3 fermionic families naturally arise from
hyperionic formalism

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May 1, 2014

Abstract

We’ll show as a quantum theory based on hyperions (said also ‘complex quaternions’) gives reason straightforwardly for the existence of three fermionic families, dark matter and superfields which don’t require never seen particles.

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1 Hyperions

We give briefly a definition which we’ll use repeatedly. Hyperions, sometimes called ‘complex quaternions’ are hyper-complex numbers with seven imaginary unities \(i, j, k, I, iI, jI, kI\) and commutation rules:

\[
ij = -ji = k \quad jk = -kj = i \quad ki = -ik = j \quad i^2 = j^2 = k^2 = I^2 = -1
\]

\[
iI = Ii \quad jI = Ij \quad kI = Ik \quad (iI)^2 = (jI)^2 = (kI)^2 = 1
\]

The algebra so defined is associative.

2 Introduction

It is well known that in a grand-unified theory with \(SU(6)\) as gauge group, all fermionic fields can be arranged in a skew-symmetric matrix \(\chi\) which transforms in the skew-symmetric representation. Every entry in the matrix will be a Dirac spinor:

\[
\chi \rightarrow e^{iT} \chi e^{iT^*} \quad \text{with} \quad T \in su(6)
\]  

(1)

This formalism doesn’t give any explanation about the experimentally verified existence of three fermionic families which are identical except for masses. Moreover, the transformation law \(\Pi\) makes very hard an unification with bosonic fields \(F\) which transform in the adjoint representation:

\[
F \rightarrow e^{iT} Fe^{-iT} \quad \text{with} \quad T \in su(6)
\]  

(2)

In what follows we demonstrate that a generic hyperionic field \(\psi\), which transforms in the adjoint representation, takes into account for three fermionic families which transform in the skew symmetric representation.
3 From ‘adjoint’ to ‘skew-symmetric’

Without loss of generality, we can write a hyperionic field as

$$\psi = \psi_0 + \psi_1 k + \psi_2 \hat{i} + \psi_3 \hat{j}$$

with

$$\begin{align*}
\psi_0 & = \psi_0^1 + I\psi_0^2 \\
\psi_1 & = \phi_1^1 - i\xi_1^1 + I(\phi_2^1 - i\xi_2^1) \\
\psi_2 & = \phi_1^2 - j\xi_2^1 + I(\phi_2^2 - j\xi_2^2) \\
\psi_3 & = \phi_1^3 - k\xi_1^3 + I(\phi_2^3 - k\xi_2^3) \\
\phi_n^m, \xi_n^m & \in \mathbb{R} \quad m = 1, 2, 3; \quad n = 1, 2
\end{align*}$$

We have three ways to construct $SU(6)$ transformations

$$[SU(6)]^{(1)} \approx e^{iT} \quad [SU(6)]^{(2)} \approx e^{jT'} \quad [SU(6)]^{(3)} \approx e^{kT''}$$

where $T$ contains only $i$ as imaginary unity, $T'$ contains only $j$ and $T''$ contains only $k$. Consider now a hyperionic field $\psi$ which transforms in the adjoint representation for any class of $SU(6)$:

$$[SU(6)]^{(1)} \approx e^{iT}$$

$$\psi \to e^{iT}\psi e^{-iT} \Rightarrow \psi_1 k \to e^{iT}\psi_1 k e^{-iT} = e^{iT}\psi_1 e^{iT^*} k$$

**Adjoint rep**

$$\Rightarrow \psi_1 \to e^{iT}\psi_1 e^{iT^*}$$

**Skew sym rep**

$$\begin{pmatrix} T \text{ and } \psi_1 \text{ contain} \\
\text{only } i \text{ as imaginary unit} \end{pmatrix}$$
\[ \text{[SU}(6)\text{]}^{(2)} \approx e^{iT'} \]

\[
\psi \rightarrow e^{iT'} \psi e^{-iT'} \Rightarrow \psi_2 \rightarrow e^{iT'} \psi_2 e^{-iT'} = e^{iT'} \psi_2 e^{iT' \ast} \]

\( \text{Adjoint rep} \)

\[
\Rightarrow \psi_2 \rightarrow e^{iT'} \psi_2 e^{iT' \ast} \]

\( \text{Skew sym rep} \)

\[
\left( T' \text{ and } \psi_2 \text{ contain only } j \text{ as imaginary unit} \right)
\]

\[ \text{[SU}(6)\text{]}^{(3)} \approx e^{kT''} \]

\[
\psi \rightarrow e^{kT''} \psi e^{-kT''} \Rightarrow \psi_3 \rightarrow e^{kT''} \psi_3 e^{-kT''} = e^{kT''} \psi_3 e^{kT'' \ast} \]

\( \text{Adjoint rep} \)

\[
\Rightarrow \psi_3 \rightarrow e^{kT''} \psi_3 e^{kT'' \ast} \]

\( \text{Skew sym rep} \)

\[
\left( T'' \text{ and } \psi_3 \text{ contain only } k \text{ as imaginary unit} \right)
\]

In all the cases:

\[
\begin{align*}
\psi_0 & \xrightarrow{(1)} e^{iT} \psi_0 e^{-iT} = \psi_0' \\
\psi_0 & \xrightarrow{(2)} e^{iT'} \psi_0 e^{-iT'} = \psi_0'' \\
\psi_0 & \xrightarrow{(3)} e^{kT''} \psi_0 e^{-kT''} = \psi_0'''
\end{align*}
\]

where \( \psi_0', \psi_0'', \psi_0''' \) contain only \( I \) as imaginary unit. We can say that transformations in \( SU(6) \) maintain \( \psi_0 \) in its family, i.e. it doesn’t interact with ordinary matter via \( SU(6) \) gauge fields. This means that \( \psi_0 \) is a good candidate for dark matter. Conversely, if we apply a transformation in \( [SU(6)]^{(n)} \) to \( \psi_m \), with \( m \neq n \), we obtain a field \( \psi_q \) in the remaining family \( q \neq m, n \).
4 Gamma representation

What remains to be demonstrated is the equality between a hyperionic field and three families of spinor (Dirac) fields. Let’s start by considering the correspondence between hyperionic imaginary unities and gamma matrices. Explicitly:

\[
\begin{align*}
\hat{i} &= \gamma_2 \gamma_1 \quad \hat{j} = \gamma_1 \gamma_3 \\
\hat{k} &= \gamma_3 \gamma_2 \quad I = \gamma_0 \gamma_1 \gamma_2 \gamma_3 \\
iI &= I\hat{i} = \gamma_0 \gamma_3 \\
jI &= I\hat{j} = \gamma_0 \gamma_2 \\
kI &= I\hat{k} = \gamma_0 \gamma_1
\end{align*}
\]

We take gamma matrices in Dirac representation:

\[
\begin{align*}
\gamma_0 &= \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} & \\
\gamma_1 &= \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{pmatrix} \\
\gamma_2 &= \begin{pmatrix}
0 & 0 & 0 & -i \\
0 & 0 & i & 0 \\
0 & i & 0 & 0 \\
-i & 0 & 0 & 0
\end{pmatrix} & \\
\gamma_3 &= \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 \\
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}
\end{align*}
\]

This gives:

\[
\begin{align*}
\hat{i} &= \begin{pmatrix}
i & 0 & 0 & 0 \\
0 & -i & 0 & 0 \\
0 & 0 & i & 0 \\
0 & 0 & 0 & -i
\end{pmatrix} & \\
\hat{j} &= \begin{pmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0
\end{pmatrix} \\
\hat{k} &= \begin{pmatrix}
0 & i & 0 & 0 \\
i & 0 & 0 & 0 \\
0 & 0 & 0 & i \\
0 & 0 & i & 0
\end{pmatrix} & \\
kI &= \begin{pmatrix}
0 & 0 & 0 & -1 \\
0 & 0 & -1 & 0 \\
0 & -1 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{pmatrix}
\end{align*}
\]
\[ I = -kI_k = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix} \quad \hat{i}I = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \]

\[ \hat{j}I = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \]

By expanding \( \psi \):

\[
\psi = \psi_0^1 + I\psi_0^2 + \bar{k}\phi_1^1 + \bar{j}\xi_1^1 + \bar{k}\phi_2^1 + \bar{j}\xi_2^1 + i\phi_1^2 + k\phi_2^2 + j\xi_2^2 + i\phi_3^2 + \bar{k}\xi_3^2 + \bar{j}\xi_3^2
\]

\[
\psi = \begin{pmatrix} \psi_0^1 + i(\phi_1^1 + \xi_1^1) \\ i(\phi_1^1 + \xi_1^1) - (\xi_1^1 + \phi_1^1) \\ i\psi_0^2 - (\phi_1^2 + \xi_2^2) \\ -(\phi_1^2 + \xi_2^2) - i(\xi_1^2 + \phi_1^2) \end{pmatrix} = \begin{pmatrix} (\xi_1^1 + \phi_1^1) + i(\phi_1^1 + \xi_1^1) \\ \psi_0^1 - i(\phi_2^1 + \xi_1^1) \\ -(\phi_2^1 + \xi_2^1) + i(\xi_1^2 + \phi_1^2) \\ (\phi_2^1 + \xi_2^1) + i\psi_0^2 \end{pmatrix}
\]

\[
\begin{vmatrix}
\psi_0^1 + i(\phi_1^1 + \xi_1^1) & (\xi_1^1 + \phi_1^1) + i(\phi_1^1 + \xi_1^1) \\
\psi_0^1 - i(\phi_2^1 + \xi_1^1) & \psi_0^1 - i(\phi_2^1 + \xi_1^1) \\
-(\phi_2^1 + \xi_2^1) + i(\xi_1^2 + \phi_1^2) & -(\phi_2^1 + \xi_2^1) + i(\xi_1^2 + \phi_1^2) \\
(\phi_2^1 + \xi_2^1) + i\psi_0^2 & (\phi_2^1 + \xi_2^1) + i\psi_0^2
\end{vmatrix}
\]

\[
\begin{vmatrix}
i\psi_0^2 - (\phi_2^2 + \xi_3^2) & -(\phi_2^2 + \xi_3^2) + i(\xi_1^2 + \phi_3^2) \\
-(\phi_1^2 + \xi_2^2) - i(\xi_1^2 + \phi_2^2) & -(\phi_1^2 + \xi_2^2) - i(\xi_1^2 + \phi_2^2) \\
i(\phi_1^2 + \xi_3^2) + \psi_0^1 & (\xi_1^1 + \phi_3^1) + i(\phi_1^1 + \xi_2^2) \\
-(\xi_1^1 + \phi_3^1) + i(\phi_1^1 + \xi_2^1) & -(\xi_1^1 + \phi_3^1) + i(\phi_1^1 + \xi_2^1)
\end{vmatrix}
\]

The structure is then
\[
\psi = \begin{pmatrix}
a & -b^* & c & d^* \\
b & a^* & d & -c^* \\
c & d^* & a & -b^* \\
d & -c^* & b & a^*
\end{pmatrix}
\]
\[
\psi^\dagger = \begin{pmatrix}
a^* & b^* & c^* & d^* \\
-b & a & d & -c \\
c^* & d^* & a & b^* \\
d & -c & -b & a
\end{pmatrix}
\]

Consider now the Dirac derivative operator (noting that \(\gamma_0^2 = 1\)):

\[
M = \gamma_0 \gamma^\mu \partial_\mu = \begin{pmatrix}
-\partial_0 & 0 & -\partial_3 & -\partial_1 + i\partial_2 \\
0 & -\partial_0 & -\partial_1 - i\partial_2 & \partial_3 \\
-\partial_3 & -\partial_1 + i\partial_2 & \partial_0 & 0 \\
-\partial_1 - i\partial_2 & \partial_3 & 0 & \partial_0
\end{pmatrix}
\]

We see that \(M\) uses only four unities (1 = \(\gamma_0^2\), \(\gamma_0 \gamma_1 = \gamma_0 \gamma_2 = \gamma_0 \gamma_3 = 1\)) omitting \(I, j, k\). The complete extension is obtained by introducing complex dimensions

\[
x^\mu = \text{Re}(x^\mu) + I \text{Im}(x^\mu) = \text{Re}(x^\mu) + \gamma_0 \gamma_1 \gamma_2 \gamma_3 \text{Im}(x^\mu)
\]

and complex derivatives

\[
\frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial \text{Re}(x^\mu)} - I \frac{\partial}{\partial \text{Im}(x^\mu)} = \frac{\partial}{\partial \text{Re}(x^\mu)} - \gamma_0 \gamma_1 \gamma_2 \gamma_3 \frac{\partial}{\partial \text{Im}(x^\mu)}
\]

However we proceed with the simpler case, leaving the complete one to future works.

Our target now is to introduce a new lagrangian \(\mathfrak{L} = \text{Re}(\psi^\dagger M \psi)\) (or \(\mathfrak{L} = \text{Tr}(\psi^\dagger M \psi)\) in the Gamma-representation), demonstrating its equivalence with ordinary Dirac lagrangian \(\mathfrak{L} = \chi^\dagger M \chi + \text{c.c.}\) for some ordinary Dirac spinor \(\chi\).
Consider now the identity $\text{Tr}(\psi^\dagger M\psi) = (\psi^\dagger M\psi)^{00} + (\psi^\dagger M\psi)^{11} + (\psi^\dagger M\psi)^{22} + (\psi^\dagger M\psi)^{33}$ and proceed one term at a time:

$$M\psi = \begin{pmatrix}
-\partial_0 a - \partial_3 c - \partial_1 d + i\partial_2 d & \partial_0 b^* - \partial_3 d^* + \partial_1 c^* - i\partial_2 c^* \\
-\partial_0 b - \partial_1 c - i\partial_2 c + \partial_3 d & -\partial_0 a^* - \partial_1 d^* - i\partial_2 d^* - \partial_3 c^* \\
-\partial_3 a - \partial_1 b + i\partial_2 b + \partial_0 c & \partial_3 b^* - \partial_1 a^* + i\partial_2 a^* + \partial_0 d^* \\
-\partial_1 a - i\partial_2 a + \partial_3 b + \partial_0 d & \partial_1 b^* + i\partial_2 b^* + \partial_3 a^* - \partial_0 c^* \\
\end{pmatrix}$$

\[-\partial_0 c - \partial_3 a - \partial_1 b + i\partial_2 b \\ -\partial_0 d - \partial_1 a - i\partial_2 a + \partial_3 b \\ -\partial_3 c - \partial_1 d + i\partial_2 d + \partial_0 a \\ -\partial_1 c - i\partial_2 c + \partial_3 d + \partial_0 b \]

\[-\partial_0 a - \partial_3 c - \partial_1 d + i\partial_2 d \\ -\partial_0 b - \partial_1 c - i\partial_2 c + \partial_3 d \\ -\partial_3 a - \partial_1 b + i\partial_2 b + \partial_0 c \\ -\partial_1 a - i\partial_2 a + \partial_3 b + \partial_0 d \]

$$(\psi^\dagger M\psi)^{00} = -a^*\partial_0 a - a^*\partial_3 c - a^*\partial_1 d + ia^*\partial_2 d$$

$$-b^*\partial_0 b - b^*\partial_1 c - ib^*\partial_2 c + b^*\partial_3 d$$

$$-c^*\partial_3 a - c^*\partial_1 b + ic^*\partial_2 b + c^*\partial_0 c$$

$$-d^*\partial_1 a - id^*\partial_2 a + d^*\partial_3 b + d^*\partial_0 d$$

$$(\psi^\dagger M\psi)^{11} = -b\partial_0 b^* + b\partial_3 d^* - b\partial_1 c^* + i\partial_2 c^*$$

$$-a\partial_0 a^* - a\partial_1 d^* - ia\partial_2 d^* - a\partial_3 c^*$$

$$+d\partial_3 b^* - d\partial_1 a^* + id\partial_2 a^* + d\partial_0 d^*$$

$$-c\partial_1 b^* - ic\partial_2 b^* - c\partial_3 a + c\partial_0 c^*$$
(\psi^\dagger M\psi)^{22} = -c^*\partial_0 c - c^*\partial_3 a - c^*\partial_1 b + ic^*\partial_2 b \\
-d^*\partial_0 d - d^*\partial_1 a - id^*\partial_2 a + d^*\partial_3 b \\
-a^*\partial_3 c - a^*\partial_1 d + ia^*\partial_2 d + a^*\partial_0 a \\
-b^*\partial_1 c - ib^*\partial_2 c + b^*\partial_3 d + b^*\partial_0 b

(\psi^\dagger M\psi)^{33} = -d\partial_0 d^* + d\partial_3 b^* - d\partial_1 a^* + id\partial_2 a^* \\
-c\partial_0 c^* - c\partial_1 b^* - ic\partial_2 b^* - c\partial_3 a^* \\
+b\partial_3 d^* - b\partial_1 c^* + ib\partial_2 c^* + b\partial_0 b^* \\
-a\partial_1 d^* - ia\partial_2 d^* - a\partial_3 c^* + a\partial_0 a^*

Hence:

\text{Tr}(\psi^\dagger M\psi) = -2a^*\partial_3 c - 2a^*\partial_1 d + 2ia^*\partial_2 d - 2b^*\partial_1 c \\
-2ib^*\partial_2 c + 2b^*\partial_3 d - 2c^*\partial_3 a - 2c^*\partial_1 b \\
+2ic^*\partial_2 b - 2d^*\partial_1 a - 2id^*\partial_2 a + 2d^*\partial_3 b \\
+2b\partial_3 d^* - 2b\partial_1 c^* + 2ib\partial_2 c^* - 2a\partial_1 d^* \\
-2ia\partial_2 d^* - 2a\partial_3 c^* + 2d\partial_3 b^* - 2d\partial_1 a^* \\
+2id\partial_2 a^* - 2c\partial_1 b^* - 2ic\partial_2 b^* - 2c\partial_3 a^*

Let’s repeat calculation for Dirac lagrangian:
A quick comparison leads to the following result:

\[
M\chi = \begin{pmatrix}
-\partial_0\chi_0 - \partial_3\chi_2 - \partial_1\chi_3 + i\partial_2\chi_3 \\
-\partial_0\chi_1 - \partial_1\chi_2 - i\partial_2\chi_2 + \partial_3\chi_3 \\
-\partial_3\chi_0 - \partial_1\chi_1 + i\partial_2\chi_1 + \partial_0\chi_2 \\
-\partial_1\chi_0 - i\partial_2\chi_0 + \partial_3\chi_1 + \partial_0\chi_3
\end{pmatrix}
\]

\[
\chi^\dagger M\chi = -\chi_0^*\partial_0\chi_0 - \chi_0^*\partial_3\chi_2 - \chi_0^*\partial_1\chi_3 + i\chi_0^*\partial_2\chi_3 \\
-\chi_1^*\partial_0\chi_1 - \chi_1^*\partial_1\chi_2 - i\chi_1^*\partial_2\chi_2 + \chi_1^*\partial_3\chi_3 \\
-\chi_2^*\partial_3\chi_0 - \chi_2^*\partial_1\chi_1 + i\chi_2^*\partial_2\chi_1 + \chi_2^*\partial_0\chi_2 \\
-\chi_3^*\partial_1\chi_0 - i\chi_3^*\partial_2\chi_0 + \chi_3^*\partial_3\chi_1 + \chi_3^*\partial_0\chi_3
\]

\[
\chi^\dagger M\chi + \text{c.c.} = -\chi_0^*\partial_3\chi_2 - \chi_0^*\partial_1\chi_3 + i\chi_0^*\partial_2\chi_3 \\
-\chi_1^*\partial_1\chi_2 - i\chi_1^*\partial_2\chi_2 + \chi_1^*\partial_3\chi_3 \\
-\chi_2^*\partial_3\chi_0 - \chi_2^*\partial_1\chi_1 + i\chi_2^*\partial_2\chi_1 \\
-\chi_3^*\partial_1\chi_0 - i\chi_3^*\partial_2\chi_0 + \chi_3^*\partial_3\chi_1 \\
-\chi_0\partial_3\chi_2^* - \chi_0\partial_1\chi_3^* - i\chi_0\partial_2\chi_3^* \\
-\chi_1\partial_1\chi_2^* + i\chi_1\partial_2\chi_2^* + \chi_1\partial_3\chi_3^* \\
-\chi_2\partial_3\chi_0^* - \chi_2\partial_1\chi_1^* - i\chi_2\partial_2\chi_1^* \\
-\chi_3\partial_1\chi_0^* + i\chi_3\partial_2\chi_0^* + \chi_3\partial_3\chi_1^*
\]

A quick comparison leads to the following result:

\[
\chi = \sqrt{2} \begin{pmatrix}
a \\
b \\
c \\
d
\end{pmatrix} = \sqrt{2} \begin{pmatrix}
\psi_0^1 + i(\phi_2^1 + \xi_3^1) \\
-(\xi_1^1 + \phi_3^1) + i(\phi_1^1 + \xi_2^1) \\
-(\phi_2^2 + \xi_3^2) + i\psi_0^2 \\
-(\phi_1^2 + \xi_2^2) - i(\xi_1^2 + \phi_3^2)
\end{pmatrix}
\]
So we have

FAMILY-1

\[ \chi^{(1)} = \sqrt{2} \begin{pmatrix} 0 \\ -\xi_1^1 + i\phi_1^1 \\ 0 \\ -\phi_1^2 - i\xi_1^2 \end{pmatrix} \]

FAMILY-2

\[ \chi^{(2)} = \sqrt{2} \begin{pmatrix} i\phi_2^1 \\ i\xi_2^1 \\ -\phi_2^2 \\ -\xi_2^2 \end{pmatrix} \]

FAMILY-3

\[ \chi^{(3)} = \sqrt{2} \begin{pmatrix} i\xi_3^1 \\ -\phi_3^1 \\ -\xi_3^2 \\ -i\phi_3^2 \end{pmatrix} \]

DARK-MATTER

\[ \chi^{(DM)} = \sqrt{2} \begin{pmatrix} \psi_0^1 \\ 0 \\ i\psi_0^2 \\ 0 \end{pmatrix} \]

5 Superfield

Let consider gauge fields \( A_\mu \) for gauge transformations in \( U(6, \mathbf{H}) \), where a matrix \( U \) belongs to \( U(6, \mathbf{H}) \) if and only if

\[ UU^\dagger = U^\dagger U = 1 \]

where \( \dagger \) represents ordinary hermitian conjugation (transposition plus complex conjugation), while \( - \) is an extra conjugation restricted to imaginary unit \( I \). In the Cartan classification this group corresponds to \( Sp(12, \mathbf{C}) \).

Fields \( A_\mu \) are real linear combinations of elements in the corresponding Lie algebra \( u(6, \mathbf{H}) \), i.e. \( A_\mu^\dagger = -A_\mu \). The same algebra is filled exactly by all the already known fermionic fields, pigeonholed in three families inside \( \psi \). Note that

\begin{align*}
\text{Explicitly } & i^* = -i, j^* = -j, k^* = -k, I^* = -I, (iI)^* = iI, (jI)^* = jI, (kI)^* = kI, \bar{I} = i, \\
& \bar{j} = j, \bar{k} = k, \bar{I} = -I, (iI) = -iI, (jI) = -jI, (kI) = -kI.
\end{align*}
This means that $U(6,\mathbf{H})$ includes $SO(1,3)$, i.e. the gauge group of gravity! The six generators are obviously

$$i(\gamma_2\gamma_1), j(\gamma_1\gamma_3), k(\gamma_3\gamma_2), iI(\gamma_0\gamma_3), jI(\gamma_0\gamma_2), kI(\gamma_0\gamma_1)$$

Moreover, is precisely $SO(1,3)$ (hence gravity) which regulates transitions between homologous fields in different families.

Being both $A_\mu$ and $\psi$ in the adjoint representation, we can join them in a unique superfield:

$$\Psi = e^\mu A_\mu + \theta \psi$$

$e^\mu$ is a hyperionic tetrad for the ordinary space. The internal dimensions are in one to one correspondence with the unities $1, iI, jI, kI$ (or with all the unities $1, i, j, k, I, iI, jI, kI$ if we consider complex dimensions). Thus the internal index disappear. $\theta$ is the analogous of $e^\mu$ for one grassmannian dimension.

6 Conclusion

By concluding, we can say that hyperionic formalism not only justifies the existence of three fermionic families, but also it permits an easy unification with gauge fields and dark matter.
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