Stochastic model of self-driven two-species objects in the context of the pedestrian dynamics.

Roberto da Silva$^1$, Agenor Hentz$^1$, Alexandre Alves$^2$

$^1$Instituto de Física, Universidade Federal do Rio Grande do Sul, Porto Alegre, RS, Brasil
$^2$Depto. de Ciências Exatas e da Terra, Universidade Federal de São Paulo, Diadema, SP, Brasil

In this work we propose a model to describe the statistical fluctuations of the self-driven objects (species A) walking against an opposite crowd (species B) in order to simulate the regime characterized by stop-and-go waves in the context of pedestrian dynamics. By using the concept of single-biased random walks (SBRW), this setup is modeled both via partial differential equations and by Monte-Carlo simulations. The problem is non-interacting until the opposite particles visit the same cell of the considered particle. In this situation, delays on the residence time of the particles per cell depends on the concentration of particles of opposite species. We analyzed the fluctuations on the position of particles and our results show a non-regular diffusion characterized by long-tailed and asymmetric distributions which is better fitted by some chromatograph distributions found in the literature. We also show that effects of the reverse crowd particles is able to enlarge the dispersion of target particles in relation to the non-biased case ($\alpha = 0$) after observing a small decrease of this dispersion.

PACS numbers: 02.50.-r; 05.40.-a; 89.65.-s

A large number of stochastic phenomena in literature are related to the passage of particles through random media generated, for example, by imperfections of the environment. Examples can be found as disordered linear chains generated by arbitrary mass and spring-constant, random walks in random environments and diffusion (see for example$^2$–$^4$), charge trapping phenomena (electron transport) in semiconductor devices (see for example$^6$–$^{11}$), and transport of molecules (chromatography) in Chemistry ($^{12}$–$^{15}$).

On the other hand, the environment can be "perfect" but the particles can interact by occupying the same region in the space and statistical fluctuations can be generated not because imperfections but from the interaction among those particles. The attempts to give explanations about the dynamics of particles in these situations can be translated into problems related to the movement of human beings in corridors, crosswalks, sidewalks and public places in general. But which are the minimal physical aspects necessary to explain the concentration phenomena of human beings as a phenomena of concentration of interacting particles or hard bodies?

The corresponding literature in this case is highly concentrated into evacuation rooms in the context of phenomena related to crowd stampeded induced by panic, driven naturally by the huge importance of the problem. Results related to optimal strategy for the escape from a smoke-filled room were, for example, studied in$^{16}$–$^{18}$. Such problem is deeply related to occurrences of tragedies. A recent example was the love parade occurred in the Germany, in 2010$^{19}$ when a bad estimate of number of people in this electronic music party generated more than twenty deaths and several injured. However no modeling were developed to study, for example, statistical effects of people in contrary flux although some works have already explored the problem under other point of view. In$^{20}$ comparisons of intersecting pedestrian flows were analyzed based on experiments.

For the modeling of pedestrian dynamics some models were explored in order to study the influence of several effects for the interaction of pedestrian on the resulting velocity-density relation. An interesting point on those models is the transition from laminar flow regime to the regime of known stop-and-go wave phenomena that occurs when density of pedestrian increases above critical value of density (see for example$^{21}$ and$^{22}$).

We believe that peculiar characteristics of these "crowd" effects can be modeled in terms of a simple stochastic approach by adding some important ingredients. Let us imagine a problem that considers a straight line divided into cells where there are two crowds, denoted by groups (or species) A and B. Without loss of generality all participants of group A are initially placed in the far left-hand cell and all participants of group B are initially placed in the far right-hand cell.

The idea is that group A has the aim to arrive at the far right-side (starting point of species B) and group B to arrive at the far left-side (starting point of species A). Since the population A and B disturb each other, a natural approach for the problem is to describe it in terms of modified random walks in which there are only two possibilities of movement to each element at any given time-step: to stay still or take a step to the nearest neighbor cell towards its target.

Naturally, many similar phenomena can be imagined even from a theoretical point of view: the flux of molecules in chromatographic columns or the electronic transport in non-homogeneous media. Surely we are not simply comparing human beings crossing a corridor with molecules crossing a chromatographic column, but we are calling the attention to some similar aspects from concentration phenomena that can also occur when many bodies directed by some field (for example to arrive at
In the context of mean field regime, Montroll and West described a problem which has some relation to the present one: the problem of "clannish random walks". In that case two species, $A$ and $B$, execute concurrent random walks characterized by the intensification of the clannishness of the members of one species as a function of the concentration of the other species. However our problem has the opposite sense of that one arising from clannishness since in our model a particle can still in same cell or step towards for the next cell (directed random walk) in the direction of its target and its aim is simply to cross the corridor and not to make part of a clan.

By capturing this idea, we propose in this paper a model where species $A(B)$, with opposite targets, has a decrease in the probability to step to the right(left) proportional to the concentration of other particles $B(A)$ offering resistance during the passage. Therefore we define the following probabilities:

$$P_t^{(A,B)}(k \to k \pm 1|\rho_{B,A}) = p - \alpha(1 - \rho_{A,B})$$

$$\quad = p - \alpha(1 - \rho) - \alpha \rho_{B,A}$$

$$\quad = 1 - P_t^{(A,B)}(k \to k|\rho_{B,A})$$

(1)

Here $P_t^{(A,B)}(k \to k \pm 1|\rho_{B,A})$ denotes the probability of particle $A(B)$ at position $x = ka$ to move to the nearest neighbor $(k \pm 1)a$ cell given that the concentration of species $A(B)$ in its cell is $\rho_A(\rho_B)$ where $\rho = \rho_A + \rho_B$. Naturally, $P_t^{(A,B)}(k \to k|\rho_{B,A})$ denotes the probability of this particle to stay still at the cell under same restrictions. Here $p = 1 - q$ define the probability of a given particle to move to the nearest neighbor cell towards its respective target when the current cell contains only particles of its specie. The parameter $\alpha$, such that $0 < \alpha < p < 1$ measures the resistance level bias which, as well as $p$, at least in a first analysis, assumes the same value for all players. The problem is symmetric, and therefore we use the referential of the particles $A$, since particles $B$ present a similar behavior.

We define $n(ka, \tau) = n_{k,t}$ as the number of particles at cell $k$ after $l$ steps. Here $a$ is the size of each cell and $\tau$ the time-step. In mean field regime we can write the recurrence relation $n_{k,t+1} = P_t^{(A)}(k \to k|\rho_{A,B}) \cdot n_{k-1,t} + P_t^{(A)}(k \to k|\rho_{A,B}) \cdot n_{k+1,t}$ and so

$$n_{k,t+1} = [p - \alpha(1 - \frac{n_{k-1,t}}{N})] n_{k-1,t} + [q + \alpha(1 - \frac{n_{k+1,t}}{N})] n_{k,t}$$

(2)

So

$$n_{k,t+1} = (p - \alpha(1 - \frac{n_{k-1,t}}{N})] n_{k-1,t} + [1 - p + \alpha(1 - \frac{n_{k+1,t}}{N})] n_{k,t}$$

and so

$$n_{k,t+1} - n_{k,t} = -p(n_{k-1,t} - n_{k-1,t}) +$$

$$\alpha \left[ (1 - \frac{n_{k,t}}{N}) n_{k,t} - (1 - \frac{n_{k-1,t}}{N}) n_{k-1,t} \right]$$

(3)

Here the number $N$ deserves some discussion. In the strict sense of random walkers, and not for the adaptation of this model to explain the phenomena of human beings crossing corridors, we postulate that the number of walkers in each cell in mean-field approximation is supposed to remain constant, hence $N = \langle n \rangle = \rho_a$, with $\rho = \rho_A + \rho_B$ being the total density of particles per cell. However, for computer simulations we can study the problem by using the $\rho_A$ and $\rho_B$ correctly calculated per cell and compare it with the results obtained for the mean field approximation.

FIG. 1: Distribution of particles $A$ for $p = 1/2$. We can observe a stronger deformation from the quadratic behavior in mono-log scale (Gaussian behavior) for $\alpha = 0.4$ than for $\alpha = 0.1$. Continuous curves correspond to the solution of PDE while points correspond to MC simulations.

So by completing our mean-field results, since $n_{k,t+1} - n_{k,t}$ in first approximation is $N \frac{\partial c_A}{\partial x}$, $(n_{k,t} - n_{k-1,t})$ is $Na \frac{\partial c_A}{\partial x}$ and $(1 - \frac{n_{k-1,t}}{N}) n_{k-1,t} - (1 - \frac{n_{k+1,t}}{N}) n_{k+1,t}$ is $Na \frac{\partial c_A}{\partial x} (1 - a \alpha)$, we have $\tau \frac{\partial c_A}{\partial t} = -a \frac{\partial c_A}{\partial x} + \alpha a \frac{\partial c_A}{\partial x}$, which results in

$$\frac{\partial c_A(t, x)}{\partial t} = -A_1 \frac{\partial c_A(t, x)}{\partial x} - A_2 c_A(t, x) \frac{\partial c_A(t, x)}{\partial x}$$

(4)

where $A_1 = \lim_{a, \tau \to 0} \frac{a}{2} (p - \alpha)$ and $A_2 = \lim_{a, \tau \to 0} 2 \alpha a / \tau$.

An "ansatz" for the solution is $c_A(t, x) = f(x - t(A_1 + A_2 c_A))$. Deriving both sides with respect to $t$ one obtains

$$\frac{\partial c_A}{\partial t} = -(A_1 + A_2 c_A + A_2 \frac{\partial c_A}{\partial x}) f'(z)$$

where $z = x + t(A_1 + A_2 c_A)$. Similarly, deriving both sides with respect to $x$ one obtains

$$\frac{\partial c_A}{\partial x} = (1 - A_2 \frac{\partial c_A}{\partial x}) f'(z).$$

(5)
From Eq. 5 and the expression for $\partial c_A / \partial t$ we have
\[
(1 - A_2 \frac{\partial}{\partial x}) \frac{\partial c_A}{\partial t} = -(A_1 + A_2 c_A + A_2 \frac{\partial}{\partial x}) \frac{\partial c_A}{\partial x},
\]
which leads to Eq. 4. So given an initial distribution $c_A(0, x) = f(x)$ the time evolution of concentrations can be determined. However for the present case $f(x) = \delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i k x} dk$ such solution is not suitable. Moreover, by keeping $N$ constant, as suggested by Montroll in the context of clannish random walks, is not interesting since given our initial condition such solution does not capture the essential behavior of the problem. So changing $N$ by its real value: $n_{k+1} = m_{k, l}$ where $m_{k, l}$ denotes the number of particles of specie $B$ in the $k$-th cell at time $t = \tau$, instead of Eq. 3 we have now the concentrations of $A$ and $B$ described by two coupled equations:
\[
\begin{align*}
\alpha \left[ 1 - \frac{m_{k, l}}{n_{k, l} + m_{k, l}} \right] m_{k, l} - \left( 1 - \frac{m_{k, l}}{n_{k, l} + m_{k, l}} \right) m_{k-1, l},
\end{align*}
\tag{6}
\]
and
\[
\begin{align*}
\alpha \left[ 1 - \frac{n_{k, l}}{n_{k+1, l} + n_{k, l}} \right] n_{k, l} - \left( 1 - \frac{n_{k, l}}{n_{k+1, l} + n_{k, l}} \right) n_{k-1, l}.
\end{align*}
\tag{7}
\]
We end up with two coupled PDE equations which describe the problem instead of the uncoupled equation 4:
\[
\frac{\partial c_A}{\partial t} = -k_1 \frac{\partial c_A}{\partial x} + k_2 \frac{\partial}{\partial x} \left( \frac{c_A c_B}{c_A + c_B} \right),
\tag{8}
\]
\[
\frac{\partial c_B}{\partial t} = k_1 \frac{\partial c_B}{\partial x} - k_2 \frac{\partial}{\partial x} \left( \frac{c_A c_B}{c_A + c_B} \right)
\]
resulting in $\frac{\partial(c_A + c_B)}{\partial x} = -k_1 \frac{\partial(c_A + c_B)}{\partial x}$, where $k_1 = \lim_{a, \tau \to 0} \frac{2p}{p}$ and $k_2 = \lim_{a, \tau \to 0} \frac{2.1}{a}$. We numerically solved the recurrences 6 and 7 and concurrently we also performed Monte Carlo simulations of the problem monitoring 3 aspects of the $c_A$ distribution (since $c_B$ presents symmetric behavior):
1. Since $c_A$ reflects the residence time of particles in the medium, which are the aspects of distribution $c_A$?
2. Which are the temporal aspects $\langle x \rangle$ and $\sigma(t) = \left( \langle x^2(t) \rangle - \langle x(t) \rangle^2 \right)^{1/2}$ as well as of the skewness and kurtosis?
3. What about the effects of $\alpha$?

Our MC simulations were performed synchronously, in the context of cellular automata, i.e., all particle actions are taken simultaneously. The simulations have the initial condition as required by our aims: a given number of particles $A$ ($N_{part}$) in the position $x = 0$ and the same number of particles $B$ in the position $x = a N_{cel}$, where $N_{cel}$ is the number the total number of cells. The other cells are initially empty which means an initial delta distribution as desired. On the other hand the similar condition for the integration of equations 6 and 7 is to set: $n_{0, 0} = 1$ and $m_{N_{cel}, 0} = 1$ and $n_{k, 0} = m_{k, 0}$ for all $k \neq 0$ and $k \neq N_{cel}$.

Here, we keep our analysis restricted to the fundamental case $p = 1/2$. It is important to emphasize that the analysis for $p \neq 1/2$ follows in a similar fashion respecting the natural scale of the problem: $T = N_{cel}/p$. Fig. 1 shows the distribution of particles for two different values of the resistance parameter $\alpha = 0.1$ (upper panel) and $\alpha = 0.4$ (lower panel). We analyzed the concentration of particles at 8 different times ($t = (k - 1)/T$), for $k = 1, 2, \ldots, 8$. The reason is simple: since the particle $A$ leaps to the right with probability $p$, the probability that $m$ trials are necessary for a given particle to jump $N_{cel}$ is a negative binomial distribution:
\[
p_{m, N_{cel}} = \binom{m + N_{cel} - 1}{m} p^m (1-p)^{N_{cel} - m},
\]
which leads to $T = \langle m \rangle = \sum_{m=N_{cel}} m p_{m, N_{cel}} = N_{cel}/p$. This is the average crossing time for a particle when $\alpha = 0$, which is always larger than $T_a \neq 0$ and therefore our baseline. We used $N_{cel} = 100$ for all experiments in this paper by default, except when stated otherwise.

For a small value ($\alpha = 0.1$) we observe something next of a Gaussian behavior (quadratic function in mono-log scale). For ($\alpha = 0.4$) we observe a strong deformation of such behavior. The continuous curves correspond to the solution of PDE while the points correspond to MC simulations.

In order to describe with more details the dynamic behavior of the particle distribution along the environment, we compute the average position, its standard deviation as well as two other important quantities: the skewness and kurtosis of particle distribution for $\alpha = 0.1, 0.2, 0.3$, and 0.4, which can be calculated via the equations:
\[
\langle x(t) \rangle = \sum_{x=0}^{N_{cel}} x c_A(x, t) / \sum_{x=0}^{N_{cel}} c_A(x, t),
\]
\[
\sigma(t) = \sqrt{\langle x^2(t) \rangle - \langle x(t) \rangle^2} = \left[ \sum_{x=0}^{N_{cel}} x^2 c_A(x, t) / \sum_{x=0}^{N_{cel}} c_A(x, t) - \langle x(t) \rangle^2 \right]^{1/2},
\]
\[\text{skew}(t) = \frac{1}{\sigma(t)} \sum_{x=0}^{N_{cel}} (x - \langle x \rangle)^3 c_A(x, t)\]
and
\[\text{kurt}(t) = \frac{1}{\sigma(t)^4} \sum_{x=0}^{N_{cel}} (x - \langle x \rangle)^4 c_A(x, t) - 3\].

The temporal monitoring of such quantities has showed important aspects about the fluctuations of the particle concentration.

For example in Fig. 2 the plots (a) and (b) show respectively the behavior of the average $\langle x(t) \rangle$ and variance $\sigma(t)$ of the particle position as a function of $t$. The inset plots show respectively the same plots in log-log scale. We can observe the particle position (plot a) has the same behavior for all values of $\alpha$ up to a branching point, which corresponds to the moment where particles $A$ and $B$ start encountering each other on their way to their targets. From that, the bigger is $\alpha$, the bigger is the decrease of $\langle x(t) \rangle$ in relation to the non-interacting problem ($\alpha = 0$). More interesting effects emerge from the behavior of the variance (plot b). In this case particles $A$ have a decreasing in dispersion as they start to
interact with particles $B$, then increasing again arriving at a "iso-variance" point, i.e., the variance of the particle distribution is the same independent of the $\alpha$ value. Following, the dispersion becomes higher than the non-interacting case arriving at a peak that is higher as $\alpha$ increases. After this point the particles start to leave the environment and dispersion starts to decrease again.

Following we measured the asymmetry and tail weight of the particle distribution. The results are surprising and very interesting (see Fig. 3). In the upper panel of Fig. 3 we firstly check the peculiar behavior of skewness. After the confrontation of particles we have a cross-over of the behavior of skewness and very interesting (see Fig. 3). In the upper panel of Fig. 3, we observe an analogous cross-over of the behavior and fits with two important chromatographic functions from literature were checked:

1. **Gram-Charlier peak function**

   $$f(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}\sigma^2} \left[ 1 + \frac{a_1}{3!} H_1(z) + \frac{a_2}{4!} H_2(z) \right]$$

   where $H_1(z) = z^3 - 3z$, $H_2(z) = z^4 - 6z^3 + 3$ with $z = (x - \mu)/\sigma$.

2. **Edgeworth-Cramer peak function**

   $$f(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}\sigma^2} \left[ 1 + \frac{a_1}{3!} H_1(z) + \frac{a_2}{4!} H_2(z) + \frac{10a_3}{6!} H_3(z) \right]$$

   where $H_3(z) = z^6 - 15z^4 + 45z^2 - 15$. In both cases, $\mu$ is the center, $\sigma$ is the width and $a_1$ and $a_2$ are the unknown parameters.

By using the Levenberg Maquardt method for non-linear regression we obtain the convergent $R^2$ (i.e., the coefficient of determination) after the iterations. We start by reporting the case where the interaction among particles is less intense, i.e., $t = 2T/3$. We obtain respectively intuitively expected.

Finally we observe the possible fits for the particle distribution due to nature of phenomena. So considering the larger resistance factor studied ($\alpha = 0.4$) we fit the particle distribution at two different times: $t = 2T/3$ and $t = 3T/4$ where the effects of resistance can be better observed (Fig. 4). We can check the deviation from normal behavior and fits with two important chromatographic functions from literature were checked:

FIG. 2: Temporal description of average (upper panel) and variance (lower panel) of particle position along the environment. The inset plot in upper panel describes a zoom of selected region and in the lower panel one corresponds to the log-log plot of the dispersion vs time.

FIG. 3: Temporal description of skewness (upper panel) and kurtosis excess (lower panel) of particle position along the environment.
for Gaussian, ECS and GCAS: 0.9905, 0.9987 and 0.9977. For $t = 3T/4$ we can observe a difference even bigger between the chromatographic ones and the normal: 0.9703, 0.9977 and 0.9992.

The deviation of gaussian transport occurs in many contexts including chromatography and noise flicker in semiconductors devices, for example. One can suppose that a similar mechanism responsible for generating such distributions could be related to our problem since that in transport phenomena of molecules, or by thinking in electrons oriented by a field, a similar resistance mechanism occurs: looking at the capture/emission of electrons by traps given a Fermi level in semiconductor devices in the first case, or by the capture of molecules and their reemission in the chromatographic column as we reported in.

So in this paper we propose a stochastic model to describe the movement of particles against a contrary flux. This model is promising in order to understand some peculiarities in the pedestrian dynamics alternatively to the interesting social force models since one analyzes the non-normal behavior in particle density generated by interaction among the particles. Such interaction was modeled in a simple way by a decreasing term on the probabilities of a special random walk oriented by an ‘external field’ which does not allow the return of particles to previous cell. The interesting phenomena of reduction/increasing of dispersion of the particles is accentuated by the resistance term ($\alpha$) and a non-trivial behavior for the symmetry and tail of distribution was monitored. We are sure that such studies brings out an interesting class of stochastic process with future universalities to be explored in the transport physics in random mediums with applications in pedestrian dynamics under specific conditions.

Acknowledgments – This research was partially supported by the Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq), under the grant 11862/2012-8 (R.S.). The work of A.A. was supported by Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP), under the grant 2013/22079-8.

---

1. F. J. Dyson, Phys. Rev. 92, 1331 (1953)
2. V. V. Anshelevich, A. V. Vologodskii, J. Stat. Phys. 25, 419 (1981)
3. S. Alexander J. Bernasconi and R. Orbach, Phys. Rev. B 17, 4311 (1978)
4. D. H. U. Marchetti, R. da Silva, Braz. Journ. Phys., 29, 492 (1999)
5. J. Bernasconi, W. R. Schneider and W. Wyss, Z. Physik B 37, 175 (1980)
6. S. Machlup, J. Appl. Phys. 35, 341 (1954).
7. M.J. Kirton, M.J. Uren Adv. Phys., 38, 367 (1989)
8. R. da Silva, L. C. Lamb, G. I. Wirth, Philos. T. Roy. Soc. A. 369, 307-321 (2011).
9. R. da Silva, L. Brusamarello, G. Wirth, Physica. A 389, 2687-2699 (2010).
10. R. da Silva, G. I. Wirth, J. Stat. Mech., P04025, (2010).
11. R. da Silva, G. I. Wirth, L. Brusamarello, Int. J. Mod. Phys. B, 24, 5885 (2010).
12. J.C. Giddings, Chem. Rev., 89, 277 (1989)
13. J.C. Giddings, H. Eyring, J. Phys. Chem., 59, 416 (1955)
14. R. da Silva, L. C. Lamb, E. C. Lima, J. Dupont, Physica. A, 391, 1-7 (2012).
15. D. Helbing, I. Farkas, T. Vicsek, Nature 407, 487 (2000)
16. J. Zhang, A. Seyfried, Physica A 405, 316 (2014)
17. D. Helbing, A. Johansson, H. Z. Al-Avideen, Phys. Rev. E 75, 046109 (2007)
18. A. Portz, A. Seyfried, arXiv:1001.3283v1 (2010)
19. B. Krausz, C. Bauchhage, Comput. Vis. Image Und. 116, 307 (2012)
20. E. W. Montroll, B. West, On Enriched Collection of Stochastic Process: in Fluctuation Phenomena, Eds. E. W. Montroll and J. Lebowitz (1979)
21. D. Helbing, P. Molnár, Phys. Rev. E 51, 4282 (1995)
22. We are better exploring the possible interesting analytical solutions in other contribution in progress. In this same contribution we perform a classification of the problem and its ramifications.