An Empirical Quantile Estimation Approach to Nonlinear Optimization Problems with Chance Constraints

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Abstract
We investigate an empirical quantile estimation approach to solve chance-constrained nonlinear optimization problems. Our approach is based on the reformulation of the chance constraint as an equivalent quantile constraint to provide stronger signals on the gradient. In this approach, the value of the quantile function is estimated empirically from samples drawn from the random parameters, and the gradient of the quantile function is estimated via a finite-difference approximation on top of the quantile-function-value estimation. We establish a convergence theory of this approach within the framework of an augmented Lagrangian method for solving general nonlinear constrained optimization problems. The foundation of the convergence analysis is a concentration property of the empirical quantile process, and the analysis is divided based on whether or not the quantile function is differentiable. In contrast to the sampling-and-smoothing approach used in the literature, the method developed in this paper does not involve any smoothing function and hence the quantile-function gradient approximation is easier to implement and there are less accuracy-control parameters to tune. We demonstrate the effectiveness of this approach and compare it with a smoothing method for the quantile-gradient estimation. Numerical investigation shows that the two approaches are competitive for certain problem instances.

1 Introduction
We investigate methods for solving nonlinear optimization problems with chance constraints:

\[
\minimize_{x \in S} f(x) \text{ s.t.: } \mathbb{P}[c_1(x, \xi) \leq 0] \geq 1 - \alpha, \quad c_2(x) \leq 0, \quad (CCP)
\]

where \( \xi : \Omega \rightarrow \mathbb{R}^{n_0} \) is a random parameter with associated probability space \((\Omega, \mathcal{F}, P)\), and \( f : \mathbb{R}^n \rightarrow \mathbb{R} \), \( c_1 : \mathbb{R}^n \times \mathbb{R}^{n_0} \rightarrow \mathbb{R}^l \), and \( c_2 : \mathbb{R}^n \rightarrow \mathbb{R}^l \) are three differentiable functions. The parameter \( \alpha \in (0, 1) \) is a risk-tolerance value. For example, \( \alpha = 0.05 \) indicating that the constraint \( c_1 \leq 0 \) is allowed to be violated with probability at most 0.05. The closure \( \overline{S} \) of an open set \( S \) is an implicit closed subset in \( \mathbb{R}^n \) for further definitions of regularity conditions, which can be assumed to be sufficiently large. The constraints in \( c_2(x) \) do not involve any random parameters. For the sake of clarity, we focus on analyzing the case of \( l_1 = 1 \), namely the case with a single disjoint chance constraint. It is straightforward to generalize the analysis and the algorithms presented in this work to deal multiple disjoint chance constraints. If there are joint chance constraints, our results can still be applicable provided regularity conditions hold on the constraint functions, which is discussed in Section 2.1. In this paper \( \| \cdot \| \) denotes the \( \ell_2 \) norm in the Euclidean space.

1.1 Literature review
Optimization with chance constraints was first investigated in [8] as a modeling option that requires the probability of an event (formulated by a set of constraints) to satisfy a certain threshold. The introducing of

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chance constraints extends the formulation power in data-driven decision-making problems. Imposing chance constraints, however, may not preserve convexity of the feasible region, and hence the chance-constrained programs (CCPs) are very difficult to solve in general if not computationally intractable. The reason is that a chance-constraint function is a compound random variable, and the constraint complexity is determined by the dependency of the cumulative distribution function of the random variable on the decision variables that parameterize it, which could be highly non-convex. As an important special case of (CCP), when the function $c_1(x, \xi)$ can be decoupled as $c_1^k(x) - \xi$, the probability function $P[c_1^k(x, \xi) \leq 0]$ can be quasi-concave under certain conditions (e.g. $\xi$ follows a joint normal distribution), and hence admitting efficient algorithms for the problem given that all deterministic functions are convex. The properties of this special case have been investigated in the seminal work [37]. The differentiability and the formula of derivatives of the probability function are investigated in [13, 20, 21, 36, 40], especially for the case that the random vector follows a multivariate normal distribution [15, 41].

In the case that the random parameters follow a general probability distribution, the finite sample approximation of the continuous (possibly unknown) probability distributions is often used as an empirical approach to solving such CCPs in this case. Luedtke and Ahmed [29] investigated the effectiveness of a sample-based approximation of chance constraints and the sample complexity for achieving certain probabilistic error bounds. Additional statistical properties of the sample-based approximation approach to CCPs have been studied [4–6, 33]. In the scheme of finite-sample approximation or for the case of scenario-based CCPs, reformulation and decomposition techniques have been developed to establish computationally tractable formulations of the CCPs with the help of discrete variables, when the scenario constraints are convex [25, 26, 28]. Valid inequalities have been derived to strengthen the mixed-integer linear program reformulation of a CCP with linear scenario constraints [22, 30, 44]. Chance constraints have been incorporated into a distributionally robust (DR) optimization framework when the information about the underline probability distribution is given through a finite set of samples, and tractable reformulations of the DR chance-constrained optimization have been investigated [18, 24, 43] under different families of divergence or distance functions, for example the $\phi$-divergence and the Wasserstein distance.

For the case of general nonlinear programs with chance constraints induced by general probability distributions, a variety of numerical methods based on sampling and smoothing have been developed to identify high-quality locally optimal solutions [7, 10, 14, 19, 34]. Campi and Garatti [6] proposed a sampling-and-discarding approach to select a finite set of samples to induce constraints that are required to be satisfied, and provided sample complexity results for satisfying the chance constraint with a given probability. Along this direction, a primal-dual stochastic gradient method developed in [45] can be applied to handle the approximated problem with the large number of constraints induced by samples. (In each iteration of such an approach, only a single constraint is randomly sampled to compute the gradient of the augmented Lagrangian.) Geletu et al. [14] proposed a smooth approximation scheme that approximates the chance-constraint function with (smooth) parametric lower and upper estimation functions and then solves the parametric approximation problems with nonlinear programming (NLP) solvers. Curtis et al. [10] developed a sequential algorithm for solving the sample approximation problem of a nonlinear CCP, in which the outer iteration updates the penalty parameter while the inner sub-problem is reformulated as a mixed binary quadratic program. Kannan and Luedtke [19] proposed an approach of constructing the efficient frontier (i.e., solving for optimal objectives as the risk value continuously changes) of nonlinear CCPs instead of solving the original chance-constrained problem with a pre-specified risk value. In their formulation, the original chance-constraint function is transformed to be the objective, and hence a projected stochastic sub-gradient algorithm [11] can be applied to solve the reformulated problem with smoothing. Note that the chance-constraint function has a universal range $[0, 1]$, and hence it can be flat in a certain domain, which can slow down the progress of a gradient-descent method. Motivated by this observation, Peña-Ordieres et al. [34] suggested recasting the chance constraint as a quantile constraint because the later could have high-magnitude gradients. In their work, smoothing is applied to the quantile function to achieve numerical stability. A CCP can sometimes be equivalently formulated as a quantile function optimization problem, and for this case, Hu et al. [17] proposed a recursive algorithm developed upon the gradient-based maximum likelihood estimation method [35] to estimate the gradient of a differentiable quantile function with respect to
the parameter and used it as an ingredient in a gradient-descent algorithm for minimizing the parameterized quantile function. As a special case, Tong et al. [39] developed conservative formulations for NLPs with rare chance constraints induced by Gaussian random parameters using tools from large deviation theory.

The algorithm developed in this paper also benefits from the augmented Lagrangian methods developed for deterministic nonlinear optimization with equality and inequality constraints [1–3]. In particular, we established a probabilistic augmented Lagrangian method to handle quantile constraint(s) that are only accessible based on samples.

1.2 Contributions

Inspired by [34], our approach reformulates the chance constraint into a quantile constraint. We develop a derivative-free approach to handle the evaluation of the quantile-function gradients. Specifically, we use a sample-approximation method to estimate the value of the quantile function based on a concentration theory of the empirical process. We also use a finite-difference approach to estimate the gradients of the quantile function, which are used to build local approximation models. Local models are used in an augmented Lagrangian method (ALM) to solve the NLP with quantile constraints and other deterministic nonlinear constraints, while the inner optimization problems (with given values of penalty parameters) are solved using a trust-region method. We have developed a convergence theory when the quantile function is differentiable.

2 Quantile constraint reformulation

Let $\Xi_x = c_1(x, \xi)$ be a random variable parameterized by $x$ and the random vector $\xi$. As noted in [34], the chance constraint $P[c_1(x, \xi) \leq 0] \geq 1 - \alpha$ is equivalent to

$$Q^{1-\alpha}(x) \leq 0,$$

where $Q^{1-\alpha}(x)$ is the $1 - \alpha$ quantile of $\Xi_x$. If the $1 - \alpha$ quantile of $\Xi_x$ is not unique, we set $Q^{1-\alpha}(x) = \inf_{q \in \mathcal{Q}^{1-\alpha}(x)} q$, where $Q^{1-\alpha}(x)$ is the set of all $1 - \alpha$ quantiles of $\Xi_x$. With the reformulation of the chance constraint, (CCP) can be reformulated as the quantile-constrained problem

$$\minimize_{x \in \mathbb{R}^n} f(x) \text{ s.t.: } Q^{1-\alpha}(x) \leq 0, \quad c_2(x) \leq 0.$$

(QCP)

We assume that the gradients of the functions $f(\cdot)$ and $c_2(\cdot)$ are accessible at any $x \in \mathbb{R}^n$, and i.i.d. samples of $\xi$ can be drawn as needed. Note that the essential difficulty of (QCP) is that the derivative of the quantile function $Q^{1-\alpha}(\cdot)$ is not available in general although $\nabla c_1$ is computable. Furthermore, a sampling method is needed in order to obtain zeroth-order information about (i.e., the value of) $Q^{1-\alpha}(\cdot)$. This paper develops an approach to find a stationary point of (QCP) using sample-based estimators of $Q^{1-\alpha}(\cdot)$ and $\nabla Q^{1-\alpha}(\cdot)$. These approximations are incorporated in an augmented Lagrangian method to solve the constrained problem (QCP).

We first focus on the case where $Q^{1-\alpha}(x)$ is continuously differentiable within a bounded domain (Assumptions 2.2 and 2.3). We also assume common regularity conditions (Assumption 2.1) hold for the functions $f$, $c_1$ and $c_2$.

**Assumption 2.1.** The objective $f$ and the constraint functions $c_2$ are continuously differentiable at any $x \in \mathbb{S}$, and their gradients are Lipschitz continuous. The corresponding Lipschitz constants are denoted as $L_f$ and $L_{c_2}$.

**Assumption 2.2.** For any $x \in \mathbb{S}$, the random variable $c_1(x, \xi)$ (or $\Xi_x$) has a continuously differentiable probability density function w.r.t. the Lebesgue measure. The inverse function $F_x^{-1}$ exists and it is nonzero in a neighborhood of $F_x^{-1}(1 - \alpha)$, where $F_x$ is the cumulative distribution function of $\Xi_x$ for any $x \in \mathbb{S}$.

**Assumption 2.3.** The quantile function $Q^{1-\alpha}(\cdot)$ is continuously differentiable at any $x \in \mathbb{R}^n$, and $\nabla Q^{1-\alpha}(\cdot)$ is Lipschitz continuous with the Lipschitz constant $L_Q$. 

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Assumption 2.4. Assume that for every $x \in \mathbb{R}$, the cumulative density function $F_{\xi_x}(\cdot)$ is twice differentiable in $(a_x, b_x)$, where $a_x = \sup \{ t : F_{\xi_x}(t) = 0 \}$ and $b_x = \inf \{ t : F_{\xi_x}(t) = 1 \}$.

Note that verifying the differentiability of a quantile function with respect to the decision variable can be difficult in practice. Therefore, Assumption 2.3 requires some justification: Sufficient conditions for Assumption 2.3 to hold are discussed in [16]. To ensure this paper is self-contained, these conditions are stated with the notation of this paper in Assumption 2.5. Assumption 2.3 is implied by Assumption 2.5, but the later may appear to be easier to verify.

Assumption 2.5. The disjoint chance constraint satisfies the following:

1. The sample-wise gradient $\nabla_x c_1(x, \xi)$ exists w.p.1 for any $x \in \mathbb{S}$, and there exists a function $k(\xi)$ with $\mathbb{E}[k(\xi)] < \infty$ such that $|c_1(x_1, \xi) - c_1(x_2, \xi)| \leq k(\xi) \|x_1 - x_2\|$ for all $x_1, x_2 \in \mathbb{S}$.

2. For any $x \in \mathbb{S}$, the random variable $c_1(x, \xi)$ has a continuous density function $\rho(t; x)$ and $\nabla_x F(t; x)$ exists and is continuous with respect to both $x$ and $t$, where $F(t; x)$ is the cumulative distribution function of $c_1(x, \xi)$.

3. For any $x \in \mathbb{S}$, the function $g(t; x) = \mathbb{E}[\nabla_x c_1(x, \xi)|c_1(x, \xi) = t]$ is continuous with respect to $t$.

We will denote the three items in Assumption 2.5 as Assumption 2.5(i) for $i = 1, 2, 3$, respectively from hereafter. We note that the algorithms in this paper might also be applied to some CCPs for which the above assumptions are not necessarily satisfied. These assumptions are for convergence analysis.

2.1 Differentiability of the quantile function of a joint chance constraint

For the case of having a joint chance constraint, it requires additional procedures to reformulate the chance constraint into a quantile constraint. In this case the definition of the quantile function should be adjusted. Specifically, consider the following joint chance constraint:

$$\mathbb{P}[\psi_i(x, \xi) \leq 0, \ i = 1, \ldots, l] \geq 1 - \alpha.$$  

(2)

We define the function $\psi$ as $\psi(x, \xi) = \max\{\psi_1(x, \xi), \ i = 1, \ldots, l\}$, which is a random variable parameterized by $x$. Ideally, the joint chance constraint (2) can be reformulated as a quantile constraint $Q_{\psi}^{1-\alpha}(x) \leq 0$, where $Q_{\psi}^{1-\alpha}(x)$ denotes the $1 - \alpha$ quantile of the random variable $\psi(x, \xi)$. The following proposition provides conditions under which the quantile function $Q_{\psi}^{1-\alpha}(\cdot)$ is differentiable and hence our convergence analysis and algorithms can be applied to joint chance constraints.

Proposition 2.1. If the following conditions hold

1. Each $\psi_i$ satisfies Assumption 2.5(1),

2. For any $x \in \mathbb{S}$, the random vector $[\psi_1(x, \xi), \ldots, \psi_l(x, \xi)]$ has a continuous joint density function $\rho(t_1, \ldots, t_l; x)$ and $\nabla_x F(t_1, \ldots, t_l; x)$ exists and is continuous with respect to $x$ and all $t_i$, where $F(t_1, \ldots, t_l; x)$ is the cumulative distribution function of $[\psi_1(x, \xi), \ldots, \psi_l(x, \xi)]$, and

3. For any $x \in \mathbb{S}$, the function $g_i(t_1, \ldots, t_l; x) = \mathbb{E}[\nabla_x \psi_i(x, \xi)|\psi_j(x, \xi) = t_j, \forall j]$ is continuous with respect to $t$ for every $i \in [l]$, where $[l] := \{1, \ldots, l\}$.

then the quantile function $Q_{\psi}^{1-\alpha}(x)$ is differentiable for all $x \in \mathbb{S}$.

Proof of Proposition 2.1. It suffices to show that $\psi$ satisfies the conditions in Assumption 2.5. First, we note that condition 2 implies $\mathbb{P}[\psi_i(x, \xi) = \psi_j(x, \xi)] = 0$ for all $i < j$. Notice that $\nabla_x \psi(x, \xi) = \nabla_x \psi_i(x, \xi)$ for $\xi$ in the region $\{ \xi : \psi_i(x, \xi) > \psi_j(x, \xi) \forall j \in [l] \setminus \{i\} \}$ for all $i \in [l]$, and since $\mathbb{P}[\psi_i(x, \xi) = \psi_j(x, \xi)] = 0$ for all $i < j$ (due to the continuity of the probability density function), the gradient $\nabla_x \psi(x, \xi)$ exists w.p.1. Furthermore, $|\psi(x_1, \xi) - \psi(x_2, \xi)| \leq \max_i \{k_i(\xi)\} \|x_1 - x_2\|$, where $k_i(\xi)$ is the function in Assumption 2.5(1)
Proposition 2.1. We see that Assumption 2.5 is satisfied by the function \( f \). This verifies that Assumption 2.5 is satisfied by \( f \).

\[ F(t; x) = \mathbb{P}[\psi(x, \xi) \leq t] = \mathbb{P}[\psi_1(x, \xi) \leq t, \ldots, \psi_l(x, \xi) \leq t] = F(t, \ldots, t; x). \] (3)

It follows that

\[
\frac{dF(t; x)}{dt} = \sum_{i=1}^{l} \partial_i F(t, \ldots, t; x)
\]

\[ = \sum_{i=1}^{l} \int_{-\infty}^{t} \cdots \int_{-\infty}^{t} \rho(t_1, \ldots, t_{i-1}, t, t_{i+1}, \ldots, t_l; x) dt_1 \cdots dt_{i-1} dt_{i+1} \cdots dt_l,
\]

which is clearly continuous in \( t \). The gradient \( \nabla_x F(t; x) = \nabla_x F(t, \ldots, t; x) \) which is continuous in \( x \) and \( t \) by condition 2. To verify that \( \psi \) satisfies Assumption 4.3, we notice that for any \( x \in S \), the function \( g(t; x) \) can be computed as

\[
g(t; x) = \mathbb{E}[
abla_x \psi(x, \xi) \big| \psi(x, \xi) = t]\]

\[ = \sum_{i=1}^{l} \mathbb{E}[\nabla_x \psi_i(x, \xi) \big| \psi_i(x, \xi) = t, \psi_i(x, \xi) \geq \psi_j(x, \xi) \forall j \in [t] \setminus \{i\}].
\] (5)

Note that the second inequality of (5) holds since we have \( \mathbb{P}[\psi_i(x, \xi) = \psi_j(x, \xi)] = 0 \) for all \( i \neq j \). For \( i = 1 \), the first function in the summation can be represented as the following integral

\[
\mathbb{E}[\nabla_x \psi_1(x, \xi) \big| \psi_1(x, \xi) = t, \psi_1(x, \xi) \geq \psi_j(x, \xi) \forall j \geq 2]
\]

\[ = \int_{-\infty}^{t} \cdots \int_{-\infty}^{t} g_1(t, t_2, \ldots, t_l; x) dt_2 \cdots dt_l,
\]

which exists by condition 3. Note that in (6) the conditional expectation on \( \psi_1(x, \xi) = t, \psi_1(x, \xi) \geq \psi_j(x, \xi) \forall j \geq 2 \) has been achieved by setting the first argument in \( g_1 \) as \( t \) for achieving \( \psi_1(x, \xi) = t \) and setting the range of integral to be from \(-\infty\) to \( t \) for all the other arguments. Now we have verified that all the three conditions in Assumption 2.5 are satisfied by the function \( \psi(x, \xi) \), and hence the quantile function \( Q^{1-\alpha}_\psi(x) \) is differentiable for all \( x \in S \). \( \square \)

3 Sample-Based Quantile Approximation

Our approach for solving (QCP) uses estimates \( Q^{1-\alpha}_\psi(x) \) and \( \nabla Q^{1-\alpha}_\psi(x) \) at a given point \( x \) with samples of \( \xi \), and a finite-difference approximation is used for the estimation of \( \nabla Q^{1-\alpha}_\psi(x) \). Consider the problem of estimating the \( 1 - \alpha \) quantile of a random variable \( X \) using \( N \) i.i.d. samples \( \{X_i\}_{i=1}^{N} \) drawn from the probability distribution of \( X \). Denote the quantile as \( X^{(1-\alpha)} \). An asymptotic unbiased estimator of \( X^{(1-\alpha)} \) is the \( 1 - \alpha \) quantile \( \hat{X}^{(1-\alpha)} \) of the sequence \( \{X_i\}_{i=1}^{N} \), which is computed by

1. Sorting \( \{X_i\}_{i=1}^{N} \) in ascending order, and letting \( \{X'_i\}_{i=1}^{N} \) be the sorted sequence;
2. Letting \( \hat{X}^{(1-\alpha)} \) be the \(((1-\alpha)N)\)th element in the sorted sequence.

The following theorem connects the quantile process with a Brownian bridge.

**Theorem 3.1** ([9, Theorem 6]). Let \( \{X_i\}_{i=1}^{N} \) be i.i.d. random variables with a cumulative distribution function \( F_X \) that is twice differentiable on \((a, b)\) where \( a = \sup\{x : F_X(t) = 0\}, b = \inf\{x : F_X(t) = 1\} \) \((a \text{ and } b \text{ can be } -\infty \text{ and } +\infty \text{ respectively in general})\) and the density function \( \rho_X = F'_X \) is continuously
differentiable and greater than zero on \((a, b)\). Let \(N\) be the sample size and \(\hat{Q}_{N}^{1-\alpha} = N^{-\frac{1}{2}}(\hat{Q}_{N}^{1-\alpha} - Q^{1-\alpha})\), where \(\hat{Q}_{N}^{1-\alpha}\) is the empirical \((1 - \alpha)\)-quantile of the sample set \(\{X_i\}_{i=1}^{N}\) determined by the above two steps and \(Q^{1-\alpha} = F_{X}^{-1}(1 - \alpha)\) is the unique \((1 - \alpha)\)-quantile of the distribution \(F_{X}\). The following concentration inequality holds for every \(z > 0\):

\[
P\left(\left|\rho_{X}(Q^{1-\alpha})\hat{Q}_{N}^{1-\alpha} - \sqrt{\alpha(1-\alpha)}W\right| > (C \log N + z) \log N\right) < e^{-dz},
\]

where \(W\) is a standard normal random variable, \(C\) and \(d\) are constants that depend on \(|\rho_{X}(Q^{1-\alpha})|\) and \(|\rho'_{X}(Q^{1-\alpha})|\), where \(\rho'_{X}\) is the derivative of the density function \(\rho_{X}\).

We can now provide a probabilistic bound on the quantile estimation’s gradient.

**Theorem 3.2.** Suppose Assumptions 2.2–2.4 hold. Let \(\rho(q; x)\) and \(\rho'(q; x)\) be the density function and the derivative of the density function of \(\Xi_{x}\) evaluated at \(q\). Let \(\{\Xi_{i}\}_{i=1}^{N}\) be i.i.d. samples of \(\Xi_{x}\) and \(\hat{Q}_{N}^{1-\alpha}(x)\) be the empirical \(1 - \alpha\) quantile of the samples. Consider the sample-based quantile gradient estimator \(\hat{D}_{N}(x)\) defined by

\[
\hat{D}_{N}(x) = \sum_{k=1}^{n} \frac{\hat{Q}_{N}^{1-\alpha}(x + \beta \hat{e}_k) - \hat{Q}_{N}^{1-\alpha}(x - \beta \hat{e}_k)}{2\beta} \hat{e}_k,
\]

where \(\beta > 0\) is the finite-difference parameters, and \(\hat{e}_k\) is the basis vector in \(\mathbb{R}^n\) with a one in entry \(k\) and zeros in all the other entries. If \(\beta \leq \frac{\delta}{nL_{Q}}\) and the sample size \(N\) satisfies the following condition:

\[
N \geq O \left(\frac{C_{d}^{2} \alpha(1 - \alpha)n^{2}(\log \frac{n}{\delta})^{3}}{d^{2}\delta^{2}}\right),
\]

the following probabilistic bound on the gradient of the quantile estimation holds for every \(\delta, \gamma > 0\) and \(x \in \mathbb{R}^{n}\):

\[
P\left(\left\|\nabla Q^{1-\alpha}(x) - \hat{D}_{N}(x)\right\| \geq \delta\right) \leq \gamma,
\]

where \(C_{d}\) and \(d_{d}\) are positive constants that depend on the bounds on \(\rho(Q^{1-\alpha}(x); x)\) and \(\rho'(Q^{1-\alpha}(x); x)\) for all \(x \in \mathbb{R}^{n}\).

**Proof.** The estimation error of the quantile gradient can be bounded as follows

\[
\left\|\nabla Q^{1-\alpha}(x) - \hat{D}_{N}(x)\right\| \leq \sum_{k=1}^{n} \left|\partial_{k}Q^{1-\alpha}(x) - \frac{\hat{Q}_{N}^{1-\alpha}(x + \beta \hat{e}_k) - \hat{Q}_{N}^{1-\alpha}(x - \beta \hat{e}_k)}{2\beta}\right|
\]

\[
\leq \sum_{k=1}^{n} \left|\partial_{k}Q^{1-\alpha}(x) - \frac{Q^{1-\alpha}(x + \beta \hat{e}_k) - Q^{1-\alpha}(x - \beta \hat{e}_k)}{2\beta}\right|
\]

\[
+ \sum_{k=1}^{n} \left|\frac{Q^{1-\alpha}(x + \beta \hat{e}_k) - Q^{1-\alpha}(x - \beta \hat{e}_k)}{2\beta} - \frac{\hat{Q}_{N}^{1-\alpha}(x + \beta \hat{e}_k) - \hat{Q}_{N}^{1-\alpha}(x - \beta \hat{e}_k)}{2\beta}\right|
\]

\[
\leq \frac{1}{2} nL_{Q}\beta + \frac{1}{2\beta} \sum_{k=1}^{n} \left|Q^{1-\alpha}(x - \beta \hat{e}_k) - \hat{Q}_{N}^{1-\alpha}(x - \beta \hat{e}_k)\right|
\]

\[
+ \frac{1}{2\beta} \sum_{k=1}^{n} \left|Q^{1-\alpha}(x + \beta \hat{e}_k) - \hat{Q}_{N}^{1-\alpha}(x + \beta \hat{e}_k)\right|,
\]

where \(\partial_{k}\) denotes the derivative with respect to the \(k\)th component of \(x\), and the first term \(\frac{1}{2} nL_{Q}\beta\) in the last inequality is due to Assumption 2.3 on the \(L_{Q}\)-smoothness of the \(Q^{1-\alpha}\) function. We choose \(\beta = \frac{\delta}{nL_{Q}}\),

\[6\]
We now impose the following conditions to bound each term in \( (16) \):

\[
\mathbb{P} \left( \| \nabla Q^{1-\alpha}(x) - \hat{D}_N(x) \| \geq \delta \right) \leq \mathbb{P} \left( \frac{1}{2\beta} \sum_{k=1}^{n} (T_{1k} + T_{2k}) \geq \frac{\delta}{2} \right) \leq \sum_{k=1}^{n} \mathbb{P} \left( T_{1k} \geq \frac{\beta\delta}{2n} \right) + \sum_{k=1}^{n} \mathbb{P} \left( T_{2k} \geq \frac{\beta\delta}{2n} \right),
\]

(11)

where \( T_{1k} \) and \( T_{2k} \) are defined as

\[
T_{1k} = \left| Q^{1-\alpha}(x) - \beta \hat{e}_k \right| - \hat{Q}^{1-\alpha}_N(x - \beta \hat{e}_k),
\]

\[
T_{2k} = \left| Q^{1-\alpha}(x + \beta \hat{e}_k) - \hat{Q}^{1-\alpha}_N(x + \beta \hat{e}_k) \right|.
\]

Applying Theorem 3.1, for a fixed \( x_0 \) and any fixed sample size \( N \) the following inequality holds with probability at least \( 1 - e^{-d_{x_0} y} \):

\[
\left| z_{x_0} N^{\frac{1}{2}} (\hat{Q}^{1-\alpha}_N(x_0) - Q^{1-\alpha}(x_0)) - \sqrt{\alpha(1-\alpha)} W_{x_0} \right| \leq (C_{x_0} \log N + y) \log N,
\]

(12)

where \( z_{x_0} = \rho(Q^{1-\alpha}(x_0); x_0) \) is the density function of \( \Xi(x_0) \) evaluated at the quantile \( Q^{1-\alpha}(x_0) \). Note that the constant \( d_{x_0}, C_{x_0} \) and the standard normal variable \( W_{x_0} \) are all \( x_0 \) dependent, and the parameter \( y \) will be chosen later. It leads to the following inequalities with probability at least \( 1 - e^{-d_{x_0} y} \):

\[
\hat{Q}^{1-\alpha}_N(x_0) - Q^{1-\alpha}(x_0) \leq \sqrt{\alpha(1-\alpha)} z_{x_0} N^{-1/2} W_{x_0} + z_{x_0} N^{-1/2} (C_{x_0} \log N + y) \log N,
\]

\[
\hat{Q}^{1-\alpha}_N(x_0) - Q^{1-\alpha}(x_0) \geq \sqrt{\alpha(1-\alpha)} z_{x_0} N^{-1/2} W_{x_0} - z_{x_0} N^{-1/2} (C_{x_0} \log N + y) \log N
\]

(13)

We now impose the following conditions to bound each term in (13):

\[
\mathbb{P} \left( \sqrt{\alpha(1-\alpha)} z_{x_0} N^{-1/2} W_{x_0} \geq \frac{\beta\delta}{6n} \right) < \gamma',
\]

\[
C_{x_0} z_{x_0} N^{-1/2} (\log N)^2 < \frac{\beta\delta}{6n},
\]

\[
C_{x_0} y z_{x_0} N^{-1/2} \log N < \frac{\beta\delta}{6n},
\]

(14)

where \( \gamma' \) is a parameter which will be chosen later. Using the tail bound of the Gaussian variable, the above three inequalities are guaranteed to hold if the sample size \( N \) should satisfy

\[
N \geq O \left( \frac{\alpha(1-\alpha)n^2 C_{x_0}^2 y^2 \log \frac{1}{\gamma'}}{\beta^2 \delta^2 z_{x_0}} \right).
\]

(15)

Combining (13) and (14) concludes that

\[
\mathbb{P} \left( \left| Q^{1-\alpha}(x_0) - \hat{Q}^{1-\alpha}_N(x_0) \right| \geq \frac{\beta\delta}{2n} \right) < e^{-d_{x_0} y + \gamma'}.
\]

(16)

Then we substitute \( x_0 \leftarrow x \pm \beta \hat{e}_k \) for \( k = 1, \ldots, n \) and incorporate them into (11). We then obtain the following inequality by applying the union bound if (15) and (16) are satisfied with universal constants \( C_{x_0} \) and \( d_{x_0} \) that depend on the uniform bounds for \( \rho(Q^{1-\alpha}(x); x) \) and \( \rho'(Q^{1-\alpha}(x); x) \) over \( x \in \mathbb{F} \):

\[
\mathbb{P} \left( \left\| \nabla Q^{1-\alpha}(x) - \hat{D}_N(x) \right\| \geq \delta \right) \leq 2n(e^{-d_{x_0} y + \gamma'}).
\]

(17)

We set \( y = \frac{1}{d_{x_0}} \log \frac{1}{\gamma} \) and \( \gamma' = \frac{\gamma}{\beta^2} \), the right-hand side of (17) becomes \( \gamma \), and the complexity condition becomes (9).
The following corollary is directly implied from the proof of Theorem 3.2.

**Corollary 3.1.** Suppose Assumptions 2.2–2.4 hold. If \( N \geq O \left( \frac{C_2 \alpha (1 - \alpha) (\log \frac{1}{\delta})^3}{d^2 \delta^2} \right) \), the following inequality holds for all \( x \in \mathcal{S} \):

\[
P \left( \left| Q_{1-\alpha}(x) - \tilde{Q}_{1-\alpha}(x) \right| \geq \delta \right) \leq \gamma.
\]  

(18)

Note that compared to (9) the sample complexity in the above corollary is independent of \( n \) and \( \beta \) since it is for the estimation of the quantile itself rather than the gradient of the quantile function (which involves the step size \( \beta \) and dimension \( n \)). Clearly, the sample complexity for (18) to hold is dominated by (9).

## 4 Stochastic Merit Function Based on the Quantile Approximation

We now analyze an augmented Lagrangian method (ALM) to solve (QCP). To simplify the analysis, we unify the notation for constraint functions and reformulate (QCP) as

\[
\text{minimize } f(x) \quad \text{s.t. } g_i(x) \leq 0 \quad \forall i \in \mathcal{I}_0,
\]

where \( \mathcal{I}_0 = \{0, 1, \ldots, l_2\} \), \( \mathcal{I} = \{1, \ldots, l_2\} \), \( g_0(x) := Q^1_{1-\alpha}(x) \), and \( g_i(x) = c_{2,i}(x) \) for \( i \in [l_2] \). Definition 4.1 restates the two standard constraint qualifications based on problem setting of (NLP), which will be used for characterizing its optimality conditions.

**Definition 4.1.** (1) The Mangasarian-Fromovitz constraint qualification (MFCQ) condition [32] is satisfied by (NLP) at a point \( x^* \) if there exists \( d \in \mathbb{R}^n \) such that \( \langle \nabla g_i(x^*), d \rangle < 0 \) for all \( i \in \mathcal{A}(x^*) \), where \( \mathcal{A}(x^*) := \{ i \in \mathcal{I}_0 : g_i(x^*) = 0 \} \) is the set of active constraints at \( x^* \).

(2) The constant positive linear independence (CPLD) condition [38] is satisfied by (NLP) at a point \( x^* \), if there exists a neighborhood \( U \) of \( x^* \) such that for any \( J \subseteq \mathcal{I}_0 \) the set of gradients \( \{ \nabla g_i(x^*) : i \in J \} \) are linearly dependent, \( \{ \nabla g_i(x) : i \in J \} \) are linearly dependent for all \( x \in U \).

Given \( x \in \mathbb{R}^n \), \( \mu \in \mathbb{R}_{+}^{\left| \mathcal{I}_0 \right|} \), \( \rho > 0 \), we define the Powell-Hestenes-Rockafellar augmented Lagrangian [1] by

\[
\Phi(x, \rho, \mu) = f(x) + \rho \sum_{i \in \mathcal{I}_0} \max \left\{ 0, g_i(x) + \frac{\mu_i}{\rho} \right\}^2.
\]

(19)

The augmented Lagrangian (19) will be use as the merit function in the augmented Lagrangian method for solving (NLP), and it is initially proposed for deterministic constrained optimization problem with inequality constraints. For fixed parameters \( \rho, \mu \), the gradient of \( \Phi(x, \rho, \mu) \) is given by

\[
\nabla \Phi(x, \rho, \mu) = \nabla f(x) + \rho \sum_{i \in \mathcal{I}_0} \max \left\{ 0, g_i(x) + \frac{\mu_i}{\rho} \right\} \nabla g_i(x).
\]

(20)

Because the analytical form of \( \nabla g_0(x) \) is assumed not to be available, we use the following estimator to approximate \( \nabla \Phi(x, \lambda, \mu) \):

\[
\phi_N(x, \rho, \mu, \beta) = \nabla f(x) + \rho \max \left\{ 0, g_0(x) + \frac{\mu_i}{\rho} \right\} G_N(x)
\]

\[
+ \rho \sum_{i \in \mathcal{I}} \max \left\{ 0, g_i(x) + \frac{\mu_i}{\rho} \right\} \nabla g_i(x),
\]

(21)
where \( g_{0,N}(x) = \hat{Q}^{1-\alpha}_N(x) \) and \( G_N(x) \) is an estimator of \( \nabla g_0(x) \) defined as

\[
G_N(x) = \sum_{j=1}^n \frac{\hat{Q}^{1-\alpha}_N(y + \beta e_j) - \hat{Q}^{1-\alpha}_N(y - \beta e_j)}{2\beta} e_j.
\] (22)

We can build a local model to approximate the merit function in a small neighborhood \( B(x_0, \Delta) \) of a point \( x_0 \). This local model can utilize the estimated zeroth-order, first-order, and (when available) second-order information. The local model has the form

\[
m_N(x, \rho, \mu, \beta) = \Phi_N(x_0, \rho, \mu) + \phi_N(x_0, \rho, \mu, \beta)^\top (x - x_0) + \frac{1}{2} (x - x_0)^\top H(x - x_0),
\] (23)

where \( \phi_N \) is as in (21) and \( \Phi_N(x_0, \rho, \mu) \) is the \( N \)-sample approximation of \( \Phi(x_0, \rho, \mu) \):

\[
\Phi_N(x_0, \rho, \mu) = f(x_0) + \frac{\rho}{2} \max \left\{ 0, g_{0,N}(x_0) + \frac{\mu_0}{\rho} \right\}^2 + \frac{\rho}{2} \sum_{i \in I} \max \left\{ 0, g_i(x_0) + \frac{\mu_i}{\rho} \right\}^2
\] (24)

The parameters \( \beta, N, \Delta, \) and the matrix \( H \) control the accuracy of the approximation. Note that in the algorithm analysis, we do not require additional conditions on \( H \) other than that its norm is bounded. In the implementation, we use a neighborhood sampling approach to build a local quadratic model and extract the Hessian of the local model as an approximation of \( H \). See Section 7 for further details. We also define the quantity

\[
\sigma_N(x, \mu) = \sqrt{\min \{-g_{0,N}(x), \mu_0\}^2 + \sum_{i \in I} \min \{-g_i(x), \mu_i\}^2},
\] (25)

which will be used to quantify the level of feasibility.

5 Algorithms

We now discuss our approach for optimizing (NLP) where the problem is decomposed into an outer problem and an inner problem that are solved repeatedly. The inner problem is to minimize the sample-based merit function for fixed penalty parameter \( \rho \) and Lagrangian multipliers \( \mu \):

\[
\text{minimize} \Phi(x, \rho, \mu).
\] (26)

Since the analytical form of quantile function is unknown, the \( \Phi \) is approximated by a sample-based estimation \( \Phi_N \) (defined in (24)) in each iteration for solving the inner problem with an increasing sample size \( N \). We seek a local minimizer for the inner problem with this process being terminated once a certain criterion is met. In the outer problem, the ALM is applied to update \( \rho \) and \( \mu \).

For solving the inner problem (26), we apply a trust-region algorithm (Algorithm 1) with probabilistic ingredients. It is similar to a traditional trust-region algorithm, which enlarges or shrinks the trust region based on the relative improvement ratio. Of course, because the quantile function’s value and gradient are based on samples approximation, the algorithmic procedures (i.e., decisions on enlarging or shrinking the trust region and the termination criteria) built on top of such information are all stochastic. Probabilistic trust-region algorithms have been developed (e.g., in [23]) for derivative-free unconstrained optimization problems where the form of the objective is unknown and its value can be approximated only from sampling. In contrast to the problems studied in [23], in our objective \( \Phi_N \), only the form of the quantile function is assumed unavailable and must be estimated with a probabilistic model.

A probabilistic ALM for the outer problem is given as Algorithm 2. This algorithm updates the Lagrangian multipliers \( \mu \) and the penalty parameter \( \rho \) in (27).
Algorithm 1: Trust-region method for merit functions with penalized chance constraints

**Data:** Initial point $x_{in}$ and parameters $\mu, \rho, r, \varepsilon$ such that $\mu \geq 0, \rho > 0$ and $0 < r, \varepsilon < 1$.

**Internal parameter:** Pick $0 < \gamma_{dec} < 1 < \gamma_{inc}$, $0 < \eta_1, \eta_2$, $\tau_0 < 1$, $0 < \Delta_0$.

1. **for** $k = 0, 1, \ldots$ **do**
   2. Choose $\beta \leftarrow r_0 \Delta_k$ for each iteration. Build a $(1 - \varepsilon)$-probabilistically $\kappa$-fully linear model $m_{N_k}$ (See Definition 6.1) on $B(x^k, \Delta_k)$ with sample size $N_k$ satisfying (41).
   3. Compute $s^k \leftarrow \arg \min_{\|s\| \leq \Delta_k} m_{N_k}(x^k + s, \rho, \mu, \beta)$
   4. **if** $m_{N_k}(x^k, \rho, \mu, \beta) - m_{N_k}(x^k + s^k, \rho, \mu, \beta) \geq \eta_1 \min(\Delta_k, \Delta_k^2)$ **then**
   5. Calculate $\rho_k \leftarrow \Phi_{N_k}(x^k, \rho, \mu, \beta) - \Phi_{N_k}(x^k + s^k, \rho, \mu, \beta)$
   6. **if** $\rho_k \geq \eta_2$ **then**
   7. $x^{k+1} \leftarrow x^k + s^k; \Delta_{k+1} \leftarrow \gamma_{inc} \Delta_k$
   8. **else**
   9. $x^{k+1} \leftarrow x^k; \Delta_{k+1} \leftarrow \gamma_{dec} \Delta_k$
 10. **else**
 11. $x^{k+1} \leftarrow x^k; \Delta_{k+1} \leftarrow \gamma_{dec} \Delta_k$
 12. **if** $\Delta_{k+1} \leq r$ **then**
 13. **return** $x^k$.
 14. **else**
 15. Continue

6 Convergence Analysis

We analyze the convergence of Algorithm 2 when the quantile function $Q^{1-\alpha}(\cdot)$ is differentiable. We first analyze the probabilistic properties of local model used in the trust-region method for approximating the merit function (Section 6.1). Then use these results to analyze the convergence of Algorithm 1 for solving the merit-function minimization problem (Section 6.2). We provide a global convergence result of Algorithm 2 (Section 6.3). The parameter updating rule in this algorithm is similar to [2, 3], which is a well recognized augmented Lagrangian method for solving deterministic nonlinear programs with equality and inequality constraints. Note that the convergence of Algorithm 2 also relies on the parameter setting, in particular, the sample size drawn from $\xi$ at every main iteration. The conditions on the sample size are specified in theorems of convergence.

6.1 Probabilistic properties of local model approximation

In each iteration of Algorithm 1, a quadratic model is constructed as a local approximation of the merit function. Putative iterates are produced by minimizing this model in a trust region. Clearly, the sufficient approximation accuracy is needed to ensure convergence of the algorithm. A notion of $\kappa$-fully linearity is introduced to characterize the approximation accuracy within a neighborhood, which is formally given in Definition 6.1. The probabilistic counterpart is given in Definition 6.2 for a random local model. These definitions are brought from [23].

**Definition 6.1.** Let $f$ be continuously differentiable, let $\kappa = (\kappa_{eg}, \kappa_{ef})$ be a given vector of absolute constants, and let $\Delta > 0$ be given. A function $m_f \in C^1$ is a $\kappa$-fully linear model of $f$ on $B(x, \Delta)$ if $\nabla m_f$ is Lipschitz continuous and for all $x \in \mathbb{S}$ and $y \in B(x, \Delta)$,

$$\|\nabla f(y) - \nabla m_f(y)\| \leq \kappa_{eg} \Delta, \text{ and } |f(y) - m_f(y)| \leq \kappa_{ef} \Delta^2, \quad (28)$$

where $\kappa$ and $\Delta$ are independent of $x$.

**Definition 6.2.** Consider running an algorithm to generate an infinite sequence of points $\{x^k\}_{k=1}^\infty$, and let $F_k$ be the $\sigma$-algebra representing the information available at iteration $k$. Let $\kappa = (\kappa_{ef}, \kappa_{eg})$ be a given vector of constants, let $\varepsilon \in (0, 1)$, and let $\Delta > 0$ be given. A random model $m_f$ generated based on samples of random parameters is $(1 - \varepsilon)$-probabilistically $\kappa$-fully linear on $B(x, \Delta)$ if

$$\mathbb{P}(m_f \text{ is a } \kappa\text{-fully linear model of } \Phi \text{ on } B(x, \Delta)|F_{k-1}) \geq 1 - \varepsilon, \quad (29)$$
Algorithm 2: A probabilistic augmented Lagrangian method for solving (NLP)

| Data: List of input parameters: θ_1, θ_2, ε ∈ (0, 1), μ_{max} > 0, θ_1 > 1, and convergence control parameters \{γ_k, η_k\}_{k=1}^{∞} satisfying η_k → 0. |
|---|
| 1. Set μ_i = μ_{init} for some μ_{init} > 0 and i ∈ I_0, and initialize x_0. |
| 2. Set ρ_i = ρ_{init} for some ρ_{init} > 0. |
| for k = 1, . . . do |
| 4. Step 1. merit function minimization: Apply Algorithm 1 with the initial point x_k^{-1} obtained from the previous iteration when k ≥ 1 and with the input parameters ρ_k, μ_k, γ_k, ε to get a termination point x_k as the output. |
| 5. Step 2. convergence-control parameters reduction: Generate N_k independent samples (The conditions of N_k to guarantee convergence will be given in Section 6.), evaluate g_0, N_k(x_k^k), and update |
| \[
| μ_k^{k+1} ← \max\left\{0, μ_k^k + ρ_k^k g_0, N_k(x_k^k)\right\}, \\
| μ_i^{k+1} ← \max\left\{0, μ_i^k + ρ^k g_i(x_k^k)\right\} \quad \forall i ∈ I, \\
| μ_i^{k+1} ← \min\{μ_{max}, μ_i^{k+1}\} \quad \forall i ∈ I_0, \\
| \]
| if σ_N_k(x_k^k, μ_k^{k+1}) > η_k^k then |
| \[ \rho_i^{k+1} ← θ_0 ρ_i^k \] |
| else |
| \[ \rho_i^{k+1} ← ρ_i^k \] |

where κ and Δ are independent of x.

**Definition 6.3.** For fixed parameters ρ, μ, the quadratic models in the trust-region Algorithm 1 satisfy the (ε, θ)-probabilistic local approximation accuracy if there exists a K such that for any iteration k > K the following two conditions are satisfied:

\[
\begin{align*}
\mathbb{P}\left[ |V_k| > η_1 η_2 ∆_k^2 |F_{k-1}\right] & \leq ε, \\
\mathbb{P}\left[ |V_k| > (η_1 η_2 + w) ∆_k^2 |F_{k-1}\right] & \leq \frac{θ}{w} \quad ∀w > 0, \\
with V_k = Φ_{N_k}(x_k) - Φ(x_k) + Φ(x_k + s_k) - Φ_{N_k}(x_k + s_k),
\end{align*}
\]

where the Lagrangian and penalty parameters in the functions Φ_{N_k} and Φ are omitted here for simplicity, ∆_k ≤ 1 is the trust-region radius at iteration k, s_k is trust-region subproblem solution at iteration k, F_k is the σ-algebra representing the information available at iteration k, and η_1, η_2 are trust-region updating parameters in Algorithm 1.

**Proposition 6.1.** Suppose Assumptions 2.1–2.4 hold. Let L_g be a shared Lipschitz constant of ∇g_i for all i ∈ I_0. For any constants a ≥ 0, γ ∈ (0, 1), and any x_0, x ∈ S satisfying \|x - x_0\| ≤ Δ ≤ 1, suppose β = r_0 Δ with \(r_0 < \frac{2L_g}{\Delta} \) and N ≥ O \(\left(\frac{C_2^2(1-α)^2(\log δ)^2}{αL_g^2\Delta^2}\right)\), then the following three inequalities hold:

\[
\begin{align*}
\mathbb{P}\left[ \|\nabla g_0(x) - G_N(x_0)\| \geq 2L_g \Delta \right] & ≤ γ, \\
\mathbb{P}\left[ \|\max\{0, g_i(x) + a\} \nabla g_0(x) - \max\{0, g_0, N(x_0) + a\} G_N(x_0)\| \\
& \geq 6 (\|\nabla g_0(x_0)\| + |g_0(x_0)| + L_g)^2 + 3aL_g \right] \Delta \right] \leq γ, \\
\|\max\{0, g_i(x) + a\} \nabla g_i(x) - \max\{0, g_i(x_0) + a\} \nabla g_i(x_0)\| \\
& \leq 2(\|g_i(x)\| + \|\nabla g_i(x_0)\| + L_g)^2 + aL_g) \Delta.
\end{align*}
\]
Proof. To prove the first inequality, we notice that
\[ \| \nabla g_0(x) - G_N(x_0) \| \leq \| \nabla g_0(x) - \nabla g_0(x_0) \| + \| \nabla g_0(x_0) - G_N(x_0) \| \]
\[ \leq L_g \| x - x_0 \| + \| \nabla Q^{1-\alpha}(x_0) - \widehat{D}_N(x_0) \|, \]
where \( \widehat{D}_N(\cdot) \) is defined in (8). Because \( N \geq O \left( \frac{C^2\alpha(1-\alpha)n^2\log N}{\epsilon^2L_\gamma^2\Delta^2} \right) \), we can apply Theorem 3.2 to conclude that
\[ \mathbb{P} \left( \| \nabla g_0(x) - G_N(x_0) \| \geq L_g\Delta \right) \leq \gamma, \quad \mathbb{P} \left( \| \nabla g_0(x) - G_N(x_0) \| \geq 2L_g\Delta \right) \leq \gamma, \]
where the condition \( r_0 < \frac{2L_g\Delta}{nL_Q} \) ensures that \( \beta < \frac{2L_g\Delta}{nL_Q} \) (a condition required by Theorem 3.2). This proves the first inequality of the proposition. The second inequality of the proposition can be bounded as follows
\[
\| \max \{ 0, g_0(x) + a \} \nabla g_0(x) - \max \{ 0, g_0,N(x_0) + a \} G_N(x_0) \| \\
\leq \| \max \{ 0, g_0(x) + a \} \nabla g_0(x) - \max \{ 0, g_0(x) + a \} \nabla g_0(x_0) \| \\
+ \| \max \{ 0, g_0(x) + a \} \nabla g_0(x_0) - \max \{ 0, g_0,N(x_0) + a \} \nabla g_0(x_0) \| \\
+ \| \max \{ 0, g_0,N(x_0) + a \} G_N(x_0) - \max \{ 0, g_0,N(x_0) + a \} G_N(x_0) \| \\
\leq \left( |g_0(x)| + a \right) \| \nabla g_0(x) - \nabla g_0(x_0) \| + |g_0(x) - g_0(x_0)| \| \nabla g_0(x_0) \| \\
+ \left( |g_0(x)| + a \right) \| \nabla g_0(x_0) - G_N(x_0) \| + |g_0(x) - g_0,N(x_0)| \| G_{N}(x_0) \|, \]
where we use the property \( | \max \{ 0, a \} - \max \{ 0, b \} | \leq |a - b| \). We know that with probability at least \( 1 - 2\gamma \) the following inequalities hold jointly
\[
\| \nabla g_0(x) - G_N(x_0) \| \leq 2L_g\Delta, \\
\| G_N(x_0) \| \leq \| \nabla g_0(x_0) \| + \| \nabla g_0(x_0) - G_N(x_0) \| \leq \| \nabla g_0(x_0) \| + 2L_g\Delta, \]
\[
|g_0(x) - g_0,N(x_0)| \leq L_g\Delta \quad \text{(Corollary 3.1)}. \]
Substituting (32) into (31), we conclude that with probability at least \( 1 - 2\gamma \), the following inequality holds:
\[
\| \max \{ 0, g_0(x) + a \} \nabla g_0(x) - \max \{ 0, g_0,N(x_0) + a \} G_N(x_0) \| \\
\leq \left( |g_0(x)| + a \right) L_g\Delta + \left( \| \nabla g_0(x_0) \| + \frac{1}{2}L_g\Delta^2 \right) \| \nabla g_0(x_0) \| \\
+ 2 \left( |g_0(x)| + a \right) L_g\Delta + \| G_N(x_0) \| L_g\Delta \\
\leq \left( |g_0(x)| + a \right) L_g\Delta + \left( \| \nabla g_0(x_0) \| + \frac{1}{2}L_g\Delta^2 \right) \| \nabla g_0(x_0) \| \\
+ 2 \left( |g_0(x)| + a \right) L_g\Delta + \left( \| \nabla g_0(x_0) \| + 2L_g\Delta \right) L_g\Delta \\
\leq \left( |g_0(x)| + a \right) L_g\Delta + 2 \left( |g_0(x)| + a \right) L_g\Delta \\
+ \left( \| \nabla g_0(x_0) \| + \frac{1}{2}L_g \right) \| \nabla g_0(x_0) \| \Delta + \left( \| \nabla g_0(x_0) \| + 2L_g \right) L_g\Delta. \]
Applying the inequality
\[
|g_0(x)| \leq |g_0(x)| + \| \nabla g_0(x_0) \| \Delta + L_g\Delta^2/2 \\
\leq |g_0(x)| + \| \nabla g_0(x_0) \| + L_g/2 \]
to bound \( |g_0(x)| \) in the first term of (33) and enlarging the coefficients leads to the following inequality with probability at least \( 1 - 2\gamma \):
Assumptions 2.1 on 2021–
notations in the proof. First, applying the model definitions (Proposition 6.2).
Suppose By applying the second and the third inequalities from Proposition 6.2.

\[ \Phi(x, \rho, \mu, \beta) = \text{a fully linear model} \]

on \( B(x_0, \Delta) \) with the parameter \( \kappa = (\kappa_{eg}, \kappa_{ef}) \) where \( \kappa_{eg} = 8K_S|I_0| (1 + \sum_{i \in I_0} \mu_i + \rho) \), \( \kappa_{ef} = 6K_S|I_0| (1 + \sum_{i \in I_0} \mu_i + \rho) \), and \( K_S \) is a constant satisfying

\[ K_S \geq L_f + L_g + \|H\|, \]
\[ K_S \geq \max_{x \in S}(\|\nabla g_i(x)\| + |g_i(x)| + L_g)^2 \forall i \in I_0. \] (35)

\[ \|\nabla \Phi(x) - \nabla m_N(x)\| \]
\[ = \left\| \nabla f(x) + \rho \sum_{i \in I_0} \max \left\{ 0, g_i(x) + \frac{\mu_i}{\rho} \right\} \nabla g_i(x) - \phi_N(x_0) - H(x - x_0) \right\| \]
\[ \leq \|\nabla f(x) - \nabla f(x_0)\| + \|H(x - x_0)\| \]
\[ + \rho \left\| \max \left\{ 0, g_0(x) + \frac{\mu_0}{\rho} \right\} \nabla g_0(x) - \nabla f(x_0) \right\| \]
\[ + \sum_{i \in I} \left\| \max \left\{ 0, g_i(x) + \frac{\mu_i}{\rho} \right\} \nabla g_i(x) - \nabla f(x_0) \right\|. \] (36)

By applying the second and the third inequalities from Proposition 6.1, we conclude that with probability at least \( 1 - 2\gamma \) the following inequalities hold:

\[ \|\nabla \Phi(x) - \nabla m_N(x)\| \]
\[ \leq (\|H\| + L_f)\Delta + \left\{ 6\rho (\|\nabla g_0(x_0)\| + |g_0(x_0)| + L_g)^2 + 3\mu_0 L_g \right\}\Delta \]
\[ + \sum_{i \in I} (2\rho|g_i(x)| + \|\nabla g_i(x_0)\| + L_g)^2 + \mu_i L_g \Delta \]
\[ \leq K_S \Delta + (6\rho K_S + 3\mu_0 K_S)\Delta + \sum_{i \in I} (2\rho K_S + \mu_i K_S)\Delta \]
\[ \leq 8|I_0| \left( 1 + \rho + \sum_{i \in I_0} \mu_i \right) K_S \Delta := \kappa_{eg}\Delta. \] (37)

Using integral representation, we can rewrite \( \Phi(x) - m_N(x) \) as:

\[ \Phi(x) - m_N(x) = \Phi(x_0) - m_N(x_0) \]
\[ + \int_0^1 \left[ \nabla \Phi((1-t)x_0 + tx)^\top - \nabla m_N((1-t)x_0 + tx)^\top \right](tx - x_0)dt. \]
Then we can obtain the following inequality to bound $|\Phi(x) - m_N(x)|$:

$$
|\Phi(x) - m_N(x)| \leq |\Phi(x_0) - m_N(x_0)| \\
+ \int_0^1 \|\nabla \Phi((1-t)x_0 + tx) - \nabla m_N((1-t)x_0 + tx)\| \|x - x_0\| \, dt.
$$

The two terms on the right side of the above inequality can be bounded as follows with joint probability at least $1 - 2\gamma$:

$$
|\Phi(x_0) - m_N(x_0)| = |\Phi(x_0) - \Phi_N(x_0)| \\
= \left| \frac{\rho}{2} \max \left\{ 0, g_0(x_0) + \frac{\mu_0}{\rho} \right\} - \frac{\rho}{2} \max \left\{ 0, g_{0,N}(x_0) + \frac{\mu_0}{\rho} \right\} \right|^2 \\
\leq \frac{\rho}{2} \max \left\{ 0, g_0(x_0) + \frac{\mu_0}{\rho} \right\} + \max \left\{ 0, g_{0,N}(x_0) + \frac{\mu_0}{\rho} \right\} \cdot |g_0(x_0) - g_{0,N}(x_0)| \\
\leq \frac{\rho}{2} \max \left\{ 0, g_0(x_0) + \frac{\mu_0}{\rho} \right\} + \max \left\{ 0, g_{0,N}(x_0) + \frac{\mu_0}{\rho} \right\} \cdot |g_0(x_0) - g_{0,N}(x_0)| \\
\leq \left( \rho |g_0(x_0)| + \mu_0 + \frac{\rho}{2} L_g \Delta^2 \right) \cdot L_g \Delta^2 \quad \text{(Corollary 3.1 and condition on } N) \\
\leq \left( \rho |g_0(x_0)| + \mu_0 + \frac{\rho}{2} L_g \Delta^2 \right) \cdot L_g \Delta^2,
$$

$$
\int_0^1 \|\nabla \Phi((1-t)x_0 + tx) - \nabla m_N((1-t)x_0 + tx)\| \|x - x_0\| \, dt \leq \frac{1}{2} \kappa_{eg} \Delta^2,
$$

where (37) is used to get the second inequality. Therefore, we have that with probability at least $1 - 2\gamma$ the following inequality holds

$$
|\Phi(x) - m_N(x)| \leq \left( \rho |g_0(x_0)| + \mu_0 + \frac{\rho}{2} L_g \right) \cdot L_g \Delta^2 + \frac{1}{2} \kappa_{eg} \Delta^2 \\
\leq \frac{1}{2} \kappa_{eg} \Delta^2 + 2K_S \left( 1 + \sum_{i \in I_0} \mu_i + \rho \right) \Delta^2 \\
\leq 6|I_0| \left( 1 + \sum_{i \in I_0} \mu_i + \rho \right) K_S \Delta^2 := \kappa_{ef} \Delta^2.
$$

Inequalities (37) and (40) conclude the proof by absorbing the constant factor 2 associated with $\gamma$ into the sample complexity. \square

**Remark 6.1.** The $N \sim 1/\Delta^4$ relationship between sample size and the trust-region size matches that in [23].

**Theorem 6.1.** Suppose Assumptions 2.1–2.4 hold. Consider Algorithm 1 for given parameters $\rho, \mu$ and internal parameters $\gamma_{dec}, \gamma_{inc}, \eta_1, \eta_2, \rho_0, \Delta_0$. The following properties hold jointly

(a) $m_{N_k}$ from (33) is a $(1-\varepsilon)$-probabilistically $\kappa$-fully linear model, where $\kappa = (\kappa_{eg}, \kappa_{ef})$, $\kappa_{eg} = 8K_S|I_0| \left( 1 + \sum_{i \in I_0} \mu_i + \rho \right)$, $\kappa_{ef} = 6K_S|I_0| \left( 1 + \sum_{i \in I_0} \mu_i + \rho \right)$;

(b) Algorithm 1 satisfies the $(\varepsilon, \theta)$-probabilistic local approximation accuracy condition (Definition 6.3) provided that the number of samples $N_k$ drawn from the distribution of the random parameters $\xi$ satisfies

$$
N_k \geq O \left( \frac{C_8^2 n^2 \alpha(1 - \alpha) (\log \frac{n}{\varepsilon})^3}{A_k \Delta_k^4} \right).
$$
\[
A_k = \min \left\{ d_2^2 \rho \eta_1 \eta_2, \frac{d_2 \rho \sqrt{\eta_1 \eta_2}}{(\rho_0 + \rho K_S)^2}, \frac{d_2^2 \eta_1 \eta_2 \theta}{(\rho_0 + \rho K_S)^2} \right\}.
\]

**Proof.** To simplify notation in the proof, we omit the iteration index \(k\) in Algorithm 1 and the arguments \(\rho, \mu, \beta\) in the functions \(\Phi(x, \rho, \mu)\) and \(\Phi_N(x, \rho, \mu)\).

(a) From Proposition 6.2, if \(N \geq O \left( \frac{C_2^2 \alpha(1-\alpha) \eta_2^2 (\log \frac{1}{\Delta})^3}{d_2^2 \rho \Delta^2 \eta_1^2} \right)\), \(m_N\) is a \((1 - \varepsilon)\)-probabilistically \(\kappa\)-fully linear model, with \(\kappa_{\text{ef}} = 8K_S|I_0| (1 + \sum_{i \in I_0} \mu_i + \rho)\) and \(\kappa_{\text{sf}} = 6K_S|I_0| (1 + \sum_{i \in I_0} \mu_i + \rho)\). This sample complexity is dominated by (41).

(b) It suffices to show that the model \(m_N\) in Algorithm 1 satisfies the \((\varepsilon, \theta)\)-probabilistic local approximation accuracy condition. Note that the probability in the first condition of (30) can be estimated using the union bound:

\[
P \left[ |\Phi_N(x) - \Phi(x) + \Phi(x + s) - \Phi_N(x + s)| > \eta_1 \eta_2 \Delta^2 \right| F_{k-1} \right]
\leq P \left[ |\Phi_N(x) - \Phi(x)| > \frac{1}{2} \eta_1 \eta_2 \Delta^2 \right| F_{k-1} \right] + P \left[ |\Phi_N(x + s) - \Phi(x + s)| > \frac{1}{2} \eta_1 \eta_2 \Delta^2 \right| F_{k-1} \right],
\]
where \(\Phi_N\) is defined in (24). Similar as the first a few steps in the derivation of (38), the term \(|\Phi_N(x) - \Phi(x)|\) can be bounded deterministically as:

\[
|\Phi_N(x) - \Phi(x)| \leq \left| \frac{\rho}{2} \max \left\{ 0, g_0(x) + \frac{\mu_0}{\rho} \right\} \right|^2 - \left| \frac{\rho}{2} \max \left\{ 0, g_0(x) + \frac{\mu_0}{\rho} \right\} \right|^2
\leq \left( \rho |g_0(x)| + \mu_0 + \frac{\rho}{2} |g_0(x) - g_0(x)| \right) \cdot |g_0(x) - g_0(x)|.
\]

Corollary 3.1 implies that if \(N \geq O \left( \frac{C_2^2 \alpha(1-\alpha) \eta_2^2 (\log \frac{1}{\Delta})^3}{d_2^2 \rho \Delta^2 \eta_1^2} \right)\), then \(P \left( |g_0(x) - g_0(x)| \leq \Delta^2 \right) \geq 1 - \frac{\varepsilon}{2}\). (The constant 1/2 in the probability can be absorbed into the complexity.) Then with probability at least 1 - \(\frac{\varepsilon}{2}\):

\[
|\Phi_N(x) - \Phi(x)| \leq \left( \mu_0 + \rho |g_0(x)| + \frac{\rho}{2} \mu_0 \Delta^2 \right) \cdot \Delta^2 \leq \left( \mu_0 + \rho K_S + \frac{\rho}{2} \mu_0 \Delta^2 \right) \Delta^2,
\]
where \(t\) is a parameter to be determined. We can set \(t\) to satisfy \((\mu_0 + \rho K_S) t \leq \frac{1}{4} \eta_1 \eta_2\) and \(\frac{\rho}{2} \mu_0 \Delta^2 \leq \frac{1}{4} \eta_1 \eta_2\) to ensure that

\[
P \left( |\Phi_N(x) - \Phi(x)| \leq \frac{1}{2} \eta_1 \eta_2 \Delta^2 \right| F_{k-1} \right) \geq 1 - \frac{\varepsilon}{2}.
\]

This means if \(N\) satisfies

\[
N \geq O \left( \min \left\{ \frac{C_2^2 \eta_2 \Delta^2 \theta}{(\mu_0 + \rho K_S)^2 \eta_1^2}, \frac{d_2^2 \rho \eta_1 \eta_2 \Delta^2}{(\mu_0 + \rho K_S)^2} \right\} \right),
\]
then

\[
P \left( |\Phi_N(x) - \Phi(x)| \geq \frac{1}{2} \eta_1 \eta_2 \Delta^2 \right| F_{k-1} \right) \leq \frac{\varepsilon}{2},\]

and hence

\[
P \left[ \Phi_N(x) - \Phi(x) + \Phi(x + s) - \Phi_N(x + s) \geq \eta_1 \eta_2 \Delta^2 \right| F_{k-1} \right] \leq \varepsilon,
\]
i.e., the first condition of (30) holds, where we have used the fact that the complexity condition (43) is independent of the position and hence it can be applied to the position \(x\) and \(x + s\). Next, we show that the second condition of (30) holds. Similarly, we have

\[
P \left[ \Phi_N(x) - \Phi(x) + \Phi(x + s) - \Phi_N(x + s) > (\eta_1 \eta_2 + w) \Delta^2 \right| F_{k-1} \right]
\leq P \left[ \Phi_N(x) - \Phi(x) > \frac{1}{2} (\eta_1 \eta_2 + w) \Delta^2 \right| F_{k-1} \right]
\leq P \left[ \left| \Phi_N(x + s) - \Phi(x + s) \right| > \frac{1}{2} (\eta_1 \eta_2 + w) \Delta^2 \right| F_{k-1} \right].
\]
two functions of bound the denominator in (47)

It remains to find an upper bound for \( \eta_1 \eta_2 + w \), which is denoted as \( F_k(x^k) \) and \( F_k(x^k + s^k) \), respectively. Calculate \( \rho_k \leftarrow F_k(x^k - F_k(x^k + s^k)) \). If \( \rho_k \geq \eta_2 \) then \( x^{k+1} \leftarrow x^k + s^k \); \( \Delta_{k+1} \leftarrow \min\{1, \gamma_{inc} \Delta_k \} \). Otherwise, \( x^{k+1} \leftarrow x^k \); \( \Delta_{k+1} \leftarrow \gamma_{dec} \Delta_k \). When \( \Delta_{k+1} = 0 \), then set \( k \leftarrow k + 1 \).

It suffices to find the condition for \( N \) such that

\[
P \left\{ \Phi_N(x) - \Phi(x) \geq \frac{1}{2} \eta_1 \eta_2 + w \Delta^2 \, \big| \mathcal{F}_{k-1} \right\} \leq \frac{\theta}{2w}.
\]  

Similarly as (44), a sufficient condition of \( N \) for (45) to hold is:

\[
N \geq O \left( \frac{C_5^2 \alpha(1 - \alpha)(\log \frac{2w}{\eta_1 \eta_2})^3}{\min \left\{ \frac{d_2^2 (\eta_1 \eta_2 + w)^2}{\mu \rho \Delta^2}, \frac{d_2^2 \rho^2 (\eta_1 \eta_2 + w) \Delta^2}{\sigma \rho K_3} \right\} \right) \quad \forall w \geq \theta/2.
\]  

The above complexity bound can be strengthened by applying the inequalities \( \eta_1 \eta_2 + w \geq 2 \sqrt{\eta_1 \eta_2 w} \) to lower bound the denominator in (46) as (up-to a constant factor)

\[
\min \left\{ \frac{d_2^2 \eta_1 \eta_2 w}{(\mu \rho \Delta^2)} \Delta^4, \frac{d_2^2 \rho^2 \sqrt{\eta_1 \eta_2 w} \Delta^2}{\sigma \rho K_3} \right\}.
\]  

It remains to find an upper bound for \( \frac{\log \frac{2w}{\eta_1 \eta_2}}{\rho K_3} \) and \( \frac{\log \frac{2w}{\eta_1 \eta_2}}{\sigma \rho K_3} \) with \( w > \theta/2 \). One can verify that the above two functions of \( w \) achieve the maximum value at \( w = \frac{\rho K_3}{\eta_1 \eta_2} \) and \( w = \frac{\sigma \rho K_3}{\eta_1 \eta_2} \), respectively. This indicates that in order to ensure the second condition in (30) holds we can set \( N \) to be

\[
N \geq O \left( \frac{C_5^2 \alpha(1 - \alpha)}{\min \left\{ \frac{d_2^2 \eta_1 \eta_2 \theta^4}{(\mu \rho \Delta^2)}, \frac{d_2^2 \rho^2 \sqrt{\eta_1 \eta_2 \theta} \Delta^2}{\sigma \rho K_3} \right\}} \right)
\]  

Notice that the sample complexity condition (41) dominates (43) and (48), which concludes the proof.

### 6.2 Probability guarantee of the trust-region method for minimizing the merit function

The work [23] analyzes a probabilistic derivative-free trust-region method for solving an stochastic unconstrained optimization problem. The convergence result from Theorem 1 in [23] can be applied to our merit function when the Lagrangian parameters are fixed. Therefore, it serves as a base of convergence analysis for the quantile constrained problem concerned here. The main result presented in [23] is summarized in Theorem 6.2.
Theorem 6.2 ([23]). Let $h$ be a general smooth function that has bounded level sets and $\nabla h$ is Lipschitz continuous with constant $L_h$. Suppose Algorithm 3 is applied to minimize $h$ and the function values of $h$ can only be accessed via the stochastic estimation model $F_k$ at any iteration $k$ with Lipschitz constant bounded by $L_F$. Suppose for every iteration $k$ that is greater than some threshold, the following two conditions hold:

$$
\mathbb{P}[|F_k(x^k) - h(x^k) + h(x^k + s^k) - F_k(x^k + s^k)| > \eta_1 \eta_2 \Delta_k^2 | F_{k-1}] \leq \varepsilon 
$$

(49)

and there is a constant $\theta > 0$ such that

$$
\mathbb{P}[|F_k(x^k) - h(x^k) + h(x^k + s^k) - F_k(x^k + s^k)| > (\eta_1 \eta_2 + w) \Delta_k^2 | F_{k-1}] \leq \frac{\theta}{w} \quad \forall w > 0.
$$

(50)

Then

$$
\sum_{k=1}^{\infty} \Delta_k^2 < \infty \quad \text{and} \quad \lim_{k \to \infty} \| \nabla h(x^k) \| = 0,
$$

(51)

almost surely and the quantity

$$
\Psi_k = \max \left\{ \frac{\| \nabla h(x^k) \|}{\Delta_k}, \mathcal{L}_2 \right\}
$$

(52)

is a supermartingale under the natural filtration, where

$$
\mathcal{L}_1 = \max \left\{ 2L_F + \kappa_{eg}, \frac{(3 - \eta_2) \kappa_{eg} + 4 \kappa_{ef}}{1 - \eta_2}, \frac{2L_h}{\gamma_{inc}} \frac{L_h}{\gamma_{inc}} \left( \frac{1 - \gamma_{dec}}{\gamma_{dec}} - \frac{1 - \gamma_{inc}}{\gamma_{inc}} \right)^{-1} \right\},
$$

$$
\mathcal{L}_2 = \max \left\{ \gamma_{dec}, L_h + \mathcal{L}_1 \right\}.
$$

Theorem 6.3 (convergence of the trust-region algorithm). Suppose Assumptions 2.1–2.4 hold. Suppose Algorithm 1 is applied with constants $\gamma_{inc}, \gamma_{dec}$, fixed parameters $\rho, \mu$, and the sample size $N_k$ satisfies (41) at each iteration. Let $\{x_k\}_{k=1}^{\infty}$ be the sequence generated from this algorithm when the termination criterion is disregarded, and $\{\Delta_k\}_{k=1}^{\infty}$ be the corresponding trust-region radii. Then

$$
\sum_{k=1}^{\infty} \Delta_k^2 < \infty, \quad \text{and} \quad \lim_{k \to \infty} \| \nabla \Phi(x^k, \rho, \mu) \| = 0
$$

(53)

almost surely. Furthermore, there exists a parameter $\mathcal{L}(\rho, \mu, K_3)$, which only depends on $\rho, \mu, K_3$ (defined in (35)), all the Lipschitz constants $L_f, L_g, \| H \|$ (the Hessian used for the local approximation model (23)) and parameters $\eta_1, \eta_2, \gamma_{inc}, \gamma_{dec}$ in Algorithm 1, such that the sequence $\{\Psi_k\}_{k=1}^{\infty}$ is a supermartingale under the natural filtration, where

$$
\Psi_k(\rho, \mu, K_3) = \max \left\{ \frac{\| \nabla \Phi(x^k, \rho, \mu) \|}{\Delta_k}, \mathcal{L}(\rho, \mu, K_3) \right\}
$$

(54)

Proof. The proof is a straightforward application of Theorem 6.2 on the merit function $\Phi(\cdot, \rho, \mu)$ with fixed penalty and Lagrangian parameters $\rho$ and $\mu$. By Theorem 6.1, the assumptions and sample size condition in the theorem guarantee that the local models in Algorithm 1 satisfy the $(\varepsilon, \theta)$-probabilistic local approximation accuracy (Definition 6.3), which is required by Theorem 6.2 as (49) and (50).

Note that in this case, $\kappa_{eg} = 8K_3|Z_0| \left( 1 + \sum_{i \in Z_0} \mu_i + \rho \right)$ and $\kappa_{ef} = 6K_3|Z_0| \left( 1 + \sum_{i \in Z_0} \mu_i + \rho \right)$ from Theorem 6.1. The Lipschitz constant $L_\Phi$ of $\Phi$ can be bounded as

$$
L_\Phi \leq \left( 1 + \sum_{i \in Z_0} \mu_i + \rho \right) \max \{ L_f, L_g, L_g^2 \}
$$

and hence the supermartingale result follows with $\Psi$ defined in (54).
Theorem 6.3 ensures the norm of the gradient of $\Phi$ converges to zero almost surely as the number of iterations goes to infinity. However, we cannot run Algorithm 1 without termination since the penalty parameters need to be adjusted in the outer iterations and Algorithm 1 will be called repeatedly. When the termination radius $r$ is imposed, the algorithm will stop in finitely many iterations almost surely. In this case, the following theorem characterizes the quality of the solution.

**Theorem 6.4.** Suppose Assumptions 2.1–2.4 hold. Suppose Algorithm 1 is applied with fixed Lagrangian parameters $\rho, \mu$ and a termination radius $r$. If the sample size $N_k$ satisfies (41) at every iteration, then the algorithm will terminate in finitely many iterations almost surely. Let $x^\tau$ be the returned solution of the algorithm. Then

$$
P(\|\nabla \Phi(x^\tau, \rho, \mu)\| \leq \delta) \geq 1 - \frac{r}{\delta} \Psi_0(\rho, \mu, K_S) \quad \forall \delta > 0,
$$

where $\Psi_0(\rho, \mu, K_S)$ is defined in (54) with $k = 0$, which depends on $\rho$, $\mu$, $i \in I_0$, $K_S$, all the Lipschitz constants and internal parameters of Algorithm 1.

**Proof.** We prove by contradiction. Suppose the algorithm does not terminate. Given that Assumptions 2.1–2.4 hold, by Theorem 6.1(b), Algorithm 1 satisfies the $(\varepsilon, \theta)$-probabilistic local approximation accuracy condition. Then by Theorem 6.3, we have $\sum_{k=1}^{\infty} \Delta_k^2 < \infty$ almost surely. It follows that there exists a first iteration $\tau$ such that $\Delta_\tau < r$ almost surely, which meets the termination criterion. Therefore, the algorithm terminates in finitely many iterations almost surely. Let $\Psi_k = \max \{\|\nabla \Phi(x^k, \rho, \mu)\| / \Delta_k, L(\rho, \mu, K_S)\}$, which is a super-martingale according to Theorem 6.3. The super-martingale property implies that $E[\Psi_k] \leq \Psi_0$. Therefore, it follows that

$$
E\left[\frac{\|\nabla \Phi(x^k, \rho, \mu)\|}{\Delta_k} \right] \leq \Psi_0 \: \forall k.
$$

The termination criterion and the definition of $\tau$ implies

$$
\Delta_k > r \quad \forall k < \tau, \quad \text{and} \quad r \geq \Delta_\tau = \gamma_{\text{dec}} \Delta_{\tau-1} \geq \gamma_{\text{dec}} r.
$$

Using the optional stopping theorem [12] with respect to $\tau$, we have

$$
E\left[\frac{\|\nabla \Phi(x^\tau, \rho, \mu)\|}{r} \right] \leq E\left[\frac{\|\nabla \Phi(x^\tau, \rho, \mu)\|}{\Delta_\tau} \right] \leq \Psi_0,
$$

which implies that

$$
E[\|\nabla \Phi(x^\tau, \rho, \mu)\|] \leq r\Psi_0.
$$

Using the Markov inequality, we have

$$
P(\|\nabla \Phi(x^\tau, \rho, \mu)\| \leq \delta) = 1 - P(\|\nabla \Phi(x^\tau, \rho, \mu)\| > \delta)
\geq 1 - \frac{E[\|\nabla \Phi(x^\tau, \rho, \mu)\|]}{\delta} \geq 1 - \frac{r}{\delta} \Psi_0,
$$

which concludes the proof.

### 6.3 Almost surely convergence

We now analyze the convergence of Algorithm 2. We first present a technical lemma that will be frequently used in the remainder of the manuscript. Lemma 6.1 summarizes [27, Lemma 3.4], which is implied by the Borel–Cantelli lemma and the Markov inequality.

**Lemma 6.1.** Let $X$ be a random variable and $\{X_k\}_{k=1}^\infty$ be a sequence of random variables. If for some $r > 0$, $\sum_{k=1}^\infty E[|X_k - X|^r] < \infty$, then $X_k \to X$ a.s. If $r > 0$, $\sum_{k=1}^\infty E[|X_k|^{2r}] < \infty$, then $X_k \to X$ a.s.

The following lemma will be used to prove the almost sure convergence of the merit-function gradient later in Lemma 6.4.
Lemma 6.2. Let \( \{X_m\}_{m=1}^\infty \) be a sequence of random variables and let \( F_m = \sigma(\{X_i\}_{i=1}^m) \) be the natural filtration. Let \( \{\delta_m\}_{m=1}^\infty \) and \( \{\varepsilon_m\}_{m=1}^\infty \) be sequences of positive real numbers that converge to zero. If \( E[X_m^2|F_{m-1}] \) is almost surely bounded uniformly by a constant and there exists \( \varepsilon > 0 \) such that \( \delta_m \leq O(1/m^{1+\varepsilon}) \), \( \varepsilon_m \leq O(1/m^{2+\varepsilon}) \), and \( P(|X_{m+1}| > \delta_{m+1}|F_m) \leq \varepsilon_{m+1} \) a.s. for every \( m \), then \( X_m \to 0 \) a.s.

Proof. We first apply Lemma 6.1 to \( \{X_m\}_{m=1}^\infty \) with \( r = 1 \), \( X_0 = 0 \). So it suffices to verify \( \sum_{m=1}^\infty E[|X_m|] < \infty \). The value \( E[|X_m|] = E[E[|X_m|F_{m-1}]] \) can be bounded:

\[
E[|X_m||F_{m-1}] = E[|X_m|1_{\{|X_m| > \delta_m\}|F_{m-1}}] + E[|X_m|1_{\{|X_m| \leq \delta_m\}|F_{m-1}}] \\
\leq \sqrt{E[|X_m|^2|F_{m-1}] \cdot P(|X_m| > \delta_m|F_{m-1})} + \delta_m \leq O(\sqrt{\varepsilon_m} + \delta_m) \text{ a.s.} \tag{56}
\]

Since \( \varepsilon_m \leq O(1/m^{2+\varepsilon}) \) and \( \delta_m \leq O(1/m^{1+\varepsilon}) \), we have \( E[|X_m||F_{m-1}] \leq O(\sqrt{\sum_{m=1}^\infty 1/m^{1+\varepsilon}} + \sum_{m=1}^\infty 1/m^{1+\varepsilon}) < \infty \) and the result is shown.

Lemma 6.3. Let \( \{F_m\}_{m=1}^\infty \) be a filtration and \( \{E_m\}_{m=1}^\infty \) be a sequence of events such that \( E_k \) is \( F_m \)-measurable for all \( k < m \). Suppose \( P(E_m|F_m) \leq \delta_m \) for all \( m \). Then

\[
P(\cap_{i=k}^m E_i) \leq \prod_{i=k}^m \delta_i.
\]

Proof. Let \( 1_{E_i} \) be the indicator random variable of \( E_i \). It follows that

\[
P(\cap_{i=k}^m E_i) = E \left[ \prod_{i=k}^m 1_{E_i} \right] = E \left[ \prod_{i=k}^m 1_{E_i} | F_m \right]
\]

\[
= E \left[ 1_{E_m} | F_m \right] \cdot \prod_{i=k}^{m-1} 1_{E_i}
\]

\[
= E \left[ P(E_m|F_m) \cdot \prod_{i=k}^{m-1} 1_{E_i} \right] \\
\leq \delta_m E \left[ \prod_{i=k}^{m-1} 1_{E_i} \right].
\]

Using induction, we can finally get

\[
P(\cap_{i=k}^m E_i) \leq \prod_{i=k}^m \delta_i.
\]

We can now show that the gradient of the merit function converges to zero.

Lemma 6.4. Suppose Assumptions 2.1–2.4 hold. Suppose Algorithm 2 is applied to (NLP) to generate the sequences \( \{x^m\}_{m=1}^\infty \) and \( \{\rho^m, \mu^m, \bar{\mu}^m\}_{m=1}^\infty \). Suppose the sample size in Algorithm 1 satisfies (41) in each iteration. Suppose the termination trust-region radius \( r^m \) in Algorithm 2 satisfies \( r^m \leq o \left( \frac{1}{\Psi^m} \right) \) for some \( \sigma > 0 \), where \( \Psi^m = \Psi_0(\rho^m, \bar{\mu}^m, K_\Sigma) \) is defined in (54) with \( k = 0 \). Then for any subsequence \( \mathcal{M} \), almost surely there exists a subsequence \( \mathcal{M}' \subseteq \mathcal{M} \) such that

\[
\lim_{m \in \mathcal{M}'} \|\nabla \Phi(x^m, \rho^m, \bar{\mu}^m)\| = 0.
\]

Proof. By Theorem 6.4,

\[
P \left( \|\nabla \Phi(x^m, \rho^m, \bar{\mu}^m)\| > \delta_m | F_{m-1} \right) \leq \frac{r_m}{\delta_m} \Psi^m
\]
holds for all \( \delta^m > 0 \) for every iteration \( m \in \mathcal{M} \). Since \( r^m \leq o \left( \frac{1}{m^2} \right) \), let \( r^m = \frac{\delta^m}{\sqrt{a_m}} \) for some \( a_m \to 0 \). Set \( \delta^m = \sqrt{a_m} \) and substitute the expression of \( r^m \) and \( \delta^m \) into the above inequality, we obtain

\[
P \left( \| \nabla \Phi(x^m, \rho^m, \hat{\mu}^m) \| > \sqrt{a_m} \right| \mathcal{F}_{m-1} \right) \leq \sqrt{a_m}.
\]

(58)

Let \( E_m \) be the event:

\[
\| \nabla \Phi(x^m, \rho^m, \hat{\mu}^m) \| \leq \sqrt{a_m}.
\]

The probability inequality (58) together with Lemma 6.3 further imply that almost surely there exist an infinite number of events from the sequence \( \{E_m : m \in \mathcal{M} \} \) that happen. Let \( \{E_m : m \in \mathcal{M}' \} \) be such a subsequence of events from \( \{E_m : m \in \mathcal{M} \} \) that happen. In this subsequence, we have

\[
\lim_{m \in \mathcal{M}'} \| \nabla \Phi(x^m, \rho^m, \hat{\mu}^m) \| = \lim_{m \in \mathcal{M}'} \sqrt{a_m} = 0.
\]

\( \square \)

The following lemma provides conditions on the sample size \( N^{(m)} \) in each iteration of Algorithm 2 to ensure the convergence of the quantile-constraint evaluation based on samples.

**Lemma 6.5.** Suppose Assumptions 2.1–2.4 hold. Let \( \{x^m\}_{m=1}^{\infty} \) be the points generated from Algorithm 2. Suppose the sample size in Algorithm 1 satisfies (41) in each iteration. If the sample size \( N^m \) from Step 2 of Algorithm 2 satisfies the condition

\[
N^m \geq O \left( \frac{C_2^2 \alpha(1 - \alpha)(\log \frac{1}{\delta^m})^3}{d_2^2 \cdot (\delta^m)^2} \right),
\]

(59)

with \( \delta^m \leq O(1/m^{1 + \sigma}) \), \( \gamma^m \leq O(1/m^{2 + \sigma}) \) holding for some \( \sigma > 0 \), then

\[
\lim_{m \to \infty} |g_{0,N^m}(x^m) - g_0(x^m)| = 0 \quad a.s.
\]

(60)

**Proof.** Define a sequence of random variables

\[
X_m = |g_{0,N^m}(x^m) - g_0(x^m)| = \left| \hat{Q}^{1-\alpha}_N(x^m) - Q^{1-\alpha}(x^m) \right|.
\]

We will apply Lemma 6.2 to \( \{X_m\}_{m=1}^{\infty} \) for the desired result. We first show that the quantity \( \mathbb{E}[X_m^2 | \mathcal{F}_{m-1}] \) is bounded. Note that \( \mathbb{E}[X_m^2 | \mathcal{F}_{m-1}] \) has the form

\[
\mathbb{E}[|g_{0,N^m}(x^m) - g_0(x^m)|^2 | \mathcal{F}_{m-1}] = \mathbb{E}[|\hat{Q}^{1-\alpha}_N(x^m) - Q^{1-\alpha}(x^m)|^2 | \mathcal{F}_{m-1}].
\]

Corollary 3.1 (given Assumptions 2.2–2.4) shows that if \( N \geq O \left( \frac{C_2^2 \alpha(1 - \alpha)(\log \frac{1}{\delta^m})^3}{d_2^2 \cdot (\delta^m)^2} \right) \), the following tail bound holds for all \( x \in S \):

\[
P \left( \left| \hat{Q}^{1-\alpha}_N(x) - Q^{1-\alpha}(x) \right|^2 \geq \delta^2 \right) \leq \gamma.
\]

If we let \( Y = \left| \hat{Q}^{1-\alpha}_N(x) - Q^{1-\alpha}(x) \right|^2 \), the above inequality is equivalent to \( 1 - F_Y(\delta^2) \leq \gamma \), where \( F_Y \) represents the CDF of the random variable \( Y \). The sample condition \( N \geq O \left( \frac{C_2^2 \alpha(1 - \alpha)(\log \frac{1}{\delta^m})^3}{d_2^2 \cdot (\delta^m)^2} \right) \) implies that \( \gamma_0 \leq \exp(-C \delta^{2/3} N^{1/3}) \) for some constant \( C > 0 \). It follows that for any \( x \in S \):

\[
\mathbb{E}[|\hat{Q}^{1-\alpha}_N(x) - Q^{1-\alpha}(x)|^2] = \int_0^\infty [1 - F_Y(t)] dt \leq \int_0^\infty \exp(-CN^{1/3} t^{1/3}) dt < \infty.
\]

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The above inequality indicates that the expectation is uniformly bounded by a constant. We conclude that 
\[ E[X_m^2 | \mathcal{F}_{m-1}] \] is bounded. Then by Corollary 3.1, because \( N^m \geq O \left( \frac{C^2(1-\alpha)(\log \delta)}{d^2(\delta^m)^2} \right) \), it follows that 
\[ \mathbb{P}(\|X_m\| > \delta^m | \mathcal{F}_{m-1}) \leq \gamma^m. \]

Applying Lemma 6.2 to \( \{X_m\}_{m=1}^\infty \) concludes the proof. \( \square \) \( \square \)

Lemma 6.6 and Theorem 6.5 are probabilistic counterparts of Lemma 2.3 and Theorem 2.1 in [2], respectively.

Lemma 6.6. Suppose Assumptions 2.1–2.4 hold and Algorithm 2 has been applied to solve (NLP). Let 
\( \{x^m, \rho^m, \mu^m, \overline{\mu}^m\}_{m=1}^\infty \) be the (probabilistic) sequence generated from Algorithm 2. Let \( x^* \) be a point and let \( E^* \) be the event 
\[ E^* = \{ \omega \in \Omega : \{x^m(\omega)\} \text{ has a limit point } x^* \}. \] 

Let \( J = \{ i \in \mathcal{I}_0 : g_i(x^*) < 0 \} \). Suppose the sample size in Algorithm 1 satisfies (41) in each iteration, 
the trust-region radius \( r^m \) in Step 1 of Algorithm 2 satisfies the condition in Lemma 6.4 and the number of samples \( N^m \) in Step 2 of Algorithm 2 satisfies the condition in Lemma 6.5 for all iterations. Then the following properties hold:

1. Almost surely on \( E^* \), there exists a subsequence \( \mathcal{M} \) satisfying 
\[ \lim_{m \in \mathcal{M}} \left\| \nabla f(x^m) + \sum_{i \in \mathcal{I}_0 \setminus J} \mu_i^m \nabla g_i(x^m) \right\| = 0. \] 

2. If \( x^* \) is feasible and the MFCQ constraint qualification is satisfied by (NLP) at \( x^* \), then almost surely on \( E^* \), the \( \{\mu_i^{m+1} : m \in \mathcal{M}\} \) are bounded. In this case, \( x^* \) satisfies the KKT conditions, and if there is only one \( \mu^* \) satisfying 
\[ \nabla f(x^*) + \sum_{i \in \mathcal{I}_0} \mu_i^* \nabla g_i(x^*) = 0 \] and \( \mu^* \geq 0 \), we have 
\[ \lim_{m \in \mathcal{M}} \mu_i^{m+1} = \mu^* \text{ a.s. on } E^*. \] 

Proof. Part (1). By Lemma 6.4, almost surely on \( E^* \) there exists a subsequence indexed by \( \mathcal{M} \) (depending on the sample path \( \omega \)) such that 
\[ \lim_{m \in \mathcal{M}} \| \nabla \Phi(x^m, \rho^m, \mu^m) \| = 0. \] 

Since 
\[ \nabla \Phi(x^m, \rho^m, \mu^m) = \nabla f(x^m) + \sum_{i \in \mathcal{I}_0} \max \{ 0, \overline{\mu}_i^m + \rho^m g_i(x^m) \} \nabla g_i(x^m) \]
\[ = \nabla f(x^m) + \sum_{i \in \mathcal{I}_0} \mu_i^{m+1} \nabla g_i(x^m), \] 
(64) can be equivalently written as 
\[ \lim_{m \in \mathcal{M}} \left\| \nabla f(x^m) + \sum_{i \in \mathcal{I}_0} \mu_i^{m+1} \nabla g_i(x^m) \right\| = 0. \] 

We split the discussion for two cases.

Case 1. Suppose \( \{\rho^m : m \in \mathcal{M}\} \) are bounded. Then by the condition in Line 7 of Algorithm 2 for updating \( \rho^m \), for large enough \( m \in \mathcal{M} \) we should have 
\[ \sigma_N(x^m, \mu^{m+1}) \leq \eta^m, \]
and hence 
\[ \lim_{m \in \mathcal{M}} \sigma_N(x^m, \mu^{m+1}) = 0. \]
Since we also have
\[
\langle \sum_{i \in I} \mu_i^{m+1}, \mu_i \rangle = 0 \quad \forall i \in I.
\] (67)

It implies the following limits hold:
\[
\lim_{m \to \infty} \min \{ -g_{0,N}(x^m), \mu_0^{m+1} \} = 0,
\]
\[
\lim_{m \to \infty} \min \{ -g_i(x^m), \mu_i^{m+1} \} = 0 \quad \forall i \in I.
\] (68)

Combining (67) and (68) gives
\[
\lim_{m \to \infty} \min \{ -g_i(x^m), \mu_i^{m+1} \} = \lim_{m \to \infty} \min \{ -g_i(x^*), \mu_i^{m+1} \} = 0 \quad \forall i \in I_0.
\] (69)

Since \(g_i(x^*) < 0\) for \(i \in J\), the above limit implies that
\[
\lim_{m \to \infty} \mu_i^{m+1} = 0 \quad \text{a.s. on } E^* \forall i \in J.
\] (70)

Substituting (70) into (66) shows that (62) holds in Case 1.
Case 2. \(\{\rho_m : m \in M\}\) is unbounded, i.e., \(\rho_m \to \infty\). Since \(g_i(x^*) < 0\) for \(i \in J\) and \(0 \leq \tilde{\mu}_i^m \leq \mu_{max}\), using (68) shows that for all \(m \in M\) large enough we have
\[
\tilde{\mu}_i^m + \rho_m g_i(x^m) < 0 \quad \forall i \in I \cap J,
\]
\[
\tilde{\mu}_i^m + \rho_m g_i(x^m) < 0 \quad \text{a.s. on } E^* \quad \forall 0 \in J.
\]

Therefore, \(\lim_{m \to \infty} \mu_i^{m+1} = 0\) for \(i \in J\), which shows that (62) holds in Case 2. This concludes the proof of Part (1).

Part (2). Assume the sequence \(\{\mu_i^{m+1} : m \in M\}\) is not bounded. Let \(B_m = \|\mu_i^{m+1}\|_\infty\). Then the sequence \(\{\mu_i^{m+1}/B_m : m \in M\}\) is bounded and hence there exists a subsequence \(M_1 \subseteq M\) such that
\[
\lim_{m \to \infty} B_m = \infty, \quad \lim_{m \to \infty} \mu_i^{m+1}/B_m = \mu_i' \quad \text{for some } \mu_i' \geq 0.
\] (71)

Combining (62) and (71) gives
\[
0 = \lim_{m \to \infty} \frac{1}{B_m} \sum_{i \in I_0 \setminus J} \mu_i^{m+1} \nabla g_i(x^m)
\]
\[
= \sum_{i \in I_0 \setminus J} \mu_i' \nabla g_i(x^*)
\] (72)

Since the MFCQ holds at \(x^*\), there exists a vector \(d\) such that \(\langle \nabla g_i(x^*), d \rangle < 0\) for all \(i \in I_0 \setminus J\) (Definition 4.1). Therefore,
\[
\left\langle \sum_{i \in I_0 \setminus J} \mu_i' \nabla g_i(x^*), d \right\rangle = \sum_{i \in I_0 \setminus J} \mu_i' \langle \nabla g_i(x^*), d \rangle < 0,
\]
which contradicts (72). It shows that \(\{\mu_i^{m+1} : m \in M\}\) are bounded. Since \(\{\mu_i^{m+1} : m \in M\}\) are bounded and \(x^*\) is feasible, any limit point \(\mu_i'\) of \(\{\mu_i^{m+1} : m \in M\}\) is a set of KKT multipliers associated with \(x^*\), where the complementary slackness condition is ensured by (69). If \(\mu_i' = \mu_i^*\) is unique, it follows that \(\{\mu_i^{m+1} : m \in M\}\) must be a convergent sequence and \(\lim_{m \to \infty} \mu_i^{m+1} = \mu_i^*\). This concludes the proof of Part (2). \(\square\)
The following theorem gives the properties of a limit point \( x^* \) of the probabilistic sequence \( \{x^m\} \) generated by Algorithm 2 in different cases.

**Theorem 6.5.** Suppose Assumptions 2.1–2.4 hold and Algorithm 2 has been applied to solve (NLP). Let \( \{x^m, \rho^m, \mu^m, \bar{\mu}^m\} \) be the sequence generated from Algorithm 2. Let \( x^* \) be a point and let \( E^* \) be the event

\[
E^* = \{ \omega \in \Omega : \{x^m(\omega)\} \text{ has a limit point } x^* \}.
\]

Suppose the sample size in Algorithm 1 satisfies (41) in each iteration, the trust-region radius \( r^m \) in Step 1 of Algorithm 2 satisfies the condition in Lemma 6.4 and the number of samples \( N^m \) in Step 2 of Algorithm 2 satisfies the condition in Lemma 6.5 for all iterations. Then the following statements hold almost surely on \( E^* \):

1. If \( \{\rho^m\} \) is bounded, \( x^* \) is feasible.
2. \( x^* \) is a stationary point of the function \( \| \max \{0, g(x)\} \|^2 \), where the max operator acts on every component of \( g \).
3. If \( x^* \) is feasible and satisfies the CPLD constraint qualification of (NLP), then \( x^* \) fulfills the KKT conditions of (NLP).

**Proof.** Part (1). Since \( \{\rho^m\} \) is bounded, using the same argument as in Case 1 of Lemma 6.6 Part (1), we can obtain (69), which states that almost surely on \( E^* \) there exists a subsequence indexed by \( \mathcal{M} \) such that

\[
\lim_{m \in \mathcal{M}} \min \{ -g_i(x^*), \mu^{m+1}_i \} = 0 \quad \forall i \in I_0.
\]

Given that \( \mu^{m+1} \geq 0 \) by the updating rule from (27), the above limit implies that

\[
g_i(x^*) \leq 0 \quad \forall i \in I_0,
\]

which proves the feasibility of \( x^* \).

Part (2). Let \( F(x) := \| \max \{0, g(x)\} \|^2 = \sum_{i \in I_0} | \max \{0, g_i(x)\} |^2 \). The gradient of \( F \) has the form

\[
\nabla F(x) = \sum_{i \in I_0} \max \{0, g_i(x)\} \nabla g_i(x).
\]

If \( x^* \) is feasible, we have \( \max \{0, g_i(x^*)\} = 0 \) and hence \( \nabla F(x^*) = 0 \). Suppose \( x^* \) is infeasible. By the result of Part (1), \( \{\rho^m\} \) must be unbounded. Using (65) and (65), we conclude that almost surely on \( E^* \) there exist a subsequence \( \mathcal{M} \) satisfying

\[
0 = \lim_{m \in \mathcal{M}} \| \nabla \Phi(x^m, \rho^m, \bar{\mu}^m) \|
\]

\[
= \lim_{m \in \mathcal{M}} \left\| \nabla f(x^m) + \sum_{i \in I_0} \max \{0, \bar{\mu}^m_i + \rho^m g_i(x^m)\} \nabla g_i(x^m) \right\|.
\]

Divide both sides of (76) by \( \rho^m \) yields

\[
\lim_{m \in \mathcal{M}} \left\| \frac{1}{\rho^m} \nabla f(x^m) + \sum_{i \in I_0} \max \left\{0, \frac{\bar{\mu}^m_i}{\rho^m} + g_i(x^m)\right\} \nabla g_i(x^m) \right\| = 0.
\]

Since \( x^m \to x^* \), \( \bar{\mu}^m_i \leq \mu_{\max} \) and \( \{\rho^m\} \) is unbounded, the above limit gives

\[
\left\| \sum_{i \in I_0} \max \{0, g_i(x^*)\} \nabla g_i(x^*) \right\| = 0.
\]

Part (3). Let \( \mathcal{M} \) be the index set of a subsequence satisfying \( \lim_{m \in \mathcal{M}} x^m = x^* \) and (66). We have that almost surely on \( E^* \),

\[
\lim_{m \in \mathcal{M}} \left\| \nabla f(x^m) + \sum_{i \in I_0} \mu^{m+1}_i \nabla g_i(x^m) \right\| = 0.
\]
Let
\[ a^m = \nabla f(x^m) + \sum_{i \in I_0} \mu_i^{m+1} \nabla g_i(x^m). \]  

Case 1. The subsequence \( \{\|\mu_i^{m+1}\|_{\infty} : m \in M\} \) is bounded. Then there exists a convergent subsequence indexed by some \( M_1 \) such that \( \lim_{m \in M_1} \mu_i^{m+1} = \mu' \) for some \( \mu' \geq 0 \). The limit (77) implies that
\[ \nabla f(x^*) + \sum_{i \in I_0} \mu'_i \nabla g_i(x^*) = 0. \]

The above limit (for stationary condition) and (74) (for complementary slackness) point out that \( x^* \) fulfills the KKT condition in Case 1.

Case 2. The subsequence \( \{\|\mu_i^{m+1}\|_{\infty} : m \in M\} \) is unbounded. Define
\[ G^m = \sum_{i \in I_0} \mu_i^{m+1} \nabla g_i(x^m). \]

Since \( \|G^m\| \leq \|\nabla f(x^m)\| + \|v^m\| \), \( \{G^m\} \) is bounded. By Caratheodory’s theorem, there exist a subset \( I^m \subset I_0 \) and a vector \( \hat{\mu}^{m+1} \in \mathbb{R}^{|I^m|} \) such that \( \{\nabla g_i(x^m) : i \in I^m\} \) are linearly independent and
\[ G^m = \sum_{i \in I^m} \hat{\mu}_i^{m+1} \nabla g_i(x^m) \quad \forall m \in M. \]  

Taking an appropriate subsequence \( M_1 \), we may assume that \( I' = I^m \) for large enough \( m \in M_1 \), as because there are only finitely many subsets of \( I_0 \). Let \( B^m = \max \{1, \|\hat{\mu}_i^{m+1}\|_{\infty}\} \). Divide (79) by \( B^m \) gives that
\[ \frac{G^m}{B^m} = \sum_{i \in I'} \hat{\mu}_i^{m+1} \nabla g_i(x^m) \quad \forall m \in M. \]  

Since \( \{\hat{\mu}_i^{m+1}/B^m : m \in M\} \) is bounded, it has a convergent subsequence \( M_1 \) such that \( \hat{\mu}_i^{m+1}/B^m \to \mu' \) for some \( \mu' = 1 \geq 0 \) but \( \|\mu'_i\|_{\infty} > 0 \). Taking the limit for \( m \in M_\infty \) on both sides of (80) and using \( \|G^m/B^m\| \to 0 \) leads to
\[ \sum_{i \in I'} \mu'_i \nabla g_i(x^*) = 0. \]  

It shows that \( \{\nabla g_i(x^*) : i \in I'\} \) are linearly dependent, and hence due to the CPLD condition, \( \{\nabla g_i(x) : i \in I'\} \) are linearly dependent for any \( x \) in a neighborhood \( U \) of \( x^* \). But this contradicts (79). \( \square \)

7 Preliminary Numerical Investigation

Algorithms 1 and 2 proposed in this work have been implemented in MATLAB and are available online [31]. We have tested the numerical performance of our methods on instances of three benchmark CCPs: a non-convex quantile optimization problem, a portfolio optimization problem, and a joint chance-constrained optimization problem.

Example 7.1 (nonconvex1D [34]). The quantile optimization problem can be reformulated as a CCP:
\[ \min_{x,y} \quad y \quad \text{s.t.:} \quad \mathbb{P}[c(x,\xi) \leq y] \geq 1 - \alpha, \]  

where \( c(x,\xi) = 0.25x^4 - 1/3x^3 - ax^2 + 0.2x - 19.5 + \xi_1 x + \xi_2 \) is a non-convex univariate function, with \( \xi_1 \sim N(0,3) \) and \( \xi_2 \sim N(0,144) \) as independent random parameters. This problem is equivalent to minimizing the \((1-\alpha)\)-quantile of \( c(x,\xi) \) over \( x \).
Example 7.2 (portfolio). Consider the portfolio optimization problem instance with a single individual linear chance constraint:

$$\max t \ s.t.: \mathbb{P}\{\xi^\top x \geq t\} \geq 1 - \alpha, \sum_{i=1}^n x_i = 1, \ x_i \geq 0,$$

where $x_i$ denotes the fraction of investment in stock $i \in \{1, \ldots, n\}$, $\xi_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$ is a normally distributed random variable of return with $\mu_i = 1.05 + 0.3\frac{t_i}{n}$ and $\sigma_i = \frac{1}{2} \left(0.05 + 0.6\frac{t_i}{n}\right)$. Because the quantity $\xi^\top x$ is also a Gaussian random variable with the mean and standard deviation being analytical functions of $x$, the above problem can be reformulated as the second-order-cone program for the case $\alpha < 0.5$:

$$\max \sum_{i=1}^n \mu_i x_i + q^\alpha \sqrt{\sum_{i=1}^n \sigma_i^2 x_i^2} \ s.t.: \sum_{i=1}^n x_i = 1, \ x_i \geq 0 \ \forall i \in [n],$$

where $q^\alpha$ is the $\alpha$-quantile of the standard Gaussian distribution.

Example 7.3 (jointChance [19]). This is an example of the $\ell_1$-norm optimization with a joint chance constraint

$$\max \sum_{i=1}^n x_i \ s.t. \ \mathbb{P}\left\{\sum_{i=1}^n \xi_{ij}^2 x_i^2 \leq U, \ j = 1, \ldots, m\right\} \geq 1 - \alpha, \ x_i \geq 0 \ \forall i \in [n],$$

where $\xi_{ij}$s are dependent normal random variables with mean $j/m$ and variance 1, and $\text{cov}(\xi_{ij}, \xi_{i'j}) = 0.5$, $\text{cov}(\xi_{ij}, \xi_{i'j'}) = 0$ if $j \neq j'$. We set $m = 5$ in our numerical experiments.

Note that the above three examples have also been used in [19,34] to test the performance of their methods. We adopted the following settings in the implementation of Algorithms 1 and 2 for practical applications:

1. Since the sample size (59) and (41) required for theoretical convergence can be prohibited, we only let the sample size grow to a pre-specified maximum level ($N = 10,000$ for example) and do not include additional samples in further iterations in the implementation.

2. The theoretical convergence of our algorithms is independent of the choice of Hessian matrix approximation. In the implementation we adopt a neighborhood sampling method used by the POUNDERS derivative-free optimization method [42] to build a local quadratic model and extract the Hessian of the quadratic model as an estimation.

3. The $\beta$ value (step size) for the finite-difference estimation of the empirical quantile function is set to a constant throughout the algorithm. Theoretical convergence of the algorithm requires $\beta \to 0$, however, it has to be supported by a sufficiently large sample size $N$. Since it is impractical to let $N \to \infty$ in the implementation, $\beta$ cannot be too small. In Table 3, we compare the results from using different values for $\beta$.

Other parameters in the two algorithms are $\theta_\mu = 0.5$ (the quadratic-penalty-reduction parameter), $\gamma_{inc} = 2.0$, $\gamma_{dec} = 0.5$ (trust-region-size-control parameters), and $\eta_1 = 0.1, \eta_2 = 0.25$ (trial move acceptance parameters). The finite-difference parameters $\delta = 1.0 \times 10^{-3}$ for experiments reported in Table 1. Note that since the two methods are very different, comparison of the computational time is not very sensible as they have very different setup in parameters which matters in determining the overall performance. Therefore the computational time provided in Table 1 is mainly for reference. The termination value of the trust-region radius $r^n$ and constraint-violation tolerance $\eta^n$ in Algorithm 2 are fixed to $10^{-5}$ for each iteration.

All problem instances are generated from the three examples above by specifying the problem dimension (for ‘portfolio’ and ‘jointChance’) and the risk value $\alpha$; each instance is specified by the first three columns in Table 3. Two sets of numerical experiments were performed. First, we study the computational performance
Table 1: Computational performance of Algorithm 2 (labeled ‘finite difference’) and quantile smoothing method [34] (labeled ‘smoothing’) on example problem instances using sample sizes $N = 5,000, 10,000,$ and $20,000$. For both methods, the final objective values reported in the table are evaluated at the returned solution based on the same set of 50,000 samples.

| Ex. dim | $\alpha$ | $N = 5,000$ | $N = 10,000$ | $N = 20,000$ |
|---------|---------|-------------|-------------|-------------|
|         |         | finite difference | smoothing     | finite difference | smoothing     | finite difference | smoothing     |
|         |         | time  | obj   | time  | obj   | time  | obj   | time  | obj   | time  | obj   |
| 7.1     | 1       | 0.05  | 0.1   | 0.0075| 0.1   | 0.0078| 0.1   | -1.3069| 0.1   | -1.396| 0.1   | -1.2983|
| 7.1     | 1       | 0.1   | -4.5711| -4.5704| 0.1   | -4.5788| 0.1   | -4.5632| 0.1   | -4.5787| 0.1   | -4.1222|
| 7.1     | 1       | 0.15  | -8.8185| -8.8184| 0.1   | -7.5501| 0.1   | -7.0628| 0.1   | -8.8633| 0.1   | -7.0628|
| 7.2     | 50      | 0.05  | 0.5   | 1.2266| 0.1   | 1.2259| 0.8   | 1.2711| 0.1   | 1.2240| 1.1   | 1.2278|
| 7.2     | 50      | 0.1   | 0.5   | 1.2445| 0.1   | 1.2436| 0.6   | 1.2451| 0.1   | 1.2444| 1.2   | 1.2455|
| 7.2     | 50      | 0.15  | 0.4   | 1.2568| 0.2   | 1.2558| 0.7   | 1.2576| 0.1   | 1.2574| 1.2   | 1.2587|
| 7.2     | 100     | 0.05  | 0.9   | 1.2502| 0.3   | 1.2497| 1.5   | 1.2513| 0.4   | 1.2497| 2.6   | 1.2508|
| 7.2     | 100     | 0.1   | 0.8   | 1.2630| 0.3   | 1.2605| 1.3   | 1.2645| 0.3   | 1.2606| 2.3   | 1.2639|
| 7.2     | 100     | 0.15  | 0.9   | 1.2741| 0.3   | 1.2728| 1.5   | 1.2754| 0.3   | 1.2724| 2.3   | 1.2759|
| 7.2     | 150     | 0.05  | 1.9   | 1.2615| 0.6   | 1.2603| 2.6   | 1.2623| 0.5   | 1.2601| 4.2   | 1.2621|
| 7.2     | 150     | 0.1   | 1.6   | 1.2738| 0.5   | 1.2738| 2.7   | 1.2751| 0.5   | 1.2739| 3.8   | 1.2747|
| 7.2     | 150     | 0.15  | 1.7   | 1.2831| 0.5   | 1.2816| 2.6   | 1.2844| 0.5   | 1.2838| 5.2   | 1.2840|
| 7.2     | 200     | 0.05  | 3.1   | 1.2687| 1.0   | 1.2681| 4.3   | 1.2697| 0.8   | 1.2694| 6.2   | 1.2700|
| 7.2     | 200     | 0.1   | 2.0   | 1.2811| 1.2   | 1.2812| 3.4   | 1.2814| 0.8   | 1.2879| 5.8   | 1.2816|
| 7.2     | 200     | 0.15  | 2.1   | 1.2900| 0.9   | 1.2887| 4.0   | 1.2896| 0.7   | 1.2896| 6.0   | 1.2894|
| 7.3     | 10      | 0.05  | 4.6   | 10.6254| 5.4   | 10.6518| 12.4  | 10.6457| 11.2  | 10.6443| 13.6  | 10.6109|
| 7.3     | 10      | 0.1   | 2.5   | 9.7844| 5.2   | 9.8126| 6.9   | 9.7675| 14.3  | 9.7819| 14.4  | 9.7727|
| 7.3     | 10      | 0.15  | 4.0   | 9.2730| 5.4   | 9.2873| 15.9  | 9.2666| 10.2  | 9.2747| 16.6  | 9.2644|
| 7.3     | 20      | 0.05  | 5.3   | 22.1048| 8.9   | 22.1455| 11.3  | 22.0936| 16.5  | 22.1071| 37.3  | 22.0292|
| 7.3     | 20      | 0.1   | 2.7   | 20.5364| 7.5   | 20.5118| 6.2   | 20.5726| 15.2  | 20.5503| 20.7  | 20.5280|
| 7.3     | 20      | 0.15  | 3.5   | 19.5481| 8.2   | 19.5869| 5.0   | 19.5241| 18.2  | 19.5604| 12.0  | 19.4399|
| 7.3     | 30      | 0.05  | 9.6   | 33.1786| 9.7   | 33.4028| 12.8  | 33.2837| 24.2  | 33.4344| 16.0  | 33.3910|
| 7.3     | 30      | 0.1   | 4.0   | 30.9151| 7.7   | 31.0496| 6.2   | 30.9307| 15.3  | 31.0203| 15.4  | 31.0541|
| 7.3     | 30      | 0.15  | 4.4   | 29.4587| 8.2   | 29.5399| 5.6   | 29.4262| 8.6   | 29.4464| 14.4  | 29.4763|
| 7.3     | 40      | 0.05  | 6.2   | 44.3838| 15.2  | 44.6468| 19.6  | 44.5109| 19.2  | 44.6539| 27.9  | 44.4231|
| 7.3     | 40      | 0.1   | 5.9   | 41.8184| 10.7  | 41.8876| 12.9  | 41.7944| 15.1  | 41.7689| 17.3  | 41.5963|
| 7.3     | 40      | 0.15  | 6.1   | 39.7211| 9.8   | 39.8218| 21.0  | 39.6720| 13.6  | 39.7234| 15.1  | 39.5076|
of the augmented Lagrangian method with the empirical quantile value estimation and finite-difference estimation of the quantile gradient \((\text{ALM-quant})\) on the problem instances with different sample sizes \((N = 5,000, 10,000\ \text{and} \ 20,000)\). We compare the performance of our method with a recasted version of the method from [34], which is the most relevant one in terms of reformulating chance constraints. In [34], a smoothing method is developed to estimate the gradient of the quantile function \(\nabla Q^{1-\alpha}(x)\). In particular, it estimates \(\nabla Q^{1-\alpha}(x)\) as

\[
\nabla q(x) = \frac{\sum_{i=1}^{N} \Gamma_\epsilon(c(x, \xi_i) - Q_{\epsilon,N}(x))\nabla c(x, \xi_i)}{\sum_{i=1}^{N} \Gamma_\epsilon(c(x, \xi_i) - Q_{\epsilon,N}(x))},
\]

where \(\Gamma_\epsilon\) is a smoothing function defined as

\[
\Gamma_\epsilon(y) = \begin{cases} 
1, & y \leq -\epsilon \\
\gamma_\epsilon(y), & -\epsilon < y < \epsilon \\
0, & y \geq \epsilon
\end{cases}
\]

and \(\Gamma'_\epsilon\) is the derivative of \(\Gamma_\epsilon\). The \(Q_{\epsilon,N}(x)\) in (86) is a smoothed quantile value. To obtain \(Q_{\epsilon,N}(x)\), one needs to numerically solve a nonlinear equation involving \(\Gamma_\epsilon\). This is not convenient to implement, and therefore we replace \(Q_{\epsilon,N}(x)\) with the empirical quantile \(Q^{1-\alpha}_N(x)\) in the computation of \(\nabla q(x)\). Note that (86) for computing the gradient of the smoothed quantile function can be used in many different optimization algorithms. In [34], it is used in an interior-point method to carry out the optimization. To compare it with Algorithm 2, we use it in the augmented Lagrangian method (the same base method as in Algorithm 2), and hence only the way of computing the quantile gradient is different between Algorithm 2 and the smoothing method. The candidate values of \(\epsilon\) is the same as \(\beta\) in our comparison. The computational results of our method and the smoothing method are in Table 1.

For each method of estimating the gradient of the quantile function (finite difference vs. smoothing), it turns out that the final objective values yielded by the method are not very sensitive to the increase of sample size from 5000 to 20,000. However, a larger sample size often leads to longer computational time, which is expected because the quantile estimation and gradient calculation can take more time. Overall, the two methods are competitive in terms of the computational time and the quality of the solution. The finite-difference method gives better results for the nonconvex1D problem instances. For the portfolio problem instances, the objective values obtained by the two methods are very close, but the results generated by the finite-difference method are slightly better. The smoothing method gives better results for the joint-chance problem in 30 out of 36 instances. It means that the smoothing method has some advantage in joint-chance constrained problems with higher-dimensional random parameters.

Since the portfolio problem can be reformulated as a convex optimization problem (84), its global optimal objective is obtainable, and we can further study the optimality gap of the best objective identified by the ALM-quant. The results are summarized in Table 2. The optimality gap is in the range of 0.06% ~ 0.18%, indicating that the ALM-quant method leads to high-quality solutions for this specific category of problem instances.

For the second batch of experiments, we focus on testing the impact of the finite-difference parameter \(\beta\) on the quality of solution for the problem instances with \(\alpha\) being 0.05, 0.1 and 0.15 and sample size \(N\) being 5,000 and 10,000, respectively. For each problem instance, five different finite-difference parameters were considered \((\beta = 1.0 \times 10^{-4}, 5.0 \times 10^{-4}, 1.0 \times 10^{-3}, 5.0 \times 10^{-3}\ \text{and} \ 1.0 \times 10^{-2})\), and the objective values identified by the ALM-quant under these \(\beta\) values are reported in Table 3. All other parameters are set the same as the previous experiments. It can be observed that for most of the portfolio and joint-chance problem instances, the objective values yielded by smaller \(\beta\) values (i.e., \(1 \times 10^{-4}\) and \(5 \times 10^{-4}\) are less competitive as that yielded by larger \(\beta\) values (i.e., \(1 \times 10^{-3}, 5 \times 10^{-3}\) and \(1 \times 10^{-2}\)). To interpret this outcome, we realize that there are two sources of errors: the randomness error of quantile evaluation and the numerical error of finite differencing for the gradient estimation. The randomness error by itself only depends on the sample size.
Table 2: Comparison between the optimal objective and the objective value identified by the ALM-quant method with the setting the same as for the Table 1 and $N = 10,000$. The final objective values reported in the table are evaluated at the returned solution based on the same set of 50,000 samples.

| Ex. dim | $\alpha$ | $\beta$ | $\gamma$ | $\delta/\beta$ | $\varepsilon$ | $\zeta$ |
|---------|----------|--------|---------|----------------|-------------|-------|
| 7.1     | 1        | 0.05   | 1        | 4              | 1            | 0.05  |
| 7.1     | 1        | 0.1    | 1        | 4              | 1            | 0.05  |
| 7.1     | 1        | 0.15   | 1        | 4              | 1            | 0.05  |
| 7.2     | 50       | 0.05   | 1        | 4              | 1            | 0.05  |
| 7.2     | 50       | 0.1    | 1        | 4              | 1            | 0.05  |
| 7.2     | 50       | 0.15   | 1        | 4              | 1            | 0.05  |
| 7.2     | 100      | 0.05   | 1        | 4              | 1            | 0.05  |
| 7.2     | 100      | 0.1    | 1        | 4              | 1            | 0.05  |
| 7.2     | 100      | 0.15   | 1        | 4              | 1            | 0.05  |
| 7.2     | 150      | 0.05   | 1        | 4              | 1            | 0.05  |
| 7.2     | 150      | 0.1    | 1        | 4              | 1            | 0.05  |
| 7.2     | 150      | 0.15   | 1        | 4              | 1            | 0.05  |
| 7.2     | 200      | 0.05   | 1        | 4              | 1            | 0.05  |
| 7.2     | 200      | 0.1    | 1        | 4              | 1            | 0.05  |
| 7.2     | 200      | 0.15   | 1        | 4              | 1            | 0.05  |
| 7.3     | 10       | 0.05   | 1        | 4              | 1            | 0.05  |
| 7.3     | 10       | 0.1    | 1        | 4              | 1            | 0.05  |
| 7.3     | 10       | 0.15   | 1        | 4              | 1            | 0.05  |
| 7.3     | 20       | 0.05   | 1        | 4              | 1            | 0.05  |
| 7.3     | 20       | 0.1    | 1        | 4              | 1            | 0.05  |
| 7.3     | 20       | 0.15   | 1        | 4              | 1            | 0.05  |
| 7.3     | 30       | 0.05   | 1        | 4              | 1            | 0.05  |
| 7.3     | 30       | 0.1    | 1        | 4              | 1            | 0.05  |
| 7.3     | 30       | 0.15   | 1        | 4              | 1            | 0.05  |

Table 3: Comparison of objective values identified by Algorithm 2 with five different $\beta$ parameters for computing the finite difference. We consider sample size $N = 5000$ and $N = 10,000$ for all the three numerical examples in this investigation.

| Ex. dim | $\alpha$ | $\beta$ | $\gamma$ | $\delta/\beta$ | $\varepsilon$ | $\zeta$ |
|---------|----------|--------|---------|----------------|-------------|-------|
| 7.1     | 1        | 0.05   | 1        | 4              | 1            | 0.05  |
| 7.1     | 1        | 0.1    | 1        | 4              | 1            | 0.05  |
| 7.1     | 1        | 0.15   | 1        | 4              | 1            | 0.05  |
| 7.2     | 50       | 0.05   | 1        | 4              | 1            | 0.05  |
| 7.2     | 50       | 0.1    | 1        | 4              | 1            | 0.05  |
| 7.2     | 50       | 0.15   | 1        | 4              | 1            | 0.05  |
| 7.2     | 100      | 0.05   | 1        | 4              | 1            | 0.05  |
| 7.2     | 100      | 0.1    | 1        | 4              | 1            | 0.05  |
| 7.2     | 100      | 0.15   | 1        | 4              | 1            | 0.05  |
| 7.2     | 150      | 0.05   | 1        | 4              | 1            | 0.05  |
| 7.2     | 150      | 0.1    | 1        | 4              | 1            | 0.05  |
| 7.2     | 150      | 0.15   | 1        | 4              | 1            | 0.05  |
| 7.2     | 200      | 0.05   | 1        | 4              | 1            | 0.05  |
| 7.2     | 200      | 0.1    | 1        | 4              | 1            | 0.05  |
| 7.2     | 200      | 0.15   | 1        | 4              | 1            | 0.05  |
| 7.3     | 10       | 0.05   | 1        | 4              | 1            | 0.05  |
| 7.3     | 10       | 0.1    | 1        | 4              | 1            | 0.05  |
| 7.3     | 10       | 0.15   | 1        | 4              | 1            | 0.05  |
| 7.3     | 20       | 0.05   | 1        | 4              | 1            | 0.05  |
| 7.3     | 20       | 0.1    | 1        | 4              | 1            | 0.05  |
| 7.3     | 20       | 0.15   | 1        | 4              | 1            | 0.05  |
| 7.3     | 30       | 0.05   | 1        | 4              | 1            | 0.05  |
| 7.3     | 30       | 0.1    | 1        | 4              | 1            | 0.05  |
| 7.3     | 30       | 0.15   | 1        | 4              | 1            | 0.05  |

but it will be magnified according to the step size when it is propagated into the quantile-gradient estimation. That is, the contribution of the randomness error in the quantile-gradient estimation is roughly $\delta/\beta$, where $\delta$ is the randomness error of the quantile estimation. Therefore, while reducing the finite-difference parameter can decrease the numerical error of gradient estimation in the finite difference calculation (in the case that the function value is error free), the randomness error can be magnified significantly in the end. This implies that a relatively larger finite-difference parameter can lead to a better quantile gradient estimation and hence a better solution. From a different angle, having a larger finite-difference parameter can be interpreted as an implicit smoothing.

### 8 Concluding Remarks

The finite-difference estimation of the quantile gradient has been incorporated into an augmented Lagrangian method coupled with a trust-region algorithm to approach the nonlinear optimization problem with chance constraints. Convergence analysis has been established for this approach and numerical results show that...
this approach is competitive with the explicitly smoothing method for estimating the gradient of the quantile function for the problem instances considered. It is worth remarking that the augmented Lagrangian method serves as a carrier for the estimation of quantile-function values and gradients. The estimation can certainly be used in other algorithms for constrained optimization such as the interior-point method, and it can be directly used in NLP solvers. The performance of solving nonlinear chance-constrained problem instances in practice is a combination of the solver performance, the estimation accuracy of quantile values and quantile gradients, the sampling techniques, smoothing techniques, and other ad-hoc strategies, which require additional empirical investigation.

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References

[1] E. G. Birgin, R. A. Castillo, and J. M. Martínez, *Numerical comparison of augmented Lagrangian algorithms for nonconvex programs*, Computational optimization and applications, 31 (2012), pp. 31–55, https://doi.org/10.1007/s10589-005-1066-7.

[2] E. G. Birgin, D. Fernández, and J. M. Martínez, *The boundedness of penalty parameters in an augmented Lagrangian method with constrained subproblems*, Optimization Methods & Software, 27 (2012), pp. 1001–1024, https://doi.org/10.1080/10556788.2011.556634.

[3] E. G. Birgin and J. M. Martínez, *Practical augmented Lagrangian methods for constrained optimization*, Society for Industrial and Applied Mathematics, 2014, https://doi.org/10.1137/1.9781611973365.

[4] G. Calafiore and M. C. Campi, *Uncertain convex programs: Randomized solutions and confidence levels*, Mathematical Programming, 102 (2005), pp. 25–46, https://doi.org/10.1007/s10107-003-0499-y.

[5] G. Calafiore and M. C. Campi, *The scenario approach to robust control design*, IEEE Transactions on Automatic Control, 51 (2006), pp. 742–753, https://doi.org/10.1109/TAC.2006.875041.

[6] M. C. Campi and S. Garatti, *A sampling-and-discarding approach to chance-constrained optimization: Feasibility and optimality*, Journal of Optimization Theory and Applications, 148 (2011), pp. 257–280, https://doi.org/10.1007/s10957-010-9754-6.

[7] Y. Cao and V. M. Zavala, *A sigmoidal approximation for chance-constrained nonlinear programs*, tech. report, arXiv:2004.02402, 2020, https://doi.org/10.48550/arXiv.2004.02402.

[8] A. Charnes, W. W. Cooper, and G. H. Symonds, *Cost horizons and certainty equivalents: An approach to stochastic programming of heating oil*, Management Science, 4 (1958), pp. 235–263, https://doi.org/10.1287/mnsc.4.3.235.

[9] M. Csörgő and P. Révész, *Strong approximation of the quantile process*, The Annals of Statistics, 6 (1978), pp. 882–894, https://doi.org/10.1214/aos/1176344261.

[10] F. E. Curtis, A. Wächter, and V. M. Zavala, *A sequential algorithm for solving nonlinear optimization problems with chance constraints*, SIAM Journal on Optimization, 28 (2018), pp. 930–958, https://doi.org/10.1137/19M1261985.
[11] D. Davis and D. Drusvyatskiy, *Stochastic subgradient method converges at the rate $O(k^{-1/4})$ on weakly convex functions*, tech. report, arXiv:1802.02988, 2018, https://doi.org/10.48550/arXiv.1802.02988.

[12] R. Durrett, *Probability: Theory and Examples*, Cambridge University Press, fifth ed., 2019, https://doi.org/10.1017/9781108591034.

[13] J. Garnier, A. Omrane, and Y. Rouchi, *Asymptotic formulas for the derivatives of probability functions and their Monte Carlo estimations*, European Journal of Operational Research, 198 (2009), pp. 848–858, https://doi.org/10.1016/j.ejor.2008.09.026.

[14] A. Geletu, A. Hoffmann, M. Klöppel, and P. Li, *An inner-outer approximation approach to chance constrained optimization*, SIAM Journal on Optimization, 27 (2017), pp. 1834–1857, https://doi.org/10.1137/19M1261985.

[15] R. Henrion and A. Möller, *A gradient formula for linear chance constraints under Gaussian distribution*, Mathematics of Operations Research, 37 (2012), pp. 475–488, https://doi.org/10.1287/moor.1120.0544.

[16] L. J. Hong, *Estimating quantile sensitivities*, Operations Research, 57 (2009), pp. 118–130, https://doi.org/10.1287/opre.1080.0531.

[17] J. Hu, Y. Peng, G. Zhang, and Q. Zhang, *A stochastic approximation method for simulation-based quantile optimization*, INFORMS Journal on Computing, (2022), https://doi.org/10.1287/ijoc.2022.1214.

[18] R. Jiang and Y. Guan, *Data-driven chance constrained stochastic program*, Mathematical Programming, 158 (2016), pp. 291–327, https://doi.org/10.1007/s10107-015-0929-7.

[19] R. Kannan and J. R. Luedtke, *A stochastic approximation method for approximating the efficient frontier of chance-constrained nonlinear programs*, Mathematical Programming Computation, 13 (2021), pp. 705–751, https://doi.org/10.1007/s12532-020-00199-y.

[20] A. I. Kibzun and S. Uryas’ev, *Differentiation of probability functions: the transformation method*, Computers & Mathematics with Applications, 30 (1995), pp. 361–382, https://doi.org/10.1016/0898-1221(95)00113-1.

[21] A. I. Kibzun and S. Uryas’ev, *Differentiability of probability function*, Stochastic Analysis and Applications, 16 (1998), pp. 1101–1128, https://doi.org/10.1080/07362999808809581.

[22] S. Küçükyavuz, *On mixing sets arising in chance-constrained programming*, Mathematical Programming, 132 (2012), pp. 31–56, https://doi.org/10.1007/s10107-010-0385-3.

[23] J. Larson and S. C. Billups, *Stochastic derivative-free optimization using a trust region framework*, Computational Optimization and Applications, 64 (2016), pp. 619–645, https://doi.org/10.1007/s10589-016-9827-z.

[24] J. Liu, A. Lisser, and Z. Chen, *Distributionally robust chance constrained geometric optimization*, Mathematics of Operations Research, (2022), https://doi.org/10.1287/moor.2021.1233.

[25] X. Liu, S. Küçükyavuz, and J. Luedtke, *Decomposition algorithms for two-stage chance-constrained programs*, Mathematical Programming, 157 (2016), pp. 219–243, https://doi.org/10.1007/s10107-014-0832-7.

[26] A. Lodi, E. Malaguti, G. Nannicini, and D. Thomopoulos, *Nonlinear chance-constrained problems with applications to hydro scheduling*, Mathematical Programming, 191 (2022), pp. 405–444, https://doi.org/10.1007/s10107-019-01447-3.
[27] M. Loève, *On almost sure convergence*, Berkeley Symposium on Mathematical Statistics and Probability, 2 (1951), pp. 279–303, https://projecteuclid.org/ebook/Download?urlid=bsmsp/1200500235&isFullBook=false.

[28] J. Luedtke, *A branch-and-cut decomposition algorithm for solving chance-constrained mathematical programs with finite support*, Mathematical Programming, 146 (2014), pp. 219–244, https://doi.org/10.1007/s10107-013-0684-6.

[29] J. Luedtke and S. Ahmed, *A sample approximation approach for optimization with probabilistic constraints*, SIAM Journal on Optimization, 19 (2008), pp. 674–699, https://doi.org/10.1137/070702928.

[30] J. Luedtke, S. Ahmed, and G. L. Nemhauser, *An integer programming approach for linear programs with probabilistic constraints*, Mathematical Programming, 122 (2010), pp. 247–272, https://doi.org/10.1007/s10107-008-0247-4.

[31] F. Luo and J. Larson, *Quantile optimization software*, 2023, https://doi.org/10.5281/zenodo.10044073.

[32] O. Mangasarian and S. Fromovitz, *The Fritz John necessary optimality conditions in the presence of equality and inequality constraints*, Journal of Mathematical Analysis and Applications, 17 (1967), pp. 37–47, https://doi.org/10.1016/0022-247X(67)90163-1.

[33] A. Nemirovski and A. Shapiro, *Scenario approximation of chance constraints*, in Probabilistic and Randomized Methods for Design Under Uncertainty, G. Calafiore and F. Dabbene, eds., Springer, London, 2005, pp. 3–48, https://doi.org/10.1007/1-84628-095-8_1.

[34] A. Peña-Ordieres, J. R. Luedtke, and A. Wächter, *Solving chance-constrained problems via a smooth sample-based nonlinear approximation*, SIAM Journal on Optimization, 30 (2020), pp. 2221–2250, https://doi.org/10.1137/19M1261985.

[35] Y. Peng, M. C. Fu, B. Heidergott, and H. Lam, *Maximum likelihood estimation by Monte Carlo simulation: Toward data-driven stochastic modeling*, Operations Research, 68 (2020), pp. 1896–1912, https://doi.org/10.1287/opre.2019.1978.

[36] G. C. Pflug and H. Weisshaupt, *Probability gradient estimation by set-valued calculus and applications in network design*, SIAM Journal on Optimization, 15 (2005), pp. 898–914, https://doi.org/10.1137/S1052623403431639.

[37] A. Prékopa, *On probabilistic constrained programming*, in Proceedings of the Princeton Symposium on Mathematical Programming, 1970, pp. 113–138, https://doi.org/10.1515/9781400869930-009.

[38] L. Qi and Z. Wei, *On the constant positive linear independence condition and its application to sqp methods*, SIAM Journal on Optimization, 10 (2000), pp. 963–981, https://doi.org/10.1137/S1052623497326629.

[39] S. Tong, A. Subramanyam, and V. Rao, *Optimization under rare chance constraints*, SIAM Journal on Optimization, 32 (2022), pp. 930–958, https://doi.org/10.1137/20M1382490.

[40] S. Uryas’ev, *Derivatives of probability functions and integrals over sets given by inequalities*, Journal of Computational and Applied Mathematics, 56 (1994), pp. 197–223, https://doi.org/10.1016/0377-0427(94)90388-3.

[41] W. van Ackooij and R. Henrion, *Gradient formulae for nonlinear probabilistic constraints with Gaussian and Gaussian-like distributions*, SIAM Journal on Optimization, 24 (2014), pp. 1864–1889, https://doi.org/10.1137/130922689.
[42] S. M. Wild, *Solving derivative-free nonlinear least squares problems with POUNDERS*, in Advances and Trends in Optimization with Engineering Applications, T. Terlaky, M. F. Anjos, and S. Ahmed, eds., SIAM, 2017, pp. 529–540, https://doi.org/10.1137/1.9781611974683.ch40.

[43] W. Xie, *On distributionally robust chance constrained programs with Wasserstein distance*, Mathematical Programming, 186 (2021), pp. 115–155, https://doi.org/10.1007/s10107-019-01445-5.

[44] W. Xie and S. Ahmed, *On quantile cuts and their closure for chance constrained optimization problems*, Mathematical Programming, 172 (2018), pp. 621–646, https://doi.org/10.1007/s10107-017-1190-z.

[45] Y. Xu, *Primal-dual stochastic gradient method for convex programs with many functional constraints*, SIAM Journal on Optimization, 30 (2020), pp. 1664–1692, https://doi.org/10.1137/18M1229869.