Quantum Spin Dynamics (QSD)

T. Thiemann*
Physics Department, Harvard University,
Cambridge, MA 02138, USA

Preprint HUTMP-96/B-351

Abstract

An anomaly-free operator corresponding to the Wheeler-DeWitt constraint of Lorentzian, four-dimensional, canonical, non-perturbative vacuum gravity is constructed in the continuum. This operator is entirely free of factor ordering singularities and can be defined in symmetric and non-symmetric form.

We work in the real connection representation and obtain a well-defined quantum theory. We compute the complete solution to the Quantum Einstein Equations for the non-symmetric version of the operator and a physical inner product thereon.

The action of the Wheeler-DeWitt constraint on spin-network states is by annihilating, creating and rerouting the quanta of angular momentum associated with the edges of the underlying graph while the ADM-energy is essentially diagonalized by the spin-network states. We argue that the spin-network representation is the “non-linear Fock representation” of quantum gravity, thus justifying the term “Quantum Spin Dynamics (QSD)”.

1 Introduction

Attempts at defining an operator corresponding to the Wheeler-DeWitt constraint of Lorentzian, four-dimensional canonical vacuum gravity have been made first in the framework of the metric (or ADM) variables (see, for instance, [3]). The formulation of the theory seemed hopelessly difficult because of the complicated, non-polynomial algebraic form of the Wheeler-DeWitt constraint and it was therefore thought to be mandatory to first obtain a polynomial formulation of the theory to complete the programme.

The celebrated observation due to Ashtekar [4] is that the canonical constraints of general relativity can indeed be cast into polynomial form if one performs a certain complex canonical transformation on the gravitational phase space. This opened, for the first time, the hope that one can actually rigorously define the quantum Hamiltonian constraint (or Wheeler-DeWitt equation).

*thiemann@math.harvard.edu
The major roadblock after this was the difficult reality structure of the theory so obtained: general relativity, when written in Ashtekar’s variables, is a dynamical theory of complex-valued connections for the non-compact gauge group $SL(2,\mathbb{C})$, however, standard mathematical constructions and techniques usually used for Yang-Mills theory apply only if the gauge group is compact.

A solution to this problem has been recently proposed in form of a phase space Wick rotation transform [5] (see also [6]) which enables one to formulate general relativity as a $SU(2)$ gauge theory while keeping the polynomial algebraic form of Wheeler-DeWitt constraint and incorporating the correct reality conditions into the quantum theory. This theory can be, not surprisingly, recognized as Euclidean four-dimensional gravity as written in terms of real valued Ashtekar variables [7]. This then opens access to the recently developed, powerful calculus on the space of (generalized) connections modulo gauge transformations for compact gauge groups [8, 9, 10, 11, 12]. This calculus provides a rigorous kinematical framework by means of which constraint operators can be regularized in a well-defined and unambiguous manner. In particular, this framework has already been successfully employed to arrive at the general solution of the Gauss and Diffeomorphism constraints [13].

In order to complete the programme one still needed to construct a rigorously defined operator corresponding to the classical generator of the transform. This seems a hopeless problem to solve as the classical generator is a non-polynomial, not even analytic function of the phase space variables.

There is, however, an even more severe problem: the Hamiltonian constraint of Euclidean or Lorentzian gravity as defined by Ashtekar is a density of weight two. Namely, in order to obtain a polynomial form of those constraints one needs to rescale the constraint functional and to absorb a factor of $1/\sqrt{\det(q)}$ into the Lagrange multiplier (the lapse function; here $q = (q_{ab})$ is the intrinsic metric of an initial data hypersurface). On general grounds, an operator corresponding to a classical function on the phase space with density weight different from one needs to be renormalized. While this poses no, a priori, problems for, say, Yang-Mills theories in a classical background geometry, such a procedure is unacceptable for quantum gravity since a renormalization unavoidably introduces a length scale and thus breaks diffeomorphism invariance.

A solution to this problem was first suggested in [14] and is currently reconsidered in [15]: to produce an operator of density weight one one takes the square root of the rescaled Euclidean Hamiltonian constraint. While classically this does not alter the theory and while this seems to be a necessary step to do in quantum theory in order to keep diffeomorphism invariance, there remain problems that have to do with taking the square root of an infinite number of non-self adjoint, non-positive, non-commuting operators.

In the present article we suggest a technique to solve both problems, namely the complicated reality structure of the Lorentzian theory as well as the problem connected with the density weight, in one stroke: what was thought to create problems in the quantization process is precisely the reason for why we are able to find a finite operator: the factor $1/\sqrt{\det(q)}$ is needed.

- We show that it is indeed possible to construct a finite, (symmetric) operator
corresponding to the original, non-rescaled, Lorentzian Wheeler-DeWitt constraint whose quantum constraint algebra is non-anomalous. Since the original Wheeler-DeWitt constraint has density weight one, no renormalization is necessary.

- Amazingly, the resulting operator is not messy at all and the problem of finding exact solutions to the quantum constraint is conceivable.

- We always work with real-valued Ashtekar variables, the reality structure of the theory is very simple, the complex-valued Ashtekar variables are never introduced.

- In the construction of the Wheeler-DeWitt operator the unrescaled Euclidean Hamiltonian operator as well as the generator of the Wick transform arise in a natural way so that both operators are also constructed rigorously as a side result. This is important since the Wick rotation transform simplifies the problem of finding solutions to the quantum constraint.

- Using the same technique one can give rigorous meaning to a whole bunch of other operators in a representation where the metric is not diagonal, including but not exhausting a) the operator corresponding to the length of a curve, b) the generators of the Poincaré group for asymptotically flat topologies and c) matter contributions to the Hamiltonian constraint. It might be that it is in this sense that quantum gravity arises as the “natural regulator of the matter field theories”. By this we mean the following: in the canonical quantization programme a natural regularization procedure is point splitting. With the exception of spinorial matter, all matter Hamiltonians of, say, the standard model, are quadratic in the momenta which means that they display a density weight of two when neglecting the gravitational interaction while they have density weight one when taking gravity into account. When removing the regulator the density weight shows up in the form of a product of delta distributions evaluated at the same point which is singular. On the other hand, with gravity the singularity is removed, in the limit one arrives at a well-defined operator-valued distribution.

The article is organized as follows:

After fixing the notation and explaining the main idea we construct first the unrescaled Euclidean Hamiltonian constraint operator. We motivate our choices involved in the regularization step. The freedom in our choices is severely restricted by the requirement that the resulting operator be diffeomorphism-covariantly defined and non-anomalous. We provide a solution to both requirements. If one adds the requirement that the operator be at least symmetric (preferrably, it should possess a self-adjoint extension) then the regularization involves an additional structure. We will stick with a non-symmetric operator in the main text and provide a symmetric operator in which is non-anomalous and diffeomorphism-covariantly defined as well.

Next we construct operators corresponding to the generator of the Wick rotation transform and finally the Lorentzian Wheeler-DeWitt operator.
For the non-symmetric operator we are able to find the complete kernel of the Wheeler-DeWitt constraint operator in [1]. The physical Hilbert space turns out to be the one already given in [13].

For the symmetric operator on the other hand we do not have the complete solution yet, although solutions can be easily computed by a case by case analysis. In principle, since, expectedly, on diffeomorphism invariant states the constraint algebra is Abelian we are able to find solutions to the constraint as well as constructing a physical inner product by the group averaging method [19, 20]. We also comment on finding observables along the line of argument in [13]. A lot of the mathematical problems of quantum gravity can be solved, on so-called cylindrical subspaces, with well-known Hilbert space techniques familiar from quantum mechanics.

We have complete control over the space of solutions to both versions of the constraint and the intuitive picture that arises is the following: The Hamiltonian constraint acts by annihilating, creating and re-routing the quanta of angular momentum (with which the graphs of so-called spin-network [21, 22, 23] states are “coloured”) in units of ±ℏ, ±ℏ/2, 0. On the other hand, linear combinations of such states diagonalize the ADM energy operator very much in the same way as Fock states diagonalize the Maxwell Hamiltonian, the role of the occupation number being played by the spins of the spin-network state. Thus, the spin-network representation is the “non-linear Fock representation for quantum gravity” and this motivates to call the quantum theory we obtain “Quantum Spin Dynamics (QSD)” in analogy with QED or QCD.

2 Notation and the main idea

Let the triad on the spacelike, smooth, hypersurface Σ be denoted by \( e^a_i \), where \( a, b, c, ... \) are tensorial and \( i, j, k, ... \) are SU(2) indices. The relation with the intrinsic metric is given by \( q_{ab} = e^a_i e^b_j \delta_{ij} \). It follows that \( \text{det}(q) := \text{det}((q_{ab})) = [\text{det}((e^a_i))]^2 \geq 0 \). The densitized triad is then defined by \( E^a_i := \text{det}((e^a_i)) e^a_i \) where \( e^a_i \) is the inverse of \( e^a_i \). We also need the field \( K^i_a = \epsilon K_{ab} e^b_i \), \( \epsilon = \text{sgn(}\text{det}((e^a_i))\) where \( K_{ab} \) is the extrinsic curvature of \( \Sigma \). It turns out that the pair \((K^i_a, E^a_i)\) is a canonical one, that is, these variables obey canonical brackets \( \{K^i_a(x), E^b_j(y)\} = \kappa \delta^{(3)}(x, y) \delta^a_d \delta^b_j \) where \( \kappa \) is Newton’s constant.

Let the spin-connection (which annihilates the triad) be denoted by \( \Gamma^i_a \). Then one can show that \((A^i_a := \Gamma^i_a + K^i_a, E^a_i)\) is a canonical pair\(^1\) on the phase space of Lorentzian gravity subject to the SU(2) Gauss constraint, the diffeomorphism constraint and the Wheeler-DeWitt constraint (neglecting a term proportional to the Gauss constraint)

\[
H := \sqrt{\text{det}(q)}[K_{ab} K^{ab} - (K^a_a)^2 - R] = \frac{1}{\sqrt{\text{det}(q)}} \text{tr}((F_{ab} - 2R_{ab})[E^a_b, E^b_b])
\]  

(2.1)

where \( F_{ab} \) and \( R_{ab} \) respectively are the curvatures of the SU(2) connection \( A^i_a \) and the triad \( e^a_i \) respectively.

\(^1\)If we had chosen \( E^a_i = \sqrt{\text{det}(q)} e^a_i \) instead of \( E^a_i = \text{det}(e) e^a_i \) then this pair is not canonical, a fact often overlooked in the literature [4]. Roughly speaking, we would spoil the integrability of the spin connection.
What has been gained by reformulating canonical gravity as a dynamical theory of $SU(2)$ connections is the following: if, as we do in the sequel, one makes the assumption that there exists a phase for quantum gravity in which the excitations of the gravitational field can be probed by loops rather than, say, test functions of rapid decrease, then one has access to a powerful calculus on the space of (generalized) connections modulo gauge transformations $\mathcal{A}/\mathcal{G}$ and, in particular, there is a natural choice of a diffeomorphism invariant, faithful measure $\mu_0$ thereon which equips us with a Hilbert space $\mathcal{H} := L^2(\mathcal{A}/\mathcal{G}, d\mu_0)$, appropriate for a representation in which $A$ is diagonal. Moreover, Gauss and diffeomorphism constraints can be solved (see [10] and references therein for an introduction to these concepts).

The remaining step then is to give a rigorously defined quantum operator corresponding to the Wheeler-DeWitt constraint and to project the scalar product on its kernel.

Let us explain elements of the underlying kinematical framework [13]:

We begin by explaining the notion of a “cylindrical function” on $\mathcal{A}/\mathcal{G}$. In brief terms, gauge invariant cylindrical functions on the space of (generalized) $SU(2)$ connections are just finite linear combinations of traces of the holonomy around closed analytic loops in $\Sigma$. Each such function therefore may equally well be labelled by the closed, piecewise analytic graph $\gamma$ consisting of the union of all loops involved in that linear combination. Such a graph consists of a finite number of edges $e_1, .., e_n$ and vertices $v_1, .., v_m$. So, a function cylindrical with respect to a graph $\gamma$ typically looks like $f(A) = f_\gamma(h_{e_1}(A), .., h_{e_n}(A))$, where $h_e(A)$ is the holonomy along $e$ for the connection $A$ and $f_\gamma$ is a complex-valued function on $SU(2)^n$ such that $f(A)$ is gauge-invariant. The functions cylindrical with respect to a graph that are $n$-times differentiable with respect to the standard differentiable structure on $SU(2)^m$ for some $m$ will be denoted by $\text{Cyl}^n_\nu(\mathcal{A}/\mathcal{G})$ and $\text{Cyl}^n(\mathcal{A}/\mathcal{G}) := \cup_\nu \text{Cyl}^n_\nu(\mathcal{A}/\mathcal{G})$. One and the same cylindrical function can be represented on different graphs leading to cylindrically equivalent representants of that function. It is understood in the above union that such kind of functions are identified.

The measure $\mu_0$ referred to above is entirely characterized by its cylindrical projections defined by

$$\int_{\mathcal{A}/\mathcal{G}} d\mu_0(A) f(A) = \int_{\mathcal{A}/\mathcal{G}} d\mu_{0,\gamma}(A) f_\gamma(\{h_{e_i}(A)\}) = \int_{SU(2)^n} d^n \mu_H(g_1, .., g_n) f_\gamma(g_1, .., g_n).$$

An orthonormal basis on $\mathcal{H}$ is given by the so-called spin-network states [21, 22, 23]: given a graph $\gamma$, “colour” each of its edges $e$ with a non-trivial irreducible representation $\pi_{j_e}$ of $SU(2)$, that is, $j_e$ is the spin associated with $e$. With each vertex we associate a contraction matrix $c_v$ which contracts all the matrices $\pi_j(h_e)$ for $e$ incident at $v$ in a gauge-invariant way. In this paper we will denote a spin-network state by $T_{\gamma,j,\bar{c}}\;\bar{c}$ if we wish to stress the dependence on $\gamma, j = (j_e), \bar{c} = (c_v)$ where the vectors $j, \bar{c}$ have indices corresponding to the edges and vertices of $\gamma$ respectively. Consider the set of smooth cylindrical functions $\Phi := \text{Cyl}^\infty(\mathcal{A}/\mathcal{G})$ which can be shown to be dense in $\mathcal{H}$. By a distribution $\psi \in \Phi'$ on $\Phi$ we mean a generalized function on $\mathcal{A}/\mathcal{G}$ such that for any $\phi \in \Phi$ the number $\psi[\phi] := \int_{\mathcal{A}/\mathcal{G}} d\mu_0 \psi \phi < \infty$ is

---

2The extension of the framework to the smooth category is possible but not entirely straightforward [24]. For simplicity we restrict ourselves to the analytic category.
finite. It turns out that the solutions of the diffeomorphism constraint are elements of $\Phi'$. Suppose we have a quantization $\hat{H}(N)$ of the Hamiltonian constraint operator, that is, it is densely defined on $\mathcal{H}$ and its classical limit reduces to $H(N)$, where $N$ is the lapse function. Then its adjoint $\hat{H}(N)\dagger$ also has the same classical limit $H(N)$ (a reordering of terms gives only higher orders in $\hbar$) because $H(N)$ is real-valued. Therefore, if $\hat{H}(N)$ is not self-adjoint, then it is an option of whether we impose $\hat{H}(N)\psi = 0$ or $\hat{H}(N)\dagger\psi = 0$. In both cases, the solution $\psi$ is typically not an $L_2$ function any longer but a distribution, that is, in our case an element of $\Phi'$. Thus, given the framework of generalized eigenvectors and the associated triple $\Phi \subset \mathcal{H} \subset \Phi'$, we choose to find the generalized eigenvectors with eigenvalue 0 corresponding to the kernel of the (self-adjoint) Hamiltonian constraint operator $\hat{H}(N)\dagger$ as follows: Let $\psi \in \Phi'$ be a distribution. We say that $\psi$ is in the kernel of $\hat{H}(N)\dagger$ whenever $(\hat{H}(N)\dagger\psi)(f) := \psi(\hat{H}(N)f) = 0$ for each lapse and each cylindrical function $f$ in the (dense) domain of $\hat{H}(N)$. Note that we cannot require that $\psi$ is diffeomorphism invariant if we impose the condition $\hat{H}(N)\dagger\psi = 0$. This is because the Hamiltonian constraint does not leave the subspace of $\Phi'$, corresponding to diffeomorphism invariant elements, invariant so that one would be forced to solve the Hamiltonian constraint before the diffeomorphism constraint. On the other hand, we will see that if $\psi$ is diffeomorphism invariant and $\hat{H}(N)$ is diffeomorphism covariantly defined, then solving $\psi(\hat{H}(N)f) = 0$ is meaningful.

After this preparation we are now ready to explain the main idea of our approach. Suppose that we can give meaning, in a representation in which $A$ is diagonal, to two operators corresponding to

1) The total volume of $\Sigma$ given by

$$V := V(\Sigma) := \int_\Sigma d^3x \sqrt{|\det(q)|}$$

(2.2)

2) the integrated trace of the (densitized) extrinsic curvature of $\Sigma$

$$K := \int_\Sigma d^3x \sqrt{\det(q)} K_{ab} q^{ab} = \int_\Sigma d^3x K_i^a E_a^i.$$  

(2.3)

This means that their quantizations $\hat{V}, \hat{K}$ are densely defined on suitable subspaces of cylindrical functions (in case that we wish to obtain a symmetric operator we will also require that they are self-adjoint on $\mathcal{H}$). The motivation for these two assumptions comes from considering the following two key identities

$$\frac{[E^a, E^b]}{\sqrt{\det(q)}} = \epsilon^{abc} e_c^i(x) = 2\epsilon^{abc} \frac{\delta V}{\delta E_c^i(x)} = 2\epsilon^{abc} \left\{ \frac{A_c^i}{\kappa}, V \right\}$$

(2.4)

$$K_i^a = \frac{\delta K}{\delta E_i^a} = \left\{ \frac{A_i^a}{\kappa}, K \right\}.$$  

(2.5)

In case that $\Sigma$ is not compact, as we will need only the variation of $V$ in the sequel, is to be understood in the following way: consider any nested one-parameter family of compact manifolds $\Sigma_r \subset \Sigma_{r'}$, $\forall 0 \leq r < r' < \infty$ such that $\lim_{r \to \infty} \Sigma_r = \Sigma$. Then $\delta V(\Sigma) := \lim_{r \to \infty} \delta V(\Sigma_r)$ and this is well-defined.
The last equality relies on the observation that \( \{ \Gamma^i_v, K \} = 0 \) and it is this identity underlying the ideas developed in [5]. The importance of these identities becomes clear when we get rid of the complicated curvature term \( R_{ab} \) involved in (2.4) in favour of \( K \). We have

\[
H + H^E = \frac{2}{\sqrt{\det(q)}} \text{tr}(\{K_a, K_b\}[E^a, E^b]) = \frac{2}{\sqrt{\det(q)}} \text{tr}(\{\frac{A_a}{\kappa}, K\}, \{\frac{A_b}{\kappa}, K\}[E^a, E^b])
\]

\[
= \frac{4}{\kappa^3} \epsilon^{abc} \text{tr}(\{A_a, K\}, \{A_b, K\}\{A_c, V\}) = \frac{8}{\kappa^3} \epsilon^{abc} \text{tr}(\{A_a, K\}\{A_b, K\}\{A_c, V\})
\]

(2.6)

where we have introduced the (unrescaled) Euclidean Hamiltonian constraint

\[
H^E := \frac{1}{\sqrt{\det(q)}} \text{tr}(F_{ab}[E^a, E^b]) = \frac{2}{\kappa^3} \epsilon^{abc} \text{tr}(F_{ab}\{A_c, V\})
\]

(2.7)

So what we have achieved is to hide the non-polynomiality of the theory as determined by \( 1/\sqrt{\det(q)} \) in a Poisson bracket. Classically this is not helpful at all (except, possibly, for Hamilton-Jacobi methods or semi-classical approximations), however, we will show that in the quantum theory it is of advantage. Namely, it is now clear where we are driving at: in any regularization of the Wheeler-DeWitt constraint operator we will have to approximate the connection \( A_a \) and the curvature \( F_{ab} \) respectively by cylindrical functions given by the holonomies \( h \) along some edges and closed loops respectively. Now, one obvious quantization of (2.3) would be to replace \( V, K \) by \( \hat{V}, \hat{K} \) and the Poisson brackets \( \{..,\} \) by \( [..,]/(i\hbar) \). It then follows, given our assumption, that this quantization of (2.6) has a chance to result in a finite operator on cylindrical functions since no singular terms appear when computing the bracket and we would have managed to produce a densely defined operator.

Is it then true that there exist quantizations of \( V, K \) meeting our assumption? The answer is, surprisingly, in the affirmative:

First, it is a fact that there has already been constructed a well-defined, self-adjoint operator \( \hat{V} \) on \( \mathcal{H} \) corresponding to \( V \) [23, 26] whose action on cylindrical functions is perfectly finite: (we follow [26])

\[
\hat{V} f = \ell_p^3 \sum_{v \in V(\gamma)} \hat{V}_v f \quad \text{with} \quad \ell_p^3 \sum_{v \in V(\gamma)} \left| \frac{i}{8 \cdot 3!} \sum_{e_j, e_k, \epsilon} \epsilon(e_I, e_J, e_K) \epsilon_{ijk} X^i_I X^j_J X^K_K \right| f_\gamma(g_1, \ldots, g_n)
\]

(2.8)

where \( \epsilon(e_I, e_J, e_K) = \text{sgn}(\text{det}(\hat{e_I}(0), \hat{e_J}(0), \hat{e_K}(0))) \). We have abbreviated \( g_I = h_{e_I}(A) \) and \( X_I = X(g_I) \) is the right invariant vector field on \( SU(2) \) (we have chosen orientations such that all edges are outgoing at \( v \)). \( V(\gamma) \) is the set of vertices of \( \gamma \) and \( \ell_p := \sqrt{\hbar \kappa} \) is the Planck length. The symbol \( \hat{V}_v \) is defined as follows:

\[
\hat{V}_v f_\gamma = \left\{ \begin{array}{ll} 
\sqrt{\frac{i}{8 \cdot 3!} \sum_{e_j, e_k, \epsilon} \epsilon(e_I, e_J, e_K) \epsilon_{ijk} X^i_I X^j_J X^K_K} f_\gamma & \text{if } v \in V(\gamma) \\
0 & \text{if } v \notin V(\gamma)
\end{array} \right.
\]

that this, it is an operator which acts on the edges of any graph meeting at the point \( v \) in the way displayed in (2.8). This demonstrates that \( \hat{V} \) is a densely defined operator on thrice differentiable cylindrical functions. From this it follows already
that we have a chance of giving meaning to an operator corresponding to $H^E$.

Secondly, it is a well-known fact that $K$ is, up to a multiplicative constant, just
the time derivative of the total volume with respect to the integrated Hamiltonian
constraint (which is a signature invariant statement)

$$K = -\{\frac{V}{\kappa}, \int_{\Sigma} d^3 x H^E(x)\}$$

(2.9)

which of course can also be verified immediately. So, if we (again) replace $V, H^E$
by their quantizations and Poisson brackets by $(1/(i\hbar))$ times commutators we also
find that we have a chance of giving meaning to an operator corresponding to $K$.

This completes our explanation of the main idea. The rest of this paper is devoted
to a precise construction of the operators sketched in this section. We do this in a
series of three steps:

Step A) We begin by giving meaning to an operator corresponding to the Euclidean
Hamiltonian constraint (2.7). The result will be a (self-adjoint) operator on $\mathcal{H}$ whose
constraint algebra is anomaly-free.

Step B) We quantize $K$ along the lines sketched above using the known quantizations
$\hat{V}, \hat{H}^E$.

Step C) We quantize the Wheeler-DeWitt constraint using the known quantizations
$\hat{H}^E, \hat{K}$ and exploiting (2.6). We show that its constraint algebra is anomaly free
and that the result is a (symmetric) densely defined operator on $\mathcal{H}$ as well.

Recalling that $C = (\pi/2)K$ is the classical generator of the Wick rotation transform
we naturally obtain its quantization, as well as that of the Euclidean Hamiltonian
constraint, for free in our procedure.

3 Quantization of the Euclidean Hamiltonian constraint

The method applied in [14, 15] is to absorb the prefactor $1/\sqrt{\det(q)}$ in (2.7) into
the lapse function and to give meaning to the operator corresponding to the square
root of the trace. We will not do this. By employing the method described below we
can avoid the complications that arise in connection with these two steps. Also, it
would be less straightforward to define $K$ with this rescaled form of the constraint
since then $K$ is not the time derivative of the volume any longer.

3.1 Regularization

The regularization and all computations will be performed in an arbitrary but fixed
standard frame for $\Sigma$ as usual. The end result will be independent of that choice of
frame.

We start from the classical expression for the Euclidean constraint functional

$$H^E[N] = \frac{2}{\kappa} \int_{\Sigma} d^3 x N(x)c^{abc}\text{tr}(F_{ab}[A_c, V])$$

(3.1)

where $N$ is the lapse function divided by $\kappa$.

We now triangulate $\Sigma$ into elementary tetrahedra $\Delta$ where each of its edges are
analytic. For each tetrahedron we single out one of its vertices and call it \( v(\Delta) \). Let \( s_i(\Delta), \ i = 1, 2, 3 \) be the three edges of \( \Delta \) meeting at \( v(\Delta) \). Let \( \alpha_{ij}(\Delta) := s_i(\Delta) \circ a_{ij}(\Delta) \circ s_j(\Delta)^{-1}, \ \alpha_{ji} = \alpha_{ij}^{-1}, \) be the loop based at \( v(\Delta) \) where \( a_{ij} \) is the obvious other edge of \( \Delta \) connecting those endpoints of \( s_i, s_j \) which are distinct from \( v(\Delta) \). Then it is easy to see that

\[
H^E_\Delta[N] := -\frac{2}{3}N_v \epsilon^{ijk}\text{tr}(h_{\alpha_{ij}(\Delta)}h_{s_k(\Delta)}[h^{-1}_{s_k(\Delta)}, V])
\]

(3.2)
tends to the correct value \( 2\int_\Delta[N\text{tr}(F \wedge \{A, V\})] \) as we shrink \( \Delta \) to the point \( v(\Delta), \ N_v := N(v(\Delta)) \). Moreover, \( H^E_\Delta[N] \) is manifestly gauge-invariant since \( \alpha_{ij} \circ s_k \circ s_k^{-1} \) is a “loop with a nose”.

Let the triangulation be denoted by \( T \). Then

\[
H^E_T[N] = \sum_{\Delta \in T} H^E_\Delta[N]
\]

(3.3)
is an expression which has the correct limit \( (3.1) \) as all \( \Delta \) shrink to their basepoints (of course the number of tetrahedra filling any bounded subset of \( \Sigma \) grows to infinity under this process).

As we have said before, we now simply replace \( V \) by \( \hat{V} \) and the Poisson bracket by \( 1/ih \) times the commutator and

\[
\hat{H}^E_T[N] := \sum_{\Delta \in T} \hat{H}^E_\Delta[N], \ \hat{H}^E_\Delta[N] := -2\frac{N(v(\Delta))}{3it_p^2}\epsilon^{ijk}\text{tr}(h_{\alpha_{ij}(\Delta)}h_{s_k(\Delta)}[h^{-1}_{s_k(\Delta)}, \hat{V}]) =: N_v\hat{H}^E_\Delta
\]

(3.4)
is an operator with the correct classical limit.

It is obvious that the properties of the operator \( (3.4) \) are largely determined by the choice of triangulation \( T \) and therefore we devote the subsequent paragraph to a preliminary investigation of those properties of \( (3.4) \) which hold for any choice of triangulation. These considerations will then motivate our choices.

### 3.1.1 Motivation

The first property of \( (3.4) \) that we wish to prove is that its action on cylindrical functions is indeed finite no matter how fine the triangulation \( T \) is, provided that a certain criterion is satisfied which we derive now.

In order to see this it is sufficient to consider the operator \( [h_{s_k(\Delta)}, \hat{V}]f \) where \( f \) is a cylindrical function with respect to some graph \( \gamma \). The first case is that \( s_k(\Delta) \cap \gamma = \emptyset \). Then \( V(\gamma \cup s_k(\Delta)) = V(\gamma) \cup V(s_k(\Delta)) \) and it follows from \( (2.3) \) that \( [h_{s_k(\Delta)}, \hat{V}]f = \sum_{v \in V(\gamma)}h_{s_k(\Delta)}V_{\hat{V}}f - \sum_{v \in V(\gamma \cup s_k(\Delta))}V_{\hat{V}}h_{s_k(\Delta)}f = -f \sum_{v \in V(s_k(\Delta))}V_{\hat{V}}h_{s_k(\Delta)} = 0 \) since \( \hat{V} \) annihilates any cylindrical function, whether gauge-invariant or not, whose underlying graph is not at least three-valent\(^4\).

The next case is that \( s_k(\Delta) \cap \gamma \neq \emptyset \) but does not contain a vertex of \( \gamma \). That is, the set \( V(\gamma \cup s_k(\Delta)) - V(\gamma) \) consists of vertices which are at most four-valent, however, the tangents of the edges incident at each of those vertices lie in a two dimensional vector space. It then follows that again \( [h_{s_k(\Delta)}, \hat{V}]f = -\sum_{v \in V(\gamma \cup s_k(\Delta)) - V(\gamma)}V_{\hat{V}}h_{s_k(\Delta)}f = 0 \)

\(^4\)We say that a graph is \( n \)-valent if there are no more than \( n \) edges ingoing or outgoing at each vertex
because of the signature factor \( \epsilon(s_i, s_j, s_k) \) involved in (2.8). Notice that this property would no longer be true had we used the volume operator as defined in [25]; according to the second reference in [26], that operator does not vanish for vertices which involve only edges with co-planar tangents. Therefore, had we used this operator, as we make the triangulation finer and finer we would get more and more contributions and thus the resulting continuum operator could not even be densely defined.

It follows that (3.4) reduces to

\[
\hat{H}_E[T]\{N\}f = \sum_{\Delta \cap V(\gamma) \neq \emptyset} \hat{H}_E[\Delta]\{N\}f = \sum_{v \in V(\gamma)} N_v \sum_{\Delta \in \gamma} \hat{H}_E[\Delta]\{N\}f =: \sum_{v \in V(\gamma)} N_v \hat{H}_E[v]\{N\}f
\]

(3.5)

and we see that this expression is finite no matter how “fine” the triangulation is, provided the number of tetrahedra intersecting the vertices of \( \gamma \) stays bounded as we go to finer and finer triangulations. These considerations motivate to construct a triangulation \( T(\gamma) \) assigned to a graph \( \gamma \) which meets this criterion and we get an operator \( \hat{H}_E[\gamma] \). This furnishes our preliminary analysis.

### 3.1.2 Requirements for a triangulation adapted to a graph

Certainly, there are an infinite number of possible assignments. We choose a particular assignment guided by the following principles:

- **Non-triviality**:
  We could choose \( T(\gamma) \) in such a way that there are no intersections with \( \gamma \) at all, giving automatically a trivial result. This is inappropriate as the number of degrees of freedom should be genuinely reduced by the Hamiltonian constraint.

- **Diffeomorphism-Covariance**:
  Remember that we want to impose the constraint on a distribution \( \psi \) as outlined in section 2 and that the measure \( \mu_0 \) is diffeomorphism invariant. This fact enables us to get rid of a huge amount of ambiguity arising in the assignment of a triangulation as follows: the classical Hamiltonian constraint is not diffeomorphism invariant but it is diffeomorphism covariant. If we could carry over this classical property to the quantum theory then diffeomorphic vectors would be mapped by \( \hat{H}_E \) into diffeomorphic vectors. Therefore, provided the state \( \psi \) is diffeomorphism invariant, the number \( \psi[\hat{H}_E f] \) would only depend on the diffeomorphism invariant properties of the triangulation assignment (this was first observed in [14]). The assignment should therefore move with the graph \( \gamma \) under diffeomorphisms of \( \Sigma \).

More precisely, we have the following\footnote{This more precise formulation of the intuitive idea of diffeomorphism covariance was communicated to the author by Jurek Lewandowski}:

Let \( \gamma \) be a graph and \( \phi(\gamma) \) its diffeomorphic image for some smooth diffeomorphism \( \phi \in \text{Diff}(\Sigma) \) (the diffeomorphism does not need to be analytic but it must keep the graph analytic). Then \( \hat{H}_E[\gamma]f_\gamma \) will depend on a graph \( \hat{\gamma} \) and likewise \( \hat{H}_E[T(\phi(\gamma))]f_{\phi(\gamma)} \) will depend on a graph \( \hat{\phi}(\gamma) \). Then the requirement is that \( \hat{\gamma}, \hat{\phi}(\gamma) \) are diffeomorphic.

That is, we want that there exists \( \phi' \in \text{Diff}(\Sigma) \) such that \( \hat{U}(\phi')[\hat{H}_E[T(\gamma)]f_\gamma] = \)
\[ \hat{H}_E(\gamma) \] where \( \hat{U}(\phi')f_\gamma = f_{\phi'(\gamma)} \) is a unitary representation of the diffeomorphism group \( \text{Diff}(\Sigma) \) on \( \mathcal{H} \). Notice that we have been dealing with the Hamiltonian constraint at a point \( \hat{H}_E \) rather than with its smeared version \( \hat{H}_E^E[N] \) which is appropriate because we need to impose the constraint at every point of \( \Sigma \). We will see that in our case the notion of a constraint operator at a point makes perfect sense.

c) Cylindrical consistency:
If \( \gamma' \) is bigger than \( \gamma \) then \( T(\gamma) \) and \( T(\gamma') \) will in general differ from each other. However, if \( f \) is cylindrical with respect to \( \gamma \) then the vectors \( \hat{H}_E T(\gamma')f \) and \( \hat{H}_E T(\gamma)f \) should be diffeomorphic to each other. That is, we have cylindrical consistency up to a diffeomorphism. The reason why we do not need to require exact cylindrical consistency is because the assignment of the triangulation is only defined up to a diffeomorphism if we care only about the evaluation of a diffeomorphism invariant state \( \psi \) on the states \( \hat{H}_E f \).

d) Symmetry and Self-Adjointness:
The classical Hamiltonian constraint is a real-valued function on the phase space of general relativity. It is therefore compatible with the principles of quantum theory to construct an operator corresponding to it which is self-adjoint or at least symmetric. While one is not forced to do so as the symmetric and non-symmetric operators have the same classical limit and as we are only interested in the point 0 of the spectrum, rather than the full spectrum, there are practical reasons, among others the applicability of the group averaging method and the possibility of being able to get rid of a quantization ambiguity, which motivate to have the constraint in symmetric form. We will therefore propose two quantizations of the Hamiltonian constraint: in the main text we will stick with a non-symmetric operator as it turns out that it is technically much easier to handle and in \[ \text{[1]} \] we will treat a symmetric version of the operator.

e) Efficiency:
The result of applying \( \hat{H}_E^E[N] \) to cylindrical functions will be a cylindrical function that depends on additional edges. We want to choose an assignment which introduces as less additional structure (edges) as possible.

f) Naturality:
The assignment should be uniform, that is, it treats all the edges of the graph \( \gamma \) incident at a vertex on equal footing.

g) Anomaly-freeness:
The assignment should be free of anomalies, that is, the constraint algebra should close, otherwise we are reducing the number of degrees of freedom too much and we do not obtain a quantization of the classical theory.

h) Non-Emptyness:
The assignment should not be such that the kernel of the constraint operator

\[ ^6 \text{This observation again was first communicated to the author by Jurek Lewandowski} \]
is empty. The space of solutions to the classical Einstein equations has a rich
structure and so an empty kernel is not appropriate as it would not correspond
to the non-empty classical reduced phase space.

3.1.3 Choice of a triangulation adapted to a graph

The number of choices meeting these requirements is certainly still infinite. The
assignment $T(\gamma)$ we choose is as follows: we give in this subsection an assignment
which is appropriate only for the non-symmetric regularization of the operator and
will modify it at a later stage in order to make it appropriate for the symmetric
version.

0) Two-valent vertices:
If a vertex $v$ is two-valent, adjoin one more edge to $\gamma$ incident at $v$ and not
intersecting $\gamma$ in any other point such that its tangent at $v$ is transversal to
all the tangents of edges of $\gamma$ at $v$. We will see later that the end result of the
regularized operator is independent of that additional edge, even better, that
functions on graphs with only divalent vertices are annihilated by $\hat{H}^E(N)$.

With this preparation we will assume from now on that all vertices are at
least trivalent. Numerate all the edges $e_I$ of the (so possibly extended) graph
by some index $I, J, K, \ldots$. Also we take all edges to be outgoing at a vertex
as follows: by definition an analytical edge is an analytical embedding of a
compact interval into $\Sigma$ and a vertex $v$ is a point of $\gamma$ such that there is no
open neighbourhood $U \subset \Sigma$ of $v$ such that $\gamma \cap U$ is an embedded interval.

By definition an edge is bounded by two vertices. Given an edge $\tilde{e}$ of $\gamma$ with
endpoints $v, v'$ we subdivide it into two parts $\tilde{e} = e \circ e'$ where $e, e'$ respectively
are outgoing at $v, v'$ respectively. Note that $e \cap e'$ is not a vertex of $\gamma$ so that
$e, e'$ are strictly speaking no edges any longer but we will continue to call them
deges again as it simplifies the notation.

1) Segments and arcs:
Given an edge $e_I$ incident at a vertex $v$ choose $s_I$ to be a segment of $e_I$ which
is such that it
a) is incident at $v$ with outgoing orientation and
b) which does not include the other endpoint $v_I$ of $e_I$ distinct from $v$.

Also, given an unordered pair of edges $e_I, e_J$ incident at a vertex $v$, let $a_{IJ}$ be
a curve which is such that it
i) intersects $\gamma$ in the endpoints of $s_I, s_J$ distinct from $v$ and
ii) does not intersect $\gamma$ anywhere else.

The diffeomorphism invariant properties of the position of the arc $a_{IJ}$ will be
specified more precisely below. We will see later that in order to meet the
requirement that the constraint operator be symmetric on $H$ we have to relax
the prescriptions b),ii) above.

2) Tetrahedra saturating a vertex:
For each vertex $v$ of $\gamma$ and each unordered triple of mutually distinct edges
$(e_I, e_J, e_K)$ incident at $v$ (the number of such triples is given by $E(v) =
n(v)(n(v) - 1)(n(v) - 2)/6$ where $n(v)$ is the valence of the vertex) construct
We have a map \((s_I, s_J, s_K; a_{IJ}, a_{JK}, a_{KI}) \rightarrow (s_1(\Delta), s_2(\Delta), s_3(\Delta); a_{12}(\Delta), a_{23}(\Delta), a_{31}(\Delta))\) where the labelling is such that the orientation of the tangents at \(v\) is positive and we have indexed these six segments by the obvious tetrahedron \(\Delta\) that they form. Choose the basepoint of \(\Delta\) to be the vertex of \(\gamma\) under consideration, that is, \(v(\Delta) = v\).

We are now ready to construct the eight tetrahedra saturating \(v\) for the triple \((e_I, e_J, e_K)\): Let \([0, 1] \rightarrow s_i(\Delta)(t)\) and \([0, 1] \rightarrow a_{ij}(\Delta)(t)\) be parameterizations of \(s_i(\Delta)\) and \(a_{ij}(\Delta)\) respectively. Define their “mirror images” by

\[
  s_i^a(\Delta)(t) := 2v^a - s_i(\Delta)^a(t),
  a_{ij}^a(\Delta)(t) := 2v^a - a_{ij}(\Delta)^a(t),
  a_{ij}^{\alpha}(\Delta)(t) := a_{ij}(\Delta)^a(t) + 2t[v - s_j(\Delta)(1)]^a
\]

respectively where \(v^a\) are the coordinates of the vertex \(v\). Here we have assumed that all objects lie in a chart containing \(v\) which is always possible by choosing the basic quantities \(s_i(\Delta), a_{ij}(\Delta)\) small enough. Notice that by definition \(s_i(\Delta)(0) = v, s_i(\Delta)(1) = a_{ij}(\Delta)(0), s_j(\Delta)(1) = a_{ij}(\Delta)(1)\) so that \(s_i(\Delta)(0) = v, s_i(\Delta)(1) = a_{ij}(\Delta)(0) = a_{ij}(\Delta)(0), s_j(\Delta)(1) = a_{ij}(\Delta)(1) = a_{ij}(\Delta)(1), a_{ij}(\Delta)(1) = a_{ij}(\Delta)(1), s_i(\Delta)(1) = a_{ij}(\Delta)(0), s_j(\Delta)(1) = a_{ij}(\Delta)(1)\). We did not make use of any background metric. We now form loops \(\alpha_{ij}, \alpha_{ij}, \alpha_{ij}\) and combine them with \(s_k, s_k\) to form seven more right oriented tetrahedra. Together with \(\Delta\) these are the eight tetrahedra that we looked for. We will see that the seven “mirror” tetrahedra do not play any role at the end of the day so that the choice of adapted frame is irrelevant to define them.

Although we will only be concerned later with the one-dimensional edges of the tetrahedra constructed, we will need also their two- and three-dimensional properties in an intermediate step:

Choose any surfaces bounded by the edges of these eight tetrahedra and define the closed subset of \(\Sigma\) bounded by those faces of a tetrahedron \(\Delta\) to be the region assigned to \(\Delta\). The only property of the region assigned to \(\Delta\) which will be important is that \(\Delta\) and its mirror images saturate \(v\) which is a diffeomorphism invariant property and therefore everything will be independent of these regions.

3) Diffeomorphism invariant prescription for the position of the arcs \(a_{ij}(\Delta)\):

The following lemma, whose proof to the best of our knowledge is unpublished, shows that one can always choose two curves to lie in the \(x/y\) plane of an adapted coordinate system in which they take a standard form.

**Lemma 3.1** Let \(s_1, s_2\) be two distinct analytic curves which intersect only in their starting point \(v\). There exist parameterizations of these curves, a number \(\delta > 0\) and an analytic diffeomorphism such that in the corresponding frame the curves are given by

\[
  a) s_1(t) = (t, 0, 0), s_2(t) = (0, t, 0), t \in [0, \delta] \text{ if their tangents are linearly independent at } v
\]
b) $s_1(t) = (t, 0, 0)$, $s_2(t) = (t, t^n, 0)$, $t \in [0, \delta]$ for some $n \geq 2$ if their tangents are co-linear at $v$.

We will call the associated frame a frame adapted to $s_1, s_2$.

Proof:
Given a frame, denote by $b_i$, $i = 1, 2, 3$ the standard vector with entry 1 at the $i$-th index and zero otherwise.

First we show that any curve $s$ can be mapped into a straight line by an analytic diffeomorphism. To that end, let us expand $s(t) = f^i(t)b_i$ where $f^i$ are analytic functions of $t$. Since $\dot{s}$ is nowhere vanishing, at least one of the functions, say $f^1$, has the property $\dot{f}^1(0) \neq 0$ and so it does not in a neighbourhood of 0. Choose $b'_1 := b_1 - b_2, b_1 - b_3, b_1 - b_2 - b_3$ and $(f^2, f^3) = (f^2, f^3), (f^2 + f^1, f^3), (f^2, f^3 + f^1), (f^2 + f^1, f^3 + f^1)$ whenever $f^2(0), f^3(0)$ are $(\neq 0, \neq 0), (= 0, \neq 0), (\neq 0, 0), (= 0, = 0)$. We conclude that we can write $s(t) = f^1(t)b'_1 + f^2(t)b'_2 + f^3(t)b'_3 =: g^i(t)b'_i$ where $b'_i$ form a basis and $\hat{g}_i(0) \neq 0$. It follows by the inverse function theorem that the equation $x^i = g^i(t)$ can be inverted in a neighbourhood of 0 and that $(g^i)^{-1}(x^i)$ is analytic in a neighbourhood of 0 because $g^i(t)$ is of order $o(t)$.

We now see that the following diffeomorphism $x''(x^1, x^2, x^3) := (g^i)^{-1}(x^i)$ is analytic and maps $s(t)$ to $s'(t) = x'(s(t)) = t(b'_1 + b'_2 + b'_3)$. Upon performing a constant diffeomorphism (that is, a $GL(3)$ transformation) we can achieve that $s'(t) = tb_1$.

So we can assume that we have already mapped $s_1, s_2$ so that $s_1(t) = tb_1$. Now consider $s_2(t) = f^i(t)b_i$.

Case a) Since $\dot{s}_1(0), \dot{s}_2(0)$ are not co-linear (which is a diffeomorphism invariant statement) it follows that not both of $\dot{f}^2(0), \dot{f}^3(0)$ can vanish. So let us assume that for instance $\dot{f}^2(0) \neq 0$. By a similar argument as above we can write $s_2(t) = g^1(t)b_1 + g^2(t)b_2 + g^3(t)b_3$ where $b_1, b'_2, b_3$ are linearly independent and $\dot{g}(0) \neq 0$. The analytic diffeomorphism $x''(x^1, x^2, x^3) := (g^i)^{-1}(x^i)$ maps $s'_1(t) = x'(s(t)) = g^i(t)b'_1, s'_2(t) = t(b'_1 + b'_2 + b'_3)$. Now, a change of parameterization $t' = g^1(t)$ for $s_1$ and a final $GL(3)$ transformation proves the assertion.

Case b) Since $\dot{s}_1(0), \dot{s}_2(0)$ are co-linear it follows that $\dot{f}^1(0) \neq 0, \dot{f}^2(0) = \dot{f}^3(0) = 0$. Let $n \geq 2$ be the smaller of the two numbers $n_2, n_3$ defined by $\dot{f}^2(t) = o(t^{n_2}), \dot{f}^3(t) = o(t^{n_3})$ ($n$ is finite because the curves are not identical).

Without loss of generality we may assume $n_2 = n$ and now it follows by an already familiar argument that we can write $s_2(t) = g^1(t)b_1 + g^2(t)b_2 + g^3(t)b_3$ where $b_1, b'_2, b_3$ are linearly independent and $\dot{g}(0), (g^2)^{(n)}(0), (g^3)^{(n)}(0) \neq 0$. It follows that $g^2(t) =: \bar{g}^2(s), g^3(t) =: \bar{g}^3(s)$ are analytic functions of the analytic coordinate $s := t^n$ and that they are invertible in a neighbourhood of 0.

We now define the analytic diffeomorphism $x''(x^1, x^2, x^3) := (\bar{g}^i)^{-1}(x^i)$ where $\bar{g}^i = \bar{g}^1$ which maps $s'_1(t) = b_1((\bar{g}^1)^{-1}(t)), s'_2(t) = b_2t + t^n(b'_2 + b'_3)$ and again a reparameterization for $s_1$ and a $GL(3)$ transformation proves the assertion.

\[\square\]

Using this lemma we will now describe more precisely the choice of the arcs $a_{ij}(\Delta)$. We follow and extend (to the case of a pair of edges with co-linear tangents at their intersection) an elegant prescription due to Jurek Lewandowski to which one is driven quite naturally. For the sake of self-containedness
of the present paper we repeat the argument here to the extent we need it. The discussion is rather technical and lengthy and the reader not interested in the details may skip the rest of the present item and just should assume that there exists a diffeomorphism invariant prescription of the topology of the routing.

Let \( s_1, s_2 \) be two segments of edges \( e_1, e_2 \) of \( \gamma \), incident at the vertex \( v \). Their other endpoints \( v_1, v_2 \) are connected by an arc \( a \). Basically we wish to avoid that the arc \( a \) intersects any other point of \( \gamma \) different from \( v_1, v_2 \). Clearly, by choosing \( a \) to lie in a small enough neighbourhood of \( v \) the only danger is that \( a \) can intersect some other edge \( e \) different from \( e_1, e_2 \), also incident at \( v \). With rising valence of \( v \) the number of topologically different possibilities of routing \( a \) between the edges \( e \) incident at \( v \) becomes rather complex so that we need to give a diffeomorphism invariant description of the choice of routing.

By choosing a frame adapted to \( s \) necessary for the case that \( a \) can intersect some other edge \( e \) different from \( e_1, e_2 \), also incident at \( v \). With the idea is that we wish to choose the arc \( a \) to lie in the coordinate plane \( x, y \) associated with analyticity preserving diffeomorphisms. Consider an arc \( \gamma \) that at least \( \dot{\gamma}(0) \neq 0 \). Without loss of generality we may assume that at least \( \dot{e}(0) \neq 0 \) (otherwise \( \dot{e}(0) = 0 \); switch the role of \( x \) and \( y \) if necessary for the case that \( s_1, s_2 \) do not have co-linear tangents at \( v \). If they
do have co-linear tangents at \( v \) and \( \dot{e}^z(0) = 0 \) then \( \dot{e}^y(0) \neq 0 \) and it follows that a segment incident at \( v \) of the projection of the curve \( e \) into the \( x/y \) plane lies above the parabola \( y = x^n \) which describes \( s_2 \) and so \( a \) cannot intersect \( e \) anyway).

We can make a distinction between two situations.

**Situation A** : There exists a finite number \( m' \) such that \( (e^{(m')})^z(0) \neq 0 \) and \( m \geq m' \), \( m \leq \infty \) such that \( (e^{(m)})^y(0) \neq 0 \) are the first non-vanishing derivatives. The combination \( m' = m = \infty \) is excluded as otherwise \( s_1 \) and \( e \) would overlap in a finite segment (here we have used the analyticity of the edges). We then readily verify that under a change of adapted frame we have that the first non-vanishing derivative of the \( z \)-component of \( e \) is given by \( (e^{(m')})^z(0) = z' \cdot (e^{(m')})^z(0) \) so that the sign of \( (e^{(m')})^z(0) \) is again preserved and \( e \) is curved away from the \( x/y \) plane so as not to intersect \( a \).

**Situation B** : There exists a finite number \( m \) such that \( (e^{(m)})^y(0) \neq 0 \) and \( m' > m \), \( m' \leq \infty \) such that \( (e^{(m')})^z \neq 0 \) are the first non-vanishing derivatives. Again \( m = m' = \infty \) is an excluded possibility.

**Case that \( \dot{s}_1(0), \dot{s}_2(0) \) are not co-linear.**

If \( e \) does not point into an octant of the frame where both \( x/y \) are positive then again there is no danger that \( a \) intersects \( e \) since we choose \( a \) to lie in the \( x/y \) plane with positive \( x/y \) components as said above. Since \( x'_x, y'_y \neq 0 \), \( \partial_k y' = 0 \forall k \) it follows that the sign of the first non-vanishing derivatives of \( e^x, e^y \) are preserved under changes of the adapted frame.

So let us assume that \( e \) does point into an octant where both \( x/y \) are positive. This means that \( \dot{e}^x(0), (\dot{e}^{(m)})^y(0) > 0 \).

The first possibly non-vanishing derivative of the \( z \)-component of \( e \) is given by \( (e^{(m+1)})^z(0) = z'_y \cdot \dot{e}^x(0)(\dot{e}^{(m)})^y(0) + z'_z \cdot (e^{(m+1)})^z(0) \) and we see that this can have any sign for a suitable choice of \( z' \). We use this freedom to further fix the frame such that this sign is positive. Choose \( z'(z, y, z) = z + \beta xy \). This satisfies all requirements on \( z' \) at \( x = y = z = 0 \) and we see that upon choosing one and the same \( \beta(s_1, s_2) \) large enough we can beat the terms \( z'_z(e^{(m+1)})^z(0) \) for an arbitrary (but finite) number of edges \( e \) as to make \( (e^{(n+1)})^z(0) > 0 \).

**Case that \( \dot{s}_1(0), \dot{s}_2(0) \) are co-linear.**

If \( e \) does not point into the wedge \( y \leq x^n, x \geq 0 \) then again the arc \( a \) cannot possibly intersect \( e \) if we choose it close enough to the vertex. This time we only have that \( x'_x, y'_y \neq 0 \), \( \partial_k y' \forall k \) which implies only that the sign of the first non-vanishing derivative of \( e^y \) is preserved, we even find that \( x'_x = y'_y = 1 \) so that even its value is preserved. Now, if \( m \geq 2 \) then the sign of \( \dot{e}^x(0) \) is preserved. If \( m = 1 \) then it is not necessarily preserved but then it is true that, since \( n \geq 2 \), the projection of \( e \) into the \( x/y \) plane, which lay outside the wedge in a small enough neighbourhood of the vertex, is still outside the wedge. Therefore the condition that \( e \) does or does not point into the wedge is preserved. So let us assume that \( e \) enters into the wedge, in particular, \( \dot{e}^x(0), (\dot{e}^{(m)})^y(0) > 0 \).

We notice that the first derivative of \( e^z' \) at \( t = 0 \) which involves \( A := \partial_k \partial_z z' \) at \( v \) is of order \( n + m \) and given by a term \( c_A A(\dot{e}^x(0))^n(\dot{e}^{(m)})^y(0) \), \( c_A > 0 \). Likewise, the first derivative which involves \( B := \partial_k z' \) at \( v \) is of order \( 2m \) and
given by a term \( c_B B[(e^{(m)})^y(0)]^2 \), \( c_B > 0 \).

Subcase I) \( m' \leq \min(2m, m + n) \)

In this case the first non-vanishing derivative of \( e^{z'} \) at \( t = 0 \) is of order \( m' \) and involves a term \( z'\epsilon_z'(e^{(m')})\epsilon^z(0) \). By choosing the coefficient \( z'\epsilon_z' \) large and positive enough we can beat any possible contribution involving \( A, B \) and preserve the sign of \( (e^{(m')})\epsilon^z(0) \).

Subcase II) \( m' > \min(2m, m + n) \)

The first non-vanishing derivative of \( e^{z'} \) at \( t = 0 \) is of order \( m + n \) if \( n < m \) and involves only the term proportional to \( e^{z'} \). In this case the first non-vanishing derivative of \( v \) at \( t = 0 \) is of order \( n \) as stated above. By choosing \( e^x(t) \) as a parameter we may assume that \( e^x(t) = t \). Then we may assume \( (e^{(n)})^y(0) < n! \). Namely, if \( (e^{(n)})^y(0) > n! \) then again a segment incident at \( v \) of the projection of the curve \( e \) into the \( x/y \) plane, whose \( y \)-component looks like \( e^y(t) = t^n + o(t^{n+1}) \), lies above the parabola \( y = x^n \) which describes \( s \) and cannot possibly intersect the arc \( a \). If \( (e^{(n)})^y(0) = n! \) then we must have that either \( \alpha y \) has another higher order non-vanishing derivative \( (e^{(m)})^y(0) \neq 0, m > n \) or, if that is not the case, \( m' < \infty \) for otherwise \( e \) would coincide with \( s \) which we excluded. In the latter case, the first non-vanishing derivative of \( e^{z'} \) at \( t = 0 \) would be again \( z'(e^{(m')})\epsilon^z(0) \) because by definition of a diffeomorphism preserving the adaptedness of the frame, all derivatives of \( z'(e(t)) = z'(t, t^n, e^z(t)) \) at \( t = 0 \) which do not involve at least one partial derivative with respect to \( z \) must vanish. So again the sign of \( (e^{(m')})\epsilon^z(0) \) would be preserved. In the former case, the first non-vanishing derivative of \( e^{z'} \) at \( t = 0 \) and which is proportional to \( \partial_k^e \partial_y^e \) for some \( k \geq 0 \) and some \( l > 0 \) at \( v \) is of order \( m + n \) and proportional to \( \partial_z^e \partial_y^e (e^{(m)})^y(0) \) which follows from the fact that all contributions which do not contain at least one factor of \( (e^{(m)})^y(0) \) must vanish due to the adaptedness of the frame. Now we are back to either \( m' \leq n + m = \min(m + n, 2m) \) or \( m' > m + n \) and we have the case \( m > n \).

It follows then that if we choose any positive number \( \beta \) and some large enough positive number \( \alpha > 0 \) and the diffeomorphism (which satisfies the condition between \( A, B \) as given above) \( z' := \alpha x + \beta (y^2/2 - 2yx^n/(n!)^2) \) then the image of the edge \( e \) under this diffeomorphism will be such that it is curved away from the \( x/y \) plane as before for \( m' \leq \min(m + n, 2m) \) (where it is understood that we take \( m \) to be the next to leading order of \( e^y(t) \), with positive coefficient, in case that \( e^x(t), e^y(t) = t^n + o(t^{n+1}) \) and \( m = \infty \) otherwise) and for \( m' > \min(m + n, 2m) \) it is curved into the upper half space if \( n > m \), into the lower half space if \( n < m \) and into the lower half space if \( e^x(t) = t, e^y(t) = kt^n + o(t^{n+1}), k < 1 \) (for the other cases there is no routing to be chosen). This can be achieved for an arbitrary number of edges \( e \) with the same \( \alpha(s_1, s_2), \beta(s_1, s_2) \). Notice that the topology of the routing is diffeomorphism invariantly defined as the numbers \( n, m, m' \) are diffeomorphism invariant.
Concluding, we choose an adapted frame and a small enough neighbourhood of \( v \) and an arc \( a_{ij}(\Delta) \) going through that neighbourhood, which lies in the plane bounded by \( s_i(\Delta), s_j(\Delta) \) such that the routing through all other edges incident at \( v \) is the one described above.

4) **Tetrahedra away from the vertices**:
Denote by \( D(\Delta) \) the closed region in \( \Sigma \) filled by the eight tetrahedra constructed from a triple \( e_I, e_J, e_K \) as outlined in 1). Also consider their union \( \bar{D}(\Delta) := \Sigma \setminus \bigcup_{v \in V(\gamma)} D(v) \) and their complement with respect to \( D(v) \), that is, \( \bar{D}(\Delta) := D(v) \setminus D(\Delta) \). We triangulate \( D(\Delta) \) as outlined in 1) by the eight tetrahedra constructed. The sets \( \bar{D}(\Delta) \) are triangulated arbitrarily. As we have argued above, the final result will not depend on these tetrahedra because they do not intersect a vertex of \( \gamma \). This follows from the fact that all the tetrahedra different from the ones constructed in 1) have a basepoint different from any of the vertices of \( \gamma \) since by construction the tetrahedra described in 1) saturate them.

5) **Closeness to a triangle**:
Having fixed the topology of the routing of the arcs as in 2) we now choose, in the standard frame, the arc \( a_{ij}(\Delta) \) to be as straight as possible so that \( \alpha_{ij}(\Delta) \) looks like a triangle, as much as the routing allows. This will then justify the approximation \( h_{ij}(\Delta) \approx 1 + \frac{(\delta t)^2}{2} \dot{s}_i(\Delta) a_{ij}(0) \dot{s}_j(\Delta) b_{ij}(0) F_{ab}(\gamma) \) where \([0, \delta t]\) is the subinterval of \([0, 1]\) corresponding to \( s_i(\Delta) \) as compared to the whole edge of \( \gamma \) of which it is a segment.

### 3.2 Final Regularization of the Euclidean constraint

Let us now write the integral over \( \Sigma \) (3.1) for the classical theory as follows

\[
\int_{\Sigma} = \int_{\Sigma \setminus \bigcup_{v \in V(\gamma)} D(v)} + \sum_{v \in V(\gamma)} \int_{D(v)} = \int_{\bar{D}} + \sum_{v \in V(\gamma)} \frac{1}{E(v)} \sum_{v(\Delta) = v} [\int_{\bar{D}(\Delta)} + \int_{D(\Delta)}]
\]

\[
= \sum_{\Delta' \in \bar{D}} \int_{\Delta'} + \sum_{v \in V(\gamma)} \frac{1}{E(v)} \sum_{v(\Delta) = v} \int_{\Delta'} + \sum_{\Delta' \in D(\Delta)} \int_{\Delta'} \]  

(3.6)

In the classical expression (3.3) we can replace, for each \( \Delta \), \( \sum_{\Delta' \in D(\Delta)} \int_{\Delta'} \) by \( 8 \int_{\Delta} \) as all the tetrahedra shrink to their basepoints so that the dependence on the mirror images of \( \Delta \) drops out as promised.

Now we quantize the resulting expression (3.3) as outlined above and it follows from the considerations in section 3.1.1 that on the cylindrical subspace labelled by the graph \( \gamma \) the action of the Euclidean Hamiltonian constraint reduces to

\[
\hat{H}^E_{T(\gamma)}[N] f := \hat{H}_{\gamma}^E [N] f = \sum_{v \in V(\gamma)} N_v \frac{8}{E(v)} \sum_{v(\Delta) = v} \hat{H}^E_{\Delta} f =: \sum_{v \in V(\gamma)} N_v \hat{H}_v^E f
\]

(3.7)

This furnishes the regularization step. In particular, the expression (3.7) is finite (i.e. a cylindrical function) independently of \( T \) because the number \( E(v) \) is determined
by the graph and not by the triangulation and so does not change as we make the triangulation finer and finer. Let us now show that divalent vertices do not contribute: if $v$ is a divalent vertex of $\gamma$, if $s_3$ is the segment of the edge added in requirement 0) and if $\alpha_{12}$ is the corresponding loop along segments $s_1, s_2$ of the two edges $e_1, e_2$ of $\gamma$ incident at $v$ then

$$\hat{H}_v^E = -\frac{16}{3iL_p^2} \text{tr}((h_{\alpha_{12}} - h_{\alpha_{12}}^{-1})h_{s_3}[h_{s_3}^{-1}, \hat{V}]). \quad (3.8)$$

The terms involving $\alpha_{23}, \alpha_{31}$ drop out which follows from the fact that the volume operator annihilates divalent vertices as is obvious from the expression (2.8). Notice that if $s_3$ was not transversal to $s_1, s_2$ at $v$ then (3.8) would vanish trivially because the volume operator annihilates vertices which are such that all incident edges have co-planar tangents. But even so, (3.8) vanishes: since the end result of applying (3.8) to a gauge invariant function $f$ cylindrical with respect to a graph $\gamma$ must be gauge invariant, it cannot depend on $s_3$ (to see this, expand $\hat{H}_v^E f$ into spin-network states. Each of these spin-network states can colour $s_3$ only with spin 0 because there is no edge of $\gamma$ available in order to combine with the other endpoint of $s_3$ different from $v$ in a gauge invariant way). Moreover, it is easy to see that only a term proportional to $h_{s_3}\hat{V}h_{s_3}^{-1}$ survives. Now from the fact that the space of vertex contractors for divalent vertices is one-dimensional we see that $h_{s_3}\hat{V}h_{s_3}^{-1}f = \lambda f$ so that $\text{tr}([h_{\alpha_{12}} - h_{\alpha_{12}}]h_{s_3}\hat{V}h_{s_3}^{-1})f = \lambda \text{tr}([h_{\alpha_{12}} - h_{\alpha_{12}}])f = 0$ where we used the $SU(2)$ Mandelstam identity. The argument actually extends to the case of vertices of arbitrary valence but such that all incident edges have co-planar tangents. So we see already that functions cylindrical on graphs with only such degenerate vertices are annihilated by $\hat{H}^E(N)$. This is a feature which is shared with previous regularizations [28, 29].

It is amazing that one got expression (3.7) almost for free once one knows that the volume operator is well-defined on holonomies, no ill-defined products of distributions arise, we do not encounter any singularities, no renormalization of the operator is necessary. Note that we have no problems in ordering $\hat{V}$ to the left or to the right of the holonomies involved as $\hat{V}$ has a finite action on holonomies of $A$ as is clear from (2.8).

### 3.3 Cylindrical consistency

We have now produced an uncountable family of Euclidean Hamiltonian constraint operators $(\hat{H}^E(\gamma))_\gamma$, one operator for each graph $\gamma$ given in (3.7). What we need to make sure is that these operators are the projections to cylindrical subspaces of one and the same operator on $H$. This requirement will lead to some modifications of (3.7) (while keeping the classical limit to be still $H^E(N)$).

Also we would like to construct one version of $\hat{H}^E(N)$ which is symmetric. The way it stands, not even the projections of $\hat{H}^E(N)$ given in (3.7) are symmetric operators on $H$. Therefore, for the symmetric operator, we first order each term associated with a tetrahedron symmetrically. The result is (using that $\hat{V}$ is symmetric on $H$ and $h_e^I = T_e = (h_e^T)^{-1}$ on $H$) that, in case we wish to construct a symmetric
operator, we replace (3.7) by

\[ \hat{H}^E_N[f] := \sum_{v \in \gamma} N_v \hat{h}_v^E f, \quad \hat{h}_v^E = \frac{8}{E(v)} \sum_{v(\Delta) = v} \hat{h}_\Delta^E, \]

\[ \hat{h}_\Delta^E := -\frac{1}{3i\ell_p^2} \epsilon^{ijk} \text{tr}\{(h_{\alpha_{ij}(\Delta)}, h_{s_k(\Delta)}[h_{s_k(\Delta)}^{-1}, \hat{V}])\} \]

(3.9)

where \{..\} denotes the anti-commutator, while we stick with (3.7) if we do not wish to construct a symmetric operator. With this choice, each \( \hat{h}_v^E \) in (3.9) separately becomes a symmetric operator on \( \mathcal{H} \). Notice that this is far from guaranteeing that \( \hat{H}^E(N) \) itself is symmetric (see [1]). Both operators in (3.7), (3.9) have the dense domain \( \text{Cyl}^3(\mathcal{A}/\mathcal{G}) \) inherited from \( \hat{V} \) [26], the thrice differentiable cylindrical functions on \( \mathcal{A}/\mathcal{G} \).

We now come to make both (3.7), (3.9) cylindrically consistent. Let \( \Delta_e = X^i(h_e)X^i(h_e) \) be the Casimir operator associated with an edge e of \( \gamma \) incident at the outgoing endpoint of e. We can now define an edge projector \( \hat{p}_e := \theta(\hat{j}_e) \) where \( \hat{j}_e := \sqrt{1/4 - \Delta_e} - 1/2 \) has spectrum 0, 1/2, 1, 3/2,.. and \( \theta \) is some smooth function on \( \mathbb{R} \) which vanishes on \( (-\infty, 1/8] \) and equals 1 on \( [3/8, \infty) \). The operators \( \hat{p}_e \) are all commuting among each other and symmetric. The effect of this operator when applied to a function f cylindrical with respect to a graph \( \gamma \) is to annihilate that function if \( \gamma \) and e do not intersect in a finite segment at the outgoing endpoint of e and to leave it invariant otherwise. From these projectors we construct a tetrahedron projector \( \hat{p}_\Delta := \hat{p}_{s_1(\Delta)}\hat{p}_{s_2(\Delta)}\hat{p}_{s_3(\Delta)} \) and a vertex operator \( \hat{E}(v) := \sum_{v(\Delta) = v} \hat{p}_\Delta \). We then define the following self-consistent (up to a diffeomorphism) family of (symmetric) operators

\[ \hat{H}^E_N[\gamma] := \sum_{v \in \mathcal{V}(\gamma)} 8N_v \sum_{v(\Delta) = v} \begin{cases} \hat{h}_\Delta^E \hat{p}_\Delta / \sqrt{E(v)} & \text{for (3.7)} \\ \hat{h}_\Delta^E \hat{p}_\Delta / \sqrt{E(v)} & \text{for (3.9)} \end{cases} \]

(3.10)

We note that (3.10) still has the correct classical limit : up to terms of higher order in \( \hbar \) its action on cylindrical functions is still given by (3.7) which was shown to have the correct classical limit.

The symmetry of each member of the family defined in the second line of (3.10) is obvious. The self-consistency of both families defined in (3.10) can be checked as follows:

A graph \( \gamma \subset \gamma' \) can be obtained from \( \gamma' \) by a finite sequence of steps consisting of the following two basic ones: a) delete an edge of \( \gamma' \) and b) after removing an edge e, if one (or both) of the former endpoints v of e is now divalent and v = \( e_1 \cap e_2 \) where \( e_1, e_2 \) are analytic extensions of each other then combine \( e_{12} := e_1 \circ e_2^{-1} \) to an edge of \( \gamma \), that is, delete a vertex. In case a) it is clear that each term in (3.10) corresponding to a tetrahedron which involves the removed edge vanishes identically while in case b) all terms corresponding to the removed vertex vanish when applied to a function f cylindrical with respect to the graph \( \gamma \). Moreover, \( \hat{E}(v) \) reduces to the correct value on f. Finally, it follows from our manifestly diffeomorphism-invariant prescription of a loop-assignment that \( \hat{H}_\gamma^E f \) and \( \hat{H}_\gamma^E f \) are related by a diffeomorphism. In more detail, we have the following : if \( \gamma' \) is bigger than \( \gamma \) and if \( e'_1, e'_2 \) are edges of \( \gamma' \) which
are the parts of the edges $e_1, e_2$ of $\gamma$ incident at the same vertex $v$ at which $e_1, e_2$ are incident then the loop made from the corresponding segments $s_1', s_2'$ and $s_1, s_2$ are diffeomorphic thus guaranteeing cylindrical consistency up to a diffeomorphism. This is enough to show consistency.

In the sequel we will prove that both operators in (3.10) share the properties of diffeomorphism covariance and anomaly-freeness. To see that the second operator in (3.10) is actually symmetric requires a modification of the regularization which does not spoil those properties. We will come back to the modification in \[1\].

### 3.4 Diffeomorphism covariance

According to the programme of algebraic quantization proposed in \[13\] the solutions to the Euclidean Hamiltonian constraint are diffeomorphism invariant distributions $\psi$ on $\Phi := \text{Cyl}^\infty(\mathcal{A}/\mathcal{G})$ such that for all lapse functions $N$

$$\Psi[\hat{H}_E[N]\phi] = 0 \quad \forall \phi \in \Phi. \quad (3.11)$$

Now take $\phi = f$ to be any function, cylindrical with respect to some graph $\gamma$, in the domain of $\hat{H}_E$ then (3.11) amounts to $\sum_{v \in V(\gamma)} \Psi[\hat{H}_E f] = 0$ for all $N_v, v \in V(\gamma)$ and therefore we find that we need to satisfy

$$\Psi[\hat{H}_E f_\gamma] = 0 \quad \forall \gamma, \ f_\gamma \in \text{Cyl}_E^\gamma(\mathcal{A}/\mathcal{G}), \ v \in V(\gamma). \quad (3.12)$$

Equation (3.12) is actually quite unexpected: in section 3.1.3 we formulated the requirement of diffeomorphism-covariance in terms of the constraint at a point and is was far from clear that such an operator actually makes sense. Equation (3.12) is the precise formulation of that concept and it is manifestly well-defined.

Our triangulation adapted to a graph was geared at being diffeomorphism covariant for each of its vertices separately meaning that each of the operators $\hat{H}_E$ and $H_E$ and each of the operators $\hat{p}_\Delta, \hat{E}(v), \hat{H}_E, \hat{H}_E$ separately is covariantly defined. For $\hat{p}_\Delta, \hat{E}(v)$ this follows from the manifest covariance of $\hat{p}_e$, namely $\hat{U}(\varphi)\hat{p}_e\hat{U}(\varphi)^{-1} = \hat{p}_{\varphi(e)}$ for any $\varphi \in \text{Diff}(\Sigma)$ which in turn is a consequence of the fact that $\hat{p}_e$ is defined in terms of $h_e(A)$. For the operators $\hat{H}_E, \hat{H}_E$ we argue as follows: let $f$ be cylindrical with respect to a graph $\gamma$ and let $\varphi \in \text{Diff}(\Sigma)$. Then the tetrahedra $\Delta(\varphi(\gamma)), \varphi(\Delta(\gamma))$ are not necessarily equal to each other. However, the graphs $\gamma, \gamma' = \varphi(\gamma)$ are diffeomorphic, therefore the topology of the routing of the arcs $a_{ij}(\Delta(\gamma))$ through the edges of $\gamma$ and of the arcs $a_{ij}(\Delta(\gamma'))$ through the edges of $\gamma'$ as specified in section 3.1.3 is the same since that prescription depended only on the topology of the graph. More specifically, this prescription was shown to be independent of the frame and therefore coincides for any two vertices $v, v'$ of graphs $\gamma, \gamma'$ for which neighbourhoods $U, U'$ exist such that $U \cap \gamma$ and $U' \cap \gamma'$ are diffeomorphic. Now just choose a diffeomorphism $\varphi'$ such that $\varphi'(\gamma') = \gamma'$ and such that $\varphi'(\Delta(\gamma')) = \varphi'(\Delta(\gamma))$ for that specific $\Delta(\gamma)$ (notice that (3.12) is a linear combination of terms, each of which depends on only one specific $\Delta$ so that we can adapt $\varphi'$ to $\Delta$). Such a diffeomorphism clearly exists: there are diffeomorphisms which leave the image of the graph invariant while moving its points and off the graph it can put the arc $a_{ij}(\Delta)$ into any diffeomorphic shape. It follows from these considerations that $\hat{U}(\varphi')\hat{H}_E[\varphi(\Delta(\gamma))]\hat{U}(\varphi')^{-1} = \hat{H}_E$ which is what we wanted to show.
Then, obviously, the number $\Psi[\hat{H}^E f_s]$ depends only on the diffeomorphism class of the loop assignments $\alpha_{ij}(\Delta)$ (this was first observed in [14]). Therefore, in this diffeomorphism invariant context, the loops $\alpha_{ij}(\Delta)$ can be chosen as “small” and the triangulation as “fine” as we wish, the value of (3.12) remains invariant and in that sense the continuum limit has already been taken.

Diffeomorphism covariance is therefore a sufficient requirement for our quantum theory to correspond to a continuum theory.

One might wonder what happens if one actually takes the limit and sends $\Delta \to v(\Delta)$. It is easy to see that the result vanishes trivially which is not what we want. This happens due to the fact that after applying $\hat{H}^E[N]$ to a cylindrical function only a finite number of terms survive: since a cylindrical function is already determined on smooth connections we can, in the limit, actually replace $\hat{H}^E[N]$ by an integral over $\Delta$ as in (3.1) but the limit corresponds to a point which has zero Lebesgue measure. The fact that the limit $\Delta \to v(\Delta)$ is trivial is strange at first sight because one is used, from the lattice regularization of, say, $\lambda\phi^4$ theory, that the continuum theory is only recovered if we take the lattice spacing to zero, that is, one takes a continuous cut-off parameter to its continuum value. In our regularization such a parameter simply does not exist and the reason for that is the underlying diffeomorphism invariance of the theory. This shows that the limit $\Delta \to v(\Delta)$ is in fact inappropriate.

### 3.5 Anomaly-freeness

Although our assignment is covariant and therefore the continuum limit is already taken (in the sense explained above) so that it seems that the regulator is entirely removed, the operator (3.10) still carries a sign of the regularization procedure: it depends on the diffeomorphism class $[T]$ of the triangulation assignment which labels the freedom that we have in our regularization scheme. It is therefore not an entirely trivial task to check whether our operator $\hat{H}^E[N]$ is anomaly-free, meaning that $[\hat{H}^E[M], \hat{H}^E[N]]f$ vanishes for any cylindrical function $f$ and lapse functions $M, N$ when evaluated on a diffeomorphism invariant state $\psi$. The reason why we do not check the commutator on $\psi$ immediately is because $\psi$ is a distribution [13] and so does not lie in the domain of $(\hat{H}^E(M))^\dagger$. Therefore, the formal anomaly-freeness condition $([\hat{H}^E(N))^\dagger, (\hat{H}^E(M))^\dagger]\psi = 0$ has to be interpreted in the usual weak sense $((\hat{H}^E(N))^\dagger, (\hat{H}^E(M))^\dagger)\psi(f) = 0$ for each test function $f$, that is, every smooth cylindrical function $f \in \Phi$.

If the theory is anomaly-free, then, since $\psi$ is a generalized eigenvector [13] for the exponentiated diffeomorphism constraint $\hat{U}(\phi), \phi \in \text{Diff}(\Sigma)$ with eigenvalue 1, we expect that in the last equation the argument of $\psi$ is identically zero if some finite diffeomorphisms can be removed. This expectation turns out to be precisely correct.

We show now that a solution to the anomaly freedom problem is obtained

a) for the non-symmetric operator in case we attach the edges $a_{ij}(\Delta)$ irrespective of their differentiability at their endpoints with respect to $s_i(\Delta), s_j(\Delta)$ as prescribed in section 3.1.3

b) for the symmetric operator only if all the loops $\alpha_{ij}(\Delta)$ are chosen to be kinks with vertex at $v(\Delta)$! That is, the arc $a_{ij}(\Delta)$ joins the endpoints of $e_i(\Delta), e_j(\Delta)$ in at least a $C^1$ fashion (see [1] for the details of the attachment). An arbitrary attachment of
$a_{ij}(\Delta)$ is insufficient to guarantee anomaly-freeness. Consider for simplicity a graph $\gamma$ which only has one vertex $v$ and that it is two-valent, for instance a kink (we ignore for the moment that functions on such graphs are actually annihilated in order not to veil the argument, the problem shows up on higher valent graphs). Acting once with the Hamiltonian constraint on a function $f$ cylindrical with respect to $\gamma$ we get a function cylindrical with respect to a graph $\gamma'$ which contains $\gamma$ and an additional edge $e$ which intersects $\gamma$ in vertices $v_1, v_2$ and such that the tangents of $e, \gamma$ at these new vertices are, a priori, linearly independent. Acting once again with the Hamiltonian constraint it acts now non-trivially at all three vertices. Therefore, the commutator becomes now schematically $[\hat{H}^E(M), \hat{H}^E(N)]f = (M_v \hat{H}^E_v + M_{v_1} \hat{H}^E_{v_1} + M_{v_2} \hat{H}^E_{v_2})N_v \hat{H}^E_v f - (N_v \hat{H}^E_v + N_{v_1} \hat{H}^E_{v_1} + N_{v_2} \hat{H}^E_{v_2})M_v \hat{H}^E_v f = (M_v N_v - N_{v_1} M_{v_1}) \hat{H}^E_v \hat{H}^E_v f + (M_{v_2} N_v - N_{v_2} M_v) \hat{H}^E_{v_2} \hat{H}^E_{v_2} f$ which does not manifestly vanish even if $[\hat{H}^E_v, \hat{H}^E_{v'}] = 0$ for $v \neq v'$. One sees that what has to be avoided is that the Hamiltonian has non-trivial action at $v_1, v_2$. The idea is to exploit that the volume operator acts trivially on vertices which are such that the tangents of all edges incident at it lie in a common plane.

It is here where the volume operator as defined in the second reference of [27] is selected by the dynamics of the theory while the operator as defined in [27] has to be rejected (if one follows the approach advertised here)!

Namely, the volume operator [20] does not annihilate co-planar vertices. It is very appealing that the requirement of anomaly-freeness provides us with a selection rule among the operators [25], [26] which, a priori, from a purely kinematical point of view, are both bona fide quantizations of the classical volume functional.

Now, for the non-symmetric operator there is no problem at all because by construction of the triangulation assignment the endpoints of $a_{ij}(\Delta)$ form always new three-valent vertices of $\gamma \cup a_{ij}(\Delta)$ but the edges incident at them only have two independent tangent directions there, namely those of $e_i(\Delta)$ and $a_{ij}(\Delta)$. But for the symmetric operator we need to adjoin the smooth exceptional edges (see [1]) in at least a $C^1$ fashion because all the segments $a_{ij}(\Delta)$ which intersect the same edge $e$ of the skeleton of the given graph at all, intersect it at the same point and in general for higher than valence two the volume operator is non-vanishing on (not necessarily gauge-invariant) cylindrical functions unless the tangents of triples of edges at vertices are linearly dependent.

In [14, 15] the loop assigned does not have the topology of a kink but, the topology of a triangle. But because these operators do not involve the volume operator, rather they depend on an operator corresponding to $\epsilon_{abc} \epsilon_{ijk} E^a_i E^b_j$ which does not vanish if there are only two independent tangent directions in the problem, we expect these operators to be anomalous. However, if the loop assigned would have the topology of a kink as well we believe that the anomaly could be removed once these operators are rigorously defined.

**Theorem 3.1** The Euclidean Hamiltonian operator $\hat{H}^E(N) = (\hat{H}^E_\gamma(N), D_\gamma)$ as defined by (3.14) is non-anomalous.

Proof: Let $f$ be a function cylindrical with respect to a graph $\gamma$ and let $\Delta(\gamma)$ denote
the various tetrahedra attached to it in applying $\hat{H}^E(N)$. We may, without loss of generality, assume that $f$ is a spin-network function. Clearly the functions $\hat{H}^E_{\Delta(\gamma)} f, \hat{h}^E_{\Delta(\gamma)} f$ depend on the graph $\gamma \cup \Delta(\gamma)$. We are being very explicit here in the dependence of the tetrahedra on the graph because this will be essential in what follows. We may assume that all the tetrahedron projectors $\hat{p}_\Delta$ are non-vanishing on $f$, that is, the dependence of $f$ on all the edges of $\gamma$ is non-trivial. Then $\hat{E}(v)f = E(v, \gamma)f = n(n-1)(n-2)/6f$ where $n$ is the valence of $v$ in the graph $\gamma$. Now consider $\Delta'$ with $v' := v(\Delta') \neq v(\Delta) =: v$. Then $\hat{p}_\Delta \hat{H}^E f = \hat{H}^E f$ since the segments of edges of $\gamma$ which may have been removed in $\hat{H}^E f$ are at a vertex different from the vertex $v(\Delta')$. It follows that $\hat{E}(v')\hat{H}^E f = E(v', \gamma)\hat{H}^E f$. If, however, $v = v'$ then it is possible that $\hat{p}_\Delta g = 0$ where $g$ is a spin-network state appearing in the decomposition of $\hat{H}^E f$. Now, for the symmetric operator, since we project with $\hat{p}_\Delta$ before and after applying $\hat{H}^E f$, such $g$ are automatically removed so that either $g = 0$ or $\hat{E}(v')g = E(v, \gamma)g$ again. For the non-symmetric operator we get instead $\hat{E}(v')g = E(v, \gamma, g)g$. With this preparation we compute for the non-symmetric operator (it is understood that only terms with $\hat{p}_\Delta(\gamma \cup \Delta(\gamma))\hat{H}^E(\gamma) \neq 0$ are kept)

$$\hat{H}^E[\gamma] \hat{H}^E[N] f = \sum_{v \in V(\gamma)} \frac{N_v}{E(v, \gamma)} \sum_{v(\Delta(\gamma)) = v} \hat{H}^E[\gamma] \hat{H}^E_{\Delta(\gamma)} f$$

$$= \sum_{v \in V(\gamma)} \frac{N_v}{E(v, \gamma)} \sum_{v(\Delta(\gamma)) = v} \sum_{v' \in V(\gamma \cup \Delta(\gamma))} M_{v'v} \sum_{v(\Delta(\gamma) \cup \Delta(\gamma)) = v'} \hat{H}^E_{\Delta(\gamma) \cup \Delta(\gamma)} f \frac{1}{E(v')} \hat{H}^E_{\Delta(\gamma)} f$$

(3.13)

and similar for the symmetric operator, where the essential step has been the last one where all the contributions from $V(\gamma \cup \Delta(\gamma)) - V(\gamma)$, where $v(\Delta(\gamma)) \in V(\gamma)$ have been removed due to the fact that all these vertices are co-planar (for the symmetric operator this holds because of the kink property of those vertices as explained in Lemma 3.1 and the subsequent Collorary). We now compute (3.13) again with the roles of $M, N$ interchanged, subtract from (3.13) and obtain

$$[\hat{H}^E[\gamma], \hat{H}^E[N]] f$$

$$= \sum_{v, v' \in V(\gamma)} (M_{v'v}N_v - M_vN_{v'}) \frac{1}{E(v, \gamma)} \sum_{v(\Delta(\gamma)) = v, v(\Delta(\gamma) \cup \Delta(\gamma)) = v'} \hat{H}^E_{\Delta(\gamma) \cup \Delta(\gamma)} f \frac{1}{E(v')} \hat{H}^E_{\Delta(\gamma)} f$$

$$\times \sum_{v(\Delta(\gamma)) = v(\Delta(\gamma) \cup \Delta'(\gamma)) = v, v(\Delta'(\gamma)) = v'} [\hat{H}^E_{\Delta(\gamma) \cup \Delta(\gamma)} \hat{H}^E_{\Delta(\gamma)} - \hat{H}^E_{\Delta(\gamma) \cup \Delta(\gamma)} \hat{H}^E_{\Delta(\gamma)}] f$$

(3.14)

where in the second step the notation $v < v'$ assumes that we have ordered the vertices of $\gamma$ somehow which allowed us to write a sum over unordered pairs of vertices. Also we used that the antisymmetric product of lapse functions vanishes at equal vertices which was crucial in replacing $\hat{E}(v')$ by $E(v', \gamma) =: E(v')$ so that we may imagine to absorb this number into the lapses, $N_v/E(v) \rightarrow N_v$ (as explained above, this happens for the symmetric operator already before taking the commutator).
Formula (3.14) is valid for the symmetric operator as well if we just replace $\hat{H}_\Delta^E$ by $\hat{h}_\Delta^E$.

It is far from obvious whether (3.14) vanishes or not. Indeed, for genuine $\gamma$ and genuine choices of loop assignments depending on the graph, (3.14) is a non-vanishing cylindrical function of positive $L_2$ norm. We now evaluate a diffeomorphism invariant state on (3.14). We can take the antisymmetric product of the lapse functions evaluated at different vertices out of the integral over $\mathcal{A}/\mathcal{G}$ and it will be sufficient to show that

$$\psi[(\hat{H}_\Delta^E(\gamma \cup \Delta(\gamma)) \hat{H}_\Delta^E(\gamma) - \hat{H}_\Delta^E(\gamma \cup \Delta(\gamma)) \hat{H}_\Delta^E(\gamma))] f = 0$$

(3.15)

for each choice of $v(\Delta(\gamma)) = v(\Delta(\gamma \cup \Delta(\gamma))) = v, v(\Delta'(\gamma)) = v(\Delta'(\gamma \cup \Delta(\gamma))) = v'$ separately. To see this, notice first that the members of the first pair of tetrahedra given by $(\Delta(\gamma), \Delta(\gamma \cup \Delta(\gamma)))$ as well as the member of the second pair of tetrahedra given by $(\Delta'(\gamma), \Delta'(\gamma \cup \Delta(\gamma)))$ are diffeomorphic. This follows immediately from the fact that $v \neq v'$ so that there are disjoint neighbourhoods $U$ of $v$ and $U'$ of $v'$ where $U$ and $U'$ respectively contain both members of the first and second pair of tetrahedra respectively. Let $\varphi \in \text{Diff}(\Sigma)$ be chosen such that $\gamma \cap U'$ is left invariant and but that $\Delta(\gamma \cup \Delta(\gamma)) = \varphi(\Delta(\gamma))$. Likewise, choose $\varphi' \in \text{Diff}(\Sigma)$ such that $\gamma \cap U$ is left invariant but that $\Delta'(\gamma \cup \Delta(\gamma)) = \varphi'(\Delta'(\gamma))$. Then, using diffeomorphism invariance of $\psi$ we find that the left hand side of the (3.13) becomes

$$\psi[[\hat{U}(\varphi) \hat{H}_\Delta^E(\gamma) \hat{H}_\Delta^E(\gamma) - \hat{U}(\varphi) \hat{H}_\Delta^E(\gamma) \hat{H}_\Delta^E(\gamma))] f]$$

$$= \psi[[\hat{U}(\varphi) \hat{H}_\Delta^E(\gamma) \hat{H}_\Delta^E(\gamma) \hat{U}(\varphi^{-1}) - \hat{U}(\varphi) \hat{H}_\Delta^E(\gamma) \hat{H}_\Delta^E(\gamma) \hat{U}(\varphi')^{-1}) f]$$

$$= \psi[[\hat{H}_\Delta^E(\gamma) \hat{H}_\Delta^E(\gamma)] f].$$

(3.16)

That is, we were able to “match” the tetrahedra using diffeomorphism invariance. In the second equality we used the invariance $\hat{U}(\varphi) f = f \Rightarrow f = \hat{U}^{-1}(\varphi) f$ and similar for $\varphi'$ in order to write the commutator in such a way that it becomes manifestly antisymmetric if we replace $\hat{H}_\Delta^E \rightarrow \hat{h}_\Delta^E$.

Now we just need to use Collorary 3.1 of [1] (the non-symmetric operator is treated in a comment after this collorary) to see that the commutator in (3.14) vanishes identically because the part $V_\nu$ of the volume operator involved in $\hat{H}_\Delta^E(\gamma)$ does not act on the holonomies along edges incident at $v'$ involved in $\hat{H}_\Delta^E(\gamma)$ and vice versa since the vertices $v, v'$ are different. That completes the proof of anomaly-freeness.

A couple of remarks are in order:

- The classical constraint algebra is given by $\{H^E[M], H^E[N]\} = \int_\Sigma (MN_a - NM_a) q^{ab} V_b$, where $V_a$ is the classical diffeomorphism constraint and $q^{ab}$ is the inverse metric tensor. Naively, one would expect that the quantum version of that would be

$$\frac{1}{i\hbar} [\hat{H}^E[M], \hat{H}^E[N]] = \int_\Sigma (MN_a - NM_a) \ast q^{ab} V_b \ast$$

(3.17)

where the $\ast \ldots \ast$ is to indicate that the operators that appear have to be regularized and ordered appropriately. The immediate problem with (3.17) as, widely discussed in the literature, is that the constraint algebra is not a proper algebra, the structure functions depend on the canonical variables through $q^{ab}$ and if in the quantum theory, following the Dirac approach, the generator of the diffeomorphism constraint
does not appear to the right of $\hat{q}^{ab}$ in (3.17) then one would not expect a genuine diffeomorphism invariant state to be annihilated by the commutator of two Hamiltonian constraints: One says that the constraint algebra has an anomaly and that the quantum theory does not correspond to the classical theory because any element of the kernel of both the diffeomorphism constraint and the Hamiltonian constraint would have to satisfy the additional requirement that the right hand side of (3.17) annihilates it which reduces the number of degrees of freedom more than the classical theory would do.

One can then ask the question whether there exists a consistent quantization (regularization and factor ordering) of $\hat{H}^E(N)$ such that there is no anomaly. In [27] the authors investigate a wide class of finite-dimensional theories (gauge systems) with a Hamiltonian quadratic in the momenta and a constraint algebra which mimics (3.17) algebraically in the sense that $V_b$ is replaced by the generator of a gauge transformation and $q^{ab}$ is replaced by some non-constant function of the canonical configuration coordinates. The authors find that a consistent quantization can be obtained but never in such a way that the Hamiltonian constraint is a symmetric operator. The intuitive reason for this is clear: since the generator $\hat{V}_a$ of the unitary representation of the Diffeomorphism group has to be self-adjoint and since the metric tensor should be at least symmetric as well, then the left hand side of (3.17) should appear in the symmetric ordering $\hat{q}^{ab}\hat{V}_b + \hat{V}_a\hat{q}^{ab}$ (or an even more complicated symmetric operator) whenever $\hat{H}^E(N)$ is a symmetric operator. Now, in one of its versions, we actually did order $\hat{H}^E(N)$ symmetrically! Why, then, is that not in contradiction to the arguments given in [27] and references therein?

As is often the case with “no-go theorems”, one needs to carefully check the assumptions. The assumptions underlying the considerations in [27], when applied to our case, contain at least the following list (we do not list the assumption that one only has a finite number of degrees of freedom since we do not want to blame the failure of the theorem on that):

1) The operators $\hat{H}^E(N), \hat{q}^{ab}(x), \hat{V}_b(x)$ can be regulated and densely defined on $\mathcal{H}$.
2) The classical Hamiltonian constraint is a bilinear form in the canonical momenta.
3) The classical Diffeomorphism constraint is a linear form in the canonical momenta.

Only assumption 3) is satisfied here, the rest is violated:

1) As was shown in [9] the representation of (one-parameter subgroups of) the diffeomorphism group on $\mathcal{H}$ is not strongly continuous. Therefore, by Stone’s theorem, there does not exist a self-adjoint generator. Also a operator corresponding to $q^{ab}(x)$ is entirely meaningless in our representation: one could write it as $q^{ab} = E_i^a E_i^b / \det(q)$, however, both nominator and denominator are meaningless [13].
2) The dependence of $H^E$ on the momenta is through $\{A_a, V\}$ which is not even an analytic function of $E_i^a$.

We conclude that the considerations of [27] are not applicable in to our case.

How then can we even define a theory to be anomaly-free if we cannot define the quantizations of the generators of the symmetries of the classical theory? The answer is that the quantization of a classical theory does not force us to make sense out of every classical function as an operator. The most general question we can ask is the following:
a) Can we make sense out of the right hand side of (3.17), that is, can we make sense out of

\[ \int_\Sigma (M_N - N_{M_N})q^{ab}V_b \]  

(3.18)

The answer is, trivially, in the affirmative: up to higher order in \( \bar{\hbar} \) the commutator

\[ \left[ \hat{H}^E(M), \hat{H}^E(N) \right]/(i\hbar) \]

reproduces \( f_\Sigma (M_N - N_{M_N})q^{ab}V_b \) so that we can just define (3.18) by that commutator because by construction it is well-defined.

b) The fact that the classical Poisson bracket vanishes on the constraint surface of the phase space defined by the diffeomorphism constraint translates in the quantum theory into the requirement that (3.18) should vanish on diffeomorphism invariant states and this we checked to be the case.

To conclude, in our case the question of whether a factor ordering can be found such that \( \hat{V}_a \) stands to the right in (3.18) cannot even be asked. Therefore the quantum constraint algebra cannot have any close analogy with the classical constraint algebra. Given the fact that we can only define an operator corresponding to finite diffeomorphisms the structure of the constraint algebra that we do expect is precisely the one displayed in (3.16), namely that the commutator on a state \( f \) is a sum of terms of the form \( \hat{U}(\phi) \hat{A} \hat{B} f - \hat{U}(\phi') \hat{B} \hat{A} f \) where \( [\hat{A}, \hat{B}] f = 0 \) and \( \hat{A}, \hat{B} \) are symmetric operators. The diffeomorphisms \( \phi, \phi' \in \text{Diff}(\Sigma) \) are highly ambiguous since their action has only been specified on a finite graph induced from the one underlying \( f \), in particular \( \hat{U}(\phi)f = \hat{U}(\phi')f = f \). Using this ambiguity we can write the right hand side of the commutator \( \left[ \hat{H}^E(M), \hat{H}^E(N) \right] f \) again in manifestly anti-symmetric form \( \hat{U}(\phi) \hat{A} \hat{B} f - \hat{U}(\phi') \hat{B} \hat{A} f \) as in (3.18) so that there are no factor-ordering contradictions of the kind discussed in [27] at all!

• It will turn out that we do not need to apply any group averaging with respect to the non-symmetric Hamiltonian constraint operator but rather can compute the solutions by direct methods. This is important because if the constraint operator is not symmetric then the group averaging method cannot be immediately applied. However, for the symmetric operator, to which we can apply the method, direct methods are not possible to apply and so group averaging becomes important. Expectedly, in the diffeomorphism invariant context it is true that the constraint algebra becomes Abelian which makes group averaging especially attractive (see [3]).

4 Generator of the Wick rotation transform

As explained in section 3, since classically the generator is given by \( C = (\pi/2)K \), almost no work is needed to do. We just notice that \( \hat{K} = -\{V/\kappa, \hat{H}^E[1]\} \) and now simply define

\[ \hat{\omega} := -\frac{1}{i\ell_p^2} [\hat{V}, \hat{H}^E[1]] \] and \( \hat{C} := \frac{\pi}{2} \hat{K} \).  

(4.1)

Notice that \( \hat{V}, \hat{H}^E \) have both dense domain and range in \( \text{Cyl}^3(\overline{\mathcal{A}/G}) \). Therefore, \( \hat{K} \) has also dense domain and range in \( \text{Cyl}^3(\overline{\mathcal{A}/G}) \). Moreover, consider the case that we are using the symmetric version of \( \hat{H}^E(N) \). Then \( \hat{K} \) is also symmetric and since \( \hat{H}^E[1] \) is an imaginary-valued operator, \( \hat{K} \) and \( \hat{V} \) are both real-valued operators. We now choose complex conjugation as the conjugation in von Neumann’s theorem (see [1]) and see that both \( \hat{V}, \hat{K} \) have then self-adjoint extensions.
Remark:

Notice that the method of getting $\hat{K}$ from $\hat{V}, \hat{H}^E[1]$ simply through the commutator of these two operators is not possible to apply using any of the other operators corresponding to $H^E[1]$ as defined so far in the literature: the classical identity $K = -\{V, H^E[1]\}$ holds only for $H^E$ given by (2.7), it does not hold either for $\sqrt{\det(q)}H^E[28, 29]$ or for $\sqrt{\det(q)}H^E[14, 15]$ and thus for those approaches the operator corresponding to the generator of the Wick rotation transform is far from easy to define.

For later purposes we wish to derive a more explicit expression for $\hat{K}$. Let $f$ be a function cylindrical with respect to a graph $\gamma$. Then we have

$$\hat{K}f = -\frac{1}{i\ell^2} \sum_{v,v' \in V(\gamma)} [\hat{V}_{v'}, \hat{H}^E_v]f = -\frac{1}{i\ell^2} \sum_{v \in V(\gamma)} [\hat{V}_{v}, \hat{H}^E_v]f =: \sum_{v \in V(\gamma)} \hat{K}_vf \quad (4.2)$$

where in the first step we again used the fact that the volume operator does not see the vertices $V(\gamma \cup \Delta(\gamma)) - V(\gamma)$ and in the second step we exploited that $\hat{V}_v$ only acts on the edges incident at $v$ so that the commutator with $\hat{H}^E_{v'}$, which contains only holonomies of edges incident at $v'$, vanishes if $v' \neq v$. So, each term $\hat{K}_v$ in the sum over vertices contains only the part $\hat{V}_v$ of the full volume operator.

5 The Wheeler-DeWitt constraint operator

As outlined in section 3, we will now first derive a regulated operator corresponding to the expression $\text{tr}([K_a, K_b][E^a, E^b]) / \sqrt{\det(q)}$ which is consistently defined, whose action is diffeomorphism covariant and, in case we are dealing with a symmetric $\hat{H}^E(N)$, is symmetric as well. In the symmetric case the idea is then to treat it as a perturbation of some self-adjoint extension of $\hat{H}^E$ in the expression of $\hat{H}$ and to try to invoke the Kato-Rellich theorem to conclude that it is self-adjoint on the domain of $\hat{H}^E$ (see [1]). Finally we will prove that both versions of the operator are anomaly-free.

5.1 Regularization

We will not repeat all the arguments here as the derivation is completely analogous to the one for $\hat{H}^E$.

We begin with the classical expression for the “kinetic term”

$$T[N] := 8 \int_\Sigma d^3x \frac{N}{\kappa^3} \epsilon^{abc} \text{tr}(\{A_a, K\} \{A_b, K\} \{A_c, V\}) \quad (5.1)$$

and introduce the same triangulation $T$ as used for $H^E$ to regulate the integral with the result

$$T[N]_T := -\frac{8}{3\kappa^3} \sum_{\Delta \in T} N(v(\Delta)) \epsilon^{ijk} \text{tr}(h_{s_i(\Delta)}^{-1}\{h_{s_i(\Delta)}^{-1}, K\}h_{s_j(\Delta)}^{-1}\{h_{s_j(\Delta)}^{-1}, K\}h_{s_k(\Delta)}^{-1}\{h_{s_k(\Delta)}^{-1}, V\}). \quad (5.2)$$

Here, $N$ is again the lapse function divided by $\kappa$. Each term in the sum (5.2) labelled by a tetrahedron $\Delta$ tends to $8/\kappa^3 \int \Delta N \text{tr}(\{A, K\} \wedge \{A, K\} \wedge \{A, V\})$ as we shrink...
the tetrahedra to their basepoints, that is, we get the correct continuum limit.
The quantization of this expression now merely consists in adapting $T$ to a graph as outlined for the regularization of $\hat{H}^E$ and in replacing Poisson brackets by commutators and functions by operators. The result, when applied to a function cylindrical with respect to a graph $\gamma$, reads

$$\hat{T}[f] = -\frac{64}{3(i\hbar)^3} \sum_{\alpha \in V(\gamma)} N_v \frac{1}{E(v)} \sum_{e(\Delta) = v} \epsilon^{ijk} \times \text{tr}(h_{s_1(\Delta)} h_{s_2(\Delta)}^{-1} K_v h_{s_3(\Delta)}^{-1} K_v v) f$$

$$= \sum_{v \in V(\gamma)} N_v \frac{1}{E(v)} \sum_{e(\Delta) = v} \hat{T}_\Delta f = \sum_{v \in V(\gamma)} N_v \frac{1}{E(v)} \hat{T}_v f. \quad (5.3)$$

In order to arrive at this expression we use the same arguments as before to conclude that only tetrahedra intersecting a vertex of $\gamma$ give a contribution and we realize that all the commutators with holonomies along edges incident at $v$ are non-vanishing only for the parts $\hat{K}_v$ and $\hat{V}_v$ of $\hat{V}$ and $\hat{K}$ respectively. To prove the latter we merely need to observe the following:

Let $s$ be any segment of an edge of $\gamma$ incident at a vertex $v$ then $g := [h_s, \hat{K}_v] = \sum_{s' \in V(\gamma)} h_{s'} \hat{K}_v f - \sum_{s' \in V(\gamma) \cup (\gamma)} h_{s'} h_s f = [h_s, K_v] f$ because of the already familiar argument that $K_{v'}$ only involves the volume operator at $v'$ which commutes with $h_s$ unless $v = v'$ and we also used $s \cup \gamma = \gamma$. Next we have already seen that $h := h_{s_1(\Delta)} h_{s_2(\Delta)}^{-1} \hat{V} g = h_{s_1(\Delta)} h_{s_2(\Delta)}^{-1} \hat{V} g$ which is a cylindrical function with respect to $\gamma \cup \Delta(\gamma)$. Now, since for $v \in \gamma$ and $v' \in V(\gamma \cup \Delta(\gamma)) - V(\gamma)$ we have trivially $[h_{s_i(\Delta)}, \hat{K}_{v'}] h = 0$ it follows that $h_{s_i(\Delta)} h_{s_i(\Delta)}^{-1} \hat{K} h = h_{s_i(\Delta)} h_{s_i(\Delta)}^{-1} \hat{K} h$ as claimed. This furnishes the regularization part.

### 5.2 Cylindrical Consistency

Notice that in (3.4) there appear factors of $h_{s_i(\Delta)}$, $\Delta = \Delta(\gamma)$ which are holonomies along segments of edges of the original graph $\gamma$. Now, according to the definition of $\hat{K}$ one is supposed to adapt the triangulation associated with $\hat{K}$ (which is of course the same as the one associated with $\hat{H}^E$) to the graph $\gamma \cup s_i(\Delta(\gamma))$ when acting with $\hat{K}$ on $h_{s_i(\Delta(\gamma))} f_s$ for instance. However, $\hat{K}$ is consistently defined up to a diffeomorphism and $\gamma \cup e_i(\Delta(\gamma)) = \gamma$ so that (3.3) is cylindrically consistently defined up to a diffeomorphism, provided we introduce again the projectors $\hat{p}_\Delta$. So we redefine

$$\hat{T}_v \to \hat{T}_v := \sum_{e(\Delta) = \gamma} \hat{T}_\Delta \hat{p}_\Delta \frac{\hat{p}_\Delta}{E(v)} \quad (5.4)$$

for the non-symmetric version of $\hat{H}(N)$ while for the symmetric version we refer to (3.5), (3.6) and (3.7) of [I].

### 5.3 Diffeomorphism-covariance

Diffeomorphism covariance is immediate because $\hat{T}$ is covariantly defined if and only $\hat{H}^E$ is. There are no loop assignments different from the ones made for $\hat{H}^E$ involved in the definition of $\hat{T}$. 

29
5.4 Anomaly-freeness

**Theorem 5.1** The complete (symmetric or non-symmetric) Lorentzian Wheeler-Dewitt operator

\[
\hat{H}_\gamma[N] := \hat{T}_\gamma[N] - \hat{H}^E_\gamma[N] = \sum_{v \in V(\gamma)} N_v [\hat{T}_v - \hat{H}^E_v]
\]

is anomaly-free.

Proof:
To see this we use the fact that neither \(\hat{H}^E\) nor \(\hat{T}\) act at the additional vertices introduced by acting with \(\hat{H}^E, \hat{T}\) (in the symmetric case this requires Lemmata 3.1 and 3.2 and find that for a spin-network function \(f\) cylindrical with respect to \(\gamma\) we have

\[
[\hat{H}[M], \hat{H}[N]] f = \sum_{v \neq v' \in V(\gamma)} \frac{M_v N_{v'} - M_{v'} N_v}{E(v) E(v')} \{[\hat{T}_v, \hat{T}_{v'}] + [\hat{H}^E_{v}, \hat{H}^E_{v'}]\} = [\hat{T}_v, \hat{H}^E_v] f
\]

that is, again the vertices of every \(V(\gamma \cup \Delta(\gamma)) - V(\gamma)\) were irrelevant. In (5.5) it is understood that the operators \(\hat{T}_v, \hat{H}^E_v\) still depend on the graph on which the function depends that they are acting on in complete analogy with the considerations made in (3.14). Now we see that for \(v = v'\) the factor consisting of the antisymmetric product of the lapse functions vanishes (which was crucial in replacing \(\tilde{E}(v')\) by \(E(v') = E(v', \gamma)\)) while for \(v \neq v'\) the commutators vanishes when evaluated on a diffeomorphism invariant state as in (3.10) after getting rid of some diffeomorphism operators, again because all operators involved in the commutators only contain the parts \(\tilde{V}_v, \tilde{V}_{v'}\) of the volume operator which do not act on edges incident at different vertices.

Let us be more specific. Notice that the action of the operator \(\hat{T}\) at a vertex of \(\gamma\) such that \(v(\tilde{\Delta}) = v, \tilde{\Delta} = \tilde{\Delta}(\gamma)\), on \(f\) looks in full detail like this (\(c\) is a universal constant)

\[
\hat{T}_{\tilde{\Delta}} = \frac{c}{E(v, \gamma)^2} \sum_{v(\Delta(\gamma)) = v} \sum_{v(\Delta'(\gamma \cup \Delta(\gamma))) = v} \epsilon^{ijk} \times
\]

\[
\times \text{tr}(h_{s_i(\tilde{\Delta})} h_{s_j(\tilde{\Delta})}^{-1}) [\tilde{V}_v, \epsilon^{lmn} \text{tr}(h_{s_l(\Delta')} h_{s_n(\Delta')}^{-1}) h_{s_n(\Delta')}^{-1}] \frac{\tilde{p}_{\Delta'}}{E(v)} \times
\]

\[
\times h_{s_j(\tilde{\Delta})} h_{s_i(\tilde{\Delta})}^{-1} [\tilde{V}_v, \epsilon^{pqrs} \text{tr}(h_{s_p(\Delta')} h_{s_q(\Delta')}^{-1}) h_{s_q(\Delta')}^{-1}] h_{s_i(\Delta')}^{-1} \hat{V}_v f.
\]

While this is a complicated expression, what counts is that the tetrahedron projectors \(\tilde{p}_{\Delta'(\gamma \cup \Delta(\gamma))}\) can be replaced by \(\tilde{p}_{\Delta(\gamma)}\), that is, a projector with respect to the original graph \(\gamma\) and therefore also \(\tilde{E}(v, \gamma \cup \Delta(\gamma))\) can be replaced by \(\tilde{E}(v, \gamma)\).

The same is true for the projectors that are involved in the operator \(\hat{T}_{\tilde{\Delta}}\) or \(\hat{H}^E_{\tilde{\Delta}}\) when applying it to \(\hat{T}_{\tilde{\Delta}} f\) or \(\hat{H}^E_{\tilde{\Delta}} f\) where \(v(\Delta') = v' \neq v\). The associated tetrahedra involved depend on a nested sequence of graphs, that is, we have terms involving tetrahedra depending on \(\gamma \subset \gamma \cup \Delta(\gamma) \subset \gamma \cup \Delta(\gamma) \cup \Delta'(\gamma \cup \Delta(\gamma)) \subset \gamma \cup \Delta(\gamma) \cup \Delta'(\gamma \cup \Delta(\gamma)) \cup \Delta''(\gamma \cup \Delta(\gamma)) \cup \Delta'(\gamma \cup \Delta(\gamma)) \cup \Delta''(\gamma \cup \Delta(\gamma)) \cup \Delta'(\gamma \cup \Delta(\gamma))\). However, just as in the case of computing the commutator between two Euclidean Hamiltonian constraints, we
see that we get a sum of terms each of which looks like $\hat{U}(\varphi)\hat{A}\hat{B} - \hat{U}(\varphi')\hat{B}\hat{A}$. This happens for each pair of values that the pair of involved operators $(\hat{p}_\Delta E(v), \hat{p}_\Delta E(v'))$ may take because they act at different vertices and so it is irrelevant which acts first, they commute. This, together with the fact that the volume operator at one vertex does not act on the tetrahedra incident at another vertex shows that $\hat{A}, \hat{B}$ commute upon removing the diffeomorphisms $\varphi, \varphi'$. This furnishes the proof.

In the companion paper [1] we will compute the complete solution to both the non-symmetric Euclidean and Lorentzian as well as the Diffeomorphism constraint. Furthermore we define the symmetric version of the Lorentzian operator show that it is diffeomorphism-covariant and non-anomalous as well, however, we do not have the complete kernel in this case. Also we will report on the status of the Wick rotation transform in the light of these two articles. Finally, we will conclude with an interpretation of the results obtained, in particular, why the series consisting of these two papers was named “Quantum Spin Dynamics (QSD)”.

Acknowledgments

I am indebted to Jurek Lewandowski who taught me, in the framework of his version of the Euclidean Hamiltonian constraint derived in collaboration with Abhay Ashtekar, a) how to combine the requirement of cylindrical consistency with the diffeomorphism covariance of the loop assignment of the regularization and b) how to prescribe the topology of the routing of the arcs in a diffeomorphism invariant fashion.

This research project was supported in part by DOE-Grant DE-FG02-94ER25228 to Harvard University.

References

[1] T. Thiemann, “Quantum Spin Dynamics (QSD) II”, Harvard Preprint HUTMP-96/B-353, The preceding paper in this volume or in the gr-qc archive.
[2] R. M. Wald, “General Relativity”, The University of Chicago Press, Chicago, 1989
[3] B. S. DeWitt, Phys. Rev. 160 (1967) 1113, Phys. Rev. 162 (1967) 1195, Phys. Rev. 162 (1967) 1239
[4] A. Ashtekar, Phys. Rev. Lett. 57 2244 (1986), Phys. Rev. D36, 1587 (1987).
[5] T. Thiemann, preprint HUTMP-95/B-348, gr-qc/9511057, Class. Quantum Grav. 13 (1996) 1383-1403
[6] A. Ashtekar, “A generalized Wick transform for gravity”, preprint CGPG-95/12-1, gr-qc/9511083
[7] A. Ashtekar, in “Mathematics and General Relativity”, AMS, Providence, 1987 F. Barbero, Phys. Rev. D 51 (1995) 5507
[8] A. Ashtekar and C.J. Isham, Class. & Quan. Grav. 9, 1433 (1992).
[9] A. Ashtekar and J. Lewandowski, “Representation theory of analytic holonomy C* algebras”, in Knots and quantum gravity, J. Baez (ed), (Oxford University Press, Oxford 1994)

[10] J. Baez, Lett. Math. Phys. 31, 213 (1994); “Diffeomorphism invariant generalized measures on the space of connections modulo gauge transformations”, hep-th/9305045, in the Proceedings of the conference on quantum topology, D. Yetter (ed) (World Scientific, Singapore, 1994).

[11] D. Marolf and J. M. Mourão, “On the support of the Ashtekar-Lewandowski measure”, Commun. Math. Phys. 170 (1995) 583-606

[12] A. Ashtekar, J. Lewandowski, Journ. Geo. Physics 17 (1995) 191, J. Math. Phys. 36, 2170 (1995)

[13] A. Ashtekar, J. Lewandowski, D. Marolf, J. Mourão, T. Thiemann, “Quantization for diffeomorphism invariant theories of connections with local degrees of freedom”, Journ. Math. Phys. 36 (1995) 519-551

[14] C. Rovelli, L. Smolin, Phys. Rev. Lett. 72 (1994) 446

[15] A. Ashtekar, J. Lewandowski, “Regularization of the Hamiltonian constraint”, in preparation

[16] T. Thiemann, “The length operator in canonical quantum gravity”, Harvard Preprint HUTMP-96/B-354

[17] T. Thiemann, “The ADM Hamiltonian operator for canonical quantum gravity”, Harvard University Preprint

[18] T. Thiemann, “A regularization of canonical Yang-Mills quantum theory”, Harvard University Preprint

[19] A. Higuchi Class. Quant. Grav. 8, 1983 (1991), Class. Quant. Grav. 8, 2023 (1991)

[20] D. Marolf, “The spectral analysis inner product for quantum gravity,” preprint gr-qc/9409030, to appear in the Proceedings of the VIIth Marcel-Grossman Conference, R. Ruffini and M. Keiser (eds) (World Scientific, Singapore, 1995), Class. Quant. Grav. (1995)

“Almost Ideal Clocks in Quantum Cosmology: A Brief Derivation of Time,” preprint gr-qc/9412016.

[21] C. Rovelli, L. Smolin, “Spin networks and quantum gravity” pre-print CGPG-95/4-1.

[22] J. Baez, “Spin network states in gauge theory”, Adv. Math. (in press); “Spin networks in non-perturbative quantum gravity,” pre-print gr-qc/9504036.

[23] T. Thiemann, “The inverse Loop Transform”, preprint HUTMP-95/B-346, hep-th/9601105

[24] J. Baez, S. Sawin, “Functional Integration on Spaces of Connections”, q-alg/9507023

[25] C. Rovelli, L. Smolin, “Discreteness of volume and area in quantum gravity” Nucl. Phys. B 442 (1995) 593, Erratum : Nucl. Phys. B 456 (1995) 734

[26] A. Ashtekar, J. Lewandowski, “Quantum Geometry I : area operators”, Preprint CGPG-96/2-4, gr-qc/9602046

J. Lewandowski, “Volume and Quantization”, Potsdam Preprint, gr-qc/9602033

[27] P. Hajicéck, K. Kuchař, Phys. Rev. D41 (1990) 1091, Journ. Math. Phys. 31 (1990) 1723
[28] B. Brügmann, J. Pullin, Nucl. Phys. B 363 221
[29] B. Brügmann, J. Pullin, R. Gambini, Phys. Rev. Lett. 68 (1992) 431, Nucl. Phys. B 385 (1992) 1199