Time-Dependent Strain in Graphene

Anha Bhat\textsuperscript{1}, Salwa Alsaleh\textsuperscript{2}, Davood Momeni\textsuperscript{3}, Atikur Rehman\textsuperscript{1}, Zaid Zaz\textsuperscript{4}, Mir Faizal\textsuperscript{5,6}, Ahmed Jellal\textsuperscript{7}, Lina Alasfar\textsuperscript{8}

\textsuperscript{1}Department of Metallurgical and Materials Engineering, National Institute of Technology, Srinagar, Kashmir-190006, India
\textsuperscript{2}Department of Physics and Astronomy, College of Science, King Saud University, Riyadh 11451, Saudi Arabia
\textsuperscript{3}Department of Physics, College of Science, Sultan Qaboos University, P.O.Box 36, P.C.123, Al-Khoud, Muscat, Sultanate of Oman
\textsuperscript{4}Department of Electronics and Communication Engineering, University of Kashmir, Srinagar, Kashmir-190006, India
\textsuperscript{5}Department of Physics and Astronomy, University of Lethbridge, Lethbridge, AB T1K 3M4, Canada
\textsuperscript{6}Irving K. Barber School of Arts and Sciences, University of British Columbia - Okanagan, 3333 University Way, Kelowna, British Columbia V1V 1V7, Canada
\textsuperscript{7}Theoretical Physics Group, Faculty of Sciences, Chouaib Doukkali University, PO Box 20, 24000 El Jadida, Morocco
\textsuperscript{8}Université Clermont Auvergne, 4, Avenue Blaise Pascal 63178 Aubière Cedex, France,

Abstract

We will analyse the effect of time-dependent strain on a sheet of graphene by using the field theory approach. It will be demonstrated that in the continuum limit, such a strain will induce a non-abelian gauge field in graphene. We will analyse the effective field theory of such system near the Dirac points and study its topological properties.

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1 Introduction

Graphene is by far the most investigated two-dimensional material both on theoretical and experimental frontiers with wide applications which bridge condensed matter to quantum mechanics when seen in the perspective of physics [1] because of its remarkable emergent properties owning to different nanoscale phenomena which take place in its lattice. One such aspect is the nanomechanics, which is the most important contribution is that of the strain which is a natural phenomena that can appear under different mechanisms in graphene systems and therefore affecting its topology. For example, graphene when grown on a surface usually experiences a moderate strain due to the surface corrugations of the substrate [2] or the strain can also be developed when the graphene lattice is not complimentary to the substrate on which it is grown [3]. This can bring about the variations in the physical properties of the low dimensional systems and in this case the electronic properties are affected which again can pave a way to develop various applications.

The effective field theory describing graphene is a massless fermionic theory in three dimensions [1], which has also been verified experimentally [4, 5]. In such theory, the deterministic step is that the velocity of light \( c \) is replaced by Fermi velocity \( v_F \approx \frac{c}{300} \). It has been showed that a sheet of graphene when under strain induce the effective non-abelian gauge field [6, 7], which occur in the continuum limit of the standard tight binding approach. This is because the spin connection of the graphene deformed metric can be related to the non-abelian gauge field. The strain in graphene sheets effects the Landau levels due to induced gauge fields [8] and it can be comprehended that these gauge fields have a role to play in the physical behavior of graphene lattice. It has been reported that the time-dependent strain in graphene has been used to investigate the topological electric current [9] and has also been used to obtain the fractional topological phases in its two dimensional arrangement [10]. It is expected that the time-dependent strain will induce a three dimensional non-abelian gauge field theory in a deformed sheet of graphene, and we will construct such a theory in the present work. It may be noted that interesting deformations of graphene have been studied using Dirac equation in curved spacetime [12]- [15]. Thus, it is possible to analyse graphene using the Dirac equation in curved spacetime, which produces an effective gauge field in it.

Motivated by different developments on the subject cited above, we will analyse the gauge theory induced by a time-dependent strain in graphene. As the strain will be time-dependent, we shall consider the dynamics of the non-abelian gauge field and hence introduce its kinetic term. It is possible to analyse geometry of curved spacetime by using the translation group as a gauge group [16]- [17] or Lorentz group as a gauge group [18]. These will be done by considering the metric of spacetime for the effective field theory describing the strained graphene and introducing a transformation in terms of the Lorentz group generators \( SO(2, 1) \).

The present paper is organized as follows. In section 2, we review some relevant tools that include the tight-binding and the continuum models as a mathematical framework to describe the graphene. In section 3, we construct our effective theory describing the strained graphene in terms of the non-abelian gauge field based on deformed spacetime. The topological properties of the strained graphene will be done in section 4 where the instanton solutions will be obtained numerically. Finally, we conclude our work and give some outlooks.
2 Basic model

Revisiting the basic information [5], the graphene sheet is a two-dimensional system made up of carbon atoms, which are arranged in an hexagonal honeycomb structure. This arrangement is a triangular lattice with a basis of two atoms per unit cell where the lattice vector for this structure are \( \mathbf{a}_1 = \frac{3}{2} (3, \sqrt{3}) \), \( \mathbf{a}_2 = \frac{3}{2} (3, -\sqrt{3}) \), and the carbon-carbon distance is \( a \approx 1.42 \AA \). The reciprocal lattice vector for this structure are

\[
\mathbf{b}_1 = \frac{2\pi}{3a} (1, \sqrt{3}) \quad \text{and} \quad \mathbf{b}_2 = \frac{2\pi}{3a} (1, -\sqrt{3}) .
\]

The two points at the corners of graphene Brillouin zone are given by

\[
K = \left( \frac{2\pi}{3a}, \frac{2\pi}{3\sqrt{3}a} \right) \quad \text{and} \quad K' = \left( \frac{2\pi}{3a}, -\frac{2\pi}{3\sqrt{3}a} \right)
\]

which are called the Dirac points. The three nearest neighbors are located at

\[
\delta_1 = \frac{a}{2} (1, \sqrt{3}) , \quad \delta_2 = \frac{a}{2} (1, -\sqrt{3}) , \quad \delta_3 = -a (1, 0) \quad (2)
\]

and the six next to the nearest neighbors are located at

\[
\delta_1 = \pm \mathbf{a}_1 , \quad \delta_2 = \pm \mathbf{a}_2 , \quad \delta_3 = \pm (\mathbf{a}_1 - \mathbf{a}_2) . \quad (3)
\]

The electrons in the graphene can be described by the tight-binding approach where the electrons can hop to both nearest and next to the nearest atoms. Now the Hamiltonian for this system can be represented by

\[
H = -t \sum (a_\sigma^\dagger b_\sigma + h.c) - t' \sum (a_\sigma^\dagger a_\sigma + b_\sigma^\dagger b_\sigma + h.c.) \quad (4)
\]

where \( \sigma \) denotes the spin, \( t \sim 2.8 eV \) is the hopping energy to the nearest neighbor, and \( t' \) is the hopping energy to the next to the nearest neighbor. The energy bands for this Hamiltonian can be written as [25]

\[
E_\pm = \pm t\sqrt{3 + f(k)} - t'f(k) \quad (5)
\]

and \( f(k) \) is a function of the wave vector

\[
f(k) = 2 \cos \sqrt{3} (k_y a) + 4 \cos \left( \frac{\sqrt{3}}{2} k_y a \right) \cos \left( \frac{3}{2} k_x a \right) . \quad (6)
\]

Now expanding (5) around the Dirac point \( k = K + q \) (similarly for \( K' \)) to obtain the linear dispersion relation [25]

\[
E_\pm \sim \pm v_F |q| \quad (7)
\]

where the Fermi velocity is \( v_F = \frac{3a}{2} \). Here the velocity Expanding the operator in \( \delta_1 = \delta_1, \delta_2, \delta_3, \) and using the approximation

\[
\sum_i \exp (\pm K \delta_i) = \sum_i \exp (\pm K' \delta_i) = 0 \quad (8)
\]

to obtain the Hamiltonian near it in the continuum limit making the Hamiltonian for this system

\[
H_K = v_F (\sigma_x p_x + \sigma_y p_y) \quad (9)
\]
which is the usually denoted as Dirac Hamiltonian, with the velocity of light replaced by the Fermi velocity [26]. Now from this Hamiltonian, the effective field theory action describing graphene, in the continuum limit, can be demonstrated as [26]

\[ S_{\text{eff}} = \int d^3x \bar{\psi} (i\gamma^\mu \partial_\mu) \psi = \int d^3x \bar{\psi} i(\gamma^i \partial_i \psi - v_F \gamma^0 \partial_0 \psi) \psi. \] (10)

Having reviewed most of the fundamentals needed to describe the graphene system, it shall be in lieu of our interest to witness as to how the deformation in terms of strains in graphene will effectuate the above results. More precisely, we would construct the possible implication of a deformed spacetime in a strained graphene.

3 Strained graphene

As reported, there are different experimental ways to realise the strain effect in graphene, curved spacetime. The metric for the effective field theory of a sheet of graphene under strain can be represented by [27]

\[ ds^2 = dt^2 - g_{ij}(x,y) dx^i dx^j. \] (11)

It may be noted that in \((2+1)\)d-spacetime, the Riemann tensor has only one independent component, which is proportional to the Gaussian curvature. So, we can represent the metric for strained graphene as [28]

\[ ds^2 = dt^2 - e^{2\sigma(x,y)} dx^2 - e^{2\sigma(x,y)} dy^2. \] (12)

So, the effect of strain on graphene is represented by a single function, \(\sigma(x,y)\). However, it is also possible to take a time-dependent strain in graphene. In fact, time-dependent strain in graphene has been used to analyse the a topological electric current in graphene [9] and also to obtain fractional topological phases in graphene [10]. Now we can use the general consideration that the system does not contain terms which mix spatial and temporal coordinates, such as \(dtdx\) and \(dtdy\), and written a general time-dependent version of the metric for strain in graphene, without such terms. So, if \(\sigma(x,y,t)\) is a general function of time, then the metric describing graphene with time-dependent strain can be written as

\[ ds^2 = dt^2 - e^{2\sigma(x,y,t)} dx^2 - e^{2\sigma(x,y,t)} dy^2. \] (13)

It may be noted that here we have not specified the form of \(\sigma(x,y,t)\), and kept it very general. Now the time can be corroborated as complex in terms of \(\tau = it\) to obtain the new form for the metric

\[ ds^2 = -d\tau^2 - e^{2\sigma(x,y,\tau)} dx^2 - e^{2\sigma(x,y,\tau)} dy^2. \] (14)

To go further, we need to introduce the following transformation \(x^a \rightarrow x^b \Lambda^a_b\). This allows a fermion in three dimension to transform as

\[ \psi \rightarrow U \psi = \exp \left( \frac{i}{2} \Lambda^{ab}(x) \Sigma_{ab} \right) \psi \] (15)
where $\Sigma_{ab}$ describes the generators of Lorentz group $SO(2,1)$, for $(t,x,y)$, and it can be represented as $\Sigma_{ab} = -i[\gamma_a, \gamma_b]/4$. However, after complex equation, when $(\tau,x,y)$ is used, the generators emerge as $SO(3)$. To obtain the geometry of a sheet of graphene using this transformation, the metric is defined as $g_{\mu\nu} e^a_\mu e^b_\nu = \eta_{ab}$, and $\omega^a_\mu$ be allowed to represent the spin connection associated with it. So, using the explicit expression for this spin connection $\omega^a_\mu$ [11], we could define an effective non-abelian gauge field as

$$A_\mu = \omega^a_\mu \Sigma_{ab} = \left[2 e^a_\mu \partial_\nu e^b_\nu - 2 e^b_\nu \partial_\mu e^a_\nu - 2 e^a_\nu \partial_\mu e^b_\nu + 2 e^b_\nu \partial_\mu e^a_\nu \right] + e_{\mu c} e^a_\nu \partial_\nu e^c_\rho - e_{\mu c} e^b_\nu \partial_\nu e^a_\rho |\Sigma_{ab}.$$  

In line with above equations, the covariant derivatives of effective non-abelian gauge field can be elucidated as

$$D_\mu = \partial_\mu I + \frac{i}{2} A_\mu.$$  

(17)

The gauge field (16) obtained from the spin connection transforms under $SO(3)$ as

$$A_\mu \rightarrow [U \omega_\mu U^{-1} - (\partial_\mu U)U^{-1}]^{ab} = [UA_\mu U^{-1} - (\partial_\mu U)U^{-1}]^{ab}.$$  

(18)

Now the effective field theory of graphene under strain can be written as

$$S_{\text{eff}} = \int d^3x e^{\bar{\psi}} (i\gamma^a e^a_\mu D_\mu) \psi$$

$$= \int d^3x e^{\bar{\psi}} \left( i\gamma^a e^a_\mu \left( \partial_\mu I + \frac{i}{2} A_\mu \right) \right) \psi$$

$$= \int d^3x e^{2\sigma(\tau,x,y)} e^{\bar{\psi}} \left( i\gamma^a e^a_\mu \left( \partial_\mu I + \frac{i}{2} \omega^a_\mu \Sigma_{ab} \right) \right) \psi.$$  

(19)

which is invariant under the $SO(2,1)$ gauge symmetry. It is possible for the strain to fluctuate in time. In order to analyse its effect the kinetic part of this effective non-abelian gauge theory has to be calculated keeping in view that the detailed study of electronic and mechanical properties of graphene have been already accomplished. [19,20]. In fact, it has been proposed that the strain can be used to generate large effective gauge fields, which can have direct consequences on the electronic properties of graphene [21,22], which can be harnessed to develop quantum electronic pump based applications based on this proposal [23,24].

An important property for the above strain is that the covariant derivatives do not commute when present in a sheet of graphene. In fact, the commutator of these two covariant derivatives can be expressed as

$$iF^{\sigma\mu}_{\rho\nu} \Sigma_{ab} = [D_\mu, D_\nu] = T_{\mu\nu} D_c + \frac{i}{2} R^{ab}_{\mu\nu} \Sigma_{ab}.$$  

(20)

If it is assumed that there is no contribution pertaining to the torsion $T^c_{\mu\nu}$ term, $T^c_{\mu\nu} = 0$, then the commutator of two covariant derivatives for graphene can be expressed using the Riemann tensor as

$$iF^{\sigma\mu}_{\rho\nu} \Sigma_{ab} = R^{ab}_{\rho\mu} \Sigma_{ab}$$

$$= \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu].$$  

(21)
The curvature tensor transforms as
\[
iF_{ab\mu\nu} = F_{ab\mu\nu} \rightarrow [U]^a_i F_{cd\mu\nu}[U^{-1}]^b_d
= i[U]^a_i F_{cd\mu\nu}[U^{-1}]^b_d.
\] (22)

As the strain is time dependent, it changes with time, which necessitates the introduction of a Kinetic term for the gauge field produced by strain. The simplest action, that can produce dynamics, without higher derivative ghost states, is the Yang-Mills action. Then, the Kinetic action for the gauge fields produced by strain can be written as
\[
S_g = -\frac{1}{4g^2} \int d^3x e Tr(F_{\mu\nu} F_{\mu\nu})
= \frac{1}{4g^2} \int d^3x e^{2\sigma(x,y,\tau)} \left[ 2 \left( \frac{\partial^2}{\partial \tau^2} \sigma(x,y,\tau) + \left( \frac{\partial}{\partial \tau} \sigma(x,y,\tau) \right)^2 \right)^2
+ \left( \frac{\partial^2}{\partial y^2} \sigma(x,y,\tau) + \left( \frac{\partial}{\partial y} \sigma(x,y,\tau) \right)^2 \right)^2 e^{2\sigma(x,y,\tau)}
- \left( \frac{\partial^2}{\partial x^2} \sigma(x,y,\tau) + \left( \frac{\partial}{\partial x} \sigma(x,y,\tau) \right)^2 \right)^2 \right],
\]
\[
\text{where } g \text{ is a suitable constant describing this theory, where physically denotes the strength by coupling the fluctuations in the geometry to the effective field theory describing fermions in graphene under strain. Now the effective field theory for system can be stated as}
S_t = S_{\text{eff}} + S_g
\] (24)
which is the effective field theory of a sheet of graphene under time dependent strain, such that the dynamics is associated with the strain.

4 Topological properties

In view of the above effective field theory, there is a possibility to study its implications regarding the topological properties of graphene. It may be noted that topological defects have been observed in graphene [29]-[34]. In fact the bosonic symmetry protected topological state has been used to study the transport in graphene. The instanton tunneling is important to understand the shot noise measurement for such topological state in the strong interaction limit [35] with respect to the instanton solution and the in-gap fluctuation states in biased bilayer graphene [36]. Therefore, it will be interesting to understand the behavior of the instanton solution of the action \( S_g \). We can complexify time, and with such Euclidean time, our theory can be analyzed as a theory on a three dimensional manifold \( M \) with Euclidean signature with a discussion about the topological properties of the instanton for this manifold.

Since the action is in fact an Euclidean action, the instanton solution is divided into two components, the self-dual and anti self-dual solutions.

\[
F_{\mu\nu} = \pm \ast F_{\mu\nu}
\] (25)
with \(*\) denoting the hodge star operator. Using the explicit expression for \(F_{\mu\nu}\) from (21), it is found to have three independent components:

\[
F_{\tau x} = \frac{1}{2} \left( e^{2\sigma} (\partial_\tau \sigma)^2 + \partial_y^2 \sigma + \partial_x^2 \sigma \right) [\gamma_\tau, \gamma_x] 
+ \frac{1}{2} (\partial_\tau \partial_y \sigma) [\gamma_\tau, \gamma_y] + \frac{1}{2} (\partial_\tau \partial_x \sigma) [\gamma_x, \gamma_y] 
\]

\[
F_{\tau y} = \frac{1}{2} (\partial_\tau \partial_y \sigma) [\gamma_\tau, \gamma_x] + \frac{1}{2} \left( (\partial_\tau \sigma)^2 + \partial_y^2 \sigma \right) [\gamma_\tau, \gamma_y] 
\]

\[
F_{\tau y} = \frac{1}{2} (\partial_\tau \partial_x \sigma) [\gamma_\tau, \gamma_x] + \frac{1}{2} \left( e^{2\sigma} ((\partial_\tau \sigma)^2 + \partial_x^2 \sigma) \right) [\gamma_x, \gamma_y].
\]

The self-dual (instanton) solution, yields the following differential equations

\[
e^{2\sigma} (\partial_\tau \sigma)^2 + \partial_y^2 \sigma + \partial_x^2 \sigma = -(\partial_\tau \partial_y \sigma) - (\partial_\tau \partial_x \sigma) 
\]

\[
(\partial_\tau \partial_x \sigma) = -e^{2\sigma} ((\partial_\tau \sigma)^2 + \partial_x^2 \sigma) 
\]

\[
(\partial_\tau \partial_y \sigma) = (\partial_\tau \sigma)^2 - \partial_x^2 \sigma.
\]

Substituting the LHS of (30) and (31) in (29), to end up with the partial differential equation for the strain function \(\sigma(\tau, x, y)\) for the instanton

\[
(1 + e^{2\sigma}) \partial_\tau^2 \sigma + (\partial_\tau \sigma)^2 + \partial_x^2 \sigma + \partial_y^2 \sigma = 0
\]

as well as the differential equation for the anti-instanton solution

\[
(1 - e^{2\sigma}) \partial_\tau^2 \sigma - (\partial_\tau \sigma)^2 + \partial_x^2 \sigma + \partial_y^2 \sigma = 0.
\]

These PDE’s can be solved for different boundary conditions, depending on the characteristics of the graphene under consideration. For example, if the Dirichlet boundary conditions are introduced on the 3D Euclidean plane, letting \(\sigma(\tau = 0, x, y) = 0\) and \(\partial_\tau \sigma(\tau = 0, x, y) = 0\), the instanton and anti-instanton solutions, near \(\tau \approx 0\), become

\[
2\partial_\tau^2 \sigma + \partial_x^2 \sigma + \partial_y^2 \sigma = 0
\]

\[
\partial_\tau^2 \sigma + \partial_x^2 \sigma + \partial_y^2 \sigma = 0
\]

which can be solved numerically, see Figure 1:

We can complexify time, and with such an Euclidean time, this theory can be analyzed as a theory on a three dimensional manifold \(M\) with Euclidean signature, such as

\[
S_g^E = \frac{1}{2g^2} \int_M \text{Tr} (F \wedge F).
\]

We can now discuss the topological properties of the instanton for this manifold. It is possible to compactify \(M\) into a closed surface, \(M’\) by assuming a characteristic length \(L\) and periodic boundary conditions, such that \(x \rightarrow L\) [40]. Now for this compact manifold, we have

\[
A_\mu = U(x)^{-1} \partial_\mu U(x)
\]

where \(U(x)\) are elements of the \(SO(3)\) gauge symmetry. By this compactification and Poincare lemma, we can write \(F^2\) as an exact form

\[
\text{Tr} F^2 = dK
\]
and $K$ is a local three form, which implies that $\text{Tr} F^2$ is an element of the cohomology group $H^3(M')$. Moreover, it is possible to write $K$ as a Chern Simons form \cite{41}
\[
K = \text{Tr} \left( A dA + \frac{2}{3} A^2 \right). \quad (39)
\]
Now by using the Stokes theorem, we can write the action (36) after compactification as
\[
S^E_g = -\frac{1}{4g^2} \int_{\partial M'} \text{Tr} \left( A^3 \right). \quad (40)
\]
For the instanton solution, we have the expression for the gauge field $A$ in terms of the mappings $U : S^2 \longrightarrow SO(2,1)$. Then the above integral is reduced to the degree of that mapping divided by the area of the unit $S^2$
\[
\frac{1}{6\pi g^2} \int_{\partial M'} \text{Tr} \left( A^3 \right) = \frac{n}{g^2}, \quad (41)
\]
which can be rewritten as
\[
\frac{1}{2} \int_{M'} \text{Tr} \left( \frac{iF}{2\pi} \right)^2 \quad (42)
\]
By using the definition of the field strength tensor in terms of the Riemann curvature given in (21), the above integral reduces to the Euler characteristic of the compactified manifold $M'$ \cite{42,43}
\[
\frac{1}{2} \int_{M'} \text{Tr} \left( \frac{iF}{2\pi} \right)^2 = \frac{1}{8\pi^2} \int_{M'} \text{Tr} (R \wedge R) = \chi(M'). \quad (43)
\]
Hence, we observe that the degree of the mapping corresponding to instanton solution is proportional to a topological invariant of the theory. Therefore, it is possible to classify corrugated sheet of graphene under dynamic time dependent strain by using these topological instantons. It clear that, these topological instantons can produce measurable effects.
5 Conclusion

Using the field theory approach, a sheet of graphene was analysed with a time-dependent strain. The spin connection of this theory was used to obtain a non-abelian gauge field. As the strain was time-dependent, the kinetic term for this effective non-abelian gauge theory was also considered which motivated the study of the topological properties of the system of interest given its deformations and shows that the topological instantons could map the considerable effects as the result of such a proposed study. There are various interesting deformations of the effective field theory of graphene, and these can be analysed using various interesting and equally effective geometries. The surface of revolution with constant negative curvature has been constructed using graphene [37]. We investigated the instanton solution for the effective Yang-Mills theory, and analyzed the Euler characteristic of such a sheet of graphene.

It is possible to analyse the analogous black hole like solution in the effective field theory describing graphene [38], using a BTZ (Banados Teitelboim Zanelli) [39]- [44] like metric. It may be noted that this is only an effective solution and in which the velocity of light is replaced by the Fermi velocity. Thus, just as a real black hole forms the horizon for light, these analogous black hole like solutions form effective horizons for Fermi velocity. It has been demonstrated that a deformed sheet of graphene can also be analysed using negative constant curvature in an externally applied magnetic field, which can be done by using a stationary optical metric of the Zermelo form that is conformal to the BTZ black hole [38]. Thus, interesting analogous geometries have been studied using the effective field theory of graphene. Therefore, an analogous black hole like solution can form in a deformed sheet of graphene. However, it is known that the usual black hole physics leads to the existence of the generalized uncertainty principle [45]- [46]. It would be interesting to analyse this relation between analogous black holes in graphene and generalized uncertainty principle which would further develop investigations at the crossroads of physics pertaining to photonics and optoelectronic applications.

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References

[1] K. S. Novoselov, A. K. Geim, S. V. Morozov, D. Jiang, Y. Zhang, S. V. Dubonos, I. V. Gregorieva and A. A. Firsov, Science 306, 666 (2004).

[2] M. Teague, A. Lai, J. Velasco, C. Hughes, A. Beyer, M. Bockrath, C. Lau and N.-C. Yeh, Nano Lett. 9, 2542 (2009).

[3] Z. Ni, W. Chen, X. Fan, J. Kuo, T. Yu, A. Wee and Z. Shen, Phys. Rev. B 77, 115416 (2008).

[4] A. K. Geim, Science 324, 1530 (2009).

[5] A. H. Castro Neto, F. Guinea, N. M. R. Peres, K. S. Novoselov and A. K. Geim, Rev. Mod. Phys. 81, 109 (2009).
[6] E. Arias, A. R. Hernandez and C. Lewenkopf, Phys. Rev. B 92, 245110 (2015).

[7] B. Yang, Phys. Rev. B 91, 241403(R) (2015).

[8] F. Guinea, A. G. Geim, M. I. Katsnelson and K. S. Novoselov, Phys. Rev. B 81, 035408 (2010).

[9] A. Vaezi, N. Abedpour, R. Asgari, A. Cortijo and M. A. H. Vozmediano, Phys. Rev. B 88, 125406 (2013).

[10] P. Ghaemi, J. Cayssol, D. N. Sheng and A. Vishwanath, Phys. Rev. Lett. 108, 266801 (2012).

[11] M. Kober, Phys. Rev. D 82, 085017 (2010).

[12] A. Iorio, Ann. Phys. 326, 1334 (2011).

[13] A. Cortijo and M. A. H. Vozmediano, Eur. Phys. J. ST 148, 83 (2007).

[14] J. Smotlacha and R. Pincak, Eur. Phys. J. B 86, 480 (2013).

[15] A. Iorio, Eur. Phys. J. Plus 127, 156 (2012).

[16] Y. M. Cho, Phys. Rev. D 14, 2521 (1976).

[17] J. W. Maluf, Phys. Rev. D 67, 108501 (2003).

[18] F. W. Hehl, P. Von Der Heyde, G. D. Kerlick and J. M. Nester, Rev. Mod. Phys. 48, 393 (1976).

[19] S. Bunch, A. M. van der Zande, S. S. Verbridge, I. W. Frank, D. M. Tanenbaum, J. M. Parpia, H. G. Craighead and P. L. McEuen, Science 315, 490 (2007).

[20] D. Garcia-Sanchez, A. M. van der Zande, A. S. Paulo, B. Lassagne, P. L. McEuen and A. Bachtold, Nano Lett. 8, 1399 (2008).

[21] F. Guinea, M. I. Katsnelson and A. K. Geim, Nature Phys. 6, 30 (2009).

[22] V. M. Pereira and A. H. Castro Neto, Phys. Rev. Lett. 103, 046801 (2009).

[23] E. Prada, P. San-Jose and H. Schomerus, Phys. Rev. B 80, 245414 (2009).

[24] T. Low, Y. Jiang, M. I. Katsnelson, and F. Guinea, Nano Lett. 12, 850 (2012).

[25] P. R. Wallace, Phys. Rev. 71, 622 (1947).

[26] G. W. Semenoff, Phys. Rev. Lett. 53, 2449 (1984).

[27] A. Iorio, Ann. Phys. 326, 1334 (2011).

[28] A. Iorio and P. Pais, Phys. Rev. D 92, 125005 (2015).

[29] A. Hashimoto, K. Suenaga, A. Gloter, K. Urita and S. Iijima, Nature 430, 870 (2004).

[30] E. Cockayne, Phys. Rev. B 85, 125409 (2012).
[31] S. Ryu, C. Mudry, C. Y. Hou and C. Chamon, Phys. Rev. B 80, 205319 (2009).
[32] M. P. L. Sancho, F. de Juan and M. A. H. Vozmediano, Phys. Rev. B 79, 075413 (2009).
[33] Y. A. Sitenko and N. D. Vlasii, Nucl. Phys. B 787, 241 (2007).
[34] O. V. Yazyev and S. G. Louie, Phys. Rev. B 81, 195420 (2010).
[35] R. X. Zhang and C. X. Liu, Phys. Rev. Lett. 118, 216803 (2017).
[36] V. V. Mkhitaryan and M. E. Raikh, Phys. Rev. B 78, 195409 (2008).
[37] A. Iorio and G. Lambiase, Phys. Lett. B716, 334 (2012).
[38] M. Cvetic and G. W. Gibbons, Annals Phys. 327, 2617 (2012).
[39] M. Banados, C. Teitelboim and J. Zanelli, Phys. Rev. Lett. 69, 1849 (1992).
[40] M. Nakahara, Geometry, topology and physics, CRC Press (2003).
[41] V. Gerald, Aspects topologiques de la physique en basse dimension, Springer Berlin Heidelberg (1999).
[42] T. Fujiwara and T. Fukui, Phys. Rev. D 85, 125034 (2012).
[43] P. A. Cano, T. Ortin and P. F. Ramirez, arXiv:1704.00504.
[44] M. Banados, C. Teitelboim and J. Zanelli, Phys. Rev. Lett. 69, 1849 (1992).
[45] M. Maggiore, Phys. Lett. B 304, 65 (1993).
[46] M. I. Park, Phys. Lett. B 659, 698 (2008).