CONVERGENCE OF NONAUTONOMOUS MULTIVALUED PROBLEMS WITH LARGE DIFFUSION TO ORDINARY DIFFERENTIAL INCLUSIONS

Dedicated to Professor Tomás Caraballo on occasion of his 60th Birthday

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ABSTRACT. In this work we consider a family of nonautonomous partial differential inclusions governed by $p$-laplacian operators with variable exponents and large diffusion and driven by a forcing nonlinear term of Heaviside type. We prove first that this problem generates a sequence of multivalued nonautonomous dynamical systems possessing a pullback attractor. The main result of this paper states that the solutions of the family of partial differential inclusions converge to the solutions of a limit ordinary differential inclusion for large diffusion and when the exponents go to 2. After that we prove the upper semicontinuity of the pullback attractors.

1. Introduction. It is well known that the solutions of reaction-diffusion systems with suitable boundary conditions and large diffusion are close to the solutions of a suitable ordinary differential system (see e.g. the classical papers [17, 26, 27, 28, 24, 14, 15, 13] and, more recently, [1, 44, 11, 37, 38, 12]). Such property is very important, as in particular it allows us to extrapolate information about the dynamics inside the global attractor for the reaction-diffusion system once we know the dynamics of the ordinary differential equation. Due to the complexity of studying the structure of attractors in infinite dimensions, this method is very helpful.

In the last years some authors have considered this problem for parabolic equations generated by a $p$-laplacian operator with variable exponent (see [34, 40, 23, 36]), which appear in models of electrorheological fluids and image processing (see e.g. [20, 25] and the references therein).

As far as we know, little is done so far in this direction concerning parabolic differential inclusions with large diffusion. For differential parabolic inclusions with multivalued right-hand side of Lipschitz type the convergence of solutions and global attractors to the corresponding ones of a limit ordinary differential inclusion was proved in [35].

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In particular, we are interested in a type of differential inclusions generated by a nonlinear function having a discontinuity, which can be expressed as a differential inclusion making use of a Heaviside function. Such inclusions have been used for modelling processes of combustion in porous media [22], the conduction of electrical impulses in nerve axons (see [42, 43]) or the surface temperature on Earth (see [7, 19]) among others.

It is quite interesting, challenging and very difficult to study the structure of the global attractors for this type of inclusions. Nevertheless, some progress has been done already in the papers [2, 4, 5].

In this paper we study a parabolic differential inclusion governed by an elliptic \( p \)-laplacian operator with variable exponent, driven by a nonautonomous Heaviside forcing term and with homogeneous Neumann boundary conditions. As in the aforementioned papers, when the diffusion becomes large the solutions of the inclusion tend to be constant in space, and in this way we are able to prove that they converge to the solutions of the scalar ordinary differential inclusion considered in [10] when the exponents of the \( p \)-laplacian operator go to 2. Moreover, we prove that this problem generates a nonautonomous multivalued dynamical system possessing a pullback attractor and that the sequence of attractors behaves upper semicontinuously with respect to the pullback attractor of the limit inclusion. We observe that the structure of the pullback attractor of this ordinary inclusion is detaily described in [10]. We can expect that the dynamics in the attractor of the partial differential inclusion is similar to some extent to the dynamics inside the attractor of the limit problem. This question will be addressed in a future work.

2. Setting of the problem. Throughout the paper for any two sets \( A, B \) in a Banach space \( X \) we denote the Hausdorff semidistance from \( A \) to \( B \) by \( \text{dist}(A, B) \), and the Hausdorff distance between \( A \) and \( B \) by \( \text{dist}_H(A, B) \).

Let \( \Omega \subset \mathbb{R}^n, n \geq 1 \), be a smooth bounded domain with boundary \( \partial \Omega \) and \( \mathcal{H} := L^2(\Omega) \), with norm \( \| \cdot \|_{\mathcal{H}} \) and scalar product \( (\cdot, \cdot) \). The aim of this work is to study the asymptotic behavior of the solutions as \( s \to \infty \) for the multivalued initial value problem

\[
\begin{cases}
\frac{\partial u}{\partial t}(t) - \text{div}(D_s|\nabla u(t)|^{p_s(x)-2}\nabla u(t)) + \lambda |u(t)|^{p_s(x)-2}u(t) \in F(t, u(t)), \\
u(\tau) = u_{\tau} \in \mathcal{H},
\end{cases}
\tag{1}
\]

with homogeneous Neumann boundary conditions, where \( \lambda > 0, D_s \in [1, \infty), p_s(\cdot) \in C(\Omega), p_s^- := \min_{x \in \Omega} p_s(x) \geq 2 \), and there exists a constant \( p_0 \geq 2 \) such that \( p_s^+ := \max_{x \in \Omega} p_s(x) \leq p_0 \), for all \( s \in \mathbb{N} \). We assume that \( p_s(\cdot) \to 2 \) in \( L^\infty(\Omega) \) and \( D_s \to \infty \) as \( s \to \infty \).

Let \( b: \mathbb{R} \to \mathbb{R}^+ \) be a continuous function satisfying

\[ 0 < b_0 \leq b(t) \leq b_1, \]

and consider the Heaviside function \( H \) given by

\[
H(u) = \begin{cases} 
-1, & \text{if } u < 0, \\
[-1, 1], & \text{if } u = 0, \\
1, & \text{if } u > 0.
\end{cases}
\]

The function \( F \) in problem (1) is defined as \( F: \mathbb{R} \times \mathcal{H} \to 2^\mathcal{H} \) where

\[ F(t, y) = \{ u \in \mathcal{H} : u(x) \in b(t)H(y(x)) \text{ for a.a. } x \in \Omega \}. \tag{2} \]
We denote by \( f : \mathbb{R} \times \mathbb{R} \to P(\mathbb{R}) \) the multivalued function given by \( f(t,u) = b(t)H(u) \). Then \( f \) has nonempty, closed, bounded and convex values, and for all \( t \in \mathbb{R} \) the map \( f(t,\cdot) : \mathbb{R} \to P(\mathbb{R}) \) is upper semicontinuous. Moreover, for any \( t,s \in \mathbb{R}^+ \), \( u \in \mathbb{R} \),

\[
\text{dist}_H(f(t,u),f(s,u)) = |b(t) - b(s)|,
\]

and

\[
|f(t,u)|_+ := \sup_{y \in f(t,u)} |y| = b(t).
\]

Following the same arguments as in the proof of Lemma 6.28 in [29] we obtain:

**Lemma 2.1.** The map (2) satisfies the following properties:

i) \( F : \mathbb{R} \times H \to 2^H \) has nonempty, closed, bounded and convex values;

ii) For any \( t \in \mathbb{R} \) the map \( F(t,\cdot) : H \to P(H) \) is \( u \)-upper semicontinuous, that is, for all \( \epsilon > 0 \), \( y_0 \in H \), there is \( \delta = \delta(t,\epsilon,y_0) > 0 \) such that \( F(t,y) \subset O_\epsilon(F(t,y_0)) \), for all \( y \in O_\delta(y_0) \):

iii) For all \( y \in H \), \( \tau \in \mathbb{R} \) the map \( F(\cdot,y) : [\tau, +\infty) \to P(H) \) has a measurable selection, that is, there exists a measurable function \( h : [\tau, +\infty) \to H \) such that \( h(t) \in F(t,y) \) for a.a. \( t \in \mathbb{R}^+ \);

iv) For all \( y \in H \) and for a.a. \( t \in \mathbb{R}^+ \),

\[
\|F(t,y)\|_+ := \sup_{z \in F(t,y)} \|z\| \leq |\Omega|^{1/2}b_1.
\]

Now we recall the properties of elliptic operators in variable spaces in order to establish existence of solutions for problem (1).

For \( p(\cdot) \in L^\infty_+ (\Omega) := \{ q \in L^\infty(\Omega) : \text{essinf } q \geq 1 \} \) we recall that

\[
L^{p(\cdot)}(\Omega) := \left\{ u : \Omega \to \mathbb{R} : u \text{ is measurable, } \int_\Omega |u(x)|^{p(x)}dx < \infty \right\}
\]

and

\[
W^{1,p(\cdot)}(\Omega) = \{ u \in L^{p(\cdot)}(\Omega) : \nabla u \in L^{p(\cdot)}(\Omega) \}.
\]

The norms in the spaces \( L^{p(\cdot)}(\Omega), W^{1,p(\cdot)}(\Omega) \) are given by

\[
\rho(u) := \int_\Omega |u(x)|^{p(x)}dx, \quad \|u\|_{L^{p(\cdot)}(\Omega)} := \inf \left\{ \gamma > 0 : \rho\left(\frac{u}{\gamma}\right) \leq 1 \right\},
\]

\[
\|u\|_{W^{1,p(\cdot)}(\Omega)} = \|u\|_{L^{p(\cdot)}(\Omega)} + \|\nabla u\|_{L^{p(\cdot)}(\Omega)}.
\]

It is well known that \( W^{1,p(\cdot)}(\Omega) \subset H \subset (W^{1,p(\cdot)}(\Omega))^* \) with continuous and dense embeddings if \( p^- \geq 2 \). We refer the reader to [20, 21] and the references therein to see the properties of the Lebesgue and Sobolev spaces with variable exponents.

In the sequel for \( p_s(\cdot) \) as defined above we shall use the family of spaces \( Y_s = W^{1,p_s(\cdot)}(\Omega) \). In this case the modular \( p \) will be denoted by \( p_s \).

Consider the operator \( A_s \) defined in \( Y_s \) in such a way that to each \( u \in Y_s \) we associate the element \( A_s u : Y_s \to \mathbb{R} \) of \( Y_s^* \) given by

\[
A_s u(v) := D_s \int_\Omega |\nabla u(x)|^{p_s(x)-2}\nabla u(x) \cdot \nabla v(x)dx + \lambda \int_\Omega |u(x)|^{p_s(x)-2}u(x)v(x)dx.
\]

The authors in [39] (see also [23, 30]) proved that the operator \( A_s : Y_s \to Y_s^* \), with domain \( Y_{s^*} \), is maximal monotone, \( A_s(Y_s) = Y_{s^*} \), the realization operator of \( A_s \) at \( H = L^2(\Omega) \), denoted by \( A^*_H \), is maximal monotone in \( H \) and \( A^*_H \) is the
subdifferential $\partial \varphi_{p_s(\cdot)}$ of the convex, proper and lower semicontinuous map $\varphi_{p_s(\cdot)} : L^2(\Omega) \to \mathbb{R} \cup \{+\infty\}$ given by

$$
\varphi_{p_s(\cdot)}(u) := \begin{cases} 
D_s \int_\Omega \frac{1}{p_s(x)} |\nabla u|^2 dx + \lambda \int_\Omega \frac{1}{p_s(x)} |u|^2 dx, & \text{if } u \in Y_s,

\infty, & \text{otherwise}.
\end{cases}
$$

(3)

Moreover, as our operator is of subdifferential type, the level sets

$$
M^R_H = \{ u \in \mathcal{H} : \|u\|_{\mathcal{H}} \leq R, \varphi_{p_s(\cdot)}(u) \leq R \}
$$

are compact in $\mathcal{H}$ for all $R > 0$ and $cl_{\mathcal{H}}(D(\varphi_{p_s(\cdot)})) = \mathcal{H}$.

Hence, since $F$ satisfies the conditions given in Lemma 2.1, we can obtain the existence of global strong solutions for problem (1) using the results in Chapter 6 of [29].

We recall the definition of strong solutions for problem (1). Consider a multivalued initial value problem of the form

$$
\begin{cases}
\frac{du}{dt}(t) + \partial \varphi_{p_s(\cdot)}(u(t)) \in F(t,u(t)), & t \in (\tau,T),

u(\tau) = \xi, & \xi \in \mathcal{H}.
\end{cases}
$$

(4)

**Definition 2.2.** By a strong solution of (4) we mean a continuous function $u : [\tau,T] \to \mathcal{H}$ such that $u(\cdot)$ is absolutely continuous on any closed subinterval of $(\tau,T)$ and with the properties:

1) $u(t) \in D(\partial \varphi_{p_s(\cdot)})$, a.e. on $(\tau,T)$;

2) There exists $f \in L^1(\tau,T;\mathcal{H})$ such that $f(t) \in F(t,u(t))$ for a.a. $t \in (\tau,T)$ and

$$
\frac{du}{dt}(t) + \partial \varphi_{p_s(\cdot)}(u(t)) = f(t), \text{ a.e. on } (\tau,T);
$$

3) $u(\tau) = \xi$.

We put

$$
D_s(u_\tau,t,\tau) := \{ u(t) : u(\cdot) \text{ is a strong solution of (1) with } u(\tau) = u_\tau \},
$$

(5)

By Theorem 6.11, Lemma 6.16 and 6.17 in [29] we obtain the existence of a global strong solution for problem (1).

Let us denote by $B_r$ the ball of radius $r$ centered at 0.

**Theorem 2.3.** If $F$ is as in (2), then the multivalued problem (1) has a strong solution for every $u_\tau \in \mathcal{H}$. Moreover, for arbitrary $r \geq 0$, $T > \tau$, $t \in [\tau,T]$ the set $D_s(B_r,t,\tau) := \bigcup_{u_\tau \in B_r} D_s(u_\tau,0,\tau)$ is connected in $\mathcal{H}$ and for arbitrary $\epsilon > 0$ the set $\bigcup_{t \in [\tau+\epsilon,T]} D_s(B_r,t,\tau)$ is precompact in $\mathcal{H}$.

**Definition 2.4.** The continuous function $u : [\tau,T] \to \mathcal{H}$ is called an integral solution of problem (4) on $[\tau,T]$, if $u(\tau) = \xi$ and there exists $f \in L^1(\tau,T;\mathcal{H})$ such that $f(t) \in F(t,u(t))$, for a.a. $t \in (\tau,T)$, and for any $\eta \in D(\partial \varphi_{p_s(\cdot)})$, $v \in -\partial \varphi_{p_s(\cdot)}(\eta)$ we have that

$$
\|u(t) - \eta\|^2 \leq \|u(s) - \eta\|^2 + 2 \int_s^t (f(r) + v(u(r) - \eta)) dr, \forall t \geq s \geq \tau.
$$

(6)

It is well known that if the selection $f$ satisfies $f \in L^2(\tau,T;\mathcal{H})$, then $u(\cdot)$ is a strong solution of (4) if and only if it is an integral solution (see for example the proof of Lemma 6.16 [29]). For our problem (1) due to property iv) in Lemma 2.1 the fact that $f \in L^2(\tau,T;\mathcal{H})$ is true for arbitrary strong or integral solutions. Therefore, the sets of strong and integral solutions coincide.
The following lemma establishes that the translation and concatenation of strong solutions are also strong solutions.

**Lemma 2.5.** If \( u(\cdot) \) is a strong solution of (1) on \([\tau, T]\), then for any \( 0 < s < T - \tau \) the function \( y(\cdot) = u(\cdot) \mid_{[\tau + s, T]} \) is a strong solution.

If \( u(\cdot) \) is a strong solution of (1) on \([\tau, s]\) and \( v(\cdot) \) is a strong solution of (1) on \([s, T]\) such that \( v(s) = u(s) \), then the function

\[
z(t) = \begin{cases} u(t) & \text{if } \tau \leq t \leq s, \\ v(t) & \text{if } s \leq t \leq T,
\end{cases}
\]

is a strong solution.

**Proof.** The proof of these results for integral solutions is quite similar to the ones in Lemmas 6.31 and 6.32 of [29]. Taking into account that in our case strong solutions and integral solutions are the same, both statements follow. \( \square \)

3. Existence and properties of pullback attractors. In this section we will define a multivalued nonautonomous dynamical systems from the solutions of the family of problems (1). We will start by obtaining some estimates of the solutions which are uniform on \( s \), proving after that the existence of a sequence of pullback attractors and the compactness of its union with respect to the parameter \( s \) and the time \( t \).

It is worth noticing that by concatenation every strong solution can be extended to a globally defined one, that is, which exists for any \( t \geq \tau \). Denote by \( R_s(u_\tau, \tau) \) the set of all globally defined strong solutions with initial condition \( u_\tau \) at time \( \tau \).

First, we will prove the existence of an absorbing ball which does not depend on \( s \). We start with an auxiliary lemma.

**Lemma 3.1.** Let \( p_s^-, p_s^+ \in [2, p_0] \), \( p_0 > 2 \), for any \( s \). Then there exists \( K(\Omega) \) such that

\[
\| u \|_H \leq K(\Omega) \| u \|_{L^{p_s(\cdot)}}, \quad \text{for any } u \in L^{p_s(\cdot)} \mbox{ and } s \in \mathbb{N}. \tag{7}
\]

**Proof.** Let \( q_s(x) \) be such that

\[
\frac{1}{p_s(x)} + \frac{1}{q_s(x)} = \frac{1}{2} \mbox{ for a.a. } x \in \Omega.
\]

Hence,

\[
q_s(x) = \frac{2p_s(x)}{p_s(x) - 2}
\]

and

\[
\bar{q} := \frac{4}{p_0 - 2} \leq \frac{2p_s^-}{p_s^+ - 2} \leq q_s(x).
\]

By the Hölder inequality [20] we have that

\[
\| u \|_H \leq 2 \| u \|_{L^{p_s(\cdot)}} \| 1 \|_{L^{q_s(\cdot)}}.
\]

We estimate now \( \| 1 \|_{L^{q_s(\cdot)}} \) by a constant \( K(\Omega) \) independent of \( s \). We assume that \( \| 1 \|_{L^{q_s(\cdot)}} > 1 \), as otherwise we are done. From

\[
\int_{\Omega} \frac{1}{\gamma q_s(x)} \, dx \leq \int_{\Omega} \frac{1}{\gamma} \, dx = \frac{1}{\gamma} |\Omega|, \quad \text{if } \gamma > 1,
\]

we have

\[
\| 1 \|_{L^{q_s(\cdot)}} = \inf \{ \gamma > 0 : \int_{\Omega} \frac{1}{\gamma q_s(x)} \, dx \leq 1 \} \leq |\Omega|^{\frac{1}{\bar{q}}}.
\]
Hence,
\[ \|1\|_{L^s(\cdot)} \leq \max\{1, |\Omega|^s\} = K(\Omega). \]

Finally,
\[ \|u\|_H \leq 2K(\Omega) \|u\|_{L^p_s(\cdot)} = K(\Omega) \|u\|_{L^p_s(\cdot)}. \]

\[ \Box \]

**Corollary 1.** Let \( p_s^-, p_s^+ \in [2, p_0], \ p_0 > 2, \) for any \( s. \) Then there exists \( K(\Omega) \) such that
\[ \|u\||_H \leq K(\Omega) \|u\|_{Y_s}, \] (8)
\[ \|u\||_{H^1(\Omega)} \leq K(\Omega) \|u\|_{Y_s}, \] for any \( u \in Y_s. \) (9)

**Theorem 3.2.** Let \( p_s^-, p_s^+ \in [2, p_0], \ p_0 > 2, \) for any \( s. \) Then there exists a ball \( B_0 \) of radius \( R_0, \) which does not depend on \( s, \) such that for any bounded set \( B \) in \( \mathcal{H} \) there exists a time \( T = T(B) \) such that \( u(t) \in B_0 \) if \( t - \tau \geq T \) for any \( u \in \mathcal{R}_s(u_\tau, \tau), \ u_\tau \in B. \)

Also, for any bounded set \( B \) there exists a constant \( R = R(B) \) such that \( \|u(t)\|_H \leq R, \) for all \( t \geq \tau \) and any \( u \in \mathcal{R}_s(u_\tau, \tau), \ u_\tau \in B. \)

**Proof.** As \( u \) is a solution of (1), there exists \( f \in L^1([\tau, T]; \mathcal{H}) \) such that \( f(t) \in F(t, u(t)) \) and
\[ \frac{du}{dt}(t) + \partial \varphi_{p_s}(u(t)) = f(t), \] (10)
a.e. on \([\tau, T], \) where \( T > \tau \) is arbitrary. Multiplying the equation (10) by \( u(t), \) we obtain
\[ \frac{1}{2} \frac{d}{dt} \|u(t)\|^2_H + \langle \partial \varphi_{p_s}(u(t)), u(t) \rangle = \langle f(t), u(t) \rangle. \]

If \( \|u(t)\|_{L^p_s(\cdot)(\Omega)} \geq 1 \) and \( \|\nabla u(t)\|_{L^p_s(\cdot)(\Omega)} \geq 1, \) then by Lemma 2.3 in [30]
\[ \langle \partial \varphi_{p_s}(u(t)), u(t) \rangle \geq \frac{1}{2p_s^{-1}} \|u(t)\|_{Y_s}^{p_s^-}, \]
and then
\[ \frac{1}{2} \frac{d}{dt} \|u(t)\|^2_H \leq -\frac{1}{2p_s^{-1}} \|u(t)\|_{Y_s}^{p_s^-} + \langle f(t), u(t) \rangle. \]

Using the Cauchy-Schwarz inequality and that \( \|f(t)\|_H \leq |\Omega|^{1/2} b_1, \) we obtain
\[ \frac{1}{2} \frac{d}{dt} \|u(t)\|^2_H \leq -\frac{1}{2p_s^{-1}} \|u(t)\|_{Y_s}^{p_s^-} + \|f(t)\|_H \|u(t)\|_H \]
\[ \leq -\frac{1}{2p_s^{-1}} \|u(t)\|_{Y_s}^{p_s^-} + |\Omega|^{1/2} b_1 \|u(t)\|_H. \]

If \( p_s^- > 2 \) the standard Young inequality
\[ a^2 \leq \frac{2}{p_s} a^{p_s^-} + \frac{p_s - 2}{p_s} \]
and (8) imply that
\[ \frac{1}{2} \frac{d}{dt} \|u(t)\|^2_H \leq -\frac{1}{2p_s^{-1}} \left( \frac{p_s^-}{2} \|u(t)\|^2_{Y_s} - \frac{p_s^- - 2}{2} \right) + |\Omega|^{1/2} b_1 \|u(t)\|_H \]
\[ \leq -\frac{p_s^-}{K^2 2p_s} \|u(t)\|^2_H + \frac{p_s^- - 2}{2p_s} + \frac{K^2 2p_s^{-1}}{p_s} |\Omega| b_1^2 + \frac{p_s^-}{K^2 2p_s^{-1}} \|u(t)\|^2_H, \]
\[ \frac{d}{dt} \|u(t)\|_H^2 + \frac{p_s^-}{K^{2p_s^-}} \|u(t)\|_H^2 \leq \frac{p_s^- - 2}{2p_s^- - 1} + \frac{K^{2p_s^-}}{p_s^+} |\Omega| b_1^2. \]

Since \( p_s^- \in [2, p_0] \), there exist constants \( \gamma_{11}, \delta_{11} > 0 \) such that
\[ \gamma_{11} \leq \frac{p_s^-}{K^{2p_s^-}} \leq \frac{2p_s^- - 2}{2p_s^- - 1} + \frac{K^{2p_s^-}}{p_s^+} |\Omega| b_1^2 \leq \delta_{11} \] for any \( s \).

If \( p_s^- = 2 \), then using (8) we have
\[ \frac{1}{2} \frac{d}{dt} \|u(t)\|_H^2 \leq - \frac{1}{K^{2p_s^- - 1}} \|u(t)\|_H^2 + 2p_s^- - 2K^2 |\Omega| b_1^2 + \frac{1}{K^{2p_s^-}} \|u(t)\|_H^2, \]
so
\[ \frac{d}{dt} \|u(t)\|_H^2 + \frac{1}{K^{2p_s^- - 1}} \|u(t)\|_H^2 \leq 2p_s^- - 2K^2 |\Omega| b_1^2, \]
and we chose \( \gamma_{12}, \delta_{12} \) such that
\[ \gamma_{12} \leq \frac{1}{K^{2p_s^- - 1}} \leq 2p_s^- - 2K^2 |\Omega| b_1^2 \leq \delta_{12}. \]

Then putting \( \gamma_1 = \min\{\gamma_{11}, \gamma_{12}\} \), \( \delta_1 = \max\{\delta_{11}, \delta_{12}\} \) we have
\[ \frac{d}{dt} \|u(t)\|_H^2 + \gamma_1 \|u(t)\|_H^2 \leq \delta_1. \] (12)

If \( \|u(t)\|_{L^{p_s^+}(\Omega)} \leq 1 \) and \( \|\nabla u(t)\|_{L^{p_s^+}(\Omega)} \geq 1 \), then by Lemma 2.3 in [30]
\[ \langle \partial \varphi_{p_s^+}(u(t)), u(t) \rangle \geq \|\nabla u(t)\|_{L^{p_s^+}(\Omega)}^2 + \|u(t)\|_{L^{p_s^+}(\Omega)}^2, \]
and then
\[ \frac{1}{2} \frac{d}{dt} \|u(t)\|_H^2 \leq - (\|\nabla u(t)\|_{L^{p_s^+}(\Omega)}^2 + \|u(t)\|_{L^{p_s^+}(\Omega)}^2) + \langle f(t), u(t) \rangle \]
\[ \leq - (\|\nabla u(t)\|_{L^{p_s^+}(\Omega)}^2 + \|u(t)\|_{L^{p_s^+}(\Omega)}^2) + |\Omega|^{1/2} b_1 \|u(t)\|_H. \]

If \( p_s^- > 2 \), then using (8) and (11) for both \( p_s^- \) and \( p_s^+ \) we have
\[ \frac{1}{2} \frac{d}{dt} \|u(t)\|_H^2 \leq \left( \frac{p_s^-}{2} - \frac{p_s^- - 2}{2} - \frac{p_s^-}{2} \right) \|u(t)\|_{L^{p_s^+}(\Omega)}^2 \]
\[ + \frac{p_s^- - 2}{2} + |\Omega|^{1/2} b_1 \|u(t)\|_H \]
\[ \leq - \frac{p_s^-}{4K^2} \|u(t)\|_H^2 + \frac{p_s^- + p_s^+}{2} \|u(t)\|_H^2 + \frac{K^2 |\Omega| b_1^2}{p_s^-}. \]

Proceeding in a similar way for \( p_s^- = 2, p_s^+ > 2 \) and \( p_s^- = p_s^+ = 2 \) we obtain the existence of \( \gamma_2, \delta_2 > 0 \) such that (12) is satisfied but replacing \( \gamma_1, \delta_1 \) by \( \gamma_2, \delta_2 \).

In a similar way we can obtain this inequality for the other cases, so for \( \gamma = \min\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}, \delta = \max\{\delta_1, \delta_2, \delta_3, \delta_4\} \) the inequality
\[ \frac{d}{dt} \|u(t)\|_H^2 + \gamma \|u(t)\|_H^2 \leq \delta \]
is satisfied for almost every \( t > \tau \). Therefore, Gronwall’s lemma gives
\[ \|u(t)\|_H^2 \leq e^{-\gamma(t-\tau)} \|u_\tau\|_H^2 + \frac{\delta}{\gamma}, \]
for any \( t \geq \tau \).

Putting \( R_0 = \sqrt{1 + \frac{\delta}{\gamma}} \), we obtain that for any bounded set \( B \) there is a constant \( T(B) \) such that
\[ \|u(t)\|_H^2 \leq 1 + \frac{\delta}{\gamma} = R_0^2 \]
for \( t \geq T(B) \).
The second statement follows for $R(B) = \sqrt{\sup_{y \in B} \|y\|_{\mathcal{H}}^{2} + \frac{\xi}{s}}$. This completes the proof.

Now, we will establish the existence of a compact absorbing set which does not depend on $s$.

**Theorem 3.3.** Let $p_{s}^{-}, p_{s}^{+} \in [2, p_{0}]$, $p_{0} > 2$, for any $s$. Then there exist constants $r_{1}, R_{1} > 0$ and a compact set $K_{0}$ in $\mathcal{H}$, which do not depend on $s$, such that for any bounded set $B$ in $\mathcal{H}$ there exists a time $T = T(B)$ such that

$$\varphi_{p_{s}^{+}}(u(t)) \leq r_{1},$$

$$\|u(t)\|_{\mathcal{Y}_{s}} \leq R_{1},$$

$$u(t) \in K_{0},$$

for $t - \tau \geq T$ and any $u \in \mathcal{R}_{s}(u_{\tau}, \tau)$, $u_{\tau} \in B$.

Also, for any bounded set $B$ in $\mathcal{H}$ and any $r > 0$ there exists $r_{2}(B, r)$, $R_{2}(B, r) > 0$ such that

$$\varphi_{p_{s}^{+}}(u(t)) \leq r_{2},$$

$$\|u(t)\|_{\mathcal{Y}_{s}} \leq R_{2},$$

for $t - \tau \geq r$ and any $u \in \mathcal{R}_{s}(u_{\tau}, \tau)$, $u_{\tau} \in B$.

Moreover, if $\varphi_{p_{s}^{+}}(u_{\tau_{s}}) \leq D$, for any $u_{\tau_{s}} \in B$ and $s \in \mathbb{N}$, then there exists a constant $\kappa = \kappa(D, B)$ such that $\|u(t)\|_{\mathcal{Y}_{s}} \leq \kappa$, for all $t \geq \tau$, $s \in \mathbb{N}$ and any $u \in \mathcal{R}_{s}(u_{\tau_{s}}, \tau)$.

**Proof.** If $u$ is a solution of (1) then, there exists $f \in L^{1}([\tau, T]; \mathcal{H})$ such that $f(t) \in F(t, u(t))$ and

$$\frac{du}{dt}(t) + \partial \varphi_{p_{s}^{+}}(u(t)) = f(t),$$

a.e. on $[\tau, T]$, where $T > \tau$ is arbitrary. Furthermore, by Theorem 3.2

$$\|u(t)\|_{\mathcal{H}} \leq R_{0}, \quad \text{if } t - \tau \geq T(B).$$

We observe that by Lemma 2.1 the map $f$ belongs to $L^{2}([\tau, T]; \mathcal{H})$. Hence, using the equality [3, p.189]

$$\frac{d}{dt} \varphi_{p_{s}^{+}}(u(t)) = \left(\partial \varphi_{p_{s}^{+}}(u(t)), \frac{du}{dt}(t)\right),$$

we have

$$\frac{d}{dt} \varphi_{p_{s}^{+}}(u(t)) \leq \left(f(t) - \frac{du}{dt}(t), \frac{du}{dt}(t)\right)$$

$$= -\|f(t) - \frac{du}{dt}(t)\|_{\mathcal{H}}^{2} + \left(f(t) - \frac{du}{dt}(t), f(t)\right)$$

$$\leq -\frac{1}{2}\|f(t) - \frac{du}{dt}(t)\|_{\mathcal{H}}^{2} + \frac{1}{2}\|f(t)\|_{\mathcal{H}}^{2}. $$

Thus

$$\frac{d}{dt} \varphi_{p_{s}^{+}}(u(t)) + \frac{1}{2}\|f(t) - \frac{du}{dt}(t)\|_{\mathcal{H}}^{2} \leq \frac{1}{2}\|f(t)\|_{\mathcal{H}}^{2},$$

and we obtain

$$\frac{d}{dt} \varphi_{p_{s}^{+}}(u(t)) \leq \frac{1}{2}\|f(t)\|_{\mathcal{H}}^{2} \leq \frac{\Omega}{2} b_{1}^{2},$$

(19)
Thus, using Theorem 3.2 we have
\[
\frac{1}{2} \frac{d}{dt} \|u(t)\|_{\mathcal{H}}^2 + \varphi_{p_s}(u(t)) \leq \left\langle \partial \varphi_{p_s}(u(t)), u(t) \right\rangle.
\] (20)

Fixing \( r > 0 \) and integrating both sides of (20) over \((t, t + r)\) where \(t - \tau \geq T(B)\) we get
\[
\int_t^{t+r} \varphi_{p_s}(u(s)) ds \leq \frac{1}{2} \|u(t)\|_{\mathcal{H}}^2 + \int_t^{t+r} |\Omega|^{1/2} b_1 R_0 ds \leq \frac{1}{2} R_0^2 + |\Omega|^{1/2} b_1 R_0 r.
\]

For \( s \in (t, t + r) \) we integrate (19) over the interval \((s, t + r)\) and obtain that
\[
\varphi_{p_s}(u(t + r)) \leq \varphi_{p_s}(u(s)) + r \frac{|\Omega|}{2} b_1^2.
\]

Integrating again but now over \((t, t + r)\) we have
\[
r \varphi_{p_s}(u(t + r)) \leq \int_t^{t+r} \varphi_{p_s}(u(s)) ds + r^2 \frac{|\Omega|}{2} b_1^2 \leq \frac{1}{2} R_0^2 + |\Omega|^{1/2} b_1 R_0 r + r^2 \frac{|\Omega|}{2} b_1^2.
\]

Thus
\[
\varphi_{p_s}(u(t + r)) \leq \frac{1}{2r} R_0^2 + |\Omega|^{1/2} b_1 R_0 + r \frac{|\Omega|}{2} b_1^2 = r_1, \quad \text{if } t - \tau \geq T(B).
\]

Therefore,
\[
D_s \int_{\Omega} \frac{1}{p_s(x)} |\nabla u(t, x)|^{p_s(x)} dx + \lambda \int_{\Omega} \frac{1}{p_s(x)} |u(t, x)|^{p_s(x)} dx \leq r_1,
\]

if \( t - \tau \geq T(B) + r \). Then
\[
\frac{\min\{\lambda, 1\}}{p_s^+} [\rho_s(\nabla u(t)) + \rho_s(u(t))] \leq r_1
\]

if \( t - \tau \geq T(B) \), and hence,
\[
\rho_s(\nabla u(t)) + \rho_s(u(t)) \leq \frac{p_s^+}{\min\{\lambda, 1\}} r_1, \quad \text{(21)}
\]

if \( t - \tau \geq T(B) \).

There are four cases to analyze:

**Case 1:** If \( \|\nabla u(t)\|_{L^{p_s}(\Omega)} \geq 1 \) and \( \|u(t)\|_{L^{p_s}(\Omega)} \geq 1 \) we know that
\[
\|\nabla u(s)\|_{L^{p_s}(\Omega)} \leq \rho_s(\nabla u(t)) \leq \|\nabla u(t)\|_{L^{p_s}(\Omega)}^p,
\]

and
\[
\|u(t)\|_{L^{p_s}(\Omega)} \leq \rho_s(u(t)) \leq \|u(t)\|_{L^{p_s}(\Omega)}^p.
\]

Using (21) this gives
\[
\|u(t)\|_{Y_s} \leq R_1, \quad \text{if } t - \tau \geq T(B),
\]
where \( R_1 \) is such that \( 2 \left[ \frac{p_+^s}{\min(L,T)} \right]^{1/p_+} \leq R_1 \) for any \( s \).

**Case 2:** If \( \| \nabla u(t) \|_{L^p(\Omega)} \geq 1 \) and \( \| u(t) \|_{L^p(\Omega)} \leq 1 \) we know that
\[
\| \nabla u(t) \|_{L^p(\Omega)}^{p_+} \leq \rho_s(\nabla u(t)) \leq \| \nabla u(t) \|_{L^p(\Omega)}^{p_+},
\]
and
\[
\| u(t) \|_{L^p(\Omega)}^{p_+} \leq \rho_s(u(t)) \leq \| u(t) \|_{L^p(\Omega)}^{p_+}.
\]
Using (21) we obtain
\[
\| u(t) \|_{Y_s} \leq R_2, \quad \text{if } t - \tau \geq T(B),
\]
where \( R_2 \) is such that \( \left[ \frac{p_+^s}{\min(L,T)} \right]^{1/p_+} + \left[ \frac{p_+^s}{\min(L,T)} \right]^{1/p_+} \leq R_2 \).

**Case 3:** If \( \| \nabla u(t) \|_{L^p(\Omega)} \leq 1 \) and \( \| u(t) \|_{L^p(\Omega)} \geq 1 \) we know that
\[
\| \nabla u(t) \|_{L^p(\Omega)}^{p_+} \leq \rho_s(\nabla u(t)) \leq \| \nabla u(t) \|_{L^p(\Omega)}^{p_+},
\]
and
\[
\| u(t) \|_{L^p(\Omega)}^{p_+} \leq \rho_s(u(t)) \leq \| u(t) \|_{L^p(\Omega)}^{p_+}.
\]
By (21) it follows that
\[
\| u(t) \|_{Y_s} \leq R_3, \quad \text{if } t - \tau \geq \overline{T}(B),
\]
where \( R_3 = R_2 \).

**Case 4:** If \( \| \nabla u(t) \|_{L^p(\Omega)} \leq 1 \) and \( \| u(t) \|_{L^p(\Omega)} \leq 1 \) then we know that
\[
\| \nabla u(t) \|_{L^p(\Omega)}^{p_+} \leq \rho_s(\nabla u(t)) \leq \| \nabla u(t) \|_{L^p(\Omega)}^{p_+},
\]
and
\[
\| u(t) \|_{L^p(\Omega)}^{p_+} \leq \rho_s(u(t)) \leq \| u(t) \|_{L^p(\Omega)}^{p_+}.
\]
Using (21) this gives
\[
\| u(t) \|_{Y_s} \leq R_4, \quad \text{if } t - \tau \geq T(B),
\]
where \( R_4 \) is such that \( 2 \left[ \frac{p_+^s}{\min(L,T)} \right]^{1/p_+} \leq R_4 \).

In summary,
\[
\| u(t) \|_{Y_s} \leq R, \quad \text{if } t - \tau \geq \overline{T}(B),
\]
where \( R = \max\{ R_1, R_2, R_3, R_4 \} \).

Taking into account (9), we obtain that the ball \( K_0 \) of radius \( \widetilde{R} = K(\Omega)R \) in the space \( H^1(\Omega) \) defined by
\[
K_0 = \{ y \in H^1(\Omega) : \| y \|_{H^1(\Omega)} \leq \widetilde{R} \}
\]
is absorbing for any \( s \). Since the embedding \( H^1(\Omega) \subset \mathcal{H} \) is compact, we obtain that \( K_0 \) is a compact set of \( \mathcal{H} \).

If we replace in the previous arguments the constants \( R_0, T(B) \) by the constant \( R(B) \) from Theorem 3.2 and 0, respectively, then we obtain (16)-(17).

Let us prove now the last statement. As \( B \) is bounded in \( \mathcal{H} \), there exists \( \overline{T}(B) \) such that (22) holds for any \( u \in \mathcal{R}_s(u_\tau, \tau), \ u_\tau \in B \). Integrating (19) over the
interval $(\tau, t)$ with $t - \tau \leq T(B)$ and using that $\varphi_{p,()}(u_{\tau s}) \leq D$, for any $u_{\tau s} \in B$ and $s \in \mathbb{N}$, we obtain

$$\varphi_{p,()}(u(t)) \leq \varphi_{p,()}(u(\tau)) + \frac{|\Omega|}{2} b_1^2 T(B)$$

$$\leq D + \frac{|\Omega|}{2} b_1^2 T(B) = \gamma(D, B), \quad \forall t \in [\tau, \tau + T(B)].$$

Arguing as in the proof of the first statement we obtain the existence of a constant $\kappa_1 = \kappa_1(D, B)$ such that

$$\|u(t)\|_{Y_s} \leq \kappa_1, \quad \forall t \in [\tau, \tau + T(B)].$$

Hence, using (22) we have

$$\|u(t)\|_{Y_s} \leq \kappa \text{ for all } t \geq \tau,$$

where $\kappa = \kappa(D, B) = \max\{\kappa_1, R\}$. \qed

**Remark 1.** (15) implies that the set $K_0$ is both forward and pullback attracting.

Now we recall some definitions concerning multivalued nonautonomous dynamical systems, see e.g. [8, 9, 33].

**Definition 3.4.** Let $X$ be a complete metric space, $P(X)$ the set of all nonempty subsets of $X$ and $\mathbb{R}^d := \{(t, r) \in \mathbb{R}^2 : t \geq r\}$. The map $U : \mathbb{R}^d \times X \to P(X)$ is called a multivalued evolution process on $X$ if:

1. $U(t, t, \cdot) = 1$ is the identity map;
2. $U(t, r, x) \subset U(t, \tau, U(\tau, r, x))$, for all $x \in X$, $r \leq \tau \leq t$, where

$$U(t, \tau, U(\tau, r, x)) = \bigcup_{y \in U(\tau, r, x)} U(t, \tau, y).$$

The multivalued evolution process $U$ is called strict if

$$U(t, r, x) = U(t, \tau, U(\tau, r, x)), \quad \text{for all } x \in X, \ r \leq \tau \leq t.$$  

**Definition 3.5.** Let $U$ be a multivalued evolution process on $X$ and $t \in \mathbb{R}$. The set $D(t) \subset X$ attracts (pullback) the nonempty bounded subset $B$ of $X$ at time $t$ if:

$$\lim_{\tau \to \tau \to -\infty} \text{dist}_H(U(t, \tau)B, D(t)) = 0. \quad (23)$$

The set $D(t)$ is said to be (pullback) attracting at time $t$ if (23) is satisfied for any nonempty bounded subset $B \subset X$.

For a nonempty and bounded subset $B \subset X$ and $t \in \mathbb{R}$, let us put $\gamma^r(t, B) = \bigcup_{\tau \leq r} U(t, \tau, B)$ and $\omega(t, B) = \bigcap_{r \leq t} \gamma^r(t, B)$. The set $\omega(t, B)$ is called the pullback $\omega$-limit set of $B$ at time $t$ with respect to the multivalued evolution process $U$.

**Theorem 3.6** (Theorem 6 in [9]). Suppose that for $t \in \mathbb{R}$ and $B$ a nonempty and bounded subset of $X$ there exists a nonempty compact subset $D(t, B)$ of $X$ such that

$$\lim_{r \to -\infty} \text{dist}_X(U(t, r)B, D(t, B)) = 0.$$  

Then $\omega(t, B)$ is nonempty, compact and the minimal closed set attracting $B$ at time $t$. 

Definition 3.7. A family of sets \( \{A(t) : t \in \mathbb{R}\} \) of \( X \) is called a pullback attractor for the multivalued evolution process \( U \) if:

1. \( A(t) \) is pullback attracting at time \( t \) for all \( t \in \mathbb{R} \);
2. it is semi-invariant (or negatively invariant), that is, \( A(t) \subset U(t,r,A(r)) \), for any \((t,r) \in \mathbb{R} \times \mathbb{R} \);
3. it is minimal, that is, for any closed attracting set \( Y \) at time \( t \), we have \( A(t) \subset Y \).

It is called strictly invariant if, moreover, \( A(t) = U(t,r,A(r)) \), for any \((t,r) \in \mathbb{R} \times \mathbb{R} \).

Theorem 3.8 (Theorem 18 in [9]). Let us suppose that for all \((t,r) \in \mathbb{R} \times \mathbb{R} \) the map \( x \mapsto U(t,r,x) \in P(X) \) is closed. If, moreover, for any \( t \in \mathbb{R} \) there exists a nonempty compact set \( D(t) \) which is attracting, then the set \( A = \{A(t)\}_{t \in \mathbb{R}} \), with

\[
A(t) = \bigcup_{B \in B(X)} \omega(t,B)
\]

where \( B(X) = \{B \in P(X) : B \text{ is bounded}\} \), is the pullback attractor of \( U \). Moreover, the sets \( A(t) \) are compact.

We come back now to our problem (1). Using Lemma 2.5 it is easy to check that the map \( U_s : \mathbb{R} \times \mathcal{H} \rightarrow P(\mathcal{H}) \) defined by

\[
U_s(t,\tau,\xi) = \{z : \text{there exists } u(\cdot) \in \mathcal{R}_s(\xi,\tau) \text{ such that } u(t) = z\}
\]

is a strict multivalued evolution process. We observe that as every strong solution can be extended to a globally defined one, the set \( D_s(u_\tau,t,\tau) \) defined in (5) is equal to \( U_s(t,\tau,u_\tau) \).

Theorem 3.9. Let \( t \in \mathbb{R} \) and let \( B \) be a nonempty and bounded subset of \( \mathcal{H} \). Then the \( \omega \)-limit set \( \omega_s(t,B) \) corresponding to the multivalued evolution process associated with problem (1) is nonempty, compact and the minimal closed set attracting \( B \) at time \( t \).

Proof. Note that by Theorem 3.3 the constant family \( K(t) \equiv K_0 \) of compact sets of \( \mathcal{H} \) pullback attracts bounded sets of \( \mathcal{H} \) at time \( t \). Hence, by Theorem 3.6, \( \omega_s(t,B) \) is nonempty, compact and the minimal closed set attracting \( B \) at time \( t \).

Proposition 1. Let \( \xi_n \rightarrow \xi \) in \( \mathcal{H} \) and \( u_n(\cdot) \in \mathcal{R}_s(\xi_n,\tau) \). Then there exists a subsequence \( u_{nk}(\cdot) \) and \( u \in \mathcal{R}_s(\xi,\tau) \) such that \( u_{nk} \rightarrow u \) in \( C([\tau,T];\mathcal{H}) \) for any \( T > \tau \).

Proof. Fix \( T > \tau \). As \( u_n(\cdot) \in \mathcal{R}(\xi_n,\tau) \) there exists a sequence \( f_n \in L^1([\tau,T];\mathcal{H}) \) such that \( f_n(t) \in F(t,u_n(t)) \) a.e. on \([\tau,T]\) and

\[
\begin{cases}
\frac{\partial u_n(t)}{\partial t} + \partial \varphi_{p_s}(\cdot)(u_n(t)) = f_n(t) \quad \text{a.e. on } [\tau,T], \\
u_n(\tau) = \xi_n.
\end{cases}
\]  

(24)

In view of \( f_n(t) \in F(t,u_n(t)) \) a.e. on \([\tau,T]\), we have

\[
\|f_n(t)\|_{\mathcal{H}} \leq |\Omega|^{1/2} b_1 \quad \text{a.e. on } [\tau,T].
\]

(25)

Hence, \( \{f_n\} \) is a bounded sequence in the reflexive Banach space \( L^2([\tau,T];\mathcal{H}) \), and we obtain that there exists a subsequence such that

\[
f_n \rightarrow f \quad \text{weakly in } L^2([\tau,T];\mathcal{H}).
\]
Let \( v_n(\cdot) \) be the unique solution to
\[
\begin{cases}
\frac{\partial v_n}{\partial t}(t) + \partial \varphi_{P,\cdot}(v_n(t)) = f_n(t) & \text{a.e. on } [\tau, T], \\
v_n(\tau) = \xi.
\end{cases}
\] (26)
By Lemma 1 in [31] we have that \( v_n \) converges in \( C([\tau, T]; \mathcal{H}) \) to the solution \( u \) of
\[
\begin{cases}
\frac{\partial u}{\partial t}(t) + \partial \varphi_{P,\cdot}(u(t)) = f(t) & \text{a.e. on } [\tau, T], \\
u(\tau) = \xi.
\end{cases}
\]
Now, using (24) and (26) we have
\[
\|u_n(t) - v_n(t)\|_\mathcal{H} \leq \|\xi_n - \xi\|_\mathcal{H}, \quad \forall t \in [\tau, T].
\]
So we conclude that \( u_n \to u \) in \( C([\tau, T]; \mathcal{H}) \). Now, to finish the proof, it follows by Theorem 3.3 in [18] that \( f(t) \in F(t, u(t)) \) a.e. on \([\tau, T]\), so \( u(\cdot) \) is a strong solution of problem (1) on \([\tau, T]\).

By a diagonal argument we obtain the convergence for any \( T > \tau \). \(\Box\)

**Corollary 2.** For all \((T, \tau) \in \mathbb{R}_d\) the map \( \mathcal{H} \ni \xi \mapsto U_s(T, \tau, \xi) \in \mathcal{P}(\mathcal{H}) \) has closed graph.

**Theorem 3.10.** The multivalued evolution process associated with problem (1) has a pullback strictly invariant attractor \( \mathcal{A}_s = \{ \mathcal{A}_s(t) : t \in \mathbb{R} \} \). Moreover, the sets \( \mathcal{A}_s(t) \) are compact and \( \mathcal{A}_s(t) \subseteq K_0 \) for every \( t \in \mathbb{R} \).

**Proof.** By Theorem 3.3 the constant family \( K(t) \equiv K_0 \) of compact sets of \( \mathcal{H} \) pullback attracts bounded sets of \( \mathcal{H} \) at time \( t \), and by Corollary 2 for all \((T, \tau) \in \mathbb{R}_d\) the map \( \mathcal{H} \ni \xi \mapsto U_s(T, \tau, \xi) \in \mathcal{P}(\mathcal{H}) \) is closed. Hence, the existence and compactness of the pullback attractor follows from Theorem 3.8. Since the pullback attractor by definition is the minimal closed pullback attracting family, we obtain that \( \mathcal{A}_s(t) \subseteq K_0 \) for every \( t \in \mathbb{R} \). Finally, the invariance follows from Lemma 2.5 in [10]. \(\Box\)

As a consequence of \( \mathcal{A}_s(t) \subseteq K_0 \), where \( K_0 \) is independent of \( s \) and \( t \), we obtain the following important property.

**Corollary 3.** The set \( \bigcup_{s \in \mathbb{N}} \bigcup_{t \in \mathbb{R}} \mathcal{A}_s(t) \) is compact in the space \( \mathcal{H} \).

We will finish this section by giving a characterization of the pullback attractor as the union of all bounded complete trajectories.

Let us denote \( W_\tau = C([\tau, \infty); \mathcal{H}) \) and let \( \mathcal{R}_s = \{ \mathcal{R}_s(\tau) \}_{\tau \in \mathbb{R}} \), where \( \mathcal{R}_s(\tau) = \bigcup_{u_\tau \in \mathcal{H}} \mathcal{R}_s(u_\tau, \tau) \). Then it follows from Theorem 2.3, Lemma 2.5 and Proposition 1 that for any \( s \) the set of solutions \( \mathcal{R}_s \) satisfies the following axiomatic properties:

(H1) For any \( \tau \in \mathbb{R} \) and \( u_\tau \in \mathcal{H} \) there exists \( u \in \mathcal{R}_s(\tau) \) such that \( u(\tau) = u_\tau \).

(H2) \( u_\tau = u \upharpoonright_{[\tau, \tau + \infty)} \in \mathcal{R}_s(\tau + \tau) \) for any \( \tau \geq 0, u \in \mathcal{R}_s(\tau) \) (translation property).

(H3) Let \( u, v \in \mathcal{R}_s \) be such that \( u \in \mathcal{R}_s(\tau) \), \( v \in \mathcal{R}_s(\tau) \) and \( u(p) = v(p) \) for some \( p \geq \tau \geq \tau \). Then the function \( z \) defined by
\[
z(t) := \begin{cases} 
  u(t), & t \in [\tau, p], \\
  v(t), & t \in [p, +\infty), 
\end{cases}
\]
belongs to \( \mathcal{R}_s(\tau) \) (concatenation property).

(H4) For any sequence \( u^n \in \mathcal{R}_s(\tau) \) such that \( u^n(\tau) \to u_\tau \) in \( \mathcal{H} \), there exists a subsequence \( u^{n_k} \) and \( u \in \mathcal{R}_s(\tau) \) such that
\[
u^{n_k}(t) \to u(t), \forall t \geq \tau.
\]
A map \( \gamma : \mathbb{R} \to \mathcal{H} \) is called a complete trajectory of \( \mathcal{R}_s \) if
\[
\gamma = \gamma_{|[\tau,+\infty)} \in \mathcal{R}_s (\tau), \quad \text{for all } \tau \in \mathbb{R}.
\]
A complete trajectory is called bounded if the set \( \cup_{t \in \mathbb{R}} \gamma(t) \) is bounded in \( \mathcal{H} \).

It is well known [10, Corollaries 2.10 and 2.12] (see also [41]) that if either \((H1)-(H3)\) or \((H1)-(H2), \ (H4)\) hold and the multivalued process \( U_s \) generated by \( \mathcal{R}_s \) possesses a bounded pullback attractor \( \mathcal{A}_s = \{ A_s(t) : t \in \mathbb{R} \} \), which means that \( \cup_{t \in \mathbb{R}} A_s(t) \) is bounded, then \( \mathcal{A}_s \) can be characterized by the union of all bounded complete trajectories of \( \mathcal{R}_s \).

Therefore, as Corollary 3 implies that \( \cup_{t \in \mathbb{R}} A_s(t) \) is bounded, we obtain the following result for the pullback attractor of problem (1).

**Theorem 3.11.** The pullback attractor of problem (1) is characterized by
\[
\mathcal{A}_s (t) = \{ \gamma(t) : \gamma \text{ is a bounded complete trajectory of } \mathcal{R}_s \}.
\]

4. **The limit problem.** The following differential inclusion was considered in [10]:
\[
\begin{cases}
\frac{du}{dt} + \lambda u \in b(t)H(u), \ t \geq \tau, \\
u(\tau) = u_{\tau},
\end{cases}
\]
where \( \lambda > 0 \) and the maps \( H(u), \ b(t) \) are the same as for problem (1).

When \( D_s \to \infty \) and \( p_s (\cdot) \to 2 \), that is, when the diffusion becomes large and the variable exponent tends to 2, it is natural to expect that the solutions of problem (1) converge to the solutions of problem (28) in some sense. This will be proved rigorously in the following section. Moreover, in the last section we will prove that the pullback attractors of the family of problems (1) are upper semicontinuous on \( s \) at \( \infty \) with respect to the pullback attractor of problem (28). We observe that in [10] a precise characterization of the pullback attractor for problem (28) was given. Therefore, if the structure of the pullback attractor were robust, the attractors of problem (1) would inherit it for large diffusion.

We summarize briefly the results proved in [10].

The function \( u : [\tau, T] \to \mathbb{R} \) is called a solution of (28) if \( u \in C ([\tau, T], \mathbb{R}) \), \( \frac{du}{dt} \in L^\infty (\tau, T; \mathbb{R}) \) and there exists \( h \in L^\infty (\tau, T; \mathbb{R}) \) such that \( h(t) \in H(u(t)) \), for a.a. \( t \in (\tau, T) \), and
\[
\frac{du}{dt} + \lambda u = b(t)h(t) \text{ for a.a. } t \in (\tau, T).
\]

It is easy to see that this definition is equivalent to the one of strong solution given in Definition 2.2 if we replace the operator \( \partial \varphi_{p_s(\cdot)}(u) \) by the linear operator \( \lambda u \) and the space \( \mathcal{H} \) by \( \mathbb{R} \).

The solutions of problem (28) generate a strict multivalued process denoted by \( U_\infty : \mathbb{R}_d \times \mathbb{R} \to P(\mathbb{R}) \), which possesses a strictly invariant pullback attractor \( \mathcal{A}_\infty = \{ A_\infty(t) \}_{t \in \mathbb{R}} \).

Moreover, this attractor is bounded in the sense that the union of all sections \( \cup_{t \in \mathbb{R}} A_\infty(t) \) is bounded and it is characterized by the set of all bounded complete trajectories, that is,
\[
A_\infty(t) = \{ \gamma(t) : \gamma \text{ is a bounded complete trajectory} \}.
\]

We recall that \( \gamma : \mathbb{R} \to \mathbb{R} \) is a complete trajectory if \( u = \gamma_{|[\tau, \infty)} \) is a solution of (28) for any \( \tau \in \mathbb{R} \). A complete trajectory is bounded if \( \cup_{t \in \mathbb{R}} \gamma(t) \) is bounded.
More precisely, the pullback attractor is characterized by three nonautonomous equilibria (one of which is 0) and the heteroclinic connections which go from 0 to the two non-zero equilibria. We describe this structure in more detail.

There exists a maximal bounded complete trajectory $\xi_M(\cdot)$, which means that for any bounded complete trajectory $\psi(\cdot)$ we have

$$-\xi_M(t) \leq \psi(t) \leq \xi_M(t) \text{ for all } t \in \mathbb{R}.$$

It is defined by

$$\xi_M(t) = \int_{-\infty}^{t} e^{-\lambda(t-\tau)} b(\tau) \, d\tau.$$

This trajectory is the unique bounded complete positive trajectory, which means that, if $\psi(\cdot)$ is a bounded complete trajectory such that $\psi(t) > 0 \ (\psi(t) < 0)$ for all $t \in \mathbb{R}$, then $\psi(t) \equiv \xi_M(t) \ (\psi(t) \equiv -\xi_M(t))$. Moreover, every solution with positive (negative) initial data approaches to the complete solution $\xi_M(t) \ (-\xi_M(t))$ asymptotically as $t \to +\infty$, that is, if $u_r > 0$, $v_r < 0$, then

$$|U_\infty(t, \tau, u_r) - \xi_M(t)| \to 0, \quad |U_\infty(t, \tau, v_r) + \xi_M(t)| \to 0, \text{ as } t \to +\infty.$$

We call the three bounded complete trajectories $0$, $\xi_M(\cdot)$, $-\xi_M(\cdot)$ nonautonomous equilibria. Apart from them the only bounded complete trajectories are given by

$$\phi_r^+(t) = \begin{cases} 0 & \text{if } t \leq r, \\ \int_r^t e^{-\lambda(t-\tau)} b(\tau) \, d\tau & \text{if } t \geq r, \end{cases}$$

$$\phi_r^-(t) = \begin{cases} 0 & \text{if } t \leq r, \\ -\int_r^t e^{-\lambda(t-\tau)} b(\tau) \, d\tau & \text{if } t \geq r, \end{cases}$$

for any $r \in \mathbb{R}$, and

$$|\phi_r^+(t) - \xi_M(t)| \to 0 \text{ as } t \to +\infty,$$

$$|\phi_r(t) + \xi_M(t)| \to 0 \text{ as } t \to +\infty.$$

Therefore, $\phi_r^+$ are heteroclinic connections from 0 to $\xi_M(\cdot)$ and $\phi_r^-$ are heteroclinic connections from 0 to $-\xi_M(\cdot)$. Since the pullback attractor is characterized by all bounded complete trajectories, it follows that it consists of the three nonautonomous equilibria $0$, $\xi_M$, $-\xi_M$ and the heteroclinic connections $\phi_r^\pm$.

5. **Stability: continuity of the flows**. Our objective in this section is to prove that the limit problem of (1) as $D_s$ increases to infinity and $p_s(\cdot) \to 2$ in $L^\infty(\Omega)$ as $s \to \infty$ is described by an ordinary differential inclusion. Firstly we observe that the gradients of the solutions $u_s$ of problem (1) converge in norm to zero as $s \to \infty$, which allows us to guess the limit problem (28).

**Theorem 5.1.** If $u_s$ is a solution of (1), where $u_s(\tau) = u_{rs}$ belongs to a bounded set of $\mathcal{H}$, then for each $t > \tau$ the sequence of real numbers $\{\|\nabla u_s(t)\|_H\}$ has a subsequence $\{\|\nabla u_s(t)\|_H\}$ which converges to zero as $\ell \to \infty$.

**Proof.** We fix $t > \tau$ and choose some $T > t$. As $u_s$ is a solution of (1), there exists $\xi_s \in L^1(\tau, T; \mathcal{H})$, $\xi_s(t) \in F(t, u_s(t)) \ t - a.e. \ in \ (\tau, T)$ such that

$$\frac{\partial u_s}{\partial t}(r) + \partial \varphi_{p_s(\cdot)}(u_s(r)) = \xi_s(r).$$
Multiplying the equation by \( u_s(r) \) and using that by Theorem 3.2
\[
(\xi_s(r), u_s(r)) \leq |\Omega|^{\frac{1}{2}} b_1 R = K_1, \quad \text{for a.a.} \ r \in (\tau, T),
\]
where \( K_1 > 0 \) is independent of \( s \), we obtain
\[
\frac{1}{2} \frac{d}{dt} \|u_s(r)\|_{H}^2 + D_s \int_{\Omega} |\nabla u_s(r)|^{p_s(r)} dx \leq K_1.
\]
Repeating the same procedure as in [34], where we have to use inequality (16) in particular, we obtain a subsequence such that
\[
\|\nabla u_{s_\ell}(t)\|_{H} \leq C\|\nabla u_{s_\ell}(t)\|_{p_s(x)} \to 0 \text{ as } \ell \to \infty.
\]

The next result guarantees that (28) is in fact the limit problem for the family of problems (1) as \( s \to \infty \).

**Theorem 5.2.** Let \( u_s \) be a strong solution of (1) with \( u_s(\tau) = u_{s_\ell} \to u_\tau \) in \( H \) as \( s \to \infty \), where \( u_\tau \) is a constant function. Then there exist a strong solution \( u \) of problem (28) with \( u(\tau) = u_\tau \) and a subsequence \( \{u_{s_\ell}\}_n \) of \( \{u_s\}_s \) such that, for each \( T > \tau \), \( u_{s_\ell} \to u \) in \( C([\tau, T]; \mathcal{H}) \) as \( n \to +\infty \).

**Proof.** Let \( T > \tau \) be fixed. There exists \( f_s \in L^1(\tau, T; \mathcal{H}) \), with \( f_s(t) \in F(t, u_s(t)) \), a.e. in \( (\tau, T) \), and such that
\[
\frac{du_s}{dt} + \partial \varphi_{p_s(.)}(u_s) = f_s, \quad \text{for a.a. } t \in (\tau, T).
\]
Take an arbitrary \( \varepsilon > 0 \). Multiplying this equality by \( \frac{du_s}{dt} \), using (18) and integrating over \( (\tau + \varepsilon, T) \) we obtain
\[
\frac{1}{2} \int_{\tau+\varepsilon}^{T} \left\| \frac{du_s}{dt} \right\|_{H}^2 dt + \varphi_{p_s(.)}(u_s(T)) \leq \varphi_{p_s(.)}(u_s(\tau + \varepsilon)) + \frac{1}{2} \int_{\tau+\varepsilon}^{T} \|f_s(t)\|_{H}^2 dt.
\]
Further, by (16) there exists an \( \varepsilon \)-depending constant \( c_\varepsilon \) such that
\[
\int_{\tau+\varepsilon}^{T} \left\| \frac{du_s}{dt} \right\|_{H}^2 dt \leq c_\varepsilon + \frac{b_1^2 |\Omega|}{2} (T - \tau), \quad \text{for all } t \in [\tau + \varepsilon, T].
\]
Thus, \( \frac{du_s}{dt} \) is bounded in \( L^2(\tau + \varepsilon, T; \mathcal{H}) \). Also, by Theorem 3.2 \( u_s \) is bounded in \( L^\infty(\tau, T; \mathcal{H}) \), so there exists a subsequence \( \{u_{s_\ell}\}_n \) and a function \( u \) such that
\[
\begin{align*}
\lim_{n \to \infty} u_{s_\ell} &\to u \text{ weakly star in } L^\infty(\tau, T; \mathcal{H}), \\
\lim_{n \to \infty} u_{s_\ell} &\to u \text{ weakly in } L^2(\tau, T; \mathcal{H}), \\
\frac{du_{s_\ell}}{dt} &\to \frac{du}{dt} \text{ weakly in } L^2(\tau + \varepsilon, T; \mathcal{H}).
\end{align*}
\]
By (17) and (9) there is an \( \varepsilon \)-depending constant \( R_\varepsilon \) such that
\[
\|u_s(t)\|_{H^1(\Omega)} \leq K\|u_s(t)\|_{Y_s} \leq R_\varepsilon, \quad \text{for all } t \in [\tau + \varepsilon, T].
\]
Hence, the compact embedding \( H^1(\Omega) \subset \mathcal{H} \) implies that for any \( t \in [\tau + \varepsilon, T] \) the sequence \( u_{s_\ell}(t) \) is relatively compact in \( \mathcal{H} \). Thus, by Ascoli-Arzelà theorem there exists a subsequence (relabeled the same) such that
\[
\begin{align*}
u_{s_\ell} &\to u \text{ in } C([\tau + \varepsilon, T], \mathcal{H}), \\
\text{also, the function } u &\text{ is absolutely continuous in } [\tau + \varepsilon, T].
\end{align*}
\]
We will prove that \( u \) is an integral solution (and then a strong solution) of the limit problem \((28)\) in the interval \((\tau + \varepsilon, T)\).

First, the sequence \( f_{s_n} \) is bounded in \( L^\infty(\tau, T; H) \), so there is \( g \in L^\infty(\tau, T; H) \) such that
\[
  f_{s_n} \to g \text{ weakly star in } L^\infty(\tau, T; H),
  \quad f_{s_n} \to g \text{ weakly in } L^2(\tau, T; H).
\]

From Theorem 3.3 in [18] we have that \( g(t) \in F(u(t)) \) for a.a. \( t \in (\tau, T) \).

Second, from Theorem 5.1 we can obtain that \( u(t) \) is constant for any \( t \in (\tau, T] \).
Indeed, for \( t > \tau \) we known that \( \|\nabla u_{s_n}(t)\|_H \to 0 \). By the Poincaré-Wirtinger inequality (see p.194 in [6]), we obtain (up to a subsequence) that \( \|u(t) - \overline{u}_{s_n}(t)\|_H \to 0 \) as \( s \to +\infty \), where \( \overline{u}_{s_n}(t) := \frac{1}{|\Omega|} \int_\Omega u_{s_n}(t, x)\,dx \). It is easy to see that \( u_{s_n}(t) \to u(t) \) in \( H \) implies \( \overline{u}_{s_n}(t) \to \overline{u}(t) \) in \( H \). So, we conclude that \( u(t) = \overline{u}(t) \) and then \( u(t) \) is a constant function.

If \( u(t) > 0 \) (\( < 0 \)), then \( F(u(t)) = b(t) (-b(t)) \), so \( g(t) \) is constant. We do not know whether \( g(t) \) is constant or not when \( u(t) = 0 \).

Let \([s, t] \subseteq [\tau + \varepsilon, T] \) be an interval such that \( u(\ell) \neq 0 \) for a.a. \( \ell \in (s, t) \). Then \( g(\ell) \) is constant for a.a. \( \ell \in (s, t) \) and it is clear that
\[
  \int_s^t (f_{s_n}(\ell) - \overline{u}, u_{s_n}(\ell) - \overline{u})\,d\ell \to \int_s^t (g(\ell) - \overline{u}, u(\ell) - \overline{u})\,d\ell,
\]
for all \( \overline{u}, \overline{h} \in H \). Now, consider \( \theta \in \mathbb{R} \) and let \( \overline{h}(x) \equiv \theta, \overline{h} = \lambda \overline{h}, y_{s_n} := \partial\varphi_{p_n, x}(\overline{h}) \).

We obtain
\[
  \frac{1}{2} \|u_{s_n}(t) - \overline{h}\|_H^2 \leq \frac{1}{2} \|u_{s_n}(s) - \overline{h}\|_H^2 + \int_s^t (f_{s_n}(\ell) - \overline{u}, u_{s_n}(\ell) - \overline{u})\,d\ell
  + \int_s^t (\overline{h} - y_{s_n}, u_{s_n}(\ell) - \overline{u})\,d\ell.
\]

Using \( \nabla \overline{h} = 0 \) in the definition of \( y_{s_n} := \partial\varphi_{p_n, x}(\overline{h}) \), we have
\[
  (\overline{h} - y_{s_n}, u_{s_n}(\ell) - \overline{u}) \leq \text{ess sup}_{x \in \Omega} |\lambda \overline{h} - \lambda |\overline{h}|p_n(x) - 2\overline{h}| |\Omega|^{1/2} \|u_{s_n}(\ell) - \overline{u}\|_H.
\]

By Theorem 3.2 there exists a positive constant \( C \) such that
\[
  \|u_{s_n}(\ell) - \overline{u}\|_H \leq C, \forall \ell \geq \tau.
\]

Therefore,
\[
  \int_s^t (\overline{h} - y_{s_n}, u_{s_n}(\ell) - \overline{u})\,d\ell \leq \text{ess sup}_{x \in \Omega} |\lambda \overline{h} - \lambda |\overline{h}|p_n(x) - 2\overline{h}| |\Omega|^{1/2} C(T - \tau) \to 0
\]
as \( n \to +\infty \). Taking the limit in the inequality (31) as \( n \to +\infty \), we obtain
\[
  \frac{1}{2} \|u(t) - \overline{h}\|_H^2 \leq \frac{1}{2} \|u(s) - \overline{h}\|_H^2 + \int_s^t (g(\ell) - \overline{u}, u(\ell) - \overline{u})\,d\ell.
\]

Since \( u(t), g(t) \) are constant in \([s, t] \), this means that \( u \) is an integral (and then also strong) solution of problem \((28)\) in \([s, t] \).

Since \( u(t) \) is a continuous function which is constant for any \( t, [\tau + \varepsilon, T] \) is the countable union of intervals \([s_i, t_i] \) in which either \( u(t) \equiv 0 \) or \( u(t) \neq 0 \) for a.a. \( \ell \in (s_i, t_i) \). We define the function
\[
  f(t) = \begin{cases} 
  0 & \text{if } u(t) = 0, \\
  g(t) & \text{if } u(t) \neq 0.
\end{cases}
\]
Then \( f(t) \in F(u(t)) \) for a.a. \( t \in (\tau + \varepsilon, T) \). Since \( u(\cdot) \) is a strong solution of problem (28) in each interval \([s, t]\), it is obvious that \( u(\cdot) \) is a strong solution of problem (28) in the interval \([\tau + \varepsilon, T]\) for any \( \varepsilon > 0 \). Also, by a diagonal argument we can choose a common subsequence \( u_{s_n} \) satisfying \( u_{s_n} \to u \) in \( C([\tau + \varepsilon, T], \mathcal{H}) \) for all \( \varepsilon > 0 \).

In order to check that \( u(\cdot) \) is a strong solution of problem (28) in the interval \([\tau, T]\) it remains to prove that \( u(t) \to u_\tau \) as \( t \to \tau^+ \). Let \( v(t) = e^{-\lambda(t-\tau)}u_\tau \). The difference \( v_{s_n}(t) = u_{s_n}(t) - v(t) \) satisfies the equality
\[
\frac{dv_{s_n}}{dt} = -A_{s_n}u_{s_n} + \lambda v + f_{s_n}(t)
\]
Multiplying it by \( v_{s_n} \) and using that the operators \( A_{s_n} \) are monotone we get
\[
\frac{d}{dt} \|v_{s_n}\|^2_{\mathcal{H}} \leq 2\lambda \|v(t)\|^2_{\mathcal{H}} + 2\lambda^2 \int_{\Omega} |v(t)|^{2p_n(x) - 2} dx + 2 \|f_{s_n}(t)\|^2_{\mathcal{H}} + \frac{3}{2} \|v_{s_n}(t)\|^2_{\mathcal{H}} \leq K.
\]
Integration gives
\[
\|v_{s_n}(t)\|^2_{\mathcal{H}} + K(t - \tau).
\]
As
\[
\|u(t) - v(t)\|_{\mathcal{H}} = \lim_{n \to \infty} \|v_{s_n}(t)\|_{\mathcal{H}} \leq K(t - \tau) \text{ for } t > \tau,
\]
we obtain that
\[
\|u(t) - u_\tau\|_{\mathcal{H}} \leq \|u(t) - v(t)\|_{\mathcal{H}} + \|v(t) - u_\tau\|_{\mathcal{H}} < \delta
\]
if \( 0 < t - \tau < \eta(\delta) \). Thus, \( u(t) \to u_\tau \) as \( t \to \tau^+ \).

Finally, we need to establish that \( u_{s_n} \to u \) in \( C([\tau, T], \mathcal{H}) \). In light of the previous convergences, for this aim it is enough to prove that for any sequence of times \( t_n \to 0^+ \) we have \( u_{s_n}(t_n) \to u_\tau \). Indeed,
\[
\|u_{s_n}(t_n) - u_\tau\|_{\mathcal{H}} \leq \|u_{s_n}(t_n) - v(t_n)\|_{\mathcal{H}} + \|v(t_n) - u_\tau\|_{\mathcal{H}} \leq \sqrt{\|v_{s_n}(\tau)\|^2_{\mathcal{H}} + K(t_n - \tau) + \|v(t_n) - u_\tau\|_{\mathcal{H}} \to 0}.
\]

A standard diagonal arguments extends the result to an arbitrary \( T > \tau \).

**Lemma 5.3.** Let \( t \in \mathbb{R} \) be fixed. If for each \( s \in \mathbb{N} \), \( w_s \in A_s(t) \) and \( w_0 = \lim_{s \to \infty} w_s \) in \( \mathcal{H} \), then \( w_0 \) is a constant function.

**Proof.** Using for each fixed \( s \in \mathbb{N} \) the negative invariance of the pullback attractors we have \( A_s(t) \subset U_s(t, \tau)A_s(\tau) \) for all \( \tau \leq t \). Hence, by Poincaré-Wirtinger inequality (see p.194 in [6]) and Theorem 5.1, we obtain (up to a subsequence) that \( \|w_0 - \overline{w_s}\|_{\mathcal{H}} \to 0 \) as \( s \to +\infty \), where \( \overline{w_s} := \frac{1}{|\Omega|} \int_{\Omega} w_s(x)dx \). It is easy to see that \( w_s \to w_0 \) in \( \mathcal{H} \) implies \( \overline{w_s} \to \overline{w_0} \) in \( \mathcal{H} \). So, we conclude that \( w_0 = \overline{w_0} \) and then \( w_0 \) is a constant function.

6. Robustness: upper semicontinuity of the family of pullback attractors.
We will prove now the main result concerning upper semicontinuity of pullback attractors as \( s \to \infty \). As the dynamics inside the attractor of the limit problem (28) is known and quite simple, as a consequence we can guess that the dynamics inside the attractors \( A_s \) becomes simple for large diffusion.
Theorem 6.1. The family of global attractors \( \{ A_s; \ s \in \mathbb{N} \} \) associated with problem (1) is upper semicontinuous on \( s \) at infinity, in the topology of \( \mathcal{H} \), i.e., for each \( \tau \in \mathbb{R} \), \( \lim_{s \to +\infty} \text{dist}(A_s(\tau), A_{\infty}(\tau)) = 0 \), where \( A_{\infty} \) is the pullback attractor of the limit problem (28).

Proof. Let \( \tau \in \mathbb{R} \) and \( \{ v_j \} \) be an arbitrary sequence with \( v_j \in A_j(\tau), \ \forall \ j \in \mathbb{N} \). By Corollary 3, there exists a subsequence, that we still denote in the same way, such that \( v_j \to v_0 \) in \( \mathcal{H} \) as \( j \to +\infty \). By [16, Lemma 1.2] it is enough to prove that \( v_0 \in A_{\infty}(\tau) \). Using the characterization of the pullback attractor given in (30), what we have to do is to construct a complete bounded trajectory through \( v_0 \). From Lemma 5.3 \( v_0 \) is constant, so we can assume that \( v_0 \in \mathbb{R} \).

By Theorem 3.11 there exist bounded complete trajectories \( \gamma_j(\cdot) \) of problem (1) with \( s = s_j, \gamma_j(\tau) = v_j \) such that \( \gamma_j(\tau) = v_j \to v_0 \) in \( \mathcal{H} \) as \( j \to +\infty \). Denote \( u_j^0 := \gamma_j|_{[\tau, +\infty)} \in \mathcal{R}_j(\tau) \). By Theorem 6.1 there exists a solution \( g_0 \) of the limit problem (28) with \( g_0(\tau) = v_0 \) and a subsequence of \( \{ u_j^0 \} \), that we still denote the same, such that \( u_j^0(\tau) \to g_0(\tau) \) in \( \mathcal{H} \) as \( j \to +\infty \), for any \( t \geq \tau \).

Now we consider \( u_j^1 := \gamma_j|_{[\tau - 1, +\infty)} \in \mathcal{R}_j(\tau - 1) \). Again by Corollary 3 up to a subsequence \( u_j^1(\tau - 1) \to v_{-1} \) in \( \mathcal{H} \) as \( j \to +\infty \). From Lemma 5.3 we can consider that \( v_{-1} \in \mathbb{R} \). Using Theorem 6.1 we obtain that there exists a solution \( g_1 \) of the limit problem (28) with \( g_1(\tau - 1) = v_{-1} \) and a subsequence of \( \{ u_j^1 \} \) (relabeled the same) such that \( u_j^1(t) \to g_1(t) \) in \( \mathcal{H} \) as \( j \to +\infty \) for all \( t \geq \tau - 1 \). Now note that \( g_1(t) = g_0(t) \) for each \( t \geq \tau \). Indeed,

\[
g_1(t) = \lim_{j \to +\infty} u_j^1(t) = \lim_{j \to +\infty} u_j^0(t) = g_0(t).
\]

Proceeding in this way inductively, we find for each \( r = 0, 1, 2, \cdots \), a solution \( g_r \) of the limit problem (28) with \( g_r(\tau - r) = v_{-r} \) such that \( g_{r+1}(t) = g_r(t) \) for \( t \geq \tau - r \). Given \( t \in \mathbb{R} \), we define \( g(t) \) as the common value of \( g_r(t) \) for \( t \geq \tau - r \). Then we have that \( g \) is a complete trajectory with \( g(\tau) = v_0 \). Note that for each \( r = 0, 1, 2, \cdots \) we have that \( g_r(t) = \lim_{j \to +\infty} u_j^r(t) \) and \( u_j^r(t) \in A_j(t) \), for any \( j \in \mathbb{N} \) and \( t \geq \tau - r \). As \( A_j(t) \subset \bigcup_{s \in \mathbb{N}} A_s(t) \), for all \( j \in \mathbb{N} \), we obtain that there exists a constant \( C > 0 \) such that

\[
\| g_r(t) \|_{\mathcal{H}} \leq C, \ \forall \ t \geq \tau - r \quad \text{and} \quad r = 0, 1, 2, \cdots.
\]

Hence, we have that \( \cup_{t \in \mathbb{R}} g(t) \) is bounded in \( \mathcal{H} \). Then, there exists a constant \( \tilde{C} > 0 \) such that

\[
\| g(t) \| = \frac{1}{|\Omega|^{1/2}} \| g(t) \|_{\mathcal{H}} \leq \tilde{C}, \ \forall \ t \in \mathbb{R}.
\]

We conclude that \( g : \mathbb{R} \to \mathbb{R} \) is a complete bounded trajectory through \( v_0 \). \( \square \)

If we consider the particular case where \( p_s(x) \equiv 2 \), then it is obvious that any solution of the limit problem (28) is a solution of (1). This implies readily that every complete trajectory of (28) is a complete trajectory of (1). Thus,

\[
A_{\infty}(\tau) \subset A_s(\tau) \quad \text{for all} \ \tau \in \mathbb{R}, \ s \in \mathbb{N}.
\]

Hence, the family of global attractors \( \{ A_s; \ s \in \mathbb{N} \} \) associated with problem (1) is lower semicontinuous on \( s \) at infinity. Therefore, we obtain the continuity of the pullback attractors with respect to the Hausdorff distance.

Corollary 4. Let \( p_s(x) \equiv 2 \). Then

\[
\lim_{s \to +\infty} \text{dist}_H(A_s(\tau), A_{\infty}(\tau)) = 0.
\]
7. The autonomous case. If the function $b(\cdot) \equiv b$ does not depend on $t$, that is, it is a constant, then problems (1) and (28) are autonomous. As a particular case of the previous results we obtain the upper semicontinuity of global attractors as $s \to \infty$.

We study problem (1) with $\tau = 0$ and denote by $\mathcal{R}_s(\xi)$ the set of all globally defined strong solutions $u(\cdot)$ with $u(0) = \xi$. We define the multivalued maps $G_s : \mathbb{R}^+ \times \mathcal{H} \to P(\mathcal{H})$ by

$$G_s(t, \xi) = \{ z : \text{there exists } u(\cdot) \in \mathcal{R}_s(\xi) \text{ such that } u(t) = z \},$$

which are strict multivalued semiflows, that is, $G_s(0, \xi) = \xi$ and $G_s(t+s, \xi) = G(s, G(s, \xi))$ for any $t \in \mathbb{R}^+$, $\xi \in \mathcal{H}$.

We notice that in this autonomous setting, as the solution does not depend on the initial moment of time $\tau$, the operators $U_s$, $G_s$ satisfy the following relationship:

$$U_s(t, \tau, \xi) = U_s(t - \tau, 0, \xi) = G_s(t - \tau, \xi).$$

Hence, the pullback attractor $A_s = \{ A_s(t) : t \in \mathbb{R} \}$ given in Theorem 3.10 does not depend on time, that is, $A_s(t) \equiv A_s$. It follows also that $A_s$ is compact and

$$\text{dist}(G_s(t, B), A_s) \to 0 \quad \text{as } t \to +\infty, \quad \text{for any bounded set } B,$$

which means that $A_s$ is a compact global attractor for the semiflow $G_s$. Moreover, it follows from [32, Remark 8] that it is invariant, that is, $A_s = G(t, A_s)$ for any $t \geq 0$.

In a similar way, for problem (28) we define a strict multivalued semiflow $G_\infty$ possessing the compact invariant global attractor $A_\infty$, which satisfies that $A_\infty(t) \equiv A_\infty$, where $A_\infty = \{ A_\infty(t) : t \in \mathbb{R} \}$ is its pullback attractor.

Problem (28) has three fixed points given by $z^+_1 = \frac{\lambda}{3}$, $z^-_1 = -\frac{\lambda}{3}$, $z_0 = 0$. The global attractor is characterized by these three equilibria and the heteroclinic connections which go from 0 to either $z^+_1$ or $z^-_1$.

As a particular case of Theorem 6.1 and Corollary 4 we obtain the upper semi-continuity (respectively, continuity) of global attractors for large diffusion.

**Theorem 7.1.** \( \lim_{s \to +\infty} \text{dist}(A_s, A_\infty) = 0. \)

**Corollary 5.** If $p_s(x) \equiv 2$, then \( \lim_{s \to +\infty} \text{dist}_H(A_s, A_\infty) = 0. \)

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REFERENCES

[1] J. M. Arrieta, A. N. Carvalho and A. Rodríguez-Bernal, Upper semicontinuity for attractors of parabolic problems with localized large diffusion and nonlinear boundary conditions, *J. Differential Equations*, **168** (2000), 35–59.

[2] J. M. Arrieta, A. Rodríguez-Bernal and J. Valero, Dynamics of a reaction-diffusion equation with a discontinuous nonlinearity, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, **16** (2006), 2695–2984.

[3] V. Barbu, *Nonlinear Semigroups and Differential Equations in Banach Spaces*, Editura Academiei, Bucuresti, 1976.

[4] S. Bensid and J. I. Díaz, Stability results for discontinuous nonlinear elliptic and parabolic problems with a s-shaped bifurcation branch of stationary solutions, *Discrete Contin. Dyn. Syst., Ser. B*, **22** (2017), 1757–1778.

[5] S. Bensid and J. I. Díaz, On the exact number of monotone solutions of a simplified Budyko climate model and their different stability, *Discrete Contin. Dyn. Syst., Ser. B*, **24** (2019), 1033–1047.

[6] H. Brézis, *Analyse Fonctionale*, Paris, Masson Editeur, 1983.

[7] M. I. Budyko, The effects of solar radiation variations on the climate of the Earth, *Tellus*, **21** (1969), 611–619.

[8] T. Caraballo, P. E. Kloeden and P. Marín-Rubio, Weak pullback attractors of setvalued processes, *J. Math. Anal. Appl.*, **288** (2003), 602–707.

[9] T. Caraballo, J. A. Langa, V. S. Melnik and J. Valero, Pullback attractors for nonautonomous and stochastic multivalued dynamical systems, *Set-Valued Anal.*, **11** (2003), 153–201.

[10] T. Caraballo, J. A. Langa and J. Valero, Structure of the pullback attractor for a non-autonomous scalar differential inclusion, *Discrete Contin. Dyn. Syst. Ser. S*, **9** (2016), 979–994.

[11] V. L. Carbone, A. N. Carvalho and K. Schiabel-Silva, Continuity of attractors for parabolic problems with localized large diffusion, *Nonlinear Anal.*, **68** (2008), 515–535.

[12] V. L. Carbone, C. Gentile and K. Schiabel-Silva, Asymptotic properties in parabolic problems dominated by a p-Laplacian operator with localized large diffusion, *Nonlinear Anal.*, **74** (2011), 4002–4011.

[13] A. N. Carvalho, Infinite dimensional dynamics described by ordinary differential equations, *J. Differential Equations*, **116** (1995), 338–404.

[14] A. N. Carvalho and J. K. Hale, Large diffusion with dispersion, *Nonlinear Anal.*, **17** (1991), 1139–1151.

[15] A. N. Carvalho and A. L. Pereira, A scalar parabolic equation whose asymptotic behavior is dictated by a system of ordinary differential equations, *J. Differential Equations*, **112** (1994), 81–130.

[16] A. N. Carvalho and S. Piskarev, A general approximation scheme for attractors of abstract parabolic problems, *Numer. Funct. Anal. Optim.*, **27** (2006), 785–829.

[17] E. Conway, D. Hoff and J. Smoller, Large time behavior of solutions of systems of nonlinear reaction-diffusion equations, *SIAM J. Appl. Math.*, **35** (1978), 1–16.

[18] J. I. Díaz and I. I. Vrabie, Existence for reaction diffusion systems. A compactness method approach, *J. Math. Anal. Appl.*, **188** (1994) 521–540.

[19] J. I. Díaz, J. Hernández and L. Tello, Some results about multiplicity and bifurcation of stationary solutions of a reaction diffusion climatological model, *Rev. R. Acad. Cien. Serie A. Mat.*, **96** (2002), 357–366.

[20] L. Diening, P. Harjulehto, P. Hästö and M. Růžička, *Lebesgue and Sobolev Spaces with Variable Exponents*, Springer-Verlag, Berlin, Heidelberg, 2011.

[21] X. L. Fan and Q. H. Zhang, Existence of solutions for p(x)-laplacian Dirichlet problems, *Nonlinear Anal.*, **52** (2003), 1843–1852.

[22] E. Feireisl and J. Norbury, Some existence and nonuniqueness theorems for solutions of parabolic equations with discontinuous nonlinearities, *Proc. Roy. Soc. Edinburgh, Edinburgh*, **111A** (1991), 1–17.

[23] A. C. Fernandes, M. C. Gonçalves and J. Simsen, Non-autonomous reaction-diffusion equations with variable exponents and large diffusion, *Discrete Contin. Dyn. Syst. Ser. B*, **24** (2019), 1485–1510.

[24] G. Fusco, On the explicit construction of an ODE which has the same dynamics as a scalar parabolic PDE, *J. Differential Equations*, **69** (1987), 85–110.
Z. Guo, Q. Liu, J. Sun and B. Wu, Reaction-diffusion systems with $p(x)$-growth for image denoising, *Nonlinear Anal. Real World Appl.*, **12** (2011), 2904–2918.

J. K. Hale, Large diffusivity and asymptotic behavior in parabolic systems, *J. Math. Anal. Appl.*, **118** (1986), 455–466.

J. K. Hale and C. Rocha, Varying boundary conditions with large diffusivity, *J. Math. Pures Appl.*, **66** (1987), 139–158.

J. K. Hale and C. Rocha, Interaction of diffusion and boundary conditions, *Nonlinear Anal.*, **11** (1987), 633–64.

O. V. Kapustyan, V. S. Melnik, J. Valero and V. V. Yasinski, *Global Attractors of Multi-Valued Dynamical Systems and Evolution Equations Without Uniqueness*, National Academy of Sciences of Ukraine, Naukova Dumka, 2008.

P. E. Kloeden and J. Simsen, Pullback attractors for non-autonomous evolution equations with spatially variable exponents, *Commun. Pure Appl. Anal.*, **13** (2014), 2543–2557.

P. E. Kloeden, J. Simsen and M. S. Simsen, Asymptotically autonomous multivalued Cauchy problems with spatially variable exponents, *J. Math. Anal. Appl.*, **445** (2017), 513–531.

V. S. Melnik and J. Valero, On attractors of multi-valued semi-flows and differential inclusions, *Set-Valued Anal.*, **6** (1998), 83–111.

V. S. Melnik and J. Valero, On global attractors of multivalued semiprocesses and nonautonomous evolution inclusions, *Set-Valued Anal.*, **8** (2000), 375–403.

J. Simsen, M. S. Simsen and M. R. T. Primo, Reaction-diffusion equations with spatially variable exponents and large diffusion, *Commun. Pure Appl. Anal.*, **15** (2016), 495–506.

J. Simsen, Partial differential inclusions with spatially variable exponents and large diffusion, *Mathematics in Engineering, Science and Aerospace MESA*, **7** (2016), 479–489.

J. Simsen, Weak upper semicontinuity of pullback attractors for nonautonomous reaction-diffusion equations, *Electron. J. Qual. Theory Differ. Equ.*, **68** (2019), 1–14.

J. Simsen and C. B. Gentile, Well-posed $p$-laplacian problems with large diffusion, *Nonlinear Anal.*, **71** (2009), 4609–4617.

J. Simsen and C. B. Gentile, Systems of $p$-Laplacian differential inclusions with large diffusion, *J. Math. Anal. Appl.*, **368** (2010), 525–537.

J. Simsen, M. S. Simsen and F. B. Rocha, Existence of solutions for some classes of parabolic problems involving variable exponents, *Nonlinear Stud.*, **21** (2014), 113–128.

J. Simsen, M. S. Simsen and A. Zimmermann, Study of ODE limit problems for reaction-diffusion equations, *Opuscula Math.*, **38** (2018), 117–131.

J. Simsen and J. Valero, Characterization of pullback attractors for multivalued nonautonomous dynamical systems, in *Advances in Dynamical Systems and Control* (V. A. Sadovnichiy and Z. Zgurovsky eds.), Studies in Systems, Decision and Control 659, Springer (2016), pp. 179–195.

D. Terman, A free boundary problem arising from a bistable reaction-diffusion equation, *SIAM J. Math. Anal.*, **14** (1983), 1107–1129.

D. Terman, A free boundary arising from a model for nerve conduction, *J. Differential Equations*, **58** (1985), 345–363.

R. Willie, Large diffusivity stability of attractors in the $C$-topology for a semilinear reaction and diffusion system of equations, *Dynamics of PDE*, **3** (2006), 173–197.

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