Outer-independent $k$-rainbow domination

Qiong Kang $^{a,}$, Vladimir Samodivkin $^{b,}$, Zheui Shao $^{c,}$, Seyed Mahmoud Sheikholeslami $^{d,}$ and Marzieh Soroudi $^{d}$

$^{a}$School of Computer Science, Yangtze University, Jingzhou, People’s Republic of China; $^{b}$Department of Mathematics, University of Architecture, Civil Engineering and Geodesy, Sofia, Bulgaria; $^{c}$Institute of Computing Science and Technology, Guangzhou University, Guangzhou, People’s Republic of China; $^{d}$Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, Iran

ABSTRACT
An outer-independent $k$-rainbow dominating function of a graph $G$ is a function $f$ from $V(G)$ to the set of all subsets of $\{1, 2, \ldots, k\}$ such that both the following hold: (i) $\bigcup_{v \in V(G)} f(v) = V(G)$ whenever vertex $v$ is a vertex with $f(v) = \emptyset$, and (ii) the set of all vertices $v \in V(G)$ with $f(v) = \emptyset$ is independent. The outer-independent $k$-rainbow domination number of $G$ is the invariant $\gamma^k_{oi}(G)$, which is the minimum sum (over all the vertices of $G$) of the cardinalities of the subsets assigned by an outer-independent $k$-rainbow dominating function. In this paper, we initiate the study of outer-independent $k$-rainbow domination. We first investigate the basic properties of the outer-independent $k$-rainbow domination and then we focus on the outer-independent 2-rainbow domination number and present sharp lower and upper bounds for it.

ARTICLE HISTORY
Received 2 June 2019
Revised 29 July 2019
Accepted 3 August 2019

KEYWORDS
$k$-rainbow dominating function; $k$-rainbow domination; outer-independent domination

1. Introduction
In general, we follow the notation and graph theory terminology in [1]. Specifically, let $G = (V(G), E(G))$ be a finite simple graph. For any vertex $u$ in $G$, the open neighbourhood of $u$, written $N(u)$, is the set of vertices adjacent to $u$ and the closed neighbourhood of $u$ is the set $N[u] = N(u) \cup \{u\}$. The degree of a vertex $u \in V(G)$ is $\deg(v) = |N(v)|$. The minimum and maximum degrees of $G$ are denoted by $\delta(G)$ and $\Delta(G)$, respectively. A leaf is a vertex of degree one, and a support vertex is a vertex adjacent to a leaf. We denote the sets of all leaves and all support vertices of $G$ by $L(G)$ and $S(G)$, respectively. For a vertex $v \in V(G)$, the set of leaf neighbours of $v$ is denoted by $L(v)$. If $A \subseteq V(G)$, then $N(A)$ (respectively, $N[A]$) denotes the union of (closed) neighbourhoods of all vertices of $A$. (If the graph $G$ under consideration is not clear we write $N_G(u)$, and so on.) We denote by $P_n$ and $C_n$ the path and cycle on $n$ vertices, respectively. The distance between two vertices $u$ and $v$ in a connected graph $G$ is the length of a shortest $uv$-path in $G$. The diameter of a graph $G$, denoted by $\text{diam}(G)$, is the greatest distance between two vertices of $G$. For a vertex $v$ in a rooted tree $T$, let $C(v)$ and $D(v)$ denote the set of children and descendants of $v$, respectively and let $D[v] = D(v) \cup \{v\}$. Also, the depth of $v$, $\text{depth}(v)$, is the largest distance from $v$ to a vertex in $D(v)$. The maximal subtree at $v$ is the subtree of $T$ induced by $D[v]$, and is denoted by $T_v$. By $K_{p,q}$ we denote a complete bipartite graph with partite sets of cardinalities $p$ and $q$. A star is a $K_{1,q}$ and a double star $DS_{q,p}$, where $q \geq p \geq 1$, is a tree containing exactly two non-leaf vertices which one is adjacent to $p$ leaves and the other is adjacent to $q$ leaves. By $(X)$ we denote the induced subgraph of a graph $G$ with vertex set $X \subseteq V(G)$.

A set $I \subseteq V(G)$ is independent if no two vertices in $I$ are adjacent. The maximum cardinality of an independent set in $G$ equals the independence number $\beta_0(G)$. A vertex cover of a graph $G$ is a set of vertices that covers all the edges. The minimum cardinality of a vertex cover is denoted by $\alpha_0(G)$. The following theorem due to Gallai.

**Theorem 1.1** ([2]): Let $G$ be a graph. A subset $I$ of $V(G)$ is independent if and only if $\text{V}(G) − I$ is a vertex cover of $G$. In particular, $\beta_0(G) = |V(G)| − \alpha_0(G)$.

A set $D \subseteq V(G)$ in $G$ is called a dominating set if $N[D] = V(G)$. The domination number $\gamma(G)$ equals the minimum cardinality of a dominating set in $G$. For many applications, it is not possible to use an arbitrary dominating set $D$ of $G$. One possible form of restriction is based on imposing some conditions on the set $V(G) − D$. Here we concentrate on the property of being outer-independent, i.e. $V(G) − D$ is independent. Results on outer-independent domination parameters can be found e.g. in [3–7].

For a positive integer $k$ we denote the set $\{1, 2, \ldots, k\}$ by $[k]$. The power set (that is, the set of all subsets) of $[k]$ is denoted by $2^k$. Let $G$ be a graph and let $f$ be a function that assigns to each vertex a subset
of \([k]\); that is, \(f: V(G) \to 2^{|k|}\). The weight, \(\omega(f)\), of \(f\) is defined as \(\omega(f) = \sum_{v \in V(G)} |f(v)|\). The function \(f\) is called a \(k\)-rainbow dominating function (a \(kR\)-function) on \(G\) if for each vertex \(v \in V(G)\) with \(f(v) \neq \emptyset\) the condition \(\bigcup_{u \in N(v)} f(u) = \{1, \ldots, k\}\) is fulfilled. Given a graph \(G\), the minimum weight of a \(k\)-rainbow dominating function is called the \(k\)-rainbow domination number of \(G\), which we denote by \(\gamma^k_{\text{oir}}(G)\). The concept of rainbow domination was introduced in [8] and has been studied extensively [9–15].

Here we introduce and study a new variant of a \(k\)-rainbow dominating function. A \(k\)-rainbow dominating function \(f: V(G) \to 2^{|k|}\) is an outer-independent \(k\)-rainbow dominating function (an OI-RD-function) on \(G\) if the set \([v \in V(G) \mid f(v) = \emptyset]\) is independent. The outer-independent \(k\)-rainbow domination number \(\gamma^k_{\text{oir}}(G)\) is the minimum weight of an OI-RD-function on \(G\). An OI-RD-function of weight \(\gamma^k_{\text{oir}}(G)\) is called a \(\gamma^k_{\text{oir}}\)-function. Since any OI-RD-function is a KRD-function, we have
\[
\gamma^k_{\text{oir}}(G) \leq \gamma^k_{\text{oir}}(G).
\]

In this paper, we initiate the study of outer-independent \(k\)-rainbow domination. We first investigate the basic properties of the outer-independent \(k\)-rainbow domination and then we focus on the outer-independent \(2\)-rainbow domination number and present sharp lower and upper bounds for it.

2. Preliminary results

In this section, we present basic properties of outer-independent \(k\)-rainbow domination number \(\gamma^k_{\text{oir}}(G)\). We begin with three simple observations.

**Observation 2.1:** If \(G_1, G_2, \ldots, G_r\) are all components of a graph \(G\), then \(\gamma^k_{\text{oir}}(G) = \gamma^k_{\text{oir}}(G_1) + \gamma^k_{\text{oir}}(G_2) + \cdots + \gamma^k_{\text{oir}}(G_r)\).

**Observation 2.2:** For any graph \(G\) of order \(n\), \(\min\{n, k\} \leq \gamma^k_{\text{oir}}(G) \leq n\). In particular, \(\gamma^k_{\text{oir}}(G) = n\) whenever \(k \geq n\).

**Observation 2.3:** For any OI-RD-function \(f\) on a graph \(G\), the set \([v \in V(G) \mid f(v) = \emptyset]\) is a vertex cover of \(G\). Hence \(\alpha_0(G) + l(G) \leq \omega(f)\), where \(l(G)\) is the set of all isolated vertices of \(G\). In particular, \(\alpha_0(G) \leq \alpha_0(G) + l(G) \leq \gamma^k_{\text{oir}}(G)\).

Notice also that the outer-independent domination is the same as the outer-independent \(1\)-rainbow domination if we view an outer-independent dominating set \(D\) as an outer-independent \(1\)-rainbow dominating function \(f\) defined by \(f(v) = \{1\}\) when \(v \in D\) and \(f(v) = \emptyset\) otherwise. Therefore here we concentrate on the case when a graph \(G\) is connected and \(n - 1 \geq k \geq 2\).

Now we characterize all connected graphs of order \(n \geq k + 1\) attaining the lower bound in Observation 2.2.

**Theorem 2.1:** Let \(k \geq 2\) be a positive integer and let \(G\) be a connected graph of order \(n \geq k + 1\). Then \(\gamma^k_{\text{oir}}(G) = k\) if and only if \(G = H_G \cup \overline{K_{n-k}}\), where \(H_G\) is a graph of order \(h \leq k\).

**Proof:** First assume that \(G = H_G \cup \overline{K_{n-k}}\), where \(H_G\) is a graph of order \(h \leq k\). Define a function \(f: V(G) \to 2^{|k|}\) by \(f(x) = \emptyset\) for \(x \in V(H)\) and \(f(x) \neq \emptyset\) for \(x \in V(G)\) such that \(\bigcup_{x \in V(H)} f(x) = \{1, 2, \ldots, k\}\).\(\sum_{x \in V(G)} |f(x)| = k\). Obviously, \(f\) is an OI-RD-function on \(G\) with \(\omega(f) = k\). Therefore \(\gamma^k_{\text{oir}}(G) = k\).

Conversely, assume that \(\gamma^k_{\text{oir}}(G) = k\). Let \(f\) be a \(\gamma^k_{\text{oir}}\)-function on \(G\). Since \(n \geq k + 1\), there exists a vertex \(v\) with \(f(v) = \emptyset\). It follows that \(\bigcup_{x \in N(v)} f(x) = \{1, 2, \ldots, k\}\) and \(f(x) \neq \emptyset\) for all \(x \in N(v)\). Moreover, \(k = \gamma^k_{\text{oir}}(G) = \omega(f) = \sum_{x \in V(G)} |f(x)| \geq \sum_{x \in N(v)} |f(x)| = |\{1, 2, \ldots, k\}| = k\). Hence \(f(u) = \emptyset\) for all \(u \in V(G) \setminus N(v)\). Since \(v\) was chosen arbitrarily, \(G = (N(v)) \cup (V(G) \setminus N(v))\), where a graph \((V(G) \setminus N(v))\) has no edges.

**Theorem 2.2:** Let \(k \geq 2\) be a positive integer and let \(G\) be a connected graph of order \(n \geq k + 2\). Then \(\gamma^k_{\text{oir}}(G) = k + 1\) if and only if the following holds:

(i) there is no \(h\)-order graph \(H_G, h \leq k\), such that \(G = H_G \cup \overline{K_{n-k}}\).

(ii) there exist two nonempty disjoint vertex sets \(A, B\) such that: (i) \(|A| + |B| \leq k + 1\) and \(|B| \leq 2\), (ii) every vertex of \(V(G) - (A \cup B)\) is adjacent to every vertex of \(A \cup B\) except at most one vertex in \(B\), (iii) \(V(G) - (A \cup B)\) is independent, and (iv) for each \(x \in B, N(x) \subseteq A \cup B\) or \(A \subseteq N(x)\).

**Proof:** First assume that \(G\) satisfies (i) and (ii). It follows from Theorem 1.2 and (i) that \(\gamma^k_{\text{oir}}(G) \geq k + 1\). Now define the function \(f: V(G) \to 2^{|k|}\) by \(f(x) = \emptyset\) for \(x \in V(G) \setminus A \cup B, f(x) = \{k\}\) for \(x \in B, f(x) \neq \emptyset\) for \(x \in A\) such that \(\bigcup_{x \in A} f(x) = \{1, 2, \ldots, k\}\) and \(\sum_{x \in A} |f(x)| = k\) when \(|B| = 1\) and by \(f(x) = \emptyset\) for \(x \in V(G) - A \cup B, f(x) = \{k\}\) for \(x \in B, f(x) \neq \emptyset\) for \(x \in A\) such that \(\bigcup_{x \in A} f(x) = \{1, 2, \ldots, k-1\}\) and \(\sum_{x \in A} |f(x)| = k - 1\) when \(|B| = 2\). Obviously, \(f\) is an OI-RD- function of \(G\) with \(\omega(f) = k + 1\) and thus \(\gamma^k_{\text{oir}}(G) = k + 1\).

Conversely, assume that \(\gamma^k_{\text{oir}}(G) = k + 1\). It follows from Theorem 1.2 that \(G\) satisfies (i). Now we show that \(G\) satisfies (ii). Let \(f\) be a \(\gamma^k_{\text{oir}}\)-function on \(G\). Choose \(f\) so that \(D_f = \{u \in V(G) \mid f(u) \neq \emptyset\}\) is as small as possible. Since \(\omega(f) = k + 1\), there exists a colour, say \(k\), which appears exactly twice and each other colour appears exactly once. Hence there are two vertices, say \(z\) and \(w\), such that \(f(z) \cap f(w) = \{k\}\). Since \(n \geq k + 2\), there exists a vertex \(v\) with \(f(v) = \emptyset\) which implies that \(\bigcup_{x \in N(v)} f(x) = \{1, 2, \ldots, k\}\) and \(f(x) \neq \emptyset\) for each \(x \in N(v)\).

Now let \(A = D_f - \{x \in \{z, w\} \mid f(x) = \{k\}\}\) and \(B = \{z, w\} - A\). Since \(k \geq 2\), we have \(A \neq \emptyset\). On the other hand, if \(B = \emptyset\), then the function \(g\) defined by \(g(z) =...
\[ f(z) - \{k\} \] and \( g(x) = f(x) \) otherwise, is an O\(k\)RD-function of \( G \) with weight less that \( \gamma^k_{oir}(G) \) which is a contradiction. Thus \( A \) and \( B \) are non-empty sets. Since \( f \) is an O\(k\)RD-function, the set \( V(G) - (A \cup B) \) is independent, that is \( G \) satisfies (iii).

It follows from \( \gamma^k_{oir}(G) = k + 1 \) that \(|A| + |B| \leq k + 1\) and so (i) holds. Since for each vertex \( u \in A \), \( f(u) \) has a colour which is not appeared in other vertices, every vertex in \( V(G) - (A \cup B) \) is adjacent to every vertex of \( A \). Also since the colour \( k \) appears exactly in \( f(z) \) and \( f(w) \), each vertex of \( V(G) - (A \cup B) \) must be adjacent to one of the vertices \( z \) and \( w \). Hence (ii) holds.

Finally, if for some \( w \in B \), \( N(w) \subseteq A \cup B \) and \( A \subseteq N(w) \), then the function \( g \) defined on \( G \) by \( g(w) = \emptyset \), \( g(a) = f(a) \cup \{k\} \) for some \( a \in A \) and \( g(x) = f(x) \) otherwise, is a \( \gamma^k_{oir}(G) \)-function which contradicts the choice of \( f \). Thus (iv) holds and the proof is complete.

**Proposition 2.1:** For any graph \( G \) of order \( n \), \( \alpha(G) + l(G) = \gamma^1_{oir}(G) \leq \gamma^2_{oir}(G) \leq \cdots \leq \gamma^k_{oir}(G) \leq n \). If \( G \) has no isolated vertices, then \( \alpha(G) = \gamma^1_{oir}(G) \).

**Proof:** Let \( f_{s+1} \) be a \( \gamma^{s+1}_{oir} \)-function on \( G \), \( 1 \leq s \leq k - 1 \). Define the function \( h_s : V(G) \rightarrow 2^{|l(G)|} \) as follows: \( h_s(u) = f_{s+1}(u) \) when \( s + 1 \notin f_{s+1}(u), h_s(u) = \{1\} \) when \( f_{s+1}(u) = \{s + 1\} \) and \( h_s(u) = f_{s+1}(u) - \{s + 1\} \) otherwise. Clearly \( h_s \) is an O\(s\)RD-function and so \( \gamma^{s+1}_{oir}(G) \leq \omega(h_s) \leq \gamma^{s+1}_{oir}(G) \).

If \( C \) is a minimum vertex cover of \( G \), then the function \( g : V(G) \rightarrow \{\emptyset, \{1\}\} \) defined by \( g(v) = \{1\} \) when \( v \in C \cup l(G) \) and \( g(v) = \emptyset \) otherwise, is an O\(1\)RD-function on \( G \) with weight \(|C| + |l(G)| = \alpha(G) + |l(G)|\). The equality \( \gamma^s_{oir}(G) = \alpha(G) + |l(G)| \) now follows by Observation 2.3.

Finally, the right side inequality follows by Observation 2.2.

**Proposition 2.2:** For any graph \( G \), \( \gamma^k_{oir}(G) \leq k\alpha(G) + |l(G)| \). If \( \delta(G) \geq 1 \) then \( \gamma^k_{oir}(G) \leq k\alpha(G) \).

**Proof:** Let \( C \) be any minimum vertex cover set of \( G \) and define the function \( f : V(G) \rightarrow 2^{|l(G)|} \) by \( f(u) = \{k\} \) for \( u \in C \), \( f(u) = \{1\} \) when \( u \in l(G) \), and \( f(u) = \emptyset \) otherwise. Clearly \( f \) is an O\(k\)RD-function on \( G \) which immediately implies the required.

The bounds in Proposition 2.2 are attainable. Let \( G \) be a graph such that each vertex is either a leaf or a support vertex and let each support vertex of \( G \) is adjacent to at least \( k + 1 \) leaves. Then clearly \( S(G) \) is a minimum vertex cover set and the function \( f : V(G) \rightarrow 2^{|l(G)|} \) defined as \( f(u) = \{k\} \) when \( u \) is a support vertex and \( f(u) = \emptyset \) when \( u \) is a leaf, is an O\(k\)RD-function on \( G \) of minimum weight. Thus \( \gamma^k_{oir}(G) = k\alpha(G) \).

We will say that a graph \( G \) is a vertex cover outer independent \( k \)-rainbow graph, a V\(C\)O\(k\)-rainbow graph for short, if \( \gamma^k_{oir}(G) = k\alpha(G) \).

**Proposition 2.3:** A graph \( G \) with no isolated vertex, is V\(C\)O\(k\)-rainbow if and only if it has a \( \gamma^k_{oir} \)-function \( f \) such that for each vertex \( x \), either \( f(x) = \emptyset \) or \( f(x) = \{k\} \).

**Proof:** Assume that \( G \) is a V\(C\)O\(k\)-rainbow graph and let \( D \) be a minimum vertex cover set of \( G \). Then the function \( f : V(G) \rightarrow 2^{|l(G)|} \) defined by \( f(x) = \{k\} \) for \( x \in D \) and \( f(x) = \emptyset \) otherwise, is an O\(k\)RD-function on \( G \) which implies that \( k\alpha(G) = \gamma^k_{oir}(G) \leq \omega(f) = k|D| = k\alpha(G) \). Thus all inequalities in this chain must be equality and so \( \gamma^k_{oir}(G) = \omega(f) \), yielding \( f \) is a \( \gamma^k_{oir} \)-function satisfying that for each vertex \( x \) either \( f(x) = \emptyset \) or \( f(x) = \{k\} \).

Conversely, assume that there exists a \( \gamma^k_{oir} \)-function \( h \) such that for each vertex \( x \), either \( h(x) = \emptyset \) or \( h(x) = \{k\} \). Since the set \( A_h = \{v \in V(G) \mid h(v) = \emptyset\} \) is a vertex cover set of \( G \), we have \( k\alpha(G) \leq k|A_h| = \gamma^k_{oir}(G) \). By Proposition 2.2 we deduce that \( \gamma^k_{oir}(G) = k\alpha(G) \) and this implies that \( G \) is a V\(C\)O\(k\)-rainbow graph.

**Proposition 2.4:** Let \( H \) be an induced subgraph of a graph \( G \). Then \( \gamma^k_{oir}(G) \leq \gamma^k_{oir}(H) + |V(G)| - |V(H)| \).

**Proof:** Let \( f \) be a \( \gamma^k_{oir} \)-function on \( H \). Define an O\(k\)RD-function \( h \) on \( G \) as follows: \( h(x) = f(x) \) when \( x \in V(H) \) and \( h(x) = \{1\} \) otherwise. Since \( \omega(h) = \omega(f) + |V(G)| - |V(H)| \), we have the desired inequality.

**Observation 2.4:** Let \( f \) be an O\(k\)RD-function on a graph \( G \) and \( a_i = |\{v \in V(G) \mid i \notin f(v)\}| \) for each \( 1 \leq i \leq k \). Then \( \omega(f) = a_1 + a_2 + \cdots + a_k \).

**Theorem 2.3:** Let \( G \) be a graph of order at least two and \( k' > k \). Then \( \gamma^k_{oir}(G) \leq \gamma^{k'}_{oir}(G) + (k' - k) \gamma^k_{oir}(G) \).

**Proof:** Let \( f \) be a \( \gamma^k_{oir} \)-function on \( G \), and \( a_i = |\{v \in V(G) \mid i \notin f(v)\}| \) for each \( 1 \leq i \leq k \). Assume without loss of generality that \( a_1 \geq a_2 \geq \cdots \geq a_k \).

Define \( g : V(G) \rightarrow 2^{|l(G)|} \) by \( g(v) = f(v) \cup \{k + 1, \ldots, k'\} \) when \( k \in f(v) \) and \( g(v) = f(v) \) otherwise. Clearly \( g \) is an O\(k\)RD-function on \( G \). This fact and Observation 2.4 lead to
\[ \gamma^k_{oir}(G) \leq \omega(g) = \gamma^k_{oir}(G) + (k' - k)a_k \]
\[ \leq \gamma^k_{oir}(G) + (k' - k) \gamma^k_{oir}(G) / k \]

**Corollary 2.1:** Let \( k' > k \) be two positive integers and \( G \) a graph of order at least two. Then
\[ \gamma^{k'}_{oir}(G) \leq k' \gamma^k_{oir}(G) / k \]

3. Outer-independent 2-rainbow domination number

In this section, we focus on outer-independent 2-rainbow domination. An O2RD-function \( f \) on a graph \( G \)
can be represented by the ordered 4-tuple $\langle V_0, V_1, V_2, V_{1,2} \rangle$ (or $\langle V'_0, V'_1, V'_2, V'_{1,2} \rangle$ to refer f) of $V(G)$, where $V'_0 = \{ v \in V(G) \mid f(v) = 0 \}$, $V_1 = \{ v \in V(G) \mid f(v) = 1 \}$, $V_2 = \{ v \in V(G) \mid f(v) = 2 \}$ and $V_{1,2} = \{ v \in V(G) \mid f(v) = 1,2 \}$. In this representation, its weight is $\omega(f) = |V_1| + |V_2| + 2|V_{1,2}|$.

By Theorem 2.1 we immediately obtain

**Corollary 3.1:** For any graph $G$ of order $n \geq 2, 2 \leq \gamma^2_{oir}(G) \leq n$. Moreover $\gamma^2_{oir}(G) = 2$ if and only if $G$ is $K_2$ or $G = K_{1,n-1}$ or $G = K_{2,n-2}$ or $G = K_2 \lor K_{n-2}$.

### 3.1. *Outer-independent 2-rainbow domination versus domination parameters*

A function $f : V(G) \to \{0, 1, 2\}$ is an outer-independent Roman dominating function (OIRD-function) on $G$ if every vertex $u \in V$ for which $f(u) = 0$ is adjacent to at least one vertex $v$ for which $f(v) = 2$ and $|\{ v \mid f(v) = 0 \}|$ is an independent set. The outer-independent Roman domination number $\gamma_{oir}(G)$ is the minimum weight of an OIRD-function on $G$. Outer-independent Roman domination was introduced by Abdollahzadeh Ahangar et al. in [3]. Clearly, if $f = (V_0, V_1, V_2)$ is a $\gamma_{oir}(G)$-function, then the function $g = (V_0, V_1, 0, V_2)$ is an outer-independent 2-rainbow dominating function on a graph $G$ and so

$$\gamma_{oir}(G) \geq \gamma^2_{oir}(G). \quad (2)$$

Abdollahzadeh Ahangar et al. proved the following bounds on $\gamma_{oir}(G)$.

**Proposition 3.1 ([3]):** If $G$ is a connected triangle-free graph of order $n \geq 2$ and maximum degree $\Delta$, then $\gamma_{oir}(G) \leq n - \Delta + 1$.

**Proposition 3.2 ([3]):** Let $G$ be a connected graph of order $n$. If $G$ has girth $g < \infty$, then $\gamma_{oir}(G) \leq n + \lceil \frac{n}{2} \rceil - g$.

Next results are immediate consequences of Propositions 3.1, 3.2 and inequality (2).

**Corollary 3.2:** If $G$ is a connected triangle-free graph of order $n \geq 2$ and maximum degree $\Delta$, then $\gamma^2_{oir}(G) \leq n - \Delta + 1$. This bound is sharp for all stars $K_{1,n-1}, n \geq 2$.

**Corollary 3.3:** Let $G$ be a connected graph of order $n$. If $G$ has girth $g < \infty$, then $\gamma^2_{oir}(G) \leq n + \lceil \frac{n}{2} \rceil - g$.

In the following, we provide an upper bound on $\gamma_{oir}(G)$ in terms of $\gamma^2_{oir}(G)$ for arbitrary graphs $G$.

**Theorem 3.1:** For any graph $G$, $\gamma_{oir}(G) \leq \frac{3}{2} \gamma^2_{oir}(G)$. This bound is sharp for the family $\mathcal{F}$ of graphs illustrated in Figure 1.

![Figure 1. The graph $\mathcal{F}$.](image)

**Proof:** Let $f = (V_0, V_1, V_2, V_{1,2})$ be a $\gamma^2_{oir}(G)$-function and without loss of generality $|V_1| \geq |V_2|$. Then $g = (V_0, V_1, V_2 \cup V_{1,2})$ is an OIRD-function on $G$ implying that

$$\gamma_{oir}(G) \leq \omega(g) = |V_1| + 2|V_2| + 2|V_{1,2}| = \omega(f) + |V_2| \leq \frac{3}{2} \gamma^2_{oir}(G).$$

The notion of outer-independent Italian domination in graphs was introduced in [16]. An outer-independent Italian dominating function (OI2D-function) on a graph $G$ is a function $f : V(G) \to \{0, 1, 2\}$ such that every vertex $v \in V(G)$ with $f(v) = 0$ has at least two neighbours assigned 1 under $f$ or one neighbour $w$ with $f(w) = 2$ and the set of all vertices assigned 0 under $f$ is independent. The weight of an OI2D-function is the value $\omega(f) = \sum_{v \in V(G)} f(u)$. The minimum weight of an OI2D-function on a graph $G$ is called the outer-independent Italian domination number $\gamma_{oi2d}(G)$ of $G$. Clearly, if $f = (V_0, V_1, V_2, V_{1,2})$ is a $\gamma^2_{oir}(G)$-function, then the function $g = (V_0, V_1, V_2, V_{1,2})$ is an outer-independent Italian dominating function of $G$ and so

$$\gamma^2_{oir}(G) \geq \gamma_{oi2d}(G). \quad (3)$$

In [16], the authors proved that $\gamma_{oi2d}(C_n) = \lceil \frac{n}{2} \rceil$ for $n \geq 3$ and $\gamma_{oi2d}(K_{p,q}) = q$ for $p \geq q \geq 2$. Using these we obtain the next results.

**Proposition 3.3:** For $n \geq 3$, $\gamma^2_{oir}(C_n) = \lceil \frac{n}{2} \rceil$.

**Proof:** By (3), we have $\gamma^2_{oir}(C_n) \geq \lceil \frac{n}{2} \rceil$. Let $C_n : v_1 \cdots v_{2n} v_1$ be a cycle and define $f : V(G) \to 2^{\{0,1,2\}}$ by $f(v_{i-1}) = \{1\}$, $f(v_{i+1}) = \{2\}$ for $i \geq 0$ and $f(v) = \emptyset$ otherwise. Clearly $f$ is an OI2D-function of $C_n$ and hence $\gamma^2_{oir}(C_n) \leq \lceil \frac{n}{2} \rceil$. Thus $\gamma^2_{oir}(C_n) = \lceil \frac{n}{2} \rceil$.

**Proposition 3.4:** For $p \geq q \geq 2$, $\gamma^2_{oir}(K_{p,q}) = q$.
**Theorem 3.2:** For any connected graph $G$ of order $n \geq 2$ with minimum degree $\delta$ and maximum degree $\Delta$, 

$$\gamma_{oi}(G) \geq \lceil n\delta/(\delta + \Delta) \rceil.$$ 

Next result is an immediate consequence of Theorem 3.2 and inequality (3).

**Corollary 3.4:** For any connected graph $G$ of order $n \geq 2$ with minimum degree $\delta$ and maximum degree $\Delta$, 

$$\gamma_{oi}^2(G) \geq \lceil n\delta/(\delta + \Delta) \rceil.$$ 

This bound is sharp for cycles.

Fan et al. [16] proved the following Nordhaus–Gaddum type result for outer-independent Italian domination number.

**Theorem 3.3:** For any graph $G$ on $n$ vertices,

$$n - 1 \leq \gamma_{oi}(G) + \gamma_{oi}(G').$$

As an immediate consequence we have:

**Corollary 3.5:** For any graph $G$ on $n$ vertices,

$$n - 1 \leq \gamma_{oi}^2(G) + \gamma_{oi}^2(G').$$

The upper bound is sharp for $K_2$ and the lower bound is sharp for a graph $G$ obtained from $K_{10}$ with vertex set $\{u_1, u_2\}$ by adding new vertices $v_1, v_2, v_3, v_4, v_5$ and joining $v_i$ to $u_j$ for $2 \leq j \leq 5$ and joining $v_i$ to all vertices in $K_{10}$.

### 3.2. Trees

Here we present sharp upper and lower bounds on the outer-independent 2-rainbow domination number of trees. First we show that the outer-independent 2-rainbow domination number and the outer-independent Italian domination number of a tree are equal.

**Theorem 3.4:** For any tree $T$, $\gamma_{oi}(T) = \gamma_{oi}^2(T)$.

**Proof:** Consider a $\gamma_{oi}$-function $f = (V_0, V_1, V_2)$ on $T$ and let $U_1, U_2, \ldots, U_k$ be the components of the graph $(V_1 \cup V_2)$. Let $T_f$ be the graph whose vertex set is $\{U_1, U_2, \ldots, U_k\}$ and two vertices $U_i$ and $U_j$ are adjacent if and only if there are vertices $u_i \in U_i$ and $u_j \in U_j$ and $u_i, u_j \in V_0$ such that $u_i u_j$ is a path in $T$. If $V_1 \neq \emptyset$, then we may assume that $V_1 \cap U_1 \neq \emptyset$. Define $g : V(G) \rightarrow \{2\}$ as follows: $g(x) = \{1, 2\}$ for $x \in V_2$, $g(x) = \emptyset$ for $x \in V_0$, $g(x) = \{1\}$ whenever $f(v) = 1$ and either $v \in V_1 \cap U_1$ or $v \in U_j$, where the distance $U_1$ and $U_j$ in $T_f$ is even, and $g(x) = \{2\}$ otherwise. Clearly $g$ is an OI2RD-function on $T$ with weight $\omega(f)$ and so $\gamma_{oi}^2(T) \leq \gamma_{oi}(T)$. The result now immediately follows from (3). 

Using the results given in [16] and Theorem 3.4, we obtain the following results.

**Proposition 3.5:** For $n \geq 1$, $\gamma_{oi}^2(P_n) = \lceil \frac{n + 1}{2} \rceil$.

**Proposition 3.6:** For any tree $T$ of order $n$, $\gamma_{oi}^2(G) \leq n - \Delta + 1$.

**Theorem 3.5:** For any tree $T$ of order $n \geq 2$, 

$$\gamma_{oi}^2(T) \geq \frac{n + 3 - \ell(T)}{2},$$

where $\ell(T)$ is the number of leaves of $T$. This bound is sharp for stars and paths.

As a consequence of Propositions 2.4 and 3.5 we obtain the following result.

**Corollary 3.6:** For any connected graph $G$ of order $n$,

$$\gamma_{oi}^2(G) \leq n - \left\lceil \frac{\text{diam}(G) + 1}{2} \right\rceil.$$ 

In the sequel we will use the following observation.

**Observation 3.1:** Let $G$ be a graph.

1. If $u$ is a strong support vertex of $G$, then there is a $\gamma_{oi}^2(G)$-function $f$ with $f(u) = \{1, 2\}$. 
2. If $v_3 v_2 v_1$ is a path in $G$ such that $\text{deg}_G(v_2) = 2$ and $\text{deg}_G(v_1) = 1$, then there is a $\gamma_{oi}^2(G)$-function $f$ with $f(v_1) = \{1\}$, $f(v_2) = \emptyset$ and $f(v_3) = \emptyset$.

**Proof:** (1) is trivial. To prove (2), let $g$ be a $\gamma_{oi}^2(G)$-function. If $g(v_2) = \emptyset$, then let $f = g$. Assume then that $g(v_2) \neq \emptyset$. If $|g(v_2)| = 1$, then since $g$ is a $\gamma_{oi}^2(G)$-function, it follows that $|g(v_1)| = 1$. By the minimality of $g$, we have $g(v_3) = \emptyset$, for otherwise we may assume that $1 \in g(v_3)$ and then the function $h$ defined by $h(v_2) = \emptyset$, $h(v_1) = \{2\}$ and $h(x) = g(x)$ otherwise, is an OI2RD-function of $G$ with smaller weight than $g$, a contradiction. Hence we may assume that $g(v_2) = \emptyset$. But then the function $f : V(G) \rightarrow \{2\}$ defined by $f(v_3) = \{2\}$, $f(v_2) = \emptyset$, $f(v_1) = \{1\}$ and $f(x) = g(x)$ for all $x \in V(G) - \{v_1, v_2, v_3\}$ is a $\gamma_{oi}^2(G)$-function with desired property. If $|g(v_2)| = 2$, that is, $g(v_2) = \{1, 2\}$, then by the minimality we have $g(v_1) = g(v_3) = \emptyset$. Now the function $f$ defined above is a $\gamma_{oi}^2(G)$-function with desired property. 


Next we present an upper bound on outer-independent 2-rainbow domination number of a tree \( T \) in terms of the order and its number of support vertices. For any tree \( T \), let \( s(T) \) denote the number of its support vertices.

**Theorem 3.6:** If \( T \) is a tree of order at least 3, then

\[
\gamma^2_{\text{air}}(T) \leq \left\lfloor \frac{n(T) + s(T)}{2} \right\rfloor.
\]

This bound is sharp for all paths \( P_{2k} \) (\( k \geq 1 \)).

**Proof:** The proof is by induction on \( n(T) \). It is easy to verify that the statement is true for \( n(T) \leq 4 \). Hence, let \( n(T) \geq 5 \) and assume that every \( T' \) of order \( n(T') < n(T) \) with \( s(T') \) support vertices satisfies \( \gamma^2_{\text{air}}(T') \leq \left\lfloor \frac{n(T') + s(T')}{2} \right\rfloor \). Let \( T \) be a tree of order \( n(T) \). If \( T \) is a star, then \( \gamma^2_{\text{air}}(T) = 2 < \left\lfloor \frac{n(T)+1}{2} \right\rfloor \). Likewise, if \( T \) is a double star, then \( \gamma^2_{\text{air}}(T) = 3 \) or 4 and so \( \gamma^2_{\text{air}}(T) \leq \left\lfloor \frac{n(T)+2}{2} \right\rfloor \) with equality if and only if \( T \in \{DS_{1,2},DS_{2,2},DS_{2,3}\} \).

Henceforth, we assume that \( \text{diam}(T) \geq 4 \).

If \( T \) has a strong support vertex \( u \) with \( \ell(u) \geq 3 \), then let \( T' = T - w \) where \( w \in L(u) \). Clearly, there exists a \( \gamma^2_{\text{air}}(T') \)-function \( f \) such that \( f(u) = \{1,2\} \) and \( f \) can be extended to an O2IRD-function of \( T \) by assigning \( \varnothing \) to \( w \) and this implies that \( \gamma^2_{\text{air}}(T) \leq \gamma^2_{\text{air}}(T') \). Now the result follows by using the induction on \( T' \) and the facts \( n(T') = n(T) - 1 \) and \( s(T') = s(T) \). Henceforth, we assume that every support vertex of \( T \) is adjacent to at most two leaves.

Let \( v_1,v_2,\ldots,v_k \) be a diametrical path in \( T \) such that \( \text{deg}_T(v_2) \) is as large as possible and root \( T \) at \( v_k \). We consider the following cases.

**Case 1.** \( \text{deg}_T(v_2) = 3 \). If \( \text{deg}_T(v_2) \geq 3 \), then any \( \gamma^2_{\text{air}}(T - v_2) \)-function can be extended to an O2IRD-function of \( T \) by assigning \( \{1,2\} \) to \( v_2 \) and \( \varnothing \) to the leaves adjacent to \( v_2 \) and so \( \gamma^2_{\text{air}}(T \setminus v_2) \leq \gamma^2_{\text{air}}(T - v_2) + 2 \). Using the induction on \( T \setminus v_2 \) and the facts \( n(T - v_2) = n(T) - 3 \) and \( s(T - v_2) = s(T) - 1 \), we obtain \( \gamma^2_{\text{air}}(T) \leq \gamma^2_{\text{air}}(T \setminus v_2) + 2 \leq \floor{\frac{n(T) + s(T)}{2}} + 2 = \floor{\frac{n(T) + s(T)}{2}} + 2 \). Hence we assume that \( \text{deg}_T(v_2) = 2 \). If \( \text{diam}(T) = 4 \), then we have \( \gamma^2_{\text{air}}(T) = 4 \leq \floor{\frac{n(T) + s(T)}{2}} + 2 \). Let \( \text{diam}(T) \geq 5 \). We distinguish the followings.

**Subcase 1.1.** \( \text{deg}_T(v_4) \geq 3 \).

If \( v_4 \) is a support vertex, then any \( \gamma^2_{\text{air}}(T \setminus v_4) \)-function can be extended to an O2IRD-function of \( T \) by assigning \( \{1,2\} \) to \( v_2 \) and \( \varnothing \) to the leaves adjacent to \( v_2 \) and it follows from the induction hypothesis on \( T \setminus T_2 \) and the facts \( n(T - T_2) = n - 3 \) and \( s(T - T_2) = s(T) - 1 \) that

\[
\gamma^2_{\text{air}}(T) \leq \gamma^2_{\text{air}}(T - v_2) + 2 \leq \floor{\frac{n(T) + s(T)}{2}} + 2 = \floor{\frac{n + s}{2}}.
\]

Assume next that \( v_4 \) is not a support vertex. We consider the following situations.

(i) \( v_4 \) has a child \( w_2 \) with depth one. Let \( T' = T - T_{w_2} \) and let \( f \) be a \( \gamma^2_{\text{air}}(T') \)-function such that \( f(v_2) = \{1,2\} \). Without loss of generality, we may assume that \( f(v_4) \) and that \( 1 \in f(v_4) \). If \( \text{deg}_T(w_2) = 2 \), then the function \( f \) can be extended to an O2IRD-function of \( T \) by assigning \( \varnothing \) to \( w_2 \) and \( \varnothing \) to its leaf-neighbours, and using the induction hypothesis on \( T' \) and the facts \( n(T') = n(T) - 2 \) and \( s(T') = s(T) - 1 \) we obtain

\[
\gamma^2_{\text{air}}(T) \leq \gamma^2_{\text{air}}(T') + 1 \leq \floor{\frac{n(T) + s(T)}{2}} + 1 < \floor{\frac{n + s}{2}}.
\]

If \( \text{deg}_T(w_2) = 3 \), then the function \( f \) can be extended to an O2IRD-function of \( T \) by assigning a \( \{1,2\} \) to \( w_2 \) and \( \varnothing \) to its leaf-neighbours, and using the induction hypothesis on \( T' \) and the facts \( n(T') = n(T) - 3 \) and \( s(T') = s(T) - 1 \) we obtain

\[
\gamma^2_{\text{air}}(T) \leq \gamma^2_{\text{air}}(T') + 2 \leq \floor{\frac{n(T) + s(T)}{2}} + 2 = \floor{\frac{n + s}{2}}.
\]

(ii) \( v_4 \) has a child \( w_3 \) with depth two different from \( v_2 \). Suppose \( v_4w_3w_2v_1 \) be a path in \( T \) such that \( \text{deg}_T(w_1) = 1 \). First let \( \text{deg}_T(w_2) = 3 \). As in the first paragraph of Case 1, we may assume that \( \text{deg}_T(w_3) = 2 \). Let \( T' = T - T_3 \) and let \( f \) be a \( \gamma^2_{\text{air}}(T') \)-function such that \( f(w_2) = \{1,2\} \). Without loss of generality, we may assume that \( f(v_4) \geq 1 \). Then \( f \) can be extended to an O2IRD-function of \( T \) by assigning \( \{1,2\} \) to \( v_2 \) and \( \varnothing \) to \( v_3 \) and all leaves of \( v_2 \). Using the induction on \( T' \) and the facts \( n(T') = n(T) - 3 \) and \( s(T') = s(T) - 1 \) we obtain

\[
\gamma^2_{\text{air}}(T) \leq \gamma^2_{\text{air}}(T') + 1 \leq \floor{\frac{n(T) + s(T)}{2}} + 2 \leq \floor{\frac{n(T) + s(T)}{2}}.
\]

Assume now that \( \text{deg}_T(w_2) = 2 \) and that all children of \( w_3 \) with depth 1 has degree 2. Let \( t_1 \) be the number of children of \( w_3 \) of depth 0 and \( t_2 \) be the number of children of \( w_3 \) with depth 1 and \( t \) be the number of children of \( w_3 \) with depth 1 and let \( t = 0 \) if \( t_1 = 0 \) and \( t = 1 \) if \( t_1 \geq 1 \). Suppose \( T' = T - W_3 \). Clearly, any \( \gamma^2_{\text{air}}(T') \)-function can be extended to an O2IRD-function of \( T \) by assigning \( \{1\} \) to all leaves at distance 2 from \( w_3 \) and \( \varnothing \) to all children of \( w_3 \) and \( \varnothing \) to all leaves of \( w_3 \) if \( t_1 \geq 1 \) and \( \{2\} \) to \( w_3 \) if \( t_1 \geq 0 \). Using the induction on \( T' \) and the facts \( n(T') = n(T) - 1 - 2t_2 - t_1 \) and \( s(T') = s(T) - t_2 - t \) we obtain

\[
\gamma^2_{\text{air}}(T) \leq \gamma^2_{\text{air}}(T') + t_2 + 1 + t \leq \floor{\frac{n(T) + s(T)}{2}} + t_2 + 1 + t.
\]
Subcase 1.2. deg_{Γ}(v_{4}) = 2.

If deg_{Γ}(v_{5}) ≥ 3, then any γ_{oir}^{2}(T - T_{v_{4}})-function can be extended to an OI2RD-function of T by assigning a {1, 2} to v_{2}, a {1} to v_{4} and an ∅ to other vertices in T_{v_{4}} and it follows from the induction hypothesis and the facts n(T - T_{v_{4}}) = n(T) - 5 and s(T - T_{v_{4}}) = s(T) - 1 that

\[
\gamma_{oir}^{2}(T) \leq \left\lfloor \frac{n(T) - s(T)}{2} \right\rfloor + \left\lfloor \frac{n(T) + s(T)}{2} \right\rfloor + 3,
\]

Assume that deg_{Γ}(v_{5}) = 2. Let T' = T - v_{2} and f be a γ_{oir}^{2}(T')-function. By Observation 3.1 (item (2)) we may suppose that f(v_{4}) = {1}. Then f can be extended to an OI2RD-function of T by assigning a {1, 2} to v_{2} and an ∅ to other vertices in T_{v_{2}} and it follows from the induction hypothesis and the facts n(T') = n(T) - 4 and s(T') = s(T) that γ_{oir}^{2}(T') ≤ γ_{oir}^{2}(T') + 2 ≤ \left\lfloor \frac{n(T) + s(T')}{2} \right\rfloor.

Case 2. deg_{Γ}(v_{2}) = 2.

By the choice of diametrical path, we deduce that every child of v_{2} with depth one has degree two. Consider the following subcases.

Subcase 2.1. deg_{Γ}(v_{5}) ≥ 3.

First suppose that v_{3} is a strong support vertex or adjacent to a support vertex except v_{2}. Let T' = T - {v_{1}, v_{2}} and f a γ_{oir}^{2}(T')-function. By Observation 3.1, we may assume without loss of generality that 1 ∈ f(v_{3}). Now f can be extended to an OI2RD-function of T by assigning a {2} to v_{1} and an ∅ to v_{2}. Now we deduce from the induction hypothesis on T' and the facts n(T') = n(T) - 2 and s(T') ≤ s(T) that

\[
\gamma_{oir}^{2}(T) \leq \gamma_{oir}^{2}(T') + 1 \leq \left\lfloor \frac{n(T') + s(T')}{2} \right\rfloor + 1 \leq \left\lfloor \frac{n + s}{2} \right\rfloor.
\]

Suppose next that v_{3} is a support vertex with deg_{Γ}(v_{3}) = 3. Let T' = T - v_{1} and f a γ_{oir}^{2}(T')-function. Since v_{3} is a strong support vertex we may assume that f(v_{3}) = {1, 2}. Now f can be extended to an OI2RD-function of T by assigning a {1} to v_{1}, and as above we have γ_{oir}^{2}(T) ≤ \left\lfloor \frac{n + 3}{2} \right\rfloor.

Subcase 2.2. deg_{Γ}(v_{5}) = 2.

First assume that deg_{Γ}(v_{4}) ≥ 3. Let T' = T - (v_{1}, v_{2}, v_{3}). Then any γ_{oir}^{2}(T')-function f can be extended to an OI2RD-function of T by assigning a {1} to v_{1}, a {2} to v_{3} and an ∅ to v_{2} and it follows from the induction hypothesis and the facts n(T') = n(T) - 3 and s(T') = s(T) - 1 that

\[
\gamma_{oir}^{2}(T) \leq \gamma_{oir}^{2}(T') + 2 \leq \left\lfloor \frac{n(T') + s(T')}{2} \right\rfloor + 2 = \left\lfloor \frac{n(T) + s(T)}{2} \right\rfloor.
\]

Assume next that deg_{Γ}(v_{4}) = 2. Let T' = T - {v_{1}, v_{2}} and f a γ_{oir}^{2}(T')-function. Observation 3.1 (item (2)), we may assume that 1 ∈ f(v_{4}). Now f can be extended to an OI2RD-function of T by assigning a {2} to v_{1} and an ∅ to v_{2}, and we deduce from the induction hypothesis and the facts n(T') = n(T) - 2 and s(T') = s(T) that γ_{oir}^{2}(T) ≤ γ_{oir}^{2}(T') + 1 ≤ \left\lfloor \frac{n + 2}{2} \right\rfloor and this completes the proof.

Next result in an immediate consequence of Theorems 3.4 and 3.6 since the number of support vertices of a tree on n ≥ 3 vertices is at most \( \lceil \frac{n}{2} \rceil \).

**Corollary 3.7:** For any tree T of order n ≥ 3, γ_{oir}(T) = γ_{oir}^{2}(T) ≤ \frac{3n}{4}.

A constructive instruction of trees attaining the bound given in Corollary 3.7 is given in [16].

We close this section by establishing a lower bound on the outer-independent 2-rainbow domination number of a tree in terms of the order and the total outer-independent domination number. Recall that a set S of vertices of a graph G is a total outer-independent dominating set if every vertex from V(G) has a neighbour in S and the complement of S is an independent set. The total outer-independent domination number γ_{oir}(G) of G is the smallest possible cardinality of any total outer-independent dominating set of G. The total outer-independent domination was introduced in [17, 18]. It was observed in [17] that

\[
γ_{oir}(P_{n}) = \begin{cases} 2 & \text{if } n = 2, \\ \lfloor 2n/3 \rfloor & \text{if } n ≥ 3. \\ \end{cases}
\]

**Theorem 3.7:** For any nontrivial tree T,

\[
γ_{oir}^{2}(T) ≥ γ_{oir}(T) - \frac{(n(T))}{6} + 1.
\]

Furthermore, this bound is sharp for paths P_{6k} (k ≥ 1).

**Proof:** The proof is by induction on the order n(T). If n(T) = 2 or n(T) = 3, then γ_{oir}^{2}(T) = γ_{oir}(T) = 2 and the bound is sharp. Let n(T) ≥ 4 and assume that for any tree T' of order n(T') < n(T), γ_{oir}^{2}(T') ≥ γ_{oir}(T') - \lceil n(T') \rceil + 1. Let T be a tree of order n(T) and f = (V_{0}, V_{1}, V_{2}, V_{1,2}) be a γ_{oir}^{2}(T)-function. Since for stars and double stars T we have γ_{oir}(T) = 2, so the statement holds. Hence, we assume that T has diameter at least 4. If V_{0} = ∅, then by the fact that \( \lceil \frac{n(T)}{6} \rceil ≥ 1 \) we have γ_{oir}^{2}(T) = n(T) ≥ γ_{oir}(T). Therefore we assume that V_{0} ≠ ∅. First suppose that T has a support vertex x which
is adjacent to two or more leaves. Let \( u, v \) be two leaves adjacent to \( x \) and \( T' \) be the tree obtained from \( T \) by removing \( u \). Obviously, \( y_{\text{oi2r}}(T) = y_{\text{oi2r}}(T') \). Also by Observation 3.1, we may assume that \( f(x) = \{1, 2\} \) and so all leaves adjacent to \( x \) belong to \( V_0 \). Hence, the function \( f' \) restricted to \( T' \), is an OI2RD-function on \( T' \) implying that \( y_{\text{oi2r}}(T) \geq y_{\text{oi2r}}(T') \). Applying the inductive hypothesis to \( T' \), we get \( y_{\text{oi2r}}(T) = y_{\text{oi2r}}(T') \geq y_{\text{oi2r}}(T) - \left\lfloor \frac{n(T')}{6} \right\rfloor + 1 = \left\lfloor \frac{n(T - x)}{6} \right\rfloor + 1 \geq y_{\text{oi2r}}(T) - \left\lfloor \frac{n(T)}{6} \right\rfloor + 1 \), as desired. Therefore, we may assume that every support vertex of \( T \) is adjacent to exactly one leaf. Suppose diam\((T) = k - 1 \), and let \( P = v_1, v_2, \ldots, v_k \) be a diametrical path of \( T \). Root \( T \) at \( v_k \). Thus, \( v_1 \) is a leaf of \( T \) and \( \text{deg}_T(v_2) = 2 \).

By item (2) of Observation 3.1, there is a \( y_{\text{oi2r}}(T) \) function \( g \) such that \( g(v_1) = \{1\} \), \( g(v_2) = \emptyset \) and \( 2 \in g(v_3) \). Consider the following cases.

**Case 1.** \( \text{deg}_T(v_3) = 3 \).

Let \( T' = T - \{v_1, v_2\} \). Since \( v_3 \) is a support vertex or is adjacent to a support vertex of degree two in \( T' \), any \( y_{\text{oi2r}}(T') \)-set containing no leaf, contains \( v_3 \) and so it can be extended to an OITD-set of \( T \) by adding \( v_2 \) implying that \( y_{\text{oi2r}}(T) \leq y_{\text{oi2r}}(T') + 1 \). On the other hand, the function \( g \) restricted to \( T' \) is an OI2RD-function of \( T' \) of weight at most \( \omega(g) - 1 \) and we conclude from the induction hypothesis that

\[
y_{\text{oi2r}}^2(T) \geq y_{\text{oi2r}}^2(T') + 1 \geq y_{\text{oi2r}}(T) - 1 + \left\lfloor \frac{n(T - v_2)}{6} \right\rfloor + 2 \geq y_{\text{oi2r}}(T) - \left\lfloor \frac{n(T)}{6} \right\rfloor + 1.
\]

**Case 2.** \( \text{deg}_T(v_3) = 2 \).

If \( |g(v_4)| \geq 1 \) or \( \bigcup_{v \in N(v_4) - \{v_3\}} g(v) = \{1, 2\} \), then let \( T' = T - T_v \). Clearly the function \( g \) restricted to \( T' \) is an OI2RD-function of \( T' \) of weight at most \( \omega(g) - 2 \) and so \( y_{\text{oi2r}}^2(T) \geq y_{\text{oi2r}}^2(T') + 2 \). On the other hand, any \( y_{\text{oi2r}}(T) \)-set can be extended to an OITD-set of \( T \) by adding \( v_3, v_2 \) yielding \( y_{\text{oi2r}}(T) \leq y_{\text{oi2r}}(T') + 2 \). It follows from the induction hypothesis that

\[
y_{\text{oi2r}}^2(T) \geq y_{\text{oi2r}}^2(T') + 2 \geq y_{\text{oi2r}}(T') - 3 - \left\lfloor \frac{n(T - 2)}{6} \right\rfloor + 3 \geq y_{\text{oi2r}}(T) - \left\lfloor \frac{n(T)}{6} \right\rfloor + 1.
\]

Assume that \( g(v_4) = \emptyset \) and that \( \bigcup_{v \in N(v_4) - \{v_3\}} g(v) \neq \{1, 2\} \). Since \( V_0 \) is independent and \( g(v_4) = \emptyset \), we may assume without loss of generality that \( g(x) = \{1\} \) for each \( x \in N(v_4) - \{v_3\} \). First let \( \text{deg}_T(v_4) \geq 3 \). Assume \( T' = T - \{v_1, v_2, v_3\} \) and let \( T_1 \) be the components of \( T - \{v_3, v_4, v_5, v_6\} \) containing \( v_4 \). Define \( h : V(T') \rightarrow \{2\} \) by \( h(x) = \{1, 2\} \setminus g(x) \) if \( x \in V(T_1) \) and \( g(x) = \{1\} \), and \( h(x) = g(x) \) otherwise. Clearly, \( h \) is a \( y_{\text{oi2r}}^2(T) \)-function and we are in above situation and so we have \( y_{\text{oi2r}}^2(T) \geq y_{\text{oi2r}}(T) - \left\lfloor \frac{n(T)}{6} \right\rfloor + 1 \).

Now let \( \text{deg}_T(v_2) = 2 \). Assume that \( T' = T - \{v_1, v_2, v_3, v_4\} \). Then the function \( g \) restricted to \( T' \) is an OI2RD-function of \( T' \) implying that \( y_{\text{oi2r}}^2(T) \geq y_{\text{oi2r}}^2(T') + 2 \). On the other hand, any \( y_{\text{oi2r}}(T) \)-set can be extended to an total outer-independent set of \( T \) by adding \( v_2, v_3, v_4 \), yielding \( y_{\text{oi2r}}(T) \leq y_{\text{oi2r}}(T') + 3 \). We deduce from the induction hypothesis on \( T' \) that

\[
y_{\text{oi2r}}^2(T) \geq y_{\text{oi2r}}^2(T') + 2 \geq y_{\text{oi2r}}(T') - \left\lfloor \frac{n(T')}{6} \right\rfloor + 1 + 2 \geq y_{\text{oi2r}}(T) - 3 - \left\lfloor \frac{n(T - 4)}{6} \right\rfloor + 3 \geq y_{\text{oi2r}}(T) - \left\lfloor \frac{n(T - 6)}{6} \right\rfloor + 1.
\]

This completes the proof. \( \blacksquare \)

We conclude this paper with an open problem.

**Problem.** Prove or disprove: for any non-trivial connected graph \( G \) of order \( n \), \( y_{\text{oi2r}}^2(G) \geq y_{\text{oi2r}}(G) - \left\lfloor \frac{n}{6} \right\rfloor + 1 \).

**Disclosure statement**

No potential conflict of interest was reported by the authors.

**Funding**

This work was supported by the Natural Science Foundation of Guangdong Province under grant 2018A0303130115.

**ORCID**

Qiong Kang \( \text{http://orcid.org/0000-0001-8123-9184} \)  
Vladimir Samodivkin \( \text{http://orcid.org/0000-0001-7934-5789} \)  
Marzieh Soroudi \( \text{http://orcid.org/0000-0002-6677-6835} \)

**References**

[1] West DB. Introduction to graph theory. 2nd ed. Upper Saddle River: Prentice-Hall, Inc.; 2000.

[2] Gallai T. Uber extreme Punkt- und Kantenmengen. Ann Univ Sci Budapest Eotvos Sect Math. 1959;2:133–138.

[3] Ahangar HA, Chellali M, Samodivkin V. Outer independent Roman dominating functions in graphs. Int J Comput Math. 2017;94:2547–2557.

[4] Krzykowski M, Venkatakrishnan YB. Bipartite theory of graphs: outer-independent domination. Natl Acad Sci Lett. 2015;38:169–172.

[5] Krzykowski M. An upper bound on the 2-outer-independent domination number of a tree. C R Math. 2011;349:1123–1125.
[6] Li Z, Shao Z, Lang F, et al. Computational complexity of outer-independent total and total Roman domination numbers in trees. IEEE Access. 2018;6:35544–35550.

[7] Rad NJ, Krzywkowski M. 2-Outer-independent domination in graphs. Natl Acad Sci Lett. 2015;38:263–269.

[8] Brešar B, Henning MA, Rall DF. Rainbow domination in graphs. Taiwanese J Math. 2008;12:201–213.

[9] Amjadi J, Dehgardi N, Furuya M, et al. A sufficient condition for large rainbow domination number. Int J Comput Math Comput Syst Theory. 2017;2:53–65.

[10] Brešar B, Kraner Šumenjak T. On the 2-rainbow domination in graphs. Discrete Appl Math. 2007;155:2394–2400.

[11] Chang GJ, Wu J, Zhu X. Rainbow domination on trees. Discrete Appl Math. 2010;158:8–12.

[12] Chellali M, Haynes TW, Hedetniemi ST. Bounds on weak roman and 2-rainbow domination numbers. Discrete Appl Math. 2014;178:27–32.

[13] Meierling D, Sheikholeslami SM, Volkmann L. Nordhaus–Gaddum bounds on the k-rainbow domatic number of a graph. Appl Math Lett. 2011;24:1758–1761.

[14] Sheikholeslami SM, Volkmann L. The k-rainbow domatic number of a graph. Discuss Math Graph Theory. 2012;32:129–140.

[15] Xu G. 2-rainbow domination in generalized Petersen graphs P(n, 3). Discrete Appl Math. 2009;157:2570–2573.

[16] Fan W, Ye A, Miao F, et al. Outer-independent Italian domination in graphs. IEEE Access. 2019;7:22756–22762.

[17] Krzywkowski M. Total outer-independent domination in graphs. Manuscript.

[18] Krzywkowski M. A lower bound on the total outer-independent domination number of a tree. C R Math. 2011;349:7–9.