Charged relativistic particles in non-commutative space

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Abstract

In analogy to Wong’s equations describing the motion of a charged relativistic point particle in the presence of an external Yang-Mills field, we discuss the motion of such a particle in non-commutative space subject to an external $U_1(1)$ gauge field. This formulation, which is based on an action written in Moyal space, provides a coarser level of description than full QED on non-commutative space. The results are compared with those obtained from the different Hamiltonian approaches. Furthermore, a continuum version for Wong’s equations and for the motion of a particle in non-commutative space is derived.
1 Introduction

Over the last twenty years, non-commuting spatial coordinates have appeared in various contexts in the framework of quantum gravity and superstring theories. This fact contributed to the motivation for studying classical and quantum theories with a finite or infinite number of degrees of freedom on non-commutative spaces. Different mathematical approaches have been pursued and various physical applications have been explored, e.g. see references [1–3] for some partial reviews. Beyond the applications, classical and quantum mechanics on non-commutative space are of interest as toy models for field theories which are more difficult to handle, in particular in the case of interactions with gauge fields. In this respect, we recall a similar situation concerning the coupling of matter to Yang-Mills fields on ordinary space: A coarser level of description for the latter theories has been proposed by Wong [4] who considered the motion of charged point particles in an external gauge field [5–7]. The latter equations allow for various physical applications, e.g. to the dynamics of quarks and their interaction with gluons [6]. Somewhat similar equations, known as Mathisson-Papapetrou-Dixon equations [8] appear in general relativity for a spinning particle in curved space. In this spirit, we will consider in the present work the dynamics of a relativistic “point” particle in non-commutative space subject to an external $U_1(1)$ gauge field (thereby implementing a suggestion made in an earlier work [9], see also [10] for some related studies based on a first order expansion in the non-commutativity parameter).

More precisely, we are motivated by the motion in Moyal space, the latter space having been widely discussed as the arena for field theory in non-commutative spaces.
By proceeding along the lines of Wong’s equations (which are discussed in Section 2), we derive a set of equations for the dynamics of the particle in Moyal space in Section 3. For the description of the coupling of a particle to a gauge field, the relativistic setting is the most natural one, but our discussions (of Wong’s equations in commutative space or of a particle in non-commutative space) could equally well be done within the non-relativistic setting. Subsequently in Section 4, we briefly recall the different Hamiltonian approaches which have previously been pursued for the formulation of classical mechanics in non-commutative space and we compare the resulting equations governing the dynamics of particles coupled to an electromagnetic field. The particular case of a constant field strength is the most tractable one and will be considered by way of illustration in Section 5. In the appendix we present a continuum formulation of Wong’s equations on a generic space-time manifold which formulation readily generalizes to Moyal space.

For the motion of a relativistic point particle in four-dimensional (commutative or non-commutative) space-time, the following notations will be used. The metric tensor is given by

\[(\eta_{\mu\nu})_{\mu,\nu\in\{0,1,2,3\}} = \text{diag} (1, -1, -1, -1)\]

and we choose the natural system of units \((c = \hbar = 1)\). The proper time \(\tau\) for the particle is defined (up to an additive constant) by

\[d\tau^2 = ds^2, \quad \text{with} \quad ds^2 = dx^\mu dx_\mu = (dt)^2 - (d\vec{x})^2,\]

and for the massive particle we have \(ds^2 > 0\). From (1) it follows that \(\dot{x}^2 = 1\) where \(\dot{x}^2 \equiv \dot{x}^\mu \dot{x}_\mu\) and \(\dot{x}_\mu \equiv dx_\mu / d\tau\).

\section{Reminder on particles in commutative space}

\textbf{Abelian gauge field in flat space:} We consider the interaction of a charged massive relativistic particle with an external electromagnetic field given by the \(U(1)\) gauge potential \((A^\mu)\). The motion of this particle along its space-time trajectory \(\tau \mapsto x^\mu(\tau)\) is described by the action\footnote{To be more precise, in the integral (2) the variable \(\tau\) is viewed as a purely mathematical parameter which is only identified with proper time after deriving the equations of motion from the action. Thus, the relation \(\dot{x}^2 = 1\) is only to be used at the latter stage.}

\[S[x] = -m \int ds - q \int dx^\mu A_\mu(x(\tau)) = -m \int d\tau \sqrt{\dot{x}^2} - q \int d\tau \dot{x}^\mu A_\mu(x(\tau)).\]  

(2)

Here, \(q\) denotes the conserved electric charge of the particle associated with the conserved current density \((j^\mu)\):

\[j^\mu(y) = q \int d\tau \delta^4(y - x(\tau)), \quad \partial_\mu j^\mu = 0, \quad \int d^3y j^0(y) = q.\]  

(3)
We note that the interaction term in the functional \( (2) \) may be rewritten in terms of the above current according to
\[
q \int d\tau \dot{x}^\mu A_\mu(x(\tau)) = \int d^4y j^\mu(y)A_\mu(y).
\]
(4)

This expression is invariant under infinitesimal gauge transformations, i.e. \( \delta A_\mu = \partial_\mu \lambda \), thanks to the conservation of the current:
\[
\delta \lambda \int d^4y j^\mu A_\mu = \int d^4y j^\mu \partial_\mu \lambda = - \int d^4y (\partial_\mu j^\mu) \lambda = 0.
\]
(5)

We note that the parameter \( q \) which describes the coupling of the particle to the gauge field might in principle depend on the world line parameter \( \tau \): if this assumption is made, \( q(\tau) \) appears under the \( \tau \)-integral in \( (3) \), the current \( j^\mu \) is no longer conserved and the coupling \( (4) \) is no longer gauge invariant. Thus, \( q \) is necessarily constant along the path.

Variation of the action \( (2) \) with respect to \( x^\mu \) and substitution of the relation \( \dot{x}^2 = 1 \) leads to the familiar equation of motion
\[
m\ddot{x}^\mu = q F^{\mu\nu} \dot{x}_\nu, \quad \text{with } F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu.
\]
(6)

As is well known, the free particle Lagrangian \( L_{\text{free}} = -m\sqrt{\dot{x}^2} \) which represents the first term of the action \( (2) \) and which is non-linear in \( \dot{x}^2 \) can be replaced by \( \tilde{L}_{\text{free}} = \frac{m}{2} \dot{x}^2 \) which is linear in \( \dot{x}^2 \) since both Lagrangians yield the same equation of motion. Indeed, we will consider the latter Lagrangian in equation \( (14) \) and in Section 4 where we go over to the Hamiltonian formulation.

The aim of this work is to generalize this setting to a non-commutative space-time. Since a gauge field on non-commutative space entails a non-Abelian structure for the field strength tensor \( F_{\mu\nu} \) due to the star product, it is worthwhile to understand first the coupling of a spinless particle to a non-Abelian gauge field on commutative space-time.

**Non-Abelian gauge field in flat space:** We consider a compact Lie Group \( G \) (e.g. \( G = SU(N) \) for concreteness) with generators \( T^a \) satisfying
\[
[T^a, T^b] = i f^{abc} T^c \quad \text{and} \quad \text{Tr}(T^a T^b) = \delta^{ab}.
\]
Just as for the Abelian gauge field, the source \( j^\mu_\mu(x) \) of the non-Abelian gauge field \( A_\mu(x) \equiv A_\mu^a(x) T^a \) (e.g. a field theoretic expression like \( \bar{\psi} \gamma_\mu \tilde{T}^a \psi \) involving a multiplet \( \psi \) of spinor fields) is considered to be given by the current density \( j^\mu_\mu(x) \) of a relativistic point particle \( (3) \). Instead of an electric charge, the particle moving in an external Yang-Mills field is thus assumed to carry a color-charge or isotopic spin \( \vec{q} \equiv (q^a)_{a=1,...,\text{dim}G} \) which transforms under the adjoint representation of the structure group \( G \). Henceforth, one considers the Lie algebra-valued variable \( \vec{q}(\tau) \equiv q^a(\tau) T^a \) which is assumed to be \( \tau \)-dependent. The particle is then described in terms of its
space-time coordinates \( x^\mu(\tau) \) and its isotopic spin \( q(\tau) \), i.e. it is referred to with respect to geometric space and to internal space. For the moment, we assume \( q(\tau) \) to represent a given non-dynamical (auxiliary) variable and we will comment on a different point of view below. Its coupling to an external non-Abelian gauge field \( (A_\mu) \) is now described by the action

\[
S[x] = -m \int ds - \int d\tau \dot{x}^\mu \text{Tr} \left\{ q(\tau) A_\mu(x(\tau)) \right\}
\]

\[= -m \int d\tau \sqrt{\dot{x}^2} - \int d\tau \dot{x}^\mu q^i(\tau) A^a_\mu(x(\tau)) .
\] (7)

The current density \([3]\) presently generalizes to a Lie algebra-valued expression \( j_\mu \equiv j^a_\mu T^a \) given by

\[
j_\mu(y) = \int d\tau \dot{x}^\mu(\tau) q(\tau) \delta^4(y - x(\tau)) ,
\] (8)

and thereby the interaction term in the functional \([7]\) can be rewritten as

\[- \int d^4 y \text{Tr} \left\{ j^\mu A_\mu \right\} .
\]

For an infinitesimal gauge transformation with Lie algebra-valued parameter \( \lambda \), i.e. \( \delta_\lambda A_\mu = D_\mu \lambda \equiv \partial_\mu \lambda - ig [A_\mu, \lambda] \), we have

\[
\delta_\lambda \int d^4 y \text{Tr} \left\{ j^\mu A_\mu \right\} = \int d^4 y \text{Tr} \left\{ j^\mu D_\mu \lambda \right\} = - \int d^4 y \text{Tr} \left\{ (D_\mu j^\mu) \lambda \right\} .
\] (9)

Thus, gauge invariance of the action \([7]\) requires the current to be covariantly conserved, i.e. \( D_\mu j^\mu = 0 \). From \([8]\) we can deduce by a short calculation that

\[
(D_\mu j^\mu)^a(y) = \int d\tau \frac{Dq^a}{d\tau} \delta^4(y - x(\tau)) , \quad \text{with} \quad \frac{Dq^a}{d\tau} = \frac{dq^a}{d\tau} - ig \dot{x}^\mu [A_\mu(x(\tau)), q]^a ,
\] (10)

hence \( j^\mu \) is covariantly conserved if the charge \( q \) is covariantly constant along the world line: \( Dq^a/d\tau = 0 \) (subsidiary condition).

Variation of the action with respect to \( x^\mu \) (and use of \( \dot{x}^2 = 1 \)) yields the equations of motion

\[
m \ddot{x}^\mu = \text{Tr} \left( q F^{\mu\nu} \right) \dot{x}_\nu , \quad \text{where} \quad \frac{Dq^a}{d\tau} = 0 ,
\] (11)

and \( F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu - ig [A_\mu, A_\nu] \). Equations \([11]\), which represent the Lorentz-Yang-Mills force equation for a relativistic particle in an external Yang-Mills field, are known as Wong’s equations \([4]\). More specifically, the relation \( Dq^a/d\tau = 0 \) may be viewed as charge transport equation and it is the geometrically natural generalization (to the charge vector \( (q^a) \)) of the constancy of the charge \( q \) in electrodynamics. We should point out that the equation of motion for \( x^\mu \) had already been obtained earlier
in curved space by Kerner [11], but the charge transport equation was not established in that setting.

If one regards \( q^a \) as a dynamical variable which satisfies the equation of motion \( Dq^a/d\tau = 0 \) and which transforms under gauge variations with the adjoint representation, i.e.

\[
\delta_\lambda q(\tau) = -i g [q(\tau), \lambda(x(\tau))], \tag{12}
\]

then equations (11) are obviously gauge invariant since \( F_{\mu\nu} \) also transforms with the adjoint representation. Equations (11) and their classical solutions have been investigated in the literature and applied for instance to the study of the quark gluon plasma [6].

We emphasize that we started from the action (7) to obtain the equation of motion of \( x^\mu \), the one of \( q^a \) following from the requirement of gauge invariance of the initial action. An action which yields the equations of motion of both \( x^\mu \) and \( q^a \) has been constructed in the non-relativistic setting in reference [13]. The relativistic generalization of this approach proceeds as follows. (For simplicity, we put the coupling constant \( g \) equal to one.) One introduces a Lie algebra-valued variable \( \Lambda(\tau) \equiv \Lambda^a(\tau)T^a \) where the functions \( \Lambda^a(\tau) \) are Grassmann odd, the charge \( q \) being defined as an expression which is bilinear in \( \Lambda \) (i.e. a description which is familiar for the spin):

\[
q \equiv -\frac{1}{2} [\Lambda, \Lambda], \quad \text{i.e.} \quad q^a = -\frac{i}{2} f^{abc} \Lambda^b \Lambda^c. \tag{13}
\]

The Lagrangian

\[
L(x, \dot{x}, \Lambda, \dot{\Lambda}) \equiv m^2 \dot{x}^2 + \frac{i}{2} \text{Tr} (\Lambda D\Lambda d\tau)
= m^2 \dot{x}^2 + \frac{i}{2} \text{Tr} (\Lambda \dot{\Lambda}) + \dot{x}^\mu \text{Tr} (qA_\mu), \tag{14}
\]

is invariant under gauge transformations for which \( \delta_\lambda A_\mu = D_\mu \lambda \) and

\[
\delta_\lambda \Lambda(\tau) = -i [\Lambda(\tau), \lambda(x(\tau))], \tag{15}
\]

which implies the transformation law (12). Moreover, the Lagrangian leads to the following equations of motion for \( x^\mu \) and \( \Lambda^a \):

\[
m \ddot{x}_\mu = \dot{x}^\nu \text{Tr} \{q(\partial_\nu A_\mu - \partial_\mu A_\nu)\} + \text{Tr} (q A_\mu), \quad 0 = \frac{DA^a}{d\tau}. \tag{16}
\]

From (13) it follows that \( \frac{Dq^a}{d\tau} = -[\Lambda, \frac{D\Lambda^a}{d\tau}] \), hence (16) implies that \( \frac{Dq^a}{d\tau} = 0 \), i.e. the charge \( q \) is covariantly conserved. Substitution of the latter result into the first of

\[\text{We refer to the works [5, 6] for a treatment of dynamics involving the Lagrangian } -\frac{i}{2} \int d^3 x \text{ Tr}(F_{\mu\nu}F_{\mu\nu}) \text{ of the gauge field: the latter approach yields the covariant conservation law } D_\mu j^\mu = 0 \text{ as a consequence of the equation of motion } D_\mu F^{\mu\nu} = j^\nu \text{ and the relation } [D_\mu, D_\nu]F^{\mu\nu} = 0. \text{ For a general discussion of the issue of gauge invariance for the coupling of gauge fields to non-dynamical external sources, we refer to [12].} \]
equations (16) yields the equation of motion $m \ddot{x} = \text{Tr} (q F_{\mu\nu}) \dot{x}_{\nu}$. We note that the Hamiltonian associated to the Lagrangian (14) reads

$$H = \frac{1}{2m} \left[ p^\mu - \text{Tr} (q A^\mu) \right] \left[ p_\mu - \text{Tr} (q A_\mu) \right],$$

or $H = \frac{m}{2} \dot{x}^2$ if expressed in terms of the velocity. The Poisson brackets

$$\{ x^\mu, p_\nu \} = \delta^\mu_\nu, \quad \{ \Lambda_a, \Lambda_b \} = -i \delta_{ab},$$

which imply the non-Abelian algebra of charges

$$\{ q_a, q_b \} = f^{abc} q_c,$$

again allow us to recover all previous equations of motion from the evolution equation of dynamical variables $F$, i.e. from $\dot{F} = \{ F, H \}$. In terms of the kinematical momentum $\Pi_\mu \equiv p_\mu - \text{Tr} (q A_\mu)$, the Hamiltonian reads $H = \frac{1}{2m} \Pi^2$ and the Poisson brackets take the form

$$\{ x^\mu, \Pi_\nu \} = \delta^\mu_\nu, \quad \{ \Pi_\mu, \Pi_\nu \} = \text{Tr} (q F_{\mu\nu}), \quad \{ q_a, q_b \} = f^{abc} q_c.$$

The dynamical variable $\Lambda$ which allowed for the Lagrangian formulation is well hidden in the latter equations.

**Continuum formulation of the dynamics:** The field strength $F_{\mu\nu}$ manifests itself physically by the force field, i.e. by an exchange of energy and momentum between the charge carrier and the field [14]. In the framework of field theory, the physical entities are described by local fields, i.e. one has a continuum formulation. In order to obtain such a formulation for the particle’s equations of motion (11), we have to integrate these relations over the variable $\tau$ with a delta function concentrated on the particle’s trajectory. The resulting expressions then involve the current density $j^{a\mu}(y)$ defined in equation (8) as well as the energy-momentum tensor (density) of the point particle which is given by (see Appendix A)

$$T^{\mu\nu}(y) = \int d\tau m \dot{x}^\mu \dot{x}^\nu \delta^4 (y - x(\tau)).$$

More explicitly, by using

$$\dot{x}^\nu \partial_\nu \delta^4 (y - x(\tau)) = -\dot{x}^\nu \partial_\nu \delta^4 (y - x(\tau)) = -\frac{d}{d\tau} \delta^4 (y - x(\tau)),$$

we have

$$\partial_\mu T^{\mu\nu}(y) = \int d\tau m \dot{x}^\mu \dot{x}^\nu \delta^4 (y - x(\tau)) = \int d\tau m \dot{x}^\mu \frac{d}{d\tau} \delta^4 (y - x(\tau))$$

$$= \int d\tau m \dot{x}^\mu \delta^4 (y - x(\tau)).$$
and substitution of the particle’s equation of motion \( m\ddot{x}^\mu = \text{Tr} (qF^{\mu\nu}) \dot{x}_\nu \) then yields
\[
\partial_\nu T^{\nu\mu}(y) = \int d\tau \dot{x}_\nu q^\alpha F^{\alpha\nu}_a(x(\tau)) \delta^4(y - x(\tau)) = F^{\mu\nu}_a(y) \int d\tau \dot{x}_\nu q^\alpha \delta^4(y - x(\tau)) = F^{\mu\nu}_a(y) j^\alpha_\nu(y).
\]
The continuum version of Wong’s equations \(^{11}\) thus reads
\[
\partial_\nu T^{\nu\mu} = \text{Tr} (F^{\mu\nu} j_\nu), \quad \text{where } D_\mu j^\mu = 0. \quad (23)
\]
These equations describe the exchange of energy and momentum between the field \( F^{\mu\nu} \) and the current \( j^\mu \) (i.e. the matter). They admit an obvious generalization to Moyal space, see equation (43) below. In Appendix A we show that they also admit a natural extension to curved space (endowed with a metric tensor \( g_{\mu\nu}(x) \)). Moreover, we will prove there that they have to hold for arbitrary dynamical matter fields \( \phi \) whose dynamics is described by a generic action \( S[\phi; g_{\mu\nu}, A^a_\mu] \) which is invariant under both gauge transformations and general coordinate transformations \( g_{\mu\nu} \) and \( A^a_\mu \) representing fixed external fields).

**Curved space:** Finally, we also point out that equations which are somewhat similar to Wong’s equations appear in general relativity for a spinning particle in curved space, for which case the contraction of the Riemann tensor \( R^\alpha_\sigma \partial_\mu \partial_\nu \) with the spin tensor \( S^\mu_\alpha \) plays a role which is similar to the field strength \( F^{\mu\nu} \) in Yang-Mills theories. The explicit form of these equations of motion, which are known as the Mathisson-Papapetrou-Dixon equations \(^8\), is given by
\[
\nabla_d\frac{d\tau}{m u^\alpha} + \frac{1}{2} S^{\mu\nu} u^\sigma R^\alpha_\sigma \partial_\mu \partial_\nu = 0, \quad \nabla S^{\alpha\beta} = 0, \quad (24)
\]
where \( \nabla \) denotes the covariant derivative along the trajectory and \( (u^\mu) \) is the particle’s four-velocity.

### 3 Lagrangian approach to particles in NC space

**Moyal space and distributions:** We consider four dimensional Moyal space, i.e. we assume that the space-time coordinates fulfill a Heisenberg-type algebra (for a review see \(^1\) and references therein). Thus, the star product of functions is defined by
\[
(f \star g)(x) = \left. \left( e^{\frac{i}{2} \theta^{\mu\nu} \partial^\mu \partial^\nu} f(x)g(y) \right) \right|_{x=y},
\]
where the parameters \( \theta^{\mu\nu} = -\theta^{\nu\mu} \) are constant, and their star commutator is defined by \([f \star g] \equiv f \star g - g \star f\), which implies that \([x^\mu \star x^\nu] = \theta^{\mu\nu}\). In the sequel we will repeatedly use the following fundamental properties of the star product:
\[
\int d^4x f \star g = \int d^4x f \cdot g, \quad \int d^4x f \star g \star h = \int d^4x h \star f \star g. \quad (26)
\]
For a detailed discussion of the algebras of functions and of distributions on Moyal space in the context of non-commutative spaces and of quantum mechanics in phase space, we refer to [15] and [16], respectively. Here, we only note that the star product of the delta distribution $\delta_y$ (with support in $y$) with a function $\psi$ may be defined by application to a test function $\varphi$:

$$\langle \delta_y \star \psi, \varphi \rangle \equiv \int d^4x (\delta_y \star \psi)(x) \varphi(x) = \int d^4x \delta_y(x) (\psi \star \varphi)(x) = (\psi \star \varphi)(y). \quad (27)$$

Hence, the action of the distribution $\delta_y \star \psi$ on the test function $\varphi$ is equal to the action of the distribution $\delta_y$ on the test function $\psi \star \varphi$. Similarly, we find

$$\langle \psi \star \delta_y, \varphi \rangle \equiv \int d^4x (\psi \star \delta_y)(x) \varphi(x) = \int d^4x (\psi \star \delta_y \star \varphi)(x) = \int d^4x (\delta_y \star \varphi \star \psi)(x) = (\varphi \star \psi)(y). \quad (28)$$

The following considerations hold for an arbitrary antisymmetric matrix ($\theta^{\mu\nu}$), but for the physical applications it is preferable to assume that $\theta^{\mu 0} = 0$, i.e. assume the time to be commuting with the spatial coordinates. This choice is motivated by the fact that the parameters $\theta^{ij}$ have close analogies with a constant magnetic field both from the algebraic and dynamical points of view [2], and by the fact that a non-commuting time leads to problems with time-ordering in quantum field theory [17].

**Charged particle in Moyal space:** Since a $U_s(1)$ gauge field ($A^\mu$) on Moyal space entails a non-Abelian structure for the field strength tensor ($F_{\mu\nu}$) due to the star product [1],

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu - ig [A_\mu \star, A_\nu], \quad (29)$$

one expects that the treatment of a source for this gauge field as given by a “point” particle should allow for a description of such particles on non-commutative space which is quite similar to the one found by Wong in the case of Yang-Mills theory. If the matter content in field theory is given by a spinor field $\psi$, then the interaction term with the gauge field reads

$$\int d^4y J^\mu \star A_\mu, \quad \text{with } J^\mu \equiv g \bar{\psi} \gamma^\mu \star \psi,$$

i.e. it involves two star products. By virtue of the properties (26) one of these star products can be dropped under the integral, but not both of them. If we consider the particle limit (i.e. $J^\mu$ representing the current density of the particle), the star product between $J^\mu$ and $A_\mu$ should be maintained, otherwise the non-commutative nature of the underlying space is no longer taken into account. In fact, the pairing $\langle J, A \rangle \equiv \int d^4y J^\mu \star A_\mu$ represents the analogue of the pairing $\langle j, A \rangle \equiv \int d^4y \text{Tr} (j^\mu A_\mu)$ in Yang-Mills theory on commutative space. Accordingly, we will require the action for
the particle to be invariant under non-commutative gauge transformations defined at the infinitesimal level by \( \delta \lambda A_\mu = D_\mu \lambda \equiv \partial_\mu \lambda - ig [A_\mu ; \lambda] \) where the parameter \( \lambda \) is an arbitrary function. As was done for Wong’s equations, we assume that the charge \( q \) of the relativistic particle in non-commutative space depends on the parameter \( \tau \) parametrizing the particle’s world line and that it represents a given non-dynamical variable. The coupling of this particle to an external \( U_n(1) \) gauge field \( (A^\mu) \) can be described by the action

\[
S[x] \equiv S_{\text{free}}[x] - S_{\text{int}}[x] \equiv -m \int d\tau \sqrt{x^2} - \int d^4y \left( J^\mu \ast A_\mu \right)(y), \tag{30}
\]

where

\[
J^\mu(y) \equiv \int d\tau q(\tau) \dot{x}^\mu(\tau) \delta^4(y - x(\tau)). \tag{31}
\]

Keeping in mind the formulation of field theory on non-commutative space \( \Pi \), we will argue directly with the action rather than the Lagrangian function and require this action to be invariant under non-commutative gauge transformations. For such an infinitesimal transformation we have

\[
\delta \lambda \int d^4y J^\mu \ast A_\mu = \int d^4y J^\mu \ast D_\mu \lambda = - \int d^4y (D_\mu J^\mu) \ast \lambda. \tag{32}
\]

Hence, gauge invariance of the action \((30)\) requires the current to be \textit{covariantly conserved}, i.e. \( D_\mu J^\mu = 0 \). By virtue of equation \((31)\) we now infer that

\[
0 = (D_\mu J^\mu)(y) = \int d\tau q \dot{x}^\mu D^\mu_\mu \delta^4(y - x(\tau)) = \int d\tau q \dot{x}^\mu \left\{ \partial^\mu_\mu \delta^4(y - x(\tau)) - ig [A_\mu(y) ; \delta^4(y - x(\tau))] \right\}. \tag{33}
\]

Equation \((32)\) entails that the first term in the last line can be rewritten as

\[
\int d\tau q \dot{x}^\mu \partial^\mu_\mu \delta^4(y - x(\tau)) = - \int d\tau q \frac{d}{d\tau} \delta^4(y - x(\tau)) = \int d\tau \frac{dq}{d\tau} \delta^4(y - x(\tau)).
\]

From condition \((33)\) it thus follows that the charge \( q \) has to be \textit{covariantly conserved} along the world line in the sense that

\[
0 = \int d\tau \frac{Dq}{d\tau} \delta^4(y - x(\tau)) \equiv \int d\tau \left\{ \frac{dq}{d\tau} \delta^4(y - x(\tau)) - ig q \dot{x}^\mu \left[ A_\mu(y) ; \delta^4(y - x(\tau)) \right] \right\}. \tag{34}
\]

For later reference, we note that this relation yields the following equality after star multiplication with \( A_\nu(y) \delta y^\nu \) and integration over \( y \):

\[
\int d^4y \int d\tau \delta x^{\nu}(\tau) \frac{dq}{d\tau} \delta^4(y - x(\tau)) \ast A_\nu(y)
= ig \int d^4y \int d\tau \delta x^{\nu}(\tau) q \dot{x}^\mu \left[ A_\mu(y) ; \delta^4(y - x(\tau)) \right] \ast A_\nu(y)
= -ig \int d^4y \int d\tau \delta x^{\nu}(\tau) q \dot{x}^\mu \delta^4(y - x(\tau)) \ast \left[ A_\mu(y) ; A_\nu(y) \right]. \tag{35}
\]
In order to derive the equation of motion for the particle determined by the action \(S_{\text{int}}\), we vary the latter with respect to \(x^\mu\). The variation of \(S_{\text{int}}\) being the same as in commutative space, we only work out the variation of the interaction part \(S_{\text{int}}\), all star products being viewed as functions of the variable \(y\):

\[
\delta S_{\text{int}} = \delta \int d^4y \int d\tau \left\{ \dot{x}^\mu q \delta^4(y - x(\tau)) \star A_\mu(y) \right\}
\]

\[
= \int d^4y \int d\tau \left\{ \dot{x}^\mu q \delta^4(y - x(\tau)) \star A_\mu(y) + \dot{x}^\mu q \delta \left[ \delta^4(y - x(\tau)) \right] \star A_\mu(y) \right\}
\]

\[
= \int d^4y \int d\tau \left\{ \frac{d(\dot{x}^\mu)}{d\tau} q \delta^4(y - x(\tau)) \star A_\mu(y) + \dot{x}^\mu q \left( \dot{x}^\nu \right) \partial_\nu \delta^4(y - x(\tau)) \star A_\mu(y) \right\}
\]

\[
= \int d^4y \int d\tau (\delta x^\nu) \left\{ -\frac{d}{d\tau} \left[ q \delta^4(y - x(\tau)) \right] \star A_\nu(y) + \dot{x}^\mu q \delta^4(y - x) \star \partial_\mu A_\nu(y) \right\}.
\]

By virtue of the product rule, the first term in the last line yields two terms, one involving \(\frac{d}{d\tau}q\) which can be rewritten using relation (35), and one involving \(\frac{d}{d\tau}\delta^4(y - x(\tau))\) which can be rewritten using (22):

\[
- \int d^4y \int d\tau (\delta x^\nu) \frac{d}{d\tau} \delta^4(y - x(\tau)) \star A_\nu(y)
\]

\[
= \int d^4y \int d\tau (\delta x^\nu) \dot{x}^\mu q \left( \partial_\mu \delta^4(y - x(\tau)) \right) \star A_\nu(y)
\]

\[
= -\int d^4y \int d\tau (\delta x^\nu) \dot{x}^\mu q \delta^4(y - x(\tau)) \star \partial_\mu A_\nu(y).
\]

Hence, we arrive at

\[
\delta S_{\text{int}} = \int d^4y \int d\tau (\delta x^\nu) \dot{x}^\mu q \delta^4(y - x(\tau)) \star \left\{ \partial_\nu A_\mu(y) - \partial_\mu A_\nu(y) + i g \left[ A_\mu(y) \star A_\nu(y) \right] \right\}
\]

\[
= \int d^4y \int d\tau (\delta x^\nu) \dot{x}^\mu q \delta^4(y - x(\tau)) \star F_{\nu\mu}(y)
\]

\[
= \int d\tau (\delta x^\nu) \dot{x}^\mu q F_{\nu\mu}(x(\tau)).
\]

In conclusion, we obtain the following equation of motion for the charged relativistic particle in non-commutative space:

\[
m\dddot{x}^\mu = qF^{\mu\nu} \dot{x}_\nu, \quad \text{with} \quad F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu - ig \left[ A_\mu \star A_\nu \right].
\]

We note that the antisymmetry of \(F_{\mu\nu}\) with respect to its indices implies

\[
0 = \dddot{x}^\mu \dot{x}_\mu = \frac{1}{2} \frac{d\dot{x}^2}{d\tau},
\]

which is consistent with \(\dot{x}^2 = 1\). The gauge invariance of the equation of motion (39) follows from\(^3\)

\[
\delta_\lambda (qF^{\mu\nu} \dot{x}_\nu) = q(\delta_\lambda F^{\mu\nu}) \dot{x}_\nu = -igq \left[ F^{\mu\nu} \star \lambda \right] \dot{x}_\nu = -ig \left[ qF^{\mu\nu} \dot{x}_\nu \star \lambda \right] = -ig \left[ m\dddot{x}^\mu \star \lambda \right] = 0.
\]

\(^3\)For a discussion of \(U_\star(1)\) gauge fields coupled to external currents and the related issues of gauge invariance, we refer to the recent work \[18\].
In summary, the coupling of a relativistic particle to a gauge field \((A^\mu)\) is described in general by the Lagrangian

\[
L(x, \dot{x}) = -m\sqrt{\dot{x}^2} - qA_\mu \dot{x}^\mu, \\
\text{or} \quad L(x, \dot{x}) = \frac{m}{2} \dot{x}^2 + qA_\mu \dot{x}^\mu.
\] (40)

The non-commutativity of space-time can be implemented in the Lagrangian framework by rewriting the interaction term of the action as an integral \(\int d^4y \left( J^\mu \star A^\mu \right)(y) \) where the current \(J^\mu\) is defined by (31) and the charge \(q\) is assumed to be \(\tau\)-dependent and covariantly conserved along the world line of the particle.

**Continuum formulation of the dynamics:** By following the same lines of arguments as for the Lorentz-Yang-Mills force equation (see equations (21)-(23)), we can obtain a continuum version of the equation of motion (39) by multiplying this equation with \(\delta^4(y - x(\tau)) \star \varphi(y)\), where \(\varphi(y)\) is a suitable test function, and integrating over \(\tau\) and over \(y\). More explicitly, by starting from the energy-momentum tensor (21) for the point particle and using relation (22), we get

\[
\int d^4y \left( \partial_\nu T^{\nu\mu}(y) \right) \star \varphi(y) = \int d^4y \int d\tau m \dot{x}^\mu \dot{x}^\nu \partial_\nu \delta^4(y - x(\tau)) \star \varphi(y) \\
= -\int d^4y \int d\tau m \dot{x}^\mu \frac{d}{d\tau} \delta^4(y - x(\tau)) \star \varphi(y) \\
= \int d^4y \int d\tau m \dot{x}^\mu \delta^4(y - x(\tau)) \star \varphi(y). \tag{41}
\]

Substitution of equation (39) then yields the expression

\[
\int d^4y \int d\tau q F^{\mu\nu}(x(\tau)) \dot{x}_\nu \delta^4(y - x(\tau)) \star \varphi(y) = \int d^4y \int d\tau q F^{\mu\nu}(y) \dot{x}_\nu \delta^4(y - x(\tau)) \star \varphi(y) \\
= \int d^4y \left( F^{\mu\nu} \star J_\nu \star \varphi(y) \right), \tag{42}
\]

where we considered the current density (31) in the last line. Thus, we have the result

\[
\partial_\nu T^{\nu\mu} = F^{\mu\nu} \star J_\nu, \quad \text{where} \quad D_\mu J^\mu = 0, \tag{43}
\]

and these relations are completely analogous to the continuum equations (23) which correspond to Wong’s equations.

## 4 Hamiltonian approaches to particles in NC space

To start with, we briefly review the Hamiltonian approaches in commutative space before considering the generalization to the non-commutative setting. In the latter setting, we will notice that various approaches yield different results since several expressions which coincide in commutative space no longer agree.
4.1 Reminder on the Poisson bracket approach

The Hamiltonian formulation of relativistic (as well as non-relativistic) mechanics is based on two inputs (e.g. see reference [19]): a Hamiltonian function and a Poisson structure (or equivalently a symplectic structure). If one starts from the Lagrangian formulation, the Hamiltonian function is obtained from the Lagrange function by a Legendre transformation. E.g. the Lagrangian $L(x, \dot{x}) = \frac{m}{2} \dot{x}^2 + eA_\mu \dot{x}^\mu$ (involving the constant charge $e$) yields the Hamiltonian

$$H(x, p) = \frac{1}{2m} (p - eA)^2 = \frac{1}{2m} (p^\mu - eA^\mu)(p_\mu - eA_\mu).$$

(44)

The trajectories in phase space are parametrized by $\tau \mapsto (x(\tau), p(\tau))$ where $\tau$ denotes a real variable to be identified with proper time after the equations of motion have been derived. The Poisson brackets $\{\cdot, \cdot\}$ of the phase space variables $x^\mu$, $p^\mu$ are chosen in such a way that the evolution equation $\dot{F} = \{F, H\}$ (where $\dot{F} \equiv dF/d\tau$) yields the Lagrangian equation of motion for $x^\mu$, though written as a system of first order differential equations. For instance, if we consider the usual form of the Poisson brackets, i.e the canonical Poisson brackets

$$\{x^\mu, x^\nu\} = 0, \quad \{p^\mu, p^\nu\} = 0, \quad \{x^\mu, p^\nu\} = \eta^{\mu\nu},$$

(45)

then substitution of $F = x^\mu$ and $F = p^\mu$ into $\dot{F} = \{F, H\}$ (with $H$ given by (44)) yields the system of equations

$$m \ddot{x}^\mu = p^\mu - eA^\mu, \quad m \dot{p}^\mu = e(p_\nu - eA_\nu)(\partial^\mu A^\nu - \partial^\nu A^\mu),$$

(46)

from which we conclude that

$$m \ddot{x}^\mu = p^\mu - e\dot{A}^\mu = \frac{e}{m} (p_\nu - eA_\nu)(\partial^\mu A^\nu - \partial^\nu A^\mu) = e \dot{x}^\nu (\partial^\mu A^\nu - \partial^\nu A^\mu) \equiv e f^{\mu\nu} \dot{x}_\nu.$$

This equation coincides with the Euler-Lagrange equation for $x^\mu$ following from the Lagrangian $L(x, \dot{x}) = \frac{m}{2} \dot{x}^2 + eA_\mu \dot{x}^\mu$.

Concerning the gauge invariance, we emphasize a result [19] which does not seem to be very well known. The Hamiltonian (44) is not invariant under a gauge transformation $A_\mu \rightarrow A_\mu + \partial_\mu \lambda$ which is quite intriguing. However, it is invariant if this transformation is combined with the phase space transformation $(x^\mu, p^\mu) \rightarrow (x^\mu, p^\mu + e\dot{A}^\mu)$, the latter being a canonical transformation since it preserves the fundamental Poisson brackets (45). Indeed, under this combined transformation the kinematical momentum $\Pi^\mu \equiv p^\mu - eA^\mu$ (which coincides with $m \ddot{x}^\mu$) is invariant.

We note that the Hamiltonian (44) can be rewritten in terms of the variable $\Pi_\mu \equiv p_\mu - eA_\mu$ as $H \equiv \frac{1}{2m} \Pi^2$. Thereby $H$ has the form of a free particle Hamiltonian, but the Poisson brackets are now modified: from $\Pi_\mu = p_\mu - eA_\mu$ and (15) it follows that

$$\{x^\mu, x^\nu\} = 0, \quad \{\Pi^\mu, \Pi^\nu\} = e f^{\mu\nu}(x), \quad \{x^\mu, \Pi^\nu\} = \eta^{\mu\nu},$$

(47)
with \( f^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu \). Since the electromagnetic field strength \( f^{\mu\nu} \) is gauge invariant, the latter invariance is manifestly realized in this formulation.

In summary, the coupling of a charged particle to an electromagnetic field can either be described by the canonical Poisson brackets (45) and the minimally coupled Hamiltonian (44) or by introducing the field strength into the Poisson brackets (as a non-commutativity of the momenta) and considering a Hamiltonian which has the form of a free particle Hamiltonian.

### 4.2 Reminder on the symplectic form approach

If we gather all phase space variables into a vector \( \vec{\xi} \equiv (\xi^I) \equiv (x^0, \ldots, x^3, p^0, \ldots, p^3) \), the fundamental Poisson brackets (45) read

\[
\{\xi^I, \xi^J\} = \Omega^{IJ},
\]

with \( (\Omega^{IJ}) \equiv \begin{bmatrix} 0 & \eta^{\mu\nu} \\ -\eta^{\mu\nu} & 0 \end{bmatrix} \).

The inverse of the Poisson matrix \( \Omega \equiv (\Omega^{IJ}) \) is the matrix with entries \( \omega^{IJ} \equiv (\Omega^{-1})_{IJ} \) which defines the symplectic 2-form

\[
\omega \equiv \frac{1}{2} \sum_{I,J} \omega^{IJ} d\xi^I \wedge d\xi^J = dp^\mu \wedge dx_\mu,
\]

(49)
e.g. see reference [19] for mathematical details. The Hamiltonian equations of motion can be written as

\[
\dot{\xi}^I = \{\xi^I, H\} = \Omega^{JK} \partial_K \xi^I \partial_J H, \quad \text{i.e.} \quad \dot{\xi}^I = \Omega^{IJ} \partial_J H,
\]

or equivalently as \( \omega_{IJ} \dot{\xi}^J = \partial_I H \).

In terms of the phase space variables \((x^\mu, \Pi^\mu)\) appearing in the non-canonical Poisson brackets (47), the symplectic 2-form (49) reads

\[
\omega = d\Pi^\mu \wedge dx_\mu + \frac{1}{2} \epsilon_{\mu\nu\rho} dx^\mu \wedge dx^\nu.
\]

This formulation of the electromagnetic interaction based on the symplectic 2-form and the evolution equation \( \omega_{IJ} \dot{\xi}^J = \partial_I H \) goes back to the seminal work of Souriau [20].

### 4.3 Standard (Poisson bracket) approach to NC space-time

In order to introduce a non-commutativity for the configuration space, one generally starts from a function \( H \) on phase space to which one refers as the Hamiltonian without any reference to a Lagrangian, e.g. we can consider the function \( H \) given in equation (44). The non-commutativity of space-time is then implemented by virtue of the Poisson brackets

\[
\{x^\mu, x^\nu\} = \delta^{\mu\nu}, \quad \{p^\mu, p^\nu\} = 0, \quad \{x^\mu, p^\nu\} = \eta^{\mu\nu},
\]

(50)
where $\theta^{\mu\nu} = -\theta^{\nu\mu}$ is again assumed to be constant. (For an overview of the description of non-relativistic charged particles in non-commutative space we refer to \[2, 3, 21\], the pioneering work being \[22, 23\], see also \[24–26\] for some subsequent early work. We also mention that dynamical systems in non-commutative space can be constructed by applying Dirac’s treatment of constrained Hamiltonian systems to an appropriate action functional, see \[27\] and references therein.)

As in the commutative setting, we gather all phase space variables into a vector $\vec{\xi} \equiv (\xi^I) \equiv (x^0, \ldots, x^3, p^0, \ldots, p^3)$, the fundamental Poisson brackets (50) being now given by

$$\{\xi^I, \xi^J\} = \Omega^{IJ}, \quad \text{with } (\Omega^{IJ}) \equiv \begin{bmatrix} \theta^{\mu\nu} & \eta^{\mu\nu} \\ -\eta^{\mu\nu} & 0 \end{bmatrix}. \quad (51)$$

Quite generally, the Poisson bracket of two arbitrary functions $F, G$ on phase space reads

$$\{F, G\} \equiv \sum_{I,J} \Omega^{IJ} \partial_I F \partial_J G = \theta^{\mu\nu} \frac{\partial F}{\partial x^{\mu}} \frac{\partial G}{\partial x^{\nu}} + \frac{\partial F}{\partial p_{\mu}} \frac{\partial G}{\partial p^{\nu}} - \frac{\partial F}{\partial p^{\mu}} \frac{\partial G}{\partial p_{\nu}}. \quad (52)$$

Substitution of $F = x^\mu$ and $F = p^\mu$ into $\dot{F} = \{F, H\}$ (with $H$ given by (44)) yields the system of equations

$$m \dot{x}^\mu = (p_\nu - eA_\nu)(\eta^{\mu\nu} - e\theta^{\mu\rho}\partial^\rho A^\nu),$$
$$m \dot{p}^\mu = e(p_\nu - eA_\nu)\partial^\mu A^\nu. \quad (53)$$

In the present case, the phase space transformation $(x^\mu, p^\mu) \to \mathcal{F}(x^\mu, p^\mu + e\partial^\mu \lambda)$ does not represent a canonical transformation since it does not preserve the Poisson brackets (50) if $\theta^{\mu\nu} \neq 0$. Hence, the resulting Hamiltonian equations of motion (53) are not gauge invariant, as has already been pointed out in reference \[2\] by considering different gauges.

In the next two subsections, we recall how this problem can be overcome for the particular case of a constant field strength as well as more generally, and we compare with the results obtained in Section 3 from the action involving star products. Here, we only note that a non-Abelian structure of the field strength is hidden in equation (53). To illustrate this point, we consider the particular case where the only non-vanishing components of $\theta^{\mu\nu}$ are $\theta^{ij} = \varepsilon^{ij}\theta$ (with $i, j \in \{1, 2\}$ and $\varepsilon^{12} \equiv -\varepsilon^{21} \equiv 1$) and where the only non-vanishing components of $A^\mu$ are $A^i(x^1, x^2)$ (with $i \in \{1, 2\}$). For this situation which describes a time-independent magnetic field perpendicular to the $x^1x^2$-plane, the first of equations (53) yields

$$m \dot{x}_i = (p_k - eA_k)(\delta_{ik} - e\theta\varepsilon_{ij}\partial_j A_k),$$

and implies

$$m \frac{d}{dt}(x_i + e\theta\varepsilon_{ij}A_j) = (1 + e\mathcal{F}_{12})(p_i - eA_i). \quad (54)$$
with

$$\mathcal{F}_{12} \equiv \partial_1 A_2 - \partial_2 A_1 + e \{ A_1, A_2 \} = \partial_1 A_2 - \partial_2 A_1 + e \theta_{\rho\sigma} (\partial_{\rho} A_1)(\partial_{\sigma} A_2).$$

(55)

Thus, we find a non-Abelian structure for the generalized field strength, but in the present approach the field $F_{\mu\nu}$ is only linear in the non-commutativity parameters in contrast to the field $F_{\mu\nu}$ in (39) which involves the star commutator

$$-i [A_\mu, A_\nu] = \theta_{\rho\sigma} (\partial_{\rho} A_\mu)(\partial_{\sigma} A_\nu) + \mathcal{O}(\theta^2).$$

If the gauge potential is linear in $x$, the field strengths $F_{\mu\nu}$ and $F_{\mu\nu}$ as defined by equations (55) and (39), respectively, coincide with each other (if one identifies the coupling constant $g$ with $e$).

To conclude, we note that (54) can be solved for $p_i - e A_i$ in terms of $m\dot{x}_i$: the system of first order differential equations (53) can then be written as a second order equation for $x^\mu$, but the resulting equations of motion are not gauge invariant and they cannot be derived from a Lagrangian [21].

4.4 Standard approach to NC space-time continued

The reasoning presented concerning the brackets (47) suggests to consider a Hamiltonian which has a free form and to introduce a field strength $B^{\mu\nu}(x)$ as a non-commutativity of the momenta, i.e. consider phase space variables $(x^\mu, p^\mu)$ satisfying the non-canonical Poisson algebra

$$\{ x^\mu, x^\nu \} = \theta^{\mu\nu}, \quad \{ p^\mu, p^\nu \} = e B^{\mu\nu}, \quad \{ x^\mu, p^\nu \} = \eta^{\mu\nu},$$

(56)

with $\theta^{\mu\nu}$ constant. As pointed out in reference [28], the Jacobi identities for the algebra (56) are only satisfied if the field strength is constant:

$$\{ x^\mu, \{ p^\nu, p^\lambda \} \} + \text{cyclic permutations of } \mu, \nu, \lambda = e \theta^{\mu\nu} \partial_{\rho} B^{\nu\lambda}.$$

Thus, the dynamics of a charged particle coupled to a constant field $B^{\mu\nu}$ on non-commutative space-time can be described in terms of phase space variables $(x^\mu, p^\mu)$ satisfying the non-canonical Poisson algebra (56), the Hamiltonian being given by

$$H(p) = \frac{1}{2m} p^2 = \frac{1}{2m} p^\mu p_\mu. \quad \text{The Hamiltonian equations of motion}

m \ddot{x}^\mu = p^\mu, \quad m \ddot{p}^\mu = e B^{\mu\nu} p_\nu,$$

then imply the second order equation

$$m \ddot{x}^\mu = e B^{\mu\nu} \dot{x}_\nu.$$

(57)

This equation of motion for $x^\mu$ coincides with the one that one encounters for $\theta^{\mu\nu} = 0$ since the Hamiltonian only depends on $p$ and not on the coordinates $x^\mu$ whose Poisson brackets do not vanish. However the non-commutativity parameters $\theta^{\mu\nu}$ appear in
quantities like the volume form on phase space which is the 4-fold exterior product of
the symplectic form with itself,

\[ dV \equiv \frac{1}{4!} \omega^4 = \frac{1}{\sqrt{\det \Omega}} \, d\xi^1 \cdots d\xi^8, \quad (58) \]

where \( \Omega \) denotes the Poisson matrix and where we suppressed the exterior product
symbol.

4.5 “Exotic” (symplectic form) approach to NC space-time

The Hamiltonian approach to mechanics on non-commutative space based on the
simple form (56) of the Poisson algebra (in which the Poisson bracket \( \{ x^\mu, p^\nu \} \) has
the canonical form) has been nicknamed the standard approach. As we just recalled,
it does not allow for the inclusion of a non-constant field strength. By contrast, the
so-called exotic approach [3, 29] which is based on a simple form of the symplectic 2-
form allows us to describe generic field strengths \( B_{\mu\nu}(\vec{x}) \). In this setting, the constant
non-commutativity parameters \( \theta^{\mu\nu} \) are introduced into the symplectic 2-form\(^4\) defined
on the phase space parametrized by \( (x^\mu, p^\mu) \):

\[ \omega = dp^\mu \wedge dx_\mu + \frac{1}{2} B_{\mu\nu} \, dx^\mu \wedge dx^\nu + \frac{1}{2} \theta^{\mu\nu} \, dp^\mu \wedge dp^\nu. \quad (59) \]

The Poisson matrix is obtained by the inversion of the symplectic matrix (e.g. see
reference [29] for the case of a space-time of arbitrary dimension), and therefore it has
a more complicated form than the one corresponding to (56). By way of illustration,
we recall the result that one obtains for the simplest instance [29] where one has only
two spatial coordinates, i.e. \( \vec{x} \equiv (x_1, x_2) \):

\[ \{ x_1, x_2 \} = \kappa^{-1} \theta, \quad \{ p_1, p_2 \} = \kappa^{-1} eB, \quad \{ x_i, p_j \} = \kappa^{-1} \delta_{ij}, \quad (60) \]

where \( \kappa(\vec{x}) \equiv 1 - e\theta B(\vec{x}) \) with \( \theta_{12} \equiv \theta \) and \( B_{12} \equiv B \). None of the brackets now has
a canonical form. The equations of motion following from the Hamiltonian \( H(\vec{x}, \vec{p}) \equiv \frac{1}{2m} \vec{p}^2 + eV(\vec{x}) \) read

\[ \dot{p}_i = eE_i + eB\varepsilon_{ij}\dot{x}_j, \quad \text{with} \quad E_i \equiv -\partial_i V, \quad i \in \{1, 2\} \]
\[ m\dot{x}_i = p_i - em\theta\varepsilon_{ij}E_j, \quad \text{with} \quad m^* \equiv \kappa m, \quad \kappa(\vec{x}) \equiv 1 - e\theta B(\vec{x}), \quad (61) \]

where \( \varepsilon_{ij} \) denotes the components of the constant antisymmetric tensor normalized
by \( \varepsilon_{12} = 1 \). The parameter \( m^*(\vec{x}) \equiv m\kappa(\vec{x}) \) may be viewed as an effective mass
depending on the position of the particle. Various physical applications of this system
of evolution equations have been found in recent years, see [3] and references therein.

For \( V \equiv 0 \), we have \( p_i = m^* \dot{x}_i = m\kappa \dot{x}_i \), hence

\[ \dot{p}_i = m\kappa \dot{x}_i + m\kappa \dot{x}_i, \quad \text{with} \quad \kappa = -e\theta B = -e\theta \dot{x}_j \partial_j B. \]

\(^4\)One may as well consider \( \vec{p} \)-dependent parameters \( \theta^{\mu\nu} \).
Substitution of this expression into the first of equations (61) yields a second order differential equation for $x_i$:

$$m^*(\vec{x}) \ddot{x}_i = e \varepsilon_{ij} \dot{x}_j B^*, \quad \text{with} \quad B^* \equiv B + \frac{1}{2} m \theta \varepsilon_{ij} \dot{x}_i (\partial_j B). \quad (62)$$

This equation looks somewhat exotic and there does not appear to be a simple relationship with the second order differential equation (39). A common feature of these equations is the occurrence of $\theta$-dependent terms depending on higher order derivatives of the gauge potential. While the equation of motion (62) involves an $\vec{x}$-dependent mass, equation (39) involves a charge which depends on the particle’s trajectory, hence for both equations there is a dependence of parameters on the localization of the particle in the space in which it evolves.

The expressions in (62) simplify greatly in the case of a constant magnetic field: equation (62) then reduces to

$$m \ddot{x}_i = e \kappa \varepsilon_{ij} \dot{x}_j, \quad \text{with} \quad \kappa = 1 - e \theta B = \text{const}. \quad (63)$$

As was pointed out earlier [30], this equation of motion coincides with the “standard approach” equation (57) after a rescaling of time $t \rightarrow \kappa t$. We note that the value $\tilde{B} \equiv \frac{B}{1 - e \theta B}$ coincides with the one obtained for a constant magnetic field in two dimensions from the Seiberg-Witten map in non-commutative gauge field theory [2], but it differs from the constant non-commutative field strength

$$F_{12} \equiv \partial_1 A_2 - \partial_2 A_1 - ie [A_1 ; A_2] = \partial_1 A_2 - \partial_2 A_1 + e \{A_1, A_2\}, \quad (64)$$

e.g. in the symmetric gauge $(A_1, A_2) = (-\frac{B}{2} x_2, \frac{B}{2} x_1)$, where one finds $F_{12} = B(1 + \frac{e \theta B}{4 \kappa})$.

In conclusion, different Hamiltonian formulations for a charged “point” particle in a non-commutative space lead to different results. Furthermore, the equations of motion derived from the Lagrangian of Section 3 differ from the Hamiltonian equations of motion. However, for the special case of a constant magnetic field strength we have seen in the previous two subsections that the different Hamiltonian formulations lead to the same results (or to results that are related to each other by a redefinition of the magnetic field). So does the Lagrangian formulation of Section 3 as we will show in the following section.

5 Case of a constant field strength

We will now discuss the dynamics of charged particles coupled to a constant field strength on the basis of the results obtained in Section 3. Indeed this case represents a mathematically tractable and physically interesting application of the general formalism.
The non-commutative field strength $F_{\mu\nu}$ defined in equation (39) is constant if the gauge potential is linear in $x$. More precisely, for

$$A_\mu = -\frac{1}{2} \bar{B}_{\mu\nu} x^\nu,$$  \hspace{1cm} (65)

where the coefficients $\bar{B}_{\mu\nu} \equiv -\bar{B}_{\nu\mu}$ are constant, we obtain

$$F_{\mu\nu} = \bar{B}_{\mu\nu} - \frac{g}{4} \bar{B}_{\mu\rho} \theta^{\rho\sigma} \bar{B}_{\sigma\nu}. $$ \hspace{1cm} (66)

This field strength is constant, but dependent on the non-commutativity parameters $\theta^{\mu\nu}$. If we interpret $\bar{B}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ as the physical field strength, then (66) means that the non-commutativity parameters $\theta^{\mu\nu}$ modify in general the trajectories of the particle as compared to its motion in commutative space.

We note that the latter dependence on the non-commutativity parameters can be eliminated mathematically if one assumes that $\bar{B}_{\mu\nu}$ depends in a specific way on the parameters $\theta^{\mu\nu}$ and some $\theta$-independent constants $B_{\mu\nu}$. To illustrate this point [2], we assume that the only non-vanishing components of $\bar{B}_{\mu\nu}$ and $\theta^{\mu\nu}$ are as follows:

$$\bar{B}_{12} = -\bar{B}_{21} \equiv \bar{B}, \quad \theta^{12} = -\theta^{21} \equiv \theta, $$ \hspace{1cm} (67)

i.e.

$$F_{12} = \bar{B}(1 + \frac{g}{4} \theta \bar{B}) = -F_{21}, $$ \hspace{1cm} (68)

If $\bar{B}$ depends on $\theta$ and on a $\theta$-independent constant $B$ according to

$$B = B(B; \theta) = \frac{2}{g\theta} \left( \sqrt{1 + g\theta B} - 1 \right) = B(1 - \frac{g}{4} \theta B) + \mathcal{O}(\theta^2), $$ \hspace{1cm} (69)

then relation (66) implies that the non-commutative field strength $F_{12}$ is a $\theta$-independent constant: $F_{12} = B$.

Let us again come back to the expression (65) for the gauge field. Substitution of this expression into the subsidiary condition (34) yields

$$0 = \int d\tau \{ \dot{q} \delta^4(y - x(\tau)) - ig \dot{q} \hat{x}^\mu [A_\mu(y) \ast \delta^4(y - x(\tau))].$$

If the matrix $(\bar{B}_{\mu\nu})$ is the inverse of the matrix $(\theta^{\mu\nu})$, i.e. $\bar{B}_{\mu\rho} \theta^{\rho\sigma} = \delta^{\sigma}_{\mu}$, then $F_{\mu\nu} = (1 - \frac{g}{4}) \bar{B}_{\mu\nu}$ and, by virtue of (22) and an integration by parts, condition (70) takes the form

$$0 = \int d\tau \dot{q} \delta^4(y - x(\tau)) \left\{ 1 - \frac{g}{2} \right\}. $$ \hspace{1cm} (71)

The latter relation is obviously satisfied for a constant $q$. In this case, the equation of motion (39) for the particle in non-commutative space, i.e. $m\ddot{x}^\mu = qF^{\mu\nu}\dot{x}_\nu$, has the same form as the one of an electrically charged particle in ordinary space. This result is analogous to the one obtained for a constant magnetic field in $x^3$-direction within the Hamiltonian approaches, see equations (57) and (63).
6 Concluding remarks

Just as there exist different approaches to the formulation of gauge field theories on non-commutative spaces (e.g. the star product approach [1], the approach of spectral triples [31], of matrix models [32], . . .), there appear to exist different approaches to the dynamics of relativistic or non-relativistic particles in non-commutative space which are subject to a background gauge field. It is plausible that these approaches yield essentially the same results in the particular case of a constant magnetic field, i.e. a field strength which does not depend on the non-commuting coordinates. For general gauge fields, we also found some features which are shared by the dynamical equations resulting from the discussed Hamiltonian and Lagrangian approaches although the equations have quite different forms in this case.

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A Continuum formulation on a generic manifold

In this appendix, we show that Wong’s equations, as formulated on a generic space-time manifold, admit a simple continuum version. Moreover, we will prove that the latter formulation has to hold for arbitrary dynamical matter fields \( \phi \) whose dynamics is described by a generic action \( S[\phi; g_{\mu\nu}, A_{\mu}^a] \) which is invariant under both gauge transformations and general coordinate transformations (\( g_{\mu\nu} \) and \( A_{\mu}^a \) representing fixed external fields). These arguments generalize to Moyal space.

Let \( M \) be a four dimensional space-time manifold endowed with a fixed metric tensor \( (g_{\mu\nu}) \) of signature \((+,-,-,-)\). We denote the covariant derivative of a tensor field with respect to the Levi-Civita-connection by \( \nabla_\mu \) (e.g. \( \nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma_{\mu\rho}^\nu V^\rho \) where the coefficients \( \Gamma_{\mu\rho}^\nu \) are the Christoffel symbols) and the gauge covariant derivative as before by \( D_\mu \) (e.g. \( \delta_\lambda A_{\mu}^a = D_\mu \lambda^a \equiv \partial_\mu \lambda^a - ig[A_\mu, \lambda]^a \) for the infinitesimal gauge transformation of the Yang-Mills gauge field \( (A_\mu^a) \)). Since we used the notation \( \frac{Dq^a}{dt} \equiv \dot{x}^\mu D_\mu q^a \) in the main body of the text, we will write \( \frac{\Sigma V_\mu}{d\tau} \equiv \dot{x}^\nu \nabla_\nu V^\mu \) for the derivative of the vector field \( V^\mu(x(\tau)) \) along the trajectory \( \tau \mapsto x(\tau) \).

Lorentz-force and its non-Abelian generalization: The Lorentz-force equation on the space-time manifold \( M \) reads

\[
m \frac{dV^\mu}{d\tau} = eF_{\mu\nu}V^\nu
\]
where \( u^\mu \equiv \dot{x}^\mu \) denotes the 4-velocity of the particle of constant charge \( q \equiv e \) and where \( F_{\mu\nu} \) represents a given electromagnetic field strength. This equation of motion follows from the point particle action

\[
S[x] = \frac{m}{2} \int d\tau g_{\mu\nu}(x(\tau)) \dot{x}^\mu \dot{x}^\nu + e \int d\tau A_\mu(x(\tau)) \dot{x}^\mu
\]

(73)
on upon variation with respect to \( x^\mu \).

The natural generalization of (72) to non-Abelian Yang-Mills theory is given by Wong’s equations as written on the space-time manifold \( M \):

\[
m \frac{\nabla^2 x^\mu}{d\tau^2} = q^a F^a_{\mu\nu} \dot{x}^\nu, \quad \text{where} \quad \frac{Dq^a}{d\tau} = 0.
\]

(74)

Here, the covariant constancy of the charge-vector \( q^a \) represents the geometrically natural generalization of the ordinary constancy of the charge \( e \) appearing in the Abelian gauge theory. The equation of motion of \( x^\mu \) follows from the action functional

\[
S_W[x] = \frac{m}{2} \int d\tau g_{\mu\nu}(x(\tau)) \dot{x}^\mu \dot{x}^\nu + \int d\tau q^a A^a_\mu(x(\tau)) \dot{x}^\mu.
\]

(75)

**Continuum formulation:** The components \( T^{\mu\nu} \) of the energy-momentum tensor (density) and the components of the current density may be defined as functional derivatives of the action,

\[
T^{\mu\nu}(x) \equiv 2 \frac{\delta S_W}{\delta g_{\mu\nu}(x)}, \quad j^a_\mu(x) \equiv \frac{\delta S_W}{\delta A^a_\mu(x)}
\]

so that expression (75) implies

\[
T^{\mu\nu}(y) = \int d\tau \delta^4(y - x(\tau)) m \dot{x}^\mu(\tau) \dot{x}^\nu(\tau),
\]

\[
j^a_\mu(y) = \int d\tau \delta^4(y - x(\tau)) q^a(\tau) \dot{x}^\mu(\tau).
\]

(76)

We note that the energy-momentum 4-vector is then given by \( P^\mu = \int_{\mathbb{R}^3} d^3 x T^{\mu0} \) which yields the standard expressions:

\[
P^0 = \int_{\mathbb{R}^3} d^3 x T^{00} = m \dot{x}^0 = m \frac{dt}{d\tau} = \frac{m}{\sqrt{1 - \vec{v}^2}}, \quad P^i = m \dot{x}^i = \frac{mv^i}{\sqrt{1 - \vec{v}^2}}.
\]

The 4-divergence of the energy-momentum tensor can be evaluated by substituting the equation of motion \( m \frac{\nabla^2 x^\mu}{d\tau^2} = q^a F^a_{\mu\nu} \dot{x}^\nu \):

\[
\nabla_\mu T^{\mu\nu}(y) = \int d\tau \left( \dot{x}^\mu \nabla_\mu^g \right) \delta^4(y - x(\tau)) m \dot{x}^\nu(\tau)
\]

\[
= \int d\tau m \nabla^2 x^\nu(\tau) \delta^4(y - x(\tau))
\]

\[
= \int d\tau q^a(\tau) F_{\mu\nu}^a(x(\tau)) \dot{x}^\mu(\tau) \delta^4(y - x(\tau))
\]

\[
= F_{\mu\nu}^a(y) j^a_\mu(y).
\]
Similarly, substitution of the charge transport equation \( \frac{Dq^a}{d\tau} = 0 \) into the gauge covariant divergence of the current density gives

\[
D_\mu j^a_\mu(y) = \int d\tau (x^\mu D_\mu^a) \delta^4(y - x(\tau)) q^a(\tau) = \int d\tau \delta^4(y - x(\tau)) \frac{Dq^a(\tau)}{d\tau} = 0.
\]

Therefore the continuum version of equations (74) reads

\[
\nabla_\nu T^\nu\mu = F^a_{\nu\mu} j^a_\nu, \quad \text{where} \quad D_\mu j^a_\mu = 0.
\]

These relations may be called continuum Lorentz-Yang-Mills equations.

**General derivation of the continuum equations:** Actually equations (77) do not only hold for point particles but in a rather general context as will be shown in the sequel. To this end let us consider an arbitrary action functional

\[
S = S[\phi; g_{\mu\nu}, A^a_\mu]
\]

where \((g_{\mu\nu})\) and \((A^a_\mu)\) denote a fixed 4-geometry and Yang-Mills potential respectively, whereas \(\phi\) denotes arbitrary dynamical matter fields. Taking the action \(S\) to be gauge invariant entails the vanishing of its gauge variation:

\[
0 = \delta_\lambda S = \int \left( \frac{\delta S}{\delta \phi} \delta_\lambda \phi + \frac{\delta S}{\delta A^a_\mu} \delta_\lambda A^a_\mu \right).
\]

Together with the matter field equations of motion \(\delta S/\delta \phi = 0\) and the gauge variation of the Yang-Mills connection, \(\delta_\lambda A^a_\mu = D_\mu \lambda^a\), this implies

\[
D_\mu j^a_\mu = 0, \quad \text{where} \quad j^a_\mu(x) \equiv \frac{\delta S}{\delta A^a_\mu(x)}, \quad (78)
\]

i.e. the second of equations (77).

The fact that \(S\) is geometrically well defined is reflected by its invariance under general coordinate transformations (diffeomorphisms). The latter are generated by a generic vector field \(\xi \equiv \xi^\mu \partial_\mu\). Thus, we have

\[
0 = \delta_\xi S = \int \left( \frac{\delta S}{\delta \phi} \delta_\xi \phi + \frac{\delta S}{\delta g_{\mu\nu}} \delta_\xi g_{\mu\nu} + \frac{\delta S}{\delta A^a_\mu} \delta_\xi A^a_\mu \right), \quad (79)
\]

where the matter field equations again imply the vanishing of the first term. The metric tensor field and the Yang-Mills connection 1-form \(A \equiv A_\mu dx^\mu \equiv A^a_\mu T^a dx^\mu\) transform with the Lie derivative with respect to the vector field \(\xi\):

\[
\delta_\xi g_{\mu\nu} = (L_\xi g)_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu
\]

\[
\delta_\xi A_\mu = (L_\xi A)_\mu \equiv ((i\xi d + di\xi)A)_\mu
\]

\[
= (i\xi(dA - i\frac{g}{2}[A,A]) - ig[A,i\xi A] + di\xi A)_\mu
\]

\[
= \xi^\nu F_{\nu\mu} + D_\mu(\xi^\nu A_\nu).
\]

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Here, $i_\xi$ denotes the inner product of differential forms with the vector field $\xi$ and $F \equiv dA - \frac{i}{2}[A, A] \equiv \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu$ the Yang-Mills curvature 2-form. Substitution of the variations (80) into (79) and use of relation (78) now yields

$$\nabla_\mu T^{\mu\nu} = F^a_{\nu}\mu_j \mu^a,$$

i.e. the first of equations (77), thereby completing the proof of our claim.

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