Highly dispersive optical solitons and other solutions for the Radhakrishnan–Kundu–Lakshmanan equation in birefringent fibers by an efficient computational technique

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Abstract
In this article, we are interested to discuss the exact optical soiltons and other solutions in birefringent fibers modeled by Radhakrishnan–Kundu–Lakshmanan equation in two component form for vector solitons. We extract the solutions in the form of hyperbolic, trigonometric and exponential functions including solitary wave solutions like multiple-optical soliton, mixed complex soliton solutions. The strategy that is used to explain the dynamics of soliton is known as generalized exponential rational function method. Moreover, singular periodic wave solutions are recovered and the constraint conditions for the existence of soliton solutions are also reported. Besides, the physical action of the solution attained are recorded in terms of 3D, 2D and contour plots for distinct parameters. The achieved outcomes show that the applied computational strategy is direct, efficient, concise and can be implemented in more complex phenomena with the assistant of symbolic computations. The primary benefit of this technique is to develop a significant relationships between NLPDEs and others simple NLODEs and we have succeeded in a single move to get and organize various types of new solutions. The obtained outcomes show that the applied method is concise, direct, elementary and can be imposed in more complex phenomena with the assistant of symbolic computations.

Keywords Optical soliton · RKL equation · Generalized exponential rational function method

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1 Introduction

In the science and technology fields like engineering, circuit analysis, fluid mechanics, solid state physics, chemical physics, plasma physics, geochemistry, optical fiber, quantum field theory and biological sciences, NLPDEs are used as a governing model to explain the complexity of the physical phenomena. To know the behaviour of intricate physical phenomena, it is necessary to calculate the solutions of the governing NLPDEs. Generally, the solutions of the NLPDEs are categorized into three types as exact solutions, analytic solutions and numerical solutions. Finding the exact solutions of NLPDEs has the importance to discuss the stability of numerical solutions and also development of a broad range of new scholar to simplify the routine calculation. Exact solutions to NLPDEs play an important role in nonlinear science, since they can provide much physical information and more insight of the physical aspects of the problem and thus lead to further applications. Wave phenomena in dispersion, dissipation, diffusion, reaction and convection are very much important (Biswas et al. 2018a, c; Ozkan et al. 2020a, 2021; Celik et al. 2021; Ali et al. 2018a; Nighat et al. 2020; Seadawy et al. 2018; Cheemaa et al. 2018; Khater et al. 2000).

Furthermore, various specialists and mainstream researchers are giving more consideration to create and improve the optical transmission frameworks through optical fibers instead of birefringent fibers. However, the propagation of soliton through optical fibers governs next generation technology but there are several factors in birefringent fibers as well that are used to produce the soliton propagations (Ahmed et al. 2019; Ali et al. 2018b; Arshad et al. 2017; Ghanbari et al. 2018; Seadawy et al. 2019b). It is polarization of light that prompts bunch speed mismatch, which is at last liable for differential gathering delay and numerous other negative impacts and consequently the examination of optical soliton is one of the most intriguing and interesting zones of exploration in nonlinear optics. There are well known computational and powerful techniques to find the optical exact soliton solutions of the differential equations (Youhas and Younis 2020; Seadawy et al. 2019a, 2020a; Younis et al. 2017, 2020; Donne et al. 2020; Rehman et al. 2019a; Iqbal et al. 2019a, b, 2020; Cheemaa et al. 2019; Rizvi et al. 2020; Lu et al. 2019; Seadawy and Abdullah 2019).

Therefore, in this study we focus on constructing the exact optical soliton solutions in different structures to the RKL equation for birefringent fibers without 4WM terms in media with Kerr law nonlinearity which is known as the basic case of fiber nonlinearity. Most optical fibers which has been quite popular recently comply with this law nonlinearity. Also, this media indicates itself as self-phase modulation, a self-induced phase- and frequency-shift of a pulse of light when it moves along with any fiber nonlinearity. In case of birefringent fibers the pulses are polarized. Naturally, vector solitons are studied in birefringent fibers. We apply an efficient computational approach known as generalized exponential rational function method (GERFM) to find the various kinds of soliton solutions. The restraint relations are also observed during the mathematical analysis.

The RKL with Kerr law nonlinearity is given below (Biswas et al. 2018b; Ozkan et al. 2020b; Rehman et al. 2019b; Sulaiman et al. 2018; Seadawy et al. 2020b)

\[ i\Theta_t + a\Theta_{xx} + \beta|\Theta|^2\Theta = i\lambda(\Theta^4\Theta_x) - i\Theta_{xxx}, \]  

(1)

where the dependent variable \( \Theta(x,t) \) is complex-valued wave profile with two independent variables of \( x \) and \( t \) that represents spatio-temporal component, respectively. The first term characterizes temporal evolution whereas the parameters \( a \) denotes group velocity dispersion (GVD) while, the coefficients \( \beta \) is Kerr nonlinearity. Moreover, on right hand side the
coefficients $\delta$ and $\lambda$ sequentially accounts the third order dispersion (3OD) which induces soliton radiation and the effect of self-steepening to eliminate the formulation of shock waves. Thus, these compensatory effects of dispersion and nonlinearity provide the necessary balance to sustain soliton propagation.

Upon splitting the Eq. (1) for birefringent fibers into two components without 4WM (Jhangeera et al. 2020), we arrive at:

\[ i\Psi_t + a_1 \Psi_{xx} + (\beta_1 |\Psi|^2 + \gamma_1 |\varphi|^2)\Psi = i\left(\lambda_1 (|\Psi|^2\Psi)_x + \theta_1 (|\varphi|^2\Psi)_x\right) - i\delta_1 \Psi_{xxx} \]  

(2)

\[ i\varphi_t + a_2 \varphi_{xx} + (\beta_2 |\varphi|^2 + \gamma_2 \Psi^2)\varphi = i\left(\lambda_2 (|\varphi|^2\varphi)_x + \theta_2 (|\Psi|^2\varphi)_x\right) - i\delta_2 \varphi_{xxx}. \]  

(3)

The complex valued functions $\Psi(x, t)$ and $\varphi(x, t)$ represent the wave profiles and the Eqs. (2) and (3) represent the governing model for soliton transmission through birefringent fibers without 4WM. In the above coupled system, the coefficient $\beta_j$ accounts for self-phase modulation (SPM) and the coefficient $\gamma_j$ is the cross-phase modulation terms respectively, for $j = 1, 2$. The coefficients $\lambda_j$ and $\theta_j$ correspond to self-steepening terms, while the the four-wave mixing effect is discarded.

This piece of article is discussed as sequence: In Sect. 2, the summary of the GERFM. In Sect. 3, optical solitons. In Sect. 4, results and discussion and finally paper comes at conclusions in Sect. 5.

2 The summary of GERFM

Here, we give a brief description of GERFM (Younas et al. 2020). Let us consider a nonlinear partial differential equation (PDE)

\[ \Xi(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, \ldots) = 0, \]  

(4)

where $\Xi$ is a polynomial in its arguments.

The essence of GERFM can be presented in the following steps

**Step 1.** We introduce traveling wave transformation as:

\[ u(x, t) = u(\eta) \quad \text{and} \quad \eta = B(x - ct), \]

where $B$ and $c$ represent the amplitude component and velocity respectively. After substituting this transformation into Eq. (4), we get nonlinear ODE in the following form.

\[ \Upsilon(\Lambda, \Lambda', \Lambda'', \Lambda''', \ldots) = 0, \]  

(5)

where $\Upsilon$ is in general a polynomial function of its arguments and $'$ denotes the derivative w.r.t $\eta$.

**Step 2.** Suppose that the solution of Eq. (5) can be expressed as follows

\[ \Lambda(\eta) = d_0 + \sum_{k=1}^{n} d_k \Omega(\eta)^k + \sum_{k=1}^{n} f_k \Omega(\eta)^{-k}, \]  

(6)
where
\[
\Omega(\eta) = \frac{r_1 e^{i\eta} + r_2 e^{i2\eta}}{r_3 e^{i3\eta} + r_4 e^{i4\eta}}.
\] (7)

The unknown coefficients \(d_0, d_k, f_k (1 \leq k \leq n)\) and constants \(r_i, s_i (1 \leq i \leq 4)\) are determined and the value of \(n\) will be evaluated by using homogeneous balance principle.

Step 3. After putting Eq. (6) into Eq. (5) we get an algebraic equation
\[
R(\eta, e^{i\eta}, e^{i2\eta}, e^{i3\eta}, e^{i4\eta}) = 0.
\]
Setting each coefficient of \(R\) equal to zero, we get a system of nonlinear equations in the form of \(d_0, d_k, f_k (1 \leq k \leq n)\) and \(r_i, s_i (1 \leq i \leq 4)\) is yielded.

Step 4. On solving the above system of equations we will get the values of \(d_0, d_k, f_k (1 \leq k \leq n)\) and \(r_i, s_i (1 \leq i \leq 4)\) with the aid of any symbolic computation packages. Substituting these values in Eq. (6) we attain the soliton solutions of Eq. (4).

3 Optical solitons

For solving the above couple system of Eqs. (2)—(3), we suppose the traveling wave transformation as follows:
\[
\Psi(x,t) = Q_j(\eta) e^{i\psi}.
\] (8)
\[
\varphi(x,t) = Q_2(\eta) e^{i\psi},
\] (9)
where
\[
\eta = B(x - vt) \quad \text{and} \quad \psi = -kx + \omega t + \theta_0.
\] (10)

And \(Q_j(\eta)\) for \((j = 1, 2)\), \(B, v, \theta_0, \omega\) and \(k\) represent the amplitude component of the soliton, velocity of soliton, phase constant, soliton wave number and soliton frequency respectively. Substituting above transformations of Eqs. (8)—(10) into Eqs. (2) and (3), we get real and imaginary parts, respectively of the form
\[
B^2(\alpha_j + 3k\delta_j)Q_j'' - (\omega + \alpha_j k^2 + \delta_j k^3)Q_j + (\beta_j - k\lambda_j)Q_j^3 + r_j Q_j Q_j^2 - k\theta_j Q_j Q_j^2 = 0,
\] (11)
\[
B^3\delta_j Q_j''' - B(\nu + 2k\alpha_j + 3k^2 \delta_j)Q_j' - 3B\lambda_j Q_j^2 - \theta_j BQ_j Q_j^2 - 2\theta_j BQ_j Q_j' = 0.
\] (12)

In order to use the balancing rule, the important results emerged from Eqs. (11)—(12) by using of \(Q_j = Q_j\) are
\[
B^2(\alpha_j + 3k\delta_j)Q_j'' - (\omega + \alpha_j k^2 + \delta_j k^3)Q_j + (\beta_j - k\lambda_j + \gamma_j - k\theta_j)Q_j^3 = 0,
\] (13)
\[
B^2\delta_j Q_j''' - (\nu + 2k\alpha_j + 3k^2 \delta_j)Q_j - (\lambda_j + \theta_j)Q_j^3 = 0.
\] (14)

We can obtain
\[
\omega = \frac{8k^3 \delta_j^2 + 8k^2 \alpha_j \delta_j + 3k\nu \delta_j + 2k\alpha_j^2 + \nu \alpha_j}{\delta_j}.
\] (15)
\[
\beta_j = -\frac{2k\delta_j \theta_j + 2k \delta_j \lambda_j + \alpha_j \theta_j + \alpha_j \lambda_j + \delta_j \gamma_j}{\delta_j},
\]
\[(16)\]

which yields from
\[
\frac{\alpha_j + 3k \delta_j}{\delta_j} = \frac{\omega + \alpha_j k^2 + \delta_j k^3}{v + 2k \alpha_j + 3k^2 \delta_j} = -\frac{\beta_j - k \lambda_j + \gamma_j - k \theta_j}{\lambda_j + \theta_j}.
\]
\[(17)\]

As the function \(Q_j\) satisfies Eqs. (13)–(14). We will use Eq. (13) with Eqs. (15)–(16).

By balancing the highest order derivative and nonlinear term appear in Eq. (13) we attain \(n = 1\). Using \(n = 1\) along with Eqs. (6)–(7), then
\[
Q_j(\eta) = d_0 + d_1 \Omega(\eta) + f_1 \Omega(\eta)^{-1}.
\]
\[(18)\]

**Family-1** If we take \(r = [-1, -1, 1, -1]\) and \(s = [1, -1, 1, -1]\), then Eq. (7) changes into
\[
\Omega(\eta) = -\frac{\cosh(\eta)}{\sinh(\eta)}.
\]
\[(19)\]

**Case 1**

\[
d_0 = 0, \ d_1 = 0, \ f_1 = -\frac{\sqrt{k^3(-\delta_j) - k^2 \alpha_j - \omega}}{-\beta_j - \gamma_j + k \theta_j + k \lambda_j}, \ B = \frac{\sqrt{k^3(-\delta_j) - k^2 \alpha_j - \omega}}{2 \sqrt{\alpha_j + 3k \delta_j}}.
\]

Inserting these values in Eqs. (18) and (19), then we obtain

The optical dark soliton solution as
\[
\Psi(x, t) = \frac{\sqrt{k^3(-\delta_1) - k^2 \alpha_1 - \omega}}{\sqrt{-\beta_1 - \gamma_1 + k \theta_1 + k \lambda_1}} \tan \left[ \frac{\sqrt{k^3(-\delta_1) - k^2 \alpha_1 - \omega}}{2 \sqrt{\alpha_1 + 3k \delta_1}} (x - vt) \right]
\times e^{i \left( \frac{k^3 x^2}{2 \alpha_1} + \frac{k^2 x y}{\alpha_1} + \frac{k x z}{\alpha_1} + \frac{2 k \delta_1 y z}{\alpha_1} \right) t + \theta_1},
\]
\[(20)\]

\[
\varphi(x, t) = \frac{\sqrt{k^3(-\delta_2) - k^2 \alpha_2 - \omega}}{\sqrt{-\beta_2 - \gamma_2 + k \theta_2 + k \lambda_2}} \tan \left[ \frac{\sqrt{k^3(-\delta_2) - k^2 \alpha_2 - \omega}}{2 \sqrt{\alpha_2 + 3k \delta_2}} (x - vt) \right]
\times e^{i \left( \frac{k^3 x^2}{2 \alpha_2} + \frac{k^2 x y}{\alpha_2} + \frac{k x z}{\alpha_2} + \frac{2 k \delta_2 y z}{\alpha_2} \right) t + \theta_2}.
\]
\[(21)\]

Here \((k^3(-\delta_j) - k^2 \alpha_j - \omega)(\alpha_j + 3k \delta_j) > 0\) and \(\delta_j \neq 0\) with \(j = 1, 2\) for valid solution. The graphical representations of the solutions are shown for different values of parameters.

**Case 2**

\[
d_0 = 0, \ d_1 = \frac{\sqrt{k^3(-\delta_j) - k^2 \alpha_j - \omega}}{-\beta_j - \gamma_j + k \theta_j + k \lambda_j}, \ f_1 = 0, \ B = \frac{\sqrt{k^3(-\delta_j) - k^2 \alpha_j - \omega}}{2 \sqrt{\alpha_j + 3k \delta_j}}.
\]
Substituting these values in Eqs. (18) and (19), we attain

The singular optical soliton solution as

$$
\Psi(x, t) = -\sqrt{k^3(\delta_1) - k^2\alpha_1 - \omega \over \sqrt{-\beta_1 - \gamma_1 + k\theta_1 + k\lambda_1}} \times e^{i\left(-kx + {k^3x_1^2 + k^2x_1^2 + 3k\delta_1 + 2ax_1^2 + a_1 \over \alpha_1}t + \theta_0\right)}
$$

The combined optical soliton solution as

$$
\varphi(x, t) = -\sqrt{k^3(\delta_2) - k^2\alpha_2 - \omega \over \sqrt{-\beta_2 - \gamma_2 + k\theta_2 + k\lambda_2}} \times e^{i\left(-kx + {k^3x_2^2 + k^2x_2^2 + 3k\delta_2 + 2ax_2^2 + a_2 \over \alpha_2}t + \theta_0\right)}.
$$

Here \((k^3(\delta_1) - k^2\alpha_j - \omega)(\alpha_j + 3k\delta_j) > 0\) and \(\delta_j \neq 0\) with \(j = 1, 2\) for valid solution. The graphical representations of the solutions are shown for different values of parameters.

**Case 3**

\(d_0 = 0, d_1 = \sqrt{k^2(\alpha_j + k\delta_j) + \omega \over -2\beta_j - 2\gamma_j + 2k(\theta_j + \lambda_j)}, f_1 = -\sqrt{k^2(\alpha_j + k\delta_j) + \omega \over -2\beta_j - 2\gamma_j + 2k(\theta_j + \lambda_j)}, B = \sqrt{k^2(\alpha_j + k\delta_j) + \omega \over 2\sqrt{\alpha_j + 3k\delta_j}}.\)

Imposing these values in Eqs. (18) and (19), then we derive

The combined optical soliton solution as

$$
\Psi(x, t) = -{\sqrt{k^2(\alpha_1 + k\delta_1) + \omega \over \sqrt{-2\beta_1 - 2\gamma_1 + 2k(\theta_1 + \lambda_1)}}} \times {1 \over \sinh \left[{\sqrt{k^2(\alpha_1 + k\delta_1) + \omega \over 2\sqrt{\alpha_1 + 3k\delta_1}}}(x - vt)\right]} \cosh \left[{\sqrt{k^2(\alpha_1 + k\delta_1) + \omega \over 2\sqrt{\alpha_1 + 3k\delta_1}}}(x - vt)\right] \times e^{i\left(-kx + {k^3x_1^2 + k^2x_1^2 + 3k\delta_1 + 2ax_1^2 + a_1 \over \alpha_1}t + \theta_0\right)}.
$$
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\[ \phi(x, t) = - \frac{\sqrt{k^2(\alpha_2 + k\delta_2) + \omega}}{\sqrt{-2\beta_2 - 2\gamma_2 + 2k(\theta_2 + \lambda_2)}} \times \frac{1}{\sin \left[ \frac{\sqrt{k^2(\alpha_1 + k\delta_1) + \omega}}{2\sqrt{\alpha_1 + 3k\delta_1}}(x - vt) \right]} \cosh \left[ \frac{\sqrt{k^2(\alpha_2 + k\delta_2) + \omega}}{2\sqrt{\alpha_2 + 3k\delta_2}}(x - vt) \right] \times e^{i\left(-kx + \frac{\delta_1^2}{\nu_1} + \frac{\delta_2^2}{\nu_2} + \frac{\delta_1\delta_2}{\nu_1}\nu_2 + i\theta_1\right)}. \]  

(25)

Here \((k^2(\alpha_j + k\delta_j) + \omega)(\alpha_j + 3k\delta_j) > 0\) and \(\delta_j \neq 0\) with \(j = 1, 2\) for valid solution. The graphical representations of the solutions are shown for different values of parameters.

**Family-2** If we chose \(r = [-i, -i, 1, 1]\) and \(s = [i, -i, i, -i]\), then Eq. (7) modify into

\[ \Omega(\eta) = -\frac{\sin(\eta)}{\cos(\eta)}. \]  

(26)

The following trigonometric and combined trigonometric traveling wave solutions are

**Case 1**

\[ d_0 = 0, \ d_1 = 0, \ f_1 = -\frac{\sqrt{k^3\delta_j + k^2\alpha_j + \omega}}{\sqrt{-\beta_j - \gamma_j + k\theta_j + k\lambda_j}}, \ B = \frac{\sqrt{k^3\delta_j + k^2\alpha_j + \omega}}{2\sqrt{\alpha_j + 3k\delta_j}}. \]

Substituting these values in Eqs. (18) and (26), we attain

\[ \Psi(x, t) = \sqrt{k^3\delta_1 + k^2\alpha_1 + \omega} \cot \left[ \frac{\sqrt{k^3\delta_1 + k^2\alpha_1 + \omega}}{2\sqrt{\alpha_1 + 3k\delta_1}}(x - vt) \right] \times e^{i\left(-kx + \frac{\delta_1^2}{\nu_1} + \frac{\delta_2^2}{\nu_2} + \frac{\delta_1\delta_2}{\nu_1}\nu_2 + i\theta_1\right)}. \]  

(27)

\[ \phi(x, t) = \sqrt{k^3\delta_2 + k^2\alpha_2 + \omega} \cot \left[ \frac{\sqrt{k^3\delta_2 + k^2\alpha_2 + \omega}}{2\sqrt{\alpha_2 + 3k\delta_2}}(x - vt) \right] \times e^{i\left(-kx + \frac{\delta_1^2}{\nu_1} + \frac{\delta_2^2}{\nu_2} + \frac{\delta_1\delta_2}{\nu_1}\nu_2 + i\theta_1\right)}. \]  

(28)

Here \((k^3\delta_j + k^2\alpha_j) + \omega(\alpha_j + 3k\delta_j) > 0\) and \(\delta_j \neq 0\) with \(j = 1, 2\) for valid solution. The graphical representations of the solutions are shown for different values of parameters.

**Case 2**

\[ d_0 = 0, \ d_1 = -\frac{\sqrt{k^3\delta_j + k^2\alpha_j + \omega}}{\sqrt{-\beta_j - \gamma_j + k\theta_j + k\lambda_j}}, \ f_1 = 0, \ B = \frac{\sqrt{k^3\delta_j + k^2\alpha_j + \omega}}{2\sqrt{\alpha_j + 3k\delta_j}}. \]

Replacing these values in Eqs. (18) and (26), we get
Here \((k^3 \delta_j + k^2 \alpha_j) + \omega) (\alpha_j + 3k \delta_j) > 0\) and \(\delta_j \neq 0\) with \(j = 1, 2\) for valid solution.

**Case 3**

\[
d_0 = 0, \quad d_1 = -\frac{\sqrt{k^2 (\alpha_j + k \delta_j) + \omega}}{2 \sqrt{-\beta_j - \gamma_j + k (\theta_j + \lambda_j)}}, \quad f_1 = \frac{\sqrt{k^2 (\alpha_j + k \delta_j) + \omega}}{2 \sqrt{-\beta_j - \gamma_j + k (\theta_j + \lambda_j)}}.
\]

\[
B = \frac{\sqrt{k^2 (\alpha_j + k \delta_j) + \omega}}{2 \sqrt{2 \alpha_j + 3k \delta_j}}.
\]

Inserting these values in Eqs. (18) and (26), we obtained

\[
\Psi(x, t) = \frac{\sqrt{k^2 (\alpha_j + k \delta_1) + \omega}}{2 \sqrt{-\beta_j - \gamma_j + k (\theta_1 + \lambda_1)}} \times 
\left[ 1 - 2 \cos^2 \left( \frac{\sqrt{k^2 (\alpha_j + k \delta_1) + \omega}}{2 \sqrt{2 \alpha_j + 3k \delta_1}} (x - vt) \right) \right] 
\times \sin \left( \frac{\sqrt{k^2 (\alpha_j + k \delta_1) + \omega}}{2 \sqrt{2 \alpha_j + 3k \delta_1}} (x - vt) \right) \cos \left( \frac{\sqrt{k^2 (\alpha_j + k \delta_1) + \omega}}{2 \sqrt{2 \alpha_j + 3k \delta_1}} (x - vt) \right) 
\times e^{i \left(-\frac{k^3 \delta_1 + k^2 \alpha_1 + \omega}{\alpha_1} - \frac{\sqrt{k^3 \delta_1 + k^2 \alpha_1 + \omega}}{\sqrt{2 \alpha_1 + 3k \delta_1}} (x - vt) \right)},
\]

(31)

\[
\varphi(x, t) = \frac{\sqrt{k^2 (\alpha_2 + k \delta_2) + \omega}}{2 \sqrt{-\beta_2 - \gamma_2 + k (\theta_2 + \lambda_2)}} \times 
\left[ 1 - 2 \cos^2 \left( \frac{\sqrt{k^2 (\alpha_2 + k \delta_2) + \omega}}{2 \sqrt{2 \alpha_2 + 3k \delta_2}} (x - vt) \right) \right] 
\times \sin \left( \frac{\sqrt{k^2 (\alpha_2 + k \delta_2) + \omega}}{2 \sqrt{2 \alpha_2 + 3k \delta_2}} (x - vt) \right) \cos \left( \frac{\sqrt{k^2 (\alpha_2 + k \delta_2) + \omega}}{2 \sqrt{2 \alpha_2 + 3k \delta_2}} (x - vt) \right) 
\times e^{i \left(-\frac{k^3 \delta_2 + k^2 \alpha_2 + \omega}{\alpha_2} - \frac{\sqrt{k^3 \delta_2 + k^2 \alpha_2 + \omega}}{\sqrt{2 \alpha_2 + 3k \delta_2}} (x - vt) \right)}.
\]

(32)
Here \((k^2(\alpha_j + k\delta_j) + \omega)(\alpha_j + 3k\delta_j) > 0\) and \(\delta_j \neq 0\) with \(j = 1, 2\) for valid solution.

**Family-3** If we take \(r = [2, 1, 1, 1]\) and \(s = [1, 0, 1, 0]\), then Eq. (7) transform into

\[
\Omega(\eta) = \frac{2e^n + 1}{e^n + 1}.
\]  

**(33)**

**Case 1**

\[
d_0 = \frac{3\sqrt{k^2(-2(\alpha_j + k\delta_j)) - \omega}}{\sqrt{-\beta_j - \gamma_j + k(\theta_j + \lambda_j)}}, \quad d_1 = -\frac{2\sqrt{k^2(-2(\alpha_j + k\delta_j)) - \omega}}{\sqrt{-\beta_j - \gamma_j + k(\theta_j + \lambda_j)}}, \quad f_1 = 0,
\]

\[
B = \frac{\sqrt{2\sqrt{k^2(-2(\alpha_j + k\delta_j)) - \omega}}}{\sqrt{\alpha_j + 3k\delta_j}}.
\]

Inserting these values in Eqs. (18) and (33), then

The exponential function solution can be expressed as

\[
\Psi(x,t) = \left(\frac{\sqrt{k^2(-2(\alpha_1 + k\delta_1)) - \omega}}{\sqrt{-\beta_1 - \gamma_1 + k(\theta_1 + \lambda_1)}}\right)^{1 - e^n} e^{i(-kx + \frac{8k^2\alpha_2^2 + 8k^2\alpha_1\alpha_2 + 3k\delta_1 + 2k\alpha_2^2 + 2\alpha_1^2 + \alpha_1 + \theta_1)t + \theta_0}),
\]

**(34)**

where \(\eta = \sqrt{\frac{2\sqrt{k^2(-2(\alpha_1 + k\delta_1)) - \omega}}{\sqrt{\alpha_1 + 3k\delta_1}}}(x - vt)\).

\[
\phi(x,t) = \left(\frac{\sqrt{k^2(-2(\alpha_2 + k\delta_2)) - \omega}}{\sqrt{-\beta_2 - \gamma_2 + k(\theta_2 + \lambda_2)}}\right)^{1 - e^n} e^{i(-kx + \frac{8k^2\alpha_2^2 + 8k^2\alpha_1\alpha_2 + 3k\delta_2 + 2k\alpha_2^2 + 2\alpha_1^2 + \alpha_2 + \theta_1)t + \theta_0}),
\]

**(35)**

where \(\eta = \sqrt{\frac{2\sqrt{k^2(-2(\alpha_2 + k\delta_2)) - \omega}}{\sqrt{\alpha_2 + 3k\delta_2}}}(x - vt)\).

Here \((k^2(-2(\alpha_j + k\delta_j)) - \omega)(-\beta_j - \gamma_j + k(\theta_j + \lambda_j)) > 0 \) and \(\delta_j \neq 0\) with \(j = 1, 2\) for valid solution. The graphical representations of the solutions are shown for different values of parameters.

**Case 2**

\[
d_0 = -\frac{3\sqrt{k^2(-2(\alpha_j + k\delta_j)) - \omega}}{\sqrt{-\beta_j - \gamma_j + k(\theta_j + \lambda_j)}}, \quad d_1 = 0, \quad f_1 = \frac{4\sqrt{k^2(-2(\alpha_j + k\delta_j)) - \omega}}{\sqrt{-\beta_j - \gamma_j + k(\theta_j + \lambda_j)}},
\]

\[
B = \frac{\sqrt{2\sqrt{k^2(-2(\alpha_j + k\delta_j)) - \omega}}}{\sqrt{\alpha_j + 3k\delta_j}}.
\]

Putting these values in Eqs. (18) and (33), then

The exponential function solution can be formulated as
\[ \Psi(x, t) = \left( \frac{\sqrt{k^2(-\alpha_1 + k\delta_1)} - \omega}{\sqrt{-\beta_1 - \gamma_1 + k(\theta_1 + \lambda_1)}} \right) \left(1 - 2e^{\eta} \right) \frac{1 - 2e^{\eta}}{2e^{\eta} + 1} e^{i\left(-kx + \frac{\Omega^2 + \Omega_2^2 + \Delta_x^2 + \Delta_2^2 + 2\alpha_1^2 + \alpha_2^2}{\delta_1} \right) \tau + \theta_0}, \]

where \( \eta = \frac{\sqrt{2}k^2(-\alpha_1 + k\delta_1) - \omega}{\sqrt{\alpha_1 + 3k\delta_1}}(x - vt). \)

\[ \varphi(x, t) = \left( \frac{\sqrt{k^2(-\alpha_2 + k\delta_2)} - \omega}{\sqrt{-\beta_2 - \gamma_2 + k(\theta_2 + \lambda_2)}} \right) \left(1 - 2e^{\eta} \right) \frac{1 - 2e^{\eta}}{2e^{\eta} + 1} e^{i\left(-kx + \frac{\Omega^2 + \Omega_2^2 + \Delta_x^2 + \Delta_2^2 + 2\alpha_2^2 + \alpha_2^2}{\delta_2} \right) \tau + \theta_0}, \]

where \( \eta = \frac{\sqrt{2}k^2(-\alpha_2 + k\delta_2) - \omega}{\sqrt{\alpha_2 + 3k\delta_2}}(x - vt). \)

Here \((k^2(-\alpha_j + k\delta_j)) - \omega)(-\beta_j - \gamma_j + k(\theta_j + \lambda_j)) > 0 \) and \( \delta_j \neq 0 \) with \( j = 1, 2 \) for valid solution.

**Case 1**

\[
\begin{align*}
    d_0 &= \frac{\sqrt{k^2(\alpha_j + k\delta_j) + \omega}}{-\beta_j - \gamma_j + k(\theta_j + \lambda_j)}, \\
    d_1 &= -\frac{\sqrt{k^2(\alpha_j + k\delta_j) + \omega}}{-\beta_j - \gamma_j + k(\theta_j + \lambda_j)}, \\
    f_1 &= 0, \\
    B &= \frac{\sqrt{k^2(\alpha_j + k\delta_j) + \omega}}{2\sqrt{\alpha_j + 3k\delta_j}}.
\end{align*}
\]

Imposing these values in Eqs. (18) and (38), then

The singular periodic wave solution can be expressed as

\[
\begin{align*}
    \Psi(x, t) &= -\frac{\sqrt{k^2(\alpha_1 + k\delta_1) + \omega}}{-\beta_1 - \gamma_1 + k(\theta_1 + \lambda_1)} \tan \left[ \frac{\sqrt{k^2(\alpha_1 + k\delta_1) + \omega}}{\sqrt{2}\sqrt{\alpha_1 + 3k\delta_1}}(x - vt) \right] \\
    &\times e^{i\left(-kx + \frac{\Omega^2 + \Omega_2^2 + \Delta_x^2 + \Delta_2^2 + 2\alpha_1^2 + \alpha_2^2}{\delta_1} \right) \tau + \theta_0}, \\
\end{align*}
\]

\[
\begin{align*}
    \varphi(x, t) &= -\frac{\sqrt{k^2(\alpha_2 + k\delta_2) + \omega}}{-\beta_2 - \gamma_2 + k(\theta_2 + \lambda_2)} \tan \left[ \frac{\sqrt{k^2(\alpha_2 + k\delta_2) + \omega}}{\sqrt{2}\sqrt{\alpha_2 + 3k\delta_2}}(x - vt) \right] \\
    &\times e^{i\left(-kx + \frac{\Omega^2 + \Omega_2^2 + \Delta_x^2 + \Delta_2^2 + 2\alpha_2^2 + \alpha_2^2}{\delta_2} \right) \tau + \theta_0}.
\end{align*}
\]

Here \((k^2(\alpha_j + k\delta_j) + \omega)(\alpha_j + 3k\delta_j) > 0 \) and \( \delta_j \neq 0 \) with \( j = 1, 2 \) for valid solution.
Case 2

\[ d_0 = \frac{\sqrt{k^2(\alpha_j + k\delta_j)} + \omega}{\sqrt{-\beta_j - \gamma_j + k(\theta_j + \lambda_j)}}, \quad d_1 = 0, \quad f_1 = -\frac{2\sqrt{k^2(\alpha_j + k\delta_j)} + \omega}{\sqrt{-\beta_j - \gamma_j + k(\theta_j + \lambda_j)}}, \]

\[ B = \frac{\sqrt{k^3\delta_j + k^2\alpha_j + \omega}}{2\alpha_j + 6k\delta_j}. \]

Inserting these values in Eqs. (18) and (38), then the combined singular periodic wave solution can be written as

\[ \Psi(x,t) = \sqrt{\frac{k^2(\alpha_1 + k\delta_1)}{-\beta_1 - \gamma_1 + k(\theta_1 + \lambda_1)}} \times \left( \frac{\sqrt{k^2(\alpha_1 + k\delta_1)} + \omega}{\sqrt{2\alpha_1 + 6k\delta_1}}(x - vt) - \cos \left[ \frac{\sqrt{k^2(\alpha_1 + k\delta_1)} + \omega}{\sqrt{2\alpha_1 + 6k\delta_1}}(x - vt) \right] \right) \times e^{i\left(\frac{8k^2\Delta_1}{\alpha_1^2 + 8k^2\Delta_1 + 8k^2\Delta_1 + 2k\Delta_1^2 + \Omega}{\delta_j + 1}\right)} (41) \]

\[ \phi(x,t) = \sqrt{\frac{k^2(\alpha_2 + k\delta_2)}{-\beta_2 - \gamma_2 + k(\theta_2 + \lambda_2)}} \times \left( \frac{\sqrt{k^2(\alpha_2 + k\delta_2)} + \omega}{\sqrt{2\alpha_2 + 6k\delta_2}}(x - vt) - \cos \left[ \frac{\sqrt{k^2(\alpha_2 + k\delta_2)} + \omega}{\sqrt{2\alpha_2 + 6k\delta_2}}(x - vt) \right] \right) \times e^{i\left(\frac{8k^2\Delta_2}{\alpha_2^2 + 8k^2\Delta_2 + 8k^2\Delta_2 + 2k\Delta_2^2 + \Omega}{\delta_j + 1}\right)} (42) \]

Here \((k^2\alpha_j + k^3\delta_j) + \omega)(2\alpha_j + 6k\delta_j) > 0\) and \(\delta_j \neq 0\) with \(j = 1, 2\) for valid solution.

**Family-5** If we take \(r = [-3, -1, 1, 1]\) and \(s = [1, -1, 1, -1]\), then Eq. (7) converts into

\[ \Omega(\eta) = \frac{-2 \cosh(\eta) - \sinh(\eta)}{\cosh(\eta)} (43) \]

Case 1

\[ d_0 = -\frac{2\sqrt{k^2(-\alpha_j + k\delta_j)} - \omega}{\sqrt{-\beta_j - \gamma_j + k(\theta_j + \lambda_j)}}, \quad d_1 = -\frac{\sqrt{k^2(-\alpha_j + k\delta_j)} - \omega}{\sqrt{-\beta_j - \gamma_j + k(\theta_j + \lambda_j)}}, \quad f_1 = 0, \]

\[ B = \frac{\sqrt{k^2(-\alpha_j + k\delta_j)} - \omega}{\sqrt{2\alpha_j + 3k\delta_j}}. \]
Replacing these values in Eqs. (18) and (43), then

The hyperbolic function solution can be derived as

\[ \Psi(x, t) = \frac{\sqrt{k^2(- (\alpha_1 + k\delta_1)) - \omega}}{\sqrt{-\beta_1 - \gamma_1 + k(\theta_1 + \lambda_1)}} \tan\left[ \frac{\sqrt{k^2(- (\alpha_1 + k\delta_1)) - \omega}}{\sqrt{2\alpha_1 + 3k\delta_1}} (x - vt) \right] \times e^{i(kx + \frac{\alpha_1 x^2 k_1^2 + 2\alpha_1^2 + 2\alpha_1 k_1 + k_1^2 t^2 + \alpha_1^2 t + \alpha_1 k t + \alpha_1^2 t^2 + \alpha_1 k t^2 + \alpha_1^2 t^2}{\alpha_1})}, \] (44)

\[ \varphi(x, t) = \frac{\sqrt{k^2(- (\alpha_2 + k\delta_2)) - \omega}}{\sqrt{-\beta_2 - \gamma_2 + k(\theta_2 + \lambda_2)}} \tan\left[ \frac{\sqrt{k^2(- (\alpha_2 + k\delta_2)) - \omega}}{\sqrt{2\alpha_2 + 3k\delta_2}} (x - vt) \right] \times e^{i(kx + \frac{\alpha_2 x^2 k_2^2 + 2\alpha_2^2 + 2\alpha_2 k_2 + k_2^2 t^2 + \alpha_2^2 t + \alpha_2 k t + \alpha_2^2 t^2 + \alpha_2 k t^2 + \alpha_2^2 t^2}{\alpha_2})}. \] (45)

Here \((k^2(- (\alpha_j + k\delta_j)) - \omega)(\alpha_j + 3k\delta_j) > 0\) and \(\delta_j \neq 0\) with \(j = 1, 2\) for valid solution.

**Case 2**

\[ d_0 = -2\frac{\sqrt{k^2(- (\alpha_j + k\delta_j)) - \omega}}{\sqrt{-\beta_j - \gamma_j + k(\theta_j + \lambda_j)}}, \quad d_1 = 0, \quad f_1 = -3\frac{\sqrt{k^2(- (\alpha_j + k\delta_j)) - \omega}}{\sqrt{-\beta_j - \gamma_j + k(\theta_j + \lambda_j)}}, \]

\[ B = \frac{\sqrt{k^2(- (\alpha_j + k\delta_j)) - \omega}}{\sqrt{2\alpha_j + 3k\delta_j}}. \]

Substituting these values in Eqs. (18) and (43), then

The mixed hyperbolic function solution can be formulated as

\[ \Psi(x, t) = \frac{\sqrt{k^2(- (\alpha_1 + k\delta_1)) - \omega}}{\sqrt{-\beta_1 - \gamma_1 + k(\theta_1 + \lambda_1)}} \left\{ \begin{array}{c} 2 \sinh \left[ \frac{\sqrt{k^2(- (\alpha_1 + k\delta_1)) - \omega}}{\sqrt{2\alpha_1 + 3k\delta_1}} (x - vt) \right] + \cosh \left[ \frac{\sqrt{k^2(- (\alpha_1 + k\delta_1)) - \omega}}{\sqrt{2\alpha_1 + 3k\delta_1}} (x - vt) \right] \\ 2 \cosh \left[ \frac{\sqrt{k^2(- (\alpha_1 + k\delta_1)) - \omega}}{\sqrt{2\alpha_1 + 3k\delta_1}} (x - vt) \right] + \sinh \left[ \frac{\sqrt{k^2(- (\alpha_1 + k\delta_1)) - \omega}}{\sqrt{2\alpha_1 + 3k\delta_1}} (x - vt) \right] \end{array} \right\} \times e^{i(-kx + \frac{\alpha_1 x^2 k_1^2 + 2\alpha_1^2 + 2\alpha_1 k_1 + k_1^2 t^2 + \alpha_1^2 t + \alpha_1 k t + \alpha_1^2 t^2 + \alpha_1 k t^2 + \alpha_1^2 t^2}{\alpha_1})}, \] (46)
The exponential function solution can be shown as
\[
\Psi(x, t) = e^{-\frac{\sqrt{k^3(-\delta_j) - k^2\alpha_j - \omega}}{\sqrt{-\beta_j - \gamma_j + k\theta_j + k\lambda_j}}(x - vt)}
\]
where \( \eta = \sqrt[3]{\frac{k^3(-\delta_j) - k^2\alpha_j - \omega}{\sqrt{-\beta_j - \gamma_j + k\theta_j + k\lambda_j}}} (x - vt) \).

\[
\varphi(x, t) = e^{-\frac{\sqrt{k^3(-\delta_j) - k^2\alpha_j - \omega}}{\sqrt{-\beta_2 - \gamma_2 + k\theta_2 + k\lambda_2}}(x - vt)}
\]
where \( \eta = \sqrt[3]{\frac{k^3(-\delta_j) - k^2\alpha_j - \omega}{\sqrt{-\beta_2 - \gamma_2 + k\theta_2 + k\lambda_2}}} (x - vt) \).

Here \( k^3(-\delta_j) - k^2\alpha_j - \omega)(-\beta_j - \gamma_j + k(\theta_j + \lambda_j)) > 0 \) and \( \delta_j \neq 0 \) with \( j = 1, 2 \) for valid solution.

**Family-6** If we take \( r = [-1, 0, 1, 1] \) and \( s = [0, 1, 0, 1] \), then Eq. (7) transform into
\[
\Omega(\eta) = -\frac{1}{e^\eta + 1}.
\]
\[ \Omega(\eta) = -\frac{\sinh(\eta)}{\cosh(\eta)} \]  

(51)

Case 1

\[ d_0 = 0, \quad d_1 = \frac{\sqrt{k^2(-(\alpha_j + k\delta_j)) - \omega}}{2\sqrt{-\beta_j - \gamma_j + k(\theta_j + \lambda_j)}}, \quad f_1 = \frac{\sqrt{k^2(-(\alpha_j + k\delta_j)) - \omega}}{2\sqrt{-\beta_j - \gamma_j + k(\theta_j + \lambda_j)}}, \]

\[ B = \frac{\sqrt{k^2(-(\alpha_j + k\delta_j)) - \omega}}{2\sqrt{2\sqrt{-\alpha_j - 3k\delta_j}}}. \]

Inserting these values in Eqs. (18) and (51), then

The combo optical soliton solution can be transformed as

\[ \Psi(x, t) = -\frac{\sqrt{k^2(-(\alpha_1 + k\delta_1)) - \omega}}{2\sqrt{-\beta_1 - \gamma_1 + k(\theta_1 + \lambda_1)}} \times \left( \frac{2\cosh^2 \left[ \frac{\sqrt{k^2(-(\alpha_1 + k\delta_1)) - \omega}}{2\sqrt{2\sqrt{-\alpha_1 - 3k\delta_1}}} (x - vt) \right] - 1}{\cosh \left[ \frac{\sqrt{k^2(-(\alpha_1 + k\delta_1)) - \omega}}{2\sqrt{2\sqrt{-\alpha_1 - 3k\delta_1}}} (x - vt) \right] \sinh \left[ \frac{\sqrt{k^2(-(\alpha_1 + k\delta_1)) - \omega}}{2\sqrt{2\sqrt{-\alpha_1 - 3k\delta_1}}} (x - vt) \right] \right) \times e^{i \left( \frac{8\delta^2 \epsilon^2 + 32\delta^2 \epsilon_1 \delta_1 + 32\delta \epsilon_2 \epsilon_2 + 28\delta^2 \epsilon_2 + \epsilon_2}{\delta_1} + t + \theta_1 \right)}, \]

\[ \varphi(x, t) = -\frac{\sqrt{k^2(-(\alpha_2 + k\delta_2)) - \omega}}{2\sqrt{-\beta_2 - \gamma_2 + k(\theta_2 + \lambda_2)}} \times \left( \frac{2\cosh^2 \left[ \frac{\sqrt{k^2(-(\alpha_2 + k\delta_2)) - \omega}}{2\sqrt{2\sqrt{-\alpha_2 - 3k\delta_2}}} (x - vt) \right] - 1}{\cosh \left[ \frac{\sqrt{k^2(-(\alpha_2 + k\delta_2)) - \omega}}{2\sqrt{2\sqrt{-\alpha_2 - 3k\delta_2}}} (x - vt) \right] \sinh \left[ \frac{\sqrt{k^2(-(\alpha_2 + k\delta_2)) - \omega}}{2\sqrt{2\sqrt{-\alpha_2 - 3k\delta_2}}} (x - vt) \right] \right) \times e^{i \left( \frac{8\delta^2 \epsilon^2 + 32\delta^2 \epsilon_1 \delta_1 + 32\delta \epsilon_2 \epsilon_2 + 28\delta^2 \epsilon_2 + \epsilon_2}{\delta_2} + t + \theta_2 \right)}. \]

Here \((k^2(-(\alpha_j + k\delta_j)) - \omega)(-\alpha_j - 3k\delta_j) > 0\) and \(\delta_j \neq 0\) with \(j = 1, 2\) for valid solution.
Highly dispersive optical solitons and other solutions for the…

Substituting these values in Eqs. (18) and (51), then

The mixed optical soliton solution can be derived as

$$\Psi(x, t) = -\frac{\sqrt{k^2(-\left(\alpha_1 + k\delta_1\right)) - \omega}}{2\sqrt{-\beta_1 - \gamma_1 + k(\theta_1 + \lambda_1)}} \times \left( 2 \sinh^2 \left[ \frac{\sqrt{k^2(-\left(\alpha_1 + k\delta_1\right)) - \omega}}{2\sqrt{2}\sqrt{-\alpha_1 - 3k\delta_1}} (x - vt) \right] + 1 \right) \cosh \left[ \frac{\sqrt{k^2(-\left(\alpha_1 + k\delta_1\right)) - \omega}}{2\sqrt{2}\sqrt{-\alpha_1 - 3k\delta_1}} (x - vt) \right] \sinh \left[ \frac{\sqrt{k^2(-\left(\alpha_1 + k\delta_1\right)) - \omega}}{2\sqrt{2}\sqrt{-\alpha_1 - 3k\delta_1}} (x - vt) \right] \right)

\times e^{i \left(-kx + \frac{s_1 3k^2 + 6k^2\alpha_2 s_1 + 3k\delta_2 + 2k^2\alpha_2 + s_1 + t + \theta_0}{s_1} \right)}.

$$

$$\varphi(x, t) = -\frac{\sqrt{k^2(-\left(\alpha_2 + k\delta_2\right)) - \omega}}{2\sqrt{-\beta_2 - \gamma_2 + k(\theta_2 + \lambda_2)}} \times \left( 2 \cosh^2 \left[ \frac{\sqrt{k^2(-\left(\alpha_2 + k\delta_2\right)) - \omega}}{2\sqrt{2}\sqrt{-\alpha_2 - 3k\delta_2}} (x - vt) \right] - 1 \right) \cosh \left[ \frac{\sqrt{k^2(-\left(\alpha_2 + k\delta_2\right)) - \omega}}{2\sqrt{2}\sqrt{-\alpha_2 - 3k\delta_2}} (x - vt) \right] \sinh \left[ \frac{\sqrt{k^2(-\left(\alpha_2 + k\delta_2\right)) - \omega}}{2\sqrt{2}\sqrt{-\alpha_2 - 3k\delta_2}} (x - vt) \right] \right)

\times e^{i \left(-kx + \frac{s_2 3k^2 + 6k^2\alpha_2 s_2 + 3k\delta_2 + 2k^2\alpha_2 + s_2 + t + \theta_0}{s_2} \right)}.

Here \((k^2(-\left(\alpha_j + k\delta_j\right)) - \omega)(-\alpha_j - 3k\delta_j) > 0\) and \(\delta_j \neq 0\) with \(j = 1, 2\) for valid solution. The graphical representations of the solutions are shown for different values of parameters.

**Family-8** If we take \(r = \{1, 2, 1, 1\}\) and \(s = \{1, 0, 1, 0\}\), then Eq. (7) converts into

$$\Omega(\eta) = \frac{e^\eta + 2}{e^\eta + 1}.$$
Case 1

\[ d_0 = -\frac{3\sqrt{k^2(-\alpha_j + k\delta_j) - \omega}}{-\beta_j - \gamma_j + k(\theta_j + \lambda_j)}, \quad d_1 = 0, \quad f_1 = \frac{4\sqrt{k^2(-\alpha_j + k\delta_j) - \omega}}{-\beta_j - \gamma_j + k(\theta_j + \lambda_j)}, \]

\[ B = \frac{-\sqrt{2k^2(-\alpha_j + k\delta_j) - \omega}}{\sqrt{\alpha_j + 3k\delta_j}}. \]

Insert these values in Eqs. (18) and (56), then

The solitary wave solution can be written as

\[ \Psi(x, t) = \left[ \sqrt{k^2(-\alpha_1 + k\delta_1) - \omega} \frac{\eta e^{-2\eta}}{\eta e^{+2\eta}} \right] \left[ \begin{array}{c} e^{i(\delta x + \frac{\alpha_1^3 \delta_1^3 + 3k\delta_1 \delta_1 + 2k^2 \delta_1 \delta_1 + 2k^2 \alpha_1 + 1}{\delta_1})} \end{array} \right], \quad (57) \]

where \( \eta = \sqrt{\frac{2\sqrt{k^2(-\alpha_1 + k\delta_1) - \omega}}{\sqrt{\alpha_j + 3k\delta_1}}}(x - \nu t) \).

\[ \varphi(x, t) = \left[ \sqrt{k^2(-\alpha_2 + k\delta_2) - \omega} \frac{\eta e^{-2\eta}}{\eta e^{+2\eta}} \right] \left[ \begin{array}{c} e^{i(\delta x + \frac{\alpha_2^3 \delta_2^3 + 3k\delta_2 \delta_2 + 2k^2 \delta_2 \delta_2 + 2k^2 \alpha_2 + 1}{\delta_2})} \end{array} \right], \quad (58) \]

where \( \eta = \sqrt{\frac{2\sqrt{k^2(-\alpha_2 + k\delta_2) - \omega}}{\sqrt{\alpha_j + 3k\delta_2}}}(x - \nu t) \).

Here \( (k^2(-\alpha_j + k\delta_j) - \omega)(-\beta_j - \gamma_j + k(\theta_j + \lambda_j)) > 0 \) and \( \delta_j \neq 0 \) with \( j = 1, 2 \) for valid solution.

Case 2

\[ d_0 = -\frac{3\sqrt{k^2(-\alpha_j + k\delta_j) - \omega}}{-\beta_j - \gamma_j + k(\theta_j + \lambda_j)}, \quad d_1 = 0, \quad f_1 = \frac{2\sqrt{k^2(-\alpha_j + k\delta_j) - \omega}}{-\beta_j - \gamma_j + k(\theta_j + \lambda_j)}, \]

\[ B = \frac{-\sqrt{2k^2(-\alpha_j + k\delta_j) - \omega}}{\sqrt{\alpha_j + 3k\delta_j}}. \]

Replacing these values in Eqs. (18) and (56), then

The solitary wave solution can be obtained as

\[ \Psi(x, t) = \left( \sqrt{k^2(-\alpha_1 + k\delta_1) - \omega} \frac{1 - e^{\eta}}{1 + e^{\eta}} \right) \left[ \begin{array}{c} e^{i(\delta x + \frac{\alpha_1^3 \delta_1^3 + 3k\delta_1 \delta_1 + 2k^2 \delta_1 \delta_1 + 2k^2 \alpha_1 + 1}{\delta_1})} \end{array} \right], \quad (59) \]

where \( \eta = \sqrt{\frac{2\sqrt{k^2(-\alpha_1 + k\delta_1) - \omega}}{\sqrt{\alpha_j + 3k\delta_1}}}(x - \nu t) \).
\[
\varphi = \left( \frac{\sqrt{k^2 \left( -\left( \alpha_2 + k\delta_2 \right) \right) - \omega}}{\sqrt{-\beta_2 - \gamma_2 + k\left( \theta_2 + \lambda_2 \right)}} \right) \frac{1 - e^{\eta}}{1 + e^{\eta}} \left( -kx + \frac{8k^3 - 8k^2z_2 + 3kz_2^2 + 2akz_1 + az_1}{z_2} t + \theta_0 \right),
\]

where \( \eta = \frac{\sqrt{2 \left( k^2 \left( -\left( \alpha_j + k\delta_j \right) \right) - \omega \right)}}{\sqrt{\alpha_j + 3k\delta_j}} \)\((x - vt)\). Here \( (k^2 \left( -\left( \alpha_j + k\delta_j \right) \right) - \omega)\left( -\beta_j - \gamma_j + k\left( \theta_j + \lambda_j \right) \right) > 0 \) and \( \delta_j \neq 0 \) with \( j = 1, 2 \) for valid solution.

### 4 Results and discussion

The results of this paper will be beneficial for learners to analyze the most attractive applications of the RKL equation, which describes the propagation of waves without 4WM in birefringent fibers. Figures 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 and 12 clearly demonstrates the surfaces of the solution obtained for 3D, 2D and the contour plots, with choices of parameters for the RKL model. We can capture the behaviour of the of the acquired solution with the assistant of contour plots. In the same way, 3D figures tell us to model and demonstrate accurate physical behaviour. Through this study, we consider the exact optical soliton solutions to the nonlinear RKL model using generalized exponential rational function approach. The authors proposed different analytic approach in newly issued article and reported some fascinating findings. We can grasp from all the graphs that the GERFM is very efficient and more precise in evaluating the equation under consideration.

![Fig. 1](image.png)

**Fig. 1** The a–c show the 3D, 2D and contour physical behaviour of solution (20), respectively with the parameters \( k = 1, \omega = -3, \beta_1 = 3, \alpha_1 = -2.9, \gamma_1 = -4, \delta_1 = 2, \lambda_1 = 3, \theta_1 = 4, \theta_0 = -2, v = .8 \)
Fig. 2 The a–c show the 3D, 2D and contour physical behaviour of solution (21), respectively with the parameters $k = 1, \omega = -4, \beta_2 = 1, \alpha_2 = -2, \gamma_2 = -4.6, \delta_2 = 3, \lambda_2 = 7, \theta_2 = -3, \theta_0 = -1, v = .3$
Fig. 3  The a–c show the 3D, 2D and contour physical behaviour of solution (22), respectively with the parameters \( k = 1, \omega = -6, \beta_1 = 1, \alpha_1 = -4, \gamma_1 = -2, \delta_1 = 3, \lambda_1 = 2, \theta_1 = 6, \theta_0 = -1, \nu = .7 \)
Fig. 4 The a–c show the 3D, 2D and contour physical behaviour of solution (23), respectively with the parameters $k = 1, \omega = -3.6, \beta_2 = 2, \alpha_2 = -3, \gamma_2 = -5, \delta_2 = 4, \lambda_2 = 8, \theta_2 = -2, \theta_0 = -3, \nu = .3$.
Fig. 5 The a–c show the 3D, 2D and contour physical behaviour of solution (24), respectively with the parameters $k = 2$, $\omega = 5$, $\beta_1 = .5$, $\alpha_1 = 1$, $\gamma_1 = -5$, $\delta_1 = 7$, $\lambda_1 = 3$, $\theta_1 = 5$, $\theta_0 = -3$, $\nu = 2$
Fig. 6 The a–c show the 3D, 2D and contour physical behaviour of solution (25), respectively with the parameters \( k = 2, \omega = 1, \beta_2 = -3, \alpha_2 = 4, \gamma_2 = 0, \delta_2 = 5, \lambda_2 = 6, \theta_2 = 7, \theta_0 = 2, \nu = 1.5 \).
Fig. 7 The a–c show the 3D, 2D and contour physical behaviour of solution (27), respectively with the parameters $k = 5, \omega = -1, \beta_i = -2, \alpha_i = 1, \gamma_i = -4, \delta_i = 3, \lambda_i = 2, \theta_i = 5, \theta_0 = 3, \nu = .02$.
Fig. 8 The a–c show the 3D, 2D and contour physical behaviour of solution (28), respectively with the parameters $k = 4, \omega = 3, \beta_2 = 0, a_2 = 3, \gamma_2 = 0, \delta_2 = 8, \lambda_2 = 2, \theta_2 = 3, \theta_0 = 3, \nu = .3$
Fig. 9 The a–c show the 3D, 2D and contour physical behaviour of solution (34), respectively with the parameters $k = 0, \omega = -1, \beta_1 = -2, \alpha_1 = 1, \gamma_1 = 0, \delta_1 = -3, \lambda_1 = 5, \theta_1 = 4, \theta_0 = 2, \nu = 3$
Fig. 10 The a–c show the 3D, 2D and contour physical behaviour of solution (35), respectively with the parameters $k = 1$, $\omega = -4$, $\beta_2 = -2$, $\alpha_2 = 0$, $\gamma_2 = -5$, $\delta_2 = -2$, $\lambda_2 = 3$, $\theta_2 = 4$, $\theta_0 = 2$, $\nu = 1$
Fig. 11 The a–c show the 3D, 2D and contour physical behaviour of solution (54), respectively with the parameters $k = 8, \omega = -2, \beta_1 = -3, \alpha_1 = -6, \gamma_1 = -5, \delta_1 = -4, \lambda_1 = 4, \theta_1 = 0, \theta_0 = 1, v = 1.5$
Conclusions

This paper extracted the dynamics of optical solitons in the RKL equation without 4WM in birefringent fibers. The solutions are achieved in the shape of exponential, trigonometric and hyperbolic functions as well as exact optical soliton solutions by the mechanism of GERFM under different constraint conditions which provide the guaranty and validity of solutions are also listed. In addition, we attained the singular periodic wave solutions. The model must be extended to govern DWDM networks so that parallel communication of soliton dynamics can be addressed. These new families of solutions are shown the power, effectiveness and fruitfulness of this method. This article shelters the application of optical fibers. Also, these fresh solutions have many applications in physics and other branches of physical sciences.

References

Ahmed, I., Seadawy, A.R., Lu, D.: M-shaped rational solitons and their interaction with kink waves in the Fokas–Lenells equation. Phys. Scr. 94, 055205 (2019) (7pp)
Highly dispersive optical solitons and other solutions for the...

Ali, A., Seadawy, A.R., Lu, D.: Computational methods and traveling wave solutions for the fourth-order nonlinear Ablowitz–Kaup–Newell–Segur water wave dynamical equation via two methods and its applications. Open Phys. 16, 219–226 (2018a)

Ali, A., Seadawy, A.R., Lu, D.: New solitary wave solutions of some nonlinear models and their applications. Adv. Differ. Equ. 2018(32), 1–12 (2018b)

Arshad, M., Seadawy, A., Lu, D.: Bright-Dark Solitary Wave Solutions of generalized higher-order nonlinear Schrodinger equation and its applications in optics. J. Electromagn. Waves Appl. 31(16), 1711–1721 (2017)

Biswas, A., Yildirim, Y., Yasar, E., Zhou, Q., Moshokoa, S.P., Belic, M.: Optical solitons for Lakshmanan–Porsezian–Daniel model by modified simple equation method. Optics 160, 24–32 (2018a)

Biswas, A., Yildirim, Y., Yasar, E., Mahmood, M.F., Alshomrani, A.S., Zhou, Q., Moshokoa, S.P., Belic, M.: Optical soliton perturbation for Radhakrishnan–Kundu–Lakshmanan equation with a couple of integration schemes. Optik 163, 126–136 (2018b)

Biswas, A., Yildirim, Y., Yasar, E., Zhou, Q., Alshomrani, A.S., Moshokoa, S.P., Mi, B.: Solitons for perturbed Gerdjikov–Ivanov equation in optical fibers and PCF by extended Kudryashov’s method. Opt. Quantum Electron. 50(3), 1–13 (2018c)

Celik, N., Seadawy, A.R., Ozkan, Y.S., Yasar, E.: A model of solitary waves in a nonlinear elastic circular rod: abundant different type exact solutions and conservation laws. Chaos Solitons Fractals 143, 110486 (2021)

Cheemaa, N., Seadawy, A.R., Chen, S.: More general families of exact solitary wave solutions of the nonlinear Schrodinger equation with their applications in nonlinear optics. Eur. Phys. J. Plus 133, 547 (2018)

Cheemaa, N., Seadawy, A.R., Chen, S.: Some new families of solitary wave solutions of generalized Schamel equation and their applications in plasma physics. Eur. Phys. J. Plus 134, 117 (2019)

Donne, G.D., Hubert, M.B., Seadawy, A.R., Etienne, T., Betchewe, G., Doka, S.Y.: Chirped soliton solutions of Fokas–Lenells equation with perturbation terms and the effect of spatio-temporal dispersion in the modulational instability analysis. Eur. Phys. J. Plus 135(2), 212 (2020)

Ghanbari, B., Inc, M., Yusuf, A., Baleanu, D.: New solitary wave solutions and stability analysis of the Benney-Luke and the Phi-4 equations in mathematical physics. AIMS Math. 6, 1532–1539 (2018)

Iqbal, M., Seadawy, A.R., Lu, D., Xianwe, X.: Construction of a weakly nonlinear dispersion solitary wave solution for the Zakharov–Kuznetsov-modified equal width dynamical equation. Indian J. Phys. 1–10 (2019)

Iqbal, M., Seadawy, A.R., Lu, D., Xianwe, X.: Construction of bright-dark solitons and ion-acoustic solitary wave solutions of dynamical system of nonlinear wave propagation. Mod. Phys. Lett. A 34(37), 1950309 (2019b)

Iqbal, M., Seadawy, A.R., Khalil, O.H., Lu, D.: Propagation of long internal waves in density stratified ocean for the (2 + 1)-dimensional nonlinear Nizhnik–Novikov–Vesselinov dynamical equation. Results Phys. 16, 102838 (2020)

Jhangeera, A., Seadawy, A.R., Ali, F., Ahmed, A.: New complex waves of perturbed Shodinger equation with Kerr law nonlinearity and Kundu–Mukherjee–Naskar equation. Results Phys. 16, 102816 (2020)

Khater, A.H., Helal, M.A., Seadawy, A.R.: General soliton solutions of n-dimensional nonlinear Schrödinger equation. IL Nuovo Cimento 115B, 1303–1312 (2000)

Lu, D., Seadawy, A.R., Ahmed, I.: Peregrine-like rational solitons and their interaction with kink wave for the resonance nonlinear Schrödinger equation with Kerr law of nonlinearity. Mod. Phys. Lett. B 33(24), 1950292 (2019)

Farah, N., Seadawy, A.R., Ahmad, S., Rizvi, S.T.R., Younis, M.: Interaction properties of soliton molecules and Painleve analysis for nano bioelectronics transmission model. Opt. Quantum Electron. 52(329), 1–15 (2020)

Ozkan, Y.G., Yasar, E., Seadawy, A.: A third-order nonlinear Schrödinger equation: the exact solutions, group-invariant solutions and conservation laws. J. Taibah Univ. Sci. 14(1), 585–597 (2020a)

Ozkan, Y.G., Yasar, E., Seadawy, A.: On the multi-waves, interaction and Peregrine-like rational solutions of perturbed Radhakrishnan–Kundu–Lakshmanan equation. Phys. Scr. 95(8), 085205 (2020b)

Ozkan, Y.S., Seadawy, A.R., Yasar, E.: On the optical solitons and local conservation laws of Chen-Lee-Liu dynamical wave equation. Optik Int. J. Light Electron Opt. 227, 165392 (2021)

Rehman, H.U., Saleem, M.S., Zubair, M., Jafar, S., Latif, I.: Optical solitons with Biswas–Arshed model using mapping method. Optik 194, 163091 (2019a)

Rehman, H.U., Saleem, M.S., Sultan, A.M., Iftekhar, M.: Comments on “Dynamics of optical solitons with Radhakrishnan–Kundu–Lakshmanan model via two reliable integration schemes”. Optik 178, 557–566 (2019b)

Rehman, H.U., Saleem, M.S., Sultan, A.M., Iftekhar, M.: Comments on “Dynamics of optical solitons with Radhakrishnan–Kundu–Lakshmanan model via two reliable integration schemes”. Optik 178, 557–566 (2019b)
Rizvi, S.T.R., Seadawy, A.R., Ali, I., Bibi, I., Younis, M.: Chirp-free optical dromions for the presence of higher order spatio-temporal dispersions and absence of self-phase modulation in birefringent fibers. Mod. Phys. Lett. B 34(35), 2050399 (2020) (15 pages)

Seadawy, A.R., Abdullah, A.: Nonlinear complex physical models: optical soliton solutions of the complex Hirota dynamical model. Indian J. Phys. 1–10 (2019)

Seadawy, A., Kumar, D., Hosseini, K., Samadani, F.: The system of equations for the ion sound and Langmuir waves and its new exact solutions. Results Phys. 9, 1631–1634 (2018)

Seadawy, A.R., Arshad, M., Lu, D.: Modulation stability analysis and solitary wave solutions of nonlinear higher-order Schrödinger dynamical equation with second-order spatiotemporal dispersion. Indian J. Phys. 93(8), 1041–1049 (2019a)

Seadawy, A.R., Iqbal, M., Lu, D.: Application of mathematical methods on the ion sound and Langmuir waves dynamical systems. Pramana J. Phys. 93, Article number: 10 (2019b)

Seadawy, A.R., Arshad, M., Lu, D.: Dispersive optical solitary wave solutions of strain wave equation in micro-structured solids and its applications. Physica A 540, 123122 (2020a)

Seadawy, A.R., Alamri, S.Z., Al-Sharari, H.M.: Construction of optical soliton solutions of the generalized nonlinear Radhakrishnan–Kundu–Lakshmanan dynamical equation with power law nonlinearity. Int. J. Mod. Phys. B 34(13), 2050139 (2020b) (17 pages)

Sulaiman, T.A., Bulut, H., Yel, G., Atas, S.S.: Optical solitons to the fractional perturbed Radhakrishnan–Kundu–Lakshmanan model. Opt. Quantum Electron. 50, 372 (2018). https://doi.org/10.1007/s11082-018-1641-7

Younas, B., Younis, M.: Chirped solitons in optical monomode fibres modelled with Chen–Lee–Liu equation. Pramana J. Phys. 94, 3 (2020). https://doi.org/10.1007/s12043-019-1872-6

Younas, U., Seadawy, A.R., Younis, M., Rizvi, S.T.R.: Dispersive of propagation wave structures to the Dullin–Gottwald–Holm dynamical equation in a shallow water waves. Chin. J. Phys. 68, 348–364 (2020)

Younis, M., Younas, U., Rehman, S.U., Bilal, M., Waheed, A.: Optical bright-dark and Gaussian soliton with third order dispersion. Optik 134, 233–238 (2017)

Younis, M., Bilal, M., Rehman, S., Younas, U., Rizvi, S.T.R.: Investigation of optical solitons in birefringent polarization preserving fibers with four-wave mixing effect. Int. J. Mod. Phys. B 2050113 (2020)

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