HOMOLOGY OF EQUIVARIANT VECTOR FIELDS

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Abstract. Let $K$ be a compact Lie group. We compute the abelianization of the Lie algebra of equivariant vector fields on a smooth $K$-manifold $X$. We also compute the abelianization of the Lie algebra of strata preserving smooth vector fields on the quotient $X/K$.

1. Introduction

1.1. K. Abe and K. Fukui [AbFu2] have considered the first homology group (abelianization) of the group of equivariant smooth diffeomorphisms of a smooth $K$-manifold $X$, where $K$ is finite. They also computed the abelianization for the diffeomorphisms of the quotient orbifold $X/K$. Our results below are the analogues of their results for vector fields in the case that $K$ is a compact Lie group. The vector fields are, in a sense, the Lie algebras of the relevant diffeomorphism groups, so, hopefully, our results indicate that one should be able to generalize the Abe-Fukui results. There are already generalizations in some cases [AbFu1].

1.2. Let $X$ be a smooth $K$-manifold where $K$ is compact. Let $\mathcal{X}(X)$ denote the Lie algebra of smooth vector fields on $X$ and let $\mathcal{X}_c(X)$ denote the subalgebra of vector fields with compact support. If $X$ is algebraic, then $\mathcal{X}(X)$ will denote the polynomial vector fields on $X$. By $\mathcal{X}_c(X)^K$, etc. we mean the $K$-invariant elements in $\mathcal{X}_c(X)$, etc. We will state most of our results for $\mathcal{X}_c(X)^K$; the corresponding results for $\mathcal{X}(X)^K$, etc. follow easily from our techniques.

1.3. Let $x \in X$. Then we have the isotropy group $K_x$ and its slice representation on $W_x := T_xX/T_x(Kx)$ where $Kx$ denotes the $K$-orbit through $x$. We say that the orbit $Kx$ is isolated if $W_x$ is a point. It follows from the differentiable slice theorem that $Kx$ is isolated if and only if all isotropy groups $K_y$ of points $y$ near $x$, $K_y \neq Kx$, are conjugate to a proper subgroup of $K_x$. There is then a discrete subset $\{x_i\}_{i \in I}$ of $X$ (possibly empty) where we choose one point from each isolated orbit. Let $H_i$ denote $K_{x_i}$ and set $W_i := W_{x_i}, i \in I$.

Theorem 1.4. Let $X$ and the $x_i, H_i$ and $W_i$ be as above Then

$$\mathcal{H}(\mathcal{X}_c^\infty(X)^K) \cong \bigoplus_i \mathcal{H}(\mathfrak{k}^{H_i}/\mathfrak{h}^{H_i}) \bigoplus \mathcal{H}(\text{End}(W_i)^{H_i}).$$

Theorem 1.5. Let $H$ be a compact Lie group and $V$ an $H$-module where $V^H = \{0\}$. Write $V = \bigoplus_{j=1}^m n_j V_j$ where the $V_j$ are irreducible and pairwise non-isomorphic and $n_j V_j$ denotes the direct sum of $n_j$ copies of $V_j$. Let $l$ denote the number of $V_j$ such that $\text{End}(V_j)^H \cong \mathbb{C}$ and let $Z(\text{End}(V)^H)$ denote the center of $\text{End}(V)^H$. Then

$$\mathcal{H}(\text{End}(V)^H) \cong Z(\text{End}(V)^H) = \bigoplus_j Z(\text{End}(n_j V_j)^H) \cong \mathbb{R}^{m-l} \oplus \mathbb{C}^l.$$
Let $\mathcal{X}^\infty_c(X/K)$ denote the Lie algebra of compactly supported smooth strata preserving vector fields on $X/K$ (see §4 for definitions).

**Theorem 1.6.** Let $X$ and the $x_i$, $H_i$ and $W_i$ be as above. Then

$$\mathcal{H}(\mathcal{X}^\infty_c(X/K)) \simeq \bigoplus_i (Z(\End(W_i)^{H_i}))/\mathfrak{s}_i$$

where each $\mathfrak{s}_i$ is the Lie algebra of a torus $S_i$ lying in $Z(\End(W_i)^{H_i})$.

We will say more about the $S_i$ in §4.

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## 2. Vanishing of abelianizations

2.1. In the following, let $\mathcal{B}^\infty_c(X)^K$ denote $[\mathcal{X}^\infty_c(X)^K, \mathcal{X}^\infty_c(X)^K]$ and let $\mathcal{C}^\infty_c(X)^K$ denote the compactly supported smooth functions on $X$. Our first goal is to show that $\mathcal{H}(\mathcal{X}^\infty_c(X \times \mathbb{R})^K)$ is zero.

**Lemma 2.2.** Let $A \in \mathcal{X}^\infty_c(X)^K$ and $B \in \mathcal{X}^\infty_c(X)^K$. Then $[A, B] \in \mathcal{B}^\infty_c(X)^K$.

**Proof.** Let $g \in \mathcal{C}^\infty_c(X)^K$ be identically 1 on a neighborhood of supp $A$. Then $[A, gB] = g[A, B] + A(g)B = [A, B] \in \mathcal{B}^\infty_c(X)^K$.

**Proposition 2.3.** Let $K$ act on $X \times \mathbb{R}$ with the given action on $X$ and the trivial action on $\mathbb{R}$. Then $\mathcal{H}(\mathcal{X}^\infty_c(X \times \mathbb{R})^K) = 0$.

**Proof.** Let $t$ denote the usual coordinate function on $\mathbb{R}$ and let $g \in \mathcal{C}^\infty_c(X \times \mathbb{R})^K$. We show that $g \frac{d}{dt} \in \mathcal{B}^\infty_c(X \times \mathbb{R})^K$. For $x \in X$ and $s \in \mathbb{R}$ set $h(x, s) = \int_0^s g(x(u), u) du$. Then $h$ is smooth and $K$-invariant. Let $f \in \mathcal{C}^\infty_c(X \times \mathbb{R})^K$. Then

$$\left[ f \frac{d}{dt}, h \frac{d}{dt} \right] = f \frac{dh}{dt} \frac{d}{dt} - h \frac{df}{dt} \frac{d}{dt} \quad \text{and} \quad \left[ \frac{d}{dt}, fh \frac{d}{dt} \right] = f \frac{dh}{dt} \frac{d}{dt} + h \frac{df}{dt} \frac{d}{dt}.$$ 

Hence $2fg \frac{d}{dt} \in \mathcal{B}^\infty_c(X \times \mathbb{R})^K$. If $f$ equals 1/2 on a neighborhood of supp $g$, we obtain that $g \frac{d}{dt} \in \mathcal{B}^\infty_c(X \times \mathbb{R})^K$.

Now suppose that $A \in \mathcal{X}^\infty_c(X \times \mathbb{R})^K$. By our result above, we can assume that $A$ annihilates $t$. Set $B(x, s) = \int_0^s A(x(u), u) du$ and let $g \in \mathcal{C}^\infty_c(X \times \mathbb{R})^K$ equal 1 on a neighborhood of supp $A$. Then $g \frac{d}{dt}, B = ga - B(g) \frac{d}{dt}$. We already know that $B(g) \frac{d}{dt} \in \mathcal{B}^\infty_c(X \times \mathbb{R})^K$, hence $A \in \mathcal{B}^\infty_c(X \times \mathbb{R})^K$. Thus $\mathcal{H}(\mathcal{X}^\infty_c(X \times \mathbb{R})^K) = 0$.

2.4. Let $H$ be a closed subgroup of $K$ and $W$ an $H$-module. Then we have the twisted product $K \ast^H W$ which is the quotient $(K \times W)/H$ where $h(k, w) = (kh^{-1}, hw), h \in H, k \in K$ and $w \in W$. We denote the image of $(k, w) \in K \times W$ in $K \ast^H W$ by $[k, w]$. Note that $K \ast^H W$ is naturally a $K$-vector bundle and a real algebraic $K$-variety [Schw3].

Let $H \to \text{GL}(W)$ be the slice representation at a point $x \in X$. By the differentiable slice theorem, a $K$-neighborhood of $Kx$ in $X$ is $K$-diffeomorphic to $K \ast^H W$. By Proposition 2.3, $\mathcal{H}(\mathcal{X}^\infty_c(K \ast^H W)^K) = 0$ if $W^H \neq \{0\}$.

Let $F$ be a closed $K$-stable subset of $X$. We say that $\mathcal{H}(\mathcal{X}^\infty_c(X)^K)$ is supported on $F$ if $\mathcal{H}(\mathcal{X}^\infty_c(X \setminus F)^K) = 0$. Using a partition of unity argument we can show
Corollary 2.5. Let \( F = \{ x \in X \mid W_x^K = 0 \} \). Then \( \mathcal{H}(\mathcal{X}_c^\infty(X)^K) \) is supported on \( F \).

3. Local computations

3.1. Our results above show that there is a discrete set of orbits \( \{ Kx_i \} \) such that
\[
\mathcal{H}(\mathcal{X}_c^\infty(X)^K) \simeq \bigoplus_i \mathcal{H}(\mathcal{X}_c^\infty(K \ast H, W_i)^K)
\]
where \( H_i = Kx_i \) and \( W_i \) is the slice representation of \( H_i \) at \( x_i \). Thus it suffices to compute \( \mathcal{H}(\mathcal{X}_c^\infty(K \ast H V)^K) \) where \( H \) is a closed subgroup of \( K \), \( V \) is an \( H \)-module and \( V^H = (0) \). This computation is the content of the following theorem.

Theorem 3.2. Let \( H \) and \( V \) be as above. Then
\[
\mathcal{H}(\mathcal{X}_c^\infty(K \ast H V)) \simeq \mathcal{H}(\mathfrak{h}^H / \mathfrak{h}^H) \oplus \mathcal{H}(\text{End}(V)^H).
\]

3.3. Our proof of the theorem requires several lemmas. Set \( Y := K \ast H V \). Then
\[
\mathcal{X}(Y)^K \simeq \mathcal{X}(K \times V)^{K \times H} / (\mathcal{O}(K \times V)\mathfrak{h})^{K \times H}
\]
(see [Schw2, §4]) where \( H \) has the diagonal action (see 2.4) on \( K \times V \) (inducing an action of \( \mathfrak{h} \)) and \( \mathcal{O}(K \times V) \) denotes the polynomial functions on \( K \times V \). Now
\[
\mathcal{X}(K \times V)^{K \times H} \simeq (\mathcal{X}(K) \otimes \mathcal{O}(V) \oplus \mathcal{O}(K) \otimes \mathcal{X}(V))^{K \times H} \simeq (\mathfrak{h} \otimes \mathcal{O}(V))^H \oplus (1 \otimes \mathcal{X}(V)^H)
\]
while
\[
(\mathcal{O}(K \times V)\mathfrak{h})^{K \times H} \simeq (\mathfrak{h} \otimes \mathcal{O}(V))^H.
\]

3.4. We have the Euler operator \( E \in \mathcal{X}(V)^H \), where if \( x_1, x_2, \ldots \) are coordinate functions on \( V \), then \( E = \sum_i x_i \frac{\partial}{\partial x_i} \). By the isomorphisms above, \( E \) can be considered as a \((K \times H)\)-invariant vector field on \( K \times V \) and as a \( K \)-invariant vector field on \( Y \).

Lemma 3.5. Let \( f \in \mathcal{C}^\infty(Y)^K \). Then \( f = E(h) \) for some \( h \in \mathcal{C}^\infty(Y)^K \) if and only if \( f([e, 0]) = 0 \).

Proof. Clearly the condition on \( f \) is necessary. Suppose that \( f([e, 0]) = 0 \). Since \( f \) is \( K \)-invariant, it is determined by its restriction \( g \) to \( \{ [e, v] \mid v \in V \} \simeq V \), where \( g \) is \( H \)-invariant. Set \( h(v) = \int_0^1 (1/t)g(tv) \, dt \). Then \( h \in \mathcal{C}^\infty(V)^H \) since \( g(0) = 0 \). We have
\[
E(h)(v) = \int_0^1 \frac{1}{t} \sum_i x_i \frac{\partial g}{\partial x_i}(tv) t \, dt = \int_0^1 \sum_i x_i \frac{\partial g}{\partial x_i}(tv) \, dt = \int_0^1 \frac{d}{dt} g(tv) \, dt = g(v) - g(0) = g(v).
\]

Corollary 3.6. Let \( g \in \mathcal{C}^\infty_c(Y)^K \) such that \( g([e, 0]) = 0 \). Then \( gE \in \mathcal{B}_c^\infty(Y)^K \).

Proof. By Lemma 3.5, \( g = E(h) \) for some \( h \in \mathcal{C}^\infty(Y)^K \). Let \( f \in \mathcal{C}^\infty_c(Y)^K \) such that \( f \) is 1/2 in a neighborhood of \( \text{supp} g \). Then, as in Proposition 2.3,
\[
[E, fhE] + [fE, hE] = 2fE(h)E = 2fgE,
\]
so that \( gE \in \mathcal{B}_c^\infty(Y)^K \).

3.7. Since \( Y \) is real algebraic, the results in [Schw1, §6] show that \( \mathcal{X}_c^\infty(Y) \simeq \mathcal{C}^\infty(Y) \otimes \mathcal{O}(Y) \mathcal{X}(Y) \). For compactly supported sections we clearly have that \( \mathcal{X}_c^\infty(Y) = \mathcal{C}^\infty_c(Y) \mathcal{X}(Y) \).
3.8. We have an $E$-eigenspace decomposition

$$\mathcal{X}(K \times V)^K \cong \bigoplus_{m \geq 0} (\mathfrak{t} \otimes \mathcal{O}(V)_m)^H \oplus (1 \otimes \mathcal{X}(V)^H_m)$$

and similarly for $(\mathfrak{h} \otimes \mathcal{O}(V))^H$. The weights that occur in $\mathcal{X}(V)^H$ are all positive since $V^H = (0)$. We have an induced decomposition

$$\mathcal{X}(Y)^K = \bigoplus_{m \geq 0} \mathcal{X}(Y)^K_m.$$ 

Remark 3.9. Since the sum only contains terms for $m \geq 0$, an element of $\mathcal{X}(Y)^K$ applied to an element of $C^\infty(Y)^K \cong C^\infty(V)^H$ always vanishes at $[e, 0]$.

Lemma 3.10. Let $A \in \mathcal{X}(Y)^K_m$ and let $f \in C^\infty_c(Y)^K$. Then $fA \in B_c^\infty(Y)^K$ if

1. $m > 0$ or
2. $f([e, 0]) = 0$.

Proof. Suppose that $m > 0$. Then $[(1/m)fE, A] = fA - (1/m)A(f)E$ where $A(f)E \in B_c^\infty(Y)^K$ by Corollary 3.6. Hence $fA \in B_c^\infty(Y)^K$. If $m = 0$ and $f([e, 0]) = 0$, then let $h \in C^\infty(Y)^K$ be such that $E(h) = f$, and let $g \in C^\infty_c(Y)^K$. Then

$$[gE, hA] = gE(h)A - hA(g)E = gfA - hA(g)E,$$

where $hA(g)E \in B_c^\infty(Y)^K$ by Corollary 3.6. We may arrange that $gfA = fA$, so $fA \in B_c^\infty(Y)^K$. □

Proof of Theorem 3.2. We first define a map of Lie algebras $\varphi: \mathcal{X}_c^\infty(Y)^K \to \mathcal{X}(Y)^K_0$. Let $B = \sum_{i=1}^m f_iB_i \in \mathcal{X}_c^\infty(Y)^K$ where $f_i \in C^\infty_c(Y)^K$ and $B_i \in \mathcal{X}(Y)^K_{m_i}$, $i = 1, \ldots, m$. Define $\varphi(B) := \sum_{m_i=0} f_i([e, 0])B_i \in \mathcal{X}(Y)^K_0$. It is obvious that $\varphi$ is surjective. Suppose that $C, D \in \mathcal{X}(Y)^K$ are eigenvectors for $E$ and that $g, f \in C_c^\infty(Y)^K$. Then $[fC, gD] = fC(g)D - gD(f)C + fg[C, D]$ where $C(g)$ and $D(f)$ vanish at $[e, 0]$. Thus $\varphi([fC, gD]) = (fg)(0)[C, D] = (fg)(0)[\varphi(C), \varphi(D)] = [\varphi(fC), \varphi(gD)]$. Now $\varphi$ induces $\tilde{\varphi}: \mathcal{H}(\mathcal{X}_c^\infty(Y)^K) \to \mathcal{H}(\mathcal{X}(Y)^K_0)$, which is again surjective. Suppose that $B = \sum_i f_iB_i \in \text{Ker}(\tilde{\varphi})$ where the $B_i$ are in $\mathcal{X}(Y)^K_0$. Then $\varphi(B) = \sum_j [C_j, D_j]$ where $C_j, D_j \in \mathcal{X}(Y)^K_0$ for all $j$. Let $f \in C_c^\infty(Y)^K$ such that $f$ is 1 on a neighborhood of $[e, 0]$. Then $B - \sum_j [fC_j, fD_j] \in B_c^\infty(Y)^K$. Hence $\tilde{\varphi}$ is an isomorphism. From our equations in 3.3 it follows that $\mathcal{H}(\mathcal{X}(Y)^K_0) \cong \mathcal{H}(\mathfrak{t}^H/\mathbf{h}^H) \oplus \mathcal{H}(\text{End}(V)^H)$. □

Proof of Theorem 1.4. The theorem is immediate from 3.1 and Theorem 3.2 □

Proof of Theorem 1.5. Let $V = \bigoplus_{j=1}^m n_jV_j$ and $H$ be as in 1.5. Then $\text{End}(V)^H \cong \bigoplus_j \text{End}(n_jV_j)^H$.

There are three cases to consider.

Case 1: $\text{End}(V_j)^H \cong \mathbb{R}$. Then $\text{End}(n_jV_j)^H \cong \mathfrak{gl}(n_j, \mathbb{R})$ and $\mathcal{H}(\mathfrak{gl}(n_j, \mathbb{R})) \cong Z(\mathfrak{gl}(n_j, \mathbb{R})) \cong \mathbb{R}$.

Case 2: $\text{End}(V_j)^H \cong \mathbb{C}$. Then $\text{End}(n_jV_j)^H \cong \mathfrak{gl}(n_j, \mathbb{C})$ and $\mathcal{H}(\mathfrak{gl}(n_j, \mathbb{C})) \cong Z(\mathfrak{gl}(n_j, \mathbb{C})) \cong \mathbb{C}$.

Case 3: $\text{End}(V_j)^H \cong \mathbb{H}$, the quaternions. Then $\text{End}(n_jV_j)^H \cong \mathfrak{gl}(n_j, \mathbb{H})$ and we have that $\mathcal{H}(\mathfrak{gl}(n_j, \mathbb{H})) \cong Z(\mathfrak{gl}(n_j, \mathbb{H})) \cong \mathbb{R}$. The theorem follows. □

4. Computations on the quotient

We now consider the abelianization of the strata preserving vector fields on the quotient $X/K$. We recall a few facts about $X/K$ from [Schw1]. Let $\pi: X \to X/K$ denote the canonical map, where $X/K$ is given the quotient topology. Then $X/K$ has a differentiable structure where for $U$ an open subset of $X/K$, $C^\infty(U) = C^\infty(\pi^{-1}(U))^K$. Let $H$ be a closed subgroup of $K$. Then we have the corresponding stratum $X^{(H)} := \{ x \in X \mid K_x \text{ is conjugate to } H \}$ and its image $(X/K)^{(H)} \subset X/K$. 

The isotropy strata \((X/K)^{(H)}\) \(\subset X/K\) and \(X^{(H)}\) \(\subset X\) are smooth and locally closed submanifolds and \(\pi: X^{(H)} \rightarrow (X/K)^{(H)}\) is naturally a smooth fiber bundle (with structure group \(N_K(H)/H\)). The number of isotropy strata is locally finite on \(X\) and \(X/K\). Let \(\text{Der}(C^\infty(X/K))\) denote the derivations of \(C^\infty(X/K)\) and let \(\mathcal{X}^\infty(X/K)\) denote those derivations that preserve the ideals of functions \(I_H\), vanishing on the isotropy strata \((X/K)^{(H)}\) of \(X/K\). Each element of \(\mathcal{X}^\infty(X)^K\) restricts to a derivation of \(C^\infty(X/K)\), so there is a canonical map \(\pi_*: \mathcal{X}^\infty(X)^K \rightarrow \text{Der}(C^\infty(X/K))\).

The main theorem of [Schw1] is that \(\text{Im} \pi_* \subset \mathcal{X}^\infty(X/K)\) and that \(\pi_*\) is surjective. Clearly \(\pi_*\) is a homomorphism of Lie algebras so we have an induced surjection \(\mathcal{H}(\mathcal{X}^\infty(X)^K) \rightarrow \mathcal{H}(\mathcal{X}^\infty(X/K))\).

We only need to compute what happens in the case of \(X = K \ast^HV\) where \(H\) is a closed subgroup of \(K\) and \(V\) is an \(H\)-module such that \(V^H = (0)\). Let \(V = \oplus_{j=1}^m n_j V_j\) as in Theorem 1.5. The following has Theorem 1.6 as a corollary.

**Theorem 4.1.** Assume that \(\text{End}(V_j)^H \simeq \mathbb{C}\) if and only if \(j \leq l\) where \(l \leq m\). Let \(T\) be the corresponding torus \((S^1)^l \subset \prod_{j=1}^l Z(\text{End}(V_j)^H)\). Then \(T\) acts on \(V\) commuting with the action of \(H\), and we have an induced map \(T \rightarrow \text{Aut}(V/H)\). Let \(S\) denote the kernel where \(\dim S = k\). Then

\[\mathcal{H}(\mathcal{X}^\infty((K \ast^HV)/K)) \simeq \mathcal{H}(\mathcal{X}(V/H)) \simeq \mathbb{R}^{m-l+k} \oplus \mathbb{C}^{l-k}.\]

**Proof.** We have the canonical surjection of Lie algebras \(\pi_*: \text{End}(V)^H \rightarrow \mathcal{X}_0(V/H)\) and \(\pi_*\) induces a surjection of \(\mathcal{H}(\text{End}(V)^H)\) onto \(\mathcal{H}(\mathcal{X}(V/H))\). For every \(j\) we have the identity \(\text{Id}_j \in \text{End}(n_j V_j)^H\) and clearly these elements give linearly independent derivations of \(\mathcal{O}(V)^H\). Now consider the action of \(T\) on \(V/H\) and its kernel \(S\). Then \(\mathfrak{s}\) is the kernel of the restriction of \(\pi_*\) to the center of \(\text{End}(V)^H\), so that \(\mathfrak{s}\) is the kernel on homology. \(\square\)

**Example 4.2.** Suppose that \(H\) is a torus acting faithfully on \(V\) and \(V = \sum_{j=1}^m n_j V_j\) where \(V^H = (0)\) as in Theorem 1.5. Then \(\mathfrak{s} \simeq \mathfrak{h}\) and \(\mathcal{H}(\mathcal{X}(V/H)) \simeq \mathbb{R}^{k} \oplus \mathbb{C}^{m-k}\) where \(k = \dim H\).

**Example 4.3.** Let \(V = \mathbb{C}^n \oplus \wedge^2 \mathbb{C}^n\) with the canonical action of \(\text{SU}(n, \mathbb{C})\), \(n \geq 3\). Then \(T\) has dimension 2 and \(S\) has dimension 1. See [Schw1, Table I].

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