CAB: Continuous Adaptive Blending Estimator for Policy Evaluation and Learning

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Abstract

The ability to perform offline A/B-testing and off-policy learning using logged contextual bandit feedback is highly desirable in a broad range of applications, including recommender systems, search engines, ad placement, and personalized health care. Both offline A/B-testing and off-policy learning require a counterfactual estimator that evaluates how some new policy would have performed, if it had been used instead of the logging policy. This paper proposes a new counterfactual estimator – called Continuous Adaptive Blending (CAB) – for this policy evaluation problem that combines regression and weighting approaches for an effective bias/variance trade-off. It can be substantially less biased than clipped Inverse Propensity Score weighting and the Direct Method, and it can have less variance compared with Doubly Robust and IPS estimators. Experimental results show that CAB provides excellent and reliable estimation accuracy compared to other blended estimators, and – unlike the SWITCH estimator – is sub-differentiable such that it can be used for learning.

1 Introduction

Contextual bandit feedback is ubiquitous in a wide range of intelligent systems that interact with their users through the following process. The system observes a context, takes an action, and then observes feedback under the chosen action. The logs of search engines, recommender systems, ad-placement systems, and many other systems contain terabytes of this partial-information feedback data, and it is highly desirable to use this historic data for offline evaluation and learning. For instance, one may want to evaluate or learn a new policy for placing display ads. Here the context describes the user and the page, the action is the ad shown to the user, and the feedback is whether the user clicks on the ad. What makes evaluation and learning challenging, however, is that we only get to see the feedback for the chosen ad, but we do not observe how the user would have clicked, if the system had presented a different ad. This partial and biased nature of the feedback makes batch learning from contextual bandit feedback substantially different from typical supervised learning, where the correct label and a loss function provide full-information feedback.

Both learning and evaluation can be viewed as an example of counterfactual reasoning, where we would like to estimate from the historic log data how well some other policy would have performed,
We propose a new Continuous Adaptive Blending (CAB) estimator that aims to take advantage of the complementary strengths of DM and IPS, providing an effective tool for optimizing the bias of DM against the variance of IPS. Compared to existing estimators, we find that CAB has several advantages. First, we will show that CAB is typically substantially less biased than clipped IPS and DM while having smaller variance compared to IPS and Doubly Robust (DR) estimators. Second, compared to estimators that perform static blending (SB) [15], CAB is adaptive to the inverse propensity weights and often shows improved estimation accuracy. Third, unlike SWITCH [17], CAB is sub-differentiable which allows its use as the training objective in off-policy learning algorithms like POEM [13] and BanditNet [9]. Finally, unlike the DR estimator, CAB is specifically designed to control the bias/variance trade off, and it can be used in off-policy Learning to Rank (LTR) algorithms like [8]. We evaluate CAB both theoretically and empirically. In particular, we present theoretical results that characterize the bias and variance of CAB. Furthermore, we provide an extensive empirical comparison of CAB on both partial-information ranking problems and contextual-bandit problems.

2 Off-policy Evaluation in Contextual Bandits

Since counterfactual estimators for policy evaluation lie at the core of both off-policy evaluation and learning, we begin with a formal definition of the problem and an overview of the existing approaches. To keep notation and exposure simple, we focus on the contextual bandit setting and do not explicitly consider other partial-information settings like counterfactual LTR [8]. However, most estimators can be translated into that setting as well, and we discuss further details in the context of the ranking experiments in Section 4.2.

2.1 Contextual-Bandit Setting and Learning

In the contextual bandit setting, a context \( x \in \mathcal{X} \) (e.g., user profile, query) is drawn i.i.d. from some unknown \( P(\mathcal{X}) \), the deployed (stochastic) policy \( \pi_0(y|x) \) then selects an action \( y \in \mathcal{Y} \), and the system receives feedback \( r \sim D(r|y,x) \) for this particular context action pair. However, we do not observe feedback for any of the other actions. The logged contextual bandit data we get from logging policy \( \pi_0 \) is of the form

\[
\mathcal{S} = \{(x_i, y_i, p_i, r_i)\}_{i=1}^n,
\]

where \( p_i = \pi_0(y_i|x_i) \) if we know the logging policy \( \pi_0 \). Otherwise, we use an estimate of the logging policy \( \hat{\pi}_0(y_i|x_i) \). The observed reward is denoted as \( r_i \equiv r(y_i, x_i) \). Off-policy evaluation addresses the issue of estimating the expected reward \( R \) of a new policy \( \pi \)

\[
R(\pi) = E_{x \sim P(x)} E_{y \sim \pi(y|x)} E_{r \sim D(r|y,x)} [r].
\]

Off-policy learning aims to find a policy in some policy space \( \mathcal{\Pi} \) that maximizes the reward (or minimizes the loss)

\[
\pi_* = \text{argmax}_{\pi \in \mathcal{\Pi}} R(\pi).
\]

The expected reward \( R(\pi) \) is not known, and it is typically replaced with a counterfactual estimate \( \hat{R}(\pi) \) that can be computed from the logged bandit feedback \( \mathcal{S} \). This enables Empirical Risk Minimization (ERM) for batch learning from bandit feedback (BLBF), where the algorithm finds the policy in \( \mathcal{\Pi} \) that maximizes the estimated expected reward

\[
\hat{\pi}_* = \text{argmax}_{\pi \in \mathcal{\Pi}} \hat{R}(\pi),
\]

possibly subject to capacity and variance regularization [13]. It is expected that learning with an improved counterfactual estimator will also lead to improved learning performance [12]. In particular,
2.2 Counterfactual Estimators

There are three broad classes of counterfactual estimators for the contextual bandit setting. The first class follows a “Model the World” approach [45], which means learning a model of the reward. In particular, the DM learns a reward model from \( \mathcal{S} \) via regression \( \hat{\delta}(y, x) \), which serves an estimate of \( \delta(y, x) = \mathbb{E}_r[r|y, x] \), and then imputes any missing feedback needed to compute the expected reward in (2). In particular, DM estimates the reward as

\[
\hat{R}_{DM}(\pi) = \frac{1}{n} \sum_{i=1}^{n} \sum_{y \in \mathcal{Y}} \pi(y|x_i) \hat{\delta}(y, x_i).
\]

Typically the reward regressor does not have very large variability and we know \( \pi \), so the variance of DM is typically small. However, due to often unavoidable misspecification of the reward model and confounding, DM is often inconsistent and can have a large bias.

The second class of estimators employs a “Model the Bias” approach for the assignment mechanism, which is particularly attractive in many applications where we control the assignment mechanism by design [61316]. In particular, it is the logging policy \( \pi_0 \) that is often known and from which we can log propensities. The inverse propensity score (IPS) weighting estimator [6] exploits this knowledge of the logged propensities

\[
\hat{R}_{IPS}(\pi) = \frac{1}{n} \sum_{i=1}^{n} \frac{\pi(y_i|x_i)}{p_i} r(y_i, x_i),
\]

where \( M \) is a clipping parameter that allows trading-off between bias and variance.

The third class of estimators combines the previous two approaches. The most prominent one is the Doubly Robust estimator (DR) [1134] that treats DM as a baseline and corrects the baseline when data is available.

\[
\hat{R}_{DR}(\pi) = \frac{1}{n} \sum_{i=1}^{n} \sum_{y \in \mathcal{Y}} \pi(y|x_i) \left[ \hat{\delta}(y, x_i) + \frac{r(y_i, x_i) - \hat{\delta}(y, x_i)}{p_i} \mathbb{1}_{\{y=y_i\}} \right].
\]

It is unbiased when either the reward model or the propensity model is correct. However, by maintaining unbiasedness, we will discuss in Section 3.3 that it can still suffer from excessive variance. Furthermore, DR cannot be used for LTR from implicit feedback, as discussed in Section 4.2.

The work in [15] studies off-policy evaluation in the more general setting of reinforcement learning. When we translate the key idea behind their MAGIC estimator to the contextual bandit setting, we see that it also interpolates between DM and IPS

\[
\hat{R}_{SB}(\pi) = (1 - \tau_{SB}) \hat{R}_{DM} + \tau_{SB} \hat{R}_{IPS}.
\]

We call this estimator Static Blending (SB), since it merely takes a convex combination between DM and IPS with a fixed weight \( \tau \). Note that \( \tau \) is static and does not depend on the propensity weight.

The SWITCH Estimator [17], in contrast, is more adaptive, and it is arguably the closest existing work to the CAB estimator we will introduce in the next section. As the name implies, it switches between using DM or IPS depending on a hard threshold \( M \) on the inverse propensity weights

\[
\hat{R}_{SWITCH}(\pi) = \frac{1}{n} \sum_{i=1}^{n} \sum_{y \in \mathcal{Y}} \pi(y|x_i) \left[ \mathbb{1}_{\{\frac{\pi(y_i|x_i)}{p_i} > M\}} \hat{\delta}(y, x_i) + \mathbb{1}_{\{\frac{\pi(y_i|x_i)}{p_i} \leq M\}} \frac{\mathbb{1}_{\{y=y_i\}}}{p_i} r(y_i, x_i) \right].
\]
We now theoretically characterize the behaviour of CAB. In particular, we analyze the bias and variance of CAB, and we will compare it to those of IPS, CIPS, DR, and DM. Many of the other estimators can be seen as special cases of CAB. In particular, DM emerges as the special case of CAB for \( M = 0 \), where model misspecification typically induces a substantial bias but where variance is low. On the other extreme, IPS is the special case of \( M = \infty \), where the estimator is unbiased for logged propensities and logging policies with full support, but where the variance can be large. Note that the CAB estimator smoothly interpolates between DM and IPS which allows trading off between bias and variance by controlling \( M \). Furthermore, CIPS is the special case of CAB where \( \delta(y, x) = 0 \) for all \( x \) and \( y \).

The SB estimator can also be written in a format similar to CAB, and it differs from CAB merely in its choice of \( \tau \). In particular, the SB estimator uses a universal \( \tau^{SB} \) for all data points without any adaptation to the actual inverse propensity weight. This has the effect that all weights are clipped proportionally independent of their size, which makes SD less adaptive than CAB. Finally, the SWITCH estimator is equivalent to using a threshold \( \tau^{SWITCH} = \min\left\{ \frac{p_i}{\pi(y|x_i)}, M, 1 \right\} \), making a hard choice between DM and IPS for each \((x, y)\) pair. This is the reason why SWITCH is discontinuous and why it cannot be used as an objective in gradient-based learning algorithms for BLBF. Furthermore, we find that, compared to CAB, the hard switching creates more erratic behavior when the threshold \( M \) is changed.

### 3.1 Bias and Variance Analysis

We now theoretically characterize the behaviour of CAB. In particular, we analyze the bias and variance of CAB, and we will compare it to those of IPS, CIPS, DR, and DM.

As typical in the analysis of IPS estimators, we assume common support (i.e. positivity) of the logging policy towards the target policy.

**Assumption 1 (Common Support).** The logging policy \( \pi_0 \) should have full support for the target policy \( \pi \) which means \( \pi(y|x) > 0 \) \( \rightarrow \) \( \pi_0(y|x) > 0 \) for all \( x \).

Following the characterization of the error in [4], let \( \zeta(x, y) \) denote the multiplicative deviation of the propensity estimate from the true propensity model, and \( \Delta(x, y) \) be the additive deviation of the regression from the true reward.

\[
\zeta(x, y) = 1 - \frac{\pi_0(y|x)}{\hat{\pi}_0(y|x)}
\]

\[
\Delta(x, y) = \hat{\delta}(x, y) - \delta(x, y).
\]

Note that \( \zeta(x, y) \) is zero when the logging policy \( \pi_0 \) is known. For brevity, we denote the (possibly estimated) inverse propensity weight \( \frac{\hat{\pi}(y|x)}{\pi_0(y|x)} \) as \( c(x, y) \).
Theorem 1 (Bias of CAB). For contexts $x$ drawn i.i.d from some distribution $P(X)$ and for actions $y \sim \pi_0(y|x)$, under Assumption\(\square\) the bias of $\hat{R}_{CAB}(\pi)$ is

$$
\text{Bias}(\hat{R}_{CAB}(\pi)) = \mathbb{E}_x[\mathbb{E}_\pi[-\delta(x, y)\zeta(x, y)\mathbb{I}_{\{c(x, y) \leq M\}}] + \mathbb{E}_x\mathbb{E}_\pi[\Delta(x, y)\mathbb{I}_{\{c(x, y) > M\}}] - M\mathbb{E}_x\mathbb{E}_\pi_0[\frac{1}{1-\zeta(x,y)}\Delta(x, y)\mathbb{I}_{\{c(x, y) > M\}}]
- M\mathbb{E}_x\mathbb{E}_\pi_0[\delta(x, y)\frac{\zeta(x,y)}{1-\zeta(x,y)}\mathbb{I}_{\{c(x, y) > M\}}]
$$

(15)

Theorem 2 (Variance of CAB). Under the same conditions as in Theorem\(\square\), the variance of $\hat{R}_{CAB}(\pi)$ is

$$
\text{Var}(\hat{R}_{CAB}(\pi)) = \frac{1}{n}\left(\mathbb{V}_x(\mathbb{E}_\pi[\delta(x, y) + \Delta(x, y)]\mathbb{I}_{\{c(x, y) > M\}}) - M\mathbb{E}_\pi_0[\frac{\Delta(x, y)}{1-\zeta(x,y)}\mathbb{I}_{\{c(x, y) > M\}}]
- M\mathbb{E}_\pi_0(\delta(x, y)\frac{\zeta(x,y)}{1-\zeta(x,y)}\mathbb{I}_{\{c(x, y) > M\}}) + \mathbb{E}_\pi_0[(1-\zeta(x,y))\delta(x, y)\mathbb{I}_{\{c(x, y) \leq M\}}] + \mathbb{E}_\pi_0[\pi(y|x)\mathbb{I}_{\{c(x, y) > M\}} + M\delta(x, y)\mathbb{I}_{\{c(x, y) < M\}}]}
+ M^2\mathbb{E}_x\mathbb{E}_\pi_0[(r(x, y) - \delta(x, y))^2\mathbb{I}_{\{c(x, y) > M\}}]
$$

(16)

The first term of the variance can be understood as the randomness from sample $x$, and the second term can be treated as the randomness of the reward compounded with the inverse propensity weighting penalty when we use IPS. The third term mainly measures how the expected reward $\delta$ varies when we switch between DM and IPS. The fourth term is analogous to the second term, and it is the variance that is inherited from the DM estimator.

3.2 Bias improvements over CIPS and DM

The motivation of CAB originates from the CIPS estimator. CAB tries to impute the clipped part with DM instead of naively imputing 0 values as in CIPS. By imputing with DM, CAB aims to alleviate the high bias inherent in the CIPS estimator.

For the CIPS estimator

$$
\hat{R}_{CIPS}(\pi) = \frac{1}{n}\sum_{i=1}^{n} \min(M, \frac{\pi(y_i|x_i)}{\pi_0(y_i|x_i)})r(y_i, x_i),
$$

(17)

the bias is a special case of Theorem\(\square\)

$$
\text{Bias}(\hat{R}_{CIPS}(\pi)) = \mathbb{E}_x[\mathbb{E}_\pi[-\delta(x, y)\zeta(x, y)\mathbb{I}_{\{c(x, y) \leq M\}}] + \mathbb{E}_x\mathbb{E}_\pi_0[\mathbb{I}_{\{c(x, y) > M\}}\delta(x, y)(\frac{\pi(y|x)}{\pi_0(y|x)} - M)]
$$

(18)

and it collapses to the following if the true propensities are logged

$$
\text{Bias}(\hat{R}_{CIPS}(\pi)) = \mathbb{E}_x\mathbb{E}_\pi_0[\mathbb{I}_{\{c(x, y) > M\}}\delta(x, y)(\frac{\pi(y|x)}{\pi_0(y|x)} - M)].
$$

(19)

In comparison, the bias of CAB for this case is

$$
\text{Bias}(\hat{R}_{CAB}(\pi)) = \mathbb{E}_x\mathbb{E}_\pi_0[\mathbb{I}_{\{c(x, y) > M\}}\Delta(x, y)(\frac{\pi(y|x)}{\pi_0(y|x)} - M)].
$$

(20)

It can be seen that if we have a moderately good predictor of the reward $\delta(x, y)$, CAB will have an advantage as long as the predictor is better than imputing the constant 0 everywhere. In practice, it is
sensible to assume that the reward estimation error $\Delta(x, y)$ is substantially smaller than $\delta(x, y)$, such that CAB enjoys a substantial amount of bias reduction.

Even in the case when we do not have a perfect propensity model, we argue that the additional bias term $M \mathbb{E}_x \mathbb{E}_{\pi_0}[\delta(x, y) \frac{\pi(x, y)}{\pi_0(y|x)} \mathbb{1}_{\{c(x, y) > M\}}]$ can be small if the propensity model is reasonably good after $\zeta$ is close to 0.

In comparison to DM, CAB can also enjoy smaller bias when the propensity is known. In particular, DM has bias $\text{Bias}(\hat{R}_{DM}(\pi)) = \mathbb{E}_x \mathbb{E}_\pi(\Delta)$, while $\text{Bias}(\hat{R}_{CAB}(\pi)) = \mathbb{E}_x \mathbb{E}_\pi(\Delta \mathbb{1}_{\{c(x, y) > M\}})$.

### 3.3 Variance improvements over IPS and Doubly Robust

IPS is known to suffer from high variance, since the propensity $\pi_0(y|x)$ could be arbitrarily small ($\pi_0(y|x) \approx 0$) and thus the inverse propensity weights would be arbitrarily large. CAB alleviates this issue by controlling the inverse propensity weights through a clipping parameter $M$, and substituting the DM estimator which typically has much smaller variance. In this section, we will analyze the variance reduction compared with IPS, and state formally what we gain by substituting with the regression estimates.

To better illustrate the key differences among these estimators, we start the comparison by assuming we have a perfect reward prediction model and that the true propensities are being logged. Then the variance of the IPS estimator is

$$\text{Var}(\hat{R}_{IPS}(\pi)) = \frac{1}{n} \left\{ \mathbb{V}_x(\mathbb{E}_\pi[\delta(x, y)]) + \mathbb{E}_x \mathbb{E}_{\pi_0}[\left( \frac{\pi(y|x)}{\pi_0(y|x)} \right)^2 (r(x, y) - \delta(x, y))^2] \right\} + \mathbb{E}_x[\mathbb{V}_{\pi_0}(\frac{\pi(y|x)}{\pi_0(y|x)} \delta(x, y))]$$

while the variance for CAB is

$$\text{Var}(\hat{R}_{CAB}(\pi)) = \frac{1}{n} \left\{ \mathbb{V}_x(\mathbb{E}_\pi[\delta(x, y)]) + \mathbb{E}_x \mathbb{E}_{\pi_0}[\left( \frac{\pi(y|x)}{\pi_0(y|x)} \right)^2 (r(x, y) - \delta(x, y))^2 \mathbb{1}_{\{c(x, y) \leq M\}}] + \mathbb{E}_x[\mathbb{V}_{\pi_0}(\frac{\pi(y|x)}{\pi_0(y|x)} \delta(x, y)) \mathbb{1}_{\{c(x, y) \leq M\}}] \right\}$$

The first term results from the randomness of contexts $x$ and it is the same for both estimators. However, for IPS, the second term is the variance of reward $r(x, y)$ compounded with the inverse propensity weights squared, which could be extremely large when the logging policy $\pi_0$ and target policy $\pi$ are very different. For the third term, it is similar to the second one which is the variance of the expected reward $\delta(x, y)$ times the inverse propensity weights. These two terms together could blow up the variance of the IPS estimator and make it unbounded. However, for the variance of CAB, we have the variance of these two terms bounded by $M \mathbb{E}_x[(r(x, y) - \delta(x, y))^2] + M^2 \mathbb{E}_x[\mathbb{V}_{\pi_0}(\delta(x, y))]$, which can be much smaller than IPS. Even though CAB has two more variance terms coming from the direct model, $M^2 \mathbb{E}_{\pi_0}[(r(x, y) - \delta(x, y))^2 \mathbb{1}_{\{c(x, y) > M\}}] + \mathbb{E}_x[\mathbb{V}_{\pi_0}(M \delta(x, y) \mathbb{1}_{\{c(x, y) > M\}} + \frac{\pi(y|x)}{\pi_0(y|x)} \delta(x, y)) \mathbb{1}_{\{c(x, y) \leq M\}}]],$ these two terms are of smaller order since they are just the variance of the reward function compounded with $M$.

IPS is a special case of the DR estimator with $\hat{r}(x, y) = 0$ for all $(x, y)$. DR can enjoy smaller variance than IPS by combining the direct model with the IPS estimator. Through simple rearrangement, DR
We empirically examine and compare the accuracy of CAB in two settings. Our goal is to understand when designing new learning algorithms for both BLBF [13] and partial-information LTR [8]. In this setting, the source of the high variance is reflected in the second term.

This setting is extensively used in the off-policy evaluation literature [4, 17]. In the LTR setting, to overcome this problem of DR, in the past practitioners have resorted to also use clipping in validity of the experiments, while at the same time providing ground truth for a bias/variance analysis.

\[ \pi \]

the full-information data to train a multiclass logistic regression model that serves as the stochastic classification dataset, we split it equally into train and test sets. For the train set, we use

\[ S \]

estimators.

\[ S \]

bandit data is simulated by sampling

\[ \tau \]

\[ \tau \]

\[ \tau \]

\[ \tau \]

\[ \hat{R}_{DR}(\pi) = \frac{1}{n} \sum_{i=1}^{n} \sum_{y} (\pi(y|x_i) \hat{\delta}(y, x_i) (1 - \frac{\mathbb{1}_{\{y=y_i\}}}{\hat{\pi}_0(y|x_i)}) + r(y_i, x_i) \mathbb{1}_{\{y=y_i\}}) \]

In this form, we can see that DR can still suffer from high variance when the propensities are small, since the inverse propensity weights are compounded with both the observed loss and the estimated loss. When writing CAB in a similar format

\[ \hat{R}_{CAB}(\pi) = \frac{1}{n} \sum_{i=1}^{n} \sum_{y} (\pi(y|x_i) \hat{\delta}(y, x_i) (1 - \tau_{iy}) + \tau_{iy} r(y_i, x_i) \mathbb{1}_{\{y=y_i\}}) \]

In contrast with DR, it is apparent that the inverse propensity weights are bounded by \( M \) in CAB. To overcome this problem of DR, in the past practitioners have resorted to also use clipping in DR. In this case DR and CABBecome more similar, but they use different weighting schemes and DR still cannot be used for Learning to Rank (see Section 12). Finally, note that for the estimated loss \( \hat{\delta}(y, x_i) \) the variation is likely to be small, since we have the loss compounded with max(\( \pi(y|x_i) - M \hat{\pi}_0(y|x_i) \), 0) instead of the inverse propensity weights. Overall, the DR estimator has variance

\[ \text{Var}(\hat{R}_{DR}(\pi)) = \frac{1}{n} \left( \mathbb{E}_x(\mathbb{E}_y[\delta(x, y)]) + \mathbb{E}_x\mathbb{E}_{y,z}[\frac{\pi(y|x)}{\hat{\pi}_0(y|x)}]^2 (r(x, y) - \delta(x, y))^2 \right), \]

where the source of the high variance is reflected in the second term.

4 Experiments

We empirically examine and compare the accuracy of CAB in two settings. Our goal is to understand its bias-variance trade-off, which will inform how the estimator can be used as a training objective when designing new learning algorithms for both BLBF [13] and partial-information LTR [8]. In the BLBF setting, we conduct the evaluation on bandit feedback data for multi-class classification. This setting is extensively used in the off-policy evaluation literature [4, 17]. In the LTR setting, the evaluation is based on user feedback with position bias for ranking [8]. In both cases, we use real datasets from which we sample synthetic bandit or click data. This increases the external validity of the experiments, while at the same time providing ground truth for a bias/variance analysis. Furthermore, it allows us to vary the properties of both data and \( \pi_0 \) to explore the robustness of the estimators.

4.1 Policy Evaluation for Contextual Bandit Feedback

Following [4, 17], we conduct experiments on multiclass classification datasets from the UCI repository [2] using the standard supervised \( \rightarrow \) bandit conversion [1]. Specifically, given is a supervised dataset \( \{(x_i, y_i^*)\}_{i=1}^{n} \), where \( x \) is i.i.d drawn from a certain fixed distribution \( P(X) \) and \( y^* \in \{1, 2, \cdots, k\} \) denotes the true class label. For a particular logging policy \( \pi_0 \), the logged bandit data is simulated by sampling \( y_i \sim \pi_0(Y|x_i) \) and a deterministic loss \( r(y_i, x_i) \) is revealed. In our experiments, the loss is defined as \( r(y_i, x_i) = \mathbb{1}_{\{y_i \neq y_i^*\}} - 1 \). The resulting logged contextual bandit data \( S = \{x_i, y_i, \pi_0(y_i|x_i), r(y_i, x_i)\} \) is then used to evaluate the performance of different estimators.

Our off-policy evaluation experiments are designed as follows. For each supervised multiclass classification dataset, we split it equally into train and test sets. For the train set, we use 20% of the full-information data to train a multiclass logistic regression model that serves as the stochastic logging policy \( \pi_0 \). Similarly, we use 10% of the full-information train set to train a deterministic
multiclass logistic regression model $\pi_r(x)$ as the reward predictor, simulating that there is typically a small amount of labeled data available in most applications. The policy $\pi$ we want to evaluate is a stochastic multiclass logistic regression trained on the whole train set. Finally we use the full-information test set to generate the contextual bandit datasets $S$ for off-policy evaluation of sizes $n = 100, 200, 500...$ until we reach two passes through the whole test set. We evaluate the policy $\pi$ with different estimators on the logged bandit feedback of different size and treat the performance on the full-information test set as ground truth $R(\pi)$. We adopt the most common measure for the performance of the estimator, namely mean squared error (MSE), which is defined as $E_S[(\hat{R}(\pi) - R(\pi))^2]$. For each experiment, we repeat the bandit data generating process 500 times for statistical stability, and calculate the bias, variance and MSE.

Can CAB achieve improved estimation accuracy by trading bias for variance through $M$? Figure 1 shows how the choice of $M$ affects the bias/variance trade-off of CAB. Each plot reflects a particular size of bandit-feedback dataset for the SATIMAGE dataset (other datasets give qualitatively similar results). For each dataset size, the bias decreases as we increases $M$ as expected, since CAB moves towards the unbiased IPS estimator. However, the variance increases since more data points are relying on inverse propensity weights. For all dataset sizes, the MSE curve looks like a $U$ shape curve where the minimum lies in the middle. This confirms that CAB can decrease MSE compared to IPS and DM.

Analyzing Figure 1 in more detail, the bias stays nearly unchanged as we increase the dataset size, while variance decreases substantially. This mimics the real life scenario that sometimes we have a better IPS model, while sometimes the direct model dominates. As we see from all these curves, the CAB could achieve a reduction in MSE over both methods under all scenarios.

How does CAB compare to other off-policy estimators? Figure 2 compares different off-policy estimators on 9 UCI datasets. For each dataset, in order to show the performance comparison clearly, we present the results for moderate amounts of data. In particular, we compare the MSE of the proposed CAB with DM, CIPS, DR, SWITCH, and SB. Notice that CAB, CIPS and SWITCH all have a clipping parameter $M \in [0, \infty)$, while for SB the blending is achieve through the static weight parameter $\tau_{SB} \in [0, 1]$. To be able to plot SB together with the other estimators, we rescale $\tau_{SB}$ in the plot for comparison. DM and DR do not have any hyperparameter, so we use two horizontal lines to represent them.

For both SB and CAB and across all datasets, we observe a $U$ shape curve for MSE with the optimum value in the middle. However, on most datasets CAB substantially outperforms SB. Furthermore, CAB outperforms CIPS in the full range of M on all datasets, indicating that imputing a reasonably good regression estimate is indeed consistently better than naively imputing zero. For the SWITCH estimator, the MSE curve is substantially more erratic than that of CAB, which we conjecture is due to the hard switch it makes and the discontinuities this implies. Furthermore, CAB performs at least comparable to SWITCH across all nine datasets, and on some it can be substantially more accurate than SWITCH.

For all the datasets, DR outperforms IPS and DM as expected. However, CAB still outperforms DR in most datasets, which validates the idea that estimators outside the class of unbiased estimators can have advantages on this problem. Furthermore, when used in learning one already faces a bias/variance trade-off due to the capacity of the policy space, such that it seems unjustified to insist on the unbiasedness of the empirical risk to begin with.

4.2 Ranking Evaluation for Biased User Feedback

We now consider the related problem of ranking evaluation based on implicit feedback (e.g. clicks, duel time). Here the selection bias on the feedback signal is strongly influenced by position bias, since items lower in the ranking are less likely to be discovered by the user. However, it can be shown that this bias can be estimated [8], and that the resulting estimates can serve as propensities in IPS-style estimators.

To connect the ranking setting with the contextual bandit setting more formally, now each context $x_i$ represents query and/or user profile. However, we no longer consider actions atomic, but instead treat rankings as combinatorial actions where the reward decomposes a weighted sum of component rewards. The observable reward per component $c_{ij} \in \{0, 1\}$ may be whether the user clicks the
Figure 1: Bias, Variance and MSE graph for CAB on the SATIMAGE dataset.

Figure 2: MSE comparison of off-policy estimators for different UCI datasets.
Figure 3: Bias, variance and MSE for CAB with different amount of user data averaged over 100 runs for ranking evaluation. We pass through the dataset for 0.1, 0.2, 0.5, 1 sweeps to simulate different amounts of feedback data.

document or not. Note that we typically don’t observe all component rewards, and there is inherent ambiguity whether lack of a click means lack of relevance or lack of discovery. We thus use a latent variable \( o_{ij} \in \{0, 1\} \) to represent whether the user observes the document \( d_{ij} \) in ranking \( j \), which then leads to the following click model: a user clicks a document when the user observes it and the document is relevant, \( c_{ij} = r_{ij} \cdot o_{ij} \) where \( r_{ij} \in \{0, 1\} \) represents whether document \( d_{ij} \) is relevant in context \( x_{i} \).

Note that \( o_{ij} \) is not observed by the system, but one can estimate a missingness model [8]. In particular, let \( p_{ij} \) be the (estimated) probability of \( 1_{(o_{ij}=1)} \), which will serve as propensity. In the following experiments we use \( p_{ij} = \frac{1}{2} \) to sample clicks and as propensities in our estimators. We can then sample data \( S = \{(x_{i}, d_{ij}, p_{ij}, c_{ij})\}_{i=1}^{m} \) where \( m_{i} \) is the number of candidates for context \( x_{i} \). We denote the regression model of the DM as \( \hat{P}(r_{ij} = 1|x_{i}, d_{ij}) \). The various estimators for additive ranking metrics can be written in the form

\[
\hat{R} = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{m_{i}} \left[ (1 - \tau_{ij}) \hat{P}(r_{ij} = 1) + \tau_{ij} \frac{1_{(c_{ij}=1)}}{p_{ij}} \right] \lambda(j) \tag{26}
\]

where \( \lambda() \) is a function that maps a rank to a score and \( \tau_{ij}^{IPS} = 1, \tau_{ij}^{DM} = 0, \tau_{ij}^{CAB} = \min(p_{ij} M, 1), \tau_{ij}^{SWITCH} = 1_{(\frac{1}{p_{ij}} \leq M)} \), and \( \tau_{ij}^{SB} = \tau_{ij}^{SB} \).

Note that DR in this setting is

\[
\hat{R}_{DR} = \frac{1}{n} \sum_{i} \sum_{j} \left[ \hat{P}(r_{ij} = 1|x_{i}, d_{ij}) + \frac{1_{(r_{ij}=1)} - \hat{P}(r_{ij} = 1|x_{i}, d_{ij})}{p_{ij}} \right] 1_{(o_{ij}=1)} \lambda(j) \tag{27}
\]

which, unfortunately, is not applicable in the ranking setting. The reason is that DR depends on \( o_{ij} \), which is neither observed nor fully captured by \( c_{ij} \).

We conduct experiments on the Yahoo LTR Challenge corpus (set 1), which comes with a train/validation/test split. We use 10% of the training set for learning a DM, which reflects that we typically have a small amount of manual relevance judgements. The DM is a binary Gradient Boosted Decision Tree Classifier calibrated by Sigmoid Calibration [10]. We use \( \lambda(j) = 1 \) as the performance metric which can be interpreted as the average rank of the relevant results. To get a ranking policy for evaluation, we train a ranking SVM [7] on the remaining. As input to the estimators, different amounts of click data are generated from the test set. For each data set size, we generate the log data 100 times and report the bias, variance, and MSE with respect to the estimated ground truth from the full-information test set.

Can CAB achieve improved estimation accuracy by trading bias for variance through \( M \)?

The bias, variance and MSE under different choices of \( M \) and different amounts of user feedback for CAB are shown in Figure [3]. For each dataset size, as we increase \( M \), the bias decreases while the variance increases, which are expected since we use less of the biased DM and more of the large-variance IPS. For all data set sizes, the best MSE always falls in the middle of the range of \( M \) which confirms that CAB can effectively trade-off between bias and variance through controlling \( M \) for a range of data-set sizes.
How does CAB compare to other off-policy estimators? Figure 4 compares the MSE of the different estimators on different data-set sizes. We rescale $\tau_{SB}$ in the plot similarly as in Section 4.1. CAB is consistently better than CIPS which again confirms that, with a reasonably good DM, imputing an estimate outperforms naively imputing zero. The SB estimator is generally worse than the adaptive blending methods CAB and SWITCH. The SWITCH estimator and CAB have similar optimal performance as expected. However, SWITCH is again somewhat more erratic than CAB when $M$ is varied. While SWITCH can be used in LTR algorithms like Propensity SVM-Rank [8], we conjecture that this behavior may make model selection during learning more stable for CAB than for SWITCH.

5 Conclusion

The paper proposed and analyzed the CAB estimator, which is designed to provide a controllable bias/variance trading-off for off-policy evaluation and learning. We show theoretically that CAB is typically less biased than CIPS and DM and often enjoys smaller variance than IPS and DR. Experiment results on two commonly used partial-information settings show that it can consistently achieve improved MSE over the two special cases it generalizes, namely IPS and DM. In addition, it is consistently better than CIPS at different levels of clipping for reasonably good regression models. In most cases it outperforms DR which suggests that allowing some bias can be beneficial for MSE reduction. Compared to static blending via SB, CAB generally achieves improved performance. Finally, unlike SWITCH, CAB is sub-differentiable which allows its use for learning in gradient-descent algorithms for the contextual bandit setting. The use of CAB for learning is part of our future work.

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Appendices

A Proof of Theorem 1

Theorem 1 (Bias of CAB). For contexts \( x \) drawn i.i.d from some distribution \( P(\mathcal{X}) \) and for actions \( y \sim \pi_0(y|x) \), under Assumption 1 the bias of \( \hat{R}_{CAB}(\pi) \) is

\[
\text{Bias}(\hat{R}_{CAB}(\pi)) = \mathbb{E}_x \mathbb{E}_y [\delta(x, y) \zeta(x, y) \mathbb{1}_{(c(x, y) \leq M)}] + \mathbb{E}_x \mathbb{E}_y [\Delta(x, y) \mathbb{1}_{(c(x, y) > M)}]
\]

\[
- M \mathbb{E}_x \mathbb{E}_{\pi_0} \left[ \frac{1}{1 - \zeta(x, y)} \Delta(x, y) \mathbb{1}_{(c(x, y) > M)} \right]
\]

\[
- M \mathbb{E}_x \mathbb{E}_{\pi_0} \left[ \delta(x, y) \frac{\zeta(x, y)}{1 - \zeta(x, y)} \mathbb{1}_{(c(x, y) > M)} \right]
\]

Proof. To shorten the notations, recall that the importance weights \( c(x, y) := \frac{\pi(y|x)}{\pi_0(y|x)} \), the deviations from propensity and reward models are \( \zeta(x, y) = 1 - \frac{\pi(y|x)}{\pi_0(y|x)} \), and \( \Delta(x, y) = \delta(x, y) - \delta(x, y) \). For simplicity, let \( \delta := \delta(x, y) \) and \( \zeta := \zeta(x, y) \) in the following proof. Moreover, we use \( x, y \sim \pi \) to denote the data are generated by \( x \sim \mathcal{X}, y \sim \pi(\mathcal{Y}|x) \). Let \( Y_x \) and \( Y_x^c \) denote the disjoint sets with \( Y_x := \{y \in \mathcal{Y} : c(x, y) \leq M\} \) and \( Y_x^c := \mathcal{Y} \setminus Y_x \). Here we consider the random reward model, where \( r(x, y) \sim P_r(x, y) \), with \( \delta(x, y) = \mathbb{E}_{P_r}[r(x, y)] \). Let \( v(\pi) := \mathbb{E}_{x \sim \mathcal{X}, y \sim \pi(\mathcal{Y}|x)}[r(x, y)] \), then the bias for CAB is given by: \( \mathbb{E}[\hat{R}_{CAB}(\pi)] - v(\pi) \).

\[
\mathbb{E}[\hat{R}_{CAB}(\pi)] - v(\pi) = \mathbb{E}_{x, y \sim \pi_0}[\mathbb{1}_{y \in Y_x} \frac{\pi(y|x)}{\pi_0(y|x)} \delta(x, y)] - \mathbb{E}_{x}[\sum_{y \in Y_x} \pi(y|x) \delta(x, y)]
\]

\[
+ \mathbb{E}_{x, y \sim \pi_0}(M \frac{\hat{\pi}_0(y|x)}{\pi(y|x)} \frac{\pi(y|x)}{\pi_0(y|x)} \delta(x, y) \mathbb{1}_{y \in Y_x^c})
\]

\[
+ \mathbb{E}_x \sum_{y \in Y_x^c} \pi(y|x)(1 - M \frac{\hat{\pi}_0(y|x)}{\pi(y|x)}) \delta(x, y) + \Delta(x, y))]
\]

\[
- \mathbb{E}_x \sum_{y \in Y_x^c} \pi(y|x) \delta(x, y)
\]

Separating the terms in two parts based on \( Y_x \) and \( Y_x^c \), and analyzing the parts based on \( Y_x \) gives us:

\[
A := \mathbb{E}_{x, y \sim \pi_0}[\mathbb{1}_{(y \in Y_x)} \frac{\pi(y|x)}{\pi_0(y|x)} \delta(x, y)] - \mathbb{E}_x \sum_{y \in Y_x} \pi(y|x) \delta(x, y)
\]

\[
= \mathbb{E}_x \sum_{y \in Y_x} \pi(y|x) \frac{\pi_0(y|x)}{\pi_0(y|x)} \delta(x, y) - \sum_{y \in Y_x} \pi(y|x) \delta(x, y)
\]

\[
= \mathbb{E}_{x, y \sim \pi}[\delta(x, y) \mathbb{1}_{(c(x, y) \leq M)}]
\]
Analyzing the expectation on $Y^x_\pi$ works in a similar fashion:

$$B := \mathbb{E}_{x,y \sim \pi_0} [M \hat{\pi}_0(y|x) \pi(y|x) \delta(x,y) \mathbb{I}_{y \in Y^x_\pi}]$$

$$+ \mathbb{E}_x \left[ \sum_{y \in Y^x_\pi} \pi(y|x) (1 - M \hat{\pi}_0(y|x)) \delta(x,y) + \Delta \right] - \mathbb{E}_x \left[ \sum_{y \in Y^x_\pi} \pi(y|x) \delta(x,y) \right]$$

$$= M \mathbb{E}_{x,y \sim \pi_0} [\delta(x,y) \mathbb{I}_{\{c(x,y) > M\}}] + \mathbb{E}_{x,y \sim \pi} [(1 - M \hat{\pi}_0(y|x)) \frac{1}{1 - \zeta} \Delta \mathbb{I}_{\{c(x,y) > M\}}]$$

$$- M \mathbb{E}_{x,y \sim \pi} [\delta(x,y) \frac{\pi_0(y|x)}{\hat{\pi}_0(y|x)} \frac{1}{1 - \zeta} \mathbb{I}_{\{c(x,y) > M\}}]$$

$$= M \mathbb{E}_{x,y \sim \pi_0} [\delta(x,y) \mathbb{I}_{\{c(x,y) > M\}}] + \mathbb{E}_{x,y \sim \pi} [\Delta \mathbb{I}_{\{c(x,y) > M\}}] - M \mathbb{E}_{x,y \sim \pi_0} \left[ \frac{1}{1 - \zeta} \Delta \mathbb{I}_{\{c(x,y) > M\}} \right]$$

$$- M \mathbb{E}_{x,y \sim \pi_0} [\delta(x,y) \frac{1}{1 - \zeta} \mathbb{I}_{\{c(x,y) > M\}}]$$

$$= \mathbb{E}_{x,y \sim \pi} [\Delta \mathbb{I}_{\{c(x,y) > M\}}] - M \mathbb{E}_{x,y \sim \pi_0} \left[ \frac{1}{1 - \zeta} \Delta \mathbb{I}_{\{c(x,y) > M\}} \right] - M \mathbb{E}_{\pi_0} [\delta(x,y) \frac{\zeta}{1 - \zeta} \mathbb{I}_{\{c(x,y) > M\}}]$$

(30)

Combining all above, we have:

$$Bias(\hat{R}_{CAB}(\pi)) = \mathbb{E}_{x,y \sim \pi} [-\delta(x,y)\zeta(x,y) \mathbb{I}_{\{c(x,y) \leq M\}}] + \mathbb{E}_{x,y \sim \pi} [\Delta(x,y) \mathbb{I}_{\{c(x,y) > M\}}]$$

$$- M \mathbb{E}_{x,y \sim \pi_0} \left[ \frac{1}{1 - \zeta(x,y)} \Delta(x,y) \mathbb{I}_{\{c(x,y) > M\}} \right]$$

$$- M \mathbb{E}_{x,y \sim \pi_0} [\delta(x,y) \frac{\zeta(x,y)}{1 - \zeta(x,y)} \mathbb{I}_{\{c(x,y) > M\}}]$$

(31)

**Remark.** Here we recover the bias of DM and IPS as a special case of CAB estimator, and discuss special cases when we have logged propensities, or perfect reward model.

1. when $M$ goes to $\infty$, we have $y \in Y_x$ for all $x$, then

$$Bias(\hat{R}_{CAB}(\pi)) = \mathbb{E}_{x,y \sim \pi} [-\delta(x,y)\zeta(x,y)]$$

in which we recover the bias of IPS.

2. when $M = 0$, we have

$$Bias(\hat{R}_{CAB}(\pi)) = \mathbb{E}_{x,y \sim \pi} (\Delta(x,y))$$

in which we recover the bias of DM.

3. when propensity model is known or we have perfect propensity estimation, then $\zeta(x,y) = 0$ for all $(x,y)$ pair, and

$$Bias(\hat{R}_{CAB}(\pi)) = \mathbb{E}_{x,y \sim \pi} [\Delta(x,y) \mathbb{I}_{\{c(x,y) > M\}}] - M \mathbb{E}_{x,y \sim \pi_0} [\Delta(x,y) \mathbb{I}_{\{c(x,y) > M\}}]$$

(32)

4. when regression model is perfect, then we have $\Delta(x,y) = 0$ for all $(x,y)$ pair, then the bias becomes:

$$Bias(\hat{R}_{CAB}(\pi)) = \mathbb{E}_{x,y \sim \pi} [-\delta(x,y)\zeta(x,y) \mathbb{I}_{\{c(x,y) \leq M\}}]$$

$$- M \mathbb{E}_{x,y \sim \pi_0} [\delta(x,y) \frac{\zeta(x,y)}{1 - \zeta(x,y)} \mathbb{I}_{\{c(x,y) > M\}}]$$

(33)
B Proof of Theorem 2

Theorem 2 (Variance of CAB). Under the same conditions as in Theorem 1, the variance of $R_{CAB}$ ($\pi$) is

$$\text{Var}(R_{CAB}(\pi)) = \frac{1}{n} \left( \sum_a \pi(a|x) \frac{\delta(x,a) + \Delta(x,a)}{\tau_{xy}(r(x,y))} \right) - \frac{\Delta(x,y)}{\eta - \zeta}
\sum_a \pi(a|x) \frac{\delta(x,a) + \Delta(x,a)}{\tau_{xy}(r(x,y))} \cdot \mathbb{1}_{\{c(x,y) > M\}}
$$

$\hat{R}_{CAB} = \frac{1}{n} \sum_{i=1}^{n} Z_i(x,y,r)$, and $E(\hat{R}_{CAB}) = E(x,y) \sim \pi \cdot Z_i(x,y,r)$. Since $\text{Var}(\hat{R}_{CAB}) = \frac{1}{n} \sum_{i=1}^{n} \text{Var}(Z_i(x,y,r))$, we will calculate $\text{Var}(Z)$ first, by using the formula $\text{Var}(Z) = E(Z^2) - E(Z)^2$.

We will start with the term $E(Z^2)$.

$$E(x,y) \sim \pi \cdot Z_i(x,y,r) = E(x,y) \sim \pi \cdot \left[ \sum_a \pi(a|x) \frac{\delta(x,a) + \Delta(x,a)}{\tau_{xy}(r(x,y))} \right]$$

$$= \frac{\sum_a \pi(a|x) \frac{\delta(x,a) + \Delta(x,a)}{\tau_{xy}(r(x,y))}}{\sum_a \pi(a|x) \frac{\delta(x,a) + \Delta(x,a)}{\tau_{xy}(r(x,y))}} + \sum_a \pi(a|x) \frac{\delta(x,a) + \Delta(x,a)}{\tau_{xy}(r(x,y))} \cdot \mathbb{1}_{\{c(x,y) > M\}}$$

(34)

and we simplify the three terms one by one.

$$A := E_x((\sum_a \pi(a|x) \frac{\delta(x,a) + \Delta(x,a)}{\tau_{xy}(r(x,y))})^2) = E_x((\sum_a \pi(a|x) (1 - \tau_{xy}(r(x,y))))^2)$$

$$= E_x((\sum_a \pi(a|x) (1 - \tau_{xy}(r(x,y))))^2) = E_x((\sum_a \pi(a|x) (1 - \tau_{xy}(r(x,y))))^2)$$

(35)

$$B := E_x((\sum_a \pi(a|x) \frac{\delta(x,a) + \Delta(x,a)}{\tau_{xy}(r(x,y))} \cdot \mathbb{1}_{\{c(x,y) > M\}})^2)$$

(36)
\[ C := 2 \mathbb{E}_x \left[ \sum_a \pi(a|x)(1 - \tau_{ax}) \delta(x, a) \right] \mathbb{E}_{y \sim \pi_{0, r}} \left( \frac{\pi(y|x)}{\pi_0(y|x)} r \right) \]

\[ = 2 \mathbb{E}_x \left[ \sum_a \pi(a|x)(1 - \frac{\pi_0(a|x)}{\pi(a|x)} \frac{M}{1 - \zeta}) (\delta + \Delta) \mathbb{I}_{(c \geq M)} \right] \]

\[ \cdot (\mathbb{E}_{y \sim \pi,r}[|(1 - \zeta) r \mathbb{I}_{(c < M)} + (1 - \zeta) \frac{\pi_0(y|x)}{\pi(y|x)} \frac{M}{1 - \zeta} r \mathbb{I}_{(c \geq M)}]) \]

\[ = 2 \mathbb{E}_x \left[ (\mathbb{E}_{\pi}(\delta + \Delta) \mathbb{I}_{(c \geq M)}) - \mathbb{E}_{\pi_0}[\frac{M}{1 - \zeta} (\delta + \Delta) \mathbb{I}_{(c \geq M)}] \right] \]

\[ \cdot [\mathbb{E}_r[(1 - \zeta) \delta \mathbb{I}_{(c < M)}] + \mathbb{E}_{\pi_0}[M \delta \mathbb{I}_{(c \geq M)}]] \]

Combining all the three terms above, we have:

\[ \mathbb{E}[Z^2] = \mathbb{E}_x \left[ (\mathbb{E}_{\pi}(\delta + \Delta) \mathbb{I}_{(c \geq M)}) - \mathbb{E}_{\pi_0}[\frac{M(\delta + \Delta)}{1 - \zeta} \mathbb{I}_{(c \geq M)}]^2 \right] \]

\[ + M^2 \mathbb{E}_{x,y \sim \pi_0,r}[(r^2 \mathbb{I}_{(c \geq M)}) + \mathbb{E}_{x,y \sim \pi_0}(\frac{\pi(y|x)}{\pi_0(y|x)} r^2 (1 - \zeta)^2 r^2 \mathbb{I}_{(c < M)}))] \]

\[ + 2 \mathbb{E}_x \left[ (\mathbb{E}_{\pi}(\delta + \Delta) \mathbb{I}_{(c \geq M)}) - \mathbb{E}_{\pi_0}[\frac{M}{1 - \zeta} (\delta + \Delta) \mathbb{I}_{(c \geq M)}] \right] \]

\[ \cdot [\mathbb{E}_r[(1 - \zeta) \delta \mathbb{I}_{(c < M)}] + \mathbb{E}_{\pi_0}[M \delta \mathbb{I}_{(c \geq M)}]] \]

From Theorem 1, it is given that

\[ \mathbb{E}_{x,y \sim \pi_0,r}[Z] = \mathbb{E}_{x,y \sim \pi}[(1 - \zeta) \delta \mathbb{I}_{(c < M)}] + \mathbb{E}_{x,y \sim \pi}[(\Delta + \delta) \mathbb{I}_{(c \geq M)}] - M \mathbb{E}_{x,y \sim \pi_0}[\frac{\Delta}{1 - \zeta} \mathbb{I}_{(c \geq M)}] \]

\[ - M \mathbb{E}_{x,y \sim \pi_0}[\delta \frac{\zeta}{1 - \zeta} \mathbb{I}_{(c \geq M)}] \]

For the term \( \mathbb{E}[Z^2] \), with some simple transformations, we have:

\[ \mathbb{E}[Z^2] := (\mathbb{E}_{x,y \sim \pi_0}(\frac{\pi(y|x)}{\pi_0(y|x)} (1 - \zeta) \delta \mathbb{I}_{(c < M)})) + M(\mathbb{E}_{x,y \sim \pi_0}(\delta \mathbb{I}_{(c \geq M)}))^2 \]

\[ + (\mathbb{E}_{x,y \sim \pi}(\Delta + \delta) \mathbb{I}_{(c \geq M)}) - \mathbb{E}_{x,y \sim \pi_0}[\frac{M(\delta + \Delta)}{1 - \zeta} \mathbb{I}_{(c \geq M)}])^2 \]

\[ + 2(\mathbb{E}_{x,y \sim \pi}(\delta + \Delta) \mathbb{I}_{(c \geq M)}) - \mathbb{E}_{x,y \sim \pi_0}[\frac{M}{1 - \zeta} (\delta + \Delta) \mathbb{I}_{(c \geq M)}]) \]

\[ \cdot (\mathbb{E}_{x,y \sim \pi}((1 - \zeta) \delta \mathbb{I}_{(c < M)}) + \mathbb{E}_{x,y \sim \pi_0}(M \delta \mathbb{I}_{(c \geq M)})) \]

where we use the fact that:

\[ - M \mathbb{E}_{x,y \sim \pi_0}[\frac{\Delta}{1 - \zeta} \mathbb{I}_{(c \geq M)}] - M \mathbb{E}_{x,y \sim \pi_0}[\delta \frac{\zeta}{1 - \zeta} \mathbb{I}_{(c \geq M)}] \]

\[ = M \mathbb{E}_{x,y \sim \pi_0}[\delta \mathbb{I}_{(c \geq M)}] - \mathbb{E}_{x,y \sim \pi_0}[\frac{M(\delta + \Delta)}{1 - \zeta} \mathbb{I}_{(c \geq M)}] \]

Now based on the formula \( \mathbb{V}[Z] = \mathbb{E}[Z^2] - \mathbb{E}[Z]^2 := A + B + C - D - E - F. \)
For the term $A - E$, we have

\[
\mathbb{E}_x[(\mathbb{E}_{y \sim \pi}[(\delta + \Delta) \mathbb{I}_{\{c \geq M\}}] - \mathbb{E}_{y \sim \pi_0}[M(\delta + \Delta) \mathbb{I}_{\{c \geq M\}}])]^2 - (\mathbb{E}_{x, y \sim \pi}[(\Delta + \delta) \mathbb{I}_{\{c \geq M\}}] - \mathbb{E}_{x, y \sim \pi_0}[M(\delta + \Delta) \mathbb{I}_{\{c \geq M\}}])^2
\]

(42)

For the term $B - D$, we have

\[
M^2 \mathbb{E}_{x, y \sim \pi, r}(r^2 \mathbb{I}_{\{c \geq M\}}) + \mathbb{E}_{x, y \sim \pi, r}(\frac{\pi(y|x)}{\pi_0(y|x)}(1 - \zeta)^2 r^2 \mathbb{I}_{\{c < M\}}) - (\mathbb{E}_{x, y \sim \pi, r}(\pi(y|x) \mathbb{I}_{\{c < M\}}) + M \mathbb{E}_{x, y \sim \pi_0}[\delta \mathbb{I}_{\{c \geq M\}}])^2
\]

(43)

where \(1\) follows from:

\[
M^2 \mathbb{V}_{x, y \sim \pi_0}(\delta \mathbb{I}_{\{c \geq M\}}) + \mathbb{V}_{x, y \sim \pi_0}(\frac{\pi(y|x)}{\pi_0(y|x)}(1 - \zeta)^2 \mathbb{I}_{\{c < M\}}) - M^2 \mathbb{E}_{x, y \sim \pi_0}[\delta \mathbb{I}_{\{c \geq M\}}] + \mathbb{E}_{x, y \sim \pi_0}[\frac{\pi(y|x)}{\pi_0(y|x)}(1 - \zeta)^2(\delta - \mathbb{E}_{x, y \sim \pi_0}[\delta \mathbb{I}_{\{c \geq M\}}])]
\]

(44)
and

\[ M^2 V_x(\mathbb{E}_{y \sim \pi_0}[\delta \mathbb{1}_{(c \geq M)}]) + V_x(\mathbb{E}_{y \sim \pi_0} \mathbb{E}_{(y|x)} \frac{\pi(y|x)}{\pi_0(y|x)} (1 - \zeta) \delta \mathbb{1}_{(c < M)}) \]

\[ - 2M \mathbb{E}_{x,y \sim \pi}[(1 - \zeta) \delta \mathbb{1}_{(c < M)}] \mathbb{E}_{x,y \sim \pi_0} \delta \mathbb{1}_{(c \geq M)} \]

\[ = V_x(\mathbb{E}_{y \sim \pi_0}[M \delta \mathbb{1}_{(c \geq M)}]) + V_x(\mathbb{E}_{y \sim \pi_0} \mathbb{E}_{(y|x)} \frac{\pi(y|x)}{\pi_0(y|x)} (1 - \zeta) \delta \mathbb{1}_{(c < M)}) \]

\[ - 2C_x(\mathbb{E}_{y \sim \pi_0}[M \delta \mathbb{1}_{(c \geq M)}]) + V_x(\mathbb{E}_{x,y \sim \pi_0} \mathbb{E}_{(y|x)} \frac{\pi(y|x)}{\pi_0(y|x)} (1 - \zeta) \delta \mathbb{1}_{(c < M)}) \]

\[ = 2C_x(\mathbb{E}_{y \sim \pi_0}[M \delta \mathbb{1}_{(c \geq M)}]) + V_x(\mathbb{E}_{y \sim \pi_0} \mathbb{E}_{(y|x)} \frac{\pi(y|x)}{\pi_0(y|x)} (1 - \zeta) \delta \mathbb{1}_{(c < M)}) \]  

For the term \( C - F \), we have:

\[ 2 \mathbb{E}_x(\mathbb{E}_\pi[(\delta + \Delta) \mathbb{1}_{(c \geq M)}]) - \mathbb{E}_\pi_0[M \frac{1}{1 - \zeta} (\delta + \Delta) \mathbb{1}_{(c \geq M)}] - \mathbb{E}_x(\mathbb{E}_\pi[(1 - \zeta) \delta \mathbb{1}_{(c < M)}] + \mathbb{E}_\pi_0[M \delta \mathbb{1}_{(c \geq M)}]) \]

\[ - 2(\mathbb{E}_x,y \sim \pi_0[(\delta + \Delta) \mathbb{1}_{(c \geq M)}]) - \mathbb{E}_x,y \sim \pi_0[M \frac{1}{1 - \zeta} (\delta + \Delta) \mathbb{1}_{(c \geq M)}]) \cdot (\mathbb{E}_x,y \sim \pi_0[(1 - \zeta) \delta \mathbb{1}_{(c < M)}] + \mathbb{E}_x,y \sim \pi_0[M \delta \mathbb{1}_{(c \geq M)}]) \]

\[ = 2C_x(\mathbb{E}_\pi[(\delta + \Delta) \mathbb{1}_{(c \geq M)}]) - \mathbb{E}_\pi_0[M \frac{1}{1 - \zeta} (\delta + \Delta) \mathbb{1}_{(c \geq M)}]) - \mathbb{E}_x(\mathbb{E}_\pi[(1 - \zeta) \delta \mathbb{1}_{(c < M)}] + \mathbb{E}_\pi_0[M \delta \mathbb{1}_{(c \geq M)}]) \]

(46)

Combining the above three terms, we have:

\[ \forall Z = \mathbb{E}_x [\mathbb{E}_\pi(\delta + \Delta) \mathbb{1}_{(c \geq M)}] - M \mathbb{E}_\pi_0[\frac{\Delta}{1 - \zeta} \mathbb{1}_{(c < M)}] - M \mathbb{E}_\pi_0(\delta \frac{\Delta}{1 - \zeta} \mathbb{1}_{(c \geq M)}) + \mathbb{E}_\pi[1 - \zeta) \delta \mathbb{1}_{(c < M)}] \]

+ \mathbb{E}_x \mathbb{E}_\pi_0(\mathbb{E}_{y \sim \pi_0} \mathbb{E}_{(y|x)} \frac{\pi(y|x)}{\pi_0(y|x)} (1 - \zeta) \delta \mathbb{1}_{(c < M)})

+ M^2 \mathbb{E}_x \mathbb{E}_\pi_0[(\delta - \Delta)^2 \mathbb{1}_{(c \geq M)}] + M^2 \mathbb{E}_x(\mathbb{E}_\pi_0 \mathbb{1}_{(c \geq M)})

- 2 \mathbb{E}_x(\mathbb{E}_{x,y \sim \pi_0}[M \delta \mathbb{1}_{(c \geq M)}]) \mathbb{E}_y \sim \pi_0[(1 - \zeta) \delta \mathbb{1}_{(c < M)}]

= \mathbb{E}_x \mathbb{E}_\pi_0[\Delta x \mathbb{1}_{(c > M)}] - M \mathbb{E}_\pi_0[\frac{\Delta}{1 - \zeta} \mathbb{1}_{(c < M)}] - M \mathbb{E}_\pi_0(\delta \frac{\Delta}{1 - \zeta} \mathbb{1}_{(c \geq M)}) + \mathbb{E}_\pi_0[(1 - \zeta) \delta (x, y) \mathbb{1}_{(c < M)}]

+ \mathbb{E}_x \mathbb{E}_\pi_0(\mathbb{E}_{y \sim \pi_0} \mathbb{E}_{(y|x)} \frac{\pi(y|x)}{\pi_0(y|x)} (1 - \zeta) \delta \mathbb{1}_{(c < M)}) + M^2 \mathbb{E}_x \mathbb{E}_\pi_0[(\delta - \Delta)^2 \mathbb{1}_{(c \geq M)}]

+ \mathbb{E}_x \mathbb{E}_\pi_0(\mathbb{E}_{y \sim \pi_0} \mathbb{E}_{(y|x)} \frac{\pi(y|x)}{\pi_0(y|x)} (1 - \zeta) \delta \mathbb{1}_{(c < M)} + M \delta \mathbb{1}_{(c \geq M)}]

(47)

where the second equality follows from the fact that:

\[ \forall(a) + \forall(b) - 2 \mathbb{E}(a) \mathbb{E}(b) = \forall(a + b) - 2 \mathbb{E}(ab) \]

(48)

and

\[ \mathbb{E}_x \mathbb{E}_\pi_0(\mathbb{E}_{y \sim \pi_0} \mathbb{E}_{(y|x)} \frac{\pi(y|x)}{\pi_0(y|x)} (1 - \zeta) \delta \mathbb{1}_{(c < M)} \times M \delta \mathbb{1}_{(c \geq M)}) = 0 \]

(49)

Dividing by \( n \) gives the variance for the CAB estimator.

\[ \square \]

**Remark.** Here we recover the variance of DM and IPS as a special case of CAB estimator.

1. when \( M = 0 \), we have:

\[ \text{Var}(\hat{R}_{\text{CAB}}(\pi)) = \mathbb{E}_x(\mathbb{E}_{y \sim \pi_0}[\delta (x, y) + \Delta (x, y)]) \]

(50)

which we recover the variance of DM.
2. when $M$ goes to $\infty$, we have:

$$\text{Var}(\hat{R}_{\text{CAB}}(\pi)) = \mathbb{E}_\pi \left[ \frac{\pi(y|x)}{\pi_0(y|x)} (1-\zeta(x,y))^2 \right] - \left\{ \mathbb{E}_\pi [(1-\zeta(x,y))\delta(x,y)] \right\}^2$$

which we recover the variance of IPS after some transformations.