Strong and Weak Solutions to the Hasegawa-Mima Equation with Periodic Boundary Conditions

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Abstract

The two dimensional Hasegawa-Mima (HM) equation

$$\Delta u_t + u_t = \{u, \Delta u\} + ku_y$$

describes the time evolution of drift waves in magnetically-confined plasma. Several authors have treated the HM equation theoretically [1, 2] and numerically [3], with difficulties arising when handling the non-linear Poisson’s bracket $$\{u, \Delta u\} := u_x \Delta u_y - u_y \Delta u_x$$. In this paper, we introduce a new decoupling approach that avoids the Poisson’s bracket term by reformulating the HM equation as a system of two linear PDEs, a solution of which is a pair $$(u, w)$$ such that

$$(HM) \quad \begin{cases} w_t + \vec{V}(u) \cdot \nabla w = ku_y \\ -\Delta u + u = w, \end{cases}$$

where $$\vec{V}(u) = -u_y \vec{i} + u_x \vec{j}$$ is a divergence-free vector field. Based on this coupled hyperbolic-elliptic system, we derive several variational frames, all propitious for finding weak solutions with spatial periodic boundary conditions and lower regularity assumptions on the initial data. More precisely, for initial data $$u_0 \in H^2_P(\Omega)$$ and $$w_0 := (I - \Delta)u_0 \in L^2(\Omega)$$, we prove the existence of a weak solution that is global in time. And for initial data $$u_0 \in H^3_P(\Omega)$$ with $$w_0 := (I - \Delta)u_0 \in H^1_P(\Omega) \cap L^\infty(\Omega)$$, we prove the existence of a unique strong solution that is local in time. Our proofs are based on the existence of fixed-point ordered pairs $$\{u_N, w_N\}$$ that solve Petrov-Galerkin HM systems, constructed using spatial Fourier basis. Through appropriate a-priori estimates combined with compactness arguments, we reach when $$N \to \infty$$ limit point solutions $$(u, w)$$ to the (HM) system.

Keywords: Drift Waves; Hasegawa-Mima; Periodic Sobolev Spaces; Petrov-Galerkin Approximations; Plasma Fusion; Schauder Fixed Point Theorem; Spectral Analysis.

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1 Introduction

Magnetic plasma confinement is one of the most promising ways in future energy production. To understand the phenomena, several mathematical models can be found in literature [4, 5, 6, 7], of which the simplest and powerful two dimensional turbulent system model is the Hasegawa-Mima (HM) equation that describes the time evolution of drift waves in magnetically-confined plasma. Originally derived by Akira Hasegawa and Kunioki Mima during late 70s [5, 6], but when normalized, it can [8, 3] be put as the following PDE that is third order in space and first order in time:

$$-\Delta u_t + u_t = \{u, \Delta u\} + ku_y$$

(1)

where $$\{u, v\} = u_x v_y - u_y v_x$$ is Poisson’s bracket, $$u(x, y, t)$$ describes the electrostatic potential, $$k = \partial_x (\ln \omega_{ci})$$ depends on the background particle density $$n_0$$ and the ion cyclotron frequency $$\omega_{ci}$$, which in turn depends on the initial magnetic field. Here, $$k = 0$$ refers to homogeneous plasma. As a cultural note, equation (1) is also referred to...
as the Charney-Hasegawa-Mima equation in geophysical context that models the time-evolution of Rossby waves in the atmosphere[8].

Using semi-group theory combined with compactness methods, local existence and uniqueness of the Cauchy problem for (1) have been obtained in [1] for \( u_0 \in H^m(\mathbb{R}^2) \) with \( m \geq 4 \), and global existence for \( u_0 \in H^2(\mathbb{R}^2) \). Similarly, local existence and uniqueness of (1) with periodic boundary conditions on a square domain \( \Omega \) have been obtained in [9] for \( u_0 \in H^m_p(\Omega) \) with \( m \geq 4 \) (see (3) for definition of \( H^m_p(\Omega) \)).

In this paper, we circumvent the highly non-linear Poisson’s bracket \( \{u, \Delta u\} \), both theoretically and computationally. We introduce a new variable \( w = -\Delta u + u \) and write

\[
\{u, \Delta u\} = \{u, u - w\} = \{u, u\} + \{u, -w\} = -\{u, w\} = \{w, u\} = w_x u_y - w_y u_x.
\]

This allows us to obtain an equivalent formulation of the HM equation as the following coupled system of hyperbolic-elliptic PDEs:

\[
\begin{cases}
  w_t + \bar{V}(u) \cdot \nabla w = ku_y \\
  -\Delta u + u = w,
\end{cases}
\]

where \( \bar{V}(u) = -u_y \hat{j} + u_x \hat{i} \) is a divergence-free vector field. This new formulation will be naturally amenable not only to obtain solutions for lower regularity assumptions on the initial data \( u_0 \), but also to provide a finite element scheme for obtaining a numerical simulation.

Following [3, 9], we also take into account the toroidal shape of plasma confinement chambers, such as Tokamaks, and consider the HM equation on \( \Omega = (0, L) \times (0, L) \) with the following periodic boundary conditions: \( u \in H^1(\Omega) \) with

\[
(PBC's) \begin{cases}
  u(0, y) = u(L, y) \quad \text{a.e. } y \in [0, L] \\
  u(x, 0) = u(x, L) \quad \text{a.e. } x \in [0, L]
\end{cases}
\]

where values on the boundary are defined via the trace operator \( \text{Tr}: H^1(\Omega) \longrightarrow H^{1/2}(\partial \Omega) \) given by \( \text{Tr}(u) = u|_{\partial \Omega} \).

Accordingly, we define a new frame of periodic Sobolev spaces as follows:

\[
\begin{align*}
  H^0_p(\Omega) &= L^2(\Omega) \\
  H^1_p(\Omega) &= \left\{ u \in H^1(\Omega) \mid u \text{ satisfies } PBC's \right\} \\
  H^m_p(\Omega) &= \left\{ u \in H^m(\Omega) \mid u, u_x, u_y \in H^{m-1}_p(\Omega) \right\}
\end{align*}
\]

We remark here that \( H^m_p(\Omega) \) is a closed subspace of \( H^m(\Omega) \), and so itself is a Hilbert space. A significant consequence of the PBC’s is the vanishing integrals over \( \partial \Omega \) when integrating by parts, which in turn implies the skew-symmetry of \( \partial_x \) and \( \partial_y \). More precisely,

\[
\int_\Omega f_x g - \int_\Omega f g_x = f g|_{x=L} - f g|_{x=0} = 0, \quad \forall f, g \in H^1_p(\Omega).
\]

In this paper, we consider and solve the following formulations of the HM coupled system (2) with PBC’s according to different lower regularity assumptions on the initial data:

**Strong \( H^1 \) Formulation of the HM Coupled System**

Given \( u_0 \in H^1_p(\Omega) \) with \( w_0 := u_0 - \Delta u_0 \in H^1_p(\Omega) \cap L^\infty(\Omega) \), seek \( T > 0 \) and \((u, w)\) such that:

\[
\begin{cases}
  w_t + \bar{V}(u) \cdot \nabla w = ku_y & \text{in } L^\infty(0, T; L^2(\Omega)) \quad (1) \\
  -\Delta u + u = w & \text{in } H^1_p(\Omega), \forall t \in (0, T) \quad (2) \\
  \text{PBC's on } u, u_x, u_y, \text{ and } w & \text{a.e. on } \partial \Omega \times (0, T) \quad (3) \\
  u(0) = w_0 \text{ and } w(0) = 0 & \text{a.e. on } \Omega \quad (4)
\end{cases}
\]

Taking the \( L^2 \) inner-product of equations in (4.1) and (4.2) with a test function \( v \in H^1_p(\Omega) \) leads us to an equivalent \( H^1 \) variational form:

\[
\begin{cases}
  \langle w_t, v \rangle_{L^2} + \langle \bar{V}(u) \cdot \nabla w, v \rangle_{L^2} = \langle ku_y, v \rangle_{L^2}, \forall v \in H^1_p(\Omega), \forall t \in (0, T) \quad (1) \\
  \langle u, v \rangle_{H^1} = \langle w, v \rangle_{L^2}, \forall v \in H^1_p(\Omega), \forall t \in (0, T) \quad (2) \\
  w(0) = w_0 \quad (3) \\
  u(0) = u_0 \quad (4)
\end{cases}
\]
**Weak \( L^2 \) Formulation of the HM Coupled System**

Due to the divergence-free property of \( \vec{V}(u) \), Green’s formula yields

\[
\left\langle \vec{V}(u) \cdot \nabla w, v \right\rangle_{L^2} = - \left\langle \vec{V}(u) \cdot \nabla v, w \right\rangle_{L^2}, \quad \forall v, w \in H^1_p(\Omega), \tag{6}
\]

which for \( v = w \), gives us an essential identity

\[
\left\langle \vec{V}(u) \cdot \nabla w, w \right\rangle_{L^2} = 0, \quad \forall w \in H^1_p(\Omega). \tag{7}
\]

Identity (6) and integration of (5.1) over the temporal interval \([0, t]\), with \(0 < t < T\), allow us to relax spacial and temporal regularity requirements on \((u, w)\) and reach to the following weak \( L^2 \) formulation of the HM coupled system:

Given \( u_0 \in H^2_p(\Omega) \) with \( w_0 := u_0 - \Delta u_0 \in L^2(\Omega) \), seek \((u, w)\) such that:

\[
\begin{align}
&w(t) - w_0, v\right\rangle_{L^2} = \int^t_0 \left\langle \vec{V}(u(s)) \cdot \nabla v, w(s) \right\rangle_{L^2} + \left\langle ku_y(s), v \right\rangle_{L^2} ds, \forall v \in H^1_p(\Omega), \quad \forall t \in (0, T] \tag{1.1} \\
&\left\langle u, v \right\rangle_{H^1} = \left\langle u, v \right\rangle_{L^2}, \forall v \in H^1_p(\Omega), \quad \forall t \in (0, T] \tag{2.2} \\
w(0) = w_0 \tag{3.3} \\
u(0) = u_0 \tag{4.4}
\end{align}
\]

Formulation (8) has been used in [10] for obtaining a robust algorithm for simulating the HM model.

**Main Results**

Our main results are the following two theorems:

**Theorem 1.1.** Given initial data \( u_0 \in H^2_p(\Omega) \) with \( w_0 := (I - \Delta)u_0 \in H^1_p(\Omega) \cap L^\infty(\Omega) \), the weak \( L^2 \) formulation of the HM coupled system (8) has a solution pair \((u, w)\) with the following regularities:

\[
u \in L^\infty(0, T; H^2_p(\Omega)) \cap C([0, T], H^1_p(\Omega))
\]

and

\[
w \in L^\infty(0, T; L^2(\Omega)),
\]

where the existence time

\[
T := (C_E \|k\|_{L^\infty} + 1)^{-1}, \tag{9}
\]

independent of the initial data, allowing us to extend \((u, w)\) globally to \( T = \infty \). Here, \( C_E > 0 \) is the elliptic regularity constant from the elliptical equation.

Theorem 1.1 is proved in section 3.

**Theorem 1.2.** Given initial data \( u_0 \in H^1_p(\Omega) \) with \( w_0 := (I - \Delta)u_0 \in H^1_p(\Omega) \cap L^\infty(\Omega) \), the strong \( H^1 \) formulation of the HM coupled system (4) has a unique local solution pair \((u, w)\) on \([0, T]\), where the existence time \( T = T(u_0, w_0) \) is inversely dependent on \((u_0, w_0)\) as in equation (25), with the following regularities for the solution:

\[
u \in L^\infty(0, T; H^1_p(\Omega)) \cap C([0, T], H^1_p(\Omega)) \text{ with } u_t \in L^\infty(0, T; H^2_p(\Omega)),
\]

and

\[
w \in L^\infty(0, T; H^1_p(\Omega) \cap L^\infty(\Omega)) \cap C([0, T], L^2(\Omega)) \text{ with } w_t \in L^\infty(0, T; L^2(\Omega)).
\]

Theorem 1.2 is proved in section 4.

As a byproduct and solely for the hyperbolic equation:

(i) Global existence is obtained in Corollary 3.4 for \( w_0 \in L^2(\Omega) \), and,

(ii) Global existence and uniqueness are obtained in Corollary 4.3 for \( w_0 \in H^1_p(\Omega) \cap L^\infty(\Omega) \).
2 Methodology

To establish our main results, we study the elliptic and hyperbolic equations, each on its own. Our results are summarized as follows:

Proposition 2.1. If \( w \in H^m_P(\Omega) \), with \( m \geq 0 \), then the elliptic equation (4.2) has a unique strong solution \( u = \mathcal{E}(w) := (I - \Delta)^{-1}w \in H^{m+2}_P \) in the sense that:

\[
\begin{cases}
-\Delta u + u = w & \text{in } H^m_P(\Omega) \\
\|w\|_{H^{m+2}} \leq C_E \|w\|_{H^m_P}
\end{cases}
\]  

(10)

where \( C_E > 0 \) is an elliptic regularity constant depending only on \( m \). We note here that the temporal regularity of \( w \) carries over to \( u \).

This result is well-known for Dirichlet boundary value problems. However, for the sake of completeness, we provide a proof in Appendix A.1 using an equivalent Hilbert-Fourier basis \( \{ \phi_j / \sqrt{\lambda_j} \}_{j=1}^{\infty} \) for \( H^m_P(\Omega) \), where \( \phi_j \)'s and \( \lambda_j \)'s are the eigenvectors and eigenvalues (in non-decreasing order) of \( I - \Delta \) on \( L^2(\Omega) \) satisfying PBC's on \( \partial \Omega \).

We note here that the operator \( (I - \Delta) \) is also invertible over all finite dimensional subspaces

\[ E_N := \text{span} \{ \phi_1, \phi_2, \ldots, \phi_N \} \]

of \( H^1_0(\Omega) \). Namely if \( w = \sum_{j=1}^{N} c_j \phi_j \in E_N \), then \( u = \sum_{j=1}^{N} \lambda_j^{-1} c_j \phi_j \in E_N \). Motivated by this fact, we give a Galerkin formulation of the hyperbolic equation (5.1) on \( E_N \), and establish its existence and uniqueness in the following proposition.

Proposition 2.2. Given \( u \in C([0, T], H^2_P(\Omega)) \), then for every \( N \in \mathbb{Z}^+ \), there exists a unique function \( w_N := \mathcal{H}_N(u) \in C^1((0, T), E_N) \cap C([0, T], E_N) \) satisfying:

\[
\begin{align*}
&\frac{d}{dt} \left( \langle w'_N, v \rangle_{L^2} \right) = \left( \langle \nabla (u(t)) \cdot \nabla v, w_N \rangle_{L^2} + \langle ku_y, v \rangle_{L^2} \right) \forall v \in E_N, \forall t \in (0, T) \quad (1.1) \\
&w_N(0) = (I - \Delta)\text{proj}_{E_N} u(0) \quad (1.2)
\end{align*}
\]

where ' denotes the derivative with respect to \( t \).

Proof. Write \( w_N = \sum_{j=1}^{N} c_j(t) \phi_j \) and set \( v = \phi_j \) in (11.1) for \( j = 1, \ldots, N \), then

\[
\begin{align*}
c'_j(t) + \sum_{i=1}^{N} \left( \langle \nabla (u(t)) \cdot \nabla \phi_i, \phi_j \rangle_{L^2} \right) + \sum_{i=1}^{N} \left( \langle ku_y(t), \phi_j \rangle_{L^2} \right) &= \langle ku_y(t), \phi_j \rangle_{L^2} \forall j = 1, \ldots, N.
\end{align*}
\]

Setting \( \tilde{C}_{0,j} := \langle \text{proj}_{E_N} u_0, \phi_j \rangle_{L^2} \), this becomes the system of \( N \) ODE's:

\[ \tilde{C}'(t) + A(t)\tilde{C}(t) = \tilde{F}(t) \text{ with } \tilde{C}(0) = \tilde{C}_0. \]

Since for \( u \in C([0, T], H^2_P(\Omega)) \), \( \{ A_j(t) \} \) and \( \{ F_j(t) \} \) are defined and continuous on \([0, T]\), then by a usual Picard iteration, we get a unique solution \( \tilde{C}(t) \in C^1([0, T], \mathbb{R}^N) \cap C([0, T], \mathbb{R}^N) \), which completes the proof.

\[ \square \]

We will call \( \mathcal{H}_N \) the Galerkin solution operator for the hyperbolic equation \( (w_N = \mathcal{H}_N(u)) \), and consider a sequence of fixed-point problems, each consisting of finding a fixed-point \( u_N \) for \( \mathcal{E} \circ \mathcal{H}_N(u_N = \mathcal{E} \circ \mathcal{H}_N(u_N)) \), visualized by:

\[ u_N \in \mathcal{X}_N \quad \text{Solver operator } \mathcal{H}_N \quad \text{of the Galerkin hyperbolic PDE} \]

\[ \text{Solver Operator } \mathcal{E} \quad \text{of the elliptic PDE} \]

\[ w_N \in \mathcal{Y}_N \]

where \( \mathcal{X}_N \) and \( \mathcal{Y}_N \) are non-empty, closed, bounded, convex subsets of \( C([0, T], E_N) \) that will be specified according to the regularity assumptions on the initial data \( u_0 \) and \( u_1 \). Once specified, we will have \( \mathcal{E} \) continuous and compact,
and each $\mathcal{H}_N$ continuous, thus making each $\mathcal{E} \circ \mathcal{H}_N$ a continuous compact mapping, which according to Schauder’s theorem will have a fixed-point $u_N = \mathcal{E} \circ \mathcal{H}_N(u_N)$ satisfying (10) and (11).

Via this setup, we derive a priori estimates for the sequence of pairs $(u_N, w_N)$ in selected Bochner Hilbert spaces. Due to uniform boundedness and compact embeddings, we then extract a subsequence of fixed-point pairs and show they converge in the weak-star topology in time and weak topology in space to the solution pairs $(u, w)$ claimed in Theorems 1.1 and 1.2.

3 Proof of Theorem 1.1

A Priori $L^2$ Estimate for the Galerkin Solution

Lemma 3.1. The Galerkin solution $w_N = \mathcal{H}_N(u)$ associated to $u \in C([0, T], H^2(\Omega))$ satisfies the following estimate:

$$\|w_N\|_{L^\infty(L^2)} \leq \|k\|_{L^\infty} \|u\|_{L^\infty(H^1)} T + \|w_0\|_{L^2}$$  \hspace{1cm} (12)

Proof. Setting $v = w_N(t) \in E_N$ in (11.1) and using (7) we obtain

$$\frac{1}{2} \frac{d}{dt} \|w_N\|^2_{L^2} = \int_{\Omega} ku_tw_N,$$

which via Cauchy-Schwarz inequality becomes

$$\|w_N\|^2_{L^2} \frac{d}{dt} \|w_N\|_{L^2} \leq \|k\|_{L^\infty} \|u\|_{L^\infty(H^1)} \|w_N\|_{L^2}$$

Cancelling out $\|w_N\|_{L^2}$, integrating over $[0, t]$, with $0 < t \leq T$, and taking the sup over $[0, T]$ completes the proof.

Setup of the Sequence of Fixed-Point Problems

Given an initial data $u_0 \in H^2(\Omega)$ with $w_0 := (I - \Delta)u_0$, we pick $T := (C_E \|k\|_{L^\infty} + 1)^{-1}$ as in (9), and set

$$C_X = \frac{C_E \|u_0\|_{L^2}}{1 - C_E \|k\|_{L^\infty} T} \geq 0, \hspace{1cm} (13)$$

and for each $N \in \mathbb{Z}^+$, consider $\mathcal{X}_N$ and $\mathcal{Y}_N$ to be the following non-empty, closed, bounded, convex sets:

$$\mathcal{X}_N := \left\{ u \in C([0, T], E_N) \mid \|u\|_{L^\infty(H^2)} \leq C_X \right\}$$

$$\mathcal{Y}_N := \left\{ w \in C([0, T], E_N) \mid C_E \|w\|_{L^\infty(L^2)} \leq C_X \right\}$$

Observe that each solution operator is well-defined due to Propositions 2.1 and 2.2, and the inclusions:

(i) $\mathcal{H}_N(\mathcal{X}) \subseteq \mathcal{Y}$ through estimate (12) as

$$C_E \|\mathcal{H}_N(u)\|_{L^\infty(L^2)} \leq C_E \|k\|_{L^\infty} T \|u\|_{L^\infty(H^1)} + C_E \|w_0\|_{L^2} \leq C_E \|k\|_{L^\infty} TC_X + C_E \|w_0\|_{L^2} = C_X.$$

(ii) $\mathcal{E}(\mathcal{Y}) \subseteq \mathcal{X}$ through estimate (10) as

$$\|\mathcal{E}(w)\|_{L^\infty(H^2)} \leq C_E \|w\|_{L^\infty(L^2)} \leq C_X.$$

Also due to finite-dimensionality of $E_N$ that rises the norm-equivalence

$$\|v\|_{L^2} \leq \|v\|_{H^1} \leq \sqrt{\Lambda_N} \|v\|_{L^2} \hspace{1cm} \forall v \in E_N$$
on $E_N$, we obtain the continuity of $\mathcal{H}_N$ as follows.
Lemma 3.2. \( \mathcal{H}_N : \mathcal{X}_N \to \mathcal{Y}_N \) is continuous.

Proof. Let \( \epsilon > 0 \) be given, set \( \delta = \epsilon / (C_2^2 \lambda_N C_T / C_E + \|k\|_{L^\infty} \|T\) , and let \( u_1, u_2 \in \mathcal{X}_N \) be such that \( \|u_1 - u_2\|_{L^\infty} < \delta \). Subtracting (11.1) with \( w_2 = \mathcal{H}_N(u_2) \) from that of \( w_1 = \mathcal{H}_N(u_1) \) and setting \( v = w_1 - w_2 \), we obtain, via (7),

\[
\frac{1}{2} \frac{d}{dt} \|w_1 - w_2\|_{L^2}^2 = \left\langle \mathring{V}(u_2 - u_1) \cdot \nabla w_1, w_2 - w_1 \right\rangle_{L^2} + \left\langle k(u_1 - u_2)_y, w_2 - w_1 \right\rangle_{L^2}.
\]

Now applying Hölder’s inequality to the left hand side and cancelling \( \|w_1 - w_2\|_{L^2} \) from both sides, we obtain

\[
\frac{d}{dt} \|w_1 - w_2\|_{L^2} \leq \|\nabla w_1\|_{L^4} + \|k\|_{L^\infty} \|u_1 - u_2\|_{L^2}.\]

Integrating this over \([0, t]\), with \( t \leq T \), and taking the sup over \([0, T]\), we finally obtain

\[
\|\mathcal{H}_N(u_1) - \mathcal{H}_N(u_2)\|_{L^\infty} = \|w_1 - w_2\|_{L^\infty} \leq \epsilon.
\]

\[\Box\]

Obtaining a Candidate Solution Pair \((u, w)\)

At this point, due to Schauder fixed-point theorem, we have a sequence of fixed-point pairs \( \{u_N, w_N\} \) that are uniformly bounded in \( L^\infty(0, T; H^2_T(\Omega)) \times L^\infty(0, T, L^2(\Omega)) \) satisfying (10) and (11). Moreover:

Lemma 3.3. The sequence of temporal derivative \( u_N' \) is also uniformly bounded in \( L^\infty(0, T; L^2(\Omega)) \) as

\[
\|u_N'\|_{L^\infty(T; L^2(\Omega))} \leq C_E [4C_2^2 \lambda_N C_T + \|w_0\|_{L^2} + \|k\|_{L^\infty}] C_N.
\]

Proof. For convenience, we drop the index \( N \) from \( u_N \) and \( w_N \). Setting \( v = (I - \Delta)^{-1} u' \) in (11.1) and using the self-adjointness of \((I - \Delta)\) on \( L^2(\Omega)\), we obtain

\[
\|u'\|^2_{L^2} = \left\langle \mathring{V}(u) \cdot \nabla [(I - \Delta)^{-1} u'], w \right\rangle_{L^2} + \left\langle ku_y, (I - \Delta)^{-1} u' \right\rangle_{L^2}
\]

\[
\leq \|\mathring{V}(u)\|_{L^4} \|\nabla [(I - \Delta)^{-1} u']\|_{L^4} + \|k\|_{L^\infty} \|u_y\|_{L^2} \|\nabla [(I - \Delta)^{-1} u']\|_{L^2}
\]

\[
\leq 4C_2^2 \|w\|_{L^2} + \|k\|_{L^\infty} \|u\|_{H^2} \|u\|_{H^2} \leq C_E [4C_2^2 \|w\|_{L^2} + \|k\|_{L^\infty} \|u\|_{H^2} \|u\|_{H^2}.
\]

Finally, simplifying and taking the sup over \([0, T]\) and using estimate (12), estimate (14) follows.

Thus the sequence \( \{u_N\} \) is uniformly bounded in

\[
\{v \in L^\infty(0, T; H^2_T(\Omega)) \mid \forall v \in L^\infty(0, T; L^2(\Omega)), v \subset C([0, T], H^1_T(\Omega)).
\]

This last compact embedding is due to Aubin-Lions lemma, which in itself is based on Rellich’s Theorem [11].

Now along with Banach–Alaoglu theorem, we can eventually extract a subsequence of pairs \( (u_N, w_N) \) (re-indexed for convenience) such that:

\[
u_N \overset{\ast}{\to} u \in L^\infty(0, T; H^2_T(\Omega)), \text{ i.e. } \int_0^T \langle u_N, v \rangle_{H^2_T} dt \longrightarrow \int_0^T \langle u, v \rangle_{H^2_T} dt \forall v \in L^1(0, T; H^2_T(\Omega)),
\]

\[
w_N \overset{\ast}{\to} w \in L^\infty(0, T; L^2(\Omega)), \text{ i.e. } \int_0^T \langle w_N, v \rangle_{L^2} dt \longrightarrow \int_0^T \langle w, v \rangle_{L^2} dt \forall v \in L^1(0, T; L^2(\Omega)),
\]

\[
u_N \longrightarrow u \in C([0, T], H^1_T(\Omega)), \text{ i.e. } \|u_N - u\|_{L^\infty(H^1_T)} \longrightarrow 0.
\]

Note that \( u \) in equation (17) is equal to that of (15) due to uniqueness of weak star limits in \( L^\infty(0, T; L^2(\Omega)) \).
Passing to the Limit

We now show that this pair \((u, w)\) is a solution to the weak \(L^2\) form of the HM coupled system (8).

We have \(-\Delta u_N + u_N = w_N\) where the RHS converges to \(w\), and the LHS converges to \(-\Delta u + u\), both weakly star in \(L^\infty(0, T; L^2(\Omega))\). Thus uniqueness of weak star limits implies that \(-\Delta u + u = w\) in \(L^\infty(0, T; L^2(\Omega))\).

Regarding the hyperbolic equation, let \(t \in (0, T)\) and \(v \in H^2_0(\Omega)\). Then there exist a sequence \(\{v_M\}\) with \(v_M \in E_M\) such that \(\|v_M\|_{H^2} \leq \|v\|_{H^2}\) and \(v_M \to v\) strongly in \(H^2_0(\Omega)\). Now fix \(M \geq 1\), then for every \(N \geq M\), integrating (11.1) over \([0, t]\), we get (due to \(E_M \subset E_N\))

\[
\langle w_N(t) - w_N(0), v_M \rangle_{L^2} = \int_0^t \left\langle \tilde{V}(u_N(s) \cdot \nabla v_M, w_N(s) \right\rangle_{L^2} ds + \int_0^t \left\langle k u_{N,y}(s), v_M \right\rangle_{L^2} ds
\]

where as \(N \to \infty\), we have:

(i) \(w_N(0)\) converges strongly to \(w_0\) in \(L^2(\Omega)\) by definition.

(ii) \(w_N\) converges to \(w\) weakly in \(L^2(0, t; L^2(\Omega))\) \(\forall t \in (0, T)\) via (16) with \(v = \chi_{[0,t]} \omega \tilde{v}\) where \(\tilde{v} \in L^2(0, T; L^2(\Omega))\).

(iii) A subsequence (depending on \(v_M\)) of the continuous functions \(f_N(t) := \langle w_N(t), v_M \rangle_{L^2}\) converges uniformly to \(\langle w(t), v_M \rangle_{L^2}\) on \([0, T]\) by Arzela-Ascoli theorem. To see this, observe that \(f_N(t)\) is uniformly bounded on \([0, T]\) as

\[
|f_N(t) - f_N(s)| \leq C_N \|v_M\|_{L^2} / C_E,
\]

and equi-continuous on \((0, T)\) as

\[
\|f_N(t) - f_N(s)\|_{L^2} \leq \left| \int_s^t \langle w'_N(\tau), v_M \rangle_{L^2} \ d\tau \right| \leq \int_s^t \|w'_N\|_{L^2} \|v_M\|_{L^2} ds
\]

where \(C_L > 0\) is the constant in the Ladyzhenskaya’s inequality

\[
\|v\|_{L^4} \leq C_L \|v\|_{L^2}^{1/2} \cdot \|v\|_{H^1}^{1/2} \quad \forall v \in H^1(\Omega).
\]

Hence a subsequence \(\{f_{N_k}(t)\}\) converges uniformly to \(\langle \tilde{w}(t), v_M \rangle_{L^2}\) on \([0, T]\). By the uniqueness of weak limits in \(L^2(0, t; L^2(\Omega))\), \(\tilde{w} = w\).

(iv) Re-indexing according to the subsequence found in part (iii), we have

\[
\int_0^t \left\langle \tilde{V}(u_N(s) \cdot \nabla v_M, w_N(s) \right\rangle_{L^2} ds \to \int_0^t \left\langle \tilde{V}(u(s) \cdot \nabla v_M, w(s) \right\rangle_{L^2} ds
\]

as

\[
\left| \int_0^t \left\langle \tilde{V}(u_N(s) \cdot \nabla v_M, w_N(s) \right\rangle_{L^2} ds - \int_0^t \left\langle \tilde{V}(u(s) \cdot \nabla v_M, w(s) \right\rangle_{L^2} ds \right| \leq \int_0^t \left\langle \tilde{V}(u_N(s) - u(s) \cdot \nabla v_M, w_N(s) \right\rangle_{L^2} ds + \int_0^t \left\langle \tilde{V}(u(s) \cdot \nabla v_M, w_N(s) - w(s) \right\rangle_{L^2} ds
\]

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where
\[ A_1 \leq \sum_{\text{fixed}} \|\nabla v_M\|_{L^\infty(L^\infty)} \|w_N\|_{L^2(L^2)} \left\| \nabla \left( u_N(s) - u(s) \right) \right\|_{L^2(L^2)} \rightarrow 0, \]

and \( B_1 \rightarrow 0 \) by the weak convergence \( w_N \rightarrow w \) in \( L^2(0, t; L^2(\Omega)) \).

(v) Finally through \( u_N \rightarrow u \) in \( L^2(0, t; H^1_0(\Omega)) \) or strongly in \( C([0, T]; H^1_0(\Omega)) \), we have
\[
\int_0^t \langle ku_N, v_M \rangle_{L^2} \, ds \rightarrow \int_0^t \langle ku, v_M \rangle_{L^2} \, ds.
\]

Putting all together, for each \( M \geq 1 \), equation (18) converges to
\[
\langle w(t) - w_0, v_M \rangle_{L^2} = \int_0^t \left\langle \nabla \left( u(s) \right) \cdot \nabla v_M, w(s) \right\rangle_{L^2} \, ds + \int_0^t \langle ku_y, v_M \rangle_{L^2} \, ds \quad \text{for all } t \in (0, T),
\]
where the choice of the subsequence depends on \( M \), but ultimately, equation (19) holds for all \( M \). Finally letting \( M \rightarrow \infty \), (19) converges to (8.1).

**Initial Conditions**

Finally, we show that the initial conditions are satisfied. Taking the limit of (8.1) as \( t \rightarrow 0 \), we obtain
\[
\lim_{t \rightarrow 0} \langle w(t) - w_0, v \rangle_{L^2} = 0 \quad \forall v \in H^2_0(\Omega),
\]
which, via the elliptical equation and the self-adjointness of \( I - \Delta \), is equivalent to
\[
\lim_{t \rightarrow 0} \langle u(t) - u_0, v \rangle_{L^2} = 0 \quad \forall v \in L^2(\Omega).
\]

This completes the proof of Theorem 1.1. As a byproduct, we obtain the existence sole for the hyperbolic equation:

**Corollary 3.4.** For every \( T > 0 \), if \( u \in C([0, T], H^2_0(\Omega)) \) and \( w_0 \in L^2(\Omega) \), then the hyperbolic equation has a weak solution \( w \in L^\infty(0, T; L^2(\Omega)) \) in the sense of (8.1) and (8.3).

**Proof.** Mimic proof of Theorem 1.1 as follows: Directly after a priori estimate (12), extract the subsequence \( w_N \) and go to “Passing to the Limit” section with \( u_N = u \).

\[ \square \]

## 4 Proof of Theorem 1.2

**A Priori \( H^1 \) Estimates for the Galerkin Solution**

**Lemma 4.1.** If \( u \in C([0, T], H^2_0(\Omega)) \cap L^\infty(0, T; H^3_0(\Omega)) \) with \( w_0 \in H^2_0(\Omega) \cap L^\infty(\Omega) \), then
\[
\|w_N\|_{L^\infty(L^\infty)} \leq 21C\|k\|_{W^{2, \infty}} \|u\|_{L^\infty(H^3)} T + 2\|w_0\|_{L^\infty}, \quad (20)
\]
\[
\|w_N\|_{L^\infty(H^3)} \leq 168C\|k\|_{W^{2, \infty}} \left\| T \|u\|_{L^\infty(H^3)} \right\|^2 + (16\|w_0\|_{L^\infty} + 2\|k\|_{L^\infty}) \left\| T \|u\|_{L^\infty(H^3)} \right\| + \|w_0\|_{H^1}, \quad (21)
\]
and
\[
\|w_N\|_{L^\infty(L^2)} \leq (2C\|w_N\|_{L^\infty(H^3)} + \|k\|_{L^\infty}) \|u\|_{L^\infty(H^3)}, \quad (22)
\]
where \( C_\infty > 0 \) is the constant from the embedding \( H^2(\Omega) \hookrightarrow L^\infty(\Omega) \).
**Proof.** For convenience, we will write \( w \) instead of \( w_N \).

**Derivation of (20):** Let \( p \geq 4 \) be an even integer. Setting \( v = \text{proj}_{E_N}[pw^{p-1}](t) \) in (11.1) and using Lemma A.4 (i) on the left-hand-side term, we obtain

\[
\frac{d}{dt} \|w\|_{L^p}^p = - \langle \nabla \langle \nabla w, \nabla v \rangle \rangle_{L^2} + \langle \text{proj}_{E_N}[ku_y], pw^{p-1} \rangle_{L^2},
\]

where

\[
Z = - \langle \text{proj}_{E_Q}[ux], \partial_y w \rangle_{L^2} + \langle \text{proj}_{E_Q}[uy], \partial_x w \rangle_{L^2}.
\]

Now applying Lemma A.4 (iv) on \( w^{p-1} \), \( w_y \text{proj}_{E_N}[w^{p-1}] \) and \( w_x \text{proj}_{E_N}[w^{p-1}] \), there exists a large enough integer \( Q \geq N \) such that

\[
w_y \text{proj}_{E_N}[w^{p-1}] = \text{proj}_{E_Q}[w_y w^{-1}] \quad \text{and} \quad w_x \text{proj}_{E_N}[w^{p-1}] = \text{proj}_{E_Q}[w_x w^{-1}],
\]

so that

\[
Z = - \langle \text{proj}_{E_Q}[ux], \partial_y w \rangle_{L^2} + \langle \text{proj}_{E_Q}[uy], \partial_x w \rangle_{L^2}
= \langle \partial_y \text{proj}_{E_Q}[ux] - \partial_x \text{proj}_{E_Q}[uy], w \rangle_{L^2}
= \langle \text{proj}_{E_Q}[uxy] - \text{proj}_{E_Q}[uyx], w \rangle_{L^2} = 0.
\]

Hence (23) implies

\[
\|w\|_{L^p} \leq \langle \text{proj}_{E_N}[ku_y] \rangle_{L^\infty} \|w\|_{L^{p-1}},
\]

where

\[
\|\text{proj}_{E_N}[ku_y]\|_{L^\infty} \leq C_{\infty} \|ku_y\|_{L^{\infty}(H^2)} \leq 21C_{\infty} \|k\|_{W^{2,\infty}} \|u\|_{L^{\infty}(H^3)}
\]

and

\[
\|w\|_{L^{p-1}} \leq L^{2/p} \|w\|_{L^p}^{p-1}
\]

using Lemma A.4 (ii) and Hölder’s inequality, respectively. Cancelling out \( \|w\|_{L^p}^{p-1} \), we obtain

\[
\frac{d}{dt} \|w\|_{L^p} \leq 21C_{\infty} \|k\|_{W^{2,\infty}} \|u\|_{L^{\infty}(H^3)} L^{2/p}.
\]

Integrating this over \([0, t]\) gives

\[
\|w(t)\|_{L^p} \leq 21C_{\infty} \|k\|_{W^{2,\infty}} \|u\|_{L^{\infty}(H^3)} L^{2/p} t + L^{2/p} \|w(0)\|_{L^\infty}.
\]

Finally, letting \( p \to \infty \) and then taking the sup over \([0, T]\), estimate (20) follows.

**Derivation of (21):** Setting \( v = (I - \Delta)w \) in (11.1) and using (7), we obtain

\[
\frac{1}{2} \frac{d}{dt} \|w\|_{H^1}^2 = - \langle \nabla \langle \nabla w, \nabla v \rangle, \nabla w \rangle_{L^2} + \langle ku_y, \nabla w \rangle_{L^2},
\]

where

\[
- \langle \nabla \langle \nabla w, \nabla v \rangle, \nabla w \rangle_{L^2}
\]

simplifies through integration by parts to

\[
\langle u_{xx} w_{yy}, w \rangle_{L^2} + \langle u_{xx} w_{yx}, w \rangle_{L^2} + \langle u_{xy} w_y, w \rangle_{L^2} + \langle u_{xy} w_{yy}, w \rangle_{L^2}
- \langle u_{xy} w_{xx}, w \rangle_{L^2} - \langle u_{yy} w_{yx}, w \rangle_{L^2} - \langle u_{yy} w_{yy}, w \rangle_{L^2}
\]

whose terms can be bounded above by \( \|w\|_{L^\infty} \|u\|_{H^3} \|w\|_{H^1} \) directly or through Lemma A.4 (v). Thus after cancelling \( \|w\|_{H^1} \) out, (24) implies

\[
\frac{d}{dt} \|w\|_{H^1} \leq 8 \|w\|_{L^{\infty}(L^{\infty})} + 2 \|k\|_{L^\infty} \|u\|_{L^{\infty}(H^3)}.
\]
Now integrating over \([0, t]\), with \(t \leq T\), and taking the sup over \([0, T]\), we obtain
\[
\|w\|_{L^\infty(T)} \leq (8 \|w\|_{L^\infty(L^\infty)} + 2 \|k\|_{L^\infty}) \|w\|_{L^\infty(H^3)} T + \|w(0)\|_{H^3}.
\]
Finally via estimate (20) and \(\|w(0)\|_{H^1} \leq \|w_0\|_{H^1}\), estimate (21) follows.

**Derivation of (22):** Setting \(v = w'\) in (11.1), we obtain
\[
\|w'\|^2_{L^2} = -\langle u_y w_x, w' \rangle_{L^2} + \langle u_x w_y, w' \rangle_{L^2} + \langle k u_y, w' \rangle_{L^2}
\leq (\|u_y\|_{L^\infty} \|w_x\|_{L^2} + \|u_x\|_{L^\infty} \|w_y\|_{L^2} + \|k\|_{L^\infty} \cdot \|u_y\|_{L^2}) \|w'\|_{L^2},
\]
so that
\[
\|w'\|_{L^2} \leq (2C_{\infty} \|w\|_{H^1} + \|k\|_{L^\infty}) \|u\|_{H^3}
\]
Finally taking the sup over \([0, T]\), estimate (22) follows.

**Setup of the Sequence of Fixed-Point Problems**

Given a non-zero initial data \(u_0 \in H^3_0(\Omega)\) with \(w_0 := (I - \Delta) u_0 \in H^3_0(\Omega) \cap L^\infty(\Omega)\), we consider:
\[
\mathcal{X}_N := \left\{ u \in C([0, T], E_N) \mid T \|u\|_{L^\infty(H^3)} \leq 1 \right\}
\]
\[
\mathcal{Y}_N := \left\{ w \in C([0, T], E_N) \mid C_E T \|w\|_{L^\infty(H^1)} \leq 1 \right\}
\]
where \(T\) is the least existence time given by
\[
T := \left[ C_E (168C_{\infty} \|k\|_{W^2, \infty} + 16 \|w_0\|_{L^\infty} + 2 \|k\|_{L^\infty} + \|w_0\|_{H^1} + 1) \right]^{-1}. \tag{25}
\]
Observe that each solution operator is well-defined because:

(i) \(\mathcal{H}_N(\mathcal{X}_N) \subseteq \mathcal{Y}_N\) through estimate (21) as
\[
C_E T \|\mathcal{H}_N(u)\|_{L^\infty(H^1)} \leq C_E T (168C_{\infty} \|k\|_{W^2, \infty} + 16 \|w_0\|_{L^\infty} + 2 \|k\|_{L^\infty} + \|w_0\|_{H^1}) \leq 1.
\]

(ii) \(\mathcal{E}(\mathcal{Y}_N) \subseteq \mathcal{X}_N\) through estimate (10) as
\[
T \|\mathcal{E}(w)\|_{L^\infty(H^3)} \leq C_E T \|w\|_{L^\infty(H^1)} \leq 1.
\]

Moreover,

**Lemma 4.2.** \(\mathcal{H}_N : X_N \rightarrow Y_N\) is continuous.

**Proof.** Let \(\epsilon > 0\) be given, set \(\delta = \epsilon/[C_E^2 \sqrt{\lambda_N}/C_E + \|k\|_{L^\infty} T]\), and let \(u_1, u_2 \in \mathcal{X}_N\) be such that \(\|u_1 - u_2\|_{L^\infty(H^3)} < \delta\). Continuing similarly as in the proof of Lemma 3.2, we arrive to
\[
\frac{d}{dt} \|w_1 - w_2\|_{L^2} \leq C_E^2 \|u_1 - u_2\|_{H^2} \|w_1\|_{H^2} + \|k\|_{L^\infty} \|u_1 - u_2\|_{H^1}
\leq (C_E^2 \sqrt{\lambda_N} \|w_1\|_{H^1} + \|k\|_{L^\infty}) \|u_1 - u_2\|_{H^3}
< [C_E^2 \sqrt{\lambda_N}/(C_E T) + \|k\|_{L^\infty}] \delta = \epsilon/T.
\]
Integrating this over \([0, t]\), with \(t \leq T\), and taking the sup over \([0, T]\), we finally obtain
\[
\|\mathcal{H}_N(u_1) - \mathcal{H}_N(u_2)\|_{L^\infty(H^1)} = \|w_1 - w_2\|_{L^\infty(H^1)} \leq \sqrt{\lambda_N} \|w_1 - w_2\|_{L^\infty(L^2)} < \epsilon.
\]
\[
\square
\]
Obtaining a Candidate Solution Pair \((u, w)\)

At this point, due to Schauder fixed-point theorem, we have a sequence of fixed-point pairs \(\{u_N, w_N\}\) that are uniformly bounded in

\[
L^\infty(0, T; H^3_p(\Omega)) \times L^\infty(0, T; H^1_p(\Omega)) \cap L^\infty(\Omega),
\]

with their temporal derivatives \(\{u'_N, w'_N\}\) uniformly bounded in

\[
L^\infty(0, T; H^2_p(\Omega)) \times L^\infty(0, T; L^2(\Omega)),
\]

satisfying (10) and (11). And so by Banach–Alaoglu theorem, we can eventually extract a subsequence of pairs (re-indexed for convenience) such that:

\[
\begin{align*}
\quad u_N & \overset{\ast}{\rightharpoonup} u & \quad \text{in } L^\infty(0, T; H^1_p(\Omega)), & \text{i.e.} & \int_0^T \langle u_N, v \rangle_{H^1} \, dt \to \int_0^T \langle u, v \rangle_{H^1} \, dt & \forall v \in L^1(0, T; H^1_p(\Omega)), \quad (26) \\
\quad w_N & \overset{\ast}{\rightharpoonup} w & \quad \text{in } L^\infty(0, T; L^\infty(\Omega)), & \text{i.e.} & \int_0^T \int_\Omega w_N v \, d\Omega \to \int_0^T \int_\Omega w v \, d\Omega & \forall v \in L^1(0, T; L^1(\Omega)), \quad (28) \\
\quad w'_N & \overset{\ast}{\rightharpoonup} w' & \quad \text{in } L^\infty(0, T; H^2(\Omega)), & \text{i.e.} & \int_0^T \langle w'_N, v \rangle_{H^2} \, dt \to \int_0^T \langle w', v \rangle_{H^2} \, dt & \forall v \in L^1(0, T; H^2(\Omega)). \quad (30)
\end{align*}
\]

Note that \(w\) in (28) is equal to that of (27) due to uniqueness of weak limits in \(L^2(0, T; L^2(\Omega))\). Also \(u_t\) and \(w_t\) in (29) and (30) are the temporal derivatives of \(u\) and \(w\) in (26) and (27), respectively, due to uniqueness of weak derivatives. Also due to Aubin-Lions lemma, we have the compact embeddings:

\[
\begin{align*}
\{v \in L^\infty(0, T; H^3_p(\Omega)) \mid v_t \in L^\infty(0, T; H^2_p(\Omega))\} & \subset C([0, T], H^2_p(\Omega)), \\
\{v \in L^\infty(0, T; H^1_p(\Omega)) \mid v_t \in L^\infty(0, T; L^2(\Omega))\} & \subset C([0, T], L^2(\Omega)),
\end{align*}
\]

through which we obtain the following strong convergences:

\[
\begin{align*}
\quad u_N & \to u & \quad \text{in } C([0, T]; H^2_p(\Omega)), & \text{i.e.} & \|u_N - u\|_{L^\infty(H^2_p)} & \to 0 \\
\quad w_N & \to w & \quad \text{in } C([0, T]; L^2(\Omega)), & \text{i.e.} & \|w_N - w\|_{L^\infty(L^2)} & \to 0.
\end{align*}
\]

Passing to the Limit

We now show that this pair \((u, w)\) is a solution to the Hasegawa-Mima Coupled System (4).

We have \(-\Delta u_N + u_N = w_N\) where the RHS converges to \(w\), and the LHS converges to \(-\Delta u + u\), both weakly star in \(L^\infty(0, T; H^1_p(\Omega))\). Thus uniqueness of weak star limits implies that \(-\Delta u + u = w\) in \(L^\infty(0, T; H^1_p(\Omega))\).

Regarding the hyperbolic equation, we let \(t \in (0, T)\) and \(v \in L^2(\Omega)\) and consider a sequence \(\{v_M\}\) with \(v_M \in E_M\) such that \(\|v_M\|_{L^2} \leq \|v\|_{L^2}\) and \(v_M \to v\) strongly in \(L^2(\Omega)\). Now fix \(M \geq 1\), then for every \(N \geq M\), integrating (11.1) over \([0, t]\), we get (due to \(E_M \subset E_N\) and (6)),

\[
\int_0^t \langle w'_N, v_M \rangle_{L^2} \, ds + \langle \vec{V}(u_N) \cdot \nabla w_N, v_M \rangle_{L^2} \, ds = \int_0^t \langle ku_N, v_M \rangle_{L^2} \, ds \quad (31)
\]
Now using the weak convergence of \( w_N' \) to \( w_t \) in \( L^2(0, t; L^2(\Omega)) \), and the claims (i), (ii), (iv) and (v) in the “Passing to the Limit” section in the proof of Theorem 1.1, equation (31) converges to

\[
\int_0^t \langle w_t, v_M \rangle_{L^2} + \langle \tilde{V}(u) \cdot \nabla w, v_M \rangle_{L^2} dt = \int_0^t \langle ku_y, v_M \rangle_{L^2} dt
\]

as \( N \to \infty \), with \( w(0) = w_0 \). Letting \( M \to \infty \), we obtain

\[
\int_0^t \langle w_t + \tilde{V}(u) \cdot \nabla w - ku_y, v \rangle_{L^2} dt = 0 \quad \forall v \in L^2(\Omega) \text{ and } \forall t \in (0, T]
\]  

Subtracting equation (32) with \( t - h \) instead of \( t \) from the original, dividing both sides by \( h \), and letting \( h \to 0 \), Lebesgue’s Differentiation Theorem implies

\[
\langle w(t) + \tilde{V}(u(t)) \cdot \nabla w(t) - ku_y(t), v \rangle_{L^2} = 0 \quad \forall v \in L^2(\Omega) \text{ and } \text{a.e. } t \in (0, T]
\]

Finally, setting \( v = w(t) + \tilde{V}(u(t)) \cdot \nabla w(t) - ku_y(t) \in L^2(\Omega) \) in (33), equation (4.1).

**Uniqueness of the Solution Pair \((u, w)\)**

Finally suppose \((u_1, w_1)\) and \((u_2, w_2)\) are two solution pairs in the context of Theorem 1.2. Then subtracting the hyperbolic equation with \((u_2, w_2)\) from that of \((u_1, w_1)\), and taking the \(L^2\)-innerproduct of the resulting equation with \( w_1 - w_2 \), we obtain

\[
\frac{1}{2} \frac{d}{dt} \|w_1 - w_2\|^2_{L^2} = \langle \tilde{V}(u_1 - u_2) \cdot \nabla (w_1 - w_2), w_1 \rangle_{L^2} + \langle k(u_1 - u_2)y, w_1 - w_2 \rangle_{L^2}
\]

where by Cauchy-Schwarz inequality and Lemma A.4 (v),

\[
A_2 \leq \|w_1\|_{L^\infty} (\|((u_1 - u_2)x)|, |(w_1 - w_2)y|)_{L^2} + \|((u_1 - u_2)y), |(w_1 - w_2)x)|_{L^2})
\]

\[
\leq \|w_1\|_{L^\infty} (\|((u_1 - u_2)y)|, |w_1 - w_2|)_{L^2} + \|((u_1 - u_2)x)|, |w_1 - w_2)|_{L^2})
\]

\[
\leq 2 \|w_1\|_{L^\infty} \cdot \|w_1 - w_2\|_{H^2} \cdot \|w_1 - w_2\|_{L^2}
\]

\[
\leq 2CE \|w_1\|_{L^\infty} \cdot \|w_1 - w_2\|^2_{L^2},
\]

and

\[
B_2 \leq \|k\|_{L^\infty} \cdot \|w_1 - w_2\|_{H^2} \cdot \|w_1 - w_2\|_{L^2} \leq \|k\|_{L^\infty} CE \|w_1 - w_2\|^2_{L^2}.
\]

Now putting all together and cancelling the term \( \|w_1 - w_2\|_{L^2} \), we obtain

\[
\frac{d}{dt} \|w_1 - w_2\|_{L^2} \leq (2 \|w_1\|_{L^\infty(L^\infty)} + \|k\|_{L^\infty})CE \|w_1 - w_2\|_{L^2},
\]

where \( \|w_1 - w_2\|_{L^2} \) is continuous in \( t \) and \( w_1(0) = w_2(0) \), so that by Gronwall’s inequality, we have \( \|w_1(t) - w_2(t)\|^2_{L^2} = 0 \) where \( t \) was arbitrary. Thus, \( w_1 = w_2 \) in \( C([0, T], L^2(\Omega)) \), and via the uniqueness of the solution to the elliptic equation, \( u_1 = u_2 \) in \( C([0, T], H^2_P(\Omega)) \).

This completes the proof of Theorem 1.2. As a byproduct, we obtain the existence sole for the hyperbolic equation:

**Corollary 4.3.** For every \( T > 0 \), if \( u \in C([0, T], H^2_P(\Omega)) \cap L^\infty(0, T; H^3_P(\Omega)) \) and \( w_0 \in H^1_P(\Omega) \cap L^\infty(\Omega) \), then the hyperbolic equation has a unique strong solution

\[
\begin{aligned}
w &\in L^\infty(0, T; H^1_P(\Omega) \cap L^\infty(\Omega)) \cap C([0, T], L^2(\Omega)) \text{ with } w_t \in L^\infty(0, T; L^2(\Omega)),
\end{aligned}
\]

satisfying (4.1) and \( w(0) = w_0 \).

**Proof.** Mimic proof of Theorem 1.2 as follows: Directly after a priori estimates (20), (21) and (22), extract the subsequences involving \( w_N \) and \( w'_N \), then go to “Passing to the Limit” section with \( u_N = u \). \( \square \)
5 Concluding Remarks

Thus far, we have shown global weak existence for the $L^2$ formulation (8) to and unique local existence for the $H^1$ formulation (4) to the HM equation using a priori estimates and compactness methods to extract convergent subsequences to the sequence of fixed-points $u_N = E \circ H_N(u_n)$. To our knowledge, the following problems remain open:

1. Uniqueness of a global weak solution pair $(u, w)$ in Theorem 1.1.
2. Existence of a global strong solution pair $(u, w)$ in Theorem 1.2.
3. Uniqueness of a global weak solution $w$ solely for the hyperbolic equation in Corollary 3.4.

On the computational side, the procedures used in this paper can be implemented using spectral methods.
A Appendix

A.1 Proof of Proposition 2.1 Using an Equivalent Hilbert-Fourier Basis for $H_{P}^{m}(\Omega)$

We begin with a lemma that allows us to obtain a Hilbert-Fourier basis for $L^{2}(\Omega)$ that satisfy PBC’s. For that purpose, let $H_{P}^{-1}(\Omega)$ be the dual of $H_{P}^{1}(\Omega)$ and denote its action on $H_{P}^{1}(\Omega)$ by $\langle \cdot, \cdot \rangle_{H_{P}^{-1},H_{P}^{1}}$. Then,

Lemma A.1. The operator $(I - \Delta)^{-1} : H_{P}^{-1}(\Omega) \longrightarrow H_{P}^{1}(\Omega)$ exists and is self-adjoint and compact on $L^{2}(\Omega)$.

Proof. By Riesz-Frèchet representation theorem, for every $w \in H_{P}^{-1}(\Omega)$, there exists a unique $u \in H_{P}^{1}(\Omega)$, such that

$$\langle w, v \rangle_{H_{P}^{-1},H_{P}^{1}} = \langle u, v \rangle_{H_{P}^{1}} = \langle w, v \rangle_{L^{2}} + \langle \nabla u, \nabla v \rangle_{L^{2}} \quad \forall v \in H_{P}^{1}(\Omega)$$

(34)

with

$$\|w\|_{H_{P}^{-1}} = \|u\|_{H_{P}^{1}}.$$  

(35)

Observe that (34) is the variational formulation of $(I - \Delta)u = w \in H_{P}^{-1}(\Omega)$ through Green’s formula and use of PBC’s on $u$. And so, the operator $(I - \Delta) : H_{P}^{1}(\Omega) \longrightarrow H_{P}^{-1}(\Omega)$ is invertible with $w = (I - \Delta)^{-1}u$.

Now using (35) and Cauchy-Schwarz inequality, it can be shown directly that $\|w\|_{H_{P}^{-1}} \leq \|w\|_{L^{2}}$ whenever $w \in L^{2}(\Omega)$. And so we have

$$(I - \Delta)^{-1}|_{L^{2}(\Omega)} : L^{2}(\Omega) \xrightarrow{\text{continuously}} H_{P}^{-1}(\Omega) \xrightarrow{(I-\Delta)^{-1}} H_{P}^{1}(\Omega) \xrightarrow{\text{compactly}} L^{2}(\Omega)$$

where the compact embedding is due to Rellich’s theorem [11]. Hence $(I - \Delta)^{-1}$ is compact on $L^{2}(\Omega)$.

Considering the context of the triplet $H_{P}^{1}(\Omega) \subset L^{2}(\Omega) \simeq L^{2}(\Omega)^{*} \subset H_{P}^{-1}(\Omega)$, where all inclusions are continuous and dense, we have that

$$\langle w, v \rangle_{H_{P}^{-1},H_{P}^{1}} = \langle w, v \rangle_{L^{2}} \quad \forall w \in L^{2}(\Omega), \forall v \in H_{P}^{1}(\Omega).$$

(36)

This allows us to show that $(I - \Delta)^{-1}$ is self-adjoint on $L^{2}(\Omega)$ as

$$\langle w, (I - \Delta)^{-1}v \rangle_{L^{2}} = \langle w, (I - \Delta)^{-1}v \rangle_{H_{P}^{-1},H_{P}^{1}} = \langle (I - \Delta)^{-1}w, (I - \Delta)^{-1}v \rangle_{H_{P}^{1}} \quad \forall w, v \in L^{2}(\Omega)$$

(37)

and the last term is symmetric. \hfill \Box

Now by Theorem 6.11 of [12], $L^{2}(\Omega)$ has a Hilbert basis consisting of eigenvectors $\{\phi_{j}\}_{j=1}^{\infty}$ of $(I - \Delta)^{-1}$ with eigenvalues $\{\eta_{j}\}_{j=1}^{\infty}$. Now since the eigenvalues of an invertible operator cannot be zero, then

$$(I - \Delta)^{-1}\phi_{j} = \eta_{j}\phi_{j} \iff (I - \Delta)\phi_{j} = \frac{1}{\eta_{j}}\phi_{j}$$

(38)

so that $\{\phi_{j}\}_{j=1}^{\infty}$ are eigenvectors of $I - \Delta$ with eigenvalues $\lambda_{j} := 1/\eta_{j}$. It’s not hard to see that

$$\phi_{j} = \frac{e^{2\pi i k \xi_{j}/L}}{L} \quad \text{and} \quad \lambda_{j} = 1 + \frac{4\pi^{2} |\xi_{j}|^{2}}{L^{2}} \quad \text{with} \quad \xi_{j} \in \mathbb{N} \times \mathbb{N},$$

(39)

and all $\phi_{j}$’s satisfy the PBC’s. This basis allows us to characterize the spaces $H_{P}^{m}(\Omega)$ as follows.

Proposition A.2. Let $m \geq 0$ be an integer. There exists an innerproduct $\langle \cdot, \cdot \rangle_{m}$ on $H_{P}^{m}(\Omega)$ with

$$\langle \phi_{i}, \phi_{j} \rangle_{m} = \lambda_{i}^{m} \delta_{ij} = \begin{cases} 
\lambda_{i}^{m} & \text{for } i = j \\
0 & \text{for } i \neq j
\end{cases}.$$  

(40)
Thus making \( \{ \phi_i / \sqrt{\lambda_i^m} \}_{i=1}^{\infty} \) a Hilbert basis for \( H^{m}_P(\Omega) \) and inducing the Hilbert norm

\[
\| \cdot \|_m^2 = \sum_{i=1}^{\infty} \lambda_i^m \langle \cdot, \phi_i \rangle_{L^2}^2
\]

(41)

that is equivalent to the usual Sobolev norm:

\[
\| \cdot \|_{H^m} \leq \| \cdot \|_m \leq (1 + \kappa_m) \| \cdot \|_{H^m}
\]

(42)

where \( \kappa_m > 0 \) is a constant depending on \( m \).

**Proof.** Suppose \( m \geq 1 \) and consider the sequence

\[
Q_m : H^{2m-2}_P(\Omega) \rightarrow L^2(\Omega)
\]

of differential operators given recursively as follows

\[
Q_m = Q_{m-1}(I - \Delta) - \sum_{|\alpha| \leq m-2} (-1)^{|\alpha|} D^\alpha D^\alpha \Delta \quad m \geq 2,
\]

(43)

where \( \alpha = (\alpha_1, \alpha_2) \) is a bi-index of length \( |\alpha| = \alpha_1 + \alpha_2 \) and \( D^\alpha = \partial_{x_1}^{\alpha_1} \partial_{y_2}^{\alpha_2} \).

It is worth to note here that the domain of \( Q_m \) can extended by density to all of \( L^2(\Omega) \) as \( H^{2m-2}_P(\Omega) \) contains a Fourier basis of \( L^2(\Omega) \).

**Lemma A.3.** \( Q_m \) is a positive self-adjoint operator on \( L^2(\Omega) \), and so it has a positive self-adjoint square root

\[
Q_m^{1/2} : H^{m-1}_P(\Omega) \rightarrow L^2(\Omega)
\]

such that \( \left( Q_m^{1/2} \right)^2 = Q_m \).

**Proof.** Self-adjointness of \( Q_m \) follows immediately from \( \partial_{y_2}^2 = \partial_{y_2}^2 \) and the self-adjointness of \( \partial_{x_1}^2, \partial_{y_2}^2 \) and \( \partial_{y_2}^2 \). Now to show that \( Q_m \) is positive, we proceed by induction as follows: For \( f \in H^{2m-2}_P(\Omega) \),

Base cases: \( \langle Q_1 f, f \rangle_{L^2} = 0 \geq 0 \) and \( \langle Q_2 f, f \rangle_{L^2} = -\langle \Delta f, f \rangle_{L^2} = \| \nabla f \|_{L^2}^2 \geq 0 \)

Inductive step: Assume that \( Q_{m-1} \) is positive for some \( m \geq 2 \), then

\[
\langle Q_m f, f \rangle_{L^2} = \langle Q_{m-1} f, f \rangle_{L^2} + \langle Q_m - \Delta f, \nabla f \rangle_{L^2} + \| \nabla f \|_{H^{m-2}}^2 \geq 0.
\]

Finally, as \( Q_m \) is a differential operator that lowers spacial regularity by \( 2(m - 1) \), then \( Q_m^{1/2} \) will be a differential operator that lowers spacial regularity by \( m - 1 \). For example: \( Q_2^{1/2} = \nabla \) and \( Q_3^{1/2} = 2\nabla(\nabla - \Delta)^{1/2} \).

**Proof of Proposition A.2 continued.** Consider the innerproducts:

\[
\begin{aligned}
\langle \cdot, \cdot \rangle_0 &:= \langle \cdot, \cdot \rangle_{L^2} & & \text{on } L^2(\Omega) \\
\langle \cdot, \cdot \rangle_1 &:= \langle \cdot, \cdot \rangle_{H^1} & & \text{on } H^1_P(\Omega) \\
\langle \cdot, \cdot \rangle_m &:= \langle \cdot, \cdot \rangle_{H^m} + \langle Q_m^{1/2} \cdot, Q_m^{1/2} \cdot \rangle_{L^2} & & \text{on } H^m_P(\Omega) \text{ for } m \geq 2
\end{aligned}
\]

(44)

and denote the induced norm by

\[
\| \cdot \|_m := \sqrt{\langle \cdot, \cdot \rangle_m}.
\]

Since \( Q_m^{1/2} \) lowers spacial regularity by \( m - 1 \), then there exists \( \kappa_m > 0 \) such that

\[
\| Q_m^{1/2} \cdot \|_{L^2} \leq \kappa_m \| \cdot \|_{H^{m-1}}
\]

and so (42) follows.
Observe that $\langle \phi_i, \phi_j \rangle_0 = \langle \phi_i, \phi_j \rangle_{L^2} = \lambda_i^2 \delta_{ij}$ and
\[
\langle \phi_i, \phi_j \rangle_1 = \langle \phi_i, \phi_j \rangle_{L^2} + \langle \nabla \phi_i, \nabla \phi_j \rangle_{L^2} = \langle (I - \Delta) \phi_i, \phi_j \rangle_{L^2} = \lambda_i \langle \phi_i, \phi_j \rangle_{L^2} = \lambda_i^2 \delta_{ij}.
\]

Now for $m \geq 2$, we have
\[
\langle \phi_i, \phi_j \rangle_m = \langle \phi_i, \phi_j \rangle_{H^m} + \langle Q_m \phi_i, \phi_j \rangle_{L^2}
\]
\[
= \langle \phi_i, \phi_j \rangle_{H^{m-1}} + \langle \nabla \phi_i, \nabla \phi_j \rangle_{H^{m-2}} + \lambda_i \langle Q_m \phi_i, \phi_j \rangle_{L^2}
\]
\[
= \langle \phi_i, \phi_j \rangle_{H^{m-1}} + \langle \nabla \phi_i, \nabla \phi_j \rangle_{H^{m-1}} + \lambda_i \langle Q_m \phi_i, \phi_j \rangle_{L^2}
\]
\[
= \langle \phi_i, \phi_j \rangle_{H^{m-1}} + \lambda_i - 1 \langle \phi_i, \phi_j \rangle_{H^{m-1}} + \lambda_i \langle Q_m \phi_i, \phi_j \rangle_{L^2}
\]
\[
= \lambda_i \langle \phi_i, \phi_j \rangle_{m-1}
\]
from which by induction (40) follows. Moreover, because $\{\phi_i/\sqrt{\lambda_i}\}_{i=1}^\infty$ spans $L^2(\Omega)$ which contains a dense subset of $H^2_p(\Omega)$, then $\{\phi_i/\sqrt{\lambda_i}\}_{i=1}^\infty$ is a Hilbert basis for $H^2_p(\Omega)$.

Now for $f \in H^2_p(\Omega)$, write
\[
\sum_{i=1}^\infty \frac{1}{\lambda_i} \langle \phi_i, \phi_i \rangle_m \phi_i \xrightarrow{H^2_p} f \xrightarrow{L^2} \sum_{i=1}^\infty \langle f, \phi_i \rangle_{L^2} \phi_i,
\]
so that by orthogonality of $\phi_i$’s, we have
\[
\langle \phi_i, \phi_i \rangle_m = \lambda_i^m \langle f, \phi_i \rangle_{L^2}
\]
from which (41) follows by Parseval’s identity.

Finally, to finish the proof of Proposition 2.1, we write for $w \in H^m_p(\Omega)$, with $m \geq 0$,
\[
w = \sum_{i=1}^\infty \langle w, \phi_i \rangle_{L^2} \phi_i = \sum_{i=1}^\infty \langle u, \phi_i \rangle_{L^2} + \langle \nabla u, \nabla \phi_i \rangle_{L^2} \phi_i = \sum_{i=1}^\infty \langle u, (I - \Delta) \phi_i \rangle_{L^2} \phi_i = \sum_{i=1}^\infty \lambda_i \langle u, \phi_i \rangle_{L^2} \phi_i,
\]
so that by orthogonality of $\phi_i$’s, we have
\[
\lambda_i \langle u, \phi_i \rangle_{L^2} = \langle w, \phi_i \rangle_{L^2} \quad \forall i \geq 1.
\]
Hence
\[
\|w\|_{m+2}^2 = \sum_{i=1}^\infty \lambda_i^{m+2} \langle u, \phi_i \rangle_{L^2}^2 = \sum_{i=1}^\infty \lambda_i^{m+2} \langle u, \phi_i \rangle_{L^2}^2 = \|w\|_m^2,
\]
so that
\[
u \in H^2_{m+2}(\Omega) \text{ and } \|u\|_{H^{m+2}} \leq (1 + \kappa_m) \|u\|_{H^m}.
\]
Setting $C_E := (1 + \kappa_m)$ completes the proof.

### A.2 Auxiliary Results For the a Priori Estimates (14), (20), (21), and (22)

**Lemma A.4.** The following assertions hold:

(i) For all $u, v \in L^2(\Omega)$, we have $\langle \text{proj}_{E_N} u, v \rangle_{L^2} = \langle u, \text{proj}_{E_N} v \rangle_{L^2}$.

(ii) For all $u \in H^m_p(\Omega)$, with $m \geq 0$, we have $\|\text{proj}_{E_N} u\|_{H^m} \leq \|u\|_{H^m}$.

(iii) For all $u \in H^2_{p}(\Omega)$, we have $\text{proj}_{E_N}[(I - \Delta)u] = (I - \Delta)\text{proj}_{E_N} u$.

(iv) For all $N, M \in \mathbb{Z}^+$, there exists an integer $Q \geq M, N$ such that
\[
u v \in E_Q \text{ for all } u \in E_M \text{ and } v \in E_N.
\]
(v) For all \( u \in H_1^1(\Omega) \) and \( \varphi \in E_N \), we have
\[
\int_\Omega |u| \cdot |D\varphi| \leq \int_\Omega |Du| \cdot |\varphi|,
\]  
(46)
where \( D \) denotes \( \partial_x \) or \( \partial_y \). By density, the results also holds for \( \varphi \in H_1^1(\Omega) \).

**Proof.** Parts (i)-(iii) follow from direct computations. Part (iv) is based on the fact that the product of two Fourier basis functions can be written as the sum of multiples of Fourier basis functions with larger eigenvalues. For part (v), observe that \( u \) and \( Du \) are in \( L^1_{\text{loc}}(\Omega) \), so that \( |u| \in L^1_{\text{loc}}(\Omega) \) and is weakly differentiable with \( D|u| = \text{sign}(u)Du \).

When \( \varphi \in E_N \), we compute
\[
\int_\Omega |u| \cdot |D\varphi| = \int_{\{D\varphi \geq 0\}} |u| \cdot D\varphi - \int_{\{D\varphi < 0\}} |u| \cdot D\varphi.
\]
Since \( D\varphi \) is a finite sum of Fourier basis functions, and so the domains \( \{D\varphi \geq 0\} \) and \( \{D\varphi < 0\} \) have finite number of connected components with piecewise smooth boundaries satisfying
\[
\partial \{D\varphi \geq 0\} \setminus \partial \{D\varphi < 0\} \subset \partial \Omega.
\]  
(47)
Now integrating by parts, we obtain
\[
\int_\Omega |u| \cdot |D\varphi| = \int_{\partial \{D\varphi \geq 0\}} |u| \cdot D\varphi - \int_{\partial \{D\varphi < 0\}} |u| \cdot D\varphi + \int_{\{D\varphi \geq 0\}} |D|u| \cdot |\varphi| + \int_{\{D\varphi < 0\}} |D|u| \cdot |\varphi|
\]
\[\leq \int_{\partial \{D\varphi \geq 0\}} |u| \cdot |D\varphi| - \int_{\partial \{D\varphi < 0\}} |u| \cdot D\varphi + \int_{\{D\varphi \geq 0\}} |D|u| \cdot |\varphi| + \int_{\{D\varphi < 0\}} |D|u| \cdot |\varphi|
\]
\[\leq \int_{\partial \Omega} |u| \cdot D\varphi + \int_{\Omega} |D|u| \cdot |\varphi|,
\]
where the boundary integral vanishes as \( |u| \) and \( D\varphi \) satisfy PBC’s, and so (46) follows.

When \( \varphi \in H_1^1(\Omega) \), there is a sequence \( \{\varphi_N\} \) such that \( \varphi_N \in E_N \) and \( \varphi_N \longrightarrow \varphi \) in \( H_1^1(\Omega) \). Now we compute
\[
\int_\Omega |u| \cdot |D\varphi| \leq \int_\Omega |u| \cdot |D\varphi_N| + \int_\Omega |u| \cdot |D(\varphi - \varphi_N)|
\]
\[\leq \int_\Omega |Du| \cdot |\varphi_N - \varphi| + \int_\Omega |u| \cdot |D(\varphi - \varphi_N)|
\]
\[\leq \int_\Omega |Du| \cdot |\varphi| + |u|_{H^1} \cdot |\varphi_N - \varphi|_{L^2} + |u|_{L^2} \cdot |\varphi - \varphi_N|_{H^1}.
\]
Letting \( N \longrightarrow \infty \), the result follows.
References

[1] L. Paumond, “Some remarks on a hasegawa-mima-charney-obukhov equation,” Physica D: Nonlinear Phenomena, vol. 195, pp. 379–390, Aug 2004.

[2] B. Guo and Y. Han, “Existence and uniqueness of global solution of the hasegawa-mima equation,” Journal of Mathematical Physics, vol. 45, pp. 1639–1647, Apr 2004.

[3] F. A. Hariri, “Simulating bi-dimensional turbulence in fusion plasma with linear geometry,” Master’s thesis, American University of Beirut, 2010.

[4] ITER Physics Expert Groups on Confinement and Transport and Confinement Modeling and Database, “Plasma confinement and transport,” Nuclear Fusion, vol. 39, no. 12, pp. 2175–2249, 1999.

[5] A. Hasegawa and K. Mima, “Stationary spectrum of strong turbulence in magnetized nonuniform plasma,” Physics of Fluids, vol. 39, pp. 205–208, Jul 1977.

[6] A. Hasegawa and K. Mima, “Pseudo-three-dimensional turbulence in magnetized nonuniform plasma,” Physics of Fluids, vol. 21, pp. 87–92, Jan 1978.

[7] A. Hasegawa and M. Wakatani, “Plasma edge turbulence,” Phys. Rev. Lett., vol. 50, pp. 682–686, Feb 1983.

[8] B. Shivamoggi, “Charney-hasegawa-mima equation: A general class of exact solutions,” Physics Letters A, vol. 138, pp. 37–42, Jun 1989.

[9] H. Karakazian, “Local existence and uniqueness of the solution to the 2d hasegawa-mima equation with periodic boundary conditions,” Master’s thesis, American University of Beirut, 2016.

[10] H. Karakazian, S. Moufawad, and N. Nassif, “A finite-element model for the hasegawa–mima wave equation,” Applied Mathematics and Computation, vol. 412, p. 126550, 2022.

[11] J. Nečas, Direct Methods in the Theory of Elliptic Equations. Berlin, Heidelberg: Springer Berlin Heidelberg, 2nd ed., 2012.

[12] H. Brezis, Functional Analysis, Sobolev Spaces and Partial Differential Equations. Springer, 2011.