KÄHLER-EINSTEIN METRICS AND DING FUNCTIONAL ON Q-FANO GROUP COMPACTIFICATIONS.

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Abstract. Let $G$ be a complex, connect reductive Lie group which is the complexification of a compact Lie group $K$. Let $M$ be a Q-Fano $G$-compactification. In this paper, we first prove the uniqueness of $K \times K$-invariant (singular) Kähler-Einstein metric. Then we show the existence of (singular) Kähler-Einstein metric implies properness of the reduced Ding functional. Finally, we show that the barycenter condition is also necessary of properness.

1. Introduction

Let $G$ be an $n$-dimensional connect, complex reductive group which is the complexification of a compact Lie group $K$, with complex structure $J_G$. A projective normal variety $M$ is called a (bi-equivariant) compactification of $G$ (or $G$-compactification for simplicity) if it admits a holomorphic $G \times G$-action with an open and dense orbit isomorphic to $G$ as a $G \times G$-homogeneous space (cf. [1, 2, 24]). If there is in addition a $G \times G$-linearized ample (Q-Cartier) line bundle $L$ on $M$, then $(M, L)$ is called a polarized compactification of $G$ (cf. [3, Section 2.1]). In particular, when $K^{-1}_M$ is an ample Q-Cartier line bundle, we call $M$ a Q-Fano $G$-compactification. We refer the reader to [24, 3, 11, 12], etc. for further knowledge.

Fix a maximal complex torus $T^C$ of $G$. Up to $G \times G$-equivariant isomorphisms, polarized $G$-compactifications are in one-one correspondence with its associated polytopes (see Section 2.1 below for detail) $P$ (cf. [3, Theorem 2.4]), which lies in $J^*_G$. Let $\Phi$ be the root system of $(G, T^C)$ and $\Phi^+ \subset \Phi$ be a chosen set of positive roots. We defined $P_+$ to be the intersection of $P$ with the positive Weyl chamber defined by $\Phi^+$. In the following we take $P$ to be the polytope of $(M, K^{-1}_M)$. In terms of a weighted barycenter $b(2P_+)$ of twice the polytope $P_+$ (cf. (2.1) below), Li-Tian-Zhu in [19] proved the following criterion of existence of (singular) Kähler-Einstein metric on a Q-Fano group compactification:

Theorem 1.1. Let $M$ be a Q-Fano $G$-compactification whose associated polytope satisfies the fine condition. Then $M$ admits a Kähler-Einstein metric if and only if

$$b(2P_+) \in 4\rho + \Xi,$$

where $2\rho = \sum_{\alpha \in \Phi^+, \alpha}$ and $\Xi$ is the relative interior of the cone generated by $\Phi^+$.
Theorem [11] was first proved by Delcroix [11] for smooth Fano compactifications (see also [16] for general polarized cases). In [19], Li-Tian-Zhu proved that (1.1) implies properness of Ding functional (modulo group action) on the $(\mathcal{E}^1_{K\times K}(M, K^1))$-space introduced by [6]. Hence prove the sufficiency of (1.1) by using the variation method. In fact, the author proved that when (1.1) holds, the reduced Ding functional $D(\cdot)$ is proper on a set of convex functions $\mathcal{E}^1_{K\times K}(2P)$ on $2P$, which consists the reduction of $\mathcal{E}^1_{K\times K}(M, K^1)$-space (see Section 2.2 for detail).

On the other hand, for the direction of necessity, they showed that (1.1) is necessary for existence of (singular) Kähler-Einstein metric by checking $K$-stability, using the argument of [3] Section 3]. It is unknown whether (1.1) is also necessary for the reduced Ding functional being proper.

In this paper, we will prove that the existence of (singular) Kähler-Einstein metric implies properness of reduced Ding functional. And this forces $b(2P_\ast)$ to satisfy (1.1). Namely,

**Theorem 1.2.** Let $M$ be a Q-Fano $G$-compactification. If $M$ admits a (singular) Kähler-Einstein metric. Then

1. The reduced Ding functional $D(\cdot)$ is proper on $\mathcal{E}^1_{K\times K}(2P)$, i.e.
   \[
   D(u) \geq c_0 \int_{2P_\ast} u \omega dy - C_0, \quad \forall u \in \mathcal{E}^1_{K\times K}(2P)
   \]
   holds for suitable constants $c_0, C_0 > 0$.

2. The barycenter $b(2P_\ast)$ satisfies (1.1). Consequently, $M$ is $K$-stable.

As it is also worth mentioning that for a general smooth Fano manifold, Zhu [23] proposed the following conjecture (see also [18] Conjecture 7.7]):

**Conjecture 1.3.** [23] Conjecture 4.9 Let $(M, g)$ be an $n$-dimensional Kähler-Einstein manifold with $\omega_g \in 2\pi c_1(M)$. Denote by $\mathcal{R}$ the maximal compact subgroup of the automorphism group $\text{Aut}(M)$. Suppose that $\omega_g$ is $\mathcal{R}$-invariant. Then there are constants $c_0, C_0 > 0$ such that

\[
D(u) \geq c_0 \inf_{\sigma \in Z(\text{Aut}(M))} I(\phi_\sigma) - C_0, \quad \forall \mathcal{R}\text{-invariant Kähler potential } \phi.
\]

Here $I(\cdot)$ is the Aubin’s I-functional and $Z(\text{Aut}(M))$ denotes the center of $\text{Aut}(M)$.

Very recently, Hisamoto [15] Theorem 4.3] confirmed this conjecture. His approach uses deep results from [7] to check hypotheses of the properness principle in [10] Section 3.2]. While by a reduction progress [17] Lemma 4.10-4.11], it is direct to derive from Theorem [12] that an analogous of (1.3) holds if we replace $Z(\text{Aut}(M))$ by a possibly larger group $Z(\mathcal{R})$, but on a larger space which consists of all $K \times K$-invariant Kähler potentials. It seems to be more nature and practical to consider $K \times K$ than $\mathcal{R}$ on group compactifications, since in many cases, the full automorphism group $\text{Aut}(M)$, as well as $\mathcal{R}$, is still unknown.

Our method is direct by studying the reduced Ding functional on a space of convex functions $\mathcal{E}^1_{K\times K}(2P)$. From existence to properness we want to use the argument from [10]. For this purpose, we need first to prove a uniqueness of $K \times K$-invariant (singular) Kähler-Einstein metrics (see Theorem [5.1] below). Then we show the existence of (singular) Kähler-Einstein metric implies properness of the reduced Ding functional $D(\cdot)$. Once this shown, (1.1) then follows from an estimate of $D(\cdot)$ along a special ray in $\mathcal{E}^1_{K\times K}(2P)$ (cf. Proposition [4.1] (2) below).
The paper is organized as follows: in Section 2 we collect basic definitions and properties concerning polarized compactifications and reduced Ding functional on them. Section 3 is devoted to the uniqueness theorem for a general $\mathbb{Q}$-Fano variety, namely Theorem 3.1. In section 4 we prove Theorem 1.2. The two Appendixes collect useful properties concerning structure of $\text{Aut}(M)$.

**Notations.** Now we fix the notations in the following sections except the Appendix. We denote by

- $K$ - a connected, compact Lie group;
- $G = K^\mathbb{C}$ - the complexification of $K$, which is a complex, connected reductive Lie group;
- $J_G$ - the complex structure of $G$;
- $T$ - a fixed maximal torus of $K$ and $T^\mathbb{C}$ its complexification;
- $a := J_G t$ - the non-compact part of $t^\mathbb{C}$;
- $\Phi$ - the root system with respect to $G$ and $T^\mathbb{C}$;
- $\Phi_+$ - a chosen system of positive roots in $\Phi$;
- $W$ - the Weyl group with respect to $G$ and $T^\mathbb{C}$;
- $\text{Ad}_\sigma(\cdot) := \sigma(\cdot)\sigma^{-1}$ - the conjugate of some subgroup or Lie algebra by some element $\sigma$.

**Acknowledgement.** The authors would like to sincerely thank Professor M. Brion for telling them Proposition 6.1 in [8], which plays a crucial role in proving Theorem 3.1. They would also like to thank Professor X. Zhu for introducing us this problem, Professor D. A. Timashëv and Professor T. Delcroix for valuable comments.

## 2. Preliminaries

### 2.1. Polarized compactifications and associated polytopes.

Let $(M, L)$ be a polarized compactification of $G$. It is known that the closure $Z$ of $T^\mathbb{C}$ in $M$, together with $L|_Z$ is a polarized toric variety. Indeed, $L|_Z$ is $W T^\mathbb{C}$-linearized. The polytope associated to $(M, K^\mathbb{C})$ is defined as the associated polytope of $(Z, K^{-1}_M|_Z)$ (cf. [3, Section 2.1] and [11, Section 2.2]). It is a strictly convex, $W$-invariant rational polytope in $a^\ast = J_t^\ast$. Denote by $a^\ast_+$ the corresponding positive Weyl chamber.

$$a^\ast_+ := \{ y \in a^\ast | \langle \alpha, y \rangle \geq 0, \ \forall \alpha \in \Phi_+ \}.$$

Choose a $W$-invariant inner product $\langle \cdot, \cdot \rangle$ on $a^\ast$ which extends the Cartan-Killing form on the semisimple part $a^\ast_{ss}$ (cf. [11] Introduction). Let $P_+$ be the positive part of $P$ defined by $P_+ = P \cap a^\ast_+$. The weighted barycenter of $2P_+$ is defined by

$$b(2P_+) = \frac{\int_{2P_+} y \pi(y) \, dy}{\int_{2P_+} \pi(y) \, dy},$$

where $\pi(y) = \prod_{\alpha \in \Phi_+} \langle \alpha, y \rangle^2$.

### 2.2. The $\mathcal{E}^1$-space and reduced Ding functional.

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2When $G = T^\mathbb{C}$, we take $a^\ast_+ = a^\ast$. 

2.2.1. Singular Kähler-Einstein metric and the $E^1$-space. For a $\mathbb{Q}$-Fano variety $M$, by Kodaira’s embedding Theorem, there is an integer $\ell > 0$ such that we can embed $M$ into a projective space $\mathbb{CP}^N$ by a basis of $H^0(M, K_M^{-\ell})$. Then we have a metric $\omega_0 = \frac{1}{\ell} \omega_{FS}|_M \in 2\pi c_1(M)$, where $\omega_{FS}$ is the Fubini-Study metric of $\mathbb{CP}^N$. Moreover, there is a Ricci potential $h_0$ of $\omega_0$ such that

$$\text{Ric}(\omega_0) - \omega_0 = \sqrt{-1} \partial \bar{\partial} h_0,$$

on the regular part $M_{\text{reg}}$.

In case that $M$ has only klt-singularities, $e^{h_0}$ is $L^p$-integrate for some $p > 1$ (cf. [13, 6]).

For a general (possibly unbounded) Kähler potential $\varphi$ and $\omega_\varphi := \omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi$, define its complex Monge-Ampère measure $\omega_\varphi^n$ by

$$\omega_\varphi^n = \lim_{j \to \infty} \omega_{\varphi_j}^n,$$

where $\varphi_j = \max\{\varphi, -j\}$. According to [6], we say that $\varphi$ (or $\omega_\varphi^n$) has full Monge-Ampère (MA) mass if

$$\int_M \omega_\varphi^n = \int_M \omega_0^n.$$

The MA-measure $\omega_\varphi^n$ with full MA-mass has no mass on the pluripolar set of $\varphi$ in $M$. Thus we only need to consider the measure on the regular part $M_{\text{reg}}$.

**Definition 2.1.** We call $\omega_\varphi$ a (singular) Kähler-Einstein metric on $M$ with full MA-mass if $\omega_\varphi^n$ has full MA-mass and $\varphi$ satisfies the following complex Monge-Ampère equation,

$$\omega_\varphi^n = e^{h_0 - \varphi} \omega_0^n. \tag{2.2}$$

As in the smooth case, there is a well-known Euler-Lagrange functional for Kähler potentials associated to (2.2), often referred as the Ding functional or $F$-functional, defined by (cf. [14])

$$F(\phi) = -\frac{1}{(n+1)V} \sum_{k=0}^n \int_M \phi \omega_\phi^k \wedge \omega_0^{n-k} - \log \left( \frac{1}{V} \int_M e^{h_0 - \varphi} \omega_0^n \right). \tag{2.3}$$

On a $\mathbb{Q}$-Fano manifold with klt-singularities, Berman-Boucksom-Eyssidieux-Guedj-Zeriahi [6] showed that $F(\cdot)$ can be defined on the space $E^1(M, -K_M)$ given by

$$E^1(M, -K_M) = \{ \phi \mid \phi \text{ has full MA mass and } \sup_M \phi = 0, I(\phi) = \int_M -\phi \omega_0^n < \infty \}.$$  

They also showed that $E^1(M, -K_M)$ is compact in certain weak topology. By a result of Davas [13], $E^1(M, -K_M)$ is in fact compact in the topology of $L^1$-distance. This provides a variational approach to (2.2).

It has been shown in [6] that if $\varphi$ is a solution of (2.2), then it is $C^\infty$ on $M_{\text{reg}}$. Thus $\omega_\varphi$ satisfies the usual Kähler-Einstein equation $\text{Ric}(\omega_\varphi) = \omega_\varphi$ on $M_{\text{reg}}$. 


2.2.2. $\mathcal{E}^1$-space and reduced Ding functional on $G$-compactifications. By the standard $KAK$-decomposition [25] Section 3.5.3, there is a bijection between $K \times K$-invariant functions $\Psi$ on $G$ and $W$-invariant functions $\psi$ on $a$ which is given by

$$\Psi(\exp(\cdot)) = \psi(\cdot) : a \rightarrow \mathbb{R}.$$ 

Clearly, when a $W$-invariant $\psi$ is given, $\Psi$ is well-defined, and vice versa. From now on, for simplicity, we will not distinguish $\psi$ and $\Psi$, and we will call $\Psi$ convex on $G$ if $\psi$ is convex on $a$.

Such a correspondence can be extended to $K \times K$-invariant quantities on whole $M$ (see [21] Section 3.4). For example, let $\omega \in 2\pi c_1(L)$ be a $K \times K$-invariant Kähler metric of $M$. Then $\omega|_Z$ is a toric Kähler metric in $2\pi c_1(L|_Z)$ and there is a $W$-invariant, strictly convex function $\psi$ on $a$ such that (cf. [4])

$$\omega = \sqrt{-1}\partial\bar{\partial}\psi, \text{ on } G.$$ 

Conversely, any $W$-invariant toric Kähler metric $\omega = \sqrt{-1}\partial\bar{\partial}\psi \in 2\pi c_1(L|_Z)$ on $Z$ extends to a $K \times K$-invariant Kähler metric $\omega \in 2\pi c_1(L)$ on $M$.

Denote by $O$ the origin. In the following, we take $L = K_M^{-1}$ and fix a background metric $\omega_0 = \sqrt{-1}\partial\bar{\partial}\psi_0$ such that

$$\inf_a \psi_0 = \psi_0(O) = 0,$$

We restrict ourself to $\mathcal{E}_K^1(M, K_M^{-1})$, the space of $K \times K$-invariant functions $\phi$ in $\mathcal{E}^1(M, K_M^{-1})$, such that $\psi_0 = \psi_0 + \phi$ is normalized as [23]. For any such $\phi$, consider the Legendre function $u_\phi$ of $\psi_0$. As in [9], it is showed in [19] Section 4] that

**Theorem 2.2.** A Kähler potential $\phi \in \mathcal{E}_K^1(M, -K_M)$ if and only if the Legendre function $u_\phi$ of $\psi_0$ lies in

$$\mathcal{E}_K^1(2P) = \{u | u \text{ is convex, } W \text{-invariant on } 2P \text{ which satisfies}\}$$

$$\inf_{2P} u = u(O) = 0 \text{ and } \int_{2P^+} u \pi dy < +\infty\}.$$ 

Hence the Legendre transformation gives a bijection between $\mathcal{E}_K^1(M, -K_M)$ and $\mathcal{E}_K^1(2P)$. As a consequence, $u_\phi$ is locally bounded in $\text{Int}(2P)$ if $\phi \in \mathcal{E}_K^1(M, -K_M)$. 

For any $u \in \mathcal{E}_K^1(2P)$, its Legendre function

$$\psi_u(x) = \sup_{y \in 2P} \{ \langle x, y \rangle - u(y) \} \leq v_{2P}(x)$$ 

corresponds to a $K \times K$-invariant weak Kähler potential $\phi_u = \psi_u - \psi_0$ which belongs to $\mathcal{E}_K^1(M, -K_M)$. As we know, $e^{-\phi_u} \in L^p(\omega_0)$ for any $p \geq 0$. Thus

$$\int_{\mathbb{R}^+} e^{-\psi_u} J(x) dx$$

is well-defined.

Define the following functional on $\mathcal{E}_K^1(2P)$ by

$$D(u) = \mathcal{L}(u) + \mathcal{F}(u),$$

where

$$\mathcal{L}(u) = \frac{1}{V} \int_{2P^+} u \pi dy - u(4\rho)$$

(2.5)
and

\begin{equation}
F(u) = -\log \left( \int_{\mathbb{R}_+} e^{-\psi_u} J(x) dx \right) + u(4\rho).
\end{equation}

It is easy to see that on a $\mathbb{Q}$-Fano compactification of $G$,

\begin{equation}
\mathcal{L}(u_\phi) + u_\phi(4\rho) = -\frac{1}{(n+1)V} \sum_{k=0}^n \int_M \phi \omega_k^* \wedge \omega_0^{n-k}
\end{equation}

and $\mathcal{D}(u_\phi)$ is just the Ding functional $F(\phi)$ for $\phi \in \mathcal{E}_{K \times K}(M, K_M^{-1})$ (2.3). Hence in the following we call $\mathcal{D}(\cdot)$ the reduced Ding functional.

It is showed in [19, Section 6] that under the condition (1.1), the minimizer of (2.2) Uniqueness of (singular) Kähler-Einstein metrics

Let $M$ be any two metrics that can only be different from a $\mathbb{Q}$-Fano variety. It is well-known that when $M$ admits a (singular) Kähler-Einstein metrics, the neutral component $\text{Aut}^0(M)$ of its automorphism group $\text{Aut}(M)$ is reductive, and is the complexification of the isometry group of this (singular) Kähler-Einstein metrics [6, Theorem 5.2]. Furthermore, the (singular) Kähler-Einstein metrics is unique up to an $\text{Aut}^0(M)$-action [6, Theorem 5.1].

In this section, we will further discuss the uniqueness of $K \times K$-invariant (singular) Kähler-Einstein metrics on a $\mathbb{Q}$-Fano $G$-compactification and prove that any such two metrics can only be different from a $Z(G)$-action.

**Theorem 3.1.** Let $M$ be a $\mathbb{Q}$-Fano $G$-compactification. Then the $K \times K$-invariant (singular) Kähler-Einstein metric, if exists, is unique up to a $Z(G)$-action.

**Proof.** Denote by $\mathfrak{G} = \text{Aut}^0(M)$ and fix a maximal compact subgroup of $M$. Let $\omega_1, \omega_2$ be two $K \times K$-invariant (singular) Kähler-Einstein metrics and denote by $\mathfrak{R}_1, \mathfrak{R}_2$ their groups of isometry, respectively. Then $\mathfrak{G} = \mathfrak{R}_1^C = \mathfrak{R}_2^C$. Hence there is a $\sigma \in \mathfrak{G}$ such that $\text{Ad}_\sigma \mathfrak{R}_1 = \mathfrak{R}_2$.

Since $\mathfrak{G}$ is reductive, by Proposition [6,1] $T \times T(\subset K \times K)$ extends to a unique maximal torus $\Xi$ of $\mathfrak{G}$. Hence $\Xi \subset \mathfrak{R}_1 \cap \mathfrak{R}_2$ and up to replacing $\sigma$ by $\sigma \cdot \tau$ for some $\tau \in \mathfrak{R}_2$, we have $\text{Ad}_\sigma \tau = \tau$. Thus both $\text{Ad}_\sigma : \mathfrak{T}$ and $\Xi$ are contained in $\mathfrak{R}_1$.

Since $\Xi$ is maximal, there are $\sigma' \in N_{\mathfrak{R}_1}(\Xi), \sigma'' \in \mathfrak{R}_1$ and a $t \in \Xi^C$ such that $\sigma'' \cdot \sigma^{-1} = \sigma' \cdot t$. We have $t = (\sigma'^{-1} \cdot \sigma'') \cdot \sigma^{-1}$ Hence $\text{Ad}_{\tau^{-1}} \mathfrak{R}_1 = \mathfrak{R}_2$.

On the other hand, as both $\mathfrak{R}_1$ and $\mathfrak{R}_2 = \text{Ad}_{\tau^{-1}} \mathfrak{R}_1$ contains $K \times K$, we see that $K \times K, \text{Ad}_{\tau^{-1}}(K \times K) \subset \mathfrak{R}_1$.

By Lemma [5,1] there is a $\tilde{\sigma} \in \Xi$ such that $t' := t^{-1} \cdot \tilde{\sigma} \in C_{K \times K}(\mathfrak{G})$, and

\begin{align*}
\omega_2 &= \sigma^* \omega_1 = (t^{-1} \cdot (\sigma'^{-1} \cdot \sigma''))^* \omega_1 \\
&= (t^{-1})^* \omega_1 = t'^* (\tilde{\sigma}^{-1})^* \omega_1 = t'^* \omega_1.
\end{align*}

Recall that $G \times G = (K \times K)^C$. Hence $t' \in C_G(G)$. By [22, Proposition 1.8], $t'$ can be realized by a $Z(G)$-action.

\hfill \Box
4. Properness of Modified Ding functional

In this section, we will prove that when a $\mathbb{Q}$-Fano $G$-compactification $M$ admits a (singular) Kähler-Einstein metric, the reduced Ding functional defined in [19] will be proper. Moreover, this implies (1.1) and consequently by [12], $M$ is K-stable.

**Proposition 4.1.** Suppose that the barycenter $b(2P_+) \in a_\text{ss}$. Then:

1. If $D(\cdot)$ is bounded from below on $E_{K \times K}(2P)$. Then
   \[ b(2P_+) - 4\rho \in \Xi. \]  
   \[ (4.1) \]

2. If (1.2) holds for some uniform constants $c_0, C_0 > 0$. Then we have (1.1).

**Proof.** For item (1). Suppose that (4.1) is not true. Without loss of generality we may assume

\[ b(2P_+) - 4\rho = \sum_{i=1}^r c_i \alpha_i \]

with $c_1 < 0$. Here $\Phi_{+,s} = \{\alpha_1, \ldots, \alpha_r\}$ are the simple roots. Let $\{w_i\}_{i=1}^r$ be the corresponding fundamental weights such that

\[ \langle \alpha_i, w_j \rangle = \frac{1}{2} |\alpha_i|^2 \delta_{ij}. \]

For $\lambda > 0$, set

\[ u_\lambda(y) = \begin{cases} \max \{\langle w(\varpi_1), y \rangle | w \in W\}, & y \in 2P, \\ +\infty, & \text{otherwise}. \end{cases} \]

Then by (2.5),

\[ L(u_\lambda) = \lambda \cdot \frac{1}{2} c_1 |\alpha_1|^2. \]

Let $\psi_\lambda$ be the Legendre function of $u_\lambda$ and

\[ \bar{\psi}_\lambda = \psi_\lambda - 4\rho(x). \]

Then by (2.6) and [17] Lemma 4.8,

\[ F(u_\lambda) = -\log \int_{2P} e^{-(\bar{\psi}_\lambda - \inf_{2P_+} \bar{\psi}_\lambda)} \prod_{\alpha \in \Phi_+} \left( \frac{1 - e^{-2\alpha(x)}}{2} \right)^2 dx. \]  
   \[ (4.3) \]

We want to compute out $\psi_\lambda$. Divide $a^*$ into cones $\sigma_1, \ldots, \sigma_{s_0}$ so that each $\sigma_i$ is a linear domain of the function $u_\lambda$, that is

\[ u_\lambda|_{\sigma_i} = \lambda \langle w_i(\varpi), y \rangle \quad \text{for some } w_i \in W. \]

Clearly there is a unique one that contains $a^*_+$ and we denote it by $\sigma_+$. For a convex domain $\Omega \subseteq \mathbb{R}^n$, denote by $\nu_\Omega$ its support function. By definition,

\[ \psi_\lambda(x) = \sup_y \{ \langle x, y \rangle - u_\lambda(y) \} \]

\[ = \max_{i=1, \ldots, s_0} \sup_{y \in 2P \cap \sigma_i} \{ \langle x, y \rangle - \lambda \langle w_i(\varpi), y \rangle \} \]

\[ = \max_{i=1, \ldots, s_0} \nu_{2P \cap \sigma_i}(x - \lambda w_i(\varpi)). \]

Since $4\rho \in \text{Int}(2P \cap \sigma_+)$, it is direct to check that

\[ \nu_{2P \cap \sigma_+}(x - \lambda \varpi) - 4\rho(x - \lambda \varpi) - 4\lambda \rho(\varpi) \geq 0. \]
It then follows
\[
\tilde{\psi}_\lambda(x) \geq \tilde{\psi}(\lambda \varpi_1) = -4\lambda \rho(\varpi_1).
\]
Consequently,
\[
\tilde{\psi}_\lambda(x) - \inf_{a_+} \tilde{\psi}_\lambda \leq v_2 P(x - \lambda \varpi_1) - 4\lambda \rho(x - \varpi_1) =: \tilde{v}_2 P(x - \lambda \varpi_1).
\]
Note that for some constant\(4\rho \in \text{Int}(2P)\), thus\(\tilde{v}_2 P(x - \lambda \varpi_1)\) is proper on\( a \). Fix a \(\delta_1 > 0\), there is a \(\lambda_1(\delta_1)\) and a convex domain \(\Omega(\delta_1)\) such that for any \(\lambda \geq \lambda_1(\delta_1)\), we have\(\lambda\)
\[
(4.4) \quad \tilde{\psi}_\lambda(x) - \inf_{a_+} \tilde{\psi}_\lambda \leq \tilde{v}_2 P(x - \lambda \varpi_1) \leq \delta_1, \quad \forall x \in (\lambda \varpi_1 + \Omega(\delta_1)) \subset a_+.
\]
Here \(\Omega(\delta_1)\) can be taken as
\[
\Omega(\delta_1) = \{\tilde{v}_2 P(x) = v_2 P(x) - 2 \sum_{\alpha \in \Phi_+} (x) \leq \delta_1\} \cap (\cap_{i \geq 2}\{\alpha_i (x) \geq 0\}).
\]
On the other hand, for any \(\delta_0 > 0\), as \(\lambda \to +\infty\), we have \(1 - e^{-2\lambda_0(x)} \geq \delta_0\) whenever \(\alpha(x) \geq -\frac{1}{2\lambda} \log(1 - \delta_0)\).
Thus, we can choose a sufficiently small \(\delta_0\) which does not depend on \(\lambda\), so that the domain
\[
\Omega_0 := (\lambda \varpi_1 + \Omega(\delta_1)) \cap (\cap_{\alpha \in \Phi_+} \{1 - e^{-2\lambda_0(x)} \geq \delta_0\})
\]
satisfies
\[
(4.5) \quad \Vol(\Omega_0) \geq V_0, \quad \forall \lambda \geq \lambda_1(\delta_1),
\]
for some constant \(V_0 > 0\). By\(4.3\), \(4.4\) and \(4.5\), we have
\[
\mathcal{F}(u_\lambda) \leq -\log \int_{(\lambda \varpi_1 + \Omega_0)} e^{-\delta_1 (\delta_0^{-r})^n} dx = -\log(V_0 e^{-\delta_0 (\delta_0^{-r})^n}), \quad \lambda \to +\infty.
\]
Combining the above estimate with \(2.5\), we see that
\[
(4.6) \quad \mathcal{D}(u_\lambda) \leq \frac{1}{2} |\alpha_1|^2 c_1 \lambda + C_0, \quad \lambda \to +\infty,
\]
where \(C_0 = -\log(V_0 e^{-\delta_0 (\delta_0^{-r})^n})\). The right-hand side goes to \(-\infty\) since \(c_1 < 0\). A contradiction.
The proof of item (2) goes in a similar way. Suppose that \(1.1\) is not true, we may assume in \(4.2\) that \(c_1 \leq 0\). It then follows that \(4.0\) contradicts with \(4.2\). \(\square\)

**Remark 4.2.** The condition \(b(2P_+) \in a_+\) is equivalent to that the Futaki invariant of the compactification vanishes. This is because if \(u_0 \in \mathcal{E}_{K \times K}(2P)\) whose Legendre function corresponds to a K"ahler metric \(\omega_0 \in 2\pi c_1(M)\) and \(\theta \in a \cap \mathfrak{g}\), then \(\omega_{\phi_{\lambda}} := (e^{\lambda \phi})^* \omega_0\) is a family of \(K \times K\)-invariant metrics. Following the argument of \([17]\) Lemma 4.1, it is easy to see that
\[
Fut(\xi) = \frac{d}{d\lambda} \bigg|_{\lambda=0} F(\phi_{\lambda}) = b(2P_+)(\xi).
\]
Remark 4.3. It is showed in [19] Section 6] that \( D(\cdot) \) is convex along any linear path in \( \mathcal{E}^1_{K \times K}(2P) \). Thus when \( M \) admits a (singular) Kähler-Einstein metric \( \omega_0 \) (whose symplectic potential is \( u_0 \)), then

\[
D(u) \geq D(u_0).
\]

From Proposition 4.1 (1) and [12], we conclude that \( M \) is K-semistable. This is proved for a general \( \mathbb{Q} \)-Fano variety by Berman [5].

To prove Theorem 1.2 we need the following lemmas on geodesics in \( \mathcal{E}^1_{K \times K}(M, K^{-1}_M) \).

**Lemma 4.4.** Let \( \hat{K} \subset \text{Aut}(M) \) be a compact group. Suppose that \( \phi_0, \phi_1 \in \mathcal{E}^1_{\hat{K}}(M, K^{-1}_M) \) are two \( \hat{K} \)-invariant Kähler potentials and \( \{ \phi(x, t) \}_{t \in [\{ \mathbb{C} \} \cap \mathbb{R}^\mathbb{Z}) \in [0, 1]} \subset \mathcal{E}^1(M, K^{-1}_M) \) is a geodesic connecting them. Then for every \( t \), the function \( \phi(\cdot, t) : M \to \mathbb{C} \) is \( \hat{K} \)-invariant.

**Proof.** By definition,

\[
\phi(x, t) = \sup_{\tilde{\phi}} \{ \tilde{\phi}(x, t) | \tilde{\phi}|_{\text{Re}(t)=0}(z, t) \leq \phi_0(z), \tilde{\phi}|_{\text{Re}(t)=1}(z, t) \leq \phi_1(z), \forall z \in M \},
\]

where \( \tilde{\phi} \) is a family of continuous \( \omega_0 \)-psh functions. Thus for any \( \tilde{k} \in \hat{K} \)

\[
\phi(\tilde{k}x, t) = \sup_{\tilde{\phi}} \{ \tilde{\phi}(\tilde{k}x, t) | \tilde{\phi}|_{\text{Re}(t)=0}(z, t) \leq \phi_0(z), \tilde{\phi}|_{\text{Re}(t)=1}(z, t) \leq \phi_1(z) \}
\]

\[
= \sup_{\tilde{\phi}} \{ \tilde{\phi}(\tilde{k}x, t) | \tilde{\phi}|_{\text{Re}(t)=0}(\hat{\tilde{k}}^{-1}z, t) \leq \phi_0(\hat{\tilde{k}}^{-1}z), \tilde{\phi}|_{\text{Re}(t)=1}(\hat{\tilde{k}}^{-1}z, t) \leq \phi_1(\hat{\tilde{k}}^{-1}z) \}
\]

\[
= \sup_{\tilde{\phi}} \{ \tilde{\phi}(\tilde{x}, t) | \tilde{\phi}|_{\text{Re}(t)=0}(\hat{\tilde{x}}, t) \leq \phi_0(\hat{\tilde{x}}, t), \tilde{\phi}|_{\text{Re}(t)=1}(\hat{\tilde{x}}, t) \leq \phi_1(\hat{\tilde{x}}) \}
\]

\[
= \phi(x, t).
\]

Hence \( \{ \phi(x, t) \}_{t \in [\{ \mathbb{C} \} \cap \mathbb{R}^\mathbb{Z}) \in [0, 1]} \subset \mathcal{E}^1_{\hat{K}}(M, K^{-1}_M) \) as desired. \( \square \)

**Lemma 4.5.** Suppose that \( \phi_0, \phi_1 \in \mathcal{E}^1_{\hat{K} \times K}(M, K^{-1}_M) \). Let \( \{ \phi(x, t) \}_{t \in [\{ \mathbb{C} \} \cap \mathbb{R}^\mathbb{Z}) \in [0, 1]} \) be a variational family of \( \{ \phi(x, t) \} \) in \( \mathcal{E}^1_{\hat{K} \times K}(M, K^{-1}_M) \). By [19] Lemma 4.5], we have the length of \( \tilde{\phi}(x, t) \) is

\[
L[\phi] = \left( \int_0^1 \int_M |\phi_s(x, t)|^2 \omega_{\phi_s}^n \wedge dt \right)^{\frac{1}{2}}
\]

\[
\left( \int_0^1 \int_{\mathbb{A}_+^n} |\phi_s(x, t)|^2 \omega_{\mathbb{A}_+} \wedge dt \right)^{\frac{1}{2}},
\]

where the weighted Monge-Ampère measure \( \omega_{\mathbb{A}_+} \) is defined by [19] Definition 4.5] and \( \phi_s(x, t) \) denotes the derivative with respect to \( t \). Taking Legendre
transformation, we get
\[ L[\phi_s] = \left( \int_0^1 \int_{2P_+} |\dot{u}_s(x,t)|^2 \pi dy \wedge dt \right)^{\frac{1}{2}}. \]
It is then direct to conclude
\[ \ddot{u}_s = 0. \]
since \( s = 0 \) is a variational minima. \((4.7)\) then follows from the above equation. □

Proof of Theorem 1.2. We first show (1). We will use an argument of [10, Section 3]. Suppose that the Legendre function \( \psi_0 \) of some \( u_0 \in \mathcal{E}_{K \times K}^1(2P) \) defines a (singular) Kähler-Einstein metric \( \omega_0 = \sqrt{-1} \partial \bar{\partial} \psi_0 \) on \( M \). By [19, Lemma 6.4], we see that \( D(\cdot) \) is convex along and geodesics in \( \mathcal{E}_{K \times K}^1(2P) \).

On the other hand, by \([19, Claim 6.7]\), \( u_0 \) is a critical point of \( D(\cdot) \) on \( \mathcal{E}_{K \times K}^1(2P) \). Thus
\[ D(u) \geq D(u_0), \quad \forall u \in \mathcal{E}_{K \times K}^1(2P). \]
\((4.8)\)

Consider the ratio
\[ C := \inf \left\{ \frac{D(u) - D(u_0)}{d(u_0, u)} \mid u \in \mathcal{E}_{K \times K}^1(2P), \ d(u_0, u) \geq 1 \right\}, \]
where for any \( u, v \in \mathcal{E}_{K \times K}^1(2P) \), the distance
\[ d(u, v) := \left( \int_{2P_+} |u - v|^2 \pi dy \right)^{\frac{1}{2}}. \]

Let \( \psi_u, \psi_v \) be Legendre functions of \( u, v \), respectively, by Lemma 4.5 we see that \( d(u, v) \) coincides with the Mabuchi distance \( d_M \) of the two Kähler metrics \( \omega_u = \sqrt{-1} \partial \bar{\partial} \psi_u \) and \( \omega_v = \sqrt{-1} \partial \bar{\partial} \psi_v \).

It suffices to show that the constant \( C \) above is positive. Otherwise, we can find a sequence \( \{u_k\} \in \mathbb{N}^+ \) such that
\\[ \lim_{k \to +\infty} \frac{D(u_k) - D(u_0)}{d(u_0, u_k)} = 0. \]
For each \( k \in \mathbb{N}^+ \), consider the path
\[ u_k(t) = (1 - t)u_0 + tu_k, \quad t \geq 0. \]
Since for each \( k \), \( D(u_k(t)) \) is convex on \([0, +\infty)\). By \((4.8)\) we have
\[ 0 \leq \frac{D(u_k(t)) - D(u_0)}{t} \leq \frac{D(u_k) - D(u_0)}{d(u_0, u_k)}. \]
Take \( t_k = \left( \int_{2P_+} |u_k - u_0|^2 \pi dy \right)^{-1/2} \) and \( \dot{u}_k = u_k(t_k) \). Then
\[ d(u_0, \dot{u}_k) = 1, \]
and by \((4.11)\),
\[ 0 \leq D(\dot{u}_k(t)) - D(u_0) \leq \frac{D(u_0) - D(u_0)}{d(u_0, u_k)}. \]
By [19, Proposition 6.5], \( D(\cdot) \) is semi-continuous on \( \mathcal{E}_{K \times K}^1(2P) \). Thus by \((4.10)\), up to a subsequence,
\[ \dot{u}_k \to \dot{u}_0, \quad k \to +\infty. \]
for some $\hat{u}_0 \in E^1_{K \times K}(2P)$ which also gives a (singular) Kähler-Einstein metric $\hat{\omega}_0$ on $M$. Also,

$$d(\hat{u}_0, u_0) = d_M(\hat{\omega}_0, \omega_0) = 1.$$  

(4.12)

On the other hand, by Theorem 3.1, there is some $\sigma \in Z(G)$ such that $\hat{\omega}_0 = \sigma^* \omega_0$. Hence the corresponding Legendre function $\hat{u}_0$ of $\psi_0 = \sigma^* \psi_0$ is

$$\hat{u}_0 = u_0 - Z_\sigma^* y_i,$$

where $\sigma = e^{Z_\sigma}$ for some $Z_\sigma = (Z_\sigma^1, ..., Z_\sigma^n) \in \mathfrak{z}(\mathfrak{g})$. Since $u_0 \in E^1_{K \times K}(2P)$ is normalized at $O \in 2P$, the assumption that $\hat{u}_0 \in E^1_{K \times K}(2P)$ implies $Z_\sigma = 0$. Hence $\hat{u}_0 = u_0$, which contradicts to (4.12). We see that (4.9) can not hold and Theorem 1.2 (1) is true.

Since $M$ admits a (singular) Kähler-Einstein metric, it must have vanishing Futaki invariant (see Remark 4.2 above) and hence $b(2P_+) \in a_{ss}$. The relation (1.1) then follows from Proposition 4.1 (2). The stability result follows from (1.1) and a result in [14].

$$\square$$

5. Appendix 1: An algebraic lemma

In this appendix, we prove an elementary algebraic lemma for reductive groups.

**Lemma 5.1.** Suppose that $K$ is a compact, connected Lie group and $G = K^\mathbb{C}$ be its complexification. Let $T$ be a maximal torus of $K$ and $K'$ be a Lie subgroup of $K$. Suppose that there is a $t \in T^\mathbb{C}$ such that $Ad_t K' \subset K$. Then $t \in (C_G(K') \cap T^\mathbb{C}) \cdot T$.

**Proof.** Recall the Cartan decomposition

$$\mathfrak{g} = \mathfrak{t}^\mathbb{C} \oplus (\bigoplus_{\alpha \in \Phi_+} (V_\alpha \oplus V_{-\alpha})),$$

where $\Phi_+$ is a set of positive roots corresponding to $(G, T^\mathbb{C})$. Furthermore, for each root $\beta \in \Phi$, there is a $X_\beta$ such that

$$V_\beta = \mathbb{C} \cdot X_\beta.$$

Denote by $J$ the complex structure of $G$, we have

$$\mathfrak{t} = \mathfrak{t} \oplus (\bigoplus_{\alpha \in \Phi_+} \mathfrak{t}_\alpha),$$

where

$$\mathfrak{t} = \text{Span}_\mathbb{R}\{E_1, ..., E_r\}$$

and

$$\mathfrak{t}_\alpha = \text{Span}_\mathbb{R}\{(X_\alpha - X_{-\alpha}), J(X_\alpha + X_{-\alpha})\}.$$ 

Let

$$t = e^{T_1 + JT_2},$$

where $T_i \in \mathfrak{t}$. Then for any root $\beta \in \Phi$,

$$\text{Ad}_t \left(\begin{array}{c} X_\beta \\ JX_\beta \end{array} \right) = \left(\begin{array}{cc} e^{-\beta(T_2)} \cos \beta(T_1) & e^{-\beta(T_2)} \sin \beta(T_1) \\ -e^{-\beta(T_2)} \sin \beta(T_1) & e^{-\beta(T_2)} \cos \beta(T_1) \end{array} \right) \left(\begin{array}{c} X_\beta \\ JX_\beta \end{array} \right).$$

Thus for any root $\beta \in \Phi_+$,

$$\text{Ad}_t(X_\beta - X_{-\beta}) = e^{-\beta(T_2)}[(\cos \beta(T_1))(X_\beta - X_{-\beta}) + \sin \beta(T_1)J(X_\beta + X_{-\beta})] - 2 \sinh \beta(T_2)(\cos \beta(T_1)X_{-\beta} - \sin \beta(T_1)JX_{-\beta}),$$

where $\Phi_+$ is a set of positive roots corresponding to $(G, T^\mathbb{C})$. Furthermore, for each root $\beta \in \Phi$, there is a $X_\beta$ such that
and
\[ \text{Ad}_t J (X_\beta + X_{-\beta}) = e^{\beta(T_2)} [\sin \beta(T_1) (-X_\beta + X_{-\beta}) + \cos \beta(T_1) J (X_\beta + X_{-\beta})] \\
+ 2 \sinh \beta(T_2) (\sin \beta(T_1) X_\beta - \cos \beta(T_1) J X_\beta). \]

Hence we get for any
\[ X = \sum_{j=1}^r a_j E_j + \sum_{\alpha \in \Phi^+} (c_\alpha (X_\alpha - X_{-\alpha}) + d_\alpha J (X_\alpha + X_{-\alpha})), \]

where \( a_i, c_\alpha, d_\beta \in \mathbb{R} \), we get
\[ \text{Ad}_t X = 2 \sinh \beta(T_2) \left[ \sum_{\beta \in \Phi^+} \left( -c_\beta \cos \beta(T_1) + d_\beta \sin \beta(T_1) \right) X_{-\beta} \right. \\
\left. + \sum_{\beta \in \Phi^+} \left( c_\beta \sin \beta(T_1) + d_\beta \cos \beta(T_1) \right) J X_{-\beta} \right] \pmod{t}. \]

Then we conclude that \( X \in \mathfrak{t} \) if and only if
\[ \sinh \beta(T_2) \begin{pmatrix} -\cos \beta(T_1) & \sin \beta(T_1) \\ \sin \beta(T_1) & \cos \beta(T_1) \end{pmatrix} \begin{pmatrix} c_\beta \\ d_\beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \forall \beta \in \Phi^+. \]

Thus \( X \in \mathfrak{t} \) if and only if
\[ (5.2) \quad \sinh \beta(T_2) (c_\beta, d_\beta) = (0, 0). \]

By the assumption that \( \text{Ad}_t K' \subset K \), we see the Lie algebra \( \text{Ad}_t \mathfrak{t}' \subset \mathfrak{t} \). If the decomposition of \( \mathfrak{t}' \) according to (5.1) has non-zero component on some \( \mathfrak{t}_\alpha \oplus \mathfrak{t}_{-\alpha} \), then we can take \( X \in \mathfrak{t}' \) such that \((c_\alpha, d_\alpha) \neq (0, 0)\) in (5.2). Hence \( \alpha(T_2) = 0 \) for any such \( \alpha \). We conclude that \( e^{JT_2} \in C_G(K') \), which proves the lemma.

\[ \square \]

6. Appendix 2: Maximal torus in Aut(M)

In this Appendix, we will prove that when Aut(M) is reductive, the maximal torus of Aut(M) containing \( T \times T \) is unique. We learned this result and its proof from Professor M. Brion [8].

**Proposition 6.1.** Let \( M \) be a polarized \( G \)-compactification such that Aut\(^0\)(M) is reductive. Then the maximal torus containing \( T \times T \) (as a subgroup of Aut\(^0\)(M)) is unique.

Before the proof of Proposition 6.1, we shall first show the following lemma, which gives a characterization of torus:

**Lemma 6.2.** Let \( H \) be a connected reductive algebraic group acting faithfully on a normal projective variety \( M \). Assume that every closed \( H \)-orbit is an \( H \)-fixed point. Then \( H \) is a torus.

**Proof.** By a result of Sumihiro [20, Theorem 1], there is an \( H \)-equivariant embedding of \( M \) into some projective space \( \mathbb{P}(V) \), where \( V \) is a finite-dimensional representation of \( H \). As \( H \) is reductive, \( V \) can be decomposed as direct sums of finitely many irreducible representations of \( H \).
By assumption, every closed $H$-orbit in $M$ corresponds to an $H$-stable line in $V$, which is a 1-dimensional representation of $H$. Let $V'$ be the direct sum of all the 1-dimensional representations and $V''$ the direct sum of remaining ones. Then we decompose $V$ as sum $H$-invariant spaces of

$$V = V' \oplus V''.$$ 

The resulting projection $V \to V'$ is $H$-equivariant, and hence gives an $H$-equivariant rational map

$$\Pi : \mathbb{P}(V) \dashrightarrow \mathbb{P}(V').$$

This rational map is defined at any closed $H$-orbit in $M$ by construction. We claim that

$$M \cap \mathbb{P}(V'') = \emptyset \quad (6.1)$$

so that $f := \Pi|_X : X \to \mathbb{P}(V')$ is a morphism. Suppose that (6.1) is not true, then $M \cap \mathbb{P}(V'')$ is an $H$-stable closed projective variety. By Borel’s fixed point theorem it contains an $H$-fixed point $x_0$. This implies that $V''$ contains a 1-dimensional $H$-invariant space, contradicts to the construction of $V''$.

By (6.1) we in fact conclude that $f$ is a finite morphism. By definition, the derived subgroup $[H,H]$ acts trivially on $V'$. Since $[H,H]$ is connected and $f$ is finite, we see that $[H,H]$ acts trivially on $M$. Since $H$ acts faithfully there, $[H,H]$ is trivial and $H$ is a torus.

Proof of Proposition 6.1. By our assumption, $\text{Aut}^0(M)$ is a connected reductive algebraic group. Thus, the centralizer $H$ of $T \times T$ in $\text{Aut}^0(M)$ is a connected reductive algebraic group as well.

It suffices to show that $H$ is a torus. Consider the fixed point set $M^{T \times T}$ in $M$. It is known that $M^{T \times T}$ is finite and contained in the union of the closed $G \times G$-orbits (cf. [2, Theorem 2.7]). Since $H$ commutes with $T \times T$, we conclude that it acts on $M^{T \times T}$. However, $M^{T \times T}$ is finite and $H$ is connected, we conclude that

$$M^H = M^{T \times T}.$$ 

On the other hand, let $Y$ be any closed $H$-orbit in $M$. By Borel’s fixed point theorem, $Y$ contains a fixed point $y_0$ of the subtorus $T \times T$. Thus $y_0$ is fixed by $H$, and hence

$$Y = \{y_0\}$$

is a single point. By Lemma 6.2, $H$ is a torus as desired.

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