The Complexity of Equilibria for Risk-Modeling Valuations

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(November 2, 2015)

Abstract

We study the complexity of deciding the existence of mixed equilibria for minimization games where players use valuations other than expectation to evaluate their costs. We consider risk-averse players seeking to minimize the sum $V = E + R$ of expectation $E$ and a risk valuation $R$ of their costs; $R$ is non-negative and vanishes exactly when the cost incurred to a player is constant over all choices of strategies by the other players. In a $V$-equilibrium, no player could unilaterally reduce her cost.

Say that $V$ has the Weak-Equilibrium-for-Expectation property if all strategies supported in a player’s best-response mixed strategy incur the same conditional expectation of her cost. We introduce $E$-strict concavity and observe that every $E$-strictly concave valuation has the Weak-Equilibrium-for-Expectation property. We focus on a broad class of valuations shown to have the Weak-Equilibrium-for-Expectation property, which we exploit to prove two main complexity results, the first of their kind, for the two simplest cases of the problem:

- **Two strategies:** Deciding the existence of a $V$-equilibrium is strongly $\mathcal{NP}$-hard for the restricted class of player-specific scheduling games on two ordered links [22], when choosing $R$ as (1) $\text{Var}$ (variance), or (2) $\text{SD}$ (standard deviation), or (3) a concave linear sum of even moments of small order.

- **Two players:** Deciding the existence of a $V$-equilibrium is strongly $\mathcal{NP}$-hard when choosing $R$ as (1) $\gamma \cdot \text{Var}$, or (2) $\gamma \cdot \text{SD}$, where $\gamma > 0$ is the risk-coefficient, or choosing $V$ as (3) a convex combination of $E + \gamma \cdot \text{Var}$ and the concave $\nu$-valuation $\nu^{-1}(E(\nu(\cdot)))$, where $\nu(x) = x^r$, with $r \geq 2$. This is a concrete consequence of a general strong $\mathcal{NP}$-hardness result that only needs the Weak-Equilibrium-for-Expectation property and a few additional properties for $V$: its proof involves a reduction with a single parameter, which can be chosen efficiently so that each valuation satisfies the additional properties.

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*Partially supported by the German Research Foundation (DFG) within the Collaborative Research Centre “On-the-Fly-Computing” (SFB 901), and by funds for the promotion of research at University of Cyprus.

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1 Introduction

1.1 The Pros and Cons of Expectation

In a game, each player is using a mixed strategy, a probability distribution over her strategies; her cost depends on the choices of all players and is evaluated by a valuation: a function from probability distributions to reals. The most prominent valuation in Non-Cooperative Game Theory is expectation; each player minimizes her expected cost.

A drawback of expectation is that it may not accommodate risk and its impact on strategic decision; this inadequacy of expectation was addressed as early as 1738 by Bernoulli [4]. Indeed, risk-averse players [1] are willing to accept a larger amount of payment rather than gambling and taking the risk of a larger cost; according to [15], “a risk-averse player is willing to pay something for certainty”. So, valuations other than expectation have been sought (cf. [1, 10, 21, 27]). Concave valuations, such as variance and standard deviation, are well-suited to model risk-averse minimizing players. Already in 1906, Fisher [14] proposed attaching standard deviation to expectation as an additive measure of risk.

In his seminal paper [21], Markowitz introduced the Mean-Variance approach to portfolio maximization, advocating the minimization of variance constrained on some lower bound on the expected return. This way, instead of a single optimal solution, a class of “efficient” solutions, termed as the Efficient frontier [10], is defined, incurring the lowest risk for a given level of expected return. Popular valuations for determining a single maximizing solution from the Efficient frontier are (i) $E - \gamma \cdot \text{Var}$, where $E$ and $\text{Var}$ stand for expectation and variance, respectively, and $\gamma > 0$ describes the risk tolerance (see [10]), and (ii) the Sharpe Ratio $\text{SR} = \frac{E}{\text{SD}}$ [27], where $\text{SD}$ stands for standard deviation. The Mean-Variance paradigm [21] created Modern Portfolio Theory [10] as a new field and initiated a tremendous amount of research — see the surveys [20, 28] for an overview. However, in the Mean-Variance paradigm [21], only expectation and variance were used for evaluating the return; this choice is justified only if the return is normally distributed [17]. Subsequently this inadequacy led to risk models involving higher moments so as to accommodate returns with a more general distribution [19].

We now switch back to the minimization setting. A significant advantage of expectation is that it guarantees the existence of a Nash equilibrium [24, 25], where each player is playing a best-response mixed strategy and could not unilaterally reduce her expected cost. Existence of equilibria (for minimization games) extends to convex valuations [8, 11], but may fail for non-convex and even for concave ones. Crawford’s game [7, Section 4] was the first counterexample game with no equilibrium for a certain valuation; for more counterexamples, see [9, 22]. The view that mixed equilibria get “endangered” in games where players are not expectation-optimizers has been put forward in [7] and adopted further in [12, 22].
Fiat and Papadimitriou [12] introduced the equilibrium computation problem in games where risk-averse players use valuations more general than expectation, and addressed the complexity of deciding the existence of such equilibria. Subsequently, Mavronicolas and Monien [22] focused on the concave valuation expectation plus variance, for which they established structural and complexity results for their introduced class of player-specific scheduling games [22, Section 3]; their results provided a solid basis for the study of more general concave valuations.

1.2 Valuations More General than Expectation

In this work, we shall consider minimization games. We model the valuation of each player as the sum $V = E + R$, where $E$ and $R$ are the expectation and risk valuation, respectively. The formulation of $(E + R)$-valuations draws motivation from the Mean-Variance paradigm [21], and from the Variance Principle and the Standard Deviation Principle, two standard premium principles in Actuarial Risk Theory (cf. [18, Section 5.3]), by which $V = E + \gamma \cdot \text{Var}$ (resp., $V = E + \gamma \cdot \text{SD}$), where $\gamma > 0$ is the risk-coefficient. We focus on the associated decision problem, denoted as $\exists V$-EQUILIBRIUM, asking, given a game $G$, whether $G$ has a $V$-equilibrium, where no player could unilaterally reduce her cost (as evaluated by $V$). What is the impact of properties of $V$ on the complexity of $\exists V$-EQUILIBRIUM?\footnote{The work in [12] considered the dual setting where risk-averse players maximize non-concave valuations; they focused on the convex valuation expectation minus variance.}

We stipulate a very basic property for $R$, called Risk-Positivity: the value of $R$ is non-negative, en par with the Non-negative Loading property of the premium principles in [18, Section 5.3.1]; it is 0, yielding no risk, exactly when the cost incurred to a player is constant over all choices of strategies by the other players.

We shall focus on concave valuations. A key property of a concave valuation, called Optimal-Value, we prove and exploit is that it maintains the same optimal value over all convex combinations of strategies supported in a given best-response mixed strategy (Proposition 3.1). Unfortunately, unlike Var, moments of order higher than 2 are not concave. But on the positive side, all even moments have the Risk-Positivity property. Besides, recent work in Portfolio Theory [19] motivates using higher moments to model risk.

To obtain an enhanced class of interesting concave valuations, we introduce into the context of equilibrium computation valuations prominent for evaluating risk in Actuarial Risk Theory, which are transferred from the Mean Value Principle (see [18, Section 5.3]). Specifically, we shall consider $\nu$-valuations of the form $V^\nu = \nu^{-1}(E^\nu)$, for any increasing and strictly convex $\nu$.\footnote{Strictly speaking, some of the considered properties of $V$ are rather properties of the equilibria of some game $G$ whose players minimize $V$. For ease of presentation, we shall omit reference to $G$ since $G$ will be either fixed or clear from context in the settings we shall consider, so that the property depends only on $V$.}
function \( \nu \), so that \( \nu^{-1} \) is concave.\(^3\) It is good news that for \( \nu \) increasing and strictly convex, \( R^\nu := V^\nu - E \) has the Risk-Positivity property (Lemma 2.1); so, \( V^\nu \) is an \((E + R)\)-valuation.

1.3 Weak-Equilibrium-for-Expectation and E-Strict Concavity

As our main tool, we shall exploit the Weak-Equilibrium-for-Expectation property [22, Section 2.6]: all strategies supported in a player’s best-response mixed strategy induce the same conditional expectation, taken over all random choices of the other players, for her cost; thus, the player is holding a unique expectation for her cost no matter which of her supported strategies she ends up choosing. This property holds vacuously for Nash equilibria [24, 25]. The Weak-Equilibrium-for-Expectation property formalizes the most natural intuition for the players’ expectations; hence, it is a very natural property to seek and employ in the setting of risk. We aim at an enhanced class of valuations with the Weak-Equilibrium-for-Expectation property.

We introduce an E-strictly concave valuation as a concave valuation which, viewed as a function of a single mixed strategy, fulfills the definition of a strictly concave function for any pair of mixed strategies inducing different expectations (Definition 2.2). We observe that a convex combination of an E-strictly concave valuation and a concave valuation is E-strictly concave (Corollary 2.2); furthermore, \( \text{Var} \) and \( \text{SD} \) are E-strictly concave (Lemma 2.3). Hence, a wide class of concrete instances of E-strictly concave valuations results by plugging in the convex combination (i) \( E + \gamma \cdot \text{Var} \) for an E-strictly concave valuation, with \( \gamma > 0 \), and (ii) a \( \nu \)-valuation, with an increasing and strictly convex function \( \nu \), for a concave valuation (Corollary 2.4). We establish the key fact that every E-strictly concave valuation has the Weak-Equilibrium-for-Expectation property (Proposition 3.2). E-strictly concave valuations make the largest subclass of concave valuations we know of with the Weak-Equilibrium-for-Expectation property.

An obstacle to extending the Weak-Equilibrium-for-Expectation property to moments of order higher than 2 is their non-concavity. Instead we consider concave linear sums of even moments. (Even order is needed to guarantee the Risk-Positivity property.) We use the Optimal-Value property (Proposition 3.1) to establish that such concave valuations have the Weak-Equilibrium-for-Expectation property (Corollary 3.4); this property renders such concave linear sums of even moments sufficiently interesting to consider.

1.4 Complexity Results for More General Valuations

By exploiting the Weak-Equilibrium-for-Expectation property for an \((E + R)\)-valuation \( V \), we shall show the strong \( \mathcal{NP} \)-hardness of \( \exists V \text{-EQUILIBRIUM} \) for the two simplest cases: games with

\(^3\)For the special case \( \nu(x) = e^x \), \( V^\nu \) corresponds to the moment generating function (cf. [18, Section 2.4]) and has gained special attention as the Exponential Principle premium in Actuarial Risk Theory [18, Section 5.3].
two strategies and games with two players; these are the first complexity results for deciding the existence of equilibria in the context of risk-modeling valuations (cf. Section 1.5).

1.4.1 Two Strategies

We discover that the complexity of \( \exists V\text{-EQUILIBRIUM} \) is captured by player-specific scheduling games on two ordered links \(^{22}\) Section 5.2.2, where the cost of player \( i \) on a link \( \ell \) she chooses is a sum of weights \( \omega(i, i', \ell) \), each corresponding to player \( i \), link \( \ell \) and a player \( i' \) choosing the same link. Two ordered links \(^{22}\) Section 5.2.2 means that link 1 incurs less weight than link 2 to a player due to another player choosing the same link unless both weights are 0. We show that for an \((E + R)\)-valuation \( V \), \( \exists V\text{-EQUILIBRIUM} \) is strongly \( \mathcal{NP} \)-hard for the class of player-specific scheduling games on two ordered links when \( R \) is (1) \( \text{Var} \), or (2) \( \text{SD} \), or (3) a concave linear sum of even moments of order \( k \in \{2, 4, 6, 8\} \) (Theorem 4.6).

Instrumental to the proof of Theorem 4.6 is the key property we show that a concave linear sum of even moments of order \( k \in \{2, 4, 6, 8\} \) enjoys the Mixed-Player-Has-Pure-Neighbors property: each player \( i \) either is pure or has all of her neighbors (that is, players \( i' \) with \( \omega(i, i', 1) \neq 0 \)) pure (Proposition 4.3). This property is a quantitative expression of the view that mixed equilibria get “endangered” when players are not expectation-optimizers (cf. \cite{7, 12, 22}).

The class of concave linear sum of even moments of order \( k \in \{2, 4, 6, 8\} \) is the largest class of valuations we were able to identify with the Mixed-Player-Has-Pure-Neighbors property. We conjecture that every concave linear sum of even moments enjoys the property.

1.4.2 Two Players

We show that \( \exists V\text{-EQUILIBRIUM} \) is strongly \( \mathcal{NP} \)-hard when \( V \) is an \((E + R)\)-valuation \( V \) with the Weak-Equilibrium-for-Expectation property provided that there is a polynomial time computable \( \delta, 0 < \delta \leq \frac{1}{4} \), such that three additional conditions hold (Theorem 5.1); the requirement that \( \delta \) be polynomial time computable stems from the fact that \( \delta \) enters the reduction as a parameter. The first two such conditions ((2/a) and (2/b)) stipulate a particular inequality and a particular monotonicity property, respectively. The third condition (2/c) refers to the Crawford game \( G_C(\delta) \), a generalization of a bimatrix game from \cite{7} Section 4, whose bimatrix involves \( \delta \); it is required that \( G_C(\delta) \) has no \( V \)-equilibrium. The game \( G_C(\delta) \) is used as a “gadget” in the reduction. The proof of Theorem 5.1 involves a reduction with a single parameter \( \delta \), is

\(^{1}\)In the well-known model of weighted congestion games with player-specific latency functions \cite{23}, each weighted player may use a different, player-specific cost function of the total weight on her selected link; in this model, it is the weights that are player-specific, while each cost function is the identity one.
very general since it refers to no particular valuation but to a class of valuations enjoying two natural properties, Risk-Positivity and Weak-Equilibrium-for-Expectation.

Concrete strong \( NP \)-hardness results follow as instantiations of Theorem 5.1 for three particular valuations (1) \( V = E + \gamma \cdot \text{Var} \), (2) \( V = E + \gamma \cdot \text{SD} \), and (3) \( V = \lambda \cdot (E + \gamma \cdot \text{Var}) + (1 - \lambda) \cdot V'' \), with \( 0 < \lambda \leq 1 \), where \( \nu \) is the increasing and strictly convex function \( \nu(x) = x^r \), with \( r \geq 2 \), and with \( \gamma > 0 \) (Theorem 5.12). For all three valuations in Theorem 5.12 we prove that the three additional conditions in Theorem 5.1 hold. In particular, for Condition (2/c), we prove that \( G_C(\delta) \) has no V-equilibrium for any value \( 0 < \delta < 1 \); for the valuation (3), this holds in the more general case \( \nu \) is any increasing and strictly convex function (Lemma 5.11).

1.5 Summary, Significance and Related Work

For a concave valuation \( V \), there may or may not exist a V-equilibrium; we have identified \( V'' \), with an increasing and strictly convex function \( \nu \), as an example of a concave valuation \( V \) for which every game has a V-equilibrium (cf. Section 5). However, restricting to a strictly concave valuation excludes the existence of a mixed V-equilibrium. Restricting instead to an E-strictly concave valuation \( V \), a mixed V-equilibrium may or may not exist; what this work is revealing is that it then becomes strongly \( NP \)-hard to decide if there is one already for the two simplest cases, games with two strategies or two players (Theorems 4.6, 5.1 and 5.12).

While bringing concave valuations from Actuarial Risk Theory \[18\] into play, our framework encompasses general classes of (E + R)-valuations, assuming the Risk-Positivity property, that also enjoy the Weak-Equilibrium-for-Expectation property, and a few additional properties. The (E + Var)-equilibria on which \[12, 22\] focused are a special case of our general framework.

Fiat and Papadimitriou \[12\] (Theorem 5) presented a proof sketch to claim that it is \( NP \)-hard to decide the existence of an (E + Var)-equilibrium for games with two players; unfortunately, their proof had been flawed, containing several gaps and errors. In personal communication with Fiat and Papadimitriou \[13\], they state: “The proof, as is, has gaps and errors, which we believe can be fixed to yield a proof with the same architecture, but we have not done it yet.” In lack of a proof, the complexity of deciding the existence of an (E + Var)-equilibrium for games with two players has remained open, and Theorems 5.1 and 5.12 represent new results, with Theorem 5.12 encompassing (E + Var)-equilibria as a special case. Our reduction adapts techniques originally used by Conitzer and Sandholm \[6\] (Section 3) to show that deciding the existence of Nash equilibria with certain properties for games with two players is \( NP \)-complete.

Fiat and Papadimitriou \[12\] (Section 2) coined a notion termed as strict concavity, denoted here as FP-strict concavity, which is similar to but different than E-strict concavity. It turns out that their difference is essential since E + Var is not FP-strictly concave while it is E-strictly
The sparsity of mixed valuations was given in [12]. Fiat and Papadimitriou [12, Theorem 3 & Observation 4] proved the sparsity of mixed V-equilibria, when V is FP-strictly concave: games with a mixed V-equilibrium have measure 0. The sparsity of mixed (E+Var)-equilibria follows from [5, Theorem 1] where it is established that (E+Var) is a Mean-Variance Preference Function [5, Claim 1]. Contrary to their sparsity, this work establishes that deciding the existence of mixed (E+Var)-equilibria is strongly NP-hard (Theorems 4.6 and 5.12).

It was known that E + Var, as well as E + SD, have the Weak-Equilibrium-for-Expectation property [22, Theorem 3.5]; they were also known to have the Mixed-Player-Has-Pure-Neighbors property [22, Theorem 3.5]; they were also known to have the Mixed-Player-Has-Pure-Neighbors property for player-specific scheduling games on two ordered links [22, Theorem 5.13]. It was established in [5, Theorem 4] that there is always a correlated equilibrium [2] for E + Var.

1.6 Paper Organization

Section 2 presents the game-theoretic framework and introduces (E + R)-valuations and E-strict concavity. Equilibria and their properties are articulated in Section 3. The N^P-hardness results for two strategies and two players are presented in Sections 4 and 5, respectively. We conclude, in Section 6, with a discussion of the results and some open problems.

2 Games, (E + R)-Valuations and E-Strict Concavity

2.1 Games

For an integer n ≥ 2, an n-players game G, or game, consists of (i) n finite sets \( \{S_i\}_{i \in [n]} \) of strategies, and (ii) n cost functions \( \{\mu_i\}_{i \in [n]} \), each mapping \( S = \times_{i \in [n]} S_i \) to the reals. So, \( \mu_i(s) \) is the cost of player \( i \in [n] \) on the profile \( s = \langle s_1, \ldots, s_n \rangle \) of strategies, one per player.

A mixed strategy for player \( i \in [n] \) is a probability distribution \( p_i \) on \( S_i \); the support of player \( i \) in \( p_i \) is the set \( \sigma(p_i) = \{ \ell \in S_i \mid p_i(\ell) > 0 \} \). Denote as \( \Delta_i = \Delta(S_i) \) the set of mixed strategies for player \( i \). Player \( i \) is pure if for each strategy \( s_i \in S_i \), \( p_i(s_i) \in \{0,1\} \); else she is non-pure. Denote as \( p_i^\ell \) the pure strategy of player \( i \) choosing the strategy \( \ell \) with probability 1.

A mixed profile is a tuple \( p = (p_1, \ldots, p_n) \) of n mixed strategies, one per player; denote as \( \Delta = \Delta(S) = \times_{i \in [n]} \Delta_i \) the set of mixed profiles. The mixed profile \( p \) induces probabilities \( p(s) \) for each profile \( s \in S \) with \( p(s) = \prod_{i' \in [n]} p_i'(s_{i'}) \). For a player \( i \in [n] \), the partial profile \( s_{-i} \) (resp., partial mixed profile \( p_{-i} \)) results by eliminating the strategy \( s_i \) (resp., the mixed strategy

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1. See the Appendix for the definition of FP-strict concavity and a proof that E + Var is not FP-strictly concave.
2. See Section 3.4 for the definition of Mean-Variance Preference Functions and their relation to this work.
on a convex set T. For a player $i$, \( \Delta V_{i} \delta \) two points \( t_{1}, t_{2} \in T \) and any number \( \delta \in [0, 1] \) (resp., \( \delta \in (0, 1) \)), \( V(\delta t_{1} + (1 - \delta) t_{2}) \geq \delta V(t_{1}) + (1 - \delta) V(t_{2}) \) (resp., \( V(\delta t_{1} + (1 - \delta) t_{2}) > \delta V(t_{1}) + (1 - \delta) V(t_{2}) \)). A function \( V : T \to \mathbb{R} \) on a convex set \( T \) is concave (resp., strictly concave) if \( -V \) is concave (resp., strictly concave).

\section{2.2 \( (E + R) \)-Valuations}

For a player \( i \in [n] \), a valuation function, or valuation for short, \( V_{i} \) is a real-valued function on \( \Delta(S) \), yielding a value \( V_{i}(p) \) to each mixed profile \( p \), so that in the special case where \( p \) is a profile \( s \), \( V_{i}(s) = \mu_{i}(s) \). A valuation \( V = (V_{1}, \ldots, V_{n}) \) is a tuple of valuations, one per player; \( G^{V} \) denotes G together with \( V \).

\begin{definition}[(E + R)-Valuation] An \((E + R)\)-valuation is a valuation of the form \( V = E + R \), where \( E \) is the expectation valuation with \( E_{i}(p) = \sum_{s \in S} p(s) \mu_{i}(s) \) for \( i \in [n] \), and \( R \) is the risk valuation, a continuous valuation with the Risk-Positivity property: For each player \( i \in [n] \) and mixed profile \( p \), (C.1) \( R_{i}(p) \geq 0 \) and (C.2) \( R_{i}(p) = 0 \) if and only if for each profile \( s \in S \) with \( p(s) > 0 \), \( \mu_{i}(s) \) remains constant over all choices of strategies by the other players; in such case, \( V_{i}(p) = E_{i}(p) = \mu_{i}(s) \) for any profile \( s \in S \) with \( p(s) > 0 \).
\end{definition}

For each integer \( k \geq 0 \), the \( k \)-moment valuation is given by
\[
\kappa_{i}(p) = \sum_{s \in S} p(s) (\mu_{i}(s) - E_{i}(p))^{k},
\]
for each player \( i \in [n] \); so, \( 1M = 0 \). Furthermore, \( 2M \), known as variance and denoted as \( \text{Var} \), is concave; hence, also is the square root of variance, known as standard deviation and denoted as \( \text{SD} \). However, \( k \)-moments of order higher than 2 are not concave.

We shall consider \( \nu \)-valuations \( V_{\nu} = \nu^{-1}(E(\nu(.))) \), for an increasing and strictly convex function \( \nu \); so, \( \nu^{-1} \), and hence \( V_{\nu} \), is concave. So, for a player \( i \),
\[
V_{i}(\nu)(p) = \nu^{-1} \left( \sum_{s \in S} p(s) \cdot \nu(\mu_{i}(s)) \right);
\]
set also \( R_{\nu} := V_{\nu} - E \). We observe:

\begin{lemma}
For an increasing and strictly convex function \( \nu \), the risk valuation \( R_{\nu} \) has the Risk-Positivity property.
\end{lemma}

*We shall mostly treat a valuation function \( V_{i} \) and a valuation \( V \) interchangeably for an easier notation; we shall use \( V_{i} \) only when \( p_{i} \) has some special property.
Proof: Fix a player $i \in [n]$ and a profile $p$. Then, $R_i^\nu(p) \geq 0$ if and only if $V_i^\nu(p) \geq E_i(p)$ if and only if

$$\nu^{-1}\left(\sum_{s \in S} p(s) \cdot \nu(\mu_i(s))\right) \geq \sum_{s \in S} p(s) \cdot \mu_i(s)$$

if and only if (since $\nu$ is increasing)

$$\sum_{s \in S} p(s) \cdot \nu(\mu_i(s)) \geq \nu\left(\sum_{s \in S} p(s) \cdot \mu_i(s)\right).$$

Now (C.1) follows since $\nu$ is convex; (C.2) follows since $\nu$ is strictly convex.

By Lemma 2.1 for an increasing and strictly convex function $\nu$, $V^\nu$ is an $(E + R)$-valuation.

We shall deal with cases where for a player $i \in [n]$ and a mixed profile $p$, \{\mu_i(s) \mid p(s) > 0\} = \{a, b\}$ with $a < b$, so that $R_i(p)$ depends on the three parameters $a$, $b$ and $q$, where $q := \sum_{s \in S} |\mu_i(s) = b| p(s)$. Then, denote

$$\hat{V}_i(a, b, q) := a + q(b - a) + \hat{R}_i(a, b, q).$$

### 2.3 E-Strict Concavity

We introduce a refinement of concavity:

**Definition 2.2 (E-Strict Concavity)** Fix a player $i$. The $(E + R)$-valuation $V_i$ is E-strictly concave if for every game $G$, the following conditions hold for a fixed partial mixed profile $p_{-i}$:

1. $V_i$ is concave in the mixed strategy $p_i$.

2. For a pair of mixed strategies $p'_i, p''_i \in \Delta(S_i)$, if $E_i(p'_i, p_{-i}) \neq E_i(p''_i, p_{-i})$, then for any $\lambda$ with $0 < \lambda < 1$,

$$V_i(\lambda p'_i + (1 - \lambda)p''_i, p_{-i}) > \lambda V_i(p'_i, p_{-i}) + (1 - \lambda) V_i(p''_i, p_{-i}).$$

Note that E-strict concavity is different from the strict concavity formulated in [12, Section 2], and denoted here as FP-strict concavity, by using the payoff distribution $P_i$, which is the probability distribution on the costs induced by a mixed strategy $p_i$ and a partial mixed profile $p_{-i}$[1]. A closure property of E-strict concavity follows.

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[1] In Appendix A, we provide a counterexample to demonstrate that the valuation $E + Var$ is not FP-strictly concave.
Corollary 2.2 Fix an arbitrary pair of an E-strictly concave valuation \( V^{(1)} \) and a concave valuation \( V^{(2)} \). Then, for any \( \lambda \), with \( 0 < \lambda \leq 1 \), the valuation \( V = \lambda V^{(1)} + (1 - \lambda) V^{(2)} \) is E-strictly concave.

We observe:

Lemma 2.3 The valuations \( E + \gamma \cdot \text{Var} \) and \( E + \gamma \cdot \text{SD} \), with \( \gamma > 0 \), are E-strictly concave.

Proof: Fix a game \( G \) and a player \( i \in [n] \). Recall that \( \text{Var} \) and \( \text{SD} \) are concave in the mixed strategy \( p_i \). Also,

\[
\text{Var}_i (p_i, p_{-i}) = \sum_{s \in S} p(s) \mu_i^2(s) - \left( \sum_{\ell \in \sigma(p_i)} p_i(\ell) E_i(p_i^\ell, p_{-i}) \right)^2;
\]

so, the non-linear term in the mixed strategy \( p_i \) is a function in the variables \( E_i(p_i^\ell, p_{-i}) \), with \( \ell \in \sigma(p_i) \). Assume that there are strategies \( r, t \in \sigma(p_i) \) such that \( E_i(p_r^\ell, p_{-i}) \neq E_i(p_t^\ell, p_{-i}) \). Since the function \( \tilde{\nu}(x) = -x^2 \) is strictly concave, we get that

\[
\begin{align*}
\text{Var}_i (p_i, p_{-i}) &> \sum_{s \in S} p(s) \mu_i^2(s) - \sum_{\ell \in \sigma(p_i)} p_i(\ell) E_i^2(p_i^\ell, p_{-i}) \\
&= \sum_{\ell \in \sigma(p_i)} p_i(\ell) \left( \sum_{s_{-i} \in S_{-i}} p(s_{-i}) \mu_i^2(p_i^\ell, s_{-i}) - \sum_{\ell \in \sigma(p_i)} p_i(\ell) E_i^2(p_i^\ell, p_{-i}) \right) \\
&= \sum_{\ell \in \sigma(p_i)} p_i(\ell) \text{Var}_i (p_i^\ell, p_{-i}),
\end{align*}
\]

as needed. Now, \( \text{SD} \) is E-strictly concave as the square root of \( \text{Var} \).

By Corollary 2.2 and Lemma 2.3, it follows:

Corollary 2.4 The valuation \( V = \lambda (E + \gamma \cdot \text{Var}) + (1 - \lambda) V^\nu \), with \( 0 < \lambda \leq 1 \), where \( \nu \) is increasing and strictly convex, and with \( \gamma > 0 \), is E-strictly concave.

3 Equilibria and Their Properties

3.1 \( V \)-Equilibrium

Fix a player \( i \in [n] \). The pure strategy \( p_i^\ell \) is a \( V_i \)-best pure response to a partial mixed profile \( p_{-i} \) if

\[
V_i (p_i^\ell, p_{-i}) = \min \left\{ V_i (p_i^{\ell'}, p_{-i}) \mid \ell' \in S_i \right\}.
\]
so, the pure strategy $p_i^*$ minimizes the valuation $V_i(\cdot, p_{-i})$ of player $i$ over her pure strategies. The mixed strategy $p_i$ is a $V_i$-best response to $p_{-i}$ if

$$V_i(p_i, p_{-i}) = \min \{ V_i(p_i', p_{-i}) | p_i' \in \Delta(S_i) \};$$

so, the mixed strategy $p_i$ minimizes the valuation $V_i(\cdot, p_{-i})$ of player $i$ over her mixed strategies. The mixed profile $p$ is a $V$-equilibrium if for each player $i$, the mixed strategy $p_i$ is a $V_i$-best response to $p_{-i}$; so, no player could unilaterally deviate to another mixed strategy to reduce her cost. Denote as $\exists V$-EQUILIBRIUM the algorithmic problem of deciding, given a game $G$, the existence of a $V$-equilibrium for $G^V$.

### 3.2 The Optimal-Value Property

We show:

**Proposition 3.1 (The Optimal-Value Property)** Fix a game $G$, a player $i \in [n]$ and a partial mixed profile $p_{-i}$. Assume that (A.1) the valuation $V_i(p_i, p_{-i})$ is concave in $p_i$, and (A.2) $\hat{p}_i$ is a $V_i$-best response to $p_{-i}$. Then, for any mixed strategy $q_i$ with $\sigma(q_i) \subseteq \sigma(\hat{p}_i)$, $V_i(q_i, p_{-i}) = V_i(\hat{p}_i, p_{-i})$.

**Proof:** Set $A := V_i(\hat{p}_i, p_{-i})$. Since $\hat{p}_i$ is a $V_i$-best response to $p_i$, it follows that $V_i(q_i, p_{-i}) \geq A$ for each mixed strategy $q_i \in \Delta(S_i) \subseteq \sigma(\hat{p}_i)$. Assume, by way of contradiction, that there is a mixed strategy $q_i \in \Delta(S_i) \subseteq \sigma(p_i)$ such that $V_i(q_i, p_{-i}) > A$.

Denote as $\Delta(\sigma(\hat{p}_i))$ the set of mixed strategies for player $i$ with supports contained in $\sigma(\hat{p}_i)$; so, $\Delta(\sigma(\hat{p}_i))$ is a subspace in $[0, 1]^{\sigma(\hat{p}_i)}$.

For each $\lambda \in [0, 1]$, the strategies $\lambda \hat{p}_i + (1 - \lambda)q_i$ form a line segment in $\Delta(\sigma(\hat{p}_i))$. Extend this line segment to some strategy $\hat{q}_i \in \Delta(\sigma(\hat{p}_i))$ with $\hat{q}_i \neq \hat{p}_i$ so that $\hat{q}_i$ is an interior point on the line segment connecting $q_i$ and $\hat{p}_i$. The extension is possible since $\hat{p}_i(j) > 0$ for each strategy $j \in \sigma(\hat{p}_i)$, so that $\hat{p}_i$ is an interior point in $\Delta(\sigma(\hat{p}_i))$. Since $\hat{p}_i$ is a $V_i$-best response to $p_{-i}$, it follows that $V_i(\hat{q}_i, p_{-i}) \geq A$. So, there are points $\hat{p}_i, q_i, \hat{q}_i$ so that:

- $\hat{p}_i$ is an interior point on the line segment connecting $q_i$ and $\hat{q}_i$.
- $V_i(\hat{p}_i, p_{-i}) = A$, $V_i(q_i, p_{-i}) > A$ and $V_i(\hat{q}_i, p_{-i}) \geq A$.

A contradiction to the concavity of $V_i(\cdot, p_{-i})$. □
3.3 The Strong Equilibrium and Weak Equilibrium Properties

The mixed profile \( p \) has the Strong Equilibrium property [22, Section 2.6] for player \( i \) in the game \( G^V \) if for each strategy \( \ell \in \sigma(p_i) \),

\[
V_i(p_i^\ell, p_{-i}) = \min \left\{ V_i(p_i^{\ell'}, p_{-i}) \mid \ell' \in S_i \right\};
\]

so, each strategy in the support of player \( i \) is a \( V_i \)-best pure response to the partial mixed profile \( p_{-i} \). Clearly, the Optimal-Value property for player \( i \) implies the Strong Equilibrium property for player \( i \); Proposition 3.1 extends [22, Proposition 2.1], establishing the Strong Equilibrium property with the same assumptions.

The mixed profile \( p \) has the Weak Equilibrium property [22, Section 2.6] for player \( i \in [n] \) in the game \( G^V \) if for each pair of strategies \( \ell, \ell' \in \sigma(p_i) \),

\[
V_i(p_i^\ell, p_{-i}) = V_i(p_i^{\ell'}, p_{-i}).
\]

The mixed profile \( p \) has the Weak Equilibrium property [22, Section 2.6] in \( G^V \) if it has the Weak Equilibrium property for each player \( i \in [n] \) in \( G^V \). (So, Strong Equilibrium implies Weak Equilibrium.)

3.4 The Weak-Equilibrium-for-Expectation Property

We introduce:

**Definition 3.1 (The Weak-Equilibrium-for-Expectation Property)** The valuation \( V \) has the Weak-Equilibrium-for-Expectation property if the following condition holds for every game \( G \): For each player \( i \in [n] \), if \( p_i \) is a \( V_i \)-best-response to \( p_{-i} \), then \( p \) has the Weak Equilibrium property for player \( i \) in the game \( G^E \): for each pair of strategies \( \ell, \ell' \in \sigma(p_i) \),

\[
E_i(p_i^\ell, p_{-i}) = E_i(p_i^{\ell'}, p_{-i}).
\]

We now prove that \( E \)-strict concavity implies the Weak-Equilibrium-for-Expectation property:

**Proposition 3.2** Take a player \( i \in [n] \) where \( V_i \) is \( E \)-strictly concave. Then, \( V \) has the Weak-Equilibrium-for-Expectation property for player \( i \).

**Proof:** Assume, by way of contradiction, that \( V \) does not have the Weak-Equilibrium-for-Expectation property for player \( i \). Then, there is a game \( G \), a partial mixed profile \( p_{-i} \) and a mixed strategy \( p_i \) which is a \( V_i \)-best-response to \( p_{-i} \) such that for some strategies \( r, t \in \sigma(p_i) \), \( E_i(p_i^r, p_{-i}) \neq E_i(p_i^t, p_{-i}) \). Since \( V_i \) is \( E \)-strictly concave, this implies that

\[
V_i\left(\frac{1}{2}p_i^r + \frac{1}{2}p_i^t, p_{-i}\right) > \frac{1}{2}V_i(p_i^r, p_{-i}) + \frac{1}{2}V_i(p_i^t, p_{-i}).
\]

Since \( V_i \) is concave, the Optimal-Value property (Proposition 3.1) implies that

\[
V_i\left(\frac{1}{2}p_i^r + \frac{1}{2}p_i^t, p_{-i}\right) = V_i(p_i^r, p_{-i}) = V_i(p_i^t, p_{-i}).
\]

A contradiction.
By Lemma 2.3 and Corollary 2.4, Proposition 3.2 immediately implies:

**Corollary 3.3** Fix an \((E + R)\)-valuation \(V\), where (1) \(R = \gamma \cdot \text{Var}\), or (2) \(R = \gamma \cdot \text{SD}\), or (3) \(V = \lambda (E + \gamma \cdot \text{Var}) + (1 - \lambda) V'\), with \(0 < \lambda \leq 1\), where \(V'\) is increasing and strictly convex, and with \(\gamma > 0\). Then, \(V\) has the Weak-Equilibrium-for-Expectation property.

We now turn to a particular concave valuation and exploit the Optimal-Value property (Proposition 3.1) to prove:

**Proposition 3.4** Fix a player \(i \in [n]\), and consider the concave valuation

\[
V_i = \alpha_0 \cdot E_i + \sum_{2 \leq k \leq \ell \text{ is even}} \alpha_k \cdot kM_i,
\]

for some constants \(\alpha_k \geq 0\), \(0 \leq k \leq \ell\). Then, \(V_i\) has the Weak-Equilibrium-for-Expectation property for player \(i\).

**Proof:** Fix a game \(G\). Clearly, for a mixed profile \(p\),

\[
2kM_i(p) = \sum_{s \in S} (\mu_i(s) - E_i(p))^{2k} p(s)
\]

\[
= \sum_{s \in S} \left( \sum_{t=0}^{2k} (-1)^t \binom{2k}{t} (\mu_i(s))^t (E_i(p))^{2k-t} \right) p(s)
\]

\[
= \sum_{t=0}^{2k} (-1)^t \binom{2k}{t} \left( \sum_{s \in S} (\mu_i(s))^t p(s) \right) (E_i(p))^{2k-t} + \binom{2k}{2k-1} (E_i(p))^{2k}.
\]

Since

\[
V_i(p) = \alpha_0 \cdot E_i(p) + \sum_{2 \leq k \leq \ell} \alpha_{2k} \cdot 2kM_i,
\]

so that \(V_i(p)\) is the sum of (i) the highest-degree term \(- (2\ell - 1) (E_i(p))^{2\ell}\), which is a polynomial of degree \(2\ell\) in \(p\), and (ii) a polynomial of degree bounded by \(2\ell - 1\) in \(p\). Since \(V_i(p)\) is a concave polynomial in \(p\), the Optimal-Value property (Proposition 3.1) implies that \(V_i(p)\) is a constant polynomial in \(p\). Hence, it follows that \(- (2\ell - 1) (E_i(p))^{2\ell}\) is a constant polynomial in \(p\); thus, so is \(E_i(p)\). The Weak-Equilibrium-for-Expectation property for player \(i\) follows.
Brautbar et al. [5] Section 3.1] study a class of valuations, coming from the Mean-Variance paradigm of Markowitz [21] and termed as Mean-Variance Preference Functions; we rephrase their definition [5 Definition 1] to fit into the adopted setting of minimization games:

Definition 3.2 ([5]) Fix a player \( i \in [n] \). A Mean-Variance Preference Function is a valuation \( V_i(p_i, p_{-i}) := G_i(E_i(p_i, p_{-i}), \text{Var}_i(p_i, p_{-i})) \) which satisfies:

1. \( V_i(p_i, p_{-i}) \) is concave in \( p_i \).
2. \( G_i \) is non-decreasing in its first argument \( (E_i(p_i, p_{-i})) \).
3. Fix a partial mixed profile \( p_{-i} \) and a nonempty convex subset \( \Delta \subseteq \Delta_i = \Delta(S_i) \) such that \( V_i(p_i, p_{-i}) \) is constant on \( \Delta \). Then, both \( E_i(p_i, p_{-i}) \) and \( \text{Var}_i(p_i, p_{-i}) \) are constant on \( \Delta \).

So, a Mean-Variance Preference Function simultaneously generalizes and restricts the \((E + R)\)-valuations; it generalizes sum to \( G \) but restricts \( R \) to \( \text{Var} \). Note that Condition (3) in Definition 3.2 may be seen as a generalization of the Weak-Equilibrium-for-Expectation property conditioned on the assumption that \( V_i(p_i, p_{-i}) \) is constant on a nonempty convex set \( \Delta \subseteq \Delta_i \). It is proved in [5 Claim 1] that \( E + \text{Var} \) is a Mean-Variance Preference Function. Since \( E + \text{Var} \) is \( E \)-strictly concave (Lemma 2.3) and has the Optimal-Value property (Proposition 3.1), their established Condition (3) is a special case of our general result that every \( E \)-strictly concave valuation has the Weak-Equilibrium-for-Expectation property (Proposition 3.2).

4 Two Strategies

4.1 Player-Specific Scheduling Games

A player-specific scheduling game [22] Section 3] is equipped with an integer weight \( \omega(i, i', \ell) \) for each triple of a player \( i \in [n] \), a player \( i' \in [n] \) and a strategy \( \ell \in S_i \), also called link, with \( S_1 = \ldots = S_n = [m] \); \( \omega(i, i', \ell) \) represents the load due to player \( i' \) incurred to player \( i \) on link \( \ell \). Given the collection of weights \( \{\omega(i, i', \ell)\}_{i, i' \in [n], \ell \in [m]} \), the cost function \( \mu_i \) is defined by \( \mu_i(s) = \sum_{i' \in [n]} s_\ell \omega(i, i', s_i) \). In a player-specific scheduling game on two ordered links 1 and 2 [22 Section 5.2.2], for each pair of players \( i, i' \in [n] \), either \( \omega(i, i', 1) = \omega(i, i', 2) = 0 \) or \( \omega(i, i', 1) < \omega(i, i', 2) \).

We derive a combinatorial formula for the \( k \)-moment valuation of the cost of a player choosing a link \( \ell \) in a player-specific scheduling game. The formula takes the form of a partition polynomial [23]: a multivariable polynomial defined by a sum over partitions of the integer \( k \). The formula uses the function \( f : [0, 1] \times N_0 \rightarrow \mathbb{R} \) with

\[
f(x, j) := (-x)^j(1-x) + (1-x)^j x.
\]
Note that
\[ f(x, j) = \begin{cases} 
  x(1 - x) \left( x^{j-1} + (1 - x)^{j-1} \right), & \text{for an even integer } j \geq 2 \\
  x(1 - x) \left( -x^{j-1} + (1 - x)^{j-1} \right), & \text{for an odd integer } j \geq 3 
\end{cases} \]

The following simple claim follows by inspection.

**Lemma 4.1** The function \( f \) has the following properties:

1. \( f(x, 0) = 1 \) for all \( x \in [0, 1] \).
   \( f(x, 1) = 0 \) for all \( x \in [0, 1] \).
   \( f(0, j) = f(1, j) = 0 \) for all integers \( j \geq 0 \).

2. For an even integer \( j \geq 2 \):
   \( f(x, j) > 0 \) for all \( 0 < x < 1 \).
   \( f(x, j) = f(1 - x, j) \) for all \( x \in [0, 1] \).

3. For an odd integer \( j \geq 3 \):
   \( f \left( \frac{1}{2}, j \right) = 0 \).
   \( f(x, j) > 0 \) for all \( 0 < x < \frac{1}{2} \).
   \( f(x, j) < 0 \) for all \( \frac{1}{2} < x < 1 \).
   \( f(x, j) = -f(1 - x, j) \) for all \( x \in [0, 1] \).

We show:

**Proposition 4.2** Consider a player-specific scheduling game with \( n \) players. Then, for each player \( i \in [n] \), for a link \( \ell \in [m] \) and a mixed profile \( \mathbf{p} \),
\[
kM_i \left( \mathbf{p}_i^\ell, \mathbf{p}_{-i} \right) = \sum_{\langle r_1, \ldots, r_n \rangle \in \mathbb{N}_0^n \mid \sum_{j \in [n]} r_j = k, r_i = 0} \frac{k!}{r_1! \cdots r_n!} \prod_{j \in [n] \setminus \{i\}} f(p_j(\ell), r_j) (\omega(i, j, \ell))^{r_j}.
\]

**Proof:** We shall first consider the special cases \( k = 0 \) and \( k = 1 \).

The case \( k = 0 \):
Clearly, \( 0M_i \left( \mathbf{p}_i^\ell, \mathbf{p}_{-i} \right) = \sum_{s \in S} \left( \mu_i(s) - E_i(\mathbf{p}) \right) \mathbf{p}(s) = \sum_{s \in S} \mathbf{p}(s) = 1 \). The formula has value
\[
\sum_{\langle r_1, \ldots, r_n \rangle \in \mathbb{N}_0^n \mid \sum_{j \in [n]} r_j = 0, r_i = 0} \frac{0!}{r_1! \cdots r_n!} \prod_{j \in [n] \setminus \{i\}} f(p_j(\ell), r_j) (\omega(i, j, \ell))^{r_j} = 1.
\]
The case $k = 1$: Clearly,

$$1M_1(p_i^*, p_{-i}) = \sum_{s \in S} (\mu_i(s) - E_i(p))^1 p(s) = 0.$$ 

The value of the formula is

$$\sum_{r_1, \ldots, r_n \mid \sum_{j \in [n]} r_j = 1, r_i = 0 \quad \& \quad \forall j \in [n]: r_j \neq 1} \frac{1!}{r_1! \ldots r_n!} \prod_{j \in [n] \setminus \{i\}} f(p_j(\ell), r_j) (\omega(i, j, \ell))^{r_j} = 0,$$

since there is no term to add.

Assume now that $k \geq 2$. The proof is by induction on the number of players $n$. For the basis case where $n = 1$, $kM_1(p_i^*, .) = 0$, and 0 is also the value given by the formula (since $r_1 = 0$ implies $\sum_j r_j \neq k$, and there is no term to add).

Assume inductively that the formula holds for $n - 1$ players. For the induction step, we shall establish the formula for $n$ players. Without loss of generality, fix $i := 1$. For simplicity, write $p_j$ and $\omega_j$ for $p_j(\ell)$ and $\omega(i, j, \ell)$, respectively. For any integer $t \leq k$, denote as $p|_t$ the restriction of $p$ to the players $1, \ldots, t$; so, $p|_1 = p$. Set $kM_1(n) := kM_1(p_i^*, (p|_t)_{-1})$; so, $kM_1(n) = kM_1(p_i^*, p_{-i})$. Clearly,

$$kM_1(n)$$

$$= \sum_{s \in S} p(s) \left( \mu_1(s) - E_1(p) \right)^k$$

$$= \sum_{s_{-n} \in S_{-n}} p(s_{-n}) \cdot \left( (\mu_1(s_{-n}) + w_n - E_1(p))^k \right) p_n + (\mu_1(s_{-n}) - E_1(p))^k (1 - p_n)$$

$$= \sum_{s_{-n} \in S_{-n}} p(s_{-n}) \cdot$$

$$\left( (\mu_1(s_{-n}) + w_n - E_1(p|_{n-1}) - p_n w_n)^k \right) p_n + (\mu_1(s_{-n}) - E_1(p|_{n-1}) - p_n w_n)^k (1 - p_n)$$

$$= \sum_{s_{-n} \in S_{-n}} p(s_{-n}) \cdot$$

$$\left( (\mu_1(s_{-n}) - E_1(p|_{n-1}) + w_n (1 - p_n))^k \right) p_n + (\mu_1(s_{-n}) - E_1(p|_{n-1}) - p_n w_n)^k (1 - p_n).$$
Set $B = B(s_{-n}) := \mu_1(s_{-n}) - E_1(p \mid (n - 1))$. Then,

$$kM_1(n) = \sum_{s_{-n} \in S_{-n}} p(s_{-n}) \left( (B + (1 - p_n) w_n)^k p_n + (B - p_n w_n)^k (1 - p_n) \right)$$

$$= \sum_{s_{-n} \in S_{-n}} p(s_{-n}) \left( \left( \sum_{t=0}^{k} \binom{k}{t} B^t (1 - p_n)^{k-t} (w_n)^{k-t} \right) p_n + \left( \sum_{t=0}^{k} \binom{k}{t} B^t (-p_n)^{k-t} (w_n)^{k-t} \right) (1 - p_n) \right)$$

$$= \sum_{s_{-n} \in S_{-n}} p(s_{-n}) \sum_{t=0}^{k} \binom{k}{t} B^t (p_n (1 - p_n)^{k-t} + (1 - p_n) (-p_n)^{k-t}) (w_n)^{k-t}$$

$$= \sum_{s_{-n} \in S_{-n}} p(s_{-n}) \sum_{t=0}^{k} \binom{k}{t} B^t f(p_n, k - t) (w_n)^{k-t}$$

$$= \sum_{t=0}^{k} \binom{k}{t} tM_1(n - 1) f(p_n, k - t) (w_n)^{k-t}.$$

By induction hypothesis, it follows that

$$kM_1(n)$$

$$= \sum_{t=0}^{k} \binom{k}{t} \left( \sum_{r_1, \ldots, r_{n-1} \mid \sum_{j \in [n-1]} r_j = t, r_1 = 0 \& \forall j \in [n-1] : r_j \neq 1} \frac{t!}{r_1! \ldots r_{n-1}!} \prod_{j \in [n-1] \setminus \{1\}} f(p_j, r_j) (\omega_j)^{r_j} \right) f(p_n, k - t) (w_n)^{k-t}$$

$$= \sum_{r_1, \ldots, r_{n-1} \mid \sum_{j \in [n-1]} r_j = t, r_1 = 0 \& \forall j \in [n-1] : r_j \neq 1} \sum_{t=0}^{k} \binom{k}{t} \frac{t!}{r_1! \ldots r_{n-1}!} \prod_{j \in [n-1] \setminus \{1\}} f(p_j, r_j) f(p_n, k - t) (\omega_j)^{r_j} (w_n)^{k-t}$$

$$= \sum_{r_1, \ldots, r_{n-1} \mid \sum_{j \in [n-1]} r_j = t, r_1 = 0 \& \forall j \in [n-1] : r_j \neq 1} \sum_{t=0}^{k} \frac{k!}{r_1! \ldots r_{n-1}!(k-t)!} \prod_{j \in [n-1] \setminus \{1\}} f(p_j, r_j) f(p_n, k - t) (\omega_j)^{r_j} (w_n)^{k-t}$$

$$= \sum_{r_1, \ldots, r_{n-1}, r_n \mid \sum_{j \in [n]} r_j = k, r_1 = 0 \& \forall j \in [n-1] : r_j \neq 1} \frac{k!}{r_1! \ldots r_{n-1}! r_n!} \prod_{j \in [n] \setminus \{1\}} f(p_j, r_j) (\omega_j)^{r_j}.$$
Since \( f(p_n, r_n) = 0 \) when \( r_n = 1 \), it follows that

\[
kM_1(n) = \sum_{r_1, \ldots, r_{n-1}, r_n} \frac{k!}{r_1! \cdots r_n!} \cdot \prod_{j \in [n] \setminus \{1\}} f(p_j, r_j) \cdot (\omega_j r_j),
\]
as needed. By Lemma 4.1

\[
f(p_j(1), r_j) = \begin{cases} f(1 - p_j(1), r_j), & \text{for even } r_j \\ -f(1 - p_j(1), r_j) & \text{if } r_j \text{ is odd} \end{cases}
\]

Since \( \sum_{j \in [n] \setminus \{i\}} r_j = k \) and \( k \) is even, the number of odd \( r_j \)'s is even, and this implies that \( \alpha_1^i = \alpha_2^i \).

### 4.2 The Mixed-Player-Has-Pure-Neighbors Property

We show:

**Proposition 4.3 (The Mixed-Player-Has-Pure-Neighbors Property)** Fix a player-specific scheduling game on two ordered links 1 and 2. Fix an \((E + R)\)-valuation \( V \), where \( R \) is either (1) \( \text{Var} \), or (2) \( \text{SD} \), or (3) a concave linear sum \( \sum_{k \in \{2,4,6,8\}} \alpha_k \cdot kM \), with \( \alpha_k \geq 0 \) for \( k \in \{2, 4, 6, 8\} \). Fix a \( V \)-equilibrium \( p \) and a player \( i \in [n] \). Then, either (C.1) player \( i \) is pure, or (C.2) all players \( i' \in [n] \setminus \{i\} \) with \( \omega(i, i', 1) \neq 0 \) are pure.

**Proof:** We first prove a key property of moment valuations \( kM \), where \( k \in \{2, 4, 6, 8\} \).

**Lemma 4.4** Fix a player-specific scheduling game on two ordered links 1 and 2. Fix the valuation \( kM \), for an even integer \( k \in \{2, 4, 6, 8\} \). Fix a mixed profile \( p \) and a player \( i \in [n] \). Then, one of the conditions (C.1), (C.2) and (C.3) holds: (C.1) Player \( i \) is pure. (C.2) All players \( i' \in [n] \setminus \{i\} \) with \( \omega(i, i', 1) \neq 0 \) are pure. (C.3) Player \( i \) is mixed with \( kM_i(p_i^1, p_{-i}) < kM_i(p_i^2, p_{-i}) \).

For the proof of Lemma 4.4, we shall employ a combinatorial Embracing Lemma (Lemma 4.5) to establish that the \( k \)-moment valuation increases strictly monotone for \( k \in \{6,8\} \); it seems that a different technique is needed to extend Lemma 4.4 beyond \( k = 8 \).

**Lemma 4.5 (Embracing Lemma)** Fix a pair of odd integers \( r \geq 3 \) and \( s \geq 3 \). Fix probabilities \( p \) and \( q \) with \( 0 < p < \frac{1}{2} \) and \( \frac{1}{2} < q < 1 \). Fix a weight \( w \) and a pair of numbers \( \alpha, \beta \in \mathbb{R}^+ \)
with $\alpha \cdot \beta \geq \frac{1}{2}$. Then, the function

$$F(y) := \alpha \cdot \frac{1}{(r-1)!} \cdot \frac{1}{(s+1)!} f(p, r-1) \cdot f(q, s+1) w^{s+1} y^{r-1}$$

$$+ \frac{1}{r!} f(p, r) \cdot f(q, s) w^r y^r$$

$$+ \beta \cdot \frac{1}{(r+1)!} \cdot \frac{1}{(s-1)!} f(p, r+1) \cdot f(q, s-1) w^{s-1} y^{r+1}$$

increases strictly monotone in $y$.

The *Embracing Lemma* establishes that a triple of “adjacent” partitions in the formula from Proposition 4.2 increases strictly monotone; intuitively, the terms corresponding to two of the partitions in the triple are positive and “help out” by embracing the third negative term to counterbalance its negative effect to increasing monotonicity.

**Proof:** The proof of Lemma 4.5 uses an elementary observation:

**Observation 4.1** Consider an odd integer $r \geq 3$. Then, for all $x \in [0, 1]$,

$$f(x, r-1) \cdot f(x, r+1) > (f(x, r))^2.$$

**Proof:** The claim amounts to

$$x^2 (1-x)^2 \left( x^{r-2} + (1-x)^{r-2} \right) \cdot (x^r + (1-x)^r) > x^2(1-x)^2 \left( (1-x)^r - x^r \right),$$

which holds trivially.

We continue with the proof of the *Embracing Lemma*. Write

$$F(y) := a \cdot y^{r-1} - b \cdot y^r + c y^{r+1},$$

with

$$a := \alpha \cdot \frac{1}{(r-1)!} \cdot \frac{1}{(s+1)!} f(p, r-1) \cdot f(q, s+1) w^{s+1},$$

$$b := -\frac{1}{r!} f(p, r) \cdot f(q, s) w^r,$$

$$c := \beta \cdot \frac{1}{(r+1)!} \cdot \frac{1}{(s-1)!} f(p, r+1) \cdot f(q, s-1) w^{s-1}.$$

Note that $a, b, c > 0$. Clearly,

$$F'(y) = (r-1) a y^{r-2} - r b y^{r-1} + (r+1) c y^r$$

$$= y^{r-2} \cdot \left( (r-1) a - r b y + (r+1) c y^2 \right) := h(y).$$
We shall prove that \( h'(y) > 0 \) for all \( y \geq 0 \). Clearly,
\[
h'(y) = -rb + 2(r + 1)c y,
\]
and
\[
h''(y) = 2(r + 1)y > 0.
\]
Thus, \( h(y) \) takes its minimum value for \( y_0 = \frac{rb}{2(r + 1)c} \). Hence, it suffices to prove that \( h(y_0) > 0 \).

Clearly,
\[
h(y_0) = (r - 1)a - rb \frac{rb}{2(r + 1)c} + (r + 1)c \left( \frac{rb}{2(r + 1)c} \right)^2
\]
\[= (r - 1)a - \frac{r^2}{4(r + 1)} \frac{b^2}{c};\]
thus, \( h(y_0) > 0 \) if and only if
\[
\frac{4ac}{b^2} > \frac{r^2}{r^2 - 1}.
\]
We now verify the latter condition. Note that
\[
\frac{4ac}{b^2} = 4\alpha \beta \frac{r!}{(r - 1)! (r + 1)!} \frac{r! (s - 1)! (s + 1)!}{(f(p, r - 1))^2 (f(p, r + 1))^2 (f(q, s - 1))^2 (f(q, s + 1))^2}.
\]
By Observation 4.1, it follows that
\[
\frac{4ac}{b^2} > 2 \cdot \frac{r}{r + 1} \cdot \frac{s}{s + 1}
\]
\[\geq \frac{r}{r + 1} \cdot \frac{3}{2}
\]
\[\geq \frac{r^2}{r^2 - 1},
\]
since \( s \geq 3 \) and \( r \geq 3 \).

We now continue with the proof of Lemma 4.4.

Proof: If Condition (C.1) holds, then we are done. So, assume that Condition (C.1) does not hold, so that player \( i \) is mixed. If Condition (C.3) holds, then we are done. So, assume that Condition (C.3) does not hold. Since player \( i \) is mixed, this implies that \( kM_i(p_i^1, p_{-i}) \geq kM_i(p_i^2, p_{-i}) \). We shall establish Condition (C.2). We proceed by case analysis.

The cases \( k = 2 \) and \( k = 4 \): By Proposition 4.2, for each link \( \ell \in [2] \),
\[
2M_i(p_i^\ell, p_{-i}) = \sum_{j \in [n] \setminus \{i\} | \omega(i, j, \ell) \neq 0} f(p_j(\ell), 2) (\omega(i, j, \ell))^2.
\]
and

\[ 4M_i \left( p_i^1, p_{-i} \right) = \sum_{j \in [n] \setminus \{i\}} f((p_j(\ell), 4) (\omega(i, j, \ell))^4 \]
+ \sum_{j, k \in [n] \setminus \{i\}, j \neq k, \omega(i, j, \ell) \neq 0, \omega(i, k, \ell) \neq 0} f(p_j(\ell) - 4M \cdot f(p_k(\ell), 2) (\omega(i, j, \ell) \omega(i, k, \ell))^2. \]

Since (i) \( p_j(\ell) = 1 - p_j(\ell) \), (ii) \( f(x, 2) = f(1 - x, 2) \) for \( x \in [0, 1] \), & (iii) for each player \( j \in [n] \setminus \{i\}, \omega(i, j, 1) \neq 0 \) if and only if \( \omega(i, j, 2) \neq 0 \), it follows that

\[ \frac{2M_i \left( p_i^1, p_{-i} \right) - 2M_i \left( p_i^2, p_{-i} \right)}{\geq 0} = \sum_{j \in [n] \setminus \{i\}} f(p_j(\ell), 2) \cdot \left( (\omega(i, j, 1))^2 - (\omega(i, j, 2))^2 \right). \]

and

\[ 4M_i \left( p_i^1, p_{-i} \right) - 4M_i \left( p_i^2, p_{-i} \right) = \sum_{j \in [n] \setminus \{i\}} f((p_j(1), 4) \cdot \left( (\omega(i, j, 1))^4 - (\omega(i, j, 2))^4 \right) \]
+ \sum_{j, k \in [n] \setminus \{i\}, j \neq k, \omega(i, j, 1) \omega(i, j, 2) \omega(i, k, 1) \omega(i, k, 2) \neq 0} f(p_j(1), 2) \cdot f(p_k(1), 2) \cdot \left( (\omega(i, j, 1))^2 (\omega(i, k, 1))^2 - (\omega(i, j, 2))^2 (\omega(i, k, 2))^2 \right). \]

Since the two links are ordered, it follows that for each player \( j \in [n] \setminus \{i\}, f(p_j(1), 2) = f(p_j(1), 4) = 0 \), which implies that \( p_j(1) = 1 \) and player \( j \) is pure, as needed for Condition (C.2).

For the remaining cases \( k \in \{6, 8\} \), we shall be establishing that \( kM_i \) increases strictly monotone in the weights. Since the two links are ordered, the inequality \( kM_i \left( p_i^1, p_{-i} \right) \geq kM_i \left( p_i^2, p_{-i} \right) \) implies that for each player \( j \in [n] \setminus \{i\} \) with \( \omega(i, j, 1) \neq 0 \) and \( \omega(i, j, 2) \neq 0 \), \( f(p_j(1), r_j) = f(p_j(1), r_j) = 0 \), which implies that \( p_j(1) = 1 \) and player \( j \) is pure, as needed for Condition (C.2). We shall be referring again to the formula from Proposition [4.2].

**The case \( k = 6 \)**: Note that 6 can be partitioned as 6, (4, 2), (3, 3), (2, 4) and (2, 2, 2). The terms of the formula corresponding to the partitions 6 and (2, 2, 2) have strictly positive coefficients; so, they are increasing strictly monotone in the weights. Now group together the terms of the
formula corresponding to the partitions with \((r_j, r_k) \in \{(4, 2), (3, 3), (2, 4)\}\) in a single sum. Lemma 1.5 (with \(\alpha = \beta = 1\)) implies that the sum increases strictly monotone in the weights.

The case \(k = 8\): The only partitions of 8 that use odd numbers are \((i)\) \((5, 3)\) and \((3, 5)\), and \((ii)\) \((3, 3, 2), (3, 2, 3)\) and \((2, 3, 3)\). Partitions in \((i)\) involve two strategies, while partitions in \((ii)\) involve three strategies.

Case \((i)\): Let \(j\) and \(k\) be the two strategies with \(0 < p_j(\ell) < \frac{1}{2}\) and \(\frac{1}{2} < p_k(\ell) < 1\) for a link \(\ell \in [2]\); so, \(f(p_j(\ell), 3) \cdot f(p_k(\ell), 5) < 0\) and \(f(p_j(\ell), 5) \cdot f(p_k(\ell), 3) < 0\).

Consider now the terms of the formula corresponding to the partitions with

\[(r_j, r_k) \in \{(6, 2), (5, 3), (4, 4), (3, 5), (2, 6)\}\]

First group together the terms corresponding to the partitions with

\[(r_j, r_k) \in \{(6, 2), (5, 3), (4, 4)\}\]

in a single sum, and invoke Lemma 1.5 with \(\alpha = 1\) and \(\beta = \frac{1}{2}\); it follows that the sum increases strictly monotone in the weights. Then, group together the terms corresponding to the partitions with

\[(r_j, r_k) \in \{(2, 6), (3, 5), (4, 4)\}\]

in a single sum, and invoke Lemma 1.5 with \(\alpha = 1\) and \(\beta = \frac{1}{2}\); it follows that the sum increases strictly monotone in the weights.

Case \((ii)\): Let \(j, k\) and \(\ell\) be the three strategies with \(0 < p_j(1) < \frac{1}{2}\) and \(\frac{1}{2} < p_k(1) < 1\) for a link \(\ell \in [2]\); assume, without loss of generality, that \(\frac{1}{2} < p_k(\ell) < 1\); so, \(f(p_j(\ell), 3) \cdot f(p_k(\ell), 3) < 0\) and \(f(p_j(\ell), 3) \cdot f(p_k(\ell), 3) < 0\), while \(f(p_j(\ell), 3) \cdot f(p_k(\ell), 3) > 0\). Consider now the terms of the formula corresponding to the partitions with

\[(r_j, r_k, r_\ell) \in \{(4, 2, 2), (3, 3, 2), (2, 4, 2), (3, 2, 3), (2, 2, 4)\}\]

In the same way as for Case \((i)\), Lemma 1.5 implies that the sum of the terms corresponding to these partitions increases strictly monotone in the weights.

We continue with the proof of Proposition 1.3. Consider the \(k\)-moment valuation \(kM\) with \(k \in \{2, 4, 6, 8\}\). So, Lemma 1.3 applies. Assume, by way of contradiction, that Condition (C.3) holds; then, \(kM_i(p_i^1, p_{-i}) < kM_i(p_i^2, p_{-i})\), which implies, by definition of \(R\), that \(R_i(p_i^1, p_{-i}) < R_i(p_i^2, p_{-i})\). By the Weak-Equilibrium-for-Expectation property, \(E_i(p_i^1, p_{-i}) = E_i(p_i^2, p_{-i})\). Hence, \(V_i(p_i^1, p_{-i}) < V_i(p_i^2, p_{-i})\). Since \(V\) is concave, the Optimal-Value property (Proposition 3.1) implies that \(V_i(p_i^1, p_{-i}) = V_i(p_i^2, p_{-i})\). A contradiction. Hence, by Lemma 1.4 either (C.1) player \(i\) is pure, or (C.2) all players \(i' \in [n] \setminus \{i\}\) with \(\omega(i, i', 1) \neq 0\) are pure. 

\[22\]
4.3 $NP$-Hardness Result

We give an example of a player-specific scheduling game $G$ with $n = 3$ players $0$, $1$ and $2$ on two ordered links $1$ and $2$, with no $V$-equilibrium for an $(E + R)$-valuation $V$, where $R$ is (1) Var, or (2) SD, or (3) a concave linear sum $\sum_{k \in \{2,4,6,8\}} \alpha_k \cdot kM$, with $\alpha_k \geq 0$ for $k \in \{2,4,6,8\}$. For $i \in \{0,1,2\}$ and $\ell \in [2]$, set

$$\omega(i, i, \ell) := 0;$$
$$\omega(i, (i + 1) \mod 3, \ell) := \delta_{\ell 2};$$
$$\omega(i, (i + 2) \mod 3, \ell) := 2 + \delta_{\ell 2}. $$

($\delta$ is the Kronecker delta: $\delta_{\ell 2} = 1$ if $\ell = 2$ and 0 otherwise.) Assume, by way of contradiction, that $G$ has a pure equilibrium. If the three players are on the same link, then each player has cost greater than 0 and can reduce her cost by switching to the other link. If two of the players are on the same link $\ell \in [2]$ while the third player is on link $\ell$; her cost on link $\ell$ is $2 + \delta_{\ell 2}$, and she can reduce her cost to $\delta_{\ell 2}$ by switching to link $\ell$. A contradiction in both cases. Finally, assume that there is a $V$-equilibrium $p$ where some player $i \in \{0,1,2\}$ is mixed. By the Mixed-Player-Has-Pure-Neighbors property (Proposition 4.3), players $(i + 1) \mod 3$ and $(i + 2) \mod 3$ are pure. By the Weak-Equilibrium-for-Expectation property (Corollary 3.3 and Proposition 3.4), $E_i (p^1_i, p_{-i}) = E_i (p^2_i, p_{-i})$ or $\sum_{i' \neq i | p_{i'} = p^1_i} \omega(i, i', 1) = \sum_{i' \neq i | p_{i'} = p^2_i} \omega(i, i', 2)$. By the definition of weights, a contradiction follows. Thus, $\exists V$-EQUILIBRIUM is non-trivial in the considered setting. We show:

**Theorem 4.6** Fix an $(E + R)$-valuation $V$, where $R$ is (1) Var, or (2) SD, or (3) a concave linear sum $\sum_{k \in \{2,4,6,8\}} \alpha_k \cdot kM$, with $\alpha_k \geq 0$ for $k \in \{2,4,6,8\}$. Then, $\exists V$-EQUILIBRIUM is strongly $NP$-hard for player-specific scheduling games on two ordered links.

The proof will use a reduction from MULTIBALANCED PARTITION, a problem we introduce and show strongly $NP$-complete:

**I.** $\langle n, m, A \rangle$, with integers $n$, $m$ and a set $A = \{a_{ij} | i \in [n], j \in [m]\}$.

**Q.** Is there a subset $I \subset [n]$ such that for each $j \in [m]$, $\sum_{i \in I} a_{ij} = 3 + 2 \sum_{i \notin I} a_{ij}$?

We show:

**Proposition 4.7** MULTIBALANCED PARTITION is strongly $NP$-complete.

The proof of Proposition 4.7 employs a reduction from 3-DIMENSIONAL MATCHING [16 SP1]:
\[ \langle W, X, Y, M \rangle, \text{ where } M \text{ is a set with } M \subseteq W \times X \times Y \text{ and } W, X \text{ and } Y \text{ are disjoint sets with } |W| = |X| = |Y| = q. \]

Q.: Does \( M \) contain a matching, i.e., a subset \( M' \subset M \) with \( |M'| = q \) such that no two elements of \( M' \) agree in any coordinate?

**Proof:** Clearly, \textsc{Multibalanced Partition} belongs to \( \mathcal{NP} \). For the \( \mathcal{NP} \)-hardness we shall employ a reduction from \textsc{3-Dimensional Matching}.

Given an instance \( \langle W, X, Y, M \rangle \) of \textsc{3-Dimensional Matching}, with \( W = \{w_1, \ldots, w_q\}, X = \{x_1, \ldots, x_q\}, Y = \{y_1, \ldots, y_q\} \) and \( M = \{m_1, \ldots, m_k\} \), where for each \( i \in [k], m_i = (w_{f(i)}, x_{g(i)}, y_{h(i)}) \) with functions \( f, g, h : [k] \to [q] \) giving the first, second and third coordinate, respectively, of each element \( m_i \in M \), we construct an instance \( \langle n, m, A \rangle \) of \textsc{Multibalanced Partition} as follows:

- \( n := k + 1 \) and \( m := 3q. \)
- For \( 1 \leq i \leq k \) and \( 1 \leq j \leq m, \)
  \[ a_{ij} := \begin{cases} 1, & \text{if } (1 \leq j \leq q \text{ and } j = f(i)) \\ \text{or } (q + 1 \leq j \leq 2q \text{ and } j - q = g(i)) \\ \text{or } (2q + 1 \leq j \leq 3q \text{ and } j - 2q = h(i)) \end{cases}, \]
  and for \( 1 \leq j \leq m, \)
  \[ a_{k+1,j} := 2b_j, \]
  where
  \[ b_j := \sum_{i \in [k]} a_{ij}. \]

We prove:

**Lemma 4.8** \( \langle W, X, Y, M \rangle \) has a solution if and only if \( \langle n, m, A \rangle \) has a solution.

**Proof:** \( \Rightarrow \): Assume first that \( \langle W, X, Y, M \rangle \) has a solution \( M' \). Set
  \[ l' := \{ i \in [k] \mid m_i \in M' \}. \]
Then, clearly, for each \( j \in [m], \sum_{i \in l'} a_{ij} = 1. \) Hence, for each \( j \in [m], \)
  \[ \sum_{i \in [k] \setminus l'} a_{ij} = \sum_{i \in [k]} a_{ij} - \sum_{i \in l'} a_{ij} = b_j - 1. \]

\(^{14}\text{The reduction is very similar to the one used in the proof of [16, Theorem 3.5].}\)
Set now \( I := I' \cup \{ k + 1 \} \). Then, for each \( j \in [m] \),

\[
\sum_{i \in I} a_{ij} = \sum_{i \in I'} a_{ij} + a_{k+1,j} \\
= 1 + 2 b_j \\
= 3 + 2 (b_j - 1) \\
= 3 + 2 \sum_{i \in [k]\setminus I'} a_{ij} \\
= 3 + 2 \sum_{i \not\in I} a_{ij},
\]

so that \( I \) is a solution of \( \langle n, m, A \rangle \).

"\( \Leftarrow \)" Assume now that \( \langle n, m, A \rangle \) has a solution \( I \). We claim that \( k + 1 \in I \). Assume, by way of contradiction, that \( k + 1 \not\in I \). Then, for each \( j \in [m] \),

\[
3 + 2 \sum_{i \not\in I} a_{ij} \\
\geq 3 + 2 a_{k+1,j} \quad \text{(since \( k + 1 \not\in I \))} \\
= 3 + 4 b_j \quad \text{(by definition of \( a_{k+1,j} \))} \\
> b_j \\
= \sum_{i \in [k]} a_{ij} \quad \text{(by definition of \( b_j \))} \\
\geq \sum_{i \in I} a_{ij},
\]

a contradiction to the assumption that \( I \) is a solution of \( \langle n, m, A \rangle \). So, \( k + 1 \in I \).

Fix now an arbitrary \( j \in [m] \). Set \( \Delta_j := \sum_{i \in [k] \setminus \{k+1\}} a_{ij} \). Since \( I \) is a solution of \( \langle n, m, A \rangle \),

\[
\sum_{i \in I} a_{ij} = 3 + 2 \sum_{i \not\in I} a_{ij}.
\]

Since \( k + 1 \in I \), this implies that

\[
a_{k+1,j} + \sum_{i \in [k] \setminus \{k+1\}} a_{ij} = 3 + 2 \left( \sum_{i \in [k]} a_{ij} - \sum_{i \in I \setminus \{k+1\}} a_{ij} \right).
\]

Hence,

\[
2 b_j + \Delta_j = 3 + 2 (b_j - \Delta_j).
\]

It follows that \( \Delta_j = 1 \), so that \( I \setminus \{k + 1\} \) is a solution of \( \langle W, X, Y, M \rangle \).

Lemma 4.8 establishes the reduction for the \( \mathcal{NP} \)-hardness; since the number involved in the reduction are polynomially bounded, strong \( \mathcal{NP} \)-hardness follows.
For the proof of the reduction for Theorem 4.6, we shall use the Mixed-Player-Has-Pure-Neighbors property (Proposition 4.3) to identify the mixed players; in turn, we shall apply the Weak-Equilibrium-for-Expectation property to the mixed players to obtain a solution to MULTIBALANCED PARTITION or a V-equilibrium. We continue with the proof of Theorem 4.6.

Proof: We shall employ a reduction from MULTIBALANCED PARTITION. Given an instance \( \langle n, m, A \rangle \) of MULTIBALANCED PARTITION, we construct an instance \( \langle G \rangle \) of \( \exists V \)-EQUILIBRIUM as follows. \( G \) is a player-specific scheduling game on two ordered links with \( n + 5m \) players in the player set \( \Pi := [n] \cup \{ [k, j] | k \in [m], j \in \{0, \ldots, 4\} \} \). Set \( M := \max_k \sum_{j \in [n]} a_{kj} \). We assume that \( M \geq 4 \). We now define the weights \( \omega(\pi_1, \pi_2, \ell) \) where \( \pi_1, \pi_2 \in \Pi \) and \( \ell \in [2] \), where \( \delta \) is the Kronecker delta: \( \delta_{\ell \ell'} = 1 \) if \( \ell = \ell' \) and 0 otherwise.

1. \( \pi_1 = [k, j] \) with \( k \in [m] \) and \( j \in \{0, \ldots, 3\} \): Then,
   \[
   \omega([k, j], \pi_2, \ell) := \begin{cases} 
   M + \delta_{\ell 2}, & \text{if } \pi_2 = [k, i] \text{ with } i = 4 \text{ or } i \neq (j + 1) \text{ mod } 4 \\
   M - 4 + \delta_{\ell 2}, & \text{if } \pi_2 = [k, (j + 1) \text{ mod } 4] \\
   0, & \text{otherwise}
   \end{cases}
   \]

2. \( \pi_1 = [k, 4] \) with \( k \in [m] \): Then,
   \[
   \omega([k, 4], \pi_2, \ell) := \begin{cases} 
   M + \delta_{\ell 2}, & \text{if } \pi_2 = [k, i] \text{ with } i \in [4] \\
   a_{kj}, & \text{if } \pi_2 = [k, j] \text{ and } \ell = 1 \\
   2a_{kj}, & \text{if } \pi_2 = [k, j] \text{ and } \ell = 2 \\
   0, & \text{otherwise}
   \end{cases}
   \]

3. \( \pi_1 = i \in [n] \): Then, \( \omega(i, \pi_2, \ell) = 0 \) for all \( \pi_2 \in \Pi \) and \( \ell \in [2] \).

Note that \( G \) is a player-specific scheduling game on two ordered links. We now prove:

Lemma 4.9 In a V-equilibrium \( p \), for each \( k \in [m] \), there is an index \( j \in \{0, \ldots, 4\} \) such that player \([k, j]\) is non-pure.

Proof: Assume, by way of contradiction, that there is an index \( \hat{k} \in [m] \) such that all players \([\hat{k}, j]\) with \( j \in \{0, \ldots, 4\} \) are pure; so, \( p_{[\hat{k}, j]}[1] \in \{0, 1\} \) for all \( j \in \{0, \ldots, 4\} \). If there are at least four players \([\hat{k}, j]\) with \( j \in \{0, \ldots, 4\} \) choosing the same link, then at least one player \([\hat{k}, j]\) with \( j \in \{0, \ldots, 3\} \) can improve her cost by switching to the other link. So, assume that there are three players \([\hat{k}, j]\) with \( j \in \{0, \ldots, 4\} \) choosing some link \( \ell \in [2] \), while the remaining two players choose the other link. Then, there is an index \( j_0 \in \{0, \ldots, 3\} \) such that player \([\hat{k}, j_0]\)
chooses link $\ell$ while player $[k, (j_0 + 1) \mod 4]$ chooses link 7. So, $\mu_{[k, j_0]}(p) = 3M + 3\delta_{\ell_2}$, and player $[k, j_0]$ can improve by switching to link 7 where her cost becomes $3M - 4 + 3\delta_{\ell_2} < 3M + 3\delta_{\ell_2}$. A contradiction to the assumption that $p$ is a $V$-equilibrium.

We next prove:

**Lemma 4.10** In a $V$-equilibrium, each player $[k, j]$, with $k \in [m]$ and $j \in \{0, \ldots, 3\}$, is pure.

**Proof:** Assume, by way of contradiction, that there is a non-pure player $\pi = [k, j]$ with $k \in [m]$ and $j \in \{0, \ldots, 3\}$. Then, by the Mixed-Player-Has-Pure-Neighbors property (Proposition 4.3), all players $[k, j]$ with $j \in \{0, \ldots, 4\} \setminus \{j\}$ are pure. By the Weak-Equilibrium-for-Expectation property for player $\pi$, $E(\pi^1, p_{-\pi}) = E(\pi^2, p_{-\pi})$. For each link $\ell \in \{1, 2\}$, denote as $\Phi_\ell$ the set of players $[k, j]$ with $j \in \{0, \ldots, 4\} \setminus \{j\}$ choosing link $\ell$; set $y_\ell := 1$ if $[k, (j + 1) \mod 4] \in \Phi_\ell$ and 0 otherwise. Clearly, $|y_1 - y_2| = 1$. Then,

$$
E(\pi^1, p_{-\pi}) = |\Phi_1| \cdot M - 4y_1, \\
$$

and

$$
E(\pi^2, p_{-\pi}) = |\Phi_2| \cdot (M + 1) - 4y_2.
$$

We proceed by case analysis.

1. Assume first that $|\Phi_1| > |\Phi_2|$. So, $|\Phi_1| \geq 3$ and $|\Phi_2| \leq 1$. Then,

$$
E(\pi^1, p_{-\pi}) \geq 3M - 4y_1 > M + 1 - 4y_2 \geq E(\pi^2, p_{-\pi}),
$$

since $3M > M + 5 \geq M + 1 + 4(y_1 - y_2)$. A contradiction.

2. Assume now that $|\Phi_1| < |\Phi_2|$. So, $|\Phi_2| \geq 3$ and $|\Phi_1| \leq 1$. Then,

$$
E(\pi^2, p_{-\pi}) \geq 3M + 3 - 4y_2 > M - 4y_1 \geq E(\pi^1, p_{-\pi}),
$$

since $3M + 3 > M + 4 \geq M + 4(y_2 - y_1)$. A contradiction.

3. Assume finally that $|\Phi_1| = |\Phi_2|$. Then,

$$
E(\pi^1, p_{-\pi}) = 2M - 4y_1,
$$

and

$$
E(\pi^2, p_{-\pi}) = 2M + 2 - 4y_2.
$$

Since $E(\pi^1, p_{-\pi}) = E(\pi^2, p_{-\pi})$, it follows that $4(y_2 - y_1) = 2$. A contradiction.
The claim follows.

We are now ready to prove:

**Lemma 4.11** If $G$ has a \( V \)-equilibrium, then \( (n,m,A) \) has a solution.

**Proof:** Consider a \( V \)-equilibrium \( p \). By Lemmas 4.9 and 4.10 all players \([k,4]\) with \( k \in [m] \) are non-pure. Hence, the Mixed-Player-Has-Pure-Neighbors property (Proposition 3.3) and the definition of the weights imply together that all players \( i \in [n] \) and \([k,j]\) with \( k \in [m] \) and \( j \in \{0,\ldots,3\} \) are pure. Fix now an arbitrarily chosen player \( \pi = [k,4] \). For each link \( \ell \in [2] \), denote \( l_\ell := \{ i \in [n] \mid p_i(\ell) = 1 \} \) and \( \Phi_\ell := \{ [k,j] \mid j \in \{0,\ldots,3\} \} \) and \( p_{[k,j]}(\ell) = 1 \). Clearly,

\[
E_\pi(p_1^1, p_{-\pi}) = (1 + |\Phi_1|) \cdot M + \sum_{i \in l_1} a_i,
\]

and

\[
E_\pi(p_2^2, p_{-\pi}) = (1 + |\Phi_2|) \cdot (M + 1) + 2 \sum_{i \in l_2} a_i.
\]

The Weak-Equilibrium-for-Expectation property (Corollary 3.3 and Proposition 3.4) implies that \( E_\pi(p_1^1, p_{-\pi}) = E_\pi(p_2^2, p_{-\pi}) \). By the choice of \( M \), this implies that \( |\Phi_1| = |\Phi_2| = 2 \), so that

\[
\sum_{i \in l_1} a_i = 3 + 2 \sum_{i \in l_2} a_i.
\]

Hence, \( l \) is a solution of \( (n,m,A) \).

We finally prove:

**Lemma 4.12** If \( (n,m,A) \) has a solution, then $G$ has a $V$-equilibrium.

For the proof of Lemma 4.12, we shall use a particular monotonicity property of the risk valuation \( R = kM \) with an even integer \( k \geq 2 \). Note that

\[
\tilde{R}(a,b,q) = (1-q) \cdot (a - ((1-q) \cdot a + q \cdot b))^k + q \cdot (b - ((1-q) \cdot a + q \cdot b))^k
\]

\[
= (1-q) \cdot q^k \cdot (b-a)^k + q \cdot (1-q)^k \cdot (b-a)^k
\]

\[
= q \cdot (1-q) \cdot (b-a)^k \cdot (q^{k-1} + (1-q)^{k-1}).
\]

This implies that all valuations $R$ addressed in Theorem 4.6 incur $\tilde{R}(a,b,q)$ which increases monotonically in $b-a$ for a fixed probability $q \in (0,1)$. In fact, the function $\tilde{R}(a,b,q)$ with $a \leq b$ is a function $\tilde{\tilde{R}}(b-a,q) := \tilde{R}(a,b,q)$ in $b-a$; the function $\tilde{\tilde{R}}(b-a,q)$ increases monotonically in $b-a$ for a fixed $q \in (0,1)$ and satisfies $\tilde{\tilde{R}}(b-a,q) = \tilde{\tilde{R}}(b-a,1-q)$. For the proof of Lemma 4.12 we shall refer to these two properties together as the Two-Values Risk-Monotonicity property.

We continue with the proof of Lemma 4.12.
Proof: Consider a solution \( I \subset [n] \) of \( \langle n, m, A \rangle \). Define a mixed profile \( p \) as follows:

- For each \( i \in [n] \), \( p_i(1) := 1 \) if \( i \in I \), and 0 otherwise.
- For \( 1 \leq k \leq m \): \( p_{[k,0]}(1), p_{[k,2]}(1) := 1; p_{[k,1]}(1), p_{[k,3]}(1) := 0; p_{[k,4]}(2) := x \in (0,1) \).

We now prove that \( x \) can be chosen so that \( p \) is a \( V \)-equilibrium. We proceed by case analysis.

1. Players \( i \in [n] \) cannot improve since \( \omega(i, \pi, \ell) = 0 \) for all \( \pi \in \Pi \) and \( \ell \in [2] \).

2. Consider now players \([k, r]\) with \( k \in [m] \) and \( r \in \{0, 1, 2, 3\} \). Note that for fixed \( k \) and \( r \), \( \mu_{[k,r]}(s) \) takes only two values over all profiles \( s \in \mathcal{S} \) with \( p(s) > 0 \). (Observe for this that player \( \pi = [k, 4] \) is the only player with \( 0 < p_{\pi}(1) < 1 \) and \( \omega([k, r], \pi, \ell) \neq 0 \) for \( r \in \{0, 1, 2, 3\} \) and \( \ell \in [2] \).) Denote as \( A_{kr} \) and \( B_{kr} \) the two values taken by \( \mu_{[k,r]}(s) \) over all profiles \( s \in \mathcal{S} \) with \( p(s) > 0 \), with \( A_{kr} > B_{kr} \). By the cost functions and the definition of \( p \), we get:

   - For \( r \in \{0, 2\} \): \( A_{kr} = 3M \) and \( B_{kr} = 2M \), so that
     \[
     E_{[k,r]}(p) = (1 - x) \cdot A_{kr} + x \cdot B_{kr} = (3 - x) \cdot M ,
     \]
     and, using the Two-Values Risk-Monotonicity property,
     \[
     R_{[k,r]}(p) = \hat{R}_{[kr]}(A_{kr}, B_{kr}, 1 - x) = \hat{R}_{[kr]}(3M, 2M, 1 - x) = \tilde{R}_{[kr]}(M, x) .
     \]

   - For \( r \in \{1, 3\} \): \( A_{kr} = 3(M + 1) \) and \( B_{kr} = 2(M + 1) \), so that
     \[
     E_{[k,r]}(p) = x \cdot A_{kr} + (1 - x) \cdot B_{kr} = (2 + x) \cdot (M + 1) ,
     \]
     and, using the Two-Values Risk-Monotonicity property,
     \[
     R_{[k,r]}(p) = \hat{R}_{[kr]}(A_{kr}, B_{kr}, x) = \hat{R}_{[kr]}(3(M + 1), 2(M + 1), x) = \tilde{R}_{[kr]}(M + 1, x) .
     \]
Consider now the cost of player \([k, r]\) with \(r \in \{0, 1, 2, 3\}\) when she switches to the other link. Denote as \(\hat{p}\) the corresponding mixed profile. As in the case of the mixed profile \(p\), \(\mu_{[k, r]}(s)\) takes only two values over all profiles \(s \in S\) with \(\hat{p}(s) > 0\). Denote as \(C_{kr}\) and \(D_{kr}\) the two values taken by \(\mu_{[k, r]}(s)\) over all profiles \(s \in S\) with \(\hat{p}(s) > 0\), with \(C_{kr} > D_{kr}\).

By the cost functions and the definition of \(\hat{p}\), we get:

- For \(r \in \{0, 2\}\): \(C_{kr} = 4(M + 1) - 4 = 4M\) and \(D_{kr} = 3(M + 1) - 4 = 3M - 1\), so that
  \[
  \mathbb{E}_{[k, r]}(\hat{p}) = x \cdot C_{kr} + (1 - x) \cdot D_{kr} = 3M - 1 + x \cdot (M + 1),
  \]
  and, using the Two-Values Risk-Monotonicity property,
  \[
  R_{[k, r]}(\hat{p}) = \tilde{R}_{[k, r]}(C_{kr}, D_{kr}, 1 - x) = \tilde{R}_{[k, r]}(4M, 3M - 1, 1 - x) = \tilde{R}_{[k, r]}(M + 1, x).
  \]

- For \(r \in \{1, 3\}\): \(C_{kr} = 4M - 4\) and \(D_{kr} = 3M - 4\), so that
  \[
  \mathbb{E}_{[k, r]}(\hat{p}) = x \cdot D_{kr} + (1 - x) \cdot C_{kr} = 4M - 4 - x \cdot M,
  \]
  and, using the Two-Values Risk-Monotonicity property,
  \[
  R_{[k, r]}(\hat{p}) = \tilde{R}_{[k, r]}(C_{kr}, D_{kr}, x) = \tilde{R}_{[k, r]}(4M - 4, 3M - 4, x) = \tilde{R}_{[k, r]}(M, x).
  \]

We now determine a probability \(x \in (0, 1)\) so that for all players \([k, r]\) with \(r \in \{0, 1, 2, 3\}\), \(V_{[k, r]}(p) \leq V_{[k, r]}(\hat{p})\). We proceed by case analysis.

- For \(r \in \{0, 2\}\): Then, \(V_{[k, r]}(p) \leq V_{[k, r]}(\hat{p})\) if and only if
  \[
  (3 - x) \cdot M + \tilde{R}_{[k, r]}(M, x) \leq 3M - 1 + x \cdot (M + 1) + \tilde{R}_{[k, r]}(M + 1, x)
  \]
  if and only if
  \[
  1 \leq x \cdot (2M + 1) + \tilde{R}_{[k, r]}(M + 1, x) - \tilde{R}_{[k, r]}(M, x).
  \]
For \( r \in \{1, 3\} \): Then, \( V_{[k,r]}(p) \leq V_{[k,r]}(\hat{p}) \) if and only if

\[
(2 + x) \cdot (M + 1) + \tilde{R}_{[k,r]}(M + 1, x) \leq 4M - 4 - x \cdot M + \tilde{R}_{[k,r]}(M, x)
\]

if and only if

\[
x \cdot (2M + 1) + \tilde{R}_{[k,r]}(M + 1, x) - \tilde{R}_{[k,r]}(M, x) \leq 2M - 6.
\]

Set

\[ h(x) := x \cdot (2M + 1) + \tilde{R}_{[k,r]}(M + 1, x) - \tilde{R}_{[k,r]}(M, x). \]

We shall determine a probability \( x \) so that \( 1 \leq h(x) \leq 2M - 6 \). (Since \( M \geq 4, 2M - 6 \geq 1 \).)

Observe that \( h(0) = 0 \). Set

\[ \hat{x} := \frac{1}{2M + 1}. \]

By the Two-Values Risk-Monotonicity property, \( \tilde{R}_{[k,r]}(M + 1, x) > \tilde{R}_{[k,r]}(M, x) \). This implies that \( h(\hat{x}) > 1 \). If \( h(\hat{x}) \leq 2M - 6 \), then set \( x := \hat{x} \) and we are done. If \( h(\hat{x}) > 2M - 6 \), then the continuity of \( R \) implies that there is a probability \( \bar{x} \in (0, \hat{x}) \) such that \( 1 \leq h(\bar{x}) \leq 2M - 6 \), and we are done.

3. Finally consider players \([k, 4]\) with \( k \in [m] \). By the definition of \( p \),

\[
E_{[k, 4]} \left( p_{[k, 4]}^1, p_{-[k, 4]} \right) = 3M + \sum_{i \in I} a_{ki},
\]

and

\[
E_{[k, 4]} \left( p_{[k, 4]}^2, p_{-[k, 4]} \right) = 3(M + 1) + 2 \sum_{i \not\in I} a_{ki}.
\]

Since \( I \) is a solution of \( \langle n, m, A \rangle \), we get that

\[
E_{[k, 4]} \left( p_{[k, 4]}^1, p_{-[k, 4]} \right) = E_{[k, 4]} \left( p_{[k, 4]}^1, p_{-[k, 4]} \right).
\]

By the Risk-Positivity property, this implies that \( R_{[k, 4]}(p) = 0 \) so that \( V_{[k, 4]}(p) = E_{[k, 4]}(p) \). Since \( p \) is a fully mixed profile (since \( 0 < x < 1 \)) and \( V \) has the Weak-Equilibrium-for-Expectation property (Corollary 3.3 and Proposition 3.4), \( E_{[k, 4]}(p) \) cannot decrease; hence, neither \( V_{[k, 4]}(p) \) can.

It follows from the case analysis that \( p \) is a \( V \)-equilibrium.

Lemmas 4.11 and 4.12 establish the reduction for the \( \mathcal{NP} \)-hardness; since the numbers involved in the reduction are polynomially bounded, strong \( \mathcal{NP} \)-hardness follows.
5 Two Players

Consider a concave valuation $V^\nu$, for an increasing and strictly convex function $\nu$. Since $\nu^{-1}$ is also increasing, a mixed profile $p$ is a $V^\nu$-equilibrium for a game $G$ if and only if $p$ is an $E$-equilibrium for the game $G^{\nu}$ constructed from $G$ by setting for each player $i \in [n]$ and profile $s \in S$, $\mu^\nu_i(s) := \nu(\mu_i(s))$. Since every game has an $E$-equilibrium \cite{24, 25}, this implies that there is a $V^\nu$-equilibrium for $G$, and the associated search problem for a $V^\nu$-equilibrium is total; it is in $\text{PPPAD}$ \cite{26} for 2-players games. Nevertheless, we shall show that there are other (concave) valuations $V$ for which deciding the existence of a $V$-equilibrium is strongly $\text{NP}$-hard for 2-players games.

5.1 General $\text{NP}$-Hardness Result

We show:

\textbf{Theorem 5.1} Fix an $(E + R)$-valuation $V$ such that:

1. $V$ has the Weak-Equilibrium-for-Expectation property.
2. There is a polynomial time computable $\delta$ with $0 < \delta \leq \frac{1}{4}$ such that:
   1. $\hat{R}(1, 1 + 2\delta, q) < \frac{1}{2}$ for each probability $q \in [0, 1]$.
   2. $\hat{V}(1, 1 + 2\delta, r) < \hat{V}(1, 2, q)$ for all $0 \leq r \leq q \leq 1$.
   3. The Crawford game $G_C(\delta)$ with bimatrix
      \[
      \begin{pmatrix}
      \langle 1 + \delta, 1 + \delta \rangle & \langle 1, 1 + 2\delta \rangle \\
      \langle 1, 1 + 2\delta \rangle & \langle 1 + 2\delta, 1 \rangle
      \end{pmatrix}
      \]
      has no $V$-equilibrium.

Then, $\exists V$-EQUILIBRIUM is strongly $\text{NP}$-hard for 2-players games.

We present a general proof with a reduction involving the parameter $\delta$ from Condition (2), required to be polynomial time computable. The reduction uses the Crawford game $G_C(\delta)$ as a “gadget”; for any $\delta$ with $0 < \delta < 1$, $G_C(\delta)$ is an adapted generalization of a bimatrix game from \cite{7} Section 4]. The parameter $\delta$ enters the reduction through $G_C(\delta)$.

Specifically, the proof of Theorem 5.1 employs a reduction from SAT \cite{16 L01}. An instance of SAT is a propositional formula $\phi$ in the form of a conjunction of clauses $C = \{c_1, \ldots, c_k\}$ over a set of variables $V = \{v_1, \ldots, v_m\}$. Denote as $L = \{\ell_1, \overline{v}_1, \ldots, \ell_m, \overline{v}_m\}$ the set of literals corresponding to the variables in $V$. We shall use lower-case letters $c, c_1, c_2, \ldots, v, v_1, v_2, \ldots$, and $\ell, \ell_1, \ell_2, \ldots$ to denote clauses from $C$, variables from $V$ and literals from $L$, respectively.
Denote $\Lambda := C \cup V \cup L$. We shall use the Crawford set $\mathcal{F} = \{f_1, f_2\}$ with two strategies $f_1$ and $f_2$; $f$ denotes either $f_1$ or $f_2$. The cost values are chosen judiciously so as to carefully assure or exclude the existence of a $V$-equilibrium. We continue with the proof of Theorem 5.1.

**Proof:** Given an instance $\phi$ of SAT, construct a game $G = G(\phi) = \langle [2], \{S_i\}_{i \in [2]}, \{\mu_i\}_{i \in [2]} \rangle$ as follows. For each player $i \in [2]$, $S_i := \Lambda \cup \mathcal{F}$. The cost functions $\{\mu_i\}_{i \in [2]}$ are given in Figure 1.

![Figure 1: The cost functions for the game $G$.](image)

For each player $i \in [2]$, $S_i := \Lambda \cup \mathcal{F}$. The cost functions $\{\mu_i\}_{i \in [2]}$ are given in Figure 1.

For a player $i \in [2]$, denote $p_i(\mathcal{F}) := \sum_{\ell \in \mathcal{F}} p_i(\ell)$, $p_i(L) := \sum_{\ell \in L} p_i(\ell)$ and $p_i(\Lambda) := \sum_{\lambda \in \Lambda} p_i(\lambda)$; note that $p_i(\mathcal{F}) + p_i(\Lambda) = 1$. We prove a sequence of technical claims:

**Lemma 5.2** In a $V$-equilibrium $\langle p_1, p_2 \rangle$ for $G$, $p_1(\Lambda) \cdot p_2(\Lambda) > 0$.

**Proof:** Assume, by way of contradiction, that $p_i(\Lambda) = 0$ for some player $i \in [2]$; so, $p_i(\mathcal{F}) = 1$. For easier notation, fix $i := 1$. (The proof is the same for $i := 2$.) By Condition (2/c), the Crawford game $G_C(\delta)$ has no $V$-equilibrium. Hence, it follows that $p_2(\Lambda) > 0$. We proceed by case analysis on $p_2(\mathcal{F})$.

1. Assume first that $p_2(\mathcal{F}) > 0$; so, $p_2(\Lambda) < 1$. Fix a strategy $f \in \mathcal{F}$ with $p_2(f) > 0$. By the cost functions, $\mu_2(f', f) \leq 1 + 2\delta < 2$ for each strategy $f' \in \mathcal{F}$. Hence, $E_2(p_1, p_2' \lambda) < 2$ since $\delta < \frac{1}{2}$. Fix now a strategy $\lambda \in \Lambda$ with $p_2(\lambda) > 0$. By the cost functions, $\mu_2(f_1, \lambda) = 2$ for
each strategy $f_1 \in \mathcal{F}$. Hence, $E_2(p_1, p^f_2) = 2$. By the Weak-Equilibrium-for-Expectation
property for player 2, $E_2(p_1, p^f_2) = E_2(p_1, p^\lambda_2)$. A contradiction.

2. Assume now that $p_2(\mathcal{F}) = 0$; so, $p_2(\Lambda) = 1$. By the cost functions, $\mu_2(f, \lambda) = 2$ for each pair of strategies $f \in \mathcal{F}$ and $\lambda \in \Lambda$. Hence, $E_2(p_1, p_2) = 2$. It follows by the Risk-Positivity
property that $R_2(p_1, p_2) = 0$, so that $V_2(p_1, p_2) = 2$. Consider now player 2 switching to the strategy $f_2 \in \mathcal{F}$. By Condition (2/a),

$$R_2\left(p_1, p^f_2\right) = \hat{R}_2(1, 1 + 2\delta, p_1(f_1)) < \frac{1}{2},$$

Thus,

$$E_2\left(p_1, p^f_2\right) = \mu_2(f_2, f_2) \cdot p_1(f_2) + \mu_2(f_1, f_2) \cdot p_1(f_1)
\begin{array}{l}
= \lfloor 1 \land 1 + 2\delta \rfloor
\leq 1 + 2\delta,
\end{array}

so that

$$V_2\left(p_1, p^f_2\right)
\begin{array}{l}
= E_2\left(p_1, p^f_2\right) + R_2\left(p_1, p^f_2\right)
\leq 1 + 2\delta + \frac{1}{2}
\leq 1 + 2\delta
\end{array}

\begin{array}{l}
\text{(since } \delta \leq \frac{1}{4}).
\end{array}

A contradiction to the assumption that $\langle p_1, p_2 \rangle$ is a $V$-equilibrium.

The claim follows.

Lemma 5.3 In a $V$-equilibrium $\langle p_1, p_2 \rangle$ for G, if $p_i(\Lambda) = 1$ then $p^*_i(\Lambda) = 1$.

Proof: Set $i := 1$ so that $p_1(\Lambda) = 1$. By Lemma 5.2, $p_2(\Lambda) > 0$. If $p_2(\Lambda) = 1$, then we are
done. So assume $p_2(\Lambda) < 1$. This implies that $p_2(\mathcal{F}) > 0$.

By the cost functions, for each strategy $f \in \mathcal{F}$, $\mu_2(\lambda, f) = 1$ for each strategy $\lambda \in \Lambda$. Hence, for each strategy $f \in \mathcal{F}$ with $p_2(f) > 0$, (i) $E_2\left(p_1, p^f_2\right) = 1$, so that the Weak-Equilibrium-for-Expectation
property for player 2 implies that $E_2\left(p_1, p^\lambda_2\right) = 1$ for each strategy $\lambda \in \Lambda$ with
$p_2(\lambda) > 0$, and (ii) $R_2(p_1, p^f_2) = 0$, by the Risk-Positivity property. Hence, $V_2\left(p_1, p^f_2\right) = 1$.

Assume, by way of contradiction, that there is a strategy $\lambda' \in \Lambda$ with $p_2(\lambda') > 0$ such that
$\mu_2(\lambda, \lambda') \neq 1$ for some $\lambda \in \Lambda$ with $p_1(\lambda) > 0$. It follows by the Risk-Positivity property that
$R_2(p_1, p_2) > 0$. Then,

$$V_2(p_1, p_2) = E_2(p_1, p_2) + R_2(p_1, p_2)
\begin{array}{l}
= \sum_{\lambda \in \Lambda} E_2(p_1, p^\lambda_2) \cdot p_2(\lambda) + \sum_{f \in \mathcal{F}} E_2(p_1, p^f_2) \cdot p_2(f) + R_2(p_1, p_2)
\leq 1 + R_2(p_1, p_2).
\end{array}$$

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Lemma 5.3

So, that By Lemma 5.2, for each strategy \( p \in \Lambda \), \( \lambda \in \Lambda \), \( \lambda' \in \Lambda \), with \( p_1(\lambda) > 0 \) and \( p_2(\lambda') > 0 \). By the cost functions, it follows that for each pair of distinct strategies \( \lambda, \lambda' \in \Lambda \) with \( p_1(\lambda) > 0 \) and \( p_2(\lambda') > 0 \), \( \lambda, \lambda' \in \Lambda \) with \( \lambda \neq \lambda' \), \( \mu_1(\lambda, \lambda') = 1 \).

Consider now player 1. Note that for each strategy \( \lambda \in \Lambda \), \( \mu_1(\lambda, \lambda') = 1 \) for each strategy \( \lambda' \in \Lambda \) with \( p_2(\lambda') > 0 \), and \( \mu_1(\lambda, f) = 2 \) for each strategy \( f \in \mathcal{F} \) with \( p_2(f) > 0 \). Hence, \( \{ \mu_1(s) \mid p(s) > 0 \} = \{ 1, 2 \} \), so that \( \widehat{V}_1(p_1, p_2) = \widehat{V}_1(1, 2, q) \), where \( q = p_2(\mathcal{F}) \).

Consider player 1 switching to the pure strategy \( p_1^{f_1} \). By the cost functions, \( \mu_1(f_1, \lambda) = 1 \) for each strategy \( \lambda \in \Lambda \), \( \mu_1(f_1, f_2) = 1 \) and \( \mu_1(f_1, f_1) = 1 + \delta \). Hence, for \( \widehat{p} = \langle p_1^{f_1}, p_2 \rangle \), \( \{ \mu_1(s) \mid \widehat{p}(s) > 0 \} = \{ 1, 1 + \delta \} \), so that \( \widehat{V}_1(p_1^{f_1}, p_2) = \widehat{V}_1(1, 1 + \delta, r) \), with \( r = p_2(f_1) \). So,

\[
\widehat{V}_1(1, 1 + \delta, r) < \widehat{V}_1(1, 2, q) \quad (\text{by Condition (2/b), since } r \leq q \text{ and } 1 + \delta < 2)
\]

So, player 1 improves her cost by switching to the pure strategy \( p_1^{f_1} \). A contradiction to the assumption that \( \langle p_1, p_2 \rangle \) is a V-equilibrium. The proof for \( i := 2 \) is identical except that in the last stage it uses the inequality \( \widehat{V}_2(1, 1 + 2\delta, r) < \widehat{V}_2(1, 2, q) \), holding by Condition (2/b).

Lemma 5.4

In a V-equilibrium \( \langle p_1, p_2 \rangle \) for \( G \), \( p_1(\Lambda) = p_2(\Lambda) = 1 \).

Proof: By Lemma 5.2, \( p_1(\Lambda) > 0 \) and \( p_2(\Lambda) > 0 \). If \( p_i(\Lambda) = 1 \) for some player \( i \in [2] \), then, by Lemma 5.3, \( p_\Lambda(\Lambda) = 1 \), and we are done. So assume that \( p_1(\Lambda) < 1 \) and \( p_2(\Lambda) < 1 \); this implies that \( p_1(\mathcal{F}) > 0 \) and \( p_2(\mathcal{F}) > 0 \). By the cost functions, for each pair \( \lambda, \lambda' \in \Lambda \),

\[
\mu_1(\lambda, \lambda') + \mu_2(\lambda, \lambda') \geq 2.
\]

So,

\[
\frac{1}{p_1(\Lambda) p_2(\Lambda)} \sum_{\lambda, \lambda' \in \Lambda} (\mu_1(\lambda, \lambda') + \mu_2(\lambda, \lambda')) p_1(\lambda) p_2(\lambda') \geq 2.
\]

So, there is a player \( i \in [2] \) with

\[
b := \frac{1}{p_1(\Lambda) p_2(\Lambda)} \sum_{\lambda, \lambda' \in \Lambda} \mu_i(\lambda, \lambda') p_1(\lambda) p_2(\lambda') \geq 1.
\]

Without loss of generality, set \( i := 1 \). By the Weak-Equilibrium-for-Expectation property for player 1, \( \mathbb{E}_1(p_1^{\lambda_1}, p_2^{\lambda_2}) = \mathbb{E}_1(p_1^{\lambda_2}, p_2^{\lambda_1}) \) for all \( \lambda_1, \lambda_2 \in \Lambda \) with \( p_1(\lambda_1) > 0 \) and \( p_1(\lambda_2) > 0 \). For \( \lambda \in \Lambda \), it holds that

\[
\mathbb{E}_1(p_1^{\lambda}, p_2) = 2 p_2(\mathcal{F}) + \sum_{\lambda' \in \Lambda} \mu_1(\lambda, \lambda') p_2(\lambda') .
\]

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Hence,
\[ \sum_{\lambda' \in \Lambda} \mu_1(\lambda, \lambda') p_2(\lambda') = \sum_{\lambda' \in \Lambda} \mu_2(\lambda, \lambda') p_2(\lambda') \]
for all \( \lambda, \lambda' \in \Lambda \) with \( p_1(\lambda) > 0 \) and \( p_1(\lambda') > 0 \). Hence, it follows that for each \( \lambda \in \Lambda \) with \( p_1(\lambda) > 0 \),
\[ \sum_{\lambda' \in \Lambda} \mu_1(\lambda, \lambda') p_2(\lambda') = b \cdot p_2(\Lambda) . \]
So,
\[ E_1(p_1, p_2) = 2 p_2(\mathcal{F}) + \frac{b}{2} \cdot p_2(\Lambda) . \]
But for each strategy \( f \in \mathcal{F} \) with \( p_1(f) > 0 \),
\[ E_1\left(p_1^f, p_2\right) = a \cdot p_2(\mathcal{F}) + p_2(\Lambda) , \]
for some \( a \) with \( |a| < 2 \). Hence, the Weak-Equilibrium-for-Expectation property for player 1 implies that \( E_1(p_1^f, p_2) = E_1(p_1, p_2) \), or
\[ (2 - a) p_2(\mathcal{F}) = (1 - b) \cdot p_2(\Lambda) . \]
A contradiction. \( \blacksquare \)

We now establish some stronger properties for a \( \mathcal{V} \)-equilibrium.

**Lemma 5.5** Consider a \( \mathcal{V} \)-equilibrium \( (p_1, p_2) \) for the game \( G \). Then, \( p_1(\mathcal{L}) = p_2(\mathcal{L}) = 1 \), and for every player \( i \in [2] \), for every literal \( \lambda \in \mathcal{L} \), \( p_i(\lambda) \cdot p_i(\lambda') = 0 \). Moreover, \( E_1(p_1, p_2) = E_2(p_1, p_2) = 1 \) and \( R_1(p_1, p_2) = R_2(p_1, p_2) = 0 \).

**Proof:** By Lemma 5.4, \( p_1(\Lambda) = p_2(\Lambda) = 1 \), so that \( p_1(\mathcal{F}) = p_2(\mathcal{F}) = 0 \). By the cost functions, for each pair \( \lambda, \lambda' \in \Lambda \), \( \mu_1(\lambda, \lambda') + \mu_2(\lambda, \lambda') \geq 2 \). So, \( E_1(p_1, p_2) + E_2(p_1, p_2) \geq 2 \). It follows that there is a player \( i \in [2] \) such that \( E_i(\Lambda, \Lambda) \geq 1 \). For easier notation, take \( i := 1 \).

By the cost functions, for each \( \lambda \in \Lambda \), \( \mu_1(f_1, \lambda) = 1 \). If player 1 switches to the pure strategy \( p_1^f \), then (i) \( E_1(p_1^f, p_2) = 1 \leq E_1(p_1, p_2) \), and (ii) \( R_1(p_1^f, p_2) = 0 \), by the Risk-Positivity property. Since \( R_1(p_1, p_2) \geq 0 \) (by the Risk-Positivity property), it follows that if either \( E_1(p_1, p_2) > 1 \) or \( R_1(p_1, p_2) > 0 \), then player 1 improves her cost by switching to \( p_1^f \). Since \( (p_1, p_2) \) is a \( \mathcal{V} \)-equilibrium, it follows that both \( E_1(p_1, p_2) = 1 \) and \( R_1(p_1, p_2) = 0 \).

Now, \( E_1(p_1, p_2) + E_2(p_1, p_2) \geq 2 \) and \( E_1(p_1, p_2) = 1 \) together imply that \( E_2(p_1, p_2) \geq 1 \). In the same way as above, \( E_2(p_1, p_2) = 1 \) and \( R_2(p_1, p_2) = 0 \) follow.
By the cost functions, for each pair \( \lambda, \lambda' \in \Lambda \), \( \mu_1(\lambda, \lambda') + \mu_1(\lambda, \lambda') \geq 2 \). We have just shown that \( E_1(p_1, p_2) + E_2(p_1, p_2) = 2 \). This implies that \( \mu_1(\lambda, \lambda') = \mu_2(\lambda, \lambda') = 1 \) for each pair \( \lambda, \lambda' \in \Lambda \) with \( p_1(\lambda) > 0 \) and \( p_2(\lambda') > 0 \). Thus, by the cost functions, \( \lambda, \lambda' \in \Lambda \), so that \( p_1(L) = p_2(L) = 1 \), and \( p_i(\lambda) > 0 \) implies that \( p_2(\lambda) = 0 \) for all pairs of a player \( i \in [2] \) and a literal \( \lambda \in \Lambda \).

We continue to prove:

**Lemma 5.6** In a \( V \)-equilibrium \( \langle p_1, p_2 \rangle \) for \( G \), for each player \( i \in [2] \) and each literal \( \ell \in L \), \( p_i(\ell) + p_i(\overline{\ell}) > 0 \).

**Proof:** Assume, by way of contradiction, that there is a player \( i \in [2] \) and a literal \( \ell \in L \) such that \( p_i(\ell) = p_i(\overline{\ell}) = 0 \). Without loss of generality, set \( i := 2 \). Recall that, by Lemma 5.5, \( V_1(p_1, p_2) = 1 \) and \( p_2(L) = 1 \).

Consider player 1 switching to the pure strategy \( p_1^* \) for the variable \( v \) such that \( \ell \) and \( \overline{\ell} \) are literals for \( v \). By the cost functions, \( \mu_1(v, \lambda) = m \) if \( \lambda \in \{\ell, \overline{\ell}\} \), and 0 if \( \lambda \in L \setminus \{\ell, \overline{\ell}\} \). Since \( p_2(L) = 1 \) and \( p_2(\ell) = p_2(\overline{\ell}) = 0 \), it follows that \( \mu_1(v, \lambda) = 0 \) for all strategies \( \lambda \) with \( p_2(\lambda) > 0 \). Thus, (i) \( E_1(p_1^*, p_2) = 0 \), and (ii) \( R_1(p_1^*, p_2) = 0 \) (by the Risk-Positivity property). Hence, \( V_1(p_1^*, p_2) = 0 \), so that player 1 improves her cost by switching to the pure strategy \( p_1^* \). A contradiction to the assumption that \( \langle p_1, p_2 \rangle \) is a \( V \)-equilibrium.

Finally, we prove:

**Lemma 5.7** A \( V \)-equilibrium \( \langle p_1, p_2 \rangle \) for \( G \) induces a unique truth assignment for \( \phi \).

**Proof:** For each pair of a player \( i \in [2] \) and a literal \( \ell \in L \), it holds that (i) \( p_1(\ell) + p_1(\overline{\ell}) > 0 \) (by Lemma 5.6), and (ii) if \( p_i(\ell) > 0 \), then \( p_1(\overline{\ell}) = 0 \) (by Lemma 5.5). Thus, for each variable \( v \), there is a literal \( \ell \) for \( v \) such that \( p_1(\ell), p_2(\ell) > 0 \) and \( p_1(\overline{\ell}) = p_2(\overline{\ell}) = 0 \).

We are now ready to prove:

**Lemma 5.8** \( \phi \) is satisfiable if and only if \( G(\phi) \) has a \( V \)-equilibrium.

**Proof:** “\( \Leftarrow \)” Assume first that \( \phi \) is not satisfiable. Assume, by way of contradiction, that \( G(\phi) \) has a \( V \)-equilibrium \( \langle p_1, p_2 \rangle \). By Lemma 5.7, \( \langle p_1, p_2 \rangle \) induces a unique truth assignment \( \gamma \) for \( \phi \). Since \( \phi \) is not satisfiable, there is a clause \( c \) such that for each literal \( \ell \) with \( p_1(\ell), p_2(\ell) > 0 \), \( \ell \not\in c \). Consider now player 1 switching to the pure strategy \( p_1^* \). Then,

\[
E_1(p_1^*, p_2) = \sum_{\ell \in L : p_1(\ell) > 0} \mu_1(c, \ell) p_2(\ell) = 0,
\]
and \( R_i (p_1^c, p_2) = 0 \) (by the Risk-Positivity property), so that \( V_i (p_1^c, p_2) = 0 \). By Lemma 5.5, \( V_1(p_1, p_2) = 1 \). So, player 1 improves her cost by switching to the pure strategy \( p_1^c \). A contradiction to the assumption that \( \langle p_1, p_2 \rangle \) is a \( \mathcal{V} \)-equilibrium.

“\( \Rightarrow \)” Assume now that \( \phi \) is satisfiable. For a satisfying assignment \( \gamma \) of \( \phi \), set \( p_i(\ell) := \frac{1}{m} \) for each literal \( \ell \in L \) with \( \gamma(\ell) = 1 \). We shall prove that \( \langle p_1, p_2 \rangle \) is a \( \mathcal{V} \)-equilibrium. Fix a player \( i \in [2] \). By the cost functions, \( \mu_i(\ell_j, \ell_k) = 1 \) for each pair of literals \( \ell_j, \ell_k \) with \( p_1(\ell_j) \cdot p_2(\ell_k) > 0 \). Hence, the Risk-Positivity property implies that \( R_i(p_1, p_2) = 0 \). Furthermore,

\[
E_i(p_1, p_2) = \sum_{(\ell_j, \ell_k) \in \mathcal{L} | \gamma(\ell_j) = 1} \mu_i(\ell_j, \ell_k) p_1(\ell_j) \cdot p_2(\ell_k) = m^2 \cdot \frac{1}{m} \cdot \frac{1}{m} = 1.
\]

So, player \( i \) may decrease \( V_i \) only if she decreases \( E_i \). It suffices to prove that player \( i \) cannot decrease \( E_i \) by switching to a pure strategy. We proceed by case analysis.

1. Consider player \( i \) switching to the pure strategy \( p_i^c \) for a variable \( v \in \mathcal{V} \) with literals \( \ell, \ell' \). By the cost functions, for each literal \( \lambda \in L \), \( \mu_i(v, \lambda) = m \) if \( \lambda \in \{\ell, \ell'\} \) and 0 otherwise. By construction, \( p_i(\ell) + p_i(\ell') = \frac{1}{m} \). So,

\[
E_i (p_i^c, p_\gamma) = 0 \cdot \left( 1 - \frac{1}{m} \right) + m \cdot \frac{1}{m} = 1.
\]

2. Consider player \( i \) switching to the pure strategy \( p_i^c \) for a clause \( c \in \mathcal{C} \). By the cost functions, for each literal \( \lambda \in L \), \( \mu_i(c, \lambda) = m \) if \( \lambda \in c \) and 0 otherwise. Since \( \phi \) is satisfiable, there is at least one literal \( \ell \in c \) with \( \gamma(\ell) = 1 \); hence, by construction of \( p_\gamma \), there is at least one literal \( \ell \in c \) with \( p_\gamma(\ell) = \frac{1}{m} \). Thus,

\[
E_i (p_i^c, p_\gamma) = \sum_{\ell \in \mathcal{E}_i | p_\gamma(\ell) > 0} \mu_i(c, \ell) \cdot p_\gamma(\ell) \geq m \cdot \frac{1}{m} = 1.
\]

3. Consider player \( i \) switching to the pure strategy \( p_i^f \) for some \( f \in \mathcal{F} \). By construction of the cost functions, for each literal \( \lambda \in L \), \( \mu_i(f, \ell) = 1 \). It follows that \( E_i (p_i^f, p_\gamma) = 1 \).

4. Finally, consider player \( i \) switching to the pure strategy \( p_i^f \) for some literal \( \ell \in L \). Assume first that \( \gamma(\ell) = 1 \). Then, \( p_\gamma(\ell) = 0 \). Hence, by the cost functions, \( \mu_i(\ell, \ell') = 1 \) for each literal \( \ell' \in L \) with \( p_\gamma(\ell') > 0 \). It follows that \( E_i (p_i^f, p_\gamma) = 1 \). Assume now that \( \gamma(\ell) = 0 \). Then, \( p_\gamma(\ell) = \frac{1}{m} \). By the cost functions, \( \mu_i(\ell, \ell') = 2 \) and \( \mu_i(\ell, \ell') = 1 \) for \( \ell' \in L \setminus \{\ell\} \) with \( p_\gamma(\ell') > 0 \). It follows that

\[
E_i (p_i^f, p_\gamma) = 2 \cdot \frac{1}{m} + 1 \cdot \left( 1 - \frac{1}{m} \right) = 1 + \frac{1}{m}.
\]
The claim now follows.

Lemma 5.8 establishes the reduction for the \( \mathcal{NP} \)-hardness; since the numbers involved in the reduction are polynomially bounded, strong \( \mathcal{NP} \)-hardness follows.

### 5.2 Concrete \( \mathcal{NP} \)-Hardness Result

We remark that the proof of the reduction for Theorem 5.1 is modular in treating \( V \) in an abstract way through using Risk-Positivity, Weak-Equilibrium-for-Expectation, and the properties in Condition (2). This modularity yields an extension of Theorem 5.1 to concrete (\( E + R \))-valuations enjoying these properties. We shall verify Conditions (1) and (2) from Theorem 5.1 for the (\( E + R \))-valuations \( V \), where (1) \( R = \gamma \cdot \text{Var} \), or (2) \( R = \gamma \cdot \text{SD} \), or a valuation (3) \( V = \lambda (E + \gamma \cdot \text{Var}) + (1 - \lambda) V' \), with \( 0 < \lambda \leq 1 \), where \( \nu(x) = x^r \), with \( r \geq 2 \), and with \( \gamma > 0 \).

The Weak-Equilibrium-for-Expectation property in Condition (1) follows from Corollary 3.3. For Condition (2), we shall prove three technical claims associated with Conditions (2/a), (2/b) and (2/c), respectively. We start with Condition (2/a). We remark that the existence of a \( \delta \) such that \( \hat{R}(1, 1 + 2\delta, q) < \frac{1}{2} \) for each probability \( q \in [0, 1] \) follows already from the continuity of \( R \); but its polynomial time computation is bound to depend on each particular \( \hat{R} \). We prove:

**Lemma 5.9** Fix an (\( E + R \))-valuation \( V \), where (1) \( R = \gamma \cdot \text{Var} \), or (2) \( R = \gamma \cdot \text{SD} \), or (3) \( V = \lambda (E + \gamma \cdot \text{Var}) + (1 - \lambda) V' \), with \( 0 < \lambda \leq 1 \), where \( \nu(x) = x^r \) with \( r \geq 2 \), and with \( \gamma > 0 \).

Then, there is a polynomial time computable \( \Delta \) with \( 0 < \Delta \leq \frac{1}{4} \) such that \( \hat{R}(1, 1 + 2\delta, q) < \frac{1}{2} \) for all \( q \in [0, 1] \) and \( 0 \leq \delta < \Delta \).

**Proof:** For each valuation \( V \), we shall choose a suitable \( \Delta \).

1. **\( R = \gamma \cdot \text{Var} \):** Then, \( \gamma \cdot \hat{R}(1, 1 + 2\delta, q) = \gamma \cdot q(1 - q) \cdot 4\delta^2 \leq \gamma \cdot \frac{1}{4} \cdot 4\delta^2 = \gamma \cdot \delta^2 \). Choose \( \Delta = \frac{1}{4} \cdot \min \left\{ \frac{1}{\sqrt{\gamma}}, 1 \right\} \). This choice satisfies that for each \( \delta < \Delta \), \( \gamma \cdot \delta^2 < \frac{1}{2} \).

2. **\( R = \gamma \cdot \text{SD} \):** Then, \( \gamma \cdot \hat{R}(1, 1 + 2\delta, q) = \gamma \cdot \sqrt{q(1 - q)} \cdot 2\delta \leq \gamma \cdot \frac{1}{2} \cdot 2\delta = \gamma \cdot \delta \). Choose \( \Delta := \frac{1}{4} \cdot \min \left\{ \frac{1}{\sqrt{\gamma}}, 1 \right\} \). This choice satisfies that for each \( \delta < \Delta \), \( \gamma \cdot \delta < \frac{1}{2} \).

3. **\( V = \lambda (E + \gamma \cdot \text{Var}) + (1 - \lambda) V' \) where \( \nu(x) = x^r \) with \( r \geq 2 \):**

Consider first the risk valuation \( \hat{R} = V' - E \), where \( \nu(x) = x^r \) with \( r \geq 2 \). Note that

\[
\hat{R}(1, 1 + 2\delta, q) = \sqrt{q \cdot (1 + 2\delta)}^r + (1 - q) - 1 - q \cdot (2\delta).
\]
Set $\Delta := \frac{1}{4}$. Then, for every $\delta < \Delta$, the fact that $\hat{R}(1, 1 + 2\delta, q)$ increases monotonically in $\delta$ implies that

$$\hat{R}(1, 1 + 2\delta, q) \leq \sqrt{1 + q \cdot \left(\frac{3}{2}\right)^r - 1} - 1 - q \cdot \frac{1}{2};$$

for $q = 0$, $\sqrt{1 + q \cdot \left(\frac{3}{2}\right)^r - 1} - 1 - 0 \cdot \frac{1}{2} = 0 < \frac{1}{2}$. So assume that $q > 0$. Then, by the fact that $\sqrt{1 + q \cdot \left(\frac{3}{2}\right)^r - 1}$ increases monotonically in $q$ and $0 < q \leq 1$, we get that $\hat{R}(1, 1 + 2\delta, q) \leq \frac{3}{2} - 1 - q \cdot \frac{1}{2} = \frac{1}{2} - q \cdot \frac{1}{2} < \frac{1}{2}$ since $q > 0$.

Combined with the choice of $\Delta$ for (1), the required property for the convex combination $V = \lambda (E + \gamma \cdot \text{Var}) + (1 - \lambda) V''$, where $\nu(x) = x^r$ with $r \geq 2$, holds by choosing $\Delta$ as a rational number no larger than $\min \left\{ \frac{1}{4}, \min \left\{ \frac{1}{\sqrt{r}}, 1 \right\} \right\} = \frac{1}{4} \cdot \min \left\{ \frac{1}{\sqrt{r}}, 1 \right\}$.

We continue with Condition (2/b). We prove:

**Lemma 5.10** Fix an $(E + R)$-valuation $V$, where (1) $R = \gamma \cdot \text{Var}$, or (2) $R = \gamma \cdot \text{SD}$, or (3) $V = \lambda (E + \gamma \cdot \text{Var}) + (1 - \lambda) V''$, with $0 < \lambda \leq 1$, where $\nu$ is increasing and strictly convex, and with $\gamma > 0$. Then, there is a polynomial time computable $\Delta$ with $0 < \Delta \leq \frac{1}{4}$ such that $\hat{V}(1, 1 + 2\delta, r) < \hat{V}(1, 2, q)$ for all $q \in (0, 1)$, with $0 \leq r \leq q$, and $0 \leq \delta < \Delta$.

**Proof:** For each valuation $V$, we shall choose a suitable $\Delta$.

**Case (1):** $R = \gamma \cdot \text{Var}$: Note that

$$\hat{V}(1, 1 + 2\delta, r) = (1 + 2\delta) \cdot r + 1 \cdot (1 - r) + \gamma \cdot r(1 - r) \cdot 4\delta^2$$

$$= 2\delta \cdot r + 1 + \gamma \cdot r(1 - r) \cdot 4\delta^2,$$

and

$$\hat{V}(1, 2, q) = 2 \cdot q + 1 \cdot (1 - q) + \gamma \cdot q(1 - q)$$

$$= q + 1 + \gamma \cdot q(1 - q).$$

Note also that

$$\max_{0 \leq r \leq q} r(1 - r) = \begin{cases} \frac{1}{4}, & \text{if } q \geq \frac{1}{2} \\ q(1 - q), & \text{otherwise} \end{cases}.$$

If $q \leq \frac{1}{2}$, then $r(1 - r) \leq q(1 - q)$. Choosing $\Delta := \frac{1}{4}$, this implies immediately that $\hat{V}(1, 1 + 2\delta, r) < \hat{V}(1, 2, q)$ for $0 \leq \delta < \Delta$. Consider now $q > \frac{1}{2}$ and let $\Delta := \min \left\{ \frac{1}{4}, \frac{1}{2(1 + \gamma)} \right\}$. Denote $A := 1 + 2\delta(2q + \gamma)$. Then, for $\delta < \Delta$,

$$\hat{V}(1, 1 + 2\delta, r) \leq 2\delta \cdot q + 1 + \gamma \cdot \delta^2 < A,$$

as required.
while

\[
A < 1 + \Delta \cdot (2q + \gamma) \quad \text{(since } \delta < \Delta) \\
= 1 + \frac{2q + \gamma}{2(1 + \gamma)} < 1 + \frac{2q(1 + \gamma)}{2(1 + \gamma)} \quad \text{(since } q > \frac{1}{2}) \\
= 1 + q < \tilde{V}(1, 2, q).
\]

**Case (2):** \( R = \gamma \cdot SD \). Then,

\[
\tilde{V}(1, 1 + 2\delta, r) = 2\delta \cdot r + 1 + \gamma \cdot \sqrt{r(1 - r)} \cdot 2\delta,
\]

and

\[
\tilde{V}(1, 2, q) = q + 1 + \gamma \cdot \sqrt{q(1 - q)}.
\]

For \( q \leq \frac{1}{2} \), the argument is identical to the one for Case (1). Consider now \( q > \frac{1}{2} \), and set again \( \Delta := \min \left\{ \frac{1}{4}, \frac{1}{2(1 + \gamma)} \right\} \). Denote again \( A := 1 + \delta(2q + \gamma) \). By arguments identical to those for Case (1), we derive that \( \tilde{V}(1, 1 + 2\delta, r) < A \) and \( A \leq q + 1 < \tilde{V}(1, 2, q) \).

**Case (3):** \( V = \lambda \cdot (E + \gamma \cdot \text{Var}) + (1 - \lambda) \cdot V^\nu, \) \( \nu \) increasing and strictly convex.

Consider first the valuation \( V = V^\nu \). Note that

\[
\tilde{V}^\nu(1, 1 + 2\delta, r) = \nu^{-1}(\nu(1) \cdot (1 - r) + \nu(1 + 2\delta) \cdot r),
\]

and

\[
\tilde{V}^\nu(1, 2, q) = \nu^{-1}(\nu(1) \cdot (1 - q) + \nu(2) \cdot q).
\]

Thus,

\[
\tilde{V}^\nu(1, 1 + 2\delta, r) < \tilde{V}^\nu(1, 2, q)
\]

if and only if

\[
\nu(1 + 2\delta) - \nu(1) < q \cdot (\nu(2) - \nu(1)).
\]

Set \( \Delta := \frac{1}{4} \). Since \( r \leq q \), \( \nu \) is strictly increasing and \( 1 + 2\delta < 2 \) for \( \delta < \Delta \), the last inequality holds, and we are done.

Combined with the choice of \( \Delta \) for (1), the required property for the convex combination

\[
V = \lambda \cdot (E + \gamma \cdot \text{Var}) + (1 - \lambda) \cdot V^\nu,
\]

where \( \nu \) is increasing and strictly convex, holds by choosing

\[
\Delta := \min \left\{ \min \left\{ \frac{1}{4}, \frac{1}{7} \right\}, \frac{1}{4} \right\} = \min \left\{ \frac{1}{4}, \frac{1}{7} \right\}.
\]
Last, for Condition (2/c), we use the Weak-Equilibrium-for-Expectation property to prove:

**Lemma 5.11** Fix an \((E + R)\)-valuation \(V\), where (1) \(R = \gamma \cdot \text{Var}\), or (2) \(R = \gamma \cdot \text{SD}\), or (3) \(V = \lambda (E + \gamma \cdot \text{Var}) + (1 - \lambda) V'\), with \(0 < \lambda \leq 1\), where \(\nu\) is increasing and strictly convex, and with \(\gamma > 0\). Then, for any \(\delta\), \(0 < \delta < 1\), the Crawford game \(G_C(\delta)\) has no \(V\)-equilibrium.

**Proof:** \(G_C(\delta)\) has no pure equilibrium due to the following cycle of improvement steps:

\[
(f_1, f_1) \xrightarrow{1} (f_2, f_1) \xrightarrow{2} (f_2, f_2) \xrightarrow{1} (f_1, f_2) \xrightarrow{2} (f_1, f_1)
\]

Assume, by way of contradiction, that \(G_C(\delta)\) has a mixed \(V\)-equilibrium \(p\). Denote \(p_1 := \langle x, 1 - x \rangle\) and \(p_2 := \langle y, 1 - y \rangle\). Note that

\[
E_1(p^f_1, p_2) = y \cdot (1 + \delta) + (1 - y) \cdot 1 = 1 + y \cdot \delta,
\]

and

\[
E_1(p^f_2, p_2) = y \cdot 1(1 + \delta) + (1 - y) \cdot (1 + \delta) = 1 + 2\delta - 2y \cdot \delta.
\]

By the Weak-Equilibrium-for-Expectation property,

\[
E_1(p^f_1, p_2) = E_1(p^f_2, p_2),
\]

or \(1 + y \cdot \delta = 1 + 2\delta - 2y \cdot \delta\), yielding \(y = \frac{2}{3}\). So, \(p_2 = \langle \frac{2}{3}, \frac{1}{3} \rangle\).

**Case (3):** Clearly,

\[
E_1(p) = E_1(p^f_1, p_2) = \frac{1}{3} \cdot (3 + 2\delta),
\]

\[
\text{Var}_1(p) = x \cdot \frac{2}{3} \cdot (1 + \delta)^2 + x \cdot \frac{1}{3} \cdot 1^2 + (1 - x) \cdot \frac{2}{3} \cdot 1^2 + (1 - x) \cdot \frac{1}{3} \cdot (1 + 2\delta)^2 - \left(\frac{1}{3} \cdot (3 + 2\delta)\right)^2
\]

\[
= \frac{2\delta^2}{3} \cdot \left(\frac{4}{3} - x\right),
\]

and

\[
V_1^f(p) = \nu^{-1} \left( x \cdot \frac{2}{3} \cdot \nu(1 + \delta) + (1 - x) \cdot \frac{1}{3} \cdot \nu(1) + x \cdot \frac{2}{3} \cdot \nu(1) + (1 - x) \cdot \frac{1}{3} \cdot \nu(1 + 2\delta) \right),
\]

so that

\[
V_1(p)
\]

\[
= \lambda \cdot (E_1(p) + \gamma \cdot \text{Var}_1(p)) + (1 - \lambda) \cdot V_1^f(p)
\]

\[
= \lambda \cdot \frac{1}{3} \cdot (3 + 2\delta) + \gamma \cdot \frac{2\delta^2}{3} \cdot \left(\frac{4}{3} - x\right) + (1 - \lambda) \cdot \nu^{-1} \left( x \cdot \frac{2}{3} \cdot \nu(1 + \delta) + (1 - x) \cdot \frac{1}{3} \cdot \nu(1) + x \cdot \frac{2}{3} \cdot \nu(1) + (1 - x) \cdot \frac{1}{3} \cdot \nu(1 + 2\delta) \right).
\]
Since (i) $p_1$ is a $V_1$-best-response to $p_2$ and (ii) $V_1$ is concave in the mixed strategy $p_1$, the Optimal-Value property (Proposition 3.1) implies that there is a constant $A$ such that $V_1(p) = A$ for all $x \in [0,1]$. Since $\delta \neq 0$, this yields a contradiction for $\lambda = 1$. So assume $0 < \lambda < 1$. Then, for all $x \in [0,1],$

$$A = \lambda \cdot \left( \frac{1}{3} \cdot (3 + 2\delta) + \frac{2\delta^2}{3} \cdot \left( \frac{4}{3} - x \right) \right) + (1 - \lambda) \cdot \gamma^{-1} \left( x \cdot \frac{2}{3} \cdot \nu(1 + \delta) + (1 - x) \cdot \frac{1}{3} \cdot \nu(1) + x \cdot \frac{2}{3} \cdot \nu(1) + (1 - x) \cdot \frac{1}{3} \cdot \nu(1 + 2\delta) \right).$$

Rearranging yields $\nu(c_1 + d_1 x) = c_2 + d_2 x$, for some constants $c_1$, $d_1$, $c_2$ and $d_2$ with $d_1 := \frac{\lambda - 1}{\lambda} \cdot \frac{2\delta^2}{3} \neq 0$, for all $x \in [0,1]$. Hence,

$$\nu(x) = c_2 - c_1 \cdot \frac{d_2}{d_1} + \frac{d_2}{d_1} \cdot x$$

for all $x \in [0,1]$, so that $\nu$ is not strictly convex. A contradiction.

**Case (1):** This is the special case of Case (3) with $\lambda = 1$.

**Case (2):** Note that

$$(E_1 + \gamma \cdot SD_1) (p) = \frac{1}{3} \cdot (3 + 2\delta) + \gamma \cdot \delta \cdot \sqrt{\frac{2}{3} \cdot \left( \frac{4}{3} - x \right)}.$$

Since (i) $p_1$ is a $(E_1 + \gamma \cdot SD_1)$-best-response to $p_2$ and (ii) $E_1 + \gamma \cdot SD_1$ is concave in the mixed strategy $p_1$, the Optimal-Value property (Proposition 3.1) implies that there is a constant $A$ such that $(E_1 + \gamma \cdot SD_1)(p) = A$ for all $x \in [0,1]$. Since $\delta \neq 0$, this yields a contradiction.

Now, for Condition (2) in Theorem 5.1 choose $\delta$ as a rational number no larger than the minimum of the $\Delta$ from Lemma 5.9 and the $\Delta$ from Lemma 5.10, both of which are polynomial time computable for the $(E + R)$-valuations in Theorem 5.12. This choice guarantees that all Conditions (2/a), (2/b) and (2/c) in Theorem 5.1 are satisfied by the chosen $\delta$. Hence, it follows:

**Theorem 5.12** Fix an $(E + R)$-valuation $V$, where (1) $R = \gamma \cdot Var$, or (2) $R = \gamma \cdot SD$, or (3) $V = \lambda \cdot (E + \gamma \cdot Var) + (1 - \lambda) \cdot V^\nu$, with $0 < \lambda \leq 1$, where $\nu(x) = x^r$ with $r \geq 2$, and with $\gamma > 0$. Then, $\exists V$-EQUILIBRIUM is strongly $NP$-hard for 2-players games.

**6 Epilogue**

We embarked on a research direction making different-from-classical assumptions on the behavior of the players in order to model the real-world in a more accurate way and gain insights
that explain reality better. We have developed a framework for games with players minimizing an \((E + R)\)-valuation \(V\). Our framework enabled proving the strong \(NP\)-hardness of \(\exists V\)-EQUILIBRIUM in the simplest cases of games with two strategies or two players, respectively, and for many significant choices of \((E + R)\)-valuations \(V\).

Besides these central results, our study is making a number of additional conceptual and technical contributions through introducing several new analytical and combinatorial tools and techniques, which are of wider applicability and interest; we summarize here the main ones.

- Our proof techniques have relied heavily on the Weak-Equilibrium-for-Expectation property, which indicates its computational power.
- We introduced \(E\)-strict concavity as the most general known class of valuations with the Weak-Equilibrium-for-Expectation property.
- We imported \(\nu\)-valuations from Actuarial Risk Theory [18] into the realm of equilibrium computation and revealed some of their algorithmic properties; ditto for higher moments, generalizing the variance considered before in [12, 22], which were used before as risk-modeling valuations in Portfolio Theory [19].
- We established the Mixed-Player-Has-Pure-Neighbors property, which explicitly identifies a class of games and a corresponding class of valuations where mixed equilibria get “endangered” (cf. [7, 12]).

Our work opens up a wide avenue for future research towards revealing the complexity of \(\exists V\)-EQUILIBRIUM for other valuations \(V\). Most obviously, the modularity of the proof of Theorem 5.1 may allow, similarly to Theorem 5.12 direct derivation of new concrete complexity results for other valuations that will be shown to have the assumed properties. Enhancing the class of \(E\)-strictly concave valuations may yield such new valuations, which may be directly accommodated into the general framework we developed. But new tools and techniques may be required for settling the complexity of \(\exists V\)-EQUILIBRIUM when \(V\) is not concave or not \(E\)-strictly concave, or when it lacks the Weak-Equilibrium-for-Expectation property. It is not clear whether and how our framework could be extended to accommodate even quasiconcave valuations. We conclude with such an example, cast into the context of maximization games. The Sharpe ratio valuation, formulated as \(SR = \frac{E}{\text{SD}}\) [27], is the ratio of a convex over a concave function; although it is not convex, it is quasiconvex as shown in [29]. Does \(SR\) have the Weak-Equilibrium-for-Expectation property? What is the complexity of \(\exists SR\)-EQUILIBRIUM?
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A FP-Strict Concavity

We recall the following definition from [12, Section 2], rendering their notation and vocabulary.

**Definition A.1 (FP-Strict Convexity)** Fix a player $i \in [n]$. The valuation $V_i$ is FP-strictly convex if for any fixed partial mixed profile $x_{-i}$, for any pair of payoff distributions $P_i(x'_i, x_{-i}) \neq P_i(x''_i, x_{-i})$, where $x'_i$ and $x''_i$ are mixed strategies, and for any $\alpha$ with $0 < \alpha < 1$, it holds that

$$V_i(\alpha x'_i + (1-\alpha)x''_i, x_{-i}) < \alpha V_i(x'_i, x_{-i}) + (1-\alpha)V_i(x''_i, x_{-i}).$$

We quote from [12, Section 2] that the payoff distribution $P_i = P_i(p)$ is the probability distribution induced from a mixed profile $p$ on the range of possible payoffs for player $i$ over all profiles; so, the probability that player $i$ receives payoff $a$ is $\sum_{s \in S|\mu_i(s)=a} p(s)$. Recall also that $V_i$ is FP-strictly concave if $-V_i$ is FP-strictly convex. We observe:

**Observation A.1** The valuation $V = E - \operatorname{Var}$ is not FP-strictly convex.

**Proof:** By counterexample. Consider the game $G$ with two players 1 and 2, with $S_1 = \{s_1, s_2\}$ and $S_2 = \{t_1, t_2, t_3, t_4\}$. The utilities for player 1 are given by

$$p_1(s_1, t_j) = \begin{cases} 
\frac{9}{2}, & \text{if } j = 1 \\
\frac{7}{2}, & \text{if } j = 2 \\
0, & \text{if } j \in \{3, 4\}
\end{cases},$$

and

$$p_1(s_2, t_j) = \begin{cases} 
0, & \text{if } j \in \{1, 2\} \\
5, & \text{if } j = 3 \\
\frac{15}{4}, & \text{if } j = 4
\end{cases}.$$

Player 2 has chosen the mixed strategy $x_2 = \left(\frac{4}{7}, \frac{1}{7}, \frac{1}{10}, \frac{2}{5}\right)$. For player 1, $x'_1$ is the pure strategy $s_1$; $x''_1$ is the pure strategy $s_2$. Then, clearly, the induced payoff distributions $P_1(x'_1, x_2)$ and $P_1(x''_1, x_2)$ are different.

Note that for every mixed strategy $x_1$ of player 1,

$$V_1(x_1, x_2) = E_1(x_1, x_2) - \operatorname{Var}_1(x_1, x_2)$$

$$= E_1(x_1, x_2) - \hat{E}_1(x_1, x_2) + (E_1(x_1, x_2))^2,$$
where $\hat{E}_1$ denotes the expectation for the square game $G^2$ (where the utilities are the squares of the utilities for $G$). It is

$$E_1(x'_1, x_2) = \frac{1}{4} \cdot \frac{9}{2} + \frac{1}{4} \cdot \frac{7}{2} = 2,$$
$$E_1(x''_1, x_2) = \frac{1}{10} \cdot 5 + \frac{2}{5} \cdot \frac{15}{4} = 2,$$

and

$$\hat{E}_1(x'_1, x_2) = \frac{1}{4} \cdot \left(\frac{9}{2}\right)^2 + \frac{1}{4} \cdot \left(\frac{7}{2}\right)^2 = 8 + \frac{1}{8},$$
$$\hat{E}_1(x''_1, x_2) = \frac{1}{10} \cdot 5^2 + \frac{2}{5} \cdot \left(\frac{15}{4}\right)^2 = 8 + \frac{1}{8}.$$

Thus,

$$V_1(x'_1, x_2) = V_1(x''_1, x_2).$$

Now, set $x_1 = \alpha x'_1 + (1 - \alpha)x''_1$, where $0 < \alpha < 1$. By linearity of expectation,

$$E_1(x_1, x_2) = E_1(x'_1, x_2) = E_1(x''_1, x_2)$$

and

$$\hat{E}_1(x_1, x_2) = \hat{E}_1(x'_1, x_2) = \hat{E}_1(x''_1, x_2).$$

Hence,

$$V_1(x_1, x_2) = V_1(x'_1, x_2) = V_1(x''_1, x_2).$$

So, $V_1$ is not FP-strictly convex.