Extremal weight modules of quantum affine algebras

Hiraku Nakajima

Abstract.

Let $\hat{\mathfrak{g}}$ be an affine Lie algebra, and let $U_q(\hat{\mathfrak{g}})$ be the quantum affine algebra introduced by Drinfeld and Jimbo. In [11] Kashiwara introduced a $U_q(\hat{\mathfrak{g}})$-module $V(\lambda)$, having a global crystal base for an integrable weight $\lambda$ of level 0. We call it an extremal weight module. It is isomorphic to the Weyl module introduced by Chari-Pressley [6]. In [12, §13] Kashiwara gave a conjecture on the structure of extremal weight modules. We prove his conjecture when $\hat{\mathfrak{g}}$ is an untwisted affine Lie algebra of a simple Lie algebra $\mathfrak{g}$ of type $ADE$, using a result of Beck-Chari-Pressley [5]. As a by-product, we also show that the extremal weight module is isomorphic to a universal standard module, defined via quiver varieties by the author [16, 18]. This result was conjectured by Varagnolo-Vasserot [19] and Chari-Pressley [6] in a less precise form. Furthermore, we give a characterization of global crystal bases by an almost orthogonality property, as in the case of global crystal base of highest weight modules.

§1. Introduction

In the conference, I gave a survey on quiver varieties and finite dimensional representations of quantum affine algebras. Since I already wrote a survey article [17] on this subject, I will discuss a different one in this paper. But it is related to my talks since I will study extremal weight modules which turn out to be isomorphic to universal standard modules, which was one of the main objects in my talk.

Let us describe Kashiwara’s conjecture [12, §13] on extremal weight modules when $\hat{\mathfrak{g}}$ is the untwisted affine Lie algebra of a simple Lie algebra $\mathfrak{g}$ of type $ADE$. The notation will be explained in the next section.

Let $\lambda$ be a dominant integral weight of $\mathfrak{g}$. We write $\lambda = \sum_{i \in I} m_i \varpi_i$, where $\varpi_i$ is the $i$-th fundamental weight of $\mathfrak{g}$. We consider $\lambda$, $\varpi_i$ as level 0 weights of $\hat{\mathfrak{g}}$ by identifying them with $\sum_i m_i a_i^{\vee} \Lambda_0$, $a_i^{\vee} \Lambda_0$ respectively, where $c = \sum_i a_i^{\vee} h_i$ is the central element, and $\Lambda_i$ is the $i$th fundamental weight of $\hat{\mathfrak{g}}$. Let $V(\lambda)$ be the extremal weight module of extremal weight $\lambda$ with a global crystal base $(L(\lambda), B(\lambda), V^Z(\lambda))$ (see §2.2 for definition).
Let us define a $U_q(\hat{g})$-module
\[
\tilde{V}(\lambda) \overset{\text{def}}{=} \bigotimes_{i \in I} V(\varpi_i)^{\otimes m_i},
\]
where we take and fix any ordering of $I$ to define the tensor product. It has $U'_q(\hat{g})$-module automorphisms $z_{i,\nu}$ ($i \in I$, $\nu = 1, \ldots, m_i$) (see §3.2). Set $\tilde{L}(\lambda) \overset{\text{def}}{=} \bigotimes_{i \in I} L(\varpi_i)^{\otimes m_i}$, $\tilde{u}_\lambda \overset{\text{def}}{=} \bigotimes_{i \in I} u_{\varpi_i}^{m_i}$. Let $\tilde{B}_0(\lambda)$ be the connected component of the crystal $\bigotimes_{i \in I} B(\varpi_i)^{\otimes m_i}$ containing $\tilde{u}_\lambda \mod q\tilde{L}(\lambda)$. There is a (subset of) global base $\{G(b) \mid b \in B_0(\lambda)\}$ (see §3.2).

Let $\Psi_\lambda \overset{\text{def}}{=} \{s(z)b \mid b \in \tilde{B}_0(\lambda)\}$ where $s(z) = \prod_{i \in I} s_{\lambda_i}(z_{i,1}, \ldots, z_{i,m_i})$ is a product of Schur functions.

There exists a unique $U_q(\hat{g})$-linear homomorphism
\[
\Phi_\lambda : V(\lambda) \to \tilde{V}(\lambda)
\]
sending $u_\lambda$ to $\tilde{u}_\lambda$ (see §3.2).

**Theorem 1.**
1. $\Phi_\lambda$ is injective.
2. $\Phi_\lambda(L(\lambda)) \subset \tilde{L}(\lambda)$.
3. $\Phi_\lambda^0$ gives a bijection between $B(\lambda)$ and $\tilde{B}(\lambda)$.
4. $\Phi_\lambda$ maps the global crystal base $\{G(b) \mid b \in B(\lambda)\}$ to $\{s(z)G(b) \mid b \in \tilde{B}_0(\lambda)\}$.

While the author was preparing this article, he learned that Kashiwara also noticed that his conjecture follows from [5] when $g$ is of type $ADE$. In fact, some arguments (the proof of the injectivity of $\Phi_\lambda$, the definition of $\langle , \rangle$, etc.) has been improved from the original form after the discussion with him. After the author posted the first version of this paper to the network archive, he was informed that Jonathan Beck also proved a part of Kashiwara’s conjecture [4].

§2. Preliminaries

2.1. Affine Lie algebra

Let us fix notations for the untwisted affine Lie algebra $\hat{g}$. (For a moment we do not assume that $g$ is of type $ADE$.)

1. $\hat{I}$ : the index set of simple roots,
2. $\{\alpha_i\}_{i \in \hat{I}}$ : the set of simple roots; $\{h_i\}_{i \in \hat{I}}$ : the set of simple coroots,
(3) $\hat{P}^* \overset{\text{def.}}{=} \bigoplus_{i \in I} \mathbb{Z}h_i \oplus \mathbb{Z}d$: the dual weight lattice; $\hat{P} = \text{Hom}_\mathbb{Z}(\hat{P}^*, \mathbb{Z})$: the weight lattice.

(4) $\hat{h} \overset{\text{def.}}{=} \hat{P}^* \otimes_{\mathbb{Z}} \mathbb{Q}$: the Cartan subalgebra.

(5) the simple root $\alpha_i \in \hat{P}$ defined by $\langle h_i, \alpha_j \rangle = \delta_{ij}$, $\langle d, \alpha_j \rangle = \delta_{0j}$, where $\alpha_i$ is the Cartan matrix of $\hat{g}$.

(6) the fundamental weight $\Lambda_i \in \hat{P}$ defined by $\langle h_i, \Lambda_j \rangle = \delta_{ij}$, $\langle d, \Lambda_j \rangle = 0$.

(7) $\hat{Q} \overset{\text{def.}}{=} \bigoplus_{i \in I} \mathbb{Z}a_i$: the root lattice; $\hat{Q}^* \overset{\text{def.}}{=} \bigoplus_{i \in I} \mathbb{Z}h_i$: the coroot lattice.

(8) $\hat{Q}_+ \overset{\text{def.}}{=} \sum_{i \in I} \mathbb{Z}_{\geq 0}a_i; \hat{P}_+ \overset{\text{def.}}{=} \{ \lambda \in \hat{P} \mid \langle h_i, \lambda \rangle \geq 0 \text{ for all } i \in I \}$: the set of integral dominant weights.

(9) the unique element $c = \sum_{i \in I} a_i^\vee h_i$ $(a_i^\vee \in \mathbb{Z}_{\geq 0})$ satisfying

$$\{ h \in \hat{Q}^* \mid \langle h, \alpha_j \rangle = 0 \text{ for all } j \in \hat{I} \} = \mathbb{Z}c,$$

(10) the unique element $\delta = \sum_{i \in I} a_i \alpha_i$ $(a_i \in \mathbb{Z}_{\geq 0})$ satisfying

$$\{ \lambda \in \hat{Q} \mid \langle h_i, \lambda \rangle = 0 \text{ for all } i \in \hat{I} \} = \mathbb{Z}\delta,$$

(11) the symmetric bilinear form $(\ , \ )$ on $\hat{h}^*$, uniquely characterized by $\langle h_i, \lambda \rangle = \frac{2(\alpha_i, \lambda)}{(\alpha_i, \alpha_i)}$, $(c, \lambda) = (\delta, \lambda)$, for $\lambda \in \hat{h}^*$.

(12) $h_0 \overset{\text{def.}}{=} \sum_{i \in I} a_i$: the Coxeter number; $h^* \overset{\text{def.}}{=} \sum_{i \in I} a_i^\vee$: the dual Coxeter number.

The symmetric bilinear form $(\ , \ )$ is known to be nondegenerate, and defines an isomorphism $\nu: \hat{h} \rightarrow \hat{h}^*$ by $\langle h, \lambda \rangle = (\nu(h), \lambda)$ for $\lambda \in \hat{h}^*$. For example, $\nu(\alpha) = \delta$. This coincides with one in [6, §6].

For $\beta \in \hat{h}^*$ with $(\beta, \beta) \neq 0$, we set $\beta^\vee \overset{\text{def.}}{=} \frac{2\beta}{(\beta, \beta)}$. We have $\nu(h_i) = \alpha_i^\vee$.

We have an element $0 \in \hat{I}$ such that $\{ \alpha_i \mid i \neq 0 \}$ is the set of simple roots of $g$. It is known $a_i^\vee = a_0 = 1$ for the untwisted affine Lie algebra $\hat{g}$. We denote $\hat{I} \setminus \{ 0 \}$ by $I$.

Let $\text{cl}: \hat{h}^* \rightarrow \hat{h}^*/\mathbb{Q}\delta$ be the natural projection. Let $\hat{h}^{*0} \overset{\text{def.}}{=} \{ \lambda \in \hat{h}^{*0} \mid (c, \lambda) = 0 \}$, $\hat{P}_0 \overset{\text{def.}}{=} \hat{P} \cap \hat{h}^{*0}$ (level 0 weights). We identify $\text{cl}(\hat{h}^{*0}) \subset \hat{h}^*/\mathbb{Q}\delta$ with the dual of the Cartan subalgebra $\hat{h}$ of the finite dimensional Lie algebra $g$, which is $\bigoplus_{i \in I} \mathbb{Q}h_i$. Similarly we identify $\text{cl}(\hat{P}_0)$ with the weight lattice $P$ of $g$. We define the root lattice of $g$ by $Q \overset{\text{def.}}{=} \bigoplus_{i \in I} \mathbb{Z}a_i$. For $i \in I$, we set $\varpi_i \overset{\text{def.}}{=} \Lambda_i - a_i^\vee \Lambda_0 \in \hat{P}_0$. Then $\text{cl}(\varpi_i)$ is identified with the $i$th fundamental weight of $g$. Let $\hat{P}_0^+ \overset{\text{def.}}{=} \{ \lambda \in \hat{P}_0 \mid \langle h_i, \lambda \rangle \geq 0 \text{ for all } i \in \hat{I} \}$. Its projection $\text{cl}(\hat{P}_0^+)$ is the set of dominant integrable weights of $g$. Let $P^\vee \overset{\text{def.}}{=} \text{Hom}_\mathbb{Z}(Q, \mathbb{Z})$. The fundamental coweights $\varpi_i^\vee$ are defined by
$(\varpi_i^\vee, \alpha_j) = \delta_{ij}$ for $i, j \in I$. We extend $\varpi_i^\vee$ to a homomorphism $\hat{Q} \to \mathbb{Z}$ by setting $(\varpi_i^\vee, \delta) = 0$.

Let $\Delta$ (resp. $\Delta_+$) be the set of roots (resp. positive roots) of $\mathfrak{g}$. The set of roots $\hat{\mathcal{R}}$ of $\hat{\mathfrak{g}}$ is given by $\hat{\mathcal{R}} = \hat{\mathcal{R}}_+ \sqcup \hat{\mathcal{R}}_-$, where

$$\hat{\mathcal{R}}_+ = \{ k\delta + \alpha \mid k \geq 0, \alpha \in \Delta_+ \} \cup \{ k\delta - \alpha \mid k > 0, \alpha \in \Delta_+ \}, \quad \hat{\mathcal{R}}_- = -\hat{\mathcal{R}}_+.$$

The roots of the form $k\delta \pm \alpha$ ($k \in \mathbb{Z}$, $\alpha \in \Delta$) are called real roots, while roots $k\delta$ are called imaginary roots. The multiplicities of real roots are 1, and those of imaginary roots are equal to the rank of $\mathfrak{g}$, i.e., $\# I$.

Set

$\mathcal{R}_> \overset{\text{def}}{=} \{ k\delta + \alpha \mid k \geq 0, \alpha \in \Delta_+ \}$, \quad $\mathcal{R}_- \overset{\text{def}}{=} \{ k\delta - \alpha \mid k > 0, \alpha \in \Delta_+ \}$,

$\mathcal{R}_0 \overset{\text{def}}{=} \{ k\delta \mid k > 0 \} \times I$, \quad $\mathcal{R} \overset{\text{def}}{=} \mathcal{R}_> \sqcup \mathcal{R}_0 \sqcup \mathcal{R}_-.$

These are sets of roots, counted with multiplicities.

For $i \in \hat{I}$, we define the reflection $s_i$ acting on $\hat{\mathfrak{h}}^*$ by $s_i(\lambda) = \lambda - \langle h_i, \lambda \rangle \alpha_i$. Moreover, $s_i$ acts also on $\hat{\mathfrak{h}}$ by $s_i(h) = h - \langle h, \alpha_i \rangle h_i$. The actions of $s_i$ preserve $\hat{P}$, $\hat{Q}$, $\hat{Q}^\vee$ and $\hat{\mathfrak{h}}^*\overset{0}{}$. We have $s_i \delta = \delta$, $s_i \mathfrak{c} = \mathfrak{c}$. If $i \in I$, the corresponding reflection $s_i$ preserves $\mathfrak{h}$, $P$, $P^\vee$ and $Q$. The Weyl group $W$ (resp. affine Weyl group $\hat{W}$) of $\mathfrak{g}$ (resp. $\hat{\mathfrak{g}}$) is the subgroups of $\text{GL}(\hat{\mathfrak{h}}^*)$ (resp. $\text{GL}(\mathfrak{h}^*)$) generated by $s_i$ for $i \in \hat{I}$ (resp. $i \in I$). We define the extended Weyl group $\hat{W}$ as the semidirect product $\hat{W} \overset{\text{def}}{=} W \ltimes P^\vee$, using the $W$-action on $P^\vee$. It is known that $\hat{W}$ is a normal subgroup of $\hat{W}$, and the quotient $\tau \overset{\text{def}}{=} \hat{W}/\hat{W}$ is a finite group isomorphic to a subgroup of the group of the diagram automorphisms of $\hat{\mathfrak{g}}$, i.e., bijections $\tau : I \to I$. Moreover, $\hat{W}$ is isomorphic to $\tau \ltimes \hat{W}$.

When we consider $\xi \in P^\vee$ as an element of $\hat{W}$, we denote it by $t_\xi$. We have $t_\xi(\lambda) = \lambda - \langle \xi, \lambda \rangle \delta$ for $\xi \in P^\vee$, $\lambda \in \mathfrak{h}^*\overset{0}{}$.

**Lemma 2.1.** We have

$$\sum_{\alpha \in \hat{\mathcal{R}}_+ \cap \varpi_i^{-1}(\hat{\mathcal{R}}_-)} (\alpha, \xi) = h^\vee(\varpi_i^\vee, \xi), \quad \sum_{\alpha \in \hat{\mathcal{R}}_+ \cap \varpi_i^{-1}(\hat{\mathcal{R}}_-)} (\alpha^\vee, \xi) = h(\varpi_i^\vee, \xi).$$

**Proof.** From the above description of the root system $\hat{\mathcal{R}}_+$, we have

$$\hat{\mathcal{R}}_+ \cap \varpi_i^{-1}(\hat{\mathcal{R}}_-) = \{ \beta + n\delta \mid \beta \in \Delta_+, 0 \leq n < \langle \varpi_i^\vee, \beta \rangle \}.$$
Therefore

\[ \sum_{\alpha \in \hat{R}^\times (\hat{R}^\times)} (\alpha, \xi) = \sum_{\beta \in \Delta_+} (\beta, \xi)(\varpi_i', \beta) = \sum_{\beta \in \Delta_+} \frac{a_i}{a_i'} (\beta, \xi)(\beta, \varpi_i). \]

We consider the bilinear form on \( \mathfrak{h}^* \) defined by

\[ \Phi(\xi, \eta) \overset{\text{def}}{=} \sum_{\beta \in \Delta_+} (\beta, \xi)(\beta, \eta). \]

From the definition, it is invariant under the Weyl group \( W \). So there is a constant \( c \) such that \( \Phi(\xi, \eta) = c(\xi, \eta) \). Let \( \theta = \delta - \alpha_0 \) be the highest root of \( g \). Then we have

\[ (\theta, \theta) = (\alpha_0, \alpha_0) = 2. \]

On the other hand, we have

\[ \Phi(\theta, \theta) = \sum_{\beta \in \Delta_+} (\beta, \theta)(\beta, \theta). \]

If \( \beta = \sum_i n_i \alpha_i \in \Delta_+ \), we have \( 0 \leq n_i \leq a_i \). So we have

\[ (\beta, \theta) = -\sum_i n_i (\alpha_i, \alpha_0) > 0, \]

\[ (\beta, \theta) = (\theta, \theta) - \sum_i (n_i - a_i)(\alpha_i, \alpha_0) \leq 2, \]

where the equality holds when \( \beta = \theta \). (Note that \( (\alpha_i, \alpha_0) = a_{0i} \) is a negative integer.) Therefore

\[ \Phi(\theta, \theta) = \sum_{\beta \in \Delta_+} (\beta, \theta) + 2 = 2(\rho, \theta) + 2 \]

\[ = 2 \sum_{i \in I} (\varpi_i, \theta) + 2 = 2 \sum_{i \in I} a_i^\vee + 2 = 2 h^\vee, \]

where \( \rho \) is the half sum of the positive roots of \( \mathfrak{h} \), which is known to be equal to \( \sum_{i \in I} \varpi_i \). Therefore we have \( c = h^\vee \) and get the first equation. A similar calculation shows the second equation. Q.E.D.
2.2. Quantum affine algebra

Let $U_q(\hat{\mathfrak{g}})$ be the quantum affine algebra. We follow the notation in [1, 12]. We choose a positive integer $d$ such that $(\alpha_i, \alpha_i)/2 \in \mathbb{Z}d^{-1}$ for any $i \in \widehat{I}$. We set $q_s = q^{1/d}$. (Later we assume $\mathfrak{g}$ is of type $ADE$. Then $d = 1$ and $q_s = q$.) Then the quantum affine algebra $U_q(\hat{\mathfrak{g}})$ is the associative algebra over $\mathbb{Q}(q_s)$ with 1 generated by elements $e_i, f_i$ ($i \in \widehat{I}$), $q^h$ ($h \in d^{-1}\widehat{P}^*$), $q^{c/2}$ with certain defining relations. As customary, we set $q_i = q^{(\alpha_i, \alpha_i)/2}$, $t_i = q^{(\alpha_i, \alpha_i)h_i/2}$, $e_i^{(p)} = e_i^p/[p]_q^1$, $f_i^{(p)} = f_i^p/[p]_q^1$.

Let $U'_q(\hat{\mathfrak{g}})$ be the quantized enveloping algebra with $\text{cl}(\hat{P})$ as a weight lattice. It is the subalgebra of $U_q(\hat{\mathfrak{g}})$ generated by elements $e_i$'s (resp. $f_i$'s). Let $U_q(\hat{\mathfrak{g}})^0$ be the $\mathbb{Q}(q_s)$-subalgebra generated by elements $q^h$ ($h \in d^{-1}\widehat{P}^*$). We have the triangular decomposition $U_q(\hat{\mathfrak{g}}) \cong U_q(\hat{\mathfrak{g}})^+ \otimes U_q(\hat{\mathfrak{g}})^0 \otimes U_q(\hat{\mathfrak{g}})^-$. For $\xi \in \mathbb{Q}$, we define the root space $U_q(\hat{\mathfrak{g}})_\xi$ by

$$U_q(\hat{\mathfrak{g}})_\xi \overset{\text{def}}{=} \{ x \in U_q(\hat{\mathfrak{g}}) \mid q^h x q^{-h} = q^{(h, \xi)} x \text{ for all } h \in \widehat{P}^* \}.$$

Let $U_q^{Z}(\hat{\mathfrak{g}})$ be the $\mathbb{Z}[q, q_s^{-1}]$-subalgebra of $U_q(\hat{\mathfrak{g}})$ generated by elements $e_i^{(n)}, f_i^{(n)}, q^{h}$ for $i \in \widehat{I}, n \in \mathbb{Z}_{>0}, h \in d^{-1}\widehat{P}^*$.

Let us introduce a $\mathbb{Q}(q_s)$-algebra involutive automorphism $\vee$ and $\mathbb{Q}(q_s)$-algebra involutive anti-automorphisms $*$ and $\psi$ of $U_q(\hat{\mathfrak{g}})$ by

$$e_i^\vee = f_i, \quad f_i^\vee = e_i, \quad (q^h)^\vee = q^{-h},$$

$$e_i^* = e_i, \quad f_i^* = f_i, \quad (q^h)^* = q^{-h},$$

$$\psi(e_i) = q_i^{-1}t_i^{-1}f_i, \quad \psi(f_i) = q_i^{-1}t_i e_i, \quad \psi(q^h) = q^h.$$

We define a $\mathbb{Q}$-algebra involutive automorphism $\overline{-}$ of $U_q(\hat{\mathfrak{g}})$ by

$$\overline{e_i} = e_i, \quad \overline{f_i} = f_i, \quad \overline{q^h} = q^{-h},$$

$$\overline{a(q_s)u} = a(q_s^{-1})\overline{u} \quad \text{for } a(q_s) \in \mathbb{Q}(q_s) \text{ and } u \in U_q(\hat{\mathfrak{g}}).$$

In this article, we take the coproduct $\Delta$ on $U_q(\hat{\mathfrak{g}})$ given by

$$\Delta q^h = q^h \otimes q^h, \quad \Delta e_i = e_i \otimes t_i^{-1} + 1 \otimes e_i,$$

$$\Delta f_i = f_i \otimes 1 + t_i \otimes f_i.$$  

(2.2)
Let us denote by $\Omega$ the $\mathbb{Q}$-algebra anti-automorphism $\ast \circ \circ \lor$ of $U_q(\widehat{\mathfrak{g}})$. We have

$$\Omega(e_i) = f_i, \quad \Omega(f_i) = e_i, \quad \Omega(q^h) = q^{-h}, \quad \Omega(q_s) = q_s^{-1}.$$ 

A $U_q(\widehat{\mathfrak{g}})$-module $M$ is called integrable if

1. all $e_i, f_i$ ($i \in I$) are locally nilpotent, and
2. it admits a weight space decomposition:

$$M = \bigoplus_{\lambda \in \hat{P}} M_\lambda,$$

where $M_\lambda = \{ u \in M \mid q^h u = q^{\langle h, \lambda \rangle} u \text{ for all } h \in \hat{P}^* \}$.

Let $\check{U}_q(\widehat{\mathfrak{g}})$ be the modified enveloping algebra \cite[Part IV]{[13]}. It is defined as

$$\check{U}_q(\widehat{\mathfrak{g}}) \overset{\text{def}}{=} \bigoplus_{\lambda \in \hat{P}} U_q(\widehat{\mathfrak{g}})a_{\lambda}, \quad U_q(\widehat{\mathfrak{g}})a_{\lambda} \overset{\text{def}}{=} U_q(\widehat{\mathfrak{g}}) / \sum_{h \in \hat{P}^*} U_q(\widehat{\mathfrak{g}})(q^h - q^{\langle h, \lambda \rangle}).$$

Here the multiplication is given by

$$a_{\lambda} x = x a_{\lambda - \xi} \quad \text{for } \xi \in U_q(\widehat{\mathfrak{g}}), \quad a_{\lambda} a_{\mu} = \delta_{\lambda \mu} a_{\lambda},$$

where $a_{\lambda}$ is considered as the image of $1$ in the above definition of $U_q(\widehat{\mathfrak{g}})a_{\lambda}$.

Let $\lambda, \mu \in \hat{P}_+$. Let $V(\lambda)$ (resp. $V(-\mu)$) be the irreducible highest (resp. lowest) weight module of weight $\lambda$ (resp. $-\mu$) \cite[§3.5]{[13]}. Then there is a surjective homomorphism

$$(2.3) \quad U_q(\widehat{\mathfrak{g}})a_{\lambda - \mu} \ni u \mapsto u(u_{\lambda} \otimes u_{-\mu}) \in V(\lambda) \otimes V(-\mu),$$

where $u_{\lambda}$ (resp. $u_{-\mu}$) is a highest (resp. lowest) weight vector of $V(\lambda)$ (resp. $V(-\mu)$).

### 2.3. Braid group action

For each $w \in \hat{W}$, there exists an $\mathbb{Q}(q)$-algebra automorphism $T_w$ \cite[§39]{[13]} (denoted there by $T_{w,1}$). Also, for any integrable $U_q(\widehat{\mathfrak{g}})$-module $M$, there exists $\mathbb{Q}(q)$-linear map $T_w : M \to M$ satisfying $T_w(xu) = T_w(x)T_w(u)$ for $x \in U_q(\widehat{\mathfrak{g}})$, $u \in M$ \cite[§5]{[13]}. We denote $T_s$, by $T_1$ hereafter. By \cite[39.4.5]{[13]} we have

$$(2.4) \quad \Omega \circ T_w \circ \Omega = T_w.$$
**Lemma 2.5.** We have 

\[
(\psi \circ T_w \circ \psi)(x) = (-1)^{N'} q^{-N} T_{w^{-1}}^{-1}(x)
\]

for all \( w \in \hat{W} \), \( x \in U_q(\hat{g})_\xi \),

where 

\[
N = \sum_{\alpha \in \hat{R}^+ \cap w^{-1}(\hat{R}^-)} (\alpha, \xi), \quad N' = \sum_{\alpha \in \hat{R}^+ \cap w^{-1}(\hat{R}_0)} (\alpha^\vee, \xi).
\]

**Proof.** Let \( T''_{i-1} \) be the automorphism defined in [13, §37]. A direct calculation shows \( \psi \circ T_i \circ \psi = T''_{i-1} \). By [loc. cit., 37.2.4] we have 

\[
T''_{i-1}(x) = (-1)^{(h_i, \xi)} q^{-(\alpha_i \xi)} T_{i-1}^{-1}(x)
\]

for \( x \in U_q(\hat{g})_\xi \). Let \( w = s_{i_m} \ldots s_{i_1} \) be a reduced expression of \( w \). Then 

\[
(\psi \circ T_w \circ \psi)(x) = (-1)^{N'} q^{-N} (T_{i_m}^{-1} \ldots T_{i_1}^{-1})(x),
\]

where 

\[
N' = (h_{i_1} + s_{i_1} h_{i_2} + \cdots + s_{i_1} \ldots s_{i_{m-1}} h_{i_m}, \xi), \\
N = (\alpha_{i_1} + s_{i_1} \alpha_{i_2} + \cdots + s_{i_1} \ldots s_{i_{m-1}} \alpha_{i_m}, \xi).
\]

Since we have \( \hat{R}^+ \cap w^{-1}(\hat{R}^-) = \{ s_{i_k} \ldots s_{i_{k-1}} \alpha_{i_k} \mid k = 1, \ldots, m \} \), we are done.

Q.E.D.

As in [2, 5], the definition of the automorphism \( T_w \) of \( U_q(\hat{g}) \) can be extended to the case \( w \in \hat{W} \) by setting 

\[
\tau e_i = e_{\tau(i)}, \quad \tau f_i = f_{\tau(i)}, \quad \tau q^h = q^{h_{\tau(i)}}, \quad \tau q^d = q^d.
\]

### 2.4. Crystal base

We shall briefly recall the notion of crystal bases. For the notion of (abstract) crystals, we refer to [11, 1].

For \( n \in \mathbb{Z} \) and \( i \in \hat{I} \), let us define an operator acting on any integrable \( U_q(\hat{g}) \)-module \( M \) by 

\[
\tilde{F}_i^{(n)} \overset{\text{def.}}{=} \sum_{k \geq \max(0, -n)} f_i^{(n+k)} e_i^{(k)} a_k^n(t_i),
\]

where 

\[
a_k^n(t_i) \overset{\text{def.}}{=} (-1)^k q_i^{k(1-n)} \prod_{\nu=1}^{k-1} (1 - q_i^{n+2\nu}).
\]

And we set 

\[
\tilde{e}_i \overset{\text{def.}}{=} F_i^{(-1)}, \quad \tilde{f}_i \overset{\text{def.}}{=} F_i^{(1)}.
\]

These operators are different from those used for the definition of crystal bases in [10], but it gives us the same crystal bases by [12, Proposition 6.1].
A direct calculation shows

\[(2.6) \quad \psi(\bar{e}_i) = \bar{f}_i.\]

Let \(A_0 = \{ f(q_s) \in \mathbb{Q}(q_s) \mid f \text{ is regular at } q_s = 0 \}. \)

**Definition 2.7.** Let \(M\) be an integrable \(U_q(\mathfrak{g})\)-module. A pair \((\mathcal{L}, \mathcal{B})\) is called a crystal base of \(M\) if it satisfies

1. \(\mathcal{L}\) is a free \(A_0\)-submodule of \(M\) such that \(M \cong \mathbb{Q}(q_s) \otimes_{A_0} \mathcal{L},\)
2. \(\mathcal{L} = \bigoplus_{\lambda \in P} \mathcal{L}_\lambda\) where \(\mathcal{L}_\lambda = \mathcal{L} \cap M_\lambda\) for \(\lambda \in P,\)
3. \(\mathcal{B}\) is a \(\mathbb{Q}\)-basis of \(\mathcal{L}/q\mathcal{L} \cong \mathbb{Q} \otimes_{A_0} \mathcal{L},\)
4. \(\bar{e}_i \mathcal{L} \subseteq \mathcal{L}, \bar{f}_i \mathcal{L} \subseteq \mathcal{L}\) for all \(i \in I,\)
5. if we denote operators on \(\mathcal{L}/q\mathcal{L}\) induced by \(\bar{e}_i, \bar{f}_i\) by the same symbols, we have \(\bar{e}_i \mathcal{B} \subseteq \mathcal{B} \cup \{0\}, \bar{f}_i \mathcal{B} \subseteq \mathcal{B} \cup \{0\},\)
6. for any \(b, b' \in \mathcal{B}\) and \(i \in I,\) we have \(b' = \bar{f}_i b\) if and only if \(b = \bar{e}_i b'.\)

We define functions \(\varepsilon_i, \varphi_i: \mathcal{B} \to \mathbb{Z}_{\geq 0}\) by \(\varepsilon_i(b) = \max\{n \geq 0 \mid \bar{e}_i^n b \neq 0\}, \varphi_i(b) = \max\{n \geq 0 \mid \bar{f}_i^n b \neq 0\}.\) We set \(\bar{e}_i^{\max b} = \bar{e}_i^{\varphi_i(b)} b, \bar{f}_i^{\max b} = \bar{f}_i^{\varepsilon_i(b)} b.\)

Let \(\mathcal{L}\) be an automorphism of \(\mathbb{Q}(q_s)\) sending \(q_s\) to \(q_s^{-1}\). Let \(A_0 = \mathbb{Q}(q_s)\) consisting of rational functions regular at \(q_s = \infty.\)

**Definition 2.8.** Let \(M\) be an integrable \(U_q(\mathfrak{g})\)-module with a crystal base \((\mathcal{L}, \mathcal{B})\). Let \(\mathcal{L}\) be an involution of an integrable \(U_q(\mathfrak{g})\)-module \(M\) satisfying \(\mathcal{L} = \mathcal{L}\) for any \(x \in U_q(\mathfrak{g}), u \in M.\) Let \(M^2\) be a \(U_q^2(\mathfrak{g})\)-submodule of \(M\) such that \(M^2 = M^2, u - \mathcal{L} \in (q_s - 1)M^2\) for \(u \in M^2.\) We say that \(M\) has a global base \((\mathcal{L}, \mathcal{B}, M^2)\) if the following conditions are satisfied

1. \(M \cong \mathbb{Q}(q_s) \otimes_{\mathbb{Z}[q_s, q_s^{-1}]} M^2 \cong \mathbb{Q}(q_s) \otimes_{A_0} \mathcal{L} \cong \mathbb{Q}(q_s) \otimes_{A_0} \mathcal{L},\)
2. \(\mathcal{L} \cap \mathcal{L} \cap M^2 \to \mathcal{L} / q_s \mathcal{L} \) is an isomorphism.

As a consequence of the definition, natural homomorphisms

\[A_0 \otimes_{\mathbb{Z}} (\mathcal{L} \cap \mathcal{L} \cap M^2) \to \mathcal{L}, \quad A_0 \otimes_{\mathbb{Z}} (\mathcal{L} \cap \mathcal{L} \cap M^2) \to \mathcal{L},\]
\[\mathbb{Z}[q_s, q_s^{-1}] \otimes_{\mathbb{Z}} (\mathcal{L} \cap \mathcal{L} \cap M^2) \to M^2,
\]
are isomorphisms.

Let \(G\) be the inverse isomorphism \(\mathcal{L} / q_s \mathcal{L} \to \mathcal{L} \cap \mathcal{L} \cap M^2.\) Then \(\{G(b) \mid b \in \mathcal{B}\}\) is a base of \(M.\) It is called a global crystal base of \(M.\)

The above conditions imply \(G(b) = G(b).\)
For a dominant weight $\lambda \in \hat{P}_+$, the irreducible highest weight module $V(\lambda)$ has a global crystal base \cite{10}. If $\lambda, \mu \in \hat{P}_+$, then the tensor product $V(\lambda) \otimes V(-\mu)$ also has a global crystal base. Moreover, $\hat{U}_q(\hat{g})$ has a global crystal base $\left( \mathcal{L}(\hat{U}_q(\hat{g})), \mathcal{B}(\hat{U}_q(\hat{g})), \hat{U}_q^Z(\hat{g}) \right)$ such that the homomorphism (2.3) maps a global base of $\hat{U}_q(\hat{g})$ to the union of that of $V(\lambda) \otimes V(-\mu)$ and 0 \cite{15} Part IV]. Furthermore, the global base is invariant under $\ast$ \cite{11, 4.3.2}.

2.5. Extremal vectors

A crystal $\mathcal{B}$ over $U_q(\hat{g})$ is called regular if, for any $J \subseteq \hat{I}$, $\mathcal{B}$ is isomorphic (as a crystal over $U_q(g_J)$) to the crystal associated with an integrable $U_q(g_J)$-module. (It was called normal in \cite{11}.) Here $U_q(g_J)$ is the subalgebra generated by $e_j, f_j$ ($j \in J$), $q^h$ ($h \in d^{-1}P^*$). By \cite{11}, the affine Weyl group $\hat{W}$ acts on any regular crystal. The action $S$ is given by

\[
S_{s_i}b = \begin{cases} 
\tilde{f}^{(h_i, \text{wt } b)}_i b & \text{if } \langle h_i, \text{wt } b \rangle \geq 0, \\
\tilde{e}^{-(h_i, \text{wt } b)}_i b & \text{if } \langle h_i, \text{wt } b \rangle \leq 0
\end{cases}
\]

for the simple reflection $s_i$. We denote $S_{s_i}$ by $S_i$ hereafter.

**Definition 2.9.** Let $M$ be an integrable $U_q(\hat{g})$-module. A vector $u \in M$ with weight $\lambda \in P$ is called extremal, if the following holds for all $w \in \hat{W}$:

\[
\begin{align*}
\langle e_i T_w u, w \lambda \rangle &= 0 & \text{if } \langle h_i, w \lambda \rangle &\geq 0, \\
\langle f_i T_w u, w \lambda \rangle &= 0 & \text{if } \langle h_i, w \lambda \rangle &\leq 0.
\end{align*}
\]

(2.10)

In this case, we define $S_w u$ so that

\[
S_i S_w u = \begin{cases} 
f_i^{(h_i, w \lambda)} S_w u & \text{if } \langle h_i, w \lambda \rangle \geq 0, \\
e_i^{-(h_i, w \lambda)} S_w u & \text{if } \langle h_i, w \lambda \rangle \leq 0.
\end{cases}
\]

This is well-defined, i.e., $S_w u$ depends only on $w$.

Similarly, for a vector $b$ of a regular crystal $B$ with weight $\lambda$, we say that $b$ is extremal if it satisfies

\[
\begin{align*}
\langle \tilde{e}_i S_w b, w \lambda \rangle &= 0 & \text{if } \langle h_i, w \lambda \rangle &\geq 0, \\
\langle \tilde{f}_i S_w b, w \lambda \rangle &= 0 & \text{if } \langle h_i, w \lambda \rangle &\leq 0.
\end{align*}
\]

**Lemma 2.11.** Suppose that an integrable $U_q(\hat{g})$-module $M$ has a crystal base $(\mathcal{L}, \mathcal{B})$. If $u \in \mathcal{L} \subseteq M$ is an extremal vector of weight $\lambda$
satisfying $b \equiv u \mod q\mathcal{L} \in \mathcal{B}$, then $b$ is an extremal vector, and we have

$$S_wu = (-1)^{N_q}q^{-N_w}T_wu, \quad S wb = S_wu \mod q\mathcal{L} \quad \text{for all } w \in \hat{W},$$

where $N_q = \sum_{\alpha \in \hat{\mathbb{R}}_+ \cap w^{-1}(\hat{\mathbb{R}}_-)} \max((\alpha, \lambda), 0)$, and $N_q^\vee$ is given by replacing $\alpha$ by $\alpha^\vee$.

Proof. The equation $S_wb = S_wu \mod q\mathcal{L}$ follows from the definition of $S_w$. If $v \in M_\xi$ satisfies $e_i v = 0$ (resp. $f_i v = 0$), we have

$$T_i v = (-q_i)^{\xi_i}f_i^{(\xi_i)}v \quad \text{(resp. } T_i v = e_i^{(\xi_i)}v),$$

where $\xi_i = (h_i, \xi)$. The rest of the proof is the same as that of Lemma 2.5. Q.E.D.

The following follows from a formula for the crystal $\mathcal{B}(\tilde{U}_q(\hat{\mathfrak{g}}))$ (see [12, App. B]):

**Lemma 2.12.** Let $\lambda \in P^0$. The followings hold for $b = b_1 \otimes t_\lambda \otimes u_{-\infty} \in \mathcal{B}(\mathcal{U}_q(\hat{\mathfrak{g}})a_\lambda) = \mathcal{B}(\infty) \otimes T_\lambda \otimes \mathcal{B}(-\infty)$ with $\text{wt } b_1 \in \mathbb{Z}\delta$:

- $e_i b = 0$ or $f_i b = 0$ if and only if $\varepsilon_i(b_1) \leq \max((-h_i, \lambda), 0)$.

For $\lambda \in P$, Kashiwara defined the $\mathcal{U}_q(\hat{\mathfrak{g}})$-module $V(\lambda)$ generated by $u_\lambda$ with the defining relation that $u_\lambda$ is an extremal vector of weight $\lambda$ [11]. It is written as

$$V(\lambda) = \mathcal{U}_q(\hat{\mathfrak{g}})a_\lambda/I_\lambda, \quad I_\lambda \overset{\text{def}}{=} \bigoplus_{b \in \mathcal{B}(\mathcal{U}_q(\hat{\mathfrak{g}})a_\lambda) \setminus \mathcal{B}(\lambda)} \mathbb{Q}(q)G(b),$$

where $\mathcal{B}(\lambda) \overset{\text{def}}{=} \{ b \in \mathcal{B}(\mathcal{U}_q(\hat{\mathfrak{g}})a_\lambda) \mid b^* \text{ is extremal} \}$. Thus $V(\lambda)$ has a crystal base $(\mathcal{L}(\lambda), \mathcal{B}(\lambda))$ together with a $\mathcal{U}_q^\mathcal{Z}(\hat{\mathfrak{g}})$-submodule $V^2(\lambda)$ with a global crystal base, naturally induced from that of $\mathcal{U}_q(\hat{\mathfrak{g}})a_\lambda$. If $\lambda$ is dominant or anti-dominant, then $V(\lambda)$ is isomorphic to the highest weight module or the lowest weight module. So there is no fear of the confusion of the notation.

\footnote{He denoted it by $V^{\text{max}}(\lambda)$.}
2.6. Drinfeld realization

The quantum affine algebra $U_q(\hat{g})$ has another realization, due to \[8, 2\]. It is isomorphic to an associative algebra over $\mathbb{Q}(q^{1/2})$ with generators $x^\pm_{i,r} (i \in I, r \in \mathbb{Z})$, $q^h (h \in \mathfrak{d}^\times \hat{P}^*)$, $h^\pm_{i,m} (i \in I, m \in \mathbb{Z} \setminus \{0\})$ with certain defining relations (see \[2, \S 4\]). The isomorphism depends on the choice of $o: I \to \{\pm 1\}$, and is given by

$$x^+_{i,r} = o(i) T^r_{-\infty}(e_i), \quad x^-_{i,r} = o(i) T^r_{\infty}(f_i),$$

$$[x^+_{i,r}, x^-_{j,s}] = \delta_{ij} \frac{q^{(r-s)c/2} \psi_{i,r+s}^+ - q^{-(r-s)c/2} \psi_{i,r+s}^-}{q - q^{-1}},$$

where $\psi^\pm_i(u) \equiv \sum_{r=0}^{\infty} \psi^\pm_{i,r} u^{\pm r} \overset{\text{def.}}{=} t_i^{\pm 1} \exp \left( \pm(q_i - q_i^{-1}) \sum_{m=1}^{\infty} h_{i,\pm m} u^{\pm m} \right)$.

By (2.4) we have

$$\Omega(x^+_{i,r}) = x^-_{i,-r}, \quad \Omega(h_{i,m}) = h_{i,-m} \quad \text{for } i \in I, r \in \mathbb{Z}, m \in \mathbb{Z} \setminus \{0\}.$$ 

2.7. The crystal base of $U_q(\hat{g})$

Let us recall results in \[5\]. We assume $g$ is of type ADE hereafter. We choose a reduced expression $s_i \cdots s_{i_N}$ of $2 \rho = 2 \sum_{i \in I} \varpi_i$ in a suitable way (see loc. cit. for detail), and consider a periodic doubly infinite sequence $(\ldots, i_{-1}, i_0, i_1, \ldots)$ of $\hat{I}$ by setting $i_k = i_{k \mod N}$. Let

$$\beta_k \overset{\text{def.}}{=} \begin{cases} s_{i_0} s_{i_{-1}} \cdots s_{i_{k+1}} (\alpha_{i_k}) & \text{if } k \leq 0, \\ s_{i_1} s_{i_2} \cdots s_{i_{k-1}} (\alpha_{i_k}) & \text{if } k > 0. \end{cases}$$

We have

$$(2.13) \quad \mathcal{R}_> = \{\beta_k \mid k \leq 0\}, \quad \mathcal{R}_< = \{\beta_k \mid k > 0\}.$$

We define

$$E^{(n)}_{\beta_k} \overset{\text{def.}}{=} \begin{cases} T_{i_0}^{-1} T_{i_{-1}}^{-1} \cdots T_{i_{k+1}}^{-1} (e_{i_k}) & \text{if } k \leq 0, \\ T_{i_1} T_{i_2} \cdots T_{i_{k-1}} (e_{i_k}) & \text{if } k > 0. \end{cases}$$

We denote $E^{(1)}_{\beta_k}$ by $E_{\beta_k}$. These are root vectors for $\mathcal{R}_>$ and $\mathcal{R}_<$. By \[13, 40.1.3\] we have $E^{(n)}_{\beta_k} \in U_q(\hat{g})^+$. Explicit relations among $E^{(n)}_{\beta_k}$ and $x^\pm_{i,r}$ can be found in \[8, \text{Lemma 1.5}\].
We define $P_{m,i}$ ($m > 0, i \in I$) by

$$1 + \sum_{m > 0} P_{m,i} u^m = \exp \left( - \sum_{m > 0} \frac{(o(i)q^{c/2}u)^r h_{i,r}}{[r]q_i} \right).$$

We also define $\tilde{P}_{m,i} \in U_q(\mathfrak{g})^+$ by replacing $h_{i,r}$ by $-h_{i,r}$. These are root vectors for $R_0$. We also set $P_{-m,i} = \Omega(P_{m,i}) (m > 0, i \in I)$.

Let $c : R \to \mathbb{Z}_{\geq 0}$ be a map such that $c(\alpha) = 0$ except for finitely many $\alpha$. We denote its restrictions to $R_>, R_\geq, R_0$ by $c_>, c_\geq, c_0$ respectively. We define $E_{c_>, E_{c_\leq}} \in U_q(\mathfrak{g})^+$ by

$$E_{c_>} \overset{\text{def.}}{=} E_{\beta_0}^{(c(\beta_0))} E_{\beta_{-1}}^{(c(\beta_{-1}))} \cdots, \quad E_{c_<} \overset{\text{def.}}{=} \cdots E_{\beta_2}^{(c(\beta_2))} E_{\beta_1}^{(c(\beta_1))}.$$

Next, given $c_0$, we associate an $I$-tuple of partitions $(\lambda^{(i)})_{i \in I}$ as

$$\lambda^{(i)} \overset{\text{def.}}{=} (1^c_0(\delta,i) 2^c_0(2\delta,i) \cdots k^c_0(k\delta,i) \cdots).$$

As in [15], we denote it also in another notation:

$$\lambda^{(i)} = (\lambda_1^{(i)}, \lambda_2^{(i)}, \ldots).$$

We define the corresponding Schur function

$$S_{c_0} \overset{\text{def.}}{=} \prod_{i \in I} \det \left( \tilde{P}_{\lambda^{(i)}-k+l,i} \right)_{1 \leq k,l \leq t},$$

where $t \geq l(\lambda^{(i)})$. Note that $\tilde{P}_{m,i}$ corresponds to a complete symmetric function, while $P_{m,i}$ corresponds to an elementary symmetric function, up to sign.

Now a main result of [15] says that

1. $B_c \overset{\text{def.}}{=} E_{c_>} \cdot S_{c_0} \cdot E_{c_<}$ is contained in $L(\infty) \cap U_q(\mathfrak{g})^+$,
2. $\{ B_c \mod qL(\infty) \mid c \in \mathbb{Z}_{\geq 0}^R \}$ is the crystal base of $U_q(\mathfrak{g})^+$.

Set $(Z^R)[\lambda] \overset{\text{def.}}{=} \left\{ c_0 \in \mathbb{Z}_{\geq 0}^R \mid \langle l(\lambda^{(i)}), \langle h_i, \lambda \rangle \text{ for all } i \in I \right\}$, where $(\lambda^{(i)})_{i \in I}$ is the $I$-tuple of partition corresponding to $c_0$ as above.

We apply $\vee$ to the above crystal base to get

$$F_{c_>} \overset{\text{def.}}{=} (E_{c_>})^\vee, \quad F_{c_<} \overset{\text{def.}}{=} (E_{c_<})^\vee, \quad S_{c_0}^\vee \overset{\text{def.}}{=} (S_{c_0})^\vee.$$
2.8. Extremal weight modules and the Drinfeld realization

Extremal weight modules are defined in terms of Chevalley generators. We shall rewrite the definition in terms of Drinfeld generators, and derive several easy consequences in this subsection.

The following is a consequence of [12, Theorem 5.3].

**Lemma 2.14.** Let $u$ be a vector of an integrable $U_q(\hat{\mathfrak{g}})$-module $M$ with weight $\lambda \in \hat{P}^{0,+}$. Then the following conditions are equivalent:

1. $u$ is an extremal vector.
2. $x_i^{r}u = 0$ for all $i \in I, r \in \mathbb{Z}$.

**Remark 2.15.** The extremal weight module $V(\lambda)$ is isomorphic to the Weyl module $W_q(\lambda)$ introduced by Chari-Pressley [6]. This result was referred as ‘an unpublished work’ of Kashiwara in [loc. cit., Proposition 4.5]. Let us give Kashiwara’s proof here. Let $\lambda = \sum_{i \in I} m_i \varpi_i \in \hat{P}^{0,+}$. Then $W_q(\lambda)$ is integrable and contains a vector $w_\lambda$ of weight $\lambda$ which satisfies the above condition (2). Therefore, there is a unique $U_q(\hat{\mathfrak{g}})$-linear homomorphism $V(\lambda) \rightarrow W_q(\lambda)$, sending $v_\lambda$ to $w_\lambda$. (The integrability of $W_q(\lambda)$ was proved via the isomorphism $V(\lambda) \cong W_q(\lambda)$ in [loc. cit.]. So one must give another proof of the integrability as sketched in [loc. cit.].) Since $W_q(\lambda)$ is generated by $w_\lambda$ by definition, the homomorphism is surjective. On the other hand, any integrable $U_q(\hat{\mathfrak{g}})$-module generated by a vector $u$ of weight $\lambda$ satisfying the above condition (2) is a quotient of $W_q(\lambda)$ [loc. cit., Proposition 4.6]. Therefore $V(\lambda)$ and $W_q(\lambda)$ are isomorphic.

**Corollary 2.16.** Let $u$ be an extremal vector with weight $\lambda \in \hat{P}^{0,+}$. Then $S_{c_0}^{-1} u = S_{c_0}^+ u = 0$ for $c_0 \notin (\mathbb{Z}_{\geq 0})^I(\lambda)$.

**Proof.** It is enough to show the assertion for $u = u_\lambda \in V(\lambda)$. We have a $\mathbb{Q}(q)$-vector space isomorphism

$$V(\lambda) \ni xu_\lambda \mapsto x^{\vee} u_{-\lambda} \in V(-\lambda).$$

Therefore it is enough to show $S_{c_0}^{-1} u_{-\lambda} = \Omega(S_{c_0}) u_{-\lambda} = 0$. By [6, Proposition 4.3], which is applicable thanks to Lemma 2.14, we have

$$P_{m,i} u_{-\lambda} = 0 \quad \text{for } |m| > \langle h_i, \lambda \rangle.$$  

(More precisely, we apply [loc. cit.] after composing an automorphism $x_i^{\pm} \mapsto -x_i^{\mp}, h_i, m \mapsto -h_i, -m$.) Now the assertion follows from a standard result in the theory of symmetric polynomials. Q.E.D.
§3. A study of extremal weight modules

3.1. Fundamental representations

By [12, §5.2] there is a unique $U'_q(\hat{g})$-linear automorphism $z_i$ of $V(\varpi_i)$ with weight $\delta$, which sends $u_{\varpi_i}$ to $u_{\varpi_i}$. (Note that $d_i$ in [12, §5.2] is equal to 1 for untwisted $\hat{g}$.)

**Proposition 3.1.** $h_{i,1}u_{\varpi_i} = o(i)(-1)^{h-q^{-h^\vee}}z_iu_{\varpi_i}$.

**Proof.** We have

$h_{i,1}u_{\varpi_i} = t_i^{-1}[x_{i,1}, x_{i,0}] u_{\varpi_i} = o(i)t_i^{-1}T_{\varpi_i}^{-1}(e_i)f_iu_{\varpi_i}$.

Let us write $T_{\varpi_i} = \tau T_w$ with $w \in W$. Then Lemma 2.11 implies

(3.2) $T_{\varpi_i}^{-1}(e_i)f_iu_{\varpi_i} = (-1)^{N'_i}q^{N'_i} S_w^{-1} (e_{r_i^{-1}(i)}S_w(f_iu_{\varpi_i}))$,

where $N'_i = \sum_{\alpha \in \hat{R}_+ \cap w^{-1}(\hat{R}_-)} \max((\alpha, s_i\varpi_i), 0) - \max(\alpha, \varpi_i, 0)$, and $N'_i$ is given by replacing $\alpha$ by $\alpha^\vee$. Since $\hat{R}_+ \cap w^{-1}(\hat{R}_-) = \hat{R}_+ \cap t_{\varpi_i}^{-1}(\hat{R}_-) = \{ \beta + n\delta \mid \beta \in \Delta_+ \}$, we have

$\max((\alpha, \varpi_i), 0) = (\alpha, \varpi_i)$, $\max((\alpha, s_i\varpi_i), 0) = \begin{cases} 0 & \text{if } \alpha = \alpha_i, \\ (\alpha, s_i\varpi_i) & \text{otherwise.} \end{cases}$

Therefore

$N'_+ = (\alpha_i, \varpi_i) - \sum_{\alpha \in \hat{R}_+ \cap w^{-1}(\hat{R}_-)} (\alpha, \alpha_i) = (\alpha_i, \varpi_i) - h^\vee$,

where we have used Lemma 2.11. Similarly we have $N'^-_i = 1 - h$. Now the assertion follows from the definition of the Weyl group action $S$. Q.E.D.

**Remark 3.3.** Let $W(\varpi_i) \equiv V(\varpi_i)/(z_i - 1)V(\varpi_i)$. This is a finite dimensional irreducible $U'_q(\hat{g})$-module [12, §5.2]. The above proposition says that $W(\varpi_i)$ has the Drinfeld polynomial

$P_j(u) = \begin{cases} 1 & \text{if } j \neq i, \\ 1 + o(i)(-1)^{h-q^{-h^\vee}} & \text{if } j = i. \end{cases}$

**Proposition 3.4.** $(\hat{P}_{\pm 1,i})^\vee u_{\varpi_i} = z_i^\pm u_{\varpi_i}$. 
Proof. Let us endow a new $U_q(\mathfrak{g})$-module structure on $V(-\varpi_i)$ by

$$x \cdot u \overset{\text{def}}{=} x^\vee \cdot u, \quad (x \in U_q(\mathfrak{g}), u \in V(-\varpi_i)).$$

We denote it by $V(-\varpi_i)^\vee$. Then there is a $U_q(\mathfrak{g})$-module isomorphism $V(\varpi_i) \cong V(-\varpi_i)^\vee$ sending $u_{\varpi_i}$ to $u_{-\varpi_i}$. Using this isomorphism, we can calculate $(P_{\pm i,1})^\vee u_{\varpi_i}$ exactly as in the above proposition (in fact, more easily) to get the assertion. Q.E.D.

### 3.2. Tensor product modules

Let $\lambda = \sum_{i \in I} m_i \varpi_i \in \tilde{P}^{0,+}$. We define a $U_q(\mathfrak{g})$-module $\tilde{V}(\lambda)$, $\tilde{L}(\lambda)$, $\tilde{B}(\lambda)$, $\tilde{u}_\lambda$ as in the introduction. Let $z_{i,\nu}$ ($i \in I$, $\nu = 1, \ldots, m_i$) be the $U_q(\mathfrak{g})$-linear automorphism of $\tilde{V}(\lambda)$ obtained by the action of $z_i : V(\varpi_i) \to V(\varpi_i)$ on the $\nu$-th factor. Obviously they are commuting: $z_{i,\nu} z_{j,\mu} = z_{j,\mu} z_{i,\nu}$. Let

$$\tilde{V}(\lambda) \overset{\text{def}}{=} U_q(\mathfrak{g})[z_{i,\nu}^{\pm 1}]_{i \in I, \nu = 1, \ldots, m_i} \cdot \tilde{u}_\lambda,$$

$$\tilde{L}(\lambda) \overset{\text{def}}{=} \tilde{L}(\lambda) \cap \tilde{V}(\lambda),$$

$$\tilde{B}(\lambda) \overset{\text{def}}{=} \bigotimes_{i \in I} B(\varpi_i)^{\otimes m_i}, \quad \tilde{V}^Z(\lambda) \overset{\text{def}}{=} \bigotimes_{i \in I} (V(\varpi_i)^Z)^{\otimes m_i} \cap \tilde{V}(\lambda).$$

By [17 §8], the submodule $\tilde{V}(\lambda)$ has

1. the unique bar involution $\tilde{\pi}$ satisfying

$$\tilde{\pi} x \tilde{\pi} = x$$

for $x \in U_q(\mathfrak{g})[z_{i,\nu}^{\pm 1}]_{i \in I, \nu = 1, \ldots, m_i}$, $u \in \tilde{V}(\lambda)$,

2. the crystal base $(\tilde{L}(\lambda), \tilde{B}(\lambda))$, and

3. the $U_q(\mathfrak{g})$-submodule $\tilde{V}^Z(\lambda)$ and the global crystal base $\{ G(\tilde{b}) \mid \tilde{b} \in \tilde{B}(\lambda) \}$.

The module $\tilde{V}(\lambda)$ contains the extremal vector $\tilde{u}_\lambda$ of weight $\lambda$. Therefore there exists a unique $U_q(\mathfrak{g})$-linear homomorphism $\Phi_\lambda : V(\lambda) \to \tilde{V}(\lambda)$ sending $u_\lambda$ to $\tilde{u}_\lambda$. The image is contained in $\tilde{V}(\lambda)$.

Recall that a function $c_0 \in \mathbb{R}_0 \to \mathbb{Z}_{\geq 0}$ defines an $I$-tuple of partitions $\lambda^{(i)}_{c_0}$ as [27]. We define an endomorphism of $\tilde{V}(\lambda)$ by

$$s_{c_0}(z^{\pm 1}) \overset{\text{def}}{=} \prod_{i \in I} s_{\lambda^{(i)}}(z_{i,1}^{\pm 1}, \ldots, z_{i,m_i}^{\pm 1}),$$

where $s_{\lambda^{(i)}}$ is the Schur polynomial corresponding to the partition $\lambda^{(i)}$. If $l(\lambda^{(i)}) > m_i$, it is understood us 0.

**Proposition 3.5.** $\Phi_\lambda(S_{c_0} u_\lambda) = s_{c_0}(z) \cdot \tilde{u}_\lambda$, $\Phi_\lambda(S_{c_0}^{\text{op}} u_\lambda) = s_{c_0}(z^{-1}) \cdot \tilde{u}_\lambda$. 


Proof. On level 0 modules, we have
\[ \Delta h_{i, \pm m} = h_{i, \pm m} \otimes 1 + 1 \otimes h_{i, \pm m} + \text{a nilpotent term} \]
by \[^7\]. Up to sign, the transition between \( h_{i, m} \)'s and \( P_{k, i} \)'s is the same as that between power sums and elementary symmetric functions. The above equation means that \( \Delta \) coincides with the standard coproduct on symmetric polynomials modulo nilpotent terms \[^{15, \text{Chap. I, §5, Ex. 25}}\]. Therefore we have
\[ \Delta P_{k, i} = \sum_{s=0}^{k} P_{s, i} \otimes P_{k-s, i} + \text{a nilpotent term} \]
Using Corollary \[^{2.16}\] and Proposition \[^{3.4}\], we have the assertion. Q.E.D.

3.3. Determination of extremal vectors

Proposition 3.6. Suppose \( \lambda \in \hat{P}^{0,+} \). Consider \( B_c = F_{c_\geq} \cdot S_{c_0} \cdot F_{c_\leq} \) with \( \wt B_c \in \mathbb{Z}\delta \), and set \( b_1 \overset{\text{def.}}{=} B_c \mod q\mathcal{L}(\infty) \in \mathcal{B}(\infty) \) and \( b \overset{\text{def.}}{=} b_1 \otimes t_\lambda \otimes u_{-\infty} \in \mathcal{B}(\widehat{U}_q(\mathfrak{g}) a_\lambda) \). If \( b \) and \( b^* \) are extremal, then we have \( c_\geq \equiv 0 \equiv c_\leq \) and \( c_0 \in (\mathbb{Z}_{\geq 0})^{\lambda} \).

Proof. Assume \( c_\geq \neq 0 \) and take the largest number \( k \leq 0 \) satisfying \( c(\beta_k) \neq 0 \). Let \( w = s_{i_0} s_{i_{-1}} \cdots s_{i_k} \).

Since \( b^* \) is extremal, we can consider \( b \) as an element of \( \mathcal{B}(\lambda) \). We have
\[ b = B_c u_\lambda \mod q\mathcal{L}(\lambda). \]
By Lemma \[^{2.11}\], we have
\[ S_w^{-1} b = (-1)^{N^\vee} q^N T_w^{-1}(B_c) \cdot S_w^{-1}(u_\lambda) \mod q\mathcal{L}(\lambda) \]
for some integers \( N^\vee, N \). By \[^{11, 8.2.2}\] there exists a \( \widehat{U}_q(\mathfrak{g}) \)-linear isomorphism
\[ V(\lambda) \rightarrow V(w^{-1}\lambda); \quad S_w^{-1}(u_\lambda) \mapsto u_{w^{-1}\lambda}, \]
respecting the crystal bases. Therefore we have
\[ (-1)^{N^\vee} q^N T_w^{-1}(B_c)u_{w^{-1}\lambda} \mod q\mathcal{L}(w^{-1}\lambda) \in \mathcal{B}(w^{-1}\lambda). \]
(In fact, this is equal to \( S_w^* u_{w^{-1}\lambda} \).) Let us denote this by \( b'_1 \otimes t_{w^{-1}\lambda} \otimes b'_2 \).
We have
\[ T_w^{-1}(B_c) = T_w^{-1}(F_{c_\geq}) \cdot T_w^{-1}(S_{c_0}) \cdot T_w^{-1}(F_{c_\leq}). \]
It is clear that $T_w^{-1}(\overline{T_{c,c^0}}) \in \text{U}_q(\tilde{\frak{g}})^- \cap T_{ik} \text{U}_q(\tilde{\frak{g}})^-$. We also have $T_w^{-1}(\overline{S_{c^0}}) \in \text{U}_q(\tilde{\frak{g}})^- \cap T_{ik} \text{U}_q(\tilde{\frak{g}})^-$ by [3, Lemma 2]. (More precisely, we apply [loc. cit.] after composing $\overline{\circ \circ \circ \circ}$ by [13, 39.4.5].) Moreover, by our choice of $k$, we have

$$T_w(\overline{T_{c,c^0}}) = f_i^{(\beta_k)} T_{ik} (f_i^{(\beta_k-1)}) \cdots (\overline{\text{U}_q(\tilde{\frak{g}})^-} \cap T_{ik} \text{U}_q(\tilde{\frak{g}})^-)$$

Therefore we have

$$b'_2 = u_{-\infty}, \quad b'_1 = T_w^{-1}(B_c) \mod q\mathcal{L}(\infty), \quad e_{\ell k}(b'_1) = c(\beta_k),$$

where the last equality follows from [13, 38.1.6]. Since $b'_1 \otimes t_{w^{-1}} \otimes u_{-\infty}$ is extremal, Lemma 2.12 implies

$$(3.7) \quad c(\beta_k) \leq \max(-\langle h_{ik}, w^{-1} \lambda \rangle, 0).$$

However, we have $\langle h_{ik}, w^{-1} \lambda \rangle = (w\alpha_k^\vee, \lambda) \geq 0$ for $\lambda \in \hat{\frak{h}}^{0,+}$, because $w\alpha_k \in \hat{\frak{h}}_>$ by (2.13). So the right hand side of (3.7) is 0, and this contradicts with the choice of $k$. Therefore $c_\geq \equiv 0$. Applying $\ast$, we similarly get $c_\leq \equiv 0$. Now the last assertion is a consequence of Corollary 2.16.\hspace{1cm}\text{Q.E.D.}

**Proof of Theorem 4.** We first prove (2), (3), (4) and then (1).

(2) Recall that any vector $b \in \mathcal{B}(\lambda)$ is connected to an extremal vector $[11, 9.3.3]$. Moreover, an extremal vector can be mapped by $f_i^{\text{max}}$ to an extremal vector of the form $b_1 \otimes t_\lambda \otimes u_{-\infty}$. (See [12, Proof of Theorem 5.1].) Therefore

$$\mathcal{B}(\lambda) = \left\{ X_1 \cdots X_s \overline{S_{c_0}} \mod q\mathcal{L}(\lambda) \mid c_0 \in (\mathbb{Z}_{\geq 0})^n(\lambda), \ X_\mu \text{ is } \overline{e}_i \text{ or } \overline{f}_i \right\} \setminus \{0\}$$

by Proposition 3.6. Then $\mathcal{L}(\lambda)$ is spanned by $\{X_1 \cdots X_s \overline{S_{c_0}}\}$ over $\mathbb{Z}[q]$, by Nakayama’s lemma. Note that $\Phi_\lambda$ commutes with the operators $\overline{e}_i, \overline{f}_i$ and $\mathcal{L}(\lambda)$ is invariant under $\overline{e}_i, \overline{f}_i$. Therefore it is enough to show that $\Phi_\lambda(\overline{S_{c_0}}) \in \mathcal{L}(\lambda)$. But this follows from Proposition 3.3.

(3) By Proposition 3.3, we have

$$\Phi_{\lambda}^0(\overline{S_{c_0}}) \mod q\mathcal{L}(\lambda) \in \overline{\mathcal{B}(\lambda)} \quad \text{for } c_0 \in (\mathbb{Z}_{\geq 0})^n(\lambda).$$

As in the proof of (1), we conclude that $\Phi_{\lambda}^0(\mathcal{B}(\lambda)) \subset \overline{\mathcal{B}(\lambda)} \cup \{0\}$. From the definition, it is obvious that the image contains $\overline{\mathcal{B}(\lambda)}$. Consider $\text{Ker} \Phi_{\lambda}^0 \cap \mathcal{B}(\lambda)$. It is invariant under $\overline{e}_i, \overline{f}_i$. Since any vector is connected
to an extremal vector, \( \text{Ker} \Phi^0 \cap \mathcal{B}(\lambda) \) contains an extremal vector if it is nonempty. But we already checked that every extremal vector is mapped to a nonzero vector. Hence \( \Phi^0|_{\mathcal{B}(\lambda)} \) is injective.

(4) By the uniqueness, \( \Phi_\lambda \) respects the bar involutions on \( V(\lambda) \) and \( \tilde{V}(\lambda) \). Since \( V^Z(\lambda) = U^Z_q(\mathfrak{g})u_\lambda \), we have \( \Phi_\lambda(V^Z(\lambda)) \subset \tilde{V}^Z(\lambda) \). Therefore we have

\[
\Phi_\lambda \left( \mathcal{L}(\lambda) \cap \tilde{\mathcal{L}}(\lambda) \cap V^Z(\lambda) \right) \subset \tilde{\mathcal{L}}(\lambda) \cap V^Z(\lambda).
\]

Now the assertion follows from (3).

(1) It is easy to see that \( \tilde{\mathcal{B}}(\lambda) \) is linearly independent. Therefore \( \Phi^0_\lambda : \mathcal{L}(\lambda)/q\mathcal{L}(\lambda) \to \tilde{\mathcal{L}}(\lambda)/q\tilde{\mathcal{L}}(\lambda) \) is injective.

Let \( \{ G(b) \} \) be the global crystal base of \( V(\lambda) \). Let \( 0 \neq \sum f_b(q)G(b) \in \text{Ker} \Phi_\lambda \). Multiplying a power of \( f_b(q) \), we may assume \( f_b(q) \in A_0 \) for all \( b \) and \( f_b(0) \neq 0 \) for some \( b_0 \). Then \( \sum f_b(0)b \in \mathcal{L}(\lambda)/q\mathcal{L}(\lambda) \) is mapped to 0 by \( \Phi^0_\lambda \). The injectivity of \( \Phi^0_\lambda \) implies that \( f_b(0) = 0 \) for all \( b \). This is a contradiction.

Q.E.D.

Remark 3.8. Theorem \([11]\) together with Proposition \([12,13]\) implies that \( S^c_\infty u_\lambda \) and \( \tilde{S}^c_\infty u_\lambda \) are elements of the global base.

3.4. Standard modules

Let us briefly recall the properties of the universal standard module \( M(\lambda) \) with a weight \( \lambda = \sum m_i\varpi_i \in \hat{P}^{0,+} \) introduced in \([11,13]\). (We do not review its definition, which is based on quiver varieties.) Let \( G_\lambda \stackrel{\text{def}}{=} \prod_i \text{GL}_{m_i}(\mathbb{C}) \). Its maximal torus consisting of diagonal matrices is denoted by \( H_\lambda \). Their representation rings are denoted by \( R(G_\lambda) \), \( R(H_\lambda) \) respectively. They are isomorphic to \( \bigotimes_i \mathbb{Z}[x_{i,1}^\pm, \ldots, x_{i,m_i}^\pm]^{S_{m_i}} \) and \( \bigotimes_i \mathbb{Z}[x_{i,1}^\pm, \ldots, x_{i,m_i}^\pm] \) respectively. The universal standard module \( M(\lambda) \) is a \( U^{'\circ}q(\hat{\mathfrak{g}}) \otimes_{\mathbb{Z}} R(G_\lambda) \)-module which is integrable (in fact, it satisfies a stronger condition ‘\( t \)-integrability’) and contains a vector \( [0]_\lambda \) with

\[
x_i^+(r)[0]_\lambda = 0 \quad \text{for any} \quad i \in I, \ r \in \mathbb{Z}, \quad q^h[0]_\lambda = q^{(h,\lambda)}[0]_\lambda,
\]

\[
M(\lambda) = \left( U^{'\circ}q(\hat{\mathfrak{g}}) \otimes_{\mathbb{Z}} R(G_\lambda) \right) [0]_\lambda,
\]

\[
\psi^+_i(u)[0]_\lambda = q^{m_i} \left( \prod_{\nu=1}^{m_i} \frac{1 - q^{-1}x_{i,\nu}u}{1 - qx_{i,\nu}u} \right)^{\pm} [0]_\lambda,
\]

where \( (\ )^{\pm} \) denotes the expansion at \( u = 0 \) and \( \infty \) respectively. (In fact, we have \( M(\lambda) = U^{'\circ}q(\hat{\mathfrak{g}})[0]_\lambda \) by the proof of Theorem \([11]\)). Moreover, \( M(\lambda) \) is free of finite rank as an \( R(G_\lambda) \)-module. And \( M(\lambda) \) is simple if we tensor the quotient field of \( \mathbb{Z}[q,q^{-1}] \otimes R(G_\lambda) \).
On the other hand, we have a \( \bigotimes_{i \in I} \mathbb{Z}[z_{i,1}^{\pm, \ldots, z_{i,m_i}^{\pm}}]^{S_{m_i}} \) \(-\)module structure on \( V(\lambda) \) given by \( s_{e_0}(z)u_\lambda = S_{e_0}u_\lambda \) and \( s_{e_0}(z^{-1})u_\lambda = \sum_{i \in I} u_\lambda \) by the above discussion. We make it a \( R(G_\lambda) = \bigotimes_{i \in I} \mathbb{Z}[x_{i,1}^{\pm, \ldots, x_{i,m_i}^{\pm}}]^{S_{m_i}} \) \(-\)module structure by setting \( x_{i,\nu} = o(i)(-1)^{1-h_q-h^\vee}z_{i,\nu} \).

**Theorem 2.** There exists a unique \( \mathbf{U}_q(Z) \otimes \mathbb{Z} \mathbb{R}(\mathbf{G}_\lambda) \) \(-\)isomorphism \( \mathbf{V}_\lambda(\mathbf{Z}) \rightarrow M(\lambda) \) sending \( u_\lambda \) to \([0]_\lambda\).

This result follows from Theorem 1 as explained in [18, 1.23]. The calculation of Drinfeld polynomial, which was not given there, is done in Proposition 3.1.

**Correction to [18]:**
- Delete \( S_{\lambda_1} \times \cdots \times S_{\lambda_n} \) in Theorem 1.22.
- Replace \( R(G_\lambda) \) in page 411, line 5 by \( R(H_\lambda) \).
- Delete ‘and forgetting the symmetric group invariance’ in Remark 1.23.
- Replace ‘the submodule above’ in line 8, ‘the submodule \( \mathbf{U}_q(\mathbf{Z})[x_{k,\nu}]_{k \in I, \nu = 1, \ldots, \lambda_k} \otimes [0]_{\lambda_k}^{\otimes} \)’.

§4. A bilinear form

Kashiwara proved that the crystal base \( \mathbf{B}(\lambda) \) is an orthonormal base with respect to a natural bilinear form when \( \lambda \) is dominant [10, 5.1.1]. We prove a similar result for \( \lambda \in \hat{\mathbb{P}}^{0,+} \) in this section. This generalizes a result of Varagnolo-Vasserot [20, Theorem A] from fundamental representations to arbitrary \( \lambda \).

**Proposition 4.1** (Kashiwara). The extremal weight module \( V(\lambda) \) has a unique bilinear form \( (\cdot, \cdot) \) satisfying

\[
(4.2) \quad (u_\lambda, G(b)) = \begin{cases} 1 & \text{if } G(b) = u_\lambda, \\ 0 & \text{otherwise} \end{cases}
\]

\[
(4.3) \quad (xu, v) = (u, \psi(x)v) \quad \text{for } x \in \mathbf{U}_q(Z), u, v \in V(\lambda).
\]

**Proof.** We define a \( \mathbf{U}_q(Z) \)-module structure on \( \text{Hom}(V(\lambda), \mathbb{Q}(q)) \) by

\[
\langle xf, u \rangle \overset{\text{def}}{=} \langle f, \psi(x)u \rangle, \quad x \in \mathbf{U}_q(Z), f \in \text{Hom}(V(\lambda), \mathbb{Q}(q)), u \in V(\lambda).
\]

This defines a \( \mathbf{U}_q(Z) \)-module structure since \( \psi: \mathbf{U}_q(Z) \rightarrow \mathbf{U}_q(Z)^{opp} \) is an algebra homomorphism. Let \( u^\lambda \) be the unique linear form such that

\[
\langle u^\lambda, G(b) \rangle = \begin{cases} 1 & \text{if } G(b) = u_\lambda, \\ 0 & \text{otherwise}. \end{cases}
\]
Then $u^\lambda$ has a weight $\lambda$. We claim that $u^\lambda$ is an extremal vector. From the definition all elements in a weight space $\text{Hom}(V(\lambda), \mathbb{Q}(q))_\xi$ vanish on $V(\lambda)_\xi$. Since weights of $V(\lambda)$ are contained in the convex hull of $W\lambda$ \textit{[2] Theorem 5.3], the weights of $V'(\lambda)$ also have the same property. Therefore $u^\lambda$ is an extremal vector. Now we have a $U_q(\widehat{\mathfrak{g}})$-algebra homomorphism $V(\lambda) \to V'(\lambda) \subset \text{Hom}(V(\lambda), \mathbb{Q}(q))$ sending $u_\lambda$ to $u^\lambda$. This defines a bilinear form satisfying the desired properties. The uniqueness follows from the uniqueness of the above homomorphism. Q.E.D.

\textit{Remark 4.4.} The uniqueness holds even if (\textbf{1.3}) holds only for $x \in U_q'(\widehat{\mathfrak{g}})$. In fact, this condition together with (\textbf{4.2}) automatically implies (\textbf{4.3}) for $x = q^d$ as follows. When $u = u_\lambda$, (\textbf{4.2}) implies (\textbf{4.3}) for $x = q^d$. For a general case, we write $u = xu_\lambda$ with $x \in U_q'(\widehat{\mathfrak{g}})_\xi$. Then

$$
(q^d u, v) = q^{\langle d, \xi \rangle}(qx^d u_\lambda, v) = q^{\langle d, \xi \rangle}(q^d u_\lambda, \psi(x)v) = q^{\langle d, \xi \rangle}(u_\lambda, q^d \psi(x)v) = (u_\lambda, \psi(x)q^d v) = (xu_\lambda, q^d v) = (u, q^d v),
$$

where we have used $\psi(x) \in U_q'(\widehat{\mathfrak{g}})_\xi$.

\textbf{Lemma 4.5.} Let $M$ be an integrable $U_q'(\widehat{\mathfrak{g}})$-module with a bilinear form $(\cdot, \cdot)$ satisfying (\textbf{1.3}) for $x \in U_q'(\widehat{\mathfrak{g}})$. Then

$$(T_w u, v) = (-1)^N q^N (u, T_{w^{-1}} v) \quad \text{for all } w \in \widehat{W}, u \in M_\xi, v \in M,$$

where $N$ and $N'$ are as in Lemma \textbf{2.5}

\textit{Proof.} Let $T'_{i,1}$ be the operator defined in \textbf{[1.3] 5.2.1}. A direct calculation shows $(T_w u, v) = (u, T'_{i,1} v)$ for $u \in M_\xi, v \in M$. (We may assume that $v$ is contained in a weight space. Thanks to \textbf{[1.3]} for $x \in U_q'(\widehat{\mathfrak{g}})$, both hand sides are 0 unless the weight of $v$ is $s_i \xi + m\delta$ for some $m \in \mathbb{Z}$.) By [loc. cit., 5.2.3], we have $T'_{i,1} v = (-1)^{\langle h_i, \xi \rangle} q^{\langle \alpha_i, \xi \rangle} T_i v$. The rest of the proof is the same as that of Lemma \textbf{2.5}. Q.E.D.

\textbf{Lemma 4.6.} Let $M$ and $(\cdot, \cdot)$ be as above. Let $u, v \in M$ be extremal vectors. Then

$$(S_w u, v) = (u, S_{w^{-1}} v).$$

\textit{Proof.} Let $\xi$ be the weight of $u$. Using Lemmas \textbf{2.11} \textbf{1.3}, we have

$$(S_w u, v) = (-1)^{N_\xi + N_\xi'} q^{-N_\xi - N_\xi'} (u, S_{w^{-1}} v),$$
where

\[
N = \sum_{\alpha \in \hat{R}_+ \cap w^{-1}(\hat{R}_-)} (\alpha, \xi), \quad N_+ = \sum_{\alpha \in \hat{R}_+ \cap w^{-1}(\hat{R}_-)} \max((\alpha, \xi), 0), \quad N'_+ = \sum_{\alpha' \in \hat{R}_+ \cap w(\hat{R}_-)} \max((\alpha', w\xi), 0),
\]

and \(N_\vee, N_+^\vee, N_+^\vee'\) are defined in similar ways. Noticing \(\alpha' \in \hat{R}_+ \cap w(\hat{R}_-) \iff -w^{-1}\alpha' \in \hat{R}_+ \cap w^{-1}(\hat{R}_-),\) we have \(N = N_+ + N'_+\). Similarly \(N_\vee = N_+^\vee + N_+^\vee'\). Therefore we have the assertion. Q.E.D.

In order to study \((\cdot, \cdot)\) on \(V(\lambda)\) we need to relate it to a bilinear form on the tensor product module \(\tilde{V}(\lambda)\).

**Lemma 4.7.** We have \((z_i u, z_i v) = (u, v)\) for \(u, v \in V(\varpi_i)\).

**Proof.** By the uniqueness, it is enough to show that \((z_i u, z_i v)\) satisfies (4.3). The property (4.3) is clear. If \(x \in U'_q(\hat{g})\), then it holds since \(z_i\) is \(U'_q(\hat{g})\)-linear. It also holds for \(x = q^{-a_0 d_i} q^d\).

Let us check (4.3). Since \(\dim V(\varpi_i)_{\varpi_i} = 1\) by [12, Proposition 5.10], it is enough to show that \((z_i u_{\varpi_i}, z_i u_{\varpi_i}) = 1\). But this follows from the previous lemma. Q.E.D.

We define a \(Q(q)[z_i]^{\pm}\)-valued bilinear form \((\cdot, \cdot)\) on \(V(\varpi_i)\) by

\[
(\cdot, \cdot) =\begin{cases} \langle z_i^m(z_i^{-m} u, v) \rangle & \text{if } \text{wt}(u) = \text{wt}(v) + md_i \delta \text{ for } m \in \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases}
\]

Since \(z_i\) is \(U'_q(\hat{g})\)-linear, we have

\[
(\cdot x u, v) = (\cdot, \psi(x)v) \quad \text{for } x \in U'_q(\hat{g}), u, v \in V(\varpi_i).
\]

By Lemma 4.7 we have

\[
(\langle z_i^m u_{\varpi_i}, z_i^n u_{\varpi_i} \rangle) = z_i^{m-n}.
\]

We define a \(Q(q)[z_i]^{\pm}\) \(i \in I, \nu = 1, \ldots, m_i\)-valued bilinear form \((\cdot, \cdot)\) on \(\tilde{V}(\lambda)\) by

\[
(\cdot, \cdot) \overset{\text{def}}{=} \prod_{i, \nu} (\langle u_{i, \nu}, v_{i, \nu} \rangle),
\]
where $u_{i,\nu}$, $v_{i,\nu}$ is the $\nu$-th $V(\pi_i)$-factor of $u, v \in \tilde{V}(\lambda)$. We define a bilinear form $(\ , \ )^\sim$ on $\tilde{V}(\lambda)$ by

$$(u, v)^\sim \text{def.} \prod_{i \in I} \frac{1}{m_i!} \left( (u, v) \prod_{\mu \not= \nu} (1 - z_{i,\mu} z_{i,\nu}^{-1}) \right)\lambda,$$

where $[f]_1$ denote the constant term in $f$.

**Lemma 4.9.** Let $c_0, c'_0 \in (\mathbb{Z}_{\geq 0})(\lambda)$. Then $(s_{c_0}(z) \bar{u}_\lambda, s_{c'_0}(z) \bar{u}_\lambda)^\sim = \delta_{c_0, c'_0}$.

**Proof.** Let $f = f(z)$ and $g = g(z)$ be polynomials in $z_{i,\nu}$'s ($i \in I$, $\nu = 1, \ldots, m_i$). By (4.3) we have

$$(f(z) \bar{u}_\lambda, g(z) \bar{u}_\lambda)^\sim \prod_{i \in I} \frac{1}{m_i!} \left[ f g \prod_{\mu \not= \nu} (1 - z_{i,\mu} z_{i,\nu}^{-1}) \right]\lambda,$$

where $g = (\ldots, z_{i,\nu}^{-1}, \ldots)$. Considered as a bilinear form on the Laurent polynomial ring, it coincides with one in [3, Chap.VI, §9] with $q = t$. The Schur functions give an orthogonal base with respect to that bilinear form. Therefore we have the assertion. Q.E.D.

**Proposition 4.10.** Let $u, v \in V(\lambda)$. Then $(u, v) = (\Phi_\lambda(u), \Phi_\lambda(v))^\sim$.

**Proof.** It is enough to show that $(\Phi_\lambda(u), \Phi_\lambda(v))^\sim$ satisfies conditions in Proposition 4.1. It is clear that the condition (4.3) holds for $x \in U'_d(\tilde{g})$. By Remark 4.1, it is enough to check (4.2). From (4.3) for $x \in U'_d(\tilde{g})$, it is enough to check (4.2) when $\text{cl}(\text{wt}(b)) = \text{cl}(\lambda)$, i.e., $\text{wt}(b) = \lambda + m\delta$ for some $m \in \mathbb{Z}$. Since weights of $V(\lambda)$ is contained in the convex hull of $W\lambda$, $b$ is an extremal vector. We have

$$(\Phi_\lambda(u_\lambda), \Phi_\lambda(G(b)))^\sim = (\Phi_\lambda(S_w u_\lambda), \Phi_\lambda(S_w G(b)))^\sim$$

by Lemma 4.1. We take $S_w$ as sufficiently many compositions of $f_1^{\text{max}}$, we may assume $S_w u_\lambda = S_{c_0} u_\lambda$, $S_w G(b) = S_{c'_0} u_\lambda$. (Recall that $S_{c_0} u_\lambda$ is an element of the global basis as we explained in Remark 3.8.) Then

$$(\Phi_\lambda(u_\lambda), \Phi_\lambda(G(b)))^\sim = (s_{c_0} \bar{u}_\lambda, s_{c'_0} \bar{u}_\lambda)^\sim = \delta_{c_0, c'_0} = \delta_{u_\lambda, G(b)}$$

where we have used Proposition 3.3 and Lemma 4.3. Q.E.D.

From the proof of Proposition 4.10 the bilinear form $(\ , \ )$ on $V(\lambda)$ defined in Proposition 4.1 also has the following characterization: it satisfies (1.3) and $(s_{c_0} u_\lambda, s_{c'_0} u_\lambda) = \delta_{c_0, c'_0}$. Since these conditions are symmetric, we have the following:
Corollary 4.11. The bilinear form $(\cdot,\cdot)$ on $V(\lambda)$ is symmetric, i.e., $(u,v) = (v,u)$.

Proposition 4.12. (1) $(\mathcal{L}(\lambda),\mathcal{L}(\lambda)) \subseteq A_0$.
Let $(\cdot,\cdot)_0$ be the $\mathbb{Q}$-valued bilinear form on $\mathcal{L}(\lambda)/q\mathcal{L}(\lambda)$ induced by $(\cdot,\cdot)_{|q=0}$ on $\mathcal{L}(\lambda)$.
(2) $(\overline{e}_i u,v)_0 = (u,\overline{f}_i v)_0$ for $u,v \in \mathcal{L}(\lambda)/q\mathcal{L}(\lambda)$.
(3) $\mathcal{B}(\lambda)$ is an orthonormal base with respect to $(\cdot,\cdot)_0$. In particular, $(\cdot,\cdot)_0$ is positive definite.
(4) $\mathcal{L}(\lambda) = \{u \in V \mid (u,u) \in A_0\}$.

Proof. We shall prove
- there exist representatives $\overline{b}$ for all $b \in \mathcal{B}(\lambda)_{\xi} \subseteq \mathcal{L}(\lambda)_{\xi}/q\mathcal{L}(\lambda)_{\xi}$
such that $(\overline{b},\overline{b}') \equiv \delta_{bb'} \mod qA_0$ for $b,b' \in \mathcal{B}(\lambda)_{\xi}$
by the induction on $(\xi,\xi)$. Since $\mathcal{L}(\lambda)_{\xi}$ is spanned by $\overline{b}$'s over $A_0$, this implies the above equations for any representatives $\overline{b}$. It also implies (1) and (3). Recall $(\overline{e}_i \overline{b},\overline{b}') = (\overline{b},\overline{f}_i \overline{b}')$ by (2.6). Therefore the above assertion also implies (2).

First suppose that $b$ is extremal. Since we may assume that $\text{wt}(b) = \text{wt}(b')$ by [11, 5.3], we may assume $b'$ is also extremal by [12, 5.3]. Then we may assume $\overline{b} = S_{\hat{e}_\alpha} u_\lambda$, $\overline{b}' = S_{\hat{e}_\alpha} u_\lambda$ by applying $S_w$ for some $w \in \hat{W}$. But, in this case, the assertion has been already shown in Lemma 1.9 and Proposition 1.10.

Now we start the induction. Recall that $(\xi,\xi)$ is bounded from above and $b \in \mathcal{B}(\lambda)$ is extremal if $(\text{wt} b, \text{wt} b)$ is maximal ([11, §9.3]). Therefore when $(\xi,\xi)$ is maximal, both $b$ and $b'$ are extremal. We have already proved the assertion this case.

Now assuming the above for $\xi$ such that $(\xi,\xi) > a$, let us prove it for $\xi$ with $(\xi,\xi) = a$. For $i \in I$, suppose that $(\text{wt} \overline{e}_i b, \text{wt} \overline{e}_i b) > (\xi,\xi)$. We consider $\overline{e}_i b$. If $\overline{e}_i b \neq 0$, then we have

$$(\text{wt}(\overline{e}_i b), \text{wt}(\overline{e}_i b)) = (\xi + \alpha_i, \xi + \alpha_i) > (\xi,\xi).$$

Therefore we have

$$(\overline{f}_i \overline{e}_i b, \overline{b}') = (\overline{e}_i \overline{b}, \overline{e}_i b') \equiv \delta_{\overline{e}_i b, \overline{e}_i b'} \equiv \delta_{bb'} \mod qA_0$$

by the induction hypothesis. Hence the assertion holds if we replace the representative $\overline{b}$ by another representative $\overline{f}_i \overline{e}_i b$. Similarly, if $(\text{wt} \overline{e}_i, \xi) \leq 0$ and $\overline{f}_i b \neq 0$, we replace $\overline{b}$ by $\overline{e}_i \overline{f}_i b$ to get the assertion.

Since we may suppose that $b$ is not extremal, there exists $w \in \hat{W}$ such that $S_w b$ satisfies $\overline{e}_i S_w b \neq 0$ if $(\text{wt} \overline{e}_i, w\xi) > 0$ and $\overline{f}_i S_w b \neq 0$ if

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\[ (h, w \xi) \leq 0. \] Then we have \((\tilde{f}_i \bar{e}_i S_w \tilde{b}, S_w \tilde{b}')\) or \((\bar{e}_i \tilde{f}_i S_w \tilde{b}, S_w \tilde{b}')\) is in \(\delta_{bb'} + q\mathbb{Z}[q]\). Therefore we are done.

The statement (4) follows from [13, 14.2.2]. Q.E.D.

The followign result generalizes [20, Theorem A] from fundamental representations to arbitrary \(\lambda\):

**Theorem 3.** (1) \(\{ G(b) \}_{b \in \mathcal{B}(\lambda)} \) is almost orthonormal for \(( , )\), that is, \((G(b), G(b')) \equiv \delta_{bb'} \mod q\mathbb{Z}[q]\).

(2) \(\{ \pm G(b) \mid b \in \mathcal{B}(\lambda) \} = \{ u \in V^Z(\lambda) \mid \overline{u} = u, (u, u) \equiv 1 \mod q\mathbb{Z}[q] \}\).

**Proof.** We claim \((u, v) \in \mathbb{Z}[q, q^{-1}] \) for \(u, v \in V^Z(\lambda)\).

The assertion is obvious for the special case \(u = u_\lambda\) by (4.2). For general case, we may assume \(u = xu_\lambda\) for \(x \in \mathbb{U}_q^Z(\mathfrak{g})\). Then \((xu_\lambda, v) = (u_\lambda, \psi(x)v)\). Since \(\psi(x) \in \mathbb{U}_q^Z(\mathfrak{g})\) and \(V^Z(\lambda)\) is stable under the action of \(\mathbb{U}_q^Z(\mathfrak{g})\), the assertion follows from the special case.

Combining with Proposition 4.12, we have \((G(b), G(b')) - \delta_{bb'} \in \mathbb{Z}[q, q^{-1}] \cap q\mathbb{A}_0 = q\mathbb{Z}[q]\).

This is the statement (1). The statement (2) follows from the argument of [13, 14.2.3]. Q.E.D.

**Remark 4.13.** Lusztig conjectures that the universal standard module \(M(\lambda)\), more precisely its tensor product of \(\otimes_{R(G, \lambda)} R(H, \lambda)\), which is isomorphic to \(\tilde{V}^Z(\lambda)\), has a signed base characterized by the almost orthogonality property Theorem 3(2), with respect to geometrically defined bilinear form and bar involution [14]. (See \S 3.4 for notations.) Recently Varagnolo-Vasserot [20] give a proof of the conjecture by showing that \(\{ G(b) \mid b \in \tilde{B}(\lambda) \} \) satisfies the property. They also conjecture that the global base \(\{ G(b) \mid b \in \mathcal{B}(\lambda) \} \) of \(V(\lambda)\) satisfies the almost orthogonality property with respect to the geometric bilinear form and bar involution. Their conjecture follows from Theorem 3(2) since the geometric bilinear form and bar involution coincide with ones used in this paper, as Varangnolo and Vasserot proved that the forms satisfy the conditions in Proposition 4.1 (more precisely (4.3) and \((S_{c_0} u_\lambda, S_{c_0} u_\lambda) = \delta_{e_0, c_0}\) and the equality \(xu = \overline{x} \overline{u}\). Remark that these hold only after an appropriate normalization of standard modules so that we have \(x_i, \nu = \pm z_i, \nu\). This is the normalization in \[21\] different from ours. This point is clarified during discussion with Varagnolo-Vasserot in February 2002.
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Department of Mathematics
Kyoto University
Kyoto 606-8502
Japan