Can inhomogeneities solve the horizon problem?

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Abstract

We show how inhomogeneous cosmological models can naturally explain the large angle correlation we observe in the CMB (cosmological microwave background) radiation without invoking any inflationary stage, but simply considering the effects of inhomogeneities on the propagation of photons from the last scattering surface.
I. INTRODUCTION

High redshift luminosity distance measurements \[19, 25, 26, 27, 28, 29\] and WMAP measurement \[30, 31\] of cosmic microwave background (CMB) interpreted in the framework of standard FLRW cosmological models have strongly disfavored a matter dominated universe, and strongly supported a dominant dark energy component, corresponding to a positive cosmological acceleration, which we will denote as \(a^{\text{FLRW}}\) (not to be confused with the scale factor \(a\)). As an alternative to dark energy, it has been proposed \[6\] that we may be at the center of a inhomogeneous isotropic universe described by a Lemaitre-Tolman-Bondi (LTB) solution of Einstein’s fields equations, where spatial averaging over one expanding and one contracting region is producing a positive averaged acceleration \(a_D\), even if it was shown that quantities constructed in this way can be unobservable \[7\] because they violate the causal structure of the underlying space. Another more general approach to map luminosity distance as a function of redshift \(D_L(z)\) to LTB models has been recently proposed \[21\], where an inversion method was applied to successfully reproduce the observed \(D_L(z)\), also showing that the freedom in the choice the LTB geometry poses some degeneracy problem, which could be solved by imposing other cosmological constraints such as CMB radiation, or RSSE (redshift spherical shell energy) \[8\].

The main point is that the luminosity distance is in general sensitive to the geometry of the space through which photons are propagating along light geodesics, and therefore arranging appropriately the geometry of a given cosmological model it is possible to reproduce a given \(D_L(z)\). For FLRW models this correspond to set constraints on \(\Omega_\Lambda\) and \(\Omega_m\) and for LTB models it allows to determine the functions \(E(r), M(r), t_b(r)\).

In a similar way the inhomogeneities determine a different causal structure from the homogeneous case, in particular a spatial dependency of the Hubble radius \(H(r)^{-1}\), which can allow causal contact of points at the surface of last scattering, without invoking a period of exponential expansion.

A previous attempt to solve the horizon problem in LTB spaces was investigated in \[9, 10\], but this approach involved a special choice of the bang function \(t_b(r)\), while in our case we consider a homogeneous big bang scenario, i.e. \(t_b(r) = 0\). Since the time scale of observation of CMB is very short on cosmological scale, of the order of few years, its observed isotropy is compatible with a spherically symmetric radially inhomogeneous model, but it is not a sufficient evidence of homogeneity.
II. LEMAITRE-TOLMAN-BONDI (LTB) SOLUTION

Lemaitre-Tolman-Bondi solution can be written as\[3, 4, 5\] as:

\[ ds^2 = -dt^2 + \frac{(R, r)^2 dr^2}{1 + 2E(r)} + R^2 d\Omega^2, \] (1)

where \( R \) is a function of the time coordinate \( t \) and the radial coordinate \( r \), \( E(r) \) is an arbitrary function of \( r \), and \( R, r \) denotes the partial derivative of \( R \) with respect to \( r \).

Einstein’s equations give:

\[ \left( \frac{\dot{R}}{R} \right)^2 = \frac{2E(r)}{R^2} + \frac{2M(r)}{R^3}, \] (2)

\[ \rho(t, r) = \frac{M, r}{R^2 R, r}, \] (3)

with \( M(r) \) being an arbitrary function of \( r \) and the dot denoting the partial derivative with respect to \( t \). The solution of Eq. (2) can be written parametrically by using a variable \( \eta = \int dt/R \), as follows

\[
\begin{align*}
\tilde{R}(\eta, r) &= \frac{M(r)}{-2E(r)} \left[ 1 - \cos \left( \sqrt{-2E(r)}\eta \right) \right], \\
t(\eta, r) &= \frac{M(r)}{-2E(r)} \left[ \eta - \frac{1}{\sqrt{-2E(r)}} \sin \left( \sqrt{-2E(r)}\eta \right) \right] + t_b(r),
\end{align*}
\] (4, 5)

where \( \tilde{R} \) has been introduced to make clear the distinction between the two functions \( R(t, r) \) and \( \tilde{R}(\eta, r) \) which are trivially related by

\[ R(t, r) = \tilde{R}(\eta(t, r), r) \] (6)

and \( t_b(r) \) is another arbitrary function of \( r \), called bang function, which corresponds to the fact that big-bang/crunches happen at different times in this space. This inhomogeneity of the location of the singularities is the origin of the possible causal separation between the central observer and the spatially averaged region for models with positive \( a_V \).

We can introduce the following variables

\[ a(t, r) = \frac{R(t, r)}{r}, \quad k(r) = -\frac{2E(r)}{r^2}, \quad \rho_0(r) = \frac{6M(r)}{r^3}, \] (7)

so that Eq. (1) and the Einstein equations (2) and (3) can be written in a form which is more similar to FRW models.
The solution in Eqs. (4) and (5) can now be written as

\[
\tilde{a}(\tilde{\eta}, r) = \frac{\rho_0(r)}{6k(r)} \left[ 1 - \cos \left( \sqrt{k(r)} \tilde{\eta} \right) \right],
\]

(11)

\[
t(\tilde{\eta}, r) = \frac{\rho_0(r)}{6k(r)} \left[ \tilde{\eta} - \frac{1}{\sqrt{k(r)}} \sin \left( \sqrt{k(r)} \tilde{\eta} \right) \right] + t_b(r),
\]

(12)

where \( \tilde{\eta} \equiv \eta r = \int dt/a \).

III. PROPAGATION OF PHOTONS AFTER LAST SCATTERING IN A INHOMOGENEOUS SPACE

When we study cosmology in a not homogeneous space it is important to adopt the appropriate distance and time redshift relations and use them consistently. In particular the time of last scattering \( t_{ls}^{FRLW} \) and \( t_{ls}^{LTB} \), are different, but in the following we will omit the upper script for the cosmological model, assuming we are referring to a LTB model.

We will assume that the Universe was homogeneous at the time of last scattering and that inhomogeneities affected only the propagation of photons from the last scattering surface until us today. In order to compute \( t_{ls} \) and \( r_{ls} \), is necessary to integrate backward in time the null geodesic equation as a function of redshift. The geodesic equations expressed in terms of the redshift can be written as \([13, 14]\)

\[
\frac{dr}{dz} = \frac{\sqrt{1 + 2E(r(z))}}{(1 + z)\partial_t \partial_r R(t(z), r(z))},
\]

(13)

\[
\frac{dt}{dz} = -|\partial_t R(t(z), r(z))| \left/ \left(1 + z\right)\partial_t \partial_r R(t(z), r(z)) \right|,
\]

(14)

where \( t(z) \) and \( r(z) \) physically represent the coordinates along the null geodesic of the photon coming to us (located at \( r = 0, z = 0 \)). After integration we get:

\[
r_{ls} = r(z_{ls})
\]

(15)

\[
t_{ls} = t(z_{ls})
\]

(16)
which in general will be different from those obtained in a FRLW space.

Points which are today at an angular distance $\alpha_c$ were at the time of last scattering $t_{ls}$ at a distance:

$$d_{ls} = \alpha_c R(t_{ls}, r_{ls})$$

If we focus on a constant $r$ spatial hyper surface, we can see that the notion of event horizon is the same as in a homogeneous space, but it depends on $r$:

$$H(r) = \frac{\dot{a}}{a} = \frac{R(t, r)}{R(t, r)}$$

As long as

$$d_{ls} < H^{-1}(r_{ls})$$

$$\alpha_c < \left(\frac{dR(t_{ls}, r_{ls})}{dt}\right)^{-1}$$

the correlation of CMB on angular scale $\alpha_c$ will compatible with the causal structure of inhomogeneous space.

A stronger constraints would come by imposing that the Hubble horizon is less than the sound horizon, but we will limit to impose the upper limit given by (20).

IV. FLAT UNIVERSE

We will now study the case of a flat LTB space, i.e. when $E(r) = 0$. Einstein equations can be solved analytically:

$$R(t, r) = \left(\frac{3}{2}\sqrt{2M(r)[t - t_b(r)]}\right)^{2/3}$$

Imposing the homogeneous bang condition $t_b(r) = 0$, eq. (20) gives the following constraint on $M$:

$$M(r_{ls}) < \frac{3t_{ls}}{4\pi^3}$$

The energy density in this case is

$$\rho(t, r) = \frac{2}{3t^2}$$

which corresponds to a FRW homogeneous matter dominated universe. This upper limit is in agreement with the general case discussed in the next section, which requires $M(r)$ to
be a decreasing function of $r$ as a general feature of LTB models which can solve the Horizon problem.

V. CURVATURE DOMINATED UNIVERSE

The other limiting case which can be treated exactly is the curvature dominated Universe in which $M(r) = 0$. Also in this case it is possible to find an exact solution for $R$:

$$R(t, r) = \sqrt{2E(r)} \left[ t - t_b(r) \right]$$ (24)

The condition (20) in this case is:

$$E(r_{ls}) < \frac{1}{2\pi^2}$$ (25)

This upper limit is in agreement with the general case discussed in the next section, which requires $E(r)$ to be a decreasing function of $r$ as a general feature of LTB models which can solve the Horizon problem.

VI. GENERAL CASE

Using the implicit solution (4) we can find a general expression for the condition (20), in terms of $\eta$.

Defining $\alpha(r)$:

$$\alpha(r) = \left( \frac{dR(\eta, r)}{d\eta} \right)^{-1} \left|_{ls} \frac{dt(\eta, r)}{d\eta} \right|_{ls} = \frac{1}{\sqrt{-2E(r_{ls})}} \tan \left[ \eta_{ls} \sqrt{-E(r_{ls})/2} \right]$$ (26)

we obtain:

$$\alpha(r_{ls}) > \alpha_c$$ (27)

For any given LTB model there will be a different $\eta_{ls}$ corresponding to the time of last scattering, which can be found by solving numerically the equation (5)

$$t_{ls} = t(\eta_{ls}, r_{ls})$$ (28)

The quantity $\alpha(r)$ is the present angular projection of the Horizon scale $H[r, t(r)]^{-1}$, where $t(r)$ is the time along a null geodesic which arrive at $r = 0$ at present. In other words it is
the maximum present angular scale which could be in causal contact at co-moving coordinate \( r \) and time \( t(r) \). The \( r \) dependence is the crucial feature of LTB spaces, and it is what can be used to solve the Horizon problem with an appropriate choice of \( E(r) \) and \( M(r) \).

The general inequality (27) is apparently independent of \( M(r) \), but it in fact depends implicitly through \( \eta_{ls} \), since \( t(\eta, r) \) contains \( M(r) \). For \( E(r) = 0 \) and \( M(r) = 0 \) equation (28) is indeterminate, so the general condition (27) cannot be applied to these limit cases, which justify our previous separate treatment.

As it can be seen in figure (3) the solution of eq. (28) is inversely proportional to \( M(r) \), and \( \alpha(r) \) is proportional to \( \eta_{ls} \) and inversely proportional to \( E(r) \). This implies that decreasing \( E(r) \) and \( M(r) \) are the right type of functional dependence on the comoving coordinate to solve Horizon problem.

![Figure 1](image1.png)

**FIG. 1:** \( \alpha(r) \) is plotted for \( \eta_{ls} = 2\pi \) and \( E(r) = r \)

![Figure 2](image2.png)

**FIG. 2:** \( \alpha(r) \) is plotted for \( \eta_{ls} = 2\pi \) and \( E(r) = \frac{1}{r} \)
FIG. 3: In order to show the dependence of $\eta_{ls}$ on LTB geometry the solution of eq.(28) is plotted for $t_{ls} = 10$, $0.3 < M(r) < 10$ and $0.3 < E(r) < 5$.

In order to provide further insight on the dependence of the $\alpha(r)$ and its dependence on $\eta_{ls}$ and $E(r)$ we plotted it for different cases.

The crucial feature of $\alpha(r)$ is that it is a decreasing function of $E(r)$, and its upper limit is $\eta_{ls}/2$ that is reached as $E(r)$ goes to zero, which implies that in order to solve the Horizon problem $E(r)$ has to be a decreasing function of $r$.

As an example, fig.(2) shows how the choice $E(r) = 1/r$ can solve the Horizon problem assuming $\eta_{ls} = 2\pi$, if $r_{ls}$ is sufficiently large. From this example it is clear that the important parameters to determine if a given model explains the observed large angle correlation of CMB radiation are $r_{ls}, t_{ls}, \eta_{ls}$, which depend implicitly on the the LTB geometry through the geodesic equations(14). This pose in general a non trivial numerical problem, but our qualitative analytical analysis provides the main feature that the functions $E(r), M(r)$ should satisfy, and which could be used as general directions in a more systematic numerical investigation.

VII. CONCLUSION

We have shown that inhomogeneous cosmological models can in principle not only be alternatives to dark energy but also provide a potential alternative solution to the horizon problem, and we have set up the constraints that solutions have to satisfy.

Both dark energy and inflation correspond in the framework of FRLW cosmology to a stage of positive acceleration, and it consequently possible that if homogeneities can mimic dark
energy they could also provide the causal structure necessary to solve the Horizon problem, which could then be considered an artifact originating from neglecting inhomogeneities effects on the propagation of photons from the last scattering surface.

It must be observed though, that there may not be any model satisfying the different cosmological constraints and the conditions we have derived, and in general the two sets of conditions could be incompatible. Despite that, it is important to observe that inhomogeneities could play an important to explain any physical situation which treated in a FRLW framework involves a stage of positive acceleration.

The extensive search for a model which successfully solves the horizon problem and is compatible with other cosmological observables will be the subject of a future investigation.

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