Numerical Solutions of Matrix Differential Models using Cubic Matrix Splines II

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Abstract
This paper presents the non-linear generalization of a previous work on matrix differential models \cite{1}. It focusses on the construction of approximate solutions of first-order matrix differential equations $Y'(x) = f(x, Y(x))$ using matrix-cubic splines. An estimation of the approximation error, an algorithm for its implementation and illustrative examples for Sylvester and Riccati matrix differential equations are given.

Keywords and phrases. First order matrix differential equations, Cubic-matrix splines, Sylvester and Riccati differential equations.

1 Introduction
A great variety of phenomena in physics and engineering can be modelled in the form of matrix-differential equations. Although linear matrix-differential equations, whose numerical solutions using cubic matrix splines were presented in \cite{1}, are valid for a wide range of applications, non-linear equations are also of great interest. This work generalizes the approach of \cite{1}, providing a novel scheme to numerically solve non-linear differential matrix equations of the first-order. Concretely, in this work we will develop a method for the numerical integration of the first order matrix differential equation given by

$$Y'(x) = f(x, Y(x)) \quad \begin{cases} \quad Y(a) = Y_a \end{cases} \quad a \leq x \leq b,$$

where $Y_a, Y(t) \in \mathbb{C}^{r \times q}$, $f : [a, b] \times \mathbb{C}^{r \times q} \rightarrow \mathbb{C}^{r \times q}$.

Different examples of problem (1.1) can be found in \cite{2}. Numerical schemes to obtain approximate solutions for (1.1) by means of linear multistep methods with constant steps have been devised in \cite{3}. Although there exist a priori error bounds for these methods expressed in function of the data problem, these error bounds are given in terms of an exponential which depends on the integration step $h$. Therefore, in practice, $h$ will take too small values. Furthermore, these methods require some interpolation techniques in order to get a continuous solution \cite{3}.

Generalizing the method proposed for the linear case in \cite{1}, here we elaborate an extension using cubic-matrix splines in the numerical approximation for the solutions of (1.1). In the scalar case, cubic splines were used in \cite{4} for the resolution of ordinary differential equations obtaining approximations that, among other advantages, were of class $C^1$ in the interval $[a, b]$. These splines are easy to compute and produce an approximation error of only $O(h^4)$. Recently, this

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method has been used in the resolution of other scalar problems as discussed in [5], and even linear matrix problems (see [1]). The present work extends this powerful scheme to the resolution of matrix problems of the non-linear type (1.1).

This paper is organized as follows. In section 2 we develop the proposed method, whose algorithm is then given in Section 3. Finally, in Sections 4, 5 and 6 practical examples are presented.

Throughout this work, we will adopt the notation for norms and matrix cubic splines as in the previous work [1] and common in matrix calculus. Following this nomenclature, we define the Kronecker product of \( A = (a_{ij}) \in \mathbb{C}^{m \times n} \) and \( B \in \mathbb{C}^{r \times s} \), denoted by \( A \otimes B \), as the block matrix

\[
A \otimes B = \begin{bmatrix}
a_{11}B & \cdots & a_{1n}B \\
\vdots & \ddots & \vdots \\
a_{m1}B & \cdots & a_{mn}B
\end{bmatrix}.
\]

The column-vector operator on a matrix \( A \in \mathbb{C}^{m \times n} \) is given by

\[
\text{vec}(A) = \begin{bmatrix} A_{1} \\ \vdots \\ A_{n} \end{bmatrix}, \text{ where } A_{k} = \begin{bmatrix} a_{1k} \\ \vdots \\ a_{mk} \end{bmatrix}.
\]

If \( Y = (y_{ij}) \in \mathbb{C}^{p \times q} \) and \( X = (x_{ij}) \in \mathbb{C}^{m \times n} \), then the derivative of a matrix with respect to a matrix is defined by [6, p.62 and 81]:

\[
\frac{\partial Y}{\partial X} = \begin{bmatrix} \frac{\partial Y}{\partial x_{11}} & \cdots & \frac{\partial Y}{\partial x_{1n}} \\
\vdots & \ddots & \vdots \\
\frac{\partial Y}{\partial x_{m1}} & \cdots & \frac{\partial Y}{\partial x_{mn}} \end{bmatrix}, \text{ where } \frac{\partial Y}{\partial x_{rs}} = \begin{bmatrix} \frac{\partial y_{11}}{\partial x_{rs}} & \cdots & \frac{\partial y_{1q}}{\partial x_{rs}} \\
\vdots & \ddots & \vdots \\
\frac{\partial y_{p1}}{\partial x_{rs}} & \cdots & \frac{\partial y_{pq}}{\partial x_{rs}} \end{bmatrix}.
\]

If \( X \in \mathbb{C}^{m \times n}, Y \in \mathbb{C}^{n \times r}, Z \in \mathbb{C}^{p \times q} \), then the following rule for the derivative of a matrix product with respect to another matrix applies [6, p.84]:

\[
\frac{\partial XY}{\partial Z} = \frac{\partial X}{\partial Z} [I_{q} \otimes Y] + [I_{p} \otimes X] \frac{\partial Y}{\partial Z}, \quad (1.2)
\]

where \( I_{q} \) and \( I_{p} \) denote the identity matrices of dimensions \( q \) and \( p \), respectively. If \( X \in \mathbb{C}^{m \times n}, Y \in \mathbb{C}^{n \times r}, Z \in \mathbb{C}^{p \times q} \), the following chain rule [6, p.88] is valid:

\[
\frac{\partial Z}{\partial X} = \left[ \frac{\partial \text{vec}(Y)}{\partial X} \right]^{t} \otimes I_{n} \left[ I_{n} \otimes \frac{\partial \text{vec}(Y)}{\partial Z} \right]. \quad (1.3)
\]

If \( A = (a_{ij}) \in \mathbb{C}^{m \times n} \), the Frobenius norm of \( A \) is [12] given by:

\[
\|A\|_{F} = \sqrt{\sum_{ij} a_{ij}^2}.
\]

The following relationship between the 2-norm and Frobenius norm holds [12]:

\[
\|A\|_{2} \leq \|A\|_{F} \leq \sqrt{n} \|A\|_{2}.
\]

2 Proposed general method

Let us consider the problem
\[ Y'(x) = f(x, Y(x)) \quad \text{and} \quad Y(a) = Y_a \]  
\[ a \leq x \leq b , \]  
(2.1)

where \( Y_a, Y(t) \in \mathbb{C}^{r \times q}, f : [a, b] \times \mathbb{C}^{r \times q} \rightarrow \mathbb{C}^{r \times q}, f \in \mathcal{C}^1(T) \), with

\[ T = \{ (x, Y) : a \leq x \leq b , Y \in \mathbb{C}^{r \times q} \} , \]  
(2.2)

and \( f \) fulfills the global Lipschitz’s condition

\[ \| f(x, Y_1) - f(x, Y_2) \| \leq L \| Y_1 - Y_2 \| , \quad a \leq x \leq b , Y_1, Y_2 \in \mathbb{C}^{r \times q} ; \]  
(2.3)

which guarantees the existence and uniqueness of the continuously differentiable solution \( Y(x) \) of problem (2.1), see [7 p.99].

Let us consider \( h = (b - a)/n \), \( n \) being a positive integer, so that the partition of the interval \([a, b]\) is given by

\[ \Delta_{[a,b]} = \{ a = x_0 < x_1 < \ldots < x_n = b \} , \quad x_k = a + kh , \quad k = 0, 1, \ldots , n . \]  
(2.4)

We will construct in each subinterval \([a + kh, a + (k + 1)h]\) a matrix-cubic spline approximating the solution of problem (2.1). For the first interval \([a, a + h]\), we consider that the matrix-cubic spline is defined by

\[ S_{[a,a+h]}(x) = Y(a) + Y'(a)(x-a) + \frac{1}{2!} Y''(a)(x-a)^2 + \frac{1}{3!} A_0(x-a)^3 , \]  
(2.5)

where the matrix \( A_0 \in \mathbb{C}^{r \times q} \) is a parameter to be determined. It is straightforward to check:

\[ S_{[a,a+h]}(a) = Y(a) , \quad S_{[a,a+h]}'(a) = Y'(a) = f(a,Y(a)) . \]

To fully determine the matrix-cubic spline we still must obtain \( Y''(a) \) and \( A_0 \). We consider the functions \( h_1 \) and \( h_2 \) defined by

\[ h_1 : [a,b] \rightarrow [a,b] \quad \text{and} \quad h_2 : [a,b] \rightarrow \mathbb{C}^{r \times q} , \]

\[ h_1(x) = x , \quad h_2(x) = Y(x) , \]

where \( Y(x) \) is the theoretical solution of (2.1). We describe now \( f(x,Y(x)) \) as a composition of functions \( f \) and \((h_1,h_2)\), that is, let \( \phi : [a,b] \rightarrow \mathbb{C}^{r \times q} \) be defined by

\[ \phi(x) = [ f \circ (h_1, h_2) ](x) = f(h_1(x), h_2(x)) = f(x,Y(x)) . \]

Thus, \( \phi \) is a real variable function of \( x \), and applying theorem 8.9.2 of [6 p.170] its derivative takes the form:

\[ D\phi = D(f \circ (h_1, h_2)) = ((D_1f) (h_1, h_2)) \cdot Dh_1 + ((D_2f) (h_1, h_2)) \cdot Dh_2 , \]

where the partial derivatives of \( f, D_1(f), D_2(f) \) exist and are continuous since it is assumed that \( f \in \mathcal{C}^1(T) \). By (2.1) it is clear that

\[ \frac{d(\text{vec} Y(x))}{dx}^T = [\text{vec} f(x,Y(x))]^T . \]

Next, applying the chain rule for matrix functions (1.2) and then taking the derivative of a matrix with respect to a matrix, (1.3), one obtains

\[ Y''(x) = \frac{\partial f(x,Y(x))}{\partial x} + [\text{vec} f(x,Y(x))]^T \otimes I_r \frac{\partial f(x,Y(x))}{\partial \text{vec} Y(x)} . \]  
(2.6)

We are now in the position to evaluate \( Y''(a) \) using (2.6).
By imposing that $S$ is a solution of problem (2.1) in $x = a + h$, we have:

$$S'_{|[a,a+h]|} (a + h) = f \left(a + h, S_{|[a,a+h]|} (a + h)\right), \quad (2.7)$$

and obtain from (2.7) the matrix equation with only one unknown matrix $A_0$:

$$A_0 = \frac{2}{h^2} \left[f \left(a + h, Y(a) + Y'(a)h + \frac{1}{2}Y''(a)h^2 + \frac{1}{6}A_0h^3\right) - Y'(a) - Y''(a)h\right]. \quad (2.8)$$

Assuming that the matrix equation (2.8) has only one solution $A_0$, the matrix-cubic spline is totally determined in the interval $[a,a + h]$.

Now, in the interval $[a + h, a + 2h]$, the matrix-cubic spline takes the form

$$S_{|[a+h,a+2h]|} (x) = S_{|[a+h,a+2h]|} (a + h) + S'_{|[a+h,a+2h]|} (a + h)(x - (a + h)) + \frac{1}{2}S''_{|[a+h,a+2h]|} (a + h)(x - (a + h))^2 + \frac{1}{6}A_1(x - (a + h))^3, \quad (2.9)$$

so that $S(x)$ is of class $\mathcal{C}^2([a,b])$ on $[a, a + h] \cup [a + h, a + 2h]$, and all coefficients of the matrix-cubic spline $S_{|[a+h,a+2h]|} (x)$ are determined with the exception of $A_1 \in \mathbb{C}^{n \times n}$. By construction, matrix-cubic spline (2.9) satisfies the differential equation (2.1) in $x = a + h$. We can obtain $A_1$ by requiring that the differential equation (2.1) holds at point $x = a + 2h$:

$$S'_{|[a+2h,a+2h]|} (a + 2h) = f \left(a + 2h, S_{|[a+2h,a+2h]|} (a + 2h)\right).$$

Expanding, we obtain the matrix equation with only one unknown matrix $A_1$:

$$A_1 = \frac{2}{h^2} \left[f \left(a + 2h, S_{|[a+a+h]|} (a + h) + S'_{|[a,a+h]|} (a + h)h + \frac{1}{2}S''_{|[a,a+h]|} (a + h)h^2 + \frac{1}{6}A_1h^3\right) - S'_{|[a,a+h]|} (a + h) - S''_{|[a,a+h]|} (a + h)\right]. \quad (2.10)$$

Let us assume that the matrix equation (2.10) has only one solution $A_1$. This way the spline is totally determined in the interval $[a + h, a + 2h]$.

Iterating this process, let us construct the matrix-cubic spline taking $[a + (k - 1)h, a + kh]$ as the last subinterval. For the next subinterval $[a + kh, a + (k + 1)h]$, we define the corresponding matrix-cubic spline as

$$S_{|[a+kh,a+(k+1)h]|} (x) = \beta_k (x) + \frac{1}{3!}A_k (x - (a + kh))^3, \quad (2.11)$$

where

$$\beta_k (x) = \sum_{k=0}^{2} \frac{1}{k!} S^{(k)}_{|[a+(k-1)h,a+kh]|} (a + kh)(x - (a + kh))^k. \quad (2.12)$$

With this definition, the matrix-cubic spline is $S(x) \in \mathcal{C}^2 \left( \bigcup_{j=0}^{k} [a + jh, a + (j + 1)h]\right)$ and fulfills the differential equation (2.1) at point $x = a + kh$. As an additional requirement, we assume that $S(x)$ satisfies the differential equation (2.1) at the point $x = a + (k + 1)h$:

$$S'_{|[a+(k+1)h,a+(k+1)h]|} (a + (k + 1)h) = f \left(a + (k + 1)h, S_{|[a+kh,a+(k+1)h]|} (a + (k + 1)h)\right).$$
and expanding this equation with the unknown matrix \( A_k \) yields

\[
A_k = \frac{2}{h^2} \left[ f \left( a + (k + 1)h, S_{|a+k|}[a+(k-1)h]h, \right) + S'_{|a+k|}[a+(k-1)h]h, \right) \right) + \frac{1}{2} S''_{|a+k|}[a+(k-1)h]h, \right) \right) + \frac{1}{6} A_k h^3 \right) \\
- S'_{|a+k|}[a+(k-1)h]h - S''_{|a+k|}[a+(k-1)h]h, \right) \right) .
\]

(2.13)

Note that this matrix equation (2.13) is analogous to equations (2.8) and (2.10), when \( k = 0 \) and \( k = 1 \), respectively. We will show that these equations have an unique solution using a fixed-point argument.

For a fixed \( h \), we will consider the matrix function of matrix variable \( g : \mathbb{C}^{r \times q} \rightarrow \mathbb{C}^{r \times q} \) defined by

\[
g(T) = \frac{2}{h^2} \left[ f \left( a + (k + 1)h, S_{|a+k|}[a+(k-1)h]h, \right) + S'_{|a+k|}[a+(k-1)h]h, \right) \right) + \frac{1}{2} S''_{|a+k|}[a+(k-1)h]h, \right) \right) + \frac{1}{6} T h^3 \right) \\
- S'_{|a+k|}[a+(k-1)h]h - S''_{|a+k|}[a+(k-1)h]h, \right) \right) .
\]

(2.14)

Relation (2.13) holds if and only if \( A_k = g(A_k) \), that is, if \( A_k \) is a fixed point for function \( g(T) \).

Observe that by using (2.12) and applying the global Lipschitz’s condition (2.3) it follows that

\[
\| g(T_1) - g(T_2) \| \leq \frac{Lh}{3} \| T_1 - T_2 \| .
\]

Taking \( h < 3/L \), \( g(T) \) yields a contractive matrix function, which guarantees that equation (2.13) has unique solutions \( A_k \) for \( k = 0, 1, \ldots, n - 1 \). Hence, the matrix-cubic spline is completely determined. Taking into account [4, Theorem 5], the following result can be established.

**Theorem 2.1** Let be \( L \) the Lipschitz constant defined by (2.3). If \( h \leq 3/L \), then the matrix-cubic spline \( S(x) \) exists in each subinterval \([a + kh, a + (k + 1)h], k = 0, 1, \ldots, n - 1\), as defined in the previous construction. Furthermore, if \( f \in \mathcal{C}^3(T) \), then \( \| Y(x) - S(x) \| = O(h^4) \) \( \forall x \in [a, b] \), where \( Y(x) \) is the theoretical solution of (2.1).

### 3 Algorithm

The following algorithm is designed to compute the approximate solution of (2.1) by means of matrix-cubic splines in the interval \([a, b]\) with an error of the order \( O(h^4) \) under conditions of theorem (2.1).

- Determine the constant \( Y''(a) \) given by (2.6). Take \( n > L(b - a)/3, h = (b - a)/n \) and the partition \( \Delta[a, b] \) defined by Eq. (2.3).
- Solve the matrix equation (2.8) for \( k = 0 \) and determine \( S_{|a+h|}[a+h] \) of Eq. (2.5).
- Solve the matrix equation (2.13) iteratively for \( k = 1, \ldots, n - 1 \), and then compute the splines \( S_{|a+kh|}[a+kh] \) of Eq. (2.11).

Depending on the function \( f(r, Y) \), matrix equations (2.8) and (2.13) can be solved explicitly (see [8]) or using the iterative method (see for example [9]):

\[
T_{k+1} = g(T_k), \quad \text{where} \quad T_0 = \text{an arbitrary matrix in } \mathbb{C}^{r \times q}, \quad s = 0, 1, \ldots, n - 1
\]

and \( g(T) \) is given for (2.14). In the following section, we will test the algorithm proposed.
4 Example: A non-linear vector system

We consider the next non-linear vector differential system
\[
\begin{align*}
y_1'(x) &= -1 + e^x - \sin(x) + \sin(y_2(x)) \\
y_2'(x) &= \frac{1}{4+y_1(x)^2} - \frac{1}{5+e^{2x}+2e^x\cos(x) - \sin^2(x)} \\
y_1(0) &= 2, \quad y_2(0) = \frac{\pi}{2}
\end{align*}
\]  
\(0 \leq x \leq 1,
\]
(4.1)

It is easy to check that this problem has the exact solution \(y_1(x) = e^x + \cos(x), y_2(x) = \frac{\pi}{2}\), so in this particular case we will be able to obtain the exact error of our numerical estimates.

We can rewrite (4.1) in the compact form
\[
\begin{align*}
Y'(x) &= F(x, Y) \\
Y(0) &= \begin{pmatrix} 2 \\ \frac{\pi}{2} \end{pmatrix}
\end{align*}
\]

\(0 \leq x \leq 1, \quad Y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} \in \mathbb{R}^2, \quad F(x, Y) = \begin{pmatrix} -1 + e^x - \sin(x) + \sin(y_2(x)) \\ \frac{1}{4+y_1(x)^2} - \frac{1}{5+e^{2x}+2e^x\cos(x) - \sin^2(x)} \end{pmatrix} \in \mathbb{R}^2,
\]
(4.2)

thus \(Y'(0) = F \left( 0, \begin{pmatrix} 2 \\ \frac{\pi}{2} \end{pmatrix} \right) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\). We calculate \(Y''(0)\) using (2.6), in this case, one gets
\[
\begin{align*}
\vec{Y}(x) &= Y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}, \\
\frac{\partial F(x, Y(x))}{\partial x} &= \begin{pmatrix} e^x - \cos(x) \\ \frac{2e^{2x}+2e^x\cos(x) - 2e^x\sin(x) - 2\cos(x)\sin(x)}{(5+e^{2x}+2e^x\cos(x) - \sin^2(x))^2} \end{pmatrix}.
\end{align*}
\]
(4.3)

On the other hand, we have
\[
\begin{align*}
[\vec{F}(x, Y(x))]^T \otimes I_2 &= \begin{pmatrix} -1 + e^x - \sin(x) + \sin(y_2(x)) \\ 0 \\ -1 + e^x - \sin(x) + \sin(y_2(x)) \end{pmatrix} \begin{pmatrix} \frac{1}{4+y_1(x)^2} - \frac{1}{5+e^{2x}+2e^x\cos(x) - \sin^2(x)} \\ \frac{1}{4+y_1(x)^2} - \frac{1}{5+e^{2x}+2e^x\cos(x) - \sin^2(x)} \end{pmatrix} \otimes I_2 \\
&= \begin{pmatrix} -1 + e^x - \sin(x) + \sin(y_2(x)) \\ 0 \\ -1 + e^x - \sin(x) + \sin(y_2(x)) \end{pmatrix} \begin{pmatrix} \frac{1}{4+y_1(x)^2} - \frac{1}{5+e^{2x}+2e^x\cos(x) - \sin^2(x)} \\ \frac{1}{4+y_1(x)^2} - \frac{1}{5+e^{2x}+2e^x\cos(x) - \sin^2(x)} \end{pmatrix} \otimes I_2
\end{align*}
\]
(4.4)

and
\[
\frac{\partial F(x, Y(x))}{\partial \vec{Y}(x)} = \begin{pmatrix} \frac{\partial F(x, Y(x))}{\partial y_1} \\ \frac{\partial F(x, Y(x))}{\partial y_2} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial y_1} (-1 + e^x - \sin(x) + \sin(y_2(x))) \\ \frac{\partial}{\partial y_2} (-1 + e^x - \sin(x) + \sin(y_2(x))) \end{pmatrix} = \begin{pmatrix} \frac{0}{(4+y_1(x))^2} \\ \cos(y_2(x)) \end{pmatrix},
\]
(4.5)

Therefore
\[
\begin{align*}
\left[ [\vec{F}(x, Y(x))]^T \otimes I_2 \right] \frac{\partial F(x, Y(x))}{\partial \vec{Y}(x)} &= \begin{pmatrix} \frac{1}{4+y_1(x)^2} - \frac{9+2e^{2x}+4e^x\cos(x)+2e^x\sin(x)}{5+e^{2x}+2e^x\cos(x) - \sin^2(x)} \\ \frac{2}{2y_1(x)(-1 + e^x - \sin(x) + \sin(y_2(x)))} \end{pmatrix} \frac{1}{(4+y_1(x))^2} \cos(y_2(x)),
\end{align*}
\]
(4.6)

and by (4.3), (4.6) one concludes
### Approximation

| Interval | Approximation | Max. Error |
|----------|---------------|------------|
| [0, 0.1] | \[
\frac{2 + x + 0.177917x^3}{5 \cdot 6.2424 \times 10^{-6} x^3}
\] | 2.83337 \times 10^{-6} |
| [0.1, 0.2] | \[
1.99995 + 1.00138x - 0.0131342e^x + 0.224031x^4 + 0.667857 \times 10^{-6} x^3 + 0.0000166377x^3
\] | 2.83337 \times 10^{-6} |
| [0.2, 0.3] | \[
1.99975 + 1.00445x - 0.0291822e^x + 0.249611x^3 + 1.5708 - 4.57386 \times 10^{-6} x^3 + 0.00001953x^3 - 0.0000270433x^3
\] | 2.94712 \times 10^{-6} |
| [0.3, 0.4] | \[
1.99841 + 1.01783x - 0.0737602e^x + 0.299142e^x + 1.57079 + 0.0000126873x - 0.0000380073x^2 + 0.0000368871x^3
\] | 2.94712 \times 10^{-6} |
| [0.4, 0.5] | \[
1.99655 + 1.0318x - 0.108685x^3 + 0.328246x^3 + 1.5708 - 0.9000227163x^3 + 0.0000616192x^2 + 0.0000461351x^3
\] | 3.0698 \times 10^{-6} |
| [0.5, 0.6] | \[
1.99899 + 1.07171x - 0.188509x^3 + 0.381462x^3 + 1.57079 + 0.0000485499x - 0.000898071x^2 + 0.0000548159x^3
\] | 3.0698 \times 10^{-6} |
| [0.6, 0.7] | \[
1.98277 + 1.10736x - 0.247933x^3 + 0.414475x^3 + 1.57081 - 0.0000785694x + 0.000122058x^2 - 0.0000628872x^3
\] | 3.20977 \times 10^{-6} |
| [0.7, 0.8] | \[
1.96307 + 1.19177x - 0.368517x^3 + 0.471896x^3 + 1.57077 + 0.00011733x - 0.000157802x^2 + 0.0000703796x^3
\] | 3.20977 \times 10^{-6} |
| [0.8, 0.9] | \[
1.94382 + 1.26395x - 0.45874x^3 + 0.59489x^3 + 1.57084 - 0.0001661x + 0.00019649x^2 - 0.0000772419x^3
\] | 3.37764 \times 10^{-6} |
| [0.9, 1.0] | \[
1.89829 + 1.41574x - 0.627395x^3 + 0.571954x^3 + 1.57073 + 0.0000224533x - 0.0000625482x^2 + 0.0000835127x^3
\] | 3.37764 \times 10^{-6} |

Table 1: Approximation for vector differential system \([4.1]\) in the interval \([0, 1]\) with step size \(h = 0.1\).

\[
Y''(x) = \frac{\partial F(x, Y(x))}{\partial x} + \left[ \text{vec } F(x, Y(x)) \right]^T I_2 \frac{\partial F(x, Y(x))}{\partial \text{vec } Y(x)}
\]

\[
= \left( \frac{e^x - \cos(x)}{2e^x + 2e^x \cos(x) - 2e^x \sin(x) - 2e^x \sin(x) - 2e^x \sin(x) - 2e^x \sin(x) - 2e^x \sin(x) - 2e^x \sin(x) - 2e^x \sin(x) - 2e^x \sin(x)} \right) \cos(y_2(x))
\]

\[
\left( \frac{5 + e^x + 2e^x \cos(x) + e^x \sin(x)}{(5 + e^x + 2e^x \cos(x) - \sin^2(x))^2} \right)^2
\]

Taking into account that \(y_1(0) = 2, y_2(0) = \frac{5}{2}\) and evaluating \(Y''(x)\) of \([4.7]\) when \(x = 0\), one gets \(Y''(0) = \left( \begin{array}{c} 0 \\ 0 \end{array} \right)\).

It is straightforward to show that \(F\), defined by \([4.2]\), fulfills the global Lipschitz's condition

\[
\| f(x, Y) - f(x, Z) \| \leq \| Y - Z \| , \quad 0 \leq x \leq 1 , Y, Z \in \mathbb{R}^2
\]

thus, we can take \(L\) given by \([4.3]\) as \(L = 1\). Therefore, we need to take \(h < 3/L\) and thus \(h = 0.1\) for example. The results are generated with Mathematica using \(\text{FindRoot}\) function to solve the emerging algebraic equations, and are summarized in Table 1. In each interval, we evaluated the difference between the estimates of our numerical approach and the exact solution, and then take the Frobenius norm of this difference. The maximum of these errors are indicated in the third column for each subinterval.

## 5 Example: Sylvester matrix differential equation

Linear matrix differential equations of the type
Figure 1: Representing the Frobenius error margins for vector differential system (4.1) in the interval \([0, 1]\) with step size \(h = 0.1\).

\[
Y'(x) = A(x)Y(x) + Y(x)B(x) + C(x) \\
Y(a) = Y_a
\]

\(a \leq x \leq b\), \(Y(x), A(x), B(x), C(x) \in \mathbb{C}^{r \times r}\), (5.1)

arise in many fields of science and engineering. In the case of constant coefficients has been studied by several authors (see for example [10]). However, the variable-coefficient case has so far received little numerical treatment in the literature. We can observe that the proposed method require the matrix functions \(A(x), B(x)\) and \(C(x)\) to be differen-
tiable, while, for example, in the method proposed in [11], it is necessary that \(A(x), B(x)\) have continuous second-order derivatives and \(C(x)\) continuous in the domain \(a \leq x \leq b\).

As an example, here let us consider the Sylvester problem (5.1) with

\[
A(x) = \begin{pmatrix} 0 & xe^{-x} \\ x & 0 \end{pmatrix}, \\
B(x) = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}, \\
C(x) = \begin{pmatrix} -e^{-x}(1 + x^2) & -2e^{-x}x \\ 1 - e^{-x}x & -x^2 \end{pmatrix}
\]

\(Y(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\), \(Y(x) \in \mathbb{C}^{2 \times 2}\), \(0 \leq x \leq 1\) (5.2)

This problem has an exact solution \(Y(x) = \begin{pmatrix} e^{-x} & 0 \\ x & 1 \end{pmatrix}\), so in this particular case we will be able to obtain the exact error of our numerical estimates.

As we have \(\max_{x \in [0, 1]} \left( \left\| \begin{pmatrix} 0 & xe^{-x} \\ x & 0 \end{pmatrix} \right\| + \left\| \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \right\| \right) \leq 1.69443\), one can take the constant \(L\) given for (2.3) as \(L = 2\).

Taking the derivative of \(Y'(x) = A(x)Y(x) + Y(x)B(x) + C(x)\), gives:
Approximation of matrix-valued functions arises frequently in important applications to classical control theory [13] and as decoupling techniques for both the analytic and numerical study of boundary value problems [14]. The Riccati equation (6.1)
Figure 2: Representing the absolute error margins for the Sylvester matrix differential equation \((5.2)\) in the interval \([0, 1]\) with step size \(h = 0.1\).

has been studied extensively, and different resolution techniques have been introduced (see [15] and references therein).

The study of the Riccati equation \((6.1)\) is closely related to the underlying linear system

\[
\begin{align*}
X'(x) & = S(x)X(x) \\
X(0) & = \begin{bmatrix} I_q \\ Y_0 \end{bmatrix}
\end{align*}
\]

where

\[
X(x) = \begin{bmatrix} U(x) \\ V(x) \end{bmatrix}, \quad S(x) = \begin{bmatrix} A(x) & B(x) \\ C(x) & -D(x) \end{bmatrix}.
\]

(6.2)

Specifying the solution of \((6.1)\) is given by

\[
Y(x) = V(x)U^{-1}(x)
\]

(6.3)

where \(Y(x)\) is defined in the interval where \(U(x)\) is invertible, see [16].

Taking into account lemma 1 and 2 of [17], \(U(x)\) is invertible in the interval \([0, \delta]\) and the solution \(Y(x)\) of problem \((6.1)\) satisfies

\[
\|Y(x)\| \leq M, \quad M = (1 - \delta q_0 \exp(\delta k_0)w_0)^{-1} w_0 \exp(\delta k_0),
\]

(6.4)

where \(\delta\) is a positive number satisfying

\[
\delta k_0 + \log(\delta) < -\log(q_0w_0),
\]

(6.5)

and

\[
\begin{align*}
k_0 & = \max \left\{ \| \begin{bmatrix} A(x) & B(x) \\ C(x) & -D(x) \end{bmatrix} \| ; \ 0 \leq x \leq c \right\} \\
q_0 & = \max \{ \| A(x)B(x) \| ; \ 0 \leq x \leq c \} \\
w_0 & = \| \begin{bmatrix} I_q \\ Y_0 \end{bmatrix} \|
\end{align*}
\]

(6.6)
In accordance with [18, p.1064], we consider the matrix-valued function
\[ F(x,Y) = C(x) - D(x)Y - YA(x) - YB(x)Y, \]  
we consider the matrix-valued function
\[ F(x,Y) = C(x) - D(x)Y - YA(x) - YB(x)Y, \]  
then, if we define
\[ \begin{align*}
  a &= \sup \{ ||A(x)|| : 0 \leq x \leq \delta \} \\
  b &= \sup \{ ||B(x)|| : 0 \leq x \leq \delta \} \\
  c &= \sup \{ ||C(x)|| : 0 \leq x \leq \delta \} \\
  d &= \sup \{ ||D(x)|| : 0 \leq x \leq \delta \}
\end{align*} \tag{6.8} \]
and \( ||Y|| \leq M, ||\bar{Y}|| \leq M, \) with \( M \) gives by (6.3), the following local Lipschitz’s condition holds
\[ ||F(x,Y) - F(x,\bar{Y})|| \leq L ||Y - \bar{Y}||, \quad L = a + d + 2bM. \]  
In addition, if \( ||Y|| \leq N, \)
\[ ||F(x,Y)|| \leq c + N(a + d + bN). \]  
Using the proposed spline method, the only one solution of the matrix equations (2.8) and (2.13) for \( k = 1, \ldots, n - 1 \) is guaranteed using a fixed-point argument and the global Lipschitz’s condition (2.3). In our case, we need to prove the only one solution of the matrix equations (2.8) and (2.13) using a fixed point argument and the local Lipschitz’s condition (6.9).

We start with the matrix equation (2.8). Let us suppose that \( ||T|| \leq N_1. \) Taking into account (6.10), we take
\[ \begin{align*}
  N_2 &= ||Y(a)|| + h ||Y'(a)|| + \frac{h^2}{2} ||Y''(a)|| + \frac{h^3}{6} N_1 \\
  N_3 &= c + N_2(a + d + bN_2) \\
  N_4 &= \frac{2}{h^2} (N_3 + ||Y'(a)|| + h ||Y''(a)||)
\end{align*} \tag{6.11} \]
with \( a, b, c \) given by (6.8), and let be \( N = \max \{ N_1, N_2, N_3, N_4, M \} \) with \( M \) gives by (6.3). Let be \( \mathcal{A} = \{ Y \in \mathbb{C}^{r \times q} ; ||Y|| \leq N \} \) and we consider the continuous matrix-valued function of matrix variable \( g : \mathbb{C}^{r \times q} \rightarrow \mathbb{C}^{r \times q} \) defined by (2.14) for \( k = 0. \)

It is simple to verify that if \( T \in \mathcal{A}, \) by (6.11) and (6.10) then \( g(T) \in \mathcal{A}. \) Thus, \( g : \mathcal{A} \rightarrow \mathcal{A} \) and \( A_0 \) is a fixed point of \( g. \) In addition, if \( T_1, T_2 \in \mathcal{A}, \) \( ||T_1|| \leq M, ||T_2|| \leq M, \) has then that for \( f \) defined by (6.7), \( f \) fulfills the local Lipschitz’s condition (6.9) and taking \( h < 3/L, \) \( g(T) \) yields a contractive matrix function, which guarantees that \( g(T) \) has unique solutions \( A_0. \) Hence, the matrix-cubic spline is completely determined in \([a, a + h].\)

For \( k = 1, \ldots, n - 1, \) fixed, supposed construct cubic-matrix spline \( S(x) \) taking \([a + (k - 1)h, a + kh]\) as the last subinterval, for the next subinterval \([a + kh, a + (k + 1)h], \) to define the corresponding spline we need determine \( A_k \in \mathbb{C}^{r \times q} \) as the only one solution of the matrix equation (2.13). Let us suppose that \( ||T|| \leq N_1. \) Taking into account (6.10), we take
\[ \begin{align*}
  \tilde{N}_2 &= \left| \left| S'_{\left| a + (k - 1)h, a + kh \right|} (a + kh) \right| + h \left| \left| S''_{\left| a + (k - 1)h, a + kh \right|} (a + kh) \right| + \frac{h^2}{2} \left| \left| S'''_{\left| a + (k - 1)h, a + kh \right|} (a + kh) \right| \right| + \frac{h^3}{6} \tilde{N}_1 \right| \\
  \tilde{N}_3 &= c + \tilde{N}_2(a + d + b\tilde{N}_2) \\
  \tilde{N}_4 &= \frac{2}{h^2} \left( \tilde{N}_3 + \left| \left| S'_{\left| a + (k - 1)h, a + kh \right|} (a + kh) \right| + h \left| \left| S''_{\left| a + (k - 1)h, a + kh \right|} (a + kh) \right| \right| \right) \right)
\end{align*} \tag{6.12} \]
with \( a, b, c \) given by (6.8), and let be \( \tilde{N} = \max \{ \tilde{N}_1, \tilde{N}_2, \tilde{N}_3, \tilde{N}_4, M \} \) with \( M \) gives by (6.3). Let be \( \mathcal{A} = \{ Y \in \mathbb{C}^{r \times q} ; ||Y|| \leq \tilde{N} \} \) and we consider the continuous matrix-valued function of matrix variable \( g : \mathbb{C}^{r \times q} \rightarrow \mathbb{C}^{r \times q} \) defined by (2.14).

It is simple to verify that if \( T \in \mathcal{A}, \) by (6.12) and (6.10) then \( g(T) \in \mathcal{A}. \) Thus, \( g : \mathcal{A} \rightarrow \mathcal{A} \) and \( A_k \) is a fixed point of \( g. \) In addition, if \( T_1, T_2 \in \mathcal{A}, \) \( ||T_1|| \leq M, ||T_2|| \leq M, \) has then that for \( f \) defined by (6.7), \( f \) fulfills the local Lipschitz’s
error for all our numerical estimates. A short computation using expressions (6.4)–(6.9) yields the following constants

\[
\text{Max. Error} = 1.39903 \times 10^{-10}
\]

maximum of these errors are indicated in the third column foreach subinterval.

FindRoot function to solve the emerging algebraic equations, and is summarized in Table 3, where the numerical estimates have been rounded to the fourth relevant digit. In each interval, we evaluated the difference between the estimates of our numerical approach and the exact solution, and then take the Frobenius norm of this difference. The maximum of these errors are indicated in the third column for each subinterval.

| Interval     | Approximation                                                                 | Max. Error     |
|--------------|-------------------------------------------------------------------------------|----------------|
| [0, 0.01]    | \( \begin{pmatrix} 1 + x + 0.5x^2 + 0.167224x^3 \\ x \end{pmatrix} \)     | 1.39903 \times 10^{-10} |
| [0.01, 0.02] | \( \begin{pmatrix} 1 + x + 0.499933x^2 + 0.169461x^3 \\ x \end{pmatrix} \)   | 1.39903 \times 10^{-10} |
| [0.02, 0.03] | \( \begin{pmatrix} 1 + x + 0.499864x^2 + 0.17061x^3 \\ x \end{pmatrix} \)   | 1.41977 \times 10^{-10} |
| [0.03, 0.04] | \( \begin{pmatrix} 1 + 1.00001x + 0.49966x^2 + 0.172877x^3 \\ x \end{pmatrix} \) | 1.41977 \times 10^{-10} |
| [0.04, 0.05] | \( \begin{pmatrix} 1 + 1.00001x + 0.499518x^2 + 0.174063x^3 \\ x \end{pmatrix} \) | 1.44084 \times 10^{-10} |
| [0.05, 0.06] | \( \begin{pmatrix} 1 + 1.00003x + 0.499173x^2 + 0.176362x^3 \\ x \end{pmatrix} \) | 1.44084 \times 10^{-10} |
| [0.06, 0.07] | \( \begin{pmatrix} 0.999999 + 1.00004x + 0.498952x^2 + 0.177587x^3 \\ x \end{pmatrix} \) | 1.46223 \times 10^{-10} |
| [0.07, 0.08] | \( \begin{pmatrix} 0.999999 + 1.00008x + 0.498463x^2 + 0.179918x^3 \\ x \end{pmatrix} \) | 1.46223 \times 10^{-10} |
| [0.08, 0.09] | \( \begin{pmatrix} 0.999998 + 1.00001x + 0.49816x^2 + 0.181181x^3 \\ x \end{pmatrix} \) | 1.48391 \times 10^{-10} |
| [0.09, 0.1]  | \( \begin{pmatrix} 0.999996 + 1.00016x + 0.497521x^2 + 0.183546x^3 \\ x \end{pmatrix} \) | 1.48391 \times 10^{-10} |

Table 3: Approximation for Riccati matrix differential equation (6.13) in the interval [0, 0.1] with step size \( h = 0.01 \).

condition (6.9) and taking \( h < 3/L \), \( g(T) \) yields a contractive matrix function, which guarantees that equation (2.13) has unique solutions \( A_k \). Hence, the matrix-cubic spline is completely determined.

As an additional example for our proposed method, we consider the Riccati matrix differential equation (6.1) with

\[
A(x) = \begin{pmatrix} -x & 0 \\ -x & x \end{pmatrix}, \quad B(x) = \begin{pmatrix} -x^2 & -2 \\ 0 & 1 \end{pmatrix}, \quad D(x) = \begin{pmatrix} -1 & -x^2 \\ x & x \end{pmatrix},
\]

\[
C(x) = \begin{pmatrix} x(-e^x + e^x - x^3) \\ (1-x)(2+x+2x^2) + 1(3-2x)^x + e^x(x-x^4) \end{pmatrix}, \quad Y(0) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{6.13}
\]

In this case, the problem has an exact solution given by \( Y(x) = \begin{pmatrix} 0 & e^x \\ x^2 & x \end{pmatrix} \), which will permit us to obtain the total error for all our numerical estimates. A short computation using expressions (6.4)–(6.9) yields the following constants

\[
\begin{align*}
   k_0 &= 6.13866 \\
   w_0 &= \sqrt{2} \\
   M &= 12.0883 \\
   b &= 2.23609 \\
   d &= 1.01 \\
   q_0 &= 3 \\
   \delta &= 0.115758 \\
   \alpha &= 0.173205 \\
   c &= 1.17928 \\
   L &= 55.2443
\end{align*}
\]

which are necessary for the spline approximation in the interval [0, 0.1], where \( \delta = 0.1 \) is taken for convenience. Therefore, we need to take \( h < 3/L = 0.0543042 \) and thus \( h = 0.01 \). The results are generated with Mathematica using FindRoot function to solve the emerging algebraic equations, and are summarized in Table 3, where the numerical estimates have been rounded to the fourth relevant digit. In each interval, we evaluated the difference between the estimates of our numerical approach and the exact solution, and then take the Frobenius norm of this difference. The maximum of these errors are indicated in the third column for each subinterval.
7 Conclusions

This article develops a new method for the numerical integration of first-order matrix differential equations of the non-linear type \( Y'(x) = f(x, Y(x)), x \in [a, b] \) using matrix-cubic splines, and thereby generalizing the approach for the linear case in previous work [1]. An important advantage of the proposed method is that the approximated solution is continuous in the interval under consideration, is easy to evaluate, and has an error of the order \( O(h^4) \).

Our method is well-suited for implementation on numerical and/or symbolical computer systems (Mathematica, Matlab, etc.) as we have shown in Section 3 giving the explicit algorithm. For a full demonstration of our approach and its advantages, we conclude with two numerical examples for the Sylvester and Riccati matrix differential equations.

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