D-Branes, Π-Stability and θ-Stability

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Abstract

We investigate some aspects of Π-stability of D-branes on Calabi-Yau threefolds in cases where there is a point in moduli space where the grades nearly or completely align. We prove that an example of complete alignment is the case of a collapsed del Pezzo surface. It is shown that there is an open neighbourhood of such a point for which Π-stability reduces to θ-stability of quiver representations. This should be contrasted to the case of the large radius limit where μ-stability of sheaves cannot be extended out over an open neighbourhood.
1 Introduction

$\Pi$-stability of B-type BPS D-branes on a Calabi–Yau threefolds $X$ captures many interesting aspects of “stringy geometry”. At large radius limit such D-branes correspond to Hermitian–Yang–Mills connections on bundles over holomorphically embedded subspaces in $X$. The instant one moves away from this limit, nonperturbative string tension (i.e., $\alpha'$) corrections change this picture quite drastically. One must pass to the bounded derived category $\mathcal{D}(X)$ of coherent sheaves to understand all physical D-branes.

Objects in the derived category are, roughly speaking, equivalence classes of complexes of coherent sheaves. A complex which is nonzero in only one position is called a “stalk complex”. If an object in $\mathcal{D}(X)$ can be represented by a stalk complex then we may regard this object as simpler in some sense. Indeed, the classical D-branes at large radius limit correspond to stalk complexes whose nonzero entry is a coherent sheaf supported on a subspace of $X$.

It was shown in [1–3] that for arbitrarily large but finite $X$, one may always find a $\Pi$-stable object in $\mathcal{D}(X)$ that is not isomorphic to a stalk complex. Therefore, any method of analyzing B-type D-branes near the large radius limit in terms of bundles and connections (such as in [4–6]) is always fated to miss some of the physical objects.

The category of B-type D-branes does not “know” it is associated specifically with complexes of coherent sheaves. It just happens that coherent sheaves become important at large radius. There may be other “abelian categories” which, when “derived”, correctly yield the same D-brane category in passing to complexes.

There are other points in the moduli space where another abelian category becomes useful. Of interest to us in this lecture is the case where the “gradings” of the stable D-branes within this abelian category lie in a narrow range as spelt out more precisely below. In this case we will have an open neighbourhood of the moduli space where all stable objects can be given by stalk complexes. Our ideas are not particularly new. Much of what we discuss has been observed in [1,7], for example, and is close to some ideas of Bridgeland [8]. Our goal here is to present a systematic approach to the problem which lets us arrive at some theorems which clarify the generality of the situation.

We shall prove that this “alignment” of gradings happens if a del Pezzo surface within $X$ is shrunk. That is, when the del Pezzo surface is completely contracted to a point, the grades will align completely. If $S$ is shrunk down to something small (relative to the intrinsic $\alpha'$ scale) the grades will align sufficiently, so that all $\Pi$-stable objects are stalk complexes. In this open region of moduli space one may then use a simpler notion of stability, namely $\theta$-stability, to accurately determine the stable spectrum.

Such a simplification appears to be very special. As mentioned above, $\mu$-stability (or probably any variant thereof) for vector bundles is not valid in an open set in moduli space. Similarly, the gradings of stable objects in Gepner models do not appear to align in any way, and so we should not expect any such simplification here either.

The derived category in question will be based on quiver representations rather than coherent sheaves. Quivers have played an important rôles in the theory of D-branes in the context of quiver gauge theories and del Pezzo surfaces as explored in [9–20] for example. We will not pursue these quiver gauge theories in this paper except to say that we prove...
that these theories become appropriate precisely when the del Pezzo surface shrinks down to a point as had been suspected.

In section 2 we review the necessary concepts from quivers and their representations. Then, in section 3 we focus on D-branes that correspond to points in $S$. By looking at the moduli space of such objects we retrieve $S$ itself. In section 4 we analyze the case where the gradings are partially aligned and we prove that $\Pi$-stability reduces to $\theta$-stability in this case. Finally in section 5 we consider a complete alignment of the gradings and show that the del Pezzo surface collapses in this case.

2 Quivers and del Pezzo Surfaces

This section is a quick review of some well-known material. See [20, 21], for example, for more details. Let $Q$ be a quiver. The path algebra $A$ (over $\mathbb{C}$) of $Q$ is constructed as follows. To each node of $Q$ we associate an element $e_j \in A$ which is idempotent, $e_j^2 = e_j$. These elements are viewed as “paths of zero length” beginning and ending at node $j$. The rest of $A$ is then generated by paths of nonzero length in $Q$. If the end of path $p$ is the same node as the beginning of path $q$, then the product $qp$ is simply the composition of the paths. Otherwise the product is zero. A quiver may contain relations imposed between the paths.

A representation of $Q$ is defined by associating a vector space $V_j$ over $\mathbb{C}$ with each node. In addition, a matrix yielding a linear map is then associated to each arrow. By using matrix multiplication we thus associate a matrix to any path in $Q$. If $Q$ has relations, the matrices must obey these relations.

It is not hard to show (see [21], for example) that quiver representations are equivalent to representations of $A$, or left $A$-modules. If $V$ is a vector space and a left $A$-module, one sets $V_j = e_j V$. The dimension vector of a quiver representation is simply the list of numbers $(\dim V_0, \dim V_1, \ldots)$.

A morphism between two quiver representations can then be defined to correspond to an $A$-module homomorphism. This means that, given two representations $W$ and $V$ with underlying vector spaces $W_j$ and $V_j$ at each node, we must specify a collection of linear maps

$$f_j : W_j \rightarrow V_j, \quad (1)$$

such that for any arrow $a$,

$$f_{h(a)} W_a = V_a f_{t(a)}, \quad (2)$$

where $V_a$ is the matrix $V$ associates to $a$, and $h(a)$ and $t(a)$ are the head and tail of $a$ respectively. From these morphisms we define the category of quiver representations which is equivalent to the category of left $A$-modules.

There are two particularly useful sets of quiver representations, $L_j$ and $P_j$, each of which is labeled by a node $j$ in the quiver. $L_j$ is defined simply as the one-dimensional representation with dimension vector $(0, \ldots, 0, 1, 0, \ldots)$ with the one in the $j$th position.

$P_j$ is defined as $A e_j$, i.e., the space generated by paths starting at node $j$. It is clear that
\[ \sum_{j} e_j \] is the identity element in \( A \), and thus
\[
A = A \sum_{j} e_j = \bigoplus_{j} P_j. \tag{3}
\]

By a standard theorem of algebra, each \( P_j \) is therefore a projective \( A \)-module since it is a direct summand of \( A \) (which is obviously a free \( A \)-module).

We may now construct the derived category \( D(A\mod) \), which we will also denote \( D(Q) \), from bounded complexes of quiver representations in the usual way (see [22] for a review of this). A “stalk complex” is a complex which is nonzero in precisely one position. As always \( a[n] \) denotes the complex \( a \) shifted \( n \) places to the left. When used as an object of \( D(A\mod) \), \( V \) will represent a stalk complex with quiver representation \( V \) in position zero. Similarly, \( V[n] \) represents a stalk complex with \( V \) in position \( -n \).

Suppose \( S \) is a del Pezzo surface and \( \{ \mathcal{E}_0, \ldots, \mathcal{E}_{n-1} \} \) forms a strong exceptional collection of sheaves on \( S \) as in [23]. That is,
\[
\text{Ext}_S^q(\mathcal{E}_j, \mathcal{E}_j) = \begin{cases} 
\mathbb{C} & \text{if } q = 0, \\
0 & \text{otherwise}, 
\end{cases} \tag{4}
\]
\[
\text{Ext}_S^q(\mathcal{E}_j, \mathcal{E}_k) = 0 \quad \text{for any } q \text{ and } j > k.
\]

Furthermore, assume that the number of sheaves in this collection, \( n \), is equal to the rank of \( K_0(S) \). Bondal [24] then proved that

**Theorem 1** The bounded derived category of coherent sheaves on \( S \), \( D(S) \), is equivalent to \( D(A\mod) \), the bounded derived category of left \( A \)-modules, where
\[
A = \text{End}(\mathcal{E}_0 \oplus \ldots \oplus \mathcal{E}_{n-1})^{\text{op}}. \tag{5}
\]

Given such an exceptional collection it is easy to construct the quiver \( Q \) for which \( A \) is the path algebra. There will always be relations. For example, consider \( S = \text{dP}_1 \) given by \( \mathbb{P}^2 \) with the single point \([z_0, z_1, z_2] = [0, 0, 1] \) blown up. Let \( C_1 \) be the exceptional curve and let \( H \) be a hyperplane not intersecting \( C_1 \). Using the strongly exceptional collection \( \{ \mathcal{O}, \mathcal{O}(C_1), \mathcal{O}(H), \mathcal{O}(2H) \} \), the corresponding quiver is given by

\[
\begin{align*}
v_0 & \quad a \\
v_1 & \quad b_1 \\
v_2 & \quad c \\
v_3 & \quad d_3 \\
& \quad d_1 \\
& \quad d_2 \\
& \quad d_0 \\
& \quad b_0 \\
& \quad b_1 \\
& \quad b_0 \\
\end{align*}
\]

subject to the relations \( b_0d_1 - b_1d_0 = 0, \ ab_0d_2 - cd_0 = 0, \) and \( ab_1d_2 - cd_1 = 0 \).

Given that \( A = \text{End}(A)^{\text{op}} = \text{End}(\bigoplus_j P_j)^{\text{op}} \), there is a natural identification of the exceptional sheaves \( \mathcal{E}_j \) with the projective representations \( P_j \) which yields the equivalence of theorem \( \Pi \). Note that the definition of the exceptional collection implies that there are no
paths in $Q$ from node $i$ to node $j$ if $i < j$. This means the quiver is “directed” and contains no oriented loops. This, in turn, implies that $A$, and thus all the $P_j$’s, are finite-dimensional.

Another consequence of the fact that the quiver is directed is that we may decompose any quiver representation $E$ into a sum of $L_j$’s via a sequence of short exact sequences. We can represent this by the following triangles in $\mathbf{D}(S)$:

$$L_0 \oplus N_0 = E_0 \to E_1 \to \cdots \to E_{n-2} \to E_{n-1} = E \quad (7)$$

In this way, the $L_j$’s are the “fundamental” representations into which any representation may be decomposed in the sense of (7).

Viewing the $L_j$’s as D-branes, any representation may be interpreted as a quiver gauge theory in which the field content is given by open strings, i.e., $\operatorname{Ext}^q(L_j, L_k)$. The details of this quiver gauge theory are not important for us here and we refer to [20] and references therein for more information for the interested reader. An important condition for this quiver gauge theory to be physically meaningful is that $\operatorname{Ext}^q(L_j, L_k) = 0$, for $q \neq 1$ or 2, and any $j, k$. (8)

This condition amounts to there being no relations between relations within the quiver. We will assume $Q$ satisfies this condition from now on.

Suppose we have an embedding $i : S \hookrightarrow X$ of our del Pezzo surface $S$ into a Calabi–Yau threefold $X$. The induced map $i_* : \mathbf{D}(S) \to \mathbf{D}(X)$ maps $\mathbf{D}(S)$ into a subcategory of $\mathbf{D}(X)$ but this is not full subcategory. That is, for quiver representations $A$ and $B$ it need not be true that $\operatorname{Ext}^q_S(A, B)$ is equal to $\operatorname{Ext}^q_X(i_* A, i_* B)$. We may remedy this by adding more arrows to the quiver $Q$ as follows. First we note that a spectral sequence associated to the embedding of $S$ into $X$ leads to the following relation:

$$\operatorname{Ext}^q_X(i_* A, i_* B) = \operatorname{Ext}^q_S(A, B) \oplus \operatorname{Ext}^{3-q}_S(B, A). \quad (9)$$

In particular,

$$\operatorname{Ext}^1_X(i_* L_j, i_* L_k) = \operatorname{Ext}^1(L_j, L_k) \oplus \operatorname{Ext}^2(L_k, L_j). \quad (10)$$

Since the number of arrows from node $j$ to node $k$ is given by the dimension of $\operatorname{Ext}^1(L_j, L_k)$ (see, [22], for example for an explanation of this), we may create a “completed” quiver $\bar{Q}$ which yields the correct values of $\operatorname{Ext}^1_X(i_* L_j, i_* L_k)$ by adding $\dim \operatorname{Ext}^2(L_k, L_j)$ arrows from node $j$ to node $k$. One can show that $\dim \operatorname{Ext}^2(L_k, L_j)$ is precisely the number of independent relations between paths from node $k$ to node $j$. Our example of dP$_1$ above therefore becomes:

$$\quad \quad \quad \quad $$

We will always use dotted arrows to represent the new arrows added in.
Once we have the correct value for the Ext\(_1\)'s between the \(L_j\)'s, it follows from Serre duality Ext\(_q\)_X(A, B) = Ext\(_{3-q}\)_X(B, A), that the Ext\(_2\)'s are also correct. Since all other Ext's vanish, this quiver also yields the correct value for all Ext's between the \(L_j\)'s. Furthermore, since we can decompose any quiver representation (and thus any object in \(D(S)\)) into \(L_j\)'s, it follows that the quiver \(\bar{Q}\) yields the correct values for all the Ext's between all objects. In other words, if \(\bar{A}\) is the path algebra of the completed quiver \(\bar{Q}\), \(D(\bar{A-mod})\) maps to a full subcategory of \(D(X)\).

The category \(D(\bar{A-mod})\), which we will also denote \(D(\bar{Q})\), represents local information of \(X\) near \(S\). As we will see, it appears to represent the derived category of coherent sheaves with compact support on the total space of the normal bundle of \(S\) in \(X\). We will not attempt to prove this rigorously here though.

### 3 Points and Stability

In this section we focus attention on the objects in \(D(S)\) which correspond to points, i.e., skyscraper sheaves \(\mathcal{O}_p\), \(p \in S\). In [25] Bondal and Orlov gave a purely categorical method of determining which elements of \(D(S)\) correspond to such skyscrapers. They showed

**Theorem 2** Assuming \(S\) is a smooth del Pezzo surface, an object \(a\) in \(D(S)\) corresponds to a stalk complex given by a skyscraper sheaf if and only if the following three conditions hold

1. \(\Psi(a) = a[2]\),
2. \(\text{Hom}(a, a[n]) = 0\), for \(n < 0\),
3. \(\text{Hom}(a, a) = \mathbb{C}\).

Here \(\Psi\) is the “Serre-functor” [26] which is unique up to isomorphism and, for derived categories of quiver representations, is given by the “Nakayama functor”:[1]

\[
\Psi(a) = R \text{Hom}(a, A)^*. \quad (12)
\]

We will assume that \(\mathcal{O}_p\) is not only a stalk complex in \(D(S)\), but is also a stalk complex when viewed as an object in the derived category of quiver representations. This assumption is discussed further in section 4. It was proven in [20] that the dimension vector of a quiver representation corresponding to \(\mathcal{O}_p\) is given by

\[
(\text{rank } \mathcal{E}_0, \text{rank } \mathcal{E}_1, \ldots, \text{rank } \mathcal{E}_{n-1}). \quad (13)
\]

This is a weaker constraint than theorem 2.

Let us analyze the example of dP\(_1\) and the quiver given in (6). The quiver representation corresponding to \(\mathcal{O}_p\) will have a dimension vector \((1, 1, 1, 1)\). Note that

\[
e_j \Psi(a) = R \text{Hom}(a, Ae_j)^* = R \text{Hom}(a, P_j)^*. \quad (14)
\]

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1. \(A\) is a left-right-\(A\)-module. Therefore, if \(V\) is a left \(A\)-module, \(\text{Hom}(V, A)\) is a right \(A\)-module. Taking the dual of this restores to us a left \(A\)-module.
Suppose $V$ is a quiver representation with dimension $(1, 1, 1, 1)$ with generic maps associated to the arrows. This has a projective resolution

$$0 \to P_0 \to P_2 \to P_3 \to V \to 0. \quad (15)$$

In other words, in $\text{D}(Q)$, $V$ is isomorphic to the complex

$$P_0 \to P_2 \to P_3, \quad (16)$$

where the underline represents position zero. So

$$e_0 \Psi(V) = \mathbb{R} \text{Hom}(P_0 \to P_2 \to P_3, P_0)^*$$

$$= \left( \mathbb{C} \to 0 \to 0 \right)$$

$$= \mathbb{C}[2]. \quad (17)$$

Similarly $e_j \Psi(V) = \mathbb{C}[2]$ for $j = 1, 2, 3$. Thus the correspondence of $V$ to $\mathcal{O}_p$ is consistent with the first condition in theorem 2. It is not hard to show that the other two conditions are also satisfied. However, suppose we impose $b_0 = b_1 = 0$ on the maps in (16). Now $P_3 \to V$ is no longer a surjective map and so $V$ becomes isomorphic to the complex

$$P_0 \to P_1 \oplus P_2 (0,-) \to P_1 \oplus P_3, \quad (18)$$

which, in turn, yields

$$e_1 \Psi(V) = \left( \mathbb{C} \to \mathbb{C} \to \mathbb{C} \right), \quad (19)$$

which is not isomorphic to $\mathbb{C}[2]$. Thus $b_0 = b_1 = 0$ cannot correspond to a skyscraper sheaf $\mathcal{O}_p$ for any $p$. The general rule is that there must be a nonzero path of nonzero length to each node. We therefore also rule out $a = c = 0$ and $d_0 = d_1 = d_2 = 0$.

In this case it is easy to see $\text{dP}_1$ as the moduli space of isomorphism classes of all valid $V$’s. First note that two quiver representations are isomorphic if and only the map between the two representations is an element of the “gauge group”

$$\prod_j \text{GL}(V_j). \quad (20)$$

This means we have an overall $(\mathbb{C}^*)^4$ gauge group acting on our representations, although, as always, a diagonal $\mathbb{C}^*$ acts trivially. Suppose $d_0$ and $d_1$ are assigned values not both zero. The remaining $(\mathbb{C}^*)^3$ action together with the quiver relations and above non-vanishing conditions are then sufficient to fix $a, b_0, b_1$ and $c$. If $d_0 = d_1 = 0$ then $a = 0$, and $c$ and $d_2$ are fixed, but the ratio of $b_0$ and $b_1$ remains undetermined. This explicitly realizes $[d_0, d_1, d_2]$ as the homogeneous coordinates of the original $\mathbb{P}^2$ and $[b_0, b_1]$ as the homogeneous coordinates of the exceptional $\mathbb{P}^1$ when $d_0 = d_1 = 0$.  

6
4 \( \theta \)-stability

The key step in obtaining \( S \) as the moduli space of quiver representations in section 3 is using the criteria of theorem 2 to rule out some representations as valid skyscraper sheaves. There is an alternative way of doing this. This concerns the idea of stability either in the mathematical sense or the physical sense.

If these quiver representations are to correspond to physical 0-branes, they must be \( \Pi \)-stable in the sense of [1, 3, 7, 8, 22]. In this picture one puts a stability condition on \( D(X) \) as follows. Each nonzero object in \( D(X) \) is either stable or unstable. The stable objects \( \mathbf{a} \) have “grades” \( \xi(a) \in \mathbb{R} \) such that \( \xi(a[n]) = \xi(a) + n \) and any two stable objects satisfy the unitarity constraint

\[
\xi(a) > \xi(b) \Rightarrow \text{Hom}(a, b) = 0. \tag{21}
\]

Every unstable object \( \mathbf{c} \) then has a “decay chain” given by a sequence of distinguished triangles

\[
\begin{array}{c}
\mathbf{a}_0 = \mathbf{c}_0 \quad \mathbf{c}_1 \quad \cdots \quad \mathbf{c}_{n-2} \quad \mathbf{c}_{n-1} = \mathbf{c},
\end{array}
\tag{22}
\]

where the \( \mathbf{a}_j \)'s are stable objects satisfying

\[
\xi(a_0) \geq \xi(a_1) \geq \cdots \geq \xi(a_{n-1}). \tag{23}
\]

These decay chains are unique up to isomorphism [8] and so, in particular, no decay chain (for \( n > 1 \)) may exist for a stable object.

Now, let us suppose we are working in the derived category \( D(Q) \). For a given point in the moduli space of complexified Kähler forms, trying to determine stability of a given object using purely the definition of \( \Pi \)-stability is difficult, if not impossible. In addition, one must usually assume that the objects of some basic collection are stable for some given set of gradings. We will assume that the objects \( L_j \) are stable for all \( j \). This is a very natural assumption given the D-brane world-volume approach to \( \theta \)-stability, as outlined in [7], but we will not try to further justify this assumption here. For now let us also assume that the grades satisfy the following “partial alignment” property:

\[
\psi < \xi(L_j) < \psi + 1, \text{ for all } j, \tag{24}
\]

for some fixed real number \( \psi \). We will now show that, under these conditions, any non-trivial object which cannot be represented by a stalk complex is necessarily unstable. Suppose we have a two-term complex

\[
W^\bullet = \left( W^{-1} \xrightarrow{f} W^0 \right). \tag{25}
\]
We have a morphism between complexes

\[
\begin{array}{ccccccccc}
0 & \rightarrow & \ker(f) & \rightarrow & 0 & \rightarrow & 0 \\
\downarrow & & \uparrow & & \downarrow & & \downarrow \\
0 & \rightarrow & W^{-1} & \overset{f}{\rightarrow} & W^0 & \rightarrow & 0,
\end{array}
\]

(26)

which is a quasi-isomorphism if and only if \(f\) is surjective. Similarly

\[
\begin{array}{ccccccccc}
0 & \rightarrow & W^{-1} & \overset{f}{\rightarrow} & W^0 & \rightarrow & 0 \\
\downarrow & & \uparrow & & \downarrow & & \downarrow \\
0 & \rightarrow & 0 & \rightarrow & \coker(f) & \rightarrow & 0,
\end{array}
\]

(27)

is a quasi-isomorphism if and only if \(f\) is injective. Therefore, in order to avoid an isomorphism to a single term complex we should assume that both \(\ker(f)\) and \(\coker(f)\) are non-trivial.

In (26), \(\ker(f)\) is a non-trivial quiver representation. Let \(j\) be the minimum number for which \(\dim(e_j \ker(f)) > 0\). Then there is a non-trivial quiver morphism \(L_j \rightarrow \ker(f)\). This, in turn, implies that there is non-trivial morphism from \(L_j\) to \(W^{-1}\). That is, \(\text{Hom}(L_j[1], W^\bullet)\) is nonzero. If \(W^\bullet\) were stable then the unitarity constraint (21) would yield

\[\xi(L_j) + 1 \leq \xi(W^\bullet).\]

(28)

Similarly there must exist a maximum number \(k\) for which \(\dim(e_k \coker(f)) > 0\) yielding a map \(\coker(f) \rightarrow L_k\) resulting in \(\xi(W^\bullet) \leq \xi(L_k)\). Therefore, from (24), we have

\[\psi + 1 < \xi(L_j) + 1 \leq \xi(W^\bullet) \leq \xi(L_k) < \psi + 1,\]

(29)

which is absurd and so \(W^\bullet\) cannot be stable.

It is a simple matter to rule out translations of the above and complexes of length greater than two by the same argument. So we need only consider quiver representations themselves, rather than complexes, in order to determine the stable spectrum of objects. In particular, if \(c\) in (22) is a stalk complex then it follows (from, for example, the Grothendieck group) that every object in this diagram must be a stalk complex concentrated in the same position. This turns the sequence of triangles into a sequence of short exact sequences in the category of quiver representations. This simplifies the analysis considerably.

Let \(K_0(Q)\) be the Grothendieck group of quiver representations of \(Q\). This is simply the lattice \(\mathbb{Z}^n\) of dimension vectors where negative dimensions are allowed. Recall that the grading \(\xi\) is determined by a central charge \(Z\), where

\[Z : K_0(Q) \rightarrow \mathbb{C},\]

(30)

is homomorphism. We then have

\[\xi = \frac{1}{\pi} \arg(Z) \mod 2.\]

(31)
It follows that, for any stable representation $V$,

$$\xi(V) = \frac{1}{\pi} \arg \left( \sum_j e^{i\pi \xi(L_j)} \dim V_j \right) \mod 2. \quad (32)$$

We may fix the mod 2 ambiguity as follows. Suppose we have a stable two-dimensional representation $V$ whose dimension vector is $(0,0,\ldots,0,1,0,\ldots,0,1,0,\ldots,0)$ where the ones appear in position $i$ and $j$ with $i < j$. If $\xi(V) < \psi - 1$ then the triangle

$$L_j[-1] \rightarrow L_i \leftarrow V \quad (33)$$

destabilizes $L_i$, which we know is supposed to be stable. Similarly, if $\xi(V) > \psi + 2$ then $L_j$ would be unstable. We therefore conclude, using (24), that $\psi < \xi(V) < \psi + 1$. We may iterate this process to build up any finite-dimensional representation, and so all such representations have their gradings restricted to this unit interval thereby resolving the ambiguity in (32).

Given a short exact sequence of quiver representations

$$0 \rightarrow U \rightarrow V \rightarrow W \rightarrow 0, \quad (34)$$

if $U$ and $W$ are stable then a necessary condition for the $\Pi$-stability of $V$ is that $\xi(W) > \xi(U)$. From (32) it is clear that $\xi(V)$ lies between $\xi(U)$ and $\xi(W)$ and so we require $\xi(U) < \xi(V)$. Define

$$\theta(U) = -\text{Im} \frac{Z(U)}{Z(V)}. \quad (35)$$

This necessary condition for the stability of $V$ then becomes $\theta(U) > 0$ for all $U \subset V$.

This turns out to be a sufficient condition too as can be seen as follows. If $c = V$ in (24) is an unstable object, then, because the triangles have turned into short exact sequences, $a_0$ is subobject of $c$. In the special case of partial alignment we may extend the definition of grading so that it is defined for unstable objects as well as stable objects. We simply compute the grade of an object from (32) using the alignment to fix the mod 2 ambiguity. From this, it follows from (22) that $\xi(a_0) \geq \xi(V)$, i.e., $\theta(a_0) \leq 0$.

We have arrived at precisely King’s $\theta$-stability criterion of [27]. We have proved the following statement that $\Pi$-stability reduces the $\theta$ stability if the grades are sufficiently aligned:

**Theorem 3** If all the objects $L_j$ are stable and their grades lie within an interval of width one, then a necessary condition for $\Pi$-stability of an object in $D(Q)$ is that it be a stalk complex, i.e., of the form $V[n]$ for some quiver representation $V$. This object is then stable if and only if $\theta(U) > 0$ for all subrepresentations $U \subset V$, where $\theta$ is defined by (35).
The fact that $\Pi$-stability reduces to $\theta$-stability was also proven in [7] using a D-brane world-volume field theory argument and looking for unbroken supersymmetries. In the latter case it is difficult to know rigorously how far away from alignment one may go before $\theta$-stability fails (see [28] for more on this). We believe the above argument is more straightforward and clearly shows that $\theta$-stability is valid for a nonzero-sized region in moduli space since the grades are continuous functions of the moduli.

In the case of $dP_1$, we can now show that the condition for $\theta$-stability coincides with the criteria laid out in theorem 2. Suppose that

$$\psi < \xi(L_0) < \xi(L_1) < \xi(L_2) < \xi(p) < \xi(L_3) < \psi + 1,$$

(36)

where $p$ is a representation with dimension vector $(1, 1, 1, 1)$. It is not hard to see that $p$ will be stable if the maps in the quiver are generic. However, if, for example, we have $b_0 = b_1 = 0$ then there is a nontrivial map $p \rightarrow L_1$. The kernel of this map has dimension $(1, 0, 1, 1)$ and is a subrepresentation of $p$ with $\theta < 0$ and thus destabilizes $p$. Similarly, the other cases of $p$’s not corresponding to skyscraper sheaves are ruled out.

The moduli space of $\Pi$-stable objects of type $p$ is therefore the same space as the moduli space of skyscraper sheaves and is therefore $dP_1$ itself. Note that another choice of grades can affect this result. For example, if

$$\psi < \xi(L_0) < \xi(p) < \xi(L_1) < \xi(L_2) < \xi(L_3) < \psi + 1,$$

(37)

then the moduli space of $p$’s is $\mathbb{P}^2$. This choice of grades effectively blows the exceptional curve down.

We may generalize this result to the following

**Theorem 4** Assume that the objects $L_i$ are stable and that all skyscraper sheaves correspond to stalk complexes in $\mathbf{D}(Q)$ associated to representations $p$ with a fixed dimension vector (given by (13)). Let the nodes numbered $l, l+1, \ldots, n-1$ be the nodes of $Q$ on which no arrow has a head. If the grades satisfy

$$\psi < \xi(L_0) < \xi(L_1) < \xi(L_2) < \ldots < \xi(L_{l-1}) < \xi(p) < \xi(L_l) \ldots < \xi(L_{n-1}) < \psi + 1,$$

(38)

then the moduli space of $\Pi$-stable (or, equivalently, $\theta$-stable) objects $p$ is the del Pezzo surface $S$.

To see this one proves that the condition both for a failure of Bondal and Orlov’s condition in theorem 2 and the failure of $\Pi$-stability is that there should be a node $k$, where $k < l$, to which the representation has no nonzero paths of nonzero length. We are assuming that the generic representation $p$ satisfies the Bondal and Orlov condition and one shows that the projective resolution analogous to (15) can only jump to something violating the condition, like (18) under these circumstances. The $\Pi$-stability condition occurs since we would have a non-trivial morphism $p \rightarrow L_k$ which induces the decay of $p$. Furthermore, if $p$ can decay in any way, then it will be able to decay by this channel because of the inequalities (38).
Note that we have not proven that only stalk complexes can satisfy the conditions of theorem 2. However, we found this to be true for dP$_1$ above, and it is easy enough to check for other examples. The physics of 0-branes certainly implies that the result of theorem 4 is correct and so this assumption should be true.

We may extend the above results to D(\bar{Q}), i.e., the embedding of S into X. Once we include the dotted arrows we have directed loops in the quiver and so we are no longer guaranteed to have maps either to or from specific L$_j$'s. Unfortunately we needed this property above to make the assertion that the grades of all representations lay in the same unit interval as those of the L$_j$'s. However, in this case one can argue that any representation of \bar{Q} is a deformation of a stable representation of Q and so has the same grade. This fixes the grade in the desired interval and the remaining arguments above go through unchanged.

In particular, \Pi-stability for objects in D(\bar{Q}) descends to \theta-stability for representations of \bar{Q}. Clearly the moduli space for points on S is extended by the ability for points to move “off” S \subset X once we pass to \bar{Q}. Thus, the moduli space of \theta-stable representations of the right dimension vector should represent the normal bundle of S \subset X.

5 Complete alignment of the gradings

We now want to go to the extreme case of alignment of gradings when

$$\xi(L_0) = \xi(L_1) = \xi(L_2) = \ldots = \xi(L_{n-1}).$$

(39)

Using theorem 3 and comparing (7) to (22) we see that the only \Pi-stable objects in D(Q) are the L$_j$'s.

What happens to the moduli space of skyscraper sheaves on S? It is typical in moduli space problems to look for the set of objects in a given class which are either themselves stable, or are a direct sum of stable objects (“poly-stable”). This happens, for example, in the Donaldson–Uhlenbeck–Yau theorem [29, 30] for the case of Hermitian–Yang–Mills connection for which one uses \mu-stability. Similarly, one may analyze a symplectic quotient problem in the context of quivers, which was shown in [31] to yield the classical moduli space of certain fields in the D-Brane world-volume field theory. This was shown by King [27] to be associated to \theta-stability.

Again, when describing a D-Brane moduli space, one expects points in the moduli space to be given either by stable D-Branes, or by direct sums of stable D-branes of identical grading representing a marginally bound state. One might worry that quantum corrections can be important when analyzing bound states at threshold (see [32], for example). However, we are doing a strictly classical analysis here. Indeed, the moduli space itself is a strictly classical concept.

Skyscraper sheaves correspond to quivers whose dimension vector is given by (13). Given (39), no such quiver is stable.\footnote{Given the analysis of exceptional sheaves in [33], at least three members of the exceptional collection must have rank \geq 1.} The only object in the moduli space of quiver representations...
with this dimension vector can be the direct sum
\[ L_0^{\text{rank } E_0} \oplus L_1^{\text{rank } E_1} \oplus \ldots \oplus L_{n-1}^{\text{rank } E_{n-1}}. \] (40)

That is, the moduli space of skyscraper sheaves on $S$ is a single point.

Let us try to extend this result to $D(Q)$. Adding in the dotted arrows to the quiver, the moduli space of skyscraper sheaves is enlarged. Associating nonzero maps to the dotted arrows excludes the nonzero homomorphisms which gave the decomposition (4). Thus, generically one would expect such D-branes to remain stable when we impose (39). Indeed, we would expect points away from $S$ inside $X$ to be unaffected by our manipulations. However, any skyscraper sheaf in $D(Q)$ corresponding to a point on $S$ will become unstable and we replace $S$ by a single point corresponding to (40). This yields the following

**Theorem 5** Given the assumptions of theorem 4 and the genericity assumption above, if $X_0$ represents the moduli space of (direct sums) of $\Pi$-stable quivers in $D(Q)$ with dimension vector (13) subject to the gradings (39) then we have a birational map
\[ X \to X_0, \] (41)
in which the del Pezzo surface $S$ is contracted to a point.

In other words, totally aligned gradings make $S$ collapse to a point. Conversely, a generic skyscraper sheaf in $D(Q)$ will be stable if $\xi(L_i) \leq \xi(L_j)$ for all $i < j$ unless we have the strict alignment (39). In this sense $S$ will only shrink down to a point if we have complete alignment.

Such a contraction has been observed directly by computing periods for the case of the $S = \mathbb{P}^2$ in [31, 34] and $S = \mathbb{P}^1 \times \mathbb{P}^1$ in [20]. Similarly, it was proven in [22] that the periods align at any orbifold point. Note that in these latter cases, $X$ is defined as the target space of the string $\sigma$-model whereas our general view in this paper is that $X$ is the moduli space of 0-branes. We will not attempt here to delve into the profound question of whether these two points of view are equivalent!

It is precisely when the grades are perfectly aligned that the open strings between the D-branes $L_j$ form the field content of a quiver gauge theory (see [20], for example). Thus, the quiver gauge theory description (unperturbed by the addition of Fayet–Iliopoulos terms) is relevant when, and only when, the del Pezzo surface collapses to a point.

So far we have always assumed $\xi(L_i) \leq \xi(L_j)$ for $i < j$. The extreme converse case $\xi(L_0) > \xi(L_1) > \ldots > \xi(L_{n-1})$ represents a situation where the moduli space of skyscraper sheaves becomes completely empty. In such cases the embedding of $D(S)$ into $D(X)$ becomes geometrically meaningless. Generally one can then describe the geometry by a different embedding $D(S') \subset D(X)$ of another del Pezzo surface $S'$ (which might be equivalent to $S$) into $X$. This may represent some kind of flop transition, or a re-identification of $D(S)$ under some quantum symmetry of $X$. 

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