Soft noncommutative schemes via toric geometry and morphisms from an Azumaya scheme with a fundamental module thereto — (Dynamical, complex algebraic) D-branes on a soft noncommutative space

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Abstract

A class of noncommutative spaces, named soft noncommutative schemes via toric geometry, are constructed and the mathematical model for (dynamical/nonsolitonic, complex algebraic) D-branes on such a noncommutative space, following arXiv:0709.1515 [math.AG] (D(1)), is given. Any algebraic Calabi-Yau space that arises from a complete intersection in a smooth toric variety can embed as a commutative closed subscheme of some soft noncommutative scheme. Along the study, the notion of soft noncommutative toric schemes associated to a (simplicial, maximal cone of index 1) fan, invertible sheaves on such a noncommutative space, and twisted sections of an invertible sheaf are developed and Azumaya schemes with a fundamental module as the world-volumes of D-branes are reviewed. Two guiding questions, Question 3.12 (soft noncommutative Calabi-Yau spaces and their mirror) and Question 4.2.14 (generalized matrix models), are presented.

Key words: monoid algebra; soft noncommutative toric scheme, soft noncommutative scheme, invertible sheaf, twisted section; D-brane, Azumaya scheme, morphism

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0. Introduction and outline

The realization of the noncommutative feature of D-brane world-volumes ([H-W], [Po1], [Wi]; [L-Y1], [Liu]) makes D-branes a good candidate as a probe to noncommutative geometry. Unfortunately, all the three building blocks of modern Commutative Algebraic Geometry — the notion of localizations of a ring, the method of associating a topology to a ring via Spec, and the approach to understand a ring by studying its category of modules — have their limitation when extending to noncommutative rings and Noncommutative Algebraic Geometry. (See, e.g., [Ro1], [Ro2], [B-R-S-S-W] to get a feel.) Unlike a (fundamental) superstring world-sheet, which can have a very abundant class of geometry as its target-spaces — most notably Calabi-Yau spaces — it is not immediate clear what class of noncommutative spaces can serve as the target-space of a dynamical super D-string world-sheet and how to construct them, let alone the even deeper issue of Mirror Symmetry phenomenon when two different target-spaces can give rise to isomorphic 2-dimensional supersymmetric quantum field theories on the superstring world-sheet.

In this work we introduce a class of noncommutative spaces, named ‘soft noncommutative schemes’ via toric geometry (Sec. 2 and Sec. 3), to partially take care of the persistent difficulty of gluing in general Noncommutative Algebraic Geometry. This by construction is only a very limited class of noncommutative spaces, yet abundant enough that all the algebraic Calabi-Yau spaces that arise from complete intersections in a smooth toric variety can show up as a commutative subscheme of some soft noncommutative scheme (Corollary 3.11). A mathematical model for (dynamical, complex algebraic) D-brane on such a noncommutative space is then given (Sec. 4) as morphisms from Azumaya schemes with a fundamental module thereto. From this, one is certainly very curious as to how Mirror Symmetry stands when these soft noncommutative spaces are taken to be the target-spaces of D-string world-sheets.

Convention. References for standard notations, terminology, operations and facts are
(1) Azumaya/matrix algebra: [Ar], [Az], [A-N-T];
(2) commutative monoid: [Og];
(3) toric geometry: [Fu];
(4) aspects of noncommutative algebraic geometry: [B-R-S-S-W];
(5) commutative algebra and (commutative) algebraic geometry: [Ei], [E-H], [Ha];
(6) string theory and D-branes: [Ba], [Jo], [Po1], [Po2], [Sz].

· All commutative schemes are over \( \mathbb{C} \) and Noetherian.
· For a sheaf \( \mathcal{F} \) on a gluing system of charts, the notation ‘\( s \in \mathcal{F} \)’ means a local section \( s \in \mathcal{F}(U) \) for some chart \( U \).

Outline

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1 Monoids and monoid algebras

Basic definitions of monoids and monoid algebras we need for the current work are collected in this section to fix terminology and notation. Readers are referred to [De], [Ge], [G-H-V] and [Og], from where these definitions are adapted, for more details.

Definition 1.1. [monoid] A monoid is a triple $(M,\ast,e_M)$, abbreviated by $(M,\ast)$ or $M$, that consists of a set $M$, an associative binary operation $\ast$, and a (two-sided) identity element $e_M$ of $(M,\ast)$: $e_M \ast m = m = m \ast e_M$ for all $m \in M$. An element $m \in M$ is invertible if there exists an element $m' \in M$ such that $m \ast m' = m' \ast m = e_M$. When $m$ is invertible, such $m'$ is unique and is denoted $m^{-1}$, called the inverse of $m$. The center $\text{Center}(M)$ of $M$ is the submonoid $\{c \in M \mid c \ast m = m \ast c, \text{for all } m \in M\}$ of $M$. A homomorphism of monoids (or monoid homomorphism) is a function $\theta : M \to N$ between monoids such that $\theta(e_M) = e_N$ and $\theta(m \ast m') = \theta(m) \ast \theta(m')$. $\theta$ is called a monomorphism if it is injective, and an epimorphism if surjective.

Given a set $S$, the free monoid $F(S)$ associated to $S$ is the monoid that consists of an element $e$ and all formal words $s_1 \cdots s_k$ of finitely many elements of $S$, with the product rules $s_1 \cdots s_k \ast s'_1 \cdots s'_k = s_1 \cdots s_k s'_1 \cdots s'_k$ and $e \ast s_1 \cdots s_k = s_1 \cdots s_k \ast e = s_1 \cdots s_k$.

Let $S \subseteq M$ be a subset of a monoid. We say that $M$ is generated by $S$ if the monoid homomorphism $F(S) \to M$ specified by the inclusion $S \hookrightarrow M$, with $e \mapsto e_M$ and $s_1 \cdots s_k \mapsto s_1 \ast \cdots \ast s_k$, is surjective. We say that a monoid $M$ is finitely generated if $M$ is generated by a finite subset $S \subseteq M$. We say that a generating set $S$ of $M$ is inverse-closed if $s \in S$ and $s$ is invertible in $M$, then $s^{-1} \in S$.

By convention, when a monoid is written multiplicatively and the monoid operation $\ast$ is clear from the content, $m_1 \ast m'$ will be denoted simply $m \cdot m'$ or $mm'$. Commutative monoids are often written additively, with $\ast$ denoted $+$ and $e_M$ denoted $0$.

Definition 1.2. [product and central extension] Let $(C,\cdot,1)$ and $(M,\ast,e_M)$ be monoids. The product of $(C,\cdot,1)$ and $(M,\ast,e_M)$ is the monoid with elements $(c,m) \in C \times M$ with $(c_1,m_1) \ast (c_2,m_2) = (c_1 \cdot c_2,m_1 \ast m_2)$ and identity $(1,e_M)$. When $C$ is commutative, we’ll denote the product $C \cdot M$ or $CM$ and elements $cm$. In this case, $C \hookrightarrow C \cdot M \twoheadrightarrow M$, with $c \mapsto ce_M$ and $cm \mapsto m$, is a central extension of $M$.

Definition 1.3. [Cayley graph of monoid with finite generating set] Let $(M,\ast,e_M)$ be a monoid with an inverse-closed, finite generating set $S$. Then the Cayley graph $\Gamma_{\text{Cayley}}(M,S)$ of $(M,S)$ is a directed graph with the set of vertices $M$ and, for every pair $(m_1,m_2)$ of vertices a directed edge labelled by $s \in S$ from $m_1$ to $m_2$ if $m_2 = m_1 \ast s$. We take the convention that if both $s$ and $s^{-1}$ are in $S$, then the edge associated to $s$ and that associated to $s^{-1}$ coincide with opposite directions; i.e. the corresponding edge is bi-directed. Since every $m \in M$ can be expressed as a finite word in letters from $S$, $\Gamma_{\text{Cayley}}(M,S)$ is a connected graph.

Definition 1.4. [edge-path monoid] (Continuing Definition 1.3.) Endow the Cayley graph $\Gamma := \Gamma_{\text{Cayley}}(M,S)$ of $(M,S)$ with the topology from the geometric realization of $\Gamma$ as a 1-dimensional simplicial complex. An edge-path at the vertex $m_0$ on $\Gamma$, with the terminal vertex $m_t$, is a direction-preserving continuous map $\gamma : [0,1] \to \Gamma$, with $\gamma(0) = m_0$ and $\gamma(1) = m_t$, that is piecewise linear after a subdivision of the interval $[0,1]$ as a 1-dimensional simplicial complex. Thus, associated to each edge-path $\gamma$ at $m_0$ is a word $w_\gamma = s_1 \cdots s_k$ in letters from $S$, where $s_1$ is the initial edge $\gamma$ takes from the vertex $m_0$ and $s_k$ is the final edge $\gamma$ takes to the terminal
associative binary operation, still denoted by $\star$. Given an edge-path $\gamma$ at $m_0$ and a vertex $m \in M$, there is a translation $T_m : \gamma \mapsto m\gamma$, where $m\gamma$ is the edge-path at $m$ that takes the same word $w_\gamma$ as $\gamma$ and is piecewise linear under the same subdivision of $[0, 1]$. The constant edge-path $\gamma : [0, 1] \to \Gamma$ such that $\gamma([0, 1]) = m \in M$ will be denoted by the vertex $m$.

There is an operation $\star$ on the set of edge-paths:

- For edge-paths $\gamma_1$ and $\gamma_2$ on $\Gamma$ such that $\gamma_1(1) = \gamma_2(0)$, define
  $$(\gamma_1 \star \gamma_2)(t) = \begin{cases} 
  \gamma_1(2t), & \text{for } t \in [0, \frac{1}{2}]; \\
  \gamma_2(2t - 1), & \text{for } t \in [\frac{1}{2}, 1].
  \end{cases}$$
- For general edge-paths $\gamma_1$ and $\gamma_2$ on $\Gamma$, define
  $$\gamma_1 \star \gamma_2 := \gamma_1 \star T_{\gamma_1(1)}(\gamma_2).$$

In particular, $m \star \gamma = T_m(\gamma)$, $\gamma(0) \star \gamma = \gamma$, and $\gamma \star m = \gamma$, after a subdivision of $[0, 1]$ and a simplicial map on the subdivided $[0, 1]$ that fixes $\{0, 1\}$, for all vertex $m$ and edge-path $\gamma$.

Consider the space $\tilde{M}$ of homotopy classes of edge-paths at the vertex $e_M$ of $\Gamma$ relative to the boundary $\{0, 1\}$ of the interval $[0, 1]$. The operation $\star$ on edge-paths on $\Gamma$ descends to an associative binary operation, still denoted by $\star$, on $\tilde{M}$ with the identity the constant path $e_M$. The monoid $(\tilde{M}, \star, e_M)$ will be called the edge-path monoid associated to $(M, S)$.

**Definition 1.5.** [monoid algebra] Let $(M, \star, e_M)$ be a monoid. The $\mathbb{C}$-vector space $\bigoplus_{m \in M} \mathbb{C} \cdot m$ with the binary operation $\star$ generated by $\mathbb{C}$-linear expansion of

$$(c_1m_1) \star (c_2m_2) = (c_1c_2) \cdot (m_1 \star m_2)$$

is called the monoid algebra over $\mathbb{C}$ associated to $M$, in notation $\mathbb{C}(M)$ and the identity $1 \cdot e_M =: 1$. When $M$ is commutative, we will denote $\mathbb{C}(M)$ by $\mathbb{C}[M]$.

**Lemma 1.6.** [natural homomorphism] There is a built-in monoid epimorphism $\pi : \tilde{M} \to M$, which extends to an epimorphism $\pi : \mathbb{C}(\tilde{M}) \to \mathbb{C}(M)$ of monoid algebras over $\mathbb{C}$.

**Proof.** This follows from the surjective map $\pi : \{\text{edge-paths } \gamma \text{ at } e_M\} \to M$, with $\gamma \mapsto \gamma(1)$. \qed

**Example 1.7.** [lattice of rank $n$] Let $M$ be the lattice of rank $n$ (i.e. $\simeq \mathbb{Z}^n$ as a $\mathbb{Z}$-module). As a commutative monoid written multiplicatively, $M \simeq \times_{i=1}^n \mathbb{Z}$ and generated by the inverse-closed subset $S = \{z_1, z_1^{-1}, \ldots, z_n, z_n^{-1}\}$. The associated $\mathbb{C}$-algebra is a polynomial ring

$$\mathbb{C}[M] = \mathbb{C}[z_1, z_1^{-1}, \ldots, z_n, z_n^{-1}] \simeq \frac{\mathbb{C}[z_1, z_1', \ldots, z_n, z_n']}{(z_1z_1' - 1, \ldots, z_nz_n' - 1)}.$$ 

The Cayley diagram $\Gamma$ of $(M, S)$ is a homogeneous graph of constant valence $2n$ for all vertices, with the base vertex $1 \in M$. Under the above presentation of $M$, the edge-path monoid $\tilde{M}$ is $\langle z_1, z_1^{-1}, \ldots, z_n, z_n^{-1} \rangle$ (the noncommutative monoid generated by $z_1, z_1^{-1}, \ldots, z_n, z_n^{-1}$ subject to relations $z_i z_i^{-1} = z_i^{-1} z_i = 1$, $i = 1, \ldots, n$ and the associated monoid algebra

$$\mathbb{C}(\tilde{M}) = \mathbb{C}(z_1, z_1^{-1}, \ldots, z_n, z_n^{-1}) \simeq \frac{\mathbb{C}(z_1, z_1', \ldots, z_n, z_n')}{(z_1z_1' - 1, z_1'z_1 - 1, \ldots, z_nz_n' - 1, z_n'z_n - 1)}.$$
Here, $\mathbb{C}(z_1, z'_1, \ldots, z_n, z'_n)$ is the free associative $\mathbb{C}$-algebra with $2n$ generators and $(z_1 z'_1 - 1, z'_1 z_1 - 1, \ldots, z_n z'_n - 1, z'_n z_n - 1)$ is the two-sided ideal generated by the elements indicated. The monoid homomorphism $\pi : \tilde{M} \to M$ from the terminal-vertex-of-edge-path-at-1 map and the associated monoid-algebra homomorphism $\pi : \mathbb{C}(\tilde{M}) \to \mathbb{C}[M]$ coincide with the quotient by the commutator ideal $[\tilde{M}, \tilde{M}]$. Cf. Figure 1-1.

The complex torus $T := \mathbb{T}^n := (\mathbb{C}^\times)^n$ action on the generators

$$(z_1, \ldots, z_n) \mapsto (t_1 z_1, \ldots, t_n z_n), \quad t = (t_1, \ldots, t_n) \in T,$$

induces a $\mathbb{T}^n$-action on $\mathbb{C} \cdot M$ and $\mathbb{C} \cdot \tilde{M}$, leaving each $\mathbb{C}$-factor of $\mathbb{C} \cdot M$ and $\mathbb{C} \cdot \tilde{M}$ invariant, that render $\pi : \mathbb{C} \cdot \tilde{M} \to \mathbb{C} \cdot M$ $\mathbb{T}^n$-equivariant. This extends to a $\mathbb{T}^n$-action on $\mathbb{C}(\tilde{M})$ and $\mathbb{C}[M]$ under which $\pi : \mathbb{C}(\tilde{M}) \to \mathbb{C}[M]$ is $\mathbb{T}^n$-equivariant.

The lattice $M$ of rank $n$ in Example 1.7 is the starting point of constructions of the current work. As a commutative monoid, there are occasions when it is more convenient for $M$ to be written additively, rather than multiplicatively. We shall switch between these two pictures freely and use whichever suits better.

2 Soft noncommutative toric schemes

Soft noncommutative toric schemes are introduced in this section. The setting and the terminology used indicate our focus on the function ring, rather than the point-set and topology, of a noncommutative space while striving to retain enough underlying geometric picture for the purpose of studying (dynamical) D-branes on a noncommutative target-space in string theory within the realm of Algebraic Geometry. (Cf. [L-Y1] (D(1)), [L-L-S-Y] (D(2)), and Sec. 4.2 of the current work.)
2.1 The noncommutative affine $n$-space $ncA^n_C$ and its 0-dimensional subschemes

Definition 2.1.1. [affine scheme, function ring, noncommutative affine $n$-space over $C$] Let $\text{Ring}$ be the category of rings. An object $X$ in the opposite category $\text{Ring}^{op}$ of $\text{Ring}$ is called an affine scheme. The ring that underlies $X$ is called the function ring of $X$. If the function ring of $X$ is noncommutative (resp. commutative), then we say that $X$ is a noncommutative affine scheme (resp. commutative affine scheme). In particular, we define a noncommutative affine $n$-space over $C$ to be an object in $\text{Ring}^{op}$ that corresponds to a free noncommutative associative $C$-algebra generated by $n$ letters, in notation $C\langle z_1, \cdots, z_n \rangle$. Since any two free noncommutative associative $C$-algebras generated by $n$ letters are isomorphic for a fixed $n$, we shall denote a noncommutative affine $n$-space over $C$ commonly by $ncA^n_C$ or simply $ncA^n$. $ncA^n$ is smooth in the sense of the following extension/lifting property:

- For any $C$-algebra homomorphism $C\langle z_1, \cdots, z_n \rangle \rightarrow A$ and any $C$-algebra epimorphism $B \twoheadrightarrow A$, there exists a $C$-algebra homomorphism $C\langle z_1, \cdots, z_n \rangle \rightarrow B$ that makes the following diagram commute

\[
\begin{array}{ccc}
C\langle z_1, \cdots, z_n \rangle & \rightarrow^\sim & A \\
\downarrow & & \downarrow \\
B & \rightarrow & A
\end{array}
\]

Definition 2.1.2. [master commutative subscheme of $ncA^n$] The natural quotient

\[
C\langle z_1, \cdots, z_n \rangle \twoheadrightarrow C[z_1, \cdots, z_n], \quad z_i \mapsto z_i
\]

of $C$-algebras, with kernel the two-sided ideal $(z_iz_j - z_jz_i | 1 \leq i < j \leq n)$, defines an embedding $A^n \hookrightarrow ncA^n$, called the master commutative subscheme of $ncA^n$.

Definition 2.1.3. [closed subscheme of $ncA^n$] A closed subscheme of $ncA^n$ is a $C$-algebra quotient $C\langle z_1, \cdots, z_n \rangle \longrightarrow A$. When $A$ is commutative, then the closed subscheme is called a commutative closed subscheme of $ncA^n$. Since when $A$ is commutative, such a quotient always factors as

\[
\begin{array}{ccc}
C\langle z_1, \cdots, z_n \rangle & \longrightarrow & A \\
\downarrow & & \downarrow \\
C[z_1, \cdots, z_n] & \longrightarrow & A
\end{array}
\]

a commutative closed subscheme of $ncA^n$ is always contained in the master commutative subscheme $A^n \subset ncA^n$. When $A$ is a finite-dimensional $C$-algebra, the closed subscheme is called a 0-dimensional subscheme of $ncA^n$. In particular, a $C$-point on $ncA^n$ is a $C$-algebra quotient $C\langle z_1, \cdots, z_n \rangle \longrightarrow C$. Since $C$ is commutative, the set of $C$-points of $ncA^n$ coincide naturally with the set of $C$-points of $A^n$ via the built-in embedding $A^n \hookrightarrow ncA^n$.

Definition 2.1.4. [punctual versus nonpunctual 0-dimensional subscheme] A 0-dimensional subscheme $C\langle z_1, \cdots, z_n \rangle \rightarrow A$ of $ncA^n$ is called punctual if $A$ admits a $C$-algebra quotient

\[
A \twoheadrightarrow \text{Center}(A)/\text{Nil}_{\text{Center}(A)}
\]

Here, $\text{Center}(A) \subset A$ is the center of $A$ and $\text{Nil}_{\text{Center}(A)}$ is the ideal of nilpotent elements of $\text{Center}(A)$. Else, the 0-dimensional subscheme is called nonpunctual (i.e. “fuzzy without core”).
Example 2.1.5. [Grassmann points, Azumaya/matrix points, other special points on $ncA^n$] (1) All commutative 0-dimensional subschemes of $ncA^n$ are punctual.

(2) A Grassmann point (resp. Azumaya or matrix point) on $ncA^n$ is a $C$-algebra quotient $\mathbb{C}\langle z_1, \ldots, z_n \rangle \to A$ such that $A$ is isomorphic to a Grassmann algebra over $\mathbb{C}$

$$\mathbb{C}[\theta_1, \ldots, \theta_r]^{anti-c} := \mathbb{C}[\theta_1, \ldots, \theta_r]/(\theta_\alpha \theta_\beta + \theta_\beta \theta_\alpha | 1 \leq \alpha, \beta \leq r)$$

for some $r$ (resp. a matrix algebra $M_r(\mathbb{C})$ over $\mathbb{C}$ for some rank $r \geq 2$). Both are noncommutative points on $ncA^n$. However, Grassmann points are punctual while Azumaya/matrix points are nonpunctual since there exists no $\mathbb{C}$-algebra homomorphism from $M_r(\mathbb{C})$ to $\mathbb{C}$ for $r \geq 2$.

(3) Other special points on $ncA^n$ include points $\mathbb{C}\langle z_1, \ldots, z_n \rangle \to A$ where $A$ is isomorphic to the $\mathbb{C}$-algebra $T_{r, upper}(\mathbb{C})$ of upper triangular matrix for some rank $r \geq 2$ (resp. the $\mathbb{C}$-algebra $T_{r, lower}(\mathbb{C})$ of lower triangular matrix for some rank $r \geq 2$.) They are called upper-triangular matrix points (resp. lower-triangular matrix points). Unlike Azumaya/matrix points on $ncA^n$, both are punctual under the correspondence that sends an upper-or-lower triangular matrix to its diagonal.

Definition 2.1.6. [nested master $l$-commutative closed subscheme of $ncA^n$] Given two two-sided ideals $I$, $J$ in $\mathbb{C}\langle z_1, \ldots, z_n \rangle$. Denote by $[I, J]$ the two-sided ideal in $\mathbb{C}\langle z_1, \ldots, z_n \rangle$ generated by elements of the form $[f, g] := fg - gf$, where $f \in I$ and $g \in J$. Now let $m_0$ be the two-sided ideal $(z_1, \ldots, z_n)$ of $\mathbb{C}\langle z_1, \ldots, z_n \rangle$. Then note that $[m_0, m_0]$ and $(z_i z_j - z_j z_i | 1 \leq i < j \leq n)$ coincide. Consider the closed scheme $A^n_{(l)}$ of $ncA^n$, $l = 1, 2, \cdots$, associated the $\mathbb{C}$-algebra quotient

$$\mathbb{C}\langle z_1, \ldots, z_n \rangle \twoheadrightarrow \mathbb{C}\langle z_1, \ldots, z_n \rangle/[m_0, m_0]^l.$$ 

Since $[m_0, m_0] \subset [m_0, m_0]^2 \subset [m_0, m_0]^3 \subset \cdots$, one has a nested sequence of closed subschemes in $ncA^n$

$$A^n = A^n_{(1)} \hookrightarrow A^n_{(2)} \hookrightarrow A^n_{(3)} \hookrightarrow \cdots.$$ 

Any function on $ncA^n$ of the form $f_1 \cdots f_i$, where $f_i \in m_0$ for all $i$, becomes commutative when restricted to $A^n_{(l')}$ for $l' \leq l$. Furthermore, any closed subscheme of $ncA^n$ with this property must be a closed subscheme of $A^n_{(l)}$. Thus, we name $A^n_{(l)}$ the master $l$-commutative subscheme of $ncA^n$ (with respect to the choice of coordinate functions $z_1, \ldots, z_n$ on $ncA^n$). The restriction of any function $f$ on $ncA^n$ of degree $d \geq 1$ to any $A^n_{(l)}$ with $l \leq d$ is commutative on $A^n_{(l)}$.

Example 2.1.7. [0-dimensional subscheme and $A^n_{(l)}$] How a 0-dimensional subscheme in $ncA^n$ intersects the nested sequence $A^n_{(l)}$, $l = 1, 2, \ldots$, gives one a sense of a depth of noncommutativity the 0-dimensional subscheme of $ncA^n$ is located at. (1) A commutative 0-dimensional subscheme of $ncA^n$ lies in $A^n = A^n_{(1)}$ and hence in $A^n_{(l)}$ for all $l$.

(2) A punctual 0-dimensional subscheme of $ncA^n$ contains a commutative subscheme and hence must have nonempty intersection with $A^n$ and hence has nonempty intersection with $A^n_{(l)}$ for all $l$. They thus lie in an infinitesimal neighborhood of $A^n$ in $ncA^n$.

(3) Consider the Azumaya/matrix point on $ncA^n$ given by

$$\mathbb{C}\langle z_{ij} | 1 \leq i, j \leq r \rangle \twoheadrightarrow M_r(\mathbb{C}) , \quad z_{ij} \mapsto e_{ij},$$

where $e_{ij}$ is the $r \times r$ matrix with $ij$-entry 1 and elsewhere 0. The kernel of this $\mathbb{C}$-algebra epimorphism is the two-sided ideal $I$ generated by

$$z_{ij}z_{i'j'} - \delta_{jj'}z_{ij'} , \ 1 \leq i, j, i', j' \leq r , \text{ where } \delta \text{ is the Kronecker delta.}$$
In particular, \( z_{ij} - z_{ij}'z_{ij}z_{ij}' \cdots z_{ij_{l-1}}z_{ij_{l}}z_{ij} \in I \) for all \( l = 1, 2, \ldots \). Recall the two-sided ideal \( m_0 = (z_{ij} | 1 \leq i, j \leq r) \) of \( \mathbb{C}(z_{ij} | 1 \leq i, j \leq r) \). Then the two-sided ideal generated by \( I + [m_0, m_0^l] \) in \( \mathbb{C}(z_{ij} | 1 \leq i, j \leq r) \) is the whole \( \mathbb{C}(z_{ij} | 1 \leq i, j \leq r) \). This shows that this Azumaya point on \( ncA^{r^2} \) has no intersection with \( A^1_{ij} \) for all \( l \). In a sense, this Azumaya point is located deep in the noncommutative part of \( ncA^n \).

**Definition 2.1.8. [connectivity and hidden disconnectivity of 0-dimensional subscheme]** Given a 0-dimensional subscheme \( \mathbb{C}(z_1, \ldots, z_n) \rightarrow A \) on \( ncA^n \). We say that the 0-dimensional subscheme is *connected* if \( A \) is not a product \( A_1 \times A_2 \) of two \( \mathbb{C} \)-algebras. Else, we say that the 0-dimensional subscheme is *disconnected*. If \( A \) is not a product, yet it admits a decomposition of 1 by non-zero orthogonal idempotents

\[
1 = e_1 + \cdots + e_k,
\]

for some \( k \), where \( e_i \neq 0 \) and \( e_i^2 = e_i \) for all \( i \) and \( e_i e_j = e_j e_i = 0 \) for all \( i \neq j \), then we say that the connected 0-dimensional subscheme has *hidden disconnectivity*.

The phenomenon of hidden disconnectivity can occur for noncommutative points — for example, Azumaya/matrix points, upper-triangular matrix points, lower-triangular matrix points — on \( ncA^n \) whether they are punctual or not. (Cf. [L-Y1] (D(1)) for hidden disconnectivity of Azumaya schemes via the notion of surrogates.)

Any \( \mathbb{C} \)-algebra homomorphism

\[
u^t : \mathbb{C}[t] \rightarrow \mathbb{C}(z_1, \ldots, z_n)
\]

with \( \nu^t(t) \notin \mathbb{C} \) is a monomorphism and, by definition/tautology, induces a dominant morphism

\[
u : ncA^n \rightarrow \mathbb{A}^1.
\]

In this way, a 0-dimensional subscheme \( \mathbb{C}(z_1, \ldots, z_n) \rightarrow A \) on \( ncA^n \) is mapped to a 0-dimensional, now commutative, subscheme on the affine line \( \mathbb{A}^1 \). One may use such projections to get a sense as to how the 0-dimensional subscheme sits in \( ncA^n \) and some of its properties. (Cf. Radom transformation in analysis.)

**Example 2.1.9. [hidden disconnectivity of Azumaya/matrix point on \( ncA^n \) manifested via projection to \( A^1 \)]** Recall the matrix point on \( ncA^{r^2} \) in Example 2.1.7 (3). Consider the projection \( u : ncA^{r^2} \rightarrow \mathbb{A}^1 \) specified by the \( \mathbb{C} \)-algebra monomorphism

\[
u^t : \mathbb{C}[t] \rightarrow \mathbb{C}(z_{ij} | 1 \leq i, j \leq r), \quad t \mapsto z_{ii'} \text{ for some } i';
\]

and let \( \hat{\nu}^t \) be a \( \mathbb{C} \)-algebra homomorphism from the composition

\[
\mathbb{C}[t] \xrightarrow{\nu^t} \mathbb{C}(z_{ij} | 1 \leq i, j \leq r) \xrightarrow{M_r(\mathbb{C})}
\]

induced by the given matrix point on \( ncA^{r^2} \). Then the matrix point, though connected, is projected under \( u \) to two distinct closed points \( \{ p = 0, \ p = 1 \} \) on \( \mathbb{A}^1 \), described by the ideal \( (t(t - 1)) \), i.e. the kernel \( \text{Ker}(\hat{\nu}^t) \) of \( \hat{\nu}^t \), of \( \mathbb{C}[t] \). In this way, \( u \) reveals the hidden disconnectivity of the given matrix point on \( ncA^{r^2} \). As a (left) \( \mathbb{C}[t] \)-module through \( \hat{\nu}^t \), \( M_r(\mathbb{C}) \) is pushed forward under \( u \) to a torsion \( \mathcal{O}_{\mathbb{A}^1} \)-module on \( \mathbb{A}^1 \), with fiber-dimension \( r^2 - r \) at \( p = 0 \) and \( r \) at \( p = 1 \).

For reference, if one instead considers the projection \( v : ncA^{r^2} \rightarrow \mathbb{A}^1 \) specified by the \( \mathbb{C} \)-algebra monomorphism \( v^t : \mathbb{C}[t] \rightarrow \mathbb{C}(z_{ij} | 1 \leq i, j \leq r) \) with \( t \mapsto z_{ii'} \) for some \( i' \neq j' \). Then the given matrix point is projected to a nonreduced point on \( \mathbb{A}^1 \) associated to the ideal \( (t^2) \) of \( \mathbb{C}[t] \).
Let $N$ be a lattice isomorphic to $\mathbb{Z}^n$, with a fixed basis $(e_1, \cdots, e_n)$, and $M = \text{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ be the dual lattice, with the evaluation pairing denoted by $\langle \cdot, \cdot \rangle : M \times N \rightarrow \mathbb{Z}$ and the dual basis $(e^*_1, \cdots, e^*_n)$. Denote the associated $\mathbb{R}$-vector spaces $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$ and $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$.

Let $\sigma$ be the cone in $N_{\mathbb{R}}$ generated by $e_1, \cdots, e_n$. Then the commutative monoid $M_{\sigma} := \sigma^\vee \cap M$ is generated by $\{e^*_1, \cdots, e^*_n\}$. Let $\hat{M}$ be the associated noncommutative monoid from edge-paths at 0, cf. Definition 1.4. Then the correspondence $e^*_i \mapsto z_i$, $i = 1, \cdots, n$, specifies a $\mathbb{C}$-algebra isomorphism from the monoid algebra $\mathbb{C}[M_{\sigma}]$ to the function ring $\mathbb{C}[z_1, \cdots, z_n]$ of $\text{ncA}^n$. Under the built-in monoid homomorphism $\pi : \hat{M}_{\sigma} \rightarrow M_{\sigma}$, this isomorphism descends to an isomorphism $\mathbb{C}[M_{\sigma}] \rightarrow \mathbb{C}[z_1, \cdots, z_n]$ that gives the standard realization of $\text{A}^n$ as a toric variety. Furthermore, since the multiplicative group $\mathbb{C}^\times := \mathbb{C} - \{0\}$ in $\mathbb{C}$ lies in the center of $\mathbb{C}[\hat{M}_{\sigma}]$, the $(\mathbb{C}^\times)$-action on $\mathbb{C}[M_{\sigma}]$ generated by $(z_1, \cdots, z_n) \mapsto (t_1z_1, \cdots, t_nz_n)$ for $(t_1, \cdots, t_n) \in (\mathbb{C}^\times)^n$ lifts canonically to a $(\mathbb{C}^\times)^n$-action on $\mathbb{C}[\hat{M}_{\sigma}]$ that makes $\pi : (\mathbb{C}^\times)^n$-equivariant.

**Definition 2.1.10. [\text{ncA}^n as noncommutative affine toric scheme]** We shall call the above construction the realization of $\text{ncA}^n$ as a smooth noncommutative affine toric scheme over $\mathbb{C}$, associated to the cone $\sigma \subset N_{\mathbb{R}}$.

By construction, the restriction of the toric scheme structure on $\text{ncA}^n$ to the master commutative subscheme $\text{A}^n \hookrightarrow \text{ncA}^n$ recovers the ordinary realization of $\text{A}^n$ as a toric variety.

### 2.2 Soft noncommutative toric schemes associated to a fan

Recall the lattice $N \simeq \mathbb{Z}^n$, with a fixed basis $(e_1, \cdots, e_n)$, the dual lattice $M = \text{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$, the evaluation pairing denoted by $\langle \cdot, \cdot \rangle : M \times N \rightarrow \mathbb{Z}$, the dual basis $(e^*_1, \cdots, e^*_n)$ of $M$, and the associated $\mathbb{R}$-vector spaces $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$ and $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$.

Let $\Delta$ be a fan in $N$. (That is, $\Delta$ is a set of rational strongly convex polyhedral cones $\sigma$ in $N_{\mathbb{R}}$ such that (1) Each face of a cone in $\Delta$ is also a cone in $\Delta$; (2) The intersection of two cones in $\Delta$ is a face of each; cf. [Fu: Sec. 1.4].) For $\sigma, \tau \in \Delta$, denote $\tau \preceq \sigma$ or $\sigma \succeq \tau$ if $\tau$ is a face of $\sigma$; and denote $\tau \prec \sigma$ if $\tau$ is a face of $\sigma$ and $\tau \neq \sigma$.

**Definition 2.2.1. [\Delta-system, inverse \Delta-system of objects in a category]** Let $\mathcal{C}$ be a category, with the object set denoted $\text{ObjectC}$ and the set of morphisms of objects denoted $\text{MorphismC}$. Then a $\Delta$-system (resp. inverse $\Delta$-system) of objects in $\mathcal{C}$ is a correspondence $F : \Delta \rightarrow \text{ObjectC}$ and a choice of morphisms $h_{\tau \sigma} : F(\tau) \rightarrow F(\sigma)$ (resp. $h_{\sigma \tau} : F(\sigma) \rightarrow F(\tau)$) for all pairs of $(\sigma, \tau) \in \Delta \times \Delta$ with $\tau \prec \sigma$ such that $h_{\tau \sigma} \circ h_{\rho \tau} = h_{\rho \sigma}$ (resp. $h_{\sigma \rho} \circ h_{\sigma \tau} = h_{\sigma \rho}$) for all $\rho \prec \tau \prec \sigma$.

Recall how a variety $Y(\Delta)$ can be associated to $\Delta$, (e.g. [Fu]) as an inverse $\Delta$-system of monoid algebras or a $\Delta$-system of affine schemes: Associated to each cone $\sigma \in \Delta$ is a commutative monoid

$$M_{\sigma} := \sigma^\vee \cap M = \{u \in M \mid \langle u, v \rangle \geq 0 \text{ for all } v \in \sigma\}.$$  

By construction, there is a built-in monoid inclusion $M_{\sigma} \hookrightarrow M_{\tau}$ for $\tau \prec \sigma$. This gives rise to a $\Delta$-system of affine schemes

$$\{ U_{\sigma} := \text{Spec}(\mathbb{C}[M_{\sigma}]) \}_{\sigma \in \Delta},$$
where $\mathbb{C}[M_\sigma]$ is the monoid algebra over $\mathbb{C}$ determined by $M_\sigma$, that glue to a toric variety $Y(\Delta)$ through the system of inclusions of Zariski open sets

$$\iota_{\tau\sigma} : U_\tau \hookrightarrow U_\sigma,$$

induced from the monoid inclusion $M_\sigma \hookrightarrow M_\tau$, that by construction satisfy the gluing conditions:

$$\iota_{\tau\sigma} \circ \iota_{\rho\tau} = \iota_{\rho\sigma}, \quad \text{for all } \rho < \tau < \sigma \in \Delta.$$

The built-in $(\mathbb{C}^\times)^n$-action on $U_\sigma$ and the built-in torus embedding

$$T^n := T^n_c := \text{Spec}(\mathbb{C}[M]) \hookrightarrow U_\sigma$$

for each $\sigma \in \Delta$ glues to a $(\mathbb{C}^\times)^n$-action on $Y(\Delta)$ and a torus embedding $T^n \hookrightarrow Y(\Delta)$.

In this section, we shall construct a class of noncommutative spaces, named soft noncommutative toric schemes, associated to $\Delta$ by constructing an inverse $\Delta$-system $\{\hat{M}_\sigma\}_{\sigma \in \Delta}$ of submonoids in $\hat{M} := \langle e_1^*, e_1^{-1}, \cdots, e_n^*, e_n^{-1} \rangle$ via lifting the above construction to the edge-path monoids of commutative monoids with generators and propose soft gluings to bypass the generally unsolvable issue of localizations of noncommutative rings when trying to gluing.

Assumption 2.2.2. [on fan $\Delta$] Recall Sec. 2.1, theme: $\text{ncA}^n$ as a smooth noncommutative affine toric scheme. To ensure that we have at least one uncomplicated chart in the atlas to begin with for the noncommutative toric scheme to be constructed, we shall assume that

- The cone generated by $e_1, \cdots, e_n$ is in $\Delta$.

The corresponding chart $\simeq \text{ncA}^n$ is called the reference toric chart, cf. Definition 2.2.13. Furthermore, to ensure that we have enough good fundamental charts in the atlas before gluings, we shall assume in addition that

- All the maximal cones in $\Delta$ are $n$-dimensional, simplicial, and of index 1.

In particular, $Y(\Delta)$ is smooth.

Remark 2.2.3. [other noncommutative toric variety] Based on various aspects of toric varieties and the language used, there are other notions and constructions of “noncommutative toric varieties”; for example, the work [K-L-M-V1], [K-L-M-V2] of Ludmil Katzarkov, Ernesto Lupercio, Laurent Meersseman, and Alberto Verjovsky. It is not clear if there is any connection among them. See Introduction for our motivation from D-branes in string theory.

Inverse $\Delta$-systems of $\Delta$-admissible submonoids of $\hat{M}$

Recall from Example 1.7 the edge-path monoid $\hat{M} := \langle e_1^*, e_1^{-1}, \cdots, e_n^*, e_n^{-1} \rangle$ (cf. slightly different notation here) and the built-in monoid epimorphisms $\pi : \hat{M} \to M$, $\pi : \mathbb{C} \cdot \hat{M} \to \mathbb{C} \cdot M$ and monoid-algebra epimorphism $\pi : \mathbb{C}(\hat{M}) \to \mathbb{C}[M]$ via commutatization.

Definition 2.2.4. [submonoid of $\hat{M}$ admissible to cone in $\Delta$] Let $\sigma \in \Delta$. Then a submonoid $\hat{M}'$ of $\hat{M}$ is called admissible to $\sigma$ if it satisfies: (0) $\hat{M}'$ is finitely generated. (1) The monoid epimorphism $\pi : \hat{M} \to M$ induces a monoid epimorphism $\hat{M}' \to M_\sigma$ by restriction. (2) An $\hat{m} \in \hat{M}'$ is invertible in $\hat{M}'$ if $\pi(\hat{m})$ is invertible in $M_\sigma$. 

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Theorem 2.2.5. [existence of inverse $\Delta$-system of admissible submonoids of $\tilde{M}$] There exists an inverse $\Delta$-system $\{\tilde{M}_\sigma\}_{\sigma \in \Delta}$ of submonoids of $\tilde{M}$ such that $\tilde{M}_\sigma$ is admissible to $\sigma$ for all $\sigma \in \Delta$. We shall call such a system an inverse $\Delta$-system of admissible submonoids of $\tilde{M}$.

Proof. We proceed in three steps.

Step (a) : Submonoids of $\tilde{M}$ admissible to maximal cones in $\Delta$

Let $\sigma$ be a maximal cone in $\Delta$. By assumption, $\sigma$ is $n$-dimensional, simplicial, and of index 1. Thus, $\sigma^\lor$ is $n$-dimensional, simplicial, of index 1 in $M_\mathbb{R}$ and the generators $u_1, \ldots, u_n$ of the monoid $\sigma^\lor \cap M$ generates $M$ as well.

Lemma 2.2.6. [submonoid admissible to maximal cone] Let $\tilde{u}_i \in \pi^{-1}(u_i) \subset \tilde{M}$, for $i = 1, \ldots, n$. Then the submonoid $\langle \tilde{u}_1, \ldots, \tilde{u}_n \rangle \subset \tilde{M}$ is monoid-isomorphic to the free associative monoid of $n$ letters $\langle z_1, \ldots, z_n \rangle$.

Proof. Recall the basis $(e_1^*, \ldots, e_n^*)$ for $M$. Since $(u_1, \ldots, u_n)$ generates $M$ as well, up to a relabelling of indices one may assume that $u_i = \sum_{j=1}^n a_{ij} e_j^*$ with the coefficient $a_{ij} \neq 0$, for $i = 1, \ldots, n$. Since $(\tilde{u}_1, \ldots, \tilde{u}_n)$ projects to $(u_1, \ldots, u_n)$ under $\pi$, as a word in $2n$ letters $e_1^*, e_1^{-1}, \ldots, e_n^*, e_n^{-1}$, $\tilde{u}_i$ must contain a letter $z_i^1 := e_i^*$, if $a_{ii} > 0$, or $e_i^{-1}$, if $a_{ii} < 0$. The correspondences $\tilde{u}_i \mapsto z_i^1 \mapsto z_i$ induce monoid-isomorphisms $\langle \tilde{u}_1, \ldots, \tilde{u}_n \rangle \sim \langle z_1^1, \ldots, z_n^1 \rangle \sim \langle z_1, \ldots, z_n \rangle$.

Step (b) : Submonoids of $\tilde{M}$ admissible to lower-dimensional cones in $\Delta$

For each maximal cone $\sigma \in \Delta$, fix a submonoid $\tilde{M}_\sigma \subset \tilde{M}$ admissible to $\sigma$ as constructed in Step (a). For a cone $\tau \in \Delta$ of dimension $< n$, let $\Delta_\tau := \{\sigma \mid \tau \subsetneq \sigma \subset \sigma \text{ maximal cone} \} \subset \Delta$ and consider the submonoid of $\tilde{M}$ generated by all $\tilde{M}_\sigma$, where $\sigma$ is a maximal cone in $\Delta$ that contains $\tau$:

$$\tilde{M}'_{\tau} := \langle \tilde{M}_\sigma \mid \sigma \in \Delta_\tau \rangle \subset \tilde{M}.$$ 

Since each $\sigma \in \Delta_\tau$ is strongly convex, the submonoid $M_{\tau} := \tau^\lor \cap M$ of $M$ is generated by $M_\gamma$, $\gamma \in \Delta_\tau$. It follows that the built-in monoid epimorphism restricts to a monoid epimorphism $\pi : \tilde{M}'_{\tau} \to M_{\tau}$. The submonoid of $M_{\tau}$ that consists of invertible elements in $M_{\tau}$ is $\tau^\bot \cap M$.

Lemma 2.2.7. [finite generatedness] The submonoid $\pi^{-1}(\tau^\bot \cap M) \cap \tilde{M}'_{\tau}$ of $\tilde{M}'_{\tau}$ is finitely generated.

Proof. For $\sigma$ a maximal cone in $\Delta$, let $u_{\sigma,1}, \ldots, u_{\sigma,n}$ be the generators of $M_\sigma$ and $\tilde{u}_{\sigma,1}, \ldots, \tilde{u}_{\sigma,n}$ be the generators of the monoid $\tilde{M}_{\tau}$ with $\pi(\tilde{u}_{\sigma,i}) = u_{\sigma,i}$. Then, by construction, $\{\tilde{u}_{\sigma,i} \mid \sigma \in \Delta_\tau, i = 1, \ldots, n\} \subset \tilde{M}$ generates the submonoid $\tilde{M}'_{\tau}$ of $\tilde{M}$. Since $\tau^\bot \subset \partial \tau^\lor := \bigcup_{\rho <_\tau \rho} \rho$ and $\tau^\lor = \sum_{\sigma \in \Delta_\tau} \sigma^\lor$ is convex with $\tau^\bot$ a linear stratum of $\partial \tau^\lor$, if $m_1, m_2 \in M_{\tau}$ satisfy the condition $m_1 + m_2 \in \tau^\bot \cap M$, then both $m_1$ and $m_2$ must be in $\tau^\bot \cap M$. If follows that the submonoid $\pi^{-1}(\tau^\bot \cap M) \cap \tilde{M}'_{\tau}$ of $\tilde{M}'_{\tau}$ is generated by $\{\tilde{u}_{\sigma,i} \mid \sigma \in \Delta_\tau, i = 1, \ldots, n; u_{\sigma,i} \in \tau^\bot\}$.

\[ \square \]
Let
\[ \tilde{S}_\tau := \langle \bar{u}, u^{-1} \rangle \quad \sigma \in \Delta_\tau, \ i = 1, \ldots, n; \ u_{\sigma,i} \in \tau^+ \]
and augment \( \tilde{M}_\tau \) to
\[ \tilde{M}_\tau := \langle \tilde{M}_\tau, \tilde{S}_\tau \rangle \subset \tilde{M}. \]
Then, \( \tilde{M}_\tau \) is now a submonoid of \( \tilde{M} \) admissible to \( \tau \in \Delta \). By construction, note that the submonoid \( \tilde{M}_\tau \) of \( \tilde{M} \) that consists of all the invertible elements of the monoid \( M_\tau \) coincides with \( \tilde{S}_\tau \); and that \( \tilde{M}_0 = \tilde{M} \).

**Step (c). The inverse \( \Delta \)-system \( \{ \tilde{M}_\sigma \}_{\sigma \in \Delta} \)**

Consider the category of submonoids of \( \tilde{M} \) with morphisms of objects given by inclusions of submonoids of \( \tilde{M} \) and the collection \( \{ \tilde{M}_\sigma \}_{\sigma \in \Delta} \) of submonoids of \( \tilde{M} \) that are admissible to cones in \( \Delta \) constructed in Steps (a) and (b). For \( \tau \prec \sigma \), \( \Delta_\tau \supset \Delta_\sigma \) and hence \( \tilde{M}_\tau \subset \tilde{M}_\sigma \). For \( \rho \prec \tau \prec \sigma \), \( \tilde{M}_\sigma \subset \tilde{M}_\tau \subset \tilde{M}_\rho \) naturally. This shows that the collection \( \{ \tilde{M}_\sigma \}_{\sigma \in \Delta} \) is an inverse \( \Delta \)-system of submonoids of \( \tilde{M} \) and concludes the proof of the theorem.

\[ \square \]

**Definition 2.2.8. [augmentation and diminishment]** Let \( \{ \tilde{M}_\sigma \}_{\sigma \in \Delta} \) and \( \{ \tilde{M}'_\sigma \}_{\sigma \in \Delta} \) be two inverse \( \Delta \)-systems of \( \Delta \)-admissible submonoids of \( \tilde{M} \) such that \( \tilde{M}_\sigma \subset \tilde{M}'_\sigma \) for all \( \sigma \in \Delta \). Then we say that \( \{ \tilde{M}_\sigma \}_{\sigma \in \Delta} \) is a diminishment of \( \{ \tilde{M}'_\sigma \}_{\sigma \in \Delta} \) and \( \{ \tilde{M}'_\sigma \}_{\sigma \in \Delta} \) is an augmentation of \( \{ \tilde{M}_\sigma \}_{\sigma \in \Delta} \).

In notation, \( \{ \tilde{M}_\sigma \}_{\sigma \in \Delta} \preceq \{ \tilde{M}'_\sigma \}_{\sigma \in \Delta} \) and \( \{ \tilde{M}'_\sigma \}_{\sigma \in \Delta} \succeq \{ \tilde{M}_\sigma \}_{\sigma \in \Delta} \).

Similar arguments to the proof of Theorem 2.2.5 give the following:

**Proposition 2.2.9. [completion to inverse \( \Delta \)-system]** Let \( \Delta(n) \) be the set of \( n \)-dimensional cones in \( \Delta \). Given \( \{ \tilde{M}'_\sigma \}_{\sigma \in \Delta(n)} \) such that \( \tilde{M}_\sigma \) is a submonoid of \( \tilde{M} \) admissible to \( \sigma \), there exists an inverse \( \Delta \)-system \( \{ \tilde{M}_\tau \}_{\tau \in \Delta} \) of admissible submonoids of \( \tilde{M} \) such that \( \tilde{M}_\tau = \tilde{M}'_\tau \) for \( \sigma \in \Delta(n) \). We shall call \( \{ \tilde{M}_\tau \}_{\tau \in \Delta} \) a completion of \( \{ \tilde{M}'_\sigma \}_{\sigma \in \Delta(n)} \) to an inverse \( \Delta \)-system of admissible submonoids of \( \tilde{M} \).

**Proof.** The same proof as the proof of Theorem 2.2.5 goes through as long as \( \tilde{M}_\sigma, \sigma \in \Delta(n) \), in Step (a) is admissible to \( \sigma \).

\[ \square \]

**Proposition 2.2.10. [augmentation by \( \Delta \)-indexed submonoids]** Given an inverse \( \Delta \)-system \( \{ \tilde{M}'_\sigma \}_{\sigma \in \Delta} \) of admissible submonoids of \( \tilde{M} \) and a collection \( \{ \tilde{S}_\sigma \}_{\sigma \in \Delta} \) of finitely generated submonoids of \( \tilde{M} \) such that \( \pi(\tilde{S}_\sigma) \subset \tilde{M}_\sigma \), there exists an augmentation \( \{ \tilde{M}_\sigma \}_{\sigma \in \Delta} \) of \( \{ \tilde{M}'_\sigma \}_{\sigma \in \Delta} \) such that \( \tilde{S}_\sigma \subset \tilde{M}_\sigma \) for all \( \sigma \in \Delta \).

**Proof.** We begin with maximal cones \( \sigma \in \Delta(n) \) and define \( \tilde{M}_\sigma := \langle \tilde{M}_\sigma, \tilde{S}_\sigma \rangle \). Since \( \sigma' \) in this case is strongly convex in \( M_R, \tilde{M}_\sigma \) must already be admissible to \( \sigma \). We then proceed by induction by the dimension of cones in \( \Delta \) and assume that \( \tilde{M}_\sigma \) for \( \sigma \in \Delta(i), 1 < k \leq i \leq n \), are constructed such that (1) \( \tilde{M}_\tau \) is admissible to \( \sigma \), (2) \( \tilde{M}_\sigma \supset \langle \tilde{M}'_\sigma, \tilde{S}_\sigma \rangle \), (3) \( \tilde{M}_\tau \supset \tilde{M}_\sigma \) if \( \tau \prec \sigma \), for \( \sigma, \tau \in \Delta(i), 1 < k \leq i \leq n \). Let \( \tau \in \Delta(k-1) \) and
\[ \tilde{M}'_\tau := \langle \tilde{M}'_\tau, \tilde{S}_\tau, \tilde{M}_\sigma \mid \sigma \in \Delta_\tau \cap \Delta(k) \rangle. \]
Then \( \tilde{M}'_\tau \cap \pi^{-1}(\tau^+) \) is finitely generated and hence so is the submonoid
\[ \tilde{S}'_\tau := \{ s \in \tilde{M} \mid s \text{ or } s^{-1} \text{ is in } \tilde{M}'_\tau \cap \pi^{-1}(\tau^+) \} \]
of \( \tilde{M} \). Define

\[
\tilde{M}_\tau := (\tilde{M}'_\tau, \tilde{S}'_\tau).
\]

Then (1) \( \tilde{M}_\tau \) is admissible to \( \tau \), (2) \( \tilde{M}_\tau \supset (\tilde{M}'_\tau, \tilde{S}'_\tau) \), (3) \( \tilde{M}_\tau \supset \tilde{M}_\sigma \) if \( \tau \prec \sigma \) for all \( \sigma \in \Delta \). Finally we set \( \tilde{M}_0 = \tilde{M} \). By construction, \( \{ \tilde{M}_\sigma \}_{\sigma \in \Delta} \) is an augmentation of \( \{ \tilde{M}'_\sigma \}_{\sigma \in \Delta} \) that satisfies \( \tilde{S}_\sigma \subset \tilde{M}_\sigma \) for all \( \sigma \in \Delta \).

Corollary 2.2.11. [from \( \Delta \)-indexed to inverse \( \Delta \)-system] Let \( \{ \tilde{S}_\sigma \}_{\sigma \in \Delta} \) be a collection of finitely generated submonoids of \( \tilde{M} \) such that \( \pi(\tilde{S}_\sigma) \subset M_\sigma \). Then there exists an inverse \( \Delta \)-system \( \{ M_\sigma \}_{\sigma \in \Delta} \) of admissible submonoids of \( \tilde{M} \) such that \( \tilde{S}_\sigma \subset M_\sigma \) for all \( \sigma \in \Delta \).

Proof. Choose any inverse \( \Delta \)-system of admissible submonoids of \( \tilde{M} \) from Theorem 2.2.5 and apply Proposition 2.2.10.

Soft noncommutative toric schemes associated to \( \Delta \)

Definition 2.2.12. [soft noncommutative toric schemes associated to \( \Delta \)] (Continuing the previous notations.) Let \( \{ M_\sigma \}_{\sigma \in \Delta} \) be an inverse \( \Delta \)-system of admissible submonoids of \( \tilde{M} \) such that \( M_\sigma \simeq \langle z_1, \cdots, z_n \rangle \) for all maximal cones \( \sigma \in \Delta(n) \) and \( M_0 = \tilde{M} \) and \( \{ R_\sigma \}_{\sigma \in \Delta} := \{ \mathbb{C}(M_\sigma) \}_{\sigma \in \Delta} \) be the associated inverse \( \Delta \)-system of monoid algebras over \( \mathbb{C} \). Then the corresponding \( \Delta \)-system of noncommutative affine schemes

\[
\tilde{Y}(\Delta) := \tilde{U}(\Delta) := \{ U_\sigma \}_{\sigma \in \Delta}
\]

is called a soft noncommutative toric scheme with smooth fundamental charts associated to \( \Delta \). Here, \( U_\sigma \) is the noncommutative affine scheme associated to \( R_\sigma \) (cf. Definition 2.1.1) for \( \sigma \in \Delta \), the notation \( U(\Delta) \) emphasizes that this is a gluing system of some particular charts, and the notation \( Y(\Delta) \) emphasizes that this is treated as a noncommutative space as a whole.\(^1\)

With an abuse of language, we shall call \( \tilde{Y}(\Delta) \) \( n \)-dimensional since the function ring of each fundamental chart is freely generated by \( n \) coordinate functions.

For \( \sigma \in \Delta \), the monoid algebra \( \mathbb{C}(\tilde{M}_\sigma) \) is called interchangeably the function ring or the local coordinate rings of the chart \( \tilde{U}_\sigma \). The assignment

\[
U_\sigma \mapsto R_\sigma := \mathbb{C}(M_\sigma), \quad \sigma \in \Delta,
\]

is called the structure sheaf of \( \tilde{Y}(\Delta) \) and is denoted \( \mathcal{O}_{\tilde{Y}(\Delta)} \) as in ordinary Commutative Algebraic Geometry. By construction, for cones \( \tau \prec \sigma \) in \( \Delta \), one has a dominant morphism of noncommutative affine schemes \( \tilde{U}_\tau \to \tilde{U}_\sigma \) and a monoid-algebra monomorphism\(^2\)

\[
\iota^\sharp_{\sigma} : \mathcal{O}_{\tilde{Y}(\Delta)}(\tilde{U}_\sigma) = \mathbb{C}(M_\sigma) \hookrightarrow \mathcal{O}_{\tilde{Y}(\Delta)}(\tilde{U}_\tau) = \mathbb{C}(\tilde{M}_\tau).
\]

\(^1\)Here, we use the notation \( \tilde{Y}(\Delta) \), rather than \( X(\Delta) \) following [Fu], due to that such a space will be used as a target space for D-branes, whose world-volume is generally denoted \( X^k \).

\(^2\)Though our focus in this work is on the rings and their homomorphisms for Noncommutative Algebraic Geometry, we try to keep the contravariant underlying geometry as in Commutative Algebraic Geometry in mind whenever possible. This is why we use the notation \( \iota_{\sigma}^\sharp \) here and reserve \( \iota_{\sigma} \) for the corresponding morphism of underlying “spaces”. Here, there is no indication that \( \iota_{\sigma} : \tilde{U}_\tau \to \tilde{U}_\sigma \) in any sense is an inclusion, though it is true that, when restricted to \( Y(\Delta) \), \( \iota_{\sigma} : U_\tau \to U_\sigma \) is an open set inclusion in the sense of affine schemes in Commutative Algebraic Geometry.
These replace the role of ‘open sets’ and ‘restriction to open sets’ respectively in the usual definition of the ‘structure sheaf’ of a scheme in Commutative Algebraic Geometry.

An ideal sheaf $\mathcal{I}$ on $Y$ is an inverse $\Delta$-system, where $I_\sigma$ is a two-sided ideal of $\mathcal{R}_\sigma$, with $I_\sigma := \mathcal{R}_\sigma/\mathcal{I}_\sigma$. Given an ideal sheaf $\mathcal{I} = \{I_\sigma\}_{\sigma \in \Delta}$ on $\tilde{Y}(\Delta)$, one can form an inverse $\Delta$-system of $\mathbb{C}$-algebras $\{\mathcal{R}_\sigma / I_\sigma\}_{\sigma \in \Delta}$, with the inclusion $\mathcal{I}_\sigma : \mathcal{R}_\sigma / I_\sigma \to \mathcal{R}_\tau / I_\tau$, for $\tau < \sigma \in \Delta$, naturally induced from $i_{\tau \sigma}$. We will think of this inverse $\Delta$-system as defining a soft noncommutative toric scheme $Y(\Delta)$ and denote its structure sheaf $\mathcal{O}_Y$ and write the quotient as $\mathcal{O}_{\tilde{Y}(\Delta)} \to \mathcal{O}_{\tilde{Z}}$. Denote $\mathcal{I}$ by $\mathcal{I}_Z$. Then one has a short exact sequence

$$0 \to I_Z \to O_{\tilde{Y}(\Delta)} \to O_{\tilde{Z}} \to 0.$$  

Let $N'$ be another lattice (isomorphic to $\mathbb{Z}^{n'}$ for some $n'$ via a fixed basis $(e'_1, \ldots, e'_{n'})$ of $N'$), $\Delta'$ be a fan in $N'_Z$ that satisfies Assumption 2.2.2, and $Y(\Delta')$ be a soft noncommutative toric scheme associated to $\Delta'$, with the underlying inverse $\Delta'$-system of submonoids of $M'$ denoted $\{M'_{\sigma'}\}_{\sigma' \in \Delta'}$. A toric morphism

$$\varphi : \tilde{Y}(\Delta) \to \tilde{Y}(\Delta')$$

is a homomorphism $\varphi : N \to N'$ of lattices that satisfies the conditions:

1. $\varphi$ induces a map between fans (in notation, $\varphi : \Delta \to \Delta'$); i.e., for each $\sigma \in \Delta$ there exists a $\sigma' \in \Delta'$ such that $\varphi(\sigma) \subset \sigma'$.

2. The induced monoid-homomorphism $\varphi^\sharp : M' \to M$ restricts to a monoid homomorphism $\varphi^\sharp_{\sigma \sigma'} : M'_{\sigma'} \to M_\sigma$ whenever $\varphi(\sigma) \subset \sigma'$. (Denote the induced $\mathbb{C} \langle M'_{\sigma'} \rangle \to \mathbb{C} \langle M_\sigma \rangle$ also by $\varphi^\sharp_{\sigma \sigma'}$.)

Thus, a toric morphism $\varphi : \tilde{Y}(\Delta) \to \tilde{Y}(\Delta')$ is a map of lattices $\varphi : N \to N'$ that induces a $(\Delta', \Delta)$-system of monoid-algebra homomorphisms $\{\varphi_{\sigma' \sigma}^\sharp : \mathbb{C} \langle M'_{\sigma'} \rangle \to \mathbb{C} \langle M_\sigma \rangle\}_{\sigma' \in \Delta', \sigma \in \Delta, \varphi(\sigma) \subset \sigma'}$.

The nature of this collection justifies it be denoted

$$\varphi^\sharp : O_{\tilde{Y}(\Delta')} \to O_{\tilde{Y}(\Delta)}.$$  

**Definition 2.2.13. [reference toric chart and fundamental toric charts of $\tilde{Y}(\Delta)$]** (Continuing Definition 2.2.12.) A chart $\tilde{U}_\sigma$ of $\tilde{Y}(\Delta)$ that is associated to a maximal cone $\sigma \in \Delta(n)$ is called a fundamental toric chart of $\tilde{Y}(\Delta)$. Among them, the one associated to the cone generated by the given basis $(e_1, \ldots, e_n)$ of the lattice $N$ is called the reference toric chart of $\tilde{Y}(\Delta)$.

Fundamental toric charts are all isomorphic to the smooth noncommutative affine $n$-space $nc\mathbb{A}^n$. These charts should be thought of as the basic, good pieces to be glued via the subcollection $\{\tilde{U}_\tau\}_{\tau \in \Delta - \Delta(n)}$. The reference chart ensures that we have at least one fundamental toric chart $\tilde{U}_\sigma$ such that the morphism $\tilde{U}_0 \to \tilde{U}_\sigma$ of noncommutative affine schemes resembles the inclusion of an Zariski open set. Together, this allows one to think of $\tilde{Y}(\Delta)$ as a "partial compactification" of $nc\mathbb{A}^n$ via soft gluings of additional copies of $nc\mathbb{A}^n$’s.

**Definition 2.2.14. [softening of noncommutative toric scheme]** Let $\tilde{Y}(\Delta)$ and $\tilde{Y}'(\Delta)$ be two soft noncommutative toric schemes associated to a fan $\Delta$, with the underlying inverse $\Delta$-system of monoid algebras $\{\mathbb{C} \langle M_\sigma \rangle\}_{\sigma \in \Delta}$ and $\{\mathbb{C} \langle M'_{\sigma} \rangle\}_{\sigma \in \Delta}$ respectively. Then $\tilde{Y}'(\Delta)$ is called a softening of $\tilde{Y}(\Delta)$ if $M_\sigma = M'_{\sigma}$ for maximal cones $\sigma \in \Delta$ (i.e. $\tilde{Y}(\Delta)$ and $\tilde{Y}'(\Delta)$ have identical fundamental toric charts) and $M_\sigma \subset M'_{\sigma}$ for nonmaximal cones $\sigma \in \Delta$ (i.e. the gluings in $\tilde{Y}'(\Delta)$ is softer than those in $\tilde{Y}(\Delta)$)). By construction, there is a built-in toric morphism $\varphi : \tilde{Y}'(\Delta) \to \tilde{Y}(\Delta)$, called the softening morphism.
Remark 2.2.15. [meaning of softening: comparison to Isom-functor for moduli stack] Introducing the notion of ‘softening’ in the play is meant to remedy the fact that we don’t have a good theory of localizations of a noncommutative ring to apply to our construction of noncommutative schemes. Rather we keep a collection of charts as principal ones and the remaining charts serve for the purpose of gluing these principal charts. These ‘secondary charts’ are then allowed to be ‘dynamically adjusted’ to fulfill the purpose of serving as a medium between principal charts. One may compare this to the gluing in the construction of a moduli stack. There one cannot ‘dynamically adjusted’ to fulfill the purpose of serving as a medium between principal charts. These ‘secondary charts’ are then allowed to be schemes. Rather we keep a collection of charts as principal ones and the remaining charts serve for the purpose of gluing these principal charts. These ‘secondary charts’ are then allowed to be schemes. Rather we keep a collection of charts as principal ones and the remaining charts serve for the purpose of gluing these principal charts.

(c) The \( T^n \)-action

The \( T^n \)-action on \( \mathbb{C}[M] \) leaves each \( \mathbb{C} \cdot m, m \in M \), invariant. As \( \mathbb{C} = Center(\mathbb{C}(\hat{M})) \), the \( T^n \)-action on \( \mathbb{C}[M] \) naturally lifts to a \( T^n \)-action on \( \mathbb{C}(\hat{M}) \):

\[
\begin{array}{ccc}
\mathbb{C}(\hat{M}) & \xrightarrow{t} & \mathbb{C}(\hat{M}) \\
\pi \downarrow & & \pi \downarrow \\
\mathbb{C}[M] & \xrightarrow{t} & \mathbb{C}[M],
\end{array}
\]

that leaves each \( \mathbb{C} \cdot \hat{m}, \hat{m} \in \hat{M} \), invariant. It follows that \( T^n \) leaves each \( \mathbb{C}(\hat{M}_\sigma), \sigma \in \Delta \), invariant and that \( i^\sigma_\tau : \mathbb{C}(\hat{M}_\sigma) \hookrightarrow \mathbb{C}(\hat{M}_\tau), \tau < \sigma \), are \( T^n \)-equivariant. This defines the \( T^n \)-action on \( \hat{Y}(\Delta) \).

(d) The noncommutative toroidal morphism from the inclusion of the 0-cone \( 0 < \sigma \in \Delta \)

Since \( 0 \preceq \sigma \), for all \( \sigma \in \Delta \), and \( \hat{M}_0 = \mathbb{C}(\hat{M}) \), one has a built-in \( T^n \)-equivariant morphism

\[
\hat{T}^n := \text{Scheme}(\mathbb{C}(\hat{M})) \longrightarrow \hat{Y}(\Delta).
\]

Which plays the role of the toroidal embedding \( \hat{T}^n \hookrightarrow Y(\Delta) \) in the commutative case.

(e) The built-in \( T^n \)-equivariant embedding \( Y(\Delta) \hookrightarrow \hat{Y}(\Delta) \)

The monoid epimorphism \( \pi : \hat{M} \twoheadrightarrow M \) restricts to monoid epimorphisms \( \pi : \hat{M}_\sigma \twoheadrightarrow M_\sigma \), for all \( \sigma \in \Delta \). Which in turn gives a \( \Delta \)-collection of \( T^n \)-equivariant monoid-algebra epimorphisms \( \{\mathbb{C}(\hat{M}_\sigma) \twoheadrightarrow \mathbb{C}[M_\sigma]\}_{\sigma \in \Delta} \) that commute with inclusions:

\[
\begin{array}{ccc}
\mathbb{C}(\hat{M}_\sigma) & \xleftarrow{\pi} & \mathbb{C}(\hat{M}_\tau) \\
\pi \downarrow & & \pi \\
\mathbb{C}[M_\sigma] & \xleftarrow{\pi} & \mathbb{C}[M_\tau],
\end{array}
\]

\( \tau < \sigma \in \Delta \).

This defines a built-in \( T^n \)-equivariant embedding \( Y(\Delta) \hookrightarrow \hat{Y}(\Delta) \).

(f) The master commutative subscheme of \( \hat{Y}(\Delta) \)

The inclusions \( \mathbb{C}(\hat{M}_\sigma) \hookrightarrow \mathbb{C}(\hat{M}_\tau), \tau < \sigma \in \Delta \), induce \( \mathbb{C} \)-algebra homomorphisms

\[
j^\tau_{\tau \sigma} : \mathbb{C}(\hat{M}_\sigma)/[\mathbb{C}(\hat{M}_\sigma), \mathbb{C}(\hat{M}_\sigma)] \longrightarrow \mathbb{C}(\hat{M}_\tau)/[\mathbb{C}(\hat{M}_\tau), \mathbb{C}(\hat{M}_\tau)], \quad \tau < \sigma \in \Delta.
\]

This defines another inverse \( \Delta \)-system of commutative \( \mathbb{C} \)-algebras

\[
\{\mathbb{C}(\hat{M}_\sigma)/[\mathbb{C}(\hat{M}_\sigma), \mathbb{C}(\hat{M}_\sigma)]\}_{\sigma \in \Delta}, \{j^\tau_{\tau \sigma}\}_{\tau < \sigma \in \Delta}
\]

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and hence a (generally singular) commutative scheme $\tilde{Y}(\Delta)^\bullet$ over $\mathbb{C}$. By construction any morphism from a commutative scheme $X$ to $\tilde{Y}$ factors through a morphism $X \to \tilde{Y}(\Delta)^\bullet \hookrightarrow \tilde{Y}(\Delta)$. In particular, one has commutative diagrams

\[
\begin{array}{ccc}
\mathbb{C} \langle \tilde{M}_\sigma \rangle/\mathbb{C} \langle \tilde{M}_\sigma \rangle & \xrightarrow{f_\sigma} & \mathbb{C}\langle \tilde{M}\rangle/\mathbb{C}\langle \tilde{M}\rangle \\
\xrightarrow{g_\sigma} & & \xrightarrow{g_\tau} \\
\mathbb{C}[M_\sigma] & \xrightarrow{h_\sigma} & \mathbb{C}[M_\tau]
\end{array}, \quad \tau \prec \sigma \in \Delta,
\]

and hence $Y(\Delta) \subset \tilde{Y}(\Delta)^\bullet \subset \tilde{Y}(\Delta)$. Note that, by construction, $\mathbb{C} \langle \tilde{M}_\sigma \rangle/\mathbb{C} \langle \tilde{M}_\sigma \rangle \xrightarrow{\sim} \mathbb{C}[M_\sigma]$ for $\sigma \in \Delta(n) \cup \{0\}$. However, such isomorphism on local charts may not holds for $\tau \in \Delta(k)$, $1 \leq k \leq n - 1$.

**Definition 2.2.16.** [master commutative subscheme of $\tilde{Y}(\Delta)$] $\tilde{Y}(\Delta)^\bullet$ is called the master commutative subscheme of $\tilde{Y}(\Delta)$.

### 3 Invertible sheaves on a soft noncommutative toric scheme, twisted sections, and soft noncommutative schemes via toric geometry

We construct in this section a class of noncommutative spaces, named ‘soft noncommutative schemes’ from invertible sheaves and their twisted sections on $\tilde{Y}(\Delta)$.

**Modules and invertible sheaves on a soft noncommutative toric scheme**

**Definition 3.1.** [sheaf on $\tilde{Y}(\Delta)$ and $\mathcal{O}_{\tilde{Y}(\Delta)}$-module] (1) An inverse $\Delta$-system of objects in a category $\mathcal{C}$ (cf. Definition 2.2.1) is also called a sheaf (of objects in $\mathcal{C}$) on $\tilde{Y}(\Delta)$. For example, $\mathcal{O}_{\tilde{Y}(\Delta)}$ is a sheaf of $\mathbb{C}$-algebras on $\tilde{Y}(\Delta)$. (Which justifies its name: the structure sheaf of $\tilde{Y}(\Delta)$).

(2) Let $\tilde{Y}(\Delta) = \{\tilde{U}_\sigma\}_{\sigma \in \Delta}$ be a soft noncommutative toric scheme with the underlying inverse $\Delta$-system of monoid algebras $(\{\tilde{R}_\sigma\}_{\sigma \in \Delta}, \{i_{\tau,\sigma}\}_{\tau,\sigma \in \Delta, \tau \prec \sigma})$. A left $\mathcal{O}_{\tilde{Y}(\Delta)}$-module is an inverse $\Delta$-system

\[
\tilde{F} := (\{\tilde{F}_\sigma\}_{\sigma \in \Delta}, \{h_{\sigma\tau}\}_{\tau,\sigma \in \Delta, \tau \prec \sigma}),
\]

where $\tilde{F}_\sigma$ is a left $\tilde{R}_\sigma$-module and $h_{\sigma\tau} : i_{\tau,\sigma}^\ast \tilde{F}_\sigma := \tilde{R}_\tau \otimes_{\tilde{R}_\sigma} \tilde{F}_\sigma \to \tilde{F}_\tau$ on $\tilde{U}_\tau$ such that the following diagram commutes

\[
\begin{array}{ccc}
i_{\tau,\rho}^\ast \tilde{F}_\rho & \xrightarrow{i_{\tau,\rho}^\ast h_{\tau,\rho}} & i_{\tau,\rho}^\ast \tilde{F}_\tau \\
\downarrow h_{\sigma\rho} & & \downarrow h_{\sigma\rho} \\
i_{\tau,\rho}^\ast \tilde{F}_\sigma & & i_{\tau,\rho}^\ast \tilde{F}_\sigma
\end{array}
\]
on $\tilde{U}_\rho$ for $\rho \prec \sigma$.

An element of $\tilde{F}_\sigma$ is called a local section of $\tilde{F}$ over $\tilde{U}_\sigma$. A section (or global section) of $\tilde{F}$ is a collection $\{s_\sigma\}_{\sigma \in \Delta}$ of local sections $s_\sigma \in \tilde{F}_\sigma$, $\sigma \in \Delta$, of $\tilde{F}$ such that $h_{\sigma\tau}(s_\sigma) = s_\tau$, $\tau \prec \sigma \in \Delta$. In the last expression, $s_\sigma$ is identified as $1 \otimes s_\sigma \in \tilde{R}_\tau \otimes_{\tilde{R}_\sigma} \tilde{F}_\sigma$. The $\mathbb{C}$-vector space of sections of $\tilde{F}$ on $\tilde{Y}(\Delta)$ is denoted $H^0(\tilde{Y}(\Delta), \tilde{F})$ or simply $H^0(\tilde{F})$. 

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A homomorphism
\[ f : \mathcal{F} := (\{ F_\sigma \}_{\sigma \in \Delta}, \{ h_{\sigma \tau} \}_{\tau, \sigma \in \Delta, \tau < \sigma}) \rightarrow \mathcal{F}' := (\{ F'_\sigma \}_{\sigma \in \Delta}, \{ h'_{\sigma \tau} \}_{\tau, \sigma \in \Delta, \tau < \sigma}) \]
of left \( \mathcal{O}_{\bar{Y}(\Delta)} \)-modules is a collection \( \{ f_\sigma \}_{\sigma \in \Delta} \), where \( f_\sigma : F_\sigma \rightarrow F'_\sigma \) is a homomorphism of left \( R_\sigma \)-modules, such that the following diagrams commute

\[
\begin{array}{ccc}
t'_\sigma F_\sigma & \xrightarrow{h_{\sigma \tau}} & F'_\tau \\
\downarrow t'_\sigma(f_\sigma) & & \downarrow f_\tau \\
t'_\sigma F'_\sigma & \xrightarrow{h'_{\sigma \tau}} & F'_\tau
\end{array}
\]
for \( \tau < \sigma \in \Delta \). \( f \) is called injective (i.e. \( f \) a monomorphism) if in addition \( f_\sigma \) is injective for all \( \sigma \in \Delta \); surjective (i.e. \( f \) an epimorphism) if in addition \( f_\sigma \) is surjective for all \( \sigma \in \Delta \); an isomorphism if in addition \( f_\sigma \) is an isomorphism of left \( R_\sigma \)-modules for all \( \sigma \in \Delta \).

(3) Recall the built-in \( T^n \)-action on \( \bar{Y}(\Delta) \). Then \( \mathcal{F} \) is called \( T^n \)-equivariant if there exist isomorphisms \( \phi_t : g^t_\ast \mathcal{F} \xrightarrow{\sim} \mathcal{F} \), \( t \in T^n \), of \( \mathcal{O}_{\bar{Y}(\Delta)} \)-modules such that \( \phi_{t_2} \circ \phi_{t_1} = \phi_{t_1+t_2} \):

for all \( t_1, t_2 \in T^n \). Explicitly, note that the \( T^n \)-action on \( \bar{Y}(\Delta) \) leaves each chart \( U_\sigma, \sigma \in \Delta \) invariant and \( \phi_t \) restricts to an \( R_\sigma \)-module homomorphism

\[ \phi_t : F^t_\sigma \rightarrow F_\sigma, \text{ with } \bar{s} \mapsto \phi_t(\bar{s}) = g_t \cdot \bar{s} \]
on each chart \( U_\sigma, \sigma \in \Delta \). Here \( F^t_\sigma := g^t_\ast F_\sigma \), which is the same \( \mathbb{C} \)-vector space as \( F_\sigma \) but with the \( R_\sigma \)-module structure defined by \( \bar{r} \cdot \bar{s} := g^t_\ast(\bar{r}) \cdot \bar{s} \). This defines a \( T^n \)-action on \( \mathcal{F} \) that satisfies \( \phi_t(g^t_\ast(\bar{r}) \cdot \bar{s}) = \bar{r} \cdot (g_t \cdot \bar{s}) \) by tautology, for \( t \in T^n, \bar{r} \in R_\sigma, \) and \( \bar{s} \in F_\sigma \). The \( T^n \)-action on \( F_\sigma, \sigma \in \Delta \), is equivariant under gluings:

\[ h_{\sigma \tau}(g_t \cdot \bar{s}) = g_t \cdot h_{\sigma \tau}(\bar{s}), \]
for \( \bar{s} \in F_\sigma, t \in T^n, \) and \( \tau < \sigma \in \Delta \).

(4) Similarly, for right and two-sided \( \mathcal{O}_{\bar{Y}(\Delta)} \)-modules.

(5) Let \( \mathcal{A} = (\{ A_\sigma \}_{\sigma \in \Delta}, \{ i_\sigma \}_{\tau < \sigma \in \Delta}) \) be a sheaf of \( \mathbb{C} \)-algebras, \( \mathcal{F} = (\{ F_\sigma \}_{\sigma \in \Delta}, \{ h^F_{\sigma \tau} \}_{\tau < \sigma \in \Delta}) \) a right \( \mathcal{A} \)-module, and \( \mathcal{G} = (\{ G_\sigma \}_{\sigma \in \Delta}, \{ h^G_{\sigma \tau} \}_{\tau < \sigma \in \Delta}) \) a left \( \mathcal{A} \)-module, all three on \( Y(\Delta) \). Then define the tensor product \( \mathcal{F} \otimes \mathcal{A} \mathcal{G} \) of \( \mathcal{F} \) and \( \mathcal{G} \) over \( \mathcal{A} \) to be the following (right-on-\( \mathcal{F} \), left-on-\( \mathcal{G} \)) \( \mathcal{A} \)-module on \( Y(\Delta) \):

\[ \mathcal{F} \otimes \mathcal{A} \mathcal{G} := (\{ F_\sigma \otimes _{\mathcal{A}} G_\sigma \}_{\sigma \in \Delta}, \{ h^F_{\sigma \tau} \otimes h^G_{\sigma \tau} \}_{\tau < \sigma \in \Delta}). \]

(6) Let

\[ \phi : \bar{Y}(\Delta) = \{ \bar{R}_\sigma \}_{\sigma \in \Delta} \rightarrow \bar{Y}(\Delta') = \{ \bar{S}_{\sigma'} \}_{\sigma' \in \Delta'} \]
be a toric morphism (cf. Definition 2.2.12), \( \mathcal{F} := (\{ F_\sigma \}_{\sigma \in \Delta}, \{ h^F_{\sigma \tau} \}_{\tau < \sigma \in \Delta}) \) a left \( \mathcal{O}_{\bar{Y}(\Delta)} \)-module on \( \bar{Y}(\Delta) \), and \( \mathcal{G} := (\{ G_{\sigma'} \}_{\sigma' \in \Delta'}, \{ h^G_{\sigma' \tau'} \}_{\tau' < \sigma' \in \Delta'}) \) a left \( \mathcal{O}_{\bar{Y}(\Delta')} \)-module on \( \bar{Y}(\Delta') \). Recall the
underlying homomorphism \( N \rightarrow N' \) of lattices and the induced \( \mathbb{R} \)-linear map \( N_{\mathbb{R}} \rightarrow N'_{\mathbb{R}} \), both denoted still by \( \varphi \), (cf Definition 2.2.12).

For each \( \sigma' \in \Delta' \),
\[
\Delta_{\sigma'} := \{ \sigma \in \Delta \mid \varphi(\sigma) \subset \sigma' \}
\]
is a subfan of \( \Delta \) and specifies a subsystem \( O_{\bar{Y}(\Delta_{\sigma'})} := \{ \bar{R}_{\sigma} \}_{\sigma \in \Delta_{\sigma'}} \) of \( O_{\bar{Y}(\Delta)} \). Through the restriction of \( \varphi^\sharp : O_{\bar{Y}(\Delta')} \rightarrow O_{\bar{Y}(\Delta)} \) to \( \bar{S}_{\sigma'} \rightarrow O_{\bar{Y}(\Delta')} \), the \( \mathbb{C} \)-vector space \( H^0(\bar{Y}(\Delta'), F|_{\bar{Y}(\Delta_{\sigma'})}) \) is rendered a left \( \bar{S}_{\sigma'} \)-module. The gluing data \( \{ h_{\sigma\tau}^\sharp \}_{\tau < \sigma} \) of \( \bar{F} \) gives rise to a gluing data for the \( \Delta' \)-collection \( \{ H^0(\bar{Y}(\Delta'), F|_{\bar{Y}(\Delta_{\sigma'})}) \}_{\sigma' \in \Delta'} \) and turn the \( \Delta' \)-collection into a left \( O_{\bar{Y}(\Delta')} \)-module, denoted \( \varphi_* \bar{F} \) and named the direct image sheaf or pushforward of \( \bar{F} \) under \( \varphi \).

For each \( \sigma \in \Delta \), there exists a unique \( \rho'_\sigma \in \Delta' \) such that \( \varphi(\sigma) \subset \rho'_\sigma \) and that \( \rho'_\sigma \leq \sigma' \) for all \( \sigma' \in \Delta' \) with \( \varphi(\sigma) \subset \sigma' \). The property \( \rho'_\tau \leq \rho'_\sigma \) for \( \tau < \sigma \in \Delta \) and the gluing data \( \{ h_{\sigma\tau}^\sharp \}_{\tau < \sigma} \) of \( \bar{G} \) render the \( \Delta \)-collection \( \{ \bar{G}_{\rho'_\sigma} \}_{\sigma \in \Delta} \) a sheaf of \( \mathbb{C} \)-vector spaces on \( \bar{Y}(\Delta) \), denoted \( \varphi^{-1}\bar{G} \) and named the inverse image sheaf of \( \bar{G} \) under \( \varphi \). In particular, \( \varphi^{-1}O_{\bar{Y}(\Delta)} \) is a sheaf of \( \mathbb{C} \)-algebras on \( \bar{Y}(\Delta) \) and, by construction, \( \varphi^{-1}\bar{G} \) is a left \( \varphi^{-1}O_{\bar{Y}(\Delta')} \)-module. Since \( \bar{F} : O_{\bar{Y}(\Delta')} \rightarrow O_{\bar{Y}(\Delta)} \) renders \( O_{\bar{Y}(\Delta)} \) a two-sided \( \varphi^{-1}O_{\bar{Y}(\Delta')} \)-module, one can define further a left \( O_{\bar{Y}(\Delta)} \)-module \( \varphi^\ast \bar{G} := O_{\bar{Y}(\Delta)} \otimes_{\varphi^{-1}O_{\bar{Y}(\Delta)}} \varphi^{-1}\bar{G} \) on \( \bar{Y}(\Delta) \), named the pullback of \( \bar{G} \) under \( \varphi \).

Similarly, for right modules and two-sided modules on \( \bar{Y}(\Delta) \) and \( \bar{Y}(\Delta') \).

**Example 3.2. [ideal sheaf and structure sheaf of closed subscheme]** Recall Definition 2.2.12. Then, an ideal sheaf \( \mathcal{I}_\bar{Z} \) on \( \bar{Y}(\Delta) \) and the associated structure sheaf \( O_{\bar{Z}} \) of a soft noncommutative closed subscheme \( \bar{Z} \) of \( \bar{Y}(\Delta) \) are both two-sided \( O_{\bar{Y}(\Delta)} \)-modules.

**Definition 3.3. [invertible \( O_{\bar{Y}(\Delta)} \)-module/invertible sheaf/line bundle on \( \bar{Y}(\Delta) \)]** (Continuing Definition 3.1.) A left (resp. right, two-sided) invertible \( O_{\bar{Y}(\Delta)} \)-module (or invertible sheaf or line bundle) on \( \bar{Y}(\Delta) \) is a left (resp. right, two-sided) \( O_{\bar{Y}(\Delta)} \)-module
\[
\mathcal{L} = (\{ \bar{F}_\sigma \}_{\sigma \in \Delta}, \{ h_{\sigma\tau} \}_{\tau, \sigma \in \Delta, \tau < \sigma})
\]
such that \( \bar{F}_\sigma \simeq \bar{R}_\sigma \) as left (resp. right, two-sided) \( \bar{R}_\sigma \)-modules for \( \sigma \in \Delta \) and \( h_{\sigma\tau} \) is an isomorphism of left (resp. right, two-sided) \( \bar{R}_\tau \)-modules for \( \tau, \sigma \in \Delta \), \( \tau < \sigma \).

**Lemma 3.4. [invertible \( O_{\bar{Y}(\Delta)} \)-module: basic description]** (Continuing the notation from Sec. 2.2.) Let \( \mathcal{L} \) be a (left) invertible \( O_{\bar{Y}(\Delta)} \)-module. Then \( \mathcal{L} \) can be described by a collection
\[
\{ c_{\sigma \tau} \tilde{m}_{\sigma \tau} \}_{\tau < \sigma} := \left\{ c_{\sigma \tau} \tilde{m}_{\sigma \tau} \in \mathbb{C}^\times \tilde{M}_\tau \mid \sigma, \tau \in \Delta, \tau < \sigma, \pi(\tilde{m}_{\sigma \tau}) \in \tau \cap M, \quad c_{\sigma \rho} \tilde{m}_{\sigma \rho} = c_{\sigma \tau} c_{\tau \rho} \tilde{m}_{\sigma \rho} \tilde{m}_{\sigma \tau} \text{ for } \rho < \tau < \sigma \right\}
\]
Here, \( \tilde{m}_{\sigma \rho} \) acts on \( \bar{F}_{\sigma \rho} \simeq \bar{R}_{\sigma \rho} \) as a left \( \bar{R}_{\sigma \rho} \)-module homomorphism given by ‘the multiplication by \( \tilde{m}_{\sigma \rho} \) from the right’. In particular, \( \tilde{m}_{\tau \rho} \circ \tilde{m}_{\sigma \tau} = \tilde{m}_{\sigma \tau} \tilde{m}_{\tau \rho} \). Two such collections \( \{ c_{\sigma \tau} \tilde{m}_{\sigma \tau} \}_{\tau < \sigma} \) and \( \{ c'_{\sigma \tau} \tilde{m}'_{\sigma \tau} \}_{\tau < \sigma} \) define isomorphic invertible \( O_{\bar{Y}(\Delta)} \)-modules if and only if there exists a collection
\[
\{ c_{\sigma} \tilde{m}_{\sigma} \}_{\sigma} := \{ c_{\sigma} \tilde{m}_{\sigma} \in \mathbb{C}^\times \tilde{M}_{\sigma} \mid \sigma \in \Delta, \pi(\tilde{m}_{\sigma}) \in \sigma \cap M \}
\]
such that
\[
c'_{\sigma \tau} = c_{\sigma} c_{\sigma \tau} c_{\tau} \quad \text{and} \quad \tilde{m}'_{\sigma \tau} = \tilde{m}_{\sigma} \tilde{m}_{\sigma \tau} \tilde{m}_{\tau} .
\]
Proposition 3.5. [existence of invertible sheaf after passing to softening] \textit{Let }\tilde{Y}(\Delta)\textit{ be an }\mathbb{C}\textit{-dimensional} \noncommutative\textit{ toric scheme associated to a fan }\Delta\textit{. Recall the built-in inclusion }\overline{Y}(\Delta)\subset\tilde{Y}(\Delta)\textit{ and let }\mathcal{L}\textit{ be an invertible }\mathcal{O}_{\tilde{Y}(\Delta)}\textit{-module on }\tilde{Y}(\Delta)\textit{. Then, there exists a \textit{softening} }\tilde{Y}'(\Delta)\rightarrow\tilde{Y}(\Delta)\textit{ of }\tilde{Y}(\Delta)\textit{ such that, now on }\overline{Y}(\Delta)\subset\tilde{Y}'(\Delta),\textit{ extends to an invertible }\mathcal{O}_{\tilde{Y}'(\Delta)}\textit{-module }\mathcal{L}\textit{.}

\begin{proof}
Under Assumption 2.2.2, \( Y(\Delta) \) is smooth and thus \( \mathcal{L} \simeq \mathcal{O}_{Y(\Delta)}(D) \) for some \( \mathbb{T}^n \)-invariant Cartier divisor \( D \) on \( Y(\Delta) \) and can be specified by a collection \( \{ m_{\sigma} \in M \}_{\sigma \in \Delta(n)} \), where \( m_{\sigma} \in M \) and \( \Delta(n) \) is the collection of maximal cones in \( \Delta \), that satisfies \( m_{\sigma}^{-1}m_{\tau} \in (\sigma \cap \tau)^{\perp} \cap M \), for \( \sigma, \tau \in \Delta(n) \). \text{(Cf. [Fu: Sec. 3.4], with the monoid }M\text{ here presented multiplicatively for convenience.) For each }\tau \in \Delta(k),\text{ }k \leq n-1,\text{ fix an }\sigma \in \Delta(n)\text{ such that }\tau \prec \sigma\text{ and set }m_{\tau} = m_{\sigma}.\text{ This extend }\{ m_{\sigma} \}_{\sigma \in \Delta(n)}\text{ to a collection }\{ m_{\sigma} \}_{\sigma \in \Delta}\text{ such that }m_{\sigma}^{-1}m_{\tau} \in (\sigma \cap \tau)^{\perp} \cap M.\text{ Let }\{ \tilde{M}_{\sigma} \}_{\sigma \in \Delta}\text{ be the underlying inverse }\Delta\text{-system of monoids associated to }\tilde{Y}(\Delta)\text{ and, for each }\sigma \in \Delta,\text{ fix an }\tilde{m}_{\sigma} \in \tilde{M}_{\sigma}\text{ such that }\pi(\tilde{m}_{\sigma}) = m_{\sigma}.\text{ Set }\{ \tilde{m}_{\sigma} \}_{\tau \prec \sigma \in \Delta} = \{ \tilde{m}_{\sigma}^{-1}\tilde{m}_{\tau} \}_{\tau \prec \sigma \in \Delta}.\text{ Then, }$

$$
\pi(\tilde{m}_{\sigma}) \in \tau^{\perp} \cap M \text{ for }\tau \prec \sigma \in \Delta, \quad \text{and} \quad \tilde{m}_{\sigma} \cdot \tilde{m}_{\rho} = \tilde{m}_{\rho} \text{ for }\rho \prec \tau \prec \sigma \in \Delta.
$$

This is almost the gluing data in Lemma 3.4 for an invertible \( \mathcal{O}_{\tilde{Y}(\Delta)} \)-module \textit{except} that in general \( \tilde{m}_{\sigma} \notin \tilde{M}_{\tau} \). To remedy this, let

\[
\begin{align*}
\tilde{S}_{\sigma} &= \text{the empty set for }\sigma \in \Delta(n), \\
\tilde{S}_{\tau} &= \{ \tilde{m}_{\sigma} \mid \tau \prec \sigma \in \Delta \} \text{ for }\tau \in \Delta(k),\text{ }k \leq n-1.
\end{align*}
\]

This defines a collection of \( \Delta \)-indexed finitely generated submonoids of \( \tilde{M} \). It follows from Proposition 2.2.10 that one can augment \( \{ \tilde{M}_{\sigma} \}_{\sigma \in \Delta} \) to an inverse \( \Delta \)-system \( \{ \tilde{M}'_{\sigma} \}_{\sigma \in \Delta} \) of submonoids of \( \tilde{M} \) such that \( \tilde{S}_{\sigma} \subset \tilde{M}'_{\sigma} \). Furthermore, since \( \tilde{S}_{\sigma} = \text{empty set for }\sigma \in \Delta(n) \), it follows from the proof of Proposition 2.2.10 that it can be made that \( \tilde{M}'_{\sigma} = \tilde{M}_{\sigma} \text{ for }\sigma \in \Delta(n) \). Thus, \( \{ \tilde{M}' \}_{\sigma \in \Delta} \) defines a softening \( \tilde{Y}'(\Delta) \) of \( \tilde{Y}(\Delta) \) and \( \mathcal{L} \text{ extends to an invertible }\mathcal{O}_{\tilde{Y}'(\Delta)}\text{-module }\tilde{\mathcal{L}}\text{ on }\tilde{Y}'(\Delta).\)

\end{proof}

Since a soft noncommutative toric scheme associated to \( \Delta \) always exists (cf. Theorem 2.2.5, Proposition 2.2.9, Definition 2.2.12), Proposition 3.5 can be rephrased as:
Proposition 3.5'. [existence of invertible sheaf] Let $L$ be an invertible sheaf on the smooth toric variety $Y(\Delta)$. Then there exists a soft noncommutative toric scheme $\hat{Y}(\Delta) \supset Y(\Delta)$ associated to $\Delta$ such that $L$ extends to an invertible sheaf $\hat{L}$ on $\hat{Y}(\Delta)$.

Note that the pullback of an invertible sheaf under a morphism of noncommutative toric schemes is an invertible sheaf. In particular, any invertible sheaf on $\hat{Y}(\Delta)$ pulls back to invertible sheaves on softenings of $\hat{Y}(\Delta)$.

Twisted sections of an invertible sheaf on $\hat{Y}(\Delta)$

Let $L$ be an invertible sheaf on the $n$-dimensional smooth toric variety $Y(\Delta)$. Then $L$ is isomorphic to a $T$-Cartier divisor $O_{Y(\Delta)}(D) = O_{Y(\Delta)}(\sum_{\tau \in \Delta(1)} a_\tau D_\tau)$, where $\Delta(1) \subset \Delta$ is the set of rays (i.e. 1-dimensional cones) in $\Delta$, $D_\tau$ is the $T$-invariant Weil divisor on $Y(\Delta)$ associated to $\tau \in \Delta(1)$, and $a_\tau \in \mathbb{Z}$. The divisor $D$ determines an $m_\sigma \in M$ for each maximal cone $\sigma \in \Delta(n)$: 

$$\langle m_\sigma, v_\tau \rangle = -a_\tau, \quad \text{for all } \tau \in \Delta(1) \text{ such that } \tau \prec \sigma.$$ 

Here $v_\tau \in \tau \cap N$ is the first lattice point on the ray $\tau$. Writing the commutative monoid $M$ multiplicatively as a multiplicatively closed subset of $R_0 = \mathbb{C}[M]$ and passing to the isomorphism $L \cong O_{Y(\Delta)}(D)$, then $L(U_\sigma) = R_\sigma \cdot m_\sigma = \mathbb{C}[\sigma^\vee \cap M] \cdot m_\sigma$, $\sigma \in \Delta(n)$. It follows that, if defining the (possibly empty) polytope in $M_\mathbb{R}$ associated to $D$

$$P_D := \bigcap_{\sigma \in \Delta(n)} \sigma^\vee \cdot m_\sigma = \{ u \in M_\mathbb{R} | \langle u, v_\tau \rangle \geq -a_\tau, \text{ for all } \tau \in \Delta(1) \} \subset M_\mathbb{R},$$

then the $\mathbb{C}$-vector space $H^0(Y(\Delta), L)$ of sections of $L$ is realized as

$$H^0(Y(\Delta), O_{Y(\Delta)}(D)) = \bigoplus_{u \in P_D \cap M} \mathbb{C} \cdot u.$$ 

(Cf. [Fu: Sec. 3.3 & Sec. 3.4].)

Definition 3.6. [twisted section of invertible sheaf] Let $\mathcal{L} = (\{ \hat{F}_\sigma \}_{\sigma \in \Delta}, \{ h_{\sigma \tau} \}_{\tau \prec \sigma \in \Delta})$ be a (left) invertible sheaf on a soft noncommutative toric scheme $\hat{Y}(\Delta) = \{ \hat{R}_\sigma \}_{\sigma \in \Delta}$. Two local sections $\hat{s}_1, \hat{s}_2 \in \hat{F}_\sigma, \sigma \in \Delta$, are said to be equivalent, in notation, $\hat{s}_1 \sim \hat{s}_2$, if there exists an invertible element $\hat{\tau} \in \hat{R}_\sigma$ such that $\hat{s}_1 = \hat{\tau} \hat{s}_2$. This is clearly an equivalence relation on $\hat{F}_\sigma, \sigma \in \Delta$. A twisted section of $\mathcal{L}$ is a collection

$$\hat{s} = \{ \hat{s}_\sigma \}_{\sigma \in \Delta}$$

of local sections $\hat{s}_\sigma \in \hat{F}_\sigma$ of $\mathcal{L}$ such that

$$h_{\sigma \tau}(\hat{s}_\sigma) \sim \hat{s}_\tau, \quad \text{for } \tau \prec \sigma \in \Delta.$$ 

Fix a local trivializing isomorphism $\mathcal{F}_\sigma \simeq \hat{R}_\sigma$ of left $\hat{R}_\sigma$-modules, $\sigma \in \Delta$. Then, $\hat{s}$ is specified uniquely by $\{ \hat{r}_\sigma \}_{\sigma \in \Delta}$, with $\hat{r} \in \hat{R}_\sigma$. We will called $\{ \hat{r}_\sigma \}_{\sigma \in \Delta}$ the presentation of $\hat{s}$ with respect to the local trivialization of $\mathcal{L}$. Different choices of local trivializations of $\mathcal{L}$ give presentations of $\hat{s}$ that differ by a right multiplication of unit (i.e. invertible element) chart by chart.

Lemma 3.7. [twisted section under pullback] Let $\phi : \hat{Y}'(\Delta) \rightarrow \hat{Y}(\Delta)$ be a morphism of soft noncommutative toric schemes, $\mathcal{L}$ be an invertible sheaf on $\hat{Y}(\Delta)$, and $\hat{s}$ be a twisted section of $\mathcal{L}$. Then, $\phi^* \hat{s}$ is a twisted section of the pullback $\phi^* \mathcal{L}$ of $\mathcal{L}$ to $\hat{Y}'(\Delta)$. In particular, a twisted section pulls back to a twisted section under softening.
Proof. This follows from the fact that $\phi^*$ takes an invertible element to an invertible element chart by chart.

\[ \]

Proposition 3.8. [extension to twisted section after passing to softening] Let $\tilde{\mathcal{L}}$ be an invertible sheaf on $\tilde{Y}(\Delta)$, $\mathcal{L}$ be the restriction of $\tilde{\mathcal{L}}$ to $\tilde{Y}(\Delta) \subset \tilde{Y}(\Delta)$, and $s_1, \cdots, s_k \in H^0(\tilde{Y}(\Delta), \mathcal{L})$ be sections of $\mathcal{L}$ on $\tilde{Y}(\Delta)$. Then, there exists a softening $\tilde{\mathcal{L}}'(\Delta) \to \tilde{Y}(\Delta)$ such that all $s_1 \cdots, s_k$ of $\mathcal{L}$ extend to twisted sections of the pullback $\tilde{\mathcal{L}}'$ of $\tilde{\mathcal{L}}$ to $\tilde{Y}'(\Delta)$.

Proof. Present $\mathcal{L}$ as a $T$-Cartier divisor $\mathcal{O}_{\tilde{Y}(\Delta)}(D)$, recall the polytope $P_D$ in $M_\mathbb{R}$ associated to $D$, and denote the section of $\mathcal{L}$ corresponding to $m \in P_D \cap M$ by $s_m$. Since each $s_i$ is a $\mathbb{C}$-linear combination of finitely many $s_m$'s, $m \in P_D \cap M$, it follows from Lemma 3.7 that we only need to prove the proposition for a section of $\mathcal{L}$ in the form $s_m$ for some $m \in P_D \cap M$.

Write out more explicitly: $\tilde{Y}(\Delta) = \{ \mathbb{C} \langle \tilde{M}_\sigma \rangle \}_{\sigma \in \Delta}$ and $\tilde{\mathcal{L}} = (\{ \mathbb{C} \langle \tilde{M}_\sigma \rangle \}_{\sigma \in \Delta}, \{ c_{\sigma \tau} \tilde{m}_{\sigma \tau} \}_{\tau < \sigma \in \Delta})$ under a local trivialization (cf. Lemma 3.4). Recall the collection $\{ m_{\sigma} \}_{\sigma \in \Delta(n)} \subset M$ associated to $D$ from the beginning of the theme. For any $\tau \in \Delta - \Delta(n)$, fix an $\sigma \in \Delta(n)$ such that $\tau < \sigma$ and assign $m_{\tau}$ to be $m_{\sigma}$. This enlarges the collection $\{ m_{\sigma} \}_{\sigma \in \Delta(n)}$ to a collection $\{ m_{\sigma} \}_{\sigma \in \Delta}$ with the property that

$$ m_{\sigma} m_{\tau}^{-1} \in \tau^{-1} \cap M, \text{ for } \tau < \sigma \in \Delta. $$

Here, we write the operation of the commutative monoid $M$ multiplicatively.

Choose $\tilde{m}_{\sigma}, \tilde{m}_{\sigma}' \in \tilde{M}$ such that

$$ \pi(\tilde{m}_{\sigma}) = m_{\sigma}, \quad \pi(\tilde{m}_{\sigma}') = m', \quad \tilde{m}_{\sigma}' \in \tilde{M}_\sigma \cdot \tilde{m}_{\sigma}, $$

for all $\sigma \in \Delta$. Since $\pi : \tilde{M} \to M$ restricts to a surjection $\tilde{M}_\sigma \cdot \tilde{m}_{\sigma} \to M_{\sigma} \cdot m_{\sigma}$ and $m' \in M_{\sigma} \cdot m_{\sigma}$, such a choice always exists. It follows that $\tilde{m}_{\sigma}' = \tilde{r}_{\sigma}' \cdot \tilde{m}_{\sigma}$ for some $\tilde{r}_{\sigma}' \in \tilde{M}_\sigma \subset \tilde{R}_\sigma$, $\sigma \in \Delta$. Consider now the collection

$$ \tilde{s}' := \{ \tilde{r}_{\sigma}' \}_{\sigma \in \Delta} $$

of local sections of $\tilde{\mathcal{L}}$. By construction, it recovers the section $s_m'$ of $\mathcal{L}$ when restricted to $Y(\Delta) \subset \tilde{Y}(\Delta)$. For $\tau < \sigma \in \Delta$,

$$ h_{\sigma \tau}(\tilde{r}_{\sigma}') = \tilde{r}_{\sigma}' \cdot c_{\sigma \tau} \tilde{m}_{\sigma \tau} \in \tilde{R}_\tau. $$

As elements in $\tilde{M}$,

$$ \tilde{r}_{\sigma}' \tilde{m}_{\sigma \tau} = (\tilde{r}_{\sigma}' \tilde{m}_{\sigma \tau} \tilde{r}_{\tau}^{-1}) \cdot \tilde{r}_{\tau}'. $$

Since

$$ \pi(\tilde{r}_{\sigma}' \tilde{m}_{\sigma \tau} \tilde{r}_{\tau}^{-1}) = (m_{\sigma} m_{\tau}^{-1}) \cdot \pi(\tilde{m}_{\sigma \tau}) \in \tau^{-1} \cap M, $$

$\pi(\tilde{r}_{\sigma}' \tilde{m}_{\sigma \tau} \tilde{r}_{\tau}^{-1})$ is invertible in $\mathbb{C}[M_\tau]$. Thus, $\tilde{s}'$ is almost a twisted section of $\tilde{\mathcal{L}}$ except that the twisting factor $\tilde{r}_{\sigma}' \tilde{m}_{\sigma \tau} \tilde{r}_{\tau}^{-1}$ may not yet be in $\tilde{M}_\tau$. The same argument as in the proof of Proposition 3.5 and applying Proposition 2.2.10 show that there exists a softening $\phi : \tilde{Y}'(\Delta) \to \tilde{Y}(\Delta)$ such that the pullback collection $\phi^* \tilde{s}'$ is indeed a twisted section of the pullback invertible sheaf $\phi^* \tilde{\mathcal{L}}$ on $\tilde{Y}'(\Delta)$. This proves the proposition.

\[ \]

Since a soft noncommutative toric scheme associated to $\Delta$ with an invertible sheaf that extends $\mathcal{L}$ on $Y(\Delta)$ exists (cf. Proposition 3.5'), Proposition 3.8 is equivalent to:
Proposition 3.8’. [extension to twisted section] Let $\mathcal{L}$ be an invertible sheaf on $Y(\Delta)$ and $s_1, \cdots, s_k \in H^0(Y(\Delta), \mathcal{L})$ be sections of $\mathcal{L}$. Then, there exists a soft noncommutative toric scheme $\tilde{Y}(\Delta)$ with an invertible sheaf $\tilde{\mathcal{L}}$ that restricts to $\mathcal{L}$ on $Y(\Delta) \subset \tilde{Y}(\Delta)$ such that $s_i$ extends to a twisted section $\tilde{s}_i$ of $\tilde{\mathcal{L}}$, for $i = 1, \ldots, k$.

Soft noncommutative closed subschemes of $\tilde{Y}(\Delta)$ associated to twisted sections of invertible sheaves on $\tilde{Y}(\Delta)$

Continuing the notation and discussion of the previous theme. Let $\tilde{s}_i = \{\tilde{s}_{i, \sigma}\}_{\sigma \in \Delta}$ be a (non-zero) twisted section of an invertible sheaf $\tilde{\mathcal{L}}_i$ on $\tilde{Y}(\Delta)$, for $i = 1, \ldots, k$. With respect to a local trivialization of each $\tilde{\mathcal{L}}_i$, $\tilde{s}_i$ has a presentation $\{\tilde{r}_{i, \sigma}\}_{\sigma \in \Delta}$, with $\tilde{r}_{i, \sigma} \in \tilde{R}_\sigma$, $\sigma \in \Delta$. A different local trivialization of $\tilde{\mathcal{L}}_i$ gives rise to a different presentation of $\tilde{s}_i$ of the form $\{\tilde{r}_{i, \sigma} \tilde{u}_{i, \sigma}\}_{\sigma \in \Delta}$, where $\tilde{u}_{i, \sigma}$ is a unit (i.e. invertible element) of $\tilde{R}_\sigma$. Denote $\Gamma = \{\tilde{s}_1, \cdots, \tilde{s}_k\}$. It follows that the two-sided ideal $I_{\Gamma, \sigma} := (\tilde{r}_{1, \sigma}, \cdots, \tilde{r}_{k, \sigma})$ of $\tilde{R}_\sigma$, $\sigma \in \Delta$, associated to $\Gamma$ is independent of the local trivialization of $\tilde{\mathcal{L}}_i$, $i = 1, \ldots, k$. This defines an ideal sheaf $\mathcal{I}_\Gamma := \{I_{\Gamma, \sigma}\}_{\sigma \in \Delta}$ on $\tilde{Y}(\Delta)$ and hence a soft noncommutative closed subscheme $\mathcal{Z}_\Gamma$ of $\tilde{Y}(\Delta)$.

Definition 3.9. [soft noncommutative closed subscheme associated to twisted sections] $\mathcal{Z}_\Gamma$ is called the soft noncommutative closed subscheme of $\tilde{Y}(\Delta)$ associated to $\Gamma$. When $\Gamma = \{\tilde{s}\}$, $\tilde{Z}_\tilde{s} := \mathcal{Z}_\Gamma$ is called the soft noncommutative hypersurface of $\tilde{Y}(\Delta)$ associated to $\tilde{s}$.

Definition 3.10. [soft noncommutative scheme/space (via toric geometry)] A soft noncommutative toric scheme $\tilde{Y}(\Delta)$ associated to a fan $\Delta$ or a soft noncommutative closed subscheme $\mathcal{Z}$ of a $\tilde{Y}(\Delta)$ for some $\Delta$ will be called a soft noncommutative scheme via toric geometry, or simply a soft noncommutative scheme, or even a soft noncommutative space. The structure sheaf $\mathcal{O}_{\tilde{Y}}$ of such a noncommutative space $\tilde{Y}$ is defined to the inverse $\Delta$-system of $\mathbb{C}$-algebras that describes $\tilde{Y}$.

It is this class of noncommutative spaces that we will study further in sequels. Definition 2.2.12 and Definition 3.1 can be generalized routinely to soft noncommutative schemes and their modules. A straightforward generalization of Proposition 3.8’ to a finite collection of invertible sheaves with sections $\{(\mathcal{L}_1, s_1), \cdots, (\mathcal{L}_k, s_k)\}$ on $Y(\Delta)$ implies that:

Corollary 3.11. [commutative complete intersection subscheme] Any complete intersection subscheme $Z$ of a smooth toric variety $Y(\Delta)$ embeds as a commutative closed subscheme of a soft noncommutative scheme $\tilde{Z}$ in $\tilde{Y}(\Delta)$ such that $\tilde{Z} \cap Y(\Delta) = Z$. In particular, this applies to complete intersection Calabi-Yau spaces in a smooth toric variety.

Indeed it is the attempt to make this corollary holds that the notion of soft noncommutative schemes via toric geometry is discovered. Let us spell out a guiding question behind the NCS spin-off of the D-project before leaving this section:

Question 3.12. [soft noncommutative Calabi-Yau space and mirror] What is the correct notion/definition of soft noncommutative Calabi-Yau spaces in the current context? From pure mathematical generalization of the commutative case? From world-volume conformal invariance of supersymmetry of D-branes (cf. Sec. 4.2)? What is the mirror symmetry phenomenon in this context?
4 (Dynamical) D-branes on a soft noncommutative space

The notion of morphisms from an Azumaya scheme with a fundamental module over $\mathbb{C}$ to a soft noncommutative scheme over $\mathbb{C}$ is developed in this section.

4.1 Review of Azumaya/matrix schemes with a fundamental module over $\mathbb{C}$

The notion of Azumaya schemes (or equivalently matrix schemes) with a fundamental module and morphisms therefrom were developed in [L-Y1] (D(1)) and [L-L-S-Y] (D(2)) in the realm of Algebraic Geometry and in [L-Y2] (D(11.1)) and [L-Y3] (D(11.2)) in the realm of Differential Geometry to capture the Higgsing/unHiggsing feature of D-branes when they coincide or separate and to address dynamical D-branes moving in a space-time, [L-Y4] (D(13.1)) and [L-Y5] (D(13.3)). The generalization of such notion to cover fermionic D-branes in Superstring Theory has been a major focus in more recent years; cf. the SUSY spin-off of the D-project, e.g. [L-Y6] (SUSY(2.1)). The key setup of Azumaya/matrix schemes in the realm of Algebraic Geometry that is relevant to the current work is reviewed below for the terminology and notation. Readers are referred to the above works for more details and to [Liu] for a review of the first four years (fall 2007 – fall 2011) of the D-project.

Definition 4.1.1. [D-brane world volume: Azumaya/matrix scheme over $\mathbb{C}$] Let $X$ be a (Noetherian) scheme over $\mathbb{C}$ and $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module of rank $r$. Then the sheaf $\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}) := \text{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E})$ of endomorphisms of $\mathcal{E}$ is a noncommutative $\mathcal{O}_X$-algebra with center $\mathcal{O}_X$. The fiber of $\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})$ over each $\mathbb{C}$-point on $X$ is isomorphic to the Azumaya algebra (or interchangeably matrix algebra) of $r \times r$ matrices over $\mathbb{C}$. $\mathcal{E}$ is in the fundamental representation of $\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})$. The new ringed space from the enhanced structure sheaf, with its fundamental representation encoded, $X^{\mathbb{A}} := (X, \mathcal{O}_X^{\mathbb{A}} := \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}), \mathcal{E})$, is called an Azumaya scheme (or interchangeably matrix scheme) with a fundamental module.

The ringed space $(X, \mathcal{O}_X^{\mathbb{A}})$ serves as the world-volume of a D-brane and the fundamental (left) $\mathcal{O}_X^{\mathbb{A}}$-module $\mathcal{E}$ serves as the Chan-Paton sheaf on the world-volume. For a complete description of the data on the world-volume of a D-brane, there is also a connection $\nabla$ on $\mathcal{E}$ and, in this case, one may introduce a Hermitian structure on $\mathcal{E}$ and require that $\nabla$ be unitary. These two additional structures on $X^{\mathbb{A}}$ are irrelevant to the discussion of the current work and hence will be dropped.

As the category of (left) $\mathcal{O}_X^{\mathbb{A}}$-modules is equivalent to the category of $\mathcal{O}_X$-modules (cf. Morita equivalence), the noncommutativity of $X^{\mathbb{A}}$ may look only of a mild kind — yet important for the correct mathematical description of dynamical D-branes in string theory.

Example 4.1.2. [Azumaya/matrix point with fundamental module/$\mathbb{C}$] To get a feeling of such a mildly noncommutative space, consider the simplest case: $X = \mathbb{C}$-point $p$ and

$$p^{\mathbb{A}} := (p, M_r(\mathbb{C}), \mathbb{C}^{\mathbb{C}^r})$$

where $M_r(\mathbb{C})$ is the algebra of $r \times r$ matrices over $\mathbb{C}$. One may attempt to assign some sensible topology “$\text{Space}(M_r(\mathbb{C}))$” to the noncommutative $\mathbb{C}$-algebra, in the role of $\text{Spec}(R)$ to a commutative ring $R$. However, all the existing methods only lead to that “$\text{Space}(M_r(\mathbb{C}))$”
Definition 4.2.1. [quasi-homomorphism of Idempotents and gluing of quasi-homomorphisms apart/separate in a space-time.]

Now, the D-brane world-volume is only half of the story. The other half is how the world-volume gets mapped into a target space-time \( Y \) (i.e. morphisms from Azumaya schemes to \( Y \)). With this in mind, if \( A \subset M_r(\mathbb{C}) \) is a \( \mathbb{C} \)-subalgebra and we do know how to make sense of \( \text{Space} (A) \) and morphisms \( \text{Space} (A) \to Y \) — for example, \( A \) is commutative \( \mathbb{C} \)-subalgebra of \( M_r(\mathbb{C}) \) and \( Y \) is an ordinary Noetherian commutative scheme in Commutative Algebraic Geometry ([Ha]) — then we should expect a morphism \( p^{\mathbb{C}} \to Y \) from the composition

\[
\text{“Space} (M_r(\mathbb{C}))” \longrightarrow \text{Space} (A) \longrightarrow Y.
\]

Here, \( \longrightarrow \) indicates a dominant morphism since \( A \hookrightarrow M_r(\mathbb{C}) \).

It follows that the correct way to unravel the geometry behind \( p^{\mathbb{C}} \) is through \( \mathbb{C} \)-subalgebras of \( M_r(\mathbb{C}) \). For example, for \( r \geq 2 \), \( M_r(\mathbb{R}) \) contains \( \mathbb{C} \)-subalgebra in product form \( A = A_1 \times A_2 \). This means \( p^{\mathbb{C}} \) has hidden disconnectivity. Indeed, the consideration of commutative \( \mathbb{C} \)-subalgebras of \( M_r(\mathbb{C}) \) alone is enough to indicate that \( p^{\mathbb{C}} \) has a very rich geometry behind. Cf. [Liu: Figure 1-2, Figure 3-1, and Figure 3-2].

Example 4.1.2 motivates the following definition:

Definition 4.1.3. [surrogate of \( X^{\mathbb{C}} \)] (Continuing Definition 4.1.1.) Let \( A \subset O_X^{\mathbb{C}} \) be an \( O_X \)-subalgebra. Then, the ringed space

\[
X_A := (X, A)
\]

is called a surrogate of \( X^{\mathbb{C}} \). When \( A \) is commutative, \( X_A \) is simply \( \text{Spec} (A)/X \). The inclusion \( O_X \subset A \) specifies a dominant morphism \( X_A \to X \) over \( X \) while the inclusion \( A \subset O_X^{\mathbb{C}} \) specifies a dominant morphism \( X^{\mathbb{C}} \to X_A \) over \( X \). Cf. Figure 4-1.5

4.2 Morphisms from an Azumaya scheme with a fundamental module to a soft noncommutative scheme \( \hat{Y} \): (Dynamical) D-branes on \( \hat{Y} \)

We study in the subsection the notion of morphisms from an Azumaya scheme with a fundamental module to a soft noncommutative scheme \( \hat{Y} \). The design is guided by the requirement to reproduce the Higgsing/unHiggsing phenomenon when D-branes collide/coincide or fall apart/separate in a space-time.

Idempotents and gluing of quasi-homomorphisms

Definition 4.2.1. [quasi-homomorphism of \( \mathbb{C} \)-algebras] Given two (generally noncommutative) \( \mathbb{C} \)-algebras \( R \) and \( A \), a quasi-homomorphism from \( R \) to \( A \) is a \( \mathbb{C} \)-vector-space homomorphism \( f : R \to A \) such that \( f(r_1 r_2) = f(r_1) f(r_2) \). Note that it is not required that \( f(1_R) = 1_A \), where \( 1_R \) and \( 1_A \) are the identity of \( R \) and \( A \) respectively. Since \( f(1_R) f(1_R) = f(1_R) \), \( f(1_R) \) is an idempotent of \( A \). Since \([f(1_R), f(r)] = 0 \) for all \( r \in R \), \( f(R) \subset \text{Centralizer}(f(1_R)) \). A \( \mathbb{C} \)-algebra quasi-homomorphism \( f : R \to S \) with \( f(1_R) = 1_S \) is called as usual a \( \mathbb{C} \)-algebra homomorphism. In the other extreme, the zero-map \( 0 \to 0 \in A \) is a quasi-homomorphism.

\[^{3}\text{Reproduced from: Chien-Hao Liu and Shing-Tung Yau, D-branes and synthetic/C^\infty-algebraic symplectic/calibrated geometry, I. Lemma on a finite algebraicness property of smooth maps from Azumaya/matrix manifolds, arXiv:1504.01841 [math.SG] (D(12.1)), Figure 1-2.}\]
Figure 4-1-1. The Azumaya/matrix scheme $X^{Az}$ is indicated by a noncommutative cloud sitting over $X$. In-between are illustrated examples of surrogates (a), (b), (c), (d), (e), (f), (g) of $X^{Az}$. The vertical arrows $X^{Az} \to X_A$ and $X_A \to X$ are the built-in dominant morphisms associated to the inclusions $O_X \subset A \subset O_X^{Az}$ of $O_X$-algebras.

Let $A$, $R$, $S$ be $\mathbb{C}$-algebras with a fixed $\mathbb{C}$-algebra homomorphism $h : R \to S$. It is standard that two $\mathbb{C}$-algebra homomorphisms $f_R : R \to A$ and $f_S : S \to A$ are defined to be glued under $h$ if $f_R = f_S \circ h$. But if, instead, $f_R$ and $f_S$ are only $\mathbb{C}$-algebra quasi-homomorphisms, it takes some thought as to what the “correct” definition of ‘$f_R$ and $f_S$ glue under $h$’ should be. As a generally noncommutative $\mathbb{C}$-algebra, it can happen that $A$ is not a product of $\mathbb{C}$-algebras and yet still contains idempotents other than 0 and 1. In terms of the objects in the opposite category of the category of $\mathbb{C}$-algebras, the geometry $\text{Scheme}(A)$ behind the $\mathbb{C}$-algebra $A$ could have hidden disconnectivity (cf. Definition 2.1.8 and Example 2.1.9) and in terms of geometry it can happen that some connected components in the hidden disconnectivity of $\text{Scheme}(A)$ is mapped to the complement of the image of $\text{Scheme}(S)$ in $\text{Scheme}(R)$ under $h$. Since such hidden disconnectivity of $\text{Scheme}(A)$ is inherited from idempotents of $A$ other than 0 and 1, we need to keep track of $f_R(1_R)$ and $f_S(1_S)$, allowing the situation $f_R(1_R) \neq f_S(1_S)$, as well when considering gluing of $\mathbb{C}$-algebra quasi-homomorphisms $f_R$ and $f_S$ under $h$.

**Definition 4.2.2. [subordination relation of idempotents]** Let $A$ be a $\mathbb{C}$-algebra and $e_1, e_2 \in A$ are idempotents: $e_1^2 = e_1, e_2^2 = e_2$. We say that $e_1$ is subordinate to $e_2$, in notation $e_1 \prec e_2$, or equivalently $e_2$ is superior to $e_1$, in notation $e_2 \succ e_1$, if

$$e_1 \neq e_2 \quad \text{and} \quad e_1 e_2 = e_2 e_1 = e_1.$$

Denote ‘$e_1 \prec e_2$ or $e_1 = e_2$’ by $e_1 \preceq e_2$ or equivalently $e_2 \succeq e_1$. Note that $0 \preceq e \preceq 1$ for any idempotent $e$ of $A$.

When $e_1 \prec e_2$, $e'_2 := e_2 - e_1 \in A$ is also an idempotent subordinate to $e_2$ and it satisfies $e'_2 e_1 = e_1 e'_2 = 0$. Thus, $e_2 = e_1 + e'_2$ gives an orthogonal decomposition of $e_2$ into subordinate idempotents.
**Definition 4.2.3. [gluing of quasi-homomorphisms]** Let $A$, $R$, $S$ be $\mathbb{C}$-algebras with a fixed $\mathbb{C}$-algebra homomorphism $h : R \to S$ and $f_R : R \to A$ and $f_S : S \to A$ be quasi-homomorphisms of $\mathbb{C}$-algebras. We say that $f_R$ and $f_S$ glue under $h$, denoted $f_R \sim_h f_S$, if

$$f_S(1_S) \preceq f_R(1_R), \quad f_R(R) \subset \text{Centralizer}(f_S(1_S)),$$

and $f_S(1_S) \cdot f_R = f_R \cdot f_S(1_S) = f_S \circ h$.

In terms of a commutative diagram:

$$
\begin{array}{ccc}
R & \xrightarrow{f_R} & S \\
\downarrow{h} & & \downarrow{f_S} \\
S & \xrightarrow{f_S(1_S)} & A
\end{array}
$$

**Explanation 4.2.4. [gluing of quasi-homomorphisms - geometry behind]** Refine the commuting diagram in Definition 4.2.3 to

$$
\begin{array}{ccc}
R & \xrightarrow{f_R} & f_R(R) \subset A \\
\downarrow{h} & & \downarrow{f_S(1_S) = f_S(1_S) \cdot = : g} \\
S & \xrightarrow{f_S} & f_S(S) \subset A
\end{array}
$$

Then, in addition to $h$, $f_S$ and $f_S$ are now homomorphisms of $\mathbb{C}$-algebras and so is $g$, since $f_S(1_S) \preceq f_R(1_R)$ in $A$, while the two horizontal inclusions are only quasi-homomorphisms of $\mathbb{C}$-algebras. Let

$$e_1 := f_R(1_R) \quad \text{and} \quad e_2 := f_S(1_S).$$

Then $e_1 = e'_1 + e_2$ is an orthogonal decomposition of $e_1$ by idempotents. Since $f_R(R) \subset \text{Centralizer}(e_2)$,

$$f_R(R) = f_R(R) \cdot e'_1 + f_R(R) \cdot e_2 \quad \text{in } A$$

is an orthogonal decomposition of the $\mathbb{C}$-algebra $f_R(R)$; i.e. $f_R(R) \simeq f_R(R) \cdot e'_1 \times f_R(R) \cdot e_2$ as abstract $\mathbb{C}$-algebras. Thus, when two quasi-homomorphisms $f_R : R \to A$ and $f_S : S \to A$ glue under $h$ in the sense of Definition 4.2.3, one has a commuting diagram

$$
\begin{array}{ccc}
R & \xrightarrow{f_R} & f_R(R) \cdot e'_1 \times f_R(R) \cdot e_2 \subset A \\
\downarrow{h} & & \downarrow{f_S(1_S) = e_2 \cdot = : g} \\
S & \xrightarrow{f_S} & f_S(S) \subset A
\end{array}
$$

Or, equivalently in terms of the objects in the opposite category of the category of $\mathbb{C}$-algebras, a diagram of morphisms of (generally noncommutative) schemes:

$$
\begin{array}{ccc}
\text{Scheme}(R) & \leftarrow & \text{Scheme}(f_R(R) \cdot e'_1) \amalg \text{Scheme}(f_R(R) \cdot e_2) \\
\text{Scheme}(S) & \leftarrow & \text{Scheme}(f_S(S))
\end{array}
$$

Assume in addition that the idempotent $e_3 := 1 - e_1$ is also orthogonal to $e_2$ (i.e. $e_2 e_3 = e_3 e_2 = 0$ in $A$) and let

$$A_e := \{a \in A \mid ae = ea = a\}$$
for an idempotent \( e \in A \). Note that \( A_e \) is a \( \mathbb{C} \)-algebra with a built-in \( \mathbb{C} \)-algebra quasi-homomorphism \( A_e \hookrightarrow A \). Then, \( 1 = e'_1 + e_2 + e_3 \) is an orthogonal decomposition of 1 by idempotents in \( A \) and

\[
A' \times A_2 \times A_3 \xrightarrow{\sim} A' + A_2 + A_3 \hookrightarrow A
\]
is now an honest \( \mathbb{C} \)-subalgebra inclusion. Since \( f_R(R) \subset A' + A_2 \) and \( f_S(S) \subset A_3 \), one has now a commuting diagram of \( \mathbb{C} \)-algebra homomorphisms

\[
\begin{array}{c}
A' \times A_2 \times A_3 \\
\downarrow \pi_{12} \\
A'_1 \times A_2 \\
\downarrow \pi_2 \\
A_2
\end{array}
\]

Here \( \pi_{12} \) and \( \pi_2 \) are the projection homomorphisms of product \( \mathbb{C} \)-algebras to the indicated components. The corresponding commuting diagram of morphisms of noncommutative affine schemes in the opposite category is then: (\( \Pi = \text{disjoint union} \))

\[
\begin{array}{ccc}
\text{Scheme}(A'_1) \coprod \text{Scheme}(A_2) \coprod \text{Scheme}(A_3) & \xleftarrow{\iota_{12}} & \text{Scheme}(A) \\
\downarrow \iota_2 \downarrow & & \downarrow \iota_1 \\
\text{Scheme}(R) & \xleftarrow{\iota_1} & \text{Scheme}(A'_1) \coprod \text{Scheme}(A_2) \\
\downarrow \iota_2 \downarrow & & \downarrow \iota_1 \\
\text{Scheme}(S) & \xleftarrow{\iota_2} & \text{Scheme}(A_3)
\end{array}
\]

Here, \( \iota_{12} \) and \( \iota_2 \) are the inclusion morphism of the connected components as indicated.

Thus, a quasi-homomorphism as defined in Definition 4.2.1 is nothing but a roof of ordinary homomorphisms and, with an additional assumption of the completeness of idempotents involved, a gluing of quasi-homomorphisms in the sense of Definition 4.2.3 is nothing but the ordinary gluing of homomorphisms after the replacement by roofs.

**Remark 4.2.5.** [transitivity of subordination relation of idempotents] Caution that for a general noncommutative \( \mathbb{C} \)-algebra \( A \), idempotents \( e_1 \prec e_2 \) and \( e_2 \prec e_3 \) do not imply \( e_1 \prec e_3 \). (E.g. \( A = \mathbb{C}\langle z_1, z_2, z_3 \rangle/(z_1^2 - z_1, z_2^2 - z_2, z_3^2 - z_3, z_1 z_2 - z_1, z_2 z_3 - z_2, z_3 z_2 - z_2) \).) However, transitivity of \( \prec \) holds for idempotents of the Azumaya/matrix algebra \( M_r(\mathbb{C}) \) of rank \( r \) over \( \mathbb{C} \), for all \( r \). (For idempotents \( e_1 \prec e_2 \), \( e_2 \prec e_3 \) in \( M_r(\mathbb{C}) \), the three \( e_1, e_2, e_3 \) must be commuting and hence simultaneously diagonalizable. Which implies \( e_1 \prec e_3 \) in the end.)

**Homomorphisms from a \( \Delta \)-system of \( \mathbb{C} \)-algebras to a \( \mathbb{C} \)-algebra \( A \)**

Let \( \Delta \) be a fan in \( N_\mathbb{R} \) that satisfies Assumption 2.2.2 and \( A \) a \( \mathbb{C} \)-algebra.

**Definition 4.2.6.** [\( \Delta \)-system of idempotents in \( A \)] Let \( e_\Delta^A := \{ e_\sigma \}_{\sigma \in \Delta} \) be a collection of idempotents in \( A \) labelled by cones in \( \Delta \). \( e_\Delta^A \) is said to be
(i) a weak $\Delta$-system of idempotents in $A$ if $e_\tau \preceq e_\sigma$ for all $\tau \prec \sigma \in \Delta$;
(ii) a strong $\Delta$-system of idempotents in $A$ if $e_\sigma e_{\sigma'} = e_{\sigma \cap \sigma'}$ for all $\sigma, \sigma' \in \Delta$.

Note that since $\sigma \cap \sigma' = \sigma' \cap \sigma$, all the idempotents in a strong $\Delta$-system of idempotents in $A$ commute with each other. Furthermore, since $\tau \cap \sigma = \tau$ for $\tau \prec \sigma \in \Delta$, a strong $\Delta$-system of idempotents in $A$ is naturally a weak $\Delta$-system of idempotents in $A$.

**Lemma/Definition 4.2.7. [reduced idempotents in strong $\Delta$-system]** Let $e^A_\Delta = \{e_\sigma\}_{\sigma \in \Delta}$ be a strong $\Delta$-system of idempotents in $A$. Then, there exists a unique collection $\{e_\sigma\}_{\sigma \in \Delta}$ of orthogonal (i.e. $e_\sigma e_{\sigma'} = 0$ for $\sigma \neq \sigma'$) idempotents in $A$ such that

$$e_\sigma = \sum_{\tau \leq \sigma} e_\tau.$$  

$e_\sigma$ is called the reduced idempotent associated to $\sigma \in \Delta$ from the strong $\Delta$-system $e^A_\Delta$. It is the contribution to $e_\sigma$ after being trimmed away all the boundary contributions.

**Proof.** Explicitly, the reduced idempotent $e_\sigma$, for $\sigma \in \Delta(k)$, is given by:

$$e_\sigma := e_\sigma - \sum_{\tau \in \Delta(k-1), \tau < \sigma} e_\tau + \sum_{\tau \in \Delta(k-2), \tau < \sigma} e_\tau - \sum_{\tau \in \Delta(k-3), \tau < \sigma} e_\tau + \cdots + (-1)^{k-1} \sum_{\tau \in \Delta(1), \tau < \sigma} e_\tau + (-1)^k e_0.$$  

Equivalently, let $\{\tau_1, \ldots, \tau_k\}$ be the set of facets (i.e. codimension-1 faces) of $\sigma \in \Delta(k)$, then

$$e_\sigma = e_\sigma - \sum_i e_{\tau_i} + \sum_{i_1 < i_2} e_{\tau_{i_1}} e_{\tau_{i_2}} - \sum_{i_1 < i_2 < i_3} e_{\tau_{i_1}} e_{\tau_{i_2}} e_{\tau_{i_3}} + \cdots + (-1)^k e_{\tau_1} \cdots e_{\tau_k}.$$  

Such inclusion-exclusion formula is akin and applies to simplicial cones.

That $e^2_\sigma = e_\sigma$, for $\sigma \in \Delta$, and that $e_\sigma e_{\sigma'} = 0$, for $\sigma \neq \sigma'$, follow by induction on ‘up to $k$’, $1 \leq k \leq n - 1$.

**Definition 4.2.8. [complete strong $\Delta$-system of idempotents]** Let $e^A_\Delta := \{e_\sigma\}_{\sigma \in \Delta}$ be a strong $\Delta$-system of idempotents in $A$ and $e^A_\Delta := \{e_\sigma\}_{\sigma \in \Delta}$ be the associated collection of reduced idempotents. We say that $e^A_\Delta$ is complete if $\sum_{\sigma \in \Delta} e_\sigma = 1$.

With all these preparations, we are now ready for a key definition:

**Definition 4.2.9. [homomorphisms from inverse $\Delta$-system of $\mathbb{C}$-algebras to $A$]** Let $Q := \{Q_\sigma\}_{\sigma \in \Delta}, \{i^\tau_\sigma : Q_\sigma \to Q_\tau\}_{\tau < \sigma \in \Delta\}$ be an inverse $\Delta$-system of $\mathbb{C}$-algebras. Denote the identity element of $Q_\sigma$ by $1_{Q_\sigma}$ and that of $A$ by $1$. Then a collection

$$\varphi^\tau := \{f^\tau_\sigma : Q_\sigma \to A\}_{\sigma \in \Delta}$$  

of $\mathbb{C}$-algebra quasi-homomorphisms indexed by cones in $\Delta$ is called a homomorphism from $Q$ to $A$, denoted now $\varphi^\tau : Q \to A$, if
(i) (Inverse $\Delta$-system of quasi-homomorphisms of $\mathbb{C}$-algebras)

\[ f_\sigma \overset{\iota_\sigma}{\sim} f_\tau, \text{ for } \tau \prec \sigma \in \Delta, \]

Cf. Definition 4.2.3. In particular, $e_{\varphi^\sharp} := \{ e_\sigma := f_\sigma^\sharp(1_{Q_\sigma}) \}_{\sigma \in \Delta}$ is a weak $\Delta$-system of idempotents in $\mathcal{A}$, called the $\Delta$-system of idempotents in $\mathcal{A}$ associated to $\varphi^\sharp$.

(ii) (Completeness) $e_{\varphi^\sharp}$ is a complete strong $\Delta$-system of idempotents in $\mathcal{A}$.

For the simple uniformness of terminology but with slight abuse, we shall call such homomorphisms also $\mathbb{C}$-algebra homomorphisms when need to distinguish with homomorphisms of other algebraic structures. The set of homomorphisms from $\mathcal{Q}$ to $\mathcal{A}$ is thus denoted $\text{Hom}_{\mathbb{C}-\text{Alg}}(\mathcal{Q}, \mathcal{A})$, or simply $\text{Hom}(\mathcal{Q}, \mathcal{A})$.

**Explanation 4.2.10. [gluings of $\Delta$-system of (local) quasi-homomorphisms to a “global” homomorphism]** Given a homomorphism $\varphi^\sharp : \mathcal{Q} \rightarrow \mathcal{A}$ as defined in Definition 4.2.9. Let $e_{\varphi^\sharp} := \{ e_\sigma := f_\sigma^\sharp(1_{Q_\sigma}) \}_{\sigma \in \Delta}$, $\sigma \in \Delta$, be the collection of the reduced idempotents from $e_{\varphi^\sharp}$ and

\[ A_{e_\sigma} := \{ a \in \mathcal{A} \mid ae_{e_\sigma} = e_{e_\sigma}a = a \}, \quad \sigma \in \Delta. \]

Then, a generalization of Explanation 4.2.4 gives the inclusions of $\mathbb{C}$-algebras

\[ f_\sigma^\sharp(Q_\sigma) \subset \bigoplus_{\tau \prec \sigma} A_{e_\tau}, \text{ (as a multiplicatively closed } \mathbb{C} \text{-vector subspace of } \mathcal{A}) \]

\[ \cong \times_{\tau \prec \sigma} A_{e_\tau}, \text{ (as an abstract } \mathbb{C} \text{-algebra), } \sigma \in \Delta, \]

and commuting diagrams of homomorphisms of $\mathbb{C}$-algebras:

\[ A_\varphi := \times_{\sigma \in \Delta} A_{e_\sigma} \xrightarrow{\pi_{A_\varphi}} \mathcal{A} \]

\[ Q_\sigma \xrightarrow{f_\sigma^\sharp} A_{\varphi, \sigma} := \times_{\rho \leq \sigma \in \Delta} A_{e_\rho} \]

\[ A_{\varphi, \tau} := \times_{\rho \leq \tau \in \Delta} A_{e_\rho}, \text{ for } \tau \prec \sigma \in \Delta. \]

Here $f_\sigma^\sharp : A_\varphi \rightarrow A_{\varphi, \sigma}$, $\sigma \in \Delta$, and $f_\tau^\sharp : A_{\varphi, \sigma} \rightarrow A_{\varphi, \tau}$ are the projection maps onto factors of product $\mathbb{C}$-algebras as indicated. The underlying geometry of the diagrams is revealed in terms of commuting diagrams of morphisms of noncommutative affine schemes in the opposite category: ($\Pi = \text{disjoint union}$)

\[ \text{Scheme}(A_\varphi) = \Pi_{\sigma \in \Delta} \text{Scheme}(A_{e_\sigma}) \xrightarrow{\pi_{A_\varphi}} \text{Scheme}(\mathcal{A}) \]

\[ \text{Scheme}(Q_\sigma) \xleftarrow{f_\sigma} \text{Scheme}(A_{\varphi, \sigma}) = \Pi_{\rho \leq \sigma \in \Delta} \text{Scheme}(A_{e_\rho}) \]

\[ \text{Scheme}(Q_\tau) \xleftarrow{f_\tau} \text{Scheme}(A_{\varphi, \tau}) = \Pi_{\rho \leq \tau \in \Delta} \text{Scheme}(A_{e_\rho}) \]

for $\tau \prec \sigma \in \Delta$. Think of $\mathcal{Q}$ as defining contravariantly a noncommutative space “$\text{Space}(\mathcal{Q})$”. $\varphi^\sharp$ then defines a morphism $\varphi : \text{Scheme}(\mathcal{A}) \rightarrow \text{Space}(\mathcal{Q})$ of noncommutative spaces.
(Dynamical, complex algebraic) D-branes on a soft noncommutative scheme

Let $X$ be a (commutative) scheme over $\mathbb{C}$, $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module of rank $r$,

$$X^{Az} = (X, \mathcal{O}_X^{Az} := \text{End}_{\mathcal{O}_X}(\mathcal{E}), \mathcal{E})$$

be the Azumaya/matrix scheme over $\mathbb{C}$ with the underlying topology $X$ and the fundamental module $\mathcal{E}$, and $\tilde{Y}$ be a soft noncommutative scheme via toric geometry, with the structure sheaf

$$\mathcal{O}_\tilde{Y} = \tilde{\mathcal{O}}_{\Delta} := (\{\tilde{\mathcal{O}}_\sigma\}_{\sigma \in \Delta}, \{i^{\mathbb{Z}}_{\tau}\}_{\tau < \alpha \in \Delta}).$$

an inverse $\Delta$-system of $\mathbb{C}$-algebras, cf. Definition 3.10. Consider the sheaf of sets

$$\text{Hom}_{\mathbb{C}-\text{Alg}}(\mathcal{O}_\tilde{Y}, \mathcal{O}_X^{Az})$$

associated to the presheaf defined by the assignment $U \mapsto \text{Hom}_{\mathbb{C}-\text{Alg}}(\tilde{\mathcal{O}}_{\Delta}(U), \mathcal{O}_X^{Az}(U))$ for $U$ open in $X$ and restriction maps $\text{Hom}_{\mathbb{C}-\text{Alg}}(\tilde{\mathcal{O}}_{\Delta}(U), \mathcal{O}_X^{Az}(U)) \rightarrow \text{Hom}_{\mathbb{C}-\text{Alg}}(\tilde{\mathcal{O}}_{\Delta}(V), \mathcal{O}_X^{Az}(V))$ via the post-composition with the $\mathbb{C}$-algebra homomorphism $\mathcal{O}_X^{Az}(U) \rightarrow \mathcal{O}_X^{Az}(V)$ for open sets $V \subset U \subset X$.

**Definition 4.2.11.** [D-brane on $\tilde{Y}$: morphism $X^{Az} \rightarrow \tilde{Y}$] A morphism $\varphi : X^{Az} \rightarrow \tilde{Y}$ is by definition a global section, in notation

$$\varphi^\sharp : \mathcal{O}_\tilde{Y} \rightarrow \mathcal{O}_X^{Az}$$

of $\text{Hom}_{\mathbb{C}-\text{Alg}}(\mathcal{O}_\tilde{Y}, \mathcal{O}_X^{Az})$. Explicitly, $\varphi^\sharp$ is represented by the following data

$$(\mathcal{U} := \{U_\alpha\}_{\alpha \in I}, \varphi^\sharp_\mathcal{U} := \{\varphi^\sharp_\alpha : \tilde{\mathcal{O}}_{\Delta} \rightarrow \text{End}_{\mathcal{O}_X}(\mathcal{E}|_{U_\alpha})\}_{\alpha \in I}),$$

where $\mathcal{U}$ is an open covering of $X$ and $\varphi^\sharp_\mathcal{U}$ a collection of $\mathbb{C}$-algebra homomorphisms such that the following diagrams commute

$$\begin{array}{ccc}
\text{End}_{\mathcal{O}_X}(\mathcal{E}|_{U_\alpha}) & \xrightarrow{\varphi^\sharp_\alpha} & \tilde{\mathcal{O}}_{\Delta} \\
\downarrow \varphi^\sharp_\beta & & \\
\text{End}_{\mathcal{O}_X}(\mathcal{E}|_{U_\beta}) & \xleftarrow{\pi^{\mathbb{Z}}_{\beta}} & \tilde{\mathcal{O}}_{\Delta} \\
\end{array}$$

for all $\alpha, \beta \in I$.

With the additional data of a connection $\nabla$ on $\mathcal{E}$ (the Chan-Paton sheaf with a gauge field) this gives a mathematical model for dynamical, complex algebraic D-branes as an extended object moving on the soft noncommutative target-space $\tilde{Y}$ over $\mathbb{C}$. (One may also introduce a Hermitian structure on $\mathcal{E}$ and consider only unitary connections $\nabla$ on $\mathcal{E}$.)

**Definition 4.2.12.** [surrogate of $X^{Az}$ associated to $\varphi$] The completeness condition on idempotents (cf. Definition 4.2.9: Condition (ii)) implies that the inclusion

$$\mathcal{A}_\varphi := \mathcal{O}_X(\varphi^\sharp(\mathcal{O}_\tilde{Y})) \hookrightarrow \mathcal{O}_X^{Az}$$

is an $\mathcal{O}_X$-algebra homomorphism (rather than only a quasi-homomorphism of $\mathcal{O}_X$-algebras). The corresponding noncommutative scheme $X_\varphi$ over $X$ is called the surrogate of $X^{Az}$ associated to $\varphi$. The sequence of $\mathcal{O}_X$-algebra inclusions $\mathcal{O}_X \hookrightarrow \mathcal{A}_\varphi \hookrightarrow \mathcal{O}_X^{Az}$ gives a sequence of dominant morphisms $X^{Az} \rightarrow X_\varphi \rightarrow X$. By construction, $\varphi$ factors to a composition of $X^{Az} \rightarrow X_\varphi \rightarrow \tilde{Y}$.
Definition 4.2.13. [image of \( \varphi \) and push-forward Chan-Paton sheaf] The two-sided ideal sheaf \( \text{Ker}(\varphi^\sharp) \) of \( \mathcal{O}_{\hat{Y}} \) defines a soft closed subscheme of \( \hat{Y} \), denoted \( \text{Im} \varphi \) and called the image of \( \varphi \). \( \varphi^\sharp \) renders \( \mathcal{E} \) an \( \mathcal{O}_{\hat{Y}} \)-module, denoted \( \varphi_* \mathcal{E} \) and called the push-forward Chan-Paton sheaf on \( \hat{Y} \). By construction, \( \varphi_* \mathcal{E} \) is supported on \( \text{Im} \varphi \).

**Figure 4-2-1**

![Diagram showing a morphism \( \varphi : X^{Az} \to \hat{Y} \) with \( \text{Im} \varphi \) as the image.]

We conclude this subsection with a second guiding question:

**Question 4.2.14. [generalized matrix model]** For \( X \) a \( \mathbb{C} \)-point \( pt \), \( \mathcal{E} = \mathbb{C}^r \), and \( \Delta = \{ \sigma \} \), where \( \sigma \) is a simplicial cone of \( N_\mathbb{R} \) of index 1, one has that \( \hat{Y}(\Delta) = n \mathbb{C} \mathbb{A}^n \) and morphisms from \( pt^{Az} \) to \( n \mathbb{C} \mathbb{A}^n \) are given by tuples \( \mathbf{m} = (m_1, \cdots, m_n) \), \( m_i \in M_r(\mathbb{C}) \). Such \( \mathbf{m} \)'s (with possible Hermitian constraints) are fields in a matrix model. From this aspect, a theory with fields morphisms from \( pt^{Az} \) to a soft noncommutative toric scheme \( \hat{Y}(\Delta) \) for a fan \( \Delta \) satisfying Assumption 2.2.2 gives a generalization of matrix models via gluing simple matrix models on morphisms \( pt^{Az} \to n \mathbb{C} \mathbb{A}^n \). Details of some examples? Consequences? Relations to Topological String Theory?

### 4.3 Two equivalent descriptions of a morphism \( \varphi : X^{Az} \to \hat{Y} \)

The notion of a soft noncommutative toric scheme \( \hat{Y}(\Delta) \) over \( \mathbb{C} \) associated to a fan \( \Delta \) can be generalized to the notion of a soft noncommutative toric scheme \( \hat{Y}_X(\Delta) \) over \( X \) associated to \( \Delta \) by generalizing the notion of an inverse \( \Delta \)-system of \( \mathbb{C} \) monoid algebras \( \{ \mathbb{C} \langle M_\sigma \rangle \}_{\sigma \in \Delta} \) to the notion of an inverse \( \Delta \)-system of \( \mathcal{O}_X \) monoid algebras \( \{ \mathcal{O}_X \langle \hat{M}_\sigma \rangle \}_{\sigma \in \Delta} \). Similarly, for close

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4 Revised from: Chien-Hao Liu and Shing-Tung Yau, More on the admissible condition on differentiable maps \( \varphi : (X^{Az}, E; \nabla) \to Y \) in the construction of the non-Abelian Dirac-Born-Infeld action \( S_{DBI}(\varphi, \nabla) \), arXiv:1611.09439 [hep-th] (D(13.2.1)), Figure 2-1.
subspaces \( \tilde{Z} \) of \( \tilde{Y} \). This defines the product space \( X \times \tilde{Y} \) of the commutative scheme \( X \) and the soft noncommutative scheme \( \tilde{Y} \) and its structure sheaf \( O_{X \times \tilde{Y}} \). The built-in inclusion

\[
O_X \hookrightarrow \text{Center}(O_{X \times \tilde{Y}}) \subset O_{X \times \tilde{Y}}
\]

of \( O_X \)-algebras defines the projection map \( pr_X : X \times \tilde{Y} \to X \). In particular, an \( O_{X \times \tilde{Y}} \)-module \( \bar{F} \) on \( X \times \tilde{Y} \) is automatically an \( O_X \)-module, denoted \( pr_{X, \ast} \bar{F} \) or simply \( \bar{F} \). The built-in inclusion

\[
O_{\tilde{Y}} \hookrightarrow O_{\tilde{Y}} \cdot 1 \subset O_{X \times \tilde{Y}}
\]

of \( \mathbb{C} \)-algebras defines the projection map \( pr_{\tilde{Y}} : X \times \tilde{Y} \to \tilde{Y} \).

Let \( \bar{F} \) be a two-sided \( O_{X \times \tilde{Y}} \)-module. The support of \( \bar{F} \), in notation \( \text{Supp}(\bar{F}) \), is by definition the closed subscheme of \( X \times \tilde{Y} \) associated to the two-sided ideal sheaf of \( O_{X \times \tilde{Y}} \) generated by both left annihilators and right annihilators of \( \bar{F} \)

\[
\text{Ann}(\bar{F}) := \{ f \in O_{X \times \tilde{Y}} \mid f \cdot \bar{F} = 0 \text{ or } \bar{F} \cdot f = 0 \}.
\]

We say that \( \bar{F} \) is of relative dimension 0 over \( X \) if the restriction of \( \bar{F} \) to each \( \{ p \} \times \tilde{Y} \) for \( p \) a \( \mathbb{C} \)-point on \( X \) is a finitely dimensional \( \mathbb{C} \)-vector space and \( O_{\text{Supp}(\bar{F})} := O_{X \times \tilde{Y}} / \text{Ann}(\bar{F}) \) is a coherent \( O_X \)-module under \( pr_{X, \ast} \).

**Definition/Lemma 4.3.1. [graph of morphism \( \varphi : X^A \to \tilde{Y} \)]** Let \( \varphi : X^A \to \tilde{Y} \) be a morphism, defined via \( \varphi^\sharp : O_{\tilde{Y}} \to O_X^A := \text{End}_{O_X}(E) \). Then, \( \varphi^\sharp \) determines a homomorphism of \( O_X \)-algebras

\[
\varphi^\sharp : O_{X \times \tilde{Y}} \to O_X^A, \quad \text{with } f : \tilde{r} \mapsto f : \varphi^\sharp(\tilde{r}) \text{ for all } f \in O_X \text{ and } \tilde{r} \in O_{\tilde{Y}}.
\]

This defines the morphism \( \bar{\varphi} : X^A \to X \times \tilde{Y} \). The two-sided ideal sheaf \( \text{Ker}(\varphi^\sharp) \) of \( O_{X \times \tilde{Y}} \) defines a closed subscheme \( \Gamma_{\varphi} \) of \( X \times \tilde{Y} \) that is isomorphic to the surrogate \( X_\varphi \) of \( X^A \) associated to \( \varphi \), as noncommutative schemes over \( X \). By construction, \( \bar{E} := \varphi_* E \) is an \( O_{X \times \tilde{Y}} \)-module supported on \( \Gamma_{\varphi} \). The \( O_{X \times \tilde{Y}} \)-module \( \bar{E} \) is called the graph of \( \varphi \). It has the property that

\[
\bar{E} \cdot \tilde{r} \text{ is of relative dimension 0 over } X \text{ and } pr_{X, \ast} \bar{E} = E \quad (\text{i.e. } pr_{X, \ast} \bar{E} \simeq E \text{ canonically}).
\]

The converse also holds: Let \( \bar{E} \) be an \( O_{X \times \tilde{Y}} \)-module of relative dimension 0 over \( X \) such that

\[
\mathcal{E} := pr_{X, \ast} \bar{E} \text{ is a coherent locally free } O_X \text{-module, say of rank } r \text{ and } X^A := (X, O_X^A := \text{End}_{O_X}(E), \mathcal{E}) \text{ be the Azumaya scheme/}\mathbb{C} \text{ with the fundamental module } \mathcal{E}.
\]

Then, \( \bar{E} \) specifies a morphism \( \varphi : X^A \to \tilde{Y} \).

Cf. [L-L-S-Y: FIGURE 2-2-1] (D(2)).

**Proof.** Via \( pr_{\tilde{Y}}^\sharp \), the \( O_{\tilde{Y}} \)-module structure on \( \bar{E} \) induces a \( O_{\tilde{Y}} \)-module structure on \( \mathcal{E} \). This defines a homomorphism \( \varphi^\sharp : O_{\tilde{Y}} \to O_X^A \).

\[\square\]

From Definition/Lemma 4.3.1, one realizes that

**Lemma 4.3.2. [\( \varphi \) and moduli stack of 0-dimensional \( O_{\tilde{Y}} \)-modules]** Let \( \mathcal{M}_r^{0\varphi}(\tilde{Y}) \) be the moduli stack of 0-dimensional \( O_{\tilde{Y}} \)-modules that are of complex dimension \( r \) as \( \mathbb{C} \)-vector spaces. Then a morphism from an Azumaya scheme over \( X \) with the fundamental module of rank \( r \) to \( \tilde{Y} \) gives a morphism \( X \to \mathcal{M}_r^{0\varphi}(\tilde{Y}) \). Cf. [L-L-S-Y: FIGURE 3-1-1] (D(2)).
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