Two-player stochastic games (sometimes referred to as $2 \frac{1}{2}$-player games) are games with two players and a randomised entity called “nature”. A natural question to ask in this framework is the existence of strategies that ensure that an event happens with probability 1 (almost-sure strategies). In the case of Markov decision processes ($1 \frac{1}{2}$-player games), when the event of interest is given as a parity condition, we can replace the “nature” by two more players that play according to the rules of what is known as Banach-Mazur game [1]. In this paper we continue this research program by extending the above result to two-player stochastic parity games. As in the paper [1], the basic idea is that, under the correct hypothesis, we can replace the randomised player with two players playing a Banach-Mazur game. This requires a few technical observations, and a non trivial proof, that this paper sets out to do.

1 Introduction

In the fields of control and design of reactive systems, one often faces the problem of verifying whether a system, which is interacting with its environment, has the desired behaviour or not. The mathematical interpretation of this problem, also known as Church synthesis problem, is usually modelled by a game played on graphs. Such a game involves two players, the first one shall be called Eve represents the controller and the second one shall be called Adam and represents the environment. The behaviour we want the controller to ensure, usually called the objective, is given by a set of infinite paths over a graph. In order to check whether Eve can ensure the objective, the graph is partitioned into two sets of states; Eve’s state and Adam’s state. When the play is in Eve’s states, she chooses the next state of the play, and when it is in Adam’s states, he chooses the successor. Therefore, a play generates an infinite path and Eve wins the play if it is in the objective. Finally, the problem is to synthesise (compute) a strategy for Eve to ensure that the generated play is always in the objective.

Two-player stochastic games are a generalisation of the former model in which the graph is partitioned into three sets of states; Eve’s, Adam’s, and Nature’s. The role of Nature is to add an element of surprise. When the play is in a Nature’s state, the successor is chosen according to a coin flip. In this case the outcome of a play is no more a unique path but a set of paths. The synthesis problem becomes then whether there exists a strategy for Eve such that the measure of the set of generated paths is larger than a given threshold. In [5], it is shown that under the appropriate assumptions, the core analysis amounts to deciding whether there exists a strategy
such that the measure of the generated paths is 1. Call such a strategy an *almost-surely winning* strategy.

From a system design point of view, as already mentioned in [8], the randomisation adds some kind of fairness in the general behaviour of the system. For instance, consider a game where Eve wins if she repeats a self loop infinitely. If even a little randomisation is added so that Eve is not sure that she can stay in the loop, she cannot follow this strategy any longer. Thus, one can say that winning in the framework of Stochastic games is somehow more realistic as one cannot count on a contrived behaviour to win.

A different approach for implementing fairness is by means of topology. The main idea is that a strategy is winning if the set paths it induces is *topologically large*. In some rather general cases, sets with measure 1 coincide with those topologically large sets. In particular, Staiger [7] and Varacca and Volzer [8] in a separate work have shown that on finite Markov chains, \( \omega \)-regular sets of infinite words have measure one if and only if they have the topological property of being *co-meager*. This property can be equivalently characterised by the notion of Banach-Mazur games [6], where two players alternately play finite sequences of paths on the graph structure of the Markov chain. Brihaye et. al. have shown that this result extends to countable intersection of \( \omega \)-regular sets [3].

To further the game-theoretic intuition of [8], Asarin et. al. [1] have shown an equivalence between finite Markov decision processes (one-player stochastic games) on \( \omega \)-regular objectives, and a new notion of three-player games, where the players of the Banach-Mazur game alternately try to help the controller satisfy its objective, or spoil it. What that paper reveals is that, while the basic idea of replacing a probabilistic notion with a topological one is sound and intuitive, some technicalities have to be spelled out correctly in order for the framework to work.

Our contributions are as follows. We introduce the notion of two-player BM parity games which is a generalisation of Banach-Mazur games (c.f. Section 3). We show that these games are positionally determined (c.f. Theorem 15) for parity objectives using an approach à la Zielonka. We also show that a slight change in the order of quantifier of the definition of the game is enough to lose determinacy (c.f. Example 12). Finally, we draw a link between two-player stochastic parity games and two-player BM parity games. In particular, we show that there exists an almost-surely winning strategy in a two-player stochastic parity game if and only if there exists a winning strategy in a well chosen two-player BM game (c.f. Theorem 14), this last result subsumes the one of [1], although the proof there, being applied to a simpler case, is simpler.

## 2 Preliminaries

### 2.1 Stochastic games

For a finite set \( S \) we denote \( \text{Distr}(S) \) the set of all discrete probability distributions over \( S \), that is the set of functions \( d : S \to [0,1] \) such that \( \sum_{s \in S} d(s) = 1 \). For a distribution \( d \) in \( \text{Distr}(S) \), we denote by \( \text{Supp}(d) \) the set \( \{ s \in S \mid d(s) > 0 \} \).

Given a graph \( (S,E) \), and an element \( s \in S \), we denote \( \text{Paths}(S,E,s) \) as the set of infinite sequences \( w \in S^\omega \) such that \( w(0) = s \), and for any \( i \), \( (w(i),w(i+1)) \in E \).

**Games and plays** A *stochastic game* is a tuple \( G = (S,(S_E,S_A),A,P) \) where \( S \) is a finite set of states, \( (S_E,S_A) \) is a partition of \( S \) such that \( S_E \) is the set of states controlled by \( E \) (Eve)
and $S_A$ is the set of states controlled by $A$ (Adam), $A$ is a finite nonempty set of actions, and $\mathcal{P} : S \times A \rightarrow \text{Distr}(S)$ is a total transition function.

A play is an infinite sequence $s_0a_0s_1a_1 \cdots \in (SA)^\omega$. We denote by $S_n$ the random variable with values in $S$ that maps each play to its $n$th state and by $A_n$ the random variable with values in $A$ that maps each play to its $n$th action. Formally $S_n(s_0a_0s_1a_1 \cdots) = s_n$, and $A_n(s_0a_0s_1a_1 \cdots) = a_n$.

Strategies and Measures A strategy for $E$ is a function that tells her what is the next action to play, given a partial play of the game. Formally it is a function $\sigma : (SA)^*S_E \rightarrow A$. We define strategies for $A$ as $\tau : (SA)^*S_A \rightarrow A$.

Once a pair of strategies is chosen ($\sigma, \tau$) and an initial state $s$ is fixed, we associate the probability measure $\mathbb{P}_s^{\sigma,\tau}$ over $S^\omega$ as the only measure over the Borel sets of $S^\omega$ such that:

$$\mathbb{P}_s^{\sigma,\tau}(S_0 = s) = 1,$$

$$\mathbb{P}_s^{\sigma,\tau}(S_{n+1} = s \mid S_n = s_n) = \begin{cases} \mathcal{P}(s_n, \sigma(s_0a_0 \cdots s_n))(s_{n+1}) & \text{if } s_n \in S_E, \\ \mathcal{P}(s_n, \tau(s_0a_0 \cdots s_n))(s_{n+1}) & \text{if } s_n \in S_A. \end{cases}$$

The existence and uniqueness of such a measure is a consequence of Carathéodory’s extension theorem.

Objectives An objective is a measurable subset of plays $\Phi \subseteq S^\omega$. We say that $E$ wins almost-surely from a state $s$ if she has a strategy $\sigma$ such that for every strategy $\tau$, $\mathbb{P}_s^{\sigma,\tau}(\Phi) = 1$.

2.2 Banach-Mazur Games

The notion of Banach-Mazur game [6] can be presented with different levels of generality. Here we choose to present it in a form that is most suitable to our needs.

Let $T$ be a set, and $X \subseteq T^\omega$ a set of infinite words, and $\Phi \subseteq X$ an objective. The Banach-Mazur game on $X$ with objective $\Phi$ is played as follows. There are two players, that we can call Banach (le “bon”) and Mazur (le “méchant”). Mazur begins by playing a finite prefix $w_0$ of some word in $X$. Then Banach extends $w_0$ with another finite prefix $w_1$ of some word in $X$. The play continues, generating an infinite sequence $w_0 < w_1 < w_2 \ldots$ If the limit of this sequence belongs to the objective, then Banach wins, otherwise Mazur wins. It was proven by (the real) Banach and Mazur that Banach wins if and only if the objective has the topological property of being co-meager in the Cantor topology induced on $X$ by $T^\omega$ [6].

A special case is when there is a graph structure $(S, E)$, an initial element $s \in S$ and $X$ is $\text{Paths}(S, E, s)$. In this case we talk about the Banach-Mazur game on a graph.

Banach-Mazur games and Markov chains A Markov chain on a set of states $S$ is given by a function $\mathcal{P} : S \rightarrow \text{Distr}(S)$. It induces a graph $(S, E_\mathcal{P})$, where $(s, s') \in E_\mathcal{P}$ if and only $s' \in \text{Supp}(\mathcal{P}(s))$. A Markov chain can be seen as a stochastic game where the players always have exactly one available choice. Therefore, similarly to what we have described above, given an initial state $s$, a Markov chain generates a Borel probability measure on $\text{Paths}(S, E_\mathcal{P}, s)$. It is well known that $\omega$-regular sets are measurable [2]. Varacca and Völzer [8] (see also [7]) have shown the following result:
Theorem 1. Let $\mathcal{P}$ be a Markov chain on a finite set $S$. Let $s$ be an initial state, and let $\Phi$ be an $\omega$-regular subset of $\text{Paths}(S, E_{\mathcal{P}}, s)$. Then $\Phi$ has measure 1 under $\mathcal{P}$ if and only if Banach wins the Banach-Mazur game on $(S, E_{\mathcal{P}}, s)$ with objective $\Phi$.

2.3 Parity games

Definition 2. Let $G$ be a game, and $\chi : S \to C$ be a priority function where $C \subseteq \mathbb{N}$. The parity objective $\text{Par}$ is given by the following set

$$\text{Par} = \{ s_0s_1s_2\cdots \in S^\omega \mid \limsup(\chi(s_0)\chi(s_1)\chi(s_2)\cdots) \text{ is even} \} .$$

In the sequel, we call games equipped with parity objectives parity games.

In the setting of stochastic parity games, a natural question to ask is whether there exists a strategy that ensures the parity objective with probability 1? Formally,

Problem 3 (Almost-sure Parity). Given a stochastic parity game with initial state $s$, compute a strategy $\sigma$ if it exists such that

$$\forall \tau, \mathbb{P}_s^{\sigma, \tau}(\text{Par}) = 1 .$$

A strategy for a player is positional (or memoryless) if the choices depend only on the last state of the current play. Formally, a strategy is positional if it defines a mapping $\sigma : S \to A$.

An instrumental result in our subsequent development is the following theorem due to [9, 4].

Theorem 4 (Positional determinacy). For finite stochastic games with parity objectives, from every state $s$, either $E$ has a positional almost-surely winning strategy, or $A$ has a positional positively winning strategy.

3 Two-player BM games

In this section we present the proposal for a notion of four-player games, that correspond to two-player stochastic games, in a formal sense that we show below.

3.1 The game and the plays

Arenas A two-player BM game is a tuple $\mathcal{G} = (S, (S_E, S_A), A, \text{BM})$ where $S$ is a finite set of states, $(S_E, S_A)$ is a partition of $S$, $A$ is a finite nonempty set of actions, and $\text{BM} \subseteq S \times A \times S$ is a transition relation. We will assume, for simplicity, that for every $s \in S$ and every $a \in A$ there exists at least a state $s'$ for which $(s, a, s') \in \text{BM}$.

The game is played by four players: the Arena players $E$ (Eve) and $A$ (Adam), and the Nature players $B$ (Banach) and $M$ (Mazur).

$S_E$ is the set of states controlled by Eve while $S_A$ is the set of states controlled by Adam. When it is her turn to play, in a state $s$, $E$ chooses an action $a$. Similarly for $A$. Once an Arena player has made his or her choice, it is the turn of the Nature players to choose a next state according to the transition relation. They can also choose to pass their turn. Formally, after $E$ (or $A$) has played in state $s$ choosing an action $a$, $B$ chooses a state $s'$ such that $(s, a, s') \in \text{BM}$. He can also choose a special action $\bot$ that means that he is passing the turn. In such a case it is immediately $M$ that has to make a choice. (And dually exchanging the roles of the Nature players). $B$ and $M$ must pass their turn at some point during a play. At the beginning of the play, it is always $M$’s turn.

Next we formalise these intuitions.
Winning plays A (legal) play is an infinite word in $S((AS)^*A\bot(AS))^\omega$. It must contain infinitely many occurrences of $\bot$, and no adjacent occurrences of $\bot$. We denote the set of plays in a game $G$ by $\text{Plays}(G)$. A flattening of a play is obtained by projecting the play on $S^\omega$, forgetting the occurrences of $A$ and of $\bot$. The flattening of a play is always an infinite word. An objective is a set of infinite words $\Phi \subseteq S^\omega$.

**Definition 5.** A play $w$ on a game $G$ with objective $\Phi$ is winning for $E$ and $B$ if its flattening belongs to the objective. Otherwise the play is winning for $A$ and $M$. When clear from the context, we will say that the play is winning for $E$ (resp. $A$).

### 3.2 Turn-based Strategies

In order to define strategies, we first need to figure out which information the players are allowed to have. Nature players can try to help their allied (Banach helps Eve, while Mazur helps Adam) but Arena players must not know against which one of the Nature players they are playing at any given moment. We will explain why at the end of the section (c.f. Example 12). Therefore, Arena players are not allowed to see the occurrences of $\bot$, forgetting the occurrences of $A$ and of $\bot$.

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Once all the players have chosen their strategies, a play is obtained. In order to give the formal definition, we need the following notation:

- given a finite sequence $w$, we denote by $\text{Last}(w)$ its last element;
- given a finite sequence $w$ possibly containing occurrences of $\bot$, we denote by $\pi_\bot(w)$ the sequence obtained by eliminating these occurrences.

Formally, given a play $\bar{w}$ in $\text{Plays}(G)$ we say that it respects the strategies $\sigma$ for $E$, $\tau$ of $A$, $\bar{\sigma}$ for $B$, $\bar{\tau}$ for $M$ if for every finite prefix $w$ of $\bar{w}$:

1. if $\text{Last}(w) \in S_E$ and $\sigma(\pi_\bot(w)) = a$ then $wa$ is a prefix of $\bar{w}$
2. if $\text{Last}(w) \in S_A$ and $\tau(\pi_\bot(w)) = a$ then $wa$ is a prefix of $\bar{w}$
3. if $\text{Last}(w) \in A \cup \bot$, $w$ contains an odd number of occurrences of $\bot$, and $\bar{\sigma}(w) = x$, then $wx$ is a prefix of $\bar{w}$.
4. if $\text{Last}(w) \in A \cup \bot$, $w$ contains an even number of occurrences of $\bot$, and $\bar{\tau}(w) = x$, then $wx$ is a prefix of $\bar{w}$.

**Remark 6.** Given four strategies for the four players, and an initial state $s$, there is a unique play of the game $G$ beginning in $s$ that respects them that we call the induced play.
**Definition 7.** We say that $E$ and $B$ win structurally from a state $s$, if there exists a strategy $\sigma$ for $E$ such that for any strategy $\tau$ of $A$ there exists a strategy $\bar{\sigma}$ for $B$ such that for any strategy $\bar{\tau}$ for $M$ the induced play starting from $s$ is winning for $E$ and $B$.

### 3.3 Global Strategies

We have presented this turn based way of playing the game, as it is intuitive. However it is hard to work with it, as the definition of strategy for the Nature Players is quite involved. We propose here an alternative point of view of the strategies for Banach and Mazur, that is equivalent, but more suitable for mathematical proofs.

The intuition is that we let Adam and Eve play their strategies first in order to generate a residual tree. Then Banach and Mazur play their game as usual. Formally, given a strategy $\sigma$ for $E$ and a strategy $\tau$ of $A$, and a state $s$ we build a subset $G_f(\sigma, \tau)$ of $S(AS)^*$ as follows:

- the initial state $s$ is in $G_f(\sigma, \tau)$
- if $ws$ is in $G_f(\sigma, \tau)$ and $s \in S_E$, and $\sigma(ws) = a$, then for all $s'$ such that $(s, a, s') \in BM$, we have that $wsas'$ is in $G_f(\sigma, \tau)$
- if $ws$ is in $G_f(\sigma, \tau)$ and $s \in S_A$, and $\tau(ws) = a$, then for all $s'$ such that $(s, a, s') \in BM$, we have that $wsas'$ is in $G_f(\sigma, \tau)$

The residual tree $G(\sigma, \tau)$ is the set of infinite words obtained as limits of sequences in $G_f(\sigma, \tau)$.

**Definition 8.** We say that $E$ and $B$ win strategically from a state $s$ if from $s$ there exists a strategy $\sigma$ for $E$ and such that for any strategy $\tau$ of $A$, Banach wins the Banach-Mazur game on the residual tree $G(\sigma, \tau)$.

**Theorem 9.** $E$ and $B$ win structurally if and only if they win strategically.

From the winning strategy $\bar{\sigma}$ for $B$ it is very easy to extract the winning strategy in the Banach-Mazur Game. Conversely, given a winning strategy on the residual tree, we can define several $\bar{\sigma}$, by just choosing as we want in the branches not belonging to the residual tree.

### 3.4 Determinacy and positionality

If Eve and Banach do not win, it means that for each strategy of Eve there is a winning counterstrategy of Adam. A stronger case is when Adam has one strategy that wins against all strategies of Eve.

**Definition 10.** We say that $A$ and $M$ win strategically from a state $s$ if from $s$ there exists a strategy $\tau$ for $A$ and such that for any strategy $\sigma$ of $E$, Banach wins the Banach-Mazur game on the residual tree $G(\sigma, \tau)$.

**Definition 11** (determinacy). A game is determined if from every state $s$, either $E$ and $B$ win strategically, or $A$ and $M$ win strategically.

Positional strategies can be seen as a "pruning" of the arena of the game, by removing all the actions that have not been chosen. Formally: given two positional strategies $\sigma$ for $E$ and $\tau$ of $A$ we build a directed graph on $S$, that we also call $G(\sigma, \tau)$ as follows. For each state $s$:

- if $s \in S_E$, and $\sigma(s) = a$ then for all $s'$ such that $(s, a, s') \in BM$, we have an edge from $s$ to $s' G(\sigma, \tau)$
• if \( s \in S_A \), and \( \tau(s) = a \) then for all \( s' \) such that \( (s, a, s') \in BM \), we have an edge from \( s \) to \( s' \) in \( G(\sigma, \tau) \).

In the next section we are going to show the main technical result of this paper: that a finite BM game with parity objective is positionally determined. This means that either \( E \) and \( B \) win strategically with a positional strategy for \( E \), or \( A \) and \( M \) win strategically with a positional strategy for \( A \).

### 3.5 Don’t let your right hand know what your left hand is doing

The reader may wonder if the complex alternation of quantifier in the definition of structural win is necessary. She may have preferred the following definition:

We say that \( E \) and \( B \) win from a state \( s \) if from \( s \) there exists a strategy \( \sigma \) for \( E \) and a strategy \( \bar{\sigma} \) for \( B \) such that against any strategy \( \tau \) of A and any strategy \( \bar{\tau} \) for \( M \) the induced pre-play is a play whose flattening is in \( \Phi \).

If we considered this definition, we would give too much power to \( A \). \( A \) and \( M \) could join forces and win. Indeed, they could take advantage of the knowledge of \( B \)'s strategy, to infer which one of the Nature players has the lead. These ideas are presented in the following example.

**Example 12.** Consider the arena depicted in Figure 1 (which is essentially taken from [1]). The only state where there is a choice for an Arena player is \( s_0 \) and we suppose it belongs to \( A \). If in this state \( A \) always chooses to go visit \( s_1 \), he will lose. His only chance to win is to move the play in the bottom component of the arena. He does this by playing action \( b \) when in \( S_0 \) from time to time. if he was playing against a randomised nature, then he will almost-surely lose as he cannot always avoid state \( s_2 \).

However, if we suppose both \( A \) and \( M \) know the strategy of \( B \) i.e. they chose their strategy after \( B \), they can agree on a joint strategy as follows: \( A \) chooses to visit \( s_1 \) as long as \( B \) is playing. Once \( B \) passes his turn by playing \( \perp \), immediately \( A \) chooses the action \( b \) and then \( M \) helps him by choosing \( s_3 \). Thus, the play would visit \( s_3 \) infinitely often and \( A \) would win. Notice that \( A \) needs an infinite memory to implement this strategy. In particular, he needs to remember the entire history so he can know when does \( B \) passes his turn.

This shows that with the alternative definition, we cannot have a correspondence between the two notions of the game.

### 4 Main Theorem

In this section we prove the main result of the paper i.e. a transfer theorem between a stochastic game and the BM game induced by the following definition.

![Figure 1: A two-player BM game. The relation BM is defined by the edges of the graph. The dashed edges are the one where B and M can make important choices.](image-url)
Definition 13. Given a two-player stochastic parity games \( G \), we define \( \bar{G} \) to be the two-player BM game induced by \( G \) by simply defining the relation \( \text{BM}(s,a) \) as \( \text{Supp}(P(s,a)) \) for any state \( s \) and action \( a \).

4.1 A transfer theorem

Theorem 14 (Transfer theorem). On a finite arena with parity objective, Eve wins almost-surely the stochastic game, if and only if Eve wins the two-player parity BM game induced.

The proof of the above theorem uses the following theorem, interesting in itself.

Theorem 15. Two-player Parity BM games are positionaly determined.

Proof of Theorem 14. Assume that \( E \) wins from some state \( s \) in \( \bar{G} \). By Theorem 15 there exists a positional winning strategy of \( E \). Suppose toward a contradiction that \( E \) does not have an almost-surely winning strategy in the Stochastic game \( G \). By positional determinacy, there exists a positional strategy \( \tau \) that is positively winning for \( A \). Let us play this strategy against any positional strategy of \( E \). Let \( \sigma \) be a positional strategy for \( E \), the pair \( (\sigma, \tau) \) induces a Markov chain where the objective of \( A \) is satisfied positively. Hence, in the residual tree \( \bar{G}(\sigma, \tau) \) we have that \( M \) wins [8]. In particular, \( A \) and \( M \) have strategies to win against any positional strategy of \( E \) and any strategy of \( B \). Thus, we have concluded that all the positional strategies of \( E \) are not winning, contradiction.

Let us prove the other direction. Assume that \( E \) wins almost-surely, then by Theorem 4 she has a positional almost-surely winning strategy. Suppose toward a contradiction that \( E \) and \( B \) do not win in the game \( \bar{G} \). By positional determinacy, there exists a positional winning strategy \( \tau \) for \( A \). Let \( \sigma \) be a positional strategy for \( E \), in the residual tree \( \bar{G}(\sigma, \tau) \), \( M \) wins and because both \( \sigma \) and \( \tau \) are positional, it follows that \( M \) wins in the Markov chain induced \( G(\sigma, \tau) \). Using Theorem 1, it follows that \( A \) wins positively against any positional strategy of \( E \), a contradiction. \( \square \)

4.2 Winning States

We now turn our attention to the proof Theorem 15. In order to prove this theorem, we introduce a ew technical tools.

We also introduce the notion of lead. Intuitively, during a play of the game, we say that \( M \) (resp. \( B \)) has the lead if we are following the strategy of \( M \) (resp. \( B \)). Formally,

Definition 16. Let \( \mathcal{G}(\sigma, \tau) \) be a residual tree induced by the pair \( (\sigma, \tau) \) and let \( \rho \) be a finite play in \( \mathcal{G}(\sigma, \tau) \). Then, \( M \) has the lead along \( \rho \) if \( \rho \) contains an even number of occurrences of \( \bot \). Otherwise \( B \) has the lead.

An important structural notion is the one of subgames

Definition 17 (Subgame). Let \( Q \) be a subset of \( S \), \( Q \) induces a subgame \( \mathcal{G}(Q) \) if
\[
\forall q \in Q, \exists a \in A, \text{BM}(q,a) \subseteq Q.
\]

Definition 18 (Attractor). The attractor to \( U \) for Eve, denoted \( \text{Attr}_E(U,S) \subseteq S \), is the limit of the following sequence:
\[
\text{Attr}_E^0(U,S) = U,
\]
and for any $i \geq 0$

$$\text{Attr}_{E}^{i+1}(U,S) = \text{Attr}_{E}^{i}(U,S) \cup \{s \in S_{E} \mid \exists a \in A, BM(s,a) \cap \text{Attr}_{E}^{i}(U,S) \neq \emptyset\} \cup \{s \in S_{A} \mid \forall a \in A, BM(s,a) \cap \text{Attr}_{E}^{i}(U,S) \neq \emptyset\}.$$  

The set $\text{Attr}_{A}(U,S)$ is defined similarly for $E$. The states of $\text{Attr}_{E}(U,S)$ enjoy the following property:

**Proposition 19.** Let $s$ be in $\text{Attr}_{E}(U,S)$, there exists a strategy $\sigma$ for $E$ such that against any strategy $\tau$ for $A$, $B$ can reach $U$ in $G(\sigma,\tau)$ if he has the lead from $s$.

Obviously, the same claim holds for $A$ if stated accordingly.

**Definition 20 (Trap).** A trap for $A$ is a subset $Q$ of $S$, such that:

$$\forall q \in Q \cap S_{A}, \forall a \in A, BM(q,a) \subseteq Q,$$

$$\forall q \in Q \cap S_{E}, \exists a \in A, BM(q,a) \subseteq Q.$$  

Intuitively, a trap for $A$, $G(Q)$ is a subgame where $E$ has a strategy to force the play to never leave $Q$. We define traps for $E$ similarly.

**Lemma 21.** The complement of $\text{Attr}_{E}(U,S)$ is a trap for $E$.

That is if we let $V$ be the set $S \setminus \text{Attr}_{E}(U,S)$, then $E$ is trapped away from $U$ in $G(V)$. We will denote such a subgame by $\text{Away}_{E}(U,S)$.

Let $U$ be a subset of $U$ and $(\sigma, \tau)$ a couple of strategies, we define the set $\text{Safe}(U,S)$ For a subset of states $U$, we define the set $\text{Safe}(U,S)$ as follows:  

$$\text{Safe}(U,S) = \{s \in S \mid \exists \rho \in \text{Paths}(G(\sigma,\tau)), \text{First}(\rho) = s \wedge \text{Last}(\rho) \in \text{Attr}(U,S)\},$$

where $\text{First}(\rho)$ is the first element of the path $\rho$. Intuitively, this is the set of states in $S$ from where $B$ has a move to reach $U$. We denote its complement by $\overline{\text{Safe}(U,S)}$.

Finally, we define the set $S_{d}$ as the set of states with priority $d$.

![Figure 2](image)

(a) Largest priority $d$ is even  
(b) Largest priority $d$ is odd

**Figure 2:** Wining region in a two-player BM parity game

In Figure 2, we depicted the main ideas behind this construction. In particular we consider two cases: the first one (c.f. Figure 2a) is when the largest priority $d$ is even. In this case we claim that the winning set is given by the largest trap $T$ for $A$ such that the subgame $G(T)$ satisfies the following property; $E$ wins in $\text{Away}(S_{d},T)$. Basically, this follows from the fact that in this trap if $E$ applies the attraction strategy when in $\text{Attr}_{E}(S_{d},T)$ (c.f. gray area in 2a) and applies her winning strategy in the subgame $\text{Away}(S_{d},T)$, then we can show that under
any strategy τ of A, B has a strategy to force any play ρ in $G(\sigma, \tau)$ to enter infinitely often in Safe($S_d, T$) or to always stay in in Safe($S_d, T$). This is formalized in the proof of Proposition 22.

The second case is when the largest priority $d$ is odd (c.f. Figure 2b). We claim that the winning set of states is given by the largest trap $T$ such that $T$ can be partitioned into a sequence of subgames such that each one of them is winning for $E$. The key argument in this construction is that one can define a total order on the subgames obtained, such that a play can only escape a subgame to visit a subgame that is smaller, thus eventually any play eventually remains forever in a subgame that is winning for $E$. Details of the correctness are exposed in the proof of Proposition 23.

Let us first describe a procedure to obtain the set winning states for $E$ in a two-player parity BM game. This is done thanks to Algorithm 1. The inner loop that starts in Line 6 is the formalization of Figure 2a; it inductively constructs traps for $A$ and check that is satisfies the desired condition. The inner loop that starts in Line 16 constructs in an iterative manner the sequence of subgames.

**Algorithm 1** Procedure to compute the set of winning states in a two-player parity BM game

**Require:** Two-player parity BM game $G$ with state space $S$.

**Ensure:** Outputs the winning region for $E$.

1. Let $d$ be the largest priority of $G$.
2. Let $U_d$ be the set of states with priority $d$.
3. $U \leftarrow S$
4. if $d$ is even then
   5. repeat
   6. Compute $\text{Away}_E(U_d, U)$
   7. Compute $R$, the winning region for $E$ in the subgame $\text{Away}(U_d, U)$
   8. $R' \leftarrow U \setminus R$
   9. Compute $\text{Away}_A(R', U)$
   10. $U \leftarrow \text{Away}_A(R', U)$
   11. until $R' = \emptyset$
   12. return $U$
5. else if $d$ is odd then
6. $R' \leftarrow \emptyset$
7. repeat
8. Compute $\text{Away}_A(U_d, U)$
9. Compute $R$, the winning region for $E$ in the subgame $\text{Away}(U_d, U)$
10. Compute $\text{Attr}_E(R, U)$, the attractor of $E$ to $R$ in $G(U)$
11. $R' \leftarrow R' \cup \text{Attr}_E(R, U)$
12. $U \leftarrow \text{Away}_A(R, U)$
13. until $R = \emptyset$
14. return $R'$
8. end if

Notice that the above algorithm contains to inductive calls one in Line 7 and one in Line 17. These inductive calls are made on games where the top priority is smaller than $d$, therefore the recursion terminates. Indeed, if $d = 0$, $\text{Away}_E(U_d, U)$ is empty, and there is no further recursive call.
4.3 Correctness of Algorithm 1

We are going to present the most technical part of the paper. The proofs are inspired from the ones presented in [?] for stochastic games. While the structure of the proofs are similar, we underline that reasoning in terms of the strategies for Banach and Mazur is more intuitive.

We argue that this is one of the basic contribution of our approach: to concentrate the hard and numeric probabilistic reasoning in one place [8], and then deal more easily with the structural arguments.

Proposition 22. Let $\mathcal{G}$ be a parity game where the largest priority $d$ is even. All the states in $\mathcal{G}$ are winning for $E$ if and only if all the states in $\text{Away}(S_d, S)$ are winning for $E$ in $\mathcal{G}(\text{Away}(S_d, S))$.

Proof. Denote by $X$ the set $\text{Attr}_E(S_d, S)$ and by $Y$ the set $\text{Away}(S_d, S)$. Assume that $Y$ is winning for $E$ then $S$ is winning for $E$. Let $\sigma_X$ the strategy induced by the attractor set $X$, and $\sigma_Y$ the winning strategy for $E$ over $\mathcal{G}[Y]$. We define the positional strategy $\sigma$ as follows

$$
\sigma : S_E \rightarrow A
$$

$$
s \mapsto \begin{cases}
\sigma_X(s) & \text{if } s \in X, \\
\sigma_Y(s) & \text{if } s \in Y.
\end{cases}
$$

We show now that $\sigma$ is winning. Let $\tau$ be an arbitrary strategy for $A$, and consider the (potentially infinite) residual tree $\mathcal{G}(\sigma, \tau)$. We will show that in $\mathcal{G}(\sigma, \tau)$, $B$ wins.

Define the strategy $\bar{\sigma}$ for $B$ as follows:

$$
\bar{\sigma} : \mathcal{G}(\sigma, \tau) \rightarrow S \cup \bot
$$

$$
\rho \mapsto \begin{cases}
\rho \perp \text{ s.t. } \text{Last}(\rho') \in S_d & \text{if } \text{Last}(\rho) \in \text{Safe}(S_d, S), \\
\bar{\sigma}_Y(\rho) & \text{if } \text{Last}(\rho) \in \text{Safe}(S_d, S),
\end{cases}
$$

where $\rho$ is the longest suffix of $\rho$ that only contains states from $\text{Safe}(S_d, S)$.

Notice that $\text{Safe}(S_d, S)$ is a subset of $Y$ and that any play that starts in $\text{Safe}(S_d, S)$, remains in $\text{Safe}(S_d, S)$.

To see that $\bar{\sigma}$ is winning, let $\rho$ be a finite play in $\mathcal{G}(\sigma, \tau)$ that respects $\bar{\sigma}$. and assume that $\text{Last}(\rho)$ is in $\text{Safe}(S_d, S)$, then the subsequent play from $\bar{\rho}$ is winning because it respects $\bar{\sigma}_Y$ which is winning, the fact that Par is prefix independent entails that the subsequent play from $\rho$ is winning as well.

Now assume that $\text{Last}(\rho)$ is in $\text{Safe}(S_d, S)$, then

- if it is $B$’s turn, by playing according to $\sigma$ he will visit a state with priority $d$ and passes the lead in a state in $S_d$, thus visiting a state with priority $d$.

- If it is $M$’s turn, he either plays and passes the lead in $\text{Safe}(S_d, S)$, in which case $B$ can visit $S_d$ again, or he passes the lead in $\text{Safe}(S_d, S)$, in which case the previous case applies.

Finally, notice that $\sigma$ is positional, thus $E$ wins using a positional strategy.

Let us prove the converse, assume that all states in $\mathcal{G}$ are winning. By Lemma 20 we know that $Y$ is a trap for $E$, thus if $E$ does not win in $Y$ it cannot win in $\mathcal{G}$. $\square$

Proposition 23. Let $\mathcal{G}$ be a parity game where the largest priority $d$ is odd. All the states in $\mathcal{G}$ are winning for $E$, if and only if there exists a partition $\{Z_i\}_{1 \leq i \leq k}$ of $S$ and non empty sets $R_i, U_i$ for $i = 1, \ldots, k$, such that $U_1 = S$ and for all $1 \leq i \leq k$...
1) \( R_i \subseteq U_i \setminus (U_i)_d \) is a trap for \( A \) in \( G(U_i) \) and all \( R_i \) are winning in \( G(U_i) \);

2) \( Z_i = \text{Attr}_E(R_i, U_i) \);

3) \( U_{i+1} = U_i \setminus Z_i \).

**Proof.** Let \( \sigma_i \) be the winning strategy for \( E \) in \( G(R_i) \), let also \( \bar{\sigma}_i \) be the positional strategy induced by the attractor \( Z_i \). Define the following strategy \( \sigma \) for \( E \),

\[
\sigma : S_E \to A,
\]

\[
s \mapsto \begin{cases} 
\sigma_i(s) & \text{if } s \in R_i, \\
\bar{\sigma}_i(s) & \text{if } s \in Z_i. 
\end{cases}
\]

Let \( \tau \) be an arbitrary strategy for \( A \), and let us show that \( B \) wins in the (potentially infinite) residual tree \( G(\sigma, \tau) \). We define the following strategy \( \bar{\sigma} \) for \( B \):

\[
\bar{\sigma} : G(\sigma, \tau) \to S' \cup \perp
\]

\[
\bar{\rho} \mapsto \begin{cases} 
\rho' \perp \text{ s.t. } \text{Last}(\rho') \in R_i & \text{if } \text{Last}(\rho) \in Z_i, \\
\bar{\sigma}_i(\bar{\rho}) & \text{if } \text{Last}(\rho) \in R_i,
\end{cases}
\]

where \( \bar{\rho} \) is the longest suffix of \( \rho \) that only contains states from \( R_i \).

Let us show that \( \bar{\sigma} \) is winning in the residual tree \( G(\sigma, \tau) \). Let \( \rho \) be a finite play that respects \( \bar{\sigma} \). Since \( Z_i \) is a partition, it follows that \( \text{Last}(\rho) \) is in some \( Z_i \). Assume that it is \( M \)'s turn, then whatever action he plays, he will either \( i \) pass the lead in a state in \( Z_i \) or \( ii \) pass the lead in some \( Z_j \) such that \( i \neq j \).

If \( i \) holds, then \( B \) will move the play to \( R_i \) and all the subsequent plays will remain there since \( R_i \) is a trap for \( A \). Moreover, since \( R_i \) is winning for \( E \) over \( G(U_i) \) it follows that \( B \) wins.

If \( ii \) holds, then since there are only finitely many such \( j \) and because by construction we have \( i < j \leq k \), the play will settle in some \( Z_j \) and \( B \) can move the play to \( R_j \) and the previous arguments apply.

The converse implication now. Assume that \( E \) wins from every state in \( G_i \), we construct a partition of \( S \) that meets the requirements of the statement. Let \( X \) be the set \( \text{Attr}_A(S_d, S) \) and let \( Y \) be \( S' \setminus X \). We claim that since all states in \( G \) are winning for \( E \), then the winning region of \( E \) in \( G(Y) \) is non-empty. Assume toward a contradiction that it is not the case, then it means that all states in \( Y \) are loosing c.f. whenever \( E \) chooses a strategy, there exists a strategy \( \tau \) in \( G(Y) \) such that in the residual tree \( G(Y) \), \( B \) does not win). Let us show that this implies that all states in \( S \) are loosing for \( E \). Consider the set of states \( \text{Safe}(S_d, S) \) and \( \overline{\text{Safe}}(S_d, S) \) as defined in the proof of Proposition 22. As long as the current play is in \( \text{Safe}(S_d, S) \) then \( M \) can visit states in \( S_d \), if the play moves to \( \overline{\text{Safe}}(S_d, S) \) then \( M \) wins, a contradiction. Thus the winning region in \( G[Y] \) for \( E \) is non-empty (it is also a trap for \( A \), let this region be \( R_1 \), and \( Z_1 \) be \( \text{Attr}_E(R_1, S) \). If \( S \setminus Z_1 \) is empty, then we are done. Otherwise we repeat the construction over \( S \setminus Z_1 \). Since \( Z_1 \) is non-empty, this construction terminates.

\[\square\]

### 4.4 Positional strategies

**Proposition 24.** If \( E \) wins, then she has a positional winning strategy
Proof. We prove this by induction over the largest priority $d$ available in the game.

If $d = 0$, then any positional strategy is winning.

Assume now that $d$ is even, then from the proof of Proposition 22, we now that the winning strategy $\sigma$ uses two strategies; the attraction strategy $\sigma_X$ which is positional, and $\sigma_Y$ defined in a subgame with less priorities thus it is positional by induction.

If $d$ is odd, then thanks to the proof of Proposition 23 we know that each $\tilde{\sigma}_i$ is positional because it is induced from an attractor, and each $\sigma_i$ is positional again by induction. \qed

Proposition 25. If $A$ wins, then he has a positional winning strategy

Proof. By induction on the number of states. If $|S| = 1$ is then the result follows.

Assume it is the case for any game with state space $S$ We will again consider two cases; when the highest priority is even and when it is odd.

In former case, thanks to Algorithm 1-Line 9, we know that the winning region of $A$ is a finite union of sets of the form $\text{Attr}^A(R', U)$ where $A$ wins the parity game played in $R'$, thus since $M$ has the lead in the initial state, it suffices for $A$ to play a positional strategy to reach with the help of $M$, $R'$ is reached in one move. Once in $R'$ $A$ applies a positional winning strategy that exists by induction. Any play that respects this strategy will stay forever in $R'$ since it is a trap for $E$.

In the latter case, from Algorithm 1-Line 16, we know that the winning region is a finite union of sets of the form $\text{Attr}^A(U_d, U)$ such that $i)$ $U$ induces a subgame and $ii)$ $d$ is the largest priority in $U$. Thus, $A$ can always apply an attraction strategy and with the help of $M$ continuously visit states in $U_d$. This strategy is clearly positional. \qed

These two last propositions yield Theorem 15.

We just mention here that Theorem 14 can be easily extended as usual from parity objectives to $\omega$-regular objectives, by just making the product of the game and the parity automaton that accepts the objective. We omit the straightforward details here.

5 Conclusions

What we have shown in this paper, while technically non trivial, can still be considered only as a "sanity" check - the Banach-Mazur game can replace the probabilities in a suitable setting. While the structure of the proofs mimics the corresponding probabilistic proofs, the advantage is that all probabilistic reasoning is formally "factored out", so that following the proof becomes easier.

However, we consider that our result can be made to go further. In particular, we note that in this paper, we use the fact that stochastic parity games are positionaly determined. This result is needed in our proof of Theorem 14. What we would like to achieve is to use positional determinacy of EBAM games, to prove positional determinacy of stochastic games. This what we set out to do next.

Another contribution of this paper is to simplify the presentation of [1]. Notably there the authors introduced the heavy concept of "move tree" which we have simplified here.
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