HARMONIC MEAN, RANDOM POLYNOMIALS AND STOCHASTIC MATRICES

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Abstract. Motivated by a problem in learning theory, we are led to study the dominant eigenvalue of a class of random matrices. This turns out to be related to the roots of the derivative of random polynomials (generated by picking their roots uniformly at random in the interval [0,1], although our results extend to other distributions). This, in turn, requires the study of the statistical behavior of the harmonic mean of random variables as above, and that, in turn, leads us to delicate question of the rate of convergence to stable laws and tail estimates for stable laws.

Introduction

The original motivation for the work in this paper was provided by the first-named author’s research in learning theory, specifically in various models of language acquisition (see [KNN2001, NKN2001, KN2001]) and more specifically yet by the analysis of the speed of convergence of the memoryless learner algorithm. The setup is described in some detail in Section 4.1; here we will just recall the essentials. There is a collection of concepts $R_1, \ldots, R_n$ and words which refer to these concepts, sometimes ambiguously. The teacher generates a stream of words, referring to the concept $R_1$. This is not known to the student, but he must learn by, at each steps, guessing some concept $R_i$ and checking for consistency with the teacher’s input. The memoryless learner algorithm consists of picking a concept $R_i$ at random, and sticking by this choice, until it is proven wrong. At this point another concept is picked randomly, and the procedure repeats. It is clear that once the student hits on the right answer $R_1$, this will be his final answer, so the question is then:

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How quickly does this method converge to the truth?

Since the method is memoryless, as the name implies, it is clear that the learning process is a Markov chain, and as is well-known the convergence rate is determined by the gap between the top (Perron-Frobenius) eigenvalue and the second largest eigenvalue. However, we are also interested in a kind of a generic behavior, so we assume that the sizes of overlaps between concepts are random, with some (sufficiently regular) probability density function supported in the interval $[0, 1]$, and that the number of concepts is large. This makes the transition matrix random (that is, the entries are random variables) – the precise model is described in Section 4.1. The analysis of convergence speed then comes down to a detailed analysis of the size of the second largest eigenvalue and also of the properties of the eigenspace decomposition (the contents of Section 4.3.) Our main results for the original problem (which is presented in Section 4.4) can be summarized in the following:

**Theorem A.** Let $N\Delta$ be the number of steps it takes for the student to have probability $1 - \Delta$ of learning the concept. Then we have the following estimates for $N\Delta$:

- if the distribution of overlaps is uniform, or more generally, the density function $f(1 - x)$ at 0 has the form $f(x) = c + O(x^\delta)$, $\delta, c > 0$, then there exist positive constants $C_1, C_2$ such that

  $$\lim_{n \to \infty} P\left( C_1 < \frac{N\Delta}{|\log \Delta|n \log n} < C_2 \right) = 1,$$

- if the probability density function $f(1 - x)$ is asymptotic to $cx^\beta + O(x^{\beta+\delta})$, $\delta, \beta > 0$, as $x$ approaches 0, then we have

  $$\lim_{n \to \infty} P\left( C'_1 < \frac{N\Delta}{|\log \Delta|n} < C'_2 \right) = 1$$

  for some positive constants $C'_1$ and $C'_2$,

- if the asymptotic behavior is as above, but $-1 < \beta < 0$, then

  $$\lim_{x \to \infty} P\left( \frac{1}{x} < \frac{N\Delta}{|\log \Delta|n^{1/(1+\beta)}} < x \right) = 1.$$

It should be said that our methods give quite precise estimates on the constants in the asymptotic estimate, but the rate of convergence is rather poor – logarithmic – so these precise bounds are of limited practical importance.

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1Here and throughout this section, the reference to the relevant theorem (lemma) in the main body of the paper is given in square brackets.
Notation. We shall use the notation $a \asymp b$ to mean that $a$ is asymptotically the same as $b$. We say that $a \sim b$ if $a$ and $b$ have the same order of growth (in other words, there exist constants $c_1, c_2, d_1, d_2$, with $c_1, c_2 > 0$, so that $c_1 a + d_1 \leq b \leq c_2 a + d_2$). In addition we denote the expectation of a random variable $x$ by $E(x)$.

Eigenvalues and polynomials. In order to calculate the convergence rate of the learning algorithm described above, we need to study the spectrum of a class of random matrices. The matrix $T = (T_{ij})$ is an $n \times n$ matrix with entries (see Section 4.1):

$$T_{ij} = \begin{cases} a_i, & i = j, \\ \frac{1-a_i}{n-1}, & \text{otherwise}. \end{cases}$$

Let $B = \frac{n-1}{n} (I - T)$, so that the eigenvalues of $T$, $\lambda_i$, are related to the eigenvalues of $B$, $\mu_i$, by $\lambda_i = 1 - n/(n - 1) \mu_i$. In Section 4.2 we show the following result:

Lemma B. [4.7] Let $p(x) = (x-x_1) \ldots (x-x_n)$, where $x_i = 1 - a_i$. Then the characteristic polynomial $p_B$ of $B$ satisfies:

$$p_B(x) = \frac{x}{n} \frac{dp(x)}{dx}.$$  

Lemma B brings us to the following question:

**Question 1:** Given a random polynomial $p(x)$ whose roots are all real, and distributed in a prescribed way, what can we say about the distribution of the roots of the derivative $p'(x)$?

And more specifically, since the convergence behavior of $T^N_t$ is controlled by the top of the spectrum:

**Question 1′:** What can we say about the distribution of the smallest root of $p'(x)$, given that the smallest root of $p(x)$ is fixed?

For Question 1′ we shall clamp the smallest root of $p(x)$ at 0. Letting $H_{n-1}$ be the harmonic mean of the other roots of $p(x)$ (which are all greater than zero with probability 1), our first observation will be

Lemma C. [3.3] The smallest root $\mu_*$ of $p'(x)$ satisfies:

$$\frac{1}{2} H_{n-1} \leq (n - 1) \mu_* \leq H_{n-1}.$$  

We will henceforth assume that the roots of the polynomial $p(x)$ are a sample of size $\deg p(x)$ of a random variable, $x$, distributed in the interval $[0, 1]$. In this stochastic setting, it will be shown that $(n - 1) \mu_*$
tends to the harmonic mean of the non-zero roots of \( p \) with probability 1, when \( n \) is large. It then follows that the study of the distribution of \( \mu_* \) entails the study of the asymptotic behavior of the harmonic mean of a sample drawn from a distribution on \([0, 1]\).

**Statistics of the harmonic mean.** In view of the long and honorable history of the harmonic mean, it seems surprising that its limiting behavior has not been studied more extensively than it has. Such, however, does appear to be the case. It should also be noted that the arithmetic, harmonic, and geometric means are examples of the “conjugate means”, given by

\[
m_F(x_1, \ldots, x_n) = F^{-1}\left(\frac{1}{n} \sum_{i=1}^{n} F(x_i)\right),
\]

where \( F(x) = x \) for the arithmetic mean, \( F(x) = \log(x) \) for the geometric mean, and \( F(x) = 1/x \) for the harmonic mean. The interesting situation is when \( F \) has a singularity in the support of the distribution of \( x \), and this case seems to have been studied very little, if at all. Here we will devote ourselves to the study of harmonic mean.

Given \( x_1, \ldots, x_n \) — a sequence of independent, identically distributed in \([0, 1]\) random variables (with common probability density function \( f \)), the nonlinear nature of the harmonic mean leads us to consider the random variable

\[
X_n = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{x_i}.
\]

Since the variables \( 1/x_i \) are easily seen to have infinite expectation and variance, our prospects seem grim at first blush, but then we notice that the variable \( 1/x_i \) falls straight into the framework of the “stable laws” of Lévy – Khintchine ([FellerV2]). Stable laws are defined and discussed in Section 1.2. Which particular stable law comes up depends on the distribution function \( f(x) \). If we assume that

\[
f(x) \propto cx^\beta,
\]
as \( x \to 0 \) (for the uniform distribution \( \beta = 0, \quad c = 1 \)), we have

**Theorem D.** If \( \beta = 0 \), then let \( Y_n = X_n - \log n \). The variables \( Y_n \) converge in distribution to the unbalanced stable law \( G \) with exponent \( \alpha = 1 \). If \( \beta > 0 \), then \( X_n \) converges in distribution to \( \delta(x - \mathcal{E}) \), where \( \mathcal{E} = \mathbb{E}(1/x) \), and \( \delta \) denotes the Dirac delta function. If \(-1 < \beta < 0\), then \( n^{1-1/(1+\beta)}X_n \) converges in distribution to a stable law with exponent \( \alpha = 1 + \beta \).
The above result points us in the right direction, since it allows us to guess the form of the following results ($H_n$ is the harmonic mean of the variables):

**Theorem E.** [1.2, 1.3] Let $H_n = 1/X_n$ and $\beta = 0$. Then there exists a constant $C_1$ such that

$$\lim_{n \to \infty} E(H_n \log n) = C_1.$$ 

**Theorem F.** [2.1] Suppose $\beta > 0$, let $y = 1/x$, and let $E$ be the mean of the variable $y$. Then

$$\lim_{n \to \infty} E(EH_n) = 1.$$ 

Finally,

**Theorem G.** [2.4] Suppose $\beta < 0$. Then there exists a constant $C_2$ such that

$$E(H_n/n^{1-1/(1+\beta)}) = C_2.$$ 

**Theorem H** (1.4, 2.2, Law of large numbers for harmonic mean). Let $\beta = 0$ and let $a > 0$. Then

$$\lim_{n \to \infty} P(|H_n \log n - C_1| > a) = 0,$$

where $C_1$ is as in the statement of Theorem E. If $\beta > 0$, and $E$ is as in the statement of Theorem F, then

$$\lim_{n \to \infty} P(|H_n - E| > a) = 0.$$ 

The proofs of the results for $\beta = 0$ require estimates of the speed of convergence in Theorem D. The speed of convergence results we obtain (in Section B) are not best possible, but the arguments are simple and general. The estimates can be summarized as follows:

**Theorem I.** [B.1] Assume $\beta = 0$. Let $g_n$ be the density associated to $X_n - \log n$, and let $g$ be the probability density of the unbalanced stable law with exponent $\alpha = 1$. Then we have (uniformly in $x$):

$$g_n(x) = g(x) + O(\log^2 n/n).$$

In addition to the laws of large numbers we have the following limiting distribution results:

**Theorem J.** [1.5, 1.6] For $\alpha = 1$, the random variable $\log n(H_n \log n - C_1)$ converges to a variable with the distribution function $1 - G(-x/C_1^2)$, where $G$ is the limiting distribution (of exponent $\alpha = 1$) of variables $Y_n = X_n - c \log n$ and $C_1 = 1/c$. 


Theorem K. [2.3] For $\alpha > 1$, the random variable $n^{1-1/\alpha}(H_n - \frac{1}{n})$ converges in distribution to a variable with distribution function $1 - G(-xe^2)$, where $G$ is the unbalanced stable distribution of exponent $\alpha$.

Theorem L. [2.15] For $0 < \alpha < 1$, the random variable $H_n/n^{1-1/\alpha}$ converges in distribution to the variable with distribution function $1 - G(1/x)$, where $G$ is the unbalanced stable distribution of exponent $\alpha$.

The paper is organized as follows. In Section 1 we study some statistical properties of a harmonic mean of $n$ variables and in particular, find the expected value of its mean as $n \to \infty$. In Section 3 we explore the connection between the harmonic mean and the smallest root of the derivative of certain random polynomials. In Section 4 we uncover the connection between the rate of convergence of the memoryless learner algorithm, eigenvalues of certain stochastic matrices and the harmonic mean. The more technical material can be found in the Appendix. In Section A of the Appendix we present an explicit derivation of the stable law for a particular example with $\alpha = 1$. In Section B we evaluate the rate of convergence of the distribution of the inverse of the harmonic mean to its stable law.

1. Harmonic mean

1.1. Preliminaries. Let $x_1, \ldots, x_n$ be positive real numbers. The harmonic mean, $H_n$, is defined by

$$\frac{1}{H_n} = \frac{1}{n} \left( \sum_{i=1}^{n} \frac{1}{x_i} \right).$$

Let $x_1, \ldots, x_n$ be independent random variables, identically uniformly distributed in $[0, 1]$. We will study statistical properties of their harmonic mean, $H_n$, with emphasis on limiting behavior as $n$ becomes large.

We will use auxiliary variables $X_n$ and $Y_n$, defined as

$$X_n = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{x_i} = \frac{1}{H_n}, \quad Y_n = X_n - \log n,$$

and also variables $y_i = \frac{1}{x_i}$. The distribution of $y_i$ is easily seen to be given by

$$\bar{F}(z) = P(y_i < z) = \begin{cases} 0 & z < 1 \\ 1 - \frac{1}{z} & \text{otherwise}. \end{cases}$$

A quick check reveals that $y_i$ has infinite mean and variance, so the Central Limit Theorem is not much help in the study of $X_n$. Luckily,
however, $X_n$ converges to a stable law, as we shall see. A very brief introduction to stable laws is given in the next section.

1.2. Stable limit laws. Consider an infinite sequence of independent identically distributed random variables $y_1, \ldots, y_n, \ldots$, with some probability distribution function, $F$. Typical questions studied in probability theory are the following.

Let $S_n = \sum_{j=1}^{n} y_j$. How is $S_n$ distributed? What can we say about the distribution of $S_n$ as $n \to \infty$?

The best known example is one covered by the Central Limit Theorem of de Moivre-Laplace: if $F$ has finite mean $E$ and variance $\sigma^2$, then $(S_n - nE)/(\sqrt{n}\sigma)$ converges in distribution to the normal distribution (see, e.g., [FellerV2]). In view of this result, one says that the variable $X$ belongs to the domain of attraction of a non-singular distribution $G$, if there are constants $a_1, \ldots, a_n, \ldots$ and $b_1, \ldots, b_n, \ldots$ such that the sequence of variables $Y_n \equiv a_nS_n - b_n$ converges in distribution to $G$. It was shown by Lévy and by Khintchine that having a domain of attraction constitutes severe restrictions on the distribution as well as the norming sequences $\{a_n\}$ and $\{b_n\}$. To wit, one can always pick $a_n = n^{-1/\alpha}$, $0 < \alpha \leq 2$. It turns out that $\alpha$ is determined by the limiting behavior of the distribution $F$ so that

$$
\lim_{|x|\to\infty} \frac{1}{|x|^{\alpha+1}} \frac{dF(x)}{dx} = \begin{cases} 
Cp, & x > 0 \\
Cq, & x < 0,
\end{cases}
$$

(4)

where $p + q = 1$. In that case, $G$ is called a stable distribution of exponent $\alpha$. Note that the case when $\alpha = 2$ corresponds to the Central Limit Theorem. If the variable $y$ belongs to the domain of attraction of a stable distribution of exponent $\alpha > 1$, then $y$ has a finite expectation $E$; just as in the case $\alpha = 2$, we can choose $b_n = n^{1-1/\alpha}E$. When $\alpha < 1$, the variable $y$ does not have a finite expectation, and it turns out that we can take $b_n \equiv 0$; for $\alpha = 1$, we can take $b_n = c\log n$, where $c$ is a constant depending on $F$. Thus, the normal distribution is a stable distribution of exponent 2 (and it is also unique, up to scale and shift).

This is one of the few cases where we have an explicit expression for the density of a stable distribution; in other cases we only have expressions for their characteristic functions. The characteristic function $\Psi(k)$ of a distribution function $G(x)$ is defined to be $\int_{-\infty}^{\infty} \exp(\imath k x) dG(x)$, that is, as the Fourier transform of the density function. Levy and Khinchine showed that the characteristic functions of stable distributions can be
parameterized as follows:

\[
\log \Psi(k) = \begin{cases} 
C \frac{\Gamma(3-\alpha)}{\alpha(\alpha-1)} \left[ \cos \frac{\pi\alpha}{2} - \text{sign} \, k \left( p - q \right) \sin \frac{\pi\alpha}{2} \right] |k|^\alpha, & \alpha \neq 1 \\
- C \left[ \frac{1}{2} \pi - \text{sign} \, k \left( p - q \right) \log |k| \right] |k| + \text{const}, & \alpha = 1,
\end{cases}
\]

where the constants \( C, p \) and \( q \) can be defined by the following limits:

\[
\lim_{x \to \infty} \frac{1 - \mathcal{F}(x)}{1 - \mathcal{F}(x) + \mathcal{F}(-x)} = Cp,
\]

\[
\lim_{x \to \infty} \frac{\mathcal{F}(-x)}{1 - \mathcal{F}(x)} = Cq,
\]

and \( p + q = 1 \); the quantities \( p, q \) and \( C \) here are the same as in formula (4). We will say that the stable law is unbalanced if \( p = 1 \) or \( q = 1 \) above. This will happen if the support of the variable \( y \) is positive – this will be the only case we will consider in the sequel.

If \( \chi(k) \) is the characteristic function of our variable \( y \), then the characteristic function of the stable distribution, \( \Psi(k) \), satisfies

\[
\Psi(k) = \lim_{n \to \infty} \Psi_n(k),
\]

where

\[
\Psi_n = \exp(-ib_nk)\chi^n(a_nk).
\]

**Notation.** Throughout the paper we will use the notation \( G_n \) for the distribution function of the random variable \( Y_n \) and \( G \) for the corresponding stable distribution; \( g_n \) for the density of \( Y_n \) and \( g \) for the stable density; \( \Psi_n \) for the characteristic function of \( G_n \) and \( \Psi \) for the characteristic function of the stable distribution.

1.3. Limiting distribution of the harmonic mean \( H_n \) for \( \alpha = 1 \).

Let us go back to the example of Section 1.1, where the random variables \( x_i \) were uniformly distributed in \([0, 1]\). We will study the limiting behavior of the distribution of quantities related to \( S_n = \sum_{j=1}^n 1/x_j \).

The distribution function of the variables \( y_i = 1/x_i \) is given by (3), which implies \( p = 1 \) and \( q = 0 \), see formulas (3) and (4). From the behavior of the tails of the distribution \( \mathcal{F} \) we see that \( \alpha = 1 \), so the norming sequence should be taken \( a_n = 1/n, b_n = \log n \). Then the distribution \( G_n \) of the variable \( Y_n \) (given by equation (2)) converges to a stable distribution \( G \). The explicit form of the corresponding stable density, \( g \), can be obtained by taking the Fourier transform of the characteristic function \( \Psi \) in formula (3) (see also FellerV2, Chapter
A direct derivation of formula (10) is given in Appendix A:

\[ g(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iky} e^{-|k|/2 - i|k|/2} e^{-ik\log|k| - \gamma} \, dk, \]

where \( \gamma \) is Euler’s constant.

Remark 1.1. Results of this section can be easily generalized to any density \( f \) of the random variable \( x \) which satisfies \( \lim_{x \to 0} f(x) > 0 \). For any such distribution we obtain a stable law with exponent \( \alpha = 1 \).

Next, let us analyze the limiting behavior of the harmonic mean, \( H_n \).

To begin we will compute the behavior of the mean of \( H_n \),

\[ E(H_n) \approx \int_{-\infty}^{\infty} \frac{1}{x + \log n} dG_n. \]

It turns out that for this, we do not need the explicit form of the stable distribution \( G \) of \( Y_n \); it is enough to use the following information about the behavior of the tails:

\[ \lim_{x \to -\infty} xG(x) = 0, \quad \lim_{x \to \infty} x(1 - G(x)) = 1. \]

These equations can be obtained from (3) and (4), see XVII.5 [FellerV2]. The exact asymptotics of the tails are computed in [BLLin1973], Chapter 2.

Let us pick a large cutoff \( c_n \); we take \( c_n \) to tend to \( \infty \), but in such a way that \( c_n = o(\log n) \), e.g. \( c_n = \sqrt{\log n} \), and rewrite equation (11) as

\[ E(H_n) = I_1(c_n) + I_2(c_n) + I_3(c_n), \]

where

\[ I_1(c_n) = \int_{-\infty}^{-c_n} \frac{1}{x + \log n} dG_n, \]

\[ I_2(c_n) = \int_{-c_n}^{c_n} \frac{1}{x + \log n} dG_n, \]

\[ I_3(c_n) = \int_{c_n}^{\infty} \frac{1}{x + \log n} dG_n. \]

We estimate these integrals separately, using equation (12), the observation that \( G_n(c_n) = 0 \) for \( c_n < 1 - \log n \) and the estimate on the convergence speed of \( G_n \) to \( G \) as obtained in Section 3. Since we are integrating over an interval of length bounded by a constant times \( \log n \), it is more than sufficient for the speed of convergence to the stable density to be of order \( \log^2 n/n \), see Theorem 3.1. Integrating by parts,
we obtain
\[ I_1(c_n) \propto \int_{1-log n}^{-c_n} \frac{1}{x + \log n} dG = \frac{G(-c_n)}{-c_n + \log n} + \int_{1-log n}^{-c_n} \frac{G(x)}{(x + \log n)^2} dx. \] 

The first term is seen to be \( o \left( \frac{1}{c_n(-c_n + \log n)} \right) \), so given our choice of \( c_n \), it is \( o(1/\log n) \). The integral in the right hand side of equation (17) is asymptotically (in \( c_n \)) smaller than
\[ \int_{1-log n}^{-c_n} \frac{1}{-x(x + \log n)^2} dx = O \left( \frac{1}{\log n} \right), \]
and therefore, \( I_1 = o(1/\log n) \). To show that \( I_3 = o(1/\log n) \), we note that the integrand is dominated by \( 1/\log n \), while \( \lim_{c_n \to \infty} \int_{c_n}^{\infty} dG_n = 0 \).

For \( I_2 \) we have the trivial estimate (since \( 1/(x + \log n) \) is monotonic for \( x > -\log n \)):
\[ \frac{G(c_n) - G(-c_n)}{\log n + c_n} \leq I_2 \leq \frac{G(c_n) - G(-c_n)}{\log n - c_n}, \]
from which it follows that \( \lim_{n \to \infty} I_2 \log n = 1 \). To summarize, we have shown

**Theorem 1.2.** For the variable \( x \) uniformly distributed in \([0, 1] \), \( \lim_{n \to \infty} E(\log n H_n) = 1 \).

**Remark 1.3.** For a general density of \( x \) satisfying \( \lim_{x \to 0} f(x) > 0 \), we have \( Y_n = 1/H_n - c \log n \), and Theorem 1.2 generalizes to \( \lim_{n \to \infty} E(\log n H_n) = C_1 = 1/c \).

In addition, we have the following weak law of large numbers for \( H_n \):

**Theorem 1.4.** For any \( \epsilon > 0 \), \( \lim_{n \to \infty} P(|H_n \log n - 1| > \epsilon) = 0 \).

**Proof.** Note that
\[ P(H_n \log n - 1 > \epsilon) = P(X_n < \frac{\log n}{1 + \epsilon}), \]
while
\[ P(H_n \log n - 1 < -\epsilon) = P(X_n > \frac{\log n}{1 - \epsilon}). \]

Both probabilities decrease roughly as \( 1/(\epsilon \log n) \) using the estimates (12).

The above weak law indicates that if we are to hope for a limiting distribution for \( H_n \), we need to normalize it differently than by multiplying by \( \log n \). An examination of the argument above shows that
the appropriate normalization is $H_n \log^2 n - \log n$. Indeed, we have the following

**Theorem 1.5.** The distributions of the random variable $H_n \log^2 n - \log n$ converges to the variable with distribution function $1 - G(-x)$, where $G$ is the limiting (stable) distribution (of exponent $\alpha = 1$) of variables $Y_n = X_n - \log n$.

*Proof.* The proof is quite simple. Indeed, since

$$H_n = \frac{1}{Y_n + \log n},$$

we write

$$P(H_n \log^2 n - \log n < a) = P(\frac{\log^2 n}{Y_n + \log n} - \log n < a) = P(\frac{-Y_n \log n}{Y_n + \log n} < a).$$

Since $Y_n + \log n > 0$, we can continue:

$$P(\frac{-Y_n \log n}{Y_n + \log n} < a) = P(\frac{-a \log n}{a + \log n} < Y_n) \rightarrow P(Y_n > -a) \rightarrow 1 - G(-a),$$

where we have assumed that $n$ is large enough that $a + \log n > 0$. ☐

**Remark 1.6.** For a general density of $x$ satisfying $\lim_{x \to 0} f(x) > 0$, Theorem 1.4 can be generalized in the following way: the random variable $\log n(H_n \log n - C_1)$ converges in distribution to a variable with distribution function $1 - G(-x/C_2^1)$, where $G$ is the limiting distribution (of exponent $\alpha = 1$) of variables $Y_n = X_n - c \log n$ and $C_1 = 1/c$.

Theorem 1.5 could be viewed as a kind of an extension of Zolotarev’s identity (see [FellerV2, Chapter XVII, Section 6] and [IbLi 1971, Theorem 2.3.4]):

Let $\alpha > 1$. Then the density $p(x; \alpha)$ of the unbalanced stable law satisfies

$$xp(x; \alpha) = x^{-\alpha} p(x^{-\alpha}; \frac{1}{\alpha}).$$

(20)

**2. Limiting distribution of the harmonic mean $H_n$ for $\alpha \neq 1$**

Let us consider other types of the distribution of the variable $x$ and study the limiting behavior of the corresponding harmonic mean. If the density $f$ of the variable $x$ behaves as

$$f(x) \sim x^\beta$$

(21)
near \( x = 0 \), then we have for the density of \( y = 1/x \): \( d\mathcal{F}(y)/dy \sim |y|^{-(\beta+2)} \) as \( |y| \to \infty \), which gives \( \alpha = \beta + 1 \) as the exponent of the stable law. Using the material of Section 1.2 and the definition of \( H_n \), we obtain:

\[
Y_n = \begin{cases} 
\frac{n^{1-1/\alpha}}{H_n} & 0 < \alpha < 1, \\
\frac{1}{n^{1-1/\alpha}} \left( \frac{1}{H_n} - \mathcal{E} \right) & \alpha > 1
\end{cases}
\]

(here \( \mathcal{E} \equiv E(y) \)).

2.1. The case \( \beta > 0 \).

**Theorem 2.1.** If \( \beta > 0 \), then \( \lim_{n \to \infty} E(H_n) = 1/\mathcal{E} \).

**Proof.** If \( \beta > 0 \) (i.e. \( \alpha > 1 \)), then we have

\[
\lim_{n \to \infty} E(H_n) = \lim_{n \to \infty} \mathbf{E} \left( \frac{Y_n}{n^{1-1/\alpha}} + \mathcal{E} \right)^{-1} = \lim_{n \to \infty} \int_{-\infty}^{\infty} \frac{dG_n}{x^{1-1/\alpha} + \mathcal{E}} = \frac{1}{\mathcal{E}}.
\]

There is also the following Weak Law of Large Numbers:

**Theorem 2.2.** If \( \beta \) in equation (21) is positive, then

\[
\lim_{n \to \infty} P(\left| H_n - \frac{1}{\mathcal{E}} \right| > \epsilon) = 0.
\]

**Proof.** We have

\[
P \left( \left| H_n - \frac{1}{\mathcal{E}} \right| > \epsilon \right) = P \left( \frac{1}{\mathcal{E}} - \frac{1}{n^{1-1/\alpha}} > \epsilon \right)
\]

\[
= P \left( Y_n > n^{1-1/\alpha} \frac{\mathcal{E}^2 \epsilon}{1-\mathcal{E}} \right).
\]

Since \( \alpha = \beta + 1 > 1 \), then in the limit \( n \to \infty \) this quantity tends to zero.

In fact, we can use a manipulation akin to that in the proof of Theorem 1.5 to show:

**Theorem 2.3.** The random variable \( n^{1-1/\alpha} (H_n - \frac{1}{\mathcal{E}}) \) converges in distribution to a variable with distribution function \( 1 - G(-x\mathcal{E}^2) \), where the distribution \( G \) is the unbalanced stable distribution of exponent \( \alpha \).

**Proof.**

\[
P \left( n^{1-1/\alpha} (H_n - \frac{1}{\mathcal{E}}) < a \right) = P \left( n^{1-1/\alpha} \left( \frac{1}{\mathcal{E}} + \frac{1}{Y_n n^{1-1/\alpha}} - \frac{1}{\mathcal{E}} \right) < a \right)
\]

\[
= P \left( \frac{Y_n}{\mathcal{E} + Y_n n^{1-1/\alpha}} > -a \mathcal{E} \right).
\]
The quantity $E + Y_n n^{1/\alpha - 1}$ is positive because $Y_n \geq n^{1-1/\alpha} (1 - E)$, so we can write

\[ P \left( \frac{Y_n}{E + Y_n n^{1/\alpha - 1}} > -aE \right) = P \left( Y_n > \frac{-aE^2}{1 + aE n^{1/\alpha - 1}} \right) \rightarrow P \left( Y_n > -aE^2 \right), \]

where we have assumed that $n$ is large enough that $1 + aE n^{1/\alpha - 1} > 0$.

2.2. The case $1 < \beta < 0$.

**Theorem 2.4.** For $-1 < \beta < 0$, there is a constant $C_2$ such that

\[ E \left( \frac{H_n}{n^{1/(\beta+1)}} \right) = C_2. \]

**Proof.** For $-1 < \beta < 0$ (or $0 < \alpha < 1$) we would like to reason as follows:

\[ \lim_{n \to \infty} E \left( \frac{H_n}{n^{1-1/\alpha}} \right) = \lim_{n \to \infty} E \left( \frac{1}{Y_n} \right) = \lim_{n \to \infty} \int_{-\infty}^{\infty} \frac{dG_n}{x} = \int_{-\infty}^{\infty} \frac{dG}{x}. \tag{23} \]

Since the function $1/x$ is unbounded, the weak convergence of the distributions $G_n$ to the stable distribution $G$ is not enough to justify the last step equality in the sequence (23) above. To justify it we need the following Lemmas:

**Lemma 2.5.** Let $y_1, \ldots, y_n$ be positive independent identically distributed random variables. Let $S_n = \sum_{i=1}^{n} y_i$. Then,

\[ P(S_n < a) \leq \left[ P(y_1 < a) \right]^n. \]

**Proof.** Note that $S_n \geq \max_{1 \leq i \leq n} y_i$. \hfill \qed

Now, in our case

\[ G_n(a) = P \left( \sum_{i=1}^{n} y_i < an^{\frac{\alpha}{2}} \right) \leq \left[ P \left( y_1 < an^{\frac{\alpha}{2}} \right) \right]^n = \left[ P \left( x_1 > \frac{1}{an^{\frac{\alpha}{2}}} \right) \right]^n, \]

where the inequality follows from Lemma 2.5 (and recall that $x_i = 1/y_i$).

The probability $P(x_1 > b)$ has the following properties:

A) $P(x_1 > b) = 0$ for $b \geq 1$,

B) $1 - P(x_1 > b) \sim cb^\alpha$, for $b \ll 1$,

C) $P(x_1 > b) < 1$ for $b > 0$.

**Lemma 2.6.** $G_n(a) = 0$ for $a \leq n^{-\frac{1}{\alpha}}$.

**Proof.** Follows from the definition of $G_n$ and Property A. \hfill \qed
Lemma 2.7. There exists a \( b_0 \) such that \( 1 - P(x_1 > b) < 2c'b^\alpha \) for all \( b < b_0 \) for some \( c' > 0 \).

Proof. This follows from Property B, with \( c' = 2c \). \( \square \)

Lemma 2.8. If \( an^{1/\alpha} > 1/b_0 \) (\( b_0 \) as in the statement of Lemma 2.7), then
\[
G_n(a) \leq (1 - c'a^{-1/\alpha}n^{-1})n \sim \exp(-c'a^{-1/\alpha}).
\]

Proof. Follows from Lemma 2.5. \( \square \)

Lemma 2.9. \( G(1/(b_0n^{1/\alpha})) \leq [P(x_1 > b_0)]^n \).

Proof. Follows immediately from Lemma 2.5. \( \square \)

Now we write:
\[
\int_0^\infty \frac{dG_n}{x} = \left( \int_0^{n^{-1/\alpha}} + \int_{n^{-1/\alpha}}^{n^{-1/\alpha}/b_0} + \int_{n^{-1/\alpha}/b_0}^C + \int_C^\infty \right) \frac{dG_n}{x}.
\]

To analyze the above decomposition, we should first belabor the obvious:

Lemma 2.10.
\[
\int_a^b \frac{dG_n}{x} = \frac{G_n(b)}{b} - \frac{G_n(a)}{a} + \int_a^b \frac{G_n}{x^2}dx.
\]

Proof. Integration by parts. \( \square \)

Lemma 2.11. \( I_0(n) = 0 \).

Proof. The integrand vanishes in the interval by Lemma 2.6. \( \square \)

Lemma 2.12.
\[
\lim_{n \to \infty} I_1(n) = 0.
\]

Proof. By Lemma 2.9, \( G_n < [P(x_1 > b_0)]^n \). The result follows by integration by parts (Lemma 2.10). \( \square \)

Lemma 2.13.
\[
\lim_{n \to \infty} I_2(n) \leq \frac{\exp\left(C^{-1/\alpha}\right)}{C} + \int_C^\infty \frac{\exp\left(x^{-1/\alpha}\right)}{x^2}dx.
\]

Proof. Follows from Lemma 2.10 and Lemma 2.8. \( \square \)
Lemma 2.14.

\[
\lim_{n \to \infty} I_3(n) = \int_C \frac{dG}{x}.
\]

Proof. This follows from the weak convergence of \( G_n \) to \( G \).

The derivation (23) is justified. Indeed, if we make the constant \( C \) above large, we see that the integral of \( dG_n/x \) is bounded, hence so is the integral of \( dG/x \). Convergence follows from the dominated convergence theorem (or by making \( C \) small). We have incidentally shown that the density of the stable law decays exponentially as \( x \to 0^+ \) (exact expression can be found in \([IbLin1971, \text{Chapter } 2]\)).

Theorem 2.15. The quantity \( H_n/n^{1-1/\alpha} \) converges in distribution to the variable with distribution function \( 1 - G(1/x) \), where \( G \) is the unbalanced stable law of exponent \( \alpha \).

Proof. The proof is immediate.

3. A CLASS OF RANDOM POLYNOMIALS

Let \( x_1, \ldots, x_n \) be independent identically distributed random variables with values between zero and one. Let us consider polynomials whose roots are located at \( x_1, \ldots, x_n \):

\[
p(x) = \prod_{i=1}^{n} (x - x_i) = x^n + \sum_{i=0}^{n-1} c_i x^i.
\]

Given the distribution of \( x_i \), we would like to know the distribution law of the roots of the derivatives of \( p(x) \).

3.1. Uniformly distributed roots. Let us denote the roots of \( \frac{dp(x)}{dx} \equiv p'(x) \) by \( \mu_i, 1 \leq i \leq n - 1 \), and assume that \( \mu_i \leq \mu_{i+1} \) for all \( i \). It is convenient to denote the smallest of \( x_j \) by \( m_1 \), i.e. \( m_1 \equiv \min_j x_j \), the second smallest of \( x_j \) as \( m_2 \) and so forth. It is clear that

\[
m_i \leq \mu_i \leq m_{i+1}, \quad 1 \leq i \leq n - 1.
\]

We now assume that the \( x_j \) are independently uniformly distributed in \([0, 1]\). The distribution of \( m_1 \) is easy to compute: the probability that \( m_1 > \alpha \) is simply the probability that all of the \( x_j \) are greater than \( \alpha \), which is to say,

\[
P(m_1 > \alpha) = (1 - \alpha)^n.
\]

Using this distribution function, one can show that

\[
\mathbb{E}(m_1) = \frac{1}{n+1}.
\]
In fact, it is not hard to see that $E(m_i) = i/(n+1)$; the reader may wish to consult [FellerV2] (page 34). We thus have:

\begin{equation}
\frac{i}{n+1} \leq E(\mu_i) \leq \frac{i+1}{n+1}, \quad 1 \leq i \leq n-1.
\end{equation}

In particular, for large values of $n$ we have the estimate

$$E(\mu_*) \sim \frac{1}{n},$$

where the notation $\mu_*$ is used for the smallest root of the derivative.

### 3.2. More precise locations of roots of the derivative, given that the smallest root of the polynomial is fixed.

In the previous section we have noted that if the roots of $p(x)$ are distributed uniformly in $[0,1]$, then so are the roots of $p'(x)$. In order to understand better the distribution of the roots of $p'(x)$, first let

$$p(x) = (x - x_1) \ldots (x - x_n),$$

then we can write

$$p'(x) = p(x) \sum_{j=1}^{n} \frac{1}{x - x_j}.$$ 

In the generic case where $p(x)$ has no multiple roots, a root $\mu$ of $p'(x)$ satisfies the equation

\begin{equation}
\sum_{j=1}^{n} \frac{1}{x_j - \mu} = 0.
\end{equation}

This was interpreted by Gauss (in the more general context of complex roots) as saying that $\mu$ is in equilibrium in a force field where force is proportional to the inverse of distance, and the “masses” are at the points $x_1, \ldots, x_n$. Gauss used this simple observation to deduce the Gauss-Lucas theorem to the effect that the zeros of the derivative lie in the convex hull of the zeros of the polynomial (see [Marden1966]). We will use it to get more precise location information on the zeros. In particular, consider the smallest root $\mu_*$ of $p'(x)$. It is attracted from the left only by the root $x_1$ of $p$, and from the right by all the other roots, so we see

**Lemma 3.1.** For all $2 \leq i \leq n$, $(m_1 + m_i)/2 \geq \mu_*$, with equality if and only if $n = 2$.

**Remark 3.2.** In the sequel, we shall assume that the smallest root of $p$ equals zero (i.e. $m_1 = 0$; for simplicity of notation we assume that $x_n = 0$).
Inequalities (25) still give a good estimate for the roots \( \mu_2, \ldots, \mu_{n-1} \). One can similarly show that

\[
\frac{i - 1}{n} \leq \mathbb{E}(\mu_i) \leq \frac{i}{n}, \quad 2 \leq i \leq n - 1.
\]

However, for \( \mu_1 = \mu_* \), inequalities (25) give \( 0 \leq \mu_* \leq m_2 \). For uniformly distributed \( x_i \), this tells us that \( \mu_* \) decays like \( 1/n \) or faster. We would like to obtain a more precise estimate for the large \( n \) behavior of \( \mu_* \).

The random polynomial, \( p(x) \), now has the form

\[
p(x) = x \prod_{i=0}^{n-1} (x - x_i) = x \sum_{i=0}^{n-1} c_i x^i.
\]

We need to estimate the smallest root of \( p' \), \( \mu_* \).

**Theorem 3.3.**

\[
\frac{1}{2} \left( \sum_{i=1}^{n-1} \frac{1}{x_i} \right)^{-1} \leq \mu_* \leq \left( \sum_{i=1}^{n-1} \frac{1}{x_i} \right)^{-1}.
\]

**Proof.** The smallest root \( \mu_* \) satisfies the equation

\[
1/\mu_* = \sum_{i=1}^{n-1} 1/(x_i - \mu_*).
\]

By Lemma 3.1,

\[
\frac{x_i}{2} \leq x_i - \mu_* \leq x_i, \quad 1 \leq i \leq n - 1.
\]

The result follows immediately from equations (29) and (30).

Now it is clear that in order to find an estimate for \( \mu_* \), we need to study the behavior of \( \left( \sum_{i=1}^{n-1} 1/x_i \right)^{-1} \). In terms of the harmonic mean of independent random variables \( x_1, \ldots, x_{n-1} \), we have

\[
\frac{1}{2(n - 1)} H_{n-1} \leq \mu_* \leq \frac{1}{n - 1} H_{n-1},
\]

or, for large values of \( n \),

\[
\mathbb{E}(\mu_*) \sim \frac{1}{n} \mathbb{E}(H_n).
\]

Using Theorems 1.2 and 1.4, we readily obtain

\[
\mathbb{E}(\mu_*) \sim \frac{1}{n \log n}.
\]
4. A class of stochastic matrices

Let $T$ be an $n$ by $n$ matrix constructed as follows:

$$T_{ij} = \begin{cases} (1 - a_i)/(n - 1), & i \neq j, \\ a_i, & i = j, \end{cases}$$

where the numbers $a_i$ are independently distributed between 0 and 1. We want to study the large $n$ behavior of the second largest eigenvalue of $T$ (the largest eigenvalue is equal to 1). We will denote this eigenvalue as $\lambda_*$. In the next section we will provide some motivation for this choice of stochastic matrices.

4.1. Motivation: the memoryless learner algorithm. The following is a typical learning theory setup (see [Niyogi1998]). We have $n$ sets (which we can think of as concepts), $R_1, \ldots, R_n$. Each set $R_i$ is equipped with a probability measure $\nu_i$. The similarity matrix $A$ is defined by $a_{ij} = \nu_i(R_j)$. Since the $\nu_i$ are probability measures, we see that $0 \leq a_{ij} \leq 1$ and $a_{ii} = 1$ for all $i, j$. Now, the teacher generates a sequence of $N$ examples referring to a single concept $R_k$, and the task of the student is to guess the $k$ (i.e. to learn the concept $G_k$), hopefully with high confidence.

The learner has a number of algorithms available to him. For instance, the student may decide in advance that the concept being explained is $R_1$, and ignore the teacher’s input, insisting forever more that the concept is $R_1$. While this algorithm occasionally results in spectacular success, the probability of this is independent of the number of examples, and is inversely proportional to the number $n$ of available concepts. Here we will consider a more practical and mathematically interesting algorithm, namely,

**Memoryless learner algorithm.** The student picks his initial guess at random, as before. However, now he evaluates the teacher’s examples, and if the current guess is incorrect (i.e. if the teacher’s example is inconsistent with the current guess), he switches his guess at random. The name of the algorithm stems from the fact that the student keeps no memory of the history of his guesses, and will occasionally switch his guess to one previously rejected.

It is clear that with the memoryless learner algorithm, the student will never be able to learn the set $R_k$ if $R_k \subset R_l$. We call such a situation unlearnable, and do not consider it in the sequel. In terms of the similarity matrix, this can be rephrased as the assumption that $a_{ij} < 1$, $i \neq j$.

To define our mathematical model further, we will assume that the student picks the initial guess uniformly: $p^{(0)} = (1/n, \ldots, 1/n)^T$. The
discrete time evolution of the vector $p^{(t)}$ is a Markov process with transition matrix $T^{(k)}$, which depends on the teacher’s concept, $R_k$, and the similarity matrix, $A$. That is:

$$T^{(k)}_{ij} = \begin{cases} 
(1 - a_{ki})/(n - 1), & i \neq j, \\
 a_{ki}, & i = j.
\end{cases}$$ (34)

After $N$ examples, the probability that the student believes that the correct concept is $R_j$ is given by the $j$th component of the vector $(p^{(N)})^T = (p^{(0)})^T (T^{(k)})^N$. In particular, the probability that the student’s belief corresponds to reality (that is, $j = k$) is given by:

$$Q_{kk}(N) = [(p^{(0)})^T (T^{(k)})^N]_k.$$ (35)

It is clear that the dynamics of the memoryless learner algorithm is completely encoded by the matrix $T$ defined above by (34).

We are interested in the rate of convergence as a function of $n$, the number of possible concepts. We define the convergence rate of the algorithm as the rate of the convergence to 0 of the difference

$$1 - Q_{kk}(N).$$

In order to simplify notation, let us set $k = 1$ and skip the corresponding subscript/superscript. In order to evaluate the convergence rate of the memoryless learner algorithm, let us represent the matrix $T^{(1)} \equiv T$ as follows:

$$T = V \Lambda W,$$ (36)

where the diagonal matrix $\Lambda$ consists of the eigenvalues of $T$, which we call $\lambda_i$, $1 \leq i \leq n$; representation (36) is possible in the generic case. The columns of the matrix $V$ are the right eigenvectors of $T$, $v_i$. The rows of the matrix $W$ are the left eigenvectors of $T$, $w_i$, normalized to satisfy $\langle w_i, v_j \rangle = \delta_{ij}$, where $\delta_{ij}$ is the Kronecker symbol (so that $VW = WV = I$). The eigenvalues of $T$ satisfy $|\lambda_i| \leq 1$. We have

$$T^N = V \Lambda^N W.$$ 

Let us arrange the eigenvalues in decreasing order, so that $\lambda_1 = 1$ and $\lambda_2 \equiv \lambda_n$ is the second largest eigenvalue (we assume that it has multiplicity one). If $N$ is large, we have $\lambda_i^N \ll \lambda_2^N$ for all $i \geq 3$, so only the first two largest eigenvalues need to be taken into account. This means that in order to evaluate $T^N$ we only need the following eigenvectors: $v_1 = (1/n, 1/n, \ldots, 1/n)^T$, $v_2$, $w_1 = (n, 0, \ldots, 0)$, and $w_2$ (it is possible to check that the contribution from the other components contains multipliers $\lambda_i^N$ with $i > 2$ and thus can be neglected, see the
computation for $C_n$ in Section 4.3). It follows that
\begin{equation}
Q_{11} = 1 - C_n \lambda_*^N,
\end{equation}
where
\begin{equation}
C_n = -\frac{1}{n} \sum_{j=1}^{n} [v_2]_j [w_2]_1.
\end{equation}
The convergence rate of the memoryless learner algorithm can be found by estimating $\lambda_*$ and $C_n$. It turns out that a good understanding of $\lambda_*$ (Section 4.2), helps us also estimate $C_n$ (this is done in Section 4.3).

### 4.2. Second largest eigenvalue and the smallest root of the derivative of a random polynomial

Let $Z = I - T$, and let $x_i = 1 - a_i$. The matrix $Z$ satisfies $Z_{ii} = x_i$, while $Z_{ij} = -\frac{x_i}{n-1}$, for $j \neq i$. We have
\begin{equation}
Z = \frac{n}{n-1} D_x (I - \frac{1}{n} J_n),
\end{equation}
where $J_n$ is the $n \times n$ matrix of all ones, and $D_x$ is a diagonal matrix whose $i$-th element is $x_i$. It is convenient to introduce the matrices
\begin{equation}
M_n = I - \frac{1}{n} J_n = \begin{pmatrix}
1 & -1/n & -1/n & \ldots \\
-1/n & 1 & -1/n & \ldots \\
-1/n & -1/n & 1 & \ldots \\
\vdots & & & \\
\end{pmatrix}
\end{equation}
and
\begin{equation}
B = D_x M_n.
\end{equation}
The second largest eigenvalue of $T$, which we denote as $\lambda_*$, and the smallest nontrivial eigenvalue of $B$, $\mu_*$, are related as
\begin{equation}
\lambda_* = 1 - \frac{n}{n-1} \mu_*. 
\end{equation}
In what follows we will write down the characteristic polynomial of $B$. Let us recall the following

**Fact 4.1.** Let $A$ be an $n \times n$ matrix. Then the coefficient of $x^{n-k}$ in the characteristic polynomial $p_A(x)$ of $A$ (defined to be $\det(xI_n - A)$) is given by
\[
\sum_{S \text{ k-element subsets of } \{1, \ldots, n\}} (-1)^k \det m_S,
\]
where $m_S$ is the matrix obtained from $A$ by deleting those rows and columns whose indices are not elements of $S$ (we call $m_S$ the minor of $A$ corresponding to $S$).
We will need the following lemmas:

**Lemma 4.2.** Let $A$ be an $n \times n$ matrix, and let $D_x$ be as above. Let $m_{i_1,\ldots,i_k}$ be the $i_1,\ldots,i_k$ minor of $M$ (that is, the sub-matrix of $i_1,\ldots,i_k$ rows and $i_1,\ldots,i_k$ columns of $M$). Then the minor $\gamma_{i_1,\ldots,i_k}$ of the matrix $C = D_x A$ satisfies

$$
\det \gamma_{i_1,\ldots,i_k} = \det m_{i_1,\ldots,i_k} \prod_{l=1}^{k} x_{i_l}.
$$

**Proof.** This is immediate, since the $j$-th row of $C$ is $x_j$ times the $j$-th row of $A$. □

**Lemma 4.3.** The characteristic polynomial of $M_n = I - \frac{1}{n} J_n$ equals $x(1 - x)^{n-1}$.

**Proof.** Immediate, since the bottom eigenvalue of $M_n$ (given in equation (40)) is zero and the rest are 1. □

**Lemma 4.4.** All the $k \times k$ minors of $M_n$ (defined in the statement of Lemma 4.3) are equal.

**Proof.** By inspection – all the $k \times k$ minors of $M_n$ are $M_k + \left( \frac{1}{n} - \frac{1}{k} \right) J_k$. □

**Lemma 4.5.** The determinants $d_k$ of the $k \times k$ minors of $M_n$ are equal to $\frac{k}{n}$.

**Proof.** We know that the $\binom{n}{k} d_k = \binom{n-1}{k-1}$, from Lemmas 4.3 and 4.4. From this the assertion follows immediately. □

Now, let

$$
p_D(x) = x^n + \sum_{i=0}^{n-1} c_i x^i
$$

be the characteristic polynomial of $D_x$.

**Lemma 4.6.** The characteristic polynomial of $B$, $p_B(x)$, is given by:

$$
p_B(x) = x^n + \sum_{i=0}^{n-1} \frac{i}{n} c_i x^i,
$$

where $c_i$ are as above.
Proof. From Lemma 4.2 combined with Lemma 4.5, we see that the coefficient of $x^i$ in $p_B(x)$ is given by
\[ \frac{i}{n} \sum_{\text{$i$-element subsets $S$ of $\{1, \ldots, n\}$}} \prod_{j \in S} x_j. \]

The sum is just the $i$-th elementary symmetric function of the $x_1, \ldots, x_n$, which is equal to $c_i$. The assertion follows.

Notice that the constant term of $p_B$ vanishes, so we can write
\[ p_B(x) = xq(x), \]
where
\[ q(x) = x^{n-1} + \sum_{i=0}^{n-2} \frac{i+1}{n} x^i. \]
But obviously
\[ q(x) = \frac{1}{n} \frac{dp_D(x)}{dx}, \]
so we have

**Lemma 4.7.** The characteristic polynomials of the matrix $B$ defined in (41) and the diagonal matrix $D_x$ with elements $x_i$ are related by
\[ p_B(x) = \frac{x}{n} p'_D(x). \]

This relates the eigenvalues of the matrix $B$ and the zeros of the polynomial $q(x)$ (and $p'_D(x)$). In its turn, the smallest eigenvalue of $B$ is related to the second largest eigenvalue of our matrix $T$ by equation (42).

We can see that studying the second largest eigenvalue of a stochastic matrix of class (33) is reduced to the problem of the smallest root of the derivative of the stochastic polynomial of class (24), with $x_i = 1 - a_i$. Note that by the definition of matrix $T^{(k)}$, one of the quantities $1 - a_{ki} = x_i$ is equal to zero. This means that in order to find the distribution of the second largest eigenvalue of such a matrix, we need to refer to Section 3.2, i.e. the case where one of the roots of the random polynomial was fixed to zero, and the rest were distributed uniformly.

### 4.3. Eigenvectors of stochastic matrices

Next, let us study eigenvectors of stochastic matrices, and derive an estimate for $C_n$ in equation (38). Consider the matrix $Z$ defined in equation (39). We can write
\[ Z = W'D_{\mu}V, \]
where $V$ and $W$ are the matrices of right and left eigenvectors (respectively) of $Z$, and $D_{\mu}$ is a diagonal matrix whose entries
are the eigenvalues of $Z$. We know that the right eigenvector of $Z$ corresponding to the eigenvalue 0 is the vector $v_1 = (1, \ldots, 1)^T$, while the left eigenvector is the vector $w_1 = (1, 0, \ldots, 0)$. To write down the eigenvector $v_i$ ($i > 1$) we write $v_i = v_1 + u_i$, where $\langle v_1, u_i \rangle = 0$ — we can always normalize $v_i$ so that this is possible. If the corresponding eigenvalue is $\mu_i$, we write the eigenvalue equation:

$$\mu_i v_i = Z v_i = \frac{n}{n-1} D_x (I - \frac{1}{n} J_n) (v_1 + u_i) = \frac{n}{n-1} D_x u_i.$$ 

This results in the following equations for $u_{ij}$ — the $j$-th coordinate of $u_i$ (for $j > 1$; $u_{i1} = -1$):

$$\mu_i + \mu_i u_{ij} = \frac{n}{n-1} x_j u_{ij}, \quad (44)$$

and so

$$u_{ij} = \frac{\mu_i}{\frac{n}{n-1} x_j - \mu_i}, \quad (45)$$

On the other hand, the eigenvalue equation for $w_i$ is

$$\mu_i w_i = Z^T w_i = \frac{n}{n-1} (I - \frac{1}{n} J_n) D_x w_i,$$ 

resulting in the following equations for the coordinates:

$$\mu_i w_{ij} = \frac{n}{n-1} \left( x_j w_{ij} - \frac{1}{n} \sum_{k=1}^{n} x_k w_{ik} \right). \quad (47)$$

If we assume that $x_1 = 0$, then setting $j = 1$, we get

$$\mu_i w_{i1} = - \frac{n}{n-1} \frac{1}{n} \sum_{k=1}^{n} x_k w_{ik}, \quad (48)$$

and so

$$\frac{1}{n} \sum_{k=1}^{n} x_k w_{ik} = - \frac{n}{n-1} \mu_i w_{i1}, \quad (49)$$

and equation (47) can be rewritten (for $j > 1$) as

$$\mu_i w_{ij} = \frac{n}{n-1} \left( x_j w_{ij} + \frac{n}{n-1} \mu_i w_{i1} \right), \quad (50)$$

to get

$$w_{ij} = \frac{\frac{n}{n-1} \mu_i w_{i1}}{\frac{n}{n-1} x_j + \mu_i}, \quad (51)$$

Now, let us assume that $i = 2$, and in addition $\mu_2 \ll x_k$, $k > 1$. While it follows immediately from Lemma [3.1] that $\mu_2 < x_2$, we comment that
by our Weak Law of Large Numbers (Theorem 1.4), the probability that $\mu_2 > cn/\log n$ goes to 0 with $c$, whereas the probability that $|x_k - (k - 1)/n| > c_2/n$ goes to zero with $c_2$ (detailed results on the distribution of order statistics can be found in [FellerV2, Chapter I]).

**Remark 4.8.** The assumption that $\mu_* < x_2$ is least justified if we have reason to believe that $x_2 < 1/n$.

Thus we can write approximately:

$$v_{2j} \approx \frac{\mu_2}{x_j} + 1,$$

while

$$w_{2j} \approx -\frac{\mu_2 w_{21}}{x_j}$$

Since we must have $<w_2, v_2> = 1$, we have:

$$-w_{21} \sum_{j=2}^{n} \frac{\mu_2}{x_j} \left(\frac{\mu_2}{x_j} + 1\right) = 1,$$

which implies that

$$w_{21} \approx -\frac{1}{\mu_2 \sum_{j=2}^{n} \frac{1}{x_j}},$$

which, in turn, implies that $1/2 \leq |w_{21}| \leq 1$. This means that we have the following estimate for the quantity $C_n$ in (38):

$$1 \leq -\frac{1}{n} w_{21} \sum_{j=2}^{n} v_{2j} \leq 2,$$

i.e.

$$1 \leq C_n \leq 2.$$  

4.4. **Convergence of the memoryless learner algorithm.** Let us assume that the overlaps between concepts, $a_{ki} \equiv a_i$ in the matrix $T$, are independent random variables distributed according to density $\tilde{f}(a)$. Then the variables $x_i = 1 - a_i$ have the probability density $f(x) = \tilde{f}(1 - x)$. Our results for the rate of convergence of the memoryless learner algorithm can be summarized in the following

**Theorem 4.9.** Let us assume that the density of overlaps, $f(x)$, approaches a nonzero constant as $x \to 0$. Then in order for the learner to pick up the correct set with probability $1 - \Delta$, we need to have at least

$$N_\Delta \sim |\log \Delta|(n \log n).$$

(55)
sampling events.

Proof. Combining equations (37), (54) and (42) we can see that in order
for the learner to pick up the correct set with probability $1 - \Delta$, we
need to have at least

\begin{equation}
N_{\Delta} \sim |\log \Delta| / \mu_\ast
\end{equation}

sampling events. Since $\beta = 0$ (see equation (24)), we have $\alpha = 1$.
Using bounds (31) which relate $\mu_\ast$ to the harmonic mean, and the weak
law of large numbers (Theorem (1.4)), we obtain estimate (55). This
estimate should be understood in the following sense: as $n \to \infty$, the
probability that the ratio $\mu_\ast^{-1}/(n \log n)$ deviates from 1 by a constant
amount, tends to zero. Therefore, the right hand side of (54) behaves
like the right hand side of (53) with probability which tends to one as
$n$ tends to infinity.

For other distributions we have

\begin{theorem}
If the probability density of overlaps, $f(x)$, is asymptotic to $x^\beta + O((x)^{\beta+b})$, \( \beta > 0 \), as $x$ approaches 0, then

\begin{equation}
N_{\Delta} \sim |\log \Delta|n;
\end{equation}

if $-1 < \beta < 0$, then

\begin{equation}
\lim_{x \to \infty} P \left( \frac{1}{x} < \frac{N_{\Delta}}{|\log \Delta|n^{1/(1+\beta)}} < x \right) = 1.
\end{equation}

\end{theorem}

Proof. The proof uses the results on the harmonic mean in section

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Appendix A. Derivation of the stable law for the uniform distribution of $x_i$

Here we will provide an explicit derivation of the stable law, equation (10), in the case where the random variables $x_i$ are uniformly distributed between zero and one. The characteristic function corresponding to the distribution of $y_i = 1/x_i$, equation (3), is given by

$$\chi(k) = \int_{-\infty}^{\infty} e^{iky} \frac{1}{y^2} dy.$$ \hfill(57)

This can be evaluated explicitly; here we only present the computations for positive $k$, to avoid clutter. We have

$$\chi(k) = -|k|\pi/2 + \cos k + k\text{Si}(k) + i[\sin k - k\text{Ci}(|k|)],$$ \hfill(58)

where $\text{Si}$ and $\text{Ci}$ are sin- and cos- integrals, respectively (this expression is obtained with Mathematica). The behavior of $\chi(k)$ at 0 can be easily computed using the above formula:

$$\chi(k) = 1 - i(\gamma - 1)k - \frac{\pi}{2}k - ik\log k + \frac{1}{2}k^2 + o(k^2),$$ \hfill(59)

where $\gamma \approx 0.577216$ is Euler’s constant.

We can also obtain the asymptotics for $\chi(k)$ directly, as follows. First, we change variables, and set $u = ky$, to obtain

$$\chi(k) = k \int_k^{\infty} e^{iu} \frac{1}{u^2} du.$$ 

Let $I(k) = \int_k^{\infty} e^{iu} \frac{1}{u^2} du$ and $I_R(k) = \int_k^{R} e^{iu} \frac{1}{u^2} du.

Clearly, $I(k) = \lim_{R \to \infty} I_R(k)$. Since the integrand (call it $f(u)$) has no poles, except at 0, we see that $I_R(k) + J(k) + L(R) - K_R(k) = 0$, where $J(k)$ is the integral of $f(u)$ along the positive quadrant of the circle $|z| = k$, $L(R)$ the integral along the positive quadrant of the circle

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$|z| = R$, and $K_R(k)$ is the integral along the imaginary axis, from $ik$ to $iR$. It is easy to see that $\lim_{R \to \infty} L(R) = 0$, while

$$kJ(k) = \frac{1}{2} \int_0^{\pi/2} \exp(ik\theta - i2\theta)d\theta,$$

the Taylor series of which is easily evaluated by expanding the integrand in a Taylor series. The integral $K_R(k)$ reduces to the integral of $\exp(-u)/u^2$, the asymptotics of which can be easily obtain by repeated integration by parts.

The characteristic function of $Y_n = a_n S_n - b_n$, $\Psi_n$, can be obtained by setting $a_n = 1/n$, and $b_n = \log n$, and using equation (9):

$$\Psi_n(k) = \exp(-ik \log n)(\chi(k/n))^n$$

$$= \exp\left(-\left(\frac{\pi}{2} + i(\gamma - 1)\right)k + ik \log k\right)\left(1 + O\left(\frac{k^2 \log^2 n}{n}\right)\right),$$

where we expanded the exponential in its Taylor series. The expression for $g$ (equation (10)) follows as we take the limit $n \to \infty$ and perform the Fourier transform of (60).

**Appendix B. Rate of convergence to stable law**

Since some of the quantities we are trying to estimate depend on $n$, it is not enough for us to know that the distributions of the quantities $X_n$ converge to a stable law, but it is necessary to have good estimates on the speed of convergence (cf. Section 1.3). Our results can be summarized as follows:

**Theorem B.1.** In the case $\alpha = 1$, we have $g_n(x) = g(x) + O(\log^2 n/n).$

**Proof.** The proof falls naturally in two parts, both of which require estimates on the characteristic function of $G_n$. First, we show that we can throw away the tails of characteristic function, and then we estimate convergence in the remaining region. We introduce the following notation: let $\Psi_n(k)$ be the characteristic function of $G_n(k)$, and let $g_n^\delta$

---

2Such a result is claimed in [KK2001], where the authors find an estimate $g_n(x) = g(x)(1 + O(1/n))$, where $g_n$ is the density of $X_n$, while $g$ is the stable density, and the implicit constants are uniform in $x$. Such an estimate would be good enough for our purposes, unfortunately it is incorrect, as can be seen by noting that $g_n(x) = 0$, $x < 1 - \log n$. A correct estimate is given by [Hall1981], however the method we outline is self-contained, simple, and generalizable, so we choose to present it here.
be defined as
\[ g^n_\delta(x) = \frac{1}{2\pi} \int_{-n^{1-\delta}}^{n^{1-\delta}} \exp(-ixk)\Psi_n(k)dk. \]
In addition, let \( R_n(x) = |g_n(x) - g^n_\delta(x)|. \) Then

**Lemma B.2.**
\[ \sup_{x \in (-\infty, \infty)} R_n(x) = O(ne^{-\frac{\pi}{2}n^{1-\delta}}). \]

Lemma B.2 will be proved in Section B.1.

We will also need the following

**Lemma B.3.** Let \( \Psi(k) \) be the characteristic function of the stable density, \( g, \) and
\[ R(x) = \frac{1}{2\pi} \left( \int_{-\infty}^{-M} e^{-ikx}\Psi(k)dk + \int_{M}^{\infty} e^{-ikx}\Psi(k)dk \right). \]
Then \( R(x) = O(\exp(-\pi M/2)). \)

**Proof.** Let us recall that \( |\Psi(k)| = \exp(-\frac{\pi}{2}|k|). \) Now we write
\[ R(x) \leq \frac{1}{2\pi} \left| \int_{-\infty}^{-M} e^{-\frac{\pi}{2}|k|}dk + \int_{M}^{\infty} e^{-\frac{\pi}{2}|k|}dk \right| = \frac{2}{\pi^2} e^{-\frac{\pi}{2}M}. \]

To end the proof of Theorem B.1 we need to estimate how closely \( \Psi(k) \) is approximated by \( \Psi_n(k), \) for \( k \leq n^{1-\delta}, \) since by the above estimates we know that
\[ g(x) - g_n(x) = \frac{1}{\pi} \int_{0}^{n^{1-\delta}} (\cos(xk)(\Psi(k) - \Psi_n(k)))dk \quad \text{(plus lower order terms)} \]
(plus lower order terms). It remains only to estimate the difference \( |\Psi(k) - \Psi_n(k)|. \) From equation (60) we have
\[ g(x) - g_n(x) \leq O(\log^2 n/n) \int_{0}^{n^{1-\delta}} e^{-\frac{\pi}{2}k^2}dk = O(\log^2 n/n). \]

**B.1. An estimate on the tails of the characteristic function of** \( Y_n. \) In this section we will supply the proof of Lemma B.2; we use the notation introduced before the statement of the lemma. We write explicitly
\[ R_n(y) = \Re \int_{n^{1-\delta}}^{\infty} e^{-ik(y+\log n)}\chi^n(k/n)dk. \]
The argument given below can be easily generalized for any stable law. The only facts about the function $\chi(k)$ that we are going to use are as follows:

i) $|\chi(k)| \leq 1 - c|k|^\beta + o(k)$, $k \ll 1$, for some $\beta > 0$;

ii) $|\chi(k)| \leq C_1 < 1$, for all $k > k_0$. This holds whenever $\chi$ is not the characteristic function of a lattice distribution, essentially by the Riemann-Lebesgue lemma;

iii) $\chi \in L^1$.

Given condition (iii) we have

**Lemma B.4.** There exists an $M$ such that

$$\int_M^\infty |\chi(z)|^n dz < \frac{1}{2^n} 2 \int_M^\infty |\chi(z)| dz.$$

**Proof.** By the Riemann-Lebesgue Lemma, $\exists M_2$, such that $|\chi(z)| \leq 1/2$, for all $z \geq M_2$. Setting $M = M_2$, the assertion of the lemma follows immediately.

Let us introduce the variable $z = k/n$. We have

$$R_n(y) = n \Re \int_{n^{-\delta}}^{n} e^{-inz(y+\log n)} \chi^n(z) dz.$$

Next, we note that

$$|R_n(y)| \leq \tilde{R}_n = n \int_{n^{-\delta}}^{\infty} |\chi^n(z)| dz.$$

where the last inequality holds for $\delta < 1$ and $n$ sufficiently large. Let us write

$$\tilde{R}_n = R_{n,1} + R_{n,2} + R_{n,3},$$

where

$$R_{n,1} = n \int_{n^{-\delta}}^{z_0} |\chi(z)|^n dz,$$

$$R_{n,2} = n \int_{z_0}^{z_1} |\chi(z)|^n dz,$$

$$R_{n,3} = n \int_{z_1}^{\infty} |\chi(z)|^n dz.$$

The constant $z_0$ is chosen so that $|\chi(z)|$ is decreasing from 0 to $z_0$. Such a $z_0 > 0$ can always be found as long as $\chi(z)$ is continuous at 0, since $\chi(0) = 1$ and $|\chi(z)| < 1$ for all $z$ sufficiently close to 0 (and all $z$ if $\chi$ is not the characteristic function of a lattice distribution). If $|\chi(z)|$ is monotonically decreasing always, as seems to be the case with (58),
we take \( z_0 = 1 \). We choose \( z_1 \) in such a way that Lemma B.4 holds for \( M = z_1 \).

First we consider \( R_{n,1} \). By the choice of \( z_0 \), the function \( |\chi(z)| \) monotonically decreases on \( [n^{-\delta}, z_0] \). Therefore,

\[
|R_{n,1}| \leq nz_0|\chi(n^{-\delta})|^n.
\]

(62)

For small values of \( k \), we have (see property (i)):

\[
|\chi(k)| = 1 - \pi k / 2 + O((k \log k)^2).
\]

Therefore,

\[
|\chi(n^{-\delta})|^n \approx \exp(-\frac{\pi}{2} n^{1-\delta}),
\]

that is, it decays exponentially for any \( 0 < \delta < 1 \). Thus, from (62) we see that

\[
|R_{n,1}| \leq nz_0 \exp(-\frac{\pi}{2} n^{1-\delta}).
\]

(63)

Next, we estimate \( R_{n,2} \). Because of property (ii), we have

\[
|R_{n,2}| \leq n^{1+\delta} C_1^n,
\]

and since \( C_1 < 1 \), it also decays exponentially with \( n \). Finally, we estimate \( R_{n,3} \) by Lemma B.4. Putting everything together we conclude that \( R_n \) decays exponentially with \( n \).