Stellar equilibrium in Einstein-Chern-Simons gravity

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Abstract

We consider a spherically symmetric internal solution within the context of Einstein-Chern-Simons gravity and derive a generalized five-dimensional Tolman-Oppenheimer-Volkoff (TOV) equation. It is shown that the generalized TOV equation leads, in a certain limit, to the standard five-dimensional TOV equation.

PACS numbers: 04.50.+h, 04.20.Jb, 04.90.+e

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I. INTRODUCTION

Some time ago was shown that the standard, five-dimensional General Relativity can be obtained from Chern-Simons gravity theory for a certain Lie algebra \( \mathfrak{B} \) [1], which was obtained from the AdS algebra and a particular semigroup \( S \) by means of the S-expansion procedure introduced in Refs. [2, 3].

The five-dimensional Chern-Simons Lagrangian for the \( \mathfrak{B} \) algebra is given by [1]

\[
L_{\text{ChS}}^{(5)} = \alpha_1 l^2 \varepsilon_{abcde} R^{ab} R^{cd} e^e + \alpha_3 \varepsilon_{abcde} \left( \frac{2}{3} R^{ab} e^c e^d e^e + 2 l^2 k^{ab} R^{cd} T^e + l^2 R^{ab} R^{cd} h^e \right),
\]

where \( l \) is a length scale in the theory (see [1]), \( R^{ab} = d\omega^{ab} + \omega^{ac} \omega^{cb} \) is the curvature two-form with \( \omega^{ab} \) the spin connection, and \( T^a = De^a \) with \( D \) the covariant derivative with respect to the Lorentz piece of the connection.

From (1) we can see that [1]:

(i) the Lagrangian is split into two independent pieces, one proportional to \( \alpha_1 \) and the other to \( \alpha_3 \). If one identifies the field \( e^a \) with the vielbein, the piece proportional to \( \alpha_3 \) contains the Einstein-Hilbert term \( \varepsilon_{abcde} R^{ab} e^c e^d e^e \) plus non-linear couplings between the curvature and the bosonic “matter” fields \( h^a \) and \( k^{ab} = -k^{ba} \), which transform as a vector and as a tensor under local Lorentz transformations, respectively.

(ii) In the strict limit where the coupling constant \( l \) equals zero we obtain solely the Einstein-Hilbert term in the Lagrangian [1].

(iii) In the five-dimensional case, the connection of Eq. (17) of Ref. [1] has two possible candidates to be identified with the vielbein (see [4]), namely, the fields \( e^a \) and \( h^a \), since both transform as vectors under local Lorentz transformations. Choosing \( e^a \), makes the Einstein-Hilbert term to appear in the action, and \( T^a = De^a \) can be interpreted as the torsion two-form. This choice brings in the Einstein equations.

It is the purpose of this letter to find the stellar interior solution of the Einstein-Chern-Simons field equations, which were obtained in Refs. [5, 6].

We derive the generalized five-dimensional Tolman-Oppenheimer-Volkoff (TOV) equation and then we show that this generalized TOV equation leads, in a certain limit, to the standard five-dimensional TOV equation.
II. EINSTEIN-CHERN-SIMONS FIELD EQUATIONS FOR A SPHERICALLY SYMMETRIC METRIC

In Ref. [5] it was found that in the presence of matter described by the Lagrangian $L_M = L_M(e^a, h^a, \omega_{ab})$, we see that the corresponding field equations are given by

$$\varepsilon_{abcde} R^{cd} T^e = 0,$$

$$\alpha_3 l^2 \varepsilon_{abcd} R^{bc} R^{de} = -\frac{\delta L_M}{\delta h^a},$$

$$\varepsilon_{abcd} \left( 2\alpha_3 R^{bc} e^d e^e + \alpha_1 l^2 R^{bc} R^{de} + 2\alpha_3 l^2 D_\omega h^{bc} R^{de} \right) = -\frac{\delta L_M}{\delta e^a},$$

$$2\varepsilon_{abcd} \left( \alpha_3 l^2 R^{cd} T^e + \alpha_3 l^2 D_\omega k^{cd} T^e + \alpha_3 l^2 R^{cd} D_\omega h^e + \alpha_3 l^2 R^{cd} k^e e^f \right) = -\frac{\delta L_M}{\delta \omega^{ab}}. \tag{2}$$

For simplicity we will assume $T^a = 0$ and $k^{ab} = 0$. In this case the field equations (2) can be written in the form [6]

$$de^a + \omega^a e^b = 0,$$

$$\varepsilon_{abcd} R^{cd} D_\omega h^e = 0,$$

$$\alpha_3 l^2 \star \left( \varepsilon_{abcd} R^{bc} R^{de} \right) = -\star \left( \frac{\delta L_M}{\delta h^a} \right),$$

$$\star \left( \varepsilon_{abcd} R^{bc} e^d e^e \right) + \frac{1}{2\alpha} l^2 \star \left( \varepsilon_{abcd} R^{bc} R^{de} \right) = \kappa_E T_{ab} e^b, \tag{3}$$

where $\alpha = \alpha_3/\alpha_a$, $\kappa_E = \kappa/2\alpha_3$, $T_{ab} = \star (\delta L_M/\delta e^a)$, “$\star$” is the Hodge star operator (see Appendix B) and $T_{ab}$ is the energy-momentum tensor of matter fields (for details see Ref. [6]).

Since we are assuming spherical symmetry the metric will be of the form

$$ds^2 = -e^{2f(r)} dt^2 + e^{2g(r)} dr^2 + r^2 d\Omega_3^2 = \eta_{ab} e^a e^b \tag{4}$$

where $d\Omega_3^2 = d\theta_1^2 + \sin^2 \theta_1 d\theta_2^2 + \sin^2 \theta_1 \sin^2 \theta_2 d\theta_3^2$ and $\eta_{ab} = \text{diag}(-1, +1, +1, +1, +1)$. The two unknown functions $f(r)$ and $g(r)$ will not turn out to be the same as in Ref. [6]. In Ref. [6] was found a spherically symmetric exterior solution, i.e. a solution where $\rho(r) = p(r) = 0$. Now $f(r)$ and $g(r)$ must satisfy the field equations inside the star, where $\rho(r) \neq 0$ and $p(r) \neq 0$. For this we need the energy-momentum tensor for the stellar material, which is taken to be a perfect fluid.

Introducing an orthonormal basis,

$$e^T = e^{f(r)} dt, \quad e^R = e^{g(r)} dr, \quad e^1 = r d\theta_1, \quad e^2 = r \sin \theta_1 d\theta_2, \quad e^3 = r \sin \theta_1 \sin \theta_2 d\theta_3.$$
Taking the exterior derivatives, using Cartan’s first structural equation \( T^a = de^a + \omega^a_b e^b = 0 \) and the antisymmetry of the connection forms we find the non-zero connection forms. The use of Cartan’s second structural equation permits to calculate the curvature matrix \( R^a_b = d\omega^a_b + \omega^a_c \omega^c_b \).

Introducing these results in (3) and considering the energy-momentum tensor as the energy-momentum tensor of a perfect fluid at rest, i.e., \( T_{TT} = \rho(r) \) and \( T_{RR} = T_{ii} = p(r) \), where \( \rho(r) \) and \( p(r) \) are the energy density and pressure (for the perfect fluid), we find

\[
\begin{align*}
\frac{e^{-2g}}{r^2} & \left( g' r + e^{2g} - 1 \right) + \text{sgn}(\alpha) \frac{e^{-2g}}{r^3} g' \left( 1 - e^{-2g} \right) = \frac{\kappa_E}{12} \rho, \\
\frac{e^{-2g}}{r^2} & \left( f' r - e^{2g} + 1 \right) + \text{sgn}(\alpha) \frac{e^{-2g}}{r^3} f' \left( 1 - e^{-2g} \right) = \frac{\kappa_E}{12} p,
\end{align*}
\]

\[
\begin{align*}
\frac{e^{-2g}}{r^2} & \left\{ \left( - f' g' r^2 + f'' r^2 + 2 f' r - 2 g' r - e^{2g} + 1 \right) \\
& + \text{sgn}(\alpha) \frac{e^{-2g}}{r^3} \left( f'' - f' g' - e^{-2g} f'' - e^{-2g} (f')^2 + 3 e^{-2g} f' g' \right) \right\} = \frac{\kappa_E}{4} p.
\end{align*}
\]

III. **THE GENERALIZED TOLMAN-OPPENHEIMER-VOLKOFF EQUATION**

Since when the torsion is null, the energy-momentum tensor satisfies the following condition (see Appendix B):

\[
D_\omega(\ast T_a) = 0,
\]

we find that, for a spherically symmetric metric, (8) yields

\[
f'(r) = - \frac{p'(r)}{\rho(r) + p(r)},
\]

an expression known as the hydrostatic equilibrium equation.

Following the usual procedure, we find that (5) has the following solution:

\[
e^{-2g(r)} = 1 + \text{sgn}(\alpha) \frac{r^2}{l^2} - \text{sgn}(\alpha) \sqrt{\frac{r^4}{l^4} + \text{sgn}(\alpha) \frac{\kappa_E}{6\pi^2 l^2} \mathcal{M}(r)},
\]

where the Newtonian mass \( \mathcal{M}(r) \) is given by

\[
\mathcal{M}(r) = 2\pi^2 \int_0^r \rho(\bar{r}) \bar{r}^3 d\bar{r}.
\]

On the other hand, from (6) we find that

\[
\frac{df(r)}{dr} = f'(r) = \text{sgn}(\alpha) \frac{\kappa_E}{12 l^2 e^{-2g(r)}} \frac{p(r) r^3 + 12 r (1 - e^{-2g(r)})}{(1 - e^{-2g(r)} + \text{sgn}(\alpha) \frac{r^2}{l^2})}.
\]
Introducing (12) into (9) we find
\[
\frac{dp(r)}{dr} = p'(r) = -\text{sgn}(\alpha) \frac{(\rho(r) + p(r)) (\kappa_E p(r)r^3 + 12r(1 - e^{-2g(r)}))}{12r^2 e^{-2g(r)} (1 - e^{-2g(r)} + \text{sgn}(\alpha) \frac{\kappa_E}{r^2})} \tag{13}
\]
and introducing (10) into (13) we obtain the generalized five-dimensional Tolman-Oppenheimer-Volkoff equation
\[
\frac{dp(r)}{dr} = -\frac{\kappa_E \mathcal{M}(r) \rho(r)}{12\pi^2 r^3} \left( 1 + \frac{p(r)}{\rho(r)} \right) \left( 1 + \text{sgn}(\alpha) \frac{\kappa_E}{6\pi^2 r^4} \mathcal{M}(r) \right)^{-1/2}
\times \left[ \frac{\pi^2 r^4 p(r)}{\mathcal{M}(r)} - \frac{12\text{sgn}(\alpha) \pi^2 r^4}{\kappa_E l^2 \mathcal{M}(r)} \left( 1 - \sqrt{1 + \text{sgn}(\alpha) \frac{\kappa_E}{6\pi^2 r^4} \mathcal{M}(r)} \right) \right]
\times \left[ 1 + \text{sgn}(\alpha) \frac{l^2}{r^2} \left( 1 - \sqrt{1 + \text{sgn}(\alpha) \frac{\kappa_E}{6\pi^2 r^4} \mathcal{M}(r)} \right) \right]^{-1}. \tag{14}
\]
From (14) we can see that in the case of small $l^2$ limit, we can expand the root to first order in $l^2$. In fact
\[
\sqrt{1 + \text{sgn}(\alpha) \frac{\kappa_E}{6\pi^2 r^4} \mathcal{M}(r)} = 1 + \text{sgn}(\alpha) \frac{\kappa_E}{12\pi^2 r^4} l^2 \mathcal{M}(r) + \mathcal{O}(l^4). \tag{15}
\]
Introducing (15) into (14) we find
\[
\frac{dp(r)}{dr} \approx -\frac{\kappa_E \mathcal{M}(r) \rho(r)}{12\pi^2 r^3} \left( 1 + \frac{p(r)}{\rho(r)} \right) \left( 1 + \frac{\pi^2 r^4 p(r)}{\mathcal{M}(r)} \right)
\times \left( 1 - \frac{\kappa_E}{12\pi^2 r^4} \mathcal{M}(r) \right) \tag{16}
\]
From (16) we can see that, in the limit where $l \to 0$, we obtain
\[
\frac{dp(r)}{dr} = p'(r) \approx -\frac{\kappa_E \mathcal{M}(r)}{12\pi^2 r^3} \left( 1 + \frac{p(r)}{\rho(r)} \right) \left( 1 + \frac{\pi^2 r^4 p(r)}{\mathcal{M}(r)} \right) \left( 1 - \frac{\kappa}{12\pi^2 r^2} \mathcal{M}(r) \right)^{-1}, \tag{17}
\]
which is the standard five-dimensional Tolman-Oppenheimer-Volkoff equation (see Eq. (A4)) (compare with the four-dimensional case shown in Ref. [7]).

To solve the generalized TOV equation (14), an equation of state relating $\rho$ and $p$ is needed. This equation should be supplemented by the boundary condition that $p(R) = 0$ where $R$ is the radius of the star.

Given an equation of state $p(\rho)$, the problem can be formulated as a pair of first-order differential equations for $p(r)$, $\mathcal{M}(r)$ and $\rho(r)$, (14) and
\[
\mathcal{M}'(r) = 2\pi^2 r^3 \rho(r), \tag{18}
\]
with the initial condition $\mathcal{M}(0) = 0$. In addition, it is necessary to provide the initial condition $\rho(0) = \rho_0$. 

Let us return to the problem of calculating the metric. Once we compute \( \rho(r) \), \( \mathcal{M}(r) \), and \( p(r) \), we can immediately obtain \( g(r) \) from (10) and \( f(r) \) from (12)

\[
f(r) = -\int_r^\infty \frac{\kappa_E \mathcal{M}(\bar{r})}{12\pi^2 r^3} \left( 1 + \text{sgn}(\alpha) \frac{\kappa_E}{6\pi^2 r^4} l^2 \mathcal{M}(\bar{r}) \right)^{-1/2} \times \left[ \pi^2 \bar{r}^2 p(\bar{r}) - \frac{12 \text{sgn}(\alpha) \bar{r}^2}{\kappa_E l^2 \mathcal{M}(\bar{r})} \left( 1 - \sqrt{1 + \text{sgn}(\alpha) \frac{\kappa_E}{6\pi^2 r^4} l^2 \mathcal{M}(\bar{r})} \right) \right]^{-1} d\bar{r} \tag{19}
\]

where we have set \( f(\infty) = 0 \), a condition consistent with the asymptotic limit from the exterior solution.

It should be noted that if \( r > R \), i.e., out of the star, the following conditions are satisfied:

\[
\mathcal{M}(r) = M, \quad p(r) = \rho(r) = 0. \tag{20}
\]

Integrating (19) we find

\[
f(r) = \frac{1}{2} \ln \left[ 1 + \text{sgn}(\alpha) \frac{r^2}{l^2} \left( 1 - \sqrt{1 + \text{sgn}(\alpha) \frac{\kappa_E}{6\pi^2 r^4} l^2 M} \right) \right], \tag{21}
\]

so that

\[
e^{2f(r)} = e^{-2g(r)} = 1 + \text{sgn}(\alpha) \frac{r^2}{l^2} - \text{sgn}(\alpha) \sqrt{\frac{r^4}{l^4} + \text{sgn}(\alpha) \frac{\kappa_E}{6\pi^2 l^2} M}, \tag{22}
\]

which coincides with the outer solution.

### A. Constant Density: \( \rho(r) = \rho_0 \)

We will now consider the solution of (14) in the case where the energy density is constant, \( \rho(r) = \rho_0 \), inside the star. In this case the hydrostatic equilibrium equation (9) can be directly integrated,

\[
\rho_0 + p(r) = C e^{-f(r)}, \tag{23}
\]

where \( C \) is an integration constant.

On the other hand, from (18) \( \mathcal{M}(r) \) is given by

\[
\mathcal{M}(r) = \frac{\pi^2}{2} \rho_0 r^4. \tag{24}
\]

Introducing (24) into (10) we have

\[
e^{-2g(r)} = 1 + \text{sgn}(\alpha) \frac{r^2}{l^2} - \text{sgn}(\alpha) \sqrt{\frac{r^4}{l^4} + \text{sgn}(\alpha) \frac{\kappa_E}{12l^2} \rho_0 r^4}. \tag{25}
\]
Now, let us add the field equations (5) and (6)

$$\frac{e^{-2g}}{r^3}(f' + g') [r^2 + \text{sgn}(\alpha)l^2 (1 - e^{-2g})] = \frac{\kappa_{E}}{12}(\rho_0 + p).$$

(26)

Using now (23), multiplying by $e^{-g}$ and integrating we have

$$e^f = \frac{\kappa_{E}}{12}C e^{-g} \int \frac{r^3 \, dr}{e^{-3g} [r^2 + \text{sgn}(\alpha)l^2 (1 - e^{-2g})]} + C_0 e^{-g},$$

(27)

where $C_0$ is the corresponding integration constant. Since

$$\int \frac{r^3 \, dr}{e^{-3g} [r^2 + \text{sgn}(\alpha)l^2 (1 - e^{-2g})]} = -\frac{\text{sgn}(\alpha)l^2 e^g(r)}{\sqrt{1 + \text{sgn}(\alpha)\frac{\kappa_{E}}{12}l^2 \rho_0} (1 - \sqrt{1 + \text{sgn}(\alpha)\frac{\kappa_{E}}{12}l^2 \rho_0})}$$

(28)

we find

$$e^f = C_1 + C_0 e^{-g},$$

(29)

where

$$C_1 := -\frac{\text{sgn}(\alpha)l^2 C}{12 \sqrt{1 + \text{sgn}(\alpha)\frac{\kappa_{E}}{12}l^2 \rho_0} (1 - \sqrt{1 + \text{sgn}(\alpha)\frac{\kappa_{E}}{12}l^2 \rho_0})}.$$  

(30)

Then, we proceed to adjust the constants $C$, $C_0$, and $C_1$, so that the interior solution and exterior must match at $r = R$. In addition one should require that the pressure vanishes at $r = R$.

The calculations give

$$C = \rho_0 \sqrt{1 + \text{sgn}(\alpha)\frac{R^2}{l^2} \left(1 - \sqrt{1 + \text{sgn}(\alpha)\frac{\kappa_{E}}{12}l^2 \rho_0}\right)},$$

(31)

$$C_1 = -\frac{\text{sgn}(\alpha)\kappa_{E} l^2 \rho_0 \sqrt{1 + \text{sgn}(\alpha)\frac{R^2}{l^2} \left(1 - \sqrt{1 + \text{sgn}(\alpha)\frac{\kappa_{E}}{12}l^2 \rho_0}\right)}}{12 \sqrt{1 + \text{sgn}(\alpha)\frac{\kappa_{E}}{12}l^2 \rho_0} (1 - \sqrt{1 + \text{sgn}(\alpha)\frac{\kappa_{E}}{12}l^2 \rho_0})},$$

(32)

and

$$C_0 = -\frac{1}{\sqrt{1 + \text{sgn}(\alpha)\frac{\kappa_{E}}{12}l^2 \rho_0}}.$$  

(33)

**IV. SUMMARY AND OUTLOOK**

We have considered a spherically symmetric internal solution within the context of Einstein-Chern-Simons gravity. We derived the generalized five-dimensional Tolman-Oppenheimer-Volkoff (TOV) equation and then we proved that this generalized TOV equation leads, in a certain limit, in the standard five-dimensional TOV equation.
Acknowledgments

This work was supported in part by Dirección de Investigación, Universidad de Concepción through Grant # 212.011.056-1.0 and in part by FONDECYT through Grant N 1130653. One of the authors (C.A.C.Q) was supported by grants from the Comisión Nacional de Investigación Científica y Tecnológica CONICYT and from the Universidad de Concepción, Chile.

Appendix A: The standard Tolman-Oppenheimer-Volkoff equation in 5D

Let us recall that the energy-momentum tensor satisfies the condition

\[ \nabla_{\mu} T^{\mu\nu} = 0. \]  \hspace{1cm} (A1)

If \( T_{TT} = \rho(r) \) and \( T_{RR} = T_{ii} = p(r) \) we find

\[ \nabla_{\mu} T^{\mu r} = \frac{f'(r)(\rho(r) + p(r)) + p'(r)}{e^{2g(r)}}, \]

so that

\[ f' = -\frac{p'}{\rho + p}, \] \hspace{1cm} (A2)

an expression known as the hydrostatic equilibrium equation.

From Eqs. (A10) and (A22) of Ref. [6] we find

\[ f'(r) = \frac{\kappa E M(r)}{12\pi^{2} r^{3}} \left( 1 + \frac{\pi^{2} r^{4} p(r)}{M(r)} \right) \left( 1 - \frac{\kappa E}{12\pi^{2} r^{2}} M(r) \right)^{-1}. \] \hspace{1cm} (A3)

Introducing (A2) into (A3) we obtain the standard five-dimensional Tolman-Oppenheimer-Volkoff equation

\[ p'(r) = -\frac{\kappa E M(r)}{12\pi^{2} r^{3}} \left( 1 + \frac{p(r)}{\rho(r)} \right) \left( 1 + \frac{\pi^{2} r^{4} p(r)}{M(r)} \right) \left( 1 - \frac{\kappa E}{12\pi^{2} r^{2}} M(r) \right)^{-1}. \] \hspace{1cm} (A4)

This may be compared with the four-dimensional case shown in equation (1.11.13) of reference [7].

Appendix B: Energy-momentum tensor

It is known that if the torsion is null, then the energy-momentum tensor is divergence-free,
\[ \nabla_{\mu} T^{\mu\nu} = 0. \] The 1-form energy-momentum is given by
\[ \tilde{T}_a := T_{\mu\nu} e^\mu_a \, dx^\nu. \] \hspace{1cm} (B1)
Theorem. If the energy-momentum tensor $T_{\mu\nu}$ and the 1-form energy-momentum $\hat{T}_a$ are related by equation $(B1)$, then in a torsion-free space-time

$$\nabla_\mu T^\mu_\nu = -e^a_\nu \star D_\omega (\star \hat{T}_a)$$

(B2)

Proof.

$$\star \hat{T}_a = \frac{\sqrt{-g}}{4!} \epsilon_{\mu\nu\rho\sigma} T^\mu_\rho dx^\nu dx^\sigma dx^\sigma.$$  \hspace{1cm} (B3)

After some algebra, we find

$$-e^a_\nu \star D_\omega (\star \hat{T}_a) = \frac{1}{\sqrt{-g}} \partial_\lambda (\sqrt{-g}) T^\lambda_\nu + \partial_\lambda T^\lambda_\nu - T^\lambda_\nu (\partial_\lambda e^a_\nu + \omega^a_\lambda b e^b_\nu),$$

and using the Weyl’s lemma

$$\partial_\lambda e^a_\nu + \omega^a_\lambda b e^b_\nu - \Gamma^{\rho}_{\lambda\nu} e^a_\rho = 0,$$

we obtain

$$-e^a_\nu \star D_\omega (\star \hat{T}_a) = \frac{1}{\sqrt{-g}} \partial_\lambda (\sqrt{-g}) T^\lambda_\nu + \partial_\lambda T^\lambda_\nu - \Gamma^{\rho}_{\lambda\nu} T^\lambda_\rho,$$

$$-e^a_\nu \star D_\omega (\star \hat{T}_a) = \partial_\lambda T^\lambda_\nu + \Gamma^{\rho}_{\lambda\nu} T^\lambda_\rho - \Gamma^{\rho}_{\lambda\nu} T^\rho_\lambda = \nabla_\lambda T^\lambda_\nu.$$  \hspace{1cm} (B7)

\[\square\]

1. The Hodge star operator

The Hodge star operator for a $p$–form $P = \frac{1}{p!} P_{\alpha_1 \cdots \alpha_p} dx^{\alpha_1} \cdots dx^{\alpha_p}$ in a $d$-dimensional manifold with a non-singular metric tensor $g_{\mu\nu}$ is defined as

$$\star P = \frac{\sqrt{|g|}}{(d-p)! p!} \epsilon_{\alpha_1 \cdots \alpha_d} g^{\alpha_1 \beta_1} \cdots g^{\alpha_p \beta_p} P_{\beta_1 \cdots \beta_p} dx^{\alpha_{p+1}} \cdots dx^{\alpha_d},$$

where $\epsilon_{\alpha_1 \cdots \alpha_d}$ is the total antisymmetric Levi-Civita tensor density of weight $-1$.

2. Hydrostatic equilibrium equation

Let us consider a spherically and static-symmetric metric in five dimensions. The 1-form energy-momentum is given by

$$\hat{T}_a = T_{ab} e^b$$

(B8)
where $T_{ab}$ is the energy-momentum tensor in a comoving orthonormal frame. So, if the matter is a perfect fluid then

$$T_{TT} = \rho(r) \quad , \quad T_{RR} = T_{ii} = p(r). \quad (B9)$$

Computing the conservation equation

$$D_\omega (\hat{T}_a) = 0 \quad (B10)$$

we have

$$D_\omega (\hat{T}_a) = D_\omega (T_{ab} \ast e^b) = \frac{1}{4!} \epsilon_{fbcde} (D_\omega T_a^f) e^b e^ce^de^e, \quad (B11)$$

where we have used the torsion-free condition $D_\omega e^a = 0$. Therefore

$$D_\omega (\hat{T}_a) = \frac{1}{4!} \epsilon_{fbcde} (dT_a^f + \omega_a^g T_g^f + \omega_f^g T_{a}^g) e^b e^ce^de^e. \quad (B12)$$

The calculations give

$$D_\omega (\hat{T}_R) = e^{-g} (p' + f' (\rho + p)) e^T e^R e^{1} e^{2} e^{3} = 0 \quad (B13)$$

from which we get the so-called hydrostatic equilibrium equation

$$p' + f' (\rho + p) = 0. \quad (B14)$$

**Appendix C: Dynamic of the field $h^a$**

We consider now the field $h^a$. Expanding the field $h^a = h^a_\mu dx^\mu$ in their holonomic index we have [6]

$$h_a = h_{\mu\nu} e^\mu_a dx^\nu \quad (C1)$$

For the space-time to be static and spherically symmetric, the field $h_{\mu\nu}$ must satisfy the Killing equation $\mathcal{L}_\xi h_{\mu\nu} = 0$ for $\xi_0 = \partial_t$ and the six generators of the sphere $S_3$ must be

$$\xi_0 = \partial_t,$$
$$\xi_1 = \partial_{\theta_3},$$
$$\xi_2 = \sin \theta_3 \partial_{\theta_2} + \cot \theta_2 \cos \theta_3 \partial_{\theta_3},$$
$$\xi_3 = \sin \theta_2 \sin \theta_3 \partial_{\theta_1} + \cot \theta_1 \cos \theta_2 \sin \theta_3 \partial_{\theta_2} + \cot \theta_1 \csc \theta_2 \cos \theta_3 \partial_{\theta_3},$$
$$\xi_4 = \cos \theta_3 \partial_{\theta_2} - \cot \theta_2 \sin \theta_3 \partial_{\theta_3},$$
$$\xi_5 = \sin \theta_2 \cos \theta_3 \partial_{\theta_1} + \cot \theta_1 \cos \theta_2 \cos \theta_3 \partial_{\theta_2} - \cot \theta_1 \csc \theta_2 \sin \theta_3 \partial_{\theta_3},$$
$$\xi_6 = \cos \theta_2 \partial_{\theta_1} - \cot \theta_1 \sin \theta_2 \partial_{\theta_2}. \quad (C2)$$
Then, we have

\[ h_T = h_{tt}(r) \ e^T + h_{tr}(r) \ e^R, \]
\[ h_R = h_{rt}(r) \ e^T + h_{rr}(r) \ e^R, \]
\[ h^i = h(r) \ e^i. \]  

(C3)

From Eq. (3) we know that the dynamic of the field \( h^a \) is given by

\[ \epsilon_{abcd} R^{cd} Dh^e = 0 \]  

(C4)

with

\[ Dh^a = dh^a + \omega^a h^b \]  

(C5)

where

\[ Dh_T = e^{-g} (-h'_{tt} - f'h_{tt} + f'h_{rr}) \ e^T e^R, \]  

(C6)
\[ Dh_R = e^{-g} (-h'_{rt} - f'h_{rt} + f'h_{tr}) \ e^T e^R, \]  

(C7)
\[ Dh^i = \frac{e^{-g}}{r} (rh' + h - h_{rr}) \ e^R e^i - \frac{e^{-g}}{r} h_{rt} \ e^T e^i. \]  

(C8)

Introducing (C6 - C8) into (C4) we have

\[ h_{tr} = h_{rt} = 0, \]  

(C9)
\[ h_r = (rh)', \]  

(C10)
\[ h'_t = f'(h_r - h_t). \]  

(C11)

To find solutions to (C9, C10, C11), we assume that \( h_t(r) \) depends on \( r \) only through \( f(r) \), namely

\[ h_t(r) = h_t(f(r)) \]  

(C12)

Introducing (C12) into (C11) we have

\[ \frac{dh_t(f)}{df} f'(r) = f'(h_r - h_t) \]  

(C13)

from which we obtain the following linear differential equation, which is of first order and inhomogeneous:

\[ \dot{h}_t + h_t = h_r, \]  

(C14)
where $\dot{h}_t := \frac{dh_t}{df}$. The homogeneous solution is given by

$$h^h_t(f) = Ae^{-f(r)}, \quad (C15)$$

where $A$ is a constant to be determined.

The particular solution depends on the shape of $h_r$. If we assume a functional relationship $h$ with $f$, then the linearity of differential equation suggests the following ansatz:

$$h_r(r) = h_r(f(r)) = \sum_{n=0}^{\infty} B_n e^{nf(r)} + \sum_{m=2}^{\infty} C_m e^{-mf(r)}, \quad (C16)$$

where $B_n$ and $C_m$ are real constants. So that the particular solution is given by

$$h^p_t(f) = \sum_{n=0}^{\infty} \frac{B_n}{n+1} e^{nf(r)} - \sum_{m=2}^{\infty} \frac{C_m}{m-1} e^{-mf(r)}. \quad (C17)$$

Therefore the general solution is of the form

$$h_t(f(r)) = Ae^{-f(r)} + \sum_{n=0}^{\infty} \frac{B_n}{n+1} e^{nf(r)} - \sum_{m=2}^{\infty} \frac{C_m}{m-1} e^{-mf(r)}. \quad (C18)$$

From (C10) we find

$$h(r) = \frac{1}{r} \left( \int h_r(r) \, dr + D \right), \quad (C19)$$

where $D$ is an integration constant. This means

$$h(r) = \frac{1}{r} \sum_{n=0}^{\infty} \left( B_n \int e^{nf(r)} \, dr \right) + \frac{1}{r} \sum_{m=2}^{\infty} \left( C_m \int e^{-mf(r)} \, dr \right) + \frac{D}{r}, \quad (C20)$$

where $A$, $B_n$, and $C_m$ are arbitrary constants, and $-e^{2f(r)}$ is the metric coefficient $g_{00}$.

1. **Field asymptotically constant**

Consider the simplest case where

$$h_r(r) = h = \text{constant} \quad (C21)$$

in this case (C19) leads

$$h(r) = h + \frac{D}{r} \quad (C22)$$

and

$$h_t(r) = Ae^{-f(r)} + h. \quad (C23)$$
Since the vielbein is regular at $r = 0$ (center of the star), $h^a$ should also be regularly at $r = 0$, i.e. we should have $D = 0$. Note that the coefficient $e^{f(r)}$ is regular at $r = 0$ as can be seen from (C19).

From (C20) we can see that the asymptotic behavior of the metric coefficients is given by

$$e^{2f(r \to \infty)} = e^{-2g(r \to \infty)} = 1.$$  (C24)

Thus the asymptotic behavior of the field $h^a$ is given by

$$h_r(r \to \infty) = h, \quad h(r \to \infty) = h, \quad h_t(r \to \infty) = A + h.$$  (C25)

2. Constant density

If the density is constant then the inner solution is given by (25) and (29). In this case the solution for the field $h^a$ is given by

$$h_r(r) = h, \quad h(r) = h$$  (C26)

and

$$h_t(r) = \begin{cases} \frac{A}{C_0 + C_1 e^{-g(r)}} + h & \text{if } r < R, \\ \frac{A}{e^{-g(r)}} + h & \text{if } r \geq R, \end{cases}$$  (C27)

where

$$e^{-g(r)} = \begin{cases} \sqrt{1 + \text{sgn}(\alpha) \frac{r^2}{l^2} - \text{sgn}(\alpha) \frac{r^4}{l^4} + \text{sgn}(\alpha) \frac{\kappa E}{6 \pi^2 l^2} M(r)} & \text{if } r < R, \\ \sqrt{1 + \text{sgn}(\alpha) \frac{r^2}{l^2} - \text{sgn}(\alpha) \frac{r^4}{l^4} + \text{sgn}(\alpha) \frac{\kappa E}{6 \pi^2 l^2} M} & \text{if } r \geq R. \end{cases}$$  (C28)

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