CONTOUR ENHANCEMENT VIA A SINGULAR FREE BOUNDARY PROBLEM

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Abstract. We study a degenerate nonlinear parabolic equation with moving boundaries which describes the technique of contour enhancement in image processing. Such problem arises from the model by Malladi and Sethian after an asymptotic expansion suggested by Barenblatt: in order to recover the phenomenon of mass concentration, a singular data is imposed at the free boundary.

1. Introduction. Modern computer vision focuses, among others, on the problems of noise removal and image enhancement in real images. From a mathematical modeling viewpoint this leads to image processing, which stands in formulating a PDE of evolution type for the image intensity. Here the unknown function $u$ (said the “image intensity” or briefly “image” in the following) is defined on a two dimensional domain and takes values between 0 (black) and 1 (white); besides time represents the scale parameter which leads from noisy input data to smoothed and enhanced images. In this setting, coarsening image patterns means smoothening the profile of the function $u$, while enhancing image edges stands in increasing the modulus of the space gradient $Du$. For instance we may receive, as an input, a vague image with grey regions shading off one into each other, and we may wish a sharp image with clear-cut divisions among different objects. Now we have a smooth initial datum and we want to evolve it into a step function, that is we wish the gradient to blow-up. We talk, here, about “complete enhancement”.

The usual evolution model carries an equation of parabolic type, possibly degenerate, with a nonlinear law relating the image flux with the image itself. It has been observed since the seminal work by Perona and Malik [12] that a suitable choice of the nonlinearity produces the two desired goals simultaneously. In particular, the diffusion has to be suppressed in areas of high gradients, that is the parabolic operator has to degenerate for large values of $|Du|$; moreover some anisotropic effect must be included, in order to increase the component of the gradient which is orthogonal to the edge and to reduce the tangential component. The Perona-Malik model has deeply influenced all the subsequent literature, that contains a number of relevant works that should be mentioned; we do not dwell on this topic and refer the interested reader to the book [3] instead.

In the present note we investigate the appearance of the edge enhancement in the model by Malladi and Sethian [10]. They proposed to move the isointensity contours under curvature dependent law, namely to solve

$$u_t = (1 + |Du|^2)^{\frac{1}{2} - \alpha} \kappa,$$

(1)

2000 Mathematics Subject Classification. Primary: 35R35, 94A08; Secondary: 35K55, 35K65.

Key words and phrases. Free boundary problems, image processing.
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where $\kappa$ is the curvature of the surface $z = u(x, y)$. The nonnegative constant $\alpha$ plays the role of an enhancement parameter, because it weights the rate at which diffusion vanishes as $|Du|$ grows up; the most studied cases are $\alpha = 0$ (mean curvature flow) and $\alpha = 1$ (Beltrami flow). In order to focus on the effect of contour enhancement, it is appropriate to follow the asymptotic analysis performed by Barenblatt in [4]. He zooms into the boundary layer arising near the edges, where the normal component of the image intensity gradient is large, and ends up with a reduced equation in one space-dimension:

$$u_t = \frac{u_{xx}}{(1 + (u_x)^2)^{1+\alpha}},$$

(2)

or, via further approximation,

$$u_t = \frac{u_{xx}}{(u_x)^{2(1+\alpha)}}.$$

(3)

Here the $x$-direction lies orthogonal to the edge. After having introduced the function

$$\Phi(v) := \int_0^v \frac{ds}{(1 + s^2)^{1+\alpha}},$$

(4)

or, respectively,

$$\Phi(v) := -\int_v^\infty \frac{ds}{s^{2(1+\alpha)}} = \frac{1}{2(1 + \alpha)} v^{-(1+2\alpha)},$$

(5)

equations (2) and (3) belong to a wider class that can be written as follows

$$u_t = \Phi(u_x)_x,$$

(6)

and is known in the literature as “nonlinear diffusion”. To our purpose, the function $\Phi$ is of class $C^2$ and strictly increasing, but has finite limit at infinity:

$$\Phi'(v) > 0 \text{ for all } v, \quad \lim_{v \to +\infty} \Phi(v) = \Phi_\infty < \infty.$$

Therefore equations of type (6) are in principle parabolic, with a degeneracy at $u_x = \infty$. Also note that the class (6) includes equation of type (2) and (3) also for some negative values of $\alpha$, actually between $-1/2$ and 0.

A common feature of diffusions (linear or nonlinear) is that the profile of the solutions, as a function of $x$, smoothen as time goes by, because the level sets spread. This property opposes the desired enhancement, which requires the profile to sharpen; therefore some other mechanism must come into play.

In the paper [4] a self-similar solution is found for equation (3), for any $\alpha > 0$. It is seen that it is a local solution: indeed, it is constantly 0 or 1 outside a transition region (the grey zone) bounded by two curves $\ell(t)$ and $r(t)$. The width of this transition region decreases with time, a token that the edge enhancement is taking place. Across the two delimiting curves, the intensity function $u$ is continuous, but its derivative $u_x$ suffers an infinite jump.

Such an intermediate-asymptotic solution suggests a free boundary problem for determining the image intensity evolution in the boundary layer:

$$\begin{cases}
  u_t = \Phi(u_x)_x, & x \in (\ell, r), \quad t \in (0, T), \\
  u = 0, \quad \Phi(u_x) = \Phi_\infty, & x = \ell(t), \quad t \in (0, T), \\
  u = 1, \quad \Phi(u_x) = \Phi_\infty, & x = r(t), \quad t \in (0, T), \\
  u = u_\infty, \quad \ell = \ell_\infty, \quad r = r_\infty, & x \in (\ell_\infty, r_\infty), \quad t = 0.
\end{cases}$$

(7)
This is a free boundary problem because the unknowns are the function \( u(x, t) \) and the moving boundaries, \( \ell(t) \) and \( r(t) \), which must be determined from the problem thanks to the extra boundary conditions. For this reason the interfaces \( x = \ell(t) \) and \( x = r(t) \) are also called free boundaries. The problem is singular at the free boundary since \( u_x \to \infty \), and degenerate as a parabolic equation since \( \Phi'(u_x) \to 0 \) as \( u_x \to +\infty \).

![Diagram](image)

**Figure 1.** At the left side, the front of the solution to the free boundary problem, respectively at time 0, after some time \( t \), and when complete enhancement occurs. At the right side, the respective sections of the grey region, i.e. the transition domain.

Applications request to deal with input data \( u_0 \) that are Lipschitz-continuous on the segment line \([\ell_0, r_0] \), with \( u_0(\ell_0) = 0, u_0(r_0) = 1 \), and \( 0 < u_0 < 1 \) inside \((\ell_0, r_0)\).

The mathematical theory for the free boundary problem (7) was treated in [5], in the case of monotone \( u_0 \). The problem (7) with monotone \( u_0 \) has an unique classical solution \((u, \ell, r)\) with \( u(t) \) monotone, as a function of \( x \). The profile of \( u(t) \), as a function of \( x \), sharpens as \( t \) grows; actually the two free boundaries \( x = \ell(t) \) and \( x = r(t) \) shrink. If the two moving boundaries meet, a vertical front is formed, representing complete enhancement. A detailed asymptotic analysis is performed for the power case (5), showing that such focusing happens after a finite time for negative \( \alpha \); otherwise the distance between the free boundaries vanishes asymptotically like \( t^{-1/2\alpha} \) if \( \alpha > 0 \), or like \( e^{-\pi^2t} \) for \( \alpha = 0 \).

For general \( u_0 \), well-posedness for problem (7) was established in [2]. Existence of solutions comes from an approximation procedure that we recall, because it is of interest by itself. The difficulty created by the singular boundary conditions \( \Phi(u_x) = \Phi_\infty \) at \( x = \ell(t) \) and \( x = r(t) \) is attacked by looking at the modified problems:

\[
\begin{aligned}
\begin{cases}
  u_t = \Phi(u_x)_x, & x \in (\ell_\varepsilon, r_\varepsilon), & t \in (0, T), \\
  u = 0, & x = \ell_\varepsilon(t), & t \in (0, T), \\
  u = 1, & x = r_\varepsilon(t), & t \in (0, T), \\
  u = u_{0\varepsilon}, & \ell_\varepsilon = \ell_0, & r_\varepsilon = r_0, & x \in (\ell_0, r_0), & t = 0.
\end{cases}
\end{aligned}
\]

(7,\varepsilon)

where \( \varepsilon > 0 \) and \( u_{0\varepsilon} \) is a suitable approximation of \( u_0 \).

This problem is “easier” because a finite boundary condition is imposed at the free boundaries, which is usually referred to as the “combustion boundary condition”. More details about combustion theory are found in the survey [14].
For any given $\varepsilon > 0$, the solution $u_\varepsilon$ to (7, $\varepsilon$) lives in the region where $\Phi'$ is bounded away from zero, hence problem (7, $\varepsilon$) can be regarded as an (uniformly) parabolic approximation of (7), a posteriori. This fact shall be exploited, for instance, to produce a classical solution for the singular problem.

Our object, here, is to investigate the qualitative properties of solutions to (7) which go into the direction of contour enhancement and speak in favor of the two dimensional Malladi-Sethian model. To this aim we exploit the approximation by combustion-type problems (7, $\varepsilon$), which is the subject of next section.

2. Approximation by combustion-type problems. We recall here some facts about the combustion-type problems that shall help in passing to the limit. Because, for any $\varepsilon > 0$, problem (7, $\varepsilon$) is no longer singular, we can give an elliptic-parabolic formulation, as it is usual practice. To this end we introduce a cut-off function $b$:

$$b(u) := \begin{cases} 
0 & \text{if } u \leq 0, \\
u & \text{if } u \in (0, 1), \\
1 & \text{if } u \geq 1.
\end{cases}$$

and a larger interval $(\ell, r)$, with $\ell < \ell_0$ and $r < r$ to be chosen later.

We next state the doubly nonlinear problem

$$\begin{cases} 
b(u)_t = \Phi(u_x)_x, & x \in (\ell, r), \quad t \in (0, T), \\
\Phi(u_x) = \Phi(\frac{1}{\varepsilon}), & x = \ell \text{ or } r, \quad t \in (0, T), \\
b(u) = b(u_0), & x \in (\ell, r), \quad t = 0,
\end{cases}$$

(8, $\varepsilon$)

which is clearly parabolic in the region where $0 < u < 1$, and elliptic elsewhere.

If $u$ solves the elliptic-parabolic problem (8, $\varepsilon$), the free boundaries of (7, $\varepsilon$) can be recovered a-posteriori by means of

$$\ell_\varepsilon(t) := \sup \{ x : u(y, t) = 0 \text{ as } y \in (\ell, x) \},$$

$$r_\varepsilon(t) := \inf \{ x : u(y, t) = 1 \text{ as } y \in (x, r) \}. (9)$$

What is missing to a solution of (8, $\varepsilon$), for being a solution to the free boundary problem (7, $\varepsilon$), is that the two interfaces $\ell_\varepsilon$ and $r_\varepsilon$ really bound a transition region where $0 < u < 1$ and $u_t = \Phi(u_x)_x$, and that the two interfaces do not touch the artificial boundaries $\ell$ and $r$. In this case, the solution to (8, $\varepsilon$) can be regarded as the solution to (7, $\varepsilon$), that has been extended in a linear way beyond the interfaces.

It is worth to emphasize that the singular problem (7) does not admit an elliptic parabolic formulation, because the linear extension outside the interfaces should be vertical. To be clear, we recall here the precise notions of solution.

Definition 2.1. A function $u \in L^2(0, T; H^1(\ell, r))$ is a weak solution to the elliptic-parabolic problem (8, $\varepsilon$) if $b(u)$ is continuous in the closed rectangle $[\ell, r] \times [0, T]$, and we have

$$\int_0^T \int_\ell^r [-b(u)\varphi_t + \Phi(u_x)\varphi_x] \, dx \, dt =$$

$$\int_\ell^r b(u_0)\varphi(x, 0) \, dx + \frac{1}{\varepsilon} \int_0^T [\varphi(r, t) - \varphi(\ell, t)] \, dt \quad (10)$$

for all $\varphi \in C^1([\ell, r] \times [0, T])$ which vanish at $t = T$. 


Let $\ell_\varepsilon < r_\varepsilon$ two continuous real curves defined on $[0, T)$ with $\ell_\varepsilon(0) = \ell_o$, $r_\varepsilon(0) = r_o$, and $u_\varepsilon$ a function defined on

$$D_\varepsilon := \{(x, t) : x \in [\ell_\varepsilon(t), r_\varepsilon(t)], t \in (0, T) \} \cup (\ell_o, r_o) \times \{0\}. \quad (11)$$

We say that the triplet $(u_\varepsilon, \ell_\varepsilon, r_\varepsilon)$ is a weak solution to the free boundary problem $(7.\varepsilon)$ if $u_\varepsilon, x \in L^\infty(D)$ and extending the function $u$ linearly outside the free boundaries by

$$u_\varepsilon(x, t) = \begin{cases} 1 + (x - r_\varepsilon(t))/\varepsilon & \text{if } r_\varepsilon(t) \leq x \leq \bar{r}, \\ (x - \ell_\varepsilon(t))/\varepsilon & \text{if } \ell \leq x \leq \ell_\varepsilon(t) \end{cases}$$

gives a weak solution to the elliptic parabolic problem $(8.\varepsilon)$, for all $\ell \leq \inf \ell_\varepsilon$ and $r \geq \sup r_\varepsilon$.

Besides, the triplet $(u_\varepsilon, \ell_\varepsilon, r_\varepsilon)$ is a classical solution to the free boundary problem $(7.\varepsilon)$ if (i) $u_\varepsilon$ is of class $C^{2,1}$ in the interior of $D_\varepsilon$, where $u_{\varepsilon,t} = \Phi(u_{\varepsilon,x})_t$ holds in classical sense, (ii) $u_\varepsilon$ is continuous on $D_\varepsilon$ with $u_\varepsilon(\ell_\varepsilon(t), t) = 0$, $u_\varepsilon(r_\varepsilon(t), t) = 1$, and $u_\varepsilon(x, 0) = u_{\o\varepsilon}(x)$, and (iii) for almost any $t \in (0, T)$, the function $x \mapsto \Phi(u_{\varepsilon,x})$ is continuous up to the free boundaries with $\Phi(u_{\varepsilon,x})(\ell_\varepsilon(t), t) = \Phi(u_{\varepsilon,x})(r_\varepsilon(t), t) = 1/\varepsilon$.

Because the singular problem $(7)$ does not admit an elliptic parabolic formulation, the notion of weak solution makes no sense; on the contrary, the definition of classical solution extends with no changes.

Elliptic-parabolic problems have been intensively studied during the last decades: the general well-posedness theory has been settled in [1, 11], moreover the qualitative behavior and the relationship with free boundary problems were investigated in several papers, among which we refer to [7, 8, 9, 13], for linear diffusion $\Phi(u_\varepsilon) = u_\varepsilon$.

We also mention [2], were some of these results have been adapted to the doubly nonlinear problem $(8.\varepsilon)$. Let us recall some useful properties obtained in that paper.

**Theorem 2.1.** For any $\varepsilon > 0$, and $T > 0$, the elliptic-parabolic problem $(8.\varepsilon)$ has a unique weak solution $u_\varepsilon$. Such solution $u_\varepsilon$ has the following additional regularity properties:

i) $u_\varepsilon \in L^\infty(0, T; W^{1,\infty}(\ell, \bar{r})) \cap L^2(0, T; H^2(\ell, \bar{r}))$.

ii) the curves $\ell_\varepsilon(t)$ and $r_\varepsilon(t)$ are uniformly continuous on $[0, T)$; moreover they are bounded, respectively, from above and from below. The bounds do not depend on $\ell, \bar{r}$. 
iii) in the interior of the set $D_{\varepsilon}$, we have $0 < u_{\varepsilon} < 1$ and $u_{\varepsilon,t} = \Phi(u_{\varepsilon,x})$ in the classical, pointwise sense.

Here we have used the notations introduced in (9) and (11).

An interesting consequence of Theorem 2.1 is the global well-posedness of the approximating problems (7.\varepsilon), for all $\varepsilon > 0$.

**Theorem 2.2.** Let $u_{o\varepsilon}$ a suitably smooth approximation of $u_o$. Then for all $\varepsilon > 0$ and $T > 0$ the free boundary problem (7.\varepsilon) admits a unique weak solution. Moreover this solution is classical.

We do not prove this theorem, which easily follows from [2, Theorem 4.1].

3. **Convergence to the singular problem and concentration of profiles.**

The passage to the limit from (7.\varepsilon) to (7) is a bit arduous. Here we only sketch the main steps, to put into light how it gives rise to enhancement, i.e. to the shrinking of the internal domain.

We begin by constructing candidates for the free boundaries of the singular problem:

$$
\ell(t) := \limsup_{(\varepsilon,s) \to (0,t)} \ell_{\varepsilon}(s) \quad \text{and} \quad r(t) := \liminf_{(\varepsilon,s) \to (0,t)} r_{\varepsilon}(s).
$$

(12)

Note that, in principle, the functions $\ell$ and $r$ are only semicontinuous. The main advantage in choosing this kind of limit is that the corresponding internal region $D$ (defined as in (11), as from $\ell$ and $r$ given by (12)) is locally contained in the intersection of the internal regions $D_{\varepsilon}$ of the combustion-type problems (see [2, Lemma 5.2]). We shall see later on that actually the free boundaries $\ell_{\varepsilon}$ and $r_{\varepsilon}$ converge locally uniformly to $\ell$ and $r$, as $\varepsilon \to 0$.

The next step in passing to the limit is that the functions $u_{\varepsilon}$ converge to a function $u$ which actually solves the differential equation inside the domain $D$. At the same time, one realizes that such a convergence may take place at all times only for a subset of diffusions $\Phi$, namely that ones satisfying $\lim_{v \to +\infty} v^2 \Phi'(v) < \infty$. This is consistent with the results obtained in [5] for the power case (5) and monotone data, showing that for $\alpha < 0$ the two frontiers $\ell(t)$ and $r(t)$ collapse to the same point after a finite time. Indeed, in the power case $v^2 \Phi'(v) = v^{-2\alpha}$.

**Theorem 3.1.** There exist a time $T > 0$ and a function $u$ defined in $D$ so that

i) $u_{\varepsilon}$ tends to $u$ locally in $C^{1+\alpha,\alpha}$, for $\alpha < 1$,

ii) $u$ is of class $C^{2,1}$ and satisfies $u_t = \Phi(u_{x})$ in classical sense inside $D$,

iii) $u$ is continuous also for $t = 0$ with $u(x,0) = u_o(x)$ for all $x$ between $\ell_o$ and $r_o$.

If, in addition, $\lim_{v \to +\infty} v^2 \Phi'(v) < \infty$, then $T$ can be any positive number.

**Sketch of the proof.** Well-known estimates from the elliptic-parabolic theory give that $u_{\varepsilon}$ converge to a function $u$ locally in $BV$ and in any $L^p$. The standard parabolic machinery could then give that $u$ actually satisfies the thesis, provided to obtain uniform estimates for $u_{\varepsilon,x}$. This is highly non-trivial, because $u_{\varepsilon}$ is actually expected to take the value $\infty$ at the side boundary. In the interior of the domain, the needed estimates are achieved after switching to new variables. By virtue of uniform estimates on the approximating problems (7.\varepsilon), $u_{\varepsilon}$ is at least bounded from below; hence there exists a constant $B$ (not depending by $\varepsilon$) so that the functions $x \mapsto u_{\varepsilon}(x,t) + Bx$ are strictly increasing, for all $t$. This allows us to invert the
functions \( u_\varepsilon \), after having rotated the \( x-u_\varepsilon \) axes, and to introduce the new free variable

\[
(\ell_\varepsilon(t), r_\varepsilon(t)) \ni x \mapsto y := u_\varepsilon(x, t) + Bx \in (B\ell_\varepsilon(t), 1 + Br_\varepsilon(t)),
\]

and unknown function

\[
p_\varepsilon(y, t) := \Phi(u_\varepsilon, x)(x, t).
\]

After computations, one gets that \( p_\varepsilon \) solves an elliptic-parabolic problem posed in a non-cylindrical domain:

\[
\begin{aligned}
c(p_\varepsilon)_t &= p_\varepsilon,_{yy}, & y &\in (B\ell_\varepsilon(t), 1 + Br_\varepsilon(t)), & t &\in (0, T), \\
p_\varepsilon &= \Phi(1/\varepsilon), & y &= B\ell_\varepsilon(t) \text{ or } 1 + Br_\varepsilon(t), & t &\in (0, T), \\
c(p_\varepsilon) &= c(p_{\varepsilon, c}), & y &\in (\ell_{\text{oc}}, 1 + Br_{\text{oc}}), & t &= 0,
\end{aligned}
\]

where

\[
c(p) := \begin{cases} 
- \frac{1}{\Phi^{-1}(p) + B} & \text{if } \Phi(-B) < p < \Phi_\infty, \\
0 & \text{if } p \geq \Phi_\infty,
\end{cases}
\]

and \( p_{\varepsilon, c}(y) := \Phi(u_{\varepsilon, c}(x)) \) if \( y = u_{\varepsilon, c}(x) + Bx \).

For any \( \varepsilon \), the function \( p_\varepsilon \) stays in the parabolic region \( (\Phi(-B), \Phi(1/\varepsilon)]) \). The proof is completed by checking that, in the interior of the domain, the limit function \( p \) does not overcome the upper value \( \Phi_\infty \) and fall into the elliptic region. We make sure of that by constructing barrier functions. Here the role of \( \lim_{\varepsilon \to +\infty} v^2\Phi'(v) \) becomes clear: if it is finite, the barrier function exists for all times, otherwise we are able to construct only a local-in-time barrier. We repeat that there is no way to improve this estimate, because there are counterexamples for \( \Phi \) of power type.

At last we zoom at the free boundaries. A preliminary remark, that helps in facing technical difficulties, is that the functions \( u_\varepsilon \) are strictly increasing near at the free boundaries. Actually, this has been proved by using some simplifying hypothesis, namely that \( u_\varepsilon \) has a finite number of local extremal points and is not flat at the end points, \( \ell_\varepsilon \) and \( r_\varepsilon \). We keep here these assumptions because they are not too restrictive. The convergence near the boundaries can be stated as follows:

**Theorem 3.2.** The free boundaries of the combustion problems \((7, \varepsilon), \ell_\varepsilon \) and \( r_\varepsilon \), converge locally uniformly to \( \ell \) and \( r \) defined by \((12)\). Moreover the functions \( \Phi(u_{\varepsilon, x}) \) converge to \( \Phi(u_\varepsilon) \) locally uniformly in the set \( \{(x, t) : \ell(t) \leq x \leq r(t), t \in (0, T)\} \).

In particular \( u = 0 \) at \( x = \ell(t), u = 1 \) at \( x = r(t) \), and \( \Phi(u_x) = \Phi_\infty \) at \( x = \ell(t) \) or \( x = r(t) \).

**Sketch of the proof.** In all what follows, we consider only the left side of the domain; of course, all arguments translate to the right side. For \( \delta > 0 \), sufficiently small, we may cut a slice of the internal domain, near the left interface, bounded by the smooth curve \( \ell_\varepsilon^0(t) := \min\{x \geq \ell_\varepsilon(t) : u_\varepsilon(x, t) = \delta\} \). By means of the so called lap-number theory, one can prove that \( u_{\varepsilon, x} > 0 \) if \( \ell_\varepsilon(t) < x < \ell_\varepsilon^0(t) \), and that the curve \( t \mapsto \ell_\varepsilon^0(t) \) is of class \( C^1 \). This monotonicity property allows us to follow \([5]\) and make use of the hodograph variable

\[
(\ell_\varepsilon(t), \ell_\varepsilon^0(t)) \ni x \mapsto y := u_\varepsilon(x, t) \in (0, \delta),
\]
to consider the conjugate formulation for \( v_\varepsilon(y, t) = x \), or better for its derivative \( w_\varepsilon(y, t) = v_{\varepsilon, y}(y, t) \):

\[
\begin{cases}
    w_t = \Psi(w)_{yy} & y \in (0, \delta), \quad t > 0, \\
    w(0) = \varepsilon, \quad w(\delta) = 1/u_{\varepsilon,x}(\ell_\varepsilon^{\varepsilon}) & y = 0 \text{ and } \delta, \quad t > 0, \\
    w = 1/u_{\varepsilon}^\prime(u_{\varepsilon}^{-1}) & y \in (0, \delta), \quad t = 0.
\end{cases}
\] (14.\varepsilon)

Here \( \Psi(w) := -\Phi(1/w) \).

The main advantage in looking at problem (14.\varepsilon) instead of (7.\varepsilon), is that it is posed in a fixed domain, with Dirichlet type boundary conditions. Moreover the limit problem for (14.\varepsilon) is

\[
\begin{cases}
    w_t = \Psi(w)_{yy} & y \in (0, \delta), \quad t > 0, \\
    w(0) = 0, \quad w(\delta) = 1/u_{\varepsilon}(\ell^\varepsilon) & y = 0 \text{ and } \delta, \quad t > 0, \\
    w = 1/u_{\varepsilon}^\prime(u_{\varepsilon}^{-1}) & y \in (0, \delta), \quad t = 0,
\end{cases}
\] (14)

which has initial datum in \( L^\infty \), and a non-singular boundary condition. Here \( u \) is the limit function produced in Theorem 3.1. Besides we can go back from the functions \( w_\varepsilon \) to the functions \( u_\varepsilon \) by setting

\[ z_\varepsilon(y, t) := \int_\gamma [w_\varepsilon dy + \Psi(w_\varepsilon)_{y} dt] - \ell_\varepsilon^\varepsilon(0), \]

where \( \gamma \) is any smooth curve joining \((\delta, 0)\) to \((y, t)\), and remarking that

\[ u_\varepsilon(x, t) = y \iff x = z_\varepsilon(y, t), \quad \ell_\varepsilon(t) = \lim_{y \to 0} z_\varepsilon(y, t), \] (15)

because of the uniqueness of solution for (7.\varepsilon). Now parabolic theory yields that \( w_\varepsilon \), the solutions to (14.\varepsilon), converge to \( w \), the classical solution to (14) and that \( z_\varepsilon \) tends to

\[ z(y, t) := \int_\gamma [w dy + \Psi(w)_{y} dt] - \ell^\varepsilon(0), \]

locally uniformly in the set \((0, \delta] \times (0, \ell)\) (see [2, Lemma 5.11, Proposition 5.7] for more details). This convergence spreads out along \( y = 0 \) and yields that \( \ell_\varepsilon \) tends to

\[ \hat{\ell}(t) = \lim_{y \to 0} z(y, t). \]

Indeed for any \( 0 < y < \eta < \delta, \quad 0 < t < \ell, \) and \( \varepsilon > 0 \) we have

\[ |z_\varepsilon(y, t) - z(y, t)| \leq |z_\varepsilon(y, t) - z_\varepsilon(\eta, t)| + |z(y, t) - z(\eta, t)| + |z_\varepsilon(\eta, t) - z(\eta, t)| \]

\[ \leq 2 \sup_{\varepsilon \geq 0} \|w_\varepsilon\|_{\infty} |y - \eta| + \|z_\varepsilon - z\|_{C(K)} \]

where \( K \) is any closed rectangle containing the point \((\eta, t)\). By sending \( y \to 0 \) we get

\[ |\ell_\varepsilon(t) - \hat{\ell}(t)| \leq 2 \sup_{\varepsilon \geq 0} \|w_\varepsilon\|_{\infty} |\eta| + \|z_\varepsilon - z\|_{C(K)}, \]

that can be made arbitrary small by choosing firstly \( \eta \), and secondly \( \varepsilon \) near to 0. Note also that this implies that we can send \( \varepsilon \to 0 \) in the relation (15) and obtain

\[ u(x, t) = y \iff x = z(y, t), \quad \ell(t) = \lim_{y \to 0} z(y, t), \]

where now \( u \) is the limit function produced in Theorem (3.1) and \( \ell \) is the limit curve introduced in (12). As a byproduct, we can extend \( u \) in a continuous way till \( x = \ell(t) \) by setting \( u = 0 \).
Concerning the convergence of $\Phi(u_{\varepsilon,x})$, the proof relies on the same arguments of [6, Lemma 4.3].

We are now ready to prove that actually the model (7) gives rise to contour enhancement, i.e. that the width of the grey region decreases with time.

**Theorem 3.3.** The width of the domain $D$, $r(t) - \ell(t)$, is a decreasing function of the time.

**Proof.** It suffices to check that the two interfaces $\ell(t)$ and $r(t)$ are monotone, actually that $\ell(t)$ is non-decreasing and that $r(t)$ is non-increasing. We only deal with the left boundary, because the right one can be dealt in the same way. In force of Theorem 3.2, the thesis follows by proving that the curve $\ell_\varepsilon(t)$ is non-decreasing for any given $\varepsilon > 0$ (sufficiently small), and then letting $\varepsilon \to 0$. Now, $\ell_\varepsilon$ is the free boundary of the non-singular problem (7.ε), which can be regarded by its elliptic-parabolic formulation (8.ε).

Let $0 \leq t_\circ < T$, we want to check that $\ell_\varepsilon(t) \geq \ell_\varepsilon(t_\circ)$ for all $t_\circ \leq t < T$, or equivalently that $b(u_{\varepsilon}) = 0$ in the rectangle $[\ell_\varepsilon(t_\circ), \ell_\varepsilon(t)] \times [t_\circ, T]$. This can be done by producing a supersolution $U$ that satisfies $b(U) = 0$ in that rectangle and then invoking the comparison result [2, Proposition 3.11]. By uniform estimates $u_{\varepsilon}(x,t_\circ) \leq (x-\ell_\varepsilon(t_\circ))/\varepsilon$ as $x \geq \ell_\varepsilon(t_\circ)$, at least if $\varepsilon$ is small enough. Therefore the stationary function $U(x,t) = (x-\ell_\varepsilon(t_\circ))/\varepsilon$, for $t \geq t_\circ$, is the needed supersolution.

Finally we check that the interfaces really start at $\ell_\circ$ and $r_\circ$.

**Proposition 3.4.** We have $\lim_{\varepsilon \to 0} \ell(t) = \ell_\circ$ and $\lim_{\varepsilon \to 0} r(t) = r_\circ$.

**Proof.** We continue to argue only for the left interface. By Theorem 3.3, there exists for sure $\lim_{\varepsilon \to 0} \ell(t) \geq \ell_\circ$. Let us say that limit $\ell_1$, and assume by contradiction that $\ell_1 > \ell_\circ$.

We next turn to the elliptic-parabolic formulation of the approximating problems. For all $t$ and $\varepsilon > 0$, by the mass balance we have

$$\int_{t_\circ}^{\ell_1} b(u_{\varepsilon})(t) dx = \int_{t_\circ}^{\ell_1} u_{\circ \varepsilon} dx + \int_0^{\ell_1} [\Phi(u_{\varepsilon,x})(\ell_1) - \Phi_\infty] d\tau \geq \int_{t_\circ}^{\ell_1} u_{\circ \varepsilon} dx - C t$$

where the constant $C$ does not depend by $\varepsilon$. We thus send $\varepsilon$ to 0 and get

$$\int_{\ell(t)}^{\ell_1} u(t) dx \geq \int_{\ell_\circ}^{\ell_1} u_\circ(x) dx - C t.$$

Finally extracting the limit as $t$ goes to 0 yields $\int_{\ell_\circ}^{\ell_1} u_\circ(x) dx \leq 0$, which imply the contradiction $u_\circ = 0$ in the segment line between $\ell_\circ$ and $\ell_1$.

**Open questions.** As we have said before, problem (7) admits classical solution for all positive time in the case $\lim_{v \to +\infty} v^2 \Phi(v) < +\infty$. For $\Phi$ in the power form, the precise asymptotic is known: the grey region vanishes at infinity, and the exact speed is known. We conjecture that this behavior is generic, but trying to extend the proof given in the power case yields to the asymptotic analysis of (13), which is a nonlinear equation settled in a non-cylindrical domain.

Otherwise, if $\lim_{v \to +\infty} v^2 \Phi(v) = +\infty$, we know that the maximal existence time is finite, and that the interfaces collapse at that time, in the case of a power non-linearity. In general, the maximal existence time could be finite also because of a
new phenomenon: \( \ell(T) < r(T) \), but \( u_x \) blows up as \( t \to T \) in some internal point. If so, \( u_x \) must blow up along a segment line, which has at least one end point in the boundary of the grey region. Let us take, for the sake of clarity, that \( u_x \) blows up along \( (\ell(T), \bar{x}) \). If \( \bar{x} < r(T) \) and \( u \to 0 \) at \( (\bar{x}, T) \), then the solution could be prolonged (in some weaker sense) after the time \( T \), but the interface become discontinuous. Actually, we do not see examples with discontinuous interface, so that it is possible, in principle, that complete enhancement happens in finite time whenever \( \lim_{v \to +\infty} v^2 \Phi'(v) = +\infty \). Of course, a better understanding of this arguments is desirable, and will be subject of future studies.

**Acknowledgements.** I am sincerely grateful to Professor Juan Luis Vázquez for having introduced me to the fascinating field of free boundary problems. I am in debt with him for his huge help in understanding and solving mathematical questions, and most of all for the vital force he sends forth and which permeates his entire work.

**REFERENCES**

[1] H. W. Alt and S. Luckhaus, *Quasilinear elliptic-parabolic differential equations*, Math. Z., 183(3) (1983), 311–341.

[2] A. L. Amadori and J. L. Vázquez, *Singular free boundary problem from image processing*, Math. Models Methods Appl. Sci., 15(5) (2005), 689–715.

[3] G. Aubert and P. Kornprobst, *Mathematical Problems in Image Processing: Partial Differential Equations and the Calculus of Variations (second edition)*, volume 147 of Applied Mathematical Sciences, Springer-Verlag, 2006.

[4] G. I. Barenblatt, *Self-similar intermediate asymptotics for nonlinear degenerate parabolic free-boundary problems that occur in image processing*, Proc. Natl. Acad. Sci. USA, 98(23) (2001), 12878–12881.

[5] G. I. Barenblatt and J. L. Vázquez, *Nonlinear diffusion and image contour enhancement*, Interfaces Free Bound., 6(1) (2004), 31–54.

[6] M. Bertsch and R. Dal Passo, *Hyperbolic phenomena in a strongly degenerate parabolic equation*, Arch. Rational Mech. Anal., 117(4) (1992), 349–387.

[7] M. Bertsch and J. Hulshof, *Regularity results for an elliptic-parabolic free boundary problem*, Trans. Amer. Math. Soc., 297(1) (1986), 337–350.

[8] J. Hulshof, *An elliptic-parabolic free boundary problem: continuity of the interface*, Proc. Roy. Soc. Edinburgh Sect. A, 106(3-4) (1987), 327–339.

[9] J. Hulshof and L. A. Peletier, *An elliptic-parabolic free boundary problem Nonlinear Anal., 10(12) (1986), 1327–1346.

[10] R. Malladi and J. A. Sethian, *Image processing: flows under min/max curvature and mean curvature*, Graph. Models Image Process., 58(2) (1996), 127–141.

[11] F. Otto, *L^1-contraction and uniqueness for quasilinear elliptic-parabolic equations*, J. Differential Equations, 131(1) (1996), 20–38.

[12] P. Perona and J. Malik, *Scale-space and edge detection using anisotropic diffusion*, IEEE Transactions on Pattern Analysis and Machine Intelligence, 12(7) (1990), 629–639.

[13] C. J. van Duyn and L. A. Peletier, *Nonstationary filtration in partially saturated porous media*, Arch. Rational Mech. Anal., 78(2) (1982), 173–198.

[14] J. L. Vázquez, *The free boundary problem for the heat equation with fixed gradient condition*, In Free boundary problems, theory and applications (Zakopane, 1995), volume 363 of Pitman Res. Notes Math. Ser., pages 277–302. Longman, Harlow, 1996.