COMPLIMENTARY SERIES REPRESENTATIONS AND QUANTUM ORBIT METHOD

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Abstract. A version of quantum orbit method is presented for real forms of equal rank of quantum complex simple groups. A quantum moment map is constructed, based on the canonical isomorphism between a quantum Heisenberg algebra and an algebra of functions on a family of quantum $G$-spaces. For the series $A$, we construct some irreducible $*$-representations of $U_q g$ which correspond to the semi-simple dressing orbits of minimal dimension in the dual Poisson Lie group. It is shown that some complimentary series representations correspond to some quantum 'tunnel' $G$-spaces which do not have a quasi-classical analog.

1. Introduction

This article lies on the intersection of two areas of mathematics. One is based on the tradition of the orbit method, pioneered by A. Kirilov and B. Kostant, to realize the unitary irreducible representations of Lie groups geometrically in certain bundles on the orbits of the coadjoint action. And the other one is the legacy of the V. Drinfeld’s approach to quantum groups, which teaches us to look at the Lie groups as quasi-classical analogs of their quantum counterparts.

An important observation that has given rise to various attempts to develop a quantum analog of the classicalorbit method is that the coadjoint action is actually a special case of the dressing action of Poisson Lie groups. Namely, if $G$ is a Poisson Lie group with the trivial Poisson structure, the dual Poisson Lie group is isomorphic to the space $U_q g^*$, dual to its Lie algebra, which is considered as an Abelian Lie group with the Poisson structure defined in terms of the Kirillov-Kostant bracket.

Thus, one can look at the coadjoint action as a part of a more general picture. In particular, if the Poisson Lie group structure arises in the quasi-classical limit from a quantum group, we can raise a question on a geometric realization of the irreducible $*$-representations of the quantum universal enveloping algebra in a way similar to how the classical orbit method works. It is especially intriguing, for the classical orbit method doesn’t allow to realize all representations. For example, already in the case of $SU(1,1)$ the complimentary series representations cannot be realized in the classical case.

The quantum analog of the dressing action is based on the fundamental principle of the Drinfeld’s duality. Namely, the same quantum algebra, say, the quantum universal enveloping algebra, has two quasi-classical limits. One being the classical

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universal enveloping algebra $U_qU_q\mathfrak{g}$, and the other one being the appropriate algebra of functions on the dual Poisson Lie group $G^∗$. A surprising manifestation of this phenomenon is that the quantum analog of the dressing action becomes the quantum adjoint action of our quantum algebra on itself, if we assume that the copy of $U_qU_q\mathfrak{g}$ that serves as the space of representation becomes the algebra of functions on $G^∗$ in the limit, while the copy that acts on it becomes the enveloping algebra instead.

As another manifestation of this duality, we construct a family of quantum Heisenberg algebras which can be considered, at the same time, as a family of quantum algebras of functions on the generalized flag manifolds for $G$. The usual morphism from $U_qU_q\mathfrak{g}$ to such quantum Heisenberg algebra, whose quasi-classical analog accounts for the realization of the elements of $UU_q\mathfrak{g}$ in terms of the differential operators, becomes, when considered as a morphism from the quantum algebra of functions on $G^∗$ into a quantum algebra of functions on a $G$-space, a quantum analog of the corresponding moment map.

In this paper we proceed to introduce quantum analogs of polarizations on the quantum $G$-spaces that arise from our construction. And eventually, we provide a geometric realization of some series of the irreducible representations on $U_qU_q\mathfrak{g}$. Namely, for the series $A$, the representations corresponding to the minimal non-zero dimension orbits of the dressing action are realized geometrically.

2. Quantum Heisenberg Algebras

Throughout the paper we suppose that $q$ is real, $0 < |q| < 1$. Suppose that $G$ is a finite-dimensional complex simple Poisson Lie group, $\mathfrak{g}$ its Lie bialgebra, and $U_q\mathfrak{g}$ the corresponding quantized universal enveloping algebra (cf. [8]). Let $V$ be a finite-dimensional simple $U_q\mathfrak{g}$-module, and $V^*$ the dual $U_q\mathfrak{g}$-module defined by

$$\langle \xi \varphi, v \rangle = \langle \varphi, S(\xi)v \rangle,$$

where $\xi \in U_q\mathfrak{g}$, $v \in V$, $\varphi \in V^*$, and $S$ is the antipode in $U_q\mathfrak{g}$. Then, we can define a quantum Heisenberg algebra $\mathcal{H}_q(V)$ as follows.

Let $R$ be the quantum $R$-matrix acting in $(V \oplus V^*)^\otimes 2$, and $\hat{R} = PR$, where $P : (V \oplus V^*)^\otimes 2 \rightarrow (V \oplus V^*)^\otimes 2$ is the usual permutation operator $a \otimes b \mapsto b \otimes a$. As is well known, the operator $\hat{R}$ is invertible and diagonalizable, has real spectrum and commutes with the action of $U_q\mathfrak{g}$.

Consider the algebra $\mathbb{C}[V\hat{R}]_q$ which is the quotient of the tensor algebra

$$\mathcal{T}(V \oplus V^*) = \mathbb{C} \oplus (V \oplus V^*) \oplus (V \oplus V^*)^\otimes 2 \oplus \cdots$$
over the two-sided ideal \( J(W) \) generated by the span \( W \subset (V \oplus V^*)^2 \) of eigenvectors of \( \hat{R} \) with negative eigen-values.

The tensor algebra \( \mathcal{T}(V \oplus V^*) \) has a canonical \( U_q \mathfrak{g} \)-module algebra structure, which means that the canonical \( U_q \mathfrak{g} \)-module structure defined by the action of \( U_q \mathfrak{g} \) on \( V \oplus V^* \) is compatible with the algebra structure so that the multiplication map \( \mathcal{T}(V \oplus V^*) \otimes \mathcal{T}(V \oplus V^*) \to \mathcal{T}(V \oplus V^*) \) is a morphism of \( U_q \mathfrak{g} \)-modules.

Since \( \hat{R} \) commutes with the \( U_q \mathfrak{g} \)-action, the two-sided ideal \( J \) is a \( U_q \mathfrak{g} \)-submodule. This follows that \( \mathbb{C}[V_{R, \mathfrak{g}}] \) has a canonical \( U_q \mathfrak{g} \)-module algebra structure as well. Note that \( \mathbb{C}[V_{R, \mathfrak{g}}] \) can be thought of as a quantum analogue of the algebra of polynomial functions on the \( G \)-space \( V \oplus V^* \), the subalgebra generated by \( V^* \) (resp. \( V \)) playing the role of the algebra of holomorphic (resp. anti-holomorphic) polynomials.

In the classical case this algebra has a canonical central extension given by

\[
ab - ba = \langle a, b \rangle C,
\]

where \( C \) is the central element, and \( \langle , \rangle \) is the canonical bilinear form on \( V \oplus V^* \) – it is the natural pairing between \( V \) and \( V^* \) and zero on both \( V \) and \( V^* \). The result is known as the Heisenberg algebra. The quantum analogue is described below.

Consider the subspace \( I \) of \( U_q \mathfrak{g} \)-invariant elements in \( V \otimes V^* \otimes V \) (that is, the elements \( v \) such that \( \xi v = \varepsilon(\xi)v \) for any \( \xi \in U_q \mathfrak{g} \), where \( \varepsilon \) is the counit). It is obviously two-dimensional, one generator in \( V \otimes V^* \) and another one in \( V^* \otimes V \). Since \( \hat{R} \) commutes with the action of \( U_q \mathfrak{g} \), \( I \) is invariant with respect to \( \hat{R} \). Since \( \hat{R} \) permutes \( V \otimes V^* \) and \( V^* \otimes V \), it must have two distinct eigen-values in \( I \). From the other side, it is easy to see that \( \hat{R}^2 \mid_I \) must be a constant, so that the eigen-values must be of opposite sign. Thus, we have proved the following proposition.

**Proposition 2.1.** The vector space \( I_0 = I \cap W \) is one-dimensional.

Note that the \( U_q \mathfrak{g} \)-module \( W \) is completely reducible, so that there is a unique \( U_q \mathfrak{g} \)-submodule \( W_0 \subset W \) such that \( W = W_0 \oplus I_0 \). Let \( J(W_0) \subset \mathcal{T}(V \oplus V^*) \) be the two-sided ideal generated by \( W_0 \).

**Definition 2.1.** The \( U_q \mathfrak{g} \)-module algebra \( \mathcal{H}_q(V) = \mathcal{T}(V \oplus V^*) / J(W_0) \) is called the quantum Heisenberg algebra corresponding to \( \mathfrak{g} \) and \( V \). The following diagram is exact, where \( C \) is the image of a generator of \( I_0 \) and \( p \) is the quotient map:

\[
0 \to \mathbb{C}[C] \to \mathcal{H}_q(V) \xrightarrow{p} \mathbb{C}[V_{R, \mathfrak{g}}] \to 0.
\]

**Remark 1.** The above definition first appeared in my preprint [1]. Later we will observe how it is related to the quantum Weyl algebra introduced in [2].

**Example 2.1.** Suppose that \( \mathfrak{g} = \mathfrak{sl}(n + 1) \) equipped with the so-called standard Lie bialgebra structure. It is defined by the Manin triple \( (\mathfrak{g} \oplus \mathfrak{g}, \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-) \), where \( \mathfrak{g} \) is embedded into \( \mathfrak{g} \oplus \mathfrak{g} \) as the diagonal, \( \mathfrak{n}_+ \) as \( (n_+, 0) \), \( \mathfrak{n}_- \) as \( (0, n_-) \), \( \mathfrak{h} \) as \( \{(a, -a) \mid a \in \mathfrak{h}\} \). Here \( \mathfrak{h} \) is the Cartan subalgebra of diagonal matrices, and \( \mathfrak{b}_+ \) (resp. \( \mathfrak{b}_- \)) the nilpotent subalgebra of nilpotent upper- (resp. lower-) triangular matrices.
Recall that $U_q\mathfrak{sl}(n+1)$ is generated by $E_i, F_i, K_i, K_i^{-1}$, $i = 1, 2, \ldots, n$ with the relations

\[
[E_i, F_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{q - q^{-1}}, \quad K_i K_j = K_j K_i,
\]

\[
K_i E_j = q^{-a_{ij}} E_j K_i, \quad K_i F_j = q^{a_{ij}} F_j K_i,
\]

\[
E_i E_j = E_j E_i, (|i - j| > 1), \quad F_i F_j = F_j F_i, (|i - j| > 1),
\]

\[
E_i^2 E_{i+1} - (q + q^{-1}) E_i E_{i+1} E_i + E_{i+1} E_i^2 = 0,
\]

\[
F_i^2 F_{i+1} - (q + q^{-1}) F_i F_{i+1} F_i + F_{i+1} F_i^2 = 0,
\]

where $a_{ij} = 2$, $a_{i,i\pm1} = -1$, and $a_{ij} = 0$ otherwise.

The Hopf algebra structure on $U_q\mathfrak{sl}(n+1)$ is given by

\[
\Delta(K_i) = K_i \otimes K_i,
\]

\[
\Delta(E_i) = E_i \otimes 1 + K_i^{-1} \otimes E_i, \quad \Delta(F_i) = F_i \otimes K_i + 1 \otimes F_i,
\]

\[
S(E_i) = -K_i E_i, \quad S(F_i) = -F_i K_i^{-1}, \quad S(K_i) = K_i^{-1},
\]

\[
\varepsilon(E_i) = 0, \quad \varepsilon(F_i) = 0, \quad \varepsilon(K_i) = 1,
\]

where $\Delta$ is the comultiplication, $S$ the antipode, and $\varepsilon$ the counit.

Let $V$ be the finite-dimensional simple $U_q\mathfrak{sl}(n+1)$-module corresponding to the first fundamental weight $\omega_1$ (that is, the defining representation). The algebra $\mathbb{C}[V_{\mathfrak{g}}]_q$ in this case is generated by $z_0, z_1, \ldots, z_n$ and $\hat{z}_0, \hat{z}_1, \ldots, \hat{z}_n$ with the relations (cf. [3])

\[
z_i z_j = q z_j z_i, (i < j), \quad \hat{z}_i \hat{z}_j = q^{-1} \hat{z}_j \hat{z}_i, (i < j),
\]

\[
z_i \hat{z}_j = q \hat{z}_j z_i, (i \neq j), \quad z_i \hat{z}_i - \hat{z}_i z_i = (q^{-2} - 1) \sum_{k>1} z_k \hat{z}_k.
\]

Here $z_0, z_1, \ldots, z_n$ and $\hat{z}_0, \hat{z}_1, \ldots, \hat{z}_n$ are nothing but the projections of the vectors of the dual canonical bases of $V$ and $V^*$ respectively.

This means that the action of $U_q\mathfrak{sl}(n+1)$ on $\mathbb{C}[V_{\mathfrak{g}}]_q$ is given by

\[
E_i : z_j \mapsto \delta_{ij} z_{j-1}, \quad \hat{z}_j \mapsto -\delta_{i-1,j} q^{-1} \hat{z}_{j+1}, \quad (2.1)
\]

\[
F_i : z_j \mapsto \delta_{i-1,j} z_{j+1}, \quad \hat{z}_j \mapsto -\delta_{ij} q \hat{z}_{j-1}, \quad (2.2)
\]

\[
K_i : z_j \mapsto \begin{cases} q^{-1} z_{i-1} & \text{if } j = i - 1, \\ q z_i & \text{if } j = i, \\ z_j & \text{if otherwise,} \end{cases} \quad \hat{z}_j \mapsto \begin{cases} q \hat{z}_{i-1} & \text{if } j = i - 1, \\ q^{-1} \hat{z}_i & \text{if } j = i, \\ \hat{z}_j & \text{if otherwise.} \end{cases} \quad (2.3)
\]

In this case the subspace $I_0$ is spanned by

\[
\sum_{k=0}^n z_k \hat{z}_k - \sum_{k=0}^n q^{-2k} \hat{z}_k \otimes z_k.
\]

The quantum Heisenberg algebra $\mathcal{H}_{q}(V)$ is generated by $z_0, z_1, \ldots, z_n, \hat{z}_0, \hat{z}_1, \ldots, \hat{z}_n$ and $C$ with the relations (cf. [4])

\[
z_i z_j = q z_j z_i, (i < j), \quad \hat{z}_i \hat{z}_j = q^{-1} \hat{z}_j \hat{z}_i, (i < j),
\]

\[
z_i \hat{z}_j = q \hat{z}_j z_i, (i \neq j), \quad z_i \hat{z}_i - \hat{z}_i z_i = C + (q^{-2} - 1) \sum_{k>1} z_k \hat{z}_k,
\]

\[
z_i C = q^{2} C z_i, \quad \hat{z}_i C = q^{2} C \hat{z}_i.
\]

The action of $U_q\mathfrak{sl}(n+1)$ on $\mathcal{H}_{q}(V)$ is given by (2.1)-(2.3) and by

\[
\xi C = \varepsilon(\xi) C
\]
for any $\xi \in U_q\mathfrak{sl}(n+1)$ (i.e., $C$ is a $U_q\mathfrak{sl}(n+1)$-invariant element).

Now suppose that $\mathfrak{g}_c$ is a compact real form of $\mathfrak{g}$, which is unique up to an inner automorphism. Then, there is an antilinear involutive automorphism $\omega$ of $\mathfrak{g}$ such that

$$\mathfrak{g}_c = \{a \in \mathfrak{g} | \omega(a) = a\}.$$

As is well known, there is a quantization of $\mathfrak{g}_c$ in the form of a Hopf $\ast$-algebra $U_q\mathfrak{g}_c = (U_q\mathfrak{g}, \flat)$, where $\flat$ is an antilinear involutive algebra anti-automorphism and coalgebra automorphism such that

$$\omega_c : \xi \mapsto (S(\xi))^\flat$$

is an involution (thus, an antilinear involutive algebra automorphism and coalgebra anti-automorphism). Then, any finite-dimensional $U_q\mathfrak{g}$-module -- in particular, our module $V$ -- has a Hilbert space structure which makes the action of $U_q\mathfrak{g}_c$ into a $\ast$-representation. Let $\iota : V \to V^\ast$ be the antilinear isomorphism of vector spaces induced by the scalar product on $V$. The following proposition is rather obvious.

**Proposition 2.2.** The map $\iota : V \to V^\ast$ can be uniquely extended to an antilinear involutive anti-automorphism $\ast$ of $\mathcal{H}_\mathfrak{g}(V)$ such that $C^\ast = C$. Then, the $\ast$-algebra $\mathcal{H}_\mathfrak{g}(V) = (\mathcal{H}_\mathfrak{g}(V), \flat)$ is a $\mathcal{H}_\mathfrak{g}_c$-module $\ast$-algebra, which means that for any $\xi \in U_q\mathfrak{g}_c$, $f \in \mathcal{H}_\mathfrak{g}(V)$ one has

$$(\xi f)^\ast = \omega_c(\xi) f^\ast.$$  

(The definition of $\mathfrak{F}$-module $\ast$-algebra -- where $\mathfrak{F}$ is a Hopf $\ast$-algebra -- can be found in [1, 2, 3].)

Now suppose that $U_q\mathfrak{g}_0$ is a Hopf $\ast$-algebra $(U_q\mathfrak{g}, \sharp)$ which is a quantization of a non-split real form $\mathfrak{g}_0$ of $\mathfrak{g}$ equipped with a standard Lie bialgebra structure given by the Manin triple

$$(\mathfrak{g}, \mathfrak{g}_0, n_+ \oplus i\mathfrak{h}_0),$$

where $\mathfrak{h}_0 = \mathfrak{h} \cap \mathfrak{g}_0$ (it depends, of course, on the choice of the maximal nilpotent subalgebra $n_+$). Let

$$\omega_0 : \xi \mapsto (S(\xi))^\sharp$$

be the corresponding antilinear involutive algebra automorphism and coalgebra anti-automorphism on $U_q\mathfrak{g}_0$.

**Proposition 2.3.** There exists a unique antilinear involutive anti-automorphism $\ast$ of $\mathcal{H}_\mathfrak{g}(V)$ such that $\mathcal{H}_\mathfrak{g}_0(V) = (\mathcal{H}_\mathfrak{g}(V), \ast)$ is a $U_q\mathfrak{g}_0$-module $\ast$-algebra, which means that for any $\xi \in U_q\mathfrak{g}_0$, $f \in \mathcal{H}_\mathfrak{g}_0(V)$ one has

$$(\xi f)^\ast = \omega_0(\xi) f^\ast.$$  

**Proof.** The composition $\tau = \omega_0 \omega_c$ is a linear Hopf algebra automorphism of $U_q\mathfrak{g}$. Then, there exists an operator $t : V \to V$ such that

$$(\tau \xi)(f) = (t^{-1} \xi t)(f)$$

for any $\xi \in U_q\mathfrak{g}$, $f \in V$. The $U_q\mathfrak{g}_0$-module $\ast$-algebra structure on $\mathcal{H}_\mathfrak{g}_0(V)$ is given by

$$f^\ast = t(f^\flat)$$

for any $f \in V$, $C^\ast = C$.

The uniqueness is obvious.
Example 2.2. Within the framework of Example 2.1 we have that the compact real form of $U_q\mathfrak{sl}(n+1)$ is $U_q\mathfrak{su}(n+1) = (U_q\mathfrak{sl}(n+1), \hat{\mathfrak{b}})$, where $\hat{\mathfrak{b}}$ is given by

$$E_i^b = K_i^{-2} F_i, \; F_i^b = E_i K_i^2, \; K_i^b = K_i.$$

The $U_q\mathfrak{su}(n+1)$-module $*$-algebra structure on $\mathcal{H}_{\mathfrak{b}_0}(V) = (\mathcal{H}_{\mathfrak{b}}(V), \hat{\mathfrak{b}})$ is given by

$$z_i^b = \hat{z}_i, \; C^b = C.$$

Consider the real forms $U_q\mathfrak{su}(\mathfrak{c}) = (U_q\mathfrak{sl}(n+1), \mathfrak{c})$ of $U_q\mathfrak{sl}(n+1)$ parameterized by a sequence $\mathfrak{c} = (t_0, t_1, \ldots, t_n)$, where $t_i = \pm 1$. These are given by

$$E_i^c = t_{i-1} t_i E_i^b, \; F_i^c = t_{i-1} t_i F_i^b, \; K_i^c = K_i.$$

They are quantizations of different Lie bilagebra structures on $\mathfrak{g}_0 = \mathfrak{su}(m, n+1-m)$, where $m$ is the number of instances when $t_i = 1$ and $n + 1 - m$ is the number of instances when $t_i = -1$.

Then, the $U_q\mathfrak{su}(\mathfrak{c})$-module $*$-algebra structure on $\mathcal{H}_{\mathfrak{g}_0}(V) = (\mathcal{H}_\mathfrak{g}(V), \hat{\mathfrak{c}})$ is given by

$$z_i^c = t_i \hat{z}_i, \; C^c = C.$$

Remark 2. We will be particularly interested in the case when $\mathfrak{c} = (-1, 1)$. We denote $U_q\mathfrak{su}(-1, 1)$ by $U_q\mathfrak{su}(1, 1)$, $\mathcal{H}_{\mathfrak{g}_0}(V)$ by $\mathcal{H}$, $E_1$ by $E$, $F_1$ by $F$, and $K_1$ by $K$.

3. Quantum Generalized Flag Manifolds

In this section we establish the connection between the quantum Heisenberg algebras and some quantum $G$-spaces. From now on we use a shorter notation $\mathcal{H}$ for $\mathcal{H}_\mathfrak{g}(V)$.

Recall a construction described in [13]. Given a finite-dimensional simple $U_q\mathfrak{g}$-module $V = L(\Lambda)$ with the highest weight $\Lambda$, we define a multiplication on

$$\mathbb{C}[\mathcal{O}_V]_q^+ = \bigoplus_{k=0}^{\infty} L(k\Lambda)$$

as follows. Given $a \in L(k\Lambda)$ and $b \in L(m\Lambda)$, we take the projection of $a \otimes b$ on $L((k+m)\Lambda) \subset L(k\Lambda) \otimes L(m\Lambda)$ as the product of $a$ and $b$ (note that it is correctly defined, as the multiplicity of $L((k+m)\Lambda)$ in $L(k\Lambda) \otimes L(m\Lambda)$ is equal to 1). It is easy to see that it defines, in fact, a $U_q\mathfrak{g}$-module algebra structure on $\mathbb{C}[\mathcal{O}_V]_q^+$.

Define similarly a $U_q\mathfrak{g}$-module algebra by applying the same construction to

$$\mathbb{C}[\mathcal{O}_V]_q^- = \bigoplus_{k=0}^{\infty} L(k\Lambda)^*.$$

The multiplication maps

$$\mathbb{C}[V]_q^+ \otimes \mathbb{C}[V]_q^- \to \mathbb{C}[V]_q,$$

$$\mathcal{H}^+ \otimes \mathcal{H}_\mathfrak{g}(V)^0 \otimes \mathcal{H}_\mathfrak{g}(V)^- \to \mathcal{H}_\mathfrak{g}(V)$$

are isomorphisms of $U_q\mathfrak{g}$-module algebras. Here $\mathbb{C}[V]_q^+ = \mathcal{H}^+$ is the subalgebra generated by $V \subset V \oplus V^*$, $\mathbb{C}[V]_q^- = \mathcal{H}^-$ the subalgebra generated by $V^* \subset V \oplus V^*$, and $\mathcal{H}^0$ the subalgebra generated by $C$. 
Note that $\mathbb{C}[\mathcal{O}_V]^+_q$ (resp. $\mathbb{C}[\mathcal{O}_V]^-_q$) is the quantum analogue of the algebra of holomorphic (resp. anti-holomorphic) polynomial functions on the $G$-orbit of $Cv_\Lambda$ where $v_\Lambda \subset L(\Lambda)$ is a highest weight vector. At the same time, $\mathcal{H}^+ = \mathbb{C}[V]_q^+$ (resp. $\mathcal{H}^- = \mathbb{C}[V]_q^-$) is a quantum analogue of the algebra of holomorphic (resp. anti-holomorphic) polynomial functions on $V = L(\Lambda)$.

In the classical case (when $q = 1$) $\mathbb{C}[\mathcal{O}_V]_1$ is a quotient of $\mathbb{C}[V]_1$ over the ideal generated by the Plücker relations. A similar situation takes place in the quantum case. Namely, it was shown in [13] that $\mathbb{C}[\mathcal{O}_V]^+_q$ is a quotient of $\mathcal{H}^+$ over an ideal $J_+$ generated by the subspace

$$E^+_\Lambda = \left(q^Z - q^{4(\Lambda + \rho, \Lambda)}\right) (L(\Lambda) \otimes L(\Lambda))$$

of quadratic relations called (holomorphic) quantum Plücker relations. Here $q^Z$ is the canonical central element of $U_q\mathfrak{g}$ defined in [1], and $\rho$ is the half of the sum of all positive roots of $\mathfrak{g}$. Similarly, we can get that $\mathbb{C}[\mathcal{O}_V]^+_q$ is a quotient of $\mathcal{H}^-$ over an ideal $J_-$ generated by the subspace

$$E^-\Lambda = \left(q^Z - q^{4(\Lambda + \rho, \Lambda)}\right) (L(\Lambda)^* \otimes L(\Lambda)^*)$$

of what may be called anti-holomorphic quantum Plücker relations.

Define the $U_q\mathfrak{g}$-module algebra $\mathbb{C}[\mathcal{O}_V]_q$ as the quotient of $\mathcal{H}$ over the ideal $J$ generated by all the quantum Plücker relations in both $J^+$ and $J^-$. Instead of $C$ introduce a new generator

$$c = \frac{1}{q^{-1} - q}.$$  

Now, if we take the quasi-classical limit $q \to 1$, keeping $c$, not $C$ constant, we will get a commutative Poisson algebra which is the algebra of homogeneous polynomials on a family of projective Poisson $G$-spaces with a homogeneous parameter $c$. Homogeneous, because $z_i$, $\tilde{z}_i$'s and $c$ are defined up to a group of automorphisms

$$\kappa_\alpha : z_i \mapsto \alpha z_i, \quad \kappa_\alpha : \tilde{z}_i \mapsto \bar{\alpha} \tilde{z}_i, \quad \kappa_c : c \mapsto |\alpha|^2 c, \quad (3.1)$$

for any $\alpha \in \mathbb{C}$. Note that the same formulas define a group of automorphisms of the $U_q\mathfrak{g}$-module algebra $\mathbb{C}[\mathcal{O}_V]_q$ in the quantum case.

The above-mentioned projective $G$-spaces are the projectivizations of the $G$-orbits of the space $Cv_\Lambda$ of highest weight vectors. They are called generalized flag manifolds. They are of the form $G/P$, where $P$ is the parabolic subgroup of $G$ whose Lie algebra is generated by $\mathfrak{b}_+ = \mathfrak{n}_+ \oplus \mathfrak{h}$ and the root vectors $E^\pm_\alpha$ such that $(\lambda, \Lambda) = 0$. In particular, if $\Lambda = \rho$, we get flag manifolds themselves. If $\Lambda$ is a fundamental weight, we get Grassmanians. This observation justifies the following definition.

**Definition 3.1.** The $U_q\mathfrak{g}$-module algebra $\mathbb{C}[\mathcal{O}_V]_q$ is called the algebra of homogeneous polynomial functions on a family of quantum generalized flag manifolds.

Let $G_c$ be the compact real form of $G$ whose quantization yields $U_q\mathfrak{g}_c$, $G_0$ the non-compact real form whose quantization yields $U_q\mathfrak{g}_0$. It is easy to see that

$$J^+ = J, \quad J^* = J.$$

Therefore, $\mathbb{C}[\mathcal{O}_V]_q$ has canonical $U_q\mathfrak{g}_c^*$- and $U_q\mathfrak{g}_0$-module *-algebra structures. The $U_q\mathfrak{g}_c$-module *-algebra $(\mathbb{C}[\mathcal{O}_V]_q, \hat{z})$ can be thought of as the quantum algebra...
of homogeneous polynomials on a family of generalized flag manifolds of the form $G/P$ considered as Poisson $G_c$-spaces.

The $U_qg_0$-module $*$-algebra $(\mathbb{C}[O_V]^q, *)$ can be thought of as the quantum algebra of homogeneous polynomials on a family of corresponding symmetric Poisson $G_0$-spaces of non-compact type. As $G_0$-spaces they are isomorphic to the $G_0$-orbit of the image of $P \subset G$ in $G/P$ with respect to the quotient map, where $P$ is the same as above.

**Example 3.1.** Return to Examples [2.1] and [2.2]. In this case there are no Plücker relations, so that the relations in $\mathbb{C}[O_V]^q$ look as follows:

$$
\begin{align*}
  z_i z_j &= q z_j z_i, \quad (i < j), \\
  \hat{z}_i \hat{z}_j &= q^{-1} \hat{z}_j \hat{z}_i, \quad (i < j), \\
  z_i \hat{z}_j &= q \hat{z}_j z_i, \quad (i \neq j), \\
  z_i \hat{z}_i - \hat{z}_i z_i &= (q^2 - 1) \left( \sum_{k > i} z_k \hat{z}_k + q c \right), \\
  z_i c &= q^2 c z_i, \\
  \hat{z}_i c &= q^{-2} c \hat{z}_i.
\end{align*}
$$

The $U_qsu(\ell)$-module $*$-algebra $(\mathbb{C}[O_V]^q, *)$ is the quantum algebra of homogeneous polynomials on a family of quantum $\mathbb{C}P^n$, while the $U_qsu(\ell)$-module $*$-algebra $(\mathbb{C}[O_V]^q, *)$ is the quantum algebra of homogeneous polynomials on a family of quantum hyperboloids which possess a complex manifold structure (inherited from $G/P$).

**Remark 3.** When $G_c = SU(2)$, the family of quantum $\mathbb{C}P^1$ is nothing but the family of quantum Podleś 2-spheres introduced in [12].

It is always nice to have a large commutative subalgebra. Let $U_qh$ be the Hopf subalgebra of $U_qsl(n + 1)$ generated by $K_i, K_i^{-1}, i = 1, 2, \ldots, n$. Consider the subalgebra $\mathbb{C}[O_V][h] \subset \mathbb{C}[O_V]^q$ of the $U_qh$-invariant elements. Denote by $\mathbb{C}[O_V][inv] \subset \mathbb{C}[O_V]^q$ the subalgebra of $U_qh$-invariant elements.

**Proposition 3.1.** (1) The algebra $\mathbb{C}[O_V][h]$ is commutative and generated by

$$
x_i = \sum_{k \geq i} z_k \hat{z}_k + q c, \quad i = 0, \ldots, n + 1.
$$

Moreover, the following relations hold:

$$
\begin{align*}
  z_ix_j &= q^2 x_j z_i, \quad (i < j), \\
  \hat{z}_ix_j &= q^{-2} x_j \hat{z}_i, \quad (i < j), \tag{3.2} \\
  z_ix_j &= x_j z_i, \quad (i \geq j), \\
  \hat{z}_ix_j &= x_j \hat{z}_i, \quad (i \geq j). \tag{3.3}
\end{align*}
$$

(2) The algebra $\mathbb{C}[O_V][inv]$ is generated by

$$
c = q^{-1}x_{n+1} \text{ and } d = qx_0.
$$

Moreover, $d$ belongs to the center of $\mathbb{C}[O_V]$. The formulas (3.2)-(3.3) allow to extend $\mathbb{C}[O_V]^q$ by adding functions of $x = (x_0, x_1, \ldots, x_{n+1})$ so that the following relations hold:

$$
\begin{align*}
  z_i f(x) &= f(x_0, \ldots, x_i, q^2 x_{i+1}, \ldots, q^2 x_{n+1}) z_i, \\
  \hat{z}_i f(x) &= f(x_0, \ldots, x_i, q^{-2} x_{i+1}, \ldots, q^{-2} x_{n+1}) \hat{z}_i.
\end{align*}
$$

We denote the extended algebra by Func$(O_V)_q$. 


Proposition 3.2. The $U_q\mathfrak{g}$- and $U_q\mathfrak{h}$-module $\ast$-algebra structures can be uniquely extended from $\mathbb{C}[O_V]_q$ on $\widehat{\text{Func}(O_V)}_q$. The action of $U_q\mathfrak{g}$ is given by

$$
E_i : f(x) \mapsto f(x) - T_i f(x) \frac{x_i - q^2 x_i}{x_i - q^2 x_i} z_{i-1} z_i,
$$

$$
F_i : f(x) \mapsto T_i^{-1} f(x) - f(x) \frac{q^2 x_i - x_i}{q^2 x_i - x_i} z_{i-1},
$$

$$
K_i : f(x) \mapsto f(x),
$$

where

$$
T_i : f(x) \mapsto f(x_0, \ldots, x_{i-1}, q^2 x_i, x_{i+1}, \ldots, x_{n+1}),
$$

with involutions given by

$$
f(x)^\ast = f(x), \quad f(x)^\ast = f(x).
$$

Example 3.2. When $G_0 = SU(1, 1)$, one can modify our construction, so that we get more quantum $SU(1, 1)$-spaces.

Recall that $z_0, z_1, \tilde{z}_0, \tilde{z}_1$ are the homogeneous coordinates and $c$ a homogeneous parameter on a family of quantum $SU(1, 1)$-spaces, and that they are defined up to the automorphisms $\kappa_0$ given by (3.1).

Consider a subalgebra $\text{Func}(X)_c \subset \text{Func}(O_V)_q$ of $\kappa_0$-invariant elements in $\text{Func}(O_V)_q$. Denote by $\widehat{\text{Func}(X)}_q$ the subalgebra in $\text{Func}(O_V)_q$ of the elements which are invariant with respect to $\kappa_0$ with $|\alpha| = 1$.

Proposition 3.3. The $U_q\mathfrak{sl}(2)$-module algebra $\widehat{\text{Func}(X)}_q$ is generated by

$$
x = z_1 \tilde{z}_1 + qc, \quad y = z_0 \tilde{z}_1, \quad \tilde{y} = z_1 \tilde{z}_0
$$

with the relations

$$
yf(x) = f(q^2 x)y, \quad \tilde{y}f(x) = f(q^{-2} x)\tilde{y}, \quad (3.4)
$$

$$
\tilde{y}y = -(q^{-1} x - c)(q^{-1} x - d), \quad y\tilde{y} = -(qx - c)(qx - d), \quad (3.5)
$$

while $c$ and $d$ belong to the center of $\widehat{\text{Func}(X)}_q$.

It turns out that, besides the involution $\ast$ given in (3.1), there exists yet another one which makes $\text{Func}(X)_q$ into a $U_q\mathfrak{sl}(1, 1)$-module $\ast$-algebra. To keep the notation shorter, we will use the somewhat larger algebra $\widehat{\text{Func}(X)}_q$.

Proposition 3.4. There are two non-equivalent $U_q\mathfrak{sl}(1, 1)$-module $\ast$-algebra structures on $\widehat{\text{Func}(X)}_q$, one of them given by

$$
y^\ast = -\tilde{y}, \quad x^\ast = x, \quad c^\ast = c, \quad d^\ast = d, \quad (3.6)
$$

and the other one given by

$$
y^\ast = -y, \quad x^\ast = x, \quad c^\ast = d. \quad (3.7)
$$

In both cases we can define the $U_q\mathfrak{sl}(1, 1)$-module $\ast$-algebra $\text{Func}(X_{c_0, d_0})_q$ as the quotient of $\widehat{\text{Func}(X)}_q$ over the ideal generated by $c - c_0$ and $d - d_0$, where $c_0, d_0 \in \mathbb{R}$ in the first case and $c_0 \in \mathbb{C}, c_0 = d_0$ in the second case.

It is clear that if $c_0, d_0 \in \mathbb{R}$ and $c_0 \neq d_0$, $\text{Func}(X_{c_0, d_0})_q$ is a quantum algebra of functions on the two-sheet hyperboloid $|y|^2 = (x - c_0)(x - d_0)$. If $c_0 = d_0$, $\text{Func}(X_{c_0, d_0})_q$ is a quantum algebra of functions on the cone given by the same
equation. Finally, if \( c_0 = \bar{d}_0, c_0 \neq d_0 \), \( Func(X_{c_0,d_0})_q \) is a quantum algebra of functions on the corresponding one-sheet hyperboloid.

4. Quantum Moment Map

Recall the definition of the classical moment map, generalized for the case when \( G \) is a Poisson Lie group with a non-trivial Poisson structure. Consider the corresponding Lie bialgebra \( \mathfrak{g} \) and the dual Poisson Lie group \( G^* \) which is defined as the connected and simply connected Poisson Lie group with the Lie bialgebra \( \mathfrak{g}^* \). For any \( \xi \in \mathfrak{g} \), let \( \alpha_\xi \) be the left invariant differential 1-form on \( G^* \) with \( \alpha_\xi(e) = \xi \). The Poisson bivector field \( \pi_{G^*} \) on \( G \) defines a map \( \tilde{\pi}_{G^*} : \Omega^1(G^*) \rightarrow \text{Vect}(G^*) \).

A vector field \( \rho_\xi = \tilde{\pi}_{G^*}(\alpha_\xi) \) is called the left dressing vector field on \( G^* \) corresponding to \( \xi \in \mathfrak{g} \). The left dressing vector fields define a local action of \( G \) on \( G^* \) which is called the left dressing action. In some cases, for example, when \( G \) is compact, it can be extended to a global action. But in general, it need not be the case, as the example of \( G = SU(1, 1) \) already shows.

Suppose now that \( M \) is a left Poisson \( G \)-manifold, that is, \( M \) is a Poisson manifold with the action of \( G \) on \( M \) such that the corresponding map \( G \times M \rightarrow M \) is a Poisson map. Let \( \sigma_\xi \) be the vector field corresponding to the infinitesimal action of \( \xi \in \mathfrak{g} \). Keeping in mind that the local dressing action in our examples will not always be integrable to a global action, we modify slightly the usual definition of the generalized moment map (cf. [10]) in order to apply it to our examples.

Definition 4.1. Let \( M' \) be a union of symplectic leaves in \( M \) such that \( M' \) is a dense subset in \( M \). A map \( J : M' \rightarrow G^* \) is called a moment map for \( M \) if

\[
\sigma_\xi = \tilde{\pi}_M(J^*(\alpha_\xi)).
\]

We see that Definition 4.1 means that \( J : M' \rightarrow G^* \) intertwines locally the \( G \)-action on \( M \) with the dressing action of \( G \) on \( G^* \). When \( G \) is a Poisson ic group with the trivial Poisson structure, the dual Poisson Lie group \( G^* \) is isomorphic to \( \mathfrak{g}^* \) as a Poisson manifold and is Abelian as a group. The corresponding dressing action always extends to a global one which is nothing but the usual coadjoint action of \( G \) on \( \mathfrak{g}^* \). Thus, in this case for any Hamiltonian \( G \)-space \( M \), there exists a moment map onto a coadjoint orbit in \( \mathfrak{g}^* \).

On the quantum level, it would have been reasonable to look for a quantum moment map in the form \( Func(G^*)_q \rightarrow Func(M)_q \). However, the Drinfeld’s duality tells us that the quantum enveloping algebra \( U_q \mathfrak{g} \) can be thought of as a quantum algebra of functions on \( G^* \). Indeed, we will obtain a quantum moment map in the form \( U_q \mathfrak{g} \rightarrow Func(M)_q \).

As is well known, the quasi-classical analogue of the quantum adjoint action of \( U_q \mathfrak{g} \) on itself given by

\[
ad_q(a) : b \mapsto \sum_k a_k^{(1)} b S(a_k^{(2)}), \quad \text{whenever} \quad \Delta(a) = \sum_k a_k^{(1)} \otimes b_k^{(2)}
\]

is nothing but the left dressing action of \( U \mathfrak{g} \) on \( Func(G^*) \). Also, it is well known that for any Hopf algebra \( A \), the quantum adjoint action of \( A \) on itself equips \( A \) with an \( A \)-module algebra structure, or an \( A \)-module \(*\)-algebra structure if \( A \) is a Hopf \(*\)-algebra.

This inspires the following definition, just slightly different from the one given in [10] (we do not have to worry about the equality \( M' = M \)).
Definition 4.2. (1) Given a $U_q\mathfrak{g}$-module algebra $\mathcal{F}$, a homomorphism $J: U_q\mathfrak{g} \to \mathcal{F}$ is called a quantum moment map if $J$ is a morphism of $U_q\mathfrak{g}$-module algebras, with $U_q\mathfrak{g}$ acting on itself by means of the quantum adjoin action (1.2).

(2) Given a $U_q\mathfrak{g}_0$-module $*$-algebra $\mathcal{F}_0$, a $*$-homomorphism $J_0: U_q\mathfrak{g}_0 \to \mathcal{F}_0$ is called a quantum moment map if $J_0$ is a morphism of $U_q\mathfrak{g}_0$-module $*$-algebras, with $U_q\mathfrak{g}_0$ acting on itself by means of the quantum adjoin action (1.2).

We see that the quantum Heisenberg algebra $\mathcal{H}$ contains the subalgebras $\mathcal{H}^+$ and $\mathcal{H}^-$ generated by $V$ and $V^*$ respectively. Of course, both are $U_q\mathfrak{g}$-module subalgebras of $\mathcal{H}$. Consider the subalgebra $\mathcal{H}_0$ generated by $\mathcal{H}^-$ and $C$. It has a one-dimensional representation $\chi$ in $\mathbb{C}_\chi$ given by

$$\chi(V) = 0, \quad \chi(C) = 1.$$ 

Consider the corresponding induced $\mathcal{H}$-module

$$W = \text{Ind}^{\mathcal{H}}_{\mathcal{H}_0}(\mathbb{C}_\chi).$$

It is spanned by monomials of the form

$$a_{i_1}^{m_1}a_{i_2}^{m_2}...a_{i_k}^{m_k}1_\chi,$$

(4.3)

where $a_j \in V \subset \mathcal{H}^+$ and $1_\chi$ is a generator of $\mathbb{C}_\chi$. Thus, we see that $W$ is isomorphic to $\mathcal{H}^+$ as a vector space, with a $\mathbb{Z}^{\dim V}$-grading defined by (4.3). This equips $W$ with a $U_q\mathfrak{g}$-module structure so that $W$ is isomorphic to $\mathcal{H}^+$ as a $U_q\mathfrak{g}$-module.

Proposition 4.1. (1) The $\mathcal{H}$-module $W$ is simple and faithful.

(2) The subalgebra $\mathcal{H}^{inv}$ of the $U_q\mathfrak{g}$-invariant elements in $\mathcal{H}$ is commutative. Moreover, any homogeneous monomial of the form (4.3) in $W$ is an eigen-vector for the action of $\mathcal{H}^{inv}$.

The statement (1) of Proposition 4.1 shows that $\mathcal{H}$ is isomorphic to its image in $\text{End} W$. Also, there exists a basis in $\tilde{W}$ (spanned by the monomials of the form

$$f = v_0^{m_0}v_1^{m_1}...v_n^{m_n}1_\chi \in \tilde{W}$$

(4.4)

which diagonalizes the action of $\mathcal{H}^{inv}$. This allows us to extend the algebra $\mathcal{H}$ by the functions on the spectrum of $\mathcal{H}^{inv}$ in $W$. Denote the corresponding algebra by $\tilde{\mathcal{H}}$. One can show that the $U_q\mathfrak{g}$-module algebra structure can be extended from $\mathcal{H}$ to $\tilde{\mathcal{H}}$.

Obviously, $\tilde{\mathcal{H}}$ is isomorphic to $\text{End} W$ as an algebra. On the other hand, $U_q\mathfrak{g}$ acts in $W$. This induces a homomorphism $J: U_q\mathfrak{g} \to \tilde{\mathcal{H}}$. It is clear that the image of $U_q\mathfrak{g}$ lies in $\text{Func}(X)_q \subset \tilde{\mathcal{H}}$ — the subalgebra of $\kappa_q$-invariant elements in $\tilde{\mathcal{H}}$.

Theorem 4.1. (1) There exists a unique (up to a $U_q\mathfrak{g}$-module algebra automorphism of $U_q\mathfrak{g}$) homomorphism

$$J: U_q\mathfrak{g} \to \text{Func}(X)_q$$

such that the composition of $J$ with the action of $\text{Func}(X)_q \subset \tilde{\mathcal{H}}$ in $W$ coincides with the action of $U_q\mathfrak{g}$ in $W$.

(2) $J$ is a morphism of $U_q\mathfrak{g}$-module algebras, with $U_q\mathfrak{g}$ acting on itself via the quantum adjoin action. In other words, $J$ is a quantum moment map for $\text{Func}(X)_q$.

Proof. The first statement has been proved above. To show that (2) holds, note that the image of $U_q\mathfrak{g}$ in $\tilde{\mathcal{H}}$ must preserve the scalar degree $m = |m| = m_0 + m_1 + ... + m_n$ of a monomial of the form (4.3). Therefore, $U_q\mathfrak{g}$ maps into the subalgebra generated...
by the elements of the form $\varphi \hat{\psi}$, where $\varphi$ (resp. $\hat{\psi}$) belongs to the subalgebra of $\kappa_\alpha$-invariant elements in the extension $H^+$ (resp. $\hat{H}^-$) of $H^+$ (resp. $\hat{H}^-$) by $\hat{H}_J(V)^{\text{inv}}$. In particular, $U_q\mathfrak{h}$ can be shown to be mapped into the subalgebra of $\kappa_\alpha$-invariant elements in $\hat{H}^{\text{inv}}$.

Given $a \in U_q\mathfrak{g}$ with $\Delta(a) = \sum_k a_k^{(1)} \otimes a_k^{(2)}$, we get that

$$a \left( \varphi \hat{\psi} \right) = \sum_k a_k^{(1)}(\varphi) \left( S \left( a_k^{(2)} \right) \right) \hat{\psi} = \sum_k J \left( a_k^{(1)} \right) \varphi \left( J \left( S \left( a_k^{(2)} \right) \right) \right).$$

Recalling that $\hat{\psi} = \psi^{-1} f$, where $f \in \hat{H}^{\text{inv}}$, one can prove (2) after some short computations.

Note that the moment map $J$ for $\text{Func}(X)_q$ was obtained first as a homomorphism from $U_q\mathfrak{g}$ into the quantum Heisenberg algebra. Thus, we see another manifestation of the Drinfeld’s duality. In this case, the same map has two quasi-classical analogues. One of them is the homomorphism from the classical universal enveloping algebra to the Heisenberg algebra which corresponds to a realization of $U\mathfrak{g}$ by differential operators on a $G$-space. Another one is a moment map for a family of generalized flag manifolds. Let us look at a few examples.

**Example 4.1.** In the context of Example 2.1 (that is, when $\mathfrak{g} = \mathfrak{sl}(n + 1)$ and $V$ being the first fundamental representation), the map $J$ of Theorem 4.1 is given by

$$J : E_i \mapsto \frac{(q^{-1} - q)\hat{z}_i}{(x_{i-1} x_{i+1})^{\frac{\gamma}{2}}},$$

$$J : F_i \mapsto \frac{(q^{-1} - q)\hat{z}_i}{x_i},$$

$$J : K_i \mapsto \frac{x_i}{(x_{i-1} x_{i+1})^{\frac{\gamma}{2}}}.$$  \hfill (4.5) \hfill (4.6) \hfill (4.7)

Moreover, given a Hopf $*$-algebra structure $U_q\mathfrak{su}(\mathfrak{k})$ on $U_q\mathfrak{sl}(n + 1)$ and the involution (5.4) on $\text{Func}(X)_q$, we see that $J$ is in fact a morphism of $U_q\mathfrak{su}(\mathfrak{k})$-module $*$-algebras, thus defining a quantum moment map for the $U_q\mathfrak{su}(\mathfrak{k})$-module $*$-algebra $\text{Func}(X)_q$.

Note that the subalgebra $\text{Func}(X)_q$ of the $\kappa_\alpha$-invariant elements in $\hat{H}$ is isomorphic to the quantum Weyl algebra constructed in 4.2. Also, the quantum moment map $J$ is equivalent to the quantum oscillator map from $U_q\mathfrak{sl}(n + 1, \mathbb{C})$ into the quantum Weyl algebra constructed there.

**Remark 4.** Of course, given any automorphism $I$ of the $U_q\mathfrak{g}$-module algebra structure on $U_q\mathfrak{g}$ (defined by the quantum adjoint action), the map $J \circ I : U_q\mathfrak{g} \to \text{Func}(X)_q$ yields another quantum moment map. In particular, in the above example the group of such automorphisms is generated modulo the center by the automorphisms given by

$$I_i : E_j \mapsto (-1)^{\delta_{ij}} E_j, \quad I_i : F_j \mapsto F_j, \quad I_i : K_j \mapsto (-1)^{\delta_{ij}} K_j.$$

It is easy to see that the corresponding moment maps $J_i = J \circ I_i$ are given by the same formulas (4.5)-(4.7) except that we take another value of $(x_{i-1} x_{i+1})^{\gamma}$. 

Example 4.2. Consider the case $\mathfrak{g} = \mathfrak{sl}(2)$. Recall that in this case $\text{Func}(X)_q$ can be described in terms of the generators $x, y, \hat{y}$ as given in Proposition 3.3. Then we have

$$J : E \mapsto \frac{(q^{-1} - q)^{\frac{1}{2}}}{(cd)^{\frac{1}{2}}} y,$$

(4.8)

$$J : F \mapsto \frac{(q^{-1} - q)^{\frac{1}{2}}}{x} \hat{y},$$

(4.9)

$$J : K \mapsto \frac{x}{(cd)^{\frac{1}{2}}}.$$

(4.10)

Moreover, given the Hopf $*$-algebra $U_q\mathfrak{su}(1,1)$, $J$ is a morphism of $U_q\mathfrak{su}(1,1)$-module $*$-algebras for any of the involutions $*$ and $\ast$ on $\text{Func}(X)_q$ defined by (3.6) and (3.7) respectively. Suppose that we fix $c = c_0$ and $d = d_0$ so that $c_0d_0 > 0$. Then $J$ is a quantum moment map for any of the quantum hyperboloids (or a quantum cone) $X_{c_0,d_0}$ defined in the previous section. It is interesting to note that the quantum quadratic Casimir element

$$C_q = \frac{1}{2}(EF + FE) + \frac{q^{-1} + q}{2(q^{-1} - q)^2} (K - 2 + K^{-1})$$

(4.11)

is mapped to

$$J : C \mapsto \frac{1}{(q^{-1} - q)^2} \left( \frac{c_0}{d_0} + \frac{d_0}{c_0} - q^{-1} - q \right).$$

(4.12)

The quasi-classical analogue $J_0$ of $J$ imbeds a dense subset of the hyperboloid (or cone) $|y|^2 = (x - c_0)(x - d_0)$ (precisely, the one defined by $x \neq 0$) into the dual Poisson Lie group $SU(1,1)^*$. The picture is as follows. One can show that $SU(1,1)^*$ is isomorphic as a Lie group to the group of translations and dilations of a plane, or to the group of the matrices of the form $\begin{pmatrix} t & z \\ 0 & t^{-1} \end{pmatrix}$, where $t > 0$ and $z \in \mathbb{C}$. We can assume without loss of generality that $c_0d_0 = 1$. Then, the piece of the manifold $|y|^2 = (x - c_0)(x - d_0)$ with $x > 0$ maps into $|z|^2 = (t - c_0)(t - d_0)$, while the piece with $x < 0$ maps into $|z|^2 = (t + c_0)(t + d_0)$, which is nothing but the reflection $t \mapsto -t$ of $|z|^2 = (t - c_0)(t - d_0)$ with $t < 0$.

Of course, these imbeddings preserve the symplectic leaves. Indeed, for the one-sheet hyperboloids, both pieces $x > 0$ and $x < 0$ are two-dimensional symplectic leaves, while any point of the circle $x = 0$ is a zero-dimensional symplectic leaf. For a two-sheet hyperboloid with $0 < c_0 < d_0$, the whole sheet $x \geq d_0$ is a symplectic leaf, and the two-dimensional pieces $x < 0$ and $0 < x \leq c_0$ of the other sheet are symplectic leaves as well, while any point of the circle $x = 0$ is a zero-dimensional symplectic leaf. Finally, for the cone with $c_0 = d_0 = 1$, the pieces with $x < 0$, $0 < x < 1$, and $x > 1$ are symplectic leaves, as are the points on the circle $x = 0$ and the vertex of the cone – the unit element of the group.

5. Quantum Polarizations

In the previous section we constructed a quantum moment map $J : U_q\mathfrak{g} \rightarrow \text{Func}(X)_q$. If we find now a way to construct an irreducible $*$-representation $\pi$ of $\text{Func}(X)_q$, the composition $\pi \circ J \circ I$ (for some $U_q\mathfrak{g}_0$-module $*$-algebra automorphism $I$ of $U_q\mathfrak{g}_0$) will give us a $*$-representation of $U_q\mathfrak{g}_0$. It will be irreducible, because
the image of $J$ coincides with the subalgebra in $\text{Func}(X)_q$ of the elements invariant with respect to the automorphisms $\kappa_\alpha$.

Recall that the classical orbit method constructs an irreducible representation of an algebra of functions on a Hamiltonian manifold in sections of a certain linear bundle with connection (whose curvature is equal to the symplectic form) which are constant along a given polarization. Our construction in the quantum case draws the ideas from that classical picture.

We have constructed $\text{Func}(X)_q$ as the subalgebra in $\tilde{H}$ which consists of the $\kappa_\alpha$-invariant elements. Recall that $\kappa_\alpha$ is a family of automorphisms parameterized by a non-zero complex number $\alpha$ (see (3.1)). If we think of $\text{Func}(X)_q$ as an algebra of functions on a quantum space $X$, then $\tilde{H}$ can be thought of as an algebra of functions on the total space of a linear bundle over $X$.

Consider the subalgebra $\text{Hol}(X)_q^+$ (resp. $\text{Hol}(X)_q^-$) of in $\text{Func}(X)_q$ generated by $v_k v_{m}^{-1}$ (resp. $\hat{v}_k \hat{v}_{m}^{-1}$), where $v_k$ are vectors of a $U_q \mathfrak{g}$-invariant basis in $V$. We will see that in the examples it is going to play the role of the algebra of holomorphic (resp. anti-holomorphic) functions in the case of a complex polarization. The following proposition reflects the fact that we deal with a quantum analog of a $G$-invariant polarization.

**Proposition 5.1.** $\text{Hol}(X)_q^+$ is a $U_q \mathfrak{g}$-module subalgebra in $\text{Func}(X)_q$.

**Example 5.1.** Suppose that, as in Example 2.1, $\mathfrak{g} = \mathfrak{sl}(n + 1, \mathbb{C})$ (equipped with the standard Lie bialgebra structure), and $V$ is the highest weight $U_q \mathfrak{g}$-module with the highest weight $\omega_1$ (the first fundamental weight). We keep the same notation as before.

Then $\text{Hol}(X)_q^+$ is generated by

$$ \zeta_i = z_i^{-1} z_{i-1}, \quad (i = 1, 2, ..., n) $$

with the relations

$$ \zeta_i \zeta_j = q^{\pm 1} \zeta_j \zeta_i, \quad \text{if } j = i \pm 1, \quad (5.1) $$

$$ \zeta_i \zeta_{i+1} = \zeta_{i+1} \zeta_i, \quad \text{otherwise.} \quad (5.2) $$

Respectively, $\text{Hol}(X)_q^-$ is generated by

$$ \hat{\zeta}_i = \hat{z}_{i-1} \hat{z}_i^{-1}, \quad (i = 1, 2, ..., n) $$

with the relations

$$ \hat{\zeta}_i \hat{\zeta}_j = q^{\mp 1} \hat{\zeta}_j \hat{\zeta}_i, \quad \text{if } j = i \pm 1, \quad (5.3) $$

$$ \hat{\zeta}_i \hat{\zeta}_{i+1} = \hat{\zeta}_{i+1} \hat{\zeta}_i, \quad \text{otherwise.} \quad (5.4) $$

It is easy to write down explicit formulas for the action of $U_q \mathfrak{sl}(n + 1, \mathbb{C})$ in $\text{Hol}(X)_q^\pm$, but we will do it only in the special case of $n = 1$ (see below).

**Example 5.2.** When $n = 1$, $\text{Hol}(X)_q^+$ is generated by a single element

$$ \zeta = z_1^{-1} z_0 = (qx - c_0)^{-1} y, \quad (5.5) $$

while $\text{Hol}(X)_q^-$ is generated by

$$ \hat{\zeta} = \hat{z}_0 \hat{z}_1^{-1} = \hat{y} (qx - c_0)^{-1}. $$
The action of $U_q\mathfrak{sl}(2, \mathbb{C})$ in $\mathcal{H}ol(X)_q^+$ is given by

$$
E: f(\zeta) \mapsto -q\zeta^2 \frac{f(\zeta) - f(\zeta q^2)}{\zeta - \zeta q^2},
$$

$$
F: f(\zeta) \mapsto \frac{f(\zeta q^2) - f(\zeta)}{\zeta q^2 - \zeta},
$$

$$
K: f(\zeta) \mapsto f(\zeta q^{-2}).
$$

The action of $U_q\mathfrak{sl}(2, \mathbb{C})$ in $\mathcal{H}ol(X)_q^-$ is given by similar formulas. One can check that the center of $U_q\mathfrak{sl}(2, \mathbb{C})$ acts trivially in $\mathcal{H}ol(X)_q^\pm$.

**Proposition 5.2.** The $U_q\mathfrak{sl}(n+1, \mathbb{C})$-module algebra $\text{Func}(X)_q$ is generated by $\zeta_i$, $\hat{\zeta}_i$ ($i = 1, 2, \ldots, n$), and the functions $f(x_1, x_2, \ldots, x_n)$ with the relations \((5.1)\)-\((5.4)\) and

$$
\zeta_i f(x_1, \ldots, x_n) = f(x_1, \ldots, x_{i-1}, q^2 x_i, x_{i+1}, \ldots, x_n) \zeta_i, \quad (5.6)
$$

$$
\hat{\zeta}_i f(x_1, \ldots, x_n) = f(x_1, \ldots, x_{i-1}, q^{-2} x_i, x_{i+1}, \ldots, x_n) \hat{\zeta}_i, \quad (5.7)
$$

$$
\zeta_i \hat{\zeta}_i = \frac{x_{i-1} - x_i}{x_i - q^2 x_{i+1}}, \quad (5.8)
$$

$$
\hat{\zeta}_i \zeta_i = \frac{q^2 x_{i-1} - x_i}{x_i - x_{i+1}}. \quad (5.9)
$$

Recall that given a real form $U_q\mathfrak{su}(\ell)$ of $U_q\mathfrak{sl}(n+1, \mathbb{C})$, the involution \((2.4)\) equips $\text{Func}(X)_q$ with a $U_q\mathfrak{su}(\ell)$-module $\ast$-algebra structure. The involution \((2.4)\) is given in the above generators by

$$
\zeta_i^\ast = t_i \zeta_i, \quad x_i^\ast = x_i.
$$

In the next section we will consider first the simplest case of $U_q\mathfrak{su}(1, 1)$ to illustrate the basic ideas. We will use them later to construct some irreducible $\ast$-representations of $U_q\mathfrak{su}(\ell)$ which correspond to the dressing orbits of the minimal dimension. Let us describe, therefore, the relations in that special case more explicitly.

**Corollary 5.1.**

1. For $c_0 d_0 = 1$, $c_0, d_0 \in \mathbb{R}$, the $U_q\mathfrak{su}(1, 1)$-module $\ast$-algebra $\text{Func}(X_{c_0, d_0})_q$ is generated by $\zeta$, $\zeta^\ast$, and the functions $f(x)$ with the relations

$$
\zeta f(x) = f(q^2 x) \zeta, \quad \zeta^\ast f(x) = f(q^{-2} x) \zeta^\ast, \quad (5.10)
$$

$$
\zeta \zeta^\ast = \frac{x - q^{-1} d_0}{x - q^{-1} c_0}, \quad \zeta^\ast \zeta = \frac{x - q d_0}{x - q c_0}. \quad (5.11)
$$

In particular, the following relation holds:

$$
\Phi(\zeta \zeta^\ast) = q^2 \Phi(\zeta^\ast \zeta),
$$

where

$$
\Phi(t) = \frac{1 - \gamma t}{1 - t}, \quad \gamma = \frac{c_0}{d_0}. \quad (5.12)
$$
(2) For $c_0d_0 = 1$, $c_0, d_0 \notin \mathbb{R}$, the $U_q\mathfrak{su}(1,1)$-module $\ast$-algebra $\text{Func}(X_{c_0,d_0})_q$ is generated by $\zeta$, $\dot{\zeta}$, and the functions $f(x)$ with the relations

$$\zeta f(x) = f(q^2 x) \zeta, \quad \dot{\zeta} f(x) = f(q^{-2} x) \dot{\zeta}, \quad (5.13)$$

$$\zeta \dot{\zeta} = \frac{x - q^{-1}d_0}{x - q^{-1}c_0}, \quad \dot{\zeta} \zeta = \frac{x - q d_0}{x - q c_0}, \quad (5.14)$$

$$\zeta \zeta^* = \zeta^* \zeta = 1, \quad \dot{\zeta} \dot{\zeta}^* = \dot{\zeta}^* \dot{\zeta} = 1. \quad (5.15)$$

In particular, the following relation holds:

$$\Phi\left(\zeta \dot{\zeta}\right) = q^2 \Phi\left(\dot{\zeta} \zeta\right),$$

where $\Phi(t)$ is given by (5.12).

The algebras $\text{Hol}(X)^{\pm}_{q\hat{\nu}}$ play the role of the algebras of functions which are constant along a polarization. In particular, in the cases of $n > 1$ and of $c_0, d_0 \in \mathbb{R}$ ($n = 1$) the corresponding polarization is complex, so that they are quantum analogs of the algebras of holomorphic and anti-holomorphic functions. For the quantum one-sheet hyperboloids $c_0, d_0 \notin \mathbb{R}$ ($n = 1$), however, the polarization is real, and $\text{Hol}(X_{c_0,d_0})^\pm_{q\hat{\nu}}$ are quantum analogs of the algebras of functions which are constant along the two families of straight lines on the corresponding one-sheet hyperboloids.

6. IRREDUCIBLE $\ast$-REPRESENTATIONS OF $U_q\mathfrak{su}(1,1)$

In this section we will construct irreducible $\ast$-representations of $U_q\mathfrak{su}(1,1)$ associated with the quantum spaces $X_{c_0,d_0}$. We assume that $c_0d_0 = 1$.

As follows from (5.11) and (5.14), $\text{Func}(X_{c_0,d_0})_q$ is generated as algebra by $\zeta$, $\zeta^{-1}$, and the functions $f(x)$ (resp. by $\zeta^*$, $(\zeta^*)^{-1}$, and the functions $f(x)$, or by $\zeta$, $\zeta^{-1}$, and $f(x)$), since we see that, for instance, $\zeta^* = \frac{x - q d_0}{x - q c_0} \zeta^{-1}$. In particular, we see that $\text{Func}(X_{c_0,d_0})_q$ is generated by two distinguished subalgebras – $\text{Hol}(X_{c_0,d_0})^\pm_{q\hat{\nu}}$ and $\text{Func}(X_{c_0,d_0})^{\ast\nu}_{q\hat{\nu}}$.

In the classical case the orbit method realizes representations in sections of a linear bundle with connection whose curvature coincides with the symplectic form on the corresponding coadjoint orbit. This is a general fact that, given a symplectic form, if there exist any linear bundles with connection and that form as its curvature, they are parameterized by the local systems on the manifold. Below we give a construction of representations of $U_q\mathfrak{su}(1,1)$. It will be shown in the section about the quasi-classical analogs that the character of the commutative subalgebra $\text{Func}(X_{c_0,d_0})^{\ast\nu}_{q\hat{\nu}}$ – the one generated by the functions $f(x)$ – plays the role of a local system on the corresponding symplectic leaf.

Consider a $\ast$-homomorphism $\nu : \text{Func}(X_{c_0,d_0})^{\ast\nu}_{q\hat{\nu}} \to \mathbb{C}$ of the form

$$\nu : f(x) \mapsto f(\nu_0), \quad \text{where} \quad \nu_0 \in \mathbb{R} \setminus \{0\}.$$

It defines a one-dimensional $\text{Func}(X_{c_0,d_0})^{\ast\nu}_{q\hat{\nu}}$-module $\mathbb{C}_\nu$. Consider the induced $\text{Func}(X_{c_0,d_0})_q$-module

$$\Pi_\nu = \text{Ind}_{\text{Func}}^{\text{Func}(X_{c_0,d_0})_q} \mathbb{C}_\nu,$$

where we use a short notation $\text{Ind} = \text{Func}(X_{c_0,d_0})_q$.
Proposition 6.1. The $\text{Func}(X_{c_0,d_0})_q$-module $\Pi_\nu$ is isomorphic to $H\text{ol}(X_{c_0,d_0})^\pm_q$ as a $H\text{ol}(X_{c_0,d_0})^\pm_q$-module (with respect to the left multiplications). Moreover, the action of $\text{Func}(X_{c_0,d_0})_q$ in $\Pi_\nu$ is given by

$$
\zeta : f(\zeta)1_\nu \mapsto \zeta f(\zeta)1_\nu, \quad (6.1)
$$
$$
x : f(\zeta)1_\nu \mapsto \nu_0 f(\zeta q^{-2})1_\nu, \quad (6.2)
$$
$$
y : f(\zeta)1_\nu \mapsto \zeta^2 q^{-1} \nu_0 f(\zeta q^{-2}) - c_0 f(\zeta)1_\nu, \quad (6.3)
$$
$$
y\hat{y} : f(\zeta)1_\nu \mapsto -q\nu_0 f(\zeta q^{-2}) - d_0 f(\zeta)1_\nu, \quad (6.4)
$$

when $\Pi_\nu$ is realized as the span of monomials of the form $\zeta^k1_\nu$, where $1_\nu$ is a generator of $C_\nu$. Similar formulas hold in the case if we realize $\Pi_\nu$ as the span of monomials of the form $(\zeta^*)^k1_\nu$ or $\zeta^k1_\nu$.

As we see from (2.2), the set of eigen-values of the action of $x$ in $\Pi_\nu$ is a part of the geometric progression

$$
\mathcal{M}_{\nu_0} = \{\nu_0 q^{2k}\}_{k\in\mathbb{Z}}.
$$

Definition 6.1. Suppose that $F$ is a $*$-algebra.

1. We call a $F$-module $\Pi$ unitarizable if there exists a positive definite Hermitian scalar product $(\ ,\ )$ in $\Pi$ such that

$$
(a\nu_1,\nu_2) = (\nu_1,a^*\nu_2)
$$

for any $a \in F$ and $\nu_1,\nu_2 \in \Pi$.

2. Suppose that $\Pi$ is a unitarizable $F$-module. Consider the Hilbert space $H$ which is the completion of $\Pi$. We say that the action of $F$ in $\Pi$ defines a $*$-representation $\pi$ of $F$ in $H$ if the action of any element $a \in F$ in $\Pi$ can be extended to a closed operator $\pi(a)$ in $H$.

Theorem 6.1. (1) Let $0 < c_0 \leq d_0$ (the case of the quantum two-sheet hyperboloids and the quantum cone). Suppose that neither $x_2 = q c_0$ nor $x_0 = q^{-1}d_0$ belongs to $\mathcal{M}_{\nu_0}$. Then there exists a scalar product $(\ ,\ )$ in $\Pi_\nu$ making it into a simple unitarizable $\text{Func}(X_{c_0,d_0})_q$-module if and only if no point of $\mathcal{M}_{\nu_0}$ lies in the interval $(qc_0, qd_0)$. The corresponding scalar product in $\Pi_\nu$ can be given by

$$
(f(\zeta)1_\nu, g(\zeta)1_\nu) = \nu \ (g(\zeta^*)f(\zeta)),
$$

where $\nu$ is extended to $\text{Func}(X_{c_0,d_0})_q$ by $\nu(\zeta) = \nu (\zeta^*) = 0$. Moreover, the action of $\text{Func}(X_{c_0,d_0})_q$ in $\Pi_\nu$ defines an irreducible $*$-representation of $\text{Func}(X_{c_0,d_0})_q$. The spectrum of the action of $x$ in the corresponding Hilbert space is equal to $\mathcal{M}_{\nu_0} \cup \{0\}$.

(2) Let $c_0 = d_0 \not\in \mathbb{R}$ (the case of the quantum one-sheet hyperboloids). Then there exists a scalar product $(\ ,\ )$ in $\Pi_\nu$ making it into a unitarizable $\text{Func}(X_{c_0,d_0})_q$-module. It can be given by (6.3). Moreover, the action of $\text{Func}(X_{c_0,d_0})_q$ in $\Pi_\nu$ defines an irreducible $*$-representation of $\text{Func}(X_{c_0,d_0})_q$. The spectrum of the action of $x$ in the corresponding Hilbert space is equal to $\mathcal{M}_{\nu_0} \cup \{0\}$.

(3) Let $0 < c_0 \leq d_0$. Suppose that $qc_0 \in \mathcal{M}_{\nu_0}$ (resp. $q^{-1}d_0 \in \mathcal{M}_{\nu_0}$). Then there exists a scalar product $(\ ,\ )$ in $\Pi_\nu$ making it into a unitarizable $\text{Func}(X_{c_0,d_0})_q$-module. It can be given by (6.5). Moreover, the action of $\text{Func}(X_{c_0,d_0})_q$ in $\Pi_\nu$
defines an irreducible \(\ast\)-representation of \(\text{Func}(X_{c_0,d_0})_q\). The spectrum of the action of \(x\) in the corresponding Hilbert space is equal to

\[
\mathcal{M}_+ = \{ c_0 q^{2k+1} \}_{k=0}^{\infty} \cup \{ 0 \} \quad \text{(resp. } \mathcal{M}_- = \{ d_0 q^{-2k-1} \}_{k=0}^{\infty} \).
\]

**Proof.** It is easy to see that any monomial of the form \(\zeta^k \mathbf{1}_\nu \in \Pi_\nu\) is an eigen-vector for the action of \(x\) with the eigen-value \(\nu_0 q^{-2k}\). At the same time (6.1)-(6.4) show that the set of eigen-values of \(x\), being a part of the geometric progression \(\mathcal{M}_\nu\), would truncate only if either \(qc_0\) or \(q^{-1}d_0\) belong to \(\mathcal{M}_\nu\). This follows also from (5.5). Therefore, we need to show only the unitarizability of \(\Pi_\nu\). We need to remind some definitions.

**Definition 6.2.** Suppose that \(A\) is a Hopf \(\ast\)-algebra, \(F\) an \(A\)-module \(\ast\)-algebra. A linear functional \(f \mapsto \int f d\mu\) defined on a linear subset \(F_0\) of \(F\) is called an invariant integral on \(F\) if the following properties are satisfied:

\[
\int a f d\mu = \varepsilon(a) \int f d\mu, \quad \int f^* d\mu = \int f d\mu, \quad f \mapsto \int f^* f d\mu \text{ is a positive definite form on } F_0,
\]

for any \(a \in A\) and \(f \in F_0\), where \(\varepsilon\) is the counit in \(A\).

The following lemma is well known.

**Lemma 6.1.** Suppose that \(A\) is a Hopf \(\ast\)-algebra, \(F\) an \(A\)-module \(\ast\)-algebra with an invariant integral \(\int f d\mu : F_0 \rightarrow \mathbb{C}\). Consider the space \(L^2(F,d\mu)\) consisting of all \(f \in F\) such that \(\int f^* f d\mu < \infty\). Then \(L^2(F,d\mu)\) is a Hilbert space with the scalar product given by

\[
(f,g) = \int g^* f d\mu, \quad (6.6)
\]

and the action of \(F\) in \(L^2(F,d\mu)\) by left multiplication defines a \(\ast\)-representation of \(F\).

**Proposition 6.2.** Under the assumptions of Theorem 6.1 (1)-(2), the linear functional

\[
\int \zeta^k f(x) d\mu = \delta_{k,0} (q^{-1} - q) \sum_{x \in \mathcal{M}_{v_0}} x f(x) \quad (6.7)
\]

is an invariant integral on \(\text{Func}(X_{c_0,d_0})_q\). Similarly, under the assumptions of Theorem 6.1 (3), the linear functional

\[
\int \zeta^k f(x) d\mu = \delta_{k,0} (q^{-1} - q) \sum_{x \in \mathcal{M}_\pm} x f(x) \quad (6.8)
\]

is an invariant integral on \(\text{Func}(X_{c_0,d_0})_q\).

**Proof.** This follows from the fact that \(J(q^\rho) = J(K) = x\), where \(\rho \in U_q g\) is the half of the sum of all positive roots. But it can also be checked by a straightforward computation.

Now we can prove Theorem 6.1. We realize \(\Pi_\nu\) as a subspace in \(\text{Func}(X_{c_0,d_0})_q\) by mapping \(\mathbf{1}_\nu\) into the function \(\delta_{v_0}\) which takes the value 1 at \(v_0 \in \mathcal{M}_{v_0}\) and 0 at any other point of \(\mathcal{M}_{v_0}\). It is easy to see that this map intertwines the action of
Theorem 6.2. Given an irreducible \( \pi \)-representation \( \pi \circ J \) of \( \text{Func}(X_{c_0,d_0})_q \), \( \pi \circ J \) is an irreducible \( \pi \)-representation of \( U_q\mathfrak{su}(1,1) \).

(1) The irreducible \( \pi \)-representations described in Theorem 6.1 (1) give rise to the complimentary series representations if \( \nu_0 > 0 \) and the strange series representations if \( \nu_0 < 0 \).

(2) The irreducible \( \pi \)-representations described in Theorem 6.1 (2) give rise to the principal continuous series representations.

(3) The irreducible \( \pi \)-representations described in Theorem 6.1 (3) give rise to the holomorphic discrete series representations if they correspond to \( \mathfrak{M}_+ \) and the anti-holomorphic discrete series representations if they correspond to \( \mathfrak{M}_- \).

Proof. The theorem follows immediately from (4.11) if we assume \( c_0 = q^{1/2} \) and \( d_0 = q^{-1/2} \) and from (4.10), since any eigen-value of \( K \) will be an eigen-value of \( x \).

Note that the principal continuous series representations correspond to the symplectic leaves that are the halves \( (x > 0 \) and \( x < 0 \) of the one-sheet hyperboloids \( X_{c_0,d_0} \) as described in Example 4.2. The holomorphic discrete series representations and the strange series representations correspond to the halves \( x < 0 \) and \( 0 < x \leq c_0 \) of a sheet of the corresponding two-sheet hyperboloids \( |y|^2 = (x - c_0)(x - d_0) \), while the anti-holomorphic series representations correspond to the other sheet of the two-sheet hyperboloid.

What is especially interesting is that the complimentary series representations correspond to the case when a geometric progression \( \mathfrak{M}_{\nu_0} \) can jump over the narrow interval \((qc_0, qd_0)\). It looks as if in the quantum case the invariant measure can be extended from one sheet of a quantum two-sheet hyperboloid onto another.
one, thus making it 'connected'. We will call such quantum hyperboloids *quantum tunnel hyperboloids*. This effect disappears in the classical limit. In particular, it is interesting to compare it with the fact that the classical orbit method fails to realize the complimentary series representations.

As we pass to the quasi-classical limit, we see that the choice of a geometric progression reflects the value of the parity \( \epsilon \) of the corresponding irreducible \(*\)-representation of \( U_q \mathfrak{su}(1,1) \). Therefore, we can think of the choice of \( \mathbb{C}_\nu \) as the choice of a local system on the corresponding symplectic leaf in \( SU(1,1)^* \). However, with the 'tunnel effect' in mind, we see that certain choices of \( \mathbb{C}_\nu \) may not have any local system as their quasi-classical analogs.

On the other hand, the observed correspondence between the symplectic leaves in \( SU(1,1)^* \) and the representations of \( U_q \mathfrak{su}(1,1) \) depends on \( q \), as \( q^{2l+1} \).

As we keep the spin \( l \) of the representation fixed and take the limit \( q \to 1 \), the corresponding symplectic leaves face two options. Those which give rise to the strange series representations will go to infinity (thus, there are no strange series representations in the classical case). The other ones will tend to the nilpotent cone \( |y|^2 = (x-1)^2 \). If we consider them as points in the corresponding orbifold, we can look at the rate with which the corresponding curve in the orbifold tends to the cone. It will be an orbit of the coadjoint action in \( \mathfrak{su}(1,1)^* \).

This gives us the usual correspondence between the representations and coadjoint orbits described by the classical orbit method. Except that the 'tunnel effect' will disappear, and so will the above geometric realization of the complimentary series representations.

### 7. Degenerate Series Of Irreducible \(*\)-Representations

Consider the quantum moment map \( J : U_q \mathfrak{g}_0 \to \text{Func}(X)_q \). If we find now a way to construct an irreducible \(*\)-representation \( \pi \) of \( \text{Func}(X)_q \), the composition \( \pi \circ J \circ I \) (for some \( U_q \mathfrak{g}_0 \)-module \(*\)-algebra automorphism \( I \) of \( U_q \mathfrak{g}_0 \)) will give us a \(*\)-representation of \( U_q \mathfrak{g}_0 \). It will be irreducible, because the image of \( J \) coincides with the subalgebra in \( \text{Func}(X)_q \) of the elements invariant with respect to the automorphisms \( \kappa_\alpha \).

Throughout the section we consider the case \( U_q \mathfrak{g}_0 = U_q \mathfrak{su}(\ell) \). We proceed in a similar way as we did when \( U_q \mathfrak{g}_0 = U_q \mathfrak{su}(1,1) \). Namely, consider a one-dimensional \(*\)-representation \( \chi : \text{Func}(X)_q^{\text{inv}} \to \mathbb{C}_\chi \) of \( \text{Func}(X)_q^{\text{inv}} \). Recall that \( \text{Func}(X)_q^{\text{inv}} \) is commutative and generated by \( x_0 = q^{-1} d, x_1, ..., x_n, x_{n+1} = q c \). Since \( x_i^* = x_i \), we can think of \( \chi \) as a triple \( (c_0, d_0, \hat{\chi}) \), where \( c_0 = \chi(e), d_0 = \chi(d) \), and \( \hat{\chi} = (\chi(x_1), ..., \chi(x_n)) \) the point in \( \mathbb{R}^n \).

Since \( x_1, ..., x_n \) are invertible in \( \text{Func}(X)_q \), \( \chi(x_i) \neq 0 \) for any \( i = 1, ..., n \). If \( \chi(x_i) < 0 \), we can choose the \( U_q \mathfrak{su}(\ell) \)-module \(*\)-algebra automorphism \( I \) so that, after replacing \( \chi \) by \( \chi \circ I \), it becomes positive (see Remark 4). Thus, without losing generality, we may assume that \( \chi(x_i) > 0 \) for any \( i = 1, ..., n \).

Consider an isomorphism of vector spaces \( \mathbb{R}^n \to \mathfrak{h}^* \) such that

\[
(\lambda_1, ..., \lambda_n) \mapsto \lambda_1 \alpha_1 + ... + \lambda_n \alpha_n,
\]

where \( \alpha_1, ..., \alpha_n \) are the positive simple roots. Thus, we can think of \( \hat{\chi} \) as a point in \( q^\mathfrak{h}^* \), so that \( \hat{\chi} = q^\alpha \), where \( \alpha \in \mathfrak{h}^* \).
Proposition 7.1. The induced $\text{Func}(X)_{q}$-module

$$W = \text{Ind}_{\text{Func}(X)_{q}^{\text{inv}}}^{\text{Func}(X)_{q}} \mathbb{C}_{X}.$$ 

is isomorphic to $\text{Hol}(X)_{q}^{\pm}$ as a $\text{Hol}(X)_{q}^{\pm}$-module.

The following proposition follows immediately from Proposition 7.1 and (5.6)-(5.7).

Proposition 7.2. $W$ is spanned by the common eigen-vectors of $x_{1},...,x_{n}$, the set $Q$ of the corresponding eigen-values forming a part of the lattice $q^{\rho+\alpha} \subset q^{\mathfrak{h}^{*}}$, where $P \subset \mathfrak{h}^{*}$ is the weight lattice.

Let $v \in W$ be such an eigen-vector, with the eigen-value $(\lambda_{1},...\lambda_{n})$. By (5.8)-(5.9), it follows that

$$\zeta_{i} \zeta_{i}^{*} : v \mapsto \iota_{i} \frac{\lambda_{i-1} - \lambda_{i}}{\lambda_{i} - q^{-2} \lambda_{i+1}} v,$$

(7.1)

$$\zeta_{i}^{*} \zeta_{i} : v \mapsto \iota_{i} q^{2} \lambda_{i-1} - \lambda_{i} \frac{\lambda_{i} - \lambda_{i+1}}{\lambda_{i} - \lambda_{i+1}} v.$$

(7.2)

Proposition 7.3. The linear functional

$$\nu_{W}(f) = \text{tr}_{W} \left( f(J \circ I) (q^{\rho}) \right),$$

(7.3)

where $\rho$ is half the sum of all positive roots, is an invariant integral on $\text{Func}(X)_{q}$.

Proof. This easily follows from the fact that $J \circ I$ is a quantum moment map and, thus, intertwines the $U_{q}\mathfrak{su}(\iota)$-action on $\text{Func}(X)_{q}$ with its quantum adjoint action on itself, and from the well-known properties of the distinguished element $\rho$. \qed

Theorem 7.1. Let $\pi$ be the representation of $\text{Func}(X)_{q}$ in $W$, and a $U_{q}\mathfrak{su}(\iota)$-module $\ast$-algebra automorphism $I$ of $U_{q}\mathfrak{su}(\iota)$ is chosen so that $\chi(x_{i}) > 0$ for any $i = 1,...,n$. Then $\pi \circ J \circ I$ is an irreducible $\ast$-representation of $U_{q}\mathfrak{su}(\iota)$ if and only if

$$\iota_{i} \frac{\lambda_{i-1} - \lambda_{i}}{\lambda_{i} - q^{-2} \lambda_{i+1}} > 0,$$

(7.4)

$$\iota_{i} q^{2} \lambda_{i-1} - \lambda_{i} \frac{\lambda_{i} - \lambda_{i+1}}{\lambda_{i} - \lambda_{i+1}} > 0$$

(7.5)

for any $i = 1,...,n$ and $(\lambda_{1},...,\lambda_{n}) \in Q$.

Proof. It is clear from (7.1)-(7.2) that if the conditions (7.4)-(7.5) are not satisfied, there is no scalar product on $W$ that can make it into a unitarizable $U_{q}\mathfrak{su}(\iota)$-module.

Suppose that the conditions (7.4)-(7.5) are satisfied. By Proposition 7.1, we can identify $W$ with a subspace in $\text{Func}(X)_{q}$ generated over $\text{Hol}(X)_{q}$ by a function $\delta_{\chi} \in \text{Func}(X)_{q}^{\text{inv}}$ that takes the value 1 at $\hat{\chi}$ and 0 at any other point in $\mathbb{Q}$. The invariant integral (7.3) defines a scalar product on $W$ by $(f,g) = \nu_{W}(g^{\ast} f)$. It is clear that it defines an irreducible $\ast$-representation of $\text{Func}(X)_{q}$ and, hence, of $U_{q}\mathfrak{su}(\iota)$, in $W$. \qed
Example 7.1. Let $U_qg_0 = U_qsu(2, 1)$, so that $\iota_1 = -1$ and $\iota_2 = 1$. Then the conditions (7.4)-(7.3) imply that either

$$\lambda_1 \geq q^{-1}d_0, \lambda_1 \geq q^{-2}\lambda_2 \geq qc_0$$

or

$$\lambda_1 \leq q^{-1}d_0, \lambda_1 \leq q^{-2}\lambda_2 \leq qc_0.$$  

(7.6)

(7.7)

It is clear that the lattice $\{(\lambda_1q^k, \lambda_2q^m)\}_{k,m=-\infty}^{\infty}$ must be truncated in both horizontal and vertical directions. It is possible only if either $\zeta_1$ or $\zeta_1^*$ annihilate a common eigen-vector of $x_1$ and $x_2$. It means that its eigen-value $(\mu_1, \mu_2)$ must belong to the boundary of the region described by either (7.6) or (7.7). Then, if $(\mu_1q^2, \mu_2)$ is not in the union of both regions, $\zeta_1$ annihilates the vector, if $(\mu_1q^2, \mu_2)$ is not in the union of the two regions, $\zeta_1^*$ does, and similarly for $\zeta_2$ with $\zeta_2^*$.

It is straightforward to check that only in three cases we get irreducible representations of $U_qg(2, 1)$. Namely,

1. when $d_0 \leq q^{-2}c_0$, $\lambda_1 = q^{-1}d_0$, and $\lambda_2 = qc_0$, we get a highest weight representation, and all the eigen-values of $x_1$, $x_2$ lie in the region described by (7.6). It belongs to the degenerate holomorphic discrete series.

2. when $d_0 \geq q^{-2}c_0$, $\lambda_1 = q^{-1}d_0$, and $\lambda_2 = qc_0$, we get a lowest weight representation, and all the eigen-values of $x_1$, $x_2$ lie in the region described by (7.7). It belongs to the degenerate anti-holomorphic discrete series.

3. when $1 < \frac{d_0}{c_0} < q^{-4}$, $\lambda_2 = qc_0$, and $\lambda_1$ is such that $(\lambda_1, \lambda_2)$ belongs to the region described by (7.7), while $(\lambda_1q^2, \lambda_2)$ belongs to the region described by (7.7). It belongs to the degenerate complimentary series.

Again, we observe the same 'tunnel effect' that the complimentary series representations arise in a situation when the set $Q$ of the eigen-values of Func$(X)_q^{nu}$ can 'jump' from the set that corresponds to the holomorphic discrete series representations onto another one that corresponds to the anti-holomorphic ones. In fact, it is clear that the regions described by (7.6) and (7.7) are nothing but the projections on $T^* \subset SU(2, 1)^*$ of the symplectic leaves of the minimal non-zero dimension.

8. Appendix: Holomorphic Realization of Some Representations of $U_qg(1, 1)$

Using the quantum polarizations described above, one can obtain realizations of the discrete series representations and some of the strange series representations of $U_qg(1, 1)$ in spaces of holomorphic functions. First it was done in [7].

The following propositions are results of straightforward computations.

Proposition 8.1. (1) For any discrete series representation $T_{l/\epsilon}^+$ of highest weight $l \leq -\frac{1}{2}$, there exists a vector-function $\Theta_+(\lambda)$, holomorphic in the unit disc $|\lambda| < 1$ and taking values in the space of $T_{l/\epsilon}^+$, such that

$$\zeta\Theta_+(\lambda) = \lambda\Theta_+(\lambda)$$

and the scalar product of two such functions is equal to

$$(\Theta_+(\lambda), \Theta_+(\mu)) = \frac{1}{(\lambda\mu; q^2)_{-2l}},$$

where $(a; t)_\alpha = (a; t)_\infty^{\alpha}$ and $(a; t)_\infty = \prod_{k=0}^{\infty}(1 - at^k)$. 


(2) For any strange series representation $T_{\alpha+\frac{2\alpha+1}{2},\varepsilon}$, where $2\alpha+1 \in \mathbb{N}$, there exists a vector-function $\Theta(\lambda)$, holomorphic in the annulus $q^{2\alpha+1} < |\lambda| < 1$ and taking values in the space of the representation, such that

$$\zeta(\Theta(\lambda)) = \lambda \Theta(\lambda)$$

and the scalar product of two such functions is equal to

$$(\Theta(\lambda), \Theta(\mu)) = \frac{(q^{2(\alpha+1-\varepsilon)}; q^2)_\infty}{(q^{-2(\alpha+\varepsilon)}; q^2)_\infty} 1\Psi_1\left( \frac{q^{-2(\alpha+\varepsilon)\lambda}; t, \lambda \mu}{q^{2(\alpha+1-\varepsilon)\lambda}; q^2} \right),$$

where

$$1\Psi_1\left( \frac{a}{b}; t, x \right) = \sum_{k=-\infty}^{+\infty} (a; t)_k (b; t)_k x^k$$

is the Ramanujan’s psi-function.

Consider the map from the space of $T_{l,\varepsilon}^+$ into the space of functions holomorphic in the unit disc given by

$$\theta_+ : f \mapsto (f, \Theta_+(\lambda)).$$

Similarly, consider the map from the space of $T_{l+\frac{2\alpha+1}{2},\varepsilon}$ into the space of functions holomorphic in the annulus $q^{2\alpha+1} < |\lambda| < 1$ given by

$$\theta : f \mapsto (f, \Theta(\lambda)).$$

Thus, we get an action of $U_q\mathfrak{su}(1, 1)$ on the functions holomorphic in the respective domain. It is given by $q$-difference operators. The precise formulas are given in [1].

Now we are looking for a measure in the respective domain that establishes an isomorphism between the corresponding Hilbert spaces – the space of the representation and the space of holomorphic functions.

**Proposition 8.2.** (1) Consider the measure $d\nu_{l,\varepsilon}^+$ on the unit disc given by

$$d\nu_{l,\varepsilon}^+ = (1 - q^{-2(l-1)}) \sum_{k=0}^{\infty} q^{2k} \frac{(q^{2(k+1)}; q^2)_\infty}{(q^{2(k-2l+1)}; q^2)_\infty} \delta_{|\lambda|=q^k} d\lambda d\bar{\lambda},$$

when $l < \frac{1}{2}$, and by

$$d\nu_{l,\varepsilon}^+ = \delta_{|\lambda|<1} d\lambda d\bar{\lambda},$$

when $l = -\frac{1}{2}$, where $\delta_{|\lambda|=t}$ is the $\delta$-function on a circle $|\lambda| = t$. Then $\theta_+$ becomes an isomorphism between the space of $T_{l,\varepsilon}^+$ and the Hilbert space of functions in the unit disc with the measure $d\nu_{l,\varepsilon}^+$ that are holomorphic in the interior of the unit disc and continuous in its closure.

(2) Consider the measure $d\nu_{\alpha+\frac{2\alpha+1}{2},\varepsilon}$, where $2\alpha+1 \in \mathbb{N}$, on the annulus $q^{2\alpha+1} \leq |\lambda| \leq 1$ given by

$$d\nu_{\alpha+\frac{2\alpha+1}{2},\varepsilon} = \sum_{k=0}^{2\alpha+1} q^{2k(\alpha+1-\varepsilon)} \frac{(q^{-2(\alpha+1)\lambda}; q^2)_k}{(q^2; q^2)_k} \delta_{|\lambda|=q^k} d\lambda d\bar{\lambda}.$$

Then $\theta$ becomes an isomorphism between the space of $T_{\alpha+\frac{2\alpha+1}{2},\varepsilon}$ and the Hilbert space of the functions in the above annulus with the measure $d\nu_{\alpha+\frac{2\alpha+1}{2},\varepsilon}$ that are holomorphic in the interior of the annulus and continuous in it closure.
Thus, we obtain the realizations of the holomorphic discrete series representations in the holomorphic functions in the unit disc. And the realization of some strange series representations in the holomorphic functions in an annulus. Similarly, one can obtain a realization of the anti-holomorphic discrete series representations in the anti-holomorphic functions in the unit disc (or in the holomorphic functions in the domain outside the unit disc).

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