ANALYTIC TORSION ON MANIFOLDS WITH FIBRED BOUNDARY METRICS

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Abstract. In this paper, we construct the renormalized analytic torsion in the setup of manifold endowed with fibred boundary metrics. The method of construction is to determine the asymptotic of heat kernel, both in short time regime and long time regime and apply these asymptotics together with renormalization to determine the renormalized zeta function and the determinant of Hodge Laplacian.

1. Introduction

Analytic torsion was introduced by Ray and Singer [RaSi71] as analytic counterpart of Reidemeister torsion in topology. Ray and Singer conjectured that these two torsions are equivalent on closed manifolds. Cheeger and Müller proved this conjecture independently. Assume that \((M,g)\) is a closed Riemannian manifold and \(e^{-t\Delta_g}(x, y) := H(t, x, y)\) is the heat kernel with respect to Hodge Laplacian \(\Delta_g^q : \Omega^q(M) \to \Omega^q(M)\), acting on the space of \(q\) forms. i.e fundamental solution to heat equation,

\[
\partial_t H(t, x, y) + \Delta^q_{g,x} H(t, x, y) = 0, \tag{1.1}
\]

\[H(t = 0, x, y) = \delta(x - y).\]

One define the heat trace to be, \(\text{Tr}(e^{-t\Delta_g^q}) = \int_M e^{-t\Delta_g^q}(x, x) \text{dvol}_g\). Assume \(\Delta^q_g\) is Hodge Laplacian acting on the space of \(q\)-forms. The corresponding zeta function is defined to be,

\[
\zeta^M_q(s) := \frac{1}{\Gamma(s)} \int_M \text{Tr}(e^{-t\Delta_g^q}) t^{s-1} dt. \tag{1.2}
\]

So defined (1.2) is defined on \(\text{Re}(s) > \frac{n}{2}\) where \(n = \text{dim}(M)\) but can be extended holomorphically to complex plane \(\mathbb{C}\) and especially with regular point at \(s = 0\). The determinant of Laplacian is defined to be,

\[
\det(\Delta^q_g) := e^{-\zeta'_q(0)},
\]

Date: February 3, 2021.
One defines the analytic torsion as,

$$\log T(M) := \frac{1}{2} \sum_{q=0}^{n} (-1)^{q} \zeta'_q(0),$$

typically one would like to generalize the definition of zeta function and analytic torsion on non compact set up or non smooth set up and obtain Cheeger Müller type statements on the new set up. we refer to the PhD thesis of Sher [She13] in which he constructed the zeta function in the set up of asymptotic conic manifolds.

In this work we consider manifold $\overline{M}$ with boundary $\partial M$ with fibration boundary structure, i.e $\partial M$ is fibred over closed base $B$ with typical closed fibre $F$. On such a topological structure one may consider metric $g_\phi$ defined as,

$$g_\phi = \frac{dx^2}{x^4} + \phi^* g_B + g_F + h(x),$$

where $x$ is a boundary defining function of $\partial M$, and $\phi$ is fibration and $g_B$ is a Riemannian metric over base $B$ and $g_F$ is a symmetric bilinear form which restricts to Riemannian metric on fibre $F$ and additional assumptions as in [GTV20]. In this set up, we are going to define the concept of analytic torsion. The main difficulty arises when we consider the heat trace,

$$\text{Tr}(e^{-t\Delta_{g_\phi}}) = \int_{\mathbb{R}^+} e^{-t\Delta_{g_\phi}}(x,x)\,d\text{vol}_\phi,$$

i.e in $\phi$ set up, the boundary is located at infinity and therefore the integration over diagonal diverges. To address this problem, the renormalized heat trace is described in the hadamard manner, which essentially takes into account the integration of the heat kernel along the diagonal on the finite component. The heat kernel structure theorem may be employed to take the finite part of this integral at zero to be heat trace renormalized . In order to describe analytic torsion in the set up of $\phi$ manifolds, we can explicitly describe renormalized zeta function and Laplacian determinant by means of renormalized heat trace.

The paper is organized as follows. In section 2 we recall the set up of $\phi$ manifolds and some definitions from geometric microlocal analysis in the sense of Melrose [Mel93], which we need in order to describe structure theorems of heat kernel in section 3. We remark that these methods are in effect microlocal analysis due to Hörmander, Nierenberg and Maslow together with manifold with corners due to Cherrf and Duady [Joy09] and the type of resolution process basically arising in the algebraic geometric sense [Cuto04] of resolution of singularities. Melrose uses these methods in order to develop elliptic theory of differential operators in several geometric set ups, as b geometry [Mel93] and
ANALYTIC TORSION

In section 3 we describe structure theorems of heat kernel both in finite time regime and long time regime on appropriate space. Namely on manifolds with corners. In finite time regime this description is explicit [TaVe20] and in long time regime we use functional calculus and asymptotic of φ resolvent of φ Hodge Laplacian at low energy [GTV20] to relate heat kernel and resolvent via,

$$H^M(t, z, z') = \int_{\Gamma} e^{-t\lambda}(\Delta_\phi - \lambda)^{-1}d\lambda,$$

where Γ is a curve around the spectrum of Hodge Laplacian Δ_φ oriented counter-clockwise.

In section 4 we use these structure theorems in order to determine the asymptotics of renormalized heat trace and show that the renormalized zeta function admits meromorphic continuation on whole of complex plane \mathbb{C}. One defines then the renormalized determinant of Laplacian and analytic torsion in the set up of φ manifolds.

This work is generalization of [She13] where the fibres are trivial. Namely we generalize the result of [She13] for closed fibre F.

Acknowledgment: The author would like to thank Boris Vertman for his supervision for PhD thesis. Further he acknowledges the constructive comments of Daniel Grieser, Julie Rowlett and Boris Vertman on improvements and corrections of this work.

2. PHI MANIFOLDS AND GEOMETRIC MICROLOCAL ANALYSIS

Assume \(\overline{M}\) is a compact manifold with boundary \(\partial M\) and \(\partial M\) has fibration structure i.e, \(\partial M = B - F\), where \(\phi\) is trivialization of fibration. B is base manifold and F is closed manifold as fibre. Near the boundary one may take the product \([0, \varepsilon) \times \partial M\) by collar neighborhood theorem, and fix local coordinates on \(\overline{M}\) to be, \((x, y = (y_1 \cdots y_b), z = (z_1, \cdots z_f))\). Here \(x = \rho_{\partial M}\) is the boundary defining function of \(\partial M\), i.e, \(\partial M = \{x = 0\}\). Consider the metric,

$$g_\phi = \frac{dx^2}{x^4} + \frac{\phi^*g_B}{dx^2} + g_F,$$

on \(\overline{M}\) where \(g_B\) is Riemannian metric on base B and \(g_F\) is symmetric bilinear form which restricts to Riemannian metric on fibre F. We assume further that \(\varphi: (\partial M, g_F + \varphi^*g_B) \rightarrow (B, g_B)\) is Riemannian submersion. Such a geometric set up is called fibred boundary \(\phi\) metric manifolds. Intuitively the boundary is fibre bundle with base B and fibre F where the boundary is viewed to
be located at infinity. The metric arises as example in gravitational instantons [THMo4]. Gravitational instantons are defined as 4 dimension complete hyperkähler manifolds. They are classified in three categories. ALE or asymptotically locally euclidean where fibre $\mathbb{F}$ is a point. The classification of ALE manifolds are given by Kronheimer in his Ph.D thesis [Kro89] and the underlying manifold is minimal resolution of $\mathbb{C}^\Gamma$ where $\Gamma$ is a finite subgroup of SU(2) of type $A_k$, $D_k$, $E_6$, $E_7$, $E_8$.

The next type of gravitational instantons are ALF or asymptotically locally flat manifold where fibre $\mathbb{F}$ is $S^1$ and the last types are ALG where the fibre is $\mathbb{F} = S^1 \times \cdots \times S^1$. We present now two simple examples of fibred boundary $\phi$ manifolds.

**Example 2.1.** Consider Euclidean space $\mathbb{R}^n$ and the Euclidean metric given in polar coordinates $(r, \theta)$, $r \in \mathbb{R}, \theta \in S^{n-1}$, $g_{\text{euc}} = dr^2 + r^2 d\theta^2$. One may compactify Euclidean space by introducing change of variable, $x = \frac{1}{r}$. The metric $g_{\text{euc}}$ takes then the form, $g_{\text{sc}} = \frac{dx^2}{x^4} + \frac{d\theta}{x^2}$, which is a simple example of $\phi$ metric with trivial fibre $\mathbb{F} = \{\text{point}\}$ and base $B = S^{n-1}$.

**Example 2.2.** Consider now $\mathbb{R}^n \times F$ where $F$ is closed Riemannian manifold $(F, g_F)$. Use the product metric $g_{\text{euc}} \oplus g_F$ and introduce change of variable $x = \frac{1}{r}$ in the euclidean metric $dr^2 + r^2 d\theta^2$ as 2.1. One gets the metric, $g = \frac{dx^2}{x^4} + \frac{d\theta}{x^2} + g_F$, which is again example of $\phi$ metric with fibre $\mathbb{F}$ and base $B = S^{n-1}$ and product $S^{n+1} \times F$ at boundary.

After introducing some examples of $\phi$ metrics, one may ask classical analytic or geometric questions on such a setup. As such a question, consider heat equation 1.1 and fundamental solution to it which we denote it as $e^{-tA_{y,z}} (x, x')$ by suppressing $y, z, y', z'$. In order to define the analytic torsion for $\phi$ manifolds, one needs to determine asymptotic of heat kernel in finite time and long time regimes. To that aim, one construct resolution space which are manifold with corners and lift the integral kernel of heat kernel to those space to obtain polyhomogeneous conormal distributions in the sense made precise below. We recall the required definitions which are needed in the section 3 in order to study heat kernel structure. Our main references are [Gri01] and [Mel93] and [Mel96]. We define first the model space.

**Definition 2.3.** (Manifold with corners) One defines,

- A manifold with corners $M$ of dimension $n$ is a Hausdorff topological space which is locally diffeomorph to $\mathbb{R}^k_+ \times \mathbb{R}^{n-k}$.

- For a manifold with corners $M$, the interior $\mathring{M}$ is the set of interior points of $M$ i.e those points with neighborhood diffeomorph to $\mathbb{R}^n$. The
boundary is $M\setminus \mathring{M}$ and is defined as union of connected boundary hypersurfaces. Any intersection of two or more boundary hypersurfaces is called a corner.

- For a boundary hypersurface $H$ one denote $\rho$ as boundary defining function for $H$ if $\rho$ is smooth in interior of $M$ and $\rho = 0$ on $H$ and $d\rho \neq 0$ on $H$.

- $p$ submanifold of manifold with corner is defined to be locally coordinate submanifold.

**Definition 2.4.** (Blow up and coordinates) Assume $P \subset M$ is a $p$ submanifold of manifold with corners $M$. For each $p \in P$ replacing $p$ by inward spherical normal space yields to a space which we call it blown up space and it is denoted by $[M; P]$. The process is called blow up and the map,

$$\beta : [M; P] \longrightarrow M,$$

$$\beta |_{M\setminus \{P\}} := \text{Id}, \quad \beta |_{P} := \text{SN}(p) \mapsto p. \quad \forall p \in P.$$

is called blow down map.

One may introduce coordinates in order to do analyse on blown up space. The coordinates vary from polar coordinate on normal sphere or projective coordinates or as example logarithmic coordinates depending on the nature of problem that we deal with.

On manifold with corners, the main object of study are distributions. In the next definition we characterise polyhomogeneous distributions.

**Definition 2.5.** (Polyhomogeneous function) Assume $M$ is a manifold with corners and $E = E((E_i, H_i))$ is a index family as explained in [Gri01]. A function $u$ defined on interior of $M$ is polyhomogeneous conormal on $M$ with index family $E$ and is denoted as, $u \in A_{phg}^E(M)$ if in any neighborhood coordinates of boundary point $p$, $u(x_1, \cdots x_n) \sim \sum_{i \in E_1} \sum_{(z_i, p_i) \in E_i} a_{z_i, p_i} x_1^{z_1}(\log x_1)^{p_1}$, where $a_{z_i, p_i}$ are polyhomogeneous with respect to index set obtained from intersection of boundary hypersurfaces.

We define now interior conormal singularity.

**Definition 2.6.** Assume $P \subset M$ is a interior $p$-submanifold. A distribution $u$ on $P$ is conormal at $P$ of order $m$ if it is smooth away from $P$ an near $P = \{z = 0\}$ can be expressed locally as,

$$u(y, z) = \int_{\mathbb{R}^{n-k}} e^{iz\cdot \xi} a(y, \xi) d\xi,$$
where $a$ is classical symbol of order $m$. Order $m$ means that,

$$a(y, \zeta) \sim \sum_{j=0}^{\infty} a_{m-j}(y, \frac{\zeta}{|\zeta|})|\zeta|^{m-j},$$

with each coefficient $a_{m-j}$ smooth in $y$ and $\frac{\zeta}{|\zeta|}$. If $P$ intersect the boundary one require $a_{m-j}$ to be polyhomogeneous.

The residue theorem may be applied in order to relate the heat kernel to the resolvent kernel. We recall residue theorem from [Ah178].

**Theorem 2.7.** (Residue theorem) Let $f(z)$ be a complex valued function which is holomorphic except for isolated singularities $\alpha_j$ in a region $\Omega \subset \mathbb{C}$. Then,

$$\frac{1}{2\pi i} \oint_{\gamma} f(z) dz = \sum_j n(\gamma, \alpha_j) \text{Res}_{z=\alpha_j} f(z),$$

where $\gamma$ is a closed path homologous to zero in $\Omega$ and $\gamma$ does not pass through any of the points $\alpha_j$ and $n(\gamma, \alpha_j)$ is winding number of $\gamma$ around $\alpha_j$.

We summarized all instruments which we need in order to study heat kernel on $\phi$ manifolds and determine the asymptotic of renormalized heat trace. In the next section 3, we study the structure theorems of heat kernel in short time and long time regimes.

3. Heat kernel in short time regime and long time regime

We start this section by defining the heat kernel on a compact Riemannian manifold and continue to determine the heat kernel asymptotic on the manifold with fibred boundary, $\phi$ metric in finite time and long time. Using these asymptotics analytic torsion with respect to $\phi$ Hodge Laplacian will be constructed in section 4.

**Definition 3.1.** Assume $(M, g)$ is a closed Riemannian manifold and $\Delta_g^q : \Omega^q(M) \to \Omega^q(M)$ is the Hodge Laplacian acting on the space of $q$ forms. A heat kernel is a function, $H_{g,q} : M \times M \times [0, \infty) \to M$ that satisfies,

- $H_{g,q}(x, y, t)$ is $C^1$ in $t$ and $C^2$ in $(x, y)$.
- $\frac{\partial H_{g,q}}{\partial t} + \Delta_g^q (H_{g,q}) = 0$, where $\Delta_g^q$ is Hodge Laplacian on manifold $(M, g)$.
- $\lim_{t \to 0^+} H_{g,q}(x, y, t) = \delta(x - y)$ where by $\delta(x - y)$ we mean delta distribution.

Then the solution of heat equation with initial condition $u(x, 0) = f(x)$ is given by, $u(x, t) = \int_M H(x, y, t)f(y)dy$.

Recall that for $(M, g)$ smooth closed manifold and $\{\lambda_i\}$ spectrum of $\Delta_g$ on $M$
and $\psi_j$ the associated eigenfunctions one may write,
\[ H(x, y, t) = \sum_i e^{-\lambda_j t} \psi_i(x) \psi_i(y), \]
and the heat trace is defined as,
\[ \text{Tr}H(t) = \int_M H(x, x, t) \, dx = \sum_i e^{-\lambda_i t}. \]
The heat trace has a asymptotic expansion as $t \to 0^+$,
\[ \text{Tr}H(t) \sim t \to 0^+ \left( 4\pi t \right)^{\dim(M)/2} \sum_{j=0}^{\infty} a_j t^j, \]
where $a_j$ are integrals over $M$ of universal homogeneous polynomials in the curvature and its covariant derivatives.

We observe that the heat kernel $H(t, x, x')$ as integral kernel is supported on $M^2 \times \mathbb{R}^+$. We switch now to manifold with fibred boundary endowed with $\phi$ metric $(M, g_\phi)$. One determines asymptotics of heat kernel in short and long time regimes by blow up process.

For finite-time regime this integral kernel lifts on so called heat space to be polyhomogeneous conormal kernel and for long-time regime the statement is to determine polyhomogeneous kernel of resolvent of $\phi$ Hodge Laplacian at low energy level and then residue Theorem 2.7 applies in order to calculate long time heat kernel asymptotics.

**Finite time regime.** In order to determine the behaviour of heat kernel in short time regime, we observe that the heat kernel initially is supported on $M^2 \times \mathbb{R}^+$. One may use resolution process 2.4 to obtain a manifold with corners which we denote it as $HM_\phi$. On $HM_\phi$ the heat kernel lifts to polyhomogeneous conormal distribution in the sense of definition 2.5. One constructs in [TaVe20] the integral kernel of heat equation which lifts to polyhomogeneous conormal distribution on the heat space $HM_\phi$.

The heat space is constructed from $\overline{M}^2 \times \mathbb{R}^+$, by resolution process. Namely, one takes the elliptic $\phi$-space in time and blow up additionally the diagonal of $\overline{M}_\phi^2 \times \mathbb{R}$ at $t = 0$. The space $HM_\phi$ can be visualized as in figure 1, where we denote $\beta_{\phi-h}$ to be the blow down map. The main result of [TaVe20] reads as,

**Theorem 3.2.** [TaVe20](Theorem 7.2) With the same assumptions as in [TaVe20], the fundamental solution of the heat equation, for finite time $t < \infty$,
\[
\partial_t H(t, x, x') + \Delta^\phi_{g_\phi, x} H(t, x, x') = 0, \\
H(t = 0, x, x') = \delta(x - x'),
\]
lifts to polyhomogeneous conormal distribution on $HM_\phi$ with leading asymptotics 0 at $fd$ and $-n$ at $td$ and vanishing to infinite order on other hypersurfaces of $HM_\phi$. Here $n = \dim M$.

Consequently, Theorem 3.2 completely determines behaviour of heat kernel in finite time regime.

**Long time regime.** The relation between heat kernel and resolvent can be read from residue Theorem 2.7. Namely, we may express heat kernel as,

$$H^M_{\phi,q}(t,z,z') = \frac{1}{2\pi i} \int_{\Gamma'} e^{-t\lambda}(\Delta^q_\phi - \lambda)^{-1} d\lambda,$$

(3.1)

Where $\Gamma$ is the curve around the spectrum oriented counter-clockwise and with $(\Delta_\phi - \lambda)^{-1}$ we understand the integral kernel of resolvent $(\Delta_\phi - \lambda)^{-1}$. Note that $\Delta_\phi$ admits positive continuous real spectrum and consequently one visualize $\Gamma'$ as, figure 2. Or with the change of variable $\lambda = -\lambda$ in (3.1) one obtains,
\[
H^M_{\phi,q}(t, z, z') = \frac{1}{2\pi i} \int_{\Gamma} e^{i\lambda(D_{\phi}^q + \lambda)^{-1}} d\lambda,
\]

(3.2)

But now \(\Gamma\) is visualized as figure 3.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure3.png}
\caption{\(\Gamma\)}
\end{figure}

In order to explain the behaviour of heat kernel as \(t \to \infty\) one applies (3.2). In [GTV20] authors constructed the polyhomogeneous integral kernel of resolvent of \(\Delta_{\phi}\) on the space \(M^2_{\lambda,\phi}\) at low energy level i.e as \(\lambda \to 0^+\). The space \(M^2_{\lambda,\phi}\) is manifold with corners and may be illustrated as in figure 4.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure4.png}
\caption{Blowup space \(M^2_{\lambda,\phi}\)}
\end{figure}

The main result of [GTV20], expresses low energy of resolvent of \(\phi\) Hodge Laplacian in terms of polyhomogeneous conormal distribution on \(M^2_{\lambda,\phi}\). The result reads as follows,

**Theorem 3.3.** [GTV20] (Theorem 7.11) Under the same assumptions as in [GTV20], the resolvent \((\Delta_{\phi} + k^2)^{-1}\) as \(k \to 0^+\) is an element of the split calculus (defined as in [GTV20]) where,

\[\begin{align*}
\mathcal{E}_{sc} &\geq 0, \quad \mathcal{E}_{\phi f_0} \geq 0, \quad \mathcal{E}_{bf_0} \geq -2, \quad \mathcal{E}_{lb_0}, \mathcal{E}_{rb_0} > 0, \quad \mathcal{E}_{zf} \geq -2.
\end{align*}\]

The leading terms at \(sc, \phi f_0, bf_0\) and \(zf\) are of order 0, 0, –2, –2.

**Remark 3.4.** Theorem 3.3 is obtained in parallel work [KoRo20].
We apply (3.2) and determine the asymptotics of heat kernel in large time regime by integration over path $\Gamma$. One parametrize $\Gamma$ as, For $\frac{\pi}{4} < \varphi < \pi$ and $a \in \mathbb{R}^+$ fixed, $\Gamma$ splits in two paths $\Gamma_{1,a}$ and $\Gamma_{2,a}$ where $\Gamma_{1,a}$ and $\Gamma_{2,a}$ are parametrised as,

$$
\Gamma_{1,a} = ae^{i\theta}, \quad -\varphi \leq \theta \leq \varphi,
\Gamma_{2,a} = re^{i\theta}, \quad a \leq r < \infty.
$$

(3.3)

Assume that $R(\lambda, z, z')$ is the Schwartz kernel of resolvent $(\Delta_{\phi} + e^{i\theta}\lambda)^{-1}, \lambda > 0$. Recall that in short time regime we used $\tau := t^{\frac{1}{2}}$. Consequently fix $\omega = \tau^{-1}$ and $a = \omega^2$. $\omega \to 0$ corresponds to $t \to \infty$ and the resolvent at low energy i.e as $a \to 0^+$. We may split (3.2) into two parts and write, (by suppressing $x, y, x', y'$),

$$
H(\omega^{-2}, z, z') = \frac{1}{2\pi i} \int_{\Gamma} e^{\lambda/\omega} R(\lambda, z, z') d\lambda
= \frac{1}{2\pi i} \left( \int_{\Gamma_{1,\omega^2}} e^{\lambda/\omega} R(\lambda, z, z') d\lambda + \int_{\Gamma_{2,\omega^2}} e^{\lambda/\omega} R(\lambda, z, z') d\lambda \right).
$$

Integration on $\Gamma_{1,\omega^2}$. One may use, $\lambda = \omega^2 e^{i\theta}, -\varphi \leq \theta \leq \varphi$, and therefore,

$$
\frac{1}{2\pi i} \int_{\Gamma_{1,\omega^2}} e^{\lambda/\omega} R(\lambda, z, z') d\lambda = \frac{\omega^2}{2\pi} \int_{-\varphi}^{\varphi} e^{i\theta} R(\theta, \omega, z, z') d\theta.
$$

(3.4)

Firstly we observed that the integral converges by the boundedness of integrand with respect to $\theta$. By Theorem 3.3 for each fixed $\theta$, the integrand in (3.4) is polyhomogeneous on $M_{\omega,\phi}$ with conormal singularity at diagonal $\Delta_{\phi}$ and all coefficients depend smoothly on $\theta \in [-\varphi, \varphi]$. Consequently the integral (3.4) is polyhomogeneous on $M_{\omega,\phi}$.

The index set of $H(\omega^{-2}, z, z')$ is computed from index sets $R(\theta, \omega, z, z')$ and from $\omega^2$, i.e as $\omega$ is boundary defining function for faces $zf, bf_0, \phi f_0, lb_0, rb_0$, on the faces $zf, bf_0, \phi f_0, lb_0, rb_0$ we add $+2$ to the index sets of the resolvent Theorem 3.3 and the index sets on other faces remain the same as those index sets of $R(\theta, \omega, z, z')$.

Integration on $\Gamma_{2,\omega^2}$. The path $\Gamma_{2,\omega^2}$ consists of two rays at angels $\varphi$ and $-\varphi$. The integration over $\Gamma_{2,\omega^2}$ refers therefore to the integration over these two rays. We claim for convergence and polyhomogeneity of integration along $\varphi$ ray. The argument for integration at $-\varphi$ ray is the same. Parametrising $\lambda = re^{i\varphi}$, we express the integral over $\Gamma_{2,\omega^2}$ to be,

$$
\int_{\Gamma_{2,\omega^2}} e^{\lambda/\omega} R(\lambda, z, z') d\lambda = \int_0^{\infty} e^{\omega^{-2} r \varphi} R(r, z, z') dr,
$$

(3.5)
And use change of variable \( s = \sqrt{r} \), \( dr = 2sds \) in (3.5) to write,
\[
\int_{\Gamma_{2,\omega}} e^{i\omega r} R(\lambda, z, z') d\lambda = \int_{\omega}^{\infty} 2se^{i(\cos \phi)\frac{s^2}{2}} e^{i(\sin \phi)\frac{s^2}{2}} R(s^2, z, z') ds. \tag{3.6}
\]

As \( \frac{\pi}{2} < \phi < \pi \) is fixed, \( \cos(\varphi) < 0 \) and therefore for \( s \to +\infty \) the \( e^{i(\cos \phi)\frac{s^2}{2}} \) and consequently entire integrand decays to infinite order at \( s = \infty \), which means that the integral (3.6) converges. In order to analyse the polyhomogeneity of (3.6), we split \( R(s^2, z, z') \) into two pieces. Namely we use partition of unity and write, \( R = R_D + R_C \), where \( R_D \) is supported in a neighborhood of \( \Delta_{\Phi} \) and \( R_C \) is supported away from \( \Delta_{\Phi} \). Recall that the diagonal \( \Delta_{\Phi} \) intersects the \( sc \), \( \Phi f_0 \) and \( zf \) faces of \( M_{2,\Phi}^2 \). Accordingly we may split \( R_D \) into three pieces and write, \( R_D = R_1 + R_2 + R_3 \). \( R_1 \) is supported away from \( sc \) and near \( zf \) and \( R_2 \) is supported away from \( zf \) and near \( sc \) and \( R_3 \) is supported in interior of \( bf_0 \) near diagonal of \( \Phi f_0 \). We argue for the polyhomogeneity of integral (3.6) for each of \( R_i, i = 1, 2, 3 \). Compare to figure 5.

**Figure 5. Diagonal of \( M_{2,\Phi}^2 \)**

**Polyhomogeneity of \( R_1 \).** On the support of \( R_1 \) near \( zf \), away from \( x \neq 0 \), we may use projective coordinates, \( \hat{X} = \frac{x-x'}{x}, \hat{Y} = \frac{y-y'}{x}, \hat{Z} = \frac{z-z'}{x}, \mu = \frac{z}{x} \). By definition of conormal singularity along diagonal \( \Delta_{\Phi} \), we may express,
\[
R_1 \sim \int_{\mathbb{R}^n} e^{i(X, Y, Z)|\zeta_1, \zeta_2, \zeta_3|} \sum_{j=0}^{\infty} a_j(\frac{s}{x}, x, y, z, \frac{\zeta}{|\zeta|}|\zeta|^2-j) d\zeta, \tag{3.7}
\]
where by ~ we mean that $\mathcal{R}_1$ may be expressed locally by (3.7) on $M_{s,\phi}^2$ up to smooth function. Thus we may absorb the smooth reminder into $\mathcal{R}_C$ and plug (3.7) into (3.6) to obtain,

$$\int_{\mathbb{R}^n} e^{i(X,Y,Z)(\zeta_1,\zeta_2,\zeta_3)} \sum_{j=0}^{\infty} \left( \int_{\omega} 2se^{i\omega} a_j(\frac{s}{\lambda},x,y,z,\zeta) |\zeta|^{j-1} d\zeta ds \right).$$

From definition of conormal singularity, one has that the coefficients $a_j$ are polyhomogeneous in $x$ and $\frac{s}{\lambda}$ with index set independent from $j$ and $a_j$ is smooth in $(y,z,\zeta,|\zeta|)$. Observe that the pullback of $a_j$ via projection to $M_{\omega}^3(s,\omega,x) \times N_y \times N_f \times S^{n-1}$, is polyhomogeneous conormal with index set independent of $j$. Therefore, $2se^{i\omega} e^{i\omega} a_j(\frac{s}{\lambda},x,y,z,\zeta,|\zeta|)$ is polyhomogeneous conormal on $M_{\omega}^3(s,\omega,x) \times N_y \times N_f \times S^{n-1}$ and it admits cut off singularity at $\frac{s}{\lambda} = 1$. The integrand has infinite decay in $s$ due to the fact that $\cos \varphi < 0$ and therefore integration is well defined. Integration in $s$ corresponds to a projection map,

$$M_{\omega}^3(s,\omega,x) \rightarrow M_{\omega}^2(\omega,x),$$

which is b-fibration [Mel96] by the following lemma from [MeSi08].

**Lemma 3.5.** If $m < n$ each of the projections off $n - m$ factors of $X, \pi: X^n \rightarrow X^m$, fixes a unique b-stretched projection $\pi_b$ giving a commutative diagram,

$$X^n_b \rightarrow X^m_b \quad X^n \rightarrow X^m$$

and furthermore $\pi_b$ is b-fibration.

Extension to $N_y \times N_z \times S^{n-1}$ by $(y,z,\zeta,|\zeta|)$ does not change the b-fibration property of this map and consequently by pushforward theorem of Melrose [Mel93],

$$\int_{\omega} 2se^{i\omega} e^{i\omega} a_j(\frac{s}{\lambda},x,y,z,\zeta,|\zeta|) ds,$$

is polyhomogeneous conormal on $M_{\omega}^2(\omega,x) \times N_y \times N_z \times S^{n-1}$, with index set independent of $j$ and consequently on $M_{\omega,\varphi}^2$.

**Polyhomogeneity of $\mathcal{R}_2$.** We need to prove the polyhomogeneity of the integration of kernel $\mathcal{R}_2$ along diagonal $\Delta_\varphi \cap \text{sc}$ of $M_{s,\phi}^2$.

$\mathcal{R}_2$ is supported near sc face of $M_{s,\phi}^2$. The coordinates we employ are in the form, $X = s(\frac{1}{\lambda} - \frac{1}{\lambda'})$, $Y = \frac{s}{\lambda}(y - y')$, $\mu = \frac{s}{\lambda}, s, Z = \frac{s}{\lambda}(z - z')$. The definition of conormality implies that,
\[ \mathcal{R}_2 \sim \int_{\mathbb{R}^n} e^{i [X,Y,Z](\zeta_1,\zeta_2,\zeta_3)} \sum_{j=0}^{\infty} b_j \left( \frac{x}{s}, s, y, z, \frac{z}{|\zeta|} \right) |\zeta|^{2j} \, \text{d}\zeta. \]  

(3.9)

Where \( b_j \) are polyhomogeneous conormal in \( \frac{x}{s} \) and \( s \) with index sets independent of \( j \) and \( b_j \) smoothly depend on \( (y, z, \frac{z}{s}) \). We plug (3.9) into (3.6) and analyse the integral by arguing in two different regimes which correspond to \( \omega > \frac{x}{s} \) and \( \omega < \frac{x}{s} \).

First we assume that \( \omega > \frac{x}{s} \) and we introduce projective coordinates, on sc face of \( \mathcal{M}^2_{\omega,\phi} \), \( \tilde{X} = \omega \left( \frac{1}{x} - \frac{1}{x'} \right), \tilde{Y} = \frac{\omega}{x}(y - y'), \tilde{Z} = \frac{\omega}{x}(z - z'), \tilde{X} = \frac{x}{\omega}. \) We need expression of conormal singularity at, \( \tilde{X}, \tilde{Y}, \tilde{Z} \) and hence all coefficients are polyhomogeneous conormal on \( \mathcal{M}^2_{\omega,\phi} \) and we introduce projective coordinates, on sc face of \( \mathcal{M}^2_{\omega,\phi} \) and take new variable \( \eta \) which is defined as, \( \eta = (\eta_1, \eta_2, \eta_3) = (\frac{\omega}{x} \zeta_1, \frac{\omega}{x} \zeta_2, \frac{\omega}{x} \zeta_3) \). We plug (3.9) in (3.6) and employ new variables \( \tilde{X}, \tilde{Y}, \tilde{Z} \). Interchanging the integral and the sum to obtain,

\[ \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i [\tilde{X},\tilde{Y},\tilde{Z}](\eta_1,\eta_2,\eta_3)} \sum_{j=0}^{\infty} 2s e^{i \frac{\omega}{x} j} e^{i \phi} b_j \left( \frac{s}{\omega}, s, y, z, \frac{n}{|\eta|} \right) \left( \frac{s}{\omega} \right)^{1 - 2 - (n)} |\eta|^{2j} \, ds \, d\eta. \]

We argue that, \( \int_{\mathbb{R}^n} 2s e^{i \frac{\omega}{x} j} e^{i \phi} b_j \left( \frac{s}{\omega}, s, y, z, \frac{n}{|\eta|} \right) \left( \frac{s}{\omega} \right)^{1 - 2 - (n)} \, ds \), is polyhomogeneous conormal in \( \frac{s}{\omega}, \omega \) independent of \( j \). The argument is similar to the last step as the integrand decays to infinite order at \( s = \infty \) and is polyhomogeneous conormal in \( \frac{s}{\omega}, s \) independent of \( j \). Integration with respect to \( s \) yields the polyhomogeneity on \( \mathcal{X}^2_{b}(x, \omega) \times \mathcal{N}_y \times \mathcal{N}_l \times \mathbb{S}_m^{n-1} \) by Melrose pushforward Theorem.

The second region corresponds to \( \omega < \frac{x}{s} \). Here one use coordinates, \( \hat{X} = \frac{x-x'}{x}, \hat{Y} = y - y', \hat{Z} = z - z', \frac{x}{s}, x, \) and one expects that conormal singularities arise on, \( \hat{X} = \hat{Y} = \hat{Z} = 0 \). One notes that \( (X, Y, Z) = (\frac{x}{s}) (\hat{X}, \hat{Y}, \hat{Z}). \) One introduce variables \( (\zeta_1', \zeta_2', \zeta_3') = \frac{x}{s} (\zeta_1, \zeta_2, \zeta_3) \) and plug these variables in (3.9) to obtain,

\[ \int_{\mathbb{R}^n} e^{i [\hat{X},\hat{Y},\hat{Z}](\zeta_1',\zeta_2',\zeta_3')} \sum_{j=0}^{\infty} \int_{\mathbb{R}^n} 2s e^{i \frac{\omega}{x} j} e^{i \phi} b_j \left( \frac{s}{\omega}, s, y, z, \frac{n}{|\eta|} \right) \left( \frac{s}{\omega} \right)^{1 - 2 - (n)} ds |\zeta'|^{2j-1} \, d\zeta'. \]

The \( j \)-coefficient may be written as,

\[ \left( \frac{\omega}{x} \right)^{1 - 2 - n} \int_{\mathbb{R}^n} 2s e^{i \frac{\omega}{x} j} e^{i \phi} b_j \left( \frac{s}{\omega}, s, y, z, \frac{n}{|\eta|} \right) \left( \frac{s}{\omega} \right)^{1 - 2 - n} ds, \]

(3.10)

which is \( \left( \frac{\omega}{x} \right)^{1 - 2 - n} \) times (3.7). The \( \left( \frac{\omega}{x} \right)^{1 - 2 - n} \) is polyhomogeneous conormal on \( \mathcal{M}^2_{\omega,\phi}(x, \omega) \) and consequently (3.10) is polyhomogeneous conormal on \( \mathcal{M}^2_{\omega}(x, \omega) \) for each \( j \). As \( \omega < \frac{x}{s} \) by increasing \( j \) the order of polyhomogeneity increase and hence all coefficients are polyhomogeneous conormal on \( \mathcal{M}^2_{\omega}(x, \omega) \) with respect to the index set of the \( j = 0 \) coefficient.

We conclude that the integration on \( \mathcal{R}_2 \) yields to the polyhomogeneity of (3.9) on \( \mathcal{M}^2_{\omega,\phi} \).

### ANALYTIC TORSION

13
Polyhomogeneity of $\mathcal{R}_3$. The polyhomogeneity integration (3.5) with respect to $\mathcal{R}_3$ follows from the fact that $\mathcal{R}_3$ is supported in compact region namely on diagonal of $\phi f_0$, and by conormality (2.6) we may express by adequate local coordinates on $\phi f_0$,

$$\mathcal{R}_3 \sim \int_{\mathbb{R}^n} e^{i(z-z')\eta} \sum_{j=0}^{\infty} c_j(s, z, \frac{\eta}{|\eta|})|\eta|^{2-j} d\eta,$$

where $c_j$ are polyhomogeneous conormal at $s = 0$ and $s = \infty$, with index sets independent of $j$ at $s = 0$ and $s = \infty$. We plug (3.11) into (3.6) and we get,

$$\int_{\mathbb{R}^n} e^{i(z-z')\eta} \sum_{j=0}^{\infty} \left( \int_{\omega} 2se^{i\frac{\eta}{|\eta|}^2} e^{i\phi} c_j(s, z, \frac{\eta}{|\eta|}) ds \right) |\eta|^{2-j} d\eta. \tag{3.12}$$

Note that the $j$-th coefficient, $\int_{\omega} 2se^{i\frac{\eta}{|\eta|}^2} e^{i\phi} c_j(s, z, \frac{\eta}{|\eta|}) ds$, is polyhomogeneous conormal on $M_s^2(s, \omega)$ with index sets independent of $j, z, \frac{\eta}{|\eta|}$ with infinite decay at $s = \infty$. Consequently (3.12) is polyhomogeneous on $M_\omega^2, \phi$.

Polyhomogeneity of $\mathcal{R}_C$. Now we argue the polyhomogeneity of $\mathcal{R}_C$ term. Explicitly one may write, by suppressing $(x, y, z)$ in $z$ and $(x', y', z')$ in $z'$,

$$\int_{\omega} 2se^{i\frac{\eta}{|\eta|}^2} e^{i\phi} \mathcal{R}_s(s^2, z, z') ds.$$ By assumption $\mathcal{R}_s(s^2, x, y, z, x', y', z')$ is polyhomogeneous conormal on $M^2_{s, \phi}$ and smooth across the diagonal $\Delta_{s, \phi}$. We apply the following lemma parallel to [She13] (Lemma 10) to the integral kernel,

$$\mathcal{R}_s(s^2, x, y, z, x', y', z'). \tag{3.13}$$

to conclude that (3.13) is polyhomogeneous on $M^2_{\omega, \phi}$.

Lemma 3.6. Assume $T(k, z, z')$ be a function which is polyhomogeneous conormal on $M^2_{k, \phi}$ and smooth in the interior and decaying to infinite order at lb, rb, and bf. Then,

$$\int_{\omega} 2ke^{i\frac{\eta}{|\eta|}^2} e^{i\phi} T(k, z, z') dk,$$

is polyhomogeneous on $M^2_{\omega, \phi}$ for $\omega$ bounded from below.

Proof. In the following proof we use explicit local coordinates in each region of $M^2_{k, \phi}$ in order to plug the polyhomogeneity expression of $T(k, z, z')$ and evaluate directly the integral (3.14) to show that the resulting function is polyhomogeneous on $M^2_{\omega, \phi}$. 
Near \( bf \cap bf_0 \cap lb \). We may use coordinates, \((\zeta = \frac{x}{x_0}, s = \frac{s'}{s_0'}, k)\) in this region and the polyhomogeneity of \( T(k, x, x') \) with respect to these coordinates becomes \( T \sim \sum a_{ijl} \zeta^i s^j k^l \). We plug this expression into (3.14) and use change of variable \( \frac{\zeta}{\omega} = t \) and obtain,

\[
\int_{\infty}^{0} 2ke^{\frac{t}{2}\zeta} e^{i\phi} T(k, x, x') dk = \sum_{i} \int_{0}^{\infty} 2ke^{\frac{t}{2}\zeta} e^{i\phi} a_{ijl} k^{i-1} x^l s^j dk =
\]

\[
\sum a_{ijl} 2x^l s^j \int_{1}^{\infty} \omega e^{t-\zeta} (\omega t)^{l-i} \omega dt = \sum_{2} a_{ijl} x^l s^j \omega^{l-2} \int_{1}^{\infty} e^{-t^2} t^{l-i+1} dt
\]

\[
\sim \sum a_{ijl} (\frac{x}{\omega})^l s^j \omega^{l+2},
\]

i.e the integral is polyhomogeneous with respect to coordinates \((\frac{\zeta}{\omega}, s, \omega)\) that corresponds to region \( bf \cap bf_0 \cap lb \) of \( M^2_{\psi, \omega} \). As the argument is symmetric near \( bf \cap bf_0 \cap rb \) we leave the proof.

Near \( lb \cap lb_0 \cap bf_0 \). In this region, we may use coordinates \((x, s' = \frac{x'}{x_0'}, k = \frac{k}{k_0})\). Plugging the definition of polyhomogeneity of \( T \sim \sum a_{ijl} x^j s^i k^l \) into integral (3.14) and using change of variable \( \frac{\zeta}{\omega} = t \) yields to,

\[
\int_{\infty}^{0} 2ke^{\frac{t}{2}\zeta} e^{i\phi} T(k, x, x') dk = \sum_{i} \int_{0}^{\infty} 2ke^{\frac{t}{2}\zeta} e^{i\phi} a_{ijl} s^{j} k^{l} dk =
\]

\[
\sum a_{ijl} 2x^l s^j \int_{1}^{\infty} \omega e^{t-\zeta} (\omega t)^{j-i} \omega dt = \sum_{2} a_{ijl} x^l s^j \omega^{l-2} \int_{1}^{\infty} e^{-t^2} t^{j-i+1} dt
\]

\[
\sim \sum a_{ijl} x^l s^j (\frac{\omega}{x})^l \omega^{j+1},
\]

which means the integral is polyhomogeneous with respect to coordinates \((x, \frac{x'}{x_0}, s_0', \omega)\). That means the polyhomogeneity on the region \( lb \cap lb_0 \cap bf_0 \). The similar argument shows also the polyhomogeneity of integral on the region \( rb \cap rb_0 \cap bf_0 \).

Near \( sc \cap bf_0 \). We use the coordinates \((S = \frac{k(x-x')}{x_0}, S' = \frac{x'}{k_0}, k)\) and the polyhomogeneity of \( T(k, x, x') \) with respect to these coordinates \( T \sim \sum a_{ijl} S^j S' k^l \) in the integral (3.14), and use change of variable \( \frac{k}{\omega} = t \),

\[
\int_{\infty}^{0} 2ke^{\frac{t}{2}\zeta} e^{i\phi} T(k, x, x') dk = \sum_{i} \int_{0}^{\infty} 2ke^{\frac{t}{2}\zeta} e^{i\phi} a_{ijl} S^j S' k^l dk =
\]

\[
\sum a_{ijl} 2(S-x') t^{i-1} x^{j-1} \int_{1}^{\infty} \omega^{i-j+1} t^{l-i+1} e^{-t^2} e^{i\phi} (\omega t)^{l-i+1} \omega dt =
\]

\[
\sum 2a_{ijl} S^j S' \omega^{l-2} \int_{1}^{\infty} e^{-t^2} t^{l-i-1} dt \sim \sum a_{ijl} S^j S' \omega^{l+1},
\]
which yields to polyhomogeneity of integral in the region \( \text{sc} \cap \text{bf}_0 \) of \( M^2_{\omega, \phi} \). Similar argument shows the polyhomogeneity near the face \( \phi f_0 \) of \( M^2_{\omega, \phi} \) as well.

In the following theorem, we summarize the polyhomogeneity of the heat kernel in long time regime.

**Theorem 3.7.** The heat kernel which is given by (3.2), i.e.

\[
H^M(t, x, x') = \frac{1}{2\pi i} \int_\Gamma e^{t\lambda}(\Delta_\phi + \lambda)^{-1} d\lambda,
\]

is polyhomogeneous conormal at \( t = \omega^{-\frac{1}{2}} \) at \( \omega \rightarrow 0 \) on \( M^2_{\omega, \phi} \) with index sets given in terms of index sets of resolvent \( (\Delta_\phi + \lambda)^{-1} \) at low energy level. More explicitly the asymptotics of heat kernel in long time regime are of leading order 0 at \( \text{sc} \) face and of order 0 at \( \text{zf} \) and \( \text{bf}_0 \) faces. More over the leading order at the face \( \phi f_0 \) is 2. In long time regime the heat kernel vanishes to infinite order at \( \text{lb}, \text{rb}, \) and \( \text{bf}_0 \) faces of \( M^2_{\omega, \phi} \).

The explicit index sets are as follows,

\[
E_{\text{sc}} \geq 0, E_{\phi f_0} \geq 2, E_{\text{bf}_0} \geq 0, E_{\text{lb}}, E_{\text{rb}} > 0, E_{\text{zf}} \geq 0.
\]

4. Analytic Torsion

For \((M, g)\) compact oriented \( C^\infty\) Riemannian manifold of dimension \( \text{dim}(M) = n \) and \( \Delta^q \) Hodge Laplacian acting on \( \Omega^q(M) \), let

\[
0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq +\infty,
\]

be the sequence of eigenvalues of \( \Delta^q \). The zeta function \( \Delta^q \) is then defined by,

\[
\zeta_q(s) = \sum_{\lambda_i > 0} \lambda_i^{-s},
\]

and it turns out that the zeta function converges in the half plane \( \text{Re}(s) > \frac{n}{2} \). One can explicitly express the zeta function by integration of heat kernel along diagonal. Namely for \( \text{Re}(s) > \frac{n}{2} \),

\[
\zeta_{M, q}(s) = \frac{1}{\Gamma(s)} \int_0^\infty \text{Tr}(H^M(t))t^{s-1} dt.
\]

\( \zeta_{M, q}(s) \) admits holomorphic continuation to \( \mathbb{C} \) and especially with regular value at 0 and one may define the determinant of Hodge Laplacian as \( \exp\left(-\frac{d}{ds}\zeta_{M, q}(s)|_{s=0}\right) \) and analytic torsion is defined to be,

\[
\text{Log} T_M = \frac{1}{2} \sum_{q=0}^n (-1)^q q \frac{d}{ds} \zeta_{M, q}(s) \big|_{s=0}.
\]

Consider now \( \phi \) metric, \( g_\phi = \frac{dx^2}{x^2} + \frac{d\phi^2}{x^2} + g_r + O(x) \). One observe that the integration along diagonal of heat kernel diverges as \( x \rightarrow 0 \). The way to
overcome to this problem is to define renormalization. In this section, we describe two ways of renormalization. Renormalization together with structure Theorems of heat kernel 3.2, 3.7 give rise to the definition of renormalized heat trace and renormalized zeta function. We show that the renormalized zeta function admits meromorphic continuation to $\mathbb{C}$. We define determinant of Laplacian and renormalized zeta function in the setup of fibred boundary manifold $(M, g_\phi)$.

4.1. **Renormalized heat trace.** We follow [AAR13] in order to define two ways of renormalization.

**Riesz renormalization.** The first method of renormalization is called the renormalization in the sense of Riesz. Assume $M$ is a manifold with boundary and $x$ is a boundary defining function. Assume further that $f$ admits an asymptotic expansion in terms of $x$ and $\log x$, i.e,

$$f \sim \sum a_{s,p} x^s (\log(x))^p \tag{4.1}$$

Assume that $\mu$ is smooth non-vanishing density on $M$. The expansion of $f$ (4.1) implies the meromorphic continuation of the function, $z \in \mathbb{C} \mapsto \int_M x^z f d\mu$, for $\text{Re}(z) > C$. Now the Riesz renormalized integral of $f$ is defined to be,

$$\int_M f d\mu = \text{FP} \int_M x^z f d\mu.$$ 

**Hadamard renormalization.** The second method is due to Hadamard and is referred in [Mel93] as $b$-integral. Assume $M$ is manifold with boundary and $x$ is boundary defining function. One may define,

$$\epsilon \mapsto \int_{x \geq \epsilon} f d\mu, \tag{4.2}$$

and show that (4.2) has asymptotic expansion as $\epsilon \to 0$ when $f$ admits expansion (4.1). We define then $\int_M f d\mu = \text{FP} \int_{\epsilon = 0}^{x \geq \epsilon} f d\mu$, where by finite part we mean taking out the divergent part.

**Heat kernel renormalization in the sense of Hadamard.**

**Definition 4.1.** Assume $\overline{M}$ is a fibred boundary manifold and assume further that $H^M_\phi(x, y, z, x', y', z')$ is the heat kernel. One defines the renormalized heat trace to be,

$$\mathcal{R} \text{Tr}(H^M_\phi(t)) = \int_M H^M_\phi(t, x, y, z, x, y, z) d\text{vol}_\phi(x, y, z).$$
Equivalently one may take a sharp cutoff function $\chi(r)$ which is supported on $[0,1]$ and is equal to 1 for $0 \leq r \leq 1$. For fix $\varepsilon > 0$ consider $\chi(\frac{\varepsilon}{\varepsilon})$. The renormalized heat trace is defined as,

$$\text{RTr}(H^M_{\phi}(t)) = \text{FP}_{\varepsilon=0} \int_M \chi(\frac{\varepsilon}{\varepsilon})H^M_{\phi}(t,x,y,z,x,y,z) d\text{vol}_\phi(x,y,z).$$

(4.3)

it needs to be justified that (4.3) make sense. We demonstrate to that end that the integrand in (4.3) is polyhomogeneous in $\varepsilon$ for fixed $t$.

**Lemma 4.2.** Assume $(\mathcal{M}, g_\phi)$ is a fibred boundary manifold and $\Delta^q_\phi$ is the Hodge Laplacian acting on the space of $q$ forms. The corresponding heat kernel is polyhomogeneous conormal along diagonal $(x,y,z;x,y,z)$ in both short time regime and long time regime. Precisely,

- For $t$ bounded above, $H^M_{\phi}(t,x,y,z,x,y,z)$, is polyhomogeneous conormal in $(\tau = \sqrt{t}, x)$ with smooth dependence on $y$ and $z$.
- For $t$ bounded below and $\omega = t^{-\frac{1}{2}}$, $H^M_{\phi}(t,x,y,z,x,y,z)$ is polyhomogeneous conormal as a function of $(\omega,x)$ on b-space $(\mathbb{R}_+ (\omega) \times \mathcal{M})_b$ with smooth dependence on $y,z$.

**Proof.** The lemma follows from the structure theorems of heat kernel in last section i.e from theorems 3.2 and 3.7.

**Theorem 4.3.** The renormalized heat trace defined by (4.3) is well defined the integrand admits expansion in $\varepsilon$ for each fixed $t$. Moreover the finite part at $\varepsilon = 0$ has leading asymptotics at $t = 0$ and $t = \infty$. More precisely for coefficients $a_i$ and $b_{jl}$,

$$\text{RTr}(H^M_{\phi}(t)) \sim_{t \to 0} \sum_{j \geq 0} a_j t^{-m+\frac{j}{2}},$$

(4.4)

$$\text{RTr}(H^M_{\phi}(t)) \sim_{t \to \infty} \sum_{j \geq 0} \sum_{l=0}^{p_l} b_{jl} t^{-\frac{j}{2} \log^l(t)}.$$  

(4.5)

**Proof.** Consider the integral,

$$\int_M \chi\left(\frac{\varepsilon}{\varepsilon}\right)H^M_{\phi}(t,x,y,z,x,y,z) d\text{vol}_\phi(x,y,z).$$

(4.6)

For fixed $t$ and $\varepsilon$ the integrand is polyhomogeneous on $(\mathbb{R} \times M_\phi)_b \times \mathbb{R}^+_t$ for short time and $(\mathbb{R}_\varepsilon \times M_\phi \times \mathbb{R}^+_t)_b$ in long time. The projection $\pi_\phi$ lift to b-fibrations,

$$\Pi_{k,b} : (\mathbb{R} \times M_\phi)_b \times \mathbb{R}^+_t \longrightarrow M_\phi \times \mathbb{R}^+_t,$$

$$\Pi_{k,b2} : (\mathbb{R}_\varepsilon \times M_\phi \times \mathbb{R}^+_t)_b \longrightarrow (M_\phi \times \mathbb{R}^+_t)_b.$$
Integration in $x$ corresponds to pushforward under $\Pi_{x,b}$ (short time) and $\Pi_{x,b_2}$ (long time), the polyhomogeneity of integral with respect to $\epsilon$ follows from Melrose push forward theorem.

**Asymptotic in (4.4).** By Lemma 4.2 one can express the diagonal of heat kernel in short time regime, for $N \in \mathbb{N}$ as,

$$\text{Tr}_x H^M_\phi(t, x) = \sum_{j=0}^{N} x^j a_j(t) + H_N(x, t), \quad (4.7)$$

where $a_j(t) \sim t^{-\frac{n}{2} - j}$ and $H_N(x, t) = O(t^{\epsilon N+1})$. As $\epsilon \to 0$ we may plug (4.7) into (4.6) and evaluate the integral to obtain (4.4).

**Asymptotic in (4.5).** The diagonal of heat kernel in long time regime is of leading order 0 at $\text{sc}$ and 2 at $\phi f_0$ and 0 at $zf$ face of $M^2_{\omega, \phi}$. As $\epsilon \to 0$ one may express explicit diagonal of heat kernel i.e Lemma 4.2 at $\omega^{-\frac{1}{2}} = t = \infty$. On $\phi f_0$ face one obtain,

$$\text{Tr}_x H^M_\phi(x, \omega) \sim \sum h_i(x, \omega),$$

for $h_i(x, \omega)$ homogeneous of order 2. Similarly on $zf$ face we have,

$$\text{Tr}_x H^M_\phi(x, \omega) \sim \sum x^j.$$

Plugging these expressions into (4.6) with respect to $\phi$-volume form $d\text{vol}_\phi = x^{b-2} dx dy dz$ we obtain (4.5).

Consider formally,

$$\frac{1}{\Gamma(s)} \int_0^\infty \text{Tr}(H^k_{M, \phi})(t) t^{s-1} dt. \quad (4.8)$$

Apriori (4.8) is not defined for any $s \in \mathbb{C}$. By breaking (4.8) at some constant $c$ we may express (4.8) as sum of two integrals. By (4.4) the integral,

$$\int_0^c \text{Tr}(H^k_{M, \phi})(t) t^{s-1} dt, \quad (4.9)$$

is defined for $\text{Re}(s) > \frac{1}{2}$. Each summand can directly be evaluated to show that (4.9) admits meromorphic extension to complex plane $\mathbb{C}$. The second integral is denoted as $\int_c^\infty \text{Tr}(H^k_{M, \phi})(t) t^{s-1} dt. \quad (4.10)$

One can apply (4.5) and show that the integral converges for $\text{Re}(s) < 0$ and by evaluating directly (4.10) the meromorphic extension to complex plane $\mathbb{C}$ follows.
Definition 4.4. Assume $(\overline{M}, g_\phi)$ is fibred boundary manifold and $\Delta_\phi^k$ is the Hodge Laplacian acting on the space of $k$ forms. One defines,

- The renormalized zeta function on $(\overline{M}, g_\phi)$ at degree $k$, denoted as $\zeta_{M,\phi}^k(s)$ is defined to be,
  \[ R_\zeta_{M,\phi}^k(s) := \zeta_{M,\phi}^k(s) + \zeta_{M,\phi}^k(s) \]

- The renormalized determinant of the Laplacian on $\overline{M}$ is denoted as
  \[ -\frac{d}{ds} \zeta_{M,\phi}^k(s) \big|_{s=0}, \] where $\frac{d}{ds} \zeta_{M,\phi}^k(s) \big|_{s=0}$ is the coefficient of $s$ in the Laurent series for $\zeta_{M,\phi}^k(s)$ at $s = 0$.

One may define renormalized analytic torsion on fibred boundary manifold as,

Definition 4.5. For $(\overline{M}, g_\phi)$ fibred boundary manifold, denote $\Delta_\phi^k$ to be Hodge Laplacian acting on the space of $k$ forms. One may define the renormalized analytic torsion by,

\[ \text{Log} R T_{M,g_\phi} := \frac{1}{2} \sum_{q=0}^{n} (-1)^q q \frac{d}{ds} \zeta_{M,\phi}^q(s) \big|_{s=0}. \]

We conclude the discussion of this section pointing out that the renormalized analytic torsion as defined in 4.5 may be studied further and one may ask for the statement similar to Cheeger Müller Theorem in the fibred boundary manifold setup.

Open problem 4.6. (Cheeger Müller in the set up of $\phi$ manifolds)

1. Define the topological torsion in the set up of manifold with fibred boundary. Is this trivial extension from closed manifolds?
2. Prove Cheeger Müller type statement.

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