SOME NEW REGULARITY CRITERIA FOR THE 3D NAVIER-STOKES EQUATIONS

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Abstract. Several types of new regularity criteria of Leray-Hopf weak solutions \( u \) to the 3D Navier-Stokes equations are obtained. Some of them are based on the third component \( u_3 \) of velocity under the Prodi-Serrin index condition. And a very recent work of the authors, based on only one of the nine entries of the gradient tensor, is renovated. At last, some regularity criteria which are dependent on some parameter \( \epsilon \) are obtained.

1. Introduction

In the present paper, we address sufficient conditions for the regularity of weak solutions of the Cauchy problem for the Navier-Stokes equations in \( \mathbb{R}^3 \times (0, T) \):

\[
\begin{aligned}
\frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla)u + \nabla p &= 0, \quad \text{in } \mathbb{R}^3 \times (0, T), \\
\nabla \cdot u &= 0, \quad \text{in } \mathbb{R}^3 \times (0, T), \\
u(x, 0) &= u_0, \quad \text{in } \mathbb{R}^3,
\end{aligned}
\]

where \( u = (u_1, u_2, u_3) : \mathbb{R}^3 \times (0, T) \rightarrow \mathbb{R}^3 \) is the velocity field, \( p : \mathbb{R}^3 \times (0, T) \rightarrow \mathbb{R}^3 \) is a scalar pressure, and \( u_0 \) is the initial velocity field, \( \nu > 0 \) is the viscosity. We set \( \nabla_h = (\partial_{x_1}, \partial_{x_2}) \) as the horizontal gradient operator and \( \Delta_h = \partial^2_{x_1} + \partial^2_{x_2} \) as the horizontal Laplacian, and \( \Delta \) and \( \nabla \) are the usual Laplacian and the gradient operators, respectively. Here we use the classical notations

\[
(u \cdot \nabla)v = \sum_{i=1}^{3} u_i \partial_{x_i} v_k, \quad (k = 1, 2, 3), \quad \nabla \cdot u = \sum_{i=1}^{3} \partial_{x_i} u_i,
\]

and for sake of simplicity, we denote \( \partial_{x_i} \) by \( \partial_i \).

It is well known that the weak solution of the Navier-Stokes equations (1.1) is unique and regular in two dimensions. However, in three dimensions, the regularity problem of weak solutions of Navier-Stokes equations is an outstanding open problem in mathematical fluid mechanics. The weak solutions are known to exist globally in time, but the uniqueness, regularity, and continuous dependence on initial data for weak solutions are still open problems. Furthermore, strong solutions in the 3D case are known to exist for a short interval of time whose length depends on the initial data. Moreover, this strong solution is known to be unique and to depend continuously on the initial data (see, for example, [22], [24]). Let us recall the definition of Leray-Hopf weak solution. We set

\[
\mathcal{V} = \{ \phi : \text{the 3D vector valued } C^\infty_0 \text{ functions and } \nabla \cdot \phi = 0 \},
\]
which will form the space of test functions. Let $H$ and $V$ be the closure spaces of $\mathcal{V}$ in $L^2$ under $L^2$-topology, and in $H^1$ under $H^1$-topology, respectively.

For $u_0 \in H$, the existence of weak solutions of (1.1) was established by Leray \cite{15} and Hopf in \cite{10}, that is, $u$ satisfies the following properties:

(i) $u \in C([0, T); H) \cap L^2(0, T; V)$, and $\partial_t u \in L^1(0, T; V')$, where $V'$ is the dual space of $V$;

(ii) $u$ verifies (1.1) in the sense of distribution, i.e., for every test function $\phi \in C_0^\infty([0, T); \mathcal{V})$, and for almost every $t, t_0 \in (0, T)$, we have

$$
\int_{\mathbb{R}^3} u(x, t) \cdot \phi(x, t) dx - \int_{\mathbb{R}^3} u(x, t_0) \cdot \phi(x, t_0) dx
\leq \int_{t_0}^{t} \int_{\mathbb{R}^3} [u(x, t) \cdot (\phi_t(x, t) + \nu \Delta \phi(x, t))] dx ds
+ \int_{t_0}^{t} \int_{\mathbb{R}^3} [(u(x, t) \cdot \nabla) \phi(x, t)] \cdot u(x, t)) dx ds
$$

(iii) The energy inequality, i.e.,

$$
\|u(\cdot, t)\|_{L^2}^2 + 2\nu \int_{t_0}^{t} \| \nabla u(\cdot, s) \|_{L^2}^2 ds \leq \|u_0\|_{L^2}^2,
$$

for every $t$ and almost every $t_0$.

It is well known, if $u_0 \in V$, a weak solution becomes strong solution of (1.1) on $(0, T)$ if, in addition, it satisfies

$$
u \in C((0, T); V) \cap L^2(0, T; H^2) \text{ and } \partial_t u \in L^2(0, T; H).$$

We know the strong solution is regular (say, classical) and unique (see, for example, \cite{22, 24}).

Researchers are interested in the classical problem of finding sufficient conditions for weak solutions of (1.1) such that the weak solutions become regular, and the first result is usually referred as Prodi-Serrin conditions (see \cite{20} and \cite{21}), which states that if a weak solution $u$ is in the class of

$$
u \in L^t(0, T; L^s(\mathbb{R}^3)), \quad \frac{2}{t} + \frac{3}{s} = 1, \quad s \in [3, \infty],
$$

then the weak solution becomes regular. Recently, H. Bae and H. Cho in \cite{1} gave a two components Prodi-Serrin index criterion. Up to now, there are many results show that one can use only one component (say $u_3$) to determine the regularity of $u$. Say, I. Kukavica and M. Ziane in \cite{12} proved a regularity criterion under the following condition

$$
u \in L^t(0, T; L^s(\mathbb{R}^3)), \quad \frac{2}{t} + \frac{3}{s} \leq \frac{5}{8}, \quad s \in \left[\frac{24}{5}, \infty\right].$$

Then, it was improved by C. Cao and E. Titi in \cite{51} to

$$
u \in L^t(0, T; L^s(\mathbb{R}^3)), \quad \frac{2}{t} + \frac{3}{s} \leq \frac{2(s + 1)}{3s}, \quad s \in \left(\frac{7}{2}, \infty\right].$$

And then, Y. Zhou and M. Pokorný in \cite{26} changed the regular criterion to

$$
u \in L^t(0, T; L^s(\mathbb{R}^3)), \quad \frac{2}{t} + \frac{3}{s} \leq \frac{3}{4} + \frac{1}{2s}, \quad s \in \left(\frac{10}{3}, \infty\right].$$

More relative results, we refer to \cite{16, 25} and the reference there in. One can see that the above mentioned results on $u_3$ cannot satisfy the Prodi-Serrin index condition, and it
seems to be a price when one reduce the components of \( u \) to one. It is nature to think about what supplement is necessary to insure the Prodi-Serrin condition based on one velocity component. For example, P. Penel and M. Pokorný in [18] proved the \( u \) was regular, if

\[
\nabla u_3 \in L^\beta(0, T; L^\alpha(\mathbb{R}^3)) \text{ with } \frac{3}{\alpha} + \frac{2}{\beta} \leq 1, \alpha \in (3, \infty], \beta \in [2, \infty),
\]

and one of the following conditions holds true:

(a) \( \partial_3 u_1, \partial_3 u_2 \) belong to \( L^\beta(0, T; L^\alpha(\mathbb{R}^3)) \) with \( \frac{3}{\alpha} + \frac{2}{\beta} \leq 2, \alpha \in (3/2, \infty], \beta \in [1, \infty); \)

(b) \( \partial_2 u_1, \partial_1 u_2 \) belong to \( L^\beta(0, T; L^\alpha(\mathbb{R}^3)) \) with \( \frac{3}{\alpha} + \frac{2}{\beta} \leq 2, \alpha \in [2, 3], \beta \in [2, \infty); \)

(c) \( \partial_3 u_2 \) belong to \( L^\beta(0, T; L^\alpha(\mathbb{R}^3)) \) with \( \frac{3}{\alpha} + \frac{2}{\beta} \leq 2, \alpha \in (3/2, \infty], \beta \in [1, \infty) \), and \( \partial_2 u_1 \) belong to \( L^\beta(0, T; L^\alpha(\mathbb{R}^3)) \) with \( \frac{3}{\alpha} + \frac{2}{\beta} \leq 2, \alpha \in [2, 3], \beta \in [2, \infty]. \)

Moreover, the authors also mentioned in the Remark 2 in [18], the condition in (a) can be replaced by \( \partial_3 u_2, \partial_2 u_2, \text{ or } \partial_3 u_2, \partial_1 u_1, \text{ or } \partial_3 u_1, \partial_2 u_1, \text{ or } \partial_3 u_1, \partial_2 u_1 \). Similarly, in (c) one can replace \( \partial_3 u_2 \) by \( \partial_3 u_1 \), and replace \( \partial_2 u_1 \) by \( \partial_1 u_2 \) respectively.

From the above, we can see that the assumptions on derivative component did not contain \( \partial_3 u_i \) \( i = 1, 2, 3 \). One purpose of this paper is to capture this work, by using the incompressibility condition. We give an estimate on velocity, which is different from [18] and then get a regularity criterion on \( \partial_i u_3, i = 1, 2, 3 \), for detail see the proof of Theorem L.1 below. On the other hand, similar to (a), (b) and (c), we also consider cases of the given conditions in terms of only one component \( \partial_i u_j \) of \( \nabla u \) such that \( u_3 \) satisfies the Prodi-Serrin condition in Theorem L.2 and Corollary L.4.

Besides, we would like to point out that the full regularity of weak solutions can also be proved under alternative assumptions on the gradient of the velocity \( \nabla u \), for instance

\[
\nabla u \in L^l(0, T; L^s(\mathbb{R}^3)), \quad \frac{2}{l} + \frac{3}{s} = 2, \quad s \in [\frac{3}{2}, \infty]. \tag{1.3}
\]

Enlightened by the above, we also want to get some better regularity criteria which are also coincident with the standard Prodi-Serrin condition based on some components of \( \nabla u \). To begin with, we mention some results in this direction at first, P. Penel and M. Pokorný in [18] proved that if

\[
\nabla \partial_3 u \in L^\beta(0, T; L^\alpha(\mathbb{R}^3)) \text{ with } \frac{3}{\alpha} + \frac{2}{\beta} \leq \frac{3}{2}, \alpha \in [2, \infty], \beta \in [1, \infty),
\]

then the weak solution was regular. After that many authors improved this result, such as I. Kukavica and M. Ziane in [13] considered the case of the condition

\[
\nabla \partial_3 u \in L^\beta(0, T; L^\alpha(\mathbb{R}^3)) \text{ with } \frac{3}{\alpha} + \frac{2}{\beta} \leq 2, \alpha \in [\frac{9}{4}, 3].
\]

As to the gradient of one velocity component \( \nabla u_3 \), M. Pokorný in [19] proved the weak solution was actually regular if \( \nabla u_3 \) satisfied

\[
\nabla u_3 \in L^\beta(0, T; L^\alpha(\mathbb{R}^3)) \text{ with } \frac{3}{\alpha} + \frac{2}{\beta} \leq \frac{3}{2}, \alpha \in [2, \infty].
\]
Y. Zhou and M. Pokorný in [26] improved the result to
\[ \nabla u_3 \in L^\beta(0, T; L^\alpha(\mathbb{R}^3)) \] with
\[ \frac{3}{\alpha} + \frac{2}{\beta} \leq \begin{cases} 
\frac{19}{12} + \frac{1}{2\alpha} & \alpha \in (\frac{30}{19}, 3] \\
3 + \frac{3}{4\alpha} & \alpha \in (3, \infty],
\end{cases} \]
moreover, Y. Zhou and M. Pokorný also proved a improved result, for more detail we refer to [27]. Motivated by the above, we consider the case of two gradient velocity components and one of them satisfies the Prodi-Serrin condition, see Theorem 1.5 and Corollary 1.6. We shall point out that two gradient velocity components are not all the diagonal elements, this is more difficult than the diagonal case, for the detail see Remark 1.7 below.

In [26], the authors also studied the regularity of the solutions of the Navier-Stokes equations under the assumption on \( \partial_3 u_3 \), namely,
\[ \partial_3 u_3 \in L^\beta(0, T; L^\alpha(\mathbb{R}^3)), \quad 3 \alpha + 2 \beta < \frac{4}{5}, \quad \alpha \in (\frac{15}{4}, \infty]. \] (1.4)
Recently, the regularity criterion in terms of only one of the gradient tensor was gotten by C. Cao and E. Titi in [4] under the assumptions
\[ \frac{\partial u_j}{\partial x_k} \in L^\beta(0, T; L^\alpha(\mathbb{R}^3)), \] when \( j \neq k \)
and where \( \alpha > 3, 1 \leq \beta < \infty \), and
\[ \frac{3}{\alpha} + \frac{2}{\beta} \leq \frac{\alpha + 3}{2\alpha}, \] (1.5)
or
\[ \frac{\partial u_j}{\partial x_j} \in L^\beta(0, T; L^\alpha(\mathbb{R}^3)), \]
and where \( \alpha > 2, 1 \leq \beta < \infty \), and
\[ \frac{3}{\alpha} + \frac{2}{\beta} \leq \frac{3(\alpha + 2)}{4\alpha}. \] (1.6)
In [7], we improved this result. And here, we again study it and get an improvement of the results of [7], which is shown in Theorem 1.8. However, it is also noted that the above conditions are not coincident with the Prodi-Serrin condition.

Now, we list our main results as follows:

**Theorem 1.1.** Let \( u \) be a Leray-Hopf weak solution to the 3D Navier-Stokes equations (1.1) with the initial value \( u_0 \in V \). Suppose
\[ u_3 \in L^{\beta_1}(0, T; L^{\alpha_1}(\mathbb{R}^3)) \] with
\[ \frac{3}{\alpha_1} + \frac{2}{\beta_1} \leq 1, \alpha_1 \in (3, \infty], \] (1.7)
and one of the following conditions holds:

(i) \( \partial_3 u_2, \partial_3 u_3 \in L^{\beta_2}(0, T; L^{\alpha_2}(\mathbb{R}^3)) \) with
\[ \frac{3}{\alpha_2} + \frac{2}{\beta_2} \leq 2, \alpha_2 \in [2, 3] \] (1.8)

(ii) \( \partial_3 u_1, \partial_3 u_3 \in L^{\beta_2}(0, T; L^{\alpha_2}(\mathbb{R}^3)) \) with
\[ \frac{3}{\alpha_2} + \frac{2}{\beta_2} \leq 2, \alpha_2 \in [2, 3] \] (1.9)

Then \( u \) is regular.
Theorem 1.2. Let \( u \) be a Leray-Hopf weak solution to the 3D Navier-Stokes equations \((1.1)\). Suppose that, for some \( i, j \) with \( 1 \leq i \leq 3 \) and \( 1 \leq j \leq 2 \), \( u \) satisfies one of the following conditions:

(a) \( i \neq j \), suppose the initial value \( u_0 \in V \cap L^q(\mathbb{R}^3) \), where \( 1 < q \leq 2 \), the solution \( u \) satisfies

\[
\begin{align*}
u_3 & \in L^{\beta_1}(0, T; L^{\alpha_1}(\mathbb{R}^3)) \quad \text{with} \quad \frac{3}{\alpha_1} + \frac{2}{\beta_1} \leq 1, \quad \alpha_1 \in (3, \infty], \\
\partial_i u_j & \in L^{\beta_2}(0, T; L^{\alpha_2}(\mathbb{R}^3)),
\end{align*}
\]

with

\[
\frac{3}{\alpha_2} + \frac{2}{\beta_2} \leq \frac{3}{\alpha_2} - \frac{q}{\alpha_2} + q - 1, \quad \alpha_2 \in \left( \frac{q}{q-1}, \infty \right].
\]

(b) \( i = j \), suppose the initial value \( u_0 \in V \) and \( u_3 \) satisfies the condition \((1.10)\), and

\[
\begin{align*}
\partial_j u_j & \in L^{\beta_3}(0, T; L^{\alpha_3}(\mathbb{R}^3)), \\
3 \alpha_3 + 2 \beta_3 & \leq 3 - 2q \alpha_2 + 9 - 2 \alpha_3 \alpha_2 \in (2 \alpha_3, \infty], \\
f(\alpha_3) & = \frac{\sqrt{24 \alpha_3^2 - 24 \alpha_2 + 9 - 2 \alpha_3}}{2 \alpha_3}, \quad \alpha_3 \in \left( \frac{2 \alpha_2 + 9}{3 \alpha_2} \right).
\end{align*}
\]

Then \( u \) is regular.

Remark 1.3. When we announced the first version of this article on the arXiv.org, we were informed by the authors of \([11]\) that they finished the same result as the part (a) of Theorem 1.2 with \( q = 2 \). The above is the improved result with a parameter \( q \) satisfying \( 1 < q \leq 2 \). For \( q > 2 \), in fact, one also can get some results, for example the condition \((1.12)\) can be replaced by

\[
\frac{3}{4 \alpha_2} + \frac{2}{\beta_2} \leq \frac{\sqrt{52 \alpha_2^2 - 60 \alpha_2 + 9 - 4 \alpha_2}}{4 \alpha_2}, \quad \alpha_2 \in \left( \frac{3 + \sqrt{7}}{2}, \infty \right].
\]

Corollary 1.4. Suppose that \( u_0 \in V \), and \( u \) is a Leray-Hopf weak solution to the 3D Navier-Stokes equations \((1.1)\). Suppose that, for some \( i, j \) with \( 1 \leq i \leq 3 \) and \( 1 \leq j \leq 2 \), \( u \) satisfies one of the following conditions:

(a) \( i \neq j \), \( u_3 \) satisfies the condition \((1.10)\) and

\[
\begin{align*}
\partial_i u_j & \in L^{\beta_2}(0, T; L^{\alpha_2}(\mathbb{R}^3)), \\
3 \alpha_2 + 2 \beta_2 & \leq 1, \quad \alpha_2 \in (2, \infty].
\end{align*}
\]

(b) \( i = j \), \( u_3 \) satisfies the condition \((1.10)\), and

\[
\begin{align*}
\partial_j u_j & \in L^{\beta_3}(0, T; L^{\alpha_3}(\mathbb{R}^3)), \\
3 \alpha_3 + 2 \beta_3 & \leq \frac{3}{2}, \quad \alpha_3 \in [2, 6].
\end{align*}
\]

Then \( u \) is regular.

If we substitute the condition on \( u_3 \) by the component of the gradient of the velocity, we have the following regularity criterion, which is a further improvement of the above mentioned results of \([18]\).
Theorem 1.5. Let $u$ be a Leray-Hopf weak solution to the 3D Navier-Stokes equations (1.1) with the initial value $u_0 \in V$. Suppose

$$
\partial_3 u_i \in L^{\beta_1}(0, T; L^{\alpha_1}(\mathbb{R}^3)) \quad \text{with} \quad \frac{3}{\alpha_1} + \frac{2}{\beta_1} \leq 2, \alpha_1 \in [2, 3], \ i = 1 \ or \ 2,
$$

and

$$
\partial_3 u_3 \in L^{\beta_2}(0, T; L^{\alpha_2}(\mathbb{R}^3)),
$$

with

$$
\frac{3}{2\alpha_2} + \frac{2}{\beta_2} \leq f(\alpha_2), \ \alpha_2 \in [2, 3],
$$

where

$$
f(\alpha_2) = \frac{\sqrt{24\alpha_2^3 - 24\alpha_2 + 9} - 2\alpha_2}{2\alpha_2}.
$$

Then $u$ is regular.

Corollary 1.6. Suppose that $u_0 \in V$, and $u$ is a Leray-Hopf weak solution to the 3D Navier-Stokes equations (1.1). Assume

$$
\partial_3 u_i \in L^{\beta_1}(0, T; L^{\alpha_1}(\mathbb{R}^3)) \quad \text{with} \quad \frac{3}{\alpha_1} + \frac{2}{\beta_1} \leq 2, \alpha_1 \in [2, 3], \ i = 1 \ or \ 2,
$$

and

$$
\partial_3 u_3 \in L^{\beta_2}(0, T; L^{\alpha_2}(\mathbb{R}^3)) \quad \text{with} \quad \frac{3}{\alpha_2} + \frac{2}{\beta_2} \leq \frac{3}{2}, \alpha_2 \in [2, 3].
$$

Then $u$ is regular.

Remark 1.7. Here we only need two components of the gradient of the velocity and one of them is not on the diagonal elements of $\nabla u$. On the case of the diagonal elements of $\nabla u$, P. Penel and M. Pokorný in [18] proved the $u$ is regular when

$$
\partial_2 u_2, \partial_3 u_3 \in L^\beta(0, T; L^\alpha(\mathbb{R}^3)), \quad \text{with} \quad \frac{3}{\alpha} + \frac{2}{\beta} \leq 2, \alpha \in \left(\frac{3}{2}, \infty\right], \beta \in [2, \infty).
$$

Moreover, the condition on $\partial_3 u_i$ satisfies the Prodi-Serrin condition, which is an improvement of the result of P. Penel and M. Pokorný in [18]. Finally, we note that $\partial_3 u_i, \ i = 1 \ or \ 2$, is not the diagonal element of $\nabla u$. Thus, we cannot use the method of by multiplying $u_i$ to the $i$th equation of (1.1) to get the form $\partial_3 u_i, \ i = 1 \ or \ 2$. Therefore, it is more difficult to get the regularity criterion based on $\partial_k u_i$ and $\partial_j u_j, \ i, j \in \{1, 2, 3\}$ with $i \neq j$.

Theorem 1.8. Let $u_0$ and $u$ be as in Theorem 1.1. Suppose in addition

$$
\partial_k u_k \in L^\beta(0, T; L^\alpha(\mathbb{R}^3)), \quad \text{for some} \ k \in \{1, 2, 3\}
$$

with

$$
\frac{3}{2\alpha} + \frac{2}{\beta} \leq g(\alpha), \ \alpha \in \left(\frac{9}{5}, \infty\right],
$$

where

$$
g(\alpha) = \frac{\sqrt{289\alpha^2 - 264\alpha + 144} - 7\alpha}{8\alpha}.
$$

Then $u$ is regular.
Remark 1.9. This theorem is an improvement of [4] and [26] (see figure 1 below), and is also an improvement of Theorem 1.2 (i) and Theorem 1.3 in [7]. Moreover, we point out that the first part of Theorem 1.2 in [7], can be simplified to the following form:

For \( j \neq k \), suppose that \( u \) satisfies

\[
\int_0^T \| \partial_j u_k \|_\alpha^{\beta} \, d\tau \leq M, \quad \text{for some } M > 0,
\]

with

\[
\frac{3}{2\alpha} + \frac{2}{\beta} \leq f(\alpha), \quad \alpha \in (3, \infty) \text{ and } 1 \leq \beta < \infty,
\]

where

\[
f(\alpha) = \sqrt{103\alpha^2 - 12\alpha + 9} - 9\alpha,
\]

then \( u \) is regular. This function shows the same line as in the Figure 1. in [7]. In the proof of Theorem 1.2 (i) of [7], if we substitute \( \sigma_1 \) and \( \beta \) by

\[
\sigma_1 = \frac{(7 - \frac{3}{\alpha}) + \sqrt{\frac{9}{\alpha^2} - \frac{12}{\alpha} + 103}}{18}, \quad \beta = \frac{2\sigma_1}{9 - 8\sigma_1},
\]

we can get the desired result.

The following theorems show the variation of the criterion with some parameter.

**Theorem 1.10.** Let \( u \) be a Leray-Hopf weak solution to the 3D Navier-Stokes equations (1.1) with the initial value \( u_0 \in V \cap L^1(\mathbb{R}^3) \). Suppose that, for some \( i, j \) with \( 1 \leq i, j \leq 3 \), \( u \) satisfies one of the following conditions:

(a) \( i \neq j \),

\[
\partial_i u_j \in L^{\beta_1}(0, T; L^{\alpha_1}(\mathbb{R}^3)) \quad \text{with} \quad \frac{3}{\alpha_1} + \frac{2}{\beta_1} = 2 - \epsilon,
\]

where

\[
\alpha_1 \in \left[ \frac{3}{2 - \epsilon}, \frac{3}{3 - 2\epsilon} \right], \quad \text{and} \quad 1 \leq \epsilon < 3/2.
\]

(b) \( i = j \),

\[
\partial_j u_j \in L^{\beta_2}(0, T; L^{\alpha_2}(\mathbb{R}^3)) \quad \text{with} \quad \frac{3}{\alpha_2} + \frac{2}{\beta_2} = 2 - \epsilon,
\]

where

\[
\alpha_2 \in \left[ \frac{3}{2 - \epsilon}, \frac{6}{5 - 4\epsilon} \right], \quad \text{and} \quad 1/2 \leq \epsilon < 5/4.
\]

Then \( u \) is regular.

**Remark 1.11.** It is sufficient to assume that \( u_0 \in V \) when we consider the endpoint case of \( \epsilon = 1 \) in part (a) and \( \epsilon = 1/2 \) in part (b) respectively. In view of the result of part (b), we can show the the line in Figure 1, which is continuous in \( \epsilon \). However, we see that this line is always under the line “(1)” in Figure 1. From the proof of this Theorem, we know that we choose an intermediate parameter \( q \), and restrict \( q \) to satisfy \( 1 < q \leq 2 \) for convenience, which is the underlying reason why the line is always below line “(1)” in Figure 1. In fact, if we choose the intermediate parameter \( q \) is larger than 2, we can get...
another better result such that the corresponding line is always above line "(1)" in Figure 1, which is stated in the following Theorem.

**Theorem 1.12.** Let \( u \) be a Leray-Hopf weak solution to the 3D Navier-Stokes equations \((1.1)\) with the initial value \( u_0 \in V \). Suppose that, for some \( i, j \) with \( 1 \leq i, j \leq 3 \), \( u \) satisfies one of the following conditions:

(a) \( i \neq j \), \[
\partial_i u_j \in L^{\beta_1}(0, T; L^{\alpha_1}(\mathbb{R}^3)) \text{ with } \frac{3}{\alpha_1} + \frac{2}{\beta_1} = 2 - \epsilon,
\]
where \[
\alpha_1 \in \left[ \frac{3}{3 - 2\epsilon}, \frac{3(11 - 2\epsilon)}{2\epsilon^2 - 26\epsilon + 33} \right], \text{ and } 1 < \epsilon \leq 21/16.
\]

(b) \( i = j \), \[
\partial_j u_j \in L^{\beta_2}(0, T; L^{\alpha_2}(\mathbb{R}^3)) \text{ with } \frac{3}{\alpha_2} + \frac{2}{\beta_2} = 2 - \epsilon,
\]
where \[
\alpha_2 \in \left[ \frac{6}{5 - 4\epsilon}, \frac{18 - 2\epsilon}{(4\epsilon - 3)(\epsilon - 5)} \right], \text{ and } 1/2 < \epsilon \leq 3/4.
\]

Then \( u \) is regular.

For convenience, we recall the following version of the three-dimensional Sobolev and Ladyzhenskaya inequalities in the whole space \( \mathbb{R}^3 \) (see, for example, [6], [9], [14]). There

![Figure 1. Case of \( j = k \)](image)

The line "(1)" is the result of C.S. Cao, E.S. Titi in [4] (see (1.6)). The line "(3)" is our result, which means (1.24). The result of Y. Zhou, M. Pokorný in [26] (see (1.4)) is showed by line "(2)".
exists a positive constant $C$ such that
\[
\|u\|_r \leq C\|u\|_2^{6-r} \|\partial_i u\|_2^{r-2} \|\partial_2 u\|_2^{r-2} \|\partial_3 u\|_2^{r-2},
\]  
(1.36)
for every $u \in H^1(\mathbb{R}^3)$ and every $r \in [2,6]$, where $C$ is a constant depending only on $r$. Taking $\nabla \text{div}$ on both sides of (1.1) for smooth $(u;p)$, one can obtain
\[
-\Delta(\nabla p) = \sum_{i,j}^3 \partial_i \partial_j(\nabla(u_i u_j)),
\]
therefore, the Calderon-Zygmund inequality in $\mathbb{R}^3$ (see [23])
\[
\|\nabla p\|_q \leq C\|\nabla u\|_2, 1 < q < \infty,
\]  
(1.37)
holds, where $C$ is a positive constant depending only on $q$. And there is another estimates for the pressure
\[
\|p\|_q \leq C\|u\|_2, 1 < q < \infty,
\]  
(1.38)

2. Proof of Main Results

In this section, under the assumptions of Theorems [1.1,1.2, Theorem 1.5, Theorem 1.8, Theorem 1.10, Theorem 1.12] in Section 1 respectively, we prove our main results. First of all, we note that, by the energy inequality, for Leray-Hopf weak solutions, we have (see, for example, [22], [24] for detail)
\[
\|u(\cdot,t)\|_{L^2}^2 + 2\nu \int_0^t \|\nabla u(\cdot,s)\|_{L^2}^2 ds \leq K_1,
\]  
(2.1)
for all $0 < t < T$, where $K_1 = \|u_0\|_{L^2}^2$.

It is well known that there exists a unique strong solution $u$ local in time if $u_0 \in V$. In addition, this strong solution $u \in C([0,T^*]; V) \cap L^2(0,T^*; H^2(\mathbb{R}^3))$ is the only weak solution with the initial datum $u_0$, where $(0,T^*)$ is the maximal interval of existence of the unique strong solution. If $T^* \geq T$, then there is nothing to prove. If, on the other hand, $T^* < T$, then our strategy is to show that the $H^1$ norm of this strong solution is bounded uniformly in time over the interval $(0,T^*)$, provided additional conditions in Theorem [1.1,1.2, Theorem 1.5, Theorem 1.8, Theorem 1.10, Theorem 1.12] in Section 1 are valid. As a result the interval $(0,T^*)$ can not be a maximal interval of existence, and consequently $T^* \geq T$, which concludes our proof.

In order to prove the $H^1$ norm of the strong solution $u$ is bounded on interval $(0,T^*)$, combing with the energy equality (2.1), it is sufficient to prove
\[
\|\nabla u\|_{L^2}^2 + \nu \int_0^t \|\Delta u\|_{L^2}^2 d\tau \leq C, \ \forall \ t \in (0,T^*)
\]  
(2.2)
where $C$ is a positive constant independent of $T^*$. We recall the following lemma (see [13]), which is useful for our proof of the Theorems.

**Lemma 2.1.** Assume $v \in L_t^\infty L_x^2(\mathbb{R}^3 \times I)$ and $\nabla v \in L_t^2 L_x^6(\mathbb{R}^3 \times I)$, where $I$ is an open interval. Then $v \in L_t^r L_x^s(\mathbb{R}^3 \times I)$ for all $r$ and $s$ such that
\[
\frac{2}{s} + \frac{3}{r} = \frac{3}{2}
\]
with $2 \leq r \leq 6$,
and it holds
\[ \|v\|_{L^2_t L^\infty_x} \leq C \left\| \frac{\partial}{\partial t} v \right\|_{L^2_t L^2_x} \left\| \nabla v \right\|_{L^\infty_t L^2_x}^{3(\nu-2)/2}. \]

**Proof of Theorem 1.1** Taking the inner product of the equation (1.1) with $-\Delta u$ in $L^2$, we obtain
\[
\frac{1}{2} \frac{d}{dt} \|\nabla u\|_2^2 + \nu \|\Delta u\|_2^2 = \sum_{i,j,k=1}^3 \int_{\mathbb{R}^3} u_i \partial_i u_j \partial_{kk} u_j dx
\]
\[= \sum_{i,j,k=1}^3 \int_{\mathbb{R}^3} u_i \partial_i u_j \partial_{kk} u_j dx + \sum_{i,j,k=1}^3 \int_{\mathbb{R}^3} u_i \partial_i u_j \partial_{kk} u_j dx + \sum_{i,j,k=1}^3 \int_{\mathbb{R}^3} u_i \partial_i u_j \partial_{kk} u_j dx
\]
\[= I_1(t) + I_2(t) + I_3(t) + I_4(t). \]

By integrating by parts a few times and using the incompressibility condition, we have
\[I_1(t) = \frac{1}{2} \sum_{i,j,k=1}^2 \int_{\mathbb{R}^3} \partial_i u_i \partial_k u_j \partial_k u_j dx - \sum_{i,j,k=1}^2 \int_{\mathbb{R}^3} \partial_k u_i \partial_i u_j \partial_k u_j dx = I_1^1(t) + I_1^2(t). \]

The terms $I_1^1(t)$, $I_1^2(t)$, $I_3(t)$ and $I_4(t)$ read as
\[I_1^1(t) = -\frac{1}{2} \sum_{j,k=1}^2 \int_{\mathbb{R}^3} \partial_3 u_3 \partial_k u_j \partial_k u_j dx, \]
\[I_1^2(t) = -\sum_{j,k=1}^2 \int_{\mathbb{R}^3} \partial_k u_i \partial_i u_j \partial_k u_j dx
\]
\[= \int_{\mathbb{R}^3} \partial_3 u_1 \partial_1 u_2 \partial_3 u_2 dx + \int_{\mathbb{R}^3} \partial_3 u_2 \partial_2 u_1 \partial_1 u_2 dx + \int_{\mathbb{R}^3} \partial_3 u_1 \partial_1 u_2 \partial_1 u_2 dx
\]
\[+ \int_{\mathbb{R}^3} \partial_2 u_3 \partial_2 u_1 \partial_1 u_2 dx + \int_{\mathbb{R}^3} \partial_2 u_1 \partial_2 u_1 \partial_2 u_1 dx + \int_{\mathbb{R}^3} \partial_2 u_1 \partial_1 u_2 \partial_1 u_2 dx
\]
\[+ \int_{\mathbb{R}^3} \partial_2 u_3 \partial_2 u_2 \partial_2 u_2 dx + \int_{\mathbb{R}^3} \partial_2 u_3 \partial_2 u_2 \partial_2 u_2 dx
\]
\[= -\int_{\mathbb{R}^3} (\partial_3 u_1 \partial_1 u_2 \partial_3 u_3 + \partial_3 u_3 \partial_1 u_2 \partial_1 u_2 + \partial_2 u_1 \partial_3 u_3 \partial_2 u_2 + \partial_3 u_3 \partial_3 u_3 \partial_2 u_2) dx
\]
\[+ \sum_{j=1}^2 \sum_{k=1}^2 \int_{\mathbb{R}^3} u_3 \partial_3 u_j \partial_k u_j dx
\]
\[= -\sum_{j=1}^2 \sum_{k=1}^2 \int_{\mathbb{R}^3} \partial_k u_3 \partial_3 u_j \partial_k u_j dx - \sum_{j=1}^2 \sum_{k=1}^2 \int_{\mathbb{R}^3} u_3 \partial_3 u_j \partial_k u_j dx
\]
\[= -\sum_{j=1}^2 \sum_{k=1}^2 \int_{\mathbb{R}^3} \partial_k u_3 \partial_3 u_j \partial_k u_j dx + \frac{1}{2} \sum_{j=1}^2 \sum_{k=1}^2 \int_{\mathbb{R}^3} \partial_3 u_3 \partial_3 u_j \partial_k u_j dx, \]
\[ I_4(t) = \sum_{i,k=1}^{3} \int_{\mathbb{R}^3} u_i \partial_i u_3 \partial_{kk} u_3 \, dx \]
\[ = - \sum_{i,k=1}^{3} \int_{\mathbb{R}^3} \partial_k u_i \partial_i u_3 \partial_k u_3 \, dx - \sum_{i,k=1}^{3} \int_{\mathbb{R}^3} u_i \partial_{ik} u_3 \partial_k u_3 \, dx \]
\[ = - \sum_{i,k=1}^{3} \int_{\mathbb{R}^3} \partial_k u_i \partial_i u_3 \partial_k u_3 \, dx. \]

The above four equalities imply that
\[ |I_i| \leq C \int_{\mathbb{R}^3} |u_3| |\nabla u| |\nabla^2 u| \, dx, \quad i = 1, 3, 4. \]

- **Case of \( \partial_3 u_2, \partial_3 u_3 \)**

As for \( I_2 \), we have
\[ I_2 = - \int_{\mathbb{R}^3} \partial_3 u_1 \partial_1 u_3 \partial_3 u_3 \, dx - \int_{\mathbb{R}^3} \partial_3 u_1 \partial_1 u_2 \partial_3 u_2 \partial_3 u_3 \, dx = I_2^1 + I_2^3 + I_2^4 + I_2^5. \]

It is obvious that
\[ |I_2^j| \leq C \int_{\mathbb{R}^3} |\partial_3 u_2| |\nabla u|^2 \, dx, \quad j = 3, 6, \tag{2.4} \]
and
\[ |I_2^j| \leq C \int_{\mathbb{R}^3} |\partial_3 u_3| |\nabla^2 u| \, dx, \quad j = 2, 4, 5. \tag{2.5} \]

For the first term, by using the incompressibility condition, and integrating by parts a few times, we note that
\[ - \int_{\mathbb{R}^3} \partial_3 u_1 \partial_1 u_3 \partial_3 u_3 \, dx = 2 \int_{\mathbb{R}^3} \partial_3 u_1 \partial_1 u_3 \partial_3 u_3 \, dx \]
\[ = -2 \int_{\mathbb{R}^3} \partial_3 u_1 \partial_3 u_2 \partial_3 u_2 \partial_3 u_3 \, dx - 2 \int_{\mathbb{R}^3} \partial_3 u_1 \partial_3 u_3 \partial_3 u_3 \, dx \]
\[ = 2 \int_{\mathbb{R}^3} \partial_3 u_1 \partial_3 u_2 \partial_3 u_2 \partial_3 u_3 \, dx + 2 \int_{\mathbb{R}^3} \partial_3 u_1 \partial_3 u_3 \partial_3 u_3 \partial_3 u_3 \, dx \]
\[ + 2 \int_{\mathbb{R}^3} \partial_3 u_1 \partial_3 u_3 \partial_3 u_2 \partial_3 u_3 \, dx + 2 \int_{\mathbb{R}^3} \partial_3 u_1 \partial_3 u_3 \partial_3 u_3 \partial_3 u_3 \, dx \tag{2.6} \]

As before, \( I_2^1 \) has the estimate
\[ |I_2^1| \leq C \int_{\mathbb{R}^3} |\partial_3 u_2| |\nabla u| |\nabla^2 u| \, dx + \int_{\mathbb{R}^3} |u_3| |\nabla u| |\nabla^2 u| \, dx \]
\[ + C \int_{\mathbb{R}^3} |\partial_3 u_2| |\nabla^2 u| |u| \, dx + C \int_{\mathbb{R}^3} |\partial_3 u_3| |\nabla^2 u| |u| \, dx. \tag{2.7} \]
Therefore, by above inequalities, we see that
\[
\frac{1}{2} \frac{d}{dt} \| \nabla u \|_2^2 + \nu \| \Delta u \|_2^2 \leq C \int_{\mathbb{R}^3} |\partial_3 u_2| |\nabla u|^2 \, dx + C \int_{\mathbb{R}^3} |u_3| |\nabla u| |\nabla^2 u| \, dx \\
+ C \int_{\mathbb{R}^3} |\partial_3 u_2| |\nabla^2 u| |u| \, dx + C \int_{\mathbb{R}^3} |\partial_3 u_3| |\nabla^2 u| |u| \, dx.
\]  
(2.8)

Let \( T' \in (0, T) \) with \( T' \leq T_* \) be arbitrary. We shall prove that \( T' \) is not a blow-up point. By decreasing of \( \alpha_i, i = 1, 2 \), if necessary, we may assume
\[
u u_3 \in L^{\beta_1}(0, T; L^{\alpha_1}(\mathbb{R}^3)) \text{ with } \frac{3}{\alpha_1} + \frac{2}{\beta_1} = 1, \alpha_1 \in (3, \infty],
\]  
(2.9)
and
\[
\partial_3 u_i \in L^{\beta_2}(0, T; L^{\alpha_2}(\mathbb{R}^3)) \text{ with } \frac{3}{\alpha_2} + \frac{2}{\beta_2} = 2, \alpha_2 \in [2, 3], i = 2, 3.
\]  
(2.10)

Choose a \( t_1 \in (0, T') \) such that
\[
\| \partial_3 u_i \|_{L^{\beta_2}(t_1, T'; L^{\alpha_2}(\mathbb{R}^3))} \leq \epsilon, \ i = 2, 3,
\]  
(2.11)
and
\[
\| u_3 \|_{L^{\beta_1}(t_1, T'; L^{\alpha_1}(\mathbb{R}^3))} \leq \epsilon,
\]  
(2.12)
where \( \epsilon \) is to be determined. Let \( t_2 \in (t_1, T') \) be arbitrary. For any \( r, s \), we abbreviate
\[
\| \cdot \|_{L^r_t L^s_x} = \| \cdot \|_{L^r((t_1, t_2); L^s(\mathbb{R}^3))}.
\]  
(2.13)

We denote
\[
L = \| \nabla u \|_{L^\infty_t L^2_x} + \nu \| \Delta u \|_{L^2_t L^2_x}.
\]  
(2.14)

Let \( t \in (t_1, t_2] \) be arbitrary. Integrating (2.8) on \( (t_1, t) \), we get
\[
\| \nabla u \|_{L^2_t}^2 + 2\nu \int_{t_1}^t \| \Delta u \|_{L^2}^2 \, d\tau \\
\leq C \int_{t_1}^t \int_{\mathbb{R}^3} |\partial_3 u_2| |\nabla u|^2 \, dx \, d\tau + C \int_{t_1}^t \int_{\mathbb{R}^3} |u_3| |\nabla u| |\nabla^2 u| \, dx \, d\tau \\
+ C \int_{t_1}^t \int_{\mathbb{R}^3} |\partial_3 u_2| |\nabla^2 u| |u| \, dx \, d\tau + C \int_{t_1}^t \int_{\mathbb{R}^3} |\partial_3 u_3| |\nabla^2 u| |u| \, dx \, d\tau \\
+ \| \nabla u(t_1) \|_{L^2}^2 \\
= L_1 + L_2 + L_3 + L_4 + L_5.
\]  
(2.15)

We estimate \( L_i, i = 1, 2, 3, 4 \) one by one. Firstly, for \( L_1 \) we have
\[
L_1 \leq C \| \partial_3 u_2 \|_{L^2_t L^2_x} \| \nabla u \|_{L^2_t L^2_x}^2.
\]  
(2.16)
where \( s_1 \) and \( r_1 \) satisfy
\[
\frac{1}{\alpha_2} + \frac{2}{r_1} = 1, \quad \frac{1}{\beta_2} + \frac{2}{s_1} = 1,
\]  
(2.17)
by (2.10) and (2.17), we have \( 3 \leq r_1 \leq 4 \) and \( 2/s_1 + 3/r_1 = 3/2 \). By Lemma 2.1, we have
\[
L_1 \leq C \epsilon L^2.
\]  
(2.18)
As for $L_2$, we have
\[
L_2 \leq C \|u_3\|_{L_t^{\beta_1}L_x^{\alpha_1}} \|\Delta u\|_{L_t^{\alpha_2}L_x^{\beta_2}} \|\nabla u\|_{L_t^{\alpha_2}L_x^{\beta_2}}
\leq C \epsilon L^2,
\]
(2.19)
where $s_2$ and $r_2$ satisfy
\[
\frac{1}{\alpha_1} + \frac{1}{r_2} = \frac{1}{2}, \quad \frac{1}{\beta_1} + \frac{1}{s_2} = \frac{1}{2},
\]
(2.20)
by (2.9) and (2.20), we have $2 \leq r_2 < 6$ and $2/s_2 + 3/r_2 = 3/2$. Thus, $s_2$ and $r_2$ satisfy Lemma 2.1. The estimate of $L_3$ is as follows
\[
L_3 \leq C \|\partial_3 u_2\|_{L_t^{\beta_2}L_x^{s_2}} \|\Delta u\|_{L_t^{\alpha_2}L_x^{\beta_2}} \|u\|_{L_t^{\alpha_3}L_x^{\beta_3}}
\]
(2.21)
where $s_3$ and $r_3$ satisfy
\[
\frac{1}{\alpha_2} + \frac{1}{r_3} = \frac{1}{2}, \quad \frac{1}{\beta_2} + \frac{1}{s_3} = \frac{1}{2},
\]
(2.22)
by (2.11) and (2.22), we have
\[
\frac{2}{s_3} + \frac{3}{r_3} = \frac{1}{2}, \quad r_3 \geq 6.
\]
(2.23)
By the Gagliardo-Nirenberg inequality
\[
\|v\|_{L^r} \leq \|\nabla v\|_{L^{\frac{3r}{2}}}, \quad \frac{3}{2} \leq r < \infty.
\]
(2.24)
Therefore, we have
\[
\|u\|_{L_t^{\alpha_3}L_x^{r_3}} \leq \|\nabla u\|_{L_t^{\alpha_3}L_x^{r_4}},
\]
(2.25)
where $r_4 = 3r_3/(3 + r_3)$, and we have $2/s_3 + 3/r_4 = 3/2$ with $2 \leq r_4 \leq 6$. Combining (2.10), (2.21) and (2.25), as well as Lemma 2.1, one has
\[
L_3 \leq C \epsilon L^2.
\]
(2.26)
The term $L_4$ is estimated the same way and we get the same result. Finally, we obtain
\[
L^2 \leq C \epsilon L^2 + \|\nabla u(t_1)\|^2_{L^2}.
\]
If $t_1$ is sufficiently close to $T'$ and $\epsilon$ is sufficiently small, we can absorb the first term into the left hand side, and then we obtain that $L$ is bounded with a bound independent of $t_2 \in (t_1, T')$. Finally, we get $\|\nabla u\|_{L_t^\infty L_x^2(\mathbb{R}^3 \times (t_1, T'))} \leq C$. Therefore, the solution cannot blow up at $T'$. We complete the proof of the case of $\partial_3 u_2, \partial_3 u_3$.

*Case of $\partial_3 u_1, \partial_3 u_3$*

This case is similar to the first case. The main difference is that we have another estimate for $L_2$. Also by using the incompressibility condition, and integrating by parts a few times, we note that $L_2$ becomes
\[
- \int_{\mathbb{R}^3} \partial_3 u_2 \partial_3 u_2 \partial_2 u_2 dx = 2 \int_{\mathbb{R}^3} \partial_{23} u_2 \partial_3 u_2 u_2 dx
\]
\[
= -2 \int_{\mathbb{R}^3} \partial_{13} u_1 \partial_3 u_2 u_2 dx - 2 \int_{\mathbb{R}^3} \partial_{33} u_3 \partial_3 u_2 u_2 dx
\]
\[
= 2 \int_{\mathbb{R}^3} \partial_{13} u_2 \partial_3 u_1 u_2 dx + 2 \int_{\mathbb{R}^3} \partial_3 u_2 \partial_3 u_1 \partial_1 u_2 dx
\]
\[
+ 2 \int_{\mathbb{R}^3} \partial_3 u_3 \partial_3 u_1 u_2 dx - 2 \int_{\mathbb{R}^3} \partial_3 u_3 \partial_3 u_2 \partial_3 u_2 dx.
\]
(2.27)
and then $I_2^5$ has the estimate
\[
|I_2^5| \leq C \int_{\mathbb{R}^3} |\partial_3 u_1||\nabla u|^2 dx + C \int_{\mathbb{R}^3} |u_3||\nabla u||\nabla^2 u| dx \\
+ C \int_{\mathbb{R}^3} |\partial_3 u_1||\nabla^2 u||u| dx + C \int_{\mathbb{R}^3} |\partial_3 u_3||\nabla^2 u||u| dx. 
\] (2.28)

Therefore, we finally get
\[
\frac{1}{2} \frac{d}{dt} \|\nabla u\|^2_2 + \nu \|\Delta u\|^2_2 \leq C \int_{\mathbb{R}^3} |\partial_3 u_1||\nabla u|^2 dx + C \int_{\mathbb{R}^3} |u_3||\nabla u||\nabla^2 u| dx \\
+ C \int_{\mathbb{R}^3} |\partial_3 u_1||\nabla^2 u||u| dx + C \int_{\mathbb{R}^3} |\partial_3 u_3||\nabla^2 u||u| dx. 
\] (2.29)

By using (2.29), we give the same method as before to get the desired result. The proof is completed.

**Proof of Theorem 1.2** From the condition of this Theorem, we split the proof into two parts.

- $i \neq j$

Firstly, we consider the case of $q = 2$, and then we see that the range of $\alpha_2$ is $\alpha_2 > 2$. For convenience of writing, we set
\[
r = \frac{3\alpha_2 - 2}{\alpha_2}. 
\] (2.30)

It is easy to check that $r > 2$ when $\alpha_2 > 2$. Without loss of generality, in the proof, we will assume that $i = 1, j = 2$, the other cases can be discussed in the same way (for details see Remark 2.2 below). We begin with (2.30), and the same process to the proof of Theorem 1.1, we firstly have
\[
|I_i| \leq C \int_{\mathbb{R}^3} |u_3||\nabla u||\nabla^2 u| dx, \ i = 1, 3, 4.
\]

As for $I_2$, also by the incompressibility condition, we have
\[
I_2 = - \int_{\mathbb{R}^3} \partial_3 u_1 \partial_1 u_1 \partial_3 u_1 + \partial_3 u_2 \partial_2 u_1 \partial_3 u_1 + \partial_3 u_3 \partial_3 u_1 \partial_3 u_1 dx \\
- \int_{\mathbb{R}^3} \partial_3 u_1 \partial_1 u_2 \partial_3 u_2 + \partial_3 u_2 \partial_2 u_2 \partial_3 u_2 + \partial_3 u_3 \partial_3 u_2 \partial_3 u_2 dx \\
= - \int_{\mathbb{R}^3} \partial_3 u_1 (-\partial_2 u_2 - \partial_3 u_3) \partial_3 u_1 + \partial_3 u_2 \partial_2 u_1 \partial_3 u_1 + \partial_3 u_3 \partial_3 u_1 \partial_3 u_1 dx \\
- \int_{\mathbb{R}^3} \partial_3 u_1 \partial_1 u_2 \partial_3 u_2 + \partial_3 u_2 \partial_2 u_2 \partial_3 u_2 + \partial_3 u_3 \partial_3 u_2 \partial_3 u_2 dx \\
\leq \int_{\mathbb{R}^3} |u_2||\nabla u||\nabla^2 u| dx + \int_{\mathbb{R}^3} |u_3||\nabla u||\nabla^2 u| dx.
\]

Therefore, we get
\[
\frac{1}{2} \frac{d}{dt} \|\nabla u\|^2_2 + \nu \|\Delta u\|^2_2 \\
\leq C \int_{\mathbb{R}^3} |u_3||\nabla u||\nabla^2 u| dx + C \int_{\mathbb{R}^3} |u_2||\nabla u||\nabla^2 u| dx, 
\] (2.31)

\[= K_1(t) + K_2(t). \]
Next, we estimate $K_1(t)$ and $K_2(t)$. Firstly, we pay attention to $K_2(t)$, applying Hölder’s inequality several times, we obtain

$$K_2(t) \leq C \int_{\mathbb{R}^2} \max_{x_1} |u_2| \left( \int_{\mathbb{R}} |\nabla u|^2 dx_1 \right)^{\frac{r}{2}} \left( \int_{\mathbb{R}} |\nabla^2 u|^2 dx_1 \right)^{\frac{r}{4}} dx_2 dx_3$$

$$\leq C \left[ \int_{\mathbb{R}^2} (\max_{x_1} |u_2|)^r dx_2 dx_3 \right]^{\frac{1}{r}} \left[ \int_{\mathbb{R}} (\int_{\mathbb{R}} |\nabla u|^2 dx_1)^{\frac{r-2}{r}} dx_2 dx_3 \right]^{\frac{r}{r-2}}$$

$$\times \left[ \int_{\mathbb{R}^2} |\nabla^2 u|^2 dx_1 dx_2 dx_3 \right]^{\frac{1}{2}}$$

$$\leq C \left[ \int_{\mathbb{R}^3} |u_2|^{r-1} |\partial_1 u_2| |dx_1 dx_2 dx_3| \right]^{\frac{1}{2}} \|\Delta u\|_2$$

$$\times \left[ \int_{\mathbb{R}} (\int_{\mathbb{R}} |\nabla u|^{\frac{2r}{r-2}} dx_2 dx_3)^{\frac{r-2}{r}} dx_1 \right]^{\frac{1}{2}}$$

$$\leq C \left[ \int_{\mathbb{R}^2} |u_2| \|\partial_1 u_2\|_{L^\infty} \|\nabla u\|_{L^2} \|\partial_2 \nabla u\|_{L^2} \|\partial_3 \nabla u\|_{L^2} \|\Delta u\|_2 \right]$$

$$\leq C \left[ \int_{\mathbb{R}^2} |u_2| \|\partial_1 u_2\|_{L^\infty} \|\nabla u\|_{L^2} \|\Delta u\|_{L^2} \right].$$

In above inequality, from (2.30), we note that $\frac{2}{3-\gamma} = \alpha_2$. Therefore, applying Young’s inequality, (2.32) immediately implies

$$K_2(t) \leq C \left[ \int_{\mathbb{R}^2} |u_2|^{\frac{2(\alpha_2-1)}{3-\gamma-2}} \|\partial_1 u_2\|_{L^2}^{\frac{\alpha_2}{3-\gamma-2}} \|\nabla u\|_{L^2}^{\frac{\alpha_2-2}{3-\gamma-2}} \|\Delta u\|_{L^2}^{\frac{3(\alpha_2-2)}{3-\gamma-2}} \right]$$

$$\leq C \left[ \int_{\mathbb{R}^2} |u_2|^{\frac{4(\alpha_2-1)}{2q-6}} \|\partial_1 u_2\|_{L^2}^{\frac{2\alpha_2}{2q-6}} \|\nabla u\|_{L^2}^{\frac{2\alpha_2-2}{2q-6}} \|\Delta u\|_{L^2}^{\frac{2q-6}{2q-6}} \right]$$

$$\leq C \left[ \int_{\mathbb{R}^2} |u_2|^{\frac{2q-6}{2q-6-2q}} \|\nabla u\|_{L^2}^{\frac{2q-6}{2q-6}} \|\Delta u\|_{L^2}^{\frac{2q-6}{2q-6}} \right].$$

As for $K_1(t)$, applying Hölder’s and Young’s inequalities, we have

$$K_1(t) = C \int_{\mathbb{R}^3} \|u_3\| \|\nabla u\| \|\nabla^2 u\| dx$$

$$\leq C \left[ \int_{\mathbb{R}^3} \|u_3\|_{L^{\gamma}} \|\nabla u\|_{L^\gamma} \|\Delta u\|_{L^2} \right]$$

$$\leq C \left[ \int_{\mathbb{R}^3} \|u_3\|_{L^{\gamma}} \|\nabla u\|_{L^2} \|\Delta u\|_{L^2} \right]$$

$$\leq C \left[ \int_{\mathbb{R}^3} \|u_3\|_{L^{\frac{\gamma}{\alpha_1}}} \|\nabla u\|_{L^2} \|\Delta u\|_{L^2} \right].$$

where $\alpha_1$ and $q$ satisfy

$$\frac{1}{q} + \frac{1}{\alpha_1} = \frac{1}{2} \text{ with } 2 \leq q < 6, \alpha_1 > 3.$$ 

(2.35)

From (2.34) and (2.33), one has

$$\frac{1}{2} \frac{d}{dt} \|\nabla u\|_{L^2}^2 + \nu \|\Delta u\|_{L^2}^2 \leq C \left[ \int_{\mathbb{R}^3} \|u_2\|^{\frac{2(\alpha_2-1)}{3-\gamma-2}} \|\partial_1 u_2\|_{L^2}^{\frac{2\alpha_2}{3-\gamma-2}} \|\nabla u\|_{L^2}^{\frac{2\alpha_2-2}{3-\gamma-2}} \right]$$

$$+ C \left[ \int_{\mathbb{R}^3} \|u_3\|_{L^{\frac{\gamma}{\alpha_1}}} \|\nabla u\|_{L^2} \|\Delta u\|_{L^2} \right].$$

(2.36)
Absorbing the last term in right hand of (2.36) and integrating the inequality on time, using the energy inequality, we obtain

\[
\|\nabla u\|^2_{L^2} + \nu \int_0^t \|\Delta u\|^2_{L^2} d\tau \leq C \int_0^t \|u_2\|^2_{L^2} \|\partial_t u_2\|^{\frac{2\alpha}{\alpha - 2}}_{L^{\alpha - 2}} \|\nabla u\|^2_{L^2} d\tau \\
+ C \int_0^t \|u_3\|^2_{L^{\frac{9}{2}}} \|\nabla u\|^2_{L^2} d\tau + \|\nabla u(0)\|^2_{L^2}
\]

(2.37)

By using Gronwall’s inequality, we obtain

\[
\|\nabla u\|^2_{L^2} + \nu \int_0^t \|\Delta u\|^2_{L^2} d\tau \leq (\|\nabla u(0)\|^2_{L^2}) \exp \left( C \int_0^t \|\partial_t u_2\|^{\frac{2\alpha}{\alpha - 2}}_{L^{\alpha - 2}} d\tau \right) \exp \left( C \int_0^t \|u_3\|^2_{L^{\frac{9}{2}}} d\tau \right)
\]

(2.38)

By condition (1.10) – (1.12), (2.38) follows that the \(H^1\) norm of the strong solution \(u\) is bounded on the maximal interval of existence \((0, T^*)\).

Now, we pay attention to the case of \(1 < q < 2\). Next, we give an estimate on \(u_2\). We use \(|u_2|^{q-1}\text{sgn}(u_2)\) with \(1 < q \leq 3/2\) as test function in the equation (1.1) for \(u_2\). By using Gagliardo-Nirenberg and Hölder’s inequalities Hölder’s inequalities and (1.37), we have

\[
\frac{1}{q} \frac{d}{dt} \|u_2\|^q_{L^q} + C(q)\nu \|\nabla |u_2|^2\|^2_{L^2} \\
= - \int_{\mathbb{R}^3} \partial_2 p |u_2|^{q-1}\text{sgn}(u_2) dx \\
\leq C \|\nabla p\|_{L^q} \|u_2\|^{q-1}_{L^q} \\
\leq C \|\nabla u\|_{L^2} \|u_2\|^{q-1}_{L^2} \|u_2\|^{1-q}_{L^q} \\
\leq C \|\nabla u\|_{L^2} \|u_2\|^{\frac{3-2q}{2}}_{L^2} \|\nabla u\|^{\frac{2q-3}{2}}_{L^2} \|u_2\|^{q-1}_{L^q} \\
= C \|u_2\|^{\frac{3-2q}{2}}_{L^2} \|\nabla u\|^{\frac{4q-3}{q}}_{L^2},
\]

where we note that \(1 < q \leq 3/2\) means \(2 < \frac{2q}{2-q} \leq 6\), and (2.39) immediately implies that

\[
\frac{d}{dt} \|u_2\|_{L^q} \leq C \|u_2\|^{\frac{3-2q}{2}}_{L^2} \|\nabla u\|^{\frac{4q-3}{q}}_{L^2}.
\]
After integrating on time, and note that $u_0 \in V \cap L^q(\mathbb{R}^3)$, by energy and Hölder’s inequalities one has

$$\|u_2\|_{L^q} \leq \|u_2(0)\|_{L^q} + C \int_0^t \|\nabla u\|_{L^2} \|\nabla u\|_{L^2}^{3-2q} \, d\tau$$

$$\leq \|u_2(0)\|_{L^q} + C \int_0^t \|\nabla u\|_{L^2}^{4q-3} \, d\tau$$

$$\leq \|u_2(0)\|_{L^q} + C \left( \int_0^t \|\nabla u\|_{L^2}^2 \, d\tau \right)^{\frac{4q-3}{2q}} T^{\frac{3-2q}{2q}}$$

(2.40)

Therefore, we get

$$u_2 \in L^\infty(0,T; L^q(\mathbb{R}^3)) \text{ with } 1 < q \leq \frac{3}{2}. \quad (2.41)$$

On the other hand, by energy inequality we know that

$$u_2 \in L^\infty(0,T; L^2(\mathbb{R}^3)). \quad (2.42)$$

Finally, by interpolation, we have

$$u_2 \in L^\infty(0,T; L^{q}(\mathbb{R}^3)) \text{ with } 1 < q < 2. \quad (2.43)$$

For every $\alpha_2 > \frac{q}{q-1}$, we set

$$r = \frac{(q+1)\alpha_2 - q}{\alpha_2},$$

then we have $r > 2$. Similar to (2.32), also by Hölder’s and Young’s inequalities, we obtain another estimate

$$K_2(t) \leq C \|u_2\|_{L^{q}}^{\frac{r-1}{2q}} \|\partial_1 u_2\|_{L^{\frac{q(q-1)}{q+1}}} \|\nabla u\|_{L^2}^{\frac{q-2}{2}} \|\Delta u\|_{L^2}^{\frac{r+2}{2}}$$

$$\leq C \|u_2\|_{L^{q}}^{\frac{2(r-1)}{r-2}} \|\partial_1 u_2\|_{L^{\frac{q}{q+1}}} \|\nabla u\|_{L^2}^2 + \frac{\nu}{4} \|\Delta u\|_{L^2}^2. \quad (2.44)$$

Applying (2.43) and integrating above inequality, one has

$$\int_0^t K_2(\tau) \, d\tau \leq C \int_0^t \|u_2\|_{L^{q}}^{\frac{2(r-1)}{r-2}} \|\partial_1 u_2\|_{L^{\frac{q}{q+1}}} \|\nabla u\|_{L^2}^2 \, d\tau + \frac{\nu}{4} \int_0^t \|\Delta u\|_{L^2}^2 \, d\tau$$

$$\leq C \int_0^t \|\partial_1 u_2\|_{L^{\frac{q}{q+1}}} \|\nabla u\|_{L^2}^2 \, d\tau + \frac{\nu}{4} \int_0^t \|\Delta u\|_{L^2}^2 \, d\tau. \quad (2.45)$$

Integrating (2.31) on time, and absorbing the last term in (2.45) and (2.31) respectively, it follows that

$$\|\nabla u\|_{L^2}^2 + \nu \int_0^t \|\Delta u\|_{L^2}^2 \, d\tau$$

$$\leq C \int_0^t \|\partial_1 u_2\|_{L^{\frac{q}{q+1}}} \|\nabla u\|_{L^2}^2 \, d\tau + C \int_0^t \|u_3\|_{L^{\frac{2q}{q+1}}} \|\nabla u\|_{L^2}^2 \, d\tau + \|\nabla u(0)\|_{L^2}^2. \quad (2.46)$$

By using Gronwall’s inequality and condition (1.10)–(1.12), we also obtain $H^1$ norm of the strong solution $u$ is bounded on the maximal interval of existence $(0,T^*)$ when
1 < q < 2. Thus we prove (a).

Without loss of generality, here, we assume i = j = 2. Similar to the proof of the part (a), we estimate the second term $K_2(t)$ of (2.31). Firstly, for every

$$\frac{9}{5} < \alpha_3 \leq \infty,$$

we choose

$$\begin{cases}
\gamma = \frac{3 + \sqrt{24\alpha_3^2 - 24\alpha_3 + 9}}{12\alpha_3}, \\
\mu = \frac{\alpha_3\gamma}{2\alpha_3 - \gamma}.
\end{cases}$$

From (2.48) and (2.47), we have $\frac{1}{\gamma} > \frac{1}{2\alpha_3}$, which means that $\gamma < 2\alpha_3$, and hence $\mu > 0$ is well defined in (2.48). Also from (2.48), we see that $\gamma$ is an increasing function with the variable $\alpha_3 \in (9/5, \infty]$, and from (2.47) we get

$$2 < \frac{9}{4} < \gamma \leq \sqrt{6},$$

and moreover, (2.48) follows

$$\frac{1}{\mu} + \frac{1}{\alpha_3} + \frac{\gamma - 2}{\gamma} = 1.$$  

Besides, again by (2.48), we see that

$$\mu = \frac{6\alpha_3}{-3 + \sqrt{24\alpha_3^2 - 24\alpha_3 + 9}},$$

by the monotonicity, we obtain

$$1 < \frac{\sqrt{6}}{2} \leq \mu < 3.$$  

We choose

$$\beta_3 = \frac{2\mu}{3 - \mu},$$

then we have

$$\frac{1}{\mu} = \frac{2}{3} \left( \frac{1}{\beta_3} + \frac{1}{2} \right),$$

by (2.48), (2.50) and (2.53), we can compute that

$$\frac{2}{\beta_3} + 1 = \frac{3}{\mu} = \frac{6}{\gamma} - \frac{3}{\alpha_3} = \frac{-3 + \sqrt{24\alpha_3^2 - 24\alpha_3 + 9}}{2\alpha_3} \Rightarrow \frac{3}{2\alpha_3} + \frac{2}{\beta_3} = f(\alpha_3).$$
Now, we use $|u_2|^{\gamma-2}u_2$ with $\gamma > 2$ as test function in the equation (1.1) for $u_2$. By using of Gagliardo-Nirenberg and Hölder’s inequalities, we have
\[
\frac{1}{\gamma} \frac{d}{dt} \|u_2\|_{L^\gamma}^\gamma + C(\gamma) \nu \|\nabla |u_2|^{\frac{\gamma}{2}}\|_{L^2}^2 = - \int_{\mathbb{R}^3} \partial_2 p |u_2|^{\gamma-2} u_2 \, dx
\]
\[
\leq C \int_{\mathbb{R}^3} |p| |u_2|^{\gamma-2} |\partial_2 u_2| \, dx
\]
\[
\leq C \|p\|_{L^\nu} \|u_2\|_{L^{\gamma-2}}^{\gamma-2} \|\partial_2 u_2\|_{L^{\alpha_3}}
\]
\[
\leq C \|u\|_{L^{2\mu}} \|u_2\|_{L^{\gamma-2}} \|\partial_2 u_2\|_{L^{\alpha_3}} \quad \text{(by (2.58))}
\]
\[
\leq C \|u\|_{L^2}^{\frac{3-\mu}{\mu}} \|\nabla u\|_{L^2}^{\frac{3(\mu-1)}{\mu}} \|u_2\|_{L^{\gamma-2}}^{\gamma-2} \|\partial_2 u_2\|_{L^{\alpha_3}},
\]
where the $\gamma, \mu$ and $\alpha_3$ satisfy (2.50). The above inequality immediately implies that
\[
\frac{1}{2} \frac{d}{dt} \|u_2\|_{L^\gamma}^2 \leq C \|u\|_{L^{2\mu}}^{\frac{3-\mu}{\mu}} \|\nabla u\|_{L^2}^{\frac{3(\mu-1)}{\mu}} \|\partial_2 u_2\|_{L^{\alpha_3}}.
\]
In view of (2.52), we have $\frac{3(\mu-1)}{\mu} < 2$, applying Young’s inequality, we have
\[
\frac{1}{2} \frac{d}{dt} \|u_2\|_{L^\gamma}^2 \leq C \|\nabla u\|_{L^2}^{3} + \|u_2\|_{L^2}^{2} \|\partial_2 u_2\|_{L^{\alpha_3}}.
\]
Integrating (2.57) on time, and by energy inequality (2.1), we obtain
\[
\|u_2\|_{L^\gamma}^2 \leq \|u_2(0)\|_{L^\gamma}^2 + C \int_0^t \|\partial_2 u_2\|_{L^{3(\mu-1)/\mu}}^{\frac{3-\mu}{\mu}} \, dt
\]
\[
= \|u_2(0)\|_{L^\gamma}^2 + C \int_0^t \|\partial_2 u_2\|_{L^{\alpha_3}}^{\alpha_3} \, dt.
\]
In view of (2.49), we have $\|u_2(0)\|_{L^\gamma} < C$ for some $C > 0$, therefore, by the condition (1.14), we get
\[
u 2 \in L^\infty \left(0; T; L^7(\mathbb{R}^3) \right).
\]
For the mentioned parameter $\gamma$ in (2.48), we set
\[
r = \frac{(\gamma + 1) \alpha_3 - \gamma}{\alpha_3}.
\]
then
\[
r = \frac{12 \alpha_3 - 9 + \sqrt{24 \alpha_3^2 - 24 \alpha_3 + 9}}{3 + \sqrt{24 \alpha_3^2 - 24 \alpha_3 + 9}},
\]
since $9/5 < \alpha_3 \leq \infty$, also by monotonicity, it is easy to see $2 < r \leq \sqrt{6} + 1$. Similar to (2.32), also by Hölder’s and Young’s inequalities, we obtain another estimate
\[
K_2(t) \leq C \|u_2\|_{L^\gamma}^{\frac{\gamma+1}{\gamma}} \|\partial_2 u_2\|_{L^{\gamma+1}} \|\nabla u\|_{L^2}^{\gamma} \|\Delta u\|_{L^2}^{\gamma}
\]
\[
\leq C \|u_2\|_{L^\gamma}^{\frac{2(\gamma-1)}{\gamma}} \|\partial_2 u_2\|_{L^{\alpha_3}}^{\gamma} \|\nabla u\|_{L^2}^{\gamma} + \frac{\nu}{4} \|\Delta u\|_{L^2}^{\gamma}.
\]
Applying (2.59) and integrating above inequality, one has
\[
\int_0^t K_2(\tau) d\tau \leq C \int_0^t ||u_2||_{L^r}^{2(c-1)} \frac{2^{c-1}}{L^{\alpha_2}} ||\partial_2 u_2||_{L^{\alpha_3}}^{2} ||\nabla u||_{L^2}^{2} d\tau + \frac{\nu}{4} \int_0^t ||\Delta u||_{L^2}^{2} d\tau.
\]
(2.62)
Integrating (2.31) on time, and absorbing the last term in (2.62) and (2.34) respectively, it follows that
\[
||\nabla u||_{L^2}^{2} + \nu \int_0^t ||\Delta u||_{L^2}^{2} d\tau \leq C \int_0^t ||\partial_2 u_2||_{L^{\alpha_3}}^{2} ||\nabla u||_{L^2}^{2} d\tau + C \int_0^t ||u_3||_{L^{\alpha_1}}^{2} ||\nabla u||_{L^2}^{2} d\tau + ||\nabla u(0)||_{L^2}^{2}.
\]
(2.63)
We claim \(\frac{2}{r-2} = \beta_3\). In fact, from (2.48), we have
\[
\gamma = \frac{-3 + \sqrt{24\alpha_3^2 - 24\alpha_3 + 9}}{2(\alpha_3 - 1)},
\]
and then by the definition (2.60), one has
\[
\frac{2}{r-2} + 1 = r - 1 = \gamma - \frac{\gamma}{\alpha_3} = \frac{-3 + \sqrt{24\alpha_3^2 - 24\alpha_3 + 9}}{2\alpha_3}.
\]
(2.64)
Comparing (2.64) with (2.54), we prove the claim. Therefore, we can apply Gronwall’s inequality to (2.63), and by condition (1.10), (1.13) and (1.14) to get that the \(H^1\) norm of the strong solution \(u\) is bounded on the maximal interval of existence \((0, T^*)\). The proof of this Theorem is completed.

**Remark 2.2.** In the process of the proof, if we want to prove the case of \(i = 3\) when \(j = 2\). For the first part, the inequality (2.32) may be replaced by
\[
K_2(t) \leq C ||u_2||_{L^2}^{\frac{r+1}{r-2}} ||\partial_3 u_2||_{L^{\frac{r+2}{r-2}}} ||\nabla u||_{L^2}^{\frac{r-1}{r-2}} ||\Delta u||_{L^2}^{\frac{r+2}{r-2}},
\]
(2.65)
and (2.33) becomes
\[
K_2(t) \leq C ||u_2||_{L^2}^{\frac{4(\alpha_3 - 1)}{\alpha_3 - 2}} ||\partial_3 u_2||_{L^{\alpha_3}}^{\frac{2\alpha_3}{\alpha_3 - 2}} ||\nabla u||_{L^2}^{2} + \frac{\nu}{4} ||\Delta u||_{L^2}^{2}.
\]
(2.66)
If we want to prove the case of \(j = 1\), we shall give an alternative proof of the term \(I_2\), also by the incompressibility condition, one has
\[
I_2 = - \int_{R^3} \partial_3 u_1 \partial_1 u_1 \partial_3 u_1 + \partial_3 u_2 \partial_2 u_1 \partial_3 u_1 + \partial_3 u_3 \partial_3 u_1 \partial_3 u_1 dx
\]
\[
- \int_{R^3} \partial_3 u_1 \partial_1 u_2 \partial_3 u_2 + \partial_3 u_2 \partial_2 u_2 \partial_3 u_2 + \partial_3 u_3 \partial_3 u_2 \partial_3 u_2 dx
\]
\[
= - \int_{R^3} \partial_3 u_1 \partial_1 u_1 \partial_3 u_1 + \partial_3 u_2 \partial_2 u_1 \partial_3 u_1 + \partial_3 u_3 \partial_3 u_1 \partial_3 u_1 dx
\]
\[
- \int_{R^3} \partial_3 u_1 \partial_1 u_2 \partial_3 u_2 + \partial_3 u_2 (-\partial_1 u_1 - \partial_3 u_3) \partial_3 u_2 + \partial_3 u_3 \partial_3 u_2 \partial_3 u_2 dx
\]
\[
\leq \int_{R^3} |u_3||\nabla u| |\nabla^2 u| dx + \int_{R^3} |u_1||\nabla u| |\nabla^2 u| dx.
\]
and then we obtain
\[
\frac{1}{2} \frac{d}{dt} \| \nabla u \|_{L^2}^2 + \nu \| \Delta u \|_{L^2}^2 \\
\leq C \int_{\mathbb{R}^3} |u_3| \| \nabla u \| \| \nabla^2 u \| dx + C \int_{\mathbb{R}^3} |u_1| \| \nabla u \| \| \nabla^2 u \| dx,
\]
\[
= K_1(t) + K_2(t),
\]
by which one can prove the case of \( i \neq j = 1 \) and \( i = j = 1 \). The term \( K_1(t) \) is the same to (2.31). As for the second term \( K_2(t) \), we shall give the inequality (2.32) as the following for \( i \neq j = 1 \),
\[
K_2(t) \leq C \| u_1 \|_{L^2}^2 \| \partial_t u_1 \|_{L^2} \| \nabla u \|_{L^2} \| \Delta u \|_{L^2}, \quad i = 2 \text{ or } 3,
\]
and then get the corresponding form of (2.33). As for \( i = j = 1 \), we will use \( |u_1|^{\gamma-2} u_1 \), as test function in the equation for \( u_1 \), and we can get the similar results to (2.55). By the same process to prove the case of \( i = j = 1 \).

**Proof of Theorem 1.5** Give the same process as in the Theorem 1.1, we have
\[
\frac{1}{2} \frac{d}{dt} \| \nabla u \|_{L^2}^2 + \nu \| \Delta u \|_{L^2}^2 \leq C \int_{\mathbb{R}^3} |\partial_3 u_2| \| \nabla u \|^2 dx + C \int_{\mathbb{R}^3} |u_3| \| \nabla u \| \| \nabla^2 u \| dx
\]
\[
+ C \int_{\mathbb{R}^3} |\partial_3 u_2| \| \Delta u \| \| u \| dx + C \int_{\mathbb{R}^3} |\partial_3 u_3| \| \Delta u \| \| u \| dx
\]
\[
= L_1 + L_2 + L_3 + L_4.
\]
Here, we only prove the case of \( \partial_3 u_2, \partial_3 u_3 \). For the case of \( \partial_3 u_1, \partial_3 u_3 \), we will begin with (2.29), and from which we can give the similar proof.

We estimate \( L_i, \ i = 1, 2, 3, 4 \) one by one. Firstly, for \( L_1 \) we have
\[
L_1 \leq C \| \partial_3 u_2 \|_{L^{\alpha_1}} \| \nabla u \|_{L^{r_1}},
\]
where \( r_1 \) satisfies
\[
\frac{1}{\alpha_1} + \frac{2}{r_1} = 1 \Rightarrow r_1 = \frac{2\alpha_1}{\alpha_1 - 1},
\]
since \( 2 \leq \alpha_1 \leq 3 \), we have \( 3 \leq r_1 \leq 4 \), by Gagliardo-Nirenberg and Young’s inequalities, on has
\[
L_1 \leq C \| \partial_3 u_2 \|_{L^{\alpha_1}} \| \nabla u \|_{L^{r_1}} \| \Delta u \|_{L^{r_1}}^{5/6}
\]
\[
\leq C \| \partial_3 u_2 \|_{L^{\alpha_1}} \| \nabla u \|_{L^{r_1}} \| \Delta u \|_{L^{r_1}}^{5/6} + \frac{\nu}{4} \| \Delta u \|_{L^2}
\]
\[
= C \| \partial_3 u_2 \|_{L^{\alpha_1}} \| \nabla u \|_{L^{r_1}} \| \Delta u \|_{L^2}.
\]
As for \( L_3 \), let \( r_2 \) satisfy
\[
\frac{1}{\alpha_1} + \frac{1}{r_2} = \frac{1}{2},
\]
then we have
\[
L_3 \leq C \| \partial_3 u_2 \|_{L^{\alpha_1}} \| \Delta u \|_{L^2} \| u \|_{L^{r_2}}.
\]
From the fact that \( 2 \leq \alpha_1 \leq 3 \), we have \( r_2 \geq 6 \). By the Gagliardo-Nirenberg inequality,
\[
\| v \|_{L^r} \leq \| \nabla v \|_{L^{\frac{3r}{3r-2}}} \leq \frac{3}{2} \leq r < \infty.
\]
Therefore, we have
\[ \|u\|_{L^2} \leq \|\nabla u\|_{L^3}, \] (2.76)
where \( r_3 = 3r_2/(3 + r_2) \) with \( 2 \leq r_3 \leq 3 \). Thus, we have
\[ \frac{1}{\alpha_1} + \frac{1}{r_3} = \frac{5}{6} \Rightarrow r_3 = \frac{6\alpha_1}{5\alpha_1 - 6}, \] (2.77)
and applying Young’s inequality, (2.74) becomes
\[
L_3 \leq C\|\partial_3 u_2\|_{L^0}^2 \|\Delta u\|_{L^2} \|\nabla u\|_{L^3}. 
\]
\[
\leq C\|\partial_3 u_2\|_{L^0} \|\Delta u\|_{L^2} \|\nabla u\|_{L^3}^{\frac{6-r_3}{L^2}} \|\Delta u\|_{L^2}^{\frac{5r_3-6}{L^2}}. 
\]
\[
\leq C\|\partial_3 u_2\|_{L^0} \|\nabla u\|_{L^2}^{\frac{4r_3}{L^2}} + \frac{\nu}{8} \|\Delta u\|_{L^2}. 
\] (2.78)
\[
= C\|\partial_3 u_2\|_{L^0} \|\nabla u\|_{L^2}^{\frac{2\alpha_1-3}{L^2}} + \frac{\nu}{8} \|\Delta u\|_{L^2}. 
\]
The term \( L_4 \) is estimated in the same way and we get
\[
L_4 \leq C\|\partial_3 u_3\|_{L^0} \|\nabla u\|_{L^2}^2 + \frac{\nu}{8} \|\Delta u\|_{L^2}^2. 
\] (2.79)
Therefore, integrating on time and absorbing the last term in (2.72), (2.78) and (2.79), we get
\[
\|\nabla u\|_{L^2}^2 + \frac{3\nu}{2} \int_0^t \|\Delta u\|_{L^2}^2 d\tau 
\]
\[
\leq \|\nabla u(0)\|_{L^2}^2 + C \int_0^t \|\partial_3 u_2\|_{L^0}^{\frac{2\alpha_1}{L^2}} \|\nabla u\|_{L^2}^{\frac{2\alpha_1-3}{L^2}} d\tau 
\]
\[
+ C \int_0^t \|\partial_3 u_3\|_{L^0} \|\nabla u\|_{L^2} d\tau + C \int_0^t L_2(\tau)d\tau. 
\] (2.80)
As for the estimation of \( L_2 \), we give the same proof as the case of \( i = j \) in Theorem 1.2, in which \( u_2 \) is replaced by \( u_3 \), \( \partial_3 u_2 \) is replaced by \( \partial_3 u_3 \) and \( \alpha_3, \beta_3 \) is replaced by \( \alpha_2, \beta_2 \). Finally, we get
\[
\int_0^t L_2(\tau)d\tau \leq C \int_0^t \|u_3\|_{L^{2\alpha_2}}^\frac{2(\alpha_2-1)}{L^2} \|\partial_3 u_3\|_{L^2} \|\nabla u\|_{L^2} d\tau \]
\[
+ \frac{\nu}{2} \int_0^t \|\Delta u\|_{L^2}^2 d\tau \] (2.81)
where \( \alpha_2 \geq 2 \) and
\[
\begin{align*}
\frac{1}{\gamma} &= \frac{3 + \sqrt{24\alpha_2^2 - 24\alpha_2 + 9}}{12\alpha_2}, \\
\mu &= \frac{\alpha_2 \gamma}{2\alpha_2 - \gamma}, \\
r &= \frac{(\gamma + 1)\alpha_2 - \gamma}{\alpha_2}.
\end{align*}
\] (2.82)
Inserting (2.81) into (2.80) and absorbing the last term in (2.81), note the boundedness of \( \|u_3\|_\gamma \), we have

\[
\|\nabla u\|_{L^2}^2 + \nu \int_0^t \|\Delta u\|_{L^2}^2 d\tau \\
\leq \|\nabla u(0)\|_{L^2}^2 + C \int_0^t \|\partial_3 u_2\|_{L^{\frac{2}{2}}_{t, \alpha_1}} \|\nabla u\|_{L^2}^2 d\tau \\
+ C \int_0^t \|\partial_3 u_3\|_{L^{\frac{2}{2}}_{t, \alpha_2}} \|\nabla u\|_{L^2}^2 d\tau + C \int_0^t \|\partial_3 u_3\|_{L^{\frac{2}{2}}_{t, \alpha_2}} \|\nabla u\|_{L^2}^2 d\tau.
\]

We also can check that \( \frac{2}{r} - \frac{2}{\beta_2} = \beta_2 \) (see (2.64)), where \( \beta_2 = \frac{2\mu_2}{3-\mu} \). Denote that \( \frac{1}{\beta'} = 2\beta_2 - 3 \).

Also from (2.64), \( r = \frac{-3+\sqrt{24\alpha_2^2-24\alpha_2+9}}{2\alpha_2} + 1 \), we see that

\[
\frac{1}{\beta'} - \frac{1}{\beta_2} = \frac{2\alpha_2 - 3}{2\alpha_2} - \frac{r - 2}{2} \\
= \frac{(4 - r)\alpha_2 - 3}{2\alpha_2} \\
= \frac{6\alpha_2+3-\sqrt{24\alpha_2^2-24\alpha_2+9}}{2\alpha_2} - 3 \\
= \frac{6\alpha_2 - 3 - \sqrt{24\alpha_2^2 - 24\alpha_2 + 9}}{4\alpha_2} \\
= \frac{12\alpha_2(\alpha_2 - 1)}{4\alpha_2 \left(6\alpha_2 - 3 + \sqrt{24\alpha_2^2 - 24\alpha_2 + 9}\right)} > 0, \ \forall \ \alpha_2 \geq 2.
\]

From above inequality, we know \( \beta' < \beta_2 \) and hence \( \beta'/\beta_2 < 1 \). Therefore, applying Hölder’s inequality, and by energy inequality, one has

\[
\int_0^t \|\partial_3 u_3\|_{L^{\frac{2}{2}}_{t, \alpha_2}} \|\nabla u\|_{L^2}^2 d\tau = \int_0^t \|\partial_3 u_3\|_{L^{\frac{2}{2}}_{t, \alpha_2}} \|\nabla u\|_{L^2}^2 d\tau \\
= \int_0^t \|\partial_3 u_3\|_{L^{\frac{2}{2}}_{t, \alpha_2}} \|\nabla u\|_{L^2}^2 d\tau \|\nabla u\|_{L^2}^2 d\tau \\
\leq C \left( \int_0^t \|\partial_3 u_3\|_{L^{\frac{2}{2}}_{t, \alpha_2}} \|\nabla u\|_{L^2}^2 d\tau \right)^{\frac{\beta'}{\beta_2}} \left( \int_0^t \|\nabla u\|_{L^2}^2 d\tau \right)^{\frac{\beta_2 - \beta'}{\beta_2}} \\
\leq C \left( \int_0^t \|\partial_3 u_3\|_{L^{\frac{2}{2}}_{t, \alpha_2}} \|\nabla u\|_{L^2}^2 d\tau \right)^{\frac{\beta'}{\beta_2}} \\
\leq C \int_0^t \|\partial_3 u_3\|_{L^{\frac{2}{2}}_{t, \alpha_2}} \|\nabla u\|_{L^2}^2 d\tau + C.
\]
Finally, we get
\[
\|\nabla u\|_{L^2}^2 + \nu \int_0^t \|\Delta u\|_{L^2}^2 d\tau \\
\leq \|\nabla u(0)\|_{L^2}^2 + C \int_0^t \|\partial_3 u_2\|_{L^{2/\alpha_1}} \|\nabla u\|_{L^2}^2 d\tau \\
+ C \int_0^t \|\partial_3 u_3\|_{L^{2/\alpha_2}} \|\nabla u\|_{L^2}^2 d\tau + C. \tag{2.86}
\]

By using Gronwall’s inequality to (2.86), and by condition (1.18), (1.19) and (1.20), we get that the \(H^1\) norm of the strong solution \(u\) is bounded on the maximal interval of existence \((0, T^*)\). The proof of the case of \(\partial_3 u_2, \partial_3 u_3\) is completed.

**Proof of Theorem 1.8** Without loss of generality, we assume \(j = 3, k = 3\). For every \(\alpha \in \left(\frac{9}{5}, \infty\right]\), we take
\[
\begin{cases}
\mu = \frac{24\alpha}{\alpha - 12 + \sqrt{144 - 264\alpha + 289\alpha^2}}, \\
q_2 = \frac{2\alpha \mu}{\alpha + \mu}, \\
r = \frac{2\mu\alpha - \mu + \alpha}{\alpha + \mu}.
\end{cases} \tag{2.88}
\]

From (2.88), we see that \(\mu\) is a decreasing function of \(\alpha\), and by (2.87), we have
\[
\frac{4}{3} \leq \mu < 3, \tag{2.89}
\]
and
\[
\frac{1}{\mu} + \frac{1}{\alpha} + \frac{q_2 - 2}{q_2} = 1. \tag{2.90}
\]

On the other hand, from (2.88), we have
\[
q_2 = \frac{48\alpha}{\alpha + 12 + \sqrt{144 - 264\alpha + 289\alpha^2}},
\]
and \(q_2\) is an increasing function with \(\alpha \in \left(\frac{9}{5}, \infty\right]\), which follows
\[
\frac{9}{4} < q_2 \leq \frac{8}{3}.
\]

We choose
\[
\beta = \frac{2\mu}{3 - \mu}, \tag{2.91}
\]
then from (2.88), (2.90) and (2.91), we can compute that
\[
\frac{2}{\beta} + 1 = \frac{3}{\mu} = \frac{\alpha + \sqrt{289\alpha^2 - 264\alpha + 144}}{8\alpha} = \frac{3}{2\alpha} \Rightarrow \frac{3}{2\alpha} + \frac{2}{\beta} = g(\alpha). \tag{2.92}
\]

We denote
\[
V_1(t) = \int_0^t \|\partial_3 u_3\|_{L^2} \|\nabla u\|_{L^2}^2 d\tau = \int_0^t \|\partial_3 u_3\|_{L^{2/\alpha}} \|\nabla u\|_{L^2}^2 d\tau. \tag{2.93}
\]
Next, we give an estimate of \( u_3 \). We use \( \| u_3 \|_L^{q_2} \) as a test function in the equation (1.1) for \( u_3 \). Similar to (2.55), we have

\[
\| u_3 \|_{L^{q_2}}^2 \leq \| u_3(0) \|_{L^{q_2}}^2 + C + C \int_0^t \| \partial_3 u_3 \|_{L_\alpha}^{2/\alpha} d\tau. \tag{2.94}
\]

By the condition (1.24), (2.88) and (2.89), we have \( q_2 < 6 \). Note that \( \| u_3(0) \|_{L^{q_2}} < C \) for some \( C > 0 \), we get

\[
u u_3 \in L^{\infty}(0, T; L^{q_2}(\mathbb{R}^3)). \tag{2.95}
\]

By (2.88), we have

\[
r = \frac{(q_2 + 1)\alpha - q_2}{\alpha} = \frac{49\alpha - 36 + \sqrt{144 - 264\alpha + 289\alpha^2}}{\alpha + 12 + \sqrt{144 - 264\alpha + 289\alpha^2}}.
\]

We can check that \( r \) is an increasing function with \( \alpha \in (9/5, \infty) \), and satisfies

\[
2 < r \leq 11/3.
\]

Therefore, for such \( q_2, r, \alpha \), we can apply Lemma 2.3 in [7], and get

\[
\| \nabla u \|_{L^2}^2 + \nu \int_0^t \| \Delta u \|_{L^2}^2 d\tau
\leq \| \nabla u(0) \|_{L^2}^2 + C \int_0^t \| u_3 \|_{L^{q_2}}^{2(r-1)} \| \partial_3 u_3 \|_{L_\alpha}^{2} \| \nabla u \|_{L^2}^2 d\tau
\]

\[
+ C \int_0^t \| u_3 \|_{L^{q_2}}^{8(r-1)} \| \partial_3 u_3 \|_{L_\alpha}^{8/3} \| \nabla u \|_{L^2}^2 d\tau + C. \tag{2.96}
\]

Moreover, by the definition of \( \mu \) and \( r \), we have

\[
\frac{8}{3(r-2)} - \frac{2\mu}{3 - \mu} = \frac{8(\mu + \alpha)}{3(2\mu\alpha - 3\mu - \alpha)} - \frac{2\mu}{3 - \mu}
\]

\[
= \frac{2[12\mu + 5\mu^2 + (\alpha + 12)\alpha]}{3(2\mu\alpha - 3\mu - \alpha)(3 - \mu)} \equiv 0. \tag{2.97}
\]

Combining (2.95) and (2.97), and the fact \( \frac{2}{r-2} < \frac{8}{3(r-2)} \), we have

\[
\| \nabla u \|_{L^2}^2 + \nu \int_0^t \| \Delta u \|_{L^2}^2 d\tau \leq CV_1(t) + \| \nabla u(0) \|_{L^2}^2 + C,
\]

and end the proof for \( \alpha \in (9/5, \infty) \) by using Gronwall’s inequality.

**Proof of Theorem 1.10** The proof of this theorem is heavily rely on the Lemma 2.3 in [7].

• \( i \neq j \)

For (a), without loss of generality, we assume \( i = 1, j = 3 \). The case of \( \epsilon = 1 \) has been proved in Theorem 1.1 in [7]. For each \( 1 < \epsilon < 3/2 \) and

\[
\alpha_1 \in \left( \frac{3}{2 - \epsilon}, \frac{3}{3 - 2\epsilon} \right],
\]

we take

\[
q = \frac{(12 - 4\epsilon)\alpha_1 - 12}{3(\alpha_1 - 1)},
\]
\[ q \text{ is an increasing function with the variable } \alpha, \text{ and } \]
\[ \frac{4}{1 + \epsilon} < q \leq 2. \]

By the initial data, using the similar argument to the proof of Theorem 1.2, we see that
\[ u_3 \in L^\infty(0, T; L^q(\mathbb{R}^3)) \text{ with } 1 < q \leq 2. \quad (2.98) \]

Now, let \( r = \frac{(q+1)\alpha_1 - q}{\alpha_1} \), then
\[ r = \frac{15\alpha_1 - 4\epsilon\alpha_1 - 12}{3\alpha_1}. \]

We claim that \( r > 7/3 \). In fact, it follows from
\[ \alpha_1 > \frac{3}{2 - \epsilon} \implies 4 \left( \frac{2 - \epsilon}{3} \alpha_1 - 1 \right) = (q - \frac{4}{3})\alpha_1 - q > 0 \iff r > \frac{7}{3}. \]

Therefore, apply Lemma 2.3 in [7], we get
\[
\| \nabla u \|^2_{L^2} + \nu \int_0^t \| \Delta u \|^2_{L^2} d\tau \\
\leq C \int_0^t \| u_3 \|_{L^{\frac{2(q-1)}{q}}} \| \partial_1 u_3 \| \| \nabla u \|^2_{L^2} d\tau + \| \nabla u(0) \|^2_{L^2} (2.99)
\]
\[
+ C \int_0^t \| u_3 \|_{L^{\frac{2r-2}{r-2}}} \| \partial_1 u_3 \| \| \nabla u \|^2_{L^2} d\tau + C.
\]

From (2.98), it follows that
\[
\| \nabla u \|^2_{L^2} + \nu \int_0^t \| \Delta u \|^2_{L^2} d\tau \\
\leq C \int_0^t \| \partial_1 u_3 \|_{L^{\frac{2r-2}{r-2}}} \| \nabla u \|^2_{L^2} d\tau + \| \nabla u(0) \|^2_{L^2} \quad (2.100)
\]
\[
+ C \int_0^t \| \partial_1 u_3 \|_{L^{\frac{2r-2}{r-2}}} \| \nabla u \|^2_{L^2} d\tau + C,
\]

it is obvious that \( \frac{8}{3r-7} > \frac{2}{r-2} \). Therefore, by Hölder’s inequality one has
\[
\| \nabla u \|^2_{L^2} + \nu \int_0^t \| \Delta u \|^2_{L^2} d\tau \leq C \int_0^t \| \partial_1 u_3 \|_{L^{\frac{2r-2}{r-2}}} \| \nabla u \|^2_{L^2} d\tau + \| \nabla u(0) \|^2_{L^2} + C. \quad (2.101)
\]

We see that
\[ \frac{3}{\alpha_1} + \frac{3r-7}{4} = \frac{3}{\alpha_1} + \frac{2\alpha_1 - \epsilon\alpha_1 - 3}{\alpha_1} = 2 - \epsilon. \]

By using Gronwall’s inequality and condition (1.29), we prove (a).

• \( i = j \)

Without loss of generality, we assume \( i = j = 3 \). Firstly, we consider \( s = 1/2 \). Let
\[ \frac{1}{\mu} + \frac{1}{2} = \frac{2}{q} \text{ with } 2 < q \leq \frac{12}{5}, \quad (2.102) \]
from (2.102), we see that \( 2 < \mu \leq 3 \). For above \( q \), we prove the following fact
\[ u_3 \in L^\infty(0, T; L^q(\mathbb{R}^3)). \quad (2.103) \]
In fact, for such $\mu, q$, we can apply the same method to (2.55) (or see (2.94)) and combine Gagliardo-Nirenberg and Hölder’s inequalities to get

$$
\frac{1}{2} \frac{d}{dt} \|u_3\|_{L^q}^2 \leq C \|u\|_{L^2}^{3-\mu} \|\nabla u\|_{L^2}^{3(\mu-1)} \|\partial_3 u_3\|_{L^2}.
$$

(2.104)

Integrating (2.104) on time, applying energy inequality and the condition (1.31), one has

$$
\|u_3\|_{L^q}^2 \leq \|u_3(0)\|_{L^q} + C \int_0^t \|u\|_{L^2}^{3-\mu} \|\nabla u\|_{L^2}^{3(\mu-1)} \|\partial_3 u_3\|_{L^2} d\tau
\leq \|u_3(0)\|_{L^q} + C \int_0^t \|u\|_{L^2}^2 d\tau + C \int_0^t \|\nabla u\|_{L^2}^2 d\tau
\leq \|u_3(0)\|_{L^q} + C(T).
$$

(2.105)

This proves (2.103). Let

$$
r = \frac{q+2}{2},
$$

then $2 < r \leq 11/5$, and $\frac{q}{q+r+1} = 2$. Taking the inner product of the equation (1.1) with $-\Delta_h u$ in $L^2$, applying Hölder’s inequality several times, we obtain

$$
\frac{1}{2} \frac{d}{dt} \|\nabla_h u\|_{L^2}^2 + \nu \|\nabla_h \nabla u\|_{L^2}^2 = \int_{\mathbb{R}^3} [(u \cdot \nabla) u] \Delta_h u \, dx
\leq C \int_{\mathbb{R}^3} |u_3| |\nabla u| |\nabla_h \nabla u| \, dx \quad (\text{see [4]})
\leq C \int_{\mathbb{R}^2} \max_x |u_3| (\int_{\mathbb{R}} |\nabla u|^2 \, dx_3)^{\frac{1}{2}} (\int_{\mathbb{R}} |\nabla_h \nabla u|^2 \, dx_1 \, dx_2)
\leq C \int_{\mathbb{R}^2} (\max_x |u_3|)^{r} |dx_1 \, dx_2|^{\frac{1}{2}} (\int_{\mathbb{R}} |\nabla u|^2 \, dx_3)^{\frac{r-2}{2}}
\times [\int_{\mathbb{R}^3} |\nabla_h \nabla u|^2 \, dx_1 \, dx_2 \, dx_3]^{\frac{1}{2}}
\leq C \int_{\mathbb{R}^3} |u_3|^{r_1-1} |\partial_3 u_3| |dx_1 \, dx_2 \, dx_3|^{\frac{1}{r}} \|\nabla_h \nabla u\|_2
\times [\int_{\mathbb{R}^2} (\int_{\mathbb{R}} |\nabla u|^{2 \frac{r-2}{r-1}} \, dx_3)^{\frac{r-2}{2}} \, dx_3]^{\frac{1}{2}}
\leq C \|u_3\|_{L^q}^{\frac{r-1}{r}} \|\partial_3 u_3\|_{L^2} \|\nabla u\|_{L^2}^{\frac{r-2}{2}} \|\partial_3 \nabla u\|_{L^2}^{\frac{1}{2}} \|\nabla_h \nabla u\|_{L^2}.
$$

(2.106)

Applying Young’s inequality to (2.106), we have

$$
\frac{1}{2} \frac{d}{dt} \|\nabla_h u\|_{L^2}^2 + \nu \|\nabla_h \nabla u\|_{L^2}^2 \leq C \|u_3\|_{L^q}^{\frac{2(r-1)}{r-2}} \|\partial_3 u_3\|_{L^2}^2 \|\nabla u\|_{L^2}^2 + \frac{\nu}{2} \|\nabla_h \nabla u\|_{L^2}^2.
$$

(2.107)
After integrating (2.107) on time, combining the energy inequality and (2.103), as well as the condition (1.31), one has
\[\|\nabla_h u\|_{L^2}^2 + \nu \int_0^t \|\nabla_h \nabla u\|_{L^2}^2 d\tau \leq C \int_0^t \|u_3\|_{L^q}^{2(\alpha_2 - 1)} \|\partial_3 u_3\|_{L^2}^2 \|\nabla u\|_{L^2}^2 d\tau + \|\nabla_h u(0)\|_{L^2}^2.\] (2.108)

On the other hand, we have (see the proof of Theorem 1.3 in [8] for detail)
\[\frac{1}{2} \frac{d}{dt} \|\nabla u\|_{L^2}^2 + \nu \|\Delta u\|_{L^2}^2 \leq C \|\nabla_h u\|_{L^2} \|\nabla u\|_{L^2}^\frac{1}{2} \|\nabla_h \nabla u\|_{L^2} \|\Delta u\|_{L^2}^\frac{1}{2}.\] (2.109)

After integrating, and using (2.108) and energy inequality, we obtain
\[\|\nabla u\|_{L^2}^2 + 2\nu \int_0^t \|\Delta u\|_{L^2}^2 d\tau \leq \|\nabla u(0)\|_{L^2}^2 + (\sup_{0 \leq s \leq t} \|\nabla_h u\|_{L^2}) \left(\int_0^t \|\nabla u\|_{L^2}^2 d\tau\right)^\frac{1}{2} \times \left(\int_0^t \|\nabla_h \nabla u\|_{L^2}^2 d\tau\right)^\frac{1}{2} \times \left(\int_0^t \|\Delta u\|_{L^2}^2 d\tau\right)^\frac{1}{2} \leq \|\nabla u(0)\|_{L^2}^2 + C(T) \left(\int_0^t \|\Delta u\|_{L^2}^2 d\tau\right)^\frac{1}{2}.\] (2.110)

By Young inequality, we get the $H^1$ norm of the strong solution $u$ is bounded on the maximal interval of existence $(0, T^*)$. This completes the proof of the case of $\epsilon = 1/2$.

For each $\alpha_2 \in \left(\frac{3}{2 - \epsilon}, \frac{6}{5 - 4\epsilon}\right]$, with $1/2 < \epsilon < 5/4$,
we take
\[q = \frac{(11 - 4\epsilon)\alpha_2 - 12}{3(\alpha_2 - 1)}.\]

It is easy to see that $q$ is an increasing function of $\alpha_2$ and
\[\frac{3}{1 + \epsilon} < q \leq 2 \quad \left(\frac{1}{2} < \epsilon < \frac{5}{4}\right).\]

By using the initial data, as before (see the proof in Theorem 1.2 for detail), we have
\[u_3 \in L^\infty(0, T; L^q(\mathbb{R}^3)) \quad \text{with} \quad 1 < q \leq 2.\] (2.111)

Let $r = \frac{(q + 1)\alpha_2 - q}{\alpha_2}$, then one has
\[r = \frac{14\alpha_2 - 4\epsilon\alpha_2 - 12}{3\alpha_2},\]
and $r > 2$. In fact, it follows from
\[r - 2 = \frac{4(2\alpha_2 - \epsilon\alpha_2 - 3)}{3\alpha_2} > 0 \quad (\alpha_2 > \frac{3}{2 - \epsilon}).\]
Therefore, we can apply Lemma 2.3 in [7], and get
\[
\|\nabla u\|_{L^2}^2 + \frac{\nu}{2} \int_0^t \|\Delta u\|_{L^2}^2 \, dt \\
\leq \|\nabla u(0)\|_{L^2}^2 + C \int_0^t \|u_3\|_{L^{\alpha_2}}^{2(r-1)} \|\partial_3 u_3\|_{L^{\alpha_2}}^2 \|\nabla u\|_{L^2}^2 \, dt \\
+ C \int_0^t \|u_3\|_{L^{\alpha_2}} \|\partial_3 u_3\|_{L^{\alpha_2}} \|\nabla u\|_{L^2}^2 \, dt + C.
\]
(2.112)

From (2.111), it follows that
\[
\|\nabla u\|_{L^2}^2 + \frac{\nu}{2} \int_0^t \|\Delta u\|_{L^2}^2 \, dt \\
\leq \|\nabla u(0)\|_{L^2}^2 + C \int_0^t \|\partial_3 u_3\|_{L^{\alpha_2}}^2 \|\nabla u\|_{L^2}^2 \, dt \\
+ C \int_0^t \|\partial_3 u_3\|_{L^{\alpha_2}} \|\nabla u\|_{L^2}^2 \, dt + C,
\]
(2.113)

it is obvious that \(r > 2\) implies \(\frac{8}{3r-6} > \frac{2}{r-2}\). Therefore, by Hölder’s inequality one has
\[
\|\nabla u\|_{L^2}^2 + \nu \int_0^t \|\Delta u\|_{L^2}^2 \, dt \leq C \int_0^t \|\partial_3 u_3\|_{L^{\alpha_2}}^{2(r-1)} \|\nabla u\|_{L^2}^2 \, dt + \|\nabla u(0)\|_{L^2}^2 + C.
\]
(2.114)

We see that
\[
\frac{3}{\alpha_2} + \frac{3r - 6}{4} = \frac{3}{\alpha_2} + \frac{2\alpha_2 - \epsilon\alpha_2 - 3}{\alpha_2} = 2 - \epsilon.
\]

By using Gronwall’s inequality and condition (1.31), we prove (b).

**Remark 2.3.** In the proof of (b) of this theorem, the result of case \(\epsilon = 1/2\) is actually obtained in Theorem 1.8, in which we note that \(\beta_2\) is not necessary to be infinity when \(\alpha_2 = 2\). However, when \(\beta_2 = \infty\) we have a clear proof, and we have shown above.

**Proof of Theorem 1.12** The method of the proof of (a) is quite similar to the Theorem 1.2 (i) in [7], therefore, we only give the outline of the proof. We also assume \(i = 1, j = 3\). For every
\[
\alpha_1 \in \left[\frac{3}{3 - 2\epsilon}, \frac{3(11 - 2\epsilon)}{2\epsilon^2 - 26\epsilon + 33}\right], \quad \text{with} \quad 1 < \epsilon \leq 21/16,
\]
we set
\[
\begin{align*}
q &= \frac{(12 - 4\epsilon)\alpha_1 - 12}{3(\alpha_1 - 1)}, \\
\sigma &= \frac{3\alpha_1 q}{6\alpha_1 - q}, \\
r &= \frac{(q + 1)\alpha - q}{\alpha},
\end{align*}
\]
(2.115)
then we have \(2 < q < 6\), and
\[
\sigma = \frac{(18 - 6\epsilon)\alpha_1^2 - 18\alpha_1}{9\alpha_1^2 - (15 - 2\epsilon)\alpha_1 + 6}, \quad r = \frac{(15 - 4\epsilon)\alpha_1 - 12}{3\alpha_1}.
\]
\[
\frac{1}{\sigma} + \frac{q - 2}{q} + \frac{1}{3\alpha_1} = 1, \quad 1 < \sigma < \frac{9}{8},
\]
much we also have \( r > \frac{7}{3} \), and

\[
\frac{8}{3r - 7} > \frac{2\sigma}{9 - 8\sigma} \quad \text{with } \alpha_1 \in \left[ \frac{3}{3 - 2\epsilon}, \frac{3(11 - 2\epsilon)}{2\epsilon^2 - 26\epsilon + 33} \right], \quad \text{and } 1 < \epsilon \leq \frac{21}{16}.
\]

For the rest, we will use these parameters \( q, \sigma, r \) to get the desired result. As given the same process to [7], we will use Lemma 2.1 of [7] to estimate \( u_3 \) with the parameters \( q, \sigma \), and then by the \( r > 7/3 \) to get Lemma 2.3 (i) in [7]. Finally, by (2.116) and condition (1.32) to finish the proof.

Now, we pay attention to \((b)\), and assume \( i = j = 3 \). The proof of this part is to imitate the proof of Theorem 1.8, and for every

\[
\alpha_2 \in \left[ \frac{6}{5 - 4\epsilon}, \frac{18 - 2\epsilon}{(4\epsilon - 3)(\epsilon - 5)} \right] \text{ with } \frac{1}{2} < \epsilon \leq \frac{3}{4},
\]

we set

\[
\begin{aligned}
q &= \frac{(11 - 4\epsilon)\alpha_2 - 12}{3(\alpha_2 - 1)}, \\
\mu &= \frac{\alpha_2q}{2\alpha_2 - q}, \\
r &= \frac{(q + 1)\alpha_2 - q}{\alpha_2}.
\end{aligned}
\]

From the definition, we can check that \( 2 < q < 6 \), and

\[
\mu = \frac{(11 - 4\epsilon)\alpha_2^2 - 12\alpha_2}{6\alpha_2^2 - (17 - 4\epsilon)\alpha_2 + 12}, \quad r = \frac{2(7 - 2\epsilon)\alpha_2 - 12}{3\alpha_2},
\]

\[
\frac{1}{\mu} + \frac{1}{\alpha_2} + \frac{q - 2}{q} = 1, \quad 1 < \mu < 3, \quad 2 < q < 6,
\]

and \( r > 2 \), moreover, we also note the index satisfies

\[
\frac{8}{3(r - 2)} > \frac{2\mu}{3 - \mu} \quad \text{with } \alpha_2 \in \left[ \frac{6}{5 - 4\epsilon}, \frac{18 - 2\epsilon}{(4\epsilon - 3)(\epsilon - 5)} \right] \text{ and } \frac{1}{2} < \epsilon \leq \frac{3}{4}.
\]

Finally, we will use these parameters \( q, \mu, r \) to give the same process as the proof of Theorem 1.8 to get this result. We will bounds the \( \|u_3\|_{L^q} \) and \( \|\nabla u\|_{L^2} \) in turn by (2.118) and condition (1.34). We complete the proof.

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