Seiberg and Witten’s solution to \( N=2 \) SU(2) Yang-Mills with \( N_f=0 \) flavors has a one-complex-dimensional Coulomb branch of degenerate vacua labeled by a coordinate \( u \). The effective U(1) theory is described in terms of two functions \( a(u) \) and \( a_D(u) \). The gauge coupling, \( \tau \equiv (\theta/2\pi) + i(4\pi^2/g^2) \) is given by \( \tau = da_D/da \); it must satisfy \( \text{Im}(\tau) > 0 \). The theory is governed by a dynamically generated strong-coupling scale which we set to 1.

The mass of a dyon hypermultiplet with electric and magnetic charges \( n_e \) and \( n_m \) is given by \( M = \sqrt{2}|n_ma_D(u) + n_ea(u)|\). Whenever \( \text{Im}(a_D/a) = 0 \), any dyon becomes marginally unstable to decay into two or more other dyons (conserving electric and magnetic charges). This note presents a simple argument that determines the shape of the curve of marginal stability \( \text{Im}(a_D/a) = 0 \).

The effective theory has a duality group that acts on \( (a_D/a) \) as a vector under \( SL(2, \mathbb{Z}) \), and on \( \tau \) in the usual way. Note that \( f(u) \equiv a_D/a \) transforms like \( \tau \).

The \( U(1) \) effective theory breaks down at \( u = \pm 1 \) and \( \infty \), where a dyon hypermultiplet becomes massless. The \( SL(2, \mathbb{Z}) \) monodromies around these points are \( ST^2S^{-1}, (TS)T^2(TS)^{-1}, \) and \( -T^2 \). These matrices generate the group \( \Gamma(2) \subset SL(2, \mathbb{Z}) \). \( u(\tau) \) is a one-to-one map of a single fundamental domain of \( \Gamma(2) \) onto the complex plane, which has cusp points at \( \tau = 0, 1, \) and \( i\infty \). These cusp points correspond to the three singularities in the \( u \)-plane, and are fixed points of the corresponding \( SL(2, \mathbb{Z}) \) monodromies—see Fig. 1.

The range of the function \( f(u) \) is a subset of the complex plane with similar properties to the fundamental domain of \( \tau \). \( \Gamma(2) \) acts identically on both the \( f \)-plane and the \( \tau \)-plane, and its generators fix the same 3 points. However, since \( \text{Im}f \) is not necessarily positive the range of \( f \) may extend below the real axis, unlike \( \tau \). Indeed, since we know (from expanding the explicit expressions\(^1\) for \( a \) and \( a_D \) around \( u = \pm 1 \)) that there are whole lines where \( f \) is real, it follows that \( f^{-1} \) must map an infinite number of \( \Gamma(2) \) domains, both above and below the real axis,
onto the $u$-plane.

There is only one possibility for the shape of the range of $f(u)$, due to the fact that the generators $ST^2S^{-1}$ and $(TS)T^2(TS)^{-1}$ are of infinite order, which implies that the opening angles of the corresponding cusps must also be of infinite order, i.e., 0 or $2\pi$. An opening angle of 0 would correspond to a single fundamental domain of $\Gamma(2)$, which we have ruled out. Opening angles of $2\pi$ correspond to the domain shown in Fig. 1, a full strip in the $f$-plane with one $\Gamma(2)$ domain removed. It is easy to see that the monodromies for this region are correct. As a check, it is easily verified using the explicit expressions\(^1\) that $f(0) = -(i \pm 1)/2$.

The curve of marginal stability is the image under $f^{-1}$ of the interval $[-1,1]$, which is a simple closed curve in the $u$-plane (with $f(-1) = \pm 1$ and $f(+1) = 0$) as conjectured in Ref. 1. Outside of this curve are the images of the infinite number of $\Gamma(2)$ domains between $\text{Re}(f) = +1$ and $-1$ and with $\text{Im}(f) \geq 0$. Inside the curve are the images of all but one of the $\Gamma(2)$ domains with $\text{Im}(f) < 0$.

The curve $\text{Im} f = 0$ has been shown by independent methods to be simple and closed.\(^2\) Also, we have numerically computed it to be the curve shown in Fig. 1.

The methods presented here are easily extended to the massless $N_f = 1, 2,$ and 3 cases.\(^3\) For nonzero masses, as well as for $N_c > 2$, the curves of marginal stability become dense in moduli space.

Fig. 1: The shaded regions are the images of the $u$-plane in the $\tau$ and $f$-planes. The dashed lines are the images of $\text{Im} f = 0$.

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