AREA DISTORTION UNDER MEROMORPHIC MAPPINGS WITH NONZERO POLE HAVING QUASICONFORMAL EXTENSION

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Abstract. Let $\Sigma_k(p)$ be the class of univalent meromorphic functions defined on $D$ with $k$-quasiconformal extension to the extended complex plane $\hat{C}$, where $0 \leq k < 1$. Let $\Sigma_0^k(p)$ be the class of functions $f \in \Sigma_k(p)$ having expansion of the form $f(z) = 1/(z - p) + \sum_{n=1}^{\infty} b_n z^n$ on $D$. In this article, we obtain sharp area distortion and weighted area distortion inequalities for functions in $\Sigma_0^k(p)$. As a consequence of the obtained results, we present a sharp estimate for the bound of the Hilbert transform.

1. Introduction

Let $C$ denote the complex plane and $\hat{C}$ be the extended complex plane $C \cup \{\infty\}$. Throughout the discussion in this article, we shall use the following notations: $D = \{z : |z| < 1\}$, $\overline{D} = \{z : |z| \leq 1\}$, $D^* = \{z : |z| > 1\}$, $\overline{D}^* = \{z : |z| \geq 1\}$. Let $\Sigma$ be the class of univalent meromorphic functions defined on $D$ having simple pole at the origin with residue 1 and therefore each $f \in \Sigma$ has the following expansion

\begin{equation}
    f(z) = z^{-1} + \sum_{n=0}^{\infty} b_n z^n, \quad z \in D.
\end{equation}

It is well-known that the univalent functions defined in $D$ that admit a quasiconformal extension to the sphere $\hat{C}$ play an important role in Teichmüller space theory. It is therefore of interest to study such class of functions. To this end, let $\Sigma_k$ be the class of functions in $\Sigma$ that have $k$-quasiconformal extension ($0 \leq k < 1$) to the whole plane $\hat{C}$. Here, a mapping $f : \hat{C} \rightarrow \hat{C}$ is called $k$-quasiconformal if $f$ is a homeomorphism and has locally $L^2$-derivatives on $C \setminus \{f^{-1}(\infty)\}$ (in the sense of distribution) satisfying $|\partial f| \leq k|\partial f|$ a.e., where $\partial f = \partial f/\partial z$ and $\bar{\partial} f = \partial f/\partial \bar{z}$. Note that such an $f$ is also called $K$-quasiconformal more often, where $K = (1 + k)/(1 - k) \geq 1$. The quantity $\mu = \bar{\partial} f/\partial f$ is called the complex dilatation of $f$. The functions in the class $\Sigma_k$ has primarily been studied by O. Lehto, (compare [1]) and later R. Kühnau and S. Krushkal continued the research in this direction. More precisely, they obtained distortion theorems, coefficient estimates, area theorem for functions in this class.

In 1955, Bojarski considered the area distortion problem for quasiconformal mappings (see f.i. [1]). Thereafter further improvements on this problem were made

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by Gehring and Reich (compare [2, Theorem 1]) in a more precise form and they conjectured that

**Theorem A.** If \( f : \mathbb{D} \to \mathbb{D} \) be a \( k \)-quasiconformal mapping with \( f(0) = 0 \), then

\[
|f(E)| \leq M(K)|E|^{1/K},
\]

for all measurable set \( E \subset \mathbb{D} \), where \(|\cdot|\) stands for the area, \( K = (1+k)/(1-k) \geq 1 \), and the constant \( M(K) = 1 + O(K-1) \), as \( K \to 1 \).

This conjecture was proved by K. Astala ([6, Theorem 1.1]) in 1994 using thermodynamic formalism and holomorphic motion theory. Later, Eremenko and Hamilton in [7, Theorem 1] gave a direct and much more simpler proof to the above problem. They assumed \( f \) to be a \( k \)-quasiconformal mapping of the plane which is conformal on \( \mathbb{C} \setminus \Delta \), where \( \Delta \) is a compact set of transfinite diameter 1 and \( f \) has the normalization \( f(z) = z + o(1) \) near \( \infty \). Here we introduce the class \( \Sigma^0_k \) that consists of functions defined on \( \mathbb{D}^* \), having \( k \)-quasiconformal extension in \( \mathbb{D} \) such that they have pole at the point \( z = \infty \) and have the following form

\[
f(z) = z + \sum_{n=1}^{\infty} b_n z^{-n}, \quad z \in \mathbb{D}^*.
\]

In [7, Theorem 1], if we assume \( \Delta = \mathbb{D} \), then \( f \in \Sigma^0_k \). We state this result below:

**Theorem B.** Let \( f \in \Sigma^0_k \) having the expansion of the form (1.2), so that \( f(z) - z \to 0 \) as \( z \to \infty \).

(i) If \( f \) is conformal on \( E \subset \mathbb{D} \), then

\[
|f(E)| \leq \pi^{1-1/K}|E|^{1/K}.
\]

(ii) If \( f \) is conformal outside \( E \subset \mathbb{D} \), then

\[
|f(E)| \leq K|E|.
\]

(iii) Hence, for any arbitrary subset \( E \) of \( \mathbb{D} \),

\[
|f(E)| \leq K\pi^{1-1/K}|E|^{1/K}.
\]

All the constants in the above inequalities are best possible.

In particular, equality holds in Theorem B(i) (see [8, p. 344]) for the function

\[
f_r(z) = \begin{cases} 
  r^{1/K-1}z, & |z| < r, \\
  z |z|^{1/K-1}, & r \leq |z| \leq 1, \\
  z, & |z| > 1,
\end{cases}
\]

where \( 0 < r < 1 \) and \( f \) is conformal on \( E = \{ z : |z| < r \} \). Next, the inequality in Theorem B(ii) is sharp for the function \( f_r^{-1} \) and \( E = \{ z : r^{1/K} \leq |z| \leq 1 \} \) (compare [9, p. 324]). Also the inequality in Theorem B(iii) is sharp as the inequalities in Theorem B(i) and Theorem B(ii) are also so. Further, Astala and Nesi proved the weighted area distortion inequality ([8, Theorem 1.6]), where they considered a non
negative weight function \(w\) defined on a measurable set \(E \subset \mathbb{D}\). We state the result below:

**Theorem C.** Suppose \(f \in \Sigma_k^0\) having expansion of the form (1.2) and \(E \subset \mathbb{D}\) such that \(f\) is conformal on \(E\). Let \(w(z) \geq 0\) be a (measurable) weight function defined on \(E\), then

\[
\pi^{1-K} \left( \int_E w(z)^{1/K} \, dm \right)^K \leq \int_E w(z) J_f(z) \, dm \leq \pi^{1-1/K} \left( \int_E w(z)^{1/K} \, dm \right)^{1/K}.
\]

The inequalities are sharp. Here, \(dm = dx dy\) denotes the two dimensional Lebesgue measure on the plane with \(z = x + iy\).

We note here that, when \(w(z) = 1\) for all \(z \in E\), second inequality of the above theorem yields Theorem B(i). Area distortion results for quasiconformal mappings have several consequences. First of all it gives the precise degree of integrability of the partial derivatives of a \(K\)-quasiconformal mapping. The precise regularity of quasiconformal mappings also controls the distortion of Hausdorff dimension of a set under a \(K\)-quasiregular mapping. Area distortion inequality also provides sharp bounds of Hilbert transformation of characteristic function of a set lying in the domain of a quasiconformal mapping. See [9, chap. 13, 14] for details.

Let \(\Sigma^0(p)\) be the class of functions that are univalent, meromorphic on \(\mathbb{D}\) having a simple pole at \(z = p\) with residue 1 with the following expansion

\[
(1.4) \quad f(z) = (z - p)^{-1} + \sum_{n=1}^{\infty} b_n z^n, \quad z \in \mathbb{D}.
\]

We emphasise here that merely considering the pole of a meromorphic function at a nonzero point not only change the normalization but provide us with the Taylor expansion of the same function inside the disc \(\{z : |z| < p\}\) along with its other Laurent expansions. In this article we consider the class \(\Sigma_k^0(p)\) which consists of functions in \(\Sigma^0(p)\) that have \(k\)-quasiconformal extension to the whole plane \(\hat{\mathbb{C}}\). Alternatively, each function in the class \(\Sigma_k^0(p)\) has the expansion of the following form

\[
(1.5) \quad f(z) = z(1 - pz)^{-1} + \sum_{n=1}^{\infty} b_n z^{-n}, \quad z \in \mathbb{D}^*.
\]

This function class \(\Sigma_k^0(p)\), defined above has been introduced recently in [10]. The area theorem, coefficient estimates and distortion inequalities for this class have also been studied recently (compare [10], [11]).

In this article, we prove an area distortion inequality for functions in the class \(\Sigma_k^0(p)\). This is discussed in Theorem 1 in the next section. Further, we obtain weighted area distortion inequality for theses functions. This is the content of Theorem 2 in the next section. We point out here that Theorem 1 and Theorem 2 coincide with Theorem B and Theorem C respectively, for \(p = 0\), i.e. when \(f \in \Sigma_k^0\).

Finally as an application of Theorem 1, we present a sharp estimate for the Hilbert transform of the characteristic function \(\chi_E\), where \(E \subset \mathbb{D}\).
2. MAIN RESULTS

We start the Section with area distortion inequality for functions in the class $\Sigma_k^0(p)$.

**Theorem 1.** Let $f \in \Sigma_k^0(p)$ has the expansion of the form (1.5).

(i) If $f$ is conformal on $E \subset \mathbb{D}^*$, then

$$
|f(E)| \leq \left[ \pi(1 - p^2)^{-2} \right]^{1-1/K} |f_0(E)|^{1/K}.
$$

(ii) If $f$ is conformal outside a compact set $E \subset \mathbb{D}^*$, then

$$
|f(E)| \leq K |f_0(E)|.
$$

(iii) Hence, for any arbitrary subset $E$ of $\mathbb{D}^*$,

$$
|f(E)| \leq K \left[ \pi(1 - p^2)^{-2} \right]^{1-1/K} |f_0(E)|^{1/K}.
$$

Here $K = (1 + k)/(1 - k)$ and $f_0(z) = 1/(z - p)$, $z \in \mathbb{C}$. The constants appearing in the theorem are best possible.

**Proof.** (i) Let us define $g(z) := f(1/z)$, so that $g \in \Sigma_k^0(p)$ with the expansion of the form (1.5) in $\mathbb{D}^*$. As $g$ is obtained by composing a M"obius transformation with a $k$-quasiconformal map $f$ in $\mathbb{C}$, it is also $k$-quasiconformal in $\mathbb{C}$. Here, since $f$ is conformal in $\mathbb{D}$, therefore $g$ is also conformal in $\mathbb{D}^*$ and hence the dilatation of $g$ has support in $\mathbb{D}$ and it has the same modulus as that of $f$. Since $f$ is conformal on $E \subset \mathbb{D}^*$, so $g$ is again conformal on $\tilde{g}(E) = E' \subset \mathbb{D}$, where $\tilde{g}(z) = 1/z$. As a result, the dilatation $\mu$ of $g$ satisfies $|\mu(z)| \leq k$ for all $z \in \overline{\mathbb{D} \setminus E'}$ and vanishes on $E'$. Now we consider the dilatation

$$
\mu_\lambda(z) = \frac{\lambda \mu(z)}{k}, \quad \lambda \in \mathbb{D}.
$$

Therefore by Measurable Riemann Mapping theorem (see [9, p.168]), there exists a unique quasiconformal mapping $g(z, \lambda) = g_\lambda(z)$ (for each $\lambda$), whose dilatation is $\mu_\lambda(z)$. Now $g_\lambda \in \Sigma_\lambda^0(p)$ as $g \in \Sigma_k^0(p)$ and also $g_\lambda$ satisfies the normalization, $g_\lambda(z) = z/(1 - pz) + o(1)$ as $z \to \infty$. Hence $g_\lambda|\mathbb{D}^* \in \Sigma^0_0(p)$, so by Chichra’s area theorem (see [3]), we have

$$
|g_\lambda(\mathbb{D})| = \pi(1 - p^2)^{-2} - \pi \sum_{n=1}^\infty n|b_n|^2 \leq \pi(1 - p^2)^{-2}.
$$

Thus

$$
\int_{\mathbb{D}} J_\lambda(z) \, dm \leq \pi(1 - p^2)^{-2}, \quad (z = x + iy),
$$

where $J_\lambda$ denotes the Jacobian of the map $g_\lambda$. As $E' \subset \mathbb{D}$, it follows that

$$
\int_{E'} (1 - p^2)^2 \pi^{-1} J_\lambda(z) \, dm \leq 1.
$$

Now by holomorphic dependence of the solution to the Beltrami equation, on parameter (see f.i. [3] II, Theorem 3.1]), the function $\lambda \to g(z, \lambda)$ is holomorphic in the variable $\lambda \in \mathbb{D}$, for each fixed $z \in \mathbb{D}$. This dependency also happens for the function $\partial g(z, \lambda)$ where $g(z, \lambda)$ is analytic in $z$. As $g(z)$ is conformal in $E'$, so is
Now for \( \lambda = 0 \) by \( J \) Using the fact that for \( z \) for each fixed \( p \) \( z \) variable Thus for \( z \) consequently we have a function \( \partial g \) is conformal on the whole sphere \( 0 \). It is now easy to see that \( g \) \( f \) hence \( h \) hence we can say that the function \( a(z, \lambda) = (1 - p^2)^2 \pi^{-1} |\partial g(z, \lambda)|^2 \), then \( \log a(z, \lambda) \) is harmonic in \( \lambda \in \mathbb{D} \), for \( z \in E' \). Thus from (2.1) we see that the function \( a(z, \lambda) \) satisfies the conditions of the continuous version of Lemma 1 in [7], consequently we have we get from the above inequality

\[
(1 - p^2)^2 \pi^{-1} \int_{E'} |\partial g(z, \lambda)|^2 \, dm \leq \left[ (1 - p^2)^2 \pi^{-1} \int_{E'} |\partial g(z, 0)|^2 \, dm \right] \frac{1}{1 + |\lambda|} 
\]

\[
= \left[ (1 - p^2)^2 \pi^{-1} \int_{E'} J_0(z) \, dm \right] \frac{1}{1 + |\lambda|} 
\]

\[
= \left[ (1 - p^2)^2 \pi^{-1} |g_0(E')| \right] \frac{1}{1 + |\lambda|} \cdot 
\]

Using the fact that for \( z \in E' \), \( J_0(z) = |\partial g(z, \lambda)|^2 \), we get from the above inequality

\[
(1 - p^2)^2 \pi^{-1} |g_0(E')| \leq \left[ (1 - p^2)^2 \pi^{-1} |g_0(E')| \right] \frac{1}{1 + |\lambda|}. 
\]

Now for \( \lambda = k \), we have \( g_k = g \), which yields after simplification

\[
(2.5) \quad |g(E')| \leq \left[ \pi (1 - p^2)^{-2} \right]^{1 - 1/K} |g_0(E')|^{1/K}. 
\]

Now since \( f(z) = g(1/z) \), we get inequality (2.1), where \( E \subset \mathbb{D}^* \) and \( g_0 \) is replaced by \( f_0 \). We now find explicitly the function \( g(z, 0) = g_0(z) \). For \( \lambda = 0 \), the function \( g_0 \) is conformal on the whole sphere \( \mathbf{C} \) onto itself as well as it satisfies the normalization of the class \( \Sigma^0(\hat{p}) \) on \( \mathbb{D}^* \), viz.

(i) \( g_0(z) - z/(1 - pz) \to 0 \) as \( z \to \infty \),

(ii) \( g_0(1/p) = \infty \),

(iii) \( (1 - pz)^2 g_0'(z) \big|_{z=1/p} = 1 \).

It is now easy to see that \( g_0(z) = z/(1 - pz) \) for all \( z \in \mathbb{C} \), is the only choice and hence \( f_0(z) = g_0(1/z) = 1/(z - p) \) for all \( z \in \mathbb{C} \), which proves the theorem.

Now we consider the equality case. We observe that equality holds in (2.1) if it does hold in (2.5) and to establish this, we consider the following function:

\[
(2.6) \quad g(z) = \begin{cases} 
\frac{z}{z-p} \frac{1}{1-p^2} \left( \frac{z-p}{1-p} \right)^{1/K-1} \bigg| \frac{z-p}{1-p}, & z \in B(r), \\
\frac{z}{z-p} \frac{1}{1-p^2} \left( \frac{z-p}{1-p} \right)^{1/K-1} \bigg| \frac{z-p}{1-p}, & z \in \overline{\mathbb{D}} \setminus B(r), \\
\frac{z}{z-p}, & z \in \mathbb{D}^*,
\end{cases}
\]
where $0 < r < 1$ and $B(r) \subset \mathbb{D}$ is the disk given by

$$B(r) = \left\{ z : \left| z - \frac{p(1 - r^2)}{1 - p^2 r^2} \right| < \frac{r(1 - p^2)}{1 - p^2 r^2} \right\}.$$ 

It is easy to verify that $g$ is a member of $\Sigma^0_k(p)$ and that $g$ is conformal on the set $E' = B(r) \subset \mathbb{D}$. To establish the equality case, we again observe that the Möbius transformations $(z - p)/(1 - pz)$ and $g_0(z) = z/(1 - pz)$ maps the above disk $B(r)$ onto the disks $\{ w : |w| < r \}$ and $\{ w : |w - p(1 - p^2)| < r(1 - p^2) \}$ respectively. Hence the right hand side of (2.5) becomes $\pi r^{2/K}(1 - p^2)^{-2}$. Again $g$ in (2.6) maps the disk $B(r)$ onto the disk $\{ w : |w - p(1 - p^2)| < r^{1/K}(1 - p^2) \}$, which yields $|g(B(r))| = \pi r^{2/K}(1 - p^2)^{-2}$. Hence equality holds in (2.5) for the above $g$ and $E' = B(r)$. Now as $f(z) = g(1/z)$, we obtain the following extremal function for the inequality (2.1):

$$f(z) = \left\{ \begin{array}{ll}
\frac{r^{1/K-1}}{1 - p^2} \left( \frac{1 - pz}{z - p} \right) + \frac{p}{1 - p^2}, & z \in \hat{B}(r), \\
\frac{1}{1 - p^2} \left( \frac{1 - pz}{z - p} \right) \frac{1 - p^2}{1 - p^2}^{1/K-1} + \frac{p}{1 - p^2}, & z \in \mathbb{D} \setminus \hat{B}(r), \\
\frac{1}{z - p}, & z \in \mathbb{D},
\end{array} \right.$$ 

where we assume $0 \leq p < r < 1$. Here $\hat{B}(r)(\subset \mathbb{D}^*)$ is the image of the disk $B(r)$ under the map $\hat{g}(z) = 1/z$, given by

$$\hat{B}(r) = \left\{ z \in \mathbb{C} : \left| z + \frac{p(1 - r^2)}{r^2 - p^2} \right| > \frac{r(1 - p^2)}{r^2 - p^2} \right\}.$$ 

Hence equality holds in (2.1) for the above $f$ and $E = \hat{B}(r)$.

(ii) As before we start the proof of this part with the transformation $g(z) = f(1/z)$. By the given condition, $g$ is conformal outside a compact set $\hat{g}(E) = E' \subset \mathbb{D}$, where $\hat{g}(z) = 1/z$. Thus dilatation $\mu$ of $g$ vanishes outside the compact set $E'$. As $g \in \Sigma^0_k(p)$ of the form (1.5) in $\mathbb{D}^*$, hence by equation (1.7) of [11] p.3, we get

$$g(z) = z/(1 - pz) + T[\overline{\partial g}](z).$$ 

Taking partial derivative of both sides w.r.t. $z$ and using $\partial T[\omega] = H[\omega]$, we have

$$\partial g(z) = 1/(1 - pz)^2 + H[\overline{\partial g}](z),$$ 

where ‘$T$’ and ‘$H$’ denote two dimensional ‘Cauchy’ and ‘Hilbert’ transform respectively (see f.1. [5] I §4.3]). Since $\overline{\partial g} = \mu \partial g$, the above equation takes the following form

$$\overline{\partial g}(z) = \mu/(1 - p z)^2 + \mu H[\overline{\partial g}](z).$$ 

It is also known that

$$w = \overline{\partial g} = \mu(1 - p z)^{-2} + \mu H \left[ \mu(1 - p z)^{-2} \right] + \mu H \left[ \mu H \left[ \mu(1 - p z)^{-2} \right] \right] + \cdots$$ 

satisfies equation (2.8) (see [11] p.5]). By our assumption, $w = \overline{\partial g}$ vanishes outside $E'$. Hence using (2.7) and the fact that the Hilbert transform is a linear isometry
on $L^2(\mathbb{C})$, we get

$$
|g(E')| = \int_{E'} |J_g(z)| \, dm
$$

$$
= \int_{E'} \left( |\partial g|^2 - |\overline{\partial} g|^2 \right) \, dm
$$

$$
= \int_{E'} \left( |(1-pz)^{-2} + H[w]|^2 - |w|^2 \right) \, dm
$$

(2.10) \quad = \int_{E'} \left( |1-pz|^{-4} + 2\text{Re} \left( (1-pz)^{-2}H[w] \right) \right) \, dm + \int_{E'} \left( |H[w]|^2 - |w|^2 \right) \, dm
$$

$$
\leq \int_{E'} |1-pz|^{-4} \, dm + 2\int_{E'} |(1-pz)^{-2}H[w]| \, dm + \int_{C} \left( |H[w]|^2 - |w|^2 \right) \, dm
$$

(2.11) \quad = |g_0(E')| + 2\int_{E'} |(1-pz)^{-2}H[w]| \, dm,
$$

where $g_0(z) = z/(1-pz)$, as mentioned earlier. Now using the fact that Hilbert transformation is linear, we get from the identity (2.9) that

$$(1-pz)^{-2}H[w] = (1-pz)^{-2}H[\mu(1-pz)^{-2}] + (1-pz)^{-2}H[\mu H[\mu(1-pz)^{-2}]] + \cdots.$$ 

This gives

(2.12) \quad \int_{E'} |(1-pz)^{-2}H[w]| \, dm \leq \int_{E'} |1-pz|^{-2} |H[\mu(1-pz)^{-2}]| \, dm

$$+ \int_{E'} |1-pz|^{-2} |H[\mu H[\mu(1-pz)^{-2}]]| \, dm + \cdots.$$ 

We now apply ‘Cauchy-Schwartz’ inequality and the isometry property of Hilbert transformation to the $n$-th term of the right hand side of (2.12) to get an upper bound for this term. We show below the computational details:

$$
\int_{E'} |1-pz|^{-2} \left| \underbrace{H[\mu H \cdots \mu H[\mu(1-pz)^{-2}]]}_{n \text{ terms}} \right| \, dm
$$

$$\leq \left( \int_{E'} |1-pz|^{-4} \, dm \right)^{1/2} \left( \int_{E'} \left| \underbrace{H[\mu H \cdots \mu H[\mu(1-pz)^{-2}]]}_{n \text{ terms}} \right|^2 \, dm \right)^{1/2}
$$

$$\leq |g_0(E')|^{1/2} \left( \int_{C} \left| \underbrace{H[\mu H \cdots \mu H[\mu(1-pz)^{-2}]]}_{n \text{ terms}} \right|^2 \, dm \right)^{1/2}$$
maps the disk \( B \) and that the function 
\begin{align*}
\mu \in \mathcal{F}' \Rightarrow \int & \mu H \left[ \mu H \cdots \mu H \left[ \mu (1-pz)^{-2} \right] \right] \, dm \\
& \leq \| \mu \|_{\infty} |g_0(E')|^{1/2} \left( \int_{E'} |1-pz|^{-4} \, dm \right)^{1/2} \\
& = k^n|g_0(E')|,
\end{align*}

where \( \| \mu \|_{\infty} = k < 1 \). Using this estimate, we get from (2.12) that 
\begin{align*}
\int_{E'} |(1-pz)^{-2} H[w]| \, dm & \leq \sum_{n=1}^{\infty} |g_0(E')|^k^n \\
& = k(1-k)^{-1}|g_0(E')|.
\end{align*}

Plugging the above estimate in (2.11), we finally obtain
\begin{equation}
(2.13) \quad |g(E')| \leq \left( \frac{1+k}{1-k} \right) |g_0(E')| = K|g_0(E')|.
\end{equation}

Now applying \( f(z) = g(1/z) \), we get inequality (2.2), where \( E \subset \mathbb{D}^* \) and \( f_0(z) = 1/(z-p), z \in \mathbb{C} \). Next we show that the constant \('K'\) in Theorem (i) is best possible. This can be verified if we can show that the constant \('K'\) in (2.13) is best possible. We consider the following example:

\begin{equation}
(2.14) \quad h(z) = \begin{cases} 
\frac{z^{-1/K}}{1-pz} + \frac{p}{1-pz}, & z \in B_0(r), \\
\frac{1}{1-pz} \left( \frac{z^{-1/K}}{1-pz} \right)^{K-1} + \frac{p}{1-pz}, & z \in \mathbb{D} \setminus B_0(r), \\
\frac{z}{z-pz}, & z \in \mathbb{D}^*,
\end{cases}
\end{equation}

where \( B_0(r) (\subset \mathbb{D}) \) is the disk given by 
\[ B_0(r) = \left\{ z : \left| z - \frac{p(1-r^{2/K})}{1-p^2r^{2/K}} \right| < \frac{r^{1/K}(1-p^2)}{1-p^2r^{2/K}} \right\}, \] for \( 0 < r < 1 \).

As similar to example (2.16), the functions \( z/(1-pz) = g_0(z) \) and \((z-p)/(1-pz)\) maps the disk \( B_0(r) \) onto the disks \( \{ w : |w - \frac{p(1-p^2)^{-1}}{1-p^2} | < r^{1/K}(1-p^2)^{-1} \} \) and \( \{ w : |w | < r^{1/K} \} \) respectively. This in turn implies \( |g_0(B_0(r))| = \pi r^{2/K}(1-p^2)^{-2} \) and that the function \( h \) in (2.14) itself maps the disk \( B_0(r) \) onto the disk \( \{ w : |w - \frac{p(1-p^2)^{-1}}{1-p^2} | < r(1-p^2)^{-1} \} \). To verify the assertion we set \('E'\) in this case, as \( E' = \mathbb{D} \setminus B_0(r) \). Then \( h \) is conformal on outside of the compact set \( E' \) and 
\[ |g_0(E')| = |g_0(\mathbb{D})| - |g_0(B_0(r))| = \pi(1-p^2)^{-2}(1-r^{2/K}). \]
On the other hand,
\[
|h(E')| = |h(\overline{D})| - |h(B_0(r))| \\
= \pi(1 - p^2)^{-2}(1 - r^2) \\
= \pi(1 - p^2)^{-2} - \pi(1 - p^2)^{-2} \left[ 1 - (1 - r^2/K) \right] \\
= \pi(1 - p^2)^{-2} \left[ K(1 - r^2/K) - (K/2)(K - 1)(1 - r^2/K)^2 + \ldots \right] \\
= K|g_0(E')| + O \left( (g_0(E'))^2 \right), \quad \text{as } |g_0(E')| \to 0.
\]
Hence the constant \( 'K' \) can not be improved as equality holds in (2.13) for \( |g_0(E')| \) small enough. Composing \( h \) with the inverse mapping \( \tilde{g}(z) = 1/z \) and taking inversion of the disk \( B_0(r) \) (for \( p < r \)), extremality of (2.2) follows easily, as similar to Theorem (i). (iii) To prove the last part of the theorem, we consider the following change of variable \( g(z) = f(1/z) \). Hence \( g \in \Sigma_k^0(p) \) such that it is conformal on \( \mathbb{D}^* \) and \( k \)-quasiconformal on \( \overline{\mathbb{D}} \). We write \( g = g_1 \circ g_2 \), where \( g_2 \) is conformal on \( E \subset \mathbb{D} \), \( k \)-quasiconformal on \( \overline{\mathbb{D}} \setminus E \) and \( g_2 \in \Sigma_k^0(p) \). We assume that the function \( g_1 \) is \( k \)-quasiconformal on \( g_2(E) \) and hence on \( g_2(\overline{E}) \) (as a set of area zero is removable for quasiconformality), so that \( g_1 \) is conformal outside the compact set \( g_2(\overline{E}) \) and satisfies the conditions of Theorem B(ii). Applying Theorem (i) to \( g_2 \) and Theorem B(ii) to \( g_1 \), we get
\[
|g(E)| = |g_1(g_2(E))| \leq K|g_2(E)| \leq K \left[ \pi(1 - p^2)^{-2} \right]^{1-1/K} |g_0(E)|^{1/K}.
\]
Putting \( f(z) = g(1/z) \) we obtain the theorem in terms of \( f \) and \( g_0 \) is replaced by \( f_0(z) = 1/(z - p) \). As the constants in corresponding theorems for \( g_1 \) and \( g_2 \) are best possible, hence for Theorem (iii) also.

Remark. For the case \( p = 0 \), i.e. whenever \( f \in \Sigma_k^0 \), the inequality (2.13) reduces to that of Theorem B(i), and the extremal function \( g \) defined in (2.6) becomes \( f_r \), as defined in (2.3). This coincidence also occurs for Theorem (ii), when \( p = 0 \), as can be seen from the inequality (2.13) and the extremal function \( h \) defined in (2.14). In this case \( h \) reduces to \( f_{r^{-1}} \) for \( p = 0 \), which is the extremal case for Theorem B(ii). Although, in our case \( h \) is not the inverse mapping of \( g \).

Next we consider the weighted area distortion problem for a function in the class \( \Sigma_k^0(p) \), where we consider a nonnegative weight function \( w \) defined on a subset \( E \) of \( \mathbb{D}^* \).

**Theorem 2.** Suppose \( f \in \Sigma_k^0(p) \) with the expansion of the form (1.4) and \( E \subset \mathbb{D}^* \), such that \( f \) is conformal on \( E \). Let \( w(z) \geq 0 \) be a (measurable) weight function defined on \( E \), then
\[
\left( \frac{\pi}{(1 - p^2)^2} \right)^{1-K} \left( \int_E w(z)^{1/K} J_0(z) \, dm \right)^K \leq \int_E w(z) J_f(z) \, dm \\
\leq \left[ \frac{\pi}{(1 - p^2)^2} \right]^{1-1/K} \left( \int_E w(z)^K J_0(z) \, dm \right)^{1/K},
\]
where $J_f$ and $J_0$ denotes Jacobian of the function $f$ and $f_0(z) = 1/(z - p), z \in \mathbb{C}$ respectively. The inequalities are sharp.

Proof. The case $w(z) = 0$ for all $z$ is trivial. So we assume $w(z) > 0$ for all $z \in E$. To establish the theorem we follow the lines of the proof of [8, Theorem 1.6]. For the sake of completeness, we provide computational details. Let $g(z) = f(1/z)$ having expansion of the form (1.5) in $\mathbb{D}^*$. Next we consider the weight function $w_0(z) = w(1/z)$ defined on $\tilde{g}(E) = E' \subset \mathbb{D}$, where $\tilde{g}(z) = 1/z$. Therefore $g$ is conformal on $E'$ and $k$-quasiconformal on $\mathbb{D} \setminus E'$. As similar to (2.3), we consider the function $g_\lambda(z)$ with the dilatation $\lambda k^{-1} \mu(z)$ for $\lambda \in \mathbb{D}$. Again $g_\lambda(z)$ is conformal on $E'$ (since $g$ is so) and

\begin{equation}
(2.16) \quad g_\lambda'(z) \neq 0, \quad \text{for all } z \in E' \quad \text{and } \lambda \in \mathbb{D}.
\end{equation}

Using the concavity of logarithm and ‘Jensen’s Inequality’, we get for any function $a(z) > 0$ defined in $E'$, that

\begin{equation}
(2.17) \quad \log \left( \int_{E'} a(z) \, dm \right) = \sup_{q(z)} \left[ \int_{E'} q(z) \log \left( \frac{a(z)}{q(z)} \right) \, dm \right],
\end{equation}

where the supremum is taken over all functions $q(z)$ defined on $E'$, such that

(i) $0 < q(z) < 1$, a.e. $z \in E'$ and

(ii) $\int_{E'} q(z) \, dm = 1$. In our case, we take

$\quad a(z) = (1 - p^2)^2 \pi^{-1} w_0(z) J_\lambda(z) = (1 - p^2)^2 \pi^{-1} w_0(z) |g'_\lambda(z)|^2, \quad z \in E'$,

since for $z \in E', J_\lambda(z) = |\partial g_\lambda(z)|^2 = |g'_\lambda(z)|^2$. Hence using (2.17), we get

\begin{align}
\log \left( \int_{E'} (1 - p^2)^2 \pi^{-1} w_0(z) |g'_\lambda(z)|^2 \, dm \right) &= \sup_{q(z)} \left[ \int_{E'} q(z) \log \left( \frac{(1 - p^2)^2 \pi^{-1} w_0(z) |g'_\lambda(z)|^2}{q(z)} \right) \, dm \right] \\
&= \sup_{q(z)} \left[ \int_{E'} q(z) \log(w_0(z)) \, dm + h_p(\lambda) \right],
\end{align}

where

\begin{equation}
(2.18) \quad h_p(\lambda) = \int_{E'} q(z) \log \left( \frac{(1 - p^2)^2 \pi^{-1} |g'_\lambda(z)|^2}{q(z)} \right) \, dm
\end{equation}

is harmonic in $\lambda \in \mathbb{D}$, by (2.16), for each $z \in E'$. Using (2.17) and (2.18) successively, we get

\begin{equation}
\log \left( \int_{E'} (1 - p^2)^2 \pi^{-1} |g'_\lambda(z)|^2 \, dm \right) \leq 0.
\end{equation}

So for each $z \in E'$, $h_p(\lambda)$ is harmonic and nonpositive in $\mathbb{D}$. Hence by using ‘Harnack’s Inequality’ and the fact that $g_0(z) = z/(1 - pz)$ (as claimed in the proof of Theorem [1][i]), we have

\begin{align}
h_p(\lambda) &\leq (1 - |\lambda|)(1 + |\lambda|)^{-1} h_p(0) \\
&= (1 - |\lambda|)(1 + |\lambda|)^{-1} \int_{E'} q(z) \log \left( \frac{(1 - p^2)^2 \pi^{-1} |g'_0(z)|^2}{q(z)} \right) \, dm.
\end{align}
For \( \lambda = k \), we have \( g_\lambda = g \) and \( (1 + k)/(1 - k) = K \). Thus using above inequality (for \( \lambda = k \)) in \((2.18)\), and also using \((2.17)\) once more, we get

\[
\log \left( \int_{E'} (1 - p^2)^{2\pi - 1} w_0(z) J_g(z) \, dm \right)
\leq \sup_{q(z)} \left[ \int_{E'} q(z) \log w_0(z) \, dm + \frac{1}{K} \int_{E'} q(z) \log \left( \frac{(1 - p^2)^{2\pi - 1} J_{g_0}(z)}{q(z)} \right) \, dm \right]
= \frac{1}{K} \sup_{q(z)} \left[ \int_{E'} q(z) \log \left( \frac{(1 - p^2)^{2\pi - 1} w_0(z)^K J_{g_0}(z)}{q(z)} \right) \, dm \right]
= \log \left( \int_{E'} (1 - p^2)^{2\pi - 1} w_0(z)^K J_{g_0}(z) \, dm \right)^{1/K}.
\]

Taking exponentiation and doing a rearrangement, we obtain

\[
(2.19) \quad \int_{E'} w_0(z) J_g(z) \, dm \leq \left[ \frac{\pi}{(1 - p^2)^2} \right]^{1-1/K} \left( \int_{E'} w_0(z)^K J_{g_0}(z) \, dm \right)^{1/K}.
\]

Now putting \( w(z) = w_0(1/z), f(z) = g(1/z) \) and observing that \( J_g(z) = J_f(1/z)|z|^{-4}, J_{g_0}(z) = J_{f_0}(1/z)|z|^{-4} \), second inequality of \((2.15)\) follows from above. Here \( E' \) and \( J_{g_0} \) is replaced by \( E \) and \( J_{f_0} = J_0 \) respectively, where \( f_0(z) = 1/(z - p) \). To obtain the first inequality we use the other part of the ‘Harnack’s Inequality’ in \((2.18)\) viz.

\[
h_p(\lambda) \geq (1 + |\lambda|)(1 - |\lambda|)^{-1} h_p(0)
\]

and proceed in a similar fashion. Next we show that the second inequality of Theorem \(2\) is sharp. To verify this, it is sufficient to show that the inequality \((2.19)\) is sharp. We follow the arguments given in \([8\), Example 2.1]. First we choose the numbers \( w_j, p_j, r_j, \rho_j \) for \( j = 1, ..., n \), suitably as \( 1 \leq w_1 < w_2 < \cdots < w_n \) and \( 0 < p_j < 1 \), such that

\[
(2.20) \quad w_j = \left( \prod_{l=1}^{j} r_l \right)^{-2/K} \quad \text{and} \quad \sum_{j=1}^{n} p_j w_j^K = 1.
\]

We now consider the function

\[
g = f_{r_1}^{p_1} \circ \cdots \circ f_{r_n}^{p_n}
\]

and \( f_r \) defined in \((1.3)\). Next we consider the weight function \( w_0(z) = \sum_{j=1}^{n} w_j E_j(z) \), where

\[
E_j = \{ z : \rho_{j+1} < |z| < \rho_{j} r_j \}, \quad 1 \leq j \leq n - 1; \quad E_n = \{ z : |z| < \rho_n r_n \}.
\]

The composition in \((2.21)\) is well defined as we have

\[
r_j^2 \rho_j^2 - \rho_{j+1}^2 = p_j, \quad 1 \leq j \leq n - 1; \quad r_n^2 \rho_n^2 = p_n.
\]

In our case, we define

\[
(2.22) \quad G(z) = (1 - p^2)^{-1} g \left( \frac{z - p}{1 - p^2} \right) + p/(1 - p^2), \quad z \in \mathbb{C},
\]
and the weight function as

\[ W_0(z) = \sum_{j=1}^{n} w_j \chi_{E_j}(z), \quad \tilde{E}_j = \tilde{f}^{-1}(E_j), \text{ where } \tilde{f}(z) = (z-p)/(1-pz). \]

Now the function \( G \) defined in (2.22) belongs to the class \( \Sigma^0_k(p) \), as the function \( g \) defined in (2.21) belongs to the class \( \Sigma^0_k \). If we now take \( \tilde{E} = \cup_{j=1}^{n} \tilde{E}_j \), then \( G \) is conformal on \( \tilde{E} \). Hence using first relation of (2.20), it is easy to see that

\[ J_{G|E} = \frac{W_0(z)}{K} |1-pz|^{-4} = \frac{W_0(z)}{K} |g_0(E)|, \quad z \in \tilde{E}. \]

Again, using second relation of (2.20), we get

\[
\int_{\tilde{E}} W_0(z) J_G(z) \, dm = \sum_{j=1}^{n} \left( w_j^K \int_{\tilde{E}_j} |1-pz|^{-4} \, dm \right)
\]

\[
= \sum_{j=1}^{n} \left( w_j^K |g_0(\tilde{E}_j)| \right)
\]

\[
= \pi (1-p^2)^{-2} \left[ \sum_{j=1}^{n-1} w_j^K (r^2_j \rho_j^2 - \rho_{j+1}^2) + w_n^K r_n^2 \rho_n^2 \right]
\]

\[
= \pi (1-p^2)^{-2} \sum_{j=1}^{n} p_j w_j^K
\]

\[
= \pi (1-p^2)^{-2} = \left[ \pi (1-p^2)^{-2} \right]^{1-1/K} \left( \int_{\tilde{E}} W_0(z) J_{g_0}(z) \, dm \right)^{1/K}.
\]

As equality holds in (2.19), hence it also holds for the second inequality in (2.15). Optimality of the other inequality in (2.15) can be established by similar construction.

**Remark.** (i) While proving Theorem C in [8], the authors first assumed \( E \) to be an open set and then proved the theorem for a general set \( E \subset \mathbb{D} \) by limiting sense. Same argument also can be applied to the proof of Theorem 2, but we omit the details.

(ii) If \( w(z) = 1 \) for all \( z \in E \), then the second inequality of Theorem 2 implies Theorem (ii).

(iii) In Theorem C, we assumed \( f \in \Sigma^0_k \) of the form (1.2) in \( \mathbb{D}^* \), as taken in [8]. But if we take \( f \in \Sigma^0_k \) of the form (1.1) (with \( b_0 = 0 \)) in \( \mathbb{D} \) and \( f \) is conformal on \( E \subset \mathbb{D}^* \), then Theorem C can be restated as

\[
\pi^{1-K} \left( \int_E w(z)^{1/K} |z|^{-4} \, dm \right)^K \leq \int_E w(z) J_f(z) \, dm \leq \pi^{1-1/K} \left( \int_E w(z)^K |z|^{-4} \, dm \right)^{1/K}.
\]

This result coincides with Theorem [2] for \( p = 0 \).

As an application of Theorem [1], we prove the next result. It deals with the bounds of the Hilbert transform of the characteristic function of a set \( E \subset \mathbb{D} \).
Theorem 3. If $E \subset \mathbb{D}$, then
\begin{equation}
(2.23) \quad \int_{\mathbb{D} \setminus E} \frac{1}{|1 - pz|^2} \left| H \left[ \frac{\lambda g}{(1 - p^2)^2} \right] \right| \, dm \leq |g_0(E)| \log \left( \frac{\pi (1 - p^2)^2}{|g_0(E)|^{-1}} \right),
\end{equation}
where $g_0(z) = z/(1 - pz)$, $z \in \mathbb{C}$. The inequality is sharp.

Proof. For any function $\mu$ with $|\mu| = 1$, supported in $\mathbb{D} \setminus E$, we define $\mu_\lambda(z) = \lambda \mu(z)$ for $\lambda \in \mathbb{D}$ and consider the corresponding family of quasiconformal mappings $g_\lambda$ in $\hat{\mathbb{C}}$, with dilatation $\mu_\lambda$. We also assume that the functions $g_\lambda$ are normalized such that they belong to the class $\Sigma^0(p)$, when restricted on $\mathbb{D}^*$, therefore each function $g_\lambda$ belongs to the class $\Sigma^0_{\lambda}(p)$, for each $\lambda \in \mathbb{D}$. Now by the assumption each $g_\lambda$ is conformal on $E$, which gives from (2.10) that
\begin{equation}
|g_\lambda(E)| = \int_E |\partial g_\lambda(z)|^2 \, dm
\end{equation}
\begin{equation}
(2.24) \quad = |g_0(E)| + 2 \text{Re} \int_E (1 - p^2)^{-2} H[\partial g_\lambda] \, dm + \int_E |H[\partial g_\lambda]|^2 \, dm.
\end{equation}
Now from (2.20), $w = \partial g_\lambda$ can be written as
\begin{equation}
(2.25) \quad \partial g_\lambda = \lambda \mu(1 - pz)^{-2} + h_\lambda(z),
\end{equation}
where $\|h_\lambda\|_2 \leq C|\lambda|^2$, $C$ is a constant. Using above identity it is easy to see that
\begin{equation}
\int_E |H[\partial g_\lambda]|^2 \, dm = O(|\lambda|^2) \text{ as } \lambda \to 0.
\end{equation}
Again from (2.25) we get
\begin{equation}
\text{Re} \int_E (1 - p^2)^{-2} H[\partial g_\lambda] \, dm = \text{Re} \int_E \lambda(1 - p^2)^{-2} H[\mu(1 - pz)^{-2}] \, dm + O(|\lambda|^2), \quad \lambda \to 0.
\end{equation}
Now upon using the last two estimates obtained above, we get from (2.23) that
\begin{equation}
(2.26) \quad |g_\lambda(E)| = |g_0(E)| + 2 \text{Re} \int_E \lambda(1 - p^2)^{-2} H[\mu(1 - pz)^{-2}] \, dm + O(|\lambda|^2).
\end{equation}
Now as $g_\lambda \in \Sigma^0_{\lambda}(p)$, by area distortion inequality (Theorem 1(i)), we get
\begin{equation}
|g_\lambda(E)| \leq \left[ \pi (1 - p^2)^2 \right]^{1-1/K} |g_0(E)|^{1/K},
\end{equation}
where $K = (1 + |\lambda|)(1 - |\lambda|)^{-1}$. Since $1 - K^{-1} = 2|\lambda| + O(|\lambda|^2)$, therefore the above inequality can be written as
\begin{equation}
|g_\lambda(E)| \leq |g_0(E)| + 2|\lambda||g_0(E)| \log \left( \pi (1 - p^2)^2 |g_0(E)|^{-1} \right) + O(|\lambda|^2).
\end{equation}
Comparing the coefficients of the terms which are linear in $|\lambda|$ of the above inequality and that of with (2.26), we get
\begin{equation}
\text{Re} \left[ \lambda \int_E (1 - p^2)^{-2} H[\mu(1 - pz)^{-2}] \, dm \right] \leq |\lambda||g_0(E)| \log \left( \pi (1 - p^2)^2 |g_0(E)|^{-1} \right).
\end{equation}
Now for a particular choice of \( \lambda \), we have

\[
\text{Re} \left[ \lambda \int_{E} (1 - p\overline{z})^{-2} H[\mu(1 - pz)^{-2}] \, dm \right] = |\lambda| \left| \int_{E} (1 - p\overline{z})^{-2} H[\mu(1 - pz)^{-2}] \, dm \right|.
\]

From above two relations, we get

\begin{equation}
(2.27) \quad \left| \int_{E} (1 - p\overline{z})^{-2} H[\mu(1 - pz)^{-2}] \, dm \right| \leq |g_0(E)| \log (\pi(1 - p^2)^{-2}|g_0(E)|^{-1}).
\end{equation}

Next using ‘symmetric property’ of ‘\( H \)’ (see [9, p.95]), we have

\[
\int_{E} (1 - p\overline{z})^{-2} H[\mu(1 - pz)^{-2}] \, dm = \int_{C} \chi_E(1 - p\overline{z})^{-2} H[\mu(1 - pz)^{-2}] \, dm
\]

\[
= \int_{C} \mu(1 - pz)^{-2} H[\chi_E(1 - p\overline{z})^{-2}] \, dm
\]

\[
= \int_{\mathbb{D}\setminus E} \mu(1 - pz)^{-2} H[\chi_E(1 - p\overline{z})^{-2}] \, dm,
\]

since \( \mu \) has support in \( \mathbb{D} \setminus E \). Using the inequality (2.27), we get

\[
\left| \int_{\mathbb{D}\setminus E} \mu(1 - pz)^{-2} H[\chi_E(1 - p\overline{z})^{-2}] \, dm \right| \leq |g_0(E)| \log (\pi(1 - p^2)^{-2}|g_0(E)|^{-1}).
\]

For a suitable choice of \( \mu \), we can take modulus inside the integral of the left hand side of the above inequality, which proves the theorem. Finally it remains to prove the sharpness of the inequality (2.23). To show this we consider

\[
E = \left\{ z : \left| \frac{z - p(1 - r^2)}{1 - p^2r^2} \right| < \frac{r(1 - p^2)}{1 - p^2r^2} \right\}, \quad 0 < r < 1.
\]

Clearly \( E \subset \mathbb{D} \). Hence \( |g_0(E)| = \pi r^2(1 - p^2)^{-2} \), so that right hand side of (2.23) reduces to \( 2\pi(1 - p^2)^{-2}r^2\log(r^{-1}) \). Next in order to find the Hilbert transform of the function \( \chi_E(1 - p\overline{z})^{-2} \), we define

\[
f(z) = \begin{cases} \frac{1}{1 - p^2} \left( \frac{z - p}{1 - pz} \right), & z \in E, \\ \frac{1}{1 - p^2} \left( \frac{1 - p}{z - p} \right), & z \in \mathbb{C} \setminus E. \end{cases}
\]

Here \( f \) is continuous on \( \mathbb{C} \) and a little calculation reveals that \( \partial f = \chi_E(1 - p\overline{z})^{-2} \) and \( \partial f = -r^2(z - p)^{-2}\chi_{\mathbb{C}\setminus E} \). Using the relation \( H[\partial f] = \partial f \), we have

\[
H[\chi_E(1 - p\overline{z})^{-2}] = -r^2(z - p)^{-2}\chi_{\mathbb{C}\setminus E}.
\]
Let $w = \tilde{f}(z) = (z - p)/(1 - pz) = u + iv$. Therefore, $\tilde{f}(\mathbb{D} \setminus E) = \{w : r \leq |w| < 1\}$ and $J_{\tilde{f}}(z) = (1 - p^2)^2|1 - pz|^{-4}$. Hence we have,

$$
\int_{\mathbb{D} \setminus E} |1 - pz|^{-2} |H[\chi_E(1 - pz)^{-2}]| \, dm = r^2 \int_{\mathbb{D} \setminus E} (|1 - pz| |z - p|)^{-2} \, dm
$$

$$
= r^2(1 - p^2)^{-2} \int_{\mathbb{D} \setminus E} \left| \frac{1 - pz}{z - p} \right|^2 \frac{(1 - p^2)^2}{|1 - pz|^4} \, dm
$$

$$
= r^2(1 - p^2)^{-2} \int_{\mathbb{D} \setminus E} |w|^{-2} \, du dv
$$

$$
= 2\pi(1 - p^2)^{-2}r^2 \log(r^{-1}).
$$

Thus the inequality (2.23) is sharp and this completes proof of the theorem. \qed

**Remark.** For $p = 0$, the functions $g_\lambda$ defined in the proof of Theorem 3 belong to the class $\Sigma_{\lambda|\lambda}$ and the function $g_0$ becomes the identity function. Hence the inequality (2.23) reads as (compare Theorem 14.6.1 of [9, p.385])

$$
\int_{\mathbb{D} \setminus E} |H[\chi_E]| \, dm \leq |E| \log \left(\frac{\pi}{|E|}\right).
$$

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