Boundary-bulk relation for topological orders as the functor mapping higher categories to their centers

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Abstract

In this paper, we study the relation between topological orders and their gapped boundaries. We propose that the bulk for a given gapped boundary theory is unique. It is actually a consequence of a microscopic definition of a (potentially anomalous) topological order on an open disk, which is referred to as a local topological order in this work. Using this uniqueness, we show that the notion of “bulk” is equivalent to the notion of center in mathematics. We achieve this by first introducing the notion of a morphism between two local topological orders of the same dimension, then proving that the bulk satisfying the same universal property as that of the center in mathematics. We also propose a macroscopic definition of a local topological order as a unitary $n$-category with a unit, and explain that the notion of a morphism between two topological orders is compatible with that of a unitary $n$-functor preserving the unit in a few low dimensional cases. In the end, we explain that above boundary-bulk relation is only the first layer of a hierarchical structure which can be summarized by the functoriality of the bulk or center. This functoriality also provides the physical interpretation of some well-known mathematical results in fusion 1-categories. This work can also be viewed as the first step towards a systematic study of the category of local topological orders, and the boundary-bulk relation actually provides a useful tool for this study.

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1 Introduction

In this work, we study the boundary-bulk relation of topological orders in any dimensions via higher categories. We avoid a lengthy introduction to topological orders [W89]. Instead, we direct readers to many online resources for this vast and fast-growing topics (see for example a long list of physics references in [KW]). Unless we specified otherwise, throughout this work, by an nD topological order we mean a (potentially anomalous [KW]) topological order defined on the open (n − 1)-disk $D^{n-1}$ as the space manifold, or on the space-time manifold $D^{n-1} \times \mathbb{R}$. We also refer to such a topological order as an nD local topological order\footnote{It can be viewed as a TQFT restricted to an open disk in the sense of Morrison and Walker [MW] [Wa].} (see Def. 2.1 and Remark 2.12). It can always be realized as a gapped boundary of a gapped local Hamiltonian system in one higher dimension [KW]. It is anomaly-free if it is realizable by a gapped local Hamiltonian system in the same dimension.
There are at least two types of problems to study in the field of topological orders. The first type is to construct and classify all topological orders. The second type is to study relations among all topological orders, such as phase transitions. To study all these relations among all topological orders as a whole amounts to study the category $\mathcal{T}O$ of topological orders. The topological orders of the same space-time dimension $n$, or $n$D topological orders, form an obvious subcategory of $\mathcal{T}O$, denoted by $\mathcal{T}O_n$. There are many natural but in-equivalent definitions of $\mathcal{T}O_n$. In this work, three different and in-equivalent definitions $\mathcal{T}O_n^{\text{closed-wall}}$, $\mathcal{T}O_n^{\text{fun}}$ and $\mathcal{T}O_n^{\text{wall}}$ appear.

One type of relation that is important in physics is the relation between the boundary physics and the bulk physics. Mathematically, the boundary-bulk relation for topological orders with gapped boundaries is a relation between $\mathcal{T}O_n$ and $\mathcal{T}O_{n+1}$ (in some sense a functor $\mathcal{T}O_n \to \mathcal{T}O_{n+1}$ see Sec.5). All three definitions of $\mathcal{T}O_n$ play interesting roles in this relation. This work can also be viewed as the first step towards a systematic study of the category $\mathcal{T}O_n$, and the boundary-bulk relation actually provides a useful tool for this study.

The topological excitations on gapped boundaries were first studied in the 2+1D toric code model ([K1]) by Bravyi and Kitaev in [BK]. It was latter generalized to Levin-Wen models ([LeW]) with gapped boundaries in [KK], where the topological excitations on a boundary of such a lattice model were shown to form a unitary fusion 1-category $\mathcal{C}$, and those in the bulk form a unitary braided fusion 1-category which is given by the monoidal center $Z(\mathcal{C})$ of $\mathcal{C}$ (see also [LaW]). But this work does not address the question if there are any other possible bulks for the same boundary theory. In [FSV1], it was shown model-independently that if there is a 2+1D bulk theory $\mathcal{D}$ with a gapped boundary theory given by a unitary fusion 1-category $\mathcal{C}$, then the bulk excitations can move to the boundary and become boundary excitations. This move defines a so-called bulk-to-boundary map $L : \mathcal{D} \to \mathcal{C}$, and was shown to be a central monoidal functor [FSV1]. As a consequence, $L$ factors through the monoidal center $Z(\mathcal{C})$ of $\mathcal{C}$. Namely, there is a unique braided monoidal functor $\tilde{L} : \mathcal{D} \to Z(\mathcal{C})$ such that its composition with the forgetful functor $Z(\mathcal{C}) \to \mathcal{C}$ is $L$. This result says that among all possible bulks associated to the same boundary theory $\mathcal{C}$, $Z(\mathcal{C})$ is the universal one (a terminal object). One way to complete the proof that the bulk must be the center $Z(\mathcal{C})$ is to view the gapped boundary as a consequence of anyon condensation [BS] of $\mathcal{D}$ to the trivial phase. This idea leads to the works [KS1, FSV1, L, BJQ], in which it is shown in abelian topological theories that all boundaries of a given bulk is classified by Lagrangian subgroups associated to the bulk theory. For non-abelian cases, we need a general anyon condensation theory developed in [Ko2], in which it was shown that such a condensation is determined by a Lagrangian algebra $A$ in $\mathcal{D}$, and $\mathcal{C}$ is monoidally equivalent to the category $\mathcal{D}_A$ of $A$-modules in $\mathcal{D}$. Moreover, we have $Z(\mathcal{D}_A) \simeq \mathcal{D}$. This completes the proof of a part of the bulk-boundary relation in 3D, which says that the 3D bulk theory $\mathcal{D}$ of a given 2D boundary $\mathcal{C}$ is unique and given by the monoidal center of $\mathcal{C}$.

Does this bulk-boundary relation hold in higher dimensions? The gapped boundary of a non-trivial topologically order should be viewed as an anomalous topological order. Following [CGW], a microscopic definition of anomalous topological order was proposed in

\footnote{In [Ko2], the bootstrap analysis shows that $\mathcal{C}$ must be a sub-fusion-category of $\mathcal{D}_A$, and the boundary-bulk relation was used as a supporting evidence for $\mathcal{C} = \mathcal{D}_A$. But it is just physically natural that all possible quasi-particles that can be confined on the boundary, i.e. objects in $\mathcal{D}_A$, should all survive on the boundary after the condensation. In other words, $\mathcal{C} = \mathcal{D}_A$ is a natural physical requirement.}
It follows immediately from this definition that the bulk of a given gapped boundary must be unique \[\text{[KW, Sec.VII.C]}\]. But this result is highly non-trivial from a macroscopic point of view. Macroscopically, the physical detectable data of a local topological order is given by the fusion-braiding (and spins) properties of its topological excitations. These properties provide a macroscopic definition of a topological order via higher categories because higher categories can encode the fusion-braiding properties of excitations in an efficient way \[\text{[KW]}\]. Whether the microscopic definition is equivalent to the macroscopic definition is a highly non-trivial and important open problem. For this reason, we would like to refer to the uniqueness of the bulk for a given gapped boundary as the \textit{unique-bulk hypothesis}, and the unique bulk is denoted by \(\text{bulk} \). Main goal of this work is to prove, under the unique-bulk hypothesis, that the \(\text{bulk} \) of a given gapped boundary is given by the center of the boundary theory in a mathematical sense.

The main idea of our proof is to introduce the notion of a morphism between two topological orders (see Def. \ref{def:morphism}). This notion is defined physically and independent of any categorical definition of a topological order, but can be viewed as a special physical realization of a unitary \(n\)-functor (see Def. \ref{def:unitary_n-functor}) preserving an additional structure called the unit (see Def. \ref{def:unit}). All \(n\)D topological orders together with such morphisms defines the category \(\mathcal{T}_\mathcal{O}^{\text{fun}} \). Using such morphisms, we are able to show in Sec. \ref{sec:universal_property} that the \(\text{bulk} \) satisfies the same universal property as that of the center (of an algebra) in mathematics. By assuming that the new morphism coincides with usual notions of morphisms in mathematics, we obtain that the \(\text{bulk} \) coincides with the usual notion of center (see Sec. \ref{sec:compatibility}). Conversely, by assuming \(\text{bulk} = \text{center} \), we also show in a few low dimensional cases that our new notion of a morphism is compatible with usual notions of morphisms in mathematics (see Sec. \ref{sec:low_dimensional_cases}).

Actually, \(\text{bulk} = \text{center} \) is only the first layer of the hierarchical structure of a rather complete boundary-bulk relation (proposed earlier in Levin-Wen models \[\text{[Ko1]}\]). In Sec. \ref{sec:stronger_hypothesis} we propose a stronger version of the \textit{unique-bulk hypothesis} (see Fig. \ref{fig:hierarchical_structure}), which allow us to define the Morita/Witt equivalence of topological orders and closed/anomalous domain walls. It also provides a natural explanation of the so-called duality-defect correspondence as a part of the second layer of the complete boundary-bulk relation. We explain how the hierarchical structure of this relation can be summarized in terms of the functoriality of the \(\text{bulk} \) or center.

\textbf{Remark 1.1.} Similar boundary-bulk relation was first discovered in 2D rational conformal field theories, where it was also called open-closed duality. In particular, the uniqueness of the bulk was first proved in \[\text{[FrFrs]} \) (in the FRS framework) and in \[\text{[KR2]} \) (in a modified Segal’s framework), duality-defect correspondence in \[\text{[FrFrs]} \text{DKR1} \) and the functoriality of the bulk or center in \[\text{[DKR2]} \text{DKR3} \).

We give some remarks in Sec. \ref{sec:condensation} on what our results suggest to a possible condensation theory in higher dimensions. The main results of this work are physical and not mathematically rigorous (except Prop. \ref{prop:unitary_n-category} and those in Sec. \ref{sec:mathematical_definitions}). In Sec. \ref{sec:mathematical_definitions} and Sec. \ref{sec:universal_property} however, we would like to borrow the mathematical terminologies of lemma, proposition and theorem to highlight our physics results, which are often supported and illustrated by mathematically rigorous results in Examples. To avoid too many mathematical complexities in the main text, we move all mathematical definitions and results to Appendix and keep all mathematical proofs brief. In Sec. \ref{sec:unitary_n-category} we define a unitary \(n\)-category and a unitary \(n\)-functor.
mathematically; in Sec. A.2, we discuss the universal property of the *bulk* with higher mor-
phisms; in Sec. A.3, we introduce the notion of a weak morphism between topological orders;
in Sec. A.4, we outline the proofs of a few new mathematical results of fusion categories.
More systematic treatment of the mathematical results will appear elsewhere [AKZ].

**Remark 1.2.** The fusion-braiding properties encoded in an higher category can only de-
scribe topological excitations living in an open disk, thus define only a local topological
order. To obtain global topological orders on closed manifo lds, we need glue local topolog-
ical orders via topological chiral homology [L2, AFT2]. We will do that in [AKZ].

**Remark 1.3.** For physics oriented readers, it is possible to get the main idea of this
paper very quickly by ignoring all examples, which are long and mathematical. Nearly zero
background knowledge (except some obvious physical intuition) is needed to understand
the main text of this paper. A recommended route for the first-time reading is Sec. 2.1, 2.3,
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### 2 The category $\mathcal{T}_n^{\text{closed–wall}}$ of local topological orders

In this section, we discuss the definitions of a local topological order and some basic struc-
tures of the category $\mathcal{T}_n^{\text{closed–wall}}$. 
Figure 2: Consider a bosonic Hamiltonian lattice model defined on the closed $n$-disk $\hat{D}^n$ and an open $(n-1)$-disk $L$ in $\partial \hat{D}^n = S^{n-1}$. The disk $L$ is depicted as an open line segment between $a$ and $b$ on $\partial \hat{D}^{n+1}$, where $a \cup b = \partial L = S^{n-2}$. The lattice model in a neighborhood of $L$ determines a local topological order on the open $(n-1)$-disk $L$ (see Def. 2.1).

2.1 A microscopic definition of a local topological order

We only briefly discuss a physical definition of an $n$D topological order on an open $(n-1)$-disk from a microscopic point of view. The focus of this work is a macroscopic one.

A (potentially anomalous) topologically order can always be realized as a gapped defect in a higher dimensional bosonic Hamiltonian lattice model defined on an open disk in space with only short range interactions. Certain gapped properties need to be satisfied when we take the large-size limit [ZW]. Using the dimensional reduction depicted in Fig. 1, we can see that any topological order can always be realized as a gapped boundary of a one-dimensional higher bosonic Hamiltonian lattice model with only short range interactions. A topological order is an equivalence class of such lattice realizations.

Let $L$ be an open $(n-1)$-disk on the boundary (a sphere) of a closed $n$-disk $\hat{D}^n$ (see Fig. 2). In the following, we define the notion of an $n$D (potentially anomalous) local topological order on the space manifold $L$ (or equivalently, on the $n$-dimensional space-time manifold $L \times \mathbb{R}$) as an equivalence class of Hamiltonian lattice models defined on $\hat{D}^n$, where $n$ is the space-time dimension.

**Definition 2.1.** Consider two bosonic Hamiltonian lattice models $H$ and $H'$ with short ranged interactions defined on a closed $n$-disk $\hat{D}^n$ with an $(n-1)$-sphere boundary depicted in Fig. 2 such that both models have liquid gapped ground states ([ZW]) and the boundary is gapped. We say that these two lattice models realize the same $n$D topological order on the open $(n-1)$-disk $L$ if for any neighborhood $U$ of the boundary $L$ with large enough but finite thickness (see Fig. 2) such that the restriction of $H$ in $U$, denoted by $H|_U$, can be deformed smoothly to $H'|_U$ without closing the gap. In other words, there is a smooth family $H_t$ for $t \in [0,1]$ without closing the gap such that $H_0 = H$, and $H_{r}$ and $H_{s}$ differ only in $U$ for $s,t \in [0,1]$ and $H_1|_U = H'|_U$.

We denote $n$D topological orders by $A_n, B_n, C_n, D_n$, etc.. If the space-time dimension is clear from the context or irrelevant to the discussion, we sometimes abbreviate $C_n$ as $C$. 

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Definition 2.2. An $n$D local topological order (on an open $(n-1)$-disk $L$) is called \textit{closed} or \textit{anomaly-free} if it can be realized by a bosonic Hamiltonian lattice model defined on $S^{n-1}$ (Fig. 2 with an empty bulk). If it is not closed, it is called \textit{anomalous}.

The topological orders defined in [CGW, ZW] are closed in our sense. In physics, topological orders are often defined on closed manifolds. One can easily generalize Def. 2.1 to topological orders on closed manifolds. For example, if we replace $L$ by the entire boundary $S^{n-1} = \partial \hat{D}^n$ in Def. 2.1 (see Fig. 2), we obtain a microscopic definition of a topological order on $S^{n-1}$. We don’t need them in this work. But to study the category of topological orders, one do need generalize Def. 2.1 to a microscopic definition of multiple local topological orders connected by gapped domain walls and walls between walls, etc. This goes beyond the scope of this paper. We hope to come back to this issue in the future.

Remark 2.3. The dimensional reduction process depicted in Fig. 1 also suggests that a topological excitation in a topological order can also be viewed as an anomalous topological order, i.e. a gapped boundary of a topological order in one higher dimension.

2.2 A macroscopic definition of a local topological order

Macroscopically, the only physically detectable data of a topological order are the fusion, braidings and spins of its topological excitations. Note that these fusion-braiding properties are localized on an open disk, thus can only describe topological excitations in a local topological order [MW, Wa]. It was well-known that these fusion-braiding properties (without the spins see Remark 2.13) can be encoded efficiently in the data and axioms of an higher category [B] (see also [KW]). In this subsection, we propose a macroscopic definition of a local topological order based on the notion of a unitary $n$-category.

A mathematical definition of a unitary $n$-category is given in Def. A.4 in Appendix. Here, we only remind readers of some basic ingredients and properties of this notion without being precise. In a unitary $n$-category, the physical meaning of a 1-morphism is a defect or an excitation of codimension 1 (also called a domain wall); an $l$-morphism is an $l$-codimensional defect; an $n$-morphism is an instanton localized on the time axis. The spaces of instantons are all finite dimensional. An $(n-1)$-morphism $h$ is simple if $\text{hom}(h, h) \simeq \mathbb{C}$ and an $l$-morphism $g$ is simple if $\text{id}_g$ is simple. For $l \geq 0$, each $l$-morphism is a direct sum of simple $l$-morphisms, and there are only finitely many simple $l$-morphisms. For $k < l$, an $l$-morphism $g^{[l]}$ in $\text{hom}(h^{[k]}, h^{[l]})$ is an excitation nested on the $k$-codimensional excitation $h^{[k]}$. The composition of two $l$-morphisms corresponds to the fusion of two excitations. The identity morphism $\text{id}_h$ can be viewed as the vacuum state localized on the excitation $h$.

Each $l$-morphism $g$ has a two-side dual $\bar{g}$, which should be viewed as the anti-excitation of $g$. For any two non-isomorphic simple $k$-morphisms $f$ and $g$, $\text{hom}(f, g) = 0$. This condition was explained in [KW], Sec. XI.G, XI.H. We define $1_n$ to be the smallest unitary $n$-category consisting of a simple object $*$ and a simple $k$-morphism $\text{id}_*^k$ for $k = 1, 2, \ldots, n$, where $\text{id}_*^k$ is defined inductively by $\text{id}_*^{k+1} := \text{id}_{\text{id}_*^k}$ and $\text{id}_*^1 = \text{id}_*$.

We distinguish two types of local topological orders: simple and composite. Most of the topological orders studied in physics are simple topological orders. Composite topological orders are direct sums of simple topological orders. They naturally occur in the process of a dimensional reduction or a fine tuning of a system near multi-phase transition points in the
phase diagram [KW, ZW]. A direct sum of two simple topological orders \( A_n \) and \( B_n \) means that the system has accidental degenerate ground states, and is in either the \( A_n \)-state or the \( B_n \)-state. Additional perturbations such as applying external fields often push the system to select one ground state from the two.

**Definition 2.4.** A simple \( n \)-D local topological order is a unitary fusion \((n-1)\)-category (see Def. A.11). A composite \( n \)-D local topological order is a pair \((C_n, \iota)\), where \( C_n \) is a unitary \( n \)-category and \( \iota \) is a distinct object in \( C_n \), or equivalently, a unit (map) \( \iota : 1_n \to C_n \).

**Remark 2.5.** It is not known if the microscopic and macroscopic definitions of a local topological order are equivalent. To find compatible microscopic and macroscopic definitions of a local topological order is a fundamental open problem in the study of this subject.

**Remark 2.6.** Note that a unitary fusion \((n-1)\)-category can be viewed as a unitary \( n \)-category with a unique simple object, which can be viewed as the canonical unit map \( \iota \). So a simple \( n \)-D local topological order is a special case of composite local topological orders. The reason for our choice is because the notion of center for a monoidal \((n-1)\)-category and for an \( n \)-category with one object might be different (see Remark 4.5). Actually, notions in Def. 2.4 are special cases of a more general notion defined by a unitary \( n \)-category together with a choice of object \( \iota[0] \) and a choice of 1-morphism \( \iota[1] \) in \( \text{hom}(\iota[0], \iota[0]) \) and a choice of 2-morphism \( \iota[2] \) in \( \text{hom}(\iota[1], \iota[1]) \), so on and so forth. Equivalently, it is a pair \((C_n, \iota)\), where \( \iota = (\iota[0], \iota[1], \iota[2], \cdots, \iota[n-1]) \) and \( \iota[k+1] : 1_{n-k-1} \to \text{hom}(\iota[k], \iota[k]) \). Physically, this \( \iota \) can be viewed as a fixed background excitation with nested sub-excitations. It naturally occurs when we try to view a given topological excitation as a local topological order (see Sec. 2.4). For the purpose of this work, it is convenient and sufficient to work with Def. 2.4.

**Remark 2.7.** For physicists who are only interested in classification of local topological orders, one can ignore the unit \( \iota \) in the pair \((C_n, \iota)\). Namely, \( n \)-D local topological orders are classified by unitary \( n \)-categories. The unit \( \iota \) only occurs and useful when we try to view a topological excitation (defined by \( \iota \)) or a (external) domain wall as a topological order (recall Remark 2.3). It is perhaps better to give a different name to the pair \((C_n, \iota)\), such as a based local topological order. But we do not use this terminology here because all topological orders in this work are based. In many cases, we abbreviate the pair \((C_n, \iota)\) as \( C_n \) if \( \iota \) is clear from the context or irrelevant to the discussion. But we want to emphasize that the unit \( \iota \) is not only physically natural but also mathematically important in our study, especially in the study of the category of local topological orders.

**Remark 2.8.** Very often in the processes of dimensional reduction, we obtain unstable phases, the categorical formulation of which goes beyond above descriptions. For example, in 2D cases, unitary multi-fusion 1-categories [ENO02] appear. But these unstable phases can flow to the stable ones under the perturbation of local operators in Hamiltonians. We give a few examples of unstable phases in Example 2.11, 2.21, 2.22.

Since all topological orders studied in this work are local, for simplicity, we abbreviate “local topological orders” to “topological orders” from now on.

Macroscopically, an \( n \)-D local topological order \( C_n \) is said to be closed if any excitation can be detected by some excitations via braiding. In particular, the only excitation that has trivial double braiding with all excitations must be the vacuum. For a 2 + 1D anyon
system, this condition means that the center of the associated braided monoidal 1-category is trivial, and is equivalent to the famous non-degeneracy condition of the $S$-matrix.

**Definition 2.9.** A simple $n$D local topological order $C_n$ is called *closed* if the unitary fusion $(n - 1)$-category has a trivial center (see Eq. (4.14) for the definition of center). It is called *anomalous* if its center is non-trivial.

Def 2.9 is compatible with Def 2.2 in the sense that the physical notion of the *bulk* is equivalent to the mathematical notion of the center as we will show.

**Remark 2.10.** Since a domain wall can not be braided with any other excitations, there is no domain wall in a closed $n$D topological order except the trivial one $1_e$. As a consequence, a closed simple $n$D topological order $C_n$ must have a unique simple object $1_C$, which is the tensor unit. The category hom$(1_e, 1_e)$ is a unitary braided fusion $(n - 2)$-category. A closed simple $n$D topological order can be equivalently defined as a unitary braided fusion $(n - 2)$-category with a trivial center. But for an anomalous simple topological order, even if there is a unique simple object $1_C$, we can not replace it by the unitary braided fusion $\text{hom}(1_C, 1_C)$, because the notion of center is different (see Remark 4.4). The relation between two different centers is explained in Remark 4.5.

**Example 2.11.** Since composite topological orders are direct sums of the simple ones, we mainly discuss simple topological orders in a few low dimensional cases below.

1. A unitary 0-category is just a finite dimensional Hilbert space over $\mathbb{C}$ (see Sec. A.1). A 0D topological order is a pair $(V, f)$, where $V$ is a finite dimensional Hilbert space and $f \in V$ or a linear map $f : \mathbb{C} \rightarrow V$. It is simple if $V \simeq \mathbb{C}$. We denote the pair $(\mathbb{C}, 1)$ by $1_{00}$. A 0D topological order can be viewed as a neighborhood of an instanton or a boundary (in time direction) of an unstable (see below) 1D topological order.

2. A 1D topological order is given by a unitary 1-category together with a distinguished object $\iota$. The trivial 1D topological order, denoted by $1_1$, is a unitary 1-category, consisting of a unique simple object (or a 0-morphism) $1^{[0]}$ and its direct sums (composite objects), such as $m^{[0]} = 1^{[0]} \oplus \cdots \oplus 1^{[0]}$ ($m$ terms), together with a distinguished object $1^{[0]}$. The hom space $\text{hom}(m^{[0]}, n^{[0]})$ is the $mn$-dimensional vector space of $m \times n$ matrices. The stacking of two $1_1$ gives the same 1D topological order back, i.e. $1_1 \boxtimes 1_1 = 1_1$. The composition of two $1_1$’s give us a new 1D topological order denoted by $2_1 := 1_1 \oplus 1_1$, which contains two simple objects $1^{[0]}_{(1)}$ and $1^{[0]}_{(2)}$, and composite objects $m^{[0]}_{(1)} + n^{[0]}_{(2)}$, which is a direct sum of $m$ $1^{[0]}_{(1)}$’s and $n$ $1^{[0]}_{(2)}$’s. Physically, the phase $1_1$ is realized by an end point of a gapped quantum system defined on a line segment $I$. The interior of $I$ is a trivial 2D phase. The boundary $\partial I = S^0$ consists of two points $p_+$ and $p_-$. The unitary 1-category $1_1$ actually describes a 0+1D effective theory on $p_+$ (or $p_-$). An object (a 0-morphism) $m^{[0]}$ of $1_1$ is a boundary “excitation” with $m$-fold degeneracy. The total degenerate space of the system is $\text{hom}(m^{[0]}, n^{[0]})$ where $m^{[0]}$ is the boundary “excitation” on $p_+$ and $n^{[0]}$ is

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4Here we have already assumed the minimal description of all elementary excitations. All the condensed excitations (those can be obtained from other elementary excitations via condensations) are discarded for this minimal description (also called the core of a BF$_n^\text{core}$-category). See [KW] Sec. XI.F for more details.
that on $p_-$. The degenerate space is $mn$-dimensional as expected. The phase $2_1$ is realized by an end of a gapped quantum system defined on the line segment $I$ with an accidental and unstable 2-fold ground state degeneracy in the interior of $I$. For example, consider the toric code model defined on a closed 2-disk (as the space manifold) with a smooth (or rough) boundary, then squeeze it to a line segment $I$. This gives a realization of $2_1$. A boundary “excitation” is now described by $m_0^{(1)} + n_0^{(2)}$. Such an excitation is not a boundary state with $(m + n)$-fold degeneracy, since $m_0^{(1)}$ and $n_0^{(2)}$ have different bulk on $I$. The total degenerate space of the system is still $\hom(m_0^{(1)} + n_0^{(2)}, k_0^{(1)} + l_0^{(2)})$ where $m_0^{(1)} + n_0^{(2)}$ is the boundary “excitation” on $p_+$ and $k_0^{(1)} + l_0^{(2)}$ is the boundary “excitation” on $p_-$. Since $\hom(1^{(1)}, 1^{(2)}) = 0$, this degenerate space is $mk + nl$-dimensional as expected.

The 1D topological order $1_1$ is simple and closed. In fact, $1_1$ is the only simple 1D topological order. All 1D topological orders are the compositions of $1_1$’s.

3. A simple 2D topological order can by physically realized on an open 1-disk in the boundary of a local Hamiltonian model on the 2-disk. It has particle-like topological excitations, which can be fused. Therefore, it can be described by a unitary fusion 1-category $\mathcal{C}$. Its tensor unit is simple. This corresponds to the uniqueness of the ground state of the local Hamiltonian model on the 2-disk. The trivial 2D topological order $1_2$ is given by the category $\mathcal{Hilb}$ of finite dimensional Hilbert spaces, i.e. $1_2 = \mathcal{Hilb}$. $1_2$ is also closed. If $\mathcal{C}$ contains a non-trivial particle-like excitation, it cannot be detected by other excitations because there is no braiding. In this case, $\mathcal{C}$ describes an anomalous topological order.

An unstable 2D phase can be described by a unitary multi-fusion 1-category, which is almost the same as a unitary fusion 1-category except that the dimension of $\hom(1_\mathcal{C}, 1_\mathcal{C})$, where $1_\mathcal{C}$ is the tensor unit, is not necessary 1. The space $\hom(1_\mathcal{C}, 1_\mathcal{C})$ is the space of ground states. So it is degenerate for a unitary multi-fusion 1-category. This degeneracy makes the phase unstable and can be lifted by local operators. We illustrate some examples of such unstable phases in Example 2.21 and Example 2.22.

4. A simple 3D topological order can be described by a unitary fusion 2-category, or equivalently, a unitary 3-category $\mathcal{C}_3$ with a unique simple object $\ast$. 1-morphisms in the 3-category are string-like excitations, 2-morphisms are particle-like excitations and 3-morphisms are instantons. When $\mathcal{C}_3$ is closed, there is no simple 1-morphisms (string-like excitations) other than the trivial one $\text{id}_\ast$ because they can not be braided with other excitations thus not detectable. Moreover, the unitary braided fusion 1-category $\hom(\text{id}_\ast, \text{id}_\ast)$ is non-degenerate. As a result, $\mathcal{C}_3$ can be equivalently described by a unitary braided fusion 1-category if $\mathcal{C}_3$ is closed. In this case, the centers of $\mathcal{C}_3$ are

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5 The interior of $I$ in this case is given by a multi-fusion 1-category with 2-fold unstable ground state degeneracy (see Example 2.21).

6 In this case, string-like excitations are still possible as constructed in [KK], but they can all be obtained from condensations of point-like excitations [Ko2]. Therefore, such string-like excitations must be excluded from a minimal categorical description of a topological order [KW]. If there is a string-like excitation that is not condensed, it is not detectable via braiding because it can not be braided with any other excitations. Therefore, such a topological order must be anomalous.
and \( \text{hom}(\text{id}_*, \text{id}_*) \) are both trivial. If the topological order \( C_3 \) is anomalous, we cannot reduce \( C_3 \) to \( \text{hom}(\text{id}_*, \text{id}_*) \) even if \( \text{id}_* \) is the unique simple 1-morphism because the notion of center is different in these two cases (see Remark 4.5).

**Remark 2.12.** Topological orders defined on space manifolds other than open disks do occur in this work but only briefly discussed from a macroscopic point of view (see Example 2.21, 2.22 and Remark 3.9). For a systematic study of topological orders on other manifolds from a macroscopic point of view, we need the theory of factorization algebra \([L3, AFT2]\) (see also Remark 2.13). We will do that in [AKZ].

**Remark 2.13.** In general, topological excitations are defined on submanifolds with certain tangential structures \([AFT1]\), such as framing. In this work, we ignore some of these tangential structures in our categorical formulations, such as topological spins, mainly because we don’t know how to add it to our \( n \)-categories. For all algebraic constructions appeared in this work, such as the construction of center, gapped domain walls and Morita/Witt equivalence, the topological spins do not play any explicit role. Local topological orders can be glued to obtained global topological orders on closed manifolds via topological chiral homology \([L3, AFT2]\), which is a powerful tool to construct TQFTs. TQFTs thus obtained often contain framing anomalies.

**Remark 2.14.** Our notion of closed/anomalous local topological order is different from but not contradicting to that of a TQFT. For example, a quantum Hall system or a modular tensor category gives a closed local topological order according to our definition, but the associated 2+1D TQFT is “anomalous” due to the framing anomaly \([RT, Tu]\). It has a non-trivial bulk 3+1D TQFT, which is invertible with no non-trivial excitations \([WWa, FT, F]\). The framing anomaly occurs when we glue local topological orders to obtain global topological orders on closed manifolds \([L2, L3]\). In other words, a local topological order or a unitary \( n \)-category can be viewed as a TQFT restricted to an open disk \([MW, Wa]\).

**Remark 2.15 (Minimal Assumption).** In physics, a topological excitation is an equivalence class of excitations. It is invariant under smooth deformation and the action of any local operators \([KW]\). So on the categorical side, it is reasonable to identify all morphisms that are isomorphic. More precisely, given an ordinary unitary \( n \)-category, we take one representative from each equivalence class of \( k \)-morphisms for \( 0 \leq k < n \). These representatives form a much smaller unitary \( n \)-category, in which distinct morphisms are not isomorphic. As a result, all \( k \)-morphisms form a set with only countable many elements. We call such a category minimal \([L1]\). In many parts of this work, we would like to assume that the unitary \( n \)-categories under consideration are minimal so that it is legitimate to say that two unitary \( n \)-categories (or topological orders) are identical, i.e. \( \mathcal{C}_n = \mathcal{D}_n \). This assumption is not absolutely necessary for our theory. But it simplifies the discussion. One of the major simplifications is that two minimal unitary \( n \)-categories are equivalent only if they are isomorphic. In this case, one can identify these two \( n \)-categories. Since this assumption is not absolutely necessary, we use it only when it is needed. For example, it is used significantly in Sec. 4.1. For mathematical results associated to \( n \)-categories, we still use the usual notion of an equivalence \( \simeq \) between categories. But for physics minded readers, we do recommend to take this assumption and regard \( \simeq \) as = throughout this work.
2.3 Time-reverse of a topological order

A topological order $\mathcal{C}_n$ can be realized by a local Hamiltonian lattice model $\Gamma$. If we mirror reflect the system along the time direction or any odd number directions (see Remark 2.17), we obtain a local Hamiltonian lattice model $\overline{\Gamma}$. We denote associated topological order by $\mathcal{C}_n$ or $\mathcal{C}_n^{op}$, which is called the time-reverse of $\mathcal{C}_n$.

**Example 2.16.** We discuss a few low dimensional cases.

1. A 0D topological order $\mathcal{C}_0$ is a pair $(V, f)$, where $V$ is a finite dimensional Hilbert space and $f \in V$. We define $\mathcal{C}_0^{op}$ to be $(V^*, f^*)$, where $V^*$ is the dual vector space and can be identified with $V$ and $f^* = (f, \cdot)_V$.

2. In 1D, a 1D topological order can be described by a unitary 1-category $\mathcal{C}$. The physical meaning of an arrow in $\mathcal{C}$ is an instanton. Reversing the time amounts to flipping all the arrows in $\mathcal{C}$. Therefore, the time-reverse of $\mathcal{C}$ is the opposite category $\mathcal{C}^{op}$, which is the same category as $\mathcal{C}$ but with arrows reversed. For an unstable simple phase described by a matrix algebra $A$ with multiplication $m(a, b) = ab$, the time-reverse of $A$ is just the opposite algebra $A^{op}$ with multiplication $m^{op}(a, b) = ba$.

3. In 2D, the time-reverse of a unitary fusion 1-category $\mathcal{C}$ is the opposite category $\mathcal{C}^{op}$ equipped with the tensor product $\otimes$. The coherence isomorphisms, i.e. the associator and unit isomorphisms, in $\mathcal{C}^{op}$ are taken to the inverse of those in $\mathcal{C}$. For example, in Example 2.22 when we fold two boundaries in Fig. 5 (a), we need flipped the orientation of the N-boundary. This explains the “op” in the first line of Eq. (2.5).

Actually, the unitary fusion 1-category $\mathcal{C}^{op}$ is monodically equivalent to the fusion 1-category $\mathcal{C}^{rev}$, which is the category as $\mathcal{C}$ but equipped with the opposite tensor product $\otimes^{rev}$, i.e. $x \otimes^{rev} y := y \otimes x$, and the coherence isomorphisms are the inverse of those in $\mathcal{C}$. This monoidal equivalence $\mathcal{C}^{rev} \simeq \mathcal{C}^{op}$ is given by taking duals $x \mapsto \overline{x}$. Note that $\mathcal{C}^{rev}$, viewed as a unitary 2-category with a simple object, is just the space-reverse of $\mathcal{C}$. So the monoidal equivalence $\mathcal{C}^{rev} \simeq \mathcal{C}^{op}$ simply says that the time-reverse is equivalent to the space-inverse.

4. In $n$D for $n > 2$, the time-reverse of a unitary $n$-category is again given by the opposite category $\mathcal{C}_n^{op}$, which is the same category as $\mathcal{C}_n$ but with reversed $n$-morphisms and all coherence isomorphisms are taken to be the inverse of those in $\mathcal{C}_n$. For example, for a simple closed $\mathcal{C}_3$ topological order, i.e. a unitary braided fusion 1-category, the braiding in $\mathcal{C}_3^{op}$ is taken to be the inverse of that in $\mathcal{C}_3$.

**Remark 2.17.** For a unitary $n$-category and $k \neq l$, flipping the arrows for all $k$-morphisms and all $l$-morphisms leaves the category unchanged (up to equivalences) because of the duality. Therefore, a mirror reflection of a topological order in any odd number directions is equivalent to the time reverse.

2.4 Dimensional reductions and tensor products

A simple $n$D topological order $\mathcal{C}_n$ naturally gives an $(n-1)$D topological order by restricting to a neighborhood of a 1-codimensional simple domain wall $x$, depicted in Fig. 3. This
neighborhood automatically include the action of other domain walls on $x$ via fusion. This action on $x$ covers the entire category $C_n$. We denote the obtained $(n-1)$D topological order by $(P_{n-1}(C_n), x)$, where $P_{n-1}(C_n)$ is the same unitary $(n-1)$-category as $C_n$ but forgetting the monoidal structures and $x$ plays the role of the unit. If $x$ is the trivial domain wall, we simply denote the pair by $P_{n-1}(C_n)$. When $(C_n, \iota)$ is composite, we define $P_{n-1}(C_n)$ by the unitary $(n-1)$-category hom$(\iota, \iota)$ (forgetting the monoidal structures).

Remark 2.18. If one takes the more general definition of an $n$D topological order given in Remark 2.6, one can define $P_k(C_n)$ for all $k < n$. We do not need them in this work.

Example 2.19. A couple of examples from 3D.

1. In the toric code model, the bulk excitations are given by the monoidal center $Z(\text{Rep}_{Z_2})$ of the unitary fusion category $\text{Rep}_{Z_2}$ with simple objects $1, e, m, \epsilon$. The trivial domain wall gives an anomalous 2D topological order $P_1(Z(\text{Rep}_{Z_2}))$, which is just the same unitary fusion 1-category as $Z(\text{Rep}_{Z_2})$ but forgetting the braiding structure. The trivial excitation 1 in the toric code model can be viewed as a composite 1D topological order by including a neighborhood of it. Therefore, one should include all the image of the action of $Z(\text{Rep}_{Z_2})$ on 1. This image is nothing but $P_{n-2}(Z(\text{Rep}_{Z_2}))$, which is the same unitary 1-category as $Z(\text{Rep}_{Z_2})$ (with the usual unit) but forgetting both the braiding and fusion structures. Similarly, an $e$-particle can be viewed as a 1D topological order $(Z(\text{Rep}_{Z_2}), e)$, where $Z(\text{Rep}_{Z_2})$ is viewed as a unitary 1-category.

2. Consider a Levin-Wen type of lattice model with two boundaries with the bulk lattice defined by a unitary fusion 1-category $C$ and the lattice near two boundaries defined by unitary semisimple indecomposable $C$-modules $M$ and $N$, respectively [KK] (see Fig. 5). Excitations on the $M$-boundary (or the $N$-boundary) is given by $C_M^\vee$ (or $C_N^\vee$), where $C_M^\vee$ is the category $C$-module functors $\text{Fun}_C(M, M)^{rev}$ with the opposite tensor product. A defect of codimension 2 between the $M$-boundary and the $N$-boundary is given by a $C$-module functor $f$ in $\text{Fun}_C(M, N)$ [KK]. When this defect $f$ is viewed as an anomalous 1D topological order, we must include a neighborhood of this defect, more precisely, include all excitations generated by the fusion of the boundaries/bulk excitations with this defect. This fusion action covers all objects in $\text{Fun}_C(M, N)$ because $\text{Fun}_C(M, N)$ is an indecomposable $C_M^\vee-C_N^\vee$-bimodule. Therefore, the 1D topological
order thus obtained is given by the pair $(\text{Fun}_C(M, N), f)$ (see the purple dot in Fig. 5 (a)). When $M = N$ and $f = \text{id}_M$, it is nothing but $P_1(\text{Fun}_C(M, M))$.

One can stack an $n$D topological order $A_n$ on the top of the another one $B_n$. This operation is denoted by $\boxtimes$, also called a tensor product. More general tensor product can be obtained by gluing $A_n$ with $B_n$ by an $(n+1)$D bulk $C_{n+1}$, denoted by $A_n \boxtimes_{C_{n+1}} B_n$. It is summarized by the following graphic equations.

\[
A_n \boxtimes B_n := A_n \quad \text{1}_{n+1} \quad B_n \quad \ , \quad A_n \boxtimes_{C_{n+1}} B_n := A_n \quad C_{n+1} \quad B_n \quad (2.1)
\]

It is clear that $\boxtimes = \boxtimes_{1_{n+1}}$. Notice that $A_n \boxtimes_{C_{n+1}} B_n$ can be viewed either as an $n$D topological order or as an $(n+1)$D physical configuration. The former one is a dimensional reduction of the later one.

**Remark 2.20.** By Def. 2.4 when $C_{n+1}$ is simple and closed, i.e. a unitary braided fusion $(n-1)$-category with the usual unit, and $A_n$ and $B_n$ are simple, i.e. a unitary fusion 1-category with the usual unit, then the tensor product $A_n \boxtimes_{C_{n+1}} B_n$ is just the usual tensor product of a right $C_{n+1}$-module and a left $C_{n+1}$-module in the category of unitary fusion $(n-1)$-categories with unitary monoidal $(n-1)$-functor as morphisms. In particular, the left action $A_n \boxtimes C_{n+1} \rightarrow A_n$ and the right action $C_{n+1} \boxtimes B_n \rightarrow B_n$ are both unitary monoidal.

**Example 2.21.** We would like to illustrate the construction $A_n \boxtimes_{C_{n+1}} B_n$ by concrete lattice models when $n = 2$. Consider two narrow bands of toric code model depicted in Figure 4. These two narrow bands can be viewed as two 1+1D topological orders. This is an example of dimensional reduction.

1. In (a), the $e$- and $m$-particle in the bulk or on one of the boundary condense to vacuum on the other boundary. Therefore, there is no non-trivial particle-like excitations in the band at all, and the 2D topological order thus obtained is trivial, i.e.

$$A_2 \boxtimes_{C_1} B_2 = \mathcal{Hilb}. $$

This fact is also supported by abstract nonsense. When the toric code model is viewed as an example of Levin-Wen models, the bulk lattice is determined by the data of the unitary fusion category $\text{Rep}_{\mathbb{Z}_2}$ (the category of representations of $\mathbb{Z}_2$ group). The smooth boundary is again determined by that of $\text{Rep}_{\mathbb{Z}_2}$ but viewed as a right module over the fusion category $\text{Rep}_{\mathbb{Z}_2}$ (or a right $\text{Rep}_{\mathbb{Z}_2}$-module); the rough boundary is determined by the category $\mathcal{Hilb}$ which is a left $\text{Rep}_{\mathbb{Z}_2}$-module. The topological excitations on the smooth boundary is given by the unitary fusion category $\text{Rep}_{\mathbb{Z}_2} \simeq \text{Fun}_{\text{Rep}_{\mathbb{Z}_2}}(\text{Rep}_{\mathbb{Z}_2}, \text{Rep}_{\mathbb{Z}_2})$, those on the rough boundary by $\text{Rep}_{\mathbb{Z}_2} \simeq \text{Fun}_{\text{Rep}_{\mathbb{Z}_2}}(\mathcal{Hilb}, \mathcal{Hilb})$. The bulk excitations are given by the unitary modular tensor category $Z(\text{Rep}_{\mathbb{Z}_2}) := \text{Fun}_{\text{Rep}_{\mathbb{Z}_2}|\text{Rep}_{\mathbb{Z}_2}}(\text{Rep}_{\mathbb{Z}_2}, \text{Rep}_{\mathbb{Z}_2})$, which is the monoidal
Figure 4: Above figures illustrate two physical configurations of toric code model bounded by two gapped boundaries discussed in Example 2.21. In (a), two boundaries are different. In (b), both boundaries are the smooth boundary.

center of $\text{Rep}_{Z_2}$. The tensor product\footnote{We have secretly used the fact that $\text{Rep}_{Z_2}$ is a symmetric fusion 1-category such that $\text{Rep}_{Z_2} \simeq \text{Rep}_{Z_2}^{\text{rev}}$. See Eq. (2.5) for a more general and precise expression of this tensor product.}:

$$A_2 \boxtimes_{\mathcal{C}_2} B_2 = \text{Fun}_{\text{Rep}_{Z_2}}(\text{Rep}_{Z_2}, \text{Rep}_{Z_2}) \boxtimes_{Z(\text{Rep}_{Z_2})} \text{Fun}_{\text{Rep}_{Z_2}}(\text{Hilb}, \text{Hilb})$$

is defined by Tambara’s tensor product $[\text{Ta}]$ between a left $Z(\text{Rep}_{Z_2})$-module and a right $Z(\text{Rep}_{Z_2})$-module (see also [ENO09]). Moreover, this tensor product has a natural monoidal structure (see Sec. A.4) and we have

$$A_2 \boxtimes_{\mathcal{C}_2} B_2 \simeq \text{Fun}_{\text{Hilb}}(\text{Rep}_{Z_2} \boxtimes_{\text{Rep}_{Z_2}} \text{Hilb}, \text{Rep}_{Z_2} \boxtimes_{\text{Rep}_{Z_2}} \text{Hilb})$$

$$\simeq \text{Hilb}, \quad (2.2)$$

where the first $\simeq$ is due to (A.15) (or Prop. A.21).

2. In (b), both boundaries are the smooth boundary. In this case, $m$-particle is clearly condensed in the 2D phase. It seems that there is only $e$-particles living in the 2D topological order. However, the string (the purple line in Fig. 4 (b)) that creates a pair of $e$-particles at its ends can be detected by a string (the dotted line) that create a pair of condensed $m$-particles. It means that the $e$-string is a 2D “vacuum” that is different from the trivial string between two null-particles. Moreover, the two $e$-particles (the red dot and the green dot in Fig. 4 (b)) should be viewed as domain walls between two different vacuums, should be treated as different types of particles. They form

\text{...}
a particle and anti-particle pair. As a consequence, the 2D phase is described by a
unitary multi-fusion 1-category

\[
\mathcal{H}ilb \times M_{2 \times 2} = \left( \begin{array}{cc} \mathcal{H}ilb & \mathcal{H}ilb \\ \mathcal{H}ilb & \mathcal{H}ilb \end{array} \right) .
\]

(2.3)

where two diagonal subcategories are the usual vacuum state (any vertical blue line
in Fig.4(b) and the 2D vacuum state given by the purple string in Fig.4(b), the two
diagonal subcategories are the domain walls between two vacuums.

Above result is also guaranteed by abstract nonsense. Indeed, by (A.15) (or Prop. A.21),
the 2D phase obtained via dimensional reduction is given by

\[
\mathcal{A}_2 \boxtimes_{C_3} \mathcal{B}_2 = \mathcal{F}un_{\text{Rep}_{Z_2}} (\text{Rep}_{Z_2}, \text{Rep}_{Z_2}) \boxtimes_{Z(\text{Rep}_{Z_2})} \mathcal{F}un_{\text{Rep}_{Z_2}} (\text{Rep}_{Z_2}, \text{Rep}_{Z_2})
\]

\[
\simeq \mathcal{F}un_{\mathcal{H}ilb} (\text{Rep}_{Z_2} \boxtimes_{\text{Rep}_{Z_2}} \text{Rep}_{Z_2}, \text{Rep}_{Z_2} \boxtimes_{\text{Rep}_{Z_2}} \text{Rep}_{Z_2})
\]

\[
\simeq \mathcal{F}un_{\mathcal{H}ilb} (\text{Rep}_{Z_2}, \text{Rep}_{Z_2}),
\]

(2.4)

which is exactly the unitary multi-fusion 1-category given in (2.3).

When the distance between two boundaries is small, the tunneling of an \( m \)-particle
from one boundary to the other is a local operator. In this case, the ground state de-
generacy can be easily lifted by introducing this tunneling effect into the Hamiltonian.
Therefore, the phase described by unitary multi-fusion 1-category (2.3) (or (2.4)) is
unstable and will flow to the only closed and stable 2D phase \( 1_2 \). On the other hand,
if we keep the distance between two boundaries large even in the thermodynamic
limit, then the physical configuration depicted in Fig.4(b) with 2-fold ground state
degeneracy is stable (under local perturbations) as a 3D topological order with two
boundaries [WW1].

Before we end this example, we would like to comment that the unitary multi-fusion 1-
category defined in (2.3) is also closed because it is realizable by a lattice model in 2D.
This is consistent with the mathematical result that the monoidal center of the unitary
multi-fusion category defined in (2.3) is trivial [ENO09].

**Example 2.22.** We give more examples of \( \mathcal{A}_n \boxtimes_{\mathcal{C}_{n+1}} \mathcal{B}_n \) for \( n = 1 \) in Levin-Wen type of
lattice models [KK]. Consider a lattice model depicted in Fig.5(a), the bulk lattice defined
by a unitary fusion 1-category \( \mathcal{C} \), the upper/lower boundary lattice is defined by a unitary
indecomposable \( \mathcal{C} \)-module \( M/N \). The excitations in the bulk are given by the monoidal
center \( Z(\mathcal{C}) \) of \( \mathcal{C} \), the excitations on the \( M \)-boundary by \( \mathcal{C}_M \), those on \( N \)-boundary by \( \mathcal{C}_N \)
and the defect junction (the purple dot) by a \( \mathcal{C} \)-module functor \( f \in \mathcal{F}un_{\mathcal{C}} (M, M) \). When
the defect junction is viewed as a 1D topological order (by including the action of nearby
excitations on \( f \)), it is given by \( (\mathcal{F}un_{\mathcal{C}} (M, M), f) \). By a dimensional reduction process, i.e.
folding the two boundaries along two dotted arrow, we obtain (b) in Fig.5 where

\[
\mathcal{E} = \mathcal{F}un_{\mathcal{C}} (N, N)^{\text{rev}} \boxtimes_{Z(\mathcal{C})} \mathcal{F}un_{\mathcal{C}} (M, M)^{\text{rev}}.
\]

(2.5)

*The action of nearby excitations of \( f \) actually form a subcategory \( \mathcal{F}un_{\mathcal{C}} (N, N)f\mathcal{F}un_{\mathcal{C}} (M, M) \) of
\( \mathcal{F}un_{\mathcal{C}} (M, N) \). This subcategory is equivalent to the data \( (\mathcal{F}un_{\mathcal{C}} (M, N), f) \).
Figure 5: Above figures illustrate the dimensional reduction process in a Levin-Wen type of lattice model discussed in Example 2.22. \( \mathcal{E} \) is given by (2.5).

On the other hand, when we fold \( \mathcal{N} \)-boundary upwards and flip its orientation, the left \( \mathcal{C} \)-module \( \mathcal{N} \) becomes the right \( \mathcal{C} \)-module \( \mathcal{N}^{\text{op}} \). According to [KK], we should also have

\[
\mathcal{E} = \text{Fun}_{\text{Hilb}}(\mathcal{N}^{\text{op}} \boxtimes_{\mathcal{C}} \mathcal{M}, \mathcal{N}^{\text{op}} \boxtimes_{\mathcal{C}} \mathcal{M})^{\text{rev}} \approx \text{Fun}_{\text{Hilb}}(\text{Fun}(\mathcal{N}, \mathcal{M}), \text{Fun}(\mathcal{N}, \mathcal{M}))^{\text{op}}
\]

where we have used the identity \( \mathcal{N}^{\text{op}} \boxtimes_{\mathcal{C}} \mathcal{M} \approx \text{Fun}(\mathcal{N}, \mathcal{M}) \) [ENO09]. Indeed, the compatibility of (2.5) and (2.6) is a consequence of (A.15) (or Prop. A.21). Note that \( \mathcal{E} \) is a unitary multi-fusion category. As a 2D topological order, it should be closed because it is the bulk of the 1D topological order \( \text{Fun}(\mathcal{N}, \mathcal{M})^{\text{op}} \) (the purple dot). This is consistent with the mathematical fact that the monoidal center of \( \mathcal{E} \) is trivial [ENO09]. Moreover, as we will show later, \( \mathcal{E} \) is also the center of \( \text{Fun}(\mathcal{M}, \mathcal{N}) \) (see Example ??). Note also that the dimension of ground state degeneracy \( \text{hom}_\mathcal{E}(1_\mathcal{E}, 1_\mathcal{E}) \) is given by the number of simple objects in \( \text{Fun}(\mathcal{M}, \mathcal{N}) \). In particular, when \( \mathcal{N} = \mathcal{M} = \mathcal{C} \), \( \mathcal{E} = \text{Fun}_{\text{Hilb}}(\mathcal{C}, \mathcal{C}) \) (see Remark A.24), and if \( \mathcal{C} = \text{Rep}_{\mathbb{Z}_2} \), \( \mathcal{E} \) is nothing but the topological phase construction in Fig. 4 (b) (recall Eq. (2.4)). The dimension of ground state degeneracy \( \text{hom}_\mathcal{C}(1_\mathcal{C}, 1_\mathcal{C}) \) is 2, which is the number of simple objects in \( \text{Rep}_{\mathbb{Z}_2} \). It describes an unstable 2D phase and can flow to the only stable and closed 2D phase \( \mathbf{1}_2 \). This is consistent with recent works [WW1, HY, LWW].

One can cook up more general examples from Levin-Wen models enriched by defects as depicted in Fig. 11 (see Example 5.4). It is also clear that the following identities (recall the Minimal Assumption Remark 2.15):

\[
\mathbf{1}_n \boxtimes \mathcal{A}_n = \mathcal{A}_n = \mathcal{A}_n \boxtimes \mathbf{1}_n,
\]

\[
(\mathcal{A}_n \boxtimes_{\mathcal{A}_{n+1}} \mathcal{C}_n) \boxtimes_{\mathcal{D}_{n+1}} \mathcal{E}_n = \mathcal{A}_n \boxtimes_{\mathcal{A}_{n+1}} (\mathcal{C}_n \boxtimes_{\mathcal{D}_{n+1}} \mathcal{E}_n)
\]
hold because there is no physical detectable difference between the multi-phase configurations. For this reason, we denote the two sides of the equation (2.7) as $A \boxtimes B \boxtimes C \boxtimes D \boxtimes E$ in the rest of this paper. Moreover, if $B_{n+1}$ and $D_{n+1}$ are closed, we have the following identity:

$$P_n(B_{n+1}) \boxtimes B_{n+1} C_n = C_n = C_n \boxtimes D_{n+1} P_n(D_{n+1}).$$

(2.8)

**Remark 2.23.** When $n = 1$, $A_1, C_1, E_1$ are unitary 1-categories. In this case, the associativity (2.7) is just the associativity of Tambara’s tensor product. When $n = 0$, $A_0, C_0, E_0$ are Hilbert spaces, the tensor product is just usual tensor product between modules over algebras. In this case, the associativity is guaranteed. For higher $n$, the associativity and unit properties of the tensor product is entirely similar (recall Remark 2.20) but not yet fully established.

### 2.5 Closed gapped domain walls

By a *closed* or *anomaly free* gapped domain wall $M_{n-1}$ between $C_n$ and $D_n$, or a $C_n$-$D_n$-wall, we mean an $(n - 1)D$ topological order connecting the $C_n$-phase and the $D_n$-phase and sharing the same bulk as depicted in Fig. 6. We postpone the precise definition of the anomalous gapped domain wall to Sec. 5. All domain walls occur before Sec. 5 are closed. Therefore, we abbreviate a closed domain wall to a domain wall until Sec. 5.

An invertible gapped domain wall between two simple topological orders $C_n$ and $D_n$ is just an invertible morphism in $\mathcal{T}O_n^{closed-wall}$, and can be defined inductively.

**Definition 2.24.** A gapped domain wall $M_{n-1}$ between $C_n$ and $D_n$ is called *invertible* if there are an $(n - 2)D$ invertible domain wall between $M \boxtimes D_n M^{\text{op}}$ and $\text{id}_{C_n}$, denoted by $M \boxtimes D_n M^{\text{op}} \simeq \text{id}_{C_n}$, and another $(n - 2)D$ invertible domain wall between $M^{\text{op}} \boxtimes C_n M$ and $\text{id}_{C_n}$, denoted by $M^{\text{op}} \boxtimes C_n M \simeq \text{id}_{C_n}$ (see Def. 3.1).
Remark 2.25. Applying the Minimal Assumption Remark 2.15, we can also replace $\simeq$ by $=$ in above definition. Note that this replacement does not trivialize the higher dimensional domain walls. Their physical meaning is still preserved.

An invertible domain wall between two phases $\mathcal{C}_n$ and $\mathcal{D}_n$ is also transparent. It means that excitations in $\mathcal{C}_n$-phase can tunnel through the wall and become excitations in $\mathcal{D}_n$-phase. This tunneling is an invertible process and preserves the fusion and braiding properties. Indeed, for an invertible $\mathcal{C}_n$-$\mathcal{D}_n$-wall $M_{n-1}$, the topological excitations on one side can tunnel through the invertible wall to the other side. For excitations of codimension higher than 1 and $n > 1$, this tunneling process is invertible and is depicted in Fig. 7, where $\times$ represents a topological excitation in $\mathcal{C}_n$-phase and the dotted line in the third graph is the trivial domain wall $\text{id}_{\mathcal{D}_n}$ depicted as the dotted line in the third graph.

This physical definition of invertible domain wall becomes a mathematical one if we identify an $n$D topological order with a unitary $n$-category with a unit, $\mathbb{X}_{\mathcal{C}_n, \mathcal{D}_n}$ with corresponding mathematical tensor products, and define the invertible domain wall mathematically in the lowest dimension. Although it makes no sense physically to talk about a domain wall between two 0D phases, we can set an invertible domain wall between two 0D topological orders to be an invertible linear map between two Hilbert spaces preserving the unit. This completes the mathematical definition of an invertible domain wall in all dimensions. We illustrate this in a couple of lower dimensional cases:

1. It is more illuminating to consider a 0D invertible domain wall between two unstable phases instead of two stable ones because the later case is rather trivial and uninteresting. Such a 0D invertible domain wall between two unstable 1D topological orders (i.e. two simple $C^*$-algebras $A$ and $B$), is given by an invertible $A$-$B$-bimodule $M$...
with a unit. It is possible only if \( A \simeq B \) as algebra\(^9\). More explicitly, let \( A = \text{End}(U) \) and \( B = \text{End}(V) \) for two vector spaces \( U \) and \( V \) of the same dimension, an invertible \( A-B \)-bimodule is given by \( (\text{hom}_C(V,U), f : V \to U) \), where \( f \) is an invertible linear map. Its inverse is given by \( (\text{hom}_C(U,V), f^{-1}) \). We obtain an algebraic isomorphism \( A \to B \) defined by \( a \mapsto f \circ a \circ f^{-1} \). Conversely, an algebra isomorphism \( h : A \to B \) automatically gives an invertible \( A-B \)-bimodule \( _hB \) with the usual unit. It is easy to check two maps are inverse to each other. Therefore, a 0D invertible domain wall between two simple 1D topological orders is equivalent to an algebra isomorphism. Invertible 0D domain walls between two composite 1D topological orders are just invertible unitary 1-functors between unitary 1-categories.

2. A 1D domain wall between two simple 2D topological orders given by two unitary fusion 1-categories \( A \) and \( B \) must be an \( A-B \)-bimodule \( \mathcal{E} \) equipped with a unit. \( \boxtimes_{A,B} \) in this case is just the Tambara’s tensor product also denoted by \( \boxtimes_{A,B} \). If \( \mathcal{E} \) is invertible, it implies that \( A \) and \( B \) are Morita equivalent. In particular, it means that we can set \( A = \text{Fun}_\mathcal{C}(M,M), \ B = \text{Fun}_\mathcal{C}(N,N) \) and \( \mathcal{E} = \text{Fun}_\mathcal{C}(M,N) \) for some fusion category \( \mathcal{C} \) and indecomposable semisimple \( \mathcal{C} \)-modules \( M \) and \( N \) (for example, one can simply take \( \mathcal{C} = A \) and \( M = A \)) [ENO08]. The inverse of \( \mathcal{E} \) is \( \mathcal{E}^{\text{op}} \simeq \text{Fun}_\mathcal{C}(N,M) \). Let \( f : M \to N \) be a \( \mathcal{C} \)-module functor. Then the domain wall \( (\text{Fun}_\mathcal{C}(M,N), f) \) is invertible if and only if there is \( g \in \text{Fun}_\mathcal{C}(N,M) \) such that \( g \circ f \simeq \text{id}_M \) and \( f \circ g \simeq \text{id}_N \). In other words, \( f \) must be an invertible functor. In this case, we obtain a unitary monoidal equivalence \( A \xrightarrow{\sim} B \) defined by \( f \circ - \circ f^{-1} \). Conversely, given a monoidal equivalence \( h : A \xrightarrow{\sim} B \), we obtain an obvious invertible domain wall \( _hB \) (viewed as an \( A-B \)-bimodule). It is easy to check that these two maps are inverse of each other. We have proved that an invertible 1D domain wall is equivalent to a unitary monoidal equivalence between unitary fusion 1-categories. Invertible 1D domain walls between two composite 2D topological orders are given by invertible unitary 2-functors.

For higher dimensional cases, we expect the proof to be similar.

**Remark 2.26.** In 3D, it was shown explicitly in Levin-Wen type of lattice models that all braided auto-equivalences between closed bulk-phases can be realized by invertible gapped domain walls [KK], [ENO09].

If the domain wall \( M_{a-1} \) is not invertible, then tunneling process depicted in Fig. 7 still make sense but with the dotted line replaced by \( M^{\text{op}} \boxtimes_{\mathcal{C}} M \simeq \oplus_i N_i \), where \( M^{\text{op}} \boxtimes_{\mathcal{C}} M \) is a composite topological order and \( N_i \) is an indecomposable component of \( M^{\text{op}} \boxtimes_{\mathcal{C}} M \). In this case, there is no real tunneling because the topological excitation, which “tunnel” through the wall, always pull a string (labeled by \( \oplus_i N_i \)) behind it (see [KK] for a discussion of this phenomenon in Levin-Wen models).

**Example 2.27.** We give some examples of domain walls in Levin-Wen models.

1. In toric-code model, any line in the bulk lattice can be viewed as a trivial (obviously invertible/transparent) domain wall. A non-trivial invertible domain wall was constructed in Figure 1 in [KK] (see also the dotted line in Fig. 10). This domain wall also gives the EM duality of the bulk phase.

\(^9\)Note that this invertible module \( M \) gives an algebra isomorphism instead of Morita equivalence in the usual sense because the unit map \( \mathcal{C} \to M \) plays a non-trivial role here.
2. In the generalization of Levin-Wen models constructed in [KK], a gapped domain wall can be constructed from a bimodule over fusion categories. More precisely, if the bulk lattices on the two side of a domain wall are defined by unitary fusion 1-categories \( A \) and \( B \), then the lattice on the domain wall can be defined by a \( A\!-\!B \)-bimodule \( E \). In this case, the bulk excitations on the two sides of the wall are given by the monoidal centers \( Z(A) \) and \( Z(B) \), respectively. The excitations on the \( E \)-wall is given by \( \mathcal{F}\text{un}_{A\!|\!B}(E,E) \). When the bulk excitations move close to the wall, they become wall excitations, this process defines two bulk-to-wall maps:

\[
Z(A) \xrightarrow{L} \mathcal{F}\text{un}_{A\!|\!B}(E,E) \xrightarrow{R} Z(B).
\]

More precisely, \( L \) and \( R \) are two monoidal functors defined by

\[
L: (A \xrightarrow{f} A) \mapsto (E \simeq A \boxtimes_A E \xrightarrow{f \boxtimes_A \text{id}_A} A \boxtimes_A E \simeq E),
\]

\[
R: (B \xleftarrow{g} B) \mapsto (E \simeq E \boxtimes_B B \xrightarrow{\text{id}_E \boxtimes_B g} E \boxtimes_B E \simeq E).
\]

If the bimodule \( E \) is invertible in the sense that \( E \boxtimes_B E^{\text{op}} \simeq A \) and \( E^{\text{op}} \boxtimes_B E \simeq B \). Then both \( L \) and \( R \) are monoidal equivalences. Then the excitations tunneling from the left side to the right side is given by the monoidal equivalence \( R^{-1} \circ L: Z(A) \to Z(B) \). Moreover, \( R^{-1} \circ L \) respects the braiding, i.e. a braided monoidal equivalence. Therefore, an invertible \( A\!-\!B \)-bimodule gives an isomorphism between \( Z(A) \) and \( Z(B) \) [ENO09] [KK].

It is known that 2D domain walls \( \mathcal{E}_2 \), i.e. unitary fusion 1-categories, between two closed phases \( C_3 \) and \( D_3 \), i.e. two unitary non-degenerate braided fusion 1-categories, are classified by Lagrangian algebras in \( C_3 \boxtimes D_3 \), where \( D_3 = D_3^{\text{op}} \) is the same braided fusion 1-category as \( D_3 \) but with the braiding defined by the anti-braiding of \( D_3 \). If the bulk-to-wall maps are \( C_3 \xrightarrow{L} \mathcal{E}_2 \xleftarrow{R} D_3 \), or equivalently, \( C_3 \boxtimes D_3 \xrightarrow{LR} \mathcal{E}_2 \), then the Lagrangian algebra is determined by \( (L \boxtimes R)^{\gamma}(1_\mathcal{E}) \), where \( (L \boxtimes R)^{\gamma} \) is the right adjoint functor of \( L \boxtimes R \) and \( 1_\mathcal{E} \) is the monoidal unit of \( \mathcal{E}_2 \). We expect that the same result holds for higher dimensional cases but with unitary (braided) fusion 1-categories replaced by unitary (braided) fusion \( n \)-categories.

**Remark 2.28.** Gapped domain walls in two closed topological orders \( A_n \) and \( B_n \) can all be obtained from condensations of lower dimensional excitations. When \( A_n = B_n \), such walls are not elementary and should be excluded in the minimal categorical description of the topological order [KW].

**Remark 2.29.** Gapped boundaries, walls or defects in TQFTs have been extensively studied from various perspectives in literature (see for example [FrFRS] [MW] [KS2] [DKR2] [FSV1] [FSV2] [FY] [FS] and references therein).

### 3 The category \( \mathcal{T}\mathcal{O}_n^\text{fun} \) of topological orders

In this section, we first explain the uniqueness of the bulk of a give gapped boundary theory in Sec.3.1. This uniqueness defines the \textbf{bulk}, denoted by \( \mathcal{Z}_n(C_n) \), of a given boundary \( C_n \). Then we use it to define the notion of a morphism between two topological orders of the same dimension in Sec.3.2. We discuss higher morphisms in Sec.3.3.
3.1 The unique-bulk hypothesis

In general, an anomalous \( n \)-D topological order \( C_n \) can always be realized as a defect in a higher dimensional (possibly trivial) topological order. This realization is almost never unique. But we can always reduce it via a process of dimensional reduction to a gapped boundary of a closed \((n+1)\)D topological order (see Fig. 1). We believe that such obtained closed \((n+1)\)D topological order is unique. We denote the unique bulk topological order by \( Z_n(C_n) \) and refer to it as the \textit{bulk} of \( C_n \).

Actually, it is an immediate consequence of Def. 2.1. The key of this proof is that when an anomalous \( n \)-D topological order is realized as a boundary of an \((n+1)\)D topological order, it should remain the same \( n \)-D phase in an arbitrary neighborhood of the boundary (see [KW, Lem. 2]). However, it is unclear if the microscopic definition Def. 2.1 is equivalent to the macroscopic definition by the fusion and braiding properties of its topological excitations. For this reason, we would like to refer to the uniqueness of the \( n \)-D phase in an arbitrary neighborhood of the boundary (see [KW, Lem. 2]) as the \textit{unique-bulk hypothesis}. We would like to provide more evidence of this hypothesis from the macroscopic point of view.

In 2+1D, an anyon system can be described by a unitary modular tensor category (UMTC) (see for example [K12]). A special kind of UMTCs, called “non-chiral”, are given by the monoidal center \( Z(C) \) of a unitary fusion 1-category \( C \). Such an anyon system can be realized as bulk excitations in a Levin-Wen type of lattice model with a gapped boundary such that the topological excitations on the boundary are given by \( C \) [KK]. If a non-degenerate braided fusion category \( D \) is another bulk of the same boundary theory \( C \), then it was shown in [FSV1] that the bulk-to-boundary map, a monoidal functor \( L : D \to C \) factors through \( Z(C) \), i.e. there exists a unique braided monoidal functor \( \tilde{L} : D \to Z(C) \) such that following diagram:

\[
\begin{array}{ccc}
D & \xrightarrow{\exists \tilde{L}} & Z(C) \\
\downarrow{L} & & \downarrow{\text{forget}} \\
\mathcal{C} & & \\
\end{array}
\]

is commutative. Due to the non-degeneracy of braiding in both \( D \) and \( Z(C) \), we must have \( Z(C) = D \boxtimes D' \) [DGNO], where \( D' \) is the centralizer of \( D \) viewed as a full subcategory of \( Z(C) \). This leads to a complete proof if a gapped boundary is a consequence of an anyon condensation.

Before we go there, notice that \( Z(C) = D \boxtimes D' \) implies immediately that if the boundary \( C \) is trivial, i.e. \( C = \mathcal{H}ilb \), then \( D \) must be trivial as well, i.e. \( D = Z(\mathcal{H}ilb) = \mathcal{H}ilb \). This argument seems to generalize to all dimensions. The notion of center is defined by its universal property (see Thm. 4.1), by which, there is a canonical map from any closed bulk \( D_{n+1} \) to a boundary \( C_n \) to the center \( Z(C_n) \) of \( C_n \). This map should preserve all the fusion and braidings. If \( C_n \) is trivial, i.e. \( C_n = 1_n \), then \( Z(C_n) = 1_{n+1} \). Since \( D_{n+1} \) is closed, all excitations except the vacuum have non-trivial braiding with other excitations. The map \( D_{n+1} \to 1_{n+1} \), preserving all the braidings, is impossible unless \( D_{n+1} = 1_{n+1} \).

It is generally believed that a gapped boundary of a topological order \( C_n \) can be obtained by a condensation of \( C_n \) such that the condensed phase is the trivial one \( 1_n \), and the excitations on the gapped boundary are those condensed excitations that has non-trivial braiding with the vacuum in \( 1_n \). In 2+1D, let \( C_3 \), or simply \( C \), be a non-degenerate braided
fusion 1-category. An anyon condensation is determined by a connected commutative separable algebra $A$ in $\mathcal{C}_3$, and the condensed phase is the given by the non-degenerate braided fusion category $\mathcal{C}^{\text{loc}}_A$ of local $A$-modules in $\mathcal{C}$, and the quasi-particles confined on the gapped domain wall by the fusion category $\mathcal{C}_A$ of $A$-modules in $\mathcal{C}$ [K02]. Moreover, we have

$$\mathcal{C} \boxtimes \mathcal{C}^{\text{loc}}_A \cong Z(\mathcal{C}_A).$$

When the condensed phase is trivial, $\mathcal{C}^{\text{loc}}_A = \text{Mod}$, we obtain $\mathcal{C} = Z(\mathcal{C}_A)$. This gives a proof that, in 2+1D, the bulk is uniquely determined by the monoidal center of the gapped boundary theory.

To generalize above results to higher dimensions, we need to develop a theory of condensation in higher dimensions. A possible approach via higher category theory is outlined in [KW]. In particular, we expect that a lot of above results also hold in higher dimensions with unitary braided fusion 1-categories replaced by unitary braided fusion $n$-categories.

From now on, we assume that the bulk is unique. As a special case of (2.8), we have

$$P_n(\mathcal{Z}_n(c_n)) \boxtimes \mathcal{Z}_n(c_n) \cong c_n = c_n.$$

An immediate consequence of the unique-bulk hypothesis is the identity

$$\mathcal{Z}_{n+1}(\mathcal{Z}_n(c_n)) = 1_{n+1},$$

which leads to some interesting complexes and cohomology groups [KW] Sec. IX]. For example, the Witt group of non-degenerate braided fusion categories [DMNO] can be understood as the 3-th cohomology group of a special complex [KW] Sec. IX]. The result given by Eq. (3.1) is somewhat dual to the well-known fact that the boundary of the boundary of a manifold is empty. We hope that it leads to more mathematical results in the future.

### 3.2 A morphism between two topological orders

In mathematics, a morphism of a mathematical structure preserves the structure. It is reasonable that a morphism of between two $n$D topological orders should preserve the universal fusion-braiding properties of topological excitations [KW]. So we expect that such a morphism is given by a unitary $n$-functor preserving the units.

There are some drawbacks of above definition. First, its physical meaning is not evident. Secondly, it depends on the categorical formulation of topological order. Thirdly, it is not convenient to use for our purpose. So in this work, we would like to propose a physical definition which is independent of the mathematical formulation of a topological order. Note that a unitary $n$-functor describes a universal process of mapping excitations in one phase into another. This universal mapping process should be physically realizable. So we would like to define a morphism to be a physical realization of the universal mapping process. One way to achieve it is to let these topological excitations in $\mathcal{C}_n$ to pass a region of spacetime in which the universal mapping process is realized. This spacetime naturally has dimension $n + 1$ or higher. What it suggests is that a morphism can be realized by physics in at least one-dimensional higher.

We first define an isomorphism between two $n$D topological orders $\mathcal{C}_n$ and $\mathcal{D}_n$. 


**Definition 3.1.** An isomorphism \( a : \mathcal{C}_n \rightarrow \mathcal{D}_n \) is an invertible gapped domain wall \( \mathcal{M}_{n-1} \), viewed as an \((n - 1)D\) topological order, between the \( \mathcal{C}_n \)-phase and the \( \mathcal{D}_n \)-phase. We also denote the trivial domain wall \( \mathcal{P}_{n-1}(\mathcal{C}_n) \) in \( \mathcal{C}_n \)-phase by \( \text{id}_{\mathcal{C}_n} \).

**Remark 3.2.** Above definition is completely physical and independent of the categorical definition of a topological order. Viewed in the categorical framework, an isomorphism is an invertible unitary \( n \)-functor preserving the unit (see Section 2.5).

By our definition, an isomorphism \( a \) itself is an \((n-1)D\) domain wall. For this reason, we also use the notation \( a_{n-1} := a \) to remind readers of its dimension. The composition of two isomorphisms \( \mathcal{C}_n \xrightarrow{a} \mathcal{D}_n \xrightarrow{b} \mathcal{E}_n \) is defined by the fusion of two gapped domain walls, i.e.

\[
 b \circ a := b_{n-1} \boxtimes_{\mathcal{D}_n} a_{n-1}.
\]  

(3.2)

Notice that the identity isomorphism indeed behaves like the usual identity maps. The composition of isomorphisms is associative.

**Definition 3.3.** A morphism \( f : \mathcal{C}_n \rightarrow \mathcal{D}_n \) is a pair \((f_n^{(0)}, f_n^{(1)})\) such that

1. \( f_n^{(0)} \) is a gapped domain wall, viewed as an \( nD \) topological order, between two \((n + 1)D\) topological orders \( \mathcal{Z}_n(\mathcal{C}_n) \) and \( \mathcal{Z}_n(\mathcal{D}_n) \),

2. \( f_n^{(1)} : f_n^{(0)} \boxtimes_{\mathcal{Z}_n(\mathcal{C}_n)} \mathcal{C}_n \xrightarrow{\sim} \mathcal{D}_n \) is an invertible domain wall, viewed as an \((n - 1)D\) topological order, between \( f_n^{(0)} \boxtimes_{\mathcal{Z}_n(\mathcal{C}_n)} \mathcal{C}_n \) and \( \mathcal{D}_n \).

The physical configuration associated to this morphism can be depicted as follows:

\[
\begin{array}{c}
\mathcal{Z}_n(\mathcal{D}) \\
\mathcal{D}_n \\
\mathcal{Z}_n(\mathcal{C}) \\
\mathcal{C}_n
\end{array}
\]

(3.3)

Note that we must have \( \mathcal{Z}_n(f_n^{(0)}) = \mathcal{Z}_n(\mathcal{C}_n) \boxtimes \mathcal{Z}_n(\mathcal{D}_n) \). For simplicity, in most parts of this work, we use the following simplified picture

\[
\begin{array}{c}
\mathcal{Z}_n(\mathcal{D}) \\
\mathcal{Z}_n(\mathcal{C}) \\
\mathcal{C}_n
\end{array}
\]

(3.4)

where \( f_n^{(1)} \) is implicit. Such simplified pictorial representation of a morphism will be used later in this work.

**Remark 3.4.** In Def. 3.3 and in (3.4), we chose to put \( \mathcal{Z}_n(\mathcal{C}_n) \) on the left side of \( \mathcal{C}_n \) according to the orientation convention in Levin-Wen type of lattice models (see Fig. 5).

**Remark 3.5.** Although our definition of a morphism between two topological orders comes from our physical intuition, it does remind us of the notion of mapping cylinder in mathematics. This coincidence is perhaps not accidental because the low-energy effective theory of topological order is generally believed to be a TQFT.
Example 3.6. We give a few examples of morphisms that will be used later. Let $C_n$ and $D_n$ be two simple $n$D topological orders.

1. We consider a special isomorphism $id_{C_n} : C_n \to C_n$. It is nothing but the trivial domain wall $id_{C_n}$ in a $C_n$-phase. It can be re-expressed as a morphism $id_{C_n} = (P_n(Z_n(C_n)), id_{C_n})$, where $(id_{C_n})_{n-1}$ is defined by the canonical isomorphism $(id_{C_n})_{n-1} : P_n(Z_n(C_n)) \otimes Z_n(C_n) \cong C_n$.

2. Let $a : C_n \to D_n$ be an isomorphism. It can be re-expressed as a morphism: $a = (P_n(Z_n(C_n)), a)$, where $a_{n-1} : C \otimes Z_n(C_n) \to C_n \cong D_n$. We will show in Prop. 3.16 that isomorphisms are the same as invertible morphisms as expected.

3. There is a unit morphism $\iota_{C_n} : 1_n \to C_n$ defined by $\iota_{C_n} = (C_n, id_{C_n})$.

4. There is a bulk-to-boundary map $Z_n(C_n) \to C_n$. This map can be defined as a morphism $r : P_n(Z_n(C_n)) \to C_n$ as follows:

$$r := (P_n(Z_n(C_n)) \otimes C_n, r^{(1)}_{n-1}).$$

Note that $Z_n(P_n(Z_n(C_n))) = Z_n(C_n) \otimes Z_n(C_n)$. The physical configuration associated to $(P_n(Z_n(C_n)) \otimes C_n) \otimes Z_n(P_n(Z_n(C_n)))$ is depicted in the following picture:

5. A bulk-to-boundary map can be enhanced to an action $Z_n(C_n) \otimes C_n \to C_n$, which can be defined by a morphism $\rho : P_n(Z_n(C_n)) \otimes C_n \to C_n$, which is given by the following pair

$$\rho := (P_n(Z_n(C_n)) \otimes P_n(Z_n(C_n)), r^{(1)}_{n-1}),$$

where $\rho^{(1)}_{n-1}$ is given by

$$(P_n(Z_n(C_n)) \otimes P_n(Z_n(C_n))) \otimes Z_n(C_n) \otimes Z_n(C_n) \otimes Z_n(C_n) \otimes (P_n(Z_n(C_n)) \otimes C_n) = C_n \xrightarrow{id_{C_n}} C_n.$$

The associated physical configuration is depicted in the following picture:

Notice that one can recover the morphism $r$ from $\rho$ by composing $\rho$ with

$$P_n(Z_n(C_n)) = P_n(Z_n(C_n)) \otimes 1_n \xrightarrow{id_{P_n(Z_n(C_n)) \otimes C_n}} P_n(Z_n(C_n)) \otimes C_n.$$
6. It is intuitively clearly that there should be a natural morphism \( f : \mathcal{C}_n \to \mathcal{B}_n \otimes_{A_{n+1}} \mathcal{C}_n \), where \( A_{n+1} \) is not necessarily closed. The definition of \( f \) is shown in the following picture.

\[
\mathcal{Z}_n(\mathcal{D}) \xrightarrow{f_n^{(0)}} \mathcal{Z}_n(\mathcal{C}) \xrightarrow{g_n^{(0)}} \mathcal{E}_n
\]

where \( f_n^{(0)} = \mathcal{B}_n \otimes_{A_{n+1}} P_n(\mathcal{Z}_n(\mathcal{C}_n)) \) and \( \mathcal{D}_n := \mathcal{B}_n \otimes_{A_{n+1}} \mathcal{C}_n \). This morphism can be obtained from a dimensional reduction process depicted in Figure 13.

The composition of morphisms can also be defined.

**Definition 3.7.** Two morphisms \( \mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{g} \mathcal{E} \) can be composed to a morphism \( g \circ f = \mathcal{C} \to \mathcal{E} \) defined by

\[
g \circ f := \left( g_n^{(0)} \otimes \mathcal{Z}_n(\mathcal{D}) \right) f_n^{(0)}, \quad g_n^{(0)} \otimes \mathcal{Z}_n(\mathcal{D}) f_n^{(0)} \otimes \mathcal{Z}_n(\mathcal{C}_n) \mathcal{E}_n \xrightarrow{g_n^{(1)} \circ (\text{id}_n \otimes \mathcal{Z}_n(\mathcal{D})) \circ f_n^{(1)}} \mathcal{E}_n. \tag{3.6}
\]

**Lemma 3.8.** The composition of morphisms is associative and unital, i.e. \( h \circ (g \circ f) = (h \circ g) \circ f \) and \( \text{id}_{\mathcal{E}_n} \circ f = f = f \circ \text{id}_{\mathcal{D}_n} \) for all composable \( f, g, h \).

In above Lemma, the associativity and the unital properties hold on the nose. It amounts to say that we assume the Minimal Assumption on the categorical side. In this case, we obtain a 1-category \( \mathcal{T}_n^{\text{fun}} \) of nD topological orders with 1-morphisms defined by Def 3.3. Notice also that the trivial phase \( \mathbf{1}_n \) is an initial object in \( \mathcal{T}_n^{\text{fun}} \), and the map \( P_{n-1}(-) \) introduced in Sec. 2.4 defines a functor \( P_{n-1} : \mathcal{T}_n^{\text{fun}} \to \mathcal{T}_{n-1}^{\text{fun}} \).

**Remark 3.9.** A morphism can be viewed as a physical realization of a particular universal process of mapping excitations in \( \mathcal{C}_n \) to \( \mathcal{D}_n \). We believe that the set of equivalence classes of physical realizations maps surjectively onto the set of universal mapping processes. Logically, there is no problem if the map is not one-to-one. It is exactly the case if we consider more general physical realizations introduced in Sec. A.3. On the other hand, the notion of a morphism is very special among all physical realizations (see Prop. A.19). Ideally, we hope that the map is indeed bijective. It amounts to show that the identity \( \mathcal{C}_n \otimes \mathcal{Z}_n(\mathcal{C}_n) \mathcal{E}_n \simeq \mathcal{D}_n \) fixes \( \mathcal{E}_n \) up to isomorphisms. In this Remark, we would like to argue physically that this is reasonable. For simplicity, we assume that \( \mathcal{C}_n \) and \( \mathcal{D}_n \) are simple. The idea is to show that the left nD boundary \( \mathcal{C}_n \) of the physical configuration \( \mathcal{C}_n \otimes \mathcal{Z}_n(\mathcal{C}_n) \mathcal{E}_n \) (recall the second picture in (2.1)) can be replaced by \( \mathcal{Z}_n(\mathcal{C}_n) \) by a physical process. Consider the physical meaning of the expression \( \mathcal{C} \otimes \mathcal{C}_n^\text{op} \mathcal{C} \). Note that the subscript \( \mathcal{C}_n \otimes \mathcal{C}_n^\text{op} \mathcal{C} \) is a two-layered nD systems and depicted as the upper/lower semi-circle in the first figure in Fig. 8. If the expression \( \mathcal{C} \otimes \mathcal{C}_n^\text{op} \mathcal{C} \) indeed has a physical meaning, then \( \mathcal{C} \) must be viewed as two \( (n - 1) \)D phases (recall Remark 2.19) and depicted as the two dark “points” in the first figure in Fig. 8.

Then the expression \( \mathcal{C} \otimes \mathcal{C}_n^\text{op} \mathcal{C} \) can be viewed as the boundary of the hole in the first figure in Fig. 8. Moreover, this hole embedded in the \( \mathcal{Z}_n(\mathcal{C}_n) \)-phase should be viewed as an excitation of codimension 2 in the \( \mathcal{Z}_n(\mathcal{C}_n) \)-phase and can be absorbed by the boundary \( \mathcal{E}_n \) (see the second figure in Fig. 8). Equivalently, regarding a neighborhood of the hole as the \( (n - 2) \)D topological order \( P_{n-2}(\mathcal{Z}_n(\mathcal{C}_n)) = \mathcal{Z}_n(\mathcal{C}_n) \) (recall Remark 2.19), above arguments suggest the following identities:

\[
\mathcal{C} \otimes \mathcal{C}_n^\text{op} \mathcal{C} \simeq \mathcal{C}_n^\text{op} \otimes \mathcal{C}_n \mathcal{E}_n, \quad \mathcal{C} \otimes \mathcal{Z}_n(\mathcal{C}_n) \mathcal{E}_n \simeq \mathcal{Z}_n(\mathcal{C}_n) \otimes \mathcal{Z}_n(\mathcal{C}_n) \mathcal{E}_n = \mathcal{E}_n. \tag{3.7}
\]
Figure 8: Above two figures illustrate the physical interpretations of $\mathcal{C}_{\mathcal{C} \otimes \mathcal{C}^\text{op}} \mathcal{C}$ and $\mathcal{C}_{\mathcal{C} \otimes \mathcal{C}^\text{op}} \mathcal{C} \otimes Z_n(\mathcal{C}_n) \mathcal{E}_n$, respectively. They are explained in Remark 3.9.

where we have used the fact that $\mathcal{C}^\text{op} \simeq \mathcal{C}$ as $\mathcal{C}_n$-bimodules due to the unitarity. This implies the uniqueness of $\mathcal{E}_n$. The evidence of above arguments can be found in the case $n = 2$. When $n = 2$, $\mathcal{C}_2$ is a unitary fusion 1-category, and $Z_2(\mathcal{C}_2)$ is the monoidal center of $\mathcal{C}_2$, and we have $\mathcal{C}^\text{op} \simeq \mathcal{C}^\text{rev}$ as unitary fusion 1-categories. Then the category $\mathcal{C}_{\mathcal{C} \otimes \mathcal{C}^\text{rev}} \mathcal{C}$, usually called Hochschild homology, coincides with the monoidal center $Z(\mathcal{C}) = \mathcal{C}^\text{op} \otimes_{\mathcal{C}^\text{rev}} \mathcal{C}$ of $\mathcal{C}$. So the identity (3.7) holds precisely in this case. More details will be given elsewhere.

We expect that the same result holds for unitary fusion $n$-categories.

The following result follows from Def. 3.3 immediately.

**Theorem 3.10.** Let $f : \mathcal{C}_n \to \mathcal{D}_n$ be a morphism. If $\mathcal{C}_n$ is closed, then $\mathcal{D}_n = \mathcal{C}_n \boxtimes \mathcal{E}_n$ for some $nD$ topological order $\mathcal{E}_n$. If $\mathcal{D}_n$ is also closed, so is $\mathcal{E}_n$.

**Remark 3.11.** For simple (unstable) 1D topological order, Thm 3.10 give a classical result. More precisely, if there is an algebraic homomorphism $f : M_{m \times m} \to M_{n \times n}$, then we must have $M_{n \times n} \simeq M_{m \times m} \otimes_{\mathcal{C}} M_{k \times k}$, where $n = mk$ and the subscript $\mathcal{C}$ of $\otimes$ should be viewed as the usual center of the algebra $M_{m \times m}$.

**Remark 3.12.** When $n = 3$, a closed 3D topological order can be described by a unitary non-degenerate braided fusion category $\mathcal{C}_3$. We believe that a morphism between two such topological orders $\mathcal{C}_3$ and $\mathcal{D}_3$ is equivalent to a braided monoidal functor $f : \mathcal{C}_3 \to \mathcal{D}_3$ preserving the units. It is known that $f$ is fully-faithful and $\mathcal{D}_3 \simeq \mathcal{C}_3 \boxtimes \mathcal{C}_3'$, where $\mathcal{C}_3'$ is the centralizer of $\mathcal{C}_3$ which is the full subcategory of $\mathcal{D}_3$ consisting of all objects $x$ such that $c_{y,x} \circ c_{x,y} = \text{id}_x \otimes y$ for all $y \in \mathcal{D}_3$ [M3, DGNO]. So Thm 3.10 can be viewed as a physical generalization of the above mathematical result for non-degenerate braided fusion 1-categories and its higher dimensional generalizations.

### 3.3 Higher morphisms and the $(n, 1)$-category $\mathcal{T}_n^{\text{fun}}$

One can also introduce the notion of a 2-isomorphism between two morphisms. We need it only in Sec 4.3 and Sec A.2.

**Definition 3.13.** Let $f, g : \mathcal{C}_n \to \mathcal{D}_n$ be two morphisms. A 2-isomorphism $\phi : f \Rightarrow g$ is a pair $(\phi^{(0)}, \phi^{(1)})$ such that...
1. \( \phi_{n-1}^{(0)} : f_n^{(0)} \rightarrow g_n^{(0)} \) is an \((n - 1)D\) invertible closed gapped domain wall between \(f_n^{(0)}\) and \(g_n^{(0)}\), both of which are domain walls between \(Z_n(\mathcal{C}_n)\) and \(Z_n(\mathcal{D}_n)\).

2. \( \phi_{n-2}^{(1)} : g_{n-1}^{(1)} \circ (\phi_{n-1}^{(0)} \boxtimes Z_n(\mathcal{C}_n) \text{id}_{\mathcal{C}_n}) \rightarrow f_{n-1}^{(1)} \) is an \((n - 2)D\) invertible gapped domain wall between two associated \((n - 1)D\) domain walls.

![Diagram](image)

In this case, we denote \( f \simeq g \).

**Example 3.14.** Every morphism \( f : \mathbb{1} \rightarrow \mathcal{C}_n \) is canonically 2-isomorphic to the unit morphism. Actually, the pair \((\Phi_{n-1}^{(1)})^{-1}, \text{id}_{\mathcal{C}_n}\) defines a 2-isomorphism \( \iota_f : \iota_{\mathcal{C}_n} \Rightarrow f \).

**Remark 3.15.** In general, two morphisms \( f \) and \( g \) are not isomorphic, even if \( f_n^{(0)} \simeq g_n^{(0)} \). However, if \( f_n^{(0)} \simeq g_n^{(0)} \), then there always exists \( f_{n-1}^{(1)} \) and \( g_{n-1}^{(1)} \) such that the diagram (3.8) is commutative, i.e. \( f \simeq g \). For example, one can simply define \( f_{n-1}^{(1)} := g_{n-1}^{(1)} \circ (\phi_{n-1}^{(0)} \boxtimes Z_n(\mathcal{C}_n) \text{id}_{\mathcal{C}_n}) \).

**Proposition 3.16.** A morphism \( f : \mathcal{C}_n \rightarrow \mathcal{D}_n \) is identical to an isomorphism if and only if there is a morphism \( g : \mathcal{D}_n \rightarrow \mathcal{C}_n \) such that \( g \circ f \simeq \text{id}_{\mathcal{C}_n} \) and \( f \circ g \simeq \text{id}_{\mathcal{D}_n} \).

**Proof.** Only the sufficiency needs to be proved. Suppose \( g \circ f \simeq \text{id}_{\mathcal{C}_n} \) and \( f \circ g \simeq \text{id}_{\mathcal{D}_n} \). Then \( f^{(0)} \) is an invertible domain wall between \( Z_n(\mathcal{D}_n) \) and \( Z_n(\mathcal{C}_n) \). It is just a matter of convention how we identify \( Z_n(\mathcal{D}_n) \) with \( Z_n(\mathcal{C}_n) \). Using \( f^{(0)} \), we can identify \( Z_n(\mathcal{D}_n) \) with \( Z_n(\mathcal{C}_n) \), then \( f^{(1)} \) can be viewed as a domain wall between \( \mathcal{D}_n \) and \( \mathcal{C}_n \) (recall Def 5.2). Moreover, it is invertible. So \( f \) is nothing but the isomorphism \( f^{(1)} \).

By induction on dimensions, we define a \( k \)-isomorphism \( \phi \xrightarrow{(k)} \psi \) between two \((k - 1)\)-isomorphisms \( \phi, \psi : f \xrightarrow{(k-1)} g \) as a pair \((\Phi_{n-k+1}^{(0)}, \Phi_{n-k}^{(1)})\) of \((n - k + 1)D\) and \((n - k)D\) invertible gapped domain walls between associated domain walls. In this way, we obtain an \((n + 1, 1)\)-category of \( nD \) topological orders, denoted still by \( \mathcal{T}\mathcal{O}_n^{\text{fun}} \). An \((m, 1)\)-category is an \( m \)-category with only invertible \( k \)-morphisms for \( k > 1 \).

## 4 The universal property of the bulk

In this section, we first prove that the bulk satisfies the same universal property as that of the center (of an algebra) in mathematics, then we explain why the universal property leads to the usual notion of the center, at last, we show that the morphism defined in Def 3.3 coincides with the usual notions of morphisms in mathematics by assuming bulk = center.
4.1 The universal property of the bulk

The action $\rho : P_n(\mathcal{Z}_n(\mathcal{C}_n)) \boxtimes \mathcal{C}_n \to \mathcal{C}_n$ is unital, i.e. $\rho \circ (\iota_{\mathcal{Z}_n(\mathcal{C}_n)} \boxtimes \text{id}_{\mathcal{C}_n}) = \text{id}_{\mathcal{C}_n}$, which is equivalent to the commutativity of the following diagram:

\[
P_n(\mathcal{Z}_n(\mathcal{C}_n)) \boxtimes \mathcal{C}_n \xrightarrow{\rho} \mathcal{C}_n \xleftarrow{\text{id}_{\mathcal{C}_n}} \mathcal{C}_n.
\]

We would like to show that the pair $(P_n(\mathcal{Z}_n(\mathcal{C}_n)), \rho)$ satisfies the universal property of center that determines the pair up to unique isomorphism. As a consequence, $\mathcal{Z}_n(\mathcal{C}_n) = (P_n(\mathcal{Z}_n(\mathcal{C}_n)), \rho)$.

**Theorem 4.1.** The pair $(P_n(\mathcal{Z}_n(\mathcal{C}_n)), \rho)$ satisfies the following universal property of center. Let $(\mathcal{X}_n, f)$ be another such a pair. In other words, $\mathcal{X}_n$ is an nD topological order and $f : \mathcal{X}_n \boxtimes \mathcal{C}_n \to \mathcal{C}_n$ a morphism such that the following diagram

\[
\mathcal{X}_n \boxtimes \mathcal{C}_n \xrightarrow{\rho} \mathcal{C}_n \xleftarrow{\text{id}_{\mathcal{C}_n}} \mathcal{C}_n.
\]

is commutative. Then there is a unique morphism $f : \mathcal{X}_n \to P_n(\mathcal{Z}_n(\mathcal{C}_n))$ such that the following two diagrams

\[
\begin{align*}
\mathcal{X}_n \xrightarrow{\iota_{\mathcal{X}_n} \boxtimes \text{id}_{\mathcal{C}_n}} & \xrightarrow{f} \mathcal{C}_n \\
\mathcal{C}_n & \xrightarrow{\iota_{\mathcal{C}_n}} \mathcal{C}_n
\end{align*}
\]

\[
\begin{align*}
P_n(\mathcal{Z}_n(\mathcal{C}_n)) \boxtimes \mathcal{C}_n \xrightarrow{\iota_{P_n(\mathcal{Z}_n(\mathcal{C}_n))} \boxtimes \text{id}_{\mathcal{C}_n}} & \xrightarrow{f \boxtimes \text{id}_{\mathcal{C}_n}} \mathcal{C}_n \\
\mathcal{C}_n & \xrightarrow{\iota_{\mathcal{C}_n}} \mathcal{C}_n
\end{align*}
\]

are commutative. The notation “$\exists !$” means “exists a unique”.

**Proof.** The physical configuration associated to the composed morphism $f \circ (\iota_{\mathcal{X}_n} \boxtimes \text{id}_{\mathcal{C}_n})$ is depicted in the following picture.

\[
\begin{align*}
\mathcal{Z}_n(\mathcal{X}_n) \xrightarrow{f^{(0)}} & \mathcal{Z}_n(\mathcal{C}_n) \\
\mathcal{Z}_n(\mathcal{C}_n) & \xrightarrow{f^{(0)}} \mathcal{C}_n
\end{align*}
\]

So, the commutativity of (4.2) amounts to the identity $\mathcal{X}_n \boxtimes \mathcal{Z}_n(\mathcal{X}_n) f^{(0)} = \text{id}_{\mathcal{C}_n} = P_n(\mathcal{Z}_n(\mathcal{C}_n))$. Notice that the pair $(f^{(0)}, \text{id}_{P_n(\mathcal{Z}_n(\mathcal{C}_n))})$ defines a morphism $f : \mathcal{X}_n \to P_n(\mathcal{Z}_n(\mathcal{C}_n))$. Clearly, we have $\iota_{P_n(\mathcal{Z}_n(\mathcal{C}_n))} = f \circ \iota_{\mathcal{X}_n}$. 

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Now we draw the physical configuration associated to the composed morphism $\rho \circ (f \boxtimes \text{id}_{C_n})$ as follows:

$$
\begin{array}{ccccccc}
\mathcal{X}_n & \xleftarrow{f_n(0)} & \mathcal{Z}_n(C_n) & \xrightarrow{\rho_n} & \mathcal{Z}_n(C_n) & \xrightarrow{\text{id}_{C_n}} & \mathcal{C}_n \\
\mathcal{X}_n & \xleftarrow{f_n(0)} & \mathcal{Z}_n(C_n) & \xrightarrow{\rho_n} & \mathcal{Z}_n(C_n) & \xrightarrow{\text{id}_{C_n}} & \mathcal{C}_n \\
\mathcal{X}_n & \xleftarrow{f_n(0)} & \mathcal{Z}_n(C_n) & \xrightarrow{\rho_n} & \mathcal{Z}_n(C_n) & \xrightarrow{\text{id}_{C_n}} & \mathcal{C}_n \\
\mathcal{X}_n & \xleftarrow{f_n(0)} & \mathcal{Z}_n(C_n) & \xrightarrow{\rho_n} & \mathcal{Z}_n(C_n) & \xrightarrow{\text{id}_{C_n}} & \mathcal{C}_n \\
\mathcal{X}_n & \xleftarrow{f_n(0)} & \mathcal{Z}_n(C_n) & \xrightarrow{\rho_n} & \mathcal{Z}_n(C_n) & \xrightarrow{\text{id}_{C_n}} & \mathcal{C}_n \\
\end{array}
$$

(4.5)

where three green “dots” are all labeled by $P_n(C_n)$. By deforming the pictures topologically, we see that $\rho \circ (f \boxtimes \text{id}_{C_n}) = f$. This proves the existence of $f$.

It remains to prove the uniqueness of such $f$. Assume a morphism $g : \mathcal{X}_n \to P_n(C_n)$ satisfies $\iota_{P_n(C_n)} = g \circ \iota_{\mathcal{X}_n}$ and $\rho \circ (g \boxtimes \text{id}_{C_n}) = f$. From the former identity, we obtain the identities $\mathcal{X}_n \boxtimes g(0) = P_n(C_n)$ and $g(1) = \text{id}_{P_n(C_n)}$; from the latter, we obtain $g(0) = f$. That is, $g = f$.

**Remark 4.2.** In Thm 4.1, we have used only strictly commutative diagrams by the Minimal Assumption (see Remark 2.15). If we want to relax this condition by considering higher isomorphisms, the universal property of the center is much more complicated [L3]. We briefly discuss it in Sec. A.2.

### 4.2 The universal property and the usual notion of center

The universal property stated in Thm 4.1 is exactly the universal property of the notion of center in mathematics. This universal property defines the center up to isomorphisms. In this subsection, we try to explain this well-known result to physical-minded readers by assuming that our definition of a morphism coincides with the usual notions of morphism.

We first explain the notion of center for an ordinary algebra over $\mathbb{C}$. Let $A$ be an algebra over $\mathbb{C}$. The center of $A$ is usually defined by

$$Z(A) := \{ z \in A | za = az, \forall a \in A \},$$

(4.6)

which is a (commutative) subalgebra of $A$. But this notion can be redefined by its universal property. Consider another algebra $B$. The tensor product $B \otimes A$ has a natural algebra structure. If there is an algebra homomorphism $f : B \otimes A \to A$, then $B$ must satisfy some special property. For example, the obvious action $m : A \otimes A \to A$, defined by $m(a \otimes b) = ab$, is not an algebra map. To see this, take $1 \otimes a, b \otimes 1 \in A \otimes A$. On the one hand, we have

$$m((1 \otimes a) \cdot (b \otimes 1)) = m(b \otimes a) = ba.$$

(4.7)

On the other hand,

$$m(1 \otimes a) \cdot m(b \otimes 1) = ab.$$

(4.8)

Therefore, the multiplication map $m : A \otimes A \to A$ is not an algebra map if $A$ is not commutative. However, $Z(A) \otimes A \xrightarrow{m} A$ is an algebra homomorphism. Moreover, the
The following diagram:

\[
\begin{array}{ccc}
Z(A) \otimes A & \xleftarrow{\iota_{Z(A)} \otimes \text{id}_A} & A \\
\downarrow{m} & & \downarrow{\text{id}_A} \\
A & \rightarrow & A
\end{array}
\]

where \(\iota_{Z(A)} : C \rightarrow Z(A)\) is the unit map, is commutative. The pair \((Z(A), Z(A) \otimes A, m)\) satisfies the following so-called universal property: if \(B\) is an algebra and \(f : B \otimes A \rightarrow A\) is a unital action and an algebra map, there is a unique algebra map \(\hat{f} : B \rightarrow Z(A)\) such that the following diagram:

\[
\begin{array}{ccc}
Z(A) \otimes A & \xrightarrow{f \otimes 1} & B \otimes A \\
\downarrow{m} & & \downarrow{f} \\
A & \rightarrow & A
\end{array}
\]

is commutative. This is true because the restriction \(g := f(-, 1) : B \rightarrow A\) is an algebra map, \(f(1 \otimes a) = a\), and we have

\[
f(b \otimes a) = f((b \otimes 1) \cdot (1 \otimes a)) = g(b) \cdot f(1 \otimes a) = g(b)a
\]

\[
= f((1 \otimes a) \cdot (b \otimes 1)) = ag(b),
\]

i.e. \(g(b) \in Z(A)\). Therefore, \(g : B \rightarrow Z(A)\) is an algebra map. If we set \(\hat{f} = g : B \rightarrow Z(A)\), then the diagram (4.10) is commutative. The uniqueness of \(\hat{f}\) is obvious.

Conversely, the universal property determines the center up to canonical isomorphisms. Namely, any algebra \(Z\), satisfying the universal property, is canonically isomorphic to the algebra \(Z(A)\) defined by Eq. (4.6).

The universal property also implies that there is a canonical isomorphism

\[
\text{hom}(B \otimes C, A) \cong \text{hom}(B, Z(A)),
\]

where both “homs” are sets of algebra homomorphisms. It provides a useful characterization of the center \(Z(A)\) as an internal hom. Another useful characterization of \(Z(A)\) is the set of \(A\)-\(A\)-bimodule maps from \(A\) to \(A\), i.e. \(Z(A) = \text{hom}_A(A, A)\).

The notion of center can be defined in very general context. Jacob Lurie introduced the notion of center for an \(E_k\)-algebra object in a symmetric monoidal \(\infty\)-category \([13]\). What we need in the study of topological orders are some special cases of Lurie’s general notion.

A 0D topological order \(C_0\) is just a Hilbert space equipped with a unit map \(C \rightarrow C_0\). This can be viewed as an \(E_0\)-algebra in \(\mathcal{Hilb}\) \([13]\). In this case, the \(\otimes\) is just usual tensor product of Hilbert spaces. We have

\[
\text{hom}(X_0 \otimes C_0, C_0) \cong \text{hom}(X_0, \text{hom}_{C}(C_0, C_0)),
\]

where both sides are sets of \(E_0\)-algebra homomorphisms, i.e. linear maps that preserves the units. Therefore, we have the center \(Z(C_0) = \text{hom}_C(C_0, C_0)\), which describes an unstable phase if \(\text{dim} C_0 > 1\), which can flow to the trivial phase \(1_1\).

A 1D topological order \(C_1\) is a unitary 1-category, which can be viewed as a module over \(\mathcal{Hilb}\), equipped with a unit functor \(\mathcal{Hilb} \rightarrow C_1\). It is again an \(E_0\)-algebra but in the
symmetric monoidal 2-category of unitary 1-categories. In this case, all arrows in diagrams (1.2) and (1.3) are unitary functors that preserve the unit, and \( \boxtimes \) is the Tambara’s tensor product over the unitary fusion category \( \mathcal{H}ilb \). Again, we have

\[
\mathcal{F}un(X_1 \boxtimes C_1, C_1) \cong \mathcal{F}un(X_1, \mathcal{F}un(C_1, C_1)),
\]

where both sides are categories of unitary 1-functors preserving the unit. Therefore, we must have \( Z(C_1) = \mathcal{F}un(C_1, C_1) \). As a special case, in Figure 5 (b), we have \( C = Z(\mathcal{F}un_C(M, N)) \).

In other words, the boundary-bulk relation also holds in this case. But \( \mathcal{F}un(C_1, C_1) \) is not a unitary fusion 1-category if \( C_1 \) has more than one simple objects. It describes an unstable phase, which can flow to the trivial phase \( 1_2 \).

A composite \( n \)-D topological order \( C_n \) is a unitary \( n \)-category with a unit, i.e. an \( E_0 \)-algebra. Its center \( Z(C_n) \) is given by the category of unitary \( n \)-functors \( \mathcal{F}un(C_n, C_n) \), which is a monoidal \( n \)-category. It describes an unstable phase in general such as \( C \) in Fig. 5 (b).

A simple 2D topological order is given by a unitary fusion 1-category, a monoidal 1-category with additional structures and properties. Let \( C \) be a monoidal 1-categories. If the pair \( (Z(C), m) \) satisfies the universal property in Thm. 4.1, i.e.

\[
\exists f \times id_x \rightarrow Z(C) \times C
\]

where \( D \) is a monoidal 1-category and \( f \) is a monoidal 1-functor, then \( g = f(\boxtimes 1_C) : D \rightarrow C \) defines a monoidal functor. It requires that each object \( g(d) \in C \) acquires a half braiding, i.e. an isomorphism \( x \otimes g(d) \xrightarrow{c_{x,g(d)}} g(d) \otimes x \) for all \( x \in C \), satisfying some natural properties. Therefore, the center \( Z(C) \) of \( C \) is given by the category of all pairs \( (x, c_{x,-}) \) for \( x \in C \). The uniqueness is obvious by this definition (recall Remark 2.15). \( Z(C) \) is naturally braided with the braiding defined by the half-braidings. Equivalently, the center \( Z(C) \) of \( C \) can be defined by \( Z(C) := \mathcal{F}un_{C,C}(\mathcal{C}, C) \), where \( \mathcal{F}un_{C,C}(\mathcal{C}, C) \) is the category of bimodule functors. In this case, the functor \( m \) is given the evaluation functor \( (F, x) \mapsto F(x) \).

A simple closed 3D topological order is given by a non-degenerate unitary braided fusion 1-category. Let \( C \) be a braided monoidal 1-category. We replace all categories in (1.3) by braided monoidal categories and all arrows by braided monoidal functors, then it is easy to show that the image of the functor \( g = f(\boxtimes 1_C) : D \rightarrow C \) consists of object \( x \in C \) that are symmetric with all objects in \( C \), more explicitly, the double braiding \( x \otimes y \xrightarrow{c_{x,y}} y \otimes x \) is trivial, i.e. \( c_{y,x} \circ c_{x,y} = id_{x \otimes y} \). Such objects \( x \) define a full subcategory \( C' \) of \( C \), sometimes called centralizer of \( C \). Then it is clear that \( Z(C) = C' \). Therefore, if \( C \) is non-degenerate, then \( Z(C) \) is trivial; if \( C \) is not non-degenerate, then \( Z(C) = C' \) is a symmetric monoidal category. Symmetric monoidal category can not describe any non-trivial closed topological phases because its braidings can not detect any non-trivial excitations. Therefore, an anomalous 3D topological order can not be described by a unitary braided fusion 1-category, even though there is no non-trivial string-like excitations. We must use unitary fusion 2-category instead. The notion of center for these two descriptions are different (see Remark 4.5). The center of a unitary fusion 2-category with a unique simple object might contain simple objects other than the tensor unit.
Remark 4.3. All above examples are special cases of a more general principle: if $C_n$ is an $E_k$-algebra, then its center $Z(C_n)$ is automatically an $E_{n+1}$-algebra. It was first formulated by Kontsevich as the generalized Deligne conjecture, and was later proved by many people (see for example [L3]). For example, a vector space equipped with a distinguished vector and a category with a distinguished object can both be viewed as $E_0$-algebras (in different categories), their centers are $E_1$-algebras. A $C$-algebra and a monoidal category can both be viewed as $E_1$-algebras (in different categories), their centers are $E_2$-algebras. A braided monoidal category is an $E_2$-algebra. Its center is a symmetric monoidal category, an $E_3$-algebra and also an $E_\infty$-algebra automatically. The center (actually $E_n$-center for $n \geq 3$) of a symmetric monoidal category $\mathcal{D}$ remains to be $\mathcal{D}$. This stabilization phenomenon does not contradict to the statement that the bulk of a bulk is trivial (see Remark 4.4).

Remark 4.4. If a simple 3D topological order $C_3$ is closed, i.e. a unitary non-degenerate braided fusion 1-category, its center is trivial, i.e. $Z(C_3) = 1_4$. If $C_3$ is anomalous, it can not be described by a unitary braided fusion 1-category. Otherwise, its center gives a non-trivial symmetric tensor category. But this is impossible because particles in this symmetric tensor category can not detect each other, contradicting to the fact that the bulk of $C_3$ is closed. There must be some non-trivial string-like excitations in the bulk of $C_3$. For example, in the $Z_n$ gauge theory in 3+1D (see for example [WW2]), all string-like excitations are mutually symmetric but have non-trivial braidings with particle-like excitations. It is possible to have all string-like excitations condensed and create an anomalous 3D topological order $C_3$ on the boundary with only particle-like excitations. In this case, the anomalous topological order $C_3$ has to be a unitary fusion 2-category with a non-trivial center even if it has a unique simple object given by the tensor unit $1^\mathbb{C}$. The center of the unitary fusion 2-category $C_3$ and that of $\text{hom}(1^\mathbb{C}, 1^\mathbb{C})$ are different. Their relation is explained in Remark 4.5. From this discussion, one can see that an $n$D topological order is different from an $E_n$-algebra, which describes only particle-like excitations. For topological orders, the non-degeneracy of the braidings in $Z_n(C_n)$ leads to a even stronger stability $Z_{n+1}(Z_n(C_n)) = 1_{n+2}$.

For a simple $n$D topological order $C_n$, i.e. a unitary fusion $(n-1)$-category, the center $Z(C_n)$ is given by

$$Z(C_n) = \mathcal{F}\text{un}_{\mathbb{C}|\mathbb{C}}(\mathbb{C}, \mathbb{C})$$

where the right hand side is the category of $\mathbb{C}$-$\mathbb{C}$-bimodule functors (see also [BN] for the center of a monoidal 2-category and [L3] for general situations). There is a natural evaluation functor $ev : Z(C_n) \otimes C_n \to C_n$ defined by $F \otimes x \to F(x)$, which is monoidal. In particular, the pair $(Z(C_n), ev)$ satisfies the universal property of center:

$$Z(C_n) \otimes C_n \xymatrix{ & \mathbb{C}_n \ar@{<-}[dl]_f \ar@{<-}[dr]^{ev} \\ \mathbb{C}_n \otimes C_n \ar[urr]^{\exists ! F \otimes \text{id}_{C_n}}}$$

for any unitary fusion $(n-1)$-category $\mathcal{X}$ and a unitary monoidal $(n-1)$-functor $f$. If $C_n$ is simple and closed, it is given by a unitary braided $(n-1)$-category with one simple object. Its center is defined by Eq. (4.14). Alternatively, one can define its center as the centralizer

\footnote{Even in this case, the center of $\mathcal{C}_3$ might have more than one simple objects.}
of \( \mathcal{C}_n \) viewed as braided monoidal \((n-2)\)-category (similar to that of a braided monoidal 1-category), i.e. the full \((n-2)\)-subcategory of \( \mathcal{C}_n \) consisting of objects that are symmetric to all objects in \( \mathcal{C}_n \). The meaning of “symmetric” in higher categories was explained in [KW] in semi-mathematical terms. Above two definitions should be compatible (see Remark 4.5).

**Remark 4.5.** This remark is due to Jacob Lurie. There is a notion of center for an \( E_k \)-algebra object in a symmetric monoidal \( \infty \)-category ([L3, Sec. 5.3.1]). When we take the \( \infty \)-category to be the \( \infty \)-category of \((\infty,n)\)-categories with some additional structures, we get a notion of center for an \( E_k \)-monoidal \((\infty,n)\)-category. This center is automatically an \( E_{k+1} \)-monoidal \((\infty,n)\)-category and given by Eq. (4.14) for \( k = 1 \), and by the same formula for \( k > 1 \) but with the bimodule replaced by \( E_k \)-module. If \( \mathcal{C} \) is an \( E_k \)-monoidal \((\infty,n)\)-category, then \( \text{hom}_\mathcal{C}(1,1) \), where \( 1 \in \mathcal{C} \) is the tensor unit of \( \mathcal{C} \), is an \( E_{k+1} \)-monoidal \((\infty,n-1)\)-category. Using the results proven in Sec. 4.8 in [L3], one can show that

\[
Z(\text{hom}_\mathcal{C}(1,1)) \simeq \text{hom}_{Z(\mathcal{C})}(1,Z(\mathcal{C}))
\]

as \( E_{k+2} \)-monoidal \((\infty,n-1)\)-categories. In general, \( Z(\text{hom}_\mathcal{C}(1,1)) \) is not enough to recover the \( E_{k+1} \)-monoidal \((\infty,n)\)-category \( Z(\mathcal{C}) \).

In a summary, we can conclude that the **bulk** is indeed the center in the mathematical sense, i.e. \( Z_n(-) = Z(-) \), under the assumption that the notion of a morphism between two topological orders of the same dimension coincides with that of a morphism in various mathematical situations. Then Eq. (3.1) leads to the following mathematical conjecture.

**Conjecture 4.6.** For any unitary fusion \((n-1)\)-category \( \mathcal{C}_n \), we have \( Z(Z(\mathcal{C}_n)) = 1_{n+2} \).

This conjecture is true for \( n = 2 \) because \( Z(\mathcal{C}_2) \) is actually a modular tensor category which has a trivial centralizer [M2]. In this case, the unitary condition is not necessary. We suspect that Conj. 4.6 is true without unitarity assumption but with certain duality and finiteness assumptions.

### 4.3 Physical morphisms coincide with mathematical ones

In this subsection, using the categorical definition of a \( n \)D topological order \( \mathcal{C}_n \), we show in a few low dimensional cases that Def. 3.3 coincide with the notion of a unitary \( n \)-functor with the assumption that the **bulk** \( Z_n(\mathcal{C}_n) \) is given by the center of \( \mathcal{C}_n \).

In 0D, let \((U,u)\) and \((V,v)\) be two 0D topological orders. Namely, \( U \) and \( V \) are finite dimensional Hilbert spaces and \( u \in U, v \in V \). \( \otimes \) is just the usual Hilbert space tensor product \( \otimes_\mathcal{C} \) and the center of \( U \) is the matrix algebra \( \text{End}(U) \). A linear map \( f : U \to V \) from \( U \) to \( V \) preserving units can be realized by an 0D topological order \((\text{hom}_\mathcal{C}(U,V),f)\) together with the isomorphism

\[
\text{hom}_\mathcal{C}(U,V) \otimes_{\text{End}(U)} U \xrightarrow{\sim} V
\]

(4.16)

defined by the evaluation map \( g \otimes_{\text{End}(U)} u_1 \mapsto g(u_1) \). Conversely, a domain wall between \( \text{End}(V) \) and \( \text{End}(U) \) is an \( \text{End}(V) \)-End(\(U\))-module, which has to be a direct sum of \( \text{hom}_\mathcal{C}(U,V) \). Therefore, \( \text{hom}_\mathcal{C}(U,V) \) is the unique domain wall such that the isomorphism (4.16) is possible. Therefore, there is a bijective correspondence between our physical morphism between 0D topological orders and linear maps preserving units.
In 1D, for simple 1D topological orders, an algebra homomorphism \( f : A \to B \) between two matrix algebras \( A \) and \( B \) is only possible if \( B = C \otimes C A \) for another matrix algebra \( C \) (see Remark 3.11), where \( C \) should be viewed as the center of \( A \). It coincides with our physical definition of a morphism between \( A \) and \( B \). For composite 1D topological orders, it is entirely similar to the 0D cases. In particular, the center of a composite 1D topological order \((A, a)\) is given by the 1-category of unitary 1-functors \( \mathcal{F}\text{un}(A, A) \) with the usual unit, the identity functor \( \text{id}_A \). A unitary 1-functor \( f \) between two unitary 1-categories \( \mathcal{C} \) and \( \mathcal{D} \) preserving the units can be realized by a composite 1D topological order \((\mathcal{F}\text{un}(\mathcal{C}), \mathcal{D}), f)\) together with the equivalence

\[
\mathcal{F}\text{un}(\mathcal{C}, \mathcal{D}) \cong \mathcal{F}\text{un}(\mathcal{C}, \mathcal{C}) \cong \mathcal{D}
\]

defined again by the evaluation map \( g \otimes_{\mathcal{F}\text{un}(\mathcal{C}, \mathcal{C})} c \to g(c) \) for \( c \in \mathcal{C} \). One can check easily that this map is bijective. Therefore, our physical definition of a morphism is equivalent to the usual notion of a monoidal functor.

In 2D, a simple 2D topological order can be described by a unitary fusion 1-category \( \mathcal{C} \) (or \( \mathcal{C}_2 \) when it is viewed as a unitary 2-category with a unique simple object). In this case, it was proved that the \textit{bulk} of \( \mathcal{C} \) is indeed given by the monoidal center \( Z(\mathcal{C}) \) of \( \mathcal{C} \) \cite{Ko2}, i.e. \( Z_2(\mathcal{C}) = Z(\mathcal{C}) \). Using this fact, we obtain a notion of a morphism from Def. 3.3. We would like to check if this notion is equivalent to usual notion of unitary monoidal 1-functors.

Before we proceed, we would like to set a new convention to make our presentation easier. Recall that the excitations on the \( M \)-boundary in a Levin-Wen type of lattice model is given by \( \mathcal{F}\text{un}(\mathcal{M}, \mathcal{M})^{\text{rev}} \) instead of \( \mathcal{F}\text{un}(\mathcal{M}, \mathcal{M}) \) according to the orientation of the boundary \cite{KK}. To avoid the annoying \textit{rev}, we would like to take a mirror reflection of the configuration \cite{KK} so that \( \mathcal{C}_n \) is sitting on the left hand side of \( Z_n(\mathcal{C}_n) \). We would also like to drop the assumption of unitarity. More precisely, we would like to show for any fusion categories\footnote{Multi-fusion cases are entirely similar.} \( \mathcal{C} \) and \( \mathcal{D} \), the physical notion of a morphism \( f = (f^{(0)}, f^{(1)} : \mathcal{C} \boxtimes Z(\mathcal{C}) f^{(0)} \cong \mathcal{D}) \), where \( f^{(0)} \) is a fusion 1-category and a \( Z(\mathcal{C}) \)-\( Z(\mathcal{D}) \)-bimodule with monoidal actions\footnote{The term “monoidal actions” means that both actions \( Z(\mathcal{C}) \boxtimes f^{(0)} \to f^{(0)} \) and \( f^{(0)} \boxtimes Z(\mathcal{D}) \to f^{(0)} \) are monoidal.}, is equivalent to the usual notion of a monoidal functor \( f : \mathcal{C} \to \mathcal{D} \).

Let \( f : \mathcal{C} \to \mathcal{D} \) be a physical morphism and \( \mathcal{C}_2 = \mathcal{C} \) and \( \mathcal{D}_2 = \mathcal{D} \) are fusion categories. By definition, there is an isomorphism, i.e. a monoidal equivalence, \( f_1^{(1)} : \mathcal{C} \boxtimes Z(\mathcal{C}) f_2^{(0)} \cong \mathcal{D} \), where \( f_2^{(0)} \) is also a fusion category. Let \( 1_{f_2^{(0)}} \) be the tensor unit of the fusion category \( f_2^{(0)} \).

The functor \( x \mapsto x \boxtimes Z(\mathcal{C}) 1_{f_2^{(0)}} \) defines a natural monoidal functor \( \mathcal{C} \to \mathcal{C} \boxtimes Z(\mathcal{C}) f_2^{(0)} \). Then we obtain a composed monoidal functor

\[
\tilde{f} : \mathcal{C} \to \mathcal{C} \boxtimes Z(\mathcal{C}) f_2^{(0)} f_1^{(1)} \to \mathcal{D}.
\]

It is easy to see that isomorphic morphisms give isomorphic monoidal functors. Therefore, we obtain a map \( f \mapsto \tilde{f} \) from the set of isomorphic classes of morphisms to the set of isomorphic classes of monoidal functors.
Conversely, let $g : \mathcal{C} \to \mathcal{D}$ be a monoidal functor. Then there is a natural $\mathcal{C}\mathcal{C}$-bimodule structure on $\mathcal{D}$, denoted by $g_\mathcal{D}$. We set
\[
\bar{g}_2^{(0)} := \mathcal{F}\text{un}_{\mathcal{C}|\mathcal{E}}(\mathcal{C}, g_\mathcal{D}).
\]
By Corollary $\ref{A.23}$ there is an invertible monoidal functor
\[
\tilde{g}_1^{(1)} : \mathcal{C} \boxtimes_{Z(\mathcal{E})} \bar{g}_2^{(0)} \simeq \mathcal{D}.
\]
Therefore, we obtain a map $g \mapsto \bar{g} = (\bar{g}_2^{(0)}, \tilde{g}_1^{(1)})$. If two monoidal functors $g$ and $h$ are isomorphic, i.e. $g \simeq h$, then $g_\mathcal{D} \simeq h_\mathcal{D}$ canonically. It further induces a monoidal equivalence $\tilde{g}_2^{(0)} \simeq \tilde{h}_2^{(0)}$ such that the diagram \ref{3.8} commutes automatically.

It is trivial to show that $\bar{g} = g$. To show that $\tilde{f} \simeq f$, we notice the following sequence of monoidal equivalences:
\[
f_2^{(0)} \simeq \mathcal{Z}(\mathcal{C}) \boxtimes_{\mathcal{Z}(\mathcal{C})} f_2^{(0)} \simeq \mathcal{F}\text{un}_{\mathcal{C}|\mathcal{E}}(\mathcal{C}, \mathcal{C} \boxtimes_{\mathcal{Z}(\mathcal{C})} f_2^{(0)}) \\
\simeq \mathcal{F}\text{un}_{\mathcal{C}|\mathcal{E}}(\mathcal{C}, \mathcal{C} \boxtimes_{\mathcal{Z}(\mathcal{C})} f_2^{(0)}, \mathcal{D}f) = (\tilde{f}_1^{(0)}).
\]
We have used Prop $\ref{A.20}$ in the third step. It remains to check that the diagram \ref{3.8} with $\phi_1^{(0)}$ defined by Eq. \ref{4.18} is commutative. This amounts to show that the composed monoidal functor
\[
\mathcal{C} \boxtimes_{\mathcal{Z}(\mathcal{C})} f_2^{(0)} \xrightarrow{\text{id} \boxtimes \mathcal{Z}(\mathcal{C}) \phi_1^{(0)}} \mathcal{C} \boxtimes_{\mathcal{Z}(\mathcal{C})} \mathcal{F}\text{un}_{\mathcal{C}|\mathcal{E}}(\mathcal{C}, f_\mathcal{D}) \xrightarrow{a} \mathcal{D},
\]
where the monoidal equivalence $a$ is defined by \ref{A.17}, is isomorphic to $f_1^{(1)} : \mathcal{C} \boxtimes_{\mathcal{Z}(\mathcal{C})} f_2^{(0)} \to \mathcal{D}$. Consider the following commutative (up to isomorphisms) diagram:
\[
\begin{array}{ccc}
\mathcal{C} \boxtimes_{\mathcal{Z}(\mathcal{C})} f_2^{(0)} & \xrightarrow{\text{id} \boxtimes \mathcal{Z}(\mathcal{C}) \phi_1^{(0)}} & \mathcal{C} \boxtimes_{\mathcal{Z}(\mathcal{C})} \mathcal{F}\text{un}_{\mathcal{C}|\mathcal{E}}(\mathcal{C}, f_\mathcal{D}) \\
\simeq & & \xrightarrow{a} \mathcal{D} \\
\mathcal{C} \boxtimes_{\mathcal{Z}(\mathcal{C})} \mathcal{F}\text{un}_{\mathcal{C}|\mathcal{E}}(\mathcal{C}, \mathcal{C} \boxtimes_{\mathcal{Z}(\mathcal{C})} f_2^{(0)}) & \xrightarrow{\text{id} \boxtimes \mathcal{Z}(\mathcal{C}) \mathcal{F}\text{un}_{\mathcal{C}|\mathcal{E}}(\mathcal{C}, f_1^{(1)})} & \mathcal{C} \boxtimes_{\mathcal{Z}(\mathcal{C})} \mathcal{F}\text{un}_{\mathcal{C}|\mathcal{E}}(\mathcal{C}, \mathcal{C} \boxtimes_{\mathcal{Z}(\mathcal{C})} f_2^{(0)}) \simeq \mathcal{C} \boxtimes_{\mathcal{Z}(\mathcal{C})} \mathcal{C} \boxtimes_{\mathcal{Z}(\mathcal{C})} f_2^{(0)}
\end{array}
\]
where $c$ is defined by the equivalence \ref{A.17}. Notice that the composition of the equivalences in the left column and the bottom row is nothing but the identity functor. Therefore, we obtain the identity $a \circ (\text{id} \boxtimes \mathcal{Z}(\mathcal{C}) \phi_1^{(0)}) \simeq f_1^{(1)}$.

Therefore, we have proved that there is an one-to-one correspondence between the isomorphism classes of monoidal functors between two fusion categories $\mathcal{C}$ and $\mathcal{D}$ and those of morphisms from the topological order $\mathcal{C}_2$ to $\mathcal{D}_2$. Actually, with careful examinations, one can prove that the groupoid of monoidal functors between two fusion categories $\mathcal{C}$ and $\mathcal{D}$ is equivalent to that of morphisms from the topological order $\mathcal{C}_2$ to $\mathcal{D}_2$.

**Remark 4.7.** In general, we expect that the $(n + 1, 1)$-category $\mathcal{F}\mathcal{C}_n^{\text{fun}}$ of pairs $(\mathcal{D}_n, \iota)$ with 1-morphisms given by unitary $n$-functors preserving the units, 2-isomorphisms by natural isomorphisms and 3-isomorphisms given by modifications, so on and so forth.
The boundary-bulk relation and the functoriality of $\mathcal{Z}_n$

The unique-bulk hypothesis and $\text{bulk} = \text{center}$ are only parts of the boundary-bulk relation. For a rather complete boundary-bulk relation, we would like to propose the strong unique-bulk hypothesis, which extends the unique bulk hypothesis in this section. (recall Def. 2.1) to the situation depicted in Fig. 9. We assume this strong unique-bulk hypothesis in this section.

5.1 Closed and anomalous domain walls

In Fig. 9 note that $X_n$ is not the bulk of $a_{n-1}$ but uniquely determined by $a_{n-1}$. We define $Z_{n-1}(a_{n-1}) := X_n$. Note that $a_{n-1}$ can be viewed as a closed domain wall between $A_n$ and $X_n \boxtimes Z_{n}(B_n) B_n$ or a wall between $A_n^{op} \boxtimes Z_{n}(A_n) X_n$ and $B_n$. We do not view $a_{n-1}$ as a closed domain wall between $A_n$ and $B_n$ unless $X_n$ is trivial.

**Definition 5.1.** A domain wall $a_{n-1}$ between the $A_n$-phase and the $B_n$-phase is called **closed** if $Z_n(A_n) = Z_n(B_n)$ and $X_n$ is trivial, i.e. $X_n = \text{id}_{Z_n(A_n)}$; it is called Morita closed if $X_n$ is invertible; it is called anomalous if either $Z_n(A_n) \neq Z_n(B_n)$ or $X_n$ is not trivial.

**Definition 5.2.** If two simple topological orders $A_n$ and $B_n$ are connected by a Morita closed gapped domain wall $a_{n-1}$, then we say that they are Morita equivalent, denoted by $A_n \sim B_n$, and the Morita equivalence is given by $a_{n-1}$. When both $A_n$ and $B_n$ are closed, the Morita equivalence is also called the Witt equivalence [DMNO, FSV1, Ko2, KW].

Mathematically, two algebra objects $A$ and $B$ in a monoidal category is Morita equivalent, i.e. $A \sim B$, if there is an $A$-$B$-bimodule $P$ and an $B$-$A$-bimodule $Q$ such that $P \otimes_B Q \simeq A$ and $Q \otimes_A P \simeq B$. We used the term of Morita equivalence here because $A_n$ and $B_n$ are unitary fusion $(n - 1)$-categories and the excitations in $A_n$ and $B_n$ (see Fig. 9) indeed acts on $a_{n-1}$ and create an indecomposable invertible bimodule $AaB$. Straightly speaking, $AaB$ is only a subcategory of an indecomposable semisimple $A$-$B$-bimodule. For example, the subcategory $\mathcal{F}un_c(N, N)f\mathcal{F}un_c(M, M)$ in $\mathcal{F}un_c(M, N)$ in Example 2.22

---

13When we say $Z_n(A_n) = Z_n(B_n)$, we mean that we have made a choice of how we identify them.
this subcategory contains all the simple objects of the latter category. By abusing the notation, we denote the latter category by $AaB$ too. It is clear that $AaB$ is indecomposable because both $A$ and $B$ are simple. To see it is invertible, let $b_{n-1}$ be a gapped $B_n$-$A_n$-wall such that $Z_{n-1}(b_{n-1}) = X_n^{-1}$ in the bulk. When we placed $b_{n-1}$ next to $a_{n-1}$, $a_{n-1}$ simply fuses with $b_{n-1}$, and $a \otimes B_n b$ becomes a 1-codimensional excitation in the $A_n$-phase. The same is true for all $a' \in AaB$ and $b' \in BbA$. Mathematically, it amounts to say $AaB \otimes B bA \simeq P_{n-1}(A_n)$ as an $A$-$A$-bimodule, i.e. $AaB$ is an invertible bimodule. Note that the pair $(AaB, a)$ is not an invertible domain wall unless there exists a domain wall $b_{n-1}$ such that $a \otimes B_n b$ is the trivial 1-codimensional excitation in the $A_n$-phase. For example, 1D phase $(\text{Fun}_c(M, N), f)$ in Fig.5 is a domain wall between the $C_{M'}^{-}$-phase and the $C_{N'}^-$-phase. The 1-category $\text{Fun}_c(M, N)$ is an invertible $C_{M'}^{-}$-$C_{N'}^-$-bimodule. But the domain wall $(\text{Fun}_c(M, N), f)$ is not invertible unless $f : M \to N$ is a $C$-module equivalence.

By the definition of Morita equivalence, we have

$$A_n \sim B_n \quad \Rightarrow \quad Z_n(A_n) \simeq Z_n(B_n).$$

(5.1)

This physical result is also natural mathematically. In mathematics, Morita equivalent algebras in a certain nice monoidal category always share the same center. Actually, it is also interesting to consider another direction. In general, $Z(A) \simeq Z(B)$ does not imply $A \simeq B$. But if we assume certain duality structures on $A$ and $B$, it is possible. We give a couple of examples below.

1. In 1D, if $A$ and $B$ are two finite dimensional $C^*$-algebras, i.e. direct sums of matrix algebras, describing composite (unstable) topological orders, then $A$ is Morita equivalent to $B$ if and only if $Z(A) \simeq Z(B)$ as algebras. This result is quite trivial. But it can be generalized to a non-trivial one in the framework of 2D rational conformal field theory, in which $A$ and $B$ are two special symmetric Frobenius algebras in a modular tensor categories, then $A \sim B$ if and only if $Z(A) \simeq Z(B)$ as algebras, where $Z(A)$ is the so-called full center of $A$ [KR1].

2. In 2D, if two unitary fusion 1-categories $A_2$ and $B_2$ are Morita equivalent if and only if their monoidal centers are equivalent as braided monoidal 1-categories [MI, KI, ENO08].

It seems natural to ask if the Morita equivalence is equivalent to the equivalence of monoidal centers for a unitary fusion $n$-category. It turns out that it is not true. In 3D, all closed 3D topological orders share the same trivial $\text{bulk}$, but they are not Morita equivalent (or Witt equivalent) in general. In this case, the Witt equivalence classes form an infinite group called Witt group [DMNO]. What happens is that one can certainly glue the topological phase $Z_n(A_n)$ (with a gapped boundary $A_n$) with the phase $Z_n(B_n)$ (with a gapped boundary $B_n$) smoothly in the $\text{bulk}$ and create a domain wall between $A_n$ and $B_n$ on the boundary. But this domain wall is gapless in general. For example, a quantum hall system share the same 4D $\text{bulk}$ with $I_3$, but the domain wall between them is gapless. However, we believe that it is still important to investigate the precise relation between the Morita equivalence and the equivalence of monoidal centers because their discrepancy can have interesting and rich structures as revealed by the Witt group in 3D.

Now we assume that $B_n = A_n$ and $A_n$ is a simple topological order. If $X_n = \text{id}_{Z_n(A_n)}$, we also call the closed domain wall $a_{n-1}$ as an internal domain wall in $A_n$-phase because such
domain wall $a_{n-1}$ can be viewed as a 1-codimensional topological excitation in $A_n$-phase. If $X_n$ is invertible and $X_n \neq \text{id}_{Z_n(A_n)}$, we also call the Morita closed domain wall $a_{n-1}$ an external domain wall in $A_n$-phase.

**Example 5.3.** We give an example of an external (or Morita closed) wall in the toric code model. Consider the toric code model with a smooth boundary, a defect line and a defect of codimension 2 on the boundary depicted in Fig. 10. This model is completely free of frustration. The complete list of mutually commutative stabilizers are given as follows:

\[ A_v = \sigma_1^x \sigma_2^x \sigma_3^x \sigma_4^x, \quad B_p = \sigma_8^z \sigma_{10}^z \sigma_{11}^z \sigma_{12}^z, \quad A_{13,14,15} = \sigma_{13}^y \sigma_{14}^y \sigma_{15}^y \]

\[ C_{2,5,3,7} = \sigma_2^z \sigma_5^z \sigma_3^z \sigma_7^z, \quad D_{3,7,8,9} = \sigma_3^y \sigma_7^y \sigma_8^y \sigma_9^y, \quad Q_{6,17,18,19,20} = \sigma_6^x \sigma_{17}^x \sigma_{18}^x \sigma_{19}^x \sigma_{20}^x. \]

The excitations on the smooth boundary are given by the unitary fusion 1-category $\mathbb{R}ep\mathbb{Z}_2$. Since the dotted line is an invertible domain wall (carrying no degree of freedom) that gives the EM-duality $[\mathbb{K}, \mathbb{K}]$, the neighborhood of the plaquette $(17, 18, 19, 20)$ can be viewed as a 0+1D external wall between two smooth boundaries. Note that running an $e/m$-particle around the corner of the edges labeled by 6 and 17 (the blue dot) turn it into an $m/e$-particle. Since an $m$-particle condenses on the smooth boundary, we obtain that this external domain wall contains no non-trivial excitations. Hence, it is just $1_1 = H\text{ilb}$. Indeed, $H\text{ilb}$ is at the same time an invertible $R\text{ep}\mathbb{Z}_2$-bimodule and gives a non-trivial Morita equivalence between two smooth boundaries $R\text{ep}\mathbb{Z}_2$. This Morita equivalence $H\text{ilb}$ between $R\text{ep}\mathbb{Z}_2$ and $R\text{ep}\mathbb{Z}_2$ determines exactly the invertible domain wall (the dotted line) or the EM-duality in the bulk, i.e. an braided auto-equivalence of $Z(R\text{ep}\mathbb{Z}_2)$.

**Example 5.4.** When $n = 1$, we consider a Levin-Wen model enriched by boundaries and defects depicted in Fig. 11. Let $\mathcal{D} = \mathcal{E}$. In this case, we have $Z_2^{(0)}(\mathcal{E}) = Z(\mathcal{E})$, where $Z(\mathcal{E})$ is the monoidal center of $\mathcal{E}$. If $M$ is an invertible $A-A$-bimodule, then the $A$-module functor $a : A \to M \otimes_A A = M$ gives an external domain wall. In this case, $AaA$ as an indecomposable $A$-bimodule is nothing but $M$. The excitations on the domain wall between $Z(\mathcal{E})$ and $Z(\mathcal{D})$ are given by

\[ Z_2^{(1)}((M,a)) = Z(M) := \mathcal{F}un_{A|\mathcal{B}}(M,M)^{\text{ev}}. \quad (5.2) \]
Figure 11: Consider a Levin-Wen model enriched by gapped boundaries and defects \[KK\]. The two bulk lattices are defined by unitary fusion categories \(A\) and \(B\), respectively. The excitations in the left/right bulk are given by the monoidal center \(Z(A)/Z(B)\) of \(A/B\). The lattice near the wall between the left-bulk and the right bulk is defined by a semisimple indecomposable \(A\)-\(B\)-bimodule \(M\), and is called the \(M\)-wall. According to \[KK\], the excitations on the \(M\)-wall are given by the unitary fusion category \(Z(M) := \text{Fun}_{A|B}(M,M)^{\text{rev}}\).

Two boundary lattices are defined by \(A\) and \(B\), viewed as a left \(A\)-module and a left \(B\)-module, respectively, and is called the \(A\)-boundary and the \(B\)-boundary. The excitations on the \(A\)-boundary are given by \(A \simeq \text{Fun}_A(A,A)\); those on the \(B\)-boundary are given by \(B\). The defect junction connecting the \(A\)-boundary, \(M\)-wall and \(B\)-boundary is given by a \(C\)-module functor \(a \in \text{Fun}_A(A,M \boxtimes_B B) \simeq M\).

This example shows that, for \(n = 1\), any nontrivial invertible \(A\)-\(A\)-bimodule is physically realizable as an external domain wall in Levin-Wen models. Also note that if we fold two boundaries upward, we obtain a vertical line with the bottom end given by the pair \((M,a)\). The vertical line is given by a unitary multi-fusion 1-category \(\text{Fun}(M,M)\), which is nothing but the center of \(M\).

**Remark 5.5.** Higher dimensional generalization of Levin-Wen type of lattice models can be constructed by replacing unitary fusion 1-categories by unitary \(n\)-categories (see a special case in [WWa] and a sketch of general construction in [Wa]). We believe that the theory of topological excitations in these higher dimensional models are completely parallel to the case \(n = 3\). In particular, Example 5.4 (Fig. 11) should also work for \(n > 1\).

When a simple domain wall \(a_{n-1}\) is viewed as an anomalous \((n-1)D\) topological order, it is given by \((AaA,a)\). The \(A\)-\(A\)-bimodule \(AaA\) is indecomposable and invertible. The set of equivalence classes of the bimodules \(AaA\) for all Morita closed domain walls \(a_{n-1}\) between \(A\) and \(A\) form a group \(\text{Pic}(A_n)\). The strong unique-bulk hypothesis immediately implies that the map

\[
Z_n^{(1)} : \text{Pic}(A_n) \rightarrow \text{Aut}(Z_n(A_n))
\]

is a group isomorphism. We believe that higher dimensional analogs of Levin-Wen models (see Remark 5.5) should give physical realizations of all invertible \(A_n\)-\(A_n\)-bimodules as Morita closed domain walls. If invertible domain walls indeed one-to-one correspond to mathematical automorphisms as in usual Levin-Wen models, then the isomorphism (5.3) suggests that, for a unitary fusion \((n-1)\)-category \(A_n\), \(\text{Pic}(A_n) \simeq \text{Aut}(Z(A_n))\), where \(\text{Pic}(A_n)\) is the group of the equivalence classes of invertible \(A_n\)-\(A_n\)-bimodules, \(Z(A_n)\) the center of \(A_n\) (defined by Eq.(4.14)) and \(\text{Aut}(\cdot)\) is the group of the equivalence classes of braided auto-equivalences. This mathematical statement is still conjectural in general but is known to be true in a few low dimensional cases.
1. In 1D, if \( A_1 \) is a simple algebra over \( \mathbb{C} \), this result is trivial. But if we consider more general framework, e.g. simple special symmetry Frobenius algebras \( A \) in a modular tensor category, then \( \text{Pic}(A) \simeq \text{Aut}(Z(A)) \), where \( Z(A) \) is the full center of \( A \), is true and non-trivial [DKRI]. The physical meaning of this result is similar to what we have discussed here but in the framework of 2D rational conformal field theories [FTFRS].

2. In 2D, if \( A_2 \) is a fusion 1-category and \( Z(A_2) \) its monoidal center, then \( \text{Pic}(A_2) \simeq \text{Aut}(Z(A_2)) \), where \( \text{Aut}(-) \) is the group of braided monoidal equivalences, is true [ENO09] (see also [KK] for its physical meaning in Levin-Wen type of lattice models).

In the framework of 3D TQFT, similar phenomenon was studied recently in [FS].

### 5.2 The functoriality of \( Z_n \)

In Fig.9 an anomalous domain wall \( a_{n-1} \) between \( A_n \) and \( B_n \) determines the “bulk” closed domain wall \( X_n \), also denoted by \( Z_n^{(1)}(a_{n-1}) \). Similarly, we can define \( Z_n^{(2)}(\cdot) \) for (possibly anomalous) walls between (possibly anomalous) walls. For example, if we fatten Fig.5 in the third directions, let us imagine an anomalous wall \( a_{n-2} \) between the anomalous wall \( a_{n-1} \) defined by the gapped boundary of a domain wall \( \chi_{n-1} \) between two \( X_n \)-phases. Then this \( \chi \) is determined by \( a \) uniquely and denoted by \( Z_n^{(2)}(a) \). Similarly, we can define \( Z_n^{(i)}(a) \) for \( 2 < i < n \) (not for instantons \( i = n \) ). In this context, we can view the bulk \( Z_n \) as \( Z_n^{(0)} \). The strong unique-bulk hypothesis depicted in Fig.9 suggests that \( Z_n^{(0)}, Z_n^{(1)}(a), Z_n^{(2)}(a), \ldots, Z_n^{(n-1)}(a) \) defines a functor, still denoted by \( Z_n \), from the category \( \mathcal{T}O_{n}^{\text{wall}} \) of nD topological orders with (possibly anomalous) walls as 1-morphisms and (possibly anomalous) walls between walls as 2-morphisms, so on and so forth, to the category \( \mathcal{T}O_{n+1}^{\text{closed-walls}}, \) i.e.

\[
Z_n : \mathcal{T}O_{n}^{\text{wall}} \to \mathcal{T}O_{n+1}^{\text{closed-walls}}
\]

Physically, this funtoriality is tautological. Note that the result (5.1) and the group isomorphism in (5.3) are just parts of the first two layers of the hierarchical structures, which can be summarized as the functoriality of \( Z_n \).

Mathematically, it suggests that the notion of the center of a unitary fusion \( n \)-category is functorial if we define the domain/target categories properly. For example, when \( X_n \) is not invertible, \( AA \otimes B \) is still an (non-invertible) \( A \)-\( B \)-bimodule. If all \( A \)-\( B \)-bimodules can be physically realized by (possibly anomalous) walls between \( A \) and \( B \), then we can choose the 1-morphisms in the domain category to be the semisimple bimodules, objects in the target category are braided fusion \( n \)-categories and 1-morphisms are monoidal bimodules, which means a unitary fusion \( n \)-category with two-side \( A \)-\( B \)-actions such that both the left and the right actions are monoidal. This mathematical funtoriality is highly non-trivial and still conjectural.

When \( n = 1 \), all \( A \)-\( B \)-bimodules can be physically realized as (possibly anomalous) walls between \( A \) and \( B \) in Levin-Wen models as shown in Fig.11 in which the \( A \)-\( B \)-bimodule \( AA \otimes B \) is just the bimodule \( M \simeq \mathcal{F}un_4(A, M \boxtimes B) \). In this case, we indeed obtain a mathematical functor. Let \( \mathcal{F}us \) be the 1-category of fusion 1-categories with 1-morphisms given by equivalence classes of bimodules and \( \mathcal{B}rd^{\dagger} \) the 1-category of non-degenerate braided fusion 1-categories with 1-morphisms given by equivalence classes of monoidal bimodules \( e \mathcal{E}_2 \) such that \( Z(\mathcal{E}) = \mathcal{E} \boxtimes \mathcal{D} \), which is the condition for the domain wall \( \mathcal{E} \) to be closed.
Proposition 5.6. For (unitary) fusion 1-categories $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{A}$-$\mathcal{B}$-bimodule 1-category $\mathcal{M}$, the following assignment

$$\mathcal{A} \mapsto Z(\mathcal{A}), \quad \mathcal{A} \mathcal{B} \mapsto Z(\mathcal{M}) := \text{Fun}_{\mathcal{A} \mid \mathcal{B}}(\mathcal{M}, \mathcal{M})^{\text{rev}}$$

defines a fully faithful functor $Z : \mathcal{Fus} \to \mathcal{Brd}^{\text{cl}}$.

Proof. The functoriality of $Z$ follows from Prop. 5.21, Remark 5.22 and the associativity of the tensor product $\boxtimes_{\mathcal{A}, \mathcal{B}, \cdots}$ between bimodules over fusion categories $\mathcal{A}, \mathcal{B}, \cdots$ [ENO09]. The fully faithfulness follows from the fact that 1-morphisms $Z(\mathcal{A}) \boxtimes Z(\mathcal{B})$ in $\mathcal{Brd}^{\text{cl}}$ one-to-one correspond to the Lagrangian algebras in $Z(\mathcal{A}) \boxtimes Z(\mathcal{B})$, which further one-to-one correspond to indecomposable semisimple $\mathcal{A}$-$\mathcal{B}$-bimodules [DMNO].

Remark 5.7. Mathematically, it is natural to extend the functor defined in Prop. 5.6 to include higher morphisms. More precisely, one can extend the domain category of $Z$ to have 1-morphisms given by bimodules, 2-morphisms given by bimodule functors, 3-morphisms by natural transformations; and extend the target category to have 1-morphisms given by monoidal bimodules, 2-morphisms by usual bimodules together with a distinguished object, and 3-morphisms by morphisms between the distinguished objects. More precisely, the functor $Z$ maps the following morphisms in the target category

![Diagram](https://via.placeholder.com/150)

to the following morphisms in the target category (to make them more pictorial, we use different but equivalent ways to define the morphisms):

![Diagram](https://via.placeholder.com/150)

where the 1-morphism $Z(\mathcal{M})$ as a monoidal bimodule is equivalently defined by the cospan of monoidal categories such that $L_N$ and $R_M$ are central functors, the 2-morphism $(Z(\mathcal{M}), N) := \text{Fun}_{\mathcal{C} \mid \mathcal{D}}(\mathcal{M}, N), \mathcal{F})$ is equivalently defined by the cospan of modules such that the push-forward $\mathcal{F}^*$ is a left $Z(\mathcal{M})$-module functor and the pull-back $\mathcal{F}^*$ a right $Z(\mathcal{M})$-module functor, and the 3-morphism $Z(\phi)$ is just the natural transformation $\phi : \mathcal{F} \to \mathcal{G}$. The functoriality of this
extended $Z$, first proposed in [Ko1], is still conjectural. Whether such extended functoriality can be interpreted as a boundary-to-bulk functor is not entirely clear to us. But it does have a clear physical meaning as the data in the domain category defines Levin-Wen models with defects and its image under $Z$ defines topological excitations in the models. The functoriality of $Z$ is tautological from this point of view. We believe that such extended functoriality also holds for the centers of unitary fusion $n$-categories as well. Similar functoriality of the notion of the center of an algebra in a monoidal category was implicit in the construction of on 2D rational conformal field theories with topological defects in [FrFRS], and was formulated precisely and proved rigorously in the context of 2D TQFTs [DKR2] and in 2D rational conformal field theories [DKR3].

For $n > 2$, we believe that all $A$-$B$-bimodules can be physically realized as (possible anomalous) walls between $A$ and $B$ by higher dimensional (yet-to-be-constructed) analogues of Levin-Wen models based on unitary fusion $n$-categories. So we expect a similar functoriality of the monoidal center $Z$ for unitary fusion $n$-categories.

**Remark 5.8.** In the Language of Jacob Lurie [L3], a unitary fusion $n$-category can be viewed as an $E_1$-algebra in the symmetric monoidal $\infty$-category of $(\infty,n)$-categories, a unitary braided fusion $n$-category as an $E_2$-algebra. An monoidal bimodule $A M_B$ is an $E_1$-algebra over the $E_2$-algebra $A_n \boxtimes B_n^{op}$. Prop. 5.6 is a special case of the lax functoriality of the center of an $E_n$-algebra over an $E_{n+1}$-algebra in a symmetric monoidal $\infty$-category. But non-laxness need additional conditions such as the unitary condition. We believe that Prop 5.6 holds for unitary fusion $n$-categories.

We have shown that $Z_n^{(0)}$ satisfies the same universal property as that of the center. Actually, the $Z_n^{(1)}(a)$-bulk of the domain wall $a_{n-1}$ is also some kind of center. To see this, one can first introduce the notion of a morphism between two domain walls between the $A_n$-phase and the $B_n$-phase similar to the notion of a morphism between two topological orders. Such a morphism $a_{n-1} \to b_{n-1}$ between domain walls $a_{n-1}$ and $b_{n-1}$ corresponds to the physical configuration depicted in Fig. 12. Then one can prove that the universal property of $Z_n^{(1)}$ is the same as that of the center considered in the category of domain walls. We will not do that in this work. Instead, we would like to point out that the right hand side of Eq. (5.2) is indeed a center but considered in the category of bimodules. An analogous situation was investigated in [DKR3]. Similarly, we can also show that $Z_n^{(2)}$, $Z_n^{(3)}$, \ldots defined for higher codimensional domain walls are some kind of center by proving their universal
properties. Therefore, the functor $Z_n$ is indeed the functoriality of the notion of center in mathematics.

Mathematically, consider a (possibly anomalous) domain wall $(M_{n-1}, a)$ between two simple anomalous phases $A_n$ and $B_n$, viewed as unitary fusion $(n-1)$-categories. $M_{n-1}$ is an $A$-$B$-bimodule. We have

$$Z_n^{(1)}((M_{n-1}, a)) = \mathcal{F}un_{A_n/B_n}(M_{n-1}, M_{n-1})^{rev},$$

In Fig. 9 if $A_n$ and $B_n$ are closed, then both $A_n$ and $B_n$ are unitary braided $(n-2)$-categories, and the (possibly anomalous) domain wall $M_{n-1}$ (with the canonical unit) between $A_n$ and $B_n$ is a monoidal bimodule. Both of the bulk-to-wall maps $A \rightarrow M$ and $B \rightarrow M$ factor through $\mathcal{F}un_{M|M}(M_{n-1}, M_{n-1})$. In this case, $Z_n^{(1)}(M_{n-1})$ can be defined as follows:

$$Z_n^{(1)}((M_{n-1}, a)) = \mathcal{F}un_{M|\hat{M}}^{A_n/B_n}(M_{n-1}, M_{n-1})^{rev},$$

where the right hand side is the full subcategory of $\mathcal{F}un_{M|M}(M_{n-1}, M_{n-1})$ containing those objects that are symmetric to the images of $A_n$ and $B_n$ in $\mathcal{F}un_{M|M}(M_{n-1}, M_{n-1})^{rev}$. The physical meaning of the definition Eq. (5.5) is quite obvious. Since we have

$$Z_n^{(0)}((M, a)) = A^{op} \boxtimes Z_n^{(1)}((M_{n-1}, a)) \boxtimes B,$$

the excitations in $A_n$, $Z_n^{(1)}((M_{n-1}, a))$ and $B_n$ can all be viewed as excitations in $Z_n^{(0)}((M, a))$. It is clear that all three sets of excitations are mutually symmetric. Namely, excitations in one of the three sets can not detect excitations in a different set.

**Remark 5.9.** In [L3], Lurie defined a notion of the center of an $E_n$-algebra over an $E_{n+1}$-algebra. We believe that Lurie’s notion of center reduces to Eq. (5.5) when $n = 1$.

The strong unique-bulk hypothesis implies an “exact” sequence of functors

$$\cdots \rightarrow \mathcal{F}O_n^{wall} \rightarrow \mathcal{F}O_{n+1}^{wall} \rightarrow \mathcal{F}O_{n+2} \rightarrow \cdots.$$

In particular, we have $Z_{n+1}^{(i)}(\mathcal{F}O_n^{i}) = 1_{n+2-i}$ hold not only for $i = 0$ but also for $i = 1, 2, \ldots$. The mathematical meaning of these later cases (for $i > 0$) have not been discussed before. We briefly discuss the case $i = 1$ below.

**Example 5.10.** The functor $Z_1$ is given by the mathematical functor in Prop. 5.6 with $Z_1^1(M) = Z(M)$. When we apply the functor $Z_2$, as unitary braided fusion 1-categories, we have

$$Z_2^1(Z_1^1(M)) = \mathcal{F}un_{Z(M)|Z(\hat{M})}^{Z(A)|Z(B)}(Z(M), Z(M))^{rev} = 1_3,$$

which follows from the fact that $Z(A) \boxtimes \overline{Z(B)} = \mathcal{F}un_{Z(M)|Z(M)}(Z(M), Z(M))$.

In general, we expect that the following mathematical result to be true.

**Conjecture 5.11.** Let $A_n$ and $B_n$ be two unitary $(n-1)$-fusion categories and $M_{n-1}$ a unitary indecomposable $A_n$-$B_n$-bimodule (also a unitary $(n-1)$-category). Let $Z(A)$, $Z(B)$ and $Z(M)$ be $\mathcal{F}un_{A|A}(A, A)$, $\mathcal{F}un_{B|\hat{B}}(B, B)$ and $\mathcal{F}un_{A|\hat{B}}(M, M)^{rev}$, respectively. Then we have

$$\mathcal{F}un^{Z(A)|Z(B)}_{Z(M)|Z(M)}(Z(M_{n-1}), Z(M_{n-1})) = 1_{n+1},$$

where $1_{n+1}$ is the trivial unitary braided fusion $(n-1)$-category.
6 Conclusions and outlooks

Although the main result in this work is the boundary-bulk relation for topological orders in any dimension, another secret goal of this work is to use this relation as a tool to determine what a proper categorical description of a local topological order should be. As far as we can reach in this work, the categorical description given in this work seems works pretty well with the boundary-bulk relation. If this categorical description of local topological orders indeed works, it suggests that many features of higher dimensional topological orders might be similar to those of 3D topological orders. One possible situation is that the condensation theory of topological excitations in high dimensions is parallel to that in 3D cases [Ko2]. More precisely, a closed simple nD topological order is given by a unitary braided fusion \((n - 2)\)-category \(C_n\), a condensation should be completely determined by a commutative separable algebra \(A\) in \(C_n\) (satisfying additional conditions possibly involving topological spins). The condensed phase can be described by the braided fusion \((n - 2)\)-category of local \(A\)-modules, and the gapped domain wall created between the original and condensed phases consists of confined excitations given by the unitary fusion \((n - 2)\)-category of \(A\)-modules. But a complete condensation theory needs topological spins.

A Appendix

A.1 The definition of a unitary \(n\)-category

In this subsection, we give the definition of a unitary \(n\)-category. Before we start, we first review some elements of an \(n\)-category (see for example [L2]).

We assume that a good definition of \(n\)-category is chosen. An object in an \(n\)-category \(C_n\) is also called a 0-morphism. We also refer to \(C_n\) itself as the unique \(-1\)-morphism. In particular, a 0-category \(C_0\) is just the set of 0-morphisms. A \(C\)-linear 0-category \(C_0\) is a vector space over \(C\). The opposite category \(C_{op}^n\) is defined by flipping all \(n\)-morphisms. For a \(C\)-linear 0-category \(C_0\), \(C_{op}^0\) is the dual vector space \(\text{hom}_C(C_0, C)\). For an \(n\)-category \(C\), we define the homotopy 1-category \(h^1C\) by the 1-category with the same set of objects as \(C\) and 1-morphisms given by the equivalence classes of 1-morphisms in \(C\); we define the homotopy 2-category \(h^2C\) by the 2-category with the same set of objects and 1-morphisms as \(C\) and 2-morphisms given by the equivalence classes of 2-morphisms in \(C\).

**Definition A.1.** Let \(C\) be a 2-category and \(f : x \to y\) a 1-morphism.

1. \(f\) is said to have a left adjoint if there exist a 1-morphism \(g : y \to x\), the unit 2-morphism \(\eta : \text{id}_x \Rightarrow g \circ f\) and the co-unit 2-morphism \(\epsilon : f \circ g \Rightarrow \text{id}_y\) such that
   \[
   \text{id}_f = (f \xrightarrow{\sim} f \circ \text{id}_x \xrightarrow{\eta} f \circ g \circ f \xrightarrow{\epsilon} f), \quad \text{id}_g = (g \xrightarrow{\sim} \text{id}_x \circ g \xrightarrow{\eta} g \circ f \circ g \xrightarrow{\epsilon} g).
   \]

2. \(f\) is said to have a right adjoint if there exist a 1-morphism \(h : y \to x\), the unit 2-morphism \(\tilde{\eta} : \text{id}_y \Rightarrow f \circ h\) and the co-unit 2-morphism \(\tilde{\epsilon} : h \circ f \Rightarrow \text{id}_x\) such that
   \[
   \text{id}_f = (f \xrightarrow{\sim} \text{id}_y \circ f \xrightarrow{\tilde{\eta}} f \circ h \circ f \xrightarrow{\tilde{\epsilon}} f), \quad \text{id}_h = (h \xrightarrow{\sim} h \circ \text{id}_y \xrightarrow{\tilde{\eta}} h \circ f \circ h \xrightarrow{\tilde{\epsilon}} h).
   \]
$\mathcal{C}$ is said to have adjoints for 1-morphisms if every 1-morphism in $\mathcal{C}$ has both a left and a right adjoint.

**Definition A.2.** Let $\mathcal{C}$ be an $n$-category for $n \geq 2$. $\mathcal{C}$ has adjoints for 1-morphisms if $\mathcal{C}$ has adjoints for 1-morphisms. For $1 < k < n$, $\mathcal{C}$ has adjoints for $k$-morphisms if, for any pair of objects $x, y \in \mathcal{C}$, the $(n-1)$-category $\text{hom}_\mathcal{C}(x, y)$ has adjoints for all $(k-1)$-morphisms. $\mathcal{C}$ is said to have adjoints if $\mathcal{C}$ has adjoints for $k$-morphisms for all $0 < k < n$.

A monoidal $n$-category can be defined by an $(n+1)$-category with a single object $\ast$. Namely, the $n$-category $\text{hom}(\ast, \ast)$ is an $n$-category with a monoidal structure. A monoidal $n$-category is said to have duals for the objects if the homotopy 1-category $\mathcal{C}_{\mathbb{H}}$ is a rigid monoidal category. Then $\mathcal{C}$ is said to have duals if $\mathcal{C}$ has duals for objects and adjoints for $k$-morphisms for all $0 < k < n$. It is equivalent to $\mathcal{C}$ having adjoints when it is viewed as an $(n+1)$-category with a single object.

**Definition A.3.** A 1-category $\mathcal{C}$ is called finite if $\mathcal{C}$ is abelian, $\mathbb{C}$-linear and has finitely many simple objects and every object is the product of finitely many simple objects and all hom spaces are finite dimensional vector spaces over $\mathbb{C}$. Here, an object $x$ is simple if $\text{hom}(x, x) = \mathbb{C}$.

**Definition A.4.** For $n \geq 0$, an $n$-category $\mathcal{C}$ is unitary if the following conditions are satisfied:

1. $\mathcal{C}$ is $\mathbb{C}$-linear. That is, for every pair of $(n-1)$-morphisms $f, g : X \to Y$ in $\mathcal{C}$, $\text{hom}(f, g)$ is a finite dimensional vector space over $\mathbb{C}$. The compositions of the morphisms in $\mathcal{C}$ respect this linear structure.

2. $\mathcal{C}$ is finite. That is, $\mathcal{C}$ is closed under finite products, has finitely many simple objects, every object is the product of finitely many simple objects and $\text{hom}(i, j)$ is a finite $(n-1)$-category for simple objects $i$ and $j$. Here, an $(n-1)$-morphism $x$ is simple if $\text{hom}(x, x) = \mathbb{C}$, and a $k$-morphism $y$, for $0 \leq k < n-1$, is simple if the $(k+1)$-identity morphism $\text{id}_y$ is simple.

3. $\mathcal{C}$ has adjoints. That is, every $k$-morphism, $1 \leq k < n$, has both a left adjoint and a right adjoint.

4. There is an equivalence $\delta : \mathcal{C} \to \mathcal{C}^{\text{op}}$ which fixes all $k$-morphisms for $0 \leq k < n$, and is antilinear, involutive and positive on $n$-morphisms, i.e.

$$\delta(\lambda f) = \overline{\lambda} \delta(f), \quad \delta(f) = f, \quad f \circ \delta(f) = 0 \Rightarrow f = 0, \quad (A.1)$$

for $n$-morphism $f : X \to Y$ and $\lambda \in \mathbb{C}$.

5. For two non-isomorphic simple $k$-morphisms $j^{[k]}$ and $j^{[k]}$, $\text{hom}(j^{[k]}, j^{[k]}) = 0_{n-k-1}$, where $0_{n-k-1}$ is the zero $(n-k-1)$-category that has only the zero object and zero morphisms.

**Remark A.5.** When $n = 0$, $\mathcal{C}_0^{\text{op}}$ is obtained from $\mathcal{C}_0$ by flipping the arrows. It makes sense if we interpret the vector space $\mathcal{C}_0$ over $\mathbb{C}$ as $\text{hom}_\mathcal{C}(\mathbb{C}, \mathcal{C}_0)$. As a result, $\mathcal{C}_0^{\text{op}} = \text{hom}_\mathcal{C}(\mathcal{C}_0, \mathbb{C})$. Then the conditions in (A.1) guarantee that $\mathcal{C}_0$ is a finite dimensional Hilbert space.
Definition A.6. An $n$-functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ between two unitary $n$-categories $\mathcal{C}$ and $\mathcal{D}$ is called unitary if $\mathcal{F} \circ \delta = \delta \circ \mathcal{F}$.

Remark A.7. If $\mathcal{C}$ is a unitary $n$-category, then $\mathcal{C}$ has a terminal object $0$ as zero product by definition. Since $\text{hom}(x,0) \simeq \text{hom}(0,x)$ by duality, we see that $0$ is a zero object. More generally, finite coproducts in $\mathcal{C}$ coincide with finite products by duality. In this case, such a (co-)product is also called a direct sum. In a unitary $n$-category, the direct sum $x \oplus y$ (or the coproduct) of two objects $x$ and $y$ is characterized by the property that $\text{hom}(z, x \oplus y) \simeq \text{hom}(z,x) \oplus \text{hom}(z,y)$ as $(n-1)$-categories for all $z$.

Proposition A.8. Let $\mathcal{C}$ be a unitary $n$-category containing a $k$-morphism $f : x \to y$, $1 \leq k < n$. Then the left adjoint and the right adjoint of $f$ are canonically isomorphic.

Proof. Let $g : Y \to X$ be the left adjoint of the $k$-morphism $f$ with unit $u : \text{id}_y \to f \circ g$ and counit $v : g \circ f \to \text{id}_x$. For $1 \leq k < n$, the left (or right) adjoint of $u$ and $v$, i.e. $u^\vee : f \circ g \to \text{id}_y$ and $v^\vee : \text{id}_x \to g \circ f$ exhibit $g$ as the right adjoint of $f$. When $k = n-1$, $\delta(u)$ and $\delta(v)$ exhibit $g$ as the right adjoint of $f$. \qed

Remark A.9. If $\mathcal{C}$ is a unitary $n$-category, then $\text{hom}(x,y)$ is a unitary $(n-1)$-category for any objects $x, y \in \mathcal{C}$.

Example A.10. Def. A.4 of a unitary $n$-category is heavily loaded. We unravel this definition in a few lower dimensional cases. Recall Sec. 2.3 for the definition of $\mathcal{C}^{op}$.

1. When $n = 1$, we recover the usual definition of a unitary 1-category (see for example Def. A.4), which is defined as an abelian $\mathbb{C}$-linear finite $*$-category, where a $*$-category means a family of maps $*: \text{hom}_\mathcal{C}(x,y) \to \text{hom}_\mathcal{C}(y,x)$ (given by $\delta$) such that

$$ (g \circ f)^* = f^* \circ g^*, \quad (\lambda f)^* = \bar{\lambda} f^*, \quad f^{**} = f, \quad \forall \lambda \in \mathbb{C}^\times, $$

satisfying the positivity condition $f \circ f^* = 0 \Rightarrow f = 0$. Note that $* = \delta(-)$.

2. When $n = 2$, a unitary 2-category is a $\mathbb{C}$-linear finite 2-category having adjoints for 1-morphisms such that all hom spaces are unitary 1-categories and all coherence isomorphisms are unitary, i.e.

$$ \delta(\alpha_{f,g,h}) = \alpha_{f,g,h}^{-1}, \quad \delta(l_f) = l_f^{-1}, \quad \delta(r_f) = r_f^{-1}, \quad \delta(\eta) = \bar{\eta}, \quad \delta(\epsilon) = \bar{\epsilon}, \quad (A.2) $$

for 1-morphisms $f, g, h$, where $\alpha, l/r$ and $\eta, \bar{\eta}, \epsilon, \bar{\epsilon}$ are the associator, the left/right unit isomorphism and duality maps, respectively. Note that Eq. (A.2) follows from the definition of coherence isomorphisms in the opposite category (see Sec. 2.3).

3. When $n > 2$, the second axiom in Def. A.4 encodes the unitarity of all coherence isomorphisms. For example, a unitary 3-category with a unique simple object $*$ and a unique simple 1-morphism $\text{id}_*$ (in $\text{hom}(*,*)$) is just the usual unitary braided fusion 1-category, where the braiding is unitary, i.e. $\delta(c_{x,y}) = c_{x,y}^{-1}$ for $x \otimes y \xrightarrow{c_{x,y}} y \otimes x$.

Definition A.11. We define a unitary fusion $n$-category by the monoidal $n$-category $\text{hom}(\star, \star)$ of a unitary $(n+1)$-category with a unique simple object $\star$. We define a a unitary braided fusion $n$-category by the braided monoidal $n$-category $\text{hom}(\text{id}_\star, \text{id}_\star)$ of a unitary $(n+2)$-category with a unique simple object $\star$ and a unique simple 1-morphism $\text{id}_\star$ (in $\text{hom}(\star, \star)$).
Remark A.12. A theory of higher algebras was developed by Jacob Lurie [L3]. A unitary fusion $n$-category can be viewed as an $E_1$-algebra object (satisfying additional unitary conditions) in the symmetric monoidal $\infty$-category $\text{Cat}_{(\infty,n)}$ of $(\infty,n)$-categories with some additional structures. A unitary braided fusion $n$-category can be viewed as an $E_2$-algebra object in $\text{Cat}_{(\infty,n)}$.

Remark A.13. In the language of [KW], a BF$_n$-category is a unitary $n$-category. A BF$_{pre}$$_n$-category is a unitary $n$-category without satisfying the 5th axiom in Def. A.4. The physical meaning of the 5th axiom in Def. A.4 was explained in [KW, Sec. XI.G, XI.H]. This axiom should be removed if we want to consider symmetry enriched to pological orders [BBCW].

A.2 The universal property of the bulk with higher morphisms

In this subsection, we would like to describe the universal property of the bulk with higher isomorphisms. Categorically, it means that we drop the Minimal Assumption (see Remark 2.15).

Now the action $\rho : P_n(\mathcal{Z}_n(\mathcal{C}_n)) \boxtimes \mathcal{C}_n \to \mathcal{C}_n$ is unital in the sense that there is a 2-isomorphism $\gamma : \rho \circ (\iota \mathcal{Z}_n(\mathcal{C}_n) \boxtimes \text{id}_{\mathcal{C}_n}) \Rightarrow \text{id}_{\mathcal{C}_n}$, which is equivalent to the commutativity (up to a 2-isomorphism) of the following diagram:

\[
\begin{array}{c}
P_n(\mathcal{Z}_n(\mathcal{C}_n)) \boxtimes \mathcal{C}_n \\
\downarrow \gamma \\
\mathcal{C}_n \end{array}
\xrightarrow{\rho} 
\begin{array}{c}
\iota \mathcal{Z}_n(\mathcal{C}_n) \boxtimes \text{id}_{\mathcal{C}_n} \\
\downarrow \\
\text{id}_{\mathcal{C}_n}
\end{array}
\]

(A.3)

Then the universal property of the bulk becomes much more complicated. A complete definition was given by Lurie [L3]. We illustrate the first two layers of the structures below.

Universal property of the bulk: The triple $(P_n(\mathcal{Z}_n(\mathcal{C}_n)), \rho, \gamma)$ is terminal among all such triples. More precisely, if $(\mathcal{X}_n, f, \phi)$ is such a triple, where $\mathcal{X}_n$ is an $nD$ topological order and $f : \mathcal{X}_n \boxtimes \mathcal{C}_n \to \mathcal{C}_n$ a morphism such that the following diagram:

\[
\begin{array}{c}
\mathcal{X}_n \boxtimes \mathcal{C}_n \\
\downarrow \phi \\
\mathcal{C}_n
\end{array}
\xrightarrow{f} 
\begin{array}{c}
\iota \mathcal{Z}_n(\mathcal{C}_n) \boxtimes \text{id}_{\mathcal{C}_n} \\
\downarrow \\
\text{id}_{\mathcal{C}_n}
\end{array}
\]

(A.4)

is commutative up to a 2-isomorphism $\phi : f \circ (\iota \mathcal{Z}_n \boxtimes \text{id}_{\mathcal{C}_n}) \Rightarrow \text{id}_{\mathcal{C}_n}$, then there is a morphism between the triangles (A.3) and (A.4) given by a triple $(f, \alpha_f, \Phi_f)$, where $f : \mathcal{X}_n \to \mathcal{Z}_n(\mathcal{C}_n)$ is a 1-morphism and $\alpha_f : \rho \circ (f \boxtimes \text{id}_{\mathcal{C}_n}) \Rightarrow f$ a 2-isomorphism such that the following diagram:

\[
\begin{array}{c}
P_n(\mathcal{Z}_n(\mathcal{C}_n)) \boxtimes \mathcal{C}_n \\
\downarrow \gamma \\
\mathcal{C}_n \end{array}
\xrightarrow{\rho} 
\begin{array}{c}
\iota \mathcal{Z}_n(\mathcal{C}_n) \boxtimes \text{id}_{\mathcal{C}_n} \\
\downarrow \\
\text{id}_{\mathcal{C}_n}
\end{array}
\]

(A.5)
is commutative up to a 3-isomorphism $\Phi_f : \phi \circ (\text{id} \boxtimes \alpha_f) \circ (\zeta_{f \circ \iota_n} \boxtimes \text{id}) \Rightarrow \gamma$. Moreover, the triple $(f, \alpha_f, \Phi_f)$ is unique in the sense that, if $(g, \alpha_g, \Phi_g)$ is another triple, then there is an 2-isomorphism $\beta^{(1)} : f \Rightarrow g$ such that the diagram consisting of $\beta^{(1)}, \alpha_f$ and $\alpha_g$ is commutative up to a 3-isomorphism $\beta^{(2)}$ such that the diagram consisting of $\beta^{(2)}, \Phi_f$ and $\Phi_g$ is commutative up to a 4-isomorphism $\beta^{(3)}$. And the triple $(\beta^{(1)}, \beta^{(2)}, \beta^{(3)})$ is unique in a similar sense, so on and so forth.

We will not prove above universal property of the bulk in this work as a complete exposition would lead us too far away, so we do not go further on this subject here.

A.3 Weak morphisms

There are more general physical realization of a universal process of mapping excitations in the $C_n$-phase to the $D_n$-phase. For this reason, we would like to introduce the notion of a weak morphism between two topological orders.

**Definition A.14.** A weak morphism $f$ from an $nD$ topological order $C_n$ to another one $D_n$ is a triple $(f^{(1)}_{n+1}, f^{(2)}_n, f^{(3)}_{n-1})$ such that

1. $f^{(1)}_{n+1}$ is an $(n+1)D$ (possibly anomalous) topological order,
2. $f^{(2)}_n$ is a gapped domain wall between $f^{(1)}_{n+1}$ and other phases,
3. $f^{(3)}_{n-1} : f^{(2)}_n \boxtimes f^{(1)}_{n+1} C_n \cong \Rightarrow D_n$ is an isomorphism.

**Example A.15.** We give some examples.

1. When $D_n = C_n$, the identity weak morphism $\text{id}_C$ is defined by

   $$\text{id}_C := (1_{n+1}, 1_n, \text{id}_{C_n}). \quad (A.6)$$

2. When $a : C_n \rightarrow D_n$ is an isomorphism, it can also expressed as a triple $a = (1_{n+1}, 1_n, a)$. Namely, the information of an isomorphism is completely encoded in the third component of the triple, i.e. $a = a_{n-1} = a^{(3)}_{n-1}$.

3. For each $nD$ topological order $C_n$, there is a natural unit weak morphism $1_n \rightarrow \iota_C : C_n$ given by

   $$\iota_C = (1_{n+1}, C_n, 1_n \boxtimes C_n = C_n \xrightarrow{\text{id}_C} C_n). \quad (A.7)$$

There are morphisms between two morphisms.

**Definition A.16.** Let $f$ and $g$ be two weak morphisms from $C_n$ to $D_n$. A 2-morphism $\phi : f \Rightarrow f'$ from $f$ to $f'$, is a pair of morphisms $(a, b)$, where

1. $a : f^{(1)}_{n+1} \rightarrow g^{(1)}_{n+1}$ is a morphism;
2. $b : f^{(2)}_n \rightarrow g^{(2)}_n$ is a morphism.
such that they can form the following triangle-shaped physical configuration

\[
\begin{align*}
(f^{(2)}_n &\boxtimes a^{(1)}_{n+2} f^{(1)}_{n+1}) \boxtimes (b^{(2)}_{n+1} f^{(1)}_{n+1} C_n) \\
&\cong (f^{(2)}_n \boxtimes a^{(1)}_{n+2} f^{(1)}_{n+1}) \boxtimes g^{(2)}_n g^{(1)}_{n+1} C \cong D, \\
\end{align*}
\]

in which all the \((n+2)\)-phases are not necessarily closed. \(\phi\) is called \textit{closed} if \(a^{(1)}_{n+2}\) is closed.

**Remark A.17.** The isomorphisms \(f^{(3)}_{n-1}, g^{(3)}_{n-1}, a^{(3)}_n\) and \(b^{(3)}_{n-1}\) is not explicit shown in (A.8). Their roles can be seen by squeezing the “triangle” in (A.8) to a single “point”, which is an \(n\D\) \(\D_n\)-phase. This collapsing process can be described by the composition of the following isomorphisms:

\[
(b^{(2)}_n \boxtimes a^{(2)}_{n+1} b^{(1)}_{n+2} f^{(1)}_{n+1} C_n) \boxtimes (f^{(1)}_{n+1}) \cong (b^{(2)}_n \boxtimes a^{(2)}_{n+1} b^{(1)}_{n+2} f^{(1)}_{n+1} C_n) \boxtimes g^{(2)}_n g^{(1)}_{n+1} C \cong D,
\]

in which the first isomorphism is induced by \(a^{(3)}_n\) and the second one by \(b^{(3)}_{n-1}\), and the last one is defined by \(g^{(3)}_{n-1}\). By choosing a different way of collapsing, we obtain the following identity:

\[
b^{(2)}_n \boxtimes a^{(2)}_{n+1} b^{(1)}_{n+2} a^{(1)}_{n+1} g^{(2)}_n g^{(1)}_{n+1} (f^{(1)}_{n+1} C_n) \cong D_n,
\]

where the first isomorphism is defined by \((f^{(3)}_{n-1})^{-1}\) and the second one is due to the independence of how we collapse the “triangle”. Above two mathdisplays summarize what roles the isomorphisms \(f^{(3)}_{n-1}, g^{(3)}_{n-1}, a^{(3)}_n\) and \(b^{(3)}_{n-1}\) play in the collapsing processes.

**Remark A.18.** Weak morphisms are not composable in general. Two morphisms between weak morphisms \(\phi : f \Rightarrow g\) and \(\psi : g \Rightarrow h\) cannot be composed either.

If \(f : \mathcal{C}_n \rightarrow \mathcal{D}_n\) be a morphism, then the triple \((Z(\mathcal{C}_n), f^{(0)}, f^{(1)})\) is automatically a weak morphism. In this case, an isomorphism \(\phi\) between two morphisms is automatically a morphism between two weak morphisms.

**Proposition A.19.** Let \(f : \mathcal{C}_n \rightarrow \mathcal{D}_n\) be a weak morphism. There is a morphism \(\tilde{f} : \mathcal{C}_n \rightarrow \mathcal{D}_n\) such that there is a unique \(2\)-morphism from \(f\) to \(\tilde{f}\).

**Proof.** Let \(f = (f^{(1)}_{n+1}, f^{(2)}_n, f^{(3)}_{n-1})\) be a weak morphism \(\mathcal{C}_n \rightarrow \mathcal{D}_n\). Using the strong unique-bulk hypothesis, find the unique bulk of the physical configuration of \(f^{(2)}_n \boxtimes f^{(1)}_{n+1} \mathcal{C}_n\) depicted in Fig.13 (a). Then we fold the vertical box anti-clockwise while keep the position of the “line segment” \([f^{(2)}_n, f^{(1)}_{n+1}, \mathcal{C}_n]\) fixed. Then we obtain an \((n+1)\D\) physical configuration as
Figure 13: We start from a physical configuration depicted in (a), in which $f_{n}^{(2)} \boxtimes f_{n+1}^{(1)} e_{n} \simeq D_{n}$. Both $X_{n+1}$ and $Y_{n+1}$ are uniqueness fixed by the unique-bulk hypothesis. By folding the topless box anti-clockwise to the horizontal line, we obtain the physical configuration depicted in (b).

shown in Fig. 13 (b), in which the $C_{n}$-phase remains the same but now becomes a gapped boundary of its bulk $Z_{n}(e_{n})$, $e_{n} = f_{n}^{(2)} \boxtimes Z_{n+1}(f^{(1)}) P_{n}(y_{n+1})$ and $Z_{n}(D_{n}) = X_{n+1} \boxtimes Z_{n+1}(f^{(1)}) Y_{n+1}$. Moreover, we must have $C \boxtimes Z_{n}(e_{n}) e_{n} \simeq D_{n}$. Indeed, the dimensional reduction is independent of how we squeezing the configuration in detail. So one can choose to squeeze the topless box in (a) horizontally, and obtain a vertical “line” given by $Z_{n}(D_{n})$ with the bottom end given by $D_{n}$. Therefore, the identity $C \boxtimes Z_{n}(e_{n}) e_{n} \simeq D_{n}$ must hold. Then there is a morphism $\tilde{f} = (e_{n}, f_{n+1}^{(1)})$ such that the above dimensional reduction process defines a 2-morphism from $(f_{n+1}^{(1)}, f_{n}^{(2)}, f_{n-1}^{(3)})$ to $(Z_{n}(e_{n}), e_{n}, f_{n+1}^{(1)})$. This 2-morphism is unique by the strong unique-bulk hypothesis.

A.4 A few mathematical results on fusion categories

In this subsection, we prove a few mathematical results on fusion categories that are needed in this work. We only sketch the proofs. Details will appear elsewhere.

Given a fusion category $\mathcal{C}$, two left $\mathcal{C}$-modules $M, N$, Tambara’s tensor product $M^{\text{op}} \boxtimes \mathcal{C} N$ is defined similar to the usual tensor product of modules over an ordinary algebra [Ta]. When both $M$ and $N$ are semisimple, one can show that $M^{\text{op}} \boxtimes \mathcal{C} N \simeq \text{Fun}_{\mathcal{C}}(M, N)$ [ENO09]. In fact, the balanced functor

$$ M^{\text{op}} \times N \to \text{Fun}_{\mathcal{C}}(M, N), \quad (x, y) \mapsto \text{Hom}_{\mathcal{M}}(x, -) \otimes y $$

induces an equivalence

$$ M^{\text{op}} \boxtimes \mathcal{C} N \simeq \text{Fun}_{\mathcal{C}}(M, N). \quad (A.9) $$

Here $\text{Hom}$ denotes the internal hom.

**Proposition A.20.** Let $\mathcal{C}$ and $\mathcal{D}$ be two fusion categories, $M$ a semisimple left $\mathcal{C}$-module, $M'$ a semisimple $\mathcal{C}$-$\mathcal{D}$-bimodule and $N$ a semisimple left $\mathcal{D}$-module. The balanced functor

$$ \text{Fun}_{\mathcal{C}}(M, M') \times N \to \text{Fun}_{\mathcal{C}}(M, M' \boxtimes_{\mathcal{D}} N), \quad (f, y) \mapsto f(-) \boxtimes y $$

induces an equivalence

$$ \text{Fun}_{\mathcal{C}}(M, M') \boxtimes_{\mathcal{D}} N \simeq \text{Fun}_{\mathcal{C}}(M, M' \boxtimes_{\mathcal{D}} N). $$

$$ (A.12) $$
Proof. We have equivalences
\[ \text{Fun}_C(M, M') \boxtimes_D N \simeq M'^{\text{op}} \boxtimes_C M' \boxtimes_D N \simeq \text{Fun}_C(M, M' \boxtimes_D N). \] (A.13)
Composing them with the balanced functor
\[ M'^{\text{op}} \times M' \times N \to \text{Fun}_C(M, M') \boxtimes_D N, \]
\[ (x, x', y) \mapsto (\text{Hom}_{C,M}(x, -) \otimes x') \otimes y \]
we obtain the balanced functor
\[ M'^{\text{op}} \times M' \times N \to \text{Fun}_C(M, M' \boxtimes_D N), \]
\[ (x, x', y) \mapsto \text{Hom}_{C,M}(x, -) \otimes (x' \boxtimes y), \]
which factors through the balanced functor \((A.11)\). Consequently, the total equivalence of \((A.13)\) is induced from \((A.11)\).

The following proposition is proved in a similar way.

Proposition A.21. Let \(C, D, E\) be fusion categories, \(M, M'\) two semisimple \(C\)-\(D\)-bimodules and \(N, N'\) two semisimple \(D\)-\(E\)-bimodules. The balanced functor
\[ \text{Fun}_{C[D]}(M, M') \times \text{Fun}_{D[E]}(N, N') \to \text{Fun}_{C[E]}(M \boxtimes_D N, M' \boxtimes_D N'), \quad (f, f') \mapsto f \boxtimes f' \] (A.14)
induces an equivalence
\[ \text{Fun}_{C[D]}(M, M') \boxtimes_{Z(D)} \text{Fun}_{D[E]}(N, N') \simeq \text{Fun}_{C[E]}(M \boxtimes_D N, M' \boxtimes_D N'). \] (A.15)

Proof. We have equivalences:
\[ \text{Fun}_{C[D]}(M, M') \boxtimes_{Z(D)} \text{Fun}_{D[E]}(N, N') \]
\[ \simeq (M'^{\text{op}} \boxtimes_{C[D]^{rev}} M') \boxtimes_{Z(D)} (N^{\text{op}} \boxtimes_{E[D]^{rev}} N') \]
\[ \simeq (M'^{\text{op}} \boxtimes_C M') \boxtimes_{D[E]^{rev}} D^{\text{op}} \boxtimes_{Z(D)} (N^{\text{op}} \boxtimes_{E[D]^{rev}} N') \]
\[ \simeq (M'^{\text{op}} \boxtimes_C M') \boxtimes_{D[E]^{rev}} (D \boxtimes D^{\text{rev}}) \boxtimes_{Z(D)} (N^{\text{op}} \boxtimes_{E[D]^{rev}} N') \]
\[ \simeq (M'^{\text{op}} \boxtimes_C M') \boxtimes_{D[E]^{rev}} (N^{\text{op}} \boxtimes_{E[D]^{rev}} N') \]
\[ \simeq (M \boxtimes D N)^{\text{op}} \boxtimes_{C[D]^{rev}} (M' \boxtimes_D N') \]
\[ \simeq \text{Fun}_{C[E]}(M \boxtimes_D N, M' \boxtimes_D N'). \] (A.16)

Observe that the monoidal equivalence \(D^{\text{op}} \boxtimes_{Z(D)} D \simeq \text{Fun}_{Z(D)}(D, D) \simeq D \boxtimes D^{\text{rev}}\) carries \(1_D \boxtimes_{Z(D)} 1_D\) to \(\bigoplus_d d' \boxtimes d\) where \(d\) runs over all simple objects of \(D\).

Consider the balanced functor
\[ M'^{\text{op}} \times M' \times N^{\text{op}} \times N' \to \text{Fun}_{C[D]}(M, M') \boxtimes_{Z(D)} \text{Fun}_{D[E]}(N, N'), \]
\[ (x, x', y, y') \mapsto (\text{Hom}_{C[D]^{rev}} M(x, -) \otimes x') \boxtimes_{Z(D)} (\text{Hom}_{D[E]^{rev}} N(y, -) \otimes y'). \]
Composing it with \((A.16)\), we obtain the balanced functor

\[
\mathcal{M}^{op} \times \mathcal{M}' \times \mathcal{N}^{op} \times \mathcal{N}' \to \text{Fun}_{\mathcal{C}}(\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N}, \mathcal{M}' \boxtimes_{\mathcal{D}} \mathcal{N}'),
\]

\[
(x, x', y, y') \mapsto \bigoplus_d \text{Hom}_{\mathcal{C} \boxtimes_{\mathcal{D}} \mathcal{N}}(x \boxtimes_{\mathcal{D}} (d'^{\vee} \otimes y), -) \otimes (x' \boxtimes_{\mathcal{D}} (d \otimes y')),
\]

which factors through the balanced functor \((A.14)\). Consequently, the total equivalence of \((A.16)\) is induced from \((A.14)\). \(\square\)

**Remark A.22.** In the special case where \(\mathcal{M} = \mathcal{M}' = \mathcal{N} = \mathcal{N}'\), \((A.15)\) is a monoidal equivalence.

Let \(f : \mathcal{C} \to \mathcal{D}\) be a monoidal functor between fusion categories. Then \(\text{Fun}_{\mathcal{C}}(\mathcal{E}, \mathcal{D})\) is a fusion category \([\text{GNN}]\) which can be described as follows. An object is a pair:

\[
(d, \beta_{-d} = \{ f(c) \otimes d \xrightarrow{\beta_{c,d}} d \otimes f(c) \}_{c \in \mathcal{C}})
\]

where \(d \in \mathcal{D}\) and \(\beta_{c,d}\) is an isomorphism in \(\mathcal{D}\) natural with respect to the variable \(c \in \mathcal{C}\) and satisfying \(\beta_{c,d} \circ \beta_{c',d'} = \beta_{c',d'} \circ \beta_{c,d}\). A morphism \((d, \beta) \to (d', \beta')\) is defined by a morphism \(\psi : d \to d'\) respecting \(\beta_{c,d} \) and \(\beta_{c,d}'\) for all \(c \in \mathcal{C}\). The monoidal structure is given by the formula \((d, \beta) \otimes (d', \beta') = (d \otimes d', \beta' \circ \beta)\).

**Corollary A.23.** Let \(\mathcal{C} \to \mathcal{D}\) be a monoidal functor between fusion categories. The evaluation functor \(\mathcal{C} \times \text{Fun}_{\mathcal{C}}(\mathcal{E}, \mathcal{D}) \to \mathcal{D}, (x, f) \mapsto f(x)\) induces a monoidal equivalence

\[
\mathcal{C} \boxtimes_{Z(\mathcal{C})} \text{Fun}_{\mathcal{C}}(\mathcal{E}, \mathcal{D}) \simeq \mathcal{D}.
\]

**Proof.** It follows from \(\text{Fun}_{\mathcal{C}}(\mathcal{C}, \mathcal{C}) \simeq \mathcal{C}\) and \(\text{Fun}_{\mathcal{C}}(\mathcal{C} \boxtimes_{\mathcal{C}} \mathcal{E}, \mathcal{E} \boxtimes_{\mathcal{C}} \mathcal{D}) \simeq \mathcal{D}\). \(\square\)

As an example of dimensional reduction, when \(\mathcal{C} = \mathcal{E} = \mathcal{Vect}\), above result implies that

\[
\text{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N}, \mathcal{N}) \simeq \text{Fun}_{\mathcal{C}}(\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N}, \mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N}),
\]

which is a multi-fusion category with a trivial monoidal center \([\text{ENO09}]\).

**Remark A.24.** In the case, \(\mathcal{D} = \mathcal{C}\) and \(\mathcal{M} = \mathcal{N} = \mathcal{C}\), \((A.18)\) gives a monoidal equivalence:

\[
\mathcal{C} \boxtimes_{Z(\mathcal{C})} \mathcal{C}^{rev} \simeq \text{Fun}_{\mathcal{C}}(\mathcal{C}, \mathcal{C}).
\]

On the other hand, there is an equivalence

\[
\text{Fun}_{\mathcal{C}}(\mathcal{C}, \mathcal{C}) \simeq \mathcal{C}^{op} \boxtimes \mathcal{C}.
\]

as categories. Therefore, the category \(\mathcal{C} \boxtimes_{Z(\mathcal{C})} \mathcal{C}^{rev}\) has another monoidal structure inherit from that of \(\mathcal{C}^{op} \boxtimes \mathcal{C}\). This monoidal structure is different from that in \((A.19)\). In particular, the right hand side of \((A.19)\) is a multi-fusion category, that of \((A.20)\) is a fusion category. The monoidal structure in \((A.19)\) is the one used in Example 2.22.
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