ALGEBRAIC CONSTRUCTION OF MULTI-SPECIES \( q \)-BOSON SYSTEM

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Abstract. We construct a stochastic particle system which is a multi-species version of the \( q \)-Boson system due to Sasamoto and Wadati. Its transition rate matrix is obtained from a representation of a deformation of the affine Hecke algebra of type \( GL \).

1. Introduction

In this article we construct a multi-species version of the \( q \)-Boson system due to Sasamoto and Wadati \[9\] by using a representation of a deformation of the affine Hecke algebra.

The \( q \)-Boson system is a stochastic particle system on the one-dimensional lattice \( \mathbb{Z} \). The particles can occupy the same site simultaneously, and one particle may move from site \( i \) to \( i-1 \) independently for each \( i \in \mathbb{Z} \). The rate at which one particle moves from a cluster with \( n \) particles is given by \( 1 - q^n \), where \( q \) is a parameter of the model.

The multi-species version which we propose in this paper is described as follows. Fix a positive integer \( N \). Each bosonic particle is colored with a positive integer which is less than or equal to \( N \). One particle may move to the left in the same way as the \( q \)-Boson system, but the rate is different. Let \( b \in \{1, 2, \ldots, N\} \) be the color of the moving particle and \( m_j (j = b, b+1, \ldots, N) \) the number of particles with color \( j \) in the cluster from which the moving particle leaves. Then the rate is given by

\[
\frac{1 - q^{m_b}}{1 - q} q^{\sum_{j=b+1}^{N} m_j}.
\]

If \( N = 1 \), the transition rate matrix is equal to that of the \( q \)-Boson system up to constant multiplication.

In the following we describe how the multi-species model arises from representation theory. Hereafter we fix a positive integer \( k \) which signifies the number of particles.

In a previous paper \[10\], we introduced a deformation of the affine Hecke algebra of type \( GL_k \) with four parameters and found that an integrable stochastic particle system can be constructed from its representation as follows. The deformed algebra has a representation on the vector space \( \mathcal{F}(L) \) of \( \mathbb{C} \)-valued functions on the \( k \)-dimensional orthogonal lattice \( L := \mathbb{Z}^k \). It is defined in terms of a generalization of the discrete integral-reflection operators due to van Diejen and Emsiz \[4\]. Using
them we define an operator $G : F(L) \to F(L)$, which is a discrete analogue of the propagation operator introduced by Gutkin [6] in order to construct eigenfunctions of the Hamiltonian with delta potential, and determine the discrete Hamiltonian $H$ by the property $HG = G \sum_{i=1}^{k} t_i$, where $t_i$ is the shift operator $(t_i f)(x_1, \ldots, x_k) := f(\ldots, x_i - 1, \ldots)$. Then the operator $H$ leaves the subspace $F(L)^{S_k}$ of symmetric functions invariant. Specializing the parameters and restricting $H$ to $F(L)^{S_k}$, we obtain the transition rate matrix of a continuous-time Markov process. The resulting system is a continuous-time limit of the $q$-Hahn system due to Povolotsky [8, 1].

In this paper we generalize the above construction in a similar way to the generalization of the periodic delta Bose gas due to Emsiz, Opdam and Stokman [5]. In this article we only consider the case where one of the parameters of the deformed algebra is equal to zero. Then the algebra, which we denote by $A_k$, essentially has two parameters $\alpha$ and $q$. The algebra $A_k$ contains the Hecke algebra $H_k$ of type $A_{k-1}$ as a subalgebra. Let $M$ be a left $H_k$-module and denote by $F(L, M)$ the vector space of functions on $L$ taking values in $M$. Then we can define an action of $A_k$ on $F(L, M)$, introduce the propagation operator $G$ and determine the discrete Hamiltonian $H$ acting on $F(L, M)$ from the property $HG = G \sum_{i=1}^{k} t_i$. Then the Hamiltonian $H$ leaves a subspace $F_0(L, M)$ (see Definition 3.4 below) of $F(L, M)$ invariant.

Now the multi-species version of the $q$-Boson system is constructed as follows. Let $U$ be an $N$-dimensional vector space. We regard $U^{\otimes k}$ as a left $H_k$-module with respect to the action found by Jimbo [7]. Then the invariant subspace $F_0(L, U^{\otimes k})$ is identified with the vector space of functions $F(S)$ on the set of configurations of $k$ bosonic particles of $N$ species on the one-dimensional lattice $\mathbb{Z}$. Setting $\alpha = -(1-q)$ and restricting the Hamiltonian $H$ to $F(S)$, we obtain the transition rate matrix of the multi-species $q$-Boson system up to an additive constant.

Using the propagation operator $G : F(L, M) \to F(L, M)$, we can construct eigenfunctions of the discrete Hamiltonian $H$ by means of the Bethe ansatz method, which we call the Bethe wave functions. We should construct Plancherel theory for them to analyze the multi-species $q$-Boson system in a similar manner to the work of Borodin, Corwin, Petrov and Sasamoto [2, 3]. We leave it as a future problem.

The paper is organized as follows. In Section 2 we introduce the deformation of the affine Hecke algebra and its representation defined by the discrete integral-reflection operators. In Section 3 we define the propagation operator and the discrete Hamiltonian, and construct the Bethe wave functions. In Section 4 we derive the transition rate matrix of the multi-species $q$-Boson system from the discrete Hamiltonian.

2. A DEFORMATION OF THE AFFINE HECKE ALGEBRA AND ITS REPRESENTATION

2.1. Preliminaries. Throughout this paper we fix an integer $k \geq 2$. Let $V := \bigoplus_{i=1}^{k} \mathbb{R}v_i$ be the $k$-dimensional Euclidean space with the orthonormal basis $\{v_i\}_{i=1}^{k}$. 

We denote by $V^*$ the linear dual of $V$ and by $\{\epsilon_i\}_{i=1}^{k}$ the dual basis of $V^*$ corresponding to $\{v_i\}_{i=1}^{k}$. For $i, j = 1, \ldots, k$ we set $\alpha_{ij} := \epsilon_i - \epsilon_j$. Then the set $R := \{\alpha_{ij} \mid i \neq j\}$ forms the root system of type $A_{k-1}$ with the simple roots $\alpha_i := \alpha_{i,i+1} (1 < i < k)$. We denote the set of the associated positive and negative roots by $R^+$ and $R^-$, respectively.

Let $s_i : V \to V (1 < i < k)$ be the orthogonal reflection
\[ s_i(v) := v - a_i(v)a_i^\vee, \]
where $a_i^\vee := v_i - v_{i+1}$ is the simple coroot. The Weyl group $W$ of type $A_{k-1}$ is generated by the simple reflections $\{s_i\}_{i=1}^{k-1}$. We denote the length of $w \in W$ by $\ell(w)$. The dual space $V^*$ is a $W$-module by $(w\lambda)(v) := \lambda(w^{-1}v)$ ($w \in W, \lambda \in V^*, v \in V$).

Let $v \in V$. The orbit $Wv$ intersects the closure of the fundamental chamber
\[ \overline{C_+} := \{v \in V \mid a_i(v) \geq 0 (i = 1, \ldots, k-1)\} \]
at one point. Take the shortest element $w \in W$ such that $wv \in \overline{C_+}$. We denote it by $w_v$. Set
\[ I(v) := \{a \in R^+ \mid a(v) < 0\}. \]
If $w_v = s_{i_1} \cdots s_{i_p}$ is a reduced expression, then $I(v) = \{s_{i_1} \cdots s_{i_{p+1}}(a_{i_p})\}_{p=1}^{\ell}$. Therefore $\ell(w_v) = \#I(v)$ and $I(v) = R^+ \cap w_v^{-1}R^-$. From the above facts, we obtain the following lemma.

**Lemma 2.1.** Suppose that $I(v_1) \subset I(v_2)$. Then $w_{v_2} = w_{v_1} v_2 w_{v_1}$ and $\ell(w_{v_2}) = \ell(w_{v_1} v_2) + \ell(w_{v_1})$.

We denote by $L$ the $k$-dimensional orthogonal lattice in $V$ defined by
\[ L = \bigoplus_{i=1}^{k} \mathbb{Z}v_i, \]
and set
\[ L_+ := L \cap \overline{C_+}. \]
Hereafter we set $\epsilon_{k+1}(x) = -\infty$ for $x \in L_+$.

For $x \in L$ and $1 \leq i \leq k$, we set
\[ d_i^+(x) := \# \{p \mid i < p \leq k, \epsilon_i(x) = \epsilon_p(x)\} , \]
\[ d_i^-(x) := \# \{p \mid 1 \leq i < p, \epsilon_i(x) = \epsilon_p(x)\}. \]
We denote by $\sigma_x \in \mathfrak{S}_k$ the permutation determined by $w_x v_i = v_{\sigma_x(i)} (1 \leq i \leq k)$. Then it holds that
\begin{equation} \tag{2.1} \end{equation}
\[ d_i^+(x) = d_{\sigma_x(i)}^+(w_x x), \quad \epsilon_i(x) = \epsilon_{\sigma_x(i)}(w_x x) \]
for $x \in L$ and $1 \leq i \leq k$.

**Proposition 2.2.** Suppose that $1 \leq i < k$. 

(i) If \( v \in V \) satisfies \( a_i(v) > 0 \), then \( w_{s_is_i} = w_is_i \) and \( \ell(w_{s_is_i}) = \ell(w_i) + 1 \).
(ii) If \( x \in L \) satisfies \( a_i(x) = 0 \), then \( \sigma_x(i + 1) = \sigma_x(i) + 1 \), \( w_xs_i = s_{\sigma_x(i)}w_x \) and \( \ell(w_xs_i) = \ell(s_{\sigma_x(i)}w_x) = \ell(w_x) + 1 \).

Proof. (i) From \( a_i(v) > 0 \) we see that \( I(s_iv) = s_iI(v) \sqcup \{a_i\} \). Hence \( \ell(w_{s_is_i}) = \ell(w_i) + 1 \). Since \( w_{s_is_i} \) moves \( s_i \) into \( C_+ \), we have \( w_{s_is_i} = w_is_i \).

(ii) Since \( a_i(x) = 0 \) and \( w_xx \in L_+ \), there exists an integer \( p \) such that \( 1 \leq p \leq k \) and \( \epsilon_{\sigma_x(i)}(w_xx) = \epsilon_{\sigma_x(i+1)}(w_xx) = \epsilon_p(w_xx) \). Then, for \( j = i \) and \( i + 1 \), it holds that \( p - \sigma_x(j) = d^+_{\sigma_x(j)}(w_xx) = d^+_i(x) \). On the other hand, \( d^+_i(x) = d^+_{i+1}(x) + 1 \) because \( a_i(x) = 0 \). Therefore

\[
\sigma_x(i + 1) = p - d^+_{i+1}(x) = p - d^+_i(x) + 1 = \sigma_x(i) + 1.
\]

Set \( z := x - v_{i+1}/2 \). Since \( |a(v_j/2)| \leq 1/2 \) \((j = i, i + 1)\) for any \( a \in R \), we have \( I(x) \subset I(z) \) and \( I(x) \subset I(s_iz) \). Moreover, \( w_{s_iz} = w_zs_i \) and \( \ell(w_{s_iz}) = \ell(w_z) + 1 \) from (i). Therefore, using Lemma 2.1, we get

\[(2.2) \quad w_{w_zx}w_x s_i = w_{w_zx}w_z, \quad \ell(w_{w_zx}) + \ell(w_z) + 1 = \ell(w_{w_zx}w_z) + \ell(w_z).
\]

Note that \( w_zx = w_xx - v_{\sigma_x(i+1)/2} \) and \( w_zs_i = w_xx - v_{\sigma_x(i)/2} \). Since \( w_xx \in L_+ \), we have

\[
I(w_zx) = \{\alpha_{\sigma_x(i+1)j}\}_{j=1}^{\sigma_x(i+1)+1}, \quad I(w_zs_i) = \{\alpha_{\sigma_x(i)j}\}_{j=1}^{\sigma_x(i)+1}.
\]

Hence

\[
w_{w_zx} = s_{p-1}s_{p-2} \cdots s_{\sigma_x(i+1)}, \quad w_{w_zx}w_z = s_{p-1}s_{p-2} \cdots s_{\sigma_x(i)},
\]

where the right hand sides are reduced expressions. Using \( \sigma_x(i + 1) = \sigma_x(i) + 1 \) and (2.2), we find that \( w_zs_i = s_{\sigma_x(i)}w_x \) and \( \ell(w_zs_i) = \ell(s_{\sigma_x(i)}w_x) = \ell(w_x) + 1 \).

We will also use the following proposition. See Lemma 3.9 and Lemma 3.10 in [10] for the proof.

Proposition 2.3. Suppose that \( x \in L \) and \( 1 \leq i \leq k \). Set \( y = x - v_i \). Then it holds that

\[
(s_{\sigma_x(i)-d^-_i}(y) \cdots s_{\sigma_y(i)-1}w_y = s_{\sigma_x(i)+d^+_i(x)-1} \cdots s_{\sigma_x(i)}w_x,
\]

\[
(2.3) \quad a_j(w_xx) = 0 \quad (\sigma_x(i) \leq j < \sigma_x(i) + d^+_i(x)),
\]

\[
a_j(w_yy) = 0 \quad (\sigma_y(i) - d^-_i(y) \leq j < \sigma_y(i))
\]

and \( d^-_i(y) + \ell(w_y) = d^+_i(x) + \ell(w_x) \).

2.2. A deformation of the affine Hecke algebra and its representation.

Definition 2.4. Let \( \alpha \) and \( q \) be complex constants. We define the algebra \( A_k \) to be the unital associative \( C \)-algebra with the generators \( X_i^{\pm 1} (1 \leq i \leq k) \) and
For any $1 \leq i < k$, the assignment $T_i (1 \leq i < k)$ satisfying the following relations:

\[
(T_i - 1)(T_i + q) = 0 \quad (1 \leq i < k), \quad T_i T_{i+1} T_i = T_i T_{i+1} T_i \quad (1 \leq i \leq k - 2),
\]

\[
T_i T_j = T_j T_i \quad (|i - j| > 1), \quad X_i X_j = X_j X_i \quad (i, j = 1, \ldots, k),
\]

\[
X_{i+1} T_i - T_i X_i = T_i X_{i+1} - X_i T_i = (1 - q) X_{i+1} + \alpha \quad (1 \leq i < k),
\]

\[
X_i X_j = T_j X_i \quad (i \neq j, j + 1).
\]

When $\alpha = 0$, the algebra $A_k$ is isomorphic to the affine Hecke algebra of type $GL_k$. The subalgebra generated by $T_i (1 \leq i < k)$ is isomorphic to the Hecke algebra of type $A_{k-1}$. We denote it by $\mathcal{H}_k$.

For a left $\mathcal{H}_k$-module $M$, we denote by $F(L, M)$ the complex vector space of functions on $L$ taking values in $M$. The Weyl group $W$ acts on $F(L, M)$ by $(wf)(x) := f(w^{-1}x) (w \in W, f \in F(L, M), x \in L)$. Let $\hat{T}_i (1 \leq i < k)$ be the $\mathbb{C}$-linear operator on $F(L, M)$ defined by $(\hat{T}_i f)(x) = T_i f(x) (f \in F(L, M), x \in L)$, where $\cdot$ signifies the action of $\mathcal{H}_k$ on $M$. Then the assignment $T_i \mapsto \hat{T}_i (1 \leq i < k)$ uniquely determines an algebra homomorphism $\mathcal{H}_k \rightarrow \text{End}(F(L, M))$. It commutes with the action of $W$.

We identify the group algebra $\mathbb{C}[L]$ with the Laurent polynomial ring $\mathbb{C}[e^{\pm v_1}, \ldots, e^{\pm v_k}]$. The Weyl group acts on $\mathbb{C}[L]$ from the right by $e^x w = e^{w^{-1} x} (x \in L, w \in W)$. Using this action we define the $\mathbb{C}$-linear map $\tilde{I}_j : \mathbb{C}[L] \rightarrow \mathbb{C}[L] (1 \leq j < k)$ by

\[
\tilde{I}_j(P) := (P - P s_j) \frac{ae^{v_{j+1}} + 1 - q}{1 - e^{-v_j + v_{j+1}}}.
\]

Consider the non-degenerate $\mathbb{C}$-bilinear pairing $(\cdot, \cdot) : \mathbb{C}[L] \times F(L, M) \rightarrow M$ uniquely determined by $(e^x, f) = f(x) (x \in L, f \in F(L, M))$. Now we define the $\mathbb{C}$-linear operator $\tilde{I}_j : F(L, M) \rightarrow F(L, M) (1 \leq j < k)$ by the property

\[
(P, \tilde{I}_j(f)) = (\tilde{I}_j(P), f)
\]

for any $P \in \mathbb{C}[L]$. It is explicitly written as follows.

\[
(\tilde{I}_j f)(x) = \begin{cases} 
\sum_{l=0}^{a_j(x)-1} (\alpha f(x - la_j^y + v_{j+1}) + (1-q)f(x - la_j^y)) & (a_j(x) > 0) \\
0 & (a_j(x) = 0) \\
-\sum_{l=1}^{-a_j(x)} (\alpha f(x + la_j^y + v_{j+1}) + (1-q)f(x + la_j^y)) & (a_j(x) < 0)
\end{cases}
\]

**Lemma 2.5.** The following relations hold in $\text{End}(F(L, M))$.

(i) $\tilde{I}_j^2 = (1 - q)\tilde{I}_j$ for $1 \leq j < k$.

(ii) $s_i \tilde{I}_j = \tilde{I}_j s_i$ if $|i - j| \geq 2$.

(iii) $s_j \tilde{I}_j + \tilde{I}_j s_j = (1 - q)(s_j - 1)$ for $1 \leq j < k$. 
(iv) \( \hat{I}_j s_{j+1}s_j = s_j s_{j+1} \hat{I}_j \) and \( \hat{I}_j s_j s_{j+1} = s_j s_{j+1} \hat{I}_j \) for \( 1 \leq j \leq k - 2 \).
(v) For \( 1 \leq j \leq k - 2 \),
\[
\begin{align*}
\hat{I}_j s_{j+1} \hat{I}_j &= (1 - q)s_{j+1} \hat{I}_j s_j + s_{j+1} \hat{I}_j \hat{I}_j + \hat{I}_j s_{j+1}, \\
\hat{I}_j s_j \hat{I}_j &= (1 - q)s_j \hat{I}_j s_{j+1} + s_j \hat{I}_j \hat{I}_j + \hat{I}_j s_j.
\end{align*}
\]
(vi) \( \hat{I}_j \hat{I}_j + q s_j \hat{I}_j s_j = \hat{I}_j s_{j+1} \hat{I}_j + q s_j \hat{I}_j s_{j+1} \) for \( 1 \leq j < k \).

**Proof.** Straightforward check.

**Proposition 2.6.** Let \( M \) be a left \( \mathcal{H}_k \)-module. For \( 1 \leq j \leq k \), let \( t_j : F(L, M) \rightarrow F(L, M) \) be the shift operator
\[
(t_j f)(x) := f(x - v_j) \quad (f \in F(L, M), x \in L).
\]
Then the assignments
\[
X_j \mapsto t_j \quad (1 \leq j \leq k), \quad T_j \mapsto \hat{I}_j s_j + \hat{I}_j \quad (1 \leq j < k)
\]
uniquely extend to a \( \mathbb{C} \)-algebra homomorphism \( \rho : A_k \rightarrow \text{End}(F(L, M)) \).

**Proof.** Note that the operators \( \hat{I}_j \) \( (1 \leq j < k) \) commute with \( \hat{I}_j \) \( (1 \leq j < k) \), and
\[
(e^{-v_j}P, f) = (P, t_j f).
\]
for \( P \in \mathbb{C}[L] \) and \( f \in F(L, M) \). Now we can check the defining relations of \( A_k \) by using Lemma 2.5 and the equality
\[
\hat{I}_j(e^{-v_j}P) - e^{-v_j} \hat{I}_j(e^x) = ((1 - q) e^{-v_j} + \alpha) e^x = \hat{I}_j(e^{x - v_j}) - e^{-v_j} \hat{I}_j(e^x)
\]
for \( 1 \leq j < k \) and \( x \in L \).

We will often use the fact below which follows from (2.4).

**Proposition 2.7.** Let \( M \) be a left \( \mathcal{H}_k \)-module. Suppose that \( 1 \leq j < k \), \( x \in L \) and \( f \in F(L, M) \). If \( a_j(x) = 0 \), then \( (\rho(T_j)f)(x) = T_j f(x) \).

### 3. Discrete Hamiltonian

3.1. **Discrete Hamiltonian and Propagation operator.** For \( 1 \leq i \leq k \) and \( x \in L \), we define \( T_i^{(\pm)}(x) \in \mathcal{H}_k \) by
\[
T_i^{(-)}(x) := T_{w_x}^{-1} \left( T_{\sigma_x(i)-1}^{-1} \cdots T_{\sigma_x(i)-d_i}^{-1} x \right) \left( T_{\sigma_x(i)-d_i}^{-1} \cdots T_{\sigma_x(i)-1}^{-1} \right) T_{w_x},
\]
\[
T_i^{(+)}(x) := T_{w_x}^{-1} \sum_{j=\sigma_x(i)}^{\sigma_x(i)+d_i} \left( T_{\sigma_x(i)-1}^{-1} \cdots T_{\sigma_x(j)-1}^{-1} \right) T_{\sigma_x(j)-1}^{-1} \left( T_{j-1} \cdots T_{\sigma_x(i)-1} \right) T_{w_x}.
\]
Definition 3.1. Let $M$ be a left $\mathcal{H}_k$-module. We define the discrete Hamiltonian $H : F(L, M) \rightarrow F(L, M)$ by

$$(Hf)(x) := \sum_{i=1}^{k} q_{d_i}^{d_i}(x) T_i(x) \left( f(x - v_i) - \alpha T_i(x).f(x) \right) \quad (f \in F(L, M), x \in L),$$

where $\cdot$ means the left action of $\mathcal{H}_k$ on $M$.

Next we define the propagation operator. Let $w = s_{i_1} \cdots s_{i_l}$ be a reduced expression of $w \in W$. Then we set $T_w := T_{i_1} \cdots T_{i_l}$.

Definition 3.2. Let $M$ be a left $\mathcal{H}_k$-module. We define the propagation operator $G : F(L, M) \rightarrow F(L, M)$ by

$$(Gf)(x) := (\rho(T_w) f)(w_x) \quad (f \in F(L, M), x \in L),$$

where $\cdot$ means the action of $\mathcal{H}_k$ on $M$.

Theorem 3.3. It holds that $HG = G(\sum_{i=1}^{k} t_i)$. Therefore if $f$ is an eigenfunction of $\sum_{i=1}^{k} t_i$, then $G(f)$ is that of the discrete Hamiltonian $H$ with the same eigenvalue.

Proof. Suppose that $f \in F(L, M)$ and $x \in L$. First we fix $i$ ($1 \leq i \leq k$) and calculate $(Gf)(x - v_i)$. Set $y = x - v_i$. From Proposition 2.3 we have

$$(3.1) \quad T^{-1}_{w_y} = T^{-1}_{w_x} (T^{-1}_{\sigma_x(i)} \cdots T^{-1}_{\sigma_x(i)+d_i^+(x)-1})(T_{\sigma_y(i)-d_i^-(y)} \cdots T_{\sigma_y(i)-1}).$$

Using Proposition 2.7 we see that

$$(Gf)(x - v_i) = T^{-1}_{w_x} (T^{-1}_{\sigma_x(i)} \cdots T^{-1}_{\sigma_x(i)+d_i^+(x)-1}) \left( \rho(T_{\sigma_x(i)+d_i^+(x)-1} \cdots T_{\sigma_x(i)}) f \right)(w_y).$$

From (2.3) it holds that $w_y = w_x x - v_{\sigma_x(i)+d_i^+(x)}$. Therefore

$$(Gf)(x - v_i) = T^{-1}_{w_x} (T^{-1}_{\sigma_x(i)} \cdots T^{-1}_{\sigma_x(i)+d_i^+(x)-1}) \left( \rho(X_{\sigma_x(i)+d_i^+(x)} T_{\sigma_x(i)+d_i^+(x)-1} \cdots T_{\sigma_x(i)}) f \right)(w_x).$$

Move $X_{\sigma_x(i)+d_i^+(x)}$ to the right using the relation $X_{j+1} T_j = T_j X_j + \alpha + (1 - q) X_{j+1}$, and use Proposition 2.7 to remove $\rho(T_j)$ ($\sigma_x(i) \leq j < \sigma_x(i) + d_i^+(x)$). As a result we get

$$(3.2) \quad (Gf)(x - v_i) = T^{-1}_{w_x} (\rho(X_{\sigma_x(i)} T_{w_x}) f)(w_x) + \alpha T_i^{(+)}(x).G(f)(x)$$

$$+ (1 - q) \sum_{j=\sigma_x(i)}^{\sigma_x(i)+d_i^+(x)-1} T^{-1}_{w_x} (T^{-1}_{\sigma_x(i)} \cdots T^{-1}_{j-1} T^{-1}_{j-1} \cdots T_{\sigma_x(i)}) (\rho(X_{j+1} T_{w_x}) f)(w_x).$$
Now let us calculate \((HG(f))(x)\). The set \(\{1, 2, \ldots, k\}\) is decomposed into a direct sum of sub-intervals \(J_1, \ldots, J_r\) so that \(i \in J_a\) and \(j \in J_a\) if and only if \(\epsilon_i(w_x) = \epsilon_j(w_x)\) for \(a = 1, 2, \ldots, r\). For an interval \(J = \{l, l+1, \ldots, m\}\), set

\[
K_J := \sum_{i=l}^{m} q^{i-l}(T_{i-1}^{-1} \cdots T_{l}^{-1})(T_{l}^{-1} \cdots T_{i-1}^{-1}) \\
\times \left\{ X_i + (1 - q) \sum_{j=l}^{m-1} (T_{j-1}^{-1} \cdots T_{i}^{-1}) T_{j}^{-1} (T_{j-1} \cdots T_{i}) X_{j+1} \right\}
\]

Change the index \(i\) in the definition of \(H\) to \(\sigma_x(i)\) and rewrite \((HG(f))(x)\) using (2.1) and (3.2). Then we obtain

\[
(HG(f))(x) = \sum_{a=1}^{r} T_{w_x}^{-1} \cdot (\rho(K_{J_a} T_{w_x})f)(w_x).
\]

On the other hand, \(K_J\) is rewritten as

\[
K_J = \sum_{i=l}^{m} \left\{ q^{i-l}(T_{i-1}^{-1} \cdots T_{l}^{-1})(T_{l}^{-1} \cdots T_{i-1}^{-1}) \\
+ (1 - q) \sum_{j=l}^{i-1} q^{j-l}(T_{j-1}^{-1} \cdots T_{i}^{-1}) T_{j}^{-1} (T_{j-1} \cdots T_{i}) \right\} X_i.
\]

From the above expression, we see that \(K_J = \sum_{i=l}^{m} X_i \) using the relation \(qT_j^{-1} + (1 - q) = T_j\). Thus we get

\[
(HG(f))(x) = T_{w_x}^{-1} \cdot (\rho(\sum_{i=1}^{k} X_i T_{w_x})f)(w_x).
\]

Since \(\sum_{i=1}^{k} X_i \) commutes with \(T_i\) (\(1 \leq i < k\)), the right hand side is equal to

\[
T_{w_x}^{-1} \cdot (\rho(T_{w_x})\rho(\sum_{i=1}^{k} X_i f)(w_x) = (G(\sum_{i=1}^{k} t_i f))(x).
\]

This completes the proof.

\[\square\]

3.2. Invariant subspace.

Definition 3.4. Let \(M\) be a left \(\mathcal{H}_k\)-module. We denote by \(F_0(L, M)\) the subspace of \(F(L, M)\) consisting of the functions \(f: L \to M\) satisfying

\[
f(s_i x) = \begin{cases} T_{i}^{-1} f(x) & (a_i(x) \geq 0) \\ T_{i} f(x) & (a_i(x) < 0) \end{cases}
\]

for \(1 \leq i < k\) and \(x \in L\).

Note that if \(f \in F_0(L, M)\) then

\[
f(x) = T_{w_x}^{-1} f(w_x)
\]

for \(x \in L\).
In this subsection we prove the following theorem.

**Theorem 3.5.** Let $M$ be a left $\mathcal{H}_k$-module. Then it holds that $H(F_0(L, M)) \subset F_0(L, M)$.

For that purpose, we rewrite the formula of $Hf (f \in F_0(L, M))$. For $x \in L_+$, we define the *cluster coordinate* $(c_1, c_2, \ldots, c_r)$ of $x$ by the property that $\sum_{a=1}^r c_a = k$, $\epsilon_{c_1}(x) > \epsilon_{c_1+c_2}(x) > \cdots > \epsilon_{c_1+\cdots+c_r}(x)$ and $\epsilon_j(x) = \epsilon_{c_1+\cdots+c_r}(x)$ if $c_1 + \cdots + c_{r-1} < j \leq c_1 + \cdots + c_r$. For example, if $k = 5$ and $x = 2v_1 + 2v_2 - v_3 - 3v_4 - 3v_5$, then the cluster coordinate of $x$ is $(2, 1, 2)$.

**Proposition 3.6.** Let $M$ be a left $\mathcal{H}_k$-module. Suppose that $f \in F_0(L, M)$ and $x \in L$. Let $(c_1, \ldots, c_r)$ be the cluster coordinate of $w_xx$. Then it holds that

$$
(3.4) \quad (Hf)(x) = T_{w_1}^{-1} \sum_{a=1}^r \sum_{l=1}^{c_a} q^{c_a-l} (T_{c_1+\cdots+c_{a-1}+l}^{-1} \cdots T_{c_1+\cdots+c_{a-1}}^{-1}).f(w_xx - v_{c_1+\cdots+c_a})
$$

$$
- \frac{\alpha}{1-q} \sum_{a=1}^r (c_a - [c_a]_q) f(x),
$$

where $[n]_q := (1-q^n)/(1-q)$ is the $q$-integer.

In the proof of Proposition 3.6 we use the following formula.

**Lemma 3.7.** Suppose that $f \in F_0(L, M), x \in L_+$ and $0 \leq p < p + c \leq k$. If $a_j(x) = 0$ for $p + 1 \leq j < p + c$, then it holds that

$$
(3.5) \quad \sum_{l=1}^{c} q^{l-1} (T_{p+l-1}^{-1} \cdots T_{p+1}^{-1}).f(x - v_{p+l})
$$

$$
= \sum_{l=1}^{c} q^{c-l} (T_{p+l}^{-1} T_{p+l+1}^{-1} \cdots T_{p+c-1}^{-1}).f(x - v_{p+c}).
$$

**Proof.** For $p + 1 \leq j < m \leq p + c$, it holds that $T_j.f(x - v_j) = f(x - v_{j+1})$ and $T_m.f(x - v_j) = f(x - v_j)$ because $a_j(x - v_j) = -1 < 0$ and $a_m(x - v_j) = 0$. Using $qT_j^{-1} = T_j - (1-q)$ repeatedly, we see that

$$
q^{l-1} (T_{p+l-1}^{-1} \cdots T_{p+1}^{-1}).f(x - v_{p+l}) = f(x - v_{p+l}) - (1-q) \sum_{j=1}^{l-1} q^{l-j-1} f(x - v_{p+j}).
$$

Hence the left hand side of (3.5) is equal to $\sum_{l=1}^{c} q^{c-l} f(x - v_{p+l})$. Since $a_j(x - v_{j+1}) = 1 \geq 0$ for $p + 1 \leq j < p + c$, we have $f(x - v_{p+l}) = (T_{p+l}^{-1} \cdots T_{p+c-1}^{-1}).f(x - v_{p+c})$ for $1 \leq l \leq c$. This completes the proof.

**Proof of Proposition 3.6.** Note that $T_i.f(x) = f(x)$ if $a_i(x) = 0$. Using (2.3) and (3.3), we see that

$$
\sum_{i=1}^{k} q^{d_i}(x) T_i^{(-)}(x) T_i^{(+)}(x).f(x) = \sum_{i=1}^{k} q^{d_i}(x) d_i^+(x).f(x) = \frac{1}{1-q} \sum_{a=1}^{r} (c_a - [c_a]_q) f(x).
$$
Hence it suffices to show that

\[
(3.6) \quad \sum_{i=1}^{k} q^{d_i} (x) T_i^{(-)}(x).f(x - v_i)
\]

\[
= T_{w_x}^{-1} \sum_{a=1}^{r} \sum_{i=1}^{c_a} q^{c_a-i} T_{c_1+c_2+\cdots+c_a+l-1}^{(-)} \cdot T_{c_1+c_2+\cdots+c_a-1}^{(-)} \cdot f(w_x x - v_{c_1+\cdots+c_a}).
\]

Fix \(1 \leq i \leq k\) and set \(y = x - v_i\). From (3.1) and (3.3) we have

\[
f(y) = T_{w_x}^{-1} (T_{\sigma_x(i)} \cdots T_{\sigma_x(i)+d_i^{-1}(x)-1}) f(w_y y).
\]

Therefore

\[
T_i^{(-)}(x).f(y) = T_{w_x}^{-1} (T_{\sigma_x(i)-1} \cdots T_{\sigma_x(i)-d_i^{-1}(x)} (T_{\sigma_x(i)-d_i^{-1}(x)} \cdots T_{\sigma_x(i)+d_i^{+}(x)-1}) \cdot f(w_x x - v_{\sigma_x(i)-d_i^{-1}(x)}).
\]

Since \(w_x y = w_x x - v_{\sigma_x(i)+d_i^{+}(x)}(x)\) and \(a_j(w_x x) = 0\) for \(\sigma_x(i) - d_i^{-1}(x) \leq j < \sigma_x(i) + d_i^{+}(x)\), the right hand side above is equal to

\[
T_{w_x}^{-1} (T_{\sigma_x(i)-1} \cdots T_{\sigma_x(i)-d_i^{-1}(x)}) \cdot f(w_x x - v_{\sigma_x(i)-d_i^{-1}(x)}).\]

Note that if \(c_1+\cdots+c_a-1 < \sigma_x(i) \leq c_1+\cdots+c_a\), then \(\sigma_x(i) - d_i^{-1}(x) = c_1+\cdots+c_a-1\), which is independent of \(i\). Thus the left hand side of (3.6) is equal to

\[
T_{w_x}^{-1} \sum_{a=1}^{r} \sum_{i=1}^{c_a} q^{c_a-i} (T_{c_1+c_2+\cdots+c_a+l-1}^{(-)} \cdot T_{c_1+c_2+\cdots+c_a-1}^{(-)} \cdot f(w_x x - v_{c_1+\cdots+c_a-1+1}).
\]

Now the equality (3.6) is an immediate consequence of Lemma 3.7. \(\square\)

Now let us prove Theorem 3.5.

**Proof of Theorem 3.5.** Suppose that \(f \in F_0(L, M)\), \(x \in L_+\) and \(1 \leq i < k\). Note that \(w_{s_i(x)} = w_x x\) and hence the cluster coordinates of \(x\) and \(s_i(x)\) are the same.

If \(a_i(x) > 0\), then \(T_{w_{s_i(x)}} = T_{w_x} T_i\) from Proposition 2.2. Using (3.4), we see that \((Hf)(s_i(x)) = T_i^{-1} (Hf)(x)\). Changing \(x\) to \(s_i x\), we find that \((Hf)(s_i x) = T_i (Hf)(x)\) if \(a_i(x) < 0\). Let us consider the case where \(a_i(x) = 0\). From Proposition 2.2, it holds that \(T_i^{-1} T_{w_x} = T_{w_x} T_{s_i(x)}\). Note that \(d_{\sigma_x(i)}(w_x x) = d_i^{+}(x) > 0\) because \(\epsilon_{i+1(x)} = \epsilon_i(x)\). Hence there exists \(1 \leq a \leq r\) such that \(c_1+\cdots+c_a-1 < \sigma_x(i) < c_1+\cdots+c_a\). Now Lemma 3.8 below implies that \(T_i^{-1} (Hf)(x) = (Hf)(x)\). \(\square\)

**Lemma 3.8.** Suppose that \(x \in L_+, f \in F_0(L, M)\) and \(0 \leq p < p + c \leq k\). Set

\[
J = \sum_{i=1}^{c} q^{c-i} (T_{p+i}^{-1} T_{p+i+1}^{(-)} \cdots T_{p+c-1}^{(-)}).
\]

If \(a_j(x) = 0\) for \(p+1 \leq j < p+c\), then

\[
T_{p+i}^{-1} J.f(x - v_{p+c}) = J.f(x - v_{p+c})
\]

for \(1 \leq i \leq c-1\).
Proof. Using the quadratic relation \(q^{-1}T_j^{-2} = 1 - (1 - q)T_j^{-1}\), we have

\[
T_{p+i}^{-1}J = \sum_{l=1}^{i-1} q^{-l}(T_{p+l}^{-1} \cdots T_{p+c-1}^{-1})T_{p+i-1}^{-1} + \sum_{l=i,i+1}^{c} q^{-l}(T_{p+l}^{-1} \cdots T_{p+c-1}^{-1})
\]

\[
+ \sum_{l=i+2}^{c} q^{-l}(T_{p+l}^{-1} \cdots T_{p+c-1}^{-1})T_{p+i}^{-1}.
\]

Note that the first and the third term in the right hand side vanish if \(i = 1\) and \(i = c - 1\), respectively. Since \(T_j^{-1}f(x - v_{p+c}) = f(x - v_{p+c})\) for \(p + 1 \leq j \leq p + c - 2\), we obtain the desired formula. \(\square\)

3.3. Bethe wave functions. Let \(M\) be a left \(H_k\)-module. We construct eigenfunctions of the restriction \(H|_{F_0(L,M)}\), which we call the Bethe wave functions.

Denote by \(F(L,M)^{H_k}\) the subspace of \(F(L,M)\) consisting of the \(\rho(H_k)\)-invariant functions, that is,

\[
F(L,M)^{H_k} := \{f \in F(L,M) \mid \rho(T_i)f = f \text{ for } 1 \leq i < k\}.
\]

**Proposition 3.9.** Let \(M\) be a left \(H_k\)-module. It holds that \(G(F(L,M)^{H_k}) \subset F_0(L,M)\). Therefore if \(h \in F(L,M)^{H_k}\) is an eigenfunction of \(\sum_{i=1}^{k} t_i\), then \(G(h)\) is that of the operator \(H|_{F_0(L,M)}\) with the same eigenvalue.

**Proof.** Let \(f\) be a function which belongs to \(F(L,M)^{H_k}\). From the definition of the propagation operator, we see that \((Gf)(x) = T_{w_x}^{-1}f(w_x)\).

Suppose that \(x \in L\) and \(1 \leq i < k\). If \(a_i(x) > 0\), we have \(T_{w_{s_i}x} = T_{w_x}T_i\) by Proposition 2.2. Hence \((Gf)(s_i x) = T_i^{-1}(Gf)(x)\) because \(w_{s_i}x = w_x\). This also implies that \((Gf)(s_i x) = T_i(Gf)(x)\) if \(a_i(x) < 0\).

Let us consider the case where \(a_i(x) = 0\). Then we have \(\sigma_x(i + 1) = \sigma_x(i) + 1\) and \(T_i^{-1}T_{w_x}^{-1} = T_{w_{s_i}x}^{-1}T_{w_{s_i}x}^{-1}\) by Proposition 2.2. Now note that

\[
a_{\sigma_x(i)}(w_x) = \epsilon_{\sigma_x(i)}(w_x) - \epsilon_{\sigma_x(i+1)}(w_x) = \epsilon_i(x) - \epsilon_{i+1}(x) = a_i(x) = 0.
\]

Hence we find that \(T_{\sigma_x(i)}^{-1}f(w_x) = (\rho(T_{\sigma_x(i)}^{-1})f(w_x) = f(w_x)\) from Proposition 2.7. Therefore \(T_{\sigma_x(i)}^{-1}(Gf)(x) = (Gf)(x)\). \(\square\)

For \(1 \leq i < k\) and \(\lambda \in V^*\), we define \(Y_i(\lambda) \in H_k\) by

\[
Y_i(\lambda) := \frac{(e^{\lambda(v_i)} - e^{\lambda(v_{i+1})})T_i - e^{\lambda(v_i)}(\alpha e^{\lambda(v_{i+1})} + 1 - q)}{\alpha e^{\lambda(v_{i+1})} + e^{\lambda(v_i)} - qe^{\lambda(v_{i+1})}}.
\]

**Lemma 3.10.** The following equalities hold.

(i) \(Y_i(s_i \lambda)Y_i(\lambda) = 1\) for \(1 \leq i < k\) and \(\lambda \in V^*\).

(ii) \(Y_i(s_{i+1} \lambda)Y_i(s_i \lambda)Y_{i+1}(\lambda) = Y_i(s_{i+1}s_i \lambda)Y_{i+1}(s_i \lambda)Y_i(\lambda)\) for \(1 \leq i \leq k - 2\) and \(\lambda \in V^*\).

**Proof.** By a direct computation. \(\square\)
From Lemma 3.10, we obtain the following proposition.

**Proposition 3.11.** Suppose that \( \lambda \in V^* \). There exists a unique \( \mathbb{C} \)-algebra homomorphism \( \phi_\lambda : \mathbb{C}[W] \to \mathcal{H}_k \) such that \( \phi_\lambda(1) = 1 \) and

\[
\phi_\lambda(s_i w) = Y_i(w \lambda) \phi_\lambda(w)
\]

for \( 1 \leq i < k \) and \( w \in W \).

**Theorem 3.12.** Let \( M \) be a left \( \mathcal{H}_k \)-module. For \( \lambda \in V^* \) and \( m \in M \), we define a function \( h^m_\lambda \in F(L, M) \) by

\[
h^m_\lambda(x) := \sum_{w \in W} e^{(w \lambda)(x)}(\phi_\lambda(w)m) \quad (x \in L),
\]

where \( . \) means the left action of \( \mathcal{H}_k \) on \( M \). Then the function \( h^m_\lambda \) belongs to \( F(L, M)^{\mathcal{H}_k} \) and is an eigenfunction of \( \sum_{i=1}^{k} t_i \) with eigenvalue \( \sum_{i=1}^{k} e^{-\lambda(v_i)} \). Therefore \( G(h^m_\lambda) \) is an eigenfunction of \( H|_{F_0(L, M)} \).

**Proof.** It is clear that \( h^m_\lambda \) is an eigenfunction of \( \sum_{i=1}^{k} t_i \). From the relation (3.7), we have

\[
T_j \phi_\lambda(w) = \frac{e^{w \lambda(v_j)}(\alpha e^{w \lambda(v_{j+1})} + 1 - q)}{e^{w \lambda(v_j)} - e^{w \lambda(v_{j+1})}} \phi_\lambda(w) + \frac{\alpha e^{w \lambda(v_j + v_{j+1})} + e^{w \lambda(v_j)} - q e^{w \lambda(v_{j+1})}}{e^{w \lambda(v_j)} - e^{w \lambda(v_{j+1})}} \phi_\lambda(s_j w)
\]

for \( 1 \leq j < k \) and \( w \in W \). Moreover, from the definition of \( \hat{I}_j \), we see that

\[
\hat{I}_j(e^\mu) = \frac{\alpha e^{\mu(v_j + v_{j+1})} + (1 - q)e^{\mu(v_j)}}{e^{\mu(v_j)} - e^{\mu(v_{j+1})}} (e^\mu - s_j e^\mu) \quad (\mu \in V^*).
\]

Combining the above equalities, we find that \( \rho(T_i) h^m_\lambda = h^m_\lambda \) for \( 1 \leq i < k \).

Let \( M \) be a left \( \mathcal{H}_k \)-module. Set

\[
\mathcal{F}(L_+, M) := \{ f : L_+ \to M \mid T_i f(x) = f(x) \text{ if } a_i(x) = 0 \}.
\]

We identify \( F_0(L, M) \) with \( \mathcal{F}(L_+, M) \) by the map \( f \mapsto f|_{L_+} (f \in F_0(L, M)) \). Denote by \( H^+ \) the restriction of the discrete Hamiltonian \( H \) to \( \mathcal{F}(L_+, M) \). Proposition 3.6 implies that the operator \( H^+ \) is given by

\[
(H^+ f)(x) = \sum_{a=1}^{r} \sum_{l=1}^{c_a} q^{c_a-l} T_{c_1+\cdots+c_{a-1}+l}^{-1} T_{c_1+\cdots+c_{a-1}+l+1}^{-1} \cdots T_{c_1+\cdots+c_{a-1}+l+r}^{-1} f(x - v_{c_1+\cdots+c_a})

- \frac{\alpha}{1 - q} \sum_{a=1}^{r} (c_a - [c_a]_{q}) f(x) \quad (f \in \mathcal{F}(L_+, M), x \in L_+),
\]

where \( (c_1, c_2, \ldots, c_r) \) is the cluster coordinate of \( x \).
Corollary 3.13. Let $M$ be a left $\mathcal{H}_k$-module. Suppose that $\lambda \in V^*$ and $m \in M$, and consider the function $h_\lambda^m$ defined by (3.8). Then $h_\lambda^m|_{L_+}$ belongs to $\mathcal{F}(L_+, M)$ and is an eigenfunction of $H^+$ with eigenvalue $\sum_{i=1}^k e^{-\lambda(v_i)}$.

Proof. It follows from Theorem 3.12 and the equality $G(h_\lambda^m)|_{L_+} = h_\lambda^m|_{L_+}$. \hfill $\square$

4. Algebraic construction of multi-species $q$-Boson system

4.1. Setting. In the rest of this article we fix a positive integer $N$. For a positive integer $c$, set

$$I_{N,c} := \{1, 2, \ldots, N\}^c,$$

$$I_{N,c}^+ := \{(\mu_1, \ldots, \mu_c) \in I_{N,c} \mid \mu_1 \leq \cdots \leq \mu_c\}.$$

Let $x \in L_+$ and $(c_1, \ldots, c_r)$ be the cluster coordinate of $x$. For $\mu \in I_{N,k}$, we define $\mu[x] \in I_{N,k}$ as follows. According to the decomposition $I_{N,k} = I_{N,c_1} \times \cdots \times I_{N,c_r}$, we write $\mu = (\mu_1, \ldots, \mu_r)$, where $\mu_a \in I_{N,c_a}$ ($1 \leq a \leq r$). Then let $\mu_a^+$ be the unique element of $I_{N,c_a}$ which is a rearrangement of $\mu_a$. Now set $\mu[x] := (\mu_1^+, \ldots, \mu_r^+)$. For example, if $k = 5$, $x = 2v_1 + 2v_2 - v_3 - v_4 - v_5$ and $\mu = (3, 1, 4, 2, 5)$, then $(c_1, c_2) = (2, 3), \mu_1 = (3, 1), \mu_2 = (4, 2, 5)$, and $\mu[x] = (1, 3, 2, 4, 5)$.

Set

$$S := \{(x, \nu) \in L_+ \times I_{N,k} \mid \nu = \nu[x]\}.$$

We identify $S$ with the set of configurations of $k$ bosonic particles of $N$ species on the one-dimensional lattice $\mathbb{Z}$ as follows. For $x = \sum_{i=1}^k m_i v_i \in L_+$ and $\nu = (v_1, \ldots, v_k)$, we assign to $(x, \nu)$ the configuration such that the particles with the color $v_1, \ldots, v_k$ are on the sites $m_1, \ldots, m_k$, respectively. For example, if $k = 6, N = 4, x = 2v_1 + 2v_2 - v_3 - 3v_4 - 3v_5 - 3v_6$ and $\nu = (1, 2, 4, 2, 2, 3)$, then $(x, \nu)$ corresponds to the configuration in Figure 1.

Denote the set of $\mathbb{R}$-valued functions on $S$ by $F(S)$. In the rest of this paper we construct the transition rate matrix $Q : F(S) \to F(S)$ of a continuous-time Markov process on $S$. 

\begin{figure}[h]
\centering
\begin{tikzpicture}
\node (1) at (0,0) {1};
\node (2) at (0,-1) {2};
\node (3) at (-0.5,-2) {3};
\node (4) at (0.5,-2) {4};
\node (5) at (1,-2) {2};
\draw [-stealth] (1) -- (2);
\draw [-stealth] (2) -- (3);
\draw [-stealth] (2) -- (4);
\draw [-stealth] (4) -- (5);
\end{tikzpicture}
\caption{Figure 1.}
\end{figure}
4.2. Derivation of the transition rate matrix. Hereafter we assume that

\[ 0 < q < 1. \]

Let \( U = \bigoplus_{\mu=1}^{N} \mathbb{C} u_\mu \) be the \( N \)-dimensional vector space with the basis \( \{ u_\mu \}_{\mu=1}^{N} \). Consider the \( \mathbb{C} \)-linear operator \( R \in \text{End}(U \otimes U) \) defined by

\[
R(u_\mu \otimes u_\mu') = \begin{cases} 
q^{1/2} u_\mu' \otimes u_\mu & (\mu > \mu'), \\
q^{1/2} (1-q) u_\mu \otimes u_\mu' + q^{1/2} u_\mu' \otimes u_\mu & (\mu < \mu').
\end{cases}
\]

Let \( R_i \in \text{End}(U \otimes U) (1 \leq i < k) \) be the linear operator acting as \( R \) on the tensor product of the \( i \)-th and \((i+1)\)-th component of \( U \otimes U \).

Theorem 4.1. \[ 7 \] The assignment \( T_i \mapsto R_i (1 \leq i < k) \) uniquely extends to a \( \mathbb{C} \)-algebra homomorphism \( H_k \to \text{End}(U \otimes U) \).

In the following we regard \( U \otimes U \) as a left \( H_k \)-module with respect to the action defined above.

For \( \mu = (\mu_1, \ldots, \mu_k) \in I_{N,k} \), we set

\[ u_\mu := u_{\mu_1} \otimes \cdots \otimes u_{\mu_k} \in U \otimes U, \]

and

\[
\ell(\mu) := \# \{ (i, j) \mid 1 \leq i < j \leq k \text{ and } \mu_i > \mu_j \}. \]

Then any function \( f : L_+ \to U \otimes U \) is uniquely written in the form

\[ f(x) = \sum_{\mu \in I_{N,k}} q^{\ell(\mu)/2} f_\mu(x) u_\mu \quad (x \in L_+), \]

where \( f_\mu \) is a \( \mathbb{C} \)-valued function on \( L_+ \). Then the space \( \mathcal{F}(L_+, U \otimes U) \) has the following description.

Proposition 4.2. In the above notation the following statements are equivalent.

(i) \( f \in \mathcal{F}(L_+, U \otimes U) \).

(ii) Suppose that \( x \in L_+ \) and \( 1 \leq i < k \). If \( a_i(x) = 0 \) then \( f(x) = f_{\mu_1,\mu_2,\ldots}(x) \) for all \( \mu = (\mu_1, \ldots, \mu_k) \in I_{N,k} \).

Proof. It follows from the definition of the operator \( R \) and (4.1) by a direct computation. \( \square \)

We define the \( \mathbb{C} \)-linear map \( \varphi : F(S) \to \mathcal{F}(L_+, U \otimes U) \) by

\[ (\varphi h)(x) := \sum_{\mu \in I_{N,k}} q^{\ell(\mu)/2} h(x, \mu[x]) u_\mu \quad (h \in F(S), x \in L_+). \]

Proposition 4.2 implies that the function defined by the right hand side above belongs to \( \mathcal{F}(L_+, U \otimes U) \). The map \( \varphi \) is an isomorphism with the inverse

\[ (\varphi^{-1} f)(x, \nu) = f_\nu(x) \quad (f \in \mathcal{F}(L_+, U \otimes U), (x, \nu) \in S), \]
where \( f_\nu \) is defined by (4.1).

Now let us write down the operator \( \varphi^{-1}H^+\varphi : F(S) \to F(S) \). For \( 1 \leq c \leq k \), we set

\[
A^{(c)} := \sum_{l=1}^{c} q^{c-l}T_l^{-1} T_{l+1}^{-1} \cdots T_{c-1}^{-1} \in \mathcal{H}_k.
\]

It acts on \( U^\otimes c \). We set the matrix element \( A^{(c)}_{\mu,\mu'} \in \mathbb{R} \) by

\[
A^{(c)}_{\mu,\mu'} \in U^\otimes c.
\]

Suppose that \( h \in F(S) \) and \( (x, \nu) \in S \). Denote by \((c_1, \ldots, c_r)\) the cluster coordinate of \( x \), and decompose \( \nu = (\nu_1, \ldots, \nu_r) \), where \( \nu_a \in I_{N,c_a}^+ \) for \( 1 \leq a \leq r \). From (3.9) we see that

\[
(4.2)
(\varphi^{-1}H^+\varphi h)(x, \nu) = \sum_{a=1}^{r} \sum_{\eta \in I_{N,c_a}} q^{\ell(\eta)/2} A^{(c_a)}_{\eta,\nu}(\varphi h)(\nu_1, \ldots, \nu_{a-1}, \eta, \nu_{a+1}, \ldots, \nu_r)(x - v_{c_1+\ldots+c_a})
- \frac{\alpha}{1-q} \sum_{a=1}^{c_a} (c_a - [c_a]q) h(x, \nu).
\]

**Proposition 4.3.** Suppose that \( 1 \leq c \leq k \) and

\[
(4.3)
\nu = \left(\underbrace{1, \ldots, 1}_{m_1}, \ldots, \underbrace{N, \ldots, N}_{m_N}\right) \in I_{N,c}^+,
\]

where \( m_1, \ldots, m_N \) are non-negative integers satisfying \( \sum_{i=1}^{N} m_i = c \). Then the matrix element \( A^{(c)}_{\eta,\nu} \) is zero unless \( \eta \) is of the form

\[
(4.4)
\left(\underbrace{1, \ldots, 1, b, \ldots, b}_{m_1}, \underbrace{N, \ldots, N}_{m_N}\right)
for some \( 1 \leq b \leq N \). If \( \eta \) is equal to (4.4), then

\[
A^{(c)}_{\eta,\nu} = \frac{1 - q^{m_b}}{1-q} q^{\sum_{i=b+1}^{N} m_i/2} = \frac{1 - q^{m_b}}{1-q} q^{\ell(\eta)/2}.
\]

**Proof.** For \( n \geq 1 \), let \((, ,)\) be the non-degenerate bilinear form on \( U^\otimes n \) defined by

\[
(u_{\mu_1} \otimes \cdots \otimes u_{\mu_n}, u_{\nu_1} \otimes \cdots \otimes u_{\nu_n}) = \delta_{\mu_1\nu_1} \cdots \delta_{\mu_n\nu_n}.
\]

Consider the linear operator \( S \in \text{End}(U^\otimes 2) \) defined by

\[
S(u_{\mu} \otimes u_{\mu'}) = \begin{cases}
(1 - q^{-1})u_{\mu} \otimes u_{\mu'} + q^{-1/2}u_{\mu'} \otimes u_{\mu} & (\mu > \mu'), \\
u_{\mu} \otimes u_{\mu} & (\mu = \mu'), \\
q^{-1/2}u_{\mu'} \otimes u_{\mu} & (\mu < \mu').
\end{cases}
\]
Then it holds that \((R^{-1}u, u') = (u, Su')\) for \(u, u' \in U^\otimes 2\). Let \(S_i \in \text{End}(U^\otimes k)\) \((1 \leq i < k)\) be the linear operator acting as \(S\) on the tensor product of the \(i\)-th and \((i+1)\)-th component of \(U^\otimes k\). Then we have

\[
A^{(c)}_{\eta, \nu} = (A^{(c)} u_\eta, u_\nu) = \sum_{i=1}^{c} q^{c-i} (u_\eta, S_{c-1} \cdots S_i u_\nu).
\]

From the definition of \(S\), we see that

\[
S_{c-1} \cdots S_i u_\nu = q^{-\sum_{i=b+1}^{N} m_i/2} u_\nu^{(b)},
\]

where \(b\) is determined by the condition \(m_1 + \cdots + m_{b-1} < l \leq m_1 + \cdots + m_b\) and \(\nu^{(b)}\) is the tuple \((4.4)\). Therefore \(A^{(c)}_{\eta, \nu} = 0\) unless \(\eta\) is of the form \((4.4)\). If \(\eta\) is equal to \((4.4)\), it holds that

\[
A^{(c)}_{\eta, \nu} = \sum_{l=m_1+\cdots+m_{b-1}+1}^{m_1+\cdots+m_b} q^{c-l-\sum_{i=b+1}^{N} m_i/2} = q^{\sum_{i=b+1}^{N} m_i/2} \sum_{l=m_1+\cdots+m_{b-1}+1}^{m_1+\cdots+m_b} q^{-l} = q^{\sum_{i=b+1}^{N} m_i/2} \frac{1-q^{m_b}}{1-q}.
\]

Here we used \(c = \sum_{i=1}^{N} m_i\). This completes the proof.

For \(\nu \in I_{N,c}^{+}\) of the form \((4.3)\) and \(1 \leq b \leq N\), we set

\[
\nu^{b, \pm} := (1, \ldots, 1, b, \ldots, b, \ldots, N, \ldots, N) \in I_{N,c, \pm}^{+}.
\]

Let \((x, \nu) \in \mathcal{S}\). Denote by \((c_1, \ldots, c_r)\) the cluster coordinate of \(x\) and write \(\nu = (\nu_1, \ldots, \nu_r)\) where \(\nu_a \in I_{N,c_a}^{+}\) \((1 \leq a \leq r)\). For \((y, \eta) \in \mathcal{S}\), we write \((x, \nu) \rightsquigarrow (y, \eta)\) if the following conditions hold:

(i) \(y = x - v_{c_1+\cdots+c_a}\) for some \(1 \leq a \leq r\).

(ii) If \(c_{c_1+\cdots+c_a}(x) - 1 > c_{c_1+\cdots+c_a+1}(x)\), then

\[
\eta = (\nu_1, \ldots, \nu_{a-1}, \nu_a^{h,-}, b, \nu_{a+1}, \ldots, \nu_r)
\]

for some \(1 \leq b \leq N\). If \(c_{c_1+\cdots+c_a}(x) - 1 = c_{c_1+\cdots+c_a+1}(x)\), then

\[
\eta = (\nu_1, \ldots, \nu_{a-1}, \nu_a^{h,-}, \nu_a^{h,+}, \nu_{a+2}, \ldots, \nu_r)
\]

for some \(1 \leq b \leq N\).

Moreover, when the above conditions are satisfied, we set

\[
c(x, \nu|y, \eta) := \frac{1 - q^{m_b}}{1 - q}\sum_{i=b+1}^{N} m_i,
\]

where \(m_i\) \((1 \leq i \leq N)\) is the number of \(i\) in \(\nu_a\). We set \(c(x, \nu|y, \eta) = 0\) unless \((x, \nu) \rightsquigarrow (y, \eta)\).

Under the identification of \(\mathcal{S}\) with the set of configurations of \(k\) bosonic particles of \(N\) species given in the previous subsection, the condition \((x, \nu) \rightsquigarrow (y, \eta)\) means
that the configuration corresponding to \((y, \eta)\) is obtained from that corresponding to \((x, \nu)\) by moving one particle to the left.

From (4.2) and the equality
\[
\sum_{b=1}^{N} \frac{1 - q^{mb}}{1 - q} q^{\sum_{i=b+1}^{N} m_i} = \frac{1 - q^{\sum_{i=1}^{N} m_i}}{1 - q},
\]
we see that
\[
(\varphi^{-1} H^+ \varphi) h(x, \nu) = \sum_{(y, \eta) \in S} c(x, \nu | y, \eta) \{ h(y, \eta) - h(x, \nu) \}
+ \left( \frac{1 - q + \alpha}{1 - q} \sum_{a=1}^{r} [c_a]_q - \frac{\alpha}{1 - q} k \right) h(x, \nu) \quad (h \in F(S), (x, \nu) \in S),
\]
where \((c_1, \ldots, c_r)\) is the cluster coordinate of \(x\). From the above formula we obtain the following result.

**Theorem 4.4.** Set \(\alpha = -(1 - q)\) and suppose that \(0 < q < 1\). Then the operator
\[
Q := \varphi^{-1} H^+ \varphi - k
\]
gives the transition rate matrix of a continuous-time Markov process on \(S\).

The resulting process is described as follows. In continuous time one particle may move from site \(i\) to \(i - 1\) independently for each \(i \in \mathbb{Z}\). The transition rate at which a particle with color \(b\) moves is given by the right hand side of (4.5), where \(m_i (1 \leq i \leq N)\) is the number of particles with color \(i\) in the cluster from which the moving particle leaves.

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