Further Results on Homogeneous Two-Weight Codes

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Abstract. The results of [1, 2] on linear homogeneous two-weight codes over finite Frobenius rings are extended in two ways: It is shown that certain non-projective two-weight codes give rise to strongly regular graphs in the way described in [1, 2]. Secondly, these codes are used to define a dual two-weight code and strongly regular graph similar to the classical case of projective linear two-weight codes over finite fields [3].

1. Introduction

A finite ring $R$ is said to be a Frobenius ring if there exists a character $\chi \in \hat{R} = \text{Hom}_Z(R, \mathbb{C}^\times)$ whose kernel contains no nonzero left (or right) ideal of $R$. The (normalized) homogeneous weight $w_{\text{hom}}: R \to \mathbb{C}$ on a finite Frobenius ring $R$ is defined by

$$w_{\text{hom}}(x) = 1 - \frac{1}{|R^\times|} \sum_{u \in R^\times} \chi(ux).$$  \hspace{1cm} (1)

(This does not depend on the choice of $\chi$.) The function $w_{\text{hom}}$ is the unique complex-valued function on $R$ satisfying $w_{\text{hom}}(0) = 0$, $w_{\text{hom}}(ux) = w_{\text{hom}}(x)$ for $x \in R$, $u \in R^\times$ and $\sum_{x \in I} w_{\text{hom}}(x) = |I|$ for all nonzero left ideals $I \leq R$ (and their right counterparts).

The homogeneous weight on a finite Frobenius ring is a generalization of both the Hamming weight on $\mathbb{F}_q$ ($w_{\text{hom}}(x) = \frac{q-1}{q} w_{\text{Ham}}(x)$ for $x \in \mathbb{F}_q$) and the Lee weight on $\mathbb{Z}_4$ ($w_{\text{hom}}(x) = w_{\text{Lee}}(x)$ for $x \in \mathbb{Z}_4$). It was introduced in [4] for the case $R = \mathbb{Z}_m$ and generalized to Frobenius rings in [6, 8].

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In [1, 2] it was shown that a linear code $C$ over a finite Frobenius ring with exactly two nonzero homogeneous weights and satisfying certain nondegeneracy conditions gives rise to a strongly regular graph with $C$ as its set of vertices. In the classical case $R = \mathbb{F}_q$ this result has been known for a long time and forms part of a more general correspondence between projective linear $[n, k]$ two-weight codes over $\mathbb{F}_q$ and certain strongly regular Cayley graphs of $(\mathbb{F}_q^k, +)$ resp. regular partial difference sets in $(\mathbb{F}_q^k, +)$, and their (appropriately defined) duals (cf. [3, 5]).

The purpose of this work is to generalize the results of [1, 2] to a larger class of homogeneous two-weight codes (so-called modular two-weight codes) and establish for these codes the classical correspondence (Theorems 3.2 and 5.7 of [3]) in full generality.

2. A Few Properties of Frobenius Rings and their Homogeneous Weights

For a subset $S$ of a ring $R$ let $\perp S = \{ x \in R; xS = 0 \}$, $S^\perp = \{ x \in R; Sx = 0 \}$.

Similarly, for $S \subseteq R^n$ let $\perp S = \{ x \in R^n; x \cdot S = 0 \}$ and $S^\perp = \{ x \in R^n; S \cdot x = 0 \}$, where $x \cdot y = x_1 y_1 + \cdots + x_n y_n$.

Proposition 1. A finite ring $R$ is a Frobenius ring iff for every matrix $A \in R^{m \times n}$ the left row space $C = \{ xA; x \in R^m \}$ and the right column space $D = \{ Ay; y \in R^n \}$ have the same cardinality.

From now on we suppose that $R$ is a finite Frobenius ring with homogeneous weight $\text{w}_{\text{hom}}$.

First we determine the set of all $x \in R$ satisfying $\text{w}_{\text{hom}}(x) = 0$. Let $S_i = Rs_i$, $1 \leq i \leq \tau$, be the different left ideals of $R$ of order 2 and $S = S_1 + \cdots + S_\tau$. The set $S$ is a two-sided ideal of $R$ of order $2^\tau$, whose elements are the subset sums of $\{ s_1, \ldots, s_\tau \}$. Define $S_0 \subseteq S$ as the set of all sums of an even number of elements from $\{ s_1, \ldots, s_\tau \}$ (“even-weight subcode of $S$”). Note that $S_0$ is a subgroup of $(R, +)$, trivial for $\tau \leq 1$ and nontrivial (of order $2^{\tau-1}$) for $\tau \geq 2$.

Proposition 2. We have $\text{w}_{\text{hom}}(x) \geq 0$ for all $x \in R$, and $\{ x \in R; \text{w}_{\text{hom}}(x) = 0 \} = S_0$. Moreover, $\text{w}_{\text{hom}}(x + y) = \text{w}_{\text{hom}}(x)$ for all $x \in R$ and $y \in S_0$.

Fact 3 ([7 Th. 2]).

$$\sum_{x \in I} \text{w}_{\text{hom}}(x + c) = |I|$$

for all nonzero left (or right) ideals $I$ of $R$ and all $c \in R$.

The following correlation property of $\text{w}_{\text{hom}}$ turns out to be crucial.
Proposition 4. For a nonzero left ideal \( I \) of \( R \) and \( r, s \in R \) we have
\[
\sum_{x \in I} w_{\text{hom}}(x) w_{\text{hom}}(xr + s) = \begin{cases} 
|I| + |I| \cdot \frac{|R^x \cap (1+I^t)|}{|R^x|} \cdot (1 - w_{\text{hom}}(s)) & \text{if } |Ir| = |I|, \\
|I| & \text{if } |Ir| < |I|.
\end{cases}
\]
In particular \( \sum_{x \in R} w_{\text{hom}}(x)^2 = |R| + \frac{|R|}{|R^x|}. \)

For vectors \( x, y \in R^k \) we write \( x \sim y \) if \( xR^x = yR^x \). By [10] Prop. 5.1 this is equivalent to \( xR = yR \).

Proposition 5. For nonzero words \( g, h \in R^k \) and \( s \in R \) we have
\[
\sum_{x \in R^k} w_{\text{hom}}(x \cdot g) w_{\text{hom}}(x \cdot h + s) = \begin{cases} 
|R|^k + \frac{|R|^k}{|gR^x|} \cdot (1 - w_{\text{hom}}(s)) & \text{if } g \sim h, \\
|R|^k & \text{if } g \nsim h.
\end{cases}
\]

3. Modular Two-Weight Codes, Partial Difference Sets and Strongly Regular Cayley Graphs

Given a positive integer \( k \), the set of nonzero cyclic submodules of the free right module \( R^k_R \) is denoted by \( \mathcal{P} \). The elements of \( \mathcal{P} \) are referred to as points of the projective geometry \( \text{PG}(R^k_R) \), and a multiset \( \alpha : \mathcal{P} \to \mathbb{N}_0 \) is referred to as a multiset in \( \text{PG}(R^k_R) \).

With a left linear code \( C \subseteq R^R^n \) generated by \( k \) (or fewer) codewords and having no all-zero coordinate we associate a multiset \( \alpha_C \) in \( \text{PG}(R^k_R) \) of cardinality \( n \) in the following way: If \( C = \{xG : x \in R^k\} \) with \( G = (g_1, g_2, \ldots, g_n) \in R^{k \times n} \), define \( \alpha_C : \mathcal{P} \to \mathbb{N}_0 \) by \( \alpha(gR) = |\{j : 1 \leq j \leq n \land g_j R = gR\}|. \) The relation \( C \leftrightarrow \alpha_C \) defines a bijection between classes of monomially isomorphic left linear codes over \( R \) generated by \( k \) codewords and orbits of the group \( \text{GL}(R^k_R) \) on multisets in \( \text{PG}(R^k_R) \).

Definition 6. A linear code \( C \subseteq R^R^n \) is said to be modular if there exists \( r \in \mathbb{Q} \) such that for all points \( gR \) of \( \text{PG}(R^k_R) \) either \( \alpha_C(gR) = 0 \) or \( \alpha_C(gR) = r|gR^x| \). The number \( r \) is called the index of \( C \).

The property of \( C \) described in Def. 5 does not depend on the choice of \( \alpha_C \) (not even on the dimension \( k \)). Hence modularity of a linear code is a well-defined concept.

If \( A \subseteq R^k \setminus \{0\} \) satisfies \( AR^x = A \), the matrix \( G \) with the vectors of \( A \) as columns generates a modular (left) linear code of length \( |A| \) and index 1.

Note that projective codes over \( \mathbb{F}_q \) are modular of index \( \frac{1}{|R^x|} \) and regular projective codes over \( R \) as defined in [11] [2] are modular of index \( \frac{1}{|R^x|}. \)
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**Lemma 9**

In the special case of index $r$, where $\Gamma$ is strongly regular with parameters $(N, K, \lambda, \mu)$, the frequencies $w_1, w_2$ and $b_0 = |C_0|$ are known, the frequencies $b_1, b_2$ can be computed from the equations $b_1 + b_2 = |C| - |C_0|$, $b_1 w_1 + b_2 w_2 = \sum_{c \in C} w_{\text{hom}}(c) = n|C|$ (assuming that $C$ has no all-zero coordinate) and are given by

$$b_1 = \frac{(w_2 - n)|C| - w_2|C_0|}{w_2 - w_1}, \quad b_2 = \frac{(n - w_1)|C| + w_1|C_0|}{w_2 - w_1}. \quad (5)$$

**Lemma 8.** For a modular code $C \leq R^n$ of index $r$ and $d \in R^n$ we have

$$\sum_{c \in C} w_{\text{hom}}(c) w_{\text{hom}}(c + d) = |C| \cdot (n^2 + rn - r \cdot w_{\text{hom}}(d)). \quad (6)$$

In the special case $d = 0$ Lemma 8 reduces to $\sum_{c \in C} w_{\text{hom}}(c)^2 = (n^2 + rn)|C|$.

**Lemma 9.** The nonzero weights $w_1, w_2$ of a modular two-weight code $C \leq R^n$ of index $r$ satisfy the relation

$$(w_1 + w_2)n|C| = (n^2 + rn)|C| + w_1 w_2(|C| - |C_0|). \quad (7)$$

**Lemma 10.** For a modular two-weight code $C \leq R^n$ of index $r$ and $d \in R^n$ we have

$$\sum_{c \in C_1} w_{\text{hom}}(c + d) = b_1 w_1 + \left(b_1 - \frac{b_1 w_1}{n}\right) w_{\text{hom}}(d) \quad (8)$$

**Remark 11.** Lemmas 8 and 10 can be generalized to

$$\sum_{c \in C} w_{\text{hom}}(c) w_{\text{hom}}(c_j + d_j) = |C| \cdot (n + r - r \cdot w_{\text{hom}}(d_j))$$

and

$$\sum_{c \in C_1} w_{\text{hom}}(c + d_j) = \frac{b_1 w_1}{n} + \left(b_1 - \frac{b_1 w_1}{n}\right) w_{\text{hom}}(d_j)$$

respectively, where $j$ is any coordinate of $R^n$ and $d_j \in R$. In particular $\sum_{c \in C_1} w_{\text{hom}}(c_j) = \frac{b_1 w_1}{n}$ is independent of $j$.

Recall that a (simple) graph $\Gamma$ is *strongly regular with parameters $(N, K, \lambda, \mu)$* if $\Gamma$ has $N$ vertices, is regular of degree $K$ and any two adjacent (resp. non-adjacent) vertices have $\lambda$ (resp. $\mu$) common neighbours. The graph $\Gamma$ is called
trivial if \( \Gamma \) or its complement is a disjoint union of cliques of the same size. This is equivalent to \( \mu = 0 \) resp. \( \mu = K \).

A subset \( D \subset G \) of an (additively written) abelian group \( G \) is said to be a regular \((N,K,\lambda,\mu)\) partial difference set in \( G \) if \( N = |G|, \ K = |D|, \ 0 \not\in D, \ -D = D, \) and the multiset \( D - D \) represents each element of \( D \) exactly \( \lambda \) times and each element of \( G \setminus (D \cup \{0\}) \) exactly \( \mu \) times; cf. [9].

If \( D \) is a regular \((N,K,\lambda,\mu)\) partial difference set in \( G \), then the graph \( \Gamma(G, D) \) with vertex set \( G \) and edge set \( \{\{x, x + d\} : x \in G, d \in D\} \), the so-called Cayley graph of \( G \) w.r.t. \( D \), is strongly regular with parameters \((N,K,\lambda,\mu)\).

We are now ready to generalize the main result of [2,1] to modular two-weight codes. For a two-weight code \( C \) we denote the Cayley graph \( \Gamma(C/C_0, C_1/C_0) \) by \( \Gamma(C) \). Thus the vertices of \( \Gamma(C) \) are the cosets of \( C_0 \) in \( C \), and two cosets \( C_0 + C_0, d + C_0 \) are adjacent iff \( w_{\text{hom}}(c - d) = w_1 \). As we have already mentioned, Prop. 2 ensures that \( \Gamma(C) \) is well-defined.

**Theorem 12.** The graph \( \Gamma(C) \) associated with a modular two-weight code over a finite Frobenius ring \( R \) is strongly regular with parameters

\[
N = \frac{|C|}{|C_0|}, \quad K = \frac{(w_2 - n)N - w_2}{w_2 - w_1},
\]

\[
\lambda = \frac{K \left( \frac{w_1^2}{n} - 2w_1 \right) + w_2(K - 1)}{w_2 - w_1}, \quad \mu = \frac{K \left( \frac{w_1w_2}{w_1 - w_2} - w_1 - w_2 \right) + w_2K}{w_2 - w_1}.
\]

The graph \( \Gamma(C) \) is trivial iff \( w_1 = n \).

**Remark 13.** Since \( \Gamma(C) \) is a Cayley graph, the preceding argument shows that \( \Gamma(C) \) is trivial iff the codewords of weight 0 and \( w_2 \) form a linear subcode of \( C \) (and the cocliques of \( \Gamma(C) \) are the cosets of \((C_0 + C_2)/C_0 \) in this case).

### 4. The Dual of a Modular Two-Weight Code

Suppose \( C \leq_R R^n \) is a two-weight code over a finite Frobenius ring with nonzero weights \( w_1 < w_2 \) and frequencies \( b_1, b_2 \). Let \( M_i \in R^{b_i \times n} \) \((i = 1,2)\) be matrices whose rows are the codewords of \( C \) of weight \( w_i \) in some order.

**Definition 14.** The right linear code \( C' \leq_R R^{b_1} \) generated by the columns of \( M_1 \) is called the dual of the two-weight code \( C \).

The code \( C' \) is modular of index 1 (no matter whether \( C \) is modular or not).

**Theorem 15.** If \( C \leq_R R^n \) is a modular two-weight code with \( C_0 = \{0\} \), its dual \( C' \) is also a (modular) two-weight code with \( C'_0 = \{0\} \) and nonzero weights

\[
w'_1 = \frac{(w_2 - n - r)|C|}{w_2 - w_1} = \frac{b_1 w_1}{n}, \quad w'_2 = \frac{(w_2 - n)|C|}{w_2 - w_1}.
\]
Theorem 16. Under the assumptions of Th. [15] the graph $\Gamma(C')$ is strongly regular with parameters

$$N' = |C|, \quad K' = \frac{n}{r}, \quad \lambda' = \frac{2n - w_1 - w_2}{r} + \frac{w_1 w_2}{r^2 |C|}, \quad \mu' = \frac{w_1 w_2}{r^2 |C|}.$$ 

The graph $\Gamma(C')$ is trivial iff $w_1 = n$ (i.e. iff $\Gamma(C)$ is trivial).

Theorem 17. Let $C \leq R^n$ be a modular linear code over a finite Frobenius ring $R$ generated by $G = (g_1 | \ldots | g_n) \in R^{k \times n}$. Let $D \leq R^k_R$ be the right column space of $G$. Suppose $C$ has no all-zero coordinate and satisfies $C_0 = \{0\}$. Then the following are equivalent:

(i) $C$ is a homogeneous two-weight code;
(ii) $\Omega = g_1 R^k \cup \cdot \cdot \cdot \cup g_n R^k$ is a regular partial difference set in $(D, +)$ and $\Omega \cup \{0\}$ is not a submodule of $R^k_R$.

Remark 18. Under the assumptions of Th. [17] the set $\Omega \cup \{0\}$ is a submodule of $R^k_R$ iff $C$ is a homogeneous one-weight code, and $D \setminus \Omega$ is a submodule of $R^k_R$ iff $C$ is a homogeneous two-weight code with $w_1 = n$.

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