SOME TRIGONOMETRIC POLYNOMIALS WITH EXTREMELY SMALL UNIFORM NORM

PAVEL G. GRIGORIEV AND ARTYOM O. RADOMSKII

Abstract. An example of trigonometric polynomials with extremely small uniform norm is given. This example demonstrates the potential limits for extension of Sidon’s inequality for lacunary polynomials in a certain direction.

Key words: Sidon’s inequality, lacunary polynomials.

We prove the following

Theorem 1. Let $q > 1$ and a sequence of naturals $\{m_j\}_{j=1}^\infty$ satisfy $m_{j+1}/m_j \geq q$ for all $j$. Let another sequence of naturals $\{d_j\}_{j=1}^\infty$ satisfy $1 \leq d_j \leq m_{j+1} - m_j$. Then there exists a sequence of trigonometric polynomials $\{\delta_j\}_{j=1}^\infty$ such that

\[ \delta_j(x) = \sum_{m_j \leq s < m_j + d_j} c_s e^{isx}, \]

\[ \frac{1}{8} \leq \|\delta_j\|_1 \leq \|\delta_j\|_\infty \leq 7, \]

\[ \left\| \sum_{j=1}^N \delta_j \right\|_\infty \leq \alpha + \beta \sqrt{N} + \gamma \max_{1 \leq j \leq N} \log_q \max \left( \frac{m_j}{d_j}, \frac{1}{\ln q}, 1 \right) \]

for all $N = 1, 2, \ldots$ with some positive absolute constants $\alpha$, $\beta$ and $\gamma$.

This result improves the examples constructed by Grigoriev \cite{Gri} and Radomskii \cite{Rad}, where roughly speaking Theorem \cite{Gri} was proved for the case $m_j = 2^j$, $d_j = [2^{j'-j}]$ and some small limitations.

Remark 1. In Theorem \cite{Gri} the example is constructed with the constants $\alpha = 316$, $\beta = 7 \sqrt{2c_H}$, $\gamma = 210$, where $c_H$ is the constant from the Carleson-Hunt inequality, see \cite{Car} below. Instead of the Carleson-Hunt result one could use a well-known simpler inequality (see \cite{Car}, Ch. 13, Th. 1.17)

\[ \left\| \sup_{1 \leq j < \infty} \left| \sum_{k=1}^{m_j} b_k e^{ikx} \right| \right\|_2 \leq A_q \sum_{k=1}^\infty |b_k|^2 \]

whenever $m_{j+1}/m_j \geq q$ with a constant $A_q$ depending on $q$. The Carleson-Hunt inequality is used here because we would like to make the constants $\alpha$, $\beta$...
and \( \gamma \) independent of \( q \). Otherwise we do not try to optimize the constants in Theorem 1, e.g. for the case \( q = 2 \) it is not difficult to repeat the arguments of our proof and get some essentially better constants.

**Remark 2.** In this paper we assume that the norms of \( L^p(0, 2\pi) \) are normalized, i.e. \( \|f\|_p = \left( \frac{1}{2\pi} \int_0^{2\pi} |f|^p \, d\mu \right)^{1/p} \) for \( 1 \leq p < \infty \), where \( \mu \) is the standard Lebesgue measure so that \( \|1\|_p = 1 \).

**Remark 3.** One can deduce as a corollary a version of Theorem 1 with real polynomials \( \delta_j \) with slight changes in the constants. (We decided to prove the result for \( \delta_j \) with positive frequencies and real coefficients.) Also with minor changes in the proof one can prove a version of Theorem 1 with \( \delta_j(x) = p_j(x) \cos m_j x \) with some real trigonometric polynomials \( p_j \) such that \( \deg p_j \leq d_j \).

Taking Remark 2 into account we derive from Theorem 1 the following

**Corollary 1.** Let \( 0 \leq \varepsilon < 1 \). Then there exists a sequence of real trigonometric polynomials \( \{p_j\}_{j=1}^\infty \) such that \( \deg p_j \leq 2^{j^\varepsilon} \), \( c_- \leq \|p_j\|_1 \leq \|p_+\|_\infty \leq c_+ \) and

\[
\left\| \sum_{j=1}^N p_j(x) \cos 2^j x \right\|_\infty \leq C N^{\max(\varepsilon, \frac{1}{4})}
\]

for all \( N = 1, 2, \ldots \) with some positive absolute constants \( c_-, c_+, C \).

The example constructed in Theorem 1 gives some limits for the attempts to extend the well-known property of lacunary polynomials (Sidon’s inequality)

\[
\left\| \sum_{j=1}^N a_j \cos m_j x \right\|_\infty \geq c(q) \sum_{j=1}^N |a_j|
\]

with some \( c(q) > 0 \) whenever \( m_{j+1}/m_j \geq q > 1 \). The question of possible extension of Sidon’s inequality with substituting \( a_j \cos m_j x \) by \( p_j(x) \cos m_j x \) with \( p_j \) being trigonometric polynomials of possibly large degree was raised by Kashin and Temlyakov and Radomskii in association with estimating the entropy numbers of certain functional classes.

The scheme of proof follows that from Grigoriev and Radomskii with minor changes. This technique invented in could be called the pseudo stopping time method since the idea was borrowed from stochastic analysis (see for explanations).

**Proof of Theorem 1.** It is easy to notice that without loss of generality we can consider only the case when

\[
\frac{m_j}{d_j} \geq \max \left( 1, \frac{1}{\ln q} \right).
\]
By the famous result of Carleson and Hunt [3] the $L_2$-norm of the majorant of a trigonometric sum can be estimated as

$$\left\| \max_{0 \leq k \leq n} \left| \sum_{s=0}^{k} b_s e^{isx} \right| \right\|_2^2 \leq c_H \sum_{s=1}^{n} |b_s|^2$$

with an absolute constant $c_H > 0$.

We construct the polynomials $\delta_{k,j}$ by induction with the constants

$$\alpha := 316, \quad \beta := 7\sqrt{2c_H}, \quad \gamma := 210.$$ (6)

Let $\delta_1(x) := \exp(im_1x)$. On each inductive step in order to construct $\delta_n$ having $\delta_1, \ldots, \delta_{n-1}$ from the previous steps we denote

$$S_j := \sum_{k=1}^{j} \delta_j$$

and define

$$E^j_n := \{ x \in [0, 2\pi) : |S_j(x)| > \beta \sqrt{n} \};$$

$$B_n := \bigcup_{j=1}^{n-1} E^j_n;$$

$$\tilde{B}_n := \bigcup_{1 \leq j \leq n-a_n} O_{2\pi (\frac{n-k}{a_n})^2} (E^j_n),$$

where $a_n := 45 + 30 \log_{10} \frac{m_n}{d_n}$ (10)

and $O_\varepsilon(X)$ denotes the $\varepsilon$-neighborhood of the set $X$ on the semi-interval $[0, 2\pi)$ with the circle metric. Note that the value $n-a_n$ need not to be neither integer nor positive. If $n-a_n < 1$, the union in (9) is understood as an empty set. Similarly below the sums like $\sum_{1 \leq j \leq n-a_n}$ are supposed equal to zero whenever $n-a_n < 1$.

Set

$$\Lambda_n := \left\{ l \in \{1, \ldots, d_n\} : 2\pi \frac{l}{d_n} \notin \tilde{B}_n \right\};$$

$$\delta_n(x) := \frac{1}{d_n} \exp \left\{ i(m_n + \left[ \frac{d_n-1}{2} \right])x \right\} \sum_{l \in \Lambda_n} K_{\frac{d_n-1}{d_n}} \left( x - 2\pi \frac{l}{d_n} \right),$$

where $K_d(x) = \frac{\sin \frac{d+1}{2} x}{2d \sin \frac{x}{2}}^2$ are the Fejér kernels. Clearly so defined $\delta_n$ satisfies the frequency constraint imposed by [4].

In order to verify the induction hypothesis we need to show that

$$|\Lambda_n| > \frac{1}{4} d_n.$$ (13)
By the induction hypothesis we have (2) for $\delta_1, \ldots, \delta_{n-1}$. Applying the Chebyshev inequality for the majorant $S_{n-1}^* := \max_{1 \leq j \leq n-1} \left| \sum_{s=1}^{j} \delta_s(x) \right|$ (see (7), (8)) and then using (3), (2) and (6) we get

$$\mu B_n \leq \frac{2\pi\|S_{n-1}^*\|_2^2}{\beta^2 n} \leq \frac{2\pi c_H\|S_{n-1}\|_2^2}{\beta^2 n} \leq \frac{2\pi}{\beta^2 n} \sum_{j=1}^{n-1} \|\delta_j\|_\infty^2 \leq \frac{2\pi}{\beta^2 n} (n-1)\tau^2 < \pi. \quad (14)$$

Let us denote by $\text{Conn}(X)$ the number of connected components of a set $X \subset [0, 2\pi)$ in the circle topology. Note that $|S_j|^2$ is a real trigonometric polynomials of degree not exceeding $2(m_j + 2(d_j-1))$ and therefore the equation $|S_j(x)|^2 = \beta^2 n$ has not more than $4(m_j + 2(d_j-1))$ roots. Note that $E_n^j$ is a finite union of open intervals which endpoints are the roots of $|S_j(x)|^2 = \beta^2 n$. Taking into account (14) we conclude

$$\text{Conn}(E_n^j) \leq \frac{1}{2} |\{ x : |S_j(x)|^2 = \beta^2 n \}| \leq 2 \left( m_j + 2 \left( \frac{d_j-1}{2} \right) \right) < 4m_j. \quad (15)$$

This implies

$$\text{Conn}(\tilde{B}_n) \leq \sum_{1 \leq j \leq n-a_n} \text{Conn}(E_n^j) < 4 \sum_{1 \leq j \leq n-a_n} m_j \leq 4 \sum_{1 \leq j \leq n-a_n} m_nq^{j-n} < 4m_n \sum_{s \geq a_n} q^{-s} = 4m_nq^{-[a_n]} \frac{q}{q-1},$$

where $[x]$ denotes the ceiling integer part of $x$. Taking into account that $a_n \leq [a_n]$, $\ln q \leq q - 1$ and using (10) and (11) we proceed as

$$\text{Conn}(\tilde{B}_n) < 4m_nq^{-a_n} \frac{q}{q-1} = 4m_nq^{-45} \left( \frac{m_n}{d_n} \right)^{-30} \frac{q}{q-1} = 4d_n \left( \frac{m_n}{d_n} \right)^{-29} \frac{q^{44}}{q-1} \leq 4d_n \left( \frac{m_n}{d_n} \right)^{-29} \frac{q^{44}}{\ln q} \leq 4d_n \left( \frac{m_n}{d_n} \right)^{-28} q^{-44} \leq 4d_n \max \left( \left( \frac{m_n}{d_n} \right)^{-28}, q^{-44} \right).$$

If $q \geq e^{1/2}$, then $q^{-44} \leq e^{-22}$. If $q \leq e^{1/2}$, then $\left( \frac{m_n}{d_n} \right)^{-28} \leq \left( \frac{1}{\ln q} \right)^{-28} \leq 2^{-28}$. So we get

$$\text{Conn}(\tilde{B}_n) < 4d_n \max \left( 2^{-28}, e^{-22} \right) < \frac{d_n}{8}. \quad (16)$$

Aggregating (14) and (15) (see also (9)) we get

$$\mu \tilde{B}_n \leq \mu B_n + 2 \sum_{1 \leq j \leq n-a_n} 2\pi \left( \frac{n-j}{d_n} \right)^2 \text{Conn}(E_n^j) \leq \pi + 16\pi \sum_{1 \leq j \leq n-a_n} \left( \frac{n-j}{d_n} \right)^2 m_j$$

$$\leq \pi + 16\pi \sum_{1 \leq j \leq n-a_n} \left( \frac{n-j}{d_n} \right)^2 m_n q^{j-n} = \pi + 16\pi \sum_{1 \leq j \leq n-a_n} \left( \frac{n-j}{d_n} \right)^2 m_n q^{j-n} = \pi + 16\pi \sum_{s \leq n-1} s^2 q^{-s}. $$
To proceed with the estimate of $\mu\tilde{B}_n$ we are going to use the inequality
$$\sum_{s=a}^{\infty} s^2 q^{-s} \leq \frac{2a^2 q^{3-a}}{(q-1)^3} \quad \text{for } a = 1, 2, \ldots.$$ 

This inequality one could easily deduce from the following identity
$$\sum_{s=a}^{\infty} s^2 q^{-s} = q^{3-a} \left( a^2 + q^{-1}(1 + 2a - 2a^2) + q^{-2}(a - 1)^2 \right).$$ 

for all $q > 1$ and $a = 0, 1, \ldots$, which is not so difficult to verify.

Now proceed with the estimate of $\mu\tilde{B}_n$ as
$$\mu\tilde{B}_n < \pi + 32\pi \frac{m_n [a_n]^2 q^{3-[a_n]}}{(q-1)^3} < \pi + 32\pi \frac{m_n [a_n]^2 q^{3-[a_n]}}{\ln^3 q}.$$ 

The function $x^2 q^{-x}$ is decreasing for $x \geq 2/\ln q$. One easily checks that (4) and (10) imply $a_n \geq 2/\ln q$ and therefore we can substitute $\lceil a_n \rceil$ for $a_n$ in the right-hand side above. So recalling (10) and (4) again we continue as
$$\mu\tilde{B}_n < \pi + 32\pi \frac{m_n a_n^2 q^{3-a_n}}{\ln^3 q}$$
$$= \pi + 32\pi \left( \frac{m_n}{d_n} \right)^{29} \left( 45 + 30 \log_q \frac{m_n}{d_n} \right)^2 \frac{q^{-42}}{\ln^3 q}$$
$$\leq \pi + 32\pi \left( \frac{m_n}{d_n} \right)^{-26} \left( 45 + 30 \log_q \frac{m_n}{d_n} \right)^2 q^{-42}$$
$$\leq \pi + 64\pi \left( \frac{m_n}{d_n} \right)^{-26} \left\{ 45^2 + 30^2 \left( \frac{\ln m_n}{\ln q} \right)^2 \right\} q^{-42}$$
$$\leq \pi + 64\pi \left\{ 45^2 \left( \frac{m_n}{d_n} \right)^{-26} + 30^2 \left( \frac{m_n}{d_n} \right)^{-22} \right\} q^{-42}.$$ 

If $q \geq e^{1/2}$, then
$$\left\{ 45^2 \left( \frac{m_n}{d_n} \right)^{-26} + 30^2 \left( \frac{m_n}{d_n} \right)^{-22} \right\} q^{-42} \leq (45^2 + 30^2) e^{-21} < (64^2 + 64^2) 2^{-21} = 2^{-8}.$$ 

If $q \leq e^{1/2}$, then $\frac{m_n}{d_n} \geq 1/\ln q \geq 2$ and
$$\left\{ 45^2 \left( \frac{m_n}{d_n} \right)^{-26} + 30^2 \left( \frac{m_n}{d_n} \right)^{-22} \right\} q^{-42} < 45^2 2^{-26} + 30^2 2^{-22} < 2^{-11}.$$ 

So we finally conclude
$$\mu\tilde{B}_n < \pi + 64\pi 2^{-8} = \frac{5}{4}\pi.$$ 

(17)

Now we are ready to prove (13). Clearly, the number of elements in $\Lambda_n^c$ does not exceed the number of the intervals of type $(2\pi l - \pi, 2\pi l + \pi) / d_n (l = 1, \ldots, d_n)$ which intersect $\tilde{B}_n$ (see (11)). Such intervals can be split into two groups: those
containing an edge point of $\tilde{B}_n$ and those being included in $\tilde{B}_n$. There are not more than $2\text{Conn}(\tilde{B}_n) \leq d_n/8$ of the intervals of the first type (see (16)). Denoting by $V$ the number of the intervals of the second type and recalling (17) we get
\[ V \frac{2\pi}{d_n} \equiv V\mu\left(-\frac{\pi}{d_n}, \frac{\pi}{d_n}\right) \leq \mu\tilde{B}_n < \frac{5}{4}\pi. \]
So we have
\[ |\Lambda_n| \geq d_n - V - d_n \frac{5}{8}d_n - \frac{1}{8}d_n = \frac{1}{4}d_n. \]
Thus we proved (13).

Our next goal is to verify (2) for $\delta_n$. Let us recall some properties of the Fejér kernels. It is well-known that for all $-\pi \leq x \leq \pi$ and $d = 1, 2, \ldots$
\[ K_{d-1}(x) \geq 0, \quad \|K_{d-1}\|_1 = 1/2, \quad (18) \]
\[ K_{d-1}(x) \leq \min\left(\frac{d}{2}, \frac{\pi^2}{2d^2x^2}\right) < 5 \min\left(d, \frac{1}{d^2x^2}\right). \quad (19) \]
Denote by $\text{dist}(x, y)$ the standard distance on the circle between $x, y \in [0, 2\pi)$ and let $\tilde{d}_n := \left\lfloor \frac{d_n - 1}{2}\right\rfloor + 1$. Clearly, $1 \leq d_n/\tilde{d}_n \leq 2$. Using (12), (13), (18) and (19) we verify (2) as follows
\[ \|\delta_n\|_1 = \frac{1}{d_n} \left\| \sum_{l \in \Lambda_n} K_{\tilde{d}_n-1}\left(x - 2\pi \frac{l}{d_n}\right) \right\|_1 = \frac{|\Lambda_n|}{d_n} \|K_{\tilde{d}_n-1}\|_1 \geq \frac{1}{8} \]
and
\[ \|\delta_n\|_\infty = \frac{1}{d_n} \left\| \sum_{l \in \Lambda_n} K_{\tilde{d}_n-1}\left(x - 2\pi \frac{l}{d_n}\right) \right\|_\infty \]
\[ \leq \frac{1}{d_n} \left\| \sum_{l=1}^{d_n} \left(\tilde{d}_n, \frac{1}{d_n \text{dist}(x, 2\pi \frac{l}{d_n})^2}\right) \right\|_\infty \]
\[ \leq \left(1 + 2\sum_{s=1}^{\infty} \frac{1}{d_n^2 (2\pi \frac{s}{d_n})^2}\right) \]
\[ = 5 \left(1 + \frac{d_n^2}{2\pi^2 d_n^2} \sum_{s=1}^{\infty} \frac{1}{s^2}\right) \]
\[ = 5 \left(1 + \frac{d_n^2}{2\pi^2 d_n^2} \frac{\pi^2}{6}\right) < 7. \]

Now to complete the proof it remains to verify (3) with the constants (6), i.e. to show that
\[ |S_n(x)| \leq \alpha + \beta \sqrt{n} + \gamma \max_{1 \leq j \leq n} \log_q \max\left(\frac{m_j}{d_j}, \frac{1}{\ln q}, 1\right) \]
for each $x \in [0, 2\pi)$. 
Now we can estimate the last term in (20) as

\[ |S_n(x)| \leq |S_{\tau(x)}(x)| + \sum_{t=\tau(x)+1}^{n} |\delta_t(x)| \leq \beta \sqrt{n} + 7a_n \]

\[ = \beta \sqrt{n} + 7 \left( 45 + 30 \log_q \frac{m_n}{d_n} \right) = \alpha - 1 + \beta \sqrt{n} + \gamma \log_q \frac{m_n}{d_n}. \]

If \( \tau(x) < n - a_n \), then

\[ |S_n(x)| \leq |S_{\tau(x)}(x)| + \sum_{t=\tau(x)+1}^{\tau(x)+a_n} |\delta_t(x)| + \sum_{t=\tau(x)+a_n+1}^{n} |\delta_t(x)| \]

\[ \leq \alpha - 1 + \beta \sqrt{n} + \gamma \log_q \frac{m_n}{d_n} + \sum_{t=\tau(x)+a_n+1}^{n} |\delta_t(x)|, \quad (20) \]

It remains to estimate the last term in (20). Since \( \tau(x) + a_n + 1 \leq t \leq n \), we have \( x \in E_{n\tau(x)+1} \subset E_t^{\tau(x)+1} \) (see (7)). Therefore, by the definition of \( B_t \) (see (9)) we conclude

\[ \inf_{y \in [0,2\pi) \setminus B_t} \mathrm{dist}(x,y) \geq \frac{2\pi}{d_t} |t - \tau(x) - 1|^2. \]

Consequently, for each \( l \in \Lambda_t \) we have (see (10))

\[ \mathrm{dist} \left( x, 2\pi \frac{l}{d_t} \right) \geq \frac{2\pi}{d_t} |t - \tau(x) - 1|^2. \]

Using (19) again and applying the trivial estimates \( \sum_{s=K+1}^{s=\infty} s^{-2} < K^{-1} \) and \( \sum_{s=K}^{s=\infty} s^{-2} < 2K^{-1} \) we conclude

\[ |\delta_t(x)| \leq \frac{1}{d_t} \sum_{l \in \Lambda_t} K_{\tilde{d}_t} \left( x - 2\pi \frac{l}{d_n} \right) \leq \frac{5}{d_t} \sum_{l \in \Lambda_t} \min \left( \frac{1}{d_t}, \frac{1}{\mathrm{dist}(x,2\pi \frac{l}{d_t})^2} \right) \]

\[ \leq 10 \sum_{s=|t-\tau(x)-1|^2}^{\infty} \frac{1}{d_t d_t (2\pi \frac{s}{d_t})^2} = \frac{5d_t}{2\pi^2 d_t} \sum_{s=|t-\tau(x)-1|^2}^{\infty} \frac{1}{s^2} \]

\[ < \sum_{s=|t-\tau(x)-1|^2}^{\infty} \frac{1}{s^2} < \frac{2}{|t - \tau(x) - 1|^2}. \]

Now we can estimate the last term in (20) as

\[ \sum_{t=\tau(x)+a_n+1}^{n} |\delta_t(x)| < \sum_{t=\tau(x)+a_n+1}^{n} \frac{2}{|t - \tau(x) - 1|^2} = \sum_{s=a_n}^{\infty} \frac{2}{s^2} < \frac{2}{a_n - 1} < 1. \]

Using the last estimate in (20) we get (3) for \( S_n \). This completes the proof.
References

[1] Grigor’ev, P.G. On a sequence of trigonometric polynomials. Math. Notes 61 (1997), no. 5-6, 780–783.

[2] Grigoriev, P.G. Random and Special Polynomials with respect to a General System of Functions. Ph.D. Dissertation, Steklov Institute of Mathematics, Moscow, 2002, (in Russian).

[3] Hunt, R.A. On the convergence of Fourier series. 1968 Orthogonal Expansions and their Continuous Analogues (Proc. Conf., Edwardsville, Ill., 1967) pp. 235-255 Southern Illinois Univ. Press, Carbondale

[4] Kashin, B.S. and Temlyakov, V.N. On a norm and approximate characteristics of classes of multivariable functions. J. Math. Sciences, 155:1 (2008), 57-80.

[5] Radomskii, A.O. On an inequality of Sidon type for trigonometric polynomials. Math. Notes 89 (2011), no. 3-4, 555–561.

[6] Radomskii, A.O. On the possibility of strengthening Sidon-type inequalities. Math. Notes 94 (2013), no. 5-6, 829–833.

[7] Zygmund, A. Trigonometric Series. Cambridge University Press, 1959.

P.G. Grigoriev. Geogracom LLC
E-mail address: thepavel@mail.ru

A.O. Radomskii. Moscow Engineering Physics Institute
E-mail address: artrad@list.ru