Boolean Hedonic Games

Haris Aziz\textsuperscript{1}, Paul Harrenstein\textsuperscript{2}, Jérôme Lang\textsuperscript{3}, and Michael Wooldridge\textsuperscript{2}

\textsuperscript{1}NICTA and University of New South Wales, Australia \\
\textsuperscript{2}Department of Computer Science, University of Oxford, UK \\
\textsuperscript{3}LAMSADE, Université Paris-Dauphine, France

Abstract
We study hedonic games with dichotomous preferences. Hedonic games are cooperative games in which players desire to form coalitions, but only care about the makeup of the coalitions of which they are members; they are indifferent about the makeup of other coalitions. The assumption of dichotomous preferences means that, additionally, each player’s preference relation partitions the set of coalitions of which that player is a member into just two equivalence classes: satisfactory and unsatisfactory. A player is indifferent between satisfactory coalitions, and is indifferent between unsatisfactory coalitions, but strictly prefers any satisfactory coalition over any unsatisfactory coalition. We develop a succinct representation for such games, in which each player’s preference relation is represented by a propositional formula. We show how solution concepts for hedonic games with dichotomous preferences are characterised by propositional formulas.

1 Introduction
Hedonic games are cooperative games in which players desire to form coalitions, but only care about the makeup of the coalitions of which they are members; they are indifferent about the makeup of other coalitions (Drèze and Greenberg, 1980; Chalkiadakis et al., 2011). Because the specification of a hedonic game requires the expression of each player’s ranking over all sets of players including him, in general, such a specification requires exponential space – and, when used by a centralised mechanism, exponential elicitation time. Such an exponential blow-up severely limits the practical applicability of hedonic games, and for this reason researchers have investigated compactly represented hedonic games. One approach to this problem has been to consider possible restrictions on the possible preferences that players have. For example, one may

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assume that each player specifies only a ranking over single players, and that her preferences over coalitions are defined according to the identity of the best (respectively, worst) element of the coalition (Cechlárová and Hajduková, 2004; Cechlárová, 2008). One may also assume that each player’s preferences depend only on the number of players in her coalition (Bogomolnaia and Jackson, 2002). These representations come with a domain restriction, i.e., a loss of expressivity: Elkind and Wooldridge (2009) consider a fully expressive representation for hedonic games, based on weighted logical formulas. In the worst case, the representation of Elkind and Wooldridge requires space exponential in the number of players, but in many cases the space requirement is much smaller.

In this paper, we consider another natural restriction on player preferences. We consider hedonic games with dichotomous preferences. The assumption of dichotomous preferences means that each player’s preference relation partitions the set of coalitions of which that player is a member into just two equivalence classes: satisfactory and unsatisfactory. A player is indifferent between satisfactory coalitions, and is indifferent between unsatisfactory coalitions, but strictly prefers any satisfactory coalition over any unsatisfactory coalition.

While to the best of our knowledge dichotomous preferences have not been previously studied in the context of hedonic games, they have of course been studied in other economic settings, such as by Bogomolnaia et al. (2005), Bogomolnaia and Moulin (2004), and Bouyer et al. (2008) in the context of fair division, by Harrenstein et al. (2001) in the context of Boolean games, by Konieczny and Pinó-Pérez (2002) in the context of belief merging, by Bogomolnaia and Moulin (2004) in the context of matching, and by Brams and Fishburn (2007) (and many others) in the context of approval voting.

When the space of all possible alternatives has a combinatorial structure, propositional formulas are a very natural representation of dichotomous preferences. In such a representation, variables correspond to goods (in fair division), outcome variables (Boolean games), state variables (belief merging), or players (coalition formation). In the latter case, which we will be concerned with in the present paper, each player $i$ can express her preferences over coalitions containing her by using propositional atoms of the form $ij$ ($j \neq i$), meaning that $j$ is in the same coalition as $i$. Thus, for example, player 1 can express by the formula $(12 \lor 13) \land \neg 14$ that he wants to be in a coalition with player 2 or with player 3, but not with player 4. Our primary aim in this paper is to present such a propositional framework for specifying hedonic games and computing various solution concepts. We will first define a propositional logic using atoms of the form $ij$, together with domain axioms expressing that the output of the game should be a partition of the set of players. Then we consider a range of solution concepts, and show that they can be characterised by some specific classes of (sometimes polysize) formulas, and solved using propositional satisfiability solvers. The result is a simple, natural, and compact representation scheme for expressing preferences, and a machinery based on satisfiability for computing partitions satisfying some specific stability criteria such as Nash stability or core stability.
2 Preliminaries

In this section, we recall some definitions relating to coalitions, coalition structures (or partitions), and hedonic games. See, e.g., Chalkiadakis et al. (2011) for an in-depth discussion of these and related concepts.

Coalitions and Partitions We consider a setting in which there is a set \( N \) of \( n \) players with typical elements \( i, j, k, \ldots \). Players can form coalitions, which we will denote by \( S, T, \ldots \). A coalition is simply a subset of the players \( N \). One may usefully think of the players as getting together to form teams that will work together. A coalition structure is an exhaustive partition \( \pi = \{S_1, \ldots, S_m\} \) of the players into disjoint coalitions, i.e., \( S_1 \cup \cdots \cup S_m = N \) and \( S_i \cap S_j = \emptyset \) for all \( S_i, S_j \in \pi \) such that \( i \neq j \). For technical convenience, we slightly deviate from standard conventions and require that every coalition structure \( \pi \) contains the empty set \( \emptyset \). We commonly refer to coalition structures simply as partitions.

In examples, we also write, e.g., \([12|34|5]\) rather than the more cumbersome \([\{1,2\},\{3,4\},\{5\},\emptyset]\). For each player \( i \) in \( N \), we let \( \mathcal{A}_i = \{S \subseteq N : i \in S\} \) denote the set of coalitions over \( N \) that contain \( i \). If \( \pi = \{S_1, \ldots, S_m\} \) is a partition, then \( \pi(i) \) refers to the coalition in \( \pi \) that player \( i \) is a member of.

The notion of players leaving their own coalition and joining another lies at the basis of many of the solution concepts that we will come to consider.

We introduce some notation to represent such situations. For \( T \) a group of players (not necessarily a coalition in \( \pi \)), by \( \pi |_T \) we refer to the partition \( \{S_1 \cap T, \ldots, S_m \cap T\} \) and we write \( \pi |_{-T} \) for \( \pi |_{N \setminus T} \). Moreover, for \( S \) a coalition in partition \( \pi |_{-T} \), we use \( \pi |_{T \rightarrow S} \) to refer to the partition that results if the players in \( T \) leave their respective coalitions in \( \pi \) and join coalition \( S \). We also allow \( T \) to form a coalition of its own, in which case we write \( \pi |_{T \rightarrow \emptyset} \). Formally, we have, for \( S \in \pi |_{-T} \),

\[
\pi |_{T \rightarrow S} = \{S_j \in \pi |_{-T} : S_j \neq S\} \cup \{S \cup T, \emptyset\}.
\]

If \( T \) is a singleton \( \{i\} \) we also write \( \pi |_{-i} \) and \( \pi |_{i \rightarrow S} \) instead of \( \pi |_{-\{i\}} \) and \( \pi |_{\{i\} \rightarrow S} \), respectively. Thus, e.g., \( S \cup \{i\} \in \pi |_{i \rightarrow S} \) and \( \pi |_{i \rightarrow \pi(i) \setminus \{i\}} \).

Finally, define \( \pi |_{i \leftrightarrow j} \) as the partition where \( i \) and \( j \) exchange their places, i.e.:

\[
\pi |_{i \leftrightarrow j} = (\pi \setminus \{\pi(i), \pi(j)\}) \cup \{\pi(i) \setminus \{i\} \cup \{j\}, \pi(j) \setminus \{j\} \cup \{i\}\}.
\]

Thus, for partition \( \pi = [123|45] \), we have \( \pi(1) = \pi(2) = \{1,2,3\} \) and \( \pi(4) = \{4,5\} \). Furthermore, \( \pi |_{1,2,4,5} = [12|45] \) and \( \pi |_{-3,4} = [12|5] \). Also, \( \pi[1 \rightarrow 4,5] = [23|145] \), \( \pi[1 \rightarrow \emptyset] = [1|23|45] \), and \( \pi[3 \leftrightarrow 4] = [124|35] \).

Hedonic games Hedonic games are the class of coalition formation games in which each player is only interested in the coalition he is a member of, and is indifferent as to how the players outside his own coalition are grouped. Hedonic games were originally introduced by Drèze and Greenberg (1980) and further developed by, e.g., Bogomolnaia and Jackson (2002). Also see Haiduková (2003).
for a survey from a more computational point of view. Formally, a hedonic game is a tuple $(N, R_1, \ldots, R_n)$, where $R_i$ represents $i$’s transitive, reflexive, and complete preferences over the set of all coalitions $\mathcal{N}_i$ containing $i$. Thus, $S R_i T$ intuitively signifies that player $i$ considers coalition $S$ at least as desirable as coalition $T$, where $S$ and $T$ are coalitions in $\mathcal{N}_i$. By $P_i$ and $I_i$ we denote the strict and the indifferent part of $R_i$, respectively. The preferences $R_i$ of a player $i$ are said to be dichotomous whenever $\mathcal{N}_i$ can be partitioned into two disjoint sets $\mathcal{N}_i^+$ and $\mathcal{N}_i^-$ such that $i$ strictly prefers all coalitions in $\mathcal{N}_i^+$ to those in $\mathcal{N}_i^-$ and is indifferent otherwise, i.e., $S P_i T$ if and only if $S \in \mathcal{N}_i^+$ and $T \in \mathcal{N}_i^-$. A coalition $S$ in $\mathcal{N}_i$ is acceptable to $i$ if $i$ (weakly) prefers $S$ to coalition $\{i\}$, where he is on his own, i.e., if $S R_i \{i\}$. By contrast, we say that a coalition $S$ is satisfactory or desirable for $i$ if $S \in \mathcal{N}_i^+$. Satisfactory partitions are thus generally acceptable to all players. The implication in the other direction, however, does not hold.

We lift preferences on coalitions to preferences on partitions in a natural way: player $i$ prefers partition $\pi$ to partition $\pi'$ whenever $i$ prefers coalition $\pi(i)$ to coalition $\pi'(i)$. We also extend the concepts of acceptability and desirability of coalitions to partitions.

**Example 1** Consider the following Boolean game with four players, 1, 2, 3, and 4, whose (dichotomous) preferences are as follows. (Indifferences are indicated by commas.)

\[
\begin{align*}
1: & \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{1,2,3,4\} P_1 \{1\}, \{1,2\}, \{1,3\}, \{1,4\} \\
2: & \{2,1,3\}, \{2,1,4\}, \{2,3,4\} P_2 \{2\}, \{2,1\}, \{2,3\}, \{2,4\}, \{2,1,3,4\} \\
3: & \{3,1\}, \{3,2\}, \{3,1,2\} P_3 \{3\}, \{3,4\}, \{3,1,4\}, \{3,2,4\}, \{3,1,2,4\} \\
4: & \{4,1\}, \{4,2\}, \{4,3\}, \{4,1,2\}, \{4,1,3\}, \{4\} P_4 \{4,2,3\}, \{4,1,2,3\}
\end{align*}
\]

Thus, player 1 wants to be in a coalition of at least three and player 2 wishes to be in a coalition of exactly three. Moreover, player 3 wants to be in the same coalition as player 1 or as 2. He does not want to be in a coalition with player 4. Finally, player 4 does not want to be with players 2 and 3 together. There is exactly one partition that is satisfactory for all four players, namely $[123\mid 4]$. For players 1, 2, and 3, all coalitions are acceptable. For player 4, however, $\{4,2,3\}$ and $\{1,2,3,4\}$ are unacceptable.

**Solution Concepts for Hedonic Games** A solution concept associates with every hedonic game $(N, R_1, \ldots, R_n)$ a (possibly empty) set of partitions of $N$. Here we review some of the most common solution concepts for hedonic games.

Individual rationality captures the idea that every player prefers the coalition he is in to being on his own, i.e., that coalitions are acceptable to its members. Thus, formally, $\pi$ is individually rational if, for all players $i$ in $N$,

$$\pi(i) R_i \{i\}.$$ 

This condition is obviously equivalent to $\pi R_i \pi[i \rightarrow \emptyset]$.
For dichotomous hedonic games, a partition \( \pi \) is said to be \textit{social welfare optimal} if it maximises the number of players who are in a satisfactory coalition, that is, if \( \pi \) maximises \( \{|i \in N : \pi(i) \in \mathcal{N}^+_i\} \). In a similar way, a partition \( \pi \) is \textit{Pareto optimal} if it maximises the set of players being in a satisfactory coalition with respect to set-inclusion, that is, if there is no partition \( \pi' \) with

\[
\{i \in N : \pi(i) \in \mathcal{N}^+_i\} \subset \{i \in N : \pi'(i) \in \mathcal{N}^+_i\}.
\]

In the extreme case in which every player is in a most preferred coalition, \( \pi \) is said to be \textit{perfect} (cf., Aziz et al., 2013). A perfect partition satisfies any other of our stability concepts.

A partition is \textit{Nash stable} if no player would like to unilaterally abandon the coalition he is in and join any other existing coalition or stay on his own, that is, if, for all \( i \in N \) and all \( S \in \pi \),

\[
\pi(i) R_i S \cup \{i\}.
\]

Observe that this condition is equivalent to \( \pi R_i \pi[i \rightarrow S] \).

Core stability concepts consider group deviations instead of individual ones. A group of players, possibly from different coalitions, is said to block a partition \( \pi \) if they would all benefit by joining together in a separate coalition. Formally, \( T \) blocks (or is blocking) partition \( \pi \) if, for all \( i \in T \),

\[
T P_i \pi(i).
\]

Thus, \( T \) blocks \( \pi \) if and only if \( \pi[T \rightarrow 0] P_i \pi \) for all \( i \in T \). A group \( T \) weakly blocks (or is weakly blocking) \( \pi \) if \( T R_i \pi(i) \) holds for all \( i \in T \) and \( T P_i \pi(i) \) holds for some \( i \in T \). Then, \( \pi \) is \textit{core stable} if no group is blocking it and \( \pi \) is \textit{strict core stable} if no group is weakly blocking it.

Partition \( \pi \) is envy-free if no player is envious of another player, that is, if no player \( i \) would prefer to change places with another player \( j \). Formally, partition \( \pi \) is \textit{envy-free} if, for all players \( i \) and \( j \),

\[
\pi R_i \pi[i \leftrightarrow j].
\]

If \( \pi[i \leftrightarrow j] P_i \pi \) we also say that player \( i \) envies player \( j \).

**Example 1 (continued)** In our example, in partition \([1,2,3|4]\) each player is in a most preferred coalition. As such \([1,2,3|4]\) is perfect as well as social welfare optimal and satisfies all solution concepts mentioned above. Moreover, all partitions except \([1,2,3,4]\) and \([1,2,3,4]\) individually rational.

Now, consider partition \( \pi = [1|2,3|4] \). Here, player 2 does not want to abandon her coalition \([2,3]\) and join another as she prefers none of the following partitions to \( \pi \): \( \pi[2 \rightarrow \{1\}] = [1,2|3,4], \pi[2 \rightarrow \{2,3\}] = [1,2,3|4], \pi[2 \rightarrow \{4\}] \), and \( \pi[2 \rightarrow 0] = [1|2,3|4] \). As, however, \( \pi[1 \rightarrow \{2,3\}] = [1,2,3|4] \) and \( [1,2,3|4] P_i \pi \), partition \( \pi \) is not Nash stable.
Also observe that for \( \pi = [1|2,3|4] \) the group \( \{1,2,3\} \) is strongly blocking, as \( \pi([1,2,4] \rightarrow 0) = [1,2,4|3] \) and \([1,2,4|3] P_i \pi \) for all \( i \in \{1,2,4\} \). Thus, \( \pi \) is not core stable. By contrast, \([1,4|2,3] \) is core stable as only player 1 and 2 are not satisfied and both of them will only be if they can form a blocking coalition of exactly three. However, \([1,2,4] \) is still weakly blocking, and as such \([1,4|2,3] \) is not strict core stable.

For envy-freeness, consider partition \( \pi' = [1|2,4|3] \). Then, player 3 envies player 4, as \( \pi'[3 \equiv 4] = [1,2,3|4] \) and \([1,2,3|4] P_i \pi' \). By contrast, player 3 does not envy player 2: we have \( \pi'[3 \equiv 2] = [1|2,3,4] \) but not \([1|2,3,4] P_i \pi' \).

### 3 A Logic for Coalition Structures

In this section, we develop a logic for representing coalition structures. We will then use this logic as a compact specification language for dichotomous preference relations in hedonic games.

**Syntax** Given a set \( N \) of \( n \) players, we define a propositional language \( L_N \) built from the usual connectives and with for every (unordered) pair \( \{i,j\} \) of distinct players a propositional variable \( p_{(i,j)} \). The set of propositional variables we denote by \( V \). Observe that \( |V| = \binom{n}{2} \). For notational convenience we will write \( ij \) for \( p_{(i,j)} \). Thus, \( ij \) and \( ji \) refer to the same symbol. The language is interpreted on coalition structures on \( N \) and the informal meaning of \( ij \) is “\( i \) and \( j \) are in the same coalition”. Formally, the formulas of the language \( L_N \), with typical element \( \varphi \) is given by the following grammar

\[
\varphi ::= \; ij \mid \neg \varphi \mid (\varphi \lor \varphi)
\]

where \( i,j \in N \) and \( i \neq j \). By \(|\varphi|\) we denote the size of \( \varphi \).

For a given coalition \( S \) of players, we write \( V_S \) for the propositional variables in which some \( i \in S \) appears, i.e.,

\[
V_S = \{ij \in V : i \in S \text{ or } j \in S \}.
\]

Note that for distinct players \( i \) and \( j \) we have \( V_i \cap V_j = \{ij\} \). The propositional language over \( V_S \) we denote by \( L_S \). We write \( V_i \) and \( L_i \) for \( V_{\{i\}} \) and \( L_{\{i\}} \), respectively. The remaining classical connectives \( \bot \), \( \top \), \( \land \), \( \lor \), and \( \leftrightarrow \) are defined in the usual way. Moreover, for formulas \( \psi_1, \ldots, \psi_k \) of formulas, we have \( \bigwedge_{1 \leq m \leq k} \psi_m \) and \( \bigvee_{1 \leq m \leq k} \psi_m \) abbreviate \( \psi_1 \land \cdots \land \psi_k \) and \( \psi_1 \lor \cdots \lor \psi_k \), respectively. We also make use of the following useful notational shorthand:

\[
i_1 \cdots i_m \bar{i}_{m+1} \cdots \bar{i}_p = \bigwedge_{1 \leq j \leq m} i_1 i_j \land \bigwedge_{m \leq k \leq p} \neg i_1 i_k.
\]

Thus, \( i_1 \cdots i_m \bar{i}_{m+1} \cdots \bar{i}_p \) conveys that \( i_1, \ldots, i_m \) are in the same coalition and each of them in another coalition than \( i_{m+1} \cdots i_p \). Thus, where \( N = \{1,2,3,4\} \),

\[
12\bar{3}4 \lor 13\bar{2}4 \lor 1\bar{2}34
\]

abbreviates \((12 \land \neg 13 \land \neg 14) \lor (13 \land \neg 12 \land \neg 14) \lor (14 \land \neg 12 \land \neg 13)\) and signifies that player 1 is in a coalition of two players.
Semantics  We interpret the formulas of \( L_N \) on partitions \( \pi \) as follows.

\[
\begin{align*}
\pi &\models ij \quad \text{if and only if} \quad \pi(i) = \pi(j) \\
\pi &\models \neg \varphi \quad \text{if and only if} \quad \pi \not\models \varphi \\
\pi &\models \varphi \rightarrow \psi \quad \text{if and only if} \quad \pi \not\models \varphi \text{ or } \pi \models \psi
\end{align*}
\]

For \( \Psi \subseteq L_N \), we have \( \Psi \models \varphi \) if \( \pi \models \psi \) for all \( \psi \in \Psi \) implies \( \pi \models \varphi \). If \( \Psi = \emptyset \), we write \( \models \varphi \) and say that \( \varphi \) is valid.

Notice that partitions play a dual role in our framework: both their initial role as coalition structures, and the role of models in our logic. This dual role is key to using formulas of our propositional language as a specification language for preference relations. Thus, e.g., partition \([1|2|345]\) satisfies the following formulas of \( L_N \): \( 345, 31, 345, 1, 2, \neg 12 \land (23 \lor 34) \), and \( 12 \leftrightarrow 23 \).

Axiomatisation  We have the following axiom schemes for mutually distinct players \( i, j, \) and \( k \),

(A0) all propositional tautologies

(A1) \( ij \land jk \rightarrow ik \) \hspace{1cm} (transitivity)

as well as modus ponens as the only rule of the system:

(MP) from \( \varphi \) and \( \varphi \rightarrow \psi \) infer \( \psi \). \hspace{1cm} (modus ponens)

The resulting logic we refer to as \( \mathcal{P} \) and write \( \Psi \vdash_{\mathcal{P}} \varphi \) if there is a derivation of \( \varphi \) from \( \Psi, [\text{A0}] \) and \( [\text{A1}] \), using modus ponens.

**Theorem 1 (Completeness)** Let \( \Psi \cup \{\varphi\} \subseteq L_N \). Then,

\[ \Psi \vdash_{\mathcal{P}} \varphi \quad \text{if and only if} \quad \Psi \models \varphi. \]

*(sketch):* Soundness is straightforward. For completeness a standard Lindenbaum construction can be used. To this end, assume \( \Psi \not\vdash_{\mathcal{P}} \varphi \). Then, \( \Psi \cup \{\neg \varphi\} \) is consistent and can as such be extended to a maximal consistent theory \( \Psi^* \). Define a relation \( \sim_{\Psi^*} \) such that for all \( i, j \in N \),

\[ i \sim_{\Psi^*} j \quad \text{if and only if} \quad ij \in \Psi^*. \]

The axiom schemes \([\text{A0}]\) and \([\text{A1}]\) ensure that \( \sim_{\Psi^*} \) is a well-defined equivalence relation. Let \( [i]_{\sim_{\Psi^*}} = \{j \in N : i \sim_{\Psi^*} j\} \) be the equivalence class under \( \sim_{\Psi^*} \) to which player \( i \) belongs. Then define the partition \( \pi_{\Psi^*} = \{[i]_{\sim_{\Psi^*}} : i \in N\} \). By a straightforward structural induction, it can then be shown that for all \( \psi \in L_N \),

\[ \pi_{\Psi^*} \models \psi \quad \text{if and only if} \quad \psi \in \Psi^*. \]

It follows that \( \pi_{\Psi^*} \models \Psi \) and \( \pi_{\Psi^*} \not\models \varphi \). Hence, \( \Psi \not\models \varphi \). \[ \square \]
Alternatively, one can reason with coalition structures in standard propositional logic, by writing the transitivity axiom directly as a propositional logic formula. Let

$$trans = \bigwedge_{i,j,k \in N} (ij \land jk \rightarrow ik).$$

Then, for any propositional formulas $\varphi$ and $\psi$ of $L_N$,

$$\varphi \vdash_P \psi \text{ if and only if } \varphi \land trans \vdash \psi$$

that is, checking whether a formula $\varphi$ implies another formula $\psi$ in $P$ is equivalent to saying that $\varphi$ together with the transitivity constraint implies $\psi$. This means that reasoning tasks in $P$ can be done with a classical propositional theorem prover. In what follows we say that two formulas $\varphi$ and $\psi$ are $P$-equivalent whenever their equivalence can be proven in $P$, i.e., $\vdash_P \varphi \leftrightarrow \psi$.

## 4 Boolean Hedonic Games

The denotation of a formula $\varphi$ of our propositional language is a set of coalition structures, and we can naturally interpret these as being the desirable or satisfactory coalition structures for a particular player. Thus, instead of writing a hedonic game with dichotomous preferences as a structure $(N, R_1, \ldots, R_n)$, in which we explicitly enumerate preference relations $R_i$, we can instead write $(N, \gamma_1, \ldots, \gamma_n)$, where $\gamma_i$ is a formula of our propositional language that acts as a specification of the preference relation $R_i$. Intuitively, $\gamma_i$ represents player $i$’s ‘goal’ and player $i$ is satisfied if his goal is achieved and unsatisfied if he is not. We refer to a structure $(N, \gamma_1, \ldots, \gamma_n)$ as a **Boolean hedonic game**. Thus, a Boolean hedonic game $(N, \gamma_1, \ldots, \gamma_n)$ represents the (standard) hedonic game $(N, R_1, \ldots, R_n)$ with for each $i$,

$$\pi(i) \ R_i \ \pi'(i) \text{ if and only if } \pi \models \gamma_i \text{ implies } \pi' \models \gamma_i.$$ 

Observe that, defined thus, the preferences of each player in a hedonic Boolean game are dichotomous.

It should be clear that every dichotomous preference relation $R_i$ can be specified by a propositional formula $\gamma_i$, and hence our propositional language forms a fully expressive representation scheme for Boolean hedonic games. In fact, formulas in $L_N$ are strictly more expressive in the sense that they can represent any dichotomous preference relation over partitions rather than just preference relations over partitions as induced by a preference relation $R_i$ for a player $i$ over *coalitions* in $\mathcal{M}_i$. We find, however, that every Boolean hedonic

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1Let $i$ be a player with dichotomous preferences $R_i$ and let $X_i$ be the set of coalitions most preferred by $i$, i.e., $S \in X_i$ if and only if $S \ R_i \ S'$ for all coalitions $S$ and $S'$ containing $i$. Then, $R_i$ is represented by following formula of $L_i$ in disjunctive normal form:

$$\bigvee_{S \in X_i} \left( \bigwedge_{j \in S} ij \land \bigwedge_{k \notin S} \neg ik \right).$$
game \((N, \gamma_1, \ldots, \gamma_n)\) represents a hedonic game with dichotomous preferences provided that every player’s goal \(\gamma_i\) is equivalent to a formula in the language \(L_i\), the sublanguage of \(L_N\) in which only variables in \(V_i = \{ij : j \in N \setminus \{i\}\}\) occur. Intuitively, formulas in \(L_i\) only convey information about the coalitions player \(i\) is in or she is not in.

**Proposition 1** If a Boolean hedonic game \((N, \gamma_1, \ldots, \gamma_n)\) represents a hedonic game with dichotomous preferences, then for every player \(i\) there is a formula \(\varphi_i \in L_i\) that is \(P\)-equivalent to \(\gamma_i\). Moreover, if for every player \(i\) there is a formula \(\varphi_i \in L_i\) that is \(P\)-equivalent to \(\gamma_i\), then \((N, \gamma_1, \ldots, \gamma_n)\) represents a hedonic game with dichotomous preferences.

*(sketch):* For a player \(i\) and \(\varphi\) a formula in \(L_i\), a straightforward inductive argument shows that
\[
\pi \models \varphi \quad \text{if and only if} \quad \pi' \models \varphi \quad \text{for all} \quad \pi' \text{ with } \pi'(i) = \pi(i).
\]
Then, the result follows as a corollary.

Often, the use of propositional formulas \(\gamma_i\) gives a ‘concise’ representation of the preference relation \(R_i\), although of course in the worst case the shortest formula \(\gamma_i\) representing \(R_i\) may be of size exponential in the number of players. In what follows, we will write \((N, \gamma_1, \ldots, \gamma_n)\), understanding that we are referring to the game \((N, R_1, \ldots, R_n)\) corresponding to this specification.

**Example 1 (continued)** The hedonic game with dichotomous preferences in Example [1] is represented by the Boolean hedonic game \((N, \gamma_1, \gamma_2, \gamma_3, \gamma_4)\) with \(N = \{1, 2, 3, 4\}\) and the players’ goals given by:
\[
\begin{align*}
\gamma_1 &= (123 \lor 124 \lor 134) \\
\gamma_2 &= (213 \lor 214 \lor 234) \\
\gamma_3 &= (31 \lor 32) \land \neg 34 \\
\gamma_4 &= \neg 423.
\end{align*}
\]
For each player \(i\) we then have that \(\pi \models \gamma_i\) if and only if \(\pi \in \mathcal{N}_i^+\).

## 5 Substitution and Deviation

We establish a formal link between substitution in formulas of our language and the possibility of players deviating from their respective coalition in a given partition and joining other coalitions.

**Substitution** We first introduce some formal notation and terminology with respect to substitution of formulas for variables in our logic.

For \(ij\) a propositional variable in \(V_N\) and \(\varphi\) and \(\psi\) formulas of \(L_N\), we denote by \(\varphi_{ij \leftarrow \psi}\) the *uniform substitution* of variable \(ij\) by \(\psi\) in \(\varphi\). If \(ij = i_1j_1, \ldots, i_kj_k\) is a sequence of \(k\) distinct variables in \(V\) and \(\tilde{\psi} = \psi_1, \ldots, \psi_k\) a sequence of \(k\) formulas,
\[
\varphi_{ij \leftarrow \tilde{\psi}} = \varphi_{i_1j_1 \leftarrow \psi_1, \ldots, i_kj_k \leftarrow \psi_k}.
\]
denotes the simultaneous substitution of each \( i_m j_m \) by \( \psi_m \) (\( 1 \leq m \leq k \)). Thus, e.g., \((ij \lor \neg jk)_{ij_1 j k_1 \lor j k_1} = jk_1 \lor \neg ik_1\). A special case, which recurs frequently in what follows, is if every \( \psi_i \) is a Boolean, i.e., if \( \psi_1, \ldots, \psi_k \in \{\top, \bot\} \). Sequences \( \bar{b} = b_1, \ldots, b_k \) where \( b_1, \ldots, b_k \in \{\top, \bot\} \) we will also refer to as Boolean vectors of length \( k \). Thus, e.g., \( \top, \bot \) is a Boolean vector of length 2 and \((ij \land jk \rightarrow kl)_{ij_2 j k_2 \lor j k_2} = \top \land jk \rightarrow \bot\).

**Characterising individual deviations** Some of the stability concepts for Boolean hedonic games we consider in this paper, e.g., Nash stability, are based on which coalitions an individual player \( i \) can join given a partition \( \pi \). Recall that these coalitions are given by \( \pi|_{-i} \). Of course, not all groups of agents are included in \( \pi|_{-i} \). For instance, let partition \( \pi \) be given by \([12345]\). Then, player 1 can join coalition \( \{3,4\} \) but cannot form a coalition with players 4 and 5 by unilaterally deviating from \( \pi \). We find that the set \( \pi|_{-i} \) can be characterised in our logic. This furthermore yields a logical characterisation of when a player \( i \) can unilaterally break loose from his coalition, join another one and thereby guarantee that a given formula \( \varphi \) will be satisfied. A particularly interesting case is if \( \varphi \) implies the respective player’s goal. We thus gain expressive power with respect to whether a player can beneficially deviate from a given partition, a crucial concept.

**Lemma 1** Let \( \pi \) be a partition, \( i \) a player, \( B \) a group of players in \( N \setminus \{i\} \). Let furthermore \( \bar{b} = b_1, \ldots, b_{n-1} \) be a Boolean vector of length \( n-1 \) and \( ij = i j_1, \ldots, i j_{n-1} \) an enumeration of \( V_i \) such that \( B = \{j : ij j \bar{b} = \top\} \). Then,

1. \( B \in \pi|_{-i} \) iff \( \pi \models \text{trans}_{i j \bar{b}} \).
2. \( B \in \pi|_{-i} \) and \( \pi[i \rightarrow B] \models \varphi \) iff \( \pi \models (\varphi \land \text{trans}_{i j \bar{b}}) \).

**Proof:** We prove \[i\] the proof for \[ii\] is by structural induction on \( \varphi \) and relies on similar principles as \[i\]. As \( \bar{b} \) and \( ij \) are fixed throughout the proof, for better readability, we write \( \varphi \) for \( \varphi_{i j \bar{b}} \).

For the “only if”-direction, assume that \( B \in \pi|_{-i} \) as well as \( \pi \not\models \text{trans}' \). Observe that \( \text{trans}' = \bigwedge_{k,l,m} (kl' \land lm' \rightarrow km') \). Accordingly, there are some (mutually distinct) \( k, l, \) and \( m \) such that \( \pi \not\models kl' \land lm' \rightarrow km' \). It suffices to consider the following three cases.

\[
\begin{align*}
(a) & \quad i \notin \{k,l,m\}; & (b) & \quad i = k; & (c) & \quad i = l.
\end{align*}
\]

Case (a) cannot occur as we would have \( kl' = kl, lm' = lm, km' = km, \) and \( kl \land lm \rightarrow km \) is a theorem of the system.

If (b), then \( \pi \not\models il' \land lm' \rightarrow im' \). It follows that \( \pi \models il', \pi \models lm', \) and \( \pi \not\models im' \). Observe that in this case \( lm' = lm \). Hence, \( \pi(l) = \pi(m) \). Also notice that \( il', im' \in \{\top, \bot\} \) and, thus, \( im' = \bot \) and \( il' = \top \). Accordingly, \( l \in B \) but \( m \notin B \). As \( i \neq m \) and having assumed \( B \in \pi|_{-i} \), a contradiction follows:

\[\pi(m) \neq \pi(i) = \pi(l) = \pi(m).\]
If (c), we have \( \pi \not\models ik' \land im' \rightarrow km' \). Thus, \( \pi \models ik', \pi \models im', \) and \( \pi \not\models km' \). Observe that \( km' = km \). Hence, \( \pi(k) \neq \pi(m) \). Moreover, \( ik', im' \in \{ \top, \bot \} \), from which follows that \( ik' = \top \) and \( im' = \top \). Accordingly, both \( k, m \in B \).

With \( B \in \pi\|_\pi \), we obtain that \( \pi(k) = \pi(m) \), a contradiction.

For the “if”-direction, assume \( B \not\in \pi\|_\pi \) and \( B \neq \emptyset \). Because of the latter, there is some \( j \in B \). Accordingly, \( ij' = \top \). As \( B \not\models \pi\|_\pi \), and thus in particular \( B \neq \pi(j) \setminus \{ i \} \), there are two possibilities:

1. there is some \( k \neq i \) with \( k \in \pi(j) \) and \( k \notin B \), or
2. there is some \( k \neq i \) with \( k \notin \pi(j) \) and \( k \in B \).

If (1), we have \( \pi(j) = \pi(k) \) as well as \( ik' = \bot \). As \( jk' = jk \), it holds that \( \pi \models ij' \land jk' \) but \( \pi \not\models ik' \). If (2), however, we have \( \pi(j) \neq \pi(k) \) and \( ik' = \top \). As \( jk' = jk \), it holds that \( \pi \models ij' \land ik' \) but \( \pi \not\models jk' \). In either case it follows that \( \pi \not\models \text{trans}' \).

The following example illustrates Lemma 1.

**Example 2** Consider the partition \( \pi = [12|34|5] \). Then, \( \pi\|_\pi = \{ \{ 2 \}, \{ 34 \}, \{ 5 \}, \emptyset \} \). Let \( 1j' = 12, 13, 14, 15 \) be a fixed enumeration of \( V_1 \). Also let \( \vec{b}_1 = \bot, \top, \bot, \bot \) and \( \vec{b}_2 = \bot, \top, \bot, \top \) be Boolean vectors (of length 4). Then,

\[
[12|34|5] \models \text{trans}_{12,13,14,15} \leftarrow \bot, \top, \bot, \top.
\]

(This may be established, somewhat tediously, by painstakingly checking all 30 conjuncts of the form \( (kl \land lm) \rightarrow km \) of \( \text{trans} \).) Now, observe that \( \{ j : 1j_j = \vec{b}_1 \} = \{ 3, 4 \} \) and that \( \{ 3, 4 \} \in \pi\|_\pi \). On the other hand, observe that \( (13 \land 15 \rightarrow 35)_{1j_j = \vec{b}_2} = (\top \land \top) \rightarrow 35 \). It is easily established, however, that \( [12|34|5] \) does not satisfy \( (\top \land \top) \rightarrow 35 \) and, hence, \( \not\models \text{trans}_{1j_j = \vec{b}_2} \). Finally, observe that \( \{ j : 1j_j = \vec{b}_1 \} = \{ 3, 5 \} \) and that \( \{ 3, 5 \} \) is not in \( \pi\|_\pi \).

We now introduce the following abbreviation, where \( ij = ij_1, \ldots, ij_{n-1} \) is assumed to be a fixed enumeration of \( V_i \).

\[
\widehat{\exists} \ i \ \varphi = \bigvee_{\vec{b} \in \{ \bot, \top \}^{n-1}} (\varphi \land \text{trans}_{ij_j = \vec{b}}).
\]

Thus, \( \widehat{\exists} \ i \ \varphi \) can be understood as the operation of forgetting everything about player \( i \) (in the sense of Lin and Reiter (1994)) while taking the transitivity constraint into account. Intuitively, \( \widehat{\exists} \ i \ \varphi \) signifies that given partition \( \pi \) player \( i \) can deviate to some coalition such that that \( \varphi \) is satisfied.

**Proposition 2** Let \( \pi \) be a partition, \( i \) a player, and \( \varphi \) a formula of \( L_N \). Then,

\[
\pi \models \widehat{\exists} \ i \ \varphi \text{ iff } \pi[i \rightarrow S] \models \varphi \text{ for some } S \in \pi\|_\pi,
\]

\[11\]
Proof: First assume $\pi \models \exists i \varphi$. Then, $\pi \models (\varphi \land \text{trans})_{ij \overrightarrow{b}}$ for some $\overrightarrow{b} \in \{\bot, \top\}^{n-1}$. Define $S = \{j : ij_{ij \overrightarrow{b}} = \top\}$. By Lemma 1(iii), we then obtain $\pi[i \to S] \models \varphi$.

For the opposite direction, assume that $\pi[i \to S] \models \varphi$ for some $S \in \pi[i \to]$. Define $\overrightarrow{b} = b_1, \ldots, b_{n-1}$ as the Boolean vector of length $n-1$ such that for every $1 \leq k \leq n-1$,

$$b_k = \begin{cases} \top & \text{if } j \in S \cup \{i\} \\ \bot & \text{otherwise.} \end{cases}$$

Then, clearly, $S = \{j : ij_{ij \overrightarrow{b}} = \top\}$. By Lemma 1(iii), it follows that $\pi \models \varphi_{ij \overrightarrow{b}}$. We may conclude that $\pi \models \exists i \varphi$. □

It is important to note, however, that the number of Boolean vectors of length $k$ is exponential in $k$. Accordingly, $\exists i \varphi$ abbreviates a formula whose size is exponential in the size of $\varphi$.

Characterising group deviations Besides a single player deviating from its coalition and joining another, multiple players (from possibly different coalitions) could also deviate together and form a coalition of their own. This concept lies at the basis of, e.g., the core stability concept. We establish a formal connection between substitution and group deviations.

Let $T = \{i_1, \ldots, i_t\}$ be a group of players. Observe that $|V_T| = \binom{n}{2} - \binom{n-t}{2}$ and let $\overrightarrow{i_T}$ be a fixed enumeration of $V_T$. By the $T$-separating Boolean vector (given $\overrightarrow{i_T}$) we define as the unique Boolean vector $\overrightarrow{b_T}$ of length $\binom{n}{2} - \binom{n-t}{2}$ such that for all $i \in T$ and all $j \in N$,

$$ij_{ij \overrightarrow{b_T}} = \begin{cases} \top & \text{if } j \in T, \\ \bot & \text{otherwise.} \end{cases}$$

Intuitively, $\overrightarrow{b_T}$ represents the choice of group $T$ to form a coalition of their own. Whenever $T$ is clear from the context we omit the subscript in $\overrightarrow{b_T}$ and $\overrightarrow{i_T}$. The following characterisation now holds.

Lemma 2 Let $(N, \gamma_1, \ldots, \gamma_n)$ be a Boolean hedonic game, $T$ a group of players, $\pi$ a partition, $ij$ a fixed enumeration of $V_T$, and $\overrightarrow{b_T}$ the corresponding $T$-separating Boolean vector. Then, for every formula $\varphi \in L_N$,  

$$\pi \models \varphi_{ij \overrightarrow{b_T}} \text{ if and only if } \pi[T \to \emptyset] \models \varphi.$$  

6 Characterising Solutions

Our task in this section is to show how the various solution concepts we introduced above can be characterised as formulas of our propositional language. Let $f$ be a function mapping each Boolean hedonic game $G$ for $N$ to a formula $f(G)$ of $L_N$. Given a solution concept $\theta$, we say that $f$ is a characterisation of $\theta$ if for every Boolean hedonic game $G$ on $N$ and every partition $\pi$, we
have that \( \pi \) is a solution according to \( \theta \) for game \( G \) if and only if \( \pi \models f(G) \). If, furthermore, there exists a polynomial \( p \) such that \(|f(G)| \leq p(|N|)\), then \( f \) is a polynomial characterisation of \( \theta \).

Once we have a characterisation of \( \theta \), we know that there is a one-to-one correspondence between the partitions of \( N \) satisfying \( \theta \) and the models of \( f(G) \). Therefore, given a Boolean hedonic game \( G \):

- checking whether there exists a partition satisfying \( \theta \) in \( G \) amounts to checking whether \( f(G) \) is satisfiable;
- computing a partition satisfying \( \theta \) in \( G \) amounts to finding a model of \( f(G) \);
- computing all partitions satisfying \( \theta \) in \( G \) amounts to finding all models of \( f(G) \).

Thus, once we have a characterisation of a solution concept, one can use a SAT solver to find (some or all) or to check the existence of partitions that satisfy it. This carries over to conjunctions of solution concepts. For instance, if individual rationality is characterised by \( f_{IR} \) and envy-freeness by \( f_{EF} \), then there is a one-to-one correspondence between the individual rational envy-free partitions for \( G \) and the models of \( f_{IR}(G) \land f_{EF}(G) \). More generally, these techniques can be used for finding or checking partitions satisfying \( \theta \) that also have certain other properties expressible in \( L_N \).

In the remainder of the section we focus on how a number of classical solution concepts, and see how they can be characterised in our logic.

**Individual rationality, perfection, and optimality**  Recall that a partition is individually rational if any player is at least as happy in her coalition as being alone, that is, no player would prefer to leave her coalition to form a singleton coalition. Now we have the following characterisation of individual rationality in our logic.

**Proposition 3** Let \((N, \gamma_1, \ldots, \gamma_n)\) be a Boolean hedonic game, let \( i \) be a player with goal \( \gamma_i \), and let \( \pi \) be a partition. Let, furthermore, \( i\vec{\jmath} \) be a fixed enumeration of \( V_i \) and let \( \vec{b} = \bot, \ldots, \bot \) be the Boolean vector of length \( n - 1 \) only containing \( \bot \). Then,

\[
\begin{align*}
(\text{i}) & \quad \pi \text{ is acceptable to } i \text{ iff } \pi \models (\gamma_i)_{i\vec{\jmath} \leftarrow \vec{b}} \rightarrow \gamma_i, \\
(\text{ii}) & \quad \pi \text{ is individually rational iff } \pi \models \bigwedge_{i \in N} ((\gamma_i)_{i\vec{\jmath} \leftarrow \vec{b}} \rightarrow \gamma_i).
\end{align*}
\]

**Proof:** We only give the proof for \( (i) \), as \( (ii) \) follows as an immediate consequence. For \( (i) \), merely consider the following equivalences, of which the third
one follows from Lemma 1(ii).

\[ \pi \text{ is acceptable to } i \iff \pi R_i \pi[i \to \emptyset] \]
\[ \iff \pi[i \to \emptyset] \models \gamma_i \text{ implies } \pi \models \gamma_i \]
\[ \iff \pi \models (\gamma_i)_{i\overleftarrow{\overrightarrow{\bar{r}}}} \text{ implies } \pi \models \gamma_i \]
\[ \iff \pi \models (\gamma_i)_{i\overleftarrow{\overrightarrow{\bar{r}}}} \to \gamma_i. \]

This concludes the proof. \( \Box \)

To illustrate Proposition 3 we consider again Example 1.

Example 1 (continued) In the game of our example, all partitions are acceptable to player 1, whose goal is given by \( \gamma_1 = 123 \lor 124 \lor 134 \). Let \( V_1 \) be enumerated by \( i\overleftarrow{\overrightarrow{\bar{r}}} = 12, 13, 14 \) and let \( \vec{b} = \bot, \bot, \bot \). Then, \( (\gamma_2)_{12,13,14\overleftarrow{\overrightarrow{\bar{r}}}} \) is P-equivalent to \( \bot \) and, hence, \( \pi \models (\gamma_2)_{12,13,14\overleftarrow{\overrightarrow{\bar{r}}}} \to \gamma_1 \) for all partitions \( \pi \). According to Proposition 3 this signifies that to player 1 every partition is acceptable.

Now consider player 4, whose goal is given by \( \neg 423 \), that is, by \( \neg(42 \land 43) \). Let \( V_4 \) be enumerated by \( 41, 42, 43 \) and let \( \vec{b} = \bot, \bot, \bot \). Then, \( \neg(42 \land 43)_{41,42,43\overleftarrow{\overrightarrow{\bar{r}}}} \) is \( \neg(\bot \land \bot) \), which is obviously P-equivalent to \( \top \). Hence,

\[ \pi \models \neg(42 \land 43)_{41,42,43\overleftarrow{\overrightarrow{\bar{r}}}} \iff \pi \models \neg(42 \land 43), \]

meaning that a partition \( \pi \) is acceptable to player 4 if and only if \( \pi \) satisfies his goal.

The logical characterisation of perfect perfect partition is immediate, as witnessed by the following proposition.

**Proposition 4** Let \( (N, \gamma_1, \ldots, \gamma_n) \) be a Boolean hedonic game. Then, a partition \( \pi \) is perfect if and only if

\[ \pi \models \bigwedge_{i \in N} \gamma_i. \]

As a consequence, a perfect partition exists if and only if the formula \( \text{trans} \land \bigwedge_{i \in N} \gamma_i \) is satisfiable. Moreover, finding a social welfare maximising partition reduces to finding valuation satisfying a maximum number of formulas \( \gamma_i \land \text{trans} \), that is, to solving a MAXSAT problem.

Leveraging the same idea of iteratively checking whether a perfect partition can be found for a subset of agents, one can compute Pareto optimal solutions for a given game. A subset \( \Psi \) of formulas is said to be a maximal trans-consistent if both

(i) \( \Psi \cup \{\text{trans}\} \) is consistent, and

(ii) \( \Psi' \cup \{\text{trans}\} \) is inconsistent for all sets of formulas \( \Psi' \) with \( \Psi \subseteq \Psi' \).

We now have the following proposition.

**Proposition 5** A partition \( \pi \) of a Boolean hedonic game is Pareto optimal if and only if \( \{\gamma_i : \pi \models \gamma_i\} \) is a maximal trans-consistent subset of \( \{\gamma_1, \ldots, \gamma_n\} \)

Algorithms for computing maximal consistent subsets are well-known and could thus be exploited for the computation of Pareto optimal partitions.
Nash stability Recall that a partition $\pi$ is Nash stable, if no player $i$ wishes to leave his coalition $\pi(i)$ and join another (possibly empty) coalition so as to satisfy his goal. Leveraging our results from Section 5, we obtain the following characterisation of this fundamental solution concept.

Proposition 6 Let $(N, \gamma_1, \ldots, \gamma_n)$ be a Boolean hedonic game and $\pi$ a partition. Then,

$$\pi \text{ is Nash stable if and only if } \pi \models \bigwedge_{i \in N} ((\exists i \gamma_i) \to \gamma_i).$$

Proof: Consider an arbitrary player $i$ and observe that following equivalences hold. The fourth equivalence holds in virtue of Proposition 2. The third one is a standard law of logic: merely observe that $\pi \models \gamma_i$ is not dependent on $S$.

$$\begin{align*}
\pi \text{ is Nash stable} & \iff \text{for all } i \in N \text{ and } S \in \pi|_{-i}: \pi R_i \pi[i \to S] \\
& \iff \text{for all } i \in N \text{ and } S \in \pi|_{-i}: \text{if } \pi[i \to S] \models \gamma_i \text{ then } \pi \models \gamma_i \\
& \iff \text{for all } i \in N: \text{if } \pi[i \to S] \models \gamma_i \text{ for some } S \in \pi|_{-i} \text{ then } \pi \models \gamma_i \\
& \iff \text{for all } i \in N: \text{if } \pi \models (\exists i \gamma_i) \text{ then } \pi \models \gamma_i \\
& \iff \text{for all } i \in N: \pi \models (\exists i \gamma_i) \to \gamma_i \\
& \iff \pi \models \bigwedge_{i \in N} ((\exists i \gamma_i) \to \gamma_i)
\end{align*}$$

This concludes the proof.

Our running example illustrates this result.

Example 1 (continued) Consider again the game of Example 1. Partition $[123|4]$ satisfies each player’s goal and, consequently, is Nash stable. We also have that $[123|4] \models \gamma_1 \wedge \gamma_2 \wedge \gamma_3 \wedge \gamma_4$ and, thus,

$$[123|4] \models \bigwedge_{i \in N} ((\exists i \gamma_i) \to \gamma_i).$$

Now recall that for partition $\pi = [123|4]$ player 2’s goal is not satisfied and that she cannot deviate and join another coalition to make this happen. In this case, $\pi|_{-2} = \{\{1\}, \{3\}, \{4\}\}$. Moreover, $\pi[2 \to \{1\}] = [123|4]$, $\pi[2 \to \{3\}] = [123|4]$, and $\pi[2 \to \{4\}] = [1|3|24]$. Since, $[123|4] \not\models \gamma_2$, $[123|4] \not\models \gamma_2$, and $[1|3|24] \not\models \gamma_2$, it follows that $\pi \not\models (\exists 2 \gamma_2) \to \gamma_2$. Hence, $\pi \models (\exists 2 \gamma_2) \to \gamma_2$. Player 1, however, could deviate from $\pi_2$ and join $\{2, 3\}$ and thus have his goal satisfied. Thus, $\pi$ is not Nash stable. Now observe that $\{2, 3\} \in \pi|_{-1}$ and that $\pi[1 \to \{2, 3\}] = [123|4]$. Moreover, $[123|4] \models \gamma_1$. As thus $\pi \models (\exists 1 \gamma_1$, also $\pi \not\models (\exists 1 \gamma_1) \to \gamma_1$. We may conclude that

$$[123|4] \not\models \bigwedge_{i \in N} ((\exists i \gamma_i) \to \gamma_i).$$
Nash stable partitions are not guaranteed to exist in Boolean hedonic games. The two-player game \(\{1, 2\}, 12, \neg 21\) witnesses this fact, as can easily be appreciated. The translation into a SAT instance gives us a way to compute all Nash stable partitions of a given Boolean hedonic game. Recall, however, that the size of \(\exists i \gamma_i\) is generally exponential in the size of \(\gamma_i\).

**Core and strict core stability** Core and strict core stability relate to group deviations much in the same way as Nash stability relates to individual deviations. Group deviations we characterised in Section 5. We thus find that Lemma 2 yields a straightforward characterisation in our logic of a specific group blocking or weakly blocking a given partition.

**Proposition 7** Let \((N, \gamma_1, \ldots, \gamma_n)\) be a Boolean hedonic game and \(T\) a group of players, and \(\pi\) be a partition. Let, furthermore, \(\vec{\vec{i}}\) a fixed enumeration of \(V_T\) and \(\vec{b}\) the corresponding \(T\)-separating Boolean vector. Then,

(i) \(T\) blocks \(\pi\) if and only if \(\pi \models \bigwedge_{i \in T} (\neg \gamma_i \land (\gamma_i)_{\vec{j} \leftarrow \vec{b}})\),

(ii) \(T\) weakly blocks \(\pi\) if and only if

\[
\pi \models \bigwedge_{j \in T} (\gamma_j \rightarrow (\gamma_j)_{\vec{j} \leftarrow \vec{b}}) \land \bigvee_{i \in T} (\neg \gamma_i \land (\gamma_i)_{\vec{j} \leftarrow \vec{b}}).
\]

**Proof:** We give the proof for (i), as the one for (ii) runs along analogous lines. Consider the following equivalences, of which the third one follows immediately from Lemma 2.

\[
\begin{align*}
T \text{ blocks } \pi & \iff \text{ for all } i \in T: \neg \gamma_i \land (\gamma_i)_{\vec{j} \leftarrow \vec{b}} \land \pi \\
& \iff \text{ for all } i \in T: \gamma_i \rightarrow (\gamma_i)_{\vec{j} \leftarrow \vec{b}} \land \pi \neq \gamma_i \\
& \iff \text{ for all } i \in T: \neg \gamma_i \land (\gamma_i)_{\vec{j} \leftarrow \vec{b}} \land \pi \neq \gamma_i \\
& \iff \pi \models \bigwedge_{i \in T} (\neg \gamma_i \land (\gamma_i)_{\vec{j} \leftarrow \vec{b}}).
\end{align*}
\]

This concludes the proof. \(\square\)

Observe that the size of \(\bigwedge_{i \in T} (\neg \gamma_i \land (\gamma_i)_{\vec{j} \leftarrow \vec{b}})\) is obviously polynomial in \(\sum_{i \in T} |\gamma_i|\) and, hence, a partition \(\pi\) being blocking by particular group \(T\) of players can be polynomially characterised. It might also be worth observing that this characterisation is reminiscent of that for individual rationality and, surprisingly, much more so than of the one for Nash stability.

As a corollary of Proposition 7 and de Morgan laws, we obtain the following characterisations of a partition being core stable and of a partition being strict core stable. The characterisations, however, involve a conjunctions over all groups of players and as such is not polynomial.
Corollary 1  Let \((N, \gamma_1, \ldots, \gamma_n)\) be a Boolean hedonic game and \(\pi\) be a partition. Let for each coalition \(T\), \(\vec{ij}\) be an enumeration of \(V_T\) and \(\vec{b}\) the corresponding \(T\)-separating Boolean vector. Then,

(i) \(\pi\) is core stable if and only if \(\pi \models \bigwedge_{T \subseteq N} \bigvee_{i \in T} ((\gamma_i)_{\vec{ij} \leftarrow \vec{b}} \rightarrow \gamma_i)\).

(ii) Then, \(\pi\) is strict core stable if and only if \(\pi \models \bigwedge_{T \subseteq N} \left( \bigvee_{j \in T} (\gamma_j \land \neg((\gamma_j)_{\vec{ij} \leftarrow \vec{b}})) \lor \bigwedge_{i \in T} ((\gamma_i)_{\vec{ij} \leftarrow \vec{b}} \rightarrow \gamma_i) \right)\).

Although core stable coalition structure are not guaranteed to exist in general hedonic games, the restriction to dichotomous preferences allows us to derive this positive result.

Proposition 8  For every Boolean hedonic game, a core stable coalition structure is guaranteed to exist.

Proof: We initialise \(N'\) to \(N\) and partition \(\pi\) to \(\{\emptyset\}\). We find a maximal subset of \(S \subset N'\) for which all players are in an approved coalition that satisfies their formulas. We modify \(\pi\) to \(\pi \cup \{S\}\) and \(N'\) to \(N' \setminus S\). The procedure is repeated until no such maximal subset \(S\) exists. If \(N' \neq \emptyset\), then \(\pi\) is set to \(\pi \cup \{\{i\} : i \in N'\}\).

We now argue that \(\pi\) is core stable. We note that each player who was in some subset \(S\) will never be part of a blocking coalition. If \(N'\) was non-empty in the last iteration, then no subset of players in \(N'\) can form a deviating coalition among themselves. \(\square\)

By contrast, a strict core stable partition is not guaranteed to exist. To see this consider the three-player Boolean hedonic game \((\{1, 2, 3\}, 12, 21 \lor 23, 32)\). It is not hard to see that each of the five possible partitions is weakly blocked by either \(\{1, 2\}\) or \(\{2, 3\}\).

Envy-freeness  Recall that a partition is envy-free if no player would strictly prefer to exchange places with another player. Observe that for the trivial partitions \(\pi^0 = [1 | \cdots | n]\) and \(\pi^1 = [1, \ldots, n]\), we have \(\pi^0[i \Leftrightarrow j] = \pi^0\) and \(\pi^1[i \Leftrightarrow j] = \pi^1\) for all players \(i\) and \(j\). Accordingly \(\pi^0\) and \(\pi^1\) are envy-free. Envy-free partitions are thus guaranteed to exist in our setting. The following lemma allows us to derive a polynomial characterisation of envy-freeness.

Lemma 3  Let \((N, \gamma_1, \ldots, \gamma_n)\) be a Boolean hedonic game and \(i\) and \(j\) players in \(N\), \(\varphi\) a formula in \(L_N\). Fix, furthermore, an enumeration \(k_1, \ldots, k_{n-2}\) of \(N \setminus \{i, j\}\) and let \(ik = ik_1, \ldots, ik_{n-2}\) and \(jk = jk_1, \ldots, jk_{n-2}\) enumerate \(V_i \setminus \{ij\}\) and \(V_j \setminus \{ji\}\), respectively. Then,

\[\pi \models \varphi_{ik,jk \leftarrow ik,jk} \text{ if and only if } \pi[i \Leftrightarrow j] \models \varphi.\]
Proof: With $i\vec{k}$ and $j\vec{k}$ being fixed we write $\varphi'$ for $\varphi_{i\vec{k},j\vec{k} \leftarrow j\vec{k},i\vec{k}}$. The proof is then by induction on $\varphi$.

For the basis, let $\varphi = lm$. There are three possibilities:

(a) $lm = ij$, (b) $lm \in (V_i \cup V_j) \setminus \{ij\}$, and (c) $lm \notin V_i \cup V_j$.

If (a), we have that $lm' = ij' = ij = lm$. Now, either $\pi(i) = \pi(j)$ or $\pi(i) \neq \pi(j)$. If the former, $\pi[i \equiv j] = \pi$ as well as both $\pi \models ij'$ and $\pi[i \equiv j] \models ij$. If the latter, however, it can easily be seen that both $\pi \not\models ij'$ and $\pi[i \equiv j] \not\models ij$.

For case (b), we may assume without loss of generality that $lm = ik$ for some $k \neq j$. Then, $ik' = jk$. In case $\pi(i) = \pi(j)$, obviously, $\pi = \pi[i \equiv j]$ as well as $k \in \pi(i)$ if and only if $k \in \pi(j)$. Hence, $\pi \models ik'$ if and only if $\pi[i \equiv j] \models ik$. So, assume $\pi(i) \neq \pi(j)$. Now, either (i) $k \in \pi(i)$ and $k \notin \pi(j)$, (ii) $k \notin \pi(k)$ and $k \in \pi(j)$, or (iii) $k \notin \pi(i)$ and $k \notin \pi(j)$. If (i), $\pi \models ik'$ as well as $\pi[i \equiv j] \models jk$. In cases (ii) and (iii), we have $\pi \not\models ik'$ and $\pi[i \equiv j] \not\models jk$.

Finally, if (c), we have $lm' = lm$. As $l, m \notin \{i, j\}$, it can then easily be seen that $\pi \models lm'$ if and only if $\pi[i \equiv j] = lm$.

The cases $\varphi = -\psi$ and $\varphi = \psi \rightarrow \chi$ follow by induction. \hfill \square

We are now in a position to state the following result.

**Proposition 9** Let $(N, \gamma_1, \ldots, \gamma_n)$ be a Boolean hedonic game. Furthermore, for every two players, $i$ and $j$, and enumeration $k_1, \ldots, k_{n-2}$ of $N \setminus \{i, j\}$, let $i\vec{k} = ik_1, \ldots, ik_{n-2}$ and $j\vec{k} = jk_1, \ldots, jk_{n-2}$ enumerate $V_i \setminus \{ij\}$ and $V_j \setminus \{ij\}$, respectively. Then,

$$\pi \text{ is envy-free if and only if } \pi \models \bigwedge_{i,j \in N} ((\gamma_i)_{i\vec{k},j\vec{k} \leftarrow j\vec{k},i\vec{k}} \rightarrow \gamma_i).$$

**Proof:** By virtue of Lemma 3 the following equivalences hold:

$$\pi \text{ is envy-free if and only if } \pi \models \bigwedge_{i,j \in N} ((\gamma_i)_{i\vec{k},j\vec{k} \leftarrow j\vec{k},i\vec{k}} \rightarrow \gamma_i)$$

This concludes the proof. \hfill \square

Observe that the size of $\bigwedge_{i,j \in N} ((\gamma_i)_{i\vec{k},j\vec{k} \leftarrow j\vec{k},i\vec{k}} \rightarrow \gamma_i)$ is clearly polynomial in $\sum_{i \in T} |\gamma_i|$. Hence, a partition $\pi$ being envy-free can be polynomially characterised.
Example 1 (continued) Recall that $\gamma_3 = (31 \lor 32) \land \neg 34$ and that player 3 envies player 4 if partition $\pi' = [1|24|3]$ obtains. To see how this is reflected by Proposition 6 let $31, 32$ and $41, 42$ enumerate $V_3 \setminus \{34\}$ and $V_4 \setminus \{43\}$, respectively. Then,

$$(31 \lor 32) \land \neg 34)_{31,32,41,42} \leftarrow 41,42,31,32 = (41 \lor 42) \land \neg 34.$$ 

Now, both $\pi' \models (41 \lor 42) \land \neg 34$ and $\pi' \not\models (31 \lor 32) \land \neg 34$, and, hence, $\pi' \not\models (\gamma_3)_{34,31,32} \leftarrow 43,41,42 \rightarrow \gamma_3$.

7 Related Work and Conclusions

Our motivation and approach is strongly reminiscent of the setting of Boolean games in the context of non-cooperative game theory [Harrenstein et al., 2001]. A major difference with Boolean games and propositional hedonic games is that in Boolean games, players have preferences over outcomes, where an outcome is a truth assignment to outcome variables, and each outcome variable is controlled by a specific player. This control assignment function, which is a central notion in Boolean games, has no counterpart here, where the outcome is a partition of the players. However, there are technical similarities with and conceptual connections to Boolean games, especially when characterising solution concepts. For instance, the characterisation of Nash stable partitions by propositional formulas (Section 4) is similar to the characterisation of Nash equilibria by propositional formulas in Boolean games as by [Bonzon et al., 2009]. The basic Boolean games model of [Harrenstein et al., 2001] was adapted to the setting of cooperative games by [Dunne et al., 2008]. However, the logic used to specify player’s goals in the work of [Dunne et al.] was not intended for specifying desirable coalition structures, as we have done in the present paper.

Our work also shares some common ground with the work of [Bonzon et al., 2012], who study the formation of efficient coalitions in Boolean games, that is, coalitions whose joint abilities allow their members to jointly achieve their goals. Our work also bears some resemblance to the work of [Elkind and Wooldridge, 2002], who were interested in using logic as a foundation upon which to build a compact representation scheme for hedonic games; more precisely, their work made use of weighted Boolean formulas, and was inspired by the marginal contribution nets representation for cooperative games in characteristic function form proposed by [Ieong and Shoham, 2005]. The focus of [Elkind and Wooldridge, 2002], however, was more on complexity issues than in finding exact characterisations for solution concepts.

Finally, our work contributes to the extensive literature on compact representations for cooperative games, which has expanded rapidly over the past decade [Chalkiadakis et al., 2011].

Our characterisations of solution concepts enable to compute, using an off-the-shelf SAT solver, a partition or all partitions satisfying a solution concept or a logical combination of solution concepts. Of course, this translation is interesting only when we cannot do better. For instance, for solution concepts
leading to a polynomial characterisation, we cannot do better if and only if the corresponding decision problem is \( \text{NP} \)-complete. Identifying the complexity of finding partitions satisfying solution concepts for Boolean hedonic games is therefore the most immediate direction of further research.

There are at least three more directions in which our work might be further developed. First, we could think of relaxing our restriction to dichotomous preferences and study more general hedonic games with compact logical representations and derive exact characterisations of solution concepts. There are several ways in which more general preferences can be incorporated in our logical framework for hedonic games. For instance, instead of a single goal, we could associate with each player a prioritised set of goals. The different possibilities in this respect, however, vary in their level of sophistication. For some of the cruder extensions our results extend naturally and straightforwardly. For the more sophisticated settings more research seems to be required, which falls beyond the scope of this paper.

Second, our restriction to hedonic preferences can also be relaxed, so that players may have preferences that do depend not only on the coalition to which they belong. This would also pave the way to a more general logic of coalition structures. Solution concepts, once generalised, can hopefully be characterised. (We have positive preliminary results that go into this direction).

A third topic of future research would be the characterisation of classes of hedonic and coalition formation games in our logic. As mentioned above, various classes of hedonic games that allow for a concise representation have been proposed in the literature. It would be interesting to see whether these classes can also be polynomially characterised in our logic.

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**References**

H. Aziz, F. Brandt, and P. Harrenstein. Pareto optimality in coalition formation. *Games and Economic Behavior*, 82:562–581, 2013.

A. Bogomolnaia and M. O. Jackson. The stability of hedonic coalition structures. *Games and Economic Behaviour*, 38:201–230, 2002.

A. Bogomolnaia and H. Moulin. Random matching under dichotomous preferences. *Econometrica*, 72(1):257–279, 2004.
A. Bogomolnaia, H. Moulin, and R. Stong. Collective choice under dichotomous preferences. *Journal of Economic Theory*, 122(2):165–184, 2005.

E. Bonzon, M.-C. Lagasquie-Schiex, J. Lang, and B. Zanuttini. Compact preference representation and Boolean games. *Autonomous Agents and Multi-Agent Systems*, 18(1):1–35, 2009.

E. Bonzon, M.-C. Lagasquie-Schiex, and J. Lang. Effectivity functions and efficient coalitions in Boolean games. *Synthese*, 187(1):73–103, 2012.

S. Bouveret and J. Lang. Efficiency and envy-freeness in fair division of indivisible goods: Logical representation and complexity. *Journal of AI Research*, 32:525–564, 2008.

S. J. Brams and P. C. Fishburn. *Approval Voting*. Springer, 2007.

K. Cechlárová. Stable partition problem. In *Encyclopedia of Algorithms*, pages 885–888. Springer, 2008.

K. Cechlárová and J. Hajduková. Stable partitions with W-preferences. *Discrete Applied Mathematics*, 138(3):333–347, 2004.

G. Chalkiadakis, E. Elkind, and M. Wooldridge. *Computational Aspects of Cooperative Game Theory*. Morgan-Claypool, 2011.

J. H. Drèze and J. Greenberg. Hedonic coalitions: Optimality and stability. *Econometrica*, 48(4):987–1003, 1980.

P. E. Dunne, S. Kraus, W. van der Hoek, and M. Wooldridge. Cooperative Boolean games. In *Proceedings of the Seventh International Joint Conference on Autonomous Agents and Multiagent Systems (AAMAS-2008)*, pages 1015–1022, 2008.

E. Elkind and M. Wooldridge. Hedonic coalition nets. In *Proceedings of the Eighth International Joint Conference on Autonomous Agents and Multiagent Systems (AAMAS-2009)*, pages 417–424, 2009.

J. Hajduková. Coalition formation games: A survey. *International Game Theory Review*, 8(4):613–641, 2006.

P. Harrenstein, W. van der Hoek, J.-J. Meyer, and C. Witteveen. Boolean games. In J. van Benthem, editor, *Proceedings of the 8th Conference on Theoretical Aspects of Rationality and Knowledge (TARK)*, pages 287–298, 2001.

S. Ieong and Y. Shoham. Marginal contribution nets: A compact representation scheme for coalitional games. In *Proceedings of the Sixth ACM Conference on Electronic Commerce (EC’05)*, Vancouver, Canada, 2005.

S. Konieczny and R. Pino-Pérez. Merging information under constraints: a logical framework. *Journal of Logic and Computation*, 12(5):773–808, 2002.

F. Lin and R. Reiter. Forget it! In *Working Notes of AAAI Fall Symposium on Relevance*, pages 154–159, 1994.