On Einstein submanifolds of Euclidean space

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Abstract

Let the warped product $M^n = L^m \times F^{n-m}$, $n \geq m + 3 \geq 8$, of Riemannian manifolds be an Einstein manifold with Ricci curvature $\rho$ that admits an isometric immersion into Euclidean space with codimension two. Under the assumption that $L^m$ is also Einstein, but not of constant sectional curvature, it is shown that $\rho = 0$ and that the submanifold is locally a cylinder with an Euclidean factor of dimension at least $n - m$. Hence $L^m$ is also Ricci flat. If $M^n$ is complete, then the same conclusion holds globally if the assumption on $L^m$ is replaced by the much weaker condition that either its scalar curvature $S_L$ is constant or that $S_L \leq (2m - n)\rho$.

A Riemannian manifold $M^n$ is said to be Einstein if its Ricci tensor is proportional to the metric, that is, if $\text{Ric}_M(X,Y) = \rho \langle X,Y \rangle$ for any vector fields $X,Y \in \mathfrak{X}(M)$ and a constant $\rho \in \mathbb{R}$. Then $\rho$ is the not normalized constant Ricci curvature of $M^n$. Einstein manifolds produced as warped product have been quite intensively studied since Besse [2], in his book, presented many results on this subject. On the contrary, this class of manifolds has seldom been considered as submanifolds of Euclidean space.

Cartan [10] and Fialkow [7] proved that any Einstein hypersurface in Euclidean space $f: M^n \to \mathbb{R}^{n+1}$, $n \geq 3$, has constant sectional curvature, namely, it is either flat or an open subset of a round sphere. In [6] we investigated the case of Einstein submanifolds $M^n$ in Euclidean space $\mathbb{R}^{n+2}$ when $M^n$ possess the metric structure of a warped product over a surface, that is, if $M^n = L^2 \times F^{n-2}$. We constructed many local examples with non-constant sectional curvature. On the other hand, we showed that if $F^{n-2}$ has constant sectional curvature, the only complete examples with dimension $n \geq 5$ are a certain product of round spheres and the Ricci flat Generalized Schwarzschild metric (GS-metric) isometrically immersed as an $(n - 2)$-rotational submanifold.
Recall that the GS-metric is a Ricci flat warped product metric on the product manifold $\mathbb{R}^2 \times F^{n-2}$ where the constant Ricci curvature of $F^{n-2}$ is positive. The warped product $M^n = \mathbb{R}^2 \times_{\varphi} S^{n-2}$ is said to be endowed with the GS-metric metric when $\mathbb{R}^2$ in polar coordinates has the rotational invariant warped metric $ds^2 = dt^2 + \varphi'(t)^2 d\theta^2$ where the warping function $\varphi \in C^\infty([0, +\infty))$ is the unique positive solution of the differential equation $\varphi'' = 1 + c/\varphi^{n-3}$ with a given initial condition and a constant $c < 0$.

The above considerations raise the following general question: Which are the Einstein submanifolds $f: M^n = L^m \times_{\varphi} F^{n-m} \rightarrow \mathbb{R}^{n+2}$ when $m \geq 3$ and $n - m \geq 2$?

In this paper we investigate the case of isometric immersions of Einstein manifolds of the form $M^n = L^m \times_{\varphi} F^{n-m}$ when the base $L^m$ has dimension $m \geq 5$. We recall that if $M^n$ is Einstein, then $F^{n-m}$ also has to be Einstein; cf. Corollary 9.107 in [2]. In addition, we have that $M^n$ is a complete manifold if and only if both factors are complete.

As already clarified in [6], in order to obtain classification results, the assumption that an Euclidean submanifold is Einstein is quite weak, thus requiring the use of additional hypothesis in order to be successful. In the local case, we assume that $L^m$ is also an Einstein manifold. This class of manifolds have already been intrinsically under consideration, for instance in [1]. In the case that the manifold is complete, we do with a quite weaker assumptions on the scalar curvature of $L^m$, for instance, that it is constant. Intrinsically, the Einstein manifolds satisfying that condition have been considered in [8].

We call nontrivial an Einstein manifold $N^n$, $n \geq 4$, if in no open subset the sectional curvature is a nonnegative constant. Due to the aforementioned result by Fialkow, the condition of non triviality implies that non open subset of $N^n$ admits an isometric immersion into $\mathbb{R}^{n+1}$.

Locally, and when the base is Einstein, we have the following result.

**Theorem 1.** Let $M^n = L^m \times_{\varphi} F^{n-m}$, $n \geq m + 3 \geq 8$, and $L^m$ be nontrivial Einstein manifolds. If there exists an isometric immersion $f: M^n \rightarrow \mathbb{R}^{n+2}$ then $M^n$ is Ricci flat and the warping function satisfies that $\|\nabla \varphi\| = c \geq 0$ is constant. Moreover, any point of an open dense subset $M^n_0$ of $M^n$ lies in an open product neighborhood $U = L^n_0 \times F^n_0$, where $L^n_0 \subset L^m$ and $F^n_0 \subset F^{n-m}$, such that on $M^n_0$ always the same of one of the following holds:
(i) We have \( c = 0 \), there is an isometric immersion \( f_0: L^m_0 \to \mathbb{R}^{m+2} \), \( F_0^{n-m} \subset \mathbb{R}^{n-m} \) is an open subset and \( f|_U = f_0 \times \text{id}_{F_0} \).

(ii) We have \( c > 0 \), \( L^m_0 = I \times N^{m-1} \) where \( I \subset \mathbb{R} \) is an open interval and \( F_0^{n-m} \subset S^{n-m}(1/\sqrt{c}) \) is an open subset. There is an orthogonal splitting \( \mathbb{R}^{n+2} = \mathbb{R}^{m+1} \oplus \mathbb{R}^{n-m+1} \) such that \( f|_U = f_0 \times j \) where \( f_0: N^{m-1} \to \mathbb{R}^{m+1} \) is an isometric immersion and the map \( j: I \times F_0^{n-m} \to \mathbb{R}^{n-m+1} \) given by \( j(t,y) = ty \) is a parametrization of an open subset of \( \mathbb{R}^{n-m+1} \).

\[
\begin{array}{c|c}
\mathbb{R}^{n-m+1} \oplus \mathbb{R}^{m+1} & = \mathbb{R}^{n+2} \\
| & \\
j & f_0 \\
(I \times S^{n-m}(1/\sqrt{c})) \supset (I \times F_0^{n-m}) \times N^{m-1} = U \subset M^n
\end{array}
\]

In case (ii) we have that \( U \) is foliated by hypersurfaces parametrized by \( I \) whose images are \((n-m)\)-rotational submanifolds. Besides, observe that the submanifold \( f(U) \) in part (ii) is an \((n-m+1)\)-cylinder and thus it is trivially an \((n-m)\)-cylinder as the submanifolds in part (i).

In the following global result the assumption that \( L^m \) is Einstein is replaced by weaker conditions on its scalar curvature.

**Theorem 2.** Let \( M^n = L^m \times \varphi F^{n-m} \), \( n \geq m+3 \geq 8 \), be a complete Einstein manifold with Ricci curvature \( \rho \). Assume that no open subset of \( L^m \) admits an isometric immersion into \( \mathbb{R}^{m+1} \) and that its scalar curvature \( S_L \) satisfies one of the following:

(i) \( S_L \) is constant,

(ii) \( S_L \leq (2m - n)\rho \).

Then any isometric immersion \( f: M^n \to \mathbb{R}^{n+2} \) is globally a cylinder as in case (i) of Theorem 1.

The above results reduce the problem to the classification of the Ricci flat submanifolds in Euclidean space with codimension two. It is known that these submanifolds always have flat normal bundle; see [3] or Exercise 3.18 in [5]. But regardless of that simplification, the task seems to be rather difficult due to the reason already pointed out, namely, the weakness of the Einstein
assumption in our context. To sense the level of difficulty see Corollary 2 in [11]. On the other hand, an abundance of examples of Ricci flat submanifolds comes out from Examples 1 in [6].

1 Some general facts

Throughout the paper $F^m(\varepsilon)$, $m \geq 2$, denotes an Einstein manifold with Ricci curvature $(m-1)\varepsilon$, $\varepsilon = 1, -1, 0$. Of course, this is the case of the unit sphere $S^m(1)$ and the unit hyperbolic space $H^m(-1)$.

**Proposition 3.** The warped product $M^n = L^m \times_{\varphi} F^{n-m}(\varepsilon)$, $n - m \geq 2$, is an Einstein manifold of Ricci curvature $\rho$ if and only if the warping function $\varphi \in C^\infty(L)$ satisfies

\[(n - m)\text{Hess}\varphi(X,Y) = (\text{Ric}_L(X,Y) - \rho\langle X,Y \rangle_L)\varphi \quad (1)\]

for any vector fields $X, Y \in \mathfrak{X}(L)$ and that

\[\Delta \varphi + \frac{n - m - 1}{\varphi}(\|\nabla \varphi\|^2 - \varepsilon) + \rho \varphi = 0 \quad (2)\]

where Hess $\varphi$ denotes the Hessian and $\Delta \varphi$ the Laplacian of $\varphi$. Then $M^n$ and $L^m$ are both Einstein manifolds with Ricci curvatures $\rho$ and $\mu$, respectively, if and only if $\varphi \in C^\infty(L)$ satisfies

\[(n - m)\text{Hess}\varphi(X,Y) = (\mu - \rho)\varphi(X,Y)_L \quad (3)\]

for any $X, Y \in \mathfrak{X}(L)$ and

\[\|\nabla \varphi\|^2 = \varepsilon - \frac{m\mu + (n - 2m)\rho}{(n - m)(n - m - 1)}\varphi^2. \quad (4)\]

**Proof.** Equations (1) and (2) are the equations 9.107b) and 9.107c) in [2].

**Proposition 4.** Let $M^n = L^m \times_{\varphi} F^{n-m}(\varepsilon)$ and $L^m$, $n \geq m + 2 \geq 5$, be Einstein manifolds with Ricci curvature $\rho$ and $\mu$, respectively. Assume that $\nabla \varphi \neq 0$ at any point. Then $L^m$ is locally a warped product $I \times_{\varphi'} N^{m-1}$ where $N^{m-1}$ is an Einstein manifold with Ricci curvature $\varepsilon \rho(m - 2)/(n - 1)$, $I \subset \mathbb{R}$ is an open interval and $\varphi \in C^\infty(I)$. Moreover, we have:

(i) If $\rho = 0$, then $\varepsilon = 1$ and $\varphi(t) = t > 0$. 

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(ii) If $\rho > 0$, then $\varepsilon = 1$ and
\[
\varphi(t) = a \cos(t\sqrt{\rho/(n-1)}) + b \sin(t\sqrt{\rho/(n-1)}
\]
where $t \in (0, \pi/2\sqrt{(n-1)/\rho})$ and $a, b > 0$ satisfy $a^2 + b^2 = (n-1)/\rho$.

(iii) If $\rho < 0$, then $\varepsilon = -1, 0, 1$ and
\[
\varphi(t) = a \cosh(t\sqrt{-\rho/(n-1)}) + b \sinh(t\sqrt{-\rho/(n-1)})
\]
where $t \in \mathbb{R}$ and $a, b \in \mathbb{R}$ satisfy $a^2 - b^2 = \varepsilon(n-1)/\rho$.

Proof. The first part of the statement follows from (3) and Brinkmann's theorem ([9, Th. 4.3.3]). Since the warping function depends only on $t \in I$ then $\text{Hess } \varphi = \varphi'' \langle \cdot, \cdot \rangle_L$ and, consequently, (3) and (4) become
\[
(n-m)\varphi'' = (\mu - \rho)\varphi
\] (5)
and
\[
(\varphi')^2 = \varepsilon - \frac{m\mu + (n-2m)\rho}{(n-m)(n-m-1)}\varphi^2.
\] (6)
Differentiating (6) and using (5) we obtain
\[
(n-1)\mu = (m-1)\rho.
\] (7)
Hence (5) and (6) can be written as
\[
(n-1)\varphi'' + \rho\varphi = 0
\] (8)
and
\[
(\varphi')^2 = \varepsilon - \frac{\rho}{n-1}\varphi^2.
\] (9)
If follows from (7) and Proposition 9.106 of [2] that
\[
\text{Ric}_N = \frac{1}{n-1} \left( (m-1)\rho(\varphi')^2 + (n-1)\varphi\varphi'' + (n-1)(m-2)(\varphi'')^2 \right) \langle \cdot, \cdot \rangle_N.
\]
Hence, taking into account (8) and (9) we obtain
\[
(n-1)\text{Ric}_N = (m-2)\varepsilon \rho \langle \cdot, \cdot \rangle_N.
\]
Finally, solving (8) and using (9) gives the expressions for $\varphi$ stated in parts (i) to (iii).
2 The proofs

The proofs of the theorems given in the introduction rely heavily on results obtained in [4] for isometric immersions of warped products into space forms. From there we extract the facts given in this section.

In the case of hypersurfaces we have the following:

**Proposition 5.** Let \( f: M^n = L^m \times_{\varphi} F^{n-m} \to \mathbb{R}^{n+1}, m \geq 1 \) and \( n \geq 2 \), be an isometric immersion with \( M^n \) free of flat points. Then \( f = \psi \circ (F \times G) \), where \( \psi: V^{m+k_1} \times_{\sigma} S^{n-m+k_2}(r) \to \mathbb{R}^{n+1} \) for \( V^{m+k_1} \subset \mathbb{R}^{m+k_1} \), is a warped product representation of \( \mathbb{R}^{n+1}, k_1 + k_2 = 1 \), and the maps \( F: L^m \to V^{m+k_1} \) and \( G: F^{n-m} \to S^{n-m+k_2}(r) \) are isometric immersions.

Recall that an \((n-m)\)-rotational submanifold \( f: M^n \to \mathbb{R}^{n+2}, n > m \), with axis \( \mathbb{R}^{m+1} \) over a submanifold \( g: L^m \to \mathbb{R}^{m+2} \) is the \( n \)-dimensional submanifold generated by the orbits of the points of \( g(L) \) (disjoint from \( \mathbb{R}^{m+1} \)) under the action of the subgroup \( SO(n-m+1) \) of \( SO(n+2) \) which keeps pointwise \( \mathbb{R}^{m+1} \) invariant. Such a submanifold can be parametrized as follows:

The manifold \( M^n \) is isometric to (an open subset of) a warped product \( L^m \times_{\varphi} S^{n-m}(1) \) and there is an orthogonal splitting \( \mathbb{R}^{m+2} = \mathbb{R}^{m+1} \oplus \text{span\{e\}}, \|e\| = 1 \), such that the profile \( g: L^m \to \mathbb{R}^{m+2} \) of \( f \) is an isometric immersion given by \( g = (h, \varphi) \) where \( h: L^m \to \mathbb{R}^{m+1} \) and \( \varphi = (g, e) > 0 \).

Then, we have that \( f: L^m \times_{\varphi} S^{n-m}(1) \to \mathbb{R}^{n+2} \) is given by

\[
  f(z, y) = (h(z), \varphi(z)\phi(y))
\]

where \( \phi: S^{n-m}(1) \to \mathbb{R}^{n-m+1} \) denotes the inclusion \( S^{n-m}(1) \subset \mathbb{R}^{n-m+1} \).

The following result deals with the case of codimension two.

**Proposition 6.** Let \( f: M^n = L^m \times_{\varphi} F^{n-m} \to \mathbb{R}^{n+2}, n \geq m + 3 \geq 5 \), be an isometric immersion. Assume that no open subset of \( L^m \) admits an isometric immersion into \( \mathbb{R}^{m+1} \). Then there is an open dense subset of \( M^n \) such that each point lies in an open product neighborhood \( U = L_0^m \times F_0^{n-m} \subset M^n \) with \( L_0^m \subset L^m \) and \( F_0^{n-m} \subset F^{n-m} \) such that one of the following holds.

(i) \( f|_U \) is an \((n - m)\)-cylinder, that is, there is an isometric immersion \( h: L_0^m \to \mathbb{R}^{m+2} \) and an isometry \( j \) of \( F_0^{n-m} \) with open subset of \( \mathbb{R}^{n-m} \) such that \( f|_U = h \times j \).

(ii) \( f|_U \) is an \((n - m)\)-rotational submanifold parametrized as in (10).
2.1 The local result

Proof of Theorem 1: Up to an homothety, we may assume that $F^{n-m}$ is just $F^{n-m}(\varepsilon)$. According to Proposition 6, there is an open dense subset of $M^n$ such that each point lies in an open product neighborhood $U = L_0^m \times F_0^{n-m}$ with $L_0^m \subset L^m$ and $F_0^{n-m} \subset F^{n-m}(\varepsilon)$ so that $f|_U$ either is an $(n-m)$-cylinder being $\varphi$ constant on $U$ and which consequently is Ricci flat, or has to be an $(n-m)$-rotational submanifold.

Hereafter, we assume that $f|_U$ is an $(n-m)$-rotational submanifold. First observe that $\varphi$ is not constant on any open subset of $L_0^m$ since, otherwise, such a subset would admit an isometric immersion into $\mathbb{R}^{m+1}$ contradicting the assumption that $L^m$ is nontrivial. Thus we may assume that $\nabla \varphi \neq 0$ on $U$. Let $\rho$ be the Ricci curvature of $M^n$. From Proposition 4 we have that $L_0^m$ is locally a warped product $I \times_\varphi N^{m-1}$ where $N^{m-1}$ is an Einstein manifold with Ricci curvature $(m-2)\rho/(n-1)$ and $\varphi \in C^\infty(I)$ is given by parts (i) to (iii).

We claim that $M^n$ is Ricci flat. At first we prove that its Ricci curvature cannot be negative. To the contrary, let us suppose that $\rho < 0$. Then Proposition 4 yields

$$\varphi(t) = a \cosh(t\sqrt{-\rho/(n-1)}) + b \sinh(t\sqrt{-\rho/(n-1)})$$

where $t \in I$ and $a, b \in \mathbb{R}$ satisfy $a^2 - b^2 = (n-1)/\rho$. Setting

$$a = \sqrt{-(n-1)/\rho \sinh(t_0 \sqrt{-\rho/(n-1)})}$$

and

$$b = \sqrt{-(n-1)/\rho \cosh(t_0 \sqrt{-\rho/(n-1)})}$$

for some $t_0$, we have

$$\varphi(t) = \sqrt{-(n-1)/\rho \sinh((t + t_0) \sqrt{-\rho/(n-1)})}$$

for all $t \in I$. Since we have that the profile $g: L_0^m = I \times_\varphi N^{m-1} \rightarrow \mathbb{R}^{m+2}$ of the $(n-m)$-rotational submanifold $f|_U$ is given by $g(t, x) = (h(t, x), \varphi(t))$ where $h: L_0^m = I \times_\varphi N^{m-1} \rightarrow \mathbb{R}^{m+1}$, then

$$1 = \langle g_* \partial_t, g_* \partial_t \rangle = \langle h_* \partial_t, h_* \partial_t \rangle + (\varphi'(t))^2,$$

which contradicts that $\varphi'(t) > 1$ for $t \neq -t_0$.  

Next we show that the Ricci curvature cannot be positive. To the contrary, if \( \rho > 0 \) then Proposition \( \mathbf{4} \) gives that

\[
\varphi(t) = a \cos(t \sqrt{\rho/(n-1)}) + b \sin(t \sqrt{\rho/(n-1)})
\]

where \( a, b \in \mathbb{R} \) satisfy \( a^2 + b^2 = (n-1)/\rho \). Setting

\[
a = \sqrt{(n-1)/\rho} \cos(t_0 \sqrt{\rho/(n-1)}) \quad \text{and} \quad b = \sqrt{(n-1)/\rho} \sin(t_0 \sqrt{\rho/(n-1)})
\]

for some \( t_0 \), we obtain

\[
\varphi(t) = \sqrt{(n-1)/\rho} \cos((t-t_0) \sqrt{\rho/(n-1)})
\]

for \( t \in I \). As above the profile of \( f|_U \) is \( g(t,x) = (h(t,x), \varphi(t)) \) where \( h: L^m_0 = I \times \varphi' N^{m-1} \to \mathbb{R}^{m+1} \). The metric induced by \( h \) is

\[
\langle \cdot, \cdot \rangle_h = (1 - (\varphi'(t))^2) dt^2 + (\varphi'(t))^2 \langle \cdot, \cdot \rangle_{N^{m-1}},
\]

\[
= \cos^2 \left( (t-t_0) \sqrt{\rho/(n-1)} \right) dt^2 + \sin^2 \left( (t-t_0) \sqrt{\rho/(n-1)} \right) \langle \cdot, \cdot \rangle_{N^{m-1}}.
\]

Then setting \( s = \sqrt{(n-1)/\rho} \sin((t-t_0) \sqrt{\rho/(n-1)}) \), we obtain

\[
\langle \cdot, \cdot \rangle_h = ds^2 + \frac{\rho s^2}{n-1} \langle \cdot, \cdot \rangle_{N^{m-1}}.
\]

Thus we have an isometric immersion \( h: J \times_s K^{m-1} \to \mathbb{R}^{m+1} \), where \( J \subset \mathbb{R} \) is an open interval and \( K^{m-1} = (N^{m-1}, \rho/(n-1) \langle \cdot, \cdot \rangle_{N^{m-1}}) \) is an Einstein manifold with Ricci curvature \( m - 2 \). Then Proposition \( \mathbf{5} \) yields that \( h \) is of the form \( h = \psi \circ (f_0 \times g_0) \), where \( \psi: V^{k_1+1} \times \sigma S^{m+k_2-1}(r) \to \mathbb{R}^{m+1} \) is a warped product representation of \( \mathbb{R}^{m+1} \) such that \( k_1 + k_2 = 1 \), the maps \( f_0: J \to V^{k_1+1} \subset \mathbb{R}^{k_1+1} \) and \( g_0: K^{m-1} \to S^{m+k_2-1}(r) \) are isometric immersions and \( s = \sigma \circ f_0 \). Recall that \( \psi \) is the map defined by

\[
\psi(p_0, p_1) = p_0 + \sigma(p_0)(p_1 - q),
\]

where \( q \) is a point in the intersection \( V^{k_1+1} \cap S^{m+k_2-1}(r) \) and \( \sigma: V^{k_1+1} \to \mathbb{R}_+ \) is the linear function given by \( \sigma(p_0) = \langle p_0, q \rangle/r^2 \).

According to the above we need to distinguish two cases. At first let \( k_1 = 1 \). Hence \( f_0 \) is just a curve in \( V^2 \subset \mathbb{R}^2 \) and \( g_0 \) an isometry of \( K^{m-1} \) to an open subset of \( S^{m-1}(r) \). Moreover, since \( K^{m-1} \) is an Einstein manifold
with Ricci curvature \( m - 2 \) then \( r = 1 \). Hence the metric induced by \( h \) is the flat metric
\[
\langle \cdot, \cdot \rangle_h = ds^2 + s^2 \langle \cdot, \cdot \rangle_{S^{m-1}(1)}.
\]
For simplicity we choose \( q = (1, 0, \ldots, 0) \) and let \( f_0(s) = (x(s), 0, \ldots, 0, y(s)) \) be a parametrization. Clearly \( f_0(s) \) is a unit speed curve. Since \( s = \sigma(f_0(s)) \) then \( x(s) = s \) and hence \( y(s) \) is constant. Thus \( h(L^m_0) \) lies in a hyperplane of \( \mathbb{R}^{m+1} \) and therefore the profile in a hyperplane of \( \mathbb{R}^{m+2} \), contradicting the assumption that the Einstein manifold \( L^m \) is nontrivial.

Now let \( k_2 = 1 \). Then \( f_0 \) is the identity and \( g_0 \) an Einstein hypersurface in \( S^m(r) \). For simplicity choosing \( q = (r, 0, \ldots, 0) \) in the warped product representation of \( \mathbb{R}^{m+1} \), then \( h: J \times_s K^{m-1} \to \mathbb{R}^{m+1} \) is the cone given by
\[
h(s, z) = \frac{s}{r} g_0(z).
\]
This implies that the induced metric by \( h \) is
\[
\langle \cdot, \cdot \rangle_h = ds^2 + \left( \frac{s}{r} \right)^2 \langle \cdot, \cdot \rangle_{K^{m-1}}.
\]
On the other hand, since we had that
\[
\langle \cdot, \cdot \rangle_h = ds^2 + s^2 \langle \cdot, \cdot \rangle_{K^{m-1}},
\]
hence \( r = 1 \) and \( h: I \times_{\varphi'} N^{m-1} \to \mathbb{R}^{m+1} \) is given by
\[
h(t, z) = \sqrt{(n-1)/\rho} \sin((t-t_0)\sqrt{\rho/(n-1)})g_0(z).
\]
It follows from the parametrization (10) that the rotational submanifold is contained in the sphere \( S^{n+1}(\sqrt{(n-1)/\rho}) \) and hence has constant sectional curvature according to the classification of Einstein spherical submanifolds due to Fialkow [7]. But this contradicts our assumption on \( M^n \) and proves the claim.

Since \( M^n \) is Ricci flat then Proposition 4 yields \( \varphi(t, x) = t \). The profile \( g: L^m_0 = I \times N^{m-1} \to \mathbb{R}^{m+2} \) of the \( (n - m) \)-rotational submanifold \( f|_U \) is \( g(t, x) = (h(t, x), t) \) where \( h: L^m_0 = I \times N^{m-1} \to \mathbb{R}^{m+1} \). That
\[
1 = \langle g_*\partial_t, g_*\partial_t \rangle = \langle h_*\partial_t, h_*\partial_t \rangle + 1,
\]
implies that \( h \) is independent of \( t \). Hence \( g = f_0 \times id_I \) where \( f_0 = h \) is a nonflat Ricci flat submanifold. Now it follows from (10) that the submanifold is as in (ii). To conclude the proof, observe that \( \|\nabla \varphi\| \) has to take the same constant value in all of \( M^n \).
Remarks 7. (1) The first assumption in Theorem 1 that \( m > 4 \) is necessary. In fact, if in part (ii) we have \( m = 4 \) then \( N^3 \) being Einstein has constant sectional curvature and the same would be the case of \( L^4_0 \), but that possibility has been excluded.

(2) The first assumption on \( L^m \) in Theorem 1 cannot be dropped as shown by the following example of an Einstein manifolds that admits an isometric immersion in codimension two but not as a cylinder.

Let \( M^n = S^m(\sqrt{(m-1)/\rho}) \times S^{n-m}(\sqrt{(n-m-1)/\rho}) \) and let \( f : M^n \to \mathbb{R}^{n+2} \) be the product of the inclusions \( S^{n_j}(r_j) \subset \mathbb{R}^{n_j+1} \), where \( n_j = m, n - m \) and \( r_j = \sqrt{(n_j - 1)/\rho} \). Notice that this example justifies the necessity of the assumption on \( L^m \) in Theorem 2.

2.2 The global result

Proof of Theorem 2: We assume that \( F^{n-m} = F^{n-m}(\varepsilon) \) and by Proposition 6 there is an open dense subset of \( M^n \) so that each point lies in an open product neighborhood \( U = L^m_0 \times F^{n-m}_0 \subset M^n \) with \( L^m_0 \subset L^m \) and \( F^{n-m}_0 \subset F^{n-m}(\varepsilon) \) such that \( f|_U \) is either an \((n-m)\)-cylinder and thus \( \varphi \) is constant on \( L^m_0 \), or it is an \((n-m)\)-rotational submanifold. In the latter case, \( f|_U \) is parametrized by (10) and hence \( \varphi = \langle g, e \rangle \), where \( g : L^m_0 \to \mathbb{R}^{m+2} \) is the profile and \( e \) is a unit vector in \( \mathbb{R}^{m+2} \). Since \( \nabla \varphi \) is the tangent component of \( e \) then \( \|\nabla \varphi\| \leq 1 \). Therefore, we have in either case that \( \|\nabla \varphi\| \leq 1 \) on \( L^m_0 \), and by continuity on all of \( L^m \).

Taking traces in (1) gives

\[
(n - m) \Delta \varphi = \varphi(S_L - m \rho). \tag{11}
\]

Then combining with (2) yields

\[
S_L + (n - 2m) \rho = (n - m)(n - m - 1) \frac{\varepsilon - \|\nabla \varphi\|^2}{\varphi^2}. \tag{12}
\]

We claim that \( \varphi \) is constant on \( L^m \). Suppose to the contrary that \( \varphi \) is not constant. It follows from Proposition 6 that \( M^n \) has an open subset \( V \) such that \( f|_V \) is an \((n-m)\)-rotational submanifold. Thus \( \varepsilon = 1 \) and (12) becomes

\[
\|\nabla \varphi\|^2 = 1 - \frac{S_L + (n - 2m) \rho}{(n - m)(n - m - 1)} \varphi^2. \tag{13}
\]
Since $\|\nabla \varphi\| \leq 1$, we have $S_L \geq (2m - n)\rho$. Clearly, we have to distinguish two cases.

Suppose that $S_L = (2m - n)\rho$. Then (13) implies that $\|\nabla \varphi\| = 1$ everywhere and hence the integral curves of $\nabla \varphi$ are geodesics. If $\gamma(t), t \in \mathbb{R}$, is such a curve then $(\varphi \circ \gamma)'(t) = 1$. Hence $(\varphi \circ \gamma)(t) = t + c_0$ for any $t \in \mathbb{R}$, which is a contradiction.

Suppose now that $S_L$ is constant and $S_L > (2m - n)\rho$. It follows from (13) that $\varphi \leq a$, where

$$a = \sqrt{\frac{(n-m)(n-m-1)}{S_L + (n-2m)\rho}}.$$

In addition, (13) implies that the set $M_c$ of the critical points of the warping function coincides with the set of points where it attains the value $a$. We consider on $M \setminus M_c$ the unit vector field $v = \nabla \varphi / \|\nabla \varphi\|$. Using (13) and

$$2\nabla v \cdot \nabla \varphi = \nabla \|\nabla \varphi\|^2,$$

where $\nabla$ stands for the Levi-Civita connection on $L^m$, we obtain that the integral curves of $v$ are unit speed geodesics. Let $\gamma(s)$ be such a curve and consider the function $u = \varphi \circ \gamma$. Then (13) yields

$$(u'(s))^2 = 1 - \frac{1}{a^2}(u(s))^2$$

for all $s \in \mathbb{R} \setminus A$, where $A = \{s \in \mathbb{R} : \gamma(s) \in M_c\}$. But besides the constant solution we have that $u(s) = a \sin(c \pm s/a), c \in \mathbb{R}$ which is defined on the whole real line and then has to be discarded since it takes negative values. Thus the claim has been proved.

Since $\varphi$ is constant then $\varepsilon = 0$. In fact, if $\varepsilon = 1$ we have by Proposition 6 that $f$ is locally $(n-m)$-rotational parametrized by (10), thus contradicting the assumption that no open subset of $L^n$ admits an isometric immersion into $\mathbb{R}^{m+1}$. Hence, from (12) and that $\varphi$ is constant, we obtain $S_L = (2m - n)\rho$. But since (11) yields that $S_L = m\rho$, then $M^n$ is Ricci flat.

We have that there in no open subset $V \subset M^n$ such that $f|_V$ is an $(n-m)$-rotational submanifold parametrized by (10). In fact, if otherwise the profile of $f|_V$ would lie in a hyperplane of $\mathbb{R}^{m+2}$, and consequently, an open subset of $L^m$ would admit an isometric immersion into $\mathbb{R}^{m+1}$. Hence $f$ is locally an $(n-m)$-cylinder, and since $\rho = 0$ then $L^m$ is Ricci flat. That $f$ is globally a cylinder over a complete Ricci flat submanifold follows from Theorem 1 and the completeness of $M^n$. \qed
Acknowledgment

The first and third authors thank the Mathematics Department of the University of Murcia where part of this work was developed for the kind hospitality during their visits.

This research is part of the grant PID2021-124157NB-I00, funded by MCIN/ AEI/10.13039/501100011033/ “ERDF A way of making Europe”

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