The Quasi-localized Einstein and Møller Energy Complex as Thermodynamic Potentials

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ABSTRACT

To begin with, in this article, I obtain the Einstein and Møller energy complex in PG coordinates. According to the difference of energy within region $\mathcal{M}$ between Einstein and Møller prescription, I could present the difference of energy of RN black hole like the formula of Legendre transformation and propose that the Møller and Einstein energy complex play the role of internal energy and Helmholtz energy in thermodynamics.

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1 Introduction

In the theory of general relativity (GR), one of the most important issues which is still unsolved is the localization of energy. According to Noether’s theorem, one would define a conserved and localized energy as a consequence of energy-momentum tensor $T^{\mu \nu}$ satisfying the differential conservation law

$$\partial_\nu T^{\mu \nu} = 0.$$ (1)

However, in a curved space-time where the gravitational field is presented, the differential conservation law becomes

$$\nabla_\nu T^{\mu \nu} = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\nu} \left( \sqrt{-g} T^{\mu \nu} \right) - \frac{1}{2} g^{\nu \rho} \frac{\partial g^{\mu \rho}}{\partial x^\lambda} T^{\mu \lambda} = 0,$$ (2)

and generally does not lead to any conserved quantity. In GR, we shall look for a new quantity $\Theta^{\mu \nu} = \sqrt{-g} \left( T^{\mu \nu} + t^{\mu \nu} \right)$ instead of $T^{\mu \nu}$, which satisfies the differential conservation equation

$$\partial_\nu \Theta^{\mu \nu} = 0,$$ (3)

if we want to maintain the localization characteristics of energy. Here, $\Theta^{\mu \nu}$ is an energy-momentum complex of matter plus gravitational fields and $t^{\mu \nu}$ is regarded as the contribution of energy-momentum from the gravitational field. It should be noted that $\Theta^{\mu \nu}$ can be expressed as the divergence of the “superpotential” $U^{\mu [\nu \rho]}$ that is antisymmetric in $\nu$ and $\rho$ as

$$\Theta^{\mu \nu} = U^{\mu [\nu \rho]}.$$. (4)

Mathematically, it is freedom on the choice of superpotential, because one can add some terms $\psi^{\mu \nu \rho}$, whose divergence or double divergence is zero, to $U^{\mu \nu \rho}$. A large number of definitions for the gravitational energy in GR have been given by many different authors, for example Einstein [1], Møller [2], Landau and Lifshitz [3], Bergmann and Thomson [4], Tolman [5], Weinberg [6], Papapetrou [7], Komar [8], Penrose [9] and Qadir and Sharif [10]. On the other hand, Chang, Nester and Chen [11] showed that every energy-momentum complex is associated with a legitimate Hamiltonian boundary term and actually quasilocal.
One of those problems for using several kinds of energy-momentum complexes is that they may give different results for the same space-time. Especially Virbhadra and his colleagues [12] showed that Einstein, Landau-Lifshitz, Papapetrou, and Weinberg prescriptions (ELLPW) lead to the same results in Kerr-Schild Cartesian coordinates for a specific class of spacetime, i.e. the general nonstatic spherically symmetric space-time of the Kerr-Schild class

\[ ds^2 = B(u, r)du^2 - 2udu - r^2d\Omega \]  

and the most general nonstatic spherically symmetric space-time

\[ ds^2 = B(t, r)dt^2 - A(t, r)dr^2 - 2F(t, r)dtdr - D(t, r)r^2d\Omega, \]

but not in Schwarzschild Cartesian coordinates. Afterward Xulu [13] presented Bergmann-Thomson complex also “coincides” with ELLPW complexes for a more general than the Kerr-Schild class metric. Mirshekari and Abbassi [14] find a unique form for a special general spherically symmetric metric in which the energy of Einstein and Møller prescriptions lead to the same result. In particular, whatever coordinates do not exist the same energy complexes associated with using definitions of Einstein and Møller in some space-time solutions, i.e. Reissner-Nordström (RN) black hole. On the other hand, Yang and Radinschi [15] attemptd to investigate the difference between the energy of Einstein prescription \( E_{\text{Einstein}} \) and Møller prescription \( E_{\text{Møller}} \), and observed the difference \( \Delta E = E_{\text{Einstein}} - E_{\text{Møller}} \) can be related to the energy density of the matter fields \( T_{0} \) as

\[ \Delta E \sim r^3 \times T_{0}. \]

Matyjasek [16] also presented two analogous relations which are

\[ \Delta E = 4\pi r^3 T_{0} \]

for the simplified stress-energy tensor of the matter field and

\[ \Delta E = 4\pi r^3 \langle T_{r}^{(s)} \rangle_{\text{ren}} \]

for the approximate renormalized stress-energy tensor of the quantized massive scalar \( s = 0 \), spinor \( s = 1/2 \) and vector \( s = 1 \) field. Later, Vagenas [17] hypothesized that \( \alpha_n^{(\text{Einstein})} \) and \( \alpha_n^{(\text{Møller})} \) are the expansion coefficients
of $E_{\text{Einstein}}$ and $E_{\text{Møller}}$ in the inverse powers of $r$, and found out an interesting relation between these two coefficients

$$\alpha_n^{(\text{Einstein})} = \frac{1}{n+1} \alpha_n^{(\text{Møller})}. \quad (10)$$

Finally, Matyjasek [16] and Yang et al. [18] pointed out the following formula respectively

$$E_{\text{Møller}} = E_{\text{Einstein}} - r \frac{dE_{\text{Einstein}}}{dr}. \quad (11)$$

It should be noted that these relations in Eq. (7)-(11) offer us the mathematical formula between $E_{\text{Einstein}}$ and $E_{\text{Møller}}$ only. The remainder of the article is organized as follows. In section 2, I will calculate the energy distribution for generalized Painlevé-Gullstrand (PG) coordinates [19] by using the Einstein and Møller complex. In section 3, the physical explanation of the difference $\Delta E$ will be given. I will summarize and conclude finally in section 4. In this article, I use geometrized units in which $c = G = \hbar = 1$ and the metric has signature $(+ - - -)$.

## 2 Using the Einstein and Møller Energy Complex in generalized PG coordinates

The continuation of black holes across the horizon is a well understood problem discussed on GR. The difficulties of the Schwarzschild coordinates $(t, r, \theta, \phi)$ at the horizons of a nonrotating black hole provide a vivid illustration of the fact that the meaning of the coordinates is not independent of the metric tensor $g_{\mu\nu}$ in GR. Several coordinate systems produce a metric that is manifestly regular at horizons, i.e. the Kruskal-Szekeres, Eddington-Finkelstein, and PG coordinates. However, PG coordinates have often been employed to study the physics of black holes. They have been applied to analyse quantum dynamical black holes [20], and used extensively in derivations of Hawking radiation as tunneling following the work of Parikh and Wilczek [21]. In this section, while using PG coordinates, I will find out the energy of static spherically symmetric black hole solutions in Einstein and Møller prescriptions. In four-dimensional theory of gravity, I can write the static spherically symmetric metrics in the form

$$ds^2 = f dt^2 - f^{-1} dr^2 - r^2 d\Omega, \quad (12)$$
where $f$ is a function of $r$, i.e. $f = f(r)$. Let me transform to generalized PG coordinates [19] and introduce the PG time $dt_p = dt + \beta dr$, thus 4-metric can be written as

$$ds^2 = f dt_p^2 - 2 \sqrt{1 - \frac{f}{A^2}} dt_p dr - \frac{1}{A^2} dr^2 - r^2 d\Omega, \quad (13)$$

where $A \equiv \sqrt{f/(1 - f^2 \beta^2)}$.

At the outset, the energy component in the Einstein prescription [1] is given by

$$E_{\text{Einstein}} = \frac{1}{16\pi} \int \frac{\partial H_{0}^{0}}{\partial x^i} d^3 x, \quad (14)$$

where $H_{0}^{0}$ is the corresponding von Freud superpotential

$$H_{0}^{0} = \frac{g_{0n}}{\sqrt{-g}} \frac{\partial}{\partial x^m} \left[ (-g)(g_{0n}^{lm} - g_{ln}^{0m}) \right], \quad (15)$$

and the Latin indices take values from 1 to 3. For performing the calculations concerning the energy component of the Einstein energy-momentum complex, I have to transform the spatial parts of above metric (13) into the quasi-Cartesian coordinates $(x, y, z)$

$$ds^2 = A^2 dt_p^2 - 2 \sqrt{1 - \frac{f}{A^2}} dt_p \left( \frac{x}{r} dx + \frac{y}{r} dy + \frac{z}{r} dz \right)$$

$$- \left( 1 - \frac{1}{A^2} \right) \left( \frac{x}{r} dx + \frac{y}{r} dy + \frac{z}{r} dz \right)^2 - (dx^2 + dy^2 + dz^2). \quad (16)$$

Then, the required nonvanishing components of the Einstein energy-momentum complex $H_{0}^{0}$ are

$$H_{0}^{01} = \frac{2C}{r} x,$$

$$H_{0}^{02} = \frac{2C}{r} y,$$

$$H_{0}^{03} = \frac{2C}{r} z,$$

and these are easily shown in spherical coordinates to be a vector

$$H_{0}^{0r} = \frac{2C}{r} \hat{r}, \quad (17)$$

where $C = 1 - f$ and $\hat{r}$ is the outward normal. Applying the Gauss theorem I obtain

$$E_{\text{Einstein}} = \frac{1}{16\pi} \int H_{0}^{0r} \cdot \hat{r} r^2 d\Omega, \quad (19)$$
and the integral being taken over a sphere of radius $r$ and the differential solid angle $d\Omega$. The Einstein energy complex within radius $r$ reads

$$E_{\text{Einstein}} = \frac{r}{2} (1 - f).$$  \hfill (20)

Next, the energy component of the Møller energy-momentum complex [2] is described as

$$E_{\text{Møller}} = \frac{1}{8\pi} \int r \frac{\partial \chi_0}{\partial x^i} d^3 x,$$  \hfill (21)

where $\chi_0$ is the Møller superpotential

$$\chi_0 = \sqrt{-g} \left( \frac{\partial g_{0\alpha}}{\partial x^\beta} - \frac{\partial g_{0\beta}}{\partial x^\alpha} \right) g^{0\beta} g^l_\alpha,$$  \hfill (22)

and the Greek indices run from 0 to 3. However, the only nonvanishing component of Møller’s superpotential is

$$\chi_0 = \frac{df}{dr} r^2 \sin \theta.$$  \hfill (23)

Applying the Gauss theorem, I evaluate the integral over the surface of a sphere within radius $r$, and find the energy distribution is

$$E_{\text{Møller}} = \frac{r^2}{2} \frac{df}{dr}.$$  \hfill (24)

Here, I consider the results of calculation for two cases of the simplest black hole solutions, i.e. Schwarzschild and RN solution. In the first case I have $f = 1 - 2M/r$, therefore the energy complex of Einstein is

$$E_{\text{Einstein}} = M,$$  \hfill (25)

and of Møller is also

$$E_{\text{Møller}} = M.$$  \hfill (26)

For the next case it is defined that $f = 1 - 2M/r + Q^2/r^2$, so the energy complex in Einstein prescription is

$$E_{\text{Einstein}} = M - \frac{Q^2}{2r},$$  \hfill (27)

and in Møller prescription is

$$E_{\text{Einstein}} = M - \frac{Q^2}{r}.$$  \hfill (28)
It should be noted that the above results of Einstein energy complex in PG Cartesian coordinates are equivalent to in Schwarzschild Cartesian and Kerr-Schild Cartesian ones [12], but the time coordinate of these three coordinates is different to each other. Using the Schwarzschild black hole as an example, the time coordinate of Schwarzschild Cartesian coordinates \( t \), of Kerr-Schild Cartesian coordinates

\[
v = t + r + 2M \ln \left| \frac{r}{2M} - 1 \right|,
\]

and of PG Cartesian coordinates

\[
t_p = t + 4M \left( \sqrt{\frac{r}{2M}} + \frac{1}{2} \ln \frac{\sqrt{r/2M} - 1}{\sqrt{r/2M} + 1} \right)
\]

are not the same. In other words, independent of the choices of these three kinds of time coordinate, the energy complex of Einstein within radius \( r \) is \( E_{\text{Einstein}} = M \). It is indefinite that the energy complex of Einstein is universal for any kinds of time coordinate. Some quasi-local energy expressions [22] and the Einstein energy-momentum pseudotensors are coordinate-independent in spherically symmetric space-time. It remains to investigate whether the coordinate-independent is the property of the spherically symmetric space-time.

3 Legendre transformation between the Einstein’s and Møller’s Energy Complex

To understand the physical meaning of difference \( \Delta E \), let me to begin with examining the RN black hole, which is a static spherically symmetric solution with two horizons, as an example. The line element of RN black hole can be written as

\[
 ds^2 = f(r)dt^2 - f^{-1}(r)dr^2 - r^2d\Omega,
\]

where

\[
 f(r) = \left( 1 - \frac{r_+}{r} \right) \left( 1 - \frac{r_-}{r} \right),
\]

\( r_+ = M + \sqrt{M^2 - Q^2} \) is the event horizon and \( r_- = M - \sqrt{M^2 - Q^2} \) is the inner Cauchy horizon. According to Eq.(19) and Eq.(23), the Einstein energy complex with radius \( r \) is

\[
 E_{\text{Einstein}} = \frac{r_+ + r_-}{2} - \frac{r_+ r_-}{2r},
\]
and the Møller energy complex is

$$E_{\text{Møller}} = \frac{r_+ + r_-}{2} - \frac{r_+ r_-}{r}. \quad (34)$$

Therefore, the difference of energies with radius $r$ between the Einstein and Møller prescription can be obtained as

$$\Delta E = \frac{r_+ r_-}{2r}. \quad (35)$$

In the article of Nester et. al. [11], they had stated that "Consequently, there are various of energy, each corresponding to a different choice of boundary condition; this situation can be compared with thermodynamics with its various energies: internal, enthalpy, Gibbs, and Helmholtz." Hence, I insert the idea of black hole thermodynamics to compare energy-momentum complex with thermodynamic potential.

Afterward, I would introduce two thermodynamic qualities of black hole, the Hawking temperature [19]

$$T_H = \frac{1}{4\pi} \frac{\partial f}{\partial r} \bigg|_{r_h} \quad (36)$$

and the Bekenstein-Hawking entropy [23]

$$S_{BH} = \frac{A}{4} \bigg|_{r_h} = \pi r_h^2. \quad (37)$$

Because those two qualities are only defined on event horizon, at $r = r_+$, the temperature is given as

$$T^+ = \frac{r_+ - r_-}{4\pi r_+^2} = \frac{\sqrt{M^2 - Q^2}}{2\pi(M + \sqrt{M^2 - Q^2})^2} \quad (38)$$

and the entropy is also given as

$$S^+ = \pi r_+^2 = \pi(M + \sqrt{M^2 - Q^2})^2, \quad (39)$$

Supposing that we consider the region between those two horizons, shown as $\mathcal{M} = \mathcal{B}^3(r_+) - \mathcal{B}^3(r_-)$, the difference of energies will be obtained in the form

$$\Delta E \bigg|_{r=r_+}^{r=r_-} = -\frac{r_+ - r_-}{2} = -\sqrt{M^2 - Q^2} = -2T^+ S^+. \quad (40)$$
Here, $B^3(r)$ is a 3-sphere within a radius $r$. The term $T^+S^+$ can be considered that the heat flow streams out the region $\mathcal{M}$ by passing the boundary of $B^3(r_+)$. Therefore, Eq. (39) would be rewritten as

$$E_{\text{Møller}}|_{r_+}^r - E_{\text{Einstein}}|_{r_+}^r = 2T^+S^+. \tag{41}$$

It is meaning that the difference of energies between Einstein and Møller prescription equal to the double of the heat flow streams out by passing the boundary of $B^3(r_+)$ in the region $\mathcal{M}$ of RN black hole.

On the other hand, to base on Zhao’s study [24], the entropy of black hole, which has two horizons, is defined as $\tilde{S} = S^+ + S^-$, where the entropy of the inner Cauchy horizon can be shown as

$$S^- = \pi r_-^2 = \pi \left(M - \sqrt{M^2 - Q^2}\right)^2, \tag{42}$$

and the temperature of the inner Cauchy horizon is given as

$$T^- = \frac{\kappa_-}{2\pi}, \tag{43}$$

where the surface gravity of the inner Cauchy horizon is [25]

$$\kappa_- = \lim_{r \to r_-} \frac{1}{2(r - r_-)} \sqrt{-\frac{g^{11}}{g^{00}}} = \frac{r_+ - r_-}{2r_-^2}. \tag{44}$$

So the difference of energy between the Einstein prescription and Møller prescription within the region $\mathcal{M}$ can be written as

$$E_{\text{Møller}}|_{r_+}^r - E_{\text{Einstein}}|_{r_+}^r = T^+S^+ + T^-S^-, \tag{45}$$

and the heat flow will be with respect to both two boundaries of $\mathcal{M}$. To rewrite Eq. (44) as

$$E_{\text{Einstein}}|_{\mathcal{M}} = E_{\text{Møller}}|_{\mathcal{M}} - \sum_{\partial\mathcal{M}} TS,$$ \tag{46}

these heat flows are exhibited on every boundary of $\mathcal{M}$. Comparing Eq. (45) with the Legendre transformation, $E_{\text{Møller}}$ and $E_{\text{Einstein}}$ in the region $\mathcal{M}$ play the role of internal energy $U$ and Helmholtz energy $F$ in thermodynamics, even so there is a puzzle where $E_{\text{Møller}}$ or $E_{\text{Einstein}}$ are not a function of $T$ or $S$. We could obtain not only a physical meaning of the difference of energies in Eq. (34), although the statement can only be used to RN black hole, but also such a result agreed with the entropy redefined in Zhao’s article.
4 Conclusion and Discussion

I have attempted to answer two questions in this article. One is whether the calculation of Einstein energy-momentum complex is acceptable in PG coordinate, and the other is whether those energy-momentum complex can be described as a thermodynamic potential. Here, the expression for energy of the static spherically symmetric space-time with the PG Cartesian coordinates, Eq.(19), is obtained $E_{\text{Einstein}} = (1 - f)r/2$. This is a reasonable and satisfactory result, because Virbhadra [12], using the Kerr-Schild Cartesian coordinates, and Yang et al. [18], using the Schwarzschild Cartesian coordinates, also got the same expression. It is interesting to investigate whether there is any coincidence between the energy expressions with those three time coordinate.

In addition, I have showed that the relational formula about $E_{\text{Einstein}}$ and $E_{\text{Møller}}$ is similar to the Legendre transformation. The $E_{\text{Møller}}$ and $E_{\text{Einstein}}$ are regarded as the equivalent of internal energy $U$ and Helmholtz energy $F$ in the region $\mathcal{M}$. Although, the transformation takes us from a function of one pair variables to the other. It means that $S_{BH}$ and $T_H$ must be the variable of $E_{\text{Møller}}$ and $E_{\text{Einstein}}$, but I do not verify that yet. On the other hand, when I set $S_\perp = \pi r^2$ to be a variable, the second term in the right-hand side of Eq.(11) can be replaced as

$$E_{\text{Møller}} = E_{\text{Einstein}} - 2S_\perp \frac{dE_{\text{Einstein}}}{dS_\perp}. \quad (47)$$

To compare with $F = U - TS$, we could obtain

$$T_H = \frac{dE_{\text{Einstein}}}{dS_\perp} \bigg|_{r_h}. \quad (48)$$

Here the formula of Eq.(44) presents that $E_{\text{Einstein}}$ and $E_{\text{Møller}}$ play the role of $U$ and $F$, and is opposite to the view of above.

In summary, I have obtained the Einstein and Møller energy complexes of static spherically symmetric black hole with generalized PG coordinates in which has been used in derivations of Hawking radiation as tunneling. Base on the calculation of energy expression in generalized PG coordinates, in Eq.(45) I have combined the difference $\Delta E$ with the temperature and entropy of black hole, but Eq.(4g) do not fit in with the Legendre transformation.
Nevertheless, it is an example to show that the energy-momentum complexes of RN black hole will compare with thermodynamic potential, and future research should be considered on more kinds of space-time.

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