MATRICES TOTALLY POSITIVE RELATIVE TO A TREE

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Introduction

**DEF:** A matrix is called totally positive (TP) if every minor of it is positive.

We will be interested in submatrices of a given matrix that are TP, or permutation similar to TP.

Thus, we will be interested in permuted submatrices, identified by ordered index lists.

Suppose that $T$ is a labelled tree on $n$ vertices and $A$ is an $n$-by-$n$ matrix.

**DEF:** If $P$ is an induced path of $T$, by $A[P]$ we mean $A[\alpha]$ in which $\alpha$ consists of the indices of the vertices of $P$ in the order in which they appear along $P$.

Since everything we discuss is independent of reversal of order, there is no ambiguity.

**DEF:** For a given labelled tree $T$ on $n$ vertices, we say that $A$ is **T-TP** if, for every path $P$ in $T$, $A[P]$ is TP.
Introduction

• For a T-TP matrix, properly less is required than for a TP matrix.

• Also, like TP matrices, T-TP matrices are entry-wise positive. This follows because there exists a path connecting vertices \( i \) and \( j \) in tree \( T \), so that every entry in the corresponding T-TP matrix is in a submatrix that is, by definition, TP.

• Since all the entries in a TP matrix are positive, then \( A \) has positive coefficients.
TP matrices properties

• Among them is the fact that the eigenvalues are real, positive and distinct.

• The largest one is the Perron root and its eigenvector may be taken to be positive.

• The fact that this property of a TP matrix holds for T-TP matrices is clear from the fact that the entries are positive.

• The eigenvectors of the remaining eigenvalues alternate in sign subject to well-defined requirements, and, in particular, the eigenvector, associated with the smallest eigenvalue, alternates in sign as: (+, −, +, −, ..., ). This is because the inverse, or adjoint, has a checkerboard sign pattern and the Perron root of the alternating sign signature similarity of the inverse is the inverse of the smallest eigenvalue of the original TP matrix.
T-TP matrices (sign pattern)

• If $T$ is a Path, the sign pattern of the eigenvector concerning the $\textit{smallest}$ eigenvalue may be viewed as alternation associated with each edge of $T$, i.e. if $\{i, j\}$ is an edge of $T$, then $v_i v_j < 0$ for the eigenvector $v$ associated with the smallest eigenvalue.

• The vector $v$ is signed according to the labelled tree $T$ on $n$ vertices if, whenever $\{i, j\}$ is an edge of $T$, then $v_i v_j < 0$. This means that $v$ is totally nonzero and that the sign pattern of $v$ is uniquely determined, up to a factor of $-1$.

• We know that the eigenvector associated with the smallest eigenvalue of a TP matrix is signed according to the standardly labelled path $T$ (relative to which the TP matrix is T-TP).

• Neumaier originally conjectured that all eigenvectors should be signed as those of a TP matrix, and J. Garloff relayed to us that for any tree $T$, the eigenvector associated with the smallest eigenvalue of a T-TP matrix should be signed according to the labelled tree $T$. We have proved that the conjecture is false.
PART I

(With Boris Tadchiev)

THE STARS and the POTCHFORK
The stars on 4 vertices

• **Theorem.** For any labelled tree $T$ on fewer than 5 vertices, any $T$-TP matrix has smallest eigenvalue that is real and a totally nonzero eigenvector that is signed according to $T$.

**Sketch of the proof**

1. Then we wish to show that if $A$ is $T$-TP, then the sign pattern of $\bar{A}$ is

\[
\begin{pmatrix}
+ & - & - & - & - \\
- & + & + & + & + \\
- & + & + & + & + \\
- & + & + & + & + \\
- & + & + & + & + \\
\end{pmatrix}
\]
\[ \tilde{a}_{3,2} = (-1)^{3+2} \det A[1, 3, 4; 1, 2, 4] \\
= - \det A[3, 1, 4; 2, 1, 4] \\
= - \frac{\det A[3, 1; 2, 1] \det A[1, 4; 1, 4] - \det A[3, 1; 1, 4] \det A[1, 4; 2, 1]}{\det A[1; 1]}. \]

\[ \tilde{a}_{3,2} = -\frac{(-)(+)-(+)(-)}{(+)} > 0. \]

\[ \tilde{A} = \begin{bmatrix}
\tilde{a}_{11} & - & - & - \\
- & + & + & + \\
- & + & + & + \\
- & + & + & + \\
\end{bmatrix}. \]

Taking into account (1.1), we get that

\[ \begin{bmatrix}
\tilde{a}_{11} & - & - & - \\
- & + & + & + \\
- & + & + & + \\
- & + & + & + \\
\end{bmatrix} \begin{bmatrix}
+ & + & + & + \\
+ & + & + & + \\
+ & + & + & + \\
+ & + & + & + \\
\end{bmatrix} = \begin{bmatrix}
\det A & 0 & 0 & 0 \\
0 & \det A & 0 & 0 \\
0 & 0 & \det A & 0 \\
0 & 0 & 0 & \det A \\
\end{bmatrix}. \]

and multiplying the first row of \( \tilde{A} \) by the second column of \( A \) we get

\[ \tilde{a}_{11} + (-) + (-) + (-) = 0, \]
The stars on n vertices

- **Theorem.** Let $T$ be a star on $n$ vertices. Suppose that $A$ is T-TP and that all the submatrices of $A$ associated with the deletion of pendant vertices are P-matrices. Then, the smallest eigenvalue of $A$ is real, has multiplicity one and has an eigenvector signed according to $T$.

**Sketch of the proof**

1. Then we wish to show that if $A$ is T-TP, then the sign pattern of $\tilde{A}$ is

$$\tilde{A} = \begin{bmatrix} + & - & \cdots & - \\ - & + & \cdots & + \\ \vdots & \vdots & \ddots & \vdots \\ - & + & \cdots & + \end{bmatrix}.$$
\[ \tilde{a}_{ij} = (-1)^{i+j} (-1)^{j-1} (-1)^{i-2} \det A[i, 1, \mathbb{N}; j, 1, \mathbb{N}] \]
\[ = (-1)^{2(i+j-1)-1} \det A[i, 1, \mathbb{N}; j, 1, \mathbb{N}] > 0. \]

\[ \det A[i, 1, \mathbb{N}; j, 1, \mathbb{N}] = \frac{\det A[i, 1, \mathbb{N}'; j, 1, \mathbb{N}'] \det A[1, \mathbb{N}; 1, \mathbb{N}] - \det A[i, 1, \mathbb{N}'; 1, \mathbb{N}] \det A[1, \mathbb{N}; j, 1, \mathbb{N}']}{\det A[1, \mathbb{N}'; 1, \mathbb{N}']} \]

\[ \tilde{A} = \begin{bmatrix}
\tilde{a}_{11} & \tilde{a}_{12} & \cdots & \tilde{a}_{1n} \\
\tilde{a}_{21} & + & \cdots & + \\
\vdots & \vdots & \ddots & \vdots \\
\tilde{a}_{n1} & + & \cdots & +
\end{bmatrix} \]

Taking into account (1.1), we get that
\[
\begin{bmatrix}
\tilde{a}_{11} & - & - & - \\
- & + & + & + \\
- & + & + & + \\
- & + & + & +
\end{bmatrix}
\begin{bmatrix}
+ & + & + & + \\
+ & + & + & + \\
+ & + & + & + \\
+ & + & + & +
\end{bmatrix}
= \begin{bmatrix}
\det A & 0 & 0 & 0 \\
0 & \det A & 0 & 0 \\
0 & 0 & \det A & 0 \\
0 & 0 & 0 & \det A
\end{bmatrix}
\]

and multiplying the first row of \( \tilde{A} \) by the second column of \( A \) we get
\[ \tilde{a}_{11} + (-) + (-) + (-) = 0, \]
Bad example: The **Pitchfork**

\[
A = \begin{bmatrix}
55 & 77 & 10 & 17 & 49 \\
40 & 84 & 3 & 1 & 8 \\
57 & 74 & 86 & 15 & 47 \\
94 & 2 & 8 & 86 & 58 \\
48 & 41 & 4 & 4 & 78 \\
\end{bmatrix}
\]

Note that \( \lambda_5 \approx -6.16 \).

\[
x \approx \begin{bmatrix}
-2.98 \\
1.21 \\
-0.02 \\
2.39 \\
1 \\
\end{bmatrix}
\]

\[
\tilde{A} = \begin{bmatrix}
42023084 & -27857784 & -2494736 & -6756454 & -17016440 \\
-18274672 & 7046528 & 1241168 & 2950496 & 7815680 \\
2070092 & 1908264 & -5017752 & 386110 & 1240248 \\
-35907780 & 21866360 & 2481608 & 951670 & 18111768 \\
-14519176 & 12220096 & 1012872 & 2538312 & 279496 \\
\end{bmatrix}
\]

\[
\lambda_5 \approx -2.54,
\]

\[
x \approx \begin{bmatrix}
-68.08 \\
32.75 \\
26.69 \\
45.57 \\
1 \\
\end{bmatrix}
\]
PART II

Pendant vertices and positive determinant
New definitions

• **DEF:** For a given labelled tree $T$ on $n$ vertices, we say that $A$ is **pendent-$P$** relative to $T$ if all principal submatrices, associated with the deletion of pendent vertices, one at a time, are $P$-matrices.

• **DEF:** For a given labelled tree $T$ on $n$ vertices, we say that $A$ is **$T$-positive** if it is $T$-TP and pendent-$P$ relative to $T$. 
**THE GENERAL RESULT**

- Let $S_\sigma = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n)$ with $\sigma$ signed according to $T$.

- **Theorem.** Let $T$ be a labelled tree on $n$ vertices and $A$ be $T$-positive with $\det A > 0$. Then

  $$\text{sign}(\det A(i;j)) = (-1)^{i+j}\sigma_i\sigma_j$$

  in which $\sigma$ is signed according to $T$.

- **Corollary** If $T$ is a tree on $n$ vertices and $A$ is $T$-positive with $\det A > 0$. Then $S_\sigma A^{-1}S_\sigma$ is entry-wise positive. Therefore, $A$ satisfies the Neumaier conclusion.
Sketch of the proof

Lemma 7. Given a matrix $A \in M_n(\mathbb{R})$, then for any three distinct integers $i, j, k$, with $1 \leq i, j, k \leq n$, we have

$$
\tilde{a}_{k,i} \det A[i, N_{i,j,k}; i, N_{i,j,k}] \\
+ \tilde{a}_{k,j} \det A[j, N_{i,j,k}; i, N_{i,j,k}] + \tilde{a}_{k,k} \det A[k, N_{i,j,k}; i, N_{i,j,k}] = 0.
$$

$$
\text{sign}(\det A(i; j)) = (-1)^{i+j} \sigma_i \sigma_j \iff \text{sign}(\tilde{a}_{i,j}) = \sigma_i \sigma_j
$$
For the last lemma we need to use Jacobi’s identity [3, (0.8.4.1)]

\[ \det A[\alpha; \beta] = (-1)^{p(\alpha, \beta)} \det A \det A^{-1}[N \setminus \beta; N \setminus \alpha], \]

where \( |\alpha| = |\beta| \), and \( p(\alpha, \beta) = \sum_{i \in \alpha} i + \sum_{j \in \beta} j. \)

\[ \det \tilde{A}[j, p; i, p] = \begin{vmatrix} \tilde{a}_{j, i} & \tilde{a}_{j, p} \\ \tilde{a}_{p, i} & \tilde{a}_{p, p} \end{vmatrix} = \tilde{a}_{j, i} \tilde{a}_{p, p} - \tilde{a}_{j, p} \tilde{a}_{p, i}, \]
Where can I find the results?

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Matrices totally positive relative to a tree, II

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CrossMark
Open problems
Thank you and Merry Christmas