Dynamic scaling in the spatial distribution of persistent sites

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(March 24, 2022)

The spatial distribution of persistent (unvisited) sites in one dimensional $A + A \rightarrow \emptyset$ model is studied. The ‘empty interval distribution’ $n(k, t)$, which is the probability that two consecutive persistent sites are separated by distance $k$ at time $t$, is investigated in detail. It is found that at late times this distribution has the dynamical scaling form $n(k, t) \sim t^{-\theta} k^{-\tau} f(k/t)$. The new exponents $\tau$ and $z$ change with the initial particle density $n_0$, and are related to the persistence exponent $\theta$ through the scaling relation $z(2 - \tau) = \theta$. We show by rigorous analytic arguments that for all $n_0$, $1 < \tau < 2$, which is confirmed by numerical results.

First passage problems in non-equilibrium systems undergoing time evolution has become an important field of research lately with the discovery of persistence. Persistence probability in general is defined as follows: Given a stochastic variable $\phi(t)$ which fluctuates about a mean value, say zero, what is the probability $P(t_1, t_2)$ that $\phi(t)$ does not change sign throughout the time interval $[t_1, t_2]$. For a large class of physical systems, persistence shows a power-law decay $P(t_1, t_2) \sim (t_2/t_1)^{-\theta}$ for $t_2 \gg t_1$, with a non-trivial persistence exponent $\theta \in [1/2, 1]$ which is, in general, unrelated to other known static and dynamic exponents.

Let us consider spatially extended systems with a stochastic field $\phi(x, t)$ at each lattice site $x$, the time evolution of which is coupled to that of its neighbouring sites. $\phi(x, t)$ could be, for instance, an Ising spin $\langle \rangle$, a phase ordering field $\langle \rangle$, a diffusing field $\langle \rangle$ or the height of a fluctuating interface $\langle \rangle$. In such cases, the system gets broken up into domains of persistent and non-persistent sites in course of time. In $d = 1$, this reduces to a set of disjoint persistent and non-persistent clusters appearing alternately. As persistence decays with time, the persistent clusters shrink in size and hence, their separation grows. The following questions arise naturally in this context, which we address here: (i) How are the persistent clusters distributed in space at a given time? (ii) How does their average separation grow with time?

In one dimension, the zeroes of the stochastic field can be viewed as a set of particles, moving about in the lattice, annihilating each other when two of them meet. When a particle moves across a lattice site for the first time, the field there flips sign, and the site becomes non-persistent. If each particle is assumed to perform purely diffusive motion, this reduces to the well-known reaction-diffusion model $A + A \rightarrow \emptyset$, with appropriate initial conditions. The simplest case is random initial distribution of particles, with average density $n_0$, for which $P(t) \sim t^{-\theta}$ with $\theta = 3/8$ $\langle \rangle$, independent of $n_0$ $\langle \rangle$. We investigate spatial ordering of persistent sites in this simple model.

Our study is centered around the Empty Interval Distribution $n(k, t) \rangle$ — the probability that two randomly chosen consecutive persistent sites are separated by distance $k$ at time $t$. This distribution is analogous to the well-studied Inter-Particle Distribution Function (IPDF) in diffusion-reaction systems $\langle \rangle$. Our numerical results show that $n(k, t)$ has a non-trivial dynamic scaling form with power-law decay in $k$ and $t$, characterized by exponents $\tau$ and $\theta$ respectively. The power law decay is valid for $k \ll L(t)$, where $L(t) \sim t^\omega$ is a new dynamic length scale, which may be interpreted as the average separation between persistent regions. The three exponents are connected by the scaling relation $z(2 - \tau) = \theta$. Although the persistence exponent $\theta$ is universal for this model, we find that $\tau$ and $z$ do change with the initial particle density. We give rigorous analytical arguments on the bounds of $\tau$, showing that $1 < \tau < 2$ for all values of $n_0$. Power-law decay of $n(k, t)$ in $k$ is a consequence of spatial correlations—a random distribution of sites would correspond to exponential decay.

Our numerical simulation is done on a 1-d lattice of size $N = 10^4$ with periodic boundary conditions. Particles are initially distributed at random on the lattice, and their positions are sequentially updated—each particle was made to move one step in either direction with probability 1/2. When two particles came on top of each other, both vanished instantaneously. The time evolution is done up to 12000 Monte-Carlo steps (1 MC step is counted after all the particles in the lattice were touched once). All simulations are repeated for three different values of initial density, $n_0 = 0.2$, $0.5$ and $0.8$. The results are averaged over 500 different initial configurations.

We observe that at large times $t$ and $k \gg 1$, $n(k, t) \sim k^{-\tau}$ for $k \ll L(t)$. Here $L(t)$ is a cut-off length scale that grows with time. In Fig. 1, we present the data for $n_0 = 0.5$ and for three values of time. The same data as presented in Fig. 2 shows that for each $k$, $n_k(t) \sim t^{-\omega}$ for all $k \ll L(t)$. (It will be shown later that $\omega = \theta$). Similar power-law decay in $k$ and $t$ has been observed for other values of $n_0$ also. These observations are fairly well represented by the following dynamic scaling form:
for \( n(k,t) \), for late times and large enough \( k \).

\[
n(k,t) \sim t^{-\omega} k^{-\tau} f(k/L(t))
\]  

where the scaling function \( f(x) \approx 1 \) for \( x \ll 1 \) and decreases faster than any power of \( x \) for \( x \gg 1 \).

![Fig. 1 Log-log plot of \( n(k,t) \) with \( k \) illustrating the power law decay for small \( k \), which crosses over to faster decay at large \( k \). The data is presented for \( t = 2000, 4000 \) and \( 10000 \) (top to bottom). The vertical separation between the curves has been enhanced for clarity. The initial density \( n_0 \) is 0.5. All the three straight lines have slope \( \tau \approx 1.27 \).](image1)

The exponents appearing in Eq. 1 are not all independent. The moments of the distribution are useful in deriving the scaling relations between them. The \( m \)-th moment is 

\[
I_m(t) = \sum_k k^m n(k,t) \approx \int_1^\infty n(s,t) s^m ds.
\]

From the definition of \( n(k,t) \), one can easily see that

\[
I_0(t) = P(t) \sim t^{-\theta} ; \quad I_1(t) = N
\]

The average separation between persistent sites is given by \( I_2(t)/I_1(t) \sim L(t) \). In Fig. 3 we have \( L(t) \) plotted against \( t \) on a logarithmic scale for three values of \( n_0 \). We find that \( L(t) \) diverges with time as

\[
L(t) \sim t^z
\]

where \( z \) is a new dynamic exponent.

![Fig. 3 The average separation between persistent sites \( L(t) \) grows as a power of time \( t \). The three logarithmic plots correspond to \( n_0 = 0.8, 0.5 \) and \( 0.2 \) (top to bottom). The lines are visibly getting flatter with increasing \( n_0 \), indicating the decrease in the exponent \( z \).](image2)

The scaling relations between the exponents are obtained by making use of the conditions in Eq. 2. First of all, we show that only \( \tau < 2 \) is physically reasonable. For, if \( \tau \geq 2 \), \( I_1(t) \sim t^{-\omega} \), and from the second part of Eq. 2 it follows that \( \omega = 0 \). But since \( \omega \geq \theta \) for reasons of convergence, we get \( \theta \leq 0 \) which is absurd. So we conclude that \( \tau < 2 \). In this case, \( I_1(t) \sim t^{-\omega + z(2-\tau)} \), which according to Eq. 2 imply that

\[
z(2-\tau) = \omega
\]

Another set of scaling relations can be derived using the condition on \( I_0(t) \) in Eq. 2. Combined with Eq. 4, this gives

\[
z = \theta ; \quad \omega = \theta(2-\tau) \quad \text{if} \quad \tau < 1
\]

\[
z = \theta ; \quad \omega = \theta(2-\tau) \quad \text{if} \quad \tau > 1
\]

We present a summary of our numerical results in Table I. It is easily seen that for all values of \( n_0 \), \( z > \theta \), \( \tau > 1 \) and the scaling relations in Eq. 6 are satisfied within numerical errors. Moreover, the exponents \( z \) and \( \tau \) show a consistent decrease with increasing \( n_0 \)— they are non-universal, unlike \( \theta \). We now present an intuitive argument which accounts for these observations fairly rigorously.

As persistence decays, the non-persistent regions grow in time (the length scale of which is set by \( L(t) \) in Eq.
while the clusters of persistent sites shrink in size and eventually disappear. Let \( p(l,t) \) be the number of persistent clusters of size \( l \), at time \( t \). The total number of persistent sites at time \( t \) is \( \sum_l l p(l,t) = P(t) \), and the total number of such clusters is \( N_c(t) = \sum_l p(l,t) \). The latter is related to \( n(k,t) \) through the exact relation \( N_c(t) = \sum_{k=2}^{\infty} n(k,t) = P(t) - n(1,t) \). The average size of a cluster at time \( t \) is

\[
\bar{l}(t) = \frac{P(t)}{N_c(t)} = \left(1 - \frac{n(1,t)}{P(t)}\right)^{-1} \quad (7)
\]

From Eq. 3 \( n(1,t) \sim t^{-\omega} \) and since \( P(t) \sim t^{-\theta} \) we have \( \bar{l}(t) \sim [1 - \gamma t^{-(\omega-\theta)}]^{-1} \) where \( \gamma \) is a numerical constant. Since \( \omega > \theta \), \( \bar{l}(t) \) is a constant for late times. Now, if \( \omega < \theta \), \( \bar{l}(t) = 1 \) strictly; only if \( \omega = \theta \) any other value is possible. We argue for the latter case as follows. When clusters of persistent sites shrink in size, the depletion happens at the two ends of the cluster, independent of its size. Let the average decrease in the size of a cluster over time \( t \) be \( \xi(t) \). Clusters of initial size \( l > \xi(t) \) shrink to size \( l - \xi(t) \) after time \( t \), while those with length \( l \leq \xi(t) \) disappear. It follows that

\[
p(l,t) = p(l + \xi(t),0) \quad (8)
\]

Here, \( p(l,0) = n_0^l (1 - n_0)^l \) since the initial distribution of particles is done at random with probability \( n_0 \). After substitution in Eq. 3 we find that the time evolution of the cluster size distribution has the extremely simple form \( p(l,t) = e^{-\lambda t^\gamma} p(l,0) \) where \( \lambda = -\ln(1 - n_0) \). This result is also supported by simulations (Fig. 4). Consequently, the average cluster size \( \bar{l}(t) = \bar{l}(0) = 1/n_0 \). This implies \( \omega = \theta \) from our arguments following Eq. 6, and thus validates Eq. 3. Furthermore, since \( P(t) \sim t^{-\theta} \) we find \( \xi(t) \approx \frac{\lambda}{\theta} \ln t \) at large \( t \). It follows that a persistent cluster of initial size \( L \) has an average life-time \( \tau_L \sim \exp \left( \frac{\lambda}{\theta} L \right) \) for large \( L \). The exponential dependence of the life-time of the cluster on its size reflects the slow algebraic decay of persistence.

Our argument can be extended to show why \( \tau \) and \( z \) are possibly non-universal. First of all, the exponent relation \( \omega = \theta \) makes it possible to write the formal relation \( n(1,t) = g(\tau, n_0) P(t) \) [14]. Combined with Eq. 6 and using the result \( \bar{l}(t) = 1/n_0 \) we obtain the relation \( g(\tau, n_0) + n_0 = 1 \), which expresses implicitly the dependence of \( \tau \) on \( n_0 \). For instance, if Eq. 6 were exact for all values of \( k \), then \( P(t) \approx \int_0^\infty n(s,t) ds = \frac{\lambda t^\gamma}{\gamma} \), so that \( g(\tau, n_0) \approx \tau - 1 \), from which it follows that \( \tau \approx 2 - n_0 \). This result, although not consistent with the bounds \( 1 < \tau < 2 \), and appears to be valid in the high density limit \( n_0 \to 1 \), as indicated by the numerical values in Table I.

![Fig. 4](image) The cluster size distribution \( p(l,t) \) decays exponentially with \( l \) at all times. The figure shows \( p(l,t) \) plotted against \( l \) for \( t = 20, 40 \) and 120 (top to bottom) and \( n_0 = 0.2 \). The vertical separations have been enhanced for clarity.

In summary, we have shown that the spatial distribution of persistent clusters in one dimension exhibits rich dynamic scaling characterised by two new exponents. We have given rigorous arguments on the bounds and universality properties of these exponents, which is well-supported by numerics. Interestingly, the normalized size distribution of persistent clusters was found to be independent of time.

Our work is the first study that brings out the non-trivial features in the spatial distribution of persistent sites in a one dimensional model. The dynamic scaling form in Eq. 3 is by no means specific to the model studied, and we have observed similar forms in other one dimensional systems—diffusion equation and kinetic Ising model, for example. Similar scaling in size distribution has been observed in entirely different contexts also—for instance, diffusion-limited cluster aggregation [13] [17] and diffusion-limited deposition [18]. The feature that is common to all these processes is the irreversible coalescence of clusters (empty intervals in our case).

We are grateful to Satya Majumdar for discussions and for pointing out the similarities to aggregation models. G. M thanks B. Derrida and P. R thanks D. Stauffer for critical reading of the manuscript and valuable suggestions.

| \( n_0 \) | 0.2 | 0.5 | 0.8 |
|---|---|---|---|
| \( \theta \) | 0.3718 ± 0.0001 | 0.3769 ± 0.0000 | 0.3729 ± 0.0001 |
| \( \omega \) | 0.3763 ± 0.0001 | 0.3767 ± 0.0000 | 0.3710 ± 0.0002 |
| \( z \) | 0.5766 ± 0.0005 | 0.5107 ± 0.0003 | 0.4392 ± 0.0005 |
| \( \tau \) | 1.3502 ± 0.0262 | 1.2596 ± 0.0357 | 1.1277 ± 0.0383 |

**TABLE I.** Exponents \( \theta, \omega, z \) and \( \tau \) as measured from the simulations, for three values of initial density \( n_0 \). The exact value of \( \theta \) is 0.375, independent of \( n_0 \). The \( \omega \) values presented correspond to \( k = 1 \) and the \( \tau \) values to \( t = 12000 \). The errors given are only statistical.
B. Derrida, A. J. Bray and C. Godrèche, J. Phys. A 27, L357 (1994); D. Stauffer, J. Phys. A 27, 5029 (1994).

B. Derrida, V. Hakim and V. Pasquier, Phys. Rev. Lett. 75, 751 (1995), and J. Stat. Phys. 85, 763 (1996).

A. J. Bray, B. Derrida and C. Godrèche, Europhys. Lett. 27, 175 (1994).

S. N. Majumdar, C. Sire, A. J. Bray and S. J. Cornell, Phys. Rev. Lett 77, 2867 (1996); B. Derrida, V. Hakim and R. Zeitak, Phys. Rev. Lett 77, 2971 (1996).

S. N. Majumdar, A. J. Bray, S. J. Cornell and C. Sire, Phys. Rev. Lett 77, 3704 (1996).

E. Ben-Naim, L. Frachebourg and P. L. Krapivsky, Phys. Rev. E 53, 3078 (1996).

H. Hinrichsen and H. M. Koduvely, Eur. Phys. J. B 5, 257 (1998); K. Oerding and F. van Wijland, J. Phys. A 31, 7011 (1998).

J. Cardy, J. Phys. A 28, L19 (1995).

P. L. Krapivsky, E. Ben-Naim and S. Redner, Phys. Rev. E 50, 2474 (1994); Cécile Monthus, Phys. Rev. E 54, 4844 (1996); E. Ben-Naim, Phys. Rev. E 53, 1566 (1996); M. Howard, J. Phys. A 29, 3437 (1996).

B. P. Lee and A. D. Rutenberg, Phys. Rev. Lett 79, 4842 (1997).

J. Krug, H. Kallabis, S. N. Majumdar, S. J. Cornell, A. J. Bray and C. Sire, Phys. Rev. E 56, 2702 (1997); H. Kallabis and J. Krug, preprint: cond-mat/9809241.

S. N. Majumdar and S. J. Cornell, Phys. Rev. E 57, 3757 (1998).

D. ben-Avraham in: Nonequilibrium Statistical Mechanics in One Dimension, ed. V. Privman (Cambridge University Press, 1997).

This follows from putting \( n(1,t) = g_1(n_0)t^{-\theta} \), \( P(t) = \sum_k n(k,t) = g_2(\tau,n_0)t^{-\theta} \) and then defining \( g = g_1/g_2 \).

T. Vicsek and F. Family, Phys. Rev. Lett. 52, 1669 (1984); P. Meakin, T. Vicsek and F. Family, Phys. Rev. B 31, 564 (1985).

P. G. J. van Dongen and M. H. Ernst, Phys. Rev. Lett. 54, 1396 (1985).

H. Takayasu, M. Takayasu, A. Provata and G. Huber, J. Stat. Phys. 65, 725 (1991); S. N. Majumdar and C. Sire, Phys. Rev. Lett. 71, 3729 (1993).

Z. Racz and T. Vicsek, Phys. Rev. Lett. 51, 2382 (1983).