CONTINUOUS FAMILIES OF ISOPHASAL SCATTERING MANIFOLDS

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Abstract. We construct continuous families of scattering manifolds with the same scattering phase. The manifolds are compactly supported metric perturbations of Euclidean $\mathbb{R}^n$ for $n \geq 8$. The metric perturbation may have arbitrarily small support.

1. Introduction

Inverse spectral geometry for compact Riemannian manifolds is the study of what geometric properties of the manifold are determined by the eigenvalues of the Laplacian. For non-compact Riemannian manifolds, there may be only finitely many $L^2$-eigenvalues of the Laplacian (or even no $L^2$-eigenvalues at all!) but there are several possible analogues of spectral data for which one can pose a similar inverse problem. A particularly attractive setting in which to study the inverse problem is Euclidean $\mathbb{R}^n$ with a compactly supported perturbation of the Euclidean metric: in what follows we will write $X = (\mathbb{R}^n, g)$ where $g$ is such a compactly supported metric perturbation. Our goal is to construct continuous families of such manifolds with the same ‘spectral’ data, appropriately defined.

For the class of manifolds that we will consider, the Laplacian has purely continuous spectrum in $[0, \infty)$ and no $L^2$-eigenvalues. Thus, the resolvent of the Laplacian is an analytic function $R(z) = (\Delta_X - z)^{-1}$ on $\mathbb{C}\setminus[0, \infty)$; as we discuss in what follows, the resolvent admits a meromorphic continuation to a double covering of the complex plane if $n$ is odd, and a logarithmic covering of the complex plane if $n$ is even (this result follows, for example, from the “black box scattering” formalism introduced by Sjöstrand and Zworski in [19] [but this only treats $n$ odd]). Resolvent resonances are poles of the meromorphically continued resolvent; they serve as discrete data analogous to the eigenvalues but are less easily studied than the eigenvalues since their presence signals the solution to a non-selfadjoint eigenvalue problem for the underlying differential operator. In the literature they are also referred to simply as resonances. For the case considered here, the resolvent resonances are identical to the poles of the meromorphically continued scattering operator, which are called scattering resonances (we discuss the scattering operator in greater detail in what follows). We will call two such manifolds isopolar if they have the same scattering resonances with multiplicities.

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For certain classes of non-compact manifolds, including the class to be studied here, one can define the *scattering phase*, a function roughly analogous to the counting function for eigenvalues in the compact problem. We define and discuss the scattering phase in section 2 of what follows. Two noncompact Riemannian manifolds are said to be *isophasal* if they have the same scattering phase. In our case, if two manifolds have the same scattering phase, they also have the same scattering poles, so isophasality is a stronger condition than isopolarity.

While the past two decades have seen an explosion of examples of isospectral compact Riemannian manifolds, there are relatively few examples known of isopolar or isophasal manifolds. In dimension greater than one, the known examples include finite-area Riemann surfaces (both isopolar and isophasal—see Berard [1] and Zelditch [22, 23]), Riemann surfaces of infinite area (isopolar and isophasal—see Guillopé-Zworski [8] and Brooks-Davidovich [2]), three-dimensional Schottky manifolds (isopolar—see Brooks-Gornet-Perry [3]), and surfaces that are isometric to Euclidean space outside a compact set (isopolar and isophasal—see Brooks-Perry [4]). In all these examples, the manifolds share a common Riemannian covering. They are constructed by the analog of a technique of T. Sunada [20], which produces compact isospectral manifolds with a common finite covering.

We will prove:

**Theorem 1.1.** For every $n \geq 8$, there exist continuous families of isophasal, non-isometric Riemannian metrics on $\mathbb{R}^n$ which are Euclidean outside of a compact set of arbitrarily small volume. There also exist pairs of such metrics on $\mathbb{R}^6$.

**Remark 1.2.** Letting $m = n - 4$, the parameter space for the continuous families of isophasal metrics on $\mathbb{R}^n$ that we will construct has dimension

$$d \geq \frac{m(m-1)}{2} - \left[ \frac{m}{2} \right] \left( \frac{m}{2} + 2 \right) > 1$$

if $n = 9$ or $n \geq 11$. (If $n = 8$ or $n = 10$, the parameter space has dimension at least 1).

In addition to proving the existence of the continuous families, we will give an explicit example of a triple of isophasal metrics on $\mathbb{R}^{12}$. We will see that these metrics have very different geometry. Indeed their isometry groups have different dimension and structure.

To our knowledge, the isophasal metrics of Theorem 1.1 differ from the other known examples of isophasal or isopolar metrics in the following ways:

- They are the first continuous families of isophasal or isopolar metrics;
- They are the first examples for which the manifolds do not share a common Riemannian cover;
- They are the first isophasal or isopolar compact metric perturbations of the Euclidean metric on $\mathbb{R}^n$.

Just as the examples cited above were based on an extension to noncompact manifolds of a technique first developed by Sunada for constructing isospectral compact manifolds, our examples use an extension of a technique involving torus actions previously developed for the construction of isospectral compact manifolds with different local geometry. In fact the metrics that we use here were first constructed in Gordon [1] and Schueth [17], where they were restricted to balls and spheres. The method of torus actions was used to show that these metrics on balls and spheres were isospectral.
One might worry that the examples constructed have trivial scattering (e.g., have no scattering poles!). We show, however, that the isosphasal metrics can always be chosen to have infinitely many resonances. For metric scattering on $\mathbb{R}^n$, Sá Barreto and Tang [13] ($n$ odd) and Tang [21] ($n$ even) proved the existence of infinitely many resonances so long as the second relative heat invariant $a_2$ is non-vanishing. They also gave various geometric hypotheses which guarantee the non-vanishing of $a_2$: one of these is that the given metric is not flat but is a compactly supported perturbation of the Euclidean metric that is close in $C^k$ topology to the Euclidean metric for sufficiently large $k$ (Theorem 1.3 of [13] and Theorem 1.1 of [21]). In our examples, it is easily verified that the metrics are not flat, and it is easy to construct examples where the metrics are arbitrarily close in $C^k$ sense to the Euclidean metric for any large fixed $k$. We can actually remove the assumption that our metrics are $C^k$ close to the Euclidean metric by computing the $a_2$ heat invariant directly, at the cost of imposing a genericity assumption on the space of metrics; we carry out this computation in Section 5 for the metrics on $\mathbb{R}^n$ with $n \geq 9$.

The plan of this paper is as follows. In section 2, we discuss basics of scattering theory for asymptotically Euclidean manifolds. In section 3, we develop a method, based on the use of torus actions, for constructing manifolds with the same scattering phase (see Theorem 3.4). In section 4, we apply the technique of the previous section to show that the metrics on $\mathbb{R}^n$ constructed in [8] and [16] are isosphasal, thus proving Theorem 4.1. We give full details of the examples in dimension $n \geq 9$, based on the examples in [8] (modified as in [16] so that the metrics are Euclidean outside of an arbitrarily small compact set). The lower-dimensional examples are given by metrics constructed in [17]. Since the methods of section 3 apply in exactly the same way to these examples, we do not include the details here. Finally, in Section 5, we carry out the explicit computation that the $a_2$ heat invariant is generically nonvanishing for our examples in dimension 9 and above.

In a second paper, in preparation, we will construct pairs of conformally equivalent isosphasal Riemannian metrics, again equal to the Euclidean metric outside of a compact set. We will also construct pairs of isosphasal potentials for the Schrödinger operator on $(\mathbb{R}^n, g)$, where again $g$ is a compact perturbation of the Euclidean metric.

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1Both of these papers rely on a result of Kuwabara [12] which states that, given a flat metric $\gamma$ on a compact manifold $X$, there is a neighborhood $U$ of $\gamma$ in the $C^\infty$ topology so that if $a_2(g) = 0$ and $g \in U$, then $g$ is flat. In fact, a close examination of [12] shows that it is sufficient for $g$ and $\gamma$ to be close in $C^k$ topology for $k$ sufficiently large: see Theorem A’ of section 6 in [12]. For the connection between Kuwabara’s result on compact manifolds and the result on metric perturbations of Euclidean $\mathbb{R}^n$, see the proof of Theorem 1.3 of [13] which uses finite propagation speed for solutions of the wave equation.

2As explained in Section 3, the metrics depend on a skew-symmetric bilinear form and a $C^\infty(\mathbb{R}^n)$ function $\varphi$ which defines the support of the perturbation. One takes the function $\varphi$ sufficiently small in $C^\infty$-sense.

3The genericity condition we impose is merely a genericity condition on the choice of cut-off function $\varphi$. 
2. Metric Scattering on $\mathbb{R}^n$

In this section we review scattering theory for manifolds $X = (\mathbb{R}^n, g)$ where $g$ is a compactly supported metric perturbation of the Euclidean metric: see especially [10] and see [13] for an expository treatment that includes the case considered here. Letting $\Delta_X$ be the positive Laplace-Beltrami operator on $X$, it follows from the classical Rellich uniqueness theorem that $\Delta_X$ has no $L^2$-eigenvalues, and it is easy to prove that $\Delta_X$ has purely absolutely continuous spectrum in $[0, \infty)$. Thus the resolvent operator $\tilde{R}(z) = (\Delta_X - z)^{-1}$, considered as a mapping from $L^2(X)$ to itself, is an operator-valued analytic function of $z$ in $\mathbb{C}\setminus[0, \infty)$. It can be shown that the mapping $R(\lambda) = \tilde{R}(\lambda^2)$, initially defined on the half-plane $\Im(\lambda) > 0$ and viewed as a map from $C_0^\infty(\mathbb{R}^n)$ to $C^\infty(\mathbb{R}^n)$, admits a meromorphic continuation to the complex $\lambda$-plane if $n$ is even, and to the logarithmic plane if $n$ is odd. At any poles $\zeta$, the resolvent admits a Laurent expansion with finite polar part of the form

$$\sum_{j=1}^{N_\zeta} \frac{A_j}{\lambda - \zeta}$$

where the $A_j$ are finite-rank operators from $C_0^\infty(\mathbb{R}^n)$ to $C^\infty(\mathbb{R}^n)$. The multiplicity of the pole $\zeta$ is defined as $\dim(\oplus_j(\text{Ran} A_j))$.

To define the scattering phase, we first recall that the absolutely continuous spectrum is parameterized by scattering solutions to the eigenvalue equation $(\Delta_X - \lambda^2)u = 0$ which are easily described. In what follows, write $x \in \mathbb{R}^n$ as $x = r\omega$ where $r \geq 0$ and $\omega \in S^{n-1}$.

**Proposition 2.1.** Fix $f_- \in C^\infty(S^{n-1})$ and $\lambda > 0$. There exists a unique solution of the equation

$$(\Delta_X - \lambda^2)u = 0$$

having the asymptotic form

$$(2.1) \quad u(r\omega) = r^{(1-n)/2}e^{i\lambda r} f_+(\omega) + r^{(1-n)/2}e^{-i\lambda r} f_-(\omega) + O(r^{-(n+1)/2})$$

as $r \to \infty$. In particular, the function $f_+ \in C^\infty(S^{n-1})$ is uniquely determined.

For a proof see [10].

The Proposition implies that the mapping $f_- \mapsto f_+$ is a well-defined mapping from $C^\infty(S^{n-1})$ to itself. We denote this map, the absolute scattering matrix for $X$, by $S(\lambda)$. From the definition, it is clear that $S(\lambda)$ is a linear mapping, and that $S(\lambda)^{-1} = S(-\lambda)$ for real $\lambda \neq 0$. In the case of $X_0 = (\mathbb{R}^n, g_0)$ (where $g_0$ is the Euclidean metric on $\mathbb{R}^n$), we have the explicit formula

$$u(x) = \int_{S^2} \exp(-i\lambda x \cdot \omega) f_-(\omega) \, d\omega$$

and a stationary phase calculation shows that the absolute scattering matrix $S_0(\lambda)$ is given by

$$(2.2) \quad (S_0(\lambda)\varphi)(\omega) = i^{n-1}\varphi(-\omega).$$

Since $X$ is a compactly supported metric perturbation of $X_0$, it is not surprising that the ‘relative scattering matrix’

$$(2.3) \quad S_r(\lambda) = S(\lambda)S_0(\lambda)^{-1}$$

has especially nice properties (see, for example, section 5.2 of [13]):
Proposition 2.2. For real $\lambda \neq 0$, the relative scattering matrix $S_r(\lambda)$ extends to a unitary operator from $L^2(S^{n-1})$ to itself. Moreover

$$S_r(\lambda) = I + T(\lambda)$$

where $T(\lambda)$ is an integral operator with integral kernel belonging to $C^\infty(S^{n-1} \times S^{n-1})$.

In particular, $T(\lambda)$ extends to a trace-class operator on $L^2(S^{n-1})$, so that the operator determinant

$$\det S_r(\lambda) = \det(I + T(\lambda))$$

is well-defined (see, for example, [18] for discussion of operator determinants). Since $S_r(\lambda)$ is unitary, it follows that $\det S_r(\lambda)$ has modulus one. We note for use later that if $A$ is a trace-class operator on a Hilbert space $H$ and $B$ is a boundedly invertible linear operator on $H$, the equality

$$\det(I + A) = \det(I + BAB^{-1})$$

holds.

It can be shown that the determinant $\det(S_r(\lambda))$ extends to a meromorphic function on the complex plane ($n$ odd) or the logarithmic plane ($n$ even) whose poles coincide, including multiplicity, with the resolvent resonances.

The real-valued function

$$\sigma(\lambda) = \frac{1}{2\pi i} \log \det(S_r(\lambda))$$

on $(0, \infty)$ is called the scattering phase and behaves in many respects analogously to the counting function for eigenvalues on a compact manifold. For example, Christiansen [5] has shown that the scattering phase for a class of scattering manifolds including those considered here obeys the asymptotic law

$$\sigma(\lambda) = -c_n \text{sc-vol}(X) \lambda^n + O(\lambda^{n-\frac{1}{2}})$$

as $\lambda \to \infty$, where, in our case,

$$\text{sc-vol}(X) = \lim_{\varepsilon \downarrow 0} \left( \int_{X_\varepsilon} dg - \frac{1}{n} \text{vol}(S^{n-1}) \varepsilon^{-n} - c_{n-1} \varepsilon^{n-1} - \cdots - c_0 \log \varepsilon \right)$$

The constant $c_n$ is the same constant that appears in Weyl’s law for the counting function of eigenvalues. The constants $c_k$ are chosen to make the limit finite, and $X_\varepsilon$ is the compact set in $\mathbb{R}^n$ with $|x| \leq \varepsilon^{-1}$ (equivalently, sc-vol(X) is the Hadamard finite part of vol$_g(X_\varepsilon)$ as $\varepsilon \downarrow 0$). Note that sc-vol(X) may be positive, negative, or zero, depending on $g$.

3. Technique for constructing isosphasal manifolds

Before presenting the method we will use for constructing isosphasal metrics, we review basic properties of group actions, in particular, torus actions. Given an action of a compact Lie group $G$ on a manifold $M$, the principal orbits are the orbits with minimal isotropy. The union of the principal orbits is an open dense subset $M'$ of $M$. There exists a subgroup $H$ of $G$ such the isotropy group of every element of $M'$ is conjugate to $H$. Moreover, the isotropy group of an arbitrary element of $M$ contains a subgroup conjugate to $H$. In case $G$ is a torus, it follows that the isotropy group of every element contains $H$ itself. In particular, if a torus action is effective, then $H$ is trivial and so the action on the principal orbits is free. Thus $M'$ is a principal $G$-bundle.
Notation 3.1. Suppose a torus $T$ acts smoothly on a connected manifold $M$. For each character $\alpha : T \rightarrow S^1$ (where $S^1$ is the unit circle in $\mathbb{C}$), write
\[ \mathcal{H}_\alpha = \{ f \in C^\infty(M) : f(z \cdot x) = \alpha(z)f(x) \text{ for all } x \in M, z \in T \}. \]
For $K$ a subtorus of $T$ of codimension at most one, let $C^\infty(M)^K$ denote the space of $K$-invariant smooth functions on $M$. Then
\[ C^\infty(M)^K = \oplus_{\mathcal{H}_\alpha} \mathcal{H}_\alpha. \]
Thus by Fourier decomposition, we may decompose $C^\infty(M)$ as
\[ C^\infty(M) = C^\infty(M)^T \oplus \left( \oplus_K \left( C^\infty(M)^K \oplus C^\infty(M)^T \right) \right) \]
where $K$ varies over all subtori of codimension one.

Proposition 3.2. \cite{3}, \cite{17} Let $T$ be a torus. Suppose $T$ acts effectively by isometries on two connected Riemannian manifolds $(M_1, g_1)$ and $(M_2, g_2)$ and that the action of $T$ on the principal orbits is free. Let $M'_i$ be the union of all principal orbits in $M_i$, so $M_i$ is an open, dense submanifold of $M'_i$ and a principal $T$-bundle, $i = 1, 2$. For each subtorus $K$ of $T$ of codimension at most one, suppose that there exists a $T$-equivariant volume-preserving diffeomorphism $F_K : M_1 \rightarrow M_2$ that induces an isometry $\mathcal{F}_K$ between the induced metrics on the quotient manifolds $K \backslash M'_1$ and $K \backslash M'_2$. With respect to the Fourier decompositions of $C^\infty(M_1)$ and $C^\infty(M_2)$ given in Notation 3.1, let
\[ Q = F^*_T \oplus (\oplus_K F^*_K). \]
Then
\[ \Delta_1 = Q \circ \Delta_2 \circ Q^{-1} \]
where $\Delta_i$ is the Laplace operator on $C^\infty(M_i)$.

Remark 3.3. Proposition 3.2 as stated here differs somewhat from the original statements in \cite{3} and \cite{17}. There the manifolds $M_i$ were assumed to be compact and the conclusion was that they were isospectral. However, the explicit intertwining operator $Q$ for the Laplacians, which was constructed in the proof, did not use the assumption that the manifolds were compact. A second difference between the statement here and that in \cite{3} is that a hypothesis involving preservation by the diffeomorphisms $F_K$ of the mean curvature of the fibers has been replaced by the condition that these diffeomorphisms be volume-preserving. Dorothée Schueth made this simplifying change in her version \cite{17} of the proposition, observing that the former and latter conditions are equivalent.

Theorem 3.4. Suppose a torus $T$ acts effectively on $\mathbb{R}^n$ by orthogonal transformations. Let $g_1$ and $g_2$ be compact perturbations of the Euclidean metric on $\mathbb{R}^n$ invariant under the action of $T$. Assume that the Riemannian measure defined by both metrics coincides with Lebesgue measure. Let $(\mathbb{R}^n)'$ be the union of all principal orbits of the torus action. For each subtorus $K$ of $T$ of codimension at most one, suppose that there exists an orthogonal transformation $F_K \in O(\mathbb{R}^n)$ commuting with $T$ which induces an isometry $\mathcal{F}_K$ between the metrics induced by $g_1$ and $g_2$ on the quotient manifold $K \backslash (\mathbb{R}^n)'$. Then $g_1$ and $g_2$ have the same scattering phase.

Remark 3.5. We have stated the theorem only in the form needed for the examples given here. However, the theorem may be generalized to other settings.
Proof. The manifolds \((\mathbb{R}^n, g_1)\) and \((\mathbb{R}^n, g_2)\) satisfy the hypotheses of Proposition 3.2. Define \(Q : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)\) as in the conclusion of the proposition so that \(Q_1 = Q \circ Q_2 \circ Q^{-1}\). Since the \(F_k\) are orthogonal maps of \(\mathbb{R}^n\), the map \(Q\) induces a map \(Q_\partial : C^\infty(S^{n-1}) \to C^\infty(S^{n-1})\) which extends to an invertible isometry of \(L^2(S^{n-1})\). Since all orthogonal maps commute with the antipodal map of \(S^{n-1}\), the map \(Q_\partial\) commutes with \(S_\partial(\lambda)\) as defined in equation 2.2.

From its construction, it is clear that the intertwining map \(Q\) preserves the form of asymptotic expansions (2.3). Moreover, if \(u\) is a solution \((\Delta_{g_1} - \lambda^2)u = 0\) having an asymptotic expansion of the form (2.3), then \(Qu\) is a solution of \((\Delta_{g_2} - \lambda^2)v = 0\) having an asymptotic expansion of the form

\[
v(r\omega) = r^{(1-n)/2}e^{i\lambda r} h_+(\omega) + r^{(1-n)/2}e^{-i\lambda r} h_-(\omega) + O(r^{-(n+1)/2})
\]
as \(r \to \infty\), where

\[
h_+(\omega) = (Q_\partial f_+)(\omega)
\]
and

\[
h_-(\omega) = (Q_\partial f_-)(\omega).
\]

Let \(S_{g_1}(\lambda)\) and \(S_{g_2}(\lambda)\) be the scattering matrices associated, respectively, to \((\mathbb{R}^n, g_1)\) and \((\mathbb{R}^n, g_2)\). Since

\[
f_+ = S_{g_1}(\lambda)f_-
\]
and

\[
h_+ = S_{g_2}(\lambda)h_-,
\]
it follows from the uniqueness statement in Proposition 2.1 that

\[
Q_\partial S_{g_1}(\lambda)f_- = S_{g_2}(\lambda)Q_\partial f_-.
\]

Since this holds for any \(f_- \in C^\infty(S^{n-1})\), and \(Q_\partial\) is an invertible linear map, we have

\[
S_{g_2}(\lambda) = Q_\partial S_{g_1}(\lambda)Q_\partial^{-1}.
\]

Since \(Q_\partial\) and \(Q_\partial^{-1}\) commute with the operator \(S_\partial(\lambda)\), we conclude that

\[
Q_\partial S_{g_1}(\lambda)S_\partial(\lambda)^{-1}Q_\partial^{-1} = S_{g_2}(\lambda)S_\partial(\lambda)^{-1},
\]
so that, on taking logarithms of determinants and using (2.4),

\[
\sigma_{g_2}(\lambda) = \sigma_{g_1}(\lambda).
\]

\(\square\)

4. Examples

In [6], the first author constructed continuous families of Riemannian metrics on \(\mathbb{R}^n\), \(n \geq 9\), which pairwise satisfy the hypotheses of Theorem 3.4 modulo the condition that the metrics be Euclidean outside of a compact set. Dorothee Schueth [17] pointed out that the metrics could be modified to satisfy this additional condition; in fact they could be flat outside of a compact set of arbitrarily small volume. Moreover, Schueth constructed new continuous families of metrics on \(\mathbb{R}^n\), \(n \geq 8\), pairwise satisfying the condition of Theorem 3.4. (Note that she lowered the minimum dimension by one.) Additionally, Schueth constructed pairs, though not continuous families, of such metrics on \(\mathbb{R}^5\). In both these papers, the focus was on compact manifolds. The metrics, once constructed, were restricted to the unit ball and sphere. Using Proposition 3.2, these restricted metrics were seen to be
isospectral. In the present context, we will conclude from Theorem 3.4 that the families of metrics on $\mathbb{R}^n$ constructed in these two papers are isosphasal.

We now review the construction of the metrics in [6] modified as in [17].

**Definition 4.1.** (i) We will say that two skew-symmetric bilinear maps $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle'$ taking $\mathbb{R}^m \times \mathbb{R}^m$ to $\mathbb{R}^k$ are *isospectral* if for each $Z \in \mathbb{R}^k$ there is an orthogonal transformation $A_Z$ with the property that for every pair of vectors $(x, y) \in \mathbb{R}^m \times \mathbb{R}^m$,

$$\langle [x, y], Z \rangle_{\mathbb{R}^k} = \langle [A_Z x, A_Z y], Z \rangle_{\mathbb{R}^k},$$

where, here and in what follows, $\langle \cdot, \cdot \rangle_{\mathbb{R}^k}$ denotes the Euclidean inner product on $\mathbb{R}^k$.

(ii) We will say that the skew-symmetric bilinear maps $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle'$ are *equivalent* if there exists an orthogonal transformation $A$ of $\mathbb{R}^m$ and an orthogonal transformation $C$ of $\mathbb{R}^k$, which preserves the lattice $(2\pi Z)^k$, such that $\langle [Ax, Ay], Z \rangle = \langle [x, y], CZ \rangle$ for all $x, y \in \mathbb{R}^m$ and $Z \in \mathbb{R}^k$. We will also say that the pair of maps $(A, C)$ is an equivalence of $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle'$ in this case.

**Remark 4.2.** (i) Our notation differs from that of [3]. The bilinear maps $\langle \cdot, \cdot \rangle : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^k$ correspond to linear maps $j : \mathbb{R}^k \to \mathfrak{so}(m)$ via

$$\langle [x, y], Z \rangle = \langle (j(Z)) x, y \rangle$$

for all $x, y \in \mathbb{R}^m$ and $Z \in \mathbb{R}^k$. We say that $j$ and $j'$ are *isospectral* if $j'(Z)$ and $j(Z)$ are isospectral linear operators for each $z \in \mathbb{R}^k$. We will say that $j$ and $j'$ are *equivalent* if there exist orthogonal maps $A$ of $\mathbb{R}^m$ and $C$ of $\mathbb{R}^k$ such that $C$ preserves the lattice $(2\pi Z)^k$ and such that $A j'(Z) A^{-1} = j(CZ)$ for all $z \in \mathbb{R}^k$. These conditions correspond to the isospectrality and equivalence conditions in Definition 4.1. The $j$ maps were used in [3] rather than the bracket maps $[\cdot, \cdot]$.

(ii) Our notion of equivalence differs slightly from that in [6] and [7] in that we require $C$ to preserve the lattice $(2\pi Z)^k$. This condition is added so that $C$ induces a transformation of the torus $(2\pi Z)^k \setminus \mathbb{R}^k$.

**Definition 4.3.** (i) Let $T$ be the $k$-torus $(2\pi Z)^k \setminus \mathbb{R}^k$ embedded in $\text{SO}(2k)$ as $\text{SO}(2) \times \cdots \times \text{SO}(2)$. Then $T$ acts on $\mathbb{R}^{2k} = \mathbb{R}^2 \times \cdots \times \mathbb{R}^2$ by the standard $\text{SO}(2)$-action in each factor. This action is not free but is inner-product preserving. The Lie algebra of $T$ is $\mathfrak{z} = \mathfrak{so}(2) \oplus \cdots \oplus \mathfrak{so}(2) \simeq \mathbb{R}^k$. Given $Z \in \mathfrak{z}$, define a vector field $Z^*$ on $\mathbb{R}^{2k}$ by

$$Z_u^* = \frac{d}{dt}{\bigg|}_{t=0} (\exp(tZ) \cdot u),$$

for $u \in \mathbb{R}^{2k}$. Observe that, for $x, y \in \mathbb{R}^m$, $\langle x, y \rangle^* = \mathfrak{z}$ so that we may use (1.1) to define a vector field $[x, y]^*$ on $\mathbb{R}^{2k}$.

(ii) Given a bilinear map $\langle \cdot, \cdot \rangle : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^k$ and a smooth, compactly supported function $\varphi : [0, \infty) \times [0, \infty) \to [0, \infty)$, we now construct a Riemannian metric $g = g^{1, \cdot}, \varphi$ on $\mathbb{R}^{m+2k}$. Denote elements of $\mathbb{R}^{m+2k}$ by $(x, u)$ with $x \in \mathbb{R}^m$ and $u \in \mathbb{R}^{2k}$. First define $\psi : \mathbb{R}^{m+2k} \to [0, \infty)$ by $\psi(x, u) = \varphi(\|x\|^2, \|u\|^2)$. For $(x, u) \in \mathbb{R}^{m+2k}$, denote by $(Y, W)$ a typical element of the tangent space $T_{(x, u)}\mathbb{R}^{m+2k}$, where, by standard identifications, $Y \in \mathbb{R}^m$ and $W \in \mathbb{R}^{2k}$. We set

$$g(0, W), (0, V)) = \langle W, V \rangle_{\mathbb{R}^{2k}}$$
and define the $g$-orthogonal complement to $\{0\} \oplus \mathbb{R}^{2k}$ in $T_{(x,u)}\mathbb{R}^{m+2k}$ as follows. For $(x,u) \in \mathbb{R}^m \times \mathbb{R}^{2k}$ and $Y \in T_x \mathbb{R}^m$, let
\[
\tilde{Y}_{x,u} = (Y, Z)
\]
with
\[
Z = \psi(x,u)[x,Y]_u^*.
\]
The $g$-orthogonal complement to $\{0\} \oplus \mathbb{R}^{2k}$ is taken to be
\[
\left\{ \tilde{Y}_{x,u} : Y \in T_x \mathbb{R}^m \right\}.
\]
We put an inner product on this space so that the map $Y \mapsto \tilde{Y}_{x,u}$ is an isometry where $\mathbb{R}^m$ has the Euclidean inner product.

Note that, for $(x,u)$ outside of the support of $\psi$, we have $\tilde{Y}_{x,u} = Y$. Thus the metric so constructed is identical to the Euclidean metric away from the support of $\psi$.

**Proposition 4.4.** Suppose that $[\cdot, \cdot]$ and $[\cdot, \cdot]'$ are isospectral in the sense of Definition $\ref{def:isospectral}$ and that $g$ and $g'$ are metrics constructed as in Definition $\ref{def:metrics}$ from the data $(\varphi, [\cdot, \cdot])$ and $(\varphi, [\cdot, \cdot]')$ for the same nonnegative function $\varphi \in C^\infty_0(\mathbb{R}^+ \times \mathbb{R}^+)$.

Then $g$ and $g'$ are isospectral.

**Proof.** We apply Theorem $\ref{thm:isospectral}$. Let $W$ denote the union of the principal orbits for the action of $T$ on $\mathbb{R}^{2k}$. By identifying $\mathbb{R}^{2k}$ with $\mathbb{C}^k$, we may write
\[
W = \left\{ (z_1, \ldots, z_k) \in \mathbb{C}^k : z_1 \neq 0, \ldots, z_k \neq 0 \right\}.
\]
The union of the principal orbits for the action of $T$ on $\mathbb{R}^{m+2k}$ is given by $\mathbb{R}^m \times W$.

If $K \subset T$ is a subtorus of codimension one, then in the Lie algebra $\mathfrak{t}$ of $T$, there is a vector $Z$ orthogonal to the Lie subalgebra $\mathfrak{k}$ of $K$. By hypothesis, there is an orthogonal transformation $A_Z \in \text{O}(m)$ so that
\[
([x,y]', Z)_{\mathbb{R}^k} = ([A_Z x, A_Z y], Z)_{\mathbb{R}^k}
\]
for any $x$ and $y$ belonging to $\mathbb{R}^m$. Letting $g_K$ and $g_K'$ be the metrics on $K \setminus (\mathbb{R}^m \times W)$ induced by $g$ and $g'$, it follows from Definition $\ref{def:metrics}$ that the orthogonal map
\[
tau_K(x,u) = (A_Z x, u)
\]
of $\mathbb{R}^{m+2k}$ induces an isometry from $(K \setminus (\mathbb{R}^m \times W), g_K)$ to $(K \setminus (\mathbb{R}^m \times W), g'_K)$. Thus the hypotheses of Theorem $\ref{thm:isospectral}$ are satisfied, and we conclude that the metrics are isospectral. \hfill \Box

**Remark 4.5.** In $\ref{def:isospectral}$, the function $\psi$ did not appear; i.e., $\varphi$ (and thus $\psi$) was identically one. As mentioned above, it was Dorothee Schueth that realized the function $\psi$ could be inserted so that the metrics are Euclidean outside of a compact set.

By referring to the proofs of Proposition $\ref{def:metrics}$ and of Theorem $\ref{def:isospectral}$, we can give an explicit description of the intertwining operators between the Laplacians of the metrics in Proposition $\ref{def:metrics}$ and between their scattering phases as follows:

**Proposition 4.6.** Define $g$ and $g'$ as in Proposition $\ref{def:metrics}$. Writing $\mathbb{R}^{2k} = \mathbb{R}^2 \times \cdots \times \mathbb{R}^2$ and letting $(r_i, \theta_i)$ denote polar coordinates on the $i$th factor, we obtain coordinates $(x,r, \theta)$ on $\mathbb{R}^{m+2k}$, where $x = (x_1, \ldots, x_m)$, $r = (r_1, \ldots, r_k)$
and \( \theta = (\theta_1, \ldots, \theta_k) \). For \( Z \in \mathbb{R}^k \), choose \( A_Z \) as in Definition 4.4. Define \( Q : C^\infty(\mathbb{R}^{m+2k}) \rightarrow C^\infty(\mathbb{R}^{m+2k}) \) by

\[
Q(f)(x,r,\theta) = \sum_{Z \in \mathbb{Z}^k} \left( \int_{[0,2\pi]^n} f(A_Z(x), r, \sigma)e^{-iZ \cdot \sigma} d\sigma \right) e^{iZ \cdot \theta}.
\]

Then \( Q \) intertwines the Laplacians of the metrics \( g \) and \( g' \) on \( \mathbb{R}^{m+2k} \), and the associated map \( Q_\theta \), defined as in Theorem 3.4, intertwines their scattering phases.

We now consider whether these metrics are isometric.

**Proposition 4.7.** Fix \( \varphi \). Let \( g \) and \( g' \) be the metrics defined as in Definition 4.4 by nontrivial maps \([ \cdot, \cdot \] and \([ \cdot, \cdot \]' : \( \mathbb{R}^m \times \mathbb{R}^m \rightarrow \mathbb{R}^k \)) together with \( \varphi \).

(i) Suppose that \( \tau \) is an isometry from \((\mathbb{R}^{m+2k}, g)\) to \((\mathbb{R}^{m+2k}, g')\) that carries \( T \)-orbits to \( T \)-orbits. Then \( \tau \) is of the form

\[
(\tau(x,u) = (A(x), \tilde{C}(u))
\]

where \( A \in O(m) \), \( \tilde{C} \in O(2k) \), and \( \tilde{C} \) normalizes \( T \). Letting \( C \) be the automorphism of the Lie algebra \( \mathfrak{R}^k = \mathfrak{so}(2) \oplus \cdots \oplus \mathfrak{so}(2) \) of \( T \) given by conjugation by \( \tilde{C} \), then the pair \((A, C)\) is an equivalence of \([ \cdot, \cdot \] and \([ \cdot, \cdot \]'\), as in Definition 4.1.

(ii) Conversely, every map \( \tau \) of this form is an isometry between the two metrics.

**Proof.** (ii) is straightforward and is left to the reader.

(i) Since the isometry \( \tau \) carries \( T \)-orbits to \( T \)-orbits, it must preserve the open dense subset \( \mathbb{R}^m \times W \), where \( W \) is given in the proof of Proposition 4.4. The submanifold \( \mathbb{R}^m \times W \) has the structure of a principal \( T \)-bundle over \( \mathbb{R}^m \times (T \setminus W) \cong \mathbb{R}^m \times (\mathbb{R}^+)^k \). The metrics \( g \) and \( g' \) both induce the standard Euclidean metric on the quotient \( \mathbb{R}^m \times (\mathbb{R}^+)^k \). The isometry \( \tau \) induces an isometry \( \tilde{\tau} \) of \( \mathbb{R}^m \times (\mathbb{R}^+)^k \). Such an isometry is the composition of a translation in \( \mathbb{R}^m \) with an orthogonal transformation of the form \( A \times P \), where \( A \in O(m) \) and \( P \) permutes the coordinates in \((\mathbb{R}^+)^k \). We claim that the translation factor is trivial. To see this, note that the metrics \( g \) and \( g' \) on \( \mathbb{R}^{m+2k} \) are Euclidean on the complement of \( \{ (x,u) \in \mathbb{R}^{m+2k} : (\|x\|, \|u\|) \in \text{supp}(\varphi) \} \). Letting \( R \) be minimal such that \( \text{supp}(\varphi) \subset \{ (s,t) : s^2 + t^2 \leq R \} \), then \( g \) and \( g' \) are Euclidean on the region \( \{ (x,u) : \|x\|^2 + \|u\|^2 \geq R \} \) and not on any translate of this region. Hence \( \tilde{\tau} \) must preserve the image of this region in \( \mathbb{R}^m \times (\mathbb{R}^+)^k \), and the claim follows.

For each \( x \in \mathbb{R}^m \), \( \tau \) restricts to an isometry from the Euclidean space \( \{ x \} \times \mathbb{R}^{2k} \) to the Euclidean space \( \{ A(x) \} \times \mathbb{R}^{2k} \). Canonically identifying both spaces with \( \mathbb{R}^{2k} \), this isometry preserves the origin, since the origin is the unique \( T \)-orbit which is a single point. Thus \( \tau \) is of the form \( \tau(x,u) = (A(x), B_x(u)) \) with \( B_x \in O(2k) \) for each \( x \in \mathbb{R}^m \). We may identify \( T \) with the maximal torus \( T = SO(2) \times \cdots \times SO(2) \) of \( O(2k) \). Since \( \tau \) carries \( T \)-orbits to \( T \)-orbits, each \( B_x \) must normalize \( T \). Noting that \( T \) has finite index in its normalizer in \( O(2k) \) and that \( B_x \) depends smoothly on \( x \), there must exist \( \tilde{C} \in O(2k) \), independent of \( x \), and \( z(x) \in T \) such that \( B_x = z(x) \circ \tilde{C} \). The permutation \( P \) in the expression for \( \tilde{\tau} \) is the map of \( T \setminus W \) induced by \( \tilde{C} \).

We next show that \((A, C)\) defines an equivalence of \([ \cdot, \cdot \] and \([ \cdot, \cdot \]'\), where \( C \) is defined from \( \tilde{C} \) as in the statement of the proposition. Observe that \( \tau \circ \psi = \tau \), since \( \tau(x,u) \) preserves the norms of the two coordinates. Since \( \tau \) maps \( g \)-horizontal vectors at each point \((x,u)\) (i.e., vectors \( g \)-orthogonal to the orbit of \( T \) through
the genericity condition of Proposition 4.9, then the metrics on $T$ from the data $(\cdot, \cdot, \varphi)$ as in Definition 4.3. Then the centralizer of $T$ in the group of all isometries of $g$ consists of all maps $\tau$ of $\mathbb{R}^{m+2k}$ of the form $\tau(x, u) = (A(x), z \cdot u)$ such that $A \in O(m)$ preserves $\langle \cdot, \cdot \rangle$ (i.e., the pair $(A, Id)$ is a self-equivalence of $\langle \cdot, \cdot \rangle$) and such that $z \in T$.

Proof. An isometry that commutes with $T$ must carry $T$-orbits to $T$-orbits. Thus the corollary follows from Proposition 4.7 and the fact that $T$ is its own centralizer in $O(2k)$.

Corollary 4.8. Let $g$ be the Riemannian metric on $\mathbb{R}^{m+2k}$ defined from the data $\langle [\cdot, \cdot], \varphi \rangle$ as in Definition 4.3. Then the centralizer of $T$ in the group of all isometries of $g$ consists of all maps $\tau$ of $\mathbb{R}^{m+2k}$ of the form $\tau(x, u) = (A(x), z \cdot u)$ such that $A \in O(m)$ preserves $\langle [\cdot, \cdot] \rangle$ (i.e., the pair $(A, Id)$ is a self-equivalence of $\langle [\cdot, \cdot] \rangle$) and such that $z \in T$.

Proof. By Corollary 4.8 and the genericity condition on $\langle [\cdot, \cdot] \rangle$, $T$ is a maximal torus in the full isometry group of $g$. Now suppose that $\rho: (\mathbb{R}^n, g) \to (\mathbb{R}^n, g')$ is an isometry. Since the metrics are isometric, $T$ must also be a maximal torus in the full isometry group of $g'$. By the conjugacy of the maximal tori in any Lie group, we may assume after composing with an isometry of $g'$ that $\rho$ carries $T$-orbits to $T$-orbits. By Proposition 4.7, it follows that $\langle [\cdot, \cdot] \rangle$ and $\langle [\cdot, \cdot]', \varphi' \rangle$ are equivalent, contradicting the hypothesis.

By Proposition 4.4 and Proposition 4.9, if $\langle [\cdot, \cdot] \rangle$ and $\langle [\cdot, \cdot]', \varphi' \rangle$ are isospectral, inequivalent maps $\mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^k$ in the sense of Definition 4.1 and if $\langle [\cdot, \cdot] \rangle$ satisfies the genericity condition of Proposition 4.9, then the metrics on $\mathbb{R}^{m+2k}$ constructed from the data $([\cdot, \cdot], \varphi)$ and $([\cdot, \cdot]', \varphi')$ as in Definition 4.3, for any fixed choice of $\varphi$, are isospectral but not isometric. The following lemma shows that such isospectral, inequivalent maps are plentiful.

Proposition 4.10. Let $k = 2$, and let $m$ be any positive integer other than $1, 2, 3, 4, or 6$. Let $W_m$ be the real vector space consisting of all anti-symmetric bilinear maps from $\mathbb{R}^m \times \mathbb{R}^m$ to $\mathbb{R}^k$. Then there is a Zariski open subset $O_m$ of
$W_m$ (i.e., $\mathcal{O}_m$ is the complement of the zero locus of some non-zero polynomial function on $W$) such that each $\langle \cdot, \cdot \rangle \in \mathcal{O}_m$ belongs to a $d$-parameter family of isospectral, inequivalent elements of $W_m$. Here $d \geq \frac{m(m-1)}{2} - \left\lfloor \frac{m}{2} \right\rfloor (\left\lfloor \frac{m}{2} \right\rfloor + 2) > 1$. In particular, $d$ is of order at least $O(m^2)$. Moreover, the elements of $\mathcal{O}_m$ satisfy the genericity condition of Proposition 4.9.

The statement of the proposition in [7] is in the language of Remark 4.2. The final statement of the proposition was not explicitly stated in [7]; however, a glance at the proof given there shows that the genericity condition is one of the defining properties of the Zariski open set $\mathcal{O}_m$ constructed there. While the proposition omits $m = 6$, an explicit example of a continuous family of isospectral, inequivalent maps $\mathbb{R}^m \times \mathbb{R}^m \rightarrow \mathbb{R}^2$ was also constructed in [7].

We have now proven Theorem 1.1 when $n \geq 9$. (For $n = 10$, we refer to the comment immediately above.)

To prove Theorem 1.1 in the case $n = 8$ for continuous families and $n = 6$ for pairs, we refer to the article [17] by Dorothee Schueth. There Schueth constructed metrics on $\mathbb{R}^n$ from data $(L, \psi)$ consisting of a particular type of linear map $L$ and a cut-off function $\psi$ on $\mathbb{R}^n$ of the same type used above. (The maps $L$, which play the role of the $\langle \cdot, \cdot \rangle$ maps in the construction above, are denoted $j$ or $c$ in the two different constructions given in [17].) She defined notions of isospectrality and equivalence of the linear maps. An argument analogous to the proof of Proposition 4.11 shows that, for fixed $\psi$, the metrics on $\mathbb{R}^n$ constructed from isospectral linear maps $L$ and $L'$ are isospectral. To discuss the condition for non-isometry, we will for simplicity require that the cut-off function $\psi$ be radial. The metrics constructed in [17] are Euclidean outside of the support of $\psi$ but not on any open set on which $\psi$ is positive. Since $\psi$ is supported on a ball about the origin, any isometry between the metrics must therefore carry this ball to itself. Under a genericity condition analogous to that in Proposition 4.4, Schueth proved that the metrics on the ball are not isometric provided that the associated linear maps are inequivalent. This completes the proof.

**Example 4.11.** We give an explicit triple of isospectral metrics on $\mathbb{R}^{12}$ and compare their geometries. We let $k = 3$ and $m = 6$. Define three maps $\langle \cdot, \cdot \rangle_i : \mathbb{R}^6 \times \mathbb{R}^6 \rightarrow \mathbb{R}^3$ as follows.

To define $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$, view $\mathbb{R}^6$ as $\mathbb{R}^3 \times \mathbb{R}^3$ and denote elements of $\mathbb{R}^6$ as ordered pairs $(x, y)$, with $x, y \in \mathbb{R}^3$. Let $\times$ denote the cross product on $\mathbb{R}^3$. Define

$$[(x, y), (x', y')]_1 = x \times x' + y \times y'$$

and

$$[(x, y), (x', y')]_2 = x \times x' - y \times y'.$$

To define $\langle \cdot, \cdot \rangle_3$, view $\mathbb{R}^6$ as $H \times \mathbb{R}^2$, where $H$ denotes the quaternions. Denote elements of $\mathbb{R}^6$ as pairs $(q, y)$, with $q \in H$, $y \in \mathbb{R}^2$. View the target space $\mathbb{R}^3$ as the purely imaginary quaternions. Define

$$[(q, y), (q', y')]_3 = \text{Im}(qq'),$$

where $qq'$ is the quaternionic product.

To see that the three bracket maps are isospectral, it is easier to consider the associated maps $j_i : \mathbb{R}^3 \rightarrow \mathfrak{so}(6)$ defined as in Remark 1.2. We have

$$j_1(z)(x, y) = (z \times x, z \times y),$$
\[ j_2(z)(x, y) = (z \times x, -z \times y), \]

and

\[ j_3(z)(q, y) = (zq, 0), \]

where in the final equation, \( zq \) denotes quaternionic multiplication of the purely imaginary quaternion \( z \) with the quaternion \( q \). In each case, the eigenvalues of \( j_i(z) \) are \( \|z\|\sqrt{-1}, -\|z\|\sqrt{-1}, \) and 0, each occurring with multiplicity 2. Thus \( j_1(z), j_2(z) \) and \( j_3(z) \) are similar transformations for each \( z \), and hence the \( j_i \) are mutually isospectral. Equivalently, the \([\cdot, \cdot]\) are mutually isospectral. Thus fixing a choice of \( \varphi \), we obtain a triple of isosphasal metrics \( g_i \) on \( \mathbb{R}^{12} \).

The isometry groups \( \text{Iso}(g_i) \) of the three metrics vary both in their dimension and structure. By Corollary 4.8 and Remark 4.2, every isometry of \( g_i \) that commutes with \( T \) is the composition of an element of \( T \) with an isometry of the form \( A \times \text{Id} \) acting on \( \mathbb{R}^{12} = \mathbb{R}^6 \times \mathbb{R}^6 \), where \( A \in O(6) \) commutes with all the \( j_i(z), z \in \mathbb{R}^k \).

The image of \( j_1 \) in \( \mathfrak{so}(6) \) is the set of all matrices of the form

\[
\begin{pmatrix}
B & 0 \\
0 & B
\end{pmatrix}
\]

with \( B \in \mathfrak{so}(3) \). The centralizer of this image in \( O(6) \) is the one-parameter subgroup (circle) generated by the skew-symmetric matrix

\[
\begin{pmatrix}
0 & \text{Id} \\
-\text{Id} & 0
\end{pmatrix}.
\]

Thus the centralizer of \( T \) in the connected component of \( \text{Iso}(g_1) \) is isomorphic to \( T \times S^1 \), a four-dimensional torus. In particular, the maximal tori in \( \text{Iso}(g_1) \) are four-dimensional.

The image of \( j_2 \) in \( \mathfrak{so}(6) \) is the set of all matrices of the form

\[
\begin{pmatrix}
B & 0 \\
0 & -B
\end{pmatrix}
\]

with \( B \in \mathfrak{so}(3) \). The centralizer of this image in \( O(6) \) is trivial. Thus the three-dimensional torus \( T \) is a maximal torus in \( \text{Iso}(g_2) \) and is its own centralizer.

The connected component of the centralizer of the image of \( j_3 \) in \( \mathfrak{so}(6) \) is isomorphic to \( \text{SU}(2) \times \text{SO}(2) \), where the 3-sphere \( \text{SU}(2) \) is identified with the unit quaternions acting on \( H \) by right multiplication and where \( \text{SO}(2) \) acts on the \( \mathbb{R}^2 \) factor. Since a maximal torus in \( \text{SU}(2) \times \text{SO}(2) \) is two-dimensional, the maximal tori in \( \text{Iso}(g_3) \) are five-dimensional. Moreover, the semisimple group \( \text{SU}(2) \times \text{SO}(2) \) acts by isometries preserving the bundle structure.

5. Existence of Resonances

To show that our examples are not trivial, we will show that the metrics that we have constructed have infinitely many scattering resonances, in contrast to the Euclidean Laplacian on \( \mathbb{R}^n \) which has none. We use the methods of Sa Barreto-Tang \[15\] if \( n \) is odd and Tang \[21\] if \( n \) is even.

In \[15\] it is shown that if the Laplacians of two metrics on \( \mathbb{R}^n \) (\( n \) odd) which differ from the Euclidean metric by a super-exponentially decaying perturbation have the same resonances, then they also have the same heat invariants \( a_k \) for \( k \geq 2 \). This implies that a metric with non-vanishing heat invariant \( a_2 \) must have at least finitely many resonances, and a closer analysis of the renormalized wave trace shows that, in fact, the number of resonances must be infinite in this case.
For the case of $n$ even, it is shown in [21] that, if a metric has only finitely many resonances, then the heat invariants $a_k$ for $k \geq 2$ must vanish. These papers also give various geometric hypotheses under which $a_2 \neq 0$.

Here, we carry out the proof that the heat invariant $a_2$ is non-zero for the specific metrics constructed in Section 4. For convenience, we will restrict to the cases that $k = 2$ or $k = 3$ in the notation of Section 4. These two cases include all the examples constructed by Proposition 4.10 as well as Example 4.11.

Fix a non-trivial bilinear map $[\cdot, \cdot] : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^k$. For each non-trivial compactly supported function $\varphi : [0, \infty) \times [0, \infty) \to [0, \infty)$, define the metric $g^{[\cdot, \cdot]}_{\varphi}$ on $\mathbb{R}^{m+2k}$ as in §3. The goal of this section is to show that, for generic $\varphi$, the scattering matrix for the metric $g^{[\cdot, \cdot]}_{\varphi}$ has infinitely many resonances. It suffices to show that the second heat invariant $a_2(g^{[\cdot, \cdot]}_{\varphi})$ is non-zero.

Recall that for any Riemannian metric $g$ on an $n$-dimensional manifold $M$, the second heat invariant is given by

$$a_2(g) = \frac{(4\pi)^{-\frac{n}{2}}}{360} \int_M (5\tau^2 - 2 \| \text{Ric} \|^2 + 2 \| R \|^2) \ d\text{vol}_g,$$

where $\tau$ denotes the scalar curvature.

**Notation 5.1.** Given $\varphi : [0, \infty) \times [0, \infty) \to [0, \infty)$, consider the family of functions $\varphi_s, s \in \mathbb{R}^+$, given by $\varphi_s(t_1, t_2) = \varphi(t_1, s^2 t_2)$. Set

$$g^s = g^{[\cdot, \cdot]}_{\varphi_s}$$

and denote by $R^{(s)}$ and $\text{Ric}^{(s)}$ the curvature tensor and Ricci tensor of the metric $g^s$.

**Theorem 5.2.** Let $k = 2$ or $3$, let $[\cdot, \cdot] : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^k$ be a non-zero bilinear map, and let $\varphi : [0, \infty) \times [0, \infty) \to [0, \infty)$ be a non-trivial compactly supported $C^\infty$ function. We use Notation 5.1. Except for possibly finitely many values of $s$, $a_2(g^s) \neq 0$.

**Definition 5.3.** Let $\{f^s\}, s \in \mathbb{R}^+$, be a one-parameter family of functions on $\mathbb{R}^{m+2k}$. We will say that $\{f^s\}$ is a homogeneous $(r, s)$-deformation of degree $d$ if $f^s(x, r, \theta) = s^d f^r(x, \tilde{r}, \theta)$ for all $s$, where $\tilde{r} = (\tilde{r}_1, \ldots, \tilde{r}_k) = (sr_1, \ldots, sr_k)$. We will say $\{f^s\}$ is an $(r, s)$-deformation of degree $d$ if $f^s = f^r_s + \ldots + f^r_k$ where each $\{f^r_s\}$ is a homogeneous $(r, s)$-deformation, say of degree $d_i$, and where $d = \max\{d_i\}$. Assuming the $d_i$'s are distinct, we will refer to $f^r_s$ as the homogeneous term of degree $d_i$ in $f^r$.

The following two lemmas are elementary.

**Lemma 5.4.** If $\{f^s\}$ is a homogeneous $(r, s)$-deformation of degree $d$, then:

(i) $\{\frac{\partial}{\partial r_j} f^s\}, 1 \leq j \leq m$, is also a homogeneous $(r, s)$-deformation of degree $d$ while

(ii) $\{\frac{\partial}{\partial s_p} f^s\}, 1 \leq p \leq k$, is a homogeneous $(r, s)$-deformation of degree $d + 1$;

(iii) If $\{h^s\}$ is a homogeneous $(r, s)$-derivation of degree $d'$, then $\{f^s h^s\}$ is a homogeneous $(r, s)$-deformation of degree $d + d'$.

**Lemma 5.5.** If $\{f^s\}$ is a homogeneous $(r, s)$-deformation of degree $d$ and if each $f^s$ is continuous with compact support, then

$$\int_{\mathbb{R}^{m+2k}} f^s = s^{d-2k} \int_{\mathbb{R}^{m+2k}} f^1$$

where the integrals are with respect to Lebesgue measure.
To prove that the function $a_2(g^s)$ is non-zero except for possibly finitely many choices of $s$, we will show that the integrand is an $(r,s)$-deformation of degree 2, and that the homogeneous term of degree two is strictly positive. Consequently, after multiplying by an appropriate power of $s$, the heat invariant $a_2(g^s)$ is a non-trivial polynomial in $s$, from which Theorem 5.2 follows.

**Notation 5.6.** (i) Write $\mathbb{R}^{2k}$ as $\mathbb{R}^2 \times \mathbb{R}^2 \cdots \times \mathbb{R}^2$ and let $(r_i, \theta_i)$, $i = 1, 2, \ldots, k$, denote the polar coordinates on the $k$ factors $\mathbb{R}^2$. We thus coordinatize a dense open subset of $\mathbb{R}^{m+2k}$ by $(x, r, \theta) = (x_1, \ldots, x_m, r_1, r_2, \ldots, r_k, \theta_1, \theta_2, \ldots, \theta_k)$.

(ii) For $\{Z_1, \ldots, Z_k\}$ the standard basis of $\mathfrak{g} = \mathbb{R}^k$, the vector field $Z_p$, $p = 1, \ldots, k$, defined in equation (4.1), is given by $\hat{\theta}_p = \frac{\partial}{\partial \theta_p}$. 

(iii) Let indices $i, j, k$ run from 1 to $m$, indices $p, q$ range over $1, 2, \ldots, k$, and Greek indices range over $1, \ldots, m+2k$. Define an orthonormal frame field as follows: For $p = 1, 2, \ldots, k$, let $\hat{r}_p = \hat{\theta}_p$ and let $\hat{\theta}_p$ be a unit vector in the direction of $\hat{\theta}_p$, i.e., $\hat{\theta}_p = \frac{1}{r_p} \theta_p$. For $\{e_1, \ldots, e_m\}$ the standard basis of $\mathbb{R}^m$, define

$$a_{ip}^s(x, r) = \varphi_s(\|x\|^2, \|r\|^2)([x, e_i], Z_p).$$

Set

$$\hat{x}_i^s = e_i + a_{i1}^s(x, r) \frac{\partial}{\partial \theta_1} + \cdots + a_{ik}^s(x, r) \frac{\partial}{\partial \theta_k} = e_i + a_{i1}^s(x, r)r_1\hat{\theta}_1 + \cdots + a_{ik}^s(x, r)r_k\hat{\theta}_k.$$  

Then

$$\{E_1^s, \ldots, E_{m+2k}^s\} = \{\hat{x}_1^s, \ldots, \hat{x}_m^s, \hat{r}_1, \ldots, \hat{r}_k, \hat{\theta}_1, \ldots, \hat{\theta}_k\}$$

is an orthonormal frame field on $(\mathbb{R}^{m+2k}, g^s)$. (Note that $E_\alpha^s$ depends trivially on $s$ when $\alpha > m$.) Set

$$I_1 = \{1, \ldots, m\},$$

$$I_2 = \{m + 1, \ldots, m + k\},$$

and

$$I_3 = \{m + k + 1, \ldots, m + 2k\}.$$

**Lemma 5.7.** We use the notation of Notation 5.6 and Definition 5.3. Let $c^{(s)\gamma}_{\alpha\beta}$ denote the structure constants given by $[E_\alpha^s, E_\beta^s] = \sum_{\gamma} c^{(s)\gamma}_{\alpha\beta} E_\gamma^s$. Then:

(i) If $\gamma \in I_3$ and $\alpha, \beta \in I_1$, then $\{c^{(s)\gamma}_{\alpha\beta}\}$ is a homogeneous $(r,s)$-deformation of degree $-1$.

(ii) If $\gamma \in I_3$, one of $\alpha, \beta$ is in $I_1$ and the other is in $I_2$, then $\{c^{(s)\gamma}_{\alpha\beta}\}$ is a homogeneous $(r,s)$-deformation of degree 0.

(iii) In all other cases $c^{(s)\gamma}_{\alpha\beta} = 0$. 
Proof. By Notation 5.6, we have for $1 \leq i, j \leq m$ and $1 \leq p \leq k$,

$$[\hat{x}_i, \hat{x}_j] = \sum_{q=1}^{k} \left( \frac{\partial}{\partial x_i} a^q_{jq} - \frac{\partial}{\partial x_j} a^q_{iq} \right) \frac{\partial}{\partial \theta_q}$$

$$= \sum_{q=1}^{k} \left( \frac{\partial}{\partial r} a^q_{iq} - \frac{\partial}{\partial r} a^q_{iq} \right) r_q \hat{\theta}_q$$

$$[\hat{r}_p, \hat{x}_i] = \sum_{q=1}^{k} \frac{\partial}{\partial r_p} a^q_{iq} \frac{\partial}{\partial \theta_q}$$

$$= \sum_{q=1}^{k} \frac{\partial}{\partial r_p} a^q_{iq} r_q \hat{\theta}_q.$$ 

All other brackets of vectors in our orthonormal frame are zero. The lemma now follows from Lemma 5.7 and the fact that for each $i, p$, $\{a^q_{ip}\}$ is a homogeneous $(r, s)$-deformation of degree 0.

\[\square\]

Lemma 5.8. Let $\Gamma^{(s)}_{\alpha\beta\gamma}$, $\alpha, \beta, \gamma = 1, \ldots, m + 2k$ denote the Christoffel symbols for the metric $g^s$ with respect to the frame field $\{E^s_1, \ldots, E^s_{m+2k}\}$, i.e., $\nabla^s_{E^s_\alpha} E^s_\beta = \sum_{\gamma} \Gamma^{(s)}_{\alpha\beta\gamma} E^s_\gamma$, where $\nabla^s$ is the Levi-Civita connection for $g^s$. Then:

(i) If two of the indices $\alpha, \beta, \gamma$ lie in $I_1$ and the other lies in $I_3$, then $\{\Gamma^{(s)}_{\alpha\beta\gamma}\}$ is a homogeneous $(r, s)$-deformation of degree $-1$.

(ii) If one of the indices $\alpha, \beta, \gamma$ lies in $I_1$, one lies in $I_2$ and one lies in $I_3$, then $\{\Gamma^{(s)}_{\alpha\beta\gamma}\}$ is a homogeneous $(r, s)$-deformation of degree 0.

(iii) If the indices $\alpha, \beta, \gamma$ do not satisfy the conditions of either (i) or (ii), then $\Gamma^{(s)}_{\alpha\beta\gamma} = 0$.

Thus in every case, $\{\Gamma^{(s)}_{\alpha\beta\gamma}\}$ is a homogeneous $(r, s)$-deformation. Moreover:

(iv) If $\{\Gamma^{(s)}_{\alpha\beta\gamma}\}$ is a homogeneous $(r, s)$-deformation of degree $d$, then $\{\Gamma^{(s)}_{\alpha\beta\gamma}\}$ given by $\Gamma^{(s)}_{\alpha\beta\gamma} := E^s_\delta (\Gamma^{(s)}_{\alpha\beta\gamma})$ is a (possibly zero) homogeneous $(r, s)$-deformation. It is of degree $d + 1$ if $\delta \in I_2$ and of degree $d$ otherwise.

Proof. We have

$$\Gamma^{(s)}_{\alpha\beta\gamma} = \frac{1}{2} \{ g^s([E_\beta, E_\alpha], E_\gamma) + g^s([E_\gamma, E_\alpha], E_\beta) + g^s([E_\gamma, E_\beta], E_\alpha) \}$$

$$= \frac{1}{2} \{ c^{(s)}_{\beta\alpha} + c^{(s)}_{\gamma\alpha} + c^{(s)}_{\gamma\beta} \}.$$ 

Thus this lemma follows from Lemma 5.7.

\[\square\]

Proposition 5.9. Let $R^{(s)}_{\alpha\beta\gamma\delta} = g^s(R^{(s)}(E^s_\gamma, E^s_\beta)E^s_\delta, E^s_\alpha)$. Then $\{R^{(s)}_{\alpha\beta\gamma\delta}\}$ is an $(r, s)$-deformation of degree at most one. Moreover, if $\{R^{(s)}_{\alpha\beta\gamma\delta}\}$ has degree one, then one of the indices $\alpha, \beta, \gamma, \delta$ lies in $I_1$, two (including at least one of $\gamma, \delta$) lie in $I_2$ and one lies in $I_3$.

Proof. We have

$$R^{(s)}_{\alpha\beta\gamma\delta} = \Gamma^{(s)}_{\mu\gamma} \Gamma^{(s)}_{\beta\delta} - \Gamma^{(s)}_{\mu\delta} \Gamma^{(s)}_{\beta\gamma} - c^{(s)}_{\gamma\delta} \Gamma^{(s)}_{\beta\mu} + \Gamma^{(s)}_{\beta\delta,\gamma} - \Gamma^{(s)}_{\beta\gamma,\delta}.$$
By Lemmas 5.4 and 5.8, each of the first three terms belong to \((r,s)\)-deformations of non-positive degree. Also by Lemma 5.8, \(\{\Gamma^{(s)\alpha}_{\beta\gamma}\}\) is a homogeneous \((r,s)\)-deformation. Its degree is one if \(\gamma \in I_2\) and if \(\Gamma^{(s)\alpha}_{\beta\delta}\) has degree zero; otherwise its degree is non-positive. Again applying Lemma 5.8, if \(\Gamma^{(s)\alpha}_{\beta\delta}\) has degree zero and is non-trivial, then each of \(I_1\), \(I_2\) and \(I_3\) contains exactly one of \(\alpha\), \(\beta\), and \(\delta\). The proposition now follows.

ProofofTheorem 5.4: It follows from Proposition 5.9 that the integrand \(\{5\tau^2 - \|\text{Ric}^{(s)}\|^2 + \|R^{(s)}\|^2\}\) in \(\{a_2(g^s)\}\) is an \((r,s)\)-deformation of degree at most 2. We now show that it in fact has degree 2. We have

\[
\text{Ric}^{(s)}_{\alpha\beta} = \sum_\gamma R^{(s)}_{\alpha\gamma\beta\gamma}
\]

and

\[
\|\text{Ric}^{(s)}\|^2 = \sum_{\alpha,\beta} |\text{Ric}^{(s)}_{\alpha\beta}|^2.
\]

By Proposition 5.9, \(\{\text{Ric}^{(s)}_{\alpha\beta}\}\) is an \((r,s)\)-deformation of degree at most one. Moreover, \(\{\text{Ric}^{(s)}_{\alpha\beta}\}\) has degree one only when one of \(\alpha\), \(\beta\) lies in \(I_1\) and the other in \(I_3\). In this case, the homogeneous term of degree one in \(\text{Ric}^{(s)}_{\alpha\beta}\) is equal to the homogeneous term of degree one in \(R^{(s)}_{\alpha \beta}(\text{m}+1)\beta(\text{m}+1) + \cdots + R^{(s)}_{\alpha \beta}(\text{m+k})\beta(\text{m+k})\). Consequently, \(\{\|\text{Ric}^{(s)}\|^2\}\) is an \((r,s)\)-deformation of degree at most two, and the homogeneous term of degree two equals

\[
2 \sum_{\alpha \in I_1} \sum_{\beta \in I_3} |h_1(R^{(s)}_{\alpha \beta}(\text{m}+1)\beta(\text{m}+1)) + \cdots + h_1(R^{(s)}_{\alpha \beta}(\text{m+k})\beta(\text{m+k}))|^2,
\]

where we are using the notation \(h_1(\cdot)\) to denote the homogeneous term of degree one. (The coefficient 2 is due to the symmetry when we interchange the roles of \(\alpha\) and \(\beta\).) From the identity \((a_1 + \ldots + a_k)^2 \leq k(\sum a_i^2)\), we thus see that the homogeneous term of degree two in \(\|\text{Ric}^{(s)}\|^2\) is no bigger than \(2kT^s\) where

\[
T^s := \sum_{\alpha \in I_1} \sum_{\beta \in I_3} \left(|h_1(R^{(s)}_{\alpha \beta}(\text{m}+1)\beta(\text{m}+1))|^2 + \cdots + |h_1(R^{(s)}_{\alpha \beta}(\text{m+k})\beta(\text{m+k}))|^2\right).
\]

Next consider \(\|R^{(s)}\|^2 = \sum_{\alpha,\beta,\gamma,\delta} |R^{(s)}_{\alpha\beta\gamma\delta}|^2\). Note that \(\{\|R^{(s)}\|^2\}\) is also an \((r,s)\)-deformation of degree at most two. The homogeneous term of degree two equals the homogeneous term of degree two in the sum of those \(R^{(s)}_{\alpha\beta\gamma\delta}\) for which two of the indices lie in \(I_2\), one lies in \(I_1\) and one in \(I_3\). Due to the symmetries of the curvature tensor, we conclude that the homogeneous term of degree two in \(\|R^{(s)}\|^2\) is given by \(8T^s\).

We conclude that the homogeneous term of degree two in \(\{-\|\text{Ric}\|^2 + \|R^{(s)}\|^2\}\) satisfies \(f^s \geq (8 - 2k)T^s \geq 0\). (The latter inequality uses the hypothesis that \(k \leq 3\).) Moreover, the homogeneous term of degree two in \(\tau^s\) equals \((\tau^s)^2 \geq 0\), where \(\tau^s\) is the homogeneous term of degree one in \(\tau^s\). By Lemma 5.5, the heat invariant \(a_2(g^s)\) is a finite real linear combination of powers of \(s\) with the coefficient of \(s^{2-2k}\) being given by \(\int_{R^{m+2k}} 5(\tau^s)^2 + f^1\). Thus the theorem will follow if we show that \(\int_{R^{m+2k}} f^1 \neq 0\). Since \(f^1 \geq 0\), we have \(\int_{R^{m+2k}} f^1 = 0\) only if \(f^1 \equiv 0\).
Choose $\alpha, \beta, \gamma, \delta$ as follows: Choose $i \in \{1, \ldots, m\}$ and $p \in \{1, \ldots, k\}$ so that the linear transformation $\mathbb{R}^m \to \mathbb{R}$ given by $x \to \langle [x, e_i], Z_p \rangle$ is not identically zero. Without loss of generality, we assume $p = 1$. Let $\beta = \delta = m + 2$, let $\alpha = m + k + 1$ and let $\gamma = i$. Then by Notation 5.6 and the proofs of Lemma 5.7, Lemma 5.8, and Proposition 5.9, the homogeneous term of degree one in $R^1_{\alpha \beta \gamma \delta}$ is given by

$$\frac{\partial^2}{\partial r_1^2} a_{11} r_1 = 4r_2^2 \phi_{22}(\|x\|^2, \|r\|^2)([x, e_i], Z_1) r_1$$

(viewed as a function on $\mathbb{R}^{m+2k}$ depending trivially on $\theta$). This function cannot be identically zero since the smooth cut-off function $\varphi$ cannot be linear in either variable. It follows that $f^1$ is not identically zero. This completes the proof of Theorem 5.2. \[\square\]

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