Holographic Brownian Motion in 1+1 Dimensions

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Introduction

Motivation

- The gauge/gravity duality has been quite successfully used to study properties of systems at finite temperature.

- Noise and dissipation have been studied by different techniques in this holographic framework.

\[ J. \ de \ Boer \ et \ al. \ ; \ Son-Teaney \ (2009) \]

- Lower dimensions are always interesting!
Figure: The gravity set up for the boundary stochastic motion of the heavy particle.
• Strongly coupled field theory ⇔ Weakly coupled gravity.

\[
\left\langle \exp \left( \int_{S^d} \phi^i_0 \mathcal{O}_i \right) \right\rangle_{\text{CFT}} = Z_{\text{QG}} (\phi^i_0)
\]  (1)
Strongly coupled field theory $\iff$ Weakly coupled gravity.

$$\left\langle \exp \left( \int_{S^d} \phi_0^i O_i \right) \right\rangle_{\text{CFT}} = Z_{\text{QG}} (\phi_0^i)$$  \hspace{1cm} (1)

Real time correlators for (scalar) field theory can be obtained by choosing appropriate boundary conditions.

$$G_R(k) = -2\mathcal{F}(k, r) \bigg|_{r_m}$$  \hspace{1cm} (2)

where $\mathcal{F}(k, r) = K \sqrt{-g} g^{rr} f_{-k}(r) \partial_r f_k(r)$ \hspace{1cm} Son-Starinets (2002)
Langevin Dynamics

- The generalized Langevin equation for a heavy particle under noise $\xi$

$$\left[-M_0^0 \omega^2 + G_R(\omega)\right] x(\omega) = \xi(\omega) \quad \langle \xi(-\omega) \xi(\omega) \rangle = G_{\text{sym}}(\omega) \quad (3)$$

- Expanding $G_R(\omega)$ for small frequencies

$$G_R(\omega) = -\Delta M \omega^2 - i\gamma \omega + \ldots$$

- Then the Langevin equation reads

$$M_{\text{kin}} \frac{d^2 x}{dt^2} + \gamma \frac{dx}{dt} = \xi \quad (4)$$

with

$$\langle \xi(t) \xi(t') \rangle = \Gamma(t - t') \quad (5)$$

- Fluctuation-Dissipation relation

$$iG_{\text{sym}}(\omega) = -(1 + 2n_B) \text{Im } G_R(\omega) \quad (6)$$
The background metric \textit{AdS}$_3$-BTZ is defined as

\[ ds^2 = \frac{\bar{r}^2}{L^2} \left[-f(b\bar{r})dt^2 + dx^2\right] + \frac{L^2 d\bar{r}^2}{f(b\bar{r})\bar{r}^2} \]  

(7)

The same metric in dimensionless coordinate, \( r \equiv b\bar{r} \)

\[ ds^2 = (2\pi T)^2 L^2 \left[-r^2 f(r)dt^2 + r^2 dx^2\right] + \frac{L^2 dr^2}{r^2 f(r)} \]  

(8)

where, \( b = \frac{1}{2\pi TL^2} \), \( f(r) = 1 - \frac{1}{r^2} \) and \( T \) is Hawking temperature.
The background metric AdS$_3$-BTZ is defined as

$$ds^2 = \frac{\bar{r}^2}{L^2} \left[ -f(b\bar{r})dt^2 + dx^2 \right] + \frac{L^2d\bar{r}^2}{f(b\bar{r})\bar{r}^2}$$  \hspace{1cm} (7)

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The Nambu-Goto action is

$$S = -\frac{1}{2\pi l_s^2} \int d\tau d\sigma \sqrt{-\det h_{ab}}$$  \hspace{1cm} (9)
For small fluctuations

\[ \sqrt{-h} = (2\pi T)L^2 \sqrt{1 + (2\pi T)^2 r^4 f(r) x'^2 - \frac{\dot{x}^2}{f(r)}} \]

\[ \approx (2\pi T)L^2 \left[ 1 + \frac{1}{2} (2\pi T)^2 r^4 f(r) x'^2 - \frac{1}{2} \frac{\dot{x}^2}{f(r)} \right] \]
For small fluctuations

\[
\sqrt{-h} = (2\pi T) L^2 \sqrt{1 + (2\pi T)^2 r^4 f(r)x'^2} - \frac{\dot{x}^2}{f(r)} \\
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\]

The string world sheet action becomes

\[
S = -\int dt dr \left[m + \frac{1}{2} T_0(\partial_r x)^2 - \frac{m}{2f} (\partial_t x)^2\right]
\]

(10)

where, \(m \equiv \frac{(2\pi T) L^2}{2\pi l_s^2} = \sqrt{\lambda} T\) and \(T_0(r) \equiv \frac{(2\pi T)^3 L^2}{2\pi l_s^2} fr^4 = 4\sqrt{\lambda} \pi^2 T^3 r^2 (r^2 - 1)\)

The EOM of the string

\[- \frac{m}{f} \partial_t^2 x + \partial_r(T_0(r) \partial_r x) = 0\]

(11)
The EOM of the string

\[
\partial_r^2 f_\omega + \frac{2(2r^2 - 1)}{r(r^2 - 1)} \partial_r f_\omega + \frac{\omega^2}{(r^2 - 1)^2} f_\omega = 0 ; \quad \omega \equiv \omega/(2\pi T) \tag{12}
\]
The EOM of the string

\[ \partial_r^2 f_\omega + \frac{2(2r^2 - 1)}{r(r^2 - 1)} \partial_r f_\omega + \frac{w^2}{(r^2 - 1)^2} f_\omega = 0 ; \quad w \equiv \omega/(2\pi T) \]  

(12)

The solution to this EOM is given by

\[ f_\omega(r) = C_1 \frac{P_{1}^{iw}}{r} + C_2 \frac{Q_{1}^{iw}}{r} \]  

(13)
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$$f_\omega(r) = C_1 \frac{P_1^{i\omega}}{r} + C_2 \frac{Q_1^{i\omega}}{r} \quad (13)$$

 Modes satisfying boundary conditions

$$f^{R}_\omega (r) = \lim_{r_m \to \infty} \left\{ \frac{(1 + r)^{i\omega/2}}{(1 + r_m)^{i\omega/2}} \cdot \frac{(1 - r)^{-i\omega/2}}{(1 - r_m)^{-i\omega/2}} \cdot \frac{r_m}{r} \cdot \frac{2F_1(-1, 2; 1 - i\omega; \frac{1-r}{2})}{2F_1(-1, 2; 1 - i\omega; \frac{1-r_m}{2})} \right\} \quad (14)$$
Generalized Langevin Equation from Holography

- The EOM of the string

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\]

- The retarded correlator \( G_R(\omega) \) is defined as

\[
G_R^0 \equiv \lim_{r_m \to \infty} T_0(r)f_{-\omega}(r)\partial_rf_\omega^R(r) = -M_Q^0\omega^2 + G_R(\omega)
\]

\[
= -\mu\omega \frac{(i\sqrt{\lambda} 4\pi^2 T^2 + \mu\omega)}{2\pi(\mu - i\sqrt{\lambda}\omega)} \quad ; \quad \text{where,} \quad \mu \equiv \frac{r_m}{l_s^2}
\]
Generalized Langevin Equation from Holography

- Zero temperature mass of the particle

\[ M_Q^0 = \frac{\mu}{2\pi} = \sqrt{\lambda} T \ r_m \]  \hspace{1cm} (15)
Generalized Langevin Equation from Holography

- Zero temperature mass of the particle

\[ M^0_Q = \frac{\mu}{2\pi} = \sqrt{\lambda T} \, r_m \]  

(15)

- Retarded propagator

\[ G_R(\omega) = -\frac{\mu \omega}{2\pi} \frac{4\pi^2 T^2}{(\omega + i\frac{\mu}{\sqrt{\lambda}})} \]  

(16)
Zero temperature mass of the particle

\[ M_Q^0 = \frac{\mu}{2\pi} = \sqrt{\lambda} T \quad r_m \quad (15) \]

Retarded propagator

\[ G_R(\omega) = -\frac{\mu \omega}{2\pi} \frac{\left(\omega^2 + 4\pi^2 T^2\right)}{\left(\omega + i \frac{\mu}{\sqrt{\lambda}}\right)} \quad (16) \]

Expanding \( G_R \) in small frequencies

\[ G_R(\omega) \approx \frac{2\lambda \pi T^2}{\mu} \omega^2 - i \left(2\sqrt{\lambda} \pi T^2 \omega + \left(\frac{\sqrt{\lambda}}{2\pi} - \frac{2(\sqrt{\lambda})^3 \pi T^2}{\mu^2}\right) \omega^3\right) \quad (17) \]
Zero temperature mass of the particle

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Retarded propagator

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(17)

Generically when \( G_R(\omega) \) is expanded in small \( \omega \) it takes the form

\[ G_R(\omega) = -i\gamma \omega - \Delta M \omega^2 - i\rho \omega^3 + \ldots \]  

(18)
Viscous drag

\[ \gamma = 2\sqrt{\lambda \pi} T^2 \]  
(19)
Generalized Langevin Equation from Holography

- Viscous drag
  \[ \gamma = 2\sqrt{\lambda\pi} T^2 \]  \hspace{1cm} (19)

- Thermal mass shift
  \[ \Delta M = -\frac{2\lambda\pi T^2}{\mu} = -\sqrt{\lambda} T \frac{1}{r_m} \]  \hspace{1cm} (20)
Generalized Langevin Equation from Holography

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\[ \Delta M = -\frac{2\lambda \pi T^2}{\mu} = -\sqrt{\lambda} T \frac{1}{r_m} \]  

(20)

- Higher order “dissipation coefficient”

\[ \rho = \frac{\sqrt{\lambda}}{2\pi} - \frac{2(\sqrt{\lambda})^3 \pi T^2}{\mu^2} \]  

(21)
Generalized Langevin Equation from Holography

- Viscous drag

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- Higher order "dissipation coefficient"

\[ \rho = \frac{\sqrt{\lambda}}{2\pi} - \frac{2(\sqrt{\lambda})^3\pi T^2}{\mu^2} \]  

(21)

- In the "large frequency" limit

\[ G_R(\omega) \bigg|_{T=0} = \frac{\mu\omega^3(\omega - i\frac{\mu}{\sqrt{\lambda}})}{2\pi(\omega^2 + (\frac{\mu}{\sqrt{\lambda}})^2)} \approx \frac{\mu\omega^2}{2\pi} - i\frac{\mu^2\omega}{2\sqrt{\lambda\pi}} + \ldots \]  

(22)
It is finite and therefore no need to renormalize by adding counterterms.

It cannot be renormalized away in the boundary theory by Hermitian counter terms.

Quark moving at constant velocity doesn’t feel any drag at $T=0$.

Some 1+1 condensed matter systems exhibit such dissipation (or decoherence) at absolute zero due to zero-point fluctuations.

A possible explanation: Energy can cascade from high frequencies to low frequencies in a nonlinear system. One can expect a large Poincare recurrence time and the energy is effectively lost for good. This would then show up as a dissipation!
Brownian Motion at Stretched Horizon
Kruskal/Keldysh Correspondence

$t = -\infty$ → field “1” → $t = +\infty$

$t = -\infty - i\beta$

$\uparrow$

field “2” → $t = +\infty - i\sigma$

$\downarrow$

$U = 0$

$V = 0$

$L$

$R$

$\uparrow t_K$

$\downarrow x_K$
Brownian Motion at Stretched Horizon

Boundary stochastic motion

\[
Z = \int [\mathcal{D}x_1^0][\mathcal{D}x_2^0] [\mathcal{D}x_1][\mathcal{D}x_2] e^{iS_1 - iS_2}
\]

\[
\equiv \int [\mathcal{D}x_1^0][\mathcal{D}x_2^0] e^{iS_{\text{eff}}^0}
\]

\[
iS_{\text{eff}}^0 = -i \int \frac{d\omega}{2\pi} x_a^0 (-\omega)[G_R^0(\omega)]x_r^0(\omega) - \frac{1}{2} \int \frac{d\omega}{2\pi} x_a^0 (-\omega)[G_{\text{sym}}(\omega)]x_a^0(\omega)
\]
Brownian Motion at Stretched Horizon

Effective Action at General $r$

\[ r = r_m \]

\[ r = 1 + \epsilon \]

\[ r = 1 \]

\[ \mathcal{D}x^{0}_{1,2} \]

\[ \mathcal{D}x^{h}_{1,2} \]

\[ \mathcal{D}x^{<}_{1,2} \]
Brownian Motion at Stretched Horizon
Effective Action at General $r$

$\mathcal{D}x_{1,2}^0$

$r = r_m$

$\mathcal{D}x_{1,2}^r$

$r = r_0$

$r = 1 + \epsilon$

$r = 1$

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Brownian Motion at Stretched Horizon

- Retarded Green function at arbitrary $r = r_0$

\[
G_{R}^{r_0}(\omega) \equiv T_0(r) \frac{f_{-\omega}(r) \partial_r f_{\omega}}{|f_{\omega}(r)|^2} \bigg|_{r=r_0}
\]

\[
= -\sqrt{\lambda \pi^2 T^3} \frac{r_0 \omega (r_0 \omega + i)}{2} \cdot \frac{r_0 \omega (r_0 \omega + i)}{r_0 - i\omega}
\]

\[
= -\mu_0 \omega \frac{(i \sqrt{\lambda \pi^2 T^2} + \mu_0 \omega)}{2\pi \mu_0 - i \sqrt{\lambda \omega}}
\]
Brownian Motion at Stretched Horizon

- Retarded Green function at arbitrary $r = r_0$

$$G_{R}^{r_0}(\omega) \equiv T_0(r) \frac{f_{-\omega}(r) \partial_r f_{\omega}}{|f_{\omega}(r)|^2} \bigg|_{r=r_0}$$

$$= -\frac{\sqrt{\lambda} \pi^2 T^3}{2} \cdot \frac{r_0 \omega (r_0 \omega + i)}{(r_0 - i\omega)}$$

$$= -\mu_0 \omega \frac{(i \sqrt{\lambda} \pi^2 T^2 + \mu_0 \omega)}{2\pi(\mu_0 - i \sqrt{\lambda} \omega)}$$

- Softening of delta function

$$\lim_{t \to t_0} \int_{t_0}^{t} dt' \gamma(t') = -\lim_{t \to t_0} \int_{t_0}^{t} dt' \int_{-\infty}^{\infty} d\omega \ e^{-i\omega t'} \frac{\mu \omega}{2\pi} \frac{(\omega^2 + \pi^2 T^2)}{(\omega + i \frac{\mu}{\sqrt{\lambda}})} \to 0$$
Conclusion and frontiers

**Results**

- Natural softening of delta function in Langevin equation.
- Temperature dependent mass correction is zero (in the extreme UV limit).
- A temperature independent dissipation at all frequencies.
- The “stretched horizon” can be placed at an arbitrary radius and an effective action obtained.

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Results

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Can be done

- Study the holographic RG interpretation in this case.
- Same problem using a charged BTZ, thereby introducing a chemical potential.
- ...
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Thank You!
There is a standard way of introducing ‘non-dynamical’ or ‘fake’ variable as following

\[
e^{-\frac{1}{2} \int \frac{d\omega}{2\pi} x_a(-\omega)[iG_{\text{sym}}(\omega)]x_a(\omega)} = \int [\mathcal{D}\xi] e^{i \int x_a(-\omega)\xi(\omega)} e^{-\frac{1}{2} \int \frac{\xi(\omega)\xi(-\omega)}{iG_{\text{sym}}(\omega)}} \frac{d\omega}{2\pi}
\]

(23)

\(\xi\) here is the ‘fake’ variable that can take any random value.

\(\xi\) : the “noise” term.
The drag force $F(t)$ is given by (in frequency space)

$$F(\omega) = G_R(\omega)x(\omega)$$

For a particle moving at constant velocity $x(t) = v.t$, this translates to

$$x(\omega) = -iv\delta'(\omega)$$

Since $G_R(\omega = 0) = G'_R(\omega = 0) = 0$, the force is zero. In more detail, since we have a distribution $\delta'(\omega)$, we should consider a smooth function $f(\omega)$ and evaluate the integral:

$$\int d\omega \ G_R(\omega)x(\omega)f(\omega) = \int d\omega \ G_R(\omega)(-iv\delta'(\omega))f(\omega) = 0$$

on integrating by parts.
The Green function obtained from holography is free from delta function ‘singularity’ which usually leads to contradiction. A simple check of our claim.

$$\lim_{t \to t_0} \int_{t_0}^{t} dt' \gamma(t') = \lim_{t \to t_0} \int_{t_0}^{t} dt' \int_{-\infty}^{\infty} d\omega \ e^{-i\omega t'} \gamma(\omega)$$

$$= - \lim_{t \to t_0} \int_{t_0}^{t} dt' \int_{-\infty}^{\infty} d\omega \ e^{-i\omega t'} \frac{\mu \omega}{2\pi} \frac{(\omega^2 + \pi^2 T^2)}{(\omega + i \frac{\mu}{\sqrt{\lambda}})}$$

$$= \lim_{t \to t_0} \int_{t_0}^{t} dt' \ 2\pi i \ e^{-\frac{\mu}{\sqrt{\lambda}} t'} \frac{\mu (-i \frac{\mu}{\sqrt{\lambda}})}{2\pi} \left\{ \left(-i \frac{\mu}{\sqrt{\lambda}} \right)^2 + \pi^2 T^2 \right\}$$

$$= 0$$

So, $\gamma(\omega)$ is a smooth function.