Ordered line and skew-fields in the Desargues affine plane

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Abstract. This paper introduces ordered skew fields that result from the construction of a skew field over an ordered line in a Desargues affine plane. A special case of a finite ordered skew field in the construction of a skew field over an ordered line in a Desargues affine plane in Euclidean space, is also considered. Two main results are given in this paper: (1) every skew field constructed over a skew field over an ordered line in a Desargues affine plane is an ordered skew field and (2) every finite skew field constructed over a skew field over an ordered line in a Desargues affine plane in $\mathbb{R}^2$ is a finite ordered skew field.

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Key words: Affine Pappus Condition; ordered line; ordered skew-field; Desargues affine plane.

1 Introduction

The foundations for the study of the connections between axiomatic geometry and algebraic structures were set forth by D. Hilbert [10], recently elaborated and extended in terms of the algebra of affine planes in, for example, [13], [5, Sec.IX.3, p.574]. E.Artin in [3] shows that any ordering of a plane geometry is equivalent to a weak ordering of its skew field. He shows that that any ordering of a Desargues plane with more than four points is (canonically) equivalent to an ordering of its field. In his paper on ordered geometries [24], P. Scherk considers the equivalence of an ordering of a Desarguesian affine plane with an ordering of its coordinatizing division ring. Considerable work on ordered plane geometries has been done (see, e.g., J. Lipman [15], V.H. Keiser [12], H. Tecklenburg [27], H. S. M. Coxeter [6] and L. A. Thomas [28, 29]).

In this paper, we utilize a method that is naive and direct, without requiring the concept of coordinates. Our results are straightforward and constructive. For this reason we begin by giving a suitable definition for our search for lines in Desargues affine planes, based on the meaning of betweenness given by Hilbert [10, §3].
In addition, we introduce ordered skew files that result from the construction of a skew field over an ordered line in a Desargues affine plane. A special case of a finite ordered skew field in the construction of a skew field over an ordered line in a Desargues affine plane in Euclidean space, is also considered.

Based on the works of E. Artin [3] and J. Lipman [15] on ordered skew fields, we prove that the skew field that the constructed over an ordered line in a Desargues affine plane is an ordered skew field. To prove this, we consider ordered Desargues affine plane based on the definition given by E. Artin [3], but in this case the ordered line is given a suitable definition without the use of coordinates.

Two main results are given in this paper, namely, every skew field constructed over a skew field over an ordered line in a Desargues affine plane is an ordered skew field 4.5. and every finite skew field constructed over a skew field over an ordered line in a Desargues affine plane in $\mathbb{R}^2$ is a finite ordered skew field 4.7.

2 Preliminaries

Let $\mathcal{P}$ be a nonempty space, $\mathcal{L}$ a nonempty subset of $\mathcal{P}$. The elements $p$ of $\mathcal{P}$ are points and an element $\ell$ of $\mathcal{L}$ is a line. Collinear points on a line $\ell$ are denoted by $[A,E,B]$, where $E$ is between $A$ and $B$. Given distinct points $A, B$, there is a unique line $\ell^{AB}$ such that $A, B$ lie on $\ell^{AB}$ and we write $\ell^{AB} = A + B$ [34, p. 52]. An affine space is a vector space with the origin removed [4, § 4.1, p. 391]. The geometric structure $(\mathcal{P}, \mathcal{L})$ is an affine plane, a subspace of an affine space, provided

1. For each $\{P, Q\} \in \mathcal{P}$, there is exactly one $\ell \in \mathcal{L}$ such that $\{P, Q\} \in \ell$.

2. For each $P \in \mathcal{P}, \ell \in \mathcal{L}, P \not\in \ell$, there is exactly one $\ell' \in \mathcal{L}$ such that $P \in \ell'$ and $\ell \cap \ell' = \emptyset$ (Playfair Parallel Axiom [22]). Put another way, if $P \not\in \ell$, then there is a unique line $\ell'$ on $P$ missing $\ell$ [23].

3. There is a 3-subset $\{P, Q, R\} \in \mathcal{P}$, which is not a subset of any $\ell$ in the plane.

Put another way, there exist three non-collinear points $P$ [23].

An affine plane is a projective plane in which one line has been distinguished [12]. For simplicity, our affine geometry is on the Euclidean plane $\mathbb{R}^2$ and incident lines $\ell, \ell'$ are represented by $\ell \cap \ell' \neq \emptyset$ (intersection). A 0-plane is a point, a 1-plane a line containing a minimum of 2 collinear points and a 2-plane is an affine plane containing a minimum of 4 points, no 3 of which are collinear. An affine geometry is a geometry defined over vector spaces $V$, field $F$ (vectors are points and subspaces of $V$) and subsets $\mathcal{P}$ (points), $\mathcal{L}$ (lines) and $\Pi$ (planes) [7, § 7.5].

Desargues’ Axiom, circa 1630 [13, § 3.9, pp. 60-61] [25]. Let $A, B, C, A', B', C' \in \mathcal{P}$ and let pairwise distinct lines $\ell_k, \ell_l, \ell_m, \ell^{AC}, \ell^{A'C'} \in \mathcal{L}$ such that

$\ell_k \parallel \ell_l \parallel \ell_m$ and $\ell^{A} \parallel \ell^{A'}$ and $\ell^{C} \parallel \ell^{C'}$.

Then $\ell^{AC} \parallel \ell^{A'C'}$. 

$A, B \in \ell^{AB}, A'B' \in \ell^{A'B'},$ and $B, C \in \ell^{BC}, B'C' \in \ell^{B'C'}$.

$A \neq C, A' \neq C'$, and $\ell^{AB} \neq \ell_l, \ell^{BC} \neq \ell_l$. 

Then $\ell^{AC} \parallel \ell^{A'C'}$. 

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Example 2.1. The parallel lines $\ell^{AC}, \ell^{A'C'} \in \mathcal{L}$ in Desargues’ Axiom are represented in Fig. 1. In other words, the base of $\triangle ABC$ is parallel with the base of $\triangle A'B'C'$, provided the restrictions on the points and lines in Desargues’ Axiom are satisfied.

A Desargues affine plane is an affine plane that satisfies Desargues’ Axiom.

Theorem 2.1. Pappus, circa 320 B.C. [5, §1.4, p. 18].
If $[ACE] \in \ell^{EA}, [BFD] \in \ell^{BD}$ and $\ell^{BD}, \ell^{CD}, \ell^{EF}$ meet $\ell^{DE}, \ell^{FA}, \ell^{BC}$, then $[NLM]$ are collinear on $\ell^{NM}$.

Example 2.2. The lines $\ell^{BD}, \ell^{CD}, \ell^{EF}, \ell^{DE}, \ell^{FA}, \ell^{BC}$ in Pappus’ Axiom are represented in Fig. 2. In that case, the points of intersection $[NLM]$ lie on the line $\ell^{NM}$.
The affine Pappus condition in Theorem 2.1 has an effective formulation relative to \([NLM]\) on line \(\ell^{NM}\) given by N.D. Lane [14], i.e.,

**Affine Pappus Condition** [14]. Let \(E, C, A, B, F, D\) be mutually distinct points as shown in Fig. 2 such that \(A, B, C\) lie on \(\ell^{EA}\) and \(B, F, D\) lie on \(\ell^{BD}\) and none of these points lie on \(\ell^{EA} \cap \ell^{BD}, \ell^{BD} \cap \ell^{NM}, \ell^{NM} \cap \ell^{EA}\). Then

\[
\begin{align*}
\ell^{CB} \cap \ell^{EF} & \text{ lies on } \ell^{NM} \\
\ell^{AF} \cap \ell^{CD} & \text{ lies on } \ell^{NM}
\end{align*}
\]  

\(\Rightarrow\) \(\ell^{AB} \cap \ell^{ED}\) lies on \(\ell^{NM}\).

This leads to the following result.

**Theorem 2.2. Affine Pappus Condition** [14]. If the affine Pappus condition holds for all pairs of lines \(\ell, \ell'\) such that \(\ell \nparallel \ell'\), then the affine Pappus condition holds for all pairs \(\ell, \ell'\) with \(\ell \parallel \ell'\).

Every Desarguesian affine plane is isomorphic to a coordinate plane over a field [26] and every finite field is commutative [16, §3, p. 351]. From this, we obtain

**Theorem 2.3.** [Tecklenburg] [26].

*Every finite Desarguesian affine plane is Pappian.*

### 3 Ordered lines and ordered Desargues affine plane

An invariant way to describe an 'order' is by means of a ternary relation: the point \(B\) lies "between" \(A\) and \(C\). Hilbert has axiomatized this ternary relation [10]. In this section, we begin by giving a suitable definition for our search for lines in Desargues affine plane, based on the meaning of *betweenness* given by D.Hilbert, i.e., if \(B\) lies "between" \(A\) and \(C\), we mark it with \([A, B, C]\).

**Definition 3.1.** An *ordered line* in a Desargues Affine plane (briefly, called the *line*) satisfies the following axioms.

**Lo.1** For \(A, B, C \in \ell\), \([A, B, C] \implies [C, B, A]\).

**Lo.2** For \(A, B, C \in \ell\) are mutually distinct, then we have exactly one, \([A, B, C]\), \([B, C, A]\) or \([C, A, B]\).

**Lo.3** For \(A, B, C, D \in \ell\), then \([A, B, C]\) and \([B, C, D]\) \(\implies [A, B, D]\) and \([A, C, D]\).

**Lo.4** For \(A, B, C, D \in \ell\), then \([A, B, C]\) and \([C, B, D]\) \(\implies [D, A, B]\) or \([A, D, B]\).

**Proposition 3.1.** For all \(A, B, C, D \in \ell\), two, of \([B, A, C]\), \([C, A, D]\) and \([D, A, B]\) exclude the third.

**Proof.** Let’s get double combinations and see how third is excluded.
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Figure 3: Possibilities of doubles combinations, of \([B, A, C]\), \([C, A, D]\) and \([D, A, B]\)

(a) Suppose we are true \([B, A, C]\) and \([C, A, D]\), (see Fig. 3, (a)) then from the order axioms in Def. 3.1, we have:

\[
\begin{align*}
[B, A, C] & \quad \text{Lo.1} \quad [B, A, D] \\
[C, A, D] & \quad \text{Lo.4} \quad [D, B, A] \vee [B, D, A].
\end{align*}
\]

From Axiom \textbf{Lo.2} loses the possibility that \([D, A, B]\) is true, so \([D, A, B]\) is false.

(b) Suppose we are true \([B, A, C]\) and \([D, A, B]\), (see Fig. 3, (b)) then from the axioms above we have:

\[
\begin{align*}
[B, A, C] & \quad \text{Lo.1} \quad [B, A, C] \\
[D, A, B] & \quad \text{Lo.4} \quad [A, D, C] \vee [A, C, D].
\end{align*}
\]

From Axiom \textbf{Lo.2} loses the possibility that \([C, A, D]\) be true, so \([C, A, D]\) is false.

but each of them \([A, D, C] \vee [A, C, D]\) derives that \([C, A, D]\) is false.

(c) Suppose we are true \([B, A, C]\) and \([D, A, B]\), (see Fig. 3, (c)) then from the axioms above we have:

\[
\begin{align*}
[C, A, D] & \quad \text{Lo.4} \quad [B, C, A] \vee [C, A, B].
\end{align*}
\]

From Axiom \textbf{Lo.2} loses the possibility that \([B, A, C]\) be true, so \([B, A, C]\) is false.

\[\Box\]

\textbf{Definition 3.2.} For three different points \(A, B, C \in \ell\), that, we say that points \(B\) and \(C\) lie on the same side of point \(A\), if we have exactly one of \([A, B, C]\), \([A, C, B]\).
Proposition 3.2. For every four different points \(A, B, C, D\) in a line \(\ell\), in Desargues affine plane. Then

\[
\begin{align*}
[A, B, C] \\ [A, C, D]
\end{align*}
\implies
\begin{align*}
[A, B, D] \\ [B, C, D]
\end{align*}
\]

Proof. By \([A, B, C]\), we have that the points \(A\) and \(B\) are in the same side of point \(C\), by \([A, C, D]\), we have that the points \(A\) and \(D\) are on the opposite side of point \(C\). Then, we have that, the point \(B\) and \(D\) are on the opposite side of point \(C\) (since otherwise we would have that points \(A\) and \(D\) would be on the same side of point \(C\), this would exclude \([A, C, D]\)), so we have that \([B, C, D]\).

By \([A, B, C]\), we have that the points \(A\) and \(C\) are in the same side of point \(B\). Since \([B, C, D]\), we have that the points \(C\) and \(D\) are in the same side of point \(B\). Then, we have that, the point \(A\) and \(D\) are on opposite sides of the point \(B\) (since otherwise we would have that points \(A\) and \(D\) would be on the same side of point \(B\), and point \(C\) this will give us \([D, B, C]\) this would exclude \([B, C, D]\)), so we have that \([A, B, D]\).

\[\square\]

Definition 3.3. For three different points \(A, B, C \in \ell\), that, we say that points \(B\) and \(C\) lie on the same side of point \(A\), if we have exactly one of \([A, B, C]\), \([A, C, B]\).

Proposition 3.3. If we have, that, for four different points \(A, B, C, D \in \ell\): points \(B\) and \(C\) lie on the same side of point \(A\), and the points \(C\) and \(D\) lie on the same side of point \(A\), then the points \(B\) and \(D\) lie on the same side of point \(A\).

Proof. Let’s look at the possible cases of ’positioning’ between the points,

If,

\[
\begin{align*}
[A, D, C] \\ [B, C, A]
\end{align*}
\implies\text{Lo.1} \quad \begin{align*}
[A, D, C] \\ [A, C, B]
\end{align*} \implies\text{Lo.2} \quad [A, D, B].
\]

If,

\[
\begin{align*}
[A, B, C] \\ [A, C, D]
\end{align*} \implies\text{Lo.2} \quad [A, B, D].
\]

If,

\[
\begin{align*}
[A, D, C] \\ [A, B, C]
\end{align*} \implies\text{Lo.1} \quad [A, B, D] \lor [A, D, B].
\]

If,

\[
\begin{align*}
[A, D, C] \\ [B, C, A]
\end{align*} \implies\text{Lo.1} \quad [A, D, C] \lor [A, C, B] \implies\text{Lo.3} \quad [A, D, B].
\]

So we see that any of the possible cases stays. The case when at least two of the four above points coincide the proof is clear. \(\square\)

Definition 3.4. The parallel projection between the two lines in the Desargues affine plane, will be called, a function,

\[P_p : \ell_1 \to \ell_2, \forall A, B \in \ell_1, AB \parallel P_p(A)P_p(B).\]

It is clear that this function is a bijection between any two lines in Desargues affine planes.
**Definition 3.5.** [3] A Desargues Affine plane, is said to be **ordered**, provided

1. All lines in this plane are **ordered lines**.

2. Parallel projection of the points of one line onto the points of another line in this plane, either preserves or reverses the ordering.

**Theorem 3.4.** Each translation (dilation which is different from idₚ and has no fixed point) in a finite Desargues affine plane \( \mathcal{A}_D = (\mathcal{P}, \mathcal{L}, \mathcal{T}) \) preserves order in a line \( \ell \) of this plane.

**Proof.** Let \( A, B, C \in \ell \), and \([A, B, C]\). Let’s have, too, a each translation \( \varphi \) on this plan. Here we will distinguish two cases regarding the direction of translation.

**Case.1** The translation \( \varphi \) has not direction according to line \( \ell \), in this case \( \varphi(\ell) \parallel \ell \), and \( \varphi(\ell) \neq \ell \), (see Fig. 4). Mark \( \varphi(\ell) = \ell' \).

From the translation properties, we have,

\[
A, B, C \in \ell \implies \varphi(A), \varphi(B), \varphi(C) \in \ell'.
\]

From the translation properties described at [37], we have the following parallelisms,

\[
AB \parallel \varphi(A) \varphi(B), AC \parallel \varphi(A) \varphi(C), BC \parallel \varphi(B) \varphi(C),
\]

and

\[
A \varphi(A) \parallel B \varphi(B) \parallel C \varphi(C).
\]

So we have the following parallelograms (see [11]):

\[
(A, B, \varphi(B), \varphi(A)), (A, C, \varphi(C), \varphi(A)), (B, C, \varphi(C), \varphi(B)),
\]

![Figure 4: The translation \( \varphi \) as a parallel projection \( P_p : \ell \rightarrow \ell' \).](image)

Clearly, we see that the translation \( \varphi \) can be seen as parallel projection \( P_p \), from line \( \ell \) to line \( \ell' \), with direction the line \( \ell^{A \varphi(A)} \).

Thus by definition and so \([A, B, C]\), we have \([\varphi(A), \varphi(B), \varphi(C)]\).
Case.2 The translation $\varphi$ has direction according to line $\ell$, in this case $\varphi(\ell) \parallel \ell$, and $\varphi(\ell) = \ell$, (see Fig. 5). By translation properties have,

$$A, B, C \in \ell \implies \varphi(A), \varphi(B), \varphi(C) \in \ell.$$  

In this case, we choose a point $E \notin \ell$ of the plan. For this point, $\exists! E \in L$, so it exists a translation $\varphi_1$, such that $\varphi_1(A) = E$, and exists a translation $\varphi_2$, such that $\varphi_2(E) = \varphi(A)$.

![Figure 5: The translations $\varphi$ as a composition of two translations $\varphi_1$ and $\varphi_2$. So $\varphi = \varphi_2 \circ \varphi_1$.](image)

Well,

$$(\varphi_2 \circ \varphi_1)(A) = \varphi(A).$$

Two translations $\varphi_1$ and $\varphi_2$ have different directions from line $\ell$. Now we repeat the first case twice, once for the translation $\varphi_1$ and once for the translation $\varphi_2$, and we have:

$$[A, B, C] \implies [\varphi_1(A), \varphi_1(B), \varphi_1(C)] \implies [\varphi_2(\varphi_1(A)), \varphi_2(\varphi_1(B)), \varphi_2(\varphi_1(C))]$$

thus,

$$[A, B, C] \implies [(\varphi_2 \circ \varphi_1)(A), (\varphi_2 \circ \varphi_1)(B), (\varphi_2 \circ \varphi_1)(C)].$$

Hence

$$[A, B, C] \implies [\varphi(A), \varphi(B), \varphi(C)]$$

Theorem 3.5. Every finite ordered Desarguesian affine plane over $\mathbb{R}^2$ is Pappian.

Proof. Let $\pi$ be a finite Desarguesian affine plane over $\mathbb{R}^2$. From Theorem 2.3, $\pi$ is Pappian. Since every line in $\pi$ satisfies Axioms Lo.1-Lo.4 of definition 3.1, so every line in $\pi$ is an ordered line, also, this plane meets the conditions of the definition 3.5. Consequently, $\pi$ is a finite ordered Desarguesian affine plane. It has been observed that every Euclidean space is Pappian [27, §1.1, p. 195]. Hence, $\pi$ is Pappian. QED
4 The ordered skew-field in a line in ordered Desargues affine plane

In [35], [9], we have shown how to construct a skew-field over a line in Desargues affine plane. Let it be \( \ell \) a line of Desargues affine plane \( \mathbb{A}_D = (\mathcal{P}, \mathcal{L}, \mathcal{I}) \).

We mark \( K = (\ell, +, \ast) \) the skew-field constructed over the line \( \ell \), in Desargues affine plane \( \mathbb{A}_D \). In previous work [9], we have shown how we can transform a line in the Desargues affine plane into an additive Group of its points. We have also shown [35], [9] and [36] how to construct a skew-field with a set of points on a line in the Desargues affine plane. In addition, for a line of in any Desargues affine plane, we construct a skew-field with the points of this line, by appropriately defined addition and multiplication of points in a line (also, another very interesting construction of skew-fields in affine plane, is presented in the works [30] and [31]).

During the construction of the skew-field over a line of a Desargues affine plane, we choose two points (each affine plan has at least two points), which we write with \( O \) and \( I \) and call them zero and one, respectively. These points play the role of unitary elements regarding the two actions addition and multiplication, respectively.

**Definition 4.1.** [3],[24] A skew-field \( K \) is said to be ordered, if, satisfies the following conditions

1. \( K = K_- \cup \{0_K\} \cup K_+ \) and \( K_- \cap \{0_K\} \cap K_+ = \emptyset \),
2. For all \( k_1, k_2 \in K_+, k_1 + k_2 \in K_+ \), (\( K_+ \) is closed under addition)
3. For all \( k_1, k_2 \in K_+, k_1 \ast k_2 \in K_+ \), (a product of positive elements is positive)

Consider a line \( \ell \) in a Desargues affine plane, by defining the affine plane so that we have at least two points in this line, which we mark \( O \) and \( I \). For this line, we have shown in the previous works (see [9],[33]) that we can construct a skew-field \( K = (\ell, +, \ast) \). We have shown [9] the independence of choosing points \( O \) and \( I \) in a line, for the construction of a skew-field over this line.

To establish a separation of the set of points in line \( \ell \), we separate, firstly the points \( \{O\} \), which we mark with

\[
K_+ = \{X \in \ell \mid [O,X,I] \text{ or } [O,I,X]\} (= \ell_+).
\]

and

\[
K_- = \{X \in \ell \mid [X,O,I]\} (= \ell_-).
\]

So, clearly, from the definition of the ordered line in a Desargues affine plane, we have

\[
K = K_- \cup \{0_K\} \cup K_+,
\]

where \( 0_K = O \), and

\[
K_- \cap \{0_K\} \cap K_+ = \emptyset.
\]

**Lemma 4.1.** For all \( A, C \in K_+ \implies A + C \in K_+ \).
Proof. Let’s have the points $A, C \in \ell$, such that $[O, A, C]$ or $[O, C, A]$. Suppose we have true $[O, A, C]$, from the construction of the $A + C$ point we have that $A + C \in \ell$, (see Fig. 6) which was built according to the algorithm

\[
\begin{align*}
\forall A, C \in \ell, & \quad \begin{cases}
\text{Step.1} \exists B \notin OI \\
\text{Step.2} \ell^B_{OI} \cap \ell^A_{OB} = D \iff A + C = E.
\end{cases} \\
\text{Step.3} \ell^D_{CB} \cap OI = E
\end{align*}
\]

Figure 6: The addition of points in a line of Desargues affine plane, as a composition of tow parallel projections.

From the construction of the $A + C$ point, we have that:

\[\ell^{OB} \parallel \ell^{AD}; \ell^{BC} \parallel \ell^{D(A+C)}; \ell^{BD} \parallel OI.\]

From these parallelisms, there is the translation $\varphi_1$ with direction, according to line $\ell$, such that, $\varphi_1(O) = A$.

\[[O, A, C] = [O, \varphi_1(O), C].\]

Since, by with Theorem 3.4, the translations preseve the line-order, we have

\[[O, A, C] = [O, \varphi_1(O), C] \implies [\varphi_1(O) , \varphi_1(\varphi_1(O)) , \varphi_1(C)] \implies [\varphi_1(O) , \varphi_1(A) , \varphi_1(C)] = [A, \varphi_1(A), \varphi_1(C)].\]

Hence, we have defined a translation (see [34],[32]), such that we have true

\[[O, \varphi_1(O) = A, \varphi_1(A)] = [O, A, \varphi_1(A)]\]

From this translation, we also have,

\[\varphi_1(B) = D \text{ and } \varphi_1(C) = A + C,\]

By by with Theorem 3.4 we have

\[[O, A, C] \implies [\varphi_1(O) , \varphi_1(A) , \varphi_1(C)] = [A, \varphi_1(A), A + C].\]

For four points $O, A, \varphi_1(A), A + C \in \ell$, we have

\[\begin{aligned}
[O, A, \varphi_1(A)] & \implies [O, A, A + C] \land [O, \varphi_1(A), A + C].
\end{aligned}\]
By \([O, A, A + C]\), we have that
\[A + C \in K_+ (= \ell_+).\]

\[\Box\]

**Corollary 4.2.** For all \(A, C \in K_- \implies A + C \in K_-.\)

**Corollary 4.3.** For all \(A \in K_+ \implies -A \in K_-\).

**Proof.** If \(-A \in K_+ \implies A + (-A) = O(= 0) \in K_+\), which is a contradiction. \(\Box\)

**Lemma 4.4.** For all \(A, C \in K_+ \implies A \ast C \in K_+\).

**Proof.** Let’s have the points \(A, C \in \ell\), such that \([O, A, C]\) or \([O, C, A]\). Suppose we have true \([O, A, C]\), and suppose also that we have \([O, I, A]\) from the construction of the \(A \ast C\) point we have that \(A + C \in \ell\), which was built according to the algorithm
\[
\begin{align*}
\text{Step 1.} & \exists B \notin OI \\
\text{Step 2.} & \ell^A_{IB} \cap OB = E \iff A \ast C = F. \\
\text{Step 3.} & \ell^E_{BC} \cap OI = F
\end{align*}
\]

By construction of the point \(A \ast C\), we have parallelisms
\[IB \parallel AE, BC \parallel E(A \ast C).\]

We take a parallel projection \(P_p\) with the direction of the line \(\ell^B\), (see Fig. 7) such that
\[P_p : \ell^O \to \ell^O, P_p(O) = O; P_p(I) = B\]
and \(P_p(A) = E\), since parallel projection preserves the order, we have

![Figure 7](image.png)

**Figure 7:** The multiplication of points in a line of Desargues affine plane, as a composition of two parallel projections.
But by the multiplication algorithm, during the construction of the $A \ast C$ point, we have $BC \parallel E(A \ast C)$.

We take a parallel projection $\tilde{P}_p$ with the direction of the line $\ell^{BC}$, (see Fig. 7) such that

$\tilde{P}_p : \ell^{OB} \to \ell^{OI}, \tilde{P}_p(O) = O; \tilde{P}_p(B) = C$ and $\tilde{P}_p(E) = A \ast C$.

Since the parallel projection preserves the order, we have

\[
\frac{[O,B,E]}{BC \parallel E(A \ast C)} \implies \left[ \tilde{P}_p(O), \tilde{P}_p(B), \tilde{P}_p(E) \right] = [O,C,A \ast C]
\]

By $[O,C,A \ast C]$, we have that

$A \ast C \in K_+ (= \ell_+)$.

If we have, $[O,A,C]$ and $[A,I,C] \implies [O,A,I]$, (see Fig. 8). By construction of the point $A \ast C$, we have parallelisms

$IB \parallel AE, BC \parallel E(A \ast C)$.

We take a parallel projection $P_p$ with the direction of the line $\ell^{IB}$, such that

$P_p : \ell^{OI} \to \ell^{OB}, P_p(O) = O; P_p(A) = E$ and $P_p(I) = B$.

Since the parallel projection preserves the order, we have

$[O,A,I] \implies [P_p(O), P_p(A), P_p(I)] \implies [O,E,B]$.

Figure 8: The case where, we have, $[O,A,C]$ and $[A,I,C]$. 

But by the multiplication algorithm, during the construction of the $A \ast C$ point, we have $BC \parallel E(A \ast C)$. We take a parallel projection $\tilde{P}_p$ with the direction of the line $\ell^{BC}$, such that

$\tilde{P}_p : \ell^{OB} \to \ell^{OI}, \tilde{P}_p(O) = O; \tilde{P}_p(E) = A \ast C$ and $\tilde{P}_p(B) = C$. 

Since the parallel projection preserves the order, we have

\[ [O, E, B] \implies [\widetilde{P}_p(O), \widetilde{P}_p(E), \widetilde{P}_p(B)] = [O, A * C, C] \implies A * C \in K_+ (= \ell_+). \]

If we have, \([O, A, C]\) and \([A, C, I]\) \implies \([O, A, I] \land [O, C, I]\), (see Fig. 9). By construction of the point \(A * C\), we have parallelisms

\[ IB \parallel AE, BC \parallel E(A * C). \]

We take a parallel projection \(P_p\) with the direction of the line \(\ell^{IB}\), such that

\[ P_p : \ell^{OI} \to \ell^{OB}, P_p(O) = O; P_p(A) = E \quad \text{and} \quad P_p(I) = B. \]

Since the parallel projection preserves the order, we have

\[ [O, A, I] \implies [P_p(O), P_p(A), P_p(I)] \implies [O, E, B]. \]

\[ \text{Figure 9: The case where we have \([O, A, C]\) and \([A, C, I]\).} \]

But by the multiplication algorithm, during the construction of the \(A * C\) point, we have \(BC \parallel E(A * C)\). We take a parallel projection \(\widetilde{P}_p\) with the direction of the line \(\ell^{BC}\), such that

\[ \widetilde{P}_p : \ell^{OB} \to \ell^{OI}, \widetilde{P}_p(O) = O; \widetilde{P}_p(E) = A * C; \widetilde{P}_p(B) = C, \]

and \(\widetilde{P}_p\), since parallel projection preserves the order, we have

\[ [O, E, B] \implies [\widetilde{P}_p(O), \widetilde{P}_p(E), \widetilde{P}_p(B)] = [O, A * C, C] \implies A * C \in K_+ (= \ell_+). \]

\[ \square \]

We have indirectly from Lemma 4.4 the case when \([O, C, A]\), since \(A * C \neq C * A\).

From Lemmas 4.1 and 4.4, we have the following result.

**Theorem 4.5.** Every skew Field that is constructed over an ordered-line of a Desargues affine plane is an ordered skewField.
If we consider the special case where ordered-line of a Desargues affine plane in $\mathbb{R}^2$, we obtain the following result.

**Corollary 4.6.** Every finite skew Field that is constructed over an ordered-line of a Desargues affine plane in $\mathbb{R}^2$ is a finite ordered skew-field.

Putting together the result from Corollary 4.6 and Theorem 3.5, we obtain the following result for an ordered line in a finite Desargues affine plane in $\mathbb{R}^2$.

**Theorem 4.7.** A finite skew Field constructed over an ordered-line on a finite Desargues affine plane in $\mathbb{R}^2$ is Pappian.

**Proof.** Let $\ell$ be an ordered-line in a finite Desargues affine plane in $\mathbb{R}^2$. By definition, $\ell$ is a Desarguesian affine 1-plane. From Corollary 4.6, $\ell$ is a finite ordered skew-field. Hence, from Theorem 3.5, $\ell$ is Pappian. \( \square \)

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