Conservation laws generated by symmetry transformations of extremals and applications in Lagrangian field theory

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Abstract

We characterize symmetry transformations of Lagrangian extremals generating ‘on shell’ conservation laws. In particular, the conserved current associated with two symmetry transformations is constructed and explicit expressions of physical interest are worked out.

Key words: symmetry transformation, conservation law, second variation, Jacobi equation, Yang–Mills theory.

2010 MSC: 81T13,53Z05,58A20,58E15,58Z05.

1 Introduction

The description of fundamental interactions in Physics as fields associated with the action of Lie groups on maps between manifolds has been the cornerstone of the last Century. Indeed within this picture fields are (local) maps between manifolds adapted to a fibration (which distinguishes independent from dependent variables and their peculiarity when changing coordinates),

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i.e. are sections of fibrations, having the additional structure of a bundle (fields take values in a manifold which is the type fiber of the bundle). In particular, it is well known that, due to their invariance properties, physical fields can be described as sections of bundles associated with principal bundles, the configuration bundles are then the so-called gauge-natural bundle, see e.g. [8, 9, 22].

It is noteworthy that the variational derivation of field equations is an intrinsic operation strictly related just to the fibration structure and its prolongation up to a given order [8, 19, 42, 45]. This approach had several important developments, in particular when combined with invariance properties (geometric formulation of the Noether Theorems, specifically).

Furthermore important is now the possibility of a systematic formulation of higher variations, interpreted as variations of suitable ‘deformed’ Lagrangians [2]; combined with symmetry considerations this approach extends to field theory the concept of so-called higher-order Noether symmetries in Mechanics [41]. These are of interest in theoretical physics, in particular concerning variations of currents [16]; for applications of the second variation in gravitational theory in this context see e.g. [11, 18].

Lagrangian symmetries and symmetries of Euler–Lagrange equations have been called by Andrzej Trautman [45] invariant transformations and generalized invariant transformations, respectively, and they were characterized as particular kinds of what he called symmetry transformations, i.e. transformations of extremals into extremals of the same Euler–Lagrange equations.

Indeed, it is well known that a symmetry of Euler–Lagrange equations (generalized invariant transformations) is also a symmetry transformation of their solutions (extremals), i.e. a transformation preserving the property of a field (a section of the configuration bundle on space-time) being an extremal [26, 45].

The inverse in general is not true: symmetry transformations of solutions of equations could not be symmetries of the equations themselves. A related result stating that a Lagrangian ‘dragged’ along symmetry transformations of its own extremals, has the same extremals as the original Lagrangian (and an inverse statement stating that a transformation dragging a given Lagrangian in a Lagrangian having the same extremals is a symmetry transformation of the extremals) was obtained in [29] (see Theorem 4.3 below).

We focus on conservation laws by explicating the relation among higher variations of Lagrangians, symmetry transformations of extremals, Jacobi fields, and conserved currents. We characterize symmetry transformations...
of extremals as particular transformations of the Euler–Lagrange forms to source forms vanishing along extremals of the original Lagrangian and specifically as Jacobi fields along extremals (Theorem 4.5).

Compared with generalized symmetry transformations (i.e. transformations leaving invariant the Euler–Lagrange form of a Lagrangian) such transformations provide a weaker invariance property, since the Euler–Lagrange form is not invariant under their action, although it is transformed to a source form having the same extremals. Therefore the equations change, but the solutions of the one equation are also solutions of the second and vice versa.

By explicating the relation among the variation of an Euler-Lagrange form with the second variation of a Lagrangian and with the Jacobi morphisms (Proposition 3.4 and Remark 4.9), we prove that with this sort of weaker invariance is anyway associated a conserved current, and in particular that this current can be identified as a Noether current for a Lagrangian ‘deformed’ by a symmetry transformation of extremals and associated with (or generated by) a symmetry transformation of extremals. More specifically, symmetry transformations of extremals generate conserved currents along the extremals themselves. Indeed Theorem 4.11 states the existence of a weak (i.e. along extremals) conservation law for any couple of (infinitesimal) generators of (vertical) symmetry transformations.

As an explicit example, we write the expressions of the on shell conserved current generated by couples of symmetry transformations of extremals.

2 Contact structure, variationality and the ‘representation’ problem

We briefly recall the modern geometric calculus of variations on finite order prolongations of fibered manifolds. We denote by $X$ a differentiable manifold of dimension $n$ and by $Y$ a differentiable manifold of dimension $m + n$; we assume that it exists a fibered manifold structure $(Y, \pi, X)$ in which $X$ is the base space, $Y$ is the total space and $\pi$ is the projection. Let $U$ be an open subset of $X$. A local section (the geometric version of a physical field) $\gamma : U \to Y$ is such that $\pi \circ \gamma = Id_U$. By $(x^i, y^\sigma)$, with $i = 1 \ldots n$ and $\sigma = 1 \ldots m$ we denote local fibered coordinates, i.e. adapted to the fibration. We set $ds_i = \frac{\partial}{\partial x^i} | ds$, where $ds = dx^1 \wedge \ldots \wedge dx^n$ is the local expression of a volume element on $X$. 


Let $\Omega_q(J^kY)$ denote the module of $q$-forms on the set $J^kY$ of equivalence classes of (local) sections of the fibration having a contact of order $k$ in a point. Note that $J^kY$ as the structure of a differentiable manifold and the structure of a fibration $\pi_k: J^kY \to X$, the prolongation of order $k$ of $\pi: Y \to X$.

Of fundamental importance in the calculus of variation is the contact structure induced by the affine bundle structure of the fibrations $\pi_{k,k-1}: J^kY \to J^{k-1}Y$ (see [42], [28]). A differential $q$-form $\alpha$ on $J^kY$ is called a contact form if, for every section $\gamma$ of $\pi$, we have $(j^k\gamma)^*(\alpha) = 0$, for any $k$ order prolongation of $\gamma$. It is easy to see that forms $\omega$ locally given as

$$\omega^\sigma_{j_1...j_h} = dy_{j_1...j_h} - y_{j_1...j_h}^\sigma dx^i$$

for $0 \leq h < k$ are indeed contact 1-forms.

In particular, $(dx^i, \omega^\sigma, \omega^\sigma_{j_1}, \ldots, \omega^\sigma_{j_{k-1}}, dy_{j_1...j_k})$ is an alternative local basis for 1-forms on $J^kY$. It is important to notice that the ideal of the exterior algebra generated by contact forms on a fixed jet order prolongation is not closed under exterior derivation, while if $\alpha$ is contact so is $d\alpha$.

For every form $\rho \in \Omega_q(J^kY)$, by the contact structure we obtain the canonical decomposition [26]

$$\pi^*_{k+1,k}(\rho) = p_0 \rho + p_1 \rho + \cdots + p_q \rho$$

where $p_0 \rho$ is a form that is horizontal on $X$ (and so is often denoted by $h\rho$) while $p_i \rho$ is an $i$-contact $q$-form, that is a form generated by wedge products containing exactly $i$ contact 1-forms. We remark that if $q > n$ every $q$-form $\rho$ is contact; then we call it strongly contact if $p_{q-n} = 0$. The contact structure induces also the splitting of the exterior differential $\pi^*_{k+1,k}d\rho = d_H \rho + d_V \rho$ in the so called horizontal and vertical differentials respectively, given by

$$d_H \rho = \sum_{l=0}^q p_l dp_\rho$$
$$d_V \rho = \sum_{l=0}^q p_{l+1} dp_\rho.$$

According to [25] we define the formal derivative with respect to the $i$-th coordinate, $i = 1, \ldots, n$, by an abuse of notation also denoted by $d_i$, as an operator acting on forms. It is defined by requiring that is commutes with the exterior derivative and that it satisfies the Leibnitz rule with respect to the wedge product. We see that $d_H \rho = (-1)^q d_i \rho \wedge dx^i$ if $\rho$ is a $q$-form. In particular, on 0-forms (functions) this operator is just the total derivative, and on the basis 1-forms we have $d_i dx^j = 0$, $d_i \omega^\sigma_{j_1...j_r} = \omega^\sigma_{j_1...j_{r-1},i}$, $d_i dy^\sigma = dy^\sigma_i$. 


By an abuse of notation, $d_i$ will also indicate the usual the vector field on $J^kY$ along $\pi_{k+1,Y}$ referred to as the formal derivative.

In the following a multi-index will be an ordered $s$-uple $I = (i_1, \ldots, i_s)$; the length of $I$ is given by the number $s$; and an expression such as $I_j$ denotes the multi-index given by the $(s+1)$-uple $(i_1, \ldots, i_s, j)$.

As much as the integration by parts procedure is concerned, we will use the local formula

$$\omega^\sigma I_i \wedge ds = -d\omega^\sigma \wedge ds_i.$$  

We also recall the properties

$$d J^\omega = \omega J$$  

and

$$\partial^j \omega^\sigma I_i = \delta_\sigma^\nu \delta_j^I$$  

(where the Kronecker symbol with multi-indices has the obvious meaning: it is 1 if the multi-indices coincide up to a rearrangement and 0 otherwise).

Finally, if $\psi$ is a projectable vector field on $Y$ (i.e. an infinitesimal automorphism preserving the fibration), $j^k \psi$ is the projectable vector field, defined on $J^kY$, associated with the prolongation of the flow of $\psi$ (see, e.g. [26, 42, 45]).

The contact structure of jet prolongations enables to define an algebro-geometric object deeply related to the calculus of variations: a differential sequence of sheaves made of equivalence classes of differential forms taking a variational meaning. We refer to [27, 30, 28, 25], and to the review [33] for the construction and the representation of finite order variational sequences. The concept of a sheaf is due to Leray [31]; a classical reference on this topic is e.g. [6]; in particular we refer to [27], where the construction of a sequence of ‘variational sheaves’ can be found.

Let $\Omega^k_q$ denote the sheaf of differential $q$-forms on $J^kY$. It can be seen as a sheaf on $Y$; in fact we assign to an open set $W \subseteq Y$ a form defined on $\pi_{r,0}^{-1}(W)$. We set $\Omega^{k}_{0,c} = \{0\}$ and denote by $\Omega^k_{q,c}$ the sheaf of contact $q$-forms, for $q \leq n$, or the sheaf of strongly contact $q$-forms if $q > n$. We define the sheaf

$$\Theta^k_q = \Omega^k_{q,c} + d\Omega^k_{q-1,c},$$

where $d\Omega^k_{q-1,c}$ is the sheaf associated with the presheaf image through $d$ of $\Omega^k_{q-1,c}$. Of course $\Theta^k_q = \{0\}$ for $q > M$, $M$ depending suitably on $n, m, k$ and we get the exact subsequence of the de Rham sequence made by soft sheaves

$$\{0\} \to \Theta^k_1 \to \Theta^k_2 \to \cdots \to \Theta^k_M \to \{0\}.$$  

The quotient sequence of the de Rham sequence of forms

$$\{0\} \to \mathbb{R}_Y \to \Omega^k_0 \to \Omega^k_{1}/\Theta^k_1 \to \cdots \to \Omega^k_M/\Theta^k_M \to \Omega^k_{M+1} \to \cdots \to \Omega^k_N \to \{0\},$$

where $N = \dim(J^kY)$ and $\mathbb{R}_Y$ is the constant sheaf over $Y$, is called the Krupka’s variational sequence of order $k$ [27]. Let us denote the quotient
sheaves by $\mathcal{V}^k_q$. Morphisms in this sequence are quotients of the exterior differential; they are denoted by $\mathcal{E}_q : \mathcal{V}^k_q \to \mathcal{V}^k_{q+1}$, i.e. $\mathcal{E}_q([\rho]) = [d\rho]$. By this construction equivalence classes of forms modulo suitable contact forms are interpreted as differential forms relevant for calculus of variation (Lagrangians, currents, source forms and so on); moreover, analogously to the de Rham sequence of forms on a differentiable manifold, we get a sequence of sheaves of ‘variational’ equivalence classes: arrows morphisms in the variational sequence are relevant operators for the calculus of variations. In particular, the Euler–Lagrange mapping can be identified with a morphism in the variational sequence (see Subsections 2.1 and 3.1). Furthermore, within this framework, the representation of the second variational derivative has been studied by different approaches in [35, 14]; the rôle of the Jacobi morphism and the relation between the Noether theorems and the second variation have been investigated in [12, 13, 15, 34, 35, 36, 37, 38].

The variational sequence is a (soft sheaf) resolution of the constant sheaf $\mathcal{R}_Y$ and the cohomology of the complex of global section of the variational sequence is the de Rham cohomology of $Y$; see [27, 28]. Dealing with exact sequence of sheaves and resolutions enables to study cohomology obstructions to \emph{variational exactness of variationally closed forms} and this turns out to be of interest in many different areas of Physics; for example an obstruction to the existence of global extremals is related to the obstruction to the existence of global Noether–Bessel-Hagen currents [39]. Various kind of ‘variationality’ problems (such as the inverse problem of calculus of variations or the local triviality problem for Lagrangians) are indeed formulated in terms of the cohomology of the configuration space.

2.1 Geometric integration by parts

Strictly related to concrete applications is then the so called \emph{representation problem}, which, roughly speaking, consists in showing that \emph{classes of forms}, i.e. elements of the quotient groups $\mathcal{V}^r_q$, can be associated with \emph{global differential forms}. By the intrinsic geometric structure of the calculus of variations on finite order prolongations of fibrations, indeed, it is possible to define an operator (called \emph{representation mapping}) which takes a differential forms on the prolongation of order $r$ and associate to it a differential form on a certain prolongation order $s \geq r$, having a meaning in the Lagrangian formalism for field theory, i.e. $R^r_q : \Omega^r_q \to \Psi^s_q$, with $\Psi^s_q$ an abelian group of forms of order $s$, such that $\ker R^r_q = \Theta^r_q$. It provides an isomorphism $\mathcal{V}^r_q \cong \Psi^s_q = R^r_q(\Omega^r_q)$.
For \( q \leq n \), \( R_q^r \) can be taken to be simply the ‘horizontalization’ \( h = p_0 \). For \( q \geq n + 1 \), it is the image of an operator denoted by \( \mathcal{I} \), which will be suitably defined below and which reflects in an intrinsic way the procedure of getting a distinguished representative of a class \([\rho] \in \Omega^k_q/\Theta^k_q \) for \( q > n \) by applying to \( \rho \) the operator \( p_{q-n} \) and then factorizing by \( \Theta^k_q \), see e.g. [33].

In this paper we will refer to the interior Euler operator defined within the finite order variational sequence according to [24, 25] and applied to the representation of variational Lie derivatives according to [33].

**Definition 2.1** In the following, differential forms which are \( \omega^\sigma \) generated \( l \)-contact \( (n+l) \)-forms will be called source forms.

Now define locally the map \( \mathcal{I} : \Omega^r_{n+k} \to \Omega^{2r+1}_{n+k} \) by

\[
\mathcal{I}(\rho) = \frac{1}{k}\omega^\sigma \wedge I_\sigma = \frac{1}{k}\omega^\sigma \wedge \sum_{|I|=0}^r (-1)^{|I|} d_I \left( \frac{\partial}{\partial y^I} \right) p_k \rho .
\]

For a given \( \rho \), \( \mathcal{I}(\rho) \) is a source form of degree \( n+k \) and it is by construction a \( k \)-contact form. It turns out that, if \( \rho \) is global, \( \mathcal{I}(\rho) \) is a globally defined form; for a proof, see [25].

The operator \( \mathcal{I} \) behaves like a projector, i.e. \( \mathcal{I} \circ \mathcal{I} = (\pi_{4n+3,2r+1})^* \circ \mathcal{I} = \mathcal{I} \circ (\pi_{2r+1})^* \). Since, for any \( \eta \in \Omega^r_{n+k} \), \( \mathcal{I} \circ p_k \circ d \circ p_k \eta = 0 \), therefore \( \Theta^r_{n+k} \subseteq \ker \mathcal{I} \). The opposite inclusion also holds true: since for any \( (n+k) \)-form \( (\pi_{2r+1})^*(\rho) - \mathcal{I}(\rho) \in \Theta^{2r+1}_{n+k} \), if \( \mathcal{I}(\rho) = 0 \) then \( (\pi_{2r+1})^*(\rho) \in \Theta^{2r+1}_{n+k} \), and this implies \( \rho \in \Theta^r_{n+k} \). In fact, for any pair of integers \( s \geq r \), the quotient map \( \chi_{q,s}^r : \mathcal{V}_q^r \to \mathcal{V}_q^s \) is injective; therefore we have \( \ker \mathcal{I} = \Theta^r_{n+k} \). These properties essentially show that the interior Euler operator is well defined and thus solves the representation problem; see also [28, 46] and references therein for other approaches.

In view of a characterization of Noether currents, we study the difference between \( (\pi_{2r+1})^*(p_k \rho) \) and \( \mathcal{I}(\rho) \). In particular, we define the residual operator \( \mathcal{R} \) by the following decomposition formula which is in fact a geometric integration parts formula

\[
(\pi_{2r+1})^*(p_k \rho) = \mathcal{I}(\rho) + p_k dp_k \mathcal{R}(\rho) . \tag{1}
\]

Note that, although the decomposition above has a global meaning, \( \mathcal{R}(\rho) \) is a strongly contact \( (n+k-1) \)-form defined only locally.
Example 2.2 Following \[25\] we can characterize \( R(\rho) \) in local coordinates. For \( k \geq 1 \), if \( \Psi^I_\sigma \) are \((k-1)\)-contact \((n+k-1)\)-forms and if \( \omega^I_\sigma \) are local generators of contact 1-forms (see Section 2), up to pull-backs, we can write (a sort of integration by parts on formal derivatives of forms)

\[
p_k \rho = \sum_{|I| = 0}^{r} \omega^I_\sigma \wedge \Psi^I_\sigma = \sum_{|I| = 0}^{r} d_I (\omega^\sigma \wedge \zeta^I_\sigma) = \mathcal{I}(\rho) + p_k dp_k R(\rho),
\]

with \( \zeta^I_\sigma = \sum_{|J| = 0}^{r-|I|} (-1)^{|J|} d_J \Psi^{JI}_\sigma \). The first term gives us the Euler–Lagrange form, while by rewriting \( \omega^\sigma \wedge \zeta^I_\sigma = \Phi^I \wedge ds \), for suitable \( k \)-contact \( k \)-forms \( \Phi^I \) on \( J^{2r} Y \), we get

\[
\sum_{|I| = 1}^{r} d_I (\omega^\sigma \wedge \zeta^I_\sigma) = d_H \left( \sum_{|I| = 0}^{r-1} (-1)^k d_I \Phi^{IJ} \wedge ds_j \right) = d_H R(\rho).
\]

This local expressions for \( R(\rho) \) will be exploited in Example 4.14 for the case \( k = 1 \), specifically for concrete 1-contact \((n+1)\)-forms \( \omega^I_\sigma \wedge \Psi^I_\sigma \) associated with the exterior differential of a suitably ‘deformed’ Yang–Mills Lagrangian. We will write explicitly the forms \( \Phi^{IJ} \) relative to this Lagrangian. Combined with results of Theorem 4.11, this approach will enable us to obtain explicit conserved currents associated with symmetry transformations of Yang–Mills extremals on Minkowski space-times.

3 (Higher) variations and related currents

The representation (in \[33\] called the Takens \[43\] representation) by the horizontalization \( h \) and the interior Euler operator \( I \) defines a sequence of sheaves of differential forms (rather than of classes of differential forms), such that both the objects and the morphisms have a straightforward interpretation in the calculus of variations. We can obtain formulae for (higher) variations of a Lagrangian based on an iteration of the first variation formula expressed through the Takens representation.

3.1 Noether currents

The formulation of the First Noether Theorem \[32\] is concerned with the representation of variational Lie derivatives of classes of degree \( n \), which illus-
trates the relation between the interior Euler operator, the Euler–Lagrange operator and the exterior differential, as well as the emerging of the divergence of the Noether currents by contact decompositions and geometric integration by parts formulae.

In the following, for any $n$-form $\rho$, $I(d\rho) = I(dh\rho) = Id(h\rho)$ is the Euler–Lagrange form $E_n(h\rho)$ obtained as the representation by the interior Euler operator of the variational class defined by $d\rho$, while for any $(n-1)$-form $\mu$, $h dh\mu = p_0 dp_0 \mu$ is the horizontal differential $d_H(h\mu)$, which can be recognized as a divergence (for the notation and the interpretation in the context of geometric calculus of variations more details can be found e.g. in [33]).

**Theorem 3.1** For any $n$-form $\rho$ and for any $\pi$-projectable vector field $\psi$ on $Y$, we have, up to pull-backs by projections,

\[ L_{J^{r+1}\psi} h\rho = \psi_V I d(h\rho) + d_H(J^{r+1}\psi_V)p_{d_V h\rho} + \psi_H h\rho \]  

(2)

where $p_{d_V h\rho} = -p_1 R(dh\rho)$.

A generalization of formula (2) to class of degree greater or lower than $n$ has been obtained [7, 33]. We stress that (2) can be regarded as the local first variation formula for the Lagrangian $h\rho$ with respect to a (variation) projectable vector field; we refer the reader to [26, 28] for details.

The formula above has been first obtained by Noether in the proof of her celebrated First Theorem (see the original Noether paper in the historical survey [23]). This suggest the definition of a Noether current.

**Definition 3.2** The Noether current for a Lagrangian $\lambda$ associated with $\psi$ is defined as

\[ \epsilon_\psi(\lambda) = J^{r+1}\psi_V p_{d_V \lambda} + \psi_H \lambda . \]

The term $p_{d_V \lambda} = -p_1 R(d\lambda)$ is called a local generalized momentum.

It should be stressed that a Noether current is defined for any projectable vector field, independently from it being a Lagrangian symmetry or not. When it is not a symmetry of course the Noether current is not conserved along critical sections.
3.2 Higher Noether currents

Now we obtain a formula for the second variation, which will be further exploited in section [4]. We note that $L_{J^r+1}h \rho = hL_{J^r}\rho$, and then apply a standard inductive reasoning. Of course, the iterated variation is pulled-back up to a suitable order, in order to suitably split the Lie derivatives [2].

**Theorem 3.3** For any $n$-form $\rho$ and any pair of $\pi$-projectable vector fields $\psi_1$ and $\psi_2$, we have, up to pull-backs by projections,

$$L_{J^r+1}\psi_2L_{J^r+1}\psi_1 h \rho = \psi_{2,V} [Id(\psi_{1,V} [Id(h \rho)]) + d_H \epsilon_{\psi_2}(\psi_{1,V} [Id(h \rho)])]$$

where we define the following (higher) Noether currents associated with $\psi_2$ for the respective new Lagrangians:

$$\epsilon_{\psi_2}(\psi_{1,V} [Id(h \rho)]) = \psi_{2,H} [\psi_{1,V} [Id(h \rho)] + J^{r+1} \psi_{2,V} [p_{dv \psi_{1,V}} Id(h \rho)],$$

$$\epsilon_{\psi_2}(d_H \epsilon_{\psi_1}(h \rho)) = \psi_{2,H} [d_H (J^{r+1} \psi_{1,V} [p_{dv} h \rho] + \psi_{1,H} [h \rho]) + J^{r+1} \psi_{2,V} [p_{dv} d_H (J^{r+1} \psi_{1,V} [p_{dv} h \rho] + \psi_{1,H} [h \rho])].$$

Note that the expression (3) is given in terms of $I$ and $R$.

Related to this formula is an identity which will suggest the definition of the Jacobi morphism, with a look to a specific characterization of symmetry transformations of extremals (see Definition 4.6).

Let then $\psi_1$, $\psi_2$ be vertical vector fields. We note that, due to the exactness of the representation sequence and linearity of the Lie derivative, (for a suitable prolongation order) we can write

$$J^s \psi_1 [L_{J^s} \psi_2 Id(h \rho)] = \psi_1 [Id(\psi_2 Id(h \rho))] =$$

$$L_{J^s} \psi_2 L_{J^s} \psi_1 h \rho - [\psi_2, \psi_1] [Id(h \rho)] - d_H \epsilon_{\psi_2}(d_H \epsilon_{\psi_1}(h \rho)).$$

From (3) we get then the following identity.

**Proposition 3.4** For every pair of vertical vector fields $\psi_1$ and $\psi_2$ it holds

$$\psi_1 [Id(\psi_2 Id(h \rho))] - \psi_2 [Id(\psi_1 Id(h \rho))] =$$

$$=[\psi_1, \psi_2] [Id(h \rho)] + d_H (\epsilon_{\psi_2}(\psi_1 Id(h \rho))).$$

Note that, being the vector fields vertical, here we have $\epsilon_{\psi_2}(\psi_1 Id(h \rho)) = J^{r+1} \psi_2 [p_{dv \psi_{1,V}} Id(h \rho)]$. Note also that this current is the Noether current for the ‘deformed’ Lagrangian $\psi_1 Id(h \rho)$ and associated to $\psi_2$. 
4 Symmetry transformations of extremals and conserved currents

Let $h\rho$ be a Lagrangian of order $r+1$ on $Y$.

**Definition 4.1** A (local) section $\gamma$ is an extremal of $h\rho$ if it satisfies
\[
\mathcal{I}d(h\rho) \circ J^{2r+1}\gamma = 0.
\]

Let now $\phi$ be an automorphism of $Y$ (i.e. a transformation preserving the fibration) with projection $\phi_0$, and let $J^{r+1}\phi$ be its prolongation.

**Definition 4.2** The automorphism $\phi$ is a symmetry transformation of an extremal $\gamma$, if the section $\phi \circ \gamma \circ \phi_0^{-1}$ is also an extremal. i.e.
\[
\mathcal{I}d(h\rho) \circ J^{2r+1}(\phi \circ \gamma \circ \phi_0^{-1}) = 0.
\]

A $\pi$-projectable vector field $\psi$ is the generator of symmetry transformations of $\gamma$, if its local one-parameter group of transformations is a flow of symmetry transformations of $\gamma$. It can be shown that a symmetry of $\mathcal{I}d(h\rho)$ is also a symmetry transformation of every extremal $\gamma$ [45, 26, 28].

According with the above references, the following relates symmetry transformations of extremals with projectable vector fields dragging $h\rho$ in such a way that $L_{J^{r+1}\psi}h\rho$ admits the same extremals.

**Theorem 4.3** Let $h\rho$ be a Lagrangian of order $r+1$ and let $\gamma$ be an extremal. Then a $\pi$-projectable vector field $\psi$ generates symmetry transformations of $\gamma$ if and only if
\[
\mathcal{I}d(L_{J^{r+1}\psi}h\rho) \circ J^{2r+1}\gamma = 0,
\]

**Remark 4.4** Note that, being the Lie derivative a natural operator, it holds $L_{J^{2r+1}\psi}\mathcal{I}d(h\rho) = \mathcal{I}d(L_{J^{r+1}\psi}h\rho)$ and $\psi$ generates symmetry transformations of $\gamma$ if and only if
\[
(L_{J^{2r+1}\psi}\mathcal{I}d(h\rho)) \circ J^{2r+1}\gamma = 0.
\]

We thus characterize vertical symmetry transformations of extremals as particular transformations of the Euler–Lagrange forms to source forms vanishing along extremals of the original Lagrangian. As we shall see soon,
basically using the characterization of the Lie derivative of Euler–Lagrangian forms in terms of the second variation (see also [35]), we shall characterize them specifically as Jacobi fields along extremals. Indeed, from equation (4), holding for any vertical vector field $\psi_1$, we get the following.

**Theorem 4.5** Let $h_\rho$ be a Lagrangian of order $r+1$, let $\gamma$ be an extremal. Then a vertical vector field $\psi$ generates vertical symmetry transformations of $\gamma$ if and only if

\[ \mathcal{I}d(\psi_1 \mathcal{I}d(h_\rho)) \circ J^{r+1} \gamma = 0. \]

We focus on higher order Noether currents and in particular on currents associated with the infinitesimal second variation formula (3) in a specific way. Roughly speaking, up to horizontal differentials, the second variation (generated by vertical vector fields) of a Lagrangian $\lambda$ is the Jacobi morphism (see [19] for first order field theory; see also [14]). Here we define the Jacobi morphism within the representation sequence, i.e. by the interior Euler operator.

**Definition 4.6** Let $X_V(Y)$ be the space of vertical vector fields on $Y$. The map

\[ \mathcal{J} : \Omega^r_{n,X}(J^rY) \rightarrow X_V^r(J^{2r+1}Y) \otimes X^*_V(Y) \otimes \Omega^r_{n,X}(J^rY) \]

\[ \lambda \rightarrow \bullet \mathcal{I}d(\mathcal{I}d(\lambda)) \]

is called the Jacobi morphism associated with the Lagrangian $\lambda$.

The Jacobi morphism is self-adjoint along critical sections of a Lagrangian field theory of any order (see also [2, 14]). This is a property of great importance in physical applications.

**Theorem 4.7** For any pair of vertical vector fields $\psi_1, \psi_2$ on $Y$, we have

\[ J^{2r+1} \psi_2 \mathcal{I}(J^{2r+1} \psi_1 \mathcal{I}(d\lambda)) = 0. \]

Along extremals the Jacobi morphism is self adjoint.

**Proof.** From the decomposition (1), since $\lambda = p_0 \lambda$, up to pull-backs,

\[ 0 = dd\lambda = dp_1 d\lambda = d\mathcal{I}(d\lambda) + dp_1 dp_1 \mathcal{R}(d\lambda), \]
holds true and we have \(-J^{2r+1}\psi dI(d\lambda) \in \Theta_{n+1}^{2r+1}\) for every vertical vector field \(\psi\); this implies that
\[
J^{2r+1}\psi_2 | I(J^{2r+1}\psi_1 | dI(d\lambda)) = 0,
\]
for any pair of vertical vector fields \(\psi_1, \psi_2\).

Let \(E_\rho(\lambda)\) be the local components of the Euler–Lagrange form associated with \(\lambda\). We therefore have the local condition
\[
I(J^{2r+1}\psi | dI(d\lambda)) = \sum_{|J|\neq 0}^{2r+1} \frac{\partial E_\rho(\lambda)}{\partial y^\rho_j} d_J \psi^\rho \omega^\sigma \wedge ds +
\]
\[
- \sum_{|J|\neq 0}^{2r+1} (-1)^{|J|} d_J \left( \frac{\partial E_\rho(\lambda)}{\partial y^\rho_j} \psi^\rho \right) \omega^\sigma \wedge ds = 0,
\]

Note that along extremals the terms of the form \(\frac{\partial E_\rho(\lambda)}{\partial y^\rho_j}\) vanish. Therefore, along extremals, for every vertical vector field \(\psi\) on \(Y\), we have the equality of the following two local expressions (the first coming from the direct calculation of \(I d(\psi | I d(\lambda))\) along extremals in local coordinates, the second coming from the identity above):
\[
I d(\psi | I d(\lambda)) = \sum_{|J|\neq 0}^{2r+1} (-1)^{|J|} d_J (\psi^\rho \frac{\partial E_\rho(\lambda)}{\partial y^\rho_j}) \omega^\sigma \wedge ds = \quad (7)
\]
\[
= \sum_{|J|\neq 0}^{2r+1} d_J \psi^\sigma \frac{\partial E_\rho(\lambda)}{\partial y^\rho_j} \omega^\rho \wedge ds. \quad (8)
\]

These two local expressions provide, indeed, two (adjoint to each other) expressions for the Jacobi morphism along extremals, which is then self-adjoint. In view of their interpretation as generators of symmetry transformations of extremals, an important rôle is played by vector fields that are in the kernel of the Jacobi morphism. In the following we use the notation \(J_\psi(\lambda)\) for short to denote \(I d(\psi | I d(\lambda))\). Note that, of course, \(I d(\psi | I d(\lambda))\) should not be confused with \(I(J^{2r+1}\psi | dI(d\lambda))\). In particular \(I(J^{2r+1}\psi | dI(d\lambda))\) is related to the Helmholtz form expressing conditions of local variationality for a source form.
Definition 4.8 Let $\lambda$ be a Lagrangian of order $r$. A Jacobi field for the Lagrangian $\lambda$ is a vertical vector field $\psi$ that belongs to the kernel of the Jacobi morphism, i.e. satisfying the Jacobi equation for the Lagrangian $\lambda$

$$J_\psi(\lambda) = 0.$$ 

The Jacobi morphism $J_\psi(\lambda)$, evaluated along an extremal $\gamma$, depends only on the values of the vector field $\psi$ along $\gamma$. Thus we can define the Jacobi equation along an extremal; about its solutions we will speak of Jacobi fields along an extremal $\gamma$.

Remark 4.9 Note that by Theorem 4.5 Jacobi fields along extremals are vertical symmetry transformations of extremals and vice versa.

Remark 4.10 Equation (8) provides the ‘adjoint expression’ for the Jacobi equation along extremals; it can be of use in order to obtain an easier characterization of the kernel of the Jacobi morphism in practical computations, see Example 4.13; see also [40] for an explicit application in $SU(3)$-Yang–Mills theories in the context of a variationally featured symmetry breaking (which was suggested could be useful e.g. for a canonical characterization of confinement phases in non-abelian gauge theories [44]).

It should be stressed that our characterization of (vertical) symmetry transformations of extremals as Jacobi fields along extremals is finalized to understanding the existence of conservation laws associated with such kind of symmetry transformations, which in principle are different from Noether or Noether–Bessel-Hagen conservation laws associated with symmetries or generalized symmetries of the Lagrangian $\lambda$.

Theorem 4.11 Let $\rho$ be an $n$-form on $J^{r-1}Y$ and $h\rho$ the associated Lagrangian on $J^rY$. Let $\psi_1$ and $\psi_2$ on $Y$ be two generators of vertical symmetry transformations of extremals. Then, along extremals of $h\rho$, the weak conservation law holds true:

$$d_{H^c\psi_2(\psi_1)}[Id(h\rho)] = 0.$$ (9)

Proof. Indeed, by Theorem 4.5 the two generators of symmetry transformations $\psi_1$ and $\psi_2$ are also Jacobi fields, i.e. they must satisfy $J_{\psi_i}(h\rho) = 0$, for $i = 1, 2$. Therefore from (5), since also $[\psi_2, \psi_1]Id(h\rho)$ vanishes along extremals, we get the result.

For the interpretation of this current as a Noether current for a ‘deformed’ Lagrangian, see the note at the end of Proposition 3.4.
Remark 4.12 Suppose that $\psi_2$ is a symmetry of the first variation of $h\rho$ generated by $\psi_1$ and that $\psi_1$ and $\psi_2$ satisfy $\psi_2 \tilde{J}_{\psi_1}(h\rho) = 0$, then we have a strong conservation law:

$$d_H \epsilon_{\psi_2}(L_{\psi_1} h\rho) = 0.$$  

Now, along extremals, taking two vertical symmetry transformations $\psi_1$ and $\psi_2$, we get two separated weak conservation laws; see also [2].

Example 4.13 Let us consider a Yang–Mills theory [47] on the bundle $(C_P, \pi, M)$ of principal connections with structure bundle $(P, p, M, G)$, $G$ being a semi-simple group. Lower Greek indices label space-time coordinates, while capital Latin indices label the Lie algebra $\mathfrak{g}$ of $G$, then, on the bundle $C_P$, we introduce coordinates $(x^\mu, \omega^A_{\sigma})$.

Let $\delta$ be the Cartan-Killing metric on the Lie algebra $\mathfrak{g}$, and choose a $\delta$-orthonormal basis $T_A$ in $\mathfrak{g}$; the components of $\delta$ will be denoted $\delta^{AB}$. The Yang-Mills Lagrangian is locally expressed by

$$\lambda_{YM} = -\frac{1}{4} F^A_{\mu\nu} g^{\mu\rho} g^{\nu\sigma} F^B_{\rho\sigma} \delta_{AB} \sqrt{g} ds,$$

where $g$ stands for the absolute value of the determinant of the metric $g_{\mu\nu}$, $c^A_{BC}$ are the structure constants of $\mathfrak{g}$, $F^A_{\mu\nu} = \omega^A_{\nu,\mu} - \omega^A_{\mu,\nu} + c^A_{BC} \omega^B_{\mu} \omega^C_{\nu}$ is the so called field strength, and we set $\omega^A_{\mu,\nu} = d_\nu \omega^A_{\mu}$.

From now on, we assume the metric $\eta$ to be Minkowskian; in this case the Euler–Lagrange expressions for the Yang–Mills Lagrangian are explicitly written as

$$E^\nu_B = \delta_{BA} \eta^{\lambda\mu} \eta^{\nu\nu} (\omega^A_{\nu,\lambda} - \omega^A_{\lambda,\nu} + c^A_{ZD} \omega^Z_{\lambda,\mu} \omega^D_{\nu,\epsilon} + c^A_{ZD} \omega^Z_{\nu,\epsilon} \omega^D_{\lambda,\mu}) +$$

$$+ \eta^{\lambda\mu} \eta^{\nu\nu} \delta_{DA} (\omega^D_{\epsilon,\lambda} - \omega^D_{\lambda,\epsilon} + c^D_{EF} \omega^E_{\epsilon} \omega^F_{\lambda} c^A_{BZ} \omega^Z_{\mu}).$$

Let $(\phi^a)$ be a set of coordinates on the group $G$. A vertical vector field over $C_P$ has the form $\psi = \psi^Z_{\sigma} \frac{\partial}{\partial \omega^Z_{\sigma}}$ and its components satisfy the transformation rule $\psi^B_\mu = Ad^B_A(\phi) \psi^A_\mu \mathcal{J}^a_\mu$ where $Ad^B_A(\phi)$ is the adjoint representation of $G$ on $\mathfrak{g}$ and $\mathcal{J}^a_\mu$ denotes the inverse of the matrix of the change of coordinates in the base space. As well known, if $L(M)$ is the frame bundle of $M$, given the vector space $V = \mathfrak{g} \otimes \mathbb{R}^n$ and the representation $\lambda$ essentially defined by the transformation rules above, the sections of the associated bundle $B = (P \times_M L(M)) \times_{\lambda} V$ are in one to one correspondence with vertical vector fields over $C_P$. 
It is also known that a principal connection on $P \times M L(M)$ is induced by any pair $(\omega, \Gamma)$, where $\omega$ is a principal connection on $P$ (for example, an extremal of the Yang-Mills Lagrangian) while $\Gamma$ is a principal connection on $L(M)$ (see [20, 21] and, for gauge-natural theories, [9, 10]). In coordinates, if $\rho^A_\mu$ are right invariant vector fields on $L(M)$, then $\bar{\Omega} = dx^\mu \otimes (\partial_\mu - \omega^A_\mu \rho^A_\mu)$ is a principal connection on $P \times M L(M)$ (recall that, since we are considering a manifold $M$ which admits a Minkowskian metric, we have $\Gamma^A_\lambda = 0$). The induced connection on $B$ is given by

$$\Omega = dx^\mu \otimes (\partial_\mu + c^B_A \psi^D_\sigma \omega^A_\mu \partial^\sigma_B) \, .$$

We now refer to Theorem 4.7 to derive the expression of the Jacobi equation along critical sections by using the explicit formula (8); this notably simplifies calculations. By some careful manipulations (see [2] for details), denoting by $\nabla$ the covariant derivative corresponding to $\Omega$, the Jacobi equation for this kind of Yang-Mills theory, due to the antisymmetry of $F^D_{\beta \sigma}$ in the lower indices, splits in the antisymmetric and symmetric parts

$$\begin{align}
\eta^{\nu[\sigma} \eta^{\beta]} \alpha \{ \nabla_\beta \left[ (\nabla_\sigma \Xi^A_\alpha - \nabla_\alpha \Xi^A_\sigma) \delta_{BA} \right] + F^D_{\beta \sigma} \delta_{AD} c^A_{CBZ} \Xi^Z_\alpha \} &= 0 , \quad (12) \\
\eta^{\nu[\sigma} \eta^{\beta]} \alpha \{ \nabla_\beta \left[ (\nabla_\sigma \Xi^A_\alpha - \nabla_\alpha \Xi^A_\sigma) \delta_{BA} \right] \} &= 0 .
\end{align}$$

for any pair $(\nu, B)$.

Note that the left hand side of these equations are the analogous, for a Minkowskian metric, of the classical expression for the Jacobi operator for Yang–Mills theories on different backgrounds, see e.g. [3, 5], and it reproduces results for first order non regular Lagrangians [19].

In this and in the following example, in order to avoid confusion, let $\chi^A_\mu, \chi^A_{\mu, \nu}, \chi^A_{\mu, \nu, \rho} \ldots$ denote generators of contact forms. Here we stress that solutions $\psi$ of the above equations are the generators of symmetry transformations of Yang–Mills extremals $\omega$, i.e. if $\psi$ is a solution of the above equation, then the source form $L_{J^3\psi}(E^\nu_B \chi^B_\nu \wedge ds)$, where $E^\nu_B$ are given by (11), also vanishes along the same extremals, i.e.

$$(L_{J^3\psi}(E^\nu_B \chi^B_\nu \wedge ds)) \circ J^3 \omega = 0 .$$

Here, by a slight abuse of notation, we denoted by $\omega$ a section of the bundle $(C_P, \pi, M)$ which is an extremal.

As we already mentioned, compared with transformations leaving invariant the Euler–Lagrange form $E^\nu_B \chi^B_\nu \wedge ds$, such transformations are involved
with a weaker invariance property, since the Euler–Lagrange form is not invariant under their action, but it is transformed to a source form having the same extremals. Note that indeed the Yang–Mills extremals $\omega$ are also solutions of the equation above and vice versa.

In the next example, as an instance of application of our main result, Theorem 4.11, we determine the conserved current associated to such a weaker invariance property.

**Example 4.14** We write down explicitly the current for two given generators of vertical symmetry transformations $\psi$ and $\tilde{\psi}$, solutions of equation (12). Being the vector fields vertical, from Theorem 4.11, equation (9), the conserved current along an extremal has the form

$$\epsilon_{\tilde{\psi}}(\psi | Id(\lambda_{YM})) = -J^3 \tilde{\psi} | p_1 R(d(\psi | Id(\lambda_{YM}))).$$

Recalling that $E^\nu_B$ are coordinate expression of the Euler–Lagrange form, we apply the coordinate characterization of the residual operator (given in Example 2.2) to the form

$$d(\psi | Id(\lambda_{YM})) =$$

$$= (\frac{\partial \psi^B_B}{\partial \omega_p^Z} E^\nu_B + \psi^B_B \frac{\partial E^\nu_B}{\partial \omega_p^Z}) \chi^Z_p \wedge ds + (\psi^B_B \frac{\partial E^\nu_B}{\partial \omega_p^Z}) \chi^Z \wedge ds + (\psi^B_B \frac{\partial E^\nu_B}{\partial \omega_p^Z}) \chi^Z \wedge ds.$$

We suitably rewrite the above in the form $\sum_{|I|=0}^2 d_I(\chi^A_\mu \wedge \zeta^{\mu I}_A)$ and note that the case $|I| = 0$ gives no contribution to the residual operator, thus

$$R(d(\psi | Id(\lambda_{YM}))) = -(\psi^B_B \frac{\partial E^\nu_B}{\partial \omega_p^Z} - d_\tau (\psi^B_B \frac{\partial E^\nu_B}{\partial \omega_p^Z})) \chi^Z_p \wedge ds\xi +$$

$$-(\psi^B_B \frac{\partial E^\nu_B}{\partial \omega_p^Z}) \chi^Z_p \wedge ds\xi,$$

and the current is given by

$$\epsilon_{\tilde{\psi}}(\psi | Id(\lambda_{YM})) = [\eta^\rho_{[\xi} \eta^{\sigma]} | \nu^\nu \delta_{BA}^C \omega^D_{\sigma} (\psi^B_B \tilde{\psi}_p^Z - \psi^Z_p \tilde{\psi}^B) +$$

$$(\eta^\rho_{[\xi} \eta^{\sigma]} - \eta^\rho_{[\sigma} \eta^{\xi]} | \nu^\nu) (\psi^Z_p \nabla_\sigma (\tilde{\psi}_p^B \delta_{BZ}) - \tilde{\psi}_p^Z \nabla_\sigma (\psi^B_B \delta_{BZ})) | ds\xi;$$

here the brackets ( ) and [ ] in the superscripts denote symmetrization and anti-symmetrization, respectively (for details see [1, 2]).
Remark 4.15 It is noteworthy that, by Proposition 3.4 and in particular by Remark 4.9, here the existence and the meaning of the above current is understood under a new light, definitely relevant from a physical point of view.

In the present paper we clarify that such a conservation law emerges by an invariance property of the set of extremals and, moreover, that the associated conserved current can be interpreted as a very specific kind of Noether current, the existence of which is related with a wide class of symmetry transformations. Indeed, we proved that this current can be identified as the Noether current for the Yang–Mills Lagrangian ‘deformed’ by the symmetry transformation of extremals $\psi$ and associated with (or generated by) the symmetry transformation of extremals $\tilde{\psi}$.

Remark 4.16 We note that Equation (5) of Proposition 3.4 says us that for any vertical vector field $\psi_1 = \psi_2 = \zeta$, the current $\epsilon_\zeta(\zeta|\mathcal{I}d(h\rho))$ is a strong conserved current (i.e. conserved ‘of shell’). However, it can be easily checked that, at least in the specific case of study, for any (vertical) symmetry transformation of extremals $\tilde{\psi} = \psi$ the weak (i.e. ‘on shell’) conserved current reduces to $\eta^\rho(\sigma, \tilde{\eta}^\nu)(\psi^B_{\tilde{\rho}} \nabla_\sigma (\psi^B_{\tilde{\nu}} \delta BZ) - \psi^B_{\tilde{\rho}} \nabla_\sigma (\psi^B_{\tilde{\nu}} \delta BZ)) ds_{\tilde{\xi}}$, which vanishes identically because $\eta^\rho(\sigma, \tilde{\eta}^\nu) = \eta^\rho(\sigma, \tilde{\eta}^\nu)$. This holds true for any couple of linearly dependent symmetry transformations.

Acknowledgements

Research partially supported by Department of Mathematics - University of Torino through the projects PALM_RILO_16_01 and FERM_RILO_17_01 (MP) and written under the auspices of GNSAGA-INdAM. The first author (LA) is also supported by a NWO-UGC project, Grant BM.00193.1. The second author (MP) would like to acknowledge the contribution of the Cost Action CA17139 - European Topology Interdisciplinary Action.

References

[1] L. Accornero: Jet prolongations and calculus of variations: second and higher order variations in the framework of the variational sequence, Master thesis, University of Torino (2017).
[2] L. Accornero, M. Palese: Higher variations and conservation laws; with applications to a Yang–Mills theory on a Minkowskian background, arXiv:1710.09100v5.

[3] M.F. Atiyah, R. Bott: The Yang–Mills equations over Riemann surfaces, Philos. Trans. Roy. Soc. London Ser. A 308 (1505) (1983) 523–615.

[4] E. Bessel-Hagen: Über die Erhaltungssätze der Elektrodynamik, Math. Ann. 84 (1921) 258–276.

[5] J.P. Bourguignon: Yang-Mills theory: the differential geometric side, Differential geometry (Lyngby, 1985), 13–54, Lecture Notes Math. 1263 Springer (Berlin, 1987).

[6] G.E. Bredon: Sheaf Theory, McGraw–Hill (New York, 1967).

[7] F. Cattafi, M. Palese, E. Winterroth: Variational derivatives in locally Lagrangian field theories and Noether–Bessel-Hagen currents, Int. J. Geom. Methods Mod. Phys. 13 (2016) 1650067, 16.

[8] D.J. Eck: Gauge-natural bundles and generalized gauge theories, Mem. Amer. Math. Soc. 247 (1981) 1–48.

[9] L. Fatibene, M. Francaviglia: Natural and gauge natural formalism for classical field theories: a geometric perspective including spinors and gauge theories, Springer Science & Business Media (2003).

[10] L. Fatibene, M. Francaviglia, M. Palese: Conservation laws and variational sequences in gauge-natural theories, Math. Proc. Cambridge Philos. Soc. 130 (3) (2001) 555–569.

[11] M. Ferraris, M. Francaviglia, M. Raiteri: Conserved quantities from the equations of motion: with applications to natural and gauge natural theories of gravitation Class. Quantum Grav. 20 (18) (2003 ) 4043.

[12] M. Ferraris, M. Francaviglia, M. Palese, E. Winterroth: Canonical connections in gauge-natural field theories, Int. J. Geom. Methods Mod. Phys. 5 (6) (2008) 973–988.

[13] M. Ferraris, M. Francaviglia, M. Palese, E. Winterroth: Gauge-natural Noether currents and connection fields, Int. J. Geom. Methods Mod. Phys. 8(1) (2011) 177–185.
[14] M. Francaviglia, M. Palese, R. Vitolo: The Hessian and Jacobi morphisms for higher order calculus of variations, *Differential Geom. Appl.* **22** (2005) 105–120

[15] M. Francaviglia, M. Palese, E. Winterroth: Second variational derivative of gauge-natural invariant Lagrangians and conservation laws, in *Differential geometry and its applications*, Matfyzpress, Prague (2005), 591–604.

[16] M. Francaviglia, M. Palese, E. Winterroth: Variationally equivalent problems and variations of Noether currents, *Int. J. Geom. Meth. Mod. Phys.* **10** (1) (2013) 1220024.

[17] M. Francaviglia, M. Palese, E. Winterroth: Cohomological obstructions in locally variational field theories, *Jour. Phys. Conf. Series* **474** (2013) art. no. 012017.

[18] M. Francaviglia, M. Raiteri: Hamiltonian, energy and entropy in general relativity with non-orthogonal boundaries *Class. Quantum Grav.* **19** (2) (2002) 237.

[19] H. Goldschmidt, S. Sternberg: The Hamilton-Cartan formalism in the calculus of variations, *Ann. Inst. Fourier (Grenoble)* **23** (1973) 203–267

[20] I. Kolár: On some operations with connections, *Math. Nachr.*, **69** (1975) 297–306.

[21] I. Kolár: Prolongations of generalized connections, *Coll. Math. Soc. János Bolyai*, (Differential Geometry, Budapest, 1979) **31** (1979) 317–325.

[22] I. Kolár, P.W. Michor, J. Slovák: *Natural Operations in Differential Geometry*, (Springer–Verlag, N.Y., 1993).

[23] Y. Kosmann-Schwarzbach: The Noether Theorems; translated from French by Bertram E. Schwarzbach (Springer 2011).

[24] M. Krbek, J. Musilová: Representation of the variational sequence by differential forms, *Rep. Math. Phys.* **51** (2-3) (2003) 251–258.

[25] M. Krbek, J. Musilová: Representation of the variational sequence by differential forms, *Acta Appl. Math.* **88** (2) (2005) 177–199.
[26] D. Krupka: Some geometric aspects of variational problems in fibred manifolds, *Folia Fac. Sci. Nat. UJEP Brunensis* **14**, J. E. Purkyně Univ. (Brno, 1973) 1–65, arXiv: math-ph/0110005.

[27] D. Krupka: Variational Sequences on Finite Order Jet Spaces, *Proc. Diff. Geom. Appl.;* J. Janyška, D. Krupka eds., World Sci. (Singapore, 1990) 236–254.

[28] D. Krupka: *Introduction to global variational geometry*, Atlantis Studies in Variational Geometry, Atlantis Press (Paris, 2015) xviii+354 pp.

[29] D. Krupka: Invariant variational structures on fibered manifolds *Int. J. Geom. Methods Mod. Phys.* **12** (02) 1550020 (2015).

[30] D. Krupka: Global variational functionals in fibered spaces, *Nonlinear Anal.* **47** (4) (2001) 2633–2642.

[31] J. Leray: L’anneau d’homologie d’une représentation, *C. R. Acad. Sci. Paris* **222** (1946) 1366–1368; Structure de l’anneau d’homologie d’une représentation, *C. R. Acad. Sci. Paris* **222** (1946) 1419–1422.

[32] E. Noether: Invariante Variationsprobleme, *Nachr. Ges. Wiss. Göttingen, Math. Phys. Kl.* II (1918) 235–257.

[33] M. Palese, O. Rossi, E. Winterroth, J. Musilová: Variational sequences, representation sequences and applications in physics, *SIGMA* **12** (2016) 045, 45 pages (2016).

[34] M. Palese, E. Winterroth: Covariant gauge-natural conservation laws, *Rep. Math. Phys.* **54** (3) (2004) 349–364.

[35] M. Palese, E. Winterroth: Global Generalized Bianchi Identities for Invariant Variational Problems on Gauge-natural Bundles, *Arch. Math. (Brno)* **41** (3) (2005) 289–310.

[36] M. Palese, E. Winterroth: The relation between the Jacobi morphism and the Hessian in gauge-natural field theories, *Theoret. Math. Phys.* **152** (2) (2007) 1191–1200.

[37] M. Palese, E. Winterroth: Lagrangian reductive structures on gauge-natural bundles, *Rep. Math. Phys.* **62** (2) (2008) 229–239.
[38] M. Palese, E. Winterroth: A variational perspective on classical Higgs fields in gauge-natural theories, *Theor. Math. Phys.* **168** (1) (2011) 1002–1008.

[39] M. Palese, E. Winterroth: Topological obstructions in Lagrangian field theories, with an application to 3D Chern–Simons gauge theory, *J. Math. Phys.* **58** (2) (2017) 023502.

[40] M. Palese, E. Winterroth: Higgs fields induced by Yang-Mills type Lagrangians on gauge-natural prolongations of principal bundles, *Int. J. Geom. Meth. Mod. Phys.* **16**(3) (2019) 1950049, 15 pp.

[41] W. Sarlet, F. Cantrijn: Higher-order Noether symmetries and constants of the motion, *J. Phys. A: Math. Gen.* **14** (1981) 479–492.

[42] D.J. Saunders: The geometry of jet bundles, *London Mathematical Society Lecture Note Series* **142** Cambridge Univ. Press (Cambridge, 1989).

[43] F. Takens: A global version of the inverse problem of the calculus of variations, *J. Diff. Geom.* **14** (1979) 543–562.

[44] G. ’t Hooft: Topology of the gauge condition and new confinement phases in non-abelian gauge theories, *Nucl. Phys.* **B190[FS3]** (1981) 455–478.

[45] A. Trautman: Noether equations and conservation laws, *Comm. Math. Phys.* **6** (1967) 248–261.

[46] J. Volná, Z. Urban: The interior Euler–Lagrange operator in field theory, *Math. Slovaca* **65** (6) (2015) 1427–1444.

[47] C.N. Yang, R.L. Mills: Conservation of Isotopic Spin and Isotopic Gauge Invariance, *Phys. Rev.* **96** (1) (1954) 191–195.