Rayleigh waves propagation in orthotropic solids with two temperature in context of thermoelasticity

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Abstract. The main objective of this paper is to study Rayleigh wave propagation in homogeneous orthotropic half space with two temperature in reference to Three Phase Lag (TPL) model of thermoelasticity. The surface wave solutions are obtained for governing equations of Rayleigh wave. The relevant boundary condition satisfies the solutions and corresponding frequency equation is derived for the Rayleigh wave in orthotropic half-space. A simulation study is conducted for numerical discussion and the results are demonstrated graphically for phase velocity and attenuation coefficient with respect to frequency in the context of Green-Nagdhi, Lord-Shulman and three phase lag models of thermoelasticity.

1. Introduction
The thermal signals with finite speed have been acquiring a lot of importance for the past few decades in the theories of thermoelasticities. Classical coupled theory of thermoelasticity discussed infinite speed of thermal signals with parabolic heat equation. In 1956, Biot introduce generalized thermoelasticity theory which predict infinite speed of propagation of thermal signals involving hyperbolic heat equation. A theory of thermoelasticity was formulated that involves heat conduction with single relaxation time and this theory was further improved by taking temperature gradient into consideration in the constitutive relations [1,2]. The generalized heat conduction model for thermoelastic solids that transmit thermal signals was developed and Surface waves propagation in thermoelastic half-space that depends upon temperature was investigated [3-6]. The semi-infinite solids for propagation of Rayleigh waves and the frequency equations at different parameters under the consideration of generalized thermoelasticity of GN Model was investigated [7] and this theory was employed in analyzing the Rayleigh waves propagation with thermal relaxation times in thermoelastic solids with voids under the influence of rotation [8-9]. The Rayleigh wave in homogenous half-space of thermo-micro stretch under influence of electromagnetic effects in isotropic medium was considered and further studied the propagation of Rayleigh waves in magneto thermoelastic medium in the existence of mass diffusion under the purview of three phase lag model [10,11]. Rayleigh wave propagation in the presence of gravity, initial stress and rotation in magneto thermoelastic medium and considered the case of thermoelastic half space with thermally insulated boundary in the purview of generalized thermoelasticity and harmonic wave propagation in thermoelastic homogeneous anisotropic half space in direction of propagation of waves along crystallographic axes and discussed that thermal effects have influenced seismic waves [12-14]. The plane waves propagation were analyzed in magneto thermoelastic medium under the influence of phase lag, rotation and temperature dependent properties [15]. The governing equations of Rayleigh wave under the effect of gravity in a...
nonhomogeneous orthotropic elastic medium were derived[16]. The proliferation of Rayleigh surface waves under the influence heat conduction and investigated the Rayleigh wave propagation under influence of phase lag of thermo elasticity[17,18]. Rayleigh waves propagation in terms of dual phase lag (DPL) for thermoelastical medium and studied the anisotropic medium for Rayleigh wave propagation under the influence of viscosity in terms of TPL(three-phase Lag) model[19,20]. The plane waves studied in anisotropic thermoelastical medium in context of dual-phase-lag(DPL) and Three-phase-lag models[21]. Rayleigh surface wave in transversely isotropic media under dual-phase-lag model was discussed and analysed orthotropic solid for Rayleigh wave propagation in context of Three-phase-lag(TPL) model of thermo elasticity[22-23].

A heat conduction theory that depends upon temperature $\phi$ and $T$ where $\phi$ represent conductive-temperature and $T$ represent thermodynamic-temperature was formulated. The material parameter $\alpha'$ involves in two temperature theory. if limit $\alpha' \to 0$ this signify that $\phi \to T$, hence two temperature theory may be transformed into classical theory[24-26]. The surface wave in thermoelastic medium with two-temperature theory and developed another hypothesis of thermo elasticity by assuming the conductive temperature and the thermodynamic temperature as key components in heat conduction of elastic bodies and the difference of two temperatures represent heat supply[27-28] and explored the proliferation of consonant plane waves in media depicted by the two-temperature hypothesis of thermo elasticity[29].

Three Phase Lag (TPL) model plays important role in different field like catalytic reactions in exothermic state, nuclear boiling, scattering of phonon, interactions of phonon-electron etc. The study surface waves propagation have great significance in astrophysical problems and seismological Problems. In this Model, $\tau_q$ represents heat flux vector, $\tau_{tr}$ represent temperature gradient and $\tau_v$ represent thermal displacement gradient replaces heat conduction equation of fourier law and surface wave’s propagation velocity is affected by the coupling between strain and temperature gradient.

In this paper, frequency equation are obtained for Rayleigh waves propagation in orthotropic thermoelastic half space medium with two temperature by using the governing equation and relevant boundary condition. Various properties of waves like attenuation coefficient, phase velocity have been obtained at different range of frequencies .The variation of Rayleigh wave phase velocity and its attenuation coefficient corresponding to frequency are represented graphically for different thermoelastic models. Special cases for thermally insulated and isothermal have been discussed and corresponding frequency equations have been derived at two temperature. This study have applications in the field of monitoring of structural health, characterization of damage in materials and designing of various surface acoustic waves devices.

2. Basic equations

The governing equations of anisotropic Three-phase-lag(TPL) thermoelasticity without heat sources are as per the following:

(i) The stress-strain and displacement relation:

$$\tau_{ij} = c_{ijkl}e_{kl} - \beta_{ij}T$$ (1)

(ii) The equation of motion:

$$\tau_{ij} = \rho \frac{\partial^2 u_j}{\partial t^2}$$ (2)

(iii) The strain displacement relation:

$$e_{ij} = \left( \frac{u_{i,j} + u_{j,i}}{2} \right)$$ (3)

(iv) The stress entropy temperature relation:
\[ \rho S = \frac{Pc_e}{T_0} T + \beta_0 e_{ij} \]  

(v) The energy equation:

\[ -q_{ij} = \rho T_0 \frac{\partial S}{\partial t} \]  

(vi) The Fourier law (modified) with Three phase lag model (TPL)

\[ \left[ 1 + \tau_q \frac{\partial}{\partial t} + \frac{\tau_q^2}{2} \frac{\partial^2}{\partial t^2} \right] q_{ij} = -K_{ij} \left[ 1 + \tau_T \frac{\partial}{\partial t} \right] T_{ij} - K'_{ij} \left[ 1 + \tau_v \frac{\partial}{\partial t} \right] v_{ij} \]  

where \( T = \theta - T_0 \) represent the increment of temperature. Here \( \theta \) represent absolute temperature and \( T_0 \) assumed as body's uniform temperature , mass density is represented by \( \rho \) , \( q_i \) is represents heat flux vector, \( K'_{ij} \) and \( K_{ij} \) represents material constant and material thermal conductivity respectively, specific heat at constant strain is represented by \( c_e \) , \( c_{ijkl} \) gives elastic constants. \( e_{ij} \) are stress-strain tensor , \( S \) represents entropy per unit mass , \( u_i \) is displacement component, \( \beta_{ij} \) represents thermal modulli, \( \tau_T \) represent temperature gradient, \( \tau_v \) represent thermal displacement gradient and \( \tau_q \) phase lag in heat at the condition \( 0 \leq \tau_T \leq \tau_q \).

3. Formulation of the Problem

Consider half space \( Z=0 \) for orthotropic solid in which plane strain is parallel to x-z plane Following [23-27] propagation of Rayleigh waves in orthotropic solid half space at two temperature in terms of Three phase lag (TPL) model can be written using equation (1)-(6), we have the following relation:

\[ C_{11} \frac{\partial^2 u_1}{\partial x^2} + C_{44} \frac{\partial^2 u_1}{\partial z^2} + \left( C_{13} + C_{44} \right) \frac{\partial^2 u_1}{\partial x \partial z} - \beta_1 \frac{\partial \theta}{\partial x} = \rho \frac{\partial^2 u_1}{\partial t^2} \]  

\[ C_{44} \frac{\partial^2 u_3}{\partial x^2} + C_{33} \frac{\partial^2 u_3}{\partial z^2} + \left( C_{13} + C_{44} \right) \frac{\partial^2 u_3}{\partial x \partial z} - \beta_3 \frac{\partial \theta}{\partial z} = \rho \frac{\partial^2 u_3}{\partial t^2} \]  

\[ K_3 \left[ 1 + \tau_T \frac{\partial}{\partial t} \right] \frac{\partial}{\partial t} \left( \frac{\partial^2 T}{\partial x^2} \right) + K_4 \left[ 1 + \tau_q \frac{\partial}{\partial t} \right] \frac{\partial}{\partial t} \left( \frac{\partial^2 T}{\partial z^2} \right) + K'_{ij} \left[ 1 + \tau_v \frac{\partial}{\partial t} \right] \frac{\partial}{\partial t} \left( \frac{\partial^2 T}{\partial x \partial z} \right) = \]  

\[ \left[ 1 + \tau_q \frac{\partial}{\partial t} + \frac{\tau_q^2}{2 \partial t^2} \right] \left[ \rho c_v T_0 \left( \beta_1 \frac{\partial u_1}{\partial x} + \beta_3 \frac{\partial u_3}{\partial z} \right) \right] \]  

Two temperature relation:

\[ \theta = T - a^2 \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2} \right) \]  

4. Boundary Conditions

As \( z = 0 \) considered as thermally stress free surface, following are the assumed mechanical and thermal boundary conditions:

1. Normal stress component vanished \( \tau_{33} = 0 \)
2. Tangential stress component vanished \( \tau_{11} = 0 \)
3. Thermal conditions \( q_{ij} + mT = 0 \) and for thermally insulated surface \( m \to 0 \) and for isothermal Surface \( m \to \infty \)
5. Solution of the problem

The relationship between displacement potentials $\phi(x,z,t), \psi(x,z,t)$ and displacements components $u_1, u_2$ are assumed as follows:

$$u_1 = \frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial z} \quad u_2 = \frac{\partial \phi}{\partial z} + \frac{\partial \psi}{\partial x}$$ \hspace{1cm} (11)

Substituting the value expressions of (11) in (7), (8) and (9), it is observed $\phi$ and $\psi$ satisfied the equations:

$$C_{11} \frac{\partial^2 \phi}{\partial x^2} + (C_{13} + 2C_{44}) \frac{\partial^2 \phi}{\partial z^2} - \beta_1 \frac{\partial^2 \phi}{\partial t^2} = \rho \frac{\partial^2 \phi}{\partial t^2}$$ \hspace{1cm} (12)

$$((C_{11} - C_{44} - C_{13}) \frac{\partial^2 \psi}{\partial x^2} + C_{44} \frac{\partial^2 \psi}{\partial z^2} = \rho \frac{\partial^2 \psi}{\partial t^2}$$ \hspace{1cm} (13)

$$C_{33} \frac{\partial^2 \phi}{\partial z^2} + (C_{13} + 2C_{44}) \frac{\partial^2 \phi}{\partial z^2} - \beta_3 \frac{\partial^2 \phi}{\partial t^2} = \rho \frac{\partial^2 \phi}{\partial t^2}$$ \hspace{1cm} (14)

$$((C_{11} - C_{44} - C_{13}) \frac{\partial^2 \psi}{\partial z^2} + C_{44} \frac{\partial^2 \psi}{\partial z^2} = \rho \frac{\partial^2 \psi}{\partial t^2}$$ \hspace{1cm} (15)

$$\frac{
abla^4 \phi}{\partial x^2 \partial z^2} + \frac{\nabla^4 \psi}{\partial x^2 \partial z^2} + \frac{\nabla^4 T}{\partial x^2 \partial z^2}$$

Equations (12) and (13) are equivalence to (14) and (15) respectively, in the context of above boundary conditions. Here equations (12), (15) and (16) are considered as the solutions.

6. Normal mode analysis

Generalized thermo elastic problem in x-z plane can be solved by Fourier transformation and Laplace transformation but these methods entails tiresome process and approximate result is obtained. The main problem about these methods is that during inverse transformation it gives discrete and truncation error. To eradicate this problem, Normal mode analysis is employed with no restrictions imposed on stress and displacement distributions provide exact solutions.

As Harmonic wave propagate along x-axis, we considered solution of (12), (15) and (16) in the following form:

$$\begin{align*}
\phi &= U(z)e^{ik(x-ct)} \\
\psi &= V(z)e^{ik(x-ct)} \\
T &= W(z)e^{ik(x-ct)}
\end{align*}$$ \hspace{1cm} (17)

Here phase velocity is represented by $c$ and wave number is represented by $k$. Using the value of $\phi, \psi$ and $T$ in the equations (12), (15) and (16)

$$\left[ (C_{13} + 2C_{44})D^2 + k^2 (\rho c^2 - C_{11}) \right] \frac{\partial^2 U}{\partial z^2} - \beta_1 \left[ 1 + a^2 k^2 - a^2 D^2 \right] W = 0$$ \hspace{1cm} (18)
\[
\left[(C_{33} - C_{44} - C_{13})D^2 + k^2(\rho \omega^2 - C_{44})\right] V = 0
\]
\[
\left[i k^2 c K_i \tau_1 - k^2 \tau_2 K_i + \rho \omega^2 (1 + \alpha^2 k^2 - \alpha^2 D^2)k^2 c^2 + (K_3^* \tau_2 - i K_3 \kappa c \tau_1)D^2\right] W
\]
\[
+ (T_0^2D^2 - T_0^2k^2)k^2 c^2 U + (ikT_0^2 - i k T_0^2)k^2 c^2 DV = 0
\]

Where

\[
D^2 = \frac{d^2}{dz^2}, \tau_1 = \frac{\tau_3}{\tau_5}, \tau_2 = \frac{\tau_4}{\tau_5}, \tau_3 = 1 - i k e^c \tau_1, \tau_4 = 1 - i k e^c \tau_2, \tau_5 = 1 - i k e^c \tau_3 - \frac{k^2 c^2 \tau_2^2}{2}
\]

Eliminating U, V and W from equations (18), (19), and (20). Hence we get
\[
(D^6 + ED^4 + FD^2 + G) [U(z), V(z), W(z)] = 0
\]

In other words it can be reduced as follows:
\[
(D^2 - t_1^2)(D^2 - t_2^2)(D^2 - t_3^2) [U(z), V(z), W(z)] = 0
\]

Where \(t_1, t_2\) and \(t_3\) are positive solutions of following characteristic equations
\[
\lambda^6 + E \lambda^4 + F \lambda^2 + G = 0
\]

Equation (22) gives the positive roots as follows:
\[
t_1 = \left[\frac{2e \sin(f) - E}{3}\right]^{1/2} \quad t_2 = \left[\frac{-E - e \sin(f + \sqrt{3}\cos f)}{3}\right]^{1/2} \quad t_3 = \left[\frac{-E + e \sin(f + \sqrt{3}\cos f)}{3}\right]^{1/2}
\]

Where \(e = (E^2 - 3F)^{1/2}\) \(f = \frac{\sin^{-1}d}{3}\) \(d = \frac{2E^3 - 9EF + 27G}{2e^3}\)

\[
E = \frac{b_1 b_2 (b_3 k^3 - b_4 k^2 + b_5 b_6 k^2)}{b_7 (b_3 k^3 - b_4 k^2 + b_5 b_6 k^2)} + (b_7 k b_8 - \alpha^2 b_7 k^2)(b_7 - b_9 k b_8 - \alpha^2 b_7 k^2)k^2 + \beta_1 (b_9 b_2 a^2 k^2 - b_7 b_9 a^2 k^2)
\]

\[
F = \frac{b_2 (b_5 k^3 - b_6 k^2 + b_7 b_8 k^2)}{b_5 (b_5 k^3 - b_6 k^2 + b_7 b_8 k^2)} + (b_5 b_6 k^2 - b_5 b_8 k^2 + b_1 b_2 k^2)k^2 + \beta_2 (b_5 b_6 b_7 k^2 + b_5 b_6 a^2 k^2)
\]

\[
G = \frac{b_3 b_4 (b_5 k^3 - b_6 k^2 + b_7 b_8 k^2)}{b_3 (b_5 k^3 - b_6 k^2 + b_7 b_8 k^2)} + (b_5 b_6 k^2 - b_5 b_8 k^2 + b_1 b_2 k^2)k^2 - \beta_3 (b_5 b_6 b_7 k^2 + b_5 b_6 a^2 k^2)
\]

\[
b_1 = C_{13} + 2C_{44} \quad b_2 = C_{33} - C_{44} - C_{13} \quad b_3 = \rho \omega^2 - C_{11} \quad b_4 = \rho \omega^2 - C_{44}
\]

\[
b_5 = icK_1 \tau_1 \quad b_6 = K_1^* \tau_2 \quad b_7 = K_3^* \tau_2 \quad b_9 = icK_3 \tau_1
\]

When \(z \to \infty\), the equation is bounded and it can be drafted as follows:
\[
U(z) = \sum_{i=1}^{3} E_i e^{-i\zeta}
\]
\[
V(z) = \sum_{i=1}^{3} F_i e^{-i\zeta}
\]
\[
W(z) = \sum_{i=1}^{3} G_i e^{-i\zeta}
\]

Where \(E_i, F_i, G_i\) are constants for \(i = 1, 2, 3\)
7. Derivation of frequency equation

Where \( d_i = \frac{b_i t_i^2 + b_k k^2}{b_{12} - a^* t_i^2} \)

\( p_i = \frac{b_i k^3 - b_k k^2 + b_{11}(b_{12} - a^* t_i^2)k^2 + (b_k - k b_k) 1}{ik t_i(b_{10} - b_k)} \)

Hence the solutions of equations (12),(15) and (16) are given by

\[
\phi = \sum_{i=1}^{3} E_i e^{-\eta x + i k(x-a)} \\
\psi = \sum_{i=1}^{3} d_i E_i e^{-\eta x + i k(x-a)} \\
T = \sum_{i=1}^{3} p_i E_i e^{-\eta x + i k(x-a)}
\]

(24)

7. Derivation of frequency equation

As from equation(4) it has been noticed that two temperature gradient is related to each other

\[
q_i = \left[ -K_s(1 + \tau_i D)D' - K_s(1 + \tau_i D) \right] \frac{\partial \theta}{\partial z}
\]

(25)

The Stress Components with reference to thermoelastic potentials at two temperature are given by:

\[
\tau_{33} = C_{13} \frac{\partial^2 \phi}{\partial x^2} - C_{33} \frac{\partial^2 \phi}{\partial z^2} - (C_{13} - C_{33}) \frac{\partial^3 \psi}{\partial x \partial z} - \beta_3 \theta
\]

(26)

\[
\tau_{33} = C_{44} \left( 2 \frac{\partial^2 \phi}{\partial x \partial z} - \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^3 \psi}{\partial x^3} \right)
\]

(27)

Using the boundary conditions in equations. (24),(25) and(26),The homogeneous system of linear equations in terms of \( E_1, E_2, E_3 \) are obtained

\[
\sum_{i=1}^{3} \left[ C_{33} t_i^2 - C_{13} k^2 + (C_{13} - C_{33})ik t_i p_i - \beta_3 (b_{12} - a^* t_i^2) \right] E_i = 0
\]

(28)

\[
\sum_{i=1}^{3} \left( p_i t_i^2 + 2ik t_i + k^2 p_i \right) E_i = 0
\]

(29)

\[
\sum_{i=1}^{3} \left( b_{12} - a^* t_i^2 \right)(\alpha t_i + m) d_i E_i = 0
\]

(30)

Where \( \alpha = \frac{-ik c_K k c_T + K_s^* \tau_{33}}{-ik c_T} \)

The non trivial solutions of equations (28), (29) and (30)exist if

\[
[C_{33} t_i^2 - C_{13} k^2 + (C_{13} - C_{33})ik t_i p_i - \beta_3 (b_{12} - a^* t_i^2) \left( b_{12} t_i^2 + 2ikt_i + k^2 b_k (b_{12} - a^* t_i^2) (\alpha t_i + m) d_i \right. \\
- (b_k t_i^2 + 2ikt_i + k^2 b_k (b_{12} - a^* t_i^2) (\alpha t_i + m) d_i] + \left[ C_{33} t_i^2 - C_{13} k^2 + (C_{13} - C_{33})ik t_i p_i - \beta_3 (b_{12} - a^* t_i^2) \right] \]

(31)
\[ d_j[(b_1 t_1^2 + 2 i k t_1 + k^2 b_2)(b_1 - \alpha^* t_1^2)(\alpha t_1 + m)d_1 - (b_1 t_1^2 + 2 i k t_1 + k^2 b_2)(b_1 - \alpha^* t_1^2)(\alpha t_1 + m)d_1] + [C_{33} t_1^2 - C_{13} k^2 + (C_{13} - C_{33}) i k t_1 p_3 - \beta_3(b_1 - \alpha^* t_1^2)d_1][(b_1 t_1^2 + 2 i k t_1 + k^2 b_2)(b_1 - \alpha^* t_1^2)d_1 - (b_1 t_1^2 + 2 i k t_1 + k^2 b_2)(b_1 - \alpha^* t_1^2)d_1] = 0 \] (31)

8. Particular cases
8.1 For thermally insulated surface
In case for thermally insulated surface, on \( z = 0 \) boundary condition is \( q_z = 0 \), equation (31) transform into
\[ [C_{33} t_1^2 - C_{13} k^2 + (C_{13} - C_{33}) i k t_1 p_3 - \beta_3(b_1 - \alpha^* t_1^2)d_1][(b_1 t_1^2 + 2 i k t_1 + k^2 b_2)(b_1 - \alpha^* t_1^2)d_1 - (b_1 t_1^2 + 2 i k t_1 + k^2 b_2)(b_1 - \alpha^* t_1^2)d_1] = 0 \] (32)

8.2 For isothermal surface:
In case for isothermal surface, on \( z = 0 \) boundary condition is \( T = 0 \) then equation(32) transform into
\[ [C_{33} t_1^2 - C_{13} k^2 + (C_{13} - C_{33}) i k t_1 p_3 - \beta_3(b_1 - \alpha^* t_1^2)d_1][(b_1 t_1^2 + 2 i k t_1 + k^2 b_2)(b_1 - \alpha^* t_1^2)d_1 - (b_1 t_1^2 + 2 i k t_1 + k^2 b_2)(b_1 - \alpha^* t_1^2)d_1] = 0 \] (33)

9. Discussion of frequency equation
We can obtain the different results by considering particular values of orthotropic media from equation: 
1. When we substitute \( \tau_q = \tau_r = 0 \) in equation(31), Then reduced the frequency equation represent classical coupled thermo elasticity case
2. When we substitute \( \tau_q = \tau_r = 0 \) and \( K^*_1 = K^*_3 = 0 \), \( \tau_q \neq 0 \) in equation(31), Hence the frequency equation is obtained that agrees to model proposed by Lord-Shulman.
3. When we substitute \( \tau_q = \tau_r = 0 \) in equation(31), hence the reduced equation agrees to GN Type-III model.

10. Solution of frequency equation
Generally phase velocity (c) and wave number (k) are considered as complex quantities. As it is assumed that \( c^{-1} = B^{-1} + i \omega Q \) As \( k = M + iQ \) here \( k \) represent wave number and \( M = \frac{\omega}{B} \) where \( B, Q \) are real. The exponential term in equation (17) becomes \( iM(x-B)Qx \). Here B represents the propagation speed, attenuation coefficient represents \( Q \) and angular frequency of waves represented by \( \omega \).

11. Special cases
We take \( C_{11} = C_{33} \) and \( 2C_{44} = C_{11} - C_{13} \), \( \beta_1 = \beta_3 = \beta \), applying these assumptions satisfied the relations and no restrictions imposed on parameter. Under these assumptions the frequency equation for thermally insulated boundaries transformed into
\[
(C_{11}\Gamma_1^2 - C_{13}k^2 - \beta g_1)\Gamma_3^2 + k^2\Gamma_3 + C_{13}\Gamma_1^2 - C_{13}k^2 - \beta g_2)\Gamma_3^2 + k^2\Gamma_3 + 2k^2(g_1 - g_2)C_{13}\Gamma_1\Gamma_2 = 0
\]

and for isothermal boundaries the frequency equation transform into
\[
(C_{11}\Gamma_1^2 - C_{13}k^2 - \beta g_1)\Gamma_3^2 + k^2\Gamma_3 + C_{13}\Gamma_1^2 - C_{13}k^2 - \beta g_2)\Gamma_3^2 + k^2\Gamma_3 + 2k^2\Gamma_2 g_2(C_{11} - C_{13}) - 2k^2\Gamma_2 g_2 = 0
\]

Where \(\Gamma_3 = \left[1 - \frac{2\rho c^2}{(C_{11} - C_{33})}\right]^{1/2}k^2\) and \(g_n = \frac{C_{11}\Gamma_1^2 - (C_{11} - \rho c^2)k^2}{\beta(b_{12} - \alpha^2t_n^2)}\)

\(\Gamma_1^2\) and \(\Gamma_2^2\) provide the roots of the equation of degree 4
\[
\Gamma_4^4 + \left[C_{11}ik^3cK_1r_1 - k^2r_2K_1^* + \rho c^2k^2c^2(1 + \alpha'k^2) - (C_{11} - \rho c^2)(K_1^*r_2 - iK_1kc\tau_1) + T_0\beta^2k^2c^2\right] = 0
\]

Again if we assume that \(C_{11} = C_{33} = \lambda + 2\mu\) and \(C_{13} = \lambda, C_{44} = \mu, \tau_1 = \tau_0 = 0, K_1^* = K_3^* = 0\) and \(K_i = K\)

\(K_0 = K\) in the equation corresponding thermally insulated boundary, we obtain
\[
\left(2 - \frac{c^2}{c_2^2}\right)\left(\zeta_1^2 + \zeta_1\zeta_2 + \zeta_2^2 - 1 + \frac{c^2}{c_2^2}\right) - 4\zeta_1\zeta_2\zeta_3 = 0
\]

Where \(\zeta_1^2 = 1 - \frac{m_1^2}{k^2}, \zeta_2^2 = 1 - \frac{m_2^2}{k^2}, \zeta_3^2 = 1 - \frac{\Delta^2}{k^2}\) and \(\Delta^2 = \frac{k^2c^2}{c_2^2}\)

The roots of bi quadratic equation are given by \(m_1^2\) and \(m_2^2\).

Hence the frequency equation is obtained for Rayleigh wave propagating in thermally insulated isotropic half space given by equation (32)

12. Numerical results and discussion

Relevant parameters for Cobalt Material are assumed for numerical discussion:

\begin{align*}
C_{11} &= 3.071 \times 10^{-11} \text{Nm}^{-2} & C_{33} &= 3.581 \times 10^{-11} \text{Nm}^{-2} & C_{13} &= 1.650 \times 10^{-11} \text{Nm}^{-2} \\
C_{44} &= 4.151 \times 10^{-11} \text{Nm}^{-2} & C_0 &= 4.270 \times 10^{2}\text{J/KgK} & T_0 &= 298\text{K} \\
\beta_3 &= 6.93 \times 10^2 \text{N/m}^2\text{K} & \rho &= 8.836 \times 10^3 \text{kg/m}^3 & \beta_1 &= 7.04 \times 10^6 \text{N/m}^2\text{K} \\
K_1 &= 6.90 \times 10^2 \text{W/mK} & K'_1 &= 1.313 \times 10^2 \text{W/mK} & K_3 &= 7.01 \times 10^2 \text{W/mK} \\
K'_2 &= 1.54 \times 10^2 \text{W/mK} & \tau_q &= 2.0 \times 10^{-7}\text{s} & \tau_T &= 1.5 \times 10^{-7}\text{s} & \tau_v &= 1.0 \times 10^{-8}\text{s}
\end{align*}

Attenuation coefficient and phase velocity of Rayleigh waves have been computed at distinctive values of frequencies for isothermal and thermally insulated surface in terms of Three-phase-Lag(TPL), Lord-Shulman(L-S) model and Green-Naghdi(G-N) Type-III of thermo elasticities by considering stress free state and two temperature parameter \(a\) is considered as 0.01.
Figure 1. Phase Velocity v/s Frequency

It represent the case of the change of phase velocity corresponding to increase in frequencies of Rayleigh waves. It has been detected that for all the model (TPL model, L-S theory) and G-N theory Type-III the phase velocity of Rayleigh waves increases correspondingly with the increase of frequencies and it also shows that for all the models phase velocity remains same.

Figure 2. Attenuation Coefficient v/s Frequency

It represents the case for thermally insulated boundaries in which attenuation coefficient varies with the change of frequency of Rayleigh waves, it has been recognized that in comparison of all three models, attenuation coefficient achieves lower value for the frequency in region between 200 and 700 for L-S and Three phase lag(TPL) model. The attenuation coefficient varies almost similarly for TPL Model and LS theory. It has been viewed that attenuation coefficient coincides with almost zero and remain constant.
It represent the variation of attenuation coefficient with respect to frequency for thermally insulated surfaces where \( \tau_T \) has been fixed at 0.015 and the values of \( \tau_q \) has been changed, it has been recognized that attenuation coefficient achieves very large value at lower region of frequency and with the increase of frequency, the attenuation coefficient decreases rapidly and converges to constant value and all the values coincide with each other.

It represents the variation of attenuation coefficient of Rayleigh waves corresponding to frequency for isothermal boundaries where \( \tau_T \) has been fixed at 0.015 and the values of \( \tau_q \) has been changed. it has been viewed that the value of attenuation coefficient attains constant value and it is coincide nearly to zero when \( \tau_q \) is at 0.02. For the other values of \( \tau_q \) at 0.0 and 0.01, the attenuation coefficient almost coincide with other and with the increase of range of frequency, attenuation coefficient increases proportionally.

12. Conclusion
The propagation of Rayleigh waves in orthotropic half space with two temperature in terms of Three phase lag model(TPL) has been examined. Frequency equations for isothermal and insulated boundary
conditions has been obtained. During propagation of Rayleigh wave, the surface particles follow elliptical path. Phase velocity, attenuation coefficient of Rayleigh wave in orthotropic solids has been computed. Variation of these properties with respect to frequency has been graphically presented in the context different thermo elastic Models.

Based upon numerical and analytical observation, we can reach on the following conclusion:

1. The phase velocity variation of Rayleigh waves corresponding to frequency remains identical for all currently existing thermo elastic models. With the increase of frequency of Rayleigh wave propagating in orthotropic solids, phase velocity also increases.

2. For the case of thermally insulated surfaces, at the lower range of frequencies, the attenuation coefficient attains high values and when isothermal boundaries are considered, the attenuation coefficient of Rayleigh waves gradually increases with the increase of frequency.

3. Although the problem considered to be theoretical one, but these analytical observations can immensely helpful in providing the vital facts for researchers in different fields of seismology and mine engineering.

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