TANGENT CONES AND $C^1$ REGULARITY OF DEFINABLE SETS

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Abstract. Let $X \subset \mathbb{R}^n$ be a connected locally closed definable set in an o-minimal structure. We prove that the following three statements are equivalent: (i) $X$ is a $C^1$ manifold, (ii) the tangent cone and the paratangent cone of $X$ coincide at every point in $X$, (iii) for every $x \in X$, the tangent cone of $X$ at the point $x$ is a $k$-dimensional linear subspace of $\mathbb{R}^n$ ($k$ does not depend on $x$) varies continuously in $x$, and the density $\theta(X,x) < 3/2$.

1. Introduction

Let $X$ be a subset of $\mathbb{R}^n$ and let $x \in \mathbb{R}^n$. The tangent cone $\operatorname{tg}_x X$ and paratangent cone $\operatorname{ptg}_x X$ of $X$ at the point $x$ are defined as follows: if $x \notin X$, $\operatorname{tg}_x X = \operatorname{ptg}_x X = \{0\}$, and otherwise,

$$\operatorname{tg}_x X := \{ au \mid a \in \mathbb{R}, a \geq 0, u = \lim_{i \to \infty} \frac{x_i - x}{\|x_i - x\|}, \{x_i\} \subset X, \{x_i\} \to x \},$$

$$\operatorname{ptg}_x X := \{ au \mid a \geq 0, u = \lim_{i \to \infty} \frac{x_i - y_i}{\|x_i - y_i\|}, X \supset \{x_i\} \to x, X \supset \{y_i\} \to x \}.$$

Note that $\operatorname{tg}_x X$ and $\operatorname{ptg}_x X$ are closed sets in $\mathbb{R}^n$. We denote by $\operatorname{tg}_X := \{(x,v), x \in X, v \in \operatorname{tg}_x X\}$ and $\operatorname{ptg}_X := \{(x,v), x \in X, v \in \operatorname{ptg}_x X\}$.

Characterizing $C^1$ submanifolds of $\mathbb{R}^n$ in terms of their tangent cones has been studied by many authors, see for example [8], [10], [2], [6], or a survey of Bigolin and Golo [1]. In this paper we restrict ourselves to this problem in the context of o-minimal structures. We first prove that a connected locally closed definable subset of $\mathbb{R}^n$ is a $C^1$ manifold if and only if its tangent cone and paratangent cone coincide at every point (Theorem 3.7). This result is a strong version of the two-cones coincidence theorem (Theorem 3.6) which was initially proved by Tierno [10]. The result is no longer true if definability is omitted (Remark 3.9).

Next, we discuss a result recently established by Ghomi and Howard [6] that if $X \subset \mathbb{R}^n$ is a locally closed set such that for each $x \in X$, $\operatorname{tg}_x X$ is a hyperplane (i.e., a $(n-1)$ linear subspace of $\mathbb{R}^n$), and varies continuously in $x$, then $X$ is a union of $C^1$ hypersurfaces. Moreover, if the lower density $\Theta(X,x)$ is at most $m < \frac{3}{2}$ for every $x \in X$ then $X$ is a $C^1$ hypersurface. A natural question thus arises here is whether the result remains true if $\operatorname{tg}_x X$ are $k$-planes with $k < n-1$.

In section 4, we show in Example 4.3 that the first statement in the result of Ghomi-Howard is not always true if $k < (n-1)$. We also prove that the second statement is still valid, more precisely that if $X$ is a locally closed definable set such that for every $x \in X$, $\operatorname{tg}_x X$ is a $k$-dimensional linear subspace ($k$ is independent of $x$) varying continuously in $x$ and the density $\theta(X,x) < 3/2$ (need not be upper
bounded by an \( m < \frac{3}{2} \), then \( X \) is a \( C^1 \) manifold (Theorem 1.7). Notice that, in general, notions of lower density \( \Theta(X,x) \) and density \( \theta(X,x) \) are different. Nevertheless, with the conditions on tangent cones as above, they coincide. Therefore, our result can be considered as a generalization of the result of Ghomi-Howard.

Throughout the paper, \( \mathbb{R}^n \) denotes the \( n \)-dimensional Euclidean space equipped with the standard norm \( \|x\| = \sqrt{x_1^2 + \ldots + x_n^2} \) where \( x = (x_1, \ldots, x_n) \); \( \mathbb{B}^n(x,r) \), \( \mathbb{B}^n(x,r) \), and \( \mathbb{S}^{n-1}(x,r) \) denote respectively the closed ball, the open ball and the sphere in \( \mathbb{R}^n \) of radius \( r \) centered at \( x \). Let \( X \) be a subset in \( \mathbb{R}^n \). Denote by \( \overline{X} \) the closure of \( X \) in \( \mathbb{R}^n \) and by \( \partial X := \overline{X} \setminus X \) the boundary of \( X \). Let \( f \) be a map. We denote by \( \Gamma_f \) the graph of \( f \).

The Grassmanian \( \mathbb{G}_n^k \) of all \( k \)-dimensional linear subspaces of \( \mathbb{R}^n \) is endowed with the metric \( \delta \) defined as follows: for \( P,Q \) in \( \mathbb{G}_n^k \),

\[
\delta(P,Q) = \sup_{v \in P, \|v\|=1} \{\|v - \pi_Q(v)\|\},
\]

where \( \pi_Q : \mathbb{R}^n \to Q \) is the orthogonal projection from \( \mathbb{R}^n \) to \( Q \). Following [6], we say that \( P \in \mathbb{G}_n^k \) and \( Q \in \mathbb{G}_n^k \) are orthogonal when \( \delta(P,Q) = 1 \). Remark that this terminology does not coincide with the usual notion of orthogonality for general subspaces in Euclidean geometry: \( P = \{(x,y,z) \mid x = 0\} \) and \( Q = \{(x,y,z) \mid y = 0\} \) are orthogonal according to our definition but not all vectors in \( P \) are orthogonal to any vector in \( Q \).

By a \( k \)-dimensional \( C^1 \) manifold in \( \mathbb{R}^n \) (or \( C^1 \) manifold for simplicity) we mean a subset of \( \mathbb{R}^n \), locally \( C^1 \) diffeomorphic to \( \mathbb{R}^k \); a hypersurface in \( \mathbb{R}^n \) is a \( C^1 \) manifold in \( \mathbb{R}^n \) of dimension \( n-1 \).

Let \( X \subset \mathbb{R}^n \). In the paper, we often denote by \( \pi_x : \mathbb{R}^n \to \operatorname{tg}_x X \) the orthogonal projection. By abuse of notation, here we identify \( \operatorname{tg}_x X \) with its translation \( \{x + \operatorname{tg}_x X\} \).

By a definable set we mean a set which is definable (with parameters) in an \( o \)-minimal expansion (\( \mathbb{R}, <, +, \ldots \)) of the ordered field of real numbers. Definable sets form a large class of subsets of \( \mathbb{R}^n \): for instance, any semi-algebraic set, any sub-analytic set is definable. We refer the reader to [11], [5] for the basic properties of \( o \)-minimal structures. In the paper, we will use Curve selection Lemma ([5], Theorem 3.2), Uniform finiteness on fibers ([5] Theorem 2.9) and Hardt’s definable triviality Theorem ([5], Theorem 5.22) without repeating the references.

2. Bundle of vector spaces

Let \( X \) be a subset of \( \mathbb{R}^n \). Let \( E \subset X \times \mathbb{R}^n \). For \( x \in X \) we denote by \( E_x := \{v \in \mathbb{R}^n : (x,v) \in E\} \) the fiber of \( E \) at the point \( x \). For \( U \subset X \), we set \( E|_U := \{(x,v) \in E : x \in U\} \) and call it the restriction of \( E \) to \( U \). If every fiber of \( E \) is a linear subspace of \( \mathbb{R}^n \) we call \( E \) a bundle of vector spaces over \( X \), or a bundle over \( X \), or just a bundle if the base \( X \) is clear from the context. We call \( E \) a trivial bundle if all its fibers have the same dimension, and a closed bundle if it is a closed set in \( \mathbb{R}^{2n} \).
Suppose $E$ is a trivial bundle over $X$. If the map $X \to \mathbb{G}_n^k$ defined by $x \mapsto E_x$ is continuous, we say that $E$ is a continuous trivial bundle.

**Lemma 2.1** ([7], Propositions I, II, pages 39, 40). Let $E \subset X \times \mathbb{R}^n$.

(i) If $E$ is a closed trivial bundle then $E$ is continuous.

(ii) If $E$ is a closed bundle then the function $x \mapsto \dim E_x$ is upper-semicontinuous, i.e., for $x \in X$ there is an open neighborhood $U_x$ of $x$ in $X$ such that

$$\dim E_x \geq \dim E_y \text{ for every } y \in U_x.$$ 

**Proof.** (i) - Suppose that $k$ is the dimension of fibers of $E$. If $E$ is not continuous, there exists a sequence $\{x_i\} \subset X$ tending to $x$, $\lim_{i \to \infty} E_{x_i} \to \tau \in \mathbb{G}_n^k$ and $\tau \not\in E_x$. By the closedness of $E$, $\tau \subset E_x$, which is a contradiction.

(ii) - Suppose the assertion is not true, i.e., there exist a point $x \in X$ and a sequence $\{x_m\}$ in $X$ tending to $x$ such that $\dim E_{x_m} > \dim E_x$. We may assume that $\lim_{m \to \infty} E_{x_m} = P \in \mathbb{G}_n^k$, since $\mathbb{G}_n^k$ is compact. Note that $k > \dim E_x$. Since $E$ is closed, $P \subset E_x$. This implies $k \leq \dim E_x$, which is a contradiction. 

**Lemma 2.2.** Let $X$ be a locally closed subset of $\mathbb{R}^n$. If $\text{ptg}X$ is a trivial bundle then $\text{ptg}X$ is continuous.

**Proof.** It follows directly from the definition of the paratangent cone that $\text{ptg}(X)$ is a closed set in $\mathbb{R}^{2n}$ and $\text{ptg}_xX = \text{ptg}_x\overline{X}$ for every $x \in X$. If $V \subset X$ is a closed set in $\mathbb{R}^n$ then $\text{ptg}X|_V = \text{ptg}\overline{X}|_V$ is a closed set in $\mathbb{R}^{2n}$.

Let $x \in X$. Since $X$ is locally closed, there is $W_x$, a neighborhood of $x$ in $X$, which is closed in $\mathbb{R}^n$. The restriction $\text{ptg}X|_{W_x}$ is then a closed set in $\mathbb{R}^{2n}$. Since $\text{ptg}X$ is a trivial bundle, so is $\text{ptg}X|_{W_x}$. By (i) in Lemma 2.1 $\text{ptg}X|_{W_x}$ is continuous, $\text{ptg}X$ is, therefore, a continuous trivial bundle. 

3. Two-cones coincidence Theorem

The aim of this section is to prove Theorem 3.7, a strong version of two-cones coincidence theorem of Tierno for definable sets.

We need the following two lemmas which generalize Lemma 3.3 and Lemma 3.1 in [6].

**Lemma 3.1.** Let $X$ be a locally closed subset of $\mathbb{R}^n$ such that $\text{tg}X$ is a continuous trivial bundle of $k$-dimensional vector spaces. Let $x \in X$ and $H$ be a $k$-plane in $\mathbb{R}^n$ which is not orthogonal to $\text{tg}_xX$. Let $\pi : \mathbb{R}^n \to H$ the orthogonal projection. Then, there exists an open set $U$ of $x$ in $X$ such that $\pi|_U : U \to H$ is an open map.

**Proof.** The proof follows closely the proof of Lemma 3.3, [6].

By the continuity of $\text{tg}X$, we can choose an open neighborhood $U$ of $x$ in $X$ such that for all $q \in U$, $\text{tg}_qX$ is not orthogonal to $H$, or equivalently that $\text{tg}_qX$ is transverse to $H^\perp$, the orthogonal complement of $H$ in $\mathbb{R}^n$. We will prove that $\pi|_U$ is an open map. Fix $q \in U$. By the local closedness of $X$, there is an $r > 0$ small enough such that $W := X \cap \overline{B^n}(q, r) \subset U$ is a compact set. Moreover, the boundary $\partial W$ is in $\partial \overline{B^n}(q, r)$, meaning $q \not\in \partial W$. With $r$ sufficiently small, we may assume that

$$\pi(q) \not\in \pi(\partial W)$$
because, otherwise, there exists a sequence of positive numbers \( \{r_i\} \) tending to 0 such that for each \( i \), there is a point \( q_i \in X \cap \partial B^k(q, r_i) \) with \( \pi(q_i) = \pi(q) \). This implies that \( \text{span}(q - q_i) \subset H^\perp \). As \( i \to \infty \) we have \( q_i \to q \) and the sequence \( \text{span}(q - q_i) \) (extracting a subsequence if necessary) tends to a line \( l \in \text{tg}_x X \). Thus, \( l \in H^\perp \cap \text{tg}_q X \); but \( \dim(H^\perp \cap \text{tg}_q X) = \dim H^\perp + \dim \text{tg}_q X - n = 0 \) since \( H^\perp \) is transverse to \( \text{tg}_q X \), which is a contradiction.

Since \( \pi(\partial W) \) is a compact set, there is \( s > 0 \) such that

\[
(3.1) \quad B^k(\pi(q), s) \cap \pi(\partial W) = \emptyset.
\]

It suffices to show that \( \pi(W) \) contains an open neighborhood of \( \pi(q) \) in \( H \).

Suppose on the contrary that \( \overline{B}^k(\pi(q), \varepsilon) \not\subset \pi(W), \forall \varepsilon > 0 \). Choose a point \( q' \in \overline{B}^k(\pi(q), s/2) \setminus \pi(W) \) and let \( s' \) be the distance from \( q' \) to \( \pi(W) \). Note that \( s' \leq s/2 \). Since \( \pi(W) \) is compact, \( \overline{B}^k(q', s') \cap \pi(W) = \emptyset \) and \( \overline{B}^k(q', s') \cap \pi(W) = \emptyset \).

For every \( p \in \overline{B}^k(q', s') \),

\[
\|p - \pi(q)\| \leq \|p - q'\| + \|q' - \pi(q)\| \leq s' + s/2 \leq s.
\]

This means that \( \overline{B}^k(q', s') \subset \overline{B}^k(\pi(q), s) \). By (3.1), \( \overline{B}^k(q', s') \cap \pi(\partial W) = \emptyset \). Take \( y' \in \overline{B}^k(q', s') \cap \pi(W) \) and \( y \in \pi^{-1}(y') \cap W \). Note that \( y \not\in \partial W \), so \( y \) is an interior point of \( W \), and hence \( \text{tg}_y W = \text{tg}_y X \) which is a linear subspace.

Since \( B^k(q', s') \cap \pi(W) = \emptyset \), no point of \( W \) is contained in the cylinder \( C := \pi^{-1}(B^k(q', s')) \). This implies that \( \text{tg}_y X \subset \text{tg}_y \partial C \). Both \( \text{tg}_y X \) and \( H^\perp \) are included in \( \text{tg}_y \partial C \), so

\[
\dim(\text{tg}_y X \cap H^\perp) \geq \dim \text{tg}_y X + \dim H^\perp - \dim \text{tg}_y \partial C = k + (n - k) - (n - 1) = 1.
\]

This shows that \( \text{tg}_y X \) is orthogonal to \( H \), which is a contradiction. \( \square \)

**Lemma 3.2.** Let \( U \subset \mathbb{R}^k \) be an open set and \( f : U \to \mathbb{R}^{n-k} \) be a map. Suppose that \( \Gamma_f \) is locally closed. If \( \text{tg}\Gamma_f \) is a continuous trivial bundle of \( k \)-dimensional vector spaces and \( \text{tg}_{(x,f(x))}\Gamma_f \) is not orthogonal to \( \mathbb{R}^k \) for every \( x \in U \), then \( f \) is \( C^1 \).

**Proof.** We first prove that \( f \) is continuous. Suppose on the contrary that \( f \) is not continuous, meaning that there are \( x \in U \) and a sequence \( \{x_i\} \in U \) tending to \( x \) such that \( \lim_{i \to \infty} f(x_i) = y \neq f(x) \).

Since \( \Gamma_f \) and the orthogonal projection \( \pi : \mathbb{R}^n \to \mathbb{R}^k \) satisfy the hypothesis of Lemma 5.1, there is an open set \( V \) of \((x, f(x))\) in \( \Gamma_f \) such that \( \pi|_V \) is an open map. Set \( W := \pi(V) \), which is an open neighborhood of \( x \) in \( \mathbb{R}^k \). Since \( \{x_i\} \) tends to \( x \), there is \( N \in \mathbb{N} \) such that \( x_i \in W \) for all \( i > N \). This implies that \((x_i, f(x_i)) \in V \) for all \( i > N \).

If \( y \neq \infty \), shrinking \( V \) so that \((x, y) \not\in \overline{V} \), there is a neighborhood \( K \) of \((x, y)\) in \( \mathbb{R}^n \) such that \( K \cap V = \emptyset \). Since \( f(x_i) \) tends to \( y \), we have \((x_i, f(x_i)) \in K \) for all \( i > N \) when \( N \) is large enough. This shows that \( K \cap V \neq \emptyset \), which is a contradiction.

If \( y = \infty \), \((x_i, f(x_i)) \not\in V \) for all \( i > N \) when \( N \) is large enough. This again gives a contradiction.

We now show that \( f \) is a \( C^1 \) map.
Let \( \{a_1, \ldots, a_k, b_1, \ldots, b_{n-k}\} \) be the canonical basis of \( \mathbb{R}^n \). For \( x \in U \), consider the function \( f^*_x(t) := f(x + ta_i) \). The graph \( \Gamma_{f^*_x} \) of \( f^*_x \) is the intersection \( \Gamma_{f^*_x} = \Gamma_f \cap (x, 0) + \text{span}(a_1, b_1, \ldots, b_{n-k}) \). This implies that
\[
\text{tg}_{(x,f(x))} \Gamma_{f^*_x} \subset \text{tg}_{(x,f(x))} \Gamma_f \cap \text{span}(a_1, b_1, \ldots, b_{n-k}) =: l_x.
\]
But \( l_x \) is a line, because \( \text{tg}_{(x,f(x))} \Gamma_f \) is not orthogonal to \( \mathbb{R}^k \times \{0\}^{n-k} \). On the other hand, since \( f \) is continuous, \( \Gamma_{f^*_x} \) is a continuous curve, so \( \text{tg}_{(x,f(x))} \Gamma_{f^*_x} \) has dimension at least 1. Then \( \text{tg}_{(x,f(x))} \Gamma_{f^*_x} \cap l_x = l_x \), so \( f^*_x \) is differentiable at \( t = 0 \). Thus, \( f \) has partial derivatives at any point.

The bundle \( \text{tg}_{(x,f(x))} \Gamma_f \) is continuous, hence \( l_x \), its transverse intersection with \( \text{span}(a_1, b_1, \ldots, b_{n-k}) \) is continuous. Therefore \( f \) has continuous partial derivatives on \( U \), so \( f \) is \( C^1 \).

\[ \square \]

**Remark 3.3.** The statement of Lemma 3.1 [6] is similar to the statement of Lemma 3.2 except the local closedness of \( \Gamma_f \) is missing. This is a gap because \( f \) might not be continuous if \( \Gamma_f \) is not locally closed. For example, consider the function \( f(x) = 0 \) if \( x \) is a rational number, and \( f(x) = 1 \) otherwise. The tangent cone to \( \Gamma_f \) is the \( x \)-axis at any point, hence \( \Gamma_f \) is a continuous trivial bundle, but \( f \) is not continuous.

**Theorem 3.4.** A locally closed set \( X \subset \mathbb{R}^n \) is a \( C^1 \) manifold if and only if \( \text{tg}X \) is a continuous trivial bundle and the restriction of the map \( \pi_x : X \to \text{tg}_x X \) to some neighborhood of \( x \) in \( X \) is injective.

**Proof.** The necessity is a trivial fact. We now prove the sufficiency. For \( x \in X \), by the hypothesis, there exists an open neighborhood \( U \) of \( x \) such that \( \varphi := \pi_x|_U : U \to \text{tg}_x X \) is injective. Moreover, \( \pi_x \) is open by Lemma 3.1. This implies \( \varphi : U \to \varphi(U) \) is a homeomorphism. Consider the map \( \psi := \varphi^{-1} : \varphi(U) \to U \subset \mathbb{R}^n \). We have \( \text{tg}\varphi = \text{tg}_X|_{\varphi(U)} \), which is a continuous trivial bundle. Shrinking \( U \) if necessary we may assume that \( \text{tg}\varphi \) is not orthogonal to \( \text{tg}_y X \) for every \( y \in U \). The map \( \psi \) then satisfies the conditions of Lemma 3.2, so it is of class \( C^1 \), meaning that \( U \) is a \( C^1 \) manifold.

\[ \square \]

**Remark 3.5.** Theorem 3.4 is slightly stronger than a similar result proved by Gluck (Theorem 10.1, [8]). In the result of Gluck, \( X \) is assumed to be a topological manifold instead of a locally closed set as in our statement.

**Theorem 3.6 (Two-cones coincidence, Tierno [10]).** A locally closed subset \( X \) of \( \mathbb{R}^n \) is a \( C^1 \) manifold if and only if \( TX \) and \( \text{ptg}X \) coincide, and both are trivial bundles of vector spaces over \( X \).

**Proof.** Since \( \text{ptg}X \) is a trivial bundle, it is continuous by Lemma 2.2. On the other hand, \( \text{tg}X = \text{ptg}X \), hence \( \text{tg}X \) is a continuous trivial bundle.

Let \( x \in X \). We denote by \( \pi_x : \mathbb{R}^n \to \text{tg}_x X \) the orthogonal projection. By Theorem 3.4, it suffices to prove that the map \( \pi_x \) is injective on some neighborhood of \( x \) in \( X \).

Suppose on contrary that there are sequences of points \( \{z_i\}_i \) and \( \{z'_i\}_i \) in \( X \) converging to \( x \) such that \( \pi_x(z_i) = \pi_x(z'_i) \) for all \( i \). This implies that \( \text{span}(z_i - z'_i) \)
accumulate to a line $l \subseteq \text{tg}_x X^\perp$. By the definition, $l \subseteq \text{ptg}_x X$. Since $\text{ptg}_x X = \text{tg}_x X$, $l \subseteq \text{tg}_x X \cap \text{tg}_x X^\perp = 0$, a contradiction. \hfill \square

**Theorem 3.7** (Definable two-cones coincidence). A connected, locally closed definable subset of $\mathbb{R}^n$ is a $C^1$ manifold if and only if $\text{tg}X$ and $\text{ptg}X$ coincide.

**Proof.** We just need to show the sufficiency. First we prove that for every $x \in X$, $\text{tg}_x X$ is a linear subspace of $\mathbb{R}^n$, or equivalently that $\text{tg}_x X$ is a bundle. Fix $x \in X$, we may identify $x$ with the origin 0. By the hypothesis, $\text{tg}_0 X = \text{ptg}_0 X$ which is symmetric, i.e., if $v \in \text{tg}_0 X$ so is $-v$. It is enough to verify that if $v, w \in \text{tg}_0 X$ then $v + w \in \text{tg}_0 X$. Since $v, -w \in \text{tg}_0 X$ and $X$ is a definable set there exist two curves $\gamma, \beta$ in $X$ starting at 0 such that $v \in \text{tg}_0 \gamma$ and $-w \in \text{tg}_0 \beta$ (see Curve selection Lemma). Choose sequences of points $\{x_i\}_i \subset \gamma$ and $\{y_i\}_i \subset \beta$ converging to 0 such that $\|x_i\| = \frac{a}{b}\|y_i\|$, where $a := \|v\|$ and $b := \|w\|$. Thus,

$$v + w = \lim_{i \to \infty} (a\frac{x_i}{\|x_i\|} - b\frac{y_i}{\|y_i\|}) = \lim_{i \to \infty} a\frac{1}{\|x_i\|}(x_i - y_i) \in \text{ptg}_0 X = \text{tg}_0 X.$$  

From now on, we set

$$O_k := \{ z \in X : \dim \text{tg}_x X \leq k \}.$$  

Fix $k$ and let $z \in O_k$. Take $V$ to be a closed neighborhood of $z$ in $X$. Since $X$ is locally closed, we may assume $V$ to be a closed set in $\mathbb{R}^n$ so $\text{ptg}X|_V$ is a closed bundle. By Lemma 2.1 (ii), for $y \in V$, the map $y \mapsto \dim \text{tg}_y X$ is upper-semicontinuous, so is the map $y \mapsto \dim \text{tg}_y X$, meaning that there exists $U \subset V$, an open neighborhood of $z$, such that $k = \dim \text{tg}_y X \geq \dim \text{ptg}_y X$ for all $y \in U$. This implies that $U \subset O_k$, hence $O_k$ is an open set in $X$.

Since $X$ is definable, $\dim \text{tg}_x X \leq \dim X = d$ for every $x \in X$ (see [9], Lemma 1.2), hence $O_d = X$. Set $O := X \setminus O_{d-1}$, which is a closed set of $X$.

We denote by $X_{\text{Sing}}$ the set of singular points of $X$, i.e., points at which $X$ fails to be a $C^1$ manifold of dimension $d$. Remark that $X_{\text{Sing}}$ is a definable set of dimension less than $d$ (see for instance [11], [6]).

Since $O_{d-1} \subset X_{\text{Sing}}$, $\dim O_{d-1} < d$. For $x \in O$,

$$\text{tg}_x X \supset \text{tg}_x O \supset \text{tg}_x X \setminus \text{tg}_x O_{d-1}.$$  

Taking the closures of all sets above,

$$\text{tg}_x X \supset \text{tg}_x O \supset \text{tg}_x X \setminus \text{tg}_x O_{d-1}.$$  

Because $\text{tg}_x X$ is a linear space of dimension $d$ and $\text{tg}_x O_{d-1}$ is a linear subspace of $\text{tg}_x X$ of dimension less than $d$, $\text{tg}_x X \setminus \text{tg}_x O_{d-1} = \text{tg}_x X$. So,

$$\text{tg}_x X = \text{tg}_x O \subset \text{ptg}_x O \subset \text{ptg}_x X = \text{tg}_x X.$$

Since $O$ is closed in the locally closed set $X$, it is also locally closed. As been shown above, $\text{tg}O = \text{ptg}O = \text{ptg}X|_O$ which is a trivial bundle. By Theorem 3.6 $O$ is a $C^1$ manifold of dimension $d$. Next we will prove that $X = O$, therefore, it is a $C^1$ manifold.

Let $x \in O$ and $\pi_x : \mathbb{R}^n \to \text{tg}_x O$ be the orthogonal projection. It follows from Theorem 3.4 and Lemma 3.1 that there is an open neighborhood $U$ of $x$ in $\mathbb{R}^n$ such that the restriction of $\pi_x$ to $U \cap O$ is injective and $V := \pi_x(U \cap O)$ is an open set in
tg_x O. Set W := π_x^{-1}(V) ∩ U, so that W is an open neighborhood of x in \( \mathbb{R}^n \) with \( \pi_x(W) = V \). This implies that

\[
\pi_x(W \cap X) = \pi_x(W \cap O) = V.
\]

Since tg_x O = ptg_x X, shrinking U if necessary, the restriction of \( \pi_x \) to \( W \cap X \) is injective. The sets \( W \cap X \) and \( W \cap O \) then are graphs of mappings over the same domain V. On the other hand, \( W \cap O \subset W \cap X \), then \( W \cap O = W \cap X \). This means that \( W \cap O \), an open neighborhood of \( x \) in \( O \), is an open neighborhood of \( x \) in \( X \). Thus, \( O \) is an open set in \( X \). Since \( O \) is both closed and open in \( X \) and \( X \) is connected, \( O \) is equal to \( X \).

\[\square\]

A direct consequence of Theorem 3.6 is:

**Corollary 3.8.** Let \( X \subset \mathbb{R}^n \) be a locally closed definable set. Suppose that \( \text{tg}_x X = \text{ptg}_x X \) for every \( x \in X \). Then, each connected component of \( X \) is a \( C^1 \) manifold.

**Remark 3.9.** The definability in the hypothesis of Theorem 3.7 is necessary. Consider the following locally closed sets,

\[
X := \{(x, \sin \frac{1}{x}), x \neq 0\} \setminus \{(0, 1), (0, -1)\} \subset \mathbb{R}^2,
\]

and

\[
Y := \{-1 - \frac{1}{n}, n \in \mathbb{N}\} \cup [-1, 1] \cup \{1 + \frac{1}{n}, n \in \mathbb{N}\}.
\]

The sets \( X \) and \( Y \) are not definable in any o-minimal structure : \( X \cap \mathbb{R} \times \{0\} \) and \( Y \) have infinitely many components. The set \( X \) is connected, \( \text{tg} X = \text{ptg} X \), but \( X \) is not a \( C^1 \) manifold. The set \( Y \) has \( \text{tg} Y = \text{ptg} Y \), but \([-1, 1]\), a connected component of \( Y \), is not a \( C^1 \) manifold.

**4. Definable sets with continuous trivial tangent cones**

Let us recall the result proved by Ghomi and Howard [6].

**Definition 4.1.** Let \( X \subset \mathbb{R}^n \), \( x \in \mathbb{R}^n \). Suppose that the Hausdorff dimension of \( \text{tg}_x X \), denoted by \( \dim_H \text{tg}_x X \), is an integer \( k \). The lower density \( \Theta(X, x) \) of \( X \) at the point \( x \) is defined as follows: if \( x \notin X \) then \( \Theta(X, x) = 0 \), and otherwise,

\[
\Theta(X, x) = \lim_{r \to 0} \inf \frac{\mathcal{H}^k(X \cap B^n(x, r))}{r^k \mu_k}
\]

where \( \mathcal{H}^k \) is the \( k \)-dimensional Hausdorff measure, \( \mu_k \) is the volume of the unit ball of dimension \( k \).

**Theorem 4.2** (Theorem 1.1, [6]). Let \( X \) be a locally closed subset of \( \mathbb{R}^n \). Suppose that \( \text{tg} X \) is a \((n - 1)\)-dimensional continuous trivial bundle. Then,

(i) \( X \) is a union of \( C^1 \) hypersurfaces;

(ii) if \( \Theta(X, x) \) is at most \( m < 3/2 \) then \( X \) is a \( C^1 \) hypersurface.

The following example shows that in general the statement (i) of Theorem 4.2 is no longer true when the hyperplanes are replaced by \( k \)-planes with \( k < n - 1 \), meaning that a locally closed subset in \( \mathbb{R}^n \) with continuous trivial tangent cone might not be a union of \( C^1 \) manifolds.
Example 4.3. We identify \( C \) with \( \mathbb{R}^2 \). Consider the map \( h : C \to C \) defined as follows:

\[
h(z) := \frac{z^2}{|z|^2} \text{ if } z \neq 0, \text{ and } h(0) = 0.
\]

Denote by \( X \subset \mathbb{R}^4 \) the graph of \( h \). We have

1. \( X \) is locally closed;
2. \( \text{tg}X \) is a 2-dimensional continuous trivial bundle;
3. \( X \) is not a union of \( C^1 \) submanifolds of dimension 2 of \( \mathbb{R}^4 \).

Proof. Remark that \( h \) is continuous, hence \( X \) is a topological manifold and the condition (1) is automatically satisfied. Moreover, \( h \) is smooth except at the origin, where statement (2) is obvious. We now calculate \( \text{tg}_0 X \). Let \( x \in X \setminus \{0\} \), we may write \( x = \text{tg}_{h(z)} \) for some \( z \in C^* \). We have

\[
\frac{x}{\|x\|} = \frac{(z,h(z))}{\|(z,h(z))\|} = \left( \frac{z}{\sqrt{|z|^2 + 1}}, \frac{1}{|z|^2}, \frac{z^2}{\sqrt{|z|^2 + 1}}, \frac{1}{|z|^2} \right).
\]

Notice that \( \sqrt{|z|^2 + 1} \to 0 \) when \( z \to 0 \). Hence \( \text{tg}_0 X \subset \{z = 0\} \). On the other hand, if \( z = r e^{i\theta} \) and \( r \to 0 \), \( \frac{z^2}{\sqrt{|z|^2 + 1}} \to e^{2i\theta} \). Thus, \( (0, e^{2i\theta}) \in \text{tg}_0 X \) for all \( \theta \). This implies that \( \text{tg}_0 X = \{z = 0\} \).

Write \( z = z_1 + iz_2 \) and \( h = h_1 + ih_2 \). For \( x = (z, h(z)) \), \( z \neq 0 \), \( \text{tg}_z X \) is generated by two vectors \( u = (1, 0, \frac{\partial h}{\partial z_1}, \frac{\partial h}{\partial z_1}) \) and \( v = (0, 1, \frac{\partial h}{\partial z_2}, \frac{\partial h}{\partial z_2}) \). Denote by \( \partial_1 \) and \( \partial_2 \) the directional derivatives in the variable \( z \) along \( z_1 \)-axis and \( z_2 \)-axis respectively. We know that

\[
\partial_1 h = \frac{\partial h_1}{\partial z_1} + i \frac{\partial h_2}{\partial z_1}, \quad \partial_2 h = -i \frac{\partial h_1}{\partial z_2} + \frac{\partial h_2}{\partial z_2}.
\]

Note that \( h = \frac{z^2}{(z \bar{z})^2} \). Computation gives,

\[
\partial_1 h = \frac{5}{4} \frac{z}{|z|^2} \partial_1 z - \frac{3}{4} \frac{z^3}{|z|^2} \partial_1 \bar{z} = \frac{5}{4} \frac{z}{|z|^2} - \frac{3}{4} \frac{z^3}{|z|^2},
\]
\[
\partial_2 h = \frac{5}{4} \frac{z}{|z|^2} \partial_2 z - \frac{3}{4} \frac{z^3}{|z|^2} \partial_2 \bar{z} = \frac{5}{4} \frac{z}{|z|^2} + \frac{3}{4} \frac{z^3}{|z|^2}.
\]

If \( z \) tends to 0, then \( |\partial_1 h| \) and \( |\partial_2 h| \) tend to \( \infty \). Therefore,

\[
|u| = (1 + |\partial_1 h|^2)^{\frac{1}{2}} \to \infty, \quad \text{and} \quad |v| = (1 + |\partial_2 h|^2)^{\frac{1}{2}} \to \infty.
\]

Hence \( \lim_{x \to 0} \text{tg}_z X = \text{tg}_0 X \). This implies (2).

Now we show (3). Denote by \( \pi_0 : \mathbb{R}^4 \to \text{tg}_0 X \) the orthogonal projection from \( \mathbb{R}^4 \) onto \( \text{tg}_0 X \):

\[
\pi_0|_X : X \ni (z, h(z)) \mapsto h(z) \in \text{tg}_0 X.
\]
This map is not injective in any neighborhood of 0 since $\forall z, h(z) = h(-z)$. By Theorem 5.4, $X$ is not a $C^1$ manifold. Since $X$ is a connected topological manifold, it cannot be the union of two or more $C^1$ manifolds of dimension 2. □

**Definition 4.4** ([3], [4], [9]). Let $X \subset \mathbb{R}^n$ be a definable set and let $x \in \mathbb{R}^n$. Suppose that $\dim X = k$. It is known that the following limit always exists

$$\theta(X, x) = \lim_{r \to 0} \frac{\mathcal{H}^k(X \cap B^n(x, r))}{r^k \mu_k}.$$ 

We call it the density of $X$ at the point $x$.

**Remark 4.5.** The notions of lower density and density are not the same even for definable sets. For example, consider $X := \{(x, y, z) \in \mathbb{R}^3 : z^2 = x^2 + y^2\}$. It is easy to see that $\Theta(X, 0) = \infty$ while $\theta(X, 0) = 0$. However, if $X$ is a definable set and $\text{tg}X$ is a trivial bundle then $\dim \text{tg}_x X = \dim \text{tg}_x X = \dim X$, and therefore $\Theta(X, x) = \theta(X, x)$ for every $x \in X$.

**Lemma 4.6.** (i) Let $X, Y$ be definable sets of the same dimension. If $\dim(X \cap Y) < \dim X$, then $\theta(X \cup Y, x) = \theta(X, x) + \theta(Y, x)$. If $X \subset Y$, then $\theta(X, x) \leq \theta(Y, x)$.

(ii) If $X$ is a definable set then $\theta(X, x) \geq \theta(\text{tg}_x X, 0)$.

**Proof.** (i) is a direct consequence of the definition of density, and (ii) of Theorem 3.8 of [9]. □

**Theorem 4.7.** Let $X$ be a locally closed definable subset of $\mathbb{R}^n$. If $\text{tg}X$ is a continuous trivial bundle and $\theta(X, x) < 3/2$ for every $x \in X$, then $X$ is a $C^1$ manifold.

**Remark 4.8.** The condition on the density above is sharp, since

$$X := \{(x, y) : y = 0\} \cup \{(x, y) : x > 0, y = x^2\}$$

satisfies all other hypothesis of Theorem 4.7 and $\theta(X, 0) = 3/2$.

**Proof.** Denote by $\pi_x : \mathbb{R}^n \to \text{tg}_x X$ the orthogonal projection. Suppose on the contrary that $X$ is not a $C^1$ manifold. By Theorem 3.4 there exists $x \in X$ such that there is no neighborhood of $x$ in $X$ to which the restriction of $\pi_x$ is injective. We may assume that $x$ coincides with the origin 0 and $\text{tg}_x X = \mathbb{R}^k$ where $k = \dim X$. The map $\pi_x$ now becomes $\pi : \mathbb{R}^n \to \mathbb{R}^k$, the orthogonal projection to the first $k$ coordinates.

By Lemma 3.1 there is an open neighborhood $U$ of 0 in $X$ such that $\pi|_U$ is an open map. Hence there exists $r > 0$ such that $\mathbb{B}^k(0, r) \subset \pi(U)$. Shrinking $U$ if necessary we assume that $\pi(U) = \mathbb{B}^k(0, r)$. We also assume that for every $x \in U$, $\text{tg}_x X$ is not orthogonal to $\mathbb{R}^k$.

By the uniform finiteness on fibres of definable sets, there exists $N \in \mathbb{N}$ such that for every $z \in B^k(0, r)$, $[\pi^{-1}(z) \cap U]$, the number of connected components of $\pi^{-1}(z) \cap U$, does not exceed $N$. In fact, in this case, $[\pi^{-1}(z) \cap U] = \text{card}(\pi^{-1}(z) \cap U)$ where card denotes the cardinality. If otherwise, there is a connected component of $\pi^{-1}(z) \cap U$, write $F$, such that $\dim F \geq 1$. Since $F$ is definable, there is a point $\bar{z} \in K$ such that $\dim \text{tg}_{\bar{z}} F = 1$. But $F \subset \{(z) \times \mathbb{R}^{n-k}\}$, so $\text{tg}_{\bar{z}} F \subset ((0) \times \mathbb{R}^{n-k})$. This implies that $\text{tg}_{\bar{z}} F \subset \text{tg}_x X \cap \{0\} \times \mathbb{R}^{n-k}$. Since $\bar{z} \in U$, $\text{tg}_x X$ is not orthogonal to $\mathbb{R}^k$, then $\text{tg}_x X \cap \{(0) \times \mathbb{R}^{n-k}\} = \{0\}$, which gives a contradiction.
Set
\[ S_\kappa := \{ z \in B^k(0, r) : \text{card}(\pi^{-1}(z) \cap U) = \kappa \}. \]

Then \( \{ S_\kappa \}_{\kappa = 1}^N \) becomes a definable partition of \( B^k(0, r) \). We may assume that 0 \( \in S_\kappa \), \( \forall \kappa \) and \( S_N \neq \emptyset \). Note that \( N \geq 2 \) since the restriction of \( \pi \) is not injective on any neighborhood of 0. We claim that

(a) \( S_N \) is an open set,
(b) \( |(\pi^{-1}(S_N) \cap U)| = N \),
(c) Each connected component of \( \pi^{-1}(S_N) \cap U \) is a \( C^1 \) manifold.

We now give a proof of the claim.

Let \( q \in S_N \). Since \( \text{card}(\pi^{-1}(q) \cap U) = N \), we may write \( \pi^{-1}(q) \cap U = \{ q_1, \ldots, q_N \} \). There is \( \varepsilon > 0 \) sufficiently small such that \( K_i \cap K_j = \emptyset \) for \( i \neq j \), where \( K_i := B^n(q_i, \varepsilon) \cap U, \ i \in \{ 1, \ldots, N \} \). Since the map \( \pi|_U \) is open, there exists an open neighborhood \( V_q \) of \( q \) in \( \mathbb{R}^k \) such that \( V_q \subset \pi(K_i), \forall i \in \{ 1, \ldots, N \} \). For \( q' \in V_q \), \( \forall i \in \{ 1, \ldots, N \} \), \( \pi^{-1}(q') \cap K_i \neq \emptyset \), hence \( \text{card}(\pi^{-1}(q') \cap U) \geq N \). By the definition of \( N \), \( \text{card}(\pi^{-1}(q') \cap U) \) cannot exceed \( N \), therefore \( \text{card}(\pi^{-1}(q') \cap U) = N \), meaning \( q' \in S_N \). This implies \( V_q \subset S_N \), or equivalently \( S_N \) is open, (a) is proved.

Denote by \( A_1, \ldots, A_m \) the connected components of \( \pi^{-1}(S_N) \cap U \). Note that for every \( i \), \( A_i \) is an open set in \( U \), so \( \pi(A_i) \) is an open set in \( \mathbb{R}^k \) since \( \pi|_U \) is an open map. Assume that (b) does not hold, i.e., \( m > N \). Then, there exists an \( i \leq m \) such that \( \pi(A_i) \) does not cover the whole of \( S_N \), for simplicity we assume \( i = 1 \). Since \( \pi(A_1) \subseteq S_N \) is open, there exists \( p \in S_N \cap \partial \pi(A_1) \). Writing \( \pi^{-1}(p) \cap U = \{ p_1, \ldots, p_N \} \), there is an \( \alpha \in \{ 1, \ldots, N \} \) such that \( p_\alpha \) belongs to \( A_1 \). But \( p_\alpha \notin A_1 \) since \( \pi(p_\alpha) \in \partial \pi(A_1) \), then \( p_\alpha \in A_\beta, \beta \neq 1 \). This implies that \( A_1 \cap A_\beta \neq \emptyset \), so \( A_1 \) and \( A_\beta \) are the same connected component, which is a contradiction.

It follows from (b) that for each \( i \in \{ 1, \ldots, N \} \) the restriction \( \pi|_{A_i} : A_i \to S_N \) is a bijection. In other words, \( A_i = \Gamma_{\xi_i} \), with \( \xi_i : S_N \to \mathbb{R}^n, \xi_i(y) = \pi^{-1}(y) \cap A_i \). Note that \( \text{tg} A_i = \text{tg} X|_{A_i} \) which is a continuous trivial bundle, its fibers are, moreover, not orthogonal to \( \mathbb{R}^k \) by the construction. This shows that the function \( \xi_i \) satisfies conditions of Lemma 3.2, hence \( \Gamma_{\xi_i} \) is a \( C^1 \) manifold. This ends the proof of (c).

Let \( z \in S_N \) such that \( ||z|| < r/4 \). Let \( z' \) be a point realizing the distance from \( z \) to the boundary \( \partial S_N \) of \( S_N \). Since \( 0 \in \partial S_N \), \( s := ||z' - z|| \leq r/4 \), and then \( ||z'|| \leq ||z' - z|| + ||z|| \leq r/4 + r/4 = r/2 \), hence \( z' \in \pi(U) = B^k(0, r) \). Since \( B^k(z, s) \subset S_N \), \( \pi^{-1}(B(z, s)) \cap U \) has exactly \( N \) connected components, denoted by \( \{ B_1, \ldots, B_N \} \). Remark that \( z' \notin S_N \) (i.e., \( \text{card}(\pi^{-1}(z') \cap U) < N \) but \( z' \in \pi(B_i), \forall i \in \{ 1, \ldots, N \} \), so there are \( i, j, i \neq j \) such that \( B_i \cap B_j \cap \pi^{-1}(z') \neq \emptyset \). Take \( w \in B_i \cap B_j \cap \pi^{-1}(z') \).

Take \( C \subset B^k(0, r) \) a small closed ball outside \( B^k(z, s) \) and tangent to \( B^k(z, s) \) at \( z' \). Denote by \( D \) the connected component of \( \pi^{-1}(C) \cap U \) which contains \( w \).

Since \( \text{tg} U \) is a \( k \)-dimensional linear subspace of \( \mathbb{R}^n \) and \( \pi|_{\text{tg} U} : \text{tg} U \to \mathbb{R}^k \) is a linear bijective map, \( D, B_1, B_2 \) are disjoint definable sets of dimension \( k \) in \( U \) and \( \text{tg} D, \text{tg} B_1, \text{tg} B_2 \) are half \( k \)-planes. By Lemma 4.6,

\[
\theta(X, w) = \theta(U, w) \geq \theta(B_1, w) + \theta(B_2, w) + \theta(D, w) \\
\geq \theta(\text{tg} B_1, 0) + \theta(\text{tg} B_2, 0) + \theta(\text{tg} D, 0) \\
= \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \frac{3}{2}
\]
This contradicts the hypothesis that $\theta(X, x) < \frac{3}{2}$ for every $x \in X$.

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