THE STRONG MASSEY VANISHING CONJECTURE FOR FIELDS WITH VIRTUAL COHOMOLOGICAL DIMENSION AT MOST 1

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Abstract. We show that a strong vanishing conjecture for $n$-fold Massey products holds for fields of virtual cohomological dimension at most 1 using a theorem of Haran. We also prove the same for PpC fields, using results of Haran–Jarden. Finally we construct a pro-2 group which satisfies the weak Massey vanishing property for every $n \geq 3$, but does not satisfy the strong Massey vanishing property for $n = 4$.

1. Introduction

Definition 1.1. Let $C^\ast$ be a differential graded associative algebra with product $\cup$, differential $\delta : C^\ast \to C^{\ast + 1}$, and cohomology $H^\ast = \text{Ker}(\delta)/\text{Im}(\delta)$. Choose an integer $n \geq 2$ and let $a_1, a_2, \ldots, a_n$ be a set of cohomology classes in $H^1$. A defining system for the $n$-fold Massey product of $a_1, a_2, \ldots, a_n$ is a set $a_{ij}$ of elements of $C^1$ for $1 \leq i < j \leq n + 1$ and $(i, j) \neq (1, n + 1)$ such that

$$\delta(a_{ij}) = \sum_{k=i+1}^{j-1} a_{ik} \cup a_{kj}$$

and $a_1, a_2, \ldots, a_n$ is represented by $a_{12}, a_{23}, \ldots, a_{n,n+1}$. We say that the $n$-fold Massey product of $a_1, a_2, \ldots, a_n$ is defined if there exists a defining system. The $n$-fold Massey product $\langle a_1, a_2, \ldots, a_n \rangle_{a_{ij}}$ of $a_1, a_2, \ldots, a_n$ with respect to the defining system $a_{ij}$ is the cohomology class of

$$\sum_{k=2}^{n} a_{1k} \cup a_{k,n+1}$$

in $H^2$. Let $\langle a_1, a_2, \ldots, a_n \rangle$ denote the subset of $H^2$ consisting of the $n$-fold Massey products of $a_1, a_2, \ldots, a_n$ with respect to all defining systems. We say that the $n$-fold Massey product of $a_1, a_2, \ldots, a_n$ vanishes if $\langle a_1, a_2, \ldots, a_n \rangle$ contains zero.

Definition 1.2. Let $p$ be a prime number, let $G$ be a profinite group, let $C^\ast$ be the differential graded algebra of $\mathbb{Z}/p$-cochains of $G$ in continuous group cohomology. The cohomology of $C^\ast$ is $H^\ast = H^\ast(G, \mathbb{Z}/p)$. We say that $G$ has the strong Massey vanishing property for $n$ with respect to $p$ if $n$ is an integer $\geq 3$, if for every $a_1, a_2, \ldots, a_n \in H^1(G, \mathbb{Z}/p)$ such that $a_i \cup a_{i+1} = 0$ for every $1 \leq i < n$, the $n$-fold Massey product of $a_1, a_2, \ldots, a_n$ vanishes. We say that $G$ has the strong Massey vanishing property with respect to $p$ if it has the strong Massey vanishing property with respect to $p$ for every integer $n \geq 3$. We say that $G$ has the weak Massey...

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vanishing property with respect to \( p \) if for every integer \( n \geq 3 \) and \( a_1, a_2, \ldots, a_n \in H^1(G, \mathbb{Z}/p) \) such that \( n \)-fold Massey product of \( a_1, a_2, \ldots, a_n \) is defined, the \( n \)-fold Massey product of \( a_1, a_2, \ldots, a_n \) vanishes.

**Definition 1.3.** For every field \( K \) let \( G(K) \) denote the absolute Galois group of \( K \). Assume that the characteristic of \( K \) is not \( p \). We say that the strong Massey vanishing conjecture with respect to \( p \) holds for \( K \) if \( G(K) \) has the strong Massey vanishing property with respect to \( p \).

**Remark 1.4.** We call our conjecture strong because it is stronger in general than the Massey vanishing conjecture formulated by Mináč and Tǎn (Conjecture 1.6 of [15] on page 259) since we do not require that the \( n \)-fold Massey product of \( a_1, a_2, \ldots, a_n \) is defined, unlike them. This is a strictly stronger requirement when \( n > 3 \), see Theorem 1.10 below. See also Remark 2.6 below, which explains the difference in terms of embedding problems.

There is quite a bit of beautiful work on this fascinating conjecture (see for example [3], [5], [10], [11], and [15]), but it remains open in general. Our aim is to prove the strong form of this conjecture for two new classes of fields whose definition we recall next.

**Definition 1.5.** Recall that a field \( K \) has virtual cohomological dimension \( \leq 1 \) if there is a finite separable extension \( L/K \) with \( cd(L) \leq 1 \) where \( cd \) denotes the cohomological dimension as defined in [17]. Since the only torsion elements in the absolute Galois group of \( K \) are the involutions coming from the orderings of \( K \), it is equivalent (by a theorem in [18]) to require \( cd(L) \leq 1 \) for any fixed finite separable extension \( L \) of \( K \) without orderings, for example for \( L = K(i) \), where \( i = \sqrt{-1} \). In particular, if \( K \) itself cannot be ordered (which is equivalent to \( -1 \) being a sum of squares in \( K \)), this condition is equivalent to \( cd(K) \leq 1 \).

**Examples 1.6.** Examples of fields \( K \) which can be ordered with \( cd(K(i)) \leq 1 \) include real closed fields, function fields in one variable over any real closed ground field (by Tsen’s Hauptsatz of [19] on page 335), PRC (pseudo real closed) fields (for definition see page 450 of [8]), the field of Laurent series in one variable over any real closed ground field (by Lang’s Theorem 10 of [13] on page 384), and the field \( \mathbb{Q}^{ab} \cap \mathbb{R} \) which is the subfield of \( \mathbb{R} \) generated by the numbers \( \cos(\frac{2\pi}{n}) \), where \( n \in \mathbb{N} \) (see Corollary 6.2 of [6] on page 410).

The first main result of this paper is the following

**Theorem 1.7.** The strong Massey vanishing conjecture holds for fields \( K \) with \( cd(K(i)) \leq 1 \) with respect to every prime number.

The proof is an easy application of earlier work of Haran and Dwyer. After we review the latter, we give a quick proof of our first main result in the next section. At the recommendation of the reviewer, we will use similar methods to prove the same claim for pseudo \( p \)-adically closed fields, whose definition we recall next.

**Definition 1.8.** A field \( K \) is called pseudo \( p \)-adically closed (abbreviation: PpC) if every absolutely irreducible variety \( V \) defined over \( K \) has a \( K \)-rational point, provided \( V \) has a \( L \)-rational simple point for each \( p \)-adic closure \( L \) of \( K \).

**Theorem 1.9.** The strong Massey vanishing conjecture holds for PpC fields with respect to every prime number.
The result follows from a result of Haran–Jarden (see Theorem 3.11 below), and the validity of the strong Massey vanishing property for Demushkin groups, which is our Theorem 3.5, and does require some work. Our last main result is the following purely group-theoretical

**Theorem 1.10.** There is a pro-$2$ group $G$ which satisfies weak Massey vanishing for $n \geq 3$ (with respect to every prime number), but does not satisfy strong Massey vanishing for $n = 4$ (with respect to 2).

The key idea of the proof of this theorem is the use of *Massey envelopes* which are infinite fibre products that can be associated to each group, satisfy weak Massey vanishing, and which are in some sense universal with respect to this property.

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2. Real embedding problems and Massey products

**Definition 2.1.** An embedding problem for a profinite group $G$ is the left hand side diagram:

$$
\begin{array}{ccc}
G & \xrightarrow{\phi} & A \\
\downarrow & & \downarrow \\
B & \xrightarrow{\alpha} & A
\end{array}
$$

where $A, B$ are finite groups, the solid arrows are continuous homomorphisms and $\alpha$ is surjective. A solution of this embedding map is a continuous homomorphism $\bar{\phi} : G \to B$ which makes the right hand side diagram commutative. We say that the embedding problem above is *real* if for every involution $t \in G$ with $\phi(t) \neq 1$ there is an involution $b \in B$ with $\alpha(b) = \phi(t)$. Following Haran and Jarden (see [3]) we say that that a profinite group $G$ is *real projective* if $G$ has an open subgroup without 2-torsion, and if every real embedding problem for $G$ has a solution.

**Theorem 2.2** (Haran). A profinite group $G$ is real projective if and only if $G$ has an open subgroup $G_0$ of index $\leq 2$ with $\text{cd}(G_0) \leq 1$, and every involution $t \in G$ is self-centralizing, that is, we have $C_G(t) = \{1, t\}$.

**Proof.** This is Theorem A of [7] on page 219.

By classical Artin–Schreier theory every involution in the absolute Galois group of a field is self-centralising, so we get the following

**Corollary 2.3.** The absolute Galois group of a field $K$ is real projective if and only if $K$ satisfies $\text{cd}(K(i)) \leq 1$.

We also need to recall Dwyer’s theorem relating the vanishing of Massey products to certain embedding problems.
Let two groups in all that follow. Given the group of continuous homomorphisms $\text{Hom}(G, Z/p)$ be the group of upper triangular matrices with coefficients in $Z/p$. Let $G$ and $C^*$ be the same as in Definition [12]. Then $H^1$ is naturally isomorphic to the group of continuous homomorphisms $\text{Hom}(G, Z/p)$, and we will identify these two groups in all that follow. Given $n$ continuous homomorphisms

$$a_i : G \rightarrow Z/p \quad (i = 1, 2, \ldots, n),$$

let $E(a_1, a_2, \ldots, a_n)$ denote the embedding problem:

$$\psi \begin{array}{ccc}
G & \rightarrow & U_{n+1}(p) \\
& \phi_{n+1} & \downarrow \\
& (Z/p)^n, & \phi_{n+1}
\end{array}$$

where $\phi_{n+1}$ is given by the rule $U \mapsto (e_{12}(U), e_{23}(U), \ldots, e_{nn+1}(U))$.

**Theorem 2.5** (Dwyer). The $n$-fold Massey product $\langle a_1, a_2, \ldots, a_n \rangle$ vanishes if and only if the embedding problem $E(a_1, a_2, \ldots, a_n)$ has a solution.

**Proof.** See Theorem 2.4 of [2] on page 182. \qed

**Remark 2.6.** For every positive integer $m$ let $Z_m(p)$ and $P_m(p)$ denote the following subgroups of $U_m(p)$:

$$Z_m(p) = \{ B \in U_m(p) \mid e_{ij}(B) = 0 \text{ if } 1 \leq i < j \leq m - 1 \text{ or } 2 \leq i < j \leq m \},$$

$$P_m(p) = \{ B \in U_m(p) \mid e_{ij}(B) = 0 \text{ if } j = i + 1, i + 2 \text{ and } 1 \leq i, j \leq m \},$$

respectively. Clearly $Z_m(p) \subset P_m(p)$ and they are different when $m > 4$. By Theorem 2.4 of [2] on page 182 quoted above the $n$-fold Massey product $\langle a_1, a_2, \ldots, a_n \rangle$ is defined if and only if the embedding problem:

$$\begin{array}{ccc}
G & \rightarrow & U_{n+1}(p)/Z_{n+1}(p) \\
& \zeta_{n+1} & \downarrow \\
& (Z/p)^n, & \zeta_{n+1}
\end{array}$$

has a solution, where $\zeta_{n+1} : U_{n+1}(p)/Z_{n+1}(p) \rightarrow (Z/p)^n$ is the unique homomorphism such that the composition of the quotient map $U_{n+1}(p) \rightarrow U_{n+1}(p)/Z_{n+1}(p)$ and $\zeta_{n+1}$ is the homomorphism $\phi_{n+1} : U_{n+1}(p) \rightarrow (Z/p)^n$ in Definition [2,4] while $a_i \cup a_{i+1} = 0$ for every $1 \leq i < n$ if and only if the embedding problem:

$$\begin{array}{ccc}
G & \rightarrow & U_{n+1}(p)/P_{n+1}(p) \\
& \kappa_{n+1} & \downarrow \\
& (Z/p)^n, & \kappa_{n+1}
\end{array}$$

has a solution, where $\kappa_{n+1} : U_{n+1}(p)/P_{n+1}(p) \rightarrow (Z/p)^n$ is the unique group homomorphism such that the composition of the quotient map $U_{n+1}(p) \rightarrow U_{n+1}(p)/P_{n+1}(p)$ and $\kappa_{n+1}$ is the homomorphism $\phi_{n+1}$ in Definition [2,4].
Proof of Theorem 1.7. By Haran’s theorem it will be enough to show that every such embedding problem with \( a_i \cup a_{i+1} = 0 \) for every \( i = 1, 2, \ldots, n - 1 \) is real, in other words the restriction of the embedding problem to any subgroup of order two has a solution. In other words, by Artin-Schreier theory, we reduced the claim to the case when \( K \) is real closed, i.e. when \( G(K) = \mathbb{Z}/2 \). The claim for the latter is trivial when \( p \) is odd, since \( U_{n+1}(p) \) is a \( p \)-group. So we may assume without the loss of generality that \( p = 2 \).

Now let \( a_1, a_2, \ldots, a_n \in \text{Hom}(G(K), \mathbb{Z}/2) \) be a set of cohomology classes in \( H^1 \) such that \( a_i \cup a_{i+1} = 0 \) for every \( i = 1, 2, \ldots, n - 1 \). Let \( g \in G(K) \) be the generator.

Lemma 2.7. For every \( i = 1, 2, \ldots, n - 1 \) the following holds: if \( a_i(g) = 1 \) then \( a_{i+1}(g) = 0 \).

Proof. Since for every \( a, b \in H^1 \) the 2-fold Massey product \( \langle a, b \rangle \) is the singleton \( a \cup b \), by Theorem 2.5 we get that the embedding problem:

\[
\begin{array}{ccc}
G(K) & \xrightarrow{\psi_i} & U_3(2)\\
\downarrow & & \downarrow \\
\phi_3 & \rightarrow & (\mathbb{Z}/2)^2,
\end{array}
\]

has a solution \( \psi_i \). Assume now that \( a_i(g) = a_{i+1}(g) = 1 \). Then either

\[
\psi_i(g) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{or} \quad \psi_i(g) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.
\]

However neither of these matrices has order two, which is a contradiction. \( \square \)

In plain English the lemma above means that we can break up the row vector \( (a_1(g) \ a_2(g) \ \ldots \ a_n(g)) \) to single entries of 1-s separated by zeros. By the above it is enough to construct a matrix \( A \in U_{n+1}(p) \) such that \( A^2 \) is the identity matrix, and

\[
\phi_{n+1}(A) = a_1 \times a_2 \times \cdots \times a_n(g).
\]

In fact the matrix \( A = (a_{ij})_{i,j=1}^{n+1} \) with

\[
a_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 1, & \text{if } i + 1 = j \text{ and } a_i(g) = 1, \\ 0, & \text{otherwise}, \end{cases}
\]

will do. Indeed it is a block matrix whose off-diagonal terms are zero matrices, and the diagonal terms are either the \( 1 \times 1 \) matrix \((1)\) or the \( 2 \times 2 \) matrix \((\frac{1}{1} 1)\). These have order dividing two, so the same holds for \( A \), too. \( \square \)

Finally we point out that Mináč and Tán actually proved the strong Massey vanishing conjecture for odd rigid fields.

Definition 2.8. We say that a field \( K \) is \( p \)-rigid for every \( \alpha, \beta \in H^1(G, \mathbb{Z}/p) \) such that \( \alpha \cup \beta = 0 \), the linear subspace \( \text{span}(\alpha, \beta) \) is at most one-dimensional.

Theorem 2.9 (Mináč–Tán). Let \( p \) be an odd prime number and let \( K \) be a \( p \)-rigid field which contains a primitive \( p \)-th root of unity. Then the strong Massey vanishing conjecture holds for \( K \).
This result has been essentially proved in [13] (see Theorem 8.5 of loc. cit. and its proof), however the authors only stated that the weak Massey vanishing conjecture holds for $K$. A minimal modification of the authors’ argument will give this stronger result. We present the modified proof for the reader’s convenience.

**Proof.** We are going to show the claim for all $n \geq 2$ by induction on $n$. The initial case $n = 2$ is trivially true. Let’s assume that $n \geq 3$ and the claim holds for $n-1$. Suppose first that there is an index $k \in \{1, 2, \ldots, n\}$ such that $a_k = 0$. If $k > 1$ there is a homomorphism $\phi : G \to U_k(p)$ lifting $-a_1 \times -a_2 \times \cdots \times -a_{k-1}$ by the induction hypothesis. Otherwise let $\phi : G \to U_1(p) = \{1\}$ be the trivial homomorphism. If $k < n$ there is a homomorphism $\phi : G \to U_{n-k}(p)$ lifting $-a_{k+1} \times -a_{k+2} \times \cdots \times -a_n$ by the induction hypothesis. Otherwise let $\phi : G \to U_1(p)$ be the trivial homomorphism.

Now let $\phi : G \to U_{n+1}(p)$ be the unique homomorphism such that

$$e_{ij}(\tilde{\phi}(g)) = \begin{cases} e_{ij}(\phi(g)), & \text{if } 1 \leq i, j \leq k, \\ e_{(i-k)(j-k)}(\phi(g)), & \text{if } k + 1 \leq i, j \leq n + 1, \\ 0, & \text{otherwise}, \end{cases}$$

for every $g \in G$. In plain English $\tilde{\phi}(g)$ is a block matrix whose off-diagonal terms are zero matrices, and the diagonal terms are the $k \times k$ matrix $\phi(g)$ and the $(n-k+1) \times (n-k+1)$ matrix $\phi_{ij}(g)$. Clearly $\tilde{\phi}$ is a lift of $-a_1 \times -a_2 \times \cdots \times -a_n$. So we may assume without the loss of generality that $a_k \neq 0$ for every $k \in \{1, 2, \ldots, n\}$.

Then there is an $a \in H^1(G, \mathbb{Z}/p)$ such that $a_k = \lambda_k a$ for some $\lambda_k \in \mathbb{Z}/p$ for every $k$, since $K$ is $p$-rigid. By Theorem 8.1 of [14] the Massey product $\langle a, a, \ldots, a \rangle$ is defined and contains zero. Now a repeated application of part (b) of Lemma 6.2.4 in [14] on page 236 concludes the proof. □

3. **Demushkin groups, $p$-adic embedding problems and Massey products**

**Definition 3.1.** The kernel $\text{Ker}(E)$ of an embedding problem $E$ as in Definition 2.1 is the kernel of $\alpha$. We say that $E$ is central if $B$ is a central extension of $A$, that is, when $\text{Ker}(E)$ lies in the centre of $B$. In this case $\text{Ker}(E)$ is abelian, so we can equip it with the trivial $G$-module structure.

**Definition 3.2.** Assume now that the embedding problem $E$ is central. The obstruction class of $E$ is defined as follows. Let $\phi : G \to B$ be a continuous map such that $\alpha \circ \tilde{\phi} = \phi$. Then the map $c : G \times G \to \text{Ker}(E)$ given by the rule:

$$c(x, y) = \tilde{\phi}(xy)\tilde{\phi}(y)^{-1}\tilde{\phi}(x)^{-1} \in \text{Ker}(E), \quad (x, y \in G)$$

is a cocycle, and its cohomology class $o(E) \in H^2(G, \text{Ker}(E))$ does not depend on the choice of $\tilde{\phi}$, only on $E$. By a well-known classical result $E$ has a solution if and only if $o(E) = 0$.

**Lemma 3.3.** Let $G$ be a profinite group such that $H^2(G, \mathbb{Z}/p) = 0$. Then $G$ has the strong vanishing $n$-fold Massey product property with respect to $p$.

**Proof.** Since $U_{n+1}(p)$ is a $p$-group, it has a filtration by normal subgroups:

$$\{1\} = N_0 < N_1 \subset \cdots \subset N_{(n+1) \over 2} = \text{Ker}(\phi)$$
such that $U_{n+1}(p)/N_k$ is a central extension of $U_{n+1}(p)/N_{k+1}$ and the kernel of the quotient map $\pi_k : U_{n+1}(p)/N_k \to U_{n+1}(p)/N_{k+1}$ is:

$$\frac{N_{k+1}}{N_k} \cong \mathbb{Z}/p$$

for every $k = 0, 1, \ldots, \binom{n-1}{2} - 1$.

Note that it will be sufficient to show that the embedding problem $E(h)$:

$$U_{n+1}(p)/N_k \xrightarrow{\pi_k} U_{n+1}(p)/N_{k+1}$$

for every every homomorphism $G \to U_{n+1}(p)/N_{k+1}$ has a solution for every $k = 0, 1, \ldots, \binom{n-1}{2}$. Indeed then we would get by descending induction on the index $k$ that $-a_1 \times \cdots \times -a_n$ has a lift to $G \to U_{n+1}(p)/N_k$. The claim is now clear from the case $k = 0$.

However $E(h)$ is a central embedding problem with kernel isomorphic to $\mathbb{Z}/p$ by the above. So its obstruction class $o(E(h))$ lies in $H^2(G, \mathbb{Z}/p)$, which is zero by assumption. So $o(E(h))$ vanishes, and hence $E(h)$ has a solution. □

**Definition 3.4.** A pro-$p$ group $G$ is said to be a Demushkin group if

1. $\dim_{\mathbb{Z}/p} H^1(G, \mathbb{Z}/p) < \infty$,
2. $\dim_{\mathbb{Z}/p} H^2(G, \mathbb{Z}/p) = 1$,
3. the cup product $H^1(G, \mathbb{Z}/p) \times H^1(G, \mathbb{Z}/p) \to H^2(G, \mathbb{Z}/p)$ is a non-degenerate bilinear form.

**Theorem 3.5.** Let $n \geq 3$ be an integer and let $p$ be a prime number. Then every pro-$p$ Demushkin group has the strong vanishing $n$-fold Massey product property with respect to $p$.

This claim above is a strengthening of Theorem 4.3 of [14] on page 265, which in turn is a generalisation of Lemma 3.5 of [11] on page 1317. Note that $\mathbb{Z}/2$ is a Demushkin group, so the theorem above generalises the key ingredient of the proof of Theorem 1.7

**Proof of Theorem 3.5.** Arguing the same way as we did at the beginning of the proof of Theorem 2.9 we may assume without the loss of generality that $a_k \neq 0$ for every $k \in \{1, 2, \ldots, n\}$. Let $M_{k,m}$ denote the subgroup

$$M_{k,m} = \{ U \in U_m(p) \mid e_{ij}(U) = 0 \text{ if } 1 \leq i < j \leq m - 1 \text{ or } j = m, k \leq i \leq m - 1 \}$$

for every $k = 1, 2, \ldots, m - 1$. Clearly $M_{k,m} \subseteq M_{k+1,m}$ for every $k = 1, 2, \ldots, m - 2$.

**Lemma 3.6.** The subgroup $M_{k,m}$ of $U_m(p)$ is normal.

**Proof.** For every pair $a \leq b$ of natural numbers let $\beta_{a,b} : U_b(p) \to U_a(p)$ be the homomorphism:

$$U \mapsto (e_{ij}(U))_{i,j=1}^a,$$

that is the map which assigns to every element of $U_b(p)$ its upper left $a \times a$ block. Similarly $\beta_{a,b} : U_b(p) \to U_a(p)$ be the homomorphism:

$$U \mapsto (e_{(b-a+i)(b-a+j)}(U))_{i,j=1}^a.$$
that is the map which assigns to every element of $U_b(p)$ its lower right $a \times a$ block. 

The subgroup $M_{k,m}$ is the intersection of the kernel of $\overline{\beta}_{m-1,k,m}$ and the kernel of $\overline{\beta}_{m+1-k,m}$, so as an intersection of normal subgroups, it is normal.

Let $Q_{k,m}$ denote the quotient group $U_m(p)/M_{k,m}$ and let $\rho_{k,m} : Q_{k,m} \to Q_{k+1,m}$ denote the quotient map induced by the inclusion $M_{k,m} \subset M_{k+1,m}$.

**Lemma 3.7.** The extension $Q_{k,m}$ of $Q_{k+1,m}$ is central, and the kernel of $\rho_{k,m}$ is isomorphic to $\mathbb{Z}/p$ for every $k = 1, 2, \ldots, m - 2$.

*Proof.* As we saw in the proof of Lemma 3.6 above $Q_{k,m}$ is the fibre product:

\[
\{(A, B) \in U_{m-1}(p) \times U_{m+1-k}(p) \mid \overline{\beta}_{m-k,m-1}(A) = \overline{\beta}_{m-k,m+1-k}(B)\},
\]

considered as a subgroup of $U_{m-1}(p) \times U_{m+1-k}(p)$. Under this identification the group $\text{Ker}(\rho_{k,m})$ is:

\[
\{(A, B) \in U_{m-1}(p) \times U_{m+1-k}(p) \mid A = I_{m-1 \times m-1}, B \in Z_{m+1-k}(p)\},
\]

where $I_{m-1 \times m-1}$ is the identity matrix and $Z_{m+1-k}(p)$ is the group defined in Remark 2.6. Since $Z_{m+1-k}(p)$ is the centre of $U_{m+1-k}(p)$, it lies in the centre of $Q_{k,m}$, so the extension $Q_{k,m}$ of $Q_{k+1,m}$ is central. The map $\iota_{k,m} : \text{Ker}(\rho_{k,m}) \to \mathbb{Z}/p$ which is the composition of the isomorphism $\text{Ker}(\rho_{k,m}) \to Z_{m+1-k}(p)$ given by the rule $(A, B) \mapsto B$, and the map $Z_{m+1-k}(p) \to \mathbb{Z}/p$ given by the rule $B \mapsto \varepsilon_{1(m+1-k)}(B)$, is an isomorphism.

We will identify the group $\text{Ker}(\rho_{k,m})$ with $\mathbb{Z}/p$ via the isomorphism $\iota_{k,m}$ in the proof of Lemma 3.7 in all that follows. For every homomorphism $\psi : G \to Q_{k+1,m}$ let $E(\psi)$ denote the embedding problem:

\[
\begin{array}{ccc}
G \\
\downarrow \psi \\
\overline{\psi} \\
Q_{k,m} \xrightarrow{\rho_{k,m}} Q_{k+1,m}.
\end{array}
\]

Let $\chi : G \to \text{Ker}(\rho_{k+1,m})$ be a homomorphism. Then the map $\psi \chi$ given by the rule $g \mapsto \psi(g)\chi(g)$ is also a homomorphism from $G$ to $Q_{k+1,m}$, since $\text{Ker}(\rho_{k+1,m})$ lies in the centre of $Q_{k+1,m}$ by Lemma 3.7. Let $\phi_{k,m+1} : Q_{k,m+1} \to (\mathbb{Z}/p)^m$ be the unique homomorphism such that the composition of the quotient map $U_{m+1}(p) \to Q_{k,m+1}$ and $\phi_{k,m+1}$ is the homomorphism $\phi_{m+1} : U_{m+1}(p) \to (\mathbb{Z}/p)^m$ in Definition 2.4 for every $k = 1, 2, \ldots, m - 1$.

**Lemma 3.8.** Let $\psi : G \to Q_{k+1,n+1}$ be a homomorphism such that $\phi_{k+1,n+1} \circ \psi = -a_1 \times -a_2 \times \cdots \times -a_n$. Then

\[
o(\psi \chi) = o(\psi^\prime) + a_k \cup \chi
\]

in $H^2(G, \mathbb{Z}/p)$ for every homomorphism $\chi : G \to \text{Ker}(\rho_{k+1,n+1})$.

*Proof.* There is a commutative diagram of group homomorphisms:

\[
\begin{array}{cccc}
Q_{k,n+1} & \xrightarrow{\rho_{k,n+1}} & Q_{k+1,n+1} \\
\downarrow \phi_{k,n+1} \chi & & \downarrow \phi_{k+1,n+1} \chi \\
Q_{1,n-k+2} & \xrightarrow{\rho_{1,n-k+2}} & Q_{2,n-k+2}
\end{array}
\]
such that the vertical arrow on the left induces an isomorphism between $\text{Ker}(\rho_{k,n+1})$ and $\text{Ker}(\rho_{1,n-k-2})$ and $\phi_{2,n-k+2} \circ \lambda \circ \psi = -a_k \times a_{k+1} \times \cdots \times -a_n$. Therefore we may assume that $k = 1$ without the loss of generality, because the obstruction classes of central embedding problems are natural. In this case the obstruction classes $o(E(\psi))$ and $o(E(\psi \chi))$ are given by the Massey products for the defining systems corresponding to $\psi$ and $\psi \chi$, respectively, by Theorem 2.4 and the remark immediately follow it in [2] on page 182. Therefore the difference between the two is $a_1 \cup \chi$ (compare with Remark 2.2 of [15] on page 261), and hence the claim follows. \(\square\)

Now we are going to prove for every $k = 1, 2, \ldots, n-1$ that the embedding problem $E(k)$:

\[
\begin{array}{c}
\psi \\
\downarrow \\
(\mathbb{Z}/p)^n,
\end{array}
\]

has a solution by descending induction on $k$. Since the case $k = 1$ is the claim, this will be sufficient to conclude the proof.

Let us first consider the case $k = n-1$ first. By the induction hypothesis there are solutions $\psi$ and $\psi$ to the embedding problems:

\[
\begin{array}{c}
\phi_n (\mathbb{Z}/p)^{n-1}, \\
\end{array}
\]

\[
\begin{array}{c}
(\mathbb{Z}/p)^2,
\end{array}
\]

respectively. The direct product $\psi \times \psi : G \to U_n(p) \times U_3(p)$ lies in $Q_{n-1,n+1}$, and it is a solution to $E(n-1)$.

Now assume that $E(k)$ has a solution $\psi$ for some $k \geq 2$. By assumption $a_{k-1} \neq 0$ and the cup product

\[
\cup : H^1(G, \mathbb{Z}/p) \times H^1(G, \mathbb{Z}/p) \to H^2(G, \mathbb{Z}/p) \cong \mathbb{Z}/p
\]

is a non-degenerate bilinear form, so there is a $\chi \in H^1(G, \mathbb{Z}/p)$ such that

\[
o(E(\psi)) + a_{k-1} \cup \chi = 0.
\]

Therefore there is a solution $\overline{\psi} : G \to Q_{k-1,n+1}$ to $E(\psi \chi)$ by Lemma 3.8. This $\overline{\psi}$ is also a solution to $E(k-1)$, since $\phi_{k-1,n+1} = \phi_{k,n+1} \circ \rho_{k-1,n+1}$.

**Lemma 3.9.** For every pair of prime numbers $l, p$, not necessarily different, the maximal $l$-adic quotient $H$ of $G(\mathbb{Q}_p)$ is either a pro-$l$ Demushkin group or we have $H^2(H, \mathbb{Z}/l) = 0$.

**Proof.** When $l = p$ then either $H$ is projective, and hence $H^2(H, \mathbb{Z}/l) = 0$, or it is a Demushkin group (see §5 of [12] on pages 130-31 for a proof). When $l \neq p$ then $H$ is $\mathbb{Z}_l$ if $l$ does not divide $p - 1$, and hence $H^2(H, \mathbb{Z}/l) = 0$, or $H$ is the semi-direct product $\mathbb{Z}_l \rtimes \mathbb{Z}_l$ such that the second copy of $\mathbb{Z}_l$ acts on the first via the character $m \mapsto p^m$ if $l$ divides $p - 1$, and hence it is a Demushkin group. \(\square\)

**Definition 3.10.** We say that an embedding problem as the one in Definition 2.3 above is a $G(\mathbb{Q}_p)$-problem if for every closed subgroup $H$ of $G$ which is isomorphic
to $G(\mathbb{Q}_p)$ there is a homomorphism $\tilde{\phi}_H : H \to B$ such that $\alpha \circ \tilde{\phi}_H = \phi|_H$. Following Haran and Jarden (see [9]) we say that a profinite group $G$ is $p$-adically projective if every $G(\mathbb{Q}_p)$-problem for $G$ has a solution, and if the collection of all closed subgroups of $G$ which are isomorphic to $G(\mathbb{Q}_p)$ is topologically closed.

By the main result of [9] (see the Theorem on page 148) we know the following

**Theorem 3.11** (Haran–Jarden). If $K$ is a \(\mathbb{PpC}\) field, then $G(K)$ is $p$-adically projective. Conversely, if $G$ is a $p$-adically projective group, then there exists a \(\mathbb{PpC}\) field $K$ such that $G(K) \cong G$. □

Now we are ready to give a

**Proof of Theorem 1.9.** By the Haran–Jarden Theorem 3.11 it will be enough to show that for every prime number $l$ every such embedding problem with $a_i \cup a_{i+1} = 0$ for every $i = 1, 2, \ldots, n-1$ is a $G(\mathbb{Q}_p)$-problem. In order to do so it will be enough to show that for every $l$ as above the maximal $l$-adic quotient of $G(\mathbb{Q}_p)$ has the strong vanishing $n$-fold Massey product property with respect to $l$. However this is immediate from Lemma 3.9, Lemma 3.3 and Theorem 3.5. □

4. **Properties of unipotent groups**

**Definition 4.1.** As usual let $E_{ij} \in \text{Mat}_n(\mathbb{Z}/p)$ denote the elementary matrix characterised by the property that

\[
e_{kl}(E_{ij}) = \begin{cases} 1, & \text{if } k = i \text{ and } l = j, \\ 0, & \text{otherwise}, \end{cases}
\]

and let $I \in \text{Mat}_n(\mathbb{Z}/p)$ be the identity matrix. They satisfy the following identity:

\[
E_{ij}E_{kl} = \begin{cases} E_{il}, & \text{if } j = k, \\ 0, & \text{otherwise.} \end{cases}
\]

**Lemma 4.2.** For every $i < j$ and $k < l$ the following hold:

(a) we have $E_{ij}^2 = 0$,

(b) we have $E_{ij}E_{kl}E_{ij} = 0$,

(c) we have:

\[
[E_{ij}, E_{kl}] = \begin{cases} E_{il}, & \text{if } j = k, \\ -E_{kj}, & \text{if } l = i, \\ 0, & \text{otherwise}, \end{cases}
\]

**Proof.** Since $i \neq j$, part (a) is immediate from (4.1.1). If $E_{ij}E_{kl}E_{ij} \neq 0$ then $j = k$ and $l = i$ by (4.1.1). Then $i < j = k < l = i$, which is a contradiction. Therefore part (b) holds. If in

\[
[E_{ij}, E_{kl}] = E_{ij}E_{kl} - E_{kl}E_{ij}
\]

both terms are non-zero, then $j = k$ and $l = i$ by (4.1.1). This is not possible, as we have just seen. Therefore at most one of the terms is non-zero, and hence part (c) claim follows. □

**Lemma 4.3.** We have:

\[
[I + E_{ij}, I + E_{kl}] = \begin{cases} I + E_{il}, & \text{if } j = k, \\ I - E_{kj}, & \text{if } l = i, \\ I, & \text{otherwise}, \end{cases}
\]
for every $i < j$ and $k < l$.

**Proof.** Note that
\[(I + E_{ij})(I - E_{ij}) = I - E_{ij} + E_{ij} - E_{ij}^2 = I\]
when $i < j$, using claim (a) of Lemma 4.2. Therefore
\[[I + E_{ij}, I + E_{kl}] = (I + E_{ij})(I + E_{kl})(I - E_{ij})(I - E_{kl})\]
\[= I + E_{ij} + E_{kl} + E_{ij}E_{kl} - E_{ij} - E_{ij}^2 - E_{ij}E_{kl} - E_{ij}E_{kl}E_{ij}\]
\[+ E_{ki}E_{kl}E_{ij} + E_{ij}E_{kl} + E_{ij}E_{kl}E_{ij}E_{kl}.\]
By part (a) of Lemma 4.2 all red terms are zero, while by part (b) of Lemma 4.2 all blue terms are zero. Therefore
\[[I + E_{ij}, I + E_{kl}] = I + E_{ij}E_{kl} - E_{ij}E_{kl} = I + [E_{ij}, E_{kl}]\]
because of the cancellations between the remaining terms. The claim follows from part (c) of Lemma 4.2. \(\square\)

**Notation 4.4.** For every positive integer $m$ let $K_m(p)$ denote the following subgroup of $U_m(p)$:
\[K_m(p) = \{ B \in U_m(p) \mid e_{ii+1}(B) = 0 \text{ if } 1 \leq i < m \}.\]
For every pair of positive integers $k, m$ let $U_{k,m}(p)$ denote the following subgroup of $U_m(p)$:
\[U_{k,m}(p) = \{ B \in U_m(p) \mid e_{ij}(B) = 0 \text{ if } 1 \leq i < j \leq \min(i + k - 1, m) \}.\]

Note that, using this notation, we have
\[Z_m(p) = U_{m-1,m}(p), \ P_m(p) = U_{3,3}(p), \ U_m(p) = U_{1,m}(p) \text{ and } K_m(p) = U_{2,m}(p).\]

Finally let $e_{im} : \mathbb{Z}/p \to \mathbb{Z}/m(p)$ be the unique isomorphism such that
\[e_{im}(1) = I + E_{im}.\]

**Definition 4.5.** For every $1 \leq i < j \leq m$ let $b_{ij} : U_m(p) \to U_{j-i+1}(p)$ be the unique homomorphism such that
\[e_{kl}(b_{ij}(B)) = e_{(i+k-1)(i+l-1)}(B) \quad (\forall B \in U_m(p), \ 1 \leq k < l \leq j - i + 1).\]

In plain English this is the $(j - i + 1) \times (j - i + 1)$ diagonal block with the right upper corner at the $(i, j)$-th entry. Note that
\[(4.5.1) \quad U_{k,m}(p) = \bigcap_{j-i=k-1}^{j-i<i\leq j\leq m} \text{Ker}(b_{ij}) \quad (\forall 2 \leq k \leq m - 1).\]

In particular $U_{k,m}(p)$ is a normal subgroup of $U_m(p)$ for every $k$.

**Proposition 4.6.** The subgroup $U_{k,m}(p)$ is generated by the elements:
\[\{ I + E_{ij} \mid k \leq i + k - 1 < j \leq m \}.\]

**Proof.** First note that these elements are actually in $U_{k,m}(p)$, so the claim actually makes sense. We are going to show the latter by descending induction on $k$. When $k \geq m$ then $U_{k,m}(p)$ is the trivial group so the claim is trivially true. Now assume that the claim is true for $1 \leq k + 1 \leq m$. Then it will be sufficient to prove the lemma below. \(\square\)
**Lemma 4.7.** The image of the set \( \{ I + E_{ii+k} \mid 1 \leq i \leq m - k \} \) with respect to the quotient map \( U_{k,m}(p) \rightarrow U_{k,m}(p)/U_{k+1,m}(p) \) is a basis of the p-torsion abelian group \( U_{k,m}(p)/U_{k+1,m}(p) \).

**Proof.** Recall that the kernel of the map:

\[
b_{k,m} = \prod_{j-i=k \atop 1 \leq i < j \leq m} b_{ij} : U_{k,m}(p) \rightarrow Z_m(p)^{m-k} \cong (\mathbb{Z}/p)^{m-k}
\]

is \( U_{k+1,m}(p) \). Moreover under the identification \( Z_m(p)^{m-k} \cong (\mathbb{Z}/p)^{m-k} \) furnished by \( \iota_m \) this homomorphism \( b_{k,m} \) maps the set \( \{ I + E_{ii+k} \mid 1 \leq i \leq m - k \} \) bijectively onto the standard basis of \( (\mathbb{Z}/p)^{m-k} \) for every \( 1 \leq k \leq m - 1 \). The claim is now clear. \( \square \)

**Corollary 4.8.** The derived subgroup \( K_m(p)'^{(1)} \) is \( U_{4,m}(p) \).

**Proof.** For every \( i, j \) with \( 4 \leq i + 3 < j \leq m \) we have

\[
I + E_{ij} = [I + E_{i(i+2)}, I + E_{(i+2)j}] \in K_m(p)'^{(1)}
\]

by Lemma 4.9. Therefore \( K_m(p)'^{(1)} \) contains \( U_{4,m}(p) \) by Proposition 4.10. On the other hand the quotient \( K_m(p)/U_{4,m}(p) \) is generated by the images \( J \)

\[
\{ I + E_{ij} \mid 2 \leq i + 1 < j \leq m, \ j \leq i + 3 \}
\]

under the quotient map \( K_m(p) \rightarrow K_m(p)/U_{4,m}(p) \) by Proposition 4.10. Since for \( i, j \) and \( k, l \) with \( 2 \leq i + 1 < j \leq m, \ j \leq i + 3 \) and \( 2 \leq k + 1 < l \leq m, \ l \leq k + 3 \) if \( j = k \) then \( i + 3 < l \), and if \( l = i \), then \( k + 3 < j \), so we have

\[
[I + E_{ij}, I + E_{kl}] \in U_{4,m}(p)
\]

using Lemma 4.3. Hence the elements of \( J \) commute. Since they generate the quotient \( K_m(p)/U_{4,m}(p) \) we get that the latter is commutative. Therefore \( U_{4,m}(p) \) contains \( K_m(p)'^{(1)} \), too. \( \square \)

**Definition 4.9.** Recall that the p-Zassenhaus filtration of a finite p-group \( G \), denoted by \( G(n,p), \ n = 1, 2, \ldots \), is defined inductively by

\[
G(1,p) = G, \quad G(n,p) = (G(n/p),p)^n \prod_{i+j=n} [G(i,p), G(j,p)] \quad \text{for } n \geq 2.
\]

(The original definition is different, but it is equivalent to this one by a theorem of Lazard, see Theorem 11.2 of [1] on page 271). As its name suggest this is a descending filtration by characteristic subgroups. It follows from the formula above that for every \( n \) the quotient \( G(n,p)/G(n+1,p) \) is abelian of exponent dividing \( p \). Consider the graded \( \mathbb{Z}/p \)-module:

\[
\text{gr}(G) = \bigoplus_{n \geq 0} G(n,p)/G(n+1,p).
\]

The commutator map and the p-power map induce on \( \text{gr}(G) \) the structure of a \( p \)-restricted Lie \( \mathbb{Z}/p \)-algebra (see §12.2 of [1] on pages 298-305).

**Proposition 4.10.** We have \( U_{m}(p)(k,p) = U_{k,m}(p) \) for every \( k, m \geq 1 \).
Let $A \in U_{k,m}(p)$ be arbitrary. Write $A = I + X$ and $B = I + Y$ such that $e_{ij}(X) = 0$, if $1 \leq j \leq \min(i + k - 1, m)$, and $e_{ij}(Y) = 0$, if $1 \leq j \leq \min(i + l - 1, m)$. For every row vector $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m) \in (\mathbb{Z}/p)^m$ let $e_\alpha$ be its $l$-th coordinate $\alpha_l$. If $\alpha \in (\mathbb{Z}/p)^m$ is a row vector such that $e_\alpha(i) = 0$ when $i \leq j$ for some $j = 0, 1, \ldots$, then $e_\alpha(iX) = 0$ when $i \leq j + k$. Similarly $e_\alpha(iY) = 0$ when $i \leq j + l$. Therefore $\alpha XY$ is zero for every row vector $\alpha$, since $k + l = m$. Hence $XY = 0$, and so $AB = (I + X)(I + Y) = I + X + Y$. A similar argument shows that $BA = I + Y + X$. Therefore $AB = BA$, so claim (a) holds.

Now let $A \in U_{[m/p],m}(p)$ be arbitrary, and write $A = I + X$ such that $e_{ij}(X) = 0$, if $1 \leq j \leq \min(i + [m/p] - 1, m)$. If $\alpha \in (\mathbb{Z}/p)^m$ is a row vector such that $e_\alpha(i) = 0$ when $i \leq j$ for some $j = 0, 1, \ldots$, then $e_\alpha(iX) = 0$ when $i \leq j + [m/p]$. We get that $e_\alpha(iX^d) = 0$ when $i \leq j + [m/p]d$ by induction on $d$. So $\alpha X^p$ is zero for every row vector $\alpha$, since $[m/p]p \geq m$. Hence $X^p = 0$, and so $A^p = (I + X)^p = I + X + Y$. A similar argument shows that $BA = I + Y + X$. Therefore $AB = BA$, so claim (b) follows.

**Notation 4.12.** Let $\mathfrak{gl}_m(p)$ denote the Lie algebra associated to the rank $m$ matrix algebra $\text{Mat}_m(\mathbb{Z}/p)$ over $\mathbb{Z}/p$. Let $u_m(p) \subset \mathfrak{gl}_m(p)$ denote the sub-Lie algebra of strictly upper triangular matrices:

$$u_m(p) = \{ B \in \mathfrak{gl}_m(p) \mid e_{ij}(B) = 0 \text{ if } 1 \leq j \leq i \leq m \}.$$
Since $u_m(p)$ is a Lie subalgebra of the Lie algebra of an associative algebra which is closed under the $p$-power map, it has the structure of a $p$-restricted Lie $\mathbb{Z}/p$-algebra. For every $A \in U_m(p)$ let $c_k(A) \in \text{gr}(U_m(p))$ denote its class in the quotient $U_m(p)(k,p)/U_m(p)(k+1,p)$. In order to distinguish it from the commutator in groups, we will let $[\cdot,\cdot]$ denote the Lie bracket in Lie algebras.

**Proposition 4.13.** There is a unique isomorphism

$$\lambda_m : \text{gr}(U_m(p)) \to u_m(p)$$

of $p$-restricted Lie $\mathbb{Z}/p$-algebras such that $\lambda_m(c_{j-i}(I + E_{ij})) = E_{ij}$ for every $i, j$ such that $1 \leq i < j \leq m$.

**Proof.** By Lemma 4.7 and Proposition 4.10 the set \{ $c_k(I + E_{ii+k}) \mid 1 \leq i \leq m - k$ \} is a basis of $U_m(p)(k,p)/U_m(p)(k+1,p)$ for every $k \geq 1$. Therefore

$$\{c_{j-i}(I + E_{ij}) \mid 1 \leq i < j \leq m\}$$

is a basis of $\text{gr}(U_m(p))$. So there is a unique $\mathbb{Z}/p$-linear map $\lambda_m : \text{gr}(U_m(p)) \to u_m(p)$ such that $\lambda_m(c_{j-i}(I + E_{ij})) = E_{ij}$ for every $i, j$ such that $1 \leq i < j \leq m$.

Since $\lambda_m$ maps a basis onto a basis, it is an isomorphism. Since $\lambda_m([c_{j-i}(I + E_{ij}), c_{l-k}(I + E_{kl})]) = [\lambda_m(c_{j-i}(I + E_{ij})), \lambda_m(c_{l-k}(I + E_{kl}))]$ for every $i < j$ and $k < l$ by part (c) of Lemma 4.12 and by Lemma 4.13, we get that $\lambda_m$ is a Lie-algebra homomorphism using the bilinearity of the Lie bracket.

Let $(\cdot)^{[p]}$ denote the $p$-operation of any $p$-restricted Lie $\mathbb{Z}/p$-algebra. Then we have $(I + E_{ij})^p = I$, so $c_{j-i}(I + E_{ij})^{[p]} = 0$, while $E_{ij}^p = E_{ij} = 0$, hence

$$\lambda_m(c_{j-i}(I + E_{ij})^{[p]}) = \lambda_m(c_{j-i}(I + E_{ij}))^{[p]}$$

for every $1 \leq i < j \leq m$. Now we only need to add the following well-known fact: if $\lambda : g \to h$ is a Lie algebra isomorphism between $p$-restricted Lie $\mathbb{Z}/p$-algebras, and it respects the $p$-operation on a basis of $g$, then it is an isomorphism between $p$-restricted Lie $\mathbb{Z}/p$-algebras. □

**Definition 4.14.** Let $\phi_{n+1} : U_{n+1}(p) \to (\mathbb{Z}/p)^n$ be the homomorphism given by the rule $U \mapsto (\epsilon_{12}(U), \epsilon_{23}(U), \ldots, \epsilon_{n(n+1)}(U))$. Similarly let $\eta_{n+1} : K_{n+1}(p) \to (\mathbb{Z}/p)^{n-1}$ be the homomorphism given by the rule $U \mapsto (\epsilon_{13}(U), \epsilon_{24}(U), \ldots, \epsilon_{n-1n+1}(U))$. Let $\langle \cdot, \cdot \rangle : (\mathbb{Z}/p)^3 \times (\mathbb{Z}/p)^2 \to \mathbb{Z}/p$ be the bilinear pairing given by the rule:

$$\langle (a_1, a_2, a_3), (b_1, b_2) \rangle = a_1b_2 - a_3b_1.$$

**Corollary 4.15.** For every $A \in U_4(p)$ and $B \in K_4(p)$ we have:

$$[A, B] = \iota_4(\{\phi_4(A), \eta_4(B)\}).$$

**Proof.** From Proposition 4.10 we know that $A \in U_4(p)(1,p)$ and $B \in U_4(p)(2,p)$, so we get that $[A, B] \in U_4(p)(3,p) = Z_4(p)$ using again Proposition 4.10. As $U_4(p)(4,p)$ is trivial, again from Proposition 4.10 the commutator $[A, B]$ is uniquely determined by its class in $\text{gr}(U_4(p))$. The claim now follows from Proposition 4.13. □

**Notation 4.16.** For every $m$ let $\phi_m, \phi_m^2, \ldots, \phi_m^{m-1}$ denote the coordinates of $\phi_m : U_m(p) \to (\mathbb{Z}/p)^{m-1}$.

**Lemma 4.17.** There is a group homomorphism $\chi : U_3(2) \to U_4(2)$ such that $\phi_4 \circ \chi = (\phi_3^1, \phi_3^2, \phi_3^3)$. 
Proof. Note that the group $U_3(2)$ is generated by the two elements:

\[ x = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \]

subject to the relations:

\[ x^2 = I, \quad y^2 = I, \quad [x, y]^2 = I, \quad [x, [x, y]] = I, \quad [y, [x, y]] = I, \]

and this system of relations give a presentation of $U_3(2)$. On the other hand the the two elements:

\[ a = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \]

of $U_4(2)$ satisfy the same relations:

\[ a^2 = I, \quad b^2 = I, \quad [a, b]^2 = I, \quad [a, [a, b]] = I, \quad [b, [a, b]] = I. \]

Therefore there is a group homomorphism $\chi : U_3(p) \to U_4(p)$ such that $\chi(x) = a$ and $\chi(y) = b$. In particular

\[ \phi_4(\chi(x)) = (\phi^1_3, \phi^2_3, \phi^3_3)(x) \quad \text{and} \quad \phi_4(\chi(y)) = (\phi^1_3, \phi^2_3, \phi^3_3)(y). \]

Since $x$ and $y$ generate $U_3(p)$, the claim follows. \qed

5. Fibre products and embedding problems

**Definition 5.1.** Let $\overline{U}_m(p)$ and $\overline{U}_m(p)$ denote the quotient groups:

\[ \overline{U}_m(p) = U_m(p)/Z_m(p) \quad \text{and} \quad \overline{U}_m(p) = U_m(p)/P_m(p), \]

respectively, and let $\omega_{n+1} : U_{n+1}(p) \to \overline{U}_{n+1}(p)$ and $\omega_{n+1} : U_{n+1}(p) \to \overline{U}_{n+1}(p)$ denote the quotient maps. Let $\zeta_{n+1} : \overline{U}_{n+1}(p) \to (\mathbb{Z}/p)^n$ be the unique homomorphism such that the composition of $\omega_{n+1}$ and $\zeta_{n+1}$ is the homomorphism $\phi_{n+1}$. Similarly let $\kappa_{n+1} : \overline{U}_{n+1}(p) \to (\mathbb{Z}/p)^n$ be the unique group homomorphism such that the composition of $\omega_{n+1}$ and $\kappa_{n+1}$ is the homomorphism $\phi_{n+1}$.

**Definition 5.2.** By a class of embedding problems $B$ we mean a homomorphism $\epsilon : \Gamma \to \Delta$ of finite groups. We say that $B$ has abelian kernel if $\text{Ker}(\epsilon)$ is abelian. We say that $B$ is central if $\text{Ker}(\epsilon)$ is a central subgroup of $\Gamma$. We say that an embedding problem for a group $G$ belongs to a class $B$ as above if it is of the form:

\[ \overline{\psi} \downarrow \overline{\phi} \quad \Gamma \xrightarrow{\epsilon} \Delta \]

We will let $B(\phi)$ denote the latter. Given a homomorphism $\chi : H \to G$ we define the pull-back of $B(\phi)$ as the embedding problem $B(\phi \circ \chi)$ belonging to the class $B$.

**Definition 5.3.** Let $E_n$ denote the class of embedding problems given by the homomorphism $\phi_{n+1} : U_{n+1}(p) \to (\mathbb{Z}/p)^n$. Let $D_n$ denote the class of embedding problems given by the homomorphism $\zeta_{n+1} : \overline{U}_{n+1}(p) \to (\mathbb{Z}/p)^n$. Let $C_n$ denote the class of embedding problems given by the homomorphism $\kappa_{n+1} : \overline{U}_{n+1}(p) \to (\mathbb{Z}/p)^n$. 

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Notation 5.4. Let $G, H, J$ be three groups and let $\gamma : G \to J$ and $\chi : H \to J$ be two homomorphisms. The fibre product $G \times_{\gamma, \chi} H$ is the group:

$$G \times_{\gamma, \chi} H = \{(g, h) \in G \times H \mid \gamma(g) = \chi(h)\} \subseteq G \times H.$$ 

For every $n \geq 3$ and pro-finite group $G$ let $D_n(G)$ denote the set of continuous homomorphisms $\underline{\alpha} : G \to (\mathbb{Z}/p)^n$ such that the embedding problem $D_n(\underline{\alpha})$ has a solution.

Notation 5.5. Now let $G$ be a $p$-group, and let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in D_n(G)$ for some $n \geq 3$. Let $H$ denote the fibre product $U_{n+1}(p) \times_{\phi_{n+1}, \alpha} G$, and let $\rho : H \to G$ be the projection onto the second factor. Let $Z$ be the subgroup

$$Z = \{(a, b) \in U_{n+1}(p) \times_{\phi_{n+1}, \alpha} G \mid a \in \mathbb{Z}_{n+1}(p), \ b = 1\}$$

of $H$. Finally let $B$ be the class of embedding problems $\epsilon : \Gamma \to \Delta$ and let $B(\phi)$ be an embedding problem for $G$ belonging to the class $B$.

Proposition 5.6. Assume that the embedding problem $B(\phi \circ \rho)$ has a solution $\psi$ whose restriction onto $Z$ is trivial. Then $B(\phi)$ has a solution, too.

Proof. Now let $\overline{\Gamma}$ denote the fibre product $\overline{U}_{n+1}(p) \times \gamma, \alpha, G$, and let $\overline{\psi} : \overline{\Gamma} \to \Gamma$ be the projection onto the second factor. Note that the quotient of $H$ by its normal subgroup $Z$ is canonically isomorphic to $\overline{\Gamma}$. Since $\text{Ker}(\psi)$ contains $Z$ by assumption, the homomorphism $\psi$ factors through the quotient map $H \to \overline{\Gamma}$, so there is a solution $\overline{\psi} : \overline{\Gamma} \to \Gamma$ of the embedding problem $B(\phi \circ \rho)$. By assumption there is a solution $\omega : G \to \overline{U}_{n+1}(p)$ of the embedding problem $D_n(\alpha)$. The direct product $\omega \times \text{id}_G : G \to \overline{U}_{n+1}(p) \times G$ maps $G$ into $\overline{\Gamma}$. We have a commutative diagram:

$$
\begin{array}{ccc}
G & \xrightarrow{\omega \times \text{id}_G} & \overline{\Gamma} \\
\downarrow{\phi} & & \downarrow{\epsilon} \\
G & \xrightarrow{\text{id}_G} & \Delta,
\end{array}
$$

The composition $\overline{\psi} \circ (\omega \times \text{id}_G)$ is the identity, therefore the composition $\overline{\psi} \circ (\omega \times \text{id}_G)$ is a solution to $B(\phi)$. $\square$

Proposition 5.7. Assume that $n \geq 4$, the class $B$ has abelian kernel, and the embedding problem $B(\phi \circ \rho)$ has a solution for $G$. Then $B(\phi)$ has a solution, too.

Proof. Let $\psi : H \to \Gamma$ be a solution of $B(\phi \circ \rho)$. Then the restriction of $\psi$ onto the subgroup

$$K = \{(a, b) \in U_{n+1}(p) \times_{\phi_{n+1}, \alpha} G \mid a \in K_{n+1}(p), \ b = 1\}$$

of $H$ lands in the kernel of $\text{Ker}(\epsilon)$. The latter is abelian, so $\psi$ is trivial on the subgroup

$$K' = \{(a, b) \in U_{n+1}(p) \times_{\phi_{n+1}, \alpha} G \mid a \in U_{n+1}(p), \ b = 1\}$$

by Corollary 4.8. Since $n \geq 4$ the group $K'$ contains $Z$ (introduced in Notation 5.5), and hence the claim follows from Proposition 5.6. $\square$

Proposition 5.8. Assume that $n = 3$ and the embedding problem $B(\phi \circ \rho)$ has a solution $\psi$. In addition suppose that one of the following conditions is also true:

(a) the embedding problem $E_3(\alpha)$ has a solution,
Notation 5.10. For every \( 1 \leq i < j \leq m \) such that \((i,j) \neq (1,m)\) let \( \overline{b}_{ij} : \overline{U}_m(p) \to U_{j-i+1}(p) \) be the unique homomorphism such that the composition of the quotient map \( \omega_m : U_m(p) \to \overline{U}_m(p) \) and \( \overline{b}_{ij} \) is the homomorphism \( b_{ij} \) in Definition 4.5. For every sequence \( c_1, c_2, \ldots, c_m \in H^1(G) = \text{Hom}(G, \mathbb{Z}/p) \) let \( \text{span}(c_1, c_2, \ldots, c_m) \subseteq H^1(G) \) denote the \( \mathbb{Z}/p \)-linear span of these elements. Now let \( \gamma : G \to (\mathbb{Z}/p)^4 \) be a homomorphism with coordinates \( \gamma = (\gamma_1, \gamma_2, \gamma_3, \gamma_4) \).

\[(b)\] either \( \alpha_1 = 0 \) or \( \alpha_3 = 0 \),

\[(c)\] we have \( p = 2 \) and \( \alpha_1 = \alpha_3 \).

Then the embedding problem \( B(\phi) \) has a solution for \( H \).

**Proof.** First assume that \((b)\) holds and let \( \omega : G \to U_4(p) \) be a solution to \( E_3(\alpha) \). The direct product \( \omega \times \text{id}_G : G \to U_4(p) \times G \) maps \( G \) into \( H \). The composition \( \overline{\sigma} \circ (\omega \times \text{id}_G) \) is the identity, therefore the composition \( \sigma \circ (\omega \times \text{id}_G) \) is a solution to \( B(\phi) \). Next assume that \((b)\) is true. It will be enough to show that \( E_3(\alpha) \) has a solution by the above. Let \( v_1 : U_3(p) \to U_4(p) \) and \( v_3 : U_3(p) \to U_4(p) \) be the homomorphisms given by the rules:

\[
\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & a & b & 0 \\ 0 & 1 & c & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
\]

respectively. First suppose that \( \alpha_1 = 0 \). Since \( D_3(\alpha) \) is solvable there is a homomorphism \( \sigma : G \to U_3(p) \) such that \( (\alpha_2, \alpha_3) = \zeta_3 \circ \sigma \). Then the composition \( \sigma \circ v_1 \) is a solution for \( E_3(\alpha) \).

The proof in the case when \( \alpha_3 = 0 \) is similar. Since \( D_3(\alpha) \) is solvable there is a homomorphism \( \sigma : G \to U_3(p) \) such that \( (\alpha_1, \alpha_2) = \zeta_3 \circ \sigma \). Then the composition \( \sigma \circ v_3 \) is a solution for \( E_3(\alpha) \). Finally we assume that \((c)\) is true. Since \( D_3(\alpha) \) is solvable there is a homomorphism \( \sigma : G \to U_3(p) \) such that \( (\alpha_1, \alpha_2) = \zeta_3 \circ \sigma \). Then the composition \( \chi \circ \sigma \), where \( \chi : U_3(2) \to U_4(2) \) is the homomorphism in Lemma 4.17, is a solution for \( E_3(\alpha) \).

Now let \( \beta : G \to (\mathbb{Z}/p)^2 \) be a homomorphism.

**Proposition 5.9.** Assume that the embedding problem \( E_2(\beta \circ \rho) \) has a solution. Then \( E_2(\beta) \) has a solution, too.

**Proof.** Note that the class \( E_2 \) has abelian kernel. Therefore the claim holds when \( n \geq 4 \) by Proposition 5.7. So we may assume that \( n = 3 \) without the loss of generality. We may also suppose that \( \alpha_3 \neq 0 \) by Proposition 5.8. Therefore there is a \( g \in G \) such that \( \alpha_3(g) = 1 \). Choose a \( u \in U_4(p) \) such that \( \phi_4(u) = \alpha(g) \). Then \( (u,g) \in H \) and \( (I + E_{13}, 1) \in H \) (where \( 1 \) is the unit of \( G \)), while

\[
[(I + E_{13}, 1), (u,g)] = [(I + E_{13}, u), [1,g]] = (I + E_{14}, 1)
\]

using Lemma 4.13. Let \( \psi : H \to U_3(p) \) be a solution to \( E_2(\beta \circ \rho) \). Then \( \psi((I + E_{13}, 1)) \) lies in \( \text{Ker}(\zeta_2) \), which is a central subgroup in \( U_3(p) \). Therefore

\[
\psi((I + E_{14}, 1)) = \psi([(I + E_{13}, 1), (u,g)]) = [\psi((I + E_{13}, 1)), \psi((u,g))] = I.
\]

The element \((I + E_{14}, 1)\) generates the subgroup \( Z \) introduced in Notation 5.5, so by Proposition 5.6 the embedding problem \( E_2(\beta) \) has a solution. \( \square \)

**Notation 5.10.** For every \( 1 \leq i < j \leq m \) such that \( (i,j) \neq (1,m) \) let \( \overline{b}_{ij} : \overline{U}_m(p) \to U_{j-i+1}(p) \) be the unique homomorphism such that the composition of the quotient map \( \omega_m : U_m(p) \to \overline{U}_m(p) \) and \( \overline{b}_{ij} \) is the homomorphism \( b_{ij} \) in Definition 4.5. For every sequence \( c_1, c_2, \ldots, c_m \in H^1(G) = \text{Hom}(G, \mathbb{Z}/p) \) let \( \text{span}(c_1, c_2, \ldots, c_m) \subseteq H^1(G) \) denote the \( \mathbb{Z}/p \)-linear span of these elements. Now let \( \gamma : G \to (\mathbb{Z}/p)^4 \) be a homomorphism with coordinates \( \gamma = (\gamma_1, \gamma_2, \gamma_3, \gamma_4) \).
Proposition 5.11. Assume that \( p = 2 \), the embedding problem \( D_4(\gamma \circ \rho) \) has a solution \( \psi \), but the problem \( D_4(\gamma) \) does not. Then \( n = 3 \) and one of the following is true:

\[
\begin{align*}
(14) \text{ we have } \ker(\overline{b}_{14} \circ \psi) \cap Z = \{1\} \text{ and } \operatorname{span}(\alpha_1, \alpha_3) = \operatorname{span}(\gamma_1, \gamma_3), \\
(25) \text{ we have } \ker(\overline{b}_{25} \circ \psi) \cap Z = \{1\} \text{ and } \operatorname{span}(\alpha_1, \alpha_3) = \operatorname{span}(\gamma_2, \gamma_4),
\end{align*}
\]

where \( Z \subseteq H \) is the subgroup introduced in Notation 5.9.

Proof. Note that the class \( D_4 \) has abelian kernel. Therefore \( n = 3 \) by Proposition 5.7. We also know that \( \alpha_1 \neq 0 \) and \( \alpha_3 \neq 0 \) by part (b) of Proposition 5.8. If \( \operatorname{span}(\alpha_1, \alpha_3) \) is one-dimensional then \( \alpha_1 = \alpha_3 \), and hence \( D_4(\kappa_5 \circ \rho) \) has a solution by part (c) of Proposition 5.8. This is a contradiction, so \( \operatorname{span}(\alpha_1, \alpha_3) \) is two-dimensional.

If \( \ker(\psi) \cap Z \neq \{1\} \), then \( \ker(\psi) \supseteq Z \), where \( Z \subseteq H \) is the subgroup introduced in Notation 5.9 since this subgroup is of order 2. By Proposition 5.3 the latter is not possible, so \( \ker(\psi) \cap Z = \{1\} \). Since \( \overline{b}_{14} \times \overline{b}_{25} : \overline{\tau}_5(2) \to U_4(2) \times U_4(2) \) is injective, we get that either \( \ker(\overline{b}_{14} \circ \psi) \cap Z = \{1\} \) or \( \ker(\overline{b}_{25} \circ \psi) \cap Z = \{1\} \). Let’s consider the first case; the second can be handled similarly. We will show that (14) holds. Since \( \operatorname{span}(\alpha_1, \alpha_3) \) is two-dimensional, it will be sufficient to show that \( \alpha_1, \alpha_3 \in \operatorname{span}(\gamma_1, \gamma_3) \) in order to conclude the proof.

First assume to the contrary that \( \alpha_1 \notin \operatorname{span}(\gamma_1, \gamma_3) \). Then there is a \( g \in G \) such that \( (\gamma_1, \gamma_2, \gamma_3)(g) = (0, *, 0) \) and \( \alpha(g) = (1, *, *) \). Choose a \( u \in U_4(p) \) such that \( \phi_4(u) = \alpha(g) \). Then \( (u, g) \in H \) and \( (I + E_{24}, 1) \in H \), while

\[
[(u, g), (I + E_{24}, 1)] = [(u, I + E_{24}, 1)] = (I + E_{14}, 1)
\]

using Corollary 5.15. On the other hand

\[
\overline{b}_{14} \circ \psi((I + E_{14}, 1)) = \overline{b}_{14} \circ \psi([(u, g), (I + E_{24}, 1)])
\]

\[
= [\overline{b}_{14} \circ \psi((u, g)), \overline{b}_{14} \circ \psi((I + E_{24}, 1))] = I
\]

using Corollary 5.15 and \( \phi_4 \circ \overline{b}_{14} \circ \psi((u, g)) = (\gamma_1, \gamma_2, \gamma_3)(g) \). Since \( (I + E_{24}, 1) \) generates \( Z \), this is a contradiction, so \( \alpha_1 \in \operatorname{span}(\gamma_1, \gamma_3) \).

Now suppose that \( \alpha_3 \notin \operatorname{span}(\gamma_1, \gamma_3) \). Then there is a \( g \in G \) such that \( \alpha(g) = (*, *, 1) \) and \( (\gamma_1, \gamma_2, \gamma_3)(g) = (0, *, 0) \). Choose a \( u \in U_4(p) \) such that \( \phi_4(u) = \alpha(g) \). Then \( (u, g) \in H \) and \( (I + E_{13}, 1) \in H \), while

\[
[(I + E_{13}, 1), (u, g)] = [(I + E_{13}, u), [1, g)] = (I + E_{14}, 1)
\]

using Corollary 5.15 and and \( \phi_4 \circ \overline{b}_{14} \circ \psi((u, g)) = (\gamma_1, \gamma_2, \gamma_3)(g) \). On the other hand using the same computation as above we get

\[
\overline{b}_{14} \circ \psi((I + E_{14}, 1)) = \overline{b}_{14} \circ \psi([(I + E_{13}, 1), (u, g)]) = I.
\]

This is a contradiction, so \( \alpha_3 \in \operatorname{span}(\gamma_1, \gamma_3) \). \( \square \)

Notation 5.12. Let \( \theta_{n+1} : \overline{U}_{n+1}(p) \to \overline{U}_{n+1}(p) \) denote the quotient map. For every \( 1 \leq i < j \leq m \) such that \( j \leq i + 3 \) let \( c_{ij} : \overline{U}_m(p) \to \overline{U}_{j-i+1}(p) \) be the unique homomorphism such that \( c_{ij} \circ \omega_{n+1} = \omega_{j-i+1} \circ b_{ij} \). For every \( m \) let \( (\kappa_1^m, \kappa_2^m, \ldots, \kappa_{m-1}^m) \) denote the coordinates of \( \kappa_m : \overline{U}_m(p) \to (\mathbb{Z}/p)^{m-1} \). Let \( G_2 \) denote the double (or iterated) fibre product:

\[
\{(u, v, v) \in U_4(p) \times \overline{U}_5(p) \times U_4(p) \mid \phi_4(u) = (\kappa_1^3, \kappa_2^3, \kappa_3^3)(g), \phi_4(v) = (\kappa_5^2, \kappa_5^3, \kappa_5^4)(g)\}.
\]

Let \( \rho : G_2 \to \overline{U}_5(p) \) denote the projection onto the second, middle factor.
Lemma 5.15. Let \( \kappa \) denote the projection onto the first and the third factor, respectively.

Let \( V \subseteq G_2 \) be the subgroup:
\[
\{(u, g, v) \in U_4(p) \times U_5(p) \times U_4(p) \mid \omega_4(u) = c_{14}(g), \omega_4(v) = c_{25}(g)\}.
\]

It is isomorphic to \( U_5(p) \), indeed
\[
\mathcal{B}_{14} \times \Theta_5 \times \mathcal{B}_{25} : U_5(p) \to U_4(p) \times U_5(p) \times U_4(p)
\]
maps \( U_5(p) \) isomorphically onto \( V \). Note that \( \kappa \circ \rho \mid V \) is \( \kappa \) under this identification, and hence it is surjective. Since the kernel of \( \kappa \) is \( U_5(p) \), we get that the composition of \( \psi \mid V \) and the quotient map \( U_5(p) \to U_5(p)/U_5(p) \) is surjective. Since \( U_5(p) \) is a \( p \)-group, we get that \( \psi \mid V \) is surjective. But \( V \) and \( U_5(p) \) have the same order, as they are isomorphic, therefore \( \psi \mid V \) is an isomorphism.

Notation 5.14. Let \( N \) denote the kernel of \( \psi \). Let \( C = Z_4(p) \times \{1\} \times Z_4(p) \subseteq G_2 \) and \( L = K_4(p) \times \text{Ker}(\kappa) \times K_4(p) \subseteq G_2 \). Let \( \sigma_1 : G_2 \to U_4(p) \) and \( \sigma_3 : G_2 \to U_4(p) \) denote the projection onto the first and the third factor, respectively.

Lemma 5.15. The following hold:

(a) the order of \( N \) is \( p^3 \);
(b) the intersection \( N \cap C \) is trivial,
(c) we have \( N \subseteq L \),
(d) the map \( \eta_4 \circ \sigma_1 \times \eta_4 \circ \sigma_3 : N \to (\mathbb{Z}/p)^2 \times (\mathbb{Z}/p)^2 \) is non-trivial.

Note that the homomorphism \( \eta_4 \circ \sigma_1 \times \eta_4 \circ \sigma_3 \mid N \) in part (d) is well-defined because of part (c).

Proof. As \( \psi \mid V \) is an isomorphism the map \( \psi \mid V \) is surjective. The order of \( U_5(p) \) is \( p^3 \), while the order of \( G_2 \) is \( p^{13} \), so (a) holds. As \( C \) is a subgroup of \( V \), and \( \psi \mid V \) is an isomorphism, part (b) is clear. Since \( \kappa \circ \rho \mid V \), the subgroup \( N \) must lie in the kernel of \( \kappa \circ \rho \), which is \( L \), so (c) is true. Assume now that \( \eta_4 \circ \sigma_1 \times \eta_4 \circ \sigma_3 \mid N \) is trivial. The kernel of \( \eta_4 \circ \sigma_1 \times \eta_4 \circ \sigma_3 \) in \( L \) is the direct sum of \( C \) and \( L \cap \text{Ker}(\sigma_1 \times \sigma_3) \). Since \( N \cap C \) is trivial by part (b), the group \( N \) injects into \( L \cap \text{Ker}(\sigma_1 \times \sigma_3) \). However \( L \cap \text{Ker}(\sigma_1 \times \sigma_3) \cong \text{Ker}(\kappa) \) via \( \rho \), so its order is \( p^3 \). But the order of \( N \) is \( p^3 \) by part (a), a contradiction. So (d) holds.

Assume now that \( \eta_4 \circ \sigma_1 \mid N \) is non-trivial; the case when \( \eta_4 \circ \sigma_3 \mid N \) is non-trivial can be handled similarly. Let \( g = (g_1, g_2, g_3) \in N \) be an element such that \( \eta_4 \circ \sigma_1(g) \) is non-trivial. Then either the first or the second coordinate of \( \eta_4 \circ \sigma_1(g) \) is non-zero. Let’s first assume the former. By taking a suitable power of \( g \), if this is necessary, we may assume without the loss of generality that \( \eta_4 \circ \sigma_1(g) = (1, *) \). Note that \( (I + E_{34}, \omega_5(I + E_{34}), I + E_{23}) \in G_2 \) and
\[
[(g_1, g_2, g_3), (I + E_{34}, \omega_5(I + E_{34}), I + E_{23})] =
[(g_1, I + E_{34}], [g_2, \omega_5(I + E_{34})], [g_3, I + E_{23}]) = (I + E_{14,1}, 1, 1) \in N
\]
using Corollary 5.13 and part (c) of Lemma 5.13. Since \( (I + E_{14,1}, 1, 1) \in C \), this contradicts part (b) of Lemma 5.13.

Now suppose the latter. We may assume without the loss of generality that \( \eta_4 \circ \sigma_1(g) = (*, 1) \), as above. Note that \( (I + E_{12}, I, I) \in G_2 \) and
\[
[(I + E_{24}, I, I), (g_1, g_2, g_3)] = [(I + E_{12}, g_1], [I, g_2], [I, g_3]) = (I + E_{14,1}, 1, 1) \in N
\]
using Corollary 4.13 and part (c) of Lemma 5.13. Since \((I + E_{14}, 1, 1) \in C\), this is again a contradiction.

\[\square\]

6. Massey envelopes

**Definition 6.1.** We say that \(G\) satisfies weak Massey vanishing for \(n\) if for every homomorphism \(\varphi : G \rightarrow (\mathbb{Z}/p)^n\) such that the embedding problem \(D_n(\varphi)\) has a solution, the problem \(E_n(\varphi)\) also has a solution. We say that \(G\) satisfies strong Massey vanishing for \(n\) if for every homomorphism \(\varphi : G \rightarrow (\mathbb{Z}/p)^n\) such that the embedding problem \(C_n(\varphi)\) has a solution, the problem \(E_n(\varphi)\) also has a solution.

Now we will concentrate on the case \(p = 2\). The main result of this section is:

**Theorem 6.2.** There is a pro-2 group \(G\) which satisfies weak Massey vanishing for \(n \geq 3\), but does not satisfy strong Massey vanishing for \(n = 4\).

The proof will occupy the rest of this section.

**Definition 6.3.** Let \(D_{\leq n}(G)\) denote the union \(\bigcup_{3 \leq k \leq n} D_k(G)\). When \(G\) is finite, the set \(D_{\leq n}(G)\) is also finite. Every homomorphism \(\alpha : G \rightarrow H\) of pro-finite groups induces a map \(\alpha^* : D_{\leq n}(H) \rightarrow D_{\leq n}(G)\) via composition with \(\alpha\) which is injective when \(\alpha\) is surjective. We will identify \(D_{\leq n}(H)\) with its image under \(\alpha^*\) in this case. Finally for every \(\beta \in D_{\leq n}(G)\) let \(d(\beta)\) denote the unique integer such that \(\beta \in D_{d(\beta)}(G)\). For every set \(S\) let \(|S|\) denote its cardinality, that is, the minimal ordinal in bijection with \(S\).

**Lemma 6.4.** For every non-trivial finite \(p\)-group \(G\) and \(n \geq 3\) we have \(|D_{\leq n}(G)| \geq n\).

**Proof.** It will be sufficient to show that \(|D_n(G)| \geq 3\) for every \(n \geq 3\), as \(3(n-2) \geq n\) when \(n \geq 3\). Since \(G\) is a finite \(p\)-group there is a non-zero homomorphism \(\mu : G \rightarrow \mathbb{Z}/p\). Now let \(m_i : G \rightarrow (\mathbb{Z}/p)^n\) be the homomorphism whose \(i\)-th coordinate is \(\mu\) and all other coordinate is the zero map for every \(i = 1, 2, \ldots, n\). These maps are pairwise different, so \(|D_n(G)| \geq n \geq 3\). \(\square\)

**Definition 6.5.** Now let \(G\) be a non-trivial \(p\)-group. We can construct three sequences of objects of the following kind:

(a) a finite group \(G_k\) for every \(k = 0, 1, \ldots\),

(b) a surjective homomorphism \(\pi_k : G_k \rightarrow G_{k-1}\) for every \(k = 1, 2, \ldots\),

(c) a bijection \(\iota_k : D_{\leq k+3}(G_k) \rightarrow |D_{\leq k+3}(G_k)|\) for every \(k = 0, 1, \ldots\),

with the following properties:

(i) we have \(G_0 = G\),

(ii) we have \(G_k = U_{d(\alpha)+1}(p) \times \phi_{d(\alpha)+1, \alpha} G_{k-1}\), where \(\alpha \in D_{\leq k+2}(G_{k-1})\) is the pre-image of \(k - 1\) with respect to \(\iota_{k-1}\) for \(k \geq 1\),

(iii) the map \(\pi_k : G_k \rightarrow G_{k-1}\) is the projection onto the second factor of the fibre product \(G_k\) for \(k \geq 1\),

(iv) the restriction of \(\iota_k\) onto \(D_{\leq k+2}(G_{k-1}) \subseteq D_{\leq k+3}(G_k)\) (where the inclusion is with respect to \(\pi_k\)) is \(\iota_{k-1}\) for \(k \geq 1\).

In the construction we only have some freedom in the choice of \(\iota_k\). Note that we can perform the construction in (ii) as \(|D_{\leq k+2}(G_{k-1})| \geq k + 2\) by Lemma 6.4 so
Theorem 6.10. Assume that the embedding problem is solvable over \( M \). Given a sequence above, let \( \mathcal{M}(G) \) denote the projective limit of the system:

\[
\cdots \xrightarrow{\pi_{k+1}} G_k \xrightarrow{\pi_k} G_{k-1} \xrightarrow{\pi_{k-1}} \cdots
\]

by slight abuse of notation. We will call \( \mathcal{M}(G) \) a Massey envelope of \( G \). It is equipped with a surjective homomorphism \( \pi : \mathcal{M}(G) \to G_k \) for every \( k = 0, 1, \ldots \).

We will let \( \pi \) denote this map when \( k = 0 \).

Remark 6.6. If \( G \) is a \( p \)-group, then it is easy to prove using induction that \( G_k \) is a \( p \)-group, too. Indeed, \( p \)-groups are closed under direct products and taking subgroups, so under fibre products, too. As a consequence we get that \( \mathcal{M}(G) \) is a pro-\( p \) group in this case.

Lemma 6.7. The Massey envelope \( \mathcal{M}(G) \) satisfies weak Massey vanishing for every \( n \geq 3 \).

Proof. Let \( \alpha \in \mathcal{D}_k(\mathcal{M}(G)) \) for some \( n \geq 3 \). Then there is an index \( k \) such that \( \alpha \) is already an element of \( (\pi^k)^* (\mathcal{D}_n(G_k)) \subset \mathcal{D}_n(\mathcal{M}(G)) \). By Lemma 6.4 we may assume that \( \alpha \) is the pre-image of \( k \) with respect to \( \iota_k \) without the loss of generality by enlarging \( k \), if this is necessary. Therefore \( G_{k+1} \) is the fibre product \( U_{d(\alpha)+1}(p) \times_{\phi_{d(\alpha)+1}, G_k} G_k \). Clearly \( E_{d(\alpha)}(\alpha) \) is solvable over \( G_{k+1} \), the solution being the projection of this fibre product onto its first factor. Therefore, the pull-back of this embedding problem is solvable over \( \mathcal{M}(G) \), too.

Lemma 6.8. Let \( H \) be a pro-finite group which satisfies weak Massey vanishing for every \( n \geq 3 \) and let \( \chi : H \to G \) be a homomorphism, where \( G \) is a non-zero \( p \)-group. Then there is a homomorphism \( \tilde{\chi} : H \to \mathcal{M}(G) \) such that \( \chi = \pi \circ \tilde{\chi} \).

Proof. We are going to construct a sequence of homomorphisms \( \chi_k : H \to G_k \) by induction on \( k \) such that

1. \( \chi_0 = \chi \)
2. \( \pi_k \circ \chi_k = \chi_{k-1} \) for every \( k = 1, 2, \ldots \)

The limit \( \tilde{\chi} \) of the homomorphisms \( \chi_k \) will have the required properties. Assume now that \( \chi_{k-1} \) has been constructed already. Let \( \alpha \in \mathcal{D}_{\leq k+2}(G_{k-1}) \) be the pre-image of \( k-1 \) with respect to \( \iota_{k-1} \), as above. Then \( \alpha \circ \chi_{k-1} \in \mathcal{D}_{\leq k+2}(H) \), and hence the embedding problem \( \mathcal{E}_{d(\alpha)}(\chi_{k-1}) \) has a solution \( \sigma : H \to U_{d(\alpha)+1}(p) \) by our assumptions. The direct product \( \sigma \times \chi_{k-1} : H \to U_{d(\alpha)+1}(p) \times G_{k-1} \) lands in the fibre product \( G_k = U_{d(\alpha)+1}(p) \times_{\phi_{d(\alpha)+1}, G_k} G_k \), since \( \sigma \) is a solution of \( \mathcal{E}_{d(\alpha)}(\alpha \circ \chi_{k-1}) \), and so it furnishes a homomorphism \( \chi_k : H \to G_k \). By construction the composition of \( \chi_k \) and the projection \( \pi_k \) of \( G_k \) onto its second factor is \( \chi_{k-1} \).

Lemma 6.9. Assume that the embedding problem \( \mathcal{D}_4(\kappa_5 \circ \pi) \) has a solution, and let \( H \) be a pro-finite group which satisfies weak Massey vanishing for every \( n \geq 3 \) equipped with a homomorphism \( \phi : H \to \overline{U}_5(p) \). Then the embedding problem \( \mathcal{D}_4(\kappa_5 \circ \phi) \) is solvable.

Proof. By Lemma 6.8 there is a homomorphism \( \tilde{\phi} : H \to \mathcal{M}(G) \) such that \( \phi = \pi \circ \tilde{\phi} \). According to our assumptions we also have a solution \( \sigma : \mathcal{M}(G) \to U_5(p) \) to \( \mathcal{D}_4(\kappa_5 \circ \pi) \). Then \( \sigma \circ \tilde{\phi} \) is a solution of \( \mathcal{D}_4(\kappa_5 \circ \phi) = \mathcal{D}_4(\kappa_5 \circ \pi \circ \tilde{\phi}) \).

Theorem 6.10. The embedding problem \( \mathcal{D}_4(\kappa_5 \circ \pi) \) has no solution for the Massey envelope \( \mathcal{M}(\overline{U}_5(2)) \).
By Lemma 6.7 this result implies Theorem 6.2 since \( \mathcal{M}(\overline{U}_5(2)) \) is a 2-group, as we already noticed in Remark 6.6. We will need some lemmas.

**Lemma 6.11.** The embedding problems \( E_2((\kappa_3^3, \kappa_3^3)) \) and \( E_2((\kappa_3^2, \kappa_3^2)) \) have no solutions for \( \overline{U}_5(p) \).

**Proof.** It will be enough to prove that the embedding problems \( E_2((\phi_3^1, \phi_3^3)) \) and \( E_2((\phi_3^2, \phi_3^2)) \) have no solutions for \( U_5(p) \). By Dwyer’s theorem it will be sufficient to show that the cup products \( \phi_3^1 \cup \phi_3^3 \) and \( \phi_3^2 \cup \phi_3^2 \) are non-zero. By Lemma 4.3 the elements \( I + E_{12} \) and \( I + E_{34} \) commute, and are also \( p \)-torsion, so the subgroup \( A \) they generate is isomorphic to \( (\mathbb{Z}/p)^2 \). The pull-back of \( \phi_3^1 \cup \phi_3^3 \) onto \( A \) is non-zero by the Künneth formula for cohomology with coefficients in \( \mathbb{Z}/p \). Therefore \( \phi_3^1 \cup \phi_3^3 \) is non-zero, too. We can argue similarly for \( \phi_3^2 \cup \phi_3^2 \) by pulling it back to the subgroup generated by \( I + E_{23} \) and \( I + E_{15} \). \( \square \)

In the next two lemmas \( G \) is an arbitrary group.

**Lemma 6.12.** Let \( \alpha_1, \alpha_2, \alpha_3 \in H^1(G) \) and \( \gamma_1, \gamma_2, \gamma_3 \in H^1(G) \) be such that

\[
\text{span}(\alpha_1, \alpha_3) = \text{span}(\gamma_1, \gamma_3) \text{ and } \langle \alpha_1, \alpha_2, \alpha_3 \rangle \cap \langle \gamma_1, \gamma_2, \gamma_3 \rangle \neq \emptyset.
\]

Then \( \langle \alpha_1, \alpha_2, \alpha_3 \rangle = \langle \gamma_1, \gamma_2, \gamma_3 \rangle \).

**Proof.** Recall that for every \( \beta_1, \beta_2, \beta_3 \in H^1(G) \) the Massey product set \( \langle \beta_1, \beta_2, \beta_3 \rangle \), if it is non-empty, is a coset of the subgroup \( \beta_1 \cup H^1(G) + H^1(G) \cup \beta_3 \leq H^2(G) \).

However

\[
\alpha_1 \cup H^1(G) + H^1(G) \cup \alpha_3 = \gamma_1 \cup H^1(G) + H^1(G) \cup \gamma_3,
\]

since \( \text{span}(\alpha_1, \alpha_3) = \text{span}(\gamma_1, \gamma_3) \) and the cup product is bilinear and alternating. We get that both \( \langle \alpha_1, \alpha_2, \alpha_3 \rangle \) and \( \langle \gamma_1, \gamma_2, \gamma_3 \rangle \), being non-empty, are cosets of the same subgroup, and as their intersection is non-empty, they are equal. \( \square \)

**Notation 6.13.** Let \( F_n \) denote the class of embedding problems given by the homomorphism \( \omega_{n+1} : \overline{U}_{n+1}(p) \to \overline{U}_{n+1}(p) \). Since \( U_{n+1}(p) \) is a central extension of \( \overline{U}_{n+1}(p) \), for every group homomorphism \( \phi : G \to \overline{U}_{n+1}(p) \) the embedding problem \( F_n(\phi) \) is central. Since \( \text{Ker}(\omega_{n+1}) = Z_{n+1}(p) \cong \mathbb{Z}/p \), the obstruction class \( o(F_n(\phi)) \) lies in \( H^2(G) \). Let \( (\alpha_1, \alpha_2, \ldots, \alpha_n) : G \to (\mathbb{Z}/p)^n \) be an arbitrary homomorphism.

**Lemma 6.14.** We have

\[
\langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle = \{ o(F_n(\phi)) \mid \phi \text{ is a solution of } D_n((\alpha_1, \alpha_2, \ldots, \alpha_n)) \}.
\]

**Proof.** Recall that for every solution \( \phi \) of \( D_n((\alpha_1, \alpha_2, \ldots, \alpha_n)) \) the obstruction class \( o(F_n(\phi)) \) is the \( n \)-fold Massey product with respect to the defining system corresponding to \( \phi \) in Dwyer’s theorem. Therefore the lemma is just a convenient reformulation of the latter. \( \square \)

**Proof of Theorem 6.10** Consider the projective system:

\[
\cdots \xrightarrow{\pi_{k+1}} G_k \xrightarrow{\pi_k} G_{k-1} \xrightarrow{\pi_{k-1}} \cdots
\]

constructed in Definition 0.9 for \( G_0 = \overline{U}_5(2) \). Set \( \rho_0 \) be the identity map of \( G_0 \) and for every \( k \geq 1 \) let \( \rho_k : G_k \to G_0 \) denote the composition:

\[
\pi_1 \circ \cdots \circ \pi_{k-1} \circ \pi_k.
\]

We are going to show by induction on \( k \) that \( E_2((\kappa_3^1, \kappa_3^3) \circ \rho_k), E_2((\kappa_3^2, \kappa_3^2) \circ \rho_k) \) and \( D_4(\kappa_3 \circ \rho_k) \) have no solutions for \( G_k \). Since every group homomorphism
\[ \mathcal{M}(U_5(2)) \to U_5(2) \] factors through \( \pi^k \) for some \( k \), this implies the theorem. With all our preparations it is easy to prove that \( \mathbf{E}_2((\kappa_1^0, \kappa_2^0) \circ \rho) \) and \( \mathbf{E}_2((\kappa_1^0, \kappa_2^0) \circ \rho) \) have no solutions for \( G_k \). Indeed the \( k = 0 \) case is just Lemma 6.11 while the induction step follows at once from Proposition 6.9.

Next we prove that \( D_4(\kappa_5 \circ \rho_k) \) has no solutions for \( G_k \). Note that Lemmas 6.7 and 6.9 together imply if \( D_4(\kappa_5 \circ \rho_k) \) has no solutions for a particular Massey envelope, then it does not have solutions for all such envelopes. Therefore we may assume without the loss of generality that \( \omega((\kappa_1^0, \kappa_2^0, \kappa_3^0)) = 0 \) and \( \omega((\kappa_1^0, \kappa_2^0, \kappa_3^0)) = 1 \). In this case \( G_2 \) is the group \( G_2 \) introduced in Notation 6.12. Therefore \( D_4(\kappa_5 \circ \rho_2) \) has no solutions for \( G_2 \) by Proposition 6.13 (Since \( G_0 \) and \( G_1 \) are quotients of \( G_2 \), we also get that \( D_4(\kappa_5 \circ \rho_0) \) and \( D_4(\kappa_5 \circ \rho_1) \) have no solutions, either.)

No assume that \( D_4(\kappa_5 \circ \rho_k) \) has no solutions for some \( k \geq 2 \) and \( k = 0 \) suppose that \( D_4(\kappa_5 \circ \rho_{k+1}) \) has no solutions, either. We will prove the claim indirectly, so let's suppose that \( D_4(\kappa_5 \circ \rho_{k+1}) \) has a solution \( \psi \). Write \( \alpha = (\alpha_1, \alpha_2, \ldots) \) for the pre-image of \( k \) with respect to \( k_t \). By Proposition 6.11 we have \( n = 3 \). The key fact we need to show is the following

**Proposition 6.15.** The Massey product \( \langle \alpha_1, \alpha_2, \alpha_3 \rangle \) contains zero.

Indeed, the proof of Theorem 6.10 is now easy; by Proposition 6.15 and Dwyer's theorem the embedding problem \( E_3(\alpha) \) has a solution. Therefore \( D_4(\kappa_5 \circ \rho_k) \) has a solution by part (a) of Proposition 6.5. But this is a contradiction. It remains to show Proposition 6.16 which we will do in several steps. By Proposition 6.11 either span(\( \alpha_1, \alpha_2 \)) = span(\( \kappa_1^0, \kappa_2^0 \)) and Ker(\( \mathbf{B}_{14} \circ \psi \)) \( \cap \) \( Z = \{1\} \), or we have span(\( \alpha_1, \alpha_3 \)) = span(\( \kappa_2^0, \kappa_3^0 \)) and Ker(\( \mathbf{B}_{25} \circ \psi \)) \( \cap \) \( Z = \{1\} \), where \( Z \subset G_{k+1} \) is the subgroup \( Z = \{(a, b) \in U_4(2) \times \phi_4, \alpha \ G_k \mid a \in Z_4(2), b = 1\} \).

Let us consider the first case; the second can be handled similarly.

**Lemma 6.16.** The homomorphism \( \mathbf{B}_{14} \circ \psi \) maps \( Z \) into \( Z_4(2) \subset U_4(2) \).

**Proof.** Note that \( \rho_k \) is surjective, since it is a composition of surjective maps. Since \( \phi_4 \circ \mathbf{B}_{14} \circ \psi = (\kappa_1^0, \kappa_2^0, \kappa_3^0) \circ \rho_k \), we get that it is surjective. Since the kernel of \( \phi_4 \) is \( U_4(2)' \), we can conclude that the composition of \( \mathbf{B}_{14} \circ \psi \) and the quotient map \( U_4(2) \to U_4(2)' \) is surjective. Since \( U_4(2) \) is a 2-group, the latter implies that \( \mathbf{B}_{14} \circ \psi \) is surjective. Therefore it maps the centre of \( G_{k+1} \) into the centre of \( U_4(2) \), which is \( Z_4(2) \). Since \( Z \) lies in the centre of \( G_{k+1} \), the claim is now clear.

**Definition 6.17.** Let \( K_m \subseteq \overline{U}_m(p) \) be the image of \( K_m(p) \) under the quotient map \( \overline{U}_m(p) \to \overline{U}_m(p) \). Let \( \overline{\eta}_m : K_m(p) \to (\mathbb{Z}/p)^{m-2} \) be the unique homomorphism such that the composition of the quotient map \( K_m(p) \to K_m(p) \) and \( \overline{\eta}_m \) is \( \eta_m \).

Let \( G_{k+1} \) denote the fibre product \( U_4(2) \times \kappa_{4, \alpha} G_k \). Note that the quotient of \( G_{k+1} \) by its normal subgroup \( Z \) is canonically isomorphic to \( G_{k+1} \). Therefore by Lemma 6.10 there is a unique homomorphism \( \overline{\psi} : \overline{G}_{k+1} \to \overline{U}_4(2) \) such that the composition of the quotient map \( G_{k+1} \to \overline{G}_{k+1} \) and \( \overline{\psi} \) is \( \mathbf{B}_{14} \circ \psi \). Let \( K \subseteq G_{k+1} \) denote the subgroup \( K = \{(a, b) \in U_4(2) \times \phi_4, \alpha \ G_k \mid a \in K_4(2), b = 1\} \), and let \( \overline{K} \subseteq \overline{G}_{k+1} \) be its image under the quotient map \( G_{k+1} \to \overline{G}_{k+1} \).

**Lemma 6.18.** The map \( \overline{\eta}_4 \circ \overline{\psi} : \overline{K} \to (\mathbb{Z}/2)^2 \) is an isomorphism.
Proof. Assume that the claim is false. Since \( K \cong (\mathbb{Z}/2)^2 \) this means that the kernel of \( \pi_1 \circ \psi|_K \) is non-trivial. Let \((g, 1) \in K\) be a lift of a non-zero element \((\overline{g}, 1) \in \text{Ker}(\overline{\pi}_1 \circ \overline{\psi}|_K)\) with respect to the quotient map \( K \rightarrow K\). Then \( \eta_4 \circ \sigma((g, 1)) \neq 0\), where \( \sigma : G_{k+1} \rightarrow U_4(2) \) is the projection onto the first factor, since \( \overline{\psi} \) is non-zero.
Then either the first or the second coordinate of \( \eta_4 \circ \sigma(g) \) is non-zero.
Let’s first assume the former. Since \( p = 2 \) we have \( \eta_4 \circ \sigma((g, 1)) = (1, *) \). Let \( h \in G_k \) be such that \( \alpha(h) = (0, *, 1) \). This is possible since \( \alpha_1 \) and \( \alpha_3 \) are linearly independent. Choose a \( u \in U_4(2) \) such that \( \phi_4(u) = \alpha(h) \). Then \((u, h) \in G_{k+1}\) and 
\[
[(g, 1), (u, h)] = [(g, u), [1, h]] = (I + E_{14}, 1)
\]
using Corollary 4.15. On the other hand
\[
\overline{b}_{14} \circ \psi((I + E_{14}, 1)) = \overline{b}_{14} \circ \psi([(u, g), (I + E_{24}, 1)])
\]
\[
= [\overline{b}_{14} \circ \psi((u, g))]. \overline{b}_{14} \circ \psi((I + E_{24}, 1)) = I
\]
using Corollary 4.15 since \( \eta_4 \circ \overline{b}_{14} \circ \psi((g, 1)) = \eta_4 \circ \overline{\psi}(g, 1) = (0, 0) \). This is not possible as \( I + E_{14} \) generates \( \mathbb{Z} \).
Now suppose the latter. Since \( p = 2 \) we have \( \eta_4 \circ \sigma((g, 1)) = (*, 1) \). Let \( h \in G_k \) be such that \( \alpha(h) = (1, *, 0) \). This is possible since \( \alpha_1 \) and \( \alpha_3 \) are linearly independent. Choose a \( u \in U_4(2) \) such that \( \phi_4(u) = \alpha(g) \). Then \((u, h) \in G_{k+1}\) and 
\[
[(u, h), (g, 1)] = [(u, h), [1, h]] = (I + E_{14}, 1)
\]
using Corollary 4.15. On the other hand using the same computation as above we get
\[
\overline{b}_{14} \circ \psi((I + E_{14}, 1)) = \overline{b}_{14} \circ \psi([(u, g), (g, 1)]) = I.
\]
This is a contradiction. \( \square \)

By assumption \( D_3(\alpha) \) has a solution \( \beta : G_k \rightarrow \overline{U}_4(2) \). Then the direct product \( \beta \times \text{id}_{G_k} : G_k \rightarrow \overline{U}_4(2) \times G_k \) maps \( G_k \) into \( \overline{G}_{k+1} \). For the sake of simple notation let \( \gamma \star \beta \) denote the composition \( \gamma \circ (\beta \times \text{id}_{G_k}) \) for every homomorphism \( \gamma : \overline{G}_{k+1} \rightarrow H \), where \( H \) is any group.

**Lemma 6.19.** There is a choice of \( \beta \) such that the obstruction class \( o(F_4(\overline{\psi} \star \beta)) \) is non-zero.

**Proof.** Let \( \sigma_1 : G_2 = G_2 \rightarrow U_4(2) \) be the projection onto the first factor, as in the proof of Proposition 6.13. Then \( \sigma_1 \circ \sigma_2 \circ \cdots \circ \pi_{k-1} \circ \pi_k \) is a solution to the embedding problem \( E_{04}((\kappa_1^2, \kappa_2^2, \kappa_3^2) \circ \rho_k) \). Therefore \( \langle \kappa_1^2 \circ \rho_k, \kappa_2^2 \circ \rho_k, \kappa_3^2 \circ \rho_k \rangle \) is the set \( \rho_k^2(\kappa_1^2) \cup H^1(G_k) + H^1(G_k) \cup \rho_k^2(\kappa_3^2) \). By Dwyer’s theorem \( \rho_k^2(\kappa_1^2) \cup \rho_k^2(\kappa_3^2) \) is non-zero, since \( E_2((\kappa_1^2, \kappa_3^2) \circ \rho_k) \) has no solutions for \( G_k \). So \( \langle \kappa_1^2 \circ \rho_k, \kappa_2^2 \circ \rho_k, \kappa_3^2 \circ \rho_k \rangle \) contains a non-zero element. Therefore it will be sufficient to show that every solution of \( D_3((\kappa_1^2, \kappa_2^2, \kappa_3^2) \circ \rho_k) \) can be written in the form \( \overline{\psi} \star \beta \) for some choice of \( \beta \) by Lemma 6.13.

Let \( B \) be a central class of embedding problems given by \( \epsilon : \Gamma \rightarrow \Delta \). If \( \gamma \) is a solution of \( B(\lambda) \) for some group homomorphism \( \lambda : G \rightarrow \Delta \), then every solution of \( B(\lambda) \) is of the form \( \lambda \circ \delta \) for a unique group homomorphism \( \delta : G \rightarrow \text{Ker}(\epsilon) \), and conversely every such product is a solution of \( B(\lambda) \). Now fix a solution \( \beta \) of \( D_3(\alpha) \). Since \( D_3 \) is a central class of embedding problems, the solutions of \( D_3(\alpha) \) are of the form \( \beta \circ \delta \), where \( \delta : G_k \rightarrow K_4(2) \) is an arbitrary homomorphism, by the above. Let \( \mu : K_4(2) \rightarrow K \) be the isomorphism given by the rule \( g \mapsto (g, 1) \). Then
\[
\overline{\psi} \star (\beta \circ \delta) = (\overline{\psi} \star \beta) \circ (\overline{\psi} \circ \mu \circ \delta)
\]
for every homomorphism $\delta : G_k \to \overline{K}_4(2)$. The claim now follows from Lemma 6.15 and the fact that $\tau_4$ is an isomorphism.

Now we can conclude the proof of Proposition 6.15. Fix a choice of $\beta$ such that $o(F_4(\overline{\psi} \star \beta)) \neq 0$ and let $\overline{\pi} : G_{k+1} \to \overline{K}_4(2)$ be the projection onto the first factor. If $o(F_4(\overline{\sigma} \star \beta))$ is zero, then $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$ contains zero by Lemma 6.14. Therefore we may assume that $o(F_4(\overline{\sigma} \star \beta)) \neq 0$ without the loss of generality. Let $\overline{G}_k$ denote the pre-image of $\overline{\beta} \times \id_{\overline{G}_k}(G_k)$ with respect to the quotient map $G_{k+1} \to \overline{G}_k$. Then the kernel of the induced projection $\tau : \overline{G}_k \to G_k$ is $Z$. Since $Z$ is a central subgroup in $G_{k+1}$, the natural outer action of $G_k$ on $Z$ induced by conjugation is trivial. Therefore the natural $G_k$-action on $H^*(Z, Z/2)$ is trivial, too. In particular the inflation-reflection exact sequence for the trivial module $Z/2$ over the pair $Z \triangleleft G_k$ is:

$$\cdots \longrightarrow H^1(Z, Z/2) \longrightarrow H^2(G_k, Z/2) \longrightarrow \tau^* H^2(\overline{G}_k, Z/2).$$

Since $Z \cong Z/2$, we get that $H^1(Z, Z/2) = \Hom(Z, Z/2) \cong Z/2$, and hence the kernel of $\tau^* : H^2(\overline{G}_k, Z/2) \to H^2(\overline{G}_k, Z/2)$ is at most one-dimensional.

Both $F_4((\overline{\psi} \star \beta) \circ \tau)$ and $F_4(\overline{\sigma} \star \beta \circ \tau)$ have a solution for $\overline{G}_k$, namely $\overline{B}_4 \circ \psi|_{\overline{G}_k}$ and $\sigma|_{\overline{G}_k}$, respectively, where $\sigma : G_k+1 \to U_4(2)$ is the projection onto the first factor. Therefore by the naturality of obstruction classes both $o(F_4(\overline{\psi} \star \beta))$ and $o(F_4(\overline{\sigma} \star \beta))$ lie in the kernel of $\tau^* : H^2(\overline{G}_k, Z/2) \to H^2(\overline{G}_k, Z/2)$. Since both $o(F_4(\overline{\psi} \star \beta))$ and $o(F_4(\overline{\sigma} \star \beta))$ are non-zero, and $\ker(\tau^*)$ is at most one-dimensional, we get that $o(F_4(\overline{\psi} \star \beta)) = o(F_4(\overline{\sigma} \star \beta))$. Therefore by Lemma 6.12 we get that $\langle \alpha_1, \alpha_2, \alpha_3 \rangle = \langle \kappa^1_3 \circ \rho_k, \kappa^2_3 \circ \rho_k, \kappa^3_3 \circ \rho_k \rangle$. Since the latter contains 0, as we saw in the proof of Lemma 6.10 we get that the former contains 0, too. □

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