Invariants of Linear Control Systems with Analytic Matrices and the Linearizability Problem

K. V. Sklyar · S. Yu. Ignatovich

Received: 13 February 2021 / Revised: 29 August 2021 / Accepted: 17 September 2021 / Published online: 27 October 2021
© The Author(s), under exclusive licence to Springer Science+Business Media, LLC, part of Springer Nature 2021

Abstract

The paper continues the authors’ study of the linearizability problem for nonlinear control systems. In the recent work (Sklyar, Syst Control Lett 134:104572, 2019), conditions on mappability of a nonlinear control system to a preassigned linear system with analytic matrices were obtained. In the present paper, we solve more general problem on linearizability conditions without indicating a target linear system. To this end, we give a description of invariants for linear nonautonomous single-input controllable systems with analytic matrices, which allow classifying such systems up to transformations of coordinates. This study leads to one problem from the theory of linear ordinary differential equations with meromorphic coefficients. As a result, we obtain a criterion for mappability of nonlinear control systems to linear control systems with analytic matrices.

Keywords Nonlinear control system · Linearizability problem · Linear control system with analytic matrices · Invariant · Linear ODE with meromorphic coefficients

Mathematics Subject Classification (2010) 93B18 · 34A30

1 Introduction

The problem of linearizability attracts a great attention of experts in the control theory during almost fifty years. Starting from A. Krener [1], in the study of linearizability the Lie bracket technique is applied. We mention several important first publications [2–6]; this list undoubtedly is far from being complete. An important analytic tool here is the Frobenius Theorem on solvability of a system of partial differential equations of the first order. Within this approach, traditionally, control systems of class $C^\infty$ were considered. An
alternative idea was introduced by V. I. Korobov [7], who proposed to consider a special class of triangular systems. This approach admitted $C^1$-smooth systems. Later, these ideas were extended to general nonlinear control systems of class $C^1$ [8–12].

However, most researchers focused on autonomous control systems when studying the linearizability. In the present paper we concentrate on a nonautonomous linearizability problem. More specifically, let us consider a nonlinear control system

$$\dot{x} = f(t, x, u), \ x \in Q \subset \mathbb{R}^n, \ u \in \mathbb{R}, \ t \in [\alpha, \beta],$$

where $Q$ is a given simply connected open domain and $[\alpha, \beta]$ is a given closed interval. We are interested in mapping such a system to a linear nonautonomous system

$$\dot{z} = A(t)z + b(t)u$$  \hspace{1cm} (1)

by use of a change of variables assuming that it is also time-dependent, i.e., $z = F(t, x)$. Following [8], we assume that the vector function $f(t, x, u)$ is of class $C^1$ and a change of variables $F(t, x)$ is of class $C^2$. We also require that a target linear system has analytic matrices, i.e., $A(t)$ and $b(t)$ are real analytic on the closed interval $[\alpha, \beta]$. Such systems were considered, e.g., in [13, 14]. It was shown that the Markov moment problem [15, 16] can be efficiently applied for solving the time-optimal control problem for such systems. Moreover, conditions were proposed under which the optimal control can be found by the successive approximation method, on each step of which a power Markov moment problem is solved. In turn, a method of explicit solving the power Markov moment problem was proposed in [17]; see also [18].

The nonautonomous statement of the linearizability problem mentioned above was proposed in [19], where conditions on mappability of a nonlinear control system to a pre-assigned linear system (1) with analytic matrices were obtained. In the present paper we solve more general problem: to find conditions under which a nonlinear system is mapped to some linear system with analytic matrices, which is not given in advance.

The precise formulation of the problem is given in Section 2. As a preparatory step, in Section 3 we study some properties of the class of linear nonautonomous systems with analytic matrices, which are closely connected with the classical study of homogeneous linear differential equations of order $n$ with meromorphic coefficients [20]. In Section 4 we apply the obtained results and obtain the theorem on nonautonomous linearizability of nonlinear control systems.

## 2 Background and Statement of the Problem

Let us consider a linear control system of the form

$$\dot{x} = A(t)x + b(t)u$$  \hspace{1cm} (2)

where the matrix $A(t)$ and the vector $b(t)$ are real analytic on the closed interval $[\alpha, \beta]$. For this system, we introduce the following matrix

$$K(t) = (\Delta^0(t), \Delta^1(t), \ldots, \Delta^{n-1}(t)), \text{ where } \Delta^k(t) = \left(-A(t) + \frac{d}{dt}\right)^k b(t), \ k \geq 0.$$

Below we assume that the system (2) is controllable on the time interval $[\alpha, \beta]$. Hence, the matrix $K(t)$ is invertible on $[\alpha, \beta]$ except maybe a finite number of points, say, $(t_i)_{i=1}^N$. Let us introduce the following vector function

$$\gamma(t) = K^{-1}(t)\Delta^n(t).$$  \hspace{1cm} (3)
Invariants of Linear Control Systems...

Obviously, its components are analytic or meromorphic functions on \([\alpha, \beta]\).

Suppose we apply a linear change of variables \(z = F(t)x\) in the system (2), where \(F(t)\) is a nonsingular analytic matrix, and the system in the new variables takes the form

\[
\dot{z} = \tilde{A}(t)z + \tilde{b}(t)u.
\]

Then \(\tilde{A}(t) = (\tilde{F}(t) + F(t)A(t))F^{-1}(t), \tilde{b}(t) = F(t)b(t)\). Denote

\[
\tilde{\gamma}(t) = (\tilde{\Delta}^0(t), \tilde{\Delta}^1(t), \ldots, \tilde{\Delta}^{n-1}(t)), \text{ where } \tilde{\Delta}^k(t) = (-\tilde{A}(t) + \frac{d}{dt})^k\tilde{b}(t), \quad k \geq 0,
\]

\[
\tilde{\gamma}(t) = \tilde{K}^{-1}(t)\tilde{\Delta}^n(t).
\]

Then \(\tilde{\Delta}^k(t) = F(t)\Delta^k(t), \quad k \geq 0\). Hence, \(\tilde{K}(t) = F(t)K(t)\) is invertible on \([\alpha, \beta]\) \(\setminus \{t_i\}_{i=1}^N\)

and

\[
\tilde{\gamma}(t) = \gamma(t), \quad t \in [\alpha, \beta] \setminus \{t_i\}_{i=1}^N.
\]

Therefore, the components of \(\gamma(t)\) are invariant w.r.t. (linear analytic) changes of variables. Following [19], we call them invariants of the system (2).

If the initial controllable system is autonomous, \(\dot{x} = Ax + bu\), then \(K(t)\) turns into the Kalman matrix \(K = (b, -Ab, \ldots, (-A)^{n-1}b)\). Then \(\gamma(t)\) is a constant vector, \(\gamma = K^{-1}(-A)^N\).

One can easily show that components of \(\gamma\) equal coefficients of the characteristic polynomial of the matrix \(A\) up to a sign. More specifically, the characteristic polynomial of the matrix \(A\) equals \(\chi(\lambda) = \lambda^n - \sum_{s=0}^{n-1}(-1)^{n-s}\gamma_{s+1}\lambda^s\). Therefore, in the autonomous case the invariants are constants and, moreover, they can be arbitrary real numbers. Thus, any collection of numbers \(\gamma_1, \ldots, \gamma_n\) uniquely defines the set of linear systems that are mapped to each other by linear analytic changes of variables.

Unlike the autonomous systems, in the general case invariants are meromorphic functions, however, not all meromorphic functions are invariants of linear systems. We illustrate this claim by the following very simple example.

**Example 1** Suppose \(n = 1\), then the system (2) turns into the equation, where \(A(t)\) and \(b(t)\) are analytic functions. By definition, \(K(t) = b(t)\) and therefore

\[
\gamma(t) = \frac{1}{b(t)}(-A(t)b(t) + \dot{b}(t)) = -A(t) + \frac{\dot{b}(t)}{b(t)}.
\]

Let us pick any point \(t_0 \in [\alpha, \beta]\). If \(b(t_0) \neq 0\), then the function \(\gamma(t)\) is analytic at \(t_0\). If \(b(t_0) = 0\), then we can write \(b(t) = (t - t_0)^rc(t)\), where \(r\) is a positive integer, \(c(t)\) is analytic, and \(c(t_0) \neq 0\). Then

\[
\gamma(t) = -A(t) + \frac{\dot{c}(t)}{c(t)} + \frac{r}{t - t_0},
\]

where \(-A(t) + \frac{\dot{c}(t)}{c(t)}\) is analytic at \(t_0\). Thus, in any case the invariant \(\gamma(t)\) is analytic or meromorphic with the pole of order 1 at \(t_0\) and, in the latter case, its residue equals a positive integer. It turns out that, in this very simple case, the mentioned conditions are necessary and sufficient for \(\gamma(t)\) to be an invariant of some system. To show sufficiency, one can set \(A(t) \equiv 0\) and \(b(t) = (t - t_0)^rc(t)\) (where \(r = 0\) if \(\gamma(t)\) is analytic) and find \(c(t)\) as a solution of the linear differential equation with an analytic coefficient \(\dot{y} = (\gamma(t) - \frac{r}{t - t_0})y\).

In the general case, the following problem arises.

**Realizability Problem** For a given set of functions \(\gamma_1(t), \ldots, \gamma_n(t)\), which are meromorphic on the closed interval \([\alpha, \beta]\), to determine if they are invariants for some linear control system of the form (2) with analytic matrices.
It turns out that this realizability problem plays an important role in linearizability conditions for nonautonomous nonlinear control systems. More specifically, let us consider an affine control system of the form
\[ \dot{x} = a(t, x) + b(t, x)u, \]  
where \( a(t, x), b(t, x) \in C^1([\alpha, \beta] \times Q), Q \subset \mathbb{R}^n \). In order to introduce an analog of invariants for the system (4), let us consider the operator \( \mathcal{R} \) acting as
\[ \mathcal{R}\phi(t, x) = \phi_t(t, x) + \phi_x(t, x)a(t, x) - a_x(t, x)\phi(t, x) \]  
for any vector function \( \phi(t, x) \in C^1([\alpha, \beta] \times Q) \). Here and below the sub-indexes \( t \) and \( x \) stand for derivatives; namely, \( c_t(t, x) \) and \( c_x(t, x) \) denote the derivatives of \( c(t, x) \) on \( t \) and \( x \) respectively. Since \( a(t, x), b(t, x) \in C^1([\alpha, \beta] \times Q) \), the vector function \( \mathcal{R}b(t, x) \) can be defined, however, it can be non-differentiable. If it turns out to be differentiable, then we can define \( \mathcal{R}^2b(t, x) \), and so on. Assuming that all vector functions
\[ \mathcal{R}b(t, x), \mathcal{R}^2b(t, x), \ldots, \mathcal{R}^nb(t, x) \]  
exist, we introduce the matrix
\[ R(t, x) = (b(t, x), \mathcal{R}b(t, x), \ldots, \mathcal{R}^{n-1}b(t, x)) \]  
and the vector function
\[ \gamma(t, x) = R^{-1}(t, x)\mathcal{R}^nb(t, x), \]  
which is defined at all points \( (t, x) \) such that the matrix \( R(t, x) \) is nonsingular.

In the particular case of a driftless control system of the form
\[ \dot{x} = g(t, x)u, \]  
the operator \( \mathcal{R}(t, x) \) reduces to the derivative on \( t \), i.e., \( \mathcal{R}^k g(t, x) = g^{(k)}(t, x), k \geq 0 \) (here and below \( g^{(k)}(t, x) \) means the \( k \)-th derivative on \( t \)). In [19], driftless systems (9) were considered as a kind of a canonical form for general affine systems (4). Under the assumptions \( a(t, x) \in C^2([\alpha, \beta] \times Q) \) and \( b(t, x) \in C^1([\alpha, \beta] \times Q) \), the system (4) can be transformed to a driftless form (9) locally, at a neighborhood of any point from \([\alpha, \beta] \times Q\), by a change of variables \( z = F(t, x) \) of class \( C^2 \). A non-local transformation requires additional conditions as well as the case \( a(t, x) \in C^1([\alpha, \beta] \times Q) \). In [19] driftless systems were used in order to obtain conditions for linearizability of nonautonomous nonlinear systems.

**Definition 1** [19] We say that a system of the form (4) is **locally analytically linearizable in the domain** \( Q \) on the time interval \([\alpha, \beta]\) if there exists a change of variables
\[ z = F(t, x) \in C^2([\alpha, \beta] \times Q) \]  
satisfying the condition
\[ \det F_t(t, x) \neq 0, \quad (t, x) \in [\alpha, \beta] \times Q, \]  
such that the system (4) in the new variables takes the form (2), where \( A(t) \) and \( b(t) \) are analytic on \([\alpha, \beta]\).

In this definition, the word “locally” recalls that the change of variables is only locally invertible in the general case and the word “analytically” means that a target nonautonomous linear system has analytic matrices.

We also mention that the condition \( a(t, x), b(t, x) \in C^1([\alpha, \beta] \times Q) \) is the minimum requirement for the Lie brackets technique commonly used in the linearizability problem,
and the condition \( F(t, x) \in C^2([\alpha, \beta] \times Q) \) is compatible with the previous one, since such a change of variables leaves the system in the class \( C^1([\alpha, \beta] \times Q) \).

**Definition 2** \([19, 21]\) We say that a system of the form (4) is **locally analytically mappable in the domain** \( Q \) **on the time interval** \([\alpha, \beta]\) **to a preassigned linear controllable system** (2) **where** \( A(t), b(t) \) **are analytic on** \([\alpha, \beta]\) **if there exists a change of variables satisfying** (10), (11) **such that the system** (4) **in the new variables takes the form** (2).

We emphasize that, unlike Definition 1, in Definition 2 the target linear system is **fixed**. This requirement is justified by the theorem on linearizability conditions, originally proposed and proved in [19]. We formulate the modification obtained in [21]. Here and below \([\cdot, \cdot]\) denotes the Lie bracket, namely, \([c(t, x), d(t, x)] = d_x(t, x)c(t, x) - c_x(t, x)d(t, x)\). Also, below we use the notation \( \text{ad}_a b(x) = b(x), \text{ad}_a^{k+1} b(x) = [a(x), \text{ad}_a^k b(x)], k \geq 0 \).

**Theorem 1** \([19, 21]\) The system (4), where \( a(t, x) \in C^2([\alpha, \beta] \times Q) \) and \( b(t, x) \in C^1([\alpha, \beta] \times Q) \), is locally analytically mappable in the domain \( Q \) on the time interval \([\alpha, \beta]\) to a preassigned linear controllable system (2) if and only if all vector functions (6) exist, belong to the class \( C^1([\alpha, \beta] \times Q) \), and satisfy the conditions

\[
[R^j b(t, x), R^k b(t, x)] = 0, \quad x \in Q, \; t \in [\alpha, \beta], \; 0 \leq j, k \leq n - 1,
\]

\[
\text{rank } R(t, x) = n, \quad t \in [\alpha, \beta] \setminus \{t_i\}_{i=1}^N, \; x \in Q,
\]

and

\[
R^{-1}(t, x) R^n b(t, x) = K^{-1}(t) \Delta^n(t), \quad t \in [\alpha, \beta] \setminus \{t_i\}_{i=1}^N.
\]

In other words, the invariants (3) are unchangeable also under nonlinear changes of variables in linear systems.

**Remark 1** A possible way to transform a nonautonomous system to an autonomous one is to introduce an additional variable \( x_0 = t \); then the new equation \( \dot{x}_0 = 1 \) should be added to the system. Hence, the new system is defined by the vector fields \( \hat{a}(t, x) = (1, a(t, x))^\top \) and \( \hat{b}(t, x) = (0, b(t, x))^\top \). Then, in particular, the Lie bracket of these extended vector fields equals \( [\hat{a}(t, x), \hat{b}(t, x)] = (0, b_t(t, x) + b_x(t, x)a(t, x) - a_x(t, x)b(t, x))^\top \), which coincides with \( R b(t, x) \), except for the first zero component. Analogously, \( R^k b(t, x) \) is obtained from \( \text{ad}_a^k \hat{b}(t, x) \). However, after such a transformation, a linear system turns into a nonlinear one and, in addition, becomes uncontrollable. Hence, we cannot apply autonomous linearization results directly. Instead, in [19], mappability conditions for driftless systems (9) were obtained and applied.

The requirements (12), (13) of Theorem 1 generalize linearizability conditions for autonomous systems. Actually, if the system (4) is autonomous, i.e., \( a(t, x) = a(x) \) and \( b(t, x) = b(x) \), then \( R^k b(t, x) = \text{ad}_a^k b(x), k \geq 0 \). The condition (14) arises since the target linear system (2) is fixed. One may suggest that, dropping the assumption (14), we get the conditions for mappability to a linear system, which is not specified in advance. However, this is not true. Namely, (14) should be substituted by the following assumption: components of the vector function \( R^{-1}(t, x) R^n b(t, x) \) depend only on \( t \) and are realizable as invariants of some linear system. So, we are led to the Realizability Problem mentioned above.

In the present paper we solve this problem and obtain the conditions of local analytic linearizability (Theorem 4).
To start with, let us reformulate the Realizability Problem. For a given linear controllable system (2) with analytic matrices, denote by $\Phi(t)$ the solution of the matrix Cauchy problem $\dot{\Phi}(t) = A(t)\Phi(t)$, $\Phi(0) = I$. The matrix $\Phi(t)$ is nonsingular and real analytic on $[\alpha, \beta]$.

Then, in the variables $z = \Phi^{-1}(t)x$, the system (2) takes a driftless form

$$\dot{z} = \hat{g}(t)u,$$

where $\hat{g}(t) = \Phi^{-1}(t)b(t)$. In this case the calculations mentioned above are simplified since $\Delta^k(t)$ reduces to the $k$-th derivative on $t$. We get

$$\gamma(t) = \hat{K}^{-1}(t)\hat{g}(n)(t), \quad \hat{K}(t) = (\hat{g}(t), \hat{g}(1)(t), \ldots, \hat{g}(n-1)(t)).$$

which can be rewritten as

$$(\hat{g}(t), \hat{g}(1)(t), \ldots, \hat{g}(n-1)(t))\gamma(t) = \hat{g}(n)(t).$$

This equality means that the column vector $\gamma(t) = (\gamma_1(t), \ldots, \gamma_n(t))^\top$ is a solution of the system of linear algebraic equation (17), where the matrix and the right hand side are defined by the known vector function $\hat{g}(t)$.

Now, suppose the contrary: let $\gamma_1(t), \ldots, \gamma_n(t)$ be known and let $\hat{g}(t) = (\hat{g}_1(t), \ldots, \hat{g}_n(t))^\top$ be unknown. Then equalities (17) become linear differential equations for the components $\hat{g}_1(t), \ldots, \hat{g}_n(t)$ of the vector function $\hat{g}(t)$. More specifically, let us consider the following differential equation

$$\gamma_1(t)y + \gamma_2(t)y(1) + \cdots + \gamma_n(t)y(n-1) = y(n).$$

Then (17) means that this differential equation has $n$ linearly independent real analytic solutions $\hat{g}_1(t), \ldots, \hat{g}_n(t)$. Therefore, our Realizability Problem can be reformulated in purely classical terms.

**Realizability Problem (Reformulated)** For a given set of functions $\gamma_1(t), \ldots, \gamma_n(t)$, which are meromorphic on the closed interval $[\alpha, \beta]$, to determine if the differential equation (18) has $n$ linearly independent real analytic solutions on $[\alpha, \beta]$.

Linear differential equations with meromorphic coefficients were studied in detail for the case $n = 2$ due to their significance for the mathematical physics [22–24]. Here the method of F.G. Frobenius [25] is applicable: try to find $n$ independent solutions in the form $y(t) = t^{a_i} \phi_i(t)$, where $a_i$ are constants and $\phi_i(t)$ are analytic functions. If this is impossible, the function $\log t$ should be involved. For the case $n > 2$, the solution becomes complicated; it is discussed in [20].

The Realizability Problem formulated above does not suggest finding solutions of differential equations. Instead, it is required to propose conditions under which the equation (18) has $n$ linearly independent analytic solutions, without finding them. We solve the Realizability Problem in Section 3 and then apply the result to get linearizability conditions for nonlinear control systems in Section 4.

### 3 Analytic Solvability of Linear Differential Equations

To simplify the notation, in this section we denote $p_s(t) = -\gamma_{n-s+1}(t)$, $s = 1, \ldots, n$, i.e., we write the equation (18) as

$$y(n) + \sum_{s=1}^{n} p_s(t)y(n-s) = 0.$$
Our nearest goal is to formulate conditions under which the equation (19) has \( n \) linearly independent analytic solutions in a neighborhood of a given point \( t = t_0 \); without loss of generality we assume \( t_0 = 0 \). The following lemma describes necessary conditions for the coefficients \( p_s(t) \).

**Lemma 1** Suppose the equation (19) has \( n \) linearly independent analytic solutions in a neighborhood of the point \( t = 0 \). Then each \( p_s(t) \) is analytic or meromorphic with a pole of order no greater than \( s \) at the point \( t = 0 \).

As is well known, the necessary condition mentioned in Lemma 1 follows from much more general requirements [23, Ch. IV, Theorem 5.2]. However, in our case the proof is easy; we give it for the sake of completeness.

**Proof** First, we note that if \( y(t) \) is analytic at the point \( t = 0 \), then \( \frac{y^{(k)}(t)}{y(t)} \) is analytic or meromorphic with a pole of order at most \( k \) at \( t = 0 \).

We argue by induction on \( n \). For \( n = 1 \), the lemma is obvious. Now, for \( n \geq 2 \), let us suppose that \( y_1(t), \ldots, y_n(t) \) are linearly independent analytic solutions of the equation (19). Without loss of generality we assume that \( y_k(t) = z_k(t)y_n(t) \), where \( z_k(t) \) are analytic functions, \( k = 1, \ldots, n-1 \). Substituting \( y_1(t), \ldots, y_{n-1}(t) \) to the equation (19), after obvious simplification we obtain that \( \dot{z}_1(t), \ldots, \dot{z}_{n-1}(t) \) are \( n-1 \) linearly independent analytic solutions of the equation

\[
y^{(n-1)} + \sum_{i=1}^{n-1} \tilde{p}_i(t) y^{(n-1-i)} = 0,
\]

with

\[
\tilde{p}_i(t) = p_i(t) + \sum_{s=1}^{i-1} C_{n-s}^i p_s(t) \frac{y_n^{(i-s)}(t)}{y_n(t)} + C_n^i \frac{y_n^{(i)}(t)}{y_n(t)}, \quad i = 1, \ldots, n-1,
\]

where \( C_n^i = \frac{j!}{i!(j-i)!} \). By the induction supposition, every \( \tilde{p}_i(t) \) has a pole up to order \( i \) at \( t = 0 \). Then, using the induction on \( i \), we obtain from (20) that every \( p_i(t) \) also has a pole of order at most \( i \) at \( t = 0 \) for all \( i = 1, \ldots, n-1 \). Finally, substituting \( y_n(t) \) to (19) and expressing \( p_n(t) \) as

\[
p_n(t) = -\frac{y_n^{(n)}(t)}{y_n(t)} - \sum_{s=1}^{n-1} p_s(t) \frac{y_n^{(n-s)}(t)}{y_n(t)},
\]

we get that \( p_n(t) \) has a pole of order at most \( n \) at \( t = 0 \).

Below we use the notation for *falling factorial* for nonnegative integers \( k, q \) [26, Subsection 2.6],

\[
k^q = \begin{cases} 
\frac{k!}{(k-q)!} & \text{if } q \leq k, \\
0 & \text{if } q > k.
\end{cases}
\]

In other words, for any integer \( k \geq 0 \),

\[
k^0 = 1 \quad \text{and} \quad k^q = k(k-1) \cdots (k-q+1) \quad \text{for } q \geq 1.
\]

Now we are ready to formulate and prove the theorem on local analytic solvability of the equation (19).
Theorem 2 The equation (19) has \( n \) linearly independent analytic solutions in a neighborhood of the point \( t = 0 \) if and only if the following conditions are satisfied:

(i) each function \( p_s(t) \) is analytic or meromorphic with a pole of order no greater than \( s \), i.e., can be represented as a series

\[
p_s(t) = \sum_{i=-s}^{\infty} p_{s,i} t^i, \quad s = 1, \ldots, n,
\]

which converges in a neighborhood of \( t = 0 \);

(ii) the polynomial equation (called the indicial equation)

\[
k^n + \sum_{s=1}^{n} k^{n-s} p_{s,-s} = 0
\]

has \( n \) different nonnegative integer roots \( 0 \leq k_1 < \cdots < k_n \);

(iii) the following condition holds,

\[
\begin{vmatrix}
V_{k_1+1,k_1} & V_{k_1+1,k_1+1} & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
V_{k_n-1,k_1} & V_{k_n-1,k_1+1} & \cdots & \cdots & V_{k_n-1,k_n-1} \\
V_{k_n,k_1} & V_{k_n,k_1+1} & \cdots & \cdots & V_{k_n,k_n-1}
\end{vmatrix} = k_n - k_1 - n + 1,
\]

where \( V_{k,j} \) are defined by

\[
V_{k,k} = k^n + \sum_{s=1}^{n} k^{n-s} p_{s,-s} \quad \text{for} \quad k \geq 0,
\]

\[
V_{k,j} = \sum_{s=1}^{j} j^{n-s} p_{s,k-j-s} \quad \text{for} \quad k \geq 0, \quad 0 \leq j \leq k - 1.
\]

If this is the case, then for an analytic solution \( y(t) = \sum_{k=0}^{\infty} y_k t^k \) of the equation (19), the components \( y_{k_1}, \ldots, y_{k_n} \) can be chosen arbitrary and all other components are defined uniquely by the recurrent equation

\[
y_k = -\frac{1}{V_{k,k}} \sum_{j=0}^{k-1} V_{k,j} y_j, \quad k \geq 0, \quad k \neq k_1, \ldots, k \neq k_n.
\]

Proof Necessity The condition (i) follows from Lemma 1. Now, suppose that \( y(t) = \sum_{k=0}^{\infty} y_k t^k \) is an analytic solution of the equation (19). Using falling factorials, we write

\[
y^{(q)}(t) = \sum_{k=0}^{\infty} k^q y_k t^{k-q}, \quad q \geq 0.
\]

Substituting the series (21) and (26) to (19), we get

\[
\sum_{k=0}^{\infty} k^n y_k t^{k-n} + \sum_{s=1}^{n} \sum_{i=-s}^{\infty} p_{s,i} t^i \sum_{j=0}^{\infty} j^{n-s} y_j t^{j-n+s} = 0.
\]

Finding the coefficients of powers of \( t \) in the left hand side and equating them to zero, we get the system of equalities

\[
k^n y_k + \sum_{j=0}^{k} \sum_{s=1}^{n} j^{n-s} p_{s,k-j-s} y_j = 0, \quad k \geq 0.
\]
Using the notation (24), we rewrite the system (27) in the form

\[ V_{k,k}y_k + \sum_{j=0}^{k-1} V_{k,j}y_j = 0, \quad k \geq 0. \]  (28)

If the equation (19) has \( n \) linearly independent analytic solutions (functions \( y(t) \)), then the system (28) also has \( n \) linearly independent solutions (sequences \( \{y_k\}_{k=0}^{\infty} \)). However, if \( V_{k,k} \neq 0 \), then \( y_k \) is uniquely defined by \( y_0, \ldots, y_{k-1} \). Hence, if (28) has \( n \) linearly independent solutions, then \( V_{k,k} = 0 \) for \( n \) different nonnegative integers. Recalling the notation (24), we see that the equation \( V_{k,k} = 0 \) is of the form (22) and is polynomial for \( k \) of degree \( n \). Hence, the equation (22) has \( n \) different nonnegative integer roots, which proves item (ii).

Denote the roots of (22) by \( 0 \leq k_1 < \cdots < k_n \). Then equalities (28) imply \( y_0 = \cdots = y_{k_1-1} = 0 \) and uniquely define all \( y_k \) for \( k \geq k_1 + 1 \). Since \( y_0 = \cdots = y_{k_1-1} = 0 \) and \( V_{k_1,k_1} = 0 \), the equation (28) for \( k = k_1 \) is trivial. Let us write down the equalities (28) for \( k = k_1 + 1, \ldots, k_n \). Taking into account that \( V_{k_n,k_n} = 0 \) and using the matrix notation, we have

\[
\begin{pmatrix}
V_{k_1+1,k_1} & V_{k_1+1,k_1+1} & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
V_{k_n-1,k_1} & V_{k_n-1,k_1+1} & \cdots & V_{k_n-1,k_n-1} & 0 \\
V_{k_n,k_1} & V_{k_n,k_1+1} & \cdots & V_{k_n,k_n-1} & V_{k_n,k_n}
\end{pmatrix}
\begin{pmatrix}
y_{k_1} \\
\vdots \\
y_{k_n-2} \\
y_{k_n-1}
\end{pmatrix} = 0. \tag{29}
\]

We can consider (29) as a system of \( k_n - k_1 \) linear equations in \( k_n - k_1 \) unknowns \( y_{k_1}, \ldots, y_{k_n-1} \). We note that this system does not include \( y_{k_n} \), which can be arbitrary. Therefore, (29) has \( n - 1 \) linearly independent solutions, which implies the condition (23) and proves (iii). Formula (25) follows from (28).

**Sufficiency** Suppose that \( k_1 < \cdots < k_n \) are different nonnegative integer roots of the equation (22) and the condition (23) holds. Since \( V_{k,k} \neq 0 \) for \( k \neq k_s, s = 2, \ldots, n - 1 \), exactly \( k_n - k_1 - n + 1 \) elements in the first super-diagonal of the matrix in (23) are nonzero. Obviously, the rows containing these elements are linearly independent. Therefore, the condition (23) implies that all other rows, namely, \( (V_{k_1,k_1}, V_{k_1,k_1+1}, \ldots, V_{k_1,k_n-1}, 0, \ldots, 0) \) for \( s = 2, \ldots, n \), linearly depend on previous ones. Thus, \( y_{k_1}, \ldots, y_{k_n} \) can be chosen arbitrarily while all the rest \( y_k \) are defined uniquely from (28) by the equalities

\[ y_k = -\frac{1}{V_{k,k}} \sum_{j=0}^{k-1} V_{k,j}y_j, \quad k \neq k_1, \ldots, k \neq k_n. \tag{30} \]

Let us prove that for any choice of \( y_{k_1}, \ldots, y_{k_n} \) the series \( \sum_{k=0}^{\infty} y_k t^k \) converges in a neighborhood of the point \( t = 0 \). Recall that the functions \( p_s(t) \) are analytic or meromorphic with a pole of order no greater than \( s \). Then there exists \( C \geq 1 \) such that

\[ |p_{s,i}| \leq C^{s+i}, \quad s = 1, \ldots, n, \quad i \geq -s + 1 \]

\((p_{s,-s} \text{ are not included into formulas for } V_{k,j}, 0 \leq j \leq k - 1, \text{ see (24)}). \) Then

\[ |V_{k,j}| \leq \sum_{s=1}^{n} j^{n-s} |p_{s,k-j-s}| \leq \sum_{s=1}^{n} j^{n-s} C^{k-j}. \tag{31} \]
Below we use the following identity
\[
\sum_{j=0}^{k-1} j^m = \frac{k^{m+1}}{m+1},
\] (32)
where \( m \geq 0 \) and \( k \geq 1 \) are integers; it can be proved easily by induction on \( k \) [26, Subsection 2.6].

Let us introduce the polynomial
\[
P(k) = \sum_{s=1}^{n} \frac{k^{n-s+1}}{n-s+1}.
\] (33)
It is of degree \( n \) and its leading coefficient equals \( \frac{1}{n} \). Hence,
\[
\frac{P(k)}{V_{k,k}} \geq \frac{n}{\prod_{i=1}^{n} (k - k_i)} \rightarrow \frac{1}{n} \text{ as } k \rightarrow \infty.
\]

Now suppose \( n \geq 2 \) (the case \( n = 1 \) is considered below). Then there exists \( k_0 \geq k_n \) such that
\[
\frac{P(k)}{V_{k,k}} \leq 1 \text{ for } k \geq k_0.
\] (34)
Let us introduce \( C_1 \) as
\[
C_1 = \max_{0 \leq k \leq k_0} |y_k|,
\]
then, since \( C \geq 1 \), we get
\[
|y_k| \leq C_1 C^k \text{ for } 0 \leq k \leq k_0.
\] (35)
We prove that \( |y_k| \leq C_1 C^k \) for all \( k \geq k_0 \), by induction on \( k \). For \( k = k_0 \), we have (35). Suppose \( k \geq k_0 + 1 \) and
\[
|y_j| \leq C_1 C^j \text{ for all } 0 \leq j \leq k - 1.
\] (36)
Then using (30)–(34) and (36), we get
\[
|y_k| \leq \frac{\sum_{j=0}^{k-1} j^{n-s} C^{k-j} C^j}{V_{k,k}} = C_1 C^k \frac{\sum_{j=0}^{k-1} C^{k-j}}{V_{k,k}} \leq C_1 C_k.
\]

In the case \( n = 1 \) we have \( P(k) = k \), therefore, (34) does not hold if \( k_1 > 0 \). However, let us take into account that \( y_0 = \cdots = y_{k_1-1} = 0 \). We choose \( C_1 = |y_{k_1}| \) and use the induction supposition (36) with \( k_0 = k_1 \) and the inequality (31), which gives \( |V_{k,j}| \leq C^{k-j} \).

Then we directly obtain from (30)
\[
|y_k| \leq \frac{\sum_{j=0}^{k-1} C^{k-j} |y_j|}{V_{k,k}} \leq \frac{\sum_{j=k_1}^{k-1} C^{k-j} C^j}{k - k_1} = C_1 C_k.
\]
Thus, we proved by induction that \( |y_k| \leq C_1 C^k \) for all \( k \geq 0 \), therefore, the series \( \sum_{k=0}^{\infty} y_k t^k \) converges if \( |t| < C^{-1} \).
Therefore, choosing \( n \) linearly independent tuples \((y_{k_1}, \ldots, y_{k_n})\), we get \( n \) linearly independent analytic solutions of the equation (19).\]
Remark 2 For \( n = 2 \), the rank in (23) is \( k_2 - k_1 - 1 \), i.e., equals the dimension of the matrix minus 1. Hence, the condition (23) for \( n = 2 \) holds if and only if the matrix in (23) is singular. In the general case the condition (23) reduces to \( \frac{n(n-1)}{2} \) equalities, see Example 3 and Remark 3. For \( n = 1 \), such a condition is not applicable.

Example 2 For \( n = 2 \), the indicial equation (22) has the form

\[
k(k - 1) + kp_{1,-1} + p_{2,-2} = 0
\]  

(37)

Suppose \( p_{1,-1} = -1 \) and (37) has two nonnegative integer roots \( k_1 < k_2 \). Then \( k_1 + k_2 = -(p_{1,-1} - 1) = 2 \), therefore, the unique possible case is \( k_1 = 0, k_2 = 2 \). Then \( p_{2,-2} = k_1 k_2 = 0 \).

Then (24) implies \( V_{1,0} = p_{2,-1}, V_{1,1} = p_{1,-1} + p_{2,-2} = -1, V_{2,0} = p_{2,0}, V_{2,1} = p_{1,0} + p_{2,-1} \), therefore, the condition (23) takes the form

\[
\begin{vmatrix}
V_{1,0} & V_{1,1} \\
V_{2,0} & V_{2,1}
\end{vmatrix} = \begin{vmatrix} p_{2,-1} & -1 \\
p_{2,0} & p_{1,0} + p_{2,-1} \end{vmatrix} = 0,
\]

which gives \( p_{2,-1}(p_{1,0} + p_{2,-1}) + p_{2,0} = 0 \). We note that for any \( p_{1,0} \) and \( p_{2,-1} \) there exists a unique \( p_{2,0} \) satisfying this equality.

For example, if \( p_{1,0} = 0 \) and \( p_{2,-1} = 1 \), then \( p_{2,0} = -1 \). If \( p_{1,k} = p_{2,k} = 0 \) for all \( k \geq 1 \), i.e., \( p_1(t) = -\frac{1}{t} \) and \( p_2(t) = -1 + \frac{1}{t} \), the equation (19) takes the form

\[
\ddot{y} - \frac{1}{t} \dot{y} - \left( 1 - \frac{1}{t} \right) y = 0.
\]  

(38)

Two linearly independent analytic solutions can be found from the formula (30), which implies

\[
y_1 = y_0, \quad y_k = \frac{y_{k-2} - y_{k-1}}{k(k-2)}, \quad k \geq 3,
\]  

(39)

and \( y_0, y_2 \) can be chosen arbitrarily. In this case \( C = 1 \), therefore, the obtained analytic solutions exist at least for \( |t| < 1 \). It is easy to check that the sequences

\[
y_k = \frac{1}{k!}, \quad k \geq 0, \quad \text{and} \quad y_k = \frac{(-1)^k (1 - 2k)}{k!}, \quad k \geq 0,
\]

satisfy the equation (39); they are coefficients of the series for \( y(t) = e^t \) and \( y(t) = e^{-t} (2t + 1) \), which are analytic linearly independent solutions of (38).

Example 3 For \( n = 3 \), the indicial equation (22) has the form

\[
k(k - 1)(k - 2) + k(k - 1) p_{1,-1} + k p_{2,-2} + p_{3,-3} = 0.
\]

Suppose this equation has three roots \( k_1 = 1, k_2 = 2, k_3 = 4 \), which is true if \( p_{1,-1} = -4, p_{2,-2} = 8, p_{3,-3} = -8 \). In this case condition (23) reads

\[
\begin{vmatrix}
V_{2,1} & 0 & 0 \\
V_{3,1} & V_{3,2} & -2 \\
V_{4,1} & V_{4,2} & V_{4,3}
\end{vmatrix} = 1,
\]

which holds if and only if

\[
V_{2,1} = 0, \quad \begin{vmatrix} V_{3,2} & -2 \end{vmatrix} = 0, \quad \begin{vmatrix} V_{3,1} & -2 \end{vmatrix} = 0.
\]  

(40)

Remark 3 As was mentioned in Remark 2, the condition (23) reduces to \( \frac{n(n-1)}{2} \) equalities. It is useful to express them as conditions on minors of the matrix from (23) analogously.
to the conditions (40) in Example 3. In order to formulate them, let us denote by \( D_{i,j} \) the determinant of the matrix formed by deleting the rows and columns containing \( V_{k_s,k_s} \) for all \( s \) such that \( i < s < j \) from the matrix

\[
\begin{pmatrix}
V_{k_i+1,k_i} & V_{k_i+1,k_i+1} & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
V_{k_j-1,k_i} & V_{k_j-1,k_i+1} & \cdots & V_{k_j-1,k_j-1} \\
V_{k_j,k_i} & V_{k_j,k_i+1} & \cdots & V_{k_j,k_j-1}
\end{pmatrix}
\]

(since \( j - i - 1 \) rows and columns should be deleted, such a matrix is of dimension \( k_j - k_i - (j - i - 1) \)). One can show that the condition (23) holds if and only if

\[ D_{i,j} = 0 \text{ for all } 1 \leq i < j \leq n. \]

So, in Example 3, \( D_{1,2}, D_{2,3}, D_{1,3} \) should vanish, which coincides with (40).

**Remark 4** Let us consider the case when all functions \( p_1(t), \ldots, p_n(t) \) are analytic, i.e., \( p_{s,j} = 0 \) for all \(-s \leq j \leq -1, s = 1, \ldots, n\). Then the indicial equation (22) takes the form \( k^n = 0 \); its solutions are \( k_i = i - 1, i = 1, \ldots, n \). Taking into account that \( j^{n-s} = 0 \) if \( n - s > j \), we conclude from (24) that \( V_{k_j,j} = 0 \) for all \( 0 \leq j \leq k - 1 \leq n - 2 \). Hence, the matrix in (23) is zero. Thus, in this case all the conditions of Theorem 2 are trivially satisfied.

**Remark 5** Let us discuss in more detail the procedure for checking the conditions of Theorem 2.

The first step is to verify if the indicial equation (22) has \( n \) different nonnegative integer roots. This condition involves only \( n \) coefficients \( p_{1,-1}, \ldots, p_{n-n} \); at least, they should be integer. Let us notice that \( p_{n-n} \) equals a product of all roots (up to a sign). Hence, one can apply the following procedure: if \( p_{n-n} \neq 0 \), one tries integer factors of \( p_{n-n} \) one by one in order to find all integer roots; if \( p_{n-n} = 0 \), one analogously considers the equation

\[
k^{n-1} + \sum_{s=1}^{n-1} k^{n-s-1} p_{s-s} = 0.
\]

Thus, the first step includes a finite number of manipulations with integers.

The second step is to check the condition (23), which also uses only finite number of coefficients: only \( p_{s,j} \) for \( s = 1, \ldots, n \) and \( j = 1 - s, \ldots, k_n - k_1 - s \) are involved. One finds \( V_{s,j} \) by the formula (24), substitutes them into the matrix (23), and finds its rank. Alternatively, one can find out if all the determinants \( D_{i,j} \) vanish, see Remark 3. Thus, this condition also can be effectively verified by finitely many operations with (real) numbers.

For example, let us consider the equation

\[
\ddot{y} - \left( \frac{5}{t} + 2 + 2t + t^2 \xi_1(t) \right) \dot{y} - \left( \frac{8}{t^2} + \frac{2}{t^2} - 2 + t \xi_2(t) \right)y = 0,
\]  

where \( \xi_1(t) \) and \( \xi_2(t) \) are arbitrary analytic functions. Here \( p_{1,-1} = -5, \ p_{2,-2} = -8 \), and the indicial equation (22) takes the form \( k(k - 1) - 5k - 8 = 0 \); its roots are not integer. On the contrast, for the equation

\[
\ddot{y} - \left( \frac{5}{t} + 2 + 2t + t^2 \xi_1(t) \right) \dot{y} + \left( \frac{8}{t^2} + \frac{2}{t} - 2 + t \xi_2(t) \right)y = 0,
\]  

we have \( p_{2,-2} = 8 \), so, the indicial equation is \( k(k - 1) - 5k + 8 = 0 \). Its roots are \( k_1 = 2 \) and \( k_2 = 4 \). Now, we use \( p_{1,0}, p_{1,1}, p_{2,-1}, p_{2,0} \) to check the condition (23). We get \( V_{3,2} = \)

\( \ddots \) Springer
2p_{1,0} + p_{2,-1} = -2, V_{3,3} = -1, V_{4,2} = 2p_{1,1} + p_{2,0} = -6, V_{4,3} = 3p_{1,0} + p_{2,-1} = -4.

Since
\[
\text{rank} \left( \begin{array}{cc} V_{3,2} & V_{3,3} \\ V_{4,2} & V_{4,3} \end{array} \right) = \text{rank} \left( \begin{array}{cc} -2 & -1 \\ -6 & -4 \end{array} \right) = 2,
\]
the condition (23) does not hold. Thus, neither the equation (41) nor the equation (42) has two linearly independent analytic solutions in a neighborhood of the point \( t = 0 \), regardless of the choice of the functions \( \xi_1(t) \) and \( \xi_2(t) \).

Let us slightly change the equation (42) and consider
\[
\ddot{y} - \left( \frac{5}{t} + 2 + 2t + t^2 \xi_1(t) \right) \dot{y} + \left( \frac{8}{t^2} + \frac{2}{t} - 4 + t \xi_2(t) \right) y = 0. \tag{43}
\]
The indicial equation and the values of \( V_{3,2}, V_{3,3}, V_{4,2} \) are the same as for (42), but \( V_{4,2} = 2p_{1,1} + p_{2,0} = -8 \). Therefore,
\[
\text{rank} \left( \begin{array}{cc} V_{3,2} & V_{3,3} \\ V_{4,2} & V_{4,3} \end{array} \right) = \text{rank} \left( \begin{array}{cc} -2 & -1 \\ -8 & -4 \end{array} \right) = 1.
\]
Hence, all the conditions of Theorem 2 are satisfied. Thus, for any pair of analytic functions \( \xi_1(t) \) and \( \xi_2(t) \), the equation (43) has two linearly independent analytic solutions in a neighborhood of the point \( t = 0 \).

4 Conditions for Local Analytic Linearizability

Now we return to linear and nonlinear control systems. First, as a corollary of Theorem 2, we obtain a solution of the Realizability Problem.

**Theorem 3** (On realizability) *Let the functions \( \gamma_1(t), \ldots, \gamma_n(t) \) be analytic or meromorphic on the closed interval \([a, b]\). Denote by \( \{t_i\}_{i=1}^N \subset [a, b] \) the set of points where at least one of them has a pole. These functions are invariants for some linear control system of the form (2) with analytic matrices on \([a, b]\) if and only if the following conditions are satisfied at any \( t_i, i = 1, \ldots, N \):

(i) each function \( \gamma_s(t) \) is analytic in a neighborhood of \( t = t_i \) or meromorphic with a pole at \( t = t_i \) of order no greater than \( n - s + 1 \), i.e., \( \gamma_s(t) \) are expanded into convergent series
\[
\gamma_s(t) = \sum_{j=-n+s-1}^{\infty} \gamma_{s,j} (t - t_i)^j, \quad s = 1, \ldots, n;
\]

(ii) the polynomial equation
\[
k^n - \sum_{s=1}^{n} k^{n-s} \gamma_{n-s+1, -s} = 0 \tag{44}
\]
has \( n \) different nonnegative integer roots \( 0 \leq k_1 < \cdots < k_n \);

(iii) the following equality holds
\[
\text{rank} \left( \begin{array}{cccc} V_{k_1+1,k_1} & V_{k_1+1,k_1+1} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ V_{k_{n-1},k_1} & V_{k_{n-1},k_1+1} & \cdots & \cdots & V_{k_{n-1},k_{n-1}} \\ V_{k_{n},k_1} & V_{k_{n},k_1+1} & \cdots & \cdots & V_{k_{n},k_{n-1}} \end{array} \right) = k_n - k_1 - n + 1, \tag{45}
\]

\( \square \) Springer
where

\[
V_{k,j} = -\sum_{s=1}^{n} j_{s}^{n-s} y_{n-s+1,j-s} \quad \text{for } k_{1} + 1 \leq k \leq k_{n} - 1,
\]

\[
V_{k,j} = -\sum_{s=1}^{n} j_{s}^{n-s} y_{n-s+1,k-s} \quad \text{for } k_{1} \leq j \leq k - 1 \leq k_{n} - 1.
\]

**Proof** Necessity follows from Theorem 2. To prove sufficiency, let us consider the equation (19) with \( p_{s}(t) = -y_{n-s+1}(t) \), \( t \in [\alpha, \beta] \), \( s = 1, \ldots, n \). Let us consider any point \( \tilde{t} \in [\alpha, \beta] \).

- If \( \tilde{t} \notin \{t_{i}\}_{i=1}^{N} \), then all \( p_{s}(t) \) are analytic in a neighborhood of \( \tilde{t} \), therefore, the equation (19) has \( n \) linearly independent analytic solutions in a neighborhood of \( \tilde{t} \). Let us denote such a neighborhood by \( U(\tilde{t}) \).

- If \( \tilde{t} \in \{t_{i}\}_{i=1}^{N} \), then, by our supposition, the functions \( \hat{p}_{s}(t) = p_{s}(t + \hat{\tilde{t}}) \), \( s = 1, \ldots, n \), satisfy the conditions of Theorem 2. Hence, the equation (19) has \( n \) linearly independent analytic solutions in a neighborhood of \( \tilde{t} \). Let us denote such a neighborhood by \( U(\hat{\tilde{t}}) \).

The collection of open intervals \( \{U(\tilde{t}) : \tilde{t} \in [\alpha, \beta]\} \) is an open cover of the compact set \( [\alpha, \beta] \), therefore, a finite subcover can be found. Namely, there exist points \( \alpha \leq \tilde{t}_{1} < \cdots < \tilde{t}_{m} \leq \beta \) such that \( [\alpha, \beta] \subset \bigcup_{r=1}^{m} U(\tilde{t}_{r}) \). Without loss of generality, we assume that none of the intervals \( U(\tilde{t}_{1}) \), \( \ldots \), \( U(\tilde{t}_{m}) \) contains another interval. In this case, \( U(\tilde{t}_{r}) \cap U(\tilde{t}_{r+1}) \neq \emptyset \) for \( r = 1, \ldots, m - 1 \).

Suppose that \( y_{1}(t), \ldots, y_{n}(t) \) are \( n \) linearly independent analytic solutions of (19) in \( U(\tilde{t}_{1}) \) and \( \tilde{y}_{1}(t), \ldots, \tilde{y}_{n}(t) \) are \( n \) linearly independent analytic solutions of (19) in \( U(\tilde{t}_{2}) \). Let us consider the interval \( U(\tilde{t}_{1}) \cap U(\tilde{t}_{2}) \neq \emptyset \). Then \( y_{1}(t), \ldots, y_{n}(t) \) and \( \tilde{y}_{1}(t), \ldots, \tilde{y}_{n}(t) \) are two sets of linearly independent solutions of the (linear) differential equation (19) in \( U(\tilde{t}_{1}) \cap U(\tilde{t}_{2}) \). Therefore, \( y_{i}(t) = \sum_{j=1}^{n} a_{ij} \tilde{y}_{j}(t) \), \( i = 1, \ldots, n \), for \( t \in U(\tilde{t}_{1}) \cap U(\tilde{t}_{2}) \), where \( a_{ij} \) are some constants and the matrix \( \{a_{ij}\}_{i,j=1}^{n} \) is nonsingular.

Let us extend the functions \( y_{i}(t) \) to the set \( U(\tilde{t}_{2}) \setminus U(\tilde{t}_{1}) \) defining \( y_{i}(t) = \sum_{j=1}^{n} a_{ij} \tilde{y}_{j}(t) \) for \( t \in U(\tilde{t}_{2}) \setminus U(\tilde{t}_{1}) \) for any \( i = 1, \ldots, n \). Then the functions \( y_{1}(t), \ldots, y_{n}(t) \) become \( n \) linearly independent analytic solutions of the equation (19) in the interval \( U(\tilde{t}_{1}) \cup U(\tilde{t}_{2}) \). Continuing this process, after a finite number of steps we obtain \( n \) linearly independent analytic solutions of (19) in \( \bigcup_{r=1}^{m} U(\tilde{t}_{r}) \). These solutions, considered as components of the vector function \( \vec{g}(t) \), generate a linear control system (15) defined on \( [\alpha, \beta] \) with invariants \( y_{1}(t), \ldots, y_{n}(t) \).

**Example 4** Consider the functions

\[
y_{1}(t) = -\frac{c}{t(1-t)}, \quad y_{2}(t) = -\frac{a+bt}{t(1-t)}, \quad t \in [0, 1],
\]

(46)

having poles at \( t = 0 \) and \( t = 1 \). For the point \( t = 0 \),

\[
y_{1}(t) = -\frac{c}{t(1-t)} = -\sum_{k=1}^{\infty} ct^{k}, \quad y_{2}(t) = -\frac{a+bt}{t(1-t)} = -\frac{a}{t} - \sum_{k=0}^{\infty} (a+b) t^{k}.
\]

Therefore, the equation (44) takes the form \( k(k-1) + ka = 0 \), hence, \( k_{1} = 0, k_{2} = 1 - a \). Thus, \( a \) should be a nonpositive integer. One easily find \( V_{k,j} = j(a+b)+c, 0 \leq j \leq k-1 \). The condition (45) reduces to

\[
\prod_{j=0}^{-a} (jb+c-j(j-1)) = 0.
\]
Therefore, an integer \(0 \leq j \leq -a\) should exist such that \(jb + c = j(j - 1)\).

For the point \(t = 1\), arguing analogously, we get that \(a + b\) should be a nonnegative integer and an integer \(0 \leq j \leq a + b\) should exist such that \(jb + c = j(j - 1)\).

As a result, the functions (46) satisfy the conditions of Theorem 3 if and only if \(a \leq 0\) and \(a + b \geq 0\) are integers and there exists an integer \(0 \leq j \leq \min[-a, a + b]\) such that \(jb + c = j(j - 1)\). For example, let \(a = -1\), \(b = 2\), \(c = -2\), then \(j = 1 = -a = a + b\) satisfies the condition mentioned above. Therefore, the functions \(\gamma_1(t) = \frac{2}{t(1-t)}\) and \(\gamma_2(t) = \frac{1-2t}{t(1-t)}\) are invariants for some linear control system of the form (2) with analytic matrices on \(t \in [0, 1]\). Namely, in this case the equation (18), which is hypergeometric, takes the form

\[
\frac{2}{t(1-t)} y + \frac{1-2t}{t(1-t)}\dot{y} = \ddot{y}.
\]

It has two linearly independent solutions, which can be chosen as \(\gamma_1(t) = 2t - 1\) and \(\gamma_2(t) = t^2\). Therefore, the functions \(\gamma_1(t)\) and \(\gamma_2(t)\) are invariants of the system (15) with \(\widehat{g}(t) = (2t - 1, t^2)\).

Remark 6 We emphasize that Theorem 3 allows us to answer the realizability question without solving the equation (18). However, suppose that we find \(n\) linearly independent solutions of the equation (18) that turn out to be analytic in an interval including one or several points from the set \(\{t_i\}_{i=1}^N\). Then the conditions of Theorem 3 for all these points are satisfied automatically, hence, we do not need to check them.

Example 5 Consider the functions

\[
\gamma_1(t) = -\frac{2}{t^2(1-t)}, \quad \gamma_2(t) = \frac{2}{t(1-t)}, \quad t \in [0, 1],
\]

having poles at \(t = 0\) and \(t = 1\). At the point \(t = 0\) we get \(\gamma_{1,-2} = -2, \gamma_{2,-1} = 2\), therefore, \(k_1 = 1, k_2 = 2\). The condition (45) takes the form \(V_{2,1} = 0\); obviously it holds since \(V_{2,1} = -\gamma_{2,0} - \gamma_{1,-1} = -2 + 2 = 0\). However, at the point \(t = 1\) we have \(\gamma_{1,-2} = 0, \gamma_{2,-1} = -2\). Hence, the indicial equation does not have two nonnegative roots. Therefore, the functions (47) are invariants for a linear control system of the form (2) with analytic matrices on any interval \([0, \beta]\) such that \(\beta < 1\) but not on the interval \([0, 1]\). In order to find such a system, let us consider the equation (18), which takes the form

\[
-\frac{2}{t^2(1-t)} y + \frac{2}{t(1-t)}\dot{y} = \ddot{y}.
\]

One can verify that \(\gamma_1(t) = t\) and \(\gamma_2(t) = \frac{1}{1-t}\) are two linearly independent solutions. Hence, the functions (47) are invariants of the system (15) with \(\widehat{g}(t) = (t, \frac{t}{1-t})\) defined on \([0, \beta]\) with \(\beta < 1\).

Remark 7 Suppose the conditions of Theorem 3 hold. To find a linearizing change of variables, one first finds \(n\) linearly independent analytic solutions of the equation (18). Let \(\gamma_1(t), \ldots, \gamma_n(t)\) be such solutions, then the components of \(\gamma(t)\) are invariants of the system (15) with \(\widehat{g}(t) = (\gamma_1(t), \ldots, \gamma_n(t))\). Then, as was shown in [19, 21], a linearizing change of variables \(z = F(t, x)\) can be found as a solution of the system of the first order partial differential equations

\[
F_t(t, x) + F_x(t, x)a(t, x) = 0, \quad F_x(t, x)R(t, x) = \widehat{K}(t),
\]

where the matrices \(R(t, x)\) and \(\widehat{K}(t)\) are defined by (7) and (16) respectively.
As a consequence of Theorems 1 and 3, we obtain conditions for local analytic linearizability.

**Theorem 4** (On local analytic linearizability) Consider a nonlinear control system of the form (4), where \( a(t, x) \in C^2([\alpha, \beta] \times Q) \), \( b(t, x) \in C^1([\alpha, \beta] \times Q) \). This system is locally analytically linearizable in the domain \( Q \) on the time interval \([\alpha, \beta]\) if and only if all vector functions (6) exist, belong to the class \( C^1([\alpha, \beta] \times Q) \), satisfy the conditions (12) and (13), and components of the vector function (8) depend only on \( t \), i.e., \( \gamma(t, x) = \gamma(t) \), and are invariants for some linear control system of the form (2) with analytic matrices on \([\alpha, \beta]\), i.e., satisfy the conditions of Theorem 3.

**Example 6** Let us consider the following nonlinear control system

\[
\dot{x}_1 = \frac{4t^2 |t|}{2 + t^3 |t|} x_1 + (2 + t^3 |t|)(1 + 3t^3 x_2^2)u, \quad \dot{x}_2 = t^3 u
\]

(49)
of the class \( C^2([-1, 1] \times \mathbb{R}^2) \). We have

\[
b(t, x) = \left( \rho(t)(1 + 3t^3 x_2^2) \right), \quad \mathcal{R}b(t, x) = \left( 9t^2 \rho(t)x_2^2 \right), \quad \mathcal{R}^2 b(t, x) = \left( \frac{18 \rho(t)x_2^2}{6t} \right),
\]

where \( \rho(t) = 2 + t^3 |t| \). Then conditions (12) and (13) are satisfied with \( N = 1 \) and \( t_1 = 0 \). Moreover,

\[
\gamma(t) = R^{-1}(t, x)\mathcal{R}^2 b(t, x) = \left( \frac{0}{\frac{2}{3}} \right)
\]
depends only on \( t \). The functions \( \gamma_1(t) = 0 \) and \( \gamma_2(t) = \frac{2}{3} \) are analytic in \([-1, 1]\) except the point \( t_1 = 0 \), where they satisfy condition (i) of Theorem 3. Since \( \gamma_2, -1 = 2 \) and all other coefficients \( \gamma_{1,i} \) and \( \gamma_{2,i} \) vanish, the equation (44) has the form \( k(k - 1) - 2k = 0 \). Its roots are \( k_1 = 0, k_2 = 3 \). Moreover, \( V_{k,0} = 0 \) for \( k \geq 1 \), which implies the equality (45).

Therefore, all conditions of Theorem 3 hold for the functions \( \gamma_1(t) \) and \( \gamma_2(t) \). Thus, due to Theorem 4, the system (49) is locally analytically linearizable in \( \mathbb{R}^2 \) on the time interval \([-1, 1]\). Obviously, in this case \( \gamma_1(t) = 1 \) and \( \gamma_2(t) = t^3 \) are solutions of the equation (18), which takes the form \( \ddot{y} = \frac{2}{3} \dot{y} \).

Hence, the system (49) can be transformed to the linear driftless system

\[
\dot{x}_1 = u, \quad \dot{x}_2 = t^3 u.
\]

(50)

Now let us find a linearizing change of variables \( F(t, x) = (F_1(t, x), F_2(t, x))^\top \) from the system (48), which takes the form

\[
\begin{pmatrix} F_1 & F_2 \\ F_1x_1 & F_2x_2 \end{pmatrix} + \begin{pmatrix} F_1x_1 & F_2x_2 \\ F_1x_1 & F_2x_2 \end{pmatrix} \begin{pmatrix} \frac{\rho(t)}{\rho(t)} x_1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\]

One can easily verify that \( F_1(t, x) = \frac{x_1}{\rho(t)} - \frac{3}{2} \) and \( F_2(t, x) = x_2 \) satisfy these equations, and therefore, define a linearizing change of variables.

**Example 7** Finally, we demonstrate how invariants can be used for classifying control systems.

Let us consider one of the simplest class of nonautonomous mechanical systems, which are described by Hill’s equation \( \ddot{x} + p(t)x = u \), where \( p(t) \) is analytic. Rewriting it in the matrix form, we easily obtain the invariants: \( \gamma_1(t) = -p(t), \gamma_2(t) = 0 \). They describe all
systems that can be transformed to Hill’s equation by a change of variables. In particular, the system (49) does not satisfy this condition. On the contrary, the system

\[
\dot{x}_1 = \sin tu, \quad \dot{x}_2 = \cos tu,
\]

has the invariants \( \gamma_1(t) = -1, \gamma_2(t) = 0 \). Hence, it can be transformed to the equation \( \ddot{x} + x = u \), which corresponds to the oscillating system

\[
\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 + u.
\]

**Remark 8** In the present paper, we have focused on linearizing of nonlinear nonautonomous control systems by changing the variables only. However, a more interesting and useful and, at the same time, more sophisticated problem is a feedback linearization for nonautonomous systems, which will be the subject of our further research.

**Acknowledgements** The authors are sincerely grateful to the anonymous reviewers for their detailed comments and constructive suggestions.

**Funding** This work was financially supported by Polish National Science Centre grant no. 2017/25/B/ST1/01892.

**References**

1. Krener A. On the equivalence of control systems and the linearization of non-linear systems, SIAM. J. Control. 1973;11:670–76.
2. Brockett RW. Feedback invariants for nonlinear systems. In: Proceedings of the Seventh World Congress IFAC, Helsinki; 1978. p. 1115–20.
3. Jakubczyk B, Respondek W. On linearization of control systems. Bull Acad Sci Polonaise Ser Sci Math. 1980;28:517–22.
4. Su R. On the linear equivalents of nonlinear systems. Systems Control Lett. 1982;2:48–52.
5. Respondek W. Geometric methods in linearization of control systems. In: Mathematical control theory, vol. 14 of Banach center publ., PWN, Warsaw; 1985. p. 453–67.
6. Respondek W. Linearization, feedback and Lie brackets. Vol. Conf. 29 of Scientific Papers of the Institute of Technical Cybernetics of the Technical University of Wroclaw. 131–66; 1985.
7. Korobov VI. Controllability, stability of some nonlinear systems (Russian). Differ Uravnenija. 1973;9:614–619,. Translation: Differential Equations 9(1975) 466–69.
8. Sklyar GM, Sklyar KV, Ignatovich SY. On the extension of the Korobov’s class of linearizable triangular systems by nonlinear control systems of the class \( C^1 \). Systems Control Lett. 2005;54:1097–108.
9. Sklyar KV, Ignatovich SY, Skoryk VO. Conditions of linearizability for multi-control systems of the class \( C^1 \). Commun Math Anal. 2014;17:359–65.
10. Sklyar KV, Ignatovich SY. Linearizability of systems of the class \( C^1 \) with multi-dimensional control. Systems Control Lett. 2016;94:92–96.
11. Sklyar KV, Ignatovich SY, Sklyar GM. Verification of feedback linearizability conditions for control systems of the class \( C^1 \). In: 2017 25th Mediterranean Conference on Control and Automation (MED); 2017. p. 163–168.
12. Sklyar KV, Sklyar GM, Ignatovich SY. Linearizability of multi-control systems of the class \( C^1 \) by additive change of controls. In: Operator theory, operator algebras, and matrix theory, vol. 267 of Oper. Theory Adv Appl, Cham: Springer; 2018. p. 359–70.
13. Korobov VI, Sklyar GM. The Markov moment min-problem and time optimality (Russian). Sibirsk Mat Zh. 1991;32(1):60–71. Translation: Siberian Math. J. 32(1)(1991) 46–55.
14. Sklyar GM, Ignatovich SY. A classification of linear time-optimal control problems in a neighborhood of the origin. J Math Anal Appl. 1996;203:791–811.
15. Markov AA. New applications of continuous fractions (Russian), Notes of the Imperial Academy of Sci. 3, translation: A. Markoff, Nouvelles applications des fractions continues. Math Ann. 1896;47(4):579–97.
16. Kreǐn MG, Nudel’man AA. The Markov moment problem and extremal problems. Ideas and problems of P. L. Čebyšev and A. A. Markov and their further development (Russian), Nauka, Moscow, 1973 translation: Translations of Mathematical Monographs, vol. 50. Providence: American Mathematical Society; 1977.

17. Korobov VI, Sklyar GM. Time-optimality and the power moment problem (Russian). Mat. Sb. (N.S.) 1987;134(176 2):186–206. Translation: Math. USSR-Sb. 62(1)(1989) 185–206.

18. Korobov VI, Sklyar GM, Ignatovich SY. Solving of the polynomial systems arising in the linear time-optimal control problem. Commun Math Anal Conf. 2011;3:153–71.

19. Sklyar K. On mappability of control systems to linear systems with analytic matrices. Syst Control Lett. 2019;134:104572.

20. Forsyth AR, Vol. IV. Theory of differential equations. Part III, Ordinary linear equations. Cambridge: Cambridge University Press; 1902.

21. Sklyar K, Ignatovich S. On linearizability conditions for non-autonomous control systems. Advanced, contemporary control. Advances in intelligent systems and computing. In: Bartoszewicz A, Kabziński J, and Kacprzyk J, editors; 2020. p. 625–37.

22. Whittaker ET, Watson GN. A course of modern analysis, 3rd ed. Cambridge: Cambridge University Press; 1920.

23. Coddington EA, Levinson N. Theory of ordinary differential equations. New York: McGraw-Hill; 1955.

24. Teschl G. Ordinary differential equations and dynamical systems, vol. 140 of graduate studies in mathematics, Amer Math Soc, Providence. 2012.

25. Frobenius FG. Über die integration der linearen differentialgleichungen durch reihen. Journal für die reine und angewandte Mathematik. 1873;76:214–35.

26. Graham RL, Knuth DE, Patashnik O. Concrete mathematics: a foundation for computer science, 2nd ed. Reading: Addison-Wesley Professional; 1994.

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.