THE AFFINE ENSEMBLE: DETERMINANTAL POINT PROCESSES ASSOCIATED WITH THE \( ax+b \) GROUP

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Abstract. We introduce the affine ensemble, a class of determinantal point processes (DPP) in the half-plane \( \mathbb{C}^+ \) associated with the \( ax+b \) (affine) group, depending on an admissible Hardy function \( \psi \). We obtain the asymptotic behavior of the variance, the exact value of the asymptotic constant, and non-asymptotic upper and lower bounds for the variance on a compact set \( \Omega \subset \mathbb{C}^+ \). As a special case one recovers the DPP related to the weighted Bergman kernel. When \( \psi \) is chosen within a finite family whose Fourier transform are Laguerre functions, we obtain the DPP associated to hyperbolic Landau levels, the eigenspaces of the finite spectrum of the Maass Laplacian with a magnetic field.

1. Introduction

Determinantal point processes (DPPs) are random point distributions with negative correlations between points determined by the reproducing kernel of some Hilbert space, usually called the correlation kernel. Because of the repulsion inherent of the model, DPPs are convenient to describe physical systems with charged-like particles, to distribute random sequences of points in selected regions while avoiding clustering, to promote diversity in selection algorithms for machine learning [20], or to improve the rate of convergence in Monte Carlo methods [8]. DPPs have been introduced by Odile Macchi to model fermion distributions [22].

In this paper we introduce and study some aspects of the affine ensemble, a family of determinantal point processes on the complex upper half-plane \( \mathbb{C}^+ \), defined in terms of a representation of the \( ax+b \) group acting on a vector \( \psi \in H^2(\mathbb{C}^+) \), the Hardy space in the upper half plane. This can be seen as a geometric hyperbolic or algebraic non-unimodular analogue of the Weyl-Heisenberg ensemble [3 2], a family of planar euclidean determinantal point processes associated with the (unimodular) Weyl-Heisenberg group. In terms of the representation \( \pi(z)\psi(t) := s^{-\frac{1}{2}} \psi(s^{-1}(t-x)) \), the kernel of the affine ensemble is defined for \( \psi \) such that \( \|\psi\|_2 = 1 \) and \( \|\mathcal{F}\psi\|_{L^2([t^{-1},t+1],dt)} < \infty \), as a normalizing constant times

\[
k_\psi(z,w) = \langle \pi(w)\psi, \pi(z)\psi \rangle_{H^2(\mathbb{C}^+)}.\]

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Using the Fourier transform isomorphism $\mathcal{F} : H^2(\mathbb{C}^+) \to L^2(0, \infty)$, the kernel (1.1) can be written in a more convenient form, for $z = x + is, w = x' + is' \in \mathbb{C}^+$,

$$k_\psi(z, w) = (ss')^{1/2} \int_0^\infty e^{-ixs'}(\mathcal{F}\psi)(s'\xi)e^{-ix\xi}(\mathcal{F}\psi)(s\xi)d\xi.$$  

(1.2)

By selecting special functions $\psi$, a number of processes arise as special cases, automatically inheriting all properties of the affine ensemble. In this paper we will consider only examples with $PSL(2, \mathbb{R})$ invariance, corresponding to invariance under the linear fractional transformations of the half plane $\mathbb{C}^+$ (dilations, rotations and translations). Consider the mother wavelets chosen from the family $\{\psi_\alpha^\alpha\}_{n \in \mathbb{N}_0}, \alpha > 0$,

$$\psi_\alpha^\alpha(z) := \xi^{\alpha/2}e^{-\xi L_\alpha^\alpha(z/\xi)}, \quad \xi > 0,$$

(1.3)

where $L_\alpha^\alpha$ denotes the generalized Laguerre polynomials

$$L_\alpha^\alpha(t) = \frac{t^{-\alpha}e^t}{n!} \left( \frac{d}{dt} \right)^n (e^{-t}t^{\alpha+n}), \quad t > 0,$$

and the following projective unitary group representation of $PSL(2, \mathbb{R})$ on $L^2(\mathbb{C}^+, \mu)$:

$$\hat{\tau}_n^\alpha \left( \begin{array}{cc} az + b \\ cz + d \end{array} \right) F(z) := \left( \begin{array}{c} cz + d \\ cz + d \end{array} \right)^{2n+\alpha+1} F \left( \begin{array}{c} az + b \\ cz + d \end{array} \right),$$

(1.4)

where $a, b, c, d$ are real numbers such that $\det \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \neq 0$. Then $\hat{\tau}_n^\alpha$ leaves the Hilbert space with reproducing kernel $k_\psi_\alpha^\alpha(z, w)$ invariant. This has been shown in [1] and also that, essentially, the choice (1.3) is the only leading to spaces invariant under representations of the form (1.4), if we assume mild reasonable restrictions on $\psi$ (see Theorem 3 in [1]). Moreover, for $\psi$ within the family (1.3) and the parameter $\alpha = 2(B - n) - 1$ we obtain reproducing kernels associated with the eigenspaces of the pure point spectrum of the Maass Laplacian with weight $B$ [21, 12, 4]:

$$H_B := s^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial s^2} \right) - 2iBs \frac{\partial}{\partial x}$$

The last part of this paper will be devoted to these special cases. The pure point spectrum eigenspaces of the Maass Laplacian $H_B$ have been used in [12, 4] to model the formation of higher Landau levels in the hyperbolic plane. A physical model, put forward by Alain Comtet in [12], describes a situation where the number of levels is constrained to be a finite number, depending on the strength of the magnetic field $B$, which must exceed a lower bound for their existence (the magnetic field has to be strong enough to capture the particle in a closed orbit). The connection to analytic wavelets was suggested by the characterization of hyperbolic Landau coherent states [23] and has been implicit used in [4] and more recently in [1].
It is reasonable to expect interesting examples arising from other special choices, namely those leading to the polyanalytic structure discovered by Vasilevski [30] (see [6, 18, 5] for the special choices leading to polyanalytic spaces) but we will not explore this direction in the present paper. One of our motivations for this research was the scarceness of examples on hyperbolic DPPs. Besides the celebrated case studied by Peres and Virág [25], we only found the higher Landau levels DPP on the disc studied recently by Demni and Lazag in [14], which strongly influenced the current paper. The affine ensemble contains an uncountable number of examples as special cases, of which we only explore a very few.

The paper is organized as follows. The next section contains the required background on analytic wavelets and hyperbolic geometry. The third section is the core of the paper, where the affine ensemble is defined and the main results are proved. Section 4 specializes the results to the class of Maass-Landau ensembles and the calculations are detailed in the last section of the paper, as an appendix.

2. Background

2.1. The continuous analytic wavelet transform. We will use the basic notation for $H^2(\mathbb{C}^+)$, the Hardy space in the upper half plane, of analytic functions in $\mathbb{C}^+$ with the norm

$$\|f\|_{H^2(\mathbb{C}^+)} = \sup_{0<s<\infty} \int_{-\infty}^{\infty} |f(x+is)|^2 \, dx < \infty.$$ 

To simplify the computations it is often convenient to use the equivalent definition (since the Paley-Wiener theorem [15] gives $\mathcal{F}(H^2(\mathbb{C}^+)) = L^2(0, \infty)$)

$$H^2(\mathbb{C}^+) = \{ f \in L^2(\mathbb{R}) : (\mathcal{F}f)(\xi) = 0 \text{ for almost all } \xi < 0 \}.$$ 

Consider the $ax+b$ group (see [11, Chapter 10] for the listed properties) $G \sim \mathbb{R} \times \mathbb{R}^+ \sim \mathbb{C}^+$ with the multiplication

$$(x, s) \cdot (x', s') = (x + sx', ss').$$

The identification $G \sim \mathbb{C}^+$ is done by setting $(x, s) \sim x + is$. The neutral element of the group is $(0, 1) \sim i$ and the inverse element is given by $(x, s)^{-1} = (-\frac{x}{s}, \frac{1}{s}) \equiv -\frac{x}{s} + \frac{i}{s}$. The $ax+b$ group is not unimodular, since the left Haar measure on $G$ is $\frac{dxds}{s^2}$ and the right Haar measure $G$ is $\frac{dxds}{s}$. The left Haar measure of a set $\Omega \subseteq G$,

$$|\Omega| = \int_{\Omega} \frac{dxds}{s^2},$$

coincides, under the identification of the $ax+b$ group with $\mathbb{C}^+$, with the hyperbolic measure

$$|\Omega| = |\Omega|_h := \int_{\Omega} s^{-2} \, d\mu_{\mathbb{C}^+}(z),$$

where $d\mu_{\mathbb{C}^+}(z)$ is the Lesbegue measure in $\mathbb{C}^+$. We will write

$$d\mu^+(z) = (\text{Im } z)^{-2} d\mu_{\mathbb{C}^+}(z).$$ (2.1)
For every \( x \in \mathbb{R} \) and \( s \in \mathbb{R}^+ \), define the translation \( T_x \) by \( T_x f(t) = f(t-x) \) and the dilation \( D_s f(t) = \frac{1}{s} f(t/s) \). Let \( z = x + is \in \mathbb{C}^+ \) and define the representation, for \( \psi \in H^2(\mathbb{C}^+) \),

\[
(2.2) \quad \pi(z) \psi(t) := T_x D_s \psi(t) = s^{-\frac{1}{2}} \psi(s^{-1}(t-x)).
\]

The theory of general wavelet transforms using group representations requires the admissibility condition [7],

\[
\int_G |\langle \psi, \pi(z) \psi \rangle|^2 \, d\mu(z) < \infty,
\]

to construct square-integrable representations for general groups \( G \) with left Haar measures \( \mu \). This will lead to an isometric transform thanks to the orthogonality relations (2.6) below. In the Weyl-Heisenberg representations used in [3, 2], this follows trivially from the square-integrability of \( \psi \) only. But here, in the affine case, we need to take into account that the \( ax + b \) group is not unimodular, and the different left and right Haar measures of the representation require a further condition on the integrability of \( \psi \), which will be restricted to the class of functions such that

\[
(2.3) \quad \begin{cases} \psi \in H^2(\mathbb{C}^+) \\ 0 < 2\pi \| \mathcal{F}\psi \|_{L^2(\mathbb{R}^+, t^{-1} \, dt)}^2 = C_\psi < \infty \end{cases}.
\]

Functions satisfying (2.3) are called admissible and the constant \( C_\psi \) is the admissibility constant. Now, we have an irreducible and unitary representation \( \pi \) of the affine group on \( H^2(\mathbb{C}^+) \) [7], defined in (2.2) for an admissible \( \psi \). By this definition any admissible function is automatically in the Hardy space and so the inner product considered for the wavelet transform or the reproducing kernel is in \( H^2(\mathbb{C}^+) \). The continuous analytic wavelet transform of a function \( f \) with respect to a wavelet \( \psi \) is defined, for every \( z = x + is \in \mathbb{C}^+ \), as

\[
(2.4) \quad W_\psi f(z) = \langle f, \pi(z) \psi \rangle_{H^2(\mathbb{C}^+)}. 
\]

More explicitly,

\[
W_\psi f(z) = \sup_{0 < s < \infty} s^{-\frac{1}{2}} \int_{-\infty}^{\infty} f(t) \psi(s^{-1}(t-x)) \, dt.
\]

Using \( \mathcal{F}(H^2(\mathbb{C}^+)) = L^2(0, \infty) \), this can also be written (and we will do it as a rule to simplify the calculations) as

\[
(2.5) \quad W_\psi f(z) = s^{\frac{1}{2}} \int_0^{\infty} f(\xi) e^{-ix\xi} (\mathcal{F}\psi)(s\xi) \, d\xi.
\]

As proven recently in [19], \( W_\psi f(z) \) only leads to analytic (Bergman) phase spaces for a very special choice of \( \psi \), but it is common practice to call it in general continuous analytic wavelet transform. The orthogonality relations

\[
(2.6) \quad \int_{\mathbb{C}^+} W_{\psi_1} f_1(x, s) \overline{W_{\psi_2} f_2(x, s)} \, d\mu^+(z) = 2\pi \, \langle \mathcal{F}\psi_1, \mathcal{F}\psi_2 \rangle_{L^2(\mathbb{R}^+, t^{-1} \, dt)} \langle f_1, f_2 \rangle_{H^2(\mathbb{C}^+)},
\]
are valid for all \( f_1, f_2 \in H^2(\mathbb{C}^+) \) and \( \psi_1, \psi_2 \in H^2(\mathbb{C}^+) \) admissible. Then, setting \( \psi_1 = \psi_2 = \psi \) and \( f_1 = f_2 \) in (2.6), gives

\[
\int_{\mathbb{C}^+} |W_\psi f(x, s)|^2 \, d\mu^+(z) = C_\psi \| f \|_{H^2(\mathbb{C}^+)}^2
\]

and the continuous wavelet transform provides an isometric inclusion \( W_\psi : H^2(\mathbb{C}^+) \rightarrow L^2(\mathbb{C}^+, d\mu^+) \). Setting \( \psi_1 = \psi_2 = \psi \) and \( f_2 = \pi(z)\psi \) in (2.6) then for every \( f \in H^2(\mathbb{C}^+) \) one has

(2.7) \( W_\psi f(z) = \frac{1}{C_\psi} \int_{\mathbb{C}^+} W_\psi f(w)\langle \pi(w)\psi, \pi(z)\psi \rangle d\mu^+(z), \quad z \in \mathbb{C}^+ \).

Thus, the range of the wavelet transform

\[
W_\psi \left( H^2(\mathbb{C}^+) \right) := \{ F \in L^2(\mathbb{C}^+, \mu^+) : F = W_\psi f, \ f \in H^2(\mathbb{C}^+) \}
\]

is a reproducing kernel subspace of \( L^2(\mathbb{C}^+, \mu^+) \) with kernel

(2.8) \( k_\psi(z, w) = \frac{1}{C_\psi} \int_{\mathbb{C}^+} W_\psi f(w)\langle \pi(w)\psi, \pi(z)\psi \rangle d\mu^+(z), \quad \text{and} \quad k_\psi(z, z) = \frac{\| \psi \|_{H^2(\mathbb{C}^+)}^2}{C_\psi} \). \]

The Fourier transform \( \mathcal{F} : H^2(\mathbb{C}^+) \rightarrow L^2(0, \infty) \) can be used to simplify computations, since

(2.9) \( \langle \pi(w)\psi, \pi(z)\psi \rangle_{H^2(\mathbb{C}^+)} = \left( \mathcal{F}^{-1}\pi(s)\mathcal{F}\pi(z)\right)_{L^2(\mathbb{R}^+, dt)} = (ss')^{\frac{1}{2}} \int_0^\infty \hat{\psi}(s')\hat{\psi}(s \xi) e^{i(s-s')\xi} d\xi \).

2.2. **Hyperbolic geometry.** We will need some elementary facts of hyperbolic geometry. The hyperbolic metric in \( \mathbb{C}^+ \) is defined as [10]

(2.10) \( d(z_1, z_2) = \log \frac{1 + \varrho(z_1, z_2)}{1 - \varrho(z_1, z_2)} = 2 \tanh^{-1}(\varrho(z_1, z_2)), \)

where \( \varrho \) is the pseudohyperbolic metric in \( \mathbb{C}^+ \),

\[
\varrho(z_1, z_2) = \left| \frac{z_1 - z_2}{\overline{z_1 - z_2}} \right|.
\]

The hyperbolic ball of center \( z \in \mathbb{C}^+ \) and radius \( R < 1 \) is

\[
D(z, R) = \{ w \in \mathbb{C}^+ : d(w, z) < R \}.
\]

By direct computation it can be checked that \( \varrho(z^{-1}, i) = \varrho(i, z) \) and that \( D(z^{-1}, R) = D(z, R) \).
3. The affine ensemble

3.1. The affine ensemble. Determinantal Point Processes (we simply list the concepts we are using; for a complete definition see [9, Chapter 4]) are defined using an ambient space \( \Lambda \), a Radon measure \( \mu \) defined on \( \Lambda \), and a reproducing kernel Hilbert space \( \mathcal{H} \) contained in \( L^2(\mathbb{C}^+,d\mu^+) \). The reproducing kernel of \( \mathcal{H} \), \( K(z,w) \), is the correlation kernel of the point process \( X \). The \( k \)-point intensities are given by

\[
\rho^k(x_1,\ldots,x_k) = \det (K(x_i,x_j))_{1\leq i,j\leq k}.
\]

Given a set \( \Omega \subset \mathbb{C}^+ \), the 1-point intensity gives the expected number of points to be found in \( \Omega \):

\[
\mathbb{E}(X(\Omega)) = \int_\Omega \rho_1(z) \, d\mu^+(z) = \int_\Omega K(z,z) \, d\mu^+(z),
\]

The normalization of the kernel \( K_\psi(z,w) \) in the following definition is different from the one in (2.8) and is chosen so that \( K_\psi(z,z) = 1 \). Recall that \( k_\psi(z,z) = \|\psi\|_2^2 C_\psi \).

**Definition 1.** The affine ensemble \( X_\psi \) associated with an admissible function \( \psi \) is the Determinantal Point Process with the normalized correlation kernel

\[
K_\psi(z,w) = \frac{k_\psi(z,w)}{k_\psi(z,z)} = \frac{W_\psi(w^{-1} \cdot z)}{\|\psi\|_2^2} = W_\tilde{\psi}(w^{-1} \cdot z),
\]

where \( \tilde{\psi} = \psi/\|\psi\|_2 \).

We will assume from now on \( \|\psi\|_2 = 1 \) (if this is not the case, we will use the notation \( \tilde{\psi} = \psi/\|\psi\|_2 \)). Then

\[
K_\psi(z,w) = k_\psi(z,w) = W_\psi(w^{-1} \cdot z), \quad K_\psi(z,z) = W_\psi(i)
\]

and, from (2.6),

\[
\int_{\mathbb{C}^+} |W_\psi(w)|^2 \, d\mu^+(w) = C_\psi.
\]

3.2. Variance estimates. We will consider an operator \( T_\Omega \) acting on a function \( f \) on the range of \( W_\psi \), which smooths out the energy of \( f \) outside \( \Omega \), by first multiplication by \( 1_\Omega \) and then projecting on the range of \( W_\psi \). Using the reproducing kernel property, \( T_\Omega \) can be written as

\[
(T_\Omega f)(z) = \int_\Omega f(w) K_\psi(z,w) \, d\mu^+(w)
\]

\[
= \int_\Omega f(z') \int_\Omega K_\psi(z,z') K_\psi(z',w) \, d\mu^+(z') \, d\mu^+(w),
\]

providing an extension of \( T_\Omega \) to the whole \( L^2(\mathbb{C}^+,s^{-2}dxds) \) vanishing in the complement of the range of \( W_\psi \). By definition of 1-point intensity,

\[
\mathbb{E}(X_\psi(\Omega)) = \int_\Omega K_\psi(z,z) \, d\mu^+(z) = \text{trace} (T_\Omega) = |\Omega|_h.
\]
while the number variance of \( X_\psi(\Omega) \) is (see [16] pg. 40) for a detailed proof:

\[
\mathbb{V}[X_\psi(\Omega)] = \mathbb{E}(X_\psi(\Omega)^2) - \mathbb{E}(X_\psi(\Omega))^2 = \text{trace}(T_{11}) - \text{trace}(T_{11}^2).
\]

Our first result gives the asymptotic behavior of the variance and the exact value of the asymptotic constant. A related formula has been obtained by Shirai for Ginibre ensembles in higher Landau levels [20]. A new proof of Shirai’s formula has been obtained by Demni and Lazag [14], using a quite flexible argument based on geometric considerations, which inspired the following result.

**Theorem 1.** Let \( \psi \) admissible with \( \|\psi\|_{H^2(C^+)}^2 = 1 \). We have the following explicit formula for the variance of the affine ensemble associated with \( \psi \):

\[
\mathbb{V}[X_\psi(D(i, R))] = \int_{C^+} |W_\psi(w)|^2 |D(i, R) \cap D(w, R)|_h d\mu^+(w).
\]

Moreover, as \( R \to 1^- \),

\[
(3.4) \quad \mathbb{V}[X_\psi(D(i, R))] \sim \frac{c_\psi}{1 - R^2},
\]

where the asymptotic constant \( c_\psi \) is given by

\[
(3.5) \quad c_\psi = \frac{1}{2} \int_{C^+} |W_\psi(w)|^2 \arccos \left( 1 - 2 \left| \frac{w - i}{w + i} \right|^2 \right) d\mu^+(w).
\]

**Proof.** In the context of the concentration operator defined in the beginning of the section, set \( \Omega = D(i, R) \). Observe that \( K_\psi(z, z) = W_\psi(\psi(z \cdot z^{-1}) = W_\psi(i) \), that \( W_\psi(w^{-1} \cdot z) = \langle \pi(w)\psi, \pi(z)\psi \rangle_{H^2(C^+)} = \overline{W_\psi(z^{-1} \cdot w)} \), and use the reproducing kernel equation, \( d\mu^+ \) as the left Haar measure on the \( ax + b \) group, and Fubini, to write:

\[
\mathbb{V}[X_\psi(D(i, R))]
\]

\[
= \int_{D(i, R)} W_\psi(i) d\mu^+(z) - \int_{D(i, R) \times D(i, R)} |W_\psi(w^{-1} \cdot z)|^2 d\mu^+(w)d\mu^+(z)
\]

\[
= \int_{D(i, R) \times C^+} |W_\psi(z^{-1} \cdot w)|^2 d\mu^+(w)d\mu^+(z) - \int_{D(i, R) \times D(i, R)} |W_\psi(z^{-1} \cdot w)|^2 d\mu^+(w)d\mu^+(z)
\]

\[
= \int_{D(i, R) \times D(i, R)} |W_\psi(z^{-1} \cdot w)|^2 d\mu^+(w)d\mu^+(z)
\]

\[
= \int_{C^+} 1_{D(i, R)}(z) \int_{C^+} 1_{D(i, R)}(w) |W_\psi(z^{-1} \cdot w)|^2 d\mu^+(w) d\mu^+(z)
\]

\[
= \int_{C^+} 1_{D(i, R)}(z) \int_{C^+} 1_{D(i, R)}(z \cdot w) |W_\psi(w)|^2 d\mu^+(w) d\mu^+(z)
\]

\[
= \int_{C^+} 1_{D(i, R)}(z) \int_{D(w^{-1}, R)}(z) |W_\psi(w)|^2 d\mu^+(w) d\mu^+(z)
\]

\[
= \int_{C^+} |W_\psi(w)|^2 \left[ \int_{D(i, R) \cap D(w^{-1}, R)} d\mu^+(z) \right] d\mu^+(w),
\]
where $1_{D(i,R)}(z \cdot w) = 1_{D(w^{-1}, R)}(z)$ follows from $g(z \cdot w, i) = g(w^{-1}, z)$. Since $D(w^{-1}, R) = D(w, R)$, we conclude that

$$
\mathbb{V} [X_\psi(D(i,R))] = \int_{\mathbb{C}^+} |W_\psi(w)|^2 |D(i,R)^c \cap D(w,R)|_h \, d\mu^+(w).
$$

To prove (3.4) and to determine the asymptotic constant (3.5), we will need to find the area $|D(i,R)^c \cap D(w^{-1}, R)|_h$ when $R \to 1$. First move the integrals conformal to the unit disc. Setting

$$
\xi(z) = \frac{z - i}{z + i} \in \mathbb{D}; \quad \xi^{-1}(w) = \frac{w + 1}{i(w - 1)} \in \mathbb{C}^+,
$$

the measures can be related by

$$
(\text{Im } z)^{\alpha} d\mu_{\mathbb{C}^+}(z) = \frac{2^{\alpha+1}(1 - |\xi(z)|^2)^\alpha}{(1 - \xi(z))^{2\alpha+4}} d\mu_\mathbb{D}(\xi(z)),
$$

d$\mu_\mathbb{D}$ being the Lesbegue measure on $\mathbb{D}$. Denoting by $\mathbb{D}(\xi(w), R)$ the hyperbolic disc on $\mathbb{D}$ resulting from conformal mapping $D(w,R)$ we have

$$
\int_{D(i,R)^c \cap D(w,R)} \frac{1}{(\text{Im } z)^2} d\mu_{\mathbb{C}^+}(z) = \frac{1}{2} \int_{\mathbb{D}(0,R)^c \cap \mathbb{D}(\xi(w), R)} (1 - |\xi(z)|^2)^{-2} d\mu_\mathbb{D}(\xi(z))
$$

and we can use the computation of Theorem 1 in [14], leading to, as $R \to 1$,

$$
|D(i,R)^c \cap D(w,R)|_h = \frac{1}{2} \left| \mathbb{D}(0,R)^c \cap \mathbb{D}(\xi(w), R) \right|_h \sim \frac{1}{2} \arccos \left( \frac{1 - 2|\xi(w)|^2}{1 - R^2} \right).
$$

Thus,

$$
|D(i,R)^c \cap D(w,R)|_h \sim \frac{1}{2} \frac{\arccos \left( \frac{1 - 2|\xi(w)|^2}{1 - R^2} \right)}{1 - R^2}
$$

It follows that, as $R \to 1$,

$$
\mathbb{V} [X_\psi(D(i,R))] \sim \frac{1}{2} \frac{1}{1 - R^2} \int_{\mathbb{C}^+} |W_\psi(w)|^2 \arccos \left( 1 - 2 \frac{|w - i|^2}{w + i} \right) \, d\mu^+(w).
$$

The next result shows with a two-sided inequality that the variance of the affine ensemble is proportional to $|\Omega|_h$. The first part of the result is essentially an interpretation of the results in [13], where the lower inequality is obtained for a large class of sets $\Omega$, assuming that, for some $c > 0$,

$$
|\langle \pi(z)\psi, \pi(w)\psi \rangle_{H(\mathbb{C}^+)}|^2 \geq \frac{c}{|D(z,R)|^2} \int_{\mathbb{C}^+} 1_{D(\xi,R)}(z) 1_{D(\xi,R)}(w) \, d\xi.
$$

For $\Omega = D(i,R)$ we provide a proof of an upper bound involving the admissibility constant $C_\psi$. 

Theorem 2. Assuming that (3.7) holds, we have

\[(3.8) \quad |\Omega|_h \lesssim \mathbb{V}[X_\psi(\Omega)] \leq |\Omega|_h.\]

If \(\Omega = D(i, R)\), then

\[
\mathbb{V}[X_\psi(D(i, R))] \leq C_\psi |D(i, R)|_h.
\]

Proof. Using the operator \(T_\Omega\) we easily obtain an upper bound for the variance

\[
\mathbb{V}[X_\psi(\Omega)] = \text{trace} \ (T_\Omega) - \text{trace} \ (T_\Omega^2) \leq \text{trace} \ (T_\Omega) = |\Omega|_h,
\]

since \(\text{trace} \ (T_\Omega^2) \leq \text{trace} \ (T_\Omega)\). The lower inequality follows from [13, Lemma 3.2], where it is shown that, under the condition (3.7),

\[
\text{trace} \ (T_\Omega) - \text{trace} \ (T_\Omega^2) \gtrsim |\Omega|_h.
\]

Now set \(\Omega = D(i, R)\). Then, from Theorem 1,

\[
\mathbb{V}[X_\psi(D(i, R))] = \int_{C^+} |W_\psi(\psi(w))^2 \left[ \int_{D(i, R) \cap D(w, R)} d\mu^+(z) \right] d\mu^+(w)
\]
\[
\leq \int_{C^+} |W_\psi(\psi(w))^2 |D(w, R)|_h d\mu^+(w)
\]
\[
= |D(i, R)|_h \int_{C^+} |W_\psi(\psi(w))^2 d\mu^+(w)
\]
\[
= C_\psi |D(i, R)|_h,
\]

using \(|D(w, R)|_h = |D(i, R)|_h\) and (3.2). □

Remark 1. A reduced DPP adapted to \(\Omega\) can be defined as a Toeplitz smooth restriction of the affine ensemble to \(\Omega\) using the operator (3.3), following a scheme similar to the one used to define the finite Weyl-Heisenberg ensembles in [3]. Denoting by \(\{p_j^\Omega\}\) the eigenfunctions of (3.3), one associates with \(\Omega\) the reduced finite dimensional Hilbert space

\[
W_{\psi}^{N_\Omega} = \text{Span}\{p_j^\Omega\}_{n=1,...,N_\Omega} \subset W_\psi,
\]

where \(N_\Omega = \lfloor |\Omega|_h \rfloor\), the least integer than or equal to \(|\Omega|_h\). The \(\Omega\)-reduced affine ensemble is the finite dimensional DPP \(X_\psi^{\Omega}\) generated by the kernel

\[
K_{\psi,\Omega}(z, w) = \sum_{j=0}^{N_\Omega} p_j^\Omega(z)p_j^\Omega(w).
\]
4. The Maass-Landau levels processes

4.1. Bergman spaces. The reproducing kernel of the space $W_{\tilde{\psi}_0}^\alpha (H^2 (\mathbb{C}^+))$ is the following weighted Bergman kernel (take $n = 0$ in Proposition 1 in the Appendix):

$$K_{\tilde{\psi}_0}^\alpha (z, w) = \alpha (4 \text{Im} \, z \, \text{Im} \, w)^{\frac{\alpha + 1}{2}} \left( \frac{1}{-i(z - w)} \right)^{\alpha + 1}.$$  

This is the ‘ground level’ case of the structure considered in the next section. For $\alpha = 1$ it is a $\mathbb{C}^+$ weighted version of the DPP studied by Peres and Virág [25].

4.2. Hyperbolic Maass-Landau levels. The Hamiltonian describing the dynamics of a charged particle moving on the Poincaré upper half-plane $\mathbb{C}^+$ under the action of the magnetic field $B$ is given by:

$$H_B := s^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial s^2} \right) - 2iBs \frac{\partial}{\partial x}$$

The operator $H_B$ was first introduced by Maass in number theory [21, 24] and its interpretation as a hyperbolic analogue of the Landau Hamiltonian has been put forward by Comtet (see [12, 4]). We list here the following important properties of $H_B$ as an operator.

(1) $H_B$ is an elliptic densely defined operator on the Hilbert space $L^2 (\mathbb{C}^+, d\mu^+)$, with a unique self-adjoint realization that we denote also by $H_B$.

(2) The spectrum of $H_B$ in $L^2 (\mathbb{C}^+, d\mu^+)$ consists of two parts: a continuous part $[1/4, +\infty[$, corresponding to scattering states and a finite number of eigenvalues with infinite degeneracy (hyperbolic Landau levels) of the form

$$\epsilon_n^B := (B - n) \left( 1 - B + n \right), \, n = 0, 1, 2, \ldots, \lfloor B - \frac{1}{2} \rfloor.$$  

The finite part of the spectrum exists provided $2B > 1$. The notation $\lfloor a \rfloor$ stands for the greatest integer not exceeding $a$.

(3) For each fixed eigenvalue $\epsilon_n^B$, we denote by

$$\mathcal{E}_n^B (\mathbb{C}^+) = \{ F \in L^2 (\mathbb{C}^+, d\mu^+) \, | \, H_B F = \epsilon_n^B F \}$$

the corresponding eigenspace, which has a reproducing kernel given by

$$K_{n,B}^\alpha (z, w) = \frac{(-1)^n \Gamma (2B - n)}{n! \Gamma (2B - 2n)} \left( \frac{4 \text{Im} \, z \, \text{Im} \, w}{|z - w|^2} \right)^{B - n} \left( \frac{\bar{z} - w}{\bar{w} - z} \right)^B F \left[ \frac{-2B - n}{2B - 2n}, -n; \frac{4 \text{Im} \, z \, \text{Im} \, w}{|z - w|^2} \right],$$

where $F$ is the Gauss hypergeometric function:

$$F \left[ \begin{array}{c} a, b \\ c \end{array} ; z \right] =_2 F_1 \left[ \begin{array}{c} a, b \\ c \end{array} ; z \right] = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n! (c)_n} z^n.$$
The condition $2B > 1$ ensuring the existence of these discrete eigenvalues in (2.) means that the magnetic field has to be strong enough to capture the particle in a closed orbit. If this condition is not fulfilled, the motion will be unbounded and the orbit of the particle will intersect the upper half-plane boundary whose points stand for ‘points at infinity’ (see [12, p. 189]). The eigenvalues in (2.) which are below the continuous spectrum have eigenfunctions called bound states since the particle in such a state cannot leave the system without additional energy. Then the number of particle layers (Landau levels), $\lfloor B - \frac{1}{2} \rfloor$, depends on the strength $B$ of the magnetic field.

To make the identification with special cases of the affine ensemble kernel, in the Appendix we compute the reproducing kernels and admissibility constants of the spaces $W_{\psi_n} (H^2 (\mathbb{C}^+))$ in terms of hypergeometric functions and Jacobi polynomials. According to Proposition 1, the reproducing kernel of $W_{\psi_n} (H^2 (\mathbb{C}^+))$ is given by

$$K_{\psi_n} (z, w) = K_{n,B} (z, w).$$

Thus, the kernels $K_{\psi_n} (z, w)$, are precisely the reproducing kernels of the eigenspaces associated with the pure point spectrum of the Maass Laplacian. As a result, all properties of the affine ensemble are automatically translated to the DPP associated with the reproducing kernels $K_{\psi_n} (z, w)$, with asymptotic constant

$$c_{\psi_n} = \frac{1}{2} \int_{\mathbb{C}^+} \left| W_{\psi_n} (w) \right|^2 \arccos(1 - \frac{w - i}{w + i})^2 d\mu^+(w)$$

and admissibility constant

$$C_{\psi_n} = \frac{4\pi}{2(B - n) - 1}.$$

Then, Theorems 1 and 2 lead to the following result for the Maass-Landau process, the DPP $\mathcal{X}_{B,n}$ generated by the reproducing kernel $K_{n,B} (z, w)$ of the eigenspace of $H_B$ associated with the Maass-Landau level eigenvalue $\xi_n := (B - n)(1 - B + n)$, for $n = 0, 1, 2, \ldots, \lfloor B - \frac{1}{2} \rfloor$.

**Corollary 1.** The variance of $\mathcal{X}_{B,n}$ is given by

$$\mathbb{V} [\mathcal{X}_{B,n}(D(i, R))] = \int_{\mathbb{C}^+} |K_{n,B}(i, w)|^2 |D(i, R)^c \cap D(w, R)|_h d\mu^+(w).$$

Moreover, when $R \to 1^-$,

$$\mathbb{V} [\mathcal{X}_{B,n}(D(i, R))] \sim \frac{c_{n,B}}{1 - R^2}.$$
where \( c_{n,B} = \frac{1}{2} \int_{\mathbb{C}^+} |K_{n,B}(i,w)|^2 \arccos(1 - 2 \frac{\bar{w}}{|w+i|^2})d\mu^+(w) \). Finally, the variance of \( X_{B,n} \) satisfies the non-asymptotic bound
\[
\mathbb{V}[X_{B,n}(D(i,R))] \leq \left( \frac{4\pi}{2(B-n) - 1} \right) |D(i,R)|_h.
\]

5. Appendix

5.1. Reproducing kernels of special affine ensembles. Let \( \alpha > -1 \). For \( n = 0, 1, 2, \ldots \) we define the normalized functions \( \tilde{\psi}_n^\alpha \) such that \( \|\tilde{\psi}_n^\alpha\|_{H^2(\mathbb{C}^+)} = 1 \):

\[
(\mathcal{F}\tilde{\psi}_n^\alpha)(t) = \sqrt{\frac{2^{\alpha+1}n!}{\Gamma(n+\alpha+1)}} t^{\frac{\alpha}{2}}e^{-t}L_n^\alpha(2t), \quad t > 0.
\]

Proposition 1. The reproducing kernel of \( \mathcal{W}_{\tilde{\psi}_n^\alpha} \) is given by

\[
K_{\tilde{\psi}_n^\alpha}(z, w) = (-1)^n \Gamma(n + 1 + \alpha) \frac{4 \text{Im} z \text{Im} w}{n! \Gamma(1 + \alpha)} \left( \frac{\pi - w}{|z - \bar{w}|^2} \right)^{\alpha+1+n} \times F \left[ \begin{array}{c} n + \alpha + 1, -n \alpha + 1 \end{array} ; \frac{4 \text{Im} z \text{Im} w}{|z - \bar{w}|^2} \right],
\]

where \( F =_2 F_1 \) denotes the hypergeometric function. Setting \( \alpha = 2(B-n) - 1 \) we obtain

\[
K_{\tilde{\psi}_n^{2(B-n)-1}}(z, w) = (-1)^n \Gamma(2B - n) \frac{4 \text{Im} z \text{Im} w}{n! \Gamma(2B - 2n)} \left( \frac{\pi - w}{|z - \bar{w}|^2} \right)^{B-n} \times F \left[ \begin{array}{c} 2B - n, -n \alpha + 1 \end{array} ; \frac{4 \text{Im} z \text{Im} w}{|z - \bar{w}|^2} \right].
\]

Proof. For simplification write \( z = x + is, w = x' + is' \in \mathbb{C}^+ \). Formula (2.3) gives:

\[
K_{\tilde{\psi}_n^\alpha}(z, w) = \langle \pi(w)\tilde{\psi}_n^\alpha, \pi(z)\tilde{\psi}_n^\alpha \rangle_{H^2(\mathbb{C}^+)} = \left\langle \pi(w)\tilde{\psi}_n^\alpha, \pi(z)\tilde{\psi}_n^\alpha \right\rangle_{L^2(\mathbb{R}^+)}
\]

\[
= \frac{2^{\alpha+1}n!}{\Gamma(n + \alpha + 1)} s^{\frac{\alpha}{2}}s'^{\frac{\alpha}{2}} \int_0^\infty e^{-itx'}(ts')^{\frac{\alpha}{2}} e^{-st}L_n^\alpha(2s')e^{ixt}(ts)^{\frac{\alpha}{2}} e^{-st}L_n^\alpha(2st)dt
\]

\[
= \frac{n!}{\Gamma(n + \alpha + 1)} s^{\frac{\alpha}{2}}s'^{\frac{\alpha}{2}} \int_0^\infty e^{-it\left(\frac{x'}{2} + \frac{x'}{2}\frac{i}{\alpha}\right)} L_n^\alpha(s't)L_n^\alpha(st)dt.
\]

To compute the integral we will use the following integral formula [17, p. 810, 7.414 (13)]:

\[
\int_0^\infty e^{-t(x + \frac{a_1 + a_2}{2})} t^\alpha L_n^\alpha(a_1 t)L_n^\alpha(a_2 t)dt = \frac{\Gamma(1 + \alpha + n) b_0^\alpha}{b_0^{1+\alpha+n} n!} P_n^{(\alpha,0)} \left( \frac{b_1^2}{b_0 b_2} \right),
\]

where

\[
b_0 = k + \frac{a_1 + a_2}{2}, b_2 = k - \frac{a_1 + a_2}{2}, b_1^2 = b_0 b_2 + 2a_1 a_2, \text{Re} \alpha > -1, \text{Re} b_0 > 0
\]
and $P_n^{(\alpha, \beta)}$ denotes the Jacobi polynomial. Setting
\[
b_0 = \frac{1}{2} i(x' - x) + s' + s = \frac{1}{2} i(x' - x - is' - is) = \frac{1}{2} i(x' - is' - (x + is)) = \frac{1}{2} i(\bar{w} - z),
\]
\[
b_2 = \frac{1}{2} i(x' - x) - s' - s = \frac{1}{2} i(x' - x + is' + is) = \frac{1}{2} i(x' + is' - (x - is)) = \frac{1}{2} i(w - \bar{z})
\]
and
\[
b_i^2 = \frac{1}{4} (|z - \bar{w}|^2 + 8s's).
\]
gives
\[
K_{\tilde{\phi}_n}(z, w) = (ss')^{\frac{\alpha + 1}{2}} \frac{\Gamma(n + \alpha + 1)}{n!\Gamma(\alpha + 1)} \frac{(w - \bar{z})^n}{(w - z)^{\alpha + n + 1}} \cdot P_n^{(\alpha, 0)} \left(1 - \frac{8s's}{|z - \bar{w}|^2}\right),
\]
where the Jacobi polynomial is defined as
\[
P_n^{(\alpha, \beta)}(x) = \frac{\Gamma(n + 1 + \alpha)}{n!\Gamma(1 + \alpha)} F \left[\begin{array}{c} n + \alpha + \beta + 1, -n \\ 1 + \alpha \end{array}; \frac{1 - x}{2}\right].
\]

Thus,
\[
K_{\tilde{\phi}_n}(z, w)
= (ss')^{\frac{\alpha + 1}{2}} \frac{1}{\Gamma(\alpha + 1)} \frac{(w - \bar{z})^n}{(w - z)^{\alpha + n + 1}} F \left[\begin{array}{c} n + 1 + \alpha, -n \\ 1 + \alpha \end{array}; \frac{4s's}{|z - \bar{w}|^2}\right].
\]

Next, notice that
\[
\frac{1}{(|z - \bar{w}|^2)^{\frac{\alpha + 1}{2}}} \frac{(w - \bar{z})^n}{(w - z)^{\alpha + n + 1}} \frac{(\bar{z} - w)^{\frac{\alpha + 1}{2}}}{(\bar{w} - z)^{\frac{\alpha + 1}{2}}} = \frac{(-1)^n (\bar{z} - w)^{\alpha + n + 1}}{(\bar{w} - z)^{\alpha + n + 1}}.
\]

Hence,
\[
K_{\tilde{\phi}_n}(z, w)
= \frac{(-1)^n \Gamma(n + \alpha + 1)}{n!\Gamma(\alpha + 1)} \frac{(4s's)^{\frac{\alpha + 1}{2}}}{(|z - \bar{w}|^2)^{\frac{\alpha + 1}{2}}} \frac{(\bar{z} - w)^{\frac{\alpha + 1}{2} + n}}{(\bar{w} - z)^{\frac{\alpha + 1}{2}}} F \left[\begin{array}{c} n + 1 + \alpha, -n \\ 1 + \alpha \end{array}; \frac{4s's}{|z - \bar{w}|^2}\right].
\]

\[\square\]

5.2. Norms and admissible constants. The orthogonality relation
\[
(5.2) \quad \int_0^{+\infty} L_n^\alpha(t) L_m^\alpha(t) t^\alpha e^{-t} dt = \frac{\Gamma(n + \alpha + 1)}{n!} \delta_{n,m}
\]
written as
\[
\int_0^{+\infty} t^\alpha e^{-t} L_n^\alpha(2t) L_m^\alpha(2t) dt = \frac{\Gamma(n + \alpha + 1)}{2^{\alpha + 1} n!} \delta_{n,m}
\]

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shows that $\mathcal{F}\tilde{\psi}_n^\alpha \in L^2(\mathbb{R}^+)$. Hence, we observe that $\tilde{\psi}_n^\alpha \in H^2(\mathbb{C}^+)$ and
\[
\left\|\tilde{\psi}_n^\alpha \right\|_{L^2(\mathbb{R}^+)}^2 = \left\|\mathcal{F}\tilde{\psi}_n^\alpha \right\|_{L^2(\mathbb{R}^+)}^2 = 1.
\]
Next, we calculate the admissibility constant $C_{\tilde{\psi}_n^\alpha}$. The formula [27, (10)] gives
\[
C_{\tilde{\psi}_n^\alpha} = 2\pi \left\|\mathcal{F}\tilde{\psi}_n^\alpha \right\|_{L^2(\mathbb{R}^+, t^{-1})} = \frac{\pi 2^{\alpha+2} n!}{\Gamma(n + \alpha + 1)} \int_0^\infty \left( t^{\alpha/2} e^{-t} L_n^\alpha(2t) \right)^2 \frac{dt}{t} = 4\pi \frac{2^{\alpha+2} n!}{\alpha}.
\]
Consequently,
\[
C_{\tilde{\psi}_n^{2(B-n)-1}} = \frac{2}{2(B-n) - 1}.
\]

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