CONFORMALLY KÄHLER, EINSTEIN–MAXWELL METRICS AND BOUNDEDNESS OF THE MODIFIED MABUCHI FUNCTIONAL

ABDELLAH LAHDILI

Abstract. We prove that if a compact smooth polarized complex manifold admits in the corresponding Hodge Kähler class a conformally Kähler, Einstein–Maxwell metric, or more generally, a Kähler metric of constant $(\xi, a, p)$-scalar curvature in the sense of [2, 31], then this metric minimizes the $(\xi, a, p)$-Mabuchi functional introduced in [31]. Our method of proof extends the approach introduced by Donaldson [21, 22] and developed by Li [35] and Sano–Tipler [39], via finite dimensional approximations and generalized balanced metrics. As an application of our result and the recent construction of Koca–Tønnesen-Friedman [30], we describe the Kähler classes on a geometrically ruled complex surface of genus greater than 2, which admit conformally Kähler, Einstein-Maxwell metrics.

1. Introduction

Let $(X, J, g)$ be a compact Kähler manifold of real dimension $2m \geq 4$. A Hermitian metric $\hat{g}$ is said to be conformally Kähler, Einstein–Maxwell metric (cKEM for short) if there exist a smooth positive function $f$ on $X$ such that $g := f^2 \hat{g}$ is a Kähler metric, and

(a) $\xi := J \text{grad}_g(f)$ is Killing for both $g$ and $\hat{g}$,
(b) $\hat{g}$ has constant scalar curvature, i.e. $\text{Scal}_{\hat{g}} = \text{const}$.

In dimensions 4, cKEM metrics have been first introduced and studied in [1, 32, 33] whereas a high dimensional extension of the theory within a formal GIT setting was found by Apostolov–Maschler [5]. A number of recent existence results appear in [23, 24, 30, 31].

The terminology refers to the fact [5] that the trace-free part of the Ricci tensor of a cKEM metric $\hat{g}$ can be written as $\omega^{-1} \circ \varphi_0$, where $\omega$ is the (closed) Kähler form of $(g, J)$ and $\varphi_0$ is a co-closed primitive $(1, 1)$-form. Notice that in dimension 4, both $\omega$ and $\varphi_0$ are then harmonic as being self-dual and anti-selfdual 2-forms, respectively, so that a cKEM metric is an example of a Riemannian metric satisfying the Einstein–Maxwell equations with cosmological constant in general relativity, see [18, 27]. We also notice that if we take in the above definition $f \equiv 1$, then $\hat{g} = g$ is simply a Kähler metric of constant scalar curvature (cscK), so that the theory of cKEM metrics naturally generalizes the theory of cscK metrics which is the subject of most active current research. It is thus natural to try to extend known results about cscK metrics to the cKEM setting.

One can put cKEM metrics in the general framework of Kähler metrics with constant $(\xi, a, p)$-scalar curvature, recently introduced in [2, 31]. Denote by $\text{Aut}_{\text{red}}(X)$ the reduced automorphism group of $X$, see e.g. [26], i.e. the closed subgroup of complex automorphisms of $X$ whose Lie algebra is the ideal of holomorphic vector fields with zeroes. Let $\xi$ be a fixed real holomorphic vector field with zeros, and suppose that $\xi$ is quasi-periodic, i.e. the flow of $\xi$ generates a real torus $\mathbb{T}_\xi \subset \text{Aut}_{\text{red}}(X)$. We let $\Omega \in H_{dR}^2(X, \mathbb{R}) \cap H^{1,1}(X, \mathbb{C})$ be a fixed Kähler class on $(X, J)$ and denote by $K_{\Omega}^{\xi}$ the space of $\mathbb{T}_\xi$-invariant Kähler forms $\omega \in \Omega$. It is well-known (see e.g. [20]) that for
any $\omega \in K^c_\Omega$ the vector field $\xi$ is Hamiltonian with respect to $\omega$, i.e. $i_\xi \omega = -df$ for a smooth function $f$ on $X$ called the Killing potential of $\xi$. We normalize $f = f(\xi,\omega,a)$ by choosing a real positive constant $a > 0$ and requiring $\int_X f(\xi,\omega,a) \text{vol}_\omega = a$. As noticed in [5, Lemma 1], we can choose $a > 0$ so that $f(\xi,\omega,a) > 0$ for any $\omega \in K^c_\Omega$. For any real number $p \in \mathbb{R}$ we then define the $(\xi,a,p)$-scalar curvature of the Kähler metric $g = \omega(\cdot,J\cdot)$, $\omega \in K^c_\Omega$, to be

$$\text{Scal}_{(\xi,a,p)}(g) := f^2(\xi,\omega,a) \text{Scal}_g - 2(p-1)f(\xi,\omega,a) \Delta_g f(\xi,\omega,a) - p(p-1)|\xi|^2,$$

where $\text{Scal}_g$ is the usual scalar curvature, $\Delta_g$ is the Riemannian Laplacian on functions, and $|\cdot|_g$ is the tensor norm induced by $g$. The point of this definition is that it extends the usual scalar curvature (which corresponds to the case $\xi = 0$ and for the special value $p = 2m$ it computes the scalar curvature of the conformal metric $\tilde{g} = f^2(\xi,\omega,a)g$, so that cKEM metrics correspond to Kähler metrics with constant $(\xi,a,2m)$-scalar curvature (where $\xi = J\text{grad}_g f$ and $a = \int_X f \text{vol}_\omega$). Other values of the parameter $p$ have interesting geometric interpretations too, see e.g. [2, 34].

For a fixed Kähler form $\omega \in K^c_\Omega$ we denote by $K^c_\omega$ the space of smooth $T_\xi$-invariant Kähler potentials with respect to $\omega$:

$$K^c_\omega = \{\phi \in C^\infty(X,\mathbb{R})^\xi \mid \omega_{\phi} := \omega + dd^c\phi > 0\},$$

where $C^\infty(X,\mathbb{R})^\xi$ is the space of $T_\xi$-invariant functions. Following [31], the Kähler metrics with constant $(\xi,a,p)$-scalar curvature are critical points of the $(\xi,a,p)$-Mabuchi energy $\mathcal{M}_{(\xi,a,p)} : K^c_\omega \to \mathbb{R}$, defined by

$$\left(\text{d}\mathcal{M}_{(\xi,a,p)}(\phi)\right)(\hat{\phi}) = -\int_X \frac{\hat{\phi} \text{Scal}_{(\xi,a,p)}(\phi)}{f^{p+1}(\xi,\phi,a)} \text{vol}_\phi,$$

$$\mathcal{M}_{(\xi,a,p)}(\omega) = 0,$$

for all $\phi \in T_\phi K^c_\omega \cong C^\infty(X,\mathbb{R})^\xi$. In the above formula $f(\xi,\phi,a)$ is the Killing potential of $\xi$ with respect to $\omega_{\phi}$ (normalized by the constant $a$ as above), $\text{vol}_\phi$ denotes the volume form $\omega_\phi^m/m!$, and

$$\text{Scal}_{(\xi,a,p)}(\phi) := \text{Scal}_{(\xi,a,p)}(g_\phi) - c(\Omega,\xi,a,p),$$

where $c(\Omega,\xi,a,p)$ is the weighted average of $\text{Scal}_{(\xi,a,p)}(\omega)$, i.e.

$$c(\Omega,\xi,a,p) := \frac{\int_X \text{Scal}_{(\xi,a,p)}(\omega)}{f^{p+1}(\xi,\omega,a)} \text{vol}_\omega,$$

which is independent of the choice of $\omega \in K^c_\Omega$, see [5, 31]. Notice that if $\xi = 0$, then $\mathcal{M}_{(\xi,a,p)}$ is the usual Mabuchi energy whose critical points are the cscK metrics.

A corner-stone result in the theory of cscK metrics is the fact such a metric minimizes the Mabuchi energy in the corresponding Kähler class [9, 16, 22, 35, 39]. In this paper, we establish an analogous result concerning Kähler metrics of constant $(\xi,a,p)$-scalar curvature in a given Hodge Kähler class $\Omega = 2\pi c_1(L)$, where $L$ is a positive holomorphic line bundle on $X$. Our proof will follow Donaldson’s method [22] via approximations with balanced metrics in the case when $\text{Aut}_{\text{red}}(X)$ is discrete, and its ramifications found by C. Li [35] and Sano–Tipler [39] in the case of a general smooth polarized complex manifold.
Theorem 1. Let \((X, L)\) be a compact smooth polarized complex manifold and \(\xi\) a quasi-periodic real holomorphic vector field on \(X\), generating a real torus \(T_\xi\) in Aut_{red}(X). Then, the \((\xi, a, p)\)-Mabuchi energy of the Kähler class \(\Omega = 2\pi c_1(L)\) attains its minimum at a \(T_\xi\)-invariant \((\xi, a, p)\)-constant scalar curvature Kähler metric in \(\Omega\).

As we have already observed, the above result extends \([35]\) to the general cKEM setting. One can also define a notion of \((\xi, a, p)\)-extremal Kähler metric (see \([31]\), by requiring that the \((\xi, a, p)\)-scalar curvature defined by \([2]\) is a Killing potential for the Kähler metric \(g\), rather than being just a constant. One can also introduce (see \([31]\)) a relative version of the functional \(\mathcal{M}_{(\xi,a,p)}\), reminiscent to the relative Mabuchi energy describing the extremal Kähler metrics \([20]\). Presumably, the methods of this article can be adapted to extend Theorem 1 to the relative \((\xi, a, p)\)-extremal case, along the lines of \([39]\), but we shall discuss this and sum related questions in a forthcoming work.

One might also hope to extend Theorem 1 beyond the polarized case. Indeed, in the cscK case such an extension have been found via a deep result of Berman-Berndtsson \([9]\) on the convexity and boundedness of the Mabuchi functional. We expect that along the method of \([11]\) similar properties can possibly be established for the the functional \(\mathcal{M}_{(\xi,a,p)}\), but the details go beyond the scope of the present article.

It is observed in \([3]\) that the analogous result of Theorem 1 about external Kähler metrics can be used in order to classify, on a geometrically ruled complex surface, the Kähler classes admitting extremal Kähler metrics. Similarly, combining Theorem 1 above with the construction of \([30]\), we show

**Corollary 1.** Let \(X = \mathbb{P}(\mathcal{O} \oplus \mathcal{L}) \to C\) be a geometrically ruled complex surface over a compact complex curve \(C\) of genus \(g \geq 2\), where \(\mathcal{L}\) is a holomorphic line bundle over \(C\) of positive degree, and \(\Omega_k = 2\pi (c_1(\mathcal{O}(2)\mathbb{P}(\mathcal{O} \oplus \mathcal{L})) + (1 + \kappa) \cdot c_1(\mathcal{L}))\), \(\kappa > 1\) is the effective parametrization of the Kähler cone of \(X\), up to positive scales, see e.g. \([8, 25]\). Then,

(a) There exists a real constant \(\kappa_0(X) > 1\), such that for each \(\kappa > \kappa_0(X)\), \(\Omega_\kappa\) admits a cKEM metric, see \([30]\);

(b) For any \(\kappa \in (1, \kappa_0(X)]\), \(\Omega_\kappa\) does not admit a cKEM metric.

Outline of the paper. Section 2 contains our main new technical observation (Corollary 2) which, in the polarized case \((X, L)\), identifies the \((\xi, a, p)\)-scalar curvature \([2]\) with a coefficient in the asymptotic expansion of a suitable \(\xi\)-equivariant weighted Bergman kernels associated to the vector spaces of holomorph sections \(H_k = H^0(X, L^k)\), where \(\xi\) denotes the lifted real holomorphic vector field on \(L\), associated to the data \((\xi, a)\). This result, which is based on the general theory of Berezin–Toeplitz operators developed by Charles \([13]\) and Zelditch–Zhou \([44]\), extends the famous Bergman kernel expansion theorem \([10, 14, 17, 36, 38, 41, 43]\) to a larger family of equivariant Bergman operators, and allows us to define a suitable notion of a \((\xi, p)\)-balanced Fubini–Study metric on \(\mathbb{P}(H^*_k)\) and hence of \((\xi, p)\)-balanced Kähler metrics on \(X\) (via the Kodaira embedding). We then show in Proposition 1 that, similarly to \([22]\), if a sequence of \((\xi, p)\)-balanced metrics on \(X\) converges as \(k \to \infty\) to a smooth Kähler metric \(\omega \in 2\pi c_1(L)\), then \(\omega\) must have constant \((\xi, a, p)\)-scalar curvature. In Section 3, we give the proof of Theorem 1 closely following the method of \([22, 33, 39]\). The main new ingredient here is the introduction of suitable functionals on the finite dimensional spaces of Fubini–Study metrics on \(\mathbb{P}(H^*_k)\), quantizing the \((\xi, p)\)-Mabuchi energy, and whose minima are the \((\xi, p)\)-balanced metrics (Proposition 2). With these in place, the proof of Theorem 1 is not materially different than the arguments in \([22, 35, 39]\). In the final Section 4, we give the proof of Corollary 1. It contains three main ingredients. The
first one is the definition in [30] of a polynomial $P_\kappa(x)$ of degree $\leq 4$, and a constant $a > 0$, associated to each normalized Kähler class $\Omega_\kappa$ on $X$, such that if $P_\kappa(x) > 0$ on $(-1, 1)$, it defines, via the Calabi construction, a Kähler metric of constant $(\xi, a, 4)$-scalar curvature in $\Omega_\kappa$, where $\xi$ is the generator of the $S^1$-action on $X$ by multiplications on $O$. As shown in [30], this yields the existence part (a) of Corollary [1]. The second ingredient, taken from [6, 7], is an expression of the $(\xi, a, 4)$-Mabuchi energy in terms of $P_\kappa(x)$: Theorem [1] above then yields the non-existence of cKEM metrics associated to $\Omega_\kappa$ in the case when $\kappa$ is rational and $P_\kappa(x)$ is negative somewhere on $(-1, 1)$. In order to deal with the limiting cases, i.e. when $\kappa$ is irrational or $P_\kappa(x)$ has a double root in $(-1, 1)$, we use the stability result [31, Theorem 2].

Acknowledgement

I am grateful to the anonymous referees for their constructive criticism which lead to improvements of the presentation. I would like to thank my supervisor V. Apostolov for his guidance and invaluable advice. I am grateful to V. Apostolov, G. Maschler and C. Tønnesen-Friedman for giving me access to [6] and allowing me to use their computation of the modified Mabuchi energy (Proposition [5] below) in order to establish Corollary [1].

2. Equivariant Bergman kernels and $(\hat{\xi}, p)$-balanced metrics

In this section, we specialize to the case of a compact smooth polarized complex manifold $(X, L)$, endowed with the Kähler class $\Omega = 2\pi c_1(L)$. We denote by $\xi$ the lift to $L$ of the holomorphic vector field $\xi$, by using the normalization constant $a$. Our main observation is the definition of a class of $\xi$-equivariant weighted Bergman kernels on the vector spaces $H_k := H^0(X, L^k)$ of holomorphic sections of $L^k$, having the $(\xi, a, p)$-scalar curvature [2] as a coefficient in their asymptotic expansions as $k \to \infty$ (see Lemma [1] below). This will allow us to introduce the notion of a $(\hat{\xi}, p)$-balanced Fubini–Study metric on $\mathbb{P}(H^*_k)$, and thus of a $(\hat{\xi}, p)$-balanced Kähler metric in $\Omega = 2\pi c_1(L)$, such that if a sequence of $(\hat{\xi}, p)$-balanced metrics on $X$ smoothly converges as $k \to \infty$ to a smooth Kähler metric $\omega \in 2\pi c_1(L)$, then $\omega$ must have constant $(\xi, a, p)$-scalar curvature (Proposition [1]).

2.1. Equivariant Bergman kernels. Let $(X, L)$ be a polarized manifold with the Kähler class $\Omega = 2\pi c_1(L)$, $\xi$ a quasi-periodic holomorphic vector field generating a real torus $T_\xi \subset \text{Aut}_{\text{red}}(X)$, and $h$ a Hermitian metric on $L$ whose Chern curvature 2-form is $F_h = \omega \in K^2_\Omega$. We identify the space of Hermitian metrics $h_\phi := e^{-2\phi}h$, $\phi \in C^\infty(X, \mathbb{R})^\xi$ with positive curvature forms $F_{h_\phi} > 0$, with the space $K^\xi_\omega$.

Let $a > 0$ be a normalization constant such that for all $\phi \in K^\xi_\omega$ we have $\int_X f(\xi, \phi, a) \text{vol}_\phi = a$ and $f(\xi, \phi, a) > 0$ on $X$ (see [5, Lemma 1]). Once the constant $a > 0$ is fixed, for any $\phi \in K^\xi_\omega$ one can consider the lift $\hat{\xi}$ of $\xi$ to the total space of $L$, given by

$$\hat{\xi} = \xi^H - f(\xi, \phi, a)\partial_\theta,$$

where $\xi^H$ denotes the horizontal lift on $L$ of $\xi$ with respect to the Chern connection $\nabla^\phi$ of $h_\phi$ on $L$, and $\partial_\theta$ stands for the vector field on $L$ generated by rotations on each fiber. It follows from the arguments in [5, Lemma 1] that with our normalization for $f(\xi, \phi, a)$, we have

$$f(\xi, \phi, a) = f(\xi, \omega, a) + df^\phi(\xi),$$

showing that $\hat{\xi}$ defined by (5) is independent of the choice of $\phi \in K^\xi_\omega$. Since $\hat{\xi}$ is uniquely determined by $(\xi, a)$, we replace the subscript $(\xi, a)$ with $\hat{\xi}$. 

Acknowledgement

I am grateful to the anonymous referees for their constructive criticism which lead to improvements of the presentation. I would like to thank my supervisor V. Apostolov for his guidance and invaluable advice. I am grateful to V. Apostolov, G. Maschler and C. Tønnesen-Friedman for giving me access to [6] and allowing me to use their computation of the modified Mabuchi energy (Proposition [5] below) in order to establish Corollary [1].
Let $\Phi^\xi_t$ (resp. $\Phi^{\hat{\xi}}_t$) denote the flow of $\xi$ (resp. $\hat{\xi}$). For any smooth section $s$ of $L$ we define
\[
(A^\xi_s)(x) := -\sqrt{-1} \frac{d}{dt} \bigg|_{t=0} \Phi^\xi_t \left( s \left( \Phi^{\xi}_t(x) \right) \right).
\]
It is well-known that (see e.g. [26, Proposition 8.8.2])
\[
(A^\xi_s)(x) = -\sqrt{-1} \nabla^\phi_x + f(\hat{\xi}, \phi).
\]
For a positive function $\Psi \in C^\infty(\mathbb{R})$, we consider the following inner product on the space $C^\infty(X, L)$ of smooth sections of $L$:
\[
\langle s, s' \rangle_{(\Psi, \xi, \phi)} := \int_X (s, s')_\phi \Psi(f(\hat{\xi}, \phi)) \text{vol}_\phi,
\]
where $(\cdot, \cdot)_\phi$ stands for the pointwise Hermitian product $h_\phi$ on $C^\infty(X, L)$. A straightforward calculation shows that $A^\xi_\xi$ is a Hermitian operator with respect to $\langle \cdot, \cdot \rangle_{(\Psi, \xi, \phi)}$, preserving the finite dimensional subspace of holomorphic sections $\mathcal{H} := H^0(X, L)$. We denote by $\Lambda(\xi)$ the spectrum of $(A^\xi_\xi)|_\mathcal{H}$, and let $f(\hat{\xi}, \phi)(X) = [a_0, a_1]$ be the (fixed) image of $X$ under the normalized Killing potential of $\xi$ (see [35, Lemma 1]). For an eigensection $s \in \mathcal{H}$ corresponding to the eigenvalue $\lambda \in \Lambda(\xi)$, we have by $\langle \cdot, \cdot \rangle_{(\Psi, \xi, \phi)}$
\[
\lambda \cdot |s|^2_{h_\phi}(x) = -\sqrt{-1} \partial \bar{\partial} |s|^2_{h_\phi}(\xi) + f(\hat{\xi}, \phi)(x) |s|^2_{h_\phi}(x),
\]
for all $x \in X$. At a point $x_0 \in X$ where $|s|^2_{h_\phi}$ attains its maximum, we have $\lambda = f(\hat{\xi}, \phi)(x_0) \in [a_0, a_1]$, showing that $\Lambda(\xi) \subset [a_0, a_1]$. For a smooth function $\Phi \in C^\infty(\mathbb{R})$, we can thus define $\Phi(A^\xi_\xi)$ using the spectral theorem.

**Definition 1.** [11, 40, 44] For a $\langle \cdot, \cdot \rangle_{(\Psi, \xi, \phi)}$-orthonormal basis $\{s_i | i = 0, \ldots, N\}$ of $\mathcal{H}$, the $(\Psi, \Phi, \hat{\xi})$-equivariant Bergman kernel of the Hermitian metric $\phi \in K_\omega$ is the smooth function on $X$ defined by
\[
B_{(\Psi, \Phi, \hat{\xi})}(\phi)(x) := \Psi(f(\hat{\xi}, \phi)(x)) \sum_{i=0}^N \left( \Phi(A^\xi_\xi)s_i(x), s_i(x) \right)_\phi.
\]
Clearly $B_{(\cdot, \cdot, \phi)}$ is independent of the choice of a $\langle \cdot, \cdot \rangle_{(\Psi, \xi, \phi)}$-orthonormal basis of $\mathcal{H}$.

For $k \in \mathbb{N}$, we denote by $A^{(k)}_\xi$ the induced operator on $C^\infty(X, L^k)$, and consider for $\phi \in K_\omega$ the $(\Psi, \Phi, \hat{\xi})$-equivariant Bergman kernel of the metric $h_\phi^k$ on $L^k$:
\[
B_{(\Psi, \Phi, \hat{\xi})}(k\phi)(x) = \Psi(f(\hat{\xi}, \phi)) \sum_{i=0}^{N_k} \left( \Phi \left( \frac{1}{k} A^{(k)}_\xi \right) s_i(x), s_i(x) \right)_\phi.
\]
where $\{s_i | i = 0, \ldots, N_k\}$ is a $\langle \cdot, \cdot \rangle_{(\Psi, \xi, \phi)}$-orthonormal basis of $\mathcal{H}_k := H^0(X, L^k)$.

Asymptotic expansions of $B_{(\cdot, \cdot, \phi)}$ for $k \gg 1$ are known to exist in many special cases, see e.g. [11, 40, 37]. In its full generality, needed for the applications in this paper, such an expansion has been shown to exist in [15, 44]. The result below is a direct corollary of the latter works. In particular, using [15, Proposition 12] we can determine explicitly the second coefficient of the expansion. We note that the coefficient appearing in [40, Proposition 7.12] seems to be incomplete.

**Theorem 2.** [15, 44] The $(\Psi, \Phi, \hat{\xi})$-equivariant Bergman kernel of the Hermitian metric $h_\phi^k$ on $L^k$, for $\phi \in K_\omega$, admits an asymptotic expansion when $k \gg 1$, given by
\[
(2\pi)^m B_{(\Psi, \Phi, \hat{\xi})}(k\phi) = \Phi(f(\hat{\xi}, \phi)) + \frac{1}{k} S_{(\Psi, \Phi, \hat{\xi})}(\phi) + O \left( \frac{1}{k^2} \right),
\]
where
\[ S_{(\Psi,\Phi,\hat{\xi})}(\phi) = \frac{1}{4} \Phi(f_{(\hat{\xi},\phi)}) \text{Scal}(\phi) + \frac{1}{2} \Phi(f_{(\hat{\xi},\phi)}) (\log \Psi)'(f_{(\hat{\xi},\phi)}) \Delta_{\phi}(f_{(\hat{\xi},\phi)}) + \left[ \frac{1}{4} \Phi''(f_{(\hat{\xi},\phi)}) - \frac{1}{2} \Phi'(f_{(\hat{\xi},\phi)}) (\log \Psi)'(f_{(\hat{\xi},\phi)}) - \frac{1}{2} \Phi(f_{(\hat{\xi},\phi)}) (\log \Psi)''(f_{(\hat{\xi},\phi)}) \right] |\xi|^{2}. \]

Moreover, the above expansion holds in \( C^\infty \), i.e. for any integer \( \ell \geq 0 \) there exist a constant \( C_\ell > 0 \) such that,
\[
\left\| (2\pi)^m B_{(\Psi,\Phi,\hat{\xi})}(k\phi) - \Phi(f_{(\hat{\xi},\phi)}) - \frac{1}{k} S_{(\Psi,\Phi,\hat{\xi})}(\phi) \right\|_{C^\ell} \leq \frac{C_\ell}{k^\ell}
\]

**Proof.** The above theorem is an application of [15, Proposition 12] and we refer the reader to this work for the relevant definitions we use below.

We consider the \((\cdot,\cdot)_{(\Psi,\Phi,\hat{\xi})}\)-self-adjoint Toeplitz operator \( T_k := \Pi_{(\Psi,\hat{\xi},k\phi)} A_{\xi} \Pi_{(\Psi,\hat{\xi},k\phi)} \) and we denote by \( T_k(x) \) the restriction to the diagonal \( \{ x = x' \} \) of its Schwartz kernel i.e.
\[
T_k(x) = \Psi(f_{(\hat{\xi},\phi)}(x)) \sum_{i=0}^{N_k} \left( \frac{1}{k}(A^{(k)}_{\xi} s_i)(x), s_i(x) \right)_{k\phi}.
\]

By a straightforward calculation using (8) and the formula \( \nabla^k_\xi (\partial \log |s_i|_{k\phi}^2)(\xi) s_i \) we have:
\[
T_k(x) = -\frac{\sqrt{-1}}{k} \left( \partial \Pi_{(\Psi,\hat{\xi},k\phi)} \right)(\xi) + \left[ f_{(\hat{\xi},\phi)}(x) + \frac{\sqrt{-1}}{k} \partial \log (\Psi(f_{(\hat{\xi},\phi)}))(\xi) \right] \Pi_{(\Psi,\hat{\xi},k\phi)}(x)
\]

where \( \Pi_{(\Psi,\hat{\xi},k\phi)}(x) \) is given by
\[
\Pi_{(\Psi,\hat{\xi},k\phi)}(x) = \Psi(f_{(\hat{\xi},\phi)}(x)) \sum_{i=0}^{N_k} |s_i|^2_{k\phi}(x).
\]

Thus, \( \Pi_{(\Psi,\hat{\xi},k\phi)}(x) \) admits the following \( C^\infty \)-asymptotic expansion (see e.g. [37, Theorem 4.1.2])

\[
(10) \quad \Pi_{(\Psi,\hat{\xi},k\phi)}(x) = 1 + \frac{1}{4k} \left[ \text{Scal}(\phi) + 2\Delta_{\phi}(\log \Psi(f_{(\hat{\xi},\phi)})) \right] + \mathcal{O}\left( \frac{1}{k^2} \right).
\]

It follows from [15, Proposition 3] and [10] that the symbol of \( T_k \) is given by
\[
\sigma(T_k) = f_{(\hat{\xi},\phi)} + \frac{1}{4} \left[ f_{(\hat{\xi},\phi)} \text{Scal}(\phi) + 2 f_{(\hat{\xi},\phi)} \Delta_{\phi}(\log \Psi(f_{(\hat{\xi},\phi)})) - 2(\log \Psi)'(f_{(\hat{\xi},\phi)}) |\xi|^2 \right] h + \ldots
\]

By [15, Proposition 12] (see also [21, Proposition 2.1]) \( \Phi(T_k) \) is a Toeplitz operator too, with symbol
\[
\sigma(\Phi(T_k)) = \Phi(f_{(\hat{\xi},\phi)}) + \frac{1}{4} S_{(\Psi,\Phi,\hat{\xi})}(\phi) h + \ldots
\]

The result follows. \( \square \)

In the special case when \( \Psi(t) := t^{-p-1} \) \( (p \in \mathbb{R}) \) and \( \Phi(t) := t^q \) \( (q \in \mathbb{R}) \), we denote the associated \((\Psi,\Phi,\hat{\xi})\)-equivariant Bergman kernel by \( B_{(\xi,p,q)}(k\phi) \). As a direct corollary of Theorem 2, we have

**Corollary 2.** We have the following \( C^\infty \) asymptotic expansion
\[
(2\pi)^m B_{(\xi,p,q)}(k\phi) = f_{(\xi,\phi)}^q + \frac{1}{k} S_{(\xi,p,q)}(\phi) + \mathcal{O}\left( \frac{1}{k^2} \right),
\]

where
\[
4S_{(\xi,p,q)}(\phi) = f_{(\xi,\phi)}^q \text{Scal}(\phi) - 2(2p-1)f_{(\xi,\phi)}^q \Delta_{\phi}(f_{(\xi,\phi)}) + (q-1)(q+2p-2)f_{(\xi,\phi)}^{q-2} |\xi|^2.
\]
2.2. \((\xi, p)\)-balanced metrics. We denote by \(\Lambda_k(\hat{\xi})\) the spectrum of \(\frac{1}{k} A^{(k)}_{\xi}\), and consider the decomposition of \(\mathcal{H}_k\) as a direct sum of eigenspaces for \(\frac{1}{k} A^{(k)}_{\xi}\):

\[
\mathcal{H}_k = \bigoplus_{\lambda \in \Lambda_k(\hat{\xi})} \mathcal{H}_k(\lambda).
\]

For \(\lambda \in \Lambda_k(\hat{\xi})\) let \(\mathcal{B}_\xi(\mathcal{H}_k(\lambda))\) denote the space of positive definite Hermitian forms which are \(\hat{\xi}\)-invariant on \(\mathcal{H}_k(\lambda)\), and

\[
\mathcal{B}_\xi(\mathcal{H}_k) := \bigoplus_{\lambda \in \Lambda_k(\hat{\xi})} \mathcal{B}_\xi(\mathcal{H}_k(\lambda)).
\]

For \(\Psi(t) = t^{-p+1}\), we denote the inner product \((\cdot, \cdot)\) by \((\cdot, \cdot)_{(\xi, k, \phi, p)}\).

Definition 2. We introduce the following quantization maps:

- The \((\hat{\xi}, p)\)-Hilbert map \(\text{Hilb}^k_{(\xi, p)}: \mathcal{K}_\xi^\omega \rightarrow \mathcal{B}_\xi(\mathcal{H}_k)\) which associates to every \(T_{\hat{\xi}}\)-invariant Kähler potential, the \(\hat{\xi}\)-invariant Hermitian inner product on \(\mathcal{H}_k\), given by

\[
\text{Hilb}^k_{(\xi, p)}(\phi) := \sum_{\lambda \in \Lambda_k(\hat{\xi})} \frac{1}{\lambda(p)} (\langle \cdot, \cdot \rangle_{(\xi, k, \phi, p)})_{\mathcal{H}_k(\lambda)},
\]

where for \(\lambda \in \Lambda_k(\xi)\), we have set

\[
\lambda(p) := \lambda^{-p+1} - \frac{c(\Omega, \hat{\xi}, p)}{4k} \lambda^{-p+1},
\]

with \(c(\Omega, \hat{\xi}, p)\) given by (4).

- The \((\xi, p)\)-Fubini–Study map \(\text{FS}^k_{(\xi, p)}: \mathcal{B}_\xi(\mathcal{H}_k) \rightarrow \mathcal{K}_\omega^\xi\) given by

\[
\text{FS}^k_{(\xi, p)}(H) := \frac{1}{2k} \log \left( \frac{C_k}{N} \sum_{i=0}^{N_k} |s_i|^2 \right),
\]

where \(C_k\) is the constant given by:

\[
C_k := \frac{w_{\xi}^{-p+1}(L^k) - \frac{c(\Omega, \hat{\xi}, p)}{4k} w_{\hat{\xi}}^{-p+1}(L^k)}{\int_X f_{(\xi, \omega)}^{-p+1} \text{vol}_{\omega}},
\]

and \(\{s_i\}\) is an adapted \(H\)-orthonormal basis of \(\mathcal{H}_k\).

Note that

\[
\omega_{\text{FS}^k_{(\xi, p)}(H)} = \Phi_k^* \omega_{\text{FS}, k},
\]

where \(\Phi_k\) is the Kodaira embedding of \(X\) to \(\mathbb{P}^{N_k}\) using the basis \(\{s_i\}\), and \(\omega_{\text{FS}, k}\) is the Fubini-Study metric on \(\mathbb{P}^{N_k}\).

Definition 3. We denote by \(\rho_{(\xi, p)}(k\phi)\) the Bergman kernel of \(\text{Hilb}^k_{(\xi, p)}(\phi)\), given by

\[
\rho_{(\xi, p)}(k\phi) := \int_{(\xi, p)} f_{(\xi, \phi)}^{-p+1} \sum_{i=1}^{N_k} |s_i|^2_{k\phi}.
\]

where \(\{s_i\}\) is a \(\text{Hilb}^k_{(\xi, p)}(\phi)\)-orthonormal basis of \(\mathcal{H}_k\).
We can easily see that \( \rho(\xi,p)(k\phi) \) is independent of the chosen basis, so we can take a basis adapted to (11), showing that

\[
(17) \quad \rho(\xi,p)(k\phi) = B(\xi,p,-p+1)(k\phi) - \frac{c_{(1,\xi,p)}}{4k} B(\xi,p,-p+1)(k\phi).
\]

Corollary [2] thus yields the following

**Lemma 1.** \( \rho(\xi,p)(k\phi) \) admits a \( C^\infty \) expansion when \( k \gg 1 \)

\[
(2\pi)^m \rho(\xi,p)(k\phi) = f^{-p+1}(\xi,\phi) + \frac{1}{4k} f^{-p+1}\cdot\text{Scal}(\xi,p)(\phi) + O\left(\frac{1}{k^2}\right),
\]

in the sense that for any integer \( \ell \geq 0 \) we have,

\[
\left\| (2\pi)^m \rho(\xi,p)(k\phi) - f^{-p+1}(\xi,\phi) - \frac{1}{4k} f^{-p+1}\cdot\text{Scal}(\xi,p)(\phi) \right\|_{C^\ell} \leq \frac{C_\ell}{k^2}.
\]

In particular,

\[
FS_{\xi,p}^k \circ \text{Hilb}_{\xi,p}^k(\phi) = \phi + O(k^{-2}).
\]

Following [19, 42, 45], we give the following definition

**Definition 4.** We say that a metric \( \phi \in K_\infty^\xi \) is \( (\xi,p) \)-balanced of order \( k \) if it satisfies:

\[
FS_{\xi,p}^k \circ \text{Hilb}_{\xi,p}^k(\phi) = \phi.
\]

Note that if a metric \( \phi \in K_\infty^\xi \) is \( (\xi,p) \)-balanced of order \( k \) then we have:

\[
\rho(\xi,p)(k\phi) = C_k f^{-p+1}(\xi,\phi),
\]

where \( C_k \) is the constant given by [14]. Similarly to [19] we have

**Proposition 1.** Let \( (\phi_j)_{j \geq 0} \) be a sequence in \( K_\infty^\xi \) such that every \( \phi_j \) is a \( (\xi,p) \)-balanced metric of order \( j \) and \( \phi_j \) converge in \( C^\infty \) to \( \phi \). Then \( \omega_\phi \) has a constant \( (\xi,p) \)-scalar curvature.

**Proof.** By Lemma [1] for \( k \gg 1 \),

\[
\left\| (2\pi)^m \rho(\xi,p)(k\phi_j) - f^{-p+1}(\xi,\phi_j) - \frac{1}{4k} f^{-p+1}\cdot\text{Scal}(\xi,p)(\phi_j) \right\|_{C^\ell} \leq \frac{C_\ell}{k^2}.
\]

Letting \( j = k \), we get

\[
(18) \quad \left\| (2\pi)^m C_k f^{-p+1}(\xi,\phi_k) - f^{-p+1}(\xi,\phi_k) - \frac{1}{4k} f^{-p+1}\cdot\text{Scal}(\xi,p)(\phi_k) \right\|_{C^\ell} \leq \frac{C_\ell}{k^2}.
\]

From (14) and (17) we get

\[
(2\pi)^m C_k = \frac{1}{X} \rho(\xi,p)(h^k\text{vol}_{\omega_\phi}) \frac{1}{X} f^{-p+1}\cdot\text{vol}_{\omega_\phi} = 1 + O(k^{-2}).
\]

Taking a limit when \( k \) goes to infinity in (18), we obtain that \( \text{Scal}(\xi,p)(\phi) = c_{(\xi,p)} \).
3. Boundedness of the \((\xi, p)\)-Mabuchi energy as an obstruction to the existence of Kähler metrics with constant \((\xi, p)\)-scalar curvature

In this section we prove Theorem 1 following the method of [22, 35, 39]. To this end, for each \(k \gg 1\), we introduce appropriate functionals on the finite dimensional space of Fubini–Study metrics on \(\mathbb{P}(H_k^\xi)\), which when identified with a subspace of \(\mathcal{K}_\omega^\xi\) via the Kodaira embedding, will quantize the \((\xi, p)\)-Mabuchi functional of \(\Omega = 2\pi c_1(L)\). Furthermore, following the main ideas of [22, 35, 39], we will show that the \((\xi, p)\)-balanced metrics are minima of these functionals, and that a Kähler metric with constant \((\xi, a, p)\)-scalar curvature induces almost \((\xi, p)\)-balanced Fubini–Study metrics on \(\mathbb{P}(H_k^\xi)\) for \(k \gg 1\), i.e. they minimize the corresponding functionals up to an error that goes to zero.

3.1. Quantization of the \((\xi, p)\)-Mabuchi energy. We start with introducing finite dimensional analogues of the \((\xi, p)\)-Mabuchi energy \((\ref{eq:Mabuchienergy})\), given by \((\ref{eq:kappa_k})\) on the spaces \(\mathcal{B}_\xi(H_k)\) and \(\mathcal{F}_{(\xi, p)}^k\left(\mathcal{B}_\xi(H_k)\right)\) (see Definition 2), respectively, thus a framework for the proof of Theorem 1 along the lines of [22, 35, 39].

We introduce the functional \(I_{(\xi, p)}^k : \mathcal{B}_\xi(H_k) \to \mathbb{R}\) by
\[
I_{(\xi, p)}^k(H) = \sum_{\lambda \in \Lambda_k(\xi)} \lambda(p) \log |\det H_\lambda|,
\]
where we recall that \(\lambda(p)\) is the expression \((\ref{eq:lambda(p)})\). This functional is a quantization of the Aubin type functional \(I_{(\xi, p)}^k\) on \(\mathcal{K}_\omega^\xi\), given by its first variation,
\[
\begin{aligned}
\left(\frac{dI_{(\xi, p)}^k}{d\phi}\right)_\phi(\phi) &= 2kC_k \int_X \phi f_{(\xi, \phi)}^{-p+1} \text{vol}_{k\omega}\phi \\
\left(\frac{dI_{(\xi, p)}^k}{d\omega}\right)_\omega &= 0
\end{aligned}
\]
where \(C_k\) is given by \((\ref{eq:Ck})\). It is then straightforward to check that

**Lemma 2.** 1. We have the following expressions for the variations of the Aubin functional \(I_{(\xi, p)}^k\) and its finite dimensional version \(I_{(\xi, p)}^k\):
\[
\begin{aligned}
\left(\frac{dI_{(\xi, p)}^k}{d\phi}\right)_\phi(\phi) = &2kC_k \int_X \phi \left(1 + \frac{\Delta_\phi}{2k}\right) f_{(\xi, \phi)}^{-p+1} \text{vol}_{k\omega}\phi \\
&- C_k \int X (d\phi, d\log f_{(\xi, \phi)}^{-p+1} g_{(\xi, \phi)} f_{(\xi, \phi)}^{-p+1} \text{vol}_{k\omega}\phi,
\end{aligned}
\]
\[
\left(\frac{dI_{(\xi, p)}^k}{dH}\right)_H(H) = \sum_{\lambda \in \Lambda_k(\xi)} \lambda(p) \text{Tr}(H_\lambda^{-1} H_\lambda),
\]
where \(\phi \in \mathcal{K}_\omega^\xi\) and \(H = (H_\lambda)_{\lambda \in \Lambda_k(\xi)} \in \mathcal{B}_\xi(H_k)\).

2. The second variation of \(I_{(\xi, p)}^k\) along a path \(\phi_t \in \mathcal{K}_\omega^\xi\) is given by
\[
\frac{d^2I_{(\xi, p)}^k}{dt^2}(\phi_t) = 2kC_k \int_X \left(\dot{\phi}_t - |d\dot{\phi}_t|_{g_{(\xi, \phi)}}^2\right) f_{(\xi, \phi)}^{-p+1} \text{vol}_{k\omega}\phi_t.
\]

3. For \(\phi \in \mathcal{K}_\omega^\xi\) and \(k \gg 1\), the Aubin functional \(I_{(\xi, p)}^k\) is concave along the path \((\phi_k(t))_{t \in [0, 1]}\) of \(\mathcal{K}_\omega^\xi\) given by:
\[
\phi_k(t) := \phi + \frac{t}{2k} \log \left(f_{(\xi, \phi)}^{p-1} \rho_{(\xi, p)}(k\phi)\right).
\]
4. The variation of $(\hat{\xi}, p)$-Hilbert map $\text{Hilb}^k_{(\hat{\xi}, p)}$ is given by,

\[
\left(\frac{d \text{Hilb}^k_{(\hat{\xi}, p)}}{d\phi}\right)(s, s') = \sum_{\lambda \in \Lambda_k(\hat{\xi})} \frac{1}{\lambda(p)} \int_X (s(\lambda), s'(\lambda))_{k\phi}[2k\dot{\phi} - (d \log f_{(\hat{\xi}, \phi)}^{-p+1}, d\dot{\phi})_{g_\phi} + \Delta_\phi \dot{\phi}]f_{(\hat{\xi}, \phi)}^{-p+1}\text{vol}_{\omega_\phi},
\]

where $\phi \in \mathcal{K}^k_\omega$ and $s, s' \in \mathcal{H}_k$.

We now consider the functionals $\mathcal{L}^k_{(\hat{\xi}, p)} : \mathcal{K}_\omega \to \mathbb{R}$ and $\mathcal{Z}^k_{(\hat{\xi}, p)} : \mathcal{B}_\xi(\mathcal{H}_k) \to \mathbb{R}$ defined by

\[
\mathcal{L}^k_{(\hat{\xi}, p)} = I^k_{(\hat{\xi}, p)} \circ \text{Hilb}^k_{(\hat{\xi}, p)} + \eta^k_{(\hat{\xi}, p)},
\]

\[
\mathcal{Z}^k_{(\hat{\xi}, p)} = \mathbb{I}^k_{(\hat{\xi}, p)} \circ \text{FS}^k_{(\hat{\xi}, p)} + \mathcal{I}^k_{(\hat{\xi}, p)},
\]

where $I^k_{(\hat{\xi}, p)}$ is given by (19) and $\mathbb{I}^k_{(\hat{\xi}, p)}$ is given by (20). In what follows we will relate these functionals to the $(\hat{\xi}, p)$-balanced metrics, similarly to [22] and [33], and we will show that they quantize the $(\hat{\xi}, p)$-Mabuchi energy.

**Proposition 2.** The $(\hat{\xi}, p)$-balanced metrics of order $k$ are critical points of the functional $\mathcal{L}^k_{(\hat{\xi}, p)}$. Furthermore, there exist real constants $b_k$ such that,

\[
\lim_{k \to \infty} \left[ \frac{2}{k^{m}} \mathcal{L}^k_{(\hat{\xi}, p)} + b_k \right] = \mathcal{M}_{(\hat{\xi}, p)},
\]

where the convergence holds in the $C^\infty$-norm.

**Proof.** For a $\text{Hilb}^k_{(\hat{\xi}, p)}$-orthonormal adapted basis $\{s_{i, \lambda} \mid \lambda \in \Lambda_k(\hat{\xi}), i = 1, \ldots, n_k(\lambda)\}$ of $\mathcal{H}_k$, by (22) and (27) we have,

\[
d \left( I^k_{(\hat{\xi}, p)} \circ \text{Hilb}^k_{(\hat{\xi}, p)} \right)_{\phi}(\dot{\phi}) = -\sum_{i, \lambda} \int_X |s_{i, \lambda}|^2[2k\dot{\phi} - (d \log f_{(\hat{\xi}, \phi)}^{-p+1}, d\dot{\phi})_{g_\phi} + \Delta_\phi \dot{\phi}]f_{(\hat{\xi}, \phi)}^{-p+1}\text{vol}_{\omega_\phi}
\]

\[
= \int_X \rho_{(\hat{\xi}, p)}(k\phi)[2k\dot{\phi} - (d \log f_{(\hat{\xi}, \phi)}^{-p+1}, d\dot{\phi})_{g_\phi} + \Delta_\phi \dot{\phi}]\text{vol}_{\omega_\phi}
\]

\[
= -2k \int_X \dot{\phi} \left( 1 + \frac{\Delta_\phi}{2k} \right) \rho_{(\hat{\xi}, p)}(k\phi)\text{vol}_{\omega_\phi} + \int_X \rho_{(\hat{\xi}, p)}(k\phi)(d \log f_{(\hat{\xi}, \phi)}^{-p+1}, d\dot{\phi})_{g_\phi} \text{vol}_{\omega_\phi}.
\]

By (24) we get

\[
\left( d\mathcal{L}^k_{(\hat{\xi}, p)} \right)_{\phi}(\dot{\phi}) = -2k \int_X \dot{\phi} \left( 1 + \frac{\Delta_\phi}{2k} \right) \left[ \rho_{(\hat{\xi}, p)}(k\phi) - C_k f_{(\hat{\xi}, \phi)}^{-p+1} \right] \text{vol}_{\omega_\phi}
\]

\[
+ \int_X \left[ \rho_{(\hat{\xi}, p)}(k\phi) - C_k f_{(\hat{\xi}, \phi)}^{-p+1} \right] (d \log f_{(\hat{\xi}, \phi)}^{-p+1}, d\dot{\phi})_{g_\phi} \text{vol}_{\omega_\phi}.
\]

From the above expression it is clear that a $(\hat{\xi}, p)$-balanced metric of order $k$ is critical point of $\mathcal{L}^k_{(\hat{\xi}, p)}$. By the asymptotic expansion in Lemma II we get

\[
\int_X \left[ \rho_{(\hat{\xi}, p)}(k\phi) - C_k f_{(\hat{\xi}, \phi)}^{-p+1} \right] (d \log f_{(\hat{\xi}, \phi)}^{-p+1}, d\dot{\phi})_{g_\phi} \text{vol}_{\omega_\phi} = O(k^{m-1}),
\]
and
\[
2k \int_X \phi \left(1 + \frac{\Delta_{\phi}}{2k}\right) \left[\rho_{(\xi,p)}(k\phi) - C_k f_{(\xi,\phi)}^{-p+1}\right] \text{vol}_{\kappa}\phi
= 2k^m \int_X \text{Scal}_{(\xi,p)}(\phi) f_{(\xi,\phi)}^{-p+1} \text{vol}_{\kappa}\phi + \mathcal{O}(k^m-1)
= 2k^m \left(\mathcal{M}_{(\xi,p)}(\phi)\right)(\phi) + \mathcal{O}(k^m-1).
\]

The proof is complete.

\[\square\]

**Remark 1.** There is a natural extension of the momentum map interpretation of balanced Fubini-Study metrics given by S. K. Donaldson in [19] to \((\hat{\xi}, p)\)-balanced metrics. Indeed, let us identify \(B_{\hat{\xi}}(H_k)\) with the space of bases of \(H_k\) compatible with the splitting [13], and denote by \(\text{Aut}\hat{\xi}(X, L)\) the Lie group of automorphisms of the pair \((X, L)\) that commutes with the flow of \(\hat{\xi}\). Let \(\theta_{(\xi, k)}\) denote the group representation of \(\text{Aut}\hat{\xi}(X, L)\) in \(\text{GL}(H_k)\), given by

\[\theta_{(\xi, k)}(\gamma) s := \gamma \circ s \circ p(\gamma)^{-1},\]

where \(p : \text{Aut}(X, L) \to \text{Aut}_{\text{equiv}}(X)\) is the natural projection. For each \(k\) we have the following group actions on \(B_{\hat{\xi}}(H_k)\):

- \(\mathbb{C}^*\) by scalar multiplications;
- \(A_{(\xi, k)} := \theta_{(\xi, k)} \left(\text{Aut}\hat{\xi}(X, L)\right)\);
- \(G_{(\xi, k)} := \left\{ H \in \prod_{\lambda \in \Lambda_k(\xi)} \mathbb{U}(H_k(\lambda)) \mid \prod_{\lambda} \det(g_{\lambda})^{\lambda(p)} = 1 \right\}\).

We consider the quotients space

\[Z_{\xi}(H_k) = B_{\xi}(H_k) \big/ \left(\mathbb{C}^* \times A_{(\xi, k)}\right),\]

on which we have a natural action of \(G_{(\xi, k)}\). The quotient \(Z_{\xi}(H_k)\) carries a natural Kähler structure, defined as follows:

- The multiplication by \(\sqrt{-1}\) defines an integrable complex structure on \(B_{\xi}(H_k)\) invariant under the action of \(\mathbb{C}^* \times A_{(\xi, k)}\), so it descends to a complex structure \(J^k_Z\) on the quotient \(Z_{\xi}(H_k)\).
- There is a natural Kähler form on \(B_{\xi}(H_k)\) given by

\[\varpi^k_B := d^c Z^k_{(\xi, p)},\]

where \(d^c := J^k_{B} d\). The form \(\varpi^k_B\) is invariant under the group actions of \(\mathbb{C}^* \times A_{(\xi, k)}\) and \(G_{(\xi, k)}\), so it defines a \(G_{(\xi, k)}\)-invariant Kähler form on \(Z_{\xi}(H_k)\).

We endow \(\text{Lie}(G_{(\xi, k)})\) with the pairing

\[\langle a, b \rangle_{(\xi, p, k)} = \sum_{\lambda \in \Lambda(\xi)} \lambda(p) \cdot \text{Tr} \left( a_\lambda b^*_\lambda \right),\]

and identify \(\text{Lie}(G_{(\xi, k)})\) with the dual vector space by using \(\langle \cdot, \cdot \rangle_{(\xi, p, k)}\). The action of \(G_{(\xi, k)}\) on \(Z_{\xi}(H_k)\) is Hamiltonian with \(\varpi^k_Z\)-moment map \(\mathcal{M}^k_{(\xi, p)} : Z_{\xi}(H_k) \to \text{Lie}(G_{(\xi, k)})\) given by

\[\mathcal{M}^k_{(\xi, p)}(s) := \sqrt{-1} \left( \bigoplus_{\lambda \in \Lambda(\xi)} \left( \text{Hilb}^k_{(\xi, p)}(\text{FS}^k_{(\xi, p)}(s)) \right)_{i, j = 1, n(\lambda)} \right)_{i, j = 1, n(\lambda)}.\]
where for any \( s \in B_{\xi}(H_k) \) we identify \( s \) with the unique positive definite Hermitian form so that \( s \) is orthonormal, and for any \( a \in \text{Lie}(G_{(\xi, k)}) \),

\[
(a)_{0} = a - \frac{\langle a, \text{Id} \rangle_{(\xi, p, k)}}{\langle \text{Id, Id} \rangle_{(\xi, p, k)}} \text{Id}.
\]

Thus the zeroes of the moment map \( M_{(\xi, p)}^{(k)} \) are the \((\xi, p)\)-balanced elements of \( Z_{\xi}(H_k) \).

**Lemma 3.** For all \( \phi \in K_{\xi}^{\mathbb{C}} \) we have (in the \( C^\infty \) sense),

\[
\lim_{k \to \infty} k^{-m} \left[ L_{(\xi, p)}^{k}(\phi) - Z_{(\xi, p)}^{k} \circ \text{Hilb}_{(\xi, p)}^{k}(\phi) \right] = 0.
\]

The functional \( Z_{(\xi, p)}^{k} \) is convex along the geodesics of \( B_{\xi}(H_k) \).

**Proof.** The proof of the above Lemma is identical to the arguments of [39] Proposition 3.2.3, and [22] Proposition 1. □

**Corollary 3.** \((\xi, p)\)-balanced metrics of order \( k \) minimizes the functional \( Z_{(\xi, p)}^{k} \) on \( B_{\xi}(H_k) \).

**Proof.** Let \( H(t), t \in \mathbb{R} \) be a geodesic in \( B_{\xi}(H_k) \) with \( H(0) = H \) and such that \( h = \text{FS}_{(\xi, p)}^{k}(H) \) is \((\xi, p)\)-balanced. For a choice of an adapted \( H \)-orthonormal basis of \( H_k \) denoted by \( \{ s_{i, \lambda} | \lambda \in \Lambda_k(\xi), i = 1, \cdots, n(\lambda) \} \) we have the following expression for \( H(t) \)

\[
H(t) = \text{diag} \left( e^{tA_{\lambda}} \right)_{\lambda \in \Lambda_k(\xi)},
\]

with \( A_{\lambda} = \text{diag}(a(\lambda))_{i=1}^{n(\lambda)} \), \( a(\lambda) \in \mathbb{R} \) and \( \text{Tr}(A_{\lambda}) = 0 \). We consider the family of Kähler potentials given by \( \phi(t) := \text{FS}_{(\xi, p)}^{k}(H(t)) \). The collection \( \{ e^{-t/a_{\lambda}(\lambda)}s_{i, \lambda} | \lambda \in \Lambda_k(\xi), i = 1, \cdots, n(\lambda) \} \) is an \( H(t) \)-orthonormal adapted base of \( H_k \), so we have

\[
Z_{(\xi, p)}^{k}(t) := Z_{(\xi, p)}^{k}(H(t)) = \|_{(\xi, p)}^{k}(\phi(t)).
\]

Using the fact that \( \sum_{\lambda \in \Lambda_k(\xi)} \sum_{i=0}^{n(\lambda)} |s_{i, \lambda}|_{h_{k}}^{2} = 1 \) (because \( h = \text{FS}_{(\xi, p)}^{k}(H) \)) we get,

\[
\phi = -\frac{1}{2k} \sum_{\lambda \in \Lambda_k(\xi)} \sum_{i=0}^{n(\lambda)} a_{\lambda}(s_{i, \lambda})^{2}_{h_{k}}.
\]

If the Hermitian metric \( h^{k} \) on \( L^{k} \) corresponds to a \((\xi, p)\)-balanced metric \( \phi \in K_{\xi}^{\mathbb{C}} \) of order \( k \), we have,

\[
\frac{dZ_{(\xi, p)}^{k}(t)}{dt}(0) = -C_{k} \int_{X} \sum_{i, \lambda} a_{\lambda}(s_{i, \lambda})^{2}_{h_{k}} f_{(\xi, p)}^{-p+1} \text{vol}_{\omega} = -C_{k} \sum_{i, \lambda} a_{\lambda}(\lambda)(p) \| s_{i, \lambda} \|_{(\xi, p)}^{2} \text{Hilb}_{(\xi, p)}^{k}(\text{FS}_{(\xi, p)}^{k}(H)) = -C_{k} \sum_{\lambda \in \Lambda_k(\xi)} \lambda(p) \text{Tr}(A_{\lambda}) = 0.
\]

Thus, \( H \) is a critical point of \( Z_{(\xi, p)}^{k} \) and by the convexity of \( Z_{(\xi, p)}^{k} \), \( H \) is a minimum. □

Now suppose that \( K_{\xi}^{\mathbb{C}} \) contains a metric \( \phi^{*} \) with \((\xi, p)\)-scalar curvature. We will show in the following proposition that the metrics \( \text{Hilb}_{(\xi, p)}^{k}(\phi^{*}) \) are almost balanced in the sense that they minimizes \( Z_{(\xi, p)}^{k} \), up to an error that goes to zero.
Proposition 3. For all $\phi \in K^\xi$, there exists a smooth function $\varepsilon_\phi(k)$, such that
\[
\lim_{k \to \infty} \varepsilon_\phi(k) = 0 \text{ in } C^\ell(X, \mathbb{R}) \text{ and,}
\]
\[
k^{-m} Z^k_{\xi, p} \circ \text{Hilb}^k_{\xi, p}(\phi) \geq k^{-m} Z^k_{\xi, p} \circ \text{Hilb}^k_{\xi, p}(\phi^*) + \varepsilon_\phi(k).
\]

Proof. We denote $H_k = \text{Hilb}^k_{\xi, p}(\phi)$ and $H^*_k = \text{Hilb}^k_{\xi, p}(\phi^*)$. For a choice of an adapted $H^*_k$-orthonormal basis $\{s_{i, \lambda}\}$ of $H_k$ we can write $H_k = \text{diag}(e^{A_\lambda})_{\lambda \in \Lambda_k(\xi)}$ with $A_\lambda = \text{diag}(a_i(\lambda))_{i=1, \ldots, n(\lambda)}$, $\text{Tr}(A_\lambda) = 0$, and consider the geodesic that joins $H^*_k$ to $H_k$,
\[
H_k(t) = \text{diag}(e^{tA_\lambda})_{\lambda \in \Lambda_k(\xi)}.
\]

Let $P_k(t) := Z^k_{\xi, p}(H_k(t))$. $P_k(t)$ is a convex function by Lemma 3. It follows that,
\[
k^{-m} \left( Z^k_{\xi, p}(H_k) - Z^k_{\xi, p}(H^*_k) \right) \geq k^{-m} P'_k(0).
\]

Letting $\varepsilon_\phi(k) := k^{-m} P'_k(0)$, we have
\[
P'_k(0) = 2kC_k \int X \phi^* f^{-p+1}_p \omega_p \omega^*, = -C_k \int X \rho_A(k \phi^*) f^{-p+1}_p \omega_p \omega^*,
\]
where
\[
(28) \quad \rho_A(k \phi^*) = \sum_{i, \lambda} a_i(\lambda) f^{-p+1}_p(s_{i, \lambda})^2.
\]

By Lemma 11 since $\phi^*$ has constant $(\xi, p)$-scalar curvature we get
\[
(29) \quad \rho_{(\xi, p)}(k \phi^*) = f^{-p+1}_p + O(k^{-2}),
\]
and therefore we obtain
\[
P'_k(0) = -C_k \int X \rho_A(k \phi^*) O(k^{-2}) \omega_p \omega^*.
\]

We have
\[
e^{a_i(\lambda)} = \| s_{i, \lambda} \|_{H_k}^2 = \sum_{\lambda \in \Lambda_k(\xi)} \lambda(p)^{-1} \int X |s_{i, \lambda}|^2 f^{-p+1}_p \omega_p \omega^*.
\]

As $h^k_{\phi} = e^{-2k(\phi^* - \phi)} h^k_{\phi^*}$, there exists a constant $C_\phi > 0$ such that
\[
(31) \quad e^{-2kC_\phi} h^k_{\phi^*} \leq h^k_{\phi} \leq e^{2kC_\phi} h^k_{\phi^*}.
\]

By the fact that $f^{-p+1}_p / f^{-p+1}_p$ is bounded by positive constants (independent from $\phi$), and $\omega_p / \omega_p$ is bounded by positive constants depending only on $\phi$, using 31 we obtain from 30 the following estimate
\[
(32) \quad -2C_\phi k + B'_\phi \leq a_i(\lambda) \leq 2C_\phi k + B_\phi,
\]
where $B_\phi, B'_\phi$ are real constants depending only on $\phi, \phi^*$. We derive from 28 and 32 that,
\[
(-2C_\phi k + B'_\phi) \rho_{(\xi, p)}(k \phi^*) \leq \rho_A(k \phi^*) \leq (2C_\phi k + B_\phi) \rho_{(\xi, p)}(k \phi^*).
\]

Using 25 we infer
\[
(-2C_\phi k + B'_\phi) f^{-p+1}_p + O(k^{-1}) \leq \rho_A(k \phi^*) \leq (2C_\phi k + B_\phi) f^{-p+1}_p + O(k^{-1}),
\]
which shows that $\lim_{k \to \infty} \varepsilon_\phi(k) = 0$. \qed
3.2. Proof of Theorem 1. Now we are in position to give the proof of Theorem 1 which is very similar to [39] Theorem 3.4.1.

Proof. Let $\phi^* \in \mathcal{K}_\omega^k$ the Kähler potential of a metric with constant $(\xi, p)$-scalar curvature. For any $\phi \in \mathcal{K}_\omega^k$, by Corollary 3, we have

$$\mathcal{L}_{(\xi, p)}^k(\phi) = Z_{(\xi, p)}^k(\text{Hilb}^k_{(\xi, p)}(\phi)) + \left[ \mathcal{L}_{(\xi, p)}^k(\phi) - Z_{(\xi, p)}^k(\text{Hilb}^k_{(\xi, p)}(\phi)) \right] \geq Z_{(\xi, p)}^k(\text{Hilb}^k_{(\xi, p)}(\phi^*)) + k^m \epsilon_\phi(k) + \left[ \mathcal{L}_{(\xi, p)}^k(\phi) - Z_{(\xi, p)}^k(\text{Hilb}^k_{(\xi, p)}(\phi)) \right].$$

Thus,

$$\frac{2}{k^m} \mathcal{L}_{(\xi, p)}^k(\phi) + b_k \geq \frac{2}{k^m} \mathcal{L}_{(\xi, p)}^k(\phi^*) + b_k + \frac{2}{k^m} \left[ Z_{(\xi, p)}^k(\text{Hilb}^k_{(\xi, p)}(\phi^*)) - \mathcal{L}_{(\xi, p)}^k(\phi^*) \right] + \epsilon_\phi(k) + \frac{2}{k^m} \left[ \mathcal{L}_{(\xi, p)}^k(\phi) - Z_{(\xi, p)}^k(\text{Hilb}^k_{(\xi, p)}(\phi)) \right].$$

Using Proposition 2 and 3 together with (27), by letting $k$ go to infinity we get,

$$\mathcal{M}_{(\xi, p)}(\phi) \geq \mathcal{M}_{(\xi, p)}(\phi^*).$$

\qed

4. The conformally Kähler, Einstein–Maxwell metrics on ruled surfaces

In this section, we give the proof of the Corollary 4 from the Introduction.

4.1. The Calabi construction of cKEM metrics on ruled surfaces. Let $X = \mathbb{P}(O \oplus \mathcal{L}) \to C$ be a geometrically ruled complex surface over a compact complex curve $C$ of genus $g \geq 2$. Following [39], cKEM metrics can be constructed by using the Calabi ansatz [7, 13, 28]. Let $(g_C, \omega_C)$ be a Kähler metric on $C$ with constant scalar curvature $4(1 - g)$, where $\ell = \text{deg}(\mathcal{L}) > 0$ is the degree of $\mathcal{L}$. We denote by $\theta$ the connection 1-form on the principal $S^1$-bundle $P$ over $C$, with curvature $d\theta = \ell \omega_C$. Notice that $P$ can be identified with the unitary bundle of $(\mathcal{L}^* h^*)$ over $C$, where $h^*$ is the Hermitian metric with Chern curvature $-\ell \omega_C$; viewing equivalently $X$ as a compactification at infinity of $\mathcal{L}^* \to C$ (i.e. $X = \mathbb{P}(\mathcal{L}^* \oplus \mathcal{O})$), we can introduce a class of Kähler metrics on $X$ by

$$g = \ell(z + \kappa)g_C + \frac{dz^2}{\Theta(z)} + \Theta(z) \theta^2, \quad \omega = \ell(z + \kappa)\omega_C + dz \wedge \theta,$$

where: $z \in [-1, 1]$ is a momentum variable for the $S^1$-action on $\mathcal{L}^*$, $\Theta(z)$ is a smooth function on $[-1, 1]$, called a profile function [28], which satisfies the first order boundary conditions

$$\Theta(\pm 1) = 0, \quad \Theta'(\pm 1) = \mp 2,$$

and the positivity condition

$$\Theta(z) > 0 \text{ on } (-1, 1).$$

Here $\kappa > 1$ is a real constant which parametrizes the Kähler class

$$\Omega_\kappa = [\omega] = 2\pi \left( c_1(O(2)_{\mathbb{P}(O \oplus \mathcal{L})}) + (1 + \kappa)\ell[\omega_C] \right).$$

Notice that for the ruled surfaces we consider $H^2(X, \mathbb{R}) \cong \mathbb{R}^2$, so that any Kähler class on $X$ can be written as $\lambda \Omega_\kappa$ for some $\lambda > 0$ and $\kappa > 1$, see [25]. Furthermore, $\Omega_\kappa$ is homothetic to a Hodge class if and only if $\kappa \in (1, +\infty) \cap \mathbb{Q}$.  

For any \(|b| > 1\), \(f = |z + b|\) is a positive Killing potential with respect to (33), which corresponds up to sign to the Killing vector field \(\xi\) generating the \(S^1\)-action on \(X = \mathbb{P}(O \oplus L)\) by multiplications of the first factor \(O\); the additive constant \(b\) is simply an affine-linear modification of the normalizing constant \(a\) for the Killing potentials of \(\xi\). The main results of [30] can be summarized as follows

**Proposition 4.** Let \(X = \mathbb{P}(O \oplus L) \to C\) be a ruled complex surface as above.

- For any \(\kappa > 1\), the Futaki invariant \(\mathfrak{f}(\Omega, \xi, b)\) of [5] vanishes if and only if \(b\) satisfies

\[
\kappa = \frac{1 + b^2}{2b}.
\]

We denote by \(b_\kappa > 1\) the unique solution of (36) satisfying \(|b| > 1\).

- There exists a polynomial \(P_\kappa(z)\) of degree \(\leq 4\) such that \(\Theta(z) = P_\kappa(z)/(z + \kappa)\) satisfies the first order boundary conditions (34) and, on any open subset when \(\Theta(z) > 0\), the metric (33) is conformal to a cKEM metric with conformal factor \((z + b_\kappa)^{-2}\).

- There exists \(\kappa_0(X) \in (1, +\infty)\) such that
  1. for each \(\kappa \in (\kappa_0(X), +\infty)\) the corresponding polynomial \(P_\kappa(z) > 0\) on \((-1, 1)\), i.e. \(\Omega_\kappa\) admits a Kähler metric of the form (33) with \(\Theta(z) = P_\kappa(z)/(z + \kappa)\), such that \((z + b_\kappa)^{-2}g\) is cKEM;
  2. for each \(\kappa \in (1, \kappa_0(X))\) the corresponding polynomial \(P_\kappa(z)\) is negative somewhere on \((-1, 1)\);
  3. for \(\kappa = \kappa_0(X)\) the corresponding polynomial \(P_\kappa(z)\) is non-negative and has a zero with multiplicity 2 on \((-1, 1)\).

**4.2. The \((\xi, b_\kappa, 4)\)-Mabuchi energy.** For a given \(\kappa > 0\), the Kähler metrics (33) define a Frechét subspace \(\text{Cal}_{\omega_\kappa}\) modelled on smooth profile functions \(\Theta(z)\) satisfying (34) and (35), of \(\omega_\kappa\)-compatible \(\xi\)-invariant Kähler metrics on \(X\). We can choose as a reference metric \((g_\kappa, \omega_\kappa)\) of the form (33) with profile function \(\Theta_0 := (1 - z^2)^3\); the corresponding complex structure \(J\) can be naturally identified with the canonical complex structure on \(X\), see e.g. [7]. Letting \(u''(z) = 1/\Theta(z)\) be the fibre-wise symplectic potential of a metric in \(\text{Cal}_{\omega_\kappa}\), the fibre-wise Legendre transform of \(u\) defines a differentiable map \(T : \text{Cal}_{\omega_\kappa} \to K^{3}_{\Theta_0}\) with differential \(dT_{\bar{g}}(\bar{u}) = -\bar{\phi}\) see [7]. Thus, a positive multiple of the pull-back of \(M_{(\xi, b, 4)}\) to \(\text{Cal}_{\omega_\kappa}\) is defined by

\[
(dM_{(\xi, b, 4)})_g(\bar{u}) = \int_{-1}^{1} \bar{u}(z)S\text{cal}_{(\xi, b, 4)}|z + b|^{-5}dz
\]

\[
M_{(\xi, b, 4)}(g_\kappa) = 0.
\]

A computation similar to [7] Prop. 7 reveals that for \(b = b_\kappa\), the solution is given by the formula

**Proposition 5.** The Mabuchi energy \(M_{(\xi, b, 4)}\) restricted to the space \(\text{Cal}_{\omega_\kappa}\) of Kähler metrics given by the Calabi ansatz (33) is up to a positive constant

\[
M_{(\xi, b, 4)}(u(z)) = \int_{-1}^{1} \frac{P_\kappa(z)}{(z + b_\kappa)^3}(u''(z) - u''_\kappa(z))dz
- \int_{-1}^{1} \frac{(z + \kappa)}{(z + b_\kappa)^3} \log \left(\frac{u''(z)}{u''_\kappa(z)}\right)dz,
\]

where \(u_\kappa(z) = \frac{1}{2} \left(1 - z\right) \log \left(1 - z\right) + (1 + z) \log \left(1 + z\right)\) is the fibre-wise symplectic potential of the canonical metric \(g_\kappa \in \text{Cal}_{\omega_\kappa}\) and \(b_\kappa\) and \(P_\kappa(z)\) are the real number and polynomial of Proposition 4.
As noticed in the proof of [2 Cor. 3], if $P_\kappa(z)$ is negative on an interval $I \subset (-1, 1)$, taking $f(z)$ to be a bump function with support in $I$ and $u_k(z)$ a sequence of symplectic potentials defined by $u_k''(z) = u_k'(z) + kf(z)$, $k > 0$, one has $\lim_{k \to \infty} \mathcal{M}_{(\xi, b_\kappa)}(u_k) = -\infty$. We thus get

**Corollary 4.** If the polynomial $P_\kappa(z)$ given in Proposition 4 is negative somewhere on $(-1, 1)$, then the Mabuchi functional $\mathcal{M}_{(\xi, b_\kappa)}$ is not bounded from below.

### 4.3. Proof of Corollary 4

There are no cscK metrics on $X$ (see e.g. [5]), so that we are looking for strictly conformally Kähler, Einstein–Maxwell metrics. As in our Proof of Corollary 1.

Ω = Ωκ ± is of the vector field generating rotations on the factor $\kappa$. As the theory is invariant under homothety of the Killing potential, without loss we assume that this multiple is ±1. Finally, as $H^2(X, \mathbb{R}) = \mathbb{R}^2$, by rescaling the Kähler class we can also assume $\Omega = \Omega_\kappa, \kappa > 1$. For a Kähler metric $g \in \Omega_\kappa$ of the form $\mathcal{M}_{(\xi, b)}$, the Killing potential of $\xi$ is $|z + b|$ with $|b| > 1$. The necessary condition $\mathcal{M}_{(\xi, b)} = 0$ then forces us to consider $b = b_\kappa$, see Proposition 4. The existence of conformally Kähler, Einstein–Maxwell metrics for $\kappa \in (\kappa_0(X), \infty)$ and conformal factor $(z + b_\kappa)^{-2}$ follows from the statement in (a) of Proposition 4.

We are left to show non-existence for $\kappa \in (1, \kappa_0(X)]$. Again, by Proposition 4, we have to take $b = b_\kappa > 1$.

Consider first the case $\kappa \in (1, \kappa_0(X))$. If $\kappa$ is rational, the result follows from Theorem 4 and Corollary 4. Otherwise, if $\kappa \in (1, \kappa_0(X)) \setminus \mathbb{Q}$, we suppose for contradiction that $\Omega_\kappa$ admits a Kähler metric of constant $(z + b_\kappa, 4)$-scalar curvature. By [5 Theorem 2], the same will hold for all $(\kappa', b_{\kappa'})$ on the rational curve $\mathcal{M}_{(\xi, b)}$ which are sufficiently close to $(\kappa, b_\kappa)$, in particular for all rational pairs $(\kappa', b_{\kappa'})$ close to $(\kappa, b_\kappa)$, a contradiction. Finally, consider $\kappa = \kappa_0(X) = \kappa_0, b_{\kappa_0} = b_0$. Again, suppose for contradiction that $\Omega_{\kappa_0}$ admits a metric of constant $(\xi, b_0, 4)$-scalar curvature. We use again [5 Theorem 2] to deduce that this holds also for all $(\kappa, b_\kappa)$ near $(\kappa_0, b_0)$ and we can find again rational valued $(\kappa, b_\kappa)$ arbitrarily close to $(\kappa_0, b_0)$ with $\kappa < \kappa_0$, and still admitting a $(\xi, b_0, 4)$-extremal Kähler metric, a contradiction.

### References

[1] V. Apostolov, D. M. J. Calderbank, and P. Gauduchon, *Ambitoric geometry I: Einstein metrics and extremal ambikähler structures*, J. Reine Angew. Math. 721 (2016), 109-147.

[2] V. Apostolov, D. M. J. Calderbank, E. Legendre and P. Gauduchon, *Levi-Kähler reduction of CR structures, products of spheres, and toric geometry*, arXiv:1708.05253.

[3] V. Apostolov, D. M. J. Calderbank, P. Gauduchon and C. Tønnesen-Friedman, *Extremal Kähler metrics on ruled manifolds and stability*. Géométrie différentielle, physique mathématique, mathématiques et société (II). Astérisque 322 (2008), 93-150.

[4] V. Apostolov, H. Huang, *A splitting theorem for extremal Kähler metrics*, J. Geom. Anal. 25 (2015), no. 1, 149-170.

[5] V. Apostolov, G. Maschler, *Conformally Kähler, Einstein-Maxwell geometry*, arXiv:1512.06391v1, to appear in JEMS.

[6] V. Apostolov, G. Maschler, C. Tønnesen-Friedman, *Weighted extremal Kähler metrics and the Einstein-Maxwell geometry of projective bundles*, arXiv:1808.02813.

[7] V. Apostolov, D. M. J. Calderbank, P. Gauduchon and C. Tønnesen-Friedman, *Hamiltonian 2-forms in Kähler geometry*, III Extremal Metrics and Stability, Invent. math. 173 (2008) 547-601.

[8] V. Apostolov and C. Tønnesen-Friedman, *A remark on Kähler metrics of constant scalar curvature on ruled complex surfaces*, Bull. London Math. Soc. 38 (2006), 494-500.

[9] R. J. Berman and B. Berndtsson, *Convexity of the K-energy on the space of Kähler metrics and uniqueness of extremal metrics*, J. Amer. Math. Soc. 30 (2017), no. 4, 1165-1196.

[10] R. J. Berman, B. Berndtsson, J. Sjstrand, *A direct approach to asymptotics of Bergman kernels for positive line bundles*, Ark. Mat. 46 (2008), no. 2, 197217.

[11] R. J. Berman and D. Witt-Nyström, *Complex optimal transport and the pluripotential theory of Kähler-Ricci solitons*, arXiv:1401.8264.
[12] A. L. Besse, Einstein Manifolds, Ergebnisse (3) \textbf{10}, Springer-Verlag, Berlin-Heidelberg-New York, 1987.
[13] E. Calabi, Extremal Kähler metrics II, in: Differential Geometry and Complex Analysis (eds. I. Chavel and H.M. Farkas), Springer, Berlin, 1985.
[14] D. Catlin, The Bergman kernel and a theorem of Tian, Analysis and geometry in several complex variables (Katata, 1997), Trends Math., 123. Birkhäuser Boston, Boston, MA, 1999.
[15] L. Charles, Berezin-Toeplitz operators, a semi-classical approach, Comm. Math. Phys. \textbf{239} (2003), no. 1–2, 1-28.
[16] X. X. Chen and G. Tian, Geometry of Kähler metrics and foliations by holomorphic discs, Publ. Math. Inst. Hautes Etudes Sci. \textbf{107} (2008), 1–107.
[17] X. Dai, K. Liu, X. Ma, On the asymptotic expansion of Bergman kernel, J. Differential Geom., \textbf{72} (2006), 141.
[18] R. Debever, N. Kamran, and R. G. McLenaghan, Exhaustive integration and a single expression for the general solution of the type D vacuum and electroweak field equations with cosmological constant for a non-singular aligned Maxwell field, J. Math. Phys. \textbf{25} (1984), 19551972.
[19] S. K. Donaldson, Scalar curvature and projective embeddings. I, I. J. Differential Geom. \textbf{59} (2001), no. 3, 479–522.
[20] S. K. Donaldson, Scalar curvature and stability of toric varieties, J. Differential Geom. \textbf{62} (2002), no. 2, 289–349.
[21] S. K. Donaldson, Lower bounds on the Calabi functional, J. Differential Geom. \textbf{70} (2005), no. 3, 453-472.
[22] S. K. Donaldson, Scalar curvature and projective embeddings. II., Q. J. Math. \textbf{56} (2005), no.3, 345–356.
[23] A. Futaki and H. Ono, Volume minimization and Conformally Kähler, Einstein–Maxwell geometry, arXiv:1706.07953.
[24] A. Futaki and H. Ono, Conformally Einstein–Maxwell Kähler metrics and structure of the automorphism group, arXiv:1708.01958.
[25] A. Fujiki, Remarks on extremal Kähler metrics on ruled manifolds, Nagoya Math. J. \textbf{126} (1992) 89–101.
[26] P. Gauduchon, Calabi’s extremal metrics: An elementary introduction, Lecture Notes.
[27] G. B. Griffiths and J. Podolsky, A new look at the Plebański-Demiański family of solutions, Internat J. Mod. Phys. D \textbf{15} (2006), 335370.
[28] A. D. Hwang and M. A. Singer, A momentum construction for circle-invariant Kähler metrics, Trans. Amer. Math. Soc. \textbf{354} (2002), 2285–2325.
[29] S. Kobayashi, Transformation groups in differential geometry, Ergebnisse der Mathematik und ihrer Grenzgebiete, \textbf{70}, Springer–Verlag, New York–Heidelberg, 1972.
[30] C. Koca, C. Tønnesen-Friedman, Strongly Hermitian Einstein–Maxwell solutions on Ruled Surfaces, Annals of Global Analysis and Geometry, \textbf{50} (2016), no.1, 29-46.
[31] A. Lahdili, Automorphisms and deformations of conformally Kähler, Einstein-Maxwell metrics, arXiv:1708.01507.
[32] C. LeBrun, The Einstein–Maxwell equations, Kähler metrics, and Hermitian geometry, J. Geom. Phys. \textbf{91} (2015), 163–171.
[33] C. LeBrun, The Einstein–Maxwell Equations and Conformally Kähler Geometry, Comm. Math. Phys. \textbf{344} (2016), 621-653.
[34] M. Lejmi, M. Upmeier, Integrability theorems and conformally constant Chern scalar curvature metrics in almost Hermitian geometry, arXiv:1703.01323.
[35] C. Li, Constant scalar curvature Kähler metrics obtains minimum of K-energy, Int. Math. Res. Not., Vol. \textbf{2011}, No. 9, 2161–2175.
[36] Z. Lu, On the lower order terms of the asymptotic expansion of Tian-Yau-Zelditch, Amer. J. Math. \textbf{122} (2000).
[37] X. Ma and G. Marinescu, Holomorphic Morse inequalities and Bergman kernels, Progress in Mathematics, 254. Birkhäuser Verlag, Basel, 2007.
[38] W. -D. Ruan, Canonical coordinates and Bergman metrics, Comm. Anal. Geom. \textbf{6} (1998), no. 3, 589-631.
[39] Y. Sano and C. Tipler, Extremal Kähler metrics and lower bound of the modified K-energy, J. Eur. Math. Soc. (JEMS) \textbf{17} (2015), no.9, 2289–2310.
[40] G. Székelyhidi, Introduction to Extremal Kähler metrics, Graduate Studies in Mathematics, vol. 152, Amer. Math. Soc., Providence, RI, 2014.
[41] G. Tian, On a set of polarized Kähler metrics on algebraic manifolds, J. Differential Geom. \textbf{32} (1990) 99130.
[42] X. Wang, *Moment map, Futaki invariant and stability of projective manifolds*, Comm. Anal. Geom. **12** (2004), 1009–1037.

[43] S. Zelditch, *Szego kernels and a theorem of Tian*, Internat. Math. Res. Notices, (1998) 317-331.

[44] S. Zelditch and P. Zhou, *Central limit theorem for spectral partial Bergman kernels*, arXiv:1708.09267.

[45] S. Zhang, *Heights and reductions of semi-stable varieties*, Compositio Math. **104** (1996), 77–105.

Lahdili Abdellah, DÉPARTEMENT DE MATHÉMATIQUES, UQAM, C.P. 8888, Succursale Centre-ville, Montréal (Québec), H3C 3P8, CANADA
E-mail address: lahdili.abdellah@courrier.uqam.ca