DEFORMATION OF THE OKUBIC ALBERT
ALGEBRA AND ITS RELATION TO THE
OKUBIC AFFINE AND PROJECTIVE PLANES

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Abstract

We present a deformation of the Okubic Albert algebra introduced
by Elduque whose rank-1 idempotent elements are in biunivocal corre-
spondence with points of the Okubic projective plane, thus extending
to Okubic algebras the correspondence found by Jordan, Von Neumann,
Wigner between the octonionic projective plane and the Albert algebra
$J_3(O)$. 

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1 Introduction

All unital composition algebras, i.e. all Hurwitz algebras such as $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$ and
all their split companions, allow the construction of projective lines and planes
which found interesting and useful applications in alternative formulations of
Quantum Mechanics [JNW]. Inspired by an algebrical definition of the Spin
group, given by Elduque in [El18], in a previous work two of the authors defined
an affine and projective plane over the Okubo algebra [CZ], noting that, with
little modifications, Veronese coordinates could still be used even though Okubo
algebra is neither alternative nor unital (a review of this construction is done
in sec. 4). Crucial to this construction is the flexibility of the algebra, i.e. for
every $x$ and $y$ in the algebra we have $x * (y * x) = (x * y) * x$, which is a softer
requirement than alternativity since all alternative algebras are also flexible, as
famously shown by Artin, but not the other way around. Flexibility, together
with the property of composition of the norm, defines a symmetric composition
algebra (whose basics are here briefly reviewed in sec. 2).

The real Okubo algebra $\mathcal{O}$, also known as pseudoctonions, is deeply linked to
octonions $\mathcal{O}$ since a deformation of the Okubic product results in the octonionic
multiplication and the converse is also true [Ok78, Ok78c]. In the octonionic
case, but more generally in any Hurwitz algebra, a simple and elegant relation
links rank-1 idempotents of the Albert algebra $J_3(\mathcal{O})$ and points in the pro-
jective plane $\mathcal{O}P^2$. We find that a similar relation still exists between rank-1
idempotents of a deformation of the Okubic Albert algebra $\mathbb{A}_q(\mathcal{O})$ introduced by Elduque in [E10] and points of the Okubic projective plane $\mathcal{O}P^2$. While the Okubic Albert algebra is a Jordan algebra (isomorphic to $3_3(\mathbb{O})$, see [E10] Thm 5.15), its deformation for $q = 1/2$ it is not, even though the algebra remains commutative and flexible. Nevertheless $\mathbb{A}_{1/2}(\mathcal{O})$ is, to the best of our current knowledge, the unique algebra that turns the following diagram over the point of the affine plane $\mathcal{O}A^2$ into a commutative diagram, i.e.

$$\begin{array}{ccc}
\mathcal{O}A^2 & \rightarrow & \mathcal{O}P^2 \\
p & & \downarrow \\
\mathbb{A}_{1/2}(\mathcal{O}) & \rightarrow & \text{Veronese vectors}
\end{array} \quad (1.1)$$

To be more explicit, we here anticipate that, while it is possible to modify the Veronese conditions in (4.1) and (4.2) in order to obtain the identification between trace-one idempotents of $\mathbb{A}_q(\mathcal{O})$ and Veronese vectors, it is not possible in general, i.e. for any value of the deformation parameter $q$, to also maintain the correspondence between the newly defined Veronese vectors and the points of the Okubic affine plane $\mathcal{O}A^2$ with a modification of (4.7).

In section 2 we present symmetric composition algebras and Okubo algebras, mainly following [E10] [E11] [E15] but introducing also some new elements. In section 3 we present for the first time the Okubic projective line showing that is the one point compactification of the Okubo algebra $\mathcal{O}$. In section 4 we present the Okubic affine and projective plane, following [CZ], and briefly showing the embedding of the affine plane into the projective plane. In section 5 we introduce the deformed Okubic Albert algebra $\mathbb{A}_q(\mathcal{O})$ and show the correspondence between its trace-one idempotents and the Veronese vectors. Finally, inspired by [Fr65], we investigate and characterise the automorphisms of $\mathbb{A}_q(\mathcal{O})$.

2 The Okubo algebra

An algebra $A$ is a vector space over a ground field $\mathbb{K}$ with a bilinear multiplication. If $A$ is also endowed with a non-degenerate quadratic norm $n$ from $A$ to $\mathbb{K}$ such that

$$n(x \cdot y) = n(x)n(y),$$

for every $x, y \in A$, then the algebra is called a composition algebra. If the algebra is also unital, i.e. if it exists an element 1 such that $x \cdot 1 = 1 \cdot x = x$, then the algebra is an Hurwitz algebra.
The interplay between the composition property of the norm and the existence of a unit element is full of interesting implications [El18, Sec. 2]. Indeed, every Hurwitz algebra is endowed with an order-two anti-automorphism called conjugation, defined as
\[ x := \langle x, 1 \rangle 1 - x, \]  
where \( \langle x, y \rangle \) is the polar form of the norm which is given by
\[ \langle x, y \rangle = n(x + y) - n(x) - n(y). \]  
Moreover, definition (2.3), combined with the composition, yields to the notable relations
\[ \langle x \cdot y, z \rangle = \langle y, x \rangle \cdot \langle z, x \rangle, \]  
\[ \langle y \cdot x, z \rangle = \langle x, y \rangle \cdot \langle z, x \rangle, \]  
that imply (cfr. [El18 Prop. 2.2]) on the conjugation that
\[ x \cdot \overline{x} = n(x) 1 \quad \text{and} \quad \overline{x \cdot y} = \overline{y} \cdot \overline{x}, \]  
and finally, a relation that is crucial for the consistent definition of the Veronese coordinates on composition algebras, i.e.
\[ x \cdot (\overline{x} \cdot y) = (x \cdot \overline{x}) \cdot y = n(x) y. \]  

It is well known that, up to isomorphisms, the only Hurwitz algebras are \( \mathbb{R}, \mathbb{C}, \mathbb{H} \) and \( \mathbb{O} \) along with their split companions [El18 Cor. 2.12]. Particularly relevant for this work are the split octonions \( \mathbb{O}_s \), which we will define as the real vector space endowed with the multiplication table in Tab. 1 over the canonical base \{\( e_1, e_2, u_1, u_2, u_3, v_1, v_2, v_3 \}\) obtained starting from two idempotents \( e_1 \) and \( e_2 \) and two three-dimensional real vector subspaces defined as \( V = \{e_1 \cdot v = 0\} \) and \( U = \{e_2 \cdot u = 0\} \).

Table 1: Multiplication table of the split octonions in the canonical base, from [El18].

|       | \( e_1 \) | \( e_2 \) | \( u_1 \) | \( u_2 \) | \( u_3 \) | \( v_1 \) | \( v_2 \) | \( v_3 \) |
|-------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| \( e_1 \) | 1 0       | \( u_1 \) | \( u_2 \) | \( u_3 \) | 0 0 0     |           |           |           |
| \( e_2 \) | 0 1       | 0 0 0     |           |           |           |           |           |           |
| \( u_1 \) | 0 0 0     | 0 \( v_3 \) | \( v_1 \) | \( v_2 \) | \( v_1 \) | \( v_2 \) | \( v_3 \) |           |
| \( u_2 \) | 0 0 0     | \( v_1 \) | 0 \( v_1 \) | 0 \( v_2 \) | \( v_1 \) | \( v_2 \) | \( v_3 \) |           |
| \( u_3 \) | 0 0 0     | \( v_2 \) | \( v_1 \) | 0 \( v_1 \) | 0 \( v_3 \) | \( v_1 \) | \( v_2 \) | \( v_3 \) |
Symmetric composition algebras

Starting from any Hurwitz algebra \((A, \cdot, n)\) with conjugation \([2.2]\) we define a new multiplication
\[ x \circ y := x \cdot y, \quad (2.8) \]
that, again, is composition with respect to the norm, i.e. \(n(x \circ y) = n(x) n(y)\). This allows to define another algebra \((A, \circ, n)\), called para-Hurwitz algebra \([OM80]\). Since \(1 \circ x = x \circ 1 = x\), then para-Hurwitz algebras are not unital (thus their existence do not contradict Hurwitz theorem \([El18]\)); nevertheless, they are composition and flexible and therefore symmetric composition \([KMRT]\), i.e. they satisfy the following relation
\[ \langle x \cdot y, z \rangle = \langle x, y \cdot z \rangle, \quad (2.9) \]
or, equivalently, the following one
\[ x \circ (y \circ x) = (x \circ y) \circ x = n(x) y. \quad (2.10) \]

A similar approach was already used by Petersson \([Pet69]\) who used, instead of an anti-homomorphism of order two, an homomorphism of order three \(\tau\) that he used to deform the product of the Hurwitz algebras \((A, \cdot, n)\) as follows
\[ x * y := \tau(x) \cdot \tau^2(y). \quad (2.11) \]
Again, the algebra \((A, *, n)\) turned out to be a symmetric composition algebra, i.e.
\[ n(x * y) = n(x) n(y), \quad (2.12) \]
\[ x * (y * x) = (x * y) * x = n(x) y. \quad (2.13) \]
The interesting case is here given when the starting Hurwitz algebra is the one of split octonions \(O_s\) and the order-three homomorphism is defined as
\[ \tau(e_i) := e_i, \quad \tau(u_i) := u_{i+1}, \quad \tau(v_i) := v_{i+1}, \quad (2.14) \]
where indices are considered modulo 3 and \(\{e_1, e_2, u_1, u_2, v_1, v_2, v_3\}\) is the canonical base of \(O_s\). In this specific case the construction yields to a non-unital symmetric composition algebra that is not obtainable through the use of the previous order-two deformation, and that was indipendently discovered by Okubo \([Ok78c]\); nowadays, this algebra is named Okubo algebra , and denoted by \(O\) \([El08]\).

Okubo and split-Okubo algebras

Let \(\eta_1\) be diag \((1, 1, 1)\) while \(\eta_2\) be diag \((-1, 1, 1)\). Then, consider the real vector spaces of three by three traceless matrices over the complex numbers \(\mathbb{C}\) such that
\[ x^\dagger := \eta_2 x \eta_1, \quad (2.15) \]
for $i = 1, 2$, where $x^\dagger$ is the transpose conjugate of $x$. The previous real vector space is an algebra if provided with the following product

$$x \ast y := \mu \cdot xy + \overline{\mu} \cdot yx - \frac{1}{3} \text{Tr}(xy) \text{Id},$$

(2.16)

where $\mu = \frac{1}{6}(3 + i\sqrt{3})$ and the juxtaposition is the ordinary associative product between matrices. Such algebra is called Okubo algebra $\mathcal{O}$ if $i = 1$ and split-Okubo algebra $\mathcal{O}_s$ if $i = 2$. In both cases the algebra is flexible and non-unital and it is also a composition algebra using the norm

$$n(x) := \frac{1}{6} \text{Tr}(x^2).$$

(2.17)

A canonical basis for both algebras is provided by an idempotent element, i.e. $e \ast e = e$, which is

$$e = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

(2.18)

together with other seven elements

$$i_1 = \begin{pmatrix} 0 & 1 & 0 \\ \gamma_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad i_2 = \begin{pmatrix} 0 & -\gamma i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$i_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad i_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ \gamma_1 & 0 & 0 \end{pmatrix},$$

$$i_5 = \begin{pmatrix} 0 & 0 & -\gamma i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad i_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad i_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix},$$

(2.19)

where $\gamma = 1$ in case of $\mathcal{O}$ and $\gamma = -1$ in the case of $\mathcal{O}_s$. Thus, the Okubo algebra $\mathcal{O}$ and its split version $\mathcal{O}_s$ are eight-dimensional real algebras. A proof that they are not isomorphic is easily obtained as corollary of the following

**Proposition 1.** The Okubo Algebra $\mathcal{O}$ is a division algebra, while the split Okubo Algebra $\mathcal{O}_s$ has non-trivial divisors of zero.

**Proof.** Suppose that $d \neq 0$ is a non-trivial left or right divisor of zero, i.e. $d \ast x = 0$ or $x \ast d = 0$, then from

$$(d \ast x) \ast d = d \ast (x \ast d) = 0 = n(d) x,$$

(2.21)

we have that $d$ is a divisor of zero if and only if $n(d) = 0$, i.e. $\text{Tr}(d^2) = 0$. But the element $d$ is of the form

$$d = \begin{pmatrix} \xi_1 & x_1 + i\gamma y_1 & x_2 + i\gamma y_2 \\ \gamma x_1 - iy_1 & \xi_2 & x_3 + iy_3 \\ \gamma x_2 - iy_2 & x_3 - iy_3 & -\xi_1 - \xi_2 \end{pmatrix},$$

(2.22)
where \( x_i, y_i, \xi_i \in \mathbb{R} \) and where \( \gamma^2 = 1 \) and is \( \gamma = 1 \) in the case of Okubo \( \mathcal{O} \) while is \( \gamma = -1 \) in the split case \( \mathcal{O}_s \). Therefore, the norm is

\[
n(d) = \frac{1}{3} (\gamma x_1^2 + \gamma x_2^2 + x_3^2 + \gamma y_1^2 + \gamma y_2^2 + y_3^2 + \xi_1^2 + \xi_2^2 + \xi_1 \xi_2) ,
\]

which yields that \( \text{Tr} \left( d^2 \right) = 0 \) is not possible in the case of \( \gamma = 1 \), where as it is easily obtained in the case \( \gamma = -1 \).

**Automorphisms** The automorphism group of Okubo algebras is a Lie group of \( A_2 \) type [Ok78, El15], more specifically \( \text{Aut} (\mathcal{O}) \cong SU (3) \) while \( \text{Aut} (\mathcal{O}_s) \cong SU (2, 1) \) and therefore \( \text{der} (\mathcal{O}) \cong \text{su} (3) \), while \( \text{der} (\mathcal{O}_s) \cong \text{su} (2, 1) \). It is here worth noting that every automorphism also preserves the norm. Indeed, if \( \varphi \) is an automorphism of \( \mathcal{O} \) then from \( \varphi (x * y) = \varphi (x) * \varphi (y) \), we have

\[
\varphi ( (x * y) * x ) = \varphi ( n(x) y ) ,
\]

but also

\[
( \varphi (x) * \varphi (y) ) * \varphi (x) = n(\varphi (x)) \varphi (y) ,
\]

so that

\[
n(x) = n(\varphi (x)) .
\]

The group of automorphisms is therefore a subgroup of the orthogonal group \( O (\mathcal{O}) \). Finally, from the analysis of the norm as a function over an eight-dimensional real vector space, it is easy to see from (2.23) that \( \text{Spin} (\mathcal{O}) = \text{Spin} (8) \) and \( \text{Spin} (\mathcal{O}_s) = \text{Spin} (4, 4) \). From a Lie theoretical point of view, while in the case of octonions, the automorphism \( \text{Aut} (\mathcal{O}) \cong G_2 \) is not a maximal subgroup in \( O (8) \), since \( G_2 \subseteq O (7) \subseteq O (8) \), in this case we are considering the maximal and non-symmetric embedding of \( A_2 \) into \( D_4 \), such that the \( 8_v, 8_s, \) and \( 8_c \) of \( D_4 \) all remain irreducible.

**From Michel-Radicati to Okubo** Okubo algebras enjoy also an interesting interpretation as deformation of the Michel-Radicati algebra which was introduced in [MR70] and whose structure constants were used by GÃŒnyadin and Zagermann to construct unified Maxwell-Einstein supergravity theories (MES-GTs) in \( D = 5 \) space-time dimensions in [GZ03]. The heuristic motivation for the introduction of the Michel-Radicati product is that, given two three by three traceless Hermitian matrix over \( \mathbb{C} \), the Jordan product does not yield to a traceless matrices and therefore traceless Hermitian matrices do not form a Jordan subalgebra of \( \mathbb{J}_3 (\mathbb{C}) \) with the Jordan product. The problem is avoided defining a new product

\[
x * y := \frac{1}{2} xy + \frac{1}{2} yx - \frac{1}{3} \text{tr} (xy) \text{Id} ,
\]

which defines the Michel-Radicati algebra (namely a consistent definition of the traceless subalgebra of \( \mathbb{J}_3 (\mathbb{C}) \)). Thus, the Okubo algebras \( \mathcal{O} \) and \( \mathcal{O}_s \), might
also be interpreted as specific cases of deformed Michel-Radicati algebras with the following product

\[
x \ast_\theta y := \left( \frac{1}{2} + i\theta \right) xy + \left( \frac{1}{2} - i\theta \right) yx - \frac{1}{3} \text{Tr}(xy) \text{Id},
\]

(2.28)

where the deformation \( \theta \in \mathbb{R} \). The \( \theta \)-deformation of the Michel-Radicati product enjoys numerous interesting properties, since the Michel-Radicati algebras is always flexible, i.e.

\[
x \ast_\theta (y \ast_\theta x) = (x \ast_\theta y) \ast_\theta x,
\]

(2.29)

and Lie-admissible, i.e. the product \([x, y]_\theta = x \ast_\theta y - y \ast_\theta x\) yields to a Lie algebra. The peculiarity of the Okubo algebras \( \mathcal{O} \) and \( \mathcal{O}_s \), is that \( \theta = \pm \frac{1}{2\sqrt{3}} \) is the unique value of the Michel-Radicati deformation parameter such that the deformed product gives rise to a composition algebra, i.e.

\[
n(x \ast_\theta y) = n(x) n(y) \text{ iff } \theta = \pm \frac{1}{2\sqrt{3}}.
\]

(2.30)

Moreover, the traceful part of the deformed product (2.28), i.e.

\[
x \circ_\theta y := \left( \frac{1}{2} + i\theta \right) xy + \left( \frac{1}{2} - i\theta \right) yx,
\]

(2.31)

gives rise to a non-commutative algebra that is again Lie-admissible, i.e. \([x, y]_\theta' = x \circ_\theta y - y \circ_\theta x\) defines a Lie algebra, but, even more interestingly, the \( \circ_\theta \) product also satisfies the Jordan identity for every \( \theta \in \mathbb{R} \), thus defining a \( \theta \)-parametrized family of non-commutative Jordan algebras. Okubo algebras \( \mathcal{O} \) and \( \mathcal{O}_s \) might therefore be interpreted as the only composition algebras resulting as the traceless sector of a non-commutative Jordan algebra obtained by deforming the Jordan product over Hermitian matrices as in (2.31).

Considering three by three matrices, it is amusing to note that the algebra \( \mathfrak{J}_3(\mathbb{C})_0 \) of traceless Hermitian matrices over \( \mathbb{C} \) provides the unique case in which the algebra itself is isomorphic to the Lie algebra of its derivations. This is actually the case \( N = 3 \) of a general result, corresponding to the realization of the generators of \( \mathfrak{su}(n) \) as Hermitian traceless \( n \times n \) matrices over \( \mathbb{C} \), i.e. \( \mathfrak{J}_n(\mathbb{C})_0 \) (e.g. see [MR70, MR73]).

**From Okubo to octonions** From now on we will leaving aside the Okubo split algebra \( \mathcal{O}_s \) and focus on the real division Okubo algebra \( \mathcal{O} \) which enjoys the important feature of being in a very tight relation with octonions \( \mathbb{O} \). Previously in this section, we mentioned how Okubo algebra \( \mathcal{O} \) can be considered as the Petersson algebra obtained from the split octonion algebra \( \mathcal{O}_s \) (cfr. [El08]). On the other side, octonions \( \mathbb{O} \) can be obtained as a deformation from the Okubo algebra \( \mathcal{O} \) through the use of the new product

\[
x \cdot y := (e \ast x) \ast (y \ast e),
\]

(2.32)
where \(x, y \in \mathcal{O}\) and \(e\) is an idempotent of \(\mathcal{O}\), such as the one given by (2.18). Nevertheless, it is important to stress out that the following construction is independent from the explicit form of the idempotent that are, in fact, all conjugate under the automorphism group [El15 Thm. 20]. Following [El15], it is easy to show that since \(e \ast e = e\) and \(n(e) = 1\), for every \(x \in \mathcal{O}\) the element \(e\) acts as a left and right identity, i.e.

\[
\begin{align*}
x \cdot e &= e \ast x \ast e = n(e) x = x, \\
e \cdot x &= e \ast x \ast e = n(e) x = x.
\end{align*}
\]

Moreover, since the Okubo algebra is a composition algebra, the same norm \(n\) enjoys the following relation

\[
n(x \cdot y) = n((e \ast x) \ast (y \ast e)) = n(x) n(y),
\]

which means that \((\mathcal{O}, \cdot, n)\) is a unital composition algebra of real dimension 8 and, being also a division algebra, is therefore isomorphic to the algebra of octonions \(\mathbb{O}\), as noted by Okubo himself [Ok78, Ok78c].

**Trivolution**

Even though Okubo algebra \(\mathcal{O}\) is not unital, some interesting features of the unity (albeit not all) are recovered by the use of an idempotent \(e\). While in unital composition algebras the canonical conjugation, i.e.

\[
\overline{x} = \langle x, 1 \rangle 1 - x,
\]

has the notable properties of \(x \cdot \overline{x} = n(x) 1\) and of \(\overline{x} \cdot \overline{y} = \overline{y} \cdot \overline{x}\), in Okubo algebras the map \(x \rightarrow (x, e) e - x\) is a map of order two but it is not an automorphism, nor an anti-automorphism. Similarly, in a para-Hurwitz algebra, which is a non-unital algebra but rather it is endowed with a paraunit \(e\), the maps

\[
\begin{align*}
x &\rightarrow L_e(x) = e \ast x, \\
x &\rightarrow R_e(x) = x \ast e,
\end{align*}
\]

are a conjugation since \(e \ast x = \overline{x}\) and

\[
\begin{align*}
x \ast L_e(x) &= n(x) e, \\
R_e(x) \ast x &= n(x) e.
\end{align*}
\]

On the other hand, in the Okubo algebra \(\mathcal{O}\), even though we still have \(R_e \circ L_e = \text{id}\), both \(L_e\) and \(R_e\) are in fact neither are automorphism nor an anti-automorphism. Even though it is not possible to have a conjugation over Okubo algebra with the desired properties, we can define something in a similar fashion such as an order-three automorphism \(\tau\), that we will call here a trivolution, and that can be defined as

\[
x \rightarrow \tau(x) := \langle x, e \rangle e - x \ast e,
\]
or equivalently as
\[ x \rightarrow \tau(x) := L_e^2(x) = e \ast (e \ast x), \quad (2.42) \]
\[ x \rightarrow \tau^2(x) := R_e^2(x) = (x \ast e) \ast e. \quad (2.43) \]

It is easy to see that the automorphism \( \tau \) is of order 3 since, applying flexibility, \( R_e^2 \circ L_e^2 = \text{id} \). It is also worth noting the stunning analogy with the conjugation expressed for unital composition algebra in (2.36) and at the same time the analogy with the one expressed for para-Hurwitz algebras in (2.37). Finally, we have to stress out that (2.42) shows the deep relation between the idempotent \( e \) and the trivolution \( \tau \) in a certain sense the choice of the idempotent \( e \) and the trivolution \( \tau \) are equivalent (e.g. see [EL15, Prop. 18]).

**Remark 2.** It is worth noting that the idempotent \( e \) and the order-three automorphism \( \tau \) turn explicit the construction of octonions \( O \) from Okubo algebra \( O \) in an elegant way allowing the definition of the octonionic product and conjugation, respectively defined by
\[ x \cdot y := L_e(x) \ast R_e(y), \quad (2.44) \]
\[ \bar{x} := L_e^3(x) = e \ast (\tau(x)). \quad (2.45) \]

The fixed part of the trivolution is given by
\[ \text{Fix}_\tau(x) := \frac{1}{3} \left( x + \tau(x) + \tau^2(x) \right) = \begin{pmatrix} \xi_1 & 0 & 0 \\ 0 & \xi_2 & x_3 + iy_3 \\ 0 & x_3 - iy_3 & -\xi_1 - \xi_2 \end{pmatrix}, \quad (2.46) \]
while \( \tau \) act on the remaining part \( \bar{x} = x - \text{Fix}_\tau(x) \) of the element as
\[ \bar{x} \rightarrow \begin{pmatrix} 0 & -\overline{\tau}(x_1 + iy_1) & -\overline{\tau}(x_2 + iy_2) \\ \alpha(x_1 - iy_1) & 0 & 0 \\ \alpha(x_2 - iy_2) & 0 & 0 \end{pmatrix}, \quad (2.47) \]
where \( \alpha = (1 + i\sqrt{3})/2 \) and \( x_1, x_2, x_3, y_1, y_2, y_3 \in \mathbb{R} \). It might be of interest to note that the r.h.s. of (2.46) corresponds to \( (\mathbb{R} \oplus \mathbb{J}_2(\mathbb{C}))_0 \simeq \mathbb{J}_2(\mathbb{C}) \), i.e. the simple Jordan algebra of rank-2 over \( \mathbb{C} \), given by two by two Hermitian matrices over \( \mathbb{C} \).

**Remark 3.** It is worth remarking that \( O \) is a Lie-admissible algebra, i.e. \( O \) is a Lie algebra, namely \( \mathfrak{su}(3) \), if endowed with the commutator \([x, y] = x \ast y - y \ast x\), and that \( \tau \) realizes a \( \mathbb{Z}_2 \)-grading of the Lie algebra, i.e. \( O = \mathfrak{g}_\tau \oplus \mathfrak{g}_\bar{\tau} \), where \( \mathfrak{g}_\tau = \{ x \in O; x = \tau(x) \} \) is the Lie subalgebra of elements fixed by \( \tau \) and \( \mathfrak{g}_\bar{\tau} \) is the complement, i.e. the real vector space generated by elements of the form (2.47). The \( \mathbb{Z}_2 \)-grading of the algebra means that, i.e. \( v \ast v', s \ast s' \in \mathfrak{g}_\tau \) and \( s \ast v, v \ast s \in \mathfrak{g}_\bar{\tau} \), for every \( v, v', s \in \mathfrak{g}_\tau \) and \( s, s' \in \mathfrak{g}_\bar{\tau} \), which obviously extends to a grading of the Lie algebra such that \([v, v'], [s, s'] \in \mathfrak{g}_\tau \), and \([s, v] \in \mathfrak{g}_\bar{\tau} \).
3 The Okubic line

If in the previous sections we were often dealing with both Okubo algebra $\mathcal{O}$ and split-Okubo algebra $\mathcal{O}_s$, it is now important to stress out that all the treatment we will develop from now on, it holds for the Okubo algebra only, and cannot be extended in a straightforward way to the split-Okubo algebra, which is not a division algebra (cfr. Prop. 1). Extensions of the subsequent treatment to split-Okubo algebra will be considered in future works but momentarily left aside.

We now construct the Okubic projective line starting from an extended quadric over $\mathcal{O}$. With this in mind, we start with a real vector space $\mathcal{O} \cong \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ and a quadratic form $b$ from $\mathcal{O}$ to $\mathbb{R}$ defined as

$$b(x, \xi_1, \xi_2) = n(x) - \xi_1 \xi_2,$$

which is an extension of $n$ over $\mathcal{O}$. We then consider the quadric

$$Q = \left\{ v \in \mathcal{O} \setminus \{0\} ; \ b(v) = 0 \right\},$$

that obviously lies in $\mathbb{P}^9$ and, since it is closed, is therefore compact. Thus, to be explicit, points on the Okubic projective line are elements of the form $R(x, n(x), 1)$, i.e., $x \in \mathcal{O}$, $\xi_1, \xi_2 \in \mathbb{R}$ fulfilling $n(x) - \xi_1 \xi_2 = 0$. Our aim now is to show that this compact set is a one-point compactification of the Okubic algebra. This proves the following

**Proposition 4.** The map from $\mathcal{O} \cup \{\infty\}$ to $Q$ given by

$$x \mapsto R(x, n(x), 1),$$

$$\infty \mapsto R(0, 1, 0),$$

is an homeomorphism, showing that $Q$ is homeomorphic to the one-point compactification of the Okubo algebra $\mathcal{O}$.

**Proof.** Since the map is obviously continuous it suffices to see that also the inverse is continuous. First of all we need to notice that, since $\mathcal{O}$ is a division algebra, $n(x) = 0$ if and only if $x = 0$ and therefore if $\xi_2 = 0$ then the only $\xi_1$ for which a vector $v \in \mathcal{O} \setminus \{0\}$ is such that $b(v) = 0$ is for multiple of $\xi_1 = 1$. This means that $R(0, 1, 0)$ is the only point of the quadric that passes through the hyperplane $H = \{\xi_2 = 0\}$ in $\mathbb{P}^9$. This means that the image of all $x \in \mathcal{O}$ are in $\mathbb{P}^9 \setminus H$, which is homeomorphic to $\mathbb{R}^9$. Moreover if $R(x, \xi_1, 1)$ is in the quadric $Q$ then $n(x) - \xi_1 = 0$, i.e. $\xi_1 = n(x)$, and therefore the map from $R(x, n(x), 1) \mapsto x$ is a continuous map. □

4 The affine and projective plane over the Okubo algebra
The Okubic affine plane.

The definition of the Okubic affine plane $\mathcal{A}_2(\mathcal{O})$ is easily achieved by defining a point on the affine plane as the two coordinates $(x, y)$ with $x, y \in \mathcal{O}$, and a line of the affine plane as $[s, t]$ with $s, t \in \mathcal{O}$ where we intended $[s, t]$ as the set of Okubic elements $\{(x, s \ast x + t) : x \in \mathcal{O}\}$. Lines on the Okubic plane are therefore determined by two Okubic elements $s$ and $t$ that are called the slope and the offset of the line, respectively. Vertical lines are identified by $[c]$ that denotes the set $\{c\} \times \mathcal{O}$, where $c \in \mathcal{O}$ represents the intersection of the line with the $x$ axis. It is easily seen that since $\mathcal{O}$ is a division algebra the Okubic affine plane satisfies all incidence axioms of the affine geometry [CZ Sec. 3].

The completion of the Okubic affine plane $\overline{\mathcal{A}_2(\mathcal{O})}$ is obtained adding a line at infinity $[\infty]$, i.e.

$$[\infty] = \{(s) : s \in \mathcal{O} \cup \{\infty\}\},$$
where \( s \) identifies the end point at infinity of a line with slope \( s \in \mathcal{O} \cup \{ \infty \} \).

Finally, we define \( \infty \) as the point at infinity of \( \{ \infty \} \). It is easy to verify that this construction preserves the property of a unique line joining two different points and that two lines intersect at infinity if and only if they are parallel, i.e. have the same slope \( s \). Resuming the whole notation, as in Fig 4.1, we have three set of coordinates that indentify all the points in the completion of the affine plane, i.e. \( (x, y), (s), (\infty) \).

The Okubic projective plane.

Resuming our construction in [CZ], inspired by [SBGHLS], let \( V \cong \mathcal{O}^3 \times \mathbb{R}^3 \) be a real vector space, with elements of the form

\[
(x_\nu; \lambda_\nu)_\nu = (x_0, x_1, x_2; \lambda_0, \lambda_1, \lambda_2)
\]

where \( x_\nu \in \mathcal{O}, \lambda_\nu \in \mathbb{R} \) and \( \nu = 0, 1, 2 \). The vector \( w \in V \) is called Okubo-Veronese if

\[
\lambda_0 x_0 = x_1 \ast x_2, \quad \lambda_1 x_1 = x_2 \ast x_0, \quad \lambda_2 x_2 = x_0 \ast x_1 \quad (4.1)
\]

\[
\nu (x_0) = \lambda_1 \lambda_2, \quad \nu (x_1) = \lambda_2 \lambda_0, \quad \nu (x_2) = \lambda_0 \lambda_1. \quad (4.2)
\]

Now we consider the subspace \( H \subset V \) be of Okubo-Veronese vectors. It is straightforward to see that if \( w = (x_\nu; \lambda_\nu)_\nu \) is an Okubo-Veronese vector then also \( \mu w = \mu (x_\nu; \lambda_\nu)_\nu \) is such a vector for every \( \mu \in \mathbb{R} \), and, therefore, \( \mathbb{R} w \subset H \). We define the Okubic projective plane \( \mathbb{P}^2 \mathcal{O} \) as the geometry having this 1-dimensional subspaces \( \mathbb{R} w \) as points, i.e.

\[
\mathbb{P}^2 \mathcal{O} := \{ \mathbb{R} w : w \in H \setminus \{0\} \}.
\]

On the projective plane \( \mathbb{P}^2 \mathcal{O} \) lines are defined through the use of a symmetric bilinear form \( \beta \) that is the extension of the polar form \( (2.3) \) of the quadratic norm \( \nu \) over the Okubo algebra \( \mathcal{O} \), i.e.

\[
\beta (v, w) := \sum_{\nu=0}^2 \left( (x_\nu, y_\nu) + \lambda_\nu \eta_\nu \right), \quad (4.4)
\]

where \( v = (x_\nu; \lambda_\nu)_\nu \) and \( w = (y_\nu; \eta_\nu)_\nu \) are Okubo-Veronese vectors in \( H \subset V \cong \mathcal{O}^3 \times \mathbb{R}^3 \). The lines \( \ell_w \) in the projective plane \( \mathbb{P}^2 \mathcal{O} \) are the orthogonal spaces of a vector \( w \in H \), i.e.

\[
\ell_w := w^\perp = \{ z \in V : \beta (z, w) = 0 \},
\]

and, clearly, a point \( \mathbb{R} v \) is incident to the line \( \ell_w \) when \( \mathbb{R} v \subset w^\perp \).

Remark 5. It is worth expliciting how the norm \( \nu \) defined over the symmetric composition algebra \( \mathcal{O} \) is intertwined with the geometry of the plane. This relations is manifest when we consider the quadratic form of the bilinear symmetric form \( \beta \), i.e.

\[
\|v\| := \beta (v, v) = 2 \nu (x_0) + 2 \nu (x_1) + 2 \nu (x_2) + \lambda_0^2 + \lambda_1^2 + \lambda_2^2, \quad (4.6)
\]
Correspondence between projective and affine plane

The identification of the affine Okubic plane with the projective one can be explicited defining the map that sends a point of the affine plane to the projective point in $V \cong O_3^3 \times \mathbb{R}^3$, i.e.

\[
\begin{align*}
(x, y) &\mapsto \mathbb{R} (x, y, x * y; n (y), n (x), 1), \\
(x) &\mapsto \mathbb{R} (0, 0, x; n (x), 1, 0), \\
(\infty) &\mapsto \mathbb{R} (0, 0, 0; 1, 0, 0).
\end{align*}
\]

(4.7)

(4.8)

(4.9)

Since the Okubo algebra is a symmetric composition algebra, then the map is well defined. Indeed, from (4.2) we note that

\[
n(x) = \lambda_1, \hspace{1em} n(y) = \lambda_0,
\]

(4.10)

and since Okubo is a composition algebra then

\[
n(x * y) = n(x) n(y).
\]

(4.11)

Since Okubo algebra is flexible we also have that (4.1) are satisfied and

\[
\begin{align*}
\lambda_0 x &= y * (x * y) = n(y)x, \\
\lambda_1 y &= (x * y) * x = n(x)y, \\
\lambda_2 (x * y) &= x * y,
\end{align*}
\]

and therefore $\mathbb{R} (x, y, x * y; n (y), n (x), 1)$ is a point in the Okubic projective plane. As for the converse, if a point $p$ of coordinates $(x_\nu; \lambda_\nu)_\nu$ is in $\mathbb{P}^2 O$ then it satisfies (4.2) and it has one of the $\lambda_\nu$ different from zero.

In [CZ, Sec. 4] we have shown that the map (4.7) that the correspondence is a bijection that sends affine points incident to an affine line into projective points incident to the same projective line.

5 A deformation of the Okubic Albert algebra

Since there is a well known geometric correspondence [CMCA] between rank-one idempotents of a simple rank-three Jordan algebra $J_3 (k)$ over a Hurwitz algebras $k \in \{ \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O} \}$ and elements of the projective planes $kP^2$ that was introduced by Jordan, Von Neumann and Wigner [JNW] in 1934, it thus appears logical to investigate the Jordan algebras of rank three over the Okubo algebra that can be potentially linked to the Okubic projective plane introduced above. Great work in this sense was done by Elduque who first introduced an equivalent of Tits-Freudenthal Magic Square for flexible composition algebras in [El04, El02] and later on defined an Okubic Albert algebra [El08] which he found to be isomorphic to the exceptional Jordan algebra over split-octonions $J_3 (O_s)$ and could be therefore used to define a $\mathbb{Z}_3$-grading over the exceptional Lie algebra $f_4$. 

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We now introduce a slight variation of the Okubic Albert algebra introduced by Elduque in [El18, Theo 5.15]. Let be \( A \) a quadratic and a cubic norm on the deformed Okubic Albert algebra Hurwitz algebra (see [El08, sec. 5.3]). It is therefore natural to define a linear, which in addition to being an extension to the Okubic algebra of the definition article, we will set a full investigation on the matter for a future article. For the purposes of this knowledge, that also article, we will set a full investigation on the matter for a future article. For the purposes of this article, we will set \( q = 1 \) we recover the Okubic Albert Algebra \( A(\mathcal{O}) \) introduced by Elduque which is a Jordan Algebra that it was shown to be isomorphic to \( \mathfrak{3}_3(\mathbb{O}) \). We also notice, for the first time to our knowledge, that also \( q = -1 \) gives a Jordan Algebra, that we would expect to be isomorphic to the exceptional Jordan algebra \( \mathfrak{3}_3(\mathbb{O}) \), even though we leave a full investigation on the matter for a future article. For the purposes of this article, we will set \( q = \frac{1}{2} \), which gives a new algebra \( A_{1/2}(\mathcal{O}) \) which is unital, with unit \((0, 0, 0; 1, 1, 1)\), commutative and flexible but is neither alternative nor Jordan.

It is worth noting that using the correspondence

\[
(x; \lambda) \mapsto \begin{pmatrix}
\lambda_0 & x_2 & \pi_1 \\
\pi_2 & \lambda_1 & x_0 \\
x_1 & \pi_0 & \lambda_2
\end{pmatrix},
\]

the product in (5.1) give rise to the usual Jordan algebras \( \mathfrak{3}_3(K) \) when \( K \) is an Hurwitz algebra (see [El08, sec. 5.3]). It is therefore natural to define a linear, a quadratic and a cubic norm on the deformed Okubic Albert algebra \( \mathfrak{3}_{1/2}(\mathcal{O}) \)

**i.e.** the trace of an element as the linear norm

\[
\text{Tr}((x; \lambda)) := \lambda_0 + \lambda_1 + \lambda_2,
\]

the quadratic norm or simply the norm of an element, i.e.

\[
\|(x; \lambda)\| := 2n(x_0) + 2n(x_1) + 2n(x_2) + \lambda_0^2 + \lambda_1^2 + \lambda_2^2,
\]

which in addition to being an extension to the Okubic algebra of the definition in the Hurwitz case, is also consistent with (4.0). From the norm we obtain its polarization, i.e. the inner product given by

\[
\langle (x; \lambda), (y; \mu) \rangle_A := \|(x + y; \lambda + \mu)\| - \|(x; \lambda)\| - \|(y; \mu)\|.
\]

Finally, we introduce the cubic norm \( N \) of an element, i.e.

\[
N((x; \lambda)) := \lambda_0\lambda_1\lambda_2 - (\lambda_0n(x_0) + \lambda_1n(x_1) + \lambda_2n(x_2)) + 2((x_0 + e) * (x_1 + x_2), e)
\]
where \( n(x) \) is the Okubic norm of \( x \) and \( e \) is the idempotent element defined in (2.18).

In analogy with the \( \mathfrak{J}_3(k) \) case, we will define an idempotent element of \( \mathfrak{A}_{1/2}(O) \) to be rank-1, an idempotent element that has cubic norm \( N((x_\nu; \lambda_\nu)) = 0 \) and \( \text{Tr}((x_\nu; \lambda_\nu)) = 1 \). Indeed, it is well known that on idempotent elements \( e \) of \( \mathfrak{J}_3(k) \) we have that \( \text{Tr}(e) = \text{rank}(e) \) thus rank-1 idempotents are those and only those with trace equal to one (e.g. see [SBGHLS, Note 16.8]).

**Correspondence with the Okubic projective plane**

Even though \( \mathfrak{A}_{1/2}(O) \) is not a Jordan algebra, it yet retains the geometrical analogy with \( \mathfrak{J}_3(k) \) when \( k \in \{ \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O} \} \) since the condition of idempotency on rank-1 elements, i.e. \( (x_\nu; \lambda_\nu) \circ (x_\nu; \lambda_\nu) = (x_\nu; \lambda_\nu) \), \( N((x_\nu; \lambda_\nu)) = 0 \) and \( \text{Tr}((x_\nu; \lambda_\nu)) = 1 \), yields to

\[
\begin{pmatrix}
(1 - \lambda_0) x_0 + x_1 * x_2 \\
(1 - \lambda_1) x_1 + x_2 * x_0 \\
(1 - \lambda_2) x_2 + x_0 * x_1 \\
\lambda_0 x_0 + n(x_1) + n(x_2) \\
\lambda_1 x_1 + n(x_0) + n(x_2) \\
\lambda_2 x_2 + n(x_0) + n(x_1)
\end{pmatrix} = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ \lambda_0 \\ \lambda_1 \\ \lambda_2 \end{pmatrix},
\]

which is satisfied iff (4.2) and (4.1) hold true along with the condition \( \lambda_0 + \lambda_1 + \lambda_2 = 1 \), which isolate a specific representative of the class \( R(x_\nu; \lambda_\nu) \). We therefore have a one-to-one correspondence between points of the Okubic projective plane as defined in (4.3) and rank-1 idempotent elements of the algebra \( \mathfrak{A}_{1/2}(O) \).

It is worth noting that conditions to (4.2) and (4.1) together with \( \text{Tr}((x_\nu; \lambda_\nu)) = 1 \), imply that the quadratic norm is identically zero since

\[
\| (x_\nu; \lambda_\nu) \| = 2n(x_0) + 2n(x_1) + 2n(x_2) + \lambda_0^2 + \lambda_1^2 + \lambda_2^2 = 2\lambda_1 \lambda_2 + 2\lambda_2 \lambda_0 + 2\lambda_0 \lambda_1 + \lambda_0^2 + \lambda_1^2 + \lambda_2^2 = (\lambda_0 + \lambda_1 + \lambda_2)^2 = 1,
\]

and, moreover that the cubic norm is identically zero since

\[
N((x_\nu; \lambda_\nu)) = \lambda_0 \lambda_1 \lambda_2 - (\lambda_0 n(x_0) + \lambda_1 n(x_1) + \lambda_2 n(x_2)) + 2 \langle (x_0 * e) * (x_1 * x_2), e \rangle = -2\lambda_0 \lambda_1 \lambda_2 - 2\lambda_0 \langle (x_0 * e) * x_0, e \rangle = -2\lambda_0 \lambda_1 \lambda_2 - 2\lambda_0 \lambda_1 \lambda_2 \langle e, e \rangle = 0
\]

**Remark 6.** In the case of the Jordan algebra \( \mathfrak{J}_3(k) \) over a Hurwitz algebras \( k \in \{ \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O} \} \), the Veronese conditions constrain the determinant to be zero exactly as for \( \mathfrak{A}_{1/2}(O) \). Moreover, in case of the Jordan algebra \( \mathfrak{J}_3(k) \) the condition of an idempotent to be trace one, i.e. \( \lambda_0 + \lambda_1 + \lambda_2 = 1 \), is equivalent to the requirement of being a rank-1 idempotent [SBGHLS, CMCA]. Therefore,
it exists a perfect analogy between the correspondence of the deformed Okubic Albert algebra $\mathbb{A}_{1/2}(\mathcal{O})$ and the Okubic projective plane $\mathcal{O}P^2$ and the correspondence between the usual projective planes $\mathbb{K}P^2$ and the simple rank-three Jordan algebras $\mathfrak{J}_3(\mathbb{K})$.

**Remark 7.** We also have to note that, while $\mathbb{A}_{1/2}(\mathcal{O})$ is not a Jordan algebra, the restriction on the idempotent elements does satisfy the Jordan identity. This is a consequence of the flexibility of the Okubo algebra. Indeed, if an algebra is flexible, i.e.

$$ (x \circ y) \circ x = x \circ (y \circ x), $$

(5.10)

then, on the idempotent elements, we have

$$
(x \circ y) \circ (x \circ x) = (x \circ y) \circ x = x \circ (y \circ x) = x \circ (x \circ (x \circ x)),
$$

(5.11)

and thus idempotent elements fulfill the Jordan identity.

**Automorphisms of $\mathbb{A}_{1/2}(\mathcal{O})$**

Freudenthal [Fr65] and Rosenfeld [Ro98, Ro93], noted how the automorphisms of $\mathfrak{J}_3(\mathcal{O})$ could be interpreted as isometries of the octonionic projective plane $\mathcal{O}P^2$, giving rise to an algebro-geometrical realization of the real compact form of the exceptional Lie group $F_4(-52)$, while the other real forms, i.e. $F_4(1)$ and $F_4(-20)$, are obtained as isometry groups of the split-octonionic projective plane $\mathcal{O}sP^2$ and the octonionic hyperbolic plane $\mathcal{O}H^2$ respectively (see [CMCAa, CMCAb] for a recent account). Rosenfeld himself in [Ro98, Ro93] proceeded in relating real forms of exceptional Lie groups with projective and hyperbolic planes over tensorial products of Hurwitz algebras. This approach had yielded to a definition of the so-called octonionic Rosenfeld planes as homogeneous (and symmetric) spaces (see [CM22] for a comprehensive treatment). It is therefore interesting to study the automorphisms of $\mathbb{A}_{1/2}(\mathcal{O})$ in order to extend Freudenthal and Rosenfeld’s description to the Okubic projective plane.

We start and notice that if $\varphi$ is an automorphism of the Okubo algebra, namely if it belongs to $\text{Aut}(\mathcal{O})$, i.e. $\varphi(x \ast y) = \varphi(x) \ast \varphi(y)$, then it exists an automorphism

$$
\Phi(x_\nu; \lambda_\nu) = (\varphi(x_0), \varphi(x_1), \varphi(x_2); \lambda_0, \lambda_1, \lambda_2) \in \text{Aut}(\mathbb{A}_{1/2}(\mathcal{O})),
$$

(5.12)

since, as shown in section 2, an automorphism of $\mathcal{O}$ is also an isometry, i.e.

$$
\langle \varphi(x), \varphi(y) \rangle = \langle x, y \rangle.
$$

It is therefore straightforward to realize that $SU(3)$ is a subgroup of $\text{Aut}(\mathbb{A}_{1/2}(\mathcal{O}))$. Moreover, we also note that $\mathbb{Z}/3\mathbb{Z} < \text{Aut}(\mathbb{A}_{1/2}(\mathcal{O}))$ since

$$
\tau((x_0, x_1, x_2; \lambda_0, \lambda_1, \lambda_2)) = (x_2, x_0, x_1; \lambda_2, \lambda_0, \lambda_1)
$$

(5.13)
is an automorphism. On the other hand, a transposition of just two coordinates
does not yield to an automorphism; this is easily seen considering the element
\[(x, 0, 0; 0, 0, 0) \circ (0, y; 0, 0, 0) = (0, 0, q(x * y); 0, 0, 0). \quad (5.14)\]
Then, suppose \(\sigma\) to be an application that switches the coordinates 0 with 1, we then have that
\[
\sigma ((0, 0, q(x * y); 0, 0, 0)) = (0, x, 0; 0, 0, 0) \circ (y, 0, 0; 0, 0, 0),
\]
which happens \(iff\) \((y * x) = (x * y)\), which would require the product \(*\) to be commutative.

In order to conclude the present investigation, we have to consider the vector
space decomposition of \(A_{1/2}(\mathcal{O})\) as
\[
A_{1/2}(\mathcal{O}) \simeq \mathcal{O}^{(0)} \oplus \mathcal{O}^{(1)} \oplus \mathcal{O}^{(2)} \oplus \mathbb{R}^{(0)} \oplus \mathbb{R}^{(1)} \oplus \mathbb{R}^{(2)}, \quad (5.17)
\]
and \(w_i\) and \(\omega_i\) are the maps from \(\mathcal{O}\) and \(\mathbb{R}\) respectively in \(A_{1/2}(\mathcal{O})\), such
that they send the element of the algebra into the \(i\) coordinate, e.g. \(w_2(x) := (0, 0, 0; 0, 0, 0)\) and \(\omega_2(\lambda) := (0, 0, 0; 0, 0, \lambda)\). The algebra \(A_{1/2}(\mathcal{O})\) has unit
\((0, 0, 0; 1, 1, 1)\) which can be obtained as the sum of three primitive, i.e. that
are not sum of other idempotents, and orthogonal, i.e. such that \(e_i \circ e_j = 0\),
idempotents given by
\[
e_0 := \omega_0(1) = (0, 0, 0; 1, 0, 0), \quad (5.18)
e_1 := \omega_1(1) = (0, 0, 0; 0, 1, 0), \quad (5.19)
e_2 := \omega_2(1) = (0, 0, 0; 0, 0, 1). \quad (5.20)
\]
We thus have that \(\{e_0, e_1, e_2\}\) is a complete system of primitive idempotents. In
fact is the only complete system of primitive idempotents with real coordinates.
We now state the following

**Lemma 8.** If \(\Phi\) is an automorphism, then
\[
\Phi (e_i) = e_{\sigma(i)}, \quad (5.21)
\]
for a suitable permutation \(\sigma \in \mathfrak{S}_3\). In other words, a complete system of primitive idempotents with only real coordinates must be sent in another complete system of primitive idempotents with only real coordinates.

**Proof.** That a complete primitive system of idempotents must go into a complete primitive system of idempotents it is self-evident. What is left to show is that a primitive real idempotent can only go in a primitive real idempotent, where we have called **real idempotents** those of the form \(e = (0, 0, 0; \alpha_0, \alpha_1, \alpha_2)\). First of all we notice that a primitive real idempotent has only one non-null real coordinate, since for \((5.7)\) all real coordinates must be idempotent and thus an idempotent of the form \(e = (0, 0, 0; \alpha_0, \alpha_1, 0)\), would be decomposable into the
sum of two idempotents \((0, 0, 0; \alpha_0, 0, 0)\) and \((0, 0, 0; 0, \alpha_1, 0)\), thus violating the condition of being primitive. Therefore, without any loss of generality, let us consider the left action of a primitive real idempotent observing the left action of \(e_0 = (0, 0, 0; 1, 0, 0)\). Since

\[
L_{e_0} (y_\eta; \mu_\eta) = e_0 \circ (y_\eta; \mu_\eta) = \left(0, \frac{1}{2} y_1, \frac{1}{2} y_2; \mu_0, 0, 0 \right),
\]  

(5.22)

we then have that the image \(L_{e_0}\) is given by \(\mathcal{O}^{(1)} \oplus \mathcal{O}^{(2)} \oplus \mathbb{R}^{(0)}\). Now let us consider the kernel of the left action of \(e_0\), i.e.

\[
\ker L_{e_0} = \left\{ (y_\eta; \mu_\eta) \in \mathbb{A}_{1/2} (\mathcal{O}) : e_0 \circ (y_\eta; \mu_\eta) = 0 \right\}.
\]  

(5.23)

We notice that for every \((y_\eta; \mu_\eta) \in \ker L_{e_0}\) we must have a corresponding element in \(\ker L_{\Phi(e_0)}\) since

\[
0 = \Phi (0) = \Phi (e_0 \circ (y_\eta; \mu_\eta)) = \Phi (e_0) \circ \Phi (y_\eta; \mu_\eta),
\]  

(5.24)

so that \(\Phi (y_\eta; \mu_\eta) \in \ker L_{\Phi(e_0)}\). We then have

\[
\dim (\ker L_{\Phi(e_0)}) \geq \dim (\ker L_{e_0}),
\]  

(5.25)

and from (5.22) it is clear that \(\dim (\ker L_{e_0}) = 10\). If \(\Phi (e_0)\) is a primitive real idempotent the condition is clearly satisfied. Now, let us suppose that \(\Phi (e_0) = \varepsilon = (x_\nu; \lambda_\nu)\) is a primitive, but not of real type, idempotent. We therefore have that at least one Okubic coordinate \(x_\nu \in \mathcal{O}\) has to be different from zero (we also notice from (5.1), that if we have more non-null Okubic coordinate, then the dimension of the kernel is smaller, so this is not a restrictive hypothesis). Let us suppose without any loss of generality that the idempotent \(\varepsilon\) has \(x_0 \neq 0\). Since Okubo algebra is a division algebra, this means that \(n (x_0) \neq 0\), and since every idempotent of non-real type satisfies Veronese conditions \(4.1\) and \(4.2\) at least \(\lambda_1, \lambda_2\) are non-null, because \(n (x_0) = \lambda_1 \lambda_2\). But then, analyzing the kernel, of the left action \(L_{\varepsilon}\), we have that

\[
L_{\varepsilon} \circ (y_\eta; \mu_\eta) = \left(\begin{array}{c}
\lambda_1 \eta_0 + \mu_1 \mu_2 \eta_0 \\
\lambda_2 \eta_1 + q (y_2 \ast x_0) \\
\lambda_1 \eta_2 + q (x_0 \ast y_1) \\
0 \\
\lambda_1 \mu_1 + \langle x_0, y_0 \rangle \\
\lambda_2 \mu_2 + \langle x_0, y_0 \rangle
\end{array} \right),
\]  

(5.26)

from which one obtain \(\dim (\ker L_{\varepsilon}) = 1 < \dim (\ker L_{e_0})\), thus ruling out the possibility of \(\Phi (e_0)\) being \(\varepsilon\).

Using the Lemma (8) we can now investigate all automorphisms. Let \(\Phi \in \text{Aut} (\mathbb{A}_{1/2} (\mathcal{O}))\), then we have that

\[
\Phi (e_i) = e_{\sigma(i)},
\]  

(5.27)

\[18\]
for some \( \sigma \in S_3 \) and for \( i \in \mathbb{Z}/3\mathbb{Z} \). Now, considering (5.22) we then have that
\[
O^{(i)} = (e_{i+1} \circ A_{1/2}(O)) \cap (e_{i+2} \circ A_{1/2}(O)),
\]
where \( i \in \mathbb{Z}/3\mathbb{Z} \). Therefore, we have that
\[
\Phi \left( O^{(i)} \right) = \Phi \left( (e_{i+1} \circ A_{1/2}(O)) \cap (e_{i+2} \circ A_{1/2}(O)) \right) = \Phi (e_{i+1} \circ A_{1/2}(O)) \cap \Phi (e_{i+2} \circ A_{1/2}(O)) = \Phi (e_{i+1} \circ A_{1/2}(O)) \cap (\Phi (e_{i+2} \circ A_{1/2}(O)) = O^{(\tau(i))},
\]
(5.28)
where \( i \in \mathbb{Z}/3\mathbb{Z} \) and for every \( i \in \mathbb{Z}/3\mathbb{Z} \). Therefore there exist three automorphisms \( \phi_0, \phi_1 \) and \( \phi_2 \) of \( O \) such that
\[
\Phi (w_i(x)) = w_{\tau(i)}(\phi_i(x)).
\]
A straightforward computation from (5.1) shows that in order for \( \Phi \) to be an automorphisms it must be that
\[
\phi_i(x) \ast \phi_{i+1}(y) = \phi_{i+2}(x \ast y),
\]
for every \( i \in \mathbb{Z}/3\mathbb{Z} \) and \( x, y \in O \). Relation (5.34) defines a \( \mathbb{Z}/3\mathbb{Z} \) graded action of three copies of \( SU(3) \) on \( A_{1/2}(O) \), from which we deduce
\[
\text{Aut} \left( A_{1/2}(O) \right) = \mathbb{Z}/3\mathbb{Z} \ltimes (SU(3) \times SU(3) \times SU(3)),
\]
(5.35)
where \( \mathbb{Z}/3\mathbb{Z} \) is a sort of “triality symmetry” which interchanges the \( SU(3) \) factors in a cyclic way.

6 Conclusions and further developments

In this article we presented a deformation of the Okubic Albert algebra \( A_q(O) \) introduced by Elduque, which for \( q = \pm 1 \) returns a Jordan algebra and for \( q = 1/2 \) constitutes an algebraic equivalent of the Okubic affine and projective planes. Even though \( A_{1/2}(O) \) is not a Jordan algebra, nevertheless it perfectly realizes, for the Okubic case, the well-known relation between points of the octonionic plane \( OP^2 \) and rank-1 idempotent elements of the exceptional Jordan algebra \( J_3(O) \). In the octonionic case, the automorphisms of \( J_3(O) \) yield to an algebro-geometrical realization of the compact real form of \( F_4(-52) \) as the isometry group of the octonionic projective plane \( OP^2 \). Analogously, in the Okubic case we have found that \( \text{Aut} \left( A_{1/2}(O) \right) \) is the group \( \mathbb{Z}/3\mathbb{Z} \ltimes SU(3) \times SU(3) \times SU(3) \). A natural development of the present article would be definition and the study of some sort of projective plane over the split-Okubo algebra \( O_s \). Indeed, even though with some key differences, similar results to those found for the octonionic projective plane do hold in the case of the split-octonionic projective plane \( O_sP^2 \) and in the case of the octonionic hyperbolic plane \( OH^2 \) yielding,
when considering the respectively isometry groups, to concrete realizations of
the real form of the Lie groups $F_{4(4)}$ and $F_{4(-20)}$. It is therefore reasonable
to expect that an analogous situation would hold for a split-Okubic projective
plane $O_s P^2$ even though, as in the case of the split-octonions, dealing with a
non-division algebra will yield to non trivial geometrical implications.

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