SCHUR FUNCTION ANALOGS FOR A FILTRATION OF THE SYMMETRIC FUNCTION SPACE

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Abstract. We consider a filtration of the symmetric function space given by $\Lambda_t^{(k)}$, the linear span of Hall-Littlewood polynomials indexed by partitions whose first part is not larger than $k$. We introduce symmetric functions called the $k$-Schur functions, providing an analog for the Schur functions in the subspaces $\Lambda_t^{(k)}$. We prove several properties for the $k$-Schur functions including that they form a basis for these subspaces that reduces to the Schur basis when $k$ is large. We also show that the connection coefficients for the $k$-Schur function basis with the Macdonald polynomials belonging to $\Lambda_t^{(k)}$ are polynomials in $q$ and $t$ with integral coefficients. In fact, we conjecture that these integral coefficients are actually positive, and give several other conjectures generalizing Schur function theory.

1. Introduction

Let $\Lambda$ be the ring of symmetric functions in the variables $x_1, x_2, \ldots$, with coefficients in $\mathbb{Q}(q,t)$, for parameters $q$ and $t$. The Schur functions, $s_\lambda[X]$, form a fundamental basis of $\Lambda$, with central roles in fields such as representation theory and algebraic geometry. For example, the Schur functions can be identified with the characters of irreducible representations of the symmetric group, and their products are equivalent to the Pieri formulas for multiplying Schubert varieties in the intersection ring of a Grassmannian. Furthermore, the connection coefficients of the Schur function basis with various bases such as the homogeneous symmetric functions, the Hall-Littlewood polynomials, and the Macdonald polynomials, are positive and have representation theoretic interpretations. In the case of the Macdonald polynomials, $H_\lambda[X;q,t]$, this expansion takes the form

$$H_\lambda[X;q,t] = \sum_{\mu} K_{\mu\lambda}(q,t) s_\mu[X] , \quad K_{\mu\lambda}(q,t) \in \mathbb{N}[q,t],$$

where $K_{\mu\lambda}(q,t)$ are known as the $q,t$-Kostka polynomials. The representation theoretic interpretation for these polynomials is given in [1, 2, 3].

Here, we consider the filtration $\Lambda_t^{(1)} \subset \Lambda_t^{(2)} \subset \cdots \subset \Lambda_t^{(\infty)} = \Lambda$, given by the linear span of Hall-Littlewood polynomials indexed by $k$-bounded partitions. That is,

$$\Lambda^k_t = \mathcal{L}\{H_\lambda[X;t]\}_{\lambda_1 \leq k} , \quad k = 1, 2, 3, \ldots .$$

We introduce a new family of symmetric functions that are indexed by $k$-bounded partitions, denoted $s^k_\lambda[X;t]$, and prove a number of properties for these functions. In particular, we show that they form bases for the subspaces, $\Lambda^k_t$. Our functions will be called the $k$-Schur functions since they appear to play a role for $\Lambda^k_t$ that is analogous to the role of the Schur functions for $\Lambda$. That is, the $k$-Schur functions give rise to the generalization of many Schur positivity properties. Details are given following a brief outline of our construction for $s^k_\lambda[X;t]$.

The characterization of $s^k_\lambda[X;t]$ relies on a $t$-generalization for Schur function products. More precisely, for any partition sequence $S = (\lambda^{(1)}, \ldots, \lambda^{(\ell)})$, a $t$-analog for the product $s_{\lambda^{(1)}}[X] \cdots s_{\lambda^{(\ell)}}[X]$
was introduced in \[8, 10, 12\]. We first prove that a very particular subset of these generalized products forms a basis for \(\Lambda^{(k)}_t\). The elements of our basis, denoted \(G^{(k)}_\lambda[X;t]\), are the generalized Schur products with sequence \(S\) obtained by splitting \(\lambda\) into pieces that depend on \(k\). \(G^{(k)}_\lambda[X;t]\) are thus called \(k\)-split polynomials. These polynomials are essential in our definition for the \(k\)-Schur functions as we use a linear operator on \(\Lambda^{(k)}_t\) defined by

\[
T_i^{(k)} G^{(k)}_\lambda[X;t] = \begin{cases} G^{(k)}_\lambda[X;t] & \text{if } \lambda_1 = i \\ 0 & \text{otherwise} \end{cases}.
\]  

(1.3)

The final ingredient needed to define the \(k\)-Schur functions is the vertex operator, \(B_i\), introduced in \[4\] to recursively build the Hall-Littlewood polynomials. More precisely, \(H_{\lambda_1, \ldots, \lambda_\ell}[X;t] = B_{\lambda_1} \cdots B_{\lambda_\ell} \cdot 1\). Analogously to this relation, we now define the \(k\)-Schur function for \(k\)-bounded \(\lambda = (\lambda_1, \ldots, \lambda_\ell)\), by

\[ s^{(k)}_\lambda[X;t] = T_{\lambda_1}^{(k)} B_{\lambda_1} \cdots T_{\lambda_\ell}^{(k)} B_{\lambda_\ell} \cdot 1. \]  

(1.5)

Our work to characterize this basis was originally motivated by two conjectures suggesting that the \(k\)-Schur functions play a central role in the understanding of the \(q, t\)-Kostka polynomials. Together, these conjectures refine relation (1.1). That is, for any \(k\)-bounded partition \(\lambda\),

\[ s^{(k)}_\lambda[X;t] = \sum_{\mu \geq \lambda} v^{(k)}_{\mu \lambda}(t) s^{(k)}_\mu[X], \quad v^{(k)}_{\mu \lambda}(t) \in \mathbb{N}[t]. \]  

(1.6)

Both conjectures hold when \(k = 2\) (Section \[3\]) and we shall prove that \(v^{(k)}_{\mu \lambda}(t) \in \mathbb{Z}[t]\) and that \(K^{(k)}_{\mu \lambda}(q,t) \in \mathbb{Z}[q,t]\) for all \(k\). Tables of coefficients \(v^{(k)}_{\mu \lambda}(t)\) and \(K^{(k)}_{\mu \lambda}(q,t)\) are included in section \[8\] to illustrate these conjectures. Our examples suggest a stronger property,

\[ 0 \subseteq K^{(k)}_{\mu \lambda}(q,t) \subseteq K^{(k)}_{\mu \lambda}(q,t) \]  

(1.8)

where for two polynomials \(P, Q \in \mathbb{Z}[t]\), \(P \subseteq Q\) means \(Q - P \in \mathbb{N}[t]\).

More generally, it develops that properties of the \(k\)-Schur functions, with a number of conjectures, provide a \(k\)-generalization for the properties that make the Schur functions important to the theory of symmetric functions. In particular, we prove that the \(s^{(k)}_\lambda[X;t]\) form a basis for \(\Lambda^{(k)}_t\) and that the \(k\)-Schur functions of \(\Lambda^{(\infty)}_t = \Lambda\) are indeed the Schur functions themselves. Conjectural evidence for the significance of the \(k\)-Schur functions includes a \(k\)-analog of partition conjugation and generalizations of Pieri and Littlewood-Richardson rules. Consequently, in the case of the multiplicative action of \(h_1[X]\), a \(k\)-analog of the Young Lattice is induced. Further, we have observed that the \(k\)-Schur functions, expanded in terms of \(k\)-Schur functions in two sets of variables, have coefficients in \(\mathbb{N}[t]\). This is a special property of Schur functions that is not shared by the Hall-Littlewood or Macdonald functions. Finally, the \(k\)-Schur functions of \(\Lambda^{(k)}_t\), when embedded in \(\Lambda^{(k)}_{k'}\) for \(k' > k\), seem to decompose positively in terms of \(k'\)-Schur functions:

\[ s^{(k)}_\lambda[X;t] = s^{(k')}_{\lambda'}[X;t] + \sum_{\mu \geq \lambda} v^{(k-k')}_{\mu \lambda}(t) s^{(k')}_{\mu}[X;t], \quad \text{where } v^{(k-k')}_{\mu \lambda}(t) \in \mathbb{N}[t]. \]  

(1.9)

It happens, remarkably, that not all of the \(k\)-Schur functions need to be constructed using \[10\]. For each \(k\), there is a subset of \(s^{(k)}_\lambda[X;t]\), called the irreducible \(k\)-Schur functions, from which all other \(k\)-Schur functions may be constructed \[8\] by simply applying a succession of certain operators. The elements of this set are the \(k\)-Schur functions indexed by partitions with no more than \(i\) parts equal
to \( k - i \), and the operators are vertex operators \([\mathcal{X}]\) associated to rectangularly shaped partitions \((\ell^{k-i}t)\) for \( t = 1, \ldots, k \). That is,
\[
\begin{align*}
\phi_{\lambda}^{(k)}[X; t] &= t^c B_{R_1} B_{R_2} \cdots B_{R_t} \phi_{\mu}^{(k)}[X; t] \quad \text{with} \quad c \in \mathbb{N},
\end{align*}
\]
for an irreducible \( \phi_{\mu}^{(k)} \), and vertex operators \( B_R \), where \( R \) is a partition of rectangular shape.

Since the Hall-Littlewood polynomials at \( t = 1 \) are the complete symmetric functions
\[
H_{\lambda}[X; 1] = h_{\lambda_1}[X] h_{\lambda_2}[X] \cdots h_{\lambda_n}[X],
\]
we see that \( \Lambda_{\lambda}^{(k)} \) reduces to the polynomial ring \( \Lambda^{(k)} = \mathbb{Q}[h_1, \ldots, h_k] \). Each of the properties held by the \( k \)-Schur functions has a specialization in this subring. In particular, since \( B_R \) is simply multiplication by the Schur function \( s_R \) when \( t = 1 \), relation (1.10) reduces to
\[
\phi_{\lambda}^{(k)}[X] = s_{R_1}[X] s_{R_2}[X] \cdots s_{R_t}[X] \phi_{\mu}^{(k)}[X].
\]
The irreducible \( k \)-Schur functions thus constitute a natural basis for the quotient ring \( \Lambda^{(k)}/\mathcal{I}_k \), where \( \mathcal{I}_k \) is the ideal generated by Schur functions indexed by partitions of the form \((\ell^{k+1-t})\).

This work is closely related to \([3]\), where symmetric functions called atoms are introduced in terms of tableaux. We conjecture that the \( k \)-Schur function \( \phi_{\lambda}^{(k)}[X; t] \) defined here, is equivalent to the atom \( A_{\lambda}^{(k)}[X; t] \) of \([3]\). Thus, a combinatorial interpretation for the \( k \)-Schur functions can be found in \([3]\), as well as many other properties that our functions appear to obey. In particular, a discussion of a \( k \)-generalization for the Young lattice is present. The reader should be aware of notational differences in \([3]\) such as \( A_{\lambda}^{(k)}[X; t] \leftrightarrow \phi_{\lambda}^{(k)}[X; t], V_k \leftrightarrow A_{\lambda}^{(k)} \), and \( S_{\lambda}[X] \leftrightarrow s_{\lambda}[X] \).

The article proceeds as follows: after the introduction of basic definitions in Section \([3]\), the case \( t = 1 \) is addressed in Sections \([4]\) and \([5]\). Although some of what appears in these sections can be obtained from the results in Sections \([3]\) and \([4]\) owing to its greater simplicity, we find it pedagogical to treat the case \( t = 1 \) independently. In Section \([6]\), an important unitriangularity property of the generalized Schur products is used. However, the proof has been relegated to the Appendix, where known properties of these products including a Morris-type recursion can also be found.

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2. Definitions

2.1. Partitions. Symmetric polynomials are indexed by partitions, sequences of non-negative integers \( \lambda = (\lambda_1, \lambda_2, \ldots) \) with \( \lambda_1 \geq \lambda_2 \geq \ldots \). The number of non-zero parts in \( \lambda \) is denoted \( \ell(\lambda) \) and the degree of \( \lambda \) is \( |\lambda| = \lambda_1 + \cdots + \lambda_n \). We say that \( \lambda \) is a partition of \( n \), denoted \( \lambda \vdash n \), if \( |\lambda| = n \). We use the dominance order on partitions with \( |\lambda| = |\mu| \), where \( \lambda \leq \mu \) when \( \lambda_1 + \cdots + \lambda_i \leq \mu_1 + \cdots + \mu_i \) for all \( i \). Given two partitions \( \lambda \) and \( \mu \), \( \lambda \cup \mu \) stands for the partition rearrangement of the parts of \( \lambda \) and \( \mu \). Note that if \( \lambda \leq \mu \) and \( \nu \leq \omega \), then \( \lambda \cup \nu \leq \mu \cup \omega \).

Any partition \( \lambda \) has an associated Ferrers diagram with \( \lambda_i \) lattice squares in the \( i^{th} \) row, from the bottom to top. For example,
\[
\lambda = (4, 2) = \begin{array}{cc}
\text{	extbullet} & \\
\text{	extbullet} & \text{	extbullet} & \text{	extbullet} & \text{	extbullet}
\end{array}
\]
For each cell \( s = (i, j) \) in the diagram of \( \lambda \), let \( \ell(s), \ell(s), a(s) \) and \( a'(s) \) be respectively the number of cells in the diagram of \( \lambda \) to the south, north, east and west of the cell \( s \). The hook-length of any

\[^1\text{The change in terminology is to prevent confusion with other objects know as atoms in tableaux combinatorics.}\]
cell in \( \lambda \), is \( h_s(\lambda) = \ell(s) + a(s) + 1 \). In the example, \( h_{(1,2)}(4,2) = 2 + 1 + 1 \). The main hook-length of \( \lambda \), \( h_M(\lambda) \), is the hook-length of the cell \( s = (1, 1) \) in the diagram of \( \lambda \). Therefore, \( h_M((4, 2)) = 5 \).

The conjugate \( \lambda' \) of a partition \( \lambda \) is defined by the reflection of the Ferrers diagram about the main diagonal. For example, the conjugate of \((4, 2)\) is

\[
\lambda' = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} = (2, 2, 1, 1).
\]

A skew shape \( \lambda/\mu \) is the diagram obtained by deleting the diagram of \( \mu \) from \( \lambda \).

\[
(6, 5, 4, 3, 1)/(4, 2) = \begin{array}{ccccccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{array}
\]

A partition \( \lambda \) is said to be \( k \)-bounded if its first part is not larger than \( k \), i.e., if \( \lambda_1 \leq k \). We associate to any \( k \)-bounded partition \( \lambda \) a sequence of partitions, \( \lambda^{\rightarrow k} = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(r)}) \), called the \( k \)-split of \( \lambda \). \( \lambda^{\rightarrow k} \) is obtained by partitioning \( \lambda \) (without rearranging the entries) into partitions \( \lambda^{(i)} \) where \( h_M(\lambda^{(i)}) = k \), for all \( i < r \). For example, \((3, 2, 2, 2, 1, 1)^{\rightarrow 3} = ((3), (2, 2), (2, 1), (1)) \) and \((3, 2, 2, 2, 1, 1)^{\rightarrow 4} = ((3, 2), (2, 2, 1), (1)) \). Equivalently, the diagram of \( \lambda \) is cut horizontally into partitions with main hook-length \( k \).

\[
\begin{array}{ccccccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{array}
\quad \text{and} \quad
\begin{array}{ccccccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{array}
\]

The last partition in the sequence \( \lambda^{\rightarrow k} \) may have main hook-length less than \( k \). It is important to note that \( \lambda^{\rightarrow k} = (\lambda) \) when \( h_M(\lambda) \leq k \).

### 2.2. Symmetric functions

The power sum \( p_i(x_1, x_2, \ldots) \) is

\[
p_i(x_1, x_2, \ldots) = x_1^i + x_2^i + \cdots ,
\]

and for a partition \( \lambda = (\lambda_1, \lambda_2, \ldots) \),

\[
p_{\lambda}(x_1, x_2, \ldots) = p_{\lambda_1}(x_1, x_2, \ldots) p_{\lambda_2}(x_1, x_2, \ldots) \cdots .
\]

We employ the notation of \( \Lambda \)-rings, needing only the formal ring of symmetric functions \( \Lambda \) to act on the ring of rational functions in \( x_1, \ldots, x_N, q, t \), with coefficients in \( \mathbb{R} \). The action of a power sum \( p_i \) on a rational function is, by definition,

\[
p_i \left[ \sum_{\alpha} c_{\alpha} u_{\alpha} \over \sum_{\beta} d_{\beta} v_{\beta} \right] = \sum_{\alpha} c_{\alpha} u_{\alpha}^{i} \over \sum_{\beta} d_{\beta} v_{\beta}^{i} ,
\]

with \( c_{\alpha}, d_{\beta} \in \mathbb{R} \) and \( u_{\alpha}, v_{\beta} \) monomials in \( x_1, \ldots, x_N, q, t \). Since the power sums form a basis of the ring \( \Lambda \), any symmetric function has a unique expression in terms of power sums, and \( (2.7) \) extends to an action of \( \Lambda \) on rational functions. In particular \( f[X] \), the action of a symmetric function \( f \) on the monomial \( X = x_1 + \cdots + x_N \), is simply \( f(x_1, \ldots, x_N) \). In the remainder of the article, we will always consider the number of variables \( N \) to be infinite, unless otherwise specified.

The complete symmetric function \( h_r[X] \) is

\[
h_r[X] = \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_r} x_{i_1} x_{i_2} \cdots x_{i_r} ,
\]

and \( h_{\lambda}[X] \) stands for the homogeneous symmetric function

\[
h_{\lambda}[X] = h_{\lambda_1}[X] h_{\lambda_2}[X] \cdots .
\]
In the same way, the elementary symmetric function \( e_r[X] \) is
\[
e_r[X] = \sum_{1 \leq i_1 < i_2 < \cdots < i_r} x_{i_1} x_{i_2} \cdots x_{i_r},
\]
and \( e_\lambda[X] \) stands for
\[
e_\lambda[X] = e_{\lambda_1}[X] e_{\lambda_2}[X] \cdots.
\]

Although the Schur functions may be characterized in many ways, here it will be convenient to
use the Jacobi-Trudi determinantal expression:
\[
s_\lambda[X] = \det \left[ e_{\lambda_i+j-1}[X] \right]_{1 \leq i,j \leq \ell(\lambda)},
\]
where \( e_r[X] = e_r[1] = 0 \) if \( r < 0 \). Note, in particular, \( s_r[X] = h_r[X] \) and \( s_1[X] = e_r[X] \).

The homomorphism \( \omega \), which is an involution on \( \Lambda \), is defined by
\[
\omega(h_r[X]) = e_r[X],
\]
and is such that \( \omega(s_\lambda[X]) = s_\lambda[X] \).

We recall that the Macdonald scalar product, \( \langle \ , \ \rangle_{q,t} \), on \( \Lambda \otimes \mathbb{Q}(q,t) \) is defined by setting
\[
\langle p_\lambda[X], p_\mu[X] \rangle_{q,t} = \delta_{\lambda\mu} z_\lambda \prod_{i=1}^{\ell(\lambda)} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}},
\]
where for a partition \( \lambda \) with \( m_i(\lambda) \) parts equal to \( i \), we associate the number
\[
z_\lambda = 1^{m_1(\lambda)} 2^{m_2(\lambda)} 3^{m_3(\lambda)} \cdots.
\]
When \( q = t \), this scalar product does not depend on a parameter and then satisfies
\[
\langle s_\lambda[X], s_\mu[X] \rangle = \delta_{\lambda\mu}.
\]

The Macdonald integral forms \( J_\lambda[X; q, t] \) are uniquely characterized by
\[
(\text{i}) \ \langle J_\lambda, J_\mu \rangle_{q,t} = 0, \quad \text{if } \lambda \neq \mu, \quad (2.17)
\]
\[
(\text{ii}) \ J_\lambda[X; q, t] = \sum_{\mu \leq \lambda} v_{\lambda\mu}(q,t) s_\mu[X] \quad \text{with} \quad v_{\lambda\mu}(q,t) \in \mathbb{Q}(q,t), \quad (2.18)
\]
\[
(\text{iii}) \ v_{\lambda\mu}(q,t) = \prod_{s \in \lambda} (1-q^{a(s)} t^{\ell(s)+1}). \quad (2.19)
\]

Here, we use a modification of the Macdonald integral forms that is obtained by setting
\[
H_\lambda[X; q, t] = J_\lambda[X/(1-t); q, t] = \sum_{\mu} K_{\mu\lambda}(q,t) s_\mu[X],
\]
with the coefficients \( K_{\mu\lambda}(q,t) \in \mathbb{N}[q,t] \) known as the \( q,t \)-Kostka polynomials. In the case \( q = 0 \),
\( J_\lambda[X; q, t] \) reduces to the Hall-Littlewood polynomial, \( J_\lambda[X; q, t] = Q_\lambda[X; t] \). Again, we shall use a modification of the Hall-Littlewood polynomials,
\[
H_\lambda[X; t] = H_\lambda[X; 0, t] = Q_\lambda[X/(1-t); t] = s_\lambda[X] + \sum_{\mu > \lambda} K_{\mu\lambda}(t) s_\mu[X],
\]
with the coefficients \( K_{\mu\lambda}(t) \in \mathbb{N}[t] \) known as the Kostka-Foulkes polynomials. Then, in the limit \( t = 1 \), we have \( H_\lambda[X; 1] = h_\lambda[X] \), giving
\[
h_\lambda[X] = H_\lambda[X; 1] = s_\lambda[X] + \sum_{\mu > \lambda} K_{\mu\lambda} s_\mu[X],
\]
with the coefficients \( K_{\mu\lambda} \in \mathbb{N} \) known as the Kostka numbers.
For partitions $\lambda$ and $\mu$, we have

$$s_\lambda[X]s_\mu[X] = \sum_{\rho: |\rho|=|\lambda|+|\mu|} c_{\mu\rho}^\rho s_\rho[X], \quad (2.23)$$

where the coefficients $c_{\mu\rho}^\rho \in \mathbb{N}$ are the Littlewood-Richardson coefficients. They satisfy

**Property 1.** $c_{\lambda\mu\rho}^{\lambda\mu} = 1$, and $c_{\lambda\mu}^\rho = 0$ unless $\lambda \cup \mu \leq \rho$ and $(\lambda' \cup \mu')' \geq \rho$.

**Proof.** The unitriangular relation (2.22) gives

$$s_\lambda[X] = h_\lambda[X] + \sum_{\mu > \lambda} K_\mu^{-1} h_\mu[X], \quad (2.24)$$

which implies

$$s_\lambda[X]s_\mu[X] = \sum_{\nu \geq \lambda, \gamma \geq \mu} K_{\nu\lambda}^{-1} K_{\gamma\mu}^{-1} h_{\nu \cup \gamma}[X] = h_{\lambda \cup \mu} + \sum_{\rho > \lambda \cup \mu} d_{\lambda\rho}^\rho h_\rho, \quad (2.25)$$

since $\nu \geq \lambda$ and $\gamma \geq \mu$ imply $\nu \cup \gamma \geq \lambda \cup \mu$. Then, from (2.22), $c_{\lambda\mu}^{\lambda\mu} = 1$ and $c_{\lambda\mu}^\rho = 0$ unless $\lambda \cup \mu \leq \rho$. Similarly, it can be shown that $(\lambda' \cup \mu')' \geq \rho$ by applying the involution $\omega$ to (2.22).

The Pieri rules are the cases $\mu = (r)$ and $\mu = (1^r)$ of expression (2.23). Then

$$h_r[X]s_\lambda[X] = \sum_\mu s_\mu[X] \quad \text{and} \quad e_r[X]s_\lambda[X] = \sum_\nu s_\nu[X], \quad (2.26)$$

where the sums run over all $\mu$‘s and $\nu$‘s such that $\mu/\lambda$ and $\nu/\lambda$ are respectively a horizontal $r$-strip and a vertical $r$-strip.

### 3. The $k$-Schur Functions

Here, and in the following section, we study the space $\Lambda_t^{(k)}$ when $t = 1$. In this case, the subspace reduces to a subring of $\Lambda$, defined by

$$\Lambda^{(k)} = \Lambda^{(k)}_1 = L \{ h_\lambda[X] \}_{\lambda, \lambda_1 \leq k} = L \{ e_\lambda[X] \}_{\lambda, \lambda_1 \leq k}. \quad (3.1)$$

The last equality holds since the determinantal expression (2.13) of $s_r[X] = h_r[X]$, in terms of elementary functions, gives only terms of the type $e_i[X]$, where $i \leq r$.

#### 3.1. $k$-split polynomials

Our construction relies on the introduction of a family of symmetric polynomials called the $k$-split polynomials. If $\lambda^{-k} = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(n)})$ is the $k$-split of a $k$-bounded partition $\lambda$, then we define the $k$-split polynomial by

$$G^{(k)}_\lambda[X] = s_{\lambda^{(1)}}[X] \cdots s_{\lambda^{(n)}}[X]. \quad (3.2)$$

A number of properties can be derived from this definition. For example, because $\lambda^{-k} = (\lambda)$ when $h_M(\lambda) \leq k$, it follows immediately that

**Property 2.**

$$G^{(k)}_\lambda[X] = s_\lambda[X], \quad \text{for} \quad h_M(\lambda) \leq k. \quad (3.3)$$

Moreover, since $G^{(k)}_\lambda[X]$ is the product $s_{\lambda^{(1)}}[X] \cdots s_{\lambda^{(n)}}[X]$, Property 1 of the Littlewood-Richardson rule applied repeatedly implies that $\lambda = \lambda^{(1)} \cup \cdots \cup \lambda^{(n)}$ must index the minimal element in the Schur function expansion of $G^{(k)}_\lambda[X]$. That is,
Property 3. For any $k$-bounded partition $\lambda$,

$$G^{(k)}_{\lambda}[X] = s_{\lambda}[X] + \sum_{\mu > \lambda} a^{(k)}_{\lambda \mu} s_{\mu}[X], \quad a^{(k)}_{\lambda \mu} \in \mathbb{N}. \quad (3.4)$$

Finally, since $(k, \lambda)^{\rightarrow k} = ((k), \lambda^{\rightarrow k})$ where $(k, \lambda) = (k, \lambda_1, \lambda_2, \ldots)$, we have

Property 4. For any $k$-bounded partition $\lambda$,

$$s_k[X] G^{(k)}_{\lambda}[X] = G^{(k)}_{(k,\lambda)}[X]. \quad (3.5)$$

In fact, it happens that these polynomials form a basis for the subring $\Lambda^{(k)}$.

Theorem 5. $\{G^{(k)}_{\lambda}[X]\}_{\lambda \leq k}$ forms a basis for $\Lambda^{(k)}$.

Proof. $\Lambda^{(k)}$ is the linear span of homogeneous symmetric functions over all $k$-bounded partitions. Since the elements $G^{(k)}_{\lambda}[X]$ are also indexed by $k$-bounded partitions and by Property 3, they are linearly independent, it suffices to show that these elements lie in $\Lambda^{(k)}$. That is, for a $k$-bounded partition $\lambda$, we must be able to expand $G^{(k)}_{\lambda}[X]$ in terms of $h_{\mu_1}[X]h_{\mu_2}[X] \cdots$ where $\mu_i \leq k$ for all $i$. Observe that any entry $h_j[X]$ in the determinantal expression (2.12) for $s_\nu$ can be indexed by $j$ no larger than $\nu_1 + \ell(\nu) - 1 = h_M(\nu)$. Since $G^{(k)}_{\lambda}[X] = s_{\lambda^{(i)}}[X] s_{\lambda^{(j)}}[X] \cdots$ with $h_M(\lambda^{(i)}) \leq k$ for all $i$, $G^{(k)}_{\lambda}[X]$ is a product of determinants having entries $h_j[X]$, with $j \leq k$. \hfill \Box

We now have that $G^{(k)}_{\lambda}[X] \in \Lambda^{(k)}$ for all $k$-bounded partitions $\lambda$. Therefore, these polynomials can be expanded in terms of $h_\mu$ with $\mu_1 \leq k$. In fact, since $G^{(k)}_{\lambda}[X]$ is unitriangularly related to $s_\lambda$ by Property 3, and $s_\lambda$ is unitriangularly related to $h_\lambda$ (3.22), we have

Property 6.

$$G^{(k)}_{\lambda}[X] = h_{\lambda}[X] + \sum_{\mu > \lambda, \mu_1 \leq k} g^{(k)}_{\lambda \mu} h_\mu[X], \quad g^{(k)}_{\lambda \mu} \in \mathbb{Z}. \quad (3.6)$$

3.2. $k$-Schur functions. Although the $k$-split polynomials form a basis for $\Lambda^{(k)}$, they do not have the fundamental role for $\Lambda^{(k)}$ that the Schur functions have for $\Lambda$. For example, since

$$s_2[X] G^{(3)}_{3,1,1}[X] = G^{(3)}_{3,2,1,1}[X] - G^{(3)}_{3,3,2,2}[X]$$

reveals a negative coefficient, the $k$-split basis does not satisfy a refinement of the Pieri Rule. Additionally, there are no partitions $\mu$ such that the involution $\omega$ sends $G^{(k)}_{\mu}[X]$ to a single element $G^{(k)}_{\mu}[X]$ as with the Schur functions. However, these polynomials do play a key role in the construction of our Schur analog, $s^{(k)}_{\lambda}[X]$. It happens that we use a projection operator, $T^{(k)}_j$, acting linearly on $\Lambda^{(k)}$ and defined by

$$T^{(k)}_j G^{(k)}_{\lambda}[X] = \begin{cases} G^{(k)}_{\lambda}[X] & \text{if } \lambda_1 = j \\ 0 & \text{otherwise} \end{cases}. \quad (3.7)$$

Note that this projection operator sends an element of $\Lambda^{(k)}$ into an element belonging to the linear span of $k$-split polynomials whose first parts all equal to $j$. We can now introduce the $k$-Schur functions, our fundamental objects in $\Lambda^{(k)}$. 7
Definition 7. For a $k$-bounded partition $\lambda$, the $k$-Schur function is defined recursively by
\[
 s_\lambda^{(k)}[X] = T_{\lambda_1}^{(k)} s_{\lambda_1}[X] s_{\lambda_2,\lambda_3,\ldots}[X], \quad \text{with} \quad s_\lambda^{(k)}[X] = 1. \tag{3.8}
\]

Explicit examples of the expansion of $k$-Schur functions in terms of Schur functions are given, letting $t = 1$, in the tables of Subsection 9.1.

Since the space $\Lambda^{(k)}$ is simply $\Lambda$ when $k \to \infty$, we expect to recover $s_\lambda$ from $s_\lambda^{(k)}$ in this case. The following property supports our assertion that the $k$-Schur functions generalize the Schur functions.

Property 8. For any $k$-bounded partition $\lambda$,
\[
s_\lambda^{(k)}[X] = s_\lambda[X], \quad \text{for} \quad h_M(\lambda) \leq k. \tag{3.9}
\]

Proof. Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ be a partition with $\lambda_1 \leq k$ and $h_M(\lambda) \leq k$. In the case that $\lambda = ()$, by definition, $s_0^{(k)}[X] = 1 = s_0[X]$. Let $\lambda = (\lambda_2, \lambda_3, \ldots)$ and assume by induction on $\ell(\lambda)$, that $s_\lambda^{(k)}[X] = s_\lambda[X]$. We proceed to show $s_\lambda^{(k)}[X] = s_\lambda[X]$. From (2.12), we have that
\[
s_\lambda[X] = \det \begin{pmatrix} h_{\lambda_1}[X] & h_{\lambda_1+1}[X] & \cdots & h_{\lambda_1+\ell-1}[X] \\ h_{\lambda_2-1}[X] & h_{\lambda_2}[X] & \cdots & h_{\lambda_2+\ell-2}[X] \\ \vdots & \vdots & \ddots & \vdots \\ h_{\lambda_{\ell-1}+1}[X] & h_{\lambda_{\ell-1}+\ell-2}[X] & \cdots & h_{\lambda_\ell}[X] \end{pmatrix}. \tag{3.10}
\]
The expansion of this determinant about the first row gives that the coefficient of $h_{\lambda_1}[X]$ is exactly $s_\lambda[X]$. Further, since all other terms in the first row are of the form $h_i[X]$, with $i > \lambda_1$, we have
\[
s_\lambda[X] = h_{\lambda_1}[X] s_\lambda[X] + \sum_{\mu: \mu_1 > \lambda_1} c_\mu h_\mu[X]. \tag{3.11}
\]
Property (a) gives that $s_\lambda[X] = G_\lambda^{(k)}[X]$ since $h_M(\lambda) \leq k$. Noting also that $s_\lambda, s_\lambda^{(k)}[X] = G_\lambda^{(k)}[X] - \sum_{\mu: \mu_1 > \lambda_1} c_\mu h_\mu[X]. \tag{3.12}$

Since $s_\lambda[X] s_\lambda^{(k)}[X]$ and $G_\lambda^{(k)}[X]$ both belong to $\Lambda^{(k)}$, $\sum_{\mu: \mu_1 > \lambda_1} c_\mu h_\mu[X]$ is in $\Lambda^{(k)}$ and can therefore be expanded in terms of $G_\mu^{(k)}$. This expansion, by the unitriangularity property (3.6), yields
\[
s_\lambda[X] s_\lambda^{(k)}[X] = G_\lambda^{(k)}[X] - \sum_{\mu: \mu_1 > \lambda_1} d_\mu G_\mu^{(k)}[X]. \tag{3.13}
\]
Since $s_\lambda^{(k)}[X] = T_{\lambda_1}^{(k)} s_{\lambda_1}[X] s_\lambda[X]$ by the induction hypothesis, applying $T_{\lambda_1}$ to this expression gives
\[
s_\lambda^{(k)}[X] = T_{\lambda_1}^{(k)} \left( G_\lambda^{(k)}[X] - \sum_{\mu: \mu_1 > \lambda_1} d_\mu G_\mu^{(k)}[X] \right) = G_\lambda^{(k)}[X] = s_\lambda[X], \tag{3.14}
\]
thus proving the property.

It also happens that there is a unitriangularity property satisfied by the $k$-Schur functions.

Property 9. For any $k$-bounded partition $\lambda$,
\[
s_\lambda^{(k)}[X] = h_\lambda[X] + \sum_{\mu > \lambda \atop \mu_1 \leq k} d_{\lambda\mu}^{(k)} h_\mu[X], \quad \text{with} \quad d_{\lambda\mu}^{(k)} \in \mathbb{Z}. \tag{3.15}
\]
Proof. Let \( \lambda \) be a \( k \)-bounded partition. If \( \lambda = (r) \) then \( h_M(\lambda) \leq k \) and Property 8 gives
\[
s_r^{(k)}[X] = s_r[X] = h_r[X].
\]
Let \( \hat{\lambda} \) denote the partition \( \lambda \) without its first part and assume by induction on \( \ell(\lambda) \) that
\[
s_{\hat{\lambda}}^{(k)}[X] = h_{\hat{\lambda}}[X] + \sum_{\gamma > \hat{\lambda} ; \gamma_1 \leq k} d_\gamma h_\gamma[X], \quad \text{with} \quad d_\gamma \in \mathbb{Z}.
\]
Substituting this expression into the definition \( s_{\lambda}^{(k)}[X] = T_{\lambda_1} s_{\lambda_2}^{(k)} \) gives
\[
s_{\lambda}^{(k)}[X] = T_{\lambda_1}^{(k)} \left( h_{\lambda}[X] + \sum_{\gamma > \lambda ; \gamma_1 \leq k} d_\gamma h_{(\lambda_1) \cup \gamma}[X] \right) = T_{\lambda_1}^{(k)} \left( h_{\lambda}[X] + \sum_{\nu > \lambda ; \nu_1 \leq k} c_\nu h_\nu[X] \right), \quad \text{with} \quad c_\nu \in \mathbb{Z}.
\]
since \( \gamma > \hat{\lambda} \implies (\lambda_1) \cup \gamma > (\lambda_1) \cup \hat{\lambda} = \lambda \). The unitriangularity relation (3.6) then gives
\[
s_{\lambda}^{(k)}[X] = T_{\lambda_1} \left( G_\lambda^{(k)}[X] + \sum_{\mu > \lambda} v_\mu G_\mu^{(k)}[X] \right) = G_\lambda^{(k)}[X] + \sum_{\mu > \lambda ; \mu_1 = \lambda_1} v_\mu G_\mu^{(k)}[X], \quad \text{with} \quad v_\mu \in \mathbb{Z}.
\]
Using formula (3.6) again then proves the claim. \( \square \)

We now have the tools to prove that in fact, the \( k \)-Schur functions do form a basis.

**Theorem 10.** The \( k \)-Schur functions form a basis for \( \Lambda^{(k)} \). That is
\[
\Lambda^{(k)} = \mathcal{L} \left\{ s_\lambda^{(k)}[X] \right\}_{\lambda_1 \leq k}.
\]

**Proof.** Since \( \Lambda^{(k)} = \mathcal{L}\{h_\lambda[X] \}_{\lambda_1 \leq k} \), the theorem follows immediately from the unitriangularity relation found in Property 8. \( \square \)

Property 8 of the \( k \)-split polynomials states that \( s_k[X] G_\lambda^{(k)}[X] = G_\lambda^{(k)}[X] \). In the next section, we prove a similar property for the \( k \)-Schur functions and generalize the phenomenon.

4. Irreducibility

Here we study the product of certain Schur functions with a \( k \)-Schur function. In particular, we are concerned with partitions of the form \((\ell^{k+1-\ell})\) for \( \ell = 1, \ldots, k \). Hereafter, such a partition is referred to as a \( k \)-rectangle. A \( k \)-bounded partition with no more than \( i \) parts equal to \( k - i \), for \( i = 0, \ldots, k - 1 \), is called an irreducible partition. Otherwise, the partition is reducible.

**Definition 11.** A \( k \)-Schur function indexed by an irreducible partition is said to be an irreducible. Otherwise, the \( k \)-Schur function is called reducible.

For example, the irreducible \( k \)-Schur functions for \( k = 1, 2, 3 \) are
\[
k = 1 : \quad s_0^{(1)}, \\
k = 2 : \quad s_0^{(2)}, s_1^{(2)}, \\
k = 3 : \quad s_0^{(3)}, s_1^{(3)}, s_2^{(3)}, s_{1,1}^{(3)}, s_{2,1}^{(3)}, s_{2,1,1}^{(3)}.
\]

These examples suggest the following property:
Property 12. There are $k!$ distinct $k$-irreducible partitions.

The concept of irreducibility arose in our study of the multiplication of a $k$-Schur function with a Schur function indexed by a $k$-rectangle. It happens that this produces a single $k$-Schur function. More precisely, it was shown that

Theorem 13. If $\lambda$ is a $k$-bounded partition, then

$$s_{(k-\ell+1)}[X] s_{\lambda}^{(k)}[X] = s_{\lambda_{(k-\ell+1)}}^{(k)}[X].$$

(4.2)

Therefore, given the $k!$ irreducible $k$-Schur functions, any reducible $s_{\lambda}^{(k)}$ is obtained simply by the multiplication of a sequence of Schur functions indexed by $k$-rectangles on the proper irreducible $k$-Schur function. The case that the $k$-rectangle is the partition $(k)$ relies only on properties of the $k$-split polynomials. We include the proof here.

Property 14. For $\lambda$ any $k$-bounded partition,

$$s_k[X] s_{\lambda}^{(k)}[X] = s_{\lambda,k}^{(k)}[X].$$  

(4.3)

Proof. The unitriangular relation between the Schur functions and $G^{(k)}_{\lambda}[X]$ given in (3.20) implies

$$s_k[X] s_{\lambda}^{(k)}[X] = s_k \left( G^{(k)}_{\lambda}[X] + \sum_{\mu > \lambda, \mu_1 = \lambda_1} v_{\mu,\lambda}^{(k)} G^{(k)}_{\mu}[X] \right).$$

(4.4)

Property 12 then gives the action of $s_k[X]$ on a $k$-split polynomial, and we have

$$s_k[X] s_{\lambda}^{(k)}[X] = G^{(k)}_{k,\lambda}[X] + \sum_{\mu > \lambda, \mu_1 = \lambda_1} v_{\mu,\lambda}^{(k)} G^{(k)}_{\mu,\ell}[X].$$

(4.5)

Since each polynomial in the right hand side of this expression is indexed by a partition with first component $k$, the expression is invariant under the action of the projection operator $T^{(k)}_k$. That is,

$$T^{(k)}_k s_k[X] s_{\lambda}^{(k)}[X] = T^{(k)}_k \left( G^{(k)}_{k,\lambda}[X] + \sum_{\mu > \lambda, \mu_1 = \lambda_1} v_{\mu,\lambda}^{(k)} G^{(k)}_{k,\mu}[X] \right) = G^{(k)}_{k,\lambda}[X] + \sum_{\mu > \lambda, \mu_1 = \lambda_1} v_{\mu,\lambda}^{(k)} G^{(k)}_{k,\mu}[X].$$

The right hand side in this expression is the same as that of (4.3) and we thus have that by definition

$$T^{(k)}_k s_k[X] s_{\lambda}^{(k)}[X] = s_k[X] s_{\lambda}^{(k)}[X] = s_{\lambda,k}^{(k)}[X].$$

□

The role of Schur functions indexed by $k$-rectangles in the subring $\Lambda^{(k)}$ leads naturally to the study of the quotient ring $\Lambda^{(k)}/I_k$, where $I_k$ denotes the ideal generated by $s_{(k+1-\ell)}[X]$. It is known that

Proposition 15. The homogeneous functions indexed by $k$-irreducible partitions form a basis of the quotient ring $\Lambda^{(k)}/I_k$.

Thus, the dimension of the quotient ring $\Lambda^{(k)}/I_k$ is $k!$. Since we have shown that the $k$-Schur functions form a basis for $\Lambda^{(k)}$, Theorem 15 implies

Theorem 16. The irreducible $k$-Schur functions form a basis of the quotient ring $\Lambda^{(k)}/I_k$.

In fact, the irreducible $k$-Schur function basis offers a very beautiful way to carry out operations in this quotient ring: first work in $\Lambda^{(k)}$ using $k$-Schur functions and then replace by zero all the $k$-Schur functions indexed by partitions which are not $k$-irreducible.
5. Analogs of Schur function properties

Computer experimentation reveals that many of the properties making the Schur function basis so important are generalized by the \( k \)-Schur functions. We now state several of these properties.

5.1. The \( k \)-conjugation of a partition. We give a generalization of partition conjugation that is an involution on \( k \)-bounded partitions, and reduces to usual conjugation of partitions for large \( k \).

A skew diagram \( D \) has hook-lengths bounded by \( k \) if the hook-length of any cell in \( D \) is not larger than \( k \). For a positive integer \( m \leq k \), the \( k \)-multiplication \( m \times \left( \begin{array}{c} k \\ k \end{array} \right) D \) is the skew diagram \( D \) obtained by prepending a column of length \( m \) to \( D \) such that the number of rows of \( D \) is as small as possible while ensuring that its hook-lengths are bounded by \( k \). For example,

\[
5 \times \left( \begin{array}{c} 3 \\ 1 \end{array} \right) = \left( \begin{array}{c} 4 \\ 3 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{array} \right) \quad \text{(5.1)}
\]

**Definition 17.** The \( k \)-conjugate of a \( k \)-bounded partition \( \lambda = (\lambda_1, \ldots, \lambda_n) \), denoted \( \lambda^{\omega_k} \), is the vector obtained by reading the number of boxes in each row of the skew diagram,

\[
D = \lambda_1 \times \left( \begin{array}{c} k \\ k \end{array} \right) \cdots \times \left( \begin{array}{c} k \\ k \end{array} \right) \lambda_n,
\]

arising by \( k \)-multiplying the entries of \( \lambda \) from right to left.

When \( k \to \infty \), \( \lambda^{\omega_k} = \lambda' = D \) since each \( k \)-multiplication step reduces to concatenating a column of height \( \lambda_i \). Further, the \( k \)-conjugate is an involution on \( k \)-bounded partitions:

**Theorem 18.** \( \omega_k \) is an involution on partitions bounded by \( k \). That is, for \( \lambda \) with \( \lambda_1 \leq k \),

\[
(\lambda^{\omega_k})^{\omega_k} = \lambda.
\]

We have observed that the \( k \)-conjugation of a partition plays a natural role in the generalization of classical Schur function properties. We now give two examples.

5.2. The involution \( \omega \). It is known that the involution in (2.13) acts on a Schur function by

\[
\omega s_\lambda[X] = s_{\lambda'}[X].
\]

We thus naturally examine the action of \( \omega \) on a \( k \)-Schur function since \( \omega \) preserves the space \( \Lambda^{(k)} \) by (3.1). Many examples support the following natural generalization of (5.4):

**Conjecture 19.** For any \( k \)-bounded partition \( \lambda \),

\[
\omega s_\lambda^{(k)}[X] = s_{\lambda^{\omega_k}}^{(k)}[X].
\]

The conjecture holds when \( h_M(\lambda) \leq k \) since, in this case, \( \lambda^{\omega_k} = \lambda' \) and by Property 8, \( s_\lambda^{(k)} = s_\lambda \).

5.3. Pieri Rules. Beautiful combinatorial algorithms are known for the Littlewood-Richardson coefficients that appear in a product of Schur functions;

\[
s_\lambda[X] s_\mu[X] = \sum_\nu c_{\lambda \mu}^\nu s_\nu[X] \quad \text{where} \quad c_{\lambda \mu}^\nu \in \mathbb{N}.
\]

Since \( \Lambda^{(k)} \) is a ring, and we have shown that \( s_\lambda^{(k)} \) forms a basis for this space, a similar expression holds for the product of two \( k \)-Schur functions. That is, for \( k \)-bounded partitions \( \lambda \) and \( \mu \),

\[
s_\lambda^{(k)}[X] s_\mu^{(k)}[X] = \sum_\nu c_{\lambda \mu}^{(k)} s_\nu^{(k)}[X] \quad \text{where} \quad c_{\lambda \mu}^{(k)} \in \mathbb{Z},
\]
where the integrality of \( c_{\lambda\mu}^{(k)} \) follows from Property 9. Further, Property 13 says that the \( k \)-Schur functions are simply the Schur functions when \( k \) is large, and therefore \( c_{\lambda\mu}^{(k)} = c_{\lambda\mu}^{(\nu)} \) for \( k \geq |\nu| \). In fact, we believe the coefficients are nonnegative for all \( k \). That is,

\[ \text{Conjecture 20.} \quad \text{For all \( k \)-bounded partitions \( \lambda, \mu, \nu \), we have} \quad 0 \leq c_{\lambda\mu}^{(k)} \leq c_{\lambda\mu}^{(\nu)}. \]

In particular, (5.7) reduces to a \( k \)-generalization of the Pieri rule when \( \lambda \) is a row (resp. column) of length \( \ell \leq k \) since \( s_{\lambda}^{(k)}[X] \) reduces to \( h_{\ell}[X] \) (resp. \( e_{\ell}[X] \)). That is, for \( \ell \leq k \),

\[ h_{\ell}[X] s_{\lambda}^{(k)}[X] = \sum_{\mu \in E_{\lambda,\ell}^{(k)}} s_{\mu}^{(k)}[X] \quad \text{and} \quad e_{\ell}[X] s_{\lambda}^{(k)}[X] = \sum_{\mu \in E_{\lambda,\ell}^{(k)}} s_{\mu}^{(k)}[X], \tag{5.8} \]

for some sets of partitions \( E_{\lambda,\ell}^{(k)} \) and \( \bar{E}_{\lambda,\ell}^{(k)} \), which we believe naturally extend the Pieri rules by:

\[ \text{Conjecture 21.} \quad \text{For any positive integer} \ \ell \leq k, \]

\[ E_{\lambda,\ell}^{(k)} = \{ \mu \mid \mu/\lambda \text{ is a horizontal} \ \ell\text{-strip and} \ \mu^{\omega_k}/\lambda^{\omega_k} \text{ is a vertical} \ \ell\text{-strip} \}, \]

\[ \bar{E}_{\lambda,\ell}^{(k)} = \{ \mu \mid \mu/\lambda \text{ is a vertical} \ \ell\text{-strip and} \ \mu^{\omega_k}/\lambda^{\omega_k} \text{ is a horizontal} \ \ell\text{-strip} \}. \tag{5.9} \]

For example, to obtain the indices of elements occurring in \( e_2 s_{3,2,1}^{(4)} \), we find \( (3, 2, 1)^{\omega_2} = (2, 2, 1, 1) \) by definition. Adding a horizontal 2-strip to \((2,2,1,1)\) in all ways, we obtain \((2,2,2,1,1),(3,2,1,1,1),(3,2,2,1)\) and \((4,2,1,1)\) of which all are 4-bounded. Our set then consists of all the 4-conjugates of these partitions that leave a vertical 2-strip when \((3, 2, 1)\) is extracted from them. The 4-conjugates are

\[(2, 2, 2, 1, 1)^{\omega_2} = \begin{array}{c|c|c} 1 & 1 & 2 \end{array}, \quad (3, 2, 1, 1, 1)^{\omega_2} = \begin{array}{c|c|c|c} 1 & 1 & 1 & 2 \end{array}, \quad (4, 2, 1, 1)^{\omega_2} = \begin{array}{c|c|c|c|c} 0 & 0 & 0 & 1 & 2 \end{array}, \tag{5.10}\]

and of these, the first three are such that a vertical 2-strip remains when \((3, 2, 1)\) is extracted. Thus

\[ e_2[X] s_{3,2,1}^{(4)}[X] = s_{3,3,2}^{(4)}[X] + s_{3,2,2,1}^{(4)}[X] + s_{3,2,1,1,1}^{(4)}[X]. \tag{5.11} \]

### 6. \( t \)-generalization

The ring of symmetric polynomials over rational functions in an extra parameter \( t \) has proven to be of interest in many fields of mathematics and physics. One natural basis of this space is given by the Hall-Littlewood polynomials, \( H_\lambda[X; t] \), which provide \( t \)-analogues of the homogeneous symmetric functions \( h_\lambda[X] \). Our approach employs vertex operators that arise in the recursive construction for the Hall-Littlewood polynomials. These operators can be defined for \( \ell \in \mathbb{Z} \), by

\[ B_\ell = \sum_{i=0}^{\infty} s_{i+\ell}[X] s_i[X(t-1)]^\perp, \tag{6.1} \]

where for \( f, g \) and \( h \) arbitrary symmetric functions, \( f^\perp \) is such that on the scalar product \( \langle 2.16 \rangle \),

\[ \langle f^\perp g, h \rangle = \langle g, fh \rangle. \tag{6.2} \]

The operators add an entry to the Hall-Littlewood polynomials, that is,

\[ H_\lambda[X; t] = B_{\lambda_1} H_{\lambda_2, \ldots, \lambda_\ell}[X; t], \quad \text{for} \quad \lambda_1 \geq \lambda_2. \tag{6.3} \]

Further, since they satisfy the relation

\[ B_m B_n = t B_n B_m + t B_{m+1} B_{n-1} - B_{n-1} B_{m+1}, \quad m, n \in \mathbb{Z}, \tag{6.4} \]

and \( B_\ell \cdot 1 = 0 \) if \( \ell < 0 \), their action on \( \Lambda \) can be computed algebraically.
We now consider a generalization of the subspace $\Lambda^{(k)}$, given by

$$\Lambda_t^{(k)} = \mathcal{L} \{ H_\lambda[X; t] \}_{\lambda: \lambda_1 \leq k}.$$  \hspace{1cm} (6.5)

It is clear that $\Lambda_t^{(1)} \subseteq \Lambda_t^{(2)} \subseteq \cdots \subseteq \Lambda_t^{(\infty)} = \Lambda$ and thus that these subspaces provide a filtration for $\Lambda$. Note that $\Lambda_t^{(k)}$ can be equivalently defined as

$$\Lambda_t^{(k)} = \mathcal{L} \{ s_\lambda[X/(1 - t)] \}_{\lambda: \lambda_1 \leq k}, \quad \text{or} \quad \Lambda_t^{(k)} = \mathcal{L} \{ H_\lambda[X; q; t] \}_{\lambda: \lambda_1 \leq k}.$$  \hspace{1cm} (6.6)

Again, we want elements that play the role in $\Lambda_t^{(k)}$ that the Schur functions play in $\Lambda$. In particular, since $\Lambda_t^{(\infty)} = \Lambda$, we want a basis for $\Lambda_t^{(k)}$ that reduces to the Schur functions when $k$ is large.

Our construction of these elements will naturally extend Definition 3 for the $k$-Schur functions in the case $t = 1$. First, multiplication by $s_{\lambda_1}[X]$ is replaced by the action of the operator $B_{\lambda_1}$. Then, the projection operator $T_{\lambda_i}^{(k)}$ is replaced by a $t$-analogous operator which, to define, requires an appropriate extension of the $k$-split polynomials. Recent developments in the theory of symmetric functions aid us with this task.

### 6.1. $k$-split polynomials

Important in our work with the $k$-Schur functions is a family of polynomials, studied in many recent papers such as [12], [13], that give a $t$-analog of the product of Schur functions. These functions, indexed by a sequence of partitions, can be built recursively using vertex operators [13]. For a partition $\lambda$ of length $m$, define

$$B_\lambda = \prod_{1 \leq i < j \leq m} (1 - t e_{ij}) B_{\lambda_1} \cdots B_{\lambda_m},$$  \hspace{1cm} (6.7)

where $e_{ij}$ acts by

$$e_{ij} (B_{\lambda_1} \cdots B_{\lambda_m}) = B_{\lambda_1} \cdots B_{\lambda_i+1} B_{\lambda_j-1} \cdots B_{\lambda_m}.$$  \hspace{1cm} (6.8)

For any sequence of partitions $(\lambda^{(1)}, \lambda^{(2)}, \ldots)$, the generalized Schur function product is then defined recursively by

$$\mathcal{H}_{(\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}, \ldots)}[X; t] = B_{\lambda^{(1)}} \mathcal{H}_{(\lambda^{(2)}, \lambda^{(3)}, \ldots)}[X; t],$$  \hspace{1cm} (6.9)

starting with $\mathcal{H}_{(\ldots)} = 1$. Note that since $B_\lambda \cdot 1 = s_\lambda[X]$, we have that

$$\mathcal{H}_{(\lambda)}[X; t] = s_\lambda[X].$$  \hspace{1cm} (6.10)

Appendix 8.1 gives an earlier formulation [12] for $\mathcal{H}_{(\lambda^{(1)}, \lambda^{(2)}, \ldots)}[X; t]$. Note in [13], $B_\lambda$ is denoted $H_\lambda^t$ and is given in terms of generating series. Formula (6.7) can be extracted from their formula (17).

It was shown [12] that for a sequence of partitions, $S = (\lambda^{(1)}, \lambda^{(2)}, \ldots)$,

$$\mathcal{H}_S[X; t] = \sum_{\mu \vdash |\lambda^{(1)}| + |\lambda^{(2)}| + \cdots} K_{\mu,S}(t) s_\mu[X], \quad \text{where} \quad K_{\mu,S}(t) \in \mathbb{Z}[t].$$  \hspace{1cm} (6.11)

Since the action of $B_\lambda$ on an arbitrary function $f$ reduces to multiplication by $s_\lambda[X]

$$B_\lambda f = s_\lambda[X] f,$$  \hspace{1cm} (6.12)

the $K_{\mu,S}(t)$ are known as generalized Kostka polynomials since, in this case, they satisfy

$$K_{\mu,S}(1) = \langle s_\mu[X], s_{\lambda^{(1)}[X]} s_{\lambda^{(2)}[X]} \cdots \rangle.$$  \hspace{1cm} (6.13)

For our purposes, we consider only the $\mathcal{H}_S[X; t]$ indexed by a dominant sequence $S$. That is, sequences of partitions $S = (\lambda^{(1)}, \lambda^{(2)}, \ldots)$ such that the concatenation of $\lambda^{(1)}, \lambda^{(2)}, \ldots$, denoted $\bar{S}$, forms a partition. In this case, we prove that $\mathcal{H}_S[X; t]$ obeys important untriangular relations.
Property 22. If $S$ is dominant, with $\bar{S} = \lambda$, then
\[
H_{S}[X; t] = s_{\lambda}[X] + \sum_{\mu \succ \lambda} K_{\mu; S}(t)s_{\mu}[X], \quad \text{where } K_{\mu; S}(t) \in \mathbb{Z}[t],
\] (6.14)
\[
H_{S}[X; t] = H_{X}[X] + \sum_{\mu \succ \lambda} C_{\mu; S}(t)H_{\mu}[X], \quad \text{where } C_{\mu; S}(t) \in \mathbb{Z}[t].
\] (6.15)

Proof. Proposition 42 with Proposition 43 prove relation (6.14) (see Appendix 3.2). The second identity then follows from unitriangularity (2.21) of $s_{\lambda}[X]$ in terms of $H_{\mu}[X]$. \qed

We have discovered that a particular subset of the $H_{S}$ not only form a basis for $\Lambda^{(k)}_{i}$, but are essential in the construction of the $k$-Schur functions.

Definition 23. The $k$-split polynomials are defined, for a $k$-bounded partition $\lambda$, by
\[
G^{(k)}_{\lambda}[X; t] = H_{S}[X; t] \quad \text{where } S = \lambda^{+k} \quad \text{is the } k\text{-split of } \lambda.
\] (6.16)

Since $H_{S}[X; t]$ reduces to the product of Schur functions indexed by elements of $S$ when $t = 1$,
\[
G^{(k)}_{\lambda}[X; 1] = G^{(k)}_{\lambda}[X],
\] (6.17)
and therefore, $G^{(k)}_{\lambda}[X; t]$ is a proper $t$-generalization of the $G^{(k)}_{\lambda}[X]$ introduced in Section 3.4.

It develops that the $k$-split polynomials satisfy properties analogous to those held by the $k$-split polynomials at $t = 1$. We first show that the $G^{(k)}_{\lambda}[X; t]$ actually lie in the space $\Lambda^{(k)}_{i}$. To this end, we start by showing that operators $B_{i}$ preserve this space.

Proposition 24. If $f \in \Lambda^{(k)}_{i}$ then $B_{i} f \in \Lambda^{(k)}_{i}$ for all $i \in \mathbb{Z}$ with $i \leq k$.

This claim follows from a preliminary result on subspaces of $\Lambda^{(k)}_{i}$. More precisely, for $a \in \mathbb{Z}$ and $b \in \mathbb{N}$, with $a \leq b$, we define
\[
\Lambda^{(a,b)}_{i} = \mathcal{L}\{H_{\lambda}[X; t]\}_{a \leq \lambda \leq b},
\] (6.18)
and $\Lambda^{(a,b)}_{i} = \Lambda^{(a,b)}_{i}$ for $a < 0$. In particular, $\Lambda^{(k)}_{i} = \Lambda^{(0,k)}_{i}$ and $\Lambda^{(a,b)}_{i} \subseteq \Lambda^{(k)}_{i}$ if $b \leq k$. Thus, Proposition 24 is an immediate consequence of

Lemma 25. If $f \in \Lambda^{(k)}_{i}$ then $B_{i} f \in \Lambda^{(i,k)}_{i}$ for all integers $i \leq k$.

Proof. The assertion holds for $i = k$ since $B_{k} H_{\lambda}[X; t] = H_{(k,\lambda_{1}, \lambda_{2}, \ldots)}[X; t] \in \Lambda^{(k,k)}_{i}$. Assume by induction that for all $f \in \Lambda^{(k)}_{i}$,
\[
B_{i} f \in \Lambda^{(i,k)}_{i}, \quad j < i \leq k.
\] (6.19)
Thus, it suffices to show that $B_{j} H_{\lambda}[X; t] \in \Lambda^{(j,k)}_{i}$.

We prove this claim by induction on $\ell(\lambda)$. First, $B_{j} H_{(j)}[X; t] = H_{(j)}[X; t] \in \Lambda^{(j,k)}_{i}$ for $j \geq 0$ and otherwise $B_{j} H_{(j)}[X; t] = 0 \in \Lambda^{(0,k)}_{i} = \Lambda^{(j,k)}_{i}$. Now, let $\lambda = (\lambda_{1}, \ldots, \lambda_{n})$ be a $k$-bounded partition and assume $B_{j} H_{\lambda}[X; t] \in \Lambda^{(j,k)}_{i}$ for all $\lambda = (\lambda_{2}, \ldots, \lambda_{n})$. If $\lambda_{1} \leq j$ then $B_{j} H_{\lambda}[X; t] = H_{(j, \lambda_{1}, \lambda_{2}, \ldots)} \in \Lambda^{(j,k)}_{i}$ by (6.13). Now consider $\lambda_{1} = j + 1$. The commutation relation (6.4) reduces to $B_{j} H_{\lambda} = B_{j} B_{j+1} H_{\lambda} = t B_{j+1} B_{j} H_{\lambda}$. Since $f = B_{j} H_{\lambda} \in \Lambda^{(j,k)}_{i}$ by assumption, $B_{j} H_{\lambda} = t B_{j+1} f \in \Lambda^{(j+1,k)}_{i} \subseteq \Lambda^{(j,k)}_{i}$ by (6.19). Finally, for $\lambda_{1} > j + 1$ we again use (6.4) to obtain $B_{j} H_{\lambda} = B_{j} B_{\lambda_{1}} H_{\lambda} = t B_{\lambda_{1}} B_{j+1} H_{\lambda} + t B_{j+1} B_{\lambda_{1}+1} H_{\lambda} - B_{\lambda_{1}-1} B_{j+1} H_{\lambda}$. Regarding the first term in the right hand side, our assumption gives that $B_{j} H_{\lambda} \in \Lambda^{(j,k)}_{i} \subseteq \Lambda^{(k)}_{i}$ and then using (6.19), $t B_{\lambda_{1}} B_{j} H_{\lambda} \in$
\[ \Lambda_i^{(\lambda_1, k)} \subseteq \Lambda_i^{(j, k)} \] since \( \lambda_1 > j \). Similar reasoning applies to the second and third term, and we thus have \( B_i H_A \in \Lambda_i^{(j, k)} \), proving our claim. \( \square \)

Now, given that \( \Lambda_i^{(k)} \) is invariant under \( B_i \), we can prove a more general statement that will imply the \( k \)-split polynomials lie in \( \Lambda_i^{(k)} \).

**Proposition 26.** If \( \lambda \) is a partition with \( h_M(\lambda) \leq k \), then \( B_\lambda f \in \Lambda_i^{(k)} \) for any \( f \in \Lambda_i^{(k)} \).

**Proof.** An operator version of the Jacobi-Trudi determinant \((2.12)\) is given by \((6.7)\). Thus, the expansion of \( B_\lambda \) in terms of products of the operators \( B_i \) yields only terms \( B_i \) with \( i \leq k \). Proposition 24 then implies the result. \( \square \)

**Property 27.** For any \( k \)-bounded partition \( \lambda \), we have that
\[ G_\lambda^{(k)}[X; t] \in \Lambda_i^{(k)}. \] (6.20)

**Proof.** Recursion \((6.9)\) for \( H_S \) allows us to restate Definition 23 as,
\[ G_\lambda^{(k)}[X; t] = B_{\lambda^{(1)}} \cdots B_{\lambda^{(n)}} \cdot 1, \quad \text{where} \quad \lambda^{\cdot k} = (\lambda^{(1)}, \ldots, \lambda^{(n)}). \] (6.21)
Since all \( h_M(\lambda^{(i)}) \leq k \), Proposition 26 implies \( B_{\lambda^{(n)}} \cdot 1 \in \Lambda_i^{(k)} \) given \( 1 \in \Lambda_i^{(k)} \). By induction, assuming \( B_{\lambda^{(2)}} \cdots B_{\lambda^{(n)}} \cdot 1 \in \Lambda_i^{(k)} \), Proposition 26 verifies our claim. \( \square \)

**Property 28.** We have
\[ G_\lambda^{(k)}[X; t] = H_\lambda[X; t] + \sum_{\mu > \lambda, \mu \leq k} g^{(k)}_{\lambda\mu}(t) H_\mu[X; t], \quad \text{where} \quad g^{(k)}_{\lambda\mu}(t) \in \mathbb{Z}[t]. \] (6.22)

**Proof.** The \( k \)-split polynomials can be expanded in terms of Hall-Littlewood polynomials indexed by \( k \)-bounded partitions since Property 27 proves they lie in \( \Lambda_i^{(k)} \). Then, by the unitriangular expansion of \( H_S \) given in \((6.15)\), \( G_\lambda^{(k)}[X; t] = H_{\lambda^{\cdot k}}[X; t] \) has the asserted unitriangularity. \( \square \)

**Property 29.** Let \( \lambda \) be such that \( h_M(\lambda) \leq k \). Then,
\[ G_\lambda^{(k)}[X; t] = s_\lambda[X]. \] (6.23)

**Proof.** \( G_\lambda^{(k)}[X; t] = H_{(\lambda)}[X; t] \) since \( \lambda^{\cdot k} = (\lambda) \) for \( h_M(\lambda) \leq k \). The claim then follows from \((6.10)\), which states that \( H_{(\lambda)}[X; t] = s_\lambda[X] \) for all \( \lambda \). \( \square \)

**Theorem 30.** The \( k \)-split polynomials form a basis of \( \Lambda_i^{(k)} \).

**Proof.** Since \( \Lambda_i^{(k)} \) is the span of Hall-Littlewood polynomials indexed by \( k \)-bounded partitions and \( G_\lambda^{(k)} \) are also indexed by \( k \)-bounded partitions, the theorem follows from Property 28. \( \square \)

6.2. \( k \)-Schur functions. As with the \( t = 1 \) case, although the \( k \)-split polynomials form a basis for \( \Lambda_i^{(k)} \), they do not play the fundamental role that the Schur functions do for \( \Lambda \). However, these polynomials are needed in the construction of our Schur analog, \( s_\lambda^{(k)}[X; t] \). We use a projection operator, \( \tilde{T}_j^{(k)} \), for \( j \leq k \), that acts linearly on \( \Lambda_i^{(k)} \) by
\[ \tilde{T}_j^{(k)} G_\lambda^{(k)}[X; t] = \begin{cases} G_\lambda^{(k)}[X; t] & \text{if } \lambda_1 = j \\ 0 & \text{otherwise} \end{cases}. \] (6.24)
Definition 31. For $k$-bounded partition $\lambda$, the $k$-Schur functions are recursively defined
\[
s^{(k)}_\lambda[X;t] = T^{(k)}_{\lambda\lambda_1} B_{\lambda_1} s^{(k)}_{\lambda_1,\lambda_2,...}[X;t], \quad \text{where } s^{(k)}_{\emptyset}[X;t] = 1.
\] (6.25)

Tables of $k$-Schur functions in terms of Schur functions can be found in Subsection 9.1.

When $t = 1$, from (6.12) and (6.17), we recover the $k$-Schur functions of Definition 3 that is,
\[
s^{(k)}_\lambda[X;1] = s^{(k)}_\lambda[X].
\] (6.26)

We now prove several other properties satisfied by the $k$-Schur functions.

Property 32. For $\lambda$ a $k$-bounded partition, we have
\[
s^{(k)}_\lambda[X;t] = G^{(k)}_\lambda[X;t] + \sum_{\mu > \lambda, \mu_1 = \lambda_1} u^{(k)}_{\mu\lambda}(t) G^{(k)}_{\mu}[X;t], \quad \text{where } u^{(k)}_{\mu\lambda}(t) \in \mathbb{Z}[t].
\] (6.27)

Proof. The assertion holds for $s^{(k)}_{\emptyset}[X;t] = 1 = G^{(k)}_{\emptyset}[X;t]$ by definition, Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a $k$-bounded partition and assume by induction on $n$ that the property is true for $\lambda = (\lambda_2, \ldots, \lambda_n)$. Definition 3 gives that
\[
s^{(k)}_\lambda[X;t] = T^{(k)}_{\lambda\lambda_1} B_{\lambda_1} s^{(k)}_\lambda[X;t].
\]
By the induction hypothesis, we have
\[
B_{\lambda_1} s^{(k)}_\lambda[X;t] = B_{\lambda_1} \left( G^{(k)}_\lambda[X;t] + \sum_{\mu > \lambda, \mu_1 = \lambda_1} u^{(k)}_{\mu\lambda}(t) G^{(k)}_{\mu}[X;t] \right), \quad u^{(k)}_{\mu\lambda}(t) \in \mathbb{Z}[t].
\] (6.28)

Recall the $k$-split polynomial is defined (6.16) by $G^{(k)}_\gamma = H_{\gamma^{-k}}$, and thus we have
\[
B_{\lambda_1} s^{(k)}_\lambda[X;t] = B_{\lambda_1} \left( H^{(k)}_\lambda[X;t] + \sum_{\mu > \lambda, \mu_1 = \lambda_1} u^{(k)}_{\mu\lambda}(t) H^{(k)}_{\mu}[X;t] \right)
\[
= H^{((\lambda_1),\lambda^{-k})}_\lambda[X;t] + \sum_{\mu > \lambda, \mu_1 = \lambda_1} u^{(k)}_{\mu\lambda}(t) H^{((\lambda_1),\mu^{-k})}_\mu[X;t].
\] (6.29)

$\mu_1 = \lambda_1 = \lambda_2 \leq \lambda_1$ implies that $((\lambda_1), \lambda^{-k})$ and $((\lambda_1), \mu^{-k})$ are dominant sequences. Further, $\lambda < \mu$ implies that $\lambda = (\lambda_1) \cup \lambda < (\lambda_1) \cup \mu$. Therefore, unitriangularity of $H_S$ (6.15) further gives that
\[
B_{\lambda_1} s^{(k)}_\lambda[X;t] = H_{\lambda}[X;t] + \sum_{\mu > \lambda} v^{(k)}_{\mu\lambda}(t) H_{\mu}[X;t], \quad v^{(k)}_{\mu\lambda}(t) \in \mathbb{Z}[t].
\] (6.30)

$s^{(k)}_\lambda[X;t] \in \Lambda^{(k)}_t$, since the induction hypothesis gives that it can be expanded in terms of $G^{(k)}_{\mu}[X;t]$. Thus, $B_{\lambda_1} s^{(k)}_\lambda[X;t] \in \Lambda^{(k)}_t$ by Proposition 24 and therefore the coefficients $v^{(k)}_{\mu\lambda}(t)$ are non-zero only for $\mu_1 \leq k$. (6.31) then becomes
\[
B_{\lambda_1} s^{(k)}_\lambda[X;t] = H_{\lambda}[X;t] + \sum_{\mu > \lambda} v^{(k)}_{\mu\lambda}(t) H_{\mu}[X;t], \quad v^{(k)}_{\mu\lambda}(t) \in \mathbb{Z}[t]
\] (6.31)

Unittriangularity between Hall-Littlewood polynomials and $k$-split polynomials gives,
\[
B_{\lambda_1} s^{(k)}_\lambda[X;t] = G^{(k)}_\lambda[X;t] + \sum_{\mu > \lambda, \mu_1 \leq k} u^{(k)}_{\mu\lambda}(t) G^{(k)}_{\mu}[X;t], \quad u^{(k)}_{\mu\lambda}(t) \in \mathbb{Z}[t].
\] (6.32)

Applying $T^{(k)}_{\lambda\lambda_1}$ to both sides of this expression, we arrive at
\[
T^{(k)}_{\lambda\lambda_1} B_{\lambda_1} s^{(k)}_\lambda[X;t] = G^{(k)}_\lambda[X;t] + \sum_{\mu > \lambda, \mu_1 = \lambda_1} u^{(k)}_{\mu\lambda}(t) G^{(k)}_{\mu}[X;t], \quad u^{(k)}_{\mu\lambda}(t) \in \mathbb{Z}[t].
\] (6.33)

Since $s^{(k)}_\lambda = T^{(k)}_{\lambda\lambda_1} B_{\lambda_1} s^{(k)}_\lambda$, we have our claim. \qed
The following theorem is an immediate consequence of this property.

**Theorem 33.** The $k$-Schur functions form a basis of $\Lambda_t^{(k)}$.

We can thus refine the expansion of Hall-Littlewood polynomials in terms of Schur functions (2.22). That is, for any $k$-bounded partition $\lambda$,

$$H_\lambda[X; t] = s_\lambda^{(k)}[X; t] + \sum_{\mu > \lambda, \mu_1 \leq k} K^{(k)}_{\mu \lambda}(t) s_\mu[X; t], \quad \text{where } K^{(k)}_{\mu \lambda}(t) \in \mathbb{Z}[t]. \quad (6.34)$$

The integrality of $K^{(k)}_{\mu \lambda}(t)$ follows from the unitriangularity and integrality in Properties 28 and 32. Moreover, by our triangularity and integrality properties and (2.21), we also have integrality of the coefficients in Conjecture (1.6):

**Property 34.** For any $k$-bounded partition $\lambda$,

$$s_\lambda^{(k)}[X; t] = s_\lambda[X] + \sum_{\mu > \lambda} v^{(k)}_{\mu \lambda}(t) s_\mu[X], \quad \text{where } v^{(k)}_{\mu \lambda}(t) \in \mathbb{Z}[t]. \quad (6.35)$$

This unitriangularity property, given that the coefficients in the Schur function expansion of the Macdonald polynomials are polynomials in $q$ and $t$ with integral coefficients, implies

**Property 35.** For any $k$-bounded partition $\lambda$,

$$H_\lambda[X; q, t] = \sum_{\mu, \mu_1 \leq k} K^{(k)}_{\mu \lambda}(q, t) s_\mu[X; t], \quad \text{where } K^{(k)}_{\mu \lambda}(q, t) \in \mathbb{Z}[q, t]. \quad (6.36)$$

Now, to further support the idea that the $s_\lambda^{(k)}[X; t]$ provide a refinement for Schur function theory, we must show that they reduce to the usual $s_\lambda[X]$ when $k \to \infty$. This result relies on a lemma.

**Lemma 36.** If $\lambda$ is a partition with $h_M(\lambda) \leq k$ then $\overline{T}_\lambda^{(k)} B_\lambda f = \overline{T}_\lambda^{(k)} B_\lambda f$ for all $f \in \Lambda_t^{(k)}$.

**Proof.** We need to show that, for $f \in \Lambda_t^{(k)}$, $\overline{T}_\lambda^{(k)} (B_\lambda - B_\lambda f) \cdot f = 0$. Definition (6.7) gives

$$B_\lambda = \prod_{2 \leq j \leq \ell(\lambda)} (1 - te_{1j})B_{\lambda_1} \prod_{2 \leq i < j \leq \ell(\lambda)} (1 - te_{ij})B_{\lambda_2} \cdots B_{\lambda_n} = \prod_{2 \leq j \leq \ell(\lambda)} (1 - te_{1j})B_{\lambda_1} B_\lambda \quad (6.37)$$

In the expansion of this product, with the exception of the term $B_{\lambda_1} B_\lambda$, each term contains at least one $e_{1j}$, which increases the index of $B_{\lambda_1}$. Therefore, using the argument of Proposition 24, we have

$$B_\lambda = B_{\lambda_1} B_\lambda + \sum_{i=1}^{\ell(\lambda)-1} c_i(t) B_{\lambda_1+i} O_i, \quad c_i(t) \in \mathbb{Z}[t], \quad (6.38)$$

where $O_i$ is a product of $B_j$’s with $j \leq k$. Proposition 24 states that $B_i f \in \Lambda_t^{(k)}$ for $f \in \Lambda_t^{(k)}$ and $i \leq k$ and it is thus clear that $O_i \cdot f \in \Lambda_t^{(k)}$. Furthermore, since $\lambda_1 + i \leq \lambda_1 + \ell(\lambda) - 1 = h_M(\lambda) \leq k$, we have $B_{\lambda_1+i} O_i \cdot f \in \Lambda_t^{(\lambda_1+i, k)} \subseteq \Lambda_t^{(\lambda_1+1, k)}$ by Lemma 25. Thus, from expression (6.38),

$$(B_\lambda - B_{\lambda_1} B_\lambda) \cdot f = \sum_{i=1}^{\ell(\lambda)-1} c_i(t) B_{\lambda_1+i} O_i \cdot f \in \Lambda_t^{(\lambda_1+1, k)}. \quad (6.39)$$

The definition of $\Lambda_t^{(\lambda_1+1, k)}$, with unitriangularity in Proposition 28 then gives the expansions

$$(B_\lambda - B_{\lambda_1} B_\lambda) \cdot f = \sum_{\mu, \mu_1+1 \leq \mu_1 \leq k} b_{\mu}(t) H_\mu[X; t] = \sum_{\mu_1 \geq \lambda_1+1} d_\mu(t) G^{(k)}_{\mu}(X; t). \quad (6.40)$$
Finally, acting with $\tilde{T}^{(k)}_{\lambda_1}$, we have $\tilde{T}^{(k)}_{\lambda_1} (B_{\lambda_1} B_{\lambda} \cdot f - B_{\lambda} \cdot f) = 0$. \hfill \Box

**Property 37.** For any $k$-bounded partition $\lambda$,
\[ s^{(k)}_{\lambda} [X; t] = s_{\lambda} [X] \quad \text{if} \quad h_{M}(\lambda) \leq k. \]  

**Proof.** We proceed by induction on $\ell(\lambda)$. First, $s^{(k)}_{\lambda_0} [X; t] = 1 = s_{\lambda_0} [X]$. Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a $k$-bounded partition with $h_{M}(\lambda) \leq k$. If $\lambda = (\lambda_2, \ldots, \lambda_n)$ then $h_{M}(\lambda) \leq h_{M}(\lambda')$ and induction implies $s^{(k)}_{\lambda} [X; t] = s_{\lambda} [X]$. Thus, $s^{(k)}_{\lambda} [X; t] = \tilde{T}^{(k)}_{\lambda_1} B_{\lambda_1} s_{\lambda} [X]$ and it suffices to show $\tilde{T}^{(k)}_{\lambda_1} B_{\lambda_1} s_{\lambda} [X] = s_{\lambda} [X]$. Since (6.10) gives that $B_{\mu} \cdot 1 = s_{\mu} [X]$, we need only show $\tilde{T}^{(k)}_{\lambda_1} B_{\lambda_1} \cdot 1 = B_{\lambda_1} \cdot 1$, or $\tilde{T}^{(k)}_{\lambda_1} (B_{\lambda_1} - B_{\lambda_1} B_{\lambda_1}) \cdot 1 = 0$. This result follows from the special case $f = 1 \in \Lambda^{(k)}_{\lambda_1}$ of the previous lemma. \hfill \Box

7. IRREDUCIBILITY AND $k$-CONJUGATION

The concept of irreducibility introduced in Section 4 extends naturally in the space $\Lambda^{(k)}_{\lambda_1}$. In fact, the following $t$-analog of Theorem 13 holds:

**Theorem 38.** For any $k$-bounded partition $\lambda$,
\[ B_{(t^k - 1)^{-1}} s^{(k)}_{\lambda} [X; t] = t^* s^{(k)}_{\lambda_{(t^k - 1)^{-1}}} [X; t], \]  

where $t^*$ stands for some power of $t$.

This theorem implies that when $k = 2$, any $k$-Schur function can be built using a sequence of operators $B_2$ and $B_{1,1}$ applied to either $s^{(2)}_{\lambda} [X; t] = s_{1} [X]$ or $s^{(2)}_{\lambda_0} [X; t] = s_{\lambda_0} [X] = 1$. Thus, there is a connection between the $k$-Schur functions and the positive functions introduced in [13], and, from [13], to the generalized Kostka polynomials of [12]. This connection implies that our refinement (17) of Macdonald’s original positivity conjecture holds when $k = 2$.

With regard to extending Conjecture 19 to the general case, we first note that the involution $\omega$ is not well defined on $\Lambda^{(k)}_{\lambda_1}$. That is, an element $f \in \Lambda^{(k)}_{\lambda_1}$ may be such that $\omega f \notin \Lambda^{(k)}_{\lambda_1}$. However, there is a simple generalization for $\omega$ that preserves $\Lambda^{(k)}_{\lambda_1}$. Let $\omega_t$ be defined on an element $f \in \Lambda$, by
\[ \omega_t f = (\omega f)|_{t \rightarrow 1/t}. \]  

For instance, if $f = \sum_{\mu} c_{\mu} (q, t) s_{\mu} [X]$, we have
\[ \omega_t f = \sum_{\mu} c_{\mu} (q, 1/t) s_{\mu} [X]. \]  

Since $\omega$ is an involution, $\omega_t$ is also. In fact, it is well defined on $\Lambda^{(k)}_{\lambda_1}$.

**Proposition 39.** If $f \in \Lambda^{(k)}_{\lambda_1}$, then $\omega_t f \in \Lambda^{(k)}_{\lambda_1}$.

**Proof.** Since $\Lambda^{(k)}_{\lambda_1}$ can also be defined as $\Lambda^{(k)}_{\lambda_1} = \mathcal{L} \{ s_{\lambda} [X/(1 - t)] \}_{\lambda; \lambda_1 \leq k}$, it suffices to show, for any partition $\lambda$,
\[ \omega_t s_{\lambda} [X/(1 - t)] = (-t)^{|\lambda|} s_{\lambda} [X/(1 - t)]. \]  

Let $\tilde{X}$ denote the alphabet $X$ where $x_i \rightarrow -x_i$ for all $i$. That is, $P[\tilde{X}] = P[\tilde{x}_1, \tilde{x}_2, \ldots]$ for an arbitrary symmetric function $P[X] = P[x_1, x_2, \ldots]$. Note, this implies that if $P[X]$ is homogeneous of degree $d$, then $P[\tilde{X}] = (-1)^d P[X]$. Since $\omega$ acts on $P[X]$ by $\omega P[X] = P[-\tilde{X}]$, we thus have
\[ \omega_t s_{\lambda} [X/(1 - t)] = s_{\lambda} [-\tilde{X} / (1 - 1/t)] = s_{\lambda} [t \tilde{X} / (1 - t)] = (-t)^{|\lambda|} s_{\lambda} [X/(1 - t)], \]  

which completes the proof. \hfill \Box

The action of $\omega_t$ on a $k$-Schur function appears to take a simple form generalizing Conjecture 19.
Conjecture 40. For any k-bounded partition $\lambda$,\
$$\omega_1 s^{(k)}_\lambda [X; t] = t^{-c(\lambda)} s^{(k)}_{\omega_\lambda} [X; t],$$
(7.6)
where $c(\lambda)$ is some nonnegative integer.

Another unique property of the Schur functions is the expansion,
$$s_\lambda[X + Y] = \sum_{|\mu| + |\rho| = |\lambda|} c_{\mu \rho}^\lambda s_\mu [X] s_\rho [Y] \quad \text{where} \quad c_{\mu \rho}^\lambda \in \mathbb{N}.$$  
(7.7)
We have found by experimentation that the k-Schur functions also satisfy a similar relation,

Conjecture 41. For any k-bounded partition,\
$$s^{(k)}_\lambda [X + Y; t] = \sum_{|\mu| + |\rho| = |\lambda|} g_{\mu \rho}^\lambda(t) s^{(k)}_\mu [X; t] s^{(k)}_\rho [Y; t] \quad \text{where} \quad g_{\mu \rho}^\lambda \in \mathbb{N}[t].$$
(7.8)

8. Appendix

8.1. Definition of $\mathcal{H}_S[X; t]$ \[2\]. Consider a sequence of partitions $S = (\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}, \ldots)$ with $\eta_i$ parts (some of which may be zero) in $\lambda^{(i)}$. If $\eta = (\eta_1, \eta_2, \ldots)$ and $\ell(S) \equiv |\eta| = n$, we set
$$\text{Roots}_n = \{(i, j) \mid 1 \leq i \leq \eta_1 + \cdots + \eta_k < j \leq n \text{ for some } l\}.$$  
(8.1)
The formal power series is then defined, for the alphabet $X_n = x_1 + \cdots + x_n$, by
$$B_\eta[X_n; t] = \prod_{(i, j) \in \text{Roots}_n} \frac{1}{1 - tx_i/x_j}.$$  
(8.2)

Given a function $f(x_1, \ldots, x_n)$, the action of a permutation $\sigma$ of $S_n$ is defined by
$$\sigma f(x_1, x_2, \ldots, x_n) = f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}).$$  
(8.3)

If $A$ is the the antisymmetrizer, $\sum_{\sigma \in S_n} \text{sign}(\sigma) \mu$, and $\delta = (n-1, n-2, \ldots, 0)$, the operator $\pi$ is defined as
$$\pi(f) = A(x^\delta f)/A(x^\delta).$$  
(8.4)

Denoting the concatenation of the partitions in $S$ by $\bar{S}$, we then have the generating series
$$\mathcal{H}_S[X_n; t] = \pi \left( x^{-\bar{S}} B_\eta[X; t] \right).$$  
(8.5)

Since $\pi$ sends a polynomial in the variables of $X_n$ to a symmetric polynomial, $\mathcal{H}_S$ is symmetric in $X_n$. Letting $P_n$ be the set of elements of $\mathbb{Z}^n$ whose entries are weakly decreasing, we can thus write
$$\mathcal{H}_S[X_n; t] = \sum_{\omega \in P_n} K_{\omega; S} t s_\omega [X_n].$$  
(8.6)

Considering only partitions of $|S| \equiv |\lambda^{(1)}| + |\lambda^{(2)}| + \cdots$, we finally define the symmetric polynomial
$$\mathcal{H}_S[X_n; t] = \sum_{\mu \vdash |S|} K_{\mu; S} t s_\mu [X_n].$$  
(8.7)

Note that if $S'$ denotes the sequence of partitions obtained by appending the partition $(0^\ell)$ to a sequence of partitions $S$, then $|S'| = |S|$. In this case, we can show that
$$\mathcal{H}_{S'}[X_{n+\ell}] = \sum_{\mu \vdash |S'|} K_{\mu; S'} t s_\mu [X_{n+\ell}] = \sum_{\mu \vdash |S|} K_{\mu; S} t s_\mu [X_{n+\ell}],$$  
(8.8)
that is, $\mathcal{H}_S[X_{n+\ell}]$ is equal to $\mathcal{H}_S[X_n \rightarrow X_{n+\ell}]$. The number of variables in $\mathcal{H}_S[X_n]$ is thus irrelevant (as long as it is large enough) and therefore, at times, we work with the infinite alphabet $X$.  


8.2. Properties of the generalized Kostka polynomials. It is known that the coefficients $K_{\mu;S}(t)$ obey a Morris-type recurrence. Starting with a one-element sequence $S = (\lambda^{(1)})$, we have

$$K_{\mu;S}(t) = \begin{cases} 1 & \text{if } \mu = \lambda^{(1)} \\ 0 & \text{otherwise} \end{cases}. \quad (8.9)$$

Then for an arbitrary sequence $S = (\lambda^{(1)}, \lambda^{(2)}, \ldots)$, letting $m = \ell(\lambda^{(1)})$ and $\hat{S} = (\lambda^{(2)}, \lambda^{(3)}, \ldots)$, $K_{\mu;S}(t)$ satisfies the recurrence

$$K_{\mu;S}(t) = \sum_{w \in S_n / (S_m \times S_{n-m})} (-1)^{\ell(w) - |\lambda^{(1)}|} \sum_{\nu \vdash |S|} c^\nu_{\alpha(w) / \lambda^{(1)}, \beta(w)} K_{\nu;\hat{S}}(t), \quad (8.10)$$

where $w$ runs over minimal length coset representatives, and where $\alpha(w)$ and $\beta(w)$ are the first $m$ and last $n - m$ parts of the weight $w^{-1}(\mu + \delta) - \delta$, respectively. Note that $\beta(w)$ is always a partition, and that the $w$-th summand is understood to be zero unless all the parts of $\alpha(w)$ are nonnegative and $\alpha(w) \supseteq \lambda^{(1)}$. Further, $c^\nu_{\alpha(w) / \lambda^{(1)}, \beta(w)}$ is the Littlewood-Richardson coefficient

$$c^\nu_{\alpha(w) / \lambda^{(1)}, \beta(w)} = (s_{\alpha(w) / \lambda^{(1)}}[X] s_{\beta(w)}[X], s_{\nu}[X]). \quad (8.11)$$

Note that any element $w$ of $S_n / (S_m \times S_{n-m})$ is of the form

$$[[i_1, \ldots, i_m]] = [[i_1, \ldots, i_m, 1, \ldots, \hat{i}_1, \ldots, \hat{i}_m, \ldots, n]] \quad 1 \leq i_1 < \cdots < i_m \leq n, \quad (8.12)$$

where $1, \ldots, i_1, \ldots, i_m, \ldots, n$ denotes the sequence $1, \ldots, n$ with the elements $i_1, \ldots, i_m$ omitted. Since any permutation $w = [w_1, \ldots, w_n]$ satisfies

$$w^{-1}(\mu + \delta) - \delta = (\mu w_1 - (w_1 - 1), \ldots, \mu w_n - (w_n - n)) \quad (8.13)$$

in the case that $w = [[i_1, \ldots, i_m]], (8.12)$ implies that

$$(\mu_1, \ldots, \mu_m) \supseteq \alpha(w) \quad (8.14)$$

and

$$\mu_j = (\beta(w))_{j-m} \quad \text{for all } j > i_m. \quad (8.15)$$

We now prove two properties held by the generalized Kostka polynomials, $K_{\mu;S}(t)$, when the sequence $S$ is dominant, that is, when $\hat{S}$ is a partition.

**Proposition 42.** If $S$ is a dominant sequence of partitions with $\hat{S} = \mu$, then $K_{\mu;S}(t) = 1$.

**Proof.** For $S$ with one element, the claim follows from (8.13). Let $S = (\lambda^{(1)}, \lambda^{(2)}, \ldots)$, with $\ell(\lambda^{(1)}) = m$. Since $\hat{S} = \mu$, we have $K_{\hat{\mu};\hat{S}} = 1$, where $\hat{\mu} = (\mu_{m+1}, \mu_{m+2}, \ldots)$, by induction on the number of elements in $S$. Consider the $w$-th summand of (8.14). Since the only non-zero terms occur when $\alpha(w) \supseteq \lambda^{(1)} = (\mu_1, \ldots, \mu_m)$, and $\alpha(w) \subseteq (\mu_1, \ldots, \mu_m)$ by (8.14), the only term is when $\alpha(w) = \lambda^{(1)}$ and $\beta(w) = (\hat{S}) \equiv \hat{\mu}$. Thus, $K_{\mu;S}(t) = c^0_{\alpha, \beta} K_{\hat{\mu};\hat{S}} = K_{\hat{\mu};\hat{S}} = 1$, since $c^0_{\alpha, \beta} = 0$ unless $\nu = \gamma$. \hfill $\square$

We are also able to prove that many of the generalized Kostka polynomials vanish. To this end, we need the following two lemmas. The first lemma concerns the Littlewood-Richardson coefficients.

**Lemma 43.** Let $\nu$, $\alpha$, $\lambda$ and $\beta$ be partitions and let $N = |\alpha| - |\lambda|$. If $\beta^N \equiv (\beta_1 + N, \beta_2, \beta_3, \ldots)$ then $c^\nu_{\alpha / \lambda, \beta} = 0$ when $\nu \nleq \beta^N$.

**Proof.** Given $s_{\alpha / \lambda} = \sum_{\rho \vdash N} d_{\rho} s_{\rho}$, we have, by definition, $c^\nu_{\alpha / \lambda, \beta} = \sum_{\rho \vdash N} d_{\rho} c^\nu_{\rho, \beta}$. Therefore, the lemma will hold if we can show that $c^\nu_{\rho, \beta} = 0$ for all $\rho \vdash N$ when $\nu \nleq \beta^N$. Since $\rho \vdash N$ implies $\rho' \geq (N)'$, we have $(\rho' \cup (\beta')') = ((N)' \cup (\beta')') = \beta^N$. But Property (1) gives that $c^\nu_{\rho, \beta} = 0$ unless $\nu \leq (\rho' \cup (\beta')')$. Thus, $c^\nu_{\rho, \beta} = 0$ unless $\nu \leq (\rho' \cup (\beta')') \leq \beta^N$, that is $c^\nu_{\rho, \beta} = 0$ when $\nu \nleq \beta^N$. \hfill $\square$
Lemma 44. Let $\mu$ and $\lambda$ be partitions such that $\mu \nleq \lambda$, and let $\alpha$ and $\beta$ denote respectively the first $m$ and last $n-m$ parts of $w^{-1}(\mu + \delta) - \delta$ for $w \in S_m/(S_m \times S_{n-m})$. If $\alpha \supseteq (\lambda_1, \ldots, \lambda_m)$ then $\beta^N \nsubseteq (\lambda_{m+1}, \ldots, \lambda_n)$, where $\beta^N$ is as defined in Lemma 44.

Proof. Since $w = [i_1, \ldots, i_m]$, (8.14) implies $(\mu_1, \ldots, \mu_m) \supseteq \alpha$. Thus, given $\alpha \supseteq (\lambda_1, \ldots, \lambda_m)$, $\lambda_r \leq \alpha_r \leq \mu_r$ for all $r \leq m$.

(8.16)

Now, $\mu \nleq \lambda$ implies that there exists some $i$ that is the smallest integer such that

$$\mu_1 + \cdots + \mu_i < \lambda_1 + \cdots + \lambda_i.$$  

(8.17)

This necessarily implies that $\mu_i < \lambda_i$, and therefore $i > m$ by (8.16). We now prove a result from which the lemma will then follow. That is, beyond the $i$th entry, $\alpha \cup \beta$ is identical to $\mu$;

$$\alpha \cup \beta = (\ldots, \ldots, \beta_{i-m+1}, \ldots, \beta_{n-m}) = (\ldots, \ldots, \mu_{i+1}, \ldots, \mu_n).$$  

(8.18)

The lemma follows since $|\alpha| + |\beta| = |\alpha \cup \beta| = |\mu|$ implies by (8.18) that $|\alpha| + \beta_1 + \cdots + \beta_{i-m} = (\alpha \cup \beta)_1 + \cdots + (\alpha \cup \beta)_i = \mu_1 + \cdots + \mu_i$. Which is to say that $|\alpha| + \beta_1 + \cdots + \beta_{i-m} < \lambda_1 + \cdots + \lambda_m + \lambda_{m+1} + \cdots + \lambda_i$, by (8.17). With $N = |\alpha| - (\lambda_1 + \cdots + \lambda_m)$, we then have $N + \beta_1 + \cdots + \beta_{i-m} < \lambda_{m+1} + \cdots + \lambda_i$, which gives $\beta^N \nsubseteq (\lambda_{m+1}, \ldots, \lambda_n)$.

Proof of (8.18). We first show that beyond the $i$th entry, $w^{-1}(\mu + \delta) - \delta$ and $\mu$ are identical;

$$w^{-1}(\mu + \delta) - \delta = (\ldots, \ldots, \beta_{i-m+1}, \ldots, \beta_{n-m}) = (\ldots, \ldots, \mu_{i+1}, \ldots, \mu_n).$$  

(8.19)

With $\mu_i < \lambda_i$ and $i > m$, we have $\mu_i < \lambda_i \leq \lambda_m$ and thus $\mu_i < \lambda_m \leq \alpha_m$ by (8.16). Further, using (8.13), we have that $\alpha_m = \mu_m - (i_m - m) \leq \mu_m$ since $i_m \geq m$. Thus, $\mu_i < \alpha_m \leq \mu_m$, which gives $i_m < i$. Hence, since $\beta_{j-m} = \mu_j$ for all $j > i_m$ by (8.13), this also holds for $j > i$, that is, $(\beta_{i-m+1}, \ldots, \beta_{n-m}) = (\mu_{i+1}, \ldots, \mu_n)$. Now, given (8.19), we will show (8.18). Note that $\alpha \cup \beta$ is the rearrangement of $w^{-1}(\mu + \delta) - \delta$. Since $\beta$ is a partition, the rearrangement will not concern the entries of $\beta$. $\mu_i < \alpha_m$ implies that $\beta_{i-m+1} = \mu_{i+1} \leq \mu_i < \alpha_m$ (i.e. $\beta_{i-m+1}$ is smaller than the smallest element of $\alpha$), and we have $\alpha \cup \beta = (\ldots, \ldots, \beta_{i-m+1}, \ldots, \beta_{n-m}) = (\ldots, \ldots, \mu_{i+1}, \ldots, \mu_n)$. \[
\square
\]

We can now prove the following statement concerning the generalized Kostka polynomials.

Proposition 45. If $S$ is dominant with $\bar{S} = \lambda$, then $K_{\mu;S}(t) = 0$ for $\mu \nleq \lambda$.

Proof. If $S = (\lambda^{(1)})$ has only one element, then (8.9) gives that $K_{\mu;S}(t) = 0$ when $\mu \nleq \lambda^{(1)}$.

Assume by induction on the number of parts of $S$ that $K_{\nu;\bar{S}}(t) = 0$ for all $\nu \nleq \bar{S}$. We now show that the right hand side of (8.10) is zero for $\mu \nleq \bar{S} = \lambda$. Since the $w$-th summand is non-zero only when $\alpha(w) \supseteq (\lambda^{(1)})$, the conditions of Lemma 44 are satisfied. Therefore, $\beta(w)^N$ is such that $\beta(w)^N \nleq (\bar{S})$. Since Lemma 44 gives that $c'_{\alpha(w)\lambda^{(1)}}, \beta(w) = 0$ when $\nu \nleq \beta(w)^N$, the only non-zero terms in the sum occur when $\nu \leq \beta(w)^N$, with $\bar{S} \nleq \beta(w)^N$. If we assumed $\nu \geq \bar{S}$, then $\bar{S} \leq \nu \leq \beta(w)^N$ would imply $\bar{S} \leq \beta(w)^N$. By contradiction, we have that the non-zero terms occur when $\nu \geq \bar{S}$. Our induction hypothesis then proves all terms are zero. \[
\square
\]
9. Tables

In the tables below, we have not included the cases when $k \geq |\lambda|$, which, from Property 37, simply correspond to the trivial cases $s_\lambda[X;t] = s_\lambda[X]$.

9.1. $k$-Schur functions in terms of Schur functions.

| $k = 2$ | $1^2$ | $21$ | $3$ | $k = 2$ | $1^2$ | $21^2$ | $2^2$ | $31$ | $4$ |
|---------|-------|------|-----|---------|-------|-------|-----|-----|-----|
| $1^3$   | 1     | $t$  |     | $1^2$   | 1     | $t$   | $t^2$|     |     |
| $21^3$  |       | $t$  |     | $21^2$  |       | 1     | $t$  |     |     |
| $2^3$   |       | 1    |     | $2^2$   |       | 1     | $t$  |     |     |

| $k = 2$ | $1^3$ | $21^4$ | $2^4$ | $31^2$ | $32$ | $41$ | $5$ |
|---------|-------|--------|-------|--------|-----|-----|-----|
| $1^3$   | 1     | $t$   | $t^2$|       |     |     |     |
| $21^3$  |       | $t$   | $t^2$|       |     |     |     |
| $2^4$   | 1     | $t$   | $t^2$|       |     |     |     |
| $31^2$  |       |       | 1    | $t$   |     |     |     |
| $32$    |       |       | 1    | $t$   |     |     |     |

| $k = 3$ | $1^3$ | $21^3$ | $2^3$ | $31^2$ | $32$ | $41$ | $5$ | $k = 3$ | $1^4$ | $21^2$ | $2^2$ | $31$ | $4$ |
|---------|-------|--------|-------|--------|-----|-----|-----|---------|-------|--------|-------|-----|-----|
| $1^3$   | 1     | $t$   | $t^2$|       |     |     |     | $1^4$   | 1     | $t$   |     |     |     |
| $21^3$  |       | $t$   | $t^2$|       |     |     |     | $21^2$  |       | 1     | $t$ |     |     |
| $2^3$   | 1     | $t$   | $t^2$|       |     |     |     | $2^2$   |       | 1     |     |     |     |
| $31^2$  |       |       | 1    | $t$   |     |     |     | $31$    |       |       |     |     |     |
| $32$    |       |       | 1    | $t$   |     |     |     | $32$    |       |       |     |     |     |

| $k = 4$ | $1^3$ | $21^3$ | $2^3$ | $31^2$ | $32$ | $41$ | $5$ |
|---------|-------|--------|-------|--------|-----|-----|-----|
| $1^3$   | 1     | $t$   |       |       |     |     |     |
| $21^3$  |       | $t$   |       |       |     |     |     |
| $2^3$   | 1     |       |       |       |     |     |     |
| $31^2$  |       |       |       |       |     |     |     |
| $32$    |       |       |       |       |     |     |     |

| $k = 4$ | $1^4$ | $21^2$ | $2^2$ | $31^2$ | $32$ | $41$ | $5$ | $k = 4$ | $1^4$ | $21^2$ | $2^2$ | $31^2$ | $32$ | $41$ | $5$ |
|---------|-------|--------|-------|--------|-----|-----|-----|---------|-------|--------|-------|--------|-----|-----|-----|
| $1^4$   | 1     | $t$   |       |       |     |     |     | $1^4$   | 1     | $t$   |       |       |     |     |     |
| $21^2$  |       |       |       |       |     |     |     | $21^2$  |       |       |       |       |     |     |     |
| $2^2$   | 1     |       |       |       |     |     |     | $2^2$   |       |       |       |       |     |     |     |
| $31^2$  |       |       |       |       |     |     |     | $31^2$  |       |       |       |       |     |     |     |
| $32$    |       |       |       |       |     |     |     | $32$    |       |       |       |       |     |     |     |
| $41^2$  |       |       |       |       |     |     |     | $41^2$  |       |       |       |       |     |     |     |
| $42$    |       |       |       |       |     |     |     | $42$    |       |       |       |       |     |     |     |
### 9.2. Macdonald polynomials in terms of $k$-Schur functions.

| $k = 2$ | $1^4$ | $21$ | $k = 2$ | $1^4$ | $21^2$ | $k = 2$ | $1^3$ | $21^2$ | $k = 2$ | $1^3$ | $21^3$ | $2^2$ |
|--------|--------|------|--------|--------|--------|--------|--------|--------|--------|--------|--------|------|
| $1^3$  | $1$    | $t^2$| $1^4$  | $1$    | $t^2 + t^3$| $t^4$  | $1$    | $t^2 + t^3$| $t^6$  |
| $21$   | $q$    | $1$  | $21^2$ | $q$    | $1 + qt^2$| $t$    | $21^3$ | $q$    | $1 + qt^3$| $t^2$  |
|        |        |      | $2^2$  | $q^2$  | $q + qt$ | $1$    | $2^4$  | $q^4$  | $q + qt$ | $1$    |      |

| $k = 3$ | $1^3$ | $21^3$ | $k = 3$ | $1^4$ | $21^3$ | $k = 3$ | $1^3$ | $21^2$ | $k = 3$ | $1^3$ | $21^3$ | $31^2$ |
|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|------|
| $1^3$  | $1$    | $t^2 + t^3 + t^5$| $1^3$  | $1$    | $t^2 + t^3 + t^5$| $1^3$  | $1$    | $t^2 + t^3 + t^5$| $3^2$  |
| $21^3$ | $q$    | $1 + qt^2 + qt^3$| $2^1$  | $q^2$  | $q + qt + q^2t^2$| $31^2$ | $q^3$  | $q + q^2t + q^3t^2$| $32$   |
| $4^2$  | $q^4$  | $q^2 + q^3t + q^4t^2$| $1 + qt^2 + qt^3$| $t$    |        | $32$   | $q^3$  | $q + q^2t + q^3t$| $t$    |

| $k = 4$ | $1^3$ | $21^3$ | $2^1$ | $31^2$ | $32$ | $41$ |
|--------|--------|--------|--------|--------|------|------|
| $1^3$  | $1$    | $t^2 + t^3 + t^5$| $2^1$  | $q$    | $1 + qt^2 + qt^3$| $31^2$ | $q^3$  | $q + q^2t + q^3t^2$| $32$   |
| $2^1$  | $q^2$  | $q + qt + q^2t^2$| $31^2$ | $q^3$  | $q + q^2t + q^3t^2$| $32$   | $q^3$  | $q + q^2t + q^3t$| $t$    |
| $4^1$  | $q^4$  | $q^2 + q^3t + q^4t^2$| $31^2$ | $q^3$  | $q + q^2t + q^3t^2$| $32$   | $q^3$  | $q + q^2t + q^3t + q^4t^2$| $t$    |

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