Information geometry of the Tojo-Yoshino’s exponential family on the Poincaré upper plane

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Abstract

We study the dually flat information geometry of the Tojo-Yoshino exponential family with sample space the Poincaré upper plane and parameter space the open convex cone of $2 \times 2$ positive-definite matrices. Using the framework of Eaton’s maximal invariant, we prove that all $f$-divergences between Tojo-Yoshino Poincaré distributions are functions of 3 simple determinant/trace terms. We report closed-form formula for the Fisher information matrix, the differential entropy and the Kullback-Leibler divergence and Bhattacharyya distance between such distributions.

1 Introduction

Since it was proven that hyperbolic geometry [4] is very well suited for embedded graph trees with low distortions [28], a recent trend in machine learning and data science is to embed discrete hierarchical graph structures into continuous spaces with low distortions for further downstream tasks [20, 27, 16, 19, 30, 29]. There exists many models of hyperbolic geometry [4] like the Poincaré disk or upper plane conformal models, the Klein non-conformal disk model, the Beltrami hemisphere model, the Weierstrass hyperbolic model, etc. Those models are in correspondences by bijections yielding isometric embeddings of one model to another [10]. For example, the hyperbolic Voronoi diagrams have been visualized\(^\dagger\) in the five main models of hyperbolic distribution [26]. As a byproduct of these low-distortion hyperbolic embeddings, many embedded data sets are available in hyperbolic space, and those data set need to be further processed. It is therefore important to build statistical models for these data sets using probability distributions with support the hyperbolic space or mixtures thereof. One of the very first proposed family of such “hyperbolic distributions” was proposed in 1981 [18] and are commonly called the hyperboloid distributions. Barbaresco defined the “Souriau-Gibbs distribution” (2019) in the Poincaré disk [7] (Eq. 57). Recently, Tojo and Yoshino described a generic method [34] to build exponential families of distributions on homogeneous spaces which are invariant under the action of a Lie group, and illustrate their method with an example of an exponential family distribution supported on the Poincaré upper plane with

\(^\dagger\)See the video online at https://www.youtube.com/watch?v=i9IUzNxeH4o
its conjugate prior in [32] (2020). Conjugate priors are a convenient mathematical machinery used in Bayesian statistics and Bayesian learning [14, 1].

In this paper, we consider the dually flat information geometry [3] of this family of Tojo-Yoshino Poincaré distributions (abbreviated as the TYP distributions in the reminder). Since the family of TYP distributions form an exponential family [8], the underlying information-geometric structure of the family viewed as a smooth manifold is dually flat [3]. Dually flat manifolds are also called Bregman manifolds [21, 22] since they admit canonical Bregman divergences [9].

We first prove that all f-divergences between TYP distributions can be expressed as functions of three terms (Proposition 1 and Corollary 1). Then we report the Fisher information matrix of the TYP family (Eq. 1) and various information-theoretic quantities like the differential entropy of a TYP distribution (Proposition 3 and Eq. 2), or the Kullback-Leibler divergence (Proposition 2 and Eq. 1) and the Bhattacharyya divergences (Eq. 3) between two TYP distributions.

2 The TYP exponential family with sample space the Poincaré upper plane

Tojo and Yoshino described a versatile method [34, 32] to build “interesting” exponential families of distributions on homogeneous spaces which are invariant under the action of a Lie group \( G \) (generalizing the construction in [11]). They exemplify their method on the upper plane

\[ \mathbb{H} = \{(x, y) \in \mathbb{R}^2 : y > 0\} \]

by constructing the an exponential family with probability density functions invariant under the action of Lie group \( G = \text{SL}(2, \mathbb{R}) \), the set of invertible matrices with unit determinant.

3 TYP exponential family: Vector and matrix parametrizations

The probability density function (pdf) of a TYP distribution [32] expressed using a 3D vector parameter \( \theta_v = (a, b, c) \in \mathbb{R}^3 \) is given by

\[
p_{\theta_v}(x, y) = \frac{\sqrt{ac - b^2}}{\pi} \exp\left(2\sqrt{ac - b^2}\right) \exp\left(-\frac{a(x^2 + y^2) + 2bx + c}{y}\right) \, dx \, dy \]

where \( \theta_v \) belongs to the parameter space

\[
\Theta_v = \{(a, b, c) \in \mathbb{R}^3 : a > 0, c > 0, \, ab - c^2 > 0\},
\]

an open 3D convex cone. Thus the parameter space of the TYP family is 3D with the 2D hyperbolic plane sample space. Figure 1 displays three examples of density profiles of TYP distributions.

In general, an exponential family [8, 24] \( \mathcal{P} = \{p_\theta(x) : \theta \in \Theta\} \) has its probability density functions which can be written canonically as

\[
p_\theta(x) = \exp\left(\theta^\top t(x) - F(\theta) + k(x)\right),
\]

where \( t(x) \) denotes the sufficient statistic vector, \( \theta \) the natural parameter, \( k(x) \) an auxiliary carrier term and \( F(\theta) \) the log-normalizer (also called free energy in statistical physics and cumulant function in statistics). Let \( h(x) = e^{k(x)} \).
Let $z = x + iy \in \mathbb{C}$ and consider the mapping $\alpha(z) = \begin{bmatrix} \sqrt{y} & x \\ 0 & \frac{1}{\sqrt{y}} \end{bmatrix}$. Then the pdf rewrites as

$$p_\theta(z) = \frac{\sqrt{\theta} e^{2\sqrt{|\theta|}}}{\pi} \exp \left( -\text{tr} \left( \theta \alpha(z) \alpha(z)^\top \right) \right) \frac{dz d\bar{z}}{\text{Im}(z)^2},$$

where $\theta = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ belongs to the space of symmetric positive-definite matrices $\text{Sym}^+(2, \mathbb{R})$, $\text{tr}(\cdot)$ denotes the matrix trace. The family $\mathcal{P} = \{ p_\theta : \theta \in \text{Sym}^+(2, \mathbb{R}) \}$ is an exponential family of order 3 with log-normalizer which can be expressed either using the vector parameter $\theta_v$ or the positive-definite matrix $\theta$:

$$F(\theta_v) = \log \left( \frac{\pi}{\sqrt{ac - b^2 \exp(2\sqrt{ac - b^2})}} \right),$$

$$F(\theta) = \log \left( \frac{\pi}{\sqrt{|\theta| \exp(2\sqrt{|\theta|})}} \right),$$

where $|\theta|$ denotes the determinant of $\theta$. It will be useful in the remainder to define the square root of the determinant of $\theta$:

$$D = \sqrt{|\theta|} = \sqrt{ac - b^2} > 0.$$

We have $F(\theta_v) = (\log \frac{\pi}{D}) - 2D$. Since Bregman generators are equivalent modulo affine terms, we have

$$F(\theta_v) \equiv - \log D = -\frac{1}{2} \log |\theta|.$$

It is well-known that $- \log |\theta|$ is a strictly convex function [13].
The sufficient statistic vector $t_v(x)$ is
\[ t_v(x, y) = -\left( \frac{x^2 + y^2}{y}, x, 1 \right), \]
or in equivalent matrix form:
\[ t(z) = -\alpha(z)\alpha(z)^\top = -\frac{1}{y} \begin{bmatrix} x^2 + y^2 & x \\ x & 1 \end{bmatrix}. \]

The auxiliary carrier measure term is $h(x, y) = e^{k(x,y)} = \frac{1}{y^2}$, and $h(x)\,dx\,dy$ is a $\text{SL}(2, \mathbb{R})$-invariant measure, or equivalently $h(z) = e^{k(z)} = \frac{1}{y^2}$ and $h(z)\,dz\,d\bar{z}$ is a $\text{SL}(2, \mathbb{R})$-invariant measure.

We can write these TYP densities in the following vector/matrix canonical forms of exponential families [24]:
\[
\begin{align*}
p_{\theta_v}(x, y) &= \exp (\langle \theta_v, t(x, y) \rangle - F(\theta_v) + k(x, y)), \\
p_\theta(z) &= \exp (\langle \theta, t(z) \rangle - F(\theta) + k(z)),
\end{align*}
\]
where $\langle v_1, v_2 \rangle = v_1^\top v_2$ denotes the vector inner product (dot product) and $\langle M_1, M_2 \rangle$ denotes the matrix inner product $\langle M_1, M_2 \rangle = \text{tr}(M_1 M_2^\top)$. As mentioned in [32], the Poincaré distributions are related to the hyperboloid distributions [18].

4 Statistical $f$-divergences

The $f$-divergence [12] induced by a convex generator $f: \mathbb{R}_{++} \to \mathbb{R}$ between two PDFs $p(x)$ and $q(x)$ defined on the support $\mathbb{H}$ is defined by
\[ I_f(p : q) := \int p(x) f \left( \frac{q(x,y)}{p(x,y)} \right) \,dx\,dy. \]

Since $I_f(p : q) \geq f(1)$, we consider convex generators $f(u)$ such that $f(1) = 0$. The class of $f$-divergences include the total variation distance, the Kullback-Leibler divergence (and its two common symmetrizations, namely, the Jeffreys divergence and the Jensen-Shannon divergence), the squared Hellinger divergence, the Pearson and Neyman sided $\chi^2$-divergences, etc.

We state the notion of maximal invariant by following [15]. Let $G$ be a group acting on a set $X$. We denote it by $(g, x) \mapsto gx$. We say that a map $\varphi$ from $X$ to a set $Y$ is maximal invariant if it is invariant, specifically, $\varphi(gx) = \varphi(x)$ for every $g \in G$ and $x \in X$, and furthermore, whenever $\varphi(x_1) = \varphi(x_2)$ there exists $g \in G$ such that $x_2 = gx_1$. Every invariant map is a function of maximal invariant. Specifically, if a map $\psi$ from $X$ to a set $Z$ is invariant, then, there exists a map $\Phi$ from $\varphi(X)$ to $Z$ such that $\Phi \circ \varphi = \psi$.

**Proposition 1** (Maximal invariant). Define a group action of $\text{SL}(2, \mathbb{R})$ to $\text{Sym}(2, \mathbb{R})^2$ by
\[
(g, (\theta_1, \theta_2)) \mapsto (g^{-\top} \theta_1 g^{-1}, g^{-\top} \theta_2 g^{-1}).
\]
Then, $S(\theta_1, \theta_2) := (|\theta_1|, |\theta_2|, \text{tr}(\theta_2 \theta_1^{-1}))$ is maximal invariant of the action.

**Corollary 1** (TYP $f$-divergence terms). Every $f$-divergence between two Poincaré distributions is a function of $(|\theta_1|, |\theta_2|, \text{tr}(\theta_2 \theta_1^{-1}))$. 

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Thus we have

\[ \text{cone, SPD cone for short).} \]

parameter space (e.g., the natural parameter space \( \Theta \) is the open convex symmetric positive-definite space). Between parametric densities (e.g., the KLD) and the corresponding parameter divergence on the same set of eigenvalues of \( \theta_3 \) and \( \theta'_3 \) are identical with each other. By this and \( \theta_3, \theta'_3 \in \text{Sym}(2, \mathbb{R}) \), there exists \( h \in \text{SO}(2, \mathbb{R}) \) such that \( h \theta_3 = \theta'_3 \). Hence \( (h \theta_3), \theta_2 = g_{\theta'_1} \). We also see that

\[
(h \theta_3), \theta_1 = g_{\theta'_1}, \theta_1 = \sqrt{\theta_1} | I_2 = \sqrt{\theta'_1} | I_2 = g_{\theta'_1}, \theta'_1.
\]

Thus we have

\[(\theta'_1, \theta'_2) = (g_{\theta'_1}), (\theta_1, \theta_2).\]

**Proof.** It is obvious that \( S \) is invariant with respect to the action, that is, \( S(\theta_1, \theta_2) = S(g \theta_1, g \theta_2) \).

Assume that \( S(\theta_1, \theta_2) = S(\theta'_1, \theta'_2) \). We see that there exists \( g_{\theta_1} \in \text{SL}(2, \mathbb{R}) \) such that \( g_{\theta_1}, \theta_1 = g_{\theta'_1}, \theta'_1 = \sqrt{\theta_1} | I_2 \). Then, \( \theta_1 = \sqrt{\theta_1} | g_{\theta_1}, g_{\theta_1} \). Let \( \theta_3 := g_{\theta_1}, \theta_2 = g_{\theta_1}^\top \theta_2 g_{\theta_1}^{-1} \). Then,

\[
\text{tr}(\theta_3) = \text{tr}(g_{\theta_1}^\top g_{\theta_1}^{-1}) = \sqrt{\theta_1} | \text{tr}(\theta_2 \theta_2^{-1}).
\]

We define \( g_{\theta'_1} \) and \( \theta'_3 \) in the same manner. Then, \( \text{tr}(\theta_3) = \text{tr}(\theta'_3) \) and \( | \theta_3 | = | \theta'_3 | \). Hence the set of eigenvalues of \( \theta_3 \) and \( \theta'_3 \) are identical with each other. By this and \( \theta_3, \theta'_3 \in \text{Sym}(2, \mathbb{R}) \), there exists \( h \in \text{SO}(2, \mathbb{R}) \) such that \( h \theta_3 = \theta'_3 \). Hence \( (h \theta_3), \theta_2 = g_{\theta'_1} \). We also see that

\[
(h \theta_3), \theta_1 = g_{\theta'_1}, \theta_1 = \sqrt{\theta_1} | I_2 = \sqrt{\theta'_1} | I_2 = g_{\theta'_1}, \theta'_1.
\]

Thus we have

\[
(\theta'_1, \theta'_2) = (g_{\theta'_1}), (\theta_1, \theta_2).
\]

## 5 Kullback-Leibler divergences from reverse Bregman divergences

The Kullback-Leibler divergence is a \( f \)-divergence obtained for the generator \( f(u) = -\log u \). Since the Kullback-Leibler divergence between two densities of an exponential family amounts to a reverse Bregman divergence \([6]\), we have

\[
D_{\text{KL}}[p_{\theta_1} : p_{\theta_2}] = \int_0^\infty \int_{y=0}^\infty p_{\theta_1}(x, y) \log \frac{p_{\theta_1}(x, y)}{p_{\theta_2}(x, y)} dxdy = B_F(\theta_2 : \theta_1),
\]

where

\[
F(\theta) \equiv -\frac{1}{2} \log |\theta| - 2 \sqrt{|\theta|}
\]

since Bregman generators are equivalent modulo affine terms. The log-normalizer can be expressed as a function of \( D \): \( F(D) = -\log D - 2D \) where \( D = D(a, b, c) \).

Figure 2 schematically illustrates the correspondence between calculating a statistical divergence between parametric densities (e.g., the KLD) and the corresponding parameter divergence on the parameter space (e.g., the natural parameter space \( \Theta \) is the open convex symmetric positive-definite cone, SPD cone for short).

The matrix gradient of \( F(\theta) \) is

\[
\nabla F(\theta) = -\left( \frac{1}{2} + \sqrt{|\theta|} \right) \theta^{-\top},
\]

where \( \theta^{-\top} = (\theta^{-1})^\top \) denotes the inverse transpose operator.

Thus we have

\[
D_{\text{KL}}[p_{\theta_1} : p_{\theta_2}] = B_F(\theta_2 : \theta_1) := F(\theta_2) - F(\theta_1) - \langle \theta_2 - \theta_1, \nabla F(\theta_1) \rangle,
\]

\[
= \frac{1}{2} \log \left| \frac{\theta_1}{\theta_2} \right| + 2 \left( \sqrt{|\theta_1|} - \sqrt{|\theta_2|} \right) + \left( \frac{1}{2} + \sqrt{|\theta_1|} \right) (\text{tr}(\theta_2 \theta_2^{-1}) - 2),
\]

\[
= \log \frac{D_1}{D_2} + 2(D_1 - D_2) + \left( D_1 + \frac{1}{2} \right) (\text{tr}(\theta_2 \theta_2^{-1}) - 2)
\]

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Figure 2: Correspondence between calculating the statistical divergence $D_{KL}$ between parametric densities and the corresponding parameter divergence on the cone parameter space

### Proposition 2. The Kullback-Leibler divergence between two TYP distributions $p_{\theta_1}$ and $p_{\theta_2}$ is

$$D_{KL}[p_{\theta_1} : p_{\theta_2}] = \frac{1}{2} \log \frac{|\theta_1|}{|\theta_2|} + 2 \left( \sqrt{|\theta_1|} - \sqrt{|\theta_2|} \right) + \left( \frac{1}{2} + \sqrt{|\theta_1|} \right) (\text{tr}(\theta_2 \theta_1^{-1}) - 2).$$

Observe that the KLD is indeed a function of $D_1 = |\theta_1|$, $D_2 = |\theta_2|$ and $\text{tr}(\theta_2 \theta_1^{-1})$ as claimed in Proposition 1.

The action of $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{SL}(2, \mathbb{R})$ with $ad - bc = 1$ (for $a, b, c, d \in \mathbb{R}$) on the sample space $z \in \mathbb{H}$ is a linear fractional transformation:

$$z \mapsto \frac{az + b}{cz + d}.$$

Furthermore, the action $g.z = \frac{az + b}{cz + d}$ corresponds to the action of $g$ on $\theta$:

$$g.\theta := g^{-T} \times \theta \times g^{-1}.$$  

We check that for any $g \in \text{SL}(2, \mathbb{R})$, we have

$$D_{KL}[p_{\theta_1} : p_{\theta_2}] = D_{KL}[g.p_{\theta_1} : g.p_{\theta_2}] = D_{KL}[p_{g.\theta_1} : p_{g.\theta_2}].$$

Notice that the only symmetric Bregman divergences are squared Mahalanobis divergences. Thus the KLD between two Poincaré distributions is asymmetric: $D_{KL}[p_{\theta_1} : p_{\theta_2}] \neq D_{KL}[p_{\theta_2} : p_{\theta_1}].$

### 6 Dual moment parameterization and Fenchel-Young divergences

The dual moment parameterization $\mathcal{M}$ is $\eta = E_{p_{\theta}}[t(x)] = \nabla F(\theta)$. Observe that $\eta$ is a symmetric negative-definite matrix, and therefore the moment parameter space is

$$H = \{ \eta(\theta) : \theta \in \Theta \} = \text{Sym}^-(2, \mathbb{R}).$$
To compute the convex conjugate function $F^*(\eta) = \sup_{\theta \in \Theta} \langle \theta, \eta \rangle - F(\theta)$, first consider inverting $\eta(\theta) = D^*(\eta)\eta^{-1}$ for a scalar function $D^*(\eta) < 0$. Since $\eta(\theta(\eta)) = \eta$, and $\eta(\theta) = \nabla F(\theta)$, we get

$$\nabla F(D^*(\eta)\eta^{-1}) = -\left(\frac{1}{2} + \sqrt{|D^*(\eta)\eta^{-1}|}\right)\frac{\eta}{D^*(\eta)} = \eta.$$ 

That is, we find that

$$D^*(\eta) = -\left(\frac{1}{2} + D(\theta)\right).$$

We obtain the convex conjugate as

$$F^*(\eta) = \langle \eta, \theta(\eta) \rangle - F(\theta(\eta)),$$

where $\langle \eta, \theta(\eta) \rangle = 2D^*(\eta)$.

The Fenchel-Young divergence

$$A_{F,F^*}(\theta_1 : \eta_2) = F(\theta_1) + F^*(\eta_2) - \langle \theta_1, \eta_2 \rangle,$$

is an equivalent of the Bregman divergence using the mixed natural/moment parameters. We have

$$B_F(\theta_1 : \theta_2) = A_{F,F^*}(\theta_1 : \eta_2),$$

where $\eta_2 = \nabla F(\theta_2)$. Thus the KLD between $p_{\theta_1}$ and $p_{\theta_2}$ can be calculated in many equivalent ways:

$$D_{KL}[p_{\theta_1} : p_{\theta_2}] = B_F(\theta_1 : \theta_2) = A_{F,F^*}(\theta_1 : \eta_1) = A_{F^*,F}(\eta_1 : \theta_2) = B_{F^*}(\eta_1 : \eta_2),$$

since $(F^*)^* = F$ (Fenchel–Moreau biconjugation theorem).

Exponential families enjoy many nice properties [8]: For example, the maximum likelihood estimator (MLE) of a set of $n$ identically and independently distributed observations $z_1, \ldots, z_n$ on $\mathbb{H}$ is given in the moment parametrization as follows:

$$\hat{\eta} = \frac{1}{n} \sum_{i=1}^{n} t(z_i).$$

The MLE is consistent, asymptotically normally distributed, and efficient [8] (i.e., matching the Cramér-Rao lower bound).

7 Fisher information matrix and Riemannian metric

The Fisher information matrix for a regular exponential family [8] is defined as

$$I_\theta(\theta) = -E_{p_\theta} \left[ \nabla^2_\theta \log p_\theta(x) \right].$$

Thus for an exponential family, we have $I_\theta(\theta) = \nabla^2 F(\theta)$. Using the dual parameterization, we have $I_\eta(\eta) = \nabla^2 F^*(\eta)$. 

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By using symbolic computing (see Appendix), we obtain the Fisher information matrix of the Poincaré distributions as follows:

\[
I_{\theta_v}(\theta_v) =
\begin{pmatrix}
\frac{c^2 \sqrt{ac-b^2} - a c^3 + b^2 c^2}{\sqrt{a c-b^2} (2 a^2 c^2 - 4 a b^2 c + 2 b^4)} & \frac{- b c \sqrt{a c-b^2} - a c^2 + b^3 c}{\sqrt{a c-b^2} (2 a^2 c^2 - 2 a b^2 c + b^4)} & \frac{(a c-2 b^2) \sqrt{a c-b^2} + b^2}{2 a^2 c^2 - 4 a b^2 c + 2 b^4} \\
\frac{- b c \sqrt{a c-b^2} - a c^2 + b^3 c}{\sqrt{a c-b^2} (2 a^2 c^2 - 2 a b^2 c + b^4)} & \frac{2 a c \sqrt{a c-b^2} - a c^2 + b^2}{\sqrt{a c-b^2} (2 a^2 c^2 - 2 a b^2 c + b^4)} & \frac{(a c-b^2) \sqrt{a c-b^2} + b^2}{2 a^2 c^2 - 4 a b^2 c + 2 b^4} \\
\frac{(a c-2 b^2) \sqrt{a c-b^2} + b^2}{a^2 c^2 - 2 a b^2 c + b^4} & \frac{- a b \sqrt{a c-b^2} - a c^2 + b^3 c}{\sqrt{a c-b^2} (2 a^2 c^2 - 2 a b^2 c + b^4)} & \frac{(a c-2 b^2) \sqrt{a c-b^2} + b^2}{2 a^2 c^2 - 4 a b^2 c + 2 b^4}
\end{pmatrix}.
\]

The determinant is

\[
|I_{\theta_v}(\theta_v)| = -\frac{4}{4 a^3 c^3 - 12 a^2 b^2 c^2 + 12 a b^4 c - 4 b^6}.
\]

Since \(I_{\theta_v}(\theta_v)\) is positive-definite, we have \(|I_{\theta_v}(\theta_v)| > 0\). For example, when \(b = 0\), we can check easily that \(|I_{\theta_v}(a, 0, c)| = \frac{1 + 4 a c - 4 \sqrt{a c}}{4 a^2 c} > 0\).

The Fisher information matrix is used to define the Fisher information metric \(g\) (expressed as the FIM \(I_{\theta_v}\) in the local coordinate system \(\theta_v\)) on the Riemannian manifold of the TYP distributions \(\mathcal{P}\). The length element \(ds^2 = d\theta_v^\top I_{\theta_v}(\theta_v) d\theta_v\) is independent of the parameterization. The Rao distance \([5]\) is the geodesic distance of the Riemannian manifold \((\mathcal{P}, g)\).

Note that this Riemannian metric is invariant under the linear fractional transformation action of \(\text{SL}(2, \mathbb{R})\). Moreover, any \(f\)-divergence yields a scaled metric \(f''(1) I_\theta(\theta)\) since

\[
I_g[p_\theta : p_{\theta+\delta\theta}] = \frac{1}{2} f''(1) I_\theta(\theta) d\theta + o(||d\theta||^2),
\]

when \(d\theta \to 0\), see \([3]\).

Many Riemannian metrics and related Riemannian distances have been investigated and classified by invariance and other properties on the open cone of positive-definite matrices. We refer the reader to the PhD thesis \([31]\) for a panorama of Riemannian metrics.

8 Differential entropy

The differential entropy can be calculated as follows \([25]\):

\[
h[p_\theta] = -F^*(\eta) - E_{p_\theta}[k(x)] = -F^*(\eta) + 2 E_{p_\theta}[\log y].
\]

Thus we need to calculate \(E_{p_\theta}[\log y]\). Let \(D := \sqrt{ac-b^2}\). By the change-of-variable \(y = \frac{D}{a} e^z\),

\[
E_{p_\theta}[\log y] = \sqrt{\frac{D}{\pi}} e^{2D} \int_\mathbb{R} \left( \log \frac{D}{a} + z \right) \exp \left(-2D \cosh(z) - \frac{z^2}{2}\right) dz.
\]

Let \(K_\nu(z)\) be the modified Bessel function of second kind of order \(\nu\). Then, it has the following integral expression \([17]\,\text{Eq. (8.432.1)}\):

\[
K_\nu(z) = \int_0^\infty \exp(-z \cosh(t)) \cosh(\nu t) dt, \quad \text{Re}(z) > 0.
\]

Then,

\[
\int_\mathbb{R} z \exp \left(-2D \cosh(z) - \frac{z^2}{2}\right) dz = -2 \int_0^\infty z \exp(-2D \cosh(z)) \sinh \left(\frac{z}{2}\right) dz = -2 \frac{\partial}{\partial \nu} \bigg|_{\nu=1/2} K_\nu(2D).
\]

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Therefore,
\[ E_{p_θ}[\log y] = 2\sqrt{\frac{D}{\pi}} e^{2D} \left( \log \left( \frac{D}{a} \right) K_{1/2}(2D) - \frac{\partial}{\partial \nu} \bigg|_{\nu=1/2} K_{\nu}(2D) \right). \]

By [17, Eq. (8.469.3) and (8.486(1.21)],
\[ E_{p_θ}[\log y] = \log \left( \frac{D}{a} \right) - \int_0^\infty \frac{e^{-t}}{t + 4D} dt. \]

That is, we have
\[ E_{p_θ}[\log y] = \log \frac{D}{a} - e^{4D} \Gamma(0, 4D), \]
where
\[ \Gamma(a, x) = \int_x^\infty t^{a-1} e^{-t} dt \]
is the upper incomplete Gamma function.

**Proposition 3.** The differential entropy \( h[p_θ] \) of a Tojo-Yoshino Poincaré distribution \( p_θ \) is
\[ h(p_θ) = 1 + \log \frac{\pi}{D} + 2 \log \frac{D}{a} - e^{4D} \Gamma(0, 4D), \tag{2} \]
where \( D = \sqrt{|θ|} \).

**Example 1.** Let us report a numerical example as follows:
\[ \theta_1 = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} \\ 1 & \frac{1}{2} \end{bmatrix}, \quad \theta_2 = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} \\ 1 & \frac{1}{2} \end{bmatrix}. \]

We have the dual parameters:
\[ \eta_1 \simeq \begin{bmatrix} -0.488 & 0.244 \\ 0.244 & -3.906 \end{bmatrix}, \quad \eta_2 \simeq \begin{bmatrix} -3.132 & 0.391 \\ 0.391 & -0.783 \end{bmatrix}. \]

The dual potential functions are evaluated as
\[ F(\theta_1) \simeq -3.114, \quad F(\theta_2) \simeq -1.904, \]
and
\[ F^*(\eta_1) \simeq -0.669, \quad F^*(\eta_2) \simeq -1.032, \]
We find that the forward and reverse Kullback-Leibler divergences are
\[ D_{KL}[p_{θ_1} : p_{θ_2}] \simeq 5.360, \quad D_{KL}[p_{θ_2} : p_{θ_1}] \simeq 8.573 \]
The differential entropies of \( p_{θ_1} \) and \( p_{θ_2} \) are
\[ h[p_{θ_1}] \simeq -0.608, \quad h[p_{θ_2}] \simeq 3.074 \]
Now choose the following transformation matrix \( g \) of \( SL(2, \mathbb{R}) \):
\[ g = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}. \]
Then we have

\[ g.\theta_1 = \begin{bmatrix} \frac{31}{4} & -\frac{31}{4} \\ -\frac{4}{3} & \frac{4}{3} \end{bmatrix}, \quad g.\theta_2 = \begin{bmatrix} \frac{9}{2} & -\frac{9}{2} \\ -\frac{4}{3} & \frac{4}{3} \end{bmatrix}. \]

We check that the invariance of the KLD by the action of \( g \):

\[ D_{KL}[g.p_{\theta_1} : g.p_{\theta_2}] = D_{KL}[p_{g.\theta_1} : p_{g.\theta_2}] = D_{KL}[p_{\theta_1} : p_{\theta_2}] \approx 5.360. \]

Figure 3 displays the distributions \( p_{\theta_1}, p_{\theta_2}, p_{g.\theta_1}, \) and \( p_{g.\theta_2} \).

9 The skew Bhattacharyya distances

The \( \alpha \)-Bhattacharyya divergence between two distributions \( p(x, y) \) and \( q(x, y) \) with support \( \mathbb{H} \) is defined by

\[ D_{\alpha}[p : q] = -\log \int_{\mathbb{H}} p^\alpha(x, y)q^{1-\alpha}(x, y)dxdy. \]

When the densities belong the same exponential family with cumulant function \( F(\theta) \), i.e., \( p = p_{\theta_1} \) and \( q = p_{\theta_2} \) of an exponential family, the \( \alpha \)-Bhattacharyya divergence amounts to a skew Jensen divergence:

\[ D_{\alpha}[p_{\theta_1} : p_{\theta_2}] = J_{F,\alpha}(\theta_1 : \theta_2), \]

where

\[ J_{F,\alpha}(\theta_1 : \theta_2) = \alpha F(\theta_1) + (1 - \alpha)F(\theta_2) - F(\alpha\theta_1 + (1 - \alpha)\theta_2). \]
Moreover, the KLD between densities of an exponential family tends asymptotically to a scaled skewed Jensen divergence [23]:

\[ D_{KL}[p_{\theta_1} : p_{\theta_2}] = \lim_{\alpha \to 0} \frac{1}{\alpha(1-\alpha)} J_{F,\alpha}(\theta_1 : \theta_2) \]

with the skewed Jensen divergence for the TYP family is

\[ J_{F,\alpha}(\theta_1 : \theta_2) = \frac{1}{2} \log \left( \frac{|(1-\alpha)\theta_1 + \alpha \theta_2|}{|\theta_1|^{1-\alpha}|\theta_2|^\alpha} \right) \]

\[ + 2 \left( \sqrt{|(1-\alpha)\theta_1 + \alpha \theta_2|} - ((1-\alpha)\sqrt{|\theta_1|} + \alpha \sqrt{|\theta_2|}) \right). \quad (3) \]

By choosing \( \alpha \) small (say, \( \alpha = 0.01 \)), we can approximate the KLD by a scaled \( \alpha \)-skewed Jensen divergence which does not require to calculate the gradient term \( \nabla F(\theta) \).

10 Concluding remarks

We have considered the Tojo-Yoshino Poincaré exponential family [33] \( \{p_{\theta(a,b,c)}\} \) defined on the hyperbolic Poincaré upper plane support. Since the information geometry of exponential families are dually flat spaces induced by the convex log-normalizer of the family, we used the canonical Bregman divergence and the log-normalizer convex conjugate to recover the Kullback-Leibler divergence and differential entropy of these distributions:

\[ D_{KL}[p_{\theta_1} : p_{\theta_2}] = \frac{1}{2} \log \frac{|\theta_1|}{|\theta_2|} + 2 \left( \sqrt{|\theta_1|} - \sqrt{|\theta_2|} \right) + \left( \frac{1}{2} + \sqrt{|\theta_1|} \right) (\text{tr}(\theta_2 \theta_1^{-1}) - 2), \]

\[ h[p_{\theta}] = 1 + \log \frac{\pi}{\sqrt{|\theta|}} + 2 \log \frac{\sqrt{|\theta|}}{a} - 2e^4 \sqrt{|\theta|} \Gamma(0, 4\sqrt{|\theta|}). \]

Moreover, the Fisher information matrix was calculated explicitly as the Hessian of the log-normalizer.

Calculations with the computer algebra system Maxima

The Fisher information metric is expressed using the natural coordinate system as the Hessian of the log-normalizer. Using Maxima [https://maxima.sourceforge.io/], we can calculate symbolically the Hessian using the following snippet code:

\[
F(a,b,c):=\text{log}( (\%pi/sqrt(a*c-b*b))*exp(2*sqrt(a*c-b*b)) );
\]

\[
hessian(F(a,b,c),[a,b,c]);
\]

\[
tex(ratsimp(%));
\]

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