
M_\pi^{(i)} t^\lambda_1 \cdots t^\lambda_n = \sum_{\gamma_i \in \{\pm\}} (-1)^{\gamma_i} \left( P_i f_w P_i^\gamma \right) M_\pi^{(i+1)} t^\lambda_1 \cdots t^\lambda_n
\end{equation}

\begin{equation}
\sum_{\gamma_i \in \{\pm\}} (-1)^{\gamma_i} \left( P_i f_w P_i^\gamma \right) M_\pi^{(i+1)} t^\lambda_1 \cdots t^\lambda_n
\end{equation}

This proves our claim for all $i$ by induction. We observe that the pre-factor $\alpha$ in each step just gets multiplied with the expression is eq. \ref{eq:7.7} giving the product in our claim.)

Next, we need to evaluate the trace of $M_\pi$ as given in eq. \ref{eq:7.6} The endomorphism is nilpotent unless
\begin{equation}
\forall i : c_{0,1} + \sum_{p=1}^{n} c_{w,p,i} = 0.
\end{equation}

We remark that $w_1, \ldots, w_n$ is just a permutation of $\{1, \ldots, n\}$, so these conditions can be rewritten as $\sum_{p=1}^{n} c_{p,i} = 0$. In the nilpotent case the trace is clearly zero. Hence, we may assume we are in the case where eq. \ref{eq:7.8} holds. Using these equations and the useful convention $w_{n+1} := 0$, our expression for $M_\pi$ simplifies to
\begin{equation}
M_\pi t^\lambda_1 \cdots t^\lambda_n = \prod_{i=1}^{n} (\delta_0 \leq \lambda_i + \sum_{p=1}^{n+1} c_{w,p,i} < -c_{w,i})
- \delta_0 \leq \lambda_i + c_{w,i} + \sum_{p=1}^{n+1} c_{w,p,i} < 0)
\end{equation}

The endomorphism $M_\pi$ is visibly diagonal of finite rank and we may reduce the computation of the trace to a (finite-dimensional) stable vector subspace. A finite subset of the $t^\lambda_1 \cdots t^\lambda_n$ ($\lambda_1, \ldots, \lambda_n \in \mathbb{Z}$) provides a basis. We see in eq. \ref{eq:7.9} that $M_\pi$ acts diagonally on these basis vectors with eigenvalues $\pm 1$ or 0. Moreover, for
each $i$ we either have $c_{w,i} \geq 0$ or $c_{w,i} < 0$, which shows that each bracket of the shape $(\delta_{0 \leq \lambda < -c} - \delta_{-c < \lambda < 0})$ in eq. $[7.9]$ either attains only values in $\{+1,0\}$ when we run through all $\lambda_1, \ldots, \lambda_n \in \mathbb{Z}$, or only values in $\{-1,0\}$. This shows that we only need to count (with appropriate sign) the non-zero eigenvalues of $M_\pi$ in order to evaluate the trace. Note that our finite subset of $t_1^{\lambda_1} \cdots t_n^{\lambda_n}$ indexes a basis, so we need to count the number of such basis vectors with non-zero eigenvalue. We introduce the non-standard shorthand $\lfloor \pm \rfloor$.

The value of a fixed bracket $(\delta_{0 \leq \lambda < -c} - \delta_{-c < \lambda < 0})$ - when non-zero - is always either $+1$, or always $-1$. Thus, the number of non-zero eigenvalues is simply the number of elements within the hypercube such that each $\lambda_i$ lies within the range of length $|\pm c_{w,i}|$ counted above, and therefore

$$\tau M_\pi = \prod_{i=1}^n (-c_{w,i}) = \prod_{i=1}^n (-c_{w,i}) = (-1)^n \prod_{i=1}^n c_{\pi(i),i}$$

(because $[a] - [a] = -a$ for all $a \in \mathbb{Z}$). We plug this into eq. $[7.2]$ and recognize the usual formula for the determinant. This finishes the proof. \qed

We are now ready to prove the remaining theorems from the introduction:

**Proof of Thms. 1, 2, 3, 4, & 5.** We use Thm. 7 to obtain Thm. 4. Then Thm. 4 follows as a special case. For Thm. 5 we use the shorthands $\pi = P_1^+ = P^+$ (following both the notation of Arbarello, de Concini and Kac and ours). On the one hand we compute

$$[\pi, f_1]f_0 = [P, f_1]f_0 = Pf_1f_0 - f_1Pf_0 = \lfloor Pf_0, f_1 \rfloor$$

$$= (P^+ + P^-)[P^+ f_0, f_1] = P^- [P^+ f_0, f_1] + P^+ [P^+ f_0, f_1]$$

and we have $[P^+ f_0, f_1] + [P^- f_0, f_1] = [f_0, f_1] = 0$, so this equals

$$= P^- [P^+ f_0, f_1] - P^+ [P^- f_0, f_1].$$

On the other hand, we unwind

$$\text{res } f_0 df_1 = (-1)^{\gamma_1} \text{tr } \sum_{\gamma_1 \in \{\pm\}} (-1)^{\gamma_1} (P_1^+)^{\gamma_1} \text{ad}(f_{\pi(i)}) P_1^{\gamma_1}) f_0$$

$$= -P^- [f_1, P_1^+ f_0] + P^+ [f_1, P_1^- f_0]$$

and these expressions clearly coincide. Finally Thm. 3 is true since we use the cocycle defined in Def. 5 i.e. it is constructed exactly as stated in Thm. 5. \qed

**8. Application to Multiloop Lie Algebras**

Suppose $k$ is a field and $g/k$ is a finite-dimensional centreless Lie algebra (e.g. $g$ finite-dimensional, semisimple). Then the adjoint representation $\text{ad}: g \to \text{End}_k(g)$ is injective. Thus, we obtain a Lie algebra inclusion

$$i: g(t_1^\pm, \ldots, t_n^\pm) \hookrightarrow E^n(\text{End}_k(g))_{\text{Lie}},$$

where $E$ is the functor described in Section 2 (the right-hand side is equipped with the Lie bracket $[a, b] = ab - ba$ based on the associative algebra structure). Thus, we have the pullback

$$i^*: H^{n+1}(E^n(\text{End}_R(g))_{\text{Lie}}, k) \to H^{n+1}(g(t_1^\pm, \ldots, t_n^\pm), k),$$
which we may apply to the class res\(^*\)(1), see Def. \[7\]

**Theorem 8.** Suppose \(k\) is a field and \(\mathfrak{g}/k\) is a finite-dimensional centreless Lie algebra. For \(Y_0, \ldots, Y_n \in \mathfrak{g}\) we call
\[
B(Y_0, \ldots, Y_n) := \text{tr}_{\text{End}_k(\mathfrak{g})}(\text{ad}(Y_0) \cdots \text{ad}(Y_n))
\]
the ‘generalized Killing form’. For \(n = 1\) and if \(\mathfrak{g}\) is semisimple, this is the classical Killing form of \(\mathfrak{g}\).

(1) Then on all Lie cycles admitting a lift under \(I\) as in eq. \[1\] the pullback \(i^* \text{res}^*(1) \in H^{n+1}(\mathfrak{g}[t_1^\pm, \ldots, t_n^\pm], k)\) is explicitly given by
\[
(i^* \phi)(Y_0 t_1^{c_0,1} \cdots t_n^{c_0,n} \wedge \cdots \wedge Y_n t_1^{c_n,1} \cdots t_n^{c_n,n})
\]
\[
= -(-1)^{\frac{\sum_i n_i^2}{2}} \sum_{\pi \in S_n} \text{sgn}(\pi)B(Y_{\pi(1)}, \ldots, Y_{\pi(n)}, Y_0) \prod_{i=1}^n c_{\pi(i),i}\]
whenever \(\forall i \in \{1, \ldots, n\}: \sum_{p=0}^n c_{p,i} = 0\) and zero otherwise.

(2) If \(\mathfrak{g}\) is finite-dimensional and semisimple and \(n = 1\), then \(i^* \text{res}^*(1) \in H^2(\mathfrak{g}[t_1^\pm], k)\) is the universal central extension of the loop Lie algebra \(\mathfrak{g}[t_1, t_1^{-1}]\) giving the associated affine Lie algebra \(\hat{\mathfrak{g}}\) (without extending by a derivation),
\[
0 \rightarrow k \langle c \rangle \rightarrow \hat{\mathfrak{g}} \rightarrow \mathfrak{g}[t_1, t_1^{-1}] \rightarrow 0.
\]

**Proof.** (1) According to Lemma \[9\] Thm. \[8\] and eq. \[5\] the cocycle is explicitly given by
\[
\text{res}^*(1)(f_0 \wedge \cdots \wedge f_n) = \otimes \text{res}^*(1)(f_0 \otimes f_1 \wedge \cdots \wedge f_n)
\]
\[
= \tau \sum_{\pi \in S_n} \text{sgn}(\pi)M_{\pi},\] where
\[
M_{\pi} = \sum_{\gamma_1, \ldots, \gamma_n \in \{\pm\}} (-1)^{\gamma_1 + \cdots + \gamma_n}
\]
\[
(P_1^{\gamma_1} f_\pi(1) P_1^{\gamma_1}) \cdots (P_n^{\gamma_n} f_\pi(n) P_n^{\gamma_n}) f_0.
\]
Note that \(M_{\pi} \in E^n(\text{End}_k(\mathfrak{g}))\). As we consider the pullback of the cohomology class along \(i : \mathfrak{g}[t_1^\pm, \ldots, t_n^\pm] \rightarrow E^n(\text{End}_k(\mathfrak{g}))_{\text{Lie}}\), it suffices to treat elements \(f_i := Y_{t_1^{c_i,1} \cdots t_n^{c_i,n}} \) with \(c_{i,1}, \ldots, c_{i,n} \in \mathbb{Z}\) (for \(i = 0, \ldots, n\) and \(Y_i \in \mathfrak{g}\)). Note that by our embedding \(i\) an element \(f_i\) is mapped to the endomorphism \(\text{ad}(Y_i) f_1^{c_{i,1}} \cdots f_n^{c_{i,n}}\) in \(E^n(\text{End}_k(\mathfrak{g}))\). Let \(\pi \in S_n\) be a fixed permutation. In order to compute the trace, it suffices to study the action of \(M_{\pi}\) on the basis elements \(X t_1^{\lambda_1} \cdots t_n^{\lambda_n}\) of \(\mathfrak{g}[t_1^\pm, \ldots, t_n^\pm]\), where \(\lambda_1, \ldots, \lambda_n \in \mathbb{Z}\) and \(X \in \mathfrak{g}\) runs through a basis of \(\mathfrak{g}\). We denote them with bold letters \(t_i\) instead of \(t_i\) to distinguish clearly between a basis element and \(t_i\) as an endomorphism \(t_i : x \mapsto t_i \cdot x\) in \(E^n(\text{End}_k(\mathfrak{g}))\). As in the proof of Thm. \[7\] we compute
\[
P_j^{-} f_k P_j^+ X t_1^{\lambda_1} \cdots t_n^{\lambda_n} = \delta_{0 \leq \lambda_j < -c_{k,j}} \text{ad}(Y_k) X t_1^{\lambda_1 + c_{k,1}} \cdots t_n^{\lambda_n + c_{k,n}}.
\]
and as a consequence we find
\[
\sum_{\gamma_j \in \{\pm\}} (-1)^{\gamma_j} (P_j^{-\gamma_j} x_k P_j^{\gamma_j}) X t_1^{\lambda_1} \cdots t_n^{\lambda_n}
\]
\[
= (\delta_{0 \leq \lambda_j < -c_{k,j}} - \delta_{-c_{k,j} \leq \lambda_j < 0}) \text{ad}(Y_k) X t_1^{\lambda_1 + c_{k,1}} \cdots t_n^{\lambda_n + c_{k,n}}.
\]
With an inductive computation entirely analogous to eq. (7.3) we find
\[
M_\tau X t_1^{\lambda_1} \ldots t_n^{\lambda_n} = \prod_{i=1}^{n} \left( \delta_{0 \leq \lambda_i + c_{0,i} + \sum_{p=i+1}^{n} c_{w_p,i}} \right) \left( -\delta_{-c_{w_i,i} \leq \lambda_i + c_{0,i} + \sum_{p=i+1}^{n} c_{w_p,i} < 0} \right) \ad(Y_{w_1}) \cdots \ad(Y_{w_n}) \ad(Y_0) X 
\]
where \( w_i := \pi(i) \). Unless \( \forall i : \sum_{p=0}^{n} c_{p,i} = 0 \) holds, \( M_\tau \) is clearly nilpotent and thus has trace \( \tau M_\tau = 0 \). This condition is clearly independent of \( \pi \), showing that \((i^* \res^*(1))(f_0 \wedge \cdots \wedge f_n) = 0 \) in this case. From now on assume \( \forall i : \sum_{p=0}^{n} c_{p,i} = 0 \).

Then \( M_\tau \) respects the decomposition
\[
\g[t_1^{\lambda_1} \ldots t_n^{\lambda_n}] = \prod_{\lambda_1, \ldots, \lambda_n} \g t_1^{\lambda_1} \ldots t_n^{\lambda_n}
\]
and therefore (as \( \tau \) is essentially a trace) \( \tau M_\tau = \sum \lambda_1, \ldots, \lambda_n \tau M_\tau |_{t_1^{\lambda_1} \ldots t_n^{\lambda_n}} \). For each summand of the latter we obtain
\[
\tau M_\tau |_{t_1^{\lambda_1} \ldots t_n^{\lambda_n}} = \prod_{i=1}^{n} \left( \delta_{0 \leq \lambda_i + c_{0,i} + \sum_{p=i+1}^{n} c_{w_p,i}} \right) \left( -\delta_{-c_{w_i,i} \leq \lambda_i + c_{0,i} + \sum_{p=i+1}^{n} c_{w_p,i} < 0} \right) \tr(\ad(Y_{w_1}) \cdots \ad(Y_{w_n}) \ad(Y_0)).
\]
The trace term is independent of \( \lambda_1, \ldots, \lambda_n \) (and in the shape of eq. (8.1)), so we may rewrite \( \tau M_\tau \) as
\[
\tau M_\tau = B(Y_{w_1} \ldots Y_{w_n}, Y_0) \sum \lambda_1, \ldots, \lambda_n \prod_{i=1}^{n} \left( \delta_{0 \leq \lambda_i + c_{0,i} + \sum_{p=i+1}^{n} c_{w_p,i}} \right) \left( -\delta_{-c_{w_i,i} \leq \lambda_i + c_{0,i} + \sum_{p=i+1}^{n} c_{w_p,i} < 0} \right).
\]
For the evaluation of the sum \( \sum \lambda_1, \ldots, \lambda_n \) we can apply the same eigenvalue count as in the proof of Thm. (7) This time instead of counting eigenvalues, we count non-zero summands. This yields
\[
\tau M_\tau = (-1)^n B(Y_{w_1} \ldots Y_{w_n}, Y_0) \prod_{i=1}^{n} c_{w_i,i}
\]
and thus our claim. (2) For \( n = 1 \) we obtain
\[
(i^* \res^*(1))(Y_0 t_1^{c_{0,1}} \wedge Y_1 t_1^{c_{1,1}}) = -c_{1,1} \delta_{c_{0,1} + c_{1,1} = 0} B(Y_1, Y_0).
\]
This is well-known to be the defining cocycle of the affine Lie algebra \( \g \) (usually with a positive sign, but the class is only well-defined up to non-zero scalar multiple anyway).

The natural further cases of the Virasoro algebra as well as affine Kac-Moody algebras (i.e. \( \g \) extended by derivations) will be discussed elsewhere. The computations become more involved, but no further ideas are needed.

References
1. E. Arbarello, C. De Concini, and V. G. Kac, The infinite wedge representation and the reciprocity law for algebraic curves, Theta functions—Bowdoin 1987, Part 1 (Brunswick, ME, 1987), Proc. Sympos. Pure Math., vol. 49, Amer. Math. Soc., Providence, RI, 1989, pp. 171–190. MR 1013132 (90i:22034)
2. A. A. Beilinson, Residues and adèles, Funktsional. Anal. i Prilozhen. 14 (1980), no. 1, 44–45. MR 565095 (81f:14010)
3. A. A. Beilinson, B. L. Feigin, and B. C. Mazur, *Notes on conformal field theory*, unpublished, available on http://www.math.sunysb.edu/~kirillov/manuscripts.html, 1991.

4. B. L. Feigin and B. L. Tsygan, *Cohomology of Lie algebras of generalized Jacobi matrices*, Funktsional. Anal. i Prilozhen. 17 (1983), no. 2, 86–87. MR 705056 (85c:17008)

5. I. B. Fesenko, *Analysis on arithmetic schemes. II*, J. K-Theory 5 (2010), no. 3, 437–557. MR 2658047 (2011k:14019)

6. I. B. Fesenko and M. Kurihara (eds.), *Invitation to higher local fields*, Geometry & Topology Monographs, vol. 3, Geometry & Topology Publications, Coventry, 2000, Papers from the conference held in Münster, August 29–September 5, 1999. MR 1804915 (2001h:11005)

7. E. Frenkel and X. Zhu, *Gerbal representations of double loop groups*, International Mathematics Research Notices (2011).

8. I. B. Frenkel, *Beyond affine Lie algebras*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986) (Providence, RI), Amer. Math. Soc., 1987, pp. 821–839. MR 934284 (89g:17018)

9. A. Huber, *On the Parshin-Beilinson adeles for schemes*, Abh. Math. Semin. Univ. Hamb. 61 (1991), 249–273 (English).

10. R. Hübl and A. Yekutieli, *Adèles and differential forms*, J. Reine Angew. Math. 471 (1996), 1–22. MR 1374916 (97d:14026)

11. M. Jimbo and T. Miwa, *Solitons and infinite-dimensional Lie algebras*, Publ. Res. Inst. Math. Sci. 19 (1983), no. 3, 943–1001. MR 723457 (85i:58060)

12. V. G. Kac and D. H. Peterson, *Spin and wedge representations of infinite-dimensional Lie algebras and groups*, Proc. Nat. Acad. Sci. U.S.A. 78 (1981), no. 6, part 1, 3308–3312. MR 619827 (82j:17019)

13. V. G. Kac and A. K. Raina, *Bombay lectures on highest weight representations of infinite-dimensional Lie algebras*, Advanced Series in Mathematical Physics, vol. 2, World Scientific Publishing Co. Inc., Teaneck, NJ, 1987. MR 1021978 (90k:17013)

14. J.-L. Loday, *Cyclic homology*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 301, Springer-Verlag, Berlin, 1992, Appendix E by María O. Ronco. MR 1217970 (94a:19004)

15. M. Morrow, *An explicit approach to residues on and dualizing sheaves of arithmetic surfaces*, New York Journal of Mathematics, vol. 16, p575-627, 2010, 2010.

16. E. Neher, *Extended affine Lie algebras and other generalizations of affine Lie algebras—a survey*, Developments and trends in infinite-dimensional Lie theory, Progr. Math., vol. 288, Birkhäuser Boston Inc., Boston, MA, 2011, pp. 53–126. MR 2743761 (2011m:17055)

17. J. Tate, *Residues of differentials on curves*, Ann. Sci. École Norm. Sup. (4) 1 (1968), 149–159. MR 0227171 (37 #2756)