THREE HIGHEST DEGREE IRREDUCIBLE REPRESENTATIONS
OF UPPER TRIANGULAR GROUPS $U_n(q)$

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Abstract. Let $U_n(q)$ denote the upper triangular group of degree $n$ over the finite field $F_q$ with $q$ elements. In this paper, we present the constructions of the three highest degree (complex) irreducible representations of $U_n(q)$ where $n \geq 7$.

1. Introduction

Let $q$ be a power of a prime $p$ and $F_q$ a field with $q$ elements. The group $U_n(q)$ of all upper triangular ($n \times n$)-matrices over $F_q$ with all diagonal entries equal to 1 is a Sylow $p$-subgroup of $GL_n(F_q)$. It was conjectured by G. Higman [4] that the number of conjugacy classes of $U_n(q)$ is given by a polynomial in $q$ with integer coefficients. Higman’s conjecture was refined using the (complex) character theory of $U_n(q)$.

I.M. Isaacs [7] showed that the degrees of the irreducible characters of $U_n(q)$ are of the form $q^e$, $0 \leq e \leq \mu(n)$ where the upper bound $\mu(n)$ is known explicitly in Section 4. G. Lehrer [10] conjectured that each number $N_{n,e}(q)$ of irreducible characters of $U_n(q)$ of degree $q^e$ is given by a polynomial in $q$ with integer coefficients. I.M. Isaacs [8] suggested a strengthened form of Lehrer’s conjecture stating that $N_{n,e}(q)$ is given by a polynomial in $(q-1)$ with nonnegative integer coefficients. So, Isaacs’ conjecture implies Higman’s and Lehrer’s conjectures.

To approach irreducible characters of $U_n(q)$, C. André [1] constructed elementary characters by using Kirilov’s orbits on the corresponding nilpotent algebras. With the definition of basic characters as tensor products of a set of elementary characters, the set of all basic characters gives a partition for the set of all irreducible characters $Irr(U_n(q))$ of $U_n(q)$. P. Diaconis and I.M. Isaacs [9] generalized these types of basic characters to supercharacters for any algebra groups.

When $q$ is an actual power of prime, $U_n(q)$ has no faithful characters. We come up with the definition of almost faithful characters.

Definition 1.1. Let $G$ be a group and $Z(G)$ its center. A character $\chi$ of $G$ is called almost faithful if $\chi|_{Z(G)} \neq \chi(1)1_{Z(G)}$.

It is clear that when $q = p$, almost faithful is faithful. Here, we approach representations of a Sylow $p$-subgroup $U$ of a finite group of Lie type $G(q)$ by its root system. For each positive root, we construct a minimal degree almost faithful irreducible character, which is written short as midafi, and the set of tensor products of midafis gives a partition of $Irr(U)$. Thanks to a series of plentifully helpful and friendly talks of C. André on MSRI in March 2008, we figured out that midafis are
elementary characters when \( U \) are the upper triangular groups \( U_n(q) \), and this gives a different construction for basic characters, also as known as supercharacters. But when \( U \) is a Sylow \( p \)-subgroup of \( G(q) \) where the Lie type of \( G(q) \) is not type \( A \), midafis are not supercharacters, see \([2, 5]\).

To explore irreducible characters of \( U_n(q) \), I.M. Isaacs \([8]\) used combinatoric methods and nilpotent algebras to compute the exact numbers of highest and second highest degree irreducible characters of \( U_n(q) \). Here, we decompose a certain tensor products of elementary characters from the view of root system of \( U_n(q) \), and we use this result to construct the three highest degree irreducible representations of \( U_n(q) \). And we obtain the number of third highest degree irreducible characters of \( U_n(q) \) in a recursion formula with coefficient in \((q - 1)\), see Theorem 4.3.4.

In this paper, we first make some significant summaries about \( U_n(q) \) and reprove Theorem 2.4 of C. André \([1]\) in the new approach to help readers get used to the construction of basic characters and easier to understand the representations of the three highest degree irreducible characters in Section 4. Furthermore, the proof of Theorem 2.4 shows the way how to decompose a tensor product of any two supercharacters into a sum of supercharacters and to compute their corresponding structure constants.

2. Some summaries of \( U_n(q) \)'s characters

Let \( \Sigma = \Sigma_{n-1} = \langle \alpha_1, ..., \alpha_{n-1} \rangle \) be the root system of \( GL_n(q) \) with respect to the maximal split torus equal to the diagonal group. Denote \( \Sigma^+ \) the set of all positive roots and \( \alpha_{i,j} = \alpha_i + \alpha_{i+1} + ... + \alpha_j \) for some \( 0 < i \leq j < n \). Let \( X_\alpha \) be the root subgroup of a root \( \alpha \in \Sigma^+ \). Hence, \( X_{\alpha_{i,j}} \) is the set of all upper triangular matrices in the form \( I_n + c \cdot \varepsilon_{i,j+1} \), where \( c \in \mathbb{F}_q \), \( I_n \) the \((n \times n)\) identity matrix, \( \varepsilon_{i,j+1} \) the zero matrix except 1 at entry \((i, j+1)\). The upper triangular group \( U_n(q) \) is generated by all \( X_\alpha, \alpha \in \Sigma^+ \). We denote \( U = U_n(q) \) if there is no special statement.

For any positive root \( \alpha \), André \([4]\) constructed \((q - 1)\) irreducible characters of \( U \), namely elementary characters, associated to \( \alpha \) from co-adjoint orbits inside \( U \)'s nilpotent algebra, which can be explained as follows by its root system only.

For any positive root \( \alpha_{i,j} \), we call the set \( \text{arm}(\alpha_{i,j}) = \{\alpha_{i,j} : l = i, ..., j - 1\} \) the arm of \( \alpha_{i,j} \), and the set \( \text{leg}(\alpha_{i,j}) = \{\alpha_{k,j} : k = i + 1, ..., j\} \) the leg of \( \alpha_{i,j} \). If \( i = j \), then \( \text{arm}(\alpha_i), \text{leg}(\alpha_i) \) are empty. We call \( h(\alpha) = \{\alpha\} \cup \text{arm}(\alpha) \cup \text{leg}(\alpha) \) the hook at \( \alpha \). And for any positive root \( \alpha \), we define the hook group \( H(\alpha) \) of \( \alpha \) by \( H(\alpha) = \{X_\beta : \beta \in h(\alpha)\} \), the base group \( V_\alpha \) of \( \alpha \) by \( V_\alpha = \{X_\beta : \beta \in \Sigma^+ \setminus \text{arm}(\alpha)\} \), the subtriangular group \( U_\alpha \) of \( \alpha \) by \( U_\alpha = \{X_\beta : \beta \in \Sigma^+, \exists \gamma \in \Sigma^+, \alpha-(\beta+\gamma) \in \Sigma^+\} \), and the radical group \( R_\alpha \) of \( \alpha \) by \( R_\alpha = \{X_\beta : \beta \in \Sigma^+, X_\beta \not\in U_\alpha\} \). It is clear that \( U = U_{\alpha_{1,n-1}} \). These groups can be demonstrated in the following diagram:

\[
\begin{align*}
\text{arm}(\alpha) & \quad \quad H(\alpha) = \begin{cases} \alpha \\ \text{leg}(\alpha) \end{cases} \\
U & = \begin{cases} U_\alpha \\ R_\alpha \end{cases}
\end{align*}
\]

Denote \( \text{Irr}(G) \) the set of all irreducible characters of a group \( G \), and \( \text{Irr}(G)^* = \text{Irr}(G) \setminus \{1_G\} \). If \( H \) is a normal group of \( G \), denote \( \text{Irr}(G/H) \) the set of all
irreducible characters of G with H in the kernel. If there is a subgroup K such that
G = H × K, and any ξ ∈ Irr(H), denote ξG the inflation of ξ to G, i.e. ξG is an
extension of ξ to G with K ⊂ ker(ξG). For any character ξ of a subgroup H of G,
denote Irr(G, ξ) the set of all irreducible constituents of ξG.

It is clear that Xα ∩ [Vα, Vα] = {1}. Hence, there exist (q − 1) linear characters
λ ∈ Irr(να/[Vα, Vα])∗ such that λ|Xα ̸= 1Xα and λ|Xβ = 1Xβ for others Xβ ⊂ Vα.
From now on, we denote Irr(να/[Vα, Vα])∗ the set of (q − 1) linear characters of Vα
as described above if there is no other special statement. We call λU an elementary
character at α (of U) where λ ∈ Irr(να/[Vα, Vα])∗.

Lemma 2.1. λU is irreducible and ((λ|Vα∩Uα)U)U = λU. Moreover, λU|H(α)∩Uα
is irreducible and extendible to U as λU.

Proof. Let λ ∈ Irr(να/[Vα, Vα])∗. Since the radical Rα always lies in the kernel of
λU and H(α) ⊂ Uα, it is enough to show that λU|H(α) is irreducible. By Mackey’s
formula for the double coset VαH(α) = U, λU|H(α) = λ|Vα ∩ U(α)U. Let α = αi,j
and µ = λ|Vαi,j ∩ U(αi,j). If i = j, H(α) = Xα, λ is a linear character, hence, it is
done. Now we suppose that i < j.

Let A = ⟨Xβ : β ∈ arm(αi,j)⟩ and L = ⟨Xβ : β ∈ leg(αi,j)⟩. It is clear that
[A, L] = Xαi,j = Z(H(αi,j)), and A normalizes (Vαi,j ∩ H(αi,j)) = Xαi,jL.

Any nonidentity element x ∈ A can be written uniquely as xβk(t1)...xβk(tk)
where h(βk) < h(βk+1) and tk ∈ F∗q. Let γ ∈ leg(αi,j) such that γ + βk = αi,j. For
all y ∈ Xγ, we have

(γ)μ(y) = μ(γy) = μ(y[y, x]) = μ([y, x])μ(y) = μ([y, x]) ≠ 1

since [y, x] ∈ Xαi,j. So the inertia group I(αi,j)(µ) = Vαi,j ∩ H(αi,j) = Xαi,jY.
Hence, µH(αi,j) = λ|Vαi,j ∩ H(αi,j))U = λU|H(αi,j) is irreducible. □

Using the same notations in the proof of Lemma 2.1 the following corollary is
obvious.

Corollary 2.2. λU|L = 1{1}L, the regular character of L, and if ξ ∈ Irr(Xαi,j, L)
such that ξ|Xαi,j = λ|Xαi,j, then ξH(αi,j) = λU|H(αi,j).

Any two roots α, β ∈ Σ+ are called separate if they are on neither the same row
nor same column. A nonempty subset D of Σ+ is called a basic set if all roots in
D are separate pairwise. For a basic set D, let ED be the set of all {λα}α∈D where
λα ∈ Irr(να/[Vα, Vα])∗. For any basic set D and φ ∈ ED, André [1] defined a basic
character as

ξD,φ = ⊗λα∈φλαU.

It is well-known that all p-groups are monomial, i.e. any irreducible character is
induced from a linear character of some subgroup, here, a basic character may not
be irreducible, but it is always induced from a linear character.

Lemma 2.3. For a basic character ξD,φ, let VD = ∩α∈DVα and λD = ⊗λα∈φλα|Vα,
we have

ξD,φ = ⊗λα∈φλαU = (∗λα∈φλα|Vα)U = λDU.

Proof. By the property |H · K| = |H|H(K) for any two subgroups H, K of a group,
it is easy to check that for any basic set D and any subsets S, I, J ⊂ D if I ∩ J = S
then ∩α∈SVα = (∩α∈I Vα) · (∩α∈J Vα).
We show the claim by induction on the size of $D$. Suppose that $D' = D \cup \{\beta\}$ is a basic set and for all $\phi \in E_D$, $\xi_{D,\phi} = \lambda_D^U$ where $\lambda_D = \otimes_{\lambda \in \phi} \lambda_{V_0}$ and $V_D = \cap_{\alpha \in D} V_\alpha$. Let $\phi' = \phi \cup \{\lambda_\beta\}$ for some $\phi \in E_D$ and $\lambda_\beta \in \text{Irr}(V_\beta/[V_\beta, V_\beta])^*$. By Mackey’s formula for the double coset $U = V_D \cdot V_\beta$, we have

$$
\xi_{D',\phi'} = \xi_{D,\phi} \otimes \lambda_\beta^U = \lambda_D^U \otimes \lambda_\beta^U = (\lambda_D^U|_{V_\beta} \otimes \lambda_\beta^U) = ((\lambda_D^U|_{V_\beta \cap V_\beta} \otimes \lambda_\beta|_{V_\beta \cap V_\beta})^U) = \lambda_D^U,
$$

where $\lambda_D = \otimes_{\lambda \in \phi} \lambda_{V_0}$, and $V_D = \cap_{\alpha \in D} V_\alpha$. □

Actually in the proof of Lemma 2.3, we only need a condition for $D$ that there are no two roots in $D$ on the same row. The main motivation to define basic characters is to get a partition for the set of all irreducible characters $\text{Irr}(U)$ of $U$, which comes from the following theorem.

**Theorem 2.4 (André [1]).** For any $\chi \in \text{Irr}(U)^*$, there exists uniquely a basic set $D$ and a set $\phi \in E_D$ such that $\chi$ is a constituent of $\xi_{D,\phi}$.

This theorem will be proved again in a different way from [1], without using the coadjoint orbits, in Section 3, which gives a geometric advantage to present the three highest degree irreducible representations of $U$.

3. ALMOST FAITHFUL CHARACTERS

When $q$ is an actual power of a prime $p$, i.e. $q = p^l$ where $l > 1$, $U$ has no faithful characters. Therefore, we come up with a definition of almost faithful as stated in the first part. It is noted that if $\chi$ is an almost faithful character of any group $G$, then $Z(G) \subset Z(\chi)$ where $Z(\chi) = \{g \in G : |\chi(g)| = |\chi(1)|\}$. Hence, there exists unique $\sigma \in \text{Irr}(Z(G))^*$ such that $\chi \in Z^+ \cdot \text{Irr}(G, \sigma)$. By Lemma 2.1 we have the following property of the hook group.

**Lemma 3.1.** The hook group $H(\alpha_{i,j})$ has $(q - 1)$ almost faithful irreducible characters of degree $q^{i-1}$ and $q^{2j-n}$ linear characters.

It is clear that $\lambda_\alpha^U|_{U_\alpha}$ is a minimal degree almost faithful character of $U_\alpha$ where $\lambda_\alpha \in \text{Irr}(V_\alpha/[V_\alpha, V_\alpha])^*$. The following theorem demonstrates all almost faithful irreducible characters of $U$.

**Theorem 3.2.** Any almost faithful irreducible character of $U$ uniquely factors as a tensor product of an almost faithful elementary character and an irreducible character of $\text{Irr}(U/H(\alpha_{1,n-1}))$.

**Proof.** It is well-known that $Z(U) = X_{\alpha_{1,n-1}}$. Let $\sigma \in \text{Irr}(X_{\alpha_{1,n-1}})^*$ and $\lambda^U \in \text{Irr}(U, \sigma)$ an elementary character at $\alpha_{1,n-1}$. Denote $H = H(\alpha_{1,n-1})$. We have $(\lambda^U|_H)^U = (\lambda^U|_H \otimes 1_H)^U = \lambda^U \otimes 1_H^U = \lambda^U \otimes \sum_{\eta \in \text{Irr}(U/H)} \eta(1) \eta$, where $1_H$ the principal character of $H$.

By Lemma 3.1, $\lambda^U|_H$ is irreducible and extendible to $U$, by Clifford’s theory, all characters $\lambda^U \otimes \eta$ are irreducible for all $\eta \in \text{Irr}(U/H)$. Since $\sigma^H = \lambda^U(1) \lambda^U|_H$, by the transitivity of induction, $\sigma^U = (\sigma^H)^U = \sum_{\eta \in \text{Irr}(U/H)} \lambda^U(1)\eta(1) \lambda^U \otimes \eta$. 


Therefore, for any almost faithful irreducible character $\chi \in \text{Irr}(U, \sigma)$, there exists unique $\eta \in \text{Irr}(U/H)$ such that $\chi = \lambda^U \otimes \eta$. □

It is clear that a tensor product of elementary character with a linear character remains irreducible. In particular, by Theorem 3.2 if $\lambda_\alpha^U$, $\lambda_\beta^U$ are elementary characters where $\alpha \in \text{arm}(\beta)$ and $\alpha$ is a simple root, i.e. $\lambda_\alpha^U = \lambda_\alpha$, then $\lambda_\alpha \otimes \lambda_\beta^U = \lambda_\beta^U$. Next, we observe the decomposition of a tensor product of any two nonlinear elementary characters.

**Lemma 3.3.** Let $\lambda_1^U$, $\lambda_2^U$ be two nonlinear elementary characters at $\alpha_{i,j}$, $\alpha_{l,k}$ respectively. The decomposition of $\lambda_1^U \otimes \lambda_2^U$ is classified as follows

(i) If $|H(\alpha_{i,j}) \cap H(\alpha_{l,k})| = 1$ then $\lambda_1^U \otimes \lambda_2^U$ is irreducible,

(ii) If $|H(\alpha_{i,j}) \cap H(\alpha_{l,k})| = q$ then $\lambda_1^U \otimes \lambda_2^U$ has $q$ pairwise distinct irreducible constituents of degree $q(j - i + (k - l) - 1)$ with multiplicity 1,

(iii) If $\alpha_{i,j} \in \text{leg}(\alpha_{l,k})$ (or $\alpha_{l,k} \in \text{leg}(\alpha_{i,j})$), then $\lambda_1^U \otimes \lambda_2^U$ decomposes into $\lambda_2^U$ and a sum of $\lambda_2^U \otimes \lambda_\alpha^U$ where all $\alpha \in \text{arm}(\alpha_{i,j})$ (or $\text{leg}(\alpha_{i,j})$) respectively; each component appears with multiplicity 1,

(iv) If $\alpha_{i,j} = \alpha_{l,k}$ and $\lambda_1^U \neq \lambda_2^U$, then $\lambda_1^U \otimes \lambda_2^U$ decomposes into $(\lambda_1 \otimes \lambda_2)^U$ with multiplicity $q + (j - i - 1)(q - 1)$ and a sum of $(\lambda_1 \otimes \lambda_2)^U \otimes \lambda_\alpha^U$ with multiplicity 1, where all $\alpha$’s satisfy $U_\alpha \subset U_{\alpha_{i,j} \cup \alpha_{l,k}}$.

(v) If $\alpha_{i,j} = \alpha_{l,k}$ and $\lambda_1^U = \lambda_2^U$, then $\lambda_1^U \otimes \lambda_2^U$ decomposes into $1_U$ and a sum of $\lambda_{\beta_1}^U$, $\lambda_{\beta_2}^U$ and $\lambda_{\beta_1}^U \otimes \lambda_{\beta_2}^U$, where all $\beta_1 \in \text{arm}(\alpha_{i,j})$, $\beta_2 \in \text{leg}(\alpha_{i,j})$.

**Proof.** (i) If $U_{\alpha_{i,j}} \subset U_{\alpha_{l,k}}$ (or $U_{\alpha_{l,k}} \subset U_{\alpha_{i,j}}$), it is clear by Theorem 3.2 and if $U_{\alpha_{i,j}} \cap U_{\alpha_{l,k}} = \{1\}$ then it follows by the property $\text{Irr}(G \times H) \cong \text{Irr}(G) \times \text{Irr}(H)$.
Hence, $K_{j-t} = K_{j-t+1}$, $K_{j-i} = \text{leg}(\alpha_{i,k})$, $K_{(j-i)+(k-l)}$ is empty, $X_{\beta} \subset \ker(\lambda)$ for all $\beta \in K_0$. And for every $M_t = M_{t-1}X_{\alpha}$ such that $M_t \neq M_{t-1}$, there exist unique $\beta \in K_t \setminus K_{t-1}$ satisfying $\alpha + \beta = \alpha_{i,j}$ if $t \leq j - i = \alpha_{i,k}$ if $t \geq j - i + 1$.

Suppose that $\lambda^L \in \text{Irr}(L)$ for some subgroup $L = M_t$, and $X_{\beta} \subset \ker(\lambda^L)$ for all $\beta \in K = K_t$. If $t = (l - k) + (j - i)$, it is done. Otherwise, the next induction step is from $L$ to $LX_{\alpha} = M_{t+1}$ where $\alpha = \text{arm}(\gamma)$, $\tau = \alpha_{i,j}$ if $t \leq j - i$ and $\tau = \alpha_{i,k}$ if $t \geq (j - i) + 1$. Here, $X_{\alpha}$ normalizes $L$ and there exists $\beta \in K$ in the leg of $\tau$ such that $\alpha + \beta = \tau$, $X_{\beta} \subset \ker(\lambda^L)$. So for any $1 \neq x \in X_{\alpha}$, all $x_{\beta} \in X_{\beta}$, we have

$$
\bar{\pi}(\lambda^L)(x_{\beta}) = \lambda^L(x_{\beta}) = \lambda^L(x_{\beta}) = \lambda^L(1),
$$

since $[x_{\beta}, x] \in X_\tau \subset Z(\lambda^L)$ and exist some $x_{\beta} \in X_{\beta}$ such that $\lambda([x_{\beta}, x]) \neq 1$.

Hence, all $1 \neq x \in X_{\alpha}$, $\bar{\pi}(\lambda^L) \neq \lambda^L$ since $X_{\beta}$ is not in the kernel of $\bar{\pi}(\lambda^L)$. By Mackey’s formula for the double coset $L \setminus LX_{\alpha}/L$ represented by $X_{\alpha}$, we have

$$
(\lambda^{LX_{\alpha}}, \lambda^{LX_{\alpha}}) = \sum_{x \in X_{\alpha}} (\pi(\lambda^L), \lambda^L) = (\lambda^L, \lambda^L) = 1, \text{ i.e. } \lambda^{LX_{\alpha}} \in \text{Irr}(LX_{\alpha}, \lambda).
$$

And it is clear that $X_{\gamma} \subset \ker(\lambda^{LX_{\alpha}})$ for all $\gamma \in K \setminus \{\beta\} = K_{t+1}$. Therefore, by induction, $\lambda^U \in \text{Irr}(U, \lambda)$ of degree $q^{(l-k) + (j-i)-1}$. Now, it is easy to show that for any two distinct extensions $\lambda$, $\lambda'$ of $\lambda_{12}$ to $V$, if $\lambda \neq \lambda'$ then $\lambda^{LX_{\alpha}} \neq \lambda^{LX_{\alpha}}$. Again, since $X_{\beta} \not\subset \ker(\bar{\pi}(\lambda^L))$ for all $1 \neq x \in X_{\alpha}$,

$$(\lambda^{LX_{\alpha}}, \lambda^{LX_{\alpha}}) = \sum_{x \in X_{\alpha}} (\pi(\lambda^L), \lambda^L) = (\lambda^L, \lambda^L) = 0.$$

(iii) Suppose $\alpha_{i,j} \neq \text{leg}(\alpha_{i,k})$. Let $\tau_1 = \alpha_{i,j}$, $\tau_2 = \alpha_{i,k}$, $V_1 = V_{\tau_1}$ and $V_2 = V_{\tau_2}$. By Mackey’s formula for $V_2V_1 = U$, $1_1^U \otimes 1_2^U = (1_1^U \otimes 1_2^U|_{V_1})U = (1_1^U \otimes 1_2^U|_{V_1 \cap V_2})U$.

By Corollary 2.2, $1_1^U \otimes 1_2^U|_{V_1 \cap V_2} = (1_1^U|_{V_1 \cap V_2} \otimes 1_2^U|_{V_1 \cap V_2})V_1 = (1_2^U|_{V_1 \cap V_2})V_1 = 1_2^U|_{V_1}$. Therefore, $1_1^U \otimes 1_2^U = (1_2^U|_{V_1})U = (1_2^U|_{V_1 \cap V_2})U = 1_2^U \otimes 1_1^U$.

Here, $1_1^U = (1_U \cap V_1)U$. Let $M_0 = U_{\tau_1} \cap V_1$, and $M_t = M_{t-1}X_{\alpha_{i,j-t}}$ for all $1 \leq t \leq j - i$. Hence, $M_{j-i} = U_{\tau_1}$ and $1_{U_{\tau_1} \cap V_1} = 1_{M_0}$. By the transitivity of inductions, $1_{M_0} \otimes 1_{U_{\tau_1}}$ is decomposed via a series of inductions along the arm of $\tau_1$ namely from $M_0$ to $M_1$, $M_2$, ..., $M_{j-i}$. Since $M_1/M_0 \cong X_{\alpha_{i,j-1}}$, $1_{M_0} \otimes 1_{U_{\tau_1}} = 1_{M_1} + \sum_{\rho \in \text{Irr}(M_1/M_0)^*} \rho$. For any $\rho \in \text{Irr}(M_1/M_0)^*$, by Lemma 2.1, $\rho^{U_{\tau_1}} \in \text{Irr}(U_{\tau_1})$ is an elementary character at $\alpha_{i,j-1}$. Therefore, $(1_{M_0} \otimes 1_{U_{\tau_1}}) = 1_{M_1} + \sum_{\rho \in \text{Irr}(M_1/M_0)^*} \rho^{U_{\tau_1}}$. The next step of inductions is on $1_{M_1} \otimes 1_{U_{\tau_1}} = 1_{M_1}$.

Similarly, for any $2 \leq t \leq j - i$, $M_t/M_{t-1} \cong X_{\alpha_{i,j-t}}$, $1_{M_t} = 1_{M_{t-1}} + \sum_{\rho \in \text{Irr}(M_t/M_{t-1})^*} \rho$, and $\rho^{U_{\tau_1}}$ is an elementary character at $\alpha_{i,j-t}$. Hence, $1_{M_t} \otimes 1_{U_{\tau_1}} = 1_{M_{t-1}} + \sum_{1 \leq t \leq j - i} \sum_{\rho \in \text{Irr}(M_t/M_{t-1})^*} \rho^{U_{\tau_1}}$. Therefore, $(1_{M_t} \otimes 1_{U_{\tau_1}}) = 1_U + \frac{1}{1_{M_t} \otimes 1_{U_{\tau_1}}} \sum_{1 \leq t \leq j - i} \sum_{\rho \in \text{Irr}(M_t/M_{t-1})^*} \rho^{U_{\tau_1}}$. By Theorem 1.2, each component $\lambda_2^U \otimes (\rho^{U_{\tau_1}}) \in \text{Irr}(U)$ appears in $\lambda_1^U \otimes \lambda_2^U$ with multiplicity 1, for all above $\rho$’s.
(iv) Suppose \( \alpha_{i,j} = \alpha_{l,k} \) and \( \lambda_1^U \neq \lambda_2^U \). Hence, \( \lambda_1, \lambda_2 \in \text{Irr}(V_{l,k} / [V_1, V_1]) \), and \( \lambda_1 \neq \lambda_2 \) where \( V_1 = V_{\alpha_{i,j}} \). We define \( V_2 = \langle X_\alpha : \alpha \in \Sigma^+ \setminus \text{leg}(\alpha_{i,j}) \rangle \). With the same argument in Lemma 2.1 on the leg, instead of the arm, there exists \( \xi \in \text{Irr}(V_2 / [V_2, V_2])^* \) such that \( \xi|_{\alpha_{i,j}} = \lambda_2|_{\alpha_{i,j}}, \xi|_{\alpha_{l,k}} = 1_{\alpha_{l,k}} \) for all others \( X_\alpha \subset V_2 \), and \( \xi^U = \lambda_2^U \) an elementary character at \( \alpha_{i,j} \).

By Mackey’s formula for \( V_2 V_1 = U \), we have

\[
\lambda_1^U \otimes \xi^U = (\lambda_1 \otimes \xi|_{V_1 \cap V_2})^U = ((\lambda_1|_{V_1 \cap V_2} \otimes \xi|_{V_1 \cap V_2})^U V_1).\]

Set \( \lambda_{12} = \lambda_1 \otimes \lambda_2 \in \text{Irr}(V_1 / [V_1, V_1])^* \). Since \( \xi|_{V_1 \cap V_2} = \lambda_2|_{V_1 \cap V_2} \), \( \lambda_1^U \otimes \xi^U = (\lambda_{12}|_{V_1 \cap V_2})^U = (\lambda_1 \otimes 1_{V_1 \cap V_2})^U \). Since \( X_{\alpha_{i,j}} \subset \ker(1_{V_1 \cap V_2}), 1_{V_1 \cap V_2} = (1_{V_1 \cap V_2})^U \), and, then \( \lambda_1^U \otimes \lambda_2^U = \lambda_{12}^U \otimes (1_{V_1 \cap V_2})^U \). Here, \( \lambda_{12}^U \) is an elementary character at \( \alpha_{i,j} \) since \( (\lambda_1 \otimes \lambda_2)|_{\alpha_{i,j}} \neq 1_{\alpha_{i,j}} \).

We decompose the permutation character \( 1_{V_1 \cap V_2} \) exactly with the same method in (iii) into \( V_1 \) and a sum of all elementary characters \( \lambda_\alpha^U \) at all \( \alpha \in \text{arm}(\alpha_{i,j}) \). At the fundamental root \( \alpha_i \), we have \( q \) copies of \( \lambda_{12}^U \) since \( \lambda_{12}^U \otimes \rho = \lambda_{12}^U \) for all linear elementary characters \( \rho \) at \( \alpha_i \). At others \( \alpha_{i,t} \), where \( i < t < j \), each tensor product of \( \lambda_{12}^U \) and an elementary character \( \lambda_{\alpha_{i,t}} \), by (iii), decomposes into \( \lambda_{12}^U \) and a sum of tensor products of \( \lambda_{12}^U \) and an elementary character \( \lambda_\alpha^U \) at \( \alpha \), for all \( \lambda_\alpha \in \text{Irr}(V_\alpha / [V_\alpha, V_\alpha])^* \), where \( \alpha \in \text{leg}(\alpha_{i,t}) \), which gives another \( (j - i - 1)(q - 1) \) multiplicities for \( \lambda_{12}^U \). All these tensor products are irreducible by Theorem 3.2.

(v) Suppose \( \alpha_{i,j} = \alpha_{l,k} \) and \( \lambda_1^U = \lambda_2^U \), which implies \( \lambda_1 = \lambda_2 \). We use the same setup in (iv), hence \( \lambda_2|_{V_1 \cap V_2} = \xi|_{V_1 \cap V_2} = \lambda_1|_{V_1 \cap V_2}, and \)

\[
\lambda_1^U \otimes \xi^U = (\lambda_1|_{V_1 \cap V_2} \otimes \xi|_{V_1 \cap V_2})^U = 1_{V_1 \cap V_2}^U = (1_{V_1 \cap V_2})^U V_1.\]

We use exactly the same method in (iii) to decompose the permutation character \( 1_{V_1 \cap V_2} \) into \( V_1 \) and a sum of all elementary characters at all \( \alpha \in \text{arm}(\alpha_{i,j}) \), each constituent appears with multiplicity 1. Call this set of constituents \( A \).

Each character \( \xi \) in \( A \) naturally inflates to \( U \), hence, \( \xi^U = \xi \otimes 1_{V_2}^U \). Again we decompose \( 1_{V_2}^U \) into \( U \) and sum of all elementary characters at all \( \alpha \in \text{leg}(\alpha_{i,j}) \), each constituent appears with multiplicity 1. Call this set of constituents \( B \).

Therefore, \( \lambda_1^U \otimes \lambda_2^U = (1_{V_1 \cap V_2})^U = (1_{V_1 \cap V_2})^U \).

We recall the definition of a basic character that for any basic set \( D \) and \( \phi \in E_D \), \( \xi_{D,\phi} = \otimes_{\lambda_\alpha \in \phi} \lambda_\alpha^U \). Lemma 3.3 (iii), (iv), (v) give the proof of the following corollary.

**Corollary 3.4.** Let \( S \) be a nonempty subset of \( \Sigma^+ \). The tensor product of a set of elementary characters at \( \alpha \in S \) decomposes into a sum of basic characters.

By the computation in Lemma 3.3 it is easy to compute the multiplicity of each basic character which appears as a constituent of the tensor product in Corollary...
Now, we are going to prove Theorem 2.4.

Proof of Theorem 2.4. We show by induction on \( n \) that the regular character \( 1^U, U = U_n(q) \), decomposes into a sum of basic characters, and for any \( \chi \in \text{Irr}(U) \), if \( \chi \) is an irreducible constituent of basic characters \( \xi_{D,\phi}, \xi_{D',\phi'} \) then \( \xi_{D,\phi} = \xi_{D',\phi'} \), i.e. \( D = D' \) and \( \phi = \phi' \).

Let \( M = H(\alpha_{1,n-1}) \cap V_{\alpha_{1,n-1}}, S = \{\alpha_{1,n-1}\} \cup \text{leg}(\alpha_{1,n-1}) \), and \( L_\alpha = \langle X_\beta : \beta \in \text{leg}(\alpha) \rangle \) for every \( \alpha \in S \). It is clear that \( M \) is elementary abelian, \( M = \{X_\alpha : \alpha \in S\} \), and \( |M| = q^{|S|} = q^{n-1} \), hence

\[
1_{\{1\}}^M = \sum_{\rho \in \text{Irr}(M)} \sum_{\alpha \in S} \sum_{\lambda \in \text{Irr}(V_\alpha/V_{\alpha,n})} \lambda M \otimes (1_{\{1\}}^{L_\alpha})_M.
\]

Here \( 1_{\{1\}}^{L_\alpha} \) is the regular character of \( L_\alpha \), by Corollary 2.2 \( \lambda M \otimes (1_{\{1\}}^{L_\alpha})_M = (\lambda^U|_{X_\alpha L_\alpha})_M = \lambda^U|_M \) for all \( \lambda \in \text{Irr}(V_\alpha/V_{\alpha,n})^* \) and \( \alpha \in S \). Therefore,

\[
1_{\{1\}}^M = 1_M + \sum_{\alpha \in S} \sum_{\lambda \in \text{Irr}(V_\alpha/V_{\alpha,n})} \lambda^U|_M.
\]

By the transitivity of inductions,

\[
1_{\{1\}}^U = \left(1_{\{1\}}^M\right)^U = \left(1_M + \sum_{\alpha \in S} \sum_{\lambda \in \text{Irr}(V_\alpha/V_{\alpha,n})} \lambda^U|_M\right)^U = 1_M^U + \sum_{\alpha \in S} \sum_{\lambda \in \text{Irr}(V_\alpha/V_{\alpha,n})} \lambda^U \otimes 1_M^U.
\]

Since \( U/M \cong U_{\alpha_{1,n-2}}, 1_M^U \) is the regular character of \( U_{\alpha_{1,n-2}}, \) by induction on \( n, 1_M^U \) is a sum of basis characters of \( U_{\alpha_{1,n-2}} \) inflated to \( U \). Therefore, by Corollary 3.4 \( 1_{\{1\}}^U \) is a sum of basis characters.

(Uniformity) We suppose that \( \chi \in \text{Irr}(U) \) is an irreducible constituent of basic characters \( \xi_{D,\phi} \) and \( \xi_{D',\phi'} \), which implies \( \langle \xi_{D,\phi}, \xi_{D',\phi'} \rangle = \langle \xi_{D,\phi} \otimes \xi_{D',\phi'}, 1_U \rangle \neq 1 \).

We start with a highest root in \( D \cup D' \), call it \( \alpha \). Without loss of generality, suppose \( \alpha \in D \). If \( \alpha \notin D' \), it is clear that \( X_\alpha \subset Z(\xi_{D,\phi} \otimes \xi_{D',\phi'}) \). Hence, \( X_\alpha \subset Z(\psi) \) and \( X_\alpha \notin \text{ker}(\psi) \) for all irreducible constituents \( \psi \) of \( \xi_{D,\phi} \otimes \xi_{D',\phi'} \), which implies \( \langle \xi_{D,\phi} \otimes \xi_{D',\phi'}, 1_U \rangle = 0 \), and this contradicts to our assumption. Therefore, \( \alpha \in D' \). By Lemma 3.3 (iv), (v), the corresponding characters \( \lambda_\alpha \in \phi \) and \( X_\alpha \phi \) must be equal.

By Lemma 3.3 (v), the tensor product \( \lambda_\alpha U \otimes \lambda_\alpha U \) decomposes into \( 1_U \) and a sum of elementary characters \( \lambda_\beta U, \lambda_\gamma U \), and tensor products of \( \lambda_\beta U \otimes \lambda_\gamma U \), for all \( \beta \in \text{arm}(\alpha) \) and \( \gamma \in \text{leg}(\alpha) \). Since all the roots in \( D \) and \( D' \) are pairwise separate respectively, the scalar product \( \langle \xi_{D,\phi} \otimes \xi_{D',\phi'}, 1_U \rangle \) equals \( (R_\alpha, 1_U) \), where \( R_\alpha = \langle \otimes_{\alpha \notin \delta \in D, \lambda_\delta \in \phi} \lambda_\delta U \otimes \otimes_{\alpha \notin \delta \in D', \lambda_\delta \in \phi} \lambda_\delta U \rangle \). We continue the process with a next highest root in \( D \cup D' \), until finally, \( D = D' \) and \( \phi = \phi' \). □

By the uniqueness property in Theorem 2.4 the set of all basic characters \( \xi_{D,\phi} \) gives a partition in \( \text{Irr}(U) \). Now, we define that a basic set \( D \) is decomposable if \( D \) has two nonempty disjoint subsets \( A, B \) such that \( A \cup B = D, (U_\alpha : \alpha \in A) \subset U_\beta, \) and \( \langle H(\alpha) : \alpha \in B \rangle \cap U_\beta = \{1\} \) for some \( \beta \in \Sigma^+ \); a basic set \( D \) is indecomposable if it is not decomposable.

For example, in \( U_7(q) \), \( D = \{\alpha_{1,2}, \alpha_{3,4}, \alpha_{2,5}, \alpha_{4,6}\} \) is indecomposable, and \( D = \{\alpha_{3,4}, \alpha_{1,4}\} \cup \{\alpha_{2,5}, \alpha_{1,6}\} \) is decomposable. Denote \( \text{Irr}(\xi_{D,\phi}) \) the set of all irreducible constituents of a basic character \( \xi_{D,\phi} \). The following property is very helpful to work with basic characters.
Lemma 3.5. Let $D$ be a decomposable basic set as $D = A \cup B$. Then $\text{Irr}(\xi_{D,\phi}) \cong \text{Irr}(\xi_{A,\phi_A}) \times \text{Irr}(\xi_{B,\phi_B})$ for all $\phi \in E_D$, $\phi_A \in E_A, \phi_B \in E_B$ such that $\phi = \phi_A \cup \phi_B$, i.e. $|\text{Irr}(\xi_{D,\phi})| = |\text{Irr}(\xi_{A,\phi_A}) \times \text{Irr}(\xi_{B,\phi_B})|$ and for any $\chi \in \text{Irr}(\xi_{D,\phi})$, there exist uniquely $\chi_A \in \text{Irr}(\xi_{A,\phi_A}), \chi_B \in \text{Irr}(\xi_{B,\phi_B})$ such that $\chi = \chi_A \otimes \chi_B$.

Proof. Suppose that $D$ decomposes into $D = A \cup B$ where $(U_\alpha : \alpha \in A) \subset U_3$, and $\langle H(\alpha) : \alpha \in B \rangle \cap U_\beta = \{1\} \text{ for some } \beta \in \Sigma^+$. For any $\phi \in E_D$, let $\phi_A = \{\lambda_\alpha \in \phi : \alpha \in A\} \in E_A$ and $\phi_B = \{\lambda_\alpha \in \phi : \alpha \in B\} \in E_B$. By the uniqueness property in Theorem 2.4 for all $\chi_A \in \text{Irr}(\xi_{A,\phi_A}), \chi_B \in \text{Irr}(\xi_{B,\phi_B})$, we show $\chi_A \otimes \chi_B \in \text{Irr}(U)$. It is clear that $\langle H(\alpha) : \alpha \in B \rangle \subset R_\beta$, the radical group of $\beta$. By Lemma 2.1, $\alpha U^{1}_{|R_\beta}$ is irreducible for all $\alpha \in B$. Hence, by Clifford’s theory, it is enough to show that $\xi_{B,\phi_B}|_{R_\beta}$ has the same number of constituents of $\xi_{B,\phi_B}$, i.e. for all $\chi \in \text{Irr}(\xi_{B,\phi_B}), \chi|_{R_\beta} \in \text{Irr}(R_\beta)$, because the number of irreducible constituents of $\xi_{B,\phi_B}|_{R_\beta}$ is greater than or equal to the number of irreducible constituents of $\xi_{B,\phi_B}$. Therefore, we are going to show that $(\chi, \xi_{B,\phi_B}) = (\chi|_{R_\beta}, \xi_{B,\phi_B}|_{R_\beta})$ for all $\chi \in \text{Irr}(\xi_{B,\phi_B})$. By Frobenius reciprocity,

$$
(\chi|_{R_\beta}, \xi_{B,\phi_B}|_{R_\beta}) = (\chi, (\xi_{B,\phi_B}|_{R_\beta})^{U}) = (\chi, (\xi_{B,\phi_B}|_{R_\beta} \otimes 1_{R_\beta})^{U}) = (\chi, \xi_{B,\phi_B} \otimes 1_{R_\beta}^{U}).
$$

Here $1_{R_\beta}^{U} \cong 1_{q_{1}^{U}}^{U}$ is the regular character of $U_\beta$. By Theorem 2.4, $1_{R_\beta}^{U}$ decomposes into $1_{U}$ and a sum of all basic characters of $U_\beta$ inflated to $U$. It is clear that $B \cup T$ is a basic set of $U$ for all basic sets $T$ of $U_\beta$. By Theorem 2.4, the uniqueness of $\chi \in \text{Irr}(\xi_{B,\phi_B})$ forces $(\chi, \xi_{B,\phi_B} \otimes 1_{R_\beta}^{U}) = (\chi, \xi_{B,\phi_B}). \square$

4. THREE HIGHEST DEGREE IRREDUCIBLE REPRESENTATIONS OF UPPER TRIANGULAR GROUPS $U_n(q)$

It is well known that the degrees of all irreducible characters of $U$ are powers of $q$ in Isaacs [7], and that, in Huppert [6], all irreducibles of $U$ have degrees $q^e$ for all $0 \leq e \leq \mu(n)$, where

$$\mu(n) = \begin{cases} m(m-1) & \text{if } n = 2m \\ m^2 & \text{if } n = 2m + 1 \end{cases}$$

For any $0 \leq e \leq \mu(n)$, $N_{n,e}(q)$ denotes the number of irreducible characters of $U_n(q)$ having degree $q^e$.

4.1. The highest degree irreducible representations of $U$. Isaacs [8] used combinatorics to compute the exact number of highest degree irreducible characters of $U$ as

$$N_{n,\mu(n)} = \begin{cases} (q-1)^m & \text{if } n = 2m + 1 \\ q(q-1)^{m-1} & \text{if } n = 2m \end{cases}.$$

By Theorem 3.2 when $n = 2m + 1$, $N_{n,\mu(n)}$ is exactly the number of basic characters $\xi_{D,\phi}$ where $D = \{\alpha_{k,n-k} : 1 \leq k \leq m\}$, and $n = 2m$, $N_{n,\mu(n)}$ equals the number of all basic characters $\xi_{D,\phi}$ where $D = \{\alpha_{k,n-k} : 1 \leq k \leq m\}$ or $D = \{\alpha_{k,n-k} : 1 \leq k < m\}$ because at $\alpha_{m,2m-m} = \alpha_m$, all elementary characters are linear. Therefore, we clarify all representations of highest degree irreducible characters of $U$.

Lemma 4.1.1. Any maximal degree irreducible character of $U$ is almost faithful and equals some basic character $\xi_{D,\phi}$, where $D = \{\alpha_{k,n-k} : 1 \leq k \leq \frac{m-1}{2}\}$ if $n$ is odd, and $D = \{\alpha_{k,n-k} : 1 \leq k \leq \frac{m}{2}\}$ or $D = \{\alpha_{k,n-k} : 1 \leq k < \frac{m}{2}\}$ if $n$ is even.
4.2. The second highest degree irreducible representations of $U$. We have the following two cases.

\begin{align*}
U_{n-2}(q) \quad &\text{(i)} \\
U_{n-4}(q) \quad &\text{(ii)}
\end{align*}

It is clear that, by Theorem 3.2 in (i), a tensor product of an almost faithful elementary character and a second highest degree irreducible character of $U_{n-2} \cong U_{n-2}(q)$ is a second highest degree almost faithful irreducible character of $U$. There are $(q-1)^2N_{n-2,\mu(n-2)-1}(q)$ characters in this case.

In case (ii), a tensor product of two elementary characters at $\alpha_{1,n-2}$ and $\alpha_{2,n-1}$ respectively, by Lemma 3.5 (ii), decomposes into $q$ pairwise distinct irreducible constituents with degree $q^{n-3+n-3-1} = q^{2n-7}$. Therefore, by Lemma 3.4, a tensor product of each constituent and a highest degree irreducible character of $U_{n-3} \cong U_{n-4}(q)$ is a second highest irreducible character of $U$. These characters are not almost faithful since $X_{\alpha_{1,n-1}}$ is in their kernel, and there are $q(q-1)^2N_{n-4,\mu(n-4)}(q)$ characters in this case.

**Lemma 4.2.1.** For $n \geq 5$, any second highest degree irreducible representation of $U$ is of the form (i) or (ii), and

$$N_{n,\mu(n)-1}(q) = (q-1)N_{n-2,\mu(n-2)-1}(q) + q(q-1)^2N_{n-4,\mu(n-4)}(q).$$

**Proof.** By Lemma 3.5 all constructed characters in (i), (ii) are irreducible of degree $q^{\mu(n)-1}$. Hence, we get the corresponding recursion formula. It is enough to show that there are no more other second highest degree representations except (i), (ii).

We prove this by induction $n$ with initial values $N_{1,0}(q) = 1$, $N_{2,0}(q) = q$, $N_{3,0}(q) = q^2$, $N_{3,1}(q) = q - 1$. Isaacs [8] calculated

$$N_{n,\mu(n)-1}(q) = \begin{cases} 
q(q-1)^m-1(m(q-1)+1) & \text{if } n = 2m + 1 \\
q(q-1)^m-1((m-1)q+1) & \text{if } n = 2m 
\end{cases}.$$

Suppose the recursion formula holds for all cases less than $n \geq 5$, we have

if $n = 2m$ then

$$\begin{align*}
(q-1)N_{n-2,\mu(n-2)-1}(q) + q(q-1)^2N_{n-4,\mu(n-4)}(q) &= q(q-1)^m-1((m-2)q+1) + q^2(q-1)^{m-1} \\
&= q(q-1)^m-1((m-2)q+1+q) \\
&= q(q-1)^m-1((m-1)q+1) \\
&= N_{n,\mu(n)-1}(q)
\end{align*}$$

if $n = 2m + 1$ then

$$\begin{align*}
(q-1)N_{n-2,\mu(n-2)-1}(q) + q(q-1)^2N_{n-4,\mu(n-4)}(q) &= q(q-1)^m-1((m-1)q-1+1) + q(q-1)^m \\
&= q(q-1)^m-1((m-1)q-1+1+q-1) \\
&= q(q-1)^m-1(m(q-1)+1) \\
&= N_{n,\mu(n)-1}(q).
\end{align*}$$

4.3. The third highest degree irreducible representations of $U$. We have the following five cases.
In (i), by Theorem 3.2, a tensor product of an almost faithful elementary character and a third highest degree irreducible character of $U_{\alpha_{2,n-2}} \cong U_{n-2}(q)$ is a third highest degree almost faithful irreducible character of $U$. In this case, there are $(q-1)N_{n-2,\mu(n-2)-2}(q)$ characters.

In (ii), as same as in the second highest degree irreducibles, a tensor product of any two elementary characters at $\alpha_{1,n-2}$ and $\alpha_{2,n-1}$ respectively decomposes into $q$ distinct irreducible constituents. Then a tensor product of each constituent with a second highest degree irreducible character of $U_{\alpha_{3,n-3}} \cong U_{n-4}(q)$ is a third highest degree irreducible character of $U$. They are not almost faithful and there are $q(q-1)^2N_{n-4,\mu(n-4)-1}(q)$ characters in this case.

In (iii), we observe a tensor product of three elementary characters at $\alpha_{2,n-3}$, $\alpha_{1,n-2}$, $\alpha_{3,n-1}$ respectively, and in (iv), a tensor product of three elementary characters at $\alpha_{1,n-3}$, $\alpha_{3,n-2}$, $\alpha_{2,n-1}$ respectively. Both case (iii) and (iv) are similar because they are symmetric by the graph automorphism of $GL_n(q)$, i.e. the map sending $g$ to $w_0^{-1}g(-1)^{-1}w_0$, where $w_0$ is the longest element in Weyl group $S_n$ and $t$ the transpose, more details in Gibbs 3.

**Lemma 4.3.1.** A tensor product of three elementary characters at $\alpha_{2,n-3}$, $\alpha_{1,n-2}$, $\alpha_{3,n-1}$ respectively, in case (iii), decomposes into $q^2$ distinct irreducible constituents of degree $q^{3n-14}$ with multiplicity 1. A tensor product of each constituent and a highest degree irreducible character of $U_{\alpha_{4,n-4}} \cong U_{n-6}(q)$ is a third highest degree irreducible character of $U$. And there are $q^2(q-1)^3N_{n-6,\mu(n-6)}(q)$ characters.

**Proof.** Let $D = \{\alpha_{2,n-3}, \alpha_{1,n-2}, \alpha_{3,n-1}\}$, $V_1 = V_{\alpha_{2,n-3}}$, $V_2 = V_{\alpha_{1,n-2}}$, $V_3 = V_{\alpha_{3,n-1}}$; and $\phi = \{\lambda_1, \lambda_2, \lambda_3\}$ where $\phi_i \in \text{Irr}(V_i/[V_i, V_i])^*$, $i = 1, 2, 3$. By Lemma 2.3, $\xi_{D,\phi} = \otimes_{\lambda_i \in \phi} \lambda_i^{U_i} = \lambda^U$ where $\phi_i = \otimes_{\lambda_i \in \phi} \lambda_i$ and $V_D = V_1 \cap V_2 \cap V_3$.

Let $X = X_{\alpha_1,2}X_{\alpha_2}$. Since $[X, V_D] = \{1\}$, $\langle X, V_D \rangle = XV_D \cong X \times V_D$. Since $X$ is abelian, $\lambda$ extends to $q^2$ linear characters of $XV_D$, i.e. $\lambda^{XV_D}$ decomposes into $q^2$ distinct linear characters. Let $\mu$ be an extension of $\lambda$ to $XV_D$.

Let $M_0 = XV_D$, $M_i = M_{i-1}X_{\alpha_{2,n-3-i}}$ for $1 \leq i \leq n-5$, $M_i = M_{i-1}X_{\alpha_{3,n-9-i}}$ for $n-4 \leq i \leq 2n-8$, and $M_i = M_{i-1}X_{\alpha_{3,n-9-i}}$, for $2n-7 \leq t \leq 3n-12$. Hence, $M_{n-4} = M_{n-5}M_{2n-6} = M_{2n-7}$, $M_{n-5} = (V_2 \cap V_3)X$, $M_{2n-8} = V_3$, $M_{3n-12} = U$.

We show that $\mu^{U_i}$ is irreducible by a sequence of inductions along the arms of $\alpha_{2,n-3}$, $\alpha_{1,n-2}$, $\alpha_{3,n-1}$ respectively, namely from $M_0$ to $M_1, ..., M_{3n-12}$.

Let $K_0 = \{\text{leg}(\alpha_{2,n-3}) \cup \text{leg}(\alpha_{1,n-2}) \cup \text{leg}(\alpha_{3,n-1})\} \setminus \{\alpha_{3,n-3}, \alpha_{3,n-2}\}$, and $K_i = K_{i-1} \setminus \{\alpha_{n-2-i,n-3}\}$ for $1 \leq i \leq n-5$, $K_i = K_{i-1} \setminus \{\alpha_{3,n-9-i,n-1}\}$ for $2n-7 \leq t \leq 3n-12$. It is clear that $K_{n-4} = K_{n-5}$, $K_{2n-6} = K_{2n-7}$, $K_{3n-14}$ is empty, $\beta \subset \text{ker} (\lambda)$ for all $\beta \in K_0$, and for every $M_i = M_{i-1}X_{\alpha_{i}}$, $M_i \neq M_{i-1}$, the unique root $\beta \in K_i \setminus K_{i-1}$ satisfies $\alpha + \beta \in \{\alpha_{2,n-3}, \alpha_{1,n-2}, \alpha_{3,n-3}\}$. Now we achieve the claim by applying exactly the same technique in Lemma 3.3 (ii). $\Box$
In both cases (iii) and (iv), there are \(2q^2(q-1)^3N_{n-6,\mu(n-6)}(q)\) characters. Next, in (v), we observe a tensor product of three elementary characters at \(\alpha_{1,n-3}, \alpha_{2,n-2}, \alpha_{3,n-1}\) respectively.

**Lemma 4.3.2.** A basic character described in (v) decomposes into \(q^2\) irreducible constituents of degree \(q^{3n-15}\) with multiplicity \(1\) and \((q-1)\) irreducible constituents of degree \(q^{3n-14}\) with multiplicity \(q\). Each constituent of degree \(q^{3n-14}\) tensors with a highest degree irreducible character of \(U_{\alpha_{4,n-4}}\) is a third highest degree irreducible character of \(U\). And there are \((q-1)^3N_{n-6,\mu(n-6)}(q)\) characters.

**Proof.** Let \(D = \{\alpha_{1,n-3}, \alpha_{2,n-2}, \alpha_{3,n-1}\}, V_1 = V_{\alpha_{1,n-3}}, V_2 = V_{\alpha_{2,n-2}}, V_3 = V_{\alpha_{3,n-1}},\) and \(\phi = \{\lambda_1, \lambda_2, \lambda_3\}\) where \(\lambda_i \in Irr(V_i/\langle V_i, V_i \rangle)^r, i = 1, 2, 3\). By Lemma 2.3, \(\xi_{D,\phi} = \lambda^U\) where \(\lambda = \otimes_{i} \lambda_i\vert_{V_D}\) and \(V_D = V_1 \cap V_2 \cap V_3\).

Let \(X = X_{\alpha_1,2}X_{\alpha_2}\). Since \([X, V_D] = \{1\}\), \(\langle X, V_D \rangle = XV_D \cong X \times V_D\). Therefore, \(\lambda\) extends naturally to \(XV_D\) as \(\lambda_{XV_D}\) with \(X \subset ker(\lambda_{XV_D})\). We have \(\lambda_{XV_D} = (\lambda_{XV_D} \otimes 1)^{XV_D} = \lambda_{XV_D} \otimes 1 - \sum_{\lambda \in Irr(XV_D/V_D)} \chi(1)\chi\).

Since \(1_{V_D}XV_D\) is the regular character of \(X\) and \(X\) is isomorphic to \(U_3(q), 1_{V_D}XV_D\) decomposes into \((q-1)\) irreducibles of degree \(q\) and \(q^2\) linear characters. Since \(Irr(XV_D) \cong Irr(X) \times Irr(V_D), \lambda_{XV_D} \otimes \chi \in Irr(XV_D), \) for all \(\chi \in Irr(XV_D/V_D)\).

Let \(M_0 = XV_D, M_i = M_{i-1}X_{\alpha_{i-3,n-3}}, \) for \(1 \leq i \leq n-4, M_i = M_{i-1}X_{\alpha_{2n-6-i,3}}, \) for \(n-3 \leq i \leq 2n-8, M_i = M_{i-1}X_{\alpha_{3n-9-i,3}}, \) for \(2n-7 \leq i \leq 3n-12\). Hence, \(M_{n-2} = M_{n-3} = M_{n-4}, M_{2n-9} = M_{2n-8}, M_{n-4} = (V_2 \cap V_3)X, M_{2n-8} = V_3, \) and \(M_{3n-12} = U\).

Let \(K_0 = (\text{leg}(\alpha_{1,n-3}) \cup \text{leg}(\alpha_{2,n-2}) \cup \text{leg}(\alpha_{3,n-1})) \setminus \{\alpha_{2,n-3, \alpha_{3,n-3, \alpha_{3,n-2}}},\) and \(K_i = K_{i-1} \setminus \{\alpha_{2n-2-i,n-3}\}\) for \(1 \leq i \leq n-4, K_i = K_{i-1} \setminus \{\alpha_{2n-5-i,n-2}\}\) for \(n-3 \leq i \leq 2n-8, K_i = K_{i-1} \setminus \{\alpha_{3n-8-i,n-1}\}\) for \(2n-7 \leq i \leq 3n-12\). Hence, \(K_{n-2} = K_{n-3} = K_{n-4}, K_{2n-9} = K_{2n-8}, \) and \(K_{3n-12}\) is empty. And for every \(M_i = M_{i-1}X_{\alpha}\) such that \(M_i \neq M_{i-1},\) there exists unique \(\beta \in K_i \setminus K_{i-1}\) satisfying \(\alpha + \beta \in \{\alpha_{1,n-3, \alpha_{2,n-2, \alpha_{3,n-1}}},\) \(\lambda_{\alpha_{1,k},k+1}\) is irreducible constituents of degree \(q^{(3n-12)-3+1} = q^{3n-14}\) with multiplicity \(q\) and \(q^2\) irreducible constituents of degree \(q^{3n-15}\) with multiplicity \(1\).

Before, we state all third highest degree irreducible representations of \(U\), we need the following property.

**Lemma 4.3.3.** For any \(1 \leq k < n-1,\) let \(T_k = \Sigma^{+} \setminus (\{\alpha_{1,k}, \alpha_{k+1,n-1}\} \cup \text{leg}(\alpha_{1,k}) \cup \text{arm}(\alpha_{k+1,n-1}))\). \(T_k\) is closed and \(\langle X_{\alpha} : \alpha \in T_k \rangle\) is isomorphic to \(U_{n-1}(q)\).
Proof. Let \( \{ \beta_i : i = 1, ..., n - 2 \} \) be the fundamental root set of \( U_{n-1}(q) \). Let \( \varphi \) be the function mapping \( \alpha_i \) to \( \beta_i \) for \( 0 < i < k \), \( \alpha_{k,k+1} \) to \( \beta_k \), and \( \alpha_{i+1} \) to \( \beta_i \) for \( k < i \leq n - 2 \). The claim is clear by mapping \( x_\alpha(c) \) to \( x_{\varphi(\alpha)}(c) \) for all \( \alpha \in T_k \), \( c \in \mathbb{F}_q \). \( \square \)

**Theorem 4.3.4.** For \( n \geq 7 \), any third highest degree irreducible representation of \( U \) is of the form (i), (ii), (iii), (iv), or (v). We have

\[
N_{n,\mu(n)-2}(q) = (q-1)N_{n-2,\mu(n-2)-2}(q) + q(q-1)N_{n-4,\mu(n-4)-2}(q) + 2q^2(q-1)^3N_{n-6,\mu(n-6)}(q) + (q-1)^3N_{n-6,\mu(n-6)}(q).
\]

Proof. Suppose that \( n \geq 7 \). It is clear that the formula that we want to prove comes from (i), ..., (v), hence we mainly show that there are no more basic characters which give irreducible constituents of degree \( q^{\mu(n)-2} \).

Case (i) lists all third highest degree almost faithful irreducible characters, cases (ii), ..., (v) correspond with cases where \( X_{\alpha_{1,n-1}} \) or \( X_{\alpha_{1,n-1}}X_{\alpha_{2,n-1}} \) is in the kernel of \( \xi_{D,\varphi} \). By Theorem 2.3 and the graph automorphism, it is enough to check two more cases: the first case where basic characters \( \xi_{D,\varphi} \) of \( \alpha_{k,n-1} \in D \), \( k \geq 4 \), i.e. \( X_{\alpha_{1,n-1}} \), \( X_{\alpha_{2,n-1}} \), \( X_{\alpha_{3,n-1}} \) is in the kernel of \( \xi_{D,\varphi} \), and the second case where basic characters \( \xi_{D,\varphi} \) with \( \langle \alpha_{1,n-3}, \alpha_{3,n-1} \rangle \) is in the kernel of \( \xi_{D,\varphi} \).

First, we consider all basic characters \( \xi_{D,\varphi} \) with \( \alpha_{k,n-1} \in D \), \( k \geq 4 \). Since \( (\prod_{i=1}^{k-1} X_{\alpha_{i,n-1}}) \subset \ker(\xi_{D,\varphi}) \), we decompose \( \xi_{D,\varphi} \) in the group \( U/\langle \prod_{i=1}^{k-1} X_{\alpha_{i,n-1}} \rangle \). To get convenience for notations, we still keep \( U \) as the quotient group.

Let \( V_k = V_{\alpha_{k,n-1}} \) and \( L_k = \langle X_\alpha : \alpha \in leg(\alpha_{k-1,n-1}) \rangle \). It is easy to check that \( L_k \) is normal in \( V_k \), i.e. \( N_{V_k}(L_k) = V_k \). Set \( U_k = \langle X_\alpha \subset V_k : X_\alpha \not\subset L_k \rangle \), we have \( U_k \cap L_k = \{1\} \), \( V_k = U_k \ltimes L_k \), and \( \text{Irr}(V_k/L_k) = \{ \xi_{V_k} : \xi \in \text{Irr}(U_k) \} \).

For any \( \xi \in \text{Irr}(U_k) \) and \( \lambda \in \text{Irr}(V_k/[V_k,V_k])^* \), using the same technique in Lemma 3.3 (ii), with \( M_0 = V_k \), \( M_i = M_{i-1}X_{\alpha_{k,n-1}} \) for \( 1 \leq i \leq n-1-k \), and \( K_0 = \text{arm}(\alpha_{k,n-1}) \), \( K_i = K_{i-1} \setminus \{\alpha_{n-i-1,n-1}\} \) for \( 1 \leq i \leq n-1-k \), the induced character \( (\xi_V \otimes \lambda)^U \) is irreducible. Therefore, for any highest degree irreducible characters \( \xi \in \text{Irr}(U_k) \), \( (\xi_V \otimes \lambda)^U \) is a highest degree irreducible constituent of \( \xi_{D,\varphi} \). Since \( (\xi_V \otimes \lambda)^U(1) = \xi(1) \cdot \lambda(1) = \xi(1) \cdot q^{n-1-k} \), we show that if \( \xi(1) = q^e \), then \( e < \mu(n) - n - 1 + k \) for all \( \xi \in \text{Irr}(U_k) \).

Let \( T_k = \langle X_\alpha \subset U_k : \alpha \not\subset (\text{leg}(\alpha_{1,k-1}) \cup \{\alpha_{1,k-1}\}) \rangle \). By Lemma 3.3, \( T_k \) is isomorphic to \( U_{n-2}(q) \). Therefore, the highest degree of irreducible characters of \( T_k \) is \( q^{\mu(n-2)} \). It is clear that \( [T_k, X_{\alpha_{1,k-1}}] = \{1\} \). Set \( Y = \langle T_k, X_{\alpha_{1,k-1}} \rangle = T_kX_{\alpha_{1,k-1}} \cong T_k \times X_{\alpha_{1,k-1}} \). Hence, \( \text{Irr}(Y) \cong \text{Irr}(T_k) \times \text{Irr}(X_{\alpha_{1,k-1}}) \).

It is clear that any irreducible character \( \xi \) of \( U_k \) is a constituent of some \( \phi^Y \), where \( \varphi \in \text{Irr}(Y) \), and \( \varphi^Y(1) = \varphi(1) \cdot q^{k-2} \). Since \( \mu(n) = \mu(n-2) + n - 2 \), we are going to show that \( e < \mu(n-2) + k - 3 \) for all \( \xi \in \text{Irr}(U_k) \), \( \xi(1) = q^e \). Therefore, we
observe the case where $\varphi \in \text{Irr}(T_k)$ is a highest or second highest degree irreducible character. By Lemma 3.5, Lemma 1.2.1, we consider three following subcases: \{\alpha_{1,n-1}, \alpha_{2,n-2}\}, \{\alpha_{1,n-1}, \alpha_{2,n-3}, \alpha_{3,n-2}\}, \{\alpha_{1,n-2}, \alpha_{2,n-1}\} \subset D$, where $D$ is a basic set that $\varphi \in \text{Irr}(T_k, \xi_{D,\varphi})$.

Subcase $D_1 = \{\alpha_{1,n-1}, \alpha_{2,n-2}\} \subset D$: By Theorem 3.2, Lemma 3.5, it is clear that $\varphi = \lambda^T_k \otimes \xi_{T_k}$ where $\eta \in \text{Irr}(U_{\alpha_{3,n-3}} \cap T_k)$ and $\lambda = \bigotimes_{\alpha \in \alpha_1} \lambda_\alpha | V$, $V = T_k \cap (\cap_{\alpha \in D_1} V_\alpha)$, $\lambda_\alpha \in \text{Irr}(V_{3\alpha}/[V_{3\alpha}, V_{3\alpha}])$, $\alpha \in D_1$. Let $X = X_{\alpha_{1,n-1}}X_{\alpha_{2,n-1}}X_{\alpha_{3,n-2}}$. Since $X$ is abelian and $[X, V] = \{1\}$, we have $(X, V) = XV \cong X \times V$ and $\lambda^XV = \lambda^XV \otimes \rho_\varphi(V)$. Since $X, (U_{\alpha_{3,n-3}} \cap T_k)] = \{1\}$, $\eta$ is extendible to $XT_k$. Hence, $\varphi^{XT_k} = (\lambda^T_k \otimes \xi_{T_k})^{XT_k} = \lambda^T_k \otimes \xi_{T_k} = \eta_{XT_k} \otimes \sum_{\rho \in \text{Irr}(X)} (\lambda^XV \otimes \rho_{\varphi})^{XT_k}$.

Therefore, any constituent of $\varphi^{U_{3\alpha}}$ has degree less than $q^{\mu(n-2)+k-3}$. Because $|X| = q^3$ and $\varphi^{U_{3\alpha}}(1) = \varphi(1) \cdot |U_k: T_k| \leq q^{\mu(n-2)} \cdot q^{k-1}$.

Subcase $D_2 = \{\alpha_{1,n-1}, \alpha_{2,n-3}, \alpha_{3,n-2}\} \subset D$: By Theorem 3.2, Lemma 3.3 (ii), Lemma 3.5, it is clear that $\varphi = \lambda^T_k \otimes \xi_{T_k}$ where $\eta \in \text{Irr}(U_{\alpha_{4,n-4}} \cap T_k)$, and $V = (T_k \cap (\cap_{\alpha \in D_2} V_\alpha))X_{\alpha_2}$, $\lambda = \bigotimes_{\alpha \in D_2} \lambda_\alpha | V \otimes \nu_\alpha, \alpha \in \text{Irr}(X_{\alpha_2})$. Again, with exactly the same technique in the first subcase, the claim holds.

Subcase $D_3 = \{\alpha_{1,n-2}, \alpha_{2,n-1}\} \subset D$: By Theorem 3.2, Lemma 3.3 (ii), Lemma 3.5, it is clear that $\varphi = \lambda^T_k \otimes \xi_{T_k}$ where $\eta \in \text{Irr}(U_{\alpha_{3,n-3}} \cap T_k)$, and $V = (T_k \cap (\cap_{\alpha \in D_3} V_\alpha))X_{\alpha_2}$, $\lambda = \bigotimes_{\alpha \in D_3} \lambda_\alpha | V \otimes \nu_\alpha, \alpha \in \text{Irr}(X_{\alpha_2})$. Again, with exactly the same technique in the first subcase, the claim holds.

Finally, we consider all basic characters $\xi_{D,\varphi}$ where $\{\alpha_{1,n-3}, \alpha_{3,n-1}\} \subset D$ and $X_{\alpha} \subset \text{ker}(\xi_{D,\varphi})$ where $\alpha \in S = \{\alpha_{1,n-2}, \alpha_{1,n-1}, \alpha_{2,n-2}, \alpha_{2,n-1}\}$. We decompose $\xi_{D,\varphi}$ in the quotient group $U/({\prod}_{\alpha \in S} X_{\alpha})$ but we still keep $U$ as the quotient group for the convenience of notations.

We repeat the above setup of $\alpha_{k,n-1}$ for $\alpha_{k,n-1}$ by letting $V_3 = V_{\alpha_{3,n-1}}$ and $L_3 = \langle X_{\alpha} : \alpha \in \text{leg}(\alpha_{2,n-1}) \rangle$. Again, $L_3$ is normal in $V_3$, i.e. $N_{V_3}(L_3) = V_3$. Next, let $U_3 = \langle X_{\alpha} \subset V_3 : X_{\alpha} \not\subset L_3 \rangle$ and $T_3 = \langle X_{\alpha} \subset U_3 : \alpha \not\in \text{leg}(\alpha_{1,2}) \cup \{\alpha_{1,2}\} \rangle$, we have $U_3 \cap L_3 = \{1\}$, $V_3 = U_3 \times L_3$, $\text{Irr}(V_3/L_3) = \{\xi_{V_3} : \xi \in \text{Irr}(U_3)\}$, and $T_3$ is isomorphic to $U_{n-2}(q)$.

By Lemma 1.2.1, Lemma 1.2.1, and assumption $X_{\alpha_{1,n-2}}X_{\alpha_{2,n-2}}$ in the quotient, $T_3$ does not have any irreducible characters of degree $q^{\mu(n-2)-1}$. Hence, the best irreducible characters of $T_3$ in this case may have degree $q^{\mu(n-2)-2}$. Since $[T_3, X_{\alpha_{1,2}}] = \{1\}$, $Y = \langle T_3, X_{\alpha_{1,2}} \rangle = T_3X_{\alpha_{1,2}} \cong T_3 \times X_{\alpha_{1,2}}$. Hence, all irreducible characters of $T_3$ extend to $Y$, which implies the best irreducible characters of $Y$ may have degree $q^{\mu(n-2)-2}$, call it $\varphi$. So $\varphi^{U_{3\alpha}}(1) \leq q^{\mu(n-2)-2} \cdot |U_3 : Y| = q^{\mu(n-2)-1}$. And, then, for any $\lambda \in \text{Irr}(V_3/[V_3, V_3])$, we have $((\varphi^{U_{3\alpha}})_{V_3} \otimes \lambda)^{U_{3\alpha}}(1) = \varphi^{U_{3\alpha}}(1) \cdot \lambda^{U_{3\alpha}}(1) \leq q^{\mu(n-2)-1} \cdot q^{|U_{3\alpha}|} = q^{\mu(n-2)} = q^{\mu(n-3)}$ because $\mu(n) = \mu(n-2) - 2$. So all irreducible characters of $U$ with $\prod_{\alpha \in S} X_{\alpha}$ in the kernel can not get degree greater than $q^{\mu(n-3)}$. By Theorem 3.4, all basic characters $\xi_{D,\varphi}$ in this case do not have any irreducible constituents of degree greater than $q^{\mu(n-3)}$.

Remark: M. Marjom [1] calculated $N_{n,\mu(n)-2}$ for $n$ even using combinatorial methods in his thesis.
THREE HIGHEST DEGREE IRREDUCIBLE REPRESENTATIONS OF UPPER TRIANGULAR GROUPS $U_n(q)$

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