Dark soliton (DS) and Josephson vortex (JV) in quasi-1D long Bose Josephson junction (BJJ) can be interconverted by tuning Josephson coupling. Rates of the interconversion as well as of the thermally activated phase-slip effect, resulting in the JV switching its vorticity, have been evaluated. The role of quantum phase-slip in creating superposition of JVs with opposite vorticities as a qubit is discussed as well. Utilization of the JV for controlled and coherent transfer of atomic Bose-Einstein condensate (BEC) is suggested.

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Solitons, in general, and the DS in particular, continue to be a fascinating subject for study. In 1D the snake instability is suppressed and the DS is a stable particle-like object (apart from the slow phonon induced decay). Vortices in atomic BEC have been studied in great details theoretically and experimentally as well (see in ref. [4]).

Superfluid current circulation can exist in two parallel quasi-1D waveguides, coupled by a uniform Josephson tunneling $\gamma > 0$ along their length (see Fig. 1), akin to the JV in superconducting long Josephson junctions. Traditionally, phase variation only is considered within the frame of the Sine-Gordon (SG) equation. As it turns out, such description is insufficient for the BEC waveguides in the quasi-1D regime, where the 1D Gross-Pitaevskii (GP) equation in the axial direction should be employed. Short BJJ have been thoroughly examined in refs. [5, 6, 9]. Two coupled waveguides were already studied as well with focus on bright solitons [10]. Quasi-1D BEC were created in a variety of magnetic traps [11]. Intriguing perspectives are offered by BECs on microchips [12, 13]. Coherence between two parallel elongated BECs has been demonstrated in the seminal MIT experiment and is currently being under intense investigation [14, 16]. Two parallel waveguides with varying separation from each other were designed as an interferometer in refs. [13, 15]. As noted in ref. [10], the considered setup has precise optical analogy – dual-core optical fiber. Similar system with point-like coupling has been considered in ref. [13] with emphasis on possible application in atomic interferometry.

Here we show that as the ratio $\nu = \gamma / \mu$, with $\mu$ being the chemical potential, becomes smaller than some critical value $\nu_c$, the DS transforms into the JV spontaneously. This breaks the time-reversal symmetry. Conversely, as $\nu$ exceeds $\nu_c$, the JV transforms into the DS, which restores the symmetry. The DS$\leftrightarrow$JV interconversion effect is a reversible 1D analog of the 3D DS snake instability [2]. In contrast to the 3D, where the DS irrecoverably decays into vortex rings, the DS in the quasi-1D BJJ can be controllably restored from the JV by tuning $\nu$ above the critical value. The thermal and quantum phase-slip effects can restore the symmetry as well.

The initially closed vortex circulation opens when the vortex encounters a boundary. The redirected BEC flux transfers BEC atoms from Reservoir 1 to Reservoir 2.

FIG. 1: Schematic representation of the long BJJ.

Model: Action. Dynamics of the bosonic fields $\psi_k$, $k = 1, 2$, is controlled by the action

$$S = \int dt dx \mathcal{L}, \quad \mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_{12},$$

where $t$ is time and $x$ is the axial coordinate; the Lagrangian density $\mathcal{L}$ consists of the parts

$$\mathcal{L}_k = i \hbar \psi_k^* \dot{\psi}_k - \frac{\hbar^2}{2m} |\nabla \psi_k|^2 + \frac{\mu_k}{2} |\psi_k|^2 - \frac{\gamma}{2} |\psi_k|^4,$$

describing the dynamics along $x$, as well as the Josephson tunneling

$$\mathcal{L}_{12} = \gamma \psi_1^* \psi_2 + c.c.$$
chemical potentials $\mu_k = \mu$ in each waveguide will be kept identical; $m$ is atomic mass; $g = 4\pi\hbar^2a/(mv^2)$ — the effective 1D interaction constant, with $a > 0$ being 3D scattering length and $r^2_1$ standing for the effective width of the waveguides. Strictly speaking, the standard Josephson coupling [3] is valid for $\nu < 1$. As $\nu \sim 1$, higher order terms in $\psi_{1,2}$ should be considered. However, while such terms may change the numerical value of $\nu$, they will not eliminate the interconversion effect. Thus, in what follows, we will not discuss the higher order couplings.

**Model: Dissipative function.** Significant damping of Josephson dynamics can occur due to tunneling of the normal component between two BECs. Phenomenologically, the dissipation can be introduced through the dissipative function $F_D$, so that

$$\frac{d}{dt}\delta\psi_k^* - \frac{\delta\mathcal{L}}{\delta\psi_k} = -\frac{\delta F_D}{\delta\psi_k^*}$$

in accordance with the standard procedure [20]. In general, $F_D$ must be positively defined function of the time derivative of physically observable quantities [20]. We choose it in the minimal form

$$F_D = \int dx \frac{\rho^2}{2\sigma},$$

where $\rho = |\psi_1|^2 - |\psi_2|^2$. All the information about normal component is included into the kinetic coefficient $\sigma$, which can be related to the dissipation rate of small Josephson oscillations considered in ref. [8]. Employing eqs. (1.3) for uniform $\rho$ and small relative phase, we obtain damped Josephson oscillations $\dot{\rho} + \omega^2_\nu \rho + \kappa \dot{\rho} = 0$, where the Josephson frequency and the damping coefficient are $\omega_j = 2\sqrt{\gamma(\mu + 2\gamma)}/\hbar$ and $\kappa = 8\gamma(\mu + \gamma)/(\hbar^2g\sigma)$, respectively. The value of $\kappa$, which determines a typical relaxation time $\tau \sim \sigma$, can be taken from the microscopic analysis [8].

We introduce the units of length $l_c = \hbar/\sqrt{\mu}$ (correlation length), with $n_0 = \mu/g$ being the average 1D density in a single uncoupled $(\gamma = 0)$ waveguide, and of time $t_0 = \hbar/\mu$. Then, setting $\psi_k \to \sqrt{n_0}\psi_k$, the action (1) becomes $S = \hbar S_0 \int dt dx L$, where $S_0 = l_c n_0$ (the validity of the GP regime is justified by $S_0 \gg 1$) [6]; $L = L_1 + L_2 + L_{12}$, with $L_k = i(\psi_k^* \dot{\psi}_k + c.c.)/2 - (\nabla\psi_k)^2/2 + |\psi_k|^2 - |\psi_k|^2/2$ and $L_{12} = \nu \dot{\psi}_1^* \dot{\psi}_2 + c.c.$ The dissipative function takes the form $\hbar S_0 \sqrt{2\sigma}/\sigma$, with $\sigma = \hbar\sigma t_0^2/n_0$. Employing eqs. (1.3) in these units, we obtain

$$i\dot{\psi}_1 - \frac{1}{\sigma}\psi_1 = -\frac{\nabla^2}{2}\psi_1 - \psi_1 + |\psi_1|^2\psi_1 - \nu \psi_2;$$

$$i\dot{\psi}_2 + \frac{1}{\sigma}\psi_2 = -\frac{\nabla^2}{2}\psi_2 - \psi_2 + |\psi_2|^2\psi_2 - \nu \psi_1.$$  

The dissipative terms $\sim \dot{\rho}$ resemble the phenomenological dissipation introduced in ref. [21]. These conserve the total number of atoms. We note, however, that they violate the Galilean invariance (given by the transformation $\partial_t \to \partial_t - V\nabla_x$, $\psi \to \exp[i(Vx + V^2t/2)]\psi$, with $V$ being the velocity of a new frame moving along $x$). In this regard, it is important to realize a limited nature of the above phenomenological approach — it can only be applied in the case when the tunneling of the normal component does not conserve linear momentum (along the waveguides), that is, when scattering on imperfections of the trapping potential is significant as in ref. [8].

**Dark soliton and Josephson vortex.** Static one soliton solutions of eqs. (1–7) in infinite medium belong to a family

$$\psi_{1,2} = \sqrt{1 + \nu \tanh(px)} \pm i\frac{B\sqrt{\rho/2}}{\cosh(px)}. (8)$$

The DS, characterized by $\psi_1 = \psi_2$ corresponds to $\rho = \sqrt{1 + \nu}$ and $B = 0$ (that is, total density has zero). The static JV satisfies combined symmetry — time reversal and reflection $(\psi_1 = \psi, \psi_2 = \psi^*)$, and, thus, the equation

$$\psi_1 = \psi_2.$$  

Its solution [3] is given by $p = 2\sqrt{\nu}$ and $B = B_p/2 = 1/3\nu^2$. Obviously, it exists for $\nu < 1/3$ only. The phases $\phi_{1,2}$ of the fields $\psi_{1,2} \sim \exp(i\phi_{1,2})$ change from $\phi_{1,2} = 0$ at $x = -\infty$ to $\phi_1 = -\phi_2 = \pi$ at $x = +\infty$.

**DS-JV interconversion.** The DS formally exists for all values of the dimensionless coupling $\nu$. The JV solution is valid only for $\nu < \nu_c = 1/3$. At the critical value $\nu_c$, the JV turns into the DS. Simple energy argument shows that the DS is an unstable state for $\nu < \nu_c$. The energies $E_{DS} = \frac{a_s}{6}(1 + \nu)^{3/2}$ and $E_{JV} = \frac{a_s}{6}(3 - \nu)$ of the DS and the JV, respectively, as well as their $\nu$-derivatives become equal at $\nu = \nu_c$. For $\nu < \nu_c$, one finds $E_{DS} > E_{JV}$, which implies absolute instability of the DS.

Despite being identical to the problems of refs. [22, 23, 24, 26, 27] in the static limit, dynamical eqs. (1–7) cannot be mapped on these systems. Dynamics in our case is essentially two-component. The interconversion can be well described within the family [8], where $p = p(t)$, $B = B(t)$ are some real and complex functions of time, respectively. Substituting [8] into eqs. (1.3) and performing the variational analysis based on the adiabatic approximation with respect to the mass flow along
the waveguides, we find for $\nu \rightarrow \nu_c$
\begin{align}
\dot{a} - \frac{16}{9} \dot{b} - \frac{1}{\sqrt{3}}(a^2 + b^2)b - \frac{1}{\bar{\rho}} \ddot{b} &= 0,
\dot{b} + 3(\nu - \nu_c)a + \frac{1}{\sqrt{3}}(a^2 + b^2)a &= 0,
\end{align}
where $B = a + ib$, and $a, b$ are real, and $\bar{\rho} = 3\bar{\sigma}/(32(1 + \nu))$. It is quite obvious that the interconversion of the DS ($a = b = 0$) and the JV ($b = 0, a = \pm 3^{1/4} \sqrt{\nu_c - \nu}$) proceeds on typical relaxation time $\tau$ of the BJJ [8]. Very close to the instability ($\nu \rightarrow \nu_c$) the "critical" slowing down $\sim \bar{\tau}/|\nu - \nu_c|$ takes place. Thus, the DS may vanish in accordance with the mechanism of ref.[3] before it decays into the JV. It is important to realize that the above result is independent of a particular mechanism of dissipation.

We have also performed direct numerical simulations of the full GP equations [6,7] with the initial conditions taken as either DS or JV (located at $x = 0$) for periodic boundary conditions, with the space period being about 10 times larger than soliton size. To accommodate the phases variation by $\pi$, two-soliton solutions were considered. On Fig 2 the results of slow evolution of the coupling $\nu$ from below critical $\nu = 1/7$ (where JV is stable) toward above critical $\nu = 2/5$ (where the DS is stable) as well as its reverse is presented for the dissipation $\bar{\sigma} = 0.5$. As perturbation, small uniform imbalance of the waveguides population has been imposed on the initial conditions. Starting from JV and increasing $\nu$, the sign of the JV vorticity is determined by sign of the initial density imbalance. We also note that the variational ansatz of [10,11] has been verified to reproduce the full numerical solution with good accuracy for the initial imbalances $\leq 10\%$.

Vortex dynamics. The moving DS solution is known analytically (as gray soliton ) [1]. Unfortunately, finding analytical solution of the moving JV seems impossible. However, in the limit $\nu \rightarrow 0$, the JV can be well approximated by the SG equation. Indeed, in this case, the variation of the total density can be ignored. Thus, the representation $\psi_{1,2} = \sqrt{1 \pm \rho}/2\nu^{1/2}$ can be employed, with $|\rho| << 1$. Substituting this into the (dimensionless) eqs. [14,15], and ignoring the gradient $\nabla \rho$, we obtain after the variation $\dot{\varphi} - \rho - \bar{\sigma}^{-1} \dot{\rho} = 0$ and $-\dot{\rho} + \nabla_2^2 \varphi - 4\nu \sin \varphi = 0$. Without dissipation ($\bar{\sigma} \rightarrow \infty$), we find $\rho = \varphi$, and, then, the SG equation $-\dot{\varphi} + \nabla^2_2 \varphi - 4\nu \sin \varphi = 0$ (see in [3]). In the case $\bar{\sigma} \rightarrow 0$, one finds $\dot{\rho} = \bar{\sigma} \dot{\varphi}$, and, then, the overdamped SG equation $-\bar{\sigma} \dot{\varphi} + \nabla^2_2 \varphi - 4\nu \sin \varphi = 0$. We note that, as $\nu \rightarrow 0$, the JV solution of eq.[3] satisfies the static SG equation.

It is important to note that, in the SG approximation, the JV is always stable. There is, simply, no room for the DS due to the imposed constraints. Similar situation has been discussed in ref.[28] for two-component BEC.

Quasi-1D Fluctuations. At finite temperatures $T$, phase-slip effects can destroy supercurrents. The corresponding life-time, however, can be very long [29]. In our case, stability of the JV is determined by the finite energy barrier $\Delta E = E_{DS} - E_{JV}$ with respect to thermal (quantum) jumps between the opposite orientations of the current circulation. The probability of the thermal jump is $P \sim \exp(-\Delta E/T)$. If $\nu \rightarrow \nu_c$, we find
\begin{equation}
P \sim \exp(-4.5\sqrt{3}(\nu_c - \nu)^2 E_*/T^4).
\end{equation}
Thus, to have long lived JV, temperature must satisfy the condition
\begin{equation}
T \ll 4.5\sqrt{3}(\nu_c - \nu)^2 E_*, \quad \nu \rightarrow \nu_c.
\end{equation}
Here $E_* = \mu S_0 = \sqrt{\mu T_c}$, with $T_c = \hbar^2 n_0^2/m$ being the temperature of the quasi-BEC [30] formation. For $\nu \rightarrow 0$, vortex-antivortex pairs can be created thermally (quantum mechanically). To suppress the thermal effect, $T$ must be less than the energy $2E_{JV}$ of the pair. Thus,
\begin{equation}
T \ll 16\sqrt{7} E_*, \quad \nu \rightarrow 0.
\end{equation}
It is important to note that, if the conditions [30,14] are met, the JV energy $L_{JV} \approx \hbar/\sqrt{D}$ is smaller than a typical phase coherence size $L_\phi = \hbar^2 n_0/(mT) = \ell L_{E_JV}/T$ of the quasi-BEC [31].

At low temperatures, quantum tunneling between two circulations of the JV will restore the symmetry for
\( \nu < \nu_c \) due to the quantum phase-slip. This results in a soliton, which, on one hand, is an essentially quantum object with respect to its internal structure — a superposition of opposite vorticities, and, on the other hand, can propagate like a heavy classical particle (\( S_0 \gg 1 \)) along the BJJ-waveguides. It can be viewed as a mobile qubit.

**Vortex pump.** The vortex can be used to transfer a portion of the BEC atoms between BEC reservoirs (see Fig.1). The rate of the atom deposition/depletion in the first reservoir is \( \dot{N}_1 = J_1 \), where \( J_1 = \rho_1 \nabla x \varphi_1 \) is the current along the first waveguide taken at the point where the coupling is zero (the right end of the waveguides). It is given by the density \( \rho_1 \) and the phase \( \varphi_1 \) at the boundary. Thus, \( \Delta N = \int \rho_1 \nabla x \varphi_1 dx/V \). For small \( V \), we substitute the static JV solution. Thus, the explicit integration gives (in the physical units) \( \Delta N = \frac{x_0}{V} \sqrt{(1 + \nu)(1 - 3\nu)} \), where \( V_0 = h/(\omega a) \) is the speed of sound. The GP regime \( S_0 \gg 1 \) implies \( \Delta N \gg 1 \).

Creation and detection of the JV. Observation of the described interconversion can be based on, first, creating the DS simultaneously in both waveguides. Then, moving them slowly apart (to have \( \nu < \nu_c \)) will result in vanishing of the DS into the JV, so that the zero in the densities will heal (see Fig.2). Bringing the waveguides back together (to have \( \nu > \nu_c \)) will cause reappearing of the DS (see Fig.2). The very fact of this reversible transformation, besides being potentially utilized for creating the JV, can serve as an unambiguous evidence of the static Josephson currents. It is important that this effect, in contrast to the suggestion of ref.1, can be observed in the overdamped regime. Direct imaging of the JV currents could be done by the Bragg spectroscopy technique 31. Analysis of the absorption imaging of the JV upon expansion (for 3D vortices, see [22]) will be presented elsewhere.

Applications. The utilisations of the JV for coherent BEC transfer and as a mobile qubit creates quite intriguing perspectives for quantum computations and coherent BEC manipulations in microtraps.

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