UNIMODULAR POLYNOMIAL MATRICES OVER FINITE FIELDS

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Abstract. We consider some combinatorial problems on matrix polynomials over finite fields. Using results from control theory we give a proof of a result of Helmke, Jordan and Lieb on the number of linear unimodular matrix polynomials over a finite field. As an application of our results we give a new proof of a theorem of Chen and Tseng which answers a question of Niederreiter on splitting subspaces. We use our results to affirmatively resolve a conjecture on the probability that a matrix polynomial is unimodular.

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1. INTRODUCTION

We denote by $\mathbb{F}_q$ the finite field with $q$ elements where $q$ is a prime power. Let $\mathbb{F}_q[x]$ denote the ring of polynomials over $\mathbb{F}_q$ in the indeterminate $x$. For any ring $R$ and positive integers $n, k$ we define $M_{n,k}(R)$ to be the set of all $n \times k$ matrices over $R$. We denote by $I_{n,k}$ the matrix in $M_{n,k}(\mathbb{F}_q)$ whose $(i,j)^{th}$ entry is zero whenever $i \neq j$ and equal to 1 for $i = j$.

The main objects of study in this paper are matrix polynomials over finite fields. A matrix polynomial over a field $F$ in the variable $x$ is a sum $\sum_{i=0}^{d} A_i x^i$, where $A_i \in M_{n,k}(F)$ for some fixed positive integers $n, k$. It is often convenient to view such a matrix polynomial as a single matrix whose entries are polynomials in $x$ (sometimes referred to as a polynomial matrix) and we freely alternate between these two points of view. We say that a matrix polynomial $A = \sum_{i=0}^{d} A_i x^i \in M_{n,k}(\mathbb{F}_q[x])$ is unimodular if the greatest common divisor...
of all $r \times r$ minors of $A$ is equal to 1 where $r = \min\{n, k\}$. The notion of unimodularity can be defined more generally for rectangular matrices over an arbitrary integral domain. A landmark result in the setting of unimodularity is the Quillen-Suslin theorem [20, 24] formerly known as Serre’s conjecture. We refer to [11, 16, 18, 25] for other contexts where unimodularity is considered. We begin with a combinatorial question concerning matrix polynomials over a finite field.

**Question 1.1.** Given positive integers $n, k$ and a prime power $q$, determine the number of matrices $A \in M_{n,k}(\mathbb{F}_q)$ for which the matrix polynomial $xI_{n,k} - A$ is unimodular.

This question was essentially considered by Kocięcki and Przyłuski [14] (also see [22, Prob. 1.2]) in an attempt to determine the number of reachable pairs of matrices over a finite field. Reachability is a fundamental notion in the control theory of linear systems. The question was fully answered only recently by Helmke, Jordan and Lieb [17, Thm. 1] who showed that the answer is equal to $\prod_{i=1}^{k}(q^n - q^i)$. We refer to the introduction of [22] for details and alternate formulations of the result of Helmke et al. Our main result is Lemma 2.4 which allows us to give a new proof (Corollary 2.7) of the theorem of Helmke et al. An essential ingredient in our main lemma is a control theoretic result of Brunovský on completely controllable pairs.

Further applications of our results appear in Sections 3 and 4. In Section 3 we consider splitting subspaces (defined below) which were introduced by Niederreiter [19, Def. 1] in the context of his work on the multiple recursive matrix method for pseudorandom number generation.

**Definition 1.2.** Let $d, m$ be positive integers and consider the vector space $\mathbb{F}_{q^{md}}$ over $\mathbb{F}_q$. For any element $\alpha \in \mathbb{F}_{q^{md}}$ we say that an $m$-dimensional subspace $W$ of $\mathbb{F}_{q^{md}}$ is $\alpha$-splitting if $\mathbb{F}_{q^{md}} = W \oplus \alpha W \oplus \cdots \oplus \alpha^{d-1} W$.

Niederreiter was interested in the following question on splitting subspaces.

**Question 1.3.** Given $\alpha \in \mathbb{F}_{q^{md}}$ such that $\mathbb{F}_{q^{md}} = \mathbb{F}_q(\alpha)$, what is the number of $\alpha$-splitting subspaces of $\mathbb{F}_{q^{md}}$ of dimension $m$?

It may be noted that the same question was also considered by Goresky and Klapper (see the remark in [10, p. 1653] and [10, Thm. 3(4)]). In addition to the evident cryptographic aspect, Niederreiter’s question also has interesting connections with group theory and finite projective geometry via block companion Singer cycles. We refer to [7, 8] for more on this topic. The case $m = 2$ of Niederreiter’s question was settled in [8] using a result that answers the following question: What is the probability that two randomly chosen polynomials of a fixed positive degree over a finite field are coprime? This question on the probability of coprime polynomials goes back to an exercise in Knuth [13, §4.6.1, Ex. 5]
and has subsequently been considered by Corteel, Savage, Wilf and Zeilberger [4] in the more general setting of combinatorial prefabs. In fact, our main result relies on Lemma 2.3 which may be viewed as a probabilistic result on coprime polynomials. Chen and Tseng [2, Cor. 3.4] eventually answered Niederreiter’s question on splitting subspaces by proving the following theorem which was initially conjectured in [8, Conj. 5.5].

**Theorem 1.4** (Splitting Subspace Theorem). For any \( \alpha \in \mathbb{F}_{q^{md}} \) such that \( \mathbb{F}_{q^{md}} = \mathbb{F}_q(\alpha) \), the number of \( \alpha \)-splitting subspaces of \( \mathbb{F}_{q^{md}} \) of dimension \( m \) is precisely

\[
\frac{q^{md} - 1}{q^m - 1} q^{m(m-1)(d-1)}.
\]

In this paper we use a control-theoretic result of Wimmer (Theorem 3.8) to prove Theorem 3.9 from which the Splitting Subspace Theorem follows as a corollary. In Section 4 we consider a generalization of Question 1.1. The answer to this question which was stated earlier can be given a probabilistic flavour as follows.

**Theorem 1.5.** If a matrix \( A \) is selected uniformly at random from \( M_{n,k}(\mathbb{F}_q) \), then the probability that \( xI_{n,k} - A \) is unimodular is given by \( \prod_{i=1}^k (1 - q^{i-n}) \).

Using results in Section 2, we prove a conjecture (Theorem 4.1) proposed in [22] on the proportion of unimodular polynomial matrices which generalizes Theorem 1.5.

## 2. Simple Linear Transformations

We begin by recalling the notion of a simple linear transformation [22, Def. 3.1].

**Definition 2.1.** Let \( V \) denote a vector space over a field \( F \) and let \( W \) be a subspace of \( V \). An \( F \)-linear transformation \( T : W \to V \) is simple if the only \( T \)-invariant subspace properly contained in \( V \) is the zero subspace.

The following proposition elucidates the connection between simple linear transformations and unimodularity.

**Proposition 2.2.** Let \( V \) be an \( n \)-dimensional vector space over \( F \) with ordered basis \( \mathcal{B}_n = \{v_1, \ldots, v_n\} \). Let \( \mathcal{B}_k = \{v_1, \ldots, v_k\} \) denote the ordered basis for the subspace \( W \) spanned by \( v_1, \ldots, v_k \). Let \( T : W \to V \) be a linear transformation and let \( A \in M_{n,k}(F) \) denote the matrix of \( T \) with respect to \( \mathcal{B}_k \) and \( \mathcal{B}_n \). Then \( T \) is simple if and only if \( xI_{n,k} - A \) is unimodular.

**Proof.** See [22, Prop. 2.5] and [22, Prop. 3.2]. \( \square \)

Let \( r \) be a positive integer and let \( d_1 \geq d_2 \geq \cdots \geq d_r \) be a nonincreasing sequence of integers with \( d_1 \geq 0 \). Let \( N(d_1, \ldots, d_r) \) denote the number of \( r \)-tuples \( (f_1, \ldots, f_r) \) of polynomials over \( \mathbb{F}_q \) such that \( f_1 \) is monic of degree \( d_1 + 1 \) and \( \deg f_i \leq d_i \) for \( 2 \leq i \leq r \) with \( \gcd(f_1, \ldots, f_r) = 1 \). We adopt the convention that the degree of the zero polynomial is \(-\infty\). For instance, \( N(5,3,-2) = \)
\(N(5, 3, -1) = N(5, 3)\). More generally, if there is some \(s\) such that \(d_i < 0\) for each \(s < i \leq r\), then \(N(d_1, \ldots, d_r) = N(d_1, \ldots, d_s)\).

We adapt an argument in the proof of [6, Thm. 4.1] to prove the following lemma which is central to our main result.

**Lemma 2.3.** Let \(r\) be a positive integer and let \(d_1 \geq d_2 \geq \cdots \geq d_r \geq 0\) be a sequence of integers. We have

\[
N(d_1, \ldots, d_r) = q^{k+r} - q^{k+1},
\]

where \(k = d_1 + \cdots + d_r\).

**Proof.** Fix a positive integer \(r\). Let \(S(d_1, \ldots, d_r)\) denote the set of ordered \(r\)-tuples \((f_1, \ldots, f_r)\) where \(f_1\) is monic of degree \(d_1 + 1\) and \(\deg f_i \leq d_i\) for \(2 \leq i \leq r\). We partition \(S(d_1, \ldots, d_r)\) into disjoint subsets \(S_0, S_1, \ldots, S_{d_1+1}\) where the set \(S_d(0 \leq d \leq d_1 + 1)\) denotes the set of \(r\)-tuples in \(S(d_1, \ldots, d_r)\) whose GCD is a monic polynomial of degree \(d\). For each monic polynomial \(h\) over \(\mathbb{F}_q\) of degree \(d\) and any coprime \(r\)-tuple \((g_1, \ldots, g_r)\) of polynomials in \(S(d_1 - d, \ldots, d_r - d)\), it is easy to see that \((g_1 h, g_2 h, \ldots, g_r h) \in S_d\). Conversely, for any tuple \((f_1, \ldots, f_r) \in S_d\), the polynomial \(h = \gcd(f_1, \ldots, f_r)\) is monic of degree \(d\) and \((f_1/h, \ldots, f_r/h)\) is an ordered \(r\)-tuple of coprime polynomials in \(S(d_1 - d, \ldots, d_r - d)\). As a result, we have \(|S_d| = q^d N(d_1 - d, \ldots, d_r - d)\) for \(0 \leq d \leq d_1 + 1\). If we set \(k = d_1 + \cdots + d_r\), we see that

\[
q^{k+r} = \sum_{d=0}^{d_1+1} |S_d| = \sum_{d=0}^{d_1+1} q^d N(d_1 - d, \ldots, d_r - d). \tag{1}
\]

Replacing \(d_i\) by \(d_i + 1\) for each \(1 \leq i \leq r\), we obtain

\[
q^{k+2r} = \sum_{d=0}^{d_1+2} q^d N(d_1 + 1 - d, \ldots, d_r + 1 - d)
\]

\[
= \sum_{d=-1}^{d_1+1} q^{d+1} N(d_1 - d, \ldots, d_r - d)
\]

\[
= N(d_1 + 1, \ldots, d_r + 1) + q \sum_{d=0}^{d_1+1} q^d N(d_1 - d, \ldots, d_r - d)
\]

\[
= N(d_1 + 1, \ldots, d_r + 1) + q(q^{k+r}),
\]

where the last equality follows from (1). It follows that \(N(d_1 + 1, \ldots, d_r + 1) = q^{k+2r}(1 - q^{1-r})\), or equivalently, \(N(d_1, \ldots, d_r) = q^{k+r} - q^{k+1}\) as desired. \(\square\)

The following lemma is our main result.

**Lemma 2.4.** Let \(Y \in M_{n,k}(\mathbb{F}_q)\) be such that the linear matrix polynomial \(xI_{n,k} - Y\) is unimodular. For each column vector \(b \in \mathbb{F}_q^n\) let \(Y_b = [Y \ b] \in M_{n,k+1}(\mathbb{F}_q)\).
Then the number of column vectors \( \mathbf{b} \in \mathbb{F}_q^n \) for which \( xI_{n,k+1} - Y_b \) is unimodular equals \( q^n - q^{k+1} \).

**Proof.** Suppose that \( Y = \begin{bmatrix} A \\ C \end{bmatrix} \) for some \( A \in M_k(\mathbb{F}_q) \) and \( C \in M_{n-k,k}(\mathbb{F}_q) \). Since \( Y = xI_{n,k} - Y \) is unimodular, it follows that \( (A^t, C^t) \) is a reachable pair [23, Def. 3.3.1]. Suppose that \( \text{rank}(C) = r \), and \( k_1 \geq k_2 \geq \cdots \geq k_r > k_{r+1} = \cdots = k_{n-k} = 0 \) are the controllability indices [23, Def. 5.2.6] of the pair \( (A^t, C^t) \). We have \( k_1 + \cdots + k_r = k \). By a result of Brunovský ([1] or [27, Lem. 2.7]) we may assume that \( A \) and \( C \) are of the following form:

\[
A = \text{diag}(A_1, A_2, \ldots, A_r), \quad \text{where } A_i \text{ is the } k_i \times k_i \text{ matrix } \begin{bmatrix} 0 & 0 \\ I_{k_i-1} & 0 \end{bmatrix};
\]

\[
C = \begin{bmatrix} C^t \\ 0 \end{bmatrix}, \quad \text{where } C^t = [E_1 \cdots E_r] \in M_{r,k}(\mathbb{F}_q) \text{ with } E_i = [0 e_i] \in M_{r,k_i}(\mathbb{F}_q),
\]

and \( e_i \) denotes the \( i \)th column of the \( r \times r \) identity matrix for \( 1 \leq i \leq r \).

Let \( \mathbf{b} = (b_1, b_2, \ldots, b_n)^T \in \mathbb{F}_q^n \). Then the matrix \( Y_b = [Y \ b] \) is of the form

\[
(2) \quad Y_b = \begin{bmatrix}
A_1 & 0 & \ldots & 0 & b_1 \\
0 & A_2 & \ldots & 0 & b_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & A_r & b_r \\
E_1 & E_2 & \ldots & E_r & b' \\
0 & 0 & \ldots & 0 & b
\end{bmatrix},
\]

where \( b_i = (b_{k_1+i-1+k_{i-1}+1}, \ldots, b_{k_{i-1}+i})^T \in \mathbb{F}_q^{k_i} \) for \( 1 \leq i \leq r \), \( b' = (b_{k+1}, \ldots, b_{k+r})^T \in \mathbb{F}_q^r \), and \( b = (b_{k+r+1}, \ldots, b_n)^T \in \mathbb{F}_q^{n-k-r} \) with the assumption that \( k_0 = 0 \).

Now consider the linear matrix polynomial \( Y_b = xI_{n,k+1} - Y_b \). We permute the rows of \( Y_b \) in the following way: for each \( 1 \leq i \leq r - 1 \), arrange the \((k+i)^{\text{th}}\) row of \( Y_b \) in between the \( i^{\text{th}} \) and \((i+1)^{\text{th}}\) block rows appearing in (2). The resulting matrix \( Z \) is of the following form:

\[
(3) \quad Z = \begin{bmatrix}
Z_1 & 0 & \ldots & 0 & b_1' \\
0 & Z_2 & \ldots & 0 & b_2' \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & Z_r & b_r' \\
0 & 0 & \ldots & 0 & b
\end{bmatrix},
\]

where \( Z_i = xI_{k_i} - [0, 0] \), \( b_1' = \begin{bmatrix} -b_1 \\ x - b_{k+1} \end{bmatrix} \) and \( b_i' = \begin{bmatrix} -b_i \\ -b_{k+i} \end{bmatrix} \) for \( 2 \leq i \leq r \). Now we apply the following sequence of elementary row operations to \( Z \) to eliminate \( x \) in the first \( k \) columns: in the first block row as shown in (3), add \( x \) times the \((i+1)^{\text{th}}\) row to the \( i^{\text{th}} \) row successively for \( i = k_1, k_1 - 1, \ldots, 1 \) in that order. Similarly we apply elementary row operations to the other block rows. By appropriate elementary column operations, the entries in the last column can be
made zero at suitable positions. Eventually we can transform the matrix to the following form:

\[
Z' = \begin{bmatrix}
Z'_1 & 0 & \ldots & 0 & \hat{b}_1 \\
0 & Z'_2 & \ldots & 0 & \hat{b}_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & Z'_r & \hat{b}_r \\
0 & 0 & \ldots & 0 & b
\end{bmatrix},
\]

where \(Z'_i = -\left[\begin{smallmatrix} 0 \\ I_{k_i} \end{smallmatrix}\right], \hat{b}_i = \left[\begin{smallmatrix} f_i \\ 0 \end{smallmatrix}\right]\) with \(f_1 = x^{k_{i+1}} - b_{k+1}x^{k_1} - \sum_{j=0}^{k_1-1} b_{1+j}x^j\) and \(f_i = -b_{k+i}x^{k_i} - \sum_{j=0}^{k_i-1} b_{k_1+\ldots+k_{i-1}+1+j}x^j\) for \(2 \leq i \leq r\).

Observe that if \(\hat{b} \neq 0\) then \(Z'\) is unimodular. In the case when \(\hat{b} = 0\), the \((k+1)^{th}\) determinantal divisor (the GCD of all \((k+1) \times (k+1)\) minors) of \(Z'\) is equal to \(g = \gcd(f_1, f_2, \ldots, f_r)\). By Lemma 2.3 it follows that the number of vectors \(b \in \mathbb{F}_q^n\) such that \(g \neq 1\) is given by \(q^{k+1}\). Therefore the number of vectors \(b \in \mathbb{F}_q^n\) such that \(Z'\) is unimodular is given by \((q^n - q^{k+1})\). Since \(Z'\) and \(Y_b\) are equivalent, the result follows.

The lemma can be recast in the setting of linear transformations as follows.

**Corollary 2.5.** Let \(n, k\) be integers with \(0 \leq k < n - 1\). Let \(V\) be an \(n\)-dimensional vector space over \(\mathbb{F}_q\) and let \(W, W'\) be fixed subspaces of \(V\) of dimensions \(k\) and \(k + 1\) respectively with \(W \subseteq W'\). Suppose \(T : W \rightarrow V\) is a simple linear transformation. Then the number of simple linear transformations \(T' : W' \rightarrow V\) such that \(T'|_W = T\) (the restriction of \(T'\) to \(W\) is \(T\)) is equal to \(q^n - q^{k+1}\).

**Proof.** First suppose \(k = 0\). In this case \(W\) is spanned by some nonzero vector \(w\). Then \(T\) is simple precisely when \(T(w)\) does not lie in the span of \(w\). The number of such linear transformations is clearly \(q^n - q\). For \(k \geq 1\), the corollary follows from Lemma 2.4 by considering the matrix of \(T\) with respect to suitable bases for \(W\) and \(V\).

We can now give an alternate proof of [22, Thm. 3.8] concerning the number of simple linear transformations with a fixed domain.

**Corollary 2.6.** Let \(V\) be an \(n\)-dimensional vector space over \(\mathbb{F}_q\) and \(W\) be a proper \(k\)-dimensional subspace of \(V\). The number of simple linear transformations \(T : W \rightarrow V\) equals

\[
\prod_{i=1}^{k} (q^n - q^i).
\]

We may use Proposition 2.2 to reformulate the corollary in terms of matrices. This allows us to answer Question 1.1 stated in the introduction.
Corollary 2.7. Let $n, k$ be positive integers with $k < n$. The number of matrices $A \in M_{n,k}(\mathbb{F}_q)$ such that $xI_{n,k} - A$ is unimodular equals
\[
\prod_{i=1}^{k}(q^n - q^i).
\]

By repeated application of Lemma 2.4 we obtain the following extension which is used later on in Sections 3 and 4.

Lemma 2.8. Let $n, k, t$ be positive integers such that $k + t < n$. Suppose that the matrix polynomial $xI_{n,k} - Y$ is unimodular for some $Y \in M_{n,k}(\mathbb{F}_q)$. The number of matrices $A \in M_{n,t}(\mathbb{F}_q)$ such that the matrix polynomial $xI_{n,k+t} - [Y A]$ is unimodular is equal to $\prod_{i=1}^{t}(q^n - q^{k+i})$.

3. Splitting Subspaces

Recall the definition of splitting subspace given earlier in the introduction.

Definition 3.1. Let $d, m$ be positive integers and consider the vector space $\mathbb{F}_{q^{md}}$ over $\mathbb{F}_q$. For any element $\alpha \in \mathbb{F}_{q^{md}}$ we say that an $m$-dimensional subspace $W$ of $\mathbb{F}_{q^{md}}$ is $\alpha$-splitting if
\[
\mathbb{F}_{q^{md}} = W \oplus \alpha W \oplus \cdots \oplus \alpha^{d-1} W.
\]

Closely related to splitting subspaces are block companion matrices which we define below.

Definition 3.2. For positive integers $m, d$, an $(m, d)$-block companion matrix over $\mathbb{F}_q$ is a matrix in $M_{md}(\mathbb{F}_q)$ of the form
\[
\begin{pmatrix}
0 & 0 & 0 & \ldots & 0 & 0 & C_0 \\
I_m & 0 & 0 & \ldots & 0 & 0 & C_1 \\
& \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
& \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & \ldots & I_m & 0 & C_{d-2} \\
0 & 0 & 0 & \ldots & 0 & I_m & C_{d-1}
\end{pmatrix}
\]

where $C_0, C_1, \ldots, C_{d-1} \in M_m(\mathbb{F}_q)$ and $I_m$ denotes the $m \times m$ identity matrix over $\mathbb{F}_q$ while 0 denotes the zero matrix in $M_m(\mathbb{F}_q)$.

Remark 3.3. It was shown (see the discussion after Conjecture 5.5 in [8] or Appendix A in [9] for an overview) that the Splitting Subspace Theorem is in fact equivalent to the following theorem on block companion matrices.
Theorem 3.4. For any irreducible polynomial \( f \in \mathbb{F}_q[x] \) of degree \( md \), the number of \((m, d)\)-block companion matrices over \( \mathbb{F}_q \) having \( f \) as their characteristic polynomial equals
\[
q^{m(m-1)(d-1)} \prod_{i=1}^{m-1} (q^m - q^i).
\]

It is noteworthy that the problem of counting specific types of block companion matrices having irreducible characteristic polynomial has been considered in other contexts \([3, 12, 21]\) where pseudorandom number generation is of interest. We now deduce Theorem 3.4 as a special case of Theorem 3.9 which we prove below, thereby providing an alternate proof of the Splitting Subspace Theorem.

Definition 3.5. For positive integers \( k, \ell \) with \( k < \ell \), let \( J_{\ell,k} \) denote the \( \ell \times k \) matrix given by
\[
J_{\ell,k} := \begin{bmatrix} 0 \\ I_k \end{bmatrix}.
\]

Lemma 3.6. The linear matrix polynomial
\[
x \begin{bmatrix} I_k \\ 0 \end{bmatrix} - J_{\ell,k}
\]
is unimodular.

Proof. Since the \( k \times k \) minor formed by the last \( k \) rows of the above matrix polynomial equals 1 it follows that the GCD of all \( k \times k \) minors is 1. \( \square \)

Definition 3.7. Let \( m, \ell \) be positive integers such that \( m < \ell \). An \( m \)-companion matrix of order \( \ell \) over \( \mathbb{F}_q \) is a square matrix \( C \) of the form
\[
C = [J_{\ell,\ell-m} A]
\]
for some \( A \in M_{\ell,m}(\mathbb{F}_q) \). We denote the set of all \( m \)-companion matrices of order \( \ell \) over \( \mathbb{F}_q \) by \( \mathcal{C}(\ell, m; q) \). Note that \( |\mathcal{C}(\ell, m; q)| = q^{\ell m} \).

Let \( \mathcal{P}(\ell, \mathbb{F}_q) \) denote the set of all monic polynomials of degree \( \ell \) over \( \mathbb{F}_q \). Now consider the map \( \Phi : \mathcal{C}(\ell, m; q) \to \mathcal{P}(\ell, \mathbb{F}_q) \) given by
\[
\Phi(C) := \det(xI_{\ell} - C).
\]
To determine the size of the fibers of \( \Phi \), we require a theorem of Wimmer.

Theorem 3.8 (Wimmer). Let \( F \) be an arbitrary field and let \( Y \in M_{\ell,k}(F) \). Suppose \( f(x) \in F[x] \) is a monic polynomial of degree \( \ell \) and let \( f_1(x) | \cdots | f_k(x) \) be the invariant factors of the polynomial matrix \( xI_{\ell,k} - Y \). There exists a matrix \( Z \in M_{\ell,\ell-k}(F) \) such that the block matrix \( [Y \ Z] \) has characteristic polynomial \( f(x) \) if and only if the product \( \prod_{i=1}^{k} f_i(x) \) divides \( f(x) \).

Proof. See Wimmer \([26]\) or Cravo \([5, \text{Thm. 15}]\). \( \square \)
Theorem 3.9. Suppose that $f \in \mathcal{P}(\ell, \mathbb{F}_q)$ is irreducible. Then
\[
|\Phi^{-1}(f)| = \prod_{t=1}^{m-1}(q^\ell - q^{\ell-t}).
\]

Proof. Let $C = [J^\ell,\ell - m] A] \in \mathcal{C}(\ell, m; q)$ with $A = [a_1 \ a_2 \ \cdots \ a_{m-1} \ a_m]$, where the $a_i$’s are the columns of $A$. Let $C_0 = J^\ell,\ell - m$ and let $C_i = [J^\ell,\ell - m \ a_1 \ a_2 \ \cdots \ a_i]$ denote the submatrix of $C$ formed by the first $\ell - m + i$ columns for $1 \leq i < m$. Suppose that $\Phi(C) = f$. Since $f$ is irreducible, it follows by Lemma 3.6 and Wimmer’s theorem that the linear matrix polynomials
\[
x \begin{bmatrix} I_{\ell - m + i} & 0 \\ \end{bmatrix} - C_i
\]
are unimodular for $0 \leq i \leq m - 1$. Conversely, if $a_1, \ldots, a_{m-1}$ are chosen such that the matrix polynomials in (6) are unimodular, then there is a unique choice of $a_m$ for which $\Phi(C) = f$. This follows since there are $q^\ell$ total choices for $a_m$ and for each monic polynomial $g$ of degree $\ell$, Wimmer’s theorem ensures that there exists some choice of $a_m$ such that the characteristic polynomial is $g$. By Lemma 2.8 it follows that the number of choices for the first $m - 1$ columns of $A$ is equal to $\prod_{i=1}^{m-1}(q^\ell - q^{\ell-m+i})$ which proves the result. \qed

Remark 3.10. In the case where $m$ divides $\ell$, say $d = \ell/m$, the set $\mathcal{C}(\ell, m; q)$ consists precisely of all $(m, d)$-block companion matrices over $\mathbb{F}_q$. This observation yields the following corollary stated earlier as Theorem 3.4.

Corollary 3.11. For any irreducible polynomial $f \in \mathbb{F}_q[x]$ of degree $md$, the number of $(m, d)$-block companion matrices over $\mathbb{F}_q$ having $f$ as their characteristic polynomial equals
\[
\prod_{i=1}^{m-1}(q^{md} - q^{m(d-1)+i}).
\]

In light of the above corollary and Remark 3.3 we can view Theorem 3.9 as a more general result than the Splitting Subspace Theorem. While our proof relies on results in control theory, it is shorter than the proofs of the theorem appearing in [2] and [15].

4. Probability of Unimodular Polynomial Matrices

We apply Lemma 2.8 to positively resolve a conjecture [22, Conj. 4.1] concerning the number of unimodular polynomial matrices. For positive integers $d, k, n$ with $k < n$, define
\[
M_{n,k}(\mathbb{F}_q[x]; d) := \left\{ A = x^d I_{n,k} + \sum_{i=0}^{d-1} x^i A_i : A_i \in M_{n,k}(\mathbb{F}_q) \text{ for } 0 \leq i \leq d - 1 \right\}.
\]
Theorem 4.1. The probability that a uniformly random element of $M_{n,k}(\mathbb{F}_q[x];d)$ is unimodular is given by $\prod_{i=1}^{k}(1 - q^{i-n})$.

Proof. To each element $A$ in $M_{n,k}(\mathbb{F}_q[x];d)$, we associate the corresponding $d$-tuple of its coefficients $(A_0, A_1, \ldots, A_{d-1}) \in [M_{n,k}(\mathbb{F}_q)]^d$. Now consider the matrix

$$B = \begin{bmatrix} 0 & 0 & \ldots & 0 & -A_0 \\ I_n & 0 & \ldots & 0 & -A_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & I_n & -A_{d-1} \end{bmatrix}$$

of dimension $nd \times (nd - n + k)$. Let $B = x [I_{(d-1)n+k}] - B$. By adding $x$ times the $i$th block row to the $(i-1)$th block row successively for $i = d, d-1, \ldots, 2$ in $B$ and using suitable column block operations, we obtain

$$B' = \begin{bmatrix} 0 & 0 & \ldots & 0 & A \\ I_n & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & I_n & 0 \end{bmatrix},$$

where $A = x^dI_{n,k} + \sum_{i=0}^{d-1} x^i A_i \in M_{n,k}(\mathbb{F}_q[x];d)$. Observe that $B$ is equivalent to $B'$. So the invariant factors of $B$ and $B'$ are the same. Therefore $B$ is unimodular if and only if $A$ is unimodular. By Lemma 2.8, the number of ways to choose the last $k$ columns of the matrix $B$ in (7) in such a way that $B$ is unimodular is

$$\prod_{i=1}^{k}(q^{nd} - q^{n(d-1)+i}).$$

On the other hand, the cardinality of $M_{n,k}(\mathbb{F}_q[x];d)$ is clearly $q^{nkd}$ and therefore the probability that a uniformly random element of $M_{n,k}(\mathbb{F}_q[x];d)$ is unimodular is precisely $\prod_{i=1}^{k}(1 - q^{i-n})$. $\square$

Note that the probability computed in the theorem is independent of $d$.

Remark 4.2. The above theorem is a generalization of Corollary 2.7 which is evidently the special case $d = 1$.

Theorem 4.1 parallels a result of Guo and Yang [11, Thm. 1] who prove that the natural density of unimodular $n \times k$ matrices over $\mathbb{F}_q[x]$ is precisely $\prod_{i=1}^{k}(1 - q^{i-n})$.

Remark 4.3. To study the invariant factors of an element $A \in M_{n,k}(\mathbb{F}_q[x];d)$, it suffices to study those of the corresponding linear matrix polynomial $B$ associated to the matrix $B$ as defined in equation (7). The matrix polynomial $B$ is called the linearization of $A$. 
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