LCQPow – A Solver for Linear Complementarity Quadratic Programs

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Abstract In this paper we introduce an open-source software package written in C++ for efficiently finding solutions to quadratic programming problems with linear complementarity constraints. These problems arise in a wide range of applications in engineering and economics, and they are challenging to solve due to their structural violation of standard constraint qualifications, and highly nonconvex, nonsmooth feasible sets. This work extends a previously presented algorithm based on a sequential convex programming approach applied to a standard penalty reformulation. We examine the behavior of local convergence and introduce new algorithmic features. Competitive performance profiles are presented in comparison to state-of-the-art solvers and solution variants in both existing and new benchmarks.

Keywords Optimization · Complementarity Constraints · Sequential Convex Programming · Hybrid Systems

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1 Introduction

This paper presents the release 1.0 of LCQPow, which is a solver designed to efficiently solve Linear Complementarity Quadratic Programs (LCQP). These problems can be expressed in the form

$$\begin{align*}
\text{minimize} & \quad \frac{1}{2} x^\top Q x + g^\top x \\
\text{subject to} & \quad b \leq A x, \\
& \quad 0 \leq L x \perp R x \geq 0,
\end{align*}$$

where $Q$ is assumed to be positive definite (a more detailed form matching the solver’s API is later introduced in (4)). Note that the objective function together with constraint (1b) define a generic convex quadratic problem. The main difficulty of the above class arises through the nonlinear and nonconvex complementarity constraints. Their compact form (1c) denotes the set of constraints

$$0 \leq L x \perp R x \geq 0 \iff \begin{cases} 0 \leq L x, \\ 0 \leq R x, \\ 0 = x^\top L^\top R x. \end{cases}$$

We refer to the matrices $L$ and $R$ as the complementarity selector matrices. The pair of rows $L_i, R_i \in \mathbb{R}^{1 \times n}$ introduces the $i$th complementarity constraint, consisting of nonnegativity $0 \leq L_i x$, $0 \leq R_i x$ and orthogonality $x^\top L_i^\top R_i x = 0$. Note that the nonnegativity constraints (2a), (2b) make the orthogonality constraint (2c) equivalent to complementarity satisfaction for each individual complementarity pair. The values of $L_i$ and $R_i$ define a weighted selection of the optimization variables to be present in the $i$th complementarity constraint. In the simplest case the selector matrices consist of unit vector rows, which imposes that at least one of the selected optimization variables vanishes for each complementarity pair. For example, this is the case in the two dimensional toy problem

$$\begin{align*}
\text{minimize} & \quad (x_1 - 1)^2 + (x_2 - 1)^2 \\
\text{subject to} & \quad 0 \leq x_1 \perp x_2 \geq 0,
\end{align*}$$

which is illustrated in Figure 1.

It is well known that problem (4) violates standard constraint qualifications such as the Linear Independence Constraint Qualification (LICQ), or even the weaker Mangasarian-Fromovitz Constraint Qualification (MFCQ) at every feasible point. Constraint regularity conditions are crucial assumptions for the concept of stationarity, e.g., for verifying Karush-Kuhn-Tucker (KKT) points. This problem implies that standard approaches to solving Nonlinear Programs (NLP) typically fail, creating a need for specialized methods. Therefore, the theory of stationarity has been adapted and statements tailored to Mathematical Programs with Complementarity Constraints (MPCC) have been developed.

For recent advances in the field of MPCC, including extensive lists of applications and methods, we refer to the surveys. Within the subclass of convex quadratic objectives and linear constraints, there only exists a small amount of research. These problems arise in a wide range of applications in engineering and economics,
Fig. 1: This illustration of the toy problem shows the feasible set $\Omega$ (solid) together with the two strongly stationary points located at $(1,0)$ and $(0,1)$, and the spurious solution in the origin. The level lines (dashed) represent the objective function on the left, and the penalty function $x_1 \cdot x_2$ (see (7)) on the right.

e.g., in optimal control problems of dynamical systems following discontinuous, but piecewise linear, dynamics. Generally, systems of such dynamics are known as hybrid systems, and they have equivalently been modeled via mixed-logical dynamics. Due to the combinatorial structure of the problem class it comes natural to investigate branch-and-bound methods in order to find global solutions. Recent advances proposed low-complexity methods for such systems, motivated by the fact that mixed-integer solvers require high computational power and memory availability. Similarly, the intention of the solver presented here is to rapidly generate good local solutions with complementarity satisfaction up to machine precision.

The remainder of this paper is structured as follows. Section 2 provides background, existing methods and solvers for LCQPs. The algorithm is described in detail in Section 3, which builds upon the design originally presented in [6]. This includes an outer penalty loop, an inner Sequential Convex Programming (SCP) loop, an analytical globalization scheme, adaptive penalty updates and a heuristic for escaping saddle points. In Section 4, we state local convergence properties of the inner and outer loops, and provide statements for merit function descent at each inner loop iterate. The performance of the solver is benchmarked in Section 5 against a variety of solvers and methods.

The contribution of this work primarily consists of the open-source software implementation written in C++. The code previously introduced provided the proof of concept for the underlying method, and is now transferred into a reliable, robust and efficient solver with extended flexibility and user options. This is supported by thorough benchmarks, which consist of the MPCC benchmark, the benchmark discussed in [6], and one benchmark created specifically for this paper. Additionally, the theoretical results are extended by a local convergence property, which states instant convergence on identification of a locally optimal active set.
2 Background

In this section we give a brief discussion of the background material, such as stationarity concepts and existing methods. We specifically lay our focus on the methods used for comparison within the numerical benchmarks in Section 5.

Let us begin by introducing a more generic form of the problem definition (1). Most importantly, this form enables users to pass arbitrary bounds on the complementarity variables. Whereas their upper bounds behave like simple linear constraints, their lower bounds have a special meaning: for each satisfied complementarity pair one of the lower bounds must be active. Hence, it is crucial to require their lower bounds to admit finite values. This form matches the solver’s API and reads as

\[
\begin{align*}
    & \text{minimize} \quad \frac{1}{2} x^\top Q x + g^\top x \\
    & \text{subject to} \quad (L x - \ell_L)^\top (R x - \ell_R) = 0, \\
    & \quad \ell_L \leq L x \leq u_L, \\
    & \quad \ell_R \leq R x \leq u_R, \\
    & \quad \ell_A \leq A x \leq u_A, \\
    & \quad \ell_x \leq x \leq u_x,
\end{align*}
\]

where \(0 \prec Q = Q^\top \in \mathbb{R}^{n \times n}\), \(g \in \mathbb{R}^n\), \(L, R \in \mathbb{R}^{n_c \times n}\), \(\ell_L, \ell_R, u_L, u_R \in \mathbb{R}^{n_c}\), \(A \in \mathbb{R}^{n_A \times n}\), \(\ell_A, u_A \in \mathbb{R}^{n_A}\), and \(\ell_x, u_x \in \mathbb{R}^n\). We denote by \(\Omega \subset \mathbb{R}^n\) the feasible set of (1).

Many QP solvers exploit the special structure of the box constraints (4f), however, in view of the theoretical analysis and the higher level algorithm, these constraints can be seen as linear constraints. We thus assume throughout this paper that the box constraints (4f) are passed via the linear constraints (4e).

Further, let \(\tilde{A}, \tilde{A}, u_A\) refer to the stacked combination of all linear constraints and their bounds (4c)-(4f). We refer to the resulting feasible set as the relaxed feasible set \(\tilde{\Omega} \supset \Omega\) of (1). Throughout this paper we assume that this relaxed feasible set satisfies LICQ in every feasible point.

2.1 Stationarity of LCQPs

As mentioned, the considered problem class violates standard constraint qualifications required in order for the KKT conditions to necessarily hold in solutions. We therefore review the adapted stationarity concept for complementarity constrained programs \[1,8,17\]. Let us first define the (in)active sets

\[
\begin{align*}
    & A^i(x) = \{ i \in J_A \mid \ell_A_i = A_i x < u_A_i \}, \\
    & A^o(x) = \{ i \in J_A \mid \ell_A_i = A_i x = u_A_i \}, \\
    & A^c(x) = \{ i \in J_A \mid A_i x = u_A_i \}, \\
    & A^f(x) = \{ i \in J_A \mid A_i x < u_A_i \},
\end{align*}
\]

where \(J_A = \{1, \ldots, n_A\}\). Analogously define the respective sets for the constraints \(\ell_L \leq L x \leq u_L\) and \(\ell_R \leq R x \leq u_R\) by \(L^i, R^i\), etc. Further, let \(W^i(x) = L^i(x) \cap R^i(x)\),
The adapted stationarity concepts are very similar to the standard KKT conditions of \cite{[1]}, with the sole difference that the signs of the dual variables associated with constraints $\mathcal{L}^l(x)$ and $\mathcal{R}^l(x)$ are not required to be nonnegative.

**Definition 1** A feasible point $x \in \Omega$ of LCQP \cite{[1]} is called strongly stationary, if there exist dual variables $y = (y_A, y_L, y_R) \in \mathbb{R}^n_A \times \mathbb{R}^{n_L} \times \mathbb{R}^{n_R}$ satisfying

$$Qx + g - A^T y_A - L^T y_L - R^T y_R = 0,$$

$$y_{A_i} = 0, \quad i \in A^l(x), \quad y_{A_i} \geq 0, \quad i \in A^l(x), \quad y_{A_i} \leq 0, \quad i \in A^u(x),$$

$$y_{L_i} = 0, \quad i \in \mathcal{L}^l(x), \quad y_{L_i} \geq 0, \quad i \in \mathcal{L}^u(x), \quad y_{L_i} \leq 0, \quad i \in \mathcal{L}^u(x),$$

$$y_{R_i} = 0, \quad i \in \mathcal{R}^l(x), \quad y_{R_i} \geq 0, \quad i \in \mathcal{R}^l(x), \quad y_{R_i} \leq 0, \quad i \in \mathcal{R}^u(x).$$

2.2.1 Penalty Reformulation

The first method replaces \eqref{eq:4b} with a penalty in the objective and reads as

$$\min_{x \in \mathbb{R}^n} \quad \frac{1}{2} x^T Qx + g^T x + \rho \cdot (Lx - \ell_L)^T (Rx - \ell_R) \quad (7a)$$

$$\text{subject to} \quad \ell_A \leq Ax \leq u_A, \quad (7b)$$

where $\rho > 0$ is the respective penalty parameter. Note that the penalty term is always nonnegative due to \eqref{eq:4c} and \eqref{eq:4d}. The right plot in Figure 1 depicts the level lines of this penalty function for the toy problem \cite{[3]}. The corresponding LCQP \cite{[1]} is then approximated by solving \cite{[3]} either a single time with a large penalty value, or sequentially with an exponentially increasing penalty value \cite{[21]} Section 4. Ralph and Wright proved that this is an exact penalty reformulation for large enough, but finite, values of $\rho$ \cite{[20]} Section 5:

**Theorem 1** Let \cite{[3]} satisfy LICQ at $x^* \in \mathbb{R}^n$. Then the following statements hold:

(i) If $(x^*, y^*_A, y^*_L, y^*_R)$ is a strongly stationary point of the LCQP \cite{[1]}, then there exist dual variables $(y_A, y_L, y_R)$ such that $x^*$ is a KKT point of \cite{[3]} for any $\rho$ satisfying

$$\rho \geq 1 + \max \left\{ 0, \max_{i \in \mathcal{L}^c(x^*)} \left\{ \frac{-y^*_L}{R_i x^* - \ell_{R_i}} \right\}, \max_{i \in \mathcal{R}^c(x^*)} \left\{ \frac{-y^*_R}{L_i x^* - \ell_{L_i}} \right\} \right\}. \quad (8)$$
The dual variables are given by

\[
\bar{y}_A = y_A^*, \\
\bar{y}_L_i = y_{L_i}^* + \rho(R_i x^* - \ell_L), \quad \text{for } i \in \mathcal{L}(x^*), \\
\bar{y}_R_i = y_{R_i}^* + \rho(L_i x^* - \ell_R), \quad \text{for } i \in \mathcal{R}(x^*),
\]

\[(9a) - (9e)\]

(ii) If \((x^*, y_A, y_L, y_R)\) is a KKT point of \((7)\) and \((Lx^* - \ell_L)\top(Rx^* - \ell_R) = 0\), then \((x^*, y_A^*, y_L^*, y_R^*)\) is a strongly stationary point of the LCQP \((4)\), where the dual variables are obtained via \((9)\).

The algorithm implemented in the presented software package is based on this penalty reformulation, and we thus focus on this specific penalty function, though other choices are conceivable as well \cite{22–24}.

2.2.2 Constraint Regularization Reformulations

The remaining two NLP methods both replace \((4b)\) by constraint regularization strategies, each of which uses a parameter \(\sigma > 0\). These methods approximate \((4)\) by

\[
\begin{align*}
\text{minimize} & \quad \frac{1}{2} x^\top Q x + g^\top x \\
\text{subject to} & \quad (L x - \ell_L)\top(R x - \ell_R) = \sigma, \\
& \quad \ell_A \leq Ax \leq u_A,
\end{align*}
\]

which we call the smoothed reformulation of \((4)\), and by

\[
\begin{align*}
\text{minimize} & \quad \frac{1}{2} x^\top Q x + g^\top x \\
\text{subject to} & \quad (L x - \ell_L)\top(R x - \ell_R) \leq \sigma, \\
& \quad \ell_A \leq Ax \leq u_A,
\end{align*}
\]

which we call the relaxation of \((4)\). In contrast to the penalty parameter \(\rho\), these methods approximate \((4)\) for \(\sigma \to 0\). As for the penalty reformulation, they naturally lend themselves to being used in a sequential scheme with exponentially decaying choices for \(\sigma\).

2.2.3 MIQP Reformulation

Finally, we address a direct reformulation into a mixed-integer quadratic program. This is straightforward under the existence of finite upper bounds on the complementarity variables, i.e., given \(u_L, u_R < \infty\). In that case we may simply introduce
two binary variables \((z^L_i, z^R_i)\) for each complementarity constraint \(i \in \{1, 2, \ldots, n_c\}\). We can then enforce complementarity via the set of constraints

\[
\begin{align*}
\ell_L i & \leq L_i x, & \quad (12a) \\
\ell_R i & \leq R_i x, & \quad (12b) \\
L_i x & \leq u_L i (1 - z^L_i) \ell_L i, & \quad (12c) \\
R_i x & \leq u_R i (1 - z^R_i) \ell_R i, & \quad (12d) \\
1 & \geq z^L_i + z^R_i. & \quad (12e)
\end{align*}
\]

Thus, for \(z^L_i = 1\) we simply regain the upper bound. On the other hand, \(z^L_i = 0\) fixes the complementarity variable to its lower bound. In total the reformulation reads as

\[
\begin{align*}
\text{minimize} \quad & \frac{1}{2} x^\top Q x + g^\top x & \quad (13a) \\
\text{subject to} \quad & \ell_A \leq \tilde{A} x \leq u_A, & \quad (13b) \\
& L_i x \leq u_L i (1 - z^L_i) \ell_L i, & \quad 1 \leq i \leq n_c, \quad (13c) \\
& R_i x \leq u_R i (1 - z^R_i) \ell_R i, & \quad 1 \leq i \leq n_c, \quad (13d) \\
& 1 \geq z^L_i + z^R_i. & \quad (13e)
\end{align*}
\]

where \(B = \{0, 1\}\). If upper bounds on the complementarity variables are not defined, then we rely on a big-\(M\) reformulation, by setting the upper bounds to a large \(M \gg 0\).

## 3 Algorithm

In this section we describe in detail the initial algorithmic development \[6\] and provide several extensions. We begin by considering the penalty reformulation \[7\] together with an exponential penalty update rule similar to the one described by Ferris et al. in \[21\]. This technique leads to a sequence of nonconvex quadratic programming problems (which we denote by the outer loop below). Solutions to each of these problems are then found via a SCP method (which provides the inner loop). The key steps of the algorithm are captured in the pseudocode Algorithm 1 at the end of this section. In this section we introduce many parameters, the matching API names of which we mention as \((\text{parameterName})\) whenever newly introduced. Table 1 summarizes their default values and feasible range. Finally, we remind the reader that, in contrast to \[6\], the complementarity constraints are generalized by allowing generic lower and upper bounds, which creates some subtle differences.

### 3.1 Penalty Homotopy

Motivated by Theorem \[1\] we desire a solution of the penalty reformulation \[7\] for a penalty large enough to satisfy the complementarity constraints. However, the required penalty value is a priori unknown. One could consider simply solving \[4\] for
a very large penalty in the hope of instantly satisfying complementarity. However, these penalty reformulations often become ill-conditioned for large penalty parameters, i.e., the largest absolute eigenvalue of $Q + \rho C$ significantly dominates the smallest absolute eigenvalue. Furthermore, a homotopy often avoids convergence to strongly suboptimal solutions, as for example shown for Optimal Control Problems (OCP) with discontinuous dynamics [25]. On the other hand, solving the penalized subproblem for a very small penalty parameter leads to a solution close to the global minimum $\hat{x}^*$ of the objective function (4a) with respect to the relaxed feasible set $\tilde{\Omega}$. By gradually increasing the penalty parameter we hope to find a solution path from the relaxed minimizer $\hat{x}^*$ to a strongly stationary point $x^* \in \Omega$ approximated from within the relaxed feasible set $\tilde{\Omega}$. Yet, this is only a heuristic and there is no guarantee of finding the global minimizer, or even any minimizer, as the original NLP (4) is nonconvex.

Let us now describe the homotopy. For a given penalty parameter $\rho_k > 0$ the respective penalty reformulation (7) is solved as described in the next section. Subsequently, the penalty parameter is updated as $\rho_{k+1} = \beta \rho_k$ with a fixed factor $\beta > 1$. This method also requires the choice of an initial penalty parameter $\rho_0 > 0$ (initialPenaltyParameter), which is typically chosen rather small. The factor $\beta$ (penaltyUpdateFactor) represents the base of the exponential growth, and one could alternatively write $\rho_k = \beta^k \rho_0$. This procedure is repeated until complementarity is satisfied, or the penalty parameter exceeds its limit (maxPenaltyParameter), in which case the convergence is assumed to have failed.

Before proceeding with the inner loop, let us refine the penalty formulation (7). We introduce the penalty function

$$
\varphi(x) = (Lx - \ell_L)^\top (Rx - \ell_R) = \frac{1}{2} x^\top C x + g_{\varphi}^\top x + \ell_{\varphi}^\top \ell_R,
$$

where $\frac{1}{2} C = \frac{1}{2} (L^\top R + R^\top L)$ is the symmetrization of the product $L^\top R$, and $g_{\varphi} = -(R^\top \ell_L + L^\top \ell_R)$ is the linear component of $\varphi$. We remark here that $C$ is typically indefinite. If it does not contain negative eigenvalues, then the penalty reformulation is convex and its unique solution satisfies complementarity. We then combine the linear components of the objective function (7a) by defining $g_k = g + \rho_k g_{\varphi}$. Finally, we obtain the following optimization problem, that is equivalent to (7)

$$
\begin{align}
\text{minimize} & \quad \frac{1}{2} x^\top Q x + g_k^\top x + \rho_k \frac{1}{2} x^\top C x \\
\text{subject to} & \quad \ell_{\tilde{A}} \leq \tilde{A} x \leq u_{\tilde{A}},
\end{align}
$$

and call the sequence of solving these problems for increasing $\rho_k$ the outer loop.

### 3.2 Sequential Convex Programming

Each outer loop problem is solved using SCP [26], resulting in an inner loop. Let $k$ and $j$ denote the outer and inner loop indices, respectively. We denote by $x_{kj}$ the most recent inner SCP iterate. The sequence is initialized with an initial guess $x_{00}$, or alternatively the global minimizer of the relaxed problem (see Section 3.3).
The penalty function, which is the only nonconvex component of (15), is approximated at \( x_{kj} \) using its first-order Taylor expansion
\[
\varphi(x) \approx \varphi(x_{kj}) + (x - x_{kj})^\top \nabla \varphi(x_{kj}) = (\varphi(x_{kj}) - x_{kj}^\top (Cx_{kj} + g_\varphi)) + x^\top (Cx_{kj} + g_\varphi).
\]

Note that \( x^\top (Cx_{kj} + g_\varphi) \) is the only term dependent on \( x \). Since the constant terms do not affect the optimizer, we omit them from here on. Replacing the penalty function by this term yields the convex inner loop subproblem
\[
\begin{align*}
\text{minimize} & \quad \frac{1}{2} x^\top Q x + (g_k + \rho_k Cx_{kj})^\top x, \\
\text{subject to} & \quad \ell_{\tilde{A}} \leq \tilde{A} x \leq u_{\tilde{A}}.
\end{align*}
\]

We denote the unique minimizer of the inner loop subproblem by \( x_{kj}^* \) and the corresponding step direction by \( p_{kj} = x_{kj}^* - x_{kj} \). Given this inner solution an optimal step length \( \alpha_{kj} \) is obtained from a globalization scheme described in Section 3.3. Finally, the step update \( x_{k,j+1} = x_{kj} + \alpha_{kj} p_{kj} \) is performed. The inner loop is terminated once a KKT point of the respective outer loop problem (15) is found.

There are two reasons why it can be attractive to replace the full penalty function by its linear approximation. First, convex subproblems are obtained at the cost of the additional inner loop. Noting that the eigenvalues of \( C \) will dominate over those of \( Q \) for large penalty parameters, we find that the penalty formulation (15) becomes more and more indefinite as the penalty parameter grows. On the other hand, convexity of the inner loop subproblem (16a) is always ensured, as the Hessian matrix is given by \( Q \) for every subproblem. This also induces the second advantage: the Hessian and constraint matrices remain constant over both the inner and outer loop iterates. Consequently, the KKT matrix factorization can be reused, and each subproblem can be solved efficiently, e.g., by making use of the warm-starting techniques employed in QP solvers such as qpOASES [27] or OSQP [28]. With the computation of factorizations being a significant expense, this advantage can outweigh the cost of inner loop iterations, as demonstrated in Section 5. We will also see that the SCP loop terminates finitely near an exact solution if \( \rho_k \) is large enough (see Theorem 3).

### 3.3 Optimal Step Length Globalization

Consider the merit function
\[
\psi_k(x) = \frac{1}{2} x^\top (Q + \rho_k C)x + g_k^\top x,
\]
which coincides with the outer loop objective function (15a). On the other hand, the inner loop objective function (16a) provides the strictly convex quadratic model
\[
\vartheta_{kj}(x) = \frac{1}{2} x^\top Q x + (g_k + \rho_k Cx_{kj})^\top x.
\]

As discussed in [6], the step length formula is obtained by minimizing the merit function along the step \( p_{kj} \), i.e., by solving
\[
\begin{align*}
\text{minimize} & \quad \psi_k(x_{kj} + \alpha_{kj} p_{kj}),
\end{align*}
\]

Note that \( x^\top (Cx_{kj} + g_\varphi) \) is the only term dependent on \( x \). Since the constant terms do not affect the optimizer, we omit them from here on. Replacing the penalty function by this term yields the convex inner loop subproblem
\[
\begin{align*}
\text{minimize} & \quad \frac{1}{2} x^\top Q x + (g_k + \rho_k Cx_{kj})^\top x, \\
\text{subject to} & \quad \ell_{\tilde{A}} \leq \tilde{A} x \leq u_{\tilde{A}}.
\end{align*}
\]

We denote the unique minimizer of the inner loop subproblem by \( x_{kj}^* \) and the corresponding step direction by \( p_{kj} = x_{kj}^* - x_{kj} \). Given this inner solution an optimal step length \( \alpha_{kj} \) is obtained from a globalization scheme described in Section 3.3. Finally, the step update \( x_{k,j+1} = x_{kj} + \alpha_{kj} p_{kj} \) is performed. The inner loop is terminated once a KKT point of the respective outer loop problem (15) is found.

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\]

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\[
\begin{align*}
\text{minimize} & \quad \psi_k(x_{kj} + \alpha_{kj} p_{kj}).
\end{align*}
\]
This yields a scalar QP and its analytical solution is given by

\[
\alpha^*_{kj} = \begin{cases} 
\n -\nabla \psi_k(x_{kj})^\top p_{kj}, & \text{if } p_{kj}^\top C_{kj} > 0, \\
1, & \text{else.}
\end{cases}
\]

(20)

**Remark 1** The step length \(\alpha^*_{kj}\) is strictly positive if \(x_{kj}\) is not already a KKT point of the outer loop problem. We will discuss this in more detail in Section 4 by proving strict merit function descent in direction \(p_{kj}\), i.e., \(\nabla \psi_k(x_{kj})^\top p_{kj} < 0\).

**Remark 2** The formula presented in [6] was derived for a slightly less generic form, as generic bounds on the complementarity pairs were not permitted. However, this only changes the linear component \(g_k\), which remains constant for each inner loop.

### 3.4 Dynamic Penalty Updates

It is possible that the inner loop requires many iterates until a satisfactory level for convergence is reached, while the progress of merit function descent might stagnate. In the context of interior point methods for MPCCs, Leyffer et al. have shown that it can be advantageous to terminate the inner loop prematurely and update the penalty parameter dynamically [29, Section 5]. This dynamic update is triggered whenever an inner loop iterate satisfies

\[
\varphi(x_{kj}) > \varepsilon \varphi \quad \text{and} \quad \varphi(x_{kj}) > \eta \max\{\varphi(x_{k,j-1}), \ldots, \varphi(x_{k,j-n})\},
\]

(21)

where \(\varepsilon \varphi > 0\) describes the numerical tolerance for the complementarity violation \((\text{complementarityTolerance})\). This method assures that, until complementarity is satisfied, each iterate reduces one of the previous \(n\) complementarity violations by at least a factor of \(\eta\). We embedded this strategy into our solver with the options \(\text{nDynamicPenalty}\) and \(\text{etaDynamicPenalty}\). This strategy can be switched off by setting \(\text{nDynamicPenalty} = 0\).

### 3.5 Initialization Strategy

The initial guess is often a crucial factor for finding good local solutions of nonlinear programs. Thus it is desirable to initialize solvers in the basis of attraction of a good — ideally global — solution. The presented solver contains the option \(\text{solveZeroPenaltyFirst}\), a flag indicating whether the sequence should be initialized by solving (15) with \(\rho = 0\) (recall that this problem is convex, and its solution is the global minimizer of the objective function over the relaxed set \(\tilde{\Omega}\)). This canonical choice makes passing an initial guess optional.

However, this method becomes disadvantageous if proximity to a good solution is known. In this case the flag should be disabled and the solver should be initialized with a large penalty parameter in order to prevent the solver from leaving the area of attraction of the local solution. This is especially the case if the active set of the global solution has been identified (see Theorem 3).
Fig. 2: The penalty reformulation of the LCQP has a stable trajectory of saddle points (minimizers for $\rho < 2$) along the points $(\frac{2}{2 + \rho}, \frac{2}{2 + \rho})$ marked with the black crosses. The red dot indicates the saddle point of the penalized objective of which the level lines are indicated.

3.6 Gradient Perturbation

Some problems might have a stable trajectory of minimizers or saddle points towards undesirable solutions. We demonstrate this issue using the toy problem. In this case the standard strategy would initialize at $(1, 1)$ and follow the saddle point trajectory into the origin and terminate at this locally maximal solution (see Figure 2). With a small zero mean random perturbation of the gradient $g_k$ at each step, we move the iterates away from the saddle points until the error is large enough for the QP solver to detect descent towards one of the strongly stationary points $(1, 0)$ or $(0, 1)$. Alternatively, one could consider applying the perturbation to the step directly. However, this would require additional safety-checks in order to dodge infeasibility. Perturbing the gradient is safe in this aspect as it only alters the objective function.

3.7 Termination Criterion

We terminate the algorithm under three different scenarios: either a solution is found, or the penalty parameter is too large, or the maximum number of iterations are exceeded. The termination criterion for a solution consists of KKT point verification of an iterate $x_{kj}$ for the penalty formulation together with sufficient complementarity satisfaction. The tolerances of both conditions can be adapted via the options stationarityTolerance and complementarityTolerance, respectively. Note that any feasible iterate $x_{kj}$ of satisfies all constraints of except for the orthog-
Thus it is sufficient to check
\[
\|(Q + \rho_k C)x_{kj} + g_k - A^\top y_A - L^\top y_L - R^\top y_R\|_\infty \leq \text{stationarityTolerance}, \\
\varphi(x_{kj}) \leq \text{complementarityTolerance}.
\]

The remaining feasibility conditions are assumed to be transferred through the QP solver. Since the underlying QP solvers use inherently different termination criteria, it is difficult to provide bounds on how to choose the termination tolerances precisely. If the termination conditions for LCQPow are chosen too small, then precision errors from the utilized QP solver may interfere with convergence. In this case, the precision for LCQPow should be decreased, i.e., the tolerance increased (or vice versa the QP solver precision should be increased). We remark here that, on successful convergence, the dual variables obtained from the penalty reformulation are translated into dual variables of the original LCQP (4a) using (9).

3.8 QP Solvers

Through the user option `qpSolver` it is possible to switch between the three modes 0, 1, and 2. The mode 0 refers to qpOASES [30] in dense mode, 1 refers to qpOASES in sparse mode, and 2 refers to OSQP [31] (in sparse mode).

The performance of mode 1 will depend on how qpOASES is compiled. If Matlab is installed on the machine one can pass the CMake option `-DQPOASES_SCHUR=ON` to compile qpOASES with the Schur Complement method [32, Chapter 8], which uses the sparse linear solver MA57 [33].

3.9 Print Level

The solver prints some information about the iterates to the command line, the amount of which can be controlled via the user option `printLevel`. If no iterate output is desired then 0 can be passed. Mode 1 will print only one iterate of each inner loop. Mode 2 will print every iterate.

3.10 Software

The open-source software package written in C++ is available through the GitHub repository

https://github.com/hallfjonas/LCQPow

Version v0.1.0 was used for this paper. This repository contains three submodules, which have to be initialized after cloning the repository. Those modules are the QP solvers qpOASES [30] and OSQP [31], and the unit test framework GoogleTest [34]. The solver can be called either directly through C++ or through its Matlab interface. The user options with default values and feasible range are listed in Table 1.
Algorithm 1: Pseudocode of the solver’s main loop.

Input: $\rho > 0$, $\beta > 1$, $\epsilon_{\text{tol}} > 0$

Output: Stationary point $(x_k, y_k)$ of LCQP (4)

1. # Create QP solver and factorize KKT matrix
   \[ \text{qp}(Q, g, A, \ell_x, u_x, \ell_A, u_A); \]
2. # Initialize solver with zero penalty QP
   \[ (x_k, y_k) = \text{qp.solve}(); \]
3. # Outer loop (penalty update loop)
   while true do
   4.   # Update outer loop linear term
   5.   \[ g_k = g - \rho(R^T \ell_L + L^T \ell_R); \]
   6.   # Inner loop (approximate penalty function)
   7.   while stationarity$(x_k, y_k, \rho) > \epsilon_{\text{tol}}$ do
   8.     # Update objective’s linear component
   9.     \[ \text{qp.update}_g(g_k + \rho C x_k); \]
   10.    # Step computation and dual variable update (solve (16))
   11.       \[ (x_n, y_k) = \text{qp.solve}(); \]
   12.       # Get optimal step length according to (20)
   13.       \[ \alpha = \text{get\_step\_length}(x_k, x_n, \rho); \]
   14.       # Update primal variables
   15.       \[ x_k = x_k + \alpha (x_n - x_k); \]
   16.       # Perform dynamic penalty update
   17.       if Condition (21) holds then
   18.         break;
   19.   end
   20.   if \[ \varphi(x_k) < \epsilon_{\text{tol}} \] then
   21.     return $(x_k, y_k); \]
   22.   end
   23.   # Increase penalty parameter
   24.   \[ \rho = \beta \cdot \rho; \]

Table 1: User options with their default values and feasible range

| Parameter Name              | Default Value   | Feasible Values | Section |
|-----------------------------|-----------------|-----------------|---------|
| stationarityTolerance       | $1e+6 \cdot \epsilon_{\text{mach}}$ | $\mathbb{R}_{>0}$ | 3.7     |
| complementarityTolerance    | $1e+3 \cdot \epsilon_{\text{mach}}$ | $\mathbb{R}_{>0}$ | 3.7     |
| initialPenaltyParameter     | $1e-2$          | $\mathbb{R}_{>0}$ | 3.1     |
| penaltyUpdateFactor         | 2               | $\mathbb{R}_{>1}$ | 3.1     |
| solveZeroPenaltyFirst       | 1               | $\{0, 1\}$     | 3.5     |
| maxIterations               | $1e+3$          | $\mathbb{N}$    | 3.7     |
| maxPenaltyParameter         | $1e+4$          | $\mathbb{R}_{>0}$ | 3.7     |
| printLevel                  | 2               | $\{0, 1, 2\}$  | 3.9     |
| qpSolver                    | 0               | $\{0, 1, 2\}$  | 3.8     |
| nDynamicPenalty             | 3               | $\mathbb{N}$    | 3.4     |
| etaDynamicPenalty           | 0.9             | $(0, 1)$        | 3.3     |
4 Convergence Analysis

We now draw our attention to the local convergence behavior of the above introduced algorithm. We begin by revisiting in more detail the properties introduced in [6]: a relationship between the minimizers of the inner loop problem with the KKT points of the outer loop problem (Lemma 1); and strict merit function descent in each inner loop iterate until convergence is reached (Theorem 2). This section is concluded with the local convergence statement Theorem 3, which shows that the algorithm converges in one step once the active sets of the complementarity pairs of the current iterate coincide with those of a strongly stationary point.

Lemma 1 Let \((x_{kj}, \rho_k)\) be a feasible iterate of (16). Then the respective inner loop minimizer \(x_{kj}^*\) is equal to \(x_{kj}\) iff \(x_{kj}\) is a KKT point of the outer loop problem (15).

Proof For a proof of this standard result we refer to [26, Lemma 4.1].

Theorem 2 Given \(x_{kj} \in \tilde{\Omega}\) with inner loop step \(p_{kj} = x_{kj}^* - x_{kj}\), the merit function at \(x_{kj}\) is nonincreasing in direction \(p_{kj}\), i.e.,

\[
\nabla \psi_k(x_{kj})^\top (x_{kj}^* - x_{kj}) \leq 0.
\]

Furthermore, if \(x_{kj}\) is not a stationary point of (15) (with respect to \(\rho_k\)), then

\[
\nabla \psi_k(x_{kj})^\top (x_{kj}^* - x_{kj}) < 0.
\]

Proof Since \(x_{kj}^*\) is the global minimum of the inner loop optimization problem, the following relation holds

\[
\vartheta_{kj}(x_{kj}^*) \leq \vartheta_{kj}(x),
\]

where \(x\) is any feasible point of (15) and \(\vartheta_{kj}\) is as defined in (18). Since \(\vartheta_{kj}\) is convex and differentiable it holds for any \(a, b \in \mathbb{R}^n\) that

\[
\nabla \vartheta_{kj}(a)^\top (b - a) \leq \vartheta_{kj}(b) - \vartheta_{kj}(a).
\]

This property provides descent for the quadratic model

\[
\nabla \vartheta_{kj}(x_{kj})^\top p_{kj} \leq \vartheta_{kj}(x_{kj}^*) - \vartheta_{kj}(x_{kj}) \leq 0.
\]

Note that this inequality becomes strict if \(x_{kj} \neq x_{kj}^*\), since (24) becomes strict. Further, we have

\[
\nabla \psi(x_{kj}, \rho_k)^\top p_{kj} = (Qx_{kj} + \rho_k Cx_{kj} + g)^\top p_{kj}
\]

where \(Q, C, g\) are the matrices and vectors defined in (16). Thus, the directional derivatives of the merit function and quadratic model at \(x_{kj}\) towards \(p_{kj}\) agree. Inequality (22) immediately follows.

Assume that \(x_{kj}\) is not outer loop stationary. Then Lemma 1 yields \(x_{kj}^* \neq x_{kj}\). As remarked before, the inequality (26) becomes strict, and plugging in (27) concludes the claim.

Theorem 3 Let \((x^*, y^*)\) be a strongly stationary point of (4) and let \(x_{kj} \in \Omega\) be an iterate of the algorithm such that \(\mathcal{L}(x_{kj}) = \mathcal{L}(x^*)\) and \(\mathcal{R}(x_{kj}) = \mathcal{R}(x^*)\), i.e., the respective active complementarity sets agree. Further, let \(\rho_k\) be sufficiently large. Then the next inner loop iterate will coincide with the strongly stationary point, i.e., \(x_{k,j+1} = x^*\), and the algorithm will return the strongly stationary point \((x^*, y^*)\).
Proof Similar to Theorem 1 we define
\[
\bar{y}_A = y_A, \quad (28a)
\]
\[
\bar{y}_L_i = y^*_L_i, \quad \text{for } i \notin \mathcal{L}(x_{kj}), \quad (28b)
\]
\[
\bar{y}_{R_i} = y^*_R_i, \quad \text{for } i \notin \mathcal{R}(x_{kj}), \quad (28c)
\]
\[
\bar{y}_{L_i} = y^*_L_i + p_k(R_i x_{kj} - \ell_{R_i}) \geq 0, \quad \text{for } i \in \mathcal{L}(x_{kj}), \quad (28d)
\]
\[
\bar{y}_{R_i} = y^*_R_i + p_k(L_i x_{kj} - \ell_{L_i}) \geq 0, \quad \text{for } i \in \mathcal{R}(x_{kj}), \quad (28e)
\]
where \(p_k\) is assumed to be large enough to satisfy the inequalities in (28d) and (28e).

The existence of such a \(p_k\) can be seen as follows. Applying Theorem 1 shows that there certainly exists such a penalty parameter if \(x_{kj}\) is replaced by \(x^*\), and let us denote an adequate choice by \(\tilde{p}_k\). Thus \(\bar{y}(x_{kj})\) respects the sign conditions required for a KKT point of (16). Strong stationarity of (16) transfers immediately from Theorem 1.

We now want to show that \(x_{kj}^* = x^*\). Note that the dual variables \((\bar{y}_A, \bar{y}_L, \bar{y}_R)\) respect the sign conditions required for a KKT point of (16). Strong stationarity of \((x^*, y^*)\) yields
\[
0 = Q x^* + g - A^T \bar{y}_A - L^T \bar{y}_L - R^T \bar{y}_R
\]
\[
= Q x^* + g - A^T \bar{y}_A - L^T (y_L - p_k(R_i x_{kj} - \ell_{R_i})) - R^T (y_R - p_k(L_i x_{kj} - \ell_{L_i}))
\]
\[
= Q x^* + g + p_k((L^T R + R^T L)x_{kj} + g_\varphi) - A^T \bar{y}_A - L^T \bar{y}_L - R^T \bar{y}_R
\]
\[
= Q x^* + g_k + p_k C x_{kj} - A^T \bar{y}_A - L^T \bar{y}_L - R^T \bar{y}_R.
\]
where \(g_\varphi = - (L^T \ell_R + R^T \ell_L)\) and \(g_k = g + p_k g_\varphi\). This shows that \((x^*, \bar{y}_A, \bar{y}_L, \bar{y}_R)\) is the unique KKT point of the strictly convex inner loop problem (16). Thus \(x^* = x_{kj}^*\).

It remains to be shown that the globalization scheme will not interfere. More precisely, we must show \(\alpha_{kj}^* = 1\). Note that \(\varphi(x_{kj} + \alpha_{kj} p_{kj}) = 0\) for all \(0 \leq \alpha_{kj} \leq 1\), as we never leave the complementarity satisfying active sets \(\mathcal{L}(x_{kj}) = \mathcal{L}(x^*)\) and \(\mathcal{R}(x_{kj}) = \mathcal{R}(x^*)\) (recall that the respective constraints are linear). Consequently, \(\varphi(x_{kj})\) can not be curved along \(p_{kj}\), i.e.,
\[
p_{kj}^T C p_{kj} = 0. \quad (31)
\]
This leads to a full step \(\alpha_{kj}^* = 1\) according to the step length formula (20). Finally
\[
x_{kj+1} = x_{kj} + \alpha_{kj}^*(x_{kj}^* - x_{kj}) = x_{kj}^* = x^*.
\]
Note that (30d) provides stationarity of the outer loop. The termination conditions of the outer loop are thus satisfied as well. \(\square\)
Table 2: Number of variables, constraints and complementarity constraints listed for each problem test set. In this table we denote the mean value of a vector $x \in \mathbb{R}^n$ by $\bar{x}$.

| Problem Set       | range ($n$) | $\bar{n}$ | range ($n_A$) | $\bar{n}_A$ | range ($n_c$) | $\bar{n}_c$ |
|-------------------|-------------|-----------|---------------|-------------|---------------|-------------|
| MacMPEC           | (2, 2000)   | 180       | (0, 1500)     | 97          | (1, 1000)     | 81          |
| IVOCP             | (151, 301)  | 226       | (50, 100)     | 75          | (100, 200)    | 150         |
| Moving Masses     | (554, 1104) | 829       | (034, 604)    | 454         | (200, 400)    | 300         |

5 Numerical Experiments

We consider three different benchmarks: the LCQP subset of the MacMPEC benchmark [16]: an initial value OCP of a discontinuous dynamic with a single switch; and an OCP with the goal of bringing a system of moving masses to a steady state, in which the complementarities arise from a Coulomb friction model. Table 2 gives an overview of the variable, constraint and complementarity dimensions of each problem set. The benchmarks are available at

https://github.com/hallfjonas/LCQPTest

and we encourage readers to reproduce the outcomes.

For each benchmark we compare the performance profile as introduced in [35]. The used performance metric is the fraction of problems solved within a time factor $\tau$ compared to the fastest solver of each problem. Let $P$ denote the set of problems of a given benchmark. Then, for a solver $s$, this fraction is denoted by

$$P_s = \left\{ p \in P : r_{p,s} \leq \tau \right\} / |P|,$$

where the ratio $r$ is defined by

$$r_{p,s} = \frac{\text{CPU time of } s \text{ to solve } p}{\text{fastest solver CPU time to solve } p}.$$

On failed convergence, we set this value to $\infty$.

We consider various methods: LCQPow with qpOASES [30]; LCQPow with qpOASES utilizing the sparse linear solver MA57 [32]; LCQPow with OSQP [31]; Gurobi [36] for solving the MIQP reformulation; IPOPT [37] for solving the respective problems (7), (10), (11); and finally IPOPT NLP, which solves (11) for a sufficiently small $\sigma > 0$ without the homotopy procedure. The solver IPOPT is called through its CasADi interface [38]. We carefully chose to use internal solver timings in order to remove as much overhead as possible, though some timings may still include some amount of overhead. We remark that the timings including the overhead yield very similar results; mostly the MIQP method varies as its models are rebuilt by Gurobi.

5.1 The LCQP subset of MacMPEC

The MacMPEC problem set [16] contains a variety of optimization problems with complementarity constraints, out of which we selected the ones fitting the LCQP
Fig. 3: Performance plot comparing various solution variants for the LCQP subset of MacMPEC \[16\].

Figure 3 shows that LCQPow with qpOASES mostly outperforms all other methods. Tables 3 and 4 show the objective values obtained by the various methods for each problem. On top of efficient solution computation, this supports that our method is able to find the global solution for many of the posed problems. However, for some problems the solver gets stuck in local solutions. In this benchmark, LCQPow with OSQP also achieves fast results for many problems, however, it is less robust and convergence fails for about a quarter of the problems.

5.2 Initial Value Problem

We now consider an initial value finding problem which was introduced by Stewart and Anitescu \[12, \text{Section 2}\]. This numerical test example contains a dynamical system with a discontinuous right hand side, in which a single switch occurs. The position of the switch is solely dependent on the initial value. The optimization
problem in continuous time is given by
\[
\begin{align*}
\text{minimize} & \quad \int_0^2 x(t)^2 \, dt + (x(2) - 5/3)^2 \\
\text{subject to} & \quad x(0) = x_0, \\
& \quad \dot{x}(t) \in 2 - \text{sgn}(x(t)), \quad t \in [0, 2].
\end{align*}
\] (33a)

The discontinuous dynamics (33c) describe a Filippov Differential Inclusion (FDI) and can be reformulated into a linear complementarity system [25]. We then discretize the system using the implicit Euler scheme with \( N \) nodes. The resulting LCQP reads as
\[
\begin{align*}
\text{minimize} & \quad \sum_{k=0}^{N-1} hx_k^2 + \left( x_N - \frac{5}{3} \right)^2 \\
\text{subject to} & \quad x_k - x_{k-1} - h(3(1 - y_k) + y_k) = 0, \quad \text{for } k = 1, \ldots, N, \\
& \quad 0 \leq x_k + \lambda_k^+ \perp 1 - y_k \geq 0, \quad \text{for } k = 1, \ldots, N, \\
& \quad 0 \leq \lambda_k^- \perp y_k \geq 0, \quad \text{for } k = 1, \ldots, N,
\end{align*}
\] (34a)

where \( h = T/N \) is the discretization step size. The benchmark is created by varying \( N \in \{50, 55, \ldots, 100\} \) and the initial guess \( x_0 \in \mathcal{X}_0 \) for the initial value, where \( \mathcal{X}_0 \) contains 10 equidistant values between \(-1.9\) and \(-0.9\).

Figure 4a presents the performance profile comparing various methods, showing that LCQPow outperforms the other methods; particularly the OSQP variant achieves fast results. This significant speed-up does not suffer from a trade-off in terms of solution quality, as it is able to find the same solutions as the MIQP reformulation as shown in Figure 4b. Most homotopy approaches solved via IPOPT find the same solutions in this benchmark. The NLP reformulation achieves convergence with similar speed compared to the qpOASES variant of LCQPow, however, its solution quality highly depends on the initialization.

5.3 Moving Masses

We present an OCP formulation of the test problem described by Stewart in [11, Section 5]. Consider a number of \( s \) springs connecting \( s \) masses: the first mass \( M_1 \) is connected to a wall and each other mass \( M_i \) is connected to its preceding mass \( M_{i-1} \).

We assume that the rest length of the spring has no influence in the dynamics and that the positions of the masses are given in their own coordinate frame. The position \( p_i \) of mass \( M_i \) is thus 0 if the spring attached left to the mass \( M_i \) is relaxed, and we assume that the masses never collide. We introduce a control \( u \in \mathbb{R} \), which represents a force applied to the last mass \( M_s \). The states are described by \( x = (p, v) \in \mathbb{R}^{2s} \), where \( p \) and \( v \) capture the positions and velocities of the respective masses. The full setup is depicted in Figure [5].

Each mass slides over the ground and introduces a frictional force. The direction of this force changes with a sign change of the respective velocity, and thus leads to
Fig. 4: Performance profile and objective function comparison for the Initial Value OCP. The vertical lines on the right indicate a change in the number of discretization intervals from $N = 50$ (experiments 1 through 10) up to $N = 100$ (experiments 101 through 110). For a fixed $N$, the initial guess for $x_0$ is varied from $-1.9$ (left most experiment) to $-0.9$ (right most experiment).
discontinuous dynamics,

\begin{align}
\dot{v}(0) &= \bar{v}_0, \\
p(0) &= \bar{p}_0, \\
\ddot{p} &= v,
\end{align}

\begin{equation}
\dot{v}_i \in \mathcal{F}_i(x) = \begin{cases} 
(p_{i-1} - p_i) + (p_{i+1} - p_i) - v_i - 0.3 \cdot \text{sgn}(v_i), & i = 1, \\
(p_s - p_{s-1}) - v_s - 0.3 \cdot \text{sgn}(v_s) + u, & i = s,
\end{cases}
\end{equation}

where the initial value \( \bar{x}_0 = (\bar{v}_0^T, \bar{p}_0^T)^T \in \mathbb{R}^{2s} \) is fixed. Again, we reformulate the FDI into the dynamic complementarity system

\begin{align}
x(0) &= \bar{x}_0, \\
\ddot{p} &= v, \\
\dot{v}_i &= \begin{cases} 
(-p_i) + (p_2 - p_1) - v_1 - 0.3 \cdot \text{sgn}(v_1), & i = 1, \\
(p_{i-1} - p_i) + (p_{i+1} - p_i) - v_i - 0.3 \cdot (2y_i(t) - 1), & 1 < i < s, \\
(p_{i-1} - p_i) + u - v_i - 0.3 \cdot (2y_i(t) - 1), & i = s,
\end{cases}
\end{align}

\begin{align}
0 &\leq \lambda^- + v \perp 1 - y \geq 0, \\
0 &\leq \lambda^- \perp y \geq 0.
\end{align}

Let \( f(x, y, \lambda, u) \) denote the right hand side of \( 36b \) - \( 36c \) such that \( \dot{x} = f(x, y, \lambda, u) \). We formulate the goal of forcing the system into the equilibrium point of the resting
position, i.e., to obtain $x(T) = 0$, while penalizing the control input and equilibrium deviation at each stage. The OCP in continuous time reads as

$$\begin{align*}
\text{minimize} & \quad \int_0^T x(t)^T x(t) + u(t)^2 \, dt \\
\text{subject to} & \quad x(0) = \hat{x}_0, \\
& \quad \dot{x}(t) = f(x(t), y(t), \lambda(t), u(t)), \\
& \quad 0 \leq v(t) + \lambda^-(t) \perp 1 - y(t) \geq 0, \\
& \quad 0 \leq \lambda^-(t) \perp y(t) \geq 0, \\
& \quad 0 = x(T).
\end{align*}$$

This OCP is again discretized using implicit Euler with $50 \leq N \leq 100$ nodes over a varying time range of $2 \leq T \leq 4$. A trajectory for a sample solution is shown in Figure 6. The performance profile for $s = 2$ masses is given in Figure 7a. LCQPow with qpOASES exploiting sparsity with MA57 clears the benchmark fastest, though the relaxed and NLP variants solved via IPOPT achieve similar results. The OSQP variant of LCQPow is less robust, as it only solves about 75%. However, the problems for which it succeeds are solved significantly faster than the other method. Due to the increased number of complementarity constraints, the MIQP variant is outperformed. Figure 7b shows that, up to a few exceptions, all methods find solutions of the same quality, which are most likely the same solutions.

6 Conclusion

In conclusion, the introduced solver LCQPow was demonstrated to provide competitive solutions for quadratic programming problems with linear complementarity constraints. It offers user flexibility, including the choice of inherently different QP solvers on the lower level, that allow choosing between the tradeoff of robustness and high performance.
Fig. 7: Performance profile and objective function comparison for the Moving Masses OCP. The vertical lines on the right indicate a change in the number of discretization intervals from $N = 50$ (experiments 1 through 10) up to $N = 100$ (experiments 101 through 110). For a fixed $N$, the experiment time $T$ is varied between 2 (left) and 4 (right).
Table 3: Objective values for MacMPEC solutions obtained via LCQPow and Gurobi.

| problem     | best known | LCQPow | qpOASES | LCQPow OSQP | Gurobi |
|-------------|------------|--------|---------|-------------|--------|
| bard1       | 17         | 25     | -       | -           | 17     |
| bard1m      | 17         | 25     | -       | -           | 17     |
| bard2       | -6598      | -6598  | -6598   | -6598       | -6598  |
| bilevel2    | -6600      | -6600  | -6600   | -6600       | -6600  |
| bilevel2m   | -6600      | -6600  | -6600   | -6600       | -6600  |
| ex9.2.1     | 17         | 25     | -       | -           | 17     |
| ex9.2.2     | 100        | 1.00e+02 | 1.00e+02 | 100        |
| ex9.2.2.4   | 5.00e-01   | 5.00e-01 | 5.00e-01 | 5.00e-01   |
| ex9.2.5     | 5          | 9      | 9       | 5           |
| ex9.2.6     | -1         | -1     | -1      | -1          |
| ex9.2.7     | 17         | 25     | -       | -           | 17     |
| flp2        | 0          | 2.34e-12 | 2.34e-12 | 2.34e-12   |
| flp4.1      | 0          | -1.55e-15 | -1.55e-15 | 0          |
| flp4.2      | 0          | -2.22e-16 | -1.11e-15 | 0          |
| flp4.3      | 0          | 1.55e-15 | 1.55e-15 | 0          |
| flp4.4      | 0          | 6.66e-16 | -2.22e-15 | 0          |
| gauvin      | 20         | 20     | 20      | 20          |
| hs044.i     | 1.56e+01   | 6.25e-06 | -       | 6.25e-06   |
| jr1         | 5.00e-01   | 5.00e-01 | 5.00e-01 | 5.00e-01   |
| jr2         | 5.00e-01   | 5.00e-01 | 5.00e-01 | 5.00e-01   |
| kth2        | 0          | -2.22e-16 | -2.22e-16 | 0          |
| kth3        | 5.00e-01   | 5.00e-01 | 5.00e-01 | 5.00e-01   |
| liswet1.050 | 1.40e-02   | 1.40e-02 | 1.40e-02 | 1.40e-02   |
| liswet1.100 | 1.37e-02   | 1.37e-02 | -       | 1.37e-02   |
| liswet1.200 | 1.70e-02   | 1.70e-02 | -       | 3.38e-02   |
| nash1a      | 7.89e-30   | 5.20e-12 | 5.20e-12 | 4.71e-12   |
| nash1b      | 7.89e-30   | 5.20e-12 | 5.20e-12 | 4.71e-12   |
| nash1c      | 7.89e-30   | 5.20e-12 | 5.20e-12 | 4.71e-12   |
| nash1d      | 7.89e-30   | 5.20e-12 | 5.20e-12 | 4.71e-12   |
| nash1e      | 7.89e-30   | 5.20e-12 | 5.20e-12 | 4.71e-12   |
| portfl1     | 1.50e-05   | 2.04e-02 | 2.04e-02 | 2.04e-02   |
| portfl2     | 1.46e-05   | 2.78e-02 | 2.78e-02 | 2.79e-02   |
| portfl3     | 6.27e-06   | 2.28e-02 | 2.27e-02 | 2.31e-02   |
| portfl4     | 2.18e-06   | 2.05e-02 | 2.05e-02 | 2.21e-02   |
| portfl6     | 2.36e-06   | 2.40e-02 | 2.40e-02 | 2.16e-01   |
| gpec1       | 80         | 80     | 80      | 80          |
| gpec2       | 45         | 4.50e+01 | 4.50e+01 | 45          |
| scholtes3   | 5.00e-01   | 5.00e-01 | 5.00e-01 | 5.00e-01   |
| sl1         | 1.00e-04   | 1.00e-04 | 1.00e-04 | 1.00e-04   |
Table 4: Objective values for MacMPEC solutions obtained by the IPOPT variants.

| problem      | best known | penalty | smoothed | relaxed | NLP  |
|--------------|------------|---------|----------|---------|------|
| bard1        | 17         | 17      | 17       | 25      | 17   |
| bard1m       | 17         | 17      | 17       | 25      | 17   |
| bard2        | -6598      | -6598   | -6.60e+03| -6598   | -6598|
| bilevel2     | -6600      | -6600   | -6600    | -6600   | -6600|
| bilevel2m    | -6600      | -6600   | -6600    | -6600   | -6600|
| ex9.2.1      | 17         | 17      | 17       | 25      | 17   |
| ex9.2.2      | 100        | 1.00e+02| 1.00e+02 | 1.00e+02| 1.00e+02|
| ex9.2.4      | 5.00e-01   | 5.00e-01| 5.00e-01 | 5.00e-01| 5.00e-01|
| ex9.2.5      | 5          | 9       | 5        | 9       | 9.80e+00|
| ex9.2.6      | -1         | -1      | -1.00e+00| -1      | -1   |
| ex9.2.7      | 17         | 17      | 17       | 25      | 17   |
| flp2         | 0          | 2.34e-12| 8.87e-12 | 2.34e-12| 7.60e-12|
| flp4.1       | 0          | -2.70e-09| 7.02e-06 | -2.70e-09| -2.70e-09|
| flp4.2       | 0          | -5.40e-09| 6.32e-06 | -5.40e-09| -5.40e-09|
| flp4.3       | 0          | -6.30e-09| 7.90e-06 | -6.30e-09| -6.30e-09|
| flp4.4       | 0          | -9.00e-09| 1.18e-05 | -9.00e-09| -9.00e-09|
| gauvin       | 20         | 20      | 325      | 20      | 20   |
| hs044.i      | 1.56e+01   | 6.25e-06| 1.56e+01 | 6.25e-06| 1.56e+01|
| jr1          | 5.00e-01   | 5.00e-01| 5.00e-01 | 5.00e-01| 5.00e-01|
| jr2          | 5.00e-01   | 5.00e-01| 5.00e-01 | 5.00e-01| 5.00e-01|
| kth2         | 0          | -9.00e-11| 1.00e-07 | -9.00e-11| -8.72e-11|
| kth3         | 5.00e-01   | 5.00e-01| 5.00e-01 | 5.00e-01| 5.00e-01|
| liswet1.050  | 1.40e-02   | 1.40e-02| 1.30e-01 | 1.40e-02| 1.40e-02|
| liswet1.100  | 1.37e-02   | 1.37e-02| 2.40e-01 | 1.37e-02| 1.37e-02|
| liswet1.200  | 1.70e-02   | 1.70e-02| 4.75e-01 | 1.70e-02| 1.70e-02|
| nash1a       | 7.89e-30   | 1.27e-11| 4.71e-12 | 5.59e-12| 4.71e-12|
| nash1b       | 7.89e-30   | 1.25e-11| 4.71e-12 | 5.59e-12| 4.71e-12|
| nash1c       | 7.89e-30   | 1.25e-11| 4.71e-12 | 5.59e-12| 4.71e-12|
| nash1d       | 7.89e-30   | 1.27e-11| 4.71e-12 | 5.59e-12| 4.71e-12|
| nash1e       | 7.89e-30   | 5.55e-12| 4.71e-12 | 5.59e-12| 4.71e-12|
| portfl1      | 1.50e-05   | 2.04e-02| 2.06e-02 | 2.04e-02| 2.04e-02|
| portfl2      | 1.46e-05   | 2.78e-02| 2.97e-02 | 2.78e-02| 2.78e-02|
| portfl3      | 6.27e-06   | 2.27e-02| 2.27e-02 | 2.27e-02| 2.27e-02|
| portfl4      | 2.18e-06   | 2.05e-02| 2.06e-02 | 2.05e-02| 2.05e-02|
| portfl6      | 2.36e-06   | 2.40e-02| 2.46e-02 | 2.40e-02| 2.40e-02|
| spec1        | 80         | 80      | 8.00e+01 | 80      | 80   |
| spec2        | 45         | 4.50e+01| 4.50e+01 | 4.50e+01| 4.50e+01|
| scholtes3    | 5.00e-01   | 5.00e-01| 5.00e-01 | 5.00e-01| 1.00e+00|
| sl1          | 1.00e-04   | 1.00e-04| 1.00e-04 | 1.00e-04| 1.00e-04|
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