M-TRACES IN (NON-UNIMODULAR) PIVOTAL CATEGORIES

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Abstract. We generalize the notion of a modified trace (or m-trace) to the setting of non-unimodular categories. M-traces are known to play an important role in low-dimensional topology and representation theory, as well as in studying the category itself. Under mild conditions we give existence and uniqueness results for m-traces in pivotal categories.

1. Introduction

1.1. Background. Tensor categories with duals have a notion of the categorical trace of a morphism. These traces and the corresponding concept of dimension are a key tool in applications to low-dimensional topology, representation theory, and other fields. However, it is often the case that categories of interest are not semi-simple, the categorical traces vanish, and these constructions become trivial.

In the past decade it became clear there exist non-trivial replacements for trace functions on non-semi-simple ribbon and, more generally, pivotal categories (e.g. see [GKP11, GKP13, GPV13]). We call these modified traces or m-traces, for short. The study of m-traces leads to new, interesting quantum invariants of links and 3-manifolds as well as applications in the study of representation theory, Hopf algebras, Deligne categories, logarithmic conformal field theory, and other fields (e.g. see [CMR16, CG17, DCP17, GPT09, CGP14, CK12, Com14, CH17, BKN12, BCGP16, BBG17, BBG18, AS17, Rup16, Mur17, Len17, Pliu16]). However, until now the existence and theory of m-traces has been limited to unimodular categories (i.e. categories in which the projective cover and injective hull of the unit object coincide). The goal of this paper is to generalize m-traces to the non-unimodular setting.

1.2. Statement of main results. In what follows we highlight the main results. For simplicity’s sake, in the introduction we assume $k$ is an algebraically closed field and $C$ is a pivotal, $k$-linear, locally-finite, tensor category. Roughly speaking, this is a category with a tensor product, duals, and the morphism sets are finite-dimensional $k$-vector spaces. See Section 2 for precise definitions. Such categories are ubiquitous. For example, they appear:

- as categories of finite-dimensional modules for finite-dimensional pivotal (quasi-)Hopf algebras;
- in the study of logarithmic conformal field theories (e.g. see [Gab03]);
- as fusion categories of categorical dimension zero (e.g. see [EGNO15]).

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Given a fixed pair of objects $\alpha$ and $\beta$ in $\mathcal{C}$ and a right ideal $I$ (a certain kind of full subcategory), we define the notion of a right $(\alpha, \beta)$-trace on $I$. This $m$-trace is a family of $k$-linear functions,

$$\{ t_V : \text{Hom}_\mathcal{C}(\alpha \otimes V, \beta \otimes V) \to k \}_{V \in I},$$

where $V$ runs over all objects of $I$, and such that certain partial trace and cyclicity properties hold. See Section 3.2 for a precise definition. In the case when $\alpha$ and $\beta$ are both the unit object of $\mathcal{C}$, then we recover the unimodular $m$-traces of [GKP11, GKP13, GPV13].

Our first main result is Theorem 4.4 in which we show, given an absolutely indecomposable object $P$ and objects $\alpha$ and $\beta$ in $\mathcal{C}$ with $\text{Hom}_\mathcal{C}(\alpha, P) = k$ and $\text{Hom}_\mathcal{C}(P, \beta) = k$, there exists a right $(\alpha, \beta)$-trace on a certain (possibly empty) right ideal $I^\beta_\alpha$. Furthermore, if either $\alpha$ or $\beta$ is the unit object then Theorem 4.5 implies this $m$-trace is unique up to scaling.

Because of the generality of these existence and uniqueness results it is difficult to explicitly describe the ideal $I^\beta_\alpha$ or the functions $t_V$. However, there is a notable case where we can say more. If $P$ is assumed to be the projective cover of the unit object, $1$, and if $\alpha$ denotes a simple subobject of $P$, then our second main result shows the above theorem defines a unique, nontrivial, right $(\alpha, 1)$-trace on $P \text{proj}(\mathcal{C})$, the full subcategory of projective objects of $\mathcal{C}$. This is already interesting and powerful in the context of unimodular categories (i.e. when $\alpha \cong 1$). It says any locally-finite, unimodular, pivotal category $\mathcal{C}$ with enough projectives has an $m$-trace on $P \text{proj}(\mathcal{C})$. Previously this was only known in special cases, such as when $\mathcal{C}$ is the category of representations for a finite group or when $\mathcal{C}$ contains a simple projective object. For example, see [BBG18, GKP13, GR17].

We end the paper with a discussion of how the notion of a right $(\alpha, \beta)$-trace on a category leads to a natural generalization of the notion of a Calabi-Yau category. Variations on Calabi-Yau categories play an important role in mathematical physics, algebraic geometry, integrable systems, the representation theory of finite-dimensional algebras, and the categorification of cluster algebras.

If $F, G$ are endofunctors of $\mathcal{C}$, then we say $\mathcal{C}$ is an $(F, G)$-twisted Calabi-Yau category if for all objects $U$ and $V$ there is a vector space isomorphism

$$\text{Hom}_\mathcal{C}(F(U), V) \cong \text{Hom}_\mathcal{C}(V, G(U))^*,$$

which is functorial in both $U$ and $V$. For example, an $(\text{Id}_\mathcal{C}, G)$-twisted Calabi-Yau structure on $\mathcal{C}$ amounts to saying $G$ is a right Serre functor in the sense of Bondal-Kapranov [BK89]. Just as a Calabi-Yau category is a categorical generalization of the notion of a symmetric Frobenius algebra, an $(F, G)$-twisted Calabi-Yau category generalizes the notion of a Frobenius extension of $k$ as defined by Morita [Mor65].

Our main theorem applied to $P \text{proj}(\mathcal{C})$ can be reformulated as saying there is a twisted Calabi-Yau structure on $P \text{proj}(\mathcal{C})$ for any pivotal, $k$-linear, locally-finite tensor category $\mathcal{C}$. As a consequence, $P \text{proj}(\mathcal{C})$ admits a right Serre functor for any such category. See Theorem 6.2.

1.3. Future applications. One motivation for the development of $m$-traces is their use in constructing invariants in low-dimensional topology. In particular, in forthcoming work the second two authors with Costantino and Turaev define generalized Kuperberg and Turaev-Viro invariants from certain unimodular pivotal
One of the main ingredients in this construction is the existence of a non-degenerate m-trace. This is one motivation for Theorem 5.5 which says, under mild conditions, a right m-trace is always non-degenerate. This work is still within the context of unimodular categories. An interesting future line of research is to use the m-traces of this paper to construct generalized Kuperberg and Turaev-Viro invariants from non-unimodular pivotal tensor categories.

1.4. Related work. While working on this paper we learned A. Fontalvo Orozco and A. M. Gainutdinov were defining a notion of module trace which is related to our m-traces, see [FG]. However, they use different techniques and their work generalizes the relation between the theory of integrals in a Hopf algebra $H$ with the modified trace on the projective ideal $\mathcal{P}roj(H\text{- Mod})$ as established in [BBG18] in the unimodular case.

Shimizu recently introduced the notion of integrals for finite tensor categories [Shi14, Shi17]. It would be interesting to generalize the results of [BBG18] and Orozco-Gainutdinov to the categorical setting by relating the m-traces introduced here to the integrals of Shimizu.

2. Preliminaries

2.1. Pivotal categories. We recall the definition of a pivotal tensor category, see for instance, [BW99]. A tensor category $\mathcal{C}$ is a category equipped with a covariant bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ called the tensor product, an associativity constraint, a unit object $1$, and left and right unit constraints such that the Triangle and Pentagon Axioms hold. When the associativity constraint and the left and right unit constraints are all identities we say that $\mathcal{C}$ is a strict tensor category. By Mac Lane’s coherence theorem for pivotal tensor categories, every such category is equivalent (as a pivotal tensor category) to a strict one (e.g. see [NS07, Theorem 2.2]). To simplify the exposition, we formulate further definitions only for strict tensor categories; the interested reader will easily extend them to arbitrary tensor categories. In what follows we adopt the convention that $fg$ will denote the composition of morphisms $f \circ g$.

A strict tensor category $\mathcal{C}$ has a left duality if for each object $V$ of $\mathcal{C}$ there is an object $V^*$ of $\mathcal{C}$ and morphisms

$$\overleftarrow{\text{coev}}_V : 1 \to V \otimes V^* \quad \text{and} \quad \overrightarrow{\text{ev}}_V : V^* \otimes V \to 1 \quad (2.1)$$

such that

$$(\text{Id}_V \otimes \overrightarrow{\text{ev}}_V)(\overleftarrow{\text{coev}}_V \otimes \text{Id}_V) = \text{Id}_V \quad \text{and} \quad (\overleftarrow{\text{ev}}_V \otimes \text{Id}_{V^*})(\text{Id}_{V^*} \otimes \overleftarrow{\text{coev}}_V) = \text{Id}_{V^*}.$$

A left duality determines for every morphism $f : V \to W$ in $\mathcal{C}$ the dual (or transpose) morphism $f^* : W^* \to V^*$ by

$$f^* = (\overleftarrow{\text{ev}}_W \otimes \text{Id}_{V^*})(\text{Id}_{W^*} \otimes f \otimes \text{Id}_{V^*})(\text{Id}_{W^*} \otimes \overleftarrow{\text{coev}}_V),$$

and determines for any objects $V, W$ of $\mathcal{C}$, an isomorphism $\gamma_{V, W} : W^* \otimes V^* \to (V \otimes W)^*$ by

$$\gamma_{V, W} = (\overleftarrow{\text{ev}}_W \otimes \text{Id}_{(V \otimes W)^*})(\text{Id}_{W^*} \otimes \overleftarrow{\text{ev}}_V \otimes \text{Id}_W \otimes \text{Id}_{(V \otimes W)^*})(\text{Id}_{W^*} \otimes \text{Id}_V \otimes \overleftarrow{\text{coev}}_{W^* \otimes V}).$$

Similarly, $\mathcal{C}$ has a right duality if for each object $V$ of $\mathcal{C}$ there is an object $V^*$ of $\mathcal{C}$ and morphisms

$$\overrightarrow{\text{coev}}_V : 1 \to V^* \otimes V \quad \text{and} \quad \overleftarrow{\text{ev}}_V : V \otimes V^* \to 1 \quad (2.2)$$
such that
\[(\text{Id}_V \otimes \overline{ev}_V)(\overline{\text{coev}}_V \otimes \text{Id}_V) = \text{Id}_V \]  
and  
\[(\overline{ev}_V \otimes \text{Id}_V)(\text{Id}_V \otimes \overline{\text{coev}}_V) = \text{Id}_V.\]

The right duality determines for every morphism \(f: V \to W\) in \(\mathcal{C}\) the dual morphism \(f^*: W^* \to V^*\) by
\[f^* = (\text{Id}_V \otimes \overline{ev}_W)(\text{Id}_V \otimes f \otimes \text{Id}_W)(\overline{\text{coev}}_V \otimes \text{Id}_W),\]
and determines for any objects \(V, W\), an isomorphism \(\gamma'_{V,W}: W^* \otimes V^* \to (V \otimes W)^*\) by
\[\gamma'_{V,W} = (\text{Id}_{V \otimes W} \otimes \overline{ev}_V)(\text{Id}_{V \otimes W} \otimes \overline{\text{coev}}_V \otimes \text{Id}_W \otimes \text{Id}_V)(\overline{\text{coev}}_{V \otimes W} \otimes \text{Id}_W \otimes \text{Id}_V).

A pivotal category is a tensor category with left duality \(\{\overline{\text{coev}}_V, \overline{ev}_V\}_{V \in \mathcal{C}}\) and right duality \(\{\text{coev}_V, \overline{ev}_V\}_{V \in \mathcal{C}}\) which are compatible in the sense that \(V^* = V^*\), \(f^* = f^*\), and \(\gamma_{V,W} = \gamma'_{V,W}\) for all \(V, W, f\) as above. Every pivotal category has natural tensor isomorphisms
\[\phi = \{\phi_V = (\overline{ev}_V \otimes \text{Id}_{V^*})(\text{Id}_V \otimes \overline{\text{coev}}_V): V \to V^{**}\}_{V \in \mathcal{C}}.\]

We remind the reader of the well-known diagrammatic calculus for pivotal tensor categories, see for example [Kas95, Chapter XIV] or [GPV13]. For brevity’s sake we choose to not use it here. Nevertheless, many of the calculations in this paper are most easily understood when done diagrammatically.

2.2. Tensor \(k\)-categories. Let \(k\) be a commutative ring. A tensor \(k\)-category is a tensor category \(\mathcal{C}\) which is enriched over the category of \(k\)-modules. That is, \(\mathcal{C}\) is additive, the hom-sets of \(\mathcal{C}\) are left \(k\)-modules, and the composition and tensor product of morphisms are \(k\)-bilinear.

An object \(V\) of a tensor \(k\)-category \(\mathcal{C}\) is absolutely irreducible (or absolutely simple) if \(\text{End}_\mathcal{C}(V)\) is a free \(k\)-module of rank one; that is, if the \(k\)-homomorphism \(k \to \text{End}_\mathcal{C}(X), k \mapsto k \text{Id}_X\) is an isomorphism. We identify \(\text{End}_\mathcal{C}(V)\) and \(k\) via this map. We always assume the unit object, \(\mathbb{1}\), is absolutely irreducible.

We call an object \(V\) of \(\mathcal{C}\) absolutely indecomposable if
\[\text{End}_\mathcal{C}(V)/\text{J(End}_\mathcal{C}(V)) \cong k.\]

Here \(\text{J(End}_\mathcal{C}(V))\) denotes the Jacobson radical of the endomorphism ring \(\text{End}_\mathcal{C}(V)\). We say an absolutely indecomposable object is end-nilpotent if the Jacobson radical of its endomorphism algebra is nilpotent.

2.3. Projective and injective objects. Let \(\mathcal{C}\) be a category. Recall that an object \(P\) of \(\mathcal{C}\) is projective if the functor \(\text{Hom}_\mathcal{C}(P, -): \mathcal{C} \to \text{Set}\) preserves epimorphisms, that is, if for any epimorphism \(p: X \to Y\) and any morphism \(f: P \to Y\) in \(\mathcal{C}\), there exists a morphism \(g: P \to X\) in \(\mathcal{C}\) such that \(f = pg\). We denote by \(\text{Proj}(\mathcal{C})\) the class of projective objects of \(\mathcal{C}\). An object of \(\mathcal{C}\) is injective if it is projective in the opposite category \(\mathcal{C}^{\text{op}}\). In other words, an object \(Q\) of \(\mathcal{C}\) is injective if for any monomorphism \(i: X \to Y\) and any morphism \(f: X \to Q\) in \(\mathcal{C}\), there exists a morphism \(g: Y \to Q\) in \(\mathcal{C}\) such that \(f = gi\).

When \(\mathcal{C}\) is pivotal the projective and injective objects coincide (e.g. see [GPV13 Lemma 17]). Thus in this case \(\text{Proj}(\mathcal{C})\) is also the class of injective objects of \(\mathcal{C}\). The projective cover of an object is unique up to non-unique isomorphism, if it exists. We say \(\mathcal{C}\) has enough projectives if every object in \(\mathcal{C}\) has a projective cover.
We call \( \mathcal{C} \) \textit{locally-finite} if, for every pair of objects \( X, Y \) in \( \mathcal{C} \), \( \text{Hom}_\mathcal{C}(X, Y) \) has a finite length composition series as a \( k \)-module. If \( \mathcal{C} \) is a locally-finite tensor category, then, for example by \cite{Kra13}, an indecomposable projective object \( P \) has a unique simple quotient (which we call the \textit{head} of \( P \), a unique simple subobject (which we call the \textit{socle} of \( P \)), and \( \text{End}_\mathcal{C}(P) \) is end-nilpotent. By definition, \( \mathcal{C} \) is \textit{unimodular} if the socle of the projective cover of \( 1 \) is isomorphic to \( 1 \).

2.4. \textbf{Invertible objects.} We call an object \( X \) in \( \mathcal{C} \) \textit{invertible} if \( \text{ev}_X : X^* \otimes X \to 1 \) and \( \text{coev}_X : 1 \to X \otimes X^* \) are isomorphisms. For example, \( 1 \) is always an invertible object and in a finite tensor category the socle of the projective cover of \( 1 \) is always an invertible object (see \cite[Section 6.4]{EGNO15}).

3. \textbf{Right \((\alpha, \beta)\)-Traces}

3.1. \textbf{Ideals.} Let \( \mathcal{C} \) be a pivotal \( k \)-category.

A \textit{right partial trace} (with respect to \( W \)) is the map \( \text{tr}^W_r : \text{Hom}_\mathcal{C}(V \otimes W, X) \to \text{Hom}_\mathcal{C}(V, X) \), defined, for \( g \in \text{Hom}_\mathcal{C}(V \otimes W, X) \), by

\[
\text{tr}^W_r(g) = (\text{Id}_X \otimes \text{ev}_W)(g \otimes \text{Id}_{W^*}):(\text{Id}_V \otimes \text{coev}_W).
\]

Similarly, a \textit{left partial trace} (with respect to \( W \)) is the map \( \text{tr}^W_l : \text{Hom}_\mathcal{C}(W \otimes V, X) \to \text{Hom}_\mathcal{C}(W, X) \), defined by

\[
\text{tr}^W_l(h) = (\text{ev}_V \otimes \text{Id}_X)(h \otimes \text{Id}_{W^*}):(\text{coev}_W \otimes \text{Id}_V).
\]

By a \textit{right} (resp. \textit{left}) ideal of \( \mathcal{C} \) we mean a full subcategory \( I \), of \( \mathcal{C} \) such that:

1. \textbf{Closed under tensor products:} If \( V \) is an object of \( I \) and \( W \) is any object of \( \mathcal{C} \), then \( V \otimes W \) (resp. \( W \otimes V \)) is an object of \( I \).

2. \textbf{Closed under retracts:} If \( V \) is an object of \( I \), \( W \) is any object of \( \mathcal{C} \), and there exists morphisms \( f : W \to V \), \( g : V \to W \) such that \( gf = \text{Id}_W \), then \( W \) is an object of \( I \).

An \textit{ideal} of \( \mathcal{C} \) is a full subcategory of \( \mathcal{C} \) which is both a right and left ideal. For example, the full subcategory whose objects are the class of projective objects, \( \text{Proj}(\mathcal{C}) \), is an ideal by \cite[Lemma 17]{GPV13}.

3.2. \textbf{Traces.} Let \( \alpha \) and \( \beta \) be objects of \( \mathcal{C} \) and \( I \) a right ideal in \( \mathcal{C} \). A \textit{right \((\alpha, \beta)\)-trace} on \( I \) (or \( m \)-trace for short) is a family of \( k \)-linear functions,

\[
\{t_V : \text{Hom}_\mathcal{C}(\alpha \otimes V, \beta \otimes V) \to k \}_{V \in I},
\]

where \( V \) runs over all objects of \( I \), and such that the following two conditions hold:

1. \textbf{Partial trace property.} If \( U \in I \) and \( W \in \mathcal{C} \), then for any \( f \in \text{Hom}_\mathcal{C}(\alpha \otimes U \otimes W, \beta \otimes U \otimes W) \) we have

\[
t_U \otimes W(f) = t_U(\text{tr}^W_r(f)).
\]

2. \textbf{\((\alpha, \beta)\)-Cyclicity.} If \( U, V \in I \), then for any morphisms \( f : \alpha \otimes V \to \beta \otimes U \) and \( g : U \to V \) in \( \mathcal{C} \) we have

\[
t_V((\text{Id}_\beta \otimes g)f) = t_U(f(\text{Id}_\alpha \otimes g)).
\]

Similarly, a \textit{left \((\alpha, \beta)\)-trace} on a left ideal \( I \) is a family of linear functions,

\[
\{t_V : \text{Hom}_\mathcal{C}(V \otimes \alpha, V \otimes \beta) \to k \}_{V \in I},
\]
which satisfies the obvious left partial trace property and the left \((α, β)\)-cyclicity property. An \((α, β)\)-trace on an ideal \(I\) is a left \((α, β)\)-trace on \(I\) which is also a right \((α, β)\)-trace.

**Remark 3.1.** When \(α = β = k\), a right (resp. left) \((α, β)\)-trace is a right (resp. left) modified trace as defined in [GPV13].

### 3.3. The dual trace.

As we now explain, there is a natural notion of the dual of a right \(m\)-trace. If \(I\) is a full subcategory of \(C\), define its dual \(I^*\) to be the full subcategory with objects \(I^* = \{V ∈ \text{Obj}(C) : V^* ∈ I\}\). It is straightforward to check if \(I\) is a right (resp. left) ideal then \(I^*\) is a left (resp. right) ideal. If \(t\) is a right \((α, β)\)-trace \(t\) on \(I\), then given \(V \in I^*\) define \(t^*_V : \text{Hom}_C(V ⊗ β^*, V ⊗ α^*) → k\) by

\[
t^*_V(f) = t_{V^*}((\text{Id}_{V^*} ⊗ φ^{-1})(f^*(\text{Id}_{V^*} ⊗ φ_α)))
\]

where \(φ\) is the pivotal structure. In light of the following result we call \(t^*_V\) the dual of \(t\).

**Lemma 3.2.** Let \(t\) be a right \((α, β)\)-trace \(t\) on a right ideal, \(I\). Then \(t^*_V\) is a left \((β^*, α^*)\)-trace on the left ideal \(I^*\).

**Proof.** This follows easily from the observing the left partial trace of the dual morphism is the dual of the right partial trace. \(\Box\)

One can analogously define the dual of a left \(m\)-trace on a left ideal \(I\) and obtain a right \(m\)-trace on \(I^*\). Furthermore, a straightforward check verifies that the dualizing a right or left \(m\)-trace twice yields the original right or left \(m\)-trace.

### 3.4. Related traces.

We next explain how to construct new \(m\)-traces from old. Assume \(α_1, β_1, α_2,\) and \(β_2\) are a fixed list of objects in \(C\) and that we have a fixed morphism \(h : α_2^* ⊗ β_2 → α_1^* ⊗ β_1\). For any object \(V \in C\), the morphism \(h\) induces a \(k\)-linear map

\[
h_* : \text{Hom}_C(α_2 ⊗ V, β_2 ⊗ V) → \text{Hom}_C(α_1 ⊗ V, β_1 ⊗ V)
\]
given by

\[
f → (ev_{α_1} ⊗ \text{Id}_{β_1} ⊗ \text{Id}_V)(\text{Id}_{α_1} ⊗ h ⊗ \text{Id}_V)(\text{Id}_{α_1} ⊗ \text{Id}_{α_2} ⊗ f)(\text{Id}_{α_1} ⊗ \text{coev}_{α_2} ⊗ \text{Id}_V).
\]

**Lemma 3.3.** Let \(t\) be a right \((α_1, β_1)\)-trace on a right ideal \(I\). Assume we have a fixed morphism \(h \in \text{Hom}_C(α_2^* ⊗ β_2, α_1^* ⊗ β_1)\). Then the family of \(k\)-linear maps \(h_* t\),

\[
\{(h_* t)_V : \text{Hom}_C(α_2 ⊗ V, β_2 ⊗ V)\}_{V ∈ I},
\]
defined by

\[
(h_* t)_V(f) = t(h_*(f)),
\]
is a right \((α_2, β_2)\)-trace on \(I\).

Furthermore, if \(h' \in \text{Hom}_C(α_3^* ⊗ β_3, α_2^* ⊗ β_2)\) then \(h'_*(h_* t) = (h'h)_* t\). \(\Box\)

**Proof.** The is a straightforward verification using the definition of \(h_*\) and \(h_* t\). \(\Box\)

Similarly, a morphism \(h : β_2 ⊗ α_2 → β_1 ⊗ α_1^*\) induces a \(k\)-linear map

\[
h_* : \text{Hom}_C(V ⊗ α_2, V ⊗ β_2) → \text{Hom}_C(V ⊗ α_1, V ⊗ β_1)
\]
and if \(t\) is a left \((α_1, β_1)\)-trace on an ideal \(I\), then we can analogously define a left \((α_2, β_2)\)-trace \(h_* t\).
Using the obvious morphisms as $h$ in the previous lemma along with Lemma 3.2 yields the following result.

**Proposition 3.4.** Let $I$ be a right ideal of $\mathcal{C}$. Then there are canonical bijections between the following families:

1. the right $(\alpha, \beta)$-traces on $I$,
2. the right $(\beta^* \otimes \alpha, \mathbf{1})$-traces on $I$,
3. the right $(\mathbf{1}, \alpha^* \otimes \beta)$-traces on $I$,
4. the left $(\beta^*, \alpha^*)$-traces on $I^*$,
5. the left $(\alpha \otimes \beta^*, \mathbf{1})$-traces on $I^*$,
6. the left $(\mathbf{1}, \beta \otimes \alpha^*)$-traces on $I^*$.

4. Existence of right and left $(\alpha, \beta)$-traces

4.1. **Trace tuples.** Let $\mathcal{C}$ be a pivotal $k$-category. In this section we require $k$ to be an integral domain. To simplify exposition, in this section we only work with right $m$-traces and right ideals. However, the interested reader can easily formulate that analogous left versions of the definitions and statements.

**Definition 4.1.** Let $P$, $\alpha$ and $\beta$ be objects of $\mathcal{C}$. Let $\eta : \alpha \to P$ and $\epsilon : P \to \beta$ be nonzero morphisms. We say $(P, \alpha, \beta, \eta, \epsilon)$ is a trace tuple if the following conditions hold:

1. The object $P$ is absolutely indecomposable and end-nilpotent.
2. The left $k$-modules $\text{Hom}_\mathcal{C}(\alpha, P)$ and $\text{Hom}_\mathcal{C}(P, \beta)$ are free and generated by $\eta$ and $\epsilon$, respectively.

Let $(P, \alpha, \beta, \eta, \epsilon)$ be a trace tuple. Consider the following classes of objects:

$I_\alpha = \{ V : \text{there exists } \sigma_V : P \otimes V \to \alpha \otimes V \text{ such that } \sigma_V(\eta \otimes \text{Id}_V) = \text{Id}_{\alpha \otimes V} \}$,

$I_\beta = \{ V : \text{there exists } \tau_V : \beta \otimes V \to P \otimes V \text{ such that } (\epsilon \otimes \text{Id}_V)\tau_V = \text{Id}_{\beta \otimes V} \}$,

$I_{\alpha}^\beta = I_\alpha \cap I_\beta$.

For each of these, we abuse notation by using the same name for the full subcategory of $\mathcal{C}$ consisting of objects isomorphic to an object in the given class. The following lemma is a straightforward check using the definitions.

**Lemma 4.2.** If $(P, \alpha, \beta, \eta, \epsilon)$ is a trace tuple, then $I_\alpha$, $I_\beta$ and $I_{\alpha}^\beta$ are right ideals.

We set the following notation. When $P$ is absolutely indecomposable write $f \mapsto \langle f \rangle$ for the canonical quotient map $\text{End}_\mathcal{C}(P) \to k$. For a trace tuple $(P, \alpha, \beta, \eta, \epsilon)$ and morphisms $g \in \text{Hom}_\mathcal{C}(\alpha, P)$ and $h \in \text{Hom}_\mathcal{C}(P, \beta)$, let $\langle g \rangle_\eta$, $\langle h \rangle_\epsilon \in k$ be defined by

$g = \langle g \rangle_\eta \eta$ and $h = \langle h \rangle_\epsilon \epsilon$.

**Lemma 4.3.** Let $(P, \alpha, \beta, \eta, \epsilon)$ be a trace tuple. For any $f \in \text{End}_\mathcal{C}(P)$ the following statements hold:

1. $\epsilon f = \langle f \rangle \epsilon$,
2. $f \eta = \langle f \rangle \eta$,
3. $\langle f \rangle = \langle \epsilon f \rangle_\epsilon = \langle f \eta \rangle_\eta$.

**Proof.** Since $P$ is absolutely indecomposable, we have $f = \langle f \rangle \text{Id}_P + n$ for $\langle f \rangle \in k$ and $n \in J(\text{End}_\mathcal{C}(P))$. The first statement then follows once we prove $\epsilon n = 0$. Since $\text{Hom}_\mathcal{C}(P, \beta)$ is a free left $k$-module generated by $\epsilon$ we have $\alpha n = \lambda \epsilon$ for some $\lambda \in k$. 

But since \( P \) is end-nilpotent, \( n \) is nilpotent and \( n^k = 0 \) for some \( k > 0 \). But then \( 0 = c n^k = \lambda^k \epsilon \) and, hence, \( \lambda^k = 0 \). Since we are assuming \( k \) is an integral domain, it follows that \( \lambda = 0 \). This proves the first statement, the second follows analogously. The first two parts of the lemma immediately imply the third statement. \( \square \)

4.2. Existence of \( m \)-traces.

**Theorem 4.4.** Let \((P, \alpha, \beta, \eta, \epsilon)\) be a trace tuple. Then there exists a right \((\alpha, \beta)\)-trace on \( I^\beta_\alpha \) defined for \( V \in I^\beta_\alpha \) and \( f \in \text{Hom}_\mathcal{C}(\alpha \otimes V, \beta \otimes V) \) by

\[
t_v(f) = \langle \tau_V(f) \rangle_\eta = \langle \tau_V(f \sigma_V) \rangle_\epsilon
\]

where \( \sigma_V : P \otimes V \rightarrow \alpha \otimes V \) and \( \tau_V : \beta \otimes V \rightarrow P \otimes V \) are any morphisms satisfying \( \sigma_V(\eta \otimes \text{Id}_V) = \text{Id}_{\alpha \otimes V} \) and \( (\epsilon \otimes \text{Id}_V) \tau_V = \text{Id}_{\beta \otimes V} \).

**Proof.** First, we note \( t_v \) is \( \mathbb{k} \)-linear and the morphisms \( \sigma_V \) and \( \tau_V \) exist because \( V \) is assumed to lie in \( I^\beta_\alpha \). Next, let \( \sigma_V \) and \( \tau_V \) be any such morphisms. Then,

\[
\langle \tau_V(f) \rangle_\eta = \langle \tau_V(f \sigma_V(\eta \otimes \text{Id}_V)) \rangle_\eta = \langle \tau_V(f \sigma_V) \rangle_\epsilon
\]

where the fourth equality comes from Lemma 4.3 part (3). Thus, \( t_v(f) \) is independent of the choice of \( \sigma_V \) or \( \tau_V \).

Next we show this family of functions satisfies the partial trace property. Let \( U \in I^\beta_\alpha, W \in \text{Ob}(\mathcal{C}) \) and \( f \in \text{Hom}_\mathcal{C}(\alpha \otimes U \otimes W, \beta \otimes U \otimes W) \). Since \( U \in I^\beta_\alpha \) there exists \( \tau_U : \beta \otimes U \rightarrow P \otimes U \) such that \( (\epsilon \otimes \text{Id}_U) \tau_U = \text{Id}_{\beta \otimes U} \). Choose \( \tau_U \otimes W \) to be equal to \( \tau_V \otimes \text{Id}_W \) then \( (\epsilon \otimes \text{Id}_{U \otimes W}) \tau_{U \otimes W} = \text{Id}_{\beta \otimes U \otimes W} \). Therefore, we can use \( \tau_{U \otimes W} \) to define \( t_{U \otimes W} \) and we see

\[
t_{U \otimes W}(f) = \langle \tau_{U \otimes W}(f) \rangle_\eta = \langle \tau_{U \otimes W}((\tau_U \otimes \text{Id}_W)f) \rangle_\eta = \langle \tau_U((\tau_U \otimes \text{Id}_W)f) \rangle_\eta = t_U((\tau_U \otimes \text{Id}_W)(f)).
\]

To prove the \((\alpha, \beta)\)-cyclicity property, let \( f : \alpha \otimes V \rightarrow \beta \otimes U \) and \( g : U \rightarrow V \). Then,

\[
t_V((\text{Id}_\beta \otimes g)f) = \langle \tau_V((\text{Id}_\beta \otimes g)f \sigma_V) \rangle_\epsilon = \langle \tau_V(f \sigma_V(\text{Id}_\beta \otimes g)) \rangle_\epsilon
\]

where the first equality comes from the definition of the trace, the second from the properties of the pivotal structure, the third from the definition of \( I^\beta_\alpha \), and the last from Lemma 4.3 part (3). \( \square \)
4.3. **Uniqueness.** It is of particular interest when one or both of \( \alpha \) and \( \beta \) are the unit object. In this case we have the following uniqueness result.

**Theorem 4.5.** Suppose \((P, \alpha, \beta, \eta, \epsilon)\) is a trace tuple with \( P \in I^2_\alpha \).

1. If \( \beta = 1 \), then the right \((\alpha, 1)\)-trace on \( I^1_\alpha \) is unique up to a scalar. Specifically, if \( t' \) is a right \((\alpha, 1)\)-trace on \( I^1_\alpha \) and \( t \) is the right \((\alpha, 1)\)-trace defined by Theorem 4.4, then
   \[
   t' = t'_P(\eta \otimes \epsilon) t.
   \]
   Moreover, \( t_P(\eta \otimes \epsilon) = 1 \).

2. If \( \alpha = 1 \), then the right \((1, \beta)\)-trace on \( I^1_\beta \) is unique up to a scalar. Specifically, if \( t' \) is a right \((1, \beta)\)-trace on \( I^1_\beta \) and \( t \) is the right \((1, \beta)\)-trace defined by Theorem 4.4, then
   \[
   t' = t'_P(\epsilon \otimes \eta) t.
   \]
   Moreover, \( t_P(\epsilon \otimes \eta) = 1 \).

**Proof.** Let \( t' \) be right \((\alpha, 1)\)-trace on \( I^1_\alpha \). If \( f \in \text{Hom}_P(\alpha \otimes V, V) \) then
\[
t'_V(f) = t'_V((\epsilon \otimes Id_V)\tau_V f) = t'_P(\tau_V f)(\eta \otimes \epsilon) = t'_P(\epsilon \otimes \eta) = t_P(\epsilon \otimes \eta).
\]
The first equality comes from the fact that \((\epsilon \otimes Id_V)\tau_V = Id_V \) by definition of \( I^1_\alpha \); the second from strictness and \((\alpha, 1)\)-cyclicity, the third from the partial trace property and the last two from the definitions of the \( \eta \)-bracket and the trace \( t \), respectively.

Finally, using the same properties along with Lemma 4.3 yields
\[
t_P(\eta \otimes \epsilon) = (t'_P(\tau_P(\eta \otimes \epsilon)))_\eta = (\langle Id_P \otimes \epsilon \rangle \tau_P \eta)_\eta = (\langle \epsilon \otimes \epsilon \rangle \tau_P)_\epsilon = (\epsilon)_\epsilon = 1.
\]

The proof of the second statement is entirely analogous.

For short, when \( \beta = 1 \) we say a right (resp. left) \((\alpha, 1)\)-trace on a right (resp. left) ideal \( I \) is a right (resp. left) \( \alpha \)-trace on \( I \).

4.4. **A handy lemma.** The following lemma will be useful in what follows.

**Lemma 4.6.** Let \( t \) be the right trace associated to a trace tuple \((P, \alpha, \beta, \eta, \epsilon)\) as in Theorem 4.4 and let \( V \) be an object in \( I^2_\alpha \). Then:

1. For any \( f \in \text{Hom}_P(P \otimes V, \beta \otimes V) \), one has \( t_V(f(\eta \otimes Id_V)) = \langle tr^V_P(f) \rangle_\epsilon \).
2. For any \( g \in \text{Hom}_P(\alpha \otimes V, P \otimes V) \), one has \( t_V((\epsilon \otimes Id_V)g) = \langle tr^V_P(g) \rangle_\eta \).
3. For any \( h \in \text{Hom}_P(P \otimes V, P \otimes V) \), \( t_V((\epsilon \otimes Id_V)h(\eta \otimes Id_V)) = \langle tr^V_P(h) \rangle_\eta \).

**Proof.** From definitions and Lemma 4.3 we have
\[
t_V(f(\eta \otimes Id_V)) = \langle tr^V_P(\tau_V f(\eta \otimes Id_V)) \rangle_\eta = \langle tr^V_P(\tau_V f) \rangle_\eta = \langle \epsilon tr^V_P(\tau_V f) \rangle_\epsilon = \langle tr^V_P(f) \rangle_\epsilon.
\]
Similarly,
\[
t_V((\epsilon \otimes Id_V)g) = \langle tr^V_P((\epsilon \otimes Id_V)g) \rangle_\epsilon = \langle \epsilon tr^V_P(g) \rangle_\epsilon = \langle tr^V_P(g) \rangle_\eta.
\]
The third statement is proven in a similar fashion.
4.5. **Left and right compatibility.** If the reader formulates the theory for left ideals and m-traces, then the result is compatible with the right version, as we next explain.

**Definition 4.7.** Let $t\mathbf{^l}$ be a left $(\alpha, \beta)$-trace on $I^l$ and $t\mathbf{^r}$ be a right $(\alpha, \beta)$-trace on $I^r$. We say that $t\mathbf{^l}$ and $t\mathbf{^r}$ are compatible if for any $(V, W) \in I^l \times I^r$, and for any $f \in \text{Hom}_\mathcal{C}(V \otimes \alpha \otimes W, V \otimes \beta \otimes W)$,

$$t_{\mathbf{^r}}(\text{tr}_V^V(f)) = t_{\mathbf{^l}}(\text{tr}_W^W(f)).$$

**Proposition 4.8.** The left and right trace associated to a trace tuple $(P, \alpha, \beta, \eta, \epsilon)$ are compatible.

**Proof.** Since $V$ is in the left ideal $L_\alpha$ and $W$ is in the right ideal $I^\beta$, there exists $\sigma_V : V \otimes P \rightarrow V \otimes \alpha$ and $\tau_W : \beta \otimes W \rightarrow P \otimes W$ such that

$$f = (\text{Id}_V \otimes \epsilon \otimes \text{Id}_W)(\text{Id}_V \otimes \tau_W)(\text{Id}_V \otimes \eta \otimes \text{Id}_W).$$

Then by Lemma 4.6 we have

$$t_{\mathbf{^r}}(\text{tr}_V^V(f)) = \langle \text{tr}_V^V(\text{tr}_W^W((\text{Id}_V \otimes \tau_W)f(\sigma_V \otimes \text{Id}_W)))) \rangle = t_{\mathbf{^l}}(\text{tr}_V^V(f)).$$

\[\square\]

5. **Examples**

In addition to the known examples of unimodular m-traces in the literature (e.g. [GKP11, GKP13, BBG18, GR17]), we have the following non-unimodular m-traces.

5.1. **The toy example.** Let $S$ be an absolutely irreducible object in $\mathcal{C}$ and set $P = \alpha = \beta = S$, and let $\epsilon : P \rightarrow \beta$ and $\eta : \alpha \rightarrow P$ be the identity maps. Then $(P, \alpha, \beta, \eta, \epsilon)$ is a trace tuple and $I_\alpha^\beta = \mathcal{C}$. Also, the $(\alpha, \beta)$-trace of Theorem 4.4 is given by $t_V(f) = \langle \text{tr}_V^V(f) \rangle$ for all $f \in \text{Hom}_\mathcal{C}(S \otimes V, S \otimes V)$.

5.2. **Quantized enveloping algebras.** In this subsection let $k = \mathbb{C}(q)$, where $q$ is an indeterminate. We follow standard conventions without elaboration. The reader may consult [Jan03, Jan96, CP94] for further details. Let $\mathfrak{g}$ be a complex semisimple Lie algebra and let $\mathfrak{b} \subseteq \mathfrak{g}$ be a fixed choice of Cartan and Borel subalgebras, respectively. Let $U_q(\mathfrak{b}) \subseteq U_q(\mathfrak{g})$ be the corresponding quantized enveloping algebras over $k$.

We order elements of the weight lattice, $\Lambda$, using the usual dominance order determined by our choice of $\mathfrak{b}$. If $L$ is a finite-dimensional simple $U_q(\mathfrak{g})$-module then there is a unique maximal nonzero weight space. Let $\lambda \in \Lambda$ be the highest weight and let $L_\lambda$ denote the corresponding 1-dimensional $\lambda$-weight space. Similarly, $L$ has a unique lowest nonzero weight space, $L_\alpha$. By restriction we can view $L$ as a $U_q(\mathfrak{b})$-module, and then $L_\lambda$ is the simple socle, $L_\alpha$ is the simple head, and $L$ is cyclically generated by $L_\alpha$. In particular, $L$ is an absolutely indecomposable $U_q(\mathfrak{b})$-module.

In short, the previous paragraph shows the canonical projection and inclusion maps $\epsilon : L \rightarrow L_\alpha$ and $\eta : L_\lambda \rightarrow L$ make $(L, L_\lambda, L_\alpha, \epsilon, \eta)$ into a trace tuple in the category of finite-dimensional $U_q(\mathfrak{b})$-modules (which is known to be a pivotal $k$-tensor category). A similar example holds in the non-quantum case as well.

Now let $U_\zeta(\mathfrak{g})$ be the restricted specialization of the quantized enveloping algebra at $\zeta \in \mathbb{C}$, a primitive, odd $\ell$th root of unity. We assume $\ell$ is greater than the Coxeter number for $\mathfrak{g}$ and is not divisible by 3 if $\mathfrak{g}$ has a direct summand of type $G_2$. For a dominant integral $\lambda \in \Lambda$, let $H_\zeta^\beta(\lambda)$, $V_\zeta(\lambda)$, and $T_\zeta(\lambda)$ denote the induced,
Weyl, and tilting $U_\zeta(g)$-modules of highest weight $\lambda$. Then $H_\zeta^0(\lambda)$ and $V_\zeta(\lambda)$ are absolutely irreducible and $T_\zeta(\lambda)$ is absolutely indecomposable. Furthermore, since $\text{Hom}_{U_\zeta(g)}(V_\zeta(\lambda), T(\lambda)) = \mathbb{C}$ and $\text{Hom}_{U_\zeta(g)}(T_\zeta(\lambda), H_\zeta^0(\lambda)) = \mathbb{C}$, there are maps $\eta : V_\zeta(\lambda) \to T(\lambda)$ and $\epsilon : T_\zeta(\lambda) \to H_\zeta^0(\lambda)$ which make $(T_\zeta(\lambda), H_\zeta^0(\lambda), V_\zeta(\lambda), \epsilon, \eta)$ into a trace tuple. A parallel example exists for semisimple algebraic groups over an algebraically closed field.

5.3. Projective objects. In this section $k$ is assumed to be an algebraically closed field and $\mathcal{C}$ is a locally-finite, pivotal, $k$-tensor category. In particular, in $\mathcal{C}$ every simple object is absolutely simple by Schur’s Lemma and every indecomposable object is absolutely indecomposable and end-nilpotent by Fitting’s Lemma. As remarked in Subsection 2.3 such categories include a wide range of examples. This subsection implies these examples all admit unique nontrivial right $m$-traces.

Lemma 5.1. If $P$ is an indecomposable projective object in $\mathcal{C}$, then there are unique absolutely irreducible objects $\alpha$ and $\beta$ and morphisms $\epsilon : P \to \beta$ and $\eta : \alpha \to P$ such that $(P, \alpha, \beta, \eta, \epsilon)$ is a trace tuple.

Proof. If $P$ is an indecomposable projective, then it is absolutely indecomposable, end-nilpotent, and has an irreducible head $\beta$ and irreducible socle $\alpha$ by Section 2.3. Set $\epsilon : P \to \beta$ and $\eta : \alpha \to P$ to be the canonical projection and inclusion, respectively. Then $(P, \alpha, \beta, \eta, \epsilon)$ is a trace tuple. \hfill $\square$

Recall $\mathcal{P}\text{proj}(\mathcal{C})$ denotes the ideal of projective objects in $\mathcal{C}$.

Lemma 5.2. Let $(P, \alpha, \beta, \eta, \epsilon)$ be an arbitrary trace tuple. Then, $\mathcal{P}\text{proj}(\mathcal{C}) \subseteq I_\alpha^\beta$. In particular, if $\mathcal{C}$ contains a projective object, then $I_\alpha^\beta$ is nonempty.

Proof. Let $Q$ be a projective object in $\mathcal{C}$. Then the morphism $\epsilon \otimes \text{id}_Q : P \otimes Q \to \beta \otimes Q$ is an epimorphism. Since $Q$ is projective and $\mathcal{P}\text{proj}(\mathcal{C})$ is an ideal, it follows $\beta \otimes Q$ is projective, and so the morphism $\epsilon \otimes \text{id}_Q$ splits. Therefore, $Q$ is an object of $I_\beta^\alpha$. Similarly, $\eta \otimes \text{id}_Q : \alpha \otimes Q \to P \otimes Q$ is a monomorphism and $\alpha \otimes Q$ is projective (hence injective), so the morphism $\eta \otimes \text{id}_Q$ again splits and $Q$ is an object of $I_\alpha^\beta$. Taken together this shows $Q \in I_\alpha^\beta$. \hfill $\square$

Lemma 5.3. Given any trace tuple $(P, \alpha, \beta, \eta, \epsilon)$ where $P$ is projective and either $\alpha$ or $\beta$ is invertible. Then, $I_\alpha^\beta = \mathcal{P}\text{proj}(\mathcal{C})$.

Proof. We do only the case when $\beta$ is invertible as the other case is similar. Let $V \in I_\beta^\alpha$. Then by definition $\epsilon \otimes \text{id}_V : P \otimes V \to \beta \otimes V$ splits. But $P$ is in the ideal $\mathcal{P}\text{proj}(\mathcal{C})$, so $\beta \otimes V$ is in $\mathcal{P}\text{proj}(\mathcal{C})$ and, hence, $\beta^* \otimes \beta \otimes V \cong V$ is an object of $\mathcal{P}\text{proj}(\mathcal{C})$. The reverse inclusion is given by the previous lemma. \hfill $\square$

Note, if $S$ is a absolutely irreducible, projective object, then the toy example of Subsection 5.1 shows the previous result could fail if there are no assumptions on $\alpha$ and $\beta$.

The following theorem summarizes the outcome of the previous lemmas.

Theorem 5.4. If $P$ is an indecomposable projective object $\mathcal{C}$, then there are unique absolutely irreducible objects $\alpha$ and $\beta$ and morphisms $\epsilon : P \to \beta$ and $\eta : \alpha \to P$ such that $(P, \alpha, \beta, \eta, \epsilon)$ is a trace tuple and $\mathcal{P}\text{proj}(\mathcal{C}) \subseteq I_\alpha^\beta$. Moreover, if either $\alpha$ or $\beta$ is invertible, then $\mathcal{P}\text{proj}(\mathcal{C}) = I_\alpha^\beta$. 

The next result demonstrates the m-trace defined by the previous result combined with Theorem 4.4 is nontrivial. Specifically, given $Q \in \mathcal{P}roj(\mathscr{C})$, one can choose $V$ (since it is arbitrary) so that $\text{Hom}_\mathscr{C}(\alpha \otimes Q, V)$ is nontrivial. Consequently, for any $Q \in \mathcal{P}roj(\mathscr{C})$ the next theorem shows both $\text{Hom}_\mathscr{C}(\alpha \otimes Q, \beta \otimes Q)$ and $t_Q$ are nonzero.

**Theorem 5.5.** Let $(\mathcal{P}, \alpha, \beta, \eta, \epsilon)$ be the trace tuple given by an absolutely indecomposable projective $\mathcal{P}$ as in the previous theorem. Let $t$ be the right trace given by Theorem 4.4. Then for any $Q \in \mathcal{P}roj(\mathscr{C}) \subseteq I^\beta_\mathcal{P}$ and $V \in \mathscr{C}$ the map

$$\text{Hom}_\mathscr{C}(V, \beta \otimes Q) \times \text{Hom}_\mathscr{C}(\alpha \otimes Q, V) \to \mathbb{k} \text{ given by } (g, f) \mapsto t_Q(gf)$$

is a non-degenerate pairing.

**Proof.** From Lemma 5.2 $\mathcal{P}roj(\mathscr{C}) \subseteq I^\beta_\mathcal{P}$ so the function exists. We next show its right kernel is trivial (the proof for the left kernel is similar). If $f \in \text{Hom}_\mathscr{C}(\alpha \otimes Q, V)$ is not zero, then $f^* = (f \otimes \text{Id}_Q)(\text{Id}_\alpha \otimes \text{coev}_Q) \in \text{Hom}_\mathscr{C}(\alpha, V \otimes Q^*)$ is a non zero map from $\alpha$ to the projective object $V \otimes Q^*$. Since projective covers (hence injective envelopes) are unique, $\mathcal{P}$ is the unique indecomposable projective object with $\alpha$ as a subobject, the map $f'$ factors through an indecomposable summand of $V \otimes Q^*$ which is isomorphic to $\mathcal{P}$. That is, there are morphisms $\iota : \mathcal{P} \to V \otimes Q^*$ and $p : V \otimes Q^* \to \mathcal{P}$ such that $\iota = \text{Id}_\mathcal{P}$ and $f' = \iota \eta$.

Let $g \in \text{Hom}_\mathscr{C}(V, \beta \otimes Q)$ be given by $g = (\epsilon \otimes \text{Id}_Q)(p \otimes \text{Id}_Q)(\text{Id}_V \otimes \text{coev}_Q)$. Then $gf = (\epsilon \otimes \text{Id}_Q)f''$ where $f'' \in \text{Hom}_\mathscr{C}(\alpha \otimes Q, \mathcal{P} \otimes Q)$ is given by $f'' = (p \otimes \text{Id}_Q)(\text{Id}_V \otimes \text{coev}_Q)f$. The first of the following equalities holds by Lemma 4.6

$$t_Q(gf) = \langle \text{tr}_Q(f''') \rangle_\eta = \langle pf' \rangle_\eta = \langle pu \rangle_\eta = \langle \eta \rangle_\eta = 1.$$ 

Combining Theorems 5.4 and 4.5 with the previous result immediately yields the following corollary. Also, there exists an analogous unique, nontrivial left m-trace on $\mathcal{P}roj(\mathscr{C})$.

**Corollary 5.6.** Let $\mathbb{k}$ be an algebraically closed field and $\mathscr{C}$ be a locally-finite, pivotal, $\mathbb{k}$-tensor category which has enough projectives. Let $\mathcal{P}$ be the projective cover of $\mathbb{1}$ and let $\alpha$ be the socle of $\mathcal{P}$. This data determines a unique (up to scalar), nontrivial right $\alpha$-trace on $\mathcal{P}roj(\mathscr{C})$.

### 5.4. Ambidextrous objects

In earlier work the authors introduced the notion of a right ambidextrous object and the associated right m-trace. We now explain how that construction is a special case of the one introduced here. Let $\mathscr{C}$ be a ribbon category, $S$ be an absolutely irreducible object, and let $\epsilon = \text{ev} : S \otimes S^* \to \mathbb{1}$ and $\eta = \text{coev} : \mathbb{1} \to S \otimes S^*$. Let $S \otimes S^* = \bigoplus_i W_i$ be the decomposition of $S$ into indecomposable objects. Then $S$ is right ambidextrous in the sense of [GKP13] if and only if there is an $i$ such that the restriction of $\epsilon$ and $\eta$ to $P := W_i$ makes $(P, \mathbb{1}, \mathbb{1}, \epsilon, \eta)$ into a trace tuple. In which case $I^\beta_P$ equals the ideal generated by $S$ and the m-trace defined here agrees with the one defined therein.

### 6. Twisted Calabi-Yau Categories

In this section we continue to assume $\mathbb{k}$ is a field and $\mathscr{C}$ is a $\mathbb{k}$-linear category.
6.1. Twisted Calabi-Yau Categories. Next we introduce the notion of a twisted Calabi-Yau category.

Definition 6.1. Let $F, G : \mathcal{C} \rightarrow \mathcal{C}$ be fixed endofunctors of a category, $\mathcal{C}$. Then $\mathcal{C}$ is an $(F, G)$-twisted Calabi-Yau category if it is equipped with a family of $k$-linear maps

$$\{ t_U : \text{Hom}_\mathcal{C}(F(U), G(U)) \rightarrow k \}_{U \in \mathcal{C}}$$

such that the following properties hold:

1. **Non-degeneracy.** For any objects $U, V$ in $\mathcal{C}$, the pairing

$$\text{Hom}_\mathcal{C}(V, G(U)) \times \text{Hom}_\mathcal{C}(F(U), V) \rightarrow k \text{ given by } (g, f) \mapsto t_U(gf)$$

is non-degenerate.

2. **Cyclicity.** For any objects $U, V$ in $\mathcal{C}$ and any morphisms $f : F(V) \rightarrow G(U)$ and $g : U \rightarrow V$ in $\mathcal{C}$, we have

$$t_V(gf) = t_U(fg).$$

The non-degeneracy condition provides a canonical vector space isomorphism,

$$\text{Hom}_\mathcal{C}(F(U), V) \cong \text{Hom}_\mathcal{C}(V, G(U))^*,$$

which is functorial in both $U$ and $V$.

This notion generalizes existing constructions. For example, a $(\text{Id}_\mathcal{C}, \text{Id}_\mathcal{C})$-twisted Calabi-Yau category is nothing but a Calabi-Yau category. If a category $\mathcal{C}$ is a $(\text{Id}_\mathcal{C}, G)$-twisted Calabi-Yau category, then $G$ is a right Serre functor in the sense of Bondal-Kapranov [BK89].

In the special case when $\mathcal{C}$ is a category with a single object, $*$, then being an $(F, G)$-twisted Calabi-Yau category is equivalent to having a $k$-linear map $t : \text{End}_\mathcal{C}(*) \rightarrow k$ which satisfies $t(g(ab)) = t(g(a)b)$ for fixed algebra endomorphisms $f, g : \text{End}_\mathcal{C}(*) \rightarrow \text{End}_\mathcal{C}(*)$ along with the requirement the induced pairing $(a, b) \mapsto t(ab)$ be nondegenerate. In this way, it generalizes the well known fact that a Calabi-Yau structure on a category with a single object is equivalent to the notion of a symmetric Frobenius algebra. As a special case, if $g$ is the identity endomorphism, then exactly have $\text{End}_\mathcal{C}(*)$ is a Frobenius extension of $k$ in the sense of Morita [Mor65] (or see [PS10] for a modern treatment).

6.2. A twisted Calabi-Yau structure on $\mathcal{P}\text{roj}(\mathcal{C})$. In this section we assume $k$ is an algebraically closed field and $\mathcal{C}$ is a locally-finite, pivotal, $k$-tensor category with enough projectives.

If $X$ is an fixed object of $\mathcal{C}$, then we write $F_X$ for the endofunctor $X \otimes -$. If $P$ is an indecomposable projective object in $\mathcal{C}$, then there is the corresponding trace tuple $(P, \alpha, \beta, \eta, \epsilon)$ and right $(\alpha, \beta)$-trace, $t$, on $\mathcal{P}\text{roj}(\mathcal{C})$ given by Theorem 5.4. Combining this with Theorem 5.5 yields the following result.

Theorem 6.2. Let $P$ be an indecomposable projective object in $\mathcal{C}$. The corresponding trace tuple $(P, \alpha, \beta, \eta, \epsilon)$ and right $(\alpha, \beta)$-trace, $t$, makes $\mathcal{P}\text{roj}(\mathcal{C})$ into an $(F_\alpha, F_\beta)$-twisted Calabi-Yau category.

As an application, if we take $P$ to be the injective hull of $1$ and $\beta$ is the simple head of $P$, then $\mathcal{P}\text{roj}(\mathcal{C})$ is an $(\text{Id}_\mathcal{C}, F_\beta)$-twisted Calabi-Yau category and, hence, $F_\beta$ is a right Serre functor on $\mathcal{P}\text{roj}(\mathcal{C})$. We also have the following special case of the previous theorem. Recently Gainutdinov-Runkel [GR17] obtained the same result under the assumption $\mathcal{C}$ is finite and factorisable.
Corollary 6.3. Assume $\mathcal{C}$ is a locally-finite, pivotal, unimodular, $k$-tensor category with enough projectives. Then $\text{Proj} (\mathcal{C})$ is a Calabi-Yau category.

We end by noting the following generalization of the single object example from the previous section. If $\mathcal{C}$ is a finite tensor category as in [EGNO15], then it has finitely many indecomposable projectives $P_1, \ldots, P_t$ and $Q = \bigoplus_{i=1}^t P_i$ is a projective generator. Let $P_0$ be the projective cover of $I$ and let $\alpha$ be the socle of $P_0$. Then $\alpha$ is invertible, $\alpha \otimes Q \cong Q$, and the right $\alpha$-trace on $\text{Proj} (\mathcal{C})$ defines a $k$-linear map $t : \text{End} \circ (Q) \to k$ which makes $\text{End} \circ (Q)$ a Frobenius extension of $k$.

References

[AS17] N. Andruskiewitsch and C. Schweigert. On unrolled Hopf algebras. ArXiv e-prints, December 2017, 1701.00153.

[BBG17] A. Beliakova, C. Blanchet, and N. Geer. Logarithmic Hennings invariants for restricted quantum $\mathfrak{sl}(2)$. ArXiv e-prints, May 2017, 1705.09083.

[BBG18] A. Beliakova, C. Blanchet, and A. M. Gainutdinov. Modified trace is a symmetrised integral. ArXiv e-prints, December 2018, 1801.0021.

[BCGP16] Christian Blanchet, Francesco Costantino, Nathan Geer, and Bertrand Patureau-Mirand. Non-semi-simple TQFTs, Reidemeister torsion and Kaschaev’s invariants. Adv. Math., 301:1–78, 2016. doi:10.1016/j.aim.2016.06.003.

[BK89] A. I. Bondal and M. M. Kapranov. Representable functors, Serre functors, and reconstructions. Izv. Akad. Nauk SSSR Ser. Mat., 53(6):1183–1205, 1337, 1989. doi:10.1070/IM1990v035n03ABEH000716.

[BKN12] Brian D. Boe, Jonathan R. Kujawa, and Daniel K. Nakano. Complexity for modules over the classical Lie superalgebra $gl(m|n)$. Compos. Math., 148(5):1561–1592, 2012. doi:10.1112/S0010437X12000231.

[BW99] John W. Barrett and Bruce W. Westbury. Spherical categories. Adv. Math., 143(2):357–375, 1999. doi:10.1006/aima.1998.1800.

[CG17] T. Creutzig and T. Gannon. Logarithmic conformal field theory, log-modular tensor categories and modular forms. Journal of Physics A Mathematical General, 50:404004, October 2017, 1605.04630. doi:10.1088/1751-8121/aa8538.

[CGP14] Francesco Costantino, Nathan Geer, and Bertrand Patureau-Mirand. Quantum invariants of 3-manifolds via link surgery presentations and non-semi-simple categories. $J.$ Topol., 7(4):1005–1053, 2014. doi:10.1112/jtopol/jtu006.

[CGPT18] Francesco Costantino, Nathan Geer, Bertrand Patureau-Mirand, and Vladimir Turaev. Kuperberg and Turaev-Viro invariants in unimodular pivotal tensor categories. In preparation, 2018.

[CH17] Jonathan Comes and Thorsten Heidersdorf. Thick ideals in Deligne’s category $\text{Rep} (O_h)$. J. Algebra, 480:237–265, 2017. doi:10.1016/j.jalgebra.2017.01.050.

[CK12] Jonathan Comes and Jonathan R. Kujawa. Modified traces on Deligne’s category $\text{Rep} (S_t)$. J. Algebraic Combin., 36(4):541–560, 2012. doi:10.1007/s10801-012-0349-1.

[CMR16] T. Creutzig, A. Milas, and M. Rupert. Logarithmic link invariants of $U_q (\mathfrak{sl}_2)$ and asymptotic dimensions of singlet vertex algebras. ArXiv e-prints, May 2016, 1605.05634.

[Com14] Jonathan Comes. Ideals in Deligne’s category $\text{Rep} (GL_\delta)$. Math. Res. Lett., 21(5):969–984, 2014. doi:10.4310/MRL.2014.v21.n5.a4.

[CP94] Vojvodic Chari and Andrew Pressley. A guide to quantum groups. Cambridge University Press, Cambridge, 1994.

[DGP17] M. De Renzi, N. Geer, and B. Patureau-Mirand. Renormalized Hennings Invariants and 2+1-TQFTs. ArXiv e-prints, July 2017, 1707.08043.

[EGNO15] Pavel Etingof, Shlomo Gelaki, Dmitri Nikshych, and Victor Ostrik. Tensor categories, volume 205 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2015. doi:10.1090/surv/205.

[FG] A. Fontalvo Orozco and A.M. Gainutdinov Module traces and Hopf group-coalgebras. Preprint.
[Gab03] Matthias R. Gaberdiel. An algebraic approach to logarithmic conformal field theory. In Proceedings of the School and Workshop on Logarithmic Conformal Field Theory and its Applications (Tehran, 2001), volume 18, pages 4593–4638, 2003. doi:10.1142/S0217751X03016860.

[GKP11] Nathan Geer, Jonathan Kujawa, and Bertrand Patureau-Mirand. Generalized trace and modified dimension functions on ribbon categories. Selecta Math. (N.S.), 17(2):453–504, 2011. doi:10.1007/s00029-010-0046-7.

[GKP13] Nathan Geer, Jonathan Kujawa, and Bertrand Patureau-Mirand. Ambidextrous objects and trace functions for nonsemisimple categories. Proc. Amer. Math. Soc., 141(9):2963–2978, 2013. doi:10.1090/S0002-9939-2013-11563-7.

[GPT09] Nathan Geer, Bertrand Patureau-Mirand, and Vladimir Turaev. Modified quantum dimensions and re-normalized link invariants. Compos. Math., 145(1):196–212, 2009. doi:10.1112/S0010437X08003795.

[GPV13] Nathan Geer, Bertrand Patureau-Mirand, and Alexis Virelizier. Traces on ideals in pivotal categories. Quantum Topol., 4(1):91–124, 2013. doi:10.4171/QT/36.

[GR17] A. M. Gainutdinov and I. Runkel. Projective objects and the modified trace in factorizable finite tensor categories. ArXiv e-prints, March 2017, 1703.00150.

[Jan96] Jens Carsten Jantzen. Lectures on quantum groups, volume 6 of Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 1996.

[Jan03] Jens Carsten Jantzen. Representations of algebraic groups, volume 107 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, second edition, 2003.

[Kas95] Christian Kassel. Quantum groups, volume 155 of Graduate Texts in Mathematics. Springer-Verlag, New York, 1995. doi:10.1007/978-1-4612-0783-2.

[Kra15] Henning Krause. Krull-Schmidt categories and projective covers. Expo. Math., 33(4):535–549, 2015. doi:10.1016/j.exmath.2015.10.001.

[Len17] S. D. Lentner. The unrolled quantum group inside Lusztig’s quantum group of divided powers. ArXiv e-prints, February 2017, 1702.05164.

[Mor65] Kiiti Morita. Adjoint pairs of functors and Frobenius extensions. Sci. Rep. Tokyo Kyoiku Daigaku Sect. A, 9:40–71 (1965), 1965.

[Mur17] Jun Murakami. Generalized Kashaev invariants for knots in three manifolds. Quantum Topol., 8(1):35–73, 2017. doi:10.4171/QT/86.

[NS07] Siu-Hung Ng and Peter Schauenburg. Higher Frobenius-Schur indicators for pivotal categories. In Hopf algebras and generalizations, volume 441 of Contemp. Math., pages 63–90. Amer. Math. Soc., Providence, RI, 2007. doi:10.1090/conm/441/08500.

[Phu16] N. Phu Ha. Topological invariants from quantum group $U_q(gl(2|1))$ at roots of unity. ArXiv e-prints, July 2016, 1607.03728.

[PS16] Jeffrey Pike and Alistair Savage. Twisted Frobenius extensions of graded superrings. Algebr. Represent. Theory, 19(1):113–133, 2016. doi:10.1007/s10468-015-9565-1.

[Rup16] M. Rupert. Logarithmic Hopf link invariants for the unrolled restricted quantum group of $sl(2)$. Master’s thesis, The school of the thesis, The address of the publisher, 7 2016.

[Shi14] K. Shimizu. On unimodular finite tensor categories. ArXiv e-prints, February 2014, 1402.3482.

[Shi17] K. Shimizu. Integrals for finite tensor categories. ArXiv e-prints, February 2017, 1702.02425.