HOMOLOGICAL AND BLOCH INVARIANTS FOR $\mathbb{Q}$-RANK ONE SPACES AND FLAG STRUCTURES

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Abstract. We use group homology to define invariants in algebraic K-theory and in an analogue of the Bloch group for $\mathbb{Q}$-rank one lattices and for some other geometric structures. We also show that the Bloch invariants of CR structures and of flag structures can be recovered by a fundamental class construction.

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1. Introduction

Fundamental class constructions and Bloch invariants are by now a classical theme in the topological study of hyperbolic 3-manifolds, going back to the work of Dupont and Sah on scissors congruences and more recently the work of Neumann, Yang and Zickert on Bloch invariants. For a hyperbolic 3-manifold \( M = \Gamma \backslash \mathbb{H}^3 \) with \( \Gamma \subset \text{PSL}(2, \mathbb{C}) \) one can on the one hand consider its PSL(2, \mathbb{C})-fundamental class \([M]_{\text{PSL}(2, \mathbb{C})}\), that is the image of the fundamental class \([M] \in H_3(M) \cong H_3(\Gamma) \) in \( H_3(\text{PSL}(2, \mathbb{C})) \), and on the other hand one can use ideal triangulations or more generally degree one ideal triangulations to define an invariant \( \beta(M) \) in the Bloch group \( B(\mathbb{C}) \). In [26] it was shown that one can recover the volume and the Chern-Simons invariant mod \( \mathbb{Q} \) from \( \beta(M) \). (In later work Neumann constructed an invariant in an extended Bloch group, from which one can recover the Chern-Simons invariant mod \( \mathbb{Z} \).)

The approach via ideal triangulations is better suited for doing practical calculations, see for example [26]. On the other hand the fundamental class approach is useful for theoretical considerations, e.g. to study the behaviour of hyperbolic volume under cut and paste in [23]. By the Bloch-Wigner Theorem (proved in more generality by Dupont-Sah in [12, Appendix A]) there is an isomorphism

\[
H_3(\text{PSL}(2, \mathbb{C}), \mathbb{Z})/\text{Torsion} \cong B(\mathbb{C}),
\]

and this isomorphism sends \([M]_{\text{PSL}(2, \mathbb{C})}\) to \( \beta(M) \). (One may pictorially think of a triangulation whose vertices are moved to infinity to produce an ideal triangulation. In some weak sense this picture can be made precise, see [22].) In particular the Bloch invariant is determined by the PSL(2, \mathbb{C})-fundamental class.

The construction of the Bloch invariant was generalized to higher-dimensional hyperbolic manifolds in [26, Section 8]. On the other hand Goncharov [16, Section 2] generalised the fundamental class construction to get - associated to an odd-dimensional hyperbolic manifold \( M^{2k-1} \) and a spinor representation \( \text{SO}(2k-1,1) \to \text{GL}(n, \mathbb{C}) \) - an element in \( H_{2k-1}(\text{GL}(n, \mathbb{C})) \) such that application of the Borel class recovers (a fixed multiple of) the volume. In [16, Section 3] he also used ideal triangulations to construct an extension \( m(M^{2k-1}) \in \text{Ext}^1_{\text{M}_{\mathbb{Q}}}(Q(0); Q(k)) \) in the category of mixed Tate motives over \( \mathbb{Q} \) and thus an element in \( K_{2k-1}(\mathbb{Q}) \otimes \mathbb{Q} \) according to Beilinson’s description of K-theory of fields. (In degree 3 one has \( K_3^{ind}(\mathbb{C}) \otimes \mathbb{Q} = B(\mathbb{C}) \otimes \mathbb{Q} \) and one recovers the Bloch invariant from this \( K \)-theoretic approach.)

In [21] the third-named author generalized Goncharov’s (first) construction to finite-volume locally symmetric spaces of noncompact type \( M = \Gamma \backslash G/K \) which are either closed or of \( \mathbb{R} \)-rank one. To each representation \( \rho : G \to \text{SL}(n, \mathbb{C}) \) the construction yields an element in \( H_*(\text{SL}(n, \mathbb{Q})) \) or, after a suitable projection, an element in

\[
PH_*(\text{GL}(\mathbb{Q})) \cong K_*(\mathbb{Q}) \otimes \mathbb{Q}
\]
such that application of the Borel class (to either of the two elements) yields a multiple $c_\rho \text{Vol}(M)$ of the volume $\text{Vol}(M)$. The factor $c_\rho$ depends only on $\rho$, in particular one can recover the volume if $c_\rho \neq 0$. Moreover, [21, Section 3] provides a complete list of fundamentals representations $\rho : G \to \text{SL}(n, \mathbb{C})$ with $c_\rho \neq 0$, for example one has $c_\rho \neq 0$ whenever $\text{dim}(G/K) \equiv 3 \text{ mod } 4$ and $\rho \neq \text{id}$. However, in the noncompact case, the only $\mathbb{R}$-rank one examples with $c_\rho \neq 0$ were (odd-dimensional) real-hyperbolic manifolds.

In this paper we further generalize Goncharov’s construction to $\mathbb{Q}$-rank one spaces. That means, for a $\mathbb{Q}$-rank one locally symmetric space of noncompact type $M = \Gamma \backslash G/K$ we construct elements

$$\gamma(M) \in H_\ast(\text{SL}(n, \mathbb{Q}))$$

and

$$\gamma(M) \in K_\ast(\mathbb{Q}) \otimes \mathbb{Q}$$

such that application of the Borel class yields again $c_\rho \text{Vol}(M)$. Compare Proposition 7.1 and Theorem 7.2 for the precise statements. Thus one can get many non-compact non-hyperbolic examples with nontrivial invariants.

The fundamental class construction shall be useful for deriving general results about the relation of topology and volume. For practical computations however the Bloch group approach appears to be more feasible, not only in 3-dimensional hyperbolic geometry [26] but also in the study of CR-structures [14] or flag structures [1].

In the 3-dimensional hyperbolic case, Neumann-Yang constructed the Bloch invariant [26, Definition 2.5] in the so-called pre-Bloch group $\mathcal{P}(\mathbb{C})$ and then proved in [26, Theorem 6] that it actually belongs to the Bloch group $\mathcal{B}(\mathbb{C}) \subset \mathcal{P}(\mathbb{C})$. The pre-Bloch group $\mathcal{P}(\mathbb{C})$ satisfies a natural isomorphism

$$\mathcal{P}(\mathbb{C}) \cong H_3(C_\ast(\partial_\infty G/K) \otimes_{\mathbb{Z}G} \mathbb{Z})$$

for $G/K = \text{SL}(2, \mathbb{C})/\text{SU}(2) = \mathbb{H}^3$. Thus it is natural to define a Bloch invariant of higher-dimensional locally symmetric spaces $M^d = \Gamma \backslash G/K$ (and representations $\rho : G \to \text{SL}(n, \mathbb{C})$) as an element in

$$H_d(C_\ast(\partial_\infty \text{SL}(n, \mathbb{C})/\text{SU}(n)) \otimes_{\mathbb{Z}\text{SL}(n, \mathbb{C})} \mathbb{Z}).$$

In [22] this was done for $\mathbb{R}$-rank one symmetric spaces and it was shown that the Bloch invariant is the image of the Goncharov invariant $\gamma(M)$ under a naturally defined evaluation homomorphism which generalizes the homomorphism from the Bloch-Wigner Theorem. The construction of the generalized Bloch invariant uses proper ideal fundamental cycles since the existence of ideal triangulations is unclear in general. The proof of well-definedness of the Bloch invariant (i.e., independence from the chosen proper ideal fundamental cycle, [22, Lemma 3.4.1]) was building on the equality $H_d(C_\ast(\partial_\infty G/K) \otimes_{\mathbb{Z}G} \mathbb{Z}) \cong \mathbb{Z}$ for lattices $\Gamma \subset G$, which in the $\mathbb{R}$-rank one case can be proved by an immediate generalization of the results of Neumann-Yang (who proved this equality in [26] for hyperbolic 3-space). However it is
unclear how to generalize this argument to the higher rank case. Therefore we avoid this point in Section 8 by directly defining the Bloch invariant

$$\beta(M) \in H_d(C_*\partial_\infty\text{SL}(n,\mathbb{C})/\text{SU}(n)) \otimes_{\mathbb{Z}} \text{SL}(n,\mathbb{C}) \otimes \mathbb{Z}$$

for locally symmetric spaces (either closed or of $\mathbb{Q}$-rank one) as the image of the Goncharov invariant $\gamma(M)$ under the evaluation homomorphism.

In Section 9 we consider a similar construction for convex projective manifolds. We hope to further exploit this in other papers.

In Section 10 we prove that also the Bloch invariants of CR structures (as defined by Falbel-Wang in [14]) and of flag structures (as defined by Bergeron-Falbel-Guilloux in [1]) can be recovered from a fundamental class construction. We apply this to prove that these Bloch invariants are preserved under certain cut-and-paste operations.

2. Basics on Group (co)homology

2.1. Group homology. For a topological group $G$, let $G^\delta$ denote the group with discrete topology. Let $BG^\delta$ denote the simplicial set whose $k$-simplices are $k$-tuples $(g_1, \ldots, g_k)$ with a natural boundary operator $\partial$:

$$\partial(g_1, \ldots, g_k) = (g_2, \ldots, g_k) + \sum_{i=1}^{k-1} (-1)^i (g_1, \ldots, g_ig_{i+1}, \ldots, g_k)$$

$$+ (-1)^k (g_1, \ldots, g_{k-1}).$$

It forms a chain complex $C_*^{\text{simp}}(BG^\delta)$ of $BG^\delta$ whose homology with a coefficient ring $R$ is defined as the group homology

$$H_*(G, R) = H_*^{\text{simp}}(BG^\delta, R) = H_*^{\text{simp}}(C_*(BG^\delta \otimes \mathbb{Z} R), \partial \otimes 1).$$

Throughout the paper, $BG$ will be understood as $BG^\delta$.

Let $M$ be a Riemannian manifold of nonpositive sectional curvature and $x_0 \in M$, a lift $\tilde{x}_0 \in \tilde{M}$ of $x_0$ be fixed. Any ordered tuple of vertices in $\tilde{M}$ determines a unique straight simplex. A singular simplex $\sigma \in C_*(M)$ is straight if some (hence any) lift $\tilde{\sigma} \in C_*(\tilde{M})$ is straight. Let $C_*^{\text{str}, x_0}(M)$ be the chain complex of straight simplices with all vertices $x_0$.

Set $\Gamma = \pi_1(M, x_0)$. Then there are two canonical homomorphisms $\Psi : C_*^{\text{simp}}(BG) \to C_*^{\text{str}, x_0}(M)$ defined by

$$\Psi(g_1, \ldots, g_k) = \pi(\text{str}(\tilde{x}_0, g_1 \tilde{x}_0, g_1g_2 \tilde{x}_0, \ldots, g_1 \cdots g_k \tilde{x}_0))$$

and $\Phi : C_*^{\text{str}, x_0}(M) \to C_*^{\text{simp}}(BG)$ defined by

$$\Phi(\sigma) = ([\sigma|_{\zeta_1}], \ldots, [\sigma|_{\zeta_k}])$$

where $e_0, \ldots, e_k$ are the vertices of the standard simplex $\Delta^k$, $\zeta_i$ is the standard sub-1-simplex with $\partial \zeta_i = e_i - e_{i-1}$, each $[\sigma|_{\zeta_i}] \in \pi_1(M, x_0) = \Gamma$ is the homotopy class of $\sigma|_{\zeta_i}$ and $\pi : \tilde{M} \to M$ is the universal covering map of $M$. It is easy to show that $\Psi$ and $\Phi$ are chain isomorphisms inverse to each other.
The inclusion
\[ i : C_{\ast \text{str},x_0}(M) \subset C_\ast(M) \]
and the straightening ([21, Section 2.1])
\[ \text{str} : C_\ast(M) \to C_{\ast \text{str},x_0}(M) \]
are chain homotopy inverses. Hence we have a chain homotopy equivalence, called the \textit{Eilenberg-MacLane map}
\[ EM : C_\ast^{\text{simp}}(B\Gamma) \to C_\ast(M). \]
We will frequently use the induced isomorphism
\[ EM^{-1} = \Phi_* \circ \text{str} : H_\ast(M,\mathbb{Z}) \to H_\ast^{\text{simp}}(B\Gamma,\mathbb{Z}). \]
The geometric realization \(|B\Gamma|\) is a \(K(\Gamma,1)\), thus there is a classifying map \(h^M : M \to |B\Gamma|\) which induces an isomorphism on \(\pi_1\) level. The inclusion map of simplices \(i : C_\ast^{\text{simp}}(B\Gamma) \to C_\ast(|B\Gamma|)\), induces an isomorphism
\[ i_* : H_\ast^{\text{simp}}(B\Gamma,\mathbb{Z}) \to H_\ast(|B\Gamma|,\mathbb{Z}) \]
such that \(h^M_* = i_* \circ EM^{-1}\) if \(M\) is aspherical and of the homotopy type of a CW complex (which is always the case for Riemannian manifolds of nonpositive sectional curvature).

For a commutative ring \(A \subset \mathbb{C}\) with unit, let \(\text{GL}(A) = \bigcup_{n=1}^\infty \text{GL}(n,A)\) be the increasing union, and \(|B\text{GL}(A)|\) its classifying space as above.

Let \(\rho : \Gamma \to \text{GL}(A)\) be a representation. This induces \(B\rho : B\Gamma \to B\text{GL}(A)\) and \(|B\rho| : |B\Gamma| \to |B\text{GL}(A)|\). The composition
\[ H_\ast(M,\mathbb{Q}) \xrightarrow{EM^{-1}} H_\ast^{\text{simp}}(B\Gamma,\mathbb{Q}) \xrightarrow{(B\rho)_*} H_\ast^{\text{simp}}(B\text{GL}(A),\mathbb{Q}) \]
induces a map
\[ (H_\rho)_* : H_\ast(M,\mathbb{Q}) \to H_\ast^{\text{simp}}(B\text{GL}(A),\mathbb{Q}). \]
If \(M\) is a closed, oriented and connected \(d\)-dimensional manifold, \((H_\rho)_d[M]\) will play an important role for us where \([M]\) is the fundamental class in \(H_d(M,\mathbb{Q}) \cong \mathbb{Q}\).

\textbf{2.2. Volume class and Borel class.} Let \(G\) be a noncompact semisimple, connected Lie group. Let \(X = G/K\) be the associated symmetric space of dimension \(d\) with a maximal compact subgroup \(K\) of \(G\). Let us denote by \(H_c^\ast(G,\mathbb{R})\) the continuous cohomology of \(G\). The comparison map \(\text{comp} : H_c^\ast(G,\mathbb{R}) \to H_\ast^{\text{simp}}(BG,\mathbb{R})\) is defined by the cochain map
\[ \text{comp}(f)(g_1, \ldots, g_k) = f(1, g_1, g_1 g_2, \ldots, g_1 g_2 \cdots g_k) \]
for a \(G\)-invariant cochain \(f : C^{k+1} \to \mathbb{R}\). Fix a point \(x \in X\). The volume class \(v_d \in H_d^\ast(G,\mathbb{R})\) is defined by the cocycle
\[ v_d(g_0, \ldots, g_d) = \text{algvol}(\text{str}(g_0x, \ldots, g_dx)) := \int_{\text{str}(g_0x, \ldots, g_dx)} dvol_X, \]
where $dvol_X$ is the $G$-invariant Riemannian volume form on $X$, and $str$ is the geodesic straightening of a simplex with those vertices. (That is $str(g_0x,\ldots, g_dx)$ is the unique straight simplex with the given ordered set of vertices.) Under the comparison map

$$comp(v_d)(g_1,\ldots, g_d) = algvol(str(x, g_1x,\ldots, g_1\cdots g_dx)).$$

Then it is not difficult to show that if $N = \Gamma \backslash X$ is a closed locally symmetric space, with $j: \Gamma \to G$ the inclusion, then

$$\text{Vol}(N) = \langle comp(v_d), B_j \circ EM_d^{-1}[N] \rangle.$$ 

For a detailed proof of this, see [21, Theorem 1].

Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebra of $G$ and $K$ respectively. If $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition, then the Lie algebra of the compact dual $G_u$ of $G$ is $\mathfrak{g}_u = \mathfrak{k} \oplus \mathfrak{p}$. Note that the relative Lie algebra cohomology $H^\ast(\mathfrak{g}, \mathfrak{k})$ is the cohomology of the complex of $G$-invariant differential forms on $G/K$ and there is the Van Est isomorphism

$$\mathcal{J}: H^\ast(G, \mathbb{R}) \to H^\ast(\mathfrak{g}, \mathfrak{k}).$$

Let $I^k_S(G_u)$ and $I^k_A(G_u)$ be the space of $ad$-invariant symmetric and anti-symmetric multilinear $k$-forms on $\mathfrak{g}_u$. There are isomorphisms

$$\Phi_A: I^k_A(G_u) \to H^k(G_u, \mathbb{R}),$$

and the Chern-Weil isomorphism

$$\Phi_S: I^k_S(G_u) \to H^{2k}(BG_u, \mathbb{R}),$$

see for example [8, Section 5].

When $G_u = U(n)$ is a unitary group, there is

$$\text{Tr}_k(A_1,\ldots, A_k) = \frac{1}{(2\pi i)^k k!} \sum_{\sigma \in S_k} \text{Tr}(A_{\sigma(1)}\cdots A_{\sigma(k)})$$

so that

$$C_k = \Phi_S(\text{Tr}_k) \in H^{2k}(BU(n), \mathbb{R})$$

is the $2k$-th component of the universal Chern character.

For the fibration $G_u \to EG_u \to BG_u$ and an associated transgression map $\tau$ from a subspace of $H^{2k-1}(G_u, \mathbb{Z})$ to $H^{2k}(BG_u, \mathbb{Z})/\text{ker}(s)$, where $s$ is the suspension homomorphism [2], there is a homomorphism [9]

$$R: I^k_S(G_u) \to I^{2k-1}_A(G_u)$$

such that $\tau \circ \Phi_A \circ R = \pi \circ \Phi_S$ where $\pi: H^{2k}(BG_u, \mathbb{Z}) \to H^{2k}(BG_u, \mathbb{Z})/\text{ker}(s)$. Then the Borel class is

$$b_{2k-1} = \Phi_A(R(\text{Tr}_k)) \in H^{2k-1}(U(n), \mathbb{R}) = H^{2k-1}(u(n), \mathbb{R}).$$

The last equality holds since $U(n)$ is a compact manifold, $H^\ast(U(n), \mathbb{R})$ is the de Rham cohomology of $U(n)$, and by averaging it is isomorphic to the cohomology of the complex of $U(n)$-invariant differential forms on $U(n)$.
On the other hands, since \((\text{GL}(n, \mathbb{C}))_u = U(n) \times U(n)\) and via Van Est isomorphism
\[
H^*_c(\text{GL}(n, \mathbb{C}), \mathbb{R}) = H^*(\text{gl}(n, \mathbb{C}), u(n)) = H^*(u(n) \oplus u(n), u(n)) = H^*(u(n), \mathbb{R})
\]
we may consider
\[
b_{2k-1} \in H^{2k-1}_c(\text{GL}(n, \mathbb{C}), \mathbb{R}).
\]

2.3. Volume for compact locally symmetric manifolds. It is a standard fact that for \(d = \text{dim}(G/K)\), \(H^d_c(G, \mathbb{R}) \cong \mathbb{R}\) via Van Est isomorphism.

If \(\Gamma\) is a uniform lattice in \(G\), then \(H^d(\Gamma \backslash G/K, \mathbb{R}) \cong H^d(\Gamma, \mathbb{R}) \cong H^d_c(G, \mathbb{R}) = \mathbb{R}\).

Hence the volume class \(v_d\) as defined in Section 2.2 can be viewed as a generator of \(H^d(\Gamma, \mathbb{R})\).

Proposition 2.1. For a symmetric space \(G/K\) of noncompact type with odd dimension \(d = 2m - 1\), and a representation \(\rho : \Gamma \to \text{GL}(n, \mathbb{C})\) with a closed manifold \(N = \Gamma \backslash G/K\), there exists a constant \(c_\rho \in \mathbb{R}\) such that
\[
\langle \text{comp}(b_d), H_\rho[N] \rangle = c_\rho \text{Vol}(N).
\]

If \(\rho : \Gamma \to \text{GL}(n, \mathbb{C})\) factors over a representation \(\rho_0 : G \to \text{GL}(n, \mathbb{C})\), then \(c_\rho\) depends only on \(\rho_0\).

Proof. A representation \(\rho : \Gamma \to \text{GL}(n, \mathbb{C})\) induces a homomorphism
\[
\rho^*_c : H^d_c(\text{GL}(n, \mathbb{C}), \mathbb{R}) \to H^d(\Gamma, \mathbb{R}).
\]

Since \(\rho^*_c(b_d) \in H^d(\Gamma, \mathbb{R}) = \mathbb{R} \cdot v_d\), there is a constant \(c_\rho \in \mathbb{R}\) such that \(\rho^*_c(b_d) = c_\rho v_d\). Hence for a fundamental cycle \(\sum_{i=1}^l a_i \sigma_i\) of \(N\),
\[
\langle \text{comp}(b_d), H_\rho[N] \rangle = \langle \text{comp}(b_d), B_\rho \circ EM_d^{-1}[N] \rangle
\]
\[
= \langle \text{comp}(\rho^*_c b_d), \Phi_\ast \circ \text{str}_\ast[N] \rangle
\]
\[
= c_\rho \sum_{i=1}^l a_i \langle \text{comp}(v_d), \Phi_\ast(\text{str}_\ast \sigma_i) \rangle
\]
\[
= c_\rho \sum_{i=1}^l a_i \cdot \text{algvol}(\text{str}_\ast \sigma_i)
\]
\[
= c_\rho \text{Vol}(N)
\]

We refer the reader to [21, Theorem 2] for the proof of the second claim. □

3. \(\mathbb{Q}\)-rank 1 locally symmetric spaces

In this section, we consider only \(\mathbb{Q}\)-rank 1 lattices \(\Gamma \subset G\). We first collect some definitions and results about \(\mathbb{Q}\)-rank 1 lattices.
3.1. Arithmetic lattices. Let \( G \) be a noncompact, semisimple Lie group with trivial center and no compact factors. Then one may define arithmetic lattices in the following way.

**Definition 3.1.** A lattice \( \Gamma \) in \( G \) is called arithmetic if there are

1. a semisimple algebraic group \( G \subset \text{GL}(n, \mathbb{C}) \) defined over \( \mathbb{Q} \) and
2. an isomorphism \( \varphi : G(\mathbb{R})^0 \to G \)

such that \( \varphi(\mathbb{G}(\mathbb{Z}) \cap G(\mathbb{R})^0) \) and \( \Gamma \) are commensurable.

It is well-known due to Margulis [25] that all irreducible lattices in higher rank Lie groups are arithmetic. The \( \mathbb{Q} \)-rank of a semisimple algebraic group \( G \) is defined as the dimension of a maximal \( \mathbb{Q} \)-split torus of \( G \). For an arithmetic lattice \( \Gamma \) in \( G \), \( \mathbb{Q} \)-rank(\( \Gamma \)) is defined by the \( \mathbb{Q} \)-rank of \( G \) where \( G \) is an algebraic group as in Definition 3.1.

A closed subgroup \( P \subset G \) defined over \( \mathbb{Q} \) is called rational parabolic subgroup if \( P \) contains a maximal, connected solvable subgroup of \( G \). For any rational parabolic subgroup \( P \) of \( G \), one obtains the rational Langlands decomposition of \( P = P(\mathbb{R}) \):

\[
P = N_P \times A_P \times M_P,
\]

where \( N_P \) is the real locus of the unipotent radical \( N_P \) of \( P \), \( A_P \) is a stable lift of the identity component of the real locus of the maximal \( \mathbb{Q} \)-torus in the Levi quotient \( P/N_P \) and \( M_P \) is a stable lift of the real locus of the complement of the maximal \( \mathbb{Q} \)-torus in \( P/N_P \).

Let \( X = G/K \) be the associated symmetric space of noncompact type with a maximal compact subgroup \( K \) of \( G \). Write \( X_P = M_P/(K \cap M_P) \). Let us denote by \( \tau : M_P \to X_P \) the canonical projection. Fix a base point \( x_0 \in X \) whose stabilizer group is \( K \). Then we have an analytic diffeomorphism

\[
\mu : N_P \times A_P \times X_P \to X, \quad (n, a, \tau(m)) \to nam \cdot x_0,
\]

which is called the rational horocyclic decomposition of \( X \). For more detail, see [5, Section III.2].

3.2. Precise reduction theory. Let \( \mathfrak{g} \) and \( \mathfrak{a}_P \) denote the Lie algebras of the Lie groups \( G \) and \( A_P \) defined above. Then the adjoint action of \( \mathfrak{a}_P \) on \( \mathfrak{g} \) gives a root space decomposition:

\[
\mathfrak{g} = \mathfrak{g}_0 + \sum_{\alpha \in \Phi(\mathfrak{g}, \mathfrak{a}_P)} \mathfrak{g}_\alpha,
\]

where

\[
\mathfrak{g}_\alpha = \{ Z \in \mathfrak{g} \mid \text{ad}(A)(Z) = \alpha(A)Z \text{ for all } A \in \mathfrak{a}_P \},
\]

and \( \Phi(\mathfrak{g}, \mathfrak{a}_P) \) consists of those nontrivial characters \( \alpha \) such that \( \mathfrak{g}_\alpha \neq 0 \). It is known that \( \Phi(\mathfrak{g}, \mathfrak{a}_P) \) is a root system. Fix an order on \( \Phi(\mathfrak{g}, \mathfrak{a}_P) \) and denote by \( \Phi^+(\mathfrak{g}, \mathfrak{a}_P) \) the corresponding set of positive roots. Define

\[
\rho_P = \sum_{\alpha \in \Phi^+(\mathfrak{g}, \mathfrak{a}_P)} (\dim \mathfrak{g}_\alpha)\alpha.
\]
Let $\Phi^{++}(g, a_P)$ be the set of simple positive roots.

Since we consider only $\mathbb{Q}$-rank 1 arithmetic lattices, we restrict ourselves from now on to the case $\mathbb{Q}$-rank($G$) = 1. Then the followings hold:

1. All proper rational parabolic subgroups of $G$ are minimal.
2. For any proper rational parabolic subgroup $P$ of $G$, $\dim A_P = 1$.
3. The set $\Phi^{++}(g, a_P)$ of simple positive $\mathbb{Q}$-roots contains only a single element.

For any proper rational parabolic subgroup $P$ of $G$ and any $t > 1$, define $A_{P,t} = \{ a \in A_P \mid \alpha(a) > t \}$, where $\alpha$ is the unique root in $\Phi^{++}(g, a_P)$. For bounded sets $U \subset N_P$ and $V \subset X_P$, the set $S_{P,U,V,t} = U \times A_{P,t} \times V \subset N_P \times A_P \times X_P$ is identified with the subset $\mu(U \times A_{P,t} \times V)$ of $X = G/K$ by the horospherical decomposition of $X$ and called a Siegel set in $X$ associated with the rational parabolic subgroup $P$. Given a $\mathbb{Q}$-rank 1 lattice $\Gamma$ in $G$, it is a well-known result due to A. Borel and Harish-Chandra that there are only finitely many $\Gamma$-conjugacy classes of rational parabolic subgroups. Recall the precise reduction theory in $\mathbb{Q}$-rank 1 case as follows (see [5, Proposition III.2.21]).

**Theorem 3.2.** Let $\Gamma$ be a $\mathbb{Q}$-rank 1 lattice in $G$. Let $G$ denote a semisimple algebraic group defined over $\mathbb{Q}$ with $\mathbb{Q}$-rank($G$) = 1 as in Definition 3.1. Denote by $P_1, \ldots, P_s$ representatives of the $\Gamma$-conjugacy classes of all proper rational parabolic subgroups of $G$. Then there exist a bounded set $\Omega_0$ in $\Gamma \backslash G/K$ and Siegel sets $U_i \times A_{P_i,t_i} \times V_i$, $i = 1, \ldots, s$, in $X = G/K$ such that

1. each Siegel set $U_i \times A_{P_i,t_i} \times V_i$ is mapped injectively into $\Gamma \backslash X$ under the projection $\pi: X \rightarrow \Gamma \backslash X$,
2. the image of $U_i \times V_i$ in $(\Gamma \cap P_i) \backslash N_{P_i} \times X_{P_i}$ is compact,
3. $\Gamma \backslash X$ admits the following disjoint decomposition

$$\Gamma \backslash X = \Omega_0 \cup \bigcup_{i=1}^{s} \pi(U_i \times A_{P_i,t_i} \times V_i).$$

Geometrically $B_P(t) = \mu(N_P \times A_{P,t} \times X_P)$ is a horoball for any proper minimal rational parabolic subgroup $P$ of $G$. Hence each $\mu(U_i \times A_{P_i,t_i} \times V_i)$ is a fundamental domain of the cusp group $\Gamma_i = \Gamma \cap P(\mathbb{R})$ acting on the horoball $B_{P_i}(t_i)$ and each $\mu(U_i \times V_i)$ is a bounded domain in the horosphere that bounds the horoball $B_{P_i}(t_i)$. Furthermore, each set $\pi(U_i \times A_{P_i,t_i} \times V_i)$ corresponds to a cusp of the locally symmetric space $\Gamma \backslash X$. We refer the reader to [5] for more details.

3.3. **Rational horocyclic coordinates.** Let $P$ be a proper minimal rational parabolic subgroup of $G$ with $\mathbb{Q}$-rank($G$) = 1. The pullback $\mu^* g$ of the
metric \( g \) on \( X \) to \( N_P \times A_P \times X_P \) is given by
\[
ds_{(n,a,\tau(m))}^2 = \sum_{\alpha \in \Phi^+(\mathfrak{a}_P)} e^{-2\alpha(\log a)} h_\alpha \oplus da^2 \oplus d(\tau(m))^2,
\]
where \( h_\alpha \) is some metric on \( \mathfrak{g}_\alpha \) that smoothly depends on \( \tau(m) \) but is independent of \( a \). Choosing orthonormal bases \( \{N_1, \ldots, N_r\} \) of \( \mathfrak{n}_P \), \( \{Z_1, \ldots, Z_l\} \) of some tangent space \( T_{\tau(m)} X_P \) and \( A \in \mathfrak{a}_P \) with \( \|A\| = 1 \), one can obtain rational horocyclic coordinates \( \eta : N_P \times A_P \times X_P \to \mathbb{R}^r \times \mathbb{R} \times \mathbb{R}^l \) defined by
\[
\eta \left( \exp \left( \sum_{i=1}^r x_i N_i \right) , \exp(yA) , \exp \left( \sum_{i=1}^l z_i Z_i \right) \right) = (x_1, \ldots, x_r, y, z_1, \ldots, z_l).
\]
We abbreviate \( (x_1, \ldots, x_r, y, z_1, \ldots, z_l) \) as \( (x, y, z) \). Then the \( G \)-invariant Riemannian volume form \( d\text{vol}_X \) on \( X \cong N_P \times A_P \times X_P \) with respect to the rational horocyclic coordinates is given by
\[
d\text{vol}_X = h(x, z) \exp^{-2\|\rho_P\|_y} dx dy dz,
\]
where \( h(x, z) \) is a smooth function that is independent of \( y \). See [4, Corollary 4.4].

Note that all proper rational minimal parabolic subgroups are conjugate under \( G(\mathbb{Q}) \). Hence the respective root systems are canonically isomorphic [3] and moreover, one can conclude \( \|\rho_P\| = \|\rho_P'\| \) for any two proper minimal rational parabolic subgroups \( P, P' \) of \( \mathbb{Q} \)-rank 1 algebraic group \( G \).

4. Straight simplices

Let \( X \) be a simply connected complete Riemannian manifold with nonpositive sectional curvature and \( \partial_\infty X \) be the ideal boundary of \( X \). For \( x_0, \ldots, x_k \in X \), the straight simplex \( \text{str}(x_0, \ldots, x_k) \) is defined inductively as follows: First, \( \text{str}(x_0) \) is the point \( x_0 \in X \), and \( \text{str}(x_0, x_1) \) is the unique geodesic arc from \( x_1 \) to \( x_0 \). In general, \( \text{str}(x_0, \ldots, x_k) \) is the geodesic cone on \( \text{str}(x_0, \ldots, x_{k-1}) \) with the top point \( x_k \). Since there is the unique geodesic connecting two points in \( X \), each ordered \((k + 1)\)-tuple \( (x_0, \ldots, x_k) \) determines the unique straight simplex.

If the sectional curvature of \( X \) is strictly negative, one can define the notion of straight simplex in \( X \cup \partial_\infty X \). For any ordered tuple \( (u_0, \ldots, u_k) \in X \cup \partial_\infty X \), the straight simplex \( \text{str}(u_0, \ldots, u_k) \) is well defined as above. A straight simplex \( \text{str}(u_0, \ldots, u_k) \) is called an ideal straight simplex if at least one of \( u_0, \ldots, u_k \) is in \( \partial_\infty X \). In general, however, an ideal straight simplex is not well defined for a simply connected Riemannian manifold with nonpositive sectional curvature. For example, let \( X \) be a higher rank symmetric space and consider two points \( \theta_1, \theta_2 \) in \( \partial_\infty X \) which cannot be connected by any geodesic in \( X \). Then, one cannot define a straight simplex \( \text{str}(\theta_1, \theta_2) \) with ideal vertices \( \theta_1, \theta_2 \). However, in the particular case
that \( x_0, \ldots, x_{k-1} \in X \) and \( \theta \in \partial_\infty X \), we can define an ideal straight simplex \( \text{str}(x_0, \ldots, x_{k-1}, \theta) \) as usual. This is because there is the unique geodesic from a point in \( X \) to a point in \( \partial_\infty X \). Hence, the geodesic cone on \( \text{str}(x_0, \ldots, x_{k-1}) \) with the top point \( \theta \) is well defined. We only need such kind of ideal straight simplex to construct our invariant in K-theory for a \( \mathbb{Q} \)-rank 1 locally symmetric space.

**Setup.** We will stick to the following notations from now on. Let \( G \) be a noncompact, semisimple Lie group with trivial center and no compact factors and \( X = G/K \) be the associated symmetric space with a maximal compact subgroup \( K \) of \( G \). Given a \( \mathbb{Q} \)-rank 1 arithmetic lattice \( \Gamma \) in \( G \), we denote by \( G \) a \( \mathbb{Q} \)-rank 1 semisimple algebraic group defined over \( \mathbb{Q} \) as in Definition 3.1. Let \( P_1, \ldots, P_s \) be the representatives of the \( \Gamma \)-conjugacy classes of all proper rational parabolic subgroups of \( G \). According to the precise reduction theory, we fix a fundamental domain \( F \subset X \) as in Theorem 3.2 as follows:

\[
F = \Omega_0 \bigcup \prod_{i=1}^{s} U_i \times A_{P_i,t_i} \times V_i
\]

Each half-geodesic \( A_{P_i,t_i} \) uniquely determines a point in \( \partial_\infty X \), denoted by \( c_i \). Write \( \Gamma_i = \Gamma \cap P_i(\mathbb{R}) \) and \( N = \Gamma \setminus X \). Note that each \( \Gamma_i \) is the stabilizer of \( c_i \) in \( \Gamma \). Since \( N \) is tame, \( N \) is homeomorphic to the interior of a compact manifold \( M \) with boundary. Let \( \partial_1 M, \ldots, \partial_s M \) be the connected components of the boundary \( \partial M \) of \( M \). Then there is a one-to-one correspondence between \( \Gamma_1, \ldots, \Gamma_s \) and \( \partial_1 M, \ldots, \partial_s M \). Indeed, we can assume that each \( \partial_i M \) is homeomorphic to the quotient space of a horosphere based at \( c_i \) by the action of \( \Gamma_i \).

**Lemma 4.1.** For any \( c \in \{c_1, \ldots, c_s\} \), the volume of the ideal straight simplex \( \text{str}(x_0, \ldots, x_{d-1}, c) \) is finite for any \( x_0, \ldots, x_{d-1} \in X \).

**Proof.** Let \( P \) be the proper minimal rational parabolic subgroup associated with \( c \). Let \( \varphi : \Delta^{d-1} \to X \) be a parametrization of \( \text{str}(x_0, \ldots, x_{d-1}) \). Choose a coordinate system \( s = (s_1, \ldots, s_{d-1}) \) in \( \Delta^{d-1} \). In the rational horocyclic coordinates of \( X = N_P \times A_P \times X_P \cong \mathbb{R}^r \times \mathbb{R} \times \mathbb{R}^l \), we can write \( \varphi(s) = (x(s), y(s), z(s)) \). Define a map \( \psi : \Delta^{d-1} \times [0, \infty) \to X \) by

\[
\psi(s, t) = (x(s), y(s) + t, z(s)).
\]

A line \( x(s) \times \mathbb{R}^+ \times z(s) \) is a geodesic representing \( c \) for any \( s \in \Delta^{d-1} \). Hence, it is easy to see that \( \psi \) is a parametrization of \( \text{str}(x_0, \ldots, x_{d-1}, c) \).

Denote by \( G(w_1, \ldots, w_k) \) the Gram determinant of \( w_1, \ldots, w_k \in \mathbb{R}^d \). It is a standard fact that \( \sqrt{G(w_1, \ldots, w_k)} \) is the \( k \)-dimensional volume of the parallelogram with edges \( w_1, \ldots, w_k \). We abbreviate \( \frac{\partial \psi}{\partial s_1}, \ldots, \frac{\partial \psi}{\partial s_{d-1}} \) as \( \frac{\partial \psi}{\partial s} \).
Then
\[
\psi^*dvol_X(s,t) = h(x(s),z(s))e^{-2\|\rho_P\|(y(s)+t)} \sqrt{G\left(\frac{\partial \psi}{\partial s}(s,t), \frac{\partial \psi}{\partial t}(s,t)\right)} ds_1 \cdots ds_{d-1} dt
\]
\[
= h(x(s),z(s))e^{-2\|\rho_P\|(y(s)+t)} \sqrt{G\left(\frac{\partial \varphi}{\partial s}(s), \frac{\partial \varphi}{\partial y}(\psi(s,t))\right)} ds_1 \cdots ds_{d-1} dt
\]
\[
\leq h(x(s),z(s))e^{-2\|\rho_P\|(y(s)+t)} \sqrt{G\left(\frac{\partial \varphi}{\partial s}(s)\right)} ds_1 \cdots ds_{d-1} dt
\]

The last inequality follows from \(\|\frac{\partial}{\partial y}(\psi(s,t))\| = 1\). Hence, we have
\[
\text{Vol} (\text{str}(x_0, \ldots, x_{d-1}, c)) = \int_{\Delta_d} \psi^*dvol_X ds_1 \cdots ds_{d-1} dt
\]
\[
\leq \int_{\Delta_d} h(x(s),z(s))e^{-2\|\rho_P\|y(s)} \sqrt{G\left(\frac{\partial \varphi}{\partial s}(s)\right)} ds_1 \cdots ds_{d-1} \cdot \int_0^\infty e^{-2\|\rho_P\|t} dt
\]
\[
= \text{Vol} (\text{str}(x_0, \ldots, x_{d-1})) \cdot \frac{1}{2\|\rho_P\|} < \infty.
\]
This completes the proof. \(\square\)

5. **Cuspidal completion**

In this section, we will define the notion of disjoint cone for \(M\) and the cuspidal completion of the classifying space \(B\Gamma\), following [21, Section 4.2.1].

As we mentioned before, one can identify each component \(\partial_i M\) with the quotient \(\Gamma_i \backslash H_i\) where \(H_i\) is the horosphere that bounds a horoball \(B_i = N\Gamma_i \times A\Gamma_i \times X\Gamma_i\). Note that such \(B_i\)'s are disjoint in \(\mathbb{Q}\)-rank 1 case. Hence we have a homeomorphism of tuples

\[(M, \partial_1 M, \ldots, \partial_i M) \rightarrow (\Gamma \backslash (X - \bigcup_{i=1}^s \Gamma B_i), \Gamma_1 \backslash H_1, \ldots, \Gamma_s \backslash H_s).\]

5.1. **Disjoint cone of topological spaces.** For a topological space \(Y\) and subspaces \(A_1, \ldots, A_s\) one can define a disjoint cone

\[\text{Dcone}(\bigcup_{i=1}^s A_i \rightarrow Y)\]

by coning each \(A_i\) to a point \(c_i\). In other words, \(\text{Dcone}(\bigcup_{i=1}^s A_i \rightarrow Y)\) is the space obtained by gluing \(Y\) and \(\bigcup_{i=1}^s \text{Cone}(A_i)\) along \(\bigcup_{i=1}^s A_i\).

**Lemma 5.1.** Let \(M\) be a compact, connected, smooth, oriented manifold with boundary \(\partial M\). Then there is an isomorphism

\[H_* (M, \partial M) \cong H_* (\text{Dcone}(\bigcup_{i=1}^s \partial_i M \rightarrow M))\]

in degrees \(* \geq 2\). In particular, \(H_d (\text{Dcone}(\bigcup_{i=1}^s \partial_i M \rightarrow M), \mathbb{R}) \cong \mathbb{R}\) if \(d = \dim (M) \geq 2\).
Proof. It is a well-known consequence of Morse theory that \((M, \partial M)\) is homotopy equivalent to a pair of CW-complexes. Since homology is preserved under homotopy equivalences, we can henceforth assume for the proof that \((M, \partial M)\) is a pair of CW-complexes. In particular [6, VII. Corollary 1.4] the inclusion \(j : \partial M \to M\) is a cofibration.

Let \(K_j\) be the mapping cone of \(j\) with vertex \(c\). Then [6, VII. Corollary 1.7] implies that we have isomorphisms

\[
H_*(K_j, v) \cong H_*(M/\partial M) \cong H_*(M, \partial M) ,
\]

thus \(H_*(K_j) \cong H_*(M, \partial M)\) for \(* \geq 1\).

Let \(c_i \in \text{Dcone} (\bigcup_{i=1}^s \partial_i M \to M)\) be the vertex of \(\text{Cone} (\partial_i M)\) for each \(i = 1, \ldots, s\). The projection \(\text{Dcone} (\bigcup_{i=1}^s \partial_i M \to M) \to K_j\) is a cellular map which maps \(c_1, \ldots, c_s\) to \(c\) and is an isomorphism of cellular chain groups in degree \(\geq 1\). Hence it induces an isomorphism of cellular homology in degree \(* \geq 2\). It is well known that cellular and singular homology of a pair of CW-complexes agree, thus the claim follows. \(\square\)

The argument actually provides an isomorphism

\[
H_*(\text{Dcone}(\bigcup_{i=1}^s A_i \to Y)) = H_*(Y, \bigcup_{i=1}^s A_i)
\]

for \(* \geq 2\) whenever \(Y\) and \(A_i\) are CW-complexes.

5.2. Disjoint cone of simplicial sets. For a simplicial set \((S, \partial S)\) and a symbol \(c\), the cone over \(S\) with the cone point \(c\) is the quasisimplicial set \(\text{Cone}(S)\) whose \(k\)-simplices are either \(k\)-simplices in \(S\) or cones over \((k-1)\)-simplices in \(S\) with the cone point \(c\). The boundary operator \(\partial\) in \(\text{Cone}(S)\) is defined by \(\partial \sigma = \partial S \sigma\) and

\[
\partial \text{Cone}(\sigma) = \text{Cone}(\partial S \sigma) + (-1)^{\dim(\sigma)+1} \sigma
\]

for \(\sigma \in S\).

If \(\{T_i \mid i \in I\}\) is a family of simplicial subsets of \(S\) indexed over a set \(I\), then define the quasisimplicial set \(\text{Dcone}(\bigcup_{i \in I} T_i \to S)\) as the pushout

\[
\begin{array}{ccc}
\bigcup_{i \in I} T_i & \to & S \\
\downarrow & & \downarrow \\
\text{Dcone} \left( \bigcup_{i \in I} T_i \right) & \to & \text{Dcone} \left( \bigcup_{i \in I} T_i \to S \right)
\end{array}
\]

Recall that in Section 2.1 we defined the simplicial set \(BG\) for a group \(G\). Now let us consider \(X = G/K\) a symmetric space of noncompact type and \(\Gamma \subset G\) a lattice. We define the cuspidal completion \(BG^{\text{comp}}\) of \(BG\) to be

\[
\text{Dcone} \left( \bigcup_{c \in \partial \infty X} BG \to BG \right).
\]
In addition, define the \textit{cuspidal completion} $B\Gamma^{\text{comp}}$ of $B\Gamma$ to be
\[
D_{\text{cone}} \left( \bigcup_{i=1}^{s} B\Gamma_i \to B\Gamma \right),
\]
where $\Gamma_i$ are parabolic groups. More precisely, $B\Gamma^{\text{comp}}$ is the quasisimplicial set whose $k$-simplices $\sigma$ are either of the form
\[
\sigma = (\gamma_1, \ldots, \gamma_k)
\]
with $\gamma_1, \ldots, \gamma_k \in \Gamma$ or for some $i \in \{1, \ldots, s\}$ of the form
\[
\sigma = (p_1, \ldots, p_{k-1}, c_i)
\]
with $p_1, \ldots, p_{k-1} \in \Gamma_i$.

\textbf{Lemma 5.2.} Let $M$ be a compact, connected, smooth, oriented manifold with boundary $\partial M$. Then there is an isomorphism
\[
H_\ast (M, \partial M) \cong H_\ast^{\text{simp}} (D_{\text{cone}} (\bigoplus_{i=1}^{s} C_\ast (\partial_i M) \to C_\ast (M)))
\]
in degrees $\geq 2$. In particular $H_\ast^{\text{simp}} (D_{\text{cone}} (\bigoplus_{i=1}^{s} C_\ast (\partial_i M) \to C_\ast (M) ; \mathbb{R})) \cong \mathbb{R}$ if $d = \text{dim} (M) \geq 2$.

\textit{Proof.} For a simplicial set $S$, we denote by $|S|$ the geometric realisation of $S$. One can think of $C_\ast (\partial_i M), C_\ast (M)$ as simplicial sets. Note that we have a natural isomorphism between the simplicial homology of the simplicial set and the singular homology of its geometric realisation. Thus to derive \textbf{Lemma 5.2} from \textbf{Lemma 5.1} it is sufficient to provide an isomorphism
\[
H_\ast (D_{\text{cone}} (\bigcup_{i=1}^{s} \partial_i M \to M)) \cong H_\ast (|D_{\text{cone}} (\bigoplus_{i=1}^{s} C_\ast (\partial_i M) \to C_\ast (M))|).
\]

There is a natural Mayer-Vietoris sequence for CW-complexes (see the remark after [6, Prop. A.5]), hence the canonical continuous map
\[
|D_{\text{cone}} (\bigoplus_{i=1}^{s} C_\ast (\partial_i M) \to C_\ast (M))| \to D_{\text{cone}} (\bigcup_{i=1}^{s} \partial_i M \to M)
\]
yields the following commutative diagram:
We note that $|\text{Cone} (C_* (\partial_{l} M))|$ and $\text{Cone} (\partial_{l} M)$ are contractible, hence their homology vanishes in degrees $\geq 1$. Moreover $H_* (|\text{Cone} (C_* (M))|) \to H_* (M)$ and $H_* (|\text{Cone} (\partial_{l} M)|) \to H_* (\partial_{l} M)$ are isomorphisms: this follows from [18, Theorem 2.27] together with the fact that $H_*(X)$ is by definition the same as $H_*^{\text{imp}} (C_* (X))$ for any topological space $X$.

Thus the five lemma implies the wanted isomorphism

$$H_* (|\text{Dcone} (\bigoplus_{i=1}^{s} C_* (\partial_{l} M) \to C_* (M))|) \to H_* (\text{Dcone} (\bigcup_{i=1}^{s} \partial_{l} M \to M)).$$

In the sequel we will consider the situation that two points $x_i, x$ in $X := \text{Dcone} (\bigoplus_{i=1}^{s} C_* (\partial_{l} M) \to C_* (M))$ are connected by a 1-simplex $e_i$ with $\partial e_i = x - x_i$. For a 1-simplex $\sigma$ with both vertices in $x_i$ we can define "conjugation with $e_i$" by $C_{e_i} (\sigma) := \overline{e_i} \ast \sigma \ast e_i$. In particular, for $x_i \in \partial_{l} M$ and if $\pi_1 (\partial_{l} M, x_i) \to \pi_1 (M, x_i)$ is injective, then $C_{e_i}$ realizes an isomorphism of $\pi_1 (\partial_{l} M, x_i)$ to a subgroup $\Gamma_i \subset \pi_1 (M, x)$. For a 1-simplex $\sigma$ with $\partial \sigma = c_i - x_i$ we define $C_{e_i} (\sigma) := \overline{e_i} \ast \sigma$.

**Definition 5.3.** Let $(M, \partial M)$ be a pair of topological spaces, $\partial_{l} M, \cdots, \partial_{s} M$ be the path components of $\partial M$. Denote by $c_i \in \text{Dcone} (\bigcup_{j=1}^{s} \partial_{l} M \to M)$ the vertex of $\text{Cone} (\partial_{l} M)$ for $i = 1, \ldots, s$. Let $x \in M, x_1 \in \partial_{l} M, \ldots, x_s \in \partial_{s} M$. For $i \in \{1, \ldots, s\}$ we define

$$\tilde{C}^{x_i} (\partial_{l} M) \subset \text{Cone} (C_* (\partial_{l} M)) \subset C_* (\text{Dcone} (\bigcup_{i=1}^{s} \partial_{l} M \to M))$$

to be the subcomplex freely generated by those simplices in $\text{Cone} (\partial_{l} M)$ for which

- either all vertices are in $x_i$,
- or all but the last vertex is in $x_i$ and the last vertex is in $c_i$.

For $i = 1, \ldots, s$ fix a path $e_i$ from $x$ to $x_i$ and the corresponding $C_{e_i}$. Define

$$\tilde{C}^{x_i} (M) \subset C_* (\text{Dcone} (\bigcup_{i=1}^{s} \partial_{l} M \to M))$$

to be the subcomplex freely generated by those simplices $\sigma$ for which

- either all vertices are in $x$,
- or for some $i \in \{1, \ldots, s\}$ there exists a simplex $\sigma' \subset \tilde{C}^{x_i} (\partial_{l} M)$ such that $C_{e_i}$ maps the 1-skeleton of $\sigma'$ to the 1-skeleton of $\sigma$ (up to homotopy fixing the 0-skeleton).

We remark that in the last case the homotopy classes (rel. $\{0, 1\}$) of all edges between all but the last vertices belong to $\Gamma_i \subset \pi_1 (M, x)$.

For the statement of the following lemma we will denote by

$$j_1 : \tilde{C}^{x} (M) \to C_* (\text{Dcone} (\bigcup_{i=1}^{s} \partial_{l} M \to M))$$

and

$$j_2 : \text{Dcone} (\bigoplus_{i=1}^{s} C_* (\partial_{l} M) \to C_* (M)) \to C_* (\text{Dcone} (\bigcup_{i=1}^{s} \partial_{l} M \to M))$$

the inclusions.
Lemma 5.4. Let $M$ be a compact, connected, smooth, oriented manifold with boundary $\partial M$. Let $x \in M$. Then there exists a chain map

$$F : C_* (\text{Dcone} (\bigoplus_{i=1}^n C_* (\partial_i M) \to C_* (M))) \to \tilde{C}^{\alpha}_* (M)$$

such that $j_1 \circ F$ is chain homotopic to $j_2$.

Proof. To write out the claim of the theorem: we want to show that there exist sequences of chain maps

$$F_n : C_n (\text{Dcone} (\bigoplus_{i=1}^n C_* (\partial_i M) \to C_* (M))) \to \tilde{C}^{\alpha}_n (M)$$

and of chain homotopies $K_n : C_n (\text{Dcone} (\bigoplus_{i=1}^n C_* (\partial_i M) \to C_* (M))) \to C_{n+1} (\text{Dcone} (\cup_{i=1}^n \partial_i M \to M))$ for $n = 0, 1, 2, \ldots$ such that

$$\partial K_n (\sigma) + K_{n-1} (\partial \sigma) = F_n (\sigma) - \sigma$$

for all $\sigma \in C_n (\text{Dcone} (\bigoplus_{i=1}^n C_* (\partial_i M) \to C_* (M)))$.

We will use the procedure for dividing $\Delta^n$ into $(n + 1)$-simplices which is described in [18, page 112]. So for each $n \in \mathbb{N}$ we let $v_{n,0}, \ldots, v_{n,n}$ and $w_{n,0}, \ldots, w_{n,n}$ be the vertices of $\Delta^n \times \{0\}$ and $\Delta^n \times \{1\}$, respectively, and for $0 \leq j \leq n$, we denote by $\kappa_{n,j} : \Delta^{n+1} \to \Delta^n \times [0, 1]$ the affine $(n + 1)$-simplex with vertices $v_0, \ldots, v_j, w_j, \ldots, w_n$. We will inductively prove a slightly stronger statement as above, namely we will show that for each $n$-simplex $\sigma$ in $\text{Dcone} (\bigoplus_{i=1}^n C_* (\partial_i M) \to C_* (M))$ one can define a continuous map $L_\sigma : \Delta^n \times [0, 1] \to \text{Dcone} (\cup_{i=1}^n \partial_i M \to M)$ such that $K_n (\sigma)$ is given by $K_n (\sigma) = \sum_{j=0}^n L_\sigma \circ \kappa_{n,j}$ (and of course that the so defined $K_n$ satisfies the above properties).

Let us first consider $n = 0$. A 0-simplex $\sigma$ in $\text{Dcone} (\bigoplus_{i=1}^n C_* (\partial_i M) \to C_* (M))$ is either a 0-simplex in $M$ or a cone point $c_i$.

If $\sigma = c_i$, then we define $F_0 (c_i) = c_i$ and $K_0 (c_i)$ is the 1-simplex mapped constantly to $c_i$.

If the 0-simplex $\sigma$ belongs to $M - \partial M$, then we define $F_0 (\sigma) = x$ and $K_0 (\sigma)$ is some (arbitrarily chosen) 1-simplex in $M \subset \text{Dcone} (\cup_{i=1}^n \partial_i M \to M)$ with $\partial_0 K_0 (\sigma) = x$ and $\partial_1 K_0 (\sigma) = \sigma$.

If the 0-simplex $\sigma$ belongs to $\partial_i M$, then we first choose some 1-simplex $e_\sigma$ in $\partial_i M$ with $\partial_0 e_\sigma = x_i, \partial_1 e_\sigma = \sigma$. (If $\sigma = x_i$, we let $e_\sigma$ be the constant 1-simplex.) Recall from Definition 2.3 that we have fixed a path $c_i$ from $x_i$ to $x$ which yields the isomorphism between $\pi_1 (\partial_i M, x_i)$ and $\Gamma_i$ by conjugation. Define then $F_0 (\sigma) = x$ and $K_0 (\sigma)$ is the 1-simplex obtained as concatenation of $e_\sigma$ and $c_i$. In particular $\partial_0 K_0 (\sigma) = x$ and $\partial_1 K_0 (\sigma) = \sigma$.

Let us now consider $n = 1$. A 1-simplex $\sigma$ in $\text{Dcone} (\bigoplus_{i=1}^n C_* (\partial_i M) \to C_* (M))$ is either a 1-simplex in $M$ or the cone (with cone point $c_i$) over a 0-simplex in $\partial_i M$. We have defined $K_0 (\partial_1 \sigma)$ and $K_0 (\partial_0 \sigma)$. Inclusion $\partial \Delta^1 \rightarrow...
$\Delta^1$ is a cofibration, hence we have a continuous map $L_\sigma : \Delta^1 \times [0,1] \to \text{Dcone } (\cup_{i=1}^n \partial_i M \to M)$ such that $L_\sigma (x,0) = x$ for $x \in \sigma$ and $L_\sigma (\partial_j \sigma, t) = K_0 (\partial_j \sigma) (t)$ for $j = 0,1$. Then define $K_1 (\sigma) := L_\sigma \circ \kappa_{1,0} + L_\sigma \circ \kappa_{1,1}$ and $F_1 (\sigma)$ by $F_1 (\sigma) (x) := L_\sigma (x,1)$ for $x \in \Delta^1$.

It is clear that $\partial K_1 (\sigma) + K_0 (\partial \sigma) = F_1 (\sigma) - \sigma$ and that $F_0 (\partial \sigma) = \partial F_1 (\sigma)$.

We still have to check that $F_1 (\sigma) \in \hat{C}^{\tau^e}_1 (M)$. If $\sigma$ is the cone over a simplex in $\partial M$, then $\partial_0 \sigma = c_i$ and $\partial_1 \sigma \in \partial_i M \subset M$, hence $F_0 (\partial_0 \sigma) = c_i$ and $F_0 (\partial_1 \sigma) = x$, thus $F_1 (\sigma) \in \hat{C}^{\tau^e}_1 (M)$. If $\sigma \in C_1 (M)$, then $\partial_i F_1 (\sigma) = F_0 (\partial_i \sigma) = x$ for $j = 0,1$, thus $F_1 (\sigma) \in \hat{C}^{\tau^e}_1 (M)$. Moreover (this will be needed in the next steps) if $\sigma \in C_1 (\partial_i M)$, then the homotopy class (rel. \{0,1\}) of $F_1 (\sigma)$ belongs to $\Gamma_i \subset \pi_1 (M,x)$. Indeed, $F_1 (\sigma)$ is in the homotopy class (rel. \{0,1\}) of $K_0 (\partial_1 \sigma) * \sigma * K_0 (\partial_0 \sigma) = \sum_{i} e_i * e_{\partial_i \sigma} * \sigma * e_{\partial_0 \sigma} * e_i$, where the bar means the 1-simplex with opposite orientation. Now $e_{\partial_0 \sigma}$ represents an element in $\pi_1 (\partial_i M, x_i)$ and by assumption conjugation with $e_i$ provides to isomorphism to $\Gamma_i$, hence $F_1 (\sigma)$ represents an element in $\Gamma_i$.

We now proceed to prove the theorem by induction. Assume that $F_k$ and $K_k$ have been defined for $k \leq n$. We will assume as part of the inductive hypothesis (and prove as part of the induction claim) that $F_n (\sigma)$ has all vertices in $x$ if $\sigma \in C_1 (M)$ and that $F_n (\sigma)$ has its last vertex in $c_i$ if $\sigma$ has its last vertex in $c_i$. (This is satisfied for $n \leq 1$ by the above construction.)

Let $\sigma : \Delta^{n+1} \to \text{Dcone } (\cup_{i=1}^n \partial_i M \to M)$ be an $(n+1)$-simplex in $\text{Dcone } (\oplus_{i=1}^n C_1 (\partial_i M) \to C_1 (M))$. By the inductive hypothesis we have for $j = 0, \ldots, n+1$ a continuous map $L_{\partial_j \sigma} : \Delta^n \times [0,1] \to \text{Dcone } (\cup_{i=1}^n \partial_i M \to M)$ such that $K_n (\partial_j \sigma)$ is given by $K_n (\partial_j \sigma) = \sum_{l=0}^n L_{\partial_j \sigma} \circ \kappa_{n,j,l}$. (In particular $L_{\partial_j \sigma} (x,0) = x$ for $x \in \partial_j \Delta^n$.) Since the inclusion $\partial \Delta^n \to \Delta^n$ is a cofibration by [6, VII. Corollary 1.4] we have a continuous map $L_\sigma : \Delta^{n+1} \times [0,1] \to \text{Dcone } (\cup_{i=1}^n \partial_i M \to M)$ such that $L_\sigma |_{\Delta^{n+1} \times \{0\}}$ agrees with $\sigma$ (after the obvious identification of $\Delta^{n+1}$ with $\Delta^{n+1} \times \{0\}$) and for $j = 0, \ldots, n+1$ $L_\sigma |_{\partial_j \Delta^{n+1} \times \{0,1\}}$ agrees with $L_{\partial_j \sigma}$. Then define

$$K_{n+1} (\sigma) := \sum_{j=0}^{n+1} L_\sigma \circ \kappa_{n+1,j}$$

and

$$F_{n+1} (\sigma) := L \circ \tau_{n+1},$$

where $\tau_{n+1} : \Delta^{n+1} \to \Delta^{n+1} \times [0,1]$ is defined by $\tau_{n+1} (x) = (x,1)$.

It is clear by construction that $\partial K_{n+1} (\sigma) + K_n (\partial \sigma) = F_{n+1} (\sigma) - \sigma$ and that $\partial F_{n+1} (\sigma) = F_n (\partial \sigma)$.

We have to check that $F_{n+1} (\sigma) \in \hat{C}_n^{\tau^e} (M)$. If $\sigma$ is an $(n+1)$-simplex in $M$, then all $\partial_j \sigma$ are $n$-simplices in $M$, hence by induction all vertices of all $F_j (\partial_j \sigma)$ are in $x$. Because of $\partial_j F_{n+1} (\sigma) = F_n (\partial_j \sigma)$ this implies that all vertices of $F_{n+1} (\sigma)$ are in $x$, hence $F_{n+1} (\sigma) \in \hat{C}_n^{\tau^e} (M)$. 

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If $\sigma$ is the cone (with cone point $c_i$) over an $n$-simplex $\tau = \partial_n \sigma$, then we have by inductive hypothesis that $F_n (\partial_{n+1} \sigma)$ has all its vertices in $x$ and moreover that all $F_n (\partial_j \sigma)$ with $0 \leq j \leq n$ have their last vertex in $c_i$. Because of $\partial_j F_n (\sigma) = F_n (\partial_j \sigma)$ this implies that $F_{n+1} (\sigma)$ has its last vertex in $c_i$ and the remaining vertices in $x$. Moreover, if $n+1 = 2$, then $\partial_2 \sigma \in C_1 (\partial_i M)$ and it follows (from the construction for $n = 1$) that the homotopy class (rel. $\{0, 1\}$) of $\partial_2 F_2 (\sigma) = F_1 (\partial_2 \sigma)$ belongs to $\Gamma_i \subset \pi_1 (M, x)$.

If $n+1 \geq 3$, then, since each edge of $\sigma$ is an edge of some $\partial_j \sigma$ and since $F_n (\partial_j \sigma) \in \hat{C}_n^x (M)$, it follows that the homotopy classes (rel. $\{0, 1\}$) of all edges between all but the last vertices belong to $\Gamma_i \subset \pi_1 (M, x)$. Thus $F_{n+1} (\sigma) \in \hat{C}_{n+1}^x (M)$.

\[ \square \]

**Corollary 5.5.** Let $M$ be a compact, connected, smooth, oriented, aspherical manifold with aspherical $\pi_1$-injective boundary $\partial M = \partial_1 M \cup \ldots \cup \partial_s M$. Let $x \in M$. Assume $\Gamma_i \cap \Gamma_j = 0$ for $i \neq j$, where $\Gamma_i \subset \pi_1 (M, x)$ for $i = 1, \ldots, s$ is defined by Definition 5.3. Then the chain map

$$F : \text{Dcone} \left( \bigoplus_{i=1}^s C_* (\partial_i M) \to C_* (M) \right) \to \hat{C}_*^x (M)$$

induces an isomorphism of homology groups.

**Proof.** Lemma 5.4 implies that $j_{1*} F_* = j_{2*}$ and Lemma 5.2 implies that $j_{2*}$ is an isomorphism. Hence $F_*$ is injective. It remains to show that $F_*$ is surjective, i.e., that every cycle in $\hat{C}_*^x (M)$ is homologous to some cycle of the form $F_* z$ with $z$ a cycle in $\text{Dcone} \left( \bigoplus_{i=1}^s C_* (\partial_i M) \to C_* (M) \right)$.

Let $\sum_{j=1}^r a_j \sigma_j \in \hat{C}_*^x (M)$ be a cycle. Let

$$J^{deg} = \{ j : \sigma_j \text{ has an edge representing } 0 \in \pi_1 (M, x) \} .$$

The same argument as in the proof of [21, Lemma 5.15] shows that $\sum_{j \in J^{deg}} a_j \sigma_j$ is a 0-homologous cycle, thus $\sum_{j \notin J^{deg}} a_j \sigma_j$ is homologous to $\sum_{j=1}^r a_j \sigma_j$. We can and will therefore without loss of generality assume that no $\sigma_j$ has an edge representing $0 \in \pi_1 (M, x)$.

Let $c_1, \ldots, c_s$ be the cone points and for $i \in \{1, \ldots, s\}$ let

$$J_i = \{ j : \sigma_j \text{ has its last vertex in } c_i \} .$$

We note that for $i \neq l$ a simplex in $J_i$ can not have a face in common with a simplex in $J_l$. Indeed such a face would have elements in $\Gamma_i \subset \pi_1 (M, x)$ and $\Gamma_l \subset \pi_1 (M, x)$ which is impossible because of $\Gamma_i \cap \Gamma_l = \emptyset$.

Now let $K$ be the simplicial complex defined as a union $K = \Delta_1 \cup \ldots \cup \Delta_s$ of homeomorphic images of the $d$-dimensional standard simplex with identifications $\partial_i \Delta_j = \partial_k \Delta_l$ if and only if $\partial_i \sigma_j = \partial_k \sigma_l$. Let $\sigma : K \to \text{Dcone} \left( \bigcup_{i=1}^s \partial_i M \to M \right)$ be defined by $\sigma |_{\Delta_j} = \sigma_j$, where the homeomorphism from $\Delta_j$ to the standard simplex is understood. By construction,

$$\sum_{j=1}^r a_j \sigma_j = \sigma_* \left( \sum_{j=1}^r a_j \Delta_j \right) .$$

We will now homotope $\sigma$ such that its image becomes a chain in
Dcone ($\bigoplus_{i=1}^{s} C_\ast (\partial_i M) \to C_\ast (M)$). First, if $j \in J_i$, then we homotope all but the last vertex of $\Delta_j$ from $x$ to $x_i$ along the path $e_i$ from Definition 5.3. Since for $i \neq l$ simplices in $J_i$ and $J_l$ have no face in common this can be done simultaneously for all $\Delta_j$ with $j \in J_1 \cup \ldots \cup J_s$. By successive application of the cofibration property this homotopy can be extended to all of $K$. For $j \in J_i$ it follows from the definition of $\Gamma_i$ in Definition 5.3 that after this homotopy the edges of $\Delta_j$ opposite to the cone point are all mapped to loops at $x_i$ homotopic rel. $\{0,1\}$ into $\partial_i M$. We may thus (using again the cofibration property to successively extend the homotopy from the 1-skeleton to $K$) further homotope $\sigma$ to have all these edges in $\partial_i M$, and the remaining edges of $\Delta_i$ mapped to $\text{Cone}(\partial_i M)$. Finally, since $M$ and $\partial_i M$ are aspherical we have $\pi_{s\geq 2}(M,\partial M) = 0$, thus we can successively further homotope $\sigma$ such that:

- for $j \in J_i$ all higher-dimensional subsimplices and finally $\Delta_j$ are mapped to $\partial_i M$ (if they don’t contain the cone point) or to $\text{Cone}(\partial_i M)$ (if they do contain the cone point).

- for $j \not\in J_1 \cup \ldots \cup J_s$ all higher-dimensional subsimplices and finally $\Delta_j$ are mapped to $M$.

Thus we obtain a cycle $c$ in $\text{Dcone} (\bigoplus_{i=1}^{s} C_\ast (\partial_i M) \to C_\ast (M))$. By construction $F_\ast c$ is homotopic, hence homologous, to $\sum_{j=1}^{r} a_j \sigma_j$.

\[ \text{Corollary 5.6.} \] Under the assumptions of Corollary 5.5 we have

\[ H_d \left( \tilde{C}_\ast^{\infty} (M); \mathbb{R} \right) \cong \mathbb{R} \]

for $d = \text{dim}(M) \geq 2$ and $x \in M$.

6. Eilenberg-MacLane map

Recall the homeomorphism of tuples as we describe in Section 5,

\[ (M, \partial_1 M, \ldots, \partial_s M) \to (\Gamma \setminus (X - \bigcup_{i=1}^{s} \Gamma B_i), \Gamma \setminus H_1, \ldots, \Gamma \setminus H_s) . \]

Let $c_i$ denote the cone point of $\text{Cone}(\partial_i M)$. Identifying each $\text{Cone}(\partial_i M) - c_i$ with $\Gamma \setminus B_i$, we have a homeomorphism

\[ \Gamma \setminus X \to \text{Dcone}(\bigcup_{i=1}^{s} \partial_i M \to M) - \{c_1, \ldots, c_s\} \]

extending the homeomorphism of tuples above. Composition of the universal covering $X \to \Gamma \setminus X$ with this homeomorphism yields a covering map

\[ X \to \text{Dcone}(\bigcup_{i=1}^{s} \partial_i M \to M) - \{c_1, \ldots, c_s\} . \]

Then we finally have a projection map

\[ \pi : X \cup \bigcup_{i=1}^{s} \Gamma \partial_{\infty} B_i \to \text{Dcone}(\bigcup_{i=1}^{s} \partial_i M \to M) \]

such that $\pi|_X : X \to \text{Dcone}(\bigcup_{i=1}^{s} \partial_i M \to M) - \{c_1, \ldots, c_s\}$ is a covering, $\pi|_{B_i} : \Gamma B_i \to \text{Cone}(\partial_i M) - C_i$ is a covering with deck group $\Gamma$ and $\pi$ maps $\Gamma \partial_{\infty} B_i$ to $c_i$ for $i = 1, \ldots, s$. Due to this projection map, we can define the notion of (ideal) straight simplex in $\text{Dcone}(\bigcup_{i=1}^{s} \partial_i M \to M)$. 


Definition 6.1. We say that a $k$-simplex $\sigma$ in $\text{Dcone}(\bigcup_{i=1}^{n} \partial_i M \to M)$ is straight if $\sigma$ is of the form $\pi(\text{str}(u_0, \ldots, u_k))$ for $u_0, \ldots, u_k \in X \cup \bigcup_{i=1}^{n} \Gamma \partial_{\infty} B_i$.

Remark. In the $\mathbb{R}$-rank 1 case, every $\partial_{\infty} B_i$ consists of a point in $\partial_{\infty} X$ and for any ordered pair $(u_0, \ldots, u_k) \in X \cup \bigcup_{i=1}^{n} \Gamma \partial_{\infty} B_i$, $\text{str}(u_0, \ldots, u_k)$ is well defined. In contrast, in the higher rank case, each $\partial_{\infty} B_i$ is not a point. More precisely, if $B_i$ is any horoball centered at $z_i$, then

$$\partial_{\infty} B_i = \left\{ w \in \partial_{\infty} X \mid Td(z_i, w) \leq \frac{1}{2} \pi \right\},$$

where $Td$ is the Tits metric on $\partial_{\infty} X$ (see [19]). Furthermore, $\text{str}(u_0, \ldots, u_k)$ may not be defined for some ordered pair $(u_0, \ldots, u_k)$.

We denote

$$\hat{C}_s(M) := C_*(\text{Dcone}(\bigcup_{i=1}^{n} \partial_i M \to M)).$$

Recall from Definition 5.3 that we choose base points $x_0, x_1, \ldots, x_s$ of $M$, $\partial_1 M, \ldots, \partial_s M$ respectively and identify $\pi_1(\partial_i M, x_i)$ with a subgroup $\Gamma_i$ of $\pi_1(M, x_0)$ by choosing a path connecting $x_0$ and $x_i$ for $i = 1, \ldots, s$.

The assumptions of Corollary 5.6 are satisfied for $\mathbb{Q}$-rank 1 spaces, thus we have

$$H_d(\hat{C}_s^{x_0}(M), \mathbb{Q}) = H_d(\hat{C}_s(M), \mathbb{Q}) = H_d(M, \partial M, \mathbb{Q}) = \mathbb{Q}.$$

We define a chain complex

$$\hat{C}_s^{\text{str},x_0}(M) := \mathbb{Z}[\{ \sigma \in \hat{C}_s^{x_0}(M) \mid \sigma \text{ is straight} \}],$$

and for the $s$-tuple $(c_1, \ldots, c_s)$ with $c_i \in \partial_{\infty} B_i$ (see the setup in Section 4), we define the subcomplex $\hat{C}_s^{\text{str},x_0,c}(M)$ of $\hat{C}_s^{\text{str},x_0}(M)$ freely generated by those simplices that are either of the form

$$\sigma = \pi(\text{str}(\tilde{x}_0, \gamma_1 \tilde{x}_0, \ldots, \gamma_k \tilde{x}_0))$$

where $\tilde{x}_0$ is a lift of $x_0$ and $\gamma_1, \ldots, \gamma_k \in \Gamma$ or of the form

$$\sigma = \pi(\text{str}(\tilde{x}_0, p_1 \tilde{x}_0, \ldots, p_1 \cdots p_{k-1} \tilde{x}_0, c_i))$$

for $p_1, \ldots, p_{k-1} \in \Gamma_i$ and $i \in \{1, \ldots, s\}$.

Lemma 6.2. The following hold.

(a) There is an isomorphism of chain complexes

$$\Phi : \hat{C}_s^{\text{str},x_0,c}(M) \to C_*(\text{B} \Gamma_{\text{comp}}).$$

(b) The inclusion $\hat{C}_s^{\text{str},x_0,c}(M) \to \hat{C}_s(M)$ induces an isomorphism

$$H_d(\hat{C}_s^{\text{str},x_0,c}(M), \mathbb{Q}) \to H_d(\text{Dcone}(\bigcup_{i=1}^{n} \partial_i M \to M), \mathbb{Q}).$$

(c) The composition of $\Phi^{-1}$ with the inclusion $\hat{C}_s^{\text{str},x_0,c}(M) \to \hat{C}_s(M)$ induces an isomorphism

$$EM_d : H_d^{\text{simp}}(\text{B} \Gamma_{\text{comp}}, \mathbb{Q}) \to H_d(\text{Dcone}(\bigcup_{i=1}^{n} \partial_i M \to M), \mathbb{Q}).$$
**Proof.** (a) We define a chain isomorphism $\Phi : \hat{C}_s^\text{str,x0,c}(M) \to C_s^\text{simp}(B\Gamma_{\text{comp}})$ as follows: For a straight $k$-simplex $\sigma$ of the form

$$\sigma = \pi(\text{str}(\bar{x}_0, \gamma_1 \bar{x}_0, \ldots, \gamma_k \bar{x}_0)),$$

define $\Phi(\sigma) = (\gamma_1, \ldots, \gamma_k)$. For a straight $k$-simplex $\sigma$ of the form

$$\sigma = \pi(\text{str}(\bar{x}_0, p_1 \bar{x}_0, \ldots, p_{k-1} \bar{x}_0, c_i))$$

with $p_1, \ldots, p_{k-1} \in \Gamma_i$ and $i \in \{1, \ldots, s\}$, define $\Phi(\sigma) = (p_1, \ldots, p_{k-1}, c_i)$. Extending this linearly to $\hat{C}_s^\text{str,x0}(M)$, we obtain a chain map

$$\Phi : \hat{C}_s^\text{str,x0,c}(M) \to C_s^\text{simp}(B\Gamma_{\text{comp}}).$$

Conversely, one can define a chain map $\Psi : C_s^\text{simp}(B\Gamma_{\text{comp}}) \to \hat{C}_s^\text{str,x0,c}(M)$ as follows: For a $k$-simplex of the form $(\gamma_1, \ldots, \gamma_k)$, define

$$\Psi(\gamma_1, \ldots, \gamma_k) = \pi(\text{str}(\bar{x}_0, \gamma_1 \bar{x}_0, \ldots, \gamma_k \bar{x}_0)).$$

For a $k$-simplex of the form $(p_1, \ldots, p_{k-1}, c_i)$ with $p_1, \ldots, p_{k-1} \in \Gamma_i$, define

$$\Psi(p_1, \ldots, p_{k-1}, c_i) = \pi(\text{str}(\bar{x}_0, p_1 \bar{x}_0, \ldots, p_{k-1} \bar{x}_0, c_i)).$$

By construction, it is clear that $\Phi$ and $\Psi$ are inverse to each other. Hence, $\Phi$ and $\Psi$ are isomorphisms between the two chain complexes.

(b) Note that one can extend the domain of $\Phi$ to $\hat{C}_s^{x_0}(M)$ in the following way. Let $\sigma$ be a $k$-simplex in $\hat{C}_s^{x_0}(M)$ and $e_0, \ldots, e_k$ be the vertices of the standard simplex $\Delta^k$. For $i = 1, \ldots, k$, let $\zeta_i$ be the standard sub-1-simplex with $\partial \zeta_i = e_i - e_{i-1}$. Define

$$\Phi(\sigma) = ([\sigma|_\zeta_1], \ldots, [\sigma|_\zeta_k]),$$

where each $[\sigma|_\zeta_i] \in \Gamma = \pi_1(M, x_0)$ is the homotopy class of $\sigma|_\zeta_i$ if all vertices of $\sigma$ are in $x_0$ and

$$\Phi(\sigma) = ([\sigma|_\zeta_1], \ldots, [\sigma|_\zeta_{k-1}], c_i),$$

if the last vertex of $\sigma$ is $C_i$. Then consider the composition of maps as follows:

$$\hat{C}_s^{\text{str,x0,c}}(M) \xrightarrow{i} \hat{C}_s^{x_0}(M) \xrightarrow{\Phi} C_s^{\text{simp}}(B\Gamma_{\text{comp}}) \xrightarrow{\Psi} \hat{C}_s^{\text{str,x0,c}}(M)$$

Obviously, $\Psi \circ \Phi \circ i = id$ and thus we have an injective homomorphism

$$i_* : H_d(\hat{C}_s^{\text{str,x0,c}}(M), \mathbb{Q}) \to H_d(\hat{C}_s^{x_0}(M), \mathbb{Q}).$$

From Corollary 5.6 we obtain $H_d(\hat{C}_s^{x_0}(M), \mathbb{Q}) \cong \mathbb{Q}$, thus $H_d(\hat{C}_s^{\text{str,x0,c}}(M), \mathbb{Q})$ as a vector space over $\mathbb{Q}$ is at most one dimensional. On the other hand Lemma 6.3 below shows that $EM_d^{-1}[M, \partial M]$ is a nontrivial element in $H_d(B\Gamma_{\text{comp}}, \mathbb{R}) \cong H_d(\hat{C}_s^{\text{str,x0,c}}(M), \mathbb{R})$, because evaluation of some cocycle is not zero. Hence, $i_*$ is actually an isomorphism. Furthermore, considering
that by Corollary 5.5 the inclusion \( \hat{\mathcal{C}}^{x_0}_s(M) \to \hat{\mathcal{C}}_s(M) \) is a homology equivalence, one can conclude that the inclusion \( \hat{\mathcal{C}}^{x_0,c}_s(M) \to \hat{\mathcal{C}}_s(M) \) induces an isomorphism

\[
H_d(\hat{\mathcal{C}}^{x_0,c}_s(M), \mathbb{Q}) \to H_d(\text{Dcone}(\cup_{i=1}^s \partial_i M \to M), \mathbb{Q}).
\]

Finally, (c) follows from (a) and (b).

\[\square\]

**Remark.** In the \( \mathbb{R} \)-rank 1 case, the geodesic straightening map \( \text{str}_*: \hat{\mathcal{C}}^{x_0}_s(M) \to \hat{\mathcal{C}}^{x_0}_s(M) \) that is a left-inverse to the inclusion \( \hat{\mathcal{C}}^{x_0}_s(M) \subset \hat{\mathcal{C}}^{x_0}_s(M) \) is well defined and \( C^\text{simp}_s(B\Gamma^{\text{comp}}) \) is isomorphic to \( \hat{\mathcal{C}}^{x_0}_s(M) \). (See [21]). However, in the higher rank case, the straightening map \( \text{str}_*: \hat{\mathcal{C}}^{x_0}_s(M) \to \hat{\mathcal{C}}^{x_0}_s(M) \) is not well defined as we mentioned in Section 4. Hence \( C^\text{simp}_s(B\Gamma^{\text{comp}}) \) is not isomorphic to \( \hat{\mathcal{C}}^{x_0}_s(M) \) but is isomorphic to its subcomplex \( \hat{\mathcal{C}}^{x_0,c}_s(M) \). Despite such differences with \( \mathbb{R} \)-rank 1 case, we obtain the homology class

\[
EM_d^{-1}[M, \partial M] \in H^\text{simp}_d(B\Gamma^{\text{comp}}, \mathbb{Q})
\]
in the same manner as in the \( \mathbb{R} \)-rank 1 case. This will enable us in Section 7 to define an invariant in K-theory for a \( \mathbb{Q} \)-rank 1 locally symmetric space.

Recall that the volume cocycle \( \text{comp}(\nu_d) = cv_d \in C^d_{\text{simp}}(BG) \) is defined by

\[
cv_d(g_1, \ldots, g_d) = \int_{\text{str}(\tilde{x}_0, g_1 \tilde{x}_0, \ldots, g_d \tilde{x}_0)} d\text{vol}_X,
\]

where \( d\text{vol}_X \) is the \( G \)-invariant Riemannian volume form on \( X = G/K \). If \( X \) is a \( \mathbb{R} \)-rank 1 symmetric space, one can extend this volume cocycle \( cv_d \) to a cocycle \( \text{cv}_d \in C^d_{\text{simp}}(BG^{\text{comp}}) \). In the higher rank case, one cannot obtain an extended volume cocycle in \( C^d_{\text{simp}}(BG^{\text{comp}}) \). However, we can extend the volume cocycle in \( C^d_{\text{simp}}(B\Gamma) \) to at least a cocycle in \( C^d_{\text{simp}}(B\Gamma^{\text{comp}}) \). Define a cocycle \( \text{cv}_d \in C^d_{\text{simp}}(B\Gamma^{\text{comp}}) \) as follows: For a \( d \)-simplex \( (\gamma_1, \ldots, \gamma_d) \) with \( \gamma_1, \ldots, \gamma_d \in \Gamma \), define

\[
\text{cv}_d(\gamma_1, \ldots, \gamma_d) = cv_d(\gamma_1, \ldots, \gamma_d).
\]

For a \( d \)-simplex \( (p_1, \ldots, p_{d-1}, c_i) \) with \( p_1, \ldots, p_{d-1} \in \Gamma_i \) and \( i \in \{1, \ldots, s\} \), define

\[
\text{cv}_d(p_1, \ldots, p_{d-1}, c_i) = \int_{\text{str}(\tilde{x}_0, p_1 \tilde{x}_0, \ldots, p_{d-1} \tilde{x}_0, c_i)} d\text{vol}_X.
\]

It follows from Lemma 4.1 that \( \text{cv}_d \) is well defined. An application of Stokes’ Theorem (to compact submanifolds with boundary of every ideal simplex exactly as in the proof of Lemma 6.3 below) shows that \( \text{cv}_d \) is a cocycle in \( C^d_{\text{simp}}(B\Gamma^{\text{comp}}) \). Hence, the cocycle \( \text{cv}_d \) determines a cohomology class in \( H^d_{\text{simp}}(B\Gamma^{\text{comp}}) \), denoted by \( \text{cv}_d \).
Lemma 6.3. If $N$ is a $\mathbb{Q}$-rank 1 locally symmetric space of dimension at least 3, then
\[
\langle \mathcal{P}_d, EM_d^{-1}[M, \partial M] \rangle = \text{Vol}(N).
\]

Proof. Let $z$ be a relative fundamental cycle in $C_d(M, \partial M)$ representing the relative fundamental class $[M, \partial M]$. We think of $M$ as a submanifold of $N$ via the homeomorphism of tuples
\[
(M, \partial_1 M, \ldots, \partial_s M) \to (\Gamma \setminus (X - \cup_{i=1}^s \Gamma B_i), \Gamma_1 \setminus H_1, \ldots, \Gamma_s \setminus H_s)
\]
and $z$ as a chain in $C_d(N)$.

Since $\partial z$ represents $[\partial M]$, we can write $\partial z = \partial_1 z + \cdots + \partial_s z$ where $\partial_i z$ is a cycle representing $[\partial_i M]$ for $i = 1, \ldots, s$. Make $z$ a chain $z^0$ in $\tilde{C}^{x_0}_d(M)$ via the chain homotopy $\tilde{C}_s(M) \to \tilde{C}^{x_0}_d(M)$ in the proof of [21, Lemma 8]. Note that $z^0$ is obtained by adding several 1-dimensional paths to $z$ and hence
\[
\text{algvol}(z^0) = \text{algvol}(z) = \text{Vol}(M).
\]

Now, consider a geodesic cone $\text{Cone}_g(\partial_i z)$ over $\partial_i z$ with the top point $c_i$ for $i = 1, \ldots, s$. Due to $\dim \mathbb{A}_p = 1$, it is not difficult to see that
\[
\text{algvol}(\text{Cone}_g(\partial_i z)) = (-1)^{d+1} \text{Vol}(\Gamma_i \setminus B_i).
\]
Since $\text{Cone}_g(\partial_i z^0)$ is obtained by adding two dimensional objects of $N$ to $\text{Cone}_g(\partial_i z)$, its algebraic volume is not changed, that is,
\[
\text{algvol}(\text{Cone}_g(\partial_i z^0)) = \text{algvol}(\text{Cone}_g(\partial_i z)) = (-1)^{d+1} \text{Vol}(\Gamma_i \setminus B_i).
\]

Now, define $c(z^0) = z^0 + (-1)^{d+1} \text{Cone}_g(\partial z^0)$. Then it can be checked that $c(z^0)$ is a cycle in $\tilde{C}^{x_0}_d(M)$ by
\[
\partial c(z^0) = \partial z^0 + (-1)^{d+1} \partial \text{Cone}_g(\partial z^0)
\]
\[
= \partial z^0 + (-1)^{d+1} \text{Cone}_g(\partial \partial z^0) + (-1)^{d+1} (-1)^d \partial z^0 = 0.
\]
Furthermore, we have
\[
\text{algvol}(c(z^0)) = \text{Vol}(M) + \sum_{i=1}^s \text{Vol}(\Gamma_i \setminus B_i)
\]
\[
= \text{Vol}(M) + \text{Vol}(N - M) = \text{Vol}(N).
\]
Note that we can straighten $c(z^0)$ because all vertices of every ideal simplex in $c(z^0)$ are in $x_0$ except for the last vertex with $c_i$ for some $i \in \{1, \ldots, s\}$. Furthermore, $\text{str}(c(z^0))$ is in $\tilde{C}^{\text{str}, x_0}_d(M)$ and represents $\Psi \circ EM_d^{-1}[M, \partial M]$.

To prove the lemma, it is sufficient to show that
\[
\text{algvol}(\text{str}(c(z^0))) = \text{Vol}(N).
\]

Let $H_0, H_1 : K \to \text{Dcone}(\cup_{i=1}^s \partial_i M \to M)$ be the simplicial maps realising the cycles $c(z^0) + \text{Cone}_g(\partial z^0)$ and $\text{str}(c(z^0) + \text{Cone}_g(\partial z^0))$, respectively. (See the construction in the proof of 5.5.) Let $H : K \times [0, 1] \to \text{Dcone}(\cup_{i=1}^s \partial_i M \to M)$ be the straight line homotopy between $c(z^0) + \text{Cone}_g(\partial z^0)$ and $\text{str}(c(z^0) + \text{Cone}_g(\partial z^0))$, such that $H_0 = H(., 0), H_1 = H(., 1)$. 


The homotopy $H$ yields a chain homotopy $L_* : \tilde{C}^{str,x_0,c}_*(M) \to \tilde{C}^{str,x_0,c}_{*+1}(M)$ from the straightening map $str$ to the identity. (See the construction in the proof of 5.4.) This satisfies $\partial L_k + L_{k-1}\partial = str - id$. Then
\[
\begin{align*}
str(c(z^0)) - c(z^0) &= str(z^0) - z^0 + (-1)^{d+1}(\text{Cone}_g(str(\partial z^0)) - \text{Cone}_g(\partial z^0)) \\
&= \partial L_d(z^0) + L_{d-1}(\partial z^0) + (-1)^{d+1}\text{Cone}_g(\partial L_{d-1}(\partial z^0) + L_{d-2}(\partial \partial z^0)) \\
&= \partial L_d(z^0) + L_{d-1}(\partial z^0) + (-1)^{d+1}\text{Cone}_g(L_{d-1}(\partial z^0)) - L_{d-1}(\partial z^0) \\
&= \partial(L_d(z^0) + (-1)^{d+1}\text{Cone}_g(L_{d-1}(\partial z^0)))
\end{align*}
\]

In order to conclude $\text{algvol}(str(c(z^0))) = \text{algvol}(c(z^0))$ we want to apply Stokes’ Theorem to show that the integral of the volume form over $\partial(L_d(z^0) + (-1)^{d+1}\text{Cone}_g(L_{d-1}(\partial z^0)))$ vanishes. It is clear that the integral of the closed form $d\text{vol}$ over $\partial L_d(z^0)$ vanishes and thus it remains to look at simplices in $\partial\text{Cone}_g(L_{d-1}(\partial z^0))$.

Since the volume form is defined on the complement of the cone points we can apply Stokes’ Theorem to compact submanifolds with boundary of every (ideal) simplex in $\text{Let } \mathcal{H}_{ik}$ be a sequence of horospheres converging towards $c_i$ for $k \to \infty$. In the following we will call simplices in $\text{Cone}_g(\partial_i z)$ proper ideal simplices if they have a vertex in $c_i$, i.e. if they are not contained in $\partial_i z$. For a proper ideal simplex in $\text{Cone}_g(\partial_i z)$ its edges are either edges of a simplex in $\partial_i z$ or otherwise they are geodesics ending in $c_i$, which therefore are transverse to the horospheres $\mathcal{H}_{ik}$. Moreover all higher-dimensional proper ideal simplices in $\text{Cone}_g(\partial_i z^0)$ and their straightenings are a union of geodesic lines ending in $c_i$. In particular all proper ideal simplices occurring in $\text{Cone}_g(\partial_i z^0)$ and $\text{str}(\text{Cone}_g(\partial_i z^0))$ are transverse to the $\mathcal{H}_{ik}$’s.

The Relative Transversality Theorem (see [17]) yields that any map $H : K \times [0,1] \to \text{Dcone}(\bigcup_{i=1}^s \partial_i M \to M)$ whose restriction to $K \times \{0,1\}$ is transverse to $\bigcup_{i=1}^s \mathcal{H}_{ik}$ can be homotoped (by an arbitrarily small homotopy, keeping $K \times \{0,1\}$ fixed) to a map which is transverse on all of $K \times [0,1]$. The homotopy can be chosen to fix subsimplices which are already transverse. Thus we can homotope (keeping $c(z^0)$ and $\text{str}(c(z^0))$ as well as $L_d(z^0)$ fixed) the simplices in $\text{Cone}_g(L_{d-1}(\partial z^0))$ to be transverse to the $\mathcal{H}_{ik}$’s.

Then, for any $(d + 1)$-dimensional simplex
\[
\kappa : \Delta^{d+1} \to \text{Dcone}(\bigcup_{i=1}^s \partial_i M \to M) \cong N \cup \{\text{cusps}\}
\]
occurring in $\text{Cone}_g(L_{d-1}(\partial z^0))$, and for each $k \in \mathbb{N}$, we conclude from transversality that $\kappa^{-1}(\mathcal{H}_{ik})$ is a $d$-dimensional submanifold $K_{ik}$ bounding a $(d + 1)$-dimensional submanifold $\Omega_{ik} \subset \Delta^{d+1}$ which does not contain the preimage of $c_i$. Thus we can apply Stokes’ Theorem to $\Omega_{ik}$ and obtain
\[
\int_{\partial \Delta^{d+1} \cap \Omega_{ik}} H^* d\text{vol}_X + \int_{K_{ik}} H^* d\text{vol}_X = \int_{\Omega_{ik}} dH^* d\text{vol}_X = 0
\]
because $H^*dvol_X$ is a closed form. We note that $H$ maps the $d$-dimensional submanifold $K_{ik}$ to the $(d - 1)$-dimensional submanifold $\mathcal{H}_{ik} \subset N$ and therefore $\int_{K_{ik}} H^*dvol_X = 0$. Thus

$$\int_{\partial \Delta^{d+1} \cap \Omega_{ik}} H^*dvol_X = 0$$

for all $k \in \mathbb{N}$. Then, since the countable union $\cup_{k \in \mathbb{N}} \partial \Delta^{d+1} \cap \Omega_{ik}$ equals $\Delta^{d+1} \setminus \kappa^{-1}(c_i)$ and since $dvol_X$ is defined to be zero on $c_i$, we conclude that

$$\int_{\partial \Delta^{d+1}} H^*dvol_X = 0.$$

Summing up over all $\kappa : \Delta^{d+1} \rightarrow Dcone (\cup_{i=1}^d \partial_i M \rightarrow M)$ occurring in $Cone_g(L_{d-1}(\partial z^0))$ we obtain

$$\int_{\partial Cone_g(L_{d-1}(\partial z^0))} dvol_X = 0.$$

Then, we have

$$algvol(str(c(z^0))) = \int_{str(c(z^0))} dvol_X = \int_{c(z^0)} dvol_X = Vol(N),$$

which completes the proof. $\square$

Assume that $d$ is odd. Let $b_d$ be the Borel class in $H^d_c(SL(n, \mathbb{C}), \mathbb{R})$. (It is obtained by restriction of the Borel class $b_d \in H^d_c(GL(n, \mathbb{C}), \mathbb{R})$ defined in Section 2.2.) According to the Van Est isomorphism, a representative $\beta_d$ of $b_d$ is given by

$$\beta_d(g_0, \ldots, g_d) = \int_{str(g_0\tilde{o}, \ldots, g_d\tilde{o})} dbol$$

where $dbol$ is an $SL(n, \mathbb{C})$-invariant differential $d$-form on $SL(n, \mathbb{C})/SU(n)$ and $\tilde{o}$ is a point in $SL(n, \mathbb{C})/SU(n)$. Recall that a comparison map

$$comp : C^*_c(SL(n, \mathbb{C}), \mathbb{R}) \rightarrow C^*_{simp}(BSL(n, \mathbb{C}), \mathbb{R})$$

is defined by $comp(f)(g_1, \ldots, g_k) = f(1, g_1, g_1 g_2, \ldots, g_1 \cdots g_k)$.

As in [21, Section 4.2.3] define $BSL(n, \mathbb{C})^{fb}_d$ as the set consisting of $d$-simplices in $BSL(n, \mathbb{C})^{comp}_d$ which are either $d$-simplices in $BSL(n, \mathbb{C})_d$ or of the form $(p_1, \ldots, p_{d-1}, c)$ satisfying

$$\left| \int_{str(\tilde{o}, p_1\tilde{o}, \ldots, p_{d-1}\tilde{o}, c)} dbol \right| < \infty.$$

We define $BSL(n, \mathbb{C})^{fb}_d$ to be the quasisimplicial set generated by $BSL(n, \mathbb{C})^{comp}_d$ under face maps. Then consider a cocycle $\overline{\beta}_d : C^*_{simp}(BSL(n, \mathbb{C})^{fb}_d, \mathbb{R}) \rightarrow \mathbb{R}$ defined by

$$\overline{\beta}_d(g_1, \ldots, g_d) = \int_{str(\tilde{o}, g_1\tilde{o}, \ldots, g_d\tilde{o})} dbol$$
for \((g_1, \ldots, g_d) \in BSL(n, \mathbb{C})^{\mathsf{fb}}\), and
\[
\overline{c^\beta d}(p_1, \ldots, p_{d-1}, c) = \int_{\text{str}(\tilde{o}, p_1 \tilde{o}, \ldots, p_{d-1} \tilde{o}, c)} dbol
\]
for \((p_1, \ldots, p_{d-1}, c) \in BSL(n, \mathbb{C})^{\mathsf{fb}}\). By the construction of \(\overline{c^\beta d}\), it is obvious that \((B_i)^* (\overline{c^\beta d}) = \text{comp}(\beta_d)\) and hence, \((B_i)^* (\overline{c^\beta d})\) is a cocycle representing \(\text{comp}(\beta_d)\) where \(B_i : BSL(n, \mathbb{C}) \to BSL(n, \mathbb{C})^{\mathsf{fb}}\) is the natural inclusion map.

**Lemma 6.4.** Let \(\rho : (G, K) \to (\text{SL}(n, \mathbb{C}), \text{SU}(n))\) be a representation. Let \(j : \Gamma \to G\) be the natural inclusion map. Then we have
\[
(B_\rho \circ B_j)^* (C^\text{simp}_d(BG^{\text{comp}})) \subset C^\text{simp}_d(\text{BSL}(n, \mathbb{C})^{\mathsf{fb}}),
\]
where \(B_\rho : BG^{\text{comp}} \to \text{BSL}(n, \mathbb{C})^{\text{comp}}\) and \(B_j : BG^{\text{comp}} \to BG^{\text{comp}}\) are the induced maps from \(\rho\) and \(j\) respectively. Furthermore, there exists a constant \(c_\rho \in \mathbb{R}\) such that \((B_\rho \circ B_j)^* (\overline{c^\beta d})\) represents \(c_\rho \overline{\nu_d}\).

**Proof.** To prove the first statement, it suffices to show that for a \(d\)-simplex of the form \((p_1, \ldots, p_{d-1}, c_i)\) with \(p_1, \ldots, p_{d-1} \in \Gamma_i\),
\[
(B_\rho \circ B_j)^* (p_1, \ldots, p_{d-1}, c_i) \subset C^\text{simp}_d(\text{BSL}(n, \mathbb{C})^{\mathsf{fb}}).
\]

The homomorphism \(\rho\) induces a \(\rho\)-equivariant map \(S_\rho : X \to Y\) and \(\rho_\infty : \partial_\infty X \to \partial_\infty Y\) where \(X = G/K\) and \(Y = \text{SL}(n, \mathbb{C})/\text{SU}(n)\). Since \(S_\rho^{\mathsf{fb}}\) is a \(G\)-invariant \(d\)-form on \(X\) and the space of \(G\)-invariant differential \(d\)-forms on \(X\) is generated by the \(G\)-invariant Riemannian volume form \(dvol_X\), there is a constant \(c_\rho \in \mathbb{R}\) such that \(S_\rho^{\mathsf{fb}} = c_\rho dvol_X\). Noting that \(S_\rho\) maps geodesics to geodesics, we have
\[
S_\rho(\text{str}(\tilde{x}_0, p_1 \tilde{x}_0, \ldots, p_1 \cdots p_{d-1} \tilde{x}_0, c_i)) = \text{str}(\tilde{o}, \rho(p_1) \tilde{o}, \ldots, \rho(p_1) \cdots \rho(p_{d-1}) \tilde{o}, \rho_\infty(c_i))
\]
where \(\tilde{x}_0, \tilde{o}\) are points in \(X\) and \(Y\) whose stabilizers are \(K\) and \(\text{SU}(n)\) respectively. For \((g_1, \ldots, g_{k-1}, c) \in BG^{\text{comp}}\), observe that the induced map \(B_\rho : BG^{\text{comp}} \to \text{BSL}(n, \mathbb{C})^{\text{comp}}\) is defined by
\[
B_\rho(g_1, \ldots, g_{k-1}, c) = (\rho(g_1), \ldots, \rho(g_{k-1}), \rho_\infty(c)).
\]

From the above observations, we have
\[
\int_{\text{str}(\tilde{o}, \rho(p_1) \tilde{o}, \ldots, \rho(p_1) \cdots \rho(p_{d-1}) \tilde{o}, \rho_\infty(c_i))} dbol = \int_{\text{str}(\tilde{x}_0, p_1 \tilde{x}_0, \ldots, p_1 \cdots p_{d-1} \tilde{x}_0, c_i)} (S_\rho)^* dbol = \int_{\text{str}(\tilde{x}_0, p_1 \tilde{x}_0, \ldots, p_1 \cdots p_{d-1} \tilde{x}_0, c_i)} c_\rho dvol_X < \infty.
\]

Furthermore, the above equation implies
\[
(B_\rho \circ B_j)^* (\overline{c^\beta d})(p_1, \ldots, p_{d-1}, c_i) = c_\rho \overline{\nu_d}(p_1, \ldots, p_{d-1}, c_i).
\]
Therefore, this implies the second statement. \(\square\)
7. Invariants in group homology and K-theory

In this section, we construct \( \gamma(N) \in K_d(\overline{Q}) \otimes \mathbb{Q} \) for a \( \mathbb{Q} \)-rank 1 locally symmetric space \( N = \Gamma \backslash G/K \).

**Proposition 7.1.** Let \( \Gamma \subseteq \mathbf{G}(\overline{\mathbb{Q}}) \) be a \( \mathbb{Q} \)-rank 1 lattice. Let \( N = \Gamma \backslash G/K \) be the interior of the manifold with boundary \( M \). Let \( \rho : \mathbf{G}(\overline{\mathbb{Q}}) \to \text{SL}(n, \overline{\mathbb{Q}}) \) be a representation. Suppose that \( \rho(\Gamma_i) \) is unipotent for all \( i \in \{1, \ldots, s\} \). Then

\[
(B_\rho \circ B_j)_* EM_{d-1}[M, \partial M] \in H_{d-\text{rank}}^s(B\text{SL}(n, \overline{\mathbb{Q}}))^\text{fib}, \mathbb{Q})
\]

has a preimage

\[
\overline{\gamma}(N) \in H_{d-\text{rank}}^s(B\text{SL}(n, \overline{\mathbb{Q}}), \mathbb{Q}),
\]

where \( j : \Gamma \to \mathbf{G}(\overline{\mathbb{Q}}) \) is the natural inclusion map.

**Proof.** The exact same argument in the proof of [21, Proposition 1] works under the assumption that every \( \rho(\Gamma_i) \) is unipotent. For the detailed proof, see [21, Proposition 1]. \( \square \)

Let \( A \) be a subring with unit of the ring of complex numbers. One can regard an element in \( H_{s-\text{rank}}^s(B\text{SL}(A), \mathbb{Q}) \) as an element in \( H_s(|B\text{SL}(A)|^+, \mathbb{Q}) \) due to the canonical identifications

\[
H_{s-\text{rank}}^s(B\text{SL}(A), \mathbb{Q}) \cong H_s(|B\text{SL}(A)|^{\mathbb{Q}}, \mathbb{Q}) \cong H_s(|B\text{SL}(A)|^+, \mathbb{Q}).
\]

By the Milnor-Moore theorem, the Hurewicz homomorphism

\[
K_s(A) = \pi_s(|B\text{GL}(A)|^+) \to H_s(|B\text{GL}(A)|^+, \mathbb{Z})
\]

gives, after tensoring with \( \mathbb{Q} \), an injective homomorphism

\[
I_s : K_s(A) \otimes \mathbb{Q} = \pi_s(|B\text{GL}(A)|^+) \otimes \mathbb{Q} \to H_s(|B\text{GL}(A)|^+, \mathbb{Q}).
\]

Its image consists of primitive elements, denoted by \( PH_s(|B\text{GL}(A)|^+, \mathbb{Q}) \).

By Quillen, inclusion \( |B\text{GL}(A)| \subset |B\text{GL}(A)|^+ \) induces an isomorphism

\[
Q_s : PH_s(|B\text{GL}(A)|, \mathbb{Q}) \to PH_s(|B\text{GL}(A)|^+, \mathbb{Q}) \cong K_s(A) \otimes \mathbb{Q}.
\]

Once the projection

\[
pr_s : H_{s-\text{rank}}^s(B\text{GL}(A), \mathbb{Q}) \to PH_{s-\text{rank}}^s(B\text{GL}(A), \mathbb{Q}) \cong PH_s(|B\text{GL}(A)|, \mathbb{Q})
\]

is fixed, we can define an element \( I_s^{-1} \circ Q_s \circ pr_s(\alpha) \in K_s(A) \otimes \mathbb{Q} \) for each \( \alpha \in H_{s-\text{rank}}^s(B\text{GL}(A), \mathbb{Q}) \). For \( h \in K_{2m-1}(A) \otimes \mathbb{Q} \), define

\[
\langle b_{2m-1}, h \rangle = \langle \text{comp}(b_{2m-1}), Q_{2m-1}^{-1} \circ I_{2m-1}(h) \rangle.
\]

In particular, when \( A = \overline{\mathbb{Q}} \), note that by [21, Corollary 2] the projection \( pr_s \)

\[
\text{can be chosen such that}
\]

\[
\langle \text{comp}(b_{2m-1}), pr_{2m-1}(\alpha) \rangle \Rightarrow \langle \text{comp}(b_{2m-1}), \alpha \rangle
\]

for all \( m \in \mathbb{N} \). We refer the reader to [21, Section 2.5] for more details about this.
Theorem 7.2. Let $G/K$ be a symmetric space of noncompact type with odd dimension $d$ and $N = \Gamma \backslash G/K$ be a $\mathbb{Q}$-rank 1 locally symmetric space. Let $\rho : G \to \text{GL}(n, \mathbb{C})$ be a representation with $\rho^{*}b_d \neq 0$. Suppose that $\rho(\Gamma_i)$ is unipotent for all $i \in \{1, \ldots, s\}$. Then there is an element

$$\gamma(N) \in K_{d}(\overline{Q}) \otimes \mathbb{Q}$$

such that the application of the Borel class $b_d$ yields

$$\langle b_d, \gamma(N) \rangle = c_{\rho} \text{Vol}(N)$$

for some constant $c_{\rho} \neq 0$.

Proof. First, note that connected, semisimple Lie groups are perfect, hence each representation $\rho : G \to \text{GL}(n, \mathbb{C})$ has image in $\text{SL}(n, \mathbb{C})$. In addition, we can assume that $\rho$ maps $K$ to $\text{SU}(n)$, which can be achieved upon conjugation.

Once $G$ is identified with $G(\mathbb{R})^0$ via an isomorphism, by Weil’s rigidity theorem, there is a $g \in G$ such that $g\Gamma g^{-1} \subset G(\overline{Q})^0$. Moreover, it follows from the classification of irreducible representations of Lie groups [15] that each representation $\rho : G(\mathbb{R})^0 \to \text{GL}(n, \mathbb{C})$ is isomorphic to a representation $\rho' : G(\mathbb{R})^0 \to \text{GL}(n, \mathbb{C})$ such that $G(\overline{Q})^0$ is mapped to $\text{GL}(n, \overline{Q})$. Thus, we can assume that $\Gamma \subset G(\overline{Q})^0$ and $\rho|_{G(\overline{Q})^0} : G(\overline{Q})^0 \to \text{SL}(n, \overline{Q})$ is a representation for which every $\rho(\Gamma_i)$ is unipotent.

According to Proposition 7.1, we obtain $\overline{\gamma}(N) \in H_{d}^{\text{simp}}(\text{BSL}(n, \overline{Q}), \mathbb{Q})$. Let us denote by $\overline{\gamma}(N)$ the image of $\overline{\gamma}(N)$ in $H_{d}^{\text{simp}}(\text{BGL}(n, \overline{Q}), \mathbb{Q})$. Now, we define

$$\gamma(N) = I_{d}^{-1} \circ Q_{d} \circ pr_{d}(\overline{\gamma}(N)) \in K_{d}(\overline{Q}) \otimes \mathbb{Q}.$$ 

Then, we have

$$\langle b_d, \gamma(N) \rangle = \langle \text{comp}(b_d), pr_{d}(\overline{\gamma}(N)) \rangle$$

$$= \langle \text{comp}(b_d), \overline{\gamma}(N) \rangle = \langle \text{comp}(\beta_d), \overline{\gamma}(N) \rangle$$

$$= \langle (B_i)^{*}c_{\beta_d}, \overline{\gamma}(N) \rangle = \langle c_{\beta_d}, (B_i)^{*} \overline{\gamma}(N) \rangle$$

$$= \langle c_{\beta_d}, (B_{\rho} \circ B_j)^{*}c_{\beta_d}, EM_{d}^{-1}[M, \partial M] \rangle$$

$$= \langle c_{\rho \circ B_j}^{*}c_{\beta_d}, EM_{d}^{-1}[M, \partial M] \rangle = c_{\rho} \text{Vol}(N),$$

where $B_{\rho} : \text{BSL}(n, \mathbb{C}) \to \text{BSL}(n, \mathbb{C})^{\text{th}}$ is the natural inclusion map. Therefore, we complete the proof.

Due to $\langle b_d, \gamma(N) \rangle = c_{\rho} \text{Vol}(N) \neq 0$, it can be checked that $\gamma(N)$ is a nontrivial element in $K_{d}(\overline{Q}) \otimes \mathbb{Q}$ if $\rho^{*}b_d \neq 0$. Kuessner [21, Theorem 3] characterizes a complete list of irreducible symmetric spaces $G/K$ of noncompact type and fundamental representation $\rho : G \to \text{GL}(n, \mathbb{C})$ with $\rho^{*}b_{2m-1} \neq 0$ for $d = 2m-1 = \dim(G/K)$. In the noncompact case, he only gets invariants for hyperbolic manifolds since there exist no such fundamental representations for the other $\mathbb{R}$-rank 1 semisimple Lie groups. However, in the list,
there are a lot of fundamental representations for higher rank semisimple Lie groups. Theorem 7.2 enables us to get invariants for \( \mathbb{Q} \)-rank 1 locally symmetric spaces, including hyperbolic manifolds, by using the fundamental representations in the list.

8. Relation to classical Bloch group

In [26], Neumann-Yang constructed an invariant of finite volume hyperbolic 3-manifolds which lie in the Bloch group \( \mathcal{B}(\mathbb{C}) \). In this section, we give an invariant of a \( \mathbb{Q} \)-rank 1 locally symmetric space, which coincides with the classical Bloch invariant of cusped hyperbolic 3-manifolds.

Let \( X \) be a symmetric space of noncompact type and \( C_k(\partial_{\infty}X) \) the free abelian group generated by \((k+1)\)-tuples of points of \( \partial_{\infty}X \) modulo the relations

1. \((\theta_0, \ldots, \theta_k) = \text{sign}(\tau)(\theta_{\tau(0)}, \ldots, \theta_{\tau(k)})\) for any permutation \( \tau \) and
2. \((\theta_0, \ldots, \theta_k) = 0\) whenever \( \theta_i = \theta_j \) for some \( i \neq j \)
3. \( \partial(\theta_0, \ldots, \theta_k) = \sum_{i=0}^{k}(-1)^i(\theta_0, \ldots, \hat{\theta}_i, \ldots, \theta_k) \)

The generalized pre-Bloch group of \( X \) is

\[ P^*_{\mathcal{G}}(X) = H^*(C_*(\partial_{\infty}X) \otimes \mathbb{Z}G, \partial \otimes \mathbb{Z}G \text{id}) \]

and the generalized pre-Bloch group of \( \mathbb{C} \) as

\[ P^n_{\mathcal{G}}(\mathbb{C}) = P^*(\text{SL}(n, \mathbb{C})/\text{SU}(n)). \]

Note that \( P_3(H^3_{\mathbb{R}}) = P_3^2(\mathbb{C}) \) is the classical pre-Bloch group \( P(\mathbb{C}) \).

**Remark.** One may wonder what happens if Condition (1) is omitted. Let \( \hat{C}_*(\partial_{\infty}X) \) be the chain complex analogously defined without condition (1) and \( \pi : \hat{C}_*(\partial_{\infty}X) \rightarrow C_*(\partial_{\infty}X) \) the projection, then (because of condition (ii)) each element of \( C_n(\partial_{\infty}X) \) has \((n+1)!\) preimages and we may define a right-inverse to the projection \( \pi \) by sending each \( c \in C_n(\partial_{\infty}X) \) to the formal sum of its preimages divided by \((n+1)!\). One checks that this defines a chain map. Thus one obtains an injection \( H_*(C_*(\partial_{\infty}X) \otimes \mathbb{Z}G, \partial \otimes \mathbb{Z}G \text{id}) \rightarrow H_*(\hat{C}_*(\partial_{\infty}X) \otimes \mathbb{Z}G, \partial \ otimes \mathbb{Z}G \text{id}) \).

Therefore, each \( G \)-equivariant cocycle on \( \hat{C}_*(\partial_{\infty}X) \) also defines a \( G \)-equivariant cocycle on \( C_*(\partial_{\infty}X) \). (One may think of this new cocycle as taking the signed average of the evaluations of the old cocycle on the \((n+1)!\) simplices obtained by permuting the vertices.) In particular, for \( \mathbb{R} \)-rank one spaces, the algebraic volume \( \text{algvol} \) yields a well-defined cocycle on \( C_*(\partial_{\infty}X) \otimes \mathbb{Z}G \). (This was not made explicit in [22].)

If \( \rho : G \rightarrow \text{SL}(n, \mathbb{C}) \) is a nontrivial (hence reductive) representation, then it induces a smooth map \( X = G/K \rightarrow \text{SL}(n, \mathbb{C})/\text{SU}(n) \) and its extension to the ideal boundary

\[ \rho_{\infty} : \partial_{\infty}X \rightarrow \partial_{\infty}(\text{SL}(n, \mathbb{C})/\text{SU}(n)). \]

Using this one can define a generalized Bloch invariant of a \( \mathbb{Q} \)-rank 1 locally symmetric space \( N = \Gamma \backslash G/K \) as follows. Fix a point \( c_0 \in \partial_{\infty}X \) and define
\( ev_{\Gamma,0_1,\ldots,c_s} : C_*(B\Gamma^{comp}) \to C_*(\partial_{\infty}X) \otimes_{\mathbb{Z}G} \mathbb{Z} \) on generators by

\[
ev_{\Gamma,0_1,\ldots,c_s}(\gamma_1, \cdots, \gamma_k) = (c_0, \gamma_1c_0, \cdots, \gamma_1 \cdots \gamma_kc_0) \otimes 1
\]

for \( \gamma_1, \ldots, \gamma_k \in \Gamma \), and

\[
ev_{\Gamma,0_1,\ldots,c_s}(p_1, \ldots, p_{k-1}, c_i) = (c_0, p_1c_0, \cdots, p_{k-1}c_0, c_i) \otimes 1
\]

for \( p_1, \ldots, p_{k-1} \in \Gamma \). It is straightforward to check that \( \ev_{\Gamma,0_1,\ldots,c_s} \) extends linearly to a chain map and thus, it induces a homomorphism

\[
(\ev_{\Gamma,0_1,\ldots,c_s})_* : H_*^{simp}(B\Gamma^{comp}, \mathbb{Z}) \to \mathcal{P}_*(X).
\]

In addition, one can define a map

\[
\rho_\infty : C_*(\partial_{\infty}X) \otimes_{\mathbb{Z}G} \mathbb{Z} \to C_*(\partial_{\infty}(\mathrm{SL}(n, \mathbb{C})/\mathrm{SU}(n))) \otimes_{\mathbb{Z}SL(n, \mathbb{C})} \mathbb{Z}
\]

defined by

\[
\rho_\infty((\theta_0, \ldots, \theta_k) \otimes 1) = (\rho_\infty(\theta_0), \ldots, \rho_\infty(\theta_k)) \otimes 1.
\]

Hence, it yields a homomorphism

\[
(\rho_\infty)_* : \mathcal{P}_*(X) \to \mathcal{P}_n^*(\mathbb{C}).
\]

To obtain an element of the generalized pre-Bloch group of \( \mathbb{C} \) for a \( \mathbb{Q} \)-rank 1 locally symmetric space \( N = \Gamma\backslash G/K \), we need an integer homology class in \( H_d^{simp}(B\Gamma^{comp}, \mathbb{Z}) \). Note that \( EM^{-1}_d[M, \partial M] \in H_d^{simp}(B\Gamma^{comp}, \mathbb{Q}) \) is a rational homology class. However, it can be easily checked that we can obtain an integer homology class \( EM^{-1}_d[M, \partial M]_{\mathbb{Z}} \in H_d^{simp}(B\Gamma^{comp}, \mathbb{Z}) \) from the relative fundamental class \( [M, \partial M]_{\mathbb{Z}} \in H^d(M, \partial M, \mathbb{Z}) \) with integer coefficients as follows. Given a relative fundamental cycle \( z \) with integer coefficients, \( c(z^0) = z^0 + (-1)^{d+1} \mathrm{Cone}_g(\partial z^0) \) defined in the proof of Lemma 6.3 is also a cycle with integer coefficients. For another relative fundamental cycle \( \bar{z} \) with integer coefficients, there are chains \( w \in C_{d+1}(M, \mathbb{Z}) \) and \( y \in C_d(\partial M, \mathbb{Z}) \) with all vertices in \( x_0 \) and such that \( z^0 - \bar{z}^0 = \partial w^0 + y^0 \). Then

\[
\begin{align*}
c(z^0) - c(\bar{z}^0) &= z^0 - \bar{z}^0 + (-1)^{d+1}(\mathrm{Cone}_g(\partial z^0) - \mathrm{Cone}_g(\partial \bar{z}^0)) \\
&= \partial w^0 + y^0 + (-1)^{d+1}\mathrm{Cone}_g(\partial y^0) \\
&= \partial w^0 + y^0 + (-1)^{d+1}(\partial \mathrm{Cone}_g(y^0) + (-1)^d y^0) \\
&= \partial(w^0 + (-1)^{d+1}\mathrm{Cone}_g(y^0))
\end{align*}
\]

It is obvious that \( \Phi(w^0 + (-1)^{d+1}\mathrm{Cone}_g(y^0)) \) is a chain in \( C^{simp}_{d+1}(B\Gamma^{comp}, \mathbb{Z}) \). Hence \( c(z^0) \) determines a homology class in \( H_d^{simp}(B\Gamma^{comp}, \mathbb{Z}) \) independent of the choice of relative fundamental cycle \( z \), denoted by \( EM^{-1}_d[M, \partial M]_{\mathbb{Z}} \).

**Definition 8.1.** For a \( \mathbb{Q} \)-rank 1 locally symmetric space \( N \) of dimension \( d \), define an element \( \beta_\rho(N) \) in the generalized pre-Bloch group \( \mathcal{P}_d^*(\mathbb{C}) \) by

\[
\beta_\rho(N) := (\rho_\infty)_d \circ (\ev_{\Gamma,0_1,\ldots,c_s})_d \circ EM^{-1}_d[M, \partial M]_{\mathbb{Z}}.
\]
Suppose that every \( \rho(\Gamma_i) \) is unipotent. Then the proof of Proposition 7.1 works for \((B_\rho \circ B_j)_*EM^{-1}_d[M, \partial M]_\mathbb{Z} \in H^{simp}_d(B\Sigma(n, \mathbb{Q})^{fb}, \mathbb{Z})\). Thus, it has a preimage \( (\overline{\gamma}(N))_\mathbb{Z} \in H^{simp}_d(B\Sigma(n, \mathbb{Q}), \mathbb{Z}) \). In the case that \( N \) is a \( \mathbb{R} \)-rank 1 locally symmetric space, Kuessner [22] showed that \( (ev_{SL(n, \mathbb{C})})_d(\overline{\gamma}(N))_\mathbb{Z} = \beta_\rho(N) \)

where the evaluation map

\[
ev_{SL(n, \mathbb{C})} : C^*_{simp}(B\Sigma(n, \mathbb{C}), \mathbb{Z}) \to C^*_\infty(SU(n)/SU(n)) \otimes_{Z\Sigma(n, \mathbb{C})} \mathbb{Z}
\]

is defined on generators by

\[
ev(g_1, \ldots, g_k) = (c_0, g_1c_0, \ldots, g_1 \ldots g_kc_0) \otimes 1.
\]

In the same way, this holds for \( \mathbb{Q} \)-rank 1 locally symmetric spaces. Specialy it recovers the classical Bloch invariant of cusped hyperbolic 3-manifolds. For this reason we will call \( \beta_\rho(N) \) the \textit{generalized Bloch invariant} for either a compact locally symmetric manifold or a finite volume \( \mathbb{Q} \)-rank 1 locally symmetric space \( N \).

One advantage of our approach is that one can define the Bloch invariant without the notion of degree one ideal triangulation. Neumann and Yang used the fact that cusped hyperbolic 3-manifolds admit a degree one ideal triangulation to define the classical Bloch invariant of cusped hyperbolic 3-manifolds. Since it is not known whether general locally symmetric spaces admit such a triangulation, it seems to be difficult to extend the definition of the classical Bloch invariant from hyperbolic 3-manifolds to general locally symmetric spaces in the same way that Neumann and Yang constructed it. However, we here use only the relative fundamental cycle \([M, \partial M]_\mathbb{Z}\) to define \( \beta_\rho(N) \) and moreover, this agrees with the classical Bloch invariant for cusped hyperbolic 3-manifolds [22]. Hence our approach makes it possible to generalize the Bloch invariant for cusped hyperbolic 3-manifolds without the existence of a degree one ideal triangulation. Furthermore, even if one defines an invariant by using a degree one ideal triangulation of \( N \), the invariant should agree with \( \beta_\rho(N) \). (This is shown for \( \mathbb{R} \)-rank 1 spaces in [22, Theorem 4.0.2].)

9. **Bloch group for convex projective manifolds**

Given a strictly convex real projective manifold \( N \), up to taking a double cover, there is a holonomy map

\[
\rho : \pi_1(N) = \Gamma \to SL(n, \mathbb{R})
\]

where \( \Gamma \) acts on a strictly convex projective domain \( \Omega \) equipped with a Hilbert metric. If \( \Omega \) is conic, then the image of \( \rho \) is in \( SO(n-1, 1) \) and the manifold is a real hyperbolic manifold.

Let’s assume that \( \dim(N) = 3 \). For a given hyperbolic representation \( \rho_0 : \Gamma \to SL(2, \mathbb{C}) \subset SL(4, \mathbb{R}) \subset SL(4, \mathbb{C}) \), assume there is a deformation of
ρ₀ to convex projective structures. The inclusion

\[ i : \text{SL}(2, \mathbb{C}) = \text{SO}(3, 1) \subset \text{SL}(4, \mathbb{R}) \subset \text{SL}(4, \mathbb{C}) \]

is not the standard one but by [21, Corollary 5] we have \( i^* b_3 \neq 0 \).

There is a canonical invariant called Bloch invariant developed by Dupont-Sah and others. We want to generalize this notion to projective manifolds.

When \( \Gamma \) acts on the strictly convex domain \( \Omega \subset \mathbb{R}^3 \), in general \( \text{Aut}(\Omega) \) does not have a Lie group structure and we do not have a volume form naturally induced from the Lie group. On the other hands, \( \Omega \) has a Finsler metric, called Hilbert metric invariant under \( \text{Aut}(\Omega) \). \( \Omega \) with a Hilbert metric behaves like a hyperbolic space, hence one can define a straight simplex. Fix a base point \( x \in \Omega \). The volume class \( v_d \in H^d(\Gamma, \mathbb{R}) \) is defined by the cocycle

\[
\nu_d(\gamma_0, \ldots, \gamma_d) = \int_{\text{str}(\gamma_0x, \ldots, \gamma_dx)} \text{dvol}_\Omega,
\]

where \( \text{dvol}_\Omega \) is a signed Finsler volume form on \( \Omega \), and \( \text{str} \) is a geodesic straightening. One can check that it is a cocycle since \( \text{str}(\gamma_0x, \ldots, \gamma_dx) \) is a top dimensional simplex. We have the following lemma similar to Lemma 4.1.

**Lemma 9.1.** The volume of the ideal straight simplex \( \text{str}(x_0, \ldots, x_{d-1}, c) \) is finite for any \( x_0, \ldots, x_{d-1} \in \Omega \) and \( c \) is a cuspidal point.

**Proof.** This follows from the Proposition 11.2 of [10].

This lemma allows us to carry out the similar constructions as in previous sections. Hence we can define a cocycle \( \tilde{\nu}_d \in \text{C}^d(\text{BT}^{\text{comp}}) \) extending \( \text{comp}(\nu_d) \). Let \( \tilde{\nu}_d \) denote the element represented by \( \tilde{\nu}_d \) in \( H^d(\text{BT}^{\text{comp}}, \mathbb{R}) \). By Lemma 6.2, \( H_d(\text{BT}^{\text{comp}}, \mathbb{R}) = H_d(M, \partial M, \mathbb{R}) = \mathbb{R} \) where \( M \) is a compact manifold with boundary whose interior is homeomorphic to \( N \). It is known that for a cusped hyperbolic 3-manifold, there exists a degree one ideal triangulation. We can use the same ideal triangulation to obtain a triangulation of \( M \) by Hilbert metric ideal tetrahedra to obtain

\[
\langle \tilde{\nu}_d, EM_d^{-1}[M, \partial M] \rangle = \text{Vol}_{\text{Finsler}}(N).
\]

For the Borel class \( b_3 \in H^3_c(\text{GL}(\mathbb{C}), \mathbb{R}) \), it is known that \( \rho^* b_3 \neq 0 \) for any nontrivial representation \( \rho : \text{SL}(2, \mathbb{C}) \to \text{GL}(\mathbb{C}) \). Then for any finite volume hyperbolic manifold \( N = \Gamma \backslash H^3_{\mathbb{R}} \), the induced representation

\[
\rho_0 : \Gamma \to \text{GL}(\mathbb{C})
\]

gives rise to a nontrivial Borel class \( \rho^*_0 b_3 = c_{\rho_0} \overline{\tau}_{\rho_0} \) by Lemma 6.4. Since a strictly convex projective structure \( \rho : \Gamma \to \text{SL}(4, \mathbb{R}) \) lies in the same component containing \( \rho_0 \) in the character variety \( \chi(\Gamma, \text{SL}(4, \mathbb{C})) \), \( H(\rho)[M, \partial M] \)
is nontrivial, indeed equal to $H(p_0)[M, \partial M] \in H_3(BGL(\mathbb{C})^3, \mathbb{Z})$. Hence
\[
\langle b_3, B(\rho) \circ EM^{-1}[M, \partial M] \rangle = \langle b_3, B(\rho_0) \circ EM^{-1}[M, \partial M] \rangle = \langle \rho \circ b_3, EM^{-1}[M, \partial M] \rangle = c_{\rho_0} \text{Vol}_{hyp}(M) = c_{\rho} \text{Vol}_{Finsler}(M).
\]
Since $H^d_{simp}(BG_{comp}, \mathbb{R}) = H_d(BG_{comp}, \mathbb{R})^* = \mathbb{R}$, $\rho^*b_3 = c_{\rho}c_{\bar{\rho}}$.

If $\sum a_i\tau_i$ is a proper ideal fundamental cycle (i.e. each simplex is a proper ideal straight simplex) of $M$, the sum of the cross-ratios
\[
\beta(M) = \sum a_i[cr(\tau_i)] \in \mathcal{P}_d(\Omega) := H_3(C_*(\partial \Omega)_{\Gamma})
\]
defines a generalized Neumann-Yang invariant. Note that if $\Omega$ is not conic, the cross-ratio is not a complex number as in $H^2_{hyp}$. It is a question how to interpret this invariant in terms of a volume and the Chern-Simons invariant.

10. Bloch invariant in $SU(2, 1)$ and $SL(3, \mathbb{C})$

10.1. Falbel-Wang invariant. Falbel constructed a discrete representation
\[
\rho : \pi_1(S^3 \setminus K) \to SU(2, 1) \subset SL(3, \mathbb{C})
\]
which is faithful and parabolic on the torus boundary where $K$ is a figure 8 knot. Falbel-Wang further constructed a Bloch invariant using this representation. In this section, we prove that their invariant can be computed from $H(\rho) EM^{-1}[M, \partial M]$.

Following [1, Section 3.8] we identify the complex hyperbolic space $H^2_{hyp}$ with $\pi(V_-) \subset CP^2$, where $V_- := \{(x, y, z) \in \mathbb{C}^3 : x\bar{x} + y\bar{y} + z\bar{z} < 0\}$ and $\pi : \mathbb{C}^3 \to CP^2$ is the canonical projection. The ideal boundary $\partial_\infty \mathbb{C}H^2 \simeq S^3$ is then identified with $\pi(V_0)$, where $V_0 := \{(x, y, z) \in \mathbb{C}^3 : x\bar{x} + y\bar{y} + z\bar{z} = 0\}$.

The following construction involves an identification of $CP^1$ with the set of (affine) complex lines through a given point in $CP^2$. There is some arbitrariness in choosing such an identification, a specific choice is given in [14, Section 2.5] with a derived explicit formula in [14, Definition 2.24].

**Definition 10.1. (Falbel-Wang construction, [14, Section 2.5]):**

- a) Fix an identification $\partial_\infty H^2_{hyp} = S^3$ and define a homomorphism $FW_{01} : \mathcal{P}_3(H^2_{hyp}) \to \mathcal{P}_3(H^3_{\mathbb{R}})$ on generators $(p_0, p_1, p_2, p_3)$ of $\mathcal{P}_3(H^2_{hyp})$ by
  \[
  FW_{01}(p_0, p_1, p_2, p_3) = (t_0, t_1, t_2, t_3),
  \]
  where we define $t_0 \in CP^1 = \partial_\infty H^2_{hyp}$ to be the complex line through $p_0$ tangent to $S^3 \subset CP^2$ and for $i = 1, 2, 3$ we define $t_i \in CP^1 = \partial_\infty H^3_{\mathbb{R}}$ to be the complex line in $CP^2$ passing through $p_0$ and $p_i$.

- b) For $a \neq b \in \{0, 1, 2, 3\}$ there are unique $k, l \in \{0, 1, 2, 3\}$ such that the ordered set $(a, b, k, l)$ is an even permutation of $(0, 1, 2, 3)$ and we define
  \[
  FW_{ab} : \mathcal{P}_3(H^2_{hyp}) \to \mathcal{P}_3(H^3_{\mathbb{R}})
  \]
on generators \((p_0, p_1, p_2, p_3)\) of \(\mathcal{P}_3 (H^2_C)\) by
\[
FW_{ab}(p_0, p_1, p_2, p_3) = FW_{01}(p_a, p_b, p_k, p_l).
\]

We will occasionally consider the \(FW_{ab}\) as maps which send \((\text{SU}(2, 1)\)-orbits of) ideal simplices in \(H^2_C\) to \((\text{SO}(3, 1)\)-orbits of) ideal simplices in \(H^3_{\mathbb{R}}\).

If \(M\) is a hyperbolic 3-manifold and \(\rho : \pi_1 M \to \text{SU}(2, 1)\) is a reductive representation (that means \(\rho(\pi_1 M) \subset \text{SU}(2, 1)\) is a reductive subgroup), then by \([11]\) (see also \([24]\)) we obtain a \(\rho\)-equivariant developing map \(D : H^3_{\mathbb{R}} = \tilde{M} \to H^2_C\) and a continuous boundary map \(\partial_\infty D : \partial_\infty H^3_{\mathbb{R}} \to \partial_\infty H^2_C\). In particular, if \(T\) is an ideal simplex in \(M\) (that is a \(\pi_1 M\)-orbit of some ideal simplex \(\tilde{T}\) with vertices \(v_0, v_1, v_2, v_3 \in \partial_\infty H^3_{\mathbb{R}}\)), then we can define \(D(T)\) to be the \(\pi_1 M\)-orbit of the ideal simplex in \(H^2_C\), whose vertices are \(\partial_\infty D(c_i)\) for \(i = 0, 1, 2, 3\).

Let \(\mathcal{P}_3^{nd}(H^3_{\mathbb{R}}) \subset \mathcal{P}_3 (H^3_{\mathbb{R}})\) be the subgroup generated by 4-tuples \((z_0, z_1, z_2, z_3)\) of \(pairwise\ \text{distinct}\) points. (By definition, simplices in an ideal triangulation are nondegenerate and thus give elements in \(\mathcal{P}_3^{nd}(H^3_{\mathbb{R}})\).)

Recall that the cross ratio
\[
X : \mathcal{P}_3^{nd}(H^3_{\mathbb{R}}) \to \mathcal{P}(\mathbb{C})
\]
is well defined and yields an isomorphism between \(\mathcal{P}_3^{nd}(H^3_{\mathbb{R}})\) and \(\mathcal{P}(\mathbb{C})\). We will use the abbreviation
\[
FW := FW_{01} + FW_{10} + FW_{23} + FW_{32} : \mathcal{P}_3 (H^2_C) \to \mathcal{P}_3 (H^3_{\mathbb{R}}).
\]

**Definition 10.2.** (Falbel-Wang invariant): Let \(M = \bigcup_{i=1}^r T_i\) be an ideal triangulation of a hyperbolic 3-manifold and \(\rho : \pi_1 M \to \text{SU}(2, 1)\) a reductive representation, then the invariant \(\beta_{FW}(M) \in \mathcal{B}(\mathbb{C}) \subset \mathcal{P}(\mathbb{C})\) is defined by
\[
\beta_{FW}(M) := \sum_{i=1}^r X (FW(\text{cr}(D(T_i)))).
\]

(It is proved in \([14, \text{Theorem 1.1}]\) that \(\beta_{FW}(M)\) lies in \(\mathcal{B}(\mathbb{C})\) and does not depend on the ideal triangulation. The latter fact will also follow from our Lemma 10.3.)

**Lemma 10.3.** Let \(M\) be a finite-volume hyperbolic 3-manifold, \(\rho : \pi_1 M \to \text{SU}(2, 1)\) a reductive representation which maps the peripheral subgroups to unipotent subgroups. Then
\[
\beta_{FW}(M) = X (FW (H (ev) (H (\rho) (EM^{-1}[M, \partial M]))))
\]

**Proof.** Let \(x_0 \in M\), \(c_0 \in \partial_\infty \tilde{M} = \partial_\infty H^3_{\mathbb{R}}\) and \(b_0 := \partial_\infty D(c_0) \in \partial_\infty H^2_C\). Lemma 3.3.4 in \([22]\) constructs a chain map \(\hat{C} : \hat{C}_*^{\text{cr}, x_0}(M) \to \hat{C}_*^{\text{cr}, c_0}(M)\).

Let \(G = \text{SU}(2, 1)\), \(K = \text{SU}(2) \times \text{U}(1)\), \(\Gamma = \pi_1 M\), \(\Gamma_i = \pi_1 \partial_i M\) for the path components \(\partial_i M\) of \(\partial M\), \(c_i\) the cusps associated to \(\Gamma_i\). We use the
whose derivation (except for the first square) is explained in the proof of [22, Theorem 4.0.2]. In the first square we have
\[ ev_{c_0} (g_1, \ldots, g_n) = (c_0, g_1 c_0, \ldots, g_1 \ldots g_n c_0) \otimes 1 \]
for \((g_1, \ldots, g_n) \in BG\) and
\[ ev_{c_0} (Cone_{c_i} (g_1, \ldots, g_{n-1})) = (c_0, g_1 c_0, \ldots, g_1 \ldots g_{n-1} c_0, c_i) \otimes 1, \]
similarly for \(ev_{c_0} \).

The diagram shows that
\[ H (ev_b) \left( H (\rho) \left( EM^{-1} [M, \partial M] \right) \right) \]
is represented by
\[ \partial_{\infty} D \left( \hat{C} \left( str \left( z + Cone \left( \partial z \right) \right) \right) \right) \]
whenever \(z \in Z_\ast (\bar{M}, \partial \bar{M})\) is a relative fundamental cycle.

If \(z \in Z_\ast (\bar{M}, \partial \bar{M})\) is a relative fundamental cycle, then \(str \left( z + Cone \left( \partial z \right) \right) \in \hat{C}_{\ast, str, x_0} (M)\) is an ideal fundamental cycle in the sense of [22, Definition 3.1.4]. By [22, Lemma 3.3.4] this implies that \(\hat{C} \left( str \left( z + Cone \left( \partial z \right) \right) \right)\) is an ideal fundamental cycle.

On the other hand, let \(M = \cup_{i=1}^r T_i\) be an ideal triangulation. Let \(p_0^i, p_1^i, p_2^i, p_3^i \in \partial_{\infty} \bar{M} = \partial_{\infty} H^3_{\mathbb{R}}\) be the ideal vertices of \(T_i\). Then
\[ FW \left( \partial_{\infty} D \left( p_0^i \right) \right), FW \left( \partial_{\infty} D \left( p_1^i \right) \right), FW \left( \partial_{\infty} D \left( p_2^i \right) \right), FW \left( \partial_{\infty} D \left( p_3^i \right) \right) \]
are the ideal vertices of \(FW (D (T_i))\). By definition we have
\[ \beta_{FW} (M) = \sum_{i=1}^r 4X \left( FW_{01} \left( D (T_i) \right) \right). \]
We have proved in the proof of [22, Lemma 3.4.1] that the homology classes of \( cr ( \hat{C} ( str ( z + \text{Cone} ( \partial z ))) ) \) and of \( \sum_{i=1}^{r} cr ( T_i ) \) are the same in \( H_* \left( C_* ( \partial_\infty \hat{M} ) \Gamma \right) \).

Thus there is some \( w \in C_* ( \partial_\infty \hat{M} ) \Gamma \) with

\[
\partial w = cr ( \hat{C} ( str ( z + \text{Cone} ( \partial z ))) ) - \sum_{i=1}^{r} cr ( T_i ).
\]

\( \partial_\infty D \) is a chain map by construction. By [14, Lemma 3.2] (following [13, Theorem 5.2]), we have that \( X ( FW ( \partial C_4 ( \partial_\infty H^2_G )) ) \subset \mathbb{Z} [ \mathbb{C} - \{ 0, 1 \} ] \) is in the subgroup generated by the 5-term relations, that is \( X ( FW ( \partial u )) = 0 \in \mathcal{B} ( \mathbb{C} ) \) for all \( u \in C_4 ( \partial_\infty H^2_G ) \). Hence we obtain

\[
X \left( FW \left( \partial_\infty D \left( cr \left( \hat{C} ( str ( z + \text{Cone} ( \partial z ))) \right) \right) \right) \right) - \sum_{i=1}^{r} X \left( FW \left( \partial_\infty D \left( cr \left( T_i \right) \right) \right) \right) = X ( FW ( \partial_\infty D ( w )) ) = 0.
\]

Hence the cycle \( \sum_{i=1}^{r} X ( FW ( \partial_\infty D ( cr ( T_i ))) ) \) represents the homology class \( X ( FW ( H \left( ev_{\partial_\infty} \right) H ( \rho ) \left( E M^{-1} [ M, \partial M ] \right) ) ) ) \). Because of \( \partial_\infty D ( cr ( T_i )) = cr ( D ( T_i )) \) this implies the claim. \( \square \)

For Corollary 10.4 and Corollary 10.8 we will consider the situation that a 3-manifold \( M^\tau \) is obtained from another 3-manifold \( M \) by cutting along some \( \pi_1 \)-injective surface \( \Sigma \subset M \) and regluing via \( \tau : \Sigma \to \Sigma \). If in this situation for a representation \( \rho : \pi_1 M \to G \) we have some \( A \in G \) with \( \rho ( \tau_* h ) = A \rho ( h ) A^{-1} \) for all \( h \in \pi_1 \Sigma \), then we get an induced representation \( \rho^\tau : \pi_1 M^\tau \to G \) by a standard application of the Seifert-van Kampen Theorem as in [23, Section 2]. This representation \( \rho^\tau \) will be used in the statements of Corollary 10.4 and Corollary 10.8. We say that the representation is parabolics-preserving if it sends \( \pi_1 \partial M \) to parabolic elements. (It is easy to see from the explicit description in the proof of [23, Proposition 3.1] that \( \rho^\tau \) is reductive and parabolics-preserving if \( \rho \) is.)

**Corollary 10.4.** Let \( M \) be a compact, orientable 3-manifold, \( \Sigma \subset M \) a properly embedded, incompressible, boundary-incompressible, 2-sided surface, \( \tau : \Sigma \to \Sigma \) an orientation-preserving diffeomorphism of finite order and \( M^\tau \) the manifold obtained by cutting \( M \) along \( \Sigma \) and regluing via \( \tau \).

If \( M \) and \( M^\tau \) are hyperbolic and if the reductive, parabolics-preserving representation \( \rho : \pi_1 M \to SU(2, 1) \) satisfies \( \rho ( \tau_* \sigma ) = A \rho ( \sigma ) A^{-1} \) for some \( A \in SU(2, 1) \) and all \( \sigma \in \pi_1 \Sigma \), then

\[
\beta_{FW} ( M ) \otimes 1 = \beta_{FW} ( M^\tau ) \otimes 1 \in \mathcal{B} ( \mathbb{C} ) \otimes \mathbb{Q}
\]

with respect to the representations \( \rho \) and \( \rho^\tau \).

**Proof.** The proof is essentially the same as the one of Corollary 10.8 below, which in turn is essentially the same as that for [23, Theorem 1]. Therefore we omit the proof at this point and just mention that literally the same
Corollary 10.8 shows that
and in view of Lemma 10.3 this implies $\beta_{FW}(M) \otimes 1 = \beta_{FW}(M^r) \otimes 1$. □

10.2. Tetrahedra of flags.

Definition 10.5. ([1, Section 2]): Let

$$F\ell(C) = \{(\{x\}, \{f\}) \in P(C^3) \times P(C^{3*}) : f(x) = 0\}$$

where $P(V)$ denotes the projectivization of $V$. For

$$T = (\{(x_0, f_0), (x_1, [f_1]), (x_2, [f_2]), (x_3, [f_3])\}) \in C_3(F\ell(C))$$

and $a \neq b \in \{0, 1, 2, 3\}$ we define $z_{ab} \in C$ as follows: choose $k, l \in \{0, 1, 2, 3\}$ such that $(a, b, k, l)$ is a positive permutation of $(0, 1, 2, 3)$ and let

$$z_{ab} := \frac{f_a(x_k)}{f_a(x_l)} \det(x_a, x_b, x_l) \frac{\det(x_a, x_b, x_k)}{\det(x_a, x_b, x_l)}.$$

Then define

$$\beta : C_3(F\ell) \rightarrow P(C)$$

by

$$\beta(T) := [z_{01}] + [z_{10}] + [z_{23}] + [z_{32}].$$

Let $H_3(F\ell) := H_3(C_3(F\ell))$ for the canonical action of $G := SL(3, C)$ on $F\ell$. Then [1, Proposition 3.3] implies that $\beta$ yields a well-defined map $\beta_* : H_3(F\ell) \rightarrow P(C).

Moreover, if $M = \Gamma \backslash H_3^3$ is a hyperbolic 3-manifold and $h : CP^1 \rightarrow F\ell$ a map equivariant with respect to some homomorphism $\Gamma \rightarrow SL(3, C)$, then one obtains a well-defined chain map $h_* : C_*(CP^1)_\Gamma \rightarrow C_*(F\ell)_{SL(3, C)}$.

The following definition is due to Bergeron-Falbel-Guilloux ([1]).

Definition 10.6. If $M = \bigcup_{i=1}^r T_i$ is an ideal triangulation of a hyperbolic 3-manifold, $\rho : \pi_1M \rightarrow SL(3, C)$ a representation and

$$h : CP^1 \rightarrow F\ell(C)$$

a $\rho$-equivariant map, then define

$$\beta_h(M) := \sum_{i=1}^r \beta_* (h_* (P_0^i, P_1^i, P_2^i, P_3^i)) \in P(C),$$

where $P_0^i, P_1^i, P_2^i, P_3^i$ are the vertices of $T_i$.

We remark (compare [7, Proposition 10.79]) that $F\ell$ corresponds to the set $SL(3, C)/P$ of Weyl chambers in $\partial_{\infty}(SL(3, C)/SU(3))$ where $P$ is a minimal parabolic subgroup. If $\rho : \pi_1 M \rightarrow SL(3, C)$ is a reductive representation, then there exists a $\rho$-equivariant harmonic map $H_3^3 = \tilde{M} \rightarrow SL(3, C)/SU(3)$ (see [11,24]). But it is not easy to prove the existence of a $\rho$-equivariant boundary map $CP^1 \rightarrow \partial_{\infty}(SL(3, C)/SU(3))$. It should be easier to show
the existence of the boundary map \( h : \mathbb{C}P^1 \to \text{SL}(3, \mathbb{C})/P = \mathcal{F}l \) instead. We will not deal with that issue here, in this paper, we always assume the existence of such a map.

**Relation to hyperbolic Bloch invariant,** [1, Section 3.7]. If \( M \) is an orientable hyperbolic manifold, then by Culler’s Theorem its monodromy representation \( \Gamma \to \text{PSL}(2, \mathbb{C}) \) lifts to \( \text{SL}(2, \mathbb{C}) \). Composition with the (unique) irreducible representation \( \text{SL}(2, \mathbb{C}) \to \text{SL}(3, \mathbb{C}) \) yields representations \( \rho : \Gamma \to \text{SL}(3, \mathbb{C}) \). In this case there is a canonically (independent of \( \Gamma \)) defined \( \rho \)-equivariant map \( \mathbb{C}P^1 \to \mathcal{F}l \) as follows.

Recall that the irreducible 3-dimensional representation of \( \text{SL}(2, \mathbb{C}) \) can be defined as follows. Consider the \( \mathbb{C} \)-vector space of complex homogeneous polynomials of degree 2 in two variables. This is a 3-dimensional vector space \( V \) generated by \( x^2, xy \) and \( y^2 \). \( \text{SL}(2, \mathbb{C}) \) acts by \( (\text{AP})(x, y) := P(A^{-1}(x, y)) \).

We may consider its projectivization \( P(V) \) and the projectivization of the dual space \( P(V^*) \), whose elements we will write as homogeneous column vectors. Then a \( \rho \)-equivariant map \( h : \mathbb{C}P^1 \to \mathcal{F}l \subset P(V) \times P(V^*) \) is given by

\[
\begin{align*}
h([x, y]) &= ([x^2, xy, y^2], \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & -1 & 0 \end{bmatrix}^T).
\end{align*}
\]

([1] gives an apparently different construction which however - after computation - yields the same map \( h \).) It turns out that the so-defined \( \beta_h(M) \) coincides with 4 times the usual Bloch invariant \( \beta(M) \) of the hyperbolic 3-manifold \( M \). Indeed, if \( T = (P_0, P_1, P_2, P_3) \in C_3(\mathbb{C}P^1) \) is an ideal simplex of cross ratio \( t \), then explicit computation shows that the simplex \( h(T) = (h(P_0), h(P_1), h(P_2), h(P_3)) \in C_3(\mathcal{F}l) \) satisfies

\[
-z_{01}(h(T)) = z_{10}(h(T)) = z_{23}(h(T)) = z_{32}(h(T)) = t.
\]

**Relation to CR Bloch invariant,** [1, Section 3.8]. If \( D \) is the developing map of a reductive representation \( \pi_1 M \to \text{SU}(2, 1) \) and \( h \) is the composition of \( \partial_{\infty} D \) with the map \( S^3 \to \mathcal{F}l(\mathbb{C}) \) given in [1, Section 3.8], then \( \beta_h(M) \) coincides with \( \beta_{FW}(M) \).

**Lemma 10.7.** Let \( M \) be a finite-volume hyperbolic 3-manifold, and \( h : \mathbb{C}P^1 \to \mathcal{F}l(\mathbb{C}) \) equivariant with respect to some homomorphism \( \pi_1 M \to \text{SL}(3, \mathbb{C}) \). Then

\[
\beta_h(M) = \beta_{*h_*H}(ev) \text{EM}^{-1}[M, \partial M].
\]

**Proof.** Let \( c_0 \in \partial_{\infty} \mathbb{H}^3_\mathbb{R} \). The commutative diagram in the proof of Lemma 10.3 shows that \( H(ev_{c_0}) \text{EM}^{-1}[M, \partial M] \) is represented by \( \text{cr} \left( \tilde{\mathcal{C}}(\text{str} (z + \text{Cone} (\partial z))) \right) \) whenever \( z \in Z_* \left( \tilde{\mathcal{M}}, \partial \tilde{\mathcal{M}} \right) \) is a relative fundamental cycle.

In the proof of Lemma 10.3 we have seen that the homology classes of \( \text{cr} \left( \tilde{\mathcal{C}}(\text{str} (z + \text{Cone} (\partial z))) \right) \) and of \( \sum_{i=1}^{r} \text{cr}(T_i) \) are the same in \( H_* \left( C_* \left( \partial_{\infty} \tilde{\mathcal{M}} \right) \right)_\Gamma \).
whenever $M = \cup_{i=1}^{r} T_i$ is an ideal triangulation. Thus there is some $w \in C_\ast \left( \partial_\infty \tilde{M} \right)_\Gamma$ with
\[ \partial w = cr \left( \hat{C} \left( str \left( z + \text{Cone} \left( \partial z \right) \right) \right) \right) - \sum_{i=1}^{r} \left( cr \left( T_i \right) \right). \]

$h_\ast$ is a chain map by construction. Moreover, by [1, Proposition 3.3] (following from [13, Theorem 5.2]) we have that $\beta$ maps boundaries to zero, thus $\beta \left( \partial h_\ast \left( w \right) \right) = 0$, which implies

\[ \beta \left( h_\ast \left( cr \left( \hat{C} \left( str \left( z + \text{Cone} \left( \partial z \right) \right) \right) \right) \right) \right) - \sum_{i=1}^{r} \beta \left( h_\ast \left( cr \left( T_i \right) \right) \right) = \beta \left( \partial h_\ast \left( w \right) \right) = 0. \]

Thus the cycle $\sum_{i=1}^{r} \beta \left( h_\ast \left( cr \left( T_i \right) \right) \right)$ represents the homology class
\[ \beta_h H \left( ev_{c_0} \right) E M^{-1} [M, \partial M], \]
which implies the claim.

\[ \square \]

For the following corollary we will use the notations introduced before in Corollary 10.4. The following corollary applies for example when $M^\tau$ is a (generalized) mutation of $M$ and $\rho : \pi_1 M \rightarrow \text{SL}(3, \mathbb{C})$ is the composition of the inclusion $\pi_1 M \subset \text{SL}(2, \mathbb{C})$ with some representation $\text{SL}(2, \mathbb{C}) \rightarrow \text{SL}(3, \mathbb{C})$. In this case $\rho^\tau$ is obtained from the composition of the inclusion $\pi_1 M^\tau \subset \text{SL}(2, \mathbb{C})$ (see [23, Section 2]) with the same representation $\text{SL}(2, \mathbb{C}) \rightarrow \text{SL}(3, \mathbb{C})$ and we have $h = h^\tau$.

**Corollary 10.8.** Let $M$ be a compact, orientable 3-manifold, $\Sigma \subset M$ a properly embedded, incompressible, boundary-incompressible, 2-sided surface, $\tau : \Sigma \rightarrow \Sigma$ an orientation-preserving diffeomorphism of finite order and $M^\tau$ the manifold obtained by cutting $M$ along $\Sigma$ and regluing via $\tau$.

If $M$ and $M^\tau$ are hyperbolic and if the parabolics-preserving representation $\rho : \pi_1 M \rightarrow \text{SL}(3, \mathbb{C})$ satisfies $\rho (\tau_* \sigma) = A \rho (\sigma) A^{-1}$ for some $A \in \text{SL}(3, \mathbb{C})$ and all $\sigma \in \pi_1 \Sigma$, then
\[ \beta_h (M) \otimes 1 = \beta_{h^\tau} (M^\tau) \otimes 1 \in B(\mathbb{C}) \otimes \mathbb{Q} \]
when $h$ and $h^\tau$ are $\rho$- resp. $\rho^\tau$-equivariant maps from $\partial_\infty H^3_G$ to $\mathcal{F}_1$.

**Proof.** The proof is essentially the same as for [23, Theorem 1]. Since $\Sigma$ is a 2-sided, properly embedded surface, it has a neighborhood $N \simeq \Sigma \times [0, 1]$ in $M$, and a neighborhood $N^\tau \simeq \Sigma \times [0, 1]$ in $M^\tau$. The complements $M - \text{int} \left( N \right)$ and $M^\tau - \text{int} \left( N^\tau \right)$ are diffeomorphic and we let $X$ be the union of $M$ and $M^\tau$ along this identification of $M - \text{int} \left( N \right)$ and $M^\tau - \text{int} \left( N^\tau \right)$. The union of $N$ and $N^\tau$ yields a copy of the mapping torus $T^\tau$ in $X$. We have
\[ (2) \quad i_{M^\ast} [M, \partial M] - i_{M^\ast} [M^\tau, \partial M^\tau] = i_{T^\ast} [T^\tau, \partial T^\tau] \in H_3 (X, \partial X, \mathbb{Z}). \]
The made assumption implies that $\rho$ and $\rho^\tau$ extend to a representation $\rho_X : \pi_1 X \rightarrow \text{SL}(3, \mathbb{C})$. 
As in the proof of [23, Theorem 1] we have a finite cyclic covering $\hat{X} \to X$ such that $\hat{X}$ contains a copy of $\Sigma \times S^1$ finitely covering $T^r \subset X$. Let $\hat{M}, \hat{M}^r \subset \hat{X}$ be the preimages of $M$ and $M^r$. Application of the transfer map yields

$$i_{\hat{M}^*} \left[ \hat{M}, \partial \hat{M} \right] - i_{\hat{M}^r*} \left[ \hat{M}^r, \partial \hat{M}^r \right] = i_{\Sigma \times S^1*} \left[ \Sigma \times S^1, \partial \Sigma \times S^1 \right].$$

Again as in the proof of [23, Theorem 1] we obtain a representation $\rho_{\hat{X}} : \pi_1\hat{X} \to \text{SL}(3, \mathbb{C})$ and - because the lift $\hat{X}$ is chosen such that $\rho_{\hat{X}}(\pi_1\partial\hat{X})$ consists of parabolics - a continuous map

$$B\rho_{\hat{X}} : \left( B\pi_1\hat{X} \right)^{\text{comp}} \to \text{BSL}(3, \mathbb{C})^{\text{comp}}.$$

The classifying map $\Psi_{\hat{X}} : \hat{X} \to |B\pi_1\hat{X}|$ extends to

$$\Psi_{\hat{X}} : \text{Dcone} \left( \bigcup_{i=1}^s \partial_i\hat{X} \to \hat{X} \right) \to |\left( B\pi_1\hat{X} \right)^{\text{comp}}|$$

and the same argument as in [23] shows that $\left( |B\rho_{\hat{X}}| \Psi_{\hat{X}} i_{\Sigma \times S^1} \right)_* \text{ factors over } H_3(\Sigma, \partial\Sigma) = 0$ and is therefore 0. Thus Equation 3 implies

$$\left( |B\rho_{\hat{X}}| \Psi_{\hat{X}} i_{\hat{M}^r} \right)_* \left[ \hat{M}, \partial \hat{M} \right] = \left( |B\rho_{\hat{X}}| \Psi_{\hat{X}} i_{\hat{M}^r} \right)_* \left[ \hat{M}^r, \partial \hat{M}^r \right]$$

$$\in H_3 \left( |\text{BSL}(3, \mathbb{C})^{\text{comp}}| \right).$$

Again following the same argument from [23] we conclude

$$H(\rho) \left( EM^{-1} [M, \partial M]_{\mathbb{Q}} \right) = H(\rho^r) \left( EM^{-1} [M^r, \partial M^r]_{\mathbb{Q}} \right).$$

The $\rho$-equivariance of $h$ implies that $h \circ ev_{\text{SL}(2, \mathbb{C})} = ev_{\text{SL}(3, \mathbb{C})} \circ \rho$, hence $h_*H(ev) = H(ev)H(\rho)$ and thus

$$\beta_h(M) \otimes 1 = \beta_*h_*H(ev)EM^{-1} [M, \partial M]_{\mathbb{Q}}$$

$$= \beta_*H(ev)H(\rho) \left( EM^{-1} [M, \partial M]_{\mathbb{Q}} \right)$$

$$= \beta_*H(ev)H(\rho^r) \left( EM^{-1} [M^r, \partial M^r]_{\mathbb{Q}} \right)$$

$$= \beta_*h_*H(ev)EM^{-1} [M^r, \partial M^r]_{\mathbb{Q}}$$

$$= \beta_h(M^r) \otimes 1$$

in view of Lemma 10.7. \qed

The so-called Bloch regulator map

$$\rho : B(\mathbb{C}) \to \mathbb{C}/\mathbb{Q}$$

is known to send the Bloch invariant $\beta(M)$ of hyperbolic 3-manifolds to $\frac{1}{\pi} \left( \text{Vol}(M) + i\text{CS}(M) \right) \mod \mathbb{Q}$, as was proved in [26, Theorem 1.3]. In other words, the imaginary part of $\rho(\beta(M))$ determines the volume (and the real part determines the Chern-Simons invariant mod $\mathbb{Q}$). Thus it is
natural to define the volume of flag structures as (a multiple of) the imaginary part of $\rho(\beta_h(M))$. Bergeron-Falbel-Guilloux in fact define in [1, Section 3.6] the volume of a flag structure to be $\frac{2\pi^2}{4} \text{Im}(\rho(\beta_h(M)))$. The analogously defined volume of CR structures is always zero by [14, Theorem 3.12] but the volume of flag structures is a nontrivial and potentially interesting invariant. Corollary 10.4 of course implies its invariance under the cut-and-paste operation described in the statement of the corollary.

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