Bose-Einstein condensate and Spontaneous Breaking of Conformal Symmetry on Killing Horizons

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Abstract. Local scalar QFT (in Weyl algebraic approach) is constructed on degenerate semi-Riemannian manifolds corresponding to Killing horizons in spacetime. Covariance properties of the $C^*$-algebra of observables with respect to the conformal group $PSL(2,\mathbb{R})$ are studied. It is shown that, in addition to the state studied by Guido, Longo, Roberts and Verch for bifurcated Killing horizons, which is conformally invariant and KMS at Hawking temperature with respect to the Killing flow and defines a conformal net of von Neumann algebras, there is a further wide class of algebraic (coherent) states representing spontaneous breaking of $PSL(2,\mathbb{R})$ symmetry. This class is labeled by functions in a suitable Hilbert space and their GNS representations enjoy remarkable properties. The states are non equivalent extremal KMS states at Hawking temperature with respect to the residual one-parameter subgroup of $PSL(2,\mathbb{R})$ associated with the Killing flow. The KMS property is valid for the two local sub algebras of observables uniquely determined by covariance and invariance under the residual symmetry unitarily represented. These algebras rely on the physical region of the manifold corresponding to a Killing horizon cleaned up by removing the unphysical points at infinity (necessary to describe the whole $PSL(2,\mathbb{R})$ action). Each of the found states can be interpreted as a different thermodynamic phase, containing Bose-Einstein condensate, for the considered quantum field. It is finally suggested that the found states could describe different black holes.

1 Introduction.

In a remarkable paper [11], among other results, Guido, Longo, Roberts and Verch show that, in a globally hyperbolic spacetime containing a bifurcate Killing horizon [17], the local algebra of observables (realized as bounded operators associated to bounded spacetime regions in a suitable Hilbert space) may induce a local algebra of observables localized at the horizon itself with interesting properties. In fact, the induced local algebra turns out to be covariant with respect to a unitary representation of Möbius group of the circle $PSL(2,\mathbb{R}) := SL(2,\mathbb{R})/\{\pm I\}$.
defined in the Hilbert space of the system. The covariance property is referred to the geometric action of the Möbius group of the circle on the horizon as explained below. The work, on a hand uses general theorems due to Wiesbrock [39, 40] establishing the existence of $SL(2,\mathbb{R})$ representations related to modular operators of von Neumann algebras. On the other hand it enjoys some interplay with several “holographic” ideas (including LightFront Holography) in QFT [34, 25, 26, 27].

The central mathematical object employed in [11] is a net of von Neumann algebras built upon a certain state which is assumed to exist and satisfy the following requirement. Its restriction to the subnet of observables which are localized at the horizon, must be KMS at Hawking temperature for the Killing flow. In that case, the net of observables localized at the future horizon $F$ (see fig.1) is shown to support a unitary representation of $PSL(2,\mathbb{R})$ giving rise to a conformal net (see for instance [4, 8, 10, 5] and references therein).

It is worth noticing that the full $PSL(2,\mathbb{R})$-covariance of the observables of the conformal net is apparent when one extends the future Killing horizon $F$ by adding points at infinity obtaining a manifold $S^1 \times \Sigma$, $\Sigma$ being the transverse manifold at the bifurcation of horizons. $S^1$ represents nothing but the history $\mathbb{R}$ of a particle of light living on the future horizon compactified into a circle by means of the addition of a point at infinity. The addition of points at infinity is necessary because $PSL(2,\mathbb{R})$ acts properly as a subgroup of the diffeomorphisms of the circle $S^1$ and not the line $\mathbb{R}$. In particular the action of $PSL(2,\mathbb{R})$ on $S^1$ includes arbitrary rotations of the circle itself which shift the point at infinity in the physical region $\mathbb{R}$.

From a physical point of view these transformations have no meaning. So it seems that the found covariance of the observables localized at the horizon under the full group $PSL(2,\mathbb{R})$ is actually too large. The problem could be traced back to the state used to construct the von Neumann net of observables.

In spite of this drawback, the results proved in [11] shows the existence of a nice interplay of Killing horizons, thermal states at the correct physical temperature, and conformal symmetry. This result is strongly remarkable in its own right.

In the first part of this paper we give an explicit procedure to built up a local algebra of observables localized on a degenerate semi-Riemannian manifold $\mathbb{M} := S^1 \times \Sigma$ (obtained from future or past Killing horizons in particular) based on Weyl quantization procedure. This is done without referring to external (bulk) algebras and states and restriction procedures. We find, in fact, a conformal net of observables relying on a $PSL(2,\mathbb{R})$-invariant vacuum $\lambda$: At algebraic level there is a representation $\alpha$ of $PSL(2,\mathbb{R})$ made of $^*$-automorphisms of the Weyl algebra $W(\mathbb{M})$ and there is a state $\lambda$ on $W(\mathbb{M})$ which is invariant under $\alpha$. In the GNS representation of $\lambda$, $\alpha$ is implemented covariantly by a unitary representation $U$ of $PSL(2,\mathbb{R})$. Moreover it is showed that $\lambda$ is KMS at Hawking temperature, with respect to the generator of conformal dilatations, in suitable regions $F_{\pm}$ of $\mathbb{M}$ (see fig.1). $F_{\pm}$ do not include points at infinity and are to the two disjoint regions in $F$ respectively in the past and in the future of the bifurcation surface.

In the second part we try to solve the problem focused above concerning the physical inappropriateness of the full $PSL(2,\mathbb{R})$ covariance whenever $\mathbb{M}$ is realized by adding (unphysical) points at infinity to a future Killing horizon $F$.

To this end, it is proven that it is possible to get rid of the unphysical action of $PSL(2,\mathbb{R})$
and single out the physical part of the horizon at quantum level, i.e. in Hilbert space, through a sort of spontaneous breaking of \( \text{PSL}(2, \mathbb{R}) \) symmetry. In fact, we establish the existence of other, unitarily inequivalent, GNS representations of \( W(M) \) based on new coherent KMS states \( \lambda_\zeta \) at Hawking temperature. Here \( \zeta \) denotes any functions in \( L^2(\Sigma) \). Those states are no longer invariant under the whole representation \( \alpha \) and in particular they are not invariant under the unphysical transformations of \( \text{PSL}(2, \mathbb{R}) \). However the residual symmetry still is covariantly and unitarily implementable and singles out the algebras \( A(F_+) \) and \( A(F_-) \) as unique invariant subalgebras. The states \( \lambda_\zeta \) represent different thermodynamical phases with respect to \( \lambda \) (this is because the states \( \lambda_\zeta \) are extremal KMS states) at Hawking temperature. Those states have different properties in relation with the appearance of a Bose-Einstein condensate localized at the horizon. Finally, we suggest that these states could, in fact, denote different black holes. In this view the bosonic field \( \phi \) generating the Weyl representations could represent a noncommutative coordinate in the physical regions \( F_\pm \), whereas its mean value represents the classical coordinate describing the parameter of integral curves of the Killing vector restricted to the horizon.

### 2 Scalar free QFT on degenerate semi-Riemannian manifolds.

#### 2.1 Basic definitions and notation

In this paper we deal with metric-degenerate semi-Riemannian manifolds of the product form \( S^1 \times \Sigma \), where \( \Sigma \) is a connected oriented \( d \)-dimensional manifold equipped with a positive metric. \( S^1 \) is assumed to be oriented and endowed with the null metric. \( S^1 \times \Sigma \) itself is oriented by the orientation induced from those of \( S^1 \) and \( \Sigma \). \( S^1 \times \Sigma \) will be called degenerate manifold in the following and it will be denoted by \( M \) throughout. A standard frame \( \theta \) on the factor \( S^1 \) of \( M \) is a positive-oriented local coordinate patch on \( S^1 \) which maps \( S^1 \setminus \{ \infty \} \) bijectively to the the segment \( -\pi < \theta < \pi, \infty \) being a point of \( S^1 \). Throughout \( C^\infty_c(M; \mathbb{R}) \) and \( C^\infty_c(M; \mathbb{C}) \) denote the space of compactly-supported real-valued, resp. complex-valued, smooth functions on \( M \) and \( \omega_\Sigma \) is the volume form on \( \Sigma \) induced by the metric of \( \Sigma \). \( C^\infty_c(M; \mathbb{C}) \) is endowed with a natural symplectic (i.e. bilinear and antisymmetric) form given by, if \( \psi, \psi' \in C^\infty_c(M; \mathbb{C}) \),

\[
\Omega(\psi, \psi') := \int_M \psi' \epsilon_\psi - \psi \epsilon_{\psi'} \quad \text{where} \quad \epsilon_\psi := d\psi \wedge \omega_\Sigma .
\]

Concerning KMS states we adopt the definition 5.3.1 in [2] (see also chapter V of [13] where the \( \sigma \)-weak topology used in the definition above in the case of a von Neumann algebra is called weak *-topology also known as ultraweak topology).

The symbol \( \mathbb{N} \) denotes the set of natural numbers \( \{0, 1, 2, \ldots\} \), whereas \( \mathbb{N}' \) means \( \mathbb{N} \setminus \{0\} \).

#### 2.2 Bifurcate Killing horizon and Kruskal cases

A simple example of three-dimensional degenerate manifold can be obtained from a submanifold of Kruskal manifold. However everything follows is valid, more generally, for any \( (d + 2) \)-dimensional globally hyperbolic spacetime containing a bifurcate Killing horizon [17] if replacing \( S^2 \) with a generic \( d \)-dimensional spacelike...
submanifold $\Sigma$. In a basis of Killing vector fields of Kruskal spacetime is made of three fields: two generating the $S^2$ symmetry and $\xi$ generating time evolution in the two static open wedges where $\xi$ is timelike. The region where $(\xi, \xi) = 0$ is made of the union of two three-dimensional submanifolds, $\mathbb{P}$ and $\mathbb{F}$, which we call, respectively the past and the future Killing horizon of the manifold in reference to fig.1. $\mathbb{P} \cap \mathbb{F}$ is the bifurcation surface, i.e. a spacelike two-dimensional oriented submanifold where $\xi = 0$, given by $S^2$ equipped with the Euclidean standard metric of a 2-sphere with radius given by Schwarzschild one $r_s$. That metric is induced from the spacetime metric. $\mathbb{F}$ is isometric to the degenerate manifold $\mathbb{R} \times S^2$. $\mathbb{R}$ is made of the orbits of the null Killing vector $\xi$ restricted to $\mathbb{F}$. We assume that the origin of $\mathbb{R}$ is arranged to belong to the bifurcation manifold $S^2$. The metric induced on $\mathbb{F}$ is degenerate along $\mathbb{R}$ and invariant under $\mathbb{R}$-displacements. A degenerate manifold $M = S^1 \times \Sigma$ can, obviously, be obtained from $\mathbb{F}$ by adding a point at infinity $\infty$ to $\mathbb{R}$ producing $S^1$. In this case $M = S^1 \times S^2$. Orientation of $S^1$ is that induced by $\mathbb{R}$. Then $\theta(V) = 2 \tan^{-1} V$, with $V \in \mathbb{R}$, is a standard frame on $S^1$.

Other examples of degenerate manifold arise from the event horizon of topological black-holes [36, 21] where $\Sigma$ is replaced by a compact two-dimensional manifold of arbitrary nonnegative genus.

![Carter-Penrose conformal diagram of Kruskal spacetime](image)

**Fig. 1.** Carter-Penrose conformal diagram of Kruskal spacetime

2.3. *Weyl/symplectic approach.* In [25, 26, 27] we have considered the limit case of a degenerate manifold $M = S^1$ where $M \setminus \{ \infty \}$ is as part of its boundary made of a bifurcate Killing horizon in $2D$ Minkowski spacetime. In that case local QFT can be induced on $M$, by means of a suitable restriction procedure of standard linear QFT in the bulk spacetime. This restriction actually enjoys some holographic properties because it preserves information about bulk quantum field theory. Here we construct QFT on a general degenerate manifold $M = S^1 \times \Sigma$ without referring to any restriction procedure. The restriction procedure with holographic properties could be generalized to more complicated manifolds (Kruskal manifold in particular) and this issue will be investigated elsewhere. The formulation of real scalar QFT on a degenerate manifold $M$ we present here is an adaptation of the theory of fields obeying linear field equations in globally hyperbolic spacetimes [2, 17, 37, 38].

The starting point of QFT is the real vector space of wavefunctions $S(M) := C^\infty_c(M; \mathbb{R})/\sim$, where $\psi \sim \psi'$ iff $\epsilon_\psi = \epsilon_{\psi'}$. $\Omega$ induces a symplectic form

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1In this paper, barring few differences, we make use of conventions and notation of [38].
on $\mathcal{S}(\mathbb{M})$ still indicated by $\Omega$ and defined by, if $[\psi], [\psi'] \in \mathcal{S}(\mathbb{M})$,

$$\Omega([\psi], [\psi']) := \Omega(\psi, \psi'). \quad (2)$$

Remarks.

(1) Two facts hold: (a) $\psi \sim \psi'$ iff $\frac{\partial \psi - \partial \psi'}{\partial \rho} = 0$ everywhere, and (b) $\epsilon_\psi = \frac{\partial \psi - \partial \psi'}{\partial \rho} \wedge \omega_\Sigma$, $\rho$ being any (local) coordinate on $S^1$. Using (a) and (b) one proves straightforwardly that $\Omega([\psi], [\psi'])$ is well-defined, that is it does not depend on the representatives $\psi, \psi'$ chosen in the classes $[\psi], [\psi']$.

(2) $\Omega$ is nondegenerate on $\mathcal{S}(\mathbb{M}; \mathbb{R})$, that is $\Omega([\psi], [\psi']) = 0$ for all $[\psi'] \in \mathcal{S}(\mathbb{M})$ implies $[\psi] = 0$. The definition $\mathcal{S}(\mathbb{M}) := C^\infty_c(\mathbb{M}; \mathbb{R})/\sim$ gets rid of degenerateness of $\Omega$ on $C^\infty_c(\mathbb{M}; \mathbb{R})$ due to functions constant in $S^1$. Nondegenerateness allows the use of standard procedure to build up QFT within the Weyl formalism as explained below. Another possibility to remove degenerateness is to define $\Omega$ on $\mathcal{S}(\mathbb{M}; \mathbb{R})$ functions with vanishing integral with respect to some measure $d\rho$ induced by a coordinate $\rho$ on $S^1$. Such a definition, differently to that given above, would break invariance under orientation-preserving diffeomorphisms of $S^1$, which is a natural physical requirement due to the absence of a metric on $S^1$. Breaking diffeomorphism invariance will enter the theory through the choice of a reference quantum state.

(3) Henceforth we indicate a wavefunction $[\psi]$ by $\psi$ if the notation is not misunderstandable.

“Wavefunction” is quite an improper term, due to the absence of any equation of motion on $\mathbb{M}$, nevertheless the “wavefunctions” introduced here play a rôle similar to that of the smooth solutions of Klein-Gordon equation in a globally hyperbolic spacetime. As $\mathcal{S}(\mathbb{M})$ is a real vector space equipped with a nondegenerate symplectic form $\Omega$, there exist a complex $C^*$-algebra (theorem 5.2.8 in [2]) generated by elements, $W(\psi)$ with $\psi \in \mathcal{S}(\mathbb{M})$ satisfying, for all $\psi, \psi' \in \mathcal{S}(\mathbb{M})$,

$$W(-\psi) = W(\psi)^*, \quad W(\psi)W(\psi') = e^{i\Omega(\psi, \psi')/2}W(\psi + \psi'). \quad (W1) \quad (W2)$$

That $C^*$-algebra, indicated by $W(\mathbb{M})$, is unique up to (isometric) $*$-isomorphisms (theorem 5.2.8 in [2]). As consequences of (W1) and (W2), $W(\mathbb{M})$ admits unit $I = W(0)$, each $W(\psi)$ is unitary and, from the nondegenerateness of $\Omega$, $W(\psi) = W(\psi')$ if and only if $\psi = \psi'$. $W(\mathbb{M})$ is called Weyl algebra associated with $\mathcal{S}(\mathbb{M})$ and $\Omega$ whereas the $W(\psi)$ are called symplectically-smeared (abstract) Weyl operators. The formal interpretation of elements $W(\psi)$ is $W(\psi) \equiv e^{i\Omega(\psi, \hat{\phi})}$ where $\Omega(\psi, \hat{\phi})$ are symplectically-smeared scalar fields as we shall see shortly.

2.4. Implementing locality: fields smeared with forms. In a globally-hyperbolic spacetime $X$ the local smearing is obtained employing real compactly supported functions $f$ instead of solutions of field equations [38] to smear field operator. In particular one gives a rigorous field meaning to:

$$\hat{\phi}(f) = \int_X \hat{\phi}(x)f(x) \, d\mu(x), \quad (3)$$

$\mu$ being the measure induced by the metric of $X$. The support of the smearing function $f$ gives a suitable notion of support of the associated observable $\hat{\phi}(f)$. In this way locality can be
implemented by stating that observables with causally disjoint supports commute. In our case it is impossible to assign a unique support to a class of equivalence \([\psi]\) and thus implementation of locality is not very straightforward in the symplectic approach. Furthermore, there is no natural measure \(\mu\) on \(\mathbb{M}\) as that present in (3) because \(S^1\) is metrically degenerate. Both problems can be solved by using compactly supported forms instead of compactly supported functions. Let us indicate by \(\mathcal{D}(\mathbb{M})\) the space of real forms \(\epsilon_\psi\) (see (1)) with \(\psi \in C^\infty_c(\mathbb{M}; \mathbb{R})\). In a globally hyperbolic spacetime [38] the relation between wavefunctions and smooth compactly supported functions (now elements of \(\mathcal{D}(\mathbb{M})\)) used in (3), is implemented by the causal propagator, \(E: \mathcal{D}(\mathbb{M}) \rightarrow \mathcal{S}(\mathbb{M})\), [38]. It is a \(\mathbb{R}\)-linear surjective map which associates a smooth function with a wavefunctions (supported in the causal set generated by the support of the smooth function) and satisfies several properties. The crucial property describing the interplay of \(E\) and \(\Omega\) reads, if \(E(\omega, \omega') := \int_\mathbb{M} E(\omega) \omega'\),
\[
\Omega(E\omega, E\omega') = E(\omega, \omega') \quad \text{for all } \omega, \omega' \in \mathcal{D}(\mathbb{M}) ,
\] (4)
In our case (4) and surjectivity determine \(E\) uniquely on \(\mathbb{M}\).

**Proposition 2.1.** On a degenerate manifold \(\mathbb{M} := S^1 \times \Sigma\) there is a unique surjective \(\mathbb{R}\)-linear map \(E: \mathcal{D}(\mathbb{M}) \rightarrow \mathcal{S}(\mathbb{M})\) satisfying (4). Moreover the following facts hold.

(a) If \(\theta\) is a standard frame on \(S^1\) and \(\omega \in \mathcal{D}(\mathbb{M})\) is realized as a \(2\pi\)-periodic form in \(\theta\) viewed as positive-oriented coordinate \(\mathbb{R}\) and \(s \in \Sigma\), \(E\) admits the representation
\[
(E(\omega))(\theta, s) = \frac{1}{4} \int_{\theta' \in [-\pi, \pi]} \int_{s' \in \Sigma} \left( \text{sign}(\theta') - \frac{\theta'}{\pi} \right) \delta(s, s') \omega(\theta - \theta', s') \right) .
\] (5)

(b) \(E\) is bijective and in particular, for \(\psi \in \mathcal{S}(\mathbb{M})\), \(\omega \in \mathcal{D}(\mathbb{M})\), one has
\[
E(\epsilon_\psi) = \frac{1}{2} \psi \quad \text{and} \quad \epsilon_{E(\omega)} = \frac{1}{2} \omega .
\] (6)
Thus \((\omega, \omega') \mapsto E(\omega, \omega')\) is a nondegenerate symplectic form on \(\mathcal{D}(\mathbb{M})\).

**Proof.** The fact that \(E\) defined in (5) satisfies (4) can be proved straightforwardly by direct computation. Direct computation shows also the validity of (6) proving injectivity and surjectivity. Any linear surjective map \(E\) satisfying (4) fulfills also \(\Omega(\psi, E\omega') = \int_\mathbb{M} \psi \omega'\) for every \(\psi \in \mathcal{S}(\mathbb{M})\) and \(\omega' \in \mathcal{D}(\mathbb{M})\). If \(E, E'\) are surjective linear maps satisfying (4), one has \(\Omega(\psi, E\omega - E'\omega) = \int_\mathbb{M} \psi(\omega - \omega) = 0\) for every \(\psi \in \mathcal{S}(\mathbb{M})\). \(\Omega\) is non degenerate and thus \(E\omega - E'\omega = 0\) for every \(\omega \in \mathcal{D}(\mathbb{M})\). Hence \(E = E'\). The final statement is now obvious. □

We shall call the bijective map \(E\) in (5) **causal propagator**, regardless of the partial inappropriateness of the name due to the lack of field equations. In spacetimes, existence of field equations is responsible for the failure of the injectivity of the causal propagator. \(\delta(s, s')\) in (5) has an evident physical meaning if \((S^1 \setminus \{\infty\}) \times \Sigma\) is thought as the future Kruskal Killing horizon and
E is interpreted as the limit case of a properly defined causal propagator: As the boundary of a causal sets \( J(S) \), for \( S \subset \Sigma \), is made of portions of the factor \( S^1 \), causal separation of sets \( S, S' \subset \Sigma \) assigned at different “times” of \( S^1 \setminus \{ \infty \} \), is equivalent to \( S \cap S' = \emptyset \).

As in spacetimes, if \( \omega \in \mathcal{D}(\mathbb{M}) \), the form-smeared (abstract) Weyl field is defined as

\[
V(\omega) := W(E\omega).
\]

With this definition one immediately gets Weyl relations once again: For all \( \omega, \eta \in \mathcal{D}(\mathbb{M}) \),

\[
(V1) \quad V(-\omega) = V(\omega)^*, \quad (V2) \quad V(\omega)V(\eta) = e^{iE(\omega, \eta)/2}V(\omega + \eta).
\]

Since \( E \) is injective, differently from the extent in a spacetime, \( V(\omega) = V(\omega') \) if and only if \( \omega = \omega' \). A notion of locality on \( \mathbb{M} \) (in a straightforward extension of original idea due to Sewell [35]) can be introduced at this point by the following proposition (the proof is in the appendix).

**Proposition 2.2.** \( [V(\omega), V(\omega')] = 0 \) for \( \omega, \omega' \in \mathcal{D}(\mathbb{M}) \) if one of the conditions is fulfilled:

(a) there are two open disjoint segments \( I, I' \subset S^1 \) with \( \text{supp} \omega \subset I \times \Sigma \) and \( \text{supp} \omega' \subset I' \times \Sigma \),

(b) there are two open disjoint sets \( S, S' \subset \Sigma \) with \( \text{supp} \omega \subset S \times S' \) and \( \text{supp} \omega' \subset S^1 \times S' \).

The \(*\)-algebra \( \mathcal{W}(\mathbb{M}) \) is local in the sense stated in the thesis of Proposition 2.2. Notice that \( \text{supp} \omega \cap \text{supp} \omega' = \emptyset \) does not imply commutativity of \( W(\omega) \) and \( W(\omega') \) in general.

2.5. **Fock representations.** Breaking invariance under orientation-preserving \( S^1 \)-diffeomorphisms, Fock representations of \( \mathcal{W}(\mathbb{M}) \) can be introduced as follows generalizing part of the construction presented in 2.4 of [12] and in [27]. From a physical point of view, the procedure resembles quantization with respect to Killing time in a static spacetime. Fix a standard frame \( \theta \) on \( S^1 \).

Any representative \( \psi \) of \( \hat{\psi} \in \mathcal{S}(\mathbb{M}) \) can be expanded in Fourier series in the parameter \( \theta \), where \( \mathbb{N}' := \mathbb{N} \setminus \{0\} \),

\[
\psi(\theta, s) \sim \sum_{n \in \mathbb{N}'} e^{-i\theta \psi(s, n)} + \sum_{n \in \mathbb{N}'} e^{i\theta \psi(s, n)} + \sqrt{4\pi n} = \psi_+(\theta, s) + \psi_-(\theta, s).
\]

\( \psi_+ \) is the \( \theta \)-positive frequency part of \( \psi \). The term with \( n = 0 \) was discarded due to the equivalence relation used defining \( \mathcal{S}(\mathbb{M}) \), the remaining terms depend on \( \psi \) only. \( \Sigma \ni s \mapsto \psi(s, n) \) is smooth, supported in a compact set of \( \Sigma \) independent from \( n \) and, using integration by parts, for any \( \gamma > 0 \), there is \( C_\gamma \geq 0 \) with \( ||\psi(s, n)\||_\infty \leq C_\gamma n^{-\gamma} \) for \( n \in \mathbb{N}' \) so that the series in (8) converges uniformly and \( \theta \)-derivative operators can be interchanged with the symbol of summation. The found estimation and Fubini’s theorem entail that the sesquilinear form

\[
\langle \psi_+, \psi_+ \rangle := -i\Omega(\psi_+, \psi_+)
\]

on the space of complex linear combinations of \( \theta \)-positive frequency parts satisfies

\[
\langle \psi_+, \psi_+ \rangle = \sum_{n=1}^\infty \int_{\Sigma} \psi_+(s, n) \psi(s, n) \omega_\Sigma(s) = \int_{\Sigma} \sum_{n=1}^\infty \psi_+(s, n) \psi(s, n) \omega_\Sigma(s).
\]
Thus it is positive and defines a Hermitian scalar product. The one-particle space $\mathcal{H}$ is now defined as the completion w.r.t $\langle \cdot , \cdot \rangle$ of the space of positive $\theta$-frequency parts $\psi_+$ of wavefunctions. Due to (10), $\mathcal{H}$ is isomorphic to $L^2(\mathbb{R}_{>0}, \frac{d\theta}{\theta})$.

The field operator symplectically smeared with $\psi \in \mathcal{D}(\mathbb{M})$ and the field operator smeared with the form $\omega \in \mathcal{D}(\mathbb{M})$ are respectively the operators:

$$\Omega(\psi, \phi) := ia(\psi)+ - ia^\dagger(\psi)+ \quad \text{and} \quad \phi(\omega) := \Omega(E\omega, \phi)$$

(11)

where the operators $a^\dagger(\psi)+$ and $a(\psi)+$ (C-linear in $\psi+$) respectively create and annihilate the state $\psi+$. The common invariant domain of all the involved operators is the dense linear manifold $F(\mathcal{H})$ spanned by the vectors with finite number of particle. $\Omega(\psi, \phi)$ and $\phi(\omega)$ are essentially self-adjoint on $F(\mathcal{H})$ (they are symmetric and $F(\mathcal{H})$ is dense and made of analytic vectors) and satisfy bosonic commutation relations (CCR):

$$[\Omega(\psi, \phi), \Omega(\psi', \phi')] = -i\Omega(\psi, \psi') I \quad \text{and} \quad [\phi(\omega), \phi(\omega')] = -iE(\omega, \omega') I .$$

The definition $\phi(\omega) := \Omega(E\omega, \phi)$ is here nothing but a rigorous interpretation of the formula $\phi(\omega) = \int_M \hat{\phi}(x)\omega(x)$. Finally the unitary operators

$$\hat{W}(\psi) := e^{i\Omega(\psi, \phi)} \quad \text{and, equivalently,} \quad \hat{V}(\omega) := \hat{W}(E\omega) = e^{i\phi(\omega)}$$

(12)

enjoy properties (W1), (W2) and, respectively (V1), (V2), so that they define a unitary representation $\hat{W}(\mathbb{M})$ of $\hat{W}(\mathbb{M})$ which is also irreducible. The proof of these properties follows from propositions 5.2.3 and 5.2.4 in [2].

If $\Pi : \hat{W}(\mathbb{M}) \to \hat{W}(\mathbb{M})$ denotes the unique (I being nondegenerate) $C^*$-algebra isomorphism between these two Weyl representations, $(\mathfrak{F}_+^+(\mathcal{H}), \Pi, \Psi)$ coincides, up to unitary transformations, with the GNS triple associated with the algebraic pure state $\lambda$ on $\hat{W}(\mathbb{M})$ uniquely defined by the requirement (see the appendix)

$$\lambda(W(\psi)) := e^{-\langle \psi+, \psi+ \rangle}/2 .$$

(13)

3 Conformal nets on degenerate manifolds.

3.1. $Diff_+(\mathbb{S}^1)$, $PSL(2, \mathbb{R})$ and associated $*$-automorphisms on $\mathbb{M}$. We recall here some basic notions of conformal representations on $\mathbb{S}^1$. Let $Vect(\mathbb{S}^1)$ be the infinite-dimensional Lie algebra of the infinite-dimensional Lie group (see Milnor [23]) of orientation-preserving smooth diffeomorphisms of the circle $Diff_+(\mathbb{S}^1)$. $Vect(\mathbb{S}^1)$ is the real linear space of smooth vector fields on $\mathbb{S}^1$.\footnote{The construction of $\mathcal{H}$ is equivalent to that performed in the approach of [38] (see also [17]) using the real scalar product on $\mathcal{S}(\mathbb{M})$, $\mu(\psi, \psi') := -Im \Omega(\psi+, \psi'+)$ and the map $K : \mathcal{S}(\mathbb{M}) \ni \psi \mapsto \psi+ \in \mathcal{H}$.}
$S^1$ and whose associated one-parameter diffeomorphisms preserve the orientation of $S^1$. $\text{Vect}^C(S^1)$ denotes the complex Lie algebra $\text{Vect}(S^1) \oplus i\text{Vect}(S^1)$ with usual Lie brackets $\{\cdot, \cdot\}$ and involution $\iota : X \mapsto -\overline{X}$ for $X \in \text{Vect}^C(S^1)$, so that $\iota(\{X,Y\}) = \{\iota(Y), \iota(X)\}$. $\text{Vect}(S^1)$ is the (real) sub-Lie-algebra of $\text{Vect}^C(S^1)$ of anti-Hermitian elements with respect to $\iota$. $a$ denotes the Lie subalgebra of $\text{Vect}^C(S^1)$ whose elements have a finite number of Fourier component with respect to a standard frame $\theta$ which is supposed to be fixed from now on. A basis for $a$ is made of fields

$$L_n := ie^{in\theta} \partial_\theta, \quad \text{with } n \in \mathbb{Z}. \quad (14)$$

They enjoy the so-called Hermiticity condition, $\iota(L_n) = L_{-n}$ and the well-known Virasoro commutation rules with vanishing central charge, $[L_n, L_m] = (n - m)L_{n+m}$.

We remind that $SL(2, \mathbb{R})$ and $SU(1, 1)$ are isomorphic through the map $SL(2, \mathbb{R}) \ni h \mapsto g \in SU(1, 1)$ where:

$$g := \begin{pmatrix} \zeta & \eta \\ \overline{\eta} & \overline{\zeta} \end{pmatrix}, \quad h := \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \quad \text{and} \quad \zeta := \frac{\alpha + \delta + i(\beta - \gamma)}{2}, \quad \eta := \frac{\delta - \alpha - i(\beta + \gamma)}{2}. \quad (15)$$

$\text{Diff}^+(S^1)$ includes the Möbius group of the circle $PSL(2, \mathbb{R}) := SU(1, 1)/\{\pm 1\}$ as a finite-dimensional subgroup: Thinking $S^1$ as the unit complex circle parametrized by $\theta$, an element

$$g : e^{i\theta} \mapsto \frac{\zeta e^{i\theta} + \eta}{\eta e^{i\theta} + \zeta}, \quad \text{with } \theta \in [-\pi, \pi]. \quad (15)$$

The corresponding inclusion of Lie algebras is illustrated by the fact that the three $\iota$-anti-Hermitian linearly-independent elements of $a$

$$K := i\mathcal{L}_0 = -\partial_\theta, \quad S := i\frac{\mathcal{L}_1 + \mathcal{L}_{-1}}{2} = -\cos \theta \partial_\theta, \quad D := i\frac{\mathcal{L}_1 - \mathcal{L}_{-1}}{2} = -\sin \theta \partial_\theta \quad (16)$$

enjoy the commutation rules of the elements $k, s, d$ of the basis of the Lie algebra $sl(2, \mathbb{R})$ with

$$k = \frac{1}{2} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad s = \frac{1}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad d = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \quad (17)$$

In particular $k$ is the generator of the subgroup of rotations $SO(2)/\{\pm 1\} \subset PSL(2, \mathbb{R})$ given by displacements in $\theta$. $\text{Diff}^+(S^1)$ acts naturally as a group of isometries on the semi-Riemannian manifold $\mathbb{M} = S^1 \times \Sigma$. If $g \in \text{Diff}^+(S^1)$, we shall use the same symbol to indicate the associated diffeomorphism of $\mathbb{M}$.

### 3.2. Invariance with respect to $PSL(2, \mathbb{R})$. From now on we use the following notation. If $g \in \text{Diff}^+(S^1)$ and $\psi \in C^\infty_c(\mathbb{M}; \mathbb{C})$, $\psi^g := \psi \circ g$. If $[\psi] \in S(\mathbb{M})$, the element $[\psi]^g \equiv [\psi^g]$ is well defined and it will be indicated by $\psi^g$ simply if the meaning is clear from the context. The usual pull-back action on forms $\omega \in \mathcal{D}(\mathbb{M})$ will be denoted similarly: $\omega^g := g^* \omega$. Notice that $g^*$ leaves $\mathcal{D}(\mathbb{M})$ fixed: Using (6), it results that if $\psi = E\omega$ with $\omega \in \mathcal{D}(\mathbb{M})$ then $\omega^g = 2\epsilon\psi^g \in \mathcal{D}(\mathbb{M})$. 


Ω and E are invariant under $Diff^+(\mathbb{S}^1)$. That is, for all $\psi, \phi \in C_c^\infty(\mathbb{M}; \mathbb{C})$, $g \in Diff^+(\mathbb{S}^1)$ and $\omega, \eta \in \mathcal{D}(\mathbb{M})$, 
\[ \Omega(\psi, \phi) = \Omega(\psi^g, \phi^g) \quad \text{and} \quad E(\omega, \eta) = E(\omega^g, \eta^g). \] 
(18)
Therefore, as a consequence of general results ((4) in theorem 5.2.8 of [2]), $Diff^+(\mathbb{M})$ admits a representation $\alpha : g \mapsto \alpha_g$ made of *-automorphisms of the algebra $\mathcal{W}(\mathbb{M})$ induced by
\[ \alpha_g(V(\omega)) := V(\omega^{(g^{-1})}). \] 
(19)
In the following we employ only the restriction of the representation $\alpha$ to the M"obius group of the circle $PSL(2, \mathbb{R}) \ni g \mapsto \alpha_g$ in terms of *-automorphisms of $\mathcal{W}(\mathbb{M})$.
The definition of the state $\lambda$ (13) is not $Diff^+(\mathbb{S}^1)$ invariant since it relies upon the choice of a preferred standard frame $\theta$. Let us show that actually a different standard frame $\theta'$ produces the same $\lambda$ provided the coordinate transformation $\theta' = \theta'(\theta)$ belongs to $PSL(2, \mathbb{R})$.

**Theorem 3.1.** Let $\theta$ be a standard frame on $\mathbb{S}^1$ of $\mathbb{M} := \mathbb{S}^1 \times \Sigma$, consider the state on $\mathcal{W}(\mathbb{M})$, $\lambda$ (13) and the representation $\alpha$ of $PSL(2, \mathbb{R})$ defined above. The following hold.

(a) $\lambda$ is invariant under $\alpha$, that is $\lambda(\alpha_g(w)) = \lambda(w)$ for all $g \in PSL(2, \mathbb{R})$ and $w \in \mathcal{W}(\mathbb{M})$.

(b) If $\theta'$ is another standard frame $\mathbb{S}^1$ such that the coordinate transformation $\theta' = \theta'(\theta)$ belongs to $PSL(2, \mathbb{R})$, then $\lambda' = \lambda$ where $\lambda'$ is the analog of $\lambda$ referred to $\theta'$.

The proof arises from (13) using the invariance of $\Omega$ under $Diff^+(\mathbb{S}^1)$ and the following lemma.

**Lemma 3.1.** Let $\theta$ be a standard frame on $\mathbb{S}^1$ of $\mathbb{M} := \mathbb{S}^1 \times \Sigma$. The action of $PSL(2, \mathbb{R}) \subset Diff^+(\mathbb{S}^1)$ preserves positive frequency parts. That is, if $g \in PSL(2, \mathbb{R})$, $\psi \in \mathcal{S}(\mathbb{M})$, $\omega \in \mathcal{D}(\mathbb{M})$, 
\[ (\psi^g)_+ = (\psi \circ g) \quad \text{and} \quad (\omega^g)_+ = g^* \omega_+, \] 
(20)
where $\omega_+ := \epsilon_{\phi_+}$ is called the $\theta$-positive-frequency part of any form $\omega := \epsilon_{\phi}$ in $\mathcal{D}(\mathbb{M})$.

**Proof.** From Remark on p. 271 of [27] one finds that $(\psi^g)_+ = (\psi \circ g)$ for all $g \in PSL(2, \mathbb{R})$ and $\psi \in \mathcal{S}(\mathbb{M})$. The result straightforwardly extends to $\omega \in \mathcal{D}(\mathbb{M})$ using the definition of $\mathcal{D}(\mathbb{M})$.
\[ \square \]

We stress that (20) does not hold for generic diffeomorphisms $g \in Diff^+(\mathbb{S}^1)$.

### 3.3. Virasoro representations and Conformal nets.
Let us investigate on the existence of operator representations of Virasoro algebra and the real sub algebra $sl(2, \mathbb{R})$ in the Fock space $\mathcal{F}_+(\mathcal{H})$ introduced above focusing, in particular, on the relationship with the algebra $\mathcal{W}(\mathbb{M})$. Fix a standard frame $\theta$ on $\mathbb{S}^1$ and build up the associated Fock space and the Weyl representation. It is possible to introduce in $\mathcal{F}_+(\mathcal{H})$ a new class of operators which generalizes chiral currents straightforwardly. If $\mathbb{N}' := \{1, 2, 3, \ldots\}$ and $\{u_j\}_{j \in \mathbb{N}'}$ is a Hilbert basis of $L^2(\Sigma, \omega_\Sigma)$ the vectors
\[ Z_{jn}(\theta, s) := \frac{u_j(s)e^{-i\theta}}{\sqrt{4\pi n}} \]
define a Hilbert basis of the one-particle space \( \mathcal{H} \). We can always reduce to the case of real vectors \( u_j \) and we assume that\(^4\) henceforth. The functions \( \mathcal{D}(\mathcal{M}) \ni \omega \mapsto a(E\omega) \) and \( \mathcal{D}(\mathcal{M}) \ni \omega \mapsto a\dagger(E\omega) \), where the operators work on the domain \( F(\mathcal{H}) \), can be proved to be distributions using the strong-operator topology (to show it essentially use (1) in prop. 5.2.3 in [2]) and the usual test-function topology on \( \mathcal{D}(\mathcal{M}) \) induced by families of seminorms referred to derivatives (of any order) in coordinates of components of forms \( \omega \) (see 2.8 in [7]). \( \mathcal{D}(\mathcal{M}) \ni \omega \mapsto \hat{\phi}(\omega) \) admits the distributional kernel

\[
\hat{\phi}(\theta, s) = \frac{1}{i\sqrt{4\pi}} \sum_{(n,j) \in \mathbb{Z} \times \mathbb{N}'} \frac{u_j(s)e^{-in\theta}}{n} J_n^{(j)},
\]

where the \textbf{(generalized) chiral currents} \( J_n^{(j)} : F(\mathcal{H}) \to F(\mathcal{H}) \) are defined as follows

\[
J_0^{(j)} = 0, \quad J_n^{(j)} = i\sqrt{n}a(Z_{jn}) \quad \text{if } n > 0 \quad \text{and} \quad J_n^{(j)} = -i\sqrt{-n}a\dagger(Z_{j,-n}) \quad \text{if } n < 0.
\]

They satisfy on \( F(\mathcal{H}) \) both the Hermiticity condition \( J_n^{(j)\dagger} \mid_{F(\mathcal{H})} = J_{-n}^{(j)} \) and the oscillator commutation relations \( \{J_n^{(j)}, J_m^{(j)}\} = n\delta^{(j)}\delta_{n,-m}I \). Introducing the usual normal order prescription \( \cdots \) : “operators \( J_p^{(j)} \) with negative index \( p \) must precede those with positive index \( p \)”, one can try to define the linearly-independent operators, with \( c \in \mathbb{N}' \cup \{\infty\} \)

\[
L_k^{(c)} := \frac{1}{2} \sum_{n \in \mathbb{Z}, j \leq c} :J_n^{(j)} J_{k-n}^{(j)}:, \quad L_k := L_k^{(\infty)}
\]

on some domain in \( \mathfrak{F}_+(\mathcal{H}) \). We shall denote the complex infinite-dimensional algebra spanned by \( L_k^{(c)} \) by \( \hat{d}_c \). One can formally show that \( L_k \) have two equivalent geometric expressions

\[
L_k = \frac{1}{2i} \Omega(\hat{\phi}, \mathcal{L}_k(\hat{\phi})); \quad L_k = \int_{\mathcal{M}} \partial_\theta \hat{\phi} \partial_\theta \hat{\phi} :((\theta, s)e^{ik\theta} d\theta \wedge \omega_{\Sigma},
\]

\( \mathcal{L}_k(\hat{\phi}) \) is the “scalar field” obtained by the action of the differential operator \( \mathcal{L}_k \) (naturally extended from \( \mathbb{S}^1 \) to the product \( \mathbb{M} = \mathbb{S}^1 \times \Sigma \)) on the “scalar field” \( \phi \). The same formulae hold if replacing \( L_k^{(c)} \) for \( L_k \) and replacing \( \hat{\phi} \) with \( \hat{\phi}^{(c)} \) given by the right-hand side of (21) with the sum over \( j \) restricted to the set \( \{1, 2, \cdots, c\} \). If \( c \) is finite the following proposition can be proved by direct inspection.

**Proposition 3.1.** Fix a standard frame \( \theta \) on \( \mathbb{S}^1 \) of \( \mathbb{M} = \mathbb{S}^1 \times \Sigma \), take \( c \in \mathbb{N}' \) and consider the real vector space \( \hat{\mathfrak{a}}_c \) generated by the operators \( L_k^{(c)} \) in (22) equipped with the commutator \( [\cdot, \cdot] \)

\(^4\) \( L^2(\Sigma, \omega_\Sigma) \) is separable since the Borel measure induced by \( \omega_\Sigma \) is \( \sigma \)-finite and the Borel \( \sigma \)-algebra of \( \Sigma \) is countably generated (the topology of \( \Sigma \) being second countable by definition of manifold). If \( \{u_j\} \) is a Hilbert basis \( \{u_j\} \) is such. Orthononormalization procedure of a maximal set of linearly independent generators in the set of all \( u_j + \overline{u}_j, i(u_j - \overline{u}_j) \) yields a real Hilbert basis.
and the involution $\hat{a}_c \ni a \mapsto a^{\dagger}|_{F(\mathfrak{H})}$. The following holds.

(a) The elements of $\hat{a}_c$ are well defined on $F(\mathfrak{H})$ which is a dense invariant space of common analytic vectors.

(b) $(\hat{a}_c, [\cdot, \cdot], \cdot^{\dagger}|_{F(\mathfrak{H})})$ is a central representation, with central charge $c$, of the algebra $(a, \{\cdot, \cdot\}, i)$ (that is a unitarizable Virasoro representation) since the following relations hold:

$$L^{(c)}_{-n} = L^{(c)}_n|_{F(\mathfrak{H})}, \quad (25)$$

$$[L^{(c)}_n, L^{(c)}_m] = (n-m)L^{(c)}_{n+m} + \frac{(n^3-n)c}{12}\delta_{n+m,0}I. \quad (26)$$

(c) The representation is positive energy, i.e. the generators of rotations $L^{(j)}_0$ is non-negative.

(d) Each operator $L^{(c)}_n$ does not depend on the choice for the real base $\{u_j\}_{j \leq c}$ (but depends on the finite dimensional subspace spanned by those vectors).

Notice that the found Virasoro representations are strongly reducible [15]. Once they are decomposed into unitarizable irreducible highest-weight representations [15], they can be exponentiated ([9, 18, 5]) obtaining unitary strongly continuous representations of $\text{Diff}^+(\mathbb{S}^1)$.

In general there is no physical reason to single out a Hilbert basis $\{u_j\}$ or equivalently a sequence of finite dimensional subspaces of $L^2(\Sigma, \omega_\Sigma)$. In the presence of particular symmetries for $\Sigma$ a class of finite dimensional subspaces can be picked out referring to the invariant subspaces with respect to a unitary representation on $L^2(\Sigma, \omega_\Sigma)$ of the symmetry group. For instance, think to $\Sigma = \mathbb{S}^2$, in that case one may decompose $\psi \in L^2(\mathbb{S}^2)$ using (real and imaginary parts of) spherical harmonics $Y^l_m$. Hence a suitable class of finite dimensional subspaces are those with fixed angular momentum $l = 0, 1, 2, \ldots$. The sphere $\mathbb{S}^2$ is reconstructed as a sequence of fuzzy spheres ([20]) with greater and greater angular momentum $l$. The associated Virasoro representations have central charges $c_l = 2l + 1$.

In the absence of symmetries only the case $c = \infty$ seems to be physically interesting. Let us turn attention on this case. Serious problems arise when trying to give a rigorous meaning to all the operators $L_n$. First of all (26) becomes meaningless due to $c = \infty$ in the right-hand side. Furthermore, by direct inspection one finds that, if $n < -1$, the domain of $L_n$ cannot include any vector of $F(\mathfrak{H})$ due to an evident divergence (this drawback would arise also for $|n| = 1$ if $J^{(j)}_0 = 0$ were false). However, by direct inspection, one finds that $L_n$ with $n \geq -1$ are well defined on $F(\mathfrak{H})$ which is, in fact a common invariant dense domain made of analytic vectors, moreover $L_n \Psi = 0$. The central charge does not appear considering commutators of those operators. The complex space (finitely) spanned by those vectors is closed with respect to the commutator but, unfortunately, it is not with respect to the Hermitean conjugation so that they cannot represent a Lie algebra of observables. However, restricting to the case $|n| \leq 1$ everything goes right and one gets a Lie algebra closed with respect to the Hermitean conjugation. Anti-Hermitean linearly-independent operators generating that Lie algebra are

$$iK := iL_0, \quad iS := i\frac{L_1 + L_{-1}}{2}, \quad iD := \frac{L_1 - L_{-1}}{2}. \quad (27)$$
They enjoy the commutation rules of the elements \( k, s, d \) of the basis of the Lie algebra \( sl(2, \mathbb{R}) \) \((17)\). As a consequence a representation \( R : sl(2, \mathbb{R}) \to \mathcal{L}(F(\mathcal{H})) \) can be realized by assuming \( iK = R(k), iS = R(s), iD = R(d) \) and \( R : \alpha k + \beta s + \gamma d \mapsto \alpha K + \beta S + \gamma D \) for all \( \alpha, \beta, \gamma \in \mathbb{R} \). One expects that this representation is associated, via exponentiation, with a strongly continuous (projective) unitary representation of the universal covering of \( SL(2, \mathbb{R}), SL(2, \mathbb{R}) \). Let us prove that such a representation does exists and enjoys remarkable properties.

**Theorem 3.2.** Fix a standard frame \( \theta \) on \( S^1 \) of \( \mathbb{M} = S^1 \times \Sigma \) and construct the GNS (Fock) realization of \( \mathcal{W}(\mathbb{M}) \) associated with the state \( \lambda \) in \((13)\) and the representation \( R \). It turns out that the Hermitean operators \( iR(x) \), with \( x \in sl(2, \mathbb{R}) \), are essentially selfadjoint on \( F(\mathcal{H}) \) and there is a unique strongly-continuous representation \( PSL(2, \mathbb{R}) \ni g \mapsto \hat{\mathcal{F}}(g) : \mathcal{F}(\mathcal{H}) \to \mathcal{F}(\mathcal{H}) \) with

\[
U(\exp(tx)) = e^{R(x)t}, \quad \text{for all } x \in sl(2, \mathbb{R}) \text{ and } t \in \mathbb{R}.
\]  

(28)

The following further facts hold.

(a) \( U \) is a positive-energy representation of \( PSL(2, \mathbb{R}) \) – that is the self-adjoint generator \( \overline{K} \) of the subgroup of rotations, has nonnegative spectrum – and furthermore \( \sigma(\overline{K}) = \{0, 1, 2, \ldots \} \).

(b) \( U \) and its generators do not depend on the choice of the basis \( \{u_j\}_{j \in \mathbb{Z}} \subset L^2(\Sigma, \omega_\Sigma) \).

In particular, \( U \) is the tensorialization of \( G \mapsto \{U_\gamma \} \). Referring to the factorization of the one-particle space \( \mathcal{H} = \ell^2(\mathbb{C}) \otimes L^2(\Sigma, \omega_\Sigma) \), it holds \( U_\gamma = V \otimes I \), where \( V \) is the restriction to the one-particle space of the representation \( U \) in the simplest case \( \mathbb{M} = S^1 \).

(c) Each subspace of \( \mathcal{F}(\mathcal{H}) \) with finite number of particles is invariant under \( U \).

(d) The GNS representative of \( \lambda, F \), is invariant under \( U \) and it is the only unit vector of \( \mathcal{F}(\mathcal{H}) \) invariant under \( \{e^{it\overline{U}}\}_{t \in \mathbb{R}} \) up to phases.

The proof of the theorem is given in the appendix. The following further theorem states that \( \mathcal{W}(\mathbb{M}) \) transform covariantly under this representation with respect to the action of the diffeomorphisms of \( PSL(2, \mathbb{R}) \subset Diff^+(S^1) \) seen in 3.2.

**Theorem 3.3.** With hypotheses and notation of theorem 3.2, the following holds.

(a) \( U \) is \( PSL(2, \mathbb{R}) \) covariant. In other words it implements unitarily the representation \( \alpha \) of \( PSL(2, \mathbb{R}) \) defined in 3.2: For all \( g \in PSL(2, \mathbb{R}) \),

\[
U(g) w U(g)^* = \alpha_g(w), \quad \text{for all } w \in \mathcal{W}(\mathbb{M}) .
\]  

(b) The one-parameter group of *-automorphisms associated with the one-parameter group of diffeomorphisms respectively generated by vector fields \( K, S, D \) correspond, through \((29)\), to the one-parameter unitary subgroups of \( U \) respectively generated by \( iK, iS, iD \).

The proof of the theorem is given in the appendix. Theorems 3.2 and 3.3 has a remarkable consequence concerning the existence of a so-called conformal net on \( S^1 \) associated with the sign conventions should be clear, anyway to fix them notice that formally \( [iK, \hat{\phi}(\theta, s)] = -\partial_\theta \hat{\phi}(\theta, s) \).
the algebra $\hat{W}(\mathcal{M})$. This fact has a wide spectrum of relevant consequences in physics and in mathematics, see for instance [4, 8, 10, 5] and references therein. We remind the reader that any weakly-closed *-subalgebra of the unital $C^*$-algebra of all bounded operators on a Hilbert space is called von Neumann algebra if it contains the unit operator. For several theoretical reasons (see [13]) the largest set of bounded observables of a quantum system represented in a Hilbert space may be assumed to be made of the self-adjoint elements of a suitable von Neumann algebra. If $X$ is a *-algebra of bounded operators over a Hilbert space, $X'$ denotes the algebra of the bounded operators which commute with each element of $X$ and it results that [13] $X$ is a von Neumann algebra if and only if $X = (X')'$. In any cases, $X'' := (X')'$ is the minimal von Neumann algebra which contains $X$. It is called the von Neumann algebra generated by $X$.

**Definition 3.1.** Let $I$ be the set of non empty, nondense, open intervals of $S^1$. Assume that $S^1$ is equipped with a standard coordinate frame $\theta$. A conformal net on $S^1$ is any triple $(\mathcal{A}, \Psi, \hat{U})$ where $\mathcal{A}$ is any family $\{\mathcal{A}(I) \mid I \in \mathcal{I}\}$ of von Neumann algebras on an infinite-dimensional separable complex Hilbert space $\mathcal{H}_\mathcal{A}$, and the following properties hold.

(C1) **Isotony.** $\mathcal{A}(I) \subset \mathcal{A}(J)$, if $I \subset J$ with $I, J \in \mathcal{I}$.

(C2) **Locality.** $\mathcal{A}(I) \subset \mathcal{A}(J)'$, if $I \cap J = \emptyset$ with $I, J \in \mathcal{I}$.

(C3) **Möbius covariance.** $U(g)\mathcal{A}(I)U(g)\dagger = \mathcal{A}(gI)$, $I \in \mathcal{I}$, $g \in PSL(2, \mathbb{R})$, where $U$ is a strongly continuous unitary representation of $PSL(2, \mathbb{R})$ in $\mathcal{H}_\mathcal{A}$ and $g$ denotes the Möbius transformation (15) associated with $\theta$.

(C4) **Positivity of the energy.** The representation $U$ is a positive-energy representation.

(C5) **$U$-invariance and uniqueness of the vacuum.** $\Psi \in \mathcal{H}_\mathcal{A}$ is the unique (up to phases) unit vector invariant under $U$.

(C6) **Cyclicity of the vacuum.** $\Psi$ is cyclic for the algebra $\mathcal{A}(S^1) := \bigvee_{I \in \mathcal{I}} \mathcal{A}(I)$.

We have the following theorem.

**Theorem 3.4.** Fix a standard frame $\theta$ on $S^1$ of $\mathcal{M} := S^1 \times \Sigma$ and define the associated Weyl algebra $\hat{W}(\mathcal{M})$ in the Fock space $\mathcal{F}_+((\mathfrak{H})$ with vacuum state $\Psi$ and the representation of $PSL(2, \mathbb{R})$, $U$ of Theorems 3.2 and 3.3. With those hypotheses the family

$$\mathcal{A} = \{\mathcal{A}(I) \mid I \in \mathcal{I}\}, \quad \text{with} \quad \mathcal{A}(I) = \{\hat{V}(\omega) \mid \text{supp} \omega \subset I \times \Sigma\}'', \quad (30)$$

together with $\Psi$ and $U$ form a conformal net on $S^1$ such that $\hat{W}(\mathcal{M}) \subset \mathcal{A}(S^1)$.

**Proof.** (C1), (C2) and (C3) are straightforward consequences of the definition (30) using the fact that (von Neumann’s density theorem) $\mathcal{A}(K)$ is the closure with respect the strong operator topology of the *-algebra generated by the elements in $\{\hat{V}(\omega) \mid \text{supp} \omega \subset K \times \Sigma\}$, employing proposition 2.2 concerning (C2) and theorem 3.3 concerning (C3). (C4) and (C5) are part of theorem 3.2. (C6) is a consequence of the fact that $\Psi$ is cyclic with respect to $\hat{W}(\mathcal{M})$ (see the appendix) and $\hat{W}(\mathcal{M}) \subset \mathcal{A}(S^1)$. This inclusion is a consequence of the fact that, if $I, J \in \mathcal{I}$ and $S^1 = I \cup J$, then, due to (W2), each element of $\hat{W}(\mathcal{M})$ has the form $c\hat{V}(\omega)\hat{V}(\omega')$ where
supp \omega \subset I \times \Sigma, \ supp \omega' \subset J \times \Sigma \text{ and } |c| = 1, \text{ so that } \hat{W}(M) \subset A(I) \lor A(J) \subset A(S^1). \quad \Box

Remarks.

1. Our construction of a conformal net for, in particular, a bifurcate Killing horizon in a globally hyperbolic spacetime, is explicit in giving the effective form of of the unitary representation of $PSL(2, \mathbb{R})$ and the relationship with the whole Virasoro algebra. It does not require any assumption on the existence of any algebra of observables in the spacetime where $(S^1 \setminus \{\infty\}) \times \Sigma$ can be viewed to be embedded, or any KMS state on that algebra. A different approach was presented in [11] where it is shown that, in a globally hyperbolic spacetime containing a bifurcate Killing horizon, a conformal net can be obtained by restriction to the horizon of a local algebra in the spacetime realized using a GNS representation with cyclic vector which satisfies the KMS condition with respect to the Killing time flow. The unitary representation of $PSL(2, \mathbb{R})$ was obtained there making use of relevant results by Weisbrod et al [40, 39, 12] on the interplay of modular theory and conformal theory. It seems plausible that our construction can be recovered also using the approach of [11] defining a bulk algebra of observables and a KMS state appropriately. This topic will be investigated elsewhere.

2. Conformal nets enjoy relevant properties [4, 8, 10, 5]:

- **Reeh-Schlieder property.** $\Psi$ is cyclic and separating for every $A(I)$.
- **Bisognano-Wichmann property.** The modular operator $\Delta_I$ associated with every $A(I)$ satisfies $\Delta_I^t = U(\exp(2\pi D_I))$ for every $t \in \mathbb{R}$, $\{\exp(tD_I)\}_{t \in \mathbb{R}} \subset PSL(2, \mathbb{R})$ being the one-parameter subgroup which leaves $I$ invariant (with $D_I$ defined as in remark (2) after theorem 4.1 below) so that $\Psi$ is a KMS state for $A(I)$ at inverse temperature $2\pi$ w.r.to $-D_I$ for $A((0, \pi))$.
- **Haag duality.** $A(I)' = A(Int(S^1 \setminus I))$ for every $A(I)$.
- **Irreducibility.** $A(S^1)$ includes all of bounded operators on $\mathcal{H}_A$.
- **Factoriality.** Each $A(I)$ is a type $\text{III}_1$ factor.
- **Additivity.** For every $A(I)$, it holds $A(I) \subset \lor_{J \in \mathcal{S}} A(J)$ if $\cup_{J \in \mathcal{S}} J \supset I$.

3. With obvious changes, theorems 3.2, 3.3, 3.4 are still valid if one considers operators $L_n^{(c)}$ with $c < \infty, |n| \leq 1$ and the real basis $u_j$ is made of smooth functions with $j \leq c$.

4 Spontaneous breaking of $SL(2, \mathbb{R})$ symmetry and thermal states.

4.1. **Back to Physics.** Consider the degenerate manifold $M = S^1 \times S^2$ obtained by the future Killing horizon $F = R \times S^2$ of the Kruskal manifold as discussed in section 2.1. (However what we go to say can be generalized to globally hyperbolic spacetimes with a bifurcate Killing horizon.) In particular the orientation of $S^1 = R \cup \{\infty\}$ is that induced on $R$ from time-orientation of the spacetime. Let $\theta$ be a standard frame on $S^1$ such that that, with $D$ given in (16),

$$\xi |_{\mathcal{F}} = -\kappa D,$$

(31)

$\xi$ being the global Killing field defining Schwarzschild time in both static wedges and $\kappa$ being the surface gravity which is constant on the Killing horizon $F, \kappa = (4GM)^{-1}, M$ being the mass...
of the black hole [37, 38]. Eq. (31) does not fix a standard frame uniquely. However the remaining freedom does not affect the construction we go to present as a consequence of theorem 4.3 below.

The requirement (31) implies that the adimensional parameter \( v \in \mathbb{R} \) of the integral curves of \(-\mathcal{D}\) on \( \mathcal{F} \) coincides, up the factor \( \kappa^{-1} \) and the choice for the origin, with the usual light-coordinate\(^6\): \( v = \kappa(t + r_*) \). There \( r_* \) is the usual Regge-Wheeler tortoise coordinate and \( t \) the Schwarzschild time, that is the parameter of the integral curves of \( \xi \) in any Schwarzschild wedge. In our picture the point \( \infty \) of \( S^1 = \mathbb{R} \cup \{ \infty \} \) corresponds to \( \theta = \pi \) whereas \( \theta = 0 \) corresponds to the bifurcation surface of \( \mathcal{F} \) (see section 2.1).

Let us illustrate the physical consequences of the choice (31) for bosonic QFT built up on the future horizon together with a Möbius-covariant representation of \( \text{PSL}(2, \mathbb{R}) \) everything associated with the preferred choice for the coordinate \( \theta \) on \( S^1 \).

A celebrated result by Kay and Wald [17] states that: Any globally-defined quasifree state on a globally hyperbolic spacetime with a bifurcate Killing horizon (Kruskal manifold in particular) which is invariant under \( \xi \) and satisfies some further requirement (Hadamard condition imposed on the two-point function of the quasifree state in particular) [17] must be unique and KMS with respect to \( \xi \) with the Hawking inverse temperature \( \beta_H = 2\pi/\kappa \). From a physical point of view, one expects that the system of the field defined on the horizon be in thermal equilibrium with the state in the bulk. More precisely, since \( \partial_{\kappa t} \) reduces to \(-\mathcal{D}\) on the future horizon due to (31), one might assume that the natural state on the Killing horizon is a KMS state with respect to \(-\mathcal{D}\) at the inverse temperature \( 2\pi \): That coincides with Hawking inverse temperature referred to the adimensional “time” \( v \) on \( \mathcal{F} \). A first-glance candidate for such a state is just the restriction to \( \lambda \) to the algebra of observables supported in the future Killing horizon (omitting the unphysical points \( \{ \infty \} \times S^2 \)). This is because \( \lambda \) enjoys the very inverse temperature \( 2\pi \) referred to \(-\mathcal{D}\). On the other hand there are physical reasons to reject that candidate. Indeed, the circle \( S^1 = \mathbb{R} \cup \{ \infty \} \) admits two physically distinguishable points: The point at infinity, which cannot be reached physically because it corresponds to a surface which does not belong to the Kruskal manifold. The other point corresponds to the bifurcation manifold where \( \xi \) vanishes. (In the general case \( M := S^1 \times \Sigma \) considered in this work, \( M \) itself cannot represent a portion of spacetime due to the presence of closed causal curves lying in \( S^1 \) and thus one point of \( S^1 \) at least must be removed to make contact with physics.) The remaining points of \( S^1 \) are physically equivalent barring the fact that they are either in the past or in the future of \( \theta = 0 \). This determines two regions \( \mathcal{F}_- \equiv (-\pi,0) \times S^2 \) and \( \mathcal{F}_+ \equiv (0,\pi) \times S^2 \) in the physical part \( \mathbb{R} \times S^1 \), of the manifold \( S^1 \times S^2 \), corresponding to respectively the future and past part – with respect to the bifurcation manifold – of the future Killing horizon of Kruskal spacetime. Conversely, the whole \( \text{PSL}(2, \mathbb{R}) \) unitary representation, referred to the Fock space \( \mathcal{F}_+(\mathcal{H}) \) built upon \( \lambda \), which, in turn, is invariant under the whole representation \( U \), cannot select those physical regions. In particular \( \text{PSL}(2, \mathbb{R}) \) includes arbitrary displacements of the coordinate \( \theta \). Those transformations connect the physical regions with the points at infinity. For these reasons \( \lambda \) seems not to be completely satisfactory from the point of view of physics in spite of its relevant thermal properties.

\(^6\)This fact is evident using well-known global Kruskal null coordinates \( U,V \) [37].
Once a reference state $\mu$ is fixed on $\mathcal{W}(M)$, the physical regions $F_{\pm}$ correspond to von Neumann algebras $\mathcal{A}(F_{+})$ and $\mathcal{A}(F_{-})$ (based upon the GNS representation of $\mu$) representing the observables in those regions.

In the following we show that it is possible to single out those physical regions at quantum, i.e. Hilbert space, level through a sort of spontaneous breaking of $SL(2,\mathbb{R})$ symmetry referring to a new state $\lambda_{x} \neq \lambda$ which preserves the relevant thermal properties. We mean that the following facts, actually valid for any manifold $M = S^{1} \times \Sigma$, hold true. At algebraic level there is a representation $\alpha$ of Möbius group of the circle $PSL(2,\mathbb{R})$ made of $^\ast$-automorphisms of the Weyl algebra $\mathcal{W}(M)$. Moreover, we have seen in theorems 3.2 and 3.3 that there is a state $\lambda$ on $\mathcal{W}(M)$ which is invariant under $\alpha$ and, in the GNS representation of $\lambda$, $\alpha$ is implemented unitarily and covariantly by a representation $U$ of $PSL(2,\mathbb{R})$. We show below that there are other, unitarily inequivalent, GNS representations of $\mathcal{W}(M)$ based on new states $\lambda_{x}$ which are no longer invariant under the whole $\alpha$, but such that, the residual symmetry is still covariantly and unitarily implementable and singles out the algebras $\mathcal{A}(F_{+})$ and $\mathcal{A}(F_{-})$ as unique invariant algebras. We show also that every $\lambda_{x}$ enjoys the same thermal (KMS) properties as $\lambda$ and it represents a different thermodynamical phase with respect to $\lambda$.

4.2. Symmetry breaking. We need some definitions to go on. Coming back to the general case $M = S^{1} \times \Sigma$ where $\Sigma$ is any Riemannian manifold, fix a standard frame $\theta \in (-\pi, +\pi)$ on $S^{1}$. The regions $F_{\pm}$ are defined as those containing the points $(0, \pi) \times \Sigma$ and $(-\pi, 0) \times \Sigma$ respectively. Consider the one-parameter subgroup of Möbius transformations $\mathbb{R} \ni t \rightarrow \exp(tD)$ where $D := -\sin \theta \frac{\partial}{\partial \theta}$ in $M$. It admits 0 and $\pi$ as unique fixed points. On the other hand, it is simply proved that (up to nonvanishing factors) $D$ is the unique nonzero vector field in the representation of $sl(2,\mathbb{R})$ which vanishes at 0 and $\pi$. As a consequence that subgroup is the unique (up to rescaling of the parameter) nontrivial one-parameter subgroup of $PSL(2,\mathbb{R})$ which admits $(0, \pi)$ and $(-\pi, 0)$ as invariant segments. The origin of the parameter $v$ of the integral curves of $-D$ can be arranged in order that

$$v = \Gamma(\theta) := \ln \left| \tan \frac{\theta}{2} \right|,$$

(32)

where $v$ ranges monotonically in $\mathbb{R}$ with $dv/d\theta > 0$ for $\theta \in (0, \pi)$, whereas it ranges monotonically in $\mathbb{R}$ with $dv/d\theta < 0$ for $\theta \in (-\pi, 0)$. In spite of its singularity at $\theta = 0$, the function $\Gamma$ in (32) is locally integrable. Thus for any fixed function $\zeta \in L^{2}(\Sigma, \omega_{\Sigma})$, $\Lambda_{x}(V(\omega)) := \lambda(V(\omega)) e^{i \int_{M} \Gamma(\zeta \omega_{\Sigma} + \overline{\zeta} \omega_{\Sigma})}$ is well defined if $\omega \in D(M)$. Let us show that $\Lambda_{x}$ extends to a state on $\mathcal{W}(M)$. It holds $\Lambda_{x}(V(0)) = 1$. Using (V1), (V2) and imposing linearity, $\Lambda_{x}$ defines a linear functional on the $^\ast$-algebra generated by all of objects $V(\omega)$. As $\lambda$ is positive, $\Lambda_{x}$ turns out to be positive too, finally $\mathbb{R} \ni t \rightarrow \Lambda_{x}(V(\omega))$ is continuous. For known theorems [19] there is a unique extension $\lambda_{x}$ of $\Lambda_{x}$ to a state on $\mathcal{W}(M)$: If the real function $\zeta \in L^{1}_{loc}(\Sigma, \omega_{\Sigma})$ is fixed, it is the unique state satisfying

$$\lambda_{x}(V(\omega)) = \lambda(V(\omega)) e^{i \int_{M} \Gamma(\zeta \omega_{\Sigma} + \overline{\zeta} \omega_{\Sigma})}$$

(33)
for all $\omega \in \mathcal{D}(M)$. Similar states, obtained by linear deformation of the vacuum state of a Fock representation of Weyl algebra, are known in the literature as coherent states. They were studied in [33] for photons in flat spacetime and in [22] (see also [32]). Several propositions presented in those work could be re-adapted to our case with some efforts. We think anyway that the shortest way consists of giving independent proofs based on more modern general results of local quantum physics [13] as the proofs of our propositions are not very complicated. Similar states for free QFT defined in globally hyperbolic spacetimes contain a bifurcate Killing horizon give rise to the failure of the uniqueness property proved in [17] (see the first footnote on p. 70 in [17]).

$\lambda_\zeta$ and its GNS triple $(\mathfrak{H}_\zeta, \Pi_\zeta, \Psi_\zeta)$ enjoy the remarkable properties stated in the theorems below.

**Theorem 4.1.** Fix a standard frame $\theta$ on $S^1$ of $M = S^1 \times \Sigma$, define $\mathcal{D}$ as in (16) and the group of *-automorphisms $\alpha$ representing $\text{PSL}(2, \mathbb{R})$ as in 3.1, $\{\alpha_t^{(\zeta)}\}_{t \in \mathbb{R}}$ being any one-parameter subgroup associated with the vector field $X$. If $\zeta \in L^2(\Sigma, \omega_\Sigma)$ and $\lambda_\zeta$ is the state defined in (33) with GNS triple $(\mathfrak{H}_\zeta, \Pi_\zeta, \Psi_\zeta)$, the following holds:

(a) The map $V(\omega) \mapsto V(\omega) e^{i \int_M \Gamma(\zeta \omega_\Sigma + \overline{\zeta \omega_\Sigma})}$, $\omega \in \mathcal{D}(M)$, uniquely extends to a *-automorphism $\gamma_\zeta$ on $\mathcal{W}(M)$ and

$$\lambda_\zeta(w) = \lambda(\gamma_\zeta w), \quad \text{for all } w \in \mathcal{W}(M),$$

$$\gamma_\zeta \circ \alpha_t^{(\zeta)} = \alpha_t^{(\zeta)} \circ \gamma_\zeta, \quad \text{for all } t \in \mathbb{R},$$

(b) (i) $\lambda_\zeta$ is pure, (ii) if $\zeta \neq \zeta'$ a.e., $\lambda_\zeta$ and $\lambda_{\zeta'}$ are not quasiequivalent, (iii) $\lambda_\zeta$ is invariant under $\{\alpha_t^{(\zeta)}\}_{t \in \mathbb{R}}$, but it is not under any other one-parameter subgroup of $\alpha$ (barring those associated with $cD$ for $c \in \mathbb{R}$ constant) when $\zeta \neq 0$ almost everywhere.

(c) $\mathfrak{H}_\zeta$ identifies with a Fock space $\mathfrak{F}_+(\mathfrak{H}_\zeta)$ with vacuum vector $\Psi_\zeta$ and, for all $\omega \in \mathcal{D}(M)$,

$$\Pi_\zeta : V(\omega) \mapsto \hat{V}_\zeta(\omega) := e^{i \hat{\phi}_\zeta(\omega)}, \quad \text{where} \quad \hat{\phi}_\zeta(\omega) := \hat{\phi}_0(\omega) + \{ \int_M \Gamma(\zeta \omega_\Sigma + \overline{\zeta \omega_\Sigma}) \} I,$$

$\hat{\phi}_0(\omega)$ being here the standard field operator in the Fock space $\mathfrak{F}_+(\mathfrak{H}_\zeta)$ as in 2.4.

(d) There is a strongly continuous one-parameter group of unitary operators $\{U_t^{(\zeta)}(\omega)\}_{t \in \mathbb{R}}$ with

$$\alpha_t^{(\zeta)}(w) = U_t^{(\zeta)}(w) U_t^{(\zeta)}(\omega) \quad \text{for all } t \in \mathbb{R} \text{ and } w \in \mathcal{W}_\zeta(M) := \Pi_\zeta(\mathcal{W}(M)).$$

Moreover (the derivative is performed in the strong sense where it exists)

$$\frac{d}{dt} |_{t=0} U_t^{(\zeta)}(\omega) = \frac{-i}{2} \Omega(\hat{\phi}_0, D\hat{\phi}_0).$$

The proof is in the appendix.
Theorem 4.2. In the hypotheses of theorem 4.1 the following holds for net of von Neumann algebras

\[ \mathcal{A}_\xi = \{ \mathcal{A}_\xi(I) \mid I \in \mathcal{I}\}, \quad \text{with} \quad \mathcal{A}_\xi(I) = \{ \hat{V}_\xi(\omega) \mid \text{supp} \, \omega \subset I \times \Sigma \}''. \quad (39) \]

(a) \( \mathcal{A}_\xi \supset \mathcal{W}_\xi(M) \) and it enjoys the following properties: (i) isotony, (ii) locality, (iii) \( \{ \exp(tD)\}_{t \in \mathbb{R}} \)-covariance, (iv) \( U^{(\mathcal{D})}\)-invariance and uniqueness of the vacuum \( \Psi_\xi \), (v) Reeh-Schlieder, (vi) Haag duality, (vii) factoriality, (viii) irreducibility, (ix) additivity.

(b) If \( \xi \neq 0 \) a.e., \( \mathcal{A}_\xi(\mathbb{F}_+) := \mathcal{A}_\xi((0, \pi)) \) and \( \mathcal{A}_\xi(\mathbb{F}_-) := \mathcal{A}_\xi((-\pi, 0)) \) are the unique \( \{ U^{(\mathcal{D})}_t \}_{t \in \mathbb{R}} \)-invariant algebras in \( \mathcal{A}_\xi \).

(c) If \( \Delta \) is the modular operator associated with \( \mathcal{A}_\xi(\mathbb{F}_+) \) then

\[ \Delta^{it} = U^{(\mathcal{D})}_\xi(2\pi t), \quad \text{for all} \quad t \in \mathbb{R}. \quad (40) \]

Thus \( \lambda_\xi \) is a KMS state on \( \mathcal{A}_\xi(\mathbb{F}_+) \) with temperature \( T = 1/2\pi \), with respect to \( \{ \alpha_t^{(-\mathcal{D})} \}_{t \in \mathbb{R}} \) (extended to \( \sigma \)-weak one-parameter group of \( * \)-automorphisms of \( \mathcal{A}_\xi(\mathbb{F}_+) \) through (37)).

Proof. (a) and (c) Since the difference between \( \hat{V}_\xi(\omega) \) and \( e^{i\overline{\phi_0(\omega)}} \) amounts to a phase only, each algebra \( \mathcal{A}_\xi(I) \) of \( \mathcal{A}_\xi \) coincides with the analog constructed starting from operators \( e^{i\overline{\phi_0(\omega)}} \) and using the same \( I \in \mathcal{I} \). Hence theorem 3.4 and subsequent remark 2 hold using the field \( \phi_0 \), replacing \( \Psi \) with \( \Psi_\xi \) and employing the representation \( U \) of \( PSL(2, \mathbb{R}) \) which leaves \( \Psi_\xi \) unchanged. Notice that \( U \) does not implement \( \alpha_0 \! \). In this way all the properties cited in the thesis turn out to be automatically proved with the exception of (iii) and (iv). However using (35), (38) and (d) of theorem 3.2 also those properties can be immediately proved. The proof of (c) is straightforward. \( \mathcal{A}_\xi((0, \pi)) \) coincides with the analog constructed starting from operators \( e^{i\overline{\phi_0(\omega)}} \). In that case the thesis holds with respect to the subgroup of \( U, e^{t\Omega(\phi_0; D(\phi_0))}/2 \) (remark (2) after theorem 3.4). Now (38) implies the validity of the thesis in our case.

(b) Since \( \mathcal{D} \) admits the only zeros at \( \theta = 0 \) and \( \theta = \pi \equiv -\pi \), the only open nonempty and nondense intervals of \( S^1 \) which are invariant under the one-parameter group \( \{ g_{\xi}^{t(\mathcal{D})}\}_{t \in \mathbb{R}} \) generated by \( \mathcal{D} \) are \( (0, \pi) \) and \( (-\pi, 0) \). \( \mathcal{D} \)-covariance reads \( U^{(\mathcal{D})}_\xi(t) \mathcal{A}_\xi(I) U^{(\mathcal{D})\dagger}_\xi(t) = \mathcal{A}_\xi(g_{\xi}^{t(\mathcal{D})}(I)) \) and thus \( \mathcal{A}_\xi((0, \pi)) \) and \( \mathcal{A}_\xi((-\pi, 0)) \) are invariant under \( \{ U^{(\mathcal{D})}_\xi(t) \}_{t \in \mathbb{R}} \). Let us prove their uniqueness. Consider the case of \( I = (a, b) \) with \( 0 \leq a < b < \pi \). There are \( t' > 0 \) and \( a' > 0 \), with \( a' < b \) and such that \( g_{\xi}^{t(\mathcal{D})}(a', b) \cap (a, b) = \emptyset \). Therefore, by locality it holds \( [U^{(\mathcal{D})}_\xi(t') \mathcal{A}_\xi((a', b)) U^{(\mathcal{D})\dagger}_\xi(t'), \mathcal{A}_\xi((a, b)) ] = 0 \), i.e. \( [\mathcal{A}_\xi((a', b)), U^{(\mathcal{D})}_\xi(-t') \mathcal{A}_\xi((a, b)) U^{(\mathcal{D})\dagger}_\xi(-t')] = 0 \). If \( \mathcal{A}_\xi((a, b)) \) were invariant under \( \{ U^{(\mathcal{D})}_\xi(t) \}_{t \in \mathbb{R}} \), the latter identity above would imply that \( [\mathcal{A}_\xi((a', b)), \mathcal{A}_\xi((a, b))] = 0 \), and thus in particular \( \mathcal{A}_\xi((a', b)) \subset \mathcal{A}_\xi((a', b))' \) which is trivially false because elements \( \hat{V}_\xi(\omega) \in \mathcal{A}_\xi((a', b)) \) generally do not commute. All the remaining cases can be reduced to that studied above with obvious adaptations. \( \square \)

Remarks.

(1) (c) in the last theorem is valid also replacing \( \mathbb{F}_- \) for \( \mathbb{F}_+ \) and \( \mathcal{D} \) for \( -\mathcal{D} \) as well. Theorems
4.1 and 4.2 hold in particular for $\Sigma = S^2$ and $M = S^1 \times S^2$. In that case one finds easily that: 

\[ \lambda_\zeta \text{ is invariant under the group of } \ast \text{-autormorphisms induced by the action of } SO(3) \text{ as isometry group on } S^2 \text{ if and only if } \zeta \text{ is constant a.e. on } S^2. \]

Generic $\Sigma$ do not admit $SO(3)$ as group of isometries, in that case $\lambda_\zeta$ is invariant under the relevant isometry group of $\Sigma$ provided $\zeta$ is so. Finally we notice that the hypotheses $\zeta \in L^2(\Sigma, \omega_\Sigma)$ can be relaxed in $\zeta \in L^1_{\text{loc}}(\Sigma, \omega_\Sigma)$ (the space of locally integrable functions on $\Sigma$ with respect to $\omega_\Sigma$) both in the theorems 4.1 and 4.2, the only result that could fail to hold is (ii) in (b) of the theorem 4.1.

(2) The theorems 4.1 and 4.2 refer to the pair of segments $(0, \pi)$ and $(-\pi, 0)$ in the circle realized as the segment $[-\pi, \pi]$ with $-\pi \equiv \pi$. From a physical point of view there is no way to distinguish between the pair of regions $(0, \pi), (-\pi, 0)$ and any other pair of open nonempty segments $I, J \subset S^1$ such that $J = \text{int}(S^1 \setminus I)$. This is because there is no way to measure segments on $S^1$ as the metric is degenerate therein. In fact the theorem can be stated for any pair of such segments. To prove it we notice that there exists a Möbius diffeomorphism $g : S^1 \to S^1$ with $I = g((0, \pi))$ and $J = g((\pi, 0))$. Hence, theorems 4.1 and 4.2 can be re-stated replacing $(0, \pi)$ and $(-\pi, 0)$ with, respectively $I$ and $J$, replacing the state (33) with the state and assuming to have fixed some $\zeta \in L^2(\Sigma, \omega_\Sigma)$,

\[ \lambda_I(V(\omega)) := \lambda(V(\omega)) e^{\Gamma_I(\omega)}, \quad \text{with } \Gamma_I(\omega) := \int_{M_I} \Gamma(\zeta g^* \omega_+ + \zeta^* g^* \omega_+) \]

and replacing $D$ with the generator $D_I$ of the one-parameter subgroup of $PSL(2, \mathbb{R}) \ni t \mapsto \exp(tD_I) := g \circ \exp(tD) \circ g^{-1}$ which leaves invariant $I$ and $J$ ($D_I$ does not depend on the choice of $g$).

Notice also that if $I, J$ is a pair of segments as said above and $h$ is any Möbius transformation, $h(I), h(J)$ still is a pair of open nonempty segments with $h(J) = \text{int}(S^1 \setminus h(I))$ and it holds (using also lemma 3.1)

\[ \lambda_{h(I)}(V(\omega)) = \lambda_I(V(h^* \omega)) . \]

This fact means that the $PSL(2, \mathbb{R})$ symmetry, broken at Hilbert-space level, is restored at algebraic level by considering the whole class of states $\lambda_I$. The residual Virasoro representation after breaking $PSL(2, \mathbb{R})$ symmetry is analyzed in the Appendix.

(3) Considering again the particular case of the Kruskal manifold, the requirement (31), that is 

\[ - \sin \theta \partial_\theta = -\kappa^{-1} \xi \big|_X, \]

fixes the standard frame only up to a coordinate transformation $\theta' = \theta'(\theta)$, where $\theta'$ being any other positive oriented coordinate frame on $S^1$ satisfying $\sin \theta' \partial_{\theta'} = \sin \theta \partial_\theta$. Since our construction of quantum field theory on $M$ relies upon the choice of a standard frame on $S^1$, a natural question is: Are quantum field theories based on $\lambda_\zeta$ and its analog $\lambda'_\zeta$ with obvious notation, unitarily equivalent? (Notice that $\zeta$ is the same for both states). The answer is strongly positive because of the following general result.

\[ \text{Assume that, in coordinates } \theta, I \text{ has length equal or shorter than } J. \text{ The diffeomorphism } g^{-1} \text{ is the composition of a rigid rotation generated by } X \text{ which maps the center of } I \text{ in } 0, \text{ a dilatation generated by } D \text{ which enlarges the transformed } I \text{ up to } (-\pi/2, \pi/2) \text{ and another anti-clockwise rigid rotation of } \pi/2. \]
Theorem 4.3. With the same hypotheses as in theorem 4.1, let $\theta'$ be another standard frame on $S^1$. Referring to the coordinate frame $\theta'$, let $D'$ be the vector field analog of $D$ and let $\lambda'_\zeta$ be the state analog of $\lambda_\zeta$ (both states defined on $\mathcal{W}(\mathbb{M})$). If $D' = D$ then, for any $\zeta \in L^2(\Sigma, \omega_\Sigma)$,

$$\lambda'_\zeta = \lambda_\zeta.$$

(41)

Proof. In our hypotheses $\theta'(\theta) = 2\tan^{-1}(e^c\tan(\theta/2))$ for some $c \in \mathbb{R}$. The transformation $\theta \to \theta'(\theta)$ interpreted as an active diffeomorphism is nothing but the action of the element $\alpha_{-c}^{(D)}$ of the one-parameter group generated by $D$. Since $\lambda_\zeta$ is invariant under that group ((b) in theorem 4.1) the thesis is true. □

5 Towards physical interpretations.

Consider the case of $\mathbb{M}$ constructed by the future Killing horizon of Kruskal manifold (however theorem 5.1 below holds true for a generic degenerate manifold $\mathbb{M} = S^1 \times \Sigma$). As is well known the complete maximal Kruskal solution of Einstein equation describes a spacetime with an eternal pair of balck hole - white hole. However, some features (e.g. Hawking radiation) of real black holes produced by collapse can be modelled by using the right Schwarzschild wedge and the region containing the future singularity in Kruskal manifold, the region about $\mathbb{F}_+$ (see [37, 38]) in particular. $\mathbb{F}_+$ itself can be considered as (an extension of the) actual event horizon of a physical black hole. The spacetime of a physical black hole obtained by stellar collapse has no white hole neither Killing bifurcate horizon. Nevertheless, in the sense stated below a physical black hole will asymptotically approach such a spacetime (at least a spacetime including a bifurcate Killing horizon). Indeed, in [31] Racz and Wald considered a globally hyperbolic, stationary spacetime containing a black hole but no white hole, assuming, further, that the event horizon $E$ of the black hole is a Killing horizon with compact cross-sections. With those hypotheses they proved that, if surface gravity is non-zero and constant throughout the horizon, one can globally extend the initial spacetime so that the image of $E$ is a proper subset of a regular bifurcate Killing horizon in the enlarged spacetime. In that paper they also provided necessary and sufficient conditions for the extendibility of matter fields to the enlarged spacetime. These results support the view that any spacetime representing the asymptotic final state of a black hole formed by gravitational collapse may be assumed to possess a bifurcate Killing horizon (see [31] for details). Therefore, from a physical point of view, it is worth investigating the physical meaning for the theory referred to the GNS representation of $\lambda_\zeta$ when restricting to the region $\mathbb{F}_+$.

5.1. Extremal KMS states: Existence of different thermodynamical phases. By construction $\lambda_\zeta$ are KMS states on the $C^*$-algebra $\mathcal{W}(\mathbb{F}_+)$, the Weyl algebra generated by Weyl operators $V(\omega)$ with $\text{supp}\ \omega \subset \mathbb{F}_+$ which is contained in $A_\zeta(\mathbb{F}_+)$. As states on $\mathcal{W}(\mathbb{F}_+)$, $\lambda_\zeta$ and $\lambda'_{\zeta'}$ can be compared also if $\zeta \neq \zeta'$ (they do not belong to a common folium if (ii) in (b) of theorem 4.1 holds, so they cannot be compared on a common von Neumann algebra of observables in that case). The next theorem, valid for the general case $\mathbb{M} = S^1 \times \Sigma$, shows that $\{\lambda_\zeta\}_{\zeta \in L^2(\Sigma, \omega_\Sigma)}$ is a
family of extremal states in the convex space of KMS states over $\mathcal{W}(\mathbb{F}_+)$ at inverse temperature $2\pi$ with respect to $-\mathcal{D}$.

**Theorem 5.1.** With the same hypotheses as in theorem 4.1 the following holds.

(a) Any state $\lambda_\zeta$ (with $\zeta \in L^2(\Sigma, \omega_\Sigma)$) defines an extremal states in the convex set of KMS states on the $C^*$-algebra $\mathcal{W}(\mathbb{F}_+)$ at inverse temperature $2\pi$ with respect to $\{\alpha_t^{(-\mathcal{D})}\}_{t \in \mathbb{R}}$.

(b) Different choices of $\zeta$ individuate different states on $\mathcal{W}(\mathbb{F}_+)$ which are not unitarily equivalent as well.

*Proof.* Let $(\Omega_\zeta, \Pi_\zeta, \Psi_\zeta)$ be the GNS representations of $\lambda_\zeta$. The GNS representations of $\lambda_\zeta\big|_{\mathcal{W}(\mathbb{F}_+)}$ must be (up to unitary equivalences) $(\Omega_\zeta, \Pi_\zeta|_{\mathcal{W}(\mathbb{F}_+)}, \Psi_\zeta)$ due to Reeh-Schlieder property ((a) in theorem 4.2) of $A_\zeta(\mathbb{F}_+)$. Since $A_\zeta(\mathbb{F}_+) = \Pi_\zeta|_{\mathcal{W}(\mathbb{F}_+)}$ is a (type $\text{III}_1$) factor, the state $\lambda_\zeta|_{\mathcal{W}(\mathbb{F}_+)}$ - namely $\Pi_\zeta|_{\mathcal{W}(\mathbb{F}_+)}$ - is primary (see III.2.2 in [13]). As a consequence, by theorem 1.5.1 in [13], the KMS state $\lambda_\zeta|_{\mathcal{W}(\mathbb{F}_+)}$ is extremal in the space of KMS states on $\mathcal{W}(\mathbb{F}_+)$ with respect to $\alpha_t^{(-\mathcal{D})}$ at the temperature of $\lambda_\zeta|_{\mathcal{W}(\mathbb{F}_+)}$ itself. Obviously $\lambda_\zeta|_{\mathcal{W}(\mathbb{F}_+)} \neq \lambda_{\zeta'}|_{\mathcal{W}(\mathbb{F}_+)}$ because, if $\zeta - \zeta'$ is not zero almost everywhere, the integrals in the exponentials defining $\lambda_\zeta$ and $\lambda_{\zeta'}$ produce different results when applied to $V(\omega)$ with $\text{supp} \omega \subset \mathbb{F}_+$ with a suitable choice of $\omega$. The proof of non equivalence is the same as done (see the appendix) for the states defined in the whole von Neumann algebras. □

The natural interpretation of this fact is that the states $\lambda_\zeta$, restricted to the observables in the physical region $\mathbb{F}_+$, are nothing but different thermodynamical phases of the same system at the Hawking temperature (see V.1.5 in [13]).

5.2. Bose-Einstein condensate and states $\lambda_\zeta$ with $\zeta$ real. In the following we assume that $\zeta$ is real. Let us examine some features of the generators $\hat{\phi}_\zeta$ of the Weyl representation associated with $\lambda_\zeta$ when restricted to the physical region $\mathbb{F}_+$. Consider $\omega \in \mathcal{D}(\mathcal{M})$ such that $\text{supp} \omega \subset \mathbb{F}_+$ and such that $\omega(v, s)$ can be rewritten as $\frac{d\psi}{dv}(v, s) \omega_{\Sigma} \wedge \omega_{\Sigma}$ where $\psi$ is smooth and compactly supported in $\mathbb{F}_+$. Similar “wavefunctions” $\psi$ have been considered in [26] building up scalar QFT on a Killing horizon ($\mathbb{F}_+$ in our case). Using (32) we can write the formal expansion

$$
\hat{\phi}_\zeta(\omega) = \int_{\mathbb{F}_+} \hat{\phi}_0(\theta_+(v)) \omega(v, s) + \int_{\mathbb{F}_+} \zeta(s) v \omega(v, s) . \tag{42}
$$

In terms of wavefunctions, if $\Omega_{\mathbb{F}_+}$ is the restriction of the right-hand side of the definition of $\Omega$ given in (1) to real smooth functions compactly supported in $\mathbb{F}_+$, it holds

$$
\Omega(\psi, \hat{\phi}_\zeta) = \Omega_{\mathbb{F}_+}(\psi, \hat{\phi}_0) - \int_{\Sigma} \left( \int_{-\infty}^{+\infty} \psi(v, s)dv \right) \zeta(s) \omega_{\Sigma}(s) . \tag{43}
$$

The group of elements $e^{itH_\zeta} := U_\zeta^{(-\mathcal{D})}(t), t \in \mathbb{R}$ generates displacements $v \mapsto v - t$ in the variable $v$ in the argument of the wavefunctions $\psi$, since $v$ is just the parameter of the integral curves
of \(-D\) which takes the form \(\frac{\partial}{\partial v}\) in \(F_+\). Using Fourier transformation with respect to \(v\) we can write down

\[
\psi(v, s) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^+} dE \, \tilde{\psi}(E, s) e^{-iEv} + \tilde{\psi}(E, s) e^{iEv}.
\]  \quad (44)

In heuristic sense \(H_\zeta\) acts on the wavefunctions \(\psi\) as the multiplicative operator \(\tilde{\psi}(E, s) \mapsto E\tilde{\psi}(E, s)\). Physically speaking, thermal properties of \(\lambda_\zeta\) are referred just to the energy notion associated with that Hamiltonian. Actually, as is well-known, this interpretation must be handled with great care: the interpretation of \(\tilde{\psi} as a representative of a one-particle quantum state can be done in a Fock space whose vacuum state does not coincide with the KMS state \(\lambda_\zeta\) (see V.1.4 and the discussion in p. 219 of [13]). Using (44), (43) can be re-written as

\[
\Omega(\psi, \hat{\phi}_\zeta) = \Omega_{F_+}(\psi, \hat{\phi}_0) - \sqrt{2\pi} \int_{\Sigma} \zeta(s) \tilde{\psi}(0, s) + \omega_\Sigma(s).
\]  \quad (45)

From (45) it is apparent that \(\hat{\phi}_\zeta\) gets contributions from zero-energy modes \((E = 0)\) as it happens in Bose-Einstein condensate. To this end see chapter 6 of [30] and 5.2.5 of [2], especially p. 72, where in the decomposition of the KMS state \(\omega\) (after the thermodynamical limit) in \(\mathbb{R}^\nu\)-ergodic states, the mathematical structure of the latter states resemble that of the states \(\lambda_\zeta\). The decomposition of the field operator (42) into a “quantum” (with vanishing expectation value) and a “classical” (i.e. commuting with all the elements of the algebra) part is typical of the theoretical description of a boson system containing a Bose-Einstein condensate; the classical part plays the role of an order parameter [6, 30].

Let us focus attention on the generator of \(U(t)^{-D}(\tau) = e^{itH_\zeta}\) in the representation of a state \(\lambda_\zeta\). Using theorem 4.1 we find (both sides are supposed to be restricted to the core \(F(3H_\zeta)\))

\[
H_\zeta = \int_M \sin(\theta) : \frac{\partial \hat{\phi}_\zeta}{\partial \theta} \frac{\partial \hat{\phi}_\zeta}{\partial \theta} : (\theta, s) d\theta \wedge \omega_\Sigma(s).
\]

Indeed, if \(\theta_{\pm}(v) = \pm 2\tan^{-1}(e^v)\) are the inverse functions of \(v = \Gamma(\theta)\) in \(F_+\) and \(F_-\) respectively, passing from coordinates \((\theta, s)\) to coordinates \((v, s)\) and employing the field \(\hat{\phi}_0\) the right-hand side of the formula above can be rearranged as

\[
H_\zeta = \lim_{N \to +\infty} \left\{ \int_{F_+} \chi_N(v) \frac{\partial \hat{\phi}_0}{\partial v} \frac{\partial \hat{\phi}_0}{\partial v} : (\theta_+(v), s) dv \wedge \omega_\Sigma(s) + ||\zeta||^2 \int_{\mathbb{R}} \chi_N(v) dv 
\right.
\]

\[
- \int_{F_-} \chi_N(v) \frac{\partial \hat{\phi}_0}{\partial v} \frac{\partial \hat{\phi}_0}{\partial v} : (\theta_-(v), s) dv \wedge \omega_\Sigma(s) - ||\zeta||^2 \int_{\mathbb{R}} \chi_N(v) dv \right\},
\]

where the function \(\chi_N\) is smooth with compact support in \([-N, N]\) and becomes the constant function 1 for \(N \to +\infty\). The two constant terms in brackets cancel out each other, they having the opposite sign, and the final form of \(H_\zeta\) is just that in (d) of theorem 4.1. The normal
ordering prescription used in the integrals is defined by subtracting \( (\Psi, \hat{\phi}_0(\theta', s') \hat{\phi}_0(\theta, s) \Psi) \) before applying derivatives and then smoothing with a product of delta in \( \theta, \theta' \) and \( s, s' \). We do not enter into mathematical details here which are quite standard procedures of applied microlocal analysis similar to that used in Hadamard regularization [3, 14, 24].

From the decomposition of \( H_\zeta \) written above, we see that it is made of two contributions \( H_\zeta^{(-)} \), \( H_\zeta^{(+)} \) respectively localized at the two disjoint regions of \( \mathbb{R} \), \( \mathbb{R}^- \) and \( \mathbb{R}^+ \). The two terms have the same value with opposite sign as one expects from the indefiniteness of the self-adjoint generator \( D \) (corresponding to the fact that the Killing vector \( -\mathcal{D} \) changes orientation passing from \( \mathbb{R}^+ \) to \( \mathbb{R}^- \)). Let us concentrate on the second term in the contribution \( H_\zeta^{(+)} \) to \( H_\zeta \) due to \( \mathbb{R}^+ \). It is a volume divergence

\[
\lambda_\zeta(H_\zeta^{(+)}) = E_\zeta := ||\zeta||^2 \int_{\mathbb{R}} dv.
\]

This can be interpreted as the energy of the BE condensate localized at \( \mathbb{R}^+ \) whose density is finite and amounts to \( ||\zeta||^2 \).

5.3. Conclusions: Can the condensate describe physical properties of a black hole? Here, to conclude, we try to give some hints to relate the properties of the condensate with spacetime, i.e. Schwarzschild black hole, properties. To do it we start from a deeper point of view. The only difference between two different Schwarzschild black holes concerns their masses, that is their Schwarzschild radii. Since we want to ascribe this difference to a feature of a state, the background and the system supporting the state must be independent from the black-hole radius. In this way the states \( \lambda_\zeta \) have to be referred to a quantum field theory on an abstract manifold \( \mathbb{M} = S^1 \times S^2 \) with a metric on \( S^2 \) which does not coincide with the actual metric of a particular black hole. We assume the hypothesis of spherical symmetry so that the metric on \( S^2 \) is determined by fixing the value of an adimensional parameter only (the radius rate for instance). In this view a state \( \lambda_\zeta \) on the scalar field \( \hat{\phi} \) must fix the geometry of the black hole under the constraints of the presence of a Killing horizon and spherical symmetry. Since we are in fact dealing with quantum gravity we adopt natural Planck units \((h = c = G = 1)\) so that we can employ pure numbers in the following. In particular, the pure number defining the radius of \( S^2 \) will be denoted by \( r_0 \).

The idea that the assignment of a (classical) scalar field fixes the metric of a spacetime (solution of Einstein equations) when other constraints are given on the metric is not new, the so-called dimensional-reduction theory for gravitation leads to such a scenario (e.g. see [29] with cited references) where the scalar field is related to the dilaton field. Now we adopt a similar point of view but, in addition, we assume also that the assignment of the configuration of the scalar field is due to the assignment of a quantum state of that field. Let us see how this idea can be implemented from the following remark.

Spherical symmetry implies that \( \zeta \) must be constant on \( S^2 \) (see remark 1 after theorem 4.2). Since the considered states are coherent the field admits a nonvanishing averaged value. Formally
it holds
\[ \lambda_\zeta(\hat{\phi}_\zeta(\theta, s)) = v_\zeta(\theta). \] (46)

(See remarks below). Hence the mean value of \( \hat{\phi}_\zeta \) with respect to \( \lambda_\zeta \) picks out a preferred coordinate frame along the light lines of \( \mathbb{F}_+ \). So, up to the choice of the origin, the mean value of the field \( \hat{\phi}_\zeta \) defines a preferred coordinate \( v_\zeta \) in the physical region \( \mathbb{F}_+ \). Now the natural hypotheses is that \( v_\zeta \) is the parameter of the Killing field \( \xi|_{\mathbb{F}_+} \) of the considered black hole as in 4.1. In other words we are saying that \( \zeta \) determines a black hole in the class of Schwarzschild ones by determining its surface gravity through the identity (both sides are pure numbers since we are employing natural Planck units):
\[ \zeta = \kappa^{-1}. \] (47)

Such a black hole must have horizon surface \( S_\zeta = \pi \zeta^2 \). As a consequence we find that
\[ ||\zeta||^2 = 4\pi\zeta^2 r_0^2 \] (48)
scales as the actual surface of the black hole horizon (and it is exactly the measure of the surface provided \( r_0 = 1/2 \)). This provides some clues for an interpretation of \( ||\zeta||^2 \) that is, equivalently, the density of energy of the condensate \( E_\zeta \int_\mathbb{R} dv \).

**Remarks** A pair of mathematical remarks are necessary interpret (46).

(1) \( \lambda_\zeta(\hat{\phi}_\zeta(\theta, s)) \) is not well defined and it could be thought as the weak limit of a sequence \( \lambda_\zeta(\hat{\phi}_\zeta(\omega_n)) \) where the forms \( \omega_n \) regularize Dirac’ s delta centered in \( (\theta, s) \in \mathbb{F} \).

(2) Furthermore, one has to take into account that the allowable forms have the shape \( \omega_n(\theta, s) = \frac{\partial f_n(\theta, s)}{\partial \theta} d\theta \wedge \omega_\Sigma \) where \( f_n \) is periodic in \( \theta \). It is not possible to produce a regularization sequence for \( \delta(s, s') \frac{\partial \delta(\theta' - \theta)}{\partial \theta} d\theta \wedge \omega_\Sigma \) in this way due to the periodic constraint. The drawback can easily be skipped by fixing an origin \( v_{\zeta 0} \) for \( v_\zeta \) (corresponding to some \( \theta_0 \)) for the coordinate \( x \). In other words one considers a sequence of forms \( \omega_n^{(\theta, s)} \) induced by smooth \( \theta \)-periodic functions \( f_n(\theta, s) = \delta_n(s' - s) [\Theta_n(\theta - \theta') + \Theta_n(\theta' - \theta_0)] \), where \( \{\delta_n(s')\} \) regularize \( \delta(s') \) and \( \{\Theta_n(\theta')\} \) regularize the step distribution whose derivative is just \( \delta(\theta') \). In this sense
\[ \lim_{n \to +\infty} \lambda_\zeta(\hat{\phi}_\zeta(\omega_n^{(\theta, s)})) = v_\zeta(\theta) - v_{\zeta 0}. \]

The presented results could lead to an interesting scenario which deserves future investigation. The Kruskal spacetime could be a classical object arising by spontaneous breaking of \( SL(2, \mathbb{R}) \) symmetry as well as Bose-Einstein condensation due to a state of a local QFT defined on a certain conformal net. In particular the abstract field operator \( \phi \) can be seen as a noncommutative coordinate on \( \mathbb{F}_+ \). (Obviously noncommutativity arises from canonical commutation relations \( [\phi(\theta, s), \phi(\theta', s')] = iE(\theta, s, \theta', s'). \)) Commutativity is restored under the choice of an appropriate coherent state on that \( * \)-algebra considering the averaged values of the field. This
A.2. Residual Virasoro representation after breaking $\lambda$ is irreducible, (up to unitary transformations) the GNS triple associated with $W$ that $\Pi(F)$ satisfies also (49), the uniqueness of the GNS triple proves that the triple $(\hat{F}, \Pi, \Psi)$ is the GNS triple associated with a particular pure algebraic state $\lambda$ (quasifree [1, 17] and invariant under the automorphism group associated with $\partial_\theta$) on $\mathcal{W}(M)$ we go to introduce. Define
\[
\lambda(W(\psi)) := e^{-\langle \psi_+, \psi_+ \rangle / 2}
\]
than extend $\lambda$ to the $^*$-algebra finitely generated by all the elements $W(\psi)$ with $\psi \in S(M)$, by linearity and using (W1), (W2). It is simply proved that, $\lambda(1) = 1$ and $\lambda(a^*a) \geq 0$ for every element $a$ of that $^*$-algebra so that $\lambda$ is a state. As the map $\mathbb{R} \ni t \mapsto \lambda(W(t\psi))$ is continuous, known theorems [19] imply that $\lambda$ extends uniquely to a state $\lambda$ on the complete Weyl algebra $\mathcal{W}(M)$. On the other hand, by direct computation, one finds that $\lambda(W(\psi)) = \langle \Psi, \hat{W}(\psi)\Psi \rangle$. Since a state on a $C^*$-algebra is continuous, this relation can be extended to the whole algebras by linearity and continuity and using (W1), (W2) so that a general GNS relation is verified:
\[
\lambda(a) = \langle \Psi, \Pi(a)\Psi \rangle \quad \text{for all } a \in \mathcal{W}(M) .
\tag{49}
\]
To conclude, it is sufficient to show that $\Psi$ is cyclic with respect to $\Pi$. Let us show it. If $\hat{\mathcal{F}}(M)$ denotes the $^*$-algebra generated by field operators $\Omega(\psi, \phi)$, $\psi \in S(M)$, defined on $F(\mathcal{H})$, $\hat{\mathcal{F}}(M)\Psi$ is dense in the Fock space (see proposition 5.2.3 in [2]). Let $\Phi \in \hat{\mathcal{F}}(M)$ be a vector orthogonal to both $\Psi$ and to all the vectors $\hat{W}(t_1 \psi_1) \cdots \hat{W}(t_n \psi_n)\Psi$ for $n = 1, 2, \ldots$ and $t_i \in \mathbb{R}$ and $\psi_i \in S(M)$. Using Stone theorem to differentiate in $t_i$ for $t_i = 0$, starting from $i = n$ and proceeding backwardly up to $i = 1$, one finds that $\Phi$ must also be orthogonal to all of the vectors $\Omega(\psi_1, \phi) \cdots \Omega(\psi_n, \phi)\Psi$ and thus vanishes because $\hat{\mathcal{F}}(M)\Psi$ is dense. This result means that $\Pi(\mathcal{W}(M))\Psi$ is dense in the Fock space too, i.e. $\Psi$ is cyclic with respect to $\Pi$. Since $\Psi$ satisfies also (49), the uniqueness of the GNS triple proves that the triple $(\hat{\mathcal{F}}(M), \Pi, \Psi)$ is just (up to unitary transformations) the GNS triple associated with $\lambda$. Since the Fock representation is irreducible, $\lambda$ is pure.

A.2. Residual Virasoro representation after breaking $PSL(2, \mathbb{R})$ symmetry. The complex Lie algebra $(\mathfrak{a}, \{\cdot, \cdot\}, i)$ of vector field on $\mathbb{S}^1$ (see discussion in 3.1) is made of vector fields
on $\mathbb{S}^1$ whose diffeomorphism groups, generated by their real and imaginary parts, do not admit (in general) $\mathbb{F}_\pm$ as invariant regions, when extended to $\mathbb{M} = \mathbb{S}^1 \times \Sigma$. This happens in particular for generators $L_n = i e^{i n \theta} \partial_\theta$. However, it is possible to rearrange that basis in order to partially overcome the problem. Consider the equivalent basis of a made of the following real vector fields $-i L_0$, $\mathcal{E}_n := (1 - \cos((2n)\theta)) \partial_\theta$, $\mathcal{O}_n := (1 + \cos((2n + 1)\theta)) \partial_\theta$, $\mathcal{S}_n := -\sin(n\theta) \partial_\theta$ with $n = 1, 2, \ldots$. Barring $-i L_0$ and $\mathcal{O}_n$, the other fields admit $\mathbb{F}_\pm$ as invariant regions. Moreover the fields $\mathcal{S}_n$ define a Lie algebra with respect to the usual Lie bracket whereas $\mathcal{E}_n$, or $\mathcal{E}_n$ together $\mathcal{S}_n$, do not so. However allowing infinite linear combinations of vector fields – using for instance $L^2$-convergence for the components of vector fields with respect to $\partial_\theta$ (the same result hold anyway using stronger notions of convergence as uniform convergence of functions and their derivatives up to some order) – one sees that each $\mathcal{E}_n$ can be expanded as an infinite linear combination of $\mathcal{S}_n$. From these considerations one might expect, at least, that fields $\mathcal{E}_n$, but not the vectors $L_n$ and $\mathcal{O}_n$, admit some operator representation in $\mathcal{H}$ in terms of the field operator $\hat{\phi}_c$. In fact this is the case if $\zeta$ is a real function in $L^2(\Sigma, d\Sigma)$. If one tries to define operators $L^{(c)}_{\zeta}$ as in (23) with $\hat{\phi}_{c}(\zeta) := \hat{\phi}_{c}(\zeta) + \zeta \Gamma$, one immediately faces ill-definiteness of those operators due to infinite additive terms and the same problem arises for formal operators $0_{n}^{(c)} := L_{\zeta 0}^{(c)} + (L_{\zeta 2n+1}^{(c)} + L_{\zeta -2n-1}^{(c)})/2$ and also for $E_{n}^{(c)} := L_{\zeta 0}^{(c)} - (L_{\zeta 2n}^{(c)} + L_{\zeta -2n}^{(c)})/2$. However these terms cancel out if considering the operators $G_{n}^{(c)} := (L_{\zeta n}^{(c)} - L_{\zeta -n}^{(c)})/(2i)$ with $n = 1, 2, \ldots$, which are well defined and essentially selfadjoint on $F(\mathcal{H})$. Moreover, the operators $G_{n}^{(c)}$ define a Lie algebra with respect to the commutator. (Direct inspection shows that if $c = \infty$ none of the considered operators is well-defined on $F(\mathcal{H})$. It is plausible that operators $G_{n}^{(c)}$ define one-parameter groups which implement covariance with respect to analogous groups of diffeomorphisms generated by associated vector fields $\mathcal{S}_n$, and that the exponentiation of the algebra of $G_{n}^{(c)}$ produces a unitary representation of a (perhaps the) subgroup of $\text{Diff}^+(\mathbb{S}^1)$ of the diffeomorphisms which leaves $\mathbb{F}_\pm$ invariant. However, it is worth stressing that, barring the case $G_{1}^{(c)}$ which generates just $U^{(D)}_{\zeta}(t)$, $\Psi_{\zeta}$ is not invariant under the remaining unitary groups.

### A.3. Proofs of some theorems.

**Proof of Proposition 2.2.** Let $\theta$ be a standard frame on $\mathbb{S}^1$. Assume the condition (a) holds. We can write $\omega = \epsilon_f$ and $\omega' = \epsilon_{f'}$ for some functions $f, f' \in C^\infty(\mathbb{S}^1 \times \Sigma; \mathbb{C})$. To use these facts we notice that, in the general case, it holds $E(\epsilon_f, \epsilon_{f'}) = \Omega(f, f')/4$ by proposition 2.1. Therefore, by (V2), to conclude the proof it is sufficient to show that $\Omega(f, f') = 0$. Let us prove it. In our hypotheses $f'$ is constant in the variable $\theta$ in $I \times \Sigma$ since $\frac{\partial f'(\theta, s)}{\partial \theta} = 0$ therein and $I \times \Sigma$ is connected by paths with $s$ constant. Moreover, if $t, t'$ are the endpoints of $I$, it must hold $f(t, s) = f(t', s)$ for every $s \in \Sigma$. Indeed $\frac{\partial f(\theta, s)}{\partial \theta} = 0$ vanishes outside $I \times \Sigma$ – and thus $f$ is constant in $\theta$ in that set as before – and $f$ is periodic in $\theta$ at $s$ fixed by hypotheses. Integrating by parts in the right-hand side of the definition of $\Omega$ given in (1) with $f$ and $f'$ in place of $\psi$ and $\psi'$,

$$
\Omega(f, f') = 2 \int_{\Sigma} \omega_{\Sigma}(s) \int_{\mathbb{S}^1} f'(\theta, s) \frac{\partial f}{\partial \theta}(\theta, s) d\theta = 2 \int_{\Sigma} \omega_{\Sigma}(s) \int_{I} f'(\theta, s) \frac{\partial f}{\partial \theta}(\theta, s) d\theta .
$$

27
$f'$ is constant in $\theta$ in $I \times \Sigma$ and $f(t', s) = f(t, s)$, $t, t'$ being the extreme points of $I$, so that

$$\frac{1}{2} \Omega(f, f') = \int_I f'(\theta, s) \frac{\partial f}{\partial \theta}(\theta, s) d\theta = f'(s) \frac{\partial f}{\partial \theta}(\theta, s) d\theta = f'(s)(f(t', s) - f(t, s)) = 0.$$  

Now suppose that (b) holds true. In this case one has

$$i\Omega(f, f') = 2 \int_{\Sigma} \omega_{\Sigma}(s) \int_{S^1} f'(\theta, s) \frac{\partial f}{\partial \theta}(\theta, s) d\theta = 2 \int_{\Sigma} \omega_{\Sigma}(s) \int_{S^1} f'(\theta, s) \frac{\partial f}{\partial \theta}(\theta, s) d\theta.$$  

Since $\frac{\partial f}{\partial \theta}(\theta, s) = 0$ in the set $S^1 \times S$ which is connected by paths with $s$ constant, $f'$ does not depend on $\theta$ in that set and thus

$$\frac{1}{2} \Omega(f, f') = 2 \int_{\Sigma} \omega_{\Sigma}(s) \int_{S^1} f'(\theta, s) \frac{\partial f}{\partial \theta}(\theta, s) d\theta = 2 \int_{\Sigma} \omega_{\Sigma}(s) f'(s) d\theta = 0.$$  

Finally (W2) or equivalently (V2) entails the thesis. $\square$

**Proof of Theorem 3.2.** The operator $L := K^2 + S^2 + D^2$ is essentially selfadjoint on $F(\mathcal{H})$ since the dense invariant space $F(\mathcal{H})$ is made of analytic vectors. The proof is straightforward by direct estimation of $\|L^n \Psi\|$ with $\Psi \in F(\mathcal{H})$ (there is a constant $C_\Psi \geq 0$ with $\|L^n \Psi\| \leq C_\Psi^n$). As a consequence of some results by Nelson (Theo. 5.2, Cor. 9.1, Lem. 9.1 and Lem. 5.1 in [28]) the Hermitian operators $iR(x)$ with $x \in sl(2, \mathbb{R})$ are essentially selfadjoint on $F(\mathcal{H})$ and there is a unique strongly-continuous representation $SL(2, \mathbb{R}) \ni g \mapsto \mathcal{F}_+(\mathcal{H}) \rightarrow \mathcal{F}_+(\mathcal{H})$ such that (28) holds true.

(a) $k$ generates the one-parameter subgroup $S^1$ in $SL(2, \mathbb{R})$ – that is $\mathbb{R} \ni t \mapsto \exp(\imath t k)$ with period $4\pi$ – as well as the one-parameter subgroup $\mathbb{R} \ni t \mapsto l(t)$ isomorphic to $\mathbb{R}$ in $SL(2, \mathbb{R})$. From the general theory of $SL(2, \mathbb{R})$ representations, a representation $SL(2, \mathbb{R}) \ni g \mapsto V(g)$ is in fact a representation of $SL(2, \mathbb{R})$ if $t \mapsto V(l(t))$ has period $4\pi/k$ for some integer $k \neq 0$. It is simply proved that the operator $K$ is the tensorialization of the operator defined on $L^2(\mathbb{C}) \otimes L^2(\Sigma, \omega_\Sigma)$ by extending

$$\{C_n\}_{n=1,2,\ldots} \otimes u_j \mapsto \{nC_n\}_{n=1,2,\ldots} \otimes u_j$$

by linearity. As a consequence the spectrum of $\mathcal{K}$ is the set $\sigma(\mathcal{K}) = \{0, 1, 2, \ldots\}$ where the eigenspace with eigenvalue 0 is one-dimensional and it is generated by the vacuum state $\Psi$. This implies that $\mathbb{R} \ni t \mapsto e^{\imath t \mathcal{K}} = U(l(t))$ has period $2\pi$. As a first consequence $U$ is a proper representation of $SL(2, \mathbb{R})$. Furthermore, since $\sigma(\mathcal{K})$ is nonnegative, the representation is a positive-energy representation. Finally, notice that $-I = e^{2\pi \imath k}$ and thus $U(-I) = e^{2\pi \imath \mathcal{K}} = I$ and so $U$ is a representation of $PSL(2, \mathbb{R}) := SL(2, \mathbb{R})/\pm I$. 

(b) and (c). From direct inspection one sees that the operators $K, S, D$ are tensorializations of the respective operators $K|_{\mathcal{G}_{\Sigma}}, S|_{\mathcal{G}_{\Sigma}}, D|_{\mathcal{G}_{\Sigma}}$ in particular their restriction to the space generated by the vacuum vector coincide with the operator 0. Moreover, decomposing $\mathcal{H} = L^2(\mathbb{C}) \otimes L^2(\Sigma, \omega_\Sigma)$, one finds

$$K|_{\mathcal{G}_{\Sigma}} = K_0 \otimes 0, \quad S|_{\mathcal{G}_{\Sigma}} = S_0 \otimes 0, \quad D|_{\mathcal{G}_{\Sigma}} = D_0 \otimes 0,$$
where $K_0, S_0, D_0$ are obtained by restricting to the one-particle space the operators $K, S, D$ defined in the case $\mathbb{M} = S^1$ (without transverse manifold). Using again Nelson results these operators give rise to a representation $\overline{SL(2, \mathbb{R})} \ni g \mapsto V(g) \otimes I$ in $\mathcal{H}$. (This representation is, in fact, an irreducible representation of $SL(2, \mathbb{R})$, see [27].) By tensorialization this representation extends to a representation $U'$ in the whole Fock space. By construction, the generators $iK', iS', iD'$ of this representation ad associated with $k, s, d$ respectively coincides with $iK, iS, iD$ on $F(\mathcal{H})$ respectively. Nelson’s uniqueness property implies that $U' = U$. By construction $U'(=U')$ admits every space with finite number of particles as invariant space, including the space with zero particles spanned by the vacuum state.

(d) First of all, as said above, $U$ leaves invariant the space generated by the vacuum vector $\Psi$ so that it is an invariant vector up to a phase. Let us show that this is the only unit vector with this property. By (b), the operator $\overline{D}$ is the tensorialization of $D_0 \oplus I = \overline{D_0} \oplus I$ where the generator of $V$, $\overline{D_0}$, is defined on the one-particle space in the case of the absence of $\Sigma$, $\ell^2(\mathbb{C})$, and $I$ acts on $L^2(\Sigma, \omega_{\Sigma})$. In [26, 27] the representation $V$ has been studied, realized, under a suitable Hilbert space isomorphism, in the space $L^2(\mathbb{R}^+, dE)$. In that space $\overline{D_0}$ is the closure of the essentially-selfadjoint operator $-i(Ed/dE + 1/2)$. The original dense, invariant domain of $-i(Ed/dE + 1/2)$ is a core for $D_0$ made of smooth functions on $(0, +\infty)$ (see [26] for details) of the form $\sqrt{E}e^{-\beta E}P(E)$ with $\beta > 0$ a constant not depending on the considered function and $P$ any polynomial. Under the unitary transformation $U$, which takes the form $(U\psi)(x) := (2\pi)^{-1/2} \int_0^{+\infty} e^{-ix \ln E} \psi(E)/\sqrt{E}dE$ on the domain of $-i(Ed/dE + 1/2)$, this operator becomes the operator position $X$ (i.e $(X\psi)(x) = x\psi(x)$) on $L^2(\mathbb{R}, dx)$ restricted to a core contained in the Schwartz space. As a consequence $\sigma(\overline{D_0}) = \sigma_c(\overline{D_0}) = \sigma(X) = \mathbb{R}$ and, similarly, $\sigma(\overline{D_0} \oplus I) = \sigma_c(\overline{D_0} \oplus I) = \mathbb{R}$. Therefore, passing to the tensorialization, $\sigma(\overline{D}) = \mathbb{R}$ and $\sigma_p(\overline{D}) = \{0\}$ with, up to phases, unique eigenvector given by the vacuum vector $\Psi$. If $\Phi$ is a unit vector which is up-to-phases invariant under $U$, it must be in particular $e^{it\overline{X}}\Phi = u_X(t)\Phi$ where $X$ is any real linear combination of $K, S, D$ and $|u_X| = 1$. As the domain of $X$ is dense, it contains a vector $\Phi'$ with $\langle \Phi', \Phi \rangle \neq 0$ and thus $u_X(t) = \langle e^{it\overline{X}}\Phi', \Phi \rangle/\langle \Phi', \Phi \rangle$ is differentiable at $t = 0$ by Stones’ theorem. As a consequence, the left-hand side $e^{it\overline{X}}\Phi = u_X(t)\Phi$ must be differentiable at $t = 0$. By Stone theorem $\Phi$ belongs to the domain of $\overline{X}$ and it holds $\overline{X}\Phi = \lambda_X \Phi$ where $\lambda_X = -i\overline{u}_X/dt|_{t=0}$. Specializing the identity to $X = D$, from the spectral structure of $\overline{D}$, one concludes that it must be $\lambda_D = 0$ and, up to phases, $\Phi = \Psi$. □

**Proof of Theorem 3.3.** (a) and (b). To establish (29) it is sufficient to prove those identities for $w = \hat{V}(\omega)$ with $\omega \in D(\mathbb{M})$ and $g \in PSL(2, \mathbb{R})$. Actually, with the said choices for $w$

$$U(g) a U^\dagger(g) = a'_\omega(a), \quad \text{for all } a \in F(\mathbb{M}).$$

(50)

implies (29). For if (50) holds, taking the adjoint twice for both sides one gets the relations for selfadjoint field operators $U(g) \hat{\omega} U^\dagger(g) = \hat{\omega}(g^{-1})$. Then (12) implies (29) for $w = \hat{V}(\omega)$ via standard spectral theory. To conclude the proof of (a) it is now sufficient to show the validity of (50) with $a = \hat{\omega} \omega$ or of the equivalent statement

$$U(g) \Omega(\psi, \hat{\omega}) U^\dagger(g) = \Omega(\psi(g^{-1}), \hat{\omega}), \quad \text{for all } \psi \in S(\mathbb{M}) \text{ and } g \in PSL(2, \mathbb{R}).$$

(51)
In turn, using the fact that $U$ preserves the vacuum vector and is the tensorialization of $U|_{\mathcal{G}}$ (theorem 3.2) as well as (11) one sees that (51) is equivalent to
\[\psi(g) = U(g^{-1})|_{\mathcal{G}} \psi_+ + \overline{U(g^{-1})}|_{\mathcal{G}} \psi_+, \quad \text{for all } \psi \in S(M) \text{ and } g \in PSL(2,\mathbb{R}). \quad (52)\]

Let us prove (52). If $\psi \in S(M)$ and $g \in Diff^+(S^1)$ the map $\psi \mapsto \psi(g)$ induces a $\mathbb{R}$-linear map from the space of $\theta$-positive frequency parts $\psi_+$ to the same space given by
\[\psi_+ \mapsto S(g)\psi_+ := ((\psi_+ + \overline{\psi_+})(g^{-1}))_+.\]

In this way the action of $g$ on the wavefunction $\psi$ is equivalent to the action of $S(g)$ on its positive frequency part $\psi_+$:
\[\psi(g^{-1}) = S(g)\psi_+ + \overline{S(g)\psi_+}. \quad (53)\]

However, in general, $S(g)$ is not $\mathbb{C}$-linear (and thus it cannot be seen as a map $\mathcal{H} \to \mathcal{H}$) since, using $\chi_+ := i\psi_+$ above, one gets $S(g)(i\psi_+) = \left(i((\psi_+ - i\psi_+)(g^{-1}))_+ = i((\psi_+ - \overline{\psi_+})(g^{-1}))_+ \neq i((\psi_+ + \overline{\psi_+})(g^{-1}))_+ = iS(g)\psi_+. \text{ Actually, if } g \in PSL(2,\mathbb{R}), \text{ it turns out that } (\overline{\psi_+ + g^{-1}})_+ = 0$ so that $S(g)\psi_+ = (\psi_+ \circ g^{-1})_+ \text{ and } S$ is $\mathbb{C}$-linear. This nontrivial result was proved in Lemma i 3.1. To conclude the proof it is sufficient to show that $S(g) = U(g)|_{\mathcal{G}}$ for all $g \in PSL(2,\mathbb{R})$. To establish such an identity we first notice that $S(g) : \mathcal{H} \to \mathcal{H}$ is a unitary representation of $PSL(2,\mathbb{R})$. The only fact non self-evident is that $S(g)$ preserve the scalar product. It is however true because, if $\chi := i\psi_+ - i\psi_-$, it holds
\[\langle \psi_+, \psi'_+ \rangle = -i\Omega(\overline{\psi_+}, \psi_+) = -\frac{i}{2}(\Omega(\psi, \psi') + i\Omega(\chi, \psi'))\]

now, due to (53) we can replace the arguments $\psi_+, \psi'_+$ by respectively $S(g)\psi_+, S(g)\psi'_+$ and the arguments $\psi, \psi', \chi$ by $\psi(g^{-1}), \psi'(g^{-1}), \chi(g^{-1})$ respectively, obtaining a similar identity; finally, since the action of positive-oriented diffeomorphisms of $S^1$ preserves the symplectic form, one has $\Omega(\psi(g^{-1}), \psi'(g^{-1})) + i\Omega(\chi(g^{-1}), \chi(g^{-1})) = \Omega(\psi, \psi') + i\Omega(\chi, \psi')$ and thus $\langle S(g)\psi_+, S(g)\psi'_+ \rangle = \langle \psi_+, \psi'_+ \rangle$. To conclude the proof it is sufficient to notice that, by direct inspection making use of Stone theorem one finds \(^8\) that, if $\psi_{nj} = \{\delta_{np}\}_{p=1,2,\ldots} \otimes u_j \in \ell^2(\mathbb{C}) \otimes L^2(\Sigma, \omega_\Sigma) = \mathcal{H}$
\[iX\psi_{nj} = \frac{d}{dt}S(exp(tx))\psi_{nj}\]
where $X = K, S, D$ and, respectively, $x = k, s, d$ ($k, d, s$ being the basis of $sl(2,\mathbb{R})$ introduced above). On the other hand the same result holds, by construction, for the representation $U|_{\mathcal{G}}$
\[iX\psi_{nj} = \frac{d}{dt}U(exp(tx))\psi_{nj}\]
Since the elements $\psi_{nj}$ span a dense space of analytic vectors for $K|_{\mathcal{G}}^2 + S|_{\mathcal{G}}^2 + D|_{\mathcal{G}}^2$, by the results by Nelson cited in the proof of theorem 3.2, $S = U|_{\mathcal{G}}$. Now (53) implies (52) and this

\(^8\)Details are very similar to those in the corresponding part of Theorem 2.4 in [27]
Proof of Theorem 4.1. (a) Consider the closure $W_\zeta(M)$ of the $*$-algebra of in $W(M)$ spanned elements $V_\zeta(\omega) := V(\omega)e^{t\int_M \Gamma(\zeta_\omega + \zeta_\omega^+)}$ with $\omega \in D(M)$. Obviously the obtained $C^*$-algebra coincides with $W(M)$ itself. On the other hand its generators $V_\zeta(\omega)$ satisfy (V1) and (V2) and thus, by theorem 5.8.8 in [2] there is a unique $*$-isomorphism $\gamma_\zeta : W(M) \to W_\zeta(M) = W(M)$ with $\gamma_\zeta(V(\omega)) = V(\omega)e^{t\int_M \Gamma(\zeta_\omega + \zeta_\omega^+)}$. Finally, by construction $\lambda(\gamma_\zeta(V(\omega))) = \lambda(\zeta(V(\omega))$ and thus, linearity and continuity imply (33). Let us proof (35). Due to linearity and continuity, it is sufficient to show the validity of the relation when restricting to elements $V_\zeta(\omega)$. In turn, since $V(\omega)$ is invariant under $g_t := \exp(tD)$ and using lemma 3.1, the validity of (35) for those elements is a consequence of the invariance of the integral $\int_M \zeta_\omega$ under the action of $g_t^*$ on the argument $\omega_+$ which we go to prove. If $D(M) \ni \omega = \frac{\partial f_+}{\partial \theta} d\theta \wedge \omega_\Sigma(s)$ and defining $\theta_\pm(v) = \pm 2\tan^{-1}(e^v)$, direct computation yields:

$$\int_M \zeta_\omega = - \lim_{N \to +\infty} \int_{-N}^{N} dv \int_{\Sigma} \omega_\Sigma(s) \zeta(s) [f_+(\theta_+(v), s) - f_+(\theta_-(v), s)] + \text{boundary terms}.$$  

Using periodicity of $f_+$ in $\theta$, boundary terms can be re-arranged into a term

$$\lim_{\Theta \to \pi} \left[ (\Theta - \pi) \ln \left( \tan \frac{\Theta}{2} \right) \int_{\Sigma} \zeta(s) \frac{f_+(\Theta, s) - f_+(\pi, s)}{\Theta - \pi} \omega_\Sigma \right]$$

and three other similar terms where $-\pi$ or 0 replaces $\pi$. The last integral can be bounded uniformly in $\Theta$ using Lagrange theorem since $\frac{\partial f_+}{\partial \theta}$ is continuous and compactly supported. As a consequence the limit vanishes and the boundary terms can be dropped. Finally, using the fact that $v$ is the parameter of the integral curves of $D$ one has,

$$\int_M \zeta_\omega g_t^* \omega_+ = - \lim_{N \to +\infty} \int_{-N}^{N} dv \int_{\Sigma} \omega_\Sigma(s) \zeta(s) [f_+(\theta_+(v - t), s) - f_+(\theta_-(v - t), s)]$$

$$= - \lim_{N \to +\infty} \int_{-N}^{N} dv \int_{\Sigma} \omega_\Sigma(s) \zeta(s) [f_+(\theta_+(v), s) - f_+(\theta_-(v), s)] = \int_M \zeta_\omega,$$

so that the invariance of the integral functional under $\exp(tD)$ is evident.

(b) Let us start from the bottom. Since $\lambda$ is invariant under the whole $PSL(2,\mathbb{R})$ group, invariance (noninvariance) of $\lambda_\zeta$ is equivalent to invariance (noninvariance) of the integral functional in the right-hand side of (33). Let us study that integral. Take $\omega(\theta, s) = \frac{\partial f_+}{\partial \theta} h(s) d\theta \wedge \omega_\Sigma(s)$ where $s$ are coordinates on $\Sigma$ and the real functions $f$ and $h$ are smooth with the latter compactly supported as well. Assume $\zeta \neq 0$ a.e. We can fix $h$ such that $\int_\Sigma \zeta h = e^{i\alpha}$. In this case

$$\int_M \zeta_\omega g_t^* \omega_+ = \int_\Sigma \Gamma(\theta)(e^{i\alpha} \frac{\partial f_+}{\partial \theta} d\theta + c.c.) .$$
As a consequence, if \( \{g_t\}_{t \in \mathbb{R}} \) denotes the one-parameter subgroup of \( \text{PSL}(2, \mathbb{R}) \) generated by \( X = (a + b \cos \theta + c \sin \theta) \partial \theta \), with \( a, b, c \in \mathbb{R} \), one has:

\[
\frac{d}{dt} |_{t=0} \int_{\mathbb{M}} \Gamma(\zeta g_t^* \omega + \zeta g_t^* \omega) = \int_{\mathbb{S}^1} \Gamma(\theta) \left( e^{i \beta \theta} \left( (a + b \cos \theta + c \sin \theta) \frac{df^+}{d\theta} \right) \right) d\theta + c.c. .
\]

The invariance of the integral implies that the left-hand must vanish no matter the choice of \( f \):

\[
\int_{\mathbb{S}^1} \Gamma(\theta) \frac{\partial}{\partial \theta} \left( (a + b \cos \theta + c \sin \theta) \frac{e^{i \beta f^+}}{\partial \theta} \right) d\theta + c.c. = 0 .
\]

Using \( f(\theta) := \cos(\theta - \alpha) \) one finds that it must be \( a = 0 \) as a consequence of the identity above. Then using \( f(\theta) := \cos(2\theta - \alpha) \) one finds that it must also be \( b = 0 \). We conclude that the integral functional is invariant at most under the group generated by \( c \sin \theta \partial / \partial \theta = -cD \). On the other hand the proof of such an invariance arises directly from (34) and (35) using the fact that \( \lambda \) is invariant under \( \gamma_\zeta^{(2)} \) as stated in (c) in theorem 3.4.

The fact that \( \lambda_\zeta \) is pure (that is extremal) is an immediate consequence of (33) using the fact that \( \gamma_\zeta \) is bijective and \( \lambda \) is pure. As the \( \lambda_\zeta \) are pure their GNS representations are irreducible. Therefore the proof of the fact that \( \lambda_\zeta \) and \( \lambda_\zeta' \) are not quasi-equivalent if \( \zeta \neq \zeta' \) a.e. reduces to the proof that, if \( \zeta \neq \zeta' \) a.e., there is no unitary transformation \( U : \mathcal{F}_+(\mathcal{H}_\zeta) \rightarrow \mathcal{F}_+(\mathcal{H}_{\zeta'}) \) such that \( U\hat{V}_\zeta(\omega) U^{-1} = \hat{V}_{\zeta'}(\omega) \) for all \( \omega \in \mathcal{D}(\mathbb{M}) \). We shall make use of the first statement in (c) which will be proved independently from the following. Suppose that there is such a unitary transformation for some choice of \( \zeta \neq \zeta' \). As a consequence one gets also the identity \( U\hat{V}_\zeta(\omega)e^{-i\int_{\mathbb{M}} (\Gamma \omega + c.c.) U^{-1} = \hat{V}_{\zeta'}(\omega)e^{-i\int_{\mathbb{M}} (\Gamma \omega + c.c.)} \). That is, re-defining \( \zeta' - \zeta \rightarrow \zeta' \neq 0 \), one has \( U e^{i \theta_\zeta(\omega)} U^\dagger = e^{i \theta_{\zeta'}(\omega)} \) where we have also identified the one-particle Hilbert spaces \( \mathcal{F}_\zeta \) and \( \mathcal{F}_{\zeta'} \) with the one-particle space \( \mathcal{H} \) of the GNS representation of \( \lambda \) (and thus the Fock spaces). Via Stone theorem (using above \( \omega = t\omega \) and \( t \in \mathbb{R} \)) one gets \( U e^{i \theta_\zeta(\omega)} = \phi_0(\omega) U \), that is \( iUa(\psi_+) - a(\psi_+) + \int_{\mathbb{M}} \xi e_{\psi_+} + c.c.)U = i a(\psi_+) - a(\psi_+) U \) where \( \psi_+ = E \omega_+ \) according with (b) in proposition 2.1. Using the analogous relation for \( \psi' := i \psi_+ - i \psi_+^\dagger \) one gets in the end

\[
U \left[ a(\psi_+) - a(\psi_+) + a(\psi_+) + a(\psi_+) \right] - \left( 4i \int_{\mathbb{M}} \xi \Gamma(\psi_+) \right) U = \left[ a(\psi_+) - a(\psi_+) + a(\psi_+) + a(\psi_+) \right] U .
\]

Applying both sides to the vacuum state \( \Psi_\zeta \) and computing the scalar product of the resulting vectors with \( \Psi_\zeta \) itself, the identity above implies that

\[
- \left( 2i \int_{\mathbb{M}} \xi \Gamma(\psi_+) \right) \langle \Psi_\zeta, U \Psi_\zeta \rangle = \langle a(\psi_+) \Psi_\zeta, U \Psi_\zeta \rangle .
\]

If \( \{\psi_+ m\}_{m \in \mathbb{N}} \) is a Hilbert base of \( \mathcal{H}_\zeta \), iteration of the procedure sketched above produces

\[
\langle \Psi_\zeta, U \Psi_\zeta \rangle \prod_n \frac{\lambda_{N_n}}{\sqrt{N_n}} = \langle N_1, N_2, \ldots, N_m, \ldots | U \Psi_\zeta \rangle .
\]
for any vector with finite number of particles $|N_1, N_2, \ldots, N_m, \ldots\rangle$, $N_m$ being the occupation number of the state $\psi_{+m}$ and where $\lambda_m := -2i \int_{\Sigma} c_{\psi_{+m}}$. It must be $\langle \Psi, U \Psi \rangle \neq 0$, otherwise all components of $U \Psi$ would vanish producing $U \Psi = 0$ which is impossible since $U$ is unitary. Conversely, as $||\Psi||^2 = 1$, it must hold $||U \Psi||^2 = 1$. This identity can be expanded with the basis of states $|N_1, N_2, \ldots, N_m, \ldots\rangle$ and a straightforward computations which employs (54) produces

$$||U \Psi||^2 = ||\Psi, U \Psi||^2 \exp \left( \sum_{m=1}^{+\infty} |\lambda_m|^2 \right).$$

The series can explicitly be computed using a basis $\psi_{(n,j)}(\theta, s) = u_j(s) e^{i\theta} \sqrt{4\pi n}$ where $u_j$ is any basis of $L^2(\Sigma, \omega\Sigma)$ made of compactly supported real smooth functions. In that case $\int_{\Sigma} \xi u_j \omega\Sigma \neq 0$ for some $j = j_0$ (otherwise the function $\xi$ on $\Sigma$ would have $L^2(\Sigma, \omega\Sigma)$-norm zero). One finds

$$|\lambda_{2n+1,j_0}|^2 = C |\int_{\Sigma} \xi u_{j_0} \omega\Sigma|^2 (2n+1)^{-1}$$

with $C > 0$ so that the series in (55) diverges and the found contradiction shows that $U$ cannot exist.

(c) By direct inspection one finds that the operators $V_\xi(\omega)$ enjoy (V1) and (V2). Therefore, (theorem 5.2.8, in [2]) the $C^*$-algebra $\hat{W}_\xi(\mathcal{M})$ given by the closure of the *-algebra generated by $V_\xi(\omega)$ is a representation of Weyl algebra and there is a *-algebra isomorphism of $C^*$ algebras, $\Pi_\xi : \mathcal{W} \rightarrow \hat{W}_\xi(\mathcal{M})$ which satisfies (36). The vacuum vector of $\hat{H}_\xi = \mathcal{F}_+(\mathcal{H}_\xi)$ is cyclic with respect to $\Pi_\xi$ because $\hat{W}_\xi(\mathcal{M}) \Psi_\xi$ is the same space as the dense space (see A.1) spanned by vectors $e^{i\phi(\omega_1)} \ldots e^{i\phi(\omega_n)} \Psi_\xi$, $n = 1, 2, \ldots$. Finally it holds

$$\lambda_\xi(V(\omega)) = \lambda(V(\omega)) e^{i\int_{\omega} c_{\Gamma_{\omega}+\text{c.c.}}} e^{i\int_{\omega} c_{\Gamma_{\omega}+\text{c.c.}}} \langle \Psi_\xi, e^{i\phi(\omega)} \Psi_\xi \rangle e^{i\int_{\omega} c_{\Gamma_{\omega}+\text{c.c.}}} \langle \Psi_\xi, e^{i\phi(\omega)} \Psi_\xi \rangle = \langle \Psi_\xi, \hat{V}_\xi(\omega) \Psi_\xi \rangle,$$

that is $\lambda_\xi(V(\omega)) = \langle \Psi_\xi, \Pi_\xi(V(\omega)) \Psi_\xi \rangle$. By linearity and continuity this relation extends to the whole algebras: $\lambda_\xi(w) = \langle \Psi_\xi, \Pi_\xi(w) \Psi_\xi \rangle$, $w \in \mathcal{W}$. We conclude that $(\mathcal{F}_+(\mathcal{H}_\xi), \Pi_\xi, \Psi_\xi)$ is the (unique, up to unitary transformations) GNS triple for $\lambda_\xi$.

(d) Let us denote by $\{g_t\}_{t \in \mathbb{R}}$ the one-parameter group of M"o"bius transformations generated by $\mathcal{D}$. The statements (a) and (b) in theorem 3.3 imply that if $D$ is defined as $(1/2i) : \Omega(\phi_0, \mathcal{D}(\phi_0))$: then $e^{itD} e^{i\phi(\omega)} e^{-itD} = e^{i\phi_0(g_t^{-1}\omega)}$. Since $\int_{\omega} c_{\Gamma_{\omega}+\text{c.c.}}$ is invariant under the action of $g_t$ on $\omega$ as seen in the proof of (a), we have also

$$e^{itD} e^{i\phi(\omega)} e^{i\int_{\omega} c_{\Gamma_{\omega}+\text{c.c.}}} e^{-itD} = e^{i\phi_0(g_t^{-1}\omega)} e^{i\int_{\omega} c_{\Gamma_{\omega}+\text{c.c.}}}$$

that can be rewritten as $e^{itD} \hat{V}_\xi(e^{-itD} \hat{V}_\xi(\omega) e^{-itD}) = \hat{V}_\xi(\omega(g_t^{-1}))$ and thus extends to the whole Weyl algebra proving (37). $\square$

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9The space $\mathcal{C}$ of smooth compactly supported functions on $\Sigma$ is dense in $L^2(\Sigma, \omega\Sigma)$. As the latter is separable $\mathcal{C}$ contains a countable subset $\mathcal{C}'$ still dense in $L^2(\Sigma, \omega\Sigma)$. In turn one may extract from $\mathcal{C}'$ a subset $\mathcal{C}''$ of linearly independent elements which span the same dense space as $\mathcal{C}'$. Usual orthonormalization procedure applied to $\mathcal{C}''$ gives a Hilbert basis for $L^2(\Sigma, \omega\Sigma)$ made of smooth compactly supported functions. Proceeding as in footnote 4 one gets the wanted basis of $u_j$.  

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