ABOUT THE NUCLEARITY OF $S_{(M_p)}$ AND $S_\omega$

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Dedicated to Prof. Luigi Rodino on the occasion of his 70\textsuperscript{th} birthday.

Abstract. We use an isomorphism established by Langenbruch between some sequence spaces and weighted spaces of generalized functions to give sufficient conditions for the (Beurling type) space $S_{(M_p)}$ to be nuclear. As a consequence, we obtain that for a weight function $\omega$ satisfying the mild condition: $2\omega(t) \leq \omega(Ht) + H$ for some $H > 1$ and for all $t \geq 0$, the space $S_\omega$ in the sense of Björck is also nuclear.

1. Introduction and preliminaries

For a sequence $(M_p)_{p \in \mathbb{N}_0}$ which satisfies Komatsu’s standard condition $(M2)'$ (stability under differential operators) and, moreover, the condition that there is $H > 0$ such that for any $C > 0$ there is $B > 0$ with

$$s^{s/2}M_p \leq BC^s H^{s+p}M_{s+p}, \quad \text{for any } s, p \in \mathbb{N}_0,$$

where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, Langenbruch \cite{8} proves that the Hermite functions are a Schauder basis in the spaces of ultradifferentiable functions of (Beurling type):

$$S_{(M_p)}(\mathbb{R}^d) := \left\{ f \in C^\infty(\mathbb{R}^d) : \text{ for any } j \in \mathbb{N}, \sup_{\alpha, \beta \in \mathbb{N}_0^d} \sup_{x \in \mathbb{R}^d} |x^\alpha D^\beta f(x)|j^{\alpha+\beta}/M_{\|\alpha+\beta\|} < +\infty \right\}.$$ 

Moreover, in \cite{8} it is also established an isomorphism between $S_{(M_p)}$ and the Köthe sequence space:

$$\Lambda_{(M_p)} := \left\{ (c_k)_{k \in \mathbb{N}_0} : \text{ for any } j \in \mathbb{N}_0, \sup_{k \in \mathbb{N}_0} |c_k| e^{M(jk^{1/2})} < +\infty \right\},$$

where

$$M(t) = \sup \log \frac{t^p M_0}{M_p}, \quad t > 0,$$

is the associated function of $(M_p)$.

In this paper we use Grothendieck-Pietsch criterion to characterize when the space $\Lambda_{(M_p)}$ is nuclear under the assumption that $(M_p/M_0)^{1/p}$ is bounded below by a positive constant and, hence, $M(t)$ is increasing and convex in $\log t$ (see \cite{7} p. 49). Indeed, we prove in Theorem 2.2 that $\Lambda_{(M_p)}$ is nuclear if and only if there is $H > 1$ such that for any $t > 0$ we have

$$M(t) + \log t \leq M(Ht) + H.$$ 

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As it is observed in [8, (2.2)], condition $(M2)'$ implies $(1.3)$. Therefore, conditions $(M2)'$ and $(1.1)$ imply that $S_{(M_p)}$ is nuclear (see Corollary 2.3). This should be compared with [11], where the authors prove that $S_{(M_p)}$ is nuclear under Komatsu’s conditions $(M1)$ and $(M2)$.

As a consequence of Theorem 2.2 we give a simple proof of the nuclearity of the space $S_{\omega}$ in the sense of Björck [1] given in Definition 3.1 under the following condition of Bonet, Meise and Melikhov [5] on the weight function $\omega$:

$$\exists H > 1 \text{ s.t. } 2\omega(t) \leq \omega(Ht) + H, \quad t \geq 0.$$  

In fact, in this case the space $S_{\omega}$ is isomorphic to the space $S_{(M_p)}$ for some suitable sequence $(M_p)$.

2. Results for the space $S_{(M_p)}$

In this section we characterize the nuclearity of $\Lambda_{(M_p)}$ and give sufficient conditions for the nuclearity $S_{(M_p)}$.

We consider a sequence $(M_p)_p$ satisfying the condition that $(M_p/M_0)^{1/p}$ is bounded from below by a positive constant, so that the associated function defined by (1.2) is increasing and convex in $\log t$.

From Grothendieck-Pietsch criterion it is easy to obtain the following

**Lemma 2.1.** The Köthe sequence space $\Lambda_{(M_p)}$ is nuclear if and only if for every $j \in \mathbb{N}$ there exists $m \in \mathbb{N}$ with $m \geq j$ such that

$$\sum_{k=0}^{+\infty} e^{M(jk^{1/2}) - M(mk^{1/2})} < +\infty.$$  

**Proof.** It follows from Proposition 28.16 of [9]. □

**Theorem 2.2.** The space $\Lambda_{(M_p)}$ is nuclear if and only if $(1.3)$ holds.

**Proof.** Let us first remark that $(1.3)$ implies

$$M(t) + 2\log t = M(t) + \log t + \log t$$  

$$\leq M(Ht) + H + \log(Ht) - \log H$$  

$$\leq M(H^2 t) + 2H - \log H$$

and, more in general,

$$M(t) + N \log t \leq M(H^N t) + C_{N,H}, \quad \forall N \in \mathbb{N},$$  

for some constant $C_{N,H} > 0$ depending on $N$ and $H$.

Let us now assume that $(1.3)$ is satisfied and prove the nuclearity of $\Lambda_{(M_p)}$, using $(2.2)$ for a fixed $N > 2$. By Lemma 2.1 it’s enough to prove the convergence of the series $(2.1)$. Indeed,
for every fixed \( j \in \mathbb{N} \), choosing \( m \geq H^N j \),

\[
e^{M(jk^{1/2}) - M(mk^{1/2})} \leq e^{M(jk^{1/2}) - M(H^N jk^{1/2})} \\
\leq e^{M(jk^{1/2}) - M(jk^{1/2}) - N \log(jk^{1/2}) + C_{N,H}} \\
= e^{C_{N,H} j - N \frac{1}{k^{N/2}}}
\]

and the series \( \sum_{k=1}^{+\infty} \frac{1}{k^{N/2}} \) converges since \( N > 2 \).

Let us now assume that the series (2.1) converges and prove (1.3). To this aim, let us first remark that, for \( m > j \),

\[
k \mapsto -M(e^{\log mk^{1/2}}) - M(e^{\log jk^{1/2}})
\]

is decreasing, because \( M(e^t) \) is convex by our assumptions (see [7, p. 49]), and therefore its difference quotient \( \frac{M(e^t) - M(e^s)}{t-s} \) is increasing with respect to both variables \( t \) and \( s \); this implies that

\[
M(mk^{1/2}) - M(jk^{1/2}) = M(e^{\log mk^{1/2}}) - M(e^{\log jk^{1/2}}) \log \frac{m}{j}
\]

is increasing with respect to \( k \).

Then the convergence of (2.1) implies that

\[
\lim_{k \to +\infty} ke^{M(jk^{1/2}) - M(mk^{1/2})} = 0
\]

and hence

\[
\sup_{k \in \mathbb{N}} ke^{M(jk^{1/2}) - M(mk^{1/2})} \leq A,
\]

for some \( A \in \mathbb{R}^+ \). Then

\[
\log k + M(jk^{1/2}) - M(mk^{1/2}) \leq \log A, \quad \forall k \in \mathbb{N},
\]

and hence

\[
M(jk^{1/2}) - M(mk^{1/2}) \leq -\log k + \log A = -2\log(jk^{1/2}) + \log(j^2A) \leq -\log(jk^{1/2}) + \log(j^2A).
\]

To prove that (2.3) implies (1.3) let us first consider \( t \geq 1 \) and choose the smallest \( k \in \mathbb{N} \) such that \( t \leq jk^{1/2} \). Since

\[
j(k + 1)^{1/2} - jk^{1/2} = \frac{j}{\sqrt{k+1} + \sqrt{k}} < j, \quad \forall k \in \mathbb{N},
\]

we have that \( jk^{1/2} \in [t, (j + 1)t] \) and therefore, from (2.3),

\[
M(t) + \log t \leq M(jk^{1/2}) + \log(jk^{1/2}) \leq M(mk^{1/2}) + \log(j^2A) = M\left(\frac{m}{j} jk^{1/2}\right) + \log(j^2A) \leq M\left(\frac{m}{j}(j + 1)t\right) + \log(j^2A), \quad \forall t \geq 1,
\]
and hence, for $H = \max \left\{ \frac{m}{j}(j+1), \log(j^2A) + M(1) \right\}$, we have that (1.3) is satisfied for all $t > 0$.

So, we automatically obtain

**Corollary 2.3.** If $(M_p)$ satisfies $(M2)'$ and (1.1), the space $S_{(M_p)}$ is nuclear.

**Proof.** The spaces $S_{(M_p)}$ and $\Lambda_{(M_p)}$ are isomorphic because $(M_p)$ satisfies $(M2)'$ and (1.1) by Theorem 3.4 of [6]. Since $(M2)'$ implies (1.3) (see for instance [6, (2.2)]), the result follows from Theorem 2.2.

**Remark 2.4.** Looking inside the proof of Theorem 3.4 of [8] we can see that in fact Langenbruch needs only (1.1) and (1.3), so that in the above corollary we could substitute the assumption $(M2)'$ with the condition that the associated function $M(t)$ satisfies (1.3).

### 3. Results for the space $S_\omega$. Examples

In this section we give a sufficient condition for the space $S_\omega$ in the sense of Björck [1] to be nuclear.

We consider continuous increasing *weight functions* $\omega : [0, +\infty) \to [0, +\infty)$ satisfying:

- $\exists L > 0$ s.t. $\omega(2t) \leq L(\omega(t) + 1)$, $\forall t \geq 0$,
- $\omega(t) = \omega(0)$, as $t \to +\infty$,
- $\exists a \in \mathbb{R}, b > 0$ s.t. $\omega(t) \geq a + b \log(1 + t)$, $\forall t \geq 0$,
- $\varphi : t \mapsto \omega(e^t)$ is convex.

Then we define $\omega(\zeta) := \omega(|\zeta|)$ for $\zeta \in \mathbb{C}^d$.

We denote by $\varphi^*$ the *Young conjugate* of $\varphi$, defined by

$$
\varphi^*(s) := \sup_{t \geq 0} (ts - \varphi(t)).
$$

We recall that $\varphi^*$ is increasing and convex, $\varphi^{**} = \varphi$ and $\varphi^*(s)/s$ is increasing. Moreover, it will be not restrictive, in the following, to assume $\omega_{|[0,1]} \equiv 0$ and hence $\varphi^*(0) = 0$.

The space $S_\omega(\mathbb{R}^d)$ of weighted rapidly decreasing functions is then defined by (see [1]):

**Definition 3.1.** $S_\omega(\mathbb{R}^d)$ is the set of all $u \in L^1(\mathbb{R}^d)$ such that $u, \hat{u} \in C^\infty(\mathbb{R}^d)$ and

1. $\forall \lambda > 0, \alpha \in \mathbb{N}_0^d : \sup_{x \in \mathbb{R}^d} e^{\lambda \omega(x)} |D^\alpha u(x)| < +\infty$,
2. $\forall \lambda > 0, \alpha \in \mathbb{N}_0^d : \sup_{\xi \in \mathbb{R}^d} e^{\lambda \omega(\xi)} |D^\alpha \hat{u}(\xi)| < +\infty$,

where $D^\alpha = (-i)^{|\alpha|} \partial^\alpha$.

Note that

$$
\omega_0(t) = \begin{cases} 
0, & 0 \leq t \leq 1 \\
\log t, & t > 1 
\end{cases}
$$

is a weight function for which $S_{\omega_0}(\mathbb{R}^d)$ coincides with the classical Schwartz class $S(\mathbb{R}^d)$.

The space $S_\omega(\mathbb{R}^d)$ is a Fréchet space with different equivalent systems of seminorms (cf. [3], [4], [2]). In particular, we shall use in what follows the family of seminorms

$$
p_\lambda(u) = \sup_{\alpha, \beta \in \mathbb{N}_0^d} \sup_{x \in \mathbb{R}^d} |x_\beta D^\alpha u(x)| e^{-\lambda \varphi^*(\frac{|\alpha + \beta|}{x})}.
$$
Given a weight function $\omega$ we construct the sequence $(M_p)$ by

\begin{equation}
M_p = e^{\varphi^*(p)}, \quad \forall p \in \mathbb{N}_0.
\end{equation}

Then the associated function of $M_p$ is equivalent to the given weight $\omega$. Indeed, on one side, since $M_0 = 1$, we have, for $t > 0$:

\[
M(t) = \sup_{p \in \mathbb{N}_0} \log \frac{t^p}{M_p} = \sup_{p \in \mathbb{N}_0} \left( \log t^p - \log e^{\varphi^*(p)} \right)
\leq \sup_{s \geq 0} (s \log t - \varphi^*(s)) = \varphi(\log t) = \omega(t).
\]

On the other side, for $t > 0$:

\[
\omega(t) = \sup_{s \geq 0} (s \log t - \varphi^*(s)) = \sup_{p \in \mathbb{N}_0} \sup_{p \leq s < p+1} (s \log t - \varphi^*(p)) = \log t + M(t) \leq 2M(t) + \log M_1
\]

since $M(t) \geq \log t - \log M_1$ by definition.

Therefore

\begin{equation}
M(t) \leq \omega(t) \leq M(t) + \log t \leq 2M(t) + A, \quad \forall t > 0,
\end{equation}

and for some $A > 0$.

Moreover,

\begin{equation}
M_p = e^{\varphi^*(p)} = \exp\{\sup_{t \geq 0} (pt - \omega(e^t))\} = \sup_{t \geq 0} \left( e^{pt} e^{-\omega(e^t)} \right)
\end{equation}

\begin{equation}
= \sup_{s \geq 1} (s^p e^{-\omega(s)}) = \sup_{s \geq 0} (s^p e^{-\omega(s)}),
\end{equation}

since $\omega|_{[0,1]} \equiv 0$.

Let us remark that the sequence $(M_p)$ satisfies $(M_p/M_0)^{1/p} \geq 1$ and the condition of logarithmic convexity

\begin{equation}
(M1) \quad M_p^2 \leq M_{p-1}M_{p+1}, \quad p \in \mathbb{N},
\end{equation}

since

\[
2\varphi^*(p) = 2 \sup_{t \geq 0} (tp - \varphi(t)) \leq \sup_{t \geq 0} (t(p-1) - \varphi(t)) + \sup_{t \geq 0} (t(p+1) - \varphi(t))
= \varphi^*(p-1) + \varphi^*(p+1).
\]

If $\omega$ satisfies condition (BMM), then also $M(t)$ satisfies condition (BMM) because, by (3.3),

\begin{equation}
2M(t) \leq \frac{1}{2} (4\omega(t)) \leq \frac{1}{2} (2\omega(Ht) + 2H)
\leq \frac{1}{2} \omega(H^2t) + \frac{3}{2} H \leq M(H^2t) + \frac{A}{2} + \frac{3}{2} H.
\end{equation}

Then, by [7 Prop. 3.6], we obtain that $(M_p)$ satisfies also the condition of stability under ultradifferential operators:

\begin{equation}
(M2) \quad \exists A, H > 0 \text{ s.t. } M_p \leq AH^p \min_{0 \leq q \leq p} M_q M_{p-q}.
\end{equation}
Moreover, the sequence \((M_p)_p\) satisfies (1.1). Indeed, since \(\omega(t) = o(t)\) as \(t \to +\infty\), we have that for every \(\varepsilon > 0\) there exists \(R_\varepsilon > 0\) such that \(\omega(t) \leq \varepsilon t + R_\varepsilon\) for all \(t \geq 0\). Therefore, for \(s \geq \varepsilon\),
\[
\phi^*(s) = \sup_{t \geq 0} \left( ts - \omega(e^t) \right) \geq \sup_{t \geq 0} \left( ts - \varepsilon e^t \right) - R_\varepsilon = s \log \frac{s}{\varepsilon} - s - R_\varepsilon,
\]
and hence
\[
\left( \frac{s}{\varepsilon} \right)^s \leq e^{s + \phi^*(s) + R_\varepsilon}, \quad \forall s \geq \varepsilon.
\]
Since, for \(s \leq \varepsilon\) we have that \(s^s \leq (\varepsilon s)^s\), we finally have that for every \(s > 0\):
\[
s^{s/2} M_p \leq s^s e^{\phi^*(p)} \leq e^{R_\varepsilon (\varepsilon s)^s} e^{\phi^*(s) + \phi^*(p)} \leq e^{R_\varepsilon (\varepsilon s)^s} s^{\phi^*(p+s)} = e^{R_\varepsilon (\varepsilon s)^s} M_{p+s}.
\]

If \(\omega\) satisfies (BMM), then \(\Lambda(M_p)\) coincides with the sequence space
\[
(3.7) \quad S_\omega := \left\{ (c_k)_{k \in \mathbb{N}_0} : \sup_{k \in \mathbb{N}_0} |c_k| e^{\omega(j^{k/2})} < +\infty \forall j \in \mathbb{N}_0 \right\},
\]
by (3.4) and (3.6).

**Theorem 3.2.** Let \(\omega\) be a weight function. Then \(S_\omega\) is nuclear if and only if \(\omega\) satisfies condition (1.3).

**Proof.** As in Theorem 2.2 we use [9, Prop. 28.16] for the sequence space \(S_\omega\). \(\square\)

**Example 3.3.** Condition (1.3) for a weight function \(\omega\) is weaker than (BMM). For instance
\[
\omega(t) = \begin{cases} 0, & 0 \leq t \leq 1 \\ \log^2 t, & t > 1 \end{cases}
\]
satisfies (1.3) but not (BMM).

**Corollary 3.4.** Let \(\omega\) be a weight function satisfying (BMM). Then \(\Lambda_\omega\) is nuclear.

**Proposition 3.5.** Let \(\omega\) be a weight function satisfying (BMM) and \((M_p)\) the sequence defined by (3.3). Then \(S_\omega(\mathbb{R}^d)\) is equal (as vector space and as locally convex space) to \(S(M_p)\) and isomorphic to \(\Lambda(M_p) = \Lambda_\omega\).

**Proof.** We endow \(S_\omega(\mathbb{R}^d)\) with the family of seminorms (3.2). It is isomorphic (and hence equal) to \(S(M_p)\) because, by [5, formulas (5), (6)], the following two conditions hold:
\[
\forall j \in \mathbb{N} \exists \lambda, c > 0 \text{ s.t. } e^{\lambda \phi^*(\xi)} \leq cj^{-p} M_p, \quad \forall p \in \mathbb{N}_0,
\]
and
\[
\forall \lambda > 0 \exists j \in \mathbb{N}, C > 0 \text{ s.t. } j^{-p} M_p \leq Ce^{\lambda \phi^*(\xi)}, \quad \forall p \in \mathbb{N}_0.
\]

Finally, \(S(M_p)\) is isomorphic to \(\Lambda(M_p)\) by Theorem 3.4 of [8], since \((M_p)\) satisfies (M2) (stronger than (M2)') and (1.1). Moreover \(\Lambda(M_p)\) coincides with \(\Lambda_\omega\), as we already remarked in the comment for formula (3.7). \(\square\)

Condition (1.3), written in terms of the weight function \(\omega\), is equivalent to the nuclearity of \(\Lambda_\omega\) by Theorem 3.2, but it is not necessary for the nuclearity of \(S_\omega\). For example, the weight \(\omega_0(t)\) defined by (3.1) does not satisfy (1.3) and hence \(\Lambda_{\omega_0}\) is not nuclear, while \(S_\omega\) is well known to be nuclear. In particular, \(\Lambda_{\omega_0}\) and \(S\) are not isomorphic. On the other hand, from the results that we have we do not know if condition (1.3) is sufficient for the nuclearity of \(S_\omega\), but we need the stronger condition (BMM), as we state in the following.
Theorem 3.6. Let $\omega$ be a weight function satisfying (BMM). Then $S_\omega$ is a nuclear space.

Proof. It follows from Proposition 3.5 and Corollary 3.4. □

Example 3.7. There exist sequences $(M_p)_p$ satisfying (1.1) and (1.3) (for the associated function), but not (M2).

Let us consider, for example, a weight function $\omega$ satisfying (1.3) but not (BMM) (see Example 3.3) and construct the sequence $(M_p)_p$ as in (3.3). Then $M_p$ satisfies (M1), (1.1) and its associated function satisfies (1.3) because, by (3.4) and (2.2):

$$M(t) + \log t \leq \omega(t) + 2 \log t - \log t \leq \omega(H^2 t) + C_{2,H} - \log t$$

$$\leq M(H^2 t) + \log(H^2 t) + C_{2,H} - \log t = M(H^2 t) + 2 \log H + C_{2,H}.$$  

However, $M(t)$ does not satisfy (BMM) because $\omega(t)$ does not satisfy (BMM) (see (3.4)), therefore $M_p$ does not satisfy (M2) by [7, Prop. 3.6].

The example above furnishes a sequence $(M_p)_p$ satisfying (M1), but not (M2), for which the space $S_{(M_p)}$ is nuclear, by Corollary 2.3 and Remark 2.4. Comparing with [11] it is then interesting the following

Corollary 3.8. Condition (M2) is not necessary for the nuclearity of $S_{(M_p)}$.

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