CONNECTED SUMS OF CLOSED RIEMANNIAN MANIFOLDS
AND FOURTH ORDER CONFORMAL INVARIANTS

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Abstract. In this note we take some initial steps in the investigation of a fourth order analogue of the Yamabe problem in conformal geometry. The Paneitz constants and the Paneitz invariants considered are believed to be very helpful to understand the topology of the underlined manifolds. We calculate how those quantities change, analogous to how the Yamabe constants and the Yamabe invariants do, under the connected sum operations.

1. Introduction

Let \((M, g)\) be a connected compact Riemannian manifold without boundary of dimension \(n \geq 5\). Let

\[
Q[g] = -\frac{n - 4}{4(n - 1)} \Delta R + \frac{(n - 4)(n^3 - 4n^2 + 16n - 16)}{16(n - 1)^2(n - 2)^2} R^2 - \frac{2(n - 4)}{(n - 2)^2} |Ric|^2
\]

be the so-called \(Q\)-curvature, where \(R\) is the scalar curvature, \(Ric\) is the Ricci curvature. And let

\[
P[g] = (-\Delta)^2 - \text{div}_g((\frac{(n - 2)^2 + 4}{2(n - 1)(n - 2)} Rg - \frac{4}{n - 2} Ric) d) + Q[g]
\]

be the so-called the Paneitz-Branson operator. It is known that

\[
P[g]u = Q[g_u]u^{\frac{n+4}{n-4}}
\]

which is called the Paneitz-Branson equation, where \(g_u = u^{\frac{4}{n-4}} g\) (cf. [P] [Br] [XY] [DHL] [DMA]). We consider the equation (1.3) as a fourth order analogue of the well-known scalar curvature equation

\[
L[g]v = R[g_v]v^{\frac{n+2}{n-2}},
\]
where
\begin{equation}
L[g] = -\frac{4(n-1)}{n-2} \Delta + R
\end{equation}
is the so-called conformal Laplacian and $g_v = v^{\frac{4}{n-2}} g$. The well-known Yamabe problem in conformal geometry is to find a metric, in a given class of conformal metrics, which is of constant scalar curvature, i.e. to solve
\[ L[g] v = Y v^{\frac{n+2}{n-2}} \]
on a given manifold $(M, g)$ for some positive function $v$ and a constant $Y$. The affirmative resolution to the Yamabe problem was given in [Sc] after other notable works [Ya] [Tr] [Au]. In fact, it was proven that there exists a so-called Yamabe metric $g_v$ in the class $[g]$ which is a minimizer for the so-called Yamabe functional
\[ Y(v) = \frac{\int_M (vL[g] v) dv_g}{(\int_M v^{\frac{2n}{n-2}} dv_g)^{\frac{n-2}{n}}} \]
In chapter one we investigate a fourth order analogue of the Yamabe problem. Let $C_+^{\infty}(M)$ be the space of smooth non-negative functions on $M$. Similar to the Yamabe problem, we define the Paneitz functional
\begin{equation}
\phi_g(u) = \frac{\int_M (uP[g] u) dv_g}{(\int_M u^{\frac{2n}{n-4}} dv_g)^{\frac{n-4}{n}}}
\end{equation}
for $u \in C_+^{\infty}(M)$ and the Paneitz constant associated with $(M, [g])$
\begin{equation}
\lambda(M, [g]) = \inf_{u \in C_+^{\infty}(M)} \phi(u).
\end{equation}
It is clear that $\lambda(M, [g])$ is a conformal invariant of the conformal class $[g]$ because of the conformally covariant property of the Paneitz-Brans operator:
\begin{equation}
P[g_w] u = w^{-\frac{4}{n-4}} P[g] (w \cdot u)
\end{equation}
where $g_w = w^{\frac{4}{n-4}} g \in [g]$. To describe the differential structure of $M$, we define
\begin{equation}
\lambda(M) = \sup_{[g]} \lambda(M, [g]).
\end{equation}
We will refer to $\lambda(M)$ as the Paneitz Invariant of the manifold $M$ as the counterpart of Yamabe invariant. In [Gi], Gil-Medrano studied the Yamabe constant for a connected sum of two closed manifolds. One interesting consequence of connected sum results in [Gi] is that every compact manifold without boundary admits a conformal class of metrics whose Yamabe constant is very negative. In Section 2 of Chapter One we calculate as Gil-Medrano did in [Gi] to verify that.
Theorem 1.1. Let \((M_1, g_1)\) and \((M_2, g_2)\) be two compact Riemannian manifolds of dimension \(n \geq 5\). Then, for each \(\epsilon > 0\), there is a conformal class \([g]\) of metrics on \(M_1 \# M_2\) such that

\[
\lambda(M_1 \# M_2, [g]) < \min\{\lambda(M_1, [g_1]), \lambda(M_2, [g_2])\} + \epsilon
\]

and there exists a conformal class \([h]\) of metrics on \(M_1 \# M_2\) such that

\[
\lambda(M_1 \# M_2, [h]) < 2^{-\frac{n-4}{n}} (\lambda(M_1, [g_1]) + \lambda(M_2, [g_2])) + \epsilon.
\]

Due to the works of Schoen and Yau [SY] (see also [GL]), one knows that there is some topological constraint for a manifold to possess a metric of positive Yamabe constant. Therefore it is interesting to see how the Yamabe invariant is effected by connected sum. It was proven in [Ko] [SY] [GL] that the Yamabe invariant of connected sum of two manifolds with positive Yamabe invariants is still positive. More precisely, Kobayashi in [Ko] showed that the Yamabe invariant of connected sum of two manifolds is greater than or equal to the smaller of the Yamabe invariants of the two. In Section 3 of Chapter 1 we obtain an analogue for the Paneitz invariant.

Theorem 1.2. If \(M_1\) and \(M_2\) are compact manifolds of dimension \(n \geq 5\), then

\[
\lambda(M_1 \# M_2) \geq \min\{\lambda(M_1), \lambda(M_2)\}.
\]

The positivity of Paneitz invariant in dimension higher than 4 should be a topological constraint, as indicated by successful researches in [CY] (references therein) for fourth order analogue of how Gaussian curvature influences the geometry of surfaces in dimension 4. Another testing ground is to consider closed locally conformally flat manifolds. Then the recent works in [CHY] [G] indicate to us that the positivity of fourth order curvature is indeed very informative about the topology of the underlined manifolds. We would also like to mention the work by Xu and Yang in [XY] where they demonstrated that positivity of the Paneitz-Branson operator is stable under the process of taking connected sums of two closed Riemannian manifolds.

In Section 1 of Chapter 1 we discuss some preliminary facts about the Paneitz functional. In Section 2 we calculate and verify Theorems 1.1. In Section 3 we prove Theorem 1.2.

2. Preliminaries

Recall that the Yamabe constant of any closed manifold of dimension greater than 2 is a finite number and the largest possible Yamabe constant is realized
and only realized by the Yamabe constant of the standard round sphere in each dimension. The difficult part is to show that the round sphere is the only one that has the largest Yamabe constant, which was the last step in the resolution of Yamabe problem solved by Schoen in [Sc] based on a positive mass theorem of Schoen and Yau. We observe that, by (1.3),

\[(2.1) \int_M (uP[g]u) dv_g = \int_M uQ[g_u]u^{\frac{n+4}{n}} dv_g = \int_M Q[g_u]u^{\frac{2n}{n-4}} dv_g = \int_M Q[g_u] dv_{g_u},\]

where \(g_u = u^{\frac{4}{n-4}} g \in [g].\) Hence

\[\int_M (uP[g]u) dv_g = \int_M \left(\frac{(n-4)(n^3-4n^2+16n-16)}{16(n-1)^2(n-2)^2} R^2 - \frac{2(n-4)}{(n-2)^2} |Ric|^2 dv\right)[g_u]
\leq \frac{(n-4)(n^3-4n^2+16n-16)}{16(n-1)^2(n-2)^2} \int_M (R^2) dv[g_u]\]

When we consider a Yamabe metric \(g_u\), i.e.

\[(2.2) \frac{\int_M (R dv)[g_u]}{\text{vol}(M, g_u)^{\frac{n}{n-2}}} = Y \frac{\text{vol}(M, g_u)^{\frac{n}{n-2}}}{n(n-1)} \leq n(n-1) \frac{\text{vol}(S^n, g_0)^{\frac{n}{n}}}{n(n-1)},\]

we have

\[\frac{\int_M (uP[g]u) dv_g}{\text{vol}(M, g_u)^{\frac{n+4}{n}}} \leq \frac{(n-4)(n^3-4n^2+16n-16)}{16(n-1)^2(n-2)^2} Y^2 \text{vol}(M, g_u)^{\frac{4}{n}}\]
\[\leq \frac{(n-4)(n^3-4n^2+16n-16)}{16(n-1)^2(n-2)^2} (n(n-1))^2 \text{vol}(S^n, g_0)^{\frac{4}{n}}\]
\[= \frac{\int_{S^n} (Q dv)[g_0]}{\text{vol}(S^n, g_0)^{\frac{n}{n-4}}} = \lambda(S^n, [g_0]).\]

Consequently we obtain

**Lemma 2.1.** Let \((M^n, g)\) be a closed Riemannian manifold of dimension greater than 4 with nonnegative Yamabe constant. Then

\[(2.4) \lambda(M^n, [g]) \leq \lambda(S^n, [g_0]),\]

and the equality holds if and only if \((M, g)\) is conformally equivalent to the standard round sphere \((S^n, g_0)\).

On the other hand, by some choices of testing functions similar to the ones used to estimate the Yamabe functional, we get
Lemma 2.2. Let \((M^n, g)\) be a closed Riemannian manifold of dimension greater than 4. Then
\[
-\infty < \lambda(M^n, [g]) \leq \lambda(S^n, [g_0]),
\]
where \(g_0\) is the standard round metric on the sphere \(S^n\).

Proof. The Paneitz constant is easily seen to be bounded from the below. Because, by (1.2),
\[
\int_M (uP[g]u)dv = \int_M |\Delta u|^2 dv + a_n \int_M R|\nabla u|^2 dv
- \frac{4}{n-4} \int_M \operatorname{Ric}(\nabla u, \nabla u) dv + \int_M Qu^2 dv,
\]
where
\[
a_n = \frac{(n-2)^2 + 4}{2(n-1)(n-2)}.
\]

It suffices to estimate (2.3) for nonnegative functions such that
\[
\int_M u^{\frac{2n}{n-4}} dv = 1.
\]
Hence, By Holder inequality,
\[
\int_M (uP[g]u)dv \geq \int_M |\Delta u|^2 dv - C_1 \int_M |\nabla u|^2 dv - C_2 \int_M u^2 dv
\geq \int_M |\Delta u|^2 dv - C_1 \int_M (-\Delta u)udv - C_2 \int_M u^2 dv
\geq \frac{1}{2} \int_M |\Delta u|^2 dv - \frac{1}{2} C_1^2 \int_M u^2 dv - C_2 \int_M u^2 dv
\geq -\left(\frac{1}{2} C_1^2 + C_2 \right) \left( \int_M u^{\frac{2n}{n-4}} dv \right)^{\frac{n-4}{n}} \operatorname{vol}(M, g)^\frac{4}{n}
\geq -\left(\frac{1}{2} C_1^2 + C_2 \right) \operatorname{vol}(M, g)^\frac{4}{n}.
\]
for some constants \(C_1, C_2 > 0\) depending on \((M^n, g)\).

To estimate the upper bound we choose to works in a geodesic normal coordinate in very small geodesic ball \(B_{2\epsilon} \subset M\) and transplant the rescaled round sphere metric. Let \(B_{2\epsilon}(0) \subset R^n\) and
\[
g_{ij}(x) = \delta_{ij} + O(|x|^2), \forall x \in B_{2\epsilon}(0).
\]
Let
\begin{equation}
(2.9) \quad u_\epsilon(x) = \begin{cases} 
\left( \frac{2\epsilon^3}{\epsilon^6 + |x|^2} \right)^{\frac{n-4}{2}} & \forall x \in B_\epsilon(0) \\
0 & \forall x \notin B_2(0) 
\end{cases}
\end{equation}
be a smooth nonnegative function on $M$. Then it is easily calculated that
\begin{equation}
(2.10) \quad \int_M (u_\epsilon P[g]u_\epsilon) dv = \int_{B_\epsilon(0)} |\Delta u_\epsilon|^2 dx + o(1)
\end{equation}
and
\begin{equation}
(2.11) \quad \int_M u_\epsilon^{\frac{2n}{n-4}} dv = \int_{B_\epsilon(0)} u_\epsilon^{\frac{2n}{n-4}} dx + o(1)
= \int_{R^n} (\frac{2\epsilon^3}{\epsilon^6 + |x|^2})^{\frac{n-4}{2}} dx + o(1)
\end{equation}
Therefore
\begin{equation}
(2.12) \quad \phi(u_\epsilon) = \frac{\int_M (u_\epsilon P[g]u_\epsilon) dv}{(\int_M u_\epsilon^{\frac{2n}{n-4}} dv)^{\frac{n-4}{n}}} = \frac{\int_{R^n} |\Delta s|^2 dx}{(\int_{R^n} s^{\frac{2n}{n-4}} dx)^{\frac{n-4}{n}}} + o(1),
\end{equation}
where $s = (\frac{2}{1+|x|^2})^{\frac{n-4}{2}}$. Thus, take $\epsilon \to 0$, we arrive at
\begin{equation}
(2.13) \quad \lambda(M, [g]) \leq \lambda(S^n, [g_0]).
\end{equation}

One interesting question would be whether $(M, g)$ is conformally equivalent to $(S^n, g_0)$ when $\lambda(M, [g]) = \lambda(S^n, [g_0])$ without assuming the Yamabe constant of $(M, g)$ is nonnegative. In other words one would be interested in searching for some analogue of a positive mass theorem of Schoen and Yau here if it make any sense.
3. Connected Sums and the Paneitz Constant

In this section we will calculate the Paneitz functional on a connected sum of two closed manifolds and verify Theorem 1.1. Let \((M, g)\) be a closed manifold of dimension higher than 4. Fix a point \(p \in M\) and let

\[
f_\delta = \begin{cases} 
0 & \forall x \in B_\delta(p) \\
1 & \forall x \in M \setminus B_{2\delta}(p)
\end{cases}
\]

be a family of smooth functions. We may ask

\[
|\nabla f_\delta| < \frac{C_0}{\delta} \quad \text{for some number } C_0 > 0. 
\]

First we calculate

**Lemma 3.1.** Let \((M, g)\) be a closed manifold of dimension greater than 4. Let \(u \in C_+^{\infty}(M)\) be given. Then \(u_\delta = f_\delta u \in C_+^{\infty}(M)\) and

\[
\varphi_g(u_\delta) = \varphi_g(u) + o(1)
\]

as \(\delta \to 0\).

**Proof.** We simply calculate, for a fixed \(\delta > 0\), by (2.6) and (3.2),

\[
\int_M (u_\delta P[g]u_\delta) dv = \int_M |\Delta u_\delta|^2 dv + a_n \int_M R|\nabla u_\delta|^2 dv \\
- \frac{4}{n-4} \int_M \operatorname{Ric}(\nabla u_\delta, \nabla u_\delta) dv + \int_M Q u_\delta^2 dv \\
= \int_M (uP[g]u) dv + o(1)
\]

and

\[
\int_M u_\delta^{\frac{2n}{n-4}} dv = \int_M u^{\frac{2n}{n-4}} dv + o(1),
\]

as \(\delta \to 0\).
Now let us consider the connected sum of two closed Riemannian manifolds. Let \((M_1, g_1)\) and \((M_2, g_2)\) be two compact Riemannian manifolds without boundary of dimension \(n \geq 5\). For \(x_1 \in M_1\) and \(x_2 \in M_2\), let \(B_{\delta_1}(x_1) \subset M_1\) and \(B_{\delta_2}(x_2) \subset M_2\) be geodesic balls respectively. To make the connected sum one simply to take off the open balls \(B_{\frac{1}{2}\delta_1}(x_1)\) and \(B_{\frac{1}{2}\delta_2}(x_2)\) from \(M_1\) and \(M_2\), identify \(\partial B_{\frac{1}{2}\delta_1}(x_1)\) with \(\partial B_{\frac{1}{2}\delta_2}(x_2)\). Hence

\[
M_1 \# M_2 = \frac{\left( (M_1 \setminus B_{\frac{1}{2}\delta_1}(x_1)) \cup (M_2 \setminus B_{\frac{1}{2}\delta_2}(x_2)) \right) / \left\{ \partial B_{\frac{1}{2}\delta_1}(x_1) \sim \partial B_{\frac{1}{2}\delta_2}(x_2) \right\}}.
\]

We may construct a metric \(g\) on the connected sum \(M_1 \# M_2\) such that \(g\) agrees with \(g_1\) on \(M_1 \setminus B_{\delta_1}(x_1)\) and \(g_2\) on \(M_2 \setminus B_{\delta_2}(x_2)\). Notice that topologically \(M_1 \# M_2\) does not depend on the value of \(\delta_i\) when they are sufficiently small. Now let us calculate and estimate the Paneitz functional on the connected sum.

**Theorem 3.2.** Let \((M_1, g_1)\) and \((M_2, g_2)\) be two closed Riemannian manifolds of dimension \(n \geq 5\). Then for each \(\varepsilon > 0\), there is a conformal structure \([g]\) on \(M_1 \# M_2\) such that

\[
\lambda(M_1 \# M_2, [g]) < \min\{\lambda(M_1, [g_1]), \lambda(M_2, [g_2])\} + \varepsilon.
\]

Alternatively, we may find a conformal structure \([g]\) on \(M_1 \# M_2\) such that

\[
\lambda(M, [g]) < \lambda(M_1, [g_1]) + \lambda(M_2, [g_2])2^{-\frac{n-4}{2}} + \varepsilon.
\]

**Proof.** Let us assume that \(\lambda(M_1, [g_1]) \leq \lambda(M_2, [g_2])\) and \(\varepsilon > 0\) fixed. By the definition of the Paneitz constant, we know that there is a real number \(\delta > 0\) and a smooth function \(u_\delta \in C_+^\infty(M)\) such that \(u_\delta\) vanishes on a geodesic ball \(B_\delta(x_1)\) of radius \(\delta\) and centered at \(x_1 \in M_1\) and such that

\[
\varphi_g(u_\delta) < \lambda(M_1, [g_1]) + \varepsilon.
\]

Let \(g\) be a metric on \(M = M_1 \# M_2\) which agrees with \(g_1\), when restricted to \(M_1 \setminus B_\delta(x_1)\). And define the function \(\tilde{u}_\delta\) on \(M_1 \# M_2\) as follows:

\[
\begin{aligned}
\tilde{u}_\delta &= u_\delta \quad \text{on} \quad M_1 \setminus B_\delta(x_1) \\
\tilde{u}_\delta &= 0 \quad \text{elsewhere}.
\end{aligned}
\]

We then have it that

\[
\varphi_g(\tilde{u}_\delta) = \frac{\int_M (\Delta \tilde{u}_\delta^2 + a_n R|\nabla \tilde{u}_\delta|^2 - \frac{4}{n-2}Ric(\nabla \tilde{u}_\delta, \nabla \tilde{u}_\delta) + Q\tilde{u}_\delta^2)dv}{(\int_M \tilde{u}_\delta^{\frac{2n}{n-2}}dv)^{\frac{n-2}{n}}}.
\]
Recalling that \( u_\delta \) vanishes on \( B_\delta(x_1) \) we see that
\[
\varphi_g(\tilde{u}_\delta) = \varphi_{g_1}(u_\delta) < \lambda(M_1, [g_1]) + \epsilon.
\]
Consequently,
\[
\lambda(M, [g]) < \lambda(M_1, [g_1]) + \epsilon = \min(\lambda(M_1, [g_1]), \lambda(M_2, [g_2])) + \epsilon.
\]

We will now proceed to prove (3.8). First notice that Lemma 3.1 can be use to say that for any fixed \( \epsilon > 0 \), \( x_1 \in M_1 \), \( x_2 \in M_2 \), we can find two positive reals \( \delta_1, \delta_2 \) and smooth functions \( u_{\delta_1}, u_{\delta_2} \), where \( u_{\delta_i} \in C^\infty(M_i) \), with the following properties:
\[
\begin{cases}
  u_{\delta_1} = 0 & \text{on } B_{\delta_1}(x_1) \\
  \varphi_{g_1}(u_{\delta_1}) < \lambda(M_1, [g_1]) + \epsilon_1
\end{cases}
\]
and
\[
\begin{cases}
  u_{\delta_2} = 0 & \text{on } B_{\delta_2}(x_2) \\
  \varphi_{g_2}(u_{\delta_2}) < \lambda(M_2, [g_2]) + \epsilon_1
\end{cases}
\]
where \( \epsilon_1 = 2^{-n+4/n}\epsilon \). Also, notice that we can assume without loss of generality that the \( L^{\frac{2n}{n-4}}(M) \) norms of \( u_{\delta_1} \) and \( u_{\delta_2} \) are normalized. Using the same reasoning as in the proof of (3.7), a metric \( \tilde{g} \) on \( M_1 \# M_2 \) can be constructed such that \( \tilde{g} \) agrees with \( g_i \) when restricted to \( M_i \setminus B_{\delta_i}(x_i) \). Let us consider now the function \( \tilde{u} \) on \( M = M_1 \# M_2 \) given by
\[
(3.9) \quad \tilde{u} = \begin{cases}
  u_{\delta_1} & \text{on } M_1 \setminus B_{\delta_1}(x_1) \\
  u_{\delta_2} & \text{on } M_2 \setminus B_{\delta_2}(x_1) \\
  0 & \text{elsewhere}
\end{cases}
\]
then
\[
\varphi_g(\tilde{u}) = \int_{M_1 \setminus B_{\delta_1}(x_1)} ((\Delta \tilde{u})^2 + a_n R |\nabla \tilde{u}|^2 - \frac{4}{n-4} \text{Ric}(\nabla \tilde{u}, \nabla \tilde{u}) + Q \tilde{u}^2) \, dv + \int_{M_2 \setminus B_{\delta_2}(x_2)} (\Delta \tilde{u}^2 + a_n R |\nabla \tilde{u}|^2 - \frac{4}{n-2} \text{Ric}(\nabla \tilde{u}, \nabla \tilde{u}) + Q \tilde{u}^2) \, dv
\]
Using (3.9) we then obtain
\[
\varphi_g(\tilde{u}) = \frac{\int_{M_1 \setminus B_{\delta_1}(x_1)} ((\Delta \tilde{u}_{\delta_1})^2 + a_n R |\nabla \tilde{u}_{\delta_1}|^2 - \frac{4}{n-2} \text{Ric}(\nabla \tilde{u}_{\delta_1}, \nabla \tilde{u}_{\delta_1}) + Q \tilde{u}_{\delta_1}^2) \, dv}{(\int_{M_1 \setminus B_{\delta_1}(x_1)} \tilde{u}_{\delta_1}^{\frac{2n}{n-4}} \, dv + \int_{M_2 \setminus B_{\delta_2}(x_2)} \tilde{u}_{\delta_2}^{\frac{2n}{n-4}} \, dv)^{\frac{n}{n-4}}}
\]
and
\[
\varphi_g(\tilde{u}) = \frac{\int_{M_2 \setminus B_{\delta_2}(x_2)} ((\Delta \tilde{u}_{\delta_2})^2 + a_n R |\nabla \tilde{u}_{\delta_2}|^2 - \frac{4}{n-2} \text{Ric}(\nabla \tilde{u}_{\delta_2}, \nabla \tilde{u}_{\delta_2}) + Q \tilde{u}_{\delta_2}^2) \, dv}{(\int_{M_1 \setminus B_{\delta_1}(x_1)} \tilde{u}_{\delta_1}^{\frac{2n}{n-4}} \, dv + \int_{M_2 \setminus B_{\delta_2}(x_2)} \tilde{u}_{\delta_2}^{\frac{2n}{n-4}} \, dv)^{\frac{n}{n-4}}}
\]
Now, recalling the above stated properties of $u_{\delta_1}$ and $u_{\delta_2}$, we may also assume
\[ \int_{M_i \setminus B_{\delta_i}(x_i)} u_{\delta_i} \frac{2n}{n-4} dv = 1, \]
and
\[ \varphi_{g_i}(u_{\delta_i}) = \int_{M_i \setminus B_{\delta_i}(x_i)} (\Delta \tilde{u}_{\delta_i})^2 + a_n R |\nabla \tilde{u}_{\delta_i}|^2 - \frac{4}{n-2} \text{Ric}(\nabla \tilde{u}_{\delta_i}, \nabla \tilde{u}_{\delta_i}) + Q \tilde{u}_{\delta_i}^2) dv < \lambda(M_i, [g_i]) + \epsilon_1. \]

Thus
\[ \lambda(M, [g]) \leq \varphi_{\bar{u}}(\bar{u}) < (\lambda(M_1, [g_1]) + \lambda(M_2, [g_2]) + 2\epsilon_1)2^{-\frac{n-4}{n}} \]
\[ = (\lambda(M_1, [g_1]) + \lambda(M_2, [g_2]))2^{-\frac{n-4}{n}} + \epsilon. \]

4. Connected Sums and the Paneitz Invariants

Kobayashi in [Ko] showed that the Yamabe invariant of connected sum of two manifolds is greater than or equal to the smaller of the Yamabe invariants of the two. The aim of this section is to generalize this result of Kobayashi to the case of compact manifolds of dimension $n \geq 5$, and with the Yamabe invariant $Y(M)$ replaced by its fourth order analogue the Paneitz invariant $\lambda(M)$. Namely, we have

**Theorem 4.1.** If $M_1$ and $M_2$ are closed manifolds of dimension $n \geq 5$. If $\lambda(M_1) > 0$ and $\lambda(M_2) > 0$ then

(4.1) \[ \lambda(M_1 \# M_2) \geq \min\{\lambda(M_1), \lambda(M_2)\}. \]

We will basically follow the approach taken by Kobayashi in [Ko]. First we consider the Paneitz invariant on the disjoint union of compact manifolds. Take two $n$-manifolds with conformal structures, say $(M_1, [g_1])$ and $(M_2, [g_2])$. We write $(M, [g]) = (M_1, [g_1]) \sqcup (M_2, [g_2])$ if $M$ is the disjoint union of $M_1$ and $M_2$, and $g_i = \{g|_{M_i}; g \in [g]\}$ for $i = 1, 2$. Let $u$ be a smooth non-negative function on $M$. Since $M$ is the disjoint union of $M_1$ and $M_2$ it follows that we can write $u = u_1 + u_2$, where $u_i = 0$ on $M_j$, where $i \neq j$ and where $u_i$ is a non-negative smooth function on $M_i$. If we assume that $\lambda(M_i, [g_i]) \geq 0$ for $i = 1, 2$, then it can easily be seen that

\[ \lambda(M, [g]) = \min\{\lambda(M_1, [g_1]), \lambda(M_2, [g_2])\}. \]
Due to Lemma 2.2, we can assume that $\lambda(M_1)$ and $\lambda(M_2)$ are finite; and we can use the above equation to conclude that

$$\lambda(M) = \min\{\lambda(M_1), \lambda(M_2)\}.$$  

Let $M$ be a compact manifold of dimension $n \geq 5$, and $p_1$ and $p_2$ two points of $M$. We take off two small balls around $p_1$ and $p_2$, and then attach a handle instead, the handle being topologically the product of a line segment and $S^{n-1}$. The new manifold obtained in this way will be denoted by $\overline{M}$. Let $M_1$ and $M_2$ be Riemannian manifolds and let $M_1 \cup M_2$ denote the disjoint union of $M_1$ and $M_2$. If $M = M_1 \cup M_2$ and $p_1$ and $p_2$ are taken from $M_1$ and $M_2$ respectively, then $\overline{M} = M_1 \# M_2$. Therefore we see that in order to prove Theorem 4.1 it suffices to show

$$\lambda(\overline{M}) \geq \lambda(M).$$

**Proof of Theorem 4.1.** Let $\epsilon$ be an arbitrary positive number, which will be fixed throughout. First, we take a metric $g$ on $M$ such that

$$(4.2) \quad \lambda(M, [g]) > \lambda(M) - \epsilon.$$  

Due to continuity considerations we may assume that $[g]$ is conformally flat around the points $p_1$ and $p_2$. Then there is a function $\gamma \in C^\infty(M \setminus \{p_1, p_2\})$ and $g \in [g]$ such that $\tilde{g} = e^\gamma g$ is a complete metric of $M \setminus \{p_1, p_2\}$ and that each of the two ends is isometric to the half infinite cylinder $[0, \infty) \times S^{n-1}(1)$. For convenience, we write

$$(M \setminus \{p_1, p_2\}, \tilde{g}) = [0, \infty) \times S^{n-1}(1) \cup (\tilde{M}, \tilde{g}) \cup [0, \infty) \times S^{n-1}(1),$$

where $\tilde{M}$ is the complement of the two cylinders. We can glue $(\tilde{M}, \tilde{g})$ and $[0, l] \times S^{n-1}(1)$, along their boundaries to get a smooth Riemannian manifold $(\overline{M}, g_l)$, where $\overline{M}$ as mentioned in the beginning of the section:

$$(\overline{M}, g_l) = (\tilde{M}, \tilde{g}) \cup [0, l] \times S^{n-1}(1).$$

We then have

$$\lambda(\overline{M}, [g_l]) = \inf_{f > 0} \frac{\int_{\overline{M}} ((\Delta f)^2 + a_n R |\nabla f|^2 - \frac{4}{n-2} \text{Ric}(\nabla f, \nabla f) + Q f^2) dv}{(\int_{\overline{M}} f^{2n \over{n-4}} dv)^{n \over{n-4}}}.$$  

So, take a positive function $f_l \in C^\infty(\overline{M})$ such that

$$(4.4) \quad \int_{\overline{M}} ((\Delta f_l)^2 + a_n R |\nabla f_l|^2 - \frac{4}{n-2} \text{Ric}(\nabla f_l, \nabla f_l) + Q f_l^2) dv < \lambda(\overline{M}, [g_l]) + \frac{1}{l + 1}$$

and

$$(4.5) \quad \int_{\overline{M}} f_l^{2n \over{n-4}} dv = 1.$$
Lemma 4.2. There is a section, say \( \{t_i\} \times S^{n-1} \), in the cylindrical part of \( \overline{M} \) such that

\[
\int_{\{t_i\} \times S^{n-1}} \left( (\Delta f_i)^2 + a_n R |\nabla f_i|^2 - \frac{4}{n-2} \text{Ric}(\nabla f_i, \nabla f_i) + Qf^2 \right) dv < \frac{B}{l},
\]

where \( B \) is a constant independent of \( l \).

Proof. Using (4.4) we have it that

\[
\int_{S^{n-1} \times [0,l]} \left( (\Delta f)^2 + a_n R |\nabla f|^2 - \frac{4}{n-2} \text{Ric}(\nabla f, \nabla f) + Qf^2 \right) dv < \lambda(M, [g_i]) + \frac{1}{1 + l} - \int_{\overline{M}} \left( (\Delta f_i)^2 + a_n R |\nabla f_i|^2 - \frac{4}{n-2} \text{Ric}(\nabla f_i, \nabla f_i) + Qf_i^2 \right) dv.
\]

It follows then that it suffices to demonstrate that there exists a constant \( D \), independent of \( l \), such that

\[
\int_{\overline{M}} \left( (\Delta f_i)^2 + a_n R |\nabla f_i|^2 - \frac{4}{n-2} \text{Ric}(\nabla f_i, \nabla f_i) + Qf_i^2 \right) dv > D.
\]

Towards this end, we first notice that we can rewrite (4.3) as follows:

\[
(\overline{M}, \overline{g}_i) = (\widetilde{M}_1, \widetilde{g}_1) \cup [0,l] \times S^{n-1}(1) \cup (\widetilde{M}_2, \widetilde{g}_2),
\]

where \( (\widetilde{M}_i, \widetilde{g}_i), i \in \{1, 2\} \), is conformal to \( (M_i, g_i) \setminus (B_i(p_i), \delta) \), where \( B_i(p_i) \) is a small ball centered at \( p_i \) and \( \delta \) is the Euclidean metric. Now, noting that \( a_n R + \frac{4}{n-4} \text{Ric} \) is a strictly positive operator on the cylindrical component of \( \overline{M} \) and that \( Q \) is a strictly positive function on the cylindrical component, we see that we can write

\[
(\widetilde{M}_i, \widetilde{g}_i) = (N_i, h_i) \cup (N_i', h_i')
\]

where \( (N_1', h_1') \cap ([0,l] \cup S^{n-1}) = S^{n-1} \times \{0\} \); \( (N_2', h_2') \cap ([0,l] \cup S^{n-1}) = S^{n-1} \times \{l\} \); \( h_i' \) is conformally flat; \( a_n R_{h_i'} - \frac{4}{n-2} \text{Ric}_{h_i'} \) is a positive operator pointwise on \( N_i' \); and \( Q_{h_i'} \) is positive on \( N_i' \). In geometric terms we can think of \( (N_i', h_i') \) as a small part of the necks of the connected sum \( \overline{M} \) adjacent to the cylindrical component. We will now use this refined decomposition of \( \overline{M} \) to decompose \( f_i \); that is, we write

\[
f_i = f_{1,i} + f_{c,i} + f_{2,i},
\]

where \( f_{1,i} \) is supported on \( \overline{M}_1; f_{2,i} \) is supported on \( \overline{M}_2 \); and \( f_{c,i} \) is supported on \( N_1' \cup ([0,l] \times S^{n-1}) \cup N_2' \). Furthermore we assume that \( f_{1,i}, f_{2,i}, \) and \( f_{c,i} \) vanish smoothly at some nonzero distance away from the boundaries of their respective supports. We will now see that the energies \( \int_{\overline{M}} f_{1,i} Pf_{1,i} dv \),
Lemma 2.1 then provides us with the existence of negative constants $\lambda$ for smooth, non-negative function $f$ that vanish near the boundaries of their respective supports, we can extend $f$ to a smooth, non-negative function $f'$ on $M_i$, by defining $f'_i$ to be zero on $M_i \setminus \overline{M}_i$. Lemma 2.1 then provides us with the existence of negative constants $D_i$ such that $\int_{M} f_i \, dv \geq D_i \left( \int_{M_i} f_i \, dv \right)^{\frac{n-2}{n-4}} \geq D_i$. Since $D_i$ is determined strictly by the conformal structure of $(M_i, g_i)$, the above bounds are independent of $l$. Putting these three energy estimates together we have it that there exists a constant $D$ such that

$$\int_{M} ((\Delta f_i)^2 + a_n R |\nabla f_i|^2 - \frac{4}{n-2} \text{Ric}(\nabla f_i, \nabla f_i) + Q f_i^2) \, dv > D.$$ 

As a consequence we have it that there is a $t_i \in [0, l]$ such that

$$\int_{\{t_i\} \times S^{n-1}} ((\Delta f_i)^2 + a_n R |\nabla f_i|^2 - \frac{4}{n-2} \text{Ric}(\nabla f_i, \nabla f_i) + Q f_i^2) \, dv < (\lambda(\overline{M}, C_i) + \frac{1}{1+l} + D)/l,$$

which gives us Lemma 4.1 with $B = (\lambda(\overline{M}) + 1 + B_1)$.

Now we cut off $\overline{M}$ on the section $\{t_1 \times S^{n-1}\}$, and attach two half-infinite cylinders to it, so $(M, \{p_1, p_2\}, \overline{g})$ reappears. But this time we describe it as follows:

$$(M, \{p_1, p_2\}, \overline{g}) = [0, \infty) \times S^{n-1}(1) \cup (\overline{M} - \{t_1\} \times S^{n-1}, g_i) \cup [0, \infty) \times S^{n-1}(1).$$

We think of the function $f_i$ as defined on $\overline{M} - \{t_i\} \times S^{n-1}$, and extend it to the whole space $M - \{p_1, p_2\}$ as follows: Let $F_i$ be Lipschitz function of $\overline{M} - \{p_1, p_2\}$ such that

$$F_i = f_i \quad \text{on} \quad \overline{M} - \{t_i\} \times S^{n-1}$$

and

$$F_i(t, x) = \begin{cases} (1-t)\tilde{f}_i(x) & \text{for} \quad (t, x) \in [0, 1] \times S^{n-1}, \\ 0 & \text{for} \quad (t, x) \in [1, \infty) \times S^{n-1}, \end{cases}$$

where $\tilde{f}_i = f_i|_{\{t_i\} \times S^{n-1}} \in C^\infty(S^{n-1})$. Now it easy to see from (4.4) and (4.6) that

$$\int_{M \setminus \{p_1, p_2\}} ((\Delta F_i)^2 + a_n R |\nabla F_i|^2 - \frac{4}{n-2} \text{Ric}(\nabla F_i, \nabla F_i) + Q F_i^2) \, dv < \lambda(\overline{M}, [g_i]) + \frac{B}{l},$$
where \( B \) is a constant independent of \( l \). Obviously from (4.5) we get

\[
\int_{\mathcal{M}\backslash\{p_1, p_2\}} F_l^{\frac{2n}{n-4}} dv > 1.
\]

Therefore, we have

(4.9)

\[
\inf_{\mathcal{M}\backslash\{p_1, p_2\}} \frac{\int_{\mathcal{M}\backslash\{p_1, p_2\}} ((\Delta F)^2 + a_n R|\nabla F|^2 - \frac{4}{n-2} \text{Ric}(\nabla F, \nabla F) + QF^2) \, dv}{(\int_{\mathcal{M}\backslash\{p_1, p_2\}} F^\frac{2n}{n-4} dv)^\frac{n}{n-4}} \leq \lambda(\mathcal{M}),
\]

where the infimum is taken over all nonnegative Lipschitz functions \( F \) with compact support. It follows from the choice of the metric \( \tilde{g} \) that the left side of (4.9) is equal to \( \lambda(M, [g]) \). Since \( \epsilon \) can be chosen arbitrarily in (4.2), we conclude \( \lambda(M) \leq \lambda(\mathcal{M}) \), which completes the proof.

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