Analysis of a SU(4) generalization of Halperin’s wave functions as an approach towards a SU(4) fractional quantum Hall effect in graphene sheets

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Inspired by the four-fold spin-valley symmetry of relativistic electrons in graphene, we investigate a possible SU(4) fractional quantum Hall effect, which may also arise in bilayer semiconductor quantum Hall systems with small Zeeman gap. SU(4) generalizations of Halperin’s wave functions [Helv. Phys. Acta \textbf{56}, 75 (1983)], which may break differently the original SU(4) symmetry, are studied analytically and compared, at \( \nu = 2/3 \), to exact-diagonalization studies.

The fractional quantum Hall effect (FQHE) in conventional semiconductor (typically GaAs) heterostructures may be understood to great extent with the help of one-component U(1) trial wave functions, such as Laughlin’s \[1\] or composite-fermion generalizations of it \[2\]. The effect occurs due to the repulsive electronic interactions when the filling factor, the ratio \( \nu = n_{el}/n_B \) between the electronic density \( n_{el} \) and the flux density \( n_B = B/(\hbar/e) \), belongs to the “magic” series \( \nu = p/(2s \pm 1) \), with integral \( s \) and \( p \), or particle-hole symmetric fillings.

Soon after Laughlin’s original proposal, Halperin \[3\] and the flux density \( B \) until extremely high magnetic fields \[3\]. Instead, he proposed a two-component generalization of Laughlin’s wave function in order to account for the spin SU(2) degree of freedom \[3\]. Indeed, polarization measurements indicate that several states, such as e.g. \( \nu = 2/3 \) and \( 2/5 \), are not spin-polarized and may sometimes undergo transitions between competing FQHE states with different spin polarization when the total magnetic field is varied at constant filling factor \[3\].

The recent observation of a relativistic integral quantum Hall (QH) effect in graphene \[5, 6\] has led to several theoretical investigations concerning an eventual FQHE in this novel carbon compound \[5, 7, 8, 9, 10, 11, 12\]. In contrast to the abovementioned GaAs heterostructures, graphene has an underlying SU(4) symmetry due to the two-fold valley degeneracy in addition to the physical spin degree of freedom. The Coulomb interaction respects this symmetry to lowest order in \( a/l_B \sim 10^{-2} \ldots 10^{-1} \), where \( a = 0.14 \)nm is the distance between nearest-neighbor carbon atoms, and \( l_B = \sqrt{\hbar/eB} = 25/\sqrt{B[T]K} \)nm is the magnetic length \[5, 12\]. In previous exact-diagonalization studies on potential candidates of a FQHE in graphene \[5, 10\], the SU(4) symmetry has been omitted and a complete spin polarization presupposed. In this case, the two-fold valley degeneracy may be mimicked by a SU(2) isospin, and graphene may be treated as a non-relativistic SU(2) QH system if one replaces the non-relativistic by the relativistic effective potentials \[8, 14\]. More recently, Töke and Jain have proposed a composite-fermion construction at \( \nu = p/(2sp + 1) \) with an internal SU(4) symmetry \[12\].

Here, we investigate a graphene SU(4) FQHE, where we omit the assumption of complete spin-polarization because the ratio \( \Delta_Z/(e^2/\ell_B) \approx 0.003 \ldots 0.014 \sqrt{B[T]K} \) remains small (the precise value depends on the effective dielectric constant which may vary from \( \epsilon \approx 1 \) for a free graphene layer to \( \epsilon \approx 5 \) on a SiO substrate \[13\]). We propose a SU(4) generalization of Halperin’s wave function, which may lead to a plethora of new FQHE states not captured in previous studies. We mainly discuss the filling factor \( \nu = 2/3 \), where our SU(4) exact-diagonalization results indicate rich physics beyond the SU(2) case \[8, 10\]. Notice that the graphene filling factor \( \nu_G \) is defined with respect to the Dirac points, and the central Landau level (LL) is therefore half-filled at \( \nu_G = 0 \), whereas in semiconductor QH systems \( \nu \) is defined with respect to the bottom of the lowest LL. In order to make the connection between the two of them, taking into account the four-fold degeneracy, one has to choose \( \nu_G = -2 + \nu \) (or \( \nu_G = 2 - \nu \), due to electron-hole symmetry).

In the spirit of Ref. \[8\], we generalize Laughlin’s wave function \[1\] to SU(\( K \)) with \( K \) components,

\[ \psi_{m_1, \ldots, m_K; n_{ij}}^{SU(K)} = \phi_{m_1, \ldots, m_K}^{L} \phi_{n_{ij}}^{\text{inter}} e^{-\sum_{j=1}^{K} \sum_{k_j=1}^{N_j} |z_{k_j}^{(j)}|^2/4}, \]

where

\[ \phi_{m_1, \ldots, m_K}^{L} = \prod_{j=1}^{K} \prod_{k_j < l_j} \left( z_{k_j}^{(j)} - z_{l_j}^{(j)} \right)^{m_j} \]

is a product of \( K \) Laughlin wave functions, and

\[ \phi_{n_{ij}}^{\text{inter}} = \prod_{i < j} \prod_{k_i, k_j} \left( z_{k_i} - z_{k_j}^{(j)} \right)^{n_{ij}} \]

is a product of \( K \times K \) Laughlin wave functions, taking into account the four-fold valley degeneracy.

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Tab. I shows examples of SU(4) N-particle wave functions (1), labeled by the set of exponents \([m_1m_2m_3m_4, n_+, n_-, n_z, n_e]\), where we have restricted the 10 exponents to only 7, \(n_+ \equiv n_{13}, n_- \equiv n_{24}, \) and \(n_e \equiv n_{12} = n_{14} = n_{23} = n_{34}.\) This means that we treat all intervalley correlations on the same footing, with an exponent \(n_e,\) and the intravalley correlations for different spin are described by the exponent \(n_+\) and \(n_-\) in the valleys + and −, respectively. Apart from the total filling factor \(\nu_T = \nu_1 + \nu_2 + \nu_3 + \nu_4,\) the \([m_1m_2m_3m_4, n_+n_-n_zn_e]\) wave functions have the spin \(S_z/(N/2) = [\nu_1 + \nu_2 - (\nu_3 + \nu_4)]/\nu_T,\) the valley \(I_z/(N/2) = [\nu_1 + \nu_3 - (\nu_2 + \nu_4)]/\nu_T,\) and a third polarization \(P_z/(N/2) = [\nu_1 + \nu_4 - (\nu_2 + \nu_3)]/\nu_T.\) These polarizations are good quantum numbers associated with the mutually commuting \(z\)-components of the SU(4) spin, \(\tau_z \otimes 1, 1 \otimes \tau_z,\) and \(\tau_z \otimes \tau_z,\) respectively, in terms of the diagonal Pauli matrix \(\tau_z.\) Although they vary in the interval \([-N/2, N/2],\) they are not independent and indeed restricted to the interior of a tetrahedron depicted in Fig. 1.

Some physical properties of the SU(4) states may be characterized by the rank \(r\) of the matrix \(M_4.\) In the case of \(r = 4,\) as e.g.
for the \([3333, 111]\) wave function at \(\nu = 2/3\) or \([5555, 222]\) at \(\nu = 4/11,\) all \(\nu_j\) are uniquely determined by Eq. (2), and the \(S_z, I_z,\) and \(P_z\) are thus fixed. This is represented by the black dot in Fig. 1 for the \([3333, 111]\) wave function. For \(r = 3,\) three rows of \(M_4\) are linearly independent. Consider the first row \([|\uparrow, +\rangle\) component] to be a multiple of the third \([|\downarrow, +\rangle\) component]. In this example, the wave function, such as \([3737, 233]\) at \(\nu = 4/11,\) represents a state with a SU(2) spin ferromagnet in the + valley. Defining a combined filling factor for the + component, \(\nu_+ = \nu_{1+} + \nu_{4+},\) one may describe this state alternatively by a SU(3) wave function (4), with an invertible \(M_3\) matrix. For \(r = 2,\) one may e.g. realize a SU(3) ferromagnet of three components, whereas the combined three-component filling factor and that of the fourth are fixed. Another possibility is that the SU(2) ferromagnet discussed for \(r = 3\) is accompanied by another SU(2) ferromagnet in the −
valley with no coherence between the two of them. Examples are the [3333, 033] and the [3535, 235] wave functions at $\nu = 2/3$ and 4/11, respectively. The polarizations are now restricted to planes with fixed $I_z$, as depicted in Fig. 1. By a simple exchange of the components, harmless in the case of an underlying SU(4) symmetry, $S_z$ or $P_z$ may play the role of $I_z$ so that the planes with fixed $I_z$, $S_z$, and $P_z$ are equivalent. Both ferromagnetic states for $r = 2$ may be described by a SU(2) Halperin wave function with invertable matrix $M_2$. The case $r = 1$ represents a SU(4) ferromagnet the polarizations of which explore the full tetrahedron depicted in Fig. 1. The only constraint is fixed by $\nu_T$, and one finds a SU(4) Laughlin wave function, where all possible states may be obtained by any SU(4) rotation of a state with $\nu_1 = 1/(2s + 1)$ and $\nu_2 = \nu_3 = \nu_4 = 0$.

As becomes apparent from Tab. I, the same filling factor may be realized by different wave functions with different matrix rank $r$. This has to be contrasted to the SU(2) case where one needs to invoke the composite-fermion theory [2] to obtain competing states with different spin polarization at the same filling factor, such as e.g. at $\nu = 2/5$. However, filling factors which do not arise in U(1) or SU(2) wave functions, such as e.g. $\nu = 8/19$ (or 10/23 and 26/47, not shown in Tab. I), may be described by SU(4) wave functions with $r = 4$. One may therefore principally expect a closer vicinity of FQHE states in SU(4) than in SU($K < 4$).

In our numerical studies, we use an exact diagonalization on the sphere with a fully implemented SU(4) invariance. All calculations are performed in the lowest LL. The various SU(4) trial wave functions are obtained by tuning the pseudopotentials in the abovementioned manner. They appear as zero-energy ground states with the lowest number of flux quanta for a given interaction (similarly we obtain the Laughlin state or the Halperin states [18]). The number of flux quanta threading the sphere is $2S$. Due to the large Hilbert spaces we need to consider, our calculations are restricted to the system sizes of $N = 8$ fermions.

In the spherical geometry, the two states [3333,111] and [3333,033] occur at $2S = (3N/2) - 3$. Fig. 2(a) shows the energy spectrum of the FQHE at $\nu = 2/3$, with $2S = 9$ and $N = 8$, for a choice of pseudopotentials $V_m^A = V_m^E = V_m^{E-s} = 0$, except for $V_1^A = V_0^E = V_0^{E-s} = 1$, for which [3333,111] is expected to be the exact ground state. The ground state is found at a total angular momentum $L = 0$, which is, together with the finite gap to all excited states, a condition for an incompressible FQHE state. Moreover, the ground state is non-degenerate and has $S_z = I_z = P_z = 0$, in agreement with the [3333, 111] state (Fig. 1). In Fig. 2(b), the energy spectrum ($2S = 9, N = 8$) is shown for a different pseudopotential choice with $V_m^A = V_m^E = V_m^{E-s} = 0$, except for $V_0^{E-s} = V_1^{E-s} = V_1^A = 1$ – a model for which the [3333,033] state expected to be the exact ground state. One finds a gapped $L = 0$ ground state with $I_z = 0$ and a degeneracy due to a free choice of $S_z$ and $P_z$ within the light gray plane in Fig. 1 as for a [3333,033] state.

The Coulomb interaction in the lowest (central) LL turns out to be more involved. We have first performed Monte-Carlo calculations for up to $N = 70$ particles, which indicate that in this case the [3333,111] state is much lower in energy than the [3333,033] state when compared to the statistical error for both states. Fig. 3(a) shows the energy spectrum for $2S = 9$ and $N = 8$, obtained from exact diagonalization of the lowest-LL Coulomb interaction. The spectrum is reminiscent of that for $2S = 11$ and $N = 8$ [Fig. 3(b)], but it is clearly different from those of the [3333,111] and [3333,033] states. Both spectra show a $L = 0$ ground state that is connected to a collective mode with a striking $L^2$ behavior (black lines). However the ground-state degeneracy is different, as shown in Tab. II. For $2S = 11$, we obtain a $[N/2, N/2] = [4,4]$ multiplet [12], in agreement with results by Töke and Jain [12]. The non-degenerate $S_z = I_z = 0$, $P_z = 4$ state, is, up to an interchange of $S_z$ and $P_z$, the state described by Apalkov and Chakraborty

![FIG. 2: Low-energy spectra, for $N = 8$ and $2S = 9$, with contact potential in the LLL, built in such a way that the trial states (a) [3333,111] and (b) [3333,033] are the only ground states with zero energy. We only show the lowest-energy state for each value of the total angular momentum $L$.](image)

![FIG. 3: Low-energy spectra with Coulomb interaction in the lowest LL (a) for $2S = 9$ and $N = 8$, (b) for $2S = 11$ and $N = 8$. We show only the lowest energy for each value of $L$ in (a), but also the spectrum for $S_z = 0, I_z = 0, P_z = 4$ (crosses), for comparison with the SU(2) case [3]. The continuous line shows a $L^2$ fit through the lowest-energy states. The only fit parameter is known with a precision of 1% (if removing the $L = 1$ point) in both cases.](image)
TABLE II: Degeneracy of the ground state for the Coulomb interaction in the different $0 \leq S_z \leq L_z \leq P_z$ sectors for $N = 8, 2S = 11$ (second row) and $N = 8, 2S = 9$ (third row). (004) and (013) are not ground states for $2S = 9$.

| $(S_z, L_z, P_z)$ | (000) | (002) | (004) | (011) | (013) | (022) | (112) |
|----------------|------|------|------|------|------|------|------|
| deg. $(2S = 11)$ | 3    | 2    | 1    | 1    | 1    | 1    | 1    |
| deg. $(2S = 9)$  | 3    | 1    | –    | 3    | –    | 1    | 1    |

who have considered an internal SU(2) valley symmetry and a complete spin polarization. Indeed, the spectra coincide, but Fig. 3(b) shows clearly that this is only the top of the iceberg and that states with different SU(4) polarization also belong to the ground-state manifold. More importantly, the low-energy properties are dominated by a $L^2$ mode (Fig. 3) we conjecture to be associated with collective spinlike waves the symmetry of which is determined by a subgroup of SU(4). Notice that such mode would not affect the incompressibility of the ground state.

In order to make a connection between the $[3333, 111]$ ground state and that obtained for Coulomb interaction in the lowest LL for $2S = 9$ and $N = 8$, we evaluate the ground states when varying the pseudopotentials $V^E_1$ and $V^E_2$ from 0 (exact model for the $[3333, 111]$ state) to 1, keeping $V^A_1 = V^A_2 = 1$. Here, we set $V^E_m = V^{E-s}_m$ for all $m$. Fig. 4 shows the resulting phase diagram, in terms of the ground state degeneracies. The $[3333, 111]$ state is stable over a large region in the phase diagram, but one also obtains doubly degenerate (at moderate $V^E_1 \sim 0.4...0.5$) and compressible ground states with $L \neq 0$. Most saliently, a $105 \times$ degenerate ground state is obtained at $V^E_1 = 1$, for $V^E_2 > 0.25$, such as that corresponding to the Coulomb interaction (dashed line in Fig. 4). This degeneracy is precisely that of a $[4, 4]$ SU(4) multiplet. The phase is critical in the sense that it is destroyed as soon as we slightly deviate from $V^E_1 = V^A_1 = V^A_2 = 1$.

In conclusion, we have analyzed a SU(4) generalization of Halperin’s wave function $\tilde{\Phi}$, which may be a promising approach towards a SU(4) FQHE eventually occurring in high-mobility graphene sheets. These trial wave functions yield incompressible states at filling factors, which may not, in some instances, be described by Laughlin $\tilde{\Phi}$, SU(2) Halperin $\tilde{\Phi}$, or SU(4) composite-fermion wave functions $\tilde{\Phi}$, such as at $\nu = 8/19$. Moreover, the rank of the exponent matrix may be related to ferromagnetic properties of the trial states within the SU(4) symmetry. Whether these FQHE states are realized, depends on the precise form of the interaction potential. At $\nu = 2/3$, two candidates, $[3333, 111]$ and $[3333, 033]$, with different residual symmetry, yield incompressible FQHE states for appropriately chosen pseudopotentials. The Coulomb interaction, however, chooses a state with a pronounced $L^2$ collective (Goldstone) mode, both for $2S = 9$ and 11.

This mode is thus likely to survive in the thermodynamic limit.

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