Curvature-based gauge-invariant perturbation theory for gravity: a new paradigm

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Abstract

A new approach to gravitational gauge-invariant perturbation theory begins from the fourth-order Einstein-Ricci system, a hyperbolic formulation of gravity for arbitrary lapse and shift whose centerpiece is a wave equation for curvature. In the Minkowski and Schwarzschild backgrounds, an intertwining operator procedure is used to separate physical gauge-invariant curvature perturbations from unphysical ones. In the Schwarzschild case, physical variables are found which satisfy the Regge-Wheeler equation in both odd and even parity. In both cases, the unphysical ”gauge” degrees of freedom are identified with violations of the linearized Hamiltonian and momentum constraints, and they are found to evolve among themselves as a closed subsystem. If the constraints are violated, say by numerical finite-differencing, this system describes the hyperbolic evolution of the constraint violation. It is argued that an underlying raison d’être of causal hyperbolic formulations is to make the evolution of constraint violations well-posed.
1 Introduction

The paradigmatic application of perturbation theory in general relativity is to describe distortions of the Schwarzschild black hole\[1\]. Moncrief’s classic gauge-invariant treatment\[2\] of this problem is a remarkable piece of work, but to many its success has an air of the magical to it. The principal motivation for this paper is to use the Schwarzschild problem to introduce a powerful and elegant new method for gravitational gauge-invariant perturbation theory. This method is practically algorithmic and follows naturally from a hyperbolic formulation of the Einstein equations\[3, 4, 5\], referred to hereafter as the Einstein-Ricci formulation, whose centerpiece is a wave equation for curvature. Other applications of this new hyperbolic formulation have been pursued in \[6, 7\].

Besides serving as an example of a general technique, several new insights into black hole perturbation theory arise: The first of these is the recognition that the natural gauge-invariant perturbations are curvature perturbations, not metric perturbations, though one can get “back to the metric” if one desires. Many people have shared the intuition that Einstein’s theory should truly be a theory of propagating curvature, but here that ideal is explicitly realized. Perhaps more surprising is the discovery that the linearized Hamiltonian and momentum constraints themselves constitute the non-physical gauge degrees of freedom. Violations of the constraints represent perturbations that take the background solution away from the constraint hypersurface defining the physical theory. Finally, the physical gauge-invariant variables satisfy Regge-Wheeler equations\[8\], coupled to constraint violations, in both odd and even parity.

The following schematic form is found in detail below for the system of equations describing the evolution of the radial part of perturbations about the Schwarzschild background

\[ ds^2 = -N^2 dt^2 + N^{-2} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2, \]

\[ N^2 = 1 - 2M/r, \]

on a \( t = \text{const.}, K_{ij} = 0 \) slice. The gauge-invariant physical degrees of freedom are \( A_u \) in odd-parity and \( A_g \) in even-parity. (Their expression in terms of the radial parts of perturbations of 3+1 variables will be given below.) Momentum constraint violations are encoded in \( c_{u\theta} \) for odd-parity and \( c_{g\theta} \) and \( c_{g\varphi} \) in even-parity. Violation of the Hamiltonian constraint is encoded in \( c_h \). These equations are valid in a neighborhood of the Schwarzschild initial data, which is a “point” on the physical constraint hypersurface in the phase space of Einsteinian initial data. They reveal how perturbations of Schwarzschild behave even when the constraints are violated perturbatively.
The equations are

\[
\left( -\partial^2_t + (N^2 \partial_r)^2 - V_{RW} \right) A_u - V_{u\theta} c_{u\theta} = 0 \quad (1)
\]

\[
\left( -\partial^2_t + (N^2 \partial_r)^2 - V_{RW} \right) A_g - V_{g\theta} c_{g\theta} - V_{g\theta} c_{g\theta} = 0 \quad (2)
\]

\[
\left( -\partial^2_t + (N^2 \partial_r)^2 - V_{u\theta} \right) c_{u\theta} = 0 \quad (3)
\]

\[
\begin{pmatrix}
-\partial^2_t + \nabla^k \nabla_k - V_{hh} & -V_{hr} & -V_{h\theta} \\
-V_{rh} & -\partial^2_t + (N^2 \partial_r)^2 - V_{rr} & -V_{r\theta} \\
-V_{\theta h} & -V_{\theta r} & -\partial^2_t + (N^2 \partial_r)^2 - V_{g\theta} \\
\end{pmatrix}
\begin{pmatrix}
 c_h \\
 c_{gr} \\
 c_{g\theta} \\
\end{pmatrix}
= 0, \quad (4)
\]

where $V_{RW}$ is the Regge-Wheeler potential[8].

There are two immediate significant observations to make about this system of equations. The constraint variables evolve among themselves under the hyperbolic subsystem of equations (3), (4). If they and their time derivatives vanish initially, they are guaranteed to vanish for all time. This is conjectured to be a general feature of constrained hyperbolic systems and is discussed further below. Next, when the constraints are satisfied, so the constraint variables vanish, both the odd- and even-parity physical perturbations satisfy the Regge-Wheeler equation[8]. This somewhat surprising result runs counter to folklore which attributes the existence of the familiar Zerilli equation[9] for even-parity perturbations somehow to parity. The new derivation reveals that parity is not the issue.

Chandrasekhar[1] has observed that because there is no difference between even and odd parity in the Newman-Penrose formalism, there is no reason to expect different equations for the different parities. This was part of the motivation that led him to construct the transformations[10] between the Zerilli and Regge-Wheeler equations and their Newman-Penrose analog, the Bardeen-Press equation[11], that make their isospectrality clear. Such isospectral transformations between equations are ubiquitous[12, 13], but transformations which relate similarly simple potentials arise only under special conditions (cf. e.g. [14]), something which could well be attributed to accident. It is clear from the present work that the existence of the isospectral transformation allows one to reach both the Regge-Wheeler and Zerilli equations, in either parity, but that the Regge-Wheeler equation is preferable on the grounds of simplicity. In particular, the Regge-Wheeler equation has regular singular points only at physically significant locations while the Zerilli equation has a further regular singular point at an angular momentum dependent location.
An intuitive argument helps to explicate the structure of the evolution equations (1)-(4). In gauge-invariant perturbation theory, the natural candidate variables for perturbation are those which vanish in the background\[15\]. In a constrained theory, the constraints themselves furnish a natural subset of such variables: The constraints are constructed from the degrees of freedom of the full unconstrained theory; they are satisfied, i.e. vanish, in the background; under arbitrary perturbations, they are generally nonzero. Indeed, just as satisfaction of the constraints defines the physical sector of the theory, their violation (partially) parametrizes the theory away from the physical sector of the theory. The variables based on the constraints will be referred to as constraint variables. In distinction, the variables which parametrize the constraint hypersurface will be referred to as physical variables. The constraint variables are a wise choice to describe some of the “gauge” degrees of freedom of a constrained theory, gauge in the sense of characterizing unphysical aspects of motion.

By a constrained hyperbolic theory, we mean a constrained theory which is well-posed, and in particular one in which the constraints are guaranteed to remain satisfied provided they are satisfied initially. It seems intuitively clear that if such a theory were expressed in terms of constraint variables and physical variables, the system of equations would split much as (1)-(4) do. There would be a subsystem in which the constraint variables evolve among themselves, and there would be further equations in which the physical variables couple to the constraint variables. Physical variables could appear in the constraint subsystem only nonlinearly, if at all, multiplied by constraint variables, or else their nonzero presence would act as a source to drive the constraint variables away from zero, given vanishing initial data. Because of this, the physical variables cannot appear in the constraint variable subsystem in the perturbative setting. That a system of equations would admit such a closed internal system of equations is obviously special, but it is the feature which distinguishes a constrained hyperbolic system from an unconstrained one. This qualitative analysis is expected to hold in the fully nonlinear theory, and work is in progress to demonstrate this\[16\].

One of the subtle issues that arises in constrained physical theories like general relativity is that while the theory is hyperbolic on physical grounds, its mathematical representation in redundant variables may not be. In other words, were the theory reduced to the true degrees of freedom, it would be a hyperbolic theory. However, when expressed in redundant variables, the theory may not impose hyperbolic evolution on unphysical combinations of variables, e.g. the constraint variables. Einstein’s equations, viewed as a system of differential equations for the metric, are an example of a constrained system which is physically hyperbolic but not mathematically so. If further restrictions are applied, e.g. through special coordinate conditions, a modified system
of equations can be made hyperbolic\[7, 18\].

The several new recent hyperbolic formulations follow a different route to hyperbolicity\[3, 4, 5, 19\]. By various extensions, the redundant sector of the theory is enlarged, so the whole theory becomes manifestly hyperbolic. A key virtue of achieving a hyperbolic formulation is that the system of equations is well-posed, so that, in particular in causal hyperbolic formulations, this guarantees that violation of the constraints evolves in a predictable, though not necessarily stable, fashion\[16\]. An ill-posed theory is vulnerable to catastrophic breakdown: arbitrarily small perturbations may lead to arbitrarily large deviations arbitrarily quickly. In a well-posed theory, there may be exponentially growing modes, which may in turn be identified as instabilities, but their growth rate is bounded.

In numerical simulations, the urgency of these considerations comes to the fore. Except in special cases, the very implementation of the constrained system will introduce constraint violation because finite-differencing generally does not respect the constraints. While there exist numerical methods for evolving ill-posed systems, there are many more highly developed methods for handling hyperbolic systems. One can choose to re-solve the constraints at each time step to try to remain near the constraint hypersurface, or one can face the full theory and address the issue of controlling constraint violation directly. In the perturbative example here, once the splitting into constraint and physical variables has been made, preservation of the constraints under differencing is no longer an issue—this reflects the fact that the theory has been reduced to the linearized true degrees of freedom. On the other hand, one can undertake a stability analysis of the system with constraint violations which will reflect on the theory in the original variables, and thereby gain insight into the nature of instabilities likely to appear in a numerical simulation of the full theory in the original variables. Work on these issues is in progress.

2 Intertwining procedure

We outline the procedure for gravitational gauge-invariant perturbation theory as follows: Consider perturbations about the Schwarzschild background. The fourth-order form of the Einstein-Ricci formulation\[3\] is a hyperbolic system of a type termed “hyperbolic non-strict”\[20\]. One equation of this system is a wave equation for the time-derivative of the extrinsic curvature, \(\hat{\partial}_0 K_{ij}\) (\(\hat{\partial}_0 \equiv \partial/\partial t - L_\beta\), where \(L_\beta\) is the Lie derivative along the shift \(\beta\)). Because \(\hat{\partial}_0 K_{ij}\) is part of the decomposition of the Riemann tensor \(R_{0i0j}\) in 3+1 variables, this wave equation propagates curvature. The time derivatives of the components of the extrinsic curvature vanish in the Schwarzschild background, so
their perturbations are necessarily gauge-invariant. They are chosen as the perturbative quantities and will be referred to hereafter as curvature perturbations. Their selection accords with the principle that the natural candidates for gauge-invariant quantities are those which vanish in the background\[15\]. Note, in contrast, that the Riemann tensor itself $R_{00ij}$ does not vanish in the background. The Einstein-Ricci system is hyperbolic for arbitrarily specified lapse and shift, which consequently do not have to be perturbed.

There are six components of $\hat{\partial}_0 K_{ij}$, and one has six coupled wave equations in terms of them. In addition, there are four “constraints,” formed from time-derivatives of the three momentum constraints and the Hamiltonian constraint. This leaves two independent equations to be found. If one makes a tensor spherical harmonic decomposition of the perturbations of $\hat{\partial}_0 K_{ij}$, the latter naturally divide into odd and even parity. The two dynamical equations and one constraint with odd parity lead to the Regge-Wheeler equation. This leaves four dynamical equations and three constraints with even parity.

The isolation of a wave equation from the even parity system is the technical crux of the gravitational perturbation calculation. An analogy will make the procedure clear. Suppose that one had four linear algebraic equations in four unknowns. To solve them, one would take linear combinations of the equations to isolate linear combinations of the unknowns. Equivalently, one diagonalizes the matrix of coefficients in the equations by transforming to a different basis.

In the present case, we have four linear differential equations in four unknowns. We construct linear differential combinations of the equations to isolate linear differential combinations of the unknowns. We do not diagonalize the matrix of differential operators, but we re-group the variables to split the differential equations into partially uncoupled form which is diagonal when the constraints hold. We do this by grouping the original variables into combinations which constitute the constraints and new variables which are independent of the constraints. This is accomplished by an intertwining transformation\[12, 13\] in which one matrix of differential operators $M_1$ is transformed into another $M_2$ by a matrix operator $\mathcal{D}$

$$M_2 \mathcal{D} = \mathcal{D} M_1.$$  \hspace{1cm} (5)

If $\mathcal{D}$ were invertible, one would have the familiar expression $\mathcal{D} M_1 \mathcal{D}^{-1} = M_2$ for a basis change. When the constraints hold, the matrix $M_2$ is diagonal; when they do not, the matrix $M_2$ is simply a matrix in a different basis.

Nonperturbatively, the momentum constraints are

$$R_{0ij} = -N \left( \nabla^k K_{jk} - \nabla_j H \right) = 0 \hspace{1cm} (6)$$
where $H = K^k_k$ is the trace of the extrinsic curvature and overbars indicate spatial quantities, here spatial covariant derivatives. The perturbed time-derivative of the momentum constraints are differential linear combinations of the curvature perturbations $\partial_0 K_{ij}$. They are thus appropriate combinations to use in re-organizing the system of four linear differential equations. As displayed in (3), (4), the momentum constraint variables evolve among themselves together with the Hamiltonian constraint variable.

The Hamiltonian constraint itself

$$G^0_0 = -\frac{1}{2}(H^2 - K^{ij} K_{ij} + \bar{R}) = 0$$

involves only metric perturbations when perturbed about a $K_{ij} = 0$ slice (where the underline indicates a background quantity). As a spatial constraint, this constraint may be violated when evaluated on a general metric and extrinsic curvature. This reflects violation of the Hamiltonian constraint. Taking two time derivatives converts the metric perturbations to curvature perturbations and gives an equation for the evolution of violation of the Hamiltonian constraint. This is simply the time derivative of the doubly-contracted Bianchi identity $\nabla^\mu G_{\mu0} = 0$, which can therefore be read as an equation evolving violations of the Hamiltonian constraint through coupling to violations of the momentum constraints (cf. [21]).

Separately, the relation

$$R^0_0 = -N^{-1} \partial_0 H + K_{ij} K^{ij} - N^{-1} \bar{\nabla}^k \bar{\nabla}_k N = 0$$

stands as a dynamical “constraint” coupling curvature and metric perturbations in perturbation theory. This equation is not part of the Einstein-Ricci system and hence may be violated. Two time derivatives again convert the metric perturbations to curvature perturbations.

The identity

$$R^k_k = -2 G^a_0 + R^a_0$$

relates the two constraints to the field equations. The second time derivative of $R^k_k$ occurs as part of the dynamical equations which define the fourth-order Einstein-Ricci theory and is therefore part of an equation which does not admit violations. Thus, two time derivatives of (7) establishes the correlation between violations of $G^0_0 = 0$ and of $R^0_0 = 0$. In view of this correlation, through a useful abuse of language, $R^0_0$ will be referred to also as the Hamiltonian constraint, and the so-called Hamiltonian constraint variable $c_h$ in (1)-(4) reflects violations of the $R^0_0$ constraint. Violations of $G^0_0$ can be eliminated from the second time derivative of (7) using the doubly-contracted Bianchi
identity mentioned above, resulting in the wave equation in (4) for violations of $R^0_{00} = 0$. Details of this will be given below.

Thus, three of the four equations have been “diagonalized” already, and it is not necessary to begin from a general matrix ansatz for the intertwining matrix $D$, though this could have been done. Let $g_k(r, t), k = 1, \ldots, 4,$ be the radial parts of the even-parity curvature perturbations, and let $\hat{L}_k(g_1, \ldots, g_4; r, t) = 0, k = 1, \ldots, 4,$ be the associated dynamical coupled second-order differential equations for them. (Note that $\hat{L}_k$ is not simply the differential operator, but the full equation; this is indicated notationally by the hat.) It is sufficient to consider a first-order differential linear combination of the four dynamical equations to achieve a single wave operator acting on the same differential linear combination of radial curvature perturbations

$$\sum_{k=1}^{4}(b_k \partial_r + a_k)\hat{L}_k = L_{RW} \sum_{k=1}^{4}(b_k \partial_r + a_k)g_k + \hat{f}(R_{00}, R_{0j}),$$

(8)

where

$$L_{RW} = -\partial_t^2 + (1 - \frac{2M}{r})\partial_r(1 - \frac{2M}{r})\partial_r - V_{RW}(r)$$

(9)

is a wave operator of Regge-Wheeler form and $\hat{f}$ is a linear differential operator applied to the constraints. The potential

$$V_{RW}(r) = (1 - \frac{2M}{r})\left(\frac{\ell(\ell + 1)}{r^2} - \frac{6M}{r^3}\right)$$

(10)

is the Regge-Wheeler potential—it could be left undetermined initially and solved for self-consistently. (Alternatively, one could attempt to reach the Zerilli potential starting from a second-order differential combination.) The $a_i$ and $b_i$ are functions of $r$ to be determined by equating like coefficients of derivatives of the $g_k$’s term-by-term.

A simplification is achieved by recognizing that the momentum constraints can be used to eliminate linear spatial derivatives of particular radial coefficients in favor of other terms. This allows one to set two of the $b_i$ to zero without loss of generality. The equations for the $a_k$ and $b_k$ obtained from (8) by equating terms are over-determined. After determining $a_k$ and $b_k$ from a sufficient set of equations, the remaining equations are consistency conditions. If the equations are not consistent, an intertwining of the chosen form does not exist, though one of a different form, e.g. higher order differential or infinite order by means of integral operators, may exist.

The operator transformation between the Regge-Wheeler and Zerilli equations that Chandrasekhar\[1\] discusses is also an example of intertwining\[14, 22\]. Because intertwining transformations compose, when diagonalizing the system above, either equation
can be reached by using an appropriate linear differential combination of the curvature perturbations. One has

\[ D_1 L_{RW} = L_Z D_1, \]
\[ D_2 L_Z = L_{RW} D_2, \]  (11)

where the Regge-Wheeler operator \( L_{RW} \) is given by (9) and the Zerilli operator is

\[ L_Z = -\partial_t^2 + (1 - \frac{2M}{r})\partial_r (1 - \frac{2M}{r})\partial_r - V_Z(r) \]  (12)

with the Zerilli potential [1, 9]

\[ V_Z(r) = \frac{2N^2}{r^3(nr + 3M)^2} \left( (n + 1)n^2r^3 + 3Mn^2r^2 + 9M^2nr + 9M^3 \right), \]  (13)

where \( N^2 = 1 - 2M/r \) and \( n = (\ell - 1)(\ell + 2)/2 \). It is important to note that one can pass in either direction by means of differential operators; the inverse transformation need not be an integral operator. The intertwining operators are [14]

\[ D_1 = (1 - \frac{2M}{r})\partial_r + \frac{3M(r - 2M)}{r^2(nr + 3M)} + \omega, \]
\[ D_2 = (1 - \frac{2M}{r})\partial_r - \frac{3M(r - 2M)}{r^2(nr + 3M)} - \omega, \]  (14)

where \( \omega = n(n + 1)/(3M) = (\ell - 1)(\ell + 1)(\ell + 2)/(12M) \). The intertwining relations (11) become more transparent when one recognizes that they are a consequence of associativity, \( D_1(D_2D_1) = (D_1D_2)D_1 \) and \( D_2(D_1D_2) = (D_2D_1)D_2 \), because the radial parts of the Regge-Wheeler and Zerilli operators factorize

\[ L_{RW} = D_2 D_1 - \partial_t^2 + \omega^2, \]
\[ L_Z = D_1 D_2 - \partial_t^2 + \omega^2. \]  (15)

This factorization property is very suggestive, but it turns out not to be the best property to generalize intertwining.

3 Einstein-Ricci formulation

As shown in [3, 4, 5], the dynamical part of Einstein’s equations can be cast in hyperbolic form in 3+1 language as the Einstein-Ricci formulation. Consider a globally hyperbolic
manifold of topology $\Sigma \times \mathbb{R}$ with the metric
\begin{equation}
 ds^2 = -N^2 dt^2 + g_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt),
\end{equation}
where $N$ is the lapse, $\beta^i$ is the shift, and $g_{ij}$ is the spatial metric. Introduce the non-coordinate co-frame,
\begin{equation}
 \theta^0 = dt, \quad \theta^i = dx^i + \beta^i dt.
\end{equation}
with corresponding dual (convective) derivatives
\begin{equation}
 \partial_0 = \partial / \partial t - \beta^i \partial / \partial x^i, \quad \partial_i = \partial / \partial x^i.
\end{equation}
The natural time derivative for evolution is
\begin{equation}
 \hat{\partial}_0 = \partial_0 + \beta_k \partial_k - \mathcal{L}_\beta = \partial / \partial t - \mathcal{L}_\beta,
\end{equation}
where $\mathcal{L}_\beta$ is the Lie derivative in a time slice $\Sigma$ along the shift vector. In combination with the lapse as $N^{-1}\hat{\partial}_0$, this is the derivative with respect to proper time along the normal to $\Sigma$. The extrinsic curvature $K_{ij}$ of $\Sigma$ is defined as
\begin{equation}
 \hat{\partial}_0 g_{ij} = -2NK_{ij},
\end{equation}
From this follows the equation for the evolution of the Christoffel connection
\begin{equation}
 \hat{\partial}_0 \Gamma^k_{ij} = -g^{mk}[\nabla_j (NK_{im}) + \nabla_i (NK_{mj}) - \nabla_m (NK_{ij})].
\end{equation}
Barred quantities are three-dimensional.

The dynamical Einstein equations, $R_{ij} = \rho_{ij}$, where $\rho_{ij}$ is a matter source, are equivalent to a third-order Einstein-Ricci system. The system is labelled “third-order” by the equivalent of the highest number of derivatives of $g_{ij}$ that can occur in the theory. (These derivatives need not and generally do not appear explicitly.) The third-order Einstein-Ricci system is obtained from (20) and a wave equation for $K_{ij}$
\begin{equation}
 N^2 \Box K_{ij} + NJ_{ij} + NS_{ij} = N\Omega_{ij} \equiv N(\hat{\partial}_0 R_{ij} - \nabla_i R_{0j} - \nabla_j R_{0i}),
\end{equation}
where
\begin{equation}
 \Box = -N^{-1}\hat{\partial}_0 N^{-1}\hat{\partial}_0 + \nabla^k \nabla_k.
\end{equation}
Here, $\Omega_{ij}$ is a matter source term which, as a consequence of the field equations, vanishes in vacuum. For this paper, we restrict attention to vacuum spacetimes. $J_{ij}$ is a nonlinear
self-interaction term. If we denote the trace of the extrinsic curvature by \( H = K^k_k \), \( J_{ij} \) is given by

\[
J_{ij} = \hat{\partial}_0 (HK_{ij} - 2K^k_k K_{jk}) + (N^{-2} \hat{\partial}_0 N + H) \bar{\nabla}_i \bar{\nabla}_j N \\
-2N^{-1}(\bar{\nabla}_k N)\bar{\nabla}_{(i}(NK_{j)}^k) + 3(\bar{\nabla}^k N)\bar{\nabla}_k K_{ij} \\
+ N^{-1} K_{ij} \bar{\nabla}^k (N\bar{\nabla}_k N) - 2\bar{\nabla}_{(i}(K_{j)}^k \bar{\nabla}_k N) + N^{-1} H \bar{\nabla}_i \bar{\nabla}_j N^2 \\
+ 2N^{-1}(\bar{\nabla}_{(i} H)(\bar{\nabla}_{j)} N^2) - 2NK^k(\bar{\nabla}_{(i} \bar{\nabla}_{j)} k) - 2N \bar{\nabla}_{i j} k K^{km},
\]

where \( M_{ij} = \frac{1}{2}(M_{ij} + M_{ji}) \). The three-curvatures can be expressed in terms of four-curvatures and then eliminated using the field equations.

Finally, \( S_{ij} \) is a slicing term,

\[
S_{ij} = -N^{-1} \bar{\nabla}_i \bar{\nabla}_j (\hat{\partial}_0 N + N^2 H).
\]

This must be equal to a functional involving fewer than third derivatives of the metric to assure the hyperbolic (wave) nature of the equation (22). Two simple ways to do this are the following. One may impose the harmonic slicing condition

\[
\hat{\partial}_0 N + N^2 H = 0,
\]

in which case

\[
S_{ij} = 0.
\]

Alternatively, one may specify the mean curvature by demanding \( H = h(x, t) \). The lapse function \( N \) is then determined by solving the elliptic equation

\[
\bar{\nabla}^k \bar{\nabla}_k N = -\hat{\partial}_0 h(x, t) + NK_{ij} K^{ij}.
\]

The special case of maximal slicing, \( H \equiv 0 \), gives

\[
S_{ij} = -N^{-1} \bar{\nabla}_i \bar{\nabla}_j \hat{\partial}_0 N.
\]

In this case, the Einstein-Ricci system is mixed hyperbolic-elliptic[24].

With harmonic slicing, the dynamical part of Einstein’s equations are given by the definition of the extrinsic curvature (20), the wave equation (22), and the harmonic slicing condition (26). (In this case, all the equations of motion are equivalent to a first-order symmetric hyperbolic system[3, 4] with characteristics the light cone and the direction orthogonal to the time slices.) With slicing given by specified mean curvature,
the dynamical equations are (20) and (22), and the lapse is determined by interleaving
the solution of the elliptic equation (28) on each time-slice.

To complete the new formulation of the Einstein equations, we specify the initial
Cauchy data. Initial $g_{ij}$ and $K_{ij}$ must be chosen compatible with the Gauss-Codazzi
(Hamiltonian and momentum) constraints. The Hamiltonian constraint is

$$G^0_0 = \frac{1}{2}(R^0_0 - R^k_k) = -\frac{1}{2}(\bar{R} + H^2 - K_{mk}K^{mk}) = 0, \quad (30)$$

and the momentum constraints are

$$NG^0_i = \bar{\nabla}_k(K^k_i - \delta^k_i H) = 0. \quad (31)$$

These are treated as an elliptic system on the initial slice by the usual methods[23, 25].
Furthermore, to guarantee that the dynamical equations produce a solution to Einstein’s
equations, it is also necessary that $\hat{\partial}_0 K_{ij}$ be specified initially so that Einstein’s equations
hold on the initial slice

$$R_{ij} = \tilde{R}_{ij} - N^{-1} \hat{\partial}_0 K_{ij} + H K_{ij} - 2K_{ik}K^k_j - N^{-1}\bar{\nabla}_i\bar{\nabla}_j N, \quad (32)$$

where $R_{ij}$ on the left hand side is replaced by its expression in terms of the matter
sources. In the case of harmonic slicing, the lapse is specified on the initial slice. The
shift, hidden in $\hat{\partial}_0$, is a freely specifiable function on spacetime and is not a dynamical
variable of the system.

To obviate the need for special handling of the slicing term $S_{ij}$ and to allow the lapse
to be specified freely, one can take another time-derivative, apply a further constraint,
and pass to the fourth-order Einstein-Ricci formulation. These equations proceed from
(20) and the equation

$$N\hat{\partial}_0(N\hat{\nabla}K_{ij}) + N\hat{\partial}_0(J_{ij} + S_{ij}) + N\hat{\nabla}_i\hat{\nabla}_j(N\hat{\partial}_0 H - N^2K_{mk}K^{mk} + N\hat{\nabla}^k\hat{\nabla}_k N) =$$

$$= N\tilde{\Omega}_{ij} \equiv N(\hat{\partial}_0^2 R_{ij} - 2\hat{\partial}_0\bar{\nabla}_i R_{j0} + \bar{\nabla}_i\bar{\nabla}_j R_{00}). \quad (33)$$

The effect of including $R_{00}$ is to incorporate the Hamiltonian constraint while cancelling
the threatening highest derivative term of $H$ in $S_{ij}$. The Cauchy data are extended by
specifying $\hat{\partial}_0^2 K_{ij}$, subject to the requirement (in vacuum) that $\hat{\partial}_0 R_{ij} = 0$ hold on the
initial slice. Both the lapse and the shift are freely specifiable functions on spacetime
and are not dynamical variables. The system (20), (33) is hyperbolic non-strict in the
sense of Leray-Ohya[20]. In particular, it is well-posed with solutions in an appropriate
Gevrey (not Sobolev) class[3].

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Let us first consider a weak-field analysis of the equations above around a flat spacetime background with a Minkowski metric. Let \( g_{ij} = \hat{g}_{ij} + g'_{ij} \) and \( K_{ij} = \hat{K}_{ij} + K'_{ij} \) where underlines denote background values and primes the first order corrections. In the present case, \( g_{ij} = \delta_{ij} \) and \( K_{ij} = 0 \). In the fourth-order Einstein-Ricci formulation, the lapse and shift can be arbitrarily specified: allow a lapse perturbation \( N = \bar{N} + N' = 1 + N' \), but set the shift perturbation to zero, \( \beta^i = \beta' = 0 \). The use of a lapse perturbation is unnecessary, but its inclusion will demonstrate this explicitly. (The lapse perturbation \( N' \) here should not be confused with \( \bar{N}' = \partial_r N \) used later in the paper.) At first order, (20) becomes

\[
\partial_t g'_{ij} = -2K'_{ij}. \tag{34}
\]

The wave equation (33) from the fourth-order Einstein-Ricci formulation is

\[
\hat{\Box} \partial_t K'_{ij} - \nabla_i \nabla_j (\partial_t^2 N' + \partial_t H') + \nabla_i \nabla_j R'_{00} = 0 \tag{35}
\]

\( \hat{\Box} = -\partial_t^2 + \nabla_k \nabla_k \), where \( \nabla_i \) is the spatial covariant derivative with respect to the background metric \( \hat{g}_{ij} \).

The weak-field form of the identity

\[
R_{000j} = N \hat{\Box} K_{ij} + N^2 K_{ik} K^k_j + N \nabla_i \nabla_j N \tag{36}
\]

is

\[
R'_{000j} = \partial_t K'_{ij} + \nabla_i \nabla_j N'. \tag{37}
\]

Its trace gives

\[
R'_{00} = \partial_t H' + \nabla^k \nabla_k N'. \tag{38}
\]

Using this in (35) gives

\[
\hat{\Box} (\partial_t K'_{ij} + \nabla_i \nabla_j N') = \hat{\Box} R'_{000j} = 0. \tag{39}
\]

This equation is clearly lapse independent. Furthermore, the perturbations are explicitly Riemann tensor perturbations.

There are six degrees of freedom to \( R'_{000j} \), but we expect only two physical gauge-invariant degrees of freedom. The resolution of this apparent paradox is to recognize that by tracing (39) one obtains an equation for \( R'_{00} \)

\[
\hat{\Box} R'_{00} = 0, \tag{40}
\]

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while taking a (spacetime) divergence and using the contracted Bianchi identity,
\[ \nabla^i R_{0i0j} = -\nabla_0 R_{0j} + \nabla_j R_{00}, \]
(41)
one reaches an equation for \( \partial_t R'_{0j} \)
\[ \hat{\square} \partial_t R'_{0j} = 0. \]
(42)
Thus, four of the six equations evolve constraints. A general perturbation may violate
the constraints, and these four equations determine the constraint violation evolution.
The physical degrees of freedom are the remaining two degrees of freedom of \( R'_{0i0j} \), the
transverse traceless parts.
To make contact with traditional analyses, one can examine (37) in more detail by
reading it as an equation for \( \partial_t K'_{ij} \) and splitting \( K'_{ij} \) into a sum of transverse traceless,
longitudinal traceless and trace parts
\[ K'_{ij} = K''_{TTij} + K''_{LTij} + \frac{1}{3} g_{ij} H'. \]
(43)
The trace of (37) is
\[ \partial_t H' = R'_{00} - \hat{\nabla}^k \hat{\nabla}_k N' \]
as above. Note that \( R'_{00} \) is not set to zero as traditionally done. This is because a
general perturbation will violate the constraint, and it is easier to work with free rather
than constrained perturbations.
The longitudinal part can be identified by using the definition that the divergence of
the transverse part vanishes. Split the extrinsic curvature perturbation and the Riemann
tensor into a sum of transverse traceless and longitudinal (with trace) parts
\[ K'_{ij} = K''_{TTij} + K''_{LTij}, \quad R'_{0i0j} = R''_{0i0j} + R''_{0j0i}, \]
(45)
where the divergence of the TT parts is assumed to vanish. The divergence of (37) is then
\[ \partial_t \hat{\nabla}^i K''_{ij} = \hat{\nabla}^i R''_{0i0j} - \hat{\nabla}^i \hat{\nabla}_i \hat{\nabla}_j N'. \]
(46)
Using the definition of the perturbative momentum constraint
\[ R'_{0j} = - (\hat{\nabla}^i K'_{ij} - \hat{\nabla}_j H') \]
and (44) leads to the perturbative form of the contracted Bianchi identity (41)
\[ - \partial_t R'_{0j} = \hat{\nabla}^i R''_{0i0j} - \hat{\nabla}_j R'_{00}. \]
(48)
This reveals the longitudinal nature of this identity. Stripping the divergence away from (46) and removing the trace gives the tracefree longitudinal equation
\[ \partial_t K'_{ij}^{LT} = R'_{0i0j} - \frac{1}{3} g_{ij} R'_{00} - \left( \nabla_{(i} \nabla_{j)} N' \right) - \frac{1}{3} g_{ij} \nabla^k \nabla_k N'. \] (49)

Finally, the transverse traceless equation is
\[ \partial_t K'_{ij}^{TT} = R'_{0i0j}. \] (50)

If one further splits (34) to obtain the transverse traceless part, then one finds
\[ -\frac{1}{2} \partial_t^2 g'_{ij}^{TT} = R'_{0i0j}, \] (51)

which is a well-known result. One of the virtues of this analysis is that it clarifies the role of the transverse and longitudinal parts of the Riemann tensor and emphasizes that the true physical degrees of the freedom of the linearized gravitational field lie in the transverse traceless part.

5 Gauge-invariant perturbation theory: Schwarzschild

The fourth-order Einstein-Ricci system is the natural one for gauge-invariant perturbation theory. Because the lapse and the shift are freely specifiable, their perturbations do not need to be considered. Furthermore, because one is working with perturbations which have the dimensions of curvature, it is easier to find variables which are gauge-invariant in the background, and hence natural candidates to perturb. It is convenient nevertheless to begin by perturbing the third-order Einstein-Ricci system (without fixing the slicing term) and later to take a time derivative to reach the perturbed fourth-order theory. This procedure organizes and simplifies the computation because the background is time-independent.

As perturbations, let \( g_{ij} = g_{ij} + g'_{ij}, \ K_{ij} = K_{ij} + K'_{ij}, \ \bar{\Gamma}^{i}_{jk} = \bar{\Gamma}^{i}_{jk} + \bar{\Gamma}'^{i}_{jk}, \) where an underline indicates the background quantity and a prime the first order perturbation. In the fourth-order theory, the lapse is unperturbed, so \( N \equiv \bar{N}, \) and the underline will be suppressed. The shift \( \beta^k \equiv \bar{\beta}^k \) is also unperturbed.

To be explicit, consider perturbations of the Schwarzschild background
\[ ds^2 = -(1 - \frac{2M}{r})dt^2 + (1 - \frac{2M}{r})^{-1}dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2. \] (52)
This example admits three useful simplifications: The background shift vanishes, \( \beta^k = 0 \), so \( \partial_0 = \partial_t \). The background lapse \( N \) is time-independent, so many terms in (22) and (33) vanish. Finally, the background extrinsic curvature \( K_{ij} \) and its derivatives (space and time) vanish on the natural slice \( (t = \text{constant}) \) to be perturbed. This implies that their Lie derivatives along an arbitrary vector \( v \) vanish, e.g.

\[
\mathcal{L}_v K_{ij} = v^k \partial_k K_{ij} + K_{kj} \partial_i v^k + K_{ik} \partial_j v^k = 0.
\]

Thus, they are perturbatively gauge-invariant variables, and therefore natural candidates for perturbation. Their selection accords with the general principle emphasized in [15] that one should always choose to perturb quantities which vanish in the background. In addition, many more terms in (22) and (33) vanish because the extrinsic curvature is zero in the background.

The result of perturbing (22) in a background sharing these properties (time-independent lapse, vanishing shift, and vanishing background extrinsic curvature) with Schwarzschild is the wave equation

\[
N \Omega'_{ij} = -\partial_t^2 K'_{ij} + N^2 \sum_k \sum_i K'_{ij} - N^2 \sum_i \sum_j H' - 4N \sum_k N \sum_i K'_{ij}k + 3N \left( \sum_i N \sum_k K'_{ij} - 2N K'_{k(i} \sum_j N - 2K'_{k(i} (\sum_j) N) \right) \sum_k N
+ NH' \sum_i \sum_j N + K'_{ij} (\sum_i N) \sum_k N - 2N^2 K'_{k(i} (\sum_j) N) - 2N^2 \sum_i R_{ij}^k - 2N^2 \sum_i R_{ij}^k m K'_{km}.
\]

(An additional term \( K'_{ij} N \sum_k \sum_k N \) vanishes in the Schwarzschild background and is not included.) The spatial Ricci and Riemann tensors are computed in the background metric.

Note the presence of the slicing term as the second derivative of \( H' = \text{tr} K' \). This spoils the hyperbolicity of the perturbative wave equation. If one wished, one could return to the full theory and impose harmonic slicing. When perturbing the theory, one would have to allow perturbations of the lapse, but this second derivative term would be removed. Interestingly, the perturbed lapse would still not appear in the perturbed wave equation. If one went on to introduce a tensor spherical harmonic multipole decomposition of the perturbations of the extrinsic curvature, as will be done below for the time-derivative of the extrinsic curvature, perturbed third-order equations would result. These form the basis of the perturbative-matching outer-boundary module being used as one method of extracting gravitational waves and imposing outer boundary conditions on the three-dimensional numerical simulations carried out by the Binary Black Hole Alliance [26].
The second derivative of $H$ in the full theory can also be removed by passing to the fourth-order theory by taking a time-derivative and adding second spatial derivatives of $R_{00}$. Because the background is time-independent and the background extrinsic curvature vanishes, the effect of taking a time-derivative is simply to replace the perturbed extrinsic curvature $K'_{ij}$ by its time derivative $\partial_t K'_{ij} = \dot{K}'_{ij}$. The additional perturbed $R_{00}$ term has the form

$$N\bar{\nabla}_i \bar{\nabla}_j R'_{00} = N^2 \bar{\nabla}_i \bar{\nabla}_j \dot{H}' + N \dot{H}' \bar{\nabla}_i \bar{\nabla}_j N + 2N (\bar{\nabla}_i N) \bar{\nabla}_j \dot{H}' +$$

$$-N \bar{\nabla}_i \bar{\nabla}_j (Ng^{mn}g'_{nj} \bar{\nabla}_m \bar{\nabla}_k N + Ng^{mk} \bar{\Gamma}'_{ml} \partial_l N)$$

(55)

One sees that by using $R'_{00}$, the second derivative of $\dot{H}'$ is cancelled, but as well metric perturbations are introduced into the perturbed fourth-order equations through $g'$ and $\bar{\Gamma}'$. This is the price that one has paid to achieve explicit hyperbolicity in the full theory. The next step is to eliminate the metric perturbations in favor of time-differentiated extrinsic curvature perturbations. This is done by using $R'_{00}$. On first sight, this appears simply to undo the step which cancelled the second derivative of $H$, but this is not so. We desire to understand the behavior of the theory away from the constraint hypersurface, so we do not set $R'_{00} = 0$, but leave it as a free variable which happens to vanish on-shell. The presence of this off-shell term marks the fact that the second derivative of $H$ has been cancelled in the full theory.

This raises a valuable point. It is important to emphasize that one starts from a well-posed hyperbolic theory before perturbing. If the $R_{00}$ term were not present in the full theory, as it is not when one simply takes a time-derivative of the third order theory (without fixing the slicing), one would have a system which is believed not to be hyperbolic. The perturbed form of that non-hyperbolic system would naively agree with the one we have just found if one were to set $R'_{00} = 0$. This perturbative theory therefore agrees with general relativity on-shell, but the Hamiltonian constraint has not been fully incorporated, so that when violations of the Hamiltonian constraint occur, the theory cannot respond properly, that is, in a manifestly hyperbolic fashion.

The $R_{00}$ term compensates behavior of the second spatial derivative of $H$ to make the theory hyperbolic. When $R_{00}$ vanishes, this compensation is evidently insignificant. As the magnitude of the second spatial derivatives of $R_{00}$ increase however, one suspects that the compensation becomes more important. Because this condition involves second derivatives, small variations in $R_{00}$ from zero can nevertheless produce arbitrarily large contributions to the equation. This reveals an important caveat: it is dangerous to study a constrained theory solely within the constraint hypersurface and not to consider its behavior when the constraints are violated. The full mathematical character of a
fundamental physical theory with constraints, in particular its hyperbolicity and well-posedness, involves off-shell information.

Having given this warning, in the case of perturbations of Schwarzschild at least, the role of the $R_{00}$ term appears to be relatively innocuous. The system of equations does not change character dramatically when this term is removed. This suggests there is further room to explore the role of the $R_{00}$ term.

The equation for the perturbed fourth-order theory is

$$N\ddot{\Omega}_{ij} = -\partial_t^2 \dot{K}'_{ij} + N^2\tilde{\nabla}^k \tilde{\nabla}_k \dot{K}'_{ij} - N^2\tilde{\nabla}_i \tilde{\nabla}_j \dot{H}' + N\tilde{\nabla}_i \tilde{\nabla}_j R'_{00}$$

$$-4N\tilde{\nabla}^k N\tilde{\nabla}_i (\dot{K}'_{j})_k + 3N\tilde{\nabla}^k N\tilde{\nabla}_k \dot{K}'_{ij} - 2N\dot{K}'_k(\tilde{\nabla}_i \tilde{\nabla}_j)\tilde{\nabla}^k N + N\dot{H}'\tilde{\nabla}_i \tilde{\nabla}_j N$$

$$-2\dot{K}'_{k}(\tilde{\nabla}_i N)\tilde{\nabla}^k N + \dot{K}'_{ij}(\tilde{\nabla}_k N)\tilde{\nabla}^k N - 2N^2 \dot{K}'_k(\tilde{\nabla}_j R'_j)^k - 2N^2 \tilde{R}'_{ij} \tilde{R}_m^{km}.$$  

(Again, an additional term $\dot{K}'_{ij} N\tilde{\nabla}^k \tilde{\nabla}_k N$ vanishes in Schwarzschild case and has been dropped.)

6 Odd-parity perturbations

Because the wave operator on the Schwarzschild background separates in spherical coordinates, perturbations in the Schwarzschild background are naturally handled by making a multipole decomposition in tensor spherical harmonics. These in turn naturally divide into odd and even parity. Following Moncrieff[4], we decompose the odd-parity perturbations as

$$\dot{K}'_{ij} = u_1(t, r)(\hat{e}_1)_{ij} + u_2(t, r)(\hat{e}_2)_{ij},$$

where $u_1(t, r)$ and $u_2(t, r)$ are radial perturbations and

$$\hat{e}_1 = \begin{pmatrix} 0 & -\frac{1}{\sin \theta} \partial_\phi Y_{\ell m} & \sin \theta \partial_\theta Y_{\ell m} \\ \text{symm} & 0 & 0 \\ \text{symm} & 0 & 0 \end{pmatrix}$$

and

$$\hat{e}_2 = \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{\sin \theta} (\partial_\theta \partial_\phi - \cot \theta \partial_\phi) Y_{\ell m} & \frac{\sin \theta}{2} \frac{1}{\sin^2 \theta} \partial_\phi^2 + \cot \theta \partial_\theta - \partial_\theta^2 Y_{\ell m} \\ \text{symm} & -\sin \theta (\partial_\theta \partial_\phi - \cot \theta \partial_\phi) Y_{\ell m} \end{pmatrix}$$

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are the odd-parity tensor spherical harmonics. In these formulae, $Y_{\ell m}(\theta, \phi)$ are the standard scalar spherical harmonics satisfying

$$(\partial_\theta^2 + \cot \theta \partial_\theta + \frac{1}{\sin^2 \theta} \partial_\phi^2) Y_{\ell m} = -\ell(\ell + 1) Y_{\ell m}.$$  

The notation “symm” indicates the matrices are symmetric.

From (56) one obtains equations for $u_1$ and $u_2$. To simplify the expressions it is useful to introduce

$$\lambda_\ell = \ell(\ell + 1)$$

as the (negative of the) eigenvalue of the spherical harmonics and to replace second and higher derivatives of $N$ by equivalent expressions in terms of $N$ and $N' \equiv \partial_r N$ (not to confuse the prime with the perturbative part). From the $r\theta$-component of (56), after dividing out the common angular factor $-\partial_\phi Y_{\ell m}/\sin \theta$, one has

$$-\partial_t^2 u_1 + N^4 \partial_r^2 u_1 + 4N^3 N' \partial_r u_1 + \frac{N^2}{r^2} (-\lambda_\ell - 4N^2 + 6rNN' + r^2N'^2) u_1 + \frac{(2 - \lambda_\ell)N^2}{r^3} u_2 = 0. \quad (60)$$

From the $\theta\phi$ equation, after dividing out

$$(- \sin \theta \partial_\theta^2 Y_{\ell m} + \cos \theta \partial_\theta Y_{\ell m} + \csc \theta \partial_\phi^2 Y_{\ell m})/2,$$

one has

$$-\partial_t^2 u_2 + N^4 \partial_r^2 u_2 + \frac{2N^3}{r} (-N + 2rN') \partial_r u_2 + \frac{N^2}{r} (-\lambda_\ell + 4N^2 - 4rNN' + r^2N'^2) u_2 + \frac{4N^3}{r} (-N + rN') u_1 = 0. \quad (61)$$

Of the constraints, only one is present in odd parity. It can be found from $\dot{R}'_{\theta\theta}$, and its radial expression is

$$\frac{\sin \theta \dot{R}'_{\theta\theta}}{\partial_\phi Y_{\ell m}} = c_{u\theta} = N^3 \partial_r u_1 + \left(\frac{2N^3}{r} + N^2 N'\right) u_1 + \frac{\lambda_\ell - 2}{2r^2} N u_2. \quad (62)$$

(A degenerate expression is found from $\dot{R}'_{\theta\phi}$.) If $c_{u\theta} = 0$, the odd-parity part of the momentum constraints hold; if not, $c_{u\theta}$ measures the violation. The variable $u_2$ can be eliminated from (60) using this constraint, and one has

$$-\partial_t^2 u_1 + N^4 \partial_r^2 u_1 + (4N^3 N' + \frac{2N^4}{r}) \partial_r u_1 + \frac{N^2}{r^2} (-\lambda_\ell + 8rNN' + r^2N'^2) u_1 - \frac{2N}{r} c_{u\theta} = 0. \quad (63)$$
By rescaling $u_1$, the differential part of the operator can be brought into the familiar Regge-Wheeler form,

$$-\partial^2_t + N^2 \partial_r N^2 \partial_r.$$  

The new variable 

$$A_u \equiv r Nu_1 \quad (64)$$

is the gauge-invariant variable which satisfies the Regge-Wheeler equation

$$\left(-\partial^2_t + N^2 \partial_r N^2 \partial_r - \frac{N^2}{r^2} (\lambda_\ell - 6r NN')\right)A_u - 2N^2 c_{u\theta} = 0. \quad (65)$$

The familiar Regge-Wheeler potential is explicitly given by 

$$V_{RW}(r) = \frac{N^2}{r^2} (\lambda_\ell - 6r NN') = \left(1 - \frac{2M}{r}\right)(\ell(\ell + 1) - 6\frac{M}{r^3}). \quad (66)$$

The $c_{u\theta}$ term reflects how the Regge-Wheeler equation is modified when the odd-parity momentum constraint is violated.

From the perturbative form of $(36)$ in the Schwarzschild background, the gauge-invariant variable $A_u$ is readily identified as the radial part of one of the components of the Riemann tensor

$$A_u = r Nu_1 = -r \frac{\sin \theta R'_{0r0\theta}}{\partial_\phi Y_{\ell m}} \quad (67)$$

The “extra” factor of $r$ is an artifact which arises because the differential part of the Regge-Wheeler operator is written as if space were one-dimensional whereas the operator actually comes from the radial part of a three-dimensional Laplacian. The factor of $r$ is just the scaling to change the apparent dimension of the background space.

The equation which evolves $c_{u\theta}$ is found by forming the differential linear combination of $(60)$ and $(61)$ implied by $(62)$ to create a second time derivative of $c_{u\theta}$. One finds

$$\left(-\partial^2_t + N^2 \partial_r N^2 \partial_r - \frac{N^2\lambda_\ell}{r^2}\right)c_{u\theta} = 0. \quad (68)$$

(The scaling of $c_{u\theta}$ was chosen earlier to put the differential part of the operator in Regge-Wheeler form.)

The odd-parity violation of the momentum constraints propagates hyperbolically on its own, and the pair $(65)$ and $(68)$ together form a hyperbolic system. If $c_{u\theta} = 0$ and $\partial_t c_{u\theta} = 0$ initially, $(68)$ guarantees that they remain so. It is a necessary feature that the constraints evolve amongst themselves without involving the physical degree of freedom. This enables zero initial data to remain zero. We will see this again in the even-parity case.
7 Even-parity perturbations

Following Moncrief, but relabelling the radial variables and rescaling the \( g_1 \) variable to have more rational dimensions, we can decompose the even-parity perturbations as

\[
\hat{K}'_{ij} = r g_1(t, r)(\hat{f}_1)_{ij} + g_2(t, r)N^{-2}(\hat{f}_2)_{ij} + r^2 g_3(t, r)(\hat{f}_3)_{ij} + r^2 g_4(\hat{f}_4)_{ij}, \tag{69}
\]

where

\[
\hat{f}_1 = \begin{pmatrix}
0 & \partial_\theta Y_{\ell m} & \partial_\phi Y_{\ell m} \\
\text{symm} & 0 & 0 \\
\text{symm} & 0 & 0
\end{pmatrix}, \tag{70}
\]

\[
\hat{f}_2 = \begin{pmatrix}
Y_{\ell m} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \tag{71}
\]

\[
\hat{f}_3 = \begin{pmatrix}
0 & Y_{\ell m} & 0 \\
0 & 0 & \sin^2 \theta Y_{\ell m}
\end{pmatrix}, \tag{72}
\]

\[
\hat{f}_4 = \begin{pmatrix}
0 & 0 & 0 \\
0 & \partial_\theta^2 Y_{\ell m} & (\partial_\theta \partial_\phi - \cot \theta \partial_\phi) Y_{\ell m} \\
0 & \text{symm} & (\partial_\phi^2 + \sin \theta \cos \theta \partial_\phi) Y_{\ell m}
\end{pmatrix}. \tag{73}
\]

are the even-parity tensor spherical harmonics. The trace of \( \hat{K}'_{ij} \) is found to be

\[
\hat{H'} = (g_2 + 2 g_3 - \lambda_\ell g_4)Y_{\ell m}. \tag{74}
\]

Finally, let

\[
R'_{00} = c_h Y_{\ell m}.
\]

The fourth-order Einstein-Ricci equations \((50)\) for even-parity radial perturbations have the following forms. From the \( r\theta \)-equation, up to a factor of \( r \partial_\theta Y_{\ell m} \), the equation for \( g_1 \) is

\[
\hat{L}_1 : -\partial_\theta^2 g_1 + N^4 \partial_r^2 g_1 + \left( \frac{2N^4}{r} + 4N^3 N' \right) \partial_r g_1 - \frac{N^2}{r} \partial_r (g_2 + 2 g_3 - \lambda_\ell g_4)
\]
\[ + \frac{N^2}{r^2} (-\lambda - 4N^2 + 10rNN + r^2N^2)g_1 \]
\[ + \frac{N}{r^2} (3N - 2rN')g_2 + \frac{N^2}{r^2} (\lambda - 2)g_4 + \frac{N}{r} \partial_r c_h - \frac{N}{r^2} c_h = 0. \] (75)

From the \(rr\)-equation, up to a factor of \(Y_{\ell m}/N^2\), the equation for \(g_2\) is
\[ \hat{L}_2 : -\partial_t^2 g_2 - N^4 \partial_r^2 (2g_3 - \lambda \ell g_4) + \left( \frac{2N^4}{r} - N^3 N' \right) \partial_r g_2 - N^3 N' \partial_r (2g_3 - \lambda \ell g_4) \]
\[ + \frac{4N^4 \lambda \ell}{r^2} g_1 + \frac{N^2}{r^2} (-\lambda - 4N^2 + 6rNN - r^2N^2)g_2 \]
\[ + \frac{2N^3}{r^2} (N - 2rN')(2g_3 - \lambda \ell g_4) + N^3 \partial_r^2 c_h + N^2 N' \partial_r c_h = 0. \] (76)

From the \(\theta \phi\)-equation, up to a factor of \(r^2(\partial_\theta \partial_\phi Y_{\ell m} - \cot \theta \partial_\phi Y_{\ell m})\), the equation for \(g_4\) is
\[ \hat{L}_4 : -\partial_t^2 g_4 + N^4 \partial_r^2 g_4 + \frac{2N^3}{r} (N + 2rN') \partial_r g_4 + \frac{4N^3}{r^2} (N - rN')g_1 - \frac{N^2}{r^2} g_2 - \frac{2N^2}{r^2} g_3 \]
\[ + \frac{N^2}{r^2} (2N^2 + 4rNN' + r^2N^2)g_4 + \frac{N}{r^2} c_h = 0. \] (77)

Finally, taking the trace of the \(\theta \theta\)- and \(\phi \phi\)-equations, dividing out the common factor \(2Y_{\ell m}\), and adding \(\lambda \ell/2\) times (77), one obtains the equation for \(g_3\)
\[ \hat{L}_3 : -\partial_t^2 g_3 + N^4 \partial_r^2 g_3 - \frac{N^4}{r} \partial_r g_2 - 4N^3 N' \partial_r g_3 + \frac{\lambda \ell N^4}{r} \partial_r g_4 \]
\[ + \frac{N^3}{r^2} (2N - 5rN')g_2 + \frac{N^2}{r^2} (-\lambda - 2N^2 + 6rNN' + r^2N^2)g_3 \]
\[ + \frac{\lambda \ell N^3}{r^2} (2N - rN')g_4 + \frac{N^3}{r} \partial_r c_h = 0. \] (78)

The (time-differentiated) perturbed momentum constraints in even-parity reduce to the two expressions
\[ \frac{\hat{r}_{0r}}{NY_{\ell m}} \equiv \dot{c}_r = \partial_r (2g_3 - \lambda \ell g_4) + \frac{1}{r} (\lambda \ell g_1 - 2g_2 + 2g_3 - \lambda \ell g_4), \] (79)
\[ \frac{\hat{r}_{0\theta}}{N\partial_\theta Y_{\ell m}} \equiv \dot{c}_\theta = -rN^2 \partial_r g_1 - g_1 (3N^2 + rNN') + g_2 + g_3 - g_4. \] (80)
(Again, a degenerate expression is found for \( \tilde{R}_{00}^{\prime} \).) These define variables \( \tilde{c}_r \) and \( \tilde{c}_{\theta \theta} \) which are useful for studying the effects of momentum constraint violations in perturbation theory. By forming the appropriate differential linear combinations of (75)-(78), one finds that the momentum constraint variables satisfy the following system of coupled wave equations

\[
\begin{align*}
- \partial_t^2 \tilde{c}_r + N^4 \partial_r^2 \tilde{c}_r + \frac{2N^3}{r} (N + 4rN') \partial_r \tilde{c}_r &+ \frac{4N^2N'}{r} \partial_r c_h = 0, \\
+ \frac{N^2}{r^2} (-2 - \lambda_\ell + 9r^2N'^2) \tilde{c}_r &+ \frac{2\lambda_\ell N^2}{r^3} \tilde{c}_{\theta \theta} - \frac{\lambda_\ell N'}{r^2} c_h = 0,
\end{align*}
\]

\[
\begin{align*}
- \partial_t^2 \tilde{c}_{\theta \theta} + N^4 \partial_r^2 \tilde{c}_{\theta \theta} &+ 4N^3N' \partial_r \tilde{c}_{\theta \theta} - N^2N' \partial_r c_h \\
+ \frac{N^2}{r^2} (-\lambda_\ell - 2rNN' + r^2N'^2) \tilde{c}_{\theta \theta} &+ \frac{2N^3}{r} (N + rN') \tilde{c}_r = 0.
\end{align*}
\]

Introducing rescaled momentum constraint variables

\[
c_r = rN^3 \tilde{c}_r = \frac{rN^2 \hat{R}_0^\prime}{Y_{\ell m}} \tag{83}
\]

and

\[
c_{\theta \theta} = N \tilde{c}_{\theta \theta} = \frac{\hat{R}_{00}^\prime}{\partial_{\theta} Y_{\ell m}} \tag{84}
\]

puts the differential operator part in Regge-Wheeler form, leaving the equations

\[
\left( -\partial_t^2 + N^2 \partial_r^2 - \frac{N^2}{r^2} (2 + \lambda_\ell + 2rNN') \right) c_r
\]

\[
+ \frac{2N^4\lambda_\ell}{r^2} \tilde{c}_{\theta \theta} + 4N^5N' \partial_r c_h - \frac{\lambda_\ell N^3N'}{r} c_h = 0 \tag{85}
\]

and

\[
\left( -\partial_t^2 + N^2 \partial_r^2 - \frac{\lambda_\ell N^2}{r^2} \right) c_{\theta \theta} + \frac{2N}{r^2} (N + rN') c_r - N^3N' \partial_r c_h = 0. \tag{86}
\]

An evolution equation for the \( R_{00} \) constraint is found by taking the trace of (53). The perturbed constraint is

\[
c_h Y_{\ell m} = R'_{00} = NH' - N\hat{g}^{mn}g'_{mn}g^{jk}\tilde{\nabla}_n\tilde{\nabla}_kN - N\hat{g}^{mk}\tilde{\Gamma}^{\prime i}_{kj} \hat{\nabla}_i N. \tag{87}
\]
This involves metric perturbations through the $g'$ and $\Gamma'$ terms. A time derivative of $\Gamma'$ only involves perturbed extrinsic curvatures and a second time derivative involves $\dot{K}'_{ij}$, 

$$\partial_t^2 \Gamma'_{i mk} = -g^{ij} \left( \bar{\nabla}_m (N\dot{K}'_{jk}) + \bar{\nabla}_k (N\dot{K}'_{jm}) - \bar{\nabla}_j (N\dot{K}'_{mk}) \right).$$  \hfill (88)

Taking two time derivatives of (87) allows second time derivatives of $\dot{H}'$ to be replaced by second derivatives of $c_h$ and leads to the equation

$$\left( -\partial_t^2 + N^4 \partial_t^2 + \left( \frac{2N^4}{r} + N^3 N' \right) \partial_t - \frac{N^2 \lambda}{r^2} \right) c_h + \frac{2NN'}{r} c_r = 0.$$  \hfill (89)

An alternative derivation which reveals the role of the Hamiltonian constraint $G^0_0$ is given as follows. Let

$$c_h Y_{\ell m} = G^0_{0} = -\frac{1}{2} \bar{R}'.$$  \hfill (90)

Two time derivatives gives

$$\partial_t^2 (c_h G Y_{\ell m}) = -\bar{\nabla}^k \dot{R}'_{k0} - N^{-1} (\bar{\nabla}^k N) \dot{R}'_{k0},$$  \hfill (91)

which is simply the time derivative of the contracted Bianchi identity $\nabla^\mu G_{\mu 0}$ as mentioned above. The trace of (56) corresponds through (33) to

$$\partial_t^2 R^k_{\ k} - 2 \bar{\nabla}^k \dot{R}'_{k0} + \bar{\nabla}^k \bar{\nabla}_k R_{00} = 0.$$  \hfill (92)

The second time derivative of the perturbed form of (7) gives

$$\partial_t^2 R^k_{\ k} = -N^{-2} \partial_t^2 (c_h Y_{\ell m}) - 2 \partial_t^2 (c_h G Y_{\ell m}).$$  \hfill (93)

Substituting this and (90) into (92) gives

$$- N^{-2} \partial_t^2 (c_h Y_{\ell m}) + \bar{\nabla}^k \bar{\nabla}_k (c_h Y_{\ell m}) + 2N^{-1} (\bar{\nabla}^k N) \dot{R}'_{k0} = 0,$$  \hfill (94)

which is readily confirmed to agree with (89).

The constraint variables $c_r$, $c_g\theta$, and $c_h$ together form a hyperbolic system. If they and their first time derivatives all vanish, the initial data for this system is identically zero and it remains zero under evolution. If any are non-zero, the constraint violations evolve hyperbolically. Below we will see that $c_r$, $c_g\theta$ and $c_h$ occur as source terms in the even-parity Regge-Wheeler equation.
8 Intertwining and the even-parity Regge-Wheeler equation

Having found how constraint violations evolve, it remains to find how the physical even-parity degree of freedom propagates. The procedure is to form an arbitrary first-order differential linear combination of (73)-(78) and require that it equal a Regge-Wheeler wave operator

\[ \square_{RW} = -\partial_r^2 + N^2 \partial_r \partial_N - V_{RW}(r) \]  

acting on differential linear combination of the \( g_k \), plus constraint variables. More precisely, one requires

\[ \sum_{k=1}^{4} \left( b_k(r) \partial_r + a_k(r) \right) \hat{L}_k = \square_{RW} \left( \sum_{k=1}^{4} b_k(r) \partial_r g_k + a_k(r) g_k \right) + \hat{f}(\tilde{c}_r, \tilde{c}_{g\theta}, c_h). \]  

As discussed in the introduction, this equation can be understood as a component of a matrix intertwining relation between a matrix differential operator in the basis of the \( g_k \) and one in the basis which isolates the physical degree of freedom \( A_g \) from the constraint variables \( \tilde{c}_r, \tilde{c}_{g\theta} \) and \( c_h \). The other components of the matrix transformation were determined above when the appropriate combinations of the \( g_k \) were found which produce equations involving only the constraint variables.

One can begin with the potential \( V_{RW}(r) \) of the wave operator undetermined and solve for it self-consistently, but this complicates some of the intermediate expressions, so the ansatz is made that the potential has the Regge-Wheeler form (66). The ansatz for the transformation (96) can be simplified slightly before beginning. Because the momentum constraint variables are built from radial derivatives of \( g_1 \) and \( 2g_3 - \lambda \ell g_4 \), one can without loss of generality set \( b_1(r) = 0 \) and \( b_3(r) = 0 \) because the momentum constraint variables can be used to eliminate the radial derivatives of \( g_1 \) and \( g_3 \).

The procedure is now to write out the equation (96) and compare like derivatives. The collection of coefficients of the derivatives of the \( g_k \), starting from the highest derivatives, form a set of recursive relations for the coefficients in the intertwining transformation. For example, one immediately sees from the \( \partial_r^3 g_2 \) and \( \partial_r^2 g_2 \) coefficients that \( b_2(r) = 0 \) and \( a_2(r) = 0 \). Thus \( g_2 \) does not appear in the physical even-parity gauge-invariant variable. As each successive intertwining coefficient is determined, the remaining relations are simplified.

In anticipation of future developments, it proves convenient to rescale the remaining coefficients as follows

\[ b_4(r) = N^3 \tilde{b}_4(r), \]  

\[ 25 \]
\[ a_1(r) = N^3 \tilde{a}_1(r), \]
\[ a_3(r) = N \tilde{a}_3(r), \]
\[ a_4(r) = N \tilde{a}_4(r). \]

It is also necessary to add zero in the form of an arbitrary multiple \( c_1(r), c_2(r) \) of the (unscaled) momentum constraint variables \( \tilde{c}_r, \tilde{c}_\theta \) minus their expressions in terms of the \( g_k \). This reflects the fact that the right-hand side of (96) may contain an arbitrary amount of momentum constraint variables. The Hamiltonian constraint variable is not added as it would mix in metric perturbations which are not present. [Note however that the Hamiltonian constraint variable appears on both sides of (96) because of its presence in (73)-(78).]

After these simplifications, reduce second and higher derivatives of \( N \) to first and no derivatives by taking derivatives of \( N^2 = 1 - 2M/r \), and denote differentiation of coefficients with respect to \( r \) by primes. Then the set of recursive equations obtained from (96) becomes

\[
\begin{align*}
\frac{2N^7}{r} \frac{\partial^2}{\partial r^2} g_4 : & \quad \tilde{b}_4 - r \tilde{b}_4' \\
\frac{N^5}{r^2} \frac{\partial}{\partial r} g_2 : & \quad r \tilde{a}_1 + r \tilde{a}_3 + \tilde{b}_4 \\
\frac{N^5}{r^2} \frac{\partial}{\partial r} g_4 : & \quad \lambda \epsilon (r \tilde{a}_1 + r \tilde{a}_3 + \tilde{b}_4) + 2r \tilde{a}_4 - 2r^2 \tilde{a}_4' \\
 & \quad + 4r \tilde{b}_4 NN' - 8r^2 \tilde{b}_4' NN' - r^2 \tilde{b}_4'' N^2 - \lambda \epsilon rc_1 / 2 \\
- \frac{2N^5}{r^2} \frac{\partial}{\partial r} g_3 : & \quad r \tilde{a}_1 + \tilde{b}_4 + r^2 \tilde{a}_3' - rc_1 / 2 \\
\frac{2N^5}{r^2} \frac{\partial}{\partial r} g_1 : & \quad r \tilde{a}_1 N^2 + 2 \tilde{b}_4 N^2 - r^2 \tilde{a}_4' N^2 - 2r^2 \tilde{a}_1 NN' - 2r \tilde{b}_4 NN' + N r^3 c_2 / 2 \\
\frac{N^3}{r^3} g_2 : & \quad -r \tilde{a}_4 + 3r \tilde{a}_1 N^2 + 2r \tilde{a}_3 N^2 + 2 \tilde{b}_4 N^2 \\
 & \quad - 2r^2 \tilde{a}_1 NN' - 5r^2 \tilde{a}_3 NN' - 2r \tilde{b}_4 NN' - r^3 c_2 N - rc_1 N^2 \\
\frac{N^3}{r^3} g_4 : & \quad r \lambda \epsilon (\tilde{a}_4 + \tilde{a}_1 N^2 + 2 \tilde{a}_3 N^2 - r \tilde{a}_3 NN' - c_1 N^2 / 2) \\
 & \quad - 2N^2 (r \tilde{a}_4 - \tilde{a}_4 + 2 \tilde{b}_4 N^2 + 2r \tilde{b}_4 NN' - 2r^2 \tilde{b}_4 N'') \\
 & \quad - 4r^3 \tilde{a}_4'' NN' - r^3 N^2 \tilde{a}_4'' + r^3 c_2 N \\
\frac{N^3}{r^3} g_3 : & \quad - 2r \tilde{a}_4 - 2r \tilde{a}_3 N^2 + 4 \tilde{b}_4 N^2 + 2r^2 \tilde{a}_3 NN' - 4r \tilde{b}_4 NN'.
\end{align*}
\]
\[ \frac{N^3}{r^3} g_1 : \quad 4N^2 (r \tilde{a}_4 - r \tilde{a}_1 N^2 - 2 \tilde{b}_4 N^2) + N^3 N'(10r^2 \tilde{a}_1 + 28r \tilde{b}_4 - 8r^3 \tilde{a}_1') \\
-8r^2 N^2 N'(r \tilde{a}_4 + \tilde{b}_4) - 4r^2 N N' \tilde{a}_4 - r^3 \tilde{a}_1'' N^4 \]
\[ + \lambda r c_1 N^2 / 2 + 3r^3 c_2 N^3 + r^4 c_2 N^2 N' \]

From the \( \partial^2_r g_4 \) coefficient, one has the relation
\[ \tilde{b}_4 = r \partial_r \tilde{b}_4 \] (99)
from which one concludes \( \tilde{b}_4 = r \). (An overall multiplicative constant can be taken to be unity without loss of generality since it represents a constant scale of the variable \( A_g \).) To cancel the \( \partial_r g_2 \) term, one must set
\[ \tilde{a}_3(r) = -\tilde{a}_1(r) - 1. \] (100)

Cancelling the \( \partial_r g_3 \) and \( \partial_r g_1 \) terms determines \( c_1 \) and \( c_2 \) to be
\[ c_1(r) = 2(1 + \tilde{a}_1 - r \partial_r \tilde{a}_1) \] (101)
\[ c_2(r) = \frac{-2}{r^2} \left( 2N - 2r N' + \tilde{a}_1(N - 2r N') - r N \tilde{a}_1' \right) \]

The coefficient of \( g_2 \) determines \( \tilde{a}_4 \) in terms of \( \tilde{a}_1 \)
\[ \tilde{a}_4 = 2N^2 - r N N' + \tilde{a}_1(N^2 - r N N'). \] (102)

After imposing this, the coefficient of \( \partial_r g_4 \) gives the equation
\[ - \lambda \xi + 4N^2 - 16r N N' + \tilde{a}_1(-\lambda \xi + 2N^2 - 8r N N') + r \tilde{a}_4' (\lambda \xi - 2N^2 + 2r N N') = 0. \] (103)

Inspection suggests the substitution
\[ \tilde{a}_1 = \alpha r - 2 \]
which leads to
\[ -6M \alpha + \lambda \xi = 0. \]

Thus, one has
\[ \tilde{a}_1 = \frac{\lambda \xi r}{6M} - 2. \] (104)
After applying this relation, all the intertwining coefficients are fixed. The remaining equations implied by the $g_1$, $g_3$ and $g_4$ coefficients are satisfied identically if the lapse takes the Schwarzschild form $N^2 = 1 - 2M/r$. These final relations are the consistency conditions of the over-determined system of equations. It is easy to see that if the lapse weren’t special, e.g. Schwarzschild, the intertwining would not have been consistent.

Assembling the above results gives the physical even-parity gauge-invariant variable

$$A_g = N\left(rN^2\partial_r g_4 + N^2\left(\frac{r\lambda_\ell}{6M} - 2\right)g_1 + (1 - \frac{r\lambda_\ell}{6M})g_3 + \left(\frac{M}{r} - \frac{\lambda_\ell}{2} + \frac{r\lambda_\ell}{6M}\right)g_4\right). \quad (105)$$

This variable satisfies the Regge-Wheeler equation

$$\left(-\partial_t^2 + N^2\partial_r N^2\partial_r - V_{RW}(r)\right)A_g + \frac{N^2}{r^2}c_r$$

$$+ \frac{N^2}{r^2}\left(\frac{2\lambda_\ell}{3} - \frac{4M}{r}\right)c_{g\theta} + \frac{N^2}{r^2}\left(\frac{2M}{r} - \frac{\lambda_\ell}{6}\right)c_h = 0. \quad (106)$$

As with the odd-parity equation, there are constraint violating terms present to show how the theory evolves off-shell (perturbatively). On-shell this is the Regge-Wheeler equation. The gauge-invariant variable $A_g$ has even-parity and is constructed from time-derivatives of the extrinsic curvature.

One can reach the Zerilli equation

$$\left(-\partial_t^2 + N^2\partial_r N^2\partial_r - V_Z(r)\right)\tilde{A}_g + \cdots = 0, \quad (107)$$

where $\tilde{A}_g = D_1A_g$ and the Zerilli potential is

$$V_Z(r) = \frac{2N^2}{r^3(nr + 3M)^2}\left((n + 1)n^2r^3 + 3Mn^2r^2 + 9M^2nr + 9M^3\right), \quad (108)$$

$(n = (\ell - 1)(\ell + 2)/2)$ by applying the appropriate intertwining operator $D_1$ to (106) and using (11). Note the Zerilli potential has an unphysical singularity at $nr + 3M = 0$. Making this final transformation is unnecessary and arguably undesirable because of the unphysical singularity structure of the potential. Both the odd- and even-parity gauge-invariant variables satisfy a Regge-Wheeler equation, so parity is not the explanation for the existence of the Zerilli equation. Whether the existence of the Zerilli equation has a deeper meaning than accidental coincidence is an open question. From the standpoint of computation, however, the physics of even-parity perturbations is fully captured in the Regge-Wheeler equation without spurious unphysical complications, and the choice of description should be clear.
9 Conclusion

For completeness, the full set of gauge-invariant perturbation equations (65), (68), (85), (86), (89), (106) is gathered here

\[
\left( -\partial^2_t + N^2 \partial_r N^2 \partial_r - \frac{N^2}{r^2} (\lambda_\ell - 6r \beta N') \right) A_u - 2N^2 c_{u\theta} = 0. \tag{109}
\]

\[
\left( -\partial^2_t + N^2 \partial_r N^2 \partial_r - \frac{N^2}{r^2} (\lambda_\ell - 6r \beta N') \right) A_g
\quad + \frac{N^2}{r^2} c_r + \frac{N^2}{r^2} \left( \frac{2\lambda_\ell}{3} - \frac{4M}{r} \right) c_{g\theta} + \frac{N^2}{r^2} \left( \frac{2M}{r} - \frac{\lambda_\ell}{6} \right) c_h = 0. \tag{110}
\]

\[
\left( -\partial^2_t + N^2 \partial_r N^2 \partial_r - \frac{N^2}{r^2} \lambda_\ell \right) c_{u\theta} = 0. \tag{111}
\]

\[
\left( -\partial^2_t + N^2 \partial_r N^2 \partial_r - \frac{N^2}{r^2} (2 + \lambda_\ell + 2r \beta N') \right) c_r + \frac{2N^4 \lambda_\ell}{r^2} c_{g\theta} + \left( 4N^5 \beta_N' \frac{\lambda_\ell N^3 N'}{r} \right) c_h = 0. \tag{112}
\]

\[
\left( -\partial^2_t + N^2 \partial_r N^2 \partial_r - \frac{\lambda_\ell N^2}{r^2} \right) c_{g\theta} + \frac{2N}{r^2} (N + r \beta N') c_r - N^3 \beta_N' c_h = 0. \tag{113}
\]

Several important lessons are learned from the example of perturbing the Schwarzschild black hole. The one specific to the Schwarzschild case is that the Regge-Wheeler equation suffices to describe the evolution of both odd and even parity physical perturbations. This result has been known mathematically for some time\cite{29, 14, 1, 22}, but a skeptic might assign significance to the observation that the Zerilli equation always seems to arise from even-parity perturbations\cite{1, 2, 9}. The calculation presented here should answer that doubt. The Regge-Wheeler equation has been obtained directly in even parity by an intertwining transformation, which decouples the equations and involves the fewest number of derivatives possible. The intertwining procedure is an effective tool which systematizes a technically involved computation and makes it transparent. The Zerilli equation can be reached by a further transformation which introduces an unphysical angular momentum-dependent singularity into the potential. Chandrasekhar\cite{1} (pp. 198-199) discusses whether one should dismiss the Zerilli equation and concludes not. “Dismiss” is probably too strong a word, but one certainly need never use the Zerilli
equation in computations, as all of the physics in it is captured more succinctly in the Regge-Wheeler equation.

Perhaps the principal lesson which follows from the Schwarzschild example is the recognition that working with the fourth-order Einstein-Ricci formulation shifts the emphasis from metric perturbations to curvature perturbations and in so doing reveals that the physical gauge-invariant quantities propagated by the Regge-Wheeler equation are formed from curvature perturbations and their spatial derivatives. Furthermore, one discovers that the gauge-invariant perturbations in unphysical directions are given by (time derivatives of) violations of the linearized Hamiltonian and momentum constraints. Additionally, the constraint violations evolve as a closed hyperbolic subsystem (111)-(114), so that if the constraints are satisfied initially, they continue to be satisfied. It is worth emphasizing that this evolution subsystem for the constraints comes directly from the dynamical equations and not from separately considering the contracted Bianchi identities. The information for evolving the constraints is embedded within the dynamical part of the theory, as it must be for consistency.

These observations are general and help one to organize the calculation of gauge-invariant perturbations in a wide class of constrained hyperbolic theories: After recasting a constrained theory in hyperbolic form without gauge-fixing, say by the procedure outlined in [4], one perturbs a complete set of independent combinations of the fundamental variables which vanish in the background. Since the combinations vanish in the background, their perturbations are necessarily gauge-invariant [15]. The constraints themselves provide a subset of these variables, and their perturbations will evolve as a closed subsystem if the equations are consistent. From a complete set of gauge-invariant variables, the physical subset which are independent of the constraint variables can be constructed in principle by using the intertwining procedure described in the text.

The fact that the gauge-invariant perturbations of Minkowski space and the Schwarzschild black hole are curvature perturbations which split into the true linearized degrees of freedom and constraints encourages speculation about the nature of the true degrees of freedom in the fully nonlinear theory. From the twenty degrees of freedom of the Riemann tensor in four dimensions, the ten components of the Ricci tensor are obtained by tracing with the spatial 3-metric ($R^0_{i0j}$ must of course be added to $R^k_{ikj}$ to obtain the spatial Ricci tensor). Respectively, this accounts for one, three and all six degrees of freedom of $R_{0i0j}$, $R_{0ijk}$ and $R_{ijkl}$. The ten Ricci components vanish (in vacuum) by equivalence of the Einstein-Ricci formulation with Einstein’s theory. Six more degrees of freedom are obtained, three each, from divergences of $R_{0i0j}$ and $R_{0ijk}$. By the Bianchi identities, these reduce to combinations of derivatives of the Ricci tensor and hence also vanish. This leaves two degrees of freedom each in $R_{0i0j}$ and $R_{0ijk}$, namely the transverse
traceless parts. Work is in progress to prove the conjecture that something related to the transverse traceless parts of the Riemann tensor are the true degrees of freedom of the gravitational field[10].

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