Index theory and non-commutative geometry on foliated manifolds

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Abstract. This paper gives a survey of the index theory of tangentially elliptic and transversally elliptic operators on foliated manifolds as well as of related notions and results in non-commutative geometry.

Keywords: non-commutative geometry, manifolds, foliations, transversally elliptic operators, tangentially elliptic operators, index, $K$-theory, operator algebras.

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Introduction

The index theory of elliptic operators is one of the most significant achievements in the mathematics of the twentieth century. It began with a question raised by Gel’fand in 1959. An arbitrary elliptic operator $A$ on a compact manifold without boundary determines a Fredholm problem in Sobolev spaces, and therefore its integer-valued index

$$\text{Ind } A = \dim \ker A - \dim \text{coker } A$$

is defined. Gel’fand observed that this index depends only on topological properties of the operator and raised the question of finding an explicit formula for the index of an elliptic operator in topological terms. An answer to this question was given by Atiyah and Singer in 1963. We refer the reader to the original papers [1]–[4] and the books [5]–[10], and to the references cited therein for the classical Atiyah–Singer theorem and its proofs.

Later on, the development of the index theory went in several directions. The present survey is devoted to two of these directions. One of them originated
in papers of Atiyah [11] and Singer [12] and concerns a class of non-Fredholm operators. This class consists of differential operators on a compact manifold $M$ which are invariant under an action of some compact Lie group $G$ on the manifold and elliptic in directions conormal to the orbits of the action. Such operators are said to be transversally elliptic. The index of a transversally elliptic operator is no longer an integer, but a distribution on the Lie group $G$. Transversally elliptic operators are naturally regarded as an analogue of elliptic operators on the space $M/G$ of orbits of the group action. In the papers [13] and [14] Connes considered transversally elliptic operators on compact foliated manifolds.

Another direction in the index theory comes from the Atiyah–Singer index theorem for families of elliptic operators [4]. The index of a family of elliptic operators parametrized by points of some topological space $X$ is defined as an element of the group $K(X)$ of the topological $K$-theory. A generalization of this theory is the index theory of tangentially elliptic operators on a foliated manifold. A differential operator on a foliated manifold $(M,F)$ is said to be tangentially elliptic if it contains differentiations only along the leaves of the foliation (and therefore can be restricted to any leaf of the foliation), and its restriction to each leaf is an elliptic operator. Any tangentially elliptic operator can be regarded as a family of elliptic operators on leaves of the foliation, parametrized by points of the leaf space $M/F$.

Both transversally elliptic and tangentially elliptic operators on a foliated manifold have a natural interpretation in terms of its leaf space. The leaf space, generally speaking, is a very singular object, and it is poorly described by means of classical tools of geometry, topology, and analysis. Here we get help from non-commutative geometry, one of the main goals of which is the development of methods for the study of geometry, topology, and analysis on singular spaces such as the leaf space of a foliation.

There are several fundamental ideas which lie at the basis of non-commutative geometry. The first of them is to pass from geometric spaces to algebras of functions on these spaces and to translate basic geometric and analytic notions and constructions into algebraic language. Such a procedure is well known and has been used for a long time, for instance, in algebraic geometry. The next idea is that in many cases, especially when the classical algebra of functions is small or has a bad structure, it is useful to consider some non-commutative algebra as an analogue of it. This necessitates extending the basic geometric and analytic definitions to the case of a general non-commutative algebra. Such a point of view has been well known since the time of the Gel’fand–Naimark theory of commutative $C^*$-algebras. For instance, the theory of $C^*$-algebras is a far-reaching generalization of the theory of topological spaces and is often called non-commutative topology, and the theory of von Neumann algebras is a generalization of the classical theory of measure and integration. These ideas have turned out to be very fruitful in the index theory as well.

In [15] Connes defined the $C^*$-algebra $C^*(M,F)$ of a foliation $(M,F)$, which it is natural to regard as an analogue of the algebra of continuous functions on the leaf space $M/F$. For instance, the index of a tangentially elliptic operator on $(M,F)$ is well defined as an element of the group $K(C^*(M,F))$ of the operator topological $K$-theory.
The main purpose of non-commutative differential geometry, which was initiated by Connes [14] and is actively developing at the present time (cf. the recent surveys [16]–[18] and the books [19]–[25] in regard to various aspects of non-commutative geometry), is to extend the methods described above to analytic objects on geometric spaces and to non-commutative algebras. Here the main attention is focused on the facts that, first, a correct non-commutative generalization applied in the classical setting, that is, to an algebra of functions on a compact manifold, must agree with its classical analogue, and second, it must inherit basic algebraic and analytic properties of its classical analogue. Nevertheless, it should be said that, as a rule, such non-commutative generalizations are quite non-trivial and have a richer structure and essentially new features in comparison with their commutative analogues.

It should be noted that the emergence of non-commutative differential geometry itself in the paper [14] is directly connected with index theory, because the notion of cyclic cohomology introduced there was invented by Connes in attempts to define the index of transversally elliptic operators on foliated manifolds. Namely, the index of a transversally elliptic operator on a foliated manifold \((M, \mathcal{F})\) is defined as a cyclic cohomology class on some subalgebra of the \(C^*\)-algebra \(C^*(M, \mathcal{F})\). On the other hand, cyclic cohomology and the corresponding non-commutative differential calculus play an important role in the index theory of tangentially elliptic operators. First of all, we observe that in the classical case the Chern character \(\text{ch}(\text{Ind} P)\) of the index of a family \(P\) of elliptic operators parameterized by points of a smooth manifold \(B\) can be considered as a de Rham cohomology class of the base \(B\). Moreover, as stated by a local index theorem proved by Bismut [26], for families of Dirac operators it is not only the cohomology class \(\text{ch}(\text{Ind} P) \in H^*(B)\) that has a geometric meaning, but also a certain differential form on \(B\) representing this class. In order to get numerical invariants from \(\text{ch}(\text{Ind} P)\), one can consider its pairings with arbitrary de Rham currents on \(B\). A non-commutative generalization of such a construction is given by the higher index theory for tangentially elliptic operators on foliated manifolds, which studies higher indices of a tangentially elliptic operator defined as pairings of its index with cyclic cohomology classes of some smooth subalgebras of the algebra \(C^*(M, \mathcal{F})\).

This paper is devoted to an exposition of the aspects of index theory and non-commutative geometry mentioned above. We begin in §1 with a survey of necessary notions of the classical index theory of elliptic operators. In §2 we give some information from non-commutative topology — the notion of \(C^*\)-algebra as a non-commutative topological space, the non-commutative analogue of a vector bundle and of a field of Hilbert spaces, the simplest (and the basic) invariant — the \(K\)-theory and the \(K\)-homology. At the end of the section we give some necessary information from the non-commutative theory of measure and integration. §3 is devoted to basic notions of non-commutative differential geometry — cyclic (co)homology and non-commutative differential calculus, spectral triples as an analogue of a non-commutative Riemannian structure and the non-commutative local index theory. In §4 we turn to foliations, starting with a brief summary of necessary information from foliation theory. In §5 we describe the construction of the operator algebras associated with a foliation, which makes use of the notion of holonomy groupoid of a foliation. Then basic objects of non-commutative topology
of foliations are discussed — holonomy equivariant bundles, fields of Hilbert spaces, K-theory and the Baum–Connes conjecture, non-commutative integration theory. §6 is devoted to non-commutative differential calculus on the leaf space of a foliation and to constructions of cyclic cocycles on the operator algebras associated with a foliation. Finally, §§7 and 8 are devoted, respectively, to the index theory of transversally elliptic and tangentially elliptic operators.

This paper has some overlap with the author’s previous survey [18], which is devoted to various aspects of non-commutative geometry of foliations. In the present survey the main emphasis is placed on the index theory of differential operators associated with a foliation, in particular, on applications of methods of non-commutative geometry to index theory.

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1. Survey of the classical index theory

1.1. Elliptic operators and the index. Let $M$ be an $n$-dimensional smooth compact manifold without boundary, and $E$ and $F$ smooth complex vector bundles on $M$ of rank $N_E$ and $N_F$, respectively. (Here and subsequently, ‘smooth’ means of class $C^\infty$. We will always assume that all objects under consideration are of class $C^\infty$.) A linear operator $D: C^\infty(M, E) \to C^\infty(M, F)$ is called a differential operator of order $m$ if in any local chart $\phi: U \subset M \to \mathbb{R}^n$ and any trivializations $E|_U \cong U \times \mathbb{C}^{N_E}$ and $F|_U \cong U \times \mathbb{C}^{N_F}$ of the bundles $E$ and $F$ over it the operator $D$ has the form

$$D = \sum_{|\alpha| \leq m} a_\alpha(x) \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}, \quad x \in \phi(U) \subset \mathbb{R}^n, \quad (1.1)$$

where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$ is a multi-index, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, and the $a_\alpha$ are smooth functions on $\mathbb{R}^n$ with values in the space $\mathcal{L}(\mathbb{C}^{N_E}, \mathbb{C}^{N_F})$ of complex $N_E \times N_F$ matrices.

For a differential operator $D$ given by the formula (1.1) in some local chart $\phi: U \subset M \to \mathbb{R}^n$ and for trivializations of the bundles $E$ and $F$ over it we define its (complete) symbol

$$\sigma(x, \xi) = \sum_{|\alpha| \leq m} a_\alpha(x)(i\xi)^\alpha, \quad x \in \phi(U) \subset \mathbb{R}^n, \quad \xi \in \mathbb{R}^n,$$

and its principal symbol

$$\sigma_m(x, \xi) = \sum_{|\alpha| = m} a_\alpha(x)(i\xi)^\alpha, \quad x \in \phi(U) \subset \mathbb{R}^n, \quad \xi \in \mathbb{R}^n.$$

The principal symbol is invariantly defined as a smooth section of the bundle $\text{Hom}(\pi^*E, \pi^*F)$ on the cotangent bundle $T^*M$, where $\pi: T^*M \to M$ denotes the canonical projection, and $\pi^*E$ and $\pi^*F$ are the pull-backs of the bundles $E$ and $F$, respectively, on $T^*M$ by the map $\pi$. A differential operator $D: C^\infty(M, E) \to C^\infty(M, F)$ is said to be elliptic if $N_E = N_F$ and the principal symbol $\sigma_m(\xi)$ is invertible for $\xi \in T^*M \setminus 0$. 


In the index theory it is necessary to consider a wider class of operators, the class of pseudodifferential operators. We recall that a linear operator $P : C^\infty(M, E) \to \mathcal{D}'(M, F)$ belongs to the class $\Psi^m(M, E, F)$ of pseudodifferential operators of order $m$ if in a coordinate domain $X \subset \mathbb{R}^n$ it can be represented as

$$Pu(x) = \int e^{(x-y)\xi} p(x, \xi)u(y) \, dy \, d\xi, \quad x \in X,$$

where $u \in C^\infty_c(X, \mathbb{C}^{N_E})$, and the function $p$, called the complete symbol of $P$, belongs to the symbol class $S^m(X \times \mathbb{R}^n, \mathcal{L}(\mathbb{C}^{N_E}, \mathbb{C}^{N_F}))$.

The principal symbol $\sigma(P)$ of a $P \in \Psi^m(M, E, F)$ is an element of the symbol space $S^m(T^*M, \text{Hom}(\pi^*E, \pi^*F))$, uniquely determined up to elements of the space $S^{m-1}(T^*M, \text{Hom}(\pi^*E, \pi^*F))$. In local coordinates $\sigma(P)$ is given by the complete symbol $p$ of $P$. We say that $P$ is elliptic if its principal symbol $\sigma(P)$ has a representative $p \in S^m(T^*M, \text{Hom}(\pi^*E, \pi^*F))$ which is pointwise invertible outside some compact subset of $T^*M$ and satisfies the estimate

$$\|p(x, \xi)^{-1}\| \leq C(1 + |\xi|)^{-m}$$

for some constant $C$ and some Riemannian metric on $M$.

For an elliptic operator $P$ there exists an operator $Q \in \Psi^{-m}(M, F, E)$ such that $QP - \text{Id}$ and $PQ - \text{Id}$ are smoothing operators. The operator $Q$ is called a parametrix for $P$. The existence of a parametrix implies that $P$ defines a Fredholm operator acting in Sobolev spaces

$$P_{(s)} : H^{s+m}(M, E) \to H^s(M, F)$$

for any $s \in \mathbb{R}$. The kernel of $P_{(s)}$ is finite-dimensional and lies in $C^\infty(M, E)$, and its image is closed in $H^s(M, F)$ and coincides with the orthogonal complement in $H^s(M, F)$ of the kernel of $P^*$. Therefore, the index

$$\text{Ind } P_{(s)} = \dim \text{Ker } P_{(s)} - \dim \text{Coker } P_{(s)} = \dim \text{Ker } P_{(s)} - \dim \text{Ker } P^*_{(s)}$$

of $P_{(s)}$ is well defined and independent of $s$; it is called the index of $P$.

Examples of elliptic operators are given by Dirac operators. Let us recall their definition (for an exposition of basic facts of spin geometry and the theory of Dirac operators see, for instance, [6], [10], [27]).

Let $M$ be a compact manifold of even dimension $n$, $g_M$ a Riemannian metric on $M$, and $\nabla$ the Levi-Civita connection on $TM$. For any $x \in M$ denote by $\text{Cl}(T_xM)$ the complex Clifford algebra of the Euclidean space $T_xM$. If one chooses an orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ in $T_xM$, then $\text{Cl}(T_xM)$ is defined as an algebra over $\mathbb{C}$ generated by the elements 1 and $e_1, e_2, \ldots, e_n$ satisfying the relations

$$e_\alpha e_\beta + e_\beta e_\alpha = -2\delta_{\alpha\beta}, \quad \alpha, \beta = 1, 2, \ldots, n.$$

The Clifford algebra $\text{Cl}(T_xM)$ has a natural $\mathbb{Z}_2$-grading. Recall that a vector space $V$ is said to be $\mathbb{Z}_2$-graded if it has a decomposition $V = V_0 \oplus V_1$ into a direct sum of subspaces. Equivalently, a $\mathbb{Z}_2$-grading on $V$ is determined by an operator $\gamma \in \mathcal{L}(V)$ such that $\gamma^2 = 1$. With respect to the decomposition $V = V_0 \oplus V_1$ the
operator $\gamma$ has a block form $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. If $V$ has a Euclidean structure, then it is natural to assume that the subspaces $V_0$ and $V_1$ are orthogonal, which is equivalent to the self-adjointness of $\gamma$. For any linear operator $T$ on a $\mathbb{Z}_2$-graded space $V$ its supertrace is defined by

$$\mathrm{Tr}_s(T) = \mathrm{Tr} \gamma T = \mathrm{Tr} T_{11} - \mathrm{Tr} T_{22},$$

where $T$ is written in the block form $T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$ determined by the decomposition $V = V_0 \oplus V_1$.

We consider the vector bundle $\mathrm{Cl}(TM)$ on $M$ whose fibre at $x \in M$ coincides with $\mathrm{Cl}(T_x M)$. This bundle is associated with the principal $SO(n)$-bundle $O(TM)$ of oriented orthonormal frames in $TM$: $\mathrm{Cl}(TM) = O(TM) \times_{O(n)} \mathrm{Cl}(\mathbb{R}^n)$. Therefore, the Levi-Civita connection $\nabla$ induces a natural connection $\nabla^{\mathrm{Cl}(TM)}$ in $\mathrm{Cl}(TM)$ which is compatible with Clifford multiplication and preserves the $\mathbb{Z}_2$-grading on $\mathrm{Cl}(TM)$. If $\{e_1, e_2, \ldots, e_n\}$ is a local orthonormal frame in $TM$ and the $\omega_{\alpha\beta}^\gamma$ are the coefficients of $\nabla$: $\nabla e_\alpha = \sum e_\gamma \omega_{\alpha\beta}^\gamma e_\beta$, then

$$\nabla_{e_\alpha}^{\mathrm{Cl}(TM)} = e_\alpha + \frac{1}{4} \sum_{\gamma=1}^n \omega_{\alpha\beta}^\gamma c(\epsilon_\beta)c(\epsilon_\gamma),$$

where for any $a \in C^\infty(M, \mathrm{Cl}(TM))$, $c(a)$ denotes the operator of pointwise left multiplication by $a$ in $C^\infty(M, \mathrm{Cl}(TM))$.

A complex vector bundle $\mathcal{E}$ on $M$ is called a Clifford module if for any $x \in M$ there is a representation of the algebra $\mathrm{Cl}(T_x M)$ on $\mathcal{E}_x$ depending smoothly on $x$. The action of an $a \in \mathrm{Cl}(T_x M)$ on an $s \in \mathcal{E}_x$ will be denoted by $c(a)s \in \mathcal{E}_x$. A Clifford module $\mathcal{E}$ is said to be self-adjoint if it is endowed with a Hermitian metric such that the operator $c(f): \mathcal{E}_x \to \mathcal{E}_x$ is skew-symmetric for any $x \in M$ and $f \in T_x M$. An arbitrary Clifford module $\mathcal{E}$ has a natural $\mathbb{Z}_2$-grading $\mathcal{E} = \mathcal{E}_+ \oplus \mathcal{E}_-$. A connection $\nabla^\mathcal{E}$ on a Clifford module $\mathcal{E}$ is called a Clifford connection if for any $f \in C^\infty(M, TM)$ and $a \in C^\infty(M, \mathrm{Cl}(TM))$ the following relation holds:

$$[\nabla^\mathcal{E}_f, c(a)] = c(\nabla^{\mathrm{Cl}(TM)}_f a).$$

A self-adjoint Clifford module $\mathcal{E}$ endowed with a Hermitian Clifford connection $\nabla^\mathcal{E}$ is called a Clifford bundle. Let $e_1, \ldots, e_n$ be a local orthonormal basis in $TM$. The Dirac operator $D_\mathcal{E}$ associated with a Clifford bundle $\mathcal{E}$ is defined by

$$D_\mathcal{E} = \sum_{i=1}^n c(e_i)\nabla_{e_i}^\mathcal{E}.$$
Example 1.1. An example of a Clifford bundle is the complexified exterior bundle $\Lambda T^*M \otimes \mathbb{C}$ on a Riemannian manifold $M$. An action of $\text{Cl}(TM)$ on it is given by the formula
\[
c(v) = \varepsilon v^* - iv, \quad v \in T_xM,
\]
where $v^* \in T_x^*M$ is the covector dual to $v$, $\varepsilon v^*$ is the exterior multiplication by $v^*$, and $iv$ is the inner multiplication by $v$. A Clifford connection is the Hermitian connection determined by the Riemannian metric. The corresponding Dirac operator is the de Rham operator acting in $C^\infty(M, \Lambda T^*M^*)$ by the formula
\[
D_{\Lambda T^*M^*} = d + d^*,
\]
where $d$ is the de Rham differential.

Example 1.2. Recall that the group Spin$(n)$ is the non-trivial double covering of the group $SO(n)$. A spin structure on a Riemannian manifold $M$ is defined to be a principal Spin$(n)$-bundle $O'(TM)$ on $M$ which is a double covering of the principal $SO(n)$-bundle $O(TM)$ of oriented orthonormal frames in $TM$ such that the map $O'(TM) \to O(TM)$ induces the double covering $\text{Spin}(n) \to SO(n)$ in each fibre. A manifold $M$ is called a spin manifold if it admits a spin structure.

There is a unique (up to an isomorphism) non-trivial irreducible unitary representation $S$ of the group Spin$(n)$, called the spin representation. The space of spinors has a natural $\mathbb{Z}_2$-grading. For a Riemannian spin manifold $M$, denote by $F(TM)$ the associated Hermitian vector bundle of spinors on $M$: $F(TM) = O'(TM) \times_{\text{Spin}(n)} S$. This bundle is a self-adjoint Clifford module. The Levi-Civita connection $\nabla$ has a lift to a Clifford connection $\nabla^{F(TM)}$ on $F(TM)$. The corresponding Dirac operator is called the spin Dirac operator or simply the Dirac operator.

More generally, one can take a Hermitian vector bundle $E$ endowed with a Hermitian connection $\nabla^E$. Then $F(TM) \otimes E$ is a Clifford module: an action of $a \in \text{Cl}(TM)$ on $F(TM) \otimes E$ is defined by the operator $c(a) \otimes 1$ ($c(a)$ denotes the action of $a$ on $F(TM)$). The connection $\nabla^{F(TM) \otimes E} = \nabla^{F(TM)} \otimes 1 + 1 \otimes \nabla^E$ on $F(TM) \otimes E$ is a Clifford connection. The corresponding Dirac operator $D_{F(TM) \otimes E} = D_E$ is called the twisted spin Dirac operator (or the spin Dirac operator with coefficients in $E$).

Example 1.3. By definition, the group Spin$^c(n)$ is the subgroup of the complex Clifford algebra $\text{Cl}(\mathbb{R}^n)$ generated by the group Spin$(n)$ and the group $S^1 = \{z \in \mathbb{C} : |z| = 1\}$.

Let $L$ be a principal $S^1$-bundle on $M$. The natural representation of $S^1$ on $\mathbb{C}$ allows one to regard $L$ as a complex line bundle on $M$. A Spin$^c$-structure on $M$ is defined to be a principal Spin$^c(n)$-bundle $O'(TM)$ on $M$ which is a double covering of the principal $SO(n) \times S^1$-bundle $O(TM) \times L$ such that the map $O'(TM) \to O(TM)$ induces the double covering Spin$^c(n) \to SO(n) \times S^1$ in each fibre. The bundle $L$ is called the fundamental line bundle associated with the Spin$^c$-structure. The corresponding Dirac operator is called the Spin$^c$ Dirac operator.

1.2. K-theory. The topological $K$-theory plays a very important role in the index theory. Let $X$ be a compact topological space. The set of isomorphism classes of
finite-dimensional complex vector bundles on $X$ endowed with the operation of direct sum is an Abelian semigroup. This semigroup generates an Abelian group $K(X)$, the Grothendieck group, which consists of formal differences of vector bundles (virtual vector bundles). A continuous map $f: X \to Y$ induces a natural homomorphism $f^*: K(Y) \to K(X)$ which depends only on the homotopy class of $f$.

In the case when $X$ is a locally compact space we will use the group $K(X)$ of $K$-theory with compact supports, which is conveniently described as follows. The group $K(X)$ is generated by complexes of the form

$$
0 \longrightarrow E_0 \xrightarrow{d_0} E_1 \xrightarrow{d_1} \cdots \xrightarrow{d_{N-1}} E_N \xrightarrow{d_N} 0,
$$

(1.2)

where $E_0, \ldots, E_N$ are vector bundles over $X$ and $d_0, \ldots, d_N$ are morphisms of bundles. The support of such a complex is the closure of the set of all $x \in X$ for which the sequence (1.2) is not exact. We denote by $S(X)$ the set of homotopy classes of complexes of the form (1.2) with compact support and by $S_\emptyset(X)$ the subset of $S(X)$ which consists of complexes of the form (1.2) with empty support. The direct sum operation defines a semigroup structure on $S(X)$. The group $K(X)$ is defined as the quotient $S(X)/S_\emptyset(X)$, which is in fact an Abelian group. Equivalently, one can consider the set $S_N(X)$ of homotopy classes of complexes with compact support of fixed length $N$ instead of $S(X)$. In particular, for $N = 1$ we obtain a description of $K(X)$ in terms of triples $(E_0, E_1, d_0)$, where the morphism $d_0: E_0 \to E_1$ is an isomorphism outside some compact set.

For any integer $n \geq 0$ put $K^n(X) = K(X \times \mathbb{R}^n)$. There is a fundamental fact (Bott periodicity) stating that

$$
K^2(X) = K(X \times \mathbb{R}^2) \cong K^0(X) = K(X).
$$

Thus, we have only two essentially different groups $K^0(X) = K(X)$ and $K^1(X) = K(X \times \mathbb{R})$ in topological $K$-theory.

One can give a definition of $K^1(X)$ in terms of the algebra $C(X)$. Consider the group $GL(N, \mathbb{C})$ of invertible complex $N \times N$ matrices, and assume that $GL(N, \mathbb{C})$ is embedded in $GL(N + 1, \mathbb{C})$ by means of the map

$$
X \mapsto \begin{pmatrix} X & 0 \\ 0 & 1 \end{pmatrix}.
$$

Let $GL_\infty(\mathbb{C}) = \lim_{\to} GL(N, \mathbb{C})$. Then

$$
K^1(X) = C(X, GL_\infty(\mathbb{C})) / C(X, GL_\infty(\mathbb{C}))(0),
$$

where $C(X, GL_\infty(\mathbb{C}))$ denotes the group of continuous functions on $X$ with values in $GL_\infty(\mathbb{C})$ and $C(X, GL_\infty(\mathbb{C}))(0)$ denotes the identity component in $C(X, GL_\infty(\mathbb{C}))$. Equivalently, one can consider the group $U(N)$ of unitary $N \times N$ matrices instead of $GL(N, \mathbb{C})$.

The role of orientation in $K$-theory is played by a complex spin ($\text{Spin}^c$) structure. A real vector bundle $V$ of rank $m$ over $X$ is said to be $K$-orientable if its structure group reduces to the group $\text{Spin}^c(m)$. Any complex vector bundle is $K$-orientable.
There is the following description of $\text{Spin}^c(m)$-structures on a bundle $V$. Choose an arbitrary Riemannian metric in the fibres of $V$. Let $\text{Cl}(V)$ be the corresponding bundle of complex Clifford algebras. If $m$ is even, then a $\text{Spin}^c(m)$-structure on $V$ is given by a choice of an orientation on $V$ along with a bundle $S$ of irreducible Clifford modules. For odd $m$ one should replace $\text{Cl}(V)$ by its even part. For any $K$-orientable bundle $V$ over a compact manifold $X$ one has the Thom isomorphism $K(V) \cong K(X)$.

The $K$-theory and the usual cohomology groups (say, singular or Čech) of a compact topological space $X$ are connected by the Chern character

$$
\text{ch}: K^0(X) \to H^{ev}(X, \mathbb{Q}) = \bigoplus_{\text{even } k} H^k(X, \mathbb{Q}),
$$

$$
\text{ch}: K^1(X) \to H^{odd}(X, \mathbb{Q}) = \bigoplus_{\text{odd } k} H^k(X, \mathbb{Q}),
$$

which becomes an isomorphism after tensoring by $\mathbb{Q}$.

If $X$ is a smooth manifold, then one has an explicit differential-geometric construction of the Chern character. The construction of the even Chern character (1.3) is a particular case of the Chern–Weil construction of characteristic classes of vector bundles. If $E$ is a smooth vector bundle on $X$, then the Chern character $\text{ch}(E) \in H^{ev}(X, \mathbb{R})$ of the corresponding class $[E]$ in $K^0(X)$ is represented by the de Rham cohomology class of the closed differential form

$$
\text{ch}(E, \nabla) = \text{Tr} \exp \left( \frac{F}{2\pi i} \right) \in \Omega^{ev}(X, \mathbb{C})
$$

for any connection $\nabla: C^\infty(X, E) \to C^\infty(X, E \otimes T^*X)$ in $E$, where $F = \nabla^2$ is the curvature of $\nabla$.

The odd Chern character (1.4) is obtained from the even Chern character (1.3) by transgression. This is a particular case of the construction of Chern–Simons classes (see [28], [29]). If $U \in C^\infty(X, U(N))$, then the Chern character $\text{ch}(U) \in H^{odd}(X, \mathbb{R})$ of the corresponding class $[U]$ in $K^1(X)$ is given by the de Rham cohomology class of the closed differential form

$$
\text{ch}(U) = \sum_{k=0}^{+\infty} (-1)^k \frac{k!}{(2k+1)!} \text{Tr}(U^{-1}dU)^{2k+1} \in \Omega^{odd}(X, \mathbb{C}).
$$

We note two more important particular cases of characteristic classes given by the Chern–Weil construction, which we will need below:

(i) $\text{Td}(E) \in H^{ev}(X, \mathbb{C})$ is the Todd class of a complex vector bundle $E$, which is represented by the de Rham cohomology class of the closed differential form

$$
\text{Td}(E, \nabla) = \text{det} \left( \frac{F}{e^F - 1} \right) = \exp \text{Tr} \left( \log \left( \frac{F}{e^F - 1} \right) \right) \in \Omega^{ev}(X, \mathbb{C});
$$

(ii) $\tilde{A}(E) \in H^{4*}(X, \mathbb{R})$ is the $\tilde{A}$-genus (the reduced Atiyah–Hirzebruch class) of a real vector bundle $E$, which is represented by the de Rham cohomology
class of the closed differential form
\[ \widehat{A}(E, \nabla) = \det^{1/2} \left( \frac{F/2}{\sinh(F/2)} \right) = \exp \left( \frac{1}{2} \log \frac{F/2}{\sinh(F/2)} \right) \in \Omega^{4*}(X, \mathbb{R}). \]

The definition of the Todd class \( Td(E) \in H^{ev}(X, \mathbb{C}) \) can be extended to an arbitrary \( K \)-oriented vector bundle \( E \) by the formula
\[ Td(E) = e^{c_1(L)/2} \widehat{A}(E), \]
where \( c_1(L) \) is the first Chern class of the fundamental line bundle \( L \) associated with the \( \text{Spin}^c \)-structure on \( E \).

### 1.3. The Atiyah–Singer theorem.
Let \( M \) be a closed oriented manifold of dimension \( n \) and consider an elliptic pseudodifferential operator \( P : C^\infty(M, E) \to C^\infty(M, F) \), where \( E \) and \( F \) are complex vector bundles over \( M \). By definition, the principal symbol \( \sigma(P) \) of \( P \) is an isomorphism of the bundles \( \pi^*E \) and \( \pi^*F \) outside some compact neighbourhood of the zero section \( M \subset TM \). Therefore, it gives rise to a well-defined element \( [\pi^*E, \pi^*F, \sigma(P)] \in K(TM) \) of the \( K \)-theory with compact supports of the tangent bundle \( \pi : TM \to M \) of \( M \) (which can be identified with the cotangent bundle \( T^*M \) by means of a Riemannian metric on \( M \)). One can prove that the index of the elliptic operator \( P \) depends only on the class in \( K(TM) \) defined by its principal symbol. Moreover, any element of \( K(TM) \) can be obtained by means of this construction from the principal symbol of some elliptic operator. Thus, a homomorphism (the analytic index) \( \text{Ind}_a : K(TM) \to \mathbb{Z} \) is well defined.

On the other hand, by using topological constructions one can define a homomorphism (the topological index) \( \text{Ind}_t : K(TM) \to \mathbb{Z} \). Let us briefly describe its definition. Choose an embedding \( i : M \to \mathbb{R}^n \) (such an \( i \) exists for sufficiently large \( n \)). Denote by \( di : TM \to T\mathbb{R}^n \) its differential, which in this case is a proper embedding. Its normal bundle coincides with the lift of the bundle \( N \oplus N \) by the map \( di \), where \( N \) is the normal bundle of \( i \). Choose a diffeomorphism \( N \oplus N \to W \), where \( W \) is a tubular neighbourhood of \( TM \) in \( T\mathbb{R}^n \). One has the Thom isomorphism \( \varphi^K : K(TM) \to K(N \oplus N) \cong K(W) \) for the Hermitian complex vector bundle \( N \oplus N \to TM \). For an open subset \( W \) of \( T\mathbb{R}^n \) there is a natural map \( K(W) \to K(T\mathbb{R}^n) \), called the Gysin homomorphism. The composition of the Thom homomorphism and the Gysin homomorphism is a map
\[ i_1 : K(TM) \to K(T\mathbb{R}^n) = K(\mathbb{R}^{2n}). \]

This construction holds for any smooth proper embedding \( M \to V \) of manifolds, and the resulting map \( i_1 : K(M) \to K(V) \), also called the Gysin homomorphism, is independent on the choice of \( W \) and other auxiliary elements of the construction. We regard \( \mathbb{R}^{2n} = \mathbb{R}^n \oplus \mathbb{R}^n = \mathbb{C}^n \to \text{pt} \) as a complex vector bundle. Then the Thom isomorphism \( K(\text{pt}) \cong K(\mathbb{R}^{2n}) \) is defined. Its inverse is the Bott periodicity isomorphism \( \beta : K(\mathbb{R}^{2n}) \cong K(\text{pt}) = \mathbb{Z} \). The topological index is defined by the formula
\[ \text{Ind}_t = \beta \circ i_1 : K(TM) \to \mathbb{Z}. \]
The Atiyah–Singer index theorem in the $K$-theoretic form [1] states the following.

**Theorem 1.4.** One has the identity

$$\text{Ind}_a = \text{Ind}_t : K(TM) \to \mathbb{Z}.$$ 

The following cohomological formula holds for $\text{Ind}_t$ [3]. Let $\pi_t : H^*(TM) \to H^*(M)$ be the map given by integration along the fibres of the bundle $\pi : TM \to M$. This map is inverse to the Thom isomorphism $\Phi : H^*(M) \to H^*(TM)$.

**Theorem 1.5.** For any $x \in K(TM)$ one has the identity

$$\text{Ind}_t(x) = (-1)^{n(n+1)/2} \int_M (\pi_t \text{ch}(x)) \text{Td}(TM \otimes \mathbb{C}).$$

An immediate consequence of this formula is that the index of any elliptic operator on a compact oriented odd-dimensional manifold equals zero.

Let $M$ be an even-dimensional oriented Riemannian spin compact manifold and $E$ a complex vector bundle on $M$ with a Hermitian structure $g^E$ and a unitary connection $\nabla^E$. The spin Dirac operator $D_{E,+} : C^\infty(M, F_+(TM) \otimes E) \to C^\infty(M, F_-(TM) \otimes E)$ with coefficients in $E$ has index

$$\text{Ind}(D_{E,+}) = \int_M A(TM) \text{ch}(E).$$

In this case one has a stronger form of the index theorem, which we will state now. To start with, we recall the McKean–Singer formula [30]

$$\text{Ind}(D_{E,+}) = \text{Tr}_s(\exp(-t(D_E)^2)), \quad t > 0.$$

Here we regard the space $L^2(M, F(TM) \otimes E)$ as a $\mathbb{Z}_2$-graded Hilbert space and denote by $\text{Tr}_s$ the supertrace of an operator.

Let $P_t(x, y)$ be the smooth kernel of the operator $\exp(-t(D_E)^2)$ with respect to the Riemannian volume form $dy$. Then

$$\text{Ind}(D_{E,+}) = \int_M \text{Tr}_s(P_t(x, x)) \, dx, \quad t > 0.$$

In [31]–[33] the following theorem is proved (the local index theorem).

**Theorem 1.6.** The following pointwise limit relation holds as $t \to 0$:

$$\text{Tr}_s(P_t(x, x)) \to \left\{ \hat{A}(TM, \nabla^TM) \text{ch}(E, \nabla^E) \right\}^{\text{max}}.$$ 

Another proof of the local index theorem, which significantly improved its geometric understanding, was given by Getzler [34]. The Atiyah–Singer index theorem in the cohomological form is an immediate consequence of the local index theorem and the McKean–Singer formula.

A generalization of the index theorem to manifolds with boundary was obtained by Atiyah, Patodi, and Singer in [35] and [36]. In [1] an equivariant index theorem for elliptic operators invariant under an action of a compact Lie group was also proved. In [2] a formula was proved which provides an expression for the $G$-index of an elliptic $G$-complex in terms of the fixed points of the action (an analogue of the Atiyah–Bott–Lefschetz formula [37], [38]).
1.4. The index theory for self-adjoint operators. Let us consider a first-order self-adjoint elliptic operator \( D : C^\infty(M, E) \to C^\infty(M, E) \) on a closed manifold \( M \). Then its index equals zero. Nevertheless, as discovered in [28] and [39], the operator \( D \) gives rise to some analytic-index type topological invariants. For this, one makes use of Toeplitz operators \( T_u \) associated with unitary multipliers \( u \in C^\infty(M, U(N)) \).

The Toeplitz operator \( T_u \) is the bounded operator on \( L^2(M, E \otimes \mathbb{C}^N) \) given by

\[
T_u = P_+ M_u P_+ ,
\]

where \( P_+ \) is the spectral projection corresponding to the positive semi-axis for the operator \( D \times I_N \) acting in \( L^2(M, E \otimes \mathbb{C}^N) \), and \( M_u \) is the operator of multiplication by \( 1 \otimes u \). The operator \( T_u \) is a Fredholm operator. For its index there is a topological formula [28] which is derived from the Atiyah–Singer index theorem:

\[
\text{Ind } T_u = \int_{ST^*M} \pi^* \text{ch}(u) \text{ch}(E_+) \pi^* \text{Td}(TM \otimes \mathbb{C}),
\]

where \( \text{ch}(u) \in H^{\text{odd}}(M) \) is the Chern character of the class \([u] \in K^1(M)\) defined by (1.4), \( \pi^* \text{ch}(u) \in H^{\text{odd}}(ST^*M) \) is its lift to the cosphere bundle \( ST^*M \) by the natural projection \( \pi : ST^*M \to M \), \( E_+ \) is the subbundle of the bundle \( \pi^* E \) on \( ST^*M \) generated by the positive eigenvectors of the principal symbol \( \sigma(D) \), and \( \text{ch}(E_+) \in K(ST^*M) \) is its Chern character.

One can show that the index of the Toeplitz operator \( T_u \) depends only on the class \([u] \in K^1(M)\) of the unitary multiplier \( u \in C^\infty(M, U(N)) \), thus yielding a map

\[
K^1(M) \to \mathbb{Z}, \quad K^1(M) \ni [u] \mapsto \text{Ind } T_u.
\]

As a result, the index theory of self-adjoint operators is often called the odd index theory.

1.5. The families index theory. The index theory for families of elliptic operators was developed in [4]. Let \( Z \) be a fibration over a Hausdorff topological space \( Y \) with fibre \( X \) and structure group \( \text{Diff}(X) \) (a manifold over \( Y \)), and let \( E \) and \( F \) be fibrations over \( Y \). Denote by \( Z_y \) the fibre of \( Z \) over \( y \) and by \( E_y \) and \( F_y \) the restrictions of the bundles \( E \) and \( F \) to \( Z_y \). We consider a continuous family \( \{ P_y \in \Psi^d(Z_y, E_y, F_y) : y \in Y \} \) of elliptic pseudodifferential operators. If \( \dim \text{Ker } P_y \) is independent of \( y \), then the family \( \{ \text{Ker } P_y : y \in Y \} \) of vector spaces defines a vector bundle \( \text{Ker } P \) over \( Y \). The same holds for \( \text{Coker } P \). In this case the index of the family \( P \) is defined by

\[
\text{Ind}_a(P) = [\text{Ker } P] - [\text{Coker } P] \in K(Y).
\]

In the general case when \( \dim \text{Ker } P_y \) varies, the definition of the index of \( P \) as an element of \( K(Y) \) is given by a slight modification of this definition.

Denote by \( TZ/Y \) the vertical tangent space (the tangent space along the fibres of the fibration). The symbol of the family \( P \) defines an element \( [\sigma(P)] \in K(TZ/Y) \). The analytic index of \( P \) depends only on \( [\sigma(P)] \in K(TZ/Y) \), thus defining an analytic index map

\[
\text{Ind}_a : K(TZ/Y) \to K(Y).
\]
The construction of a topological index $\text{Ind}_t : K(TZ/Y) \to K(Y)$ for families of elliptic operators is a direct generalization of the construction of the topological index for elliptic operators. One simply needs to perform all the constructions used there fibrewise over $Y$. The family index theorem proved in [4] establishes the coincidence of the analytic and topological indices:

**Theorem 1.7.** One has the equality

$$\text{Ind}_a = \text{Ind}_t : K(TZ/Y) \to K(Y).$$

We also formulate the index theorem for families of elliptic operators in the cohomological form.

**Theorem 1.8.** Let $P$ be a family of elliptic operators parametrized by a manifold $Y$ and $u \in K(TZ/Y)$ the class of the symbol of $P$. Then

$$\text{ch} \text{Ind} P = (-1)^n \pi_*(\text{ch} u \text{Td}(TZ \otimes \mathbb{C})) \in H^*(Y, \mathbb{Q}),$$

where $n$ is the dimension of fibres, and $\pi_* : H^*(TZ/Y) \to H^*(Y)$ is integration along the fibres of $TZ/Y \to Y$.

An important example is the index theorem for families of Dirac operators. Let $\pi : M \to B$ be a fibration of compact manifolds with compact fibres $Z_y$, $y \in B$, of even dimension $n = 2l$. Suppose that the vertical tangent bundle $TM/B$ admits a spin structure. Let $g_{M/B}$ be a smooth metric on $TM/B$. Denote by $F = F_+(TM/B) \oplus F_-(TM/B)$ the spinor bundle for $TM/B$. Let $E$ be a Hermitian bundle on $M$ endowed with a unitary connection $\nabla^E$. For any $y \in B$ one has a well-defined Dirac operator

$$D_{E,y} = \begin{bmatrix} 0 & D_{E,y,-} \\ D_{E,y,+} & 0 \end{bmatrix},$$

acting in $C^\infty(Z_y, F(TZ_y) \otimes E)$. Thus, the family of elliptic operators $\{D_{E,y} : y \in B\}$ is well defined. For the index of this family one has the formula

$$\text{ch}(\text{Ind} D_{E,+}) = \pi_* [\hat{A}(TM/B) \text{ch} E] \in H^{ev}(B, \mathbb{Q}).$$

Bismut [26] proved a stronger version of this result, the so-called local index theorem for families.

Let $T^HM$ be a subbundle of $TM$ such that $TM = T^HM \oplus TM/B$. The bundle $T^HM$ can be identified with $\pi^*TB$. For any $U \in TB$ denote by $U^H$ its lift to $T^HM$. Any metric $g_B$ on $TB$ lifts to $T^HM$.

For any $U, V \in TB$ put

$$T(U, V) = -P[U^H, V^H] \in TM/B,$$

where $P : TM \to TM/B$ is the orthogonal projection.

For $U \in C^\infty(B, TB)$ let $\text{div}_Z(U^H)$ denote the divergence of the vector field $U^H$ with respect to the vertical Riemannian volume form $dv_Z$. 

Let $\nabla^L$ be the Levi-Civita connection on $TM$ associated with $g_B \oplus g_{M/B}$. We introduce a Euclidean connection $\nabla^{TM/B}$ on $TM/B$ by the formula

$$\nabla^{TM/B} = P\nabla^L.$$  

The connection $\nabla^{TM/B}$ is independent of the choice of $g_B$ and is canonically determined by $T^H M$ and $g_{M/B}$, and $\nabla^{TM/B}$ and $\nabla^E$ determine a connection $\nabla^{F \otimes E} : C^\infty(M, F \otimes E) \to C^\infty(M, T^H M \otimes F \otimes E)$ on $F \otimes E$. We define a connection $\nabla^H : C^\infty(M, F \otimes E) \to C^\infty(M, (T^H M)^* \otimes F \otimes E)$ as follows: for $U \in C^\infty(B, TB)$ and $s \in C^\infty(M, F \otimes E)$,

$$\nabla^H_U s = \nabla^{F \otimes E}_U s + \frac{1}{2} \text{div}_Z(U^H)s.$$  

Let $f_1, \ldots, f_m$ be a local orthonormal basis in $TB$ and $f^1, \ldots, f^m$ its dual basis in $T^*B$. For any $t > 0$ we consider the operator

$$A_t : C^\infty(M, F \otimes E) \to C^\infty(M, (T^H M)^* \otimes F \otimes E),$$

given by

$$A_t = \nabla^H + \sqrt{t} \, D_E - \frac{1}{8\sqrt{t}} \, f^\alpha f^\beta c(T(f_\alpha, f_\beta)).$$

It defines a superconnection on $C^\infty(M, F \otimes E)$, regarded as an infinite-dimensional vector bundle over $B$ with fibre $C^\infty(Z_y, F(TZ_y) \otimes E)$ at $y \in B$, in the sense of the following definition.

Denote by $\Omega(M)$ the space of smooth differential forms on $M$ and, for any vector bundle $\mathcal{E}$ on $M$, by $\Omega(M, \mathcal{E})$ the space of smooth differential forms on $M$ with coefficients in $\mathcal{E}$.

**Definition 1.9.** Let $\mathcal{E}$ be a $\mathbb{Z}_2$-graded vector bundle on a manifold $M$. A superconnection on $\mathcal{E}$ is defined to be an odd first-order differential operator $\mathbb{A} : \Omega^\pm(M, \mathcal{E}) \to \Omega^\mp(M, \mathcal{E})$ satisfying the $\mathbb{Z}_2$-graded Leibniz rule: if $\alpha \in \Omega(M)$ and $\theta \in \Omega(M, \mathcal{E})$, then

$$\mathbb{A}(\alpha \wedge \theta) = d\alpha \wedge \theta + (-1)^{|\alpha|} \alpha \wedge \mathbb{A}(\theta).$$

A superconnection $\mathbb{A}$ on $\mathcal{E}$ gives rise to an action on $\Omega(M, \text{End } \mathcal{E})$ which is compatible with the Leibniz rule:

$$\mathbb{A}\alpha = [\mathbb{A}, \alpha], \quad \alpha \in \Omega(M, \text{End } \mathcal{E}).$$

The curvature of the superconnection $\mathbb{A}$ is defined as the operator $\mathbb{A}^2$ acting in the space $\Omega(M, \text{End } \mathcal{E})$.

In our case the curvature $A_t^2$ of the superconnection $A_t$ is given by a family of second-order elliptic operators with coefficients in $\Lambda^2(T^H M)^*$ acting along the fibres of $\pi : M \to B$:

$$A_t^2 : C^\infty(M, F \otimes E) \to C^\infty(M, \Lambda^2(T^H M)^* \otimes F \otimes E).$$
The operator
\[
\exp(-A_t^2) : C^\infty(M, F \otimes E) \to C^\infty(M, \Lambda^*(T^*M)^* \otimes F \otimes E)
\]
\[
\cong C^\infty(M, \Lambda^*(T^*B) \otimes F \otimes E)
\]
is given by a family of smoothing operators acting along the fibres of the fibration, with coefficients in differential forms on the base \(B\).

For any \(t > 0\) we consider the even form on \(B\)
\[
\alpha_t = \phi \text{Tr}_s[\exp(-A_t^2)],
\]
where the linear endomorphism \(\phi : \Lambda(T^*B) \to \Lambda(T^*B)\) is given by the formula
\[
\omega \mapsto (2i\pi)^{-\deg \omega/2} \omega \quad \text{and} \quad \text{Tr}_s \text{ denotes the fibrewise supertrace.}
\]

The following facts hold [26]:
1) \(\alpha_t\) is a real even closed differential form on \(B\);
2) the de Rham cohomology class \([\alpha_t] \in H^{ev}(B, \mathbb{Q})\) is equal to \(\text{ch}(\text{Ind} D_E, +)\);
3) (the family local index theorem) as \(t \to 0\),
\[
\alpha_t = \pi^* [\hat{A}(TM/B, \nabla^{TM/B}) \text{ch}(E, \nabla^E)] + O(t).
\]

Moreover, it is proved in [6], [40] that if \(\text{Ker} D_E\) is a vector bundle and \(\nabla^{\text{Ker} D_E}\) is the orthogonal projection of the connection \(\nabla^H\) on \(\text{Ker} D_E\), then as \(t \to +\infty\),
\[
\alpha_t = \text{ch}(\text{Ker} D_E, \nabla^{\text{Ker} D_E}) + O\left(\frac{1}{\sqrt{t}}\right).
\]

1.6. The higher index theory. For an even-dimensional closed connected Riemannian spin manifold \(M\) and a Hermitian vector bundle \(E\) on it, the Atiyah–Singer index theorem establishes a connection between the index of the spin Dirac operator \(D_E\) with coefficients in \(E\) and the topological expression \(\int_M \hat{A}(TM) \text{ch}(E)\). If \(M\) is not simply connected, then one can modify the index theorem, taking into account the fundamental group \(\Gamma\) of \(M\). Denote by \(p : M \to \pi_1 M\) the classifying map for the universal covering \(\tilde{M} \to M\). The higher index theory attempts to give an analytic interpretation of the expression
\[
\int_M \hat{A}(TM) \text{ch}(E)p^* [\eta],
\]
where \([\eta] \in H^*(B\Gamma, \mathbb{C})\). As examples of applications of the higher index theory, we can mention the Novikov conjecture on homotopy invariance of non-simply connected manifolds and questions on the existence of metrics of positive scalar curvature (see, for instance, the papers [41]– [44] and the references cited therein).

The higher index theory is directly connected with the \(L^2\)-index theory on coverings of compact manifolds, which originated in [45]. There Atiyah proved a \(\Gamma\)-index theorem for \(\Gamma\)-invariant elliptic operators on a covering of a compact manifold. In [46] Connes and Moscovici proposed an approach to the higher index theory based on the use of non-commutative differential geometry, in particular, of cyclic cohomology. Their approach, like some others, is based on the idea of regarding a \(\Gamma\)-invariant elliptic operator on a covering of a compact manifold as a family of
elliptic operators parameterized by points of a non-commutative space $B$ whose algebra of continuous functions is the reduced group $C^*_r\Gamma$. In [47] there is a proof of a higher index theorem using methods of the local index theory for families of elliptic operators. We also mention the paper [48], where a higher analogue of the eta-invariant for $\Gamma$-invariant operators on a $\Gamma$-covering of a compact manifold was introduced.

2. Basic notions of non-commutative topology

2.1. Non-commutative spaces and bundles. Recall that a $C^*$-algebra is an involutive Banach algebra $A$ with

$$\|a^*a\| = \|a\|^2, \quad a \in A.$$  

A simplest example of a $C^*$-algebra is the algebra $C_0(X)$ of continuous functions on a locally compact Hausdorff topological space $X$ vanishing at infinity, endowed with the operations of the pointwise addition and multiplication, with the standard involution given by complex conjugation, and with the uniform norm

$$\|f\| = \sup_{x \in X} |f(x)|, \quad f \in C_0(X).$$

The Gel’fand–Naimark theorem enables one to reconstruct from a commutative $C^*$-algebra $A$ a unique locally compact Hausdorff topological space $X$ such that $A \cong C_0(X)$. More precisely, $X$ coincides with the set $\hat{A}$ of all characters of the algebra $A$, that is, of all continuous homomorphisms $A \to \mathbb{C}$, endowed with the topology of pointwise convergence. This fact allows one to regard an arbitrary $C^*$-algebra as the algebra of continuous functions on some virtual space. For this reason, the theory of $C^*$-algebras is often called non-commutative topology.

The algebra $L(H)$ of bounded operators on a Hilbert space $H$ equipped with the involution given by taking adjoints and with the operator norm is a $C^*$-algebra. By the second Gel’fand–Naimark theorem, any $C^*$-algebra is isometrically $*$-isomorphic to some norm-closed $*$-subalgebra of the algebra $L(H)$ for some Hilbert space $H$.

We recall that a right module $E$ over a unital algebra $A$ is said to be finitely generated if it is generated by a finite family $\{x_i \in E : i = 1, 2, \ldots, k\}$ of elements, that is, the submodule of finite $A$-linear combinations of the form $\sum_{i=1}^k x_i a_i$, where $a_i \in A$, is dense in $A$. A right $A$-module $E$ is said to be projective if there exists a right $A$-module $E'$ such that the direct sum $E \oplus E'$ is isomorphic to the free module $A^N$ for some $N$.

If $E$ is a continuous complex vector bundle on a compact topological space $X$, then the space $C(X, E)$ of its continuous sections is a finitely generated projective module over the algebra $C(X)$ of continuous functions on $X$. The action of $C(X)$ on $C(X, E)$ is given by the formula

$$(a \cdot s)(x) = a(x)s(x), \quad x \in X.$$  

The Serre–Swan theorem states that any finitely generated projective $C(X)$-module is isomorphic to the $C(X)$-module $C(X, E)$ for some finite-dimensional complex vector bundle $E$. Thus, an arbitrary finitely generated projective module over
a $C^*$-algebra can be regarded as an analogue of a finite-dimensional complex vector bundle over the corresponding non-commutative space, or in other words, as a non-commutative vector bundle.

In many problems of index theory and non-commutative geometry, it is useful to consider more general objects, namely, continuous fields of Hilbert spaces. A natural example of a continuous field of Hilbert spaces arises in the index theory for families of elliptic operators. Let $Z$ be a fibration over a Hausdorff topological space $Y$, with fibre $X$ and structure group $\text{Diff}(X)$, and let $E$ be a bundle over $Y$. As above, denote by $Z_y$ the fibre of $Z$ over $y$ and by $E_y$ the restriction of $E$ to $Z_y$. Then the family $\{L^2(Z_y, E_y), y \in Y\}$ of Hilbert spaces is a continuous field of Hilbert spaces on $Y$.

A non-commutative analogue of a continuous field of Hilbert spaces is the notion of Hilbert $C^*$-module. Hilbert $C^*$-modules can be regarded as a natural generalization of Hilbert spaces which arises if one replaces the field of scalars $\mathbb{C}$ by an arbitrary $C^*$-algebra.

**Definition 2.1.** Let $B$ be a $C^*$-algebra. A pre-Hilbert $B$-module is defined to be a right $B$-module $\mathcal{E}$ equipped with a sesquilinear map $\langle \cdot, \cdot \rangle_B : \mathcal{E} \times \mathcal{E} \to B$ (linear in the second argument) satisfying the following conditions:

1) $\langle x, x \rangle_B \geq 0$ for any $x \in \mathcal{E}$;
2) $\langle x, x \rangle_B = 0$ if and only if $x = 0$;
3) $\langle y, x \rangle_B = \langle x, y \rangle_B^*$ for any $x, y \in \mathcal{E}$;
4) $\langle x, yb \rangle_B = \langle x, y \rangle_B b$ for any $x, y \in \mathcal{E}$, $b \in B$.

The map $\langle \cdot, \cdot \rangle_B$ is called a $B$-valued inner product.

For a pre-Hilbert $B$-module $\mathcal{E}$ the formula $\|x\|_B = \|\langle x, x \rangle_B \|^{1/2}$ defines a norm on $\mathcal{E}$. If $\mathcal{E}$ is complete in the norm $\|\cdot\|_B$, then $\mathcal{E}$ is called a Hilbert $C^*$-module. In the general case the action of $B$ and the inner product on $\mathcal{E}$ are extended to its completion $\widetilde{\mathcal{E}}$, making $\widetilde{\mathcal{E}}$ into a Hilbert $C^*$-module.

**Example 2.2.** If $\mathcal{H} = \{H_x : x \in X\}$ is a continuous field of Hilbert spaces over a compact topological space $X$, then the space $C(X, \mathcal{H})$ of its continuous sections is a Hilbert module over $C(X)$. The action of $C(X)$ on $C(X, \mathcal{H})$ is given by the formula

$$\langle a \cdot s \rangle(x) = a(x)s(x), \quad x \in X,$$

and the inner product by

$$\langle s_1, s_2 \rangle_{C(X)}(x) = \langle s_1(x), s_2(x) \rangle_{H_x}, \quad x \in X.$$

Let $\mathcal{E}$ be a Hilbert $B$-module. Denote by $\mathcal{B}(\mathcal{E})$ the set of all endomorphisms $T$ of $\mathcal{E}$ such that there exists an adjoint endomorphism, that is, an endomorphism $T^* : \mathcal{E} \to \mathcal{E}$ such that $\langle Tx, y \rangle_B = \langle x, T^*y \rangle_B$ for any $x, y \in \mathcal{E}$. Any operator in $\mathcal{B}(\mathcal{E})$ is a bounded operator on $\mathcal{E}$, and the algebra $\mathcal{B}(\mathcal{E})$ is a $C^*$-algebra with respect to the uniform norm.

For any $x, y \in \mathcal{E}$ denote by $\theta_{x,y}$ the operator defined in $\mathcal{E}$ by $\theta_{x,y}(z) = x(y, z)_B$, $z \in \mathcal{E}$. It is easy to see that $\theta_{x,y} \in \mathcal{B}(\mathcal{E})$. The closure $\mathcal{H}(\mathcal{E})$ of the linear span of $\{\theta_{x,y} : x, y \in \mathcal{E}\}$ is a closed ideal in $\mathcal{B}(\mathcal{E})$. Its elements are called compact endomorphisms of $\mathcal{E}$. 
Example 2.3. Let $Z$ be a fibration over a compact topological space $Y$ with fibre $X$ and structure group $\text{Diff}(X)$ and let $E$ be a bundle over $Y$. Consider the Hilbert $C(Y)$-module $\mathcal{E}$ of continuous sections of the continuous field $\{L^2(Z_y, E_y), y \in Y\}$ of Hilbert spaces on $Y$. Then an arbitrary continuous family $\{P_y \in \Psi^d(Z_y, E_y) : y \in Y\}$ of pseudodifferential operators with $d \leq 0$ defines a bounded endomorphism of the Hilbert $C(Y)$-module $\mathcal{E}$. If $d < 0$, this endomorphism is a compact endomorphism.

The notion of isometric $\ast$-isomorphism between $C^*$-algebras is a natural analogue of the notion of homeomorphism of topological spaces. There is a broader equivalence relation for $C^*$-algebras called strong Morita equivalence. It preserves many invariants of $C^*$-algebras, for instance, the K-theory, the space of irreducible representations, the cyclic cohomology, and it coincides with the relation of isometric $\ast$-isomorphism on the class of commutative $C^*$-algebras. We briefly recall some information about the strong Morita equivalence (a more detailed exposition can be found in [49]).

Definition 2.4. Let $A$ and $B$ be $C^*$-algebras.

An $A$-$B$-bimodule is a vector space $X$ endowed with structures of a left $A$-module and a right $B$-module which are compatible in the sense that $(ax)b = a(xb)$ for any $x \in X$, $a \in A$, and $b \in B$.

An $A$-$B$-equivalence bimodule is defined to be an $A$-$B$-bimodule $X$ endowed with $A$-valued and $B$-valued inner products $\langle \cdot, \cdot \rangle_A$ and $\langle \cdot, \cdot \rangle_B$, respectively, such that $X$ is a right Hilbert $B$-module and a left Hilbert $A$-module with respect to these inner products and, moreover,

1) $\langle x, y \rangle_A z = x \langle y, z \rangle_B$ for any $x, y, z \in X$,

2) the set $\langle X, X \rangle_A$ spans a dense subset of $A$, and the set $\langle X, X \rangle_B$ spans a dense subset of $B$.

Algebras $A$ and $B$ for which there is an $A$-$B$-equivalence bimodule are said to be strong Morita equivalent.

It is not difficult to show that strong Morita equivalence is an equivalence relation.

For any linear space $L$, denote by $\tilde{L}$ the complex conjugate linear space, which coincides with $L$ as a set and has the same addition operation, but has multiplication by scalars given by the formula $\lambda \tilde{x} = (\bar{\lambda}x)^\sim$. If $X$ is an $A$-$B$-equivalence bimodule, then $\tilde{X}$ is endowed with the structure of a $B$-$A$-equivalence bimodule. For instance, $b\tilde{x}a = (a^\ast xb^\ast)^\sim$.

Theorem 2.5. Let $X$ be an $A$-$B$-equivalence bimodule. Then the map $E \to X \otimes_B E$ defines an equivalence of the category of Hermitian $B$-modules and the category of Hermitian $A$-modules, with inverse given by the map $F \to F \otimes_B \tilde{X}$.

In particular, Theorem 2.5 implies that two commutative $C^*$-algebras are strongly Morita equivalent if and only if they are isomorphic.

The following theorem relates the notion of strong Morita equivalence with the notion of stable equivalence.
Theorem 2.6. Let $A$ and $B$ be $C^*$-algebras with countable approximate units. Then these algebras are strongly Morita equivalent if and only if they are stably equivalent, that is, the algebras $A \otimes \mathcal{K}$ and $B \otimes \mathcal{K}$ are isomorphic. (Here $\mathcal{K}$ denotes the algebra of compact operators on a separable Hilbert space.)

2.2. The operator $K$-theory. We begin this subsection with some facts from $K$-theory for $C^*$-algebras, the non-commutative analogue of topological $K$-theory.

Let $A$ be a unital $C^*$-algebra. The group $K_0(A)$ is defined as the Grothendieck group of the semigroup of isomorphism classes of finitely generated projective modules over $A$, with the direct sum operation. Thus, elements of $K_0(A)$ can be regarded as formal differences of isomorphism classes of finitely generated projective modules over $A$. Equivalently, one can consider isomorphism classes of orthogonal projections in the algebra of matrices over $A$.

Another definition of the group $K_0(A)$ is given as follows. Denote by $M_n(A)$ the algebra of $n \times n$ matrices with entries in $A$. Let us assume that $M_n(A)$ is embedded in $M_{n+1}(A)$ by means of the map $X \rightarrow \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix}$. Let $M_\infty(A) = \lim\limits_{\longrightarrow} M_n(A)$. The group $K_0(A)$ is defined as the set of homotopy equivalence classes of projections $(p^2 = p = p^*)$ in $M_\infty(A)$ equipped with the direct sum operation

$$p_1 \oplus p_2 = \begin{pmatrix} p_1 & 0 \\ 0 & p_2 \end{pmatrix}.$$ 

Denote by $GL_n(A)$ the group of invertible $n \times n$ matrices with entries in $A$, and assume that $GL_n(A)$ is embedded in $GL_{n+1}(A)$ by means of the map $X \rightarrow \begin{pmatrix} X & 0 \\ 0 & 1 \end{pmatrix}$. Let $GL_\infty(A) = \lim\limits_{\longrightarrow} GL_n(A)$. The group $K_1(A)$ is defined as the set of homotopy equivalence classes of unitary matrices $(u^* u = uu^* = 1)$ in $GL_\infty(A)$, equipped with the direct sum operation.

If $A$ has no unit and $A^+$ is the algebra obtained by adjoining the unit to $A$, then one has the homomorphism $i : \mathbb{C} \rightarrow A^+ : \lambda \mapsto \lambda \cdot 1$, which induces a homomorphism $i_* : K_0(\mathbb{C}) \rightarrow K_0(A^+)$, and $K_0(A)$ is defined as the kernel of this homomorphism. Moreover, $K_1(A) = K_1(A^+)$ by definition.

These definitions agree with those in the commutative case: the isomorphisms $K_i(C_0(X)) \cong K^i(X)$, $i = 0, 1$, hold for any locally compact topological space $X$.

For an arbitrary algebra $\mathcal{A}$ over $\mathbb{C}$ we will consider the groups $K_0(\mathcal{A})$ and $K_1(\mathcal{A})$ of algebraic $K$-theory. The group $K_0(\mathcal{A})$ is defined like the group $K_0(A)$ of topological $K$-theory with the use of idempotents ($e^2 = e$) in $M_\infty(\mathcal{A})$ instead of projections. The group $K_1(\mathcal{A})$ is defined as the quotient of $GL_\infty(\mathcal{A})$ by the commutator subgroup $[GL_\infty(\mathcal{A}), GL_\infty(\mathcal{A})]$.

The homological $K$-functor — the object dual to the topological $K$-theory — was introduced by purely homotopic methods by Whitehead in 1962. Atiyah [50] observed that an elliptic operator on a smooth manifold can be regarded in some sense as an element of a $K$-homology group. He formulated functional-analytic axioms for two basic properties of an elliptic pseudodifferential operator on a compact manifold. Using them, he defined, for any compact topological space $X$, a class of objects $\text{Ell}(X)$ and, in the case when $X$ is a CW-complex, an epimorphism $\text{Ell}(X) \rightarrow K_0(X)$. He proposed regarding elements of $\text{Ell}(X)$ as representing
cycles for $K_0(X)$. Atiyah’s ideas were completely realized by Kasparov in [51] (see also [52]). The analytic construction in [51] of the $K$-homology groups is applicable to an arbitrary non-commutative $C^*$-algebra and is based on the notion of Fredholm module.

**Definition 2.7.** A Fredholm module (or a $K$-cycle) over an algebra $A$ is a pair $(H, F)$, where

1) $H$ is a Hilbert space equipped with a $*$-representation $\rho$ of the algebra $A$;
2) $F$ is a bounded operator on $H$ such that for any $a \in A$ the operators $(F^2 - 1)\rho(a)$, $(F - F^*)\rho(a)$, and $[F, \rho(a)]$ are compact on $H$.

A Fredholm module $(H, F)$ is said to be even if the Hilbert space $H$ is endowed with a $\mathbb{Z}_2$-grading $\gamma$, the operators $\rho(a)$ are even, $\gamma\rho(a) = \rho(a)\gamma$, and the operator $F$ is odd, $\gamma F = -F\gamma$. Otherwise it is said to be odd.

The homology groups $K^0(A)$ (respectively, $K^1(A)$) of a $C^*$-algebra $A$ are defined as the sets of homotopy equivalence classes of even (respectively, odd) Fredholm modules over $A$. The direct sum operation defines an Abelian group structure on $K^0(A)$ and $K^1(A)$.

**Example 2.8.** Let $M$ be a compact manifold, $E_0$ and $E_1$ Hermitian vector bundles on $M$, $P \in \Psi^0(M, E_0, E_1)$ a zero-order elliptic operator whose principal symbol $\sigma_P$ satisfies the condition $\sigma_P \sigma_P^* = 1$ (for instance, if $D \in \Psi^d(M, E_0, E_1)$ is an elliptic operator of order $d > 0$, then one can take for $P$ the operator $D(1 + D^*D)^{-1/2}$), and $Q \in \Psi^0(M, E_1, E_0)$ a parametrix for $P$. We define the $\mathbb{Z}_2$-graded Hilbert space $H = L^2(M, E_0) \oplus L^2(M, E_1)$ endowed with the natural action of the algebra $C(M)$, and the bounded operator $F$ on $H$ given by the matrix $\begin{pmatrix} 0 & Q \\ P & 0 \end{pmatrix}$. Then the pair $(H, F)$ is an even Fredholm module over $C(M)$.

For $D$ one can take the Dirac operator $D_\pm^\delta : C^\infty(M, \mathcal{E}_\pm) \to C^\infty(M, \mathcal{E}_\pm)$ associated with an arbitrary Clifford bundle $\mathcal{E}$ over an even-dimensional compact Riemannian manifold $M$.

**Example 2.9.** Let $M$ be a compact manifold, $E$ a Hermitian vector bundle on $M$, and $F \in \Psi^0(M, E)$ a zero-order elliptic operator whose principal symbol $\sigma_F$ satisfies the condition $\sigma_F^* = \sigma_F$, $\sigma_F^2 = 1$ (for example, if $D \in \Psi^d(M, E)$ is a self-adjoint elliptic operator of order $d > 0$, then one can take for $F$ the operator $D(1 + D^2)^{-1/2}$). We define the $\mathbb{Z}_2$-graded Hilbert space $H = L^2(M, E) \oplus 0$ endowed with the natural action of the algebra $C(M)$. Then the pair $(H, F)$ is an odd Fredholm module over $C(M)$.

For an operator $D$ one can take the Dirac operator $D_\pm : C^\infty(M, \mathcal{E}) \to C^\infty(M, \mathcal{E})$ associated with an arbitrary Clifford bundle $\mathcal{E}$ over an odd-dimensional compact Riemannian manifold $M$.

Corresponding to a Fredholm module $(H, F)$ over an algebra $A$ is an index map $\text{ind} : K_*^+(A) \to \mathbb{Z}$. In the even case the operator $F$ takes the form

\[
F = \begin{pmatrix} 0 & F^- \\ F^+ & 0 \end{pmatrix}, \quad F_\pm : H^\pm \to H^\mp,
\]
with respect to the decomposition $H = H^+ \oplus H^-$ given by the $\mathbb{Z}_2$-grading of $H$. For an idempotent $e \in M_q(A)$ the operator $e(F^+ \otimes 1)e$ acting from $e(H^+ \otimes \mathbb{C}^q)$ to $e(H^- \otimes \mathbb{C}^q)$ is Fredholm, and its index depends only on the class of $e$ in $K_0(A)$. Therefore, a map $\text{Ind}: K_0(A) \to \mathbb{Z}$ is well defined by the formula

$$\text{Ind}[e] = \text{Ind}(F^+ \otimes 1)e. \quad (2.1)$$

In the odd case, for a unitary matrix $U \in GL_q(A)$ the operator $(P \otimes 1)U(P \otimes 1)$ with $P = (1 + F)/2$ is a Fredholm operator. Moreover, the index of $(P \otimes 1)U(P \otimes 1)$ depends only on the class of $U$ in $K_1(A)$. Thus, one obtains the map $\text{Ind}: K_1(A) \to \mathbb{Z}$ given by the formula

$$\text{Ind}[U] = \text{Ind}(P \otimes 1)U(P \otimes 1). \quad (2.2)$$

If $A$ is a $C^*$-algebra, then in both the even and odd cases the map $\text{Ind}$ determines a map of the group $K_i(A)$ of topological $K$-theory to $\mathbb{C}$, and it depends only on the class determined by the Fredholm module $(H, F)$ in the $K$-homology group $K^i(A)$.

In [28] a geometric definition of the $K$-homology groups is given. We briefly recall this definition. A $K$-cycle on a topological space $X$ is defined to be a triple $(M, E, \phi)$, where $M$ is a compact Spin$^c$ manifold without boundary, $E$ is a complex vector bundle on $M$, and $\phi$ is a continuous map from $M$ to $X$. We remark that $M$ is not necessarily connected, and the fibres of $E$ may have different dimensions on different connected components. One can define a natural notion of isomorphism of $K$-cycles. On the set of isomorphism classes of $K$-cycles on $X$ there is an equivalence relation generated by three elementary relations, called bordism, direct sum, and vector bundle modification.

The geometric $K$-homology group $K^\text{geom}(X)$ is defined as the set of equivalence classes of $K$-cycles on $X$. An Abelian group structure on $K^\text{geom}(X)$ is given by the obvious operation of disjoint union of $K$-cycles.

Denote by $K^\text{geom}_0(X)$ (respectively, $K^\text{geom}_1(X)$) the subgroup of $K^\text{geom}(X)$ which consists of equivalence classes of $K$-cycles $(M, E, \phi)$ such that each connected component of $M$ is even-dimensional (respectively, odd-dimensional).

An isomorphism of geometric and analytic $K$-homology groups is constructed as follows. Let $(M, E, \phi)$ be a $K$-cycle on $X$. Let $D_E$ be the Spin$^c$-Dirac operator on $M$ with coefficients in $E$, and let $[D_E] \in K_i(M)$, $i = \dim M \pmod 2$, be the corresponding class in the $K$-homology of $M$ (see Examples 2.8 and 2.9). The map $\phi: M \to X$ induces a map $\phi_*: K_i(M) \to K_i(X)$ in the $K$-homology. The corresponding class $(M, E, \phi) \to \phi_*[D_E] \in K_i(X)$ defines a map $K^\text{geom}_i(X) \to K_i(X)$, $i = 0, 1$, which is an isomorphism [53].

There is a natural transformation of homology theories, the homology Chern character $\text{ch}: K_*(X) \to H_*(X, \mathbb{Q})$ defined as follows (see [28]). Consider a $K$-cycle $(M, E, \phi)$ on $X$. The map $\phi: M \to X$ induces a map $\phi_*: H_*(M, \mathbb{Q}) \to H_*(X, \mathbb{Q})$ of rational homology groups. Put

$$\text{ch}(M, E, \phi) = \phi_*(\text{ch}(E) \cup \text{Td}(M) \cap [M]) \in H_*(X, \mathbb{Q}), \quad (2.3)$$

where the cap product $\text{ch}(E) \cup \text{Td}(M) \cap [M] \in H_*(M, \mathbb{Q})$ coincides with the Poincaré dual to $\text{ch}(E) \cup \text{Td}(M) \in H^*(M, \mathbb{Q})$. In particular, we observe that, for the Dirac
operator $D_E$ on a Spin$^c$-manifold $M$ with coefficients in $E$, we have

$$\text{ch}[D_E] = \text{ch}(E) \cup \text{Td}(M) \cap [M] \in H_*(M, \mathbb{Q}).$$

If $X$ is a finite CW-complex, then the Chern character $\text{ch}$ determines an isomorphism $K_*(X) \otimes \mathbb{Q} \rightarrow H_*(X, \mathbb{Q})$. For an arbitrary CW-complex $X$ we will consider singular homology $H_*(X, \mathbb{Q})$ and $K$-homology $K_*(X)$ with compact supports. Therefore, the Chern character $\text{ch}: K_*(X) \otimes \mathbb{Q} \rightarrow H_*(X, \mathbb{Q})$ is well defined and is also a rational isomorphism.

We will also need the $K$-homology groups $K_*(F)$ of $X$ twisted by a real vector bundle $F$ on $X$. They are defined by the formula

$$K_j(F) = K_j(F, F \{0\}), \quad j = 0, 1.$$

One can give an equivalent definition, choosing a Euclidean structure in the fibres of $F$ and introducing the unit ball and unit sphere bundles $BF$ and $SF$ of $F$. Then

$$K_j(F) = K_j(BF, SF), \quad j = 0, 1.$$

The $K$-homology fundamental class of a compact Spin$^c$-manifold $M$ is the class $[D] \in K_i(M)$, where $i = \dim M \pmod{2}$, defined by an arbitrary Spin$^c$ Dirac operator $D$ on $M$. The Poincaré duality in $K$-theory states that the cap product with $[D]$ gives an isomorphism

$$K^*(M) \cong K^a_*(M).$$

An analogous statement also holds for an arbitrary compact Spin$^c$-manifold $M$ with boundary $\partial M$:

$$K^*(M, \partial M) \cong K^a_*(M).$$

If $X$ is an arbitrary closed Riemannian manifold, then an application of these statements to the Spin$^c$-manifold $T^*X$ leads to the Poincaré duality isomorphisms

$$K^*(T^*X) = K^*(BT^*X, ST^*X) \cong K^a_*(X).$$

This isomorphism has a natural interpretation in terms of index theory. Namely, consider an arbitrary element $x \in K^0(BT^*X, ST^*X)$. It is given by a triple $(E_0, E_1, \sigma)$, where $E_0$ and $E_1$ are smooth vector bundles on $X$ and $\sigma$ is an isomorphism $\sigma: \pi^*E_0 \rightarrow \pi^*E_1$ of vector bundles. Without loss of generality one can assume that $\sigma$ is the principal symbol of some pseudodifferential operator $D_\sigma$. Then the class $[D_\sigma]$ in $K^a_*(X)$ determined by $D_\sigma$ coincides with the Poincaré dual of $x \in K^0(BT^*X, ST^*X)$. A similar construction holds for elements $x \in K^1(BT^*X, ST^*X)$.

For an arbitrary compact Spin$^c$-manifold $M$ with boundary $\partial M$ there is another Poincaré duality isomorphism

$$K^*(M) \cong K_*(M, \partial M).$$

If $X$ is an arbitrary closed Riemannian manifold, then an application of this isomorphism to the Spin$^c$-manifold $T^*X$ leads to the Poincaré duality isomorphisms

$$K_*, T^*X(X) = K_*(BT^*X, ST^*X) \cong K^*(X).$$
At the end of this subsection we present some information about the bivariant Kasparov $KK$-functor introduced in [39].

Let $A$ and $B$ be $C^*$-algebras. In this subsection we take an $(A, B)$-bimodule to be a $\mathbb{Z}_2$-graded countably generated Hilbert module $H$ equipped with an action of $A$ given by a representation $\rho: A \to \mathcal{B}(H)$ of it by even operators on $H$. Denote by $Q_A(H)$ the subalgebra of $\mathcal{B}(H)$ formed by operators $T \in \mathcal{B}(H)$ such that $[T, \rho(a)] \in \mathcal{K}(H)$ for any $a \in A$, and by $I_A(H)$ the ideal in $Q_A(H)$ which consists of operators $T \in \mathcal{B}(H)$ such that $T \rho(a) \in \mathcal{K}(H)$ and $\rho(a)T \in \mathcal{K}(H)$ for any $a \in A$.

We consider pairs $(H, F)$, where $H$ is an $(A, B)$-bimodule and $F \in Q_A(H)$ is an odd operator such that the operators $F - F^* + F^2 - I$ belong to $I_A(H)$. We say that pairs $(H_1, F_1)$ and $(H_2, F_2)$ are isomorphic if there exists a grading-preserving isometric isomorphism $u: H_1 \to H_2$ of $(A, B)$-bimodules such that $F_2 = uF_1u^{-1}$. The set of isomorphism classes of pairs $(H, F)$ is denoted by $E(A, B)$.

A homotopy between elements $(H_0, F_0) \in E(A, B)$ and $(H_1, F_1) \in E(A, B)$ is defined to be an element $(H, F) \in E(A, B[0, 1])$ whose restrictions to the extreme points $t = 0$ and $t = 1$ of the segment coincide with $(H_0, F_0)$ and $(H_1, F_1)$, respectively: $H_t = H \otimes_{B[0, 1]} B$, $F_t = T \otimes 1$. We define $KK(A, B)$ as the set of homotopy equivalence classes in $E(A, B)$.

The set $KK(A, B)$ is an Abelian group with respect to the direct sum operation: $(H_1, F_1) \oplus (H_2, F_2) = (H_1 \oplus H_2, F_1 \oplus F_2)$.

We denote by $KK_0(A, B)$ (respectively, $KK_1(A, B)$) the subset of $KK(A, B)$ given by the elements $(H, F) \in E(A, B)$ such that the operator $F$ is odd (respectively, even).

The groups $KK_i(\mathbb{C}, B)$ are naturally isomorphic to the groups $K_i(B)$ of topological $K$-theory. In some sense this isomorphism is an algebraic reformulation of the definition of the index of a family of elliptic operators. In the case when $B = C(X)$ is the algebra of continuous functions on a compact topological space $X$, a family $P$ of elliptic operators parameterized by the points of $X$ defines naturally an element of the group $KK(\mathbb{C}, B)$. The element of $K_0(B) = K(X)$ corresponding to this element under the isomorphism $KK(\mathbb{C}, B) \cong K(B)$ coincides with the index of the family $P$. For an arbitrary $C^*$-algebra $B$, elements of the group $KK_i(\mathbb{C}, B)$ can be constructed using elliptic pseudodifferential operators invariant under the action of $B$. An index theorem for elliptic operators over a $C^*$-algebra was proved by Mishchenko and Fomenko in [54] (see also [5]). It is also clear that in the case of $B = \mathbb{C}$ the definition of $KK_i(A, \mathbb{C})$ coincides with the definition of the $K$-homology groups $K^i(A)$.

The main technical tool is a bilinear pairing (the cup-cap product)

$$KK_i(A_1, B_1 \otimes D) \otimes KK_j(D \otimes A_2, B_2) \to KK_{i+j}(A_1 \otimes A_2, B_1 \otimes B_2)$$

which generalizes all known products in $K$-theory and $K$-homology. We will denote the cup-cap product of elements $x \in KK_i(A_1, B_1 \otimes D)$ and $y \in KK_j(D \otimes A_2, B_2)$ by $x \circ_D y$. The idea for the construction of the cup-cap product is borrowed from the theory of elliptic operators. It has many natural properties of a product: it is contravariant in $A_1$ and $A_2$, covariant in $B_1$ and $B_2$, functorial in $D$, and associative.
The latter means that
\[(x_1 \otimes D_1, x_2) \otimes D_2 x_3 = x_1 \otimes D_1 (x_2 \otimes D_2 x_3)\]
for any \(x_1 \in KK_i(A_1, B_1 \otimes D_1), x_2 \in KK_j(D_1 \otimes A_2, B_2 \otimes D_2), x_3 \in KK_\ell(D_2 \otimes A_3, B_3)\).

2.3. Non-commutative measure theory. The initial data of non-commutative measure theory are a pair \((\mathcal{M}, \phi)\) consisting of a von Neumann algebra \(\mathcal{M}\) and a weight \(\phi\) on \(\mathcal{M}\).

**Definition 2.10.** A von Neumann algebra is an involutive subalgebra of the algebra \(L(H)\) of bounded operators on a Hilbert space \(H\) which is closed in the weak operator topology.

**Definition 2.11.** A weight on a von Neumann algebra \(\mathcal{M}\) is a function \(\phi\) defined on the set \(\mathcal{M}_+\) of positive elements of \(\mathcal{M}\) with values in \(\mathbb{R}_+ = [0, +\infty]\) which satisfies the conditions
\[
\phi(a + b) = \phi(a) + \phi(b), \quad a, b \in \mathcal{M}_+, \\
\phi(\alpha a) = \alpha \phi(a), \quad \alpha \in \mathbb{R}_+, \quad a \in \mathcal{M}_+.
\]

A weight on a von Neumann algebra \(\mathcal{M}\) is called a trace if
\[
\phi(a^*a) = \phi(aa^*), \quad a \in \mathcal{M}_+.
\]

**Definition 2.12.** A weight \(\phi\) on a von Neumann algebra \(\mathcal{M}\) is
1) faithful if for any \(a \in \mathcal{M}_+\) the equality \(\phi(a) = 0\) implies that \(a = 0\);
2) normal if, for any bounded increasing net \(\{a_\alpha\}\) of elements of \(\mathcal{M}_+\) with the least upper bound \(a\), one has the equality \(\phi(a) = \sup_\alpha \phi(a_\alpha)\);
3) semifinite if the linear span of the set \(\{x \in \mathcal{M}_+: \phi(x) < \infty\}\) is \(\sigma\)-weakly dense in \(\mathcal{M}\).

Any von Neumann algebra has a faithful normal semifinite weight. A von Neumann algebra is said to be semifinite if it has a faithful normal semifinite trace.

**Example 2.13.** The usual trace \(\text{tr}\) on the von Neumann algebra \(L(H)\) of bounded linear operators on a Hilbert space \(H\) is a faithful normal semifinite trace. Moreover, for any bounded positive operator \(T\) on \(H\) the functional \(\phi_T(A) = \text{tr} A T, A \in L(H)\), is a faithful normal semifinite weight on \(L(H)\).

**Example 2.14.** If there is given a \(\sigma\)-finite measure \(\mu\) on a measurable space \(X\), then the elements of the space \(L^\infty(X, \mu)\), regarded as multiplication operators acting in the Hilbert space \(L^2(X, \mu)\), form a von Neumann algebra. Moreover, the equality
\[
\phi(f) = \int_X f(x) \, d\mu(x), \quad f \in L^\infty(X, \mu),
\]
defines a faithful normal semifinite trace on \(L^\infty(X, \mu)\).
We recall that for any subset $S$ of $L(H)$ its commutant is the set of bounded operators on $H$ which commute with all operators in $S$. An involutive subalgebra $\mathcal{M}$ of $L(H)$ is a von Neumann algebra if and only if $\mathcal{M}'' = \mathcal{M}$. An unbounded linear operator $T$ acting in a Hilbert space $H$ is said to be affiliated with a von Neumann algebra $\mathcal{M}$ acting in $H$ if $Tu = uT$ for any unitary operator on $H$ belonging to the commutant $\mathcal{M}'$ of $\mathcal{M}$.

Let $\mathcal{N}$ be a semifinite von Neumann algebra and $\tau$ a faithful normal semifinite trace on $\mathcal{N}$. The norm-closed two-sided ideal in $\mathcal{N}$ generated by elements $E \in \mathcal{N}$ with $\tau(E) < \infty$ will be denoted by $\mathcal{K}_\mathcal{N}$. Its elements are called $\tau$-compact operators. A Breuer–Fredholm operator is defined to be any operator $P \in \mathcal{N}$ whose image under the canonical map $\pi: \mathcal{N} \to \mathcal{N}/\mathcal{K}_\mathcal{N}$ is invertible in the algebra $\mathcal{N}/\mathcal{K}_\mathcal{N}$. The Breuer–Fredholm index of a Breuer–Fredholm operator $F \in \mathcal{N}$ is defined by the formula

$$\text{Ind}(F) = \tau(P_{\text{Ker}} F) - \tau(P_{\text{Coker}} F),$$

where $P_{\text{Ker}} F$ and $P_{\text{Coker}} F$ are the projections on the kernel and cokernel of $F$, respectively. The theory of Breuer–Fredholm operators was developed in the papers [55] and [56] in the case when $\mathcal{N}$ is a factor, and was extended to the case when $\mathcal{N}$ is not a factor in [57].

3. Non-commutative differential geometry

3.1. Cyclic cohomology and homology. In this subsection we give a definition of cyclic cohomology and homology, which play the role of non-commutative analogues of the homology and cohomology of topological spaces (for cyclic cohomology, see the books [19], [58], [59] and the references cited therein). It is important to remark that the definition of the de Rham homology and cohomology of a topological space requires the introduction of an additional structure on this space, for instance, the structure of a smooth manifold. In the non-commutative case this results in the fact that cyclic cocycles are usually defined not on a $C^*$-algebra, the analogue of the algebra of continuous functions, but on a certain subalgebra of it which consists of ‘smooth’ functions. For now we postpone (see § 3.4) a discussion of the non-commutative analogue of the algebra of smooth functions on a smooth manifold, the notion of smooth algebra, and turn to the definition of the cyclic cohomology for an arbitrary algebra.

Let $\mathcal{A}$ be an algebra over $\mathbb{C}$. We consider the complex $(C^*(\mathcal{A}, \mathcal{A}^*), b)$, where

(i) $C^k(\mathcal{A}, \mathcal{A}^*)$, $k \in \mathbb{N}$, is the space of $(k+1)$-linear forms on $\mathcal{A}$;

(ii) the coboundary $b\psi \in C^{k+1}(\mathcal{A}, \mathcal{A}^*)$ of an element $\psi \in C^k(\mathcal{A}, \mathcal{A}^*)$ is given by

$$b\psi(a^0, \ldots, a^{k+1}) = \sum_{j=0}^{k} (-1)^j \psi(a^0, \ldots, a^j a^{j+1}, \ldots, a^{k+1}) + (-1)^{k+1} \psi(a^{k+1} a^0, \ldots, a^k), \quad a^0, a^1, \ldots, a^{k+1} \in \mathcal{A}.$$ 

The cohomology of this complex is called the Hochschild cohomology of the algebra $\mathcal{A}$ with coefficients in the bimodule $\mathcal{A}^*$ and is denoted by $HH(\mathcal{A})$. 
Let $C^{k}_\lambda(A)$ be the subspace of $C^{k}(A, A^*)$ which consists of all $\psi \in C^{k}(A, A^*)$ satisfying the cyclicity condition
\[
\psi(a^1, \ldots, a^k, a^0) = (-1)^k \psi(a^0, a^1, \ldots, a^k), \quad a^0, a^1, \ldots, a^k \in A. \quad (3.1)
\]
The differential $b$ maps the subspace $C^{k}_\lambda(A)$ to $C^{k+1}_\lambda(A)$, and the cyclic cohomology $HC^\ast(A)$ of $A$ is defined as the cohomology of the complex $(C^{\ast}_\lambda(A), b)$.

**Example 3.1.** For $k = 0$ the space $HC^0(A)$ coincides with the space of all trace functionals on $A$. For this reason cyclic $k$-cocycles on $A$ are called $k$-traces on $A$ (for $k > 0$, higher traces).

**Example 3.2.** If $A = \mathbb{C}$, then $HC^n(\mathbb{C}) = 0$ if $n$ is odd and $HC^n(\mathbb{C}) = \mathbb{C}$ if $n$ is even. For odd $n$ a non-trivial cocycle $\phi \in C^n(A)$ is given by
\[
\phi(a^0, a^1, \ldots, a^n) = a^0 a^1 \cdots a^n, \quad a^0, a^1, \ldots, a^n \in \mathbb{C}.
\]
Equivalently, the cyclic cohomology can be described using a $(b, B)$-bicomplex. We define an operator $B: C^k(A, A^*) \to C^{k-1}(A, A^*)$ by
\[
B \psi(a^0, \ldots, a^{k-1}) = \psi(a^1, a^0, \ldots, a^{k-1} - (1)^k \psi(a^0, \ldots, a^{k-1}, 1).
\]
One has $B^2 = 0$ and $BB = -Bb$.

Consider the following double complex:
\[
C^{n,m} = C^{n-m}(A, A^*), \quad n, m \in \mathbb{Z},
\]
with differentials $d_1: C^{n,m} \to C^{n+1,m}$ and $d_2: C^{n,m} \to C^{n,m+1}$ given by
\[
d_1 \psi = (n - m + 1)b \psi, \quad d_2 \psi = \frac{1}{n - m} B \psi, \quad \psi \in C^{n,m}.
\]
For any $q \in \mathbb{N}$ consider the complex $(F^qC, d)$, where
\[
(F^qC)^p = \bigoplus_{\begin{subarray}{c}m \geq q \n m + m = p \end{subarray}} C^{n,m}, \quad p \in \mathbb{N}, \quad d = d_1 + d_2.
\]
Then one has the isomorphism
\[
HC^n(A) \cong H^p(F^qC), \quad n = p - 2q.
\]
This isomorphism associates with any $\psi \in HC^n(A)$ a cocycle $\phi \in H^p(F^qC)$ with arbitrary fixed $p$ and $q$ satisfying $n = p - 2q$ which has a single non-vanishing
component $\phi_{p,q} = (-1)^{[n/2]}\psi$. In particular, any cocycle of the complex $(F^qC, d)$ is cohomologous to a cocycle of the above form.

The periodic cyclic cohomology groups $HP^\text{ev}(\mathcal{A})$ and $HP^\text{odd}(\mathcal{A})$ can also be defined by taking the inductive limit of the groups $HC^k(\mathcal{A})$, $k \geq 0$, with respect to the periodicity operator $S: HC^k(\mathcal{A}) \to HC^{k+2}(\mathcal{A})$. For any $\psi \in C^k(\mathcal{A})$ one has

$$S\psi(a^0, \ldots, a^{k+2}) = \frac{1}{(k+1)(k+2)} \left( \sum_{j=1}^{k+1} \psi(a^0, \ldots, a^{j-1}a^j a^{j+1}, \ldots, a^{k+2}) - \sum_{0 \leq i < j \leq k+1} \psi(a^0, \ldots, a^{i-1}a^i, \ldots, a^j a^{j+1}, \ldots, a^{k+2}) \right).$$

In terms of the $(b, B)$-bicomplex the periodic cyclic cohomology is described as the cohomology of the complex

$$C^b(\mathcal{A}) \xrightarrow{\partial + B} C^b(\mathcal{A}) \xrightarrow{\partial + B} C^b(\mathcal{A}),$$

where

$$C^{ev/odd}(\mathcal{A}) = \bigoplus_{even/odd k} C^k(\mathcal{A}).$$

**Example 3.3.** Let $\mathcal{A}$ be the locally convex topological algebra $C^\infty(M)$ of smooth functions on an $n$-dimensional compact manifold $M$, and let $\mathcal{D}_k(M)$ denote the space of $k$-dimensional de Rham currents on $M$. The Hochschild cohomology and the cyclic cohomology of $\mathcal{A}$ are computed in [14]. The following map $\varphi \mapsto C_\varphi$ defines an isomorphism of the continuous Hochschild cohomology group $HH^k(\mathcal{A})$ with the space $\mathcal{D}_k(M)$:

$$\langle C_\varphi, f^0 df^1 \wedge \cdots \wedge df^k \rangle = \frac{1}{k!} \sum_{\sigma \in S_k} \varepsilon(\sigma)\varphi(f^0, f^{\sigma(1)}, \ldots, f^{\sigma(k)}),$$

$$f^0, f^1, \ldots, f^k \in C^\infty(M).$$

One has $C_{B,\varphi} = k dt C_\varphi$, where $dt$ is the de Rham boundary for currents. Therefore, for any $k$ the continuous cyclic cohomology group $HC^k(\mathcal{A})$ is canonically isomorphic to the direct sum

$$\text{Ker } dt \oplus H_{k-2}(M, \mathbb{C}) \oplus H_{k-4}(M, \mathbb{C}) \oplus \cdots,$$

where $H_k(M, \mathbb{C})$ denotes the usual de Rham homology. The continuous periodic cyclic cohomology $HP^{ev/odd}(\mathcal{A})$ is canonically isomorphic to the de Rham homology

$$H^{ev/odd}(M, \mathbb{C}) = \bigoplus_{even/odd k} H_k(M, \mathbb{C}).$$

**Example 3.4.** We recall that a (homogeneous) $k$-cocycle on a discrete group $\Gamma$ is a map $h: \Gamma^{k+1} \to \mathbb{C}$ satisfying

$$h(\gamma_0, \ldots, \gamma_k) = h(\gamma_0, \ldots, \gamma_k), \quad \gamma, \gamma_0, \ldots, \gamma_k \in \Gamma;$$

$$\sum_{i=0}^{k+1} (-1)^i h(\gamma_0, \ldots, \gamma_{i-1}, \gamma_{i+1}, \ldots, \gamma_{k+1}) = 0, \quad \gamma_0, \ldots, \gamma_{k+1} \in \Gamma.$$
With any homogeneous $k$-cocycle $h$ we can associate a (non-homogeneous) $k$-cocycle $c \in Z^k(\Gamma, \mathbb{C})$ by

$$c(\gamma_1, \ldots, \gamma_k) = h(e, \gamma_1, \gamma_1 \gamma_2, \ldots, \gamma_1 \cdots \gamma_k).$$

It is easy to check that $c$ satisfies the following condition:

$$c(\gamma_1, \gamma_2, \ldots, \gamma_k) + \sum_{i=0}^{k-1} (-1)^{i+1} c(\gamma_0, \ldots, \gamma_i \gamma_{i+1}, \gamma_i, \gamma_{i+2}, \ldots, \gamma_k) + (-1)^{k+1} c(\gamma_0, \gamma_1, \ldots, \gamma_k) = 0.$$ 

We say that a cocycle $c \in Z^k(\Gamma, \mathbb{C})$ is normalized (in the sense of Connes) if $c(\gamma_1, \ldots, \gamma_k)$ equals zero in the case when either $\gamma_i = e$ for some $i$ or $\gamma_1 \cdots \gamma_k = e$. Every cohomology class in $H^k(\Gamma, \mathbb{C})$ can be represented by a normalized cocycle.

The group ring $\mathbb{C}[\Gamma]$ consists of all functions $f : \Gamma \to \mathbb{C}$ with finite support. Multiplication in $\mathbb{C}[\Gamma]$ is given by the convolution

$$f_1 * f_2(\gamma) = \sum_{\gamma_1 \gamma_2 = \gamma} f_1(\gamma_1) f_2(\gamma_2), \quad \gamma \in \Gamma.$$ 

A normalized cocycle $c \in Z^k(\Gamma, \mathbb{C})$ determines a cyclic $k$-cocycle $\tau_c$ on $\mathbb{C}[\Gamma]$ by the formula

$$\tau_c(f_0, \ldots, f_k) = \sum_{\gamma_0 \cdots \gamma_k = e} f_0(\gamma_0) \cdots f_k(\gamma_k) c(\gamma_1, \ldots, \gamma_k), \quad f_0, f_1, \ldots, f_k \in \mathbb{C}[\Gamma].$$ 

Let us now recall the definition of cyclic homology for an algebra $\mathcal{A}$ over $\mathbb{C}$. Denote by $A^{\otimes, k+1}$ the tensor product of $k + 1$ copies of $A$ and consider the endomorphism $t$ of $A^{\otimes, k+1}$ given by

$$t(a^0 \otimes a^1 \otimes \cdots \otimes a^k) = (-1)^k a^0 \otimes \cdots \otimes a^k \otimes a^0, \quad a^0, a^1, \ldots, a^k \in A.$$ 

Let us consider also the map $b$ from $A^{\otimes, k+1}$ into $A^{\otimes, k}$ defined by the formula

$$b(a^0 \otimes \cdots \otimes a^k) = \sum_{j=0}^{k-1} (-1)^j a^0 \otimes \cdots \otimes a^j a^{j+1} \otimes \cdots \otimes a^k$$

$$+ (-1)^k a^k a^0 \otimes \cdots \otimes a^{k-1}, \quad a^0, a^1, \ldots, a^k \in \mathcal{A}.$$ 

We put

$$C^\lambda_k(\mathcal{A}) = \frac{A^{\otimes, k+1}}{\text{im}(Id - t)}.$$ 

The differential $b$ defines a map $b$ from $C^\lambda_{k+1}(\mathcal{A})$ to $C^\lambda_k(\mathcal{A})$, and the cyclic homology $HC_* (\mathcal{A})$ of $\mathcal{A}$ is defined as the homology of the complex $(C^\lambda_* (\mathcal{A}), b)$. 

If $A$ is a unital locally $m$-convex Fréchet algebra, that is, a unital algebra which is a locally convex topological Fréchet vector space such that the product is continuous, then the topological cyclic cohomology groups $HC^k(A)$ are defined in the same way as above, using continuous $(k+1)$-linear functionals. Similarly, the topological cyclic homology groups $HC_k(A)$ are defined in the same way as above, using completed projective tensor products.
3.2. Non-commutative differential forms. Let $A$ be a unital algebra. A differential graded algebra is a graded algebra

$$\Omega_*(A) = \Omega_0(A) \oplus \Omega_1(A) \oplus \Omega_2(A) \oplus \cdots$$
endowed with a linear differentiation $d$ of degree 1. Thus, for any $j$ and $k$ one has $\Omega_j(A)\Omega_k(A) \subset \Omega_{j+k}(A)$. The operator $d$ defines a map

$$d: \Omega_j(A) \to \Omega_{j+1}(A), \quad j \geq 0,$$
and satisfies the conditions $d^2 = 0$ and

$$d(\omega_j \cdot \omega_k) = d\omega_j \cdot \omega_k + (-1)^j \omega_j \cdot d\omega_k, \quad \omega_j \in \Omega_j(A), \ \omega_k \in \Omega_k(A).$$

We denote by $[\Omega_*(A), \Omega_*(A)]_l$ the linear subspace spanned by the graded commutators $[\omega_j, \omega_k] = \omega_j \cdot \omega_k - (-1)^j \omega_k \cdot \omega_j$, where $j + k = l$ and $\omega_j \in \Omega_j(A)$, $\omega_k \in \Omega_k(A)$. Let

$$\bar{\Omega}_l(A) = \frac{\Omega_l(A)}{[\Omega_*(A), \Omega_*(A)]_l}.$$

The differential $d$ induces a linear differential, also denoted by $d$, on the graded vector space $\bar{\Omega}_*(A) = \bigoplus_l \bar{\Omega}_l(A)$. We denote by $\bar{H}_*(A)$ the homology of this complex and call it the non-commutative de Rham homology of the algebra $\Omega_*(A)$.

For a unital locally $m$-convex Fréchet algebra $A$ one can define the completion $\hat{\Omega}_*(A)$ of $\Omega_*(A)$, which is a differential graded Fréchet algebra. Then one can define the non-commutative topological de Rham cohomology $\hat{H}_*(A)$ as the homology of the complex $\left(\hat{\Omega}_*(A)/[\hat{\Omega}_*(A), \hat{\Omega}_*(A)], d\right)$.

An example of a differential graded algebra is the universal differential graded algebra $\Omega A$ of a unital algebra $A$. We recall its construction. The algebra $\Omega A$ is a graded algebra: $\Omega A = \bigoplus_{p=0}^\infty \Omega^p A$. In degree 0 the space $\Omega^0 A$ coincides with $A$. In degree 1 the space $\Omega^1 A$ is generated as a left $A$-module by symbols of degree 1 of the form $\delta a$, where $a \in A$, satisfying the relations

$$\delta(ab) = \delta(a)b + a\delta(b), \quad a, b \in A,$$
$$\delta(\alpha a + \beta b) = \alpha \delta(a) + \beta \delta(b), \quad a, b \in A, \ \alpha, \beta \in \mathbb{C}.$$ 

In particular, one has $\delta(1) = 0$. Thus, a general element of the space $\Omega^1 A$ has the form $a = \sum a_i \delta b_i$, where $a_i, b_i \in A$.

The algebra $\Omega A$ is generated as an algebra by the elements of $\Omega^1 A$. In particular,

$$(a_0 \delta a_1)(b_0 \delta b_1) = a_0 \delta(a_1 b_0)\delta b_1 - a_0 a_1 \delta b_0 \delta b_1.$$ 

Therefore, an arbitrary element of $\Omega A$ is represented as a finite linear combination of elements of the form $a_0 \delta a_1 \delta a_2 \cdots \delta a_p$ with some $a_0, a_1, a_2, \ldots, a_p \in A$. Left and right multiplications by elements of $\Omega^0 A = A$ yield an $A$-$A$-bimodule structure on $\Omega A$.

A differential $\delta: \Omega^p A \to \Omega^{p+1} A$ is well defined by

$$\delta(a_0 \delta a_1 \delta a_2 \cdots \delta a_p) = \delta a_0 \delta a_1 \delta a_2 \cdots \delta a_p, \quad a_0, a_1, a_2, \ldots, a_p \in A.$$
If $A$ has an involution $a \mapsto a^*$, then the algebra $\Omega A$ also has a natural involution
algebra structure.

The non-commutative de Rham homology of the universal enveloping algebra $\Omega A$
is called the non-commutative de Rham homology of $A$ and denoted by $\text{HDR}_*(A)$. It
is closely related to the cyclic homology $HC_*(A)$ of $A$. Namely, for $n > 0$ the
non-commutative de Rham homology group $\text{HDR}_n(A)$ coincides with the kernel of
$B: HC_n(A) \to HH_{n+1}(A)$ (see (3.2)). Here $HC_n(A)$ and $HH_{n+1}(A)$ denote the
reduced cyclic homology and reduced Hochschild homology, respectively (for more
details, see [58] and [59]).

The use of differential graded algebras enables one to give a general construction
of cyclic cocycles on an arbitrary algebra [14]. We define a cycle of dimension $n$
to be a triple $(\Omega, d, f)$, where $(\Omega = \bigoplus_{j=0}^n \Omega^j, d)$ is a differential graded algebra
and $f$ is a closed graded trace on $(\Omega, d)$ of degree $n$. Here by a closed graded
trace of degree $n$ we mean a linear functional $f: \Omega^n \to \mathbb{C}$ satisfying the following
conditions:

1) $\int \omega_2 \omega_1 = (-1)^{jk} \int \omega_1 \omega_2$ for $\omega_1 \in \Omega^j$, $\omega_2 \in \Omega^k$;
2) $\int df = 0$ for $\omega \in \Omega^{n-1}$.

Let $A$ be an algebra over $\mathbb{C}$. A cycle over $A$ is defined to be a cycle $(\Omega, d, f)$
along with a homomorphism $\rho: A \to \Omega^0$. For any cycle $(A \xrightarrow{\rho} \Omega, d, f)$ over $A$ we
can define its character by the formula

$$\tau(a_0, a_1, \ldots, a_n) = \int \rho(a_0) d(\rho(a_1)) \cdots d(\rho(a_n)), \quad a_0, a_1, a_2, \ldots, a_n \in A.$$ 

It is easy to check that $\tau$ is a cyclic cocycle on $A$. Moreover, one can show that any
cyclic cocycle on $A$ is the character of some cycle over $A$.

**Example 3.5.** Let $M$ be a smooth manifold without boundary, and consider the
graded algebra $\Omega$ of smooth differential forms on $M$: $\Omega = \bigoplus_{p=0}^{\infty} \Omega^p$, where $\Omega^p = C^\infty(M, \Lambda^p TM \otimes \mathbb{C})$. The de Rham differential $d$ makes $\Omega$ into a differential graded
algebra. Finally, the linear functional $I(\omega) = \int_M \omega$ is a closed graded trace of degree
$n$ on $\Omega$. The corresponding cyclic cocycle on $C^\infty_c(M)$ is given by

$$\tau(f^0, f^1, \ldots, f^n) = \int_M f^0 df^1 \wedge \cdots \wedge df^n, \quad f^0, f^1, \ldots, f^n \in C^\infty_c(M).$$

Moreover, any closed $k$-dimensional de Rham current $C \in \mathcal{D}_k(M)$ on $M$ defines
a closed graded trace of degree $k$ on $\Omega$:

$$I_C(\omega) = \langle C, \omega \rangle, \quad \omega \in \Omega^k.$$ 

The character of the cycle $(\omega, d, I_C)$ over $C^\infty_c(M)$ coincides with the cyclic cocycle
on $C^\infty_c(M)$ given by

$$\psi_C(f^0, f^1, \ldots, f^k) = \langle C, f^0 df^1 \wedge \cdots \wedge df^k \rangle, \quad f^0, f^1, \ldots, f^k \in C^\infty_c(M). \quad (3.3)$$

**Example 3.6.** Let $\Gamma$ be a discrete group. The universal differential graded algebra
$\Omega^*(\Gamma) = \Omega \mathbb{C} \Gamma$ of the group algebra $\mathbb{C} \Gamma$ consists of finite linear combinations
of symbols of the form $\gamma_0 d \gamma_1 \cdots d \gamma_n$ with some $\gamma_0, \gamma_1, \ldots, \gamma_n \in \Gamma$. Any normalized
cyclole $c \in Z^k(\Gamma, \mathbb{C})$ (see Example 3.4) defines a closed graded trace on $\Omega^*(\Gamma)$ by the formula

$$\int \gamma_0 d\gamma_1 \cdots d\gamma_n = \begin{cases} c(\gamma_1, \ldots, \gamma_k) & \text{if } n = k \text{ and } \gamma_0 \gamma_1 \cdots \gamma_n = e, \\ 0 & \text{otherwise.} \end{cases}$$

The character of this cycle coincides with the cyclic cocycle $\tau_c$.

### 3.3. Non-commutative Chern–Weil construction.

For an algebra $\mathcal{A}$ over $\mathbb{C}$ the Chern character in $K$-homology was constructed in [58] as a map $\text{ch} : K_0(A) \to \text{HDR}_{\text{ev}}(\mathcal{A})$. This construction is a straightforward generalization of the classical Chern–Weil construction and makes use of the notions of connection and curvature for finitely generated projective $\mathcal{A}$-modules.

In the cohomological setting a non-commutative analogue of the Chern–Weil construction [14] is a construction of a pairing between $HC^*(\mathcal{A})$ and $K_*(\mathcal{A})$ for an arbitrary algebra $\mathcal{A}$.

The pairing between $HC^\text{ev}(\mathcal{A})$ and $K_0(\mathcal{A})$ is defined as follows. For any cocycle $\varphi = (\varphi_{2k})$ in $C^\text{ev}(\mathcal{A})$ and for any idempotent $e$ in $M_q(\mathcal{A})$ put

$$\langle [\varphi], [e] \rangle = \sum_{k \geq 0} (-1)^k \frac{(2k)!}{k!} \varphi_{2k} \# \text{Tr} \left( e - \frac{1}{2}, e, \ldots, e \right), \quad (3.4)$$

where $\varphi_{2k} \# \text{Tr}$ is the $(2k + 1)$-linear map on $M_q(\mathcal{A}) = M_q(\mathbb{C}) \otimes \mathcal{A}$ given by

$$\varphi_{2k} \# \text{Tr}(\mu^0 \otimes a^0, \ldots, \mu^{2k} \otimes a^{2k}) = \text{Tr}(\mu^0 \cdots \mu^{2k}) \varphi_{2k}(a^0, \ldots, a^{2k}) \quad (3.5)$$

for any $\mu^j \in M_q(\mathbb{C})$ and $a^j \in \mathcal{A}$.

The pairing between $HC^\text{odd}(\mathcal{A})$ and $K_1(\mathcal{A})$ is given by

$$\langle [\varphi], [U] \rangle = \frac{1}{\sqrt{2i\pi}} \sum_{k \geq 0} (-1)^k k! \varphi_{2k+1} \# \text{Tr}(U^{-1} - 1, U - 1, \ldots, U^{-1} - 1, U - 1), \quad (3.6)$$

where $\varphi = (\varphi_{2k+1}) \in G^\text{odd}(\mathcal{A})$ and $U \in U_q(\mathcal{A})$.

It is important in index theory that the index maps (2.1) and (2.2) associated with a Fredholm module $(H, F)$ satisfying an additional $p$-summability condition can be computed in terms of the pairing of elements of $K^* (\mathcal{A})$ with a certain cyclic cohomology class $\tau = \text{ch}_*(H, F) \in HC^n(\mathcal{A})$ called the Chern character of the Fredholm module $(H, F)$ [14].

We recall that for any $p \geq 1$ the Schatten class $L^p(H)$ consists of all compact operators $T$ on a Hilbert space $H$ such that $|T|^p$ is a trace class operator. Let $\mu_1(T) \geq \mu_2(T) \geq \cdots$ be the singular numbers (the s-numbers) of a compact operator $T$ on $H$, that is, the eigenvalues of $|T| = \sqrt{T^*T}$ taken with multiplicities. Then

$$T \in L^p(H) \iff \text{tr} |T|^p = \sum_{n=1}^{\infty} |\mu_n(T)|^p < \infty.$$

**Definition 3.7.** A Fredholm module $(H, F)$ over an algebra $\mathcal{A}$ is $p$-summable if $(F^2 - 1)\rho(a), (F - F^*)\rho(a), \text{and } [F, \rho(a)]$ belong to $L^p(H)$ for any $a \in \mathcal{A}$. 
Let \((H, F)\) be a Fredholm module over an algebra \(\mathcal{A}\), and assume that the module is \((p + 1)\)-summable and is even if \(p\) is even. It determines a cycle over the algebra \(\mathcal{A}\) as follows. First of all, one can assume without loss of generality that \(F^2 = 1\). We construct a graded algebra \(\Omega = \bigoplus_{j=0}^{n} \Omega^j\). For \(k = 0\) put \(\Omega^0 = \mathcal{A}\). For \(k > 0\) the space \(\Omega^k\) is the linear span of the bounded operators on \(H\) of the form \(\omega = a^0[F, a^1] \cdots [F, a^k]\), where \(a^0, a^1, \ldots, a^k \in \mathcal{A}\). The product in \(\Omega\) is given by the product of operators. The differential \(d: \Omega \to \Omega\) is defined by
\[
d\omega = F\omega - (-1)^k \omega F, \quad \omega \in \Omega^k.
\]

Let us define a closed graded trace \(\text{Tr}_s: \Omega^n \to \mathbb{C}, n > p\). For any operator \(T\) on \(H\) such that \(FT + TF \in \mathcal{L}^1(H)\) we put
\[
\text{Tr}'(T) = \frac{1}{2} \text{Tr}(F(FT + TF)).
\]

Note that \(\text{Tr}'(T) = \text{Tr}(T)\) if \(T \in \mathcal{L}^1(H)\). For any \(\omega \in \Omega^n\) put \(\text{Tr}_s \omega = \text{Tr}'(\omega)\) if \(n\) is odd and \(\text{Tr}_s \omega = \text{Tr}(\gamma \omega)\) if \(n\) is even, where \(\gamma\) denotes the grading operator on \(H\). The character \(\tau_n\) of the cycle described above is called the Chern character of the Fredholm module \((H, F)\). Under the condition \(F^2 = 1\), \(\tau_n\) is given for odd \(n > p\) by
\[
\tau_n(a^0, a^1, \ldots, a^n) = \lambda_n \text{tr}(a^0[F, a^1] \cdots [F, a^n]), \quad a^0, a^1, \ldots, a^n \in \mathcal{A},
\]
and for even \(n > p\) by
\[
\tau_n(a^0, a^1, \ldots, a^n) = \lambda_n \text{tr}(\gamma a^0[F, a^1] \cdots [F, a^n]), \quad a^0, a^1, \ldots, a^n \in \mathcal{A},
\]
where the \(\lambda_n\) are constants depending only on \(n\).

For different \(n > p\) the characters \(\tau_n\) are compatible in the sense that \(S\tau_n = \tau_{n+2}\). Therefore, the class \(\tau = \text{ch}_s(H, F) \in HP^*(\mathcal{A})\) is well defined in the periodic cyclic cohomology of the algebra \(\mathcal{A}\).

### 3.4. Smooth algebras.

In this subsection we give some general facts about smooth subalgebras of \(C^*\)-algebras that amount to a non-commutative analogue of the algebra of smooth functions on a smooth manifold. Let \(A\) be a \(C^*\)-algebra, and \(A^+\) the algebra obtained by adjoining a unit to \(A\). Let \(\mathcal{A}\) be a \(*\)-subalgebra of \(A\), and \(\mathcal{A}^+\) the algebra obtained by adjoining the unit to \(\mathcal{A}\).

**Definition 3.8.** We say that \(\mathcal{A}\) is a smooth subalgebra of a \(C^*\)-algebra \(A\) if \(\mathcal{A}\) is a dense \(*\)-subalgebra of \(A\) that is stable under the holomorphic functional calculus, that is, for any \(a \in \mathcal{A}^+\) and for any function \(f\) holomorphic in a neighbourhood of the spectrum of \(a\) (regarded as an element of the algebra \(A^+)\) we have \(f(a) \in \mathcal{A}^+\).

Suppose that \(\mathcal{A}\) is a dense \(*\)-subalgebra of \(A\) endowed with the structure of a Fréchet algebra whose topology is finer than the topology induced by the topology of \(A\). A necessary and sufficient condition for \(\mathcal{A}\) to be a smooth subalgebra is given by the spectral invariance condition (see [60], Lemma 1.2):

- \(\mathcal{A}^+ \cap GL(A^+) = GL(\mathcal{A}^+)\), where \(GL(\mathcal{A}^+)\) and \(GL(A^+)\) denote the groups of invertible elements in \(\mathcal{A}^+\) and \(A^+\), respectively.
This fact remains true in the case when $\mathcal{A}$ is a locally multiplicatively convex Fréchet algebra (that is, its topology is given by a countable family of submultiplicative seminorms) such that the group $GL(\mathcal{A})$ of invertibles is open ([60], Lemma 1.2).

If $\mathcal{A}$ is a smooth subalgebra of a $C^*$-algebra $A$, then for any $n$ the algebra $M_n(\mathcal{A})$ is a smooth subalgebra of the $C^*$-algebra $M_n(A)$. In particular, $GL_n(\mathcal{A}^+)$ coincides with the intersection $M_n(\mathcal{A}^+) \cap GL_n(A^+)$. Let us regard $GL_n(\mathcal{A}^+)$ as the inductive limit of the topological groups $GL_n(\mathcal{A}^+)$. Denote by $GL\infty(\mathcal{A}^+)$ the inductive limit of the topological groups $GL_n(\mathcal{A}^+)$. One of the most important properties of smooth subalgebras is the following fact, an analogue of smoothing in the operator $K$-theory (see [61], § VI.3, and [62]).

**Theorem 3.9.** If $\mathcal{A}$ is a smooth subalgebra of a $C^*$-algebra $A$, then the inclusion $\mathcal{A} \hookrightarrow A$ induces isomorphisms

$$K_0(\mathcal{A}) \cong K_0(A) \quad \text{and} \quad \pi_n(GL\infty(\mathcal{A}^+)) \to \pi_n(GL\infty(A^+)) = K_{n+1}(A).$$

For any $p$-summable Fredholm module $(H, F)$ over an algebra $\mathcal{A}$ the algebra

$$\mathcal{C} = \{ T \in \mathcal{A} : [F, T] \in \mathcal{L}^p(H) \},$$

where $\mathcal{A}$ denotes the uniform closure of $\mathcal{A}$ in $\mathcal{L}(H)$, is a smooth subalgebra of the $C^*$-algebra $\mathcal{A}$. Moreover, one can show that the cocycle $\tau_n$ on $\mathcal{A}$ defined by (3.7) and (3.8) extends by continuity to a cyclic cocycle on $\mathcal{C}$. This along with Theorem 3.9 allows us to assert that the pairing with $\text{ch}^*_p(H, F) \in HC^n(\mathcal{A})$ defines a map $K_*(\mathcal{A}) \cong K_*(\mathcal{C}) \to \mathbb{Z}$. This fact is called the topological invariance property of the Chern character $\text{ch}^*_p(H, F)$ of the Fredholm module $(H, F)$.

### 3.5. Non-commutative Riemannian geometry.

According to [63] and [64], the initial data of non-commutative Riemannian geometry is a spectral triple.

**Definition 3.10.** A spectral triple is a set $(\mathcal{A}, \mathcal{H}, D)$, where

1) $\mathcal{A}$ is an involutive algebra;
2) $\mathcal{H}$ is a Hilbert space equipped with a $*$-representation of $\mathcal{A}$;
3) $D$ is an (unbounded) self-adjoint operator acting in $\mathcal{H}$ such that
   
   (i) for any $a \in \mathcal{A}$ the operator $a(D - i)^{-1}$ is a compact operator on $\mathcal{H}$;
   (ii) $D$ almost commutes with elements of $\mathcal{A}$ in the sense that $[D, a]$ is bounded for any $a \in \mathcal{A}$.

A spectral triple is said to be even if $\mathcal{H}$ is endowed with a $\mathbb{Z}_2$-grading $\gamma \in \mathcal{L}(\mathcal{H})$, $\gamma = \gamma^*$, $\gamma^2 = 1$, and moreover, $\gamma D = -D \gamma$ and $\gamma a = a \gamma$ for any $a \in \mathcal{A}$. Otherwise the spectral triple is said to be odd.

Spectral triples were considered for the first time in the paper [65], where they were called unbounded Fredholm modules. A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ determines a Fredholm module $(\mathcal{H}, F)$ over $\mathcal{A}$, where $F = D(I + D^2)^{-1/2} [65]$. In a certain sense the operator $F$ is connected with measurement of angles and is responsible for a conformal structure, whereas $|D|$ is connected with measurement of lengths.
Definition 3.11. A spectral triple \((\mathcal{A}, \mathcal{H}, D)\) is \(p\)-summable (or \(p\)-dimensional) if for any \(a \in \mathcal{A}\) the operator \(a(D - i)^{-1}\) is an element of the Schatten class \(L^p(\mathcal{H})\).

A spectral triple \((\mathcal{A}, \mathcal{H}, D)\) is said to be finite-dimensional if it is \(p\)-summable for some \(p\).

The greatest lower bound of all \(p\) such that a finite-dimensional spectral triple is \(p\)-summable is called the dimension of the spectral triple.

The dimension of a spectral triple \((\mathcal{A}, \mathcal{H}, D)\) coincides with the dimension of the corresponding Fredholm module \((\mathcal{H}, F)\), \(F = D(I + D^2)^{-1/2}\), over \(\mathcal{A}\) [65].

Classical Riemannian geometry is described by the spectral triple \((\mathcal{A}, \mathcal{H}, D)\) associated with a compact Riemannian spin manifold \((M, g)\):

1) the involutive algebra \(\mathcal{A}\) is the algebra \(C^\infty(M)\) of smooth functions on \(M\);
2) the Hilbert space \(\mathcal{H}\) is the space \(L^2(M, F(TM))\), where the algebra \(\mathcal{A}\) acts by multiplication;
3) the operator \(D\) is the spin Dirac operator.

The Weyl asymptotic formula for eigenvalues of self-adjoint elliptic operators on a compact manifold implies at once that this spectral triple is finite-dimensional and has dimension equal to the dimension of \(M\).

Let \((\mathcal{A}, \mathcal{H}, D)\) be a spectral triple. Assume for simplicity that the algebra \(\mathcal{A}\) has a unit. We consider the operator \(|D| = (D^2)^{1/2}\). Denote by \(\delta\) the (unbounded) differentiation on \(L(\mathcal{H})\) given by

\[
\delta(T) = [[|D|, T]], \quad T \in \text{Dom} \delta \subset L(\mathcal{H}).
\] (3.9)

For any \(T \in L(\mathcal{H})\) denote by \(\delta^i(T)\) the \(i\)th commutator with \(|D|\).

Definition 3.12. The space \(\text{OP}^0\) consists of all \(T \in L(\mathcal{H})\) such that \(\delta^i(T) \in L(\mathcal{H})\) for any \(i \in \mathbb{N}\):

\[
\text{OP}^0 = \bigcap_n \text{Dom} \delta^n.
\]

The space \(\text{OP}^0\) is a smooth subalgebra of the \(C^*-\)algebra \(L(\mathcal{H})\) (for instance, see [66], Theorem 1.2). The uniform closure \(\overline{\mathcal{A}}\) of \(\mathcal{A}\) in \(L(\mathcal{H})\) can be regarded as the algebra of continuous functions on some virtual topological space. In a certain sense the algebra \(\text{OP}^0 \cap \overline{\mathcal{A}}\) consists of functions on the given space which are infinitely differentiable in the quantum sense. For the spectral triple associated with a compact Riemannian manifold, the algebra \(\text{OP}^0 \cap C(M)\) contains the algebra \(C^\infty(M)\). We refer the reader to [67] for the problem of reconstructing the smooth structure of a manifold \(M\) from the associated spectral triple \((\mathcal{A}, \mathcal{H}, D)\).

We recall that the Riemannian volume form \(d\nu\) associated with a Riemannian metric \(g\) is given in local coordinates by the formula \(d\nu = \sqrt{\det g} \, dx\). To define its non-commutative analogue one uses the trace \(\text{Tr}_\omega\) introduced by Dixmier in [68] as an example of a non-standard trace on \(L(\mathcal{H})\).

Consider the ideal \(L^{1+}(\mathcal{H})\) in the algebra \(\mathcal{K}(\mathcal{H})\) of compact operators which consists of all \(T \in \mathcal{K}(\mathcal{H})\) such that

\[
\sup_{N \in \mathbb{N}} \frac{1}{\log N} \sum_{n=1}^{N} \mu_n(T) < \infty,
\]
where \( \mu_1(T) \geq \mu_2(T) \geq \cdots \) are the singular numbers of \( T \). For any invariant mean \( \omega \) on the amenable group of upper triangular \( 2 \times 2 \) matrices, Dixmier constructed a linear form \( \lim_\omega \) on the space \( \ell^\infty(\mathbb{N}) \) of bounded sequences which has the following properties:

1) \( \lim_\omega \) coincides with the functional of taking the limit \( \lim \) on the subspace of convergent sequences,
2) \( \lim_\omega \{c_n\} \geq 0 \) if \( c_n \geq 0 \) for any \( n \),
3) \( \lim_\omega \{c'_n\} = \lim_\omega \{c_n\} \), where \( \{c'_n\} = \{c_1, c_1, c_2, c_2, c_3, c_3, \ldots \} \),
4) \( \lim_\omega \{c_{2n}\} = \lim_\omega \{c_n\} \).

For a positive operator \( T \in \mathcal{L}^{1+}(\mathcal{H}) \) the value of \( \text{Tr}_\omega(T) \) is given by

\[
\text{Tr}_\omega(T) = \lim_\omega \frac{1}{\log N} \sum_{n=1}^{N} \mu_n(T).
\]

Let \( M \) be an \( n \)-dimensional compact manifold, \( E \) a vector bundle on \( M \), and \( P \in \Psi^m(M, E) \) a classical pseudodifferential operator. Thus, in any local coordinate system its complete symbol \( p \) can be represented as an asymptotic sum \( p \sim p_m + p_{m-1} + \cdots \), where \( p_l(x, \xi) \) is a homogeneous function of degree \( l \) in \( \xi \). As shown in [69], the Dixmier trace \( \text{Tr}_\omega(P) \) does not depend on the choice of \( \omega \) and coincides with the value \( \tau(P) \) of the trace \( \tau \) introduced by Wodzicki [70] and Guillemin [71] on the algebra \( \Psi^*(M, E) \) of classical pseudodifferential operators of arbitrary order. The trace \( \tau \) is defined as follows. For \( P \in \Psi^*(M, E) \) the density \( \rho_P \) is defined in local coordinates as

\[
\rho_P = \left( \int_{|\xi|=1} \text{Tr} p_{-n}(x, \xi) \, d\xi \right) |dx|.
\]

The density \( \rho_P \) turns out to be independent of the choice of a local coordinate system, and therefore gives a well-defined density on \( M \). The integral of \( \rho_P \) over \( M \) is the value of the Wodzicki–Guillemin trace:

\[
\tau(P) = \frac{1}{(2\pi)^n} \int_M \rho_P = \frac{1}{(2\pi)^n} \int_{S^*M} \text{Tr} p_{-n}(x, \xi) \, dx \, d\xi. \quad (3.10)
\]

Wodzicki [70] showed that \( \tau \) is a unique trace on \( \Psi^*(M, E) \).

According to [69] (see also [20]), any operator \( P \in \Psi^{-n}(M, E) \) belongs to the ideal \( \mathcal{L}^{1+}(L^2(M, E)) \), and for any invariant mean \( \omega \)

\[
\text{Tr}_\omega(P) = \tau(P).
\]

The above results imply the formula

\[
\int_M f \, d\nu = c(n) \text{Tr}_\omega(f|D|^{-n}), \quad f \in \mathcal{A},
\]

where \( c(n) = 2^{n-[n/2]} \pi^{n/2} \Gamma(n/2+1). \) Thus, the Dixmier trace \( \text{Tr}_\omega \) can be regarded as a proper non-commutative generalization of the integral.
3.6. **Non-commutative local index theorem.** If one looks at a geometric space as the union of parts of different dimensions, the notion of dimension introduced in Definition 3.11 gives only the maximum of the dimensions of the parts of this space. To take into account lower-dimensional parts of the geometric space, Connes and Moscovici [63], [64] proposed taking as a more correct notion of dimension of a smooth spectral triple not a single real number \( d \), but a subset \( S_d \subset \mathbb{C} \), called its dimension spectrum.

**Definition 3.13.** A spectral triple \((\mathcal{A}, \mathcal{H}, D)\) is said to be smooth if for any \( a \in \mathcal{A} \) one has the inclusions \( a, [D, a] \in \text{OP}^0 \).

The spectral triple associated with a smooth Riemannian manifold \( M \) is smooth.

The smoothness condition of a spectral triple \((\mathcal{A}, \mathcal{H}, D)\) can be treated in the following sense: the algebra \( \mathcal{A} \) consists of smooth (in the quantum sense) functions on the corresponding non-commutative space (see the comments after Definition 3.12).

**Definition 3.14.** A spectral triple \((\mathcal{A}, \mathcal{H}, D)\) has discrete dimension spectrum \( S_d \subset \mathbb{C} \) if \( S_d \) is a discrete subset of \( \mathbb{C} \), the triple is smooth, and for any \( b \in \mathcal{B} \) the distributional zeta-function \( \zeta_b(z) \) of \( |D| \) given by

\[
\zeta_b(z) = \text{tr} \ b |D|^{-z},
\]

is defined in the half-plane \( \{z \in \mathbb{C} : \text{Re} \ z > d\} \) and extends to a holomorphic function on \( \mathbb{C} \setminus S_d \).

The dimension spectrum is said to be simple if the singularities of \( \zeta_b(z) \) at \( z \in S_d \) are at most simple poles.

**Example 3.15.** Let \( M \) be a compact manifold of dimension \( n \), \( E \) a vector bundle on \( M \), and \( D \in \Psi^1(M, E) \) a self-adjoint elliptic operator. Then the triple \((C^\infty(M), L^2(M, E), D)\) is a smooth \( n \)-dimensional spectral triple. The algebra \( \mathcal{B} \) is contained in the algebra \( \Psi^0(M, E) \) of zero-order classical pseudodifferential operators. For any classical pseudodifferential operator \( P \in \Psi^m(M, E) \), \( m \in \mathbb{Z} \), the function \( z \mapsto \text{tr} \ P |D|^{-z} \) has a meromorphic extension to \( \mathbb{C} \) with at most simple poles at integer points \( k, k \leq m + n \). The residue of this function at \( z = 0 \) coincides with the Wodzicki–Guillemin trace \( \tau(P) \) of \( P \) (see (3.10)):

\[
\text{res}_{z=0} \text{tr} \ P |D|^{-z} = \tau(P). \tag{3.11}
\]

In particular, this spectral triple has a simple discrete dimension spectrum lying in \( \{k \in \mathbb{Z} : k \leq n\} \).

In [63], [64] a definition is given of the algebra \( \Psi^*(\mathcal{A}) \) of pseudodifferential operators associated with a smooth spectral triple \((\mathcal{A}, \mathcal{H}, D)\) in the case when the algebra \( \mathcal{A} \) is unital. By the spectral theorem, for any \( s \in \mathbb{R} \) the operator \( |D|^s \) is a well-defined positive self-adjoint operator acting in \( \mathcal{H} \) which is unbounded for \( s > 0 \). For \( s \geq 0 \) we denote by \( \mathcal{H}^s \) the domain of the operator \( |D|^s \), and for \( s < 0 \) we put \( \mathcal{H}^{-s} = (\mathcal{H}^s)^* \). Also, let \( \mathcal{H}^\infty = \bigcap_{s \geq 0} \mathcal{H}^s \), \( \mathcal{H}^\infty^* = (\mathcal{H}^\infty)^* \).
We say that a bounded operator $P$ on the space $\mathcal{H}^\infty$ belongs to the class $\text{OP}^\alpha$ if $P(D)^{-\alpha} \in \text{OP}^0$.

**Definition 3.16.** We say that an operator $P : \mathcal{H}^\infty \to \mathcal{H}^{-\infty}$ belongs to the class $\Psi^*(\mathcal{A})$ if it admits an asymptotic expansion

$$P \sim \sum_{j=0}^{+\infty} b_{q-j}|D|^{q-j}, \quad b_{q-j} \in \mathcal{B},$$

which means that for any $N$

$$P - (b_q|D|^q + b_{q-1}|D|^{q-1} + \cdots + b_{-N}|D|^{-N}) \in \text{OP}^{-N-1}.$$  

It was proved in [63], Appendix B that $\Psi^*(\mathcal{A})$ is an algebra. For the spectral triple $(C^\infty(M), L^2(M, E), D)$ described in Example 3.15 the algebra $\Psi^*(\mathcal{A})$ is contained in $\Psi^*(M, E)$.

Recall that a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ defines the Fredholm module $(\mathcal{H}, F)$ over $\mathcal{A}$, where $F = D(I + D^2)^{1/2}$, and thereby the index map $\text{Ind} : K_*(\mathcal{A}) \to \mathbb{C}$ (see the formulae (2.1) and (2.2)). As shown above, this map can be expressed in terms of the pairing with the cyclic cohomology class $\text{ch}_*(\mathcal{H}, F) \in H^P(\mathcal{A})$, the Chern character of the Fredholm module $(\mathcal{H}, F)$ (see the formulae (3.7) and (3.8)). Let us call $\text{ch}_*(\mathcal{H}, F)$ the Chern character of the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ and denote it by $\text{ch}_*(\mathcal{A}, \mathcal{H}, D)$. The formulae (3.7) and (3.8) have a defect, in that they express the map $\text{Ind}$ in terms of the operator traces. In the classical case these traces are non-local functionals, and it is impossible to compute them in coordinate charts. To correct this defect, Connes and Moscovici proved for the index map another formula, which involves Wodzicki–Guillemin trace type functionals. These functionals are local in the sense of non-commutative geometry, because they vanish on any trace class operator on $\mathcal{H}$. Therefore, the Connes–Moscovici formula can be naturally called the non-commutative local index theorem.

Suppose that a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is smooth and, for simplicity, has simple discrete dimension spectrum. We define the non-commutative integral determined by this spectral triple by setting

$$\int b = \text{res} \text{tr} b |D|^{-z}, \quad b \in \mathcal{B}. \quad (3.12)$$

The functional $\int$ is a trace on $\mathcal{B}$, which is local in the sense of non-commutative geometry.

**Theorem 3.17** ([63], Theorem II.3). Suppose that $(\mathcal{A}, \mathcal{H}, D)$ is an even spectral triple which is $p$-summable and has simple discrete dimension spectrum.

1) An even cocycle $\varphi^\infty_{CM} = (\varphi_{2k})$ in the $(b, B)$-bicomplex of $\mathcal{A}$ is defined by the following formulae: for $k = 0$

$$\varphi_0(a^0) = \text{res} \text{tr} a^0|D|^{-z},$$

and for $k \neq 0$

$$\varphi_{2k}(a^0, \ldots, a^{2k}) = \sum_{\alpha \in \mathbb{N}^n} c_{k,\alpha} \int \gamma a^0 [D, a^1]^{|\alpha_1|} \cdots [D, a^{2k}]^{|\alpha_{2k}|} |D|^{-2(|\alpha|+k)},$$
where
\[ c_{k,\alpha} = \frac{(-1)^{|\alpha|} \cdot 2\Gamma(|\alpha| + k)}{\alpha! (\alpha_1 + 1) \cdots (\alpha_1 + \cdots + \alpha_{2k} + 2k)} \]
and the symbol \( T^{[j]} \) denotes the \( j \)th iterated commutator with \( D^2 \).

2) The cohomology class defined by \( \varphi_{\text{CM}}^{\text{ev}} \) in \( HP^{\text{ev}}(\mathcal{A}) \) coincides with the Chern character \( \text{ch}^* \)

**Theorem 3.18** ([63], Theorem II.2). Suppose that \( (\mathcal{A}, \mathcal{H}, D) \) is a spectral triple which is \( p \)-summable and has simple discrete dimension spectrum. Then

1) an odd cocycle \( \varphi_{\text{CM}}^{\text{odd}} = (\varphi_{2k+1}) \) in the \( (b, B) \)-bicomplex of \( \mathcal{A} \) is defined by

\[
\varphi_{2k+1}(a^0, \ldots, a^{2k+1}) = \sqrt{2i\pi} \sum_{\alpha \in \mathbb{N}^n} c_{k,\alpha} \int a^0[D, a^1][\alpha_1] \cdots [D, a^{2k+1}][\alpha_{2k+1}] \left| D \right|^{-2(|\alpha|+k)-1},
\]

where
\[ c_{k,\alpha} = \frac{(-1)^{|\alpha|} \Gamma(|\alpha| + k + 1/2)}{\alpha! (\alpha_1 + 1) \cdots (\alpha_1 + \cdots + \alpha_{2k+1} + 2k + 1)}; \]

2) the cohomology class defined by \( \varphi_{\text{CM}}^{\text{odd}} \) in \( HP^{\text{odd}}(\mathcal{A}) \) coincides with the Chern character \( \text{ch}^* \).

**Example 3.19.** Let \( M \) be a compact manifold of dimension \( n \) and \( D \) a first-order, self-adjoint, elliptic, pseudodifferential operator on \( M \) acting on sections of a vector bundle \( E \) on \( M \). Then the non-commutative integral \( \int \) defined by the spectral triple \( (C^\infty(M), L^2(M, E), D) \) coincides with the Wodzicki–Guillemin trace \( \tau \) (see (3.10) and Example 3.15).

In the case when \( D \) is the spin Dirac operator on a compact Riemannian spin manifold \( M \) in Theorem 3.17, we have for any \( f^0, f^1, \ldots, f^m \in C^\infty(M) \) that

\[
\tau(\gamma f^0[D, f^1][\alpha_1] \cdots [D, f^m][\alpha_m] \left| D \right|^{-(2|\alpha|+m)}) = 0
\]

when \( |\alpha| \neq 0 \), and

\[
\tau(\gamma f^0[D, f^1] \cdots [D, f^m] \left| D \right|^{-m}) = c_m \int_M f^0 df^1 \wedge \cdots \wedge df^m \wedge \hat{A}(TM, \nabla)
\]

when \( \alpha_1 = \alpha_2 = \cdots = \alpha_{2k} = 0 \), where \( c_m \) is some constant.

If the dimension of \( M \) is even, then the spectral triple is even and the components of the corresponding even cocycle \( \varphi_{\text{CM}}^{\text{ev}} = (\varphi_{2k}) \) are given by

\[
\varphi_{2k}(f^0, \ldots, f^{2k}) = \frac{1}{(2k)!} \int_M f^0 df^1 \wedge \cdots \wedge df^{2k} \wedge \hat{A}(TM, \nabla)^{(n-2k)},
\]

where \( f^0, f^1, \ldots, f^{2k} \in C^\infty(M) \).
If the dimension of $M$ is odd, then the spectral triple is odd and the components of the corresponding odd cocycle $\varphi_{\text{odd}} = (\varphi_{2k+1})$ are given by

$$\varphi_{2k+1}(f^0, \ldots, f^{2k}) = \sqrt{2i\pi} \frac{(2i\pi)^{-[n/2]+1}}{(2k+1)!} \times \int_M f^0 df^1 \wedge \cdots \wedge df^{2k+1} \wedge \hat{A}(TM, \nabla)^{(n-2k-1)},$$

where $f^0, f^1, \ldots, f^{2k+1} \in C^\infty(M)$.

### 3.7. Semifinite spectral triples

The study of index theory problems in various situations such as measurable foliations, Galois coverings, and almost periodic operators has served as a motivation for extending methods of non-commutative geometry to the case when the algebra of bounded operators on a Hilbert space is replaced by an arbitrary semifinite von Neumann algebra. In this subsection we give some information from semifinite non-commutative geometry (for further information see, for instance, the survey [72] and its references).

**Definition 3.20.** A semifinite spectral triple is a set $(\mathcal{A}, \mathcal{H}, D)$, where

1) $\mathcal{H}$ is a Hilbert space;
2) $\mathcal{A}$ is an involutive subalgebra of a semifinite von Neumann algebra $\mathcal{N}$ acting in $\mathcal{H}$;
3) $D$ is an (unbounded) self-adjoint operator acting in $\mathcal{H}$ and affiliated with $\mathcal{N}$ such that
   (i) the operator $(D - i)^{-1}$ is a $\tau$-compact operator in $\mathcal{N}$ (relative to some faithful normal semifinite trace $\tau$ on $\mathcal{N}$),
   (ii) the operator $[D, a]$ is a bounded operator belonging to $\mathcal{N}$ for each $a \in \mathcal{A}$.

A spectral triple is said to be even if the space $\mathcal{H}$ is equipped with a $\mathbb{Z}_2$-grading $\gamma \in L(\mathcal{H})$, $\gamma = \gamma^*$, $\gamma^2 = 1$, and moreover, $\gamma D = -D \gamma$ and $\gamma a = a \gamma$ for any $a \in \mathcal{A}$. Otherwise the spectral triple is said to be odd.

For an element $S \in \mathcal{N}$ its $t$th generalized singular number ($t \in \mathbb{R}$) is given by

$$\mu_t(S) = \inf\{\|SE\| \mid E \text{ is a projection in } \mathcal{N} \text{ with } \tau(1 - E) \leq t\}.$$  

The space $L^{(1, \infty)}(\mathcal{N})$ consists of elements $T \in \mathcal{N}$ such that

$$\|T\|_{L^{(1, \infty)}} = \sup_{t > 1} \frac{1}{\log(1 + t)} \int_0^t \mu_s(T) \, ds < \infty.$$  

For any $p > 1$ put

$$\psi_p(t) = \begin{cases} t & \text{for } 0 \leq t \leq 1, \\ t^{1-1/p} & \text{for } 1 \leq t. \end{cases}$$

The space $L^{(p, \infty)}(\mathcal{N})$ consists of $T \in \mathcal{N}$ such that

$$\|T\|_{L^{(p, \infty)}} = \sup_{t > 1} \frac{1}{\psi_p(t)} \int_0^t \mu_s(T) \, ds < \infty.$$
Definition 3.21. A semifinite spectral triple \((\mathcal{A}, \mathcal{H}, D)\) is \((p, \infty)\)-summable if for any \(a \in \mathcal{A}\) the operator \((1 + D^2)^{-1/2}\) is an element of the class \(L^{(p, \infty)}(\mathcal{N})\).

Definition 3.22. A pre-Fredholm module over a unital Banach algebra \(\mathcal{A}\) is a pair \((\mathcal{H}, F)\), where

1) \(\mathcal{A}\) has a continuous representation in a semifinite von Neumann algebra \(\mathcal{N}\) acting in a Hilbert space \(\mathcal{H}\);
2) \(F\) is a self-adjoint Breuer–Fredholm operator acting in \(\mathcal{H}\) such that \(1 - F^2 \in \mathcal{K}_\mathcal{N}\) and \([F, a] \in \mathcal{K}_\mathcal{N}\) for any \(a \in \mathcal{A}\).

If \(1 - F^2 = 0\), then \((\mathcal{H}, F)\) is called a Fredholm module.

A pre-Fredholm module \((\mathcal{H}, F)\) is said to be even if the Hilbert space \(\mathcal{H}\) is equipped with a \(\mathbb{Z}_2\)-grading \(\gamma\) such that the operators \(\rho(a)\) are even \((\gamma a = a \gamma)\) and the operator \(F\) is odd \((\gamma F = -F \gamma)\). Otherwise it is said to be odd.

Definition 3.23. A pre-Fredholm module \((\mathcal{H}, F)\) is said to be \((p, \infty)\)-summable if \(1 - F^2 \in L^{(p/2, \infty)}(\mathcal{N})\) and \([F, a] \in L^{(p, \infty)}(\mathcal{N})\) for a dense set of elements \(a \in \mathcal{A}\).

If \((\mathcal{A}, \mathcal{H}, D)\) is a semifinite spectral triple, then the pair \((\mathcal{H}, F)\), where \(F = D(1 + D^2)^{1/2}\), is a pre-Fredholm module over \(\mathcal{A}\). Using a faithful normal semifinite trace \(\tau\) on the von Neumann algebra \(\mathcal{N}\) instead of the standard trace on the algebra \(L(\mathcal{H})\), one can define by standard formulae the Chern character of an arbitrary \((p, \infty)\)-summable pre-Fredholm module \((\mathcal{H}, F)\), and hence the Chern character of any finite-dimensional semifinite spectral triple \((\mathcal{A}, \mathcal{H}, D)\). In the papers [73], [74] a non-commutative local index theorem is proved for semifinite spectral triples. Its formulation is similar to that of the non-commutative local index theorem (see Theorems 3.17 and 3.18), with the sole difference that the role of the Dixmier trace \(\text{Tr}_\omega\) in these theorems is played by its generalization to the case of an arbitrary semifinite von Neumann algebra. For Dixmier traces and general singular traces on semifinite von Neumann algebras and their applications, see the survey [72] and its references.

3.8. Non-commutative spectral geometry and type III. The authors of [75] introduced the notion of twisted (or \(\sigma\)-spectral) triple, making it possible to apply the methods of non-commutative geometry for certain type III non-commutative spaces.

Definition 3.24. Let \(\mathcal{A}\) be an algebra equipped with an automorphism \(\sigma\). An ungraded \(\sigma\)-spectral triple over \(\mathcal{A}\) is defined to be a set \((\mathcal{A}, \mathcal{H}, D)\), where

1) \(\mathcal{H}\) is a Hilbert space endowed with an action of \(\mathcal{A}\);
2) \(D\) is an (unbounded) self-adjoint operator acting in \(\mathcal{H}\) such that the operator \((D - i)^{-1}\) is a compact operator on \(\mathcal{H}\) and the operator \(Da - \sigma(a)D\) is bounded for any \(a \in \mathcal{A}\).

If the algebra \(\mathcal{A}\) is involutive and its representation on \(\mathcal{H}\) is a \(*\)-representation, then in addition we impose the unitarity condition

\[\sigma(a^*) = (\sigma^{-1}(a))^*, \quad a \in \mathcal{A}.\]

A graded \(\sigma\)-spectral triple is defined in a similar way, but in this case the space \(\mathcal{H}\) is equipped with a \(\mathbb{Z}_2\)-grading \(\gamma \in L(\mathcal{H})\), \(\gamma = \gamma^*, \gamma^2 = 1\), and we have \(\gamma D = -D \gamma\) and \(\gamma a = a \gamma\) for any \(a \in \mathcal{A}\).
Example 3.25. Let \((A, \mathcal{H}, D)\) be a spectral triple and \(h \in A\) a self-adjoint element, \(h = h^*\). Consider the automorphism \(\sigma\) of \(A\) given by

\[ \sigma(a) = e^{2h}ae^{-2h}, \quad a \in A. \]

The ‘perturbed’ spectral triple \((A, \mathcal{H}, D')\) with \(D' = e^hDe^{-h}\) is a \(\sigma\)-spectral triple.

Another example of \(\sigma\)-spectral triples arises from transverse geometry of an arbitrary codimension-one foliation (see Example 7.7). The most vital open problem is to extend the notions mentioned above to the case of higher-codimensional foliations. It is expected that the general case will require the use of dual actions of Lie groups such as \(GL(n)\) and, more generally, of quantum groups.

Definition 3.26. A \(\sigma\)-spectral triple \((A, \mathcal{H}, D)\) is said to be Lipschitz regular if the operator \(|D|a - \sigma(a)|D|\) is bounded for any \(a \in A\).

If a \(\sigma\)-spectral triple \((A, \mathcal{H}, D)\) is Lipschitz regular and \(F = D|D|^{-1}\), then \((\mathcal{H}, F)\) is a Fredholm module over \(A\). Moreover, if \((A, \mathcal{H}, D)\) is finite-dimensional, then the Fredholm module \((\mathcal{H}, F)\) is also finite-dimensional. Thus, the Chern character \(\text{ch}(\mathcal{H}, F) \in HP^*(A)\) is well defined for any finite-dimensional Lipschitz regular \(\sigma\)-spectral triple \((A, \mathcal{H}, D)\). On the other hand, for any \(\sigma\)-spectral triple \((A, \mathcal{H}, D)\) such that \(D^{-1} \in \mathcal{L}^{(n, \infty)}\) for some even \(n\) one can define a cyclic cocycle \(\Psi_{D, \sigma}\) on \(A\) by

\[ \Psi_{D, \sigma}(a^0, a^1, \ldots, a^n) = \text{tr}(\gamma d_\sigma a^0D^{-1}d_\sigma a^1 \cdots D^{-1}d_\sigma a^n), \quad a^0, a^1, \ldots, a^n \in A, \]

where \(d_\sigma a = Da - \sigma(a)D\) for any \(a \in A\). If a \(\sigma\)-spectral triple \((A, \mathcal{H}, D)\) satisfies the stronger condition

\[ |D|^{-t}(|D|^t a - \sigma^t(a)|D|^t) \in \mathcal{L}^{(n, \infty)}, \quad a \in A, \quad t \in \mathbb{R}, \]

then the cocycle \(\Psi_{D, \sigma}\) defines the same cohomology class as the Chern character \(\text{ch}(\mathcal{H}, F) \in HP^*(A)\).

An analogue of the non-commutative local index theorem in this case remains an open question. For a certain class of twisted spectral triples of type III, such a theorem was proved very recently in [76].

4. Some background material from foliation theory

4.1. Foliations: definitions and examples. In this subsection we recall the definition of a foliated manifold and some notions connected with foliations.

Definition 4.1. (1) An atlas \(\mathcal{A} = \{(U_i, \phi_i)\}\), where \(\phi_i: U_i \subset M \to \mathbb{R}^n\), of a smooth manifold \(M\) of dimension \(n\) is called an atlas of a foliation of dimension \(p\) and codimension \(q\) \((p \leq n, p + q = n)\) if, for any \(i\) and \(j\) such that \(U_i \cap U_j \neq \emptyset\), the coordinate transformations \(\phi_{ij} = \phi_i \circ \phi_j^{-1}: \phi_j(U_i \cap U_j) \subset \mathbb{R}^p \times \mathbb{R}^q \to \phi_i(U_i \cap U_j) \subset \mathbb{R}^p \times \mathbb{R}^q\) have the form

\[ \phi_{ij}(x, y) = (\alpha_{ij}(x, y), \gamma_{ij}(y)), \quad (x, y) \in \phi_j(U_i \cap U_j) \subset \mathbb{R}^p \times \mathbb{R}^q. \]

2) Two atlases of a foliation of dimension \(p\) are equivalent if their union is again an atlas of a foliation of dimension \(p\).
3) A manifold $M$ endowed with an equivalence class $\mathcal{F}$ of atlases of a foliation of dimension $p$ is called a manifold with a foliation of dimension $p$.

An equivalence class $\mathcal{F}$ of atlases of a foliation is also called a complete atlas of a foliation. We will also say that $\mathcal{F}$ is a foliation on the manifold $M$.

A pair $(U, \phi)$ belonging to some atlas of a foliation $\mathcal{F}$, and also the corresponding map $\phi: U \to \mathbb{R}^n$, are called a foliated chart of $\mathcal{F}$, and $U$ a foliated coordinate neighbourhood.

Let $\phi : U \subset M \to \mathbb{R}^n$ be a foliated chart. The connected components of the set $\phi^{-1}(\mathbb{R}^p \times \{y\})$, $y \in \mathbb{R}^q$, are called plaques of $\mathcal{F}$.

The plaques of $\mathcal{F}$ taken for all possible foliated charts form a base of a topology on $M$. This topology is called the leaf topology on $M$. We will also denote by $\mathcal{F}$ the set $M$ endowed with the leaf topology. One can introduce a $p$-dimensional smooth manifold structure on $\mathcal{F}$.

The connected components of $\mathcal{F}$ are called leaves of $\mathcal{F}$. They are (one-to-one) immersed $p$-dimensional submanifolds of $M$. For any $x \in M$ there exists a unique leaf passing through $x$. We will denote this leaf by $L_x$.

One can give an equivalent definition of a foliation by saying that there is a foliation $\mathcal{F}$ of dimension $p$ on an $n$-dimensional manifold $M$ if $M$ is represented as the union of a family $\{L_\lambda : \lambda \in \mathcal{L}\}$ of disjoint connected (one-to-one) immersed submanifolds of dimension $p$, and $M$ has an atlas $\mathcal{A} = \{(U_i, \phi_i)\}$ such that, for any coordinate chart $(U_i, \phi_i)$ with local coordinates $(x_1, x_2, \ldots, x_n)$ and for any $\lambda \in \mathcal{L}$ the connected components of the set $L_\lambda \cap U_i$ are given by equations of the form $x_{p+1} = c_{p+1}, \ldots, x_n = c_n$ for some constants $c_{p+1}, \ldots, c_n$.

**Example 4.2.** Let $M$ be an $n$-dimensional smooth manifold, $B$ a $q$-dimensional smooth manifold, and $\pi : M \to B$ a submersion (that is, the differential $d\pi_x : T_x M \to T_{\pi(x)}B$ is surjective for any $x \in M$). The connected components of the pre-images of points of $B$ under the map $\pi$ give a codimension-$q$ foliation of $M$ which is called the foliation determined by the submersion $\pi$. If, in addition, the pre-images $\pi^{-1}(b)$, $b \in B$, are connected, then the foliation is said to be simple.

**Example 4.3.** If $X$ is a non-singular (that is, non-vanishing) smooth vector field on a manifold $M$, then its phase curves form a dimension-one foliation.

More generally, let a connected Lie group $G$ act smoothly on a smooth manifold $M$ so that the dimension of the stationary subgroup $G_x = \{x \in G : gx = x\}$ is independent of $x \in M$. In particular, one can assume that the action is locally free, which means the discreteness of the isotropy group $G_x$ for any $x \in M$. Then the orbits of the action of $G$ define a foliation on $M$.

**Example 4.4** (a linear foliation on the torus). Consider the vector field $\tilde{X}$ on $\mathbb{R}^2$ given by

$$\tilde{X} = \alpha \frac{\partial}{\partial x} + \beta \frac{\partial}{\partial y}$$

with constant $\alpha$ and $\beta$. Since $\tilde{X}$ is invariant by all translations, it defines a vector field $X$ on the two-dimensional torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. The vector field $X$ defines a foliation $\mathcal{F}$ on $T^2$. The leaves of $\mathcal{F}$ are the images of parallel lines $L = \{(x_0 + t\alpha, y_0 + t\beta) : t \in \mathbb{R}\}$ with slope $\theta = \beta/\alpha$ under the projection $\mathbb{R}^2 \to T^2$. 
In the case when $\theta$ is rational all leaves of $\mathcal{F}$ are closed and are circles, and the foliation $\mathcal{F}$ is defined by the fibres of a fibration $T^2 \to S^1$. In the case when $\theta$ is irrational, all leaves of $\mathcal{F}$ are dense in $T^2$.

**Example 4.5** (homogeneous foliations). Let $G$ be a Lie group and $H \subset G$ a connected Lie subgroup of it. The family $\{gH : g \in G\}$ of right cosets of $H$ forms a foliation $\mathcal{H}$ on $G$. If $H$ is a closed subgroup, then $G/H$ is a manifold and $\mathcal{H}$ is the foliation with leaves given by the fibres of the fibration $\pi: G \to G/H$.

Moreover, suppose that $\Gamma \subset G$ is a discrete subgroup of $G$. Then the set $M = \Gamma \setminus G$ of left cosets of $\Gamma$ is a manifold of the same dimension as $G$. If $\Gamma$ is cocompact in $G$, then $M$ is compact. In any case, because $\mathcal{H}$ is invariant under left translations and $\Gamma$ acts from the left, the foliation $\mathcal{H}$ is mapped by the map $G \to M = \Gamma \setminus G$ to a well-defined foliation $\mathcal{H}_\Gamma$ on $M$, which is often denoted by $\mathcal{F}(G, H, \Gamma)$ and called a locally homogeneous foliation. The leaf of $\mathcal{H}_\Gamma$ through a point $\Gamma g \in M$ is diffeomorphic to $H/(g\Gamma g^{-1} \cap H)$.

**Example 4.6** (suspension). Let $B$ be a connected manifold and $\tilde{B}$ its universal cover equipped with the action of the fundamental group $\Gamma = \pi_1(B)$ by deck transformations. Suppose that there is given a homomorphism $\phi: \Gamma \to \text{Diff}(F)$ of $\Gamma$ to the diffeomorphism group $\text{Diff}(F)$ of a smooth manifold $F$. We define a manifold $M = \tilde{B} \times_F F$ as the quotient of the manifold $\tilde{B} \times F$ by the action of $\Gamma$ given, for any $\gamma \in \Gamma$, by

$$\gamma(b, f) = (\gamma b, \phi(\gamma)f), \quad (b, f) \in \tilde{B} \times F.$$  

There is a natural foliation $\mathcal{F}$ on $M$ whose leaves are the images of the sets $\tilde{B} \times \{f\}$, $f \in F$, under the projection $\tilde{B} \times F \to M$. If for any $\gamma \in \Gamma$ with $\gamma \neq e$ the diffeomorphism $\phi(\gamma)$ has no fixed points, then all the leaves of $\mathcal{F}$ are diffeomorphic to $\tilde{B}$.

There is defined a bundle $\pi: M \to B: [(b, f)] \mapsto b \pmod{\Gamma}$ such that the leaves of $\mathcal{F}$ are transversal to the fibres of $\pi$. The bundle $\pi: M \to B$ is often said to be a flat foliated bundle.

A foliation $\mathcal{F}$ defines a subbundle $F = T\mathcal{F}$ of the tangent bundle $TM$, called the tangent bundle of $\mathcal{F}$. It consists of all vectors tangent to the leaves of $\mathcal{F}$. Denote by $\mathcal{X}(M) = C^\infty(M, TM)$ the Lie algebra of all smooth vector fields on $M$ with respect to the Lie bracket and by $\mathcal{X}_\mathcal{F}(M) = \{F \in \mathcal{X}(M, F) \mid \text{subspace of vector fields on } M \text{ tangent to leaves of } \mathcal{F} \text{ at each point.} \}$ The subspace $\mathcal{X}_\mathcal{F}(M)$ is a subalgebra of the Lie algebra $\mathcal{X}(M)$. Moreover, by the Frobenius theorem, a subbundle $E$ of the bundle $TM$ is the tangent bundle of some foliation if and only if it is involutive, that is, the space of sections of this bundle is a Lie subalgebra of the Lie algebra $\mathcal{X}(M)$: for any $X, Y \in \mathcal{X}(M, E)$ we have $[X, Y] \in \mathcal{X}(M, E)$.

We also introduce the following objects: $\tau = TM/T\mathcal{F}$ is the normal bundle of $\mathcal{F}$; $P_\tau: TM \to \tau$ is the natural projection; $N^*\mathcal{F} = \{\nu \in T^*M : \langle \nu, X \rangle = 0 \text{ for any } X \in F\}$ is the conormal bundle of $\mathcal{F}$. Usually we will denote by $(x, y) \in I^p \times I^q$ ($I = (0, 1)$ is the open interval) the local coordinates in a foliated chart $\phi: U \to I^p \times I^q$ and by $(x, y, \xi, \eta) \in I^p \times I^q \times \mathbb{R}^p \times \mathbb{R}^q$ the local coordinates in the corresponding chart on $T^*M$. Then the subset $N^*\mathcal{F} \cap \pi^{-1}(U) = U_1$ (here $\pi: T^*M \to M$ is the bundle map) is given by the equation $\xi = 0$. Therefore,
\( \phi \) defines a natural coordinate chart \( \phi_n : U_1 \rightarrow I^p \times I^q \times \mathbb{R}^q \) on \( N^* \mathcal{F} \) with coordinates \( (x, y, \eta) \).

A \( q \)-dimensional distribution \( Q \subset TM \) such that \( TM = F \oplus Q \) is called a distribution transversal to the foliation, or a connection on the foliated manifold \( (M, \mathcal{F}) \). Any Riemannian metric \( g \) on \( M \) defines a transversal distribution \( H \) formed by the orthogonal complement of \( F \) with respect to this metric:

\[
H = F^\perp = \{ X \in TM : g(X, Y) = 0 \text{ for any } Y \in F \}.
\]

**Definition 4.7.** A vector field \( V \) on a foliated manifold \( (M, \mathcal{F}) \) is called an infinitesimal transformation of \( \mathcal{F} \) if \( [V, X] \in \mathfrak{X}(\mathcal{F}) \) for any \( X \in \mathfrak{X}(\mathcal{F}) \).

The set of infinitesimal transformations of \( \mathcal{F} \) is denoted by \( \mathfrak{X}(M/\mathcal{F}) \). If \( V \in \mathfrak{X}(M/\mathcal{F}) \) and \( T_t : M \rightarrow M, \ t \in \mathbb{R} \), is the flow of the vector field \( V \), then the diffeomorphisms \( T_t \) are automorphisms of the foliated manifold \( (M, \mathcal{F}) \), that is, they take each leaf of \( \mathcal{F} \) to a (possibly different) leaf.

**Definition 4.8.** A vector field \( V \) on a foliated manifold \( (M, \mathcal{F}) \) is said to be projectable if its normal component \( P_\tau(V) \) is locally the lift of a vector field on the local base.

In other words, a vector field \( V \) on \( M \) is projectable if, in any foliated chart with local coordinates \( (x, y), x \in \mathbb{R}^p, y \in \mathbb{R}^q \), it has the form

\[
V = \sum_{i=1}^p f^i(x, y) \frac{\partial}{\partial x^i} + \sum_{j=1}^q g^j(y) \frac{\partial}{\partial y^j}.
\]

There is a natural action of the Lie algebra \( \mathfrak{X}(\mathcal{F}) \) on the space \( C^\infty(M, \tau) \). The action of a vector field \( X \in \mathfrak{X}(\mathcal{F}) \) on \( N \in C^\infty(M, \tau) \) is given by

\[
\theta(X)N = P_\tau[X, \tilde{N}],
\]

where \( \tilde{N} \in \mathfrak{X}(M) \) is any vector field on \( M \) such that \( P_\tau(\tilde{N}) = N \). A vector field \( N \in \mathfrak{X}(M) \) is projectable if and only if its transverse component \( P_\tau(N) \in C^\infty(M, \tau) \) is invariant under the \( \mathfrak{X}(\mathcal{F}) \)-action \( \theta \). It is easy to see from this that a vector field on a foliated manifold is projectable if and only if it is an infinitesimal transformation of the foliation.

**4.2. Holonomy and transverse structures.** Let \( (M, \mathcal{F}) \) be a foliated manifold. The holonomy map is a generalization of the first-return map (or Poincaré map) for flows to the case of foliations.

**Definition 4.9.** A smooth transversal is a compact \( q \)-dimensional manifold \( T \) (possibly disconnected and with boundary) and an embedding \( i : T \rightarrow M \) whose image is everywhere transverse to the leaves of \( \mathcal{F} : T_i(t) \cap i(T) = T_i(t)M \) for any \( t \in T \).

We will identify a transversal \( T \) with the image \( i(T) \subset M \).

**Definition 4.10.** A transversal is complete if it meets every leaf of the foliation.
Take an arbitrary continuous leafwise path $\gamma$ with initial point $\gamma(0) = x$ and final point $\gamma(1) = y$. (A path $\gamma : [0, 1] \to M$ is said to be leafwise if its image $\gamma([0, 1])$ is entirely contained in one leaf of the foliation.) Let $T_0$ and $T_1$ be smooth transversals such that $x \in T_0$ and $y \in T_1$.

Choose a partition $t_0 = 0 < t_1 < \cdots < t_k = 1$ of $[0, 1]$ such that for any $i = 1, \ldots, k$ the curve $\gamma([t_{i-1}, t_i])$ is contained in some foliated chart $U_i$. Shrinking the neighbourhoods $U_1$ and $U_2$ if necessary, one can assume that for any plaque $P_1$ of $U_1$ there is a unique plaque $P_2$ of $U_2$ which meets $P_1$. Shrinking the neighbourhoods $U_1$, $U_2$, and $U_3$ if necessary, one can assume that for any plaque $P_2$ of $U_2$ there is a unique plaque $P_3$ of $U_3$ which meets $P_2$, and so on. In the end we get a family $\{U_1, U_2, \ldots, U_k\}$ of foliated coordinate neighbourhoods which covers the curve $\gamma([0, 1])$ and is such that for any $i = 1, \ldots, k$ and any plaque $P_{i-1}$ of $U_{i-1}$ there is a unique plaque $P_i$ of $U_i$ which meets $P_{i-1}$. In particular, we get a one-to-one correspondence between the plaques of $U_1$ and the plaques of $U_k$.

The smooth transversal $T_0$ determines a parametrization of the plaques of $U_1$ near $x$. Correspondingly, a smooth transversal $T_1$ determines a parametrization of the plaques of $U_k$ near $y$. Taking into account the one-to-one correspondence constructed above between the plaques of $U_1$ and $U_k$, we get a diffeomorphism $H_{T_0T_1}(\gamma)$ of some neighbourhood of $x$ in $T_0$ onto some neighbourhood of $y$ in $T_1$, which is called the holonomy map along the path $\gamma$.

It is easy to see that the germ of $H_{T_0T_1}(\gamma)$ at $x$ does not depend on the choice of a partition $t_0 = 0 < t_1 < \cdots < t_k = 1$ of $[0, 1]$ and a family $\{U_1, U_2, \ldots, U_k\}$ of foliated coordinate neighbourhoods. Moreover, the germ of $H_{T_0T_1}(\gamma)$ at $x$ is not changed if we replace $\gamma$ by any other continuous leafwise path $\gamma_1$ from $x$ to $y$ which is homotopic to $\gamma$ in the class of continuous leafwise paths from $x$ to $y$.

If $\gamma$ is a closed leafwise path starting and ending at $x$, and $T$ is a smooth transversal such that $x \in T$, then $H_{TT}(\gamma)$ is a local diffeomorphism of $T$ which leaves $x$ fixed. The correspondence $\gamma \to H_{TT}(\gamma)$ defines a group homomorphism $H_T$ from the fundamental group $\pi_1(L_x, x)$ of the leaf $L_x$ to the group Diff$_x(T)$ of germs at $x$ of local diffeomorphisms of $T$ which leave $x$ fixed. The image of $H_T$ is called the holonomy group of the leaf $L_x$ at $x$. The holonomy group of a leaf $L$ at a point $x \in L$ is independent (up to an isomorphism) of the choice of a transversal $T$ and the choice of $x$. A leaf is said to have trivial holonomy if its holonomy group is trivial.

**Example 4.11.** Let $X$ be a complete non-singular vector field on a manifold $M$ of dimension $n$, $x_0$ a (for simplicity, isolated) periodic point of the flow $X_t$ of the given vector field, and $C$ the corresponding closed phase curve. Let $T$ be an $(n-1)$-dimensional submanifold of $M$ passing through $x_0$ and transverse to the vector $X(x_0)$:

$$T_xM = T_xT \oplus \mathbb{R}X(x_0).$$

For all $x \in T$ close enough to $x_0$ there is a least $t(x) > 0$ such that the corresponding positive semi-trajectory $\{X_t(x) : t > 0\}$ of the flow meets $T$: $X_{t(x)}(x) \in T$. Thus, we get a local diffeomorphism $\phi_T : x \mapsto X_{t(x)}(x)$ of $T$ which is defined in a neighbourhood of $x_0$ and takes $x_0$ to itself. This diffeomorphism is called the first return map (or the Poincaré map) along the curve $C$. 
If $\mathcal{F}$ is the foliation on $M$ given by the trajectories of $X$, then the holonomy group of the leaf $C$ coincides with $\mathbb{Z}$, and the germ of $\phi_T$ at $x_0$ is a generator of this group.

For any smooth transversal $T$ and for any $x \in T$ there is a natural isomorphism between the tangent space $T_xT$ and the normal space $\tau_x$ of $\mathcal{F}$. Thus, the normal bundle $\tau$ plays the role of the tangent bundle of the (germs of) transversals to $\mathcal{F}$. For any continuous leafwise path $\gamma$ from a point $x$ to a point $y$ and for any smooth transversals $T_0$ and $T_1$ with $x \in T_0$ and $y \in T_1$ the differential of the holonomy map $H_{T_0T_1}(\gamma)$ at $x$ defines a linear map $dH_{T_0T_1}(\gamma)_x: \tau_x \to \tau_y$. It is easy to check that this map is independent of the choice of transversals $T_0$ and $T_1$. It is called the linear holonomy map and denoted by $dh_{\gamma}: \tau_x \to \tau_y$. Taking the adjoint of $dh_{\gamma}$, one obtains a linear map $dh_{\gamma}^*: N^*\mathcal{F}_y \to N^*\mathcal{F}_x$.

We now turn to another notion connected with holonomy, the notion of holonomy pseudogroup. First, we recall the general definition of a pseudogroup.

**Definition 4.12.** A family $\Gamma$ of diffeomorphisms between open subsets of a manifold $X$ (or, in other words, of local diffeomorphisms of $X$) is called a pseudogroup on $X$ if the following conditions hold:

1) if $\Phi \in \Gamma$, then $\Phi^{-1} \in \Gamma$;
2) if $\Phi_1: U \to U_1$ and $\Phi_2: U_1 \to U_2$ belong to $\Gamma$, then $\Phi_2 \circ \Phi_1: U \to U_2$ belongs to $\Gamma$;
3) if $\Phi: U \to U_1$ belongs to $\Gamma$, then its restriction to any open subset $V \subset U$ belongs to $\Gamma$;
4) if a diffeomorphism $\Phi: U \to U_1$ coincides on some neighbourhood of each point in $U$ with an element of $\Gamma$, then $\Phi \in \Gamma$;
5) the identity diffeomorphism belongs to $\Gamma$.

**Example 4.13.** The set of all local diffeomorphisms of a manifold $X$ forms a pseudogroup on $X$. One can also consider pseudogroups consisting of local diffeomorphisms of $X$ which preserve some geometric structure, for instance, the pseudogroup of local isometries of a Riemannian manifold, and so on.

**Definition 4.14.** Let $(M, \mathcal{F})$ be a smooth foliated manifold and $X$ the disjoint union of all smooth transversals to $\mathcal{F}$. The holonomy pseudogroup of the foliation $\mathcal{F}$ is defined to be the pseudogroup $\Gamma$ of all local diffeomorphisms of $X$ whose germ at any point coincides with the germ of the holonomy map along some leafwise path.

**Definition 4.15.** Let $(M, \mathcal{F})$ be a smooth foliated manifold and $T$ a smooth transversal. The holonomy pseudogroup induced by the foliation $\mathcal{F}$ on $T$ is defined to be the pseudogroup $\Gamma_T$ of all local diffeomorphisms of $T$ whose germ at each point coincides with the germ of the holonomy map along some leafwise path.

There is a special class of smooth transversals given by good covers of the manifold $M$.

**Definition 4.16.** A foliated chart $\phi: U \subset M \to \mathbb{R}^p \times \mathbb{R}^q$ is said to be regular if it admits an extension to a foliated chart $\bar{\phi}: \bar{V} \to \mathbb{R}^p \times \mathbb{R}^q$ such that $\overline{U} \subset \overline{V}$.
Definition 4.17. A cover of a manifold $M$ by foliated neighbourhoods $\{U_i\}$ is said to be good if:

1) any chart $(U_i, \phi_i)$ is a regular foliated chart;
2) if $\overline{U_i} \cap \overline{U_j} \neq \emptyset$, then $U_i \cap U_j \neq \emptyset$ and the set $U_i \cap U_j$ is connected, and the same is true for the corresponding foliated neighbourhoods $V_i$;
3) each plaque for $V_i$ meets at most one plaque of $V_j$, and a plaque of $U_i$ meets a plaque of $U_j$ if and only if the intersection of the corresponding plaques of $V_i$ and $V_j$ is non-empty.

Good covers always exist.

Let $\mathcal{U} = \{U_i\}$ be a good cover for the foliation $\mathcal{F}$, $\phi_i: U_i \xrightarrow{\sim} I^p \times I^q$. For any $i$ put

$$T_i = \phi_i^{-1}([0] \times I^q).$$

Then $T_i$ is a transversal and $T = \bigcup T_i$ is a complete transversal. For $y \in T_i$ denote by $P_i(y)$ the plaque of $U_i$ passing through $y$. For any pair of indices $i$ and $j$ such that $U_i \cap U_j \neq \emptyset$ we define

$$T_{ij} = \{y \in T_i : P_i(y) \cap U_j \neq \emptyset\}.$$

There is defined a transition function $f_{ij}: T_{ij} \to T_{ji}$ given for $y \in T_{ij}$ by the formula $f_{ij}(y) = y_1$, where $y_1 \in T_{ji}$ corresponds to the unique plaque $P_j(y_1)$ such that $P_i(y) \cap P_j(y_1) \neq \emptyset$. The holonomy pseudogroup $\Gamma_T$ induced by $\mathcal{F}$ on $T$ coincides with the pseudogroup generated by the maps $f_{ij}$.

Definition 4.18 (see, for instance, [77]). A transverse structure on a foliation $\mathcal{F}$ is a structure on a complete transversal $T$ that is invariant under the action of the holonomy pseudogroup $\Gamma_T$.

Using the notion of transverse structure, one can distinguish classes of foliations with specific transverse properties. For instance, if a complete transversal $T$ is equipped with a Riemannian metric and the holonomy pseudogroup $\Gamma_T$ consists of local isometries of this Riemannian metric, then we get the class of Riemannian foliations (see §4.3). Similarly, if a complete transversal $T$ is equipped with a symplectic structure and the holonomy pseudogroup $\Gamma_T$ consists of local diffeomorphisms preserving this symplectic structure, then we get the class of symplectic foliations. One can also consider Kähler foliations, measurable foliations, and so on.

In [78] Haefliger defined cohomology groups associated with the transverse structure of a foliation. As above, let $\mathcal{U} = \{U_i\}$ be a good cover for a foliation $\mathcal{F}$, $T = \bigcup T_i$ the corresponding complete transversal, and $\Gamma_T$ the holonomy pseudogroup induced by $\mathcal{F}$ on $T$. Denote by $\Omega^k_c(M/\mathcal{F})$ the quotient of the space $\Omega^k_c(T)$ of smooth compactly supported differential $k$-forms on $T$ by the subspace spanned by the forms $\alpha - h^*\alpha$ with $h \in \Gamma_T$ and the support of $\alpha \in \Omega^k_c(T)$ contained in the image of $h$. Let us endow the space $\Omega^k_c(M/\mathcal{F})$ with the quotient topology induced by the usual $C^\infty$-topology on $\Omega^k_c(T)$. In general, $\Omega^k_c(M/\mathcal{F})$ is not a Hausdorff topological space. The de Rham differential $d_T: \Omega^k_c(T) \to \Omega^{k+1}_c(T)$ induces a continuous differential $d_H: \Omega^k_c(M/\mathcal{F}) \to \Omega^{k+1}_c(M/\mathcal{F})$. It should be noted that both $\Omega^k_c(M/\mathcal{F})$ and $d_H$ are independent of the choice of a good cover $\mathcal{U} = \{U_i\}$. 
The complex \((\Omega_c(M/F), d_H)\) and its cohomology \(H^*_c(M/F)\) are called respectively the Haefliger forms and the Haefliger cohomology of the foliation \(\mathcal{F}\).

If the tangent bundle \(T\mathcal{F}\) is oriented, then there is defined a continuous open surjective linear map called integration along the leaves,\[
\int_{\mathcal{F}} : \Omega^{k+p}(M) \to \Omega^k_c(M/F),
\]
which satisfies the condition \(d_H \circ \int_{\mathcal{F}} = \int_{\mathcal{F}} \circ d\), where \(d\) is the de Rham differential on \(\Omega(M)\). Thus, one has the induced map\[
\int_{\mathcal{F}} : H^{k+p}(M) \to H^k_c(M/F).
\]

Before turning to a discussion of an analogue of the notion of measure on the leaf space of a foliation, we recall some basic facts concerning densities and integration of densities.

**Definition 4.19.** Let \(L\) be an \(n\)-dimensional linear space and \(\mathcal{B}(L)\) the set of bases in \(L\). An \(\alpha\)-density on \(L\) \((\alpha \in \mathbb{R})\) is a function \(\rho : \mathcal{B}(L) \to \mathbb{C}\) such that for any \(A = (A_{ij}) \in GL(n, \mathbb{R})\) and \(e = (e_1, \ldots, e_n) \in \mathcal{B}(L)\)
\[
\rho(e \cdot A) = |\det A|^{\alpha} \rho(e),
\]
where \((e \cdot A)_i = \sum_{j=1}^n e_j A_{ji}, i = 1, \ldots, n.\)

We will denote by \(|L|^\alpha\) the space of all \(\alpha\)-densities on \(L\). For any vector bundle \(V\) on \(M\) denote by \(|V|^\alpha\) the associated bundle of \(\alpha\)-densities, \(|V| = |V|^1\).

For any smooth compactly supported density \(\rho\) on a smooth manifold \(M\) there is a well-defined integral \(\int_M \rho\), regardless of whether \(M\) is orientable or not. This fact enables one to define a Hilbert space \(L^2(M)\) canonically associated with \(M\) and consisting of square-integrable half-densities on \(M\). The diffeomorphism group of \(M\) acts on the space \(L^2(M)\) by unitary transformations.

**Definition 4.20.** A (Borel) transversal to a foliation \(\mathcal{F}\) is a Borel subset of \(M\) which meets each leaf of the foliation in an at most countable set.

**Definition 4.21.** A transverse measure is a countably additive Radon measure \(\Lambda\) defined on the set of all transversals to the foliation.

**Definition 4.22.** A transverse measure \(\Lambda\) is said to be holonomy invariant if for any transversals \(B_1\) and \(B_2\) and for any bijective Borel map \(\phi : B_1 \to B_2\) such that for any \(x \in B_1\) the point \(\phi(x)\) belongs to the leaf through the point \(x\) we have \(\Lambda(B_1) = \Lambda(B_2)\).

**Example 4.23.** A transverse density is defined to be any section of the bundle \(|\tau|\).
Since for any smooth transversal \(T\) there is a canonical isomorphism \(T_x T \cong \tau_x\), a continuous positive transverse density \(\rho \in C(M, |\tau|)\) determines a continuous positive density on \(T\), which in turn determines a transverse measure. This transverse measure is holonomy invariant if and only if \(\rho\) is invariant under the linear holonomy action.
**Example 4.24.** Any compact leaf $L$ of the foliation $\mathcal{F}$ determines a holonomy invariant transverse measure $\Lambda$. For any transversal $T$ and any set $A \subset T$, its measure $\Lambda(A)$ equals the number of elements in $A \cap L$.

**Example 4.25.** Let $\mathcal{F}$ be the horizontal foliation of a flat foliated bundle $M = \tilde{B} \times_{\Gamma} F$ (see Example 4.6). Any measure on $F$ invariant under the action of the group $\Gamma$ determines a holonomy invariant measure for $\mathcal{F}$.

Let $\alpha \in C^\infty(M, |T\mathcal{F}|)$ be a smooth positive leafwise density on $M$. Starting from a transverse measure $\Lambda$ and the density $\alpha$ one can construct a Borel measure $\mu$ on $M$ in the following way. Take a good cover $\{U_i\}$ of $M$ by foliated coordinate neighbourhoods with the corresponding coordinate maps $\phi_i : U_i \to I^p \times I^q$, and a partition of unity $\{\psi_i\}$ subordinate to this cover. We consider the corresponding complete transversal $T = \bigcup_i T_i$, where $T_i = \phi_i^{-1}(\{0\} \times I^q)$. In any foliated chart $(U_i, \phi_i)$ the transverse measure $\Lambda$ defines a measure $\Lambda_i$ on $T_i$, and the smooth positive leafwise density $\alpha$ defines a family $\{\alpha_{i,y} : y \in T_i\}$, where $\{\alpha_{i,y}\}$ is a smooth positive density on the plaque $P_i(y)$. Observe that $\Lambda$ is holonomy invariant if and only if, for any pair of indices $i$ and $j$ such that $U_i \cap U_j \neq \emptyset$, we have the relation $f_{ij}(\Lambda_i) = \Lambda_j$.

For any $u \in C^c_\infty(M)$ put

$$\int_M u(m) \, d\mu(m) = \sum_i \int_{T_i} \int_{P_i(y)} \psi_i(x,y)u(x,y)\alpha_{i,y}(x) \, d\Lambda_i(y). \tag{4.1}$$

One can show that this formula defines a measure $\mu$ on $M$ which is independent of the choice of a cover $\{U_i\}$ and a partition of unity $\{\psi_i\}$.

A measure $\mu$ on $M$ will be said to be holonomy invariant if it is obtained from a holonomy invariant transverse measure $\Lambda$ by means of the above construction with some choice of a smooth positive leafwise density $\alpha$.

If in the above construction we take the restrictions to the leaves of an arbitrary differential $p$-form $\omega$ on $M$ instead of the leafwise density $u \cdot \alpha$, then we obtain a well-defined functional $C$ on $C^c_\infty(M, \Lambda^p T^*M)$ called the Ruelle–Sullivan current corresponding to $\Lambda$:

$$\langle C, \omega \rangle = \sum_i \int_{T_i} \int_{P_i(y)} \psi_i(x,y)\omega_{i,y}(x) \, d\Lambda_i(y), \quad \omega \in C^c_\infty(M, \Lambda^p T^*M),$$

where $\omega_{i,y}$ is the restriction of $\omega$ to the plaque $P_i(y)$, $y \in T_i$.

A transverse measure $\Lambda$ is holonomy invariant if and only if the corresponding Ruelle–Sullivan current $C$ is closed:

$$\langle C, d\sigma \rangle = 0, \quad \sigma \in C^c_\infty(M, \Lambda^{p-1} T^*M).$$

**Example 4.26.** Suppose that a transverse measure $\Lambda$ is given by a smooth positive transverse density $\rho \in C^\infty(M, |\tau|)$. Take a positive leafwise density $\alpha \in C^\infty(M, |T\mathcal{F}|)$. Then the corresponding measure $\mu$ on $M$ is given by the smooth positive density $\alpha \otimes \rho \in C^\infty(M, |TM|)$, which corresponds to $\alpha$ and $\rho$ under the canonical isomorphism $|TM| \cong |T\mathcal{F}| \otimes |\tau|$ given by the short exact sequence $0 \to T\mathcal{F} \to TM \to \tau \to 0$. 
Example 4.27. Suppose that a holonomy invariant transverse measure $\Lambda$ is given by a compact leaf $L$ of the foliation $\mathcal{F}$, and $\alpha \in C^\infty(M, |T\mathcal{F}|)$ is a smooth positive leafwise density on $M$. Then the corresponding measure $\mu$ on $M$ is a $\delta$-measure along $L$:

$$\int_M f(x) \, d\mu(x) = \int_L f(x) \alpha(x), \quad f \in C_c(M).$$

Example 4.28. Suppose that a foliation $\mathcal{F}$ is given by the orbits of a locally free action of a Lie group $H$ on a compact manifold $M$ and a smooth leafwise density $\alpha$ is given by a fixed Haar measure $dh$ on $H$. Then the corresponding measure $\mu$ on $M$ is holonomy invariant if and only if it is invariant under the action of $H$.

4.3. Transverse Riemannian geometry. An infinitesimal expression of the holonomy on a foliated manifold is the canonical flat connection

$$\tilde{\nabla}: \mathcal{X}(\mathcal{F}) \times C^\infty(M, \tau) \to C^\infty(M, \tau)$$

defined in the normal bundle $\tau$ along the leaves of $\mathcal{F}$ (the Bott connection) [79]. It is given by

$$\tilde{\nabla}_X N = \theta(X)N = P_\tau[X, \tilde{N}], \quad X \in \mathcal{X}(\mathcal{F}), \quad N \in C^\infty(M, \tau), \quad (4.2)$$

where $\tilde{N} \in C^\infty(M, TM)$ is any vector field on $M$ such that $P_\tau(\tilde{N}) = N$. Thus, the restriction of $\tau$ to any leaf of $\mathcal{F}$ is a flat vector bundle. The parallel transport defined by $\tilde{\nabla}$ in $\tau$ along any leafwise path $\gamma: x \to y$ coincides with the linear holonomy map $dh_\gamma: \tau_x \to \tau_y$.

Definition 4.29. A connection $\nabla: \mathcal{X}(M) \times C^\infty(M, \tau) \to C^\infty(M, \tau)$ in the normal bundle $\tau$ is said to be adapted if its restriction to $\mathcal{X}(\mathcal{F})$ coincides with the Bott connection $\tilde{\nabla}$.

One can construct an adapted connection starting with an arbitrary Riemannian metric $g_M$ on $M$. Denote by $\nabla^g$ the Levi-Civita connection determined by $g_M$. An adapted connection $\nabla$ is given by

$$\nabla_X N = P_\tau[X, \tilde{N}], \quad X \in \mathcal{X}(\mathcal{F}), \quad N \in C^\infty(M, \tau),$$

$$\nabla_X N = P_\tau \nabla^g_X \tilde{N}, \quad X \in C^\infty(M, F^\perp), \quad N \in C^\infty(M, \tau), \quad (4.3)$$

where $\tilde{N} \in C^\infty(M, TM)$ is any vector field such that $P_\tau(\tilde{N}) = N$. One can show that the adapted connection $\nabla$ described above is torsion-free.

Definition 4.30. An adapted connection $\nabla$ in the normal bundle $\tau$ is said to be holonomy invariant if for any $X \in \mathcal{X}(\mathcal{F})$, $Y \in \mathcal{X}(M)$, and $N \in C^\infty(M, \tau)$ we have

$$(\theta(X)\nabla)_Y N := \theta(X)[\nabla_Y N] - \nabla_{\theta(X)Y} N - \nabla_Y [\theta_Y N] = 0.$$
A fundamental property of basic connections is the fact that their curvature $R_{\nabla}$ is a basic form, that is, $i_X R_{\nabla} = 0$, $\theta(X) R_{\nabla} = 0$ for any $X \in \mathcal{X}(\mathcal{F})$. There are topological obstructions for the existence of basic connections for an arbitrary foliation.

**Definition 4.31.** A foliation $\mathcal{F}$ on a manifold $M$ is said to be Riemannian if it has a transverse Riemannian structure. In other words, $\mathcal{F}$ is Riemannian if there is a cover $\{U_i\}$ of $M$ by foliated coordinate charts, $\phi_i: U_i \rightarrow \mathbb{I}^p \times \mathbb{I}^q$, and Riemannian metrics $g^{(i)}(y) = \sum_{\alpha\beta} g_{\alpha\beta}^{(i)}(y) dy^\alpha dy^\beta$ defined on the local bases $I^q$ of $\mathcal{F}$ such that, for any coordinate transformation $\phi_{ij}(x, y) = \alpha_{ij}(x, y)$, $\gamma_{ij}(y)$, $(x, y) \in \phi_j(\bigcap U_i \cap U_j)$, the map $\gamma_{ij}$ preserves the metric on $I^q$, $\gamma_{ij}^*(g^{(j)}) = g^{(i)}$.

**Theorem 4.32.** A foliation $\mathcal{F}$ is Riemannian if and only if there exists a Riemannian metric $g^M$ on $M$ such that the induced metric $g_\tau$ on the normal bundle $\tau$ is holonomy invariant: for any $X \in \mathcal{X}(\mathcal{F})$ and for any $U, V \in C^\infty(M, \tau)$,

$$\overset{\circ}{\nabla}_X g_\tau(U, V) := X[g_\tau(U, V)] - g_\tau(\overset{\circ}{\nabla}_X U, V) - g_\tau(U, \overset{\circ}{\nabla}_X V) = 0.$$

**Definition 4.33.** Any Riemannian metric on $M$ satisfying the conditions of Theorem 4.32 is said to be bundle-like.

For a Riemannian foliation $\mathcal{F}$ and a bundle-like metric $g^M$ the adapted connection $\nabla$ on the normal bundle $\tau$ given by (4.3) is a Riemannian connection: for any $Y \in \mathcal{X}(M)$ and $U, V \in C^\infty(M, \tau)$,

$$Y[g_\tau(U, V)] = g_\tau(\nabla_Y U, V) + g_\tau(U, \nabla_Y V).$$

One can prove that a (torsion-free) Riemannian connection on the normal bundle $\tau$ of a Riemannian foliation $\mathcal{F}$ is unique. It is uniquely determined by the transverse metric $g_\tau$ and is called the transverse Levi-Civita connection for $\mathcal{F}$. Thus, the transverse Levi-Civita connection is an adapted connection. Moreover, it turns out that the transverse Levi-Civita connection is holonomy invariant, and therefore is a basic connection. In particular, this proves the existence of a basic connection for any Riemannian foliation.

The existence of a bundle-like metric on a foliated manifold imposes strong restrictions on the geometry of the foliation. There are structure theorems for Riemannian foliations obtained by Molino. Using these structure theorems, one can reduce many questions concerning Riemannian foliations to the case of Lie foliations, that is, foliations with transverse structure modelled by a finite-dimensional Lie group (see Example 4.36).

**Example 4.34.** Any foliation defined by a submersion $\pi: M \rightarrow B$ is Riemannian.

**Example 4.35.** The orbits of a locally free isometric action of a Lie group on a Riemannian manifold define a Riemannian foliation. On the other hand, flows whose orbits form a Riemannian foliation are called Riemannian flows. There are examples of Riemannian flows which are not isometric (see, for instance, [80]).
Example 4.36. Let $M$ be a smooth manifold, $\mathfrak{g}$ a real finite-dimensional Lie algebra, and $\omega$ a $1$-form on $M$ with values in $\mathfrak{g}$ satisfying the conditions
1) the map $\omega_x : T_x M \to \mathfrak{g}$ is surjective for any $x \in M$,
2) $d\omega + \frac{1}{2}[\omega, \omega] = 0$.

The distribution $F_x = \ker \omega_x$ is integrable, and hence defines a foliation of codimension $q = \dim \mathfrak{g}$ on $M$. Such a foliation is called a Lie $\mathfrak{g}$-foliation. Any Lie foliation is Riemannian.

Example 4.37. The horizontal foliation $\mathcal{F}$ on a flat foliated bundle $M$ (see Example 4.6) is Riemannian if and only if for any $\gamma \in \Gamma$ the diffeomorphism $\phi(\gamma)$ preserves some Riemannian metric on $\mathcal{F}$.

Let $\mathcal{F}$ be a transversally oriented Riemannian foliation and $g$ a bundle-like Riemannian metric. The induced metric on $\tau$ yields the transverse volume form $v_\tau \in C^\infty(M, \Lambda^q \tau^*) = C^\infty(M, \Lambda^q N^* \mathcal{F})$ which is holonomy invariant and, therefore, gives rise to a holonomy invariant transverse measure on $\mathcal{F}$.

5. Non-commutative topology of foliations

In this section we will describe the non-commutative algebras associated with the leaf space of a foliation. First, we will define an algebra consisting of very nice functions, on which all basic operations of analysis are defined; then, depending on the problem in question, we will complete this algebra and obtain an analogue of the algebra of measurable, continuous, or smooth functions. The role of a ‘nice’ algebra is played by the algebra $C^\infty_c(G)$ of smooth compactly supported functions on the holonomy groupoid $G$ of the foliation. Therefore, we start with the notion of the holonomy groupoid of a foliation.

5.1. Holonomy groupoid. A foliation $\mathcal{F}$ defines an equivalence relation $\mathcal{R} \subset M \times M$ on $M$: $(x, y) \in \mathcal{R}$ if and only if $x$ and $y$ lie on the same leaf of $\mathcal{F}$. Generally, $\mathcal{R}$ is not a smooth manifold, but one can resolve its singularities, constructing a smooth manifold $G$ called the holonomy groupoid or the graph of the foliation, which coincides ‘almost everywhere’ with $\mathcal{R}$ and which can be used in many cases as a substitute for $\mathcal{R}$. The idea of the holonomy groupoid appeared in papers of Ehresmann, Reeb, and Thom and was completely realized by Winkelnkemper in [81]. First of all, we give the general definition of a groupoid.

**Definition 5.1.** We say that a set $G$ carries the structure of a groupoid with a set $G^{(0)}$ of units if there are maps
1) $\Delta : G^{(0)} \to G$ (the diagonal map or the unit map),
2) an involution $i : G \to G$ called the inversion and written as $i(\gamma) = \gamma^{-1}$,
3) the range map $r : G \to G^{(0)}$ and the source map $s : G \to G^{(0)}$,
4) the associative multiplication $m : (\gamma, \gamma') \to \gamma \gamma'$ defined on the set

$$G^{(2)} = \{ (\gamma, \gamma') \in G \times G : r(\gamma') = s(\gamma) \},$$

satisfying the conditions
(i) $r(\Delta(x)) = s(\Delta(x)) = x$ and $\gamma \Delta(s(\gamma)) = \gamma$, $\Delta(r(\gamma)) \gamma = \gamma$,
(ii) $r(\gamma^{-1}) = s(\gamma)$ and $\gamma \gamma^{-1} = \Delta(r(\gamma))$. 
Alternatively, one can define a groupoid as a small category in which each morphism is an isomorphism. In particular, it is convenient to represent an element $\gamma \in G$ as an arrow $\gamma: x \to y$, where $x = s(\gamma)$ and $y = r(\gamma)$. We will also use the standard notation (for $x, y \in G(0)$):

$$G^x = \{ \gamma \in G : r(\gamma) = x \} = r^{-1}(x), \quad G_x = \{ \gamma \in G : s(\gamma) = x \} = s^{-1}(x),$$

$$G^x_y = \{ \gamma \in G : s(\gamma) = x, \ r(\gamma) = y \}.$$

**Definition 5.2.** A groupoid $G$ is said to be smooth (or a Lie groupoid) if $G^{(0)}$, $G$, and $G^{(2)}$ are smooth manifolds, $r$, $s$, $i$, and $m$ are smooth maps, $r$ and $s$ are submersions, and $\Delta$ is an immersion.

**Example 5.3** (trivial groupoid). Let $X$ be an arbitrary set, put $G = X$ and $G^{(0)} = X$, and let the maps $s$ and $r$ be the identity maps (in other words, each element $x \in G^{(0)} = X$ is identified with the unique element $\gamma: x \to x$).

**Example 5.4** (equivalence relations). Any equivalence relation $R \subset X \times X$ defines a groupoid if one puts $G^{(0)} = X$ and $G = R$, and lets the maps $s: R \to X$ and $r: R \to X$ be given by $s(x, y) = y$ and $r(x, y) = x$. Thus, pairs $(x_1, y_1)$ and $(x_2, y_2)$ can be multiplied if and only if $y_1 = x_2$, and then $(x_1, y_1)(x_2, y_2) = (x_1, y_2)$. Moreover,

$$\Delta(x) = (x, x), \quad x \in X,$$

$$(x, y)^{-1} = (y, x), \quad (x, y) \in R.$$

In the particular case when $R = X \times X$, we obtain a so-called principal groupoid.

**Example 5.5** (Lie groups). A Lie group $H$ defines a smooth groupoid as follows: $G = H$, $G^{(0)}$ consists of a single point, and the maps $i$ and $m$ are defined by the group operations in $H$.

**Example 5.6** (group actions). Let a Lie group $H$ act smoothly from the left on a smooth manifold $X$. The crossed product groupoid $X \rtimes H$ is defined as follows: $G^{(0)} = X$, $G = X \times H$. The maps $s: X \times H \to X$ and $r: X \times H \to X$ have the form $s(x, h) = h^{-1}x$, $r(x, h) = x$. Thus, pairs $(x_1, h_1)$ and $(x_2, h_2)$ can be multiplied if and only if $x_2 = h_1^{-1}x_1$, and then $(x_1, h_1)(x_2, h_2) = (x_1, h_1h_2)$. Moreover,

$$\Delta(x) = (x, e), \quad x \in X,$$

$$(x, h)^{-1} = (h^{-1}x, h^{-1}), \quad x \in X, \ h \in H.$$

**Example 5.7** (the fundamental groupoid). Let $X$ be a topological space, $G = \Pi(X)$ the set of homotopy classes of paths in $X$ with all possible endpoints. More precisely, if $\gamma: [0, 1] \to X$ is a path from $x = \gamma(0)$ to $y = \gamma(1)$, then we denote by $[\gamma]$ the homotopy class of $\gamma$ with fixed $x$ and $y$. Define the groupoid $\Pi(X)$ as the set of triples $(x, [\gamma], y)$ with $x, y \in X$ and $\gamma$ a path from $x = \gamma(0)$ to $y = \gamma(1)$ and with multiplication given by the product of paths. The groupoid $\Pi(X)$ is called the fundamental groupoid of $X$.

**Example 5.8** (the Haefliger groupoid $\Gamma_n$ [82], [83]). Let $M$ be a smooth manifold. The groupoid $\Gamma_M$ consists of germs of local diffeomorphisms of $M$ at various points.
of $M$. Thus, $(\Gamma_M)^{(0)} = M$. If $\gamma \in \Gamma_M$ is the germ at $x \in M$ of a diffeomorphism $f$ from some neighbourhood $U$ of $x$ onto the open set $f(U)$, then $s(\gamma) = x$ and $r(\gamma) = f(x)$. The multiplication in $\Gamma_M$ is given by the composition of maps. If $M = \mathbb{R}^n$, then the groupoid $\Gamma_M$ is denoted by $\Gamma_n$.

The holonomy groupoid $G = G(M, \mathcal{F})$ of a foliated manifold $(M, \mathcal{F})$ is defined in the following way. Let $\sim_h$ be the equivalence relation on the set of continuous leafwise paths $\gamma : [0, 1] \to M$ specifying that $\gamma_1 \sim_h \gamma_2$ if $\gamma_1$ and $\gamma_2$ have the same initial and final points and the same holonomy maps: $h_{\gamma_1} = h_{\gamma_2}$. The holonomy groupoid $G$ is the set of $\sim_h$-equivalence classes of leafwise paths. The set $G^{(0)}$ of units is the manifold $M$. The multiplication in $G$ is given by the product of paths. The corresponding source and range maps $s, r : G \to M$ are given by $s(\gamma) = \gamma(0)$ and $r(\gamma) = \gamma(1)$. Finally, the diagonal map $\Delta : M \to G$ takes any $x \in M$ to the element in $G$ given by the constant path $\gamma(t) = x$, $t \in [0, 1]$. To simplify the notation we will identify $x \in M$ with $\Delta(x) \in G$.

For any $x \in M$ the map $s$ maps $G^x$ onto the leaf $L_x$ through $x$. The group $G^x_G$ coincides with the holonomy group of $L_x$. The map $s : G^x \to L_x$ is the regular covering with covering group $G^x_G$, called the holonomy covering.

The holonomy groupoid $G$ has the structure of a smooth (generally non-Hausdorff and non-paracompact) manifold of dimension $2p + q$. We recall the construction of an atlas on $G$ [15].

Suppose that $\phi : U \to I^p \times I^q$ and $\phi' : U' \to I^p \times I^q$ are two foliated charts, and let $\pi = \text{pr}_{nq} \circ \phi : U \to \mathbb{R}^q$ and $\pi' = \text{pr}_{nq} \circ \phi' : U' \to \mathbb{R}^q$ be the corresponding distinguished maps. The charts $\phi$ and $\phi'$ are said to be compatible if for any $m \in U$ and $m' \in U'$ with $\pi(m) = \pi'(m')$ there is a leafwise path $\gamma$ from $m$ to $m'$ such that the corresponding holonomy map $h_\gamma$ takes the germ $\pi_m$ of $\pi$ at $m$ to the germ $\pi_{m'}$ of $\pi'$ at $m'$.

For any pair of compatible foliated charts $\phi$ and $\phi'$ denote by $W(\phi, \phi')$ the subset of $G$ consisting of all $\gamma \in G$ with $s(\gamma) = m = \phi^{-1}(x, y) \in U$ and $r(\gamma) = m' = \phi'^{-1}(x', y') \in U'$ such that the corresponding holonomy map $h_\gamma$ takes the germ $\pi_m$ of the map $\pi = \text{pr}_{nq} \circ \phi$ at $m$ to the germ $\pi_{m'}$ of the map $\pi' = \text{pr}_{nq} \circ \phi'$ at $m'$. There is a coordinate map

$$\Gamma : W(\phi, \phi') \to I^p \times I^p \times I^q,$$

which takes each element $\gamma \in W(\phi, \phi')$ such that $s(\gamma) = m = \mathcal{X}^{-1}(x, y)$, $r(\gamma) = m' = \mathcal{X}'^{-1}(x', y')$, and $h_{\gamma} \pi_m = \pi_{m'}$ to the triple $(x, x', y) \in I^p \times I^p \times I^q$. As shown in [15], the coordinate neighbourhoods $W(\phi, \phi')$ form an atlas of a $(2p + q)$-dimensional manifold (generally non-Hausdorff and non-paracompact) on $G$. Moreover, the groupoid $G$ is a smooth groupoid.

Non-Hausdorffness of the holonomy groupoid is related with the phenomenon of one-sided holonomy. The simplest example of a foliation with non-Hausdorff holonomy groupoid is given by the trajectories of a non-singular vector field on the plane having a one-sided limit cycle. As shown in [81], the holonomy groupoid is Hausdorff if and only if the holonomy maps $H_{T_0 T_1}(\gamma_1)$ and $H_{T_0 T_1}(\gamma_2)$ along any leafwise paths $\gamma_1$ and $\gamma_2$ from $x$ to $y$ and given by smooth transversals $T_0$ and $T_1$ passing through $x$ and $y$, respectively, coincide if they coincide on some open subset $U \subset T_0$ such that $x \in \overline{U}$. In particular, the holonomy groupoid is Hausdorff.
if the holonomy is trivial or real-analytic. Moreover, the holonomy groupoid of a Riemannian foliation is Hausdorff. In the following we will always assume that $G$ is a Hausdorff manifold.

**Example 5.9.** If $\mathcal{F}$ is a simple foliation defined by a submersion $\pi: M \to B$, then its holonomy groupoid $G$ consists of all $(x, y) \in M \times M$ such that $\pi(x) = \pi(y)$, and moreover, $G^{(0)} = M$, and the maps $s: G \to M$ and $r: G \to M$ are given by $s(x, y) = y$ and $r(x, y) = x$.

**Example 5.10.** If a foliation $\mathcal{F}$ is given by the orbits of a free smooth action of a connected Lie group $H$ on a manifold $M$, then its holonomy groupoid coincides with the crossed product groupoid $M \rtimes H$.

**Example 5.11.** Consider the horizontal foliation $\mathcal{F}$ on a flat foliated bundle $M$ (see Example 4.6). Suppose that the following condition holds: if for some element $g \in \Gamma$ there exists an open set $U$ such that $xg = x$ for any $x \in U$, then $g$ is the identity element of the group $\Gamma$. Under this condition, the holonomy groupoid $G$ of $\mathcal{F}$ is isomorphic to the orbit space of the action of $\Gamma$ given by $(b_1, b_2, f)g = (b_1g, b_2g, \phi(g)f)$ on the manifold $\tilde{B} \times \tilde{B} \times F$, where $(b_1, b_2, f) \in \tilde{B} \times \tilde{B} \times F$, $g \in \Gamma$.

$$G \cong (\tilde{B} \times \tilde{B} \times F) / \Gamma.$$ 

Denote by $[b_1, b_2, f]$ the equivalence class of an element $(b_1, b_2, f) \in \tilde{B} \times \tilde{B} \times F$ in $(\tilde{B} \times \tilde{B} \times F) / \Gamma$. Then the range and source maps in the groupoid $G$ are given by $r([b_1, b_2, f]) = [b_1, f]$, $s([b_1, b_2, f]) = [b_2, f]$.

Elements $[b_1, b_2, f]$ and $[b'_1, b'_2, f']$ can be multiplied if and only if there exists an element $g \in \Gamma$ such that $b_2 = b'_1g$ and $f = \phi(g)f'$. In this case $$[b_1, b_2, f][b'_1, b'_2, f'] = [b_1g^{-1}, b'_2, f'].$$

In addition to the holonomy groupoid there are other groupoids which can be associated with the foliation. First of all it is the groupoid given by the equivalence relation on $M$ for which points $x$ and $y$ are equivalent if they lie on the same leaf of the foliation (the coarse groupoid). As mentioned above, this groupoid is not smooth. One can consider the fundamental groupoid of the foliation $\Pi(M, \mathcal{F})$, which also consists of equivalence classes of leafwise paths, but in this case two leafwise paths are equivalent if they are homotopic in the class of leafwise paths with fixed endpoints. The fundamental groupoid of the foliation $\Pi(M, \mathcal{F})$ is a smooth groupoid (see, for instance, [84]).

There is a foliation $\mathcal{G}$ of dimension $2p$ on the holonomy groupoid. In any coordinate chart $W(\phi, \phi')$ given by a pair of compatible foliated charts $\phi$ and $\phi'$ the leaves of $\mathcal{G}$ are given by equations of the form $y = \text{const}$. The leaf of $\mathcal{G}$ through $\gamma \in G$ consists of all $\gamma' \in G$ such that $r(\gamma)$ and $r(\gamma')$ lie on the same leaf of $\mathcal{F}$, and it coincides with the holonomy groupoid of this leaf. The holonomy group of a leaf of $\mathcal{G}$ coincides with the holonomy group of the corresponding leaf of $\mathcal{F}$.

The differential of the map $(r, s): G \to M \times M$ maps the tangent bundle $T\mathcal{G}$ of $\mathcal{G}$ isomorphically to the bundle $F \boxtimes F$ on $M \times M$, therefore, there is a canonical isomorphism $T\mathcal{G} \cong r^* F \oplus s^* F$. 
A distribution $H$ on $M$ transverse to $\mathcal{F}$ determines a distribution $HG$ on $G$ transverse to $\mathcal{G}$. For any $X \in H_y$ there is a unique vector $\tilde{X} \in T_{\gamma}G$ such that $ds(\tilde{X}) = dh^{-1}_y(X)$ and $dr(\tilde{X}) = X$, where $dh_\gamma : H_x \rightarrow H_y$ is the linear holonomy map associated with $\gamma$. The space $H_\gamma G$ consists of all vectors of the form $\tilde{X} \in T_{\gamma}G$ for different $X \in H_y$. In any coordinate chart $W(\phi, \phi')$ on $G$ the tangent space $T_{\gamma}\mathcal{G}$ of $\mathcal{G}$ at some $\gamma$ with coordinates $(x, x', y)$ consists of vectors of the form $X \frac{\partial}{\partial x} + X' \frac{\partial}{\partial x'}$, and the distribution $H_\gamma G$ consists of the vectors $X \frac{\partial}{\partial x} + X' \frac{\partial}{\partial x'} + Y \frac{\partial}{\partial y}$ such that $X \frac{\partial}{\partial x} + Y \frac{\partial}{\partial y} \in H_{(x,y)}$ and $X' \frac{\partial}{\partial x'} + Y \frac{\partial}{\partial y} \in H_{(x',y)}$.

Let $g_M$ be a Riemannian metric on $M$ and $H = F^\perp$. Then a Riemannian metric $g_G$ on $G$ is defined as follows. All the components in $T_\gamma G = F_y \oplus F_x \oplus H_\gamma G$ are mutually orthogonal, and by definition $g_G$ coincides with $g_M$ on $F_y \oplus H_\gamma G \cong F_y \oplus H_y = T_y M$ and with $g_F$ on $F_x$.

If $\mathcal{F}$ is Riemannian and $g_M$ is a bundle-like Riemannian metric, then $g_G$ is bundle-like, and therefore $\mathcal{G}$ is Riemannian. Moreover, in this case the maps $s : G \rightarrow M$ and $r : G \rightarrow M$ are Riemannian submersions and locally trivial fibrations. In particular, all the holonomy coverings $G^z$ of leaves of $\mathcal{F}$ are diffeomorphic.

Let $G$ be an arbitrary Lie groupoid with the set of units $G^{(0)} = Z$. For any (possibly non-Hausdorff) smooth manifold $Z$ and any smooth map $\rho : Z \rightarrow M$ put

$$Z \times_\rho G = \{(z, \gamma) \in Z \times G : \rho(z) = s(\gamma)\}.$$  

A smooth right action of the groupoid $G$ on $Z$ is defined to be a map $Z \times_\rho G \rightarrow Z$, $(z, \gamma) \mapsto z\gamma$, satisfying the conditions

$$\rho(z\gamma) = r(\gamma), \quad (z\gamma)\gamma' = z(\gamma\gamma'), \quad z \cdot x = x.$$

As an example, one can consider the action of $G$ on $M$ defined by

$$\rho = \text{id} : M \rightarrow M, \quad M \times_\rho G \ni (y, \gamma) \mapsto y\gamma = s(\gamma) \in M.$$  

An action of the groupoid $G$ on $Z$ is said to be proper if

1. the map $Z \times_\rho G \rightarrow Z \times Z$, $(z, \gamma) \mapsto (z, z\gamma)$, is proper (that is, the pre-image of every compact set is compact);

2. the set $Z/\Gamma$ of equivalence classes of the equivalence relation $\Gamma$ on $Z$ with $z \sim z'$ whenever $z\gamma = z'$ for some $\gamma \in G$ is Hausdorff.

If $Z$ is a $G$-manifold, then the orbits of the $G$-action define a foliation on $Z$.

5.2. The $C^*$-algebra and the von Neumann algebra of a foliation. In this subsection we will describe the construction of the $C^*$-algebra associated with an arbitrary foliation. This algebra can be regarded as a non-commutative analogue of the algebra of continuous functions on the leaf space of the foliation. We will only consider Hausdorff groupoids. For the definition of the $C^*$-algebra of a foliation in the case when the holonomy groupoid is not Hausdorff, see, for instance, [13].

There are two ways to define $C^*$-algebras associated with a foliation. The first makes use of the auxiliary choice of a smooth Haar system, and the second does not require auxiliary choices and uses the language of half-densities.
5.2.1. Definitions using a Haar system. In this subsection we give the definition of the $C^*$-algebras associated with an arbitrary smooth groupoid $G$. In fact, the assumption of smoothness of the groupoid is not essential here, and all the definitions can be generalized to the case of topological groupoids.

**Definition 5.12.** A smooth Haar system on a smooth groupoid $G$ is a family $\{\nu^x : x \in G^{(0)}\}$ of positive Radon measures on $G$ satisfying the following conditions:

1) the support of $\nu^x$ coincides with $G^x$, and $\nu^x$ is a smooth positive measure on $G^x$;

2) the family $\{\nu^x : x \in G^{(0)}\}$ is left-invariant, that is, for any continuous function with compact support $f \in C_c(G^x)$, $f \geq 0$, and any $\gamma \in G$ such that $s(\gamma) = x$ and $r(\gamma) = y$ we have

$$\int_{G^y} f(\gamma_1) d\nu^y(\gamma_1) = \int_{G^x} f(\gamma_1) d\nu^x(\gamma_1);$$

3) the family $\{\nu^x : x \in G^{(0)}\}$ is smooth, that is, for any $\phi \in C^\infty_c(G)$ the function

$$G^{(0)} \ni x \mapsto \int_{G^x} \phi(\gamma) d\nu^x(\gamma)$$

is a smooth function on $G^{(0)}$.

For a compact foliated manifold $(M, \mathcal{F})$ a smooth Haar system $\{\nu^x : x \in G^{(0)}\}$ on the holonomy groupoid $G$ of $\mathcal{F}$ is given by an arbitrary smooth positive leafwise density $\alpha \in C^\infty(M, |T\mathcal{F}|)$. For any $x \in M$ the positive Radon measure $\nu^x$ on $G^x$ is defined as the lift of the density $\alpha$ by the holonomy covering $s: G^x \to M$.

Let $G$ be a smooth groupoid, $G^{(0)} = M$, and $\{\nu^x : x \in M\}$ a smooth Haar system. We introduce an involutive algebra structure on $C^\infty_c(G)$ by

$$(k_1 * k_2)(\gamma) = \int_{G^x} k_1(\gamma_1)k_2(\gamma_1^{-1}\gamma) d\nu^x(\gamma_1), \quad \gamma \in G^x,$$

$$k^*(\gamma) = \overline{k(\gamma^{-1})}, \quad \gamma \in G.$$

For any $x \in M$ there is a natural representation of $C^\infty_c(G)$ on $L^2(G^x, \nu^x)$ given for $k \in C^\infty_c(G)$ and $\zeta \in L^2(G^x, \nu^x)$ by

$$R_x(k)\zeta(\gamma) = \int_{G^x} k(\gamma_1\gamma^{-1})\zeta(\gamma_1) d\nu^x(\gamma_1), \quad r(\gamma) = x.$$ 

The completion of the involutive algebra $C^\infty_c(G)$ in the norm

$$\|k\| = \sup_x \|R_x(k)\|$$

is called the reduced $C^*$-algebra of the groupoid $G$ and denoted by $C^*_r(G)$. Also, the full $C^*$-algebra $C^*(G)$ of the groupoid is defined as the completion of $C^\infty_c(G)$ in the norm

$$\|k\|_{\text{max}} = \sup \|\pi(k)\|,$$

where the supremum is taken over the set of all $*$-representations $\pi$ of the algebra $C^\infty_c(G)$ on Hilbert spaces.
Example 5.13. In Example 5.3 the groupoid $G$ is smooth if $X$ is a smooth manifold. In this case $G^x = \{x\}$ for any $x \in X$, and a smooth Haar system is given by an arbitrary smooth function on $X$. The operator algebras $C^*_r(G)$ and $C^*(G)$ coincide with the commutative $C^*$-algebra $C_0(X)$.

Example 5.14. The principal groupoid introduced in Example 5.4 is smooth if $X$ is a smooth manifold. In this case any smooth Haar system has the following form: $\nu^x$ is a fixed smooth positive density $\mu$ in each $G^x \cong X$. The operations in $C^c_\infty(G)$ are given by

$$(k_1 * k_2)(x, y) = \int_X k_1(x, z)k_2(z, y) \, d\mu(z), \quad (x, y) \in X \times X,$$

$$k^*(x, y) = k(y, x), \quad (x, y) \in X \times X,$$

where $k, k_1, k_2 \in C^c_\infty(G)$. Thus, elements of $C^c_\infty(G)$ can be regarded as the kernels of integral operators acting in $C^\infty(X)$ (with respect to the density $\mu$). For any $x \in X$ the representation $R_x$ associates to every $k \in C^c_\infty(G) \subset C^\infty(X \times X)$ the integral operator acting in $L^2(G^x, \nu^x) \cong L^2(X, \mu)$ with integral kernel $k$:

$$R_x(k)u(y) = \int_X k(y, z)u(z) \, d\mu(z), \quad u \in L^2(X, \mu).$$

Finally, $C^*_r(G)$ and $C^*(G)$ coincide with the algebra $\mathcal{K}(L^2(X, \mu))$ of compact operators on $L^2(X, \mu)$.

Example 5.15. In Example 5.5 a smooth Haar system is given by a left-invariant Haar measure $dh$ on $H$: $\nu^x = dh$. The product in $C^c_\infty(G)$ is the classical convolution operation given for any functions $u, v \in C^\infty_c(H)$ by

$$(u * v)(g) = \int_H u(h)v(h^{-1}g) \, dh, \quad g \in H,$$

the involution is given by

$$u^*(g) = \overline{u(g^{-1})}, \quad u \in C^\infty_c(H),$$

and the operator algebras $C^*_r(G)$ and $C^*(G)$ are the group $C^*$-algebras $C^*_r(H)$ and $C^*(H)$.

Example 5.16. In Example 5.6 the manifold $G^x = \{(x, h) : h \in H\}$ is diffeomorphic to $H$ for any $x \in X$, and a smooth Haar system on $G$ can be defined by using an arbitrary Haar measure on $H$. The product in $C^c_\infty(G)$ is given for any functions $u, v \in C^\infty_c(X \times H)$ by

$$(u * v)(x, g) = \int_H u(x, h)v(h^{-1}x, h^{-1}g) \, dh, \quad (x, g) \in X \times H,$$

the involution is given, for a function $u \in C^\infty_c(X \times H)$, by

$$u^*(x, g) = \overline{u(g^{-1}x, g^{-1})}, \quad (x, g) \in X \times H.$$
The operator algebras $C^*_r(G)$ and $C^*(G)$ corresponding to the crossed product groupoid $G = X \times H$ coincide with the crossed products $C_0(X) \rtimes_r H$ and $C_0(X) \rtimes H$ of the algebra $C_0(X)$ by the group $H$ with respect to the induced action of $H$ on $C_0(X)$.

If the group $H$ is discrete, then elements of the algebra $C^\infty_c(G)$ are families \( \{a_\gamma \in C^\infty_c(X) : \gamma \in H\} \) such that \( a_\gamma \neq 0 \) for finitely many elements $\gamma$. It is convenient to write them in the form $a = \sum_{\gamma \in H} a_\gamma U_\gamma$. The product in the algebra $C^\infty_c(G)$ is written as

\[
(a_{\gamma_1} U_{\gamma_1})(b_{\gamma_2} U_{\gamma_2}) = (a_{\gamma_1} T_{\gamma_1}(b_{\gamma_2})) U_{\gamma_1 \gamma_2},
\]

where $T_\gamma$ denotes the operator acting in $C_0(X)$ induced by the action of $\gamma \in H$:

\[
T_\gamma f(x) = f(\gamma^{-1}x), \quad x \in X, \quad f \in C_0(X).
\]

The involution in the algebra $C^\infty_c(G)$ is given by

\[
(a_\gamma U_\gamma)^* = T_{\gamma^{-1}}(a_\gamma) U_{\gamma^{-1}}.
\]

Let $G$ be the holonomy groupoid of a foliation $\mathcal{F}$ on a compact manifold $M$. Elements of the algebra $C^\infty_c(G)$ can be regarded as families of the kernels of integral operators along the leaves of the foliation (more precisely, on the holonomy coverings $G^x$). Namely, each $k \in C^\infty_c(G)$ corresponds to the family $\{R_x(k) : x \in M\}$, where $R_x(k)$ is the integral operator acting in $L^2(G^x, \nu^x)$ given by the integral kernel

\[
K(\gamma_1, \gamma_2) = k(\gamma_1^{-1} \gamma_2), \quad \gamma_1, \gamma_2 \in G^x.
\]

The product of elements $k_1$ and $k_2$ in $C^\infty_c(G)$ corresponds to composition of the integral operators: $\{R_x(k_1)R_x(k_2) : x \in M\}$. The $C^*$-algebra $C^*(G)$ (respectively, $C^*_r(G)$) associated with the holonomy groupoid $G$ of the foliation $(M, \mathcal{F})$ will be called the $C^*$-algebra of the foliation $(M, \mathcal{F})$ (respectively, the reduced $C^*$-algebra of the foliation $(M, \mathcal{F})$) and denoted by $C^*(M, \mathcal{F})$ (respectively, $C^*_r(M, \mathcal{F})$).

**Example 5.17.** Let $\mathcal{F}$ be a simple foliation on a compact manifold $M$ defined by a submersion $\pi : M \to B$. Fix a smooth Haar system on it. For any $y \in B$ denote by $\Psi^{-\infty}(Z_y)$ the involutive algebra of integral operators with smooth kernel acting in the space $C^\infty(Z_y)$, where $Z_y$ is the fibre of the fibration $\pi$ at $y$. Let us consider the field $\Psi^{-\infty}(M/B)$ of involutive algebras on $B$ whose fibre at $y \in B$ is $\Psi^{-\infty}(Z_y)$. For any section $\sigma$ of the field $\Psi^{-\infty}(M/B)$ the integral kernels of the operators $\sigma_y$ give rise to a well-defined function on the holonomy groupoid $G$ of the foliation $\mathcal{F}$. We say that a section $\sigma$ is smooth if the corresponding function on $G$ is smooth. Thus, we obtain a description of the algebra $C^\infty_c(G)$ as the algebra of smooth sections of the field $\Psi^{-\infty}(M/B)$ of fibrewise integral operators with smooth kernels.

**Example 5.18.** Consider an example of the linear foliation on the torus. Thus, let $M = T^2 = \mathbb{R}^2/\mathbb{Z}^2$ be the two-dimensional torus, and suppose that a foliation $\mathcal{F}_\theta$ is given by the trajectories of the vector field $X = \frac{\partial}{\partial x} + \theta \frac{\partial}{\partial y}$, where $\theta \in \mathbb{R}$ is a fixed irrational number.
Since the foliation is given by the orbits of a free group action of $\mathbb{R}$ on $T^2$, its holonomy groupoid coincides with the crossed product groupoid $T^2 \rtimes \mathbb{R}$. Thus, $G = T^2 \rtimes \mathbb{R}$, $G(0) = T^2$, $s(x, y, t) = (x - t, y - \theta t)$, $r(x, y, t) = (x, y)$, $(x, y) \in T^2$, $t \in \mathbb{R}$, and the multiplication is given by
\[(x_1, y_1, t_1)(x_2, y_2, t_2) = (x_1, y_1, t_1 + t_2)\]
if $x_2 = x_1 - t_1$, $y_2 = y_1 - \theta t_1$.

The reduced $C^*$-algebra $C_r^*(T^2, \mathcal{F}_\theta)$ of the linear foliation $\mathcal{F}_\theta$ on $T^2$ coincides with the reduced crossed product $C(T^2) \rtimes_r \mathbb{R}$. Therefore, the product $k_1 * k_2$ of $k_1, k_2 \in C_c^\infty(T^2 \times \mathbb{R}) \subset C(T^2) \rtimes_r \mathbb{R}$ is given by
\[(k_1 * k_2)(x, y, t) = \int_{-\infty}^{\infty} k_1(x, y, t_1)k_2(x-t_1, y-\theta t_1, t-t_1) dt_1, \quad (x, y) \in T^2, \quad t \in \mathbb{R},\]
and for any $k \in C_c^\infty(T^2 \times \mathbb{R})$,
\[k^*(x, y, t) = \overline{k(x-t, y-\theta t, -t)}, \quad (x, y) \in T^2, \quad t \in \mathbb{R}.\]

For any $k \in C_c^\infty(T^2 \times \mathbb{R})$ and any $(x, y) \in T^2$ the operator $R_{(x,y)}(k)$ has the following form on $L^2(G(x,y), \nu(x,y)) \cong L^2(\mathbb{R}, dt)$: for any $u \in L^2(\mathbb{R}, dt)$,
\[R_{(x,y)}(k)u(t) = \int_{-\infty}^{\infty} k(x-t_1, y-\theta t_1, t-t_1)u(t_1) dt_1, \quad t \in \mathbb{R}.\]

If $\theta$ is rational, then the linear foliation on $T^2$ is given by the orbits of a free group action of $S^1$ on $T^2$, and its holonomy groupoid coincides with the crossed product groupoid $G = T^2 \rtimes S^1$.

We note some facts which connect the structure of the reduced $C^*$-algebra $C_r^*(M, \mathcal{F})$ of a foliation with the topology of $\mathcal{F}$ (for more details see [85], [86]).

**Theorem 5.19** [85]. Let $(M, \mathcal{F})$ be a foliated manifold.

1. The $C^*$-algebra $C_r^*(M, \mathcal{F})$ is simple if and only if $\mathcal{F}$ is minimal, that is, each leaf of it is dense in $M$.
2. The $C^*$-algebra $C_r^*(M, \mathcal{F})$ is primitive if and only if $\mathcal{F}$ is (topologically) transitive, that is, it has a leaf which is dense in $M$.
3. For $\mathcal{F}$ amenable in the sense that $C_r^*(M, \mathcal{F}) = C^*(M, \mathcal{F})$ the $C^*$-algebra $C_r^*(M, \mathcal{F})$ has a representation consisting of compact operators if and only if $\mathcal{F}$ has a compact leaf.

In the paper [85] a description is given of the space of primitive ideals of the $C^*$-algebra $C_r^*(M, \mathcal{F})$.

5.2.2. **Definition using half-densities.** In this subsection we will give the definitions of the operator algebras associated with a foliated manifold, without using a choice of Haar system. For this, we will use the language of half-densities.

Let $(M, \mathcal{F})$ be a compact foliated manifold and consider the vector bundle $|T\mathcal{F}|^{1/2}$ of leafwise half-densities on $M$. Using the source map $s$ and the range
map \( r \), lift \(|T\mathcal{F}|^{1/2}\) to the vector bundles \( s^* (|T\mathcal{F}|^{1/2})\) and \( r^*(|T\mathcal{F}|^{1/2})\) on the holonomy groupoid \( G \). We define a vector bundle \(|T\mathcal{G}|^{1/2}\) on \( G \) as

\[
|T\mathcal{G}|^{1/2} = r^*(|T\mathcal{F}|^{1/2}) \otimes s^*(|T\mathcal{F}|^{1/2}).
\]

The bundle \(|T\mathcal{G}|^{1/2}\) is naturally identified with the bundle of leafwise half-densities on the foliated manifold \((G, \mathcal{G})\).

An involutive algebra structure on \( C_c^\infty (G, |T\mathcal{G}|^{1/2})\) is defined by

\[
(\sigma_1 \ast \sigma_2)(\gamma) = \int_{\gamma_1 \gamma_2 = \gamma} \sigma_1(\gamma_1) \sigma_2(\gamma_2), \quad \gamma \in G,
\]

\[
\sigma^* (\gamma) = \sigma(\gamma^{-1}), \quad \gamma \in G,
\]

where \( \sigma, \sigma_1, \sigma_2 \in C_c^\infty (G, |T\mathcal{G}|^{1/2}) \). The formula for \( \sigma_1 \ast \sigma_2 \) should be interpreted in the following way. If we write \( \gamma: x \to y, \gamma_1: z \to y, \) and \( \gamma_2: x \to z \), then

\[
\sigma_1(\gamma_1) \sigma_2(\gamma_2) \in |T_y\mathcal{F}|^{1/2} \otimes |T_z\mathcal{F}|^{1/2} \otimes |T_z\mathcal{F}|^{1/2} \otimes |T_x\mathcal{F}|^{1/2}
\]

\[
\cong |T_y\mathcal{F}|^{1/2} \otimes |T_z\mathcal{F}|^{1/2} \otimes |T_x\mathcal{F}|^{1/2},
\]

and integrating the \(|T_z\mathcal{F}|^{1/2}\)-component \( \sigma_1(\gamma_1) \sigma_2(\gamma_2) \) with respect to \( z \in M \), we obtain a well-defined section of the bundle \( r^*(|T\mathcal{F}|^{1/2}) \otimes s^*(|T\mathcal{F}|^{1/2}) = |T\mathcal{G}|^{1/2} \).

### 5.3. Vector bundles and fields of Hilbert spaces.

A natural analogue of the notion of vector bundle on the leaf space of a foliation is the notion of holonomy equivariant vector bundle, or vector \( G \)-bundle.

**Definition 5.20.** A vector bundle \( E \) (complex or real) on a foliated manifold \((M, \mathcal{F})\) is said to be holonomy equivariant if there is given a representation \( T \) of the holonomy groupoid \( G \) on the fibres of \( E \), that is, for any \( \gamma \in G, \gamma: x \to y \), there is defined a linear operator \( T(\gamma): E_x \to E_y \) such that \( T(\gamma_1 \gamma_2) = T(\gamma_1) T(\gamma_2) \) for any \( \gamma_1, \gamma_2 \in G \) with \( r(\gamma_2) = s(\gamma_1) \).

A Hermitian (respectively, Riemannian) vector bundle \( E \) on a foliated manifold \((M, \mathcal{F})\) is said to be holonomy equivariant if it is a holonomy equivariant vector bundle and the representation \( T \) is unitary (respectively, orthogonal): \( T(\gamma^{-1}) = T(\gamma)^* \) for any \( \gamma \in G \).

For any holonomy equivariant vector bundle \( E \to M \) the action of the groupoid \( G \) on \( E \) defines a horizontal foliation \( \mathcal{F}_E \) on \( E \) of the same dimension as the foliation \( \mathcal{F} \). The leaf of \( \mathcal{F}_E \) through a point \( v \in E \) consists of all points of the form \( T(\gamma)^{-1}(v) \) with \( \gamma \in G, r(\gamma) = \pi(v) \). Thus, any holonomy equivariant vector bundle is foliated in the sense of [87].

**Definition 5.21.** A vector bundle \( p: P \to M \) is said to be foliated if there is a foliation \( \mathcal{F} \) on \( P \) of the same dimension as \( \mathcal{F} \) whose leaves are transversal to the fibres of \( p \) and mapped by \( p \) to leaves of \( \mathcal{F} \).

Equivalently, one can say that a foliated vector bundle is a vector bundle \( P \) on \( M \) such that there exists a flat connection in the space \( C_c^\infty (M, P) \) defined along the leaves of \( \mathcal{F} \), that is, an operator

\[
\nabla: \mathcal{F}(\mathcal{F}) \times C_c^\infty (M, P) \to C_c^\infty (M, P)
\]
satisfying the standard conditions
\[ \nabla f_X = f \nabla X, \quad \nabla_X(fs) = (Xf)s + f \nabla_X s \]
for any \( f \in C^\infty(M), \ X \in \mathcal{X}(\mathcal{F}), \ s \in C^\infty(M,P), \) and also the flatness condition
\[ [\nabla_X, \nabla_Y] = \nabla_{[X,Y]}, \quad X, Y \in \mathcal{X}(\mathcal{F}). \]

The parallel transport along leafwise paths associated with the connection \( \nabla \) defines an action of the fundamental groupoid \( \Pi(M, \mathcal{F}) \) of the foliation in the fibres of the foliated vector bundle \( P \). In general, the parallel transport may depend on the holonomy of the corresponding path, therefore, this action does not necessarily descend to an action of the holonomy groupoid in the fibres of \( P \), and hence a foliated vector bundle is not necessarily holonomy equivariant.

Example 5.22. The normal bundle \( \tau_x = T_x M/T_x \mathcal{F}, x \in M \), is a holonomy equivariant vector bundle if it is equipped with the action of the holonomy groupoid \( G \) by the linear holonomy map \( dh_\gamma: \tau_x \to \tau_y, \gamma: x \to y. \) The corresponding partial flat connection defined along the leaves of \( \mathcal{F} \) is the Bott connection (see (4.2)). The normal bundle \( \tau \) is a holonomy equivariant Riemannian vector bundle if \( \mathcal{F} \) is a Riemannian foliation.

The conormal bundle \( N^* \mathcal{F} \) equipped with the action of the holonomy groupoid \( G \) by the linear holonomy map \( (dh_\gamma^*)^{-1}: N^*_x \mathcal{F} \to N^*_y \mathcal{F} \) for \( \gamma: x \to y \) and, more generally, an arbitrary tensor bundle associated with the normal bundle \( \tau \) are also holonomy equivariant.

For any holonomy equivariant vector bundle \( E \) on a foliated manifold \( (M, \mathcal{F}) \) there is defined a natural representation \( R_E \) of the algebra \( C^\infty_c(G) \) on the space \( C^\infty(M,E) \) of sections of this bundle. Let \( \{ \nu^x : x \in G^{(0)} \} \) be a smooth Haar system on \( G \). For any \( u \in C^\infty(M,E) \) the section \( R_E(k)u \in C^\infty(M,E) \) is given by

\[
R_E(k)u(x) = \int_{G^x} k(\gamma)T(\gamma)[u(s(\gamma))] d\nu^x(\gamma), \quad x \in M. \quad (5.2)
\]

If \( E \) is a holonomy equivariant Hermitian vector bundle on \( M \), then the representation \( R_E \) is a \(*\)-representation, and therefore it extends to a \(*\)-representation of the \( C^* \)-algebra \( C^*(M, \mathcal{F}) \).

In [13] the notion of continuous field of Hilbert spaces on the leaf space of the foliation \( \mathcal{F} \) is introduced, and it is shown that there is a one-to-one correspondence between continuous fields of Hilbert spaces on the leaf space of \( \mathcal{F} \) and Hilbert \( C^* \)-modules over the \( C^* \)-algebra \( C^*_r(M, \mathcal{F}) \).

Let \( H = \{ H_x : x \in M \} \) be a measurable field of Hilbert spaces on \( M \) equipped with a unitary representation of the holonomy groupoid \( G \):

\[ U(\gamma): H_x \to H_y, \quad \gamma \in G, \quad \gamma: x \to y. \]

For any measurable sections \( \xi, \eta \) of \( H \) define a function \( (\xi, \eta) \) on \( G \) by

\[
(\xi, \eta)(\gamma) = \langle \xi_y, U(\gamma)\eta_x \rangle, \quad \gamma \in G, \quad \gamma: x \to y.
\]
We fix an arbitrary smooth Haar system \( \{ \nu^x : x \in M \} \) on \( G \). For any measurable section \( \xi \) of the field \( H \) denote by \( \| \xi \|_\infty \) the least number \( c \in [0, +\infty] \) such that for any \( y \in M \) and \( \alpha \in H_y \)
\[
\int_{G^y} |(\alpha, U(\gamma)\xi_x)|^2 \, dv^y(\gamma) \leq c\|\alpha\|^2.
\]

**Definition 5.23.** A field \( \{ H_x : x \in M \} \) is called a continuous field of Hilbert spaces on the leaf space of the foliation \( \mathcal{F} \) if there is a distinguished linear space \( \Gamma \) of its sections such that
1) \( \Gamma \) contains a countable total \( \| \cdot \|_\infty \)-dense subset, in particular, \( \| \cdot \|_\infty \) takes finite values on \( \Gamma \);
2) \((\xi, \eta) \in C^*_r(M, \mathcal{F})\) for any \( \xi, \eta \in \Gamma \);
3) \( \Gamma \) is closed in the \( \| \cdot \|_\infty \)-norm;
4) for any \( f \in C^*_c(G) \) and \( \xi \in \Gamma \) one has \( \xi \ast f \in \Gamma \), where (cf. (5.2))
\[
(\xi \ast f)(y) = \int_{G^y} f(\gamma)U(\gamma)\xi(x) \, dv^y(\gamma).
\]

We will not present a general construction of the Hilbert \( C^* \)-module over the \( C^* \)-algebra \( C^*_r(M, \mathcal{F}) \) associated with an arbitrary continuous field of Hilbert spaces on the leaf space of a foliation \( \mathcal{F} \), but consider only one important particular case. Let \( E \) be a vector bundle on a foliated manifold \((M, \mathcal{F})\). It defines a continuous field of Hilbert spaces on the leaf space of \( \mathcal{F} \) and therefore a Hilbert \( C^* \)-module over the \( C^* \)-algebra \( C^*_r(M, \mathcal{F}) \). It can be defined in two ways, depending on whether we wish to consider left or right modules. If we wish to work with left modules, then, as in [15], we put \( H_x = L^2(G^x, s^* E) \) for any \( x \in M \) and define for any \( \gamma : x \to y \) the operator \( L(\gamma) : L^2(G^x, s^* E) \to L^2(G^y, s^* E) \) acting on \( \xi \in L^2(G^x, s^* E) \) by the formula
\[
L(\gamma)\xi(\gamma_1) = \xi(\gamma^{-1}\gamma_1), \quad \gamma_1 \in G^y.
\]

To work with right modules, we put \( H_x = L^2(G_x, r^* E) \) for any \( x \in M \) (as in [13], [88]) and define for any \( \gamma : x \to y \) the operator \( R(\gamma) : L^2(G_x, r^* E) \to L^2(G_y, r^* E) \) acting on \( \xi \in L^2(G_x, r^* E) \) by
\[
R(\gamma)\xi(\gamma_1) = \xi(\gamma_1\gamma), \quad \gamma_1 \in G_y.
\]

Let us describe the corresponding right Hilbert \( C^* \)-module over the \( C^* \)-algebra \( C^*_r(M, \mathcal{F}) \) [13], [88]. We begin with the definition of a pre-Hilbert \( C^*_c(G) \)-module \( \mathcal{E}_\infty \). As a linear space \( \mathcal{E}_\infty \) coincides with \( C^*_c(G, r^* E) \). The module structure on \( \mathcal{E}_\infty \) is introduced as follows: the action of \( f \in C^*_c(G) \) on \( s \in \mathcal{E}_\infty \) is given by
\[
(s \ast f)(\gamma) = \int_{G^y} s(\gamma')f(\gamma'^{-1}\gamma) \, dv^y(\gamma'), \quad \gamma \in G^y,
\]
and the inner product on \( \mathcal{E}_\infty \) with values in \( C^*_c(G) \) is given by
\[
\langle s_1, s_2 \rangle(y) = \int_{G^y} \langle s_1(\gamma'^{-1}), s_2(\gamma'^{-1}) \rangle_{E(s(\gamma'))} \, dv^y(\gamma'), \quad s_1, s_2 \in \mathcal{E}_\infty.
\]
There is also a left action of the algebra $C^\infty(M)$ on $\mathcal{E}_\infty$ given by
\[
(a \cdot s)(\gamma) = a(y)s(\gamma), \quad \gamma \in G^y,
\]
where $a \in C^\infty_c(M)$ and $s \in \mathcal{E}_\infty$.

The completion of the space $\mathcal{E}_\infty$ in the norm $\|s\| = \|R((s, s))\|^{1/2}$ defines a Hilbert C*-module over $C^*_r(M, \mathcal{F})$, which we denote by $\mathcal{E} = \mathcal{E}_{M,E}$. It has a $C^*_r(M, \mathcal{F})$-valued $C^*_r(M, \mathcal{F})$-sesquilinear form $\langle \cdot, \cdot \rangle$ which is the extension by continuity of the sesquilinear form on $\mathcal{E}_\infty$. Thus, $\mathcal{E}_{M,E}$ is a $C(M)$-$C^*_r(M, \mathcal{F})$-bimodule.

If $E$ is holonomy equivariant, then there is a left action of the algebra $C^\infty_c(G)$ on $\mathcal{E}_\infty$ given by
\[
(f \ast s)(\gamma) = \int_{G^y} f(\gamma')T(\gamma') [s(\gamma'^{-1}\gamma)] \, d\nu^y(\gamma'), \quad \gamma \in G^y,
\]
where $f \in C^\infty_c(G)$ and $s \in \mathcal{E}_\infty$. Unlike the right action, the left action does not extend to an action of the algebra $C^*_r(M, \mathcal{F})$ by bounded endomorphisms of the Hilbert C*-module $\mathcal{E}$ over $C^*_r(M, \mathcal{F})$. Nevertheless, using the methods of reduction to the maximal compact subgroup developed by Kasparov [89], one can construct an element $[E] \in KK(C^*_r(M, \mathcal{F}), C^*_r(M, \mathcal{F}))$ corresponding to an arbitrary holonomy equivariant bundle $E$. This correspondence is a generalization of the map $j^G : KK(C_0(X), C_0(Y)) \to KK(C_0(X) \rtimes G, C_0(Y) \rtimes G)$ constructed by Kasparov [89] for any $G$-manifolds $X$ and $Y$.

### 5.4. Strong Morita equivalence and transversals

Consider a compact foliated manifold $(M, \mathcal{F})$. A choice of a complete transversal for the foliation enables us to reduce the holonomy groupoid $G$ of the foliation $\mathcal{F}$ to an equivalent groupoid which in many cases turns out to be simpler.

For any two subsets $A, B \subset M$ let
\[
G^A_B = \{ \gamma \in G : r(\gamma) \in A, \ s(\gamma) \in B \}.
\]
In particular,
\[
G^M_T = G_T = \{ \gamma \in G : s(\gamma) \in T \}.
\]
If $T$ is a complete transversal, then $G^T_T$ is a submanifold and a subgroupoid of $G$. It is called a reduced holonomy groupoid.

As shown in [86], if $T$ is a complete transversal, then the reduced C*-algebras $C^*_r(M, \mathcal{F})$ and $C^*_r(G^T_T)$ are strongly Morita equivalent. In particular, this easily implies that
\[
C^*_r(M, \mathcal{F}) \cong \mathcal{K} \otimes C^*_r(G^T_T).
\]

Following [90], we describe the construction of a $C^*_r(M, \mathcal{F})$-$C^*_r(G^T_T)$-equivalence bimodule which gives the strong Morita equivalence of the algebras $C^*_r(M, \mathcal{F})$ and $C^*_r(G^T_T)$.

Consider the manifold $P = G_T$. There is a natural left action of the groupoid $G$ on $P$ given by left multiplication in $G$, and a right action of the groupoid $G^T_T$ given by right multiplication in $G$. These actions commute. In the language of the paper [90] the manifold $P$ is a $(G, G^T_T)$-equivalence.
Correspondingly, there is a left action of the algebra \( C_c^\infty(G) \) on \( C_c^\infty(G_T) \) given for \( f \in C_c^\infty(G) \) and \( \varphi \in C_c^\infty(G_T) \) by
\[
(f \cdot \varphi)(p) = \int_{G_r(p)} f(\gamma) \varphi(\gamma^{-1}p) \, d\nu_r(p)(\gamma), \quad p \in G_T,
\]
and a right action of the algebra \( C_c^\infty(G_T^G) \) on \( C_c^\infty(G_T) \) given for \( g \in C_c^\infty(G_T^G) \) and \( \varphi \in C_c^\infty(G_T) \) by
\[
(\varphi \cdot g)(p) = \sum_{\gamma \in G_T^G} \varphi(p\gamma) g(\gamma^{-1}), \quad p \in G_T.
\]

The inner product on \( C_c^\infty(G_T) \) with values in the algebra \( C_c^\infty(G_T^G) \) is given by the following formula: for any \( \varphi, \psi \in C_c^\infty(G_T) \)
\[
\langle \varphi, \psi \rangle_{C_c^\infty(G_T^G)}(\gamma) = \int_{G_r(\gamma)} \overline{\varphi(\gamma^{-1}p)} \psi(\gamma^{-1}p\gamma) \, d\nu_r(p)(\gamma_1), \quad \gamma \in G_T^G,
\]
where \( p \in G_r(\gamma) \) is an arbitrary point (the right-hand side of the formula is independent of the choice of \( p \)). Similarly, the inner product on \( C_c^\infty(G_T) \) with values in the algebra \( C_c^\infty(G) \) is given by the formula: for any \( \varphi, \psi \in C_c^\infty(G_T) \)
\[
\langle \varphi, \psi \rangle_{C_c^\infty(G)}(\gamma) = \sum_{\gamma_1 \in G_T^G} \varphi(\gamma^{-1}p\gamma_1) \overline{\psi(p\gamma_1)}, \quad \gamma \in G,
\]
where \( p \in G_r(\gamma) \) is an arbitrary point (the right-hand side of the formula is independent of the choice of \( p \)).

As a result one has isomorphisms of the Hochschild homology, cyclic homology, and periodic cyclic homology of the algebras \( C_c^\infty(G) \) and \( C_c^\infty(G_T^G) \) and of the K-theory of the \( C^* \)-algebras \( C^*_r(M, \mathcal{F}) \) and \( C^*_r(G_T^G) \). It is also proved in [77] that the natural embedding \( G_T^G \subset G \) induces a homotopy equivalence \( BG_T^G \simeq BG \) of the classifying spaces of \( G_T^G \) and \( G \) (see §5.5).

**Example 5.24.** If \( \mathcal{F} \) is a simple foliation defined by a fibration \( M \to B \), then the \( C^* \)-algebra \( C^*_r(M, \mathcal{F}) \) is strongly Morita equivalent to the \( C^* \)-algebra \( C_0(B) \).

**Example 5.25.** On the manifold \( M = \tilde{B} \times \Gamma F \) we consider a foliation \( \mathcal{F} \) obtained from the manifold \( B \) and a homomorphism \( \phi: \Gamma = \pi_1(B) \to \text{Diff}(F) \) by the suspension construction (see Example 4.6). The image of the set \( \{b_0\} \times F \subset \tilde{B} \times F \) \((b_0 \in \tilde{B} \) is an arbitrary element\) under the projection \( \tilde{B} \times \Gamma F \to M \) is a complete transversal to \( \mathcal{F} \). If the condition given in Example 5.11 holds, then the algebra \( C^*_r(M, \mathcal{F}) \) is strongly Morita equivalent to the reduced crossed product \( C(F) \rtimes_r \Gamma \).

**Example 5.26.** Consider the linear foliation \( \mathcal{F}_\theta \) on the two-dimensional torus \( T^2 \), where \( \theta \in \mathbb{R} \) is a fixed irrational number, and the transversal \( T \) given by the equation \( y = 0 \). The leaf space of the foliation \( \mathcal{F}_\theta \) is identified with the orbit space of the \( \mathbb{Z} \)-action on \( S^1 = \mathbb{R}/\mathbb{Z} \) generated by the rotation
\[
R_\theta(x) = x - \theta \pmod{1}, \quad x \in S^1.
\]
The algebra \( A_\theta = C^*(G_T^T) \) coincides with the crossed product \( C(S^1) \rtimes \mathbb{Z} \) of the algebra \( C(S^1) \) by the group \( \mathbb{Z} \) with respect to the \( \mathbb{Z} \)-action \( R_\theta \) on \( C(S^1) \). One can show that the algebra \( A_\theta \) is generated by the elements \( U \) and \( V \) satisfying the relation \( VU = e^{2\pi i \theta} UV \). It has a concrete realization as the uniform closure of the subalgebra of \( L^2(S^1 \times \mathbb{Z}) \) consisting of finite sums of the form \( \sum_{(n,m) \in \mathbb{Z}^2} a_{nm} u^n v^m \), where \( a_{nm} \in \mathbb{C} \) and the operators \( u \) and \( v \) have the following form: for \( f \in L^2(S^1 \times \mathbb{Z}) \)

\[
uf(x, n) = f(x, n + 1), \quad vf(x, n) = e^{2\pi i (x-n\theta)} f(x, n), \quad x \in S^1, \quad n \in \mathbb{Z}.
\]

In many cases it is convenient to consider as a dense subalgebra of \( A_\theta \) the algebra

\[
\mathcal{A}_\theta = \left\{ a = \sum_{(n,m) \in \mathbb{Z}^2} a_{nm} U^n V^m : \{a_{nm}\} \in \mathcal{S}(\mathbb{Z}^2) \right\},
\]

where \( \mathcal{S}(\mathbb{Z}^2) \) is the space of rapidly decreasing sequences (that is, of sequences such that \( \sup_{(n,m) \in \mathbb{Z}^2} (|n| + |m|)^k |a_{nm}| < \infty \) for any natural \( k \)).

For \( \theta = 0 \), the algebra \( A_\theta \) is commutative and isomorphic to the commutative \( C^* \)-algebra of continuous functions on the two-dimensional torus \( T^2 \). Therefore, for an arbitrary \( \theta \) the algebra \( A_\theta \) is often called the algebra of continuous functions on the non-commutative torus \( T^2_\theta \), and the notation \( \mathcal{A}_\theta = C^\infty(T^2_\theta) \), \( A_\theta = C(T^2_\theta) \) is used. The algebra \( A_\theta \) was introduced in the paper [91] (see also [92]) and has found many applications in mathematics and physics (see, for instance, the survey [93]).

Thus, the \( C^* \)-algebra \( C^*_r(T^2, \mathcal{F}_\theta) \) of the linear foliation on \( T^2 \) is strongly Morita equivalent to the algebra \( A_\theta \).

An important property of the groupoid \( G_T^T \) associated with a complete transversal \( T \) is (see [94]) that it is an étale groupoid, that is, the source map \( s : G \to G(0) \) is a local diffeomorphism. According to [94], a smooth groupoid is equivalent to an étale groupoid if and only if all its isotropy groups \( G^*_x \) are discrete. In the latter case such a groupoid is called a foliation groupoid.

One can introduce an equivalence relation for groupoids that is similar to the strong Morita equivalence for \( C^* \)-algebras [77], [90] (see also [94]). Roughly speaking, two groupoids are equivalent if they have the same orbits spaces, and therefore the same transverse geometries. It was proved in [90] that if groupoids \( G \) and \( H \) are equivalent, then their reduced \( C^* \)-algebras are strongly Morita equivalent.

### 5.5. \( K \)-theory of foliation \( C^* \)-algebras and the Baum–Connes conjecture.

Computation of the \( K \)-theory for foliation \( C^* \)-algebras is not a simple problem. This problem has a sufficiently simple solution for foliations with proper holonomy groupoid. The holonomy groupoid \( G_X \) of a foliation \( \mathcal{F}_X \) on a manifold \( X \) is said to be proper if the map \((r, s) : G_X \to X \times X \) is proper. If the holonomy groupoid of a foliation \((X, \mathcal{F}_X)\) is proper, then the foliation \( \mathcal{F}_X \) is proper, that is, each leaf of it is an embedded submanifold of \( X \). The leaf space \( X/\mathcal{F}_X \) of a foliation \((X, \mathcal{F}_X)\) with proper holonomy groupoid is an orbifold, and the group \( K(C^*(X, \mathcal{F}_X)) \) coincides with the \( K \)-theory defined by \( G_X \)-equivariant bundles on \( X \) compactly supported in \( X/\mathcal{F}_X \) (see [95]).
In the general case a geometric construction of elements in \( K(C^*(M, \mathcal{F})) \) was proposed in [96] (see also [88]). It is based on a definition of the geometric K-theory \( K^*_{\text{top}}(M, \mathcal{F}) \). Before we give this definition, we introduce some auxiliary notions.

We will need the notion of smooth map \( f \) from the leaf space \( M_1/\mathcal{F}_1 \) of a foliation \( \mathcal{F}_1 \) on a manifold \( M_1 \) to the leaf space \( M_2/\mathcal{F}_2 \) of a foliation \( \mathcal{F}_2 \) on a manifold \( M_2 \). A smooth map \( f: M_1 \to M_2/\mathcal{F}_2 \) can be defined as a cocycle \((U_i, \gamma_{ij})\) on \( M_1 \) with values in the holonomy groupoid \( G_2 \) of \( \mathcal{F}_2 \). Here \( \{U_i\}_{i \in A} \) is an open cover of the manifold \( M_1 \), and the \( \gamma_{ij}: U_i \cap U_j \to G_2 \) are smooth maps satisfying the relations

\[
\gamma_{ij}(x)\gamma_{jk}(x) = \gamma_{ik}(x), \quad x \in U_i \cap U_j \cap U_k.
\]

More precisely, we say that two cocycles \((U_i, \gamma_{ij})\) and \((U'_i, \gamma'_{ij})\) are equivalent if they extend to a cocycle on the disjoint union of the coverings \( \{U_i\} \) and \( \{U'_i\} \). A map \( f: M_1 \to M_2/\mathcal{F}_2 \) is defined as an equivalence class of cocycles. This definition can be understood as follows. The maps \( \gamma_{ii}: U_i \to (G_2)^{(0)} = M_2 \) are the local lifts of the map \( f: U_i \to M_2/\mathcal{F}_2 \). The fact that, for different \( i \) and \( j \), and for \( x \in U_i \cap U_j \), the points \( f_i(x) \) and \( f_j(x) \) lie on the same leaf of \( \mathcal{F}_2 \) is described by the element \( \gamma_{ij}(x): f_j(x) \to f_i(x) \) of the holonomy groupoid \( G_2 \).

The graph of a map \( f: M_1 \to M_2/\mathcal{F}_2 \) given by a cocycle \((U_i, \gamma_{ij})\) is a smooth (not necessarily Hausdorff) manifold \( G_f \), which is defined as the set of equivalence classes on the set \( \{(x, i, \gamma) \in M_1 \times A \times G_2 : x \in U_i, \ r(\gamma) = f_i(x)\} \) with respect to the equivalence relation specifying that \((x, i, \gamma) \sim (x', j, \gamma')\) if \( x = x' \) and \( \gamma_{ji}(x) = \gamma'\gamma^{-1} \). There is a smooth map \( r_f: G_f \to M_1 \), \( r_f(x, i, \gamma) = x \), and a right action of the groupoid \( G_2 \) on \( G_f \) given by the maps \( s_f: G_f \to M_2, \ s_f(x, i, \gamma) = s(\gamma) \), and \( G_f \times_{s_f} G_2 \to G_f, \ (x, i, \gamma)\gamma' = (x, i, \gamma') \).

The description of a map in terms of its graph can be generalized to the case of maps from \( M_1/\mathcal{F}_1 \) to \( M_2/\mathcal{F}_2 \) as follows. Let \( G_1 \) and \( G_2 \) be the holonomy groupoids of the foliations \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \), respectively. The graph \( G_f \) of a map \( f: M_1/\mathcal{F}_1 \to M_2/\mathcal{F}_2 \) is a smooth (not necessarily Hausdorff) manifold equipped with smooth maps \( r_f: G_f \to M_1 \) and \( s_f: G_f \to M_2 \), a left action \( G_1 \times_{r_f} G_f \to G_f \) of the groupoid \( G_1 \), and a right action \( G_f \times_{s_f} G_2 \to G_f \) of the groupoid \( G_2 \). It is assumed that the actions commute, \( r_f: G_f \to M_1 \) is a principal bundle with structure groupoid \( G_2 \) (that is, \( r_f \) is a submersion, and for any \( x \) and \( y \) in \( G_f \) such that \( r_f(x) = r_f(y) \) there is a unique element \( \gamma \in G_1 \) with \( x\gamma = y \)), and the action of \( G_2 \) is proper.

Equivalently, a map \( f: M_1/\mathcal{F}_1 \to M_2/\mathcal{F}_2 \) can be defined either as a cocycle on \( G_1 \) with values in \( G_2 \) or as a homomorphism \( \varphi: (G_1)^{T_j}_{T_1} \to (G_2)^{T_j}_{T_2} \) of the reduced groupoids, where the \( T_j \) are complete transversals for the foliations \( (M_j, \mathcal{F}_j) \), \( j = 1, 2 \) (for more details, see [13], [88], [95]).

The composition \( f_2 \circ f_1: M_1/\mathcal{F}_1 \to M_3/\mathcal{F}_3 \) of maps \( f_1: M_1/\mathcal{F}_1 \to M_2/\mathcal{F}_2 \) and \( f_2: M_2/\mathcal{F}_2 \to M_3/\mathcal{F}_3 \) is given by the graph \( G_{f_2 \circ f_1} = G_{f_1} \times_{G_2} G_{f_2} \), which is defined as the set of equivalence classes in the set \( \{(x_1, x_2) \in G_{f_1} \times G_{f_2} : s_{f_1}(x_1) = r_{f_2}(x_2)\} \) with respect to the equivalence relation specifying that \((x_1, x_2) \sim (y_1, y_2)\) if there exists \( \gamma \in G_2 \) such that \( x_1\gamma = y_1, \ \gamma y_2 = x_2 \).

**Example 5.27.** Let \( \mathcal{F}_i \) be a simple foliation defined by a submersion \( \pi_i: M_i \to B_i, \ i = 1, 2 \). Any map \( F: B_1 \to B_2 \) defines a map \( f: M_1/\mathcal{F}_1 \to M_2/\mathcal{F}_2 \). Its
Example 5.29. The graph of the projection id: $M/\mathcal{F} \to M/\mathcal{F}$ is the holonomy groupoid $G$.

Example 5.28. The graph of the identity map id: $M/\mathcal{F} \to M/\mathcal{F}$ is the holonomy groupoid $G$. More generally, the graphs of a map $f: M_1/\mathcal{F}_1 \to M_2/\mathcal{F}_2$ and of the corresponding lift $\tilde{f}: M_1 \to M_2/\mathcal{F}_2$ coincide as sets and differ only by the fact that the graph of $f$ has the structure of a left $G_1$-manifold, but the graph of $\tilde{f}$ does not.

Example 5.30. The graph of the projection id: $M/\mathcal{F} \to \text{pt}$ is $M$.

Denote by $\text{Ml}_n(\mathbb{R})$ the metilinear group, that is, the non-trivial twofold covering of the group $G_1^c_n(\mathbb{R})$ of non-singular real $n \times n$ matrices with positive determinant. Put $\text{Ml}_n^c = \text{Ml}_n(\mathbb{R}) \times _{\mathbb{Z}_2} S^1$. The maximal compact subgroup of $\text{Ml}_n^c$ is $\text{Spin}^c(n)$. A holonomy equivariant real vector bundle $E$ on $M$ of rank $r$ is said to be $K$-orientable if its structure group (as a $G$-bundle) reduces to $\text{Ml}_n^c$.

Let $f: M_1/\mathcal{F}_1 \to M_2/\mathcal{F}_2$ and let $E$ be a holonomy equivariant bundle on $M_2$. Then $s_f^* E$ is a $G_2$-bundle on $G_f$ such that the action of $G_1$ on it is trivial. Since $G_f$ is a principal $G_1$-bundle, there exists a unique $G_1$-bundle $E'$ on $M_1$ such that $r_f^* E' = s_f^* E$. Let $E' = f^* E$.

A smooth map $f: M_1/\mathcal{F}_1 \to M_2/\mathcal{F}_2$ is said to be $K$-orientable if the $G_1$-bundle $\tau_1 \oplus f^* \tau_2$ on $M_1$ is $K$-orientable.

For any smooth $K$-oriented map $f: M_1/\mathcal{F}_1 \to M_2/\mathcal{F}_2$ one can naturally define an element $f! \in KK(C^*(M_1, \mathcal{F}_1), C^*(M_2, \mathcal{F}_2))$ generalizing the Gysin homomorphism in $K$-theory (see §1.3). In the case when the foliation $\mathcal{F}_1$ is trivial, a construction of the element $f! \in KK(C(M_1), C^*(M_2, \mathcal{F}_2))$ was given in [13] and studied more systematically in [88], where it was also proved that $(f \circ g)! = g! \otimes f!$. In [96] a construction of $f!$ was used in the case when the holonomy groupoid of $\mathcal{F}_1$ is proper. Finally, in [95] there is a definition of $f! \in KK(C^*(M_1, \mathcal{F}_1), C^*(M_2, \mathcal{F}_2))$ for an arbitrary smooth $K$-oriented map $f: M_1/\mathcal{F}_1 \to M_2/\mathcal{F}_2$ and it is proved that $(f \circ g)! = g! \otimes f!$. For a $K$-oriented tangent bundle $T\mathcal{F}$ the natural projection $p: M \to M/\mathcal{F}$ is a $K$-oriented map, and the element $p! \in KK(C(M), C^*_r(M, \mathcal{F}))$ coincides with the element $[D] \in KK(C(M), C^*_r(M, \mathcal{F}))$ defined by the corresponding tangential $\text{Spin}^c$ Dirac operator $D$ (see §8.2). For the map $\pi: M/\mathcal{F} \to \text{pt}$ the construction of $\pi! \in KK(C^*(M, \mathcal{F}), \mathbb{C})$ is directly connected with a $K$-theoretic analogue of the construction of the transverse fundamental class of a foliation (see §6.2). The difficult problem of constructing the element $\pi!$ was solved in [95]. Its solution makes use of the lift to para-Riemannian foliations, the Thom isomorphism, and transversally hypo-elliptic operators (see §§7.4 and 8.3, in particular, Theorem 8.6). Finally, in the case when a map $f: M_1/\mathcal{F}_1 \to M_2/\mathcal{F}_2$ is an embedding, the construction of $f!$ is analogous to the classical construction of the Gysin homomorphism in $K$-theory, in particular, of the topological index (see §1.3), and makes use of the Thom isomorphism and of the construction of the normal groupoid associated to a foliation.

Generators of the geometric $K$-theory group $K_{\text{top}}^* (M, \mathcal{F}) (K$-cocycles) are equivalence classes of quadruples $(X, \mathcal{F}_X, x, f)$, where $(X, \mathcal{F}_X)$ is a foliation with proper
holonomy groupoid, $x \in K_\ast(C^\ast(X, \mathcal{F}_X))$, and $f: X/\mathcal{F}_X \to M/\mathcal{F}$ is a $K$-oriented map. The equivalence relation of $K$-cocycles is given by

$$(X, \mathcal{F}_X, x, f \circ g) \sim (Y, \mathcal{F}_Y, x \otimes g!, f),$$

where $g: X/\mathcal{F}_X \to Y/\mathcal{F}_Y$ is a $K$-oriented map. The addition in $K^\ast_{\text{top}}(M, \mathcal{F})$ is given by the disjoint sum operation.

In [96] (see also [88]) a map $\mu: K^\ast_{\text{top}}(M, \mathcal{F}) \to K_\ast(C^\ast(M, \mathcal{F}))$ is defined by

$${\mu: (X, \mathcal{F}_X, x, f) \to f!(x) = x \otimes f!}.$$  

The Baum–Connes conjecture asserts that $\mu$ is an isomorphism. Composing $\mu$ with the map $K_\ast(C^\ast(M, \mathcal{F})) \to K_\ast(C^\ast_r(M, \mathcal{F}))$ induced by the canonical projection $C^\ast(M, \mathcal{F}) \to C^\ast_r(M, \mathcal{F})$, one gets a map $\mu_r: K^\ast_{\text{top}}(M, \mathcal{F}) \to K_\ast(C^\ast_r(M, \mathcal{F}))$.

For an arbitrary Lie groupoid $G$, as for any small category, the classifying space $BG$ is defined. It can be constructed by a slight modification of Milnor’s classical construction of the classifying space of a group (see, for instance, [82]). We describe Segal’s construction [97]. Consider the simplicial set $NG$ such that the set $NG_n$ of its $n$-simplices is defined as

$${NG}_n = G^{(n)} = \{(\gamma_1, \ldots, \gamma_n) \in G^n : s(\gamma_i) = r(\gamma_{i+1}), i = 1, \ldots, n-1\},$$

for $n > 1$ the boundary operators $\delta_j: NG_n \to NG_{n-1}$, $j = 0, 1, \ldots, n$, have the form

$${\delta_0(\gamma_1, \ldots, \gamma_n) = (\gamma_2, \ldots, \gamma_n), \quad \delta_j(\gamma_1, \ldots, \gamma_n) = (\gamma_1, \ldots, \gamma_j, \gamma_{j+1}, \ldots, \gamma_n), \quad 1 \leq j \leq n-1, \quad \delta_n(\gamma_1, \ldots, \gamma_n) = (\gamma_1, \ldots, \gamma_{n-1})}$$

and for $n = 0$ the boundary operators $\delta_j: NG_1 \to NG_0 = G^{(0)}$, $j = 0, 1$, have the form

$${\delta_0(\gamma_1) = r(\gamma_1), \quad \delta_1(\gamma_1) = s(\gamma_1), \quad \delta_2(\gamma_1) = r(\gamma_2), \quad \delta_3(\gamma_1) = \gamma_{1+1}, \cdots, \gamma_n}, \quad 0 \leq j \leq n.$$ 

The simplicial set $NG$ is called the nerve of the groupoid $G$. The classifying space $BG$ of $G$ is defined as the geometrical realization of the simplicial set $NG$.

As an important particular case, consider the groupoid $G_{\mathcal{U}}$ associated with an arbitrary open cover $\mathcal{U} = \{U_i\}_{i \in A}$ of a topological space $X$. It is defined by the formulae

$${G_{\mathcal{U}}}^{(0)} = \bigsqcup_{i \in A} U_i = \{(x, i) \in M \times A : x \in U_i\},$$

$${G_{\mathcal{U}}} = \bigsqcup_{i, j \in A} U_i \cap U_j = \{(x, i, j) \in M \times A \times A : x \in U_i \cap U_j\},$$

the maps $s, r: G_{\mathcal{U}} \to G_{\mathcal{U}}^{(0)}$ have the form

$${s(x, i, j) = (x, i), \quad r(x, i, j) = (x, j).}$$
The set of $n$-simplices of the nerve $NG_{\mathcal{U}}$ of this groupoid is described as

$$(NG_{\mathcal{U}})_n = \{(x, i_1, \ldots, i_n) : x \in U_{i_1} \cap \cdots \cap U_{i_n}\} = \bigcup U_{i_1} \cap \cdots \cap U_{i_n},$$

where the union is taken over all sets $(i_1, \ldots, i_n)$ such that $U_{i_1} \cap \cdots \cap U_{i_n}$ is non-empty. If a cover $\mathcal{U}$ is locally finite and all non-empty intersections $U_{i_1} \cap \cdots \cap U_{i_n}$ are contractible, then the classifying space $BG_{\mathcal{U}}$ is homotopy equivalent to $X$.

A smooth map $f: X \to M/\mathcal{F}$ given by a cocycle $(U_i, \gamma_{ij})$ defines a morphism from the groupoid $G_{\mathcal{U}}$ associated with the cover $\{U_i\}$ to the holonomy groupoid $G$ of the foliation $(M, \mathcal{F})$. This morphism takes any $(x, i) \in G_{\mathcal{U}}(0)$ to $\gamma_{ij}(x) \in G$ and any $(x, i, j) \in G_{\mathcal{U}}$ to $\gamma_{ij}(x) \in G$. The induced map $BG_{\mathcal{U}} \to BG$ of the classifying spaces is well defined. If the cover $\mathcal{U}$ is locally finite and all non-empty intersections $U_{i_1} \cap \cdots \cap U_{i_n}$ are contractible, then this construction gives rise to a map $\bar{f}: X \to BG$ defined up to homotopy equivalence. The map $\bar{f}$ is the classifying map for the principal $G$-bundle $r_f: G_f \to X$ defined by the map $f: X \to M/\mathcal{F}$.

Thus, the classifying space $BG$ of the holonomy groupoid of the foliation is an analogue of the leaf space $M/\mathcal{F}$ in homotopy theory (the homotopy leaf space). There is also the universal classifying space for all smooth codimension-$q$ foliations. It is the classifying space $BG_q$ of the Haefliger groupoid $\Gamma_q$ (see Example 5.8) introduced by Haefliger [82], [77].

For any codimension-$q$ foliation $\mathcal{F}$ on a manifold $M$ there is defined (up to homotopy equivalence) the universal map $BG \to B\Gamma_q$. It is constructed as follows. Let $\mathcal{U} = \{U_i\}$ be a good cover of $M$ by foliated charts, and $T = \bigcup_i T_i$ the corresponding complete transversal. Since every local transversal $T_i$ is diffeomorphic to $I^q$, a natural morphism $G^T_q \to \Gamma_q$ of groupoids is well defined. The universal map $BG \to B\Gamma_q$ is obtained as the composition of the induced map $BG^T \to B\Gamma_q$ and the homotopy equivalence $BG \simeq BG^T_q$.

We refer the reader to the survey [98] and its references for various questions related to the classification problem for foliations.

Let $(M, \mathcal{F})$ be a foliated manifold. The normal bundle $\tau$ on $M$, being a holonomy equivariant bundle, defines a vector bundle on $BG$ which will also be denoted by $\tau$. Let us consider the $K$-homology group $K_{*,\tau}(BG)$ of $BG$ twisted by the bundle $\tau$. Using Poincaré duality, one can show that elements of $K_{*,\tau}(BG)$ are represented by geometric cycles $(X, E, f)$, where $X$ is a smooth compact manifold, $E$ is a complex vector bundle on $X$, and $f: X \to M/\mathcal{F}$ is a $K$-oriented map.

There is defined a map

$$K^*_\text{top}(M, \mathcal{F}) \to K_{*,\tau}(BG)$$

which associates to any quadruple $(X, \mathcal{F}_X, x, f)$ the set $(X, E, \bar{f})$, where $E$ is a $G_X$-equivariant bundle on $X$ corresponding to the element $x \in K_*(C^*(X, \mathcal{F}_X))$, and $\bar{f}: X \to M/\mathcal{F}$ is a natural lift of $f: X/\mathcal{F}_X \to M/\mathcal{F}$. This map is rationally injective and, in the case when the isotropy groups $G^Z_x$ are torsion free, is an isomorphism.

Denote by $H_{*,\tau}(BG, \mathbb{Q})$ the singular homology of the pair $(B\tau, S\tau)$. One has the Chern character $\text{ch}: K_{*,\tau}(BG) \to H_{*,\tau}(BG, \mathbb{Q})$ and the Thom isomorphism
Φ: $H_{k+q,\tau}(BG,\mathbb{Q}) \rightarrow H_k(BG,\mathbb{Q})$, $q = \dim \tau$. Their composition $\Phi \text{ch}: K_{\ast,\tau}(BG) \rightarrow H_{\ast}(BG,\mathbb{Q})$ has the following form [99] (see also [100] and (2.3)): for any $y = [X,E,f] \in K_{\ast,\tau}(BG)$

$$\Phi \text{ch}(y) = \bar{f}_*(\text{ch}(E) \cup \text{Td}(TX \oplus \bar{f}^*\tau) \cap [X]) \in H_{\ast}(BG,\mathbb{Q}),$$

(5.5)

where $\bar{f}: X \rightarrow BG$ is the map defined by the map $f: X \rightarrow M/\mathcal{F}$.

We refer the reader to the bibliography given in [18] for various aspects of the Baum–Connes conjecture for foliations and related computations of the $K$-theory for foliation $C^*$-algebras.

### 5.6. Transverse integration.

The foundations of non-commutative integration theory for foliations were laid by Connes in [15] (see also [101], [102], [19]). Let $(M,\mathcal{F})$ be a compact foliated manifold. Let $\Lambda$ be a holonomy quasi-invariant transverse measure for $\mathcal{F}$, $\alpha$ a strictly positive smooth leafwise density on $M$, and $\nu = s^*\alpha$ the corresponding smooth Haar system. The measure $\Lambda$ and $\alpha$ enables one to construct a measure $\mu$ on $M$ (see (4.1)). Finally, the measure $\mu$ and the Haar system $\nu$ define a measure $m$ on $G$:

$$\int_G f(\gamma) \, d\mu(\gamma) = \int_M \left( \int_{G^x} f(\gamma) \, d\nu^x(\gamma) \right) \, d\mu(x), \quad f \in C_c(G).$$

In §5.2 we defined the representation $R_x$ of the involutive algebra $C_c^\infty(G)$ on the Hilbert space $L^2(G^x,\nu^x)$ for any $x \in M$. Consider the representation $R$ of the algebra $C_c^\infty(G)$ on $L^2(G,m) = \bigoplus_M L^2(G^x,\nu^x) \, d\mu(x)$ defined as the direct integral of $R_x$:

$$R = \int_M R_x \, d\mu(x).$$

**Definition 5.31.** The von Neumann algebra $W^*_\Lambda(M,\mathcal{F})$ of a foliation $\mathcal{F}$ is defined as the closure of the image of the algebra $C_c^\infty(G)$ under the representation $R$, in the weak topology of the space $\mathcal{L}(L^2(G,m))$.

Because the definition of the von Neumann algebra $W^*_\Lambda(M,\mathcal{F})$ depends only on the measure class of $m$ (that is, on the family of all sets of $m$-measure zero), this definition is independent of the choice of $\alpha$.

Elements of the algebra $W^*_\Lambda(M,\mathcal{F})$ can be described as families $(T_x)_{x \in M}$ of bounded operators on $H_x = L^2(G^x,\nu^x)$ such that:

1) for any $\gamma \in G$, $\gamma: x \rightarrow y$,

$$L(\gamma)T_xL(\gamma)^{-1} = T_y,$$

(5.6)

where the representation $L$ is given by the formula (5.3);

2) the function $M \ni x \mapsto \|T_x\|$ is essentially bounded with respect to the measure $\mu$;

3) for any $\xi,\eta \in L^2(G,m)$ the function $M \ni x \mapsto \langle T_x(\xi_x),\eta_x\rangle_{H_x}$ is measurable.

Operator families satisfying the condition (5.6) will be called (left-invariant) $G$-operators. We also use right-invariant $G$-operators, which are families $(T_x)_{x \in M}$.
of bounded operators on $H_x = L^2(G_x, \nu_x)$, $\nu_x = r^* \alpha$, such that $R(\gamma)T_xR(\gamma)^{-1} = T_y$ for any $\gamma \in G$, $\gamma: x \to y$, where the representation $R$ is given by (5.4).

A holonomy invariant measure $\Lambda$ on $M$ defines a normal semifinite trace $\text{tr}_\Lambda$ on the von Neumann algebra $W^*_\Lambda(M, \mathcal{F})$. For any bounded measurable function $k$ on $G$ the value $\text{tr}_\Lambda(k)$ is finite and given by

$$\text{tr}_\Lambda(k) = \int_M k(x) \, d\mu(x).$$

In [15] a description of weights on the von Neumann algebra $W^*_\Lambda(M, \mathcal{F})$ is given. As explained in [102], the construction of [15] can be interpreted as a correspondence between weights on $W^*(M, \mathcal{F})$ and operator-valued densities on the leaf space $M/\mathcal{F}$.

**Theorem 5.32.** Let $(M, \mathcal{F})$ be a foliated manifold. The von Neumann algebra $W^*_\Lambda(M, \mathcal{F})$ is a factor if and only if the foliation is ergodic, that is, any bounded measurable function constant along the leaves of $\mathcal{F}$ is constant on $M$.

It is known that von Neumann algebras are classified according to three classes: type I, II, and III. Any von Neumann algebra $\mathcal{M}$ can be canonically represented as a direct sum $\mathcal{M}_1 \oplus \mathcal{M}_{II} \oplus \mathcal{M}_{III}$ of von Neumann algebras $\mathcal{M}_1$, $\mathcal{M}_{II}$, and $\mathcal{M}_{III}$ of types I, II, and III, respectively.

**Theorem 5.33.** Let $(M, \mathcal{F})$ be a foliated manifold. The von Neumann algebra $W^*_\Lambda(M, \mathcal{F})$ is of

1) type I if and only if the leaf space is isomorphic to a standard Borel space;
2) type II if and only if there exists a holonomy invariant transverse measure and the algebra is not of type I;
3) type III if and only if there exists no holonomy invariant transverse measure.

### 6. Non-commutative differential calculus on the leaf space

#### 6.1. The transverse de Rham complex of a foliation.

Let $(M, \mathcal{F})$ be a foliated manifold. In this subsection we describe a non-commutative analogue of the de Rham complex on the leaf space $M/\mathcal{F}$ [19], [103].

Consider the space $\Omega^\infty = C_c^\infty(G, r^* \Lambda N^* \mathcal{F} \otimes |T\mathcal{G}|^{1/2})$. We define a product on it by the formula

$$(\omega_1 \ast \omega_2)(\gamma) = \int_{\gamma_1 \gamma_2 = \gamma} \omega_1(\gamma_1) \wedge H(\gamma_1)[\omega_2(\gamma_2)], \quad \gamma \in G,$$

where $\omega_1, \omega_2 \in C_c^\infty(G, |T\mathcal{G}|^{1/2})$, and $H(\gamma): \Lambda N_y^* \mathcal{F} \to \Lambda N_y^* \mathcal{F}$ is the linear holonomy map associated with $\gamma \in G$, $\gamma: x \to y$. Thus, $\Omega^\infty$ is a graded algebra.

We now define the transverse de Rham differential as a linear map $d_H: \Omega^0_{\infty} = C_c^\infty(G, |T\mathcal{G}|^{1/2}) \to \Omega^1_{\infty} = C_c^\infty(G, r^* N^* \mathcal{F} \otimes |T\mathcal{G}|^{1/2})$ satisfying the condition

$$d_H(k_1 \ast k_2) = d_H k_1 \ast k_2 + k_1 \ast d_H k_2, \quad k_1, k_2 \in C_c^\infty(G, |T\mathcal{G}|^{1/2}).$$

The construction of $d_H$ makes essential use of an auxiliary choice of a distribution $H$ on $M$ transverse to $F = T\mathcal{F}$. 

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There is a decomposition of $TM$ into the direct sum $TM = F \oplus H$. Therefore, one has the corresponding bigrading of the exterior bundle $\Lambda^*T^*M$:

$$\Lambda^k T^*M = \bigoplus_{i+j=k} \Lambda^{i,j} T^*M, \quad \Lambda^{i,j} T^*M = \Lambda^i H^* \otimes \Lambda^j F^*,$$

and also (see, for instance, [6], Proposition 10.1) the corresponding decomposition of the de Rham differential $d$ into a sum of bigraded components of the form

$$d = d_F + d_H + \theta. \quad (6.1)$$

Here

1) $d_F = d_{0,1}: C^\infty (M, \Lambda^{i,j} T^*M) \to C^\infty (M, \Lambda^{i,j+1} T^*M)$ is the tangential de Rham differential, which is a first-order tangentially elliptic operator independent of the choice of $g$;

2) $d_H = d_{1,0}: C^\infty (M, \Lambda^{i,j} T^*M) \to C^\infty (M, \Lambda^{i+1,j} T^*M)$ is the transversal de Rham differential, which is a first-order transversally elliptic operator;

3) $\theta = d_{2, -1}: C^\infty (M, \Lambda^{i,j} T^*M) \to C^\infty (M, \Lambda^{i+2,j-1} T^*M)$ is a zero-order differential operator which is the operator of inner multiplication by the 2-form $\theta$ on $M$ with values in $F$, $\theta \in C^\infty (M, F \otimes \Lambda^2 \tau^*)$, given by

$$\theta(X, Y) = p_F([X, Y]), \quad X, Y \in C^\infty (M, H),$$

where $P_F: TM \to F$ is the natural projection (the form $\theta$ is called the curvature of the transverse distribution $H$; in particular, $\theta$ vanishes if and only if $H$ is completely integrable).

The transverse distribution $H$ defines naturally a transverse distribution $HG \cong r^*H$ on the foliated manifold $(G, \mathcal{F})$ (see §5.1) and the corresponding transversal de Rham differential $d_H: C^\infty_c (G) \to C^\infty_c (G, r^* N^* \mathcal{F})$ (see (6.1)).

For an arbitrary smooth leafwise density $\lambda \in C^\infty_c (M, |T \mathcal{F}|)$ we define a 1-form $k(\lambda) \in C^\infty_c (M, H^*) \cong C^\infty (M, N^* \mathcal{F})$ as follows. Take an arbitrary point $m \in M$. In a foliated chart $\phi: U \to I^p \times I^q$ defined in a neighbourhood of $m$ ($\phi (m) = (x^0, y^0)$) the density $\lambda$ can be written as $\lambda = f(x, y) \, |dx|$, $(x, y) \in I^p \times I^q$. Then

$$k(\lambda) = d_H \lambda + \sum_{i=1}^p L_{\frac{\partial}{\partial x_i}} \, d_H x_i,$$

where for any $X = \sum_{i=1}^p X^i \frac{\partial}{\partial x_i} \in \mathfrak{X} (\mathcal{F})$ and any $\omega = \sum_{j=1}^q \omega_j \, dy_j \in C^\infty (M, H^*)$ the Lie derivative $\mathcal{L}_X \omega \in C^\infty (M, H^*)$ is given by

$$\mathcal{L}_X \omega = \sum_{i=1}^p \sum_{j=1}^q X^i \frac{\partial \omega_j}{\partial x_i} \, dy_j.$$

One can give a slightly different description of $k(\lambda)$. For any $X \in H_m$ let $\tilde{X}$ be an arbitrary projectable vector field, which coincides with $X$ at $m$:

$$\tilde{X} (x, y) = \sum_{i=1}^p X^i (x, y) \frac{\partial}{\partial x_i} + \sum_{j=1}^q Y^j (y) \frac{\partial}{\partial y_j}.$$
Put
\[
 k(\lambda)(X) = \sum_{i=1}^{p} X^i(x^0, y^0) \frac{\partial f}{\partial x_i}(x^0, y^0) \\
+ \sum_{j=1}^{q} Y^j(y^0) \frac{\partial f}{\partial x_j}(x^0, y^0) + \sum_{i=1}^{p} \frac{\partial X^i}{\partial x_i}(x^0, y^0) f(x^0, y^0).
\]

It can be checked that this definition is independent of the choice of a foliated chart \( \phi: U \to IP \times I^q \) and an extension \( \tilde{X} \).

If \( M \) is Riemannian, \( \lambda \) is given by the induced leafwise Riemannian volume form, and \( H = F^\perp \), then \( k(\lambda) \) coincides with the mean curvature 1-form of \( \mathcal{F} \).

An arbitrary leafwise half-density \( \rho \in C^\infty(M, |T\mathcal{F}|^{1/2}) \) can be written as \( \rho = f|\lambda|^{1/2} \) with \( f \in C^\infty(M) \) and \( \lambda \in C^\infty(M, |T\mathcal{F}|) \). Then \( d_H \rho \in C^\infty_c(M, N^*\mathcal{F} \otimes |T\mathcal{F}|^{1/2}) \) is defined as

\[
d_H \rho = (d_H f)|\lambda|^{1/2} + \frac{1}{2} f|\lambda|^{1/2}k(\lambda).
\]

Any \( f \in C^\infty_c(G, |T\mathcal{G}|^{1/2}) \) can be written as \( f = us^*(\rho)r^*(\rho) \), where \( u \in C^\infty_c(G) \) and \( \rho \in C^\infty_c(M, |T\mathcal{F}|^{1/2}) \). The element \( d_H f \in C^\infty_c(G, r^*N^*\mathcal{F} \otimes |T\mathcal{G}|^{1/2}) \) is defined as

\[
d_H f = d_H us^*(\rho)r^*(\rho) + us^*(d_H \rho)r^*(\rho) + us^*(\rho)r^*(d_H \rho).
\]

The operator \( d_H \) has a unique extension to a differentiation of the differential graded algebra \( \Omega^\infty = C^\infty_c(G, r^*\Lambda^qN^*\mathcal{F} \otimes |T\mathcal{G}|^{1/2}) \). By definition, for any \( f \in C^\infty_c(G, |T\mathcal{G}|^{1/2}) \) and \( \omega \in C^\infty_c(M, \Lambda^qN^*\mathcal{F}) \) one has

\[
d_H(fr^*\omega) = (d_H f)r^*\omega + fr^*(d_H \omega).
\]

We define a closed graded trace \( \tau \) on the differential graded algebra \( (\Omega^\infty, d_H) \) by the formula

\[
\tau(\omega) = \int_M \omega|_M,
\]

where \( \omega \in \Omega^q = C^\infty_c(G, r^*\Lambda^qN^*\mathcal{F} \otimes |T\mathcal{G}|^{1/2}) \). Here \( \omega|_M \) denotes the restriction of the form \( \omega \) to \( M \), which is a section of the bundle \( \Lambda^qN^*\mathcal{F} \otimes |T\mathcal{F}| \) on \( M \). Since the foliation is transversally oriented, the integral of \( \omega|_M \) over \( M \) is well defined.

### 6.2. Transverse fundamental class of a foliation.

Let \( (M, \mathcal{F}) \) be a foliated manifold and \( H \) an arbitrary distribution on \( M \) transverse to \( F = T\mathcal{F} \). In this subsection we describe, following [99] (see also [19]), the simplest construction of a cyclic cocycle on the algebra \( C^\infty_c(G, |T\mathcal{G}|^{1/2}) \), namely, the construction of the transverse fundamental class.

In the previous subsection we constructed a graded algebra \( \Omega^\infty \), a differential \( d_H \), and a closed graded trace \( \tau \) on it. The problem is that, since the distribution \( H \) is non-integrable, it is not true in general that \( d_H^2 = 0 \). Using the equality (6.1) and computing the type \((0,2)\) component in the representation of the operator \( d^2 \) as a sum of bihomogeneous components, we get that

\[
d_H^2 = -(d_F \theta + \theta d_F).
\]
The operator $-(d_F \theta + \theta d_F)$ is a tangential differential operator, so is given by exterior multiplication by a vector-valued distribution $\Theta \in \mathcal{D}'(G, r^* \Lambda^2 N^* \mathcal{F} \otimes |T\mathcal{G}|^{1/2})$ supported in $G^{(0)}$. One can show that

$$d_H^2 \omega = \Theta \wedge \omega - \omega \wedge \Theta, \quad \omega \in \Omega_\infty.$$ 

Moreover, $d_H \Theta = 0$. Using these facts, one can canonically construct a differential graded algebra $(\tilde{\Omega}_\infty, \tilde{d}_H)$ and a closed graded trace $\tilde{\tau}$ on it (see [14, [19]). The algebra $\tilde{\Omega}_\infty$ consists of $2 \times 2$ matrices $\omega = \{\omega_{ij}\}$ with entries in $\Omega_\infty$. An element $\omega \in \tilde{\Omega}_\infty$ has degree $k$ if $\omega_{11} \in \Omega_k^1$, $\omega_{12}, \omega_{21} \in \Omega_{k-1}^1$, and $\omega_{22} \in \Omega_k^{-2}$. The product in $\tilde{\Omega}_\infty$ is given by

$$\omega \cdot \omega' = \begin{bmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \Theta \end{bmatrix} \begin{bmatrix} \omega_{11}' & \omega_{12}' \\ \omega_{21}' & \omega_{22}' \end{bmatrix},$$

the differential by

$$\tilde{d}_H \omega = \begin{bmatrix} d_H \omega_{11} & d_H \omega_{12} \\ -d_H \omega_{21} & -d_H \omega_{22} \end{bmatrix} + \begin{bmatrix} 0 & -\Theta \\ 1 & 0 \end{bmatrix} \omega + (-1)^{\omega} \omega \begin{bmatrix} 0 & 1 \\ -\Theta & 0 \end{bmatrix}$$

and the closed graded trace $\tilde{\tau} : \tilde{\Omega}_\infty^q \to \mathbb{C}$ is defined by

$$\tilde{\tau} \left( \begin{bmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{bmatrix} \right) = \tau(\omega_{11}) - (-1)^\omega \tau(\omega_{22} \Theta).$$

Finally, the homomorphism $\tilde{\rho} : C_c^\infty(G, |T\mathcal{G}|^{1/2}) \to \tilde{\Omega}_\infty^0$ is given by

$$\tilde{\rho}(k) = \begin{bmatrix} k & 0 \\ 0 & 0 \end{bmatrix}.$$ 

Thus, the triple $(\tilde{\Omega}_\infty, \tilde{d}_H, \tilde{\tau})$ is a cycle over the algebra $C_c^\infty(G, |T\mathcal{G}|^{1/2})$. It is called the fundamental cycle of the transversally oriented foliation $(M, \mathcal{F})$. The character of this cycle defines a cyclic cocycle $\phi_H$ on the algebra $C_c^\infty(G, |T\mathcal{G}|^{1/2})$. The cocycle $\phi_H$ depends on the auxiliary choice of a horizontal distribution $H$, but the corresponding cyclic cohomology class is independent of this choice. The class $[M/\mathcal{F}] \in HC^q(C_c^\infty(G, |T\mathcal{G}|^{1/2}))$ defined by the cocycle $\phi_H$ is called the transverse fundamental class of the foliation $(M, \mathcal{F})$.

Let $C_c^\infty(G, |T\mathcal{G}|^{1/2})^+$ be got by adjoining a unit to the algebra $C_c^\infty(G, |T\mathcal{G}|^{1/2})$. For an even $q$ let us extend the cycle $(\tilde{\Omega}_\infty, d_H, \tilde{\tau})$ to a cycle over $C_c^\infty(G, |T\mathcal{G}|^{1/2})^+$ by putting $\tilde{\tau}(\Theta^{q/2}) = 0$. It is proved in [104] that the formula

$$\chi^r(k^0, k^1, \ldots, k^r) = \frac{(-1)^{(q-r)/2}}{((q+r)/2)!} \sum_{i_0 + \cdots + i_r = (q-r)/2} \int_M k^0 \Theta^{i_0} d_H k^1 \Theta^{i_1} \cdots d_H k^r \Theta^{i_r},$$

where $r = q, q-2, \ldots, k^0 \in C_c^\infty(G, |T\mathcal{G}|^{1/2})^+, k^1, \ldots, k^r \in C_c^\infty(G, |T\mathcal{G}|^{1/2})$, defines a cocycle in the $(b, B)$-complex of the algebra $C_c^\infty(G, |T\mathcal{G}|^{1/2})^+$. The class defined by the cocycle $\chi$ in $HC^q(C_c^\infty(G, |T\mathcal{G}|^{1/2}))$ coincides with $[M/\mathcal{F}]$. 
The pairing with the class \([M/\mathcal{F}] \in HC^q(C^\infty_\mathcal{C}(G, |\mathcal{T}\mathcal{G}|^{1/2}))\) defines an additive map \(\varphi: K(C^\infty_\mathcal{C}(G, |\mathcal{T}\mathcal{G}|^{1/2})) \to \mathbb{C}\). An important problem is the question of topological invariance of this map, that is, the question whether it is possible to extend it to an additive map from \(K(C^*_\mathcal{C}(G))\) to \(\mathbb{C}\). This problem was solved in [99].

A standard method of solving the problem of topological invariance of the map \(\varphi\) consists in constructing a smooth subalgebra \(\mathcal{B}\) of \(C^*_\mathcal{C}(M, \mathcal{F})\) which contains the algebra \(C^\infty_\mathcal{C}(G, |\mathcal{T}\mathcal{G}|^{1/2})\) and is such that the cyclic cocycle \(\phi_H\) on \(C^\infty_\mathcal{C}(G, |\mathcal{T}\mathcal{G}|^{1/2})\), which defines the transverse fundamental class \([M/\mathcal{F}]\), extends by continuity to a cyclic cocycle on \(\mathcal{B}\) and thereby defines a map in the topological \(K\)-theory \(K(\mathcal{B}) \cong K(C^*_\mathcal{C}(G)) \to \mathbb{C}\). This has been done for so-called para-Riemannian foliations (see §7.4) by using some properties of densely defined cyclic cocycles on Banach algebras.

For an arbitrary foliated manifold \((M, \mathcal{F})\) a bundle \(P\) over \(M\) is constructed in [99] whose fibres are connected spin manifolds of non-positive curvature, and then a natural lift of the foliation \(\mathcal{F}\) to a para-Riemannian foliation \(\mathcal{V}\) on \(P\) (see a more detailed exposition in §7.4). Since \(\mathcal{V}\) is para-Riemannian, its transverse fundamental class defines a map \(K(C^*_\mathcal{C}(P, \mathcal{V})) \to \mathbb{C}\). On the other hand, since the fibres of \(P\) are connected spin manifolds of non-positive curvature, there is an injective map \(K(C^*_\mathcal{C}(M, \mathcal{F})) \to K(C^*_\mathcal{C}(P, \mathcal{V}))\), making it possible to construct the desired extension \(K(C^*_\mathcal{C}(M, \mathcal{F})) \to \mathbb{C}\) for the initial foliation \((M, \mathcal{F})\). For geometric consequences of this construction, see §8.3 and also [99].

Actually, if the tangent bundle \(T\mathcal{F}\) is \(K\)-oriented, then, for any element \(P\) of the subring of the ring \(H^*(M, \mathbb{R})\) generated by Pontryagin classes of the normal bundle \(\tau\) and Chern classes of arbitrary holonomy equivariant bundles on \(E\), an additive map \(\varphi_P: K(C^*_\mathcal{C}(M, \mathcal{F})) \to \mathbb{C}\) is constructed in [99] such that for any \(E \in K^i(M)\) with \(i = \dim M \pmod 2\)

\[
\varphi_P(E \otimes p!) = \langle \text{ch}(E) \cdot \text{Td}(TM \otimes \mathbb{C}) \cdot P, [M]\rangle, \tag{6.2}
\]

where \(p! \in KK(C(M), C^*_\mathcal{C}(M, \mathcal{F}))\) is the element associated with the natural projection \(p: M \to M/\mathcal{F}\) (see a more precise statement in Theorem 8.5).

We recall that the \(K\)-homology fundamental class of a compact Spin\(^c\)-manifold \(M\) is defined to be the class \([D] \in K_i(M)\), where \(i = \dim M \pmod 2\), determined by an arbitrary Spin\(^c\) Dirac operator \(D\) on \(M\). If the foliation \((M, \mathcal{F})\) is Riemannian, the tangent bundle \(T\mathcal{F}\) is \(K\)-oriented, and the normal bundle has a holonomy invariant complex spin structure, then the corresponding transverse Spin\(^c\) Dirac operator \(\bar{D}_n\) defines a \(K\)-cohomology class \([\bar{D}_n]\) \(\in K^i(C^*(M, \mathcal{F}))\), where \(i = \dim(M/\mathcal{F}) \pmod 2\) (see §7.3), and therefore a map \(K_i(C^*_\mathcal{C}(M, \mathcal{F})) \to \mathbb{C}\). It is proved in Theorem 4.2 of [100] that for any class \([E] \in K^0(M)\) given by a bundle \(E\) on \(M\) one has

\[
(E \otimes p!) \otimes [\bar{D}_n] = \text{Ind}(D_E),
\]

where \(D_E\) is the Spin\(^c\) Dirac operator on \(M\) with coefficients in \(E\). In particular, by the Atiyah–Singer index theorem,

\[
(E \otimes p!) \otimes [\bar{D}_n] = \langle \text{ch}(E) \cdot \text{Td}(TM \otimes \mathbb{C}), [M]\rangle. \tag{6.3}
\]
Thus, the right-hand side of (6.3) coincides with the right-hand side of (6.2) with \( P = \text{Td}(\tau) \). In the general case the class \( [D^h] \in K^i(C^*_r(M, \mathcal{F})) \) is not well defined, but the construction described above makes it possible to construct the corresponding map \( \varphi_P : K(C^*_r(M, \mathcal{F})) \to \mathbb{C} \). These arguments may serve as a justification of the name ‘transverse fundamental class’ for a map of the form \( \varphi_P \).

**Definition 6.1.** A transverse current is a current \( C \) (that is, a continuous linear functional on the space of smooth, compactly supported differential forms) defined on the disjoint union of all transversals to the foliation.

**Definition 6.2.** A transverse current \( C \) is said to be holonomy invariant if, for any transversals \( B_1 \) and \( B_2 \) and any map \( \phi : B_1 \to B_2 \) belonging to the holonomy pseudogroup, \( \phi_* (C_{B_1}) = C_{B_2} \).

Let \( C \) be a closed, holonomy invariant transverse current of degree \( k \). Define a continuous linear functional \( \rho_C \) on the space \( C^\infty_c(G, \Lambda^k N^* \mathcal{F} \otimes |T\mathcal{F}|_{1/2}) \) as follows.

Take a good cover \( \{U_i\} \) of the manifold \( M \) by foliated coordinate neighbourhoods with the corresponding coordinates maps \( \phi_i : U_i \to I^p \times I^q \) and a partition of unity \( \{\psi_i\} \) subordinate to this covering. We consider the corresponding complete transversal \( T = \bigcup_i T_i \), where \( T_i = \phi_i^{-1}([0] \times I^q) \). In any foliated chart \( (U_i, \phi_i) \) the transverse current \( C \) defines a current \( C_i \) on \( T_i \). The current \( C \) is holonomy invariant if and only if \( f_{ij} (C_i) = C_j \) for any pair of indices \( i \) and \( j \) such that \( U_i \cap U_j \neq \emptyset \).

For any section \( \omega \in C^\infty_c(G, \Lambda^k N^* \mathcal{F} \otimes |T\mathcal{F}|_{1/2}) \) and any \( i \) the expression
\[
\int_L \Psi_i \omega = \int_{P_i(y)} \psi_i(x, y) \omega(x, y)
\]
gives a well-defined \( k \)-form on \( T_i \). Put
\[
\rho_C(\omega) = \sum_i \left< C_i, \int_L \Psi_i \omega \right>.
\]

One can show that the functional \( \rho_C \) is well defined by this formula, that is, the result is independent of the choice of a covering \( \{U_i\} \) and a partition of unity \( \{\psi_i\} \). The formula
\[
\tau_C(\omega) = \rho_C(\omega|_M), \quad \omega \in C^\infty_c(G, r^* \Lambda^k N^* \mathcal{F} \otimes |T\mathcal{F}|_{1/2}),
\]
gives a closed graded trace \( \tau_C \) on the differential graded algebra
\[
(\Omega_\infty = C^\infty_c(G, r^* \Lambda^k N^* \mathcal{F} \otimes |T\mathcal{F}|_{1/2}), d_H).
\]

Thus, any closed holonomy invariant transverse current of degree \( k \) defines a cyclic cocycle on the algebra \( C^\infty_c(G, |T\mathcal{F}|_{1/2}) \). Topological invariance of these cocycles follows from the results of the paper [99] (see [105], Remark 5.12).

**6.3. The cyclic cocycle defined by the Godbillon–Vey class.** We consider a smooth compact manifold \( M \) with a transversally oriented codimension-one foliation \( \mathcal{F} \). The Godbillon–Vey class is a 3-dimensional cohomology class of \( M \).
It is the simplest example of secondary characteristic classes of a foliation. We recall its definition. Since \( \mathcal{F} \) is transversally oriented, it is globally defined by a non-vanishing smooth 1-form \( \omega \) (that is, \( \ker \omega_x = T_x \mathcal{F} \) for any \( x \in M \)). It follows from the Frobenius theorem that there exists a 1-form \( \alpha \) on \( M \) such that \( d\omega = \alpha \wedge \omega \). One can check that the 3-form \( \alpha \wedge d\alpha \) is closed, and its cohomology class does not depend on the choice of \( \omega \) and \( \alpha \). This class \( GV \in H^3(M, \mathbb{R}) \) is called the Godbillon–Vey class of \( \mathcal{F} \).

Let \( T \) be a complete smooth transversal given by a good cover of \( M \) by regular foliated charts. Thus, \( T \) is an oriented one-dimensional manifold. In this subsection we construct a cyclic cocycle on \( C_c^\infty(G_T^+ ) \) corresponding to the Godbillon–Vey class. This is done in several steps. To start with, we describe the construction of the Godbillon–Vey class as a secondary characteristic class associated with the cohomology \( H^*(W_1, \mathbb{R}) \) of the Lie algebra \( W_1 = \mathbb{R}[ [x ] ] \partial_x \) of formal vector fields on \( \mathbb{R} \).

The cohomology \( H^*(W_1, \mathbb{R}) \) was computed by Gel’fand and Fuchs (for instance, see the book \([106]\) and its references). They are finite-dimensional, and the only non-trivial groups are \( H^0(W_1, \mathbb{R}) = \mathbb{R} \cdot 1 \) and \( H^3(W_1, \mathbb{R}) = \mathbb{R} \cdot gv \), where

\[
gv(p_1 \partial_x, p_2 \partial_x, p_3 \partial_x) = \begin{vmatrix} p_1(0) & p_2(0) & p_3(0) \\ p_1'(0) & p_2'(0) & p_3'(0) \\ p_1''(0) & p_2''(0) & p_3''(0) \end{vmatrix}, \quad p_1, p_2, p_3 \in \mathbb{R}[ [x ] ].
\]

Consider the bundle \( F^k_+ T \to T \) of positively oriented frames of order \( k \) on \( T \) and the bundle \( F_\infty^+ T = \lim \limits_{\rightarrow} F^k_+ T \) of positively oriented frames of infinite order on \( T \). By definition, a positively oriented frame \( r \) of order \( k \) at \( x \in T \) is the \( k \)-jet at \( 0 \in \mathbb{R} \) of an orientation-preserving diffeomorphism \( f \) which maps a neighbourhood of \( 0 \) in \( \mathbb{R} \) onto some neighbourhood of \( x = f(0) \) in \( T \). If \( y: U \to \mathbb{R} \) is a local coordinate on \( T \) defined in a neighbourhood \( U \) of \( x \), then the numbers

\[
y_0 = y(x), \quad y_1 = \frac{d(y \circ f)}{dt}(0), \quad \ldots, \quad y_k = \frac{d^k(y \circ f)}{dt^k}(0)
\]

are coordinates of the frame \( r \), and moreover, \( y_1 > 0 \).

There is a natural action of the pseudogroup \( \Gamma^+(T) \) of orientation-preserving local diffeomorphisms of \( T \) on \( F_\infty^+ T \). Let \( \Omega^*(F_\infty^+ T)^{\Gamma^+(T)} \) denote the space of differential forms on \( F_\infty^+ T \) invariant under the action of \( \Gamma^+(T) \). There is a natural isomorphism \( J: C^*(W_1) \to \Omega^*(F_\infty^+ T)^{\Gamma^+(T)} \) of differential algebras defined in the following way. First of all, let \( v \in W_1 \), and let \( h_t \) be any one-parameter group of local diffeomorphisms of \( \mathbb{R} \) such that \( v \) is the \( \infty \)-jet of the vector field \( \frac{dh_t}{dt} \bigg|_{t=0} \). Then we define a \( \Gamma^+(T) \)-invariant vector field on \( F_\infty^+ T \) whose value at \( r = j_0 f \in J_\infty^+(T) \) is given by

\[
v(r) = j_0 \left( \frac{d(f \circ h_t)}{dt} \bigg|_{t=0} \right).
\]

For any \( c \in C^q(W_1) \) put

\[
J(c)(v_1(r), \ldots, v_q(r)) = c(v_1, \ldots, v_q).
\]
One can check that this isomorphism takes the cocycle \( gv \in C^3(W_1, \mathbb{R}) \) to the three-form
\[
gv = \frac{1}{y_1^3} \, dy \wedge dy_1 \wedge dy_2 \in \Omega^3(F^2_+ T)^{\Gamma^+(T)}.
\]
Consider the bundle \( F^\infty(M/\mathcal{F}) \) on \( M \) consisting of infinite-order jets of all possible distinguished maps. There is a natural map \( F^\infty(M/\mathcal{F}) \to F^+_T/\Gamma^+(T) \). Using this map and the \( \Gamma^+(T) \)-invariance of \( gv \in \Omega^3(F^2_+ T) \), one can lift \( gv \) to a closed form \( gv(\mathcal{F}) \in \Omega^3(F^\infty(M/\mathcal{F})) \). The fibres of the fibration \( F^2(M/\mathcal{F}) \to M \) are contractible, so \( H^3(F^\infty(M/\mathcal{F}), \mathbb{R}) \cong H^3(M, \mathbb{R}) \), and the cohomology class in \( H^3(M, \mathbb{R}) \) corresponding to the cohomology class of \( gv(\mathcal{F}) \) in \( H^3(F^\infty(M/\mathcal{F}), \mathbb{R}) \) under this isomorphism coincides with the Godbillon–Vey class of \( \mathcal{F} \).

Let \( C^*(G^T_T, \Omega^*(G^T_T)) \) denote the space of cochains on \( G^T_T \) with values in the space \( \Omega^*(G^T_T) \) of differential forms on \( G^T_T \). By a Van Est type theorem (see [83]) there is an embedding
\[
\Omega^*(F^\infty_+ T)^{\Gamma^+(T)} \to C^*(G^T_T, \Omega^*(G^T_T)). \tag{6.4}
\]
This map is a homomorphism of complexes which induces an isomorphism in cohomology.

Let \( \rho \) be an arbitrary smooth positive density on \( T \). We define a homomorphism \( \delta: G^T_T \to \mathbb{R}_+ \), called the modular homomorphism, by setting \( \delta = r^* \rho/s^* \rho \), where \( r^* \rho \) (respectively, \( s^* \rho \)) denotes the lift of the density \( \rho \) to \( G^T_T \) by the map \( r: G^T_T \to T \) (respectively, \( s: G^T_T \to T \)), and also the homomorphism \( \ell = \log \delta: G^T_T \to \mathbb{R} \). The formula
\[
c(\gamma_1, \gamma_2) = \ell(\gamma_2) \, d\ell(\gamma_1) - \ell(\gamma_1) \, d\ell(\gamma_2), \quad \gamma_1, \gamma_2 \in G^T_T,
\]
defines a 2-cocycle on \( G^T_T \) with values in the space of 1-forms on \( G^T_T \). This cocycle, called the Bott–Thurston cocycle, corresponds to \( gv \in \Omega^3(F^2_+ T)^{\Gamma^+(T)} \) under the isomorphism given by the embedding (6.4).

The Bott–Thurston cocycle \( c \) defines a cyclic 2-cocycle \( \psi \) on \( C_c^\infty(G^T_T) \) by the formula (see Example 3.4)
\[
\psi(k^0, k^1, k^2) = \int_{\gamma_0 \gamma_1 \gamma_2 \in T} k^0(\gamma_0) k^1(\gamma_1) k^2(\gamma_2) c(\gamma_1, \gamma_2), \quad k^0, k^1, k^2 \in C_c^\infty(G^T_T). \tag{6.5}
\]
This is the cyclic cocycle corresponding to the Godbillon–Vey class of \( \mathcal{F} \).

Connes proved the topological invariance of this cocycle, that is, the fact that the additive map \( \varphi: K(C_c^\infty(G^T_T)) \to \mathbb{C} \) defined by the pairing with the class \( [\psi] \in HC^2(C_c^\infty(G^T_T)) \) of \( \psi \), \( \phi(e) = \langle e, [\psi] \rangle \), determines a map \( \varphi: K(C^*_c(G^T_T)) \cong K(C^*_r(M, \mathcal{F})) \to \mathbb{C} \). Moreover, one has the formula (see also the formula (8.8))
\[
\varphi(\mu_r(x)) = \langle \Phi \text{ch}(x), GV \rangle, \quad x \in K^*_\text{top}(M, \mathcal{F}). \tag{6.6}
\]
For the cyclic cocycle corresponding to the Godbillon–Vey class of \( \mathcal{F} \) there is another description which connects it with invariants of the von Neumann algebra of this foliation [99]. The formula
\[
\tau(k^0, k^1) = \int_{G^T_T} k^0(\gamma^{-1}) \, dk^1(\gamma), \quad k^0, k^1 \in C_c^\infty(G^T_T),
\]
defines a cyclic 1-cocycle on $C_c^\infty(G^T_T)$. The class of $\tau$ in $HC^1(C_c^\infty(G^T_T))$ corresponds to the transverse fundamental class of $F$ in $HC^1(C_c^\infty(G))$ under the isomorphism $HC^1(C_c^\infty(G^T_T)) \cong HC^1(C_c^\infty(G))$ defined by the strong Morita equivalence (see §5.4).

The fixed smooth positive density $\rho$ on $T$ defines a faithful normal semifinite weight $\phi_\rho$ on the von Neumann algebra $W^*(G^T_T)$ of the groupoid $G^T_T$. For any $k \in C_c^\infty(G^T_T)$ the value of the weight $\phi_\rho$ is given by

$$\phi_\rho(k) = \int_T k(x)\rho(x).$$

Let us consider the one-parameter group of automorphisms $\sigma_t$ of the von Neumann algebra $W^*(G^T_T)$ given by

$$\sigma_t(k)(\gamma) = \delta(\gamma)^{it}k(\gamma), \quad k \in C_c^\infty(G^T_T), \quad t \in \mathbb{R}.$$ 

This group is the group of modular automorphisms associated with the weight $\phi_\rho$ by the Tomita–Takesaki theory.

The significance of the group of modular automorphisms is explained, in particular, by the following characterization of it: a one-parameter group of $^*$-automorphisms $\sigma_t$ of a von Neumann algebra $M$ is the group of modular automorphisms associated with a weight $\omega$ if and only if $\omega$ satisfies the Kubo–Martin–Schwinger conditions with respect to $\sigma_t$, that is, there is a function $f$ analytic in the strip $\text{Im} \ z \in (0, 1)$ and continuous in its closure such that for any $a, b \in M, t \in \mathbb{R}$

$$f(t) = \omega(\sigma_t(a)b), \quad f(t + i) = \omega(b\sigma_t(a)).$$

Following Connes [99], we define a 1-trace on a Banach algebra $B$ to be a bilinear functional $\phi$ defined on a dense subalgebra $A \subset B$ such that

1) $\phi$ is a cyclic cocycle on $A$,

2) for any $a^1 \in A$ there is a constant $C > 0$ such that

$$|\phi(a^0, a^1)| \leq C\|a^0\|, \quad a^0 \in A,$$

and we define a 2-trace on $B$ to be a trilinear functional $\phi$ defined on a dense subalgebra $A \subset B$ such that

1) $\phi$ is a cyclic cocycle on $A$,

2) for any $a^1, a^2 \in A$ there is a constant $C > 0$ such that

$$|\phi(x^0, a^1 x^1, a^2) - \phi(x^0 a^1, x^1, a^2)| \leq C\|a^1\| \|a^2\|, \quad x^0, x^1 \in A.$$

The formula

$$\dot{\tau}(k^0, k^1) = \lim_{t \to 0} \frac{1}{t} (\tau(\sigma_t(k^0), \sigma_t(k^1)) - \tau(k^0, k^1)), \quad k^0, k^1 \in C_c^\infty(G^T_T),$$

defines a 1-trace on $C^*_\tau(G^T_T)$ with domain $C_c^\infty(G^T_T)$ and invariant under the action of the automorphism group $\sigma_t$. 
For a $C^*$-algebra $A$ and for any 1-trace $\phi$ on it that is invariant under an action of a one-parameter automorphism group $\alpha_t$ with generator $D$ such that $\text{dom} \, \phi \cap \text{dom} \, D$ is dense in $A$ one can define a 2-trace $\chi = i_D \phi$ on $C^*_r(G_T^F)$ (an analogue of the contraction) by
\[
\chi(a^0, a^1, a^2) = \phi(D(a^2)a^0, a^1) - \phi(a^0D(a^1), a^2), \quad a^0, a^1, a^2 \in \text{dom} \, \phi \cap \text{dom} \, D.
\]

**Theorem 6.3.** Suppose that $(M, \mathcal{F})$ is a manifold with a transversally oriented codimension-one foliation, $T$ is a complete smooth transversal, and $\rho$ is a smooth positive density on $T$. Then the cyclic cocycle $\psi \in HC^2\left(C^*_c(G_T^F)\right)$ corresponding to the Godbillon–Vey class of $\mathcal{F}$ coincides with $i_D \tau$.

One can naturally associate to any von Neumann algebra $M$ an action (called the flow of weights [107]) of the multiplicative group $\mathbb{R}^+$ on a certain commutative von Neumann algebra, namely, the centre of the crossed product $M \rtimes \mathbb{R}$ of $M$ by $\mathbb{R}$ relative to the action of $\mathbb{R}$ on $M$ given by the modular automorphism group $\sigma_t$. As a consequence of Theorem 6.3, Connes established the following geometric fact.

**Theorem 6.4** [99]. Suppose that $(M, \mathcal{F})$ is a manifold with a transversally oriented codimension-one foliation. If the Godbillon–Vey class $GV \in H^3(M, \mathbb{R})$ does not vanish, then the flow of weights of the von Neumann algebra of the foliation $\mathcal{F}$ has a finite invariant measure.

In particular, this implies the following, earlier result.

**Theorem 6.5** [108]. Suppose that $(M, \mathcal{F})$ is a manifold with a transversally oriented codimension-one foliation. If the Godbillon–Vey class $GV \in H^3(M, \mathbb{R})$ does not vanish, then the von Neumann algebra of the foliation $\mathcal{F}$ has a non-trivial type III component.

In [109], [110] there is another construction of the cyclic cocycle associated with the Godbillon–Vey class, as a cyclic cocycle on the $C^*$-algebra $C^*_r(M, \mathcal{F})$ in the case when the foliation $\mathcal{F}$ is the horizontal foliation of a flat foliated $S^1$-bundle.

### 6.4. General constructions of cyclic cocycles

Let $G$ be a smooth étale groupoid (for instance, the reduced holonomy groupoid $G_T^F$ of a foliated manifold $(M, \mathcal{F})$ associated with a complete transversal $T$). The tangent bundle $\tau$ on $G^{(0)}$, being a $G$-bundle, defines a vector bundle on $BG$, which will also be denoted by $\tau$. Consider the cohomology group $H^*_\tau(BG)$ of the space $BG$ twisted by $\tau$: $H^*_\tau(BG) = H^*(B\tau, S\tau)$. Connes [19] (Chapter III, § 2.δ, Theorem 14 and Remark b)) constructed a natural map
\[
\Phi_* : H^*_\tau(BG) \to H^*P(C^*_c(G)).
\]

The constructions of the transverse fundamental class of a foliation and of the cyclic cocycle associated with the Godbillon–Vey class are particular cases of this general construction.

Let us start a description of the construction of the map $\Phi_*$ with a definition of a bicomplex $(C^{n,*}, d_1, d_2)$. For $n > 0$ and $-q \leq m \leq 0$ ($q = \dim G^{(0)}$) the space $C^{n,m}$ consists of de Rham currents of degree $-m$ on the manifold
\[
G^{(n)} = \{(\gamma_1, \ldots, \gamma_n) \in G^n : s(\gamma_i) = r(\gamma_{i+1}), \ i = 1, \ldots, n-1\}
\]
which vanish if either some $\gamma_j$ belongs to $G^{(0)}$ or $\gamma_1 \cdots \gamma_n \in G^{(0)}$. The space $C^{0,m},$ $-q \leq m \leq 0$, consists of de Rham currents of degree $-m$ on the manifold $G^{(0)}$. Otherwise, put $C^{n,m} = \{0\}$. The coboundary $d_1: C^{n,m} \rightarrow C^{n+1,m}$ is given by

$$d_1 = (-1)^m \sum_{j=0}^{m} (-1)^j \delta_j^*,$$

where the pull-back maps $\delta^*_j: D_m(G^{(n-1)}) \rightarrow D_m(G^{(n)})$ are induced by the étale maps $\delta_j: G^n \rightarrow G^{(n-1)}$ defined for $n > 1$ by

$$\delta_0(\gamma_1, \ldots, \gamma_n) = (\gamma_2, \ldots, \gamma_n),$$
$$\delta_j(\gamma_1, \ldots, \gamma_n) = (\gamma_1, \ldots, \gamma_j \gamma_{j+1}, \ldots, \gamma_n), \quad 1 \leq j \leq n-1,$$
$$\delta_n(\gamma_1, \ldots, \gamma_n) = (\gamma_1, \ldots, \gamma_{n-1}).$$

For $n = 1$ the maps $\delta_j: G \rightarrow G^{(0)}$, $j = 0, 1$, are defined by

$$\delta_0(\gamma_1) = r(\gamma_1), \quad \delta_1(\gamma_1) = s(\gamma_1), \quad \gamma_1 \in G.$$

The de Rham boundary $d^t: D_m(G^{(n)}) \rightarrow D_{m-1}(G^{(n)})$ gives the coboundary $d_2: C^{n,m} \rightarrow C^{n,m+1}$. The $k$th cohomology group of the complex $(C^*, d = d_1 + d_2)$ associated with the bicomplex $(C^*, d_1, d_2)$ coincides with $H^{k+q}_H(BG)$.

We introduce a bicomplex $(\Omega^*_c(G), d', d'')$. The space $\Omega^*_c(G)$ is defined as the quotient of the space of smooth compactly supported differential forms of degree $m$ on $G^{(n+1)}$ by the subspace of forms supported in the set of all $(\gamma_0, \ldots, \gamma_n)$ such that $\gamma_j \in G^{(0)}$ for some $j > 0$. The product in $\Omega^*_c(G)$ is given by

$$(\omega_1 \omega_2)(\gamma_0, \ldots, \gamma_{n_1}, \ldots, \gamma_{n_1+n_2}) = \sum_{\gamma' = \gamma_{n_1}} \omega_1(\gamma_0, \ldots, \gamma_{n_1-1}, \gamma) \wedge \omega_2(\gamma', \gamma_{n_1+1}, \ldots, \gamma_{n_1+n_2})$$

$$+ \sum_{j=0}^{n_1-2} (-1)^{n_1-j-1} \sum_{\gamma' = \gamma_j} \omega_1(\gamma_0, \ldots, \gamma_{j-1}, \gamma, \gamma', \ldots, \gamma_{n_1-1}) \wedge \omega_2(\gamma_{n_1}, \ldots, \gamma_{n_1+n_2}),$$

where $\omega_1 \in \Omega^{n_1,m_1}_c(G)$, $\omega_2 \in \Omega^{n_2,m_2}_c(G)$. In this formula the fact that the maps $r$ and $s$ are étale is used to identify the cotangent spaces.

The differential $d': \Omega^*_c(G) \rightarrow \Omega^*_{c+1,m}(G)$ is given by

$$d'\omega(\gamma_0, \ldots, \gamma_{n+1}) = \chi_{G^{(0)}}(\gamma_0) \omega(\gamma_1, \ldots, \gamma_{n+1}),$$

where $\chi_{G^{(0)}} \in C^\infty(G)$ is the indicator function of the set $G^{(0)}$, and the differential $d'': \Omega^{n,m}_c(G) \rightarrow \Omega^{n,m+1}_c(G)$ is given by the usual de Rham differential.

Let $(\Omega^*_c(G), d = d' + d'')$ be the differential graded algebra associated with the bicomplex $(\Omega^*_c(G), d', d'')$.

For an arbitrary cochain $c \in C^{n,m}$ in the bicomplex $(C^*, d_1, d_2)$ its push-forward by the map

$$(\gamma_1, \ldots, \gamma_n) \in G^{(n)} \mapsto ((\gamma_1 \cdots \gamma_n)^{-1}, \gamma_1, \ldots, \gamma_n) \in G^{(n+1)}$$
if \( n > 0 \) and by the natural embedding \( G^{(0)} \rightarrow G \) if \( n = 0 \) defines a de Rham current of degree \(-m\) on \( G^{(n+1)}\). Denote by \( \tilde{\Phi} \) the corresponding linear functional on \( \Omega^k_c(G) \).

The morphism \( \Phi \) from the bicomplex \( (C^*, d_1, d_2) \) to the \( (b, B) \)-bicomplex of the algebra \( \mathscr{A} = C^\infty_c(G) \) is defined as follows. For any \( c \in C^{n,m} \) the corresponding element \( \Phi(c) \in C^{n-m}(\mathscr{A}, \mathscr{A}^*) \) is the \((n-m+1)\)-linear functional on \( \mathscr{A} \) given by the formula (with \( \ell = n - m + 1 \))

\[
\Phi(c)(a^0, \ldots, a^\ell) = \lambda_{n,m} \sum_{j=0}^{\ell} (-1)^{j(\ell-j)} \tilde{\Phi}(d^j a^{j+1} \cdots d^\ell a^0 da^1 \cdots da^j), \quad a^0, \ldots, a^\ell \in \mathscr{A},
\]

where \( \lambda_{n,m} = n!/(\ell+1)! \). Here we regard the algebra \( \mathscr{A} \) as a subalgebra of the algebra \( \Omega^0_c(G) \), and the product and the differential \( d \) are taken in the algebra \( \Omega^*_c(G) \).

**Example 6.6.** If \( G \) is the trivial groupoid associated with a manifold \( M \) (see Example 5.3), then any closed current \( C \) of degree \( k \) on \( M \) defines a cocycle \( c \) in the complex \( (C^*, d) \). This cocycle has the sole non-zero component \( c_{0,-k} = C \in C^{0,-k} \). The corresponding cyclic cocycle \( \Phi(c) \) on the algebra \( C^\infty_c(M) \) coincides with the cocycle given by the formula (3.3).

**Example 6.7.** More generally, if \( (M, \mathcal{F}) \) is a compact foliated manifold and \( G = G_T^\pi \) is its reduced holonomy groupoid associated with a complete transversal \( T \) defined by a good cover \( \mathcal{U} = \{ U_i \} \), then any closed holonomy invariant transverse current \( C \) of degree \( k \) on \( M \) defines a cocycle \( c \) in the complex \( (C^*, d) \). This cocycle has the sole non-zero component \( c_{0,-k} = C \in C^{0,-k} \). The class in \( HC^*(C^\infty_c(G_T^\pi)) \) of the cyclic cocycle \( \Phi(c) \) on the algebra \( C^\infty_c(G_T^\pi) \) corresponds under the isomorphism defined by Morita equivalence to the class in \( HC^*(C^\infty_c(G)) \) of the cyclic cocycle on \( C^\infty_c(G) \) given by the current \( C \) (see §6.2).

**Example 6.8.** This example is a generalization of Examples 3.3 and 3.4. It also served as a motivation for the construction of \( \Phi \). Let \( M \) be an \( n \)-dimensional oriented manifold, and \( \Gamma \) a discrete group acting on \( M \) by orientation-preserving diffeomorphisms. Consider a \( k \)-cocycle \( \omega \) on \( \Gamma \) with coefficients in the \( \Gamma \)-module \( \Omega^n(M) \) of smooth differential \( n \)-forms on \( M \), \( \omega \in Z^k(\Gamma, \Omega^n(M)) \), such that \( \omega(g_1, \ldots, g_k) = 0 \) in the case when either \( g_i = e \) for some \( i \) or \( g_1 \cdots g_k = e \). As shown in [99] (Lemma 7.1), the following equality defines a cyclic \( k \)-cocycle \( \tau \) on the algebra \( C^\infty_c(M \times \Gamma) \subset C(M) \rtimes_\tau \Gamma \) (see Example 5.16):

\[
\tau_\omega(f_0, \ldots, f_k) = \sum_{g_0, \ldots, g_k \in \Gamma} \int_M f_0(x, g_0) f_1(x g_0, g_1) \cdots f_k(x g_0 g_1 \cdots g_{k-1}, g_k) \omega(g_1, \ldots, g_k)(x).
\]

The crossed product groupoid \( G = M \times \Gamma \) is an étale groupoid. One can check that the \( k \)-cocycle \( \omega \) defines an element \( \omega \in C^{k,0} \) and that \( \Phi(\omega) = \tau_\omega \). For more details we refer the reader to [19], Chapter III, §2.δ.
The construction of the cyclic cocycle corresponding to the Godbillon–Vey class of a foliation (see the formula (6.5))) is actually a particular case of this construction (the only point is that the role of the discrete group $\Gamma$ is played by the reduced holonomy groupoid of the foliation).

Let $G$ be a Hausdorff smooth étale groupoid. Consider the space of loops in $G$: 

$$B^{(0)} = \{ \gamma \in G : r(\gamma) = s(\gamma) \}.$$ 

We say that a subset $\mathcal{O} \subset B^{(0)}$ is invariant under the action of $G$ if for any $\gamma \in \mathcal{O}$ and $g \in G$ such that $s(g) = r(\gamma)$ one has an inclusion $g\gamma g^{-1} \in \mathcal{O}$.

For any invariant subset $\mathcal{O} \subset B^{(0)}$ one can define the localized cyclic cohomology $HC^n(C_c^\infty(G))_\mathcal{O}$. Moreover, if $B^{(0)} = \bigsqcup_\alpha \mathcal{O}_\alpha$ is the representation of $B^{(0)}$ as the disjoint union of open invariant subsets, then one has the direct sum decomposition [11]

$$HC^n(C_c^\infty(G)) = \bigoplus_\alpha HC^n(C_c^\infty(G))_{\mathcal{O}_\alpha}.$$ 

A subset $\mathcal{O} \subset B^{(0)}$ is said to be elliptic if it is invariant and the order of each element $\gamma \in \mathcal{O}$ is finite. A subset $\mathcal{O} \subset B^{(0)}$ is said to be hyperbolic if it is invariant, and the order of each element $\gamma \in \mathcal{O}$ is infinite.

One can take the set of units $\mathcal{O} = G^{(0)}$ for an open and closed subset $\mathcal{O} \subset B^{(0)}$. The localizations on this set are usually denoted by the subscript [1] instead of $\mathcal{O}$.

The morphism $\Phi$ described above provides a description of the corresponding component $HP^*(C_c^\infty(G))_{[1]}$ in the periodic cyclic cohomology $HP^n(C_c^\infty(G))$. Namely, one has the isomorphism

$$HP^{ev/odd}(C_c^\infty(G))_{[1]} \cong \bigoplus_{even/odd \ k} H^{k+q}(BG).$$

A similar description in terms of homologies of some double complexes is obtained in [111] for an arbitrary elliptic component $\mathcal{O} \subset B^{(0)}$. The computation of the localized cyclic cohomology for hyperbolic components is more complicated. It uses in a greater extent the combinatorics of the groupoid.

7. The index theory of transversally elliptic operators

7.1. Equivariant transversally elliptic operators and their distributional index. Transversally elliptic operators appeared for the first time in the papers [11] and [12] in the following situation. Let $M$ be a smooth compact manifold and $G$ a compact Lie group acting on $M$. Denote by $\mathfrak{g}$ the Lie algebra of $G$. For any $X \in \mathfrak{g}$ denote by $X_M$ the corresponding fundamental vector field on $M$. Vectors of the form $X_M(x)$ with $X \in \mathfrak{g}$ span the tangent space $T_x(G, x)$ of the orbit passing through $x$. Consider the space

$$(T^*_G M)_x = \{ \xi \in T^*_x M : \langle \xi, X_M(x) \rangle = 0 \text{ for any } X \in \mathfrak{g} \}.$$

**Definition 7.1.** A classical pseudodifferential operator $P$ from $C^\infty(M, \mathcal{E}^+)$ to $C^\infty(M, \mathcal{E}^-)$ of order $m$ acting on sections of $G$-equivariant vector bundles $\mathcal{E}^\pm$ on $M$ is said to be $G$-transversally elliptic if it commutes with the action of $G$ and its principal symbol $\sigma(P)(x, \xi) : \pi^*\mathcal{E}^+ \to \pi^*\mathcal{E}^-$ is invertible for any $(x, \xi) \in T^*_G M \setminus 0$. 

The choice of a $G$-invariant Riemannian metric $g$ on $M$ and $G$-invariant Hermitian structures on $\mathcal{E}^\pm$ defines Hilbert structures in the spaces $L^2(M, \mathcal{E}^\pm)$. Let us regard a $G$-transversally elliptic operator $P \in \Psi^m(M, \mathcal{E}^+, \mathcal{E}^-)$ as a closed unbounded operator from $L^2(M, \mathcal{E}^+)$ to $L^2(M, \mathcal{E}^-)$ obtained as its closure from the initial domain $C^\infty(M, \mathcal{E}^+)$. Then the kernels $\text{Ker} P$ and $\text{Ker} P^*$ of the operators $P$ and $P^*$ in $L^2(M, \mathcal{E}^+)$ and $L^2(M, \mathcal{E}^-)$, respectively, are $G$-invariant closed subspaces.

For any $G$-equivariant vector bundle $\mathcal{E}$ on $M$ denote by $T(g)$ the induced action of $g \in G$ in $L^2(M, \mathcal{E})$. For any function $\phi \in C^\infty_c(G)$ define the operator $T(\phi)$ in $L^2(M, \mathcal{E})$ by

$$T(\phi) = \int_G \phi(g) T(g) \, dg.$$ 

The $G$-equivariant index $\text{Ind}^G(P)$ of $P$ is the $G$-invariant distribution on $G$ defined by

$$\langle \text{Ind}^G(P), \phi \rangle = \text{tr} T(\phi) \Pi_{\text{Ker} P} - \text{tr} T(\phi) \Pi_{\text{Ker} P^*}, \quad \phi \in C^\infty_c(G),$$

where $\Pi_{\text{Ker} P}$ and $\Pi_{\text{Ker} P^*}$ are the orthogonal projections onto $\text{Ker} P$ and $\text{Ker} P^*$, respectively.

The principal symbol $\sigma(P)$ of a $G$-transversally elliptic operator $P$ defines an element $[\sigma(P)]$ of the equivariant $K$-theory $K_G(T_G^* M)$ with compact supports of the space $T_G^* M$. The $G$-equivariant index $\text{Ind}^G(P) \in \mathcal{D}'(G)^G$ of $P$ depends only on the class $[\sigma(P)] \in K_G(T_G^* M)$. Thus, the $G$-equivariant index induces a homomorphism of $R(G)$-modules (the analytic index)

$$\text{Ind}^G_a : K_G(T_G^* M) \to \mathcal{D}'(G)^G.$$ 

In [11] an algorithm is given for computing the index of a $G$-transversally elliptic operator. Using this algorithm, Berline and Vergne [112], [113] obtained an explicit cohomological index formula (see also [114] and the references therein for recent advances in this direction).

Let $M$ be a smooth compact manifold and $G$ a compact Lie group acting on $M$. For a fixed $s \in G$ consider the submanifold $M(s) = \{ x \in M : sx = x \}$ of fixed points of the action of $s$ on $M$. Denote by $\mathcal{N} = \mathcal{N}(M, M(s))$ the normal bundle of $M(s)$ in $M$. Let $G(s)$ denote the centralizer of $s$, $G(s) = \{ t \in G : st = ts \}$, and $\mathfrak{g}(s)$ its Lie algebra. The action of $G$ induces an action of $G(s)$ on $M(s)$, and the bundle $\mathcal{N} = \mathcal{N}(M, M(s))$ is a $G(s)$-equivariant vector bundle on $M(s)$. One has the formula

$$T_x M \cong T_x M(s) \oplus \mathcal{N}_x, \quad x \in M(s), \quad (7.1)$$

where $T_x M(s) = \{ v \in T_x M : sv = v \}$, $\mathcal{N}_x = (1 - s)T_x M$.

The Levi-Civita connection $\nabla^M$ preserves the decomposition (7.1), which defines a representation $\nabla^M = \nabla^0 \oplus \nabla^1$, where $\nabla^0$ is a $G(s)$-invariant connection on $TM(s)$, which coincides with the Levi-Civita connection on $M(s)$, and $\nabla^1$ is a $G(s)$-invariant connection on $\mathcal{N}$ compatible with the induced metric on $\mathcal{N}$. A similar representation holds for the Riemannian curvature form $R^M$ on $TM$:

$$R^M|_{M(s)} \cong R^0 \oplus R^1,$$

where $R^0$ and $R^1$ are the curvatures of the connections $\nabla^0$ and $\nabla^1$, respectively.
Denote by $\Omega^\infty_G(\mathfrak{g}, M) = C^\infty(\mathfrak{g}, \Omega(M))^G$ the space of $G$-invariant smooth maps from $\mathfrak{g}$ to the space $\Omega(M)$ of smooth differential forms on $M$. Elements of the space $\Omega^\infty_G(\mathfrak{g}, M)$ will be called $G$-equivariant forms on $M$. The equivariant differential $d_\mathfrak{g}: \Omega^\infty_G(\mathfrak{g}, M) \to \Omega^\infty_G(\mathfrak{g}, M)$ is defined by the formula

$$(d_\mathfrak{g}\alpha)(X) = d(\alpha(X)) - \iota(X_M)(\alpha(X)), \quad X \in \mathfrak{g},$$

where $\iota(X_M)$ denotes the inner product by $X_M$. We say that a $G$-equivariant form $\alpha$ on $M$ is equivariantly closed if $d_\mathfrak{g}\alpha = 0$.

Let $\mathcal{E}$ be a $G$-equivariant vector bundle endowed with a $G$-invariant connection $\nabla$. The covariant derivative $\nabla$ defines a direct sum decomposition $T\mathcal{E} = V\mathcal{E} \oplus H\mathcal{E}$, where $H\mathcal{E}$ is the horizontal space of the connection, and the vertical space $V\mathcal{E}$ is isomorphic to $\pi^*\mathcal{E}$. We define the moment map $\mu: \mathfrak{g} \to \text{End}(\mathcal{E})$ as follows. For any $m \in M$ and $v \in \mathcal{E}_m$, the vector $(\mu(Y)v)_m \in \mathcal{E}_m$ is the projection of the vector $(Y_E)_v \in T_v\mathcal{E}$ on $V\mathcal{E}_v \cong \mathcal{E}_m$. In the case when $\nabla$ is the Levi-Civita connection on $TM$, the corresponding moment map $\mu^M(Y) \in C^\infty(M, \text{so}(TM))$, $Y \in \mathfrak{g}$, called the Riemannian moment of the manifold $M$, is given by the formula

$$\mu^M(Y)Z = -\nabla_Z Y_M, \quad Z \in TM.$$

The equivariant curvature of the connection $\nabla$ is defined by

$$R(Y) = \mu(Y) + R,$$

where $R = \nabla^2 \in \Omega^2(M, \text{End}\mathcal{E})$ is the curvature of the connection $\nabla$ and $\mu(Y): \mathcal{E} \to \mathcal{E}$ is the associated moment map.

Let $R_0(Y)$ and $R_1(Y)$ be the equivariant curvatures of the respective connections $\nabla^0$ and $\nabla^1$, and define the $G(s)$-equivariant forms $J(M(s))$ and $D_s(\mathcal{N}(M, M(s)))$ on $M(s)$ by

$$J(M(s))(Y) = \det \frac{e^{R_0(Y)/2} - e^{-R_0(Y)/2}}{R_0(Y)},$$

$$D_s(\mathcal{N}(M, M(s)))(Y) = \det(1 - se^{R_1(Y)})$$

for any $Y \in \mathfrak{g}(s)$. The forms $J(M(s))$ and $D_s(\mathcal{N}(M, M(s)))$ are equivariantly closed. Moreover, for any $Y$ in a sufficiently small neighbourhood of 0 in $\mathfrak{g}(s)$ the form $J(M(s))(Y)$ is invertible in $\Omega(M)$.

Suppose that $P: C^\infty(M, \mathcal{E}^+) \to C^\infty(M, \mathcal{E}^-)$ is a $G$-equivariant transversally elliptic operator acting on sections of $G$-equivariant vector bundles $\mathcal{E}^\pm$ on $M$, and let $\sigma: \pi^*\mathcal{E}^+ \to \pi^*\mathcal{E}^-$ be its principal symbol. Choose arbitrary $G$-invariant connections $\nabla^{\mathcal{E}^\pm}$ on $\mathcal{E}^\pm$ and define an odd endomorphism $U(\sigma)$ of the $\mathbb{Z}_2$-graded bundle $\pi^*\mathcal{E} = \pi^*\mathcal{E}^+ \oplus \pi^*\mathcal{E}^-$ by the formula

$$U(\sigma)(x, \xi) = \begin{pmatrix} 0 & \sigma(x, \xi)^* \\ \sigma(x, \xi) & 0 \end{pmatrix}.$$ 

Let $\mathcal{A}(\sigma)$ be the superconnection on $\pi^*\mathcal{E}$ given by

$$\mathcal{A}(\sigma) = iU(\sigma) + \pi^*\nabla.$$
or, equivalently,

\[ A(\sigma) = \begin{pmatrix} \pi^* \nabla^{L^+} & i\sigma^* \\ i\sigma & \pi^* \nabla^{L^-} \end{pmatrix}. \]

For any \( Y \in g \) the action of the equivariant curvature \( F^k(Y) \in \Omega(M, \text{End} \, E) \) of the superconnection \( A(\sigma) \) on \( \Omega(M, E) \) has the form

\[ F^k(Y) = (A - i(Y_M))^2 + \mathcal{L}^E(Y), \]

where \( \mathcal{L}^E(Y) \) is the first-order differential operator defined by the Lie derivative of the action of \( G \) on \( \Omega(M, E) \).

We define the equivariant Chern character \( \text{ch}(A(\sigma)) \in \Omega^\infty_G(g, M) \) by

\[ \text{ch}(A(\sigma))(Y) = \text{tr}_s \exp F^k(Y), \quad Y \in g, \]

and, for any \( s \in G \), the equivariant Chern character \( \text{ch}_s(A(\sigma)) \in \Omega^\infty_{G(s)}(g(s), M(s)) \) by

\[ \text{ch}_s(A(\sigma))(Y) = \text{tr}_s s^E \exp F^k(Y)|_{M(s)}, \quad Y \in g(s), \]

where \( s^E \) denotes the action in the fibres of the bundle \( E|_{M(s)} \) defined by the action of \( s \).

Define an equivariant extension of the canonical symplectic form \( \Omega \) on \( T^*M \) by

\[ \Omega(Y) = -d_Y \omega^M, \quad Y \in g, \]

where \( d_Y \) denotes the operator \( d - \iota(Y_M) \) on \( \Omega(M) \), or

\[ \Omega(Y)(x, \xi) = \Omega(x, \xi) + \langle \xi, Y_M(x) \rangle, \quad (x, \xi) \in T^*M. \]

Finally, we will need the notion of descent of distributions. If \( N \) is a submanifold of \( G \) transverse to \( G \)-orbits, then the restriction of an arbitrary distribution \( \Theta \in \mathcal{D}'(G)^G \) to \( N \) is well defined as a distribution on \( N \). If \( U_s(0) \) is a sufficiently small neighbourhood of 0 in \( g(s) \), then the submanifold \( s \exp U_s(0) \) is transverse to the orbits of the adjoint action of \( G \) on \( G \), and use of the above result allows us to speak about the restriction of the \( G \)-invariant distribution \( \text{Ind}^G(P) \in \mathcal{D}'(G) \) to \( s \exp U_s(0) \).

**Theorem 7.2.** Let \( s \in G \). For any \( Y \in U_s(0) \)

\[ \text{Ind}^G(P)|_{s \exp U_s(0)}(Y) = \int_{T^*M(s)} \frac{1}{(2\pi)^n_s} \exp(-id_Y^\omega^M) \text{ch}_s(A(\sigma))(Y) D_s(\mathcal{N}(M, M(s)))(Y) J(M(s))(Y). \]

In this formula \( n_s = \dim M(s) \), and the right-hand side \( \theta^s \in \mathcal{D}'(s \exp U_s(0)) \) should be understood in the following way: if \( \phi \in C^\infty_c(s \exp U_s(0)) \), then

\[ I_\phi = \int_{\theta(s)} \frac{\exp(-id_Y^\omega^M) \text{ch}_s(A(\sigma))(Y)}{D_s(\mathcal{N}(M, M(s)))(Y) J(M(s))(Y)} \phi(Y) dY \]

is rapidly decreasing along the fibres of the bundle \( T^*M(s) \) and

\[ \langle \theta^s, \phi \rangle \overset{\text{def}}{=} \int_{T^*M(s)} \frac{1}{(2\pi)^n_s} I_\phi. \]
The index theorem for transversally elliptic operators in the $K$-theoretic form was proved very recently by Kasparov [115]. In [116] the index theorem for transversally elliptic operators was proved in the case when the isotropy groups are finite. In [117] the results of [11] were applied to prove the index theorem on orbifolds. See the papers [118]–[121] for applications of index theory in the representation theory of Lie groups. In [122] the authors connected the equivariant index of transversally elliptic operators with the fractional analytic index of projective elliptic operators associated with an Azumai bundle and with the twisted $K$-theory (see [123]).

The existence of the index of a transversally elliptic operator in the case when the group $G$ is non-compact was proved by Hörmander [11] (see also [124], [125]). In this case the simplest examples show that the $G$-equivariant index $\text{Ind}^G(P) \in \mathcal{D}'(G)^G$ of a $G$-transversally elliptic operator $P$ is not a homotopy invariant of its principal symbol $\sigma(P)$. Therefore, the analytic index map is not well defined. In fact, this was one of the main motivations for Connes to introduce cyclic cohomology. Results on computing the index of a transversally elliptic operator for non-compact Lie groups will be given in the next subsection.

7.2. Transversally elliptic operators on foliations. Let $(M, \mathcal{F})$ be a compact foliated manifold and $E$ a Hermitian vector bundle on $M$. We recall that the principal symbol $p_m$ of a classical pseudodifferential operator $P \in \Psi^m(M, E)$ is a smooth section of the bundle $\text{Hom}(\pi^*E)$ on $T^*M \setminus \{0\}$. The transverse principal symbol $\sigma_P$ of $P$ is the restriction of its principal symbol $p_m$ to $\tilde{N}^*\mathcal{F} = N^*\mathcal{F} \setminus \{0\}$. Thus, $\sigma_P$ is a smooth section of the bundle $\text{Hom}(\pi^*E)$ on $\tilde{N}^*\mathcal{F}$. An operator $P \in \Psi^m(M, E)$ is transversally elliptic if its transverse principal symbol $\sigma_P(\nu)$ is invertible for any $\nu \in \tilde{N}^*\mathcal{F}$.

Suppose now that $E$ is a holonomy equivariant Hermitian vector bundle on $M$. Thus, there is given a representation $T$ of the holonomy groupoid $G$ of the foliation $\mathcal{F}$ on the fibres of $E$, that is, for any $\gamma \in G$, $\gamma : x \to y$, a linear operator $T(\gamma) : E_x \to E_y$ is defined. The transverse principal symbol $\sigma_P$ of $P \in \Psi^m(M, E)$ is holonomy invariant if, for any $\gamma \in G$, $\gamma : x \to y$, one has the equality

$$T(\gamma) \circ [\sigma_P(dh^*_\gamma(\nu))] = \sigma_P(\nu) \circ T(\gamma), \quad \nu \in N^*_y\mathcal{F},$$

where we have used the isomorphisms $(\pi^*E)_{dh^*_\gamma(\nu)} \cong E_x$ and $(\pi^*E)_\nu \cong E_y$.

Examples of transversally elliptic operators are transverse Dirac operators (see [126], [127] and the references therein).

Let $M$ be a compact manifold endowed with a Riemannian foliation $\mathcal{F}$ of even codimension $q$ and $g_M$ a bundle-like metric on $M$, and denote by $T^HM$ the orthogonal complement of $T\mathcal{F}$. Let $\nabla$ be the transverse Levi-Civita connection in $T^HM$.

For any $x \in M$ let $\text{Cl}(Q_x)$ be the Clifford algebra of the Euclidean space $Q_x$. We define a $\mathbb{Z}_2$-graded vector bundle $\text{Cl}(Q)$ on $M$ whose fibre at $x \in M$ is $\text{Cl}(Q_x)$. This bundle is associated with the principal $SO(q)$-bundle $O(Q)$ of oriented orthonormal frames in $Q$, $\text{Cl}(Q) = O(Q) \times_{O(q)} \text{Cl}(\mathbb{R}^q)$. Therefore, the transverse Levi-Civita connection $\nabla$ induces a natural leafwise flat connection $\nabla^{\text{Cl}(Q)}$ in $\text{Cl}(Q)$ which is compatible with the Clifford multiplication and preserves the $\mathbb{Z}_2$-grading on $\text{Cl}(Q)$.
A complex vector bundle $\mathcal{E}$ on $M$ endowed with an action of the bundle $\text{Cl}(Q)$ is called a transverse Clifford module. The action of an element $a \in C^\infty(M, \text{Cl}(Q))$ on an $s \in C^\infty(M, \mathcal{E})$ will be denoted by $c(a)s \in C^\infty(M, \mathcal{E})$. A transverse Clifford module $\mathcal{E}$ is said to be self-adjoint if it is endowed with a Hermitian metric such that the operator $c(f): \mathcal{E}_x \to \mathcal{E}_x$ is skew-symmetric for any $x \in M$ and $f \in Q_x$. A transverse Clifford module $\mathcal{E}$ has a natural $\mathbb{Z}_2$-grading $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$. A connection $\nabla^\mathcal{E}$ on a transverse Clifford module $\mathcal{E}$ is called a Clifford connection if, for any $f \in C^\infty(M, T^HM)$ and $a \in C^\infty(M, \text{Cl}(Q))$,

$$[\nabla^\mathcal{E}_f, c(a)] = c(\nabla^\text{Cl}(Q)_f a).$$

A self-adjoint transverse Clifford module equipped with a Hermitian Clifford connection is called a transverse Clifford bundle.

Let $\tau \in C^\infty(M, T^HM)$ the mean curvature vector of $F$. If $e_1, e_2, \ldots, e_p$ is a local orthonormal basis in $T^F$, then

$$\tau = \sum_{i=1}^p P_H(\nabla^L_{e_i} e_i).$$

Let $\mathcal{E}$ be a transverse Clifford bundle on $M$ equipped with a Hermitian Clifford connection $\nabla^\mathcal{E}$. Let $f_1, \ldots, f_q$ be a local orthonormal basis in $T^HM$. The transverse Dirac operator $D_\mathcal{E}$ is defined by

$$D_\mathcal{E} = \sum_{\alpha=1}^q c(f_\alpha) \left( \nabla^\mathcal{E}_{f_\alpha} - \frac{1}{2} g_M(\tau, f_\alpha) \right).$$

The operator $D_\mathcal{E}$ is formally self-adjoint in $L^2(M, \mathcal{E})$. We also observe that $D_\mathcal{E}$ has a holonomy invariant principal symbol, as follows from the fact that the metric $g_M$ is bundle-like.

Another example of a transversally elliptic operator is the transverse de Rham operator acting in the space $C^\infty(M, \Lambda T^HM^*)$ by the formula

$$D_H = d_H + d_H^\ast,$$

where $d_H$ is the transverse de Rham differential (see (6.1)). If the foliation admits a transversal spin structure, then the operator $D_H$ is connected with the transverse Dirac operator $D_{F(Q) \otimes F(Q)^\ast}$, where $F(Q)$ is the associated transversal spinor bundle, as follows:

$$D_{F(Q) \otimes F(Q)^\ast} = D_H - \frac{1}{2} (\varepsilon_{\tau^\ast} + i\tau).$$

Thus, these operators coincide if and only if $\tau = 0$, that is, all the leaves are minimal submanifolds [126].

In [128] formulae for the distributional index of invariant transversally elliptic operators are obtained in the case when the group $G = \mathbb{R}$ acts locally freely and isometrically on a compact Riemannian manifold $M$. In this case the action defines a non-singular isometric flow $\phi_\ast: M \to M$, and its orbits define a foliation $\mathcal{F}$.

Let $E$ and $F$ be Hermitian vector bundles on $M$. Suppose that there exists an action of $\mathbb{R}$ in the fibres of the bundles $E$ and $F$ that preserves the Hermitian
structure and covers the action of $\mathbb{R}$ on $M$. Let $D : C^\infty(M, E) \to C^\infty(M, F)$ be an arbitrary $\mathbb{R}$-invariant, first-order, transversally elliptic operator.

We recall that a periodic orbit of the flow $\phi$ is said to be non-degenerate if 1 is not an eigenvalue of the associated Poincaré map. Denote by $X$ the generator of the flow. Then there is a natural action of the flow in the normal bundle $Q$:

$$d\phi_s(x) : Q_x = T_x M/\mathbb{R}X(x) \to Q_{\phi_s(x)} = T_{\phi_s(x)} M/\mathbb{R}X(\phi_s(x)).$$

A periodic orbit $c$ with (not necessarily minimal) period $l$ of the flow $\phi$ is said to be non-degenerate (or simple) if

$$\det(\text{id} - d\phi_l(x) : Q_x \to Q_x) \neq 0,$$

where $x \in c$ is an arbitrary point on $c$, $\phi_l(x) = x$. In this case, put

$$b_l(c) = \frac{\text{Tr}(\phi_l : E_x \to E_x) - \text{Tr}(\phi_l : F_x \to F_x)}{|\det(\text{id} - d\phi_l(x) : Q_x \to Q_x)|}.$$

**Theorem 7.3** [128]. Suppose that all closed orbits of the flow $\phi_s$ are simple. The restriction of the index $\text{Ind}^\mathbb{R}(D) \in \mathscr{D}'(\mathbb{R})^\mathbb{R}$ of the operator $D$ to $\mathbb{R}\setminus\{0\}$ is given by

$$\text{Ind}^\mathbb{R}(D) = \sum_c l(c) \sum_{k \neq 0} b_{kl(c)}(c) \cdot \delta_{kl(c)},$$

where $c$ runs over the set of all primitive closed orbits of the flow $\phi$, and $l(c)$ denotes the length of $c$ (its minimal positive period).

Since the set of periods of periodic orbits of the flow is bounded away from zero, the restriction of $\text{Ind}^\mathbb{R}(D)$ to some neighbourhood of zero is a distribution supported at $\{0\}$ and, therefore, is a linear combination of the delta-function at zero $\delta_0$ and its derivatives.

There is an important particular case when the derivatives of $\delta_0$ do not contribute to the formula for $\text{Ind}^\mathbb{R}(D)$, namely, the case when $D$ is the transverse de Rham operator. Note first of all that in this case the contribution of a non-degenerate periodic orbit $c$ of $\phi$ is given by

$$\varepsilon_l(c) = \text{sgn} \det(\text{id} - d\phi_l(x) : Q_x \to Q_x).$$

Let $T^H M$ denote the orthogonal complement of $T\mathcal{F}$. The curvature of the distribution $T^H M$ is a transverse 2-form $R_H \in C^\infty(M, \Lambda^2 T^H M^*)$ (see (6.1)), and therefore the form $\text{Pf}(R_H/2\pi) \in C^\infty(M, \Lambda^q T^H M^*)$ is well defined. Denote by $X^*$ the 1-form on $M$ dual to the vector field $X$ with respect to the Riemannian metric. The product $\text{Pf}(R_H/2\pi) \wedge X^*$ is a top-degree differential form on $M$.

**Theorem 7.4** [128]. In some neighbourhood of 0 in $\mathbb{R}$,

$$\text{Ind}^\mathbb{R}(D) = \int_M \text{Pf}\left(\frac{R_H}{2\pi}\right) \wedge X^* \cdot \delta_0.$$
Let us now turn to the case when the distribution $T^H M$ is integrable. Denote by $\mathcal{H}$ the foliation generated by the distribution $T^H M$. The action of the flow $\phi$ preserves the foliation $\mathcal{H}$, that is, takes any leaf of $\mathcal{H}$ to a (possibly different) leaf. There is a holonomy invariant transverse volume form $\Lambda$ on $\mathcal{H}$ corresponding to the form $dt$ on $\mathbb{R}$. Any transverse Dirac operator $D$ is a tangentially elliptic operator with respect to $\mathcal{H}$, and therefore the $\Lambda$-index $\text{Ind}_\Lambda(D)$ of $D$ is well defined (see § 8.1).

**Theorem 7.5** [128]. In some neighbourhood of 0 in $\mathbb{R}$,

$$\text{Ind}^{\mathbb{R}}(D) = \text{Ind}_\Lambda(D) \cdot \delta_0.$$ 

In [129] the case when the flow $\phi$ is not necessarily isometric and the operator $D$ is not invariant was studied. More precisely, the following situation was considered. Let $\mathcal{M}$ be a compact manifold, let $\phi$ be a non-singular flow on $\mathcal{M}$, and let $X$ denote the generator of $\phi$. Suppose that there exists an integrable distribution $H \subset T\mathcal{M}$ which is invariant under the action of the flow and transversal to orbits of the flow. If $\mathcal{H}$ is the foliation on $\mathcal{M}$ defined by the distribution $H$, then $\phi$ preserves the foliation $\mathcal{H}$. Consider the leafwise de Rham complex $(\Omega(\mathcal{H}), d^{\mathcal{H}})$, where $\Omega(\mathcal{H}) = C^\infty(M, \Lambda T\mathcal{H}^*)$ is the space of smooth leafwise forms on $M$, and $d^{\mathcal{H}}$ is the leafwise de Rham differential. We choose an arbitrary Riemannian metric on the leaves of $\mathcal{H}$ and extend it to a Riemannian metric on $\mathcal{M}$ by setting $|X(x)| = 1$ for any $x \in \mathcal{M}$ and saying that $X$ is orthogonal to $H$, and we consider the leafwise de Rham operator

$$D^{\mathcal{H}} = d^{\mathcal{H}} + d^{\ast}_{\mathcal{H}}.$$ 

It coincides with the transverse de Rham operator for the foliation $\mathcal{F}$ defined by the orbits of the flow $\phi$.

**Theorem 7.6** [129]. In some neighbourhood of 0,

$$\text{Ind}^{\mathbb{R}}(D^{\mathcal{H}}) = \int_M \text{Pf} \left( \frac{R^{\mathcal{H}}}{2\pi} \right) \wedge X^* \cdot \delta_0. \tag{7.2}$$

If all closed orbits of the flow $\phi_s$ are simple, then away from zero

$$\text{Ind}^{\mathbb{R}}(D^{\mathcal{H}}) = \sum_c l(c) \sum_{k \neq 0} \varepsilon_{kl(c)}(c) \cdot \delta_{kl(c)} \tag{7.3},$$

where $c$ runs over the set of all primitive closed orbits of the flow $\phi_t$, $l(c)$ denotes the length of $c$, $x$ is an arbitrary point on $c$, and

$$\varepsilon_{l(c)}(c) = \text{sgn} \det (\text{id} - d\phi_{l(c)}(x) : T_x \mathcal{H} \to T_x \mathcal{H}).$$

In this case the coefficient of $\delta_0$ coincides with the Euler $\Lambda$-characteristic $\chi_\Lambda(\mathcal{F})$ of $\mathcal{F}$ introduced in [15] as the $\Lambda$-index of the leafwise de Rham operator $D^{\mathcal{H}}$. The equality

$$\chi_\Lambda(\mathcal{F}) = \int_M \text{Pf} \left( \frac{R^{\mathcal{H}}}{2\pi} \right) \wedge X^*$$
is a consequence of the Gauss-Bonnet theorem for measurable foliations proved in [15] as a particular case of the index theorem for measurable foliations (see Theorem 8.1).

We remark also that for an isometric flow $\phi$ the formula (7.3) was independently proved in [130].

We refer the reader to the paper [129] for connections of Theorem 7.6 with the reduced cohomology of $\mathcal{H}$ and the Hodge theory for foliations. Taking into account these relationships, we can understand the statement of Theorem 7.6 as a dynamical Lefschetz formula for flows, that is, a formula which connects invariants of a foliation with closed orbits (for a discussion of dynamical Lefschetz formulae for flows see, for instance, [131]). We should also mention a dynamical Lefschetz formula for flows which was proposed as a conjecture first by Guillemin [132] and later independently by Patterson [133]. This formula has a form similar to (7.3), but it is written in the case when the transversal foliation $\mathcal{H}$ has codimension greater than 1, and therefore one cannot apply analytic results of index theory for transversally elliptic operators (see a survey of results about the Guillemin–Patterson conjecture in the book [134]).

The recent interest in the index theory of transversally elliptic operators, in particular, in the situation when there is a flow on a compact manifold which preserves a foliation, and in dynamical Lefschetz formulae for flows, is closely connected with Deninger’s approach to the study of arithmetic zeta-functions, based on analogies between arithmetic geometry and the theory of dynamical systems on foliated manifolds (see, for instance, the papers [135], [136] and the references therein).

In [137] the results of [129] were extended to the case of an arbitrary Lie group action. More precisely, in this paper a more general situation when $\mathcal{H}$ is a Lie foliation of a compact manifold $M$ is considered. In this case one can define an action of the structural Lie group $G$ on $M$ ‘up to leafwise homotopy’, which enables one to define the index $\text{Ind}^G(D_{\mathcal{H}})$ of the leafwise de Rham operator $D_{\mathcal{H}}$ as a distribution on $G$. In [137] a Lefschetz formula is proved which gives an expression for $\text{Ind}^G(D_{\mathcal{H}})$ in terms of the fixed points of the action. It can be regarded as a generalization of the Selberg formula.

### 7.3. Spectral triples associated with transversally elliptic operators.

Let $(M, \mathcal{F})$ be a compact foliated manifold, and $E$ a holonomy equivariant Hermitian vector bundle on $M$. As shown in [14], the pair $(H, F)$, where the Hilbert space $H = L^2(M, E)$ is equipped with an action of the algebra $C^*(M, \mathcal{F})$ by means of the $\ast$-representation $R_E$ and $F \in \Psi^0(M, E)$ is a transversally elliptic operator with holonomy invariant transverse principal symbol $\sigma_F$ such that $\sigma_F^2 = 1$ and $\sigma_F^* = \sigma_F$, is a Fredholm module over the $C^*$-algebra $C^*(M, \mathcal{F})$. Thus, the transversally elliptic operator $F$ defines a class $[F] \in K_0(C^*(M, \mathcal{F}))$. If one regards the pair $(H, F)$ as a Fredholm module over the algebra $C^\infty_c(G)$, then this module is summable, and therefore its Chern character $\tau_n = \text{ch}_n(H, F) \in HC^n(C^\infty_c(G))$ is well defined.

Similarly, one can consider a compact $G$-manifold $M$ ($G$ an arbitrary Lie group) and a $G$-equivariant Hermitian vector bundle $E$ on $M$. Then the pair $(H, F)$, where $H = L^2(M, E)$ and $F \in \Psi^0(M, E)$ is a (not necessarily $G$-invariant) transversally elliptic operator with invariant transverse principal symbol $\sigma_F$ such that $\sigma_F^2 = 1$ and $\sigma_F^* = \sigma_F$, is a Fredholm module over the $C^*$-algebra $C^*(G)$. The corresponding
Fredholm module over the algebra $C^\infty_c(G)$ is summable, and its Chern character $\tau_n = \text{ch}_*(H, F) \in HC^n(C^\infty_c(G))$ is defined.

As pointed out in [14], if the operator $F$ is invariant, then its $G$-equivariant index $\chi = \text{Ind}_G(F) \in D'(G)^G$ defines a trace on the group algebra $C^\infty_c(G)$, that is, an element of $HC^0_c(G)$, and $\tau_{2N} = S^N \chi$ for sufficiently large $N$. In other words, the Chern character $\tau_{2N}$ and the $G$-equivariant index $\chi$ define the same element in $HP^{ev}(C^\infty_c(G))$. In the case when the operator $F$ is not invariant, the Chern character $\text{ch}_*(H, F) \in HP^*(C^\infty_c(G))$ of the Fredholm module $(H, F)$ is a well-defined homotopy invariant, unlike its distributional index. This observation served as one of the main motivations for Connes to introduce the notion of cyclic cohomology.

In [138] spectral triples associated with transversally elliptic operators on a compact foliated manifold $(M, \mathcal{F})$ were constructed. More precisely, it was proved in [138] that the triple $(\mathcal{A}, H, D)$ with

1) $\mathcal{A}$ the algebra $C^\infty_c(G)$,
2) $H$ the Hilbert space $L^2(M, E)$ of square-integrable sections of a holonomy equivariant Hermitian vector bundle $E$ on which an element $k \in \mathcal{A}$ acts by the $*$-representation $R_E$,
3) $D$ a self-adjoint first-order transversally elliptic operator with holonomy invariant transverse principal symbol such that $D^2$ is self-adjoint and has scalar principal symbol,

is a finite-dimensional spectral triple of dimension $q = \text{codim } \mathcal{F}$.

As an example, one can assume that the foliation $\mathcal{F}$ is Riemannian and $g_M$ is a bundle-like metric on $M$. Let $H$ be the orthogonal complement of $F = T\mathcal{F}$. Let $H = L^2(M, \Lambda^*H^*)$ be the space of transverse differential forms equipped with the natural action $R_{\Lambda^*H^*}$ of $\mathcal{A}$, and $D$ the transverse signature operator $d_H + d_H^*$. The triple $(\mathcal{A}, H, D)$ is a finite-dimensional spectral triple of dimension $q = \text{codim } \mathcal{F}$.

It is proved in [138] that the spectral triples associated with transversally elliptic operators are smooth and have simple discrete dimension spectrum $S_d$ which is contained in the set $\{v \in \mathbb{N} : v \leq q\}$.

One should note that in this case the algebra $\mathcal{A}$ is non-unital, which can be understood as a reflection of the fact that the space $M/\mathcal{F}$ is non-compact. Therefore, one should modify the definitions of various geometric and analytic objects given in the unital case, taking into account the behaviour of these objects ‘at infinity’. In [139] the algebra $\Psi_0(\mathcal{A})$ associated with an arbitrary smooth spectral triple $(\mathcal{A}, H, D)$ was constructed. This algebra can be regarded as an analogue of the algebra of pseudodifferential operators on a non-compact manifold whose symbols vanish at infinity along with derivatives of any order. In the same paper there is a description of the non-commutative pseudodifferential calculus for spectral triples associated with transversally elliptic operators in terms of the transversal pseudodifferential calculus on foliated manifolds, together with a description of the non-commutative geodesic flow (see also [126]).

Example 7.7. Let $\mathcal{A} = C^\infty(S^1) \times \Gamma$ be the algebraic crossed product of the algebra $C^\infty(S^1)$ by the group $\Gamma$ of orientation-preserving diffeomorphisms of the
circle $S^1$. An arbitrary element of the algebra $\mathcal{A}$ is represented as a finite sum
\[ a = \sum_{\phi \in \Gamma} a_\phi U^*_\phi, \quad a_\phi \in C^\infty(S^1), \]
the product is given by
\[ (a_\phi U^*_\phi)(b_\psi U^*_\psi) = a_\phi (b_\psi \circ \phi) U^*_\phi \]
and the involution by
\[ (a_\phi U^*_\phi)^* = U^*_{\phi^{-1}} \bar{a} \]
(see Example 5.16). Define an involutive representation of the algebra $\mathcal{A}$ on the Hilbert space $\mathcal{H} = L^2(S^1)$ by
\[ (\rho(aU^*_\phi)\xi)(x) = a(x)\phi'(x)^{1/2}\xi(\phi(x)), \quad \xi \in \mathcal{H}, \quad x \in S^1. \]
Let the operator $D$ be $D = \frac{1}{i} \frac{\partial}{\partial x}$.

The triple $(\mathcal{A}, \mathcal{H}, D)$ is a spectral triple if and only if the action of $\Gamma$ is isometric. In the general case (see [75]) the triple $(\mathcal{A}, \mathcal{H}, D)$ is a $\sigma$-spectral triple, where the automorphism $\sigma$ of $\mathcal{A}$ is defined by
\[ \sigma(aU^*_\phi) = \frac{d\phi}{dx} aU^*_\phi. \]

**Example 7.8.** We give examples of spectral triples $(\mathcal{A}, \mathcal{H}, D)$ associated with the non-commutative torus $T^2_\theta$ (see [140] and Example 5.26). These triples are parameterized by a complex number $\tau$ with $\text{Im} \tau > 0$. Put
\[ \mathcal{A} = \mathcal{A}_\theta = \left\{ a = \sum_{(n,m) \in \mathbb{Z}^2} a_{nm} U^n V^m : a_{nm} \in \mathcal{S}(\mathbb{Z}^2) \right\}. \]

Define a canonical normalized trace $\tau_0$ on $\mathcal{A}_\theta$ by
\[ \tau_0(a) = a_{00}, \quad a \in \mathcal{A}_\theta. \]
Let $L^2(\mathcal{A}_\theta, \tau_0)$ be the Hilbert space which is the completion of the space $\mathcal{A}_\theta$ in the inner product $(a, b) = \tau_0(b^* a)$, $a, b \in \mathcal{A}_\theta$. The Hilbert space $\mathcal{H}$ is defined as the sum of two copies of the space $L^2(\mathcal{A}_\theta, \tau_0)$ equipped with the grading given by the operator
\[ \gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]

The representation $\rho$ of the algebra $\mathcal{A}_\theta$ on $\mathcal{H}$ is given by left multiplication, that is, for any $a \in \mathcal{A}_\theta$
\[ \rho(a) = \begin{pmatrix} \lambda(a) & 0 \\ 0 & \lambda(a) \end{pmatrix}, \]
where the operator $\lambda(a)$ is defined on $\mathcal{A}_\theta \subset L^2(\mathcal{A}_\theta, \tau_0)$ by the formula
\[ \lambda(a)b = ab, \quad b \in \mathcal{A}_\theta. \]
We introduce differentiations \( \delta_1 \) and \( \delta_2 \) on \( A\theta \) by
\[
\delta_1(U) = 2\pi i U, \quad \delta_1(V) = 0; \quad \delta_2(U) = 0, \quad \delta_2(V) = 2\pi i V.
\]
The operator \( D \) depends explicitly on \( \tau \) and has the form
\[
D = \begin{pmatrix} 0 & \delta_1 + \tau \delta_2 \\ -\delta_1 - \bar{\tau} \delta_2 & 0 \end{pmatrix}.
\]
The triples constructed above are two-dimensional smooth spectral triples.

We refer the reader to the book [23] and its references for the index theory of elliptic operators with shifts and the corresponding non-commutative geometry. The theory of elliptic operators on the non-commutative torus \( T^2_\theta \) (see [92], [19]) is a particular example of this general theory.

7.4. Para-Riemannian foliations and transversally hypo-elliptic operators. A foliation \( \mathcal{F} \) on a manifold \( M \) is said to be para-Riemannian if there exists an integrable distribution \( V \) on \( M \) which contains the tangent bundle \( T\mathcal{F} \) of \( \mathcal{F} \) and is such that the bundles \( TM/V \) and \( V/T\mathcal{F} \) are holonomy equivariant Riemannian bundles. If \( V \) is an integrable distribution on \( M \) which defines the para-Riemannian structure, and \( \mathcal{V} \) is the corresponding foliation on \( M, \mathcal{F} \subset \mathcal{V} \), then \( \mathcal{V} \) is Riemannian, and the restriction of the foliation \( \mathcal{F} \) to each leaf \( L \) of \( \mathcal{V} \) is a Riemannian foliation on \( L \).

As mentioned already in §6.2, the interest in para-Riemannian foliations consists in the fact that in some problems of index theory the study of arbitrary foliations can be reduced to the study of para-Riemannian foliations. Let us describe the corresponding construction by Connes and Moscovici [63]. In fact, they dealt with a closely related situation (a strongly Morita equivalent one): they considered an oriented smooth manifold \( W \) endowed with an action of a discrete group \( \Gamma \) instead of a foliated manifold. We consider the fibration \( \pi: P(W) \to W \) whose fibre \( P_x(W) \) at \( x \in W \) is the space of all Euclidean metrics on the vector space \( T_xW \). Thus, a point \( p \in P(W) \) is given by a point \( x \in W \) and a non-degenerate quadratic form on \( T_xW \). Let \( F_+(W) \) be the bundle of positive frames in \( W \) whose fibre \( F_x(W) \) at \( x \in W \) is the space of orientation-preserving linear isomorphisms \( \mathbb{R}^n \to T_xW \).

Equivalently, the bundle \( P(W) \) can be described as the orbit space of the bundle \( F_+(W) \) with respect to the fibrewise action of the subgroup \( SO(n) \subset GL(n, \mathbb{R}) \), \( P(W) = F_+(W)/SO(n) \). We will use a natural invariant Riemannian metric on the symmetric space \( GL_+(n, \mathbb{R})/O(n) \), given by the matrix Hilbert–Schmidt norm on the tangent space of \( GL(n, \mathbb{R})/SO(n) \), which is identified with the space of symmetric \( n \times n \) matrices. If we transfer this metric to the fibres \( P_x \) of the bundle \( P(W) = P \), then we obtain a Euclidean structure on the vertical distribution \( V \subset TP \). The normal space \( N_p = T_pP/V_p \) is naturally identified with the space \( T_xW, x = \pi(p) \). Thus, the quadratic form on \( T_xW \) corresponding to \( p \) defines a natural Euclidean structure on \( N_p \).

There are natural actions of the group \( \Gamma \) on \( F_+(W) \) and \( P \). This action takes fibres of the bundle \( \pi: P \to W \) to fibres. Moreover, the Euclidean structures introduced above on the distributions \( V \) and \( N \) are invariant under the action of the group \( \Gamma \). In this case one says that there is a triangular structure on \( P \) invariant
under the group action. It is an analogue of a para-Riemannian foliation in this situation. A very essential aspect of this construction is the fact that the fibres of \( \pi: P \to W \) are complete Riemannian manifolds of non-positive curvature.

Another important property of an arbitrary para-Riemannian foliation \((M, \mathcal{F})\) is the existence of a Fredholm module over its \(C^*\)-algebra \(C^*(M, \mathcal{F})\). A construction of this module is given in [95]. It makes use of transversally hypo-elliptic operators and pseudodifferential operators of type \((\rho, \delta)\).

In [63] Connes and Moscovici described a spectral triple associated with an invariant triangular structure. Thus, let \( P \) be a smooth manifold equipped with an action of a discrete group \( \Gamma \). Suppose that \( P \) is the total space of a bundle \( \pi: P \to W \) over an oriented smooth manifold \( W \). The action of \( \Gamma \) leaves \( W \) invariant and takes fibres of \( \pi: P \to W \) to fibres. Finally, \( \Gamma \) preserves the Euclidean structures on the vertical distribution \( V \subset TP \) and on the horizontal distribution \( N = TP/V \).

Consider the Hermitian vector bundle \( E = \Lambda^* (V^* \otimes \mathbb{C}) \otimes \Lambda^* (N^* \otimes \mathbb{C}) \) over \( P \). The Hermitian structure in the fibres of \( E \) is determined by the Euclidean structures on \( V \) and \( N \). The bundles \( \Lambda^* (V^* \otimes \mathbb{C}) \) and \( \Lambda^* (N^* \otimes \mathbb{C}) \) have the grading operators \( \gamma_V \) and \( \gamma_N \) given by the Hodge operators of the Euclidean structures and the orientations of \( V \) and \( N \). The Euclidean structures on \( V \) and \( N \) also define a natural volume form \( v \in \Lambda^* V^* \otimes \Lambda^* N^* = \Lambda T^* P \).

Let \( \mathcal{A} \) be the crossed product \( C^*_c(P) \rtimes \Gamma \). We recall that this algebra is generated as a linear space by expressions of the form \( fU_g \), where \( f \in C^*_c(P) \) and \( g \in \Gamma \) (see Example 5.6).

Let \( \mathcal{K} \) be the space \( L^2(P, E) \) equipped by the Hilbert structure given by the volume form \( v \) and the Hermitian structure on \( E \). The action of \( \mathcal{A} \) in \( \mathcal{K} \) is given in the following way. A function \( f \in C^*_c(P) \) acts as the corresponding multiplication operator acting in \( \mathcal{K} \). For any \( g \in \Gamma \) the unitary operator \( U_g \) is given on \( \mathcal{H} \) by the natural actions of \( \Gamma \) on sections of the bundles \( V \) and \( N \).

Consider the foliation \( \mathcal{F} \) given by the fibres of the fibration \( P \). Then \( V = TV \), and \( N \) is the normal bundle of \( \mathcal{F} \). Denote by \( d_L: C^\infty(P, E) \to C^\infty(P, E) \) the tangential de Rham differential associated with the foliation \( \mathcal{F} \) (see (6.1)). Let \( Q_L \) be the second-order tangential differential operator acting in \( C^\infty(P, E) \) given by

\[
Q_L = d_L d_L^* - d_L^* d_L.
\]

As shown in [63], the principal symbol of \( Q_L \) is homotopic to the principal symbol of the signature operator \( d_L + d_L^* \).

We choose an arbitrary distribution \( H \) on \( P \) transverse to \( V \) and consider the corresponding transverse de Rham differential \( d_H \) (see (6.1)) and the transverse signature operator

\[
Q_H = d_H + d_H^*.
\]

This operator depends on the choice of \( H \), but its transverse principal symbol is independent of \( H \). Let us define a mixed signature operator \( Q \) on \( C^\infty(P, E) \) as

\[
Q = Q_L (-1)^{\partial N} + Q_N,
\]

where \((-1)^{\partial N}\) denotes the parity operator in the transverse direction, that is, it coincides with \(1\) on \( \Lambda^e N^* \) and with \(-1\) on \( \Lambda^{\text{odd}} N^* \). Assume that \( Q \) is essentially self-adjoint in \( \mathcal{K} \). Using functional calculus, we define the operator \( D \) as \( Q = D|D| \).
One should note that, although the above construction can be applied to any manifold endowed with an invariant triangular structure, the question of essential self-adjointness of the operator $Q$ is a difficult analytic question, since the manifold $P$ is non-compact. In [63] this question was answered just for the example described above of a triangular structure on the Euclidean metrics bundle $P(W)$ associated with an arbitrary group $\Gamma$ of diffeomorphisms of a smooth manifold $W$.

**Theorem 7.9** [63]. For the triangular structure associated with an arbitrary group $\Gamma$ of diffeomorphisms of a smooth manifold $W$ the operator $Q$ is self-adjoint and the triple $(A, H, D)$ constructed above is a spectral triple of dimension $\dim V + 2 \dim N$.

The proof of this theorem makes essential use of the pseudodifferential calculus constructed by Beals and Greiner [141] on Heisenberg manifolds. It is also shown in this paper that the non-commutative integral $\int$ defined by such a spectral triple coincides with the Wodzicki–Guillemin type trace defined on the Beals–Greiner algebra of pseudodifferential operators.

In [142] Connes and Moscovici computed the Chern character of the spectral triple constructed in Theorem 7.9 by using the non-commutative local index theorem, Theorem 3.18. In fact, a direct computation of the Chern character of a spectral triple associated with a triangular structure on a smooth manifold using the formulae in Theorem 3.18 is quite cumbersome even in the one-dimensional case. One gets formulae involving thousands of terms, most of which give zero contribution. To simplify the computations a priori, Connes and Moscovici introduced a Hopf algebra $H_n$ of transverse vector fields on $\mathbb{R}^n$ which plays the role of a quantum symmetry group. They constructed the cyclic cohomology $HC^*(\mathcal{H})$ for an arbitrary Hopf algebra $\mathcal{H}$ and a map

$$HC^*(\mathcal{H}_n) \to HP^*(C^\infty_c(P) \rtimes \Gamma).$$

Moreover, they showed that the cyclic cohomology $HC^*(\mathcal{H}_n)$ is canonically isomorphic to the Gel’fand–Fuchs cohomology $H^*(W_n, SO(n))$ (see §8.3). Therefore, there is defined a characteristic homomorphism

$$\chi_{SO(n)}^*: H^*(W_n, SO(n)) \to HP^*(C^\infty_c(P) \rtimes \Gamma).$$

It is the composition of the map (8.6)) and the homomorphism $\Phi_*$ (see (6.7)).

The following theorem is the main result of [142] (see also [143] and the survey [144]).

**Theorem 7.10.** Let $(\mathcal{A}, \mathcal{H}, D)$ be the spectral triple introduced in Theorem 7.9. The Chern character $ch_*(\mathcal{A}, \mathcal{H}, D) \in HP^*(C^\infty_c(P(W)) \rtimes \Gamma)$ is the image of a universal class $\mathcal{L}_n \in H^*(W_n, SO(n))$ under the characteristic homomorphism $\chi^*_{SO(n)}$:

$$ch_*(\mathcal{A}, \mathcal{H}, D) = \chi^*_{SO(n)}(\mathcal{L}_n).$$

There is one more computation, given in [145], illustrating the Connes–Moscovici local index theorem. Let $\Sigma$ be a closed Riemann surface and $\Gamma$ a discrete pseudogroup of local conformal maps of $\Sigma$ without fixed points. With use of
methods in [142] (the bundle $P$ of Kähler metrics on $\Sigma$, hypo-elliptic operators, Hopf algebras), a spectral triple is constructed in [145] which is a generalization of the classical Dolbeault complex to this setting. The Chern character of this spectral triple as a cyclic cocycle on the crossed product $C_\infty(\Sigma) \rtimes \Gamma$ is computed in terms of the fundamental class $[\Sigma]$ and a cyclic 2-cocycle which is a generalization of the class Poincaré-dual to the Euler class. This formula can be regarded as a non-commutative version of the Riemann–Roch theorem.

A Hopf algebra of the same type as $\mathcal{H}_1$ was constructed by Kreimer [146] for the study of the algebraic structure of the perturbative quantum field theory. This connection was further elaborated in [147]–[149]. One should also mention the papers [150] and [151] by Connes and Moscovici on modular Hecke algebras, where it is shown, in particular, that such an important algebraic structure on the modular forms as the Rankin–Cohen brackets has a natural interpretation in the language of non-commutative geometry in terms of the Hopf algebra $\mathcal{H}_1$.

8. The index theory of tangentially elliptic operators

Let $(M, \mathcal{F})$ be a compact foliated manifold and $E$ a smooth vector bundle on $M$. A linear differential operator $D$ of order $\mu$ acting in $C^\infty(M, E)$, is called a tangential differential operator if, in any foliated chart $\phi: U \subset M \to I^p \times I^q$ and any trivialization of $E$ over it, the operator $D$ has the form

$$D = \sum_{|\alpha| \leq \mu} a_\alpha(x, y) \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_p^{\alpha_p}}, \quad (x, y) \in I^p \times I^q,$$

where $\alpha = (\alpha_1, \ldots, \alpha_p) \in \mathbb{Z}_+^p$ is a multi-index, $|\alpha| = \alpha_1 + \cdots + \alpha_p$, and the $a_\alpha$ are smooth matrix-valued functions on $I^p \times I^q$.

For a tangential differential operator $D$ given by (8.1) in some foliated chart $\phi: U \subset M \to I^p \times I^q$ and a trivialization of $E$ over it, we define its tangential (complete) symbol

$$\sigma(x, y, \xi) = \sum_{|\alpha| \leq \mu} a_\alpha(x, y)(i\xi)^{\alpha}, \quad (x, y) \in I^p \times I^q, \quad \xi \in \mathbb{R}^p,$$

and its tangential principal symbol

$$\sigma_\mu(x, y, \xi) = \sum_{|\alpha| = \mu} a_\alpha(x, y)(i\xi)^{\alpha}, \quad (x, y) \in I^p \times I^q, \quad \xi \in \mathbb{R}^p.$$

The tangential principal symbol is invariantly defined as a section of the bundle $\text{Hom}(\pi_F^* E)$ on $T^*\mathcal{F}$ (where $\pi_F: T^*\mathcal{F} \to M$ is the natural projection).

A tangential differential operator $D$ is said to be tangentially elliptic if its tangential principal symbol $\sigma_\mu$ is invertible for $\xi \neq 0$.

Let $D: C^\infty(M, E) \to C^\infty(M, E)$ be a tangential differential operator on a compact foliated manifold $(M, \mathcal{F})$. The restrictions of $D$ to the leaves of $\mathcal{F}$ define a family $(D_L)_{L \in M/\mathcal{F}}$, where for any leaf $L$ of $\mathcal{F}$ the operator $D_L: C^\infty(L, E|_L) \to C^\infty(L, E|_L)$ is a differential operator on $L$. For any $x \in M$ the lift of the operator $D_{Lx}$ by the holonomy covering $s: G^x \to L_x$ defines a differential operator $D_{Lx}: C^\infty(G^x, s^* E) \to C^\infty(G^x, s^* E)$. The operator family $\{D_x, x \in M\}$ is
a $G$-operator (see (5.6)). Families of this form will be called differential $G$-operators. If $D$ is a tangentially elliptic operator, then the corresponding $G$-operator will be said to be elliptic. In [15] the corresponding algebra of pseudodifferential $G$-operators on a foliated manifold is constructed.

8.1. The index theory for measurable foliations. Let $D$ be a tangentially elliptic operator on a compact foliated manifold $(M, \mathcal{F})$, and \{ $D_x: x \in M$ \} the corresponding elliptic $G$-operator. Suppose that the foliation $\mathcal{F}$ has a holonomy invariant transverse measure $\Lambda$. For $x \in M$, define elements $P_{\ker D_x}$ and $P_{\ker D^*_x}$ of the foliation von Neumann algebra $W^*_\Lambda(M, \mathcal{F})$. The holonomy invariant measure $\Lambda$ defines a faithful normal semifinite trace $\text{tr}_\Lambda$ on $W^*_\Lambda(M, \mathcal{F})$. It is proved in [15] that the dimensions

$$
\text{dim}_\Lambda \ker D = \text{tr}_\Lambda P_{\ker D}, \quad \text{dim}_\Lambda \ker D^* = \text{tr}_\Lambda P_{\ker D^*}
$$

are finite, and therefore the index of the tangentially elliptic operator $D$ is well defined by

$$
\text{Ind}_\Lambda(D) = \text{dim}_\Lambda \ker D - \text{dim}_\Lambda \ker D^*.
$$

Suppose that the bundle $T\mathcal{F}$ is oriented. As in the index theory for families of elliptic operators, the tangential principal symbol $\sigma_D$ of $D$ defines an element of $K(T^*\mathcal{F})$. Denote by $\pi_{F1}: H^*(T\mathcal{F}) \to H^*(M)$ the map given by integration along the fibres of the bundle $\pi_F: T\mathcal{F} \to M$.

Theorem 8.1 [15].

$$
\text{Ind}_\Lambda(D) = (-1)^p(p+1)/2 \langle C, \pi_{F1} \text{ch}(\sigma_D) \text{Td}(T^*\mathcal{F} \otimes \mathbb{C}) \rangle,
$$

where $C$ is the Ruelle–Sullivan current corresponding to the transverse measure $\Lambda$.

This theorem is completely analogous to the Atiyah–Singer index theorem in the cohomological form, Theorem 1.5, the only difference being that here one uses the pairing with the Ruelle–Sullivan current $C$ instead of integration over the compact manifold on the right-hand side of the Atiyah–Singer formula.

An odd version of Theorem 8.1 is proved in [152] (see also [153]). Suppose that $D: C^\infty(M, E) \to C^\infty(M, E)$ is a first-order tangentially elliptic operator on a compact foliated manifold $(M, \mathcal{F})$, and let $D_x: C^\infty(G^x, s^*E) \to C^\infty(G^x, s^*E)$, $x \in M$, be the corresponding differential $G$-operator. Assume that the foliation $\mathcal{F}$ has a holonomy invariant transverse measure $\Lambda$. Assume also that the operator $D_x$ is self-adjoint in the space $L^2(G^x, s^*E)$. Denote by $P_x$ its spectral projection corresponding to the positive semi-axis. For any $\phi \in C(M, U(N))$ denote by $M_\phi$ the corresponding $G$-operator of multiplication by $\phi$. The leafwise Toeplitz operator associated with $\phi$ is a bounded $G$-operator

$$
T_\phi = \{ T_{\phi,x}: L^2(G^x, s^*E \otimes \mathbb{C}^N) \to L^2(G^x, s^*E \otimes \mathbb{C}^N), \quad x \in M \},
$$

given by

$$
T_{\phi,x} = P_x M_\phi P_x.
$$
It is proved in [152] that if \( \phi \in C(M, U(N)) \) is invertible, then the operator \( T_\phi \) is a Breuer–Fredholm operator, and the Breuer–Fredholm index of \( T_\phi \) is defined by

\[
\text{Ind}_A(T_\phi) = \dim_A(\ker T_\phi) - \dim_A(\ker T_\phi^*) .
\]

Denote by \( E_+ \) the subbundle of the bundle \( \pi_* E \) on \( ST^* \mathcal{F} \) spanned by the eigenvectors of the principal symbol \( \sigma_1(D) \) of \( D \) corresponding to positive eigenvalues. As above, denote by \( \pi_F: H^*(ST^* \mathcal{F}) \to H^*(M) \) the map given by integration along the fibres of the bundle \( \pi_F: ST^* \mathcal{F} \to M \).

**Theorem 8.2** [153], [152].

\[
\text{Ind}_A(T_\phi) = (-1)^p \langle c, \text{ch}([\phi]) \pi_F! \text{ch}(E_+) \text{Td}(T \mathcal{F} \otimes \mathbb{C}) \rangle ,
\]

where \( C \) is the Ruelle–Sullivan current corresponding to the transverse measure \( \Lambda \).

One should also note the papers [154]–[160], [92], [57], where analogous problems of index theory (in both the even and odd settings) were studied in closely related situations—for almost periodic and random operators on \( \mathbb{R}^n \).

In [161] an analogue of the Atiyah–Patodi–Singer theorem for measurable foliations was proved. In particular, the eta-invariant of tangentially elliptic operators was introduced there (see also [162]). In [163] (see also [164]) an analogue of the Atiyah–Bott–Lefschetz fixed point formula [37] was proved for maps of a compact manifold with a measurable foliation which take each leaf to itself.

### 8.2. The \( K \)-theoretic index theory

In [88] (see also [13]) a \( K \)-theoretic version of the index theorem is proved for tangentially elliptic operators on an arbitrary compact foliated manifold \( (M, \mathcal{F}) \). Let \( D \) be a tangentially elliptic operator on a compact manifold \( M \) acting on sections of a vector bundle \( E \) on \( M \). Using operator techniques, one constructs the analytic index \( \text{Ind}_a(D) \in K_0(C^*_r(M, \mathcal{F})) \) of the corresponding right-invariant elliptic \( G \)-operator \( \{ D_x: C^\infty_c(G_x, r^* E) \to C^\infty_c(G_x, r^* E) \}, \ x \in M \} \) [15], [13], and starting from the class \( [\sigma_D] \in K(T^* \mathcal{F}) \) defined by the tangential principal symbol \( \sigma_D \) of \( D \), one constructs its topological index \( \text{Ind}_t(D) \in K_0(C^*_r(M, \mathcal{F})) \).

The analytic index is constructed in the following way. Recall that the bundle \( E \) gives rise to the \( (C(M), C^*_r(M, \mathcal{F})) \)-bimodule \( \mathcal{E}_{M,E} \) defined as the completion of the pre-Hilbert \( C^\infty_c(G) \)-module \( \mathcal{E}_{\infty} = C^\infty_c(G, r^* E) \) (see §5.3). Any right-invariant pseudodifferential \( G \)-operator naturally defines an endomorphism of the Hilbert \( C^*_r(M, \mathcal{F}) \)-module \( \mathcal{E}_{M,E} \). Therefore, the operator \( D \) defines an element \( [D] \in KK(C(M), C^*_r(M, \mathcal{F})) \) given by the pair \((\mathcal{E}_{M,E}, F)\), where \( F = D(I + D^2)^{1/2} \). The image of this element under the map

\[
KK(C(M), C^*_r(M, \mathcal{F})) \to KK(\mathbb{C}, C^*_r(M, \mathcal{F})) = K_0(C^*_r(M, \mathcal{F})),
\]

is the analytic index of \( D \), \( \text{Ind}_a(D) \in K_0(C^*_r(M, \mathcal{F})) \). In [152] another construction of the element \( [D] \in KK^1(C(M), C^*_r(M, \mathcal{F})) \) corresponding to a self-adjoint tangentially elliptic operator \( D \) is given. This construction uses an extension of the \( C^* \)-algebra \( C^*_r(M, \mathcal{F}) \) generated by smoothed leafwise Toeplitz operators.

We now describe the construction of the topological index of a tangentially elliptic operator \( D \). Let \( i \) be an embedding of the manifold \( M \) into \( \mathbb{R}^{2n} \). Denote by \( N \) the
total space of the normal bundle of the leaves: \( N_x = (i_*(F_x)) \perp \subset \mathbb{R}^{2n} \). We consider the foliation \( \widetilde{F} \) on the manifold \( \widetilde{M} = M \times \mathbb{R}^{2n} \) with fibres \( L = L \times \{ t \} \), where \( L \) is a leaf of \( \mathcal{F} \) and \( t \in \mathbb{R}^{2n} \). The map \( N \ni (x, \xi) \mapsto (x, i(x) + \xi) \) takes some open neighbourhood of the zero section in \( N \) to an open transversal \( T \) of the foliation \( (\widetilde{M}, \widetilde{\mathcal{F}}) \).

For a suitable open neighbourhood \( \Omega \) of the transversal \( T \), the restriction of the foliation \( \mathcal{F} \) to \( \Omega \) is Morita equivalent to the algebra \( C_0(T) \). Hence, the embedding \( C_r^*(\Omega, \widetilde{\mathcal{F}}|_{\Omega}) \subset C^*_r(\widetilde{M}, \widetilde{\mathcal{F}}) \) defines a map \( K^0(N) \to K^0(C^*_r(\widetilde{M}, \widetilde{\mathcal{F}})) \). Since \( C^*_r(\widetilde{M}, \widetilde{\mathcal{F}}) = C^*_r(M, \mathcal{F}) \otimes C_0(\mathbb{R}^{2n}) \), the Bott periodicity implies that \( K^0(C^*_r(\widetilde{M}, \widetilde{\mathcal{F}})) = K^0(C^*_r(M, \mathcal{F})) \). With use of the Thom isomorphism, \( K^0(T^*\mathcal{F}) \) is identified with \( K^0(N) \). Thus, one gets the topological index map

\[
\text{Ind}_t : K^0(T^*\mathcal{F}) \to K_0(C^*_r(M, \mathcal{F})).
\]

**Theorem 8.3** [88], [13]. For any tangentially elliptic operator \( D \) on a compact foliated manifold \( (M, \mathcal{F}) \),

\[
\text{Ind}_a(D) = \text{Ind}_t(D).
\]

If a foliation \( \mathcal{F} \) is given by the fibres of a fibration \( M \to B \), then \( K_0(C^*_r(M, \mathcal{F})) \cong K^0(B) \) and Theorem 8.3 reduces to the Atiyah–Singer index theorem for families of elliptic operators, Theorem 1.7. If the foliation has a holonomy invariant measure \( \Lambda \), then the trace \( \text{tr}_\Lambda \) on the \( C^* \)-algebra \( C^*_r(M, \mathcal{F}) \) is well defined. In turn, it defines a map \( \text{Tr}_\Lambda : K^0(C^*_r(M, \mathcal{F})) \to \mathbb{R} \). It is not difficult to show that \( \text{Tr}_\Lambda(\text{Ind}_a(D)) = \text{Ind}_a(D) \). The composition \( \text{Tr}_\Lambda \circ \text{Ind}_t \) can be computed by topological methods, and, as a consequence of Theorem 8.3, one gets the index theorem for measurable foliations (see §8.1). We note also the paper [165], which gives a generalization of Theorem 8.3 in the twisted \( K \)-theory.

Following the ideas of the paper [28], one can give an equivalent formulation of Theorem 8.3 in terms of the map \( \mu \) [19]. First of all, observe that the principal symbol \( \sigma_D \) of a tangentially elliptic operator \( D \) determines in the geometric \( K \)-homology group \( K^0_{\text{top}}(M, \mathcal{F}) \) a class \([\sigma_D]\) given by the \( K \)-cycle \((T, \mathcal{F}, [\sigma_D], p \circ \pi)\), where \([\sigma_D] \in K^0(T^*\mathcal{F})\) is the class given by \( \sigma_D \), and the map \( p \circ \pi : T^*\mathcal{F} \to M/\mathcal{F} \) is obtained as the composition of the natural projections \( \pi : T^*\mathcal{F} \to M \) and \( p : M \to M/\mathcal{F} \). Then

\[
\mu_r([\sigma_D]) = \text{Ind}_a(D) \in K_0(C^*_r(M, \mathcal{F})). \tag{8.2}
\]

In [166] there is an equivariant generalization of Theorem 8.3 to the case of an action of a compact Lie group \( H \) taking each leaf of \( \mathcal{F} \) to itself. As a consequence, the author extended the Lefschetz theorem proved in [163] to the case of arbitrary tangentially elliptic complexes under the assumption that the diffeomorphism \( f : M \to M \) is included into an action of a compact Lie group \( H \) taking each leaf of \( \mathcal{F} \) to itself.

Finally, we note that in [167] (see also [168], [169]) semifinite spectral triples associated with tangentially elliptic operators are constructed. Let \((M, \mathcal{F})\) be a smooth compact foliated manifold whose leaves are even-dimensional spin manifolds. Denote by \( S \) the associated spinor bundle, and suppose that there is a holonomy invariant measure \( \Lambda \). We regard the involutive algebra \( \mathcal{A} = C(M) \) as a subalgebra of the semifinite von Neumann algebra \( \mathcal{N} = W^*_\Lambda(M, \mathcal{F}) \) acting in the Hilbert
As a zero-order elliptic \( G \)-pseudodifferential operator, then its principal symbol defines a class \([\sigma_A] \in K_1(C^\infty(S^*\mathcal{F}))\), and the element \( \partial[\sigma_A] \in K_0(C^*_r(M, \mathcal{F})) \) coincides with the analytic index of \( A \) defined in §8.2. Explicit algebraic construction of the connecting homomorphism in \( K \)-theory leads to the following rule for constructing the analytic index of an elliptic symbol \( a \in C^\infty(\mathcal{F}, \text{End}(\pi^*E, \pi^*F)) \). After the bundle \( E \oplus F \) is embedded in a trivial bundle, the function \( \tilde{a} = \begin{pmatrix} 0 & -a^{-1} \\ a & 0 \end{pmatrix} \in C^\infty(S^*\mathcal{F}, \text{Hom}(\pi^*(E \oplus F))) \) defines an element of \( GL_N(C^\infty(S^*\mathcal{F})) \) for sufficiently large \( N \). This element can be lifted to \( GL_N(\Psi^0(\mathcal{F})) \). For example, choose any operator \( A \in \Psi^0(\mathcal{F}; E, F) \) such that \( \sigma_0(A) = a \), and any \( B \in \Psi^0(\mathcal{F}; F, E) \) such
that \( \sigma_0(B) = a^{-1} \). Then \( S_0 = I - BA \in \Psi^{-1}(\mathcal{F}; E) \) and \( S_1 = I - AB \in \Psi^{-1}(\mathcal{F}; F) \). The operator

\[
L = \begin{pmatrix} S_0 & -B - S_0B \\ A & S_1 \end{pmatrix} \in \Psi^0(\mathcal{F}; E \oplus F)
\]

provides the desired lift. Thus, \( \sigma_0(L) = \tilde{a} \) and

\[
L^{-1} = \begin{pmatrix} S_0 & B + BS_0 \\ -A & S_1 \end{pmatrix} \in \Psi^0(\mathcal{F}; E \oplus F).
\]

By definition, put

\[
\partial[a] = [P] - [e], \tag{8.3}
\]

where \( P \) and \( e \) are the idempotents defined as follows:

\[
P = L \begin{pmatrix} I_E & 0 \\ 0 & 0 \end{pmatrix} L^{-1} = \begin{pmatrix} S_0^2 & S_0(I + S_0)B \\ S_1A & IF - S_1^2 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 0 \\ 0 & IF \end{pmatrix}. \tag{8.4}
\]

The construction of the \( G \)-pseudodifferential calculus and the definition of the analytic index have been extended to the case of an arbitrary Lie groupoid in [171], [172].

In fact, the calculus of pseudodifferential operators associated with the holonomy groupoid enables one to construct a parametrix \( B \in \Psi^0(\mathcal{F}; F, E) \) such that \( S_0 = I - BA \in \Psi^{-\infty}(\mathcal{F}; E) \) and \( S_1 = I - AB \in \Psi^{-\infty}(\mathcal{F}; F) \). Then the formulæ (8.3) and (8.4) define the analytic index of \( A \) as an element \( \text{Ind} A \in K_0(C_c^\infty(G)) \) (see [46], [19]).

The natural embedding \( j: C_c^\infty(G) \to C^*_r(M, \mathcal{F}) \) induces a map \( j_*: K_0(C_c^\infty(G)) \to K_0(C^*_r(M, \mathcal{F})) \). One can show (see [46]) that \( j_*(\text{Ind} A) = \text{Ind}_a A \). The map \( j_* \), in general, is not an isomorphism, and therefore we lose some information in passing from \( \text{Ind} A \in K_0(C_c^\infty(G)) \) to \( \text{Ind}_a A \in K_0(C^*_r(M, \mathcal{F})) \). Namely, for the analytic index with values in \( K_0(C^*_r(M, \mathcal{F})) \) there are results like vanishing or homotopy invariance. For instance, as shown in [173] (see also [174], [175]), the analytic index \( \text{Ind}_a A \in K_0(C^*_r(M, \mathcal{F})) \) of the leafwise signature operator on a compact manifold equipped with an even-dimensional oriented foliation is invariant under leafwise oriented homotopy equivalences. On the other hand, the analytic index \( \text{Ind} A \in K_0(C_c^\infty(G)) \) of an elliptic \( G \)-pseudodifferential operator \( A \) depends in general not just on the class \( [\sigma A] \in K_0(C^\infty(T^*\mathcal{F})) \) defined by its principal symbol. In particular, the analytic index does not in general define a map \( K_1(C(S^*\mathcal{F})) \to K_0(C_c^\infty(G)) \). A corresponding example for a certain Lie groupoid is given by Connes in [19] (Chap. II, §10.γ, Proposition 10; see also [176]).

The significance of the analytic index \( \text{Ind} A \) of an elliptic \( G \)-pseudodifferential operator \( A \) with values in \( K_0(C_c^\infty(G)) \) consists in the fact that it gives rise to numerical invariants (higher indices) upon taking the pairing \( \langle \text{Ind} A, \tau \rangle \in \mathbb{C} \) with an arbitrary (periodic) cyclic cocycle \( \tau \) on \( C_c^\infty(G) \).

In [19] (Chap. III, §7.γ, Corollary 13) a higher index theorem for tangentially elliptic operators is formulated. It is an analogue of the higher index \( \Gamma \)-index theorem proved in [46] for \( \Gamma \)-invariant elliptic operators on a \( \Gamma \)-covering of a compact manifold.
Theorem 8.4. Let $A$ be a tangentially elliptic operator on a compact manifold equipped with a transversally oriented foliation $(M, \mathcal{F})$. Then for any $\omega \in H^*(BG)$

$$\langle \text{Ind} A, \Phi_* (\omega) \rangle = (2\pi i)^{-q} \langle \omega, \text{ch}_\tau (\sigma_A) \rangle. \quad (8.5)$$

Here $\text{Ind} A \in K_0(C_c^\infty (G))$ is the analytic index of $A$, $[\sigma_A]$ is the class defined in the geometric $K$-homology group $K_\text{top}^*(M, \mathcal{F})$ by the principal symbol of $A$ (see §8.2), $\Phi_* : H^*(BG) \to HP^*(C_c^\infty (G))$ is the map introduced in §6.4, and $\text{ch}_\tau (\sigma_A)$ is the twisted Chern character $\text{ch}_\tau (\sigma_A) = Td(\tau) - 1 \Phi \text{ch}(\sigma_A)$.

It follows from (5.5) that for any $y = [X, E, f] \in K_{*, \tau}(BG)$ we have

$$\text{ch}_\tau (y) = \tilde{f}_* (\text{ch}(E) \cup Td(X) \cap [X]).$$

Applying this formula to the class $[\sigma_A] \in K_\text{top}^*(M, \mathcal{F})$, one can rewrite (8.5) as

$$\langle \text{Ind} A, \Phi_* (\omega) \rangle = \langle \pi_F ! \text{ch}(\sigma_A) Td(T\mathcal{F} \otimes \mathbb{C}) \tilde{p}^* \omega, [M] \rangle,$$

where $\tilde{p} : M \to BG$ corresponds to the map $p : M \to M/\mathcal{F}$, and $\pi_F : T^* \mathcal{F} \to M$ is the natural projection.

In [177] a proof of Theorem 8.4 is given by methods of algebraic topology in the case when the foliation is the horizontal foliation of a flat foliated bundle (see Example 4.6). This proof is based on a formalism of cyclic type cohomology developed in papers of Cuntz and Quillen together with results in the papers [111] and [178].

Various particular cases of Theorem 8.4 have been studied in several papers. In the case when the foliation is given by the fibres of a fibration $M \to B$, Theorem 8.4 reduces to the index theorem for families of elliptic operators, Theorem 1.8. If the foliation $\mathcal{F}$ has a holonomy invariant measure $\Lambda$ and $\omega \in H^*(BG)$ is the corresponding class, then Theorem 8.4 reduces to the measurable index theorem, Theorem 8.1 (see also the remarks after Theorem 8.3). In the case when $\omega$ corresponds to an invariant form, the given statement was proved in [179]. For the horizontal foliation of a flat foliated bundle, particular cases were considered in [153], [180], [110], [181]. We mention the paper [153], which establishes a connection between the higher index of the tangential Dirac operator on the horizontal foliation of a flat foliated bundle $M \to B$ associated with some transverse cocycle, and the relative eta-invariant of the Dirac operator on the base $B$.

We should note once more that Theorem 8.4 has no topological and geometrical consequences such as a vanishing theorem or homotopy invariance. For this, one needs to establish a statement similar to Theorem 8.4 for the analytic index with values in $K_0(C^*_r (M, \mathcal{F}))$. Topological invariance of the cocycles defined by the pairing with elements in the image of the map $\Phi_*$ was investigated in [99] (see also §6.2).

Before we formulate the basic result of the paper [99], let us recall some information about secondary characteristic classes of foliations. These classes are given by the characteristic homomorphism (see, for instance, [182])

$$\chi_\mathcal{F} : H^*(W_q; O(q)) \to H^*(M, \mathbb{R}),$$
which is defined for any codimension-$q$ foliation $\mathcal{F}$ on a smooth manifold $M$, where $H^*(W_q; O(q))$ denotes the relative cohomology of the Lie algebra $W_q$ of formal vector fields on $\mathbb{R}^q$. A basic property of the secondary characteristic classes is their functoriality: if a smooth map $f: N \to M$ is transverse to a transversally oriented foliation $\mathcal{F}$ and $f^*\mathcal{F}$ is the foliation on $N$ induced by $f$ (by definition, the leaves of $f^*\mathcal{F}$ are connected components of the pre-images of leaves of $\mathcal{F}$ under the map $f$), then

$$f^*(\chi_{\mathcal{F}}(\alpha)) = \chi_{f^*\mathcal{F}}(\alpha), \quad \alpha \in H^*(W_q; O(q)).$$

The classifying space $B\Gamma_q$ of the groupoid $\Gamma_q$ classifies codimension-$q$ foliations on a given manifold $M$ in the sense that any foliation $\mathcal{F}$ on $M$ defines a map $M \to B\Gamma_q$, and moreover, in the case when $M$ is compact a homotopy class of maps $M \to B\Gamma_q$ corresponds to a concordance class of foliations on $M$ [82]. For any codimension-$q$ foliation $\mathcal{F}$ on a manifold $M$, the classifying map $M \to B\Gamma_q$ is obtained as the composition of the map $\tilde{p}: M \to BG$ associated with the projection $p: M \to M/\mathcal{F}$ and the universal map $BG \to B\Gamma_q$ (see §5.5).

By the functoriality of the characteristic homomorphism, it suffices to know it for the universal foliation on $B\Gamma_q$. This gives the universal characteristic homomorphism

$$\chi: H^*(W_q; O(q)) \to H^*(B\Gamma_q, \mathbb{R}).$$

For any complete transversal $T$, $\chi$ is represented as the composition

$$\chi: H^*(W_q; O(q)) \to H^*(BG_T, \mathbb{R}) \to H^*(B\Gamma_q, \mathbb{R}).$$

Since the groupoids $G_T$ and $G$ are equivalent, $H^*(BG_T, \mathbb{R}) \cong H^*(BG, \mathbb{R})$, which defines a map

$$H^*(W_q; O(q)) \to H^*(BG, \mathbb{R}).$$

(8.6)

Elements of the image of $H^*(W_q; O(q))$ by the map (8.6) will be called secondary characteristic classes. For computations of $H^*(W_q; O(q))$ see, for instance, [106].

In [98] Gorokhovsky generalized the construction in §6.3 (which uses the modular automorphism group) of the cyclic cocycle associated with the Godbillon–Vey class to the case of arbitrary secondary characteristic classes. This construction makes essential use of the cyclic cohomology theory developed in the paper [142] for Hopf algebras (see §7.4).

The main result of [99] is formulated as follows.

**Theorem 8.5.** Let $(M, \mathcal{F})$ be a (not necessarily compact) foliated manifold, which is transversally oriented. Let $G$ be its holonomy groupoid and $\pi: BG \to B\Gamma_q$ the classifying map for the $\Gamma_q$-structure defined by the foliation. Let $\tau$ be the bundle on $BG$ defined by the normal bundle $\tau$ of $\mathcal{F}$. Denote by $\mathcal{R}$ the subring of $H^*(BG, \mathbb{C})$ generated by the Pontryagin classes of $\tau$, the Chern classes of holonomy equivariant bundles on $M$, and the secondary characteristic classes.

For any $P \in \mathcal{R}$ there is an additive map $\varphi_P$ from $K_\bullet(C^*_\tau(M, \mathcal{F}))$ to $\mathbb{C}$ such that

$$\varphi_P(\mu_r(x)) = \langle \Phi \text{ch}(x), P \rangle, \quad x \in K^*_\text{top}(M, \mathcal{F}).$$

(8.7)
We recall that the Chern character $K^*_{\text{top}}(M,\mathcal{F}) \to H_{*,\tau}(BG) = H_*(B\tau, S\tau)$ is denoted by $\text{ch}$, and $\Phi: H_{*,\tau}(BG) \to H_*(BG)$ is the Thom isomorphism. Using (5.5), we can write the formula (8.7) as follows for any $x = [X,E,f] \in K_{*,\tau}(BG)$:

$$\varphi_P(\mu_\tau(x)) = \langle \text{ch}(E) \text{Td}(TX \oplus \tilde{f}^*\tau)\tilde{f}^*P, [X] \rangle.$$  

(8.8)

In particular, note that the formulae (6.2) and (6.6) are particular cases of the formulae (8.7).

As a consequence, Theorem 8.5 leads to information about injectivity of the map $\mu_\tau$. It also implies that the map $\varphi_P$ takes integer values on the image of $\mu$. (One should note that $\varphi_P(K_*(C^*_r(M,\mathcal{F})))$ is not, in general, contained in $\mathbb{Z}$.)

Let $A$ be a tangentially elliptic operator on a compact manifold equipped with a transversally oriented foliation $(M,\mathcal{F})$, and let $\text{Ind}_a(A) \in K_0(C^*_r(M,\mathcal{F}))$ be its analytic index. Applying (8.8) to the class $[\sigma_A]$ in the geometric $K$-homology group $K^*_{\text{top}}(M,\mathcal{F})$ defined by the principal symbol of $A$ (see §8.2) and taking into account (8.2), we obtain the higher index theorem for $A$:

$$\varphi_P(\text{Ind}_a(A)) = \langle \pi_{\mathcal{F},1} \text{ch}(\sigma_A) \text{Td}(T\mathcal{F} \otimes \mathbb{C}) \text{Td}(\tilde{\tau} \otimes \mathbb{C})\tilde{p}^*P, [M] \rangle,$$

where $P \in \mathcal{R}$, $\tilde{p}: M \to BG$ is the map corresponding to the map $p: M \to M/\mathcal{F}$, and $\pi_{\mathcal{F},1}: T^*\mathcal{F} \to M$ is the natural projection.

In [95] a $K$-theoretic analogue of Theorem 8.5 is obtained in a particular case. Denote by $p$ the natural projection $M/\mathcal{F} \to \text{pt}$. Suppose that it is a $K$-oriented map (which is equivalent to the normal bundle $\tau$ being a $K$-oriented bundle). Let $p! \in KK^*(C^*(M,\mathcal{F}),\mathbb{C})$ be the corresponding class, which defines the Gysin homomorphism (see §5.5). A holonomy equivariant complex vector bundle $L$ on $(M,\mathcal{F})$ defines an element $[L] \in KK^*(C^*(M,\mathcal{F}),C^*(M,\mathcal{F}))$ (see §5.3). Define an element $p_L \in KK^*(C^*(M,\mathcal{F}),\mathbb{C})$ by $p_L = [L] \otimes p!$.

**Theorem 8.6 [95].** Let $L$ be a holonomy equivariant complex vector bundle on $(M,\mathcal{F})$. For any $y = [X,\mathcal{F}_X,x,f] \in K^*_{\text{top}}(M,\mathcal{F})$

$$\mu(y) \otimes p_L = x \otimes [f^*L] \otimes (p \circ f)! \in \mathbb{Z}.$$

In particular, for any $y = [X,E,f] \in K_{*,\tau}(BG)$

$$\mu(y) \otimes p_L = \langle \text{ch}(E) \text{ch}(f^*L) \text{Td}(TX), [X] \rangle.$$  

(8.9)

Theorem 8.6 implies Theorem 8.5 in the case when $p! \in KK^*(C^*_r(M,\mathcal{F}),\mathbb{C})$, for instance, when the foliation is amenable (and, therefore, $C^*_r(M,\mathcal{F}) = C^*(M,\mathcal{F})$). One should note that the homomorphisms $\varphi \circ \lambda_*$ (where $\lambda: K_*(C^*(M,\mathcal{F})) \to K_*(C^*_r(M,\mathcal{F}))$ is the natural projection) and $\otimes p_L$ do not in general coincide as homomorphisms from $K_*(C^*_r(M,\mathcal{F}))$ to $\mathbb{C}$ and coincide only on the image of $\mu$.

As shown in [100], if a foliation $\mathcal{F}$ is Riemannian and the normal bundle $\tau$ has a holonomy invariant complex spin structure, then the element $p_L$ coincides with the $K$-homology class $[D_L] \in K_*(C^*(M,\mathcal{F}))$ defined by the transverse $\text{Spin}^c$ Dirac operator $D_L$ with coefficients in $L$. Therefore, the equality (8.9) can be rewritten as follows. Denote by $L$ the bundle on $BG$ corresponding to $L$. For any $y = [X,E,f] \in K_{*,\tau}(BG)$,

$$\mu(y) \otimes [D_L] = \langle \text{ch}(E) \text{ch}(\tilde{f}^*L) \text{Td}(TX), [X] \rangle.$$
The authors of [100] propose viewing this as an index formula for the transverse Dirac operator $D_L$ and give examples where this formula could be useful for computing the distributional index of transversally elliptic operators. In [184], [185] there is a proof of Theorem 8.4 for a tangential Dirac operator which generalizes Bismut’s proof in [26] of the local index theorem for families of Dirac operators (see §1.5).

Let $(M, \mathcal{F})$ be a compact foliated manifold. Suppose that the leaves of $\mathcal{F}$ are even-dimensional spin manifolds. Choose a Riemannian metric in the fibres of the bundle $T\mathcal{F}$, and denote by $F(T\mathcal{F})$ the associated spinor bundle. Let $V$ be a Hermitian vector bundle on $M$ equipped with a Hermitian connection $\nabla^V$. Consider the Clifford bundle $\mathcal{E} = F(T\mathcal{F}) \otimes V$ over the Clifford algebra of $T\mathcal{F}$ and the associated tangential Dirac operator $D_{\mathcal{E}}$.

In [184] (see also a more precise formulation in [185]) the authors proved the local index theorem for a family of Dirac operators invariant under a free, proper, cocompact action of an étale groupoid. The case of a tangential Dirac operator is reduced to this case using the following construction. Let $T$ be a complete transversal and $G_T^T$ the associated reduced holonomy groupoid. The set $P = G_T$ is a smooth manifold which is equipped with the natural free proper action of $G_T^T$ given by right multiplication (see §5.4). We remark that the orbit space $P/G_T^T$ of this action coincides with $M$. The map $s$ defines a submersion $\pi: G_T \to (G_T^T)^{(0)} = T$. The submersion $\pi: G_T \to T$ is a $G_T^T$-equivariant map if we consider the action of $G_T^T$ on $T$ given by right multiplication. Denote by $Z_x = \pi^{-1}(x) = G_x$ the fibre of the fibration $\pi: G_T \to T$ at $x \in T$, and by $TZ$ the vertical tangent space. The projection $P = G_T \to M = P/G_T^T$ takes fibres of the fibration $\pi$ to leaves of the foliation $\mathcal{F}$ on $M$ and $TZ$ to the tangent bundle $F$ of $\mathcal{F}$.

The leafwise spin structure in $T\mathcal{F}$ lifts to a $G_T^T$-invariant spin structure in $TZ$. Denote by $F(TZ)$ the associated spinor bundle and by $\hat{V}$ the lift of $V$ to $P$. Consider the leafwise Clifford bundle $\hat{\mathcal{E}} = F(TZ) \otimes \hat{V}$ and the associated tangential Dirac operator $D_{\hat{\mathcal{E}}}$ on $P$. The operator $D_{\hat{\mathcal{E}}}$ determines a family of Dirac operators acting along the fibres of the fibration $\pi: P \to T$ and invariant under the action of $G_T^T$.

The index $\text{Ind} D_{\hat{\mathcal{E}}}$ of the operator $D_{\hat{\mathcal{E}}}$ is well defined as an element of the group $K_0(C_c^\infty(G_T^T) \otimes \mathcal{R})$, where $\mathcal{R}$ is the algebra of rapidly decreasing infinite real matrices. Consider the differential graded algebra $\Omega_c^*(G_T^T)$ introduced in §6.4. For an arbitrary closed graded trace $\eta$ on $\Omega_c^*(G_T^T)$ the authors of [184] define a pairing of the Chern character $\text{ch}(\text{Ind} D_{\hat{\mathcal{E}}}) \in HP(C_c^\infty(G_T^T))$ with $\eta$ and an element $\Phi_\eta \in H^*_c(BG)$ associated with $\eta$. As above, denote by $\bar{p}: M \to BG$ the map corresponding to the natural projection $p: M \to M/\mathcal{F}$. Then the following theorem holds.

**Theorem 8.7** [184].

$$\langle \text{ch}(\text{Ind} D_{\hat{\mathcal{E}}}), \eta \rangle = \int_M \hat{\text{A}}(T\mathcal{F}) \text{ch}(V) \bar{p}^* \Phi_\eta.$$  

The proof of this Theorem is based on a non-commutative equivariant version of the Bismut constructions (see §1.5) applied to the $G_T^T$-equivariant submersion.
Here an important role is played by a construction of a certain differenti-ation
\[ \nabla^{0,1} : C^\infty_c(P, \hat{\mathcal{E}}) \to \Omega^{0,1}_c(\mathcal{G}^T) \otimes C^\infty_c(\mathcal{G}^T) \]
in ‘non-commutative’ directions. It is the use of this non-commutative connection that enables one to include into consideration cohomology classes in dimension greater than the codimension of the foliation (such as the Godbillon–Vey class, which is a three-dimensional cohomology class for a codimension-one foliation).

In [185] a direct proof of Theorem 8.5 for the tangential Dirac operator is given in the particular case when the class \( \omega \) is defined by a holonomy invariant transverse current, without using an auxiliary choice of a complete transversal. The authors make use of a differential graded algebra similar to the algebra constructed in \( \S\ 6.2 \). In the papers [186], [187] Theorem 8.7 was extended to the case when \( M \) is a manifold with boundary and the foliation is transversal to the boundary.

As mentioned above, a tangentially elliptic operator \( D \) on a compact foliated manifold \((M, \mathcal{F})\) defines a class \([D] \in KK(C(M), C^*_c(M, \mathcal{F}))\). This class can be represented by an explicit \( p \)-summable quasi-homomorphism \( \psi_D \) from \( C^\infty_c(M) \) to \( C^\infty_c(\mathcal{G}^T) \) in the sense of the papers [188], [189]. In [190] the bivariant Chern character of this quasi-homomorphism introduced in [188], [189] is computed.

There is another approach to higher index theorems for tangentially elliptic operators based on the use of Haefliger cohomology (see \( \S\ 4.2 \)).

Let \((M, \mathcal{F})\) be a compact foliated manifold. Suppose that the dimension of \( \mathcal{F} \) is even, it is oriented, and it has a spin structure. Let \( \mathcal{E} \) be a Hermitian vector bundle on \( M \), \( D_{\mathcal{E}} \) the corresponding leafwise Dirac operator on \( M \) with coefficients in the bundle \( \mathcal{E} \), and \( \text{Ind}(D_{\mathcal{E}}) \in K_0(C^\infty_c(\mathcal{G})) \) its analytic index. In [179] the Chern character \( \overline{\text{ch}}(\text{Ind}(D_{\mathcal{E}})) \) is defined as an element of the Haefliger cohomology \( H^*_c(M/\mathcal{F}) \) of \( \mathcal{F} \). The construction of the Chern character is a direct modification of the Bismut construction (see \( \S\ 1.5 \)). It makes use of an analogue of the Bismut superconnection associated with the operator \( D_{\mathcal{E}} \) and of the heat operator determined by the curvature of this superconnection.

Under some additional restrictions on the foliation it is proved in [191] that the following equality holds in the Haefliger cohomology \( H^*_c(M/\mathcal{F}) \):
\[
\overline{\text{ch}}(\text{Ind}(D_{\mathcal{E}})) = \frac{1}{(2\pi i)^{p/2}} \int_{\mathcal{F}} \tilde{\Lambda}(T\mathcal{F}, \nabla^{T\mathcal{F}}) \text{ch}(\mathcal{E}, \nabla^{\mathcal{E}}).
\]

The Chern character \( \text{ch}_a : K_0(C^\infty_c(\mathcal{G})) \to H^*_c(M/\mathcal{F}) \) with values in the Haefliger cohomology was constructed by the authors of [105], and this enabled them to translate the Connes–Skandalis index theorem for tangentially elliptic operators to the language of Haefliger cohomology, using the results of the paper [99].

Denote by \( \hat{\mathcal{F}} \) the dimension-\( p \) foliation on the manifold \( M \times \mathbb{R}^{2k} \) induced by \( \mathcal{F} \). The holonomy groupoid \( G^{G\mathbb{R}^{2k}} \) of this foliation coincides with \( G \times \mathbb{R}^{2k} \). Taking the composition \( \text{ch}_a : K_0(C^\infty_c(G^{\mathbb{R}^{2k}})) \to H^*_c(M \times \mathbb{R}^{2k}/\hat{\mathcal{F}}) \) with integration along \( \mathbb{R}^{2k} \), one obtains a map
\[
\text{ch}_{\mathbb{R}^{2k}} : K_0(C^\infty_c(G^{\mathbb{R}^{2k}})) \to H^*_c(M/\mathcal{F}).
\]
We note that in general there is no Bott isomorphism between $K_0(C^\infty_c(G^{2k}))$ and $K_0(C^\infty_c(G))$. For sufficiently large $k$ the map $\pi_1: K^0_c(T\mathcal{F}) \to K_0(C^\infty_c(G^{2k}))$ is well defined.

**Theorem 8.8.** For any $u \in K^0_c(T\mathcal{F})$

$$ch^k_u \circ \pi_1(u) = (-1)^p \int_F \pi_{\mathcal{F}}!(ch(u)) \operatorname{Td}(T\mathcal{F} \otimes \mathbb{C}) \in H^*_c(M/\mathcal{F}),$$

where $\pi_{\mathcal{F}}!: H^*_c(T\mathcal{F}, \mathbb{R}) \to H^*(M, \mathbb{R})$ is integration along the fibres of the bundle $\pi_{\mathcal{F}}: T\mathcal{F} \to M$.

Suppose that the dimension of the foliation $\mathcal{F}$ is even, and it is oriented and has a spin structure. Let $\mathcal{E}$ be a Hermitian vector bundle on $M$, and $D_\mathcal{E}$ the corresponding leafwise Dirac operator on $M$ with coefficients in the bundle $\mathcal{E}$. In [192], assuming that the foliation $\mathcal{F}$ is Riemannian and the bundle $\operatorname{Ind}(D_\mathcal{E})$ is transversally smooth, the authors proved the coincidence of the two Chern characters defined in previous papers:

$$\overline{ch}(\operatorname{Ind}(D_\mathcal{E})) = ch_a(\operatorname{Ind}(D_\mathcal{E})) \in H^*_c(M/\mathcal{F}).$$

Finally, in [193], under the assumption that the family of projections on leafwise harmonic forms in the middle dimension is transversally smooth in a certain sense, the authors defined a higher harmonic signature of an even-dimensional oriented Riemannian foliation on a compact Riemannian manifold and proved its invariance under leafwise homotopies.

We also mention the papers [194], [195], which concern cyclic versions of the Lefschetz formula for diffeomorphisms which take each leaf of the foliation to itself.

In all the theorems mentioned above, the higher indices $\operatorname{Ind}_\tau(A)$ of an elliptic $G$-operator $A$ depend only on the class $[\sigma_A] \in K_0(T^*\mathcal{F})$ defined by its principal symbol, and this, as has already been noted above, does not hold for just any cyclic cocycle $\tau$. Following the paper [176], we say that a (periodic) cyclic cocycle $\tau$ on the algebra $C^\infty_c(G)$ can be localized if the value of the functional $A \mapsto \langle \operatorname{Ind} A, \tau \rangle \in \mathbb{C}$, defined on the set of elliptic $G$-operators, depends only on $[\sigma_A]$. In this case there is a well-defined map $\operatorname{Ind}_\tau: K_0(T^*\mathcal{F}) \to \mathbb{C}$ satisfying the condition

$$\langle \operatorname{Ind} A, \tau \rangle = \operatorname{Ind}_\tau([\sigma_A]).$$

The map $\operatorname{Ind}_\tau$ is called a higher localized index associated with $\tau$.

We say that a $(k+1)$-linear functional $\tau$ on the space $C^\infty_c(G)$ is bounded if it extends to a continuous $(k+1)$-linear functional $\tau_m$ on the space $C^m_c(G)$ for some $m \in \mathbb{N}$. Many geometric cocycles (such as the group cocycles, the transverse fundamental class, and the cocycles defined by the Godbillon–Vey class and by secondary characteristic classes of the foliation) are bounded cocycles. It is proved in [196] that any bounded cyclic cocycle on the algebra $C^\infty_c(G)$ can be localized. The proof of this fact is based on the construction of the tangent groupoid $G^T$ associated with the holonomy groupoid $G$ and of a certain algebra $\mathcal{L}_c(G^T)$ of functions on $G^T$. This algebra was constructed in [197]. It is a strict deformation quantization of the Schwartz algebra $\mathcal{L}(T^*\mathcal{F})$. One can construct an analytic index map

$$\operatorname{Ind}_a: K(T^*\mathcal{F}) \to K_0(C^*_r(M, \mathcal{F})).$$
by means of the tangent groupoid $G^T$ and its $C^*$-algebra $C^*_r(G^T)$ (see [95], [171], [172]). Using this construction, the author of [176] derived a formula for the higher localized index $\text{Ind}_\tau$ associated with a bounded cocycle $\tau$ in terms of an asymptotic limit of cocycles on the algebra $\mathcal{L}_c(G^T)$.

All the facts about higher localized indices mentioned above hold for an arbitrary Lie groupoid $G$. In this case the role of the cotangent space $T^*_F$ is played by the space $A^*_G$, where $AG$ is the Lie algebroid of the groupoid $G$. It is proved in [196] that if the groupoid $G$ is étale, then any cyclic cocycle on the algebra $C^\infty_c(G)$ is bounded. This statement can be applied, for instance, to the reduced holonomy foliation groupoid $G^T_T$ associated with some complete transversal $T$.

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