DONALDSON-THOMAS TRANSFORMATION OF GRASSMANNIAN

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Abstract. Kontsevich and Soibelman [KS08] defined the notion of Donaldson-Thomas invariants of a 3d Calabi-Yau category with a stability condition. A family of examples of such categories can be constructed from an arbitrary cluster variety. The corresponding Donaldson-Thomas invariants are encoded by a special formal automorphism of the cluster variety, known as Donaldson-Thomas transformation.

Fix two integers $m$ and $n$ with $1 < m < m+1 < n$. It is known that the configuration space $\text{Conf}_n(\mathbb{P}^{m-1})$, closely related to Grassmannian $\text{Gr}_m(n)$, is a cluster Poisson variety. In this paper we determine the Donaldson-Thomas transformation of $\text{Conf}_n(\mathbb{P}^{m-1})$ as an explicitly defined birational automorphism of $\text{Conf}_n(\mathbb{P}^{m-1})$. Its variant acts on the Grassmannian by a birational automorphism.

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1. Introduction

Donaldson-Thomas invariants were first introduced by Donaldson and Thomas [DT98] as geometric invariants on a Calabi-Yau threefold. Kontsevich and Soibelman [KS08] generalized the notion of Donaldson-Thomas invariants to 3d Calabi-Yau categories with a stability condition. Cluster varieties produce an important family of such 3d Calabi-Yau categories, and the data of Donaldson-Thomas invariants in this setting is encoded by a special formal automorphism of the cluster variety, called the Donaldson-Thomas transformation. Keller [Kel13] gave a combinatorial characterization of a subclass of Donaldson-Thomas transformations using quiver mutations. Goncharov and Shen [GS16] gave an equivalent definition using tropical points of the cluster varieties; Donaldson-Thomas transformations of this particular form are called cluster Donaldson-Thomas transformations. The main result of our paper claims that the Donaldson-Thomas
transformation of the cluster Poisson variety \( \text{Conf}_n(\mathbb{P}^{m-1}) \) is a cluster Donaldson-Thomas transformation, which we define explicitly as a birational automorphism on the space \( \text{Conf}_n(\mathbb{P}^{m-1}) \).

1.1. Main Result. Fix two integers \( m \) and \( n \) with \( 1 < m < m+1 < n \). The configuration space of \( n \) points on \( \mathbb{P}^{m-1} \) is defined as

\[
\text{Conf}_n(\mathbb{P}^{m-1}) := \text{PGL}_m \setminus (\mathbb{P}^{m-1})^n,
\]

where \( \text{PGL}_m \) acts diagonally on the product of projective spaces. Since we have quotient out the projective linear group action, configuration spaces of points in isomorphic projective spaces are canonically isomorphic. This observation is important for our definition of the transformation \( \text{DT} \) of \( \text{Conf}_n(\mathbb{P}^{m-1}) \) below.

Let \([l_1, \ldots, l_n] \) be a generic point in \( \text{Conf}_n(\mathbb{P}^{m-1}) \). Any consecutive \( m-1 \) lines \( l_i, \ldots, l_{i+m-2} \) span a hyperplane in \( \mathbb{C}^m \) and uniquely define a dual line in \( (\mathbb{C}^m)^\vee \); we will denote this dual line by \( h_{[i,i+m-2]} \). Since \( \text{Conf}_n(\mathbb{P}^{m-1}) \) and \( \text{Conf}_n((\mathbb{P}^{m-1})^\vee) \) are canonically isomorphic, we can use it to define a rational map

\[
\text{DT} : \text{Conf}_n(\mathbb{P}^{m-1}) \rightarrow \text{Conf}_n(\mathbb{P}^{m-1})
\]

\[
[l_1, \ldots, l_n] \mapsto [h_{[2-m,n]}; h_{[3-m,1]}, \ldots, h_{[1-m,n-1]}].
\]

Our main result of this paper is the following.

**Theorem 1.1.** The rational map \( \text{DT} \) is the cluster Donaldson-Thomas transformation of \( \text{Conf}_n(\mathbb{P}^{m-1}) \).

1.2. Applications. There are several applications of our main result, suggested by Goncharov.

The first application is a short proof of Keller’s reformulated version of Zamolodchikov’s periodicity conjecture [Kel12].

The original form of the periodicity conjecture states that for a pair of Dynkin diagrams \( \Delta \) and \( \Delta' \) with vertex sets \( I \) and \( I' \) and incidence matrices \( a_{ij} \) and \( a'_{ij} \), respectively, the system

\[
Y_{i,i',t-1}Y_{i,i',t+1} = \frac{\prod_{j \in I}(1 + Y_{j,j',t})^{a_{ij}}}{\prod_{j' \in I'}(1 + Y_{i,j',t})^{a'_{ij'}}}
\]

is of period dividing \( 2(h + h') \) in \( t \), where \( h \) and \( h' \) are the Coxeter numbers of \( \Delta \) and \( \Delta' \).

The periodicity conjecture has been proved by many people using various methods (c.f. [Kel12]): Frenkel and Szenes [FS95], and independently by Giozzi and Tateo [GT96] for the type \((A_n, A_1)\); Fomin and Zelevinsky for the type \((\Delta, A_1)\); independently by Volkov [Vol07], Szenes [Sze09], and Henriques [Hen07] for the type \((A_n, A_m)\). In [Kel12], Keller reformulated the periodicity conjecture and provided a proof for the general case \((\Delta, \Delta')\) using categorification of cluster algebras and Donaldson-Thomas invariants. Keller’s work links the the periodicity conjecture to cluster theories, and reduces the periodicity conjecture to showing that the order of Donaldson-Thomas transformation divides into \( 2(h + h') \).

As we will see later in our paper, one of the quivers associated to the cluster Poisson variety \( \text{Conf}_n(\mathbb{P}^{m-1}) \) is of the type \( A_{n-m-1} \boxtimes A_{m-1} \) (Example 2.11). According to Keller ([Kel12], Lemma 3.7), the periodicity conjecture for type \((A_{n-m-1}, A_{m-1})\) can be reduced to showing the following.

**Theorem 1.2.** The cluster Donaldson-Thomas transformation \( \text{DT} \) on \( \text{Conf}_n(\mathbb{P}^{m-1}) \) is of a period dividing \( 2n \).

**Proof.** It suffices to show that \( \text{DT}^{2n}[l_1, \ldots, l_n] = [l_1, \ldots, l_n] \) for a generic element \([l_1, \ldots, l_n] \). From the explicit formula of \( \text{DT} \) before Theorem 1.1 we see that \( \text{DT}^2 : [l_1, \ldots, l_n] \mapsto [l_{1-m}, l_{2-m}, \ldots, l_{n-m}] \), which is just a cyclic shift by \( m \). Therefore \( \text{DT}^{2n}[l_1, \ldots, l_n] = [l_1, \ldots, l_n] \). \( \square \)

The second application of the cluster Donaldson-Thomas transformation of \( \text{Conf}_n(\mathbb{P}^{m-1}) \) is a proof of Fock and Goncharov’s duality conjecture in the case of Grassmannian. It is an immediate corollary of our main result combining with Theorem 0.10 of [GJKK14].

Finally, the geometric realization of the cluster Donaldson-Thomas transformation closely resembles what is known as the parity conjugation for scattering amplitude in physics (see Appendix of [GGSVV13]).
1.3. A Variant of the Donaldson-Thomas Transformation for the Grassmannian. The configuration space Conf\(_n(\mathbb{P}^{m-1})\) is closely related to the Grassmannian. Indeed, consider an \(n\)-dimensional complex vector space \(V\) with a given basis \(\{e_i\}\). Let Gr\(_{n-m}(V)\) be the Zariski open subset of the Grassmannian Gr\(_{n-m}(V)\) consisting of \((n-m)\)-dimensional subspaces transverse to all coordinate hyperplanes spanned by basis vectors \(e_i\). Then there is a natural map from Gr\(_{n-m}(V)\) to the configuration space Conf\(_n(\mathbb{C}^m - \{0\}) := \text{GL}_m \backslash (\mathbb{C}^m - \{0\})^n\) given by

\[
H \mapsto (e_1/H, \ldots, e_n/H).
\]

Note that the right hand side naturally sits inside Conf\(_n((V/H) - \{0\})\), which is canonically isomorphic to Conf\(_n(\mathbb{C}^m - \{0\})\). This map is in fact an isomorphism onto its image, which is a Zariski open subset Conf\(_n(\mathbb{C}^m - \{0\})\) consisting of elements satisfying the totally generic condition (see Condition 2.5). Thus the Grassmannian Gr\(_{n-m}(V)\) is canonically birationally equivalent to the configuration space Conf\(_n(\mathbb{C}^m - \{0\})\).

The split algebraic torus \((\mathbb{C}^*)^n\) acts on both sides of the isomorphism Gr\(_{n-m}(V) \rightarrow \text{Conf}_n(\mathbb{C}^m - \{0\})\) via \((\lambda_i) : e_i \mapsto \lambda_i e_i\), and it follows that this isomorphism is a \((\mathbb{C}^*)^n\)-equivariant map. If we quotient out the torus action on the right, we obtain a Zariski open subset of Conf\(_n(\mathbb{P}^{m-1})\), which we denote as Conf\(_n^*(\mathbb{P}^{m-1})\). There is a natural lift of the Donaldson-Thomas transformation DT of Conf\(_n(\mathbb{P}^{m-1})\) to Conf\(_n(\mathbb{C}^m - \{0\})\) given by

\[
\text{DT} : [v_1, \ldots, v_n] \mapsto [\xi_1, \xi_2, \ldots, \xi_{m+1}]
\]

where each \(\xi_i\), expressed using a Hodge star operator \(*\) on \(\mathbb{C}^m\), is defined (up to a sign) as

\[
\xi_i := * \left( \bigwedge_{j \in [n-i+2, m-i]} v_j \right)
\]

(see (3.3) and (3.6) for details). Since Gr\(_{n-m}(V)\) is canonically birationally equivalent to Conf\(_n(\mathbb{C}^m - \{0\})\), we can view this lifted map DT as the Donaldson-Thomas transformation of the Grassmannian Gr\(_{n-m}(V)\).

1.4. Structure of the Paper. Section 2 contains basic definitions, constructions and propositions necessary for the development of this paper: Subsection 2.1 discusses properties of the configuration spaces Conf\(_n(\mathbb{C} - \{0\})\) and Conf\(_n(\mathbb{P}^{m-1})\); Subsection 2.2 defines minimal bipartite graph and its related notions, which is a slight modification of Postnikov’s bipartite graph used in [Pos06]; Subsection 2.3 and 2.4 are brief reviews of Fock and Goncharov’s cluster varieties and its related tropicalization, introduced in [FG03]; Subsection 2.4 will also contain the definition of cluster Donaldson-Thomas transformation given by Goncharov and Shen in [GS16]; Subsection 2.5 recalls Postnikov’s perfect orientations and boundary measurement maps in [Pos06], and focuses on a special case which we call “special perfect orientation”.

Section 3 focuses on the involution \(*\), which resembles the map \(*\) introduced by Goncharov and Shen in [GS16] and is closely related to the cluster Donaldson-Thomas transformation. For self-containedness purpose we will also include a proof of the birationally equivalence between Conf\(_n(\mathbb{P}^{m-1})\) and a cluster Poisson variety using the involutive property of \(*\). Near the end of Section 3 we will give a nice coordinate expression of the induced map \(*\) on one particular seed \(X\)-torus.

Section 4 presents a detailed proof of our main theorem, which utilizes all the tools we have developed throughout the paper.

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2. Preliminaries

2.1. Configuration Space. We will discuss some basic properties of the configuration spaces in this subsection.

Definition 2.1. Let $G$ be a group and let $X$ be a $G$-set (a set with a left $G$-action). The configuration space of $n$ points on $X$ is the quotient

$$\text{Conf}_n(X) := G \backslash X^n$$

where $G$ acts diagonally on $X^n$.

In this paper we will be mainly focusing on two configuration spaces $\text{Conf}_n(\mathbb{C}^m - \{0\}) := \text{GL}_m / (\mathbb{C}^m - \{0\})^n$ and $\text{Conf}_n(\mathbb{P}^{m-1}) := \text{PGL}_m / (\mathbb{P}^{m-1})^n$.

Remark 2.2. Strictly speaking, these configuration spaces are stacks rather than algebraic varieties. Since the maps we define in this paper are all going to be defined on the Zariski subsets satisfying the consecutively generic condition as stated below, which are algebraic varieties, for the rest of this paper we will always impose the consecutively generic condition on the configuration spaces $\text{Conf}_n(\mathbb{C}^m - \{0\})$ and $\text{Conf}_n(\mathbb{P}^{m-1})$ without further mentioning.

Condition 2.3. An element $[v_1, \ldots, v_n]$ (respectively $[l_1, \ldots, l_n]$) of configuration space $\text{Conf}_n(\mathbb{C}^m - \{0\})$ (respectively $\text{Conf}_n(\mathbb{P}^{m-1})$) is said to satisfy the consecutively generic condition if $v_1, \ldots, v_{i+m-1}$ (respectively $l_1, \ldots, l_{i+m-1}$) are in generic position for all $i$.

There is a natural $(\mathbb{C}^*)^{n-1}$-fiber bundle map

$$\text{Conf}_n(\mathbb{C}^m - \{0\}) \to \text{Conf}_n(\mathbb{P}^{m-1})$$

induced by the projectivization map $\mathbb{C}^m - \{0\} \to \mathbb{P}^{m-1}$.

There is also a cyclic symmetry present in the configuration spaces $\text{Conf}_n(\mathbb{C}^m - \{0\})$ and $\text{Conf}_n(\mathbb{P}^{m-1})$; if a point $[v_1, \ldots, v_n]$ (resp. $[l_1, \ldots, l_n]$) is in $\text{Conf}_n(\mathbb{C}^m - \{0\})$ (resp. $\text{Conf}_n(\mathbb{P}^{m-1})$), then so is the point $[v_2, \ldots, v_n, v_1]$ (resp. $[l_2, \ldots, l_{n-1}, l_1]$). Using cyclic symmetry can reduce a proof of a statement about any consecutive $m$ entries to the statement about the first $m$ consecutive entries.

Let $\det$ be the standard volume form on $\mathbb{C}^m$. For any $m$-element subset $I$ of $\{1, \ldots, n\}$ we define the corresponding Plücker coordinate on $(\mathbb{C}^m - \{0\})^n$ as

$$\Delta_I(v_1, \ldots, v_n) = \det \left( \bigwedge_{i \in I} v_i \right),$$

where the notation $\bigwedge$ indicates that we take the wedge product with ascending indices. After quotienting out the $\text{GL}_m$-action, the Plücker coordinates become “projective functions” on $\text{Conf}_n(\mathbb{C}^m - \{0\})$: they are only well-defined up to a global non-zero multiplicative constant. Note that ratios between Plücker coordinates are still well-defined rational functions on $\text{Conf}_n(\mathbb{C}^m)$.

If we would like to make Plücker coordinates more well-defined, we can view them as regular functions on the auxiliary space

$$\widetilde{\text{Conf}}_n(\mathbb{C}^m) := \text{SL}_m / (\mathbb{C}^m - \{0\})^n$$

(with consecutively generic condition imposed). We will call $\widetilde{\text{Conf}}_n(\mathbb{C}^m - \{0\})$ the affine cylinder over $\text{Conf}_n(\mathbb{C}^m - \{0\})$. There is a $\mathbb{C}^*$-fiber bundle map $\widetilde{\text{Conf}}_n(\mathbb{C}^m - \{0\}) \to \text{Conf}_n(\mathbb{C}^m - \{0\})$.

One important relation among the Plücker coordinates is the 3-term Plücker relation, which says that for any $(m-2)$-element subset $J \subset \{1, \ldots, n\}$ and any four elements $i < j < k < l$ from $J^c$, the following equation always holds:

\begin{equation}
\Delta_{J \cup \{i, k\}} \Delta_{J \cup \{j, l\}} = \Delta_{J \cup \{i, j\}} \Delta_{J \cup \{k, l\}} + \Delta_{J \cup \{i, l\}} \Delta_{J \cup \{j, k\}}.
\end{equation}

This relation can be easily remembered by the following square: the product of the two diagonals equals the sum of two products of opposite sides.
To help us carry out proofs and computations, we often need to restrict ourselves to Zariski open subsets of configuration spaces consisting of elements that satisfy the totally generic condition as stated below, and we will denote them by $\text{Conf}_n^\circ(\mathbb{C}^m - \{0\})$ and $\text{Conf}_n^\circ(\mathbb{P}^{m-1})$.

**Condition 2.5.** An element $[v_1, \ldots, v_n]$ (respectively $[l_1, \ldots, l_n]$) of configuration space $\text{Conf}_n(\mathbb{C}^m - \{0\})$ (respectively $\text{Conf}_n(\mathbb{P}^{m-1})$) is said to satisfy the **totally generic condition** if for any $m$-element subset $I$ of $\{1, \ldots, n\}$, elements of the subset $\{v_i \mid i \in I\}$ (respectively $\{l_i \mid i \in I\}$) are in generic position.

**2.2. Minimal Bipartite Graph.** Bipartite graphs on a disk were used by Postnikov in the study of the stratification of the non-negative Grassmannian in [Pos06]. We will use some modified versions of their ideas, such as perfect orientation and boundary measurement map, as tools to prove our statements in this paper, and thus we think it would be helpful to include a subsection introducing the notion of minimal bipartite graphs for future usage. Similar bipartite graph methods were also used by Goncharov in the study of moduli spaces of $G$-local systems on surfaces [Gon12].

Let $D$ be the closed unit disk on $\mathbb{C}$ with $n$ marked points at the $n$th roots of unity labeled $1, \ldots, n$ clockwise on the boundary. For any bipartite graph $\Gamma$ drawn on $D$ with exactly one edge connecting to each marked point on the boundary, we can draw the so-called **zig-zag strands** on $D$ going from one marked point on the boundary to another, following the rules below.

1. At each marked point on the boundary, the leaving zig-zag strand succeeds the arriving zig-zag strand in the clockwise direction:

![Diagram](image)

2. Zig-zag strands travel next to edges in $\Gamma$.
3. Zig-zag strands turn right at each black vertex and turn left at each white vertex; as a result, zig-zag strands circulate in the counterclockwise direction around a black vertex and circulate in the clockwise direction around a white vertex.

**Definition 2.6.** A **minimal bipartite graph** (of rank $m$) is a bipartite graph $\Gamma$ on $D$ with exactly one edge connecting to each boundary marked point, satisfying the following conditions.

1. Each of the zig-zag strands goes from a boundary marked point $i$ to $i + m$.
2. No zig-zag strand intersects itself in the interior of $D$.
3. No two zig-zag strands form a parallel bigon.
4. Black vertices are trivalent.
5. Each edge connected to a marked point on the boundary is connecting to a white vertex at the other end; we call such an edge an **external edge** and the white vertex to which an external edge connects a **boundary white vertex**.


Remark 2.7. Among the conditions above, (4) and (5) are only for auxiliary purpose to make our proof smoother; our results still hold for minimal bipartite graphs with conditions (4) and (5) dropped.

If we draw the zig-zag strands of a minimal bipartite graph $\Gamma$ on $\mathbb{D}$, tighten up the three strands surrounding a black vertex to a triple intersection as below,

and then forget $\Gamma$, what we get is an example of Thurston’s minimal triple diagram associated to the pairing $i \mapsto i + m$, introduced in his paper [Thu04]. In particular, Thurston’s results on minimal triple diagrams can be turned into the following theorem on minimal bipartite graphs.

Theorem 2.8 (Thurston). Minimal bipartite graphs exist for any fixed parameters $m$ and $n$. Any two minimal bipartite graphs with the same parameters $m$ and $n$ can be transformed into one another by a finite sequence of the following two types of $2 \leftrightarrow 2$ moves.

**Type I:**

**Type II:**

Part of Thurston’s theorem guarantees the existence of minimal bipartite graphs, but we also would like to give an explicit construction for our paper. The following is a procedure to construct one for fixed parameters $m$ and $n$ (with $1 < m < m + 1 < n$).

(1) Draw this vertical pattern with $n - m$ black vertices.

(2) Put $m - 1$ such vertical patterns side by side.
(3) Add in a boundary white vertex with edges connecting to all left most black vertices.

(4) Deform the graph by isotopy so that the external edges match up with the marked points on the boundary of $\mathbb{D}$.

We give this minimal bipartite graph the special name $\Gamma_0$. The minimal bipartite graph $\Gamma_0$ is important to many of our proofs: the general philosophy is that, whenever we want to show some property that is shared by all minimal bipartite graphs, we first show it on $\Gamma_0$, and then verify that this property is invariant under 2 ↔ 2 moves, and conclude the proof by invoking Thurston’s theorem. The following are two examples of such statements.

**Proposition 2.9.** Any minimal bipartite graph is connected.

*Proof.* This statement is obviously true for $\Gamma_0$. Notice that a 2 ↔ 2 move does not disconnect a graph. Therefore the statement is true for all minimal bipartite graphs. □

**Proposition 2.10.** In a minimal bipartite graph $\Gamma$, there is a one-to-one correspondence between marked points on the boundary of $\mathbb{D}$ and boundary white vertices.

*Proof.* This statement is obviously true for $\Gamma_0$. Notice that a 2 ↔ 2 move is a local move that never changes the boundary white vertices. Therefore this statement is true for all minimal bipartite graphs. □

Next we would like to present the canonical way of producing quivers from minimal bipartite graphs. To start, we need to define what a face of a minimal bipartite graph $\Gamma$ is: a *face* is just a connected component of $\mathbb{D} - \Gamma$ as we all have imagined; in particular if a face is not totally enclosed by edges of $\Gamma$, we call it a *boundary face*. To each minimal bipartite graph $\Gamma$ we can construct two quivers $\tilde{\mathbf{i}}$ and $\mathbf{i}$; let’s first describe the construction of $\mathbf{i}$.

1. Put a vertex for each face of $\Gamma$.
2. Put a counterclockwisely oriented 3-cycle for the faces next to a black vertex as below.

(3) Remove any 2-cycles in the quiver.
After constructing $\tilde{i}$, the quiver $i$ is then obtained from $\tilde{i}$ by removing all the vertices corresponding to boundary faces and all the arrows involving those vertices. Because $i$ does not have any vertex corresponding to boundary faces, we also call it the boundary-removed quiver associated to $\Gamma$.

**Example 2.11.** Starting from the special minimal bipartite graph $\Gamma_0$ as described above, we obtain the following corresponding quivers, and by convention we will denote them as $\tilde{i}_0$ and $i_0$ respectively. One can see right away that $i_0$ is indeed of type $A_{n-m-1} \boxplus A_{m-1}$, as claimed in our short proof of the corresponding case of the periodicity conjecture (Theorem 1.2).

If we draw the associated quivers for two minimal bipartite graphs which differ by a $2 \leftrightarrow 2$ move, we see that a type I $2 \leftrightarrow 2$ move is the same as a quiver mutation at the vertex corresponding to the center face, whereas a type II $2 \leftrightarrow 2$ move does not change the associated quiver at all. Relating this observation back to Thurston’s theorem, we obtain the following corollary of Thurston’s theorem.

**Corollary 2.12 (Thurston).** If $i$ and $i'$ (resp. $\tilde{i}$ and $\tilde{i}'$) are the quivers associated to two minimal bipartite graphs $\Gamma$ and $\Gamma'$ with the same parameters $m$ and $n$, then $i'$ (resp. $\tilde{i}'$) can be obtained from $i$ (resp. $\tilde{i}$) by a sequence of quiver mutations corresponding to the sequence of $2 \leftrightarrow 2$ moves which turns $\Gamma$ into $\Gamma'$ given by Thurston’s theorem.

Another construction of minimal bipartite graphs that is important to us is what we call duality.
Definition 2.13. Given a minimal bipartite graph $\Gamma$ we define its dual to be the minimal bipartite graph $\Gamma^\circ$ obtained by reflecting $\Gamma$ over the diameter of $\mathbb{D}$ bisecting either arc between 1 and $n$.

Remark 2.14. In this paper there are two distinct usage of the notation $\circ$: one is to denote Zariski open subsets of configuration spaces that satisfy the totally generic condition, and the other is to denote notions that are associated to dual minimal bipartite graphs. We apologize for any confusion that may be caused by this overusage.

Remark 2.15. Note that for two minimal bipartite graphs $\Gamma$ and $\Gamma^\circ$ dual to each other, there is a canonical one-to-one correspondence between faces of $\Gamma$ and $\Gamma^\circ$. If we denote the corresponding faces by $f$ and $f^\circ$ respectively, then it is not hard to see that in the associated quivers ($\tilde{i}$ or $i$), the exchange matrices are related by

$$\epsilon_{fg} = -\epsilon_{f^\circ g^\circ}.$$

A pictorial way of saying this is that the associated quiver of $\Gamma^\circ$ is obtained from that of $\Gamma$ by a reflection over the diameter of $\mathbb{D}$ bisecting the arc between 1 and $n$ plus reversing all the arrows.

Remark 2.16. Above we have mentioned a general way of proving statements about minimal bipartite graphs by using $\Gamma_0$ together with Thurston’s Theorem. Sometimes instead of verifying a property on $\Gamma_0$, it may be easier to verify it on $\Gamma^\circ_0$. After using whichever way that works better, we can still apply Thurston’s theorem to conclude the proof for all minimal bipartite graphs as well.

The last part of our introduction to minimal bipartite graphs is on the notion of dominating sets.

Definition 2.17. Let $\Gamma$ be a minimal bipartite graph. We denote the zig-zag strand going from $i - m$ to $i$ by $\zeta^i_{i-m}$. We say that a face $f$ of $\Gamma$ is dominated by the zig-zag strand $\zeta^i_{i-m}$ if $f$ lies on the left of $\zeta^i_{i-m}$, and we define the dominating set $I(f)$ to be the collection of upper indices $i$ for which $\zeta^i_{i-m}$ dominate $f$.

Proposition 2.18. Let $\Gamma$ be a minimal bipartite graph. If we denote the unshaded face next to the boundary marked point $i$ in the counterclockwise direction as $f_i$, then $I(f_i) = [i, i + m]$, where the notation $[i,j]$ is defined to be

$$[i,j] := \begin{cases} 
\{i,i+1,\ldots,j\} & \text{if } i < j; \\
\{i,i+1,\ldots,n,1,\ldots,j\} & \text{if } i > j.
\end{cases}$$

Proof. This follows from the definition of dominating sets. \qed

Proposition 2.19. Let $\Gamma$ be a minimal bipartite graph, then the dominating set of each face is of size $m$.

Proof. Note that we always cross two opposite strands to go from one face to the adjacent face across an edge; therefore all dominating sets are of the same size, which is $m$ by looking at the boundary faces. \qed

Proposition 2.20. Let $\tilde{i}$ be the quiver associated to a minimal bipartite graph $\Gamma$. If $f$ is not a boundary face of $\Gamma$, then the multiplicity of any element $i \in \{1,\ldots,n\}$ occurring in all $I(g)$ as $g$ runs over all faces pointing to $f$ in $\tilde{i}$ is equal to the multiplicity of any element $i \in \{1,\ldots,n\}$ occurring in all $I(h)$ as $h$ runs over all faces being pointed to by $f$ in $\tilde{i}$.

Proof. Just verify it carefully with pictures. \qed
Proposition 2.21. A Type II $2 \leftrightarrow 2$ move does not change the dominating set of any face, and a Type I $2 \leftrightarrow 2$ move changes the dominating sets only locally, and the change is as follows, where $J$ is a $(m-2)$-element subset of $\{1, \ldots, n\} - \{i, j, k, l\}$.

![Diagram](image)

Proof. Just verify it carefully with pictures. 

If we compare the local change of dominating sets under a Type I $2 \leftrightarrow 2$ move with the indices of the 3-term Plücker relation (2.4), we see that they actually match up! This is one of the keys to relate the configuration space $\text{Conf}_n(\mathbb{C}^m - \{0\})$ to the cluster variety $\mathcal{A}_{[i_0]}$.

2.3. Cluster Variety. We will give a brief review of Fock and Goncharov’s theory of cluster ensemble, and relate it to our geometric example, namely the configuration spaces $\text{Conf}_n(\mathbb{C}^m - \{0\})$ and $\text{Conf}_n(\mathbb{P}^{m-1})$. We will mainly follow the coordinate description presented in [FG03] with skewsymmetric exchange matrix and no frozen vertices.

Definition 2.22. A seed $i$ is an ordered pair $(I, \epsilon)$ where $I$ is a finite set and $\epsilon$ is a skewsymmetric matrix whose rows and columns are indexed by $I$.

The data of a seed as defined above is equivalent to the data of a quiver with vertex set $I$ and exchange matrix $\epsilon_{ij}$. We will make no distinctions between seeds and quivers in this paper, and hence we always use the seed notation to denote any quiver.

The notion of quiver mutation can be described by a precise formula in the seed language: for an element $k \in I$, the seed mutation $\mu_k$ gives a new seed $(I', \epsilon')$ where $I' = I$ and

$$
\epsilon'_{ij} = \begin{cases} 
-\epsilon_{ij} & \text{if } k \in \{i, j\}; \\
\epsilon_{ij} & \text{if } \epsilon_{ik}\epsilon_{kj} \leq 0 \text{ and } k \notin \{i, j\}; \\
\epsilon_{ij} + |\epsilon_{ik}\epsilon_{kj}| & \text{if } \epsilon_{ik}\epsilon_{kj} > 0, \ k \notin \{i, j\}.
\end{cases}
$$

Starting with an initial seed $i_0$, we say that a seed $i$ is mutation equivalent to $i_0$ if there is a sequence of seed mutations that turns $i_0$ into $i$; we denote the set of all seeds mutation equivalent to $i_0$ by $[i_0]$. To each seed $i$ in $[i_0]$ we associate two split algebraic tori $\mathcal{A}_i = (\mathbb{C}^*)^{|I|}$ and $\mathcal{X}_i = (\mathbb{C}^*)^{|I|}$, which are equipped with canonical coordinates $(A_i)$ and $(X_i)$ indexed by the set $I$ respectively. These two split algebraic tori are linked by a map $p_i : \mathcal{A}_i \to \mathcal{X}_i$ given by

$$
p_i(X_i) = \prod_{j \in I} A_j^{\epsilon_{ij}}.
$$

The split algebraic tori $\mathcal{A}_i$ and $\mathcal{X}_i$ are called a seed $\mathcal{A}$-torus and a seed $\mathcal{X}$-torus respectively.

A seed mutation $\mu_k : i \to i'$ gives rise to birational equivalences between the corresponding seed $\mathcal{A}$-tori and the corresponding seed $\mathcal{X}$-tori respectively, both of which by an abuse of notation we also denote as $\mu_k$; in terms of the canonical coordinates $(A_{i'}^k)$ and $(X_{i'}^k)$ they can be expressed as

$$
\mu_k^k(A_i) = \begin{cases} 
A_{i}^{-1} \left( \prod_{\epsilon_{kj} > 0} A_j^{\epsilon_{kj}} + \prod_{\epsilon_{kj} < 0} A_j^{-\epsilon_{kj}} \right) & \text{if } i = k, \\
A_i & \text{if } i \neq k,
\end{cases}
$$

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and \( \mu_k^*(X'_i) = \left\{ \begin{array}{ll} X^{-1}_k & \text{if } i = k, \\ X_i \left( 1 + X_k^{-\text{sign}(e_{ik})} \right)^{-e_{ik}} & \text{if } i \neq k. \end{array} \right. \)

These two birational equivalences are called \textit{cluster A-mutation} and \textit{cluster X-mutation} respectively. One important feature about cluster mutations is that they commute with the respective \( p \) maps.

\[
\begin{array}{ccc}
A_i & -\mu_i^* & A_i' \\
\sigma & & \sigma' \\
\mu_i & & \mu_i' \\
A_i' & = & A_i
\end{array}
\]

Besides cluster mutations between seed tori we also care about cluster isomorphisms induced by seed (quiver) isomorphisms. A \textit{seed isomorphism} \( \sigma : i \rightarrow i' \) is a bijection \( \sigma : I \rightarrow I' \) such that \( e'_{\sigma(i)\sigma(j)} = e_{ij} \).

Given a seed isomorphism \( \sigma : i \rightarrow i' \) between two seeds in \( \mathcal{I}_0 \), we obtain isomorphisms on the corresponding seed tori, which by an abuse of notation we also denote by \( \sigma \), defined by

\[
\sigma^*(A_{\sigma(i)}) = A_i \quad \text{and} \quad \sigma^*(X_{\sigma(i)}) = X_i.
\]

We call these isomorphisms \textit{cluster isomorphisms}. It is not hard to see that cluster isomorphisms also commute with the \( p \) maps.

\[
\begin{array}{ccc}
A_i & \xrightarrow{\sigma} & A_i' \\
p_i & & p_i' \\
A_i' & \xrightarrow{\sigma} & A_i
\end{array}
\]

Compositions of seed mutations and seed isomorphisms are called \textit{seed cluster transformations}, and compositions of cluster mutations and cluster isomorphisms are called \textit{cluster transformations}. A seed cluster transformation transformation \( i \rightarrow i' \) is called \textit{trivial} if it induces identity maps on the corresponding seed \( A \)-torus \( A_i \) and seed \( X \)-torus \( X_i \).

By gluing the seed tori with cluster mutations we obtain the corresponding \textit{cluster varieties}, which will be denoted as \( A_{\mathcal{I}_0} \) and \( X_{\mathcal{I}_0} \) respectively. Then cluster transformations can be seen as automorphisms on these cluster varieties. In particular, the cluster variety \( X_{\mathcal{I}_0} \) carries a natural Poisson variety structure given by

\[
\{X_i, X_j\} = \epsilon_{ij} X_i X_j.
\]

Thus a cluster variety \( X_{\mathcal{I}_0} \) is also known as a \textit{cluster Poisson variety}.

Due to the positivity of the mutation formulas, we can tropicalize these cluster varieties, which we will describe in the next subsection.

Since the maps \( p_i \) commute with cluster mutations, they naturally glue into a map \( p : A_{\mathcal{I}_0} \rightarrow X_{\mathcal{I}_0} \) of cluster varieties.

For the rest of this subsection we will link the cluster varieties \( A_{\mathcal{I}_0} \) and \( X_{\mathcal{I}_0} \) associated to the initial seeds \( \mathcal{I}_0 \) and \( \mathcal{I}_0 \) given in last subsection to the configuration spaces \( \text{Conf}_m(C^m - \{0\}) \) and \( \text{Conf}_n(\mathbb{P}^{m-1}) \) respectively.

Consider any minimal bipartite graph \( \Gamma \) with fixed parameters \( m \) and \( n \) satisfying \( 1 < m < m + 1 < n \). We know from Proposition 2.19 that the size of dominating sets \( I(f) \) is always \( m \) for every face \( f \) of \( \Gamma \).

Therefore to each face \( f \) of \( \Gamma \) we can define a rational function \( A_f : \text{Conf}_m(C^m - \{0\}) \rightarrow \mathbb{C}^* \) by

\[
A_f[v_1, \ldots, v_n] := \Delta_{I(f)}[v_1, \ldots, v_n].
\]

These rational functions \( A_f \) then give rise to a rational map

\[
\tilde{\psi}_f : \text{Conf}_m(C^m - \{0\}) \rightarrow A_i.
\]

Let’s investigate how \( \tilde{\psi}_f \) and \( \tilde{\psi}_{f'} \) relate to each other for two minimal bipartite graphs \( \Gamma \) and \( \Gamma' \) with the same parameters \( m \) and \( n \). On the one hand, Corollary 2.12 tells us that \( \Gamma \) and \( \Gamma' \) can be related by a sequence of \( 2 \leftrightarrow 2 \) moves, which induces a sequence of mutations on the corresponding quivers \( i \) and \( i' \) at non-boundary faces. On the other hand, we know from Proposition 2.21 that a Type II \( 2 \leftrightarrow 2 \) move does not change the dominating set of any face, and a Type I \( 2 \leftrightarrow 2 \) move changes the dominating sets locally within the indices of the 3-term Plücker relation; in fact if we compare the 3-term Plücker relation (2.4) with the
cluster $A$-mutation formula induced by a Type I 2 $\leftrightarrow$ 2 move, we see that they are exactly the same! This proves the following proposition.

**Proposition 2.23.** The maps $\tilde{\psi}_i : \widetilde{Conf}_n(C^m - \{0\}) \rightarrow A_i$ glue into a rational map $\tilde{\psi} : \widetilde{Conf}_n(C^m - \{0\}) \rightarrow \widetilde{A}_i$.

Next let’s put in the $\chi$-picture. Let $\tilde{p} : A_i[i_0] \rightarrow X_i[i_0]$ be the $p$ map between cluster varieties. By comparing $i$ and $i$ we see that there is a projection $q_i : X_i \rightarrow X_i$ which projects the split algebraic torus $X_i$ to its non-boundary face factors. It is not hard to see that this projection map also commutes with any cluster $\chi$-mutation $\mu_k$ taking place at a vertex $k$ of $i$. Thus the maps $q_i$ glue into a map of cluster varieties $q : X_i[i_0] \rightarrow X_i[i_0]$.

Consider the composition $\widetilde{Conf}_n(C^m - \{0\}) \xrightarrow{\tilde{\psi}} A_i[i_0] \xrightarrow{\tilde{p}} X_i[i_0] \xrightarrow{q} X_i[i_0]$. Proposition 2.20 tells us that this composition is constant along the fibers of $\widetilde{Conf}_n(C^m - \{0\}) \rightarrow \widetilde{Conf}_n(P^{m-1})$. Therefore we can pass $q \circ \tilde{p} \circ \tilde{\psi}$ to a rational map $\psi : \widetilde{Conf}_n(P^{m-1}) \rightarrow X_i[i_0]$, which fits into the following commutative diagram.

\[
\begin{array}{ccc}
\widetilde{Conf}_n(C^m - \{0\}) & \xrightarrow{\tilde{\psi}} & A_i[i_0] \\
\downarrow & & \downarrow \tilde{p} \\
\widetilde{Conf}_n(P^{m-1}) & \xrightarrow{\psi} & X_i[i_0] \\
\end{array}
\]

2.4. Tropicalization, Laminations, and Cluster Donaldson-Thomas Transformation. In this subsection we will discuss tropicalization of cluster varieties and the laminations on them. Near the end we will use laminations to give a definition of Goncharov and Shen’s cluster Donaldson-Thomas transformation.

Let’s start with a split algebraic torus $\mathcal{X}$. The semiring of positive rational functions on $\mathcal{X}$, which we denote as $P(\mathcal{X})$, is the semiring consisting of elements in the form $f/g$ where $f$ and $g$ are linear combinations of characters on $\mathcal{X}$ with positive integral coefficients. A rational map $\phi : \mathcal{X} \rightarrow \mathcal{Y}$ between two split algebraic tori is said to be positive if it induces a semiring homomorphism $\phi^* : P(\mathcal{Y}) \rightarrow P(\mathcal{X})$. It then follows that composition of positive rational maps is still a positive rational map.

One typical example of a positive rational map is a cocharacter $\chi$ of a split algebraic torus $\mathcal{X}$: the induced map $\chi^*$ pulls back an element $f/g \in P(\mathcal{X})$ to $\frac{f \phi(\chi)}{g \phi(\chi)}$ in $P(C^*)$, where $\langle f, \chi \rangle$ and $\langle g, \chi \rangle$ are understood as linear extensions of the canonical pairing between characters and cocharacters with values in powers of $z$. We will denote the lattice of cocharacters of a split algebraic torus $\mathcal{X}$ by $\mathcal{X}^t$ for reasons that will become clear in a moment.

Note that $P(C^*)$ is the semiring of rational functions in a single variable $z$ with positive integral coefficients. Thus if we let $\mathcal{Z}^t$ be the semiring $(\mathbb{Z}, \max, +)$, then there is a semiring homomorphism $\deg_z : P(C^*) \rightarrow \mathbb{Z}^t$ defined by $f(z)/g(z) \mapsto \deg_z f - \deg_z g$. Therefore a cocharacter $\chi$ on $\mathcal{X}$ gives rise to a natural semiring homomorphism

$$\deg_z(\cdot, \chi) : P(\mathcal{X}) \rightarrow \mathbb{Z}^t$$

**Proposition 2.24.** The map $\chi \mapsto \deg_z(\cdot, \chi)$ is a bijection between the lattice of cocharacters and set of semiring homomorphisms from $P(\mathcal{X})$ to $\mathbb{Z}^t$.

**Proof.** Note that $P(\mathcal{X})$ is a free commutative semiring generated by any basis of the lattice of characters, and in particular any choice of coordinates $\langle X_i, \chi \rangle_{i=1}^n$. Therefore to define a semiring homomorphism from $P(\mathcal{X})$ to $\mathbb{Z}^t$ we just need to assign to each $X_i$ some integer $a_i$. But for any collection of $(a_i)$ there exists a unique cocharacter $\chi$ such that $\langle X_i, \chi \rangle = z^{a_i}$. Therefore $\chi \mapsto \deg_z(\cdot, \chi)$ is indeed a bijection. \qed
Corollary 2.25. A positive rational map \( \phi : \mathcal{X} \to \mathcal{Y} \) between split algebraic tori gives rise to a natural map \( \phi^! : \mathcal{X}^t \to \mathcal{Y}^t \) between the respective lattice of cocharacters.

Proof. Note that \( \phi \) induces a semiring homomorphism \( \phi^* : P(\mathcal{Y}) \to P(\mathcal{X}) \). Therefore for any cocharacter \( \chi \) of \( \mathcal{X} \), the map \( f \mapsto \deg_z(\phi^* f, \chi) \) is a semiring homomorphism from \( P(\mathcal{Y}) \to \mathbb{Z}^t \). By the above proposition there is a unique cocharacter \( \eta \) of \( \mathcal{Y} \) representing this semiring homomorphism, and we assign \( \phi^!(\chi) = \eta \). \( \square \)

We may also want to give an explicit formula for the induced map \( \phi^! \). To do so, let’s fix two coordinate charts \((X_i)\) on \( \mathcal{X} \) and \((Y_j)\) on \( \mathcal{Y} \). Then \((X_i)\) gives rise to a basis \( \{e_i\} \) of the lattice of cocharacters \( \mathcal{X}^t \), which is defined by

\[
e_i^*(X_k) := \begin{cases} z & \text{if } k = i; \\ 1 & \text{if } k \neq i. \end{cases}
\]

This basis allows us to write each cocharacter \( \chi \) of \( \mathcal{X} \) as a linear combination \( \sum x_i e_i \). It is not hard to see that

\[
x_i = \deg_z(X_i, \chi).
\]

Similarly the coordinate chart \((Y_j)\) also gives rise to a basis \( \{f_j\} \) of the lattice of cocharacters \( \mathcal{Y}^t \), and we can write each cocharacter of \( \mathcal{Y} \) as a linear combination \( \sum y_j f_j \). On the other hand, for any positive rational function \( q \) in \( r \) variables \( X_1, \ldots, X_r \), we have the so-called naive tropicalization, which turns \( q \) into a map from \( \mathbb{Z}^r \) to \( \mathbb{Z} \) via the following process:

1. replace addition in \( q(X_i) \) by taking maximum;
2. replace multiplication in \( q(X_i) \) by addition and replace division in \( q(X_i) \) by subtraction;
3. replace every constant term by zero;
4. replace \( X_i \) by \( x_i \).

It is not hard to see that, given a positive rational map \( \phi : \mathcal{X} \to \mathcal{Y} \), the induced map \( \phi^! \) maps \( \sum x_i e_i \) to \( \sum y_j f_j \) where

\[
y_j := (\phi^*(Y_j))^!(x_i).
\]

From this proposition and its proof we get out many important constructions. First we define the tropicalization of a split algebraic torus \( \mathcal{X} \) to be its lattice of cocharacters \( \mathcal{X}^t \) (and hence the notation). Second given a positive rational map \( \phi : \mathcal{X} \to \mathcal{Y} \) between split algebraic tori, we define its tropicalization to be the map \( \phi^! : \mathcal{X}^t \to \mathcal{Y}^t \). Third given an isomorphism \( \mathcal{X} \cong (\mathbb{C}^*)^r \) of split algebraic tori, we not only obtain a coordinate system \((X_i)\) on \( \mathcal{X} \) but also a basis \( \{e_i\} \) of \( \mathcal{X}^t \), which we call the basic laminations associated to \((X_i)\).

Now we can go back to the cluster varieties \( \mathcal{A}_{[i_k]} \) and \( \mathcal{X}_{[i_k]} \). Since both cluster varieties are obtained by gluing seed tori via positive birational equivalences, we can tropicalize everything and obtain two new glued objects which we call tropicalized cluster varieties and denote as \( \mathcal{A}_{[i_k]}^t \) and \( \mathcal{X}_{[i_k]}^t \).

Since each seed \( \mathcal{X} \)-torus \( X_i \) is given a split algebraic torus, it has a set of basic laminations associated to the canonical coordinates \((X_i)\); we will call this set of basic laminations the positive basic \( \mathcal{X} \)-laminations and denote them as \( l^+_i \). Note that \( \{-l^+_i\} \) is also a set of basic laminations on \( X_i \); we will call them the negative basic \( \mathcal{X} \)-laminations and denote them as \( l^-_i \).

With all the terminologies developed, we can now state the definition of Goncharov and Shen’s cluster Donaldson-Thomas transformation as follows.

Definition 2.27 (Definition 2.15 in [GS16]). A cluster Donaldson-Thomas transformation (on a seed \( \mathcal{X} \)-torus \( X_i \)) is a cluster transformation \( DT : X_i \to X_i \) whose induced tropical cluster \( \mathcal{X} \)-transformation \( DT^! : \mathcal{X}_i^t \to \mathcal{X}_i^t \) maps each positive basic \( \mathcal{X} \)-laminations \( l^+_i \) to its corresponding negative basic \( \mathcal{X} \)-laminations \( l^-_i \).

Goncharov and Shen proved that a cluster Donaldson-Thomas transformation enjoys the following properties.

Theorem 2.28 (Goncharov-Shen, Theorem 2.16 in [GS16]). A cluster Donaldson-Thomas transformation \( DT : X_i \to X_i \) is unique if it exists. If \( Y \) is another seed in \([i]\) (the collection of seeds mutation equivalent to \( i \)) and \( \tau : X_{[i]} \to X_{[i]} \) is a cluster transformation, then the conjugate \( \tau DT\tau^{-1} \) is the cluster Donaldson-Thomas
transformation in $X_i$. Therefore it makes sense to say the cluster Donaldson-Thomas transformation $DT$ exists on a cluster $X$-variety without referring to any one particular seed $X$-torus.

From our discussion on tropicalization above, we can translate the definition of a cluster Donaldson-Thomas transformation into the following equivalent one, which we will use to prove the existence of cluster Donaldson-Thomas transformation on $Conf_n(P^{m-1})$ in Section 4.

**Proposition 2.29.** A cluster transformation $DT : X_i \to X_i$ is a cluster Donaldson-Thomas transformation if and only if
\[ \text{deg}_{X_i} DT^*(X_j) = -\delta_{ij} \]
where $\delta_{ij}$ denotes the Kronecker delta.

**Proof.** From Equation (2.26) we see that $DT(l_i^+) = l_i^-$ if and only if $\text{deg}_{X_i} DT^*(X_j) = -\delta_{ij}$. \qed

### 2.5. Special Perfect Orientation and Boundary Measurement Map.

Perfect orientations and boundary measurement maps are introduced by Postnikov in his work on the non-negative Grassmannian [Pos06]. What we call special perfect orientation in this paper is a special case of his perfect orientation and hence the name. To begin, we define Postnikov’s perfect orientation as below.

**Definition 2.30.** A perfect orientation on a minimal bipartite graph is a choice of orientations on the edges of the minimal bipartite graph, such that every white vertex has only one in-coming edge and every black vertex has only one out-going edge. Given a perfect orientation on a minimal bipartite graph, we call a marked point on the boundary of $D$ a source of that perfect orientation if the external edge connecting it to a white vertex is oriented towards that white vertex; otherwise we call it a sink of that perfect orientation.

Perfect orientation on a given minimal bipartite graph is highly non-unique, and they may even have oriented cycles. The following are two perfect orientations on the same minimal bipartite graph.

![Perfect orientations on a minimal bipartite graph](image)

Although perfect orientations are not unique, there are some combinatorial invariants to it. For example we have the following observation.

**Proposition 2.31.** If $\Gamma$ is a minimal bipartite graph associated to $\sigma$, then any perfect orientation on $\Gamma$ has exactly $m$ sources.

**Proof.** By simple combinatorial counting we see that the number of sources of a perfect orientation is equal to $2B + W - I$ where $B$, $W$ and $I$ are the numbers of black vertices, white vertices, and internal edges respectively. So it suffices to show that $m = 2B + W - I$.

![Zig-zag strand and part of $\Gamma$](image)

Now consider a zig-zag strand $\zeta$ and the part of $\Gamma$ that it dominates. It looks like the above picture. If we cut along the dashed arrow, then the part on the left of that dashed arrow is again a bipartite graph on a disk with external edges (not necessarily minimal, however). If we view a graph on a disk as some...
decomposition of the disk into 0-cells (vertices), 1-cells (edges) and 2-cells (faces), then Euler characteristic
tells us that
\[ V - E + F = 1 \]
where \( V \), \( E \), and \( F \) are the numbers of vertices, edges, and faces of that graph respectively (we are not
counting the boundary marked points and the boundary arcs between them because they cancel out each
other exactly). Now by considering all \( n \) zig-zag strands and the parts that we cut out as described above,
we have \( n \) bipartite graphs on disks. For these \( n \) bipartite graphs, we have counted each face \( m \) times (since
the dominating set of each face is \( m \) by Proposition 2.19), each black vertex \( m - 2 \) times, each white vertex
\( m - 1 \) times, and each edge \( m - 1 \) times. By summing up all the \( n \) Euler characteristics we get
\[ (m-2)B + (m-2)W - (m-1)(I + n) + mF = n. \]
On the other hand, computing the Euler characteristic for the minimal bipartite graph \( \Gamma \) itself, we get
\[ B + W - (I + n) + F = 1. \]
Eliminate \( F \) from these two equations we then obtain \( m = 2B + W - I. \)

For the rest of this subection we will introduce a particular type of perfect orientations on a minimal
bipartite graph, which we call the **special perfect orientation**.

**Remark 2.32.** Before we go into the details, we need to point out that because the special perfect orientation
relies on the linear ordering on \( \{1, \ldots, n\} \) inherited from \( \mathbb{Z} \), it breaks the cyclic symmetry in our story that
we have so far. But this is okay because one can replace the linear ordering \( < \) with \( \prec \) defined by
\[ t \prec t + 1 \prec \cdots \prec n \prec 1 \prec \cdots \prec t - 1. \]
Therefore all statements and constructions using special perfect orientations are cyclically equivalent.

To define a perfect orientation on a minimal bipartite graph, it suffices to state a way of assigning
orientation for the three edges connected to a black vertex (recall that black vertices in a minimal bipartite
graph is trivalent). Under our trivalent assumption, a typical black vertex is surrounded by three zig-zag
strands as follows.

Due to the non-existence of parallel bigon, \( i, j, \) and \( k \) must be in clockwise order on the boundary of \( \mathbb{D} \). Therefore once we impose the linear ordering \( < \) on \( \{1, \ldots, n\} \) inherited from \( \mathbb{Z} \), one and only one of the
following three must be true: \( i < j < k, j < k < i, \) or \( k < i < j \). We then find the middle one among \( i, j, \) and \( k \) with respect to the linear ordering \( < \), say \( j \), and declare that the edge opposite to \( \zeta_j \) to be out-going
from the black vertex. We claim that the orientations on all other edges of \( \Gamma \) are then uniquely determined
by bipartity and the conditions that each black vertex has only one out-going edge and each white vertex
has only one in-coming edge. To prove this claim, it suffices to show the following.

**Proposition 2.33.** Any white vertex indeed has exactly one in-coming edge.

**Proof.** First by the minimality condition on minimal triple diagrams, the lower indices of the zig-zag strands
passing next to a white vertex must be in a clockwise order; hence with respect to the linear order \( < \) we
must have \( i_1 < i_2 < \cdots < i_k \) in the following picture.
But as we see from the construction of the special perfect orientation, in order for an internal edge to go into a white vertex, the lower index of the zig-zag strand clockwise succeeding the internal edge must be smaller than that of the zig-zag strand clockwise proceeding the same internal edge. This happens only between $\zeta_{i_1}$ and $\zeta_{i_k}$. Therefore if the edge between $\zeta_{i_1}$ and $\zeta_{i_k}$ is internal then we have one in-coming edge to the center white vertex; otherwise the edge between $\zeta_{i_1}$ and $\zeta_{i_k}$ is external and all other edges are internal (Proposition 2.10), and we can just declare that the external edge is an in-coming edge to the center white vertex.

Next we are going to prove a couple important properties of special perfect orientations.

**Proposition 2.34.** There is no oriented cycle in the special perfect orientation.

**Proof.** First let’s show that no boundary of a face is oriented under the special perfect orientation. Assume that $f$ is a face with oriented boundary; without loss of generality let’s assume that the boundary of $f$ is clockwise oriented. Then at any black vertex at the boundary of $f$, such as the follow one, we must have $i<j$.

Since this must hold for every black vertex at the boundary of $f$, the lower indices of zig-zag strands that go counter clockwise around $f$ will be in a cyclic order with respect to the linear order $<$, which is absurd. Therefore no boundary of a single face is oriented.

Now suppose we have some oriented cycle under the special perfect orientation. Then it must bound a region with more than one face. In particular, at some vertex of this oriented cycle there must be some edge going into the bounded region inside. Fix such an edge, which may orient inward or outward under the special perfect orientation; the proof of either case is analogous, so without loss of generality let’s assume that it is oriented inward. Since each vertex, regardless of the color, has at least one out-going edge, we can
follow this inward pointing edge and continue into some path inside the oriented cycle.

But since there are only finitely many edges inside the oriented cycle, we must end with one of the following two cases: either the path closes up and give us a smaller oriented cycle bounding fewer faces (left), or it exists the interior and connects back to the oriented cycle (right). Either way we will end up with an oriented cycle bounding fewer faces, and the proof is complete by induction on the number of faces that it bounds. □

**Proposition 2.35.** If $\Gamma$ is a minimal bipartite graph, then the source set of the special perfect orientation is always $[1, m]$ (recall that we have the notation convention $[1, m] := \{1, \ldots, m\}$).

**Proof.** Let’s start by showing that the statement holds for the minimal bipartite graph $\Gamma_0$. By carefully comparing the zig-zag strands, we find that the special perfect orientation on $\Gamma_0$ takes the following form.

In fact the faces of $\Gamma_0$ are in fact in bijection with the cells in an $m \times (n - m)$ array of cells, and the special perfect orientation on $\Gamma_0$ can be easily remembered by the following orientations on the edges of $m \times (n - m)$ array of cells.

On the other hand, it can be verify that a $2 \leftrightarrow 2$ move never changes the sources and sinks (not even locally) and hence the source sets of the special perfect orientations on all minimal bipartite graphs are the same.
Then the proof is finished because we see above that \([1, m]\) is the source set of the special perfect orientation on \(\Gamma_0\).

Now we turn to the second topic of this subsection, which is the Postnikov’s boundary measurement map. Consider a minimal bipartite graph \(\Gamma\) with its special perfect orientation. A path \(\gamma\) from a source \(i\) to a sink \(j\) in the special perfect orientation is a finite collection of consecutive edges in \(\Gamma\) with compatible orientations starting with the external edge leaving \(i\) and ending with the external edge going into \(j\). We denote a path in the special perfect orientation going from \(i\) to \(j\) by \(\gamma : i \rightarrow j\). Note that since the special perfect orientation has no oriented cycles and there are only finitely many edges in a minimal bipartite graph, there are only finitely many paths from any given source to any given sink.

Given a path \(\gamma\) on \(\Gamma\), we say that a face \(f\) of \(\Gamma\) is dominated by \(\gamma\) if it lies on the right of \(\gamma\) with respect to the orientation of \(\gamma\). We denote the set of faces dominated by \(\gamma\) by \(\hat{\gamma}\).

**Definition 2.36.** Let \(\Gamma\) be a minimal bipartite graph on \(\mathbb{D}\) and let \(\hat{\Gamma}\) be the corresponding quiver. For \(i \in [1, m]\) and \(j \in [m+1, n]\), we define the boundary measurement function \(\tilde{x}_i^j\) to be the polynomial function on the seed \(\mathcal{X}\)-torus \(\mathcal{X}_i\) given by

\[
\tilde{x}_i^j(X_f) := \sum_{\gamma; i \rightarrow j \in \hat{\gamma}} \prod_{f \in \gamma} X_f.
\]

Using the boundary measurement functions, we define the boundary measurement map \(\tilde{x}_i\) as

\[
\tilde{x}_i : \mathcal{X}_i \rightarrow \text{Conf}_n(\mathbb{C}^m - \{0\}) \quad (X_f) \mapsto [M_{ij}],
\]

where

\[
M_{ij} = \begin{cases} 
\delta_{ij} & \text{if } 1 \leq j \leq m; \\
(-1)^{m-i} x_i^j(X_f) & \text{if } m+1 \leq j \leq n.
\end{cases}
\]

**Remark 2.37.** It is not clear from this construction that some \([M_{ij}]\) can possibly be inside \(\text{Conf}_n(\mathbb{C}^m - \{0\})\), so one may now just regard \(\tilde{x}_i\) as a map from \(\mathcal{X}_i\) to \(\text{Mat}_{m,n}\). Later as an immediate corollary of Proposition 3.5, we see that the image of \(\tilde{x}_i\) in fact intersect \(\text{Conf}_n(\mathbb{C}^m - \{0\})\) in a Zariski open subset, and \(\tilde{x}_i\) is actually a dominant rational map.

**Remark 2.38.** Postnikov’s original treatment of the boundary measurement functions includes a sign factor depending on the winding number of a path, and the set of dominated faces lie on the left instead of on the right. In this paper we change the set of dominated faces to the other side to better suit our construction, and since we showed that a special perfect orientation has no oriented cycles (Proposition 2.34), all paths have vanishing winding number and hence we do not need that sign factor.

### 3. The * Map

When Goncharov suggested the study of the cluster Donaldson-Thomas trannformation on \(\text{Conf}_n(\mathbb{P}^{m-1})\), the map * was the first map that the author constructed as a potential candidate. As it turns out, the map *, in some sense, is “too nice” to be the cluster Donaldson-Thomas transformation. Yet the map * has many beautiful properties and the proof of the main theorem of this paper relies heavily on properties of *.

We will introduce the map * and prove some of its properties in this section.

#### 3.1. Definition and Construction

Let’s give a geometric definition of the map * first. Recall that for a generic point \((l_1, \ldots, l_n) \in (\mathbb{P}^{m-1})^n\) we use the notation \(h_{i,i+m-2}\) to denote the defining dual line in \((\mathbb{P}^{m-1})^\vee\) of the hyperplane spanned by \(l_i, \ldots, l_{i+m-2}\). After quotienting out the projectively linear group actions, we know that \(\text{Conf}_n(\mathbb{P}^{m-1})\) is naturally isomorphic to \(\text{Conf}_n((\mathbb{P}^{m-1})^\vee)\), and hence we can define the following map.

**Definition 3.1.** The map \(* : \text{Conf}_n(\mathbb{P}^{m-1}) \rightarrow \text{Conf}_n(\mathbb{P}^{m-1})\) is defined by

\[
*: [l_1, \ldots, l_n] \mapsto [h_{[1,m-1]}l_{[n,m-2]}, \ldots, h_{[2,m]}l_{[n,m-2]}].
\]
For the rest of this subsection, we will give an alternative definition of the map $*$ using cluster language, which is the key in proving many interesting properties of the map $*$.

Fix a minimal bipartite graph $\Gamma$ with parameters $m$ and $n$. Let $\tilde{\Gamma}$ be its corresponding quiver. We have learned from the introduction that there is a rational map $\tilde{a}_\Gamma : \text{Conf}_n(\mathbb{C}^m - \{0\}) \to \mathcal{A}_i$, and there is a natural $p$ map $\tilde{p}_\Gamma : \mathcal{A}_i \to X_i$. Since every vertex in the quiver $\tilde{\Gamma}$ associated to $\Gamma$ has an equal number of in-coming arrows and out-going arrows, the composition $\tilde{p}_\Gamma \circ \tilde{a}_\Gamma$ is actually constant along the fiber $\text{Conf}_n(\mathbb{C}^m - \{0\}) \to \text{Conf}_n(\mathbb{C}^m - \{0\})$ and hence can be passed down to a rational map

$$\tilde{p}_\Gamma \circ \tilde{a}_\Gamma : \text{Conf}_n(\mathbb{C}^m - \{0\}) \to X_i.$$

Next we need a reflection map $\tilde{r} : X_i \to X_i$, where $\tilde{\Gamma}$ denotes quiver corresponding to the dual minimal bipartite graph $\Gamma^\circ$ (Definition 2.13). If we denote the corresponding faces in $\Gamma$ and $\Gamma^\circ$ by $f$ and $f^\circ$ respectively, then the reflection map $\tilde{r}$ is defined by

$$\tilde{r}^*(X_f) = X_f.$$

**Definition 3.2.** The map $*_\Gamma : \text{Conf}_n(\mathbb{C}^m - \{0\}) \to \text{Conf}_n(\mathbb{C}^m - \{0\})$ is the rational map defined to be the composition of the following sequence of maps:

$$\text{Conf}_n(\mathbb{C}^m - \{0\}) \xrightarrow{\tilde{p}_\Gamma \circ \tilde{a}_\Gamma} X_i \xrightarrow{\tilde{r}} X_i \xrightarrow{\tilde{m}_f} \text{Conf}_n(\mathbb{C}^m - \{0\}).$$

Such a definition of the $*_\Gamma$ may seem a bit arbitrary, so our next goal is to convince our readers that this definition is rationally equivalent to the geometric definition we gave in Definition 3.1.

Recall the Zariski open subset $\text{Conf}_f(\mathbb{C}^m - \{0\})$ consisting of totally generic elements. It follows that $\tilde{p}_\Gamma \circ \tilde{a}_\Gamma$ is a well-defined regular map on $\text{Conf}_f(\mathbb{C}^m - \{0\})$. Let $[v_1, \ldots, v_n]$ be a point in $\text{Conf}_f(\mathbb{V})$. By simple counting we know that the dominating set $I(\mathbb{W})$ of each white vertex $\mathbb{W}$ in $\Gamma$ is of size $m - 1$. Therefore for each white vertex $\mathbb{W}$ we can define a linear functional $\xi_\mathbb{W} \in (\mathbb{C}^m)^\vee$ by

$$\xi_\mathbb{W}(\mathbb{V}) := \det \left( \mathbb{V} \land \sum_{i \in I(\mathbb{W})} v_i \right).$$

Note that since $I(\mathbb{W}) \subseteq I(\mathbb{F})$ for any neighboring face $\mathbb{F}$, it follows that $\xi_\mathbb{W} \neq 0$ for any $\mathbb{W}$.

Then we make the following observation. Consider a black vertex in $\Gamma$ connecting to three white vertices $r, s$, and $t$, with zig-zag stands $\zeta_{i-m}^i, \zeta_{j-m}^j$, and $\zeta_{k-m}^k$, and faces $f, g$, and $h$ around, arranged as follows.

From the picture we see that all three dominating sets $I(r), I(s)$, and $I(t)$ share the same $m - 2$ element subset $J$, with $I(r) = J \cup \{i\}$, $I(s) = J \cup \{j\}$, and $I(t) = J \cup \{k\}$. Without loss of generality we assume that $i < j < k$. For $a = i, j, k$, we define $\nu_a := \{b \in J \mid b < a\}$. Then we observe the following.

$$A_f(v_1, \ldots, v_n) = (-1)^{\nu_a + 1} \xi_s(v_k) = (-1)^{\nu_i} \xi_t(v_j);$$

$$A_g(v_1, \ldots, v_n) = (-1)^{\nu_s} \xi_i(v_k) = (-1)^{\nu_i} \xi_c(v_j);$$

$$A_h(v_1, \ldots, v_n) = (-1)^{\nu_t} \xi_r(v_j) = (-1)^{\nu_i} \xi_s(v_i).$$

If we define another linear functional

$$\eta := A_f(v_1, \ldots, v_n)\xi_r - A_g(v_1, \ldots, v_n)\xi_s + A_h(v_1, \ldots, v_n)\xi_t.$$
Then obviously the inner product of \( \eta \) with any element in \( \{v_l \mid l \in J \cup \{i, j, k\}\} \) vanishes; but the right hand side is actually a spanning set of \( V \), and hence we can conclude that

\[
\xi_s = \frac{A_f(v_1, \ldots, v_n)}{A_g(v_1, \ldots, v_n)} \xi_r + \frac{A_h(v_1, \ldots, v_n)}{A_g(v_1, \ldots, v_n)} \xi_t.
\]

The coefficients look familiar: they are part of \( X_f X_g \) and \( X_f \) respectively, and this will link us to the star map.

In fact, since we have assumed that \( i < j < k \), the special perfect orientation on \( \Gamma^0 \) is already determined, which is the picture on the right above. Note that that picture is still drawn on \( \Gamma \), so please be careful that the dominated faces of a path in the special perfect orientation on \( \Gamma^0 \) now lie on the right.

Lastly if the marked point \( i \) on the boundary of \( D \) is connected to the boundary white vertex \( w_i \) in \( \Gamma \), then we associate to \( i \) the linear functional

\[
(3.3) \quad \xi_i := \begin{cases} 
(-1)^{m-i} \xi_{w_i} & \text{if } 1 \leq i \leq m; \\
\xi_{w_i} & \text{if } m + 1 \leq i \leq n.
\end{cases}
\]

**Proposition 3.4.** For any \( j \in \{m + 1, \ldots, n\} \), we have

\[
\xi_j = \sum_{i=1}^{m} (-1)^{m-i} \overline{m}_{i,j} (X_f) \xi_i,
\]

where \( (X_f) := \tilde{r} \circ \tilde{p}_1 \circ \tilde{a}_1[v_1, \ldots, v_n] \). In particular, \( [v_1, \ldots, v_n] \mapsto [\xi_1, \ldots, \xi_n] \) is precisely the map \( \ast \Gamma \) we defined earlier in this subsection.

**Proof.** If we start from a sink \( j \) and go against the special perfect orientation on \( \Gamma^0 \) and repeatedly use the identity \( \xi_s = \frac{A_f(v_1, \ldots, v_n)}{A_g(v_1, \ldots, v_n)} \xi_r + \frac{A_h(v_1, \ldots, v_n)}{A_g(v_1, \ldots, v_n)} \xi_t \) at each black vertex, we obtain an expansion of \( v_j \) in terms of the vectors \( \{v_i \mid i \in [1, m]\} \) which agrees with the expression of the boundary measurement map \( \tilde{m}_{\Gamma^0} \). \( \square \)

The above proposition is inspiring. First note that by Equation (3.3), \( \xi_i \) can be defined entirely based upon \( v_i \), so the maps \( \ast \Gamma \) actually does not depend on the choice of minimal bipartite graph \( \Gamma \), hence it makes sense to drop the index \( \Gamma \) and instead write (with an abuse of notation)

\[
\ast : \text{Conf}_n(\mathbb{C}^m - \{0\}) \rightarrow \text{Conf}_n(\mathbb{C}^m - \{0\}).
\]

Second, we see that \( \xi_i \) is in fact a dual vector corresponding to the hyperplane spanned by \( v_{2-i}, \ldots, v_{m-i} \) (indices are taken modulo \( n \)), so we almost recover the geometric definition (Definition 3.1)! All we have to do next is to projectivize \( \ast \), and then we obtain back the definition of \( \ast \) we had at the beginning of this section:

\[
\ast : \text{Conf}_n(\mathbb{P}^{m-1}) \rightarrow \text{Conf}_n(\mathbb{P}^{m-1})
\]

\[ [l_1, l_2, \ldots, l_n] \mapsto [h[l_1, \ldots, 1], h[n-m-2], \ldots, h[2, m]] .
\]

(Recall that \( h[i, m+i-2] \) denotes the dual line corresponding to the hyperplane spanned by the lines \( l_i, \ldots, l_{m+i-2} \), where the indices are taken modulo \( n \)).

As Remark 2.37 pointed out, so far we haven’t proved that the image of the boundary measurement map \( \tilde{m}_{\Gamma} \) or the image of \( \ast \), has non-trivial intersection with the configuration spaces. Thus we need to add in the following supplementary proposition.

**Proposition 3.5.** If \( [v_1, \ldots, v_n] \) is a point in \( \text{Conf}_n(\mathbb{C}^m - \{0\}) \) then so is \( [\xi_1, \ldots, \xi_n] \), where \( \xi_i \) are defined by Equation (3.3).

**Proof.** By cyclic symmetry, it suffices to show that \( \{\xi_1, \ldots, \xi_m\} \) is a basis of \( (\mathbb{C}^m)^\vee \). Let \( \langle \cdot, \cdot \rangle \) be the standard inner product on \( \mathbb{C}^m \), which gives rise to an identification between \( (\mathbb{C}^m)^\vee \) and \( \mathbb{C}^m \) (hence the volume form \( \det \) can be applied to \( (\mathbb{C}^m)^\vee \) too) as well as a Hodge dual operator \( \ast : \bigwedge^{m-k} \mathbb{C}^m \rightarrow \bigwedge^k \mathbb{C}^m \). Then up to a sign, we can write

\[
(3.6) \quad \xi_i = \ast \left( \bigwedge_{j \in [n-i+2, n-i+m]} v_j \right).
\]
The statement is then reduced to showing that $\det(\xi_1, \ldots, \xi_m) \neq 0$. To save our energy, we do the rest of the proof up to a sign as well:

$$\det(\xi_1 \wedge \cdots \wedge \xi_m)$$

$$= \det(*)((v_1 \wedge \cdots \wedge v_{m-1}) \wedge (v_n \wedge \cdots \wedge v_{m-2}) \wedge \cdots \wedge (*v_{n-m+2} \wedge \cdots \wedge v_n))$$

$$= \langle * (v_1 \wedge \cdots \wedge v_{m-1}) \wedge (v_n \wedge \cdots \wedge v_{m-2}) \wedge \cdots \wedge *(v_{n-m+3} \wedge \cdots \wedge v_1), v_{n-m+2} \wedge \cdots \wedge v_n \rangle.$$

Now we need to pair up each dual vector on the left with each vector on the right in all possible non-vanishing ways. First consider $v_n$ at the far right end; on the left, the only dual vector that does not annihilate $v_n$ is $*(v_1 \wedge \cdots \wedge v_{m-1})$. Second consider $v_{n-1}$ and the only factor other than the first factor on the left that does not annihilate $v_{n-1}$ is the second factor $*(v_1 \wedge \cdots \wedge v_{m-2} \wedge v_n)$. As we go through all the vectors from right to left, we see that there is only one possible non-vanishing pairing, which is to pair up the left and right in a reverse order.

The problem is now reduced to showing that each factor is not vanishing. Let’s take the first one $\langle * (v_1 \wedge \cdots \wedge v_{m-1}), v_n \rangle$ for example. Recall that the square of the Hodge dual operator is the identity up to a sign.

The statement is then reduced to showing that each factor is not vanishing. Let’s take the first one $\langle * (v_1 \wedge \cdots \wedge v_{m-1}), v_n \rangle$ for example. Recall that the square of the Hodge dual operator is the identity up to a sign.

The following proposition gives us a clear idea of the relation between the two $*$ maps:

**Proposition 3.7.** The following diagram commutes. The two vertical maps are $(\mathbb{C}^*)^{n-1}$ fiber bundle maps. The bottom map is an involution.

$$\begin{array}{ccc}
\text{Conf}_n(\mathbb{C}^m - \{0\}) & \overset{*}{\longrightarrow} & \text{Conf}_n(\mathbb{C}^m - \{0\}) \\
\downarrow & & \downarrow \\
\text{Conf}_n(\mathbb{P}^{m-1}) & \overset{*}{\longrightarrow} & \text{Conf}_n(\mathbb{P}^{m-1})
\end{array}$$

**Proof.** The commutativity and fiber bundle maps follows easily from the fact that the bottom map is obtained by projectivizing the top map. The fact that the bottom map is an involution follows from simple computation using Definition 3.1.

3.2. **Birational Equivalence.** As a byproduct of the involutive nature of the map $*$, we give a proof of the birational equivalence between the configuration space $\text{Conf}_n(\mathbb{P}^{m-1})$ and the cluster variety $\mathcal{X}_{\Gamma_0}$. Readers are welcome to skip this subsection if they are familiar with this fact. As we stated at the end of Subsection 2.3, it suffices to prove Proposition ?? for some minimal bipartite graph $\Gamma$, and the choice we make here is $\Gamma := \Gamma_0$.

Using the map on $\text{Conf}_n(\mathbb{P}^{m-1})$ we can fit the rational map $x_{\Gamma_0}$ into the following commutative diagram.

$$\begin{array}{ccc}
\text{Conf}_n(\mathbb{C}^m - \{0\}) & \longrightarrow & \mathcal{X}_{\Gamma_0} \\
\downarrow & & \downarrow \\
\text{Conf}_n(\mathbb{P}^{m-1}) & \longrightarrow & \text{Conf}_n(\mathbb{P}^{m-1})
\end{array}$$

In particular, the rational map $x_{\Gamma_0}^{-1}$ is defined by first lifting an element from $\mathcal{X}_{\Gamma_0}$ to $\mathcal{X}_{\Gamma_0}$, mapping it over by $D \circ \tilde{m}_{\Gamma_0} \circ \tilde{r}$ and then projecting it back down; since this definition involves a lifting, we need to show well-definedness before we do anything else.

**Proposition 3.8.** The rational map $x_{\Gamma_0}^{-1}$ is well-defined.
Proof. Note that the lifting in the definition of $x^{-1}_{\Gamma_0}$ is the same as assigning some non-zero complex numbers to the boundary face variables of $\Gamma_0$; since the map $\tilde{r}$ maps boundary face variables to boundary face variables and $i$ is a fiber bundle morphism, it suffices to show that adjusting boundary face variables in $X^0_{\Gamma_0}$ does not move the image of $\tilde{m}_{\Gamma_0}$ across different fibers of $\text{Conf}_n(\mathbb{C}^m - \{0\}) \to \text{Conf}_n(\mathbb{P}^{m-1})$. For notation simplicity we denote the face counterclockwisely next to the marked point $i$ in $\Gamma_0^*$ by $f^*_i$, and denote its corresponding face variable by $X^0_f$.

Let $[\xi_1, \ldots, \xi_n]$ be the image of $\tilde{m}_{\Gamma_0}$ (so $\xi_1, \ldots, \xi_m$ are the standard basis vectors of $\mathbb{C}^m$). We then make the following observations.

1. For $X^0_j$ with $m + 1 \leq j \leq n$, any path in the special perfect orientation on $\Gamma_0^*$ ending at $k$ with $j \leq k \leq n$ always dominates the face $f_j$, so $\xi_k$ is in the form
   $$\xi_k = X^0_j \sum_{i=1}^{n} \alpha_{ik}^j (X^0_f) \xi_i$$
   where $\alpha_{ik}^j$ is a polynomial in terms of face variables other than $X^0_j$. This implies that $X^0_j$ is a scaling factor of the vectors $\xi_j, \xi_{j+1}, \ldots, \xi_n$.

2. For $X^0_i$ with $2 \leq i \leq m$, any path in the special perfect orientation on $\Gamma_0^*$ starting at $k$ with $1 \leq k \leq i - 1$ always dominates the face $f_i$, so for any sink $j$, $\xi_j$ is in the form
   $$\xi_j = X^0_i \sum_{k=1}^{i-1} \beta_{i,j}^k (X^0_f) \xi_k + \sum_{l=i}^{m} \gamma_{i,j}^l (X^0_f) \xi_l$$
   where $\beta_{i,j}^k$ is a polynomial in terms of face variables other than $X^0_i$. This implies that $X^0_i$ is a scaling factor of the vectors $\xi_1, \ldots, \xi_i$.

3. What about $X^0_1$? Note that no path in the special perfect orientation on $\Gamma_0^*$ dominates this face.
   Therefore changing $X^0_1$ does not move the image of the boundary measurement map at all.

These observations show that changing boundary face variables of $X^0_i$ does not move the image of $\tilde{m}_{\Gamma_0}$ across different fibers of $\text{Conf}_n(\mathbb{C}^m - \{0\}) \to \text{Conf}_n(\mathbb{P}^{m-1})$, which is precisely what we want to show. \qed

Now we are finally ready to prove Proposition ??, which in turn proves that $\text{Conf}_n(\mathbb{P}^{m-1})$ and $X_{[b]}$ are birationally equivalent.

Proof of Proposition ??: We observe that the composition

$$\text{Conf}_n(\mathbb{C}^m - \{0\}) \xrightarrow{\tilde{m}_{\Gamma_0}} \text{Conf}_n(\mathbb{P}^{m-1}) \xrightarrow{\alpha_{\tilde{m}_{\Gamma_0}}} \text{Conf}_n(\mathbb{P}^{m-1})$$

restricting to $\text{Conf}_n^0(\mathbb{C}^m - \{0\})$ is a well-defined map and is equal to $\alpha^2$. Since the maps $x_{\Gamma_0}$ and $x^{-1}_{\Gamma_0}$ are defined as maps induced by the corresponding maps in the top row, it follows that $x^{-1}_{\Gamma_0} \circ x_{\Gamma_0}$ is also a well-defined map on the Zariski open subset $\text{Conf}_n^0(\mathbb{P}^{m-1})$ and is equal to $\alpha^2 = 1$.

The fact that $x^{-1}_{\Gamma_0} \circ x_{\Gamma_0}$ is well-defined and equal to the identity map on $\text{Conf}_n^0(\mathbb{P}^{m-1})$ also implies that $\text{Conf}_n^0(\mathbb{P}^{m-1})$ lies inside the domain of $x_{\Gamma_0}$ and the image of $x^{-1}_{\Gamma_0}$, and hence the composition $x_{\Gamma_0} \circ x^{-1}_{\Gamma_0}$ is a well-defined rational map that factors through $\text{Conf}_n(\mathbb{P}^{m-1})$. To show that $x_{\Gamma_0} \circ x^{-1}_{\Gamma_0}$ is rationally equivariant...
to the identity map on some Zariski open subset, it suffices to show that \( x_{\Gamma_0} \) is dominant, and we will use
an elementary argument from algebraic geometry.

Note that \( \mathcal{X}_0 \) and \( \text{Conf}_{\mathcal{X}}(\mathbb{P}^{m-1}) \) are of the same dimension. On the one hand, we know that \( \mathcal{X}_0 \) is irreducible because it is a split algebraic torus; on the other hand, being a geometric quotient of an irreducible
variety, \( \text{Conf}_{\mathcal{X}}(\mathbb{P}^{m-1}) \) is also irreducible. Since the rational involution \( *=x_{\Gamma_0}^{-1} \circ x_{\Gamma_0} \) factors through \( \mathcal{X}_0 \), it follows that the image of \( x_{\Gamma_0} \) is an irreducible subset of \( \mathcal{X}_0 \) of top dimension. Thus the closure of the image
of \( x_{\Gamma_0} \) must be the whole \( \mathcal{X}_0 \) because \( \mathcal{X}_0 \) is irreducible. \( \square \)

3.3. **Positivity of the \(*\) Map.** Since now we know that \( x_{\Gamma} : \text{Conf}_{\mathcal{X}}(\mathbb{P}^{m-1}) \to \mathcal{X}_1 \) is a birational equivalence
for any minimal bipartite graph \( \Gamma \), the map \( * \) on \( \text{Conf}_{\mathcal{X}}(\mathbb{P}^{m-1}) \) naturally induces a rational involution \( * \) on
each seed \( \mathcal{X} \)-torus \( \mathcal{X}_1 \) associated to a minimal bipartite graph \( \Gamma \). Since this induced map \( * : \mathcal{X}_1 \to \mathcal{X}_1 \) is a
rational map from a split algebraic torus back to itself, it makes sense to ask whether this rational map is
positive or not, and we will show in this subsection that the answer is positive.

Note that we can break the induced map \( * \) as the bottom row of the following commutative diagram.

\[
\begin{array}{ccc}
\mathcal{X}_1 & \xrightarrow{r} & \mathcal{X}_1 \\
\downarrow{q_r} & & \downarrow{q_r} \\
\mathcal{X}_1 & \xrightarrow{m_r} & \text{Conf}_{\mathcal{X}}(\mathbb{P}^{m-1}) - x_{\Gamma} \to \mathcal{X}_1
\end{array}
\]

Then the problem of showing that the induced star map on \( \mathcal{X}_1 \) is positive reduced to showing that the
composition of the maps in the top row is positive. But we already know that \( r \) and \( m_r \) are positive, so it
suffices to show that \( m_r \circ m_r = \mathcal{X}_1 \to \mathcal{X}_1 \) is positive. In fact, we will show something even stronger, which
is the following.

**Proposition 3.9.** Fix a minimal bipartite graph \( \Gamma \). Let \( I \) be an \( m \)-element subset of \{1, \ldots, n\}. Then the
pull-back \( m^*_I (\Delta_I/\Delta_{[1,m]}) \) is a polynomial with positive integral coefficients in terms of the face variables \( X_f \)
of \( \Gamma \).

**Proof.** This proof is due to Postnikov [Pos06]. Let \([\xi_1, \ldots, \xi_n]\) be a point in the image of \( m_r \). Break the
\( m \)-element subset \( I \) of \{1, \ldots, n\} into \( I_s \cup I_t \) where \( I_s := I \cap [1, m] \) and \( I_t := I \cap [m+1, n] \). From Definition
2.36 we know that

\[
\Delta_I(\xi_1, \ldots, \xi_n) = \det \left( \left( \sum_{j \in I_t} -1 \right) \right) \wedge \left( \sum_{i \in [1,m]-I} \right)
\]

Recall that \( m^*_I(X_f) \) is actually a polynomial for any \( i \in [1, m] \) and any \( j \in [m+1, n] \), and each term
corresponds to a path in the canonical perfect orientation on \( \Gamma \) going from \( i \) to \( j \). Therefore when we expand the right hand side of the above equation, we will get an expression \( \Phi_I(X_f)\Delta_{[1,m]} \) where \( \Phi_I(X_f) \) is
a polynomial in terms of the face variables of \( \Gamma \).

Now it remains to show that the coefficients in \( \Phi_I(X_f) \) are positive. Note that each term in \( \Phi_I(X_f) \)
corresponds to a family of \( |I| \) number of paths in the canonical perfect orientation on \( \Gamma \) going from \( [1, m] - I \)
to \( I_t \). We claim that if the family of paths are not pairwise disjoint, then there is another term in \( \Phi_I(X_f) \) to
cancel it out. Suppose \( \gamma \) and \( \eta \) are two intersecting paths in one such family (the picture looks as if they are
not intersecting but they in fact are sharing some edges together). Then we can just switch the beginning
parts of \( \gamma \) and \( \eta \) to obtain \( \gamma' \) and \( \eta' \) and keep everything else in the family unchanged; the resulting family
of paths will give another term in $\Phi_f(X_f)$

\[ \prod_{l=1}^k \prod_{f \in \gamma_l} X_f, \]

whose coefficient is obviously both positive and integral. This finishes the proof. \qed

3.4. A Special Expression for the $*$ Map. In this subsection we will focus on the minimal bipartite graph $\Gamma_0$ and show the following statement.

**Proposition 3.10.** The induced map $*: X_{i_0} \to X_{i_0}$ can be expressed in the canonical coordinates as

\[ *^*(X_{i,j}) = X_{n-m-i,m-j}^{-1}, \]

where the subscripts of $X_{i,j}$ come from the quiver $i_0$ associated to $\Gamma_0$, as presented in (2.11).

**Proof.** Let $[l_1, \ldots, l_n] := x_\Gamma^{-1}(X_{i,j})$. Once for all we fix a section $s$ of the fiber bundle map $\text{Conf}_n(\mathbb{C}^m) \to \text{Conf}_n(\mathbb{P}^{m-1})$ and let $[v_1, \ldots, v_n] := s[l_1, \ldots, l_n]$. Let $\langle \cdot, \cdot \rangle$ be the standard inner product on $\mathbb{C}^m$, which gives rise to an identification between $(\mathbb{C}^m)^\vee$ and $\mathbb{C}^m$ as well as a Hodge dual operator $* : \wedge^{m-k} \mathbb{C}^m \to \wedge^k \mathbb{C}^m$. From the definition of the $*$ we know that $* : \chi_\Gamma \to \chi_\Gamma$ can be factored as

\[ *(X_{i,j}) = x_\Gamma \circ s \circ x_\Gamma^{-1}(X_{i,j}) = q_\Gamma \circ \tilde{p}_\Gamma \circ \tilde{a}_\Gamma \circ * \circ s \circ x_\Gamma^{-1}(X_{i,j}) = q_\Gamma \circ \tilde{p}_\Gamma \circ \tilde{a}_\Gamma \circ *[v_1, \ldots, v_n]. \]

So now the problem reduces to computing certain ratios of Plücker coordinates of $[\xi_1, \ldots, \xi_n] := *[v_1, \ldots, v_n]$. Recall from Equation (3.6) that we have the following equality up to a sign:

\[ \xi_i = * \left( \bigwedge_{j \in [n-i+2,m-i]} v_j \right). \]

Depending on where $(i,j)$ is in the quiver $i_0$, we need to break the computation into five different cases. Combinatorially there are more cases to consider, but we can use the fact of $*$ being an involution to deduce
those cases. The following are the five cases drawn on part of $i_0$:

\[
\begin{array}{ccc}
(0,0) & (1,m-2) & (1,m) \\
\downarrow & \downarrow & \downarrow \\
(1,1) & (1,m-1) & (1,m) \\
\downarrow & \downarrow & \downarrow \\
(2,1) & (2,m-1) & (2,m) \\
\end{array}
\]

\[
\begin{array}{ccc}
(i-1,1) & (1,j-1) & (1,j+1) \\
\downarrow & \downarrow & \downarrow \\
(i,1) & (1,j) & (1,j+1) \\
\downarrow & \downarrow & \downarrow \\
(i+1,1) & (1,j) & (1,j+1) \\
\end{array}
\]

\[
\begin{array}{ccc}
(i-1,j-1) & (i-1,j) & (i-1,j+1) \\
\downarrow & \downarrow & \downarrow \\
(i,j-1) & (i,j) & (i,j+1) \\
\downarrow & \downarrow & \downarrow \\
(i+1,j) & (i+1,j) & (i+1,j+1) \\
\end{array}
\]

What’s left in the proof is just a lengthy computation of each case. To save time, we will just do the last case, which includes all techniques required to compute all other cases.

The range of $i$ and $j$ for the third case is $2 \leq i \leq m-2$ and $2 \leq j \leq n-m-2$. Note that the dominating set of the face corresponding to vertex $(i,j)$ is

\[
I_{i,j} = \begin{cases} 
[1,m] & \text{if } (j,i) = (0,0); \\
[1,m-j] \cup [m+i-j+1,m+i] & \text{if } (i,j) \neq (0,0).
\end{cases}
\]

To simplify notation, we will temporarily use an abuse of notation and denote

$$[a,a+m-2] := \star(v_a \wedge v_{a+1} \wedge \cdots \wedge v_{a+m-2}).$$

We also temporarily define

$$A^v_{i,j} := A_{i,j}(v_1, \ldots, v_n) \quad \text{and} \quad A^\xi_{i,j} := A_{i,j}(\xi_1, \ldots, \xi_n).$$

Using these short hand notations we can then compute

$$A^v_{i,j} = \det(\xi_1 \wedge \cdots \wedge \xi_{m-j} \wedge \xi_{m+i-j+1} \wedge \cdots \wedge \xi_{m+i})$$

$$= \det([1,m] \wedge \cdots \wedge [n-m+j+2,j])$$

$$\wedge [n-m-i+j+1,n-i+j-1] \wedge \cdots \wedge [n-m-i+2,n-i])$$

$$= ([1,m-1] \wedge \cdots \wedge [n-m+j+2,j] \wedge [n-m-i+j+1,n-i+j-1] \wedge \cdots$$

$$\wedge [n-m-i+3,n-i+1],v_{n-m-i+2} \wedge \cdots \wedge v_{n-i})$$

$$= A^v_{n-m-i+j-1,m} \cdots A^v_{n-m-j+1,m}([1,m-1] \wedge \cdots \wedge [n-m+j+2,j],v_{n-m-i+j+1} \wedge \cdots \wedge v_{n-i}).$$

It then follows that

$$\star^* (X_{i,j}) = \frac{A^\xi_{i-1,j-1} A^\xi_{i,j+1} A^\xi_{i+1,j}}{A^\xi_{i-1,j} A^\xi_{i+1,j+1} A^\xi_{i,j-1}}$$
\[
\begin{align*}
&= ([1, m - 1] \wedge \cdots \wedge [n - m + j + 1, j - 1], v_{n-m-i+j+1} \wedge \cdots \wedge v_{n-i+1}) \\
&= ([1, m - 1] \wedge \cdots \wedge [n - m + j + 1, j - 1], v_{n-m-i+j} \wedge \cdots \wedge v_{n-i}) \\
&\times ([1, m - 1] \wedge \cdots \wedge [n - m + j + 2, j], v_{n-m-i+j+2} \wedge \cdots \wedge v_{n-i+1}) \\
&\times ([1, m - 1] \wedge \cdots \wedge [n - m + j + 3, j + 1], v_{n-m-i+j+3} \wedge \cdots \wedge v_{n-i+2}) \\
&= A^v_{n-m-i+1,m-j+1} A^v_{n-m-i,m-j} A^v_{n-m-i-1,m-j-1} A^v_{n-m-i-2,m-j-2}
\end{align*}
\]

Since the expression \( *^*(X_{i,j}) = X^{-1}_{n-m-i,m-j} \) already has a positive integral coefficient, we do not need to do anything to fix the sign. \( \square \)

4. Cluster Donaldson-Thomas Transformation of Conf\(_n(P^{m-1})\)

Now let’s put the map \(*\) aside and talk about the main subject of this paper, the Donaldson-Thomas transformation of Conf\(_n(P^{m-1})\). So far we have obtain the following chain of commutative diagrams, in which each map in the bottom row is a birational equivalence (recall that \( i_0 \) is the quiver associated to the minimal bipartite graph \( \Gamma_0 \), which is described in Example 2.11).

\[
\cdots \Rightarrow X_{[i_0]} \Rightarrow X_{[i_0]} \Rightarrow \text{Conf}_n(C^m - \{0\}) \Rightarrow \cdots \\
\Downarrow \psi \quad \Downarrow \psi \quad \Downarrow \psi \quad \Downarrow \psi \quad \Downarrow \psi \\
\cdots \Rightarrow X_{[i_0]} \Rightarrow X_{[i_0]} \Rightarrow \text{Conf}_n(P^{m-1}) \Rightarrow \cdots \\
\Downarrow \chi \quad \Downarrow \chi \quad \Downarrow \chi \quad \Downarrow \chi \\
X_{[i_0]} \Rightarrow X_{[i_0]} \Rightarrow \text{Conf}_n(P^{m-1}) \Rightarrow \cdots
\]

By composing every two consecutive maps in the bottom row, we can fold the bottom row into the following diagram, which commutes with the same choice of directions for vertical arrows.

\[
\begin{array}{c}
\text{Conf}_n(P^{m-1}) \Rightarrow \text{Conf}_n(P^{m-1}) \\
\Downarrow \chi \quad \Downarrow \chi \\
X_{[i_0]} \Rightarrow X_{[i_0]}
\end{array}
\]

By definition the rational map on the top maps \([i_1, \ldots, i_n] \to [h_{[2-m,n]}, h_{[3-m,1]}, \ldots h_{[1-m,n-1]}] \), which is in fact a regular map and coincide with our description of the Donaldson-Thomas transformation DT in Theorem 1.1; when we say that DT on Conf\(_n(P^{m-1})\) is the Donaldson-Thomas transformation of Conf\(_n(P^{m-1})\), what we actually mean is that the bottom map \( \psi \circ \chi \), which by an abuse of notation we also denote as DT, is the Donaldson-Thomas transformation of the cluster Poisson variety \( X_{[i_0]} \).

By Goncharov and Shen’s theorem (Theorem 2.28), in order to prove Theorem 1.1, it suffices to show the following two things for some seed \( X \)-torus \( X \) and the restriction of the induced map DT on it:

1. DT : \( X_i \Rightarrow X_i \) is a cluster \( X \)-transformation, i.e., a composition of cluster \( X \)-mutations and cluster \( X \)-isomorphisms;
2. its tropicalization DT\( ^t \) on \( X_i \) maps positive basic \( X \)-laminations to negative basic \( X \)-laminations.

As it turns out, the easiest seed \( X \)-torus to work with is \( X_{[i_0]} \), where \( i_0 \) is the boundary-removed quiver as described in Example 2.11.

Before we start the proof of Theorem 1.1, we need one more piece of construction which we call dual minimal bipartite graph: given a minimal bipartite graph \( \Gamma \), we define its dual minimal bipartite graph \( \Gamma^* \) to be the one obtained by flipping \( \Gamma \) over the diameter bisecting the (either) arc between \( m \) and \( m+1 \). It is not hard to verify that \( \Gamma^* \) is also a minimal bipartite graph. By convention, we will denote the quiver and its corresponding boundary-removed quiver associated to \( \Gamma^* \) as \( i^* \) and \( i^* \) respectively. Take the minimal
bipartite graph $\Gamma_0$ for example. It’s dual $\Gamma_0^*$ looks like the following.

Using the dual boundary-removed quiver $i_0^*$, we can prove the following lemma, which is exactly part (1) of the proof of Theorem 1.1.

**Lemma 4.1.** The induced birational equivalence $\text{DT} : \mathcal{X}_{i_0} \rightarrow \mathcal{X}_{i_0}$ is a cluster $\mathcal{X}$-transformation.

**Proof.** Consider the dual minimal bipartite graph $\Gamma_0^*$ and its associated boundary-removed quiver $i_0^*$. Recall that the map $*$ on $\text{Conf}_n(\mathbb{P}^{m-1})$ is rationally equivalent to the composition $r \circ \chi \circ \psi$, which implies that $\chi$ is rationally equivalent to the composition $r \circ * \circ \psi^{-1}$ (note that $r$ is an involution and $\psi$ is a birational equivalence). We can then form the following commutative diagram, where $\mu$ is a composition of cluster $\mathcal{X}$-mutations given by Thurston’s theorem (Corollary 2.12).

Since we know that $\mu$ is a composition of cluster $\mathcal{X}$-mutations, it suffices to show that the composition

$$\sigma := \psi_{i_0}^{-1} \circ r \circ * \circ \psi_{i_0}^{-1}$$

is a cluster $\mathcal{X}$-isomorphism.
We will continue to use a pair of integers to represent vertices in the quiver \( \hat{\mathcal{I}}^*_0 \) as we have done in \( \hat{\mathcal{I}}_0 \) in Example 2.11, which turns out to look like the following.

If we denote the dominating set of the face \((i, j)\) in \( \Gamma^*_0 \) by \( I^*_{i,j} \), then we find that

\[
I^*_{i,j} = \begin{cases} 
[1, m] & \text{if } (i, j) = (0, 0); \\
[i + 1, m] \cup [n - j + 1, n + i - j] & \text{if } (i, j) \neq (0, 0).
\end{cases}
\]

By an abuse of notation let’s denote the permutation \((\frac{1}{m} \cdots m^m m^1 \cdots n)\), (i.e. \( i \mapsto m-i+1 \mod n \)) on \( \{1, \ldots, n\} \) by \( r \) as well. Then we see that by applying \( r \) to \( I^*_{i,j} \), we get

\[
rI^*_{i,j} = \begin{cases} 
[1, m] & \text{if } (i, j) = (0, 0); \\
[1, m - i] \cup [m - i + j + 1, m + j] & \text{if } (j, i) \neq (0, 0).
\end{cases}
\]

which is exactly the same as the dominating set \( I_{j,i} \) for faces of \( \Gamma_0 \) (c.f. Equation (3.11)). By a careful comparison we see that \( r \) not only gives a correspondence of vertices of \( \hat{\mathcal{I}}_0 \) and \( \hat{\mathcal{I}}^*_0 \) on the level of dominating sets, but also this correspondence restricts to a quiver anti-isomorphism between \( \hat{\mathcal{I}}_0 \) and \( \hat{\mathcal{I}}^*_0 \) (a quiver anti-isomorphism \( \tau \) is a bijection between the vertex sets such that the exchange matrices are related by \( \epsilon_{ij} = -\epsilon_{\tau(i)\tau(j)} \)). This implies that

\[
(\psi^\delta_{\hat{\mathcal{I}}_0} \circ r)^* (X^*_{i,j}) = \frac{1}{\psi^\delta_{\hat{\mathcal{I}}_0}(X^*_{j,i})}.
\]

in which we denote the coordinates on \( \mathcal{X}_0^* \) by \( (X^*_{i,j}) \). Now if we then apply \((\ast \circ \psi^{-1}_{\hat{\mathcal{I}}_0})^*\) to both sides, we recovers \( \sigma^* \) on the left hand side and \( \ast^* \) on \( \mathcal{X}_0^* \) on the right hand side, and by Proposition 3.10 we see that

\[
\sigma^* (X^*_{i,j}) = \frac{1}{\ast^*(X^*_{j,i})} = X_{n-m-j,m-i}.
\]

One can also verify from the quivers directly that \((i, j) \mapsto (n - m - j, m - i)\) defines a quiver isomorphism between \( \hat{\mathcal{I}}_0^* \) and \( \hat{\mathcal{I}}_0 \). Therefore we can conclude that \( \sigma \) is a cluster \( \mathcal{X} \)-isomorphism and hence the composition \( DT = \mu \circ \sigma \) from \( \mathcal{X}_0^* \) back to itself is a cluster \( \mathcal{X} \)-transformation.

\[\square\]

**Lemma 4.2.** The induced birational equivalence \( DT : \mathcal{X}_0 \rightarrow \mathcal{X}_0 \) has the property that

\[
\deg_{\mathcal{X}_0}(DT^*(X_g)) = -\delta_{fg}.
\]

**Proof.** Note that the induced birational equivalence \( DT \) on \( \mathcal{X}_0 \) is rationally equivalent to the following composition, where \( s \) is an arbitrary section of \( \text{Conf}_n(\mathbb{P}^m - \{0\}) \rightarrow \text{Conf}_n(\mathbb{P}^{m-1}) \):

\[
DT \cong \psi_{\mathcal{I}_0} \circ \chi_{\mathcal{I}_0} \cong q \circ \hat{\mathcal{I}}_0 \circ \psi_{\hat{\mathcal{I}}_0} \circ s \circ \chi_{\mathcal{I}_0}.
\]

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The right hand side tells us that $\text{DT}^*(X_f)$ can be computed as a quotient between Plücker coordinates according to the combinatorial data given by the quiver $\tilde{i}_0$ and the dominating sets of its vertices.

Recall from Equation (3.11) that the dominating sets of the vertices of $\tilde{i}_0$ are

$$I_{i,j} = \begin{cases} [1, m] & \text{if } (j,i) = (0,0); \\ [1, m-j] \cup [m+i-j+1, m+i] & \text{if } (i,j) \neq (0,0). \end{cases}$$

Also recall from Proposition 3.9 that for a generic point $[v_1, \ldots, v_n] := s \circ \chi_{i_0}(X_f)$, its Plücker coordinate $\Delta_{I_{i,j}}[v_1, \ldots, v_n]$ is a polynomial obtained by summing up families of pairwise disjoint paths in the special perfect orientation on $\Gamma_0$ going from $[1, m] - I_{i,j}$ to $[m+1, n] \cap I_{i,j}$. From the identity above we can compute the following.

\begin{align*}
[1, m] - I_{i,j} &= \begin{cases} \emptyset & \text{if } (i,j) = (0,0); \\ [m-j+1, m+i-j] & \text{if } (i,j) \neq (0,0) \text{ and } i \leq j; \\ [m-j+1, m] & \text{if } (i,j) \neq (0,0) \text{ and } i \geq j. \end{cases} \\
[m+1, n] \cap I_{i,j} &= \begin{cases} \emptyset & \text{if } (i,j) = (0,0); \\ [m+1, m+i] & \text{if } (i,j) \neq (0,0) \text{ and } i \leq j; \\ [m+i-j+1, m+i] & \text{if } (i,j) \neq (0,0) \text{ and } i \geq j. \end{cases}
\end{align*}

One great feature of the minimal bipartite graph $\Gamma_0$ is that its special perfect orientation is really special: it forms a grid that only go to the right horizontally and down vertically (see Proposition 2.35).

Thus for any dominating set $I_{i,j}$, there is a unique such family of pairwise disjoint path that maximizes the degree of all face variables, which consists of a collection of \(\Gamma\)-shaped paths going from $[1, m] - I_{i,j}$ to $[m+1, n] \cap I_{i,j}$ as follows.

Thus for any dominating set $I_{i,j}$, there is a unique such family of pairwise disjoint path that maximizes the degree of all face variables, which consists of a collection of \(\Gamma\)-shaped paths going from $[1, m] - I_{i,j}$ to $[m+1, n] \cap I_{i,j}$ as follows.

Now we just need to find out what is the degree of a given face variable $X_{k,l}$ in $\text{DT}^*(X_{i,j})$, which is a certain ratios of Plücker coordinates of $s \circ \chi_{i_0}(X_f)$. Depending on where $(i,j)$ is in the quiver $i_0$, we need to
break the computation into six different cases.

What’s left in the proof is just a lengthy computation of each case. To save time, we will just do the last case, which includes all techniques required to compute all other cases.

First we can pair up \((i, i - 1)\) with \((i + 1, i)\), \((i - 1, i - 1)\) with \((i + 1, i + 1)\), \((i - 1, i)\) with \((i, i + 1)\), based on the difference of the two coordinates within each pair. Next we make the observation that in each pair, the two families of \(\heartsuit\)-shaped paths only differ by one or two. Let’s consider the pair \((i, i - 1)\) and \((i + 1, i)\) for example. The family of \(\heartsuit\)-shaped paths in \(\Delta_{i,i-1}\) goes from \([m-i+2, m]\) to \([m+2, m+i]\); since \(\Delta_{i+1,i}\) is in the numerator and \(\Delta_{i,i-1}\) is in the denominator, the overall contribution of this pair is a single \(\heartsuit\)-shaped path going from \(m-i+1\) to \(m+i+1\) in the numerator.

Similarly, the overall contribution of the pair \(((i-1, i), (i, i + 1))\) is a single \(\heartsuit\)-shaped path going from \(m-i\) to \(m+i\) in the numerator, and the overall contribution of the pair \(((i-1, i-1), (i+1, i+1))\) is two disjoint \(\heartsuit\)-shaped paths going from \([m-i, m-i+1]\) to \([m+i, m+i+1]\) in the denominator.

Now if we draw these four \(\heartsuit\)-shaped paths on \(\Gamma_0\) and remembering whether they belong to the numerator or denominator, we see that the only non-zero degree face variable in this ratio is \(X_{i,i}\), which is of degree \(-1\).
From this we can conclude that
\[
\deg_{X_i} DT^*(X_{i,i}) = \begin{cases} 
0 & \text{if } f \neq (i,i); \\
-1 & \text{if } f = (i,i). 
\end{cases}
\]

\[\square\]

**Proof of Theorem 1.1.** By Lemma 4.1 we know that the induced birational equivalence \( DT : X_{i0} \rightarrow X_{i0} \) is a cluster \( X \)-transformation. Then by Proposition 2.29 it suffices to show that \( \deg_{X_i} DT^*(X_g) = -\delta_{fg} \). But this is precisely what Lemma 4.2 shows, so the proof is complete. \[\square\]

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