Bounds on series-parallel slowdown

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Abstract

We use activity networks (task graphs) to model parallel programs and consider series-parallel extensions of these networks. Our motivation is two-fold: the benefits of series-parallel activity networks and the modelling of programming constructs, such as those imposed by current parallel computing environments. Series-parallelisation adds precedence constraints to an activity network, usually increasing its makespan (execution time). The slowdown ratio describes how additional constraints affect the makespan. We disprove an existing conjecture positing a bound of two on the slowdown when workload is not considered. Where workload is known, we conjecture that \( \frac{4}{3} \) slowdown is always achievable, and prove our conjecture for small networks using max-plus algebra. We analyse a polynomial-time algorithm showing that achieving \( \frac{4}{3} \) slowdown is in exp-APX. Finally, we discuss the implications of our results.

1 Introduction

An approach to reducing the execution time of a computer program is to run it on multiple processors simultaneously. The study of parallel programming and architectures has seen a resurgence with the widespread adoption of multi-core processing units in computing systems. Commercial numerical software such as MATLAB\(^1\) and Mathematica\(^2\) can now take advantage of multiple processors, and OpenCL is a recently finalised standard for programming with multiple-processor systems \([15]\).

An important aspect of parallel programming is scheduling, the method by which code is allocated to processors \([12]\). Here we instead consider the inherent precedence constraints of a parallel program and the constraints imposed by transformation and by the programming constructs that are used to describe parallelism, both of which affect execution time. Our concerns are orthogonal to scheduling since we assume sufficient processors and hence the decision on what to schedule next is unimportant.

A program can be divided up into activities or tasks. This can be done in different ways depending on the granularity used. Here we do not consider granularity further but assume some reasonable approach has been used. The activities can be related to each other by the order in which they must occur for the program to work correctly. For instance, if one activity modifies a variable and another activity uses this modified value, then the modifying activity must occur before the activity uses the new value. An activity that must occur before another \textit{precedes} the other activity and there is a \textit{precedence constraint} between the two activities. Precedence is imposed by the structure of the program and is inherent to the particular set of activities.

\(^{1}\)Via the MATLAB Parallel Computing Toolbox. \url{http://www.mathworks.com/}

\(^{2}\)From version 7. \url{http://www.wolfram.co.uk/}
The formalism used to describe precedences between activities is known as an activity network, network or task graph. We use the activity-on-node variant of this model, where weights are associated with the vertices of the network. These and variants such as PERT networks or machine schedules (sometimes with edge instead of node weights) are widely used in fields such as project management, operational research, combinatorial optimization and computer science.

Activity networks can be classified by their structure. Structures of interest are series-parallel (SP), and level-constrained (LC) [13] which are a proper subset of SP and a subset of of Bulk Synchronous Programming (BSP) which has been used successfully as an approach to parallel programming [1, 18]. Analysis of activity networks is difficult but is easier for SP [4]. For instance, scheduling is NP-hard but polynomial-time for SP networks [6]. We call the addition of constraints to achieve an SP activity network series-parallelisation (SP) [7].

Programming constructs can also impose an SP structure over and above the inherent constraints. The most obvious is the sequencing of commands in a sequential programming language but the addition of constraints can also occur with parallel constructs as we show in the motivating example in Section 2.

The precedence constraints between activities determine the minimum time to execute the program. Assuming a sufficient number of processors and non-preemptive, work-conserving scheduling the fastest time for execution will be the time taken to execute slowest chains of activities, called critical paths. Chains consist of activities that are totally ordered and hence must proceed one after another, excluding the possibility of parallelism.

This paper considers the difference in execution time between activity networks, comparing a network with only inherent precedence constraints with the same network with added precedence constraints that make it an SP structure. Adding constraints results in programs that take at least as long and we consider the slowdown where slowdown is the ratio of the slower program to the faster one. We characterise the slowdown induced by LC and disprove an existing conjecture about slowdown for SP [19]. This requires demonstrating that large slowdown can occur for every possible series-parallelisation of a specific network. A new conjecture is presented, and results proved for small instances. Additionally we discuss the complexity of finding the optimal SP for a network. First we present a motivating example, followed by background and definitions of the relevant structures after which come the main results and conjecture. We finish with the implications of our results and further research.

2 Motivating example

We next consider a simple example involving computations dependent on earlier computations. In a 1-dimensional flow model of heat diffusion in a metal rod, we calculate the temperature at $m$ points for each time step. The temperature at time $\tau + 1$ at point $p_i$ is dependent on the temperature at time $\tau$ at points $p_{i-1}$, $p_i$ and $p_{i+1}$. If we view each calculation as an activity $a_{i,\tau}$, this is an example of neighbour synchronisation example.
synchronisation (NS) as illustrated in Figure 1(a) when considering the solid lines only. This network is not SP because of the edges \((a_{1,1}, a_{2,1})\), \((a_{1,1}, a_{2,2})\) and \((a_{1,3}, a_{2,2})\) and the lack of the edge \((a_{1,3}, a_{2,1})\). This is an example of the smallest non-SP activity network, the N network shown in Figure 1(b). There are many instances of N in the example activity network.

An obvious (although not necessarily the best) way to series-parallelise this activity network is to require all activities at time \(\tau\) to precede those at time \(\tau + 1\). The dashed lines in Figure 1(a) illustrate the added precedence constraints. The edge \((a_{1,3}, a_{2,1})\) is added as well as edges to remove the other N networks. Figure 1(a) is an example of a level-constrained (LC) extension.

Assume unit workloads for all activities apart from one much slower activity at each time instance \(\tau\) with duration \(t(\tau, 2\tau) = C \gg 1\). Hence for every calculation of a specific point over time, there is only one large workload. The execution time for the above series-parallelisation will be \((C - 1)(m + 1)/2 + s\) where \(s \geq n\) is the total number of timesteps. This gives large slowdown since the execution time considering only inherent constraints is \(C + s - 1\).

There may be better ways to series-parallelise this network, however a language such as MATLAB may impose a particular SP activity network through its programming constructs. If one expresses this example as parallel code using the \texttt{parfor} statement (in the obvious simple way) then one will achieve the SP network given in Figure 1(a).

An understanding of the slowdown obtained by various forms of series-parallelisation is therefore important, particularly due to the increased usage of parallel programming constructs to take advantage of multi-core processors.

### 3 Background

This section defines notation and basic concepts for activity-on-node networks.

**Definition 1.** An activity-on-node network (task graph, activity network, or simply, network) consists of:

- \(V = \{a_1, \ldots, a_n\}\) a set of activities,
- \(G = (V, E)\) a directed acyclic graph with precedence constraints \(E \subseteq V \times V\),
- \(t : V \rightarrow (0, \infty)\) a workload assigning a duration to each activity.

A precedence constraint \((a, b)\) captures the idea that activity \(a\) must complete before activity \(b\) can begin. We assume that we are working with the transitive closure of the precedence constraints, namely that the precedence relation is irreflexive and transitive. However, when drawing activity networks, we only draw the edges that appear in the transitive reduction of the network.

The makespan of an activity network \(G\), denoted \(T(G)\), is the time to complete all activities of the network. This depends on the scheduling policy and the number and configuration of processors. We make the following assumptions.

- **Scheduling:** We assume non-preemptive scheduling, namely once an activity is assigned to a processor, it will complete on that processor without interruption; and a work-conserving scheduling policy, namely no processor is left idle if there are still activities waiting to start.

- **Number and type of processors:** The processors are identical and there are sufficiently many, in the sense that any activity that is ready to execute can be started. It is sufficient to have as many processors available as the width of the activity network.

- **Overheads:** All overheads such as communication, contention and decisions about which activity to execute next are included in the workload.

Given these assumptions, we can characterise the makespan of activity networks.

**Definition 2.** Let \(G = (V, E)\) and \(G' = (V', E')\) be directed graphs.

- \(G\) is a subgraph of \(G'\), \(G \subseteq G'\) if \(V \subseteq V'\) and \(E \subseteq E'\).
• If $G$ is a subgraph of $G'$ then $G'$ is a supergraph of $G$.
• $G$, a subgraph of $G'$, is an antichain if $E$ is empty.
• $G$, a subgraph of $G'$, is a chain if $E$ is a total order over $V$.
• $G'$, a supergraph of $G$, is an extension if $E \subseteq E'$ and $V = V'$.

An extension formally defines what it means to add precedence constraints and does not permit addition of activities so $t$ remains unchanged. A subnetwork has the obvious meaning.

Definition 3. Let $G = (V, E)$ be an activity network.

• $\text{depth}(G) = \max\{|C| \mid C \text{ a chain in } G\}$.
• $\text{width}(G) = \max\{|A| \mid A \text{ an antichain in } G\}$.

A chain represents its activities occurring one after the other, and hence the time taken for a chain to execute is the sum of the durations for each activity.

Proposition 1. The makespan of a chain $C = (V, E)$ with $V = \{a_1, \ldots, a_n\}$ is

$$T(C) = \sum_{i=1}^{n} t(a_i).$$

The makespan of an activity network can be characterised as the time it takes to complete a chain in the network with the longest completion time (a critical path). The proof is straightforward, and makes essential use of the work-conserving property of the scheduling policy, and the fact that there are sufficient processors. If the number of processors is insufficient, a work-conserving approach may be sub-optimal [11].

Proposition 2. The makespan of an activity network $G = (V, E)$ is

$$T(G) = \max\{T(C) \mid C \text{ is a chain in } G\}.$$  

When we create extensions by adding constraints to obtain a specific network structure, we cannot decrease the time that the activity network will take to complete [10, 16]. We can define the ratio between the two makespans as a slowdown.

Definition 4. Let $H$ be an extension of $G$ then the slowdown is $T(H)/T(G)$.

4 Structure of activity networks

We need to define what it means for an activity network to be series-parallel. Figure 2(a) is not SP and Figure 2(b) is SP. The N network in Figure 1(b) is also not SP. An activity network is SP if it consists of a single activity or can be recursively decomposed into chains and antichains using series and parallel composition.

Definition 5. An activity network $G = (V, E)$ is series-parallel (SP) if $G$ can be expressed using the SP grammar $g ::= (g \oplus g) | g \cdot g | a$ where $a$ is an activity, and each activity appears at most once. A string generated by the SP grammar is an SP expression.

We also use juxtaposition $G_1G_2$ for $G_1 \cdot G_2$. The network in Figure 2(b) can be expressed as $(a((b \oplus c)(d \oplus e)f) \oplus (gh(i \oplus j)(k \oplus l \oplus m))) \oplus nop)$.

Definition 6. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be activity networks with $V_1 \cap V_2 = \emptyset$ and $E_1 \cap E_2 = \emptyset$.

\footnote{If we were comparing a sequential program with its parallel version, we would consider speedup, namely the ratio of the faster to the slower. Since we know that the program with additional precedence constraints will take at least as long as the original, we consider slowdown, the ratio of the slower to the faster.}
The parallel composition of $G_1$ and $G_2$ is $G_1 \oplus G_2 = (V_1 \cup V_2, E_1 \cup E_2)$.

The series composition of $G_1$ and $G_2$ is $G_1 \cdot G_2 = (V_1 \cup V_2, (V_1 \times V_2) \cup E_1 \cup E_2)$.

SP networks are exactly those that do not contain the N network [17]. If we have a network that is not SP, we can add constraints until it is SP. An SP extension of an activity network always exists since if we add sufficient constraints we obtain a chain, which is SP [5]. The activity network in Figure 2(b) is a series-parallelisation of the activity network in Figure 2(a). We can easily calculate the makespan of an SP network.

**Proposition 3.** Let $G = (V, E)$ be an SP activity network with SP expression $g$. The makespan of $G$ is $T(G) = T(g)$ where

$$T((g_1 \oplus g_2)) = \max\{T(g_1), T(g_2)\}, \quad T(g_1 \cdot g_2) = T(g_1) + T(g_2), \quad T(a) = t(a).$$

This links the SP grammar with the max-plus algebra [3]. For convenience, the symbol $\oplus$ will denote max and the symbol $\cdot$ will denote arithmetic $+$.

Level-constrained networks are a strict subset of SP. The level of an activity $a$ is the size of a maximal chain in the network which has $a$ as its last activity.

**Definition 7.** For an activity network $G = (V, E)$, the level of an activity $a$ is

$$\lambda(a) = \max\{|C| \mid C \text{ is a chain in } G, a \in C, \text{ and for all } b \in C, (b, a) \in E\}.$$  

The level of each activity in a network can be computed in polynomial time, by marking activities in a breadth-first search of the network’s transitive reduction. The depth of an activity network is the maximum level of its activities. We can now add precedence constraints to obtain an extension of the network that maintains its level structure. This is a common technique [13, 16].

**Definition 8.** For an activity network $G = (V, E)$, the level-constrained (LC) extension of $G$ is the network $G_L = (V, E_L)$, where $E_L = \{(a, b) \mid \lambda(a) < \lambda(b)\}$.

Note that $G_L$ is an extension of $G$, and that $\text{depth}(G) = \text{depth}(G_L)$. We can identify a level as $\Lambda_i = \{a \in V \mid \lambda(a) = i\}$; each level is an antichain and the levels partition the activities. $G_L$ is also in BSP form [18] since each level consists of independent chains (of size one, in this case) and all activities in one level must complete before any activity in the next level can start. LC networks have the form $\alpha_1 \ldots \alpha_d$ where $\alpha_i = (a_{i,1} \oplus \ldots \oplus a_{i,m_i})$.

We consider a structure that is non-SP for sufficiently large networks.

**Definition 9.** A neighbour synchronisation (NS) network of depth $d$, width $w$, and degree $\Delta$, denoted $\text{ns}(d, w, \Delta)$, consists of activities $a_{i,j}$ with $i \in \{1, \ldots, d\}, j \in \{1, \ldots, w\}$, and precedence constraints $(a_{i,j}, a_{i+1,j+k})$ for every $k = -[(\Delta - 1)/2], -[(\Delta - 1)/2] + 1, \ldots, [(\Delta - 1)/2]$ (as long as $1 \leq j + k \leq w$).

Figure 1(a) depicts an NS network of depth $t$, width $m$, and degree 3. The dashed precedence constraints are those added by the process of LC extension.
5 Bounding LC slowdown

There are three reasons for considering LC networks. First, they relate to BSP, a useful applied technique for parallel programming, and second, they are efficient to construct for any given activity network. Last, it is straightforward to construct an upper-bound on the slowdown for a given workload.

**Theorem 1.** Given an activity network \( G = (V, E) \) and its LC extension \( G_L = (V, E_L) \), the slowdown is bounded by the ratio \( \rho \) of the largest to the smallest duration in the workload.

\[
\frac{T(G_L)}{T(G)} \leq \frac{\max\{t(a) \mid a \in V\}}{\min\{t(a) \mid a \in V\}} = \rho.
\]

**Proof.** Given an LC extension \( G_L = (V, E_L) \) of \( G \), its makespan is \( T(G_L) = \sum_{i=1}^{\text{depth}(G)} \max\{t(a) \mid a \in \Lambda_i\} \leq \text{depth}(G) \cdot \max\{t(a) \mid a \in V\} \) since a critical path is determined by the slowest activity at each level and is bounded by the depth times the largest duration. Also \( \text{depth}(G) \cdot \min\{t(a) \mid a \in V\} \leq T(G) \) since the depth of \( G \) is the size of the longest chain and the time taken for each activity in this chain is at least as long as the activity with the shortest duration. The result follows from these two inequalities.

By Theorem 1, if all activities have similar durations, then the slowdown will be close to one. If we know in advance that \( \rho \) is small, then it is reasonable to series-parallelise using an LC extension. This is efficient to obtain, and BSP is then also an appropriate model for the computation, since any BSP can be transformed to LC by treating independent chains as single activities.

Conversely, if \( \rho \) is large, its importance depends on how tight it is. If it is tight, and we know that large values may occur because an activity could be delayed (for instance, due to a cache miss, or swapping to and from disk, or because of competition for resources), then the large value of \( \rho \) indicates that a LC extension is a poor choice for series-parallelization.

By considering \( ns(d, w, 3) \) with \( w \geq 2d - 1 \), we can demonstrate that slowdown for the LC extension can be arbitrarily close to \( \rho \).

**Proposition 4.** For any \( \epsilon > 0 \), there exists an NS activity network \( G \) and a workload \( t \) such that \( \rho - T(G_L)/T(G) < \epsilon \).

However, \( \rho \) can be pessimistic: consider \( ns(1, w, 3) \) with one large activity and many small ones. This is already SP, yet \( \rho \) can be made arbitrarily large.

The next section presents two conjectures about bounds for general series-parallelisations of activity networks.

6 Bounding SP slowdown

This section considers a conjecture by van Gemund [19]. We need to introduce a parameterised notation for makespan. Denote the makespan by \( T(G, t) \) to indicate specifically the role of the workload function \( t \). There are two different classes of algorithms that can be used to obtain a series-parallelisation. We use the notation \( S(G, t) \) to denote the SP network that is the output of some algorithm that considers both the graph and the workload, and \( S'(G) \) to denote the SP network that is the output of some algorithm that considers only the graph. Using this notation we can posit two distinct hypotheses:

**Workload-independent:** \( \exists \kappa \forall G \exists S' \forall t \left[ T(S'(G), t)/T(G, t) \leq \kappa \right] \)

**Workload-dependent:** \( \exists \kappa \forall G \forall t \exists S \left[ T(S(G, t), t)/T(G, t) \leq \kappa \right] \)

These can be understood as follows. The first states that for every graph, there is a series-parallelisation with a slowdown bound of \( \kappa \) that works for every possible workload on that graph and the second states that given a graph and a workload, there is a series-parallelisation with slowdown bound of \( \kappa \).

Van Gemund [19] conjectures that \( \kappa = 2 \) is a bound for slowdown for the workload-independent case.
Conjecture 1 ([19]). For any activity network $G = (V, E)$, it is possible to find a SP extension $G_{SP}$ of $G$, such that for every workload $t: V \rightarrow (0, \infty)$,

$$\frac{T(G_{SP}, t)}{T(G, t)} \leq 2.$$  

There is an algorithm that meets this bound under “reasonable” workloads [19]. The following result disproves Conjecture 1.

Theorem 2. For any series-parallelisation of $Q = ns(3, 8, 3)$, there exists a workload leading to slowdown greater than 2.

We need some lemmas for the proof.

Lemma 1. Any SP extension of a weakly connected network $G$ will have an SP expression of the form $\alpha \beta$, where both $\alpha$ and $\beta$ are SP expressions.

Proof. An SP expression $\alpha \oplus \beta$ has no constraints between activities in $\alpha$ and in $\beta$, so the network is disconnected. The result follows by contradiction.

Lemma 2. Suppose an NS network $G$ has SP expression $\alpha \beta$ with $\Delta$ odd.

1. If $a_{i,j}$ is in $\alpha$ then $a_{k,l}$ is also, whenever $(\Delta - 1)(i - k)/2 \geq |j - l|$.

2. If $a_{i,j}$ is in $\beta$ then $a_{k,l}$ is also, whenever $(\Delta - 1)(k - i)/2 \geq |j - l|$.

Proof. Suppose $a_{i,j}$ is in $\alpha$. By the definition of NS networks, $(\Delta - 1)(i - k)/2 \geq |j - l|$ means that $a_{k,l}$ precedes $a_{i,j}$. If $a_{k,l}$ were in $\beta$ then $a_{i,j}$ would precede $a_{k,l}$, which is impossible. The second part is symmetric.

For a network $G$ and an SP expression $\alpha$, let $G|_{\alpha}$ denote the subnetwork of $G$ consisting of only those activities that appear in $\alpha$.

Lemma 3. Suppose $d \geq 3$ and $w \geq 3$. Any SP extension of $ns(d, w, 3)$ will have an SP expression of the form $\alpha \beta$, where either $G|_{\alpha}$ or $G|_{\beta}$ is not SP.

Proof. We argue a contradiction for $ns(3, 3, 3)$; the result follows for larger $w$ and $d$ by considering any subnetwork isomorphic to $ns(3, 3, 3)$ which is not completely contained in either $\alpha$ or $\beta$. Suppose $\alpha$ and $\beta$ are both SP. Suppose activity $a_{2,1}$ and $a_{2,3}$, are both in $\alpha$ without loss of generality. By Lemma 2, $a_{1,1}$ and $a_{1,2}$ are then both in $\alpha$ or both in $\beta$. However, $\{a_{1,1}, a_{1,2}, a_{2,1}, a_{2,3}\}$ forms an N network in $G$, so $G|_{\alpha}$ cannot be SP. Now suppose activity $a_{2,1}$ is in $\alpha$ and $a_{2,3}$ is in $\beta$ (the opposite arrangement is symmetric). If $a_{2,2}$ is in $\alpha$ then $\{a_{1,1}, a_{1,3}, a_{2,1}, a_{2,2}\}$ forms an N network in $G$; if $a_{2,2}$ is in $\beta$ then $\{a_{2,2}, a_{2,3}, a_{3,1}, a_{3,3}\}$ forms an N network in $G$. Hence at least one of $G|_{\alpha}$ or $G|_{\beta}$ is not SP.

The depth of 3 in Lemma 3 is necessary, as any NS network of depth 2 can be made SP by enforcing level 1 to precede level 2, and each level is an SP network. Further, any width 2 NS network is SP, so the width of 3 is also necessary.

Proof (of Theorem 2). We show that in any SP extension $Q'$ of $Q$, there must exist three activities $a, b, c$ which form an antichain in $Q$ but a chain in $Q'$, and then construct a suitable workload using this chain. Possible arrangements of $a, b, c$ are illustrated.
By Lemma 1, any SP extension $Q'$ has an SP expression as $\alpha\beta$. Now by Lemma 3, at least one of $\alpha$ or $\beta$ is not SP. Moreover, the subnetwork of just the last three columns is isomorphic to $ns(3, 3, 3)$, so its activities that are in either $\alpha$ or $\beta$ must form a non-SP subnetwork. Without loss of generality, suppose this is $\beta$ (in the degenerate case there may then be no activities in $\alpha$ from the last three columns).

Now $a_{3, 6}, a_{3, 7}, a_{3, 8}$ must all be in $\beta$ by Lemma 2, by a similar argument to that in the proof of Lemma 3. There are now two possibilities.

The first is that at least one of $a_{1, 1}, a_{1, 2}, a_{1, 3}$ appears in $\alpha$. In this case, denote this activity by $a$. Further, at least two of $a_{2, 6}, a_{2, 7}, a_{2, 8}$ must be in $\beta$, and these two together with two of $a_{3, 6}, a_{3, 7}, a_{3, 8}$ then forms an N subnetwork $Q_N$ of $Q$. Note that in $Q$, $a$ does not precede any of the activities of $Q_N$.

The second possibility is that $a_{1, 1}, a_{1, 2}, a_{1, 3}$ are all in $\beta$. Then by Lemma 2, $a_{2, 1}$ and $a_{2, 2}$ are both in $\beta$ as well, when $a_{1, 1}, a_{1, 3}, a_{2, 1}, a_{2, 2}$ forms an N subnetwork $Q_N$ of $Q$. In this case, consider the activities $\{a_{1, 1}, a_{1, 5}, a_{1, 6}, a_{1, 7}, a_{1, 8}\}$. At least one of these must be in $\alpha$, by Lemma 2 and since $\alpha$ is non-empty. Denote this activity by $a$. In $Q$, $a$ does not precede any of the activities of $Q_N$.

In either case, in $Q'$ there must be two activities $b$ and $c$ of $Q_N$ which form an antichain in $Q$ but a chain in $Q'$. Now $a$ and $b$ forms an antichain in $Q$ but $a$ precedes $b$ in $Q'$, and the same observation holds for $a$ and $c$. Hence $\{a, b, c\}$ forms an antichain in $Q$ but a chain in $Q'$.

Let $T(a) = T(b) = T(c) = 1$ and $T(x) = \epsilon$ for every other activity $x$. The slowdown of $Q'$ is then at least $3/(1 + 2\epsilon)$, which can be made arbitrarily close to 3. In particular, if $\epsilon = 1/10$ then the slowdown is at least $5/2$.

We next state a workload-dependent conjecture, and provide evidence for it.

## 7 New conjecture

**Conjecture 2.** For any activity network $G = (V, E)$ and workload $t : V \to (0, \infty)$, there exists an SP extension $G_{SP}$ of $G$, such that

$$\frac{T(G_{SP}, t)}{T(G, t)} \leq \frac{4}{3}. $$

We now need to consider the evidence to support this conjecture. At least four activities are required to represent a non-SP network, and the only non-SP network on four activities is the N network given in Figure 1(b). We start by proving the result for the case of four activities.

**Theorem 3.** Let $G^4$ be an activity network with four activities and workload $t$, then there exists an SP extension $G_{SP}^4$ of $G^4$ such that $T(G_{SP}^4, t)/T(G^4, t) \leq 4/3$.

**Proof.** All networks with four activities except the N network are SP, for which $T(G_{SP}^4, t)/T(G^4, t) = 1 \leq 4/3$. In the case of $G^4 = N$, label the activities of $N$ so that it has edges $(a,c), (a,d)$ and $(b,d)$. There are then three minimal SP extensions (in the sense that every other SP extension contains one of these as a subnetwork):

- $(K)$: $(a,c), (a,d), (b,d), (a,b)$
- $(X)$: $(a,c), (a,d), (b,d), (b,c)$
- $(V)$: $(a,c), (a,d), (b,d), (c,d)$. 

Denote $t(x)$ by $x$ for each $x \in \{a,b,c,d\}$. A quantity such as $3(x+y)$ can be written $xxyyxy$ or using commutativity, just $xxyyxy$. Also, if $x \leq y$ and $x \leq z$ then the conclusion $x \leq \max\{y, z\}$ can instead be written as $x \leq y \oplus z$. Now $T(N) = \max\{ac, ad, bd\} = ac \oplus ad \oplus bd$, $T(K) = \max\{ac, abd\} = ac \oplus abd$, $T(X) = \max\{ac, ad, bc, bd\} = ac \oplus ad \oplus bc \oplus bd$, and $T(V) = \max\{acd, bd\} = acd \oplus bd$.

The slowdown is always at least 1, so suppose it is greater than 1 (if it is equal to 1 then the theorem is true). Then each of $T(K), T(X), \text{and } T(V)$ must exceed $T(N)$. Now if $acd \leq bd$ then $T(V) = bd \leq T(N)$, a contradiction, so $acd > bd$, and hence $ac > b$. If $abd \leq ac$ then $T(K) = ac \leq T(N)$, a contradiction,
so \( abd > ac \), and hence \( bd > c \). If \( b \leq a \) then \( T(X) = ac \oplus ad \leq T(N) \), a contradiction, so \( b > a \). If \( c \leq d \) then \( T(X) = ad \oplus bd \leq T(N) \), a contradiction, so \( c > d \). Combined, this yields \( ac > b > a \) and \( bd > c > d \). This leads to \( T(N) = ac \oplus bd \), \( T(K) = abd \), \( T(X) = bc \), and \( T(V) = acd \). Of the three possibilities for an SP extension with minimal makespan, we analyse \( K \) (symmetric to \( V \)); \( X \) is similar.

Since \( K \) has minimal makespan among SP extensions, \( abd \leq bc \) and \( abd \leq acd \), so \( ad \leq c \) and \( b \leq c \). Hence \( abd \leq cc \), so \( aaabbbdd \leq abccccc \). Therefore either \( bbbddd \leq acceccc \) or \( aaa \leq bd \). In the first case, \( aaabbbdd \leq aaaccccc \), and in the second case, \( aaabbbdd \leq bbbbbddd \). In either event, \( aaabbbdd \leq aaaccccc \oplus bbbbbddd \). However, \( abd \) is just \( T(K) \) and \( ac \oplus bd \) is just \( T(N) \), so \( T(K)/T(N) \leq 4/3 \).

Each of the three minimal SP extensions of \( N \) with the workload \( t(a) = 1, t(b) = 2, t(c) = 2 \) and \( t(d) = 1 \) has the same makespan of 4, while \( T(N,t) = 3 \), so the slowdown in this case is at least 4/3. This shows that if the workload-dependent bounded slowdown conjecture holds, then the 4/3 bound is tight.

Theorem 3 is independent of specific workloads. This is also the case for the next theorem. The five-activity case requires case analysis but it is done by contradiction rather than by the direct method used in the four-activity case, and it also uses max-plus algebra. Some additional remarks are necessary.

Directed acyclic graphs can be decomposed into modules [14]. When the edges form a transitive relation, modules have either series or parallel structure, or cannot be further decomposed. Modular decomposition for activity networks can then be thought of as an extension of the SP grammar in Section 4 by adding a terminal \( \mathcal{N} \) representing those networks that cannot be further decomposed in series or in parallel. Such indecomposable networks include the N network and \( ns(d,w,3) \) for \( d \geq 3 \) and \( n \geq 3 \).

**Theorem 4.** Let \( G^5 \) be an activity network with five activities and workload \( t \) then there exists an SP extension \( G^5_{SP} \) of \( G^5 \) such that \( T(G^5_{SP},t)/T(G^5,t) \leq 4/3 \).

**Proof.** There are 16 non-isomorphic non-SP activity networks with five activities. An activity network and its dual\(^4\) have the same slowdown results and we need only consider 9 activity networks. Six of these can be analysed using decomposition which yields an SP network with unit slowdown, together with an N network to which Theorem 3 can be applied, and the two slowdowns can then be combined [16, Theorem 5.12]. For the three remaining indecomposable networks, the minimal SP extensions are identified, and each case is checked using arguments similar to those in the proof of Theorem 3, yielding sets of inequalities which each lead to a contradiction if slowdown greater than 4/3 is assumed.

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### 8 Programmatic approach

The six-activity case has been checked using an approach that is now described. Our implementation also verified the proofs for 4 and 5 activities.

For a fixed number of activities \( n \), we want to consider all non-SP networks with \( n \) activities, and for each of these, to show that for every possible workload there is a SP extension which achieves the 4/3 bound. Working inductively, for networks with fewer than \( n \) activities we have already shown the 4/3 bound. We also only need to consider activity networks up to isomorphism. Additionally, we do not need to consider networks that can be decomposed such that there is at least one series or parallel node in the decomposition, since the slowdown is then bounded above by the slowdown of an activity network with less than \( n \) activities [16, Theorem 5.12]. Therefore we need only consider indecomposable networks and those which are decomposable but where every module is indecomposable.

The overall schema is to consider each possible activity network \( G \) in turn, assuming that it is a counterexample. Each of its SP extensions then has slowdown exceeding 4/3. This generates a system of inequalities, and we can then demonstrate that this system has no solution.

First all possible \( n \)-activity networks are generated and classified into SP, decomposable (but not SP), or indecomposable. Isomorphic activity networks are discarded, reducing the number of candidate

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\(^4\)The dual of a directed graph is the graph with its edges reversed.
counterexamples. For some candidate \( G \), consider each minimal extension \( H \). Only considering minimal extensions is valid because any non-minimal extension \( H' \) will give \( T(H) \leq T(H') \). Every SP extension \( H \) exceeds the 4/3 bound, so we require that \( 4T(G) < 3T(H) \) for each such extension. It is also necessary to consider some extensions that are not SP as these generate additional, necessary constraints. Specifically, we consider the decomposable extensions because they have slowdown of at most 4/3. If \( T(G) = T(H) \) for a decomposable extension \( H \) then by the inductive hypothesis we could find an SP extension of \( G \) that would meet the bound, hence we require that \( T(G) < T(H) \) for every decomposable extension \( H \) of \( G \).

We now need to ask which workloads can allow all these constraints to hold simultaneously. Since \( T(H) = \max\{T(C) \mid C \text{ is a chain in } H\} \), we can consider each possible chain as a critical path and generate additional constraints that \( T(C) \geq T(D) \) for all chains \( D \) in \( H \). Hence we need to consider the disjunction of the sets of inequalities

\[
\{T(G) < T(C)\} \cup \{4T(G) < 3T(C)\} \cup \{T(C) \geq T(D)\} \mid D \text{ a chain in } H, D \neq C
\]

for every maximal chain \( C \) in \( H \). We only need to consider maximal chains since non-maximal chains have lower makespan. The makespan of a chain is simply the sum of its activity durations, so each choice of critical path \( C \) generates a system of linear inequalities expressed with variables that represent the unknown activity durations.

These inequalities can now be fed to a constraint solver such as \texttt{clp(q)} [9] to check if a workload does exist that meets the constraints. If one is found then we have found a counter-example to the 4/3 conjecture. For 4, 5, and 6 activities, an exhaustive search showed that no counterexamples exist.

We have proved formally that the 4/3 bound holds for the four-activity and five-activity case, and we have a programmatic proof of the six-activity case. This provides some evidence that Conjecture 2 is true. The techniques used for smaller indecomposable networks can be applied to the seven-activity case also. However, the systems of inequalities are too large to handle with the tools currently used, so such a proof would require new techniques or tools.

9 Conclusions and further work

Series-parallelising an activity network is done implicitly when a program is expressed in an inherently series-parallel formalism, or explicitly for the purposes of aiding scheduling. We now consider the implications of the bound for LC slowdown, the disproof of the factor of 2 conjecture, and the new factor of 4/3 conjecture.

As shown in Section 5, LC slowdown is bounded above. If all activities have very similar durations, a good bound is obtained and LC extensions are useful. However, this bound is not necessarily tight when durations vary.

In the motivating example, deciding which series-parallelisation to use at the time of writing the program forces a particular series-parallelisation before the workload is known. Consider a parallel programming environment that only allows SP activity networks to be expressed. At the time of writing, MATLAB is one such environment and we believe that in practice both Mathematica and OpenCL also require activity networks to be SP\(^5\).

Theorem 2 shows that requiring the series-parallelisation to be chosen before the workload is known accurately, may result in slowdown of more than 2. Iterating the construction for larger NS networks (of greater width as well as depth) allows the slowdown to be forced to be arbitrarily large.

Neighbour-synchronised networks are common in practice and may be quite large. The workload in practice may be different to what was expected when writing the program; for instance, contention for shared resources, communication delays, and cache misses are just some of the stochastic effects that affect

\(^5\) Mathematica and OpenCL both provide SP constructs, as well as more general methods to specify synchronization between activities; unfortunately these require creating objects for each precedence constraint. Such a heavy-weight mechanism only makes sense if activities are all very large (for instance, if the program consists of just a few threads), or there are only few precedence constraints.
parallel computation and that may produce large variations in the duration of an activity. Therefore, choosing a series-parallelisation without taking into account possible variations in workload may lead to large slowdown.

If one postpones the decision, it may be possible to do automated analysis at compile time, or the scheduler may be able to work around any locally arising bottlenecks due to stochastic variation in activity durations. Hence it would seem to be worthwhile allowing sufficient expressivity in the language so that one can more closely approximate the activity network of a computation.

On a positive note, if one can find a series-parallelisation that gives one the conjectured $4/3$ bound, then the impact of adding constraints is limited – the program will only take one-third as long again as it would have taken without the additional constraints and this seems a reasonable penalty to pay to obtain a structure that makes many scheduling problems easier.

However, one needs to take into account the cost of finding a series-parallelisation that achieves the bound. Consider the optimisation problem

| MINIMUM SERIES-PARALLELISATION (MSP) |
|-------------------------------------|
| **Input:** poset $G$, workload $t: V(G) \rightarrow (0, \infty)$ |
| **Output:** poset $H$, $H$ is a SPE of $G$ |
| **Criterion:** minimise $T(H)$. |

Let $|x|$ denote the size of an instance $x$ of MSP. It is easy to show that MSP is in the complexity class NPO [2]. Computing the level-constrained extension of an activity network can be done in polynomial time as discussed in Section 4. The approximation ratio of this procedure is bounded by $2^{O(|x|^2)}$. MSP is therefore in the class exp-APX, which is strictly contained in NPO unless P = NP [2].

Conjecture 2 implies that MSP can be approximated within a factor of $4/3$, but there is not necessarily a polynomial-time algorithm that can achieve this. A branch-and-bound algorithm for solving MSP never needs to consider more than $2^{O(|x|^2)}$ possible extensions, each corresponding to a subset of edges.

So a polynomial-time algorithm achieves slowdown of at most $2^{O(|x|^2)}$. On the other hand, an SP extension with minimal slowdown can be found in $2^{O(|x|^2)}$ time, and Conjecture 2 would bound this slowdown as being at most $4/3$. It is not clear how to close this gap; it appears possible that MSP is exp-APX-hard.

MSP also seems related to the classical decision problem MINIMUM PRECEDENCE CONSTRAINED SCHEDULING (MPCS) [6], which is NP-complete. The difficulty of MPCS derives from there being only a limited number of processors. In contrast, MSP appears to be difficult because the output network must be series-parallel. The $(4/3 - \epsilon)$-inapproximability of MPCS [8] suggests that a similar inapproximability result may exist for MSP.

Several directions for future work are envisaged. The first relates to the proof of Conjecture 2, at least for 7 activities. This requires improving the implementation so its correctness could be verified and finding more powerful techniques that avoid case analysis. Second, a programming construct to specify NS networks could be added to existing programming environments and its performance established. Finally, if the decision version of MSP could be shown to be NP-complete, perhaps by reduction from MPCS, then the NP-hardness of MSP would follow. Proving that MSP is exp-APX-hard is another goal.

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