On the strong approximations of partial sums of $f(n_k x)$

Marko Raseta

Abstract

We prove a strong invariance principle for the sums $\sum_{k=1}^N f(n_k x)$, where $f$ is a smooth periodic function on $\mathbb{R}$ and $(n_k)_{k \geq 1}$ is an increasing random sequence. Our results show that in contrast to the classical Salem-Zygmund theory, the asymptotic properties of lacunary series with random gaps can be described very precisely without any assumption on the size of the gaps.

1 Introduction

Let $f: \mathbb{R} \to \mathbb{R}$ be a measurable function satisfying

$$f(x+1) = f(x), \quad \int_0^1 f(x)dx = 0, \quad \|f\|^2 = \int_0^1 f^2(x)dx < \infty. \quad (1.1)$$

It is well known that for rapidly increasing $(n_k)_{k \geq 1}$ the sequence $(f(n_k x))_{k \geq 1}$ behaves like a sequence of independent random variables. For example, if

$$n_{k+1}/n_k \to \infty \quad (1.2)$$

and $f$ is a Lipschitz function, then

$$N^{-1/2} \sum_{k=1}^N f(n_k x) \xrightarrow{d} N(0, \|f\|^2) \quad (1.3)$$

and

$$\limsup_{N \to \infty} (2N \log \log N)^{-1/2} \sum_{k=1}^N f(n_k x) = \|f\| \quad \text{a.s.} \quad (1.4)$$

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1) Department of Mathematics, University of York Email: marko.raseta@york.ac.uk
with respect to the probability space \((0, 1)\) equipped with Borel sets and Lebesgue measure (see Takahashi [22], [23]). Assuming only the Hadamard gap condition
\[
n_{k+1}/n_k \geq q > 1, \quad k = 1, 2, \ldots
\] (1.5)
the situation becomes more complicated. Kac [14] proved that \(f(n_k x)\) satisfies the CLT for \(n_k = 2^k\) and Erdős and Fortet (see also [14]) showed that this generally fails for \(n_k = 2^k - 1\). Gapsopshkin [12] showed that \(f(n_k x)\) satisfies the CLT provided the ratios \(n_{k+1}/n_k\) are integers or \(n_{k+1}/n_k \to \alpha > 1\) where \(\alpha\) is irrational for \(r = 1, 2, \ldots\).

A necessary and sufficient number-theoretic condition for the CLT for \(f(n_k x)\) under (1.5) was given by Aistleitner and Berkes [1].

For sequences \((n_k)_{k \geq 1}\) growing slower than exponentially, the asymptotic behavior of \(S_N = \sum_{k=1}^{N} f(n_k x)\) still depends on arithmetic properties of \(n_k\), but the arising number theoretical problems become essentially intractable. As a consequence, the asymptotic distribution (if it exists) of normed sums of \(f(n_k x)\) is not known even for \(f(x) = \sin x\) and simple sequences like \(n_k = k^r\) \((r = 2, 3, \ldots)\). On the other hand, as it happens frequently in analysis, the "typical" behavior of the sum \(S_N\) is much more manageable than individual cases. For example, it is an open question if \(\sin n_k x\) satisfies the central limit theorem for \(n_k = e^{ck\alpha}\) and all \(\alpha > 0\) (for more information, see Erdős [8]); on the other hand, Kaufman [15] showed that the CLT holds for \(n_k = e^{ck\alpha}\) for all \(\alpha > 0\) and almost all \(c > 0\) in the sense of Lebesgue measure. Random constructions have also been used to solve many problems in harmonic analysis, see e.g. Salem and Zygmund [17], Erdős [7], Halberstam and Roth [13], Berkes [3], [4], Bobkov and Götze [6], Fukuyama [9], [10], [11], Aistleitner and Fukuyama [2].

The purpose of this paper is to prove a strong invariance principle for \(\sum_{k=1}^{N} f(n_k x)\) in the case when \((n_k)_{k \geq 1}\) is an increasing random walk, i.e. \(n_{k+1} - n_k\) are i.i.d. positive random variables. This model was introduced by Schatte [19] and it has remarkable properties, see Schatte [19], [20], Weber [24], Berkes and Weber [5]. In this paper we will prove the following result.

**Theorem.** Let \((X_n)_{n \in \mathbb{N}}\) be a sequence of i.i.d. positive random variables defined on a sufficiently large probability space \((\Omega, \mathcal{F}, P)\) and let \(S_n = \sum_{k=1}^{n} X_k\). Assume \(X_1\) is bounded with bounded density. Let \(f\) be a Lip \((\alpha)\) function satisfying (1.1) and put
\[
A_x = \|f\|^2 + 2 \sum_{k=1}^{\infty} \mathbb{E} f(U) f(U + S_k x),
\] (1.6)
where \(U\) is a uniform \((0, 1)\) random variable, independent of \((X_j)_{j \geq 1}\). Then there exists a Brownian motion such that
\[
\sum_{k=1}^{n} f(S_k x) = W(A_x n) + O \left(n^{5/12 + \varepsilon} \log n\right) \quad \text{a.s.}
\]
for all \(\varepsilon > 0\).
Clearly, the existence of such an \( U \) can always be guaranteed by an enlargement of the probability space. The absolute convergence of the series in (1.6) will follow from the proof of our theorem.

Changing the notation slightly, our theorem and Fubini’s theorem imply immediately the LIL

\[
\limsup_{N \to \infty} \left(2N \log \log N\right)^{-1/2} \sum_{k=1}^{N} f(n_k x) = A_1^{1/2} \quad \text{a.s.} \quad (1.7)
\]

and its functional version for almost all sequences \( (n_k) \) generated by the random walk model. Note that, in contrast to the nonrandom case (1.4), the limsup in (1.7) is a function of \( x \). Other immediate consequences are Chung’s lower LIL

\[
\liminf_{N \to \infty} \left(\frac{\log \log N}{N}\right)^{1/2} \max_{1 \leq M \leq N} \left| \sum_{k=1}^{M} f(n_k x) \right| = \frac{8}{\pi^2} A_1^{1/2} \quad \text{a.s.} \quad (1.8)
\]

and the Kolmogorov-Erdős-Feller-Petrovski test stating that for any positive nondecreasing function \( \varphi \) on \((0, \infty)\) and almost all \( x \) the inequality

\[
\sum_{k=1}^{M} f(n_k x) > \sqrt{N} \varphi(N)
\]

holds for finitely or infinitely many \( N \) according as the integral

\[
\int_{1}^{\infty} \frac{\varphi(t)}{t} e^{-\varphi(t)^2/2} dt
\]

converges or diverges.

2 Proof of the theorem

We start with some preparatory results. All sums here are considered modulo 1.

**Lemma 1** (Schatte, [19]). Let the three random vectors \( X = (X_1, \ldots, X_r) \), \( U \), and \((W_1, \ldots, W_s)\) be independent and let \( W = f(X) \) with a measurable function \( f \). If \( U \) is uniformly distributed, then the two random vectors \( X \) and \((W + U + W_1, W + U + W_2, \ldots, W + U + W_s)\) are independent, too.

**Lemma 2** (Schatte, [19]). Let \( W \) and \( U \) be independent random variables, where \( U \) is uniformly distributed. Then \( W + U \) is independent of \( W \).

**Lemma 3** (Schatte, [19]). Let \( X \) be a random variable with the distribution function \( F(x) \), where \( \sup_{0 \leq x < 1} |F(x) - x| \leq \varepsilon \). Further let \( U \) be a uniformly distributed random variable being independent of \( X \). Then there exists a uniformly distributed random variable \( V \) such that
(i) \(|V - X| \leq \varepsilon\)
(ii) \(V = f(U, X)\), where \(f\) is measurable.

**Theorem 2.1** (Schatte, [18]). Let \(p_n(x)\) denote the density of \(Y_n = \sum_{i=1}^{n} X_i\), where \(X_i\'s\) are i.i.d. random variables. Then the following assertions are equivalent;

a) The density \(p_m(x)\) is bounded for some \(m\).

b) \(\sup_{0 \leq x < 1} |p_n(x) - 1| \to 0\) as \(n \to \infty\).

c) \(\sup_{0 \leq x < 1} |p_n(x) - 1| \leq Cw^n\), where \(C\) and \(w < 1\) are real constants.

Last four results and simple induction yield the following:

**Lemma 4.** Fix \(l \in \mathbb{N}\), \(x \neq 0\) and define a sequence of sets by

\[
I_1 := \{1, 2, \ldots, \beta\}
\]

\[
I_2 := \{p_1, p_1 + 1, \ldots, p_1 + \beta_1\} \text{ where } p_1 \geq \beta + l + 2
\]

\[
I_3 := \{p_2, p_2 + 1, \ldots, p_2 + \beta_2\} \text{ where } p_2 \geq \beta_1 + l + 2
\]

\[
\vdots
\]

\[
I_n := \{p_{n-1}, p_{n-1} + 1, \ldots, p_{n-1} + \beta_{n-1}\} \text{ where } p_{n-1} \geq p_{n-2} + \beta_{n-2} + l + 2
\]

\[
\vdots
\]

Then there exists a sequence \(\delta^x_1, \delta^x_2, \ldots\) of random variables satisfying the following properties:

(i) \(|\delta^x_n| \leq Cxe^{-\lambda x l}\) for all \(n \in \mathbb{N}\), where \(\lambda_x\) and \(C_x\) are some positive quantities that depend on \(x\) only.

(ii) The random variables

\[
\sum_{i \in I_1} f(S_i x), \sum_{i \in I_2} f(S_i x - \delta^x_1), \ldots, \sum_{i \in I_n} f(S_i x - \delta^x_{n-1}), \ldots
\]

are independent.

In an identical fashion to Lemma 4 we can prove the following result.

**Lemma 5.** Let \(p, q, c, s\) be 4 natural numbers listed in an increasing order. Then there exists a random variable \(T\) with \(T = O(e^{-\lambda(q-p)})\) for some constant \(\lambda\) such that

(i) \((S_q - T, S_r - T, S_s - T)\) is a random vector with all three components having uniform distribution.

(ii) Random vector in (i) is independent of \(S_p\).
Corollary 2.1. Fix $x \in \mathbb{R}$. Define a sequence of random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ by

$$Y^x_k = f(S^x_k) - \mathbb{E}F(S^x_k).$$

Then for any sequence $(a_k)_{k \in \mathbb{N}}$ we have

$$\mathbb{E} \left( \sum_{k=1}^{n} a_k Y^x_k \right)^4 \leq C_x \left( \sum_{k=1}^{n} a_k^2 \right)^2$$

for some constant $C_x$ depending on $x$ only.

**Proof.** As a direct consequence of Lemma [5] we can find a random variable $T^x_x$ such that

$$(S^x_q - T^x_q, S^x_r - T^x_r, S^x_s - T^x_s) \perp \perp S^x_p.$$  

For simplicity define

$$Z^x_{q,r,s} := (f(S^x_q - T^x_q) - \mathbb{E}f(S^x_q))(f(S^x_r - T^x_r) - \mathbb{E}f(S^x_r))(f(S^x_s - T^x_s) - \mathbb{E}f(S^x_s)).$$

Then it is clear that $Y^x_p$ is independent of $Z^x_{q,r,s}$ and that they are both zero mean random variables. Using Lemma [5] one can see that

$$|\mathbb{E}Y^x_p Y^x_q Y^x_r Y^x_s| \leq C_x e^{-D_x(q+p-s-r)}.$$

But then it follows that

$$\mathbb{E} \left( \sum_{k=1}^{n} a_k Y^x_k \right) \leq C_x \sum_{1 \leq p \leq q \leq r \leq s \leq n} |a_p| |a_q| |a_r| |a_s| e^{-D_x(q-p+s-r)}.$$

Now let $q - p = \omega$, $s - r = \zeta$. We have

$$\sum_{1 \leq p \leq q \leq r \leq s \leq n} |a_p| |a_q| |a_r| |a_s| e^{-(q-p+s-r)} =$$

$$\sum_{1 \leq p \leq \omega, r, \omega, r+\zeta \leq \omega} |a_p| |a_{p+\omega}| |a_r| |a_{r+\zeta}| e^{-\omega+\zeta} \leq$$

$$e^{-\omega+\zeta} \sum_{1 \leq p \leq r \leq \omega \leq n} |a_p| |a_{p+\omega}| a_r |a_{r+\zeta}| \leq$$

$$e^{-\omega+\zeta} \left( \sum_{1 \leq p \leq n} a_p^2 \right)^{1/2} \left( \sum_{1 \leq p \leq \omega \leq n} a_{p+\omega}^2 \right)^{1/2} \left( \sum_{1 \leq r \leq n} a_r^2 \right)^{1/2} \left( \sum_{1 \leq r+\zeta \leq n} a_{r+\zeta}^2 \right)^{1/2} \leq$$

$$e^{-\omega+\zeta} \left( \sum_{1 \leq k \leq n} a_k^2 \right)^{1/2},$$

and the proof is now complete.
Put $\tilde{m}_k = \sum_{j=1}^{k} \lfloor j^{1/2} \rfloor$, $\hat{m}_k = \sum_{j=1}^{k} \lfloor j^{1/4} \rfloor$ and let $m_k = \tilde{m}_k + \hat{m}_k$. Using Lemma 4 we can construct sequences $(\Delta_k^x)_{k \in \mathbb{N}}$, $(\Pi_k^x)_{k \in \mathbb{N}}$ of random variables such that setting

$$T_k := \sum_{j=m_k}^{m_{k-1}+\lfloor \sqrt{k} \rfloor} \left( f(S_j x - \Delta_{k-1}^x) - \mathbb{E} f(S_j x - \Delta_{k-1}) \right)$$

$$T_k^* := \sum_{j=m_k}^{m_{k-1}+\lfloor \sqrt{k} \rfloor} \left( f(S_j x - \Pi_{k-1}^x) - \mathbb{E} f(S_j x - \Pi_{k-1}) \right)$$

we have

(i) $\Delta_0^x = 0$; $|\Delta_k^x| \leq C_x e^{-\lambda_x \sqrt{k}}$, $(T_k)_{k \in \mathbb{N}}$ is a sequence of independent random variables,

(ii) $\Pi_0^x = 0$; $|\Pi_k^x| \leq C_x e^{-\lambda_x \sqrt{k}}$, $(T_k^*)_{k \in \mathbb{N}}$ is a sequence of independent random variables.

Routine stationary-type arguments yield the following:

**Lemma 6.**

$$\sum_{k=1}^{n} \text{Var}(T_k) \sim A_x \tilde{m}_n, \quad \sum_{k=1}^{n} \text{Var}(T_k^*) \sim A_x \hat{m}_n.$$ 

where $A_x$ is defined by (1.4).

The following lemma is a special case of Strassen’s strong approximation theorem.

**Lemma 7.** Let $Y_1, Y_2, \ldots$ be independent r.v.’s with finite fourth moments, let $a_n = \sum_{i=1}^{n} \mathbb{E} Y_i^2$ and assume

$$\sum_{n=1}^{\infty} \mathbb{E} Y_n^4 / a_n^{2\vartheta} < \infty$$

with $0 < \vartheta < 1$. Then the sequence $Y_1, Y_2, \ldots$ can be redefined on a new probability space together with a Wiener process $\zeta(t)$ such that

$$Y_1 + \cdots + Y_n = \zeta(a_n) + o \left( a_n^{(1+\vartheta)/4} \log a_n \right) \quad \text{a.s.}$$

We are now ready to prove our result.

As before, let

$$T_k := \sum_{j=m_k}^{m_{k-1}+\lfloor \sqrt{k} \rfloor} f(S_j x - \Delta_{k-1}^x) \quad \text{and} \quad T_k^* := \sum_{j=m_k}^{m_{k-1}+\lfloor \sqrt{k} \rfloor} f(S_j x - \Pi_{k-1}^x).$$

We will apply Lemma 7 for both $(T_k)_{k \in \mathbb{N}}$ and $(T_k^*)_{k \in \mathbb{N}}$. Clearly, $(T_k)_{k \in \mathbb{N}}$ is a sequence of independent, zero-mean random variables and $|T_k| \leq M \sqrt{k}$. Corollary 2.1 gives as:

$$\mathbb{E} \left( \sum_{k=1}^{N} c_k Y_k^4 \right) \leq C_x \cdot \left( \sum_{k=1}^{N} c_k^2 \right)^2$$

We are now ready to prove our result.
where $Y_k^x = f(S_k x) - Ef(S_k x)$. But then
\[ \mathbb{E}T_k^4 \leq C_1^x k + C_2^x k^2 e^{-\alpha k} \sqrt{k-1} \]
for some suitable constants $C_1^x$ and $C_2^x$. Now observe that
\[ \sum_{k=1}^n \mathbb{E}Y_k^2 \sim A_x m_n \sim A_x \cdot \frac{2}{3} n^{3/2} \]
by Lemma 6. These facts together imply that
\[ \sum_{k \in \mathbb{N}} \mathbb{E}T_k^4 \frac{m_k}{m^{2(2/3+\epsilon)}} < \infty \]
for all $\epsilon > 0$ and thus by Lemma 7 we get
\[ \sum_{k=1}^n T_k = \zeta(A_x m_n) + o \left( m_n^{(1+\frac{2}{3}+\epsilon)/4} \log m_n \right) \text{ a.s.} \]
Define a sequence $(p(n))_{n \in \mathbb{N}}$ of integers by
\[ m_{p(n)} \leq n < m_{p(n)+1}. \]
Clearly,
\[ \sum_{k=1}^{p(n)} T_k = \zeta \left( A_x m_{p(n)} \right) + o \left( m_{p(n)}^{(1+\frac{2}{3}+\epsilon)/4} \log m_{p(n)} \right); \text{ a.s.} \]
and similarly
\[ \sum_{k=1}^{p(n)} T_k^* = \zeta' \left( A_x \hat{m}_{p(n)} \right) + o \left( \hat{m}_{p(n)}^{(1+\frac{2}{3}+\delta)/4} \log m_{p(n)} \right); \text{ a.s.} \]
for some other Brownian motion $\zeta'$. Now
\[ \sum_{k=1}^n f(S_k x) = \sum_{k=1}^{p(n)} T_k + \sum_{k=1}^{p(n)} T_k^* + \sum_{k=1}^{p(n)} \sum_{j=m_{k-1}+1}^{m_k} \left( f(S_j x) - f(S_j x - \Delta_{k-1} x) \right) \]
\[ + \sum_{k=1}^{p(n)} \sum_{j=m_{k-1}+[\sqrt{k}]+1}^{m_k} \left( f(S_j x) - f(S_j x - \Pi_{k-1} x) \right) + \sum_{k=m_{p(n)}+1}^n f(S_k x). \]
Thus
\[ \sum_{k=1}^n f(S_k x) - \zeta(A_x n) = \sum_{k=1}^{p(n)} T_k - \zeta \left( A_x m_{p(n)} \right) + \zeta \left( A_x m_{p(n)} \right) - \zeta(A_x n) \]
\[ + \sum_{k=1}^{p(n)} T_k^* - \zeta' \left( A_x \hat{m}_{p(n)} \right) + \zeta' \left( A_x \hat{m}_{p(n)} \right). \]
We investigate each term separately.

(i) We have seen that
\[ \sum_{k=1}^{p(n)} T_k - \zeta(A_x m_{p(n)}) = o \left( m_{p(n)}^{(1+2/3+\varepsilon)/4} \log m_{p(n)} \right). \]

But \( m_{p(n)} \sim n \) and thus
\[ \sum_{k=1}^{p(n)} T_k - \zeta(A_x m_{p(n)}) = o \left( n^{5/12+\varepsilon} \log n \right) = o \left( n^{5/12+\varepsilon} \log n \right). \]

(ii) We have
\[ |\zeta(A_x m_{p(n)}) - \zeta(A_x n)| = |\zeta(A_x n) - \zeta(A_x m_{p(n)})| \]
\[ \overset{d}{=} \left| N(0, A_x (n - m_{p(n)})) \right| \]

Now
\[ n - m_{p(n)} \leq [(p(n) + 1)^{1/2}] + [(p(n) + 1)^{1/4}] \leq C n^{1/3} \]
for all \( n \) large enough. Thus
\[ \sum_{n \in \mathbb{N}} \mathbb{P} \left( \left| \zeta(A_x m_{p(n)}) - \zeta(A_x n) \right| \geq n^{7/24} \right) < \infty \]
and consequently
\[ \left| \zeta(A_x m_{p(n)}) - \zeta(A_x n) \right| \leq n^{7/24} \quad a.s. \]
for all \( n \) large enough. Hence
\[ \frac{1}{n^{5/12+\varepsilon} \log n} \left| \zeta(A_x m_{p(n)}) - \zeta(A_x n) \right| \to 0 \quad a.s. \]

(iii) We have also seen that
\[ \sum_{k=1}^{p(n)} \hat{T}_k - \zeta'(A_x \hat{m}_{p(n)}) = o \left( \hat{m}_{p(n)}^{(1+3/5+\delta)/4} \log \hat{m}_{p(n)} \right). \]

But \( \hat{m}_n \sim C n^{5/4} \) and thus \( \hat{m}_{p(n)} \sim C n^{5/6} \). Hence for sufficiently small \( \delta \) we have
\[ \frac{1}{n^{5/12+\varepsilon} \log n} \left| \sum_{k=1}^{p(n)} \hat{T}_k - \zeta'(A_x \hat{m}_{p(n)}) \right| \leq C n^{8.5+\delta} \frac{8+5\delta}{24} \to 0 \quad a.s. \]
as $n \to \infty$.

(iv) Clearly $\zeta'(A_x \hat{m}_p(n)) = \hat{\zeta}'(A_x \hat{m}_p(n)) \sim n^{1/2}$. Observe that $\hat{m}_n \sim Cn^{5/4}$, $p(n) \sim Cn^{2/3}$ and thus $\hat{m}_p(n) \sim Cn^{5/6}$. Consequently,

$$|\zeta'(A_x \hat{m}_p(n))| \leq \frac{n^{5/12 + \varepsilon/2}}{n^{5/12 + \varepsilon} \log n} \to 0 \quad \text{a.s.}$$

as $n \to \infty$.

(v) We have

$$\left| \sum_{k=1}^{p(n) m_{k-1} + [\sqrt{k}]} \sum_{j=m_{k-1}+1}^{p(n) m_{k-1} + [\sqrt{k}]} (f(S_j x) - f(S_j x - \Delta_{k-1}^{-} x)) \right| \leq \sum_{k=1}^{p(n) m_{k-1} + [\sqrt{k}]} \sum_{j=m_{k-1}+1}^{p(n) m_{k-1} + [\sqrt{k}]} |f(S_j x) - f(S_j x - \Delta_{k-1}^{-} x)|$$

$$\leq \sum_{k=1}^{p(n) m_{k-1} + [\sqrt{k}]} \sum_{j=m_{k-1}+1}^{p(n) m_{k-1} + [\sqrt{k}]} C_x^{\alpha} e^{-\alpha \lambda x_{\sqrt{k} - 1}} = \sum_{k=1}^{p(n)} C_x^{\alpha} [\sqrt{k}] e^{-\alpha \lambda x_{\sqrt{k} - 1}}$$

Thus

$$\frac{1}{n^{5/12 + \varepsilon} \log n} \sum_{k=1}^{p(n) m_{k-1} + [\sqrt{k}]} \sum_{j=m_{k-1}+1}^{p(n) m_{k-1} + [\sqrt{k}]} (f(S_j x) - f(S_j x - \Delta_{k-1}^{-} x)) \to 0 \quad \text{a.s.}$$

as $n \to \infty$.

(vi) Arguing exactly as in (v), we get

$$\frac{1}{n^{5/12 + \varepsilon} \log n} \sum_{k=1}^{P(n)} \sum_{j=m_{k-1}+1}^{m_k} (f(S_j x) - f(S_j x - \Pi_{k-1}^{-} x)) \to 0 \quad \text{a.s.}$$

(vii) We have

$$\left| \sum_{k=m_{p(n)}+1}^{n} f(S_k x) \right| \leq \sum_{k=m_{p(n)}+1}^{n} |f(S_k x)| \leq M(n - m_{p(n)})$$

$$\leq M\left(\left\lfloor (p(n) + 1)^{1/2} \right\rfloor + \left\lfloor (p(n) + 1)^{1/4} \right\rfloor \right) \leq 2M \left( (p(n) + 1)^{1/2} \right) \sim 2Mn^{1/3}.$$ 

Thus

$$\frac{1}{n^{5/12 + \varepsilon} \log n} \sum_{k=m_{p(n)}+1}^{n} f(S_k x) \to 0 \quad \text{a.s.}$$

as $n \to \infty$. Summarizing the above estimates, we obtain our result.
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