Research Article

Jin Li*

Linear barycentric rational collocation method for solving biharmonic equation

https://doi.org/10.1515/dema-2022-0151
received January 6, 2022; accepted August 1, 2022

Abstract: Two-dimensional biharmonic boundary-value problems are considered by the linear barycentric rational collocation method, and the unknown function is approximated by the barycentric rational polynomial. With the help of matrix form, the linear equations of the discrete biharmonic equation are changed into a matrix equation. From the convergence rate of barycentric rational polynomial, we present the convergence rate of linear barycentric rational collocation method for biharmonic equation. Finally, several numerical examples are provided to validate the theoretical analysis.

Keywords: linear barycentric rational, collocation method, error functional, biharmonic equation, equidistant nodes, Chebyshev nodes

MSC 2020: 65D05, 65L60, 31A30

1 Introduction

In this article, we pay our attention to the numerical solution of biharmonic equation:

$$\nabla^4 u = \nabla^2 \nabla^2 u = \frac{\partial^4 u(x, y)}{\partial x^4} + 2 \frac{\partial^4 u(x, y)}{\partial x^2 \partial y^2} + \frac{\partial^4 u(x, y)}{\partial y^4} = f(x, y), (x, y) \in \Omega,$$

with boundary condition

$$u(x, y) = g(x, y), \quad \frac{\partial u(x, y)}{\partial n} = h_1(x, y), \quad (x, y) \in \partial \Omega$$

or

$$u(x, y) = g(x, y), \quad \frac{\partial^2 u(x, y)}{\partial x^2} + \frac{\partial^2 u(x, y)}{\partial y^2} = h_2(x, y), \quad (x, y) \in \partial \Omega,$$

where $\Omega = [a, b] \times [c, d]$ and $f(x, y) \in C(\Omega), g(x, y), h_1(x, y), h_2(x, y) \in C^0(\Omega)$. Biharmonic equation [1] is widely used in electrostatics, mechanical engineering, theoretical physics, and so on. There are a lot of numerical methods to solve biharmonic equation such as finite difference method, finite element method, boundary element method, spectral method, and so on. Generally, the interpolation nodes of barycentric Lagrange
interpolation such as second kind of Chebyshev point is not the equidistant node. In order to obtain the equidistant node of the barycentric formulae, Floater and Hormann [12], Floater et al. [13], Klein and Berrut [14,15] have proposed a rational interpolation scheme which has high numerical stability and interpolation accuracy on both equidistant nodes and non-equidistant nodes. In recent articles, Wang et al. [16–19] successfully applied the barycentric interpolation collocation method to solve initial value problems, plane elasticity problems [20], incompressible plane problems, telegraph equation [21], beam force vibration equation [22], and non-linear problems, which have expanded the application fields of the collocation method. For the two-dimensional biharmonic boundary problems, a new spectral collocation method [23] and depression of order [24] are reported to numerically solve it.

Based on the one-dimensional linear barycentric rational interpolation, two-dimensional barycentric rational interpolation polynomial is constructed, then barycentric rational interpolation collocation method is obtained to solve biharmonic equation. With the help of vector form, the discrete linear equation of two-dimensional biharmonic equation is changed into matrix equation which can be coded easily. Moreover, the error estimate of linear barycentric rational interpolation for biharmonic equation is obtained and the convergence rate is also presented.

This article is organized as follows: In Section 2, the differentiation matrix and barycentric rational interpolation collocation scheme for biharmonic equation are presented, then the matrix form of collocation scheme is obtained. In Section 3, the convergence rate is proved. Finally, some numerical examples are listed to illustrate our theorem.

2 Differentiation matrix of biharmonic equation

Let \( a = x_1 < x_2 < \cdots < x_m = b, \quad h = \frac{b-a}{m-1}, \) and \( c = y_1 < y_2 < \cdots < y_n = d, \quad r = \frac{d-c}{n-1} \) with mesh point \((x_j, y_k), \quad j = 1, 2, \ldots, m; \quad k = 1, 2, \ldots, n, \quad \Omega = [x_i, x_{i+1}] \times [y_j, y_{j+1}], \) then we have

\[
u(x_j, y) = u(y), \quad j = 1, 2, \ldots, m
\]

on the interval \([a, b],\)

and

\[
u(x, y) = \sum_{j=1}^{m} L_j(x)u(y),
\]

where \( L_j(x) = \frac{w_j}{\sum_{i=1}^{m} w_i}, \quad w_j = \sum_{i \in I_j} (-1)^i \prod_{j \neq k}^{d+1} \frac{1}{x_k - x_j}, \) is the basis function, and \( I_j = \{ i \in I; \quad k - d \leq i \leq k \}, \quad I = \{ 0, 1, \cdots, n - d \}. \) Similarly on the interval \([c, d],\)

we have

\[
u(y_k) = u(x_j, y_k) = u_{jk}, \quad j = 1, 2, \ldots, m; \quad k = 1, 2, \ldots, n
\]

and

\[
u(y) = \sum_{k=1}^{n} L_k(y)u_{jk},
\]

with \( L_k(y) = \frac{w_k}{\sum_{i=1}^{n} w_i}, \quad w_i = \sum_{i \in I_k} (-1)^i \prod_{j \neq k}^{d+1} \frac{1}{y_k - y_j}. \) then we have

\[
u(x, y) = \sum_{j=1}^{m} L_j(x) \sum_{k=1}^{n} L_k(y)u_{jk} = \sum_{j=1}^{m} \sum_{k=1}^{n} L_j(x)L_k(y)u_{jk}.
\]

Taking (8) into equation (1), we obtain

\[\sum_{j=1}^{m} \sum_{k=1}^{n} L_j^{(4)}(x)L_k^{(4)}(y)u_{jk} + \sum_{j=1}^{m} \sum_{k=1}^{n} L_j^{(n)}(x)L_k^{(n)}(y)u_{jk} + \sum_{j=1}^{m} \sum_{k=1}^{n} L_j(x)L_k^{(n)}(y)u_{jk} = f(x, y).\]
By taking \( x = x_j \) in (9), then we change equation (9) into the matrix equation as
\[
\begin{bmatrix}
\sum_{k=1}^{n} L_k(y)u_{lk} \\
\vdots \\
\sum_{k=1}^{n} L_k(y)u_{mk}
\end{bmatrix}
+ 2 \begin{bmatrix}
C_{11}^{(2)} & \cdots & C_{1m}^{(2)} \\
\vdots & \ddots & \vdots \\
C_{ml}^{(2)} & \cdots & C_{mm}^{(2)}
\end{bmatrix}
\begin{bmatrix}
\sum_{k=1}^{n} L_k(y)u_{lk} \\
\vdots \\
\sum_{k=1}^{n} L_k(y)u_{mk}
\end{bmatrix}
+ \begin{bmatrix}
C_{11}^{(4)} & \cdots & C_{1m}^{(4)} \\
\vdots & \ddots & \vdots \\
C_{ml}^{(4)} & \cdots & C_{mm}^{(4)}
\end{bmatrix}
\begin{bmatrix}
\sum_{k=1}^{n} L_k(y)u_{lk} \\
\vdots \\
\sum_{k=1}^{n} L_k(y)u_{mk}
\end{bmatrix} = \begin{bmatrix} f(y) \\
\vdots \\
f_m(y) \end{bmatrix},
\tag{10}
\]
and equation (10) is found on the point \( y_k, k = 1, 2, \ldots, n \) and we obtain
\[
\begin{bmatrix}
\sum_{k=1}^{n} L_k(y)u_{lk} \\
\vdots \\
\sum_{k=1}^{n} L_k(y)u_{mk}
\end{bmatrix}
+ 2 \begin{bmatrix}
C_{11}^{(2)} & \cdots & C_{1m}^{(2)} \\
\vdots & \ddots & \vdots \\
C_{ml}^{(2)} & \cdots & C_{mm}^{(2)}
\end{bmatrix}
\begin{bmatrix}
\sum_{k=1}^{n} L_k(y)u_{lk} \\
\vdots \\
\sum_{k=1}^{n} L_k(y)u_{mk}
\end{bmatrix}
+ \begin{bmatrix}
C_{11}^{(4)} & \cdots & C_{1m}^{(4)} \\
\vdots & \ddots & \vdots \\
C_{ml}^{(4)} & \cdots & C_{mm}^{(4)}
\end{bmatrix}
\begin{bmatrix}
\sum_{k=1}^{n} L_k(y)u_{lk} \\
\vdots \\
\sum_{k=1}^{n} L_k(y)u_{mk}
\end{bmatrix} = \begin{bmatrix} f(y) \\
\vdots \\
f_m(y) \end{bmatrix},
\tag{11}
\]
where \( f(x_j, y_k) = f_j f_k, f_j = [f_1, f_2, \ldots, f_m]^T = [f_1(y_k), f_2(y_k), \ldots, f_m(y_k)]^T \) and \( C_{ik}^{(2)} = L_k^2(x_i), D_{ij}^{(2)} = L_k^2(y_j), C_{ik}^{(4)} = L_k^4(x_i), D_{ij}^{(4)} = L_k^4(y_j), u_1 = [u_{11}, u_{12}, \ldots, u_{1n}], j = 1, 2, \ldots, m, k = 1, 2, \ldots, n \).

According to mathematical induction, we obtain the recurrence formula of \( k \)-order \((k \geq 2)\) differential matrix as
\[
\begin{align*}
C_{ij}^{(k)} &= k(C_{ij}^{(k-1)})C_{ij}^{(1)} - \frac{C_{ij}^{(k-1)}}{x_i - x_j}, & i \neq j \\
C_{ii}^{(k)} &= -\sum_{j=1, j \neq i}^{n} C_{ij}^{(k)} 
\end{align*}
\tag{12}
\]
and
\[
\begin{align*}
D_{ij}^{(k)} &= k(D_{ij}^{(k-1)})D_{ij}^{(1)} - \frac{D_{ij}^{(k-1)}}{y_i - y_j}, & i \neq j \\
D_{ii}^{(k)} &= -\sum_{j=1, j \neq i}^{n} D_{ij}^{(k)} 
\end{align*}
\tag{13}
\]
First, the Kronecker product of matrix \( A = (a_{ij})_{m \times n} \) and \( B = (b_{ij})_{k \times l} \) is defined as
\[
A \otimes B = (a_{ij}B)_{mk \times nl},
\]
where the matrix \( A \otimes B \) is order of \( mk \times nl \) and
\[
a_{ij}B = \begin{bmatrix}
a_{ij}b_{11} & a_{ij}b_{12} & \cdots & a_{ij}b_{1l} \\
a_{ij}b_{21} & a_{ij}b_{22} & \cdots & a_{ij}b_{2l} \\
\vdots & \vdots & \ddots & \vdots \\
a_{ij}b_{k1} & a_{ij}b_{k2} & \cdots & a_{ij}b_{kl}
\end{bmatrix}.
\]
Then matrix \( A \) and \( B \) can be changed to \((m \times n)\) column vector as
\[
x = [x_1, x_2, \ldots, x_m], \\
y = [y_1, y_2, \ldots, y_n],
\]
With the help of the matrix equation, the linear equation systems (11) can be written as
\[
(C^{(4)} \otimes I_n) \begin{bmatrix} u_1 \\
\vdots \\
u_m \end{bmatrix} + 2(C^{(2)} \otimes D^{(2)}) \begin{bmatrix} u_1 \\
\vdots \\
u_m \end{bmatrix} + (I_m \otimes D^{(4)}) \begin{bmatrix} u_1 \\
\vdots \\
u_m \end{bmatrix} = \begin{bmatrix} f_1 \\
\vdots \\
f_m \end{bmatrix},
\tag{14}
\]
where \( u_j = u_j(y_k), f_j = f_j(y_k), j = 1, 2, \ldots, m; k = 1, 2, \ldots, n \) and \( I_m, I_n \) are identity matrices of \( m, n \), then we have

\[
(C^{(4)} \otimes I_n + 2C^{(2)} \otimes D^{(2)} + I_m \otimes D^{(4)})U = F
\]

and

\[
LU = F,
\]

where

\[
L = C^{(4)} \otimes I_n + 2C^{(2)} \otimes D^{(2)} + I_m \otimes D^{(4)}
\]

and \( U = [u_1^T, u_2^T, \ldots, u_m^T]^T = [u_{11}, \ldots, u_{1n}, u_{21}, \ldots, u_{2n}, \ldots, u_{mn}]^T, F = [f_1^T, f_2^T, \ldots, f_m^T]^T = [f_{11}, f_{12}, \ldots, f_{1n}, f_{21}, \ldots, f_{2n}, \ldots, f_{mn}]^T,\)

\( j = 1, 2, \ldots, m, k = 1, 2, \ldots, n, \) and \( \otimes \) is the Kronecker product of matrix.

For the boundary condition of (2), \( u(x, y) = g(x, y), \frac{\partial u(x, y)}{\partial x} + \frac{\partial u(x, y)}{\partial y} = h(x, y), \) we obtain the discrete formulae as

\[
\begin{align*}
\sum_{j=1}^{n} \sum_{k=1}^{m} C_{n, j}^{(1)} \delta_k u_{ij} = h(y_k), k = 1, 2, \ldots, n, \text{ and } u_{(k-1)n+1, j} = g(x_k), \sum_{j=1}^{n} \sum_{k=1}^{m} \delta_k D_{n, j}^{(1)} u_{ij} = h(x_k), k = 1, 2, \ldots, m.
\end{align*}
\]

Now we have finished the barycentric rational discrete form of the biharmonic equation, and the matrix equation of the biharmonic equation is also obtained. Some remarks of the Kronecker product and how to choose collocation points are given in the following.

Second, the equidistant node and the second kind of Chebyshev point are chosen as the collocation point. The equidistant node is

\[
x_j = a + \frac{b - a}{n - 1}(j - 1), j = 1, \ldots, n,
\]

and its weight function is

\[
w_j = (-1)^{n-j-1}C_{n-1}^{j-1},
\]

where \( C_{n-1}^{j-1} = \frac{(n-1)!}{(n-j-1)!j!(n-j)!}. \)

And the second kind of Chebyshev point is

\[
x_j = \cos \left( \frac{(j-1)\pi}{n-1} \right), j = 1, \ldots, n
\]

and its weight function is

\[
w_j = (-1)^{j}\delta_j, \quad \delta_j = \begin{cases} 1/2, & j = 1, n \\ 1, & \text{otherwise.} \end{cases}
\]

3 Convergence and error analysis

For one-dimensional function \( u(x) \) is approximated by rational function \( r(x) \). Its error functional is defined as

\[
e(x) = u(x) - r(x) = (x - x_i) \cdots (x - x_{i+d_0})u[x_i, x_{i+1}, \ldots, x_{i+d_0}, x],
\]

where \( d_0 \in [0, n], d_0 \in \mathbb{N} \), and

\[
e(x) = \frac{\sum_{i=1}^{n-1-d_0} \lambda_i(x)}{\sum_{i=1}^{n-1-d_0} \lambda_i(x)} \frac{A(x) - B(x)}{A(x)} = O\left( h^{d_0+1} \right),
\]
where
\[
A(x) = \sum_{i=1}^{n-1-d_0} (-1)^i u(x_i, \ldots, x_{i+d_0}, x) \tag{22}
\]
and
\[
B(x) = \sum_{i=1}^{n-1-d_0} \lambda_i(x), \tag{23}
\]
where
\[
\lambda_i(x) = \frac{(-1)^i}{(x - x_j) \cdots (x - x_{i+d_0})}. \tag{24}
\]

The following theorem has been proved by Jean-Paul Berrut, see [13].

**Theorem 3.1.** If \( n, d_0, d_0 \leq n - 1 \), and \( k \leq d_0 \) are positive integers and \( u(x) \in C^{d_0+1}[a, b] \). If the nodes \( x_i, i = 1, \ldots, n \), are equidistant or quasi-equidistant, then we have
\[
|e^{(k)}(x_i)| \leq Ch^{d_0+1-k}.
\]

For the barycentric rational interpolation of function \( u(x, y) \) by \( u_{n,m}(x, y) \), we can obtain the barycentric rational interpolation
\[
u_{n,m}(x, y) = \sum_{j=1}^{m} \sum_{k=1}^{n} \frac{w_{j,k}}{k(x - x_j(y - y_k))} u_{j,k}, \tag{25}\]
where \( u_{j,k} = u(x_j, y_k), d_1 \in [0, m-1], d_2 \in [0, n-1], \) and
\[
w_{j,k} = (-1)^{j-d_{1-k}d_{1-d_2}} \sum_{k=1}^{d_1} \prod_{h=1}^{k} \frac{1}{|x_j - x_h|} \prod_{h=1}^{k} \frac{1}{|y_k - y_h|}, \tag{26}\]
and \( j_1 = \{ k_1 : j_1-d_1 \leq k_1 \leq j_1 \}, I_m = \{0, \ldots, m-1-d_1 \}, J_k = \{ k_2 \in I_n : k - d_2 \leq k_2 \leq k \}, I_n = \{0, \ldots, n-1-d_2 \}. \)

By the error term of barycentric rational interpolation for two-dimensional function, we have
\[
e(x, y) = u(x, y) - u_{n,m}(x, y) = (x - x_j) \cdots (x - x_{j+d_1}) u(x_j, x_{j+1}, \ldots, x_{j+d_1}, x) + (y - y_k) \cdots (y - y_{k+d_2}) u(y_k, y_{k+1}, \ldots, y_{k+d_2}, y). \tag{27}\]

The following theorem can be proved similarly as Li and Cheng [11].

**Theorem 3.2.** For the \( e(x, y) \) defined in (27) and \( u(x, y) \in C^{d_1+1}[a, b] \times C^{d_2+1}[c, d] \), we have
\[
|e(x, y)| \leq C(h^{d_1+1} + r^{d_2+1}). \tag{28}\]

**Proof.** For the function \( u(x, y) \) with \( w_{j,k}(x, y) \) defined as (26), the error functional can be expressed as (see Theorem 1 [24])
\[
u(x, y) - u_{n,m}(x, y) = \frac{\sum_{j=1}^{m-1-d_0} \sum_{k=1}^{n-1-d_0} \lambda_j(x) \lambda_k(y) [u(x, y) - u_{n,m}(x, y)]}{\sum_{j=1}^{m-1-d_0} \sum_{k=1}^{n-1-d_0} \lambda_j(x) \lambda_k(y)}. \tag{29}\]

By the error formulae of barycentric rational interpolation
\[
u(x, y) - u_{n,m}(x, y) = u(x, y) - u_{n,m}(x_i, y) + u_{n,m}(x_i, y) - u_{n,m}(x, y) = (x - x_j) \cdots (x - x_{j+d_1}) u(x_j, x_{j+1}, \ldots, x_{j+d_1}, x, y) + (y - y_k) \cdots (y - y_{k+d_2}) u(y_k, y_{k+1}, \ldots, y_{k+d_2}, x, y). \tag{30}\]
We have
\[ u(x, y) - u_{n,m}(x, y) = \frac{\sum_{j=1}^{m-1-d_1}(-1)^j u[x_j, x_{j+1}, \ldots, x_{j+d_1}, x, y]}{\sum_{j=1}^{m-1-d_1} \lambda_j(x)} + \frac{\sum_{k=1}^{n-1-d_2}(-1)^k u[y_k, y_{k+1}, \ldots, y_{k+d_2}, x, y]}{\sum_{k=1}^{n-1-d_2} \lambda_k(y)}. \] (31)

By the similar analysis in Floater and Hormann [12], we have
\[ \left| \sum_{j=1}^{m-1-d_1} \lambda_j(x) \right| \geq \frac{1}{d_1! \tau^{d_1}}, \] (32)
and
\[ \left| \sum_{k=1}^{n-1-d_2} \lambda_k(y) \right| \geq \frac{1}{d_2! \tau^{d_2}}. \] (33)

Combining (31), (32), and (33) together, the proof of Theorem 3.2 is completed. □

**Corollary 3.3.** For the \(\varepsilon(x, y)\) defined in (27), we have
\[
\begin{align*}
|\varepsilon_x(x, y)| & \leq C(h^{d_1} + \tau^{d_1}), \quad u(x, y) \in C^{d_1}[a, b] \times C^{d_1}[c, d], \\
|\varepsilon_y(x, y)| & \leq C(h^{d_2} + \tau^{d_2}), \quad u(x, y) \in C^{d_2}[a, b] \times C^{d_2}[c, d], \\
|\varepsilon_{xx}(x, y)| & \leq C(h^{d_1} + \tau^{d_1}), \quad u(x, y) \in C^{d_1}[a, b] \times C^{d_1}[c, d], \quad d_1 \geq 2, \\
|\varepsilon_{yy}(x, y)| & \leq C(h^{d_2} + \tau^{d_2}), \quad u(x, y) \in C^{d_2}[a, b] \times C^{d_2}[c, d], \quad d_2 \geq 2, \\
|\varepsilon_{xxx}(x, y)| & \leq C(h^{d_1} + \tau^{d_1}), \quad u(x, y) \in C^{d_1}[a, b] \times C^{d_1}[c, d], \quad d_1 \geq 4, \\
|\varepsilon_{yyy}(x, y)| & \leq C(h^{d_2} + \tau^{d_2}), \quad u(x, y) \in C^{d_2}[a, b] \times C^{d_2}[c, d], \quad d_2 \geq 4.
\end{align*}
\] (34)

This corollary can be obtained similarly as Theorem 3.2, here we omit it.

Combining (8) and (1), we have
\[ T\varepsilon(x, y) = e_{xxxx}(x, y) + 2e_{xxyy}(x, y) + e_{yyyy}(x, y). \] (35)

In the following theorem, the main result is presented.

**Theorem 3.4.** Let
\[ \nabla^4 u(x, y) = \nabla^2 \nabla^2 u(x, y) = \frac{\partial^4 u(x, y)}{\partial x^4} + 2 \frac{\partial^4 u(x, y)}{\partial x^2 \partial y^2} + \frac{\partial^4 u(x, y)}{\partial y^4} = f(x, y), (x, y) \in \Omega \] (36)
and
\[ u(x, y) = g(x, y), \quad \frac{\partial u(x, y)}{\partial x} + \frac{\partial u(x, y)}{\partial y} = h_1(x, y), \quad (x, y) \in \partial(\Omega), \] (37)
where \(\Omega = [a, b] \times [c, d]\) and \(f(x, y), g(x, y), h(x, y)\) are consistent on \(\partial \Omega\). Then we have
\[ \max_{\Omega_y} |u(x, y) - u_{n,m}(x, y)| \leq C(h^{d_3} + \tau^{d_3}), \] (38)
where \(u_{n,m}(x, y)\) is defined as (25) and \(u(x, y) \in C^{d_3}[a, b] \times C^{d_3}[c, d], d_1 \geq 3, d_2 \geq 3\).

**Proof.** As \(L = C^{(4)} \otimes I_n + 2C^{(2)} \otimes D^{(2)} + I_m \otimes D^{(2)}\) and \(u(x, y) = \sum_{j=1}^{m} L_f(x) \sum_{k=1}^{n} L_k(y) u_{ik} = \sum_{j=1}^{m} \sum_{k=1}^{n} L_f(x) L_k(y) u_{ik}\). By
\[ U - U_{n,m} = U - L^{-1} F = L^{-1} (LU - F) = L^{-1} T\varepsilon(x, y), \]
where we have used $LU_{n,m} = F$, we assume that matrix $L$ is the inverse matrix which means
\begin{equation}
    u(x, y) - u_{n,m}(x, y) = \sum_{j=1}^{mn} M_{jk}(x, y) Te(x, y),
\end{equation}
where matrix $L$ is the inverse matrix and $M_{jk}(x, y)$ is the element of matrix $L^{-1}$.

By the definition of $Te(x, y)$ and
\begin{equation}
    \nabla^4 u(x, y) - \nabla^4 u_{n,m}(x, y) = \frac{\partial^4 u(x, y)}{\partial x^4} + 2 \frac{\partial^4 u(x, y)}{\partial x^2 \partial y^2} + \frac{\partial^4 u(x, y)}{\partial y^4} - \left[ \frac{\partial^4 u(x_m, y_n)}{\partial x^4} + 2 \frac{\partial^4 u(x_m, y_n)}{\partial x^2 \partial y^2} + \frac{\partial^4 u(x_m, y_n)}{\partial y^4} \right]
\end{equation}
\begin{equation}
    = \frac{\partial^4 u(x, y)}{\partial x^4} - \frac{\partial^4 u(x_m, y_n)}{\partial x^4} + 2 \frac{\partial^4 u(x, y)}{\partial x^2 \partial y^2} - \frac{\partial^4 u(x_m, y_n)}{\partial x^2 \partial y^2} + \frac{\partial^4 u(x, y)}{\partial y^4} - \frac{\partial^4 u(x_m, y_n)}{\partial y^4}
\end{equation}
\begin{equation}
    = R_1(x, y) + R_2(x, y) + R_3(x, y),
\end{equation}
we have
\begin{align*}
    R_1(x, y) &= e_{xxxx}(x, y) = u_{xxxx}(x, y) - u_{xxxx}(x_m, y_n), \\
    R_2(x, y) &= e_{yyyy}(x, y) = u_{yyyy}(x, y) - u_{yyyy}(x_m, y_n), \\
    R_3(x, y) &= 2e_{xyy}(x, y) = 2u_{xyy}(x, y) - 2u_{xyy}(x_m, y_n).
\end{align*}

Then, for $R_3(x, y)$ we have
\begin{align*}
    R_1(x, y) &= u_{xxxx}(x, y) - u_{xxxx}(x_m, y_n) = u_{xxxx}(x, y) - u_{xxxx}(x_m, y_n) + u_{xxxx}(x_m, y_n) - u_{xxxx}(x_m, y_n) \\
    &= \sum_{j=1}^{m-1-d_x} (-1)^{j} u_{xxxx}(x_{j}, x_{j+1}, \ldots, x_{j+d_x}, y) \\
    &+ \sum_{k=1}^{n-1-d_y} (-1)^{k} u_{yyyy}(y_{k}, y_{k+1}, \ldots, y_{k+d_y}, x_m, y_n) \\
    &= e_{xxxx}(x, y_n) + e_{xxxx}(x_m, y_n),
\end{align*}
where
\begin{equation}
    |R_1(x, y)| \leq |e_{xxxx}(x, y_n) + e_{xxxx}(x_m, y_n)| \leq C(h^{d_x-3} + \tau^{d_x+1}).
\end{equation}

Similarly, for $R_3(x, y)$ we have
\begin{equation}
    R_2(x, y) = u_{yyyy}(x, y) - u_{yyyy}(x_m, y_n) = e_{yyyy}(x, y_n) + e_{yyyy}(x_m, y_n)
\end{equation}
and
\begin{equation}
    |R_2(x, y)| \leq |e_{yyyy}(x, y_n) + e_{yyyy}(x_m, y_n)| \leq C(h^{d_y-1} + \tau^{d_y-3}).
\end{equation}
Similarly, for $R_3(x, y)$ we have
\begin{equation}
    |R_3(x, y)| \leq |e_{xyy}(x, y_n) + e_{xyy}(x_m, y_n)| \leq C(h^{d_x-3} + \tau^{d_x-3}).
\end{equation}
Combining (41), (43), and (44), the proof of Theorem 3.4 is completed. \qed
4 Numerical examples

In the following section, we present some examples to illustrate our theorem analysis. The boundary condition has been considered by replacing the first, last row and column of discrete equation by the replacement method.

**Example 4.1.** Consider the biharmonic equation with \( \Omega = [0, 1] \times [0, 1] \) and \( f(x, y) = (x^4 + y^4 + 2x^2y^2 + 8xy + 4)e^{xy} \), with the boundary condition

\[
u(0, y) = u(1, y) = u(x, 0) = u(x, 1) = 0
\]

and

\[
u_x(0, y) = y, \quad u_x(1, y) = ye^y, \quad u_y(x, 0) = x, \quad u_y(x, 1) = xe^x.
\]

Its analytical solution is

\[
u(x, y) = e^{xy}.
\]

In this example, we test the linear barycentric rational collocation method with the equidistant nodes, and Table 1 shows that the convergence rate is \( O(h^{d_1}) \) with \( d_1 = d_2 = 3, 4, 5 \). In Table 2, for the Chebyshev nodes, the convergence rate of times is \( O(h^{d_2}) \) with \( d_1 = d_2 = 3, 4, 5 \), which agree with our theorem analysis. For \( d_1 = d_2 = 2 \), there are convergence rates \( O(h) \) and \( O(h^{1.5}) \) in Tables 1 and 2, which is out of our goal of this article.

We choose \( m = 20; n = 20; d_1 = d_2 = 10 \); to test our algorithm. Figure 1 shows the error estimate of equidistant nodes with barycentric Lagrange interpolation collocation method, and Figure 2 shows the error estimate of Chebyshev nodes with barycentric Lagrange interpolation collocation method.

Figure 3 shows the error estimate of barycentric rational Lagrange interpolation collocation method with equidistant nodes, and Figure 4 shows the barycentric rational interpolation collocation method of the error estimate of Chebyshev nodes. From Figures 3 and 4, we know that the barycentric rational interpolation collocation method has higher accuracy under the condition of Chebyshev nodes.

Table 3 shows condition number of equidistant nodes with different \( d_1 \) and \( d_2 \). In Table 4, condition number of Chebyshev nodes with different \( d_1 \) and \( d_2 \) is shown.

| Table 1: Convergence rate of equidistant nodes with different \( d_1 \) and \( d_2 \) by barycentric rational interpolation |
|---|---|---|---|---|
| \( m \times n \) | \( d_1 = d_2 = 2 \) | \( d_1 = d_2 = 3 \) | \( d_1 = d_2 = 4 \) | \( d_1 = d_2 = 5 \) |
| 8 \times 8 | 1.5128 \times 10^{-4} | 4.7103 \times 10^{-5} | 7.2316 \times 10^{-7} | 3.7284 \times 10^{-7} |
| 16 \times 16 | 5.8896 \times 10^{-5} | 1.3610 | 2.3196 | 1.1067 \times 10^{-7} | 2.7081 | 2.2959 \times 10^{-8} | 4.0215 |
| 32 \times 32 | 2.4460 \times 10^{-5} | 1.2677 | 1.3426 \times 10^{-6} | 2.8131 | 1.3944 \times 10^{-7} | — | 1.4910 \times 10^{-7} | — |
| 64 \times 64 | 9.5489 \times 10^{-6} | 1.3570 | 8.2261 \times 10^{-6} | — | 6.7989 \times 10^{-6} | — | 6.1683 \times 10^{-6} | — |

| Table 2: Convergence rate of Chebyshev nodes with different \( d_1 \) and \( d_2 \) by barycentric rational interpolation |
|---|---|---|---|---|
| \( m \times n \) | \( d_1 = d_2 = 2 \) | \( d_1 = d_2 = 3 \) | \( d_1 = d_2 = 4 \) | \( d_1 = d_2 = 5 \) |
| 8 \times 8 | 1.8264 \times 10^{-4} | 6.7627 \times 10^{-5} | 3.5778 \times 10^{-7} | 1.0981 \times 10^{-7} |
| 16 \times 16 | 9.0841 \times 10^{-5} | 1.0076 | 4.3131 | 3.7635 \times 10^{-8} | 3.2489 | 1.1783 \times 10^{-8} | 3.2202 |
| 32 \times 32 | 3.1179 \times 10^{-5} | 1.5428 | 6.4577 \times 10^{-6} | — | 1.0106 \times 10^{-5} | — | 8.8547 \times 10^{-5} | — |
| 64 \times 64 | 2.6024 \times 10^{-3} | — | 3.4060 \times 10^{-3} | — | 9.3608 \times 10^{-2} | — | 6.9634 \times 10^{-1} | — |
Figure 1: Error estimate of equidistant nodes with Lagrange interpolation $m = 20; n = 20; d_1 = d_2 = 10$.

Figure 2: Error estimate of Chebyshev nodes with Lagrange interpolation $m = 20; n = 20; d_1 = d_2 = 10$.

Figure 3: Error estimate of equidistant nodes with barycentric rational interpolation $m = 20; n = 20; d_1 = d_2 = 10$. 
Example 4.2. Consider the biharmonic equation with \( \Omega = [-1, 1] \times [-1, 1] \) and \( f(x, y) = 0 \), with the boundary condition
\[
 u(-1, y) = u(1, y) = u(x, -1) = u(x, 1) = 0 
\]
and
\[
 u_x(-1, y) = \frac{1}{2} (\sin(-1) \cosh(y) - \cos(1) \sinh(y)) + \frac{1}{2} (\cos(1) \cosh(y) + \sin(-1) \sinh(y)),
\]
\[
 u_y(y, 1) = \frac{1}{2} (\sin(1) \cosh(y) - \cos(1) \sinh(y)) + \frac{1}{2} (\cos(1) \cosh(y) + \sin(1) \sinh(y)),
\]
\[
 u_y(x, -1) = \frac{x}{2} (\sin(x) \sinh(-1) - \cos(x) \cosh(-1))
\]
\[
 u_y(x, 1) = \frac{x}{2} (\sin(x) \sinh(1) - \cos(x) \cosh(1)).
\]
Its analytical solution is
\[
 u(x, y) = \frac{x}{2} (\sin(x) \cosh(y) - \cos(x) \sinh(y)).
\]

In this example, we test the linear barycentric rational collocation method with the equidistant nodes, Table 5 shows that the convergence rate is \( O(h^3) \) with \( d_1 = d_2 = 3, 4, 5 \). In Table 6, for the Chebyshev nodes, the convergence rate of times is \( O(\tau^d) \) with \( d_1 = d_2 = 3, 4, 5 \), which agrees with our theorem analysis. For \( d_1 = d_2 = 2 \), there are the convergence rates \( O(h^{1.5}) \) and \( O(h^2) \) in Tables 5 and 6, which is out of our goal of this article.

We choose \( m = 10; n = 10; d_1 = d_2 = 8 \); to test our algorithm. Figure 5 shows the error estimate of equidistant nodes with barycentric Lagrange interpolation collocation method, and Figure 6 shows the error estimate of Chebyshev nodes with barycentric Lagrange interpolation collocation method.

Figure 7 shows the error estimate of equidistant nodes with rational barycentric rational interpolation collocation method, and Figure 8 shows the error estimate of Chebyshev nodes. From Figures 7 and 8, we know that the barycentric rational interpolation collocation method has higher accuracy under the condition of Chebyshev nodes.

Example 4.3. Consider the biharmonic equation with the \( \Omega = [0, 1] \times [0, 1] \) and \( f(x, y) = 384\pi^4 - 640\pi^4 \cos(2ny)^2 - 640\pi^4 \cos(2nx)^2 + 1024 \cos(2nx)^2 \pi^4 \cos(2ny)^2 \), with the boundary condition
\[
 u(0, y) = u(1, y) = u(x, 0) = u(x, 1) = 0
\]
Table 3: Condition number of equidistant nodes with different $d_1$ and $d_2$ by barycentric rational interpolation

| $m \times n$ | $d_1 = d_2 = 2$ | $d_1 = d_2 = 3$ | $d_1 = d_2 = 4$ | $d_1 = d_2 = 5$ |
|--------------|-----------------|-----------------|-----------------|-----------------|
| 8 $\times$ 8 | $1.5128 \times 10^{-4}$ | $4.7103 \times 10^{-5}$ | $8.8260 \times 10^{-5}$ | $7.2316 \times 10^{-7}$ |
| 16 $\times$ 16 | $5.8896 \times 10^{-5}$ | $9.4358 \times 10^{-6}$ | $4.3390 \times 10^{-7}$ | $1.0167 \times 10^{-7}$ |
| 32 $\times$ 32 | $2.4460 \times 10^{-5}$ | $1.3426 \times 10^{-6}$ | $1.2153 \times 10^{-9}$ | $1.3944 \times 10^{-7}$ |
| 64 $\times$ 64 | $9.5489 \times 10^{-6}$ | $8.2261 \times 10^{-6}$ | $3.0609 \times 10^{-10}$ | $6.7989 \times 10^{-6}$ |

Table 4: Condition number of Chebyshev nodes with different $d_1$ and $d_2$ by barycentric rational interpolation

| $m \times n$ | $d_1 = d_2 = 2$ | $d_1 = d_2 = 3$ | $d_1 = d_2 = 4$ | $d_1 = d_2 = 5$ |
|--------------|-----------------|-----------------|-----------------|-----------------|
| 8 $\times$ 8 | $1.8264 \times 10^{-4}$ | $6.7627 \times 10^{-5}$ | $4.9504 \times 10^{-5}$ | $3.5778 \times 10^{-7}$ |
| 16 $\times$ 16 | $9.0841 \times 10^{-5}$ | $3.4022 \times 10^{-6}$ | $6.0817 \times 10^{-7}$ | $3.7635 \times 10^{-8}$ |
| 32 $\times$ 32 | $3.1179 \times 10^{-5}$ | $6.4577 \times 10^{-6}$ | $2.6232 \times 10^{-10}$ | $3.6369 \times 10^{-11}$ |
| 64 $\times$ 64 | $2.6024 \times 10^{-3}$ | $3.4060 \times 10^{-13}$ | $3.5042 \times 10^{13}$ | $9.3608 \times 10^{-2}$ |
Table 5: Convergence rate of equidistant nodes with different \( d_1 \) and \( d_2 \) by barycentric rational interpolation

| \( m \times n \) | \( d_1 = d_2 = 2 \) | \( d_1 = d_2 = 3 \) | \( d_1 = d_2 = 4 \) | \( d_1 = d_2 = 5 \) |
|---------------|-----------------|-----------------|-----------------|-----------------|
| \( 8 \times 8 \) | \( 1.1279 \times 10^{-2} \) | \( 4.3323 \times 10^{-3} \) | \( 1.0184 \times 10^{-3} \) | \( 5.8208 \times 10^{-4} \) |
| \( 16 \times 16 \) | \( 4.8796 \times 10^{-3} \) | \( 1.2088 \times 10^{-3} \) | \( 8.6515 \times 10^{-5} \) | \( 1.4807 \times 10^{-5} \) |
| \( 32 \times 32 \) | \( 1.8411 \times 10^{-3} \) | \( 1.4062 \times 10^{-3} \) | \( 3.1989 \times 10^{-4} \) | \( 3.8031 \times 10^{-7} \) |
| \( 64 \times 64 \) | \( 6.2066 \times 10^{-4} \) | \( 1.5687 \times 10^{-4} \) | \( 3.1247 \times 10^{-7} \) | \( 3.5572 \times 10^{-7} \) |

Table 6: Convergence rate of Chebyshev nodes with different \( d_1 \) and \( d_2 \) by barycentric rational interpolation

| \( m \times n \) | \( d_1 = d_2 = 2 \) | \( d_1 = d_2 = 3 \) | \( d_1 = d_2 = 4 \) | \( d_1 = d_2 = 5 \) |
|---------------|-----------------|-----------------|-----------------|-----------------|
| \( 8 \times 8 \) | \( 1.0182 \times 10^{-2} \) | \( 6.0338 \times 10^{-3} \) | \( 5.2497 \times 10^{-4} \) | \( 9.9332 \times 10^{-5} \) |
| \( 16 \times 16 \) | \( 7.7462 \times 10^{-3} \) | \( 1.6321 \times 10^{-4} \) | \( 2.5083 \times 10^{-5} \) | \( 3.8030 \times 10^{-6} \) |
| \( 32 \times 32 \) | \( 2.3755 \times 10^{-3} \) | \( 1.7053 \times 10^{-5} \) | \( 3.0277 \times 10^{-7} \) | \( 3.7729 \times 10^{-7} \) |
| \( 64 \times 64 \) | \( 6.4680 \times 10^{-4} \) | \( 1.8768 \times 10^{-5} \) | \( 2.5854 \times 10^{-7} \) | \( 2.3854 \times 10^{-2} \) |

Figure 5: Error estimate of equidistant nodes with Lagrange interpolation \( m = 10; n = 10; d_1 = d_2 = 8 \).

Figure 6: Error estimate of Chebyshev nodes with Lagrange interpolation \( m = 10; n = 10; d_1 = d_2 = 8 \).
Its analytical solution is
\[ u(x, y) = (\sin(2\pi x))^2(\sin(2\pi y))^2. \]

In this example, we test the linear barycentric rational with the equidistant nodes, and Table 7 shows the convergence rate is \( O(h^d) \) with \( d_1 = d_2 = 2, 3, 4, 5 \). In Table 8, for the Chebyshev nodes, the convergence rate of times is \( O(\tau^d) \) with \( d_1 = d_2 = 2, 3, 4, 5 \), which agree with our theorem analysis.

We choose \( m = 20; n = 20; d_1 = d_2 = 8 \) to test our algorithm. Figure 9 shows the error estimate of equidistant nodes with barycentric Lagrange interpolation collocation method, and Figure 10 shows the error estimate of Chebyshev nodes with barycentric Lagrange interpolation collocation method. Figure 11 shows the error estimate of equidistant nodes with rational barycentric rational interpolation collocation method, and Figure 12 shows the error estimate of Chebyshev nodes. From Figures 11 and 12, we know that the barycentric rational interpolation collocation method has higher accuracy under the condition of Chebyshev nodes.
Table 7: Convergence rate of equidistant nodes with different \(d_1\) and \(d_2\) by barycentric rational interpolation

| \(m \times n\) | \(d_1 = d_2 = 2\) | \(d_1 = d_2 = 3\) | \(d_1 = d_2 = 4\) | \(d_1 = d_2 = 5\) |
|-----------------|-----------------|-----------------|-----------------|-----------------|
| 8 \(\times\) 8  | \(4.4289 \times 10^{-1}\) | \(4.1219 \times 10^{-1}\) | \(6.9430 \times 10^{-1}\) | \(3.6467 \times 10^{0}\) |
| 16 \(\times\) 16 | \(1.9631 \times 10^{-1}\) | \(9.3822 \times 10^{-2}\) | \(2.1353 \times 10^{-1}\) | \(3.6911 \times 10^{-2}\) |
| 32 \(\times\) 32 | \(6.7432 \times 10^{-2}\) | \(1.6326 \times 10^{-2}\) | \(2.5272 \times 10^{-3}\) | \(4.4577\) |
| 64 \(\times\) 64 | \(1.9433 \times 10^{-2}\) | \(2.1067 \times 10^{-3}\) | \(2.9541\) | \(7.9457 \times 10^{-4}\) |

Table 8: Convergence rate of Chebyshev nodes with different \(d_1\) and \(d_2\) by barycentric rational interpolation

| \(m \times n\) | \(d_1 = d_2 = 2\) | \(d_1 = d_2 = 3\) | \(d_1 = d_2 = 4\) | \(d_1 = d_2 = 5\) |
|-----------------|-----------------|-----------------|-----------------|-----------------|
| 8 \(\times\) 8  | \(1.6656 \times 10^{0}\) | \(1.5281 \times 10^{0}\) | \(2.0820 \times 10^{-1}\) | \(4.2919 \times 10^{1}\) |
| 16 \(\times\) 16 | \(2.6816 \times 10^{-1}\) | \(3.9003 \times 10^{-2}\) | \(5.2920\) | \(7.9025 \times 10^{-3}\) |
| 32 \(\times\) 32 | \(8.6146 \times 10^{-2}\) | \(6.7989 \times 10^{-3}\) | \(2.5202\) | \(2.9583 \times 10^{-5}\) |
| 64 \(\times\) 64 | \(2.2906 \times 10^{-2}\) | \(4.6030 \times 10^{-4}\) | \(3.8847\) | \(3.1813 \times 10^{-4}\) |

Figure 9: Error estimate of equidistant nodes with Lagrange interpolation \(m = 10; n = 10; d_1 = d_2 = 8\).

Figure 10: Error estimate of Chebyshev nodes with Lagrange interpolation \(m = 20; n = 20; d_1 = d_2 = 8\).
5 Conclusion

In this article, we have presented the linear barycentric collocation methods to solve the two-dimensional elliptic boundary value problems. With the help of error estimation of error functional, the convergence rate of the biharmonic equation is proved with the constant coefficient, we have presented the convergence rate $O(h^{d_1 - 3} + \tau^{d_2 - 3})$, while the numerical examples show that the convergence rate is $O(h^{d_1 - 1} + \tau^{d_2 - 1})$ for the equidistant nodes and $O(h^{d_1} + \tau^{d_2})$ for the Chebyshev nodes with barycentric collocation methods. Particularly for the case $d_1 = d_2 = 2$, there are still the convergence rate which can reach $O(h)$ or even $O(h^2)$, this is an interesting phenomenon which will be investigated in future works.

Acknowledgments: The author gratefully acknowledges the helpful comments and suggestions of the reviewers, which have improved the presentation.
Funding information: The work of Jin Li was supported by Natural Science Foundation of Shandong Province (Grant No. ZR2016JL006), Natural Science Foundation of Hebei Province (Grant No. A2019209533), National Natural Science Foundation of China (Grant Nos 11471195 and 11771398), and China Postdoctoral Science Foundation (Grant No. 2015T80703).

Conflict of interest: The author declares that he has no conflicts of interest.

Data availability statement: The data that support the findings of this study are available from the corresponding author upon reasonable request.

References

[1] A. Cardone, D. Conte, R. D’Ambrosio, and B. Paternoster, Multivalued collocation methods for ordinary and fractional differential equations, Mathematics 10 (2022), no. 2, 185, DOI: https://doi.org/10.3390/math10020185.

[2] J. Shen, T. Tang, and L. Wang, Spectral Methods: Algorithms, Analysis and Applications, Springer, New York, 2011.

[3] J. P. Berrut, S. A. Hosseini, and G. Klein, The linear barycentric rational quadrature method for Volterra integral equations, SIAM J. Sci. Comput. 36 (2014), no. 1, 105–123, DOI: https://doi.org/10.1137/120904020.

[4] P. Berrut and G. Klein, Recent advances in linear barycentric rational interpolation, J. Comput. Appl. Math. 259 (2014), 95–107, DOI: https://doi.org/10.1016/j.cam.2013.03.044.

[5] E. Cirillo and K. Hormann, On the Lebesgue constant of barycentric rational Hermite interpolants at equidistant nodes, J. Comput. Appl. Math. 349 (2019), 292–301, DOI: https://doi.org/10.1016/j.cam.2018.06.011.

[6] A. Abdi, J. P. Berrut, and S. A. Hosseini, The linear barycentric rational method for a class of delay Volterra integro-differential equations, J. Sci. Comput. 75 (2019), 1757–1775, DOI: https://doi.org/10.1007/s10915-017-0608-3.

[7] J. Li and Y. Cheng, Linear barycentric rational collocation method for solving second-order Volterra integro-differential equation, Comput. Appl. Math. 39 (2020), 92, DOI: https://doi.org/10.1007/s40314-020-1114-z.

[8] M. Li and C. Huang, The linear barycentric rational quadrature method for auto-convolution Volterra integral equations, J. Sci. Comput. 78 (2019), no. 1, 549–564, DOI: https://doi.org/10.1007/s10915-018-0779-6.

[9] J. Y. Lee and L. Greengard, A fast adaptive numerical method for stiff two-point boundary value problems, SIAM J. Sci. Comput. 18 (1997), no. 2, 403–429, DOI: https://doi.org/10.1137/S1064827594272797.

[10] N. R. Bayramov and J. K. Kraus, On the stable solution of transient convection-diffusion equations, J. Comput. Appl. Math. 280 (2015), no. 1, 275–293, DOI: https://doi.org/10.1016/j.cam.2014.12.001.

[11] J. Li and Y. Cheng, Linear barycentric rational collocation method for solving heat conduction equation, Numer. Methods Partial Differ. Equ. 37 (2021), no. 1, 533–545, DOI: https://doi.org/10.1002/num.22539.

[12] M. S. Floater and K. Hormann, Barycentric rational interpolation with no poles and high rates of approximation, Numer. Math. 107 (2007), no. 2, 315–331, DOI: https://doi.org/10.1007/s00211-007-0493-y.

[13] J. P. Berrut, M. S. Floater, and G. Klein, Convergence rates of derivatives of a family of barycentric rational interpolants, Appl. Numer. Math. 61 (2011), no. 9, 989–1000, DOI: https://doi.org/10.1016/j.apnum.2011.05.001.

[14] G. Klein and J. P. Berrut, Linear rational finite differences from derivatives of barycentric rational interpolants, SIAM J. Numer. Anal. 50 (2012), no. 2, 643–656, DOI: https://doi.org/10.1137/110827156.

[15] G. Klein and J. P. Berrut, Linear barycentric rational quadrature, BIT Numer. Math. 52 (2012), 407–424, DOI: https://doi.org/10.1007/s10543-011-0357-x.

[16] S. Li and Z. Wang, High Precision Meshless barycentric Interpolation Collocation Method-Algorithmic Program and Engineering Application, Science Publishing, Beijing, 2012.

[17] Z. Wang and S. Li, Barycentric Interpolation Collocation Method for Nonlinear Problems, National Defense Industry Press, Beijing, 2015.

[18] Z. Wang, Z. Xu, and J. Li, Mixed barycentric interpolation collocation method of displacement-pressure for incompressible plane elastic problems, Chinese J. Appl. Mech. 35 (2018), no. 3, 195–201.

[19] Z. Wang, L. Zhang, Z. Xu, and J. Li, Barycentric interpolation collocation method based on mixed displacement-stress formulation for solving plane elastic problems, Chinese J. Appl. Mech. 35 (2018), no. 2, 304–309.

[20] M. L. Zhuang, C. Q. Miao, and S. Y. Ji, Plane elasticity problems by barycentric rational interpolation collocation method and a regular domain method, Internat. J. Numer. Methods Engrg. 121 (2020), no. 18, 4134–4156, DOI: https://doi.org/10.1002/nme.6431.

[21] J. Li, X. Su, and J. Qu, Linear barycentric rational collocation method for solving telegraph equation, Math. Method. Appl. Sci. 44 (2021), no. 14, 11720–11737, DOI: https://doi.org/10.1002/mma.7548.
[22] J. Li and S. Yu, *Linear barycentric rational collocation method for Beam force vibration equation*, Shock. Vib. *2021* (2021), 5584274, DOI: https://doi.org/10.1155/2021/5584274.

[23] N. Mai-Duy and R. I. Tanner, *A spectral collocation method based on integrated Chebyshev polynomials for two-dimensional biharmonic boundary-value problems*, J. Comput. Appl. Math. *201* (2007), no. 1, 30–47, DOI: https://doi.org/10.1016/j.cam.2006.01.030.

[24] J. Li and Y. Cheng, *Barycentric rational method for solving biharmonic equation by depression of order*, Numer. Methods Partial Differ. Equ. *37* (2021), no. 3, 1993–2007, DOI: https://doi.org/10.1002/num.22638.