Dynamical Role of Scalar Fields in K = 0 Universe

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Abstract:

It is an accepted and well examined practice in cosmology to invoke a suitable form of scalar field with potential $V(\phi)$ when the observed evolution of the universe cannot be reconciled with theoretical believes. In this article we investigate the role of a class of scalar fields in cosmology under minimal and maximal coupling with background matter. We further analyze the dynamical behavior of scalar fields through its scaling solutions by considering their constant equation of state parameter.

1 Introduction:

In standard cosmology the accelerated expansion of the universe consider as one of the fundamental aspects after its observational support in 1998 of Supernova Ia [1-2]. To explain the observed accelerated expansion concept of dark energy introduced as modified matter. The remarkable feature of this dark energy is its negative pressure provide it negative equation of state $w_{\text{dark energy}} < -1/3$ and this negative pressure dominate over its energy density gives the accelerated expansion of the universe. To follow the mechanism of accelerated expansion of the universe there are several candidate of dark energy proposed. Cosmological constant is one of the simplest candidate of dark energy bear the equation of state $(\omega_0 = -1)$. The dominated cosmological constant in the universe provide the exponential expansion of the universe and corresponding scenario known as inflation. Cosmological constant as a candidate of dark energy suffer by two serious problem one is cosmological constant problem and other is coincidence problem. To alleviate these two issue of $\Lambda$CDM model the interacting dark energy model proposed by a number of authors [3-11]. Apart from cosmological constant different class of scalar field proposed as candidate of dynamical dark energy including tachyonic scalar field proposed in string theory [12-14].

In the absence of fundamental theory of dark sector physics the choice of coupling strength in interacting dark energy model is purely phenomenological and motivated from left hand side of continuity equation of energy leads to it should by either linear function of Hubble parameter and energy density or function of first time derivative of energy density.

In this paper we discuss mainly the different scaling solution of continuity equation for different possible phenomenological choice of coupling strength in the interacting dark energy scenario. We
consider scalar field is candidate of dynamical dark energy as in development of scaling solution.

1.1 Action and Lagrangian of Tachyonic Scalar Field:

It has been suggested by Sen [12] the tachyonic condensate in a class of string theories can be described by an effective scalar field under minimally coupling with a Lagrangian of the form,

$$S = \int d^4x \sqrt{-g} \left( \frac{\mathcal{R}}{16\pi G} - V(\phi)\sqrt{1 - \partial^\mu \phi \partial_\mu \phi} \right)$$

(1)

with Lagrangian

$$L = -V(\phi)\sqrt{1 - \partial^\mu \phi \partial_\mu \phi}$$

by using the Eular-Lagrange equation, $T^{\mu\nu} = \frac{\partial L}{\partial (\partial_\mu \phi)} \partial^\nu \phi - g^{\mu\nu} L$; we get the energy density for $\mu = 0, \nu = 0$ component and the spatial components gives rise to pressure,

$$\rho = \frac{V(\phi)}{\sqrt{1 - \phi^2}}; \quad p = -V(\phi)\sqrt{1 - \phi^2}$$

respectively for spatially flat FLWR universe with signature +2.

2 Forms of Interaction Strength:

Here we study the dynamics of interacting tachyonic scalar fields for a spatially flat FLWR universe with line element $ds^2 = -dt^2 + a(t)(dx^a)^2$; where $a(t)$ is the scalar expansion factor with cosmic time $t$ [15, 16].

The Friedmann equation from the conservation of stress energy tensor is given by, $H^2 = \frac{8\pi G}{3} \sum \rho_i + \rho_\Lambda$ [23]; where $H = \left(\frac{\dot{a}}{a}\right)$ the Hubble parameter. The summation in equation of motion runs from $i = r$ for radiation, $i = m$ for matter (Baryonic + Darkmatter) energy densities and $\rho_\Lambda$ is the energy density due to dark energy. According to standard cosmological model 70% of the energy of the universe corresponds to dark energy, remaining ~ 30% matter (Baryon + Darkmatter) and almost negligible ~ 0.003% is radiation energy (duetophotons). Hence we may neglect the contribution of radiation energy density in universe at a large scale which modifies the Friedmann equation as, $H^2 = \frac{8\pi G}{3} \rho_m + \rho_\Lambda$. The nature of both dark energy and dark matter is still unknown but prevailing opinion assumes dark energy to be a cosmological constant ($\omega_\Lambda = -1$) whilst dark matter is modeled as a nonrelativistic fluid.

Equation of state for dark energy and matter is given by, $\omega_\Lambda = \frac{p_\Lambda}{\rho_\Lambda}$ and $\omega_m = \frac{p_m}{\rho_m}$. For tachyonic scalar fields the equation of state varies as $-1 \leq \omega \leq 0$ where $\omega = 0$ mimics dust matter with negative energy density and $\omega = -1$ denotes the proper cosmological constant. The scalar field obeys the continuity equation, $\dot{\rho} + 3H(1 + \omega)\rho = 0$ where for dark energy and matter the individual equation of continuity given by,

$$\dot{\rho}_\Lambda + 3H(1 + \omega_\Lambda)\rho_\Lambda = -Q; \quad \dot{\rho}_m + 3H(1 + \omega_m)\rho_m = Q$$

(2)
We considered a non zero interaction \( Q \) between dark energy and matter energy density which might be arising due to thermodynamic laws (Zeroth, First and Second) [17]. In LHS of equation (2) the dimensionality given as \( Jm^{-3}s^{-1} \) as \( H \) has a dimension of \( s^{-1} \). So keeping this in mind the possible forms of \( Q \) are,

\[
\begin{align*}
(A) \quad & Q = \alpha H \rho_\lambda \\
(B) \quad & Q = \beta H \rho_m \\
(C) \quad & Q = \gamma \dot{\rho}_\lambda \\
(D) \quad & Q = \delta \dot{\rho}_m \\
(E) \quad & Q = \alpha H \rho_\lambda + \beta H \rho_m \\
(F) \quad & Q = \gamma \dot{\rho}_\lambda + \delta \dot{\rho}_m
\end{align*}
\]

Where \( \alpha, \beta, \gamma, \delta \) are unknown interaction strength coefficients. The thermodynamics of interacting tachyonic scalar fields and the form of \( Q \) described in model (C),(D) has already been analysed [17]. Many different forms has also been proposed by several authors \( Q \) [18-21].

3 Scaling Solutions:

In the previous section, 6 possible forms of interaction strength has been discussed. In the following section we elaborated the possibility of distinct models arising due to the distinguishable forms of interaction strength.

3.1 Model A:

Considering the form of \( Q \) as, \( Q = \alpha H \rho_\lambda \),

\[
\dot{\rho}_\lambda + 3H(1 + \omega_\lambda)\rho_\lambda = -\alpha H \rho_\lambda \tag{3}
\]

\[
\dot{\rho}_m + 3H(1 + \omega_m)\rho_m = \alpha H \rho_\lambda \tag{4}
\]

from equation (3a) we get,

\[
\frac{\dot{\rho}_\lambda}{\rho_\lambda} = -3H \left(1 + \omega_\lambda + \frac{\alpha}{3}\right)
\]

Above equation gives the generalized masters equation for the evolution of \( \rho_\lambda \). It can be further simplified by considering \( \omega_\lambda = -1 \), which nothing but the well known cosmological constant.
\[ \int \rho_\lambda \frac{d\rho_\lambda}{\rho_\lambda} = -3 \int_0^a \frac{\alpha \, da}{a} \]

\[ \ln \left( \frac{\rho_\lambda}{\rho_\Lambda^0} \right) = \ln \left( \frac{a}{a_0} \right)^{-\alpha} \]

\[ \left( \frac{\rho_\lambda}{\rho_\Lambda^0} \right) = \left( \frac{a}{a_0} \right)^{-\alpha} \quad (5) \]

hence it is perfectly cleared from equation (4) that, \( \rho_\lambda \propto a^{-\alpha} \). And at \( \alpha = 0 \), \( \rho_\lambda = \text{constant} \). Which mimics the outcome of non interacting tachyonic \( \Lambda \)CDM model [22] and indicates a constant energy density of dark energy with expansion. Eventually equation (3\(b\)) gives,

\[ \dot{\rho}_m + 3H(1 + \omega_m)\rho_m = \alpha H \rho_\lambda \]

from equation (4) substituting the value of \( \rho_\lambda \) on above equation,

\[ \dot{\rho}_m + 3H(1 + \omega_m)\rho_m = \alpha H \rho_\Lambda^0 \left( \frac{a}{a_0} \right)^{-\alpha} \]

after solving the above equation and by substituting \( H = \frac{d}{a} \) we arrive at a differential equation,

\[ \frac{d\rho_m}{da} + 3(1 + \omega_m) \frac{\rho_m}{a} = \frac{\alpha \rho_\Lambda^0}{a_0^2} a^{-(\alpha + 1)} \quad (6) \]

whose solution can be given as,

\[ \rho_m \exp \left[ 3 \int_{a_0}^a (1 + \omega_m) \frac{da}{a} \right] = \frac{\alpha \rho_\Lambda^0}{a_0^2} \int_{a_0}^a a^{-(\alpha + 1)} \exp \left[ 3 \int_{a_0}^a (1 + \omega_m) \frac{da}{a} \right] \] d \alpha + c

The above equation gives the master equation which indicates evolution of \( \rho_m \) with expansion of the universe where \( c \) is integration constant throughout the article. Hence the previous equation gives,

\[ \left( \frac{\rho_m}{\rho_m^0} \right) = \left( \frac{a}{a_0} \right)^{-3(1+\omega_m)} + \frac{\alpha \rho_\Lambda^0}{\rho_m^0(\alpha-3(1+\omega_m))} \left[ \left( \frac{a}{a_0} \right)^{-3(1+\omega_m)} - \left( \frac{a}{a_0} \right)^{-\alpha} \right] \quad (7) \]

further when \( \alpha = 0 \), \( \left( \frac{\rho_m}{\rho_m^0} \right) = \left( \frac{a}{a_0} \right)^{-3(1+\omega_m)} \) which depicts the same result when there is no coupling as in \( \Lambda \)CDM model [22-23].

3.2 Model B:

Considering, \( Q = \beta H \rho_m \).

\[ \dot{\rho}_m + 3H(1 + \omega_m)\rho_m = \beta H \rho_m \quad (8) \]
\[
\dot{\rho}_\lambda + 3H(1 + \omega_\lambda)\rho_\lambda = -\beta H \rho_m
\]

From equation (7a) we get,
\[
\frac{\dot{\rho}_m}{\rho_m} = -3H \left( 1 + \omega_\lambda - \frac{\beta}{3} \right)
\]
which leads to the solution as,
\[
\int \frac{\rho_m}{\dot{\rho}_m} d\rho_m = -3 \int_{a_0}^{a} \left( 1 + \omega_m - \frac{\beta}{3} \right) \frac{da}{a}
\]
which can be generalized as,
\[
\left( \frac{\rho_m}{\dot{\rho}_m} \right) = \left( \frac{a}{a_0} \right)^{-3(1 + \omega_m - \frac{\beta}{3})}
\]
where \( \rho_m \propto a^{-3(1 + \omega_m - \frac{\beta}{3})} \). Which indicates rapidly increasing energy density with \( \beta \). But at \( \beta = 0 \), \( \rho_m \) mimics the energy density of zero coupling. We defined the expression of \( \rho_\lambda \) as from (7b) as,
\[
\dot{\rho}_\lambda + 3H(1 + \omega_\lambda)\rho_\lambda = -\beta H \rho_m
\]
putting the value of \( \rho_m \) from equation (8),
\[
\dot{\rho}_\lambda + 3H(1 + \omega_\lambda)\rho_\lambda = -\beta H \rho_m^0 \left( \frac{a}{a_0} \right)^{-3(1 + \omega_m - \frac{\beta}{3})}
\]
eventually we arrive at the differential form,
\[
\frac{d\rho_\lambda}{da} + 3(1 + \omega_\lambda) \frac{\rho_\lambda}{a} = -\frac{\beta \rho_m^0}{a_0^{-3(1 + \omega_m - \frac{\beta}{3})}} a^{-3(1 + \omega_m - \frac{\beta}{3}) - 1}
\]
(11)
after solving the above differential equation we arrive at an equation,
\[
\rho_\lambda \exp \left[ 3 \int_{-a_0}^{a} (1 + \omega_\lambda) \frac{da}{a} \right] = -\frac{\beta \rho_m^0}{a_0^{-3(1 + \omega_m - \frac{\beta}{3})}} \int_{-a_0}^{a} a^{-3(1 + \omega_m - \frac{\beta}{3}) - 1} \exp \left[ 3 \int_{-a_0}^{a} (1 + \omega_\lambda) \frac{da}{a} \right] da + c
\]
by putting \( \omega_\lambda = -1 \) we get an reduced equation
\[ \rho_\lambda = \frac{\beta \rho_\lambda^m}{3(1+\omega_m-\frac{\beta}{3})} \left( \frac{a}{a_0} \right)^{-3 \left( 1+\omega_m-\frac{\beta}{3} \right)} - 1 \right] + c \]

at \( a = a_0 \) we get, \( \rho_\lambda = \rho_\lambda^0 \) so,

\[ \left( \frac{\rho_\lambda}{\rho_\lambda^0} \right) = 1 + \frac{\beta \rho_\lambda^m}{3(1+\omega_m-\frac{\beta}{3})} \left( \frac{a}{a_0} \right)^{-3 \left( 1+\omega_m-\frac{\beta}{3} \right)} - 1 \]  

(12)

when \( \beta = 0 \), \( \rho_\lambda = \rho_\lambda^0 \). Which once again preserving the result for zero coupling.

### 3.3 Model C:

For \( Q = \gamma \dot{\rho}_\lambda \) we get the solutions as,

\[ \dot{\rho}_\lambda + 3H(1 + \omega_\lambda)\rho_\lambda = -\gamma \dot{\rho}_\lambda \]  

(13)

\[ \dot{\rho}_m + 3H(1 + \omega_m)\rho_m = \gamma \dot{\rho}_\lambda \]  

(14)

the master equation for the evolution of \( \rho_\lambda \) from (11a) is given by,

\[ \left( \frac{\rho_\lambda}{\rho_\lambda^0} \right) = \left( \frac{a}{a_0} \right)^{-3 \left( \frac{1+\omega_\lambda}{1+\gamma} \right)} \]  

(15)

when, \( \omega_\lambda = -1 \), \( \rho_\lambda \) is constant. At the same time from (11b) we get,

\[ \dot{\rho}_m + 3H(1 + \omega_m)\rho_m = \gamma \dot{\rho}_\lambda \]

from equation (12) after putting the value of \( \dot{\rho}_\lambda \) we get the differential form,

\[ \frac{d\rho_m}{da} + 3(1 + \omega_m)\frac{\rho_m}{a} = -3\gamma \kappa \rho_\lambda^0 \frac{a^{-3\kappa-1}}{a_0^{-3\kappa}} \]  

(16)

where \( \kappa = \left( \frac{1+\omega_\lambda}{1+\gamma} \right) \); which gives master equation of evolution of \( \rho_m \) as,

\[ \rho_m \exp \left[ 3 \int_{a_0}^{a} (1 + \omega_m) \frac{da}{a} \right] = -3\gamma \kappa \rho_\lambda^0 \frac{a^{-3\kappa-1}}{a_0^{-3\kappa}} \int_{a_0}^{a} a^{-3\kappa-1} \exp \left[ 3 \int_{a_0}^{a} (1 + \omega_m) \frac{da}{a} \right] da + c \]

\[ = -\frac{3\gamma \kappa \rho_\lambda^0}{[-3\kappa + 3(1 + \omega_m)]} \left( \frac{a}{a_0} \right)^{-3\kappa+3(1+\omega_m)} - 1 \right] + c \]

by using boundary conditions,
As, $\omega_\lambda = -1$ eventually makes $\kappa = 0$. Once again we can observe when there is no coupling i.e. $\gamma = 0$ then, \( \frac{\rho_\lambda}{\rho_m} = \left( \frac{a}{a_0} \right)^{-3(1+\omega_m)} \) which gives the predicted result for no coupling.

### 3.4 Model D:

For $Q = \varepsilon \dot{\rho}_m$ we get the solutions as,

\[ \dot{\rho}_m + 3H(1 + \omega_m)\rho_m = \varepsilon \dot{\rho}_m \quad (18) \]

\[ \dot{\rho}_\lambda + 3H(1 + \omega_\lambda)\rho_\lambda = -\varepsilon \dot{\rho}_m \quad (19) \]

the master equation for evolution of $\rho_m$ is given as,

\[ \left( \frac{\rho_m}{\rho_m^0} \right) = \left( \frac{a}{a_0} \right)^{-3\left(1+\omega_m\right)} \quad (20) \]

which gives $\rho_m \propto a^{-3\left(1+\omega_m\right)}$. Remove the coupling i.e. $\varepsilon = 0$ takes us to the predicted result,

\[ \frac{\rho_m}{\rho_m^0} = \left( \frac{a}{a_0} \right)^{-3(1+\omega_m)} \quad (21) \]

from equation (15b) we get,

\[ \dot{\rho}_\lambda + 3H(1 + \omega_\lambda)\rho_\lambda = -\varepsilon \dot{\rho}_m \]

which gives the master equation for the evolution of $\rho_\lambda$ as,

\[ \rho_\lambda \exp \left[ 3 \int_{a_0}^{a} (1 + \omega_\lambda) \frac{da}{a} \right] = \frac{3\varepsilon \delta \rho_\lambda^0}{a_0^{-3\delta}} \int_{a_0}^{a} a^{-3\delta-1} \exp \left[ 3 \int_{a_0}^{a} (1 + \omega_\lambda) \frac{da}{a} \right] da + c \]

where, $\delta = \left( \frac{1+\omega_\lambda}{1-\varepsilon} \right)$ for $\omega_\lambda = -1$, we get the reduced form of the master equation as,

\[ \rho_\lambda = \frac{3\varepsilon \delta \rho_\lambda^0}{a_0^{-3\delta}} \int_{a_0}^{a} a^{-3\delta-1} da + c \]

after applying boundary conditions we get the following form of the above equation,

\[ \frac{\rho_\lambda}{\rho_\lambda^0} = 1 - \frac{\varepsilon \rho_\lambda^0}{\rho_\lambda} \left[ \left( \frac{a}{a_0} \right)^{-3\left(1+\omega_\lambda\right)} - 1 \right] \quad (22) \]
when there is no coupling then, $\varepsilon = 0$, $\rho_\lambda = \text{constant}$.

### 3.5 Model E:

Considering, $Q = aH\rho_\lambda + \beta H\rho_m$

$$\dot{\rho}_\lambda + 3H(1 + \omega_\lambda)\rho_\lambda = -aH\rho_\lambda - \beta H\rho_m \quad (23)$$

$$\dot{\rho}_m + 3H(1 + \omega_m)\rho_m = aH\rho_\lambda + \beta H\rho_m \quad (24)$$

from equation (19a) we arrive at,

$$\dot{\rho}_\lambda = -3H \left(1 + \omega_\lambda\right) + \frac{\alpha}{3} \rho_\lambda - \beta H\rho_m$$

putting, $H = \frac{a}{a}$

$$\int_{\rho_\lambda}^{\rho_\lambda_0} \frac{d\rho_\lambda}{\rho_\lambda} = -3 \int_{a_0}^{a} \left(1 + \omega_\lambda\right) + \frac{\alpha}{3} + \frac{\beta \rho_m}{\rho_\lambda} \frac{da}{a}$$

when, $\omega_\lambda = -1$; we get,

$$\left(\frac{\rho_\lambda}{\rho_\lambda_0}\right) = \exp \left[-\int_{a_0}^{a} \left(\alpha + \beta \frac{\rho_m}{\rho_\lambda} \frac{da}{a}\right)\right] \quad (25)$$

and by evaluating equation (19b) we get,

$$\int_{\rho_m}^{\rho_\lambda} \frac{d\rho_m}{\rho_m} = -3 \int_{a_0}^{a} \frac{da}{a} \left(1 + \omega_m\right) - \frac{\beta}{3} - \frac{\alpha}{3} \left(\frac{\rho_\lambda}{\rho_m}\right)$$

There are three possible solution of (20) and (21);

**Solution 1:** When $\rho_m \ll \rho_\lambda$ then, equation (20) dominates as the dark energy dominates the interaction and becomes,

$$\frac{\rho_\lambda}{\rho_\lambda_0} = \exp \left[-\int_{a_0}^{a} \alpha \frac{da}{a}\right]$$

$$\left(\frac{\rho_\lambda}{\rho_\lambda_0}\right) = \left(\frac{a}{a_0}\right)^{-\alpha} \quad (27)$$
and, the master equation for $\rho_m$ by using (21) is given by,

$$
\rho_m \exp \left[ 3 \int_{a_0}^a \left( 1 + \omega_m \right) - \frac{\beta}{3} \right] \frac{da}{a} = \frac{\alpha \rho_m^0}{a_0} \int_{a_0}^a a^{-\alpha - 1} \exp \left[ 3 \int_{a_0}^a \left( 1 + \omega_m \right) - \frac{\beta}{3} \right] \frac{da}{a} da + c
$$

after calculating above integral we get,

$$
\left( \frac{\rho_m}{\rho_m^0} \right) = \left( \frac{a}{a_0} \right)^{-3[1+\omega_m]-\frac{\beta}{3}} + \frac{\alpha \rho_m^0}{\rho_m^0} \left[ \frac{a}{a_0} - \left( \frac{a}{a_0} \right)^{-3[1+\omega_m]-\frac{\beta}{3}} \right] \times \left[ \left( \frac{a}{a_0} \right)^{-\alpha} - \left( \frac{a}{a_0} \right)^{-3[1+\omega_m]-\frac{\beta}{3}} \right]
$$

\((28)\)

**Solution 2:** When $\rho_\lambda \ll \rho_m$ then the equation (21) dominates as we can observe matter domination over dark energy while interacting and eventually we arrive at,

$$
\left( \frac{\rho_m}{\rho_m^0} \right) = \left( \frac{a}{a_0} \right)^{-3[1+\omega_m]-\frac{\beta}{3}}
$$

\((29)\)

and the master equation generated for $\rho_\lambda$ by using \((19a)\) and \((20)\),

$$
\rho_\lambda \exp \left[ \int_{a_0}^a \alpha \frac{da}{a} \right] = \frac{\beta \rho_m^0}{\rho_m^0} \left[ \frac{a}{a_0} - \left( \frac{a}{a_0} \right)^{-3[1+\omega_m]-\frac{\beta}{3}} \right] \times \left[ \left( \frac{a}{a_0} \right)^{-\alpha} - \left( \frac{a}{a_0} \right)^{-3[1+\omega_m]-\frac{\beta}{3}} \right]
$$

after calculating the integral we get,

$$
\left( \frac{\rho_m}{\rho_m^0} \right) = \left( \frac{a}{a_0} \right)^{-\alpha} - \frac{\beta \rho_m^0}{\rho_m^0} \left[ \frac{a}{a_0} - \left( \frac{a}{a_0} \right)^{-3[1+\omega_m]-\frac{\beta}{3}} \right] \times \left[ \left( \frac{a}{a_0} \right)^{-\alpha} - \left( \frac{a}{a_0} \right)^{-3[1+\omega_m]-\frac{\beta}{3}} \right]
$$

\((30)\)

**Solution 3:** When $\rho_m = \rho_\lambda$ we can observe equal contribution of matter and dark energy in interaction strength which inturn gives,

$$
\left( \frac{\rho_m}{\rho_m^0} \right) = \left( \frac{a}{a_0} \right)^{-\alpha + \beta}
$$

\((31)\)

which has been generated by putting $\rho_m = \rho_\lambda$ on \((20)\). Now if we put $\rho_m = \rho_\lambda$ on \((21)\) we get,

$$
\left( \frac{\rho_m}{\rho_m^0} \right) = \left( \frac{a}{a_0} \right)^{\alpha + \beta - 3[1+\omega_m]}
$$

\((32)\)

### 3.6 Model F:

By considering, $Q = \gamma \dot{\rho}_\lambda + \varepsilon \dot{\rho}_m$
\[\dot{\rho}_\lambda + 3H(1 + \omega_\lambda)\rho_\lambda = -\gamma \dot{\rho}_\lambda - \varepsilon \dot{\rho}_m \quad (33)\]

\[\dot{\rho}_m + 3H(1 + \omega_m)\rho_m = \gamma \dot{\rho}_\lambda + \varepsilon \dot{\rho}_m \quad (34)\]

from equation (28a) we get,

\[(1 + \gamma)\dot{\rho}_\lambda + 3H(1 + \omega_\lambda)\rho_\lambda = -\varepsilon \dot{\rho}_m \]

\[\dot{\rho}_\lambda + \frac{3(1 + \omega_\lambda)}{(1 + \gamma)} H \rho_\lambda = -\frac{\varepsilon}{(1 + \gamma)} \dot{\rho}_m \]

\[\int_{\rho_\lambda^0}^{\rho_\lambda} \frac{d\rho_\lambda}{\rho_\lambda} + 3 \int_{a_0}^{a} \frac{(1+\omega_\lambda)}{(1+\gamma)} \frac{da}{a} = - \int_{\rho_m^0}^{\rho_m} \frac{\varepsilon}{(1 + \gamma)} \frac{d\rho_m}{\rho_\lambda} \quad (35)\]

similarly from (28b) we get,

\[\int_{\rho_m^0}^{\rho_m} \frac{d\rho_m}{\rho_m} + 3 \int_{a_0}^{a} \frac{(1+\omega_m)}{(1-\varepsilon)} \frac{da}{a} = \int_{\rho_\lambda^0}^{\rho_\lambda} \frac{\gamma}{(1-\varepsilon)} \frac{d\rho_\lambda}{\rho_m} \quad (36)\]

**Solution 1:** When we put \(\rho_m = \rho_\lambda\) on (29) we may say the rate at which dark energy density changes while interacting, is equal to the rate of change in matter energy density which creates a equal contribution on interaction by matter and dark energy,

\[(\frac{\rho_\lambda}{\rho_m^0}) = \left(\frac{a}{a_0}\right)^{-3(1+\omega_\lambda)} \quad (37)\]

and for \(\omega_\lambda = -1\), \(\rho_\lambda = \rho_\lambda^0\). As (29) and (30) are simultaneous equations so, we get, the expression of \(\rho_m\) by putting \(\rho_m = \rho_\lambda\) on RHS of equation (30),

\[(\frac{\rho_m}{\rho_m^0}) = \left(\frac{a}{a_0}\right)^{-3(1+\omega_m)} \quad (38)\]

**Solution 2:** When \(\rho_m \ll \rho_\lambda\); then the dark energy density changes rapidly with cosmic time compared to matter energy density which results a dark energy dominated interaction. And equation (28a) becomes,

\[\dot{\rho}_\lambda + 3H(1 + \omega_\lambda)\rho_\lambda \approx -\gamma \dot{\rho}_\lambda \quad (39)\]

\[\left(\frac{\rho_\lambda}{\rho_\lambda^0}\right) \approx \left(\frac{a}{a_0}\right)^{-3(1+\omega_\lambda)} \quad (40)\]
then, \( \omega_\lambda = -1 \) gives \( \rho_\lambda \approx \dot{\rho}_0^\lambda \) which brings back the well known result for no coupling; and the solution of \( \rho_m \) will be analogous to Model C.

**Solution 3:** When \( \rho_m \gg \rho_\lambda \) we can assume a comic state when the rate of change of matter energy density is dominating compared with dark energy while they are interacting. Which makes a matter dominated interaction and then equation (28b) will be written as,

\[
\dot{\rho}_m + 3H(1 + \omega_m)\rho_m \approx \varepsilon\dot{\rho}_m
\]

(41)

whose solution for \( \rho_\lambda \) analogous to Model D. And when \( \varepsilon = 0 \); we get the well known solution for no coupling.

### 4 Non-Minimally Coupled Scalar Fields:

While constructing the non-minimal gravitational theory of scalar field one must keep in mind that, in general we can observe various forms of these coupling. In case of four derivatives one may came across the terms [24], \( k_1 R \partial^a \phi \partial_a \phi \), \( k_2 R \partial^a \phi \partial^a \phi \), \( k_3 R \partial^a \phi \nabla^a \phi \), \( k_4 R \partial^a \phi \partial^a \phi \), \( k_5 R \partial^a \phi \partial^a \phi \), \( k_6 \nabla^a \nabla_a R \phi \), where \( k_1, k_2, \ldots k_6 \) are coupling parameters with dimensions of square of length. Without the loss of generality in this article we will mostly concern about the first two terms of the coupling. Hence due to the non minimal coupling Lagrangian will be modified as[25][26],

\[
L_{\text{eff}} = L_g + L_m;
\]

\[
L_g = \sqrt{-g} \left[ \Lambda - \frac{R}{16\pi G} + a_1 R^2 + a_2 R_{\alpha\beta} \partial^a \phi \partial^a \phi \right] \approx \sqrt{-g} \frac{R}{16\pi G};
\]

\[
L_m = \sqrt{-g} \left[ (l_{KE} - l_{PE}) + k_1 R g_{\alpha\beta} \partial^a \phi \partial^a \phi + k_2 R_{\alpha\beta} \partial^a \phi \partial^a \phi \right]
\]

considering \( k_1 = \frac{k}{2} \) and \( k_2 = -k \).

\[
L_{\text{eff}} = \sqrt{-g} \left[ \frac{R}{16\pi G} + (l_{KE} - l_{PE}) - \left( R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R \right) \partial^a \phi \partial^a \phi \right] \quad (43)
\]

Einstein Tensor defined by, \( G_{\alpha\beta} = \left( R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R \right) \);

\[
L_{\text{eff}} = \sqrt{-g} \left[ \frac{R}{16\pi G} + (l_{KE} - l_{PE}) - \left( R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R \right) \partial^a \phi \partial^a \phi \right] \quad (44)
\]

\[
L_{\text{eff}} = \sqrt{-g} \left[ \frac{R}{16\pi G} + (l_{KE} - l_{PE}) - G_{\alpha\beta} \partial^a \phi \partial^a \phi \right] \quad (45)
\]

#### 4.1 Tachyonic Scalar Field:

Analogons to equation (39) the action for Tachyonic Scalar Field can be written as,
\[
S = \int d^4x \sqrt{-g} \left( \frac{R}{16\pi} - V(\phi)\sqrt{1 - \partial^\alpha \phi \partial_\alpha \phi - k G_{\alpha\beta} \partial^\alpha \phi \partial^\beta \phi} \right) \tag{46}
\]
\[
L_{T_{SF}} = V(\phi)\sqrt{1 - \partial^\alpha \phi \partial_\alpha \phi + k G_{\alpha\beta} \partial^\alpha \phi \partial^\beta \phi} \tag{47}
\]

For further reading on non-minimally coupled tachyonic scalar fields see [27].

### 4.2 Quintessence and Phantom Scalar Fields:

\[
S = \int d^4x \sqrt{-g} \left( \frac{R}{16\pi} - (\varepsilon g_{\alpha\beta} + k G_{\alpha\beta}) \partial^\alpha \phi \partial^\beta \phi - 2V(\phi) \right) \tag{48}
\]

hence there respective Lagrangians,

\[
L_Q = \frac{1}{2}(g_{\alpha\beta} + k G_{\alpha\beta}) \partial^\alpha \phi \partial^\beta \phi + V(\phi) \tag{49}
\]
\[
L_P = \frac{1}{2}(-g_{\alpha\beta} + k G_{\alpha\beta}) \partial^\alpha \phi \partial^\beta \phi + V(\phi) \tag{50}
\]

Hence a generalised Lagrangian for quintessence and phantom can be obtained by combining (42a) and (42b);

\[
L = \frac{1}{2}(\varepsilon g_{\alpha\beta} + k G_{\alpha\beta}) \partial^\alpha \phi \partial^\beta \phi + V(\phi) \tag{51}
\]

Where \(\varepsilon = -1\) and +1 gives phantom and quintessence models. A Substitution of the parameter \(k\) by \(0\) in equation (40a) and (40b) gives us back equation (1) for minimally coupled tachyonic scalar field and if we substitute \(k = 0\) on (41), (43) we get the well known minimally coupled actions and Lagrangian for quintessence [28] and phantom [29] scalar fields. And, the scaling solutions can be obtained by the same method as discussed in section 3. For a very fine treatise on non-minimally coupled quintessence and phantom cosmology go through [30].

### 5 Conclusion

In contrast to quintessence and phantom model, a well defined treatise on relativistic model i.e. tachyonic scalar field model has been given. Due to some unknown mechanism the single tachyonic scalar field splits into matter, radiation and dark energy components. Without the loss of generality by taking only non-zero interaction between dark energy and matter, which might be arising due to thermodynamic laws; distinguishable scaling solutions has been proposed and analysed on the basis of the variation of interaction coefficient. It has been verified that, by making interaction coefficients in each model to zero we are able to reproduce the \(\Lambda CDM\) model predictions, which states that the dark energy density of the universe is constant. As \(a_0 \frac{da}{dz} = (1 + z)\) where \(z\) gives the redshift, hence it is possible to observe the variation of energy densities with redshift.
We have also discussed the forms action and Lagrangian for tachyonic, quintessence and phantom scalar fields under minimal as well non-minimal coupling with background matter. The scaling solutions can be obtained by following the same path as we have discussed in section 3.

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