Note on Permutation Sum of Color-ordered Gluon Amplitudes

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ABSTRACT: In this note we show that under BCFW-deformation the large-\(z\) behavior of permutation sum of color-ordered gluon amplitudes found by Boels and Isermann in arxiv:1109.5888 can be simply understood from the well known Kleiss-Kuijf relation and Bern-Carrasco-Johansson relation.

KEYWORDS: [BCFW-deformation, Permutation Sum]
1. Introduction

Initiated by Witten’s work on twistor theory \cite{2}, a tremendous amount of progress has been made in the calculation and understanding of scattering amplitudes. One of such is the on-shell recursion relation (BCFW recursion relation) for tree-level gluon amplitudes \cite{2,3}. The study of recursion relation originated from analyzing BCFW-deformation on a particular pair of particles \((i, j)\)

\[
p_i \rightarrow p_i - zq, \quad p_j \rightarrow p_j + zq, \quad q^2 = q \cdot p_i = q \cdot p_j = 0. \quad [\text{BCFW-def}]
\]  

(1.1)

Under the deformation, the physical on-shell amplitude \(A_n\) of \(n\)-particles becomes a rational function \(A_n(z)\) of single complex variable \(z\) with only pole structures\(^1\). The large-\(z\) behavior (or the “boundary behavior”) of \(A_n(z)\) under such a deformation turns out to be very important for our deep understanding of many properties in quantum field theories. The determinant of boundary behavior is not so easy and a naive analysis from Feynman diagrams could often lead to wrong conclusions. A very nice analysis was done by Arkani-hamed and Kaplan in \cite{4}, where because \(zq \rightarrow \infty\), whole amplitude can be considered as scattering of particles \(i, j\) from soft background constructed by other particles.

The boundary behavior can be classified into two categories. The first category is that \(A(z)\) does not vanish when \(z \rightarrow \infty\). For this category, to write down on-shell recursion relation, we need to find boundary contributions. So far there is no general theory to extract such information easily, however some progress can be found in \cite{5,6,7,8,9}. The second category is that \(A(z) \rightarrow 0\) when \(z \rightarrow \infty\). For this case, the boundary contribution is zero and the familiar on-shell recursion relation is derived based on this condition.

The vanishing behavior of second category is a little bit rough and we can make it more accurate by writing the leading behavior as \(A(z) \sim \frac{1}{z^k}\), \(k \geq 1\). If for some theories we have \(k \geq 2\), more relations can be derived in addition to the standard BCFW recursion relation. These “bonus” relations were discussed in

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\(^1\)It can be single poles at finite values of \(z\) or multiple pole at the \(z = \infty\).
where their usefulness was demonstrated from various aspects. In particular, the better vanishing behavior of color-ordered gluon amplitudes with non-adjacent deformation pair was used in \([13]\) to prove the BCJ relation \([16]\) between color-ordered gluon amplitudes. An important consequence of BCJ relation is that among all \((n-1)!\) color-ordered amplitudes, only \((n-3)!\) of them are needed and others can be written as the linear combinations of the basis.

Recently, another new better vanishing behavior under deformation \([1.1]\) was observed by Boels and Isermann in \([20]\), which can be summarized as the following two statements\(^3\)

\[
\sum_{\text{perm } \alpha} A_n(i, \{\alpha\}, j, \{\beta\}) \rightarrow \xi_{i\mu}(z) \xi_{j\nu}(z) \frac{G^{\mu\nu}(z)}{z^k}, \quad k = \begin{cases} n_\alpha, & i, j \text{ not nearby} \\ n_\alpha - 1, & i, j \text{ nearby} \end{cases} \quad [\text{Boels-1}] (1.2)
\]

and

\[
\sum_{\text{cyclic } \alpha} A_n(i, \{\alpha\}, j, \{\beta\}) \rightarrow \xi_{i\mu}(z) \xi_{j\nu}(z) \frac{G^{\mu\nu}(z)}{z^k}, \quad k = \begin{cases} 2, & i, j \text{ not nearby} \\ 1, & i, j \text{ nearby} \end{cases} \quad [\text{Boels-2}] (1.3)
\]

where \(n_\alpha\) is the number of elements in set \(\alpha\), \(\xi\) is the polarization vector and \(G_{\mu\nu}\) is given by \([4]\)

\[
G_{\mu\nu} = z\eta^{\mu\nu} f(1/z) + B^{\mu\nu}(1/z) + \mathcal{O}(1/z). \quad (1.4)
\]

With these new results, a natural question to ask is that do they provide new nontrivial relations among color-ordered amplitudes or they can be understood from known results?

In this short note we would like to answer the above question. In particular we shall show that using the familiar KK-relation and fundamental BCJ relation, \([12]\) can be easily understood. The Kleiss-Kuijf (KK) relation was first conjectured in \([22]\) and proved in \([23]\). The formula reads

\[
A_n(1, \{\alpha\}, n, \{\beta\}) = (-1)^{n_\beta} \sum_{\sigma \in \text{OP}(\alpha, \beta^T)} A_n(1, \sigma, n), \quad [\text{KK-rel}] (1.5)
\]

where the Order-Preserved (OP) sum is to be taken over all permutations of set \(\alpha \cup \beta^T\) whereas the relative ordering in sets \(\alpha\) and \(\beta^T\) (which is the reversed ordering of set \(\beta\)) are preserved. The \(n_\beta\) here is the number of elements in set \(\beta\). One non-trivial example with six gluons is given as the following

\[
A(1, \{2, 3\}, 6, \{4, 5\}) = A(1, 2, 3, 5, 4, 6) + A(1, 2, 5, 3, 4, 6) + A(1, 2, 5, 4, 3, 6) + A(1, 5, 4, 2, 3, 6) + A(1, 5, 2, 4, 3, 6) + A(1, 5, 2, 3, 4, 6) \quad [\text{KK-6-point}] (1.6)
\]

Comparing to the simple expression of KK-relation \([17]\), the general BCJ relation given in \([16]\) is complicated, where the basis with three particles fixed at three positions has been used to expand other

\(^2\)The BCJ relation has first been proved in string theory \([17, 18]\), and then in field theory \([13, 19]\).

\(^3\)Incidentally a second paper of this series of work came out during the preparation of this note, where similar argument was used \([21]\).
amplitudes. However, there is a very simple relation, which we will call the “fundamental BCJ relation”. The fundamental BCJ relation can be used to derive other BCJ relations and its expression is given by

\[ 0 = s_{21} A_n(1, 2, 3, ..., n - 1, n) + \sum_{j=3}^{n-1} (s_{21} + \sum_{t=3}^{j} s_{2t}) A(1, 3, 4, ..., j, 2, j + 1, ..., n). \]  

(1.7)

The plan of the note is following. In section two we prove the conjecture (1.2) for \(i, j\) not nearby. The nearby case is special and the proof is given in section three. Finally, a brief summation and discussion are given in section four.

2. The non-adjacent case

Without lost of generality we pick the shifted pair to be \((1, j)\) and define the following sum

\[ T_j(z) \equiv \sum_{\sigma \in S_{j-2}} A_n(\hat{1}(z), \sigma(2), ..., \sigma(j-1), \hat{j}(z), ...n - 1, n), \quad j = 2, ..., n - 1. \]  

(2.1)

where the case \(j = n\) is the special adjacent case to be discussed in next section. In the following we will encounter cases where permutation sums are performed on \((j - 2)\) elements other than \((2, ..., j - 1)\) in (2.1). For simplicity we will stick to the same notation \(T_j\), but the accurate definitions of these \(T_j\) should be clear from our discussions.

According to the conjecture (1.2), we should have

\[ \lim_{z \to \infty} T_j(z) \to \xi_{1\mu}(z) \xi_{j\nu}(z) \frac{G_{\mu\nu}(z)}{z^{j-2}}. \]  

(2.2)

For the case \(j = 2\) and \(j = 3\) we have

\[ \lim_{z \to \infty} T_2(z) = \lim_{z \to \infty} A_n(\hat{1}(z), \hat{2}(z), 3, ...n - 1, n) \to \xi_{1\mu}(z) \xi_{j\nu}(z) G_{\mu\nu}(z) \]  

\[ \lim_{z \to \infty} T_3(z) = \lim_{z \to \infty} A_n(\hat{1}(z), \hat{2}(z), \hat{3}(z), 4, ...n - 1, n) \to \xi_{1\mu}(z) \xi_{j\nu}(z) \frac{G_{\mu\nu}(z)}{z} \]  

(2.3)

which are well known to be true from [4].

For the case \(j = 4\), let us consider following two fundamental BCJ relations

\[ 0 = B_{23} = s_{2\hat{1}} A_n(\hat{1}(z), 2, 3, \hat{4}(z), 5, ..., n) + (s_{2\hat{1}} + s_{23}) A_n(\hat{1}(z), 3, 2, \hat{4}(z), 5, ..., n) \]  

\[ + \sum_{k=4}^{n-1} \left( \sum_{t=1}^{k} s_{2k} \right) A_n(\hat{1}, 3, \hat{4}, ..., k, 2, k + 1, ..., n) \]  

(2.4)

\[ ^4\text{In fact assuming the result is true for } j = 2, \text{ the case with } j = 3 \text{ can be proved using fundamental BCJ relation as is done in following paragraphs. To avoid arguing in a circle we need to establish the BCJ relation from, for example, string theory [17, 18], in stead of from the bonus relation given in [13].} \]
and its corresponding $B_{32}$ by exchanging $2 \leftrightarrow 3$. To emphasize the $z$-dependence, we have put on a “hat” for clarity, thus $s_{21} \neq s_{21}$. One key observation is that the third term in $B_{23}$ is independent of $z$ because $s_{21} + s_{24} = s_{21} + s_{24}$. Summing them up, we have

$$0 = B_{23} + B_{32} = (s_{21} + s_{31} + s_{23})T_4 + \left( \sum_{k=1}^{n-1} \sum_{t=1}^{k} s_{2k} A_n(\hat{1}, 3, \hat{4}(z), ...k, 2, k + 1, ..., n) + \{2 \leftrightarrow 3\} \right)$$

or

$$T_4 = \frac{\left( \sum_{k=1}^{n-1} \sum_{t=1}^{k} s_{2k} A_n(\hat{1}, 3, \hat{4}(z), ...k, 2, k + 1, ..., n) + \{2 \leftrightarrow 3\} \right)}{-s_{123}} \quad \text{(2.5)}$$

Since each term in the numerator of (2.5) is the form of $T_3$ defined in (2.1), using (2.3) we have immediately

$$\lim_{z \to \infty} T_4(z) \to \frac{1}{z} \lim_{z \to \infty} T_3(z) \to \xi_{1\mu}(z)\xi_{j\nu}(z) \frac{G_{\mu\nu}(z)}{z^2} \quad \text{[T4]} \quad \text{(2.6)}$$

which is the result we want.

To see more clearly our method, let us consider the case $j = 5$. For this we write down the following BCJ relations

$$B_{234} = (s_{21})A(\hat{1}, 2, 3, 4, \hat{5}, ...) + (s_{21} + s_{23})A(\hat{1}, 3, 2, 4, \hat{5}, ...) + (s_{21} + s_{23} + s_{24})A(\hat{1}, 3, 4, 2, \hat{5}, ...)
+ ... + (\sum_{t=1}^{j} s_{2t})A(\hat{1}, 3, 4, \hat{5}, ..., j, 2, j + 1, ...)
+ (s_{21} + s_{24})A(\hat{1}, 4, 2, 3, \hat{5}, ...) + (s_{21} + s_{23} + s_{24})A(\hat{1}, 4, 2, \hat{5}, ...)
+ ... + (\sum_{t=1}^{j} s_{2t})A(\hat{1}, 4, 3, \hat{5}, ..., j, 2, j + 1, ...) \quad \text{(2.7)}$$

and the corresponding $B_{324}, B_{342}, B_{423}, B_{432}$. The first observation is that from the sum $B_{234} + B_{243}$,

$$\left( \sum_{t=1}^{j} s_{2t} \right)A(\hat{1}, 3, 4, \hat{5}, ..., j, 2, j + 1, ...) + \left( \sum_{t=1}^{j} s_{2t} \right)A(\hat{1}, 4, 3, \hat{5}, ..., j, 2, j + 1, ...) \sim (\sum_{t=1}^{j} s_{2t})T_{j=4} \quad \text{(2.8)}$$

A second observation is that from the sum $(B_{234} + \text{perm})$ we get

$$(s_{21} + s_{31} + s_{41} + s_{23} + s_{24} + s_{34}) \sum_{\sigma \in S_3(2, 3, 4)} A(\hat{1}, \sigma(2), \sigma(3), \sigma(4), \hat{5}, ...) = s_{1234}T_{j=5} .$$

Now using $(B_{234} + \text{perm}) = 0$ we can solve

$$T_{j=5} = \frac{\sum_{a} c_a T_{j=4}}{-s_{1234}} \quad \text{(2.8)}$$
where $T_{j=4}^a$ are various combinations of type $T_{j=4}$ and $c_a$ are $z$-independent kinematic coefficients. Using the boundary behavior of type $T_{j=4}$ \textbf{(2.6)} we have immediately
\[
\lim_{z \to \infty} T_5(z) = \frac{1}{z} \lim_{z \to \infty} T_4(z) \to \xi_{1\mu}(z)\xi_{j\nu}(z)\frac{G_{\mu\nu}(z)}{z^3}. \tag{2.9}
\]

Having seen above two examples, we give the general proof. Assuming \textbf{(2.2)} is true for $j = k - 1$, we will show it is true for $j = k$. To do so, we consider the sum $(B_{23 \ldots (k-1)} + \text{perm}(2, \ldots, k-1))$ where $B_{23 \ldots (k-1)}$ is following fundamental BCJ relation
\[
0 = B_{23 \ldots (k-1)} = \sum_{i=2}^{k-1} (s_{2i} - \sum_{t=2}^{i} s_{2t})A_n(\hat{1}, 3, \ldots, i, 2, i + 1, \ldots, \hat{k}, \ldots, n)
\]
\[
+ \sum_{i=k}^{n-1} (s_{2i} + \sum_{t=2}^{i} s_{2t})A_n(\hat{1}, 3, 4, \ldots, \hat{k}, \ldots, i, 2, i + 1, \ldots, n).
\]

From the sum we can solve
\[
T_{j=k} = \frac{\sum a_b c_{a,b} T_{j=k-1}^{a,b}}{-s_{123\ldots(k-1)}}, \tag{2.10}
\]
where $c_{a,b} = \sum_{l=1}^{b} s_{al}$ and
\[
T_{k-1}^a(z) \equiv \sum_{\sigma \in S_{n-3}(2, 3, \ldots, a-1, a+1, \ldots, k-1)} A_n(\hat{1}(z), \{\sigma\}, \hat{k}(z), \ldots, b, a, b + 1, n - 1, n),
\]
thus we have
\[
\lim_{z \to \infty} T_k(z) \rightarrow \frac{1}{z} \lim_{z \to \infty} T_{k-1}(z) \to \xi_{1\mu}(z)\xi_{j\nu}(z)\frac{G_{\mu\nu}(z)}{z^{k-2}}. \tag{2.11}
\]

It is obvious that above method will not work for the adjacent $T_{j=n}$ because the factor $s_{12\ldots(n-1)} = \hat{s}_{n} = 0.$

3. The adjacent case

Having proved the conjecture \textbf{(1.2)} for non-adjacent case, we move on to the special adjacent case. It can happen when and only when $i, j$ are at the two ends and permutation sum is over all remaining $(n - 2)$ elements. Without losing generality, we fix $(i, j)$ to be $(1, n)$, thus we have
\[
T_n = \sum_{\sigma \in S_{n-2}} A_n(\hat{1}(z), \sigma(2), \ldots, \sigma(n-1), \hat{n}(z)), \tag{3.1}
\]
where again to emphasize the $z$-dependence we put hats on particles $(1, n)$. Before giving general argument, let us see the example $n = 5$. There are six terms in the sum and we can group them as
\[
T_5 = \left[ A(\hat{1}, 2, 3, 4, \hat{5}) + A(\hat{1}, 3, 2, 4, \hat{5}) + A(\hat{1}, 3, 4, 2, \hat{5}) \right]
\]
\[
+ \left[ A(\hat{1}, 2, 4, 3, \hat{5}) + A(\hat{1}, 4, 2, 3, \hat{5}) + A(\hat{1}, 4, 3, 2, \hat{5}) \right].
\]
For each group, using the KK-relation (1.5), we can write them as

\[ T_5 = -A(\hat{1}, 3, 4, \hat{5}, 2) - A(\hat{1}, 4, 3, \hat{5}, 2) \]  

(3.2)

where the right hand side is nothing but the non-nearby type \( T_{j=4} \). A byproduct is that all \( T_{j=4} \) are identical, no matter which element of \((2, 3, 4)\) was put at the rightmost position.

Having the example of \( n = 5 \), we explain how to regroup terms in the sum (3.1). These \((n-2)!\) terms can be divided into \((n-2)\) groups \( G_k \), where for each group particle 2 is at the fixed position \( k \) with \( k = 2, \ldots, n-1 \). Now considering a given ordering \( \sigma(3), \ldots, \sigma(n-1) \) determined by permutations over remaining \((n-3)\) elements, in each group there is an unique amplitude with the given ordering \( \sigma(3), \ldots, \sigma(n-1) \) and we sum them up to get

\[ I_\sigma = A_n(\hat{1}(z), 2, \sigma(3), \ldots, \sigma(n-1), \hat{n}(z)) + A_n(\hat{1}(z), \sigma(3), 2, \sigma(4), \ldots, \sigma(n-1), \hat{n}(z)) + \ldots \]

(3.3)

There are \((n-2)!\) terms in (3.3) and the sum is nothing but

\[ I_\sigma = -A_n(\hat{1}(z), \sigma(3), \ldots, \sigma(n-1), \hat{n}(z), 2) \]  

(3.4)

by the familiar KK-relation (1.5). Thus we have

\[ T_n = \sum_{\sigma \in S_{n-3}} I_\sigma = - \sum_{\sigma \in S_{n-3}} A_n(\hat{1}(z), \sigma(3), \ldots, \sigma(n-1), \hat{n}(z), 2) = - T_{n-1} , \]

(3.5)

and from (2.2) in previous section we have

\[ \lim_{z \to \infty} T_n(z) = - \lim_{z \to \infty} T_{n-1}(z) \to \xi_{1\mu}(z)\xi_{j\nu}(z) \frac{G_{\mu\nu}(z)}{z^{n-3}} \]

(3.6)

which is the conjecture given in (1.2). Again, from our derivation all \( T_{j=n-1} \) are identical as long as the deformation pair \((1, n)\) is same.

4. Conclusion

Recently there was a new conjecture proposed by Boels and Isermann in [20], which stated that under the BCFW-deformation the large-\( z \) behavior of partial permutation sum of color-ordered gluon amplitudes has better convergent behavior. From our previous experiences better convergent behavior could lead to new “bonus” relations among color-ordered gluon amplitudes. However, either from string theory or from BCJ relation, the minimum basis of \( n \)-point amplitudes was known to consist of \((n-3)!\) independent amplitudes, thus there is a potential contradiction between new result and our old intuition picture.

In this short note, using the KK-relation and fundamental BCJ relation, we presented a simple proof of the conjecture made in [20]. Our proof demonstrated that the better convergent behavior is a natural consequence of known results, so the “naive” contradiction does not exist.
Besides the conjecture discussed in this note, there are other combinations having better convergent behaviors as given in [20]. We feel that it would be interesting to investigate them from the new aspect. Finally working out full consequences of these results in tree and loop amplitudes would be also very important. For this direction, some progress has been made in [20].

Acknowledgements

Y. J. Du is supported in part by the NSF of China Grant No.11105118, No.11075138. B. Feng is supported by fund from Qiu-Shi, the Fundamental Research Funds for the Central Universities with contract number 2010QNA3015, as well as Chinese NSF funding under contract No.10875104, No.11031005, No.11135006, No. 11125523. CF would like to thank Yu-tin Huang for valuable discussions and the 2011 Simons workshop in Mathematics and Physics for hospitality.

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