Optimal control of a linear system with multiplicative noise at control parameter

I.P. Smirnov
Institute of Applied Physics RAS,
46 Ul’yanova Street, Nizhny Novgorod, Russia

Abstract

We investigate a control process described by a linear system of ordinary differential equations with a noise of special type acting to the control parameter. As the cost functional the probability of the final state vector to enter to a given set in the phase space is considered. Necessary conditions of optimality (of the Pontryagin maximum principle form) and existence theorems are developed. The initial control problem was trasformed to an auxiliary deterministic problem, the differentiability of the auxiliary functional was discussed.

Key words: optimal stochastic control, Pontryagin maximum principle, existence theorems, differentiability of functional

1 Introduction

Consider the optimal control problem

\[
\begin{align*}
\frac{dx}{dt} &= A(t)x + \xi(t)B(t)u(t) + f(t), \quad 0 \leq t \leq 1, \\
x(0) &= x_0, \\
\varphi(u(\cdot)) &= P(x_u(1) \in Q) \rightarrow \max_{u(\cdot) \in U}
\end{align*}
\]

(1)

\[
\varphi(u(\cdot)) = P(x_u(1) \in Q) \rightarrow \max_{u(\cdot) \in U}
\]

(2)

where \(x\) is the system state vector and \(u\) is the control one, \(Q\) is a given nonrandom goal set in the phase space, \(\xi(t)\) is a scalar random process of the given type.
The necessary conditions for optimality in the form of Pontryagin maximum principle and existence theorems are obtained in this paper keeping on our recent investigations [1–3]. In sec. 2 we state the problem, in sec. 3 reduce it to the equivalent one of deterministic type. Existence of optimal control is proved in sec. 4. Conditions of differentiability of the functional, necessary and sufficient conditions for optimality are obtained in sec. 5, 6, respectively.

2 Statement of the problem

Denote by $\langle \cdot, \cdot \rangle$ the inner product and by $|\cdot|$ the Euclidean norm of a vector (matrix) in $\mathbb{R}^n$. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. Forming the system (1) deterministic matrices $A(t)$, $B(t)$, $f(t)$ are of order $n \times n$, $n \times m$, $n \times 1$, respectively; they have summable with respect to Lebesgue measure components; the ones of matrix $B(t)$ are quadratically summable; $x_0$ is a nonrandom vector, $Q$ is a nonrandom Borel’s set in $\mathbb{R}^n$.

Let $\{\xi_1(\omega), \ldots, \xi_N(\omega)\}$ be a given set of random values on $(\Omega, \mathcal{F}, P)$,

$$\{0 = t_0 < t_1 < \ldots < t_N = 1\}$$

a finite subset of $[0, 1]$. Now we introduce the process $\xi(t, \omega)$:

$$\xi(t, \omega) = \xi_i(\omega), \ t \in [t_{i-1}, t_i), \ i = 1, N.$$ 

The set $\mathcal{U}$ of admissible controls consists of all measurable deterministic functions $u(t)$, $t \in [0, 1]$, taking their values in a fixed set $V \subset \mathbb{R}^n$. For every admissible control $u(\cdot) \in L^2(0, 1]$ there exist a unique (in the class of process with absolutely continuous trajectories) solution of the Cauchy’s problem (1) [4, p. 35]; the corresponding value of the goal functional $\varphi(u(\cdot)) \in [0, 1]$. Optimal control problem (1)-(2) can be posed as the problem of choosing of an admissible control $u^0(\cdot) \in \mathcal{U}$ provided maximal value to the functional (2):

$$\varphi(u^0(\cdot)) = \max_{u(\cdot) \in \mathcal{U}} \varphi(u(\cdot)).$$
3 Auxiliary deterministic problem

Using Cauchy’s formula for the solution of a linear differential equation system one can write

\[ x_u (1) = \Phi (1) \left( x_0 + \int_0^1 \Phi^{-1} (s) (\xi (s) B (s) u (s) + f (s)) \, ds \right) = \]

\[ = \hat{x}_0 + \Phi (1) \sum_{i=1}^N \xi_i \int_{t_{i-1}}^{t_i} \Phi^{-1} (s) B (s) u (s) \, ds = \hat{x}_0 + \sum_{i=1}^N \xi_i z_u^{(i)}, \]

where

\[ \hat{x}_0 \equiv \Phi (1) \left( x_0 + \int_0^1 \Phi^{-1} (s) f (s) \, ds \right), \]

\[ z_u^{(i)} \equiv \Phi (1) \int_{t_{i-1}}^{t_i} \Phi^{-1} (s) B (s) u (s) \, ds, \]

\( \Phi (t) \) is a fundamental matrix for the system (1) (\( \Phi (0) \) is the unitary matrix).

Note that

\[ z_u^{(i)} = \Psi_i y (t_i) - \Psi_{i-1} y (t_{i-1}), \quad \Psi_i \equiv \Phi (1) \Phi^{-1} (t_i), i = 1, N, \]

where \( y_u (t) \) is the solution of the Cauchy’s problem

\[ \begin{cases} \frac{d}{dt} y = A (t) y + B (t) u, \ t \in (0, 1), \\ y (0) = 0. \end{cases} \]

Taking into account 3) and 5) and introducing the functions of \( N \) vector variables

\[ g (z_1, \ldots, z_N) = P \left( \sum_{i=1}^N \chi_i z_i \in \hat{Q} \right), \]

\[ G (y_1, \ldots, y_N) = P \left( \sum_{i=1}^N \chi_i \Psi_i y_i \in \hat{Q} \right), \]

where \( \hat{Q} \equiv Q - x_0, \ \chi_i \equiv \xi_i - \xi_{i+1}, \ i = 1, N - 1, \ \chi_N \equiv \xi_N \), we get the following representation for the functional (2):

\[ \varphi (u (\cdot)) = P (x_u (1) \in Q) = \]

\[ = P \left( \hat{x}_0 + \sum_{i=1}^N \xi_i z_u^{(i)} \in Q \right) = \]

\[ = g \left( z_u^{(1)}, \ldots, z_u^{(N)} \right) = G (y_u (t_1), \ldots, y_u (t_N)). \]
It is clear now that the stochastic control problem (1) is equivalent to the next deterministic control problem:

\[
\begin{aligned}
    \frac{d}{dt}y &= A(t) \, y + B(t) \, u, \ t \in (0, 1), \\
    y(0) &= 0,
\end{aligned}
\]

(8)

\[\varphi(u(\cdot)) = G(y_u(t_1), \ldots, y_u(t_N)) \to \max_{u(\cdot) \in U}.\]

Note that the form of the last is unusual: the cost functional is of terminal type and it depends on control process states, taken at several moments of the time of control.

4 Existence theorem

Let \(\{y_1, \ldots, y_N\}\) be an arbitrary set of vectors in \(\mathbb{R}^n\). Construct a new vector \(Y\) in \(\mathbb{R}^M\), \(M \equiv n \cdot N\) by the following rule: \(Y = \{y_1, \ldots, y_N\} = \{y_1^1, \ldots, y_n^1; y_1^2, \ldots, y_n^N\}\). Consider the following functions

\[S(\omega, Y) = \sum_{i=1}^N \chi_i(\omega) \Psi_i y_i,\]

\[h(Y) = P \left( \omega : S(\omega, Y) \in \hat{Q} \right).\]

Lemma 1. If the set \(\hat{Q}\) is closed in \(\mathbb{R}^n\), then \(h(\cdot)\) is upper semicontinues function.

Prove. Let \(Y^* \in \mathbb{R}^M\), \(Y_k \to Y^*\) as \(k \to \infty\). Consider random events

\[A_k = \left\{ \omega : S(\omega, Y_k) \in \hat{Q} \right\},\]

\[A^* = \left\{ \omega : S(\omega, Y^*) \in \hat{Q} \right\},\]

\[B = \limsup_{k \to \infty} A_k = \bigcap_{k \geq 1} \bigcup_{j \geq k} A_j.\]

If \(\omega \in B\), then there exist a sequence \(k_m \to \infty\) such that \(\omega \in A_{k_m}\), in other words

\[S(\omega, Y_{k_m}) \in \hat{Q}\]

for every \(m\). Proceeding in this statement to the limit as \(m \to \infty\) and taking into account that the function \(S(\omega, \cdot)\) is continuous and the set \(\hat{Q}\) is closed.
we obtain $S(\omega, Y^*) \in \hat{Q}$, so $\omega \in A^*$. Hence $B \subset A^*$ and $P(B) \leq P(A)$.

On the other hand, due to measure properties \cite[5, p.34]{5}:

$$P(B) = P\left(\limsup_{k \to \infty} A_k\right) \geq \limsup_{k \to \infty} P(A_k).$$

Finally we have

$$h(Y^*) = P(A^*) \geq P(B) \geq \limsup_{k \to \infty} P(A_k) = \limsup_{k \to \infty} h(Y_k).$$

This proves the lemma.

**Theorem 1.** Suppose that $\mathfrak{U}$ is a weakly compact in $L^2[0,1]$ and the goal set $Q$ is closed in $\mathbb{R}^n$. Then the optimal problem (1)-(2) has a solution.

**Prove.** Let $\{u_l(\cdot)\}$ be a maximizing, weakly converging to $u_0(\cdot) \in \mathfrak{U}$ sequence of controls:

$$\varphi(u_l(\cdot)) \to \sup_{u(\cdot) \in \mathfrak{U}} \varphi(u(\cdot)) \leq 1.$$ 

It is easy to prove that $y_{u_l}(t_k) \to y_{u_0}(t_k), k = 1,N$. According to lemma 1 we have

$$\varphi(u_0(\cdot)) = G(y_{u_0}(t_1), \ldots, y_{u_0}(t_N)) \geq \limsup_{l \to \infty} G(y_{u_l}(t_1), \ldots, y_{u_l}(t_N)) = \lim_{l \to \infty} \varphi(u_l(\cdot)) = \sup_{u(\cdot) \in \mathfrak{U}} \varphi(u(\cdot)).$$

This proves the theorem.

Another approaches to the existence of the solutions of such kind problems were developed in \cite{1,2}.

## 5 Conditions of differentiability

Differentiability of the functional with respect to the phase coordinates plays important role in necessary and sufficient conditions of optimality. In this section we study the differentiability of functions $g, G$.

Restrict our consideration to the case of absolutely continuous with respect to Lebesque measure random vector $\xi \equiv \{\xi_1, \ldots, \xi_N\}$. Let $f$ be the

\footnote{This theorem was proved by A.Yu. Zorin}
probability density function of \( \xi, Z \equiv \{z_1^*, \ldots, z_N^*\} \) be a given set of vectors in \( \mathbb{R}^n \),

\[
O(Z) \equiv \{ r \in \mathbb{R}^N : r_1z_1^* + \ldots + r_Nz_N^* \in Q \}.
\]

If \( N > n \) then the set \( O(Z) \) is unbounded in \( \mathbb{R}^N \) even \( Q \) is bounded in \( \mathbb{R}^n \). Let \( \pi_j \) be the projection of \( O(Z) \) to the hyperplane \( r_j = 0 \). It is easily shown that for any convex \( Q \) the line passing through an arbitrary point \( p_j \in \pi_j \) in parallel to the axe \( r_j \) intersects \( O(Z) \) over interval \( r_j \in \left( r_j^{(1)}, r_j^{(2)} \right) \); the functions \( r_j = r_j^{(1,2)}(p_j) \) are bounded for a bounded \( Q \) and \( z_j^* \neq 0 \).

For a fixed \( k \in \overline{1,N} \) consider the function

\[
h_k(z_k) = g(z_1^*, \ldots, z_k^*, z_k, z_{k+1}^*, \ldots, z_N^*), \quad z_k \in \mathbb{R}^n.
\]

**Lemma 2.** Let \( j, k \in \overline{1,N} \), vector \( z_j^* \neq 0 \), the probability density \( f(r_1, \ldots, r_N) \) is differentiable with respect to \( r_j \) and the function \( \Phi_{kj}(r) \equiv \partial (r_kf) / \partial r_j \) is summable in \( \mathbb{R}^N \). Then the function \( h_k(\cdot) \) is differentiable at the point \( z_k^* \) in the direction of vector \( z_j^* \). The corresponding directional derivative

\[
\frac{\partial h_k}{\partial z_j^*} = \frac{d}{d\varepsilon} h_k(z_k + \varepsilon z_j^*) \big|_{\varepsilon=0} = - \int_{O(Z)} \Phi_{kj}(r) \, dr. \tag{9}
\]

**Prove.** We have

\[
h_k(z_k + \varepsilon z_j^*) = P \{ \xi_1z_1^* + \ldots + \xi_kz_k + \ldots + \xi_Nz_N^* \in Q \} = \int_{O(Z)} f_\eta(r) \, dr, \tag{10}
\]

where \( f_\eta \) is the distribution density of the vector

\[
\eta \equiv \begin{cases} 
\eta_i = \xi_i, & i \neq j, \\
\eta_j = \xi_j + \varepsilon \xi_k, & i = j,
\end{cases}
\]

\[
f_\eta(r) \equiv F(\varepsilon, r) = \begin{cases} 
f(r_1, \ldots, r_{j-1}, r_j - \varepsilon r_k, \ldots, r_N), & j \neq k, \\
\frac{1}{1+\varepsilon} f(r_1, \ldots, r_{j-1}, r_j, r_k, \ldots, r_N), & j = k.
\end{cases}
\]

We see that

\[
\frac{\partial F}{\partial \varepsilon}(\varepsilon, r) = \begin{cases} 
-r_k \frac{\partial f}{\partial r_j}(r_1, \ldots, r_j - \varepsilon r_k, \ldots, r_N), & j \neq k, \\
\frac{-1}{(1+\varepsilon)^2} \left[ r_j f(r_1, \ldots, \frac{r_j}{1+\varepsilon}, \ldots, r_N) \right], & j = k.
\end{cases}
\]
Consider the function
\[
\Psi (\varepsilon) \equiv \int_{O(\mathbf{Z})} \frac{\partial F}{\partial \varepsilon} (\varepsilon, \mathbf{r}) \, d\mathbf{r} = - \int_{O_{\varepsilon}(\mathbf{Z})} \Phi_{kj} (\mathbf{r}) \, d\mathbf{r},
\]
where \( O_{\varepsilon}(\mathbf{Z}) \equiv P_{\varepsilon} O(\mathbf{Z}) \), \( P_{\varepsilon} \) is a linear transformation operator in \( \mathbb{R}^N \): \( P_{\varepsilon} \equiv \|p_{il}\|, \quad p_{il} = \delta_{il} - \alpha (\varepsilon) \delta_{ij} \delta_{lk}, \)
\[
\left\{ \begin{array}{lcl}
\alpha (\varepsilon) &=& \varepsilon, \quad k \neq j, \\
\alpha (\varepsilon) &=& \varepsilon / (1 + \varepsilon), \quad k = j,
\end{array} \right.
\]
\( \delta_{ij} \) is the Kronecker delta. We have
\[
|\Psi (\varepsilon)| \leq \int_{\mathbb{R}^N} |\Phi_{kj} (\mathbf{r})| \, d\mathbf{r},
\]
so the function \( \Psi (\varepsilon) \) is integrable in a neighborhood of the point \( \varepsilon = 0 \). Let us prove that the function is continuous at this point. Let denote for \( R > 0 \)
\[
K_R \equiv \{ \mathbf{r} : |\mathbf{r}| > R \}, \quad K'_R \equiv \mathbb{R}^N \setminus K_R,
O_{\varepsilon,R}(\mathbf{Z}) \equiv O_{\varepsilon}(\mathbf{Z}) \cap K_R, \quad O'_{\varepsilon,R}(\mathbf{Z}) \equiv O_{\varepsilon}(\mathbf{Z}) \cap K'_R,
O_{0,R}(\mathbf{Z}) \equiv O(\mathbf{Z}) \cap K_R, \quad O'_{0,R}(\mathbf{Z}) \equiv O(\mathbf{Z}) \cap K'_R.
\]
As the function \( \Phi_{kj} (\mathbf{r}) \) is summable in \( \mathbb{R}^N \) then for every \( \delta > 0 \) there exist \( R > 0 \) such that
\[
\int_{E} |\Phi_{kj} (\mathbf{r})| \, d\mathbf{r} < \frac{\delta}{3}
\]
for every measurable subset \( E \subset K_R \). For \( R > 0 \) do the estimation
\[
|\Psi (\varepsilon) - \Psi (0)| = \left| \int_{O_{\varepsilon}(\mathbf{Z})} \Phi_{kj} (\mathbf{r}) \, d\mathbf{r} - \int_{O(\mathbf{Z})} \Phi_{kj} (\mathbf{r}) \, d\mathbf{r} \right| \leq
\]
\[
\leq \int_{O_{\varepsilon,R}(\mathbf{Z})} \Phi_{kj} (\mathbf{r}) \, d\mathbf{r} + \int_{O_{0,R}(\mathbf{Z})} \Phi_{kj} (\mathbf{r}) \, d\mathbf{r} + \int_{O'_{\varepsilon,R}(\mathbf{Z})} \Phi_{kj} (\mathbf{r}) \, d\mathbf{r} - \int_{O'_{0,R}(\mathbf{Z})} \Phi_{kj} (\mathbf{r}) \, d\mathbf{r}.
\]
The sum of the first two terms is less than \( 2\delta / 3 \). As the Lebesque measure of the symmetric difference between bounded sets \( O'_{\varepsilon,R}(\mathbf{Z}) \) and \( O'_{0,R}(\mathbf{Z}) \) tends
to zero as \( \varepsilon \to 0 \), then the last term is less than \( \delta/3 \) for all sufficiently small \( \varepsilon \). So we have \( |\Psi (\varepsilon) - \Psi (0)| < \delta \) for all such \( \varepsilon \). This proves the continuity of \( \Psi (\varepsilon) \) at the point \( \varepsilon = 0 \).

Listed above is sufficient for the possibility of differentiation of the integral [10] at the point \( \varepsilon = 0 \) [6, p. 132]:

\[
\frac{\partial h_k}{\partial z_j^*} = \int_{\Omega(Z)} \frac{\partial F}{\partial \varepsilon} (0, r) \ dr = - \int_{\Omega(Z)} \Phi_{kj} (r) \ dr.
\]

This proves the lemma.

For convex \( Q \) we have from (9)

\[
\frac{\partial h_k}{\partial z_j^*} = \int_{\pi_j} \left( r_k f \right) \Big|_{r_j = r_j^{(1)}(p_j)}^{r_j = r_j^{(2)}(p_j)} dp_j,
\]

where in the case of infinite values of \( r_j^{(1,2)} (p_j) \) the substitution should be settled zero. Note that using this representation of the derivative one can prove lemma 2 for more wide assumptions about \( f \) do not presuming the existence of the partial derivatives.

**Theorem 2.** Suppose that \( N \geq n \), \( k \in \overline{1, N} \), set of vectors \( \{ z_1^*, \ldots, z_n^* \} \subset \mathbb{Z} \) forms a basis in \( \mathbb{R}^n \), the density \( f \) is differentiable with respect to \( r_j \), and functions \( \Phi_{kj} (r) \equiv \partial (r_k f) / \partial r_j \) are summable in \( \mathbb{R}^N \). Then the function \( h_k (z_k) \) is differentiable at the point \( z_k^* \) and its gradient is

\[
\nabla h_k (z_k^*) = \sum_{j=1}^{n} \sum_{i=1}^{n} \frac{\partial h_k}{\partial z_j^*} \langle e^j, e^i \rangle z_i^*,
\]

(11)

where \( \{ e^1, \ldots, e^n \} \) is a dual basis for \( \{ z_1^*, \ldots, z_n^* \} \).

**Prove.** If we prove the differentiability of the function, then after decomposing of the gradient with basis \( \{ e^i \} \) and each of vectors \( e^j \) with basis \( \{ z_i^* \} \) [7, p. 229] we receive

\[
\nabla h_k = \sum_{i=1}^{n} \langle \nabla h_k, z_i^* \rangle e^i = \sum_{j=1}^{n} \frac{\partial h_k}{\partial z_j^*} \sum_{i=1}^{n} \langle e^j, e^i \rangle z_i^*,
\]

that proves formula (11).

To prove the differentiability of the function \( h_k \) it is sufficiently to check the continuity of the derivatives \( \frac{\partial h_k}{\partial z_j^*} (z_k) \) at the point \( z_k^* \). After decomposition
of an increment $\Delta z_k$ with the basis $\{z_i^*\}$, $\Delta z_k = \Delta_1 z_1^* + \ldots + \Delta_n z_n^*$, $\Delta_i \equiv \langle \Delta z_k, e_i \rangle$ we obtain by analogy with (10)

$$\frac{\partial h_k}{\partial z_j}(z_k^* + \Delta z_k) = -\int_{O_{\Delta}(Z)} \frac{\partial}{\partial r_j}(r_k f) \, dr =$$

$$= \frac{1}{1+\Delta_k} \int_{O_{\Delta}(Z)} \Phi_{kj}(r) \, dr = \Psi(\Delta_k),$$

where

$$f_{\Delta}(r) = \frac{1}{1+\Delta_k} f \left( r_1 - \frac{\Delta_1}{1+\Delta_k} r_k, \ldots, r_n - \frac{\Delta_n}{1+\Delta_k} r_k, r_{n+1}, \ldots, r_N \right),$$

$$O_{\Delta}(Z) \equiv P_{\Delta} O(z), \quad P_{\Delta} \equiv \|p_d\|,$$

$$p_d \equiv \begin{cases} \delta_{il} - \frac{\delta_{il}}{1+\Delta_k} \delta_{lk}, & i = 1, n, \\
\delta_{il}, & i = n+1, N. \end{cases}$$

We can prove that $\Psi(\Delta_k) \to \Psi(0)$ as $\Delta z_k \to 0$ similar to lemma 2; this proves the continuity of the derivatives and the theorem too.

It is clear that in the statement of the theorem one can change $\{z_i^*\}$ to any other basis $Z$ in $\mathbb{R}^n$. If dimension of the linear span $\mathcal{L}$ of $Z$ is less than $n$, then under formulated in the theorem properties of $f$ we can guarantee only differentiability of $h_k$ over subspace $\mathcal{L}$.

It is easy to construct the examples of functions $h_k(z_k)$ which are not differentiable or even not continuous at the given point $z_k^*$ (see below). The reason of these may be in absence of such properties of $Q$ as bodility, convexity, boundary smoothness.

**Example 1.**

$$n = 3, \quad N = 2, \quad Q = \left\{ x_1 = 0, (x_2 - 2)^2 + (x_3 - 2)^2 \leq 1 \right\},$$
$$z_1^* = (0, 0, 1), \quad z_2^* = (0, 1, 0).$$

**Example 2.**

$$n = 2, \quad N = 1, \quad Q = \left\{ x_1 \geq 0, x_2 \geq 0, \right\}
\left\{ 1 \leq x_1^2 + x_2^2 \leq 2 \right\}, \quad z^* = (0, 1).$$

**Example 3.**

$$n = 2, \quad N = 1, \quad Q = \left\{ (x_1 - 7)^2 + x_2^2 \leq 25, \right\}
\left\{ (x_2 - 7)^2 + x_1^2 \leq 25 \right\}, \quad z^* = (1, 1).$$
**Theorem 3.** Let function \( f \) be smooth in \( \mathbb{R}^N \), \( Q \) be a ball in \( \mathbb{R}^n \). For every set of vectors \( Z \), which does not belong to hyperplane holding the origin and touching the boundary of \( Q \), the functions \( h_k(z_k), k = 1, N \) are differentiable at points \( z_k^* \in Z \).

Using the analytical form of \( Q \), we can prove theorem 3 by direct calculations. Note that for any set of vectors belonging to tangential plane the functional equals zero; it can be maximal only for a degeneration problem (see sec. [5]). Hence the statements of the theorem are natural.

### 6 Conditions of optimality

Denote by \( Z(\mathbf{u}(\cdot)) \equiv \{z_u^{(i)}, i = 1, N\} \) the set of vectors in \( \mathbb{R}^n \), described by formulas (4) for a given control \( \mathbf{u}(\cdot) \in \mathcal{U} \). The problem (1) is called a) **degenerate** if the functional \( \varphi(\cdot) \) is constant in \( \mathcal{U} \); b) **regular for the control** \( \mathbf{u}^*(\cdot) \in \mathcal{U} \), if for any \( k = 1, N \) the functions \( h_k(z_k), \) defined for the set \( Z(\mathbf{u}^*(\cdot)) \), are differentiable at the corresponding points \( z_u^{(k)} \) (see sec. [5]).

**Lemma 3.** If \( \dot{x}_0 \in Q \), then the sufficient condition for \( u^0(\cdot) \in U \) to be optimal is that the set \( Z(\mathbf{u}^0(\cdot)) \) be trivial (i.e. consists of zero vectors only). On the contrary, if the problem is not degenerate and \( \dot{x}_0 \notin Q \), then the necessary condition for \( u^0(\cdot) \in U \) to be optimal is that the set \( Z(\mathbf{u}^0(\cdot)) \) be nontrivial.

**Prove.** If \( z_u^{(i)} = 0, i = 1, N \), then from (7) we have \( \varphi(\mathbf{u}^0(\cdot)) = P(\mathbf{x}_0 \in Q) \). The corresponding value of the probability equals \( 1 - \sup \varphi(\mathbf{u}(\cdot)) \) for \( \dot{x}_0 \in Q \) and \( 0 = \inf \varphi(\mathbf{u}(\cdot)) < \sup \varphi(\mathbf{u}(\cdot)) \) for \( \dot{x}_0 \notin Q \). This proves both statements of lemma.

**Theorem 4.** Let \( \mathbf{u}^0(\cdot) \) be optimal control, and the problem (1) is regular for the control \( \mathbf{u}^0(\cdot) \). Then for almost all \( t \in [0, 1] \)

\[
\langle \mathbf{\theta}(t), B(t) \mathbf{u}^0(t) \rangle = \max_{\mathbf{v} \in \mathbf{V}} \langle \mathbf{\theta}(t), B(t) \mathbf{v} \rangle,
\]

where the conjugate function

\[
\mathbf{\theta}(t) = \sum_{k=1}^N \chi_k(t) \mathbf{\theta}_k(t),
\]

\[
\begin{aligned}
\frac{d}{dt} \mathbf{\theta}_k &= -A'(t) \mathbf{\theta}_k \\
\mathbf{\theta}_k(1) &= \nabla h_k(z_u^{(k)}),
\end{aligned}
\]
and \( \chi_k(t) \) is the indicator of the set \( [t_{k-1}, t_k), k = 1, N \).

**Prove.** Let \( \tau \in (t_{k-1}, t_k), 0 < \varepsilon < t_k - \tau \)

\[
\begin{aligned}
u^\varepsilon(t) &= \begin{cases} v \in V, & t \in (\tau, \tau + \varepsilon), \\ u^0(t), & t \not\in (\tau, \tau + \varepsilon). \end{cases}
\end{aligned}
\]

Taking into account that for such kind of variation only vector \( z^{(k)}_{u^0} \) varies, we have

\[
\varphi (u^\varepsilon (\cdot)) - \varphi (u^0 (\cdot)) = h_k (z^{(k)}_{u^\varepsilon}) - h_k (z^{(k)}_{u^0}) =
\]

\[
= \left\langle \nabla h_k \left( z^{(k)}_{u^0} \right), \Delta \varepsilon z^{(k)} \right\rangle + o \left( |\Delta \varepsilon z^{(k)}| \right),
\]

\[
\Delta \varepsilon z^{(k)} \equiv z^{(k)}_{u^\varepsilon} - z^{(k)}_{u^0} =
\]

\[
= \Phi (1) \int_{\tau}^{\tau + \varepsilon} \Phi^{-1} (s) B (s) (v - u^0 (s)) ds.
\]

Using the Lebesgue theorem about differentiation of integrals we obtain for almost all \( \tau \in (t_{k-1}, t_k) \)

\[
\delta \varphi \equiv \lim_{\varepsilon \to 0} (\varphi (u^\varepsilon (\cdot)) - \varphi (u^0 (\cdot))) / \varepsilon =
\]

\[
= \left\langle \nabla h_k \left( z^{(k)}_{u^0} \right), \Phi (1) \Phi^{-1} (\tau) B (\tau) (v - u^0 (\tau)) \right\rangle =
\]

\[
= \left\langle \theta (\tau), B (\tau) (v - u^0 (\tau)) \right\rangle.
\]

Because the control \( u^0 (\cdot) \) is optimal, the variation \( \delta \varphi \leq 0 \); this implies (12). The theorem is proved.

**References**

[1] I.P. Smirnov, *Control of the probability of the system entry to a given region (in Russian).* Differential equations. 1990, v. 26, N 10, p. 1753-1758.

[2] I.P. Smirnov, *Necessary conditions for optimality in the problem of entrance of stochastic control process to the given region (in Russian).* Differential equations. 1990, v. 26, N 11, p. 1943-1949.

[3] I.P. Smirnov and A.Yu. Zorin, *Optimal control of the linear system with the noise at control parameter.* Proc. of the international conf. "Mathematical algorithms-1", Nizhny Novgorod, NNSU, 1995, p. 108-110.
[4] I.I. Gihman, A.V. Skorohod, *Stochastic differential equations (in Russian)*. Naukova dumka: Kiev, 1968.

[5] Jaques Neveu, *Bases mathématiques du calcul des probabilités*. Masson et cie, Paris, 1964.

[6] S.L. Sobolev, *The equations of the mathematical physics (in Russian)*. Nauka: Moscow, 1966.

[7] V.A. Il’in, A.G. Pozdn’yak, *Linear algebra (in Russian)*. Nauka: Moscow, 1984.