Carleman estimates and controllability results for fully discrete approximations of 1D parabolic equations

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Received: 18 January 2021 / Accepted: 22 July 2021 / Published online: 10 September 2021
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Abstract
In this paper, we prove a Carleman estimate for fully discrete approximations of one-dimensional parabolic operators in which the discrete parameters \(h\) and \(\Delta t\) are connected to the large Carleman parameter. We use this estimate to obtain relaxed observability inequalities which yield, by duality, controllability results for fully discrete linear and semilinear parabolic equations.

Keywords Carleman estimates · Fully discrete parabolic equations · Observability · Null controllability

1 Introduction

1.1 Controllability of the heat equation

Let \(T > 0\) and \(L > 0\). Let us consider the open interval \(\Omega = (0, L)\) and define \(Q := (0, T) \times \Omega\). Let \(\omega\) be a nonempty subset of \(\Omega\). We consider the linear control system given by

\[
\begin{align*}
    y_t - y_{xx} &= 1_{\omega} v & \text{in } Q, \\
    y(t, 0) &= y(t, L) = 0 & \text{in } (0, T), \\
    y(0, x) &= g(x) & \text{in } \Omega.
\end{align*}
\]

(C1)

Communicated by: Enrique Zuazua

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In Eq. 1, $y = y(t, x)$ is the state, $v = v(t, x)$ is the control function acting on the system on $\omega$, and $g \in L^2(\Omega)$ is a given initial data. Here, $\mathbf{1}_\omega$ stands for the indicator function of the set $\omega$.

It is well-known that for any $g \in L^2(\Omega)$ and $v \in L^2(\omega \times (0, T))$, system Eq. 1 has a unique weak solution such that $y \in C([0, T]; L^2(\Omega)) \cap L^2(0, T; H^1_0(\Omega))$. This regularity motivates the following definition.

**Definition 1.1** System Eq. 1 is said to be null-controllable at time $T$ if for any $g \in L^2(\Omega)$, there exists a control $v \in L^2(\omega \times (0, T))$ such that the corresponding solution satisfies

$$y(T, \cdot) = 0 \quad \text{in } \Omega.$$ 

It is by now well-known that Eq. 1 is indeed null-controllable for any $T > 0$ and any nonempty subset $\omega \subset \Omega$. In fact, this property holds in any dimension. The problem was addressed independently in the 1990s in the seminal works by Lebeau and Robbiano [1] and Fursikov and Imanuvilov [2]. By duality, the null-controllability of Eq. 1 is equivalent to the observability of the adjoint state. In more detail, for each $q_T \in L^2(\Omega)$, consider

$$\begin{cases} 
-q_t - q_{xx} = 0 & \text{in } Q, \\
q(t, 0) = q(t, L) = 0 & \text{in } (0, T), \\
q(T, x) = q_T(x) & \text{in } \Omega.
\end{cases} \quad (2)$$

Then, system Eq. 1 is null-controllable if and only if there exists $C_{\text{obs}} > 0$ such that the following observability inequality holds

$$|q(0)|_{L^2(\Omega)} \leq C_{\text{obs}} \left( \iint_{\omega \times (0, T)} |q|^2 \, dx \, dt \right)^{1/2}. \quad (3)$$

In this paper, our main interest is to study some controllability and observability properties for fully discrete approximations of systems Eqs. 1 and 2, respectively. As it is pointed out in [3], it is known that controllability/observability and numerics do not commute well; however, we expect to retain some control properties after discretization.

### 1.2 Full discretization by finite differences

Finite difference (FD) methods remain as one of the most popular and versatile tools for obtaining numerical solutions of PDEs due to their relative simplicity, rapid convergence, and accuracy. The theoretical framework for FD has been well established in the past years and there are several textbooks devoted to this topic, see the recent works [4–6], or the more classical ones [7, 8].

For the reasons mentioned above, we shall employ here an FD scheme to discretize our control problem Eq. 1 and prove that an observability-type inequality holds for
the FD discretization of the adjoint system Eq. \ref{eq:2}. In turn, this will yield a relaxed notion of controllability that will be made precise later.

Before continuing with our discrete control problem, we present some brief general aspects about FD schemes. To fix ideas, consider the model system

\begin{align}
\begin{cases}
y_t - y_{xx} = F & \text{in } Q, \\
y(t, 0) = y(t, L) = 0 & \text{in } (0, T), \\
y(0, x) = g(x) & \text{in } \Omega.
\end{cases}
\end{align}

(4)

where \(g\) is a given initial datum and \(F = F(t, x)\) is a given source term. We assume that \(g\) and \(F\) are regular enough.

Hereinafter, we shall use the notation \([a, b] = [a, b] \cap \mathbb{N}\) for any real numbers \(a < b\). For given \(N, M \in \mathbb{N}^*\), we set the time- and space-discretization parameters \(\Delta t = T/M\) and \(h = L/(N + 1)\), respectively. We consider the pairs \((t_n, x_i)\) with \(t_n = n \Delta t\), \(n \in [0, M]\), and \(x_i = i h\), \(i \in [0, N + 1]\). The numerical approximation of a function \(r = r(t, x)\) at a grid point \((t_n, x_i)\) will be denoted as

\[ r^n_i := r(t_n, x_i). \]

(5)

The basic idea of FD is to seek for suitable approximations of the derivatives in Eq. \ref{eq:4} and then solve the resulting difference equations. For our model problem, one of the simplest FD schemes based at the mesh point \((t_n, x_i)\) is the so-called explicit method and uses a forward difference for the time derivative, i.e., \(y_t(t_n, x_i) \approx \frac{y(t_{n+1}, x_i) - y(t_n, x_i)}{\Delta t}\), and a centered second order difference for the space derivative \(y_{xx}(t_n, x_j) \approx \frac{y(t_n, x_{i+1}) - 2y(t_n, x_i) + y(t_n, x_{i-1})}{h^2}\), whence, using notation Eq. \ref{eq:5}, the explicit FD approximation of Eq. \ref{eq:4} can be written in vector form as

\[ \begin{cases}
\frac{y^{n+1} - y^n}{\Delta t} - \Delta_h y^n = F^n & n \in [0, M - 1], \\
y_0^{n+1} = y_{N+1}^{n+1} = 0 & n \in [0, M - 1], \\
y^0 = g,
\end{cases} \]

(6)

where \(y^n = \begin{pmatrix} y_1^n \\ \vdots \\ y_N^n \end{pmatrix}\), \(g = \begin{pmatrix} g_1 \\ \vdots \\ g_N \end{pmatrix}\) and \(F^n = \begin{pmatrix} F_1^n \\ \vdots \\ F_N^n \end{pmatrix}\) stand for the sampled initial data \(g(x)\) and the source term \(F(t, x)\), respectively, and

\[ \Delta_h = \frac{1}{h^2} \begin{pmatrix} -2 & 1 \\ 1 & -2 & 1 \\ \vdots & \ddots & \ddots \\ 1 & -2 & 1 \\ 1 & \end{pmatrix}. \]

We recall the following convergence result.

**Theorem 1.2** Let \(y\) be a sufficiently smooth solution to Eq. \ref{eq:4} and let the sequence \((y^n)_{n \in [0, M]}\) be determined by the explicit scheme Eq. \ref{eq:6}. Suppose that \(\Delta t / h^2 \leq 1/2\),
then
\[
\max_{n \in [0,M]} \max_{i \in [0,N+1]} |y(t_n, x_i) - y_n^i| \leq C(\Delta t + h^2),
\]
where \( C > 0 \) only depends on \( L, T, \|y_{tt}\|_{L^\infty(\Omega \times (0,T))} \) and \( \|y_{xxxx}\|_{L^\infty(\Omega \times (0,T))} \).

The proof of this result can be found for instance in [7, Ch. 9, Sect. 4]. This result tells that the FD scheme Eq. 6 is conditionally convergent, namely it will converge towards the solution of Eq. 4 at the rate \( O(\Delta t + h^2) \) if the condition \( \Delta t/h^2 \leq 1/2 \) holds. Of course, this last condition can be impractical in applications since for finer meshes (in space), more and more time steps are needed to achieve high accuracy.

This issue can be circumvented by replacing the forward difference for the time derivative by a backward one. This scheme, known as implicit, takes the form
\[
\begin{align*}
\frac{y^{n+1} - y^n}{\Delta t} - \Delta_h y^{n+1} &= F^{n+1}, \quad n \in [0, M - 1], \\
y_0^{n+1} &= y_{N+1}^{n+1} = 0, \quad n \in [0, M - 1], \\
y^0 &= g.
\end{align*}
\]
(7)

As for the explicit scheme, we state the following convergence result.

**Theorem 1.3** Let \( y \) be a sufficiently smooth solution to Eq. 4 and let the sequence \((y^n)_{n \in [0,M]}\) be determined by the implicit scheme Eq. 7. Then
\[
\max_{n \in [0,M]} \max_{i \in [0,N+1]} |y(t_n, x_i) - y_n^i| \leq C(\Delta t + h^2)
\]
with \( C > 0 \) as in Theorem 1.

For the proof, we refer to [7, Ch. 9, Sect. 4.1]. Although very close to the previous result, we observe that in this case there is no a priori condition on \( h \) and \( \Delta t \) for the scheme to be convergent. For this reason, we call it unconditionally stable. Even though we are free to choose the discretization parameters \( \Delta t \) and \( h \) in the implicit method, practical considerations still point out that we have to choose \( \Delta t = O(h^2) \) for having good accuracy.

We conclude this section by saying that there are other implicit schemes which are unconditionally stable, one of them having the particular convergence rate of \( O(\Delta t^2 + h^2) \). For a parameter \( \theta \in (0, 1] \), let us consider the family of schemes
\[
\frac{y^{n+1} - y^n}{\Delta t} - \left[ \theta \Delta_h y^{n+1} + (1 - \theta) \Delta_h y^n \right] = \theta F^{n+1} + (1 - \theta) F^n, \quad n \in [0, M - 1],
\]
(8)
supplemented with boundary and initial conditions as before. Observe that for \( \theta = 1 \), Eq. 8 reduces to Eq. 7; while for \( \theta = 0 \), Eq. 8 becomes Eq. 6. For all \( \theta \in (0, 1] \), the FD schemes Eq. 8 are unconditionally stable and the approximation error is of order \( O(\Delta t + h^2) \). In the particular case, \( \theta = 1/2 \), Eq. 8 is the so-called Crank-Nicholson method and, as shown in [7, Ch. 9, Sect. 4.1], the approximation error is of order

\( O(\Delta t + h^2) \).
\( O(\Delta t^2 + h^2) \). This leads to the less restrictive choice of \( \Delta t = O(h) \) for achieving arbitrary accuracy.

1.3 The control problem in the discrete setting

Let us comeback to our original problem. Following the previous section, let us consider the fully discrete control system

\[
\begin{cases}
    y^{n+1} - y^n - \Delta t y^{n+1} = 1_0 v^{n+1} & n \in [0, M - 1], \\
    y_{0}^{n+1} = y_{N+1}^{n+1} = 0 & n \in [0, M - 1], \\
    y_0^n = g,
\end{cases}
\]

(9)

where \( y = (y^n)_{n \in [0, M]} \) is the (discrete) state, \( v = (v^n)_{n \in [1, M]} \) is the (discrete) control, and \( 1_0 \) stands for an approximation of the continuous indicator function \( 1_\omega \).

A natural choice is for instance the diagonal \( N \times N \) matrix with entries

\[
(1_0)_{i,i} = \begin{cases}
    1 & \text{if } x_i \in \omega, \\
    0 & \text{if } x_i \notin \omega.
\end{cases}
\]

(10)

For convenience, hereinafter, we simply denote by \( 1_\omega \) this approximation of the indicator function.

The fully discrete system Eq. 9 is the result of applying a standard implicit FD scheme to system Eq. 1. As we have mentioned before, there are other possible ways to discretize system Eq. 1, but we have made this particular choice since the method is unconditionally stable, namely, there are not a priori constraints relating \( h \) and \( \Delta t \) for guaranteeing the convergence of the method, and most importantly the computations developed in further sections are easier to handle than for the \( \theta \)-scheme.

As in the continuous case, we can introduce a notion of controllability for the fully discrete system. More precisely, Eq. 9 is said to be null controllable if for any initial datum \( g \in \mathbb{R}^N \) there exists a sequence \( v = (v^n)_{n \in [1, M]} \) such that the corresponding solution satisfies

\[
y^M = 0.
\]

(11)

Controllability results for discretized systems can be divided into three categories: space-discrete, time-discrete, and fully discrete results. Below we give a general panorama of the results available in the literature.

**Space-discrete setting.** The controllability of semi-discrete (in space) approximations of parabolic systems has deserved a lot of attention in the recent past, see, for instance, [9–17].

It comes out that when addressing controllability problems for these kind of systems, the classical notion of null-controllability Eq. 11 might be too strong since it may happen that the discrete problem is not uniformly controllable with respect to the discretization parameter. Actually, as shown in [3], there are even counter-examples in 2-D for which a time-continuous variant of Eq. 9 is not even approximately controllable for given \( h \). To handle this problem, it was proposed in [10, 18] and other related works to relax the controllability requirements and consider the so-called \( \phi(h) \)-controllability. This notion roughly consists in constructing uniformly bounded
controls (w.r.t to \( h \)) such that the norm of the space-discrete solution \( |y_h(T)| \approx \sqrt{\phi(h)} \) for some function \( h \mapsto \phi(h) \) tending to 0 as \( h \to 0 \) and amounts to prove a relaxed version of inequality Eq. 3 (cf. 48).

In this direction, it was shown in [11, 13, 15] (with three different approaches: Lebeau-Robbiano method [1], Carleman estimates [2], and moment’s method [19], respectively) that, for a finite-difference scheme in space and a continuous time variable, the uniform \( \phi(h) \)-controllability property holds for functions \( \phi(h) \) that do not tend to zero faster than some exponential \( h \mapsto e^{-C/h^\alpha} \) for some \( \alpha > 0 \). We refer to [10] for a similar discussion of Galerkin approximations.

**Time-discrete setting.** In the case where only time-discretization is used (i.e., the space variable remains continuous), the controllability results available in the literature are less compared to the previous setting. Most probably, this comes from the fact that, as pointed out by [20], system Eq. 9 is not even approximately controllable for any given \( \Delta t > 0 \), except for the trivial case \( \omega = \Omega \). At the light of this negative result and following the spirit of the space-discrete case, a natural question that arises is whether the controllability constraint Eq. 11 can be relaxed. In this direction, in [20], the controllability of Eq. 9 is addressed by employing a time-discrete Lebeau-Robbiano strategy and controlling (uniformly with respect to \( \Delta t \)) the projections of solutions over a suitable class of low frequency Fourier components. In [21], the authors prove in a quite general framework that any controllable parabolic equation is null-controllable after time discretization by applying an adequate filtering of high frequencies. Finally, in [22], the authors establish a Carleman-type estimate for time-discrete approximations of the parabolic operator \(-\partial_t - \Delta\), allowing them to obtain a \( \phi(\Delta t) \)-controllability result where a small target is reached, that is,

\[
|y^M|_{L^2_\Omega} \leq C \sqrt{\phi(\Delta t)}|g|_{L^2_\Omega},
\]

where \( C > 0 \) is uniform with respect to \( \Delta t \) and \( \Delta t \mapsto \phi(\Delta t) \) is a suitable function decaying exponentially as \( \Delta t \to 0 \). This result has been extended to fourth-order parabolic systems in [23].

**Fully discrete setting.** Regarding the controllability of fully discrete approximations of parabolic systems, the works available in the literature are far more scarce and the results are somehow limited. We begin by making the following observations.

At this point, system Eq. 9 can be understood as the discretization of a finite dimensional system of the form

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t), \quad t \in (0, T), \\
x(0) &= x_0,
\end{align*}
\]

with matrices \( A \in \mathbb{R}^{N \times N} \), \( B \in \mathbb{R}^{N \times P} \), and initial datum \( x_0 \in \mathbb{R}^N \). The function \( x : [0, T] \to \mathbb{R}^N \) represents the state and \( u : [0, T] \to \mathbb{R}^P \) the control.

Linear systems of the form Eq. 13 are said to be controllable at time \( T \) if for every initial datum \( x_0 \in \mathbb{R}^N \) and any final target \( x_1 \in \mathbb{R}^N \), there exists a control \( u \in L^2(0, T; \mathbb{R}^P) \) such that the solution of Eq. 13 satisfies \( x(T) = x_1 \). A necessary and sufficient condition for the controllability of Eq. 13 is the Kalman Rank criterion,
more precisely, system Eq. 13 is null-controllable at time \( T > 0 \) if and only if
\[
\text{rank}[B, AB, \ldots, AN^{-1}B] = N. \tag{14}
\]
Discretizing the time derivative in Eq. 13 by using an implicit scheme yields
\[
\begin{align*}
\frac{x^{n+1} - x^n}{\Delta t} &= Ax^{n+1} + Bu^{n+1} & n \in [0, M - 1],
\end{align*}
\]
\[
\begin{align*}
x^0 &= x_0.
\end{align*}
\tag{15}
\]
Notice that in our case, Eq. 9 is of the form Eq. 15 with matrices \( A = \Delta_h \) and \( B = 1_{\text{new}} \) as in Eq. 10.

The following result, originally proved in [24] (see also [25], Section 3.4, Theorem 4 for a more accessible reference), states conditions for system Eq. 15 to be controllable.

**Theorem 1.4** Assume that the pair \((A, B)\) satisfies the Kalman condition Eq. 14 and let \( \lambda_1, \ldots, \lambda_N \) be the distinct eigenvalues of \( A \). Then, system Eq. 15 is controllable if
\[
\Delta t (\lambda_\mu - \lambda_\sigma) \neq 2v\pi i, \quad v = \pm 1, \pm 2, \ldots
\]
for every two eigenvalues \( \lambda_\mu, \lambda_\sigma \) of \( A \).

In particular, as pointed out in [20], if \((A, B)\) fulfills the Kalman rank condition, system Eq. 15 is controllable if \( \Delta t > 0 \) is small enough, more precisely,
\[
\Delta t < \min_{\mu \neq \sigma} \left| \frac{2v\pi}{\lambda_\mu - \lambda_\sigma} \right|. \tag{16}
\]
For our problem, it is clear that the matrices \( A = \Delta_h \in \mathbb{R}^{N \times N} \) and \( B = 1_{\text{new}} \in \mathbb{R}^{N \times N^2} \) as in Eq. 10 verify the Kalman condition. This can be checked from the simpler case with \( B = e_i, e_i \) being the \( i \)th vector of the canonical basis of \( \mathbb{R}^N \). In such case, the Kalman condition can be readily verified since the eigenvalues of \( A = \Delta_h \) are simple (and thus \( A \) is diagonalizable), the fact that no eigenvector of \( \Delta_h \) vanishes since they are of the form \( r_k = \sin(k\pi x_i), k \in [1, N] \) (see [6, Appendix C.7]), and Vandermonde determinant.

In view of this, and recalling that the eigenvalues of \( A = \Delta_h \) are of the form \( \lambda_k(h) = -\frac{4}{h^2} \sin^2 \left( \frac{\pi kh}{2} \right), k \in [1, N] \) (see, e.g., [6, Appendix C.7]), we can see that for fixed \( h > 0 \) we can find \( \Delta t > 0 \) small enough such that the fully discrete system Eq. 9 is null-controllable in the sense of Eq. 11.

Note however that there is no hope for this result to be uniform as \( h \) and \( \Delta t \) tend to zero. Indeed, as we refine the parameter \( h > 0 \), more and more eigenvalues of \( A = \Delta_h \) are taken into account and the right hand side of Eq. 16 tends to zero as \( h \to 0 \). Thus, by keeping \( \Delta t > 0 \) fixed and taking the limit as \( h \to 0 \), the original problem becomes the time-discrete heat equation which, as we have mentioned before, is not even approximately controllable for any given \( \Delta t > 0 \). Notwithstanding, this tells us that there is hope to recover a uniform controllability result in the spirit of the works on space- and time-discrete settings presented above if additional conditions connecting \( h \) and \( \Delta t \) are added into the mix.
In this direction, as far as the author’s knowledge, there are only two works in the literature addressing the controllability of fully discrete parabolic systems. As we have mentioned, in [21], the authors prove that for any controllable parabolic equation, the discretization in time preserves some controllability properties (in the sense of Eq. 12) after some filtering process. In addition, the authors prove that a similar result holds if a suitable discretization in space is performed. With this, the authors prove relaxed observability inequalities yielding in particular a uniform controllability result where \( |y^M| = O(\sqrt{\Delta t + h^2}) \) for some \( \gamma > 0 \). On the other hand, in the work [26], the authors extend the semi-discrete Lebeau-Robbiano method used in [11, 12] and look for conditions connecting the discretization parameters \( h \) and \( \Delta t \). In this case, the authors prove that a \( \phi(h) \)-controllability result holds for Eq. 9, i.e., \( |y^M| = O(\sqrt{\phi(h)}) \) for some function \( h \mapsto \phi(h) \) tending to 0 as \( h \to 0 \) and \( \Delta t \leq C_T |\log \phi(h)|^{-1} \) for some uniform constant \( C_T > 0 \).

We remark that the above approaches rely on spectral analysis techniques and the results are thus limited to linear autonomous control systems. For this reason, in this paper, we shall look to directly prove Carleman-type estimates for fully discrete parabolic operators and employ them to prove some controllability results. The main goal and novelty of our approach will allow us to include in the analysis more general time-dependent coefficients, semi-linear systems or even coupled equations which are generally out of reach for the spectral techniques.

### 1.4 Some useful notation

In what follows, we shall introduce some notation that allow us to represent system Eq. 9 in a more compact way and also will help us to use later a formalism (differentiation, integration by parts, and so on) as close as possible to the continuous case. In particular, this will help us to clarity the exposition of our main results in the following section.

#### 1.4.1 Discretization-in-space

From the discretization points \( x_i \) introduced above, we define \( \mathcal{M} := \{x_i : i \in [1, N]\} \). We refer to this points as the primal mesh (in space). As expected, we define the boundary values as \( \partial \mathcal{M} = \{x_0, x_{N+1}\} \). Additionally, we introduce the points \( x_{i+\frac{1}{2}} := (x_{i+1} + x_i)/2 \) for \( i \in [0, N] \) (see Fig. 1). In what follows, the set of points \( \overline{\mathcal{M}} := \{x_{i+\frac{1}{2}} : i \in [0, N]\} \) will be referred as the dual mesh (in space).

We denote by \( \mathbb{R}^{\mathcal{M}} \) and \( \mathbb{R}^{\overline{\mathcal{M}}} \) the set of discrete functions defined on \( \mathcal{M} \) and \( \overline{\mathcal{M}} \), respectively. If \( u^{\mathcal{M}} \in \mathbb{R}^{\mathcal{M}} \) (resp. \( u^{\overline{\mathcal{M}}} \in \mathbb{R}^{\overline{\mathcal{M}}} \)), we denote by \( u_i \) (resp. \( u_{i+\frac{1}{2}} \)) its value at point \( x_i \) (resp. \( x_{i+\frac{1}{2}} \)).

![Fig. 1 Discretization of the space variable and its notation](image-url)
corresponding to $x_i$ (resp. $x_{i+\frac{1}{2}}$). For $u^{M} \in \mathbb{R}^{M}$, we define the discrete integral

$$
\int_{\Omega} u^{M} := \sum_{i=1}^{N} h u_i \quad (17)
$$

and, analogously, for $u^{\alpha} \in \mathbb{R}^{\alpha}$ we define

$$
\int_{\Omega} u^{\alpha} := \sum_{i=0}^{N} h u_{i+\frac{1}{2}} \quad (18)
$$

Remark 1.5 Notice that the discrete integrals Eqs. 17 and 18 are defined with the same symbol. Later, we will see that most of the time, from the context, we can infer which integral is being used. For this reason, to ease the notation, we simply write $\int$ to denote functions $u^{M}$ or $u^{\alpha}$.

For some $u \in \mathbb{R}^{M}$, we shall need to associate boundary conditions $u^{M} = \{u_0, u_{N+1}\}$. The set of such discrete functions will be denoted by $\mathbb{R}^{M \cup \alpha}$. Homogeneous Dirichlet boundary conditions consist in the choice $u_0 = u_{N+1} = 0$ and for short with write $u^{M} = 0$ or $u|_{\partial \Omega} = 0$.

With the notation above, we define the following $L^2$-inner product on $\mathbb{R}^{M}$ (resp. $\mathbb{R}^{\alpha}$)

$$(u, v)_{L^2(\Omega)} := \int_{\Omega} u v = \sum_{i=1}^{N} h u_i v_i \quad (19)$$

resp. $$(u, v)_{L^2(\Omega)} := \int_{\Omega} u v = \sum_{i=0}^{N} h u_{i+\frac{1}{2}} v_{i+\frac{1}{2}}.$$ 

The associated norms will be denoted by $|u|_{L^2(\Omega)}$. Analogously, we define the $L^\infty$-norm on $\mathbb{R}^{M}$ (resp. $\mathbb{R}^{\alpha}$) as

$$|u|_{L^\infty(\Omega)} := \sup_{i \in [1,N]} |u_i| \quad (20)$$

resp. $|u|_{L^\infty(\Omega)} := \sup_{i \in [0,N]} |u_{i+\frac{1}{2}}| \quad (21)$. 

Often times, we shall use functions restricted to (or defined on) subdomains, e.g., $\omega \subset \Omega$, where $\omega$ is a nonempty open set. Similar definitions and notations to Eqs. 19–21 will be employed for such functions. For instance, we define the discrete $L^2(\omega)$-norm on $\mathbb{R}^{M}$ (resp. $\mathbb{R}^{\alpha}$) by

$$|u|_{L^2(\omega)} := \left( \sum_{i \in [1,N], x_i \in \omega} h |u_i|^2 \right)^{1/2} \quad (22)$$

resp. $|u|_{L^2(\omega)} := \left( \sum_{i \in [0,N], x_{i+\frac{1}{2}} \in \omega} h |u_{i+\frac{1}{2}}|^2 \right)^{1/2}.$
In order to manipulate the discrete functions, we define the following translation operators for indices as the maps $\mathbb{R}^{\Omega_1,\wp_0^\Omega} \to \mathbb{R}^{\Omega_1}$ given by
\[
(s^+ u)_{i+\frac{1}{2}} := u_{i+1}, \quad (s^- u)_{i-\frac{1}{2}} := u_i, \quad i \in [0, N].
\]

A first-order difference operator $\partial_h$ and an averaging operator $m_h$ are then given by
\[
(\partial_h u)_{i+\frac{1}{2}} := \frac{u_{i+1} - u_i}{h}, \quad (s^- u)_{i+\frac{1}{2}} := \frac{1}{h} (s^+ u - s^- u)_{i+\frac{1}{2}}, \quad (23)
\]
\[
(m_h u)_{i+\frac{1}{2}} := \bar{u}_{i+\frac{1}{2}} := \frac{1}{2} (s^+ u + s^- u)_{i+\frac{1}{2}}, \quad (24)
\]

Both map $\mathbb{R}^{\Omega_1,\wp_0^\Omega} \to \mathbb{R}^{\Omega_1}$.

Likewise, we define the translation operators $\mathbb{S}^\pm$ as the maps $\mathbb{R}^{\Omega_1} \to \mathbb{R}^{\Omega_1}$ given by
\[
(\mathbb{S}^+ u)_i := u_{i+\frac{1}{2}}, \quad (\mathbb{S}^- u)_i := u_{i-\frac{1}{2}}, \quad i \in [1, N]. \quad (25)
\]

Then, a difference operator $\overline{\partial_h}$ and $\overline{m_h}$ (both mapping $\mathbb{R}^{\Omega_1} \to \mathbb{R}^{\Omega_1}$) are naturally given by
\[
(\overline{\partial_h} u)_i := \frac{u_{i+\frac{1}{2}} - u_{i-\frac{1}{2}}}{h}, \quad (\mathbb{S}^- u)_i := \frac{1}{h} (\mathbb{S}^+ u - \mathbb{S}^- u)_i, \quad (26)
\]
\[
(\overline{m_h} u)_i := \bar{u}_i := \frac{1}{2} (\mathbb{S}^+ u + \mathbb{S}^- u)_i. \quad (27)
\]

Observe that there is no need of boundary conditions at this point. Also notice, what with the above definitions, we can express $\Delta_h u_i$ as $\overline{(\partial_h \partial_h u)}_i$.

A continuous function $\psi$ defined on $\overline{\Omega}$ can be sampled on the primal mesh, that is $\psi^{\Omega_1} = \{ \psi(x_i) : i \in [1, N] \}$, which we identify to
\[
\psi^{\Omega_1} = \sum_{i=1}^{N} \mathbf{1}_{[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]} \psi_i, \quad \psi_i = \psi(x_i), \quad i \in [1, N].
\]

We also set
\[
\psi^{\partial\Omega_1} = \{ \psi(x_0), \psi(x_{N+1}) \} = \{ \psi(0), \psi(L) \},
\]
\[
\psi^{\Omega_1,\partial\Omega_1} = \{ \psi(x_i) : i \in [0, N+1] \}.
\]

The function $\psi$ can be sampled also on the dual mesh, more precisely,
\[
\psi^{\Omega_1} = \{ \psi(x_{i+\frac{1}{2}}) : i \in [0, N] \}
\]
which we identify to
\[
\psi^{\Omega_1} = \sum_{i=0}^{N} \mathbf{1}_{[x_i, x_{i+1}]} \psi_{i+\frac{1}{2}}, \quad \psi_{i+\frac{1}{2}} = \psi(x_{i+\frac{1}{2}}), \quad i \in [0, N].
\]

In the remainder of this document, we simply use the symbol $\psi$ for both the continuous function and its sampling on the primal or dual meshes. As we have mentioned before, from the context, one is able to identify the appropriate sampling. For instance, for a function $u$ defined on the primal mesh $\Omega_1$, in an expression like $\overline{\partial_h}(\rho \partial_h u)$ where $\rho : \overline{\Omega} \to \mathbb{R}$ is a continuous given function, it is clear that $\rho$ is being...
sampled on the dual mesh $\overline{\Omega}$ since $\partial_h u$ is defined on this mesh and the operator $\overline{\partial_h}$ acts on functions defined on this mesh as well.

All of these definitions, together with the integrals Eqs. 17 and 18, allow us to obtain a series of results for handling in a quite natural way the application of the derivatives $\partial_h$ and $\overline{\partial_h}$ to continuous or discrete functions. For convenience, we have summarized in Appendix A the main tools and estimates for space-discrete functions. As an example, for functions $u \in \mathbb{R}^{\Omega^h,\partial\Omega^h}$ and $v \in \mathbb{R}^{\overline{\Omega}}$, we have the following useful formula

$$\int_{\Omega} u(\overline{\partial_h} v) = - \int_{\Omega} (\overline{\partial_h} v) u + u_{N+1} v_{N+\frac{1}{2}} - u_0 v_{\frac{1}{2}},$$

which resembles classical integration-by-parts formula.

### 1.4.2 Discretization-in-time

Here, we devote to introduce some notations and definitions to handle effectively the discretization of the time variable. We recall that for given $M \in \mathbb{N}^*$, we have set $\Delta t = T / M$ and considered points $t_n = n \Delta t$, $n \in [0, M]$. We also introduce the points $t_{n+\frac{1}{2}} := (t_{n+1} + t_n)/2$ for $n \in [0, M - 1]$ (see Fig. 2).

From these, we will denote by $\mathcal{P} := \{t_n : n \in [1, M]\}$ the (primal) set of points excluding the first one and we write $\overline{\mathcal{P}} := \mathcal{P} \cup \{t_0\}$. Analogous to the space variable, to handle the approximation of time derivatives, we will work with the (dual) points $t_{n+\frac{1}{2}}$. Its collection will be defined as $\mathcal{D} := \{t_{n+\frac{1}{2}} : n \in [0, M - 1]\}$. It will be convenient to consider also an extra point $t_{M+\frac{1}{2}}$ which lies outside the time interval $[0, T]$ (see Fig. 2) and to write $\overline{\mathcal{D}} := \mathcal{D} \cup \{T_{M+\frac{1}{2}}\}$. Observe that both $\mathcal{P}$ and $\mathcal{D}$ have a total number of $M$ points.

We denote by $\mathbb{R}^{\mathcal{P}}$ and $\mathbb{R}^{\mathcal{D}}$ the sets of real-valued discrete functions defined on $\mathcal{P}$ and $\mathcal{D}$. If $u^\mathcal{P} \in \mathbb{R}^{\mathcal{P}}$ (resp. $u^\mathcal{D} \in \mathbb{R}^{\mathcal{D}}$), we denote by $u^n$ (resp. $u^{n+\frac{1}{2}}$) its value corresponding to $t_n$ (resp. $t_{n+\frac{1}{2}}$). For $u \in \mathbb{R}^{\mathcal{P}}$ we define the time-discrete integral

$$\int_0^T u^\mathcal{P} := \sum_{n=1}^{M} \Delta t \ u^n, \tag{28}$$

and for $u^\mathcal{D} \in \mathbb{R}^{\mathcal{D}}$ we define

$$\int_0^T u^\mathcal{D} := \sum_{n=0}^{M-1} \Delta t \ u^{n+\frac{1}{2}}. \tag{29}$$

![Fig. 2 Discretization of the time variable and its notation](image-url)
Remark 1.6 For the time-discrete case, we have decided to use different symbols to define the integrals. For this reason, in what follows we shall write $u$ indistinctly to refer to functions $u^P$ or $u^D$.

Let $\{X, | : |_X \}$ be a real Banach space. Obviously, $X$ can be a finite or infinite dimensional space. We denote by $X^P$ and $X^D$ the sets of vector-valued functions defined on $P$ and $D$, respectively. Using definitions Eqs. 28 and 29, we denote by $L^P_P(0, T; X)$ (resp. $L^P_D(0, T; X)$), $1 \leq p < \infty$, the space $X^P$ (resp. $X^D$) endowed with the norm

$$\|u\|_{L^P_P(0, T; X)} := \left( \int_0^T |u|^p_X \right)^{1/p} \quad \text{(resp. } \|u\|_{L^P_D(0, T; X)} := \left( \int_0^T |u|^p_X \right)^{1/p} \).$$

We also define the space $L^\infty_P(0, T; X)$ (resp. $L^\infty_D(0, T; X)$) by means of the norm

$$\|u\|_{L^\infty_P(0, T; X)} := \sup_{n \in [1, M]} |u^n|_X \quad \text{(resp. } \|u\|_{L^\infty_D(0, T; X)} := \sup_{n \in [0, M-1]} |u^{n+\frac{1}{2}}|_X \).$$

In the case where $p = 2$ and $X$ is replaced by a Hilbert space $\{H, \langle \cdot, \cdot \rangle_H \}$, $H^P$ (resp. $H^D$) becomes a Hilbert space for the norm induced by the inner product

$$\int_0^T (u, v)_H := \sum_{n=1}^M \Delta t (u^n, v^n)_H \quad \text{(resp. } \int_0^T (u, v)_H := \sum_{n=0}^{M-1} \Delta t (u^{n+\frac{1}{2}}, v^{n+\frac{1}{2}})_H \).$$

To manipulate time-discrete functions, we define translation operators $\tau^+ : X^P \to X^P$ and $\tau^- : X^D \to X^D$ as follows:

$$(\tau^+ u)^{n+\frac{1}{2}} := u^{n+1}, \quad (\tau^- u)^{n+\frac{1}{2}} := u^{n}, \quad n \in [0, M - 1].$$

With this, we can define a difference operator $D_t$ as the map from $X^D$ into $X^P$ given by

$$(D_t u)^{n+\frac{1}{2}} := \frac{u^{n+1} - u^n}{\Delta t} = \left( \frac{1}{\Delta t} (\tau^+ - \tau^-) u \right)^{n+\frac{1}{2}}, \quad n \in [0, M - 1].$$

In the same manner, we can define the translation operators $\bar{\tau}^+ : X^D \to X^P$ and $\bar{\tau}^- : X^P \to X^P$ as follows:

$$(\bar{\tau}^+ u)^n := u^{n+\frac{1}{2}}, \quad (\bar{\tau}^- u)^n = u^{n-\frac{1}{2}}, \quad n \in [1, M],$$

as well as a difference operator $D_t$ (mapping $X^D$ into $X^P$) given by

$$(D_t u)^n := \frac{u^{n+\frac{1}{2}} - u^{n-\frac{1}{2}}}{\Delta t} = \left( \frac{1}{\Delta t} (\bar{\tau}^+ - \bar{\tau}^-) u \right)^n, \quad n \in [1, M].$$

As in the previous section, these definitions allow us to readily handle the evaluation of continuous functions on the primal and dual meshes $P$ and $D$. Also, the notation here allow us to present an integration-by-parts (in time) formula in a natural 

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way. For instance, in the case of Eq. 31, for functions \( u \in H^\mathbb{F} \) and \( v \in H^\mathbb{F} \) we have
\[
\int_0^T (D_t u, v)_H = -(u^0, v^1) + (u^M, v^{M+\frac{1}{2}})_H - \int_0^T (\overline{D}_t v, u)_H.
\]

A summary with other useful formulas and estimates are presented in Appendix A.

1.4.3 Combining the effects of time- and space-discretization

With the notation introduced in the time-discrete setting, we can combine easily the effects of the discretization in space and define the functional spaces we shall work with. Let us set \( H = \mathbb{R}^m \) (resp. \( H = \mathbb{R}^m \)) endowed with the \( L^2 \) inner product Eq. 19. Observe that \( (\mathbb{R}^m)^p = \mathbb{R}^{m \times p} \) (resp. \( (\mathbb{R}^m)^p = \mathbb{R}^{m \times p} \)). Then, for fully discrete functions \( u \in \mathbb{R}^{m \times p} \) (resp. \( u \in \mathbb{R}^{m \times p} \), we can combine Eqs. 19 and 31 to define the \( L^2 \)-inner product
\[
\int_Q uv := \int_0^T (u, u)_{L^2(\Omega)} = \sum_{n=1}^M \Delta t \sum_{i=1}^N h \, u^n_i \, v^n_i
\]
resp.
\[
\int_Q uv := \int_0^T (u, u)_{L^2(\Omega)} = \sum_{n=1}^M \Delta t \sum_{i=0}^N h \, u^n_{i+\frac{1}{2}} \, v^n_{i+\frac{1}{2}} (33)
\]

As before, from the context, we will deduce which integral is being used for the space variable.

The same idea holds for functions \( u \in \mathbb{R}^{m \times \mathcal{D}} \) (resp. \( u \in \mathbb{R}^{m \times \mathcal{D}} \). In this case, we have
\[
\int_Q uv := \int_0^T (u, u)_{L^2(\Omega)} = \sum_{n=0}^{M-1} \Delta t \sum_{i=1}^N h \, u^{n+\frac{1}{2}}_i \, v^{n+\frac{1}{2}}
\]
resp.
\[
\int_Q uv := \int_0^T (u, u)_{L^2(\Omega)} = \sum_{n=0}^{M-1} \Delta t \sum_{i=0}^N h \, u^{n+\frac{1}{2}}_{i+\frac{1}{2}} \, v^{n+\frac{1}{2}} (36)
\]

Notice that the above definitions follow as close as possible the notation commonly used in the continuous case.

For short and in accordance with the notation used in the continuous case, we will denote the Hilbert space \( L^2_{\mathcal{D}}(0, T; \mathbb{R}^m) \) (resp. \( L^2_{\mathcal{D}}(0, T; \mathbb{R}^m) \)) induced by the inner product Eq. 33 (resp. Eq. 34) as \( L^2_{\mathcal{D}}(Q) \). We introduce the analogous notation \( L^2_{\mathcal{D}}(Q) \) for functions belonging to the Hilbert spaces \( L^2_{\mathcal{D}}(0, T; \mathbb{R}^m) \) and \( L^2_{\mathcal{D}}(0, T; \mathbb{R}^m) \) with norms induced by Eq. 35–Eq. 36.

As we have mentioned, often times, we shall work with functions restricted or defined on subdomains. In this case, we also adopt a standard notation used in the continuous case. For instance, using Eq. 22, we define
\[
\int_{\omega \times (0, T)} uv := \int_0^T (u, v)_{L^2(\omega)} = \sum_{n=1}^M \Delta t \left( \sum_{i \in [1, N], \, x_i \in \omega} h \, u^n_i \, v^n_i \right).
\]
We use analogous notations by changing the integral symbol \( \int \) for \( \mathcal{F} \). The corresponding spaces induced by such inner products will be denoted by \( L^2_\mathcal{P}(\omega \times (0, T)) \) and \( L^2_\mathcal{P}(\omega \times (0, T)) \).

For our purposes, another important functional space to consider is \( \mathcal{L}^\infty_\mathcal{P}(\mathcal{Q}) \). This space can be naturally obtained by considering Eq. 30 and \( X = \mathbb{R}^m \) (resp. \( X = \mathbb{R}^m \)) endowed with the norm Eq. 20 (resp. Eq. 21), this is,

\[
\| u \|_{\mathcal{L}^\infty_\mathcal{P}(\mathcal{Q})} := \| u \|_{L^\infty_\mathcal{P}(0, T; \mathbb{R}^m)} = \sup_{n \in [1, M]} \sup_{i \in [1, N]} | u^n_i | \\
\left( \| u \|_{\mathcal{L}^\infty_\mathcal{P}(\mathcal{Q})} := \| u \|_{L^\infty_\mathcal{P}(0, T; \mathbb{R}^m)} = \sup_{n \in [1, M]} \sup_{i \in [0, N]} | u^n_{i+1} | \right).
\]

Similar ideas can be used to construct and denote the space \( \mathcal{L}^\infty_\mathcal{P}(\mathcal{Q}) \).

1.5 Statement of the main results

1.5.1 Carleman estimate

To state one of our main result, we introduce several weight functions that will be used in the remainder of this paper. We begin by considering a function \( \psi \) fulfilling the following assumption.

**Assumption 1.7** Let \( \mathcal{B}_0 \) be a nonempty open set of \( \Omega = (0, L) \). Let \( \tilde{\Omega} \) be a smooth open and connected neighborhood of the interval \( \tilde{\Omega} \). The function \( x \mapsto \phi(x) \) is in \( C^k(\tilde{\Omega}) \) with \( k \) sufficiently large, and satisfies for some \( c > 0 \)

\[
\psi > 0 \quad \text{in} \quad \tilde{\Omega}, \quad |\nabla \psi| \geq c \quad \text{in} \quad \tilde{\Omega} \setminus \mathcal{B}_0, \\
\text{and} \quad \psi_x(0) > 0, \quad \psi_x(L) < 0.
\]

As an example of a function \( \psi \) verifying Assumption 1.7, we can take a point \( x_0 \in \mathcal{B}_0 \) and consider \( \psi(x) = C - (x - x_0)^2 \) for some \( C > 0 \) large enough.

Let \( K > \| \psi \|_{C(\tilde{\Omega})} \) and consider a parameter \( \lambda > 0 \). We set

\[
\varphi(x) = e^{\lambda \psi(x)} - e^{\lambda K} < 0, \quad \phi(x) = e^{\lambda \psi(x)}, \quad (37)
\]

and

\[
\theta(t) = \frac{1}{(t + \delta T)(T + \delta T - t)} \quad (38)
\]

for some \( 0 < \delta < 1/2 \). The parameter \( \delta \) is introduced to avoid singularities at time \( t = 0 \) and \( t = T \).

We state our first result, a uniform Carleman estimate for the fully discrete backward parabolic operator formally defined on the dual grid (in time) as follows

\[
(L_{\mathcal{P}} q)^n := -\langle \hat{D}_t q \rangle^n - \Delta_h (\hat{\xi} - q)^n, \quad n \in [1, M], \quad (39)
\]

for any \( q \in (\mathbb{R}^m)^{\mathcal{P}} \). The result is the following.
Theorem 1.8 Let $\Omega$ be a nonempty set of $\mathbb{R}$ and define $\varphi$ according to Eq. \ref{eq:37}. Let $\mathcal{B}$ be another open subset of $\Omega$ such that $\Omega \subset \subset \mathcal{B}$. For all $\lambda \geq 1$ sufficiently large, there exist $C > 0$, $\tau_0 \geq 1$, $h_0 > 0$, $\epsilon_0 > 0$, depending on $\mathcal{B}$, $\mathcal{B}_0$, $T$ and $\lambda$ such that

$$
\tau^{-1} \iint_Q \mathbb{E}^-(e^{2\tau\theta \varphi}) \left( |\mathcal{D}_t q|^2 + |\Delta_h (\mathbb{E}^- q)|^2 \right) + \iint_Q \mathbb{E}^-(e^{2\tau\theta \varphi}) |\partial_h (\mathbb{E}^- q)|^2 \\
+ \iint_Q \mathbb{E}^-(e^{2\tau\theta \varphi}) |\partial_h (\mathbb{E}^- q)|^2 + \tau^3 \iint_Q \mathbb{E}^-(e^{2\tau\theta \varphi})^3 (\mathbb{E}^- q)^2 \\
\leq C \left( \iint_Q \mathbb{E}^-(e^{2\tau\theta \varphi}) |\mathcal{L}_\tau q|^2 + \tau^3 \int_{\mathcal{B} \times (0, T)} \mathbb{E}^-(e^{2\tau\theta \varphi})^3 (\mathbb{E}^- q)^2 \right) \\
+ C h^{-2} \left( \int_{\Omega} |(e^{\tau\theta \varphi})^\frac{1}{2}|^2 + \int_{\Omega} |(e^{\tau\theta \varphi})^{M+\frac{1}{2}}|^2 \right) \quad \text{(40)}
$$

for all $\tau \geq \tau_0(T + T^2)$, $0 < h \leq h_0$, $\Delta t > 0$ and $0 < \delta \leq 1/2$ satisfying the conditions

$$
\frac{\tau^4 \Delta t}{\delta^4 T^6} \leq \epsilon_0 \quad \text{and} \quad \frac{\tau h}{\delta T^2} \leq \epsilon_0, \quad \text{(41)}
$$

and $q$ is any fully discrete function in $(\mathbb{R}^{\mathbb{N}})\mathbb{N}$ with $(q_{\partial \Omega})^{n-\frac{1}{2}} = 0$, $n \in [1, M]$.

To prove our fully discrete Carleman estimate, we follow as close as possible the well-known methodology introduced in [2] for the continuous setting. This method roughly consists on taking the discrete change of variable $z = rz$ where $r(t, x) = e^{\tau(t)\varphi(x)}$ with $\varphi$ and $\theta$ defined as in Eq. \ref{eq:37}–Eq. \ref{eq:38} and look for the equation verified by the variable $z$. Then, rearranging terms in a suitable way and taking the $L^2$-norm of the resulting expression, we develop some cross products and integrating by parts in time and space several times we arrive at Eq. \ref{eq:40}.

During the proof, we will use some tools and estimates that have been proved in other related works in the space- and time-discrete settings (see [13] and [22], respectively), but some results are new and specific to the fully discrete case. For instance, during the proof, the following term

$$
\iint_Q \mathbb{E}^- (r \rho \Delta_h z) \mathcal{D}_t z, \quad \text{where} \quad \rho = r^{-1}, \quad \text{(42)}
$$

arises. Normally, this term is handled by performing integration by parts in space, followed by another integration in time. In either space- or time-discrete cases, only one variable is discrete and the interaction between the time and space derivatives can handled separately. In the fully discrete case, this is not longer the case and the application of the time-discrete operator $\mathcal{D}_t$ on space-discrete operations like $\Delta_h$ has to be taken into account, see Lemma A.16.

Even though the parameter $\delta$ does not appear explicitly in Eq. \ref{eq:40} since it is hidden in the definition of Eq. \ref{eq:38}, this parameter plays a key role during the proof. We use it to avoid singularities at time $t = 0$ and $t = T$ (which correspond to the case $\delta = 0$ and are systematically used in the continuous setting, see [2, 27], but which are somehow inconvenient at the discrete level). Here, by taking $\delta > 0$, we are able to
obtain time- and space-discrete derivatives of the Carleman weights and set a suitable change of variable, which is the starting point of the proof.

**Remark 1.9** The following remarks are in order.

- We can readily recognize in Eq. 40 the structure of the continuous Carleman inequality (cf. [27, Lemma 1.3]). As we have mentioned above, the main idea of the proof is to set a change of variable, find a suitable equation and perform discrete integration by parts. This idea mimics the classical work by Fursikov and Imanuvilov [2] and the computations are similar in spirit, nevertheless, due to the discrete nature of our problem, several remainder terms of order \( \mathcal{O}(h) \) and \( \mathcal{O}(\Delta t) \) appear. Typically, these terms cannot be directly controlled by similar ones in the left-hand side of Eq. 40, but a condition like Eq. 41 allows us to take \( h \) and \( \Delta t \) sufficiently small with respect to \( \tau^{-1} \) and absorb them.

- The last two terms in Eq. 40 are specific to the discrete case. These terms, which appear while integrating by parts terms like Eq. 42 (among many others), cannot be avoided. Notice that if such terms do not appear, we would have obtained a nonrelaxed observability inequality leading to a uniform controllability result which is not true.

- Even if the last two terms of Eq. 40 have a factor \( h^{-2} \) in front, we can make them exponentially small by connecting the parameters \( \delta \) and \( h \). For controllability purposes, we can take \( \delta \) of order \( h^{\min(\delta/4,1)} \) for a parameter \( \delta \geq 1 \) to prove the smallness of these terms. Note that by doing so, a natural condition relating \( h \) and \( \Delta t \) should appear. As we will see later, we have to take \( \Delta t \) of order \( h^\delta \) to prove a controllability result for our system. A condition like this is expected and similar ones have appeared in other works addressing the controllability of fully discrete parabolic systems, see [26] and [18].

### 1.5.2 Controllability results

For a potential \( a \in L^\infty_0(Q) \), we consider the sequence \( y = \{y^n\}_{n \in [0,M]} \) verifying the recursive formula

\[
\begin{align*}
\frac{y^{n+1} - y^n}{\Delta t} - \Delta_h y^{n+1} + a^{n+1} y^{n+1} = 0 & \quad n \in [0, M - 1], \\
\left| y^{n+1} \right|_{a\Omega} = 0 & \quad n \in [0, M - 1], \\
y^0 = g.
\end{align*}
\]

With the notation introduced previously, we can compactly rewrite Eq. 43 as

\[
\begin{align*}
(D_t y)^{n+\frac{1}{2}} - \Delta_h (t^+ y)^{n+\frac{1}{2}} + (t^+ a y)^{n+\frac{1}{2}} = 0 & \quad n \in [0, M - 1], \\
y^0 = g.
\end{align*}
\]

For convenience, we shall not make explicit the boundary conditions in such compact formulas since we will devote our analysis only to systems with homogeneous Dirichlet boundary conditions.
Observe that the equation verified by $y$ is written in the dual (time) grid $\mathcal{D}$. This motivates us to look for discrete controls defined on this grid and consider controlled systems of the form

$$\begin{cases}
(D_t y)^{n+\frac{1}{2}} - \Delta_h(t^{+}y)^{n+\frac{1}{2}} + (t^{+}ay)^{n+\frac{1}{2}} = 1_\omega v^{n+\frac{1}{2}} & n \in [0, M - 1], \\
y^{0} = g.
\end{cases} \quad (45)$$

Of course, we could have used the notation $(t^{+}v)^{n+\frac{1}{2}}$ in agreement with the control appearing in system Eq. 9 but, as we will see below, this is a more comfortable and natural approach.

Following the so-called Hilbert Uniqueness Method (see [28]), it is possible to build a control function by minimizing a quadratic functional defined for the solutions to the adjoint of Eq. 43. The strategy is as follows.

The adjoint equation to Eq. 43 is given by recursive formula

$$\begin{cases}
q^{-\frac{1}{2}} - q^{n+\frac{1}{2}} - \Delta_h q^{-\frac{1}{2}} + a^n q^{-\frac{1}{2}} = 0 & n \in [1, M], \\
q^{-\frac{1}{2}} = 0 & n \in [1, M], \\
q^{M+\frac{1}{2}} = q_T,
\end{cases} \quad (46)$$

where $q_T \in \mathbb{R}^{\Omega}$ is a given initial datum. With our notation, we can rewrite Eq. 46 in the compact form

$$\begin{cases}
-(\bar{D}_t q)^n - \Delta_h (\bar{E} - q)^n + a^n (\bar{E} - q)^n = 0 & n \in [1, M], \\
q^{M+\frac{1}{2}} = q_T.
\end{cases} \quad (47)$$

Using the Carleman inequality Eq. 40, we can prove a relaxed observability inequality of the form

$$|q^\frac{1}{2}|_{L^2(\Omega)} \leq C_{obs} \left( \iint_{\omega \times (0, T)} |q|^2 + e^{-C_2/\Delta t^{\frac{1}{2}}} |q_T|^2 \right)^{\frac{1}{2}}, \quad (48)$$

where $\vartheta \geq 1$ and the positive constants $C_2$ and $C_{obs}$ only depend on $T$, $\omega$ and $\|a\|_{L^\infty_T(\Omega)}$.

Comparing with the results obtained in the time- and space-discrete cases in [13] and [22], a new parameter $\vartheta$ appears here due to the fully discrete nature of our problem. Indeed, as we will see in Proposition 3.1, inequality Eq. 48 holds for parameters $\Delta t$ and $h$ chosen sufficiently small and such that $\Delta t \leq T^{-2}h^\vartheta$. In this way, as $h$ tends to $0$, $\Delta t \rightarrow 0$ as well and we recover the well-known result in the continuous case.

With Eq. 48 at hand, we can prove that the fully discrete quadratic functional

$$J_{h, \Delta t}(q_T) = \frac{1}{2} \iint_{\omega \times (0, T)} |q|^2 + \phi(h)\frac{1}{2} |q_T|^2_{L^2(\Omega)} + (g, q^{\frac{1}{2}})_{L^2(\Omega)}$$

has a unique minimizer $\hat{q}_T$. By taking $v = 1_\omega \hat{q}$, where $\hat{q}$ is the solution to Eq. 47 with initial datum $\hat{q}_T$, we can prove that $\|v\|_{L^2_T(\omega \times (0, T))} \leq C$ uniformly with respect to $h$ and $\Delta t$ and that

$$|y^M|_{L^2(\Omega)} \leq C\sqrt{\phi(h)}|g|_{L^2(\Omega)}, \quad (49)$$
where \( h \mapsto \phi(h) \) is any given function of the discretization parameter such that

\[
\liminf_{h \to 0} \frac{\phi(h)}{e^{-C_2/h^{\min(\vartheta/4,1)}}} > 0
\]  

(50)

and

\[
\Delta t \leq T^{-\vartheta} h^\vartheta, \quad \vartheta \geq 1.
\]  

(51)

This indicates that in fact we reach a small target \( y^M \) whose size goes to zero as \( h \to 0 \) (and so \( \Delta t \)) at the prescribed rate \( \sqrt{\phi(h)} \) with controls that remain uniformly bounded with respect to \( h \) and \( \Delta t \).

Our main controllability reads as follows.

**Theorem 1.10** Let \( T > 0, \vartheta \geq 1 \) and assume that \( h > 0 \) is sufficiently small. Then, for any \( g \in \mathbb{R}^m \), any function \( \phi(h) \) verifying Eq. 50, and any \( \Delta t > 0 \) verifying Eq. 51, there exists a fully discrete control \( v \in L^2_{\mathcal{D}}(\omega \times (0, T)) \) such that

\[
\|v\|_{L^2_{\mathcal{D}}(\omega \times (0, T))} \leq C\|g\|_{L^2(\Omega)}
\]

and such that the corresponding controlled solution \( y \) to Eq. 43 satisfies Eq. 49, for a positive constant \( C \) only depending on \( \phi \), \( T \), \( \omega \) and \( \|a\|_{L^\infty_p(\Omega)} \).

**Remark 1.11** In practice, the rate \( \sqrt{\phi(h)} \) can be chosen in agreement with the accuracy of the discretization scheme, while the parameter \( \vartheta \) gives us freedom for setting the relation between \( h \) and \( \Delta t \) (see 51), which can be conveniently chosen as \( \Delta t = O(h^\vartheta) \) for any \( \vartheta \geq 1 \). See Section 4.2 for a further discussion on this direction.

One of the main advantages of proving a fully discrete Carleman estimate is that we can drop the spectral analysis techniques used in previous works (restricted to linear problems) and we can prove controllability results for semilinear systems. In fact, we can extend the previous theorem and study the controllability of the fully discrete system

\[
\begin{cases}
\frac{y^{n+1} - y^n}{\Delta t} - \Delta_h y^{n+1} + f(y^{n+1}) = 1_\omega v^{n+1/2} & n \in [0, M - 1] \\
y^{n+1}_{\partial \Omega} = 0 & n \in [0, M - 1], \\
y^0 = g,
\end{cases}
\]  

(52)

where \( f \in C^1(\mathbb{R}) \) is a globally Lipschitz function with \( f(0) = 0 \).

**Remark 1.12** As for the linear case, the implicit scheme Eq. 52 for the nonlinear equation enjoys good convergence properties. Replacing the right-hand side of Eq. 52 by the sampling of a given function \( F = F(x, t) \), Theorem 1.3 holds under the additional assumption

\[
\Delta t \leq 1/4,
\]  

(53)

where \( C_L \) is the Lipschitz constant of the function \( f \). The proof can be done by adapting the arguments in [7, Ch. 9, Sect. 4.1] and using Eq. 53 for guaranteeing the stability of the scheme.
Our controllability result reads as follows.

**Theorem 1.13** Let $T > 0$, $\vartheta \geq 1$ and assume that $h > 0$ is sufficiently small. Then, for any $g \in \mathbb{R}^{\mathbb{N}}$, any function $\phi(h)$ verifying Eq. 50, and any $\Delta t > 0$ verifying Eq. 51, there exists a uniformly bounded fully discrete control $v \in L^2_T(0, T; \mathbb{R}^{\mathbb{N}})$ such that the associated controlled solution $y$ to Eq. 52 verifies $|y^M|_{L^2(\Omega)} \leq C \sqrt{\phi(h)} |g|_{L^2(\Omega)}$, with $C > 0$ only depending on $\phi$, $T$, $\omega$ and $f$.

The proof of Theorem 1.13 follows other well-known controllability results for semilinear systems (see, for instance, [29, 30]). First, we prove the existence of a $\phi(h)$-null control for a linearized version of Eq. 52 and then, after a careful analysis on the dependence of the constants appearing in Eq. 48, we can use a standard fixed point argument to deduce the result for the nonlinear case.

### 1.6 Outline

The rest of the paper is organized as follows. In Section 2, we present the proof of Theorem 1.8. As in the continuous case, the proof roughly consists in writing a new equation after conjugation with the Carleman weight, splitting the resulting equation into two parts and then estimating the $L^2$ product between those two parts. To ease the reading, we have divided the proof in several parts and indicating clearly the procedure developed in each of them.

Section 3 is devoted to the application of the fully discrete Carleman estimate to obtain controllability results. We divide it in two parts: in the first one, using estimate Eq. 40, we derive a relaxed observability inequality, where we will pay special attention to the connection between the parameters $h$, $\Delta t$ and $\delta$. Then, this result is used to obtain the $\phi(h)$-controllability results in Theorem 1.10 and 1.13.

Finally, we devote Section 4 to present additional results, remarks and open problems regarding fully discrete Carleman estimates as well as the possible applications for handling less traditional control problems.

### 2 Fully discrete Carleman estimate

In this section, we present the proof of Theorem 1.8. For the sake of presentation, we have divided the proof in various steps and lemmas. The ideas presented here follow as close as possible the proofs presented in the classical continuous setting (see, e.g., [2, 27]).

As in other related works, we will keep track of the dependence of all constants with respect to $\lambda$, $\tau$ and $T$. Also, in accordance with the discrete nature of our problem, we will pay special attention of the dependence with respect to the discrete parameters $h$, $\Delta t$ and $\delta$. 
2.1 The change of variable

To abridge the notation, we introduce the following instrumental functions

\[
s(t) = \tau \theta(t), \quad \tau > 0, \quad t \in (-\delta T, T + \delta T),
\]

\[
r(t, x) = e^{r(t) \varphi(x)}, \quad \rho(t, x) = (r(t, x))^{-1}, \quad x \in \Omega, \quad t \in (-\delta T, T + \delta T).
\]

For \( q \in (\mathbb{R}^{\Omega}) \), we introduce the change of variables

\[
z^{n+\frac{1}{2}} = r^{n+\frac{1}{2}} q^{n+\frac{1}{2}}, \quad n \in [0, M],
\]

with \( r^{n+\frac{1}{2}} \) defined as

\[
r^{n+\frac{1}{2}} = \begin{pmatrix}
    r^{n+\frac{1}{2}}_1 \\
    \vdots \\
    r^{n+\frac{1}{2}}_N
\end{pmatrix},
\]

where we recall notation Eq. 5.

As in [13], the enlarged neighborhood \( \hat{\Omega} \) of \( \Omega \) in Assumption 1.7 allows us to apply multiple discrete operations such as \( \partial_h, m_h \) on the weight functions. In particular, from the construction of \( \psi \), we have

\[
(r \partial_h \rho)^{n+\frac{1}{2}}_0 \leq 0, \quad (r \partial_h \rho)^{n+\frac{1}{2}}_{N+1} \geq 0, \quad n \in [0, M - 1]. \tag{54}
\]

Hereinafter, we will simplify the notation in such kind of formulas by omitting the superscript \( n \) and simply write \( z = rq \) which implicitly means that the continuous function \( r \) is evaluated on the same time-grid (primal or dual) and at the same time point as the one attached to the discrete variable \( q \).

Our first task is to obtain an expression of the equation verified by \( z \). From the change of variables proposed, we have that \( q = \rho z \) and using Lemma A.1 we readily see

\[
\Delta_h [\rho z] = \bar{\partial}_h \partial_h [\rho z] = \bar{\partial}_h (\partial_h \rho) \bar{\rho} + \partial_h \partial_h (z) \bar{\rho} + 2 \partial_h z \partial_h \rho
\]

\[
= \Delta_h \rho \bar{z} + \Delta_h z \bar{\rho} + 2 \partial_h z \partial_h \rho. \tag{55}
\]

This clearly resembles the classical Leibniz formula in the continuous case. Then, using the translation operator Eq. 32, we can write the following equality on the primal grid \( \mathcal{P} \)

\[
\bar{E}^- (\Delta_h [\rho z]) = \bar{E}^- (\Delta_h \rho \bar{z}) + \bar{E}^- (\Delta_h z \bar{\rho}) + 2 \bar{E}^- (\partial_h \rho \partial_h \rho). \tag{56}
\]

On the other hand, using Eq. 149, we have

\[
\bar{D}_t (\rho z) = (\bar{E}^- \rho) \bar{D}_t z + \bar{D}_t \rho (\bar{E}^+ z). \tag{57}
\]

Thus, multiplying Eq. 56, Eq. 57 by \( (\bar{E}^- r) \), adding the resulting expressions and using identity Eq. 39, we obtain

\[
(\bar{E}^- r) \bar{D}_t \rho (\bar{E}^+ z) + \bar{E}^- (r \Delta_h \rho \bar{z}) + \bar{E}^- (r \Delta_h z \bar{\rho}) + 2 \bar{E}^- (r \partial_h \rho \partial_h z)
\]

\[
= -(\bar{E}^- r) (L_{\mathcal{D}} q). \tag{58}
\]
Using Lemma A.14 in the second term of the above expression yields
\[
\overline{D}_t z - \tau \left( \xi - \theta' \right) \varphi(\xi^+ z) + \Delta t \left( \frac{\tau}{\delta^3 T^4} + \frac{\tau^2}{\delta^4 T^6} \mathcal{O}_3(1) \right) (\xi^+ z) + \xi^-(r \Delta h \rho \overline{z}) \\
+ \xi^-(r \Delta h \Delta h \rho \overline{z}) + 2 \xi^-(r \Delta h \rho \overline{h} z) = - (\xi^- r)(L_D q).
\]
whence, using that \( \Delta t \overline{D}_t z = (\xi^+ z) - (\xi^- z) \), we get
\[
\overline{D}_t z - \tau \varphi(\xi^- z) + \xi^-(r \Delta h \rho \overline{z}) + \xi^-(r \Delta h \Delta h \rho \overline{z}) + 2 \xi^-(r \Delta h \rho \overline{h} z) \\
= - (\xi^- r)(L_D q) - \Delta t \left( \frac{\tau}{\delta^3 T^4} + \frac{\tau^2}{\delta^4 T^6} \right) \mathcal{O}_3(1)(\xi^+ z) + \tau \Delta t(\xi^- \theta') \varphi \overline{D}_t z. \tag{59}
\]
In the spirit of the continuous case, we rearrange Eq. 59 and write it in the form
\[
A z + B z = R, \tag{60}
\]
where \( A z = A_1 z + A_2 z + A_3 z, B z = B_1 z + B_2 z + B_3 z \) with
\[
A_1 z = \xi^-(r \rho \Delta h z), \quad A_2 z = \xi^-(r \Delta h \rho \overline{z}), \quad A_3 z = - \tau \varphi(\xi^- \theta'), \\
B_1 z = 2 \xi^-(r \Delta h \rho \overline{h} z), \quad B_2 z = - 2 \xi^-(s \delta_{xx} \phi z), \quad B_3 z = \overline{D}_t z, \tag{61} \tag{62}
\]
and
\[
R = - (\xi^- r)(L_D q) - \Delta t \left( \frac{\tau}{\delta^3 T^4} + \frac{\tau^2}{\delta^4 T^6} \right) \mathcal{O}_3(1)(\xi^+ z) + \tau \Delta t(\xi^- \theta') \varphi \overline{D}_t z \\
- 2 \xi^-(s \delta_{xx} \phi z). \tag{63}
\]
With the notation introduced in Section 1.3, we take the \( L^2 \)-norm in Eq. 60 and obtain
\[
\| A z \|^2_{L^2_P(Q)} + \| B z \|^2_{L^2_P(Q)} + 2 (A z, B z)_{L^2_P(Q)} = \| R \|^2_{L^2_P(Q)} . \tag{64}
\]
As in other works devoted to prove Carleman estimates, the rest of the proof consists on estimating the term \( (A z, B z)_{L^2_P(Q)} \). For clarity, we have divided it in several steps. Developing the inner product \( (A z, B z)_{L^2_P(Q)} \), we set \( I_{ij} := (A_i z, B_i z)_{L^2_P(Q)} \).

### 2.2 Estimates involving only space-discrete computations

In this step, we provide estimates for the terms \( I_{ij} \) with \( i, j = 1, 2 \). Such terms can be estimated by only operating on the discrete variable in space. Indeed, the discrete time variable plays a very minor role and the proofs can be easily adapted from similar estimates already presented in [13] for the space-discrete case. However, the computations shown in that work are made for a more general framework and with a heavier notation. For this reason, we give short and concise proofs of these estimates on Appendix B.

The first estimate reads as follows.

**Lemma 2.1** (Estimate of \( I_{11} \)) For \( \Delta t \tau (T^3 \delta^2)^{-1} \leq 1 \) and \( \tau h(\delta T^2)^{-1} \leq 1 \), we have
\[
I_{11} \geq - \tau \lambda^2 \int_Q (\xi^- \theta) \varphi |\partial_x \psi|^2 |\partial_h (\xi^- z)|^2 + Y_{11}^h - X_{11}^h,
\]
with
\[ Y_{11}^h = \int_0^T \left( 1 + (\bar{E} - sh)^2 \mathcal{O}_\lambda(1) \right) \left[ \bar{E}^- (r \bar{\partial}_h \phi)_{N+1} \left| \partial_h (\bar{E}^- z) \right|^2_{N+\frac{1}{2}} \right. \]
\[ \left. - \bar{E}^- (r \bar{\partial}_h \phi)_0 \left| \partial_h (\bar{E}^- z) \right|^2_{\frac{1}{2}} \right] \]
and
\[ X_{11}^h = \iint_Q (\bar{E}^- \nu_{11}) |\partial_h (\bar{E}^- z)|^2, \quad \nu_{11} = s \lambda \mathcal{O}(1) + s (sh)^2 \mathcal{O}_\lambda(1). \]

For the term \( I_{12} \), we have the following.

**Lemma 2.2** (Estimate of \( I_{12} \)) For \( \Delta t \tau (T^3 \delta^2)^{-1} \leq 1 \) and \( \tau h (\delta T^2)^{-1} \leq 1 \), we have
\[ I_{12} \geq 2 \tau \lambda^2 \iint_Q (\bar{E}^- \phi)^3 |\partial_x \psi| \left| \partial_h (\bar{E}^- z) \right|^2 - X_{12}^h, \]
with
\[ X_{12} = \iint_Q (\bar{E}^- \nu_{12}) |\partial_h (\bar{E}^- z)|^2 + \iint_Q (\bar{E}^- \mu_{12}) (\bar{E}^- z)^2, \]
where \( \mu_{12} = s^2 \mathcal{O}_\lambda(1) \) and \( \nu_{12} = s \lambda \phi \mathcal{O}(1) + [1 + s (sh)] \mathcal{O}_\lambda(1). \)

The term \( I_{21} \) is estimated in the following result.

**Lemma 2.3** (Estimate of \( I_{21} \)) For \( \Delta t \tau (T^3 \delta^2)^{-1} \leq 1 \) and \( \tau h (\delta T^2)^{-1} \leq 1 \), we have
\[ I_{21} \geq 3 \tau^3 \phi^4 \iint_Q (\bar{E}^- \theta)^3 |\partial_x \psi| \left( \bar{E}^- z \right)^2 + Y_{21}^h - X_{21}^h, \]
with
\[ Y_{21}^h = \int_0^T (\bar{E}^- sh)^2 \mathcal{O}_\lambda(1) \left( \bar{E}^- (r \bar{\partial}_h \phi)_0 \left| \partial_h (\bar{E}^- z) \right|^2_{N+\frac{1}{2}} \right. \]
\[ \left. + \bar{E}^- (r \bar{\partial}_h \phi)_{N+1} \left| \partial_h (\bar{E}^- z) \right|^2_{N+\frac{1}{2}} \right) \]
and
\[ X_{21}^h = \iint_Q (\bar{E}^- \mu_{21}) (\bar{E}^- z)^2 + \iint_Q (\bar{E}^- \nu_{21}) |\partial_h (\bar{E}^- z)|^2, \]
where \( \mu_{21} = (s \lambda \phi)^3 \mathcal{O}(1) + s^2 \mathcal{O}_\lambda(1) + s^3 (sh)^2 \mathcal{O}_\lambda(1) \) and \( \nu_{21} = s (sh)^2 \mathcal{O}_\lambda(1). \)

Finally, \( I_{22} \) can be estimated as follows.

**Lemma 2.4** (Estimate of \( I_{22} \)) For \( \Delta t \tau (T^3 \delta^2)^{-1} \leq 1 \) and \( \tau h (\delta T^2)^{-1} \leq 1 \), we have
\[ I_{22} \geq -2 \tau^3 \phi^4 \iint_Q (\bar{E}^- \theta)^3 |\partial_x \psi| \left( \bar{E}^- z \right)^2 - X_{22}^h, \]
with
\[ X_{22}^h = \iint_Q (\bar{E}^- \mu_{22}) (\bar{E}^- z)^2 + \iint_Q (\bar{E}^- \nu_{22}) |\partial_h (\bar{E}^- z)|^2, \]
where \( \mu_{22} = (s \lambda \phi)^3 \mathcal{O}(1) + s^2 \mathcal{O}_\lambda(1) + s^3 (sh) \mathcal{O}_\lambda(1) \) and \( \nu_{22} = s (sh)^2 \mathcal{O}_\lambda(1). \)
2.3 Estimates involving time-discrete computations

In this step, we will estimate the terms $I_{31}$, $I_{32}$, and $I_{33}$. These terms require time-discrete computations combined with some space-discrete operations but can be done without major issues. In fact, the effects of both schemes can be manipulated in a separated way and do not require any major change as compared to the works [13] or [22].

We have the following result.

**Lemma 2.5** (Estimate of $I_{31}$) For $\Delta t \tau (T^3 \delta^2)^{-1} \leq 1$ and $\tau h(\delta T^2)^{-1} \leq 1$, we have

$$I_{31} \geq -X_{31}^{h,\Delta t},$$

with

$$X_{31}^{h,\Delta t} = \iint_Q (\bar{E} - \mu_{31})(\bar{E} - z)^2 + \iint_Q (\bar{E} - v_{31})|\tilde{\partial}_h(\bar{E} - z)|^2.$$

where $\mu_{31} = Ts^2\theta^2\mathcal{O}_\lambda(1)$ and $v_{31} = T(sh)^2\theta^2\mathcal{O}_\lambda(1)$.

The proof of this result can be found in Appendix B.

For the term $I_{32}$, we have the following.

**Lemma 2.6** (Estimate of $I_{32}$) For $\Delta t \tau (T^3 \delta^2)^{-1} \leq 1$ and $\tau h(\delta T^2)^{-1} \leq 1$, we have

$$I_{32} \geq -X_{32}^{\Delta t}$$

with

$$X_{32}^{\Delta t} = \int \int_Q (\bar{E} - \mu_{32})(\bar{E} - z)^2.$$

where $\mu_{32} = Ts^2\theta^2\mathcal{O}_\lambda(1)$.

**Proof** The proof follows from a direct computation and taking into account that $|\theta'| \leq CT^2\theta$, $\|\psi\|_{C(\Omega)} = \mathcal{O}_{\lambda}(1)$ and $\|\tilde{\partial}_{xx}\psi\|_{\infty} = \mathcal{O}_{\lambda}(1)$.

For the term $I_{33}$, we have the following.

**Lemma 2.7** (Estimate of $I_{33}$) For $\Delta t \tau (T^3 \delta^2)^{-1} \leq 1$ and $\tau h(\delta T^2)^{-1} \leq 1$, we have

$$I_{33} \geq -X_{33}^{\Delta t} - W_{33}^{\Delta t},$$

with

$$X_{33}^{\Delta t} = \int \int_Q (\bar{E} - \mu_{33})(\bar{E} - z)^2$$

and

$$W_{33}^{\Delta t} = \int \int_Q (\bar{E} - \gamma_{33})(\bar{D}_t z)^2,$$

where $\mu_{33} = (\tau T^2\theta^3 + \frac{\tau \Delta t}{\delta T^2})\mathcal{O}_\lambda(1)$ and $\gamma_{33} = \Delta t\tau \theta^2\mathcal{O}_\lambda(1)$.

The proof of this result can be found in Appendix B.
2.4 A new fully discrete estimate

Here, we present a result that arises specifically in the fully discrete case. Indeed, the combined effect of time- and space-discrete computations is needed and the proof requires of some new estimates shown in Appendix A. The result reads as follows.

Lemma 2.8 (Estimate of $I_{13} + I_{23}$) For $\Delta t \tau^2 (\delta^4 T^6)^{-1} \leq \epsilon_1(\lambda)$ and $\tau h (\delta T^2)^{-1} \leq \epsilon_1(\lambda)$, there exist constants $c_\lambda, c'_\lambda > 0$ uniform with respect to $\Delta t$ and $h$ such that

\[
I_{13} + I_{23} \geq c'_\lambda \Delta t \int_Q (\partial_t (\partial_h z))^2 - c_\lambda \int_\Omega |(\partial_h z)^{M+\frac{1}{2}}|^2 - X^{h,\Delta t}_+ - W^{h,\Delta t}_+
\]

\[
- \int_\Omega (s^{M+\frac{1}{2}})\mathcal{O}_\lambda(1)|z^{M+\frac{1}{2}}|^2 - \int_\Omega (s^{\frac{1}{2}})\mathcal{O}_\lambda(1)|z^{\frac{1}{2}}|^2
\]

\[
- \int_\Omega (s^{M+\frac{1}{2}})\mathcal{O}_\lambda(1)|(\partial_h z)^{M+\frac{1}{2}}|^2 - \int_\Omega (s^{\frac{1}{2}})\mathcal{O}_\lambda(1)|(|(\partial_h z)^{\frac{1}{2}}|^2, (66)
\]

with

\[
X^{h,\Delta t}_+ = \int_Q (\mathcal{E}^- - \mu_+)(\mathcal{E}^- z)^2 + \int_Q (\mathcal{E}^- - \nu_+)|\partial_h (\mathcal{E}^- z)|^2
\]

and

\[
W^{h,\Delta t}_+ = \int_Q (\mathcal{E}^- - \gamma_+)(\partial_t z)^2,
\]

where

\[
v_+ := \left\{ T \theta (sh)^2 + s^2 (sh)^2 + \left( \frac{\tau}{\delta^3 T^4} \right) \left( \frac{\tau h}{\delta T^2} \right) + \left( \frac{\tau^2 \Delta t}{\delta^3 T^4} \right) \left( \frac{\tau h}{\delta T^2} \right)^3 \right\} \mathcal{O}_\lambda(1),
\]

\[
\mu_+ := \left\{ T s^2 \theta + \left( \frac{\tau^2 \Delta t}{\delta^3 T^4} \right) + \left( \frac{\tau \Delta t}{\delta^3 T^4} \right) \left( \frac{\tau h}{\delta T^2} \right)^3 \right\} \mathcal{O}_\lambda(1),
\]

\[
\gamma_+ := \left\{ s^{-1} (sh)^2 + \Delta t s^2 \right\} \mathcal{O}_\lambda(1).
\]

The proof of this result can be found in Appendix B. Although very similar, this result is new as compared to the works [13] and [22]. Here, we decided to estimate $I_{13}$ and $I_{23}$ together since the first term in the right-hand side of Eq. (66) (which is positive) arises in the estimation of $I_{13}$ and allows to control similar terms obtained while estimating $I_{23}$. This comes at the price of imposing new size constraints on the parameters $\tau, h$ and $\Delta t$.

Remark 2.9 We remark that in the continuous case the estimate of $I_{13} + I_{23}$ reduces to merely the first term of $\mu_+$. In particular, $I_{23} = 0$.

2.5 Towards the Carleman estimate

In this step, we start to build our fully discrete Carleman inequality. First, we present the following estimate for the right-hand side.
Lemma 2.10 (Estimate of the norm of $R$) For $\Delta t \tau (T^3 \delta^2)^{-1} \leq 1$ and $\tau h (\delta T^2)^{-1} \leq 1$, there exists a constant $C_\lambda > 0$ uniform with respect to $\tau$ and $\Delta t$ such that

$$\|R\|_{L^2_p(Q)}^2 \leq C_\lambda \left( \iint_Q (\bar{\varepsilon} - r)^2 |L_D q|^2 + X^\Delta_R + W^\Delta_R \right)$$

$$+ C_\lambda \left[ \left( \frac{\Delta t \tau}{\delta^3 T^4} \right)^2 + \left( \frac{\Delta t \tau^2}{\delta^4 T^6} \right)^2 \right] \left( \int_\Omega |z^{M + \frac{1}{2}}|^2 \right),$$

with

$$X^\Delta_R = \iint_Q (\bar{\varepsilon} - \mu_R)(\bar{\varepsilon} - z)^2$$

and

$$W^\Delta_R = \iint_Q (\bar{\varepsilon} - \gamma_R)(\bar{D} I z)^2,$$

where $\mu_R = s^2 + \left[ \left( \frac{\Delta t \tau}{\delta^3 T^4} \right)^2 + \left( \frac{\Delta t \tau^2}{\delta^4 T^6} \right)^2 \right]$ and $\gamma_R = T^2(\tau \Delta t)^2 \theta^4 + \frac{r^2(\Delta t)^4}{\delta^6 T^8}.$

The proof of this result follows from successive applications of triangle and Young inequalities and can be carried out exactly as in [22, Proof of Lemma 2.2], thus we omit the proof.

Using the estimates obtained in Lemmas 2.1 through 2.10 in identity Eq. 64, we have that for $0 < \Delta t \tau^2 (\delta^4 T^6)^{-1} \leq \epsilon_1(\lambda)$ and $0 < \tau h (\delta T^2)^{-1} \leq \epsilon_1(\lambda)$ there exists a positive constant $C_\lambda$ uniform with respect to $\tau$ and $\Delta t$ such that

$$\|Az\|_{L^2_p(Q)}^2 + \|Bz\|_{L^2_p(Q)}^2 + 2\tau \lambda^2 \iint_Q (\bar{\varepsilon} - \theta \phi |\partial_x \psi|^2 |\partial_h (\bar{\varepsilon} - z)|^2$$

$$+ 2\tau \lambda^3 \phi^2 \iint_Q (\bar{\varepsilon} - \theta)^3 |\partial_x \psi|^4 |\bar{\varepsilon} - z|^2 + 2Y$$

$$\leq C_\lambda \left( \iint_Q (\bar{\varepsilon} - r)^2 |L_D q|^2 + \int_\Omega (s^{M + \frac{1}{2}})^2 (z^{M + \frac{1}{2}})^2$$

$$+ \int_\Omega (s^{\frac{1}{2}})^2 (z^{\frac{1}{2}})^2 + \int_\Omega |(\partial_h z)^{M + \frac{1}{2}}|^2 \right)$$

$$+ C_\lambda \left( \int_\Omega (s^{M + \frac{1}{2}} h)^2 |(\partial_h z)^{M + \frac{1}{2}}|^2 + \int_\Omega (s^{\frac{1}{2}} h)^2 |(\partial_h z)^{\frac{1}{2}}|^2 \right) + 2X + 2W,$$  \hspace{1cm} (68)

where $Y := Y_{11} + Y_{21}$ and

$$W := W_{33}^\Delta + W_{33}^h + W_R^\Delta,$$  \hspace{1cm} (69)

$$X := \sum_{i,j=1}^2 X_{ij}^h + X_{31}^h + X_{32}^h + X_{33}^h + X_R + X_R^\Delta.$$  \hspace{1cm} (70)

As in other discrete Carleman works, the term $Y$ can be dropped. In fact, we have

Lemma 2.11 For all $\lambda > 0$, there exists $0 < \epsilon_2(\lambda) < \epsilon_1(\lambda)$ such that for $0 < \tau h (\delta T^2)^{-1} \leq \epsilon_2(\lambda)$, we have $Y \geq 0.$
Proof The proof of this result is straightforward. Shifting the time integral, we notice that

\[ Y = \int_0^T \left( r \partial_h \rho \right)_{N+1} |\partial_h z|_{N+\frac{1}{2}}^2 - \int_0^T \left( r \partial_h \rho \right)_0 |\partial_h z|_{\frac{1}{2}}^2 \]

\[ + O_h(1) \int_0^T (sh)^2 \left[ (r \partial_h \rho)_{N+1} |\partial_h z|_{N+\frac{1}{2}}^2 + (r \partial_h \rho)_0 |\partial_h z|_{\frac{1}{2}}^2 \right]. \]

Thanks to Eq. 54 the first two terms of the above equation are nonnegative. Then, using that \((sh) \leq \tau h(\delta T)^{-1}\) the result follows by taking \(\epsilon_2(\lambda)\) small enough. □

Using the above result and recalling that \(|\partial_h \psi| \geq C > 0\) in \(\Omega \setminus B_0\), we obtain from Eq. 68

\[ \|Az\|_{L_p^2(\Omega)}^2 + \|Bz\|_{L_p^2(\Omega)}^2 + 2\tau \lambda^2 \int_\Omega (\bar{E}-\theta)\phi|\partial_h (\bar{E}-z)|^2 \]

\[ + 2\tau^3 \lambda^4 \int_\Omega (\bar{E}-\theta)^3 \phi^3 (\bar{E}-z)^2 \]

\[ \leq C_\lambda \left( \int_\Omega (E-r)^2 |L_p q|^2 + \tau \lambda^2 \int_{B_0 \times (0,T)} (E-\theta)\phi|\partial_h (E-z)|^2 \right) \]

\[ + \tau^3 \lambda^4 \int_{B_0 \times (0,T)} (E-\theta)^3 \phi^3 (E-z)^2 \]

\[ + C_\lambda \left( \int_\Omega (s^{M+\frac{1}{2}})^2 (z^{M+\frac{1}{2}})^2 + \int_\Omega (s^{\frac{1}{2}})^2 (z^{\frac{1}{2}})^2 + \int_\Omega |(\partial_h z)^{M+\frac{1}{2}}|^2 \right) \]

\[ + C_\lambda \left( \int_\Omega (s^{M+\frac{1}{2}} h)^2 |(\partial_h z)^{M+\frac{1}{2}}|^2 + \int_\Omega (s^{\frac{1}{2}} h)^2 |(\partial_h z)^{\frac{1}{2}}|^2 \right) + 2X + 2W. \] (71)

We will use now the third term in the left-hand side of the above expression to generate a positive term containing \(|\partial_h (\bar{E}-z)|^2\) and some other terms. The precise result is as follows.

**Lemma 2.12** Let \(h_1 = h_1(\lambda)\) be sufficiently small. Then, for \(0 < h \leq h_1(\lambda)\), we have

\[ \tau \lambda^2 \int_\Omega (\bar{E}-\theta)\phi|\partial_h (\bar{E}-z)|^2 \geq \tau \lambda^2 \int_\Omega (\bar{E}-\theta)\phi|\partial_h (\bar{E}-z)|^2 + H - \tilde{X} - J, \] (72)

with

\[ H = \frac{h^2 \tau \lambda^2}{4} \int_\Omega (E-\theta)\phi|\Delta_h (E-z)|^2, \]

\[ \tilde{X} = h^2 \int_\Omega (\bar{E}-s)O_{\lambda}(1)|\partial_h (\bar{E}-z)|^2 + \int_\Omega (E-s)hO_{\lambda}(1)|\partial_h (\bar{E}-z)|^2, \]

and

\[ J = h^4 \int_\Omega (E-s)O_{\lambda}(1)|\Delta_h (\bar{E}-z)|^2. \]
The proof follows the steps of [13, Lemma 3.11], but with some simplifications due to the 1D nature of our problem. For completeness, we present a brief proof in Appendix C.

Once we reach this point, let us choose $\lambda_1 \geq 1$ sufficiently large and let us fix $\lambda = \lambda_1$ for the rest of the proof. Notice that by doing this, some of the lower order terms in the remainders $X_{ij}$, $i, j = 1, 2$, can be absorbed by its counterparts in the left-hand side of Eq. 71. Indeed, the terms of order $\mathcal{O}(1)$ can be absorbed as soon as $\lambda_1$ is large enough.

Let us take $\epsilon_3(\lambda) = \min\{\epsilon_1(\lambda_1), \epsilon_2(\lambda_1)\}$ and $0 < h \leq h_1(\lambda_1)$, then from Lemma 2.12, Eq. 71 and the discussion above, we obtain for all $0 < \Delta \tau^2(\delta T^6)^{-1} \leq \epsilon_3(\lambda)$ and $0 < \tau h(\delta T^2)^{-1} \leq \epsilon_3(\lambda)$ that

$$\|Az\|_{L^2_T(Q)}^2 + \|Bz\|_{L^2_T(Q)}^2 + \tau \int_Q (\overline{E} - \theta)|\partial_h(\overline{E} - z)|^2$$

$$+ \tau \int_Q (\overline{E} - \theta)|\partial_h(\overline{E} - z)|^2 + \tau^3 \int_Q (\overline{E} - \theta)^3(\overline{E} - z)^2 + H$$

$$\leq C_{\lambda_1} \left( \int_Q (\overline{E} - \theta)^2|L_D q|^2 + \tau \int_{\partial_0(0, T)}^3 (\overline{E} - \theta)|\partial_h(\overline{E} - z)|^2$$

$$+ \tau^3 \int_{\partial_0(0, T)}^3 (\overline{E} - \theta)^3(\overline{E} - z)^2 \right)$$

$$+ C_{\lambda_1} \left( \int_\Omega (s^{M+1} \frac{\bar{z}}{2} + \int_\Omega (s^{\frac{1}{2}})^2(\overline{E} - z)^2 + \int_\Omega |\partial_h(\overline{E} - z)|^2 \right)$$

$$+ C_{\lambda_1} \left( \int_\Omega (s^{M+1} \frac{\bar{z}}{2} + \int_\Omega (s^{\frac{1}{2}})^2(\overline{E} - z)^2 + \int_\Omega |\partial_h(\overline{E} - z)|^2 \right) + X + W + J, \quad (73)$$

where

$$H := h^2 \int_Q (\overline{E} - s)|\Delta_h(\overline{E} - z)|^2, \quad X := X_1 + X_2,$$

and

$$X_1 = \int_Q (\overline{E} - \mu_1)(\overline{E} - z)^2 + \int_Q (\overline{E} - v_{1,b})|\partial_h(\overline{E} - z)|^2 + \int_Q (\overline{E} - v_1)|\partial_h(\overline{E} - z)|^2,$$

$$X_2 = \left\{ \left( \frac{\tau \Delta t}{\delta^3 T^4} \right)^2 + \left( \frac{\Delta t^2}{\delta^4 T^4} \right)^2 + \left( \frac{\Delta t^2}{\delta^4 T^4} \right)^2 + \left( \frac{\tau h}{\delta^3 T^2} \right)^3 \right\}$$

$$\int_Q (\overline{E} - z) + \left\{ \left( \frac{\tau^2 \Delta t}{\delta^4 T^6} \right)^2 + \left( \frac{\tau h}{\delta^3 T^2} \right)^3 \right\} \int_Q |\partial_h(\overline{E} - z)|^2,$$

with $\mu_1 := [s^2 + T s^2 \theta + T^2 s \theta^2 + s^3 (sh)^2] \mathcal{O}_{\lambda_1}(1)$, $v_{1,b} := s(sh)\mathcal{O}_{\lambda_1}(1)$, $v_1 := T(sh)^2 \theta + s(sh)^2$, and where we recall that $W$ is defined in Eq. 69.
Remark 2.13  Here, we have separated the terms in $X_1$ and $X_2$ based on the following criteria. Notice that in the definition of $\mu_1$, the first three terms do not depend on $\Delta t$ or $h$. Hence, using the parameter $\tau$, we can absorb them in the left-hand side of Eq. 73 as in a classical Carleman estimate. On the other hand, notice that all of the other terms in $X_1$ have a good power of $s$ as compared to the corresponding ones in left-hand side of Eq. 73 but they are multiplied by a factor of $(sh)$. By taking $(sh)$ small enough, we will absorb them into the left-hand side. Finally, notice that all the terms in $X_2$ have some power of $\Delta t$. In a subsequent step, we will see that we can obtain a general condition for taking $\Delta t$ small enough and control them by the similar terms in the left-hand side. Similar ideas will be used to absorb the term $W$.

Let us clean up a little bit more inequality Eq. 73 by imposing some conditions on the parameter $h$ and the product $(sh)$. First, notice that the new term $H$ can control the remainder term $J$ by considering some $0 < h_0 \leq h_1(\lambda_1)$ small enough. Indeed, for $0 < h \leq h_0$, we can drop both $J$ and $H$ in Eq. 73.

As anticipated in Remark 2.13, to absorb the term $X_1$, let us choose some $0 < \epsilon_4 \leq \epsilon_3(\lambda_1)$ and some $\tau_1 \geq 1$ sufficiently large. Thus, for $\tau \geq \tau_1(T + T^2)$ and

$$\frac{\tau h}{\delta T^2} \leq \epsilon_4 \quad \text{and} \quad \frac{\Delta t \tau^2}{\delta^4 T^6} \leq \epsilon_3,$$

we have

$$\|Az\|_{L^2_p(Q)}^2 + \|Bz\|_{L^2_p(Q)}^2 + \tau \int_Q (\bar{\epsilon} - \theta) |\bar{\partial}_h (\bar{\epsilon} - z)|^2$$

$$+ \tau \int_Q |\bar{\partial}_h (\bar{\epsilon} - z)|^2 + \tau^3 \int_Q (\bar{\epsilon} - \theta)^3 (\bar{\epsilon} - z)^2$$

$$\leq C_{\lambda_1} \left( \int_Q (\bar{\epsilon} - r)^2 |L_p q|^2 + \int_{B_0 \times (0, T)} (\bar{\epsilon} - \theta) |\bar{\partial}_h (\bar{\epsilon} - z)|^2 \right. \right.$$

$$+ \tau^3 \int_{B_0 \times (0, T)} (\bar{\epsilon} - \theta)^3 (\bar{\epsilon} - z)^2 \left. \right. \left. \right. + C_{\lambda_1} \left( \int_{\Omega} (s^{M+\frac{1}{2}} h^2 |(\partial_h z)^{M+\frac{1}{2}}|^2 + \int_{\Omega} (s^{\frac{1}{2}} h^2 |(\partial_h z)^{\frac{1}{2}}|^2 + \int_{\Omega} |(\partial_h z)^M|^2 \right) \right.$$

$$+ C_{\lambda_1} \left( \int_{\Omega} (s^{M+\frac{1}{2}} h^2 |(\partial_h z)^{M+\frac{1}{2}}|^2 + \int_{\Omega} (s^{\frac{1}{2}} h^2 |(\partial_h z)^{\frac{1}{2}}|^2 + \int_{\Omega} X_2 + W.$$

To conclude this step, notice that the first term of $W_{h, \Delta t}$ (see Eq. 67) already has the good power $s^{-1}$ and factor $(sh)^2$. Hence, we can absorb this term using Eq. 74 and obtain
\[
\|Az\|^2_{L^2_p(Q)} + \|Bz\|^2_{L^2_p(Q)} + \tau \int_Q (\overline{E^{-\theta}})|\partial_h(\overline{E^{-z}})|^2 \\
+ \tau \int_Q (\overline{E^{-\theta}})|\partial_h(\overline{E^{-z}})|^2 + \tau^3 \int_Q (\overline{E^{-\theta}})^3(\overline{E^{-z}})^2 \\
\leq C_{\lambda_1} \left( \int\int_Q (\overline{E^{-r}})|L^2_v q|^2 + \tau \int_{B_0 \times (0,T)} (\overline{E^{-\theta}})|\partial_h(\overline{E^{-z}})|^2 \\
+ \tau^3 \int_{B_0 \times (0,T)} (\overline{E^{-\theta}})^3(\overline{E^{-z}})^2 \right) \\
+ C_{\lambda_1} \left( \int_{\Omega} (s^{M+\frac{1}{2}})^2(z^{M+\frac{1}{2}})^2 + \int_{\Omega} (s^{\frac{1}{2}})^2(\overline{E^{\frac{1}{2}}})^2 + \int_{\Omega} |(\partial_h z)^{M+\frac{1}{2}}|^2 \right) \\
+ C_{\lambda_1} \left( \int_{\Omega} (s^{M+\frac{1}{2}} h^2)|(\partial_h z)^{M+\frac{1}{2}}|^2 + \int_{\Omega} (s^{\frac{1}{2}} h^2)|(\partial_h z)^{\frac{1}{2}}|^2 \right) + X_2 + W, \tag{75}
\]

where \( W \) stands for
\[
W := \int\int_Q (\overline{E^{-\gamma_1}})(\overline{D_v z})^2. \tag{76}
\]

with \( \gamma_1 := \Delta t \left( \tau T\theta^2 + \frac{\tau|x|}{\delta^2 T^2} \right) + \Delta t s^2 + \left( T^2(\tau \Delta t)^2\theta^4 + \frac{\tau^2(\Delta t)^4}{\delta^2 T^8} \right). \)

2.6 Adding a term of \( \overline{D_v} \) and \( \Delta_h \) in the left-hand side and absorbing the remaining terms

Using the equation verified by \( Az \) (see Eq. 61) and since \( r\overline{\rho} = 1 + (sh)^2 \mathcal{O}_\lambda(1) \), we have
\[
\Delta_h(\overline{E^{-z}}) = Az + \mathcal{O}_\lambda(1)(sh)^2\Delta_h(\overline{E^{-z}}) + \mathcal{O}_\lambda(1)(\overline{E^{-s}})^2 \left( (\overline{E^{-z}}) + \frac{h^2}{T} \Delta_h(\overline{E^{-z}}) \right) \\
+ \mathcal{O}_\lambda(1)\tau T(\overline{E^{-\theta}})^2(\overline{E^{-z}}),
\]

where we have also used that \( r \Delta_h \rho = s^2 \mathcal{O}_\lambda(1), \) Lemma A.3 and the estimate \( |\theta'| \leq CT\theta^2 \) for all \( t \in [0, T] \). Multiplying the above equation by \( (\overline{E^{-s}})^{-1/2} \) and taking the \( L^2_p(Q) \)-norm in both sides yield
\[
\int\int_Q (\overline{E^{-s}})^{-1}|\Delta_h(\overline{E^{-z}})|^2 \\
\leq C_{\lambda_1} \left( \int\int_Q (\overline{E^{-s}})^{-1}|Az|^2 + \int\int_Q \overline{E^{-s}} \left( s^{-1}|sh|^4 \right)|\Delta_h(\overline{E^{-z}})|^2 \right) \\
+C_{\lambda_1} \left( \int\int_Q (\overline{E^{-s}})^3(\overline{E^{-z}})^2 + \int\int_Q \tau T^2(\overline{E^{-\theta}})^3(\overline{E^{-z}})^2 \right).
\]

Notice the the term containing \( \Delta_h \) in the right-hand side of the above inequality has the good power \( s^{-1} \) and the factor \( (sh)^4 \). Thus, by recalling condition Eq. 74 we can absorb it into the right-hand side. Furthermore, increasing if necessary the value of \( \tau_1 \) such that \( \tau_1 \geq 1 \) and \( s(t) \geq 1 \) for any \( t \), we get
\[
\int\int_Q (\overline{E^{-s}})^{-1}|\Delta_h(\overline{E^{-z}})|^2 \leq C_{\lambda_1} \left( \|Az\|^2_{L^2_p(Q)} + \int\int_Q (\overline{E^{-s}})^3(\overline{E^{-z}})^2 \right). \tag{77}
\]
In a similar way, using the equation verified by $B_z$ (see Eq. 62), it is not difficult to see that
\[
\int_\Omega (E^{-s-1}) (\overline{D_t z})^2 \leq C_{\lambda_1} \left( \int_\Omega (E^{-s-1}) |B_z|^2 + \int_\Omega (E^{-s}) |\partial_h (E^{-z})|^2 + \int_\Omega (E^{-s}) (E^{-z})^2 \right),
\]
where we have used that $r \overline{\partial_h \rho} = s \mathcal{O}_\lambda (1)$ (see Proposition A.7(ii)) and $\partial_{xx} \phi = \mathcal{O}_\lambda (1)$. Using again that $s(t) \geq 1$ for any $t$, we have
\[
\int_\Omega (E^{-s-1}) (\overline{D_t z})^2 \leq C_{\lambda_1} \left( \| B_z \|^2_{L^2(Q)} + \int_\Omega (E^{-s}) |\partial_h (E^{-z})|^2 + \int_\Omega (E^{-s}) (E^{-z})^2 \right). \tag{78}
\]
By combining estimates Eqs. 75, 77 and 78, we get
\[
\int_\Omega (E^{-s-1}) \left[ (\overline{D_t z})^2 + |\Delta_h (E^{-z})|^2 \right] + \int_\Omega (E^{-s}) |\partial_h (E^{-z})|^2 + \int_\Omega (E^{-s}) (E^{-z})^2 \\
+ \int_\Omega (E^{-s}) |\partial_h (E^{-z})|^2 + \int_\Omega (E^{-s}) (E^{-z})^2 \\
\leq C_{\lambda_1} \left( \int_\Omega (E^{-r})^2 |L_{D_t q}|^2 + \int_{B_0 \times (0, T)} (E^{-s}) |\partial_h (E^{-z})|^2 \\
+ \int_{B_0 \times (0, T)} (E^{-s}) (E^{-z})^2 \right) \\
+ C_{\lambda_1} \left( \int_\Omega (s^{M+\frac{1}{2}})^2 (z^{M+\frac{1}{2}})^2 + \int_\Omega (s^{\frac{1}{2}})^2 (z^{\frac{1}{2}})^2 + \int_\Omega (|\partial_h z|^{M+\frac{1}{2}})^2 \right) \\
+ C_{\lambda_1} \left( \int_\Omega (s^{M+\frac{1}{2}} h)^2 (|\partial_h z|^{M+\frac{1}{2}})^2 + \int_\Omega (s^{\frac{1}{2}} h)^2 (|\partial_h z|^{\frac{1}{2}})^2 \right) + X_2 + W. \tag{79}
\]
With this new inequality at hand, the next result gives us conditions on the parameter $\Delta t$ such that the terms $X_2$ and $W$ can be absorbed into the left-hand side. The result is as follows.

**Lemma 2.14** For any $\tau \geq 1$, there exists $\epsilon_5 = \epsilon_5(\lambda_1)$ such that for
\[
0 < \frac{\tau^4 \Delta t}{\delta^4 T^6} \leq \epsilon_5,
\]
the following estimate holds
\[
X_2 + W \leq \epsilon_5 \left( \tau^3 \int_\Omega (E^{-\theta})^3 (E^{-z})^2 + \tau^{-1} \int_\Omega (E^{-\theta})^{-1} (\overline{D_t z})^2 \right).
\]

The proof of this result is similar to that in [22, Lemma 2.3]. For completeness, we sketch it briefly in Appendix C.
Using Lemma 2.14, we take $\epsilon_5 = 1/2C_{\lambda_1}$, where $C_{\lambda_1} > 0$ is the constant appearing in Eq. 79 and set $\epsilon_6(\lambda) = \min\{\epsilon_3(\lambda_1), \epsilon_4(\lambda_1), \epsilon_5(\lambda_1)\}$. Whence, for $\tau \geq \tau_1(T + T^2), h \leq h_0$, and

$$\frac{\tau h}{\delta T^2} \leq \epsilon_6 \quad \text{and} \quad \frac{\tau^4 \Delta t}{\delta^4 T^6} \leq \epsilon_6,$$

the following estimate holds

$$\int_0^1 \int_Q (E-s)^{-1} \left[ (D^T z)^2 + |\Delta_h (E-s)|^2 \right] + \int_0^1 \int_Q (E-s) |\partial_h (E-s)|^2$$

$$+ \int_0^1 \int_Q (E-s) |\partial_h^2 (E-s)|^2 + \int_0^1 \int_Q (E-s)^3 |\partial_h (E-s)|^2$$

$$\leq C_{\lambda_1} \left( \int_0^1 \int_Q (E-s)^2 |L^\gamma q|^2 + \tau \int_{B_0 \times (0, T)} (E-s) |\partial_h (E-s)|^2 \right)$$

$$+ \tau^3 \int_{B_0 \times (0, T)} (E-s)^3 |\partial_h (E-s)|^2 + BT,$$

where we have defined

$$BT := \int_0^1 (s^{M+\frac{1}{2}})^2 (z^{M+\frac{1}{2}})^2 + \int_0^1 (s^{\frac{1}{2}})^2 (z^{\frac{1}{2}})^2 + \int_0^1 (s^{1})^2 (z^1)^2$$

$$+ \int_0^1 (s^{M+\frac{1}{2}} h)^2 (\partial_h z)^{(M+\frac{1}{2})}^2 + \int_0^1 (s^{\frac{1}{2}} h)^2 (\partial_h z)^{\frac{1}{2}}^2.$$ (82)

As in other discrete Carleman works, the terms collected in $BT$ in the above equation cannot be absorbed, but only estimated. The result reads as follows.

**Lemma 2.15** Assume that Eq. 80 holds. Then, there exists some $C > 0$ uniform with respect to $h$ and $\Delta t$ such that

$$BT \leq C(1 + \epsilon_6^2)h^{-2} \left( \int_\Omega (z^{\frac{1}{2}})^2 + \int_\Omega (z^{M+\frac{1}{2}})^2 \right).$$ (83)

**Proof** Under the hypothesis of the lemma and recalling that $\delta \leq 1/2$, we deduce that $\Delta t \leq \delta T/2$, therefore

$$\max_{t \in [0, T + \Delta t]} \theta(t) \leq 2/(\delta T^2).$$ (84)

With this estimate, we readily see that the first term in $BT$ can be bounded as

$$\int_\Omega (s^{M+\frac{1}{2}})^2 (z^{M+\frac{1}{2}})^2 = \int_\Omega \left( \tau \theta^{M+\frac{1}{2}} \right)^2 (z^{M+\frac{1}{2}})^2$$

$$\leq 4h^{-2} \int_\Omega \left( \frac{\tau h}{\delta T^2} \right)^2 (z^{M+\frac{1}{2}})^2 \leq 4\epsilon_6^2 h^{-2} \int_\Omega (z^{M+\frac{1}{2}})^2,$$

where we have used the first condition in Eq. 80. The same is true for the second term in $BT$. For the third term, we notice that $(\partial_h z)^2 \leq Ch^{-2} \{ (s^+ z)^2 + (s^- z)^2 \}$. Thus,

$$\int_\Omega |(\partial_h z)^{M+\frac{1}{2}}|^2 \leq Ch^{-2} \int_\Omega (z^{M+\frac{1}{2}})^2.$$
For the fourth term, we argue as in the previous cases to deduce
\[
\int_{\Omega} (s^{M+\frac{1}{2}}h)^2 |(\partial_h z)^{M+\frac{1}{2}}|^2 \leq C h^{-2} \left( \frac{\tau h}{\delta T^2} \right)^2 \int_{\Omega} (z^{M+\frac{1}{2}})^2 \leq C \epsilon_6^{-2} h^{-2} \int_{\Omega} (z^{M+\frac{1}{2}})^2.
\]

Note that the same is true for the last term in $BT$. Collecting the above estimates gives the desired result. \hfill \square

Combining estimates Eqs. 81 and 83 gives
\[
\begin{align*}
&\int\int_{Q} (E^{-s})^{-1} \left[ (D_{t} z)^2 + |\Delta_h (E^{-z})|^2 \right] + \int\int_{Q} (E^{-s}) |\partial_h (E^{-z})|^2 \\
&+ \int\int_{Q} (E^{-s}) |\partial_h (E^{-z})|^2 + \int\int_{Q} (E^{-s})^3 |(E^{-z})|^2 \\
&\leq C_{\lambda 1} \left( \int\int_{Q} (E^{-r})^2 |L_D q|^2 + \int\int_{B_0 \times (0,T)} (E^{-s}) |\partial_h (E^{-z})|^2 \\
&+ \int\int_{B_0 \times (0,T)} (E^{-s})^3 |(E^{-z})|^2 \right)
\end{align*}
\]
\begin{equation}
+ C_{\lambda 1} h^{-2} \left( \int_{\Omega} (E^{-z})^2 + \int_{\Omega} (z^{M+\frac{1}{2}})^2 \right)
\tag{85}
\end{equation}

for all $\tau \geq \tau_1 (T + T^2)$, $h \leq h_0$, and
\[
\frac{\tau h}{\delta T^2} \leq \epsilon_6 \quad \text{and} \quad \frac{T^4 \Delta t}{\delta^4 T^6} \leq \epsilon_6.
\tag{86}
\]

2.7 Returning to the original variable and conclusion

To conclude our proof, we will remove the local term containing $\partial_h$ in the right-hand side of Eq. 85 and then comeback to the original variable. We argue slightly different to [13] and [22] since the time and space variables are both discrete, but the overall result is the same.

We present the following.

**Lemma 2.16** For any $\gamma > 0$, there exists $C > 0$ uniform with respect to $h$ and $\Delta t$ such that
\[
\begin{align*}
&\int\int_{Q \setminus B_0} (E^{-s}) |\partial_h (E^{-z})|^2 \leq C \left( 1 + \frac{1}{\gamma} \right) \int\int_{B \times (0,T)} (E^{-s})^3 |(E^{-z})|^2 + C \int\int_{Q} |\partial_h (E^{-z})|^2 \\
&+ \int\int_{Q} (E^{-s}) |(E^{-z})|^2 + \gamma \int\int_{Q} (E^{-s})^{-1} |\Delta_h (E^{-z})|^2 \\
&+ \int\int_{Q} E^{-s} |(s^{-1}[sh])^2| \Delta_h \tau z|^2.
\end{align*}
\tag{87}
\]

The proof of this result can be found in Appendix C. This result mimics the classical procedure used in the continuous setting and tells that we can remove the local...
term of \( \partial_h(E-z) \) by paying the cost of increasing a little bit the observation set of the local term of \( (E-z) \). Notice also that some lower order terms appear; nonetheless, all of them can be absorbed in the left-hand side.

With estimate Eq. 87 in hand, we can choose \( \gamma = \frac{1}{2\varepsilon_0} \) with \( C_{\lambda_1} \) the constant appearing in Eq. 85 and set \( \tau_2 \geq \tau_1 \geq 1 \) sufficiently large to obtain

\[
\int_Q (E-s)^{-1} \left[ (D_T z)^2 + |\Delta_h(E-z)|^2 \right] + \int_Q (E-s)|\partial_h(E-z)|^2 \\
+ \int_Q (E-s)|\partial_h(E-z)|^2 + \int_Q (E-s)^3(E-z)^2 \\
\leq C_{\lambda_1} \left( \int_Q (E-r)^2 |L_D q|^2 + \int_{B \times (0,T)} (E-s)^3(E-z)^2 \right) \\
+ C_{\lambda_1} h^{-2} \left( \int_\Omega (z^{\frac{3}{2}})^2 + \int_\Omega (\tilde{z}^{M+\frac{1}{2}})^2 \right) 
\] (88)

for all \( \tau \geq \tau_2(T+T^2) \) and verifying Eq. 86. In this step, we can decrease if necessary the value of \( \varepsilon_0 \) in Eq. 86 to absorb the last term in Eq. 87 since we recall that \( (sh) \leq \tau h(\delta T^2)^{-1} \). For convenience, we still call it \( \varepsilon_0 \).

Now, we will come back to the original variable. We recall that we have set the change of variables \( z = r q \). To give appropriate bounds on the gradient terms in Eq. 88, we proceed as follows. A direct computation yields

\[
r \partial_h q = r \partial_h (\rho z) = r \tilde{\rho} \partial_h z + \tilde{z} r \partial_h \rho. \tag{89}
\]

Multiplying by \( s^{1/2} \) and taking the \( L^2(\Omega) \)-norm (since \( z \) and \( q \) are naturally defined on the dual time-grid) gives

\[
\int_Q r^2 s |\partial_h q|^2 \leq C_{\lambda_1} \left( \int_Q s |\partial_h z|^2 + \int_Q s^1 |\tilde{z}|^2 (r \partial_h \rho)^2 \right),
\]

where we have used that \( r \tilde{\rho} = O_{\lambda}(1) \). From Proposition A.7(ii), we have \( r \partial_h \rho = s O_{\lambda}(1) \); hence,

\[
\int_Q r^2 s |\partial_h q|^2 \leq C_{\lambda_1} \left( \int_Q s |\partial_h z|^2 + \int_Q s^3 |\tilde{z}|^2 \right) \\
\leq C_{\lambda_1} \left( \int_Q s |\partial_h z|^2 + \int_Q s^3 |\tilde{z}|^2 \right) \\
= C_{\lambda_1} \left( \int_Q s |\partial_h z|^2 + \int_Q s^3 | z|^2 \right),
\]

where we have used convexity in the second line and formula Eq. 147 together with zero boundary conditions in the third line. Shifting the time integral (see Eq. 152) yields

\[
\int_Q (E-s)|\partial_h (E-q)|^2 \leq C_{\lambda_1} \left( \int_Q (E-s)|\partial_h (E-z)|^2 + \int_Q (E-s)^3(E-z)^2 \right). \tag{90}
\]
The same ideas can be used to obtain a term containing \( \overline{\partial_h \bar{q}} \). Indeed, with Lemma A.2 and A.3, a straightforward computation gives

\[
\overline{\partial_h \bar{q}} = \overline{\partial_h (\rho \bar{z})} = \overline{\rho \partial_h \bar{z} + \bar{z} \partial_h \rho} \\
= \overline{\rho} \overline{\partial_h \bar{z}} + \overline{\bar{z}} \overline{\partial_h \rho} + \frac{h^2}{4} \left( \Delta_{\bar{h}} \overline{\partial_h \bar{z}} + \overline{\partial_h (\bar{z})} \Delta_{\bar{h}} \rho \right) \\
= \overline{\rho} \overline{\partial_h \bar{z}} + \overline{z} \overline{\partial_h \rho} + \frac{h^2}{4} \left( 2 \Delta_{\bar{h}} \overline{\partial_h \bar{z}} + \overline{\partial_h \bar{z}} \Delta_{\bar{h}} \rho \right),
\]

where we have used the useful identity \( \overline{\partial_h \bar{p}} = \overline{\partial_h p} \) for \( p \in \mathbb{R}^m \). Arguing as above, we multiply by \( r s^{1/2} \) the above identity and take the \( L^2(Q) \)-norm. Proposition A.7 provides the useful estimates \( r \overline{\rho} = O_{\lambda_1}(1) \), \( r \overline{\partial_h \rho} = sO_{\lambda_1}(1) \) and \( r \Delta_{\bar{h}} \rho = s^2 O_{\lambda_1}(1) \). Thus

\[
\iint_Q r^2 s |\overline{\partial_h \bar{q}}|^2 \leq C_{\lambda_1} \left( \iint_Q s^3 |\bar{z}|^2 + \iint_Q s |\overline{\partial_h \bar{z}}|^2 \right) \\
+ C_{\lambda_1} \left( \iint_Q s^{-1} (sh)^4 |\Delta_{\bar{h}} \overline{\partial_h \bar{z}}|^2 + \iint_Q s(sh^4) |\overline{\partial_h \bar{z}}|^2 \right).
\]

Note that the last two terms have the corresponding good power of \( s \) and a small factor \( (sh) \). By shifting the time integral (see Eq. 152) and recalling condition Eq. 86, we get

\[
\iint_Q \bar{E}^{-} (r^2 s) |\overline{\partial_h (\bar{E} - \bar{q})}|^2 \\
\leq C_{\lambda_1} \left( \iint_Q (\bar{E} - s)^3 (\bar{E} - z)^2 + \iint_Q (\bar{E} - s) |\overline{\partial_h (\bar{E} - z)}|^2 \right) \\
+ C_{\lambda_1} \epsilon_6^4 \left( \iint_Q (\bar{E} - s)^{-1} |\Delta_{\bar{h}} (\bar{E} - z)|^2 + \iint_Q (\bar{E} - s) |\overline{\partial_h (\bar{E} - z)}|^2 \right). \tag{91}
\]

Using similar ideas, we can give an estimate for a term containing \( \Delta_{\bar{h}} (\bar{E} - \bar{q}) \). To do so, we recall identity Eq. 55 and see that

\[
r s^{-1/2} \Delta_{\bar{h}} q = s^{-1/2} r \Delta_{\bar{h}} \rho \left( z + \frac{h^2}{4} \Delta_{\bar{h}} \rho \right) + s^{-1/2} r \overline{\partial_h \rho} \Delta_{\bar{h}} \rho + 2s^{-1/2} r \overline{\partial_h \rho} \overline{\partial_h \rho}.
\]

Arguing as we did above, we readily obtain

\[
\iint_Q \bar{E}^{-} (r^2 s^{-1}) |\Delta_{\bar{h}} (\bar{E} - \bar{q})|^2 \\
\leq C_{\lambda_1} \left( \iint_Q (\bar{E} - s)^3 (\bar{E} - z)^2 + \iint_Q (\bar{E} - s)^{-1} |\Delta_{\bar{h}} (\bar{E} - z)|^2 + \iint_Q (\bar{E} - s) |\overline{\partial_h (\bar{E} - z)}|^2 \right) \\
+ C_{\lambda_1} \epsilon_6^4 \iint_Q (\bar{E} - s)^{-1} |\Delta_{\bar{h}} (\bar{E} - z)|^2. \tag{92}
\]
Using estimates Eq. 90–Eq. 92 in Eq. 88 and decreasing (if necessary) the value of $\epsilon_6$, we obtain

$$\int_Q \bar{\xi}^{-1} (r^2s^{-1})|\Delta_h (\bar{\xi} - q)|^2 + \int_Q \bar{\xi}^{-1} (r^2s)|\bar{\partial}_h (\bar{\xi} - q)|^2$$

$$+ \int_Q \bar{\xi}^{-1} (r^2s)|\bar{\partial}_h (\bar{\xi} - q)|^2 + \int_Q \bar{\xi}^{-1} (r^2s^3) (\bar{\xi} - q)^2$$

$$\leq C_{\lambda_1} \left( \int_Q (\bar{\xi}^{-1} r)^2 |L_{Dq}|^2 + \int_{B \times (0,T)} \bar{\xi}^{-1} (r^2s^3) (\bar{\xi} - q)^2 \right)$$

$$+ C_{\lambda_1} h^{-2} \left( \int_{\Omega} |(\bar{e}^{q\bar{\psi}} q)^{\frac{1}{2}}|^2 + \int_{\Omega} |(\bar{e}^{q\bar{\psi}} q)^{M+\frac{1}{2}}|^2 \right), \quad (93)$$

where we have used that $\tau = r q$ to change variables in the terms containing $(\bar{\xi} - z)$. Notice also that we have dropped the positive term containing $D_{iz}$.

To add a term of $D_{iz} q$, we simply use the equation verified by $q$. Indeed, we have

$$\int_Q \bar{\xi}^{-1} (r^2s^{-1})|D_{iz} q|^2$$

$$\leq 2 \int_Q \bar{\xi}^{-1} (r^2s^{-1}) |L_{Dq}|^2 + 2 \int_Q \bar{\xi}^{-1} (r^2s^{-1}) |\Delta_h (\bar{\xi} - q)|^2$$

$$\leq 2 \int_Q (\bar{\xi}^{-1} r)^2 |L_{Dq}|^2 + 2 \int_Q \bar{\xi}^{-1} (r^2s^{-1}) |\Delta_h (\bar{\xi} - q)|^2, \quad (94)$$

for all $\tau \geq \tau_2 (T + T^2)$, $h \leq h_0$, and

$$\frac{\tau h}{\delta T^3} \leq \epsilon_6 \quad \text{and} \quad \frac{\tau^4 \Delta t}{\delta^4 T^6} \leq \epsilon_6.$$
3 \( \phi(h) \)-null controllability

In this section, we use the Carleman estimate Eq. 40 to deduce control properties for linear and semilinear fully discrete parabolic systems.

3.1 A fully discrete observability inequality

Let us consider the following fully discrete problem with potential \( a \in L^\infty_{loc}(Q) \)

\[
\begin{aligned}
(D_t y)^{n+\frac{1}{2}} - \Delta_h (t^+ y)^{n+\frac{1}{2}} + (t^+ ay)^{n+\frac{1}{2}} = 1_{\omega} v^{n+\frac{1}{2}} \quad n \in [0, M - 1], \\
y^{0} = g.
\end{aligned}
\] (96)

To achieve a \( \phi(h) \)-controllability result for Eq. 96, we begin by proving a relaxed observability estimate for the solutions associated to the adjoint system given by

\[
\begin{aligned}
-(D_{\tau} q)^{n} - \Delta_h (\bar{e}^- q)^n + a^n (\bar{e}^- q)^n = 0 \quad n \in [1, M], \\
q^{M+\frac{1}{2}} = q_T.
\end{aligned}
\] (97)

We have the following result.

**Proposition 3.1** For any \( \vartheta \geq 1 \), there exist positive constants \( h_0, C_0, C_1, C_2, \) and \( C_T \), such that for all \( T \in (0, 1) \), all potentials \( a \in L^\infty_{loc}(Q) \), under the conditions

\[
h \leq \min\{h_0, h_1\} \quad \text{with} \quad h_1 = C_0 \left( 1 + \frac{1}{T} + \|a\|^2_{L^\infty_{loc}(Q)} \right)^{-\max\{1, 4/\vartheta\}}
\] (98)

and

\[
\Delta t \leq \min\{T^{-2}h^\vartheta, (4 \|a\|_{L^\infty_{loc}(Q)})^{-1}\},
\] (99)

any solution to Eq. 97 with \( q_T \in \mathbb{R}^{M} \) satisfies

\[
|q^{\frac{1}{2}}|_{L^2(\Omega)} \leq C_{obs} \left( \iint_{\omega \times (0,T)} |q|^2 + e^{-\frac{C_2}{h^{\min\{3/\vartheta, 1\}}}} |q_T|^2_{L^2(\Omega)} \right)^{1/2},
\] (100)

where

\[
C_{obs} = e^{C_1(1 + \frac{1}{T} + \|a\|^2_{L^\infty_{loc}(Q)} + T\|a\|_{L^\infty_{loc}(Q)})}.
\] (101)

The proof of this result follows as close as possible the continuous case (see, e.g., [27]) in a first stage. Nonetheless, in a second part, a careful connection on the different discrete parameters (i.e., \( \Delta t, h, \) and \( \vartheta \)) should be done to obtain the uniform constants involved in Eq. 100 (see Section 4.2 for further discussion on this). Similar estimates in the discrete setting have been obtained in [13, Proposition 4.1] for the space-discrete case and [22, Proposition 3.1] in the time-discrete case.

**Proof of Proposition 3.1** For clarity, we have divided the proof in two steps. In what follows, \( C \) denotes a positive constant uniform with respect to \( h, \Delta t \) and \( \vartheta \) which may change from line to line.
Step 1. Cleaning up the Carleman estimate. Applying Eq. 40 to the solutions Eq. 97 with $B = \omega$, we readily obtain
\[
\tau^3 \iint_Q \mathcal{E} - (e^{2\tau \theta \varphi})(\mathcal{E} - q)^2 \\
\leq C \left( \iint_Q \mathcal{E} - (e^{2\tau \theta \varphi})|a(\mathcal{E} - q)|^2 + \tau^3 \iint_{\omega \times (0,T)} \mathcal{E} - (e^{2\tau \theta \varphi})(\mathcal{E} - q)^2 \right) \\
+ Ch^{-2} \left( \int_{\Omega} \left| (e^{\theta \varphi} q)^{\frac{1}{2}} \right|^2 + \int_{\Omega} \left| (e^{\theta \varphi} q)^{M+\frac{1}{2}} \right|^2 \right) 
\]
for all $\tau \geq \tau_0(T + T^2)$, $0 < h \leq h_0$, $\Delta t > 0$ and $0 < \delta \leq 1/2$ satisfying the condition
\[
\frac{\tau^4 \Delta t}{\delta^4 T^6} \leq \epsilon_0 \quad \text{and} \quad \frac{\tau h}{\delta T^2} \leq \epsilon_0. \tag{102}
\]

The first term in the right-hand side can be controlled by the term on the left-hand side by choosing $\tau$ large enough. Indeed, using that $a \in L^\infty_p (Q)$ and $\theta^{-1} \leq CT^2$ it is standard to see that by choosing
\[
\tau \geq CT^2 \|a\|_{L^\infty_p(Q)}^{2/3} \tag{103}
\]
we have
\[
\tau^3 \iint_Q \mathcal{E} - (e^{2\tau \theta \varphi})(\mathcal{E} - q)^2 \\
\leq C \left( \tau^3 \iint_{Q_\omega} \mathcal{E} - (e^{2\tau \theta \varphi})(\mathcal{E} - q)^2 \right) \\
+ Ch^{-2} \left( \int_{\Omega} \left| (e^{\theta \varphi} q)^{\frac{1}{2}} \right|^2 + \int_{\Omega} \left| (e^{\theta \varphi} q)^{M+\frac{1}{2}} \right|^2 \right). \tag{104}
\]

Notice that we can combine Eq. 103 with our initial hypothesis for $\tau$ by choosing some $\tau_1 \geq \tau_0$ large enough and setting
\[
\tau \geq \tau_1 \left( T + T^2 + T^2 \|a\|_{L^\infty_p(Q)}^{2/3} \right). 
\]

From Eq. 97, we see that $q^{n-\frac{1}{2}}$ solves the equation
\[
q^{n-\frac{1}{2}} - q^{n+\frac{1}{2}} - \Delta t\Delta_h q^{n-\frac{1}{2}} + \Delta t a^n q^{n-\frac{1}{2}} = 0, \quad n \in \{1, M\}. \tag{105}
\]

From this expression, we can take the $L^2$-inner product on $\mathbb{R}^{2n}$ (see Eq. 19) with $q^{n-\frac{1}{2}}$ and use the identity $(a - b)a = \frac{1}{2} a^2 - \frac{1}{2} b^2 + \frac{1}{2} (a - b)^2$ to deduce
\[
\frac{1}{2} \left( |q^{n-\frac{1}{2}}|^2_{L^2(\Omega)} - |q^{n+\frac{1}{2}}|^2_{L^2(\Omega)} \right) + \frac{1}{2} |q^{n-\frac{1}{2}} - q^{n+\frac{1}{2}}|^2_{L^2(\Omega)} \\
+ \Delta t \left( -\Delta_h q^{n-\frac{1}{2}}, q^{n-\frac{1}{2}} \right)_{L^2(\Omega)} \\
= -\Delta t \left( a^n q^{n-\frac{1}{2}}, q^{n-\frac{1}{2}} \right)_{L^2(\Omega)} \\
\leq \Delta t |a^n|_{L^\infty(\Omega)} |q^{n-\frac{1}{2}}|^2_{L^2(\Omega)} \leq \Delta t \|a\|_{L^\infty_p(Q)} |q^{n-\frac{1}{2}}|^2_{L^2(\Omega)}. \tag{106}
\]
Using the following uniform discrete Poincaré-type inequality (which is valid even for nonuniform meshes)

\[ |y|_{L^2(\Omega)}^2 \leq C (-\Delta y, y)_{L^2(\Omega)}, \quad \forall y \in \mathbb{R}^{|\Omega|, \partial\Omega}, \quad y = 0 \text{ on } \partial\Omega, \]

we obtain from Eq. 106, as soon as \( 2\Delta t \|a\|_{L^\infty_{t}(\Omega)} < 1 \), that

\[ |q^n + \frac{1}{2}|_{L^2(\Omega)} \leq \frac{1}{2 - 2\Delta t \|a\|_{L^\infty_{t}(\Omega)}} |q^{n+\frac{1}{2}}|_{L^2(\Omega)}, \quad n \in \llbracket 1, M \rrbracket. \quad (107) \]

From estimate Eq. 107 and the useful inequality \( e^{2x} > 1/(1-x) \) for \( 0 < x < 1/2 \), we get

\[ |q^{\frac{1}{2}}|_{L^2(\Omega)}^2 \leq e^{CT\|a\|_{L^\infty_{t}(\Omega)}} |q^{n+\frac{1}{2}}|_{L^2(\Omega)}^2, \quad n \in \llbracket 1, M \rrbracket, \quad (108) \]

for some \( C > 0 \) uniform with respect to \( \Delta t \) provided

\[ \Delta t \|a\|_{L^\infty_{t}(\Omega)} < 1/4. \quad (109) \]

Now, we turn our attention to Eq. 104. Shifting the integral in time with formula Eq. 152 and since we are adding positive terms, we see that the term in the left-hand side can be bounded as

\[ \tau^3 \iint_{Q} e^{2\tau \theta \varphi} \theta^3 q^2 \geq \sum_{n \in \llbracket M/4, 3M/4 \rrbracket} \Delta t \tau^3 \int_{\Omega} (e^{2\tau \theta \varphi} n + \frac{1}{2}) (\theta^3 n + \frac{1}{2}) |q^{n+\frac{1}{2}}|^2. \]

Recalling that \( \varphi \) is negative and independent of time, we deduce that \( (e^{2\tau \theta \varphi} n + \frac{1}{2}) \geq e^{\frac{2\tau \theta \varphi}{3\tau^2}}, \quad n \in \llbracket M/4, 3M/4 \rrbracket \), where \( K_0 := \max_{x \in \Omega} (-\varphi(x)) \). Moreover, since \( \theta \geq 1/T^2 \) for all \( t \in [0, T] \), we get

\[ \tau^3 \iint_{Q} e^{2\tau \theta \varphi} \theta^3 q^2 \geq \sum_{n \in \llbracket M/4, 3M/4 \rrbracket} \Delta t \tau^3 e^{\frac{2\tau \theta \varphi}{3\tau^2} - CT \|a\|_{L^\infty_{t}(\Omega)}} T^{-6} |q^{n+\frac{1}{2}}|_{L^2(\Omega)}^2. \]

Combining the above estimate with Eq. 108 and adding up, we get

\[ \tau^3 \iint_{Q} e^{2\tau \theta \varphi} \theta^3 q^2 \geq (T/2 - \Delta t) \tau^3 e^{-\frac{C\tau \theta \varphi}{72} - CT \|a\|_{L^\infty_{t}(\Omega)}} T^{-6} |q^{\frac{1}{2}}|_{L^2(\Omega)}^2 \geq CT e^{-\frac{C\tau \theta \varphi}{72} - CT \|a\|_{L^\infty_{t}(\Omega)}} |q^{\frac{1}{2}}|_{L^2(\Omega)}^2, \quad (110) \]

for some \( C > 0 \) uniform with respect to \( \Delta t \).

From Eq. 84 and estimate Eq. 108, we have that

\[ \int_{\Omega} \left| (e^{\tau \theta \varphi} q)^{\frac{1}{2}} \right|^2 + \int_{\Omega} \left| (e^{\tau \theta \varphi} q)^{M+\frac{1}{2}} \right|^2 \leq e^{-\frac{4k_0 \tau}{\tau^2} + CT \|a\|_{L^\infty_{t}(\Omega)}} |q^{M+\frac{1}{2}}|_{L^2(\Omega)}^2, \quad (111) \]

where we have denoted \( k_0 := \min_{x \in \Omega} (-\varphi(x)) \). On the other hand, observe that the following estimate holds

\[ e^{2\tau \theta \varphi} s^3 \leq \tau^3 2^6 T^{-6} \exp \left( -\frac{2^6 k_0 \tau}{T^2} \right) \leq C, \quad \forall (t, x) \in (0, T) \times \Omega, \]

\[ \text{Springer} \]
uniformly for $\tau \geq \frac{3}{8k_0} T^2$. This, together with estimates Eq. 110–Eq. 111, can be used in Eq. 104 to obtain

$$|q|^{1/2}_{L^2(\Omega)} \leq CT^{-1} e^{C_{\tau} T} + CT^2 \|a\|_{L^\infty(\Omega)} \int_{\omega \times (0,T)} |q|^2$$

$$+ Ch^{-2} e^{C_{\tau} T} - CT^2 \|a\|_{L^\infty(\Omega)} |q|^{M + \frac{1}{2}}_{L^2(\Omega)}.$$  

for any $\tau \geq \tau_2(T + T^2 + T^2 \|a\|_{L^\infty(\Omega)})$ with $\tau_2 = \max(\tau_1, 3/8k_0)$. Observe that for $0 < \delta \leq \delta_1 < 1/2$ small enough, we obtain

$$|q|^{1/2}_{L^2(\Omega)} \leq CT^{-1} e^{C_{\tau} T} + CT^2 \|a\|_{L^\infty(\Omega)} \int_{\omega \times (0,T)} |q|^2$$

$$+ Ch^{-2} e^{C_{\tau} T} - CT^2 \|a\|_{L^\infty(\Omega)} |q|^{M + \frac{1}{2}}_{L^2(\Omega)}.$$  

(112)

**Step 2. Connection of the discrete parameters.** To conclude the proof, we will connect the parameters $h$, $\Delta t$, and $\delta$. We recall that the following conditions should be met

$$\frac{\tau h}{T^2} \leq \epsilon_0 \quad \text{and} \quad \frac{\tau^2 \Delta t}{T^6} \leq \epsilon_0$$  

(113)

along with $h \leq h_0$, $\delta \leq \delta_1$ and $\Delta t \|a\|_{L^\infty(\Omega)} < 1/4$. Let $\delta \geq 1$. We fix $\tau = \tau_2(T + T^2 + T^2 \|a\|_{L^\infty(\Omega)})$ and define

$$h_1 := \epsilon_0 \left\{ \frac{\delta_1}{\tau_2} \left( 1 + \frac{1}{T} + \|a\|_{L^\infty(\Omega)}^{2/3} \right)^{-1} \right\}^{M_\delta}$$

and

$$\Delta t := h_1^\delta,$$  

(114)

where $M_\delta := \max\{1, 4/\delta\}$. Notice that such definitions imply that

$$\frac{h_1 \tau^{M_\delta}}{\delta_1 T^2 M_\delta} = \epsilon_0 \quad \text{and} \quad \frac{\Delta t \tau^{M_\delta}}{\delta_1 M_\delta T^2} = \epsilon_0^\delta.$$  

(115)

We choose $h \leq \min\{h_0, h_1\}$, so that $h/h_1 \leq 1$, and set

$$\delta = \left( \frac{h}{h_1} \right)^{m_\delta} \delta_1 \leq \delta_1,$$  

(116)

where $m_\delta := \min\{\delta/4, 1\}$. We note that $m_\delta M_\delta = 1$ for all $\delta \geq 1$; thus, combining Eqs. 116 and 115, we get

$$\epsilon_0 = \frac{\tau^{M_\delta} h}{\delta M_\delta T^2 M_\delta} \geq \frac{\tau h}{\delta T^2}$$  

(117)

since $M_\delta \geq 1$ for any $\delta \geq 1$ and $T$, $\delta$ are small and $\tau$ is large. This verifies the first condition of Eq. 113.

On the other hand, from Eqs. 114–116, we have

$$\epsilon_0^\delta = \frac{\Delta t \tau^{M_\delta}}{\delta_1 M_\delta T^2 M_\delta} = \frac{\tau^{M_\delta} h}{\delta M_\delta T^2}.$$
where we have used again that $m_\theta M_\theta = 1$ for $\theta \geq 1$. So, by choosing the discretization parameter $\Delta t \leq \min\{T^{-2} h^\theta, (4 \|a\|_{L^\infty(\Omega)}^{-1})\^{-1}\}$, we obtain from the above expression

$$
\frac{\Delta t \tau^4}{\delta^4 T^6} \leq \epsilon_0^\theta \leq \epsilon_0
$$

since $M_\theta \theta \geq 4$ for any $\theta \geq 1$. Note that the particular selection of $\Delta t$ implies the stability condition Eq. 109.

Thanks to Eq. 117 and the definitions of $m_\theta$, $M_\theta$, we have that $\frac{\tau}{\delta T^2} = (\frac{\epsilon_0}{\delta})^{1/M_\theta} = (\frac{\epsilon_0}{\delta})^{m_\theta}$ for $\theta \geq 1$, so from Eq. 112

$$
|q|^2 \leq C T^{-1} e^{C(1 + \frac{1}{2} + \|a\|_{L^\infty(\Omega)} + T\|a\|_{L^\infty(\Omega)})} \int_{\Omega \times (0, T)} |q|^2
$$

$$
+ Ch^{-2} e^{-C(\frac{m_\theta}{\theta})} + CT\|a\|_{L^\infty(\Omega)} |q|^2 L^2(\Omega),
$$

which yields

$$
|q|^2 \leq e^{C_1(1 + \frac{1}{2} + \|a\|_{L^\infty(\Omega)} + T\|a\|_{L^\infty(\Omega)})} \left( \int_{\Omega \times (0, T)} |q|^2 + e^{\frac{c_2}{\min(\theta \Delta x, 4)}} \int_{\Omega \times (0, T)} |q|^2 \right).
$$

The proof finishes by recalling the initial condition of system Eq. 97.

### 3.2 The linear case: proof of Theorem 1.10

With the result of Proposition 3.1, we are in position to prove our main $\phi(h)$-null controllability result. The proof follows the classical Hilbert Uniqueness Method (HUM) introduced in [31]. See also [28] and [18].

Let us fix $T \in (0, 1)$, $\theta \geq 1$ and choose $h > 0$ small enough according to Proposition 3.1. Decreasing if necessary the value of $h$, we can always assume that $\Delta t \leq T^{-2} h^\theta$. Under these conditions, Proposition 3.1 gives the relaxed observability inequality

$$
|q|^2 \leq C_{obs} \left( \int_{\Omega \times (0, T)} |q|^2 + \phi(h) |q T|^2 L^2(\Omega) \right)^{1/2},
$$

valid for all the solutions to Eq. 97, with $\phi(h) = e^{-\frac{c_2}{\min(\theta \Delta x, 4)}}$.

We introduce the fully discrete penalized functional

$$
J_{h, \Delta t}(q_T) = \frac{1}{2} \int_{\Omega \times (0, T)} |q|^2 + \phi(h) |q T|^2 L^2(\Omega) + (g, q^2) L^2(\Omega), \quad \forall q_T \in \mathbb{R}^m
$$

defined for the solutions to Eq. 97 and where we recall that $g \in \mathbb{R}^m$ is the initial datum of Eq. 96. It is not difficult to see that $J$ is continuous and strictly convex.
Moreover, from Cauchy-Schwarz and Young inequalities, we have

\[
J_{h,\Delta t}(q_T) \geq \frac{1}{2} \int_{\omega \times (0,T)} |q|^2 + \frac{\phi(h)}{2} |q_T|^2_{L^2(\Omega)} - \frac{1}{4C_{obs}^2} |q|^{1/2}_{L^2(\Omega)} - C_{obs}^2 |y_0|^2_{L^2(\Omega)} \\
\geq \frac{1}{4} \int_{\omega \times (0,T)} |q|^2 + \frac{\phi(h)}{4} |q_T|^2_{L^2(\Omega)} - C_{obs}^2 |y_0|^2_{L^2(\Omega)},
\]

where we have used inequality Eq. 118 in the second line. This allows us to conclude that \( J \) is coercive and thus the existence of a unique minimizer, that we denote by \( \hat{q}_T \), is guaranteed.

We consider \( \hat{q} \) the solution to Eq. 97 with initial datum \( \hat{q}_T \). From the Euler-Lagrange equation associated to the minimization of Eq. 119, we have

\[
\int_{\omega \times (0,T)} \hat{q} + \phi(h) \hat{q}_T \cdot q_T L^2(\Omega) = -(g, q^\frac{1}{2})_{L^2(\Omega)}
\]

for any \( q_T \in \mathbb{R}^{m_0} \). We set the control \( v = I_{h,\Delta t}^h(g) = (1_{\omega h} \hat{q}^{n+\frac{1}{2}})_{n \in \mathbb{N}} \) and consider the solution \( y \) to the controlled problem

\[
\begin{align*}
(D_t y)^{n+\frac{1}{2}} - \Delta \zeta (t+ \gamma)^{n+\frac{1}{2}} + (t+ \gamma y)^{n+\frac{1}{2}} & = 1_{\omega h} \hat{q}^{n+\frac{1}{2}} \quad n \in \mathbb{N}, \\
y^0 & = g.
\end{align*}
\]

By duality, we deduce from Eqs. 121 and 97 that

\[
\int_{\omega \times (0,T)} \hat{q} = (y^M, q_T)_{L^2(\Omega)} - (g, q^\frac{1}{2})_{L^2(\Omega)}
\]

for any \( q_T \in \mathbb{R}^{m_0} \), whence, from Eqs. 120 and 122, we conclude

\[
y^M = -\phi(h) \hat{q}_T.
\]

By taking \( q_T = \hat{q}_T \) in Eq. 120, we readily obtain

\[
\| \hat{q} \|_{L^2(\omega \times (0,T))}^2 + \phi(h) \| \hat{q}_T \|_{L^2(\Omega)}^2 = -(g, \hat{q}^\frac{1}{2}) \leq |g|_{L^2(\Omega)} |\hat{q}^\frac{1}{2}|_{L^2(\Omega)}.
\]

Hence, with the observability inequality Eq. 118 applied to \( \hat{q} \), we get

\[
\| v \|_{L^2_{T,\omega}(\omega \times (0,T))} = \| \hat{q} \|_{L^2(\omega \times (0,T))} \leq C_{obs} |g|_{L^2(\Omega)}
\]

and

\[
\sqrt{\Phi(h)} |\hat{q}_T|_{L^2(\Omega)} \leq C_{obs} |g|_{L^2(\Omega)}.
\]

In this way, the linear map

\[
L_{h,\Delta t}^h : \mathbb{R}^{m_0} \rightarrow L^2_{T,\omega}(\omega \times (0,T))
\]

\[
g \mapsto v
\]

is well-defined and continuous. From the expressions Eqs. 123 and 124, we finally get

\[
|y^M|_{L^2(\Omega)} \leq C_{obs} \sqrt{\Phi(h)} |y_0|_{L^2(\Omega)}.
\]

This ends the proof for \( \phi(h) = e^{-C_2/h^{\min(d/4,1)}} \).
The case of a general function \( \phi \) follows similarly and only a minor adjustment is required. For fixed \( \vartheta \geq 1 \) and any given \( \phi(h) \) verifying Eq. 50, we see that there exists some \( h_2 > 0 \) such that

\[
e^{-C_2 / h^{\min(0, 4, 1)}} \leq \phi(h)
\]

for all \( 0 < h \leq h_2 \). Decreasing, if necessary, the value of \( h \) and setting \( \Delta t \leq T^{-2}h^{4/\vartheta} \) we can see that inequality Eq. 118 holds for any function \( \phi(h) \) verifying Eq. 50. In this way, the rest of the proof follows by the same arguments.

Remark 3.2 To prove the general case \( T \geq 1 \), it is enough to divide the interval \([0, T]\) in two parts. First, we choose some \( T_0 < 1 \leq T \) and set \( M_0 = \lceil T_0 / \Delta t \rceil \). From the previous result, we know that there exists a fully discrete control \( v_0 = (v_0^{n+\frac{1}{2}})_{n \in [0, M_0] - 1} \) with \( \|v_0\|_{L^2_T(0, T; L^2(\mathbb{R}^n \setminus \Omega))} \leq C_{\text{obs}}^{T_0} \sqrt{\phi(h)} |g|_{L^2(\Omega)} \) such that y solution to

\[
\begin{align*}
\frac{y^{n+1} - y^n}{\Delta t} - \Delta h y^{n+1} + a^{n+1} y^{n+1} = 1_{\text{obs}} v_0^{n+\frac{1}{2}} & \quad n \in [0, M_0 - 1] \\
y^{n+1} |_{\partial \Omega} = 0 & \quad n \in [0, M_0 - 1], \\
y^0 = g, & \quad n \in [0, M_0 - 1],
\end{align*}
\tag{126}
\]

verifies

\[
|y^M|_{L^2(\Omega)} \leq C_{\text{obs}}^{T_0} \sqrt{\phi(h)} |g|_{L^2(\Omega)},
\tag{127}
\]

where \( C_{\text{obs}}^{T_0} \) is the observability constant corresponding to \( T_0 \). This defines the state \((y^n)\) for all \( n \in [0, M_0] \).

Now, we set \( v_0^{n+\frac{1}{2}} = 0 \) for \( n \in [M_0] \), \( M - 1 \) and consider the uncontrolled system

\[
\begin{align*}
\frac{y^{n+1} - y^n}{\Delta t} - \Delta h y^{n+1} + a^{n+1} y^{n+1} = 0 & \quad n \in [M_0, M - 1], \\
y^{n+1} |_{\partial \Omega} = 0 & \quad n \in [M_0, M - 1],
\end{align*}
\tag{128}
\]

with initial data \( y^{M_0} \) coming from the sequence Eq. 126. Arguing as we did in Step 1 of the proof of Proposition 3.1, we can obtain an estimate of the form

\[
|y^M|_{L^2(\Omega)}^2 \leq \kappa |y^n|_{L^2(\Omega)}^2, \quad n \in [M_0], M - 1,
\tag{129}
\]

for some \( \kappa > 0 \) only depending on \( \|a\|_{L^\infty(\Omega)} \), \( T_0 \), and \( T \). In this way, combining Eqs. 127 and 129, we have constructed a sequence \( y = (y^n)_{n \in [0, M]} \) by means of the auxiliary problems Eqs. 126 and 128 such that

\[
|y^M|_{L^2(\Omega)} \leq C \sqrt{\phi(h)} |g|_{L^2(\Omega)},
\]

which is in fact a \( \phi(h) \)-controllability constraint.
3.3 The semi-linear case: proof of Theorem 1.13

The proof of this result follows some classical arguments. For this reason, we only give a brief sketch. We define
\[
g(s) := \begin{cases} \frac{f(s)}{s} & \text{if } s \neq 0, \\ f'(0) & \text{if } s = 0. \end{cases}
\]

The assumptions on \( f \) guarantee that \( g \) and \( f' \) are well defined, continuous and bounded functions. For \( \zeta \in L^2_T(\Omega) \), we consider the linear system
\[
\begin{aligned}
\frac{y^{n+1} - y^n}{\Delta t} - \Delta h y^{n+1} + g(\zeta^{n+1})y^{n+1} &= 1_{\omega} v^{n+\frac{1}{2}} \quad n \in [0, M - 1], \\
y_{n+1}^{n+1} &= 0 \\
y^0 &= y_0.
\end{aligned}
\]  
\tag{130}

We set \( a^n_{\zeta} = g(\zeta^n) \), so that we have
\[
\|a^n_{\zeta}\|_{L^\infty_T(\Omega)} \leq K, \quad \forall \zeta \in L^2_T(\Omega),
\]  
\tag{131}

where \( K \) is the Lipschitz constant of \( f \). In view of Proposition 3.1 and Theorem 1.10, for \( \theta \geq 1 \) and \( h \) and \( \Delta t \) chosen sufficiently small, i.e., \( h \leq \min\{h_0, h_1\} \) with \( h_1 = C_1(1 + \frac{1}{\theta} + K^{2/3})^{-\max\{1, 4/\theta\}} \) and
\[
\Delta t \leq \min\{T^{-2}h^\theta, (4K)^{-1}\},
\]  
\tag{132}

we can build a control \( v_{\zeta} = L^{h,\Delta t}_{T,a^n_{\zeta}}(y_0) \) and the associated controlled solution to Eq. 130 such that
\[
|y^n_{T,a^n_{\zeta}}|_{L^2(\Omega)} \leq C e^{-\frac{C}{h^{\min\{\theta/4,1\}}}} |g|_{L^2(\Omega)}, \quad \|v_{\zeta}\|_{L^\infty_T(0,T;L^2(\omega))} \leq C |y_0|_{L^2(\Omega)},
\]  
\tag{133}

where \( C_1 > 0 \) and \( C = \exp\left[C(1 + \frac{1}{\theta} + K^{2/3} + TK)\right] \) are uniform with respect to \( \zeta \) and the discretization parameters \( h \) and \( \Delta t \). Notice that by selecting the parameter \( \Delta t \) as in Eq. 132 guarantees the existence of a solution to Eq. 130 and also the stability of the discrete scheme.

Define the map
\[
\Lambda : L^2_T(\Omega) \to L^2_T(\Omega)
\]
\[\zeta \mapsto y_{\zeta},\]

where \( y_{\zeta} \) is the solution to Eq. 130 associated to \( a^n_{\zeta} = g(\zeta^n) \), \( n \in [1, M] \), and control as in Eq. 133. Arguing as in the proof of Proposition 3.1, we can readily deduce the energy estimate
\[
\|y_{\zeta}\|_{L^2_T(\Omega)} \leq e^{CT\|a^n\|_{\infty}} \|v_{\zeta}\|_{L^2_T(\Omega)} \leq e^{CTK}|y_0|_{L^2(\Omega)},
\]  
\tag{134}

where we have used Eq. 131 and Eq. 133. Thus, we deduce that the image of \( \Lambda \) is bounded, implying that there exists a closed convex set in \( L^2_T(\Omega) \) which is fixed by \( \Lambda \). Moreover, it can be easily verified that \( \Lambda \) is a continuous map from \( L^2_T(\Omega) \).
into itself (which follows from an adaptation of [13, Lemma 5.3]), while the uniform estimate
\[ \|y_t\|_{L^2_p(Q)} \leq C|y_0|_{L^2(\Omega)} \]
for the solutions to Eq. 130 allows to conclude that \( \Lambda \) is a compact map since \( L^2_p(Q) = L^2_p(0, T; \mathbb{R}^m) \) is in fact the finite dimensional space \( (\mathbb{R}^m)^p = \mathbb{R}^m \times \mathbb{R}^m \).

All of the previous properties allow us to to apply Brouwer fixed point theorem to deduce the existence of \( y \in L^2_p(Q) \) such that \( \Lambda(y) = y \). Setting \( v = L_t^{h, \omega, \gamma} (y_0) \) we obtain
\[
\begin{align*}
\frac{y^{n+1} - y^n}{\Delta t} - \Delta y^{n+1} + f(y^{n+1}) &= 1_{\omega \omega} v^{n+\frac{1}{2}} & n &\in [0, M - 1], \\
y^{n+1}_{|\partial\Omega} &= 0 & n &\in [0, M - 1], \\
y^0 &= y_0,
\end{align*}
\]
which concludes the proof as we have found a control \( v \) that drives the solution of the semilinear parabolic system to a final state \( y^M \) satisfying estimates Eq. 133.

### 4 Further results, remarks and open problems

We devote this section to present additional discussion regarding the controllability of fully discrete systems.

#### 4.1 Some variants on the Carleman estimate

With the notation introduced in Sections 1.4.1, 1.4.2 and 1.4.3, we can readily identify and prove some additional estimates. We enumerate them below.

1. If one considers instead of \( L_\mathcal{D} \) the following forward-in-time operator
\[
(L_\mathcal{D} q)^n := (\overline{D}_t q)^n - \Delta_h (\overline{\mathcal{E}}_+ q)^n, \quad n \in [1, M],
\]
for any \( q \in (\mathbb{R}^{2\mathbb{N}_0 \times \mathbb{N}_0})^{\overline{\mathcal{D}}} \) with \( (q_{|\partial\Omega})^{n+\frac{1}{2}}, n \in [0, M - 1] \), then Theorem 1.8 also holds by replacing all the \( \overline{\mathcal{E}}^- \) operators by \( \overline{\mathcal{E}}^+ \).

2. With the tools presented in Appendix A, Theorem 1.8 can adapted without major changes to fully discrete parabolic operators acting on primal variables in time. For instance, we can consider the parabolic operator
\[
(L_\mathcal{P} y)^{n+\frac{1}{2}} = (D_t y)^{n+\frac{1}{2}} - \Delta_h (\mathcal{E}^+ y)^{n+\frac{1}{2}}, \quad n \in [0, M - 1],
\]
for all \( y \in (\mathbb{R}^{2\mathbb{N}_0 \times \mathbb{N}_0})^{\mathcal{P}} \) with \( (y_{|\partial\Omega})^{n+1} = 0, n \in [0, M] \).
Then, under similar conditions to Theorem 1.8, we can prove the following estimate

$$\tau^{-1} \iint_Q t^+ (e^{2t\varphi} \partial_{t}^{-1}) \left( |D_t y|^2 + |\Delta_h(t^+ y)|^2 \right)$$

$$+ \tau \iint_Q t^+ (e^{2t\varphi} \partial_{t}^{-1}) |\partial_h(t^+ y)|^2$$

$$+ \tau \iint_Q t^+ (e^{2t\varphi} \partial_{t}^{-1}) |\partial_h(t^+ y)|^2 + \tau^3 \iint_Q t^+ (e^{2t\varphi} \partial_{t}^{-1}) (t^+ y)^2$$

$$\leq C \left( \iint_Q t^+ (e^{2t\varphi}) |L_{\varphi} y|^2 + \tau^3 \iint_{B \times (0, T)} t^+ (e^{2t\varphi} \partial_{t}^{-1}) (t^+ y)^2 \right)$$

$$+ Ch^{-2} \left( \int_{\Omega} |(e^{t\varphi} y)^1|^2 + \int_{\Omega} |(e^{t\varphi} y)^M|^2 \right).$$

(135)

As in the previous remark, we can adapt the result for the backward-in-time operator

$$(\overline{L}_{\varphi} y)^n + \frac{1}{2} = -(D_t y)^n + \frac{1}{2} - \Delta_h(t^- y)^n + \frac{1}{2}, \quad n \in \mathbb{Z}.$$
The introduction of the parameter $\vartheta$ gives an extra degree of freedom while fixing $h$ and $\Delta t$ without altering too much the best decay rate we can achieve for the target, that is, Eq. 125 with $\phi(h) = e^{-C/h^{\min(\vartheta/4, 1)}}$ for some uniform $C > 0$.

By selecting $\vartheta \geq 4$, we impose a very restrictive condition on $\Delta t$, i.e., $\Delta t \sim h^\vartheta$, but allows to recover the best decay $|y^M|_{L^2(\Omega)} = O(e^{-C/h})$. This is comparable to the result obtained in [13] in the semi-discrete case. On the other hand, by choosing $\vartheta = 1$, we relax the condition to $\Delta t \sim h$ but at the price of modifying the size of the target and obtaining $|y^M|_{L^2(\Omega)} = O(e^{-C/h^{1/4}})$ which in turn resembles the convergence obtained for the time-discrete case in [22]. We point out that in any case, the size of the target remains exponentially small and allows to prove a more practical result for any function $\phi(h)$ verifying Eq. 50.

In the fully discrete case, conditions connecting $h$ and $\Delta t$ for controllability purposes have also appeared in [26, Theorem 3.5]. Indeed, by employing a Lebeau-Robbiano type strategy, the authors prove a controllability result for Eq. 45 with $a = \{a^n\}_{n \in [1, M]} = 0$ in which

$$|y^M|_{L^2(\Omega)} \leq C e^{-C/h^{N'}} |g|_{L^2(\Omega)}$$

for some $N > 0$ provided $\Delta t \leq C_T h^{N'}$ with $C_T > 0$. This obviously yields a better result in the linear case (without potential) and impose less restrictive conditions between $h$ and $\Delta t$.

Nonetheless, due to the spectral nature of the Lebeau-Robbiano technique, the case of a space-and-time-dependent potential $a$ or more general results for the semilinear case or coupled systems discussed below is out of reach.

### 4.3 On the practical computation of controls

The actual computation of fully discrete controls for linear systems by means of the penalized method used in Section 3.2 has been extensively discussed, for instance, in [26] and [18].

We recall that such controls are of the form $v = (1_{oh} \widehat{q}^{n+\frac{1}{2}})_{n \in [0, M-1]}$, where $(\widehat{q}^{n+\frac{1}{2}})_{n \in [0, M-1]}$ is the solution to the adjoint Eq. 97 with initial datum $\widehat{q}_T \in \mathbb{R}^{2r}$ coming from the minimization of the functional Eq. 119. Since such functional is convex, quadratic, and coercive, the conjugate gradient algorithm is a simple choice for solving the associated minimization problem. To apply this method, we just have to compute the gradient of $J_{h, \Delta t}$ at each iteration, see [18, Corollary 4.1 and Remark 4.2] for further details about the implementation.

In practice, the penalization function $h \mapsto \phi(h)$ can be chosen as $\phi(h) = h^{2p}$ where $p$ is the approximation order of the numerical scheme ($p = 2$ for our FD scheme), while the time step $\Delta t$ has to be chosen to ensure at least the same accuracy as the space discretization. This is consistent with our results and with the overall convergence of the implicit scheme (see Section 1.2). We refer to [26] for a detailed numerical analysis and further discussion of the approximation of controls for linear autonomous parabolic control systems. As far as we know, an analysis similar to
one presented in [26] for linear systems with potential or semilinear problems has yet to be carried.

We finish this remark by mentioning that there are alternative approaches for computing numerical controls for parabolic problems, see, e.g., [32–37]. These methods commonly rely on formulating the control problem as an optimization problem for a functional different than Eq. 119, which involves the weighted $L^2$-norms of the control and the solution of the equation itself. The weights are related to those appearing in the Carleman estimates. As shown in the aforementioned works, schemes derived from these strategies give satisfactory results for linear and semilinear problems.

### 4.4 On the convergence of the fully discrete controls

As the fully discrete controls obtained in Theorem 1.10 are uniformly bounded in $L^2$, then, up to a subsequence, these controls converge weakly to a function $v \in L^2(\omega \times (0, T))$ which actually drives the solution of the continuous problem to zero at the time $T$.

Note, however, that this does not guarantee that the limit control is also a least-energy control for the continuous system. In this direction, using a penalized approach as in our work, in [10], it is proved that under additional conditions, a sequence of uniformly bounded semi-discrete controls converge strongly (up to a subsequence) in $L^2(\omega \times (0, T))$ to a least-energy control for the continuous system. This analysis is carried out in the semi-discrete case (i.e., space-discrete only) but it can be used as a first step to analyze the fully discrete case. This remains as an interesting open problem.

### 4.5 Some perspectives

The approach presented in this paper can be used to deal with other less standard control problems for coupled systems in which Carleman estimates are at the heart of the proofs. We present a brief discussion on a couple of open problems that can be addressed with the tools presented here.

**Insensitizing controls.** Let $\mathcal{O} \subset \Omega$ be an observation subset and consider the functional

$$
\Psi(y) = \frac{1}{2} \sum_{n=1}^{M} \frac{\Delta t}{2} |y_n|_{L^2(\mathcal{O})}^2,
$$

and the control system

$$
\begin{cases}
    y^{n+1} - y^n - \Delta_t y^{n+1} = \mathbf{1}_\Omega v^{n+\frac{1}{2}} + \xi^{n+\frac{1}{2}} & n \in [0, M - 1], \\
    y_{|\partial\Omega}^{n+1} = 0 & n \in [0, M - 1], \\
    y^0 = g + \sigma w,
\end{cases}
$$

where $g \in L^2(\Omega)$ and $\xi \in L^2(\mathcal{O})$ are given functions and the data of Eq. 136 is incomplete in the following sense: $w \in L^2(\Omega)$ is unknown while $|w|_{L^2(\Omega)} = 1$ and $\sigma \in \mathbb{R}$ is unknown and small enough. The idea is to look for a control $v = $
such that

\[ \frac{\partial \Psi(y)}{\partial \sigma} \bigg|_{\sigma=0} = 0, \quad \forall w \in L^2(\Omega). \] (137)

This is the so-called insensitizing problem (see the seminal work [38]) and has been thoroughly studied in different contexts.

The insensitizing control problem is equivalent to study the null-controllability of a cascade system of parabolic PDEs (see, e.g., [39, Theorem 1]). At the discrete level, Eq. 137 translates into finding a control \( q \) such that

\[ q^n = 0, \] (138)

where \( q = (q^{n+\frac{1}{2}})_{n \in [0, M-1]} \) can be found from the following forward-backward cascade system

\[
\begin{cases}
\begin{aligned}
y^{n+1} - y^n - \Delta_t y^{n+1} &= 1_\omega u^{n+\frac{1}{2}} + 1_\mathcal{O} v^{n+\frac{1}{2}} & n \in [0, M-1], \\
y_{\partial \Omega}^n &= 0, \\
y^0 &= g,
\end{aligned}
\end{cases}
\]

\[
\begin{cases}
\begin{aligned}
q^{n-\frac{1}{2}} - q^{n+\frac{1}{2}} - \Delta_t q^{n-\frac{1}{2}} &= 1_\mathcal{O} y^n & n \in [1, M], \\
q_{\partial \Omega}^{n-\frac{1}{2}} &= 0, \\
q^M &= 0.
\end{aligned}
\end{cases}
\] (139)

Of course, we cannot expect to obtain such kind of result for Eq. 136 but rather a relaxed condition.

In view of previous results for space-discrete insensitizing problems (see [16, Theorem 1.4]), we think that it is possible to obtain, by means of the Carleman inequalities Eqs. 40 and 135, a fully discrete observability for the coupled system Eq. 139 (see [16, Section 4]), thus giving a relaxed notion of \( \phi(h) \)-insensitizing control for Eq. 136. Nevertheless, details remain to be given.

Hierarchic control. The notion of hierarchic control, introduced in [40], looks for a systematic way to combine the notions of optimal control and controllability. To fix ideas, let us consider the system

\[
\begin{cases}
\begin{aligned}
y^{n+1} - y^n & - \Delta_t y^{n+1} &= 1_\omega u^{n+\frac{1}{2}} + 1_\mathcal{O} v^{n+\frac{1}{2}} & n \in [0, M-1], \\
y_{\partial \Omega}^{n+1} &= 0, \\
y^0 &= g,
\end{aligned}
\end{cases}
\] (140)

where \( u \) and \( v \) are controls exerted on the sets \( \omega, \mathcal{O} \subset \Omega \) with \( \omega \cap \mathcal{O} = \emptyset \). The original idea in the work [40] is to find controls satisfying simultaneously the following control objectives

\( \phi \) Springer
Find $v$ such that

$$\min_{v \in L^2_D(0,T;\mathbb{R}^m)} \left\{ \frac{1}{2} \sum_{n=1}^{M} \Delta t |y^n - Y^n|_{L^2(\Omega)}^2 + \frac{\beta}{2} \sum_{n=0}^{M-1} \Delta t |v^{n+\frac{1}{2}}|_{L^2(\mathcal{O})}^2 \right\}. \quad (141)$$

where $Y = (Y^n)_{n \in [1,M]}$ is a given function and $\beta > 0$. Intuitively, solving Eq. 141 amounts to find the least-energy control such that $y$ remains close to the target $Y$ and is analogous to a classical optimal control problem.

Find $u$ solving the null-controllability problem

$$\min_{u \in L^2_D(0,T;\mathbb{R}^m)} \left\{ \frac{1}{2} \sum_{n=0}^{M-1} \Delta t |u^{n+\frac{1}{2}}|_{L^2(\omega)}^2 \right\}, \quad (142)$$

subject to $y^M = 0$.

Following the methodology proposed in [40], (P1)–(P2) can be solved by employing a Stackelberg strategy and a hierarchy of controls. The first step amounts to look for $v$ (the follower control) assuming that $u$ (the leader control) is fixed. Using classical optimal control tools (see, e.g., [41]) adapted to the discrete setting, we can prove that there exists $v$ solving (P1) and can be characterized by the optimality system

$$\begin{cases}
\frac{y^{n+1} - y^n}{\Delta t} - \Delta hy^{n+1} = 1_\omega u^{n+\frac{1}{2}} - \frac{1}{\beta} q^{n+\frac{1}{2}} 1_\mathcal{O}, & n \in [0, M - 1], \\
y^{n+1}_{|\partial \Omega} = 0, & n \in [0, M - 1], \\
y^0 = g, & \\
q^{n+\frac{1}{2}} - q^{n+\frac{1}{2}} = y^n - Y^n, & n \in [1, M], \\
q^{n+\frac{1}{2}}_{|\partial \Omega} = 0, & n \in [1, M], \\
q^{M+\frac{1}{2}} = 0.
\end{cases} \quad (143)$$

Notice that solving (P1) amounts to add an extra equation to the original system Eq. 140. Once this is done, according to (P2), it remains to obtain a control $h$ such that $y^M = 0$. However, as in the insensitizing case, it is not reasonable to expect this but rather the weaker notion of $\phi(h)$-controllability.

Following the spirit of the continuous case (see, for instance, [42] for a similar problem in a slightly more general framework), by duality, we should obtain a relaxed observability inequality like

$$|z^{\frac{1}{2}}|_{L^2(\Omega)}^2 + \sum_{n=1}^{M} \Delta t |\mathcal{E}^n p^n|_{L^2(\Omega)}^2 \leq C \left( \sum_{n=0}^{M-1} \Delta t |1_\omega z^{n+\frac{1}{2}}|_{L^2(\Omega)}^2 + e^{-\frac{C}{T}} |z_T|_{L^2(\Omega)}^2 \right). \quad (144)$$
for the solutions to the adjoint system

\[
\begin{align*}
\frac{z^n - z^{n+1}}{\Delta t} - \Delta_h z^{n+\frac{1}{2}} &= p^n & n \in [1, M], \\
\frac{\epsilon^{n+\frac{1}{2}}}{\Delta t} &= 0 & n \in [1, M], \\
\epsilon^{M+\frac{1}{2}} &= z_T, \\
\frac{p^{n+1} - p^n}{\Delta t} - \Delta_h p^{n+1} &= -\frac{1}{\beta} \epsilon^{n+\frac{1}{2}} \mathbf{1}_\Omega & n \in [0, M - 1], \\
p_{0\Omega}^{n+1} &= 0 & n \in [0, M - 1], \\
p_{M\Omega}^{n+1} &= 0,
\end{align*}
\]

where \( \Xi = (\Xi^n)_{n \in [1, M]} \) is a suitable exponential weight function, typically related to the Carleman weights. As in the insensitizing controls, we believe that this is doable by using the Carleman estimates Eqs. 40 and 135 and some of the procedures introduced here.

Of course, Eq. 144 will only yield a \( \phi(h) \)-controllability result instead of the null-controllability constraint in Eq. 142. Nonetheless, obtaining the relaxed inequality Eq. 144 and performing an analysis for the fully discrete case similar to [16, Section 4] will complement nicely the very recent results on numerical hierarchic control developed in [43].

**Appendix A: Discrete calculus results**

**A.1 Results for space-discrete variables**

The goal of this first section is to provide a self-contained summary of calculus rules for space-discrete operators like \( \partial_h, \overline{\partial}_h \), and to provide estimates for the successive applications of such operators on the weight functions. We present the results without a proof and refer the reader to [11] (see also [13]) for a complete discussion.

To avoid introducing cumbersome notation, we introduce the following continuous difference and averaging operators

\[
\begin{align*}
\mathfrak{s}_h^+ f(x) := f(x + \frac{h}{2}), & \quad \mathfrak{s}_h^- f(x) := f(x - \frac{h}{2}), \\
\mathfrak{d}_h f := \frac{1}{h} \left( \mathfrak{s}_h^+ - \mathfrak{s}_h^- \right) f, & \quad \mathfrak{m}_h f := \widehat{f} := \frac{1}{2} \left( \mathfrak{s}_h^+ + \mathfrak{s}_h^- \right) f.
\end{align*}
\]

With this notation, the results given below will be naturally translated to discrete versions. More precisely, for a function \( f \) continuously defined on \( \mathbb{R} \), the discrete function \( \mathfrak{d}_h f \) is in fact \( \mathfrak{d}_h f \) sampled on the dual mesh \( \mathcal{M} \), and \( \overline{\partial}_h f \) is \( \mathfrak{d}_h f \) sampled on the primal mesh \( \mathcal{M} \). Similar meanings will be used for the averaging symbols \( \mathcal{\overline{\partial}} f \) and \( \mathcal{\overline{\mathfrak{f}}} \) (see Eqs. 24 and 27, respectively) and for more intricate combinations: for instance, \( \Delta_h f = \overline{\partial}_h \mathfrak{d}_h f \) is the function \( \mathfrak{d}_h \overline{\partial}_h \mathfrak{f} \) sampled on \( \mathcal{M} \).
A.1.1 Discrete calculus formulas

**Lemma A.1** Let the functions $f_1$, $f_2$ be continuously defined over $\mathbb{R}$. We have

$$d_h(f_1 f_2) = d_h(f_1) \tilde{f}_2 + \tilde{f}_1 d_h(f_2).$$

The translation of the result to discrete functions $f_1, f_2 \in \mathbb{R}^{\Omega}$ (resp. $g_1, g_2 \in \mathbb{R}^{\Omega}$) is

$$\partial_h(f_1 f_2) = \partial_h(f_1) \tilde{f}_2 + \tilde{f}_1 \partial_h(f_2), \quad \left(\text{resp. } \overline{\partial_h}(g_1 g_2) = \overline{\partial_h}(g_1) \overline{f}_2 + \overline{f}_1 \overline{\partial_h}(g_2)\right).$$

(145)

**Lemma A.2** Let the functions $f_1$, $f_2$ be continuously defined over $\mathbb{R}$. We have

$$\tilde{f}_1 \tilde{f}_2 = \tilde{f}_1 \tilde{f}_2 + \frac{h^2}{4} d_h(f_1) d_h(f_2).$$

The translation of the result to discrete functions $f_1, f_2 \in \mathbb{R}^{\Omega}$ (resp. $g_1, g_2 \in \mathbb{R}^{\Omega}$) is

$$\tilde{f}_1 \tilde{f}_2 = \tilde{f}_1 \tilde{f}_2 + \frac{h^2}{4} \partial_h(f_1) \partial_h(f_2), \quad \left(\text{resp. } \tilde{f}_1 \tilde{f}_2 = \tilde{f}_1 \tilde{f}_2 + \frac{h^2}{4} \overline{\partial_h}(f_1) \overline{\partial_h}(f_2))\right).$$

**Lemma A.3** Let the function $f$ be continuously defined over $\mathbb{R}$. We have

$$m_h^2 f := \hat{f} = f + \frac{h^2}{4} d_h f.$$

The following result provides a discrete integration-by-parts formula and a related identity for averaged functions.

**Proposition A.4** Let $f \in \mathbb{R}^{\Omega^N \cup \Omega^J}$ and $g \in \mathbb{R}^{\Omega^J}$. Then

$$\int_{\Omega} f (\overline{\partial_h} g) = -\int_{\Omega} (\partial_h f) g + f_{N+1} g_{N+\frac{1}{2}} - f_{0} g_{\frac{1}{2}},$$

(146)

$$\int_{\Omega} \hat{f} \hat{g} = \int_{\Omega} \tilde{f} \tilde{g} - \frac{h}{2} f_{N+1} g_{N+\frac{1}{2}} - \frac{h}{2} f_{0} g_{\frac{1}{2}}.$$  

(147)

**Lemma A.5** Let $f$ be a sufficiently smooth function defined in a neighborhood of $\Omega$. We have

(i) $s_h^\pm f = f \pm \frac{h}{2} \int_{\Omega} \partial_x f (\cdot, \pm \sigma h/2) d\sigma$,

(ii) $m_h^l f = f + \frac{h^2}{4^2-l} \int_{-1}^{1} (1 - |\sigma|) \partial_x^l f (\cdot + l_j \sigma h) d\sigma$,

(iii) $\partial_h^j f = \partial_x^j f + \frac{h^2}{8^2-j(j+1)!} \int_{-1}^{1} (1 - |\sigma|)^j \partial_x^{j+2} f (\cdot + l_j \sigma h) d\sigma, \quad j = 1, 2,$

$l_1 = \frac{1}{2}, \ l_2 = 1.$
Proof The results follow from Taylor formula

\[ f(x + y) = \sum_{j=0}^{n-1} \frac{y^j}{j!} f^{(j)}(x) + y^n \int_{0}^{1} \frac{(1 - \sigma)^{n-1}}{(n-1)!} f^{(n)}(x + \sigma y) d\sigma \]  

(148)

at order \( n = 1 \) for item (i), \( n = 2 \) for item (ii) with \( j = 1, 2 \), \( n = 3 \) for item (iii) with \( j = 1 \) and \( n = 4 \) for item (iii) with \( j = 2 \). □

A.1.2 Space-discrete computations related to Carleman weights

We present here a summary of results related to space-discrete operations performed on the Carleman weight functions. The estimates presented below are at the heart of our fully discrete Carleman estimate. The proof of such results can be found on [11].

We set \( r = e^s \psi \) and \( \rho = r^{-1} \). The positive parameters \( s \) and \( h \) will be large and small, respectively. We highlight the dependence on \( s, h \) and \( \lambda \) in the following results. We always assume that \( s \geq 1 \) and \( \lambda \geq 1 \).

Lemma A.6 Let \( \alpha, \beta \in \mathbb{N} \). We have

\[
\partial^\beta_x (r \partial^\alpha_x \rho) = \alpha^\beta (-s \phi)^{\alpha + \beta} (\partial_x \psi)^{\alpha + \beta} + \alpha \beta (s \phi)^{\alpha + \beta - 1} \mathcal{O}(1) + \alpha (\alpha - 1) s^{\alpha - 1} \mathcal{O}_\lambda(1) = s^\alpha \mathcal{O}_\lambda(1).
\]

Let \( \sigma \in [-1, 1] \). We have

\[
\partial^\beta_x (r(t, \cdot) (\partial^\alpha_x \rho)(t, \cdot + \sigma h)) = \mathcal{O}_\lambda(s^\alpha (1 + (sh)^\beta)) e^{\mathcal{O}_\lambda(sh)}.
\]

Provided \( \frac{rh}{\sqrt{\mathcal{T}^2}} \leq 1 \), we have \( \partial^\beta_x (r(t, \cdot) (\partial^\alpha_x \rho)(t, \cdot + \sigma h)) = s^\alpha \mathcal{O}_\lambda(1) \). The same expressions hold with \( r \) and \( \rho \) interchanged if we replace \( s \) by \( -s \) everywhere.

Proposition A.7 Let \( \alpha \in \mathbb{N} \). Provided \( \frac{rh}{\sqrt{\mathcal{T}^2}} \leq 1 \), we have

(i) \( rm^j_h \partial^\alpha_x \rho = r \partial^\alpha_x \rho + s^\alpha (sh)^2 \mathcal{O}_\lambda(1) = s^\alpha \mathcal{O}_\lambda(1), j = 1, 2 \),

(ii) \( rm^j_h \partial^\alpha_x \rho = r \partial^\alpha_x \rho + s(s h)^j \mathcal{O}_\lambda(1), j = 0, 1, \)

(iii) \( rd^2_h \rho = r \partial^2_x \rho + s^2 (sh)^2 \mathcal{O}_\lambda(1) = s^2 \mathcal{O}_\lambda(1). \)

The same estimates hold with \( \rho \) and \( r \) interchanged.

Proposition A.8 Let \( \alpha \in \mathbb{N} \). For \( k = 0, 1, 2 \), and provided \( \frac{rh}{\sqrt{\mathcal{T}^2}} \leq 1 \) we have

(i) \( d^k_h (r \partial^\alpha_x \rho) = \partial^k_x (r \partial^\alpha_x \rho) + s^2 (sh)^{k-1} \mathcal{O}_\lambda(1) = s^2 \mathcal{O}_\lambda(1), \)

(ii) \( d^k_h (r m^j_h \rho) = (sh)^{k-1} \mathcal{O}_\lambda(1). \)

The same estimates hold with \( \rho \) and \( r \) interchanged.

Proposition A.9 Let \( \alpha, \beta \in \mathbb{N} \) and \( k, j = 0, 1, 2 \). Provided \( \frac{rh}{\sqrt{\mathcal{T}^2}} \leq 1 \), we have

(i) \( m^j_h \partial^k_x \partial^\alpha_x (r^2 \partial^2_h \rho \partial^\alpha_x \rho) = \partial^k_x \partial^\alpha_x (r^2 (\partial_x \rho) \partial^2_x \rho) + s^3 (sh)^2 \mathcal{O}_\lambda(1) = s^3 \mathcal{O}_\lambda(1), \)

(ii) \( m^j_h \partial^k_x \partial^\alpha_x (r^2 \partial^2_h \rho m^j_h \rho) = \partial^k_x \partial^\alpha_x (r \partial_x \rho) + s(s h)^2 \mathcal{O}_\lambda(1) = s \mathcal{O}_\lambda(1). \)
The same estimates hold with $\rho$ and $r$ interchanged.

Remark A.10 We set $d_{h2} := ((s^+_h)^2 - (s^-_h)^2)/2h = m_h d_h$ and $m_{h2} := ((s^+_h)^2 + (s^-_h)^2)/2$. The estimates presented in the previous results are then preserved when we replace some of the $d_h$ by $d_{h2}$ and some of the $m_h$ by $m_{h2}$.

A.2 Results for time-discrete variables

Following the spirit of the previous section, here, we devote to provide a summary of calculus rules manipulating the time-discrete operators $D_t$ and $\overline{D_t}$, and also to provide estimates for the application of such operators on the weight functions.

As we did before, to avoid introducing additional notation, we present the following continuous difference operator. For a function $f$ defined on $\mathbb{R}$, we set

$$t^+ f(t) := f(t + \frac{\Delta t}{2}), \quad t^- f(t) := f(t - \frac{\Delta t}{2}),$$

$$D_t f := \frac{1}{\Delta t} (t^+ - t^-) f.$$

As for the the space variable, we can obtain discrete versions of the results presented below quite naturally. Indeed, for a function $f$ continuously defined on $\mathbb{R}$, the discrete function $D_t f$ amounts to evaluate $D_t f$ at the mesh points $\mathcal{D}$ and $\overline{D_t} f$ is $D_t f$ sampled at the mesh points $\mathcal{P}$.

A.2.1 Time-discrete calculus formulas

Lemma A.11 Let the functions $f_1$ and $f_2$ be continuously defined over $\mathbb{R}$. We have

$$D_t(f_1 f_2) = t^+ f_1 D_t f_2 + D_t f_1 t^- f_2.$$

The same holds for

$$D_t(f_1 f_2) = t^- f_1 D_t f_2 + D_t f_1 t^+ f_2.$$

From the above formulas, if $f_1 = f_2 = f$, we have the useful identities

$$t^+ f D_t f = \frac{1}{2} D_t \left( f^2 \right) + \frac{1}{2} \Delta t (D_t f)^2, \quad t^- f D_t f = \frac{1}{2} D_t \left( f^2 \right) - \frac{1}{2} \Delta t (D_t f)^2.$$

The translation of the result to discrete functions $f, g_1, g_2 \in H^\mathcal{D}$ is

$$\overline{D_t}(g_1 g_2) = (\overline{\epsilon}^+ g_1) \overline{D_t} g_2 + \overline{D_t} g_1 (\overline{\epsilon}^- g_2),$$

$$\overline{D_t}(g_1 g_2) = (\overline{\epsilon}^- g_1) \overline{D_t} g_2 + \overline{D_t} g_1 (\overline{\epsilon}^+ g_2),$$

and

$$ (\overline{\epsilon}^+ f) \overline{D_t} f = \frac{1}{2} \overline{D_t} \left( f^2 \right) + \frac{1}{2} \Delta t (\overline{D_t} f)^2,$$

$$ (\overline{\epsilon}^- f) \overline{D_t} f = \frac{1}{2} \overline{D_t} \left( f^2 \right) - \frac{1}{2} \Delta t (\overline{D_t} f)^2.$$
The following result covers discrete integration by parts and some useful related formulas.

**Proposition A.12** Let \( \{ H, (\cdot, \cdot)_H \} \) be a real Hilbert space and consider \( u \in H^{\overline{T}} \) and \( v \in H^{\overline{T}} \). We have the following:

\[
\int_0^T ( t^+ u, v )_H = \int_0^T ( u, \tilde{\xi}^- v )_H , \tag{152}
\]
\[
\int_0^T ( \Gamma u, v )_H = \Delta t ( u^0, v^{1/2} )_H - \Delta t ( u^M, v^{M+1/2} )_H + \int_0^T ( u, \tilde{\xi}^+ v )_H . \tag{153}
\]

Moreover, combining the above identities, we have the following discrete integration by parts formula

\[
\int_0^T ( D_t u, v )_H = -( u^0, v^{1/2} )_H + ( u^M, v^{M+1/2} )_H - \int_0^T ( \overline{D}_t v, u )_H . \tag{154}
\]

**Remark A.13** If we consider two functions \( f, g \in H^{\overline{T}} \), we can combine Eqs. 152 and 154 to obtain the formula

\[
\int_0^T ( \overline{D}_t f, \tilde{\xi}^- g )_H = -( f^{1/2}, g^{1/2} )_H + ( f^{M+1/2}, g^{M+1/2} )_H - \int_0^T ( \overline{E}^+ f, \overline{D}_t g )_H . \tag{155}
\]

Analogously, for \( f, g \in H^{\overline{T}} \), the following holds

\[
\int_0^T ( D_t f, t^+ g )_H = -( f^0, g^0 )_H + ( f^M, g^M )_H - \int_0^T ( t^- f, D_t g )_H . \tag{156}
\]

Observe that in these formulas, the integrals are taken over the same discrete points. These will be particularly useful during the derivation of the Carleman estimates Eqs. 40 and 135.

### A.2.2 Time-discrete computations related to Carleman weights

We present some lemmas related to time-discrete operations applied to the Carleman weights. The proof of these results can be found in [22, Appendix B]. We recall that \( r = e^{\delta \rho} \) and \( \rho = r^{-1} \). We highlight the dependence on \( \tau, \delta, \Delta t \) and \( \lambda \) in the following estimates.

**Lemma A.14** (Time-discrete derivative of the Carleman weight) Provided \( \Delta t \tau ( T^3 \delta^2 )^{-1} \leq 1 \), we have

\[
\Gamma^- ( r ) D_t \rho = - \tau \Gamma^- ( \delta \rho ) \varphi + \Delta t \left( \frac{\tau}{\delta^3 T^4} + \frac{\tau^2}{\delta^4 T^6} \right) O_\lambda ( 1 ) .
\]

**Lemma A.15** (Discrete operations on the weight \( \theta \)) There exists a universal constant \( C > 0 \) uniform with respect to \( \Delta t, \delta \) and \( T \) such that
In addition to the results presented above, since we are dealing with a fully discrete case, we need to give an additional lemma concerning the effect of the time-discrete operator $D_t$ over some discrete operations in the space variable. This is a new result as compared to [13] and [22] but follows the arguments there. The result reads as follows.

**Lemma A.16** (Mixed derivatives) Provided $\tau h(\delta T^2) \leq 1$ and $\sigma$ is bounded, we have

(i) \[ \partial_t^\beta \left( r(t, x) \partial_x^\alpha \rho(t, x + \sigma h) \right) = T^\beta s^\alpha(t) \partial_t^\beta(\theta(t))O_\lambda(1), \ \beta = 1, 2 \text{ and } \alpha \in \mathbb{N}. \]

If in addition $\frac{\Delta t}{T^2} \leq \frac{1}{2}$, the following estimates hold

(ii) \[ D_t(r \partial_{h}^2 \rho) = T^2(\sigma^2)O_\lambda(1) + \left( \frac{\tau \Delta t}{\delta T^2} \right)(\frac{\tau h}{T^2})O_\lambda(1). \]

(iii) \[ D_t(r \partial_{x}^2 \rho) = T^2(\sigma^2)O_\lambda(1) + \left( \frac{\tau \Delta t}{\delta T^2} \right)(\frac{\tau h}{T^2})O_\lambda(1). \]

**Proof** For $\beta = 1$, the proof of item (i) can be found on [13, Proof of Proposition 2.14]. For $\beta = 2$, the proof follows exactly as in that work just by noting that $\partial_t^2(r \partial_x^\alpha \rho) = T^2 s^\alpha \partial_x^2 O_\lambda(1)$.

To prove item (ii), we shall exploit that the weights $\rho$ and $r$ can be written in separated variables. Indeed, by Lemma A.5(ii), we have

\[ r(t, x)m_h^2 \rho(t, x) = 1 + Ch^2 \int_{-1}^{1} (1 - |\sigma|)r(t, x)\partial_x^2 \rho(t, x + \sigma h)d\sigma. \] (157)

hence

\[ \frac{r(t + \Delta t, x)m_h^2 \rho(t + \Delta t, x) - r(t, x)m_h^2 \rho(t, x)}{\Delta t} = \partial_t(r(t, x)m_h^2 \rho(t, x)) + \Delta t \int_0^1 (1 - \gamma)\partial_t^2 \left( r(t + \gamma \Delta t, x)m_h^2 \rho(t + \gamma \Delta t, x) \right) d\gamma \]

\[ = A_1 + A_2. \] (158)

by a first order Taylor formula in the time variable (see formula Eq. 148). Differentiating with respect to $t$ in Eq. 157 and item (i) yield

\[ A_1 = T[s(t)h^2]O_\lambda(1). \] (159)

For estimating $A_2$, we use the change of variable $t \mapsto t + \gamma \Delta t$ for $\gamma \in [0, 1]$ in item (i) and observe that provided $\frac{\Delta t}{T^2} \leq \frac{1}{2}$, we have $\max_{t\in[0,T+\Delta t]}\theta(t) \leq \frac{1}{2}$. Therefore,

\[ A_2 = \Delta t T^2 s^2(t + \Delta t)\theta^2(t + \Delta t)h^2 O_\lambda(1) \]

\[ = \left( \frac{\tau \Delta t}{\delta T^4} \right)(\frac{\tau h}{\delta T^2})O_\lambda(1), \] (160)

where we have used that $h \ll 1$ to remove one power of $h$. Putting together Eqs. 158–160 and performing the change of variable $t \mapsto t - \frac{\Delta t}{T}$ yield the desired result.
Finally, to prove item (iii), we have from Lemma A.5(iii) that
\[
 r(t, x)\partial_t^2 \rho(t, x) = r(t, x)\partial_x^2 \rho(t, x) + C h^4 \int_{-1}^{1} (1 - |\sigma|)^3 r(t, x)\partial_t^4 \rho(t, x + \sigma h) d\sigma.
\] (161)

Therefore, arguing as above
\[
\begin{align*}
\frac{r(t + \Delta t, x)\partial_t^2 \rho(t + \Delta t, x) - r(t, x)\partial_t^2 \rho(t, x)}{\Delta t} \\
= \partial_t (r(t, x)\partial_t^2 \rho(t, x)) + \Delta t \int_{0}^{1} (1 - \gamma)\partial_t^2 \left( r(t + \gamma \Delta t, x)\partial_t^2 \rho(t + \gamma \Delta t, x) \right) d\gamma \\
= B_1 + B_2.
\end{align*}
\] (162)

Differentiating with respect to \( t \) in Eq. 161 and using item (i) yield
\[
B_1 = Ts(t)^2 \theta(t) \mathcal{O}_\lambda(1) + h^4 \left[ Ts(t)^4 \theta(t) \mathcal{O}_\lambda(1) \right].
\]

Using that \( \frac{\tau \theta}{\delta T^2} \leq 1 \) and \( h \ll 1 \), we can adjust some powers in the above expression and obtain
\[
B_1 = Ts(t)^2 \theta(t) \mathcal{O}_\lambda(1).
\] (163)

For the term \( B_2 \), we argue as for \( A_2 \). We use the change of variable \( t \mapsto t + \gamma \Delta t, \gamma \in [0, 1] \), in item (i) and observe that provided \( \frac{\Delta t}{T^\delta} \leq \frac{1}{2} \), we have \( \max_{t \in [0, t + \Delta t]} \theta(t) \leq \frac{2}{\delta T} \). In this way, we get
\[
B_2 = \Delta t T^2 s^2 (t + \Delta t) \theta^2 (t + \Delta t) \mathcal{O}_\lambda(1) + \Delta t T^2 h^4 s^4 (t + \Delta t) \theta^2 (t + \Delta t) \mathcal{O}_\lambda(1)
\]
\[
= \frac{\tau^2 \Delta t}{\delta T^6} \mathcal{O}_\lambda(1) + \left( \frac{\tau \Delta t}{\delta^3 T^4} \right)^3 \mathcal{O}_\lambda(1).\] (164)

Collecting the estimates Eq. 163–Eq. 164 in Eq. 162 and setting the change of variable \( t \mapsto t - \frac{\Delta t}{2} \) gives the desired result. \( \Box \)

### Appendix B: Estimates of the cross product in the Carleman estimate

#### B.1 Estimates that only require space discrete computations

**B.1.1 Proof of Lemma 2.1**

The proof is straightforward. First, using Eq. 152, we can relax a little bit the notation by hiding the operator \( \bar{\xi} \). More precisely we have
\[
I_{11} = 2 \iiint_Q \bar{\xi}^{-}(r^2 \bar{\rho} \Delta_h z \bar{\partial}_h \rho \bar{\partial}_h z) = 2 \iiint_Q r^2 \bar{\rho} \Delta_h z \bar{\partial}_h \rho \bar{\partial}_h z.
\]

Now, we can focus on the space variable. Noting that \( \Delta_h = \bar{\partial}_h \partial_h \) and \( \bar{\partial}_h ([\partial_h z]^2) = 2 \bar{\partial}_h (\partial_h z) \bar{\partial}_h z \) thanks to Lemma A.1, we can integrate by parts (see formula Eq. 146)
and obtain
\[
I_{11} = -2 \iint_{Q} r^2 \rho z \partial_{h} \rho \partial_{h} z
\]
\[
= - \iint_{Q} \partial_{h} \left( r^2 \rho \partial_{h} \rho \right) |\partial_{h} z|^2 + \int_{0}^{T} \left( r^2 \rho \partial_{h} \rho \right)_{N+1} |\partial_{h} z|_{N+\frac{1}{2}}^2
\]
\[
- \int_{0}^{T} \left( r^2 \rho \partial_{h} \rho \right)_{0} |\partial_{h} z|_{\frac{1}{2}}^2.
\]

Using that \( \partial_{h} (r^2 \rho \partial_{h} \rho) = -s \lambda^2 (\partial_{x} \psi)^2 \phi + s \lambda \phi \mathcal{O}(1) + s (sh)^2 \mathcal{O}_{\lambda}(1) \) (obtained from Lemma A.9(ii) and A.6) and \( r \rho = 1 + (sh)^2 \mathcal{O}_{\lambda}(1) \) (see Lemma A.7(i)), we get
\[
I_{11} = \iint_{Q} s \lambda^2 (\partial_{x} \psi)^2 \phi |\partial_{h} z|^2 - \iint_{Q} (s \lambda \phi \mathcal{O}(1) + s (sh)^2 \mathcal{O}_{\lambda}(1)) |\partial_{h} z|^2
\]
\[
+ \int_{0}^{T} (1 + (sh)^2 \mathcal{O}_{\lambda}(1)) \left[ (r \partial_{h} \rho)_{N+1} |\partial_{h} z|_{N+\frac{1}{2}}^2 - (r \partial_{h} \rho)_{0} |\partial_{h} z|_{0}^2 \right].
\]

The result follows by shifting back the time integral with formula Eq. 152.

**B.1.2 Proof of Lemma 2.2**

We proceed as in the previous proof. First, we shift the time integral and then using integration by parts in space, we obtain
\[
I_{12} = -2 \iint_{Q} s \partial_{xx} \phi r \overline{\partial_{z}} \Delta_{h} z
\]
\[
= 2 \iint_{Q} s \partial_{h} (r \overline{\theta \partial_{x} \phi}) z \partial_{h} z + 2 \iint_{Q} s (r \overline{\theta \partial_{x} \phi}) |\partial_{h} z|^2.
\]

(165)

Here, we have used that \( (z_{|\partial_{\Omega}})^{n-\frac{1}{2}} = 0 \) for \( n \in [1, M] \) so no boundary conditions appear.

Using Lemma A.5(iii), we see that \( \overline{\partial_{x} \phi} = \partial_{x} \phi + h \mathcal{O}_{\lambda}(1) \); thus, from Proposition A.7(i), we get
\[
(r \overline{\theta \partial_{x} \phi}) = \partial_{xx} \phi + \left[ h + (sh)^2 \right] \mathcal{O}_{\lambda}(1).
\]

(166)

On the other hand, by Propositions A.7(i) and A.8, we get
\[
\partial_{h} (r \overline{\theta \partial_{x} \phi}) = \left[ 1 + (sh)^2 \right] \mathcal{O}_{\lambda}(1).
\]

(167)

Using estimates Eq. 166–Eq. 167 in Eq. 165, we see that \( I_{12} \) can be written as
\[
I_{12} = 2 \iint_{Q} s \partial_{xx} \phi |\partial_{h} z|^2 + R_{12}.
\]

(168)

where
\[
R_{12} := \iint_{Q} \left[ (sh) + s (sh)^2 \right] \mathcal{O}_{\lambda}(1) |\partial_{h} z|^2 + \iint_{Q} [s + s (sh)^2] \mathcal{O}_{\lambda}(1) z \partial_{h} z.
\]
Using that
\[ \partial_{xx} \phi = \lambda^2 |\partial_x \psi|^2 \phi + \lambda \phi \mathcal{O}(1) \]
(169)
in expression Eq. 168, the result follows by Cauchy-Schwarz and Young inequalities and shifting the time integral. Notice that we have adjusted some powers of the product \((sh)\) by using that \(s \geq 1\) and \((sh) \leq \tau h(\delta T^2)^{-1} \leq 1\).

### B.1.3 Proof of Lemma 2.3

Define \( p := r^2 \Delta_h \rho \overline{\partial_h} \rho \). From Eq. 152 and noting that \( \overline{\partial_h} z = \overline{\partial_h}(\tilde{z}) \), we see that
\[ I_{21} = 2 \iint_Q p \overline{\partial_h} z \tilde{z} = \iint_Q p \overline{\partial_h}(|\tilde{z}|^2), \]
where we have used Lemma A.1. Integrating by parts in space, we get
\[ I_{21} = \iint_Q (\partial_h p) |\tilde{z}|^2 + \int_0^T \int p_{N+1} |\tilde{z}_{N+\frac{1}{2}}|^2 - \int_0^T p_0 |\tilde{z}_{\frac{1}{2}}|^2. \]

We observe that \( \tilde{z}_{\frac{1}{2}} = \frac{h}{2} (\partial_h z)_{\frac{1}{2}} \) and \( \tilde{z}_{N+\frac{1}{2}} = -\frac{h}{2} (\partial_h z)_{N+\frac{1}{2}} \). Thanks to A.6 and Proposition A.9(i) notice that \( p = \left[ s^2 \mathcal{O}_\lambda(1) + s^2 (sh)^2 \mathcal{O}_\lambda(1) \right] r \overline{\partial_h} \rho \). Thus,
\[ I_{21}^{(1)} = \int_0^T (sh)^2 + (sh)^4 \mathcal{O}_\lambda(1) \left\{ (r \overline{\partial_h})_0 (\partial_h z)_{N+\frac{1}{2}}^2 + (r \overline{\partial_h})_{N+1} (\partial_h z)_{N+\frac{1}{2}}^2 \right\} \]
\[ I_{21}^{(2)} = \int_0^T (sh)^2 \mathcal{O}_\lambda(1) \left\{ (r \overline{\partial_h})_0 (\partial_h z)_{N+\frac{1}{2}}^2 + (r \overline{\partial_h})_{N+1} (\partial_h z)_{N+\frac{1}{2}}^2 \right\} \]
by using that \( sh \leq \tau h(\delta T^2)^{-1} \leq 1 \).

From Lemma A.2, formula Eq. 147 and recalling that \((\zeta|\rho \Omega)^{n-\frac{1}{2}}\) for \( n \in [1, M] \), we have that
\[ I_{21}^{(1)} = -\iint_Q (\partial_h p) |\tilde{z}|^2 + \frac{h^2}{4} \iint_Q (\partial_h p) |\partial_h z|^2 \]
\[ = -\iint_Q \overline{\partial_h} p |\tilde{z}|^2 + \frac{h^2}{4} \iint_Q (\partial_h p) |\partial_h z|^2. \]

We claim the following.

**Claim B.1** Provided \( \tau h(\delta T^2)^{-1} \leq 1 \), we have
\[ \partial_h p = s^3 \mathcal{O}_\lambda(1) + s^3 (sh)^2 \mathcal{O}_\lambda(1), \]
\[ \overline{\partial_h} p = -s^3 \lambda^4 \phi^3 (\partial_x \psi)^4 + (s \lambda \phi)^3 \mathcal{O}_\lambda(1) + s^2 \mathcal{O}_\lambda(1) + s^3 (sh)^2 \mathcal{O}_\lambda(1). \]

**Proof** From Proposition A.9(i), we have \( \partial_h p = \partial_x (r^2 (\partial_x \rho) \partial_{xx} \rho) + s^3 (sh)^2 \mathcal{O}_\lambda(1) \). On the other hand, a straightforward computation gives
\[ \partial_x (r^2 (\partial_x \rho) \partial_{xx} \rho) = -3s^3 \lambda^4 (\partial_x \psi)^4 \phi^3 + (s \lambda \phi)^3 \mathcal{O}(1) + s^2 \mathcal{O}_\lambda(1). \]
Thus, the first result follows immediately.

For the second one, we note that \( \overline{\partial_h \rho} = \partial_{h_2} \rho \) (see Remark A.10), whence \( \overline{\partial_h \rho} = \partial_{h_2} (r^2 \Delta \rho \overline{\partial_h \rho}) \) and using Proposition A.9(i) the result follows from Eq. 172.

Using Claim B.1 to estimate in Eq. 171 and recalling Eq. 170, we readily get

\[
\begin{align*}
I_{21} &= 3 \iiint_Q s^2 \lambda^4 \phi^2 (\partial_x x \psi)^4 |z|^2 - \iiint_Q \mu_{21} |z|^2 - \iiint_Q v_{21} |z|^2 \\
&\quad + \int_0^T (s h)^2 \mathcal{O}_\lambda(1) \left\{ \left( r \overline{\partial_h \rho} \right)_0 (\partial_h z)^2 \frac{1}{N_{+1}^2} + \left( r \overline{\partial_h \rho} \right)_{N+1} (\partial_h z)^2 \frac{1}{N_{+1}^2} \right\},
\end{align*}
\]

where

\[
\mu_{21} = (s \lambda \phi)^3 \mathcal{O}(1) + s^2 \mathcal{O}_\lambda(1) + s^3 (sh)^2 \mathcal{O}_\lambda(1), \quad v_{21} = s (sh)^2 \mathcal{O}_\lambda(1).
\]

As before, we have adjusted some powers of \((sh)\) in \(v_{21}\) by recalling that \((sh) \leq r h (\delta T^2)^{-1} \leq 1\) and \(h \ll 1\). The desired result then follows by shifting the integral in time.

**B.1.3 Proof of Lemma 2.4**

By Lemma A.3, we have that \( \overline{z} = z + \frac{h^2}{4} \Delta_h z \). Thus

\[
I_{22} = -2 \iiint_Q r \Delta_h \rho s \partial_{xx \phi} z^2 - \frac{h^2}{2} \iiint_Q r \Delta_h \rho s \partial_{xx \phi} z \Delta_h z =: I_{22}^{(1)} + I_{22}^{(2)}. \tag{173}
\]

From Proposition A.7(iii) and Lemma A.6, we readily have

\[
r \Delta_h \rho = \partial_{xx} \rho + s^2 (sh) \mathcal{O}_\lambda(1) = s^2 \lambda^2 (\partial_x x \psi)^2 \phi^2 + s \mathcal{O}_\lambda(1) + s^2 (sh)^2 \mathcal{O}_\lambda(1), \tag{174}
\]

whence, combining with Eq. 169, we obtain that

\[
I_{22}^{(1)} = -2 \iiint_Q s^2 \lambda^4 \phi^2 (\partial_x x \psi)^4 |z|^2 - \iiint_Q \mu |z|^2, \tag{175}
\]

where \( \mu = s^3 \lambda^3 \phi^3 \mathcal{O}(1) + s^2 \mathcal{O}_\lambda(1) + s^3 (sh)^2 \mathcal{O}_\lambda(1) \).

For the term \( I_{22}^{(2)} \), we proceed as follows. We set \( p_\phi := r \Delta_h \rho \partial_{xx \phi} \) and by using that \( (z \partial_\Omega)^{n-1} = 0 \) for \( n \in [1, M] \), we get after integration by parts in space

\[
I_{22}^{(2)} = \frac{h^2}{2} \iiint_Q s \overline{p_\phi} \partial_h z |z|^2 + \frac{h^2}{2} \iiint_Q s \partial_h p \partial_h z z.
\]

Noting that \( \partial_h (z^2) = 2z \partial_h z \), we can integrate once again in the last term of the above to obtain

\[
I_{22}^{(2)} = \frac{h^2}{2} \iiint_Q s \overline{p_\phi} \partial_h z |z|^2 - \frac{h^2}{4} \iiint_Q s \Delta_h p_\phi z^2 =: J_1 + J_2. \tag{176}
\]

Let us estimate \( J_1 \) and \( J_2 \). For the first term, we have from Eq. 169 that \( \partial_{xx} \phi = \mathcal{O}_\lambda(1) \) and from estimate Eq. 174, we have \( p_\phi = s^2 \mathcal{O}_\lambda(1) + s^2 (sh)^2 \mathcal{O}_\lambda(1) \). The same
estimate holds for $\tilde{p}_\phi$. From this fact and adjusting some powers of $(sh)$ yields

$$J_1 = \iint_Q s(sh)^2O_\lambda(1)|\partial_hz|^2.$$  

(177)

For estimating $J_2$, we claim the following

Claim B.2 Provided $\tau h(\delta T^2)^{-1} \leq 1$ we have $h^2\Delta_h p_\phi = s^2(h + h^2)O_\lambda(1) + (sh)^4O_\lambda(1)$.

Proof From Eq. 169, notice that

$$\|\partial_{xx}\phi\|_\infty = O_\lambda(1), \quad \|\partial_h(\partial_{xx}\phi)\|_\infty = O_\lambda(1), \quad \|h\Delta_h(\partial_{xx}\phi)\|_\infty = O_\lambda(1).$$

(178)

On the other hand, a direct computation gives

$$h^2\Delta_h p_\phi = h^2\Delta_h(\partial_{xx}\phi) r\Delta_h \rho + 2h^2\partial_h \partial_{xx} \phi \partial_h (r\Delta_h \rho) + h\partial_{xx} \phi \partial_h (r\Delta_h \rho).$$

From Propositions A.7(i) and A.8, estimates Eq. 178, and the fact that $(r \partial_{xx} \phi) = \partial_x (r \partial_{xx} \phi) = \partial_{xx} (r \partial_{xx} \phi) = s^2O_\lambda(1)$, we obtain the desired result.

Using Claim B.2, we readily see that

$$J_2 = \iint_Q \left[ s^3(h + h^2) + s(sh)^4 \right] O_\lambda(1)|z|^2 = \iint_Q s^2(sh)O_\lambda(1)|z|^2.$$  

(179)

Putting together Eqs. 173, 175–176, Eqs. 177 and 179, the desired result follows by adjusting some powers of $(sh)$.

B.2 Estimates involving time-discrete computations

B.2.1 Proof of Lemma 2.5

We begin by shifting the integral in time; hence,

$$I_{31} = -2 \iint_Q \tau \varphi \theta' rz \partial_h \rho \partial_h z = -2 \iint_Q \tau \theta' \varphi \partial_h rz \partial_h z,$$

(180)

where we have used formula Eq. 147 in the second equality. Observe that no boundary conditions appear since $z_{|\partial \Omega}^{n-\frac{1}{2}} = 0$ for $n \in [1, M]$. Noting that

$$(\varphi \partial_h rz) = (\varphi \partial_h \rho) rz + \frac{h^2}{4} \partial_h (\varphi \partial_h \rho) \partial_h z$$

thanks to Lemma A.2, we rewrite Eq. 180 as

$$I_{31} = -2 \iint_Q \tau \theta' (\varphi \partial_h \rho) rz \partial_h z - \frac{h^2}{2} \iint_Q \tau \theta' \partial_h (\varphi \partial_h \rho) (\partial_h z)^2$$

$$= -2 \iint_Q \tau \theta' (\varphi \partial_h \rho) \partial_h (z^2) - \frac{h^2}{2} \iint_Q \tau \theta' \partial_h (\varphi \partial_h \rho) (\partial_h z)^2.$$
Integrating by parts in space the first term in the above expression and using that \( \overrightarrow{\partial_h p} = \partial_h p \) for \( p \in \mathbb{R}^m \), we have

\[
I_{31} = \iint_Q \tau \theta' \partial_h (\varphi r \partial_h \rho) z^2 - \frac{h^2}{2} \iint_Q \tau \theta' \partial_h (\varphi r \partial_h \rho)(\partial_h z)^2.
\]

Using that \( \|\partial_h \varphi\|_\infty = \mathcal{O}_\lambda(1) \) and \( \partial_x (r \partial_x \rho) = s \mathcal{O}_\lambda(1) \), we can prove as in Claim B.1 that

\[
\partial_h (\varphi r \partial_h \rho) = \partial_h (\varphi r \partial_h \rho) = s \mathcal{O}_\lambda(1) + s (sh)^2 \mathcal{O}_\lambda(1).
\]

Notice that since \( (sh) \leq h (\delta T)^{-1} \leq 1 \), we can further simplify the above estimate and obtain that both derivatives are of order \( s \mathcal{O}_\lambda(1) \). Thus, from this remark, we have

\[
I_{31} = \iint_Q \tau (\mathcal{E}^{-s \theta'}) \mathcal{O}_\lambda(1)(\mathcal{E}^{-z})^2 + h^2 \iint_Q \tau (\mathcal{E}^{-s \theta'}) \mathcal{O}_\lambda(1)|\partial_h (\mathcal{E}^{-z})|^2.
\]

Using that \( \theta' = (2t - T)\theta^2 \) for \( t \in [0, T] \), we obtain

\[
I_{31} = \int_Q T \mathcal{E}^{-s \theta} \mathcal{O}_\lambda(1)(\mathcal{E}^{-z})^2 + \int_Q T \mathcal{E}^{-s \theta} (sh) \mathcal{O}_\lambda(1)|\partial_h (\mathcal{E}^{-z})|^2.
\]

This ends the proof.

B.2.2 Proof of Lemma 2.7

The proof of this term can be carried out exactly as in [22], since the space variable does not play any major role. For completeness, we sketch it briefly.

Using formula Eq. 151, we write

\[
I_{33} = -\iint_Q \tau \varphi (\mathcal{E}^{-\theta'}) (\mathcal{E}^{-z}) D_{\tau} z = -\frac{1}{2} \iint_Q \tau \varphi (\mathcal{E}^{-\theta'}) D_{\tau} (z^2)
+ \frac{\Delta t}{2} \iint_Q \tau \varphi (\mathcal{E}^{-\theta'}) (D_{\tau} z)^2,
\]

and integrating by parts in time using Eq. 155, we get

\[
I_{33} = -\frac{1}{2} \int_\Omega \tau \varphi (\theta')^2 (z^2) - \frac{1}{2} \int_\Omega \tau \varphi (\theta')^{M+\frac{1}{2}} (z^{M+\frac{1}{2}})^2
+ \frac{1}{2} \int_\Omega \tau \varphi (\theta') (D_{\tau} z)^2 + \frac{\Delta t}{2} \int_\Omega \tau \varphi (\theta') (D_{\tau} z)^2.
\]

By definition, we have that

\[
\theta' = (2t - T)\theta^2,
\]

thus \( (\theta')^\frac{1}{2} < 0 \) and \( (\theta')^{M+\frac{1}{2}} > 0 \). Therefore, recalling that \( \varphi < 0 \) for all \( x \in \Omega \), we see that the first two terms in Eq. 181 are positive and therefore can be dropped. A further computation using Eq. 182 and Lemma A.15(ii) and yields

\[
I_{33} \geq -\int_Q (\mathcal{E}^{-\mu_{33}} (\mathcal{E}^{-z})^2 - \int_Q (\mathcal{E}^{-\gamma_{33}} (D_{\tau} z)^2
with \( \mu_{33} = (\tau T^2 \theta^3 + \frac{\tau \Delta t}{\delta T}) \mathcal{O}_\lambda(1) \) and \( \gamma_{33} = \Delta t \tau \theta^2 \mathcal{O}_\lambda(1) \), where we have used that \( \varphi = \mathcal{O}_\lambda(1) \).
B.3 A new estimate

B.3.1 Proof of Lemma 2.8

This is the most delicate and cumbersome estimate since it combines the action of both space and time discrete results. For clarity, we have divided the proof in three steps.

**Step 1. An estimate for** \( I_{13} \). Using integration by parts in space, we get

\[
I_{13} = \int_{Q} \tilde{e} - (r \tilde{\rho} \Delta h z) D_t z \\
= - \int_{Q} D_t (\partial_h z) \partial_h (\tilde{e} - z) \tilde{e} - (r \tilde{\rho}) - \int_{Q} \tilde{e} - (\partial_h [r \tilde{\rho}]) \partial_h (\tilde{e} - z) D_t (\tilde{z}) \\
=: I_{13}^{(1)} + I_{13}^{(2)},
\]

where we have used that \( D_t \) commutes with \( \partial_h \) and \( m_h \) in the first and second terms, respectively. Using formula Eq. 151, we have that the \( I_{13}^{(1)} \) can be written as

\[
I_{13}^{(1)} = - \frac{1}{2} \int_{Q} \tilde{e} - (r \tilde{\rho}) D_t (|\partial_h z|^2) + \frac{\Delta t}{2} \int_{Q} \tilde{e} - (r \tilde{\rho}) (D_t (\partial_h z))^2
\]

and integrating by parts in time in the first integral, we get

\[
I_{13}^{(1)} = \int_{Q} |\partial_h (\tilde{e} + z)|^2 D_t (r \tilde{\rho}) - \int_{Q} (r \tilde{\rho})^{M+\frac{1}{2}} |(\partial_h z)^{M+\frac{1}{2}}|^2 \\
+ \int_{Q} (\tilde{r} \tilde{\rho}) \frac{1}{2} |(\partial_h z)^2|^2 + \frac{\Delta t}{2} \int_{Q} \tilde{e} - (r \tilde{\rho}) (D_t (\partial_h z))^2.
\]

From Proposition A.7(i), we observe that for \( 0 < (sh) \leq \frac{r h}{\delta T^2} < \epsilon_1(\lambda) \) with \( \epsilon_1(\lambda) \) small enough, we have that \( (r \tilde{\rho}) \geq c_\lambda > 0 \); thus, the last three terms of the above equation have prescribed signs (we remark that the extra average does not affect the form of that estimate). That is,

\[
I_{13}^{(1)} \geq K - c_\lambda \int_{Q} |(\partial_h z)^{M+\frac{1}{2}}|^2 \\
+ c_\lambda \int_{Q} |(\partial_h z)^2|^2 + c_\lambda \Delta t \int_{Q} (D_t (\partial_h z))^2,
\]

where \( K := \int_{Q} |\partial_h (\tilde{e} + z)|^2 D_t (\tilde{r} \tilde{\rho}) \). We claim the following.

**Claim B.3** Provided \( \frac{\Delta t}{\delta^2 T^3} \leq \frac{1}{2} \), we have

\[
\overline{D_t (r \tilde{\rho})} = T \tilde{e}^{-(\theta [sh]^2)} O_\lambda (1) + \left( \frac{\tau \Delta t}{\delta^3 T^4} \right) \left( \frac{\tau h}{\delta T^2} \right) O_\lambda (1).
\]

**Proof** Notice that we can write \( 2 \overline{D_t (r \tilde{\rho})} = (s^+ + s^-) \overline{D_t (r \tilde{\rho})} \). The result follows by applying Lemma A.16(ii). \( \square \)
Observe that the condition on $\Delta t$ of Claim B.3 is in agreement with our initial hypothesis. Actually, the assumption of Lemma 2.8 is stronger than this condition. Using that $|\partial_h(\bar{E}^+_z)|^2 \leq C|\partial_h(\bar{E}^-z)|^2 + C(\Delta t)^2(\sigma_t(\partial_h z))^2$ and Eq. 185 we get

$$K \geq -\iint_{Q} |\partial_h(\bar{E}^-z)|^2 (\bar{E}^-v_K) - C_\lambda \Delta t \iint_{Q} \left( \frac{\tau \Delta t}{\delta^3 T^4} \right) \left( \frac{\tau h}{\delta^2 T^2} \right) (\sigma_t(\partial_h z))^2 \geq 0 \tag{186}$$

for some $C_\lambda > 0$ uniform with respect to $\Delta t$ and where $v_K := T \theta (sh)^2 \sigma_\lambda(1) + \left( \frac{\tau \Delta t}{\delta^3 T^4} \right) \left( \frac{\tau h}{\delta^2 T^2} \right) \sigma_\lambda(1)$. Notice that taking $\epsilon_1(\lambda)$ small enough in our initial hypothesis the last term of Eq. 184 controls the last term of Eq. 186. So, overall, $I_{13}^{(1)}$ can be bounded as

$$I_{13}^{(1)} \geq -c_\lambda \int_{\Omega} |(\partial_h z)^M|^\frac{1}{2} \right|^2 - \iint_{Q} |\partial_h(\bar{E}^-z)|^2 (\bar{E}^-v_K) + c_\lambda \Delta t \iint_{Q} (\sigma_t(\partial_h z))^2 \geq 0 \tag{187}$$

for a constant $c_\lambda > 0$ uniform with respect to $\Delta t$.

Let us comeback to the term $I_{13}^{(2)}$. From Proposition A.8(ii), we have that $\partial_h[\bar{r} \bar{\rho}] = (sh)^2 \sigma_\lambda(1)$ and using Cauchy-Schwarz and Young inequalities, we get

$$|I_{13}^{(2)}| \leq C \left( \iint_{Q} \bar{\xi}^- (s^{-1}[sh]^2) (\sigma_t(z))^2 + \iint_{Q} \bar{\xi}^- (s[s h]^2) |\partial_h(\bar{E}^-z)|^2 \right),$$

for some $C > 0$ only depending on $\lambda$. A direct computation shows that $|\sigma_t(z)|^2 = |\sigma_t(z)|^2 \leq |\sigma_t(z)|^2$ by convexity. This, together the fact that $z_{|\delta \Omega} = 0$ for $n \in \mathbb{N}$, we can use Eq. 147 to deduce

$$|I_{13}^{(2)}| \leq C \left( \iint_{Q} \bar{\xi}^- (s^{-1}[sh]^2) (\sigma_t(z))^2 + \iint_{Q} \bar{\xi}^- (s[s h]^2) |\partial_h(\bar{E}^-z)|^2 \right). \tag{188}$$

Combining Eqs. 187 and 188 will provide a lower bound for $I_{13}$. This concludes Step 1.

**Step 2. An estimate for $I_{23}$**. Using that $\bar{z} = z + \frac{h^2}{4} \partial_h \bar{r} \partial_h z$, we can rewrite $I_{23}$ as

$$I_{23} = \iint_{Q} \bar{\xi}^- \left( r \Delta_h \rho \bar{z} \right) \sigma_t(z)$$

$$= \iint_{Q} \bar{\xi}^- (r \Delta_h \rho z) \sigma_t(z) + \frac{h^2}{4} \iint_{Q} \bar{\xi}^- (r \Delta_h \rho) \Delta_h(\bar{E}^-z) \sigma_t(z) =: I_{32}^{(1)} + I_{32}^{(2)}.$$

Let us estimate $I_{23}^{(1)}$. Using identity Eq. 151 and the integration-by-parts formula in Eq. 155, we see that

$$I_{23}^{(1)} = \frac{1}{2} \iint_{Q} \bar{\xi}^- (r \Delta_h \rho) \sigma_t(z) - \frac{\Delta t}{2} \iint_{Q} \bar{\xi}^- (r \Delta_h \rho) (\sigma_t(z))^2$$

$$= -\frac{1}{2} \iint_{Q} \bar{\xi}^- (r \Delta_h \rho) (\sigma_t(z))^2 + \frac{1}{2} \iint_{Q} (r \Delta_h \rho)^{M+\frac{1}{2}} (\sigma_t(z))^2$$

$$- \frac{1}{2} \iint_{\Omega} (r \Delta_h \rho)^{\frac{1}{2}} |z|^2$$

$$- \frac{\Delta t}{2} \iint_{Q} \bar{\xi}^- (r \Delta_h \rho) (\sigma_t(z))^2.$$

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To estimate the last three terms of the above expression, we can use that
\[ r \Delta_h \rho = s^2 \mathcal{O}_\lambda(1) + s \lambda \phi \mathcal{O}(1) + s^2 (sh)^2 \mathcal{O}_\lambda(1) = s^2 \mathcal{O}_\lambda(1). \tag{189} \]

For the first one, we can use directly Lemma A.16(iii) sampled on the primal mesh \( \mathcal{P} \). Thus,
\[
I^{(1)}_{23} = \int_\Omega (\bar{\varepsilon} - \mu_p) (\bar{\varepsilon} - z)^2 + \int_\Omega (s^{M+\frac{1}{2}})^2 \mathcal{O}_\lambda(1) |z|^{M+\frac{1}{2}} \frac{1}{2} + \int_\Omega (s^2)^2 \mathcal{O}_\lambda(1) |z|^2 \frac{1}{2} + \Delta t \int_\Omega (\bar{\varepsilon} - s^2) \mathcal{O}_\lambda(1) (D_t z)^2, \tag{190}
\]
where \( \mu_p = T s^2 \theta \mathcal{O}_\lambda(1) + \left( \frac{\tau^2 \Delta t}{\delta^2 T^6} \right) \mathcal{O}_\lambda(1) + \left( \frac{\tau^2 \Delta t}{\delta^2 T^4} \right) \left( \frac{\tau h}{\delta T^2} \right)^3 \mathcal{O}_\lambda(1) \).

We focus now on the term \( I^{(2)}_{23} \). Using that \( \Delta_h z = \partial_h (\partial_h z) \), we can integrate by parts in space and get
\[
I^{(2)}_{23} = \frac{h^2}{4} \int_\Omega \bar{\varepsilon} - (r \Delta_h \rho) (\Delta_h (\bar{\varepsilon} - z) D_t z
\]
\[
= - \frac{h^2}{4} \int_\Omega \bar{\varepsilon} - (\Delta_h (\bar{\varepsilon} - z) \partial_h (\bar{\varepsilon} - z) D_t (\partial_h z)
\]
\[
- \frac{h^2}{4} \int_\Omega \bar{\varepsilon} - (\partial_h (r \Delta_h \rho)) \partial_h (\bar{\varepsilon} - z) D_t (\partial_h z)
\]
\[
=: J_1 + J_2, \tag{191}
\]
where we have used once again that \( D_t \) commutes with the space-discrete operations \( \partial_h \) and \( m_h \).

Arguing as we did for \( I^{(1)}_{12} \), we see that
\[
J_1 = - \frac{h^2}{8} \int_\Omega \bar{\varepsilon} - (r \Delta_h \rho) D_t (|\partial_h z|^2) + \Delta t \frac{h^2}{8} \int_\Omega \bar{\varepsilon} - (r \Delta_h \rho) |D_t (\partial_h z)|^2
\]
\[
= - \frac{h^2}{8} \int_\Omega (r \Delta_h \rho)^{M+\frac{1}{2}} |(\partial_h z)^{M+\frac{1}{2}}|^2 + \frac{h^2}{8} \int_\Omega (r \Delta_h \rho)^{\frac{1}{2}} |(\partial_h z)^{\frac{1}{2}}|^2
\]
\[
+ \frac{h^2}{8} \int_\Omega D_t (r \Delta_h \rho) |\partial_h (\bar{\varepsilon} - z)|^2 + \Delta t \frac{h^2}{8} \int_\Omega \bar{\varepsilon} - (r \Delta_h \rho) |D_t (\partial_h z)|^2.
\]

We have the following.

**Claim B.4** Provided \( \frac{\Delta t \tau}{\delta^2 T^4} \leq \frac{1}{2} \), we have
\[
D_t (r \Delta_h \rho) = T (s^2 \theta) \mathcal{O}_\lambda(1) + \left( \frac{\tau^2 \Delta t}{\delta^4 T^6} \right) \mathcal{O}_\lambda(1) + \left( \frac{\tau^2 \Delta t}{\delta^4 T^4} \right) \left( \frac{\tau h}{\delta T^2} \right)^3 \mathcal{O}_\lambda(1). \tag{192}
\]

The proof is exactly as in Claim B.3 but taking into account the estimate in Lemma A.16(iii).

Using Eq. 189 (and noting that the estimate does not change with the extra average in space), Claim B.4 and the fact that \( |\partial_h (\bar{\varepsilon} - z)|^2 \leq C|\partial_h (\bar{\varepsilon} - z)|^2 + \).
\(C(\Delta t)^2(\mathcal{D}_t(\partial_h z))^2\), we compute after a long but straightforward computation that

\[
|\mathcal{J}_1| \leq C \int_{\Omega} (s^{M+\frac{1}{2}} h)^2 |(\partial_h z)^{M+\frac{1}{2}}|^2 + C \int_{\Omega} (s^{\frac{1}{2}} h)^2 |(\partial_h z)^{\frac{1}{2}}|^2 \\
+C \int_{Q} T \bar{\varepsilon}^{-\theta} s[sh] h^2 |\partial_h(\bar{\varepsilon}^{-z})|^2 \\
+C \int_{Q} \left[ \left( \frac{\tau^2 \Delta t}{\delta^4 T^6} \right) + \left( \frac{\tau \Delta t}{\delta^3 T^4} \right) \left( \frac{\tau h}{\delta T^2} \right)^3 \right] |\partial_h(\bar{\varepsilon}^{-z})|^2
\]

\[+C \Delta t \int_{Q} \left( \frac{\tau \Delta t}{\delta^2 T^3} \right) \left( \frac{\tau h}{\delta T^2} \right) |\mathcal{D}_t(\partial_h z)|^2 + C \Delta t \int_{Q} \left( \frac{\tau h}{\delta T^2} \right)^2 |\mathcal{D}_t(\partial_h z)|^2 \\
+C \Delta t \int_{Q} \left[ \left( \frac{\tau^2 \Delta t}{\delta^4 T^6} \right) + \left( \frac{\tau \Delta t}{\delta^3 T^4} \right) \left( \frac{\tau h}{\delta T^2} \right)^3 \right] |\mathcal{D}_t(\partial_h z)|^2,
\]

(193)

for some positive \(C\) only depending on \(\lambda\). Above we used that \(h \ll 1\) to adjust some powers of \(h\). We also notice that in the above equation, all of the terms containing \(|\mathcal{D}_t(\partial_h z)|^2\) can be made small enough by the initial hypothesis of the lemma. This will be important in the next step since those terms will be absorbed by the corresponding one in Eq. 187.

We turn our attention to the term \(\mathcal{J}_2\). Estimate Eq. 189 and a quick computation yield

\[
|\mathcal{J}_2| \leq C \left( \int_{Q} \bar{\varepsilon}^{-\left(s[sh]^2\right)} |\partial_h(\bar{\varepsilon}^{-z})|^2 + \int_{Q} \bar{\varepsilon}^{-\left(s^{-1}[sh]^2\right)} |\mathcal{D}_t(\bar{z})|^2 \right) \\
\leq C \left( \int_{Q} \bar{\varepsilon}^{-\left(s[sh]^2\right)} |\partial_h(\bar{\varepsilon}^{-z})|^2 + \int_{Q} \bar{\varepsilon}^{-\left(s^{-1}[sh]^2\right)} |\mathcal{D}_t\bar{z}|^2 \right), (194)
\]

where the second inequality is obtained by arguing exactly as for the term \(I_{13}^{(2)}\).

**Step 3. Conclusion.** Now, we are in position to find a lower bound for \(I_{13} + I_{23}\). First, we shall notice that the last three terms of Eq. 193 can be controlled by the last term of Eq. 187 by decreasing (if necessary) the parameter \(\varepsilon_1(\lambda)\) in our initial hypothesis. Then, we just need to combine Eq. 187, Eq. 188, Eq. 190, Eq. 191, Eq. 193, and Eq. 194 to obtain

\[
I_{13} + I_{23} \geq -c_\lambda \int_{\Omega} |(\partial_h z)^{M+\frac{1}{2}}|^2 - \int_{Q} \bar{\varepsilon}^{- \mu_+}(\bar{\varepsilon}^{-z})^2 - \int_{Q} \bar{\varepsilon}^{- \nu_+})|\partial_h(\bar{\varepsilon}^{-z})|^2 \\
+C'_\lambda \Delta t \int_{Q} (\mathcal{D}_t(\partial_h z))^2 - \int_{Q} (\bar{\varepsilon}^{- \gamma_+})|\mathcal{D}_t\bar{z}|^2 \\
- \int_{\Omega} (s^{M+\frac{1}{2}} h)^2 \mathcal{O}_\lambda(1)|\partial_h z)^{M+\frac{1}{2}}|^2 - \int_{\Omega} (s^{\frac{1}{2}} h)^2 \mathcal{O}_\lambda(1)|\partial_h z)^{\frac{1}{2}}|^2 \\
- \int_{\Omega} (s^{M+\frac{1}{2}} h)^2 \mathcal{O}_\lambda(1)|(\partial_h z)^{M+\frac{1}{2}}|^2 - \int_{\Omega} (s^{\frac{1}{2}} h)^2 \mathcal{O}_\lambda(1)|(\partial_h z)^{\frac{1}{2}}|^2,
\]

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for some \( c_\lambda, c'_\lambda > 0 \) uniform with respect to \( h \) and \( \Delta t \) and where

\[
\nu_+ := \left\{ \theta (sh)^2 + s(sh)^2 + \left( \frac{\tau \Delta t}{\delta T^4} \right) \left( \frac{\tau h}{\delta T^2} \right) + \left( \frac{\tau^2 \Delta t}{\delta^2 T^6} \right) \left( \frac{\tau h}{\delta T^2} \right)^2 \right\} O_\lambda(1), \]

\[
\mu_+ := \left\{ \tau s^2 + \left( \frac{\tau^2 \Delta t}{\delta^2 T^6} \right) \left( \frac{\tau h}{\delta T^2} \right)^2 \right\} O_\lambda(1), \]

\[
\nu_+ := \left\{ s^{-1} (sh)^2 + \Delta t s^2 \right\} O_\lambda(1).
\]

This ends the proof.

### Appendix C: Some intermediate lemmas

#### C.1 Proof of Lemma 2.12

By shifting the integral in time (see Eq. 152) and then using Eq. 147, we have

\[
\tau^2 \lambda^2 \iint_Q (\xi^- \theta)|\partial_h (\xi^- z)|^2 = \lambda^2 \iint_Q s\phi |\partial_h z|^2 + \frac{h^2}{2} \iint_Q s\lambda^2 (\phi |\partial_h z|^2)^2 \frac{j}{2} + \frac{h}{2} \int_0^T s\lambda^2 (\phi |\partial_h z|^2)^2 N_{N+1}^2, \tag{195}
\]

where we recall that \( s = \tau \theta \). Since \( \phi \) is a positive function, notice that the last two terms of the above expression are positive.

Let us focus on the term \( D \). From Lemma A.2, we get

\[
D = \lambda^2 \iint_Q s\phi |\partial_h z|^2 + \frac{h^2}{4} \iint_Q s\lambda^2 (\phi |\partial_h z|^2) \frac{1}{2} \tag{196}
\]

where we have used that \( \overline{\partial}_h \partial_h = \Delta_h \). On the other hand, integrating by parts in the space variable, from \( D_2 \), we get

\[
D_2 = -\frac{h^2}{4} \iint_Q s\lambda^2 (\overline{\partial}_h \phi) (\partial_h z)^2 + \frac{h^2}{4} \iint_Q s\lambda^2 (\overline{\partial}_h \phi)_{N+1}^2 (\partial_h z)^2 N^2_\frac{1}{2} - \frac{h^2}{4} \iint_Q s\lambda^2 (\overline{\partial}_h \phi)_{0}^2 (\partial_h z)^2_\frac{1}{2}. \tag{196}
\]

Notice that \( \overline{\partial}_h \phi = \partial_x \phi + h^2 \mathcal{O}_\lambda(1) = O_\lambda(1) \) thanks to Lemma A.5(iii). Thus, once the parameter \( \lambda \) is fixed, we can choose \( h \) sufficiently small such that the last two terms of Eq. 195 control the last two terms of Eq. 196. Moreover, using items (ii) and (iii) of Lemma A.5, we see that \( \partial_h (\overline{\partial}_h \phi) = \partial_x \phi + h^2 \mathcal{O}_\lambda(1) = O_\lambda(1) \) and
\[ \phi = \phi + h \mathcal{O}_\lambda(1). \]

From these and putting together Eq. 195–Eq. 196, we obtain

\[
\tau \lambda^2 \iint_{Q} (\bar{e} - \theta) |\partial h (\bar{e} - z)|^2 \geq \lambda^2 \iint_{Q} s \phi |\partial h z|^2 + \iint_{Q} (s h) \mathcal{O}_\lambda(1) |\partial h z|^2 \\
+ \frac{\lambda^2 h^2}{4} \iint_{Q} s \phi (\Delta h z)^2 + h^4 \iint_{Q} s \mathcal{O}_\lambda(1) (\Delta h z)^2 \\
+ h^2 \iint_{Q} s \mathcal{O}_\lambda(1) (\partial h z)^2.
\]

Shifting the integrals in time in the right-hand side of the above expression yields the desired result.

C.2 Proof of Lemma 2.14

We begin by increasing, if necessary, the parameter \( \tau_1 \) such that \( \tau_1 \geq 1 \) and \( \tau \geq 1 \). Notice that

\[
1 \leq \tau_1 \left( \frac{1}{T} + 1 \right) \leq \tau \theta(t) = s(t), \quad t \in [0, T]. \tag{197}
\]

We repeat the definition of \( X_2 \) for convenience. We have

\[
X_2 = \left\{ \frac{\tau \Delta t}{\delta^4 T^5} \right\} + \left( \frac{\tau^2 \Delta t}{\delta^4 T^6} \right) + \left( \frac{\Delta t \tau}{\delta^3 T^4} \right)^2 + \left( \frac{\Delta t \tau^2}{\delta^3 T^4} \right)^2 + \left( \frac{\tau \Delta t}{\delta^3 T^4} \right) \left( \frac{\tau h}{\delta T^2} \right)^3 \\
\iint_{Q} (\bar{e} - z)^2 \\
+ \left\{ \frac{\tau^2 \Delta t}{\delta^4 T^6} + \left( \frac{\tau \Delta t}{\delta^3 T^4} \right) \left( \frac{\tau h}{\delta T^2} \right) + \left( \frac{\tau \Delta t}{\delta^3 T^4} \right) \left( \frac{\tau h}{\delta T^2} \right)^3 \right\} \iint_{Q} |\partial h (\bar{e} - z)|^2. \tag{198}
\]

We remark that at this point, we have the condition \( \frac{\tau h}{\delta T^2} \leq \varepsilon_3 \) for some \( \varepsilon_3 = \varepsilon_3(\lambda) \) small enough (see Eq. 74). Recalling that \( \delta \leq 1/2, 0 < T < 1 \) and Eq. 197 allows us to see that provided

\[
\frac{\tau^2 \Delta t}{\delta^4 T^6} \leq \kappa_1 \tag{199}
\]

for some \( \kappa_1 > 0 \) small enough, the term \( X_2 \) can be bounded as

\[
X_2 \leq \left( 2 \kappa_1 + 2 \kappa_1^2 + \kappa_1 \varepsilon_3^2 \right) \iint_{Q} (\bar{e} - s)^3 (\bar{e} - z)^2 \leq 5 \kappa_1 \iint_{Q} (\bar{e} - s)^3 (\bar{e} - z)^2. \tag{200}
\]

Notice that in the first inequality we have the product of the small parameters \( \kappa_1 \) and \( \varepsilon_3 \). Since \( \varepsilon_3 \) has been already chosen small we can bound it uniformly by 1.

On the other hand, from definition Eq. 76 and rewriting it as \( W = \sum_{i=1}^{3} W^{(i)} \), we proceed to bound each of the terms. For the first one, using that \( \max_{t \in [0, T]} \theta \leq
We get
\[
W^{(1)} = \iint_Q \Delta t \left( \tau T (\overline{e} \overline{- \theta})^2 + \frac{\tau \Delta t}{\delta^3 T^2} \right) (\overline{D}_t z)^2
\]
\[
\leq \left\{ \frac{\Delta t \tau^2}{\delta^3 T^5} + \left( \frac{\tau \Delta t}{\delta^2 T^3} \right)^2 \right\} \iint_Q (\overline{e} - s)^{-1} (\overline{D}_t z)^2
\]
\[
\leq (\kappa_1 + \kappa_1^2) \iint_Q (\overline{e} - s)^{-1} (\overline{D}_t z)^2. \tag{201}
\]

For \( W^{(2)} \), we have
\[
W^{(2)} = \iint_Q \Delta t (\overline{e} - s)^2 (\overline{D}_t z)^2 = \iint_Q \Delta t (\overline{e} - s)^3 (\overline{e} - s)^{-1} (\overline{D}_t z)^2
\]
\[
\leq \kappa_2 \iint_Q (\overline{e} - s)^{-1} (\overline{D}_t z)^2, \tag{202}
\]
where the condition
\[
\frac{\tau^3 \Delta t}{\delta^3 T^6} \leq \kappa_2 \tag{203}
\]
holds for some \( \kappa_2 > 0 \) small enough. Finally, using Eq. \( 197 \), we see that
\[
W^{(3)} = \iint_Q \left( T^2 (\tau \Delta t)^2 \theta^2 + \frac{T^2 (\Delta t)^4}{\delta^6 T^8} \right) (\overline{D}_t z)^2
\]
\[
\leq \tau^4 (\Delta t)^2 T^2 \iint_Q (\overline{e} - \theta)^6 (\overline{e} - s)^{-1} (\overline{D}_t z)^2 + \left( \frac{\tau \Delta t}{\delta^3 T^4} \right)^4 \iint_Q (\overline{e} - s)^{-1} (\overline{D}_t z)^2
\]
\[
\leq \left\{ \left( \frac{T^2 \Delta t}{\delta^3 T^5} \right)^2 + \left( \frac{\tau \Delta t}{\delta^3 T^4} \right)^4 \right\} \iint_Q (\overline{e} - s)^{-1} (\overline{D}_t z)^2
\]
\[
\leq (\kappa_1^2 + \kappa_1^4) \iint_Q (\overline{e} - s)^{-1} (\overline{D}_t z)^2. \tag{204}
\]

Since \( \delta \leq 1/2, \tau \geq 1 \) and \( 0 < T < 1 \), we can combine conditions Eqs. \( 199 \) and \( 203 \) into a single one verifying
\[
\frac{\tau^4 \Delta t}{\delta^4 T^6} \leq \epsilon_5
\]
for some \( \epsilon_5 = \epsilon_5(\lambda) \) small enough. Collecting estimates Eq. \( 201 \), Eqs. \( 202 \), and \( 204 \) gives the desired result.

### C.3 Proof of Lemma 2.16

We adapt the procedure used in the continuous setting. In particular, we follow [27, pp. 1409]. Let us consider \( \eta \in C_c^\infty(\Omega) \) such that
\[
0 \leq \eta \leq 1 \text{ in } \Omega, \quad \eta = 1 \text{ in a neighborhood of } B_0, \quad \text{supp } \eta \subset \subset B. \tag{205}
\]

By the properties of discretization, we can ensure the additional property
\[
\frac{\overline{D}_t \eta(\eta)}{\eta^{1/2}} \in L^\infty(\Omega) \tag{206}
\]
uniformly with respect to $h$. Obviously, the above function is supported within $B$.

By shifting the time integral and the definition of the function $\eta$, we have

$$
\iint_{Q \times _0} s |\partial_h z|^2 \leq \iint_{Q} s \eta |\partial_h z|^2 = - \iint_{Q} s \partial_h (\eta \partial_h z) z \\
= - \iint_{Q} s \partial_h (\eta \partial_h z) z - \iint_{Q} s \eta \Delta_h z =: L_1 + L_2. \quad (207)
$$

For the first term in the above expression, we have by using Eq. 206 and Cauchy-Schwarz and Young inequalities

$$
|L_1| \leq \frac{1}{2} \iint_{Q} \left| \frac{\partial_h (\eta)}{\eta^{1/2}} \right|^2 |\partial_h z|^2 + \frac{1}{2} \iint_{Q} s^2 \eta |z|^2 \\
\leq C \iint_{Q} |\partial_h z|^2 + C \iint_{Q} s^3 \eta |z|^2, \quad (208)
$$

where we have used that $s(t) \geq 1$ for all $t$ to adjust the power of $s$ in the last term of the above equation.

For the term $L_2$, using that $\eta = \eta + hO(1)$ together with Cauchy-Schwarz and Young inequalities

$$
|L_2| \leq \gamma \iint_{Q} s^{-1} \eta |\Delta_h z|^2 + C \gamma \iint_{Q} s^3 \eta |z|^2 \\
+ \frac{1}{2} \iint_{Q} s^{-1} (sh)^2 |\Delta_h z|^2 + \frac{1}{2} \iint_{Q} s |z|^2 \quad (209)
$$

for any $\gamma > 0$.

Using estimates Eq. 208–Eq. 209 in Eq. 207 and recalling the properties of $\eta$ in Eq. 205 gives the desired result after shifting the integral in time.

**Acknowledgements** The authors would like to thank Prof. Franck Boyer (Institut de Mathématiques de Toulouse) for some clarifying discussions about the works [11, 26]. They also would like to thank the anonymous referees for their careful reading and valuable comments and suggestions that helped to improve the contents and presentation of this paper.

**Funding** This work has been supported by the project A1-S-17475 of CONACyT, Mexico, and by the National Autonomous University of Mexico, grant PAPIIT, IN100919. The work of the first author has been partially supported by the program “Estancias posdoctorales por México” of CONACyT, Mexico.

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