On the Betti numbers of some Gorenstein ideals

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Abstract

Assume \( R \) is a polynomial ring over a field and \( I \) is a homogeneous Gorenstein ideal of codimension \( g \geq 3 \) and initial degree \( p \geq 2 \). We prove that the number of minimal generators \( \nu(I_p) \) of \( I \) that are in degree \( p \) is bounded above by \( \nu_0 = \binom{p+g-1}{g-1} - \binom{p+g-3}{g-1} \), which is the number of minimal generators of the defining ideal of the extremal Gorenstein algebra of codimension \( g \) and initial degree \( p \). Further, \( I \) is itself extremal if \( \nu(I_p) = \nu_0 \).

1 Introduction

Assume \( R \) is a polynomial ring over a field and \( I \) is an homogeneous Gorenstein ideal of codimension \( g \geq 3 \) and initial degree \( p \geq 2 \). We have the following conjectures on the minimal number of generators of the ideal generated by the forms of degree \( p \) in \( I \).
Conjecture 1.1 1. Always $\nu(I_p) \leq \nu_0 = \binom{p+g-1}{g-1} - \binom{p+g-3}{g-1}$, and only certain values of $\nu(I_p)$ are possible.

2. If $\nu(I_p) = \nu_0$ then $I$ is extremal in the sense of [6], or equivalently,

$$e(R/I) = e(g, p) = \binom{g + p - 1}{g} + \binom{g + p - 2}{g}.$$ 

Consequently if $\nu(I_p) = \nu_0$ then $I = (I_p)$.

These estimates involve comparisons with the numerical invariants of Schenzel’s [6] extremal Gorenstein algebras. If $I$ is a graded Gorenstein ideal of codimension $g$ and initial degree $p$, then a consequence of the Macaulay-Stanley characterization [7] of the Hilbert function of $R/I$ is that the multiplicity of $R/I$ satisfies

$$e(R/I) \geq e(g, p) = \binom{g + p - 1}{g} + \binom{g + p - 2}{g}.$$ 

(A version of this estimate for non-graded Gorenstein algebras is apparently an open problem; see [8].) Given codimension $g$ and initial degree $p$, a graded Gorenstein algebra $R/J$ with multiplicity $e(R/J) = e(g, p)$ must have a pure almost linear minimal resolution (in particular $J = (J_p)$ and $\nu_0$ is the number of generators of $J$), and there are similar formulas for the other betti numbers of $R/J$. Conversely, if the resolution of $R/J$ is pure and almost linear, or equivalently, $R/J$ is extremal, then $J = (J_p)$, $\nu(J) = \nu_0$, and $e(R/J) = e(g, p)$.

Various results in the literature dealing with Cohen-Macaulay ideals (such as [2, 3, 5]) give upper bounds for $\nu(I)$ in terms of codimension, initial degree, and multiplicity $e(R/I)$. Part of the intent of our conjectures is to elucidate the multiplicity information that is already determined (if it is at all!) by the codimension and initial degree. In general this sounds quite implausible, but for Gorenstein algebras it looks promising that something can be said along these lines. Since, however, symmetry of the $h$-vector $H(R/I)$ appears to play a major role, one would not expect any such results to generalize to the non-graded case.

One might optimistically hope for even stronger estimates than suggested by Conjecture [1.1], for instance that $\nu(I) \leq \nu_0$ or even $\beta_i(R/I) \leq \beta_i(R/J)$ for
all \( i \) (that is remove the restriction to the degree \( p \) generators of \( I \), and then pass from the first betti number of \( R/I \) to the entire minimal free resolution). We have produced a considerable amount of computational evidence by using the program MACAULAY, but the same program also enabled us to find a counterexample. We are now looking for reasonable side conditions under which these stronger estimates might hold.

2 Hilbert function techniques

**Theorem 2.1** If \( I \) is a graded Gorenstein ideal of codimension \( g \geq 3 \) and initial degree \( p \geq 2 \), then

\[
\nu(I_p) \leq \nu_0 = \binom{p + g - 1}{g - 1} - \binom{p + g - 3}{g - 1},
\]

and \( I \) is itself extremal if equality holds.

**Proof.** If either \( \nu(I_p) > \nu_0 \), or \( \nu(I_p) = \nu_0 \) and \( I \) is not extremal, then by the symmetry of the \( h \)-vector \( H(R/I) \) there is some \( j \geq p \) so that

\[
H(R/I) = (h_0, h_1, \ldots, h_{p-1}, h_p, \ldots, h_j, h_{p-1}, \ldots, h_1, h_0),
\]

where \( h_j = h_p \leq \binom{p+g-3}{g-1} = h_{p-2} \). The idea of the argument is to use the Macaulay estimate (see 2, 3, 4, 7) for \( h_{j+1} \) in terms of \( j \) and \( h_j \) to see that such a small value of \( h_j \) can not grow to such a large value of \( h_{j+1} = h_{p-1} = \binom{p+g-2}{g-1} \). We recall that this estimate is calculated from the binomial expansion for \( h_j \):

\[
h_j = \binom{a_j}{j} + \binom{a_j - 1}{j - 1} + \cdots + \binom{a_i}{i} \quad (1)
\]

where \( a_j > a_{j-1} > \cdots > a_i \geq i \geq 1 \). Then

\[
h_{j+1} \leq \binom{a_j + 1}{j + 1} + \binom{a_{j-1} + 1}{j} + \cdots + \binom{a_i + 1}{i + 1}. \quad (2)
\]

We may assume that \( h_j > j \), for if not, then \( h_j \leq j \) would imply that \( a_\ell = \ell \) for all \( \ell \), and hence \( h_j \geq h_{j+1} \), which contradicts our assumption. Notice that
by grouping the terms of (1) according to the value of $a_{\ell} - \ell$ the binomial expansion for $h_j$ can be written as

$$h_j = \sum_{n=0}^{r} \left[ \binom{j_n + k_n}{j_n} + \binom{j_n - 1 + k_n}{j_n - 1} + \ldots + \binom{j_n - i_n + k_n}{j_n - i_n} \right]$$

$$= \sum_{n=0}^{r} \left[ \binom{j_n + k_n + 1}{j_n} - \binom{j_n - i_n + k_n}{j_n - i_n - 1} \right], \quad (3)$$

where $a_j - j = k_0 > k_1 > \ldots > k_r \geq 0$, $j = j_0 > j_1 > \ldots > j_r$, $j_r - i_r = i$, and $j_n = j_{n-1} - i_{n-1} - 1$ for $1 \leq n \leq r$. Set $k = k_0$. Since $p \leq j$ and $\binom{j+k}{j} \leq h_j \leq \binom{p+q-3}{g-1}$ it follows that $k \leq g - 2$. From (1), (4), and (3), together with Pascal’s identity and $\binom{a+b+1}{b+1} = \frac{a+1}{b+1} \binom{a+b+1}{b}$, we have

$$h_{j+1} - h_j \leq \sum_{n=0}^{r} \left[ \binom{j_n + k_n + 1}{j_n + 1} - \binom{j_n - i_n + k_n}{j_n - i_n} \right]$$

$$= \sum_{n=0}^{r} \left[ \frac{k_n + 1}{j_n + 1} \binom{j_n + k_n + 1}{j_n} - \frac{k_n + 1}{j_n - i_n} \binom{j_n - i_n + k_n}{j_n - i_n - 1} \right].$$

On the other hand from the upper bound on $h_j$ and $h_{j+1} = h_{p-1}$ we see that

$$\frac{g-1}{p-1} h_j \leq \binom{p+g-3}{g-2} \leq h_{j+1} - h_j.$$ 

Since $(g-1)/(p-1) > (k+1)/(j+1)$ it follows from (3) and the last two inequalities that

$$F_0 = \sum_{n=0}^{r} \left[ \frac{k_n + 1}{j_n + 1} \binom{j_n + k_n + 1}{j_n} \right]$$

$$+ \sum_{n=0}^{r} \left[ \frac{k_n + 1}{j_n - i_n} \frac{k_n + 1}{j_n + 1} \binom{j_n - i_n + k_n}{j_n - i_n - 1} \right] < 0. \quad (4)$$

For $0 \leq s \leq r$ we set

$$F_s = \sum_{n=s}^{r} \left[ \frac{k_n + 1}{j_n + 1} \frac{k_s + 1}{j_s + 1} \binom{j_n + k_n + 1}{j_n} \right]$$

$$+ \sum_{n=s}^{r} \left[ \frac{k_n + 1}{j_n - i_n} \frac{k_s + 1}{j_s + 1} \binom{j_n - i_n + k_n}{j_n - i_n - 1} \right].$$
To derive a contradiction we are going to show the following inequalities

\[ F_0 > F_1 > \cdots > F_r = \left( \frac{k_r + 1}{j_r - i_r} - \frac{k_r + 1}{j_r + 1} \right) \left( \frac{a_i}{i - 1} \right) > 0. \]

Assume \( 1 \leq s + 1 \leq r \). Notice that

\[
0 \leq \sum_{n=s}^{r-1} \left[ \left( \frac{j_n - i_n + k_n}{j_n - i_n - 1} \right) - \left( \frac{j_{n+1} + k_{n+1} + 1}{j_{n+1}} \right) \right] + \left( \frac{j_r - i_r + k_r}{j_r - i_r - 1} \right)
= \left( \frac{j_s - i_s + k_s}{j_s - i_s - 1} \right) - \sum_{n=s+1}^{r} \left[ \left( \frac{j_n + k_n + 1}{j_n} \right) - \left( \frac{j_n - i_n + k_n}{j_n - i_n - 1} \right) \right].
\]

Therefore

\[
\left( \frac{j_s - i_s + k_s}{j_s - i_s - 1} \right) \geq \sum_{n=s+1}^{r} \left( \frac{j_{n} + k_{n} + 1}{j_{n}} \right) - \sum_{n=s+1}^{r} \left( \frac{j_{n} - i_{n} + k_{n}}{j_{n} - i_{n} - 1} \right).
\]

Let \( A_s \) denote the first summation and \( B_s \) the second in this last inequality; clearly \( A_s - B_s > 0 \). Also note that

\[
\frac{k_s + 1}{j_s - i_s} > \frac{k_s + 1}{j_s + 1} \quad \text{and} \quad \frac{k_s + 1}{j_s - i_s} > \frac{k_{s+1} + 1}{j_{s+1} + 1}.
\]

Putting these together, we compute

\[
F_s - F_{s+1} = \left( \frac{k_s + 1}{j_s - i_s} - \frac{k_s + 1}{j_s + 1} \right) \left( \frac{j_s - i_s + k_s}{j_s - i_s - 1} \right)
+ \left( \frac{k_s + 1}{j_s + 1} - \frac{k_{s+1} + 1}{j_{s+1} + 1} \right) A_s
+ \left( \frac{k_{s+1} + 1}{j_{s+1} + 1} - \frac{k_s + 1}{j_s + 1} \right) B_s
\geq \left( \frac{k_s + 1}{j_s - i_s} - \frac{k_s + 1}{j_s + 1} \right) (A_s - B_s)
+ \left( \frac{k_s + 1}{j_s + 1} - \frac{k_{s+1} + 1}{j_{s+1} + 1} \right) A_s
+ \left( \frac{k_{s+1} + 1}{j_{s+1} + 1} - \frac{k_s + 1}{j_s + 1} \right) B_s
= \left( \frac{k_s + 1}{j_s - i_s} - \frac{k_{s+1} + 1}{j_{s+1} + 1} \right) (A_s - B_s) > 0.
\]

Hence \( F_0 > F_r > 0 \), which contradicts (4). \( \square \)
Remark 2.2 We have not worked out in general which values of $\nu(I_p) < \nu_0$ are forbidden, but point out that the symmetry of $H(R/I)$ restricts the possibilities of small values of $\nu(I_p)$ just as it rules out large values. For example, if $g = 4$, then $\nu_0 = (p+1)^2$. Let us see what happens in case $p = 4$ and $\nu(I_4) < 25$. There is no apparent Hilbert function obstruction to values $\nu(I_4) \leq 15$, but in the range $15 < \nu(I_4) < 25$ one will have $10 < h_j = h_4 < 20$ growing to $h_{j+1} = 20$, and this suggests that arguments along the lines we have been giving will still be effective. Since $a_j \geq j + 2$ (notation of equation (1)) is only possible if $j = 4$, and $h_4 = \binom{6}{4} = 15$ can grow to $h_5 = 20 < \binom{7}{5}$, such arguments can only succeed for $11 \leq h_j \leq 14$. In this case we are looking for a gain $h_{j+1} - h_j \geq 6$, so the expansion of $h_j$ will have to have at least six terms of form $\binom{i+1}{i}$; this is clearly impossible for $h_j \leq 26$. We conclude that the values $21 \leq \nu(I_4) \leq 24$ do not occur. If $16 \leq \nu(I_4) \leq 20$, then $H(R/I)$ fails to be unimodal. There are no known examples of such sequences for $g = 4$ at all, and some evidence that they may not be possible. One can interpret our arguments as ruling out “extreme” failure of unimodality, leaving a grey zone of “mild” failure of unimodality open for more investigation.

Remark 2.3 It has been suggested to us that there is the possibility of a very short and elegant argument, at least for the main result that $\nu(I_p) \leq \nu_0$, based on the behavior of the combinatorial functions $f(x) = x^{(n)}$ and $g(x) = h^{(x)}$. The key to this argument rests on the observation (for which we do not know a proof) that $g$ is non-increasing.

3 Resolutions and additional conjectures

We now give an example to show that Conjecture 1.1 can not be extended to bound the number of generators in all degrees.

Example 3.1 Let

$I = (x_1^2, x_1x_2x_3 + x_3^2x_4, x_3^3, x_1x_3^2, x_4^3, x_1x_4^3, x_1x_2x_4^2 + x_3x_4^2, x_2x_3^2, x_2x_4^3)$. 

This is just the ideal quotient $(x_1^2, x_4^3, x_1x_2 - x_3x_4) : (x_1x_2 - x_3x_4)$. Then $R/I$ is a Gorenstein artin algebra with $h$-vector $(1, 4, 9, 13, 13, 9, 4, 1)$ and betti
sequence \((1, 10, 18, 10, 1)\), whereas the \(h\)-vector for an extremal Gorenstein algebra of codimension four and initial degree two is \((1, 4, 1)\) and the betti sequence is \((1, 9, 16, 9, 1)\) (notice in particular that the multiplicity \(e(R/I) = 54\) is far greater than the minimal value of six). The graded structure of the minimal free resolution is given by the MACAULAY diagram in which the \((i, j)\) entry (starting with \((0, 0)\) in the top left left hand corner) represents the rank of \(R(-i - j)\) in the \(j\)th term of the resolution.

\[
\begin{array}{cccccccc}
1 & - & - & - & - & - & - & - \\
-1 & - & - & - & - & - & - & - \\
-3 & 4 & 1 & - & - & - & - & - \\
-4 & 5 & 1 & - & - & - & - & - \\
-1 & 5 & 4 & - & - & - & - & - \\
-1 & 4 & 3 & - & - & - & - & - \\
- & - & - & 1 & - & - & - & - \\
- & - & - & - & 1 & - & - & - \\
\end{array}
\]

**Remark 3.2** If \(I\) is generated only in degree \(p\), and \(g = 4\), and \(R/J\) is an extremal algebra of the same codimension and initial degree, then the theorem shows that \(\beta_1(R/I) \leq \beta_1(R/J)\); symmetry of the minimal free resolution and \(\sum_{i=0}^{4} (-1)^i \beta_i(R/I) = 0\) then imply that \(\beta_i(R/I) \leq \beta_i(R/J)\) for all \(i\).

It seems likely that in any codimension if \(R/I\) has a pure minimal free resolution, then \(\beta_i(R/I) \leq \beta_i(R/J)\) for all \(i\). This stronger estimate is true in codimension three, without any purity assumption, thanks to the Buchsbaum-Eisenbud structure theorem. For initial degree \(p\), the extremal Gorenstein algebra of codimension three has \(\beta_1 = \beta_2 = 2p + 1\).

**Theorem 3.3** Let \(R\) be a polynomial ring over a field and \(I\) be a homogeneous Gorenstein ideal of height three. If \(p\) is the initial degree of \(I\), then \(\nu(I) \leq 2p + 1\) and \(\beta_2(R/I) \leq 2p + 1\).

**Proof.** We may assume, without loss of generality, that \(R = k[x_1, x_2, x_3]\) and \(A = R/I\) is artinian and local with socle \((A) = A_\sigma\). Then by [1] the minimal free resolution of \(A\) has the form

\[
0 \to R(-\sigma - 3) \to \bigoplus_{j=1}^{\nu} R(-n_j) \xrightarrow{\nu} \bigoplus_{i=1}^{\nu} R(-m_i) \to R,
\]
where $Y$ is an alternating matrix, and the generators $f_1, \ldots, f_\nu$ of $I$ are the maximal pfaffians of $Y$. We may take $p = m_1 \leq m_2 \leq \ldots \leq m_\nu$ and $n_1 \leq n_2 \leq \ldots \leq n_\nu = \sigma + 3 - p$. Let $d_{ij} = \deg (y_{ij}) = m_i - n_j$; notice that $y_{ij} = 0$ by minimality of the resolution if $d_{ij} \leq 0$. Each generator is a sum of monomials of degree $(\nu - 1)/2$ in the $y_{ij}$ (evident from the Laplace expansion for pfaffians); and any such term that is non-zero has degree at least $(\nu - 1)/2$ in $R$. Hence if the theorem fails every generator has degree at least $(2p + 1)/2 > p$, which is a contradiction since at least one minimal generator has degree $p$. 

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