ON THE ADDITIVE STRUCTURE OF ALGEBRAIC VALUATIONS OF CYCLIC FREE SEMIRINGS

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Abstract. In this paper, we study factorizations in the additive monoids of positive algebraic valuations \( N_0[\alpha] \) of the semiring of polynomials \( N_0[X] \) using a methodology introduced by D. D. Anderson, D. F. Anderson, and M. Zafrullah in 1990. A cancellative commutative monoid is atomic if every non-invertible element factors into irreducibles. We begin by determining when \( N_0[\alpha] \) is atomic, and we give an explicit description of its set of irreducibles. An atomic monoid is a finite factorization monoid (FFM) if every element has only finitely many factorizations (up to order and associates), and it is a bounded factorization monoid (BFM) if for every element there is a bound for the number of irreducibles (counting repetitions) in each of its factorizations. We show that, for the monoid \( N_0[\alpha] \), the property of being a BFM and the property of being an FFM are equivalent to the ascending chain condition on principal ideals (ACCP). Finally, we give various characterizations for \( N_0[\alpha] \) to be a unique factorization monoid (UFM), two of them in terms of the minimal polynomial of \( \alpha \). The properties of being finitely generated, half-factorial, and other-half-factorial are also investigated along the way.

1. Introduction

The study of the deviation of rings of integers from being a UFM in connection to their divisor class groups earned significant attention in the 1960s with the influence of the number theorists L. Carlitz [6] and W. Narkiewicz [23, 24]. Much of the divisibility theory of rings of integers, including their divisor class groups, carries over to Dedekind domains and, more generally, Krull domains. Motivated by this fact, the phenomenon of non-uniqueness of factorizations in the contexts of Dedekind and Krull domains was later investigated by A. Zaks in [27] and [28], respectively. Since then, techniques to study factorizations in the more general context of cancellative commutative monoids, specially in Krull monoids [15] and multiplicative monoids of integral domains [2], have been systematically developed, giving rise to what we know today as factorization theory.

In this paper, we are primarily concerned with factorizations in monoids of the form \((N_0[\alpha], +)\), where \( N_0[\alpha] = \{ f(\alpha) : f(X) \in N_0[X] \} \) is the homomorphic image of the semiring of polynomials \( N_0[X] \) that we obtain after evaluating at \( \alpha \in \mathbb{R} \setminus \{0\} \). These monoid valuations are cancellative and commutative. The class of cancellative commutative monoids is the most natural abstraction of that consisting of multiplicative monoids of integral domains. In addition, this class is important from the factorization-theoretical perspective because it is the most suitable abstract framework to formally define the notion of a factorization, as observed by F. Halter-Koch in [22]. From now on, every monoid we mention here is tacitly assumed to be cancellative and commutative.

Following P. M. Cohn [10], we say that a monoid is atomic if every non-invertible element factors into irreducibles. A monoid satisfies the ascending chain condition on principal ideals (ACCP) if every
increasing sequence of ideals eventually stabilizes. It follows immediately that every monoid satisfying the ACCP is atomic. There are integral domains whose multiplicative monoids are atomic but do not satisfy the ACCP; the first example was constructed by A. Grams [21]. An atomic monoid is called a finite factorization monoid (FFM) if every element admits only finitely many factorizations, and it is called a bounded factorization monoid (BFM) if for every element there is a bound for the number of irreducibles (counting repetitions) in each of its factorizations. These notions were introduced by D. D. Anderson, D. F. Anderson, and M. Zafrullah [2] in the context of Diagram (1.1) to carry out the first systematic study of factorizations in integral domains. Following Zaks [27], we call an atomic monoid half-factorial (HFM) if any two factorizations of the same element have the same number of irreducibles (counting repetitions). The implications in Diagram (1.1) hold for any monoid.

\[
\begin{array}{c@{}c@{}c@{}c@{}c@{}c}
\text{UFM} & \longleftrightarrow & \text{HFM} \\
\downarrow & \blacktriangle & \downarrow & \blacktriangle & \downarrow & \blacktriangle \\
\text{FFM} & \longleftrightarrow & \text{BFM} & \longleftrightarrow & \text{ACCP monoid} & \longleftrightarrow & \text{atomic monoid}
\end{array}
\]

(1.1)

The study of factorization theory of monoids stemming from the semiring \( \mathbb{N}_0[X] \) and its valuations has been the subject of several recent papers. Methods to factorize polynomials in \( \mathbb{N}_0[X] \) were studied in [4]. In addition, a more systematic investigation of factorizations in the multiplicative monoid of the semiring \( \mathbb{N}_0[X] \) was recently carried out by F. Campanini and A. Facchini in [5], where the similarity between the structure of \( \mathbb{N}_0[X] \) and that of Krull monoids was highlighted. On the other hand, it was proved by S. T. Chapman et al. [7] that for reasonable quadratic algebraic integers \( \alpha \), the multiplicative monoid of the valuation semiring \( \mathbb{N}_0[\alpha] \) has full infinite elasticity (the elasticity is a factorization invariant introduced in [26], and it has been extensively studied since then). The additive monoids of rational valuations \( \mathbb{N}_0[q] \) of the semiring \( \mathbb{N}_0[X] \) have also been investigated by Chapman et al. in [8], where it was proved, among other results, that the lengths of all factorizations of each element of \( \mathbb{N}_0[q] \) form an arithmetic progression. A similar result for more general additive submonoids of \( \mathbb{Q} \) was recently established in [25].

In this paper, we study atomicity and factorizations of the additive monoids of real-valuations \( \mathbb{N}_0[\alpha] \) of the cyclic free semiring \( \mathbb{N}_0[X] \), which we call (additive) monoid valuations of \( \mathbb{N}_0[X] \). The class of monoid valuations of \( \mathbb{N}_0[X] \) includes, as special cases, \( \mathbb{N}_0[X] \) (the valuation at any positive transcendental number) and all rational valuations \( \mathbb{N}_0[q] \). Rational valuations of \( \mathbb{N}_0[X] \) are special cases of Puiseux monoids, whose atomicity has been recently studied in connection to monoid rings [11] and upper triangular matrices over information semialgebras [3]. We put emphasis on algebraic valuations because the additive monoids of transcendental valuations are free and, therefore, trivial from the factorization-theoretical perspective. On the other hand, although algebraic valuations of \( \mathbb{N}_0[X] \) are natural generalizations of the rational valuations \( \mathbb{N}_0[q] \) studied in [8], most of the arithmetic and factorization properties of \( \mathbb{N}_0[q] \) do not hold or trivially generalize to algebraic valuations of \( \mathbb{N}_0[X] \). This is because, as we shall see later, most of such properties for an algebraic valuation \( \mathbb{N}_0[\alpha] \) depend on the minimal polynomial of \( \alpha \).

With our study we accomplish two goals. First, we refine Diagram (1.1) for the class of monoid valuations of \( \mathbb{N}_0[X] \). We show that for monoids in this class, being a UFM and being an HFM are equivalent conditions and also that being an FFM, being a BFM, and satisfying the ACCP are equivalent conditions. In addition, we provide examples of monoids to verify that no other implication in Diagram (1.1) becomes an equivalence in the class of monoid valuations of \( \mathbb{N}_0[X] \). Our second goal here is to investigate the properties of being finitely generated and being other-half-factorial, and then to study how these properties fit in Diagram (1.1) when restricted to monoid valuations of \( \mathbb{N}_0[X] \).
Following J. Coykendall and W. W. Smith [12], we say that an atomic monoid is an other-half-factorial monoid (OHFM) if different factorizations of the same element have different numbers of irreducibles (counting repetitions). In Diagrams (1.2) and (4.2), FGM and ATM stand for finitely generated monoids and atomic monoids, respectively. Diagram (1.2) shows as its non-obvious implications the main results we establish in this paper, where the vertical implication (in blue) holds for all algebraic monoid valuations of \( \mathbb{N}_0[X] \) while the rest of the implications hold for all monoid valuations of \( \mathbb{N}_0[X] \). The diagram also emphasizes (with red marked arrows) the implications that are not reversible: we will construct classes of algebraic monoid valuations of \( \mathbb{N}_0[X] \) witnessing the failure of such reverse implications.

\[
\begin{array}{ccc}
\text{UFM} & \leftrightarrow & \text{HFM} \\
\downarrow & & \uparrow \\
\text{FGM} & \leftrightarrow & \text{FFM} \leftrightarrow \text{BFM} \leftrightarrow \text{ACCP} & \leftrightarrow & \text{ATM}
\end{array}
\]

(1.2)

2. Notation and Background

We let \( \mathbb{N} \) and \( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \) denote the set of positive and nonnegative integers, respectively, and we let \( \mathbb{P} \) denote the set of primes. In addition, for \( X \subseteq \mathbb{R} \) and \( \beta \in \mathbb{R} \), we set \( X_{\geq \beta} := \{ x \in X : x \geq \beta \} \); in a similar way, we use the notations \( X_{< \beta}, X_{\geq \beta}, \) and \( X_{< \beta} \). For \( q \in \mathbb{Q}_{>0} \), we denote the unique \( n, d \in \mathbb{N} \) such that \( q = n/d \) and \( \gcd(n, d) = 1 \) by \( n(q) \) and \( d(q) \), respectively.

2.1. Atomic Monoids. We tacitly assume that all monoids in this paper are cancellative, commutative, and unless we specify otherwise, additively written. In addition, every monoid we treat here is either a group or a reduced monoid, that is, its only invertible element is the identity element. Let \( M \) be a reduced monoid. For \( S \subseteq M \), we let \( \langle S \rangle \) denote the submonoid of \( M \) generated by \( S \). If there exists a finite subset \( S \) of \( M \) such that \( M = \langle S \rangle \), then \( M \) is said to be a finitely generated monoid or an FGM. An element \( a \in M \setminus \{0\} \) is an atom provided that for all \( x, y \in M \) the equality \( a = x + y \) implies that either \( x = 0 \) or \( y = 0 \). The set of atoms of \( M \) is denoted by \( \mathcal{A}(M) \), and \( M \) is atomic if \( M = \langle \mathcal{A}(M) \rangle \). On the other hand, \( M \) is antimatter if \( \mathcal{A}(M) \) is empty. Finitely generated monoids are atomic.

A subset \( I \) of \( M \) is an ideal of \( M \) if \( I + M \subseteq I \), and an ideal \( I \) is principal if \( I = x + M \) for some \( x \in M \). If \( y \in x + M \), then we say that \( x \) divides \( y \) in \( M \) and write \( x \mid_M y \). An element \( p \in M \setminus \{0\} \) is a prime provided that for all \( x, y \in M \) satisfying \( p \mid_M x + y \) either \( p \mid_M x \) or \( p \mid_M y \). Clearly, every prime is an atom. The monoid \( M \) satisfies the ascending chain condition on principal ideals or the ACCP if each increasing sequence of principal ideals of \( M \) eventually stabilizes. If a monoid satisfies the ACCP, then it is atomic [16, Proposition 1.1.4]. Atomic monoids may not satisfy the ACCP, as we shall see in Proposition 4.10.

The difference group of \( M \), denoted here by \( \text{gp}(M) \), is the abelian group (unique up to isomorphism) satisfying that any abelian group containing a homomorphic image of \( M \) also must contain a homomorphic image of \( \text{gp}(M) \). The monoid \( M \) is torsion-free provided that \( \text{gp}(M) \) is a torsion-free abelian group. The monoids we are interested in this paper are torsion-free. On the other hand, the rank of \( M \), denoted by \( \text{rank} M \), is the rank of the \( \mathbb{Z} \)-module \( \text{gp}(M) \), or equivalently, the dimension of the \( \mathbb{Q} \)-vector space \( \mathbb{Q} \otimes_{\mathbb{Z}} \text{gp}(M) \). Clearly, \( \mathbb{Q} \otimes_{\mathbb{Z}} \text{gp}(M) \) contains an additive copy of the monoid \( M \) via the embedding \( M \hookrightarrow \text{gp}(M) \hookrightarrow \mathbb{Q} \otimes_{\mathbb{Z}} \text{gp}(M) \), where the last map is injective because \( \mathbb{Q} \) is a flat \( \mathbb{Z} \)-module.
For the rest of this section, assume that $M$ is atomic. The free (commutative) monoid on $\mathcal{A}(M)$ is denoted by $\mathbb{Z}(M)$. An element $z = a_1 + \cdots + a_\ell \in \mathbb{Z}(M)$, where $a_1, \ldots, a_\ell \in \mathcal{A}(M)$, is a factorization in $M$ of length $|z| := \ell$. As $\mathbb{Z}(M)$ is free, there exists a unique monoid homomorphism $\pi_M : \mathbb{Z}(M) \to M$ satisfying $\pi_M(a) = a$ for all $a \in \mathcal{A}(M)$. When there seems to be no risk of ambiguity, we write $\pi$ instead of $\pi_M$. For each $x \in M$, we set

$$Z(x) := \pi^{-1}(x) \subseteq \mathbb{Z}(M) \quad \text{and} \quad L(x) := \{|z| : z \in Z(x)|.$$ 

Since $M$ is atomic, the sets $Z(x)$ and $L(x)$ are nonempty for all $x \in M$. If $|Z(x)| < \infty$ (resp., $|L(x)| < \infty$) for all $x \in M$, then $M$ is a finite factorization monoid or an FFM (resp., a bounded factorization monoid or a BFM). Clearly, every FFM is a BFM. It follows from [16, Proposition 2.7.8] that every FGM is an FFM, and it follows from [16, Corollary 1.3.3] that every BFM satisfies the ACCP. If $|Z(x)| = 1$ (resp., $|L(x)| = 1$) for all $x \in M$, then $M$ is a unique factorization monoid or a UFM (resp., a half-factorial monoid or an HFM). It is clear from the definitions that every UFM is both an FFM and an HFM. Finally, $M$ is an other-half-factorial monoid or an OHFM provided that for all $x \in M$ and $z_1, z_2 \in Z(x)$, the equality $|z_1| = |z_2|$ implies that $z_1 = z_2$. A summary of the implications mentioned in this subsection is shown in the following diagram.

\[
\begin{array}{c}
\text{OHFM} \leftarrow \text{UFM} \rightarrow \text{HFM} \\
\downarrow \quad \downarrow \\
\text{FGM} \rightarrow \text{FFM} \rightarrow \text{BFM} \rightarrow \text{ACCP monoid} \rightarrow \text{atomic monoid}
\end{array}
\]

### 2.2. Valuation Monoids and Semirings of $\mathbb{N}_0[X]$.

As for monoids, every semiring we deal with here is cancellative and commutative. Accordingly, we say that a triple $(S, +, \cdot)$, where $(S, +)$ and $(S \setminus \{0\}, \cdot)$ are monoids, is a semiring if the multiplicative operation distributes over the additive operation, and the equality $0 \cdot x = x \cdot 0 = 0$ holds for every $x \in S$. As it is customary we shall denote a semiring $(S, +, \cdot)$ simply by $S$. The monoid $(S, +)$ is the additive monoid of $S$.

All the semirings we consider in this paper are homomorphic images of the semiring of polynomials $\mathbb{N}_0[X]$. For $\alpha \in \mathbb{R} \setminus \{0\}$, we call the semiring $\{f(\alpha) : f(X) \in \mathbb{N}_0[X]\}$ the semiring valuation of $\mathbb{N}_0[X]$ at $\alpha$, and we denote it by $S_\alpha$. It is clear that $S_\alpha$ is the subsemiring of $\mathbb{R}$ generated by $\alpha$. From now on, we reserve the notation $\mathbb{N}_0[\alpha]$ for the additive monoid of $S_\alpha$, which we call the monoid valuation of $\mathbb{N}_0[X]$ at $\alpha$ (or simply a monoid valuation of $\mathbb{N}_0[X]$). When $\alpha$ is an algebraic number (resp., a rational number), we call $S_\alpha$ an algebraic semiring valuation (resp., a rational semiring valuation) of $\mathbb{N}_0[X]$, and we call $\mathbb{N}_0[\alpha]$ an algebraic monoid valuation (resp., a rational monoid valuation) of $\mathbb{N}_0[X]$.

Assume that $\alpha \in \mathbb{C}$ is algebraic over $\mathbb{Q}$. It should not come as a surprise that in our study of an algebraic monoid valuation $\mathbb{N}_0[\alpha]$, the minimal polynomial $m_\alpha(X) \in \mathbb{Q}[X]$ of $\alpha$ plays a crucial role. We call the set of exponents of the monomials appearing in the canonical representation of a polynomial $f(X) \in \mathbb{Q}[X]$ the support of $f(X)$, and we denote it by $\text{supp} f(X)$, that is,

$$\text{supp} f(X) := \{n \in \mathbb{N}_0 : f^{(n)}(0) \neq 0\},$$

where $f^{(n)}$ denotes the $n$-th derivative of $f$. Clearly, there is a unique $\ell \in \mathbb{N}$ such that the polynomial $\ell m_\alpha(X) \in \mathbb{Z}[X]$ has content 1. In addition, there are unique polynomials $p_\alpha(X)$ and $q_\alpha(X)$ in $\mathbb{N}_0[X]$ such that $\ell m_\alpha(X) = p_\alpha(X) - q_\alpha(X)$ and $\text{supp} p_\alpha(X) \cap \text{supp} q_\alpha(X) = \emptyset$. We call the pair $(p_\alpha(X), q_\alpha(X))$ the minimal pair of $\alpha$. Finally, we recall that the conjugates of $\alpha$ (over $\mathbb{Q}$) are the complex roots of $m_\alpha(X)$.

1The rational semiring valuations of $\mathbb{N}_0[X]$ were recently investigated in [8] under the term “cyclic rational semirings”.


Descartes’ Rule of Signs states that the number of variations of sign of a polynomial \( f(X) \in \mathbb{R}[X] \) is at least and has the same parity as the number of positive roots of \( f(X) \) (counting multiplicity). In addition, it was proved by D. R. Curtiss [13] that there exists a polynomial \( \phi(X) \in \mathbb{R}[X] \), which he called a Cartesian multiplier, such that the number of variations of sign of \( \phi(X)f(X) \) equals the number of positive roots of \( f(X) \) (counting multiplicity). Using the density of \( \mathbb{Q} \) in the real line, one can readily see that Cartesian multipliers can always be taken in \( \mathbb{Z}[X] \). We proceed to record Curtiss’ result for future reference.

**Theorem 2.1.** [13, Section 5] For each \( f(X) \in \mathbb{R}[X] \), there exists \( \phi(X) \in \mathbb{Z}[X] \) such that the number of variations of sign of \( \phi(X)f(X) \) equals the number of positive roots of \( f(X) \).

3. Algebraic Considerations

In this section, we address two algebraic aspects for semiring valuations of \( \mathbb{N}_0[X] \): we determine their additive rank, and we find necessary and sufficient conditions for two such semiring valuations to be isomorphic. Although many other algebraic aspects of these valuations are rather nontrivial and worthy of study, here we only settle down the algebraic considerations that we will need in order to investigate their additive structure.

**Lemma 3.1.** For a nonzero \( \beta \in \mathbb{R} \), the following statements hold.

1. If \( \beta \) is transcendental, then \( \mathbb{N}_0[\beta] = \bigoplus_{n \in \mathbb{N}_0} \mathbb{N}_0 \beta^n \cong \bigoplus_{n \in \mathbb{N}} \mathbb{N}_0 \).
2. If \( \beta \) is algebraic and has no positive conjugates, then \( \mathbb{N}_0[\beta] = \mathbb{Z}[\beta] \).
3. If \( \beta \) is algebraic and has a positive conjugate \( \alpha \), then \( \mathbb{N}_0[\beta] \cong \mathbb{N}_0[\alpha] \), and so the monoid valuation \( \mathbb{N}_0[\beta] \) is reduced.

**Proof.** (1) It is clear that \( \mathbb{N}_0[\beta] \) is generated by \( \{ \beta^n : n \in \mathbb{N}_0 \} \) as a monoid, and the fact that \( \beta \) is transcendental immediately implies that the mentioned set of generators is integrally independent. Therefore \( \mathbb{N}_0[\beta] = \bigoplus_{n \in \mathbb{N}_0} \mathbb{N}_0 \beta^n \), which is the free commutative monoid of rank \( \mathbb{N}_0 \).

(2) Suppose now that \( \beta \) is an algebraic number having no positive conjugates. Since \( m_\beta(X) \) has no positive roots, Theorem 2.1 guarantees the existence of a nonzero polynomial \( \phi(X) \in \mathbb{Z}[X] \) such that \( \phi(X)m_\beta(X) = \sum_{i=0}^k c_i X^i \in \mathbb{N}_0[X] \). We can assume, without loss of generality, that \( \phi(0) \neq 0 \), that is, \( c_0 \neq 0 \). Since \( \beta \) is a root of \( \phi(X)m_\beta(X) \), we see that \( -c_0 = \sum_{i=1}^k c_i \beta^i \in \mathbb{N}_0[\beta] \cap \mathbb{Z}_{<0} \) and, therefore, \( -1 = -c_0 + (c_0 - 1) \in \mathbb{N}_0[\beta] \). As a consequence, \( \{ \pm \beta^n : n \in \mathbb{N}_0 \} \) is contained in \( \mathbb{N}_0[\beta] \), from which we obtain that \( \mathbb{N}_0[\beta] = \mathbb{Z}[\beta] \).

(3) Let \( \omega \) be a real conjugate of \( \beta \) over \( \mathbb{Q} \) (not necessarily positive), and consider the polynomial \( m(X) = p(X) - q(X) \in \mathbb{Z}[X] \), where \( (p(X), q(X)) \) is the minimal pair of \( \omega \). Let \( R \) be the integral domain \( \mathbb{Z}[X]/I \), where \( I \) is the prime ideal generated by \( m(X) \) in \( \mathbb{Z}[X] \). The ring homomorphism \( \mathbb{Z}[X] \to R \) consisting in evaluating at \( \omega \) induces a ring isomorphism \( \varphi : R \to \mathbb{Z}[\omega] \), namely, \( \varphi : f(X) + I \to f(\omega) \). Now observe that the set \( S(X) \) of cosets of \( \mathbb{Z}[X]/I \) having a representative in \( \mathbb{N}_0[X] \) is a subsemiring of \( R \) satisfying \( \varphi(S(X)) = S_\omega \). Note that \( \varphi(S(X)) \) does not depend on \( \omega \) but only on \( m(X) \). Hence \( S_\beta \cong S(X) \cong S_\alpha \), which implies that \( \mathbb{N}_0[\beta] \cong \mathbb{N}_0[\alpha] \).

As the factorization-theoretic aspects of groups and free commutative monoids are trivial, by virtue of Lemma 3.1 there is no loss of generality in restricting our attention to monoid valuations \( \mathbb{N}_0[\beta] \), where \( \beta \) is a positive real number. We shall do so from now on.

**Proposition 3.2.** For \( \alpha \in \mathbb{R}_{>0} \), the equality \( \text{gp}(\mathbb{N}_0[\alpha]) = \{ f(\alpha) : f(X) \in \mathbb{Z}[X] \} \) holds. In addition, the following statements hold.
(1) If $\alpha$ is transcendental, then $\text{gp}(N_0[\alpha]) \cong \bigoplus_{n \in N_0} \mathbb{Z} \alpha^n$ and rank $N_0[\alpha] = N_0$.

(2) If $\alpha$ is algebraic, then rank $N_0[\alpha] = \deg m_\alpha(X)$.

Proof. Set $G := \{ f(\alpha) : f(X) \in \mathbb{Z}[X] \}$. Since $G$ is an abelian group containing $N_0[\alpha]$, it follows that

$\text{gp}(N_0[\alpha]) \subseteq G$. Observe, on the other hand, that the generating set \{\pm \alpha^n : n \in N_0\} of $G$ is contained in $\text{gp}(N_0[\alpha])$. As a consequence, $G \subseteq \text{gp}(N_0[\alpha])$.

(1) Suppose that $\alpha$ is transcendental. As only the zero polynomial in $\mathbb{Q}[X]$ has $\alpha$ as a root, the set $\{\alpha^n : n \in N_0\}$ is integrally independent, whence $\text{gp}(N_0[\alpha]) \cong \bigoplus_{n \in N_0} \mathbb{Z} \alpha^n$. Since $\{\alpha^n : n \in N_0\}$ is a basis for the $\mathbb{Z}$-module $\text{gp}(N_0[\alpha])$, we see that rank $N_0[\alpha] = N_0$.

(2) Suppose now that $\alpha$ is algebraic, and write $m_\alpha(X) = X^d - \sum_{j=0}^{d-1} c_j X^j$ for $c_0, \ldots, c_{d-1} \in \mathbb{Q}$. Consider $N_0[\alpha]$ as an additive submonoid of the $\mathbb{Q}$-vector space $\mathbb{Q} \otimes \text{gp}(N_0[\alpha])$ via the embedding $N_0[\alpha] \hookrightarrow \text{gp}(N_0[\alpha]) \rightarrow \mathbb{Q} \otimes \text{gp}(N_0[\alpha])$. Let $V$ denote the subspace of $\mathbb{Q} \otimes \text{gp}(N_0[\alpha])$ spanned by the set $S := \{ \alpha^j : j \in [0, d-1] \}$. Since $\alpha^{d+n} = \sum_{j=0}^{d-1} c_j \alpha^{j+n}$ for every $n \in N_0$, we can argue inductively that $\alpha^n \in V$ for every $n \in N_0$. Hence $V = \mathbb{Q} \otimes \text{gp}(N_0[\alpha])$. Since $\alpha$ is an algebraic number of degree $d$, it immediately follows that $S$ is a linearly independent over $\mathbb{Q}$ and so a basis for $V$. Hence rank $N_0[\alpha] = \dim V = d$. \hfill \Box

In the following proposition we determine the isomorphism classes of semiring valuations of $N_0[X]$ whose additive monoids are atomic.

Proposition 3.3. For $\alpha \in \mathbb{R}_{>0}$, suppose that $N_0[\alpha]$ is atomic. Then the following statements hold.

(1) If $\alpha$ is transcendental, then $S_\alpha \cong N_0[X]$. In particular, any two positive transcendental numbers give isomorphic semiring valuations of $N_0[X]$.

(2) If $\alpha$ is rational, then $S_\alpha \cong S_\beta$ for some $\beta \in \mathbb{R}_{>0}$ if and only if $S_\alpha = S_\beta = N_0$ or $\alpha = \beta$.

(3) If $\alpha$ is algebraic but not rational, then $S_\alpha \cong S_\beta$ for some $\beta \in \mathbb{R}_{>0}$ if and only if $\beta$ is an algebraic conjugate of $\alpha$.

Proof. (1) The ring homomorphism $\varphi : \mathbb{Z}[X] \to \mathbb{R}$ consisting in evaluating at $\alpha$ has trivial kernel because $\alpha$ is transcendental. Since $\varphi(N_0[X]) = S_\alpha$, the restriction of $\varphi$ to $N_0[X]$ is a semiring isomorphism between $N_0[X]$ and $S_\alpha$. The last statement follows immediately.

(2) If $\alpha$ is rational, then rank $N_0[\alpha] = 1$ by Proposition 3.2. Suppose, for the direct implication, that $\varphi : S_\alpha \to S_\beta$ is a semiring isomorphism, and so $N_0[\alpha] \cong N_0[\beta]$. Since rank $N_0[\beta] = 1$, Proposition 3.2 ensures that $\beta$ is also rational. Then both $N_0[\alpha]$ and $N_0[\beta]$ are additive submonoids of $\mathbb{Q}_{>0}$, and so [20, Proposition 3.2] guarantees the existence of $q \in \mathbb{Q}_{>0}$ such that $\varphi(x) = qx$ for all $x \in N_0[\alpha]$. Now, $\varphi(1) = 1$ implies that $q = 1$, and so $S_\alpha = S_\beta$. Suppose that $S_\alpha \neq N_0$. Then it follows from [9, Proposition 4.3] that either $S_\alpha = S_\beta = N_0$ or $\{ \alpha^n : n \in N_0\} = \varphi(N_0[\alpha]) = \varphi(N_0[\beta]) = \{ \beta^n : n \in N_0 \}$. In the later case, it is clear that $\alpha = \beta$. The reverse implication is straightforward.

(3) We will only prove the direct implication because the reverse implication follows the same lines as the proof of part (3) of Lemma 3.1. To begin with, we claim that $\alpha \in \varphi(N_0[\alpha])$. Suppose, otherwise, that $\alpha \notin \varphi(N_0[\alpha])$. Then there are elements $x, y \in N_0[\alpha] \setminus \{ \} \subset N_0[\alpha]$ such that $\alpha = x + y$, and so $\alpha^n = \alpha^{n-1}x + \alpha^{n-1}y$ for every $n \in \mathbb{N}$. This implies that $\varphi(N_0[\alpha]) \subset \{ 1 \}$, which is not possible because $N_0[\alpha]$ is atomic. Let $\varphi : S_\alpha \to S_\beta$ be a semiring isomorphism for some $\beta \in \mathbb{R}_{>0}$. Since $\alpha$ is algebraic but not rational, one can use Proposition 3.2 to deduce that $\beta$ is also algebraic but not rational. Then one can argue that $\beta \in \varphi(N_0[\beta])$ in the same vein we argued that $\alpha \in \varphi(N_0[\alpha])$. Now take $f(X) \in N_0[X]$ such that $\varphi(f(\alpha)) = \beta$. As $\beta \in \varphi(N_0[\beta])$ and $\beta = f(\varphi(\alpha))$, it follows that $f(X)$ is a monic monomial, and so $\beta = \varphi(\alpha^n)$ for some $n \in \mathbb{N}$. Also, if $g(X) \in N_0[X]$ satisfies $\varphi(\alpha) = g(\beta)$, then $\varphi(\alpha) = g(\beta) = g(\varphi(\alpha^n)) = \varphi(g(\alpha^n))$, and so $\alpha = g(\alpha^n).$ Because $\alpha \in \varphi(N_0[\alpha])$, one finds that $g(X)$ must be a monic monomial, which implies that $\alpha = \alpha^{kn}$ for some $n \in \mathbb{N}$. As a
result, \( k = n = 1 \), and so \( \beta = \varphi(\alpha) \). Since \( \varphi \) is, in particular, a monoid isomorphism between \( N_0[\alpha] \) and \( N_0[\beta] \), it uniquely extends to a group isomorphism \( \varphi: \text{gp}(N_0[\alpha]) \to \text{gp}(N_0[\beta]) \) (also denoted by \( \varphi \)).

Thus, \( m_\alpha(\beta) = m_\alpha(\varphi(\alpha)) = \varphi(m_\alpha(\alpha)) = \varphi(0) = 0 \), which implies that \( m_\alpha(X) \) is also the minimal polynomial of \( \beta \).

\[ \square \]

4. Atomicity and the ACCP

In this section, we begin to study the atomic structure of monoid valuations of \( N_0[X] \) with a focus on the properties of being atomic and satisfying the ACCP.

4.1. Atomicity. To begin with, we determine the values \( \alpha \in \mathbb{R}_{>0} \) for which \( N_0[\alpha] \) is atomic, and we describe its set of atoms.

Theorem 4.1. For each \( \alpha \in \mathbb{R}_{>0} \), the following statements are equivalent.

(a) \( N_0[\alpha] \) is atomic.
(b) \( N_0[\alpha] \) is not antimatter.
(c) \( 1 \in \mathcal{A}(N_0[\alpha]) \).

In addition, if \( N_0[\alpha] \) is atomic, then the following statements hold.

(1) If \( \alpha \) is transcendental, then \( \mathcal{A}(N_0[\alpha]) = \{ \alpha^n : n \in \mathbb{N}_0 \} \).
(2) If \( \alpha \) is algebraic and \( \sigma := \min\{ n \in \mathbb{N} : \alpha^n \in \langle \alpha^j : j \in \llbracket 0, n - 1 \rrbracket \} \), then
   - if \( \sigma < \infty \), then \( \mathcal{A}(N_0[\alpha]) = \{ \alpha^n : n \in \llbracket 0, \sigma - 1 \rrbracket \} \), and
   - if \( \sigma = \infty \), then \( \mathcal{A}(N_0[\alpha]) = \{ \alpha^n : n \in \mathbb{N}_0 \} \).

Proof. (a) \( \Rightarrow \) (b): It is obvious.

(b) \( \Rightarrow \) (c): Suppose for the sake of a contradiction that \( 1 \notin \mathcal{A}(N_0[\alpha]) \). In this case, \( 1 = \sum_{i=1}^{k} c_i \alpha^i \) for some \( c_1, \ldots, c_k \in \mathbb{N}_0 \) satisfying \( \sum_{i=1}^{k} c_i \geq 2 \). Hence \( \alpha^n = \sum_{i=1}^{k} c_i \alpha^{i+n} \) for every \( n \in \mathbb{N}_0 \). As a result, \( N_0[\alpha] \) is an antimatter monoid, which is a contradiction.

(c) \( \Rightarrow \) (a): Notice first that if \( \alpha \geq 1 \), then for every \( n \in \mathbb{N} \) there are only finitely many elements of \( N_0[\alpha] \) in the interval \( [0, n] \) and, therefore, the set \( N_0[\alpha] \) can be listed increasingly. In this case, \( N_0[\alpha] \) is atomic by [18, Theorem 5.6]. Then one can reduce to the case where \( \alpha \in (0, 1) \). Since \( \alpha < 1 \) it follows that \( \alpha^i \mid N_0[\alpha] \) \( \alpha^j \) when \( i < j \). Thus, along with the fact that \( 1 \in \mathcal{A}(N_0[\alpha]) \), implies that \( \alpha^n \in \mathcal{A}(N_0[\alpha]) \) for every \( n \in \mathbb{N}_0 \). Hence \( N_0[\alpha] \) is an atomic monoid.

Now assume that \( N_0[\alpha] \) is atomic, and let us proceed to argue (1) and (2).

(1) Suppose that \( \alpha \) is transcendental. As observed in Lemma 3.1, \( N_0[\alpha] = \bigoplus_{n \in \mathbb{N}_0} N_0[\alpha^n] \) is the free commutative monoid on the set \( \{ \alpha^n : n \in \mathbb{N}_0 \} \). This implies that \( \mathcal{A}(N_0[\alpha]) = \{ \alpha^n : n \in \mathbb{N}_0 \} \).

(2) Suppose that \( \alpha \) is algebraic, and let us assume first that \( \sigma < \infty \). In this case, the inclusion \( \alpha^\sigma \in \langle \alpha^n : n \in \llbracket 0, \sigma - 1 \rrbracket \rangle \) holds, which implies that \( \alpha \geq 1 \). For each \( j \in \mathbb{N}_0 \), one can see that \( \alpha^{\sigma+j} \in \langle \alpha^{\sigma+j} : n \in \llbracket 0, \sigma - 1 \rrbracket \rangle \). Therefore \( \alpha^n \notin \mathcal{A}(N_0[\alpha]) \) for any \( n \geq \sigma \). Now suppose that \( \alpha^m = \sum_{i=0}^{k} c_i \alpha^i \) for \( m < \sigma \) and for some \( c_0, \ldots, c_k \in \mathbb{N}_0 \) with \( c_k > 0 \). Since \( \alpha \geq 1 \), it follows that \( k \leq m \). However, \( k < m \) would contradict the minimality of \( \sigma \). Hence \( k = m \), which implies that \( \alpha^m \) is an atom of \( N_0[\alpha] \). As a result, \( \mathcal{A}(N_0[\alpha]) = \{ \alpha^n : n \in \llbracket 0, \sigma - 1 \rrbracket \} \).

Let us assume now that \( \sigma = \infty \). Then \( \alpha \neq 1 \). If \( \alpha > 1 \), then \( \alpha^n \mid N_0[\alpha] \) \( \alpha^m \) when \( m > n \), whence \( \alpha^m \in \mathcal{A}(N_0[\alpha]) \) if and only if \( \alpha^m \notin \langle \alpha^n : n \in \llbracket 0, m - 1 \rrbracket \rangle \). As a consequence, \( \mathcal{A}(N_0[\alpha]) = \{ \alpha^n : n \in \mathbb{N}_0 \} \). On the other hand, assume that \( \alpha < 1 \) and fix \( m \in \mathbb{N}_0 \). Write \( \alpha^m = \sum_{i=m}^{k} c_i \alpha^i \) for some \( c_m, \ldots, c_k \in \mathbb{N}_0 \), and notice that \( c_k \) is as otherwise \( 1 = \sum_{i=m+1}^{k} c_i \alpha^{i-m} \notin \mathcal{A}(N_0[\alpha]) \) would.
that does not hold in general, as we shall see in Remark 4.6. None of the sufficient conditions for atomicity offered in Proposition 4.5, the only positive root of \( g \) is irreducible and its only positive real root is \( \alpha \).

\[ f(x) = x^3 - 1 \]

\[ \text{Corollary 4.3.} \quad \text{For } \alpha \in \mathbb{R}_{>0}, \text{ the monoid } \mathbb{N}_0[\alpha] \text{ is an FGM if and only if there is an } n \in \mathbb{N}_0 \text{ such that } \mathcal{A}(\mathbb{N}_0[\alpha]) = \{ \alpha^j : j \in [0, n] \}. \]

The reverse implication of part (1) of Theorem 4.1 does not hold in general, as we shall see in Proposition 5.7. On the other hand, the condition \( 1 \in \mathcal{A}(\mathbb{N}_0[\alpha]) \) in Theorem 4.1 may fail, resulting in \( \mathbb{N}_0[\alpha] \) being antimatter.

\[ \text{Example 4.4.} \quad \text{Consider the algebraic number } \alpha = \frac{\sqrt{5} - 1}{2}, \text{ with minimal polynomial } m_\alpha(X) = X^2 + X - 1. \text{ As } \alpha \text{ is a root of } m_\alpha(X), \text{ the equality } 1 = \alpha^2 + \alpha \text{ holds. Therefore } 1 \notin \mathcal{A}(\mathbb{N}_0[\alpha]), \text{ and it follows from Theorem 4.1 that } \mathbb{N}_0[\alpha] \text{ is an antimatter monoid.} \]

When the generator \( \alpha \) is algebraic we have the following two sufficient conditions for \( \mathbb{N}_0[\alpha] \) to be atomic in terms of the minimal polynomial of \( \alpha \).

\[ \text{Proposition 4.5.} \quad \text{Let } \alpha \in \mathbb{R}_{>0} \text{ be an algebraic number with minimal polynomial } m_\alpha(X). \text{ Then the following statements hold.} \]

1. If \( |m_\alpha(0)| \neq 1 \), then \( \mathbb{N}_0[\alpha] \) is atomic.
2. If \( m_\alpha(X) \) has more than one positive root, then \( \mathbb{N}_0[\alpha] \) is atomic.

\[ \text{Proof.} \quad \text{(1) Suppose, by way of contradiction, that } \mathbb{N}_0[\alpha] \text{ is not atomic. It follows from Theorem 4.1 that } 1 \notin \mathcal{A}(\mathbb{N}_0[\alpha]), \text{ and so there are } c_1, \ldots, c_n \in \mathbb{N}_0 \text{ such that } 1 = \sum_{i=1}^n c_i \alpha^i. \text{ Then } \alpha \text{ is a root of the polynomial } f(X) := 1 - \sum_{i=1}^n c_i X^i \in \mathbb{Q}[X]. \text{ As a result, we can write } f(X) = m_\alpha(X)g(X) \text{ for some } g(X) \in \mathbb{Q}[X], \text{ and it follows from Gauss’ Lemma that } g(X) \in \mathbb{Z}[X]. \text{ As } f(0) = 1, \text{ one obtains that } |m_\alpha(0)| = 1, \text{ which is a contradiction.} \]

\[ \text{(2) Once again, assume towards a contradiction that } \mathbb{N}_0[\alpha] \text{ is not atomic. As in the previous paragraph, we can write } 1 = \sum_{i=1}^n c_i \alpha^i \text{ for some } c_1, \ldots, c_n \in \mathbb{N}_0 \text{ and obtain a polynomial } f(X) = 1 - \sum_{i=1}^n c_i X^i \text{ having } \alpha \text{ as a root. Now Descartes’ Rule of Signs guarantees that } \alpha \text{ is the only positive root of } f(X). \text{ Since } m_\alpha(X) \text{ divides } f(X) \text{ in } \mathbb{Q}[X], \text{ the roots of } m_\alpha(X) \text{ are also roots of } f(X). \text{ As a consequence, the only positive root of } m_\alpha(X) \text{ is } \alpha, \text{ which is a contradiction.} \]

\[ \text{Remark 4.6.} \quad \text{None of the sufficient conditions for atomicity offered in Proposition 4.5 implies the other one. For instance, the polynomial } m(X) = X^2 - 4X + 1 \text{ is irreducible and has two distinct positive roots, namely, } 2 \pm \sqrt{3}; \text{ however, } |m(0)| = 1. \text{ On the other hand, the polynomial } m_\alpha(X) = X^2 + 2X - 2 \text{ is irreducible and its only positive real root is } \alpha = \sqrt{3} - 1; \text{ however, } |m_\alpha(0)| \neq 1. \]
4.2. The ACCP. A relevant class of atomic monoids is that of monoids satisfying the ACCP. In his study of Bezout rings, P. M. Cohn [10, Proposition 1.1] asserted without giving a proof that the underlying multiplicative monoid of any integral domain satisfies the ACCP provided that it is atomic. This was refuted in 1964 by A. Grans, who constructed in [21] a neat counterexample. As we shall see in this subsection, there are monoid valuations of \( N_0[X] \) that are atomic but do not satisfy the ACCP. We proceed to offer a necessary condition for an algebraic monoid valuation of \( N_0[X] \) to satisfy the ACCP.

**Theorem 4.7.** Let \( \alpha \in (0, 1) \) be an algebraic number with minimal pair \((p(X), q(X))\). If \( N_0[\alpha] \) satisfies the ACCP, then \( p(X) - X^k q(X) \notin N_0[X] \) for any \( k \in \mathbb{N} \).

**Proof.** Suppose that \( N_0[\alpha] \) satisfies the ACCP, and assume towards a contradiction that there is a \( k \in \mathbb{N} \) such that \( f(X) := p(X) - X^k q(X) \in N_0[X] \). Consider the sequence \( (q(\alpha)\alpha^{nk} + N_0[\alpha])_{n \in \mathbb{N}} \) of principal ideals of \( N_0[\alpha] \). Observe now that for every \( n \in \mathbb{N} \),

\[
q(\alpha)\alpha^{nk} = p(\alpha)\alpha^{nk} = (f(\alpha) + \alpha^k q(\alpha))\alpha^{nk} = f(\alpha)\alpha^{nk} + q(\alpha)\alpha^{(n+1)k}.
\]

As a result, \( q(\alpha)\alpha^{nk} \in q(\alpha)(\alpha^{(n+1)k} + N_0[\alpha]) \) for every \( n \in \mathbb{N} \), which means that the sequence of principal ideals \( (q(\alpha)\alpha^{nk} + N_0[\alpha])_{n \in \mathbb{N}} \) is ascending. On the other hand, since \( q(\alpha)\alpha^{nk} \) is the minimum of \( q(\alpha)\alpha^{nk} + N_0[\alpha] \) for every \( n \in \mathbb{N} \) and the sequence \( (q(\alpha)\alpha^{nk})_{n \in \mathbb{N}} \) decreases to zero, the chain of ideals \( (q(\alpha)\alpha^{nk} + N_0[\alpha])_{n \in \mathbb{N}} \) does not stabilize. This contradicts that \( N_0[\alpha] \) satisfies the ACCP. \( \square \)

From Theorem 4.7, we can easily deduce [9, Corollary 4.4].

**Corollary 4.8.** For each \( q \in \mathbb{Q} \cap (0, 1) \) with \( n(q) \geq 2 \), the monoid \( N_0[q] \) is atomic but does not satisfy the ACCP.

**Proof.** The monoid \( N_0[q] \) is atomic by Theorem 4.1. On the other hand, taking \( k = 1 \) in Theorem 4.7, we can see that \( N_0[q] \) does not satisfy the ACCP. \( \square \)

As the following example illustrates, the necessary condition in Theorem 4.7 is not sufficient to guarantee that an atomic algebraic monoid valuation satisfies the ACCP.

**Example 4.9.** The polynomial \( X^3 + X^2 + X - 2 \) is strictly increasing in \( \mathbb{R}_{\geq 0} \) and, thus, it has exactly one positive root, namely, \( \alpha \). Since \( X^3 + X^2 + X - 2 \) is irreducible, it is indeed the minimal polynomial \( m_\alpha(X) \) of \( \alpha \), and so the minimal pair of \( \alpha \) is \((p(X), q(X)) = (X^3 + X^2 + X, 2)\). Since \( |m_\alpha(0)| \neq 1 \), Proposition 4.5 guarantees that \( N_0[\alpha] \) is atomic. In addition, it is clear that \( p(X) - X^k q(X) \notin N_0[X] \) for any \( k \in \mathbb{N} \), which is the necessary condition of Theorem 4.7.

We proceed to verify that \( N_0[\alpha] \) does not satisfy the ACCP. To do this, for every \( n \in \mathbb{N} \) set \( x_n = \alpha^{n+2} + 2\alpha^{n+1} + 3\alpha^n \in N_0[\alpha] \) and consider the sequence \((x_n + N_0[\alpha])_{n \in \mathbb{N}}\) of principal ideals of \( N_0[\alpha] \). For every \( n \in \mathbb{N} \), the equality \( 2\alpha^n = \alpha^{n+3} + \alpha^{n+2} + \alpha^{n+1} \) holds and, therefore,

\[
x_n = (\alpha^{n+2} + 2\alpha^{n+1} + \alpha^n) + 2\alpha^n = (\alpha^{n+2} + 2\alpha^{n+1} + \alpha^n) + (\alpha^{n+3} + \alpha^{n+2} + \alpha^{n+1}) = x_{n+1} + \alpha^n.
\]

As a result, \( x_n + N_0[\alpha] \subseteq x_{n+1} + N_0[\alpha] \) for every \( n \in \mathbb{N} \), which means that \((x_n + N_0[\alpha])_{n \in \mathbb{N}}\) is an ascending chain of principal ideals of \( N_0[\alpha] \). As in the proof of Theorem 4.7, one can readily see that the chain of ideals \((x_n + N_0[\alpha])_{n \in \mathbb{N}}\) does not stabilize. Hence \( N_0[\alpha] \) does not satisfy the ACCP.

As an application of Theorem 4.7, we conclude this section providing, for each \( d \in \mathbb{N} \), an infinite class of atomic monoid valuations of \( N_0[X] \) of rank \( d \) that does not satisfy the ACCP.

**Proposition 4.10.** For each \( d \in \mathbb{N} \), there exist infinitely many non-isomorphic semiring valuations of \( N_0[X] \) whose additive monoids have rank \( d \), are atomic, but do not satisfy the ACCP.
Proof. Fix $d \in \mathbb{N}$. Take $q \in \mathbb{Q} \cap (0, 1)$ such that $n(q) \geq 2$ is a squarefree integer, and consider the polynomial $m(X) = X^d - q \in \mathbb{Q}[X]$. The polynomial $m(X)$ is irreducible (by Eisenstein’s Criterion) and has a root $\alpha_q$ in the interval $(0, 1)$. Observe that the algebraic monoid valuation $\mathbb{N}_0[\alpha_q]$ has rank $d$ by Proposition 3.2. Since $|m(0)| = q \neq 1$, the monoid $\mathbb{N}_0[\alpha_q]$ is atomic by part (1) of Proposition 4.5. On the other hand, it follows from Theorem 4.7 that $\mathbb{N}_0[\alpha_q]$ does not satisfy the ACCP. Varying the parameter $q$, one can obtain an infinite class of semirings $\mathbb{S}_{\alpha_q}$ with atomic additive monoids of rank $d$ that do not satisfy the ACCP. Finally, note that the semiring valuations of $\mathbb{N}_0[X]$ in this class are pairwise non-isomorphic by Proposition 3.3. □

4.3. The Bounded and Finite Factorization Properties. The primary purpose of this subsection is to prove that, for monoid valuations of $\mathbb{N}_0[X]$, the finite factorization property (and so the bounded factorization property) is equivalent to the ACCP.

Theorem 4.11. For $\alpha \in \mathbb{R}_{>0}$, the following statements are equivalent.

(a) $\mathbb{N}_0[\alpha]$ is an FFM.
(b) $\mathbb{N}_0[\alpha]$ is a BFM.
(c) $\mathbb{N}_0[\alpha]$ satisfies the ACCP.

Proof. (a) $\Rightarrow$ (b) $\Rightarrow$ (c): Each FFM is clearly a BFM, and each BFM satisfies the ACCP by [16, Corollary 1.3.3].

(c) $\Rightarrow$ (a): Assume that $\mathbb{N}_0[\alpha]$ satisfies the ACCP. In particular, $\mathbb{N}_0[\alpha]$ is atomic. We have already seen that if $\alpha$ is transcendental, then $\mathbb{N}_0[\alpha]$ is a free commutative monoid, and so it is an FFM. Therefore we assume that $\alpha$ is algebraic, and we let $m_\alpha(X)$ denote its minimal polynomial. Suppose for the sake of a contradiction that $\mathbb{N}_0[\alpha]$ is not an FFM, and let us consider the following two cases.

CASE 1: $m_\alpha(X)$ has more than one positive root (counting repetitions). Since $\mathbb{N}_0[\alpha]$ is not an FFM, it follows from [16, Proposition 2.7.8] that $\mathbb{N}_0[\alpha]$ is not an FGM. This, along with Theorem 4.1, ensures that $\mathcal{F}(\mathbb{N}_0[\alpha]) = \{\alpha^n : n \in \mathbb{N}_0\}$. For $x \in \mathbb{N}_0[\alpha]$ and $n \in \mathbb{N}_0$, set

$$Z_n(x) := \{z \in \mathbb{Z}(x) : n \in \text{supp } z(X)\}.$$  

Claim. For each $x \in \mathbb{N}_0[\alpha]$, if $|Z_n(x)| < \infty$ for every $n \in \mathbb{N}_0$, then $|Z(x)| < \infty$.

Proof of Claim. Take a nonzero element $x \in \mathbb{N}_0[\alpha]$ such that $Z_n(x)$ is a finite set for every $n \in \mathbb{N}_0$. Suppose, by way of contradiction, that $|Z(x)| = \infty$. Fix $x_0 \in Z(x)$, and set $d = \deg z_0(X)$. As $Z_n(x)$ is finite for every $n \in [0, d]$, there is a factorization $z \in Z(x)$ such that $\min \text{supp } z(X) > d$. Consider the polynomial $z(X) - z_0(X) \in \mathbb{Z}[X]$. Since $z$ and $z_0$ are factorizations of the same element, $\alpha$ is a root of $z(X) - z_0(X)$. In addition, because $\min \text{supp } z(X) > d = \deg z_0(X)$, Descartes’ Rule of Signs guarantees that $\alpha$ is indeed the only positive root of $z(X) - z_0(X)$. However, the fact that $m_\alpha(X)$ divides $z(X) - z_0(X)$ contradicts that $m_\alpha(X)$ has more than one positive root. As a result, the claim follows.

Because $\mathbb{N}_0[\alpha]$ is not an FFM, there is an $x_0 \in \mathbb{N}_0[\alpha]$ such that $|Z(x_0)| = \infty$. By the established claim, we can choose an $n_1 \in \mathbb{N}$ so that $Z_{n_1}(x_0)$ is an infinite set. Therefore $x_1 = x_0 - \alpha^{n_1} \in \mathbb{N}_0[\alpha]$ satisfies $|Z(x_1)| = \infty$. Now suppose that for some $j \in \mathbb{N}$ we have found $x_0, \ldots, x_j \in \mathbb{N}_0[\alpha]$ such that $|Z(x_i)| = \infty$ and $x_{i-1} - x_i \in \mathbb{N}_0[\alpha] \setminus \{0\}$ for every $i \in [1, j]$. Because $|Z(x_j)| = \infty$, the previous claim guarantees the existence of an $n_{j+1} \in \mathbb{N}_0$ such that the set $Z_{n_{j+1}}(x_j)$ is infinite. Then after setting $x_{j+1} = x_j - \alpha^{n_{j+1}}$, one finds that $|Z(x_{j+1})| = \infty$. Thus, we have constructed a sequence $(x_n)_{n \in \mathbb{N}_0}$ of elements in $\mathbb{N}_0[\alpha]$ satisfying $x_{n-1} - x_n \notin \mathbb{N}_0[\alpha] \setminus \{0\}$ for every $n \in \mathbb{N}$. This implies that $(x_n + \mathbb{N}_0[\alpha])_{n \in \mathbb{N}_0}$ is an ascending chain of principal ideals of $\mathbb{N}_0[\alpha]$ that does not stabilize. However, this contradicts that $\mathbb{N}_0[\alpha]$ satisfies the ACCP.
CASE 2: \( \alpha \) is the only positive root of \( m_{\alpha}(X) \) (counting repetitions). As \( \mathbb{N}_0[\alpha] \) is not an FFM, it follows from [18, Theorem 5.6] that it cannot be increasingly generated. Thus, \( \alpha \in (0, 1) \). Since \( m_{\alpha}(X) \) has only one positive root, Theorem 2.1 ensures the existence of \( \phi(X) \in \mathbb{N}_0[X] \) such that \( \phi(X)m_{\alpha}(X) \) belongs to \( \mathbb{Z}[X] \) and has exactly one variation of sign. Set \( f(X) = \sum_{i=0}^{s} c_i X^i = \phi(X)m_{\alpha}(X) \), where \( c_0, \ldots, c_s \in \mathbb{Z} \). We can assume, without loss of generality, that \( \phi(1) > 0 \). As \( f(1) = \phi(1)m_{\alpha}(1) \geq 1 \), there is a \( k \in [0, s - 1] \) such that \( \sum_{i=0}^{k+1} c_i \geq 0 \) and \( \sum_{i=0}^{j} c_i < 0 \) for every \( j \in [0, k] \). After setting \( 1_n(X) = \sum_{i=0}^{n} X^i \) for every \( n \in \mathbb{N}_{>s} \), one obtains that

\[
\alpha \in \mathbb{N}_0[\alpha].
\]

Observe that the negative coefficients of \( f(X)1_n(X) \) are precisely the coefficients of the terms of degree at most \( k \). Since \( f(\alpha)1_n(\alpha) = 0 \),

\[
(4.1) \quad x_n := \sum_{j=0}^{k} \left( \sum_{i=0}^{j} c_i \right) \alpha^j - \sum_{j=s}^{n} f(1) \alpha^j = \sum_{j=k+1}^{s-1} \left( \sum_{i=0}^{j} c_i \right) \alpha^j + \sum_{j=n+1}^{n+s} \left( \sum_{i=j-n}^{s} c_i \right) \alpha^j \in \mathbb{N}_0[\alpha].
\]

Now consider the sequence \( (x_n + \mathbb{N}_0[\alpha])_{n \in \mathbb{N}_{>s}} \) of principal ideals of \( \mathbb{N}_0[\alpha] \). It follows from (4.1) that \( x_n - x_{n+1} = f(1) \alpha^{n+1} \in \mathbb{N}_0[\alpha] \) for every \( n \geq s \). Thus, \( (x_n + \mathbb{N}_0[\alpha])_{n \in \mathbb{N}_{>s}} \) is an ascending chain of principal ideals. Since the sequence \( (x_n - x_{n+1})_{n \in \mathbb{N}_{>s}} \) strictly decreases to zero, the chain of ideals \( (x_n + \mathbb{N}_0[\alpha])_{n \in \mathbb{N}_{>s}} \) does not stabilize, which contradicts that \( \mathbb{N}_0[\alpha] \) satisfies the ACCP.

It follows from [16, Proposition 2.7.8] that every FGM is an FFM. However, there are algebraic monoid valuations of \( \mathbb{N}_0[X] \) that are FFM but not FGMs. The following simple example, which will be significantly extended in Proposition 5.9, illustrates this observation.

**Example 4.12.** Take \( q \in \mathbb{Q}_{\geq 1} \setminus \mathbb{N} \), and consider the rational monoid valuation \( \mathbb{N}_0[q] \). Since \( \mathbb{N}_0[q] \) is generated by the increasing sequence \( (q^n)_{n \in \mathbb{N}_0} \), it follows from [18, Theorem 5.6] that \( \mathbb{N}_0[q] \) is an FFM. On the other hand, \( d(q) > 1 \) implies that \( q^n \notin \langle q^j : j \in [0, n-1] \rangle \) for any \( n \in \mathbb{N} \). Hence Theorem 4.1 guarantees that \( \mathcal{A}(\mathbb{N}_0[q]) = \{ q^n : n \in \mathbb{N}_0 \} \). As a result, \( \mathbb{N}_0[q] \) is not an FGM.

The following diagram of implications summarizes the main results we have established so far on the class of monoid valuations of \( \mathbb{N}_0[X] \).

\[
(4.2) \quad \text{FGM} \xrightarrow{\text{FFM}} \text{BFM} \quad \text{ACCP} \quad \text{ATM}
\]

5. Factoriality

For every transcendental number \( \beta \in \mathbb{R}_{>0} \), the monoid valuation \( \mathbb{N}_0[\beta] \) is a UFM that is not an FGM. However, we will show in this section that the implication UFM \( \Rightarrow \) FGM holds in the class of algebraic monoid valuations of \( \mathbb{N}_0[X] \). Our primary purpose in this section is to study the properties of being half-factorial and other-half-factorial and extend the implication UFM \( \Rightarrow \) FGM to the following chain of implications: \( \text{UFM} \Leftrightarrow \text{HFM} \Rightarrow \text{OHFM} \Rightarrow \text{FGM} \). This may come as a surprise since, in general, an HFM (resp., an OHFM) may not be an FGM even in the class of torsion-free reduced atomic monoids. In addition, in the same class, there are HFM that are not UFM. The following examples shed some light upon these observations.
Example 5.1. Consider the additive submonoid $M$ of $\mathbb{Z}^2$ generated by the set $\{(1, n) : n \in \mathbb{Z}\}$. It is clear that $M$ is a torsion-free reduced atomic monoid with $\mathcal{A}(M) = \{(1, n) : n \in \mathbb{Z}\}$, from which one can deduce that $M$ is an HFM but not a UFM. However, $M$ is not an FGM. Indeed, $M$ is not even an FFM: for instance, the equalities $(2, 0) = (1, -n) + (1, n)$ (for every $n \in \mathbb{N}$) yield infinitely many factorizations of $(2, 0)$ in $M$.

Example 5.2. The support of a function $f : \mathbb{N} \to \mathbb{N}_0$ is the set $\text{supp}(f) := \{n \in \mathbb{N} : f(n) \neq 0\}$. It is clear that $F := \{f : \mathbb{N} \to \mathbb{N}_0 : |\text{supp}(f)| < \infty\}$ is a torsion-free reduced atomic monoid under addition. For each $j \in \mathbb{N}$, let $e_j : \mathbb{N} \to \mathbb{N}_0$ be the function defined by $e_j(n) = 1$ if $n = j$ and $e_j(n) = 0$ otherwise. Now consider the submonoid $M$ of $F$ generated by the set $A = \{2e_1, e_j + 2e_1 : j \in \mathbb{N}\}$. One can easily verify that $M$ is atomic with $\mathcal{A}(M) = A$. As a consequence, $M$ is not an FGM. To show that $M$ is an OHFM, suppose that

$$z := c_0(2e_1) + \sum_{j=1}^{n} c_j(e_j + 2e_1) \quad \text{and} \quad z' := c'_0(2e_1) + \sum_{j=2}^{n} c'_j(e_j + 2e_1)$$

are two factorizations of the same element in $M$ satisfying $|z| = |z'|$, that is, $\sum_{j=1}^{n} c_j = \sum_{j=0}^{n} c'_j$. It can be readily seen that, for every $j \in \mathbb{N}_{\geq 2}$, the atom $e_j + 2e_1$ is indeed a prime in $M$. As a result, $c_j = c'_j$ for every $j \in \mathbb{N}_{\geq 2}$. Thus, the equality $2c_0 + 3c_1 = 2c'_0 + 3c'_1$ holds because $z$ and $z'$ are factorizations of the same element, and the equality $c_0 + c_1 = c'_0 + c'_1$ holds because $|z| = |z'|$. Therefore $c_0 = c'_0$ and $c_1 = c'_1$, from which we obtain that $z = z'$. Hence $M$ is an OHFM.

At this point, it seems like there is no example in the factorization theory literature of an OHFM that is not an FFM. This suggests the following question.

Question 5.3. Is every OHFM an FFM?

5.1. Half-Factoriality. As mentioned in the introduction, for monoid valuations of $\mathbb{N}_0[X]$ the property of being an HFM is equivalent to that of being a UFM. Now we prove this assertion and, when the generator is algebraic, we give further characterizations of these equivalent properties.

Theorem 5.4. For $\alpha \in \mathbb{R}_{>0}$, the following statements hold.

1. $\alpha$ is transcendental if and only if $\mathbb{N}_0[\alpha]$ is a non-finitely generated UFM.
2. If $\alpha$ is algebraic of degree $d$ with minimal polynomial $m_\alpha(X)$ and minimal pair $(p(X), q(X))$, then the following statements are equivalent.
   (a) $\mathbb{N}_0[\alpha]$ is a UFM.
   (b) $\mathbb{N}_0[\alpha]$ is an HFM.
   (c) $\text{deg} \, m_\alpha(X) = |\mathcal{A}(\mathbb{N}_0[\alpha])|.$
   (d) $p(X) = X^d$ for some $d \in \mathbb{N}$.

Proof. (1) We have seen in Lemma 3.1 that $\mathbb{N}_0[\alpha]$ is the free commutative monoid of rank $\mathbb{N}_0$, which is a non-finitely generated UFM. For the reverse implication, suppose towards a contradiction that $\alpha$ is algebraic, and let $(p(X), q(X))$ be the minimal pair of $\alpha$. Since $\mathbb{N}_0[\alpha]$ is atomic but not finitely generated, $\mathcal{A}(\mathbb{N}_0[\alpha]) = \{\alpha^n : n \in \mathbb{N}_0\}$ by Theorem 4.1. As a result, $p(\alpha)$ and $q(\alpha)$ are two distinct factorizations in $\mathbb{Z}[\mathbb{N}_0[\alpha]]$ of the same element, which contradicts that $\mathbb{N}_0[\alpha]$ is a UFM.

(2) To argue this part, assume that $\alpha$ is an algebraic number with minimal polynomial $m_\alpha(X)$ and minimal pair $(p(X), q(X))$.
   (a) $\Rightarrow$ (b): This is clear.
   (b) $\Rightarrow$ (c): Suppose that $\mathbb{N}_0[\alpha]$ is an HFM. If $\alpha \in \mathbb{Q}$, then $\mathbb{N}_0[\alpha]$ is an additive submonoid of $(\mathbb{Q}_{\geq 0}, +)$, and [19, Proposition 4.2] ensures that $\mathbb{N}_0[\alpha] \cong (\mathbb{N}_0, +)$. Then $\alpha \in \mathbb{N}$, from which condition (c) follows.
immediately. So assume that \( \alpha \notin \mathbb{Q} \) and, therefore, that \( \deg m_\alpha(X) > 1 \). As in Theorem 4.1, set 
\( \sigma = \min \{ n \in \mathbb{N} : \alpha^n \in \langle \alpha^j : j \in [0, n-1] \rangle \} \). Since \( N_0[\alpha] \) is atomic, it follows from Theorem 4.1 that \( \mathcal{A}(N_0[\alpha]) = \{ \alpha^j : j \in [0, \sigma - 1] \} \). It is clear that \( \deg m_\alpha(X) \leq \sigma \). Suppose, by way of contradiction, that \( \deg m_\alpha(X) < \sigma \). In this case, both \( p(\alpha) \) and \( q(\alpha) \) are distinct factorizations in \( N_0[\alpha] \) of the same element. This, along with the fact that \( N_0[\alpha] \) is an HFM, implies that \( m_\alpha(1) = 0 \). However, this contradicts that \( m_\alpha(X) \) is an irreducible polynomial in \( \mathbb{Q}[X] \) of degree at least 2. Hence \( \deg m_\alpha(X) = \sigma = |\mathcal{A}(N_0[\alpha])| \).

\((c) \Rightarrow (d)\): Because \( \mathcal{A}(N_0[\alpha]) \) is nonempty, \( N_0[\alpha] \) is an atomic monoid by Theorem 4.1. Let \( d \) be the degree of \( m_\alpha(X) \). Since \( \mathcal{A}(N_0[\alpha]) = \{ \alpha^j : j \in [0, d-1] \} \), there are coefficients \( c_0, \ldots, c_{d-1} \in \mathbb{N} \) such that \( \alpha^d = \sum_{i=0}^{d-1} c_i \alpha^i \). Therefore \( \alpha \) is a root of the polynomial \( X^d - \sum_{i=0}^{d-1} c_i X^i \) and, as a consequence, \( m_\alpha(X) = X^d - \sum_{i=0}^{d-1} c_i X^i \) by the uniqueness of the minimal polynomial. Thus, \( p(X) = X^d \).

\((d) \Rightarrow (a)\): Let \( \sigma \) be defined as in the proof of \((b) \Rightarrow (c)\) above. Since \( p(X) = X^d \), it is clear that \( \alpha \geq 1 \) and, therefore, the monoid \( N_0[\alpha] \) is atomic by [18, Proposition 4.5]. On the other hand, the fact that \( d \) is the degree of the minimal polynomial of \( \alpha \) implies that \( \sigma = d \), and so \( \mathcal{A}(N_0[\alpha]) = \{ \alpha^n : n \in [0, d-1] \} \) by Theorem 4.1. Take two factorizations \( z_1, z_2 \in \mathbb{Z}[N_0[\alpha]] \) of the same element in \( N_0[\alpha] \). Then \( \max\{\deg z_1(X), \deg z_2(X)\} < \deg m_\alpha(X) \) and \( z_1(\alpha) = z_2(\alpha) \). Since \( \deg z_1(X) < \deg m_\alpha(X) \), the fact that \( m_\alpha(X) \) divides \( z_1(X) - z_2(X) \) in \( \mathbb{Q}[X] \) forces the equality \( z_1(X) = z_2(X) \), which implies that \( z_1 = z_2 \). Hence \( N_0[\alpha] \) is a UFM, and so Theorem 4.1 ensures that \( \mathcal{A}(N_0[\alpha]) = \{ \alpha^j : j \in [0, d-1] \} \). \( \square \)

As an immediate consequence of the characterization given in Theorem 5.4, we obtain the following result.

**Corollary 5.5.** If \( N_0[\alpha] \) is a UFM (or an HFM) for a positive algebraic \( \alpha \), then \( N_0[\alpha] \) is an FGM, and so an FFM.

The converse of Corollary 5.5 does not hold, that is, there are algebraic monoid valuations of \( N_0[X] \) that are FFMs but not HFMs.

**Example 5.6.** Take \( q \in \mathbb{Q}_{>1} \setminus \mathbb{N} \), and consider the rational monoid valuation \( N_0[q] \). Since \( q > 1 \), the monoid \( N_0[q] \) is increasingly generated, and so it follows from [18, Theorem 5.6] that it is an FFN. On the other hand, as \( N_0[q] \not\cong (\mathbb{N}, +) \), [19, Proposition 4.3.1] guarantees that \( N_0[q] \) is neither a UFM nor an HFM.

In the direction of Example 5.6, we will construct in Proposition 5.9 an infinite class of non-isomorphic algebraic monoid valuations of \( N_0[X] \) (of any possible rank) that are FFMs but not FGMs and, therefore, not HFMs by virtue of Corollary 5.5. Unlike the case of rational monoid valuations of \( N_0[X] \), we will see in Proposition 5.13 that there are infinitely many non-isomorphic algebraic monoid valuations of \( N_0[X] \) that are FGMs. The following proposition gives a necessary condition for \( N_0[\alpha] \) to be an FGM.

**Proposition 5.7.** If \( N_0[\alpha] \) is an FGM for some algebraic \( \alpha \in \mathbb{R}_{>0} \), then \( m_\alpha(X) \in \mathbb{Z}[X] \) and its only positive root is \( \alpha \) (counting multiplicity).

**Proof.** We first prove that \( m_\alpha(X) \in \mathbb{Z}[X] \). Since \( N_0[\alpha] \) is an FGM, it is an FFN. In particular, \( N_0[\alpha] \) is atomic, and it follows from Theorem 4.1 that \( \mathcal{A}(N_0[\alpha]) = \{ \alpha^j : j \in [0, n-1] \} \) for some \( n \in \mathbb{N} \). Thus, we can write \( \alpha^n = \sum_{i=0}^{n-1} c_i \alpha^i \) for some \( c_0, \ldots, c_{n-1} \in N_0 \). Since \( \alpha \) is a root of the polynomial \( h(X) := X^n - \sum_{i=0}^{n-1} c_i X^i \in \mathbb{Q}[X] \), the minimal polynomial \( m_\alpha(X) \) of \( \alpha \) divides \( h(X) \) in \( \mathbb{Q}[X] \). Then \( h(X) = m_\alpha(X) f(X) \) for some \( f(X) \in \mathbb{Q}[X] \). Since \( h(X) \) has integer coefficients, it follows from Gauss’ Lemma that both \( m_\alpha(X) \) and \( f(X) \) belong to \( \mathbb{Z}[X] \).

It only remains to check that \( \alpha \) is the only positive root of \( m_\alpha(X) \). By virtue of Descartes’ Rule of Signs, the polynomial \( h(X) \) defined in the previous paragraph has exactly one positive real root,
which must be \( \alpha \). This, together with the fact that \( m_\alpha(X) \) divides \( h(X) \) in \( \mathbb{Q}[X] \), guarantees that \( \alpha \) is the only positive real root of \( m_\alpha(X) \). \( \square \)

The necessary condition for the monoid \( \mathbb{N}_0[\alpha] \) to be finitely generated in Proposition 5.7 is not a sufficient condition, as the following example shows.

**Example 5.8.** Let \( \alpha \) be the only positive root of the polynomial \( m_\alpha(X) = X^2 + 2X - 2 \), and suppose towards a contradiction that \( \mathbb{N}_0[\alpha] \) is an FGM. The monoid \( \mathbb{N}_0[\alpha] \) is certainly atomic by part (1) of Proposition 4.5, and so it follows from Theorem 4.1 that \( \alpha^n \in \langle \alpha^2 : j \in [0, n-1] \rangle \) for some \( n \in \mathbb{N} \). As a result, \( m_\alpha(X) \) must divide a polynomial \( X^n + \sum_{i=0}^{n-1} c_i X^i \in \mathbb{Z}[X] \) with \( c_0, \ldots, c_{n-1} \in \mathbb{Z}_{\leq 0} \) for every \( i \in [0, n-1] \). Notice that \( n \geq 3 \). Take \( a_0, \ldots, a_{n-3} \in \mathbb{Q} \) such that

\[
(X^2 + 2X - 2) \left( X^{n-2} + \sum_{i=0}^{n-3} a_i X^i \right) = X^n + \sum_{i=0}^{n-1} c_i X^i.
\]

Observe that \( a_0 = -\frac{1}{2} c_0 > 0 \) and \( a_1 = \frac{1}{2} (2a_0 - c_1) > 0 \). In addition, if \( a_j > 0 \) for every \( j \in [0, k] \) for some \( k < n - 3 \), then one can compare the coefficients of \( X^{k+1} \) in both sides of (5.1) to find that \( a_{k+1} = \frac{1}{2} (2a_k + a_{k-1} - c_{k+1}) > 0 \). Hence \( a_0, \ldots, a_{n-3} \) are all positive. Now after comparing the coefficients of \( X^{n-1} \) in both sides of (5.1), one obtains that \( c_{n-1} = 2 + a_{n-3} > 0 \), which is a contradiction. Thus, \( \mathbb{N}_0[\alpha] \) cannot be an FGM.

We have seen in Example 5.6 a rank-1 monoid valuation of \( \mathbb{N}_0[X] \) that is an FFM but not an FGM. We conclude this section constructing for every \( d \in \mathbb{N} \) an infinite class of algebraic semiring valuations of \( \mathbb{N}_0[X] \) whose additive monoids are rank-\( d \) FFMs but not FGMs.

**Proposition 5.9.** For each \( d \in \mathbb{N} \), there exist infinitely many non-isomorphic algebraic semiring valuations of \( \mathbb{N}_0[X] \) whose additive monoids are rank-\( d \) FFMs that are not FGMs.

**Proof.** Suppose first that \( d = 1 \). For each \( \alpha \in \mathbb{Q}_{\geq 1} \setminus \mathbb{N}_0 \), consider the semiring valuation \( S_\alpha \). As \( S_\alpha \subseteq \mathbb{Q} \), the monoid \( \mathbb{N}_0[\alpha] \) has rank 1. Since \( \mathbb{N}_0[\alpha] \) is increasingly generated, it follows from [20, Theorem 5.6] that \( \mathbb{N}_0[\alpha] \) is an FFM. In addition, it is not hard to verify that \( \mathcal{A}(\mathbb{N}_0[\alpha]) = \{ \alpha^n : n \in \mathbb{N}_0 \} \) (see [9, Proposition 4.3]), and so \( \mathbb{N}_0[\alpha] \) is not an FGM. Lastly, part (2) of Proposition 3.3 guarantees that \( S_\alpha \not\cong S_\beta \) for any \( \beta \in \mathbb{Q}_{\geq 1} \setminus \mathbb{N}_0 \) with \( \beta \neq \alpha \).

Suppose now that \( d \geq 2 \). Take \( p \in \mathbb{P}_{\geq 5} \), and then consider the polynomial \( m(X) = (p-2)X^d + X - p \). We observe that \( m(X) \) cannot have any complex root \( \rho \) inside the closed unit disc as, otherwise, \( p = |(p - 2)| \rho^d + |\rho| \leq (p - 2) |\rho|^d + |\rho| < p - 1 \). To verify that \( m(X) \) is irreducible in \( \mathbb{Z}[X] \) suppose, by way of contradiction, that \( m(X) = f(X)g(X) \) for some \( f(X), g(X) \in \mathbb{Z}[X] \setminus \mathbb{Z} \). Hence either \( f(0) \) or \( g(0) \) divides \( p \). Assume, without loss of generality, that \( |f(0)| = 1 \) and set \( n = \deg f(X) \). After denoting the complex roots of \( f(X) \) by \( \rho_1, \ldots, \rho_n \) and its leading coefficient by \( c \), one finds that \( |\rho_1 \cdots \rho_n| = 1/c \leq 1 \). Therefore there is a \( j \in [1, n] \) such that \( |\rho_j| \leq 1 \). However, this contradicts that \( \rho_j \) is also a root of \( m(X) \). Thus, \( m(X) \) is irreducible. Since \( m(1) < 0 \), the polynomial \( m(X) \) has a root \( \alpha_p \in \mathbb{R}_{> 1} \). Now Gauss’ Lemma guarantees that \( m_{\alpha_p}(X) = X^d + \frac{1}{p^2} X - \frac{p}{p^2} \in \mathbb{Q}[X] \) is the minimal polynomial of \( \alpha_p \).

It follows from Proposition 3.2 that \( \mathbb{N}_0[\alpha_p] \) has rank \( d \). As in the case of \( d = 1 \), the fact that \( \mathbb{N}_0[\alpha_p] \) is increasingly generated ensures that it is an FMM. Because \( m_{\alpha_p}(X) \notin \mathbb{Z}[X] \), it follows from Proposition 5.7 that \( \mathbb{N}_0[\alpha_p] \) is not an FGM. Then for each prime \( p \in \mathbb{P}_{\geq 5} \) we can consider the algebraic semiring valuation \( S_{\alpha_p} \) of \( \mathbb{N}_0[X] \) whose additive monoid is a rank-\( d \) FFM that is not an FGM. In light of Proposition 3.3, distinct parameters \( p \in \mathbb{P}_{\geq 5} \) yield non-isomorphic semirings \( S_{\alpha_p} \). \( \square \)
5.2. Other-Half-Factoriality. Let us call an OHFM proper if it is not a UFM. It was proved in [12] that the multiplicative monoid of an integral domain is never a proper OHFM. There are, however, algebraic semiring valuations of \( N_0[X] \) whose additive monoids are proper OHFMs. Let us proceed to characterize such semirings.

**Theorem 5.10.** For \( \alpha \in \mathbb{R}_{>0} \), the following statements are equivalent.

(a) \( N_0[\alpha] \) is a proper OHFM.

(b) \( \alpha \) is algebraic, and \( \mathscr{A}(N_0[\alpha]) = \{ \alpha^j : j \in [0, \deg m_\alpha(X)] \} \).

In addition, for \( d \geq 3 \) there are infinitely many non-isomorphic algebraic semiring valuations of \( N_0[X] \) whose additive monoids are proper OHFMs of rank \( d \).

**Proof.** (a) \( \Rightarrow \) (b): Since \( N_0[\alpha] \) is not a UFM, it follows from Theorem 5.4 that \( \alpha \) is an algebraic number and \( \deg m_\alpha(X) < \sigma := \min\{ n \in \mathbb{N} : \alpha^n \in \mathbb{K} \} \). Let \( (p(X), q(X)) \) be the minimal pair of \( \alpha \), and suppose for the sake of a contradiction that \( \deg m_\alpha(X) + 1 < \sigma \). Consider the polynomials \( z_1(X) := p(X) + Xq(X) \) and \( z_2(X) := q(X) + Xp(X) \) of \( N_0[X] \). As \( z_2(X) - z_1(X) = (X-1)m_\alpha(X) \) and \( \deg m_\alpha(X) + 1 < \sigma \), it follows that \( z_1(\alpha) \) and \( z_2(\alpha) \) are two distinct factorizations in \( N_0[\alpha] \) of the same element. However, \( |z_1(\alpha)| = z_1(1) = z_2(1) = |z_2(\alpha)| \) contradicts that \( N_0[\alpha] \) is an OHFM. Thus, \( \deg m_\alpha(X) = \sigma - 1 \), and so (b) follows from Theorem 4.1.

(b) \( \Rightarrow \) (a): As \( 1 \in \mathscr{A}(N_0[\alpha]) \), the monoid \( N_0[\alpha] \) is atomic by Theorem 4.1. Let \( z_1, z_2 \in Z(N_0[\alpha]) \) be two factorizations of the same element such that \( |z_1| = |z_2| \). In order to show that \( z_1 = z_2 \), there is no loss in assuming that \( z_1 \) and \( z_2 \) have no atoms in common. Because \( \alpha \) is a root of the polynomial \( z_2(X) - z_1(X) \in Z[X] \), whose degree is at most \( \deg m_\alpha(X) \), there is a \( c \in \mathbb{Q} \) such that \( z_1(X) = z_2(X) = cm_\alpha(X) \). Since \( |z_1| = |z_2| \), we see that \( 1 \) is a root of \( z_2(X) - z_1(X) \). However, as \( N_0[\alpha] \) is an FGM, it follows from Proposition 5.7 that \( 1 \) is not a root of \( m_\alpha(X) \). Hence \( c = 0 \), which implies that \( z_1(\alpha) = z_2(\alpha) \). As a result, \( N_0[\alpha] \) is an OHFM. That \( N_0[\alpha] \) is not a UFM is an immediate consequence of part (2) of Theorem 5.4.

In order to prove the last statement, assume that \( d \geq 3 \). Take \( p \in \mathbb{P} \), and consider the polynomial

\[
m(X) = X^d - pX^{d-1} + pX^{d-2} - \sum_{i=0}^{d-3} pX^i.
\]

It is clear that \( m(X) \) is irreducible. Since \( m(1) = 1 - (d-2)p < 0 \), the polynomial \( m(X) \) has a real root \( \alpha_p > 1 \). Let \( (p(X), q(X)) \) be the minimal pair of \( \alpha_p \), and consider the algebraic semiring valuation \( S_{\alpha_p} \). The monoid \( N_0[\alpha_p] \) has rank \( d \) by Proposition 3.2. On the other hand, it follows from part (1) of Proposition 4.5 that \( N_0[\alpha_p] \) is atomic. In addition,

\[
m(X)(X+1) = X^{d+1} - (p-1)X^d - \left( \sum_{i=1}^{d-3} 2pX^i \right) - p
\]

implies that \( \mathscr{A}(N_0[\alpha_p]) = \{ \alpha^k_p : j \in [0, k] \} \) for some \( k \in \{ d-1, d \} \). However, notice that if \( k = d-1 \), then Theorem 5.4 would force \( p(X) \) to be a monomial, which is not the case. Therefore we obtain that \( \mathscr{A}(N_0[\alpha_p]) = \{ \alpha^k_p : j \in [0, d] \} \), and it follows from Theorem 5.10 that \( N_0[\alpha_p] \) is a proper OHFM. Finally, observe that by Proposition 3.3, different choices of \( p \in \mathbb{P} \) yield non-isomorphic algebraic semiring valuations \( S_{\alpha_p} \), whose additive monoids satisfy the desired conditions. \( \square \)

The following corollary is an immediate consequence of Theorem 5.10.

**Corollary 5.11.** If \( N_0[\alpha] \) is a proper OHFM for some \( \alpha \in \mathbb{R}_{>0} \), then it is an FGM and, therefore, an FFM.
We have just seen that if an algebraic monoid valuation $N_0[\alpha]$ is an OHFM, then it is an FGM. These two properties are equivalent when $N_0[\alpha]$ has rank 2.

**Proposition 5.12.** Let $\alpha \in \mathbb{R}_{\geq 0}$ be an algebraic number such that $N_0[\alpha]$ has rank 2. Then the following statements are equivalent.

(a) $N_0[\alpha]$ is a UFM.

(b) $N_0[\alpha]$ is an OHFM.

(c) $N_0[\alpha]$ is an FGM.

**Proof.** (a) $\Rightarrow$ (b): This is clear.

(b) $\Rightarrow$ (c): This is an immediate consequence of Theorem 5.4 and Corollary 5.11.

(c) $\Rightarrow$ (a): Assume first that $N_0[\alpha]$ is an FGM with rank 1. It follows from Proposition 3.2 that $N_0[\alpha]$ is a rational monoid valuation of $N_0[X]$. Since $N_0[\alpha]$ is an FGM, [9, Proposition 4.3] guarantees that $N_0[\alpha] = N_0$ and, therefore, it is a UFM.

Now assume that $N_0[\alpha]$ is an FGM with rank 2. It follows from Proposition 3.2 that the irreducible polynomial $m_\alpha(X)$ of $\alpha$ has degree 2, and it follows from Proposition 5.7 that $m_\alpha(X)$ belongs to $\mathbb{Z}[X]$ and has $\alpha$ as its unique positive root (counting multiplicity). Write $m_\alpha(X) = X^2 + aX - b$ for some $a, b \in \mathbb{Z}$. As $m_\alpha(X)$ has a unique positive root, Descartes’ Rule of Signs guarantees that $b > 0$.

Suppose, by way of contradiction, that $a > 0$. Since $N_0[\alpha]$ is an FGM, there is a polynomials $f(X) \in \mathbb{Z}[X]$ such that $m_\alpha(X)f(X)$ is monic and its only positive coefficient is its leading coefficient (see the proof of Proposition 5.7). Assume that the polynomial $f(X)$ has the least degree possible. Now write $f(X) = X^k + \sum_{i=0}^{k-1} c_i X^i$ for $c_0, \ldots, c_{k-1} \in \mathbb{Z}$ and

\[
m_\alpha(X)f(X) = (X^2 + aX - b) \left( X^k + \sum_{i=0}^{k-1} c_i X^i \right) = X^{k+2} + \sum_{i=0}^{k+1} d_i X^i
\]

for $d_0, \ldots, d_{k+1} \in \mathbb{Z}_{\geq 0}$. Since $a > 0$, we see that $\deg f(X) \geq 1$. As $d_0 < 0$, we obtain from (5.2) that $c_0 > 0$. Observe that $\deg f(X) \geq 2$ as, otherwise, $d_2 \leq 0$ and (5.2) would imply that $c_0 \leq -a < 0$. As $d_1 \leq 0$, we obtain from (5.2) that $c_1 \geq ac_0/b > 0$. As before, $\deg f(X) \geq 3$; otherwise, $d_3 \leq 0$ would imply that $c_1 \leq -a < 0$. For each $j \in \{2, k - 1\}$, one can compare coefficients in (5.2) to find that $bc_j \geq c_{j-2} + ac_{j-1}$. Now an immediate induction reveals that $c_j \geq 0$ for every $j \in \{0, k - 1\}$. However, comparing the coefficients of the terms of degree $k + 1$ in (5.2), we see that $c_{k-1} \leq -a < 0$, which is a contradiction. As a consequence, $a \leq 0$. Hence $N_0[\alpha]$ is a UFM by Theorem 5.4.

It follows from Proposition 5.12 that if $N_0[\alpha]$ is a proper OHFM, then its rank is at least 3. For each rank $d \in \mathbb{N}_{\geq 3}$, there are infinitely many non-isomorphic algebraic semiring valuations of $N_0[X]$ whose additive monoids are rank-$d$ FGMs that are not OHFMs.

**Proposition 5.13.** For each $d \in \mathbb{N}_{\geq 3}$, there exist infinitely many non-isomorphic semiring valuations of $N_0[X]$ whose additive monoids are rank-$d$ FGMs that are not OHFMs.

**Proof.** Fix $d \in \mathbb{N}_{\geq 3}$. Take $p \in \mathbb{P}$, and consider the polynomial

\[
m(X) = X^d - 3pX^{d-1} + 2pX^{d-2} - \sum_{i=0}^{d-3} pX^i
\]

of $\mathbb{Z}[X]$. As $m(0) = -p$, the polynomial $m(X)$ has a positive root $\alpha_p$. Since $m(X)$ is irreducible (by virtue of Eisenstein’s Criterion), it must be the minimal polynomial of $\alpha_p$. The monoid valuation $N_0[\alpha_p]$ has rank $d$ by Proposition 3.2. In addition, it follows from Proposition 3.3 that distinct choices of the parameter $p \in \mathbb{P}$ yield non-isomorphic algebraic semiring valuations $S_{\alpha_p}$ of $N_0[X]$. So proving the proposition amounts to showing that $N_0[\alpha_p]$ is an FGM that is not an OHFM.
For simplicity, set $\alpha = \alpha_p$. The monoid $\mathbb{N}_0[\alpha]$ is atomic by part (1) of Proposition 4.5. To show that $\mathbb{N}_0[\alpha]$ is an FGM, set $f(X) := m(X)(X^2 + X + 1) \in \mathbb{Z}[X]$. One can immediately verify that the only positive coefficient of $f(X)$ is its leading coefficient. Therefore $\mathcal{A}(\mathbb{N}_0[\alpha]) \subseteq \{\alpha^n : n \in [0, d+1]\}$. Since $\mathbb{N}_0[\alpha]$ is atomic, it must be an FGM. To show that $\mathbb{N}_0[\alpha]$ is not an OHFM, it suffices to consider for every $b \in \mathbb{Z}$ the polynomial $g_b(X) = m(X)(X + b)$ and observe that one of the coefficients $-p(3b - 2)$ and $-p(1 - 2b)$ of $g_b(X)$ (corresponding to degrees $d - 1$ and $d - 2$, respectively) must be positive. Thus, $\mathbb{N}_0[\alpha]$ is not an OHFM by Theorem 5.10.

Motivated by our proof of Proposition 5.13, we are inclined to believe that the following related conjecture is true.

**Conjecture 5.14.** For every $n \in \mathbb{N}$ there is an algebraic number $\alpha \in \mathbb{R}_{>0}$ such that $|\mathcal{A}(\mathbb{N}_0[\alpha])| < \infty$ and $|\mathcal{A}(\mathbb{N}_0[\alpha])| - \deg m_\alpha(X) = n$.

The main results we have established in this section are the non-obvious implications in Diagram (5.3), where the last implication (in blue) holds for all algebraic monoid valuations of $\mathbb{N}_0[X]$ and the rest of the implications hold for all monoid valuations of $\mathbb{N}_0[X]$. Observe that if we put this diagram and Diagram (4.2) together, then we obtain Diagram (1.2), the summarizing diagram presented in the introduction.

![Diagram](image)

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