Pfaffian Formulas and Schur $Q$-Function Identities

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Abstract

We establish Pfaffian analogues of the Cauchy–Binet formula and the Ishikawa–Wakayama minor-summation formula. Each of these Pfaffian analogues expresses a sum of products of subpfaffians of two skew-symmetric matrices in terms of a single Pfaffian. By using these Pfaffian formulas we give new transparent proofs to several identities for Schur $Q$-functions.

1 Introduction

The aim of this article is twofold: Firstly we establish Pfaffian analogues of the Cauchy–Binet formula and the Ishikawa–Wakayama minor summation formula for determinants. Secondly we give new transparent proofs to Schur $Q$-function identities by applying general formula for Pfaffians such as these Pfaffian analogues.

Schur $Q$-functions are a family of symmetric functions introduced by Schur in his study on the projective representations of symmetric groups. Schur $Q$-functions play the same role as Schur functions for the linear representation of symmetric groups. Later Hall (unpublished) and Littlewood introduce a family of symmetric functions with parameter $t$, as a common generalization of Schur functions (the $t = 0$ case) and Schur $Q$-functions (the $t = -1$ case).

Schur $Q$-functions appear in various situations parallel to Schur functions: the projective representations of symmetric groups, the cohomology of Lagrangian or orthogonal Grassmannians, the representations of the queer Lie super algebra $q(n)$, the BKP hierarchy. Also Schur $Q$-functions are expressed as multivariate generating functions of shifted tableaux.

In this paper we adopt Nimmo’s formula as a definition of Schur $P$- and $Q$-functions. This formula is an analogue of the bialternant definition of Schur functions.

Let $x = (x_1, \ldots, x_n)$ be a sequence of $n$ indeterminates. We put

$$A(x) = \left( \frac{x_j - x_i}{x_j + x_i} \right)_{1 \leq i, j \leq n}, \quad \text{and} \quad D(x) = \prod_{1 \leq i < j \leq n} \frac{x_j - x_i}{x_j + x_i}. \quad (1.1)$$

For a sequence $\alpha = (\alpha_1, \ldots, \alpha_l)$ of nonnegative integers of length $l$, let $V_\alpha(x)$ and $W_\alpha(x)$ be the $n \times l$ matrices given by

$$V_\alpha(x) = \left( x_i^{\alpha_j} \right)_{1 \leq i \leq n, 1 \leq j \leq l}, \quad \text{and} \quad W_\alpha(x) = \left( \chi(\alpha_j)x_i^{\alpha_j} \right)_{1 \leq i \leq n, 1 \leq j \leq l}. \quad (1.2)$$

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where \( \chi(r) = 2 \) if \( r > 0 \) and \( 1 \) if \( r = 0 \). A strict partition of length \( l \) is a strictly decreasing sequence \( \lambda = (\lambda_1, \ldots, \lambda_l) \) of positive integers. We write \( l = l(\lambda) \). We define the Schur \( P \)-function \( P_\lambda(x) \) and the Schur \( Q \)-function \( Q_\lambda(x) \) corresponding to a strict partition \( \lambda \) by putting

\[
P_\lambda(x) = \frac{1}{D(x)} \text{Pf} \left( \begin{array}{cc} A(x) & V_\alpha(x) \\ -V_\alpha(x) & O \end{array} \right),
\]

(1.3)

\[
Q_\lambda(x) = \frac{1}{D(x)} \text{Pf} \left( \begin{array}{cc} A(x) & W_\alpha(x) \\ -W_\alpha(x) & O \end{array} \right),
\]

(1.4)

where \( \alpha = (\lambda_1, \ldots, \lambda_l) \) if \( n + l \) is even, or \( \alpha = (\lambda_1, \ldots, \lambda_l, 0) \) if \( n + l \) is odd. Note that \( P_\lambda(x) = Q_\lambda(x) = 0 \) if \( l > n \).

Many of Schur function identities are easily proved by applying determinant formulas. However some of the known proofs of \( Q \)-function identities are quite different from the proofs of similar Schur function identities. For example, the Cauchy identity for Schur functions

\[
\sum_{\lambda} s_\lambda(x)s_\lambda(y) = \prod_{i,j} \frac{1}{1 - x_iy_j}
\]

(1.5)

can be proved by using the Cauchy–Binet formula for determinants and the evaluation of Cauchy determinant (see [13, I.4, Example 6]). On the other hand, no such direct proof is known for the Cauchy-type identity for Schur \( Q \)-functions

\[
\sum_{\lambda} P_\lambda(x)Q_\lambda(y) = \prod_{i,j} \frac{1 + x_iy_j}{1 - x_iy_j}.
\]

(1.6)

See [19, Abschnitt IV], [7, § 4B], [13, III.8] and [2, Chapter 7] for algebraic proofs. One of our motivations is to give an elementary linear algebraic proof to \( Q \)-function identities.

One of the main results of this paper is the following Pfaffian analogue of the Cauchy–Binet formula, which reduces to the Cauchy–Binet formula for determinants by specializing \( A = O \) and \( B = O \).

**Theorem 1.1.** (Theorem 3.2 below) Let \( m \) and \( n \) be nonnegative integers with the same parity. Let \( A \) and \( B \) be \( m \times m \) and \( n \times n \) skew-symmetric matrices, and let \( S \) and \( T \) be \( m \times l \) and \( n \times l \) matrices. Then we have

\[
\sum_K \text{Pf} \left( \begin{array}{cc} A & S([m]; K) \\ -S([m]; K) & O \end{array} \right) \text{Pf} \left( \begin{array}{cc} B & T([n]; K) \\ -T([n]; K) & O \end{array} \right) = (-1)^{|K|} \text{Pf} \left( \begin{array}{cc} A & ST \\ -T'S & -B \end{array} \right),
\]

where \( K \) runs over all subsets of \([l]\) with \( \#K \equiv m \equiv n \mod 2 \). (See Section 2 for notations.)

We can give a simple and direct proof to the Cauchy-type identity (1.6) by using this Pfaffian version of the Cauchy–Binet formula as well as the evaluation of Schur Pfaffian. Also we can use a variant to prove the Pragacz–Józefiak–Nimmo identity for skew \( Q \)-functions [17, 15]. In a forthcoming paper, we take this linear algebraic approach to study generalizations of Schur \( P \)- and \( Q \)-functions such as Ivanov’s factorial \( P \)- and \( Q \)-functions.
and the case \( t = -1 \) of Hall–Littlewood polynomials associated to the classical root systems \([14]\).

This paper is organized as follows. After reviewing basic properties of Pfaffians in Section 2, we give Pfaffian analogues of the Cauchy–Binet formula and the Ishikawa–Wakayama minor-summation formula in Section 3. In Section 4, we apply the Pfaffian analogue of the Sylvester formula to recover Schur’s original definition of \( Q \)-functions from Nimmo’s formula. In Section 5, we give a proof of the Cauchy-type formula for \( Q \)-functions by using the Pfaffian analogue of the Cauchy–Binet formula. Section 6 is devoted to a linear algebraic proof of the Pragacz–Józefiak–Nimmo formula for skew \( Q \)-functions. In Section 7 we use the Pfaffian analogue of the Ishikawa–Wakayama formula to derive a Littlewood-type formula for \( Q \)-functions.

2 Pfaffians

In this section we review basic properties of Pfaffians and give a Laplace-type expansion formula.

2.1 Basic properties of Pfaffians

Recall the definition and some properties of Pfaffians. (See [3] for some expositions) Let \( X = (x_{ij})_{1 \leq i,j \leq 2m} \) be a skew-symmetric matrix of order \( 2m \). The Pfaffian of \( X \), denoted by \( \text{Pf}(X) \), is defined by

\[
\text{Pf}(X) = \sum_{\sigma \in F_{2m}} \text{sgn}(\sigma) \prod_{i=1}^{m} x_{\sigma(2i-1), \sigma(2i)},
\]

where \( F_{2m} \) is the set of permutations \( \sigma \in S_{2m} \) satisfying \( \sigma(1) < \sigma(3) < \cdots < \sigma(2m-1) \) and \( \sigma(2i-1) < \sigma(2i) \) for \( 1 \leq i \leq m \). Such permutations are in one-to-one correspondence with set-partitions \( \pi \) of \( \{1, 2, \ldots, 2m\} \) into \( m \) disjoint 2-element subsets. If \( \sigma \in F_{2m} \) corresponds to a set-partition \( \pi = \{\{i_1, j_1\}, \ldots, \{i_m, j_m\}\} \) with \( i_k < j_k \) for \( 1 \leq k \leq m \), then we have

\[
\text{sgn}(\sigma) = (-1)^{\text{inv}(i_1, j_1, \ldots, i_m, j_m)},
\]

where \( \text{inv}(\alpha_1, \ldots, \alpha_{2m}) \) is the number of pairs \((k, l)\) such that \( k < l \) and \( \alpha_k > \alpha_l \). Note that the right hand side is independent of the ordering of blocks of \( \pi \). Since \( m \equiv \binom{2m}{2} \pmod{2} \), it follows from the definition of Pfaffians that

\[
\text{Pf}(-X) = (-1)^{\binom{2m}{2}} \text{Pf} X.
\]

Pfaffians are multilinear in the following sense. Let \( X = (x_{ij})_{1 \leq i,j \leq n} \) be a skew-symmetric matrix and fix a row/column index \( k \). If the entries of the \( k \)th row and \( k \)th column of \( X \) are written as \( x_{i,j} = \alpha x'_{i,j} + \beta x''_{i,j} \) for \( i = k \) or \( j = k \), then

\[
\text{Pf} X = \alpha \text{Pf} X' + \beta \text{Pf} X'',
\]

where \( X' \) (resp. \( X'' \)) is the skew-symmetric matrix obtained from \( X \) by replacing the entries \( x_{ij} \) for \( i = k \) or \( j = k \) with \( x'_{ij} \) (resp. \( x''_{ij} \)).
If $X$ is an $n \times n$ skew-symmetric matrix and $U$ is an $n \times n$ matrix, then we have

$$\text{Pf}(UXU) = \det(U) \text{Pf}(X).$$

It follows that Pfaffians are alternating, i.e., if $\sigma \in S_n$, we have

$$\text{Pf}(x_{\sigma(i), \sigma(j)})_{1 \leq i,j \leq n} = \text{sgn}(\sigma) \text{Pf}(x_{i,j})_{1 \leq i,j \leq n}.$$ 

Also we see that, if $Y$ is the skew-symmetric matrix obtained from $X$ by adding the $k$th row multiplied by a scalar $\alpha$ to the $l$th row and then adding the $k$th column multiplied by $\alpha$ to the $l$th column, the we have $\text{Pf} Y = \text{Pf} X$.

We use the following notations for submatrices. For a positive integer $n$, we put $[n] = \{1, 2, \ldots, n\}$. Given a subset $I \subset [n]$, we put $\Sigma(I) = \sum_{i \in I} i$. For an $M \times N$ matrix $X = (x_{i,j})_{1 \leq i \leq M, 1 \leq j \leq N}$ and subsets $I \subset [M]$ and $J \subset [N]$, we denote by $X(I; J)$ the submatrix of $X$ obtained by picking up rows indexed by $I$ and columns indexed by $J$. If $X$ is a skew-symmetric matrix, then we write $X(I)$ for $X(I; I)$. We use the convention that $\det X(\emptyset; \emptyset) = 1$ and $\text{Pf} X(\emptyset) = 1$.

For an $n \times n$ skew-symmetric matrix $X = (x_{i,j})_{1 \leq i,j \leq n}$, we have the following expansion formula along the $k$th row/column:

$$\text{Pf} X = \sum_{i=1}^{k-1} (-1)^{k+i-1} x_{i,k} \text{Pf} X([n] \setminus \{i, k\}) + \sum_{i=k+1}^{n} (-1)^{k+i-1} x_{k,i} \text{Pf} X([n] \setminus \{k, i\}).$$

Knuth [11] gave the following Pfaffian analogue of the Sylvest er identity for determinant.

**Proposition 2.1.** (Knuth [11] (2.5)) Let $n$ and $l$ be even integers and let $X$ be an $(n + l) \times (n + l)$ skew-symmetric matrix.

1. If $\text{Pf} X([n]) \neq 0$, then we have

$$\text{Pf} \left( \frac{\text{Pf} X([n] \cup \{n + i, n + j\})}{\text{Pf} X([n])} \right)_{1 \leq i,j \leq l} = \frac{\text{Pf} X}{\text{Pf} X([n])},$$

2. If $\text{Pf} X([l + 1, l + n]) \neq 0$, then we have

$$\text{Pf} \left( \frac{\text{Pf} X(\{i, j\} \cup [l + 1, l + n])}{\text{Pf} X([l + 1, l + n])} \right)_{1 \leq i,j \leq l} = \frac{\text{Pf} X}{\text{Pf} X([l + 1, l + n])},$$

where $[l + 1, l + n] = \{l + 1, l + 2, \ldots, l + n\}$.

The following evaluation formula of Schur Pfaffian is useful in various places of this paper.

**Proposition 2.2.** (Schur [19] p. 226], see also [11] Section 4]) Let $n$ be an even integer. For a sequence $x = (x_1, \ldots, x_n)$ of variables, we have

$$\text{Pf} \left( \frac{x_j - x_i}{x_j + x_i} \right)_{1 \leq i,j \leq n} = \prod_{1 \leq i < j \leq n} \frac{x_j - x_i}{x_j + x_i}.$$
2.2 Laplace-type expansion formulas for Pfaffian

The following expansion formula is stated in [1, (12)] without proof.

**Proposition 2.3.** Let $m$ and $n$ be nonnegative integers such that $m + n$ is even. For an $m \times m$ skew-symmetric matrix $Z = (z_{i,j})_{1 \leq i,j \leq m}$, an $n \times n$ skew-symmetric matrix $Z' = (z'_{i,j})_{1 \leq i,j \leq n}$, and an $m \times n$ matrix $W = (w_{i,j})_{1 \leq i \leq m, 1 \leq j \leq n}$, we have

\[
\text{Pf} \begin{pmatrix} Z & W \\ -W^T & Z' \end{pmatrix} = \sum_{I,J} \varepsilon(I,J) \text{Pf} \left( Z(I) \right) \text{Pf} \left( Z'(J) \right) \det W([m] \setminus I; [n] \setminus J),
\]

where the sum is taken over all pairs of even-element subsets $(I, J)$ such that $I \subset [m]$, $J \subset [n]$ and $m - \#I = n - \#J$, and the coefficient $\varepsilon(I, J)$ is given by

\[
\varepsilon(I, J) = (-1)^{\Sigma(I) + \Sigma(J) + \left( \frac{n}{2} \right) + \left( \frac{k}{2} \right)}, \quad k = m - \#I = n - \#J.
\]

If $m = 1$, then the formula (2.8) reduces to the expansion formula (2.4) along the first row/column.

**Proof.** We put $[n]' = \{1', 2', \ldots , n'\}$ and label the rows and columns of $\begin{pmatrix} Z & W \\ -W^T & Z' \end{pmatrix}$ by $[m] \sqcup [n]' = \{1, 2, \ldots , m, 1', 2', \ldots , n'\}$ with $1 < 2 < \cdots < m < 1' < 2' < \cdots < n'$. For an even-element subset $I$ of $[m] \sqcup [n]'$, we denote by $F_I$ the set of all set-partitions of $I$ into 2-element subsets. Given a partition $\pi \in F_{[m] \sqcup [n]'}$, we put

\[
\pi_i = \{ b \in \pi : \#(b \cap [m]) = i \} \quad \text{for} \quad i = 0, 1, 2.
\]

Then there are subsets $I \subset [m]$ and $J' \subset [n]'$ such that $\pi_2 \in F_I$, $\pi_0 \in F_{J'}$ and $m - \#I = n - \#J'$. Moreover, if $[m] \setminus I = \{r_1, \ldots , r_k\}$ and $[n]' \setminus J' = \{s_1', \ldots , s_k'\}$ with $r_1 < \cdots < r_k$, $s_1' < \cdots < s_k'$, then there exists a unique permutation $\sigma \in S_k$ such that

\[
\pi_1 = \{ \{r_1, s_{\sigma(1)}\}, \ldots , \{r_k, s_{\sigma(k)}\} \}.
\]

The correspondence $\pi \mapsto (\pi_2, \sigma, \pi_0)$ gives a bijection

\[
F_{[m] \sqcup [n]'} \rightarrow \bigsqcup_{(I,J)} F_I \times S_k \times F_{J'},
\]

where $(I, J)$ runs over all pairs of even-element subsets $I \subset [m]$ and $J \subset [n]$ such that $m - \#I = n - \#J$, and $J' = \{ j' : j \in J \}$. Let $\pi_2 = \{ \{p_1, p_2\}, \ldots , \{p_{2(m-k)}\} \}$ and $\pi_0 = \{ \{q_1', q_2'\}, \ldots , \{q_{2(n-k)}', q_{2(n-k)}'\} \}$ with $p_{2i-1} < p_{2i}$ and $q_{2j-1} < q_{2j}$. Then the inversion number of the permutation associated to $\pi$ is given by

\[
\text{inv}(p_1, p_2, \ldots , p_{2(m-k)}; r_1, s_{\sigma(1)}', \ldots , r_k, s_{\sigma(k)}') = \text{inv}(p_1, p_2, \ldots , p_{2(m-k)}) + \text{inv}(r_1, s_{\sigma(1)}', \ldots , r_k, s_{\sigma(k)}') + \text{inv}(q_1', q_2', \ldots , q_{2(n-k)}')
\]

\[
+ \# \{ (i, j) \in I \times ([m] \setminus I) : i > j \} + \left( \begin{array}{c} k \\ 2 \end{array} \right) + \# \{ (i', j') \in ([n]' \setminus J') \times J : i' > j' \}.
\]

Since $r_1 < \cdots < r_k < s_1' < \cdots < s_k'$, we have

\[
\text{inv}(r_1, s_{\sigma(1)}', \ldots , r_k, s_{\sigma(k)}) = \text{inv}(\sigma(1), \ldots , \sigma(k)).
\]
Also we have
\[
\#\{(i, j) \in I \times ([m] \setminus I) : i > j\} = \Sigma(I) - \binom{m - k + 1}{2},
\]
\[
\#\{(i', j') \in ([n'] \setminus J') \times J' : i' > j'\} = k(n - k) + \binom{n - k + 1}{2} - \Sigma(J).
\]

Since \(m \equiv n \equiv k \mod 2\) by the assumption, we see that
\[
\left(\frac{m - k + 1}{2}\right) + k(n - k) + \left(\frac{n - k + 1}{2}\right) = \left(\frac{m}{2}\right) + \left(\frac{n}{2}\right) - (k + 1)m + m + n
\]
\[
\equiv \left(\frac{m}{2}\right) + \left(\frac{n}{2}\right) \mod 2.
\]

Hence we have
\[
\text{inv}(p_1, p_2, \ldots, p_{2(m-k)}, r_1, s_{\sigma(1)}, \ldots, r_k, s_{\sigma(k)}, q'_1, q'_2, \ldots, q'_{2(n-k)})
\]
\[
\equiv \text{inv}(p_1, p_2, \ldots, p_{2(m-k)}) + \text{inv}(\sigma(1), \ldots, \sigma(k)) + \text{inv}(q'_1, q'_2, \ldots, q'_{2(n-k)})
\]
\[
+ \Sigma(I) + \Sigma(J) + \left(\frac{m}{2}\right) + \left(\frac{n}{2}\right) + \binom{k}{2}.
\]

Now (2.8) follows from the definition of Pfaffians (2.1).

By considering the case where \(Z'\) or \(W\) is the zero matrix in Proposition 2.3, we obtain the following corollary. We denote the \(p \times q\) zero matrix by \(O_{p,q}\) and write simply \(O\) for \(O_{p,q}\) if there is no confusion on the size.

**Corollary 2.4.** Suppose that \(m + n\) is even.

1. If \(Z\) is an \(m \times m\) skew-symmetric matrix and \(W\) is an \(m \times n\) matrix, then we have

   \[
   \text{Pf} \left( \begin{array}{cc} Z & W \\ -W & O_{n,n} \end{array} \right) = \begin{cases} 
   \sum_I (-1)^{\Sigma(I) + \binom{m}{2}} \text{Pf} \ Z(I) \det W([m] \setminus I; [n]) & \text{if } m > n, \\
   (-1)^{\binom{m}{2}} \det W & \text{if } m = n, \\
   0 & \text{if } m < n,
   \end{cases}
   \]

   where \(I\) runs over all \((m - n)\)-element subsets of \([n]\).

2. If \(Z\) and \(Z'\) are \(m \times m\) and \(n \times n\) skew-symmetric matrices respectively, then we have

   \[
   \text{Pf} \left( \begin{array}{cc} Z & O_{m,n} \\ O_{n,m} & Z' \end{array} \right) = \begin{cases} 
   \text{Pf} \ Z \cdot \text{Pf} \ Z' & \text{if } m \text{ and } n \text{ are even}, \\
   0 & \text{otherwise}.
   \end{cases}
   \]

**Proof.** (1) If \(Z' = O\), then we have \(\text{Pf} \ Z'(J) = 0\) unless \(J = \emptyset\).

(2) If \(W = O\), then we have \(\det W([m] \setminus I; [n] \setminus J) = 0\) unless \(I = [m]\) and \(J = [n]\). □
3 Cauchy–Binet type Pfaffian formulas

In this section we give Pfaffian analogues of the Cauchy–Binet formula and the Ishikawa–Wakayama minor-summation formula \[4\]. These are our main results of this paper.

First we consider the following special case of Proposition 2.3.

**Lemma 3.1.** Let \(m, n\) and \(l\) be nonnegative integers with \(m \equiv n \mod 2\). We put

\[
E_l^{(m+l,n+l)} = \begin{pmatrix} O_{m,n} & O_{m,l} \\ O_{l,n} & E_l \end{pmatrix},
\]

where \(E_l\) is the \(l \times l\) identity matrix. If \(Z\) and \(Z'\) are \((m+l) \times (m+l)\) and \((n+l) \times (n+l)\) skew-symmetric matrices respectively, then we have

\[
Pf \left( \begin{pmatrix} Z & E_l^{(m+l,n+l)} \\ -E_l^{(m+l,n+l)} & Z' \end{pmatrix} \right) = \sum_K (-1)^{(l-\#K)/2} Pf Z([m] \cup (m+K)) Pf Z'([n] \cup (n+K)),
\]

where \(K\) runs over all subsets of \([l]\) with \(\#K \equiv m \mod 2\) and \(m + K = \{m + k : k \in K\}\), \(n + K = \{n + k : k \in K\}\).

**Proof.** We substitute \(W = E_l^{(m+l,n+l)}\) in Proposition 2.3. Let \(I\) and \(J\) be even-element subsets of \([m+l]\) and \([n+l]\) respectively such that \(m+l - \#I = n+l - \#J\). If \([m] \not\subset I\) or \([n] \not\subset J\), then we have \(\det W([m+l] \setminus I; [n+l] \setminus J) = 0\). If \([m] \subset I\) and \([n] \subset J\), then we can write \(I = [m] \cup (m+I')\) and \(J = [n] \cup (n+J')\) for some subsets \(I', J' \subset [l]\), and we see that

\[
\det W([m+l] \setminus I; [n+l] \setminus J) = \det E_l([l] \setminus I'; [l] \setminus J') = \begin{cases} 1 & \text{if } I' = J', \\ 0 & \text{otherwise}. \end{cases}
\]

Hence \(\det W([m+l] \setminus I; [n+l] \setminus J) = 0\) unless \(I = [m] + (m+K)\) and \(J = [n] + (n+K)\) for some subset \(K \subset [l]\). In this case,

\[
\Sigma(I) = \binom{m}{2} + m + m \#K + \Sigma(K), \quad \Sigma(J) = \binom{n}{2} + n + n \#K + \Sigma(K),
\]

and

\[
\Sigma(I) + \Sigma(J) + \binom{m+l}{2} + \binom{n+l}{2} + \binom{(m+l) - (m + \#K)}{2} \equiv \binom{l - \#K}{2} \mod 2.
\]

This completes the proof. \(\Box\)

We use Lemma 3.1 to derive a Pfaffian analogue of the Cauchy–Binet formula.

**Theorem 3.2.** Let \(m\) and \(n\) be nonnegative integers with the same parity. Let \(A\) and \(B\) be \(m \times m\) and \(n \times n\) skew-symmetric matrices, and let \(S\) and \(T\) be \(m \times l\) and \(n \times l\) matrices. Then we have
\[
\sum_{K} (-1)^{\binom{\#K}{2}} \text{Pf} \left( \begin{array}{cc}
A & S([m];K) \\
-tS([m];K) & O
\end{array} \right) \text{Pf} \left( \begin{array}{cc}
B & T([n];K) \\
-T([n];K) & O
\end{array} \right) = \text{Pf} \left( \begin{array}{cc}
A & ST \\
-Ts & B
\end{array} \right), \tag{3.2}
\]

\[
\sum_{K} \text{Pf} \left( \begin{array}{cc}
A & S([m];K) \\
-tS([m];K) & O
\end{array} \right) \text{Pf} \left( \begin{array}{cc}
B & T([n];K) \\
-T([n];K) & O
\end{array} \right) = (-1)^{\binom{\#K}{2}} \text{Pf} \left( \begin{array}{cc}
A & ST \\
-Ts & -B
\end{array} \right), \tag{3.3}
\]

where \( K \) runs over all subsets of \([l] \) with \( \#K \equiv m \equiv n \mod 2 \).

**Remark 3.3.** It follows from (2.9) that both formulas (3.2) and (3.3) reduce to the Cauchy–Binet formula for determinants if we put \( A = 0 \) and \( B = 0 \):

\[
\sum_{K} \det S([m];K) \det T([m];K) = \det(ST), \tag{3.4}
\]

where \( K \) runs over all \( m \)-element subsets.

**Proof.** Apply Lemma 3.1 to the matrices

\[
Z = \left( \begin{array}{cc}
A & S \\
n \cdot S & O
\end{array} \right) \quad \text{and} \quad Z' = \left( \begin{array}{cc}
B & -T \\
T & O
\end{array} \right).
\]

Then we have

\[
\text{Pf} Z([m] + (m + K)) = \text{Pf} \left( \begin{array}{cc}
A & S([m];K) \\
n \cdot S([m];K) & O
\end{array} \right),
\]

\[
\text{Pf} Z'([n] + (n + K)) = (-1)^{\#K} \text{Pf} \left( \begin{array}{cc}
B & T([n];K) \\
-T([n];K) & O
\end{array} \right).
\]

We compute the Pfaffian on the right hand side of (3.1). By using the relation (2.3) with

\[
X = \left( \begin{array}{cccc}
A & S & O & O \\
-tS & O & O & E_t \\
O & O & B & -T \\
O & -E_t & B & T & O
\end{array} \right) \quad \text{and} \quad U = \left( \begin{array}{cccc}
E_m & O & O & O \\
O & T & E_i & O \\
O & E_n & O & O \\
E_i & S & O & O & E_t
\end{array} \right),
\]

and then by using Corollary 2.4 we see that

\[
(-1)^{nl} \text{Pf} X = \text{Pf} \left( \begin{array}{cccc}
A & ST & O & O \\
-tS & B & O & O \\
O & O & O & E \\
O & O & O & -E
\end{array} \right) = (-1)^{\binom{\#}{2}} \text{Pf} \left( \begin{array}{cc}
A & ST \\
-Ts & B
\end{array} \right).
\]

Therefore we have
\[
Pf \begin{pmatrix} A & S^T \\ -T^tS & B \end{pmatrix} = \sum_K (-1)^{(l-\#K)+\#K+n-\frac{l}{2}} Pf \begin{pmatrix} A & S([m]; K) \\ -tS([m]; K) & O \end{pmatrix} Pf \begin{pmatrix} B & T([n]; K) \\ -tT([n]; K) & O \end{pmatrix}.
\]

Since \(\#K \equiv n \mod 2\), we have
\[
\left(\frac{l-\#K}{2}\right) + \#K + nl - \left(\frac{l}{2}\right) = \left(\frac{\#K}{2}\right) + l(n-\#K) + 2\#K \equiv \left(\frac{\#K}{2}\right) \mod 2,
\]
and obtain the desired formula (3.2).

Equation (3.3) is obtained by replacing \(B\) with \(-B\) in (3.2). In fact, by multiplying the last \(k\) rows/columns by \(-1\) and then by using (2.2), we have
\[
Pf \begin{pmatrix} -B & T([n]; K) \\ -tT([n]; K) & O \end{pmatrix} = (-1)^{\#K+\left(\frac{n+\#K}{2}\right)} Pf \begin{pmatrix} B & T([n]; K) \\ -tT([n]; K) & O \end{pmatrix}.
\]
Since \(\#K \equiv n \mod 2\), we have
\[
\#K + \left(\frac{n+\#K}{2}\right) = \left(\frac{n}{2}\right) + \left(\frac{\#K}{2}\right) + (n+1)\#K \equiv \left(\frac{n}{2}\right) + \left(\frac{\#K}{2}\right) \mod 2,
\]
and obtain (3.3).

Another application of Lemma 3.1 is the following Pfaffian analogue of the Ishikawa–Wakayama minor-summation formula.

**Theorem 3.4.** Let \(m\) be an even integer and \(l\) be a positive integer. For an \(m \times m\) skew-symmetric matrix \(A\), an \(l \times l\) skew-symmetric matrix \(B\), and an \(m \times l\) matrix \(S\), we have
\[
\sum_K Pf B(K) Pf \begin{pmatrix} A & S([m]; K) \\ -tS([m]; K) & O \end{pmatrix} = Pf \begin{pmatrix} A - SBtS \\ O \end{pmatrix},
\]
where \(K\) runs over all even-element subsets of \([l]\).

**Remark 3.5.** It follows from (2.9) that (3.5) reduces to the minor-summation formula ([4, Theorem 1]) if \(A = O\):
\[
\sum_K Pf B(K) \det S([m]; K) = Pf \begin{pmatrix} SB'S \\ O \end{pmatrix},
\]
where \(K\) runs over all \(m\)-element subsets of \([l]\).

**Proof.** We apply Lemma 3.1 (with \(n = 0\)) to the matrices
\[
Z = \begin{pmatrix} A & S \\ -t'S & O \end{pmatrix}, \quad Z' = -B.
\]
Since \(\text{Pf} Z'(K) = (-1)^{\frac{\#K}{2}} \text{Pf} B(K)\) by (2.2), we have
\[
Pf \begin{pmatrix} A & S & O \\ -t'S & O & E_l \\ 0 & -E_l & -B \end{pmatrix} = \sum_{K \subset [l]} (-1)^{(l-\#K)+\frac{\#K}{2}} Pf \begin{pmatrix} A & S([m]; K) \\ -t'S([m]; K) & O \end{pmatrix} Pf B(K).
\]
By using (2.3) with
\[ X = \begin{pmatrix} A & S & O \\ -S & O & E \\ O & -E & -B \end{pmatrix}, \quad U = \begin{pmatrix} E & O & O \\ -B'S & E & O \\ t' & O & E \end{pmatrix} \]
and then by using Corollary 2.4, we obtain
\[ \text{Pf} X = \text{Pf} \begin{pmatrix} A - SB't'S & O & O \\ O & O & E \\ O & -E & -B \end{pmatrix} = (-1)^{\binom{l}{2}} \text{Pf} (A - SB't'S). \]
Hence the proof is completed by using the congruence \( \binom{l-\#K}{2} + \binom{\#K}{2} + \binom{l}{2} \equiv 0 \mod 2. \]

**Remark 3.6.** From Lemma 3.1, we can derive the following summation formula for Pfaffians [4, Theorem 3]:
\[ \sum_{I,J} (-1)^{\binom{l-I}{2}} \det T(I; J) \text{Pf} A(I) \text{Pf} B(J) = \text{Pf} \begin{pmatrix} A & E_l \\ -E_l & TBT \end{pmatrix}, \] (3.7)
where \( A \) and \( B \) are \( l \times l \) skew-symmetric matrices, \( T \) is an \( l \times l \) matrix, and the summation is taken over all pairs of even-element subsets \( I, J \subset [l] \) such that \( \#I = \#J \). In fact, if we consider the case \( m = n = 0 \) of Lemma 3.1 we obtain
\[ \sum_{K} (-1)^{\binom{l-\#K}{2}} \text{Pf} Z(K) \text{Pf} Z'(K) = \text{Pf} \begin{pmatrix} Z & E_l \\ -E_l & Z' \end{pmatrix}. \]

By taking \( Z = A \) and \( Z' = TB'T \) and using the minor-summation formula (3.6), we obtain (3.7).

### 4 Schur’s original definition of Q-functions

In this section, we recover Schur’s original definition [19] of \( Q \)-functions from Nimmo’s formula (1.4) by applying the Pfaffian analogue of the Sylvester formula (Proposition 2.1). Macdonald [13, III. 8] proves Part (3) of the following theorem by considering the generating function of Hall–Littlewood functions, And Stembridge’s derivation [21, Theorem 6.1] is based on the combinatorial definition of \( Q \)-functions and the lattice path method.

**Theorem 4.1.** (Schur [19])

1. The generating function of Schur \( Q \)-functions corresponding to partitions of length \( \leq 1 \) is given by
\[ \sum_{r \geq 0} Q_{(r)}(x)z^r = \prod_{i=1}^{n} \frac{1 + x_i z}{1 - x_i z}, \] (4.1)
where \( Q_{(0)}(x) = 1. \)
(2) The generating function of Schur $Q$-functions corresponding to partitions of length \( \leq 2 \) is given by
\[
\sum_{r,s \geq 0} Q_{(r,s)}(x) z^r w^s = \frac{z - w}{z + w} \left( \prod_{i=1}^{n} \frac{1 + x_i z}{1 - x_i z} \prod_{i=1}^{n} \frac{1 + x_i w}{1 - x_i w} - 1 \right),
\]
where $Q_{(0,0)}(x) = 0$ and
\[
Q_{(r,s)}(x) = -Q_{(r,s)}(x), \quad Q_{(r,0)}(x) = -Q_{(0,r)}(x) = Q_r(x)
\]
for positive integers $r$ and $s$.

(3) For a sequence of nonnegative integers $\alpha = (\alpha_1, \ldots, \alpha_l)$, we put
\[
S_\alpha(x) = \left( S_{(\alpha_i, \alpha_j)}(x) \right)_{1 \leq i, j \leq l}.
\]

Given a strict partition $\lambda$ of length $l$, we have
\[
Q_\lambda(x) = \begin{cases} \text{Pf} \, S_\lambda(x) & \text{if } l \text{ is even,} \\ \text{Pf} \, S_{\lambda^0}(x) & \text{if } l \text{ is odd,} \end{cases}
\]
where $\lambda^0 = (\lambda_1, \ldots, \lambda_l, 0)$.

First we show the following stability of Schur $Q$-functions.

**Lemma 4.2.** For a strict partition $\lambda$, we have
\[
Q_\lambda(x_1, \ldots, x_n, 0) = Q_\lambda(x_1, \ldots, x_n).
\]

**Proof.** Let $x = (x_1, \ldots, x_n)$ and $l = l(\lambda)$. Note that $D(x_1, \ldots, x_n, 0) = (-1)^n D(x_1, \ldots, x_n)$.

If $n + l$ is even, then by definition (1.4) we have
\[
Q_\lambda(x_1, \ldots, x_n, 0) = \frac{1}{(-1)^n D(x)} \text{Pf} \begin{pmatrix} A(x) & -1_{n,1} & W_\lambda(x) & 1_{n,1} \\ 1_{1,n} & 0 & -W_\lambda(x) & O_{l,1} \\ -1_{1,n} & O_{l,1} & 1 & O_{l,l} \\ \end{pmatrix}.
\]

where $1_{p,q}$ is the all-one matrix of size $p \times q$. By adding the $(n + 1)$st column/row to the last column/row and then expanding the resulting Pfaffian along the last column/row, we see that
\[
Q_\lambda(x, 0) = \frac{1}{(-1)^n D(x)} \cdot (-1)^n \text{Pf} \begin{pmatrix} A(x) & W_\lambda(x) \\ -W_\lambda(x) & O \\ \end{pmatrix} = Q_\lambda(x).
\]

If $n + l$ is odd, then we have
\[
Q_\lambda(x_1, \ldots, x_n, 0) = \frac{1}{(-1)^n D(x)} \text{Pf} \begin{pmatrix} A(x) & -1_{n,1} & W_\lambda(x) \\ 1_{1,n} & 0 & -W_\lambda(x) \\ -W_\lambda(x) & O_{l,1} & O_{l,l} \\ \end{pmatrix}.
\]
By pulling out the common factor $-1$ from the $(n+1)$st row/column and then moving the $(n+1)$st row/column to the last row/column, we see that

\[
Q_{\lambda}(x,0) = \frac{1}{(-1)^n D(x)} \cdot (-1)^{t+1} \operatorname{Pf} \begin{pmatrix}
A(x) & W_{\lambda}(x) & 1_{n,1} \\
-W_{\lambda}(x) & O_{l,l} & O_{l,1} \\
-1_{1,n} & O_{l,1} & 0
\end{pmatrix} = Q_{\lambda}(x).
\]

\[\Box\]

**Proof of Theorem 4.1.** (1) By the stability (Lemma 4.2), we may assume that $n$ is odd. Then we have

\[
Q_{(r)}(x) = \frac{1}{D(x)} \operatorname{Pf} \begin{pmatrix}
A(x) & W_{(r)}(x) \\
-W_{(r)}(x) & I_{r,r}
\end{pmatrix}, \quad r \geq 0,
\]

where $W_{(r)}(x)$ is the column vector $(\chi(r)x_i^r)_{1 \leq i \leq n}$. By using

\[
\sum_{r \geq 0} \chi(r)x_i^rz^r = \frac{1 + x_i z}{1 - x_i z},
\]

we see that

\[
\sum_{r \geq 0} Q_{(r)}(x)z^r = \frac{1}{D(x)} \operatorname{Pf} \begin{pmatrix}
A(x) & H_z(x) \\
-W_{z}(x) & 0
\end{pmatrix},
\]

where $H_z(x)$ is the column vector with $i$th entry $(1 + x_i z)/(1 - x_i z)$. The last Pfaffian is evaluated by using Proposition 2.2 with variables $(x_1, \ldots, x_n, -1/z)$ and we have

\[
\operatorname{Pf} \begin{pmatrix}
A(x) & H_z(x) \\
-W_{z}(x) & 0
\end{pmatrix} = D(x) \prod_{i=1}^{n} \frac{1 + x_i z}{1 - x_i z}.
\]

This completes the proof of (1).

(2) By the stability (Lemma 4.2), we may assume that $n$ is even. Then we have

\[
Q_{(r,s)}(x) = \frac{1}{D(x)} \operatorname{Pf} \begin{pmatrix}
A(x) & W_{(r)}(x) & W_{(s)}(x) \\
-W_{(r)}(x) & 0 & 0 \\
-W_{(s)}(x) & 0 & 0
\end{pmatrix}, \quad r, s \geq 0,
\]

and hence obtain

\[
\sum_{r,s \geq 0} Q_{(r,s)}(x)z^rw^s = \frac{1}{D(x)} \operatorname{Pf} \begin{pmatrix}
A(x) & H_z(x) & H_w(x) \\
-W_{z}(x) & 0 & 0 \\
-W_{w}(x) & 0 & 0
\end{pmatrix}.
\]

Applying Proposition 2.2 with variables $(x_1, \ldots, x_n, -1/z, -1/w)$, we see that

\[
\operatorname{Pf} \begin{pmatrix}
A(x) & H_z(x) & H_w(x) \\
-W_{z}(x) & 0 & 0 \\
-W_{w}(x) & 0 & 0
\end{pmatrix} = D(x) \prod_{i=1}^{n} \frac{1 + x_i z}{1 - x_i z} \frac{1 + x_i w}{1 - x_i w} \frac{z - w}{z + w}.
\]

By splitting the last row/column, we have
Hence we have Remark 4.3. We can give a direct proof to the Pfaffian identity (4.3) in the case where

$$\text{By expanding the last Pfaffian along the last row/column and using (2.7), we have}$$

$$(z - w) = \text{Pf} \left( \begin{pmatrix} A(x) & H_z(x) & 0 \\ -tH_z(x) & 0 & z - w \\ -H_w(x) & 0 & z + w \end{pmatrix} + \text{Pf} \left( \begin{pmatrix} A(x) & H_z(x) & 0 \\ -tH_z(x) & 0 & z - w \\ -0 & z - w & 0 \end{pmatrix} \right) \right).$$

Hence we have

$$\sum_{r,s \geq 0} Q_{(r,s)}(x) z^r w^s = \frac{z - w}{z + w} \frac{1 + x_i z}{1 - x_i z} \frac{1 + x_i w}{1 - x_i w}.$$  

(3) By the stability (Lemma 4.2), we may assume that $n$ is even. We apply the Pfaffian analogue of the Sylvester identity (Proposition 2.1) to the matrix $X$ given by

$$X = \begin{cases} 
A(x) & W_\lambda(x) \\
-W_\lambda(x) & O_{l,l} \\
A(x) & W_\lambda^0(x) \\
-W_\lambda^0(x) & O_{l+1,l+1}
\end{cases} \text{ if } l \text{ is even,}$$

$$\begin{cases} 
A(x) & 1_{n,1} & W_\lambda(x) \\
-1_{1,n} & 0 & O_{1,l} \\
-W_\lambda(x) & O_{l,1} & O_{l,l} \\
A(x) & 1_{n,1} & W_\lambda(x) & O_{n,1} \\
-1_{1,n} & 0 & O_{1,l} & 1 \\
-W_\lambda(x) & O_{l,1} & O_{l,l} & O_{l,l} \\
O_{1,n} & -1 & O_{1,l} & 0
\end{cases} \text{ if } l \text{ is odd.}$$

Since $\text{Pf} X([n] \cup \{n + i, n + j\})/\text{Pf} X([n]) = Q_{(\lambda_i, \lambda_j)}$ for $i < j$, Schur’s identity (4.3) immediately follows from Proposition 2.1. \hfill \Box

Remark 4.3. We can give a direct proof to the Pfaffian identity (4.3) in the case where $n$ is odd, by applying Proposition 2.1 to the matrix given by
5 Cauchy-type identity for $Q$-functions

In this section, we use the Pfaffian analogue of the Cauchy–Binet formula (Theorem 3.2) to prove the Cauchy-type identity for Schur $Q$-functions, which corresponds to the orthogonality of $Q$-functions. To prove the Cauchy-type identity, Schur [19, Abschnitt IX] (see also [7, § 4B]) used a characterization of $Q(n)$, and Macdonald [13, III.8] appeal to the theory of Hall–Littlewood functions. Also the proof given by Hoffman–Humphreys [2, Chapter 7] is based on the definition of $Q$-functions in terms of vertex operators. Bijective proofs are given by Worley [22, Theorem 6.1.1] and Sagan [18, Corollary 8.3]. Here we give a simple linear algebraic proof.

Theorem 5.1. (Schur [19, p. 231]) For $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$, we have

$$\sum_{\lambda} P_\lambda(x)Q_\lambda(y) = \prod_{i,j=1}^{n} \frac{1 + x_iy_j}{1 - x_iy_j}, \quad (5.1)$$

where $\lambda$ runs over all strict partitions.

The following lemma is obvious, so we omit the proof.

Lemma 5.2. Let $n$ be a positive integer and denote by $\mathbb{N}$ the set of nonnegative integers. To a strict partition $\lambda$ we associate the subset $I_n(\lambda) \subset \mathbb{N}$ given by

$$I_n(\lambda) = \begin{cases} \{\lambda_1, \ldots, \lambda_{l(\lambda)}\} & \text{if } n + l(\lambda) \text{ is even}, \\ \{\lambda_1, \ldots, \lambda_{l(\lambda)}, 0\} & \text{if } n + l(\lambda) \text{ is odd}. \end{cases}$$

Then the correspondence $\lambda \mapsto I_n(\lambda)$ gives a bijection from the set of all strict partitions to the set of all subsets $I$ of $\mathbb{N}$ with $\#I \equiv n \mod 2$.

Proof of Theorem 5.1. Apply the Pfaffian version of Cauchy–Binet formula (3.3) to the matrices

$$A = A(x), \quad B = A(y), \quad S = \left(x_i^k\right)_{1 \leq i \leq n, k \geq 0}, \quad T = \left(\chi(k)y_i^k\right)_{1 \leq i \leq n, k \geq 0}.$$ 

It follows from the definition of $P$- and $Q$-functions (1.3) and (1.4) that for a strict partition $\lambda$ we have

$$P_\lambda(x) = \left(\frac{-1}{D(x)}\right)^{\binom{\#I_n(\lambda)}{2}} \text{Pf} \begin{pmatrix} A(x) & S([n]; I_n(\lambda)) \\ -S([n]; I_n(\lambda)) & O \end{pmatrix},$$

$$Q_\lambda(y) = \left(\frac{-1}{D(y)}\right)^{\binom{\#I_n(\lambda)}{2}} \text{Pf} \begin{pmatrix} A(y) & T([n]; I_n(\lambda)) \\ -T([n]; I_n(\lambda)) & O \end{pmatrix}.$$ 

Hence, by using Lemma 5.2 and applying (3.3), we have

$$\sum_{\lambda} P_\lambda(x)Q_\lambda(y) = \frac{1}{D(x)D(y)} \sum_{I} \text{Pf} \begin{pmatrix} A(x) & S([n]; I) \\ -S([n]; I) & O \end{pmatrix} \text{Pf} \begin{pmatrix} A(y) & T([n]; I) \\ -T([n]; I) & O \end{pmatrix}$$

$$= \left(\frac{-1}{D(x)D(y)}\right)^{\binom{\#I_n(\lambda)}{2}} \text{Pf} \begin{pmatrix} A(x) & ST \\ -TS & A(y) \end{pmatrix}.$$
where \( \lambda \) runs over all strict partitions and \( I \) runs over all subsets of \( \mathbb{N} \) with \( \# I \equiv n \pmod{2} \). Since the \((i, j)\) entry of \( S^T \) is given by

\[
\sum_{k \geq 0} x_i^k \cdot \chi(k) y_j^k = \frac{1 + x_i y_j}{1 - x_i y_j},
\]

we can use the evaluation of the Schur Pfaffian \( \text{Pf} \) with variables \((x_1, \ldots, x_n, -1/y_1, \ldots, -1/y_n)\) to obtain

\[
\text{Pf} \left( \begin{array}{cc} A(x) & S^T \\ -T^S & -A(y) \end{array} \right) = D(x) \cdot \prod_{i,j=1}^n \frac{1 + x_i y_j}{1 - x_i y_j} \cdot (-1)^{\binom{n}{2}} D(y).
\]

This completes the proof. \( \square \)

6 Pragacz–Józefiak–Nimmo identity for skew \( Q \)-functions

In this section, we use the Pfaffian analogue of the Cauchy–Binet formula (Theorem 3.2) to prove the Pragacz–Józefiak–Nimmo identity for skew \( Q \)-functions. Pragacz–Józefiak \cite{17} and Nimmo \cite{15} used differential operators to prove this Pfaffian identity and Stembridge \cite[Theorem 6.2]{21} gave a combinatorial proof based on the lattice path method. In the course of our proof, we find a Pfaffian identity which interpolate Nimmo's identity (1.4) and Schur’s identity (4.3).

Skew \( Q \)-functions \( Q_{\lambda/\mu}(x_1, \ldots, x_n) \) are uniquely determined by the equation

\[
Q_{\lambda}(x_1, \ldots, x_n, y_1, \ldots, y_k) = \sum_{\mu} Q_{\lambda/\mu}(x_1, \ldots, x_n) Q_{\mu}(y_1, \ldots, y_k),
\]

where \( \lambda \) is a strict partition and the summation is taken over all strict partitions \( \mu \).

**Theorem 6.1.** (Pragacz–Józefiak \cite[Theorem 1]{17}, Nimmo \cite[(2.22)]{15}) For two sequences \( \alpha = (\alpha_1, \ldots, \alpha_l) \) and \( \beta = (\beta_1, \ldots, \beta_m) \) of nonnegative integers, let \( M_{\alpha/\beta}(x) \) be the \( l \times m \) matrix given by

\[
M_{\alpha/\beta}(x) = \left( Q_{(\alpha_i-\beta_{m+1-j})}(x) \right)_{1 \leq i \leq l, 1 \leq j \leq m},
\]

where \( Q_{(k)}(x) = 0 \) for \( k < 0 \). For two strict partitions \( \lambda \) and \( \mu \), we have

\[
Q_{\lambda/\mu}(x) = \begin{cases} 
\text{Pf} \left( \begin{array}{cc} S_{\lambda}(x) & M_{\lambda/\mu}(x) \\ -tM_{\lambda/\mu}(x) & O \end{array} \right) & \text{if } l(\lambda) \equiv l(\mu) \pmod{2}, \\
\text{Pf} \left( \begin{array}{cc} S_{\lambda}(x) & M_{\lambda/\mu}(x) \\ -tM_{\lambda/\mu}(x) & O \end{array} \right) & \text{if } l(\lambda) \not\equiv l(\mu) \pmod{2},
\end{cases}
\]

(6.1)

Note that

\[
\begin{pmatrix}
S_{\lambda} & M_{\lambda/\mu} \\
-tM_{\lambda/\mu} & O
\end{pmatrix}
= 
\begin{pmatrix}
S_{\lambda} & M_{\lambda/\mu} \\
-tM_{\lambda/\mu} & O
\end{pmatrix}.
\]
Hence we have

$$Q_{\lambda}(x, y) = \sum_{\mu} \tilde{Q}_{\lambda/\mu}(x)Q_{\mu}(y) .$$

By the stability (Lemma 4.2), we may assume that the length \( l = l(\lambda) \) and the number \( k \) of variables in \( y \) have the same parity.

We apply the Pfaffian analogue of the Cauchy–Binet formula (3.2) to the matrices

\[
A = S_\lambda(x), \quad S = \left(Q_{\lambda_i-r}(x)\right)_{1 \leq i \leq l, r \geq 0},
\]

\[
B = A(y), \quad T = \left(\chi(r)y_i^r \right)_{1 \leq i \leq l, r \geq 0}.
\]

Then, for a strict partition \( \mu \), we have

\[
S([l]; I_k(\mu)) = \begin{cases} M_{\lambda/\mu}(x) & \text{if } l(\mu) \equiv k \mod 2, \\ M_{\lambda/\mu^0}(x) & \text{if } l(\mu) \not\equiv k \mod 2, \end{cases}
\]

Hence we have

\[
\text{Pf}\left(\begin{array}{cc} A(x) & S([l]; I_k(\mu)) \\ -T([l]; I_k(\mu)) & O \end{array}\right) = \tilde{Q}_{\lambda/\mu}(x).
\]

And it follows from the definition (1.4) that

\[
\frac{1}{D(y)} \text{Pf}\left(\begin{array}{cc} A(y) & T([k]; I_k(\mu)) \\ -T([k]; I_k(\mu)) & O \end{array}\right) = (-1)^{\binom{\#I_k(\mu)}{2}}Q_{\mu}(y) .
\]

By applying (3.2), we see that

\[
\sum_{\mu} \tilde{Q}_{\lambda/\mu}(x)Q_{\mu}(y) = \frac{1}{D(y)} \text{Pf}\left(\begin{array}{cc} S_\lambda(x) & S^T \\ -T^*S & A(y) \end{array}\right) .
\]

Also it follows from the generating function (4.1) of \( Q_{\alpha} \)'s that the \((i,j)\) entry of \( S^T \) is given by

\[
\sum_{r \geq 0} Q_{\lambda_i-r}(x) \cdot \chi(r)y_j^r = Q_{\lambda_i}(x, y_j) .
\]

Now we can complete the proof by using the following Theorem.

**Theorem 6.2.** Let \( x = (x_1, \ldots, x_n) \) and \( y = (y_1, \ldots, y_k) \) be two sequence of variables. For a sequence \( \alpha = (\alpha_1, \ldots, \alpha_l) \) of length \( l \), let \( N_\alpha(x|y) \) be the \( l \times k \) matrix defined by

\[
N_\alpha(x|y) = \left(Q_{\alpha_i}(x, y_j)\right)_{1 \leq i \leq l, 1 \leq j \leq k}
\]

For a strict partition \( \lambda \) of length \( l \), we have

\[
Q_{\lambda}(x, y) = \begin{cases} \frac{1}{D(y)} \text{Pf}\left(\begin{array}{cc} S_\lambda(x) & N_\lambda(x|y) \\ -T^*S & A(y) \end{array}\right) & \text{if } l + k \text{ is even,} \\
\frac{1}{D(y)} \text{Pf}\left(\begin{array}{cc} S_{\lambda^0}(x) & N_{\lambda^0}(x|y) \\ -T^*S & A(y) \end{array}\right) & \text{if } l + k \text{ is odd,} \end{cases}
\]

(6.2)

where \( \lambda^0 = (\lambda_1, \ldots, \lambda_l, 0) \).
Note that the identity (6.2) reduces to Nimmo’s identity (1.4) if \( n = 0 \) and to Schur’s identity (4.3) if \( k = 0 \).

**Proof.** We denote by \( Q'_\lambda(x|y) \) the right hand side of (6.2).

First we show that \( Q'_\lambda(x|y) \) is stable with respect to \( y \), that is,

\[
Q'_\lambda(x|y_1, \ldots, y_k) = Q'_\lambda(x|y_1, \ldots, y_k).
\]

(6.3)

Let \( y = (y_1, \ldots, y_k) \). If \( l + k \) is even, then we have by using the stability (Lemma 4.2) of \( Q_\lambda(x) \),

\[
Q'_\lambda(x|y, 0) = \frac{1}{(-1)^kD(y)} \begin{pmatrix}
S_\lambda(x) & T_\lambda(x) & N_\lambda(x|y) & T_\lambda(x) \\
-N_\lambda(x|y) & 0 & 1_{k,1} & 1 \\
-N_\lambda(x|y) & -1_{k,1} & A(y) & -1_{k,1} \\
-T_\lambda(x) & -1 & 1_{1,k} & 0
\end{pmatrix}.
\]

where \( T_\lambda(x) \) is the column vector \( (Q_\lambda(x))_{1 \leq i \leq l} \). By adding the \((l + 1)\)st row/column multiplied by \(-1\) to the last row/column and then expanding the resulting Pfaffian along the last row/column, we see that

\[
Q'_\lambda(x|y, 0) = \frac{1}{(-1)^kD(y)} (-1)^l \begin{pmatrix}
S_\lambda(x) & N_\lambda(x|y) \\
T_\lambda(x) & A(y)
\end{pmatrix} = Q'_\lambda(x|y).
\]

If \( l + k \) is odd, then by moving the last row/column to the \((l + 1)\)st row/column we have

\[
Q'_\lambda(x|y, 0) = \frac{1}{(-1)^kD(y)} \begin{pmatrix}
S_\lambda(x) & N_\lambda(x|y) & T_\lambda(x) \\
-N_\lambda(x|y) & A(y) & -1_{k,1} \\
-T_\lambda(x) & 1_{1,k} & 0
\end{pmatrix} = \frac{1}{(-1)^kD(y)} (-1)^k \begin{pmatrix}
S_\lambda(x) & N_\lambda(x|y) \\
-T_\lambda(x) & A(y)
\end{pmatrix} = Q'_\lambda(x|y).
\]

Next we use the Sylvester formula for Pfaffians (Proposition 2.1) to prove

\[
Q'_\lambda(x|y) = \begin{cases}
Pf S'_\lambda(x|y) & \text{if } l(\lambda) \text{ is even}, \\
Pf S'_{\lambda^0}(x|y) & \text{if } l(\lambda) \text{ is odd},
\end{cases}
\]

(6.4)

where \( \lambda^0 = (\lambda_1, \ldots, \lambda_{l(\lambda)}, 0) \) and the matrix \( S'_\alpha(x|y) \) is defined by

\[
S'_\alpha(x|y) = \left( Q'_{(\alpha_i, \alpha_j)}(x|y) \right)_{1 \leq i, j \leq l}.
\]

By the stability (6.3), we may assume \( k \) is even. In this case the identity (6.4) can be obtained by applying (2.0) to the matrix

\[
X = \begin{cases}
S_\lambda(x) & N_\lambda(x|y) \\
-N_\lambda(x|y) & A(y) \\
S_{\lambda^0}(x) & N_{\lambda^0}(x|y) \\
-N_{\lambda^0}(x|y) & A(y)
\end{cases}
\]

if \( l(\lambda) \) is even,

\[
\begin{pmatrix}
S_\lambda(x) & -N_\lambda(x|y) \\
-N_\lambda(x|y) & A(y) \\
S_{\lambda^0}(x) & N_{\lambda^0}(x|y) \\
-N_{\lambda^0}(x|y) & -A(y)
\end{pmatrix}
\]

if \( l(\lambda) \) is odd.

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By comparing two Pfaffian identities (4.3) and (6.4), the proof of Theorem 6.2 is reduced to showing
\[ Q_{(r)}(x, y) = Q_{(r)}'(x|y), \quad (6.5) \]
\[ Q_{(r,s)}(x, y) = Q_{(r,s)}'(x|y). \quad (6.6) \]
We prove these equality by considering the generating functions. If we put
\[ F_z(u_1, \ldots, u_m) = \prod_{i=1}^{m} \frac{1 + u_i z}{1 - u_i z}, \]
then by virtue of (4.1) and (4.2), the identities (6.5) and (6.6) follow from
\[ \sum_{r \geq 0} Q_{(r)}'(x|y) z^r = F_z(x, y), \quad (6.7) \]
and
\[ \sum_{r, s \geq 0} Q_{(r,s)}'(x|y) z^r w^s = \frac{z - w}{z + w} (F_z(x, y) F_w(x, y) - 1), \quad (6.8) \]
respectively.

By the stability (6.3) we may assume k is odd for the proof of (6.7). If k is odd, then
\[ Q_{(r)}(x|y) = \frac{1}{D(y)} \text{Pf} \begin{pmatrix} 0 & N_{(r)}(x|y) \\ -N_{(r)}(x|y) & A(y) \end{pmatrix}. \]
Since \( \sum_{r \geq 0} Q(x, y_j) z^j = F_z(x)(1 + y_j z)/(1 - y_j z) \) by (4.1), we have
\[ \sum_{r \geq 0} Q_{(r)}'(x|y) z^r = \frac{1}{D(y)} \text{Pf} \begin{pmatrix} 0 & F_z(x) H_z(y) \\ -F_z(x) H_z(x) & A(y) \end{pmatrix} \]
\[ = \frac{1}{D(y)} F_z(x) \text{Pf} \begin{pmatrix} 0 & H_z(y) \\ -H_z(y) & A(y) \end{pmatrix}, \]
where \( H_z(y) \) is the column vector \( ((1 + y_i z)/(1 - y_i z))_{1 \leq i \leq k} \). By applying Proposition 2.2 with variables \(-1/z, y_1, \ldots, y_n, \) we have
\[ \text{Pf} \begin{pmatrix} 0 & H_z(y) \\ -H_z(y) & A(y) \end{pmatrix} = D(y) F_z(y), \]
and obtain (6.7).

By the stability (6.3) we may assume k is even for the proof of (6.8). If k is even, then
\[ Q_{(r,s)}'(x|y) = \frac{1}{D(y)} \text{Pf} \begin{pmatrix} 0 & Q_{(r,s)}(x) \\ -Q_{(r,s)}(x) & 0 \\ -N_{(r)}(x|y) & -N_{(s)}(x|y) \end{pmatrix} \]
for \( r, s \geq 0. \) Hence it follows from (4.1) and (4.2) that
\[ \sum_{r, s \geq 0} Q_{(r,s)}'(x|y) z^r w^s = \frac{1}{D(y)} \text{Pf} \begin{pmatrix} 0 & G_{z,w}(x) \\ -G_{z,w}(x) & 0 \\ -F_z(x) H_z(y) & -F_w(x) H_w(y) \end{pmatrix} \]
\[ F_z(x) H_z(y) \begin{pmatrix} F_z(x) H_z(y) \\ F_w(x) H_w(y) \end{pmatrix} A(y), \]
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where
\[ G_{z,w}(x) = \frac{z-w}{z+w} (F_z(x)F_w(x) - 1). \]

By splitting the first row/column and then pulling out the common factor \( F_z(x) \) and \( F_w(x) \) from the 1st and 2nd rows/columns, we see that
\[
\sum_{r,s \geq 0} Q'_{(r,s)}(x|y)z^rw^s = \frac{F_z(x)F_w(x)}{D(y)} \text{Pf} \left( \begin{array}{ccc}
0 & \frac{z-w}{z+w} & tH_z(y) \\
-\frac{z-w}{z+w} & 0 & tH_w(y) \\
-\frac{z-w}{z+w} & 0 & O_{1,k}
\end{array} \right) - \frac{1}{D(y)} \text{Pf} \left( \begin{array}{ccc}
0 & \frac{z-w}{z+w} & tH_w(y) \\
-\frac{z-w}{z+w} & 0 & -tH_w(y) \\
O_{k,1} & -tH_w(y) & A(y)
\end{array} \right).
\]

The first Pfaffian is evaluated by using the Schur Pfaffian with \((-1/z,-1/w,y_1,\ldots,y_n)\) and we see that
\[
\text{Pf} \left( \begin{array}{ccc}
0 & \frac{z-w}{z+w} & tH_z(y) \\
-\frac{z-w}{z+w} & 0 & tH_w(y) \\
-\frac{z-w}{z+w} & 0 & -tH_w(y) \\
\end{array} \right) = \frac{z-w}{z+w} F_z(y)F_w(y) D(y).
\]

By expanding the second Pfaffian along the first column/row, we see that it equals to
\[
\text{Pf} \left( \begin{array}{ccc}
0 & \frac{z-w}{z+w} & O_{1,k} \\
-\frac{z-w}{z+w} & 0 & tH_w(y) \\
O_{k,1} & -tH_w(y) & A(y)
\end{array} \right) = \frac{z-w}{z+w} D(y).
\]

Therefore we obtain
\[
\sum_{r,s \geq 0} Q'_{(r,s)}(x|y)z^rw^s = \frac{z-w}{z+w} (F_z(x,y)F_w(x,y) - 1).
\]

This completes the proof of Theorem 6.2 and hence Theorem 6.1. \( \square \)

7 Littlewood-type identity for Q-functions

In this section, we prove the following Littlewood-type identity for Q-functions. This identity is a special case \((t = \sqrt{-1})\) of [10] (1.21)] for Hall–Littlewood functions, which is essentially proved in [9] by using the representation theory of finite Chevalley groups.

**Theorem 7.1.** (Kawanaka [10]) For \( x = (x_1,\ldots,x_n) \), we have
\[
\sum_{\lambda} (1 + \sqrt{-1})^{(\lambda)} P_\lambda(x) = \prod_{i=1}^{n} \frac{1 + \sqrt{-1}x_i}{1 - x_i} \prod_{1 \leq i < j \leq n} \frac{1 + x_i x_j}{1 - x_i x_j},
\]
where \( \lambda \) runs over all strict partitions.
Remark 7.2. The right hand side of (7.1) is one of the simplest example of products involving the factor $\prod_{1 \leq i < j \leq n} (1 + x_i x_j) / (1 - x_i x_j)$ that is a(n infinite) linear combination of $P$- or $Q$-functions. Recall that a symmetric polynomial $f(x)$ is a linear combination of Schur $Q$-functions if and only if $f(t, -t, x_3, \ldots, x_n)$ is independent of $t$. (See [13] III (8.5)) for example.) Consider a symmetric power series of the form

$$f_n(x) = \prod_{i=1}^{n} \left( \prod_{j=1}^{n} \left( 1 - \frac{1 + x_i x_j}{1 - x_i x_j} \right) \right),$$

where $\{\alpha_1, \ldots, \alpha_r \} \cap \{\beta_1, \ldots, \beta_s \} = \emptyset$. Then

$$f_n(t, -t, x_3, \ldots, x_n) = \prod_{j=1}^{n} \left( 1 - \frac{1 + \alpha_j t}{1 + \beta_j t} \right) \cdot \frac{1 - t^2}{1 + t^2} \cdot f_{n-2}(x_3, \ldots, x_n)$$

is independent of $t$ if and only if

$$\{\alpha_1, \ldots, \alpha_r, -\alpha_1, \ldots, -\alpha_r, 1, -1\} = \{\beta_1, \ldots, \beta_s, -\beta_1, \ldots, -\beta_s, \sqrt{-1}, -\sqrt{-1}\}$$
as multisets. Thus $f_n(x)$ is an infinite linear combination of $P$-functions if and only if $r = s$ and $\alpha_1 = \pm \sqrt{-1}$, $\beta_1 = \pm 1$ and $\alpha_k = -\beta_k$ for $2 \leq k \leq r$ up to permutation of $\alpha_1, \ldots, \alpha_r$ and $\beta_1, \ldots, \beta_r$.

Proof of Theorem 7.1. By the stability (Lemma 4.2), we may assume $n$ is even. We apply the Pfaffian version of the minor-summation formula (Theorem 3.4) to the matrices

$$A = A(x), \quad S = \left( x_i^k \right)_{1 \leq i \leq n, k \geq 0},$$

and the skew-symmetric matrix $B$ whose $(i, j)$ entry, $0 \leq i < j$, is given by

$$B_{ij} = \begin{cases} -\alpha & \text{if } i = 0, \\ -\alpha^2 & \text{if } i > 0, \end{cases}$$

where $\alpha = 1 + \sqrt{-1}$.

By using (2.2), (2.4) and the induction on $#I$, we see that the subpfaffian $\text{Pf} B(I)$ of $B$ corresponding to a even-element subset $I \subset \mathbb{N}$ is given by

$$\text{Pf} B(I) = \begin{cases} (-1)^{\binom{#I}{2}} \alpha^{#I - 1} & \text{if } 0 \in I, \\ (-1)^{\binom{#I}{2}} \alpha^{#I} & \text{if } 0 \not\in I. \end{cases}$$

Since $n$ is even, strict partitions $\lambda$ are in bijection with even-element subsets $I_n(\lambda)$ of $\mathbb{N}$ by Lemma 5.2 and

$$\text{Pf} B(I_n(\lambda)) = (-1)^{\binom{#I_n(\lambda)}{2}} \alpha^{l(\lambda)}.$$

Also it follows from Nimmo’s identity (1.3) that

$$\text{Pf} \left( \begin{array}{c} A \\ -S([n]; I_n(\lambda)) \end{array} \right) \left( \begin{array}{c} S([n]; I_n(\lambda)) \\ O \end{array} \right) = (-1)^{\binom{#I_n(\lambda)}{2}} D(x) P_\lambda(x).$$
Hence, by using Proposition 2.2 with variables \((x_i)\) where
\[ e \]
we obtain this corollary from Theorem 7.1.

Therefore we have
\[
Pf(A - SB^lS) = \sum_I Pf B(I) Pf \left( \begin{bmatrix} A & S([n]; I) \\ -S([n]; I) & O \end{bmatrix} \right) = D(x) \sum_{\lambda} \alpha^{(\lambda)} P_{\lambda}(x),
\]
where \(I\) runs over all even-element subsets of \(\mathbb{N}\) and \(\lambda\) runs over all strict partitions.

By direct computations, we see that the \((i, j)\)-entry of \(SB^lS\) is equal to
\[
\sum_{k,l \geq 0} b_{k,l} x_i^k x_j^l = -\alpha \frac{x_j - x_i}{1 - x_i(1 - x_j)} - \alpha^2 \frac{x_i x_j (x_j - x_i)}{(1 - x_i x_j)(1 - x_i)(1 - x_j)},
\]
and the \((i, j)\) entry of \(A - SB^lS\) is equal to
\[
\frac{x_j - x_i}{x_j + x_i} - \sum_{k,l \geq 0} b_{k,l} x_i^k x_j^l = \frac{1 + \sqrt{-1} x_i}{1 - x_i} \frac{1 + \sqrt{-1} x_j (1 + x_i x_j)(x_j - x_i)}{1 - x_j (1 - x_i x_j)(x_j + x_i)}.
\]

Hence, by using Proposition 2.2 with variables \((x_1 - 1/x_1, \ldots, x_n - 1/x_n)\), we have
\[
Pf(A - SB^lS) = \prod_{i=1}^n \frac{1 + \sqrt{-1} x_i}{1 - x_i} Pf \left( \frac{(x_j - x_i)(1 + x_i x_j)}{(x_j + x_i)(1 - x_i x_j)} \right)_{1 \leq i, j \leq n}
= \prod_{i=1}^n \frac{1 + \sqrt{-1} x_i}{1 - x_i} \prod_{1 \leq i < j \leq n} \frac{(x_j - x_i)(1 + x_i x_j)}{(x_j + x_i)(1 - x_i x_j)}.
\]

This completes the proof.

By considering the real and imaginary parts of Theorem 7.1, we obtain

**Corollary 7.3.** If we put
\[
a_l = \begin{cases} 
(-1)^k 2^{2k} & \text{if } l = 4k, \\
(-1)^k 2^{2k} & \text{if } l = 4k + 1, \\
0 & \text{if } l = 4k + 2, \\
(-1)^{k+1} 2^{2k+1} & \text{if } l = 4k + 3,
\end{cases}
\]
\[
b_l = \begin{cases} 
0 & \text{if } l = 4k, \\
(-1)^k 2^{2k+1} & \text{if } l = 4k + 1, \\
(-1)^{k+1} 2^{2k+1} & \text{if } l = 4k + 2, \\
(-1)^{k+2} 2^{2k+1} & \text{if } l = 4k + 3,
\end{cases}
\]
then we have
\[
\sum_{\lambda} a_{l(\lambda)} P_{\lambda}(x) = \frac{1 - e_2 + e_4 - e_6 + \ldots}{1 - e_1 + e_2 - e_3 + \ldots} \prod_{1 \leq i < j \leq n} \frac{1 + x_i x_j}{1 - x_i x_j},
\]
\[
\sum_{\lambda} b_{l(\lambda)} P_{\lambda}(x) = \frac{e_1 - e_3 + e_5 - \ldots}{1 - e_1 + e_2 - e_3 + \ldots} \prod_{1 \leq i < j \leq n} \frac{1 + x_i x_j}{1 - x_i x_j},
\]
where \(e_k = e_k(x)\) is the \(k\)th elementary symmetric polynomial.

**Proof.** Since we have
\[
(1 + \sqrt{-1})^l = a_l + b_l \sqrt{-1},
\]
we obtain this corollary from Theorem 7.1. \(\square\)
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