Quantum realizations of Hilbert–Palatini second-class constraints

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Abstract
In a classical theory of gravity, the Barbero–Immirzi parameter ($\eta$) appears as a topological coupling constant through the Lagrangian density containing the Hilbert–Palatini term and the Nieh–Yan invariant. In a quantum framework, the topological interpretation of $\eta$ can be captured through a rescaling of the wavefunctional representing the Hilbert–Palatini theory, as in the case of the QCD vacuum angle. However, such a rescaling cannot be realized for pure gravity within the standard (Dirac) quantization procedure where the second-class constraints of Hilbert–Palatini theory are eliminated beforehand. Here, we present a different treatment of the Hilbert–Palatini second-class constraints in order to set up a general rescaling procedure (a) for gravity with or without matter and (b) for any choice of gauge (e.g. time gauge). The analysis is developed using the Gupta–Bleuler and the coherent state quantization methods.

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1. Introduction

It has been suggested [1] that the Barbero–Immirzi parameter ($\eta$) as appearing in the Holst action [2] should emerge as a topological parameter in the formally quantized theory of gravity, much like the $\theta$ parameter in QCD [1, 3]. However, the Holst term is not a topological density. This implies that one cannot think of an analogue of the Chern–Simons functional as in QCD, through which the non-perturbative vacuum structure of gravity, as anticipated, can manifest itself. Hence within this formulation, $\eta$, the coefficient of the Holst term does not qualify for a topological interpretation.

In a classical context, the topological nature of Barbero–Immirzi parameter was clarified in [4] through the Hamiltonian analysis of the following action:

$$ L = \frac{1}{2} e \Sigma_{ij} R^i_{\mu j} + \frac{\eta}{2} I_{NY} $$  

(1)
with $\Sigma_{IJ}^{\mu \nu} := \frac{1}{2}(e_I^\mu e_J^\nu - e_I^\nu e_J^\mu)$, $R^{IJ}_{\mu \nu}(\omega) := \partial_\mu \omega^I_\nu + \omega^K_\mu \omega^I_\nu \omega^L_\nu$. Here, the first term is the Hilbert–Palatini term. In the second term, $\eta$ is a coefficient of the Nieh–Yan invariant $I_{NY}$, which is a topological density [5], that is, a total divergence:

$$I_{NY} = \partial_\mu \left[ \epsilon^{\mu \nu \alpha \beta} e^I_\nu D_\alpha e^I_\beta \right]$$

and

$$R^{IJ}_{\mu \nu}(\omega) := \partial_\mu \omega^I_\nu + \omega^K_\mu \omega^I_\nu \omega^L_\nu.$$

This indicates that the two different canonical formulations, i.e. the ones with and without the Barbero–Immirzi parameter, are related through a canonical transformation generated by $j_{NY}^I$.

However, the fact that $I_{NY}$ is a topological density suggests that there must exist ‘large gauge transformations’ under which $\int d^3 x j_{NY}^I$ transforms non-trivially [6]. This would imply the existence of different inequivalent vacua parametrized by $\eta$ in the quantum theory of gravity. Then the canonical transformation as mentioned above cannot be implemented unitarily in the corresponding quantum theory. In this context, it is worthwhile to note that the appearance of $\eta$ in the spectrum of the area operator [7] also strongly hints at such a possibility.

In a quantum framework, this issue of a non-unitary realization of the canonical transformation generated by $j_{NY}^I$ can be studied through the procedure of rescaling. We recall that in the case of QCD, the interpretation of $\theta$ as a vacuum angle can be understood from this perspective alone, i.e. through the rescaling of the wavefunction $\Psi$ representing the quantum theory [3]:

$$\Psi' = e^{i \theta \int d^3 x Y_{\alpha \beta}} \Psi$$

where $Y_{\alpha \beta}$ is the Chern–Simons functional. Under large gauge transformations $\Psi$ changes by a phase which is cancelled by the phase picked up by the exponential operator, leaving $\Psi'$ to be invariant. Thus, the operator $e^{i \theta \int d^3 x Y_{\alpha \beta}}$ essentially implements the large gauge transformations. Here, $\theta$ manifestly an angular parameter characterizing the inequivalent sectors in the quantum theory. Through this rescaling one can induce a canonical transformation on the $\theta$-independent canonical operators (characterizing the standard QCD Lagrangian density without any $\theta$ term) to arrive at the $\theta$-dependent canonical operators (corresponding to the Lagrangian density containing the $\theta$ term). Note that the canonical formulation thus obtained corresponds precisely to the one coming from a Hamiltonian analysis of the QCD Lagrangian density which contains the $\theta$ term (i.e. the Pontryagin density).

Here we attempt to develop a similar rescaling procedure for gravity with or without matter. Such a construction is particularly relevant in order to study the vacuum structure of gravity. This would also allow one to go from the canonical formulation without $\eta$ (Hilbert–Palatini theory) to the one containing $\eta$ (corresponding to the Lagrangian density in (1)). To proceed, let us consider the rescaling of the wavefunctional $\Psi$ representing the formally quantized Hilbert–Palatini theory:

$$\Psi' = e^{i \eta \int d^3 x Y_{\alpha \beta}} \Psi$$

where $q$’s are the coordinates in the quantum configuration space and $Y$, the rescaling functional, is given by

$$Y = \frac{1}{2} j_{NY}^I (e, \omega)$$

and

$$\frac{1}{2} \epsilon^{abc} e^I_a D_b e^I_c.$$

However, as we demonstrate in the next section, if the Dirac procedure is used to eliminate the second-class constraints before quantization [8], the operator $j_{NY}^I$ vanishes ‘strongly’ (in the sense of Dirac in [8]). This reduces the exponential in (3) to an identity operator which clearly cannot implement the corresponding ‘large gauge transformations’. Thus, in Dirac’s
method of quantization, there is no way for $\eta$ to emerge as a vacuum angle through a rescaling as in QCD. This also means that one cannot arrive at the canonical constraints containing $\eta$ starting from the Hilbert–Palatini theory via the rescaling route.

We note that the procedure of rescaling has been applied earlier to gravity coupled to spin-$\frac{1}{2}$ fermions [9]. However, the approach in [9] uses Dirac’s method to solve the second-class constraints before quantization, or, in other words, uses the connection equation of motion. As explained already, this method cannot be applied to pure gravity, since the rescaling functional $Y$ in (4) vanishes when the connection equation is used.

Thus, one is forced to adopt a different quantization procedure other than the standard one of Dirac in order to set up a general rescaling method for gravity with or without matter. In this paper we address this issue. Here we reformulate the Hilbert–Palatini canonical constraints using the Gupta–Bleuler and the coherent state quantization methods which do not require the elimination of second-class constraints to begin with. This is then followed by a demonstration of the rescaling which essentially leads to the real Ashtekar–Barbero canonical formulation.

The general idea of the Gupta–Bleuler quantization [10] is to split the original set of first- and second-class constraints into a holomorphic and an anti-holomorphic set of first-class constraints related through the Hermitian conjugation. The physical subspace contains only those ket states which are annihilated by the holomorphic set. Here we apply this method to Hilbert–Palatini theory. The resulting space of physical wavefunctions is then used to employ the rescaling. This leads to the canonical formulation based on (1), as desired. Next we repeat the exercise using the coherent state quantization for constrained systems [11]. There we consider a squared sum of the original second-class constraints to define the physical Hilbert space. Note that such squared combinations also appear in the context of the master constraint programme [12], where the constraints are enforced in a different way than above.

In contrast to Dirac’s approach, our analysis in either cases does not require the use of the connection equation of motion for the rescaling. This particular feature is essential in order to recover a complete topological interpretation of the Barbero–Immirzi parameter, independent of any matter coupling.

In the following section we demonstrate the rescaling procedure in time gauge, first in the Gupta–Bleuler and then in the coherent state approach. We work in a representation diagonal in the densitized triad operators, as is required. In section 3, we generalize our construction without fixing any gauge. Section 4 contains a few concluding remarks.

2. Rescaling in time gauge

The Hilbert–Palatini Lagrangian density is given by

$$L = \frac{1}{2} e \Sigma_{ij} \mathcal{R}_{ij}. \tag{5}$$

We parametrize the tetrad fields as [4]

$$e_{i}^{I} = \sqrt{e} N M^{I} + N^{a} V_{a}^{I}, \quad e_{a}^{I} = V_{a}^{I}; \quad M_{I} V_{a}^{I} = 0, \quad M_{I} M^{I} = -1 \tag{6}$$

and then the inverse tetrad fields are

$$e_{I}^{j} = -\frac{M_{I}}{\sqrt{e} N}, \quad e_{a}^{I} = V_{a}^{I} + \frac{N^{a} M_{I}}{\sqrt{e} N}; \quad M^{I} V_{a}^{I} = 0, \quad V_{a}^{I} V_{b}^{J} = \delta_{a}^{b}, \quad V_{a}^{I} V_{a}^{J} = \delta_{I}^{J} + M^{I} M_{J}. \tag{7}$$

Introducing the fields

$$E_{a}^{I} = 2 e \Sigma_{a}^{I}, \quad \chi_{i} = \frac{M_{I}}{M_{0}}, \quad \bar{\omega}_{b}^{I} = \omega_{b}^{I} - \chi_{m} \omega_{m}^{I}, \quad \xi_{j} = \omega_{a}^{j} \chi_{a}. \tag{8}$$
the Lagrangian density in (5) can be written as
\[ L = E^a_i \frac{\partial}{\partial t} \tilde{\omega}^0_i - \omega^{ij} \frac{\partial}{\partial t} G_{ij} - \frac{\chi}{2} \epsilon_{ijk} G^{jk} - N H - N^a H_a. \] (9)

\( H, H_a, G^\text{boost}, \) and \( G^\text{rot} \) are the scalar, vector, boost and rotation constraints, respectively.

In terms of the canonical variables, the Nieh–Yan functional in (4) becomes
\[ Y = \frac{1}{4} \sqrt{E} \epsilon^{ijk} E^a_i E^b_j E^c_k \left[ - \chi_i \partial_a \left( \sqrt{E} \epsilon^{imb} \epsilon^{cde} E^d_m E^e_n \right) + \partial_a \left( \sqrt{E} \epsilon^{imb} \epsilon^{cde} E^d_m E^e_n \right) \right] - \frac{1}{2} M_{kk} - \frac{1}{2} \left( 1 - \chi^2 \right) \epsilon^{ijk} \epsilon^{lmn} E^i_a E^j_m E^k_n \] (10)

where we have used the identities
\[ \epsilon^{abc} V_{kc} = \sqrt{E} \epsilon^{ijk} E^a_i E^b_j, \quad \sqrt{E} V_{ck} = E^a_i \quad \text{and} \quad \sqrt{E} V_{c0} = \chi^k E^k_a \]
and also the decomposition
\[ \omega^{aij} = \frac{1}{2} E^a_i [\zeta_j + \left( 1 - \chi^2 \right) \epsilon^{ijk} E^i_a M^{kj}], \quad M^{kl} = M^{lk} \]
which is just a way to represent the nine components of \( \omega^{aij} \) in the basis of three \( \zeta \)'s and six \( M_{kl} \)'s.

One can choose the time gauge by putting \( \chi_i \approx 0 \). As this condition forms a second-class pair with the boost constraint, they have to be imposed together.

The boost constraint is given by:
\[ G^\text{boost}_i = - \partial_a E^a_i - \omega^{aij} E^a_j \]
\[ = - \partial_a E^a_i + \zeta_i \]
which is solved by the condition
\[ \zeta_i = \partial_a E^a_i. \] (11)

The first-class set of constraints are given by the following expressions:
\[ G^\text{rot}_i = \epsilon^{ijk} \omega^0_j E^a_k \]
\[ H_a = E^a_i \epsilon^{bij} R^{ij}_{ab} \]
\[ = E^a_i \delta_a^b \delta^0_j - \omega_{ij}^0 \zeta_i + [\epsilon^{ijk} E^a_j \zeta_i - E^a_k M^{kl}] G^\text{rot}_i \]
\[ H = - \frac{1}{2} E^a_i \epsilon^{bij} R^{ij}_{ab} \]
\[ = E^a_i \partial_a \zeta_i + \frac{i}{2} \zeta_i \zeta_i - \frac{1}{2} E^a_i E^b_j \zeta_i \partial_a E^a_j + \frac{1}{2} \epsilon^{ijm} E^a_i \partial_a E^a_j M^{mn} \]
\[ + \frac{i}{8} [2 \zeta_i \zeta_i + M^{kl} M^{li} - M^{kl} M^{ki}] - \frac{1}{2} E^a_i \epsilon^{bij} \omega^0_i \partial_j (R^{ij}_{ab}) \] (12)

where \( \zeta_i \) is given by (11).

From (10), we get the following expression for \( Y \) in time gauge:
\[ Y = \frac{1}{4} \sqrt{E} \epsilon^{ijk} E^a_i E^b_j E^c_k \left[ - \chi_i \partial_a \left( \sqrt{E} \epsilon^{imb} \epsilon^{cde} E^d_m E^e_n \right) + \partial_a \left( \sqrt{E} \epsilon^{imb} \epsilon^{cde} E^d_m E^e_n \right) \right] - \frac{1}{2} M_{kk}. \] (13)

1 The parametrization for \( \omega^{aij} \) here is different from that in [4].
2.1. Classical second-class constraints

As the Lagrangian density (9) is independent of the velocities associated with $M_{kl}$, we have the primary constraints involving the corresponding momenta:

$$\pi_{kl} \approx 0.$$  \hspace{1cm} (14)

These in turn imply secondary constraints, which essentially lead to the vanishing of torsion (see [4] for details):

$$\{H, \pi_{kl}\} \approx 0 = \epsilon_{ijk} E^a_i \partial_a E^b_j - \frac{1}{2}(M^{ii} \delta_{kl} - M_{kl}) + \frac{1}{2}(M^{ii} \delta_{kl} - M_{kl}) \approx 0$$  \hspace{1cm} (15)

where we have defined $F_{kl}$ as

$$F_{kl}(E^a_i) = \frac{1}{2} \epsilon_{ijk} E^a_i \partial_a E^b_j - \frac{1}{2} \epsilon_{ijk} E^a_i \partial_a E^b_j + \frac{1}{2} \epsilon_{ijm} E^a_i E^b_j \partial_a E^c_j (M_{mn} + i \alpha \pi_{mn}) + \frac{1}{8} [2 \xi_{kl} + (M_{kl} + i \alpha \pi_{kl})(M_{kl} + i \alpha \pi_{kl}) - (M_{kl} + i \alpha \pi_{kl})(M_{kl} + i \alpha \pi_{kl})]$$

Dirac’s prescription leads to the next step where the second-class constraints are solved before quantization or are eliminated through Dirac brackets. This is equivalent to imposing them ‘strongly’ as operator conditions [8]. Thus, the physical subspace of the original Hilbert space would be obtained through the states which are annihilated by the operators corresponding to the remaining set of first-class constraints. However, here the second-class pair in (14) and (15), when enforced strongly, leads to the vanishing of the rescaling functional $Y$. Thus, the Dirac quantization procedure as it is cannot provide any passage to the new set of constraints corresponding to the Lagrangian density containing the Nieh–Yan term.

Hence, one must adopt alternative quantization procedures to impose the second-class constraints in the quantum state space. Here we first employ a method which is a slight generalization of the Gupta–Bleuler approach in electrodynamics, and then repeat the exercise using the coherent state quantization. Both the cases result in a non-vanishing rescaling functional through which the canonical transformation can be carried out.

2.2. Gupta–Bleuler quantization

Following the general idea of Gupta–Bleuler quantization, we have to find suitable holomorphic and anti-holomorphic sets containing all the constraints. Here the sets can be defined as

$$C := \left(G^i_{\text{rot}}, H_i, \tilde{H}_i, (Q_{kl} + i \alpha \pi_{kl})\right)$$

$$C^\dagger := \left(G^i_{\text{rot}}, H_i, \tilde{H}_i, (Q_{kl} - i \alpha \pi_{kl})\right)$$

where $\alpha$ is a constant and $\tilde{H}_i, \tilde{H}_i, Q_{kl}$ are defined as

$$\tilde{H}_i = E^a_i \partial_a \xi_{i} + \frac{1}{2} \xi_{i} \xi_{j} - \frac{1}{2} E^a_i \partial_a \xi_{j} E^b_j + \frac{1}{2} \epsilon_{ijm} E^a_i E^b_m \partial_a E^c_j (M_{mn} + i \alpha \pi_{mn}) + \frac{1}{8} [2 \xi_{i} + (M_{kk} + i \alpha \pi_{kk})(M_{ll} + i \alpha \pi_{ll}) - (M_{kl} + i \alpha \pi_{kl})(M_{kl} + i \alpha \pi_{kl})]$$

$$\tilde{H}_i = E^a_i \partial_a \xi_{i} + \frac{1}{2} \xi_{i} \xi_{j} - \frac{1}{2} E^a_i \partial_a \xi_{j} E^b_j + \frac{1}{2} \epsilon_{ijm} E^a_i E^b_m \partial_a E^c_j (M_{mn} - i \alpha \pi_{mn}) + \frac{1}{8} [2 \xi_{i} + (M_{kk} - i \alpha \pi_{kk})(M_{ll} - i \alpha \pi_{ll}) - (M_{kl} - i \alpha \pi_{kl})(M_{kl} - i \alpha \pi_{kl})]$$

$$Q_{kl} = M_{kl} - F_{kl}(E^a_i).$$

In the presence of matter coupling this would be $[M_{kl} - F_{kl}(E^a_i, \phi_m)] \approx 0$, where $\phi_m$ denotes the matter fields, eg. fermions when gravity is coupled to fermions.
Thus, we have two sets of first-class constraints which satisfy the required algebra given by

\[ [C_A, C_B] \approx 0 \approx [C_A^\dagger, C_B^\dagger]. \]

\[ [C_A, C_B^\dagger] \approx Z_{AB} \] (18)

where \( Z_{AB} \), the central charge, is a function of \( \alpha \) in our case.

In definitions (17), instead of \( H \), one needs to take the classically equivalent constraints \( \tilde{H} \) and \( \tilde{H}^\dagger \) in order to ensure the Abelian property of the individual sets, and hence to reproduce the correct algebra as in (18). \( \tilde{H} \) and \( \tilde{H}^\dagger \) are obtained by replacing \( M_{kl} \) in \( H \) by \( (M_{kl} + i\alpha \pi_{kl}) \) and \( (M_{kl} - i\alpha \pi_{kl}) \), respectively.

Next we define a representation based on the fundamental commutation relations as

\[ \hat{E}_i^a |\psi_1\rangle = E_i^a |\psi_1\rangle, \quad \hat{\delta}_i^a |\psi_1\rangle = -i\delta E_i^a |\psi_1\rangle \]

\[ \hat{M}_{kl} |\psi_1\rangle = M_{kl} |\psi_1\rangle, \quad \hat{\pi}_{kl}^a |\psi_1\rangle = -i\delta \delta M_{kl} |\psi_1\rangle. \]

Here \( |\psi\rangle \) represents the formally quantized Hilbert–Palatini theory. The physical subspace is obtained through the realization of the set \( C \) on the ket states:

\[ \hat{C} |\psi\rangle = 0. \] (19)

Thus, the constraints involving the canonical pair \( (\hat{M}_{kl}, \hat{\pi}_{kl}) \) act as

\[ (\hat{Q}_{kl} + i\alpha \hat{\pi}_{kl}) |\psi\rangle = 0. \] (20)

The Hermitian conjugation of the above implies that the physical bra states are annihilated by \( C^\dagger \).

From (20), it follows that the original second-class constraints are satisfied individually through the expectation values with respect to the physical states:

\[ \langle \psi | \hat{Q}_{kl} | \psi \rangle = \langle \psi | \hat{\pi}_{kl}^a | \psi \rangle = 0. \]

This is how the correspondence with the classical formulation emerges in this framework.

Equation (20) completely specifies the dependence of the wavefunctional on the variables \( M_{kl} \), whereas the constraints \( G_{rot}^i, H_a \) and \( \tilde{H} \) determine the \( E^a \) dependence. Note that in \( \tilde{H} \), \( M_{kl} \) appears only through the corresponding constraint. Thus, the full wavefunctional can be written as

\[ \psi(M, E) = \hat{\phi}(M - F)\phi(E) \] (21)

where \( \hat{\phi}(M - F) \) is a Gaussian functional of \( (M_{kl} - F_{kl}) = Q_{kl} \). Thus, the Gupta–Bleuler wavefunctional differs from the one obtained through Dirac’s procedure by the vacuum of the oscillator in the \( Q \) space. The integral representation for the inner product becomes

\[ \int dM(\hat{\phi}(M - F))^2 \int dE\phi^\ast(E)\phi(E) = \int dQ(\hat{\phi}(Q))^2 \int dE\phi^\ast(E)\phi(E) \] (22)

where we have used the fact that the Jacobian corresponding to the change of variables from \( M \) to \( Q \) is identity. The \( Q \) integration can be performed trivially, leaving only the \( E \) integral. This then becomes equivalent to the reduced space integral as would be obtained by Dirac’s procedure, up to a normalization.

Importantly, the above expression contains no delta function corresponding to the constraint in (20), which usually appears as a projector in the inner product for first-class constrained systems (see, for example, chapter 13 in [13]). Also note that the presence of the Gaussian functional \( \hat{\phi}(M - F) \) in (22) leads to normalizable states in the \( M_{kl} \) sector.
2.3. Rescaling

Next we proceed to perform the rescaling in the quantized phase space. $Y$ as in (13) depends only on the operators corresponding to the configuration variables ($E^a_i, M_{kl}$). The new momenta conjugate to $\hat{E}^a_i$ are thus given by

$$\hat{\omega}'_0 = \frac{\delta Y(\hat{E}^a_i, \hat{M}_{kl})}{\delta \hat{E}^a_i}$$

where we have used the expression

$$\frac{\delta Y}{\delta E^a_i} = \epsilon_{ijk} \partial_b \left( \frac{E_{ci}}{\sqrt{E}} \right) \frac{\delta (\sqrt{E} E^{bj} E^{ck})}{\delta E^a_i}$$

with $t^a_i$ defined as

$$t^a_i = \epsilon^{abc} D_b V_{ci}$$

The new $\hat{\pi}'_{kl}$'s are obtained as

$$\hat{\pi}'_{kl} \Psi' = \epsilon^{in} / d^3x \hat{\pi}'_{kl} e^{-in} / d^3x \Psi'$$

Note that this procedure goes through in the presence of matter couplings which lead to non-vanishing torsion.

As already mentioned, the expectation value of constraint (20) among the physical states in the $\hat{M}_{kl}$ sector (i.e. the states $\tilde{\phi}(M - F)$ in (21)) leads to the relation

$$\langle \hat{M}_{kl} \rangle_M = \langle F_{kl}(\hat{E}^a_i) \rangle_M$$

which is the analogue of the classical constraint in (15). To emphasize, here torsion as an operator in (23) does not annihilate $\Psi$, rather its expectation value vanishes. This is to be contrasted with the Dirac procedure where the torsion operator vanishes 'strongly'.

New constraints. The new constraints, which annihilate the rescaled wavefunctional $\Psi'$, can be found by introducing the new momenta in the expressions as given in (12). We illustrate this for $\hat{G}^{\text{new}}_i$ below.
The rotation constraint for the action in (1) containing the Hilbert–Palatini and Nieh–Yan terms is given by
\[ \hat{G}^{rot}_i = e^{i\eta \int d^4x \hat{y}} \hat{G}^{rot}_i e^{-i\eta \int d^4x \hat{y}} = \eta \partial_a \hat{E}_a^i + \epsilon^{ijk} \hat{\omega}_{0j}^0 \hat{E}_k^a - \frac{\eta}{\sqrt{E}} \epsilon_{ijkl} \hat{E}_{ajl} \hat{I}_k^p. \]

Taking the expectation value with respect to the states \( \tilde{\phi}(M - F) \), we arrive at the familiar SU(2) Gauss’ law:
\[ \langle \hat{G}^{rot}_i \rangle_M = \eta \partial_a \hat{E}_a^i + \epsilon^{ijk} \hat{A}_j^a \hat{E}_k^a \]
where
\[ \hat{A}_d^l = \langle \hat{\omega}_{dl}^0 \rangle_M = \frac{\eta}{2} \epsilon^{ijkl} \hat{\omega}_{dlj}. \]

Without going into the detailed algebraic expressions of the remaining constraints corresponding to (1) as they are not relevant for our purpose here, we observe that they can be obtained in a similar manner as shown for \( \hat{G}^{rot}_i \).

2.4. Coherent state quantization

We now demonstrate another approach, namely the coherent state quantization for constrained systems [11]. Although this was originally designed to develop an alternative path-integral formulation using coherent states, here we use the essential idea to enforce the appropriate ‘quantum' constraints. This would allow a consistent rescaling formulation for gravity with or without matter.

Following the general construction developed by Klauder [11], we seek the states for which
\[ \langle \Psi | (\hat{M}_{kl} - \hat{F}_{kl}(\hat{E}_a^i)^2 + \hat{\pi}_{kl}^2) | \Psi \rangle = \langle \Psi | (\hat{Q}_{kl}^2 + \hat{\pi}_{kl}^2) | \Psi \rangle = 0. \] (24)

However, since we have \( \langle \hat{A}^2 \rangle = \langle \Delta \hat{A} \rangle^2 + \langle \hat{A} \rangle^2 \) for any operator \( \hat{A} \), equation (24) cannot be satisfied by any \( \Delta \hat{A} \) with non-zero uncertainty \( \Delta \hat{A} \). Hence, as suggested in [11], one has to modify the above criterion as
\[ \langle \Psi | (\hat{Q}_{kl}^2 + \hat{\pi}_{kl}^2) | \Psi \rangle \leq \lambda_0 (O(\hbar)) \] (25)
where \( \lambda_0 \) denotes the minimum eigenvalue in the spectrum of the constraint operator. The modification, being of the order of \( \hbar \), is a purely quantum feature. The rest of the constraints \( \hat{G}^{rot}_i, \hat{H}_a \), and \( \hat{H} \), as given by equation (12), are imposed as they are on the physical states:
\[ \hat{G}^{rot}_i | \Psi \rangle = \hat{H}_a | \Psi \rangle = \hat{H} | \Psi \rangle = 0. \] (26)

Now, there is a family of minimum uncertainty states, namely the canonical coherent states, defined as
\[ |M, \pi\rangle = e^{-iM\beta} e^{i\pi \hat{M}} |\beta\rangle \] (27)
where \( M = \langle \hat{M}_{kl} \rangle, \pi = \langle \hat{\pi}_{kl} \rangle \) and \( |\beta\rangle \) is some fiducial state for which \( M = 0, \pi = 0 \) (we suppress the indices just to simplify the notation). Among these, the one satisfying (25) is the coherent state for which \( Q = (M - F(E)) = 0, \pi = 0, \) with \( F(E) = \langle \hat{F}_{kl}(\hat{E}_a^i) \rangle \). Using (27), the explicit form of this state reads
\[ \Psi(M) = e^{-iF(E)\beta} \beta(M) = \beta(M - F(E)) \]
where \( \beta(M) \) is the fiducial state functional in a representation diagonal in \( \hat{M}_{kl} \).
In this formulation, one can define a projection operator $P$ onto the physical Hilbert space, requiring the following properties [11]:

$$P^\dagger = P, \quad P^2 = P.$$ 

In our case $P$ (in the $M_E$ sector) becomes simply $|\Psi_1(M)\rangle \langle \Psi_1(M)|$.

The full wavefunctional representing the physical subspace can thus be written as

$$\Psi(M, E) = \beta(M - F)\phi(E).$$

The inner product in this space reads

$$\int dM dE \Psi^{\dagger}(M, E)\Psi(M, E) = \int dQ \beta^{\dagger}(Q)\beta(Q) \int dE \phi^{\dagger}(E)\phi(E).$$

As in the Gupta–Bleuler case, here also the $Q_{kl}$ (or, $M_{kl}$) sector factors out leaving only the $E$ integral in the product. In particular, we can choose the fiducial state to be the oscillator ground state in this case. Then this expression reproduces the Gupta–Bleuler product as in (22). Thus, the two Hilbert spaces are equivalent. The rescaling can now be implemented along the lines of our previous discussion.

### 3. Rescaling for any gauge choice

Now we provide a brief outline of the rescaling procedure without choosing any gauge. For non-zero $\chi_i$, the canonical coordinates in the Lagrangian density in (9) are $\tilde{\omega}^{0l}_i, M_{kl}$ and $\chi_i$ where $\tilde{\omega}^{0l}_i$ is given by equation (8).

$Y$ in (10) can be rewritten as

$$Y = \frac{1}{4} \sqrt{E} e^{ijk} E^a_j E^b_k \left[ -\chi_i \partial_a \left( \sqrt{E} e^{ilm} e_{bcd} \chi_l E^c_m E^d_n \right) + \partial_b \left( \sqrt{E} e^{ilm} e_{bcd} E^c_m E^d_n \right) \right]$$

$$- \frac{1}{2} M_{kk} - \frac{1}{2(1 - \chi^2)} \chi_k \chi_i M_{ik} - e^{ijk} E^a_j \chi^a_k \partial_k \chi_i + \chi_k G^\text{rot}_k.$$ 

(28)

The structure of $Y$ suggests that we can choose the representation to be diagonal in the operators $\tilde{E}^a_i, \tilde{M}_{kl}$ and $\tilde{\chi}_i$. In the above equation the last term involving $G^\text{rot}_i$ commutes with all other remaining terms and acts trivially on $|\Psi\rangle$ to give zero. Hence this term can be ignored at this stage itself.

One can follow exactly the same procedure as earlier to define a suitable physical subspace using either the Gupta–Bleuler or the coherent state method, and then find the new set of canonical operators through the rescaling. Thus, the new momenta $\tilde{\omega}^{0l}_i$ conjugate to $E^a_i$ read

$$\tilde{\omega}^{0l}_i = \omega^{0l}_i - \chi_j \omega^{0l}_j - \eta \epsilon^{ijkl} \chi_k \partial_j \chi_i$$

$$+ \eta \left( \frac{E_{ai}}{2 \sqrt{E}} + \sqrt{E} e^{ilk} \epsilon_{abc} E^b_k \right) \left( t^a_{i} - \chi_i t^0_i \right)$$

(29)

where we have defined

$$\omega^{0l}_i = \omega^{0l}_i - \eta \epsilon^{ijkl} \omega_{djk}, \quad \omega^{0l}_j = \omega^{0l}_j - \eta \epsilon^{ijkl} \omega_{dk},$$

$$t^a_{i} = e^{abc} D_b V_{ci} = e^{abc} \left[ \partial_b V_{ci} + \omega_{bi} V^c_i - \omega^{0l}_b V^c_i \right],$$

$$t^0_{i} = e^{abc} D_b V_{ci} = e^{abc} \left[ \partial_b V_{ci} - \omega^{0l}_b V^c_i \right].$$

As is evident, we recover the new momenta in time gauge when $\chi_i = 0$ and $\zeta_i = \partial_0 E^i_i$. 

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The momenta corresponding to $M_{ij}$ and $\chi_i$ transform as follows:

$$\pi'_{kl} = \pi_{kl} + \eta \left( \delta_{kl} + \frac{\chi_k \chi_l}{1 - \chi^2} \right)$$

$$\zeta'_i = \zeta_i - \eta \epsilon^{ijk} E^a_j \left[ \partial_a \chi_k \sqrt{E} E^b_k \chi_m \partial_a \left( \frac{E^m_b}{\sqrt{E}} \right) \right] + \eta \left( \delta_{ij} + \frac{\chi_i \chi_j}{1 - \chi^2} \right) \chi_k M_{jk}.$$ 

The new set of constraints, which annihilate the rescaled wavefunctional $\Psi'$ can now be obtained in a manner as demonstrated for the time-gauge fixed theory.

4. Conclusion

We have illustrated how to arrive at the canonical formulation corresponding to the action containing the Hilbert–Palatini and Nieh–Yan terms starting from the Hilbert–Palatini canonical theory through a generic rescaling procedure. The constraint operators, through their action on the physical states, reproduce the real Ashtekar–Barbero formulation.

As it turns out, one cannot invoke such a rescaling to obtain the Ashtekar–Barbero constraints if the Hilbert–Palatini second-class constraints are eliminated before quantization, as in Dirac’s method. Here we have provided a remedy to this problem by using alternative approaches, namely the Gupta–Bleuler and the coherent state quantizations. These two cases result in the same physical Hilbert space. Also, here the torsional degrees of freedom as associated with the second-class constraints emerge as relevant canonical operators through $\hat{M}_{kl}$ and $\hat{\pi}^{kl}$. To emphasize, these do not appear in the Dirac-quantized phase space where the second-class constraints are eliminated beforehand.

As both the quantization methods lead to a non-vanishing rescaling functional, they apply to any arbitrary matter coupling. When such couplings lead to nonzero torsion (e.g. fermion coupled to gravity), one can obtain the new canonical constraints by writing the rescaling functional $Y$ in terms of the geometric variables (i.e. tetrads and spin connections). Using the connection equation of motion to write $Y$ in terms of matter fields there becomes purely optional. Thus, our analysis provides a complete topological interpretation of the Barbero–Immirzi parameter in a quantum framework, whether or not matter is coupled to gravity.

We have also demonstrated that the rescaling can be carried out without choosing any particular gauge (e.g. time gauge). Thus, the appearance of $\eta$ as a topological parameter in this quantum description is not an artefact of some special gauge choice, as also shown in [4] from a classical perspective.

The construction here clearly shows that $\eta$ can manifest itself as a vacuum angle through a QCD like rescaling provided the representation is chosen to be diagonal in the densitized triad operators. Also, both the quantization approaches result in a well-defined operator corresponding to the ‘large gauge transformations’. Thus, our analysis might be particularly relevant in the context of a path-integral quantization, which is the most natural arena to investigate the effects coming from a potential non-perturbative vacuum structure underlying such transformations.

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