Expansion of paranormal operator

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Abstract. Hilbert space \( H \) over the fields of \( \mathbb{R}/\mathbb{C} \) and \( B(H) \) is the set of all bounded linear operator on \( H \). This study aimed to investigate the expansion of paranormal operators and their properties in Hilbert space. The expansion in question is the definition and basic characteristics of paranormal operators developed into \( n \)th- paranormal operators and \( * \)- paranormal operators. The study resulted the properties of \( n \)th- paranormal operators and \( * \)- paranormal operators and the relationship between the two. To investigate the nature of the two operators, it requires the nature of the \( n \)- paranormal operators.

1. Introduction

Given a Hilbert space \( X \) and \( Y \) over the same field, namely \( F \). Field \( F \) which is in this paper is \( \mathbb{R}/\mathbb{C} \). The set of all finite linear operators from \( X \) to \( Y \) is written by \( \mathcal{B}(X, Y) \). Further, in the event that \( X = Y \), \( \mathcal{B}(X, X) \) is written by \( \mathcal{B}(X) \) or \( \mathcal{B}(Y) \).

Given a Hilbert space \( H \) over the field \( F \) and \( T \in \mathcal{B}(H) \), a bounded linear operator \( T \), which has property of \( \|T(x)\|^2 \leq \|T^2(x)\|^2 \) is named as paranormal operator. Then, the definition is expanded into \( \|T^{2n}(x)\| \geq \|T^n(x)\|^2 \), for \( n \in \mathbb{N} \) and \( T^n = T \circ T \circ T \circ \ldots \). The operator \( T \) with the property of \( n \) factor

\[ \|T^{2n}(x)\| \geq \|T^n(x)\|^2 \]

is called \( n \)-paranormal operator, for \( n \in \mathbb{N} \). [1] had researched about the \( n \)-paranormal operator. For investigating the property of the \( n \)-paranormal operator, needs theory of \( n \)-paranormal operator. [2] had researched about the \( n \)-paranormal operator. Further, the definition of paranormal operator is expanded into \( \|T^{2n}(x)\| \geq \|T^n(x)\|^2 \). The operator \( T \) with the property of

\[ \|T^2(x)\| \geq \|T^*(x)\|^2 \]

is called \( * \)-paranormal operator , with \( T^* \) is adjoint operator of \( T \). [3] had researched about the \( * \)-paranormal operator.

Discussion of the properties of paranormal operators begins by defining the self adjoint operator. \( T \) called self adjoint operator if \( T = T^* \) [4]. Continued to discuss the properties of
paranormal operators and hyponormal operators [5,6]. Then, discuss paranormal operators with SVEP and Weyl theorem. Discussion for the properties of $n$- paranormal operators and $n$th- paranormal focuses on algebra properties. The definition of $n$- paranormal operators and $n$th- paranormal is taken from [1], [2]. While, for properties of *-paranormal discusses definitions and properties with quasi-* -paranormal, $n$-quasi-*-paranormal, and Bishop property $\beta$ operators. The discussion begins with defining operators *-paranormal. $T$ is called to be *-paranormal operator if
\[ \|T^2(x)\| \geq \|T^* (x)\|^2 \] [3]. Then, the definition of quasi- *-paranormal operators, $n$- quasi-- * paranormal, and Bishop property $\beta$ are taken from [6-12].

Then, it takes an idea for investigating the characteristic of the operator $T$ which has properties of $\|T^{2n} (x)\| \geq \|T^n (x)\|^2$ and $\|T^2 (x)\|\|x\| \geq \|T^* (x)\|^2$. The discussion of the $n$th paranormal operator and the * paranormal operator in this paper, is more accentuated on the properties and relationships of the $n$th- paranormal operator and the * paranormal operator on the Hilbert space. Therefore, it needs to be studied more about the characteristics of the $n$th- paranormal operator and the * - paranormal operator on the Hilbert space.

2. Paranormal Operators

The following will be explained about the definition and nature of paranormal operators on Hilbert spaces. To prove the nature of paranormal operators needed a definition of self adjoint operators.

**Definition 2.1.** [13] Given a Hilbert space $H$ and $T \in B(H)$, Operator $T$ called paranormal if $\|T(x)\|^2 \leq \|T^2 (x)\|$, $\forall x \in H$.

**Example 2.2.** Well, known $\ell^2$ Hilbert spaces. Defined operator $T : \ell^2 \rightarrow \ell^2$ with :

\[ T(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, x_3, \ldots), \forall (x_1, x_2, x_3, \ldots) \in \ell^2 \]

Obtained,

1) $T$ linear. Taken any $x = (x_1, x_2, x_3, \ldots)$, $y = (y_1, y_2, y_3, \ldots) \in \ell^2$ and scalar $\alpha$ :

\[ T(x + y) = T(x_1 + y_1, x_2 + y_2, x_3 + y_3, \ldots) = (0, x_1 + y_1, x_2 + y_2, \ldots) \]
\[ = (0, x_1, x_2, \ldots) + (0, y_1, y_2, \ldots) = T(x) + T(y) \]
\[ T(\alpha x) = T(\alpha x_1, \alpha x_2, \alpha x_3, \ldots) = (0, \alpha x_1, \alpha x_2, \ldots) = \alpha(0, x_1, x_2, \ldots) = \alpha T(x). \]

So, $T$ linear.

2) $T$ finite. Taken any $(x_1, x_2, x_3, \ldots) \in \ell^2$ :

\[ \|T(x_1, x_2, x_3, \ldots)\|^2 = \|0, x_1, x_2, \ldots\|^2 = \sum_{i=2}^{\infty} |x_i|^2 = \|x\|^2 \]
\[ \Rightarrow \|T(x)\| \leq \|x\|. \]
With taken $M = 1$ so $T$ finite.

Jadi, $T \in B(\ell^2)$

3) Taken any $(x_1, x_2, x_3, \ldots) \in \ell^2$:

$$\left\| T(x_1, x_2, x_3, \ldots) \right\|^2 = \left\| (0, x_1, x_2, x_3, \ldots) \right\|^2 = \sum_{i=1}^{\infty} |x_i|^2 = \left\| (x_1, x_2, x_3, \ldots) \right\|^2$$

$$\Leftrightarrow \left\| T(x_1, x_2, x_3, \ldots) \right\| = \left\| (x_1, x_2, x_3, \ldots) \right\|$$

4) Taken any $(x_1, x_2, x_3, \ldots) \in \ell^2$

$$T(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots)$$

$$T^2(x_1, x_2, \ldots) = TT(x) = T(0, x_1, x_2, \ldots) = (0, 0, x_1, x_2, \ldots)$$

$$\left\| T^2(x_1, x_2, \ldots) \right\| = \left\| (0, 0, x_1, x_2, \ldots) \right\| = \sum_{i=1}^{\infty} |x_i|$$

5) Taken any $(x_1, x_2, x_3, \ldots) \in \ell^2$

$$T(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots)$$

$$\left\| T(x_1, x_2, \ldots) \right\|^2 = \left\| (0, x_1, x_2, \ldots) \right\|^2 = \sum_{i=1}^{\infty} |x_i|^2$$

Based on 4) and 5) obtained,

$$\left\| T(x) \right\|^2 \leq \left\| T^2(x) \right\|$$

So, $T$ paranormal operators.

To define the self adjoint operator, it is first explained about the existence of the operator. The following theorem 2.3 describes the existence of $T^*$ operator.

**Theorem 2.3.** [4] Given $H$ and $K$ Hilbert spaces. For each $T : H \to K$ for each continuous linear operator, then exist one only one continuous linear operator $T^* : K \to H$ so for each $x \in H$ and $y \in K$, consequence $\langle T(x), y \rangle = \langle x, T^*(y) \rangle$.

**Proof:**

Taken any $T \in L_\infty(H, K)$ and $y \in K$. Functionally formed $\varphi_y$ on $H$ with $\varphi_y(x) = \langle T(x), y \rangle, \forall x \in H$. Functional $\varphi_y$ is a continuous linear functional at $H$ because:

1) For each $x_1, x_2 \in H$ dan skalar $\alpha$ Obtained:

$$\varphi_y(x_1 + x_2) = \langle T(x_1 + x_2), y \rangle = \langle T(x_1), y \rangle + \langle T(x_2), y \rangle = \varphi_y(x_1) + \varphi_y(x_2)$$

$$\varphi_y(\alpha x_1) = \langle T(\alpha x_1), y \rangle = \alpha \langle T(x_1), y \rangle = \alpha \varphi_y(x_1)$$
2) For each \( x \in H \) obtained:
\[
\varphi_x(x) = \|T(x), y\| \leq \|T\| \|x\| \|y\|
\]
Because for each \( y \in K \), \( \varphi_y \) there is a single \( y' \in H \) so that for each \( x \in H \) applies \( \varphi_y(x) = \langle x, y' \rangle \). It means that for each \( y \in K \) determines singly \( y' \in H \). So there are \( T^*: K \rightarrow H \) operators with \( T^*(y) = y', \forall y \in K \). Therefore obtained:
\[
\varphi_x(x) = \langle T(x), y \rangle = \langle x, y' \rangle = \langle x, T^*(y) \rangle
\]
It is clear \( T^* \) that singular. Furthermore, \( T^* \) is linear and continuous because:

1) For each \( y_1, y_2 \in K \), \( x \in H \), dan \( \alpha, \beta \) skalar obtained:
\[
\langle x, T^*(\alpha y_1 + \beta y_2) \rangle = \langle T(x), \alpha y_1 + \beta y_2 \rangle = \langle T(x), \alpha y_1 \rangle + \langle T(x), \beta y_2 \rangle = \alpha^* \langle T(x), y_1 \rangle + \beta^* \langle T(x), y_2 \rangle = \alpha^* \langle x, T^*(y_1) \rangle + \beta^* \langle x, T^*(y_2) \rangle = \langle x, \alpha T^*(y_1) \rangle + \langle x, \beta T^*(y_2) \rangle
\]

2) For each \( x \in H \) obtained:
\begin{align*}
\text{a. If } x = \theta \text{ then, } \\
0 = \|T^*(\theta)\| = \langle T^*(\theta), T^*(\theta) \rangle = \langle \theta, TT^*(\theta) \rangle & \leq \|T\| \|\theta\| \|T^*(\theta)\| = 0.
\end{align*}
\begin{align*}
\text{b. If } x \neq \theta \text{ then, } \\
\|T^*(x)\| = \langle T^*(x), T^*(x) \rangle = \langle x, TT^*(x) \rangle & \leq \|T\| \|x\| \|T^*(x)\| \\
\Leftrightarrow \|T^*(x)\| & \leq \|T\| \|x\|.
\end{align*}

Taken \( M = \|T\| \). Obtained \( M \geq 0 \), so \( \|T^*(x)\| \leq M \|x\| \). So, \( T^* \) bounded.

So, it is evident that for each terbukti bahwa untuk setiap \( T \in L_1(H,K) \) there is one only one \( T^* \in L_1(K,H) \) so:
\[
\langle T(x), y \rangle = \langle x, T^*(y) \rangle, \forall x \in H \text{ dan } y \in K.
\]

**Definition 2.4.** [14] Continuous linear operator \( T^* \) as explained in theorem 2.3 is called adjoint operator of \( T \). 

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Afterward will be explained about the definition of self adjoint operators and existence $\|T\|$ on Hilbert spaces. The definition of self adjoint operator are required to prove the properties of paranormal operators. Definition of self adjoint operator and existence $\|T\|$ is explained in definition 2.5 and theorem 2.6.

**Definition 2.5.** [4] Given a Hilbert space $H$ and $T \in B(H)$. Operator $T$ called self adjoint if $T = T^*$

**Theorem 2.6.** [4] Given $X$, $Y$ normed spaces over filed $\mathbb{R}$ or $\mathbb{C}$ and $T : X \to Y$. For each bounded linear operator $T$, that right:

1) $\|T\| = \sup \{\|T(x)\| : x \in X, \|x\| < 1\}$

2) If $X \neq \{0\}$, then $\|T\| = \sup \{\|T(x)\| : x \in X, \|x\| = 1\}$

**Proof:**

1) Name it $K = \sup \{\|T(x)\| : x \in X, \|x\| < 1\}$. Will be proven: $K = \|T\|$. So,

$\{\|T(x)\| : x \in X, \|x\| < 1\} \subseteq \{\|T(x)\| : x \in X, \|x\| \leq 1\}$. Obtained,

$\sup \{\|T(x)\| : x \in X, \|x\| < 1\} \leq \sup \{\|T(x)\| : x \in X, \|x\| \leq 1\}$. Equivalent with $K \leq \|T\|$. Taken any $\varepsilon > 0$ and any $x \in X$. Name $y = \frac{x}{\|x\| + \varepsilon}$. The result, $\|y\| < 1$. So,

$\|T(y)\| \in \{\|T(x)\| : x \in X, \|x\| < 1\}$. So, $\|T(y)\| \leq K$. This is equivalent to:

$$\|T\left(\frac{x}{\|x\| + \varepsilon}\right)\| \leq K \Rightarrow \|T(x)\| \leq K\left(\|x\| + \varepsilon\right).$$

Because it applies to every $\varepsilon > 0$, obtained $\|T(x)\| \leq K \|x\|$. Obtained,

$\sup \{\|T(x)\| : x \in X, \|x\| \leq 1\} \leq \sup \{K \|x\| : x \in X, \|x\| \leq 1\}$

$= K \sup \{\|x\| : x \in X, \|x\| \leq 1\} = K.1 = K.$

So, $\|T\| \leq K$. Thus, $K = \|T\|$.

2) Because $X \neq \{\theta\}$ , then it can be found $x \in X$ so $\|x\| = 1$. Name $N = \sup \{\|T(x)\| : x \in X, \|x\| = 1\}$. Will be proven $N = \|T\|$. Noted that,

$\{\|T(x)\| : x \in X, \|x\| = 1\} \subseteq \{\|T(x)\| : x \in X, \|x\| \leq 1\}$. As a result,
sup \{\|T(x)\| : x \in X, \|x\| = 1\} \leq \sup \{\|T(x)\| : x \in X, \|x\| \leq 1\}. \text{Equivalent with } N \leq \|T\|. \text{ Taken any } x \in X \text{ with } \|x\| \leq 1.

If \( x = \theta \), then \( T(x) = \theta \). As a result, \( \|T(x)\| = 0 \leq N \).

If \( x \neq \theta \), then name \( y = \frac{x}{\|x\|} \). As a result, \( \|y\| = 1 \).

So, \( \|T(y)\| = \{\|T(x)\| : x \in X, \|x\| = 1\} \). SO, \( \|T(y)\| \leq N \). Tjis is equivalent to:

\[
\|T\left( \frac{x}{\|x\|} \right) \| \leq N \Rightarrow \|T(x)\| \leq N \|x\|.
\]

As a result,

\[
\sup \{\|T(x)\| : x \in X, \|x\| \leq 1\} \leq \sup \{K\|x\| : x \in X, \|x\| \leq 1\} = N \sup \{\|x\| : x \in X, \|x\| \leq 1\} = N.1 = N.
\]

So, \( \|T\| \leq N \). Thus, \( N = \|T\| \).

**Theorem 2.7.** If \( S, T \) paranormal, self-adjoint, and \( S = T \), then \( (S + T) \) and \( (ST) \) paranormal operators.

**Proof:** Taken any \( x \in H \) with \( \|x\| \leq 1 \).

Know \( \|T(x)\|^2 \leq \|T^2(x)\|, \|T(x)\|^2 \leq \|T^2(x)\|, S = S^*, \text{ dan } T = T^* \)

1. It will be shown \( (S + T) \): Noticed that:

\[
\| (S + T)(x) \|^2 = \langle (S + T)(x), (S + T)(x) \rangle = \langle (S + T)^* (S + T)(x), x \rangle = \langle (S + T)(x), (S + T)^* (S + T)(x), x \rangle
\]

\[
= \langle (S + T)(x), x \rangle \leq \| (S + T)(x) \| \|x\| \leq \| (S + T)^2 (x) \| 1 = \| (S + T)^2 (x) \|
\]

So, \( (S + T) \) paranormal operators.

2. It will be shown \( (ST) \): Noticed that:

\[
\| ST(x) \|^2 = \langle (ST)(x), (ST)(x) \rangle = \langle (ST)^* (ST)(x), x \rangle = \langle (T^* S^*)(ST)(x), x \rangle
\]

\[
= \langle (TS)(x), x \rangle \leq \| (ST)(x) \| \|x\| \leq \| (ST)^2 (x) \| 1 = \| (ST)^2 (x) \|
\]

So, \( (ST) \) paranormal operators.

**Theorem 2.8.** If \( T \) paranormal and self-adjoint, then \( T^* \) paranormal operators.

**Proof:** Taken any \( x \in H \) with \( \|x\| \leq 1 \).

\[
\| T^*(x) \|^2 = \| T(x) \|^2 = \langle T(x), T(x) \rangle = \langle T^* T(x), x \rangle \leq \| T^* T(x) \| \|x\| = \| TT(x) \| \|x\| \leq \| T^2 (x) \| 1 = \| T^2 (x) \|
\]
**Theorem 2.9.** If $T$ self-adjoint, then $T$ paranormal

**Proof:** Taken any $x \in H$ with $\|x\| \leq 1$. Known $T$ self-adjoint, it means $T = T^*$. Thus,

$$\|T(x)\|^2 = \langle T(x), T(x) \rangle = \langle T^* T(x), x \rangle \leq \|TT(x)\| \|x\| = \|TT(x)\| \leq \|T^2(x)\| = \|T^2(x)\|.$$ 

So, $T$ paranormal operators.

In [15] explained the nature of paranormal operators with $r(T)$. In this case, $r(T)$ is defined as the distance of $T$ operator.

**Corollary 2.11.** If $T$ self adjoint, then $\|T\| = r(T)$.

**Proof:** According to the theorem 2.9, obtained is $T$ paranormal operators. In [15], has been proven if $T$ paranormal operator, then $\|T\| = r(T)$.

The next characteristic that will be shown is relationship between paranormal operators and hyponormal operators. Next, the definition of the hyponormal operator will then be conveyed, then characteristics of the two operators will be continued. In [5], [6] was delivered regarding the definition of hyponormal operators.

**Definition 2.11.** Given a Hilbert space $H$ and $T \in B(H)$. Operator $T$ called hyponormal if $T^* T \geq T T^*$.

**Theorem 2.12.** If $T$ paranormal and hyponormal, then $\|T\| = \|T^*\|$.

**Proof:** Taken any $x \in H$.

$$\|T(x)\|^2 = \langle T(x), T(x) \rangle = \langle T^* T(x), x \rangle \geq \langle TT^*(x), x \rangle = \langle T^*(x), T^*(x) \rangle = \|T^*(x)\|^2$$

$$\Leftrightarrow \|T(x)\| \geq \|T^*(x)\|.$$ 

According to the theorem 2.6, obtained $\|T\| \geq \|T^*\|$.

Further analysis of the nature of paranormal operators with Weyl Theorem. The discussion requires the definition of a polaroid operator and SVEP (Single Valued Extension Property). In [7] operator $T$ is said to SVEP if for each neighborhood of $\lambda_0 \in C$ there is exactly one analytic function $f : U \rightarrow H$ that satisfies the equation $(\lambda I - T)f(\lambda) = 0$, $\forall \lambda \in U$ and $f = 0$, with $U$ is neighborhood of $\lambda_0 \in C$. While, $T$ said to polaroid operator if for each isolated point of $\sigma(T)$ is the pole of the resolvent. If $T$ paranormal operator, then $T$ polaroid [16]. If $T$ paranormal then $T$ SVEP [8]

**Theorem 2.13.** If $T$ paranormal, then $T$ Weyl Theorem.

**Proof:** Because $T$ paranormal, then polaroid and SVEP. In [17], obtained $T$ Weyl theorem.
3. Main Result

The definition of paranormal operators is developed into $n$- paranormal operators and $n$th- paranormal operators, for $n \in \mathbb{N}$. Using that definition, it will be investigated the properties of $n$- paranormal operators and $n$th- paranormal operators. This following will be presented about the properties of $n$- paranormal operators and $n$th- paranormal operators, for $n \in \mathbb{N}$ [1], [2].

**Definition 3.1.** Given a Hilbert space $H$ and $T \in B(H)$. Operator $T$ called $n$- paranormal if

$$\|T^{n+1}(x)\| \geq \|T(x)\|^{n+1}, \text{ for } a \in \mathbb{N}.$$ 

**Definition 3.2.** Given a Hilbert space $H$ and $T \in B(H)$. Operator $T$ called paranormal- $n$th if

$$\|T^{2n}(x)\| \geq \|T^n(x)\|^2, \text{ for } a \in \mathbb{N}.$$ 

**Theorem 3.3.** If $S, T$ self-adjoint, so $S + T$ $n$th- paranormal, for a $n \in \mathbb{N}$.

**Proof:** Taken $n \in \mathbb{N}$. For a $x \in H$ with $\|x\| \leq 1$.

$$\|(S + T)^n(x)\|^2 = \langle (S + T)^n(x), (S + T)^n(x) \rangle = \langle (S + T)^n(S + T)^n(x), x \rangle = \langle ((S + T)^n)^n(S + T)^n(x), x \rangle$$

$$= \langle (S^n + T^n)^n(S + T)^n(x), x \rangle = \langle (S + T)^n(S + T)^n(x), x \rangle = \langle (S + T)^{2n}(x), x \rangle$$

$$\leq \|(S + T)^{2n}(x)\| \|x\| = \|(S + T)^{2n}(x)\| \|1 = \|(S + T)^{2n}(x)\|.$$ 

So, $S + T$ $n$th- paranormal operators.

This following will be presented about the lemma of self adjoint operator on the Hilbert space over field $\mathcal{R}$. In this case, $ST$ states the composition of operators $S$ and $T$.

**Lemma 3.4.** Given $H$ Hilbert space over the field $\mathcal{R}$ and $S, T \in B(H)$. If $S, T$ self adjoint, then $ST = TS$.

**Proof:** Taken any of $x \in H$.

$$\langle ST(x), x \rangle = \langle T(x), S^*(x) \rangle = \langle x, T^* S^*(x) \rangle = \langle x, TS(x) \rangle = \langle TS(x), x \rangle.$$ 

So, $ST = TS$.

**Theorem 3.5.** Given $H$ Hilbert space over the field $\mathcal{R}$ and $S, T \in B(H)$. If $S, T$ self adjoint, then $ST$ paranormal-$n$th, for $n \in \mathbb{N}$.

**Proof:** Taken $n \in \mathbb{N}$. For every $x \in H$ with $\|x\| \leq 1$. Because $S$ and $T$ self adjoint then $S^* = S$ dan $T^* = T$. Obtained:
\[ \| (ST)^n(x) \|^2 = \langle (ST)^n(x), (ST)^n(x) \rangle = \langle (ST)^n(S + T)^n(x), x \rangle = \langle (ST)^nT^n(S + T)^n(x), x \rangle = \langle (TS)^n(S + T)^n(x), x \rangle = \langle (ST)^n(S + T)^n(x), x \rangle = \langle (ST)^{2n}(x), x \rangle \]

= \langle (T^*S^*)^{2n}(x), x \rangle = \langle (TS)^{2n}(x), x \rangle = \langle (ST)^{2n}(x), x \rangle = \langle (ST)^{2n}(x), x \rangle \]

\[ \leq \| (ST)^{2n}(x) \| \| x \| \leq \| (ST)^{2n}(x) \| 1 = \| (ST)^{2n}(x) \| \]

So, \( ST \) \( n \)-th paranormal operators.

Next, \( * \)-paranormal operators, is defined. The definition of the operator using the adjoint operator on a Hilbert space, was introduced by [3]. The following will be discussed properties of \( * \)-paranormal operators.

**Definition 3.6.** Given \( H \) Hilbert space and \( T \in B(H) \). The operator \( T \) is called to be \( * \)-paranormal if \( \| T^2(x) \| \| x \| \geq \| T^*(x) \| ^2 \).

**Theorem 3.7.** Given \( H \) Hilbert space over the field \( \mathbb{F} \) and \( S, T \in B(H) \). If \( S \), \( T \) self adjoint, then \( ST \) \( * \)-paranormal.

**Proof:** Taken any of \( x \in H \).

\[ \| (ST)^n(x) \|^2 = \langle (ST)^n(x), (ST)^n(x) \rangle = \langle ST(ST)^n(x), x \rangle = \langle STTS(x), x \rangle = \langle STT(x), x \rangle \]

= \langle (ST)^2(x), x \rangle \leq \| (ST)^2(x) \| \| x \| .

So, \( ST \) \( * \)-paranormal operators.

**Theorem 3.8.** If \( S \), \( T \) self adjoint, then \( S + T \) \( * \)-paranormal.

**Proof:** Taken any of \( x \in H \).

\[ \| (S + T)^n(x) \|^2 = \langle (S + T)^n(x), (S + T)^n(x) \rangle = \langle (S + T)^n(S + T)^n(x), x \rangle = \langle (S + T)^n(S + T)(x), x \rangle = \langle (S + T)^n(S + T)^n(x), x \rangle = \langle (S + T)^2(x), x \rangle \]

\[ \leq \| (S + T)^2(x) \| \| x \| \]

So, \( S + T \) \( * \)-paranormal operators.

In [18], it was explained that each bounded linear operator \( T \) in Hilbert space \( H \) can be decomposed into \( T = U \| T \| \) with \( U \) is unitary operator. This also applies to operators \( T^* \), can be decomposed into \( T^* = U^* \| T^* \| \).

**Theorem 3.9.** If \( T = U \| T \| \) hyponormal and self adjoint, then \( T \) \( * \)-paranormal.

**Proof:** Taken any of \( x \in H \).
\[
\left\| T^\ast(x) \right\|^2 = \left\| (U|T|)(U|T|)^\ast(x) \right\|^2 = \left\langle (U|T|)(U|T|)^\ast(x), (U|T|)(U|T|)^\ast(x) \right\rangle \leq \left\langle (U|T|)(U|T|)(U|T|)^\ast(x), (U|T|)(U|T|)^\ast(x) \right\rangle = \left\| (U|T|)(U|T|)^\ast(x) \right\|^2 = \left\| T^2(x) \right\|^2.
\]

So, \( T \) *- paranormal.

Definition 3.10. Given \( H \) Hilbert space and \( T \in B(H) \). The operator \( T \) is called to be quasi- *- paranormal if \( \left\| T^2(x) \right\| \geq \left\| T^\ast T(x) \right\|^2 \).

Definition 3.11. Given \( H \) Hilbert space and \( T \in B(H) \). The operator \( T \) is called to be \( n \)- quasi- *- paranormal if \( \left\| T^{n+2}(x) \right\| \left\| T^n(x) \right\| \geq \left\| T^\ast T^n(x) \right\|^2 \) for \( a n \in N \).

Theorem 3.12. If \( T \) *- paranormal, then \( T \) quasi- *- paranormal.

Proof: Taken any of \( x \in H \).

\[
\left\| T^\ast T(x) \right\|^2 = \left\langle T^\ast T(x), T^\ast T(x) \right\rangle \leq \left\| T^2 T(x) \right\| \left\| T^\ast T(x) \right\| = \left\| T^2(x) \right\| \left\| T(x) \right\|.
\]

So, \( T \) quasi- *- paranormal.

Theorem 3.13. If \( T \) *- paranormal, then \( T \) \( n \)- quasi- *- paranormal.

Proof: Taken any of \( x \in H \).

\[
\left\| T^\ast T^n(x) \right\|^2 = \left\langle T^\ast T^n(x), T^\ast T^n(x) \right\rangle \leq \left\| T^2 T^n(x) \right\| \left\| T^n(x) \right\| = \left\| T^{n+2}(x) \right\| \left\| T^n(x) \right\|.
\]

So, \( T \) \( n \)- quasi- *- paranormal.

The next discussion is about Bishop property \( (\beta) \). \( T \) is said to satisfy the Bishop property \( (\beta) \) if for each set \( G \subseteq C \) is open and each sequence \( f_n : G \to H \) so \( (T - z)f_n(x) \) and \( f_n(x) \) that uniform convergent to 0 are in the norm on compact subsets of \( G \), with \( C \) a complex number set, \( H \) analytic function, and \( x \in G \) [7],[8]. Then, in [6],[12] explains that the *- paranormal operator is Bishop property \( (\beta) \).

Corollary 3.14. If \( T \) hyponormal and self adjoint, then \( T \) Bishop property \( (\beta) \).
Proof: Because $T$ hyponormal and self adjoint, then according to theorema 3.13 $T^* -$ paranormal. According to [12], obtained $T$ is Bishop property ($\beta$).

Next, will be discussed regarding the relationship between the operators of the paranormal, $n$-paranormal operators, $nt$- paranormal operators, and $\ast$- paranormal operators. In addition, it will be shown some traits associated with these operators.

**Theorem 3.15.** Given $H$ Hilbert space and $T \in B(H)$. For $n \in \mathbb{N}$, if $T^n$ paranormal, then $T$ paranormal.

**Proof:** Taken $n = 1$. Obtained $T^1$ paranormal operator:

$$\|T^{1+1}(x)\| \geq \|T^1(x)\|^{1+1} \Leftrightarrow \|T^2(x)\| \geq \|T(x)\|^2$$

So, $T$ paranormal operators.

**Theorem 3.16.** Given $H$ Hilbert space and $T \in B(H)$. If $T$ paranormal, then $T^n$ paranormal, for $n \in \mathbb{N}$.

**Proof:** Taken any of $x \in H$. Using mathematical induction, obtained:

1. It will be shown, it is correct for $n = 1$. Because $T$ paranormal operator, then obtained:

$$\|T^{1+1}(x)\| = \|T^2(x)\| \geq \|T(x)\|^2 = \|T(x)\|^{1+1}$$

2. Considering that it is correct for $n = k$. Obtained,

$$\|T^{k+1}(x)\| \geq \|T(x)\|^{k+1}$$

3. It will be shown, it is correct $n = k + 1$.

$$\|T(x)\|^{(k+1)+1} = \|T(x)\|^{k+1} \|T(x)\| \leq \|T^{k+1}(x)\| \|T(x)\| = \|T^{(k+1)+1}(x)\| \Leftrightarrow \|T(x)\|^{(k+1)+1} \leq \|T^{(k+1)+1}(x)\|$$

So, $T^n$ paranormal operator

The following will be explained about the relationship between paranormal operators and $nth$-paranormal operators.

**Theorem 3.17.** If $T$ paranormal and $T^2 = T$, then $T^n$ paranormal, for $n \in \mathbb{N}$.

**Proof:** Known $\|T^2(x)\| \geq \|T(x)\|^2$, $\forall x \in H$.

Taken any of $x \in H$. Using mathematical induction, obtained:

1. It will be shown, it is correct for $n = 1$. Because $T$ paranormal operator, then obtained:
\[ \|T^{2n}(x)\| = \|T^{21}(x)\| = \|T^{2}(x)\| \geq \|T(x)\|^2 = \|T^1(x)\|^2 \]

2. Considering that it is correct for \( n = k \). Obtained,

\[ \|T^{2k}(x)\| \geq \|T^k(x)\|^2 \]

3. It will be shown, it is correct \( n = k + 1 \).

\[ \|T^{2(k+1)}(x)\| = \|T^{2k}T^2(x)\| = \|T^{2k}T^k(x)\| \geq \|T^2\|T^k(x)\| \geq \|T\||\|T^k(x)\|^2 \]

\[ = \|TT^k(x)\|^2 \geq \|T^{k+1}(x)\|^2. \]

Obtained, \( \|T^{2(k+1)}(x)\| \geq \|T^{k+1}(x)\|^2 \).

So, \( T \) \textit{nth}-paranormal operator for some \( n \in N \).

\textbf{Theorem 3.18.} Given \( H \) Hilbert space and \( T \in B(H) \). If \( T \) \( n \)-paranormal, then \( T \) \( n \)-paranormal, for \( n \in N \).

\textbf{Proof:} Taken any of \( x \in H \). Known \( \|T^{n+1}(x)\| \geq \|T(x)\|^{n+1} \), for \( n \in N \). It will be shown \( \|T^{2n}(x)\| \geq \|T^n(x)\|^2 \), for \( n \in N \). For all \( n \in N \).

Taken any of \( x \in H \). Using mathematical induction, obtained:

1. It will be shown, it is correct for \( n = 1 \). Because \( T \) paranormal operator, then obtained:

\[ \|T^{2n}(x)\| = \|T^{21}(x)\| = \|T^{2}(x)\| \geq \|T(x)\|^2 = \|T^1(x)\|^2 \]

2. Considering that it is correct for \( n = k \). Obtained,

\[ \|T^{2k}(x)\| \geq \|T^k(x)\|^2 \]

3. It will be shown, it is correct \( n = k + 1 \).

\[ \|T^{2(k+1)}(x)\| = \|T^{2k}T^2(x)\| = \|T^{2k}T^k(x)\| \geq \|T^2\|T^k(x)\| \geq \|T\||\|T^k(x)\|^2 \]

\[ = \|TT^k(x)\|^2 \geq \|T^{k+1}(x)\|^2. \]

Obtained, \( \|T^{2(k+1)}(x)\| \geq \|T^{k+1}(x)\|^2 \).

So, \( T \) \( n \)-paranormal operator for some \( n \in N \).  

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In [19], it was explained that each $T^\ast -$ paranormal applies $(n + 1) -$ paranormal. If $T$ n-$ \ast$ paranormal, then $(n + 1) -$ paranormal [20]. In this regard, the following theorem explains that opposite applies from the statement above.

**Theorem 3.19.** If $T$ paranormal, then $T^\ast$ - paranormal.

**Proof:** Taken any of $x \in H$. Known $\|T^2(x)\| \geq \|T(x)\|^2$ and $T = T^\ast$. 

\[ \|T^\ast(x)\|^2 = \|T(x)\|^2 = \langle T(x), T(x) \rangle = \langle T^\ast T(x), x \rangle = \langle TT(x), x \rangle = \langle T^2(x), x \rangle \leq \|T^2(x)\| \|x\| \]

So, $T^\ast$ - paranormal operator.

**Theorem 3.20.** [5] If $T^\ast$ - paranormal and $M$ is a $- T$ invariant closed subspace, then $T\big|_M^\ast$ - paranormal.

**Theorem 3.21.** [2] If $T$ nth- paranormal and $M$ is a $T$ invariant closed subspace, then $T\big|_M^n$ nth-paranormal.

**Proof:** Let $x \in H$ with $\|x\| \leq 1$.

\[ \left\| \left( T\big|_M^n \right)^\ast \right\| \left( x \right) \right\|^2 = \left\| T^n \right\| \left( M \right)^\ast \left( x \right) \right\|^2 \leq \left\| T^2^n \right\| \left( x \right) \right\|^2 \leq \left\| T^2^n \right\| \left( M \right)^\ast \left( x \right) \right\|^2 = \left\| \left( T\big|_M^n \right)^2 \right\| \left( x \right) \right\|^2 \]

So, $T\big|_M^n$ nth- paranormal.

**Theorem 3.22.** If $T$ nth- paranormal and $T$ is a self adjoint operator, then $T^n$ nth- paranormal.

**Proof:** Let $x \in H$ with $\|x\| \leq 1$.

\[ \left\| \left( T^n \right)^\ast \right\| \left( x \right) \right\|^2 = \left\langle \left( T^n \right)^\ast \left( x \right), \left( T^n \right)^\ast \left( x \right) \right\rangle = \left\langle \left( T^\ast \right)^n \left( T^n \right)^\ast \left( x \right), \left( T^n \right)^\ast \left( x \right) \right\rangle = \left\langle \left( T^\ast \right)^n \left( T^n \right)^\ast \left( x \right), \left( T^n \right)^\ast \left( x \right) \right\rangle \leq \left\| \left( T^\ast \right)^n \left( T^n \right)^\ast \left( x \right) \right\| \left\| x \right\| \leq \left\| \left( T^n \right)^\ast \right\| \left( x \right) \right\|^2 = \left\| \left( T^n \right)^\ast \right\|^2 \left( x \right) \right\|^2 \]

So, $T^n$ nth- paranormal.

**Theorem 3.23.** If $T^\ast$ - paranormal and $T$ is a self adjoint operator, then $T^n^\ast$ - paranormal.

**Proof:** Let $x \in H$ with $\|x\| \leq 1$.

\[ \left\| \left( T^n \right)^\ast \right\| \left( x \right) \right\|^2 = \left\langle \left( T^n \right)^\ast \left( x \right), \left( T^n \right)^\ast \left( x \right) \right\rangle = \left\langle \left( T^n \right)^\ast \left( T^n \right)^\ast \left( x \right), \left( T^n \right)^\ast \left( x \right) \right\rangle = \left\langle \left( T^n \right)^\ast \left( T^n \right)^\ast \left( x \right), \left( T^n \right)^\ast \left( x \right) \right\rangle \leq \left\| \left( T^n \right)^\ast \left( T^n \right)^\ast \left( x \right) \right\| \left\| x \right\| \leq \left\| \left( T^n \right)^\ast \right\| \left( x \right) \right\|^2 = \left\| \left( T^n \right)^\ast \right\|^2 \left( x \right) \right\|^2 \]

So, $T^n^\ast$ - paranormal.
4. Conclusion

The conclusion that can be taken is an example and algebra properties of each paranormal operator, $n$-th paranormal operator, $*$-paranormal operator, including algebra addition, multiplication operators, and relationship with self adjoint operator and hyponormal operator. In addition, other properties are obtained is relationship between each of the operators of such other description, if $T$ is $n$- paranormal, then $T$ paranormal and dependencies with invariant operator on the closed subspace of Hilbert space.

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