On Simpson’s Rule and Fractional Brownian Motion with \( H = 1/10 \)

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Abstract We consider stochastic integration with respect to fractional Brownian motion (fBm) with \( H < 1/2 \). The integral is constructed as the limit, where it exists, of a sequence of Riemann sums. A theorem by Gradinaru et al. (Ann Inst Henri Poincaré Probab Stat 41(4):781–806, 2005) holds that a sequence of Simpson’s rule Riemann sums converges in probability for a sufficiently smooth integrand \( f \) and when the stochastic process is fBm with \( H > 1/10 \). For the case \( H = 1/10 \), we prove that the sequence of sums converges in distribution. Consequently, we have an Itô-like formula for the resulting stochastic integral. The convergence in distribution follows from a Malliavin calculus theorem that first appeared in Nourdin and Nualart (J Theor Probab 23:39–64, 2010).

Keywords Itô formula · Skorohod integral · Malliavin calculus · Fractional Brownian motion

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1 Introduction

Let \( B = \{ B_t^H, t \geq 0 \} \) be a fractional Brownian motion (fBm), that is, \( B \) is a centered Gaussian process with covariance given by

\[
\mathbb{E} [B_s B_t] := R(s, t) = \frac{1}{2} \left( s^{2H} + t^{2H} - |t - s|^{2H} \right), \tag{1}
\]

for \( s, t \geq 0 \), where \( H \in (0, 1) \) is the Hurst parameter. For a smooth function \( f : \mathbb{R} \to \mathbb{R} \), we take the ‘Simpson’s rule’ Riemann sum with uniform partition,

\[
S_n^S(t) := \sum_{j=0}^{\lfloor nt \rfloor - 1} \frac{1}{6} \left( f' \left( \frac{B_{j} + B_{j+1}}{2n} \right) + 4 f' \left( \frac{B_{j} + B_{j+1}}{n} \right) / 2 \right) + f' \left( \frac{B_{j+1}}{n} \right) \left( B_{j+1} - B_{j} \right).
\]

It can be shown (see [3], or Sect. 3.1) that this sequence of sums converges in probability when \( B \) is fBm with \( H > 1/10 \), but in general it does not converge in probability when \( H \leq 1/10 \). In this paper, we consider the particular case of \( H = 1/10 \), and show that \( S_n^S(t) \) does converge weakly to a random variable. More precisely, Theorem 3.3 shows that, conditioned on the path \( \{ B_s, s \leq t \} \),

\[
S_n^S(t) \overset{\mathcal{L}}{\longrightarrow} f(B_t) - f(0) + \frac{\beta}{2880} \int_0^t f^{(5)}(B_s) dW_s, \tag{2}
\]

where \( W_t \) is a standard Brownian motion, independent of \( B \), and \( \beta \) is a constant defined in Theorem 3.3. This result allows us to write the change-of-variable formula

\[
f(B_t) \overset{\mathcal{L}}{=} f(0) + \int_0^t f'(B_s) dS B_s - \frac{\beta}{2880} \int_0^t f^{(5)}(B_s) dW_s, \tag{3}
\]

where the differential \( dS B_s \) denotes the limit of the Simpson’s rule sum.

Conditional convergence in distribution follows from a central limit theorem given in Sect. 2 (Theorem 2.3). This is a new version of a theorem that first appeared in Nourdin and Nualart [6]. This theorem uses Malliavin calculus, and applies to a random vector with components in the form of Malliavin divergence integrals. After proving Theorem 2.3, the main task in proving (3) is to verify the conditions of Theorem 2.3, which are relatively long and technical.

1.1 Background

Assuming a uniform partition, the classical Stratonovich stochastic integral is defined as
\[
\int_0^t f'(B_s) \, d B_s = \lim_{n \to \infty} S_n^T(t) := \lim_{n \to \infty} \sum_{j=0}^{[nt]-1} \frac{1}{2} \left( f' \left( B_{\frac{j}{n}} \right) + f' \left( B_{\frac{j+1}{n}} \right) \right) \left( B_{\frac{j+1}{n}} - B_{\frac{j}{n}} \right),
\]  

(4)

provided that limit exists. It has been shown that this limit exists in probability when \( B \) is a fBm with \( H > 1/6 \), but does not, in general converge in probability for \( H \leq 1/6 \) (see [2,3,8], also Sect. 3.1). Subsequently, it was proved in [8] that for \( H = 1/6 \), (4) does converge in law to a random variable that includes a Wiener–Itô integral, that is, as \( n \to \infty \)

\[
S_n^T(t) \xrightarrow{L} f(B_t) - f(0) + \gamma \int_0^t f^{(3)}(B_s) \, d W_s,
\]

where \( \gamma \) is a known constant and \( W \) is a standard Brownian motion, independent of \( B \). Hence, there is the change-of-variable formula

\[
f(B_t) \overset{L}{=} f(0) + \int_0^t f'(B_s) \, d B_s - \gamma \int_0^t f^{(3)}(B_s) \, d W_s.
\]  

(5)

The reader will recognize that (4) is the Riemann sum corresponding to the ‘Trapezoidal rule’ of basic calculus. It is certainly possible to generalize to other types of Riemann sums. The ‘Midpoint’ sum,

\[
\sum_{j=1}^{[\frac{nt}{2}]} f' \left( B_{\frac{2j-1}{2n}} \right) \left( B_{\frac{2j}{2n}} - B_{\frac{2j-2}{2n}} \right),
\]

can be shown to converge in probability for fBm with \( H > 1/4 \) (see [11]). The end point case \( H = 1/4 \) was considered in papers by Burdzy and Swanson [1], and Nourdin and Réveillac [7]. These papers proved the change-of-variable formula

\[
f(B_t) \overset{L}{=} f(0) + \int_0^t f'(B_s) \, d^* B_s + \theta \int_0^t f''(B_s) \, d W_s,
\]  

(6)

where \( \theta \) is a constant, \( W \) is a scaled Brownian motion, independent of \( B \), and the notation \( d^* B_s \) denotes the integral arising from the midpoint sum.

1.2 Extensions

Following the results (5) and (6), the present authors also wrote papers on the cases \( H = 1/4 \) and \( H = 1/6 \) [4,5]. These papers contained alternate proofs of (6) and (5),
using Malliavin calculus and a version of Theorem 2.3. An interesting difference in
the present paper, is that the sum $S_n^3(t)$ converges conditionally to a random variable
that is actually the sum of two, independent Gaussian random variables. In the cases
considered in [4,5], there was only a single random term. In those prior papers, we
also showed that the results could be extended to other Gaussian processes sufficiently
similar to fBm, for example, bifractional Brownian motion with $HK = 1/6$ in the case
of (5). It was also shown that the Midpoint and Trapezoidal Riemann sums converge
as functions in the Skorohod space $D[0, \infty)$, by proving that the sums converge in the
sense of finite-dimensional distributions. We expect that similar extensions could be
applied to the present Theorem 3.3, but we have not pursued this in the present paper.

We also expect that the techniques of this paper could be applied to the ‘Milne’s
rule’ sum for the case $H = 1/14$, see Proposition 3.1.

The organization of this paper is as follows: in Sect. 2, we give a brief description
of the Malliavin calculus definitions and identities that will be used. We also discuss
properties of fBm, and prove the central limit theorem which will be applied for the
main result. In Sect. 3, after a brief introduction we state and prove the main result,
which is Theorem 3.3. Finally, Sect. 4 contains proofs of three of the longer lemmas
from Sect. 3.

2 Notation and Theory

Let $f : \mathbb{R} \to \mathbb{R}$ be a function and $N$ be a Gaussian random variable with mean
zero and variance $\sigma^2$. We say that $f$ satisfies moderate growth conditions if there
exist constants $A$, $B$, and $\alpha < 2$ such that $|f(x)| \leq A e^{B|x|^\alpha}$. Note that this implies
$\mathbb{E} [|f(N)|^p] < \infty$ for all $p \geq 1$. We use the symbol $1_{[0,1]}$ to denote the indicator
function for a real interval $[0, t]$. The symbol $C$ denotes a generic positive constant,
which may vary from line to line. In general, the value of $C$ will depend on and the
growth conditions of a test function $f$ and the properties of a stochastic process $B$.

2.1 Elements of Malliavin Calculus

Following is a brief description of some identities that will be used in the paper. The
reader may refer to [9] for detailed coverage of this topic. Let $Z = \{Z(h), h \in \mathcal{H}\}$ be
an isonormal Gaussian process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and indexed by a
real separable Hilbert space $\mathcal{H}$. That is, $Z$ is a family of Gaussian random variables
such that $\mathbb{E}[Z(h)] = 0$ and $\mathbb{E}[Z(h)Z(g)] = \langle h, g \rangle_\mathcal{H}$ for all $h, g \in \mathcal{H}$. We will assume
that $\mathcal{F}$ is the $\sigma$-algebra generated by $Z$.

For integers $q \geq 1$, let $\mathcal{H}^{\otimes q}$ denote the $q$th tensor product of $\mathcal{H}$, and $\mathcal{H}^{\otimes q}$ denote
the subspace of symmetric elements of $\mathcal{H}^{\otimes q}$. We will also use the notation $\bigotimes_{i=1}^{q} h_i$
to denote an arbitrary tensor product, with the convention that $\bigotimes_{i=1}^{0}$ is the empty set.

Let $\{e_n, n \geq 1\}$ be a complete orthonormal system in $\mathcal{H}$. For functions $f, g \in \mathcal{H}^{\otimes q}$
and $p \in \{0, \ldots, q\}$, we define the $p$th-order contraction of $f$ and $g$ as that element of
$\mathcal{H}^{\otimes (q-p)}$ given by

\[ \otimes_{i=1}^{\frac{q-p}{2}} (f \circ g) \]
\[ f \otimes_p g = \sum_{i_1, \ldots, i_p = 1}^{\infty} \langle f, e_{i_1} \otimes \cdots \otimes e_{i_p} \rangle_{\mathcal{H} \otimes p} \otimes \langle g, e_{i_1} \otimes \cdots \otimes e_{i_p} \rangle_{\mathcal{H} \otimes p} \]  

(7)

where \( f \otimes_0 g = f \otimes g \) and \( f \otimes_q g = \langle f, g \rangle_{\mathcal{H} \otimes q} \). While \( f, g \) are symmetric, the contraction \( f \otimes_q g \) may not be. We denote its symmetrization by \( f \tilde{\otimes}_q g \).

Let \( \mathcal{H}_q \) be the \( q \)th Wiener chaos of \( Z \), that is, the closed linear subspace of \( L^2(\Omega) \) generated by the random variables \( \{ H_q(Z(h)), h \in \mathcal{H}, \| h \|_{\mathcal{H}} = 1 \} \), where \( H_q(x) \) is the \( q \)th Hermite polynomial, defined as

\[ H_q(x) = (-1)^q e^{\frac{x^2}{2}} \frac{d^q}{dx^q} e^{-\frac{x^2}{2}}. \]

and we follow the convention of Hermite polynomials with unity as a leading coefficient. For \( q \geq 1 \), it is known that the map

\[ I_q(h \otimes^q) = H_q(Z(h)) \]  

(8)

provides a linear isometry between \( \mathcal{H} \otimes^q \) (equipped with the modified norm \( \sqrt{q!} \cdot \| \cdot \|_{\mathcal{H} \otimes^q} \)) and \( \mathcal{H}_q \), where \( I_q(\cdot) \) is the generalized Wiener–Itô multiple stochastic integral. By convention, \( \mathcal{H}_0 = \mathbb{R} \) and \( I_0(x) = x \). It follows from (8) and the properties of the Hermite polynomials that for \( f \in \mathcal{H} \otimes^p \), \( g \in \mathcal{H} \otimes^q \) we have

\[ \mathbb{E} [I_p(f) I_q(g)] = \begin{cases} 
  p! \langle f, g \rangle_{\mathcal{H} \otimes^p} & \text{if } p = q \\
  0 & \text{otherwise} 
\end{cases} \]  

(9)

Let \( S \) be the set of all smooth and cylindrical random variables of the form \( F = g(Z(\phi_1), \ldots, Z(\phi_n)) \), where \( n \geq 1 \); \( g : \mathbb{R}^n \to \mathbb{R} \) is an infinitely differentiable function with compact support, and \( \phi_i \in \mathcal{H} \). The Malliavin derivative of \( F \) with respect to \( Z \) is the element of \( L^2(\Omega; \mathcal{H}) \) defined as

\[ DF = \sum_{i=1}^{n} \frac{\partial g}{\partial x_i}(Z(\phi_1), \ldots, Z(\phi_n)) \phi_i. \]

By iteration, for any integer \( q > 1 \) we can define the \( q \)th derivative \( D^q F \), which is an element of \( L^2(\Omega; \mathcal{H} \otimes^q) \).

We let \( \mathbb{D}^{q,2} \) denote the closure of \( S \) with respect to the norm \( \| \cdot \|_{\mathbb{D}^{q,2}} \) defined as

\[ \| F \|_{\mathbb{D}^{q,2}}^2 = \mathbb{E} [F^2] + \sum_{i=1}^{q} \mathbb{E} [\| D^i F \|_{\mathcal{H} \otimes^i}^2]. \]

More generally, for any Hilbert space \( V \), let \( \mathbb{D}^{k,p}(V) \) denote the corresponding Sobolev space of \( V \)-valued random variables.

We denote by \( \delta \) the Skorohod integral, which is defined as the adjoint of the operator \( D \). A random element \( u \in L^2(\Omega; \mathcal{H}) \) belongs to the domain of \( \delta \), \( \text{Dom} \delta \), if and only if,
\[ |\mathbb{E}\left[(DF, u)_{\mathcal{H}}\right]| \leq c_u \|F\|_{L^2(\Omega)} \]

for any \( F \in \mathbb{D}^{1,2} \), where \( c_u \) is a constant which depends only on \( u \). If \( u \in \text{Dom} \, \delta \), then the random variable \( \delta(u) \in L^2(\Omega) \) is defined for all \( F \in \mathbb{D}^{1,2} \) by the duality relationship,

\[ \mathbb{E}\left[F \delta(u)\right] = \mathbb{E}\left[\langle DF, u \rangle_{\mathcal{H}}\right]. \]

This is sometimes called the Malliavin integration by parts formula. We iteratively define the multiple Skorohod integral for \( q \geq 1 \) as \( \delta^{q-1}(u) \), with \( \delta^0(u) = u \). For this definition we have,

\[ \mathbb{E}\left[F \delta^q(u)\right] = \mathbb{E}\left[\langle D^q F, u \rangle_{\mathcal{H}_{\otimes q}}\right], \tag{10} \]

where \( u \in \text{Dom} \, \delta^q \) and \( F \in \mathbb{D}^{q,2} \). The adjoint operator \( \delta^q \) is an integral in the sense that for a (non-random) \( h \in \mathcal{H}_{\otimes q} \), we have \( \delta^q(h) = I_q(h) \).

The following results will be used extensively in this paper. The reader may refer to [6,9] for proofs and details.

**Lemma 2.1** Let \( q \geq 1 \) be an integer, and \( r, j, k > 0 \) be integers.

(a) Assume \( F \in \mathbb{D}^{q,2} \), \( u \) is a symmetric element of \( \text{Dom} \, \delta^q \), and \( \langle D^r F, \delta^j(u) \rangle_{\mathcal{H}_{\otimes r}} \in L^2(\Omega; \mathcal{H}_{\otimes q-r-j}) \) for all \( 0 \leq r + j \leq q \). Then \( \langle D^r F, u \rangle_{\mathcal{H}_{\otimes r}} \in \text{Dom} \, \delta^r \) and

\[ F \delta^q(u) = \sum_{r=0}^{q} \binom{q}{r} \delta^{q-r} \left( \langle D^r F, u \rangle_{\mathcal{H}_{\otimes r}} \right). \]

(b) Suppose that \( u \) is a symmetric element of \( \mathbb{D}^{j+k,2}(\mathcal{H}_{\otimes j}) \). Then we have,

\[ D^k \delta^j(u) = \sum_{i=0}^{j \wedge k} i! \binom{k}{i} \binom{j}{i} \delta^{j-i} \left( D^{k-i} u \right). \]

(c) Meyer Inequality: Let \( p > 1 \) and integers \( k \geq q \geq 1 \). Then for any \( u \in \mathbb{D}^{k,p}(\mathcal{H}_{\otimes q}) \),

\[ \| \delta^q(u) \|_{\mathbb{D}^{k-q,p}} \leq c_{k,p} \| u \|_{\mathbb{D}^{k,p}(\mathcal{H}_{\otimes q})}, \]

where \( c_{k,p} \) is a constant.

(d) Let \( u \in \mathcal{H}_{\otimes p} \) and \( v \in \mathcal{H}_{\otimes q} \). Then

\[ \delta^p(u) \delta^q(v) = \sum_{z=0}^{p \wedge q} z! \binom{p}{z} \binom{q}{z} \delta^{p+q-2z} \left( u \otimes_z v \right), \]

where \( \otimes_z \) is the contraction operator defined in (7).
2.2 A Convergence Theorem

**Definition 2.2** Assume \( F_n \) is a sequence of \( d \)-dimensional random variables defined on a probability space \((\Omega, \mathcal{F}, P)\), and \( F \) is a \( d \)-dimensional random variable defined on \((\Omega, \mathcal{G}, P)\), where \( \mathcal{F} \subseteq \mathcal{G} \). We say that \( F_n \) converges stably to \( F \) as \( n \to \infty \), if, for any continuous and bounded function \( f : \mathbb{R}^d \to \mathbb{R} \) and \( \mathcal{F} \)-measurable random variable \( M \), we have

\[
\lim_{n \to \infty} \mathbb{E}(f(F_n)M) = \mathbb{E}(f(F)M).
\]

The first version of the following central limit theorem appeared in [6]. In [4], we extended this to a multi-dimensional version, where the sequence was a vector of \( d \) components all in the same Wiener chaos. For our present paper, we need a slight modification. In this version, we lay out conditions for stable convergence of a sequence of vectors, where the vector components are not necessarily in the same Wiener chaos.

**Theorem 2.3** Let \( d \geq 1 \) be an integer, and \( q_1, \ldots, q_d \) be positive integers with \( q^* = \max\{q_1, \ldots, q_d\} \). Suppose that \( F_n \) is a sequence of random variables in \( \mathbb{R}^d \) of the form

\[
F_n = \left[ \delta_{q_1}^{(u_{1n})}, \ldots, \delta_{q_d}^{(u_{dn})} \right],
\]

where each \( u_{in} \) is a \( \mathbb{R} \)-valued symmetric function in \( D^{2q_1 \cdot 2q_i(q_i)}(\mathcal{H} \otimes_q) \). Suppose that the sequence \( F_n \) is bounded in \( L^1(\Omega) \) and that:

(a) \( \left\langle u^i_n, \bigotimes_{\ell=1}^m (D^{a_i} F^{j_i}_n) \bigotimes h_{\mathcal{H} \otimes q} \right\rangle \) converges to zero in \( L^1(\Omega) \) for all integers \( 1 \leq j, j' \leq d, \) all integers \( 1 \leq a_1, \ldots, a_m, r \leq q_j - 1 \) such that \( a_1 + \cdots + a_m + r = q_j \); and all \( h_{\mathcal{H} \otimes q} \).

(b) For each \( 1 \leq i, j \leq d \), \( \left\langle u^i_n, D^{q_i} F^j_n \bigotimes_{\mathcal{H} \otimes q} \right\rangle \) converges in \( L^1(\Omega) \) to a nonnegative random variable \( s^2_i \), and for \( i \neq j \), \( \left\langle u^i_n, D^{q_i} F^j_n \bigotimes_{\mathcal{H} \otimes q} \right\rangle \) converges to zero in \( L^1(\Omega) \).

Then \( F_n \) converges stably to a random vector in \( \mathbb{R}^d \), whose components each have independent Gaussian law \( N(0, s^2_i) \) given \( Z \).

**Proof** This proof mostly follows that given in [4], except in that case there was only a single value of \( q \). We use the conditional characteristic function. Given any \( h_1, \ldots, h_m \in \mathcal{H} \), we want to show that the sequence

\[
\xi_n = \left( F^1_n, \ldots, F^d_n, Z(h_1), \ldots, Z(h_m) \right)
\]

converges in distribution to a vector \( (F^1_\infty, \ldots, F^d_\infty, Z(h_1), \ldots, Z(h_m)) \), where, for any vector \( \lambda \in \mathbb{R}^d \), \( F_\infty \) satisfies

\[
\mathbb{E}\left( e^{i\lambda^T F_\infty | Z(h_1), \ldots, Z(h_m)} \right) = \exp\left(-\frac{1}{2} \lambda^T S \lambda \right),
\]

where \( S \) is the diagonal \( d \times d \) matrix with entries \( s^2_i \).
Since $F_n$ is bounded in $L^1(\Omega)$, the sequence $\xi_n$ is tight in the sense that for any $\varepsilon > 0$, there is a $K > 0$ such that $P\{F_n \in [-K, K]^d\} > 1 - \varepsilon$, which follows from Chebyshev’s inequality. Dropping to a subsequence if necessary, we may assume that $\xi_n$ converges in distribution to a limit $(F_\infty^1, \dots, F_\infty^d, Z(h_1), \ldots, Z(h_m))$. Let $Y := g (Z(h_1), \ldots, Z(h_m))$, where $g \in C^\infty_b(\mathbb{R}^m)$, and consider $\phi_n(\lambda) := \mathbb{E} \left[ e^{i\lambda^T F_n Y} \right]$ for $\lambda \in \mathbb{R}^d$. The convergence in law of $\xi_n$ implies that for each $1 \leq j \leq d$:

$$
\lim_{n \to \infty} \frac{\partial \phi_n}{\partial \lambda_j} = \lim_{n \to \infty} i \mathbb{E} \left[ F_n^j e^{i\lambda^T F_n Y} \right] = i \mathbb{E} \left[ F_\infty^j e^{i\lambda^T F_\infty Y} \right],
$$

(12)

where convergence in distribution follows from a truncation argument applied to $F_n^j$.

On the other hand, using the duality property of the Skorohod integral and the Malliavin derivative:

$$
\frac{\partial \phi_n}{\partial \lambda_j} = i \mathbb{E} \left[ D_{q_j} \left( u_n^j \right) e^{i\lambda^T F_n Y} \right] = i \mathbb{E} \left( \left[ u_n^j, D_{q_j} \left( e^{i\lambda^T F_n Y} \right) \right]_{\mathcal{H}^{q_j}} \right) = i \sum_{a=0}^{q_j-1} \sum_{\alpha=0}^{q_j} \left( \begin{array}{c} q_j \\ a \end{array} \right) \mathbb{E} \left[ \left[ u_n^j, D_{q_j} e^{i\lambda^T F_n \otimes D_{q_j-a} Y} \right]_{\mathcal{H}^{q_j}} \right]
$$

(13)

By condition (a), we have that $\left[ u_n^j, D_{q_j} e^{i\lambda^T F_n \otimes D_{q_j-a} Y} \right]_{\mathcal{H}^{q_j}}$ converges to zero in $L^1(\Omega)$ when $a < q_j$, so the sum term vanishes as $n \to \infty$, and this leaves

$$
\lim_{n \to \infty} i \mathbb{E} \left[ u_n^j, Y D_{q_j} e^{i\lambda^T F_n} \right]_{\mathcal{H}^{q_j}} = \lim_{n \to \infty} i \sum_{k=1}^d \mathbb{E} \left[ i \lambda_k e^{i\lambda^T F_n} \left[ u_n^j, Y D_{q_j} F_n^k \right]_{\mathcal{H}^{q_j}} \right]
$$

$$
= -\mathbb{E} \left[ \lambda_j e^{i\lambda^T F_\infty} s_j^2 Y \right]
$$

because the lower-order derivatives in $D_{q_j} e^{i\lambda^T F_n}$ also vanish by condition (a), and cross terms ($j \neq k$) terms vanish by condition (b). Combining this with (12), we obtain:

$$
i \mathbb{E} \left[ F_\infty^j e^{i\lambda^T F_\infty Y} \right] = -\lambda_j \mathbb{E} \left[ e^{i\lambda^T F_\infty} s_j^2 Y \right].
$$

This leads to the PDE system:

$$
\frac{\partial}{\partial \lambda_j} \mathbb{E} \left( e^{i\lambda^T F_\infty} | Z(h_1), \ldots, Z(h_m) \right) = -\lambda_j s_j^2 \mathbb{E} \left( e^{i\lambda^T F_\infty} | Z(h_1), \ldots, Z(h_m) \right)
$$

which has unique solution (11).
Remark 2.4 It suffices to impose condition (a) for \( h \in S_0 \), where \( S_0 \) is a total subset of \( \mathcal{H}^{\otimes r} \).

Remark 2.5 Suppose \( F_n \) is the vector sequence \( (F_n, G_n) \), where \( F_n = \delta^p (u_n) \) and \( G_n = \delta^q (v_n) \). Then to satisfy Theorem 2.3, \( F_n \) and \( G_n \) must be bounded in \( L^1(\Omega) \), and the following terms must tend to zero in \( L^1(\Omega) \):

1. \( \langle u_n, h \rangle_{\mathcal{H}^{\otimes p}} + \langle u_n, g \rangle_{\mathcal{H}^{\otimes q}} \), for arbitrary \( h \in \mathcal{H}^{\otimes p} \) and \( g \in \mathcal{H}^{\otimes q} \), respectively.
2. \( \langle u_n, \bigotimes_{i=1}^s D^{a_i} F_n \bigotimes_{i=s+1}^r D^{a_i} G_n \otimes h \rangle_{\mathcal{H}^{\otimes p}} \), where \( 0 \leq a_i < p \) and \( a_1 + \cdots + a_r < p \), and \( h \in \mathcal{H}^{\otimes p-(a_1+\cdots+a_r)} \); and \( \langle u_n, \bigotimes_{i=1}^s D^{a_i} F_n \bigotimes_{i=s+1}^r D^{a_i} G_n \rangle_{\mathcal{H}^{\otimes q}} \), where \( 0 \leq a_i < q \) and \( a_1 + \cdots + a_r = q \).
3. \( \langle v_n, \bigotimes_{i=1}^s D^{a_i} F_n \bigotimes_{i=s+1}^r D^{a_i} G_n \otimes h \rangle_{\mathcal{H}^{\otimes q}} \), where \( 0 \leq a_i < q \) and \( a_1 + \cdots + a_r < q \), and \( h \in \mathcal{H}^{\otimes q-(a_1+\cdots+a_r)} \); and \( \langle v_n, \bigotimes_{i=1}^s D^{a_i} F_n \bigotimes_{i=s+1}^r D^{a_i} G_n \rangle_{\mathcal{H}^{\otimes q}} \), where \( 0 \leq a_i < q \) and \( a_1 + \cdots + a_r = q \).
4. \( \langle u_n, D^p G_n \rangle_{\mathcal{H}^{\otimes p}} \) and \( \langle v_n, D^q F_n \rangle_{\mathcal{H}^{\otimes q}} \).

Then for condition (b), the following two terms must converge in \( L^1(\Omega) \) to nonnegative random variables: \( \langle u_n, D^p F_n \rangle_{\mathcal{H}^{\otimes p}} \) and \( \langle v_n, D^q G_n \rangle_{\mathcal{H}^{\otimes q}} \).

2.3 Fractional Brownian Motion

For some \( T > 0 \), let \( B = \{B^H_t, 0 \leq t \leq T\} \) be a fractional Brownian motion with Hurst parameter \( H \). That is, \( B \) is a centered Gaussian process with covariance \( R(s, t) \) given in (1). Let \( \mathcal{E} \) denote the set of \( \mathbb{R} \)-valued step functions on \([0, T]\). We then let \( \mathfrak{H} \) be the Hilbert space defined as the closure of \( \mathcal{E} \) with respect to the inner product

\[
\langle 1_{[0,s]}, 1_{[0,t]} \rangle_{\mathfrak{H}} = R(s, t).
\]

The mapping \( 1_{[0,t]} \mapsto B_t \) can be extended to a linear isometry between \( \mathfrak{H} \) and the Gaussian space spanned by \( B \). In this way, \( \{B(h), h \in \mathfrak{H}\} \) is an isonormal Gaussian process as in Sect. 2.1.

For an integer \( n \geq 2 \), we consider a uniform partition of \([0, \infty)\) given by \( \{j/n, j \geq 1\} \). Define the following notation:

- \( \Delta B_{j/n} = B_{j+1/n} - B_j \), and \( \overline{B}_{j/n} = \frac{1}{2} \left( B_{j/n} + B_{j+1/n} \right) \)
- \( \partial_{j/n} = 1_{[j/n, j+1/n]} \), \( \varepsilon_{j/n} = 1_{[0,j]} \), and \( \varepsilon_{j/n} = \frac{1}{2} \left( 1_{[0,j/n]} + 1_{[0,j+1/n]} \right) = \varepsilon_{j/n} + \frac{1}{2} \partial_{j/n} \).

Assume \( H < 1/2 \). The following fBm properties follow from (1).

(B.1) \( \mathbb{E} \left[ \Delta B_{j/n}^2 \right] = \left\langle \partial_{j/n}, \partial_{j/n} \right\rangle_{\mathfrak{H}} = n^{-2H} \).

(B.2) \( \mathbb{E} \left[ \Delta B_{j/n} \Delta B_{i/n} \right] = \left\langle \partial_{j/n}, \partial_{i/n} \right\rangle_{\mathfrak{H}} = (2^{2H} - 2)/2n^{2H} \).

(B.3) If \( |k - j| \geq 2 \), \( \mathbb{E} \left[ \Delta B_{j/n} \Delta B_{k/n} \right] = \left| \left\langle \partial_{j/n}, \partial_{k/n} \right\rangle_{\mathfrak{H}} \right| \leq C n^{-2H} |j - k|^{2H-2} \), where the constant \( C \) does not depend on \( j \).
For each \( j \geq 0 \), \( \sup_{t \in [0,T]} \left| \mathbb{E} \left[ \Delta B^j_t \right] \right| \leq 2n^{-2H} \).

For any \( t \in [0,T] \) and integer \( j \geq 1 \),

\[
\left| \mathbb{E} \left[ \Delta B^j_t \right] \right| = \left| \mathbb{E} \left[ \Delta B^j_t \right] \right| \leq Cn^{-2H} (j^{2H-1} + |j - nt|^{2H-1}).
\]

In particular, if \( |j - k| \geq 2 \),

\[
\left| \mathbb{E} \left[ \Delta B^j_t \right] \right| = \left| \mathbb{E} \left[ \Delta B^j_t \right] \right| \leq Cn^{-2H} (j^{2H-1} + |j - k|^{2H-1}).
\]

As a result of properties (B.1) – (B.5), we have the following technical results.

**Lemma 2.6** Let \( H < 1/2 \) and \( 0 < t \leq T \), and let \( n \geq 2 \) be an integer. Then

(a) For fixed \( 0 \leq s \leq T \) and integer \( r \geq 1 \),

\[
\sum_{j=0}^{\lfloor nt \rfloor-1} \left| \left\langle \partial_j s, \varepsilon_s \right\rangle \right|^{r} \leq Cn^{-2(r-1)H}.
\]

(b) For integer \( r \geq 1 \),

\[
\sum_{j=0}^{\lfloor nt \rfloor-1} \left| \left\langle \partial_j s, \tilde{\varepsilon}_j \right\rangle \right|^{r} \leq Cn^{-2(r-1)H}.
\]

(c) For integers \( r \geq 1 \) and \( 0 \leq k \leq \lfloor nt \rfloor \),

\[
\sum_{j=0}^{\lfloor nt \rfloor-1} \left| \left\langle \partial_j s, \partial_k s \right\rangle \right|^{r} \leq Cn^{-2rH},
\]

and consequently

\[
\sum_{j,k=0}^{\lfloor nt \rfloor-1} \left| \left\langle \partial_j s, \partial_k s \right\rangle \right|^{r} \leq C\lfloor nt \rfloor n^{-2rH}.
\]

**Proof** For (a), first note that we have \( \left| \langle \partial_0, \varepsilon_t \rangle \right| \leq T^H n^{-H} \) by (B.1) and Cauchy–Schwarz. Further, if \( \left| \frac{j}{n} - s \right| < \frac{2}{n} \), then by (B.4) we have \( \left| \langle \partial_j, \varepsilon_s \rangle \right| \leq Cn^{-2H} \). Let \( J = \{ 1 \leq j \leq \lfloor nt \rfloor, |j - ns| > 1 \} \); and note that \( |J^c| \leq 2 \). Then for the case \( r = 1 \) we have

\[
\sum_{j=0}^{\lfloor nt \rfloor-1} \left| \left\langle \partial_j s, \varepsilon_s \right\rangle \right| \leq \left| \langle \partial_0, \varepsilon_t \rangle \right| + \sum_{j \in J^c} \left| \left\langle \partial_j s, \varepsilon_s \right\rangle \right| + \sum_{j \in J} \left| \left\langle \partial_j s, \varepsilon_s \right\rangle \right| \leq T^H n^{-H} + Cn^{-2H} + \sum_{j=1}^{\lfloor nt \rfloor-1} j^{2H-1} + |j - ns|^{2H-1} \leq C\lfloor nt \rfloor^{2H} n^{-2H} \leq C.
\]
For the case $r > 1$, we have by (B.4)

$$\sum_{j=0}^{\lfloor nt \rfloor - 1} \left| \left\langle \frac{\partial}{\partial \varepsilon}, \frac{\partial}{\partial \varepsilon} \right\rangle \right| \leq \sup_{0 \leq j \leq \lfloor nt \rfloor} \left| \left\langle \frac{\partial}{\partial \varepsilon}, \frac{\partial}{\partial \varepsilon} \right\rangle \right| \sum_{j=0}^{\lfloor nt \rfloor - 1} \left| \left\langle \frac{\partial}{\partial \varepsilon}, \frac{\partial}{\partial \varepsilon} \right\rangle \right| \leq Cn^{-2(r-1)H}.$$

For (b), we have by (B.4) and (1)

$$\sum_{j=0}^{\lfloor nt \rfloor - 1} \left| \left\langle \frac{\partial}{\partial \varepsilon}, \frac{\partial}{\partial \varepsilon} \right\rangle \right| \leq \sup_{0 \leq j \leq \lfloor nt \rfloor} \left| \left\langle \frac{\partial}{\partial \varepsilon}, \frac{\partial}{\partial \varepsilon} \right\rangle \right| \sum_{j=0}^{\lfloor nt \rfloor - 1} \left| \left\langle \frac{\partial}{\partial \varepsilon}, \frac{\partial}{\partial \varepsilon} \right\rangle \right|$$

$$\leq Cn^{-2(r-1)H} \sum_{j=0}^{\lfloor nt \rfloor - 1} \frac{1}{2} \mathbb{E} \left[ \Delta B_{\frac{j}{n}} (B_{\frac{j}{n}} + B_{\frac{j+1}{n}}) \right]$$

$$= Cn^{-2(r-1)H} \sum_{j=0}^{\lfloor nt \rfloor - 1} \frac{1}{2} \mathbb{E} \left[ B_{\frac{j+1}{n}}^2 - B_{\frac{j}{n}}^2 \right]$$

$$= Cn^{-2(r-1)H} \sum_{j=0}^{\lfloor nt \rfloor - 1} \frac{1}{2} \left[ \left( \frac{j+1}{n} \right)^{2H} - \left( \frac{j}{n} \right)^{2H} \right]$$

$$\leq Cn^{-2(r-1)H} \frac{n^{1-1}}{n} \leq Cn^{-2(r-1)H}.$$

For (c), we note that $\left| \left\langle \frac{\partial}{\partial j/n}, \frac{\partial}{\partial 0} \right\rangle \right| = \left| \left\langle \frac{\partial}{\partial j/n}, \frac{\partial}{\partial \varepsilon/n} \right\rangle \right| \leq n^{-2H}$. Also note that by (B.1) and Cauchy–Schwarz we have $\left| \left\langle \frac{\partial}{\partial j/n}, \frac{\partial}{\partial k/n} \right\rangle \right| \leq n^{-2H}$ for any $1 \leq j, k \leq \lfloor nt \rfloor$. To begin the proof, we consider the case when $1 \leq k \leq \lfloor nt \rfloor - 1$ is fixed. Then

$$\sum_{j=0}^{\lfloor nt \rfloor - 1} \left| \left\langle \frac{\partial}{\partial \varepsilon}, \frac{\partial}{\partial \varepsilon} \right\rangle \right| \leq \sup_{0 \leq j \leq \lfloor nt \rfloor} \left\{ \sup_{0 \leq k \leq \lfloor nt \rfloor} \left| \left\langle \frac{\partial}{\partial \varepsilon}, \frac{\partial}{\partial \varepsilon} \right\rangle \right| \right\} \sum_{j=0}^{\lfloor nt \rfloor - 1} \left| \left\langle \frac{\partial}{\partial \varepsilon}, \frac{\partial}{\partial \varepsilon} \right\rangle \right|$$

$$\leq n^{-2(r-1)H} \left( n^{-2H} + \sum_{j=1}^{k-2} \left| \left\langle \frac{\partial}{\partial \varepsilon}, \frac{\partial}{\partial \varepsilon} \right\rangle \right| + \sum_{j=k+1}^{k+1} \left| \left\langle \frac{\partial}{\partial \varepsilon}, \frac{\partial}{\partial \varepsilon} \right\rangle \right| + \sum_{j=k+2}^{\lfloor nt \rfloor - 1} \left| \left\langle \frac{\partial}{\partial \varepsilon}, \frac{\partial}{\partial \varepsilon} \right\rangle \right| \right)$$

Then we use (B.2) and (B.3) to write

$$n^{-2(r-1)H} \left( n^{-2H} + \sum_{j=1}^{k-2} \left| \left\langle \frac{\partial}{\partial \varepsilon}, \frac{\partial}{\partial \varepsilon} \right\rangle \right| + \sum_{j=k+1}^{k+1} \left| \left\langle \frac{\partial}{\partial \varepsilon}, \frac{\partial}{\partial \varepsilon} \right\rangle \right| + \sum_{j=k+2}^{\lfloor nt \rfloor - 1} \left| \left\langle \frac{\partial}{\partial \varepsilon}, \frac{\partial}{\partial \varepsilon} \right\rangle \right| \right)$$

$$\leq n^{2(r-1)H} \left( n^{-2H} + Cn^{-2H} \sum_{j=1}^{k-2} (k-j)^{2H-2} + \sum_{j=k+1}^{k+1} n^{-2H} \right).$$
\[ + Cn^{-2H} \sum_{j=k+2}^{\lfloor nt \rfloor - 1} (j - k)^{2H-2} \]
\[ \leq Cn^{-2rH} \left( 4 + 2 \sum_{m=1}^{\infty} m^{2H-2} \right) \leq Cn^{-2rH}, \]

where we note the sum is finite because \( H < 1/2 \). For the double sum result we have

\[ \sum_{j,k=0}^{\lfloor nt \rfloor - 1} \left| \langle \partial_{\frac{\pi j}{n}}, \partial_{\frac{\pi k}{n}} \rangle^r \right| \leq \frac{1}{n} \sum_{k=0}^{\lfloor nt \rfloor - 1} \sup_{0 \leq k \leq \lfloor nt \rfloor} \left\{ \sum_{j=0}^{\lfloor nt \rfloor - 1} \left| \langle \partial_{\frac{\pi j}{n}}, \partial_{\frac{\pi k}{n}} \rangle^r \right| \right\} \leq C\lfloor nt \rfloor n^{-2rH}. \]

\[ \square \]

3 Results

3.1 Some Results for fBm with \( H > 1/14 \)

The following proposition summarizes some known results about stochastic integrals with respect to fBm, when the integrals arise from a Riemann sum construction. A comprehensive treatment can be found in an important paper by Gradinaru et al. [3].

**Proposition 3.1** Let \( g \in C^\infty(\mathbb{R}) \), such that \( g \) and its derivatives have moderate growth. The following Riemann sums converge in probability as \( n \to \infty \) to \( g(B_t) - g(0) \) for the given ranges of \( H \):

(a) **Midpoint rule**: for \( 1/6 < H < 1/2 \),

\[ \sum_{j=0}^{\lfloor nt \rfloor - 1} \frac{1}{2} \left( g' \left( \frac{B_j + B_{j+1}}{n} \right) \right) \Delta B_{\frac{j}{n}}, \]

where \( \overline{B}_{\frac{j}{n}} = \frac{1}{2} \left( \frac{B_j + B_{j+1}}{n} \right) \).

(b) **Trapezoidal rule**: For \( 1/6 < H < 1/2 \),

\[ \sum_{j=0}^{\lfloor nt \rfloor - 1} \frac{1}{2} \left( g' \left( \frac{B_j}{n} \right) + g' \left( \frac{B_{j+1}}{n} \right) \right) \Delta B_{\frac{j}{n}}. \]

(c) **Simpson’s rule**: For \( 1/10 < H < 1/2 \),

\[ \sum_{j=0}^{\lfloor nt \rfloor - 1} \frac{1}{6} \left( g' \left( \frac{B_j}{n} \right) + 4g' \left( \frac{B_j + B_{j+1}}{n} \right) + g' \left( \frac{B_{j+1}}{n} \right) \right) \Delta B_{\frac{j}{n}}. \]
(d) **Milne’s rule:** For $1/14 < H < 1/2$, 

$$
\sum_{j=0}^{\lfloor nt \rfloor - 1} \frac{1}{90} \left( 7g' \left( \frac{B_j}{n} \right) + 32g' \left( \frac{B_j}{n} + \frac{1}{4} \Delta B_j \right) + 12g' \left( \frac{\tilde{B}_j}{n} \right) \right) + 32g' \left( \frac{B_j}{n} + \frac{3}{4} \Delta B_j \right) + 12g' \left( \frac{B_{j+1}}{n} \right) \Delta B_j.
$$

Note that the ‘midpoint’ sum of part (a) is a different construction than that leading to (6). All of these results follow from Theorem 4.4 of [3], in fact they are also proved there for $H \geq 1/2$. However, here we give a different proof of part (c). By similar techniques, results (a), (b) and (d) could also be done in this way. This proof will contain some results that will be used in Sect. 3.2, and help set up the proof of Theorem 3.3.

We begin with a technical result. The proof of Lemma 3.2 is deferred to Sect. 4 due to length.

**Lemma 3.2** Let $r = 1, 3, 5, \ldots$ and $n \geq 2$ be an integer. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a $C^{2r}$ function such that $\phi$ and all derivatives up to order $2r$ have moderate growth, and let $\{B_t, t \geq 0\}$ be fBm with Hurst parameter $H$. Then for each $r$, there is a constant $C > 0$ such that

$$
\mathbb{E} \left[ \left( \sum_{j=0}^{\lfloor nt \rfloor - 1} \phi \left( \frac{\tilde{B}_j}{n} \right) \Delta B_j \right)^2 \right] \leq C \sup_{0 \leq j \leq \lfloor nt \rfloor} \left\| \phi \left( \frac{\tilde{B}_j}{n} \right) \right\|_{L^{2r,2}}^2 \lfloor nt \rfloor n^{-2rH},
$$

where $C$ depends on $r$ and $H$.

Now for the convergence of the Simpson’s rule sum. We begin with some elementary results from the calculus of deterministic functions. For $x, h \in \mathbb{R}$ and a $C^\infty$ function $g$, we have the following integral form for the Simpson’s rule sum:

$$
g(x + h) - g(x - h) = \int_{-h}^{h} g'(x + u) \, du \\
= \frac{h}{3} \left( g'(x - h) + 4g'(x) + g'(x + h) \right) \\
+ \frac{1}{6} \int_{0}^{h} \left( g^{(4)}(x - u) - g^{(4)}(x + u) \right) u(h - u)^2 \, du.
$$

See Talman [12] for a nice discussion of the Simpson’s rule error term. Next, we consider a Taylor expansion of order 7 for $g^{(4)}$. 

\( \text{Springer} \)
\[ g^{(4)}(x + u) - g^{(4)}(x) = \sum_{\ell=1}^{6} \frac{g^{(4+\ell)}(x)}{\ell!} u^\ell + \frac{g^{(11)}(\xi)}{7!} u^7; \text{ and} \]

\[ g^{(4)}(x) - g^{(4)}(x - u) = \sum_{\ell=1}^{6} \frac{(-1)^{\ell+1} g^{(4+\ell)}(x)}{\ell!} u^\ell + \frac{g^{(11)}(\eta)}{7!} u^7. \]

Adding the above equations, we obtain

\[ g^{(4)}(x + u) - g^{(4)}(x - u) = 2 \sum_{v=1}^{3} \frac{g^{(4+2v-1)}(x)}{(2v - 1)!} u^{2v-1} + \frac{g^{(11)}(\xi) + g^{(11)}(\eta)}{7!} u^7. \]

It follows that we can write

\[ g(x + h) - g(x - h) = \frac{h}{3} \left( g'(x - h) + 4g'(x) + g'(x + h) \right) \]

\[ - \frac{1}{3} \sum_{v=1}^{3} \frac{g^{(4+2v-1)}(x)}{(2v - 1)!} \int_0^h u^{2v}(h - u)^2 du \]

\[ - \frac{g^{(11)}(\xi) + g^{(11)}(\eta)}{(6)(7!)} \int_0^h u^8(h - u)^2 du \]

\[ = \frac{h}{3} \left( g'(x - h) + 4g'(x) + g'(x + h) \right) \]

\[ - \frac{g^{(5)}(x)}{90} h^5 \]

\[ - A_7 g^{(7)}(x) h^7 - A_9 g^{(9)}(x) h^9 \]

\[ - \frac{1}{6(7!)} \int_0^h \left[ g^{(11)}(\xi) + g^{(11)}(\eta) \right] u^8(h - u)^2 du, \]

where \( A_7, A_9 \) are positive constants, and \( \xi = \xi(u) \in [x - h, x + h] \), with a similar for \( \eta \).

With this relation, we now return to Proposition 3.1.c. We begin with the telescoping series,

\[ g(B_t) - g(0) = \sum_{j=0}^{\lfloor nt \rfloor - 1} \left( g \left( B_{\frac{j+1}{n}} \right) - g \left( B_{\frac{j}{n}} \right) \right) + \left( g(B_t) - g \left( B_{\frac{\lfloor nt \rfloor}{n}} \right) \right) \]

\[ = \sum_{j=0}^{\lfloor nt \rfloor - 1} \int_{B_{\frac{j+1}{n}}}^{B_{\frac{j+1}{n}}} g'(u) du + \left( g(B_t) - g \left( B_{\frac{\lfloor nt \rfloor}{n}} \right) \right). \]
By continuity, the term \( (g(B_t) - g(B_{|nt|/n})) \) tends to zero uniformly on compacts in probability (ucp) as \( n \to \infty \), and may be neglected. For each integral term, we use (*') with \( x = \bar{B}_{j/n} \) and \( h = \frac{1}{2} \Delta B_{j/n} \) to obtain

\[
\sum_{j=0}^{[nt]-1} \int_{B_{j/n}} g'(u) \, du = \sum_{j=0}^{[nt]-1} \frac{1}{6} \left( g'\left(B_{j/n}\right) + 4g'(\bar{B}_{j/n}) + g'\left(B_{j+1/n}\right) \right)
\]

\[
- \frac{1}{2^3} \frac{90}{2880} \sum_{j=0}^{[nt]-1} g(5)\left(\bar{B}_{j/n}\right) \Delta B^5_{j/n}
\]

\[
- A_{7} \sum_{j=0}^{[nt]-1} g(7)\left(\bar{B}_{j/n}\right) \Delta B^7_{j/n} - A_{9} \sum_{j=0}^{[nt]-1} g(9)\left(\bar{B}_{j/n}\right) \Delta B^9_{j/n}
\]

\[
- \frac{1}{6(7!)} \sum_{j=0}^{[nt]-1} \int_{0}^{\Delta B_{j/n}} \left( g^{(11)}(\xi) + g^{(11)}(\eta) \right) u^8 \left( \Delta B_{j/n} - u \right)^2 \, du.
\]

(14)

By Lemma 3.2, the terms

\[
\sum_{j=0}^{[nt]-1} \frac{g(5)}{2880} \left(\bar{B}_{j/n}\right) \Delta B^5_{j/n}, \quad A_{7} \sum_{j=0}^{[nt]-1} g(7)\left(\bar{B}_{j/n}\right) \Delta B^7_{j/n}, \quad A_{9} \sum_{j=0}^{[nt]-1} g(9)\left(\bar{B}_{j/n}\right) \Delta B^9_{j/n}
\]

all tend to zero in \( L^2(\Omega) \) as \( n \to \infty \). For the last term, we have the \( L^2(\Omega) \) estimate

\[
\mathbb{E} \left[ \left( \sum_{j=0}^{[nt]-1} \int_{0}^{\Delta B_{j/n}} \left( g^{(11)}(\xi) + g^{(11)}(\eta) \right) u^8 \left( \Delta B_{j/n} - u \right)^2 \, du \right)^2 \right]
\]

\[
\leq C \left( \mathbb{E} \left[ \sup_{s \in [0,t]} |g^{(11)}(B_s)|^4 \right] \right)^{1/2} \left( \sum_{j=0}^{[nt]-1} \| \Delta B_{j/n}^{11} \|_{L^4(\Omega)} \right)^{1/2}
\]

\[
\leq C |nt|^2 n^{-22H} \leq C n^{-2H},
\]

because \( \| \Delta B_{j/n}^{11} \|_{L^4(\Omega)} \leq C \left( \mathbb{E} \| \Delta B_{j/n}^2 \|_{L^4(\Omega)} \right)^{1/2} \leq C n^{-11H} \) by (B.1) and the Gaussian moments formula. Thus, we have

\[
\mathbb{P} \lim_{n \to \infty} \sum_{j=0}^{[nt]-1} \frac{1}{6} \left( g'\left(B_{j/n}\right) + 4g'(\bar{B}_{j/n}) + g'\left(B_{j+1/n}\right) \right) \Delta B_{j/n} = f(B_t) - f(0),
\]

when \( H > 1/10 \), and Proposition 3.1.c is proved. \( \square \)
As a converse to Proposition 3.1.c (and parts (a), (b) and (d) by similar computation), let \( g(x) = f(x) \) be a polynomial such that \( g^{(5)} = f^{(5)} = 1 \). Then

\[
S^S_n(t) = f\left( B_{\lfloor nt \rfloor / n} \right) - f(0) + \frac{1}{2880} \sum_{j=0}^{\lfloor nt \rfloor - 1} \Delta B^{5}_j.
\]

By Theorem 10 of Nualart and Ortiz-Latorre [10], the sequence \( \left( B_t, \sum_{j=0}^{\lfloor nt \rfloor - 1} \Delta B^{5}_{j/n} \right) \) converges in distribution to \( (B_t, W) \), where \( W \) is a Gaussian random variable, independent of \( B \). It follows that \( S^S_n(t) \) does not, in general, converge in probability when \( H \leq 1/10 \). For the critical case \( H = 1/10 \), we have the following theorem, which generalizes the result of Theorem 10 of [10] for this particular value of \( H \).

### 3.2 Main Result: fBm with \( H = 1/10 \)

Throughout the rest of this paper, we will assume that \( f : \mathbb{R} \to \mathbb{R} \) is a \( C^\infty \) function, such that \( f \) and all derivatives satisfy moderate growth conditions. Note that this implies \( \mathbb{E} \left[ \sup_{t \in [0,T]} |f^{(n)}(B_t)|^p \right] < \infty \) for all \( n = 0, 1, 2, \ldots \) and \( 1 \leq p < \infty \).

**Theorem 3.3** Let \( f : \mathbb{R} \to \mathbb{R} \) be a \( C^\infty \) function such that \( f \) and its derivatives have moderate growth conditions, and let \( \{B_t, t \geq 0\} \) be a fractional Brownian motion with \( H = 1/10 \). For \( t \geq 0 \) and integers \( n \geq 2 \), Define

\[
S^S_n(t) = \sum_{j=0}^{\lfloor nt \rfloor - 1} \frac{1}{6} \left( f' \left( B_{\frac{j}{n}} \right) + 4 f' \left( \frac{B_{\frac{j}{n}} + B_{\frac{j+1}{n}}}{2} \right) + f' \left( B_{\frac{j+1}{n}} \right) \right) \left( B_{\frac{j+1}{n}} - B_{\frac{j}{n}} \right).
\]

Then as \( n \to \infty \)

\[
\left( B_t, S^S_n(t) \right) \xrightarrow{L} \left( B_t, f(B_t) - f(0) + \frac{\beta}{25} \cdot 90 \int_0^t f^{(5)}(B_s) \, dW_s \right),
\]

where \( W = \{ W_t, t \geq 0 \} \) is a Brownian motion, independent of \( B \), and

\[
\beta = \sqrt{(5!)2^{-5}} \kappa_5 + 75 \kappa_3, \quad \text{for } \kappa_5 = \sum_{p \in \mathbb{Z}} \left( (p + 1)^{\frac{1}{5}} - 2p^{\frac{1}{5}} + (p - 1)^{\frac{1}{5}} \right)^5, \quad \text{and}
\]

\[
\kappa_3 = \sum_{p \in \mathbb{Z}} \left( (p + 1)^{\frac{1}{3}} - 2p^{\frac{1}{3}} + (p - 1)^{\frac{1}{3}} \right)^3.
\]

Consequently,

\[
f(B_t) = f(0) + \int_0^t f'(B_s) \, d^5 B_s - \frac{\beta}{2880} \int_0^t f^{(5)}(B_s) \, dW_s,
\]

where \( \int_0^t f'(B_s) \, d^5 B_s \) denotes the weak limit of the ‘Simpson’s rule’ sum \( S^S_n(t) \).
The rest of this section is given to proof of Theorem 3.3, and follows in Sects. 3.3–3.5. Following the telescoping series argument given in the proof of Proposition 3.1.c [see (14)], we can write

\[ f(B_t) - f(0) = S_n^5(t) - \frac{1}{25} \sum_{j=0}^{[nt]-1} f^{(5)}(\overline{B}_{\frac{j}{n}}) \Delta B^5_{\frac{j}{n}} - A_7 \sum_{j=0}^{[nt]-1} f^{(7)}(\overline{B}_{\frac{j}{n}}) \Delta B^7_{\frac{j}{n}} \]

\[ - A_9 \sum_{j=0}^{[nt]-1} f^{(9)}(\overline{B}_{\frac{j}{n}}) \Delta B^9_{\frac{j}{n}} \]

\[ - \frac{1}{6(7!)} \sum_{j=0}^{[nt]-1} \int_{0}^{\Delta B_{j/n}} \left( f^{(11)}(\xi) + f^{(11)}(\eta) \right) u^8 \left( \Delta B_{\frac{j}{n}} - u \right)^2 du \]

\[ + \left( f(B_t) - f\left( B_{[nt]/n} \right) \right). \]

As in the proof of Proposition 3.1.c, for \( H = 1/10 \) it follows from Lemma 3.2 that the terms including \( A_7, A_9 \) and the integral term all tend to zero in \( L^2(\Omega) \) as \( n \to \infty \), and the term \( \left( f(B_t) - f\left( B_{[nt]/n} \right) \right) \) also tends to zero ucp as \( n \to \infty \). The main task to prove Theorem 3.3, then, is to show convergence in law of the error term

\[ \sum_{j=0}^{[nt]-1} f^{(5)}(\overline{B}_{\frac{j}{n}}) \Delta B^5_{\frac{j}{n}}. \]

3.3 Malliavin Calculus Representation

In order to apply our convergence theorem (Theorem 2.3), we wish to find a Malliavin calculus representation for the term (15). Consider the Hermite polynomial identity \( H_5(x) = x^5 - 10H_3(x) - 15x \). Taking \( x = \Delta B_{j/n} / \| \Delta B_{j/n} \|_{L^2(\Omega)} = n^H \Delta B_{j/n} \), we have

\[ n^5H \Delta B^5_{\frac{j}{n}} = H_5 \left( n^H \Delta B_{\frac{j}{n}} \right) + 10H_3 \left( n^H \Delta B_{\frac{j}{n}} \right) + 15n^H \Delta B_{\frac{j}{n}}. \]

Using (8), this gives

\[ \sum_{j=0}^{[nt]-1} f^{(5)}(\overline{B}_{\frac{j}{n}}) \Delta B^5_{\frac{j}{n}} = \sum_{j=0}^{[nt]-1} f^{(5)}(\overline{B}_{\frac{j}{n}}) \delta^5 \left( \frac{\partial^{\otimes 5}}{n} \right) \]

\[ + 10n^{-2H} \sum_{j=0}^{[nt]-1} f^{(5)}(\overline{B}_{\frac{j}{n}}) \delta^3 \left( \frac{\partial^{\otimes 3}}{n} \right) \]

\[ + 15n^{-4H} \sum_{j=0}^{[nt]-1} f^{(5)}(\overline{B}_{\frac{j}{n}}) \Delta B_{\frac{j}{n}}. \]

We first show that the last term tends to zero in \( L^1(\Omega) \).
Lemma 3.4 Under the assumptions of Theorem 3.3, there is a constant \( C > 0 \) such that

\[
E \left[ \left( n^{-4H} \sum_{j=0}^{\lfloor nt \rfloor - 1} f^{(5)} \left( \frac{\tilde{B}_j}{n} \right) \Delta B_{\frac{j}{n}} \right)^2 \right] \leq C n^{-2H}.
\]

Proof We start with a 2-sided Taylor expansion of \( f^{(4)} \) of order 7. That is,

\[
f^{(4)} \left( B_{\frac{j+1}{n}} \right) - f^{(4)} \left( B_{\frac{j}{n}} \right) = \sum_{\ell=1}^{6} \frac{f^{(4+\ell)} \left( \frac{\tilde{B}_j}{n} \right)}{2^\ell \ell!} \Delta B_{\frac{j}{n}}^\ell + \frac{f^{(11)} \left( \xi_j \right)}{2^7 7!} \Delta B_{\frac{j}{n}}^7,
\]

and

\[
f^{(4)} \left( \frac{\tilde{B}_j}{n} \right) - f^{(4)} \left( B_{\frac{j}{n}} \right) = \sum_{\ell=1}^{6} \frac{(-1)^{\ell+1} f^{(4+\ell)} \left( \frac{\tilde{B}_j}{n} \right)}{2^\ell \ell!} \Delta B_{\frac{j}{n}}^\ell + \frac{f^{(11)} \left( \eta_j \right)}{2^7 7!} \Delta B_{\frac{j}{n}}^7,
\]

for some intermediate values \( \xi_j, \eta_j \) between \( B_{j/n} \) and \( B_{(j+1)/n} \). Adding the above equations, we obtain

\[
f^{(4)} \left( B_{\frac{j+1}{n}} \right) - f^{(4)} \left( B_{\frac{j}{n}} \right) = f^{(5)} \left( \frac{\tilde{B}_j}{n} \right) \Delta B_{\frac{j}{n}} + \frac{f^{(7)} \left( \frac{\tilde{B}_j}{n} \right)}{24} \Delta B_{\frac{j}{n}}^3 + \frac{f^{(9)} \left( \frac{\tilde{B}_j}{n} \right)}{2^4 5!} \Delta B_{\frac{j}{n}}^5
\]

\[+ \frac{f^{(11)} \left( \xi_j \right) + f^{(11)} \left( \eta_j \right)}{2^7 7!} \Delta B_{\frac{j}{n}}^7. \tag{16}
\]

It follows that we can write

\[
E \left[ \left( n^{-4H} \sum_{j=0}^{\lfloor nt \rfloor - 1} f^{(5)} \left( \frac{\tilde{B}_j}{n} \right) \Delta B_{\frac{j}{n}} \right)^2 \right] \leq 4 E \left[ \left( n^{-4H} \sum_{j=0}^{\lfloor nt \rfloor - 1} \frac{f^{(4)} \left( B_{\frac{j+1}{n}} \right) - f^{(4)} \left( B_{\frac{j}{n}} \right)}{2^7 7!} \Delta B_{\frac{j}{n}}^7 \right)^2 \right]
\]

\[+ 4 E \left[ \left( n^{-4H} \sum_{j=0}^{\lfloor nt \rfloor - 1} \frac{f^{(7)} \left( \frac{\tilde{B}_j}{n} \right)}{24} \Delta B_{\frac{j}{n}}^3 \right)^2 \right] \]
By growth assumptions on $f^{(4)}$, 

$$
\mathbb{E} \left[ \left( n^{-4H} \sum_{j=0}^{\lceil nt \rceil - 1} \left( f^{(4)}( B_{\frac{j+1}{n}}) - f^{(4)}( B_{\frac{j}{n}}) \right) \right)^2 \right] 
\leq C \left( \mathbb{E} \left[ \sup_{0 \leq s \leq \lceil nt \rceil} \| f^{(7)}( B_{\frac{s}{n}}) \|_{\Omega_1}^2 \right] \right)^2 \leq C n^{-14H},
$$

and

$$
\mathbb{E} \left[ \left( n^{-4H} \sum_{j=0}^{\lceil nt \rceil - 1} \left( f^{(9)}( B_{\frac{j}{n}}) \right) \right)^2 \right] 
\leq C \left( \mathbb{E} \left[ \sup_{0 \leq s \leq \lceil nt \rceil} \| f^{(9)}( B_{\frac{s}{n}}) \|_{\Omega_1}^2 \right] \right)^2 \leq C n^{-18H}.
$$

Then by (B.1),

$$
\mathbb{E} \left[ \left( n^{-4H} \sum_{j=0}^{\lceil nt \rceil - 1} \frac{f^{(11)}(\tilde{\xi}_j) + f^{(11)}(\eta_j)}{2^{17}7!} \Delta B_{\frac{j}{n}}^7 \right)^2 \right] 
\leq C \left( \mathbb{E} \left[ \sup_{s \in [0,t]} | f^{(11)}( B_s ) |^4 \right] \right)^{\frac{1}{2}} n^{-8H} \left( \sum_{j=0}^{\lceil nt \rceil - 1} \| \Delta B_{\frac{j}{n}}^7 \|_{L^4(\Omega)}^2 \right)^{\frac{1}{2}} \leq C \lceil nt \rceil^2 n^{-22H}
$$

$$
\leq C n^{-2H}.
$$

This proves the lemma.
Lemma 3.4 shows that only the terms
\[
\sum_{j=0}^{\lfloor nt \rfloor - 1} f^{(5)} \left( \frac{B_i}{n} \right) \delta^5 \left( \partial^{\otimes 5} \frac{j}{n} \right) + 10n^{-2H} \sum_{j=0}^{\lfloor nt \rfloor - 1} f^{(5)} \left( \frac{B_i}{n} \right) \delta^3 \left( \partial^{\otimes 3} \frac{j}{n} \right)
\]
are significant. Using Lemma 2.1.a, we can write the first term as
\[
\sum_{j=0}^{\lfloor nt \rfloor - 1} \delta^5 \left( f^{(5)} \left( \frac{B_i}{n} \right) \partial^{\otimes 5} \frac{j}{n} \right) + 10 \sum_{r=1}^{5} \sum_{j=0}^{\lfloor nt \rfloor - 1} \delta^{5-r} \left( f^{(5+r)} \left( \frac{B_i}{n} \right) \partial^{\otimes (5-r)} \frac{j}{n} \right) \left( \tilde{\epsilon} \frac{j}{n}, \partial \frac{j}{n} \right)_f. 
\]
By Lemma 2.1.c and (B.1), we have the estimate
\[
\left\| \delta^{5-r} \left( f^{(5+r)} \left( \frac{B_i}{n} \right) \partial^{\otimes (5-r)} \frac{j}{n} \right) \right\|_{L^2(\Omega)} \leq C \left\| \partial^{\otimes (5-r)} \right\|_{\mathcal{S}_{5^{5-r}}} \leq C n^{(r-5)H}.
\]
It follows that for \( r = 1, \ldots, 5 \), we can use Lemma 2.6.b,
\[
\mathbb{E} \left| \sum_{j=0}^{\lfloor nt \rfloor - 1} \delta^{5-r} \left( f^{(5+r)} \left( \frac{B_i}{n} \right) \partial^{\otimes (5-r)} \frac{j}{n} \right) \left( \tilde{\epsilon} \frac{j}{n}, \partial \frac{j}{n} \right)_f \right| \leq C n^{(r-5)H} \sum_{j=0}^{\lfloor nt \rfloor - 1} \left| \left( \tilde{\epsilon} \frac{j}{n}, \partial \frac{j}{n} \right)_f \right| \leq C n^{-(3+r)H}.
\]
By a similar computation,
\[
10n^{-2H} \sum_{j=0}^{\lfloor nt \rfloor - 1} f^{(5)} \left( \frac{B_i}{n} \right) \delta^3 \left( \partial^{\otimes 3} \frac{j}{n} \right) = 10n^{-2H} \sum_{j=0}^{\lfloor nt \rfloor - 1} \delta^3 \left( f^{(5)} \left( \frac{B_i}{n} \right) \partial^{\otimes 3} \frac{j}{n} \right)
\]
\[
+ 10n^{-2H} \sum_{r=1}^{3} \sum_{j=0}^{3} \delta^{3-r} \left( f^{(5+r)} \left( \frac{B_i}{n} \right) \partial^{\otimes (3-r)} \frac{j}{n} \right) \left( \tilde{\epsilon} \frac{j}{n}, \partial \frac{j}{n} \right)_f,
\]
where
\[
n^{-2H} \mathbb{E} \left| \sum_{r=1}^{3} \sum_{j=0}^{3} \delta^{3-r} \left( f^{(5+r)} \left( \frac{B_i}{n} \right) \partial^{\otimes (3-r)} \frac{j}{n} \right) \left( \tilde{\epsilon} \frac{j}{n}, \partial \frac{j}{n} \right)_f \right| \leq C n^{-4H}. 
\]
Therefore, we define
\[ F_n := \sum_{j=0}^{\lfloor nt \rfloor - 1} \delta^5 \left( f^{(5)} \left( \frac{\tilde{B}_j}{n} \right) \partial_\frac{\delta^5}{\tilde{B}_j} \right) = \delta^5 (u_n), \quad \text{where} \quad u_n = \sum_{j=0}^{\lfloor nt \rfloor - 1} f^{(5)} \left( \frac{\tilde{B}_j}{n} \right) \partial_\frac{\delta^5}{\tilde{B}_j}; \]
and
\[ G_n := 10 n^{-2H} \sum_{j=0}^{\lfloor nt \rfloor - 1} \delta^3 \left( f^{(5)} \left( \frac{\tilde{B}_j}{n} \right) \partial_\frac{\delta^3}{\tilde{B}_j} \right) = \delta^3 (v_n), \]
where \( v_n = 10 n^{-2H} \sum_{j=0}^{\lfloor nt \rfloor - 1} f^{(5)} \left( \frac{\tilde{B}_j}{n} \right) \partial_\frac{\delta^3}{\tilde{B}_j}. \)

It follows that for large \( n \), the term (15) may be represented as \( F_n + G_n + \varepsilon_n \), where \( \varepsilon_n \to 0 \) in \( L^1(\Omega) \). Then, as introduced in Remark 2.5, we will work with the vector sequence \( (F_n, G_n) \).

### 3.4 Conditions of Theorem 2.3

Our main task in this step is to show that the sequence of random vectors \( (F_n, G_n) \) satisfies the conditions of Theorem 2.3. The first condition is that \( (F_n, G_n) \) is bounded in \( L^1(\Omega) \). In fact, we have a stronger result that will also be helpful with later conditions.

**Lemma 3.5** Fix real numbers \( 0 < t \leq T \) and \( p \geq 2 \), and integer \( n \geq 2 \). Let \( \phi : \mathbb{R} \to \mathbb{R} \) be a \( C^\infty \) function such that \( \phi \) and all its derivatives have moderate growth. For integer \( 1 \leq q \leq 5 \), define
\[ w_n = \sum_{j=0}^{\lfloor nt \rfloor - 1} \phi \left( \frac{\tilde{B}_j}{n} \right) \partial_\frac{\delta^q}{\tilde{B}_j}. \]
Then for integers \( 0 \leq a \leq 5 \), there exists a constant \( c_{q,a} \) such that
\[ \left\| D^a \delta^q (w_n) \right\|_{L^p(\Omega; \mathcal{S}^{\otimes a})} \leq c_{q,a} \sup_{0 \leq j \leq \lfloor nt \rfloor} \left\| \frac{\phi \left( \frac{\tilde{B}_j}{n} \right)}{\partial^{q+a}} \right\|_{L^p(\Omega; \mathcal{S}^{\otimes a})} \left\| nt \right\| n^{-2qH} \leq Cn^{1-2qH}. \]

In particular,
\[ \left\| D^a F_n \right\|_{L^p(\Omega; \mathcal{S}^{\otimes a})} + \left\| D^a G_n \right\|_{L^p(\Omega; \mathcal{S}^{\otimes a})} \leq C. \] (17)

**Proof** This proof follows a similar result in [6], see Theorem 5.2. First, note that by Lemma 2.6.c and growth conditions on \( \phi \), for each integer \( b \geq 0 \),
\[ \| D^b w_n \|_{L^p(\Omega; \mathcal{H}^{\otimes q + b})}^2 = \sum_{j=0}^{[nt]-1} \phi^{(b)} \left( \tilde{B}_{\frac{j}{n}} \right) \varphi^{(q)} \otimes \tilde{\varepsilon}^{(b)} \|_{\mathcal{H}^{\otimes q + b}}^2 \]

\[ \leq \sup_{0 \leq j \leq [nt]} \left\| \phi^{(b)} \left( \tilde{B}_{\frac{j}{n}} \right) \right\|_{L^p(\Omega; \mathcal{H})}^2 \sum_{j,k=0}^{[nt]-1} \left\| \varphi^{(q)} \left( \tilde{E}_{\frac{j}{n}} + \tilde{E}_{\frac{k}{n}} \right) \right\| \left\| \partial \left[ \tilde{B}_{\frac{j}{n}} + \tilde{B}_{\frac{k}{n}} \right] \right\| \]

\[ \leq C[nt] n^{-2q} \sup_{0 \leq j \leq [nt]} \left\| \phi^{(b)} \left( \tilde{B}_{\frac{j}{n}} \right) \right\|^2. \]

It follows that for \( p \geq 2, \)

\[ \| D^b w_n \|_{L^p(\Omega; \mathcal{H}^{\otimes q + b})}^2 \leq C[nt] n^{-2q} \sup_{0 \leq j \leq [nt]} \left\| \phi^{(b)} \left( \tilde{B}_{\frac{j}{n}} \right) \right\|_{L^p(\Omega; \mathcal{H})}^2. \]

Then, using the Meyer inequality (see [9], Proposition 1.5.7),

\[ \| D^a \delta^q (w_n) \|_{L^p(\Omega; \mathcal{H}^{\otimes a})}^2 \leq \| \delta^q (w_n) \|_{L^{p,a}(\mathcal{H}^{\otimes a})}^2 \leq C[nt] n^{-2q} \sup_{0 \leq j \leq [nt]} \left\| \phi^{(b)} \left( \tilde{B}_{\frac{j}{n}} \right) \right\|_{L^{q+a,p}(\mathcal{H})}^2 \]

\[ \leq C[nt] n^{-2q}. \]

For (17), we have

\[ \| D^a F_n \|_{L^p(\Omega; \mathcal{H}^{\otimes a})}^2 = \| D^a \delta^5 (u_n) \|_{L^p(\Omega; \mathcal{H}^{\otimes a})}^2 \]

\[ \leq C[nt] n^{-10} \sup_{0 \leq j \leq [nt]} \left\| f^{(5)} \left( \tilde{B}_{\frac{j}{n}} \right) \right\|_{L^{q+a,p}(\mathcal{H}^{\otimes 5})} \leq C, \]

and

\[ \| D^a G_n \|_{L^p(\Omega; \mathcal{H}^{\otimes a})}^2 \leq \| n^{-2H} D^a \delta^3 (u_n) \|_{L^p(\Omega; \mathcal{H}^{\otimes a})}^2 \]

\[ \leq C[nt] n^{-10} \sup_{0 \leq j \leq [nt]} \left\| f^{(3)} \left( \tilde{B}_{\frac{j}{n}} \right) \right\|_{L^{q+a,p}(\mathcal{H}^{\otimes 3})} \leq C. \]

\[ \square \]

The fact that \((F_n, G_n)\) is bounded in \( L^1(\Omega) \) follows by taking \( a = 0 \). Next, we consider condition (a) of Theorem 2.3.

**Lemma 3.6** Under the assumptions of Theorem 3.3, \((F_n, G_n)\) satisfies condition (a) of Theorem 2.3. That is, we have

(a) For arbitrary \( h \in \mathcal{H}^{\otimes 5} \) and \( g \in \mathcal{H}^{\otimes 3} \),

\[ \lim_{n \to \infty} \mathbb{E} \left| \langle u_n, h \rangle_{\mathcal{H}^{\otimes 5}} \right| = \lim_{n \to \infty} \mathbb{E} \left| \langle v_n, g \rangle_{\mathcal{H}^{\otimes 3}} \right| = 0. \]
Lemma 3.7

Under the assumptions of Theorem 2.3, we have four terms to consider:

\[ \lim_{n \to \infty} \mathbb{E} \left| \left( u_n \otimes_i 1 D^{a_i} F_n \otimes_{i=s+1}^r D^{a_i} G_n \otimes h \right) \right|_{S^{\otimes 5}} = 0, \] where \( 0 \leq a_i < 5, \)

\( 1 \leq a_1 + \cdots + a_r < 5, \)

and \( h \in S_5 \otimes 5 - (a_1 + \cdots + a_r); \) and \( \lim_{n \to \infty} \mathbb{E} \left| \left( v_n \otimes_i 1 D^{b_i} F_n \otimes_{i=s+1}^r D^{b_i} G_n \right) \right|_{S^{\otimes 3}} = 0, \)

where \( 0 \leq b_i < 3, 1 \leq b_1 + \cdots + b_r < 3, \)

and \( g \in S_5 \otimes 3 - (b_1 + \cdots + b_r). \)

(c) \( \lim_{n \to \infty} \mathbb{E} \left| \left( u_n \otimes_i 1 D^{a_i} F_n \otimes_{i=s+1}^r D^{a_i} G_n \right) \right|_{S^{\otimes 5}} = 0, \)

where \( r \geq 2, 0 \leq a_i < 5 \) and \( a_1 + \cdots + a_r = 5; \) and \( \lim_{n \to \infty} \mathbb{E} \left| \left( v_n \otimes_i 1 D^{b_i} F_n \otimes_{i=s+1}^r D^{b_i} G_n \right) \right|_{S^{\otimes 3}} = 0, \)

where \( r \geq 2, 0 \leq b_i < 3 \) and \( b_1 + \cdots + b_r = 3. \)

The proof of this lemma is deferred to Sect. 4 due to its length. To verify condition (b) of Theorem 2.3, we have four terms to consider:

- \( \{ u_n, D^5 G_n \} \)
- \( \{ v_n, D^3 F_n \} \)
- \( \{ u_n, D^5 F_n \} \)
- \( \{ v_n, D^3 G_n \} \)

We deal with the first two terms in the following lemma. The proof is given in Sect. 4.

Lemma 3.7 Under the assumptions of Theorem 3.3, we have

(a) \( \lim_{n \to \infty} \mathbb{E} \left| \left( u_n, D^5 G_n \right) \right|_{S^{\otimes 5}} = 0. \)

(b) \( \lim_{n \to \infty} \mathbb{E} \left| \left( v_n, D^3 F_n \right) \right|_{S^{\otimes 3}} = 0. \)

This leaves the variance terms. Lemma 2.1.b allows us to write

\[
\mathbb{E} \left[ u_n, D^5 F_n \right]_{S^{\otimes 5}} = \sum_{j,k=0}^{[nt]-1} \left| f^{(5)} \left( \frac{B_j}{n} \right) \frac{\partial^{5}}{n} D^5 \delta^5 \left( f^{(5)} \left( \frac{B_k}{n} \right) \frac{\partial^{5}}{n} \right) \right|_{S^{\otimes 5}}
\]

\[
= \sum_{z=0}^{[nt]-1} \left( \frac{S}{z} \right)^2 \sum_{j,k=0}^{[nt]-1} \left| f^{(5)} \left( \frac{B_j}{n} \right) \frac{\partial^{5}}{n} \delta^5 \left( f^{(10-z)} \left( \frac{B_k}{n} \right) \frac{\partial^{5}}{n} \right) \right|_{S^{\otimes 5}}
\]

\[
+ 5! \sum_{j,k=0}^{[nt]-1} \left| f^{(5)} \left( \frac{B_j}{n} \right) \frac{\partial^{5}}{n}, f^{(5)} \left( \frac{B_k}{n} \right) \frac{\partial^{5}}{n} \right|_{S^{\otimes 5}}.
\]

We first deal with the case \( 0 \leq z \leq 4. \) We have

\[
\mathbb{E} \sum_{j,k=0}^{[nt]-1} \left| f^{(5)} \left( \frac{B_j}{n} \right) \frac{\partial^{5}}{n}, \delta^5 \left( f^{(10-z)} \left( \frac{B_k}{n} \right) \frac{\partial^{5}}{n} \right) \frac{\partial^{5}}{n} \otimes \frac{\partial^{5}}{n} \right|_{S^{\otimes 5}}
\]

\[
\leq C \sup_{0 \leq j \leq [nt]} \left| f^{(5)} \left( \frac{B_j}{n} \right) \right|_{L^2(\Omega)} \sup_{0 \leq k \leq [nt]} \left| \delta^5 \left( f^{(10-z)} \left( \frac{B_k}{n} \right) \frac{\partial^{5}}{n} \right) \right|_{L^2(\Omega)}
\]

\[
\times \sum_{j,k=0}^{[nt]-1} \left| \frac{\partial^{j}}{n}, \frac{\partial^{k}}{n} \right|_{S} \left| \frac{\partial^{z}}{n}, \frac{\partial^{z}}{n} \right|_{S}^{z}.
\]
By (B.1) and Lemma 2.1.c, we have

\[
\sup_{0 \leq k \leq |nt|} \left\| \delta^{5-z} \left( f^{(10-z)} \left( \frac{\widetilde{B}_k}{n} \right) \delta^{5-z} \right) \right\|_{L^2(\Omega)} \leq C \left\| \frac{\partial}{\partial z} \right\|_{S^0}^{5-z} \leq C n^{-(5-z)H},
\]

so for the case \( z = 0 \), we have

\[
\sup_{0 \leq j \leq |nt|} \left\| f^{(S)} \left( \frac{\widetilde{B}_j}{n} \right) \right\|_{L^2(\Omega)} \sup_{0 \leq k \leq |nt|} \left\| \delta^{5-z} \left( f^{(10-z)} \left( \frac{\widetilde{B}_k}{n} \right) \delta^{5-z} \right) \right\|_{L^2(\Omega)} \times \sum_{j,k=0}^{|nt|-1} \left\| \left( \frac{\partial}{\partial z}, \frac{\partial}{\partial z}, \frac{\partial}{\partial z} \right) \left( \frac{\partial}{\partial z}, \frac{\partial}{\partial z}, \frac{\partial}{\partial z} \right) \left( \frac{\partial}{\partial z}, \frac{\partial}{\partial z}, \frac{\partial}{\partial z} \right) \right\|_{S^0} \leq C n^{-5H} \sup_{0 \leq j \leq |nt|} \left\| \left( \frac{\partial}{\partial z}, \frac{\partial}{\partial z}, \frac{\partial}{\partial z} \right) \left( \frac{\partial}{\partial z}, \frac{\partial}{\partial z}, \frac{\partial}{\partial z} \right) \left( \frac{\partial}{\partial z}, \frac{\partial}{\partial z}, \frac{\partial}{\partial z} \right) \right\|_{S^0} \leq C n^{-8H} \text{ and } \sup_{0 \leq k \leq |nt|} \left\| \left( \frac{\partial}{\partial z}, \frac{\partial}{\partial z}, \frac{\partial}{\partial z} \right) \left( \frac{\partial}{\partial z}, \frac{\partial}{\partial z}, \frac{\partial}{\partial z} \right) \left( \frac{\partial}{\partial z}, \frac{\partial}{\partial z}, \frac{\partial}{\partial z} \right) \right\|_{S^0} \leq C,
\]

so this gives

\[
C n^{-5H} \sup_{0 \leq j \leq |nt|} \left\| \left( \frac{\partial}{\partial z}, \frac{\partial}{\partial z}, \frac{\partial}{\partial z} \right) \left( \frac{\partial}{\partial z}, \frac{\partial}{\partial z}, \frac{\partial}{\partial z} \right) \left( \frac{\partial}{\partial z}, \frac{\partial}{\partial z}, \frac{\partial}{\partial z} \right) \right\|_{S^0} \leq C |nt| n^{-13H} \leq C n^{-3H}.
\]

If \( 1 \leq z \leq 4 \), then by (B.1), (B.4) and Lemma 2.6.c we have an upper bound of

\[
\sup_{0 \leq j \leq |nt|} \left\| f^{(S)} \left( \frac{\widetilde{B}_j}{n} \right) \right\|_{L^2(\Omega)} \sup_{0 \leq k \leq |nt|} \left\| \delta^{5-z} \left( f^{(10-z)} \left( \frac{\widetilde{B}_k}{n} \right) \delta^{5-z} \right) \right\|_{L^2(\Omega)} \times \sum_{j,k=0}^{|nt|-1} \left\| \left( \frac{\partial}{\partial z}, \frac{\partial}{\partial z}, \frac{\partial}{\partial z} \right) \left( \frac{\partial}{\partial z}, \frac{\partial}{\partial z}, \frac{\partial}{\partial z} \right) \right\|_{S^0} \leq C \left\| \frac{\partial}{\partial z} \right\|_{S^0}^{5-z} \sup_{0 \leq j \leq |nt|} \left\| \left( \frac{\partial}{\partial z}, \frac{\partial}{\partial z}, \frac{\partial}{\partial z} \right) \left( \frac{\partial}{\partial z}, \frac{\partial}{\partial z}, \frac{\partial}{\partial z} \right) \right\|_{S^0} \leq C |nt| n^{-(15-z)H} \leq C n^{-H},
\]

because \( z < 5 \). It follows that the term corresponding to each \( z = 0, \ldots, 4 \) vanishes in \( L^1(\Omega) \), and we have that only the term with \( z = 5 \) is significant. For the case \( z = 5 \), we use a result from [6], see proof of Theorem 5.2.
Similarly, we have
\[
5! \sum_{j,k=0}^{[nt]-1} \left( f^{(5)} \left( B_{\frac{j}{n}} \right) \partial^{\otimes 5}_{\frac{j}{n}}, f^{(5)} \left( B_{\frac{k}{n}} \right) \partial^{\otimes 5}_{\frac{k}{n}} \right)_{\mathcal{S}^5}
\]
\[
= 5! \sum_{j,k=0}^{[nt]-1} f^{(5)} \left( B_{\frac{j}{n}} \right) f^{(5)} \left( B_{\frac{k}{n}} \right) \left( \mathbb{E} \left[ \Delta B_{\frac{j}{n}}, \Delta B_{\frac{k}{n}} \right] \right)^5
\]
\[
= \frac{5!}{25n^{10H}} \sum_{p=-\infty}^{\infty} \sum_{j=(0\vee p)} ([nt]-1)^p \left( f^{(5)} \left( B_{\frac{j}{n}} \right) f^{(5)} \left( B_{\frac{j+p}{n}} \right) \right)
\]
\[
\left( |p+1|^{2H} - 2|p|^{2H} + |p-1|^{2H} \right)^5,
\]
which (for $H = 1/10$) converges in $L^1(\Omega)$ to
\[
\frac{5!}{25} \kappa_5 \int_0^t f^{(5)}(B_s)^2 \, ds, \quad \text{where} \quad \kappa_5 = \sum_{p\in\mathbb{Z}} \left( |p+1|^{\frac{3}{2}} - 2|p|^{\frac{3}{2}} + |p-1|^{\frac{3}{2}} \right)^5. \quad (19)
\]
Hence, we have that
\[
\lim_{n \to \infty} \left\langle u_n, D^5 F_n \right\rangle_{\mathcal{S}^5} = \frac{5!}{25} \kappa_5 \int_0^t f^{(5)}(B_s)^2 \, ds. \quad (20)
\]
Similarly, we have
\[
\left\langle v_n, D^3 G_n \right\rangle_{\mathcal{S}^3} = 10^2 n^{-4H}
\]
\[
\sum_{z=0}^3 \binom{3}{z} \frac{2}{z!} \sum_{j,k=0}^{[nt]-1} \left( f^{(5)} \left( B_{\frac{j}{n}} \right) \partial^{\otimes 3}_{\frac{j}{n}}, \delta^{3-z} \left( f^{(8-z)} \left( B_{\frac{k}{n}} \right) \partial^{\otimes 3-z}_{\frac{k}{n}} \right) \partial^{\otimes z}_{\frac{s}{n}} \otimes \partial^{\otimes 3-z}_{\frac{s}{n}} \right)_{\mathcal{S}^3}.
\]
For $z = 0$,
\[
100 n^{-4H} \mathbb{E} \left[ \sum_{j,k=0}^{[nt]-1} \left( f^{(5)} \left( B_{\frac{j}{n}} \right) \partial^{\otimes 3}_{\frac{j}{n}}, \delta^3 \left( f^{(8)} \left( B_{\frac{k}{n}} \right) \partial^{\otimes 3}_{\frac{k}{n}} \right) \partial^{\otimes 3}_{\frac{s}{n}} \right) \right]_{\mathcal{S}^3}
\]
\[
\leq 100 n^{-4H} \sup_{0 \leq j \leq [nt]} \left\| f^{(5)} \left( B_{\frac{j}{n}} \right) \right\|_{L^2(\Omega)} \sup_{0 \leq k \leq [nt]} \left\| \delta^3 \left( f^{(8)} \left( B_{\frac{k}{n}} \right) \right) \right\|_{L^2(\Omega)}
\]
\[
\sup_{j,k} \left\| \left( \partial_{\frac{j}{n}}, \frac{2}{j} \right)^2 \right\|_{\mathcal{S}^3} \times \sum_{k=0}^{[nt]-1} \left\| \delta^3 \left( f^{(8)} \left( B_{\frac{k}{n}} \right) \right) \right\|_{\mathcal{S}^3}
\]
\[
\leq C [nt] n^{-11H} \leq C n^{-H}.
\]
For $z = 1$ or $z = 2$, by (B.4) and Lemma 2.6.c,
\[
100 \left( \frac{3}{z} \right)^2 z! n^{-4H} \mathbb{E} \left| \sum_{j,k=0}^{\lfloor nt \rfloor-1} \left( f^{(5)} \left( \frac{\mathbf{B}_j}{n} \right) \partial^{\otimes 3} \delta^{3-z} \left( f^{(8-z)} \left( \frac{\mathbf{B}_j}{n} \right) \partial^{\otimes 3-z} \right) \right) \right| \leq C n^{-4H} \sup_{0 \leq j \leq \lfloor nt \rfloor} \left\| f^{(5)} \left( \frac{\mathbf{B}_j}{n} \right) \right\|_{L^2(\Omega)} \sup_{0 \leq k \leq \lfloor nt \rfloor} \left\| \delta^{3-z} \left( f^{(8-z)} \left( \frac{\mathbf{B}_k}{n} \right) \partial^{\otimes 3-z} \right) \right\|_{L^2(\Omega)} \leq C \lfloor nt \rfloor n^{-(13-z)H} \leq C n^{-H},
\]
because $z \leq 2$. Then for $z = 3$, we have
\[
600 n^{-4H} \sum_{j,k=0}^{\lfloor nt \rfloor-1} \left( f^{(5)} \left( \frac{\mathbf{B}_j}{n} \right) \partial^{\otimes 3}, f^{(5)} \left( \frac{\mathbf{B}_k}{n} \right) \partial^{\otimes 3} \right) \leq 600 \left\| f^{(5)} \left( \frac{\mathbf{B}_j}{n} \right) \right\|_{L^2(\Omega)} \left\| f^{(5)} \left( \frac{\mathbf{B}_k}{n} \right) \right\|_{L^2(\Omega)} \leq 600 \left( 23 n^{10H} \right)^3.
\]
Similar to (19), this converges in $L^1(\Omega)$ to
\[
75 \kappa_3 \int_0^t f^{(5)}(B_s)^2 \, ds, \quad \text{where } \kappa_3 = \sum_{p \in \mathbb{Z}} \left( |p+1|^\frac{1}{5} - 2|p|^{\frac{1}{5}} + |p-1|^\frac{1}{5} \right)^3. \tag{21}
\]
Hence, we have that
\[
\lim_{n \to \infty} \left\langle v_n, D^3 G_n \right\rangle_{\mathcal{F}_t} = 75 \kappa_3 \int_0^t f^{(5)}(B_s)^2 \, ds. \tag{22}
\]

3.5 Proof of Theorem 3.3

By Sect. 3.3, the term (15) is dominated in probability by $\frac{1}{2880} \left( F_n + G_n \right)$. By the results of Sect. 3.4, the vector $(F_n, G_n)$ satisfies Theorem 2.3, that is, $(F_n, G_n)$ converges stably as $n \to \infty$ to a mean-zero Gaussian random vector $(F_\infty, G_\infty)$ with independent components, whose variances are given by (20) and (22), respectively. It follows that $F_n + G_n$ converges in distribution to a centered Gaussian random variable with variance
\[
s^2 = \frac{5^4}{25} \kappa_5 \int_0^t f^{(5)}(B_s)^2 \, ds + 75 \kappa_3 \int_0^t f^{(5)}(B_s)^2 \, ds = \beta^2 \int_0^t f^{(5)}(B_s)^2 \, ds,
\]
\[\square\]
where $\beta^2 = (5!)2^{-5}\kappa_5 + 75\kappa_3$. The result of Theorem 3.3 then follows from the Itô isometry. This concludes the proof.

4 Proof of Technical Lemmas

4.1 Proof of Lemma 3.2

To simplify notation, let $Y_j = \phi \left( \tilde{B}_j \right)$. Note that by (B.1), we have $\| \Delta B_j \|_{L^2(\Omega)} = \| \partial_j \|_{\bar{S}_j} = n^{-H}$. For Hermite polynomials $H_r(x)$, $r \geq 1$, it can be shown by induction on the relation $H_{q+1}(x) = xH_q(x) - qH_{q-1}(x)$ that

$$x^r = \sum_{p=0}^{\lfloor \frac{r}{2} \rfloor} C(r, p)H_{r-2p}(x),$$

where each $C(r, p)$ is an integer constant. From Sect. 2.1, we use (8) with $x = \Delta B_j / \| \Delta B_j \|_{L^2(\Omega)} = n^H \Delta B_j$ to write

$$H_r \left( n^H \Delta B_j \right) = \delta^r \left( n^H \hat{\partial}_{\Delta B_j} \right).$$

It follows that

$$n^{rH} \Delta B_j = \sum_{p=0}^{\lfloor \frac{r}{2} \rfloor} C(r, p)H_{r-2p} \left( n^H \Delta B_j \right) = \sum_{p=0}^{\lfloor \frac{r}{2} \rfloor} C(r, p)\delta^{r-2p} \left( n^{(r-2p)H} \hat{\partial}_{\Delta B_j} \right),$$

which implies

$$\Delta B_j = \sum_{p=0}^{\lfloor \frac{r}{2} \rfloor} C(r, p)n^{-2pH} \delta^{r-2p} \left( \hat{\partial}_{\Delta B_j} \right).$$

With this representation for $\Delta B_j$, we then have

$$\mathbb{E} \left[ \left( \sum_{j=0}^{\lfloor nt \rfloor - 1} Y_j \Delta B_j \right)^2 \right] = \sum_{p, p'=0}^{\lfloor r/2 \rfloor} C(r, p)C(r, p')n^{-2H(p+p')} \times \sum_{j, k=0}^{\lfloor nt \rfloor - 1} \mathbb{E} \left[ Y_j Y_k \delta^{r-2p} \left( \hat{\partial}_{\Delta B_j} \right) \delta^{r-2p'} \left( \hat{\partial}_{\Delta B_k} \right) \right]$$
\[ \sum_{p, p' = 0}^{\lfloor nt \rfloor - 1} |C(r, p)C(r, p')| n^{-2H(p + p')} \]

By Lemma 2.1.d, the product

\[ \delta^{2r - 2(p + p')}\left( \frac{\partial_i \otimes r - 2p}{\partial^r - 2p} \otimes \frac{\partial_k \otimes r - 2p'}{\partial^r - 2p'} \right) \]

consists of terms of the form

\[ C\delta^{2r - 2(p + p') - 2z} \left( \frac{\partial_i \otimes r - 2p - z}{\partial^r - 2p - z} \otimes \frac{\partial_k \otimes r - 2p'}{\partial^r - 2p'} \right) \left\{ \frac{\partial_i}{\partial^r - 2p - z}, \frac{\partial_k}{\partial^r - 2p'} \right\}^z, \]

where \( z \geq 0 \) is an integer satisfying \( 2r - 2(p + p') - 2z \geq 0 \). Using (24), we can write that (23) consists of nonnegative terms of the form

\[ Cn^{-2H(p + p')} \sum_{j, k = 0}^{\lfloor nt \rfloor - 1} \left| \mathbb{E} \left[ Y_j Y_k \delta^{2r - 2(p + p') - 2z} \left( \frac{\partial_i \otimes r - 2p - z}{\partial^r - 2p - z} \otimes \frac{\partial_k \otimes r - 2p'}{\partial^r - 2p'} \right) \left\{ \frac{\partial_i}{\partial^r - 2p - z}, \frac{\partial_k}{\partial^r - 2p'} \right\}^z \right] \right|. \]

(25)

To address terms of this type, suppose first that \( z \geq 1 \). Lemma 2.1.c implies that

\[ \left\| \delta^{2r - 2(p + p') - 2z} \left( \frac{\partial_i \otimes r - 2p - z}{\partial^r - 2p - z} \otimes \frac{\partial_k \otimes r - 2p'}{\partial^r - 2p'} \right) \right\| \leq C \left( \left\| \frac{\partial_i}{\partial^r - 2p - z} \right\| \left\| \frac{\partial_k}{\partial^r - 2p'} \right\| \right) \]

\[ \leq C \left\| \frac{\partial_1}{\partial^r - 2r - 2(p + p') - 2z} \right\| \]

\[ = Cn^{-2H(r - p - p')}. \]

Hence, for \( z \geq 1 \), (25) is bounded by

\[ Cn^{-2H(p + p')} \sup_{0 \leq j \leq \lfloor nt \rfloor} \left\| Y_j \right\|_{L^2(\Omega)} \left\| \frac{\partial_1}{\partial^r - 2r - 2(p + p') - 2z} \right\| \sum_{j, k = 0}^{\lfloor nt \rfloor - 1} \left\{ \frac{\partial_i}{\partial^r - 2p - z}, \frac{\partial_k}{\partial^r - 2p'} \right\}^z \]

\[ \leq C \sup_{0 \leq j \leq \lfloor nt \rfloor} \left\| Y_j \right\|_{L^2(\Omega)} \left\| n^{-2rH} \right\| n^{-2rH}, \]

which follows from Lemma 2.6.c.
On the other hand, for the terms with \( z = 0 \), by (10) we have

\[
\mathbb{E} \left[ Y_j Y_k \delta^{2r-2(p+p')} \left( \partial_{i}^{\otimes r-2p} \otimes \partial_{k}^{\otimes r-2p'} \right) \right] = \mathbb{E} \left[ D^{2r-2(p+p')} Y_j Y_k, \partial_{j}^{\otimes r-2p} \otimes \partial_{k}^{\otimes r-2p'} \right] \mathcal{F}^{\otimes 2r-2(p+p')}. \tag{26}
\]

By definition of the Malliavin derivative and Leibniz rule, \( D^{2r-2(p+p')} Y_j Y_k \) consists of terms of the form \( D^{a} Y_j \otimes D^{b} Y_k \), where \( a + b = 2r - 2(p + p') \). Without loss of generality, we may assume \( b \geq 1 \). By assumptions on \( \phi \) and the definition of the Malliavin derivative, we know that \( D^{b} Y_k = \phi^{(b)}(\tilde{\mathcal{B}}_{k/n}) \tilde{\epsilon}^{\otimes b} \), and we know that for each \( b \leq 2r, \; D^{b} Y_k \in L^2(\Omega; \mathcal{F}^{\otimes b}) \). It follows that we can write,

\[
\left| \mathbb{E} \left[ D^{a} Y_j \otimes D^{b} Y_k, \partial_{j}^{\otimes r-2p} \otimes \partial_{k}^{\otimes r-2p'} \right] \right| \leq C \| Y_j \|_{D^{2r-2}} \| Y_k \|_{D^{2r-2}} \left| \left[ \left( \tilde{\epsilon}_{i/n}, \partial_{j/n} \right)_{\mathcal{F}^{a-b}} \right] \right|
\times \left| \left[ \left( \tilde{\epsilon}_{k/n}, \partial_{k/n} \right)_{\mathcal{F}^{a-b}} \right] \right|,
\]

for integers \( 0 \leq \phi \leq a, \; 0 \leq \psi \leq b \). Without loss of generality, we may assume \( \psi \geq 1 \), and by implication \( b \geq 1 \). Then using (B.4),

\[
\left| \mathbb{E} \left[ D^{a} Y_j \otimes D^{b} Y_k, \partial_{j}^{\otimes r-2p} \otimes \partial_{k}^{\otimes r-2p'} \right] \right| \leq C \sup_{0 \leq j \leq [nt]} \| Y_j \|_{D^{2r-2}}^{2} n^{2H(b-1)-2H} \left| \left[ \left( \tilde{\epsilon}_{i/n}, \partial_{j/n} \right)_{\mathcal{F}^{a-b}} \right] \right|.
\]

Thus, for each pair \( (a, b) \), the corresponding term of (25) is bounded by

\[
C n^{-2H(p+p')} \sum_{j,k=0}^{[nt]-1} \left| \mathbb{E} \left[ Y_j Y_k \delta^{2r-2(p+p')} \left( \partial_{j}^{\otimes r-2p} \otimes \partial_{k}^{\otimes r-2p'} \right) \right] \right| \leq C n^{-2H(p+p'+a+b-1)} \sup_{0 \leq j \leq [nt]} \| Y_j \|_{D^{2r-2}}^{2} \sum_{j,k=0}^{[nt]-1} \left| \left[ \left( \tilde{\epsilon}_{i/n}, \partial_{j/n} \right)_{\mathcal{F}^{a-b}} \right] \right| \leq C n^{-2H(p+p'+a+b-1)} \sup_{0 \leq j \leq [nt]} \| Y_j \|_{D^{2r-2}}^{2} \sum_{j,k=0}^{[nt]-1} \left| \left[ \left( \tilde{\epsilon}_{i/n}, \partial_{j/n} \right)_{\mathcal{F}^{a-b}} \right] \right|.
\]

By Lemma 2.6.a,

\[
\sum_{j=0}^{[nt]-1} \left| \left[ \left( \tilde{\epsilon}_{i/n}, \partial_{j/n} \right)_{\mathcal{F}^{a-b}} \right] \right| \leq C [nt]^{2H} n^{-2H} \leq C.
\]
for all \(0 \leq k \leq \lfloor nt \rfloor\), so that
\[
C n^{-2H(p+p'+a+b-1)} \sup_{0 \leq j \leq \lfloor nt \rfloor} \|Y_j\|_{\mathbb{H}^2_{2r,2}}^2 \sum_{k=0}^{\lfloor nt \rfloor - 1} \left\{ \sup_{0 \leq k \leq \lfloor nt \rfloor} \sum_{j=0}^{\lfloor nt \rfloor - 1} \left| \frac{\tilde{F}(k, \theta)}{n} \right| s_{\lfloor\frac{j}{n}\rfloor}\right\} \leq C \sup_{0 \leq j \leq \lfloor nt \rfloor} \|Y_j\|_{\mathbb{H}^2_{2r,2}}^2 \lfloor nt \rfloor n^{-2H(p+p'+a+b-1)},
\]
where \(p + p' + a + b - 1 = 2r - (p + p') - 1 \geq r\), since \(p + p' + 1 \leq 2 \left\lfloor \frac{r}{2} \right\rfloor + 1 \leq r\), for odd integer \(r\). This concludes the proof.

4.2 Proof of Lemma 3.6

For \(\theta \in \{0, 2\}\) define
\[
w_n(\theta) = n^{-\theta H} \sum_{j=0}^{\lfloor nt \rfloor - 1} f^{(5)} \left( \frac{\tilde{B}_j}{n} \right) \partial_{\frac{j}{n}}^{5-\theta}; \quad \text{and} \quad \Phi_n(\theta) = \delta^{5-\theta}(w_n(\theta)).
\]
This allows us to write \(u_n = w_n(0), F_n = \Phi_n(0), v_n = 10w_n(2),\) and \(G_n = 10\Phi_n(2)\). Following Remark 2.4, we may assume that \(h \in \mathcal{S}^{5-\theta}\) has the form \(\epsilon_{t_1} \otimes \cdots \otimes \epsilon_{t_5-\theta}\), for some set of times \(\{t_1, \ldots, t_5-\theta\}\) in \([0, T]^{5-\theta}\). Then for (a), using (B.4) and Lemma 2.6.a,
\[
\mathbb{E} \left| \langle w_n(\theta), h \rangle_{\mathcal{S}^{5-\theta}} \right| = n^{-\theta H} \mathbb{E} \left| \sum_{j=0}^{\lfloor nt \rfloor - 1} f^{(5)} \left( \frac{\tilde{B}_j}{n} \right) \partial_{\frac{j}{n}}^{5-\theta}, \epsilon_{t_1} \otimes \cdots \otimes \epsilon_{t_5-\theta} \right|_{\mathcal{S}^{5-\theta}} \leq n^{-\theta H} \mathbb{E} \left( \sup_{s \in [0,1]} \left| f^{(5)}(B_s) \right| \sum_{j=0}^{\lfloor nt \rfloor - 1} \left| \partial_{\frac{j}{n}}^{5-\theta} \right|_{\mathcal{S}^{5-\theta}} \right) \leq C n^{-(8-\theta)H} \leq C n^{-6H},
\]
where the last inequality follows because \(\theta \leq 2\).

Next, for (b), consider integers \(0 \leq a_i < 5 - \theta, 0 \leq s \leq r < 5 - \theta, r \geq 1\) and \(q\), such that \(s \leq r, 1 \leq a_1 + \cdots + a_r < 5 - \theta\) and \(q = 5 - \theta - (a_1 + \cdots + a_r) \geq 1\). We have
\[
\mathbb{E} \left| \left( \prod_{i=s+1}^{r} D^{a_i} F_n \right) \left( \prod_{i=1}^{s} D^{a_i} G_n \right) \langle w_n(\theta), h \rangle_{\mathcal{S}^{5-\theta}} \right| \leq n^{-\theta H} \mathbb{E} \left| \sum_{j=0}^{\lfloor nt \rfloor - 1} f^{(5)} \left( \frac{\tilde{B}_j}{n} \right) \prod_{i=1}^{s} \left( \partial_{\frac{j}{n}}^{a_i}, D^{a_i} F_n \right) \right|_{\mathcal{S}^{5-\theta}} \times \left( \prod_{i=s+1}^{r} \left( \partial_{\frac{j}{n}}^{a_i}, D^{a_i} G_n \right) \right) \left( \partial_{\frac{j}{n}}^{q}, h \right)_{\mathcal{S}^{5-\theta}}.
\]
Using (B.1), Lemmas 3.5, and 2.6.a, this is bounded by

\[ n^{-\theta H} \sup_{0 \leq j \leq [nt]} \left\| f^{(5)}(\widetilde{B}_{j/n}) \right\|_{L^p(\Omega)} \prod_{i=1}^{r} \sup_{j} \left\| \partial_{\frac{1}{n}}^{\otimes a_i} \right\|_{\mathbb{F}_{\otimes a_i}} \prod_{i=1}^{s} \left\| D^{a_i} F_{n} \right\|_{L^p(\Omega; \mathbb{Y}_{\otimes a_i})} \times \prod_{i=s+1}^{r} \left\| D^{a_i} G_{n} \right\|_{L^p(\Omega; \mathbb{Y}_{\otimes a_i})} \]

where \( p = r + 1 \).

For (c), we want to consider terms of the form

\[ \mathbb{E} \left| \left( w_n(\theta_0), \bigotimes_{i=1}^{r} D^{a_i} \Phi_n(\theta_i) \right) \right|_{\mathbb{Y}_{\otimes 5-\theta_0}}^{2} \]

where \( \theta_i \in \{0, 2\}, 2 \leq r \leq 5 - \theta_0, 0 \leq a_i \leq 4 - \theta_0 \), and \( a_1 + \cdots + a_r = 5 - \theta_0 \). For example, the term

\[ \left\{ u_n, D^3 F_{n} \otimes D^2 G_{n} \right\}_{\mathbb{Y}_{\otimes 3}} \]

corresponds to the case \((\theta_0, \theta_1, \theta_2) = (0, 0, 2), a_1 = 3, a_2 = 2\). We will show that terms of this type tend to zero in \( L^2(\Omega) \) as \( n \to \infty \). Using the above definitions for \( w_n(\theta_i), \Phi_n(\theta_i) \), we have

\[
\mathbb{E} \left[ \left( w_n(\theta_0), \bigotimes_{i=1}^{r} D^{a_i} \Phi_n(\theta_i) \right)^{2} \right]_{\mathbb{Y}_{\otimes 5-\theta_0}}^{2} = n^{-2H(\theta_0 + \cdots + \theta_r)} \mathbb{E} \sum_{p, p' = 0}^{[nt]-1} \sum_{j_1, \ldots, j_r = 0}^{[nt]-1} \sum_{k_1, \ldots, k_r = 0}^{[nt]-1} \left( f^{(5)} \left( \frac{\widetilde{B}_{p/n}}{n} \right) \partial_{\frac{1}{n}}^{\otimes 5-\theta_0} \right) \\
\bigotimes_{i=1}^{r} D^{a_i} \delta^{5-\theta_i} \left( f^{(5)} \left( \frac{\widetilde{B}_{p'/n}}{n} \right) \partial_{\frac{1}{n}}^{\otimes 5-\theta_i} \right) \left( f^{(5)} \left( \frac{\widetilde{B}_{k_1/n}}{n} \partial_{\frac{1}{n}}^{\otimes 5-\theta_{k_1}} \right) \right) \\
\times \left( f^{(5)} \left( \frac{\widetilde{B}_{k_r/n}}{n} \right) \partial_{\frac{1}{n}}^{\otimes 5-\theta_{k_r}} \right) \bigotimes_{i=1}^{r} D^{a_i} \delta^{5-\theta_i} \left( f^{(5)} \left( \frac{\widetilde{B}_{j_1/n}}{n} \partial_{\frac{1}{n}}^{\otimes 5-\theta_{j_1}} \right) \partial_{\frac{1}{n}}^{\otimes 5-\theta_{j_r}} \right) \right)_{\mathbb{Y}_{\otimes 5-\theta_0}}^{2}. \quad (27)
\]

By Lemma 2.1.b,

\[
D^{a_i} \delta^{5-\theta_i} \left( f^{(5)} \left( \frac{\widetilde{B}_{j_i/n}}{n} \right) \partial_{\frac{1}{n}}^{\otimes 5-\theta_i} \right) \\
= \sum_{\ell_i = 0}^{(5-\theta_i) \wedge a_i} \ell_i! (5 - \theta_i) \binom{a_i}{\ell_i} \delta^{5-\theta_i - \ell_i} \left( f^{(5 + a_i - \ell_i)} \left( \frac{\widetilde{B}_{j_i/n}}{n} \right) \partial_{\frac{1}{n}}^{\otimes 5-\theta_i - \ell_i} \right) \partial_{\frac{1}{n}}^{\otimes \ell_i} \otimes \partial_{\frac{1}{n}}^{\otimes a_i - \ell_i}.
\]
Applying this to each term, we can expand the inner product
\[
\left( f^{(5)} \left( \widehat{B}_{\frac{p}{n}} \right) \partial^{\otimes 5-\theta_0} \right) \langle f^{(5)} \left( \widehat{B}_{\frac{1}{n}} \right) \partial^{\otimes 5-\theta_1} \rangle 
\otimes \cdots \otimes D^{a_r \delta^{5-\theta_r}} \left( f^{(5)} \left( \widehat{B}_{\frac{p}{n}} \right) \partial^{\otimes 5-\theta_r} \right) \rangle_{\mathcal{F}^\otimes 5-\theta_0}
\]
into terms of the form
\[
C_{\ell} f^{(5)} \left( \widehat{B}_{\frac{p}{n}} \right) \delta^{b_1} \left( f^{(\lambda_1)} \left( \widehat{B}_{\frac{1}{n}} \right) \partial^{\otimes b_1} \right) \cdots \delta^{b_r} \left( f^{(\lambda_r)} \left( \widehat{B}_{\frac{p}{n}} \right) \partial^{\otimes b_r} \right) 
\times \left( \partial_{\frac{p}{n}} \epsilon_{\frac{j_1}{n}} \right)_{\mathcal{F}_j} \left( \partial_{\frac{p}{n}} \epsilon_{\frac{j_r}{n}} \right)_{\mathcal{F}_j} a_{1-\ell_1} \cdots a_{r-\ell_r},
\]
where $C_{\ell} = C_\ell (\ell_1, \ldots, \ell_r)$ is an integer constant, each $b_i = 5 - \theta_i - \ell_i$, and each $\lambda_i = 5 + a_i - \ell_i$. It follows that (27) is a sum of terms of the form
\[
C_{\ell} C_{\nu} n^{-2H(\theta_1 + \cdots + \theta_r)} \sum_{p, p' = 0}^{[nt] - 1} f^{(5)} \left( \widehat{B}_{\frac{p}{n}} \right) f^{(5)} \left( \widehat{B}_{\frac{p'}{n}} \right) 
\times \left( \sum_{j_1 = 0}^{[nt] - 1} \delta^{b_1} \left( f^{(\lambda_1)} \left( \widehat{B}_{\frac{1}{n}} \right) \partial^{\otimes b_1} \right) \left( \partial_{\frac{p}{n}} \epsilon_{\frac{j_1}{n}} \right)_{\mathcal{F}_j} a_{1-\ell_1} \right) 
\times \cdots \times \left( \sum_{j_r = 0}^{[nt] - 1} \delta^{b_r} \left( f^{(\lambda_r)} \left( \widehat{B}_{\frac{p}{n}} \right) \partial^{\otimes b_r} \right) \left( \partial_{\frac{p'}{n}} \epsilon_{\frac{j_r}{n}} \right)_{\mathcal{F}_j} a_{r-\ell_r} \right). \quad (28)
\]
For $0 \leq j_1, \ldots, j_r \leq [nt]$ we have the estimate
\[
\left| \sum_{p = 0}^{[nt] - 1} \left( \partial_{\frac{p}{n}} \epsilon_{\frac{j_1}{n}} \right)_{\mathcal{F}_j} \left( \partial_{\frac{p}{n}} \epsilon_{\frac{j_r}{n}} \right)_{\mathcal{F}_j} a_{1-\ell_1} \cdots a_{r-\ell_r} \right| 
\leq \sup \mathcal{I} \left| \prod_{i=1}^{r} \left( \partial_{\frac{p}{n}} \epsilon_{\frac{j_i}{n}} \right)_{\mathcal{F}_j} \left( \partial_{\frac{p}{n}} \epsilon_{\frac{j_i}{n}} \right)_{\mathcal{F}_j} a_{i-\ell_i} \right|,
\]
where $\mathcal{I} = \{ 0 \leq j_1, \ldots, j_r \leq [nt] \}$. By Lemma 2.6.a and/or 2.6.c, this is bounded by $C n^{-2H(5-\theta_0)}$ if $\ell_1 + \cdots + \ell_r \geq 1$, and bounded by $C n^{-2H(5-\theta_0-1)} = C n^{-2H(4-\theta_0)}$ if and only if $\ell_1 = \cdots = \ell_r = 0$. Hence, we can write
\[
\sup_{\mathcal{I}, \mathcal{I}', p, p' = 0} \left| \sum_{j_1 = 0}^{[nt] - 1} \left( \partial_{\frac{p}{n}} \epsilon_{\frac{j_1}{n}} \right)_{\mathcal{F}_j} \cdots \left( \partial_{\frac{p'}{n}} \epsilon_{\frac{j_r}{n}} \right)_{\mathcal{F}_j} a_{r-\ell_r} \right| \leq C n^{-\Lambda H}, \quad (29)
\]
where $4H(4 - \theta_0) \leq \Lambda \leq 4H(5 - \theta_0)$. 

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It follows that terms of the form (28) can be bounded in absolute value by

\[
C n^{-2H(\theta_0 + \cdots + \theta_r)} \sup_{0 \leq p \leq \lfloor nt \rfloor} \| f^{(p)} \left( \tilde{B}_n \right) \|^2_{L^{2r+2}(\Omega)} \sup_{x,x'} \sum_{p,p'=0}^{\lfloor nt \rfloor - 1} \left| \left( \frac{\partial p}{\tilde{B}_n} , \frac{\partial p'}{\tilde{B}_n} \right) \right| \left( \partial_{r} \right) \left( \tilde{B}_n \right)^{\delta^5} \sum_{j=0}^{r-1} \delta_{b_j} \left( f^{(\lambda_j)} \left( \tilde{B}_n \right) \right) \left( \hat{f}^{(\lambda_j)} \right) \left( \tilde{B}_n \right)^{\delta^5}.
\]

By (29) and Lemma 3.5, this is bounded by

\[
C \lfloor nt \rfloor^n n^{-2H(\theta_0 + \cdots + \theta_r)} - \Lambda H - H(b_1 + \cdots + b_r + b'_1 + \cdots + b'_r).
\]

We have \( \Lambda \geq 4H(4 - \theta_0) \), and

\[
b_1 + \cdots + b_r = 5r - (\theta_1 + \cdots + \theta_r) - (\ell_1 + \cdots + \ell_r).
\]

Since \( \ell_i \leq a_i \) for each \( i \), then \( \ell_1 + \cdots + \ell_r \leq a_1 + \cdots + a_r = 5 - \theta_0 \), it follows that the exponent

\[
2H(\theta_0 + \cdots + \theta_r) + \Lambda H + H(b_1 + \cdots + b_r + b'_1 + \cdots + b'_r)
\]

\[
\geq 2H(\theta_0 + \cdots + \theta_r) + 4H(4 - \theta_0) + H(10r - 2(\theta_1 + \cdots + \theta_r) - 2(5 - \theta_0))
\]

\[
= 16H + 10(r - 1)H \geq 10rH + 6H.
\]

Hence, we have an upper bound of

\[
C \lfloor nt \rfloor^n n^{-10rH - 6H} \leq C n^{-6H}
\]

for each term of the form (28), so this term tends to zero in \( L^2(\Omega) \), and we have (c). This concludes the proof of Lemma 3.6. \( \square \)

4.3 Proof of Lemma 3.7

Starting with (a), Lemma 2.1.b gives

\[
\mathbb{E} \left| u_n, D^5 G_n \right|_{\tilde{S}^5} = n^{-2H} \mathbb{E} \left| \sum_{i=0}^{3} \binom{3}{i} \binom{5}{i} i! \sum_{j,k=0}^{\lfloor nt \rfloor - 1} f^{(5)} \left( \tilde{B}_n \right) \partial_{r}^5 \tilde{B}_n^{\delta^5} \left( \frac{\partial^i}{\tilde{B}_n} , \frac{\partial^j}{\tilde{B}_n} \right) \frac{\partial^i}{\tilde{B}_n} \hat{f}^{(5)} \left( \tilde{B}_n \right) \right|.
\]
By moderate growth conditions and (18), we have \( \| f^{(5)}(\tilde{B}_{\frac{n}{k}}) \|_{L^2(\Omega)} \leq C \) and
\[
\left\| \delta^{3-i} \left( f^{(10-i)}(\tilde{B}_{\frac{n}{k}}) \partial_{\frac{n}{k}}^{\otimes 3-i} \right) \right\|_{L^2(\Omega)} \leq C \| \partial_{\frac{n}{k}} \|_{\mathcal{S}_j} \leq C n^{(3-i)H};\]
so we have terms of the form
\[
C n^{-(5-i)H} \sum_{j,k=0}^{\lfloor nt \rfloor - 1} \left| \left\langle \partial_{\frac{n}{k}}^i, \partial_{\frac{n}{k}}^j \right\rangle \right| \left| \left\langle \partial_{\frac{n}{k}}^j, \tilde{e}_{\frac{n}{k}} \right\rangle \right| \delta^{3-i} \left( f^{(10-i)}(\tilde{B}_{\frac{n}{k}}) \partial_{\frac{n}{k}}^{\otimes 3-i} \right).
\]

If \( i > 0 \), then (B.4) and Lemma 2.6.c give an estimate of
\[
C n^{-(5-i)H} \sum_{j,k=0}^{\lfloor nt \rfloor - 1} \left| \left\langle \partial_{\frac{n}{k}}^i, \partial_{\frac{n}{k}}^j \right\rangle \right| \left| \left\langle \partial_{\frac{n}{k}}^j, \tilde{e}_{\frac{n}{k}} \right\rangle \right| \delta^{3-i} \left( f^{(10-i)}(\tilde{B}_{\frac{n}{k}}) \partial_{\frac{n}{k}}^{\otimes 3-i} \right) \leq C \left\lfloor nt \right\rfloor n^{(15-3i)H} \leq C n^{-2H},
\]
because \( i \leq 3 \). On the other hand, if \( i = 0 \), then by (B.4) and Lemma 2.6.a,
\[
C n^{-5H} \sum_{j,k=0}^{\lfloor nt \rfloor - 1} \left| \left\langle \partial_{\frac{n}{k}}^0, \tilde{e}_{\frac{n}{k}} \right\rangle \right|^5 \leq C n^{-5H} \sum_{k=0}^{\lfloor nt \rfloor - 1} \left\{ \sum_{j=0}^{\lfloor nt \rfloor - 1} \left| \left\langle \partial_{\frac{n}{k}}^j, \tilde{e}_{\frac{n}{k}} \right\rangle \right|^5 \right\} \leq C \left\lfloor nt \right\rfloor n^{-13H} \leq C n^{-3H},
\]
hence (a) is proved.

For (b), again using Lemma 2.1.b we can write
\[
\mathbb{E} \left| \left\langle v_n, D^3 F_n \right\rangle \right| \leq n^{-2H} \mathbb{E} \sum_{i=0}^{3} \binom{5}{i} \binom{3}{i} \sup_{0 \leq k \leq \lfloor nt \rfloor} \left| f^{(5)}(\tilde{B}_{\frac{n}{k}}) \partial_{\frac{n}{k}}^{\otimes 3} \right| \delta^{5-i} \left( f^{(8-i)}(\tilde{B}_{\frac{n}{k}}) \partial_{\frac{n}{k}}^{\otimes 3-i} \right) \left| \left\langle \partial_{\frac{n}{k}}^i, \partial_{\frac{n}{k}}^{\otimes 3-i} \right\rangle \right| \delta^{3-i} \left( f^{(8-i)}(\tilde{B}_{\frac{n}{k}}) \partial_{\frac{n}{k}}^{\otimes 3-i} \right).
\]
We deal with three cases. First, assume \( i = 0 \). Then we have a bound of

\[
C_n^{-2H} \sum_{j,k=0}^{[nt]-1} \mathbb{E} \left| f^{(5)} \left( \bar{B}_n \right) \delta^5 \left( f^{(8)} \left( \bar{B}_n \right) \partial \bar{B}_n \right) \delta^{\otimes 5} \right| \left| \left\{ \partial \bar{B}_n, \frac{\bar{B}_n}{n} \right\} \right|_3
\]

where, as above, we use the estimates \( \| f^{(5)} \left( \bar{B}_n \right) \|_{L^2(\Omega)} \leq C \) and \( \| \delta^5 \left( f^{(8)} \left( \bar{B}_n \right) \partial \bar{B}_n \right) \partial \bar{B}_n \|_{L^2(\Omega)} \leq C n^{-5H} \), and

\[
\sup_{j,k} \left| \left\{ \partial \bar{B}_n, \frac{\bar{B}_n}{n} \right\} \right|_{3} \sum_{k=0}^{[nt]-1} \left( \sum_{j=0}^{[nt]-1} \left| \left\{ \partial \bar{B}_n, \frac{\bar{B}_n}{n} \right\} \right|_{3} \right) \leq C [nt] n^{-4H}
\]

follows from (B.4) and Lemma 2.6.a.

The next case is for \( i = 1 \) or \( i = 2 \). Using similar estimates we have

\[
C_n^{-2H} \sum_{j,k=0}^{[nt]-1} \mathbb{E} \left| f^{(5)} \left( \bar{B}_n \right) \delta^{5-i} \left( f^{(8-i)} \left( \bar{B}_n \right) \partial \bar{B}_n \right) \delta^{\otimes 5-i} \right| \left| \left\{ \partial \bar{B}_n, \frac{\bar{B}_n}{n} \right\} \right|_{i} \left| \left\{ \partial \bar{B}_n, \frac{\bar{B}_n}{n} \right\} \right|_{3-i}
\]

where, as above, we use the estimates \( \| f^{(5)} \left( \bar{B}_n \right) \|_{L^2(\Omega)} \leq C \) and \( \| \delta^5 \left( f^{(8-i)} \left( \bar{B}_n \right) \partial \bar{B}_n \right) \partial \bar{B}_n \|_{L^2(\Omega)} \leq C n^{-5H} \), and

\[
\sup_{j,k} \left| \left\{ \partial \bar{B}_n, \frac{\bar{B}_n}{n} \right\} \right|_{i} \sum_{k=0}^{[nt]-1} \left( \sum_{j=0}^{[nt]-1} \left| \left\{ \partial \bar{B}_n, \frac{\bar{B}_n}{n} \right\} \right|_{i} \right) \leq C [nt] n^{-7-i+6H} \leq C n^{-H}.
\]

because \( 7 - i + 6 \geq 11 \) for \( i \leq 2 \).

For the case \( i = 3 \), we will use a different estimate, and show that the term with \( i = 3 \) vanishes in \( L^2(\Omega) \). Using Lemma 2.1.d we have,

\[
\mathbb{E} \left[ \left( n^{-2H} \sum_{j,k=0}^{[nt]-1} f^{(5)} \left( \bar{B}_n \right) \delta^2 \left( f^{(5)} \left( \bar{B}_n \right) \partial \bar{B}_n \right) \delta^{\otimes 2} \right)^2 \right] = n^{-4H} \sum_{j,j',k,k'=0}^{[nt]-1} \mathbb{E} \left[ f^{(5)} \left( \bar{B}_n \right) f^{(5)} \left( \bar{B}_{n'} \right) \delta^2 \left( f^{(5)} \left( \bar{B}_n \right) \partial \bar{B}_n \right) \delta^{\otimes 2} \left( f^{(5)} \left( \bar{B}_{n'} \right) \partial \bar{B}_{n'} \right) \right]
\]

\[
\left| \left\{ \partial \bar{B}_n, \frac{\bar{B}_n}{n} \right\} \right|_{3} \left| \left\{ \partial \bar{B}_{n'}, \frac{\bar{B}_{n'}}{n'} \right\} \right|_{3}
\]
\[
= n^{-4H} \sum_{p=0}^{2} \binom{2}{p}^2 p! \sum_{j,j',k,k'} \mathbb{E} \left[ g(j, j') \delta^{4-2p} \left( g(k, k') \partial_{\pi}^{2-p} \otimes \partial_{\pi'}^{2-p} \right) \right] \\
\left\langle \partial_{\pi}, \partial_{\pi'} \right\rangle \mathbb{G} \left\langle \partial_{\pi}, \partial_{\pi'} \right\rangle^{3} \mathbb{G},
\]
where \( g(j, j') = f^{(5)} \left( \tilde{B}_{j} \right) f^{(5)} \left( \tilde{B}_{j'} \right) \). Then by the Malliavin duality (10), this results in a sum of three terms of the form
\[
Cn^{-4H} \sum_{j,j',k,k'} \mathbb{E} \left[ \left( D^{4-2p} g(j, j'), g(k, k') \partial_{\pi}^{2-p} \otimes \partial_{\pi'}^{2-p} \right) \right] \\
\mathbb{G}^{4-2p},
\]
for \( p = 0, 1, 2 \). When the index \( p = 0 \), then \( \mathbb{E} \left| \left( D^{4-2p} g(j, j'), g(k, k') \partial_{\pi}^{2-p} \otimes \partial_{\pi'}^{2-p} \right) \right| \) consists of terms of the form
\[
\mathbb{E} \left| \left( \frac{\partial^{4}}{\partial x_{1}^{a} \partial x_{2}^{b}} \Psi(\tilde{B}_{j}, \tilde{B}_{j'}) \right) g(k, k') \left\langle \tilde{e}_{j}, \partial_{\pi} \right\rangle \left\langle \tilde{e}_{j'}, \partial_{\pi'} \right\rangle \left\langle \tilde{e}_{k}, \partial_{\pi} \right\rangle \left\langle \tilde{e}_{k'}, \partial_{\pi'} \right\rangle \right|,
\]
where \( \Psi(x_{1}, x_{2}) = f^{(5)}(x_{1}) f^{(5)}(x_{2}) \) and \( a + b = 4 \). By moderate growth and (B.4), we see that (31) is bounded by \( Cn^{-8H} \), and so for the case \( p = 0 \), (30) is bounded in absolute value by
\[
Cn^{-12H} \sum_{j,j',k,k'} \left\langle \partial_{\pi}, \partial_{\pi} \right\rangle^{3} \left\langle \partial_{\pi'}, \partial_{\pi'} \right\rangle^{3} = Cn^{-12H} \left( \sum_{j,k=0}^{\lfloor nt \rfloor \leq 2} \left\langle \partial_{\pi}, \partial_{\pi} \right\rangle^{3} \right)^{2} \\
\leq C \lfloor nt \rfloor^{2} n^{-24H} \leq Cn^{-4H}.
\]
By a similar estimate, when \( p = 1 \), then
\[
\mathbb{E} \left| \left( D^{2} g(j, j'), g(k, k') \partial_{\pi} \otimes \partial_{\pi'} \right) \right| \leq Cn^{-4H},
\]
so that for \( p = 1 \), then (30) is bounded in absolute value by
\[
Cn^{-8H} \sum_{j,j',k,k'} \left\langle \partial_{\pi}, \partial_{\pi'} \right\rangle \left\langle \partial_{\pi}, \partial_{\pi'} \right\rangle^{3} \left\langle \partial_{\pi'}, \partial_{\pi'} \right\rangle^{3} \\
\leq Cn^{-8H} \sup_{k,k'} \left\langle \partial_{\pi}, \partial_{\pi'} \right\rangle \left( \sum_{j,k=0}^{\lfloor nt \rfloor \leq 2} \left\langle \partial_{\pi}, \partial_{\pi} \right\rangle^{3} \right)^{2} \leq C \lfloor nt \rfloor^{2} n^{-22H} \leq Cn^{-2H}.
\]
Last, the term in (30) with \( p = 2 \) has the form

\[
Cn^{-4H} \sum_{j, j', k, k'} \mathbb{E}\left[ g(j, j')g(k, k') \left( \left\langle \frac{\partial_{j}}{\pi}, \frac{\partial_{j'}}{\pi} \right\rangle_{\mathcal{F}} \right)^2 \left( \left\langle \frac{\partial_{k}}{\pi}, \frac{\partial_{k'}}{\pi} \right\rangle_{\mathcal{F}} \right)^3 \right].
\]

This is bounded in absolute value by

\[
Cn^{-4H} \sup_{0 \leq j \leq \lfloor nt \rfloor} \left\| f^{(5)} \left( \bar{B}_{j/\pi} \right) \right\|_{L^4(\Omega)} \sum_{k, k' = 0}^{\lfloor nt \rfloor - 1} \left\langle \frac{\partial_{k}}{\pi}, \frac{\partial_{k'}}{\pi} \right\rangle_{\mathcal{F}}^2 \sum_{j, j' = 0}^{\lfloor nt \rfloor - 1} \left\langle \frac{\partial_{j}}{\pi}, \frac{\partial_{j'}}{\pi} \right\rangle_{\mathcal{F}}^3 \left\langle \frac{\partial_{k'}}{\pi}, \frac{\partial_{k}}{\pi} \right\rangle_{\mathcal{F}}^3.
\] (32)

By Lemma 2.6.c, for every \( 0 \leq k \leq \lfloor nt \rfloor \) we have

\[
\sum_{j = 0}^{\lfloor nt \rfloor - 1} \left| \left\langle \frac{\partial_{j}}{\pi}, \frac{\partial_{k}}{\pi} \right\rangle_{\mathcal{F}}^3 \right| \leq Cn^{-6H},
\]

hence (32) is bounded by

\[
Cn^{-16H} \sum_{k, k' = 0} \left\langle \frac{\partial_{k}}{\pi}, \frac{\partial_{k'}}{\pi} \right\rangle_{\mathcal{F}}^2 \leq C \lfloor nt \rfloor n^{-20H} \leq Cn^{-10H}.
\]

Lemma 3.7 is proved. 

\( \square \)

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