CONTINUOUS TIME RANDOM WALK AS A RANDOM WALK IN A RANDOM ENVIRONMENT

OFER BUSANI

Abstract. We show that for a weakly dense subset of the domain of attraction of a positive stable random variable of index $0 < \alpha < 1$ (DOA($\alpha$)) the functional stable convergence is a time-changed renewal convergence of distribution of finite mean. Applied to Continuous Time Random Walk (CTR W) à la Montroll and Weiss we show that CTR W with renewal times in a weakly dense set of DOA($\alpha$) can be realized as random walk in a random environment. We find the quenched limit and give a bound on the error of the approximation.

1. Introduction

Let $\{W_i\}_{i=1}^{\infty}$ (abbrv. $\{W_i\}$) be a sequence of i.i.d positive r.vs s.t $\mathbb{P}(W_1 > t) \sim t^{-\alpha}$ for $0 < \alpha < 1$. Then it is well known that the process $D_n^t = n^{-\frac{\alpha}{2}} \sum_{i=1}^{[nt]} W_i$, converges weakly in the $J_1$ topology to a stable subordinator, that is

$$D^n \overset{J_1}{\Rightarrow} D,$$

where $\overset{J_1}{\Rightarrow}$ denotes weak convergence w.r.t $J_1$-Skorohod topology. The fact that $W_1$ typically has big jumps carries over to the limit. This is in contrast to the SLLN of the Renewal Theorem that says that if $\{U_i\}$ is a sequence of i.i.d r.vs s.t $\mathbb{E}(U_1) = 1$ then $T_n = n^{-1} \sum_{i=1}^{[nt]} U_i$ converges in the Skorohod topology to the function $t \mapsto t$, i.e

$$T^n \overset{J_1}{\Rightarrow} t,$$

where $\overset{J_1}{\Rightarrow}$ denotes a.s convergence w.r.t $J_1$ topology. We wish to show here that these two apparently different convergences are closely related. That in fact, observing the convergence in (1.1) is essentially observing the convergence in (1.2) viewed through a sequence of random embedding of the positive real line into itself. One use of the convergence in (1.1) is in the model of Continuous Time Random Walks (CTR W) introduced in [11] by Montroll and Weiss. In the most simple setup $\{J_i\}$ and $\{W_i\}$ are two independent sequences of i.i.d r.vs. Define $(S_n, T_n) = (\sum_{i=1}^{n} J_i, \sum_{i=1}^{n} W_i)$, the (uncoupled) CTR W associated with space-time jumps $\{(J_i, W_i)\}_{i=1}^{\infty}$ (abbrv. $(J_i, W_i)$) is

$$X_t = \sum_{i=1}^{N_t} J_i,$$

where $N_t = \text{sup}\{n : T_n \leq t\}$. In order to model the microscopic behavior of a particle with long binding times to a substrate, one assumes that $W_1$ is heavy tailed, that is

$$\mathbb{P}(W_1 > t) \sim t^{-\alpha},$$
for some $0 < \alpha < 1$. The functional limit of $X_t$ for large $t$ was first considered in [10] in the mathematics literature although earlier in the physics literature ([3]).

Limits for coupled CTRW were considered in [4], and in [9] that of CTRW with space-time jumps that are infinitely divisible. It was shown that

$$n^{-1} X_{\frac{tn}{n}} \xrightarrow{d} B_{E_t},$$

where $B_t$ is a Brownian motion and $E_t$ is the inverse stable subordinator independent of $B_t$, defined by

$$E_t = \inf \{ s : D_s > t \}.$$

The process $B_{E_t}$, sometimes called the Fractional Kinetics process, is a sub-diffusion in the sense that it is self-similar with exponent $\frac{\alpha}{2}$, i.e.

$$B_{E_t} \sim c^2 B_{E_t}.$$

Our results show that the invariance principle in (1.3) where the limit is a Bm subordinated to an independent inverse subordinator, is not merely a property of the limit but is the case for the CTRW itself, even when the CTRW is coupled, i.e. when the r.v $W_i$ and $J_i$ are dependent. In fact, we show this for a larger set of CTRWs, namely CTRW with waiting times with infinite mean with some restriction on their Laplace Transform. A simple case is when $X_t$ is an uncoupled CTRW associated with the i.i.d space-time jumps $(J_i, W_i)$, where $W_i \in DOA(\alpha)$. Then we show that for every $\epsilon > 0$ one can construct a probability space where one can find a sequence of i.i.d r.vs $(J_i, U_i)$ where $E(U_1) < \infty$ and an inverse subordinator (not necessarily stable) $E_t$, independent of $\{U_i\}$, s.t if $Y_t$ is the CTRW associated with $(J_i, U_i)$ then

$$\rho_{d_{J_1}} (Y_{E_t}, X_t) < \epsilon,$$

where $\rho_{d_{J_1}}$ is the Prohorov metric on probability distributions metrizing the weak topology of distributions on the Skorohod space $D([0, \infty))$. This enables us to show that by enriching the filtration of a CTRW one may realize CTRW as an annealed process of a random walk in a random environment (RWRE). One of are two main results (Theorem 1) shows that there exists a set of distributions $A$ which is weakly dense in $DOA(\alpha)$ for which CTRW is an annealed process of RWRE. The random environment is a random time change while the quenched process is a CTRW with finite mean waiting times (independent of the environment) time-changed by the random environment. The results also show that there exists a set of distributions $B \subset A$ which can be realized as another RWRE. This time the random environment is traps in time, that is, for each time $n \in \mathbb{Z}_+$ one randomizes i.i.d trappings $\tau_n$ from a heavy tailed distribution, the quenched process will then be a CTRW with waiting times $\{\tau_n U_i\}$, where $E(U_1) < \infty$. We also show that under proper scaling of CTRW, the quenched process converges to an interesting diffusion time changed by the inverse of a stable subordinator. It shows that in the quenched limit the dynamics of the space-time jumps $(J_i, U_i)$ are translated to that of the regenerative points of the environment. Our second main result (Theorem 2) deals with trying to bound the distance in (1.4) when we scale the process’ $Y_{E_t}$ and $X_t$. We give a polynomial bound $Cn^{-c}$, however, the proof gives way to finding a better $c$ if one only finds a good way of matching the tail of a subordinator with that of $W_1$. Note that CTRW were considered in [1] as one instance of a RWRE on $\mathbb{Z}$ called a Randomly Trapped Random Walk (RTRW). However, there, the random environment is probability measures $\{\pi_z (dt)\}_{z \in \mathbb{Z}}$ on the the positive
real line. Given such a random environment, one preforms a simple random walk on \( \mathbb{Z} \) with waiting times \( \{W^z_i\}_{i \in \mathbb{Z}, z \in \mathbb{Z}} \) s.t the sequence \( \{W^z_i\}_{i \in \mathbb{Z}} \) of waiting times at site \( z \) is drawn independently from the the distribution \( \pi_z \). Reaching the site \( x \in \mathbb{Z} \) for the \( i \)’th time, the random walk waits \( W^x_i \) before moving on to the next site, i.e. traps are in space. In contrast, we show that CTRWs can, at some instances (e.g. stable distribution, Mittag-Leffler distribution), be realized as trap models where the traps are in time rather than space. Moreover, presented as a RTRW, CTRWs are essentially degenerate in the sense that the environment is deterministic, and therefore the limit is completely annealed. Here we show that by considering a larger filtration, the quenched limit retains its environment.

2. Preliminaries

Recall that a Bernstein function is a function \( f : (0, \infty) \to \mathbb{R} \) that is infinitely differentiable, \( f(s) \geq 0 \) and \((-1)^{n-1} f^{(n)}(s) \geq 0 \) for \( n \geq 1 \), where \( f^{(n)} \) is the \( n \)’th derivative of \( f \). A function \( f \) is a Bernstein function iff

\[
(2.1) \quad f(s) = a + bs + \int_0^\infty (1 - e^{-sy}) \mu(dy),
\]

where \( a, b \geq 0 \) and \( \mu \) is a measure on \((0, \infty)\) s.t \( \int_0^\infty (1 \wedge y) \mu(dy) < \infty \), one can then identify a Bernstein function with the characteristics \((a, b, \mu)\). We shall be interested in the set

\[
\mathfrak{B} := \{ f : f \text{ is an unbounded Bernstein function of characteristics } (0, b, \mu) \}.
\]

We denote the Laplace Transform (LT) of a positive measure \( \mu \) on \((0, \infty)\) by \( L \mu(s) = \int_0^\infty e^{-st} \mu(dt) \). Let \( \mathcal{CM} \) denote the space of completely monotone functions, i.e.,

\[
f \in \mathcal{CM} \iff f : (0, \infty) \to \mathbb{R} \text{ and } (-1)^{n} f^{(n)}(s) \geq 0 \text{ for } n \geq 0.
\]

Define

\[
\mathcal{L} := \{ f \in \mathcal{CM} : f(0^+) = 1 \}.
\]

Recall that \( \mathcal{L} \) is just the set of Laplace Transforms of probability measures on the positive real line. For \( \psi \in \mathfrak{B} \) we define the mapping \( \Phi_\psi : \mathcal{L} \to \mathcal{L} \) by

\[
\Phi_\psi(f)(s) = f(\psi(s)).
\]

Note that the mapping is indeed into \( \mathcal{L} \); if \( f \in \mathcal{CM} \) and \( \psi \) is a Bernstein function then \( f(\psi(s)) \in \mathcal{CM}([13, \text{Theorem } 3.6]) \). We say the function \( L : \mathbb{R}^+ \to \mathbb{R} \) is slowly varying at \( \infty \) if

\[
(2.2) \quad \lim_{x \to \infty} \frac{L(\lambda x)}{L(x)} = 1,
\]

for every \( \lambda \in \mathbb{R}^+ \). In fact it is enough to show that (2.2) holds for every \( \lambda \in \Lambda \), where \( \Lambda \subset \mathbb{R} \) is of positive Lebesgue measure. Next, define

\[
\mathcal{L}_\psi := \{ f \in \mathcal{L} : f \sim 1 - \psi(s) L(s^{-1}) \}.
\]
where $L$ is a slowly varying function, and where $f \sim 1 - \psi (s) L (s^{-1})$ means that
\[
\lim_{s \to 0} \frac{|f(s) - 1|}{\psi(s) L(s^{-1})} = 1.
\]
We also use $X \sim f$, where $X$ is a r.v and $f$ is a distribution or a r.v, to say that $X$ is distributed according to $f$, there should not be a confusion there. We denote by $\mathcal{L}$ and $\mathcal{L}_\psi$ the space of measures whose Laplace transform (LT) is in $\hat{\mathcal{L}}$ and $\hat{\mathcal{L}}_\psi$ respectively, that is, the LT $\mathcal{L}$ is a bijection between $\mathcal{L}$ and $\hat{\mathcal{L}}$ and between $\mathcal{L}_\psi$ and $\hat{\mathcal{L}}_\psi$. We also define $\Phi : \mathcal{L} \to \mathcal{L}$ as $\Phi := \mathcal{L}^{-1} \Phi_\psi L$. If $X$ is a r.v with distribution $f$, we sometimes write $\Phi_\psi (X)$ instead of $\Phi_\psi (f)$. Finally, let $\psi_1$ and $\psi_2$ be two Bernstein functions in $\mathcal{B}$, we define $\hat{\Phi}^{\psi_2}_{\psi_1} = \hat{\Phi}_{\psi_2} (\hat{\mathcal{L}}_{\psi_1})$ and $\mathcal{L}^{\psi_2}_{\psi_1} = \Phi_{\psi_2} (\mathcal{L}_{\psi_1})$.

Recall that a positive r.v $X$ is said to be in the domain of attraction of a stable (totally asymmetric) r.v $Y$ of index $0 < \alpha < 1$, i.e. $\mathbb{E} (e^{-sY}) = e^{-s^\alpha}$ (abbr. $X \in \text{DOA} (\alpha)$), if there exists a sequence of normalizing constants $a_n \to 0$ s.t
\[
a_n \sum_{i=1}^n X_i \Rightarrow Y,
\]
where $\{X_i\}$ are i.i.d copies of $X$ and $\Rightarrow$ denotes weak convergence of measures. It is well known that $X \in \text{DOA} (\alpha)$ iff $\mathbb{P} (X > t) \sim L(t) t^{-\alpha}$, where $L(t)$ is a slowly varying function. It is also known that the sequence $a_n$ is regularly varying, i.e,
\[
\lim_{n \to \infty} \frac{a_{[\lambda n]}}{a_n} = \lambda^{-\alpha} \quad \lambda > 0,
\]
and that
\[
mL (a_n^{-1} t) (a_n^{-1} t)^{-\alpha} \to \frac{t^{-\alpha}}{\Gamma (1 - \alpha)}.
\]
For convenience we let $a_1 = 1$. Our interest in Bernstein functions and the mappings $\Phi_\psi$ is in part due to the following fact: for $0 < \alpha < 1$ $\mathcal{L}_{t^{\alpha}} = \text{DOA} (\alpha)$. Moreover,
\[
\mathcal{L}_\alpha = \left\{ \mu : \mu \in \mathcal{L}, \int_0^t y \mu (dy) \text{ is slowly varying} \right\}.
\]
These are consequences of [5, Corollary 8.1.7 and Theorem 8.3.1].
Let $f : \mathbb{R} \to \mathbb{R}$ be a right continuous function with left limits. We denote
\[
f_{t-} := \lim_{\epsilon \to 0^+} f (t - \epsilon),
\]
the left limit of $f_t$. If $f$ is a left continuous function with right limits then
\[
(f_t)^+ := \lim_{\epsilon \to 0^+} f (t + \epsilon),
\]
the right limit of $f_t$. Note that whenever $g(t)$ is a continuous strictly increasing function and $f$ is right continuous with left limits $(f_{g(t)-})^+ = f_{g(t)}$ (note that we first compute $f_{g(t)-}$ and then evaluate at $g(t)$). This may not be the case when there exists $\epsilon > 0$ s.t $g(t - \epsilon) = g(t + \epsilon)$ and $f$ is not continuous at $g(t)$ . We say that $X_t$ is a CTRW with space-time jumps $\{J_t, W_t\}$ or that $X_t$ is a CTRW associated with the space-time jumps $\{J_t, W_t\}$, if
\[
X_t = \sum_{n=1}^\infty J_n 1 \{t \in [T_n, T_{n+1}]\}.
\]
where \( T_n = \sum_{i=1}^{n} W_i \). We use \( \mathbb{D}[0, T] \) to denote the subspace of \( \mathbb{R}^{[0, T]} \) for \( T > 0 \) of càdlàg functions, and \( J_t^{[0, T]} \) to denote the equivalence of law of processes in the Skorohod \( J_t \) topology on \( \mathbb{D}[0, T] \). We shall use \( \mathbb{D} \) and \( \mathbb{L} \) when we refer to \( \mathbb{D}[0, \infty) \) and \( \mathbb{L} \). We use \( X_t^n \overset{\mathcal{L}}{\rightarrow} X_t \) \( (X_t^{[0, T]} \overset{\mathcal{L}}{\rightarrow} X_t) \) to say that the law of the process \( X_t^n \) converges weakly to that of \( X_t \) w.r.t the \( J_t \) topology on \( \mathbb{D} (\mathbb{D}[0, T]) \).

Let \( d \) be a metric on the set \( V \) and let \( \mathcal{P} (V) \) be the set of all probability measures on the Borel sets (with respect to \( d \)) of \( V \). Recall that a sequence of probability measures \( \mu_n \in \mathcal{P} (V) \) converges weakly to \( \mu \in \mathcal{P} (V) \) if for every bounded continuous (with respect to \( d \)) function \( h : V \rightarrow \mathbb{R} \) we have

\[
\int h (x) \mu_n (dx) \rightarrow \int h (x) \mu (dx).
\]

Recall further that the weak topology of \( \mathcal{P} (V) \) is metrizable by the following metric

\[
\rho_d (p_1, p_2) = \inf_{\pi} \inf_{\epsilon > 0} \{ \epsilon : p_1, 2 (|X - Y| > \epsilon) < \epsilon \},
\]

where the infimum runs over all couplings of the r.v.s \( X \) and \( Y \) whose distribution is given by \( p_1 \) and \( p_2 \) respectively. For two r.v.s \( X \) and \( Y \) we sometimes write \( \rho_d (X, Y) \), which should be understood as \( \rho (p_X, p_Y) \), where \( p_X \) and \( p_Y \) are the distributions of \( X \) and \( Y \) respectively. Recall that the Skorohod \( J_t \) topology on \( \mathbb{D}[0, T] \) is metrizable in the following way: a sequence \( f_t^n \in \mathbb{R}^{[0, T]} \) converges in the \( J_t \) topology on \( f_t \in \mathbb{R}^{[0, T]} \) if there exists a sequence of homeomorphisms \( \lambda^n : [0, T] \rightarrow [0, T] \) such that

\[
\| f_t^n - f_t \| \rightarrow 0 \quad \text{and} \quad \| \lambda^n_t - t \| \rightarrow 0,
\]

as \( n \rightarrow \infty \), where \( \| \cdot \| \) is the sup norm, that is, for \( f, g \in \mathbb{R}^{[0, T]} \)

\[
\| f - g \| = \sup_{t \in [0, T]} |f_t - g_t|.
\]

Denote by \( \Lambda \) the set of all homeomorphisms form \( [0, T] \) to itself. One way to metrize the \( J_t \) topology is to use the following metric

\[
d_{J_t} (f, g) = \inf_{\lambda \in \Lambda} \{ \| g - f \| \vee \| \lambda_t - t \| \}. \]

Let \( \mathbb{D}^{\downarrow} \) be the subset in \( \mathbb{D} \) whose elements are strictly increasing. If \( d_t \in \mathbb{D} \) and increasing, we define the generalized inverse of \( d_t \) to be

\[
d_t^{-1} = \inf \{ s : d_s > t \}.
\]

Note that \( d_t^{-1} \) is continuous iff \( d_t \) is strictly increasing. Define the mapping \( \mathcal{H} : \mathbb{D} \times \mathbb{D}^{\downarrow} \rightarrow \mathbb{D} \) by

\[
\mathcal{H} (f_t, d_t) = \left( f_t d_t^{-1} \right)^+.
\]

The results in [14] show that \( \mathcal{H} \) is continuous w.r.t the \( J_t \) topology. In fact, we shall often make use of the following result by Straka and Henry ([14, Theorem 3.6]).

**Lemma 1.** ([Straka and Henry, 2011]) Suppose we have a sequence of random space-time jumps \( \{ J^n_t, W^n_t \} \) and a sequence of random increasing step process \( N^n_t \) s.t

\[
\left( J^n_{N^n_t}, W^n_{N^n_t} \right) \overset{\mathcal{L}}{\rightarrow} (A_t, D_t),
\]
where \( D_t \in \mathbb{D}_{\uparrow\uparrow} \). If \( X^n_t \) is the CTRW associated with \( \{ J^n_i, W^n_i \} \). Then

\[
X^n_t \xrightarrow{D} \mathcal{H}(A_t, D_t).
\]

As in this paper we are interested mostly in the temporal jumps of our CTRWs one may assume throughout that the spatial jumps \( \{ J^n_i \} \in \mathbb{R}^d \) for \( d \in \mathbb{N} \) are i.i.d such that

\[
\lim_{n \to \infty} \sum_{i=1}^{[nt]} J^n_i \xrightarrow{D} B_t,
\]

where \( B_t \) is a standard Bm in \( \mathbb{R}^d \). We use the term \textit{time-change} for a function \( f \) s.t \( f(0) = 0 \), \( f \) is increasing and continuous.

3. FROM RELATIVE STABILITY TO SUB-DIFFUSION

We begin with some technical lemmas that will be useful in understanding the mapping \( \Phi_\psi \).

**Lemma 2.** Let \( L \) be a slowly varying function and \( \phi(s) \) a positive function s.t for every \( \lambda > 0 \) there exist positive constants \( C_1(\lambda) \) and \( C_2(\lambda) \) s.t

\[
C_1(\lambda) \leq \frac{\phi(\lambda s)}{\phi(s)} \leq C_2(\lambda) \quad \forall s > S(\lambda),
\]

for some positive constant \( S(\lambda) \) that may depend on \( \lambda \). Then \( L(\phi(s)) \) is again slowly varying.

**Proof.** Indeed, by the Uniform Convergence Theorem (UCT) ([5, Theorem 1.2.1]) for slowly varying functions we know that

\[
\lim_{s \to \infty} \frac{L(\lambda s)}{L(s)} = 1,
\]

uniformly on any compact \( \lambda \)-set in \((0, \infty)\). Since by (3.1) there exists \( \lambda' \in [C_1, C_2] \) s.t for every \( s > S \)

\[
\frac{L(\phi(\lambda s))}{L(\phi(s))} = \frac{L(\lambda' \phi(s))}{L(\phi(s))},
\]

taking the limit while using the uniform convergence we obtain the result. \( \square \)

**Lemma 3.** Let \( \psi_1, \psi_2 \in \mathcal{B} \), then

\[
\hat{\Phi}_{\psi_2} \left( \hat{\mathcal{L}}_{\psi_1} \right) \subset \hat{\mathcal{L}}_{\psi_1(\psi_2)}.
\]

**Proof.** Suppose first that \( f \in \hat{\mathcal{L}}_{\psi_1} \). By definition \( f(s) \sim 1 - \psi_1(s) L \left( \frac{1}{\psi_1(s)} \right) \) when \( s \to 0 \) where \( L \) is a slowly varying function. It then follows that \( \hat{\Phi}_{\psi_2} f(s) \sim 1 - \psi_1(\psi_2(s)) \left( \frac{1}{\psi_2(\psi_2(s))} \right) \). Denote \( L'(s) = L \left( \frac{1}{\psi_2(s)} \right) \). We must show that \( L'(s) = L \left( \frac{1}{\psi_2(s)} \right) \) is slowly varying. By Lemma 2 it is enough to show that

\[
C_1 \leq \frac{\psi_2(s^{-1})}{\psi_2(s^{-1})} \leq C_2,
\]
for some positive constants $C_1$ and $C_2$ that may depend on $\lambda$. First assume that $\psi_2$ has representation $(0,0,\mu)$. From [13, Lemma 3.4] we see that

\begin{equation}
\frac{e - 1}{e - 1} \lambda I_\mu (s) \leq \frac{\psi_2(s^{-1})}{\psi_2(\mu s^{-1})} \leq \frac{e - 1}{e - 1} \lambda I_\mu (\lambda s),
\end{equation}

for every $s > 0$ where $I_\mu (s) = \int_0^s \mu (y, \infty) dy$. Suppose first that $\lambda \geq 1$ then by the fact that $\psi_2$ is increasing $\frac{\psi_2(s^{-1})}{\psi_2(\mu s^{-1})} \geq 1$, which shows that

\begin{equation*}
1 \leq \frac{\psi_2(s^{-1})}{\psi_2(\mu s^{-1})} \leq \frac{e - 1}{e - 1} \lambda.
\end{equation*}

Similarly, if $\lambda < 1$ we have

\begin{equation*}
\frac{e - 1}{e - 1} \lambda \leq \frac{\psi_2(s^{-1})}{\psi_2(\mu s^{-1})} \leq 1.
\end{equation*}

Now, if $\psi_2(s) = bs + \psi'(s)$, where $\psi'(s)$ has representation $(0,0,\mu)$ and $b > 0$, then

\begin{equation*}
\frac{\psi_2(s^{-1})}{\psi_2(\mu s^{-1})} = \frac{bs^{-1} + \psi'(s^{-1})}{b\lambda s^{-1} + \psi'(\lambda s^{-1})} = \frac{b + \psi'(s^{-1}) s}{b\lambda s^{-1} + \psi'(\lambda s^{-1}) s}.
\end{equation*}

We see that for $\lambda \geq 1$,

\begin{equation*}
\frac{b + \psi'(s^{-1}) s}{b\lambda s^{-1} + \psi'(\lambda s^{-1}) s} \leq \frac{b + \psi'(s^{-1}) s}{b\lambda s^{-1} + \psi'(\lambda s^{-1}) s} \leq \frac{b + \psi'(s^{-1}) s}{b\lambda s^{-1} + \psi'(\lambda s^{-1}) s}.
\end{equation*}

Note that by integration by parts and monotone convergence we see that the limit

\begin{equation*}
M = \lim_{s \to \infty} \psi'(s^{-1}) s = \lim_{s \to \infty} s \int_0^\infty \frac{e^{-s^{-1}y \mu (y, \infty)}}{y} dy
= \int_0^\infty \mu (y, \infty) dy
\end{equation*}

exists and $M \in [0, \infty]$. It follows that for some large enough $S$, for every $s > S$ we have

\begin{equation*}
C_1 (\lambda) \leq \frac{\psi_2(s^{-1})}{\psi_2(\mu s^{-1})} \leq C_2 (\lambda).
\end{equation*}

This shows that $\frac{1}{\psi_2(s^{-1})}$ satisfies (3.1), $L \left( \frac{1}{\psi_2(s^{-1})} \right)$ is slowly varying and that (3.2) holds. □
We say that a measure $\mu$ is sub-homogeneous (super-homogeneous) if for every $\lambda > 0$ there exists a constant $C(\lambda)$ s.t. $\mu(C(\lambda)x, \infty) \leq \lambda \mu(x, \infty)$ for every $x > 0$. For example, if $\mu(dx)$ is a finite measure and $\mu(x, \infty) = x^{-\alpha}L(x)$ where $L(x)$ converges to a constant at infinity, then $\mu$ is sub-homogeneous. The following is a partial uniqueness result.

**Lemma 4.** Let $\psi_1, \psi_2 \in \mathfrak{R}$, and assume that the measure $\mu_2$ of $\psi_2$ is sub-homogeneous or super-homogeneous. Then $\Phi_{\psi_2}^{-1}(\mathfrak{C}_{\psi_1(\psi_2)}) = \mathfrak{C}_{\psi_1}$. 

**Proof.** We prove this for when $\mu_2$ is sub-homogeneous as the proof for the super-homogeneous is similar. Let $f \in \mathfrak{L}$ s.t. $\Phi_{\psi_2}f \in \mathfrak{L}_{\psi_1(\psi_2)}$, or equivalently that $\Phi_{\psi_2}f \sim 1 - \psi_1(\psi_2(s))L(s^{-1})$ as $s \to 0$ where $L(s)$ is slowly varying. It follows that $\Phi_{\psi_2}^{-1}f \sim 1 - \psi_1(s)L(\frac{1}{\psi_2(s)})$, and we must show that $L'(s) = L\left(\frac{1}{\psi_2(s)}\right)$ is slowly varying. By the characterization of regularly varying function, in order to show that $L'(s)$ is slowly varying it is enough to show that

$$\frac{L'(\lambda s)}{L'(s)} \to 1,$$

for $\lambda \in \Lambda$ where $\Lambda$ is a set of positive measure. Let $\lambda \in [1, \infty)$, it is then enough to show that

$$C'_1 \leq \frac{1}{\psi_2^{-1}\left(\frac{1}{\lambda}\right)} \leq C'_2,$$

for some positive constants $C'_1, C'_2$ that may depend on $\lambda$. Since $\psi_2^{-1}(s)$ is increasing we see that

$$1 \leq \frac{\psi_2^{-1}\left(\frac{1}{\lambda}\right)}{\psi_2^{-1}\left(\frac{1}{\lambda^{-1}}\right)}.$$

It is now enough to show that $\psi_2^{-1}(t) \leq C'_2(k)\psi_2^{-1}(kt)$ for $0 < k \leq 1$ and positive $C'_2(k)$. Let $t = bs + \int_0^\infty (1 - e^{-sy}) \mu(dy)$ so that $\psi_2^{-1}(t) = s$. By the fact that $\mu$ is sub-homogeneous we see that

$$kt = kbs + \int_0^\infty (1 - e^{-sy}) \mu_2(dy)$$

$$\geq kbs + \int_0^\infty (1 - e^{-sy}) \mu_2(C(k)dy)$$

$$= kbs + \int_0^\infty (1 - e^{-sC(k)y}) \mu_2(dy)$$

$$\geq C'(k)bs + \int_0^\infty (1 - e^{-sC(k)y}) \mu_2(dy),$$

where $C'(k) = \min \{C(k), k\}$. By the fact that $\psi_2^{-1}$ is increasing we have $\psi_2^{-1}(kt) \geq sC'(k)$, or that $\psi_2^{-1}(t) \leq C'(k)^{-1}\psi_2^{-1}(kt)$. It follows that (3.4) is satisfied with
\( C_1' = 1 \) and \( C_2' = C'(\lambda^{-1})^{-1} \). Then \( L'(s) \) is slowly varying and the result follows. The proof for the super-homogeneous case follows along similar lines while taking \( \lambda \in (0, 1] \).

Combining Lemma 3 and Lemma 4 we obtain the following.

**Proposition 1.** Let \( \psi \in \mathcal{B} \), then the set of distributions \( \mathcal{L}_\psi \) is contained in \( \mathcal{L}_\psi \). Moreover, if the L\(\text{\'}\)evy measure of \( \psi \) is sub-homogeneous or super-homogeneous then \( \Phi_\psi^{-1}(\mathcal{L}_\psi) = \mathcal{L}_\psi \).

We now apply Proposition 1 to CTRW.

**Proposition 2.** Let \( Y_t \) be a CTRW with i.i.d space-time jumps \( \{ J_k, W^\psi_k \} \) where \( \{ W^\psi_k \} \in \mathcal{L}_\psi \) and \( \psi(s) \in \mathcal{B} \). Then there exists a CTRW \( X_t \) with i.i.d space-time jumps \( \{ J'_k, W^\psi_k \} \) where \( \{ W^\psi_k \} \in \mathcal{L}_\psi \) and an inverse of a subordinator with symbol \( \psi(s) \) \( E_t \) that is independent of \( \{ W^\psi_k \} \) s.t

\[
Y_t \overset{\mathcal{W}}{\sim} (X_{E_t})^+.\]

Conversely, assume \( Y_t \) is a CTRW with waiting times \( \{ W^\psi_k \} \in \mathcal{L}_\psi \) s.t \( Y_t \overset{\mathcal{W}}{\sim} (X_{E_t})^+, \) where \( X_t \) is a CTRW with waiting times \( \{ W_k \} \) and \( E_t \) is the inverse of a subordinator of symbol \( \psi(s) \) that is independent of \( \{ W^\psi_k \} \). Moreover, assume that \( \psi(s) \in \mathcal{B} \) has representation \( (0, b, \mu) \) where \( \mu \) is super-homogeneous or sub-homogeneous, then \( W^\psi_k \in \mathcal{L}_\psi \) and \( W_k \in \mathcal{L}_\psi \).

**Proof.** We note that if \( T \) is a positive r.v then \( \Phi_\psi(T) \sim D_T \) where \( D_t \) is a subordinator of symbol \( \psi \) independent of \( T \). Indeed, by the independence of \( D_t \) and \( T \) we have \( \mathbb{E}(e^{-sD_T}) = \mathbb{E}(e^{-\psi(s)T}) = \Phi_\psi(\mathcal{L}(T)) \). Let \( T^n = \sum_{k=1}^n W_k^\psi \) be the time of the \( n \)th jump of \( Y_t \). Since \( W^\psi_k \in \mathcal{L}_\psi \) there exists a distribution \( f^s \in \mathcal{L}_\psi \) s.t \( W^\psi_k \sim \Phi_\psi(f^s) \). We now generate a sequence of i.i.d r.v’s \( \{ W^\psi_k \} \) on a common probability space s.t \( W^\psi_1 \sim W^\psi_1 \). Let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be a probability space, and let \( \{ W^\psi_k \} \) be a sequence of i.i.d random variables in \( \Omega \) s.t \( \Phi_\psi(W^\psi_k) \sim W^\psi \), and let \( T^n = \sum_{k=1}^n W^\psi_k \). Let \( D_t \) be a subordinator of symbol \( \psi(s) \) in \( (\Omega, \mathcal{F}, \mathbb{P}) \) independent of \( \{ W^\psi_k \} \). Define \( W^\psi_k = D_{T_k} - D_{T_{k-1}} \), and note that \( \{ W^\psi_k \} \) are i.i.d and \( W^\psi_1 \sim W^\psi_1 \). Indeed, by the fact that \( D_t \) is a strong Markov process, independent of \( T^n \), with stationary increments we have

\[
W^\psi_k = D_{T_k} - D_{T_{k-1}}
\sim D_{T_k - T_{k-1}}
\sim D_{W^\psi_k} \sim \Phi_\psi(W^\psi_k)
\sim W^\psi_k.
\]

By the independence of increments of \( D_t \), we see that \( W^\psi_k \) are also independent. Assume now that \( \{ J'_k \} \) are i.i.d r.v’s in \( \Omega \) s.t \( \{ J'_k, W^\psi_k \} \sim (J_k, W_k^\psi) \) and that \( X_t \) is the CTRW associated with the space-time jumps \( \{ J'_k, W^\psi_k \} \). Let \( T^n = \sum_{k=1}^n W^\psi_k \), and define the process

\[
Y_t' = \sum_{n=1}^{\infty} J'_n 1(t) \{ y, T^n \leq y \}.
\]
Note that $Y_t'$ is a CTRW with space-time jumps $(J'_i, W'_k)$ and therefore $Y_t' \overset{D}{=} Y_t$.

Next we show (3.5). Since $\psi$ is unbounded we see that $D_t$ is strictly increasing and therefore $E_t$ is continuous, and it follows that a.s for every $\omega \in \Omega$ we have

$$\text{for } t \in \{ y : DT_n^a (\omega) < y \} \iff t \in \{ y : T_n^a (\omega) < E_y (\omega) \}
\iff E_t (\omega) \in \{ y : T_n^a (\omega) < y \},$$

and therefore

$$Y_t' = (Y_{t-})^+ = (\sum_{n=1}^{\infty} J'_1(t)\{ y : DT_n^a < y \})^+
= (\sum_{n=1}^{\infty} J'_1 (E_t)\{ y : T_n^a < y \})^+
= (X_{E_t} - 1)^+.$$

(3.6)

Now, suppose that $Y_t$ is a CTRW associated with the waiting times $\{ W_k \}$, where $W_k \in \Sigma$ and $\psi$ has representation $(a, b, \mu)$ with $\mu$ being super-homogeneous or sub-homogeneous and that $Y_t \sim (X_{E_t} - 1)^+,$ where $X_t$ is a CTRW with space-time jumps $(J_t, W_t)$ and $E_t$ is the inverse-subordinator of symbol $\psi$ independent of $\{ W_i \}$. Let $T_n = \sum_{i=1}^{n} W_i$, going backwards in equation (3.6), we see that $W_k \sim \Phi_\psi W_1$. It is implied by Proposition 1 that $W_1 \in \Sigma$ and the result follows. \(\Box\)

Remark 1. In [8], the mapping $\Phi_\psi$ was used implicitly to obtain fractional Poisson processes. Let $D_t$ be a subordinator of symbol $\psi$, $E_t$ its inverse and let $N_t$ be a Poisson process of intensity $1$. Then it was shown in [8, Theorem 4.1] that $N_t$ is a renewal process with waiting times $\{ W_t \}$ s.t.

$$\mathbb{P} (W_t > t) = \mathbb{E} (e^{-\lambda E_t}).$$

Remark 2. Let us say a distribution $f (dx)$ is a stable-mixture if it is of the form

$$f (dx) = \int_0^{\infty} t^{-1/\alpha} g \left( t^{-1/\alpha} x \right) p (dt) \, dx,$$

where $g (x)$ is the density of a standard stable r.v of index $0 < \alpha < 1$ and $p (dt)$ is a measure whose first moment (maybe infinite) is slowly varying. In other words, $f (dx)$ is a stable-mixture if and only if $f \in \Sigma^{\alpha}$. It is obvious that $\Sigma^{\alpha} \subseteq \Sigma$. Firstly, distributions in $\Sigma^{\alpha}$ have densities which may not be the case for distributions in $\Sigma$. Moreover, by (2.4) we see that whenever $\tilde{f} \in \Sigma$ s.t. $\tilde{f} \sim 1 - \psi (s) L (\frac{1}{s})$ with $L$ a slowly varying function s.t. $\lim_{s \to \infty} L (s)$ is zero or does not exist, $\tilde{f} \notin \Sigma^{\alpha}$. Indeed, [5, Corollary 8.1.7] states that if $L (t)$ is slowly varying then $\tilde{f} (s) \sim 1 - sL (s^{-1})$ is equivalent to $\int_0^{\infty} ydf(y) \sim L (t)$ hence $L$ must be increasing. A natural question is whether $\Sigma^{\alpha}$ is weakly dense in $\Sigma$? Unfortunately we could not answer that. We could not even answer what appears to be a simpler version of that question, namely, if $0 < a < b$ and $A^b_a = \{ \tilde{f} \in \hat{\Sigma} : \tilde{f} (s) \sim 1 - cs, a \leq c \leq b \}$, $B^b_a = \{ \tilde{f} \in \hat{\Sigma} : \tilde{f} (s) \sim 1 - c\psi (s), a \leq c \leq b \}$ is $\Phi_\psi (A^b_a)$ weakly dense in $B^b_a$?
Remark 3. In the case where \( A_t \) and \( E_t \) are independent, by the fact that \( \tilde{L}i_\text{c} \text{av} \) process are stochastically continuous we see that the \( A_{E_t} \) and \( (A_{E_t-})^+ \) have the same law.

Remark 2 underlines the possibly limited range of measures in \( \mathfrak{L}_s^\psi \) compared to \( \mathfrak{L}^\psi \). In order to extend the set \( \mathfrak{L}_s^\psi \) we may use \( \Phi_{\psi'} \) where \( \psi' (s) \in \mathfrak{B} \) s.t \( \psi'(s) \sim \psi L'(s^{-1}) \) where \( L'(t) \) is slowly varying. As the product of two slowly varying functions is a slowly varying function we must have \( \Phi_{\psi'} (\mathfrak{L}_s) \subset \mathfrak{L}^\psi \). Indeed, if \( \hat{f} (s) \sim 1 - sL(s^{-1}) \) where \( L(s^{-1}) \) is slowly varying then \( \hat{f} (\psi'(s)) \sim 1 - \psi(s)L' (s^{-1}) L \left( \psi'(s) \right)^{-1} \) and \( \hat{f} (\psi(s)) \in \mathfrak{L}^\psi \). Define the set

\[
\mathfrak{B}_\psi := \{ \psi'(s) \in \mathfrak{B} : \psi'(s) \sim \psi(s) L(s^{-1}), L \text{ is slowly varying} \},
\]

and then define

\[
\mathfrak{L}_s^{\psi} := \cup_{\psi' \in \mathfrak{B}_\psi} \Phi_{\psi'} (\mathfrak{L}_s) .
\]

Note that the mapping \( \Phi_{s^n} \) reduces the “regularity” \( s \) around \( s = 0 \) for \( \hat{f} (s) \in \hat{L}_s \) with the more coarse “regularity” \( s^n \). In order to maintain general results we make the following assumption on \( \psi(s) \).

**Assumption 1.** We assume \( \psi(s) \) satisfies

\[
\lim_{s \to 0^+} \frac{s}{\psi(s)} = 0.
\]

Note that due to the relation between the regularity of the LT \( \hat{f} \) around zero and the moments of the distribution \( f \) we see that if \( f \in \mathfrak{L}^\psi \) where \( \psi \) satisfies (3.7) then the first moment of \( f \) is infinite. It turns out that the set of distribution \( \mathfrak{L}_s^{\psi} \) is indeed rich in \( \mathfrak{L}^\psi \).

**Lemma 5.** Let \( \psi \in \mathfrak{B} \) that satisfies (3.7). Then the set of distributions \( \mathfrak{L}_s^{\psi} \) is weakly dense in \( \mathfrak{L}^\psi \).

**Proof.** Let \( Y \in \mathfrak{L}^\psi \), that is, \( \mathbb{E} (e^{-sY}) \sim 1 - \psi (s) L(s^{-1}) \). Define \( Y_n = Y 1_{[0, n]} \) and note that \( Y_n \in \mathfrak{L}_s^\psi \). Next define

\[
\psi_n (s) = s + \mu_n^{-1} \int_0^\infty \left( 1 - e^{-sy} \right) f (dy) ,
\]

where \( \mu_n = \mathbb{E} (Y_n) \) and \( f (dy) \) is the distribution of \( Y \). Since \( \psi \) satisfies (3.7) we see that \( \mu_n \to \infty \) by monotone convergence. It follows that

\[
\psi_n (s) \to s,
\]

for every \( s > 0 \). Moreover, denote by \( \hat{f} (t) = \int_t^\infty f (dy) \) the tail of the distribution \( f (dy) \). It is straightforward to verify that \( \hat{f} (s) := \int e^{-sy} f (y) dy = 1 - \hat{f}(s) \) and therefore that \( \hat{f} (s) \sim s^{-1} \psi(s) L(s^{-1}) \). Using integration by parts in (3.8) we see that for every \( n \)

\[
\psi_n (s) \sim s + \mu_n^{-1} \psi(s) L(s^{-1}) .
\]

Let \( f_n \) be the distribution of \( Y_n \). Since \( \hat{f}_n (s) \sim 1 - \mu_n s \), it follows by (3.7) that

\[
\hat{f}_n (\psi_n (s)) \sim 1 - \psi(s) L(s^{-1}) ,
\]

in particular, \( \Phi_{\psi_n} f_n \in \mathfrak{L}^\psi \). It is left to show that \( \Phi_{\psi_n} f_n \to f \) as \( n \to \infty \). But this follows easily form (3.9) and the fact that \( Y_n \) converges weakly to \( Y \). \( \square \)
Remark 4. The reason why we did not use $\psi_n = s + \mu^{-1}_n \psi(s)$ instead of the form in (3.8) is that the form in (3.8) has an advantage when $\psi(s) = s^\alpha L(s^{-1})$ where $L(t)$ is slowly varying. Indeed, by Karamata’s Theorem we know that $\mathbb{E}(e^{-sY}) \sim 1 - s^\alpha L(s^{-1}) \Gamma(1 - \alpha)$ is equivalent to $\mathbb{P}(Y > t) \sim t^{-\alpha} L(t)$. It follows that for every $n$ our approximation $\Phi_{\psi_n} Y_n$ of $Y$ satisfies

$$
\lim_{t \to \infty} \frac{\mathbb{P}(Y > t)}{\mathbb{P}(\Phi_{\psi_n} Y_n > t)} = 1.
$$

Equation (3.11) will be utilized in the sequel in order to obtain quantitative results.

Proposition 3. Let $Y^n_t$ be the CTRW associated with the i.i.d space-time jumps $(J^n_t, a_n W_t)$ where $W_1 \in \mathfrak{W}$. Then there exists a CTRW $X^n_t$ associated with the i.i.d space-time jumps $(J^n_t, n^{-1} U_t)$ s.t $U_1$ has finite mean, and a sequence of inverse-subordinators $E^n_t$ independent of $\{U_t\}$ so that for every $n$

$$
Y^n_t \overset{d}{=} \left(X^n_{E^n_t-} \right)^{+},
$$

and s.t $E^n_t$ converges in law w.r.t the $J_1$-topology to $E_t$, the inverse of a stable subordinator of index $\alpha$.

Proof. Since $W_1 \in \mathfrak{W}$, there exists $\psi \in \mathfrak{B}_a$ (we shall use the one in Lemma 5) s.t $W_1 \in \mathfrak{W}$. By Proposition 2 we see that

$$
Y^1_t = \left(X^1_{E^1_t-} \right)^{+},
$$

where $E^1_t$ is the inverse-subordinator of symbol $\psi_1 = sa_n b + \int_0^\infty (1 - e^{-sa_n y}) \mu(dy)$ and $X^1_t$ is a CTRW associated with i.i.d space-time jumps $(J^1_t, U_t)$ with $U_1 \in \mathfrak{W}$. By Proposition 2 it is enough to show that

$$
(3.12)
$$

$$
a_n W_t \sim \Phi_{\psi_n} \left(n^{-1} U_t \right),
$$

where $\psi_n$ is the symbol of a strictly increasing subordinator. Looking at the Laplace Transform of $a_n W_t$ we see that

$$
\mathbb{E}(e^{-s a_n W_t}) = \mathbb{E} \left[ e^{-U_t (sa_n b + \int_0^\infty (1 - e^{-sa_n y}) \mu(dy))} \right] = \mathbb{E} \left[ e^{-|U_t| \mu (sa_n b + \int_0^\infty (1 - e^{-sa_n y}) \mu(dy))} \right],
$$

which implies (3.12) with $\psi_n(s) = sa_n b + \int_0^\infty (1 - e^{-sa_n y}) \mu(dy)$. Letting $E^n_t$ be the inverse of a strictly increasing subordinator of symbol $\psi_n$ and invoking again Proposition 2 we see that

$$
Y^n_t = \left(X^n_{E^n_t-} \right)^{+}.
$$

We are left to show that $E^n_t$ converges in law to $E_t$, the inverse of a stable subordinator of index $\alpha$. To see that, first note that by the definition of $\psi_1$ and Karamata’s Theorem we know that $\tilde{\mu}_1(y) \sim L(y) y^{-\alpha}$. Let $h(y)$ be a smooth function with compact support $[a, b] \subset \mathbb{R}^+/\{0\}$, then by (2.3)

$$
\lim_{n \to \infty} \int_0^\infty h(y) n \mu_1 (a_n^{-1} y) dy = \lim_{n \to \infty} \int_a^b \frac{\partial h(y)}{\partial y} n \tilde{\mu}_1 (a_n^{-1} y) dy = \int_a^b \frac{\partial h(y)}{\partial y} \frac{y^{-\alpha}}{\Gamma(1 - \alpha)} dy,
$$

CTRW AS A RWRE

12
and form the fact that $a_n n \to 0$ ($a_n$ is regularly varying with parameter $-1/\alpha$) we see that $\mu_n$ converges vaguely to $\mu(dy) = \frac{\alpha}{\Gamma(\alpha)} y^{-\alpha - 1} dy$, the Lévy measure of a standard stable subordinator of index $\alpha$. However, convergence of characteristics of Feller processes implies weak convergence of their law in the $J_1$ topology. In other words, if $D^n_t$ is the subordinator whose symbol is $\psi$, we see that $D^n_t \Rightarrow D_t$.

Next we use Lemma 1 with $(n^{-1}, D^n_t)$ to obtain

$$E^n \Rightarrow E,$$

This completes the proof. □

Proposition 3 can be understood in the following way: let $A_t$ be an increasing process and let $A (\omega, T)$ be the regenerative set of $A_t$ in the interval $[0, T]$ (we may also consider $[0, \infty)$). That is,

$$A (\omega, T) = \{ u \in [0, T] : A_{u-} (\omega) < A_u (\omega) < A_{u+} (\omega), \forall \epsilon > 0 \}.
$$

Note that the mapping $\Phi_\psi$ can be viewed as a mapping on processes. Let $X_t$ be a process, then we define

$$\Phi_\psi (X_t) = X_{E_t},$$

where $E_t$ is the inverse-subordinator of symbol $\psi$ independent of $X_t$. Moreover, $\Phi_\psi$ can also be viewed as a mapping between regenerative set-valued random variables. That is, conditioned on $E (\omega, \infty)$, $\Phi_\psi (X_t) (\cdot, \infty)$ is a random regenerative set contained in $E (\omega, \infty)$. Lastly, note that conditioned on $E_t (\omega) = \xi$, $\Phi_\psi$ can be viewed as a function $\Phi_{\psi, \xi} : \mathcal{L} \to \mathcal{L}$. If $U \in \mathcal{L}$ has distribution $\mu$,

$$\Phi_{\psi, \xi} (\mu) = \xi U^{-1}.$$

$\Phi_{\psi, \xi}$ sends measures in $\mathcal{L}$ to measures whose support is in the regenerative set of $\xi$. Let $\mu \in \mathcal{L}$, and let $f_t$ be a time-change, then we define the probability measure $\mu_f$ on Borel sets of $\mathbb{R}^+$ to be

$$\mu_f (A) = \mu (A_{f^{-1}}),$$

for every Borel set in $A \subset \mathbb{R}^+$, where for an increasing $f$ $A_f$ is the set

$$A_f = \{ x \in \mathbb{R} : f (x) \in A \}.$$

We have $\Phi_{\psi, \xi} (\mu) = \mu_\xi (dx)$. If $U_1$ has finite mean then by the SLLN of Renewal Theory we know that with probability one the regenerative points of the CTRW $T^n_t$ associated with the space-time jumps $(1, n^{-1} U_1)$ 'converge' to a set that is dense in $[0, T]$, namely

$$T (\omega, T) = \cup_n T^n (\omega, T).$$

Since $D_t (\omega)$ is right continuous we deduce that the mapping $\Phi_\psi$ is 'continuous' (if $x_n \in T (\omega, T)$ s.t $x_n > x$ and $x_n \to x$ then $D_{x_n} \to D_x$) and $E (\omega, T)$ is a perfect set (closed, with no isolated points). It follows that $\Phi_\psi (T (\omega, T))$ is dense in $E (\omega, T)$. In other words, as $n \to \infty$ the trajectory $E_t (\omega)$ is delineated by the regenerative points of $T_t$. This idea holds more generally. Let $f \in \mathcal{L}$ and let $\{ U_i \}$ be i.i.d r.v.s with distribution $f$. Let us define $T_0 = 0$ and

$$T_n = \sum_{i=1}^n U_i.$$

We say that $f$ is relatively stable ([5, 8.8]) if there exist norming constants $a_n$ s.t

$$a_n T_n \to 1,$$
where convergence is in probability. Next define the renewal process
\[ N_t = \max \{ k : T_k \leq t \}. \]
Define the residual lifetime \( Z_t \) and the aging \( Y_t \) by
\[ Y_t = t - T_{N_t}, \]
\[ Z_t = T_{N_t+1} - t. \]
Finally, we let \( a_n > 0 \) be any sequence s.t
\[ 1 - \hat{f}(a_n) \sim n^{-1}. \]

The following is known [5, Theorem 8.8.1].

**Lemma 6.** Let \( f \in L_s \) and let \( Y_t \) and \( Z_t \) be the aging and the residual lifetime processes associated with \( f \). The following are equivalent:

1. \( f \in L_s \).
2. \( f \) is relatively stable.
3. \( Y_t \rightarrow 0 \) in probability.
4. \( Z_t \rightarrow 0 \) in probability.

Define \( i_n^T = \sup \{ i : T_n^i \leq T \} \) and the set
\[ A_{\delta,T}^n = \left\{ \omega : \sup_{1 \leq i \leq i_n^T} |T_n^i - T_n^{i-1}| < \delta \right\}. \]
The set \( A_{\delta,T}^n \) is the event that one cannot find two consecutive regenerative points whose distance is larger than \( \delta \). We shall need the next lemma.

**Lemma 7.** Let \( f \in L_s \) and \( T_n^i = a_n \sum_{j=1}^{i-1} U_j \) where \( \{U_i\} \) are i.i.d and \( U_1 \sim f \). Let \( T_n^i = T_{N_n^i + 1} - t \) be its residual lifetime, then for every \( \epsilon > 0 \), by Lemma 6, we have for large enough \( n \),
\[ \mathbb{P} \left( Z_n^i > 2^{-m} \right) < 2^{-m} \epsilon \quad \forall t_k, 1 \leq k < 2^m. \]
It is left to note that \( \{ \cup_{1 \leq k < 2^m} \{ Z_k > \delta/2 \} \} \subset A_{\delta,T}^n. \) \qed

**Proposition 4.** Assume
\[ (A_n^i, D_n^i) \xrightarrow{d} (A_t, D_t), \]
where \( D_t \) is a.s strictly increasing. Let \( \{U_i\} \) be i.i.d r.vs independent of the sequence \( (A_n^i, D_n^i) \) where \( U_1 \in L_s \). Let \( T_n^i = a_n \sum_{j=1}^{i-1} U_j \) be the renewal epoch and let \( \{X_n^i\}_{n=1}^\infty \) be the CTRWs associated with the space-time jumps
\[ \{J_n^i, W_n^i\} = \left\{ A_n^i - A_n^{i-1}, D_n^i - D_n^{i-1} \right\}. \]
Then
\[ X_n^i \xrightarrow{d} (A_{E_t})^+ \]
where \( E_t \) is the generalized inverse of \( D_t \).
Proof. Let $\epsilon > 0$. If $E^n$ are the generalized inverses of $D^n$, Due to (3.16) we see that $E^n_T \Rightarrow E_T$ and therefore, one can find $\bar{T} > 0$ s.t
\[
\sup_n \mathbb{P} \left( E^n_T > \bar{T} \right) < \frac{\epsilon}{3}.
\]
For every $\delta > 0$, define the event $A^n_{\delta, \bar{T}}$ as in (3.15). Consider the CTRW $Y^n_i$ associated with the time-space jumps $((J^n_i, W^n_i), a_i U_i)$ (note that $Y^n_i \in \mathbb{R}^d \times \mathbb{R}_+$). We now claim that for large enough $n$, we have
\[
\rho_{d, \bar{T}}(A^n_{\delta, \bar{T}}, (A^n_i, D^n_i), (Y^n_i)) < \epsilon.
\]
Recall that if $f \in \mathbb{D}[0, \bar{T}]$, then the modulus of continuity of $f$ is given by
\[
\omega^\bar{T}_f(\delta) = \inf \left\{ \max_{1 \leq i \leq m} \theta_f[t_{i-1}, t_i] : \exists m \geq 1, \right. \\
0 = t_0 < t_1 < \ldots < t_m = \bar{T} \text{ s.t. } t_i - t_{i-1} > \delta \text{ for all } i \leq m, \left. \right\},
\]
where
\[
\theta_f[s, t] = \sup_{s \leq u < w \leq t} |f(u) - f(w)|.
\]
Define
\[
B^n_{\delta, \bar{T}} = \left\{ \omega^\bar{T}_{(A^n_i, D^n_i)}(\delta) < \epsilon \right\},
\]
assumption (3.16), suggests that for every $\epsilon > 0$ there exists $\delta > 0$ s.t
\[
\sup_n \mathbb{P} \left( B^n_{\delta, \bar{T}} \right) > 1 - \frac{\epsilon}{3}.
\]
Define the sequence $f^n \in \mathbb{D}_{\mathbb{R}^d \times \mathbb{R}_+}[0, \bar{T}]$ by $f^n_t = (A^n_t, D^n_t)$ on $T_t \leq t < T_{i+1}$ and let $\delta' < \frac{\delta}{\bar{T}} \min(\delta, \epsilon)$. We first condition on $A^n_{\bar{T}}, B^n_{\delta, \bar{T}}, (A^n_i, D^n_i)$, and $\{E^n_T > \bar{T}\}$ i.e. we would like to show that
\[
\mathbb{P} \left( d_f(A^n_i, D^n_i, f^n) > \epsilon \left| A^n_{\bar{T}}, B^n_{\delta, \bar{T}} \right\} \right) = 0.
\]
Indeed, by (3.19), on $B^n_{\delta, \bar{T}}$ one can find $0 = t_0 < t_1 < \ldots < t_m = \bar{T}$ s.t for every $n \geq 1$, $t_i - t_{i-1} > \delta$ and $\theta_{(A^n_i, D^n_i)}[t_{i-1}, t_i] < \epsilon$ for $1 \leq i \leq m$. Let $T^n = \{ T^n_i : T^n_i \leq \bar{T} \}$. On $A^n_{\bar{T}}, B^n_{\delta, \bar{T}}$ one can find the two points
\[
T^n_{i-1} = \inf \{ T^n \cap [t_i, t_{i+1}] \}, \\
T^n_{i+1} = \sup \{ T^n \cap [t_i, t_{i+1}] \},
\]
s.t $t_i \leq T^n_{i-1} < T^n_{i+1} < t_{i+1}$ for $0 \leq i \leq m-1$. The distance between $T^n_{i-1}$ and $T^n_{i+1}$ is at most $\delta'$ and so one can find a homeomorphism $\lambda : [0, \bar{T}] \rightarrow [0, \bar{T}]$ s.t $\lambda(T^n_{i-1}) = t_i$ and s.t $\sup |\lambda(s) - s| \leq m \delta' < \frac{\delta}{\bar{T}} \delta' < \epsilon$ (one simply maps the interval $[T^n_{i-1}, t_{i+1}]$ to $[T^n_{i+1}, t_{i+1}]$ which costs no more then $\delta'$ as $|T^n_{i+1} - T^n_{i-1}| < \delta'$). Next note that by the definition of $\omega^\bar{T}_f(\delta)$ and (3.19) we see that on $A^n_{\bar{T}}, B^n_{\delta, \bar{T}}$
\[
\sup_{0 \leq s \leq \bar{T}} |f^n_{\lambda(s)} - (A^n_s, D^n_s)| < \epsilon,
\]
\[
\sup_{0 \leq s \leq \bar{T}} |\lambda(s) - s| < \epsilon.
\]
Hence (3.20) holds. By Lemma 7, for large enough \( n \)
\[ \mathbb{P}(A_n^0) > 1 - \frac{\epsilon}{3}. \]
Taking expectation in (3.20) while using independence we conclude that for large enough \( n \)
\[ \mathbb{P}\left( d_{J_1[0,T]}((A^n,D^n), f^n) > \epsilon \right) < \epsilon, \]
or (3.18) which implies that
\[ f^n_t \overset{J_1[0,T]}{\Rightarrow} (A_t,D_t). \]
From here we use Lemma 1 to obtain
\[ X^n_t \overset{J_1[0,E_T]}{\Rightarrow} (A_{E_t}-\epsilon). \]
\[ \square \]

Remark 5. If \( U_1 \) in Proposition 4 has finite mean and \( (A^n_t,D^n_t) \overset{J_1}{\rightarrow} (A_t,D_t) \) a.s, then using the SLLN of Renewal Theory and same arguments as in Proposition 4 we see that conditioned on \( \{A^n_t,D^n_t\}_{n=1}^{\infty} \) we have \( X^n_t \overset{J_1}{\rightarrow} (A_{E_t}-\epsilon) \) with probability 1.

As we have shown that CTRW with heavy tailed waiting times can be represented as CTRW with finite mean waiting times subordinated to a time-change, we see that CTRWs à la Montroll and Weiss are essentially CTRWs in random environment. Among the well known Random Walks in Random Environment (RWRE) are the so-called trap models. The most basic setup consists of a simple graph
\[ G = (V,E) \]
where \( V \) is the set of vertices and \( E \) is the set of edges.

On the graph \( G \) we preform a CTRW with exponential waiting times whose jump rate is given by
\[ w_{xy} = \begin{cases} \tau_x^{-1}, & (x,y) \in E, \\ 0, & \text{otherwise}. \end{cases} \]
and the generator is given by
\[ (3.21) \quad Lf(x) = \sum_{y \sim x} w_{x,y} (f(y) - f(x)). \]
In words, the larger \( \tau_x \) is, the deeper the trap at site \( x \) and the longer the CTRW stays at the site \( x \). In order to obtain a non-trivial (simple random walk on \( G \)) limit we assume that \( \{\tau_x\} \) are i.i.d and that \( \tau_x \in \mathbb{L}^{1} \). In [6] Fontes et al studied the Bouchaud model where \( G \) is \( \mathbb{Z} \) with nearest neighbor edges. The Markov process \( X_t \) associated with the generator (3.21) (conditioned on the environment \( \tau \)) is called the quenched process. Taking expectation w.r.t the law of \( \tau \) we obtain the annealed process. Let \( a_n \) be the sequence defined in (2.3). One is interested in the limit (in distribution) of the Bouchaud model
\[ (3.22) \quad n^{-1}X_{a_n-1} \rightarrow X_t. \]
It was proven in [6] that \( X_t \) is a Brownian motion time-changed by the generalized inverse of the local time of a standard Brownian motion integrated against a Poisson measure on \( \mathbb{R} \times \mathbb{R}^+ \) with intensity \( at^{-\alpha-1} (t)_{(0,\infty)} dt dx \). This was referred to as Singular Diffusion. It turns out that the dimension of the lattice affects the limit in (3.22)(although the scaling is different). Indeed, it was proven in [2] that under
proper scaling of the Bouchaud model on $\mathbb{Z}^d$ for $d > 1$ the limit is $B_{E_t}$, i.e. a Brownian motion time-changed by the inverse of a standard stable subordinator independent of $B_t$ (this is referred to as Fractional Kinetics). It is worth mentioning here that the scaling in dimension $d > 2$ is the same as that of the CTRW in the sense of Montroll and Wiess. The limit of the Bouchaud model for dimension larger than one is the same as that in the uncoupled Montroll and Wiess CTRW model. Proposition 3 suggests that the CTRW in the sense of Montroll and Wiess with waiting times in $\mathcal{L}_w$ has a representations as annealed process of possibly two different RWRE. Consider a probability space $(\Omega, \mathcal{F}, P)$ on which there exists a random continuous time-change (continuous increasing processes) $E_t$. We also have a CTRW $\hat{X}_t \in \mathbb{Z}^d$ associated with the i.i.d space-time jumps $(J_i, U_i)$ where $U_1$ has finite mean, $\{U_i\}$ is independent of $E_t$, and where the probability transition function $p_t((J_i, U_i) \in (dx, du))$ may depend on time and the random environment. Unless $J_i$ and $U_i$ are independent $\hat{X}_t$ need not be Markovian even if $U_1 \sim Exp(\lambda)$. Given a realization of the time change $E_t(\omega)$ (our random environment) we consider the process (the quenched process)

\begin{equation}
X^1_t = \hat{X}_{E_t}(\omega).
\end{equation}

We refer to $X^1_t$ in (3.23) as the quenched process of RWRE of type I. We will say that $\hat{X}^1_t$ is an annealed process of RWRE of type I if there exists a CTRW $\hat{X}_t$ s.t

\[ P (\hat{X}^1_t \in dx.) = \int P (\hat{X}_t \in dx.) P_E (d\xi), \]

where $P_E (d\xi)$ is the law of our random environment $E_t$, that is

\[ \int_A P_E (d\xi.) = P (E \in A), \]

with $A$ a Borel set in the Borel sigma-algebra of $\mathbb{D}$. We are interested in the limit

\begin{equation}
n^{-1} \hat{X}^1_{tn} \Rightarrow \hat{X}^1_t.
\end{equation}

Next we introduce another RWRE model which is somewhat of a temporal trap model. Let

\begin{equation}
\tau = \{ \tau_n > 0 : n \in \mathbb{Z}_+ \},
\end{equation}

be our random temporal landscape. We also assume the existence of a family of probability transition functions

\begin{equation}
p_t(s, x; y) \quad t > 0.
\end{equation}

Let $X_t$ be the CTRW who after the $n$th jump ends up at site $x$ and waits an exponential time $s$ of mean $\tau_n$, and then makes a spatial jump to one of its neighbors according to a distribution $p_{\tau_n}(s; y)$. In other words, the temporal landscape $\tau$ affects both the temporal dynamics as well as the spatial. More precisely, assume we have a sequence of positive r.v $\{\tau_i\}$ and let $\{U_i\}$ be a sequence of i.i.d waiting times s.t $E(U_1) = 1$ independent of $\{\tau_i\}$. We define $T_n = \sum_{i=1}^n \tau_i U_i$ to be the epochs of our random walk. Let $\{J_i\}$ be i.i.d r.vs in $\mathbb{Z}^d$ and $S_n = \sum_{i=1}^n J_i$ be a discrete random walk on $\mathbb{Z}^d$ s.t

\[ P (J_{n+1} = y | \tau_n = t, U_{i+1} = s) = p_t(s, x; y). \]

Then, conditioning on $\{\tau_1 = t_1, \tau_2 = t_2, \ldots\}$ we define

\[ X^\Pi_t = S_n \quad T_n \leq t < T_{n+1}. \]
We note that in general $X^I_t$ is not a Markov process, however, if $U_1$ is exponentially distributed, (3.26) is independent of $s$ and $N_t$ counts the number of jumps of $X^I_t$ until time $t$, then $(X_t, N_t)$ is a Markov process with the generator

$$L f(x, z) = \sum_{y \sim x} \tau^{-1}_z p_{x, z}(y) (f(y, z + 1) - f(x, z)).$$

We shall refer to $X^I_t$ as the quenched process of a RWRE of Type II. We define the annealed process of a RWRE of Type II similarly to that of type I. That is

$$\mathbb{P} (\tilde{X}^I_t \in dx) = \int \mathbb{P} (X^I_t \in dx) P_\tau (d\tau),$$

where $P_\tau (d\tau)$ is a probability distribution on the Borel sigma-algebra with respect to the product topology on $\mathbb{R}^N_+$ s.t for every cylinder set of the form $A = \mathbb{R} \times A_{n_1} \times A_{n_2} \cdots \times A_{n_m} \times \mathbb{R}^N_+$ with $A_{n_i} \subset \mathbb{R}_+$,

$$\int_A P_\tau (d\tau) = \mathbb{P} (\tau_{n_1} \in A_{n_1}, \tau_{n_2} \in A_{n_2}, \ldots, \tau_{n_m} \in A_{n_m}).$$

Here we shall be interested in the limit

$$n^{-1} \tilde{X}^I_{tn} \Rightarrow X^I_t$$

Let $\mathcal{M}$ be the set of probability measures whose all moments are finite. Consider the sets

$$\mathcal{A} = \cup_{\psi \in \mathcal{B}, \alpha} \Phi_\psi (\mathcal{M}),$$

$$\mathcal{B} = \Phi_{x_0} (\mathcal{M}).$$

We have seen already in Lemma 5 that $\mathcal{A}$ is weakly dense in $DOA(\alpha)$. Since $\mathcal{M}$ is dense in $\mathcal{L}$ and $\Phi_\psi$ is weakly continuous for every $\psi \in \mathcal{B}$, we conclude that $\mathcal{B}$ is weakly dense in $\mathcal{L}_{x_0}^{x_0}$.

In order to facilitate the exposition of our results we make the following technical assumption.

**Assumption 2.** Assume $\{J_i, W_i\} \in \mathbb{R}^d \times \mathbb{R}_+$ are i.i.d space-time jumps. We assume that the conditional distribution $p(dx; w) = \mathbb{P} (J_1 \in dx | W_1 = w)$ is weakly continuous in $w$, $\mathbb{E} (J_1 | W_1 = w) = 0$, $\sigma^2 (w) = \mathbb{E} (J_1^T J_1 | W_1 = w)$ is a full rank $d \times d$ matrix for $\mathbb{P} (W_1 \in dw)$ almost every $w$ and

$$\sup_{w} \| \sigma^2 (w) \| < \infty,$$

where $\| \cdot \|$ is any norm on the space of $d \times d$ matrices.

Define

$$\sigma^2_\mu = \int_{\mathbb{R}_+} \sigma^2 (t) \mu (dt) \quad \mu \in \mathcal{L}.$$
Theorem 1. Let $X_t$ be a CTRW associated with the i.i.d space-time jumps $(J_i, W_i)$ satisfying Assumption 2 where $W_1 \in \mathcal{A}$ and $J_i \in \mathbb{Z}^d$. Then $X_t$ is an annealed process of RWRE of type I. Moreover, if $W_1$ is also in $\mathcal{B}$ then $X_t$ is also an annealed process of RWRE of type II. In both cases, the limits 3.24 and 3.28 exist and equal

$$\hat{X}_t = B_{E_t},$$

where $E_t$ is the inverse of a stable subordinator, and conditioned on $E: = \xi$, $B_t$ is a time-inhomogeneous diffusion whose generator is

$$L_t (f) (x) = \frac{1}{2} \nabla_x f^T \sigma^2_{\mu(t-1)} \nabla_x f,$$

where $\mu \in \mathcal{S}_x$.

Proof. If $W_1 \in \mathcal{A}$, by Proposition 3 and the definition of $\mathcal{A}$, one can find $U_1 \in \mathcal{M}$ and an inverse-subordinator $E_t$ s.t

$$X_t \overset{J_i}{\sim} \left( \hat{X}_{E_{t-}} \right)^+,$$

where $\hat{X}_t$ is a CTRW with space-time jumps $(J_i, U_i)$ with $\mathbb{E} (U_1) < \infty$. Considering (3.13) we see that we may assume that $\mathbb{E} (U_1) = 1$ as this would only change the convergence to a standard stable subordinator by a constant time change. This proves that $X_t$ is an annealed process of RWRE of type I. Let $X^n_t$ be the CTRW associated with the space-time jumps $\left( n^{-1} J_i, n^{-\frac{d}{2}} W_i \right)$, then

$$X^n_t \overset{J_i}{\sim} n^{-1} X_{tn^{\frac{d}{2}}},$$

and by Proposition 3 we may assume w.l.o.g that there exists a sequence of inverses of subordinators $E^n_t$ s.t

$$E^n_t \overset{J_i}{\to} E_t$$

a.s. where $E_t$ is the inverse of a stable subordinator of index $\alpha$. By Proposition 3 we have

$$X^n_t \overset{J_i}{\sim} \left( \hat{X}^n_{E^n_{t-}} \right)^+,$$

where $\hat{X}^n_t$ is the CTRW associated with $\{ n^{-1} J_i, n^{-2} U_i \}$. We now wish to find $\mathbb{P} \left( (n^{-1} J_{i+1}, n^{-2} U_{i+1}) \in (dx, du) \mid E^n_t = \xi^n \right)$. Let $T_n = \sum_{i=1}^{n} U_i$ , we have

$$\mathbb{P} \left( (n^{-1} J_{i+1}, n^{-2} W_{i+1}) \in (dx, dw) \right) = \mathbb{P} \left( (n^{-1} J_{i+1}, D^n_{n^{-2} U_{i+1} + n^{-2} T_i}) - D^n_{n^{-2} T_i} \in (dx, dw) \right) = \int \mathbb{P} \left( n^{-1} J_{i+1} \in dx | n^{-2} T_i = t, n^{-2} U_{i+1} = u, D^n = (\xi^n)^{-1} \right) \times \mathbb{P} \left( n^{-2} U_{i+1} \in du \right) \mathbb{P} \left( n^{-2} T_i \in dt \right) \mathbb{P} \left( D^n \in d (\xi^n)^{-1} \right).$$

We conclude that

$$\mathbb{P} \left( n^{-1} J_{i+1} \in dx | n^{-2} T_i = t, n^{-2} U_{i+1} = u, D^n = (\xi^n)^{-1} \right) = p \left( ndx; (\xi^n)^{-1}_{n^2 (t+u)} \right).$$
Let $Y^n_t$ be the Markov process $((n^{-1}S_{N^n_t}, n^{-2}T_{N^n_t}))$ conditioned on \( \{ D_{i} = (\xi^{n})^{-1}\} \), where $S_n = \sum_{i=1}^{n} J_i$ and $N^n_t$ is a homogeneous Poisson process with intensity $n^{-2}$. $Y^n_t$ is a Markov process with generator

\[
L^n (f)(x,t) = n^2 \int p\left(dy; (\xi^n)^{-1}_{(t+u)}\right) \mathbb{P}\left(U_1 \in du\right) \left( f (x + y n^{-1}, t + un^{-2}) - f (x,t) \right),
\]

for every $f \in C^{2,1}(\mathbb{R}^d \times \mathbb{R}^+)$. Let $\xi^{-1} = \lim_{n \to \infty} (\xi^n)^{-1}$ where the limit is in $J_1$-topology. If we denote $\mu (du) = \mathbb{P}(U \in du)$, it is not hard to see that $\mu_{(\xi^{-1})'} \to \mu_{(\xi^{-1})'}$ for every $t \geq 0$ where convergence is in the weak topology of measures in $\mathcal{L}$ and where $\mu_{(\xi^{-1})'}$ is as in (3.14). By Assumption 2 it is also not hard to verify that

\[
L^n (f)(x,t) \to L (f)(x,t),
\]

where

\[
L (f)(x,t) = \frac{1}{2} \nabla_x f^T \sigma_{(\xi^{-1})'} \nabla_x f + \frac{\partial}{\partial t} f
\]

with $\nabla_x f^T = \left( \frac{\partial}{\partial x_1} f, \ldots, \frac{\partial}{\partial x_d} f \right)$. (3.29) ensures that (3.34) is indeed the generator of a Markov process on $\mathbb{D}$ (see [7, Theorem 5.4.2]). It follows that $Y^n_t \overset{d}{\to} Y_t$ where $Y_t$ is a Markov process whose generator is given by (3.34). By Lemma 1 we see that

\[
\hat{X}_t^n \overset{d}{\to} B_t,
\]

where $B_t$ is a diffusion with the generator in (3.31). Finally we conclude that

\[
\hat{X}_t^n \overset{d}{\to} (B_{\xi})^+.
\]

Since the generator in (3.31) is a local operator we conclude that $t \mapsto B_t$ is continuous a.s., and that (3.30) holds. Next we assume that $W_1 \in \mathcal{L}^c_s$. Note that this suggests that $E^n_t = E_t$ for every $n \geq 1$ and that

\[
W_t \sim D_{(U_t + T_{i-1})} - D_{T_{i-1}}
\]

(3.35)

The mapping $U \mapsto U_t^\pm$ maps the set $\mathfrak{M}$ onto $\mathfrak{M}$. It follows that $X_t$ is the CTRW associated with the space-time jumps $\{ J_i, U_t^\pm, \tau_i \}$ with $\tau = \{ \tau_i \}$ where $\tau_1 \sim D_1$. This shows that $X_t$ is an annealed process of RWRE of type II with waiting times $U_t' = \left\{ E \left( U_t^\pm \right) \right\}^{-1} U_t^\pm$ and random environment $\tau' = \left\{ E \left( U_t^\pm \right) \tau_i \right\}$. Assume for simplicity that $E \left( U_t^\pm \right) = 1$. Using (3.35) and the calculations for the RWRE of type I we conclude that the quenched limit of the RWRE of type II is $B_{\xi}$. \(\square\)

4. BOUND ON THE ERROR

In this section we give a polynomial bound on the distance between the law of a given uncoupled CTRW $Y_t$ and the law of a time changed CTRW $X_{\hat{t}}^{\pm}$ on the space $\mathbb{D} \times [0,T]$. The proof is constructive and therefore provides us with the space-time jumps of $\hat{X}_t$ as well as with the inverse-subordinators $E^n_t$. The bound relies on the following lemma.
Lemma 8. Let $X \in \mathcal{L}_{s} \alpha$ be a r.v. with tail $\tilde{f}(t) = \mathbb{P}(X > t)$. There exists a r.v $Y \in \mathcal{L}_{s}^{-\alpha}$ and a coupling $\mathbb{P}_{couple}$ of $X$ and $Y$ s.t

$$\mathbb{P}_{couple}(|X - Y| > t) = o(\tilde{f}(t))$$

Proof. Suppose $X$ is a r.v in $\mathcal{L}_{s} \alpha$ . It follows that there exists a function $L(t)$ which is positive and slowly varying s.t $\mathbb{P}(X \geq t) = L(t) t^{-\alpha}$ . By Lemma 5 and (3.10) we see that there exists $Y \in \mathcal{L}_{s}^{-\alpha}$ s.t $\mathbb{P}(Y \geq t) = L(t) t^{-\alpha} + g(t)$, where $g(t) = o(L(t) t^{-\alpha})$. We denote $F_1(t) = \mathbb{P}(Y \geq t)$, $F_2(t) = \mathbb{P}(X \geq t)$ and $I_j = [j, j + 1)$ for $j \in \mathbb{Z}_+$. We begin by coupling $X$ and $Y$ in any way on $I_j$, note that the mass that can be coupled on $I_j$ is min. ($F_1(I_j), F_2(I_j)$) where $F_1(I_j) = F_1(j) - F_1(j + 1)$ for $i \in \{1, 2\}$, and the mass that is excessive and could not be coupled is $|F_1(I_j) - F_2(I_j)| = |g(j) - g(j + 1)|$. Note also that the sign of $g(j) - g(j + 1)$ determines whether $F_1(I_j) > F_2(I_j)$ ($g(j) - g(j + 1) > 0$), $F_1(I_j) < F_2(I_j)$ ($g(j) - g(j + 1) < 0$) and $F_1(I_j) = F_2(I_j)$ ($g(j) - g(j + 1) = 0$). Next we couple the excessive mass of $X$ and $Y$ on each interval of the form $[2^n, 2^{n+1})$ in the following way: let $\{I_{ik}\}^m_{k=1}$ and $\{I_{jk}\}^m_{k=1}$ be the sets of intervals whose excessive mass from the partial coupling before is negative and non-negative respectively. More precisely, let $I_j \subset [2^n, 2^{n+1})$ then $I_j \in \{I_{ik}\}^m_{k=1} (I_j \in \{I_{jk}\}^m_{k=1})$ if $g(j) - g(j + 1) < 0$ ($g(j) - g(j + 1) \geq 0$). So $\bigcup_{k=1}^m I_{ik} \cup \bigcup_{k=1}^m I_{jk} = [2^n, 2^{n+1})$. Imagine that each $I_{ik}$ is a customer with negative mass $q_k$ and each $I_{jk}$ is a server with positive mass $s_k$. Customers enter the queue according to their original order in $[2^n, 2^{n+1})$, that is, $I_{ik}$ is in front of $I_{jk}$ iff $i < j$. The customer $I_{ik}$ leaves the queue only after he was served by $m$ servers whose total mass is at least $q_k$. Server $I_{jk}$ leaves the line as soon as he has served all its mass. For example, if in the interval $[4, 8)$ we have the following

$$I_1, I_2, I_3, I_4, I_5, I_6, I_7, I_8$$

In this case, $I_{i_1} = [4, 5), I_{i_2} = [5, 6)$ and $I_{j_1} = [6, 7), I_{j_2} = [7, 8)$. then the coupling will be

$$-0.4, -0.2|0.1, 0.7$$

and so $g(4) - g(8) = 0.2$, which is the excessive mass of $F_1([4, 8))$ over $F_2([4, 8))$ that can not be coupled in the interval $[4, 8)$. We say that the interval $I_{i_k}$ is $i$-bad if the last server $I_{j_k}$ that served him is such that $|i - j_k| > i$. For example, in (4.1) the customer $I_4$ was served by both $I_6$ and $I_7$ and since $7 - 4 = 3 < 4$ it is not 4-bad but is 2-bad. Note that if $I_j \in [2^n, 2^{n+1})$ then $I_j$ is $i$-bad iff one of the following conditions is satisfied

$$F_1([2^n, j - i)) \geq F_2([2^n, j + 1))$$

$$F_1([2^n, j + i + 1)) < F_2([2^n, j + 1)).$$
Define
\begin{equation}
\epsilon_i^1 = \sup_{j \geq 1} \sup_{1 \leq \lambda < 2} \left| \frac{L(j\lambda)}{L(j)} - 1 \right|
\end{equation}
\begin{equation}
\epsilon_i^2 = \sup_{t \geq 1} \left| \frac{g(t)}{L(t) t^{-\alpha}} \right|
\end{equation}

Note that by the UCT and the definition of \( g(t) \), \( \epsilon_i^1, \epsilon_i^2 \overset{i \rightarrow \infty}{\rightarrow} 0 \). Fix a positive integer \( i \). Note that potential \( i \)-bad intervals \( I_j \) should be looked for for \( j \geq 2^{[\log i] + 1} \), where throughout the proof we use \( \log x = \log_2 x \). Let us now check the two conditions.

Let \( t = 2^{[\log j]} \), then condition one is
\begin{equation}
\begin{align*}
F_1(t) - F_1(j-i) &\geq F_2(t) - F_2(j+1) \\
F_2(t) - F_2(j-i) + g(t) - g(j-i) &\geq F_2(t) - F_2(j+1)
\end{align*}
\end{equation}

Note that by (4.3) it is enough to look for \( j \)'s that satisfy
\begin{equation}
L(j+1)(j+1)^{-\alpha} - L(j-i)(j-i)^{-\alpha} \geq -2\epsilon_i^2 L(t) t^{-\alpha}.
\end{equation}

If \( L_{\max} = \sup_{t \leq y \leq 2t} |L(y)| \), by (4.2) we can look for \( j \)'s that satisfy
\begin{equation}
L_{\max} \left( (j+1)^{-\alpha} - (j-i)^{-\alpha} \right) \geq -2\epsilon_i^2 L_{\max} t^{-\alpha}.
\end{equation}

Using the convexity of \( t \mapsto t^{-\alpha} \) we may consider
\begin{equation}
-\alpha (j+1)^{-\alpha-1} (i+1) + \epsilon_i^1 (j-i)^{-\alpha} \geq -2\epsilon_i^2 t^{-\alpha},
\end{equation}
or
\begin{equation}
-\alpha (j+1)^{-\alpha-1} (i+1) + \epsilon_i^1 (j-i)^{-\alpha} \geq -2\epsilon_i^2 t^{-\alpha},
\end{equation}

or
\begin{equation}
\frac{(i+1)^{-\alpha} + \epsilon_i^1 (j-i)^{-\alpha}}{L(t)} \geq 2 \epsilon_i^2 t^{-\alpha} - 1.
\end{equation}

Note that \( t \) is at least \( 2^{[\log(i)]} \) and so
\begin{equation}
\frac{(i+1)^{-\alpha} + \epsilon_i^1 (j-i)^{-\alpha}}{L(t)} \geq 2^{[\log(i)]} 2^{[\log(i)]} \epsilon_i^2 t^{-[\log(i)]} - 1.
\end{equation}

It follows that for a fixed \( i \), \( i \)-bad \( j \)'s who satisfy the first condition should be looked for above a number that increases super-linearly with \( i \). Similarly, for the second condition we obtain the following condition
\begin{equation}
\frac{(i+1)^{-\alpha} + \epsilon_i^1 (j-i)^{-\alpha}}{L(t)} \geq 2^{[\log(i)]} 2^{[\log(i)]} \epsilon_i^2 t^{-[\log(i)]} - 1.
\end{equation}

Let us denote by \( ic_i \) (\( t = 2^{[\log i]} \)) the r.h.s of (4.6). It follows that one cannot find \( i \)-bad \( j \)'s between \( i \) and \( ic_i \) where the latter increases super-linearly in \( i \). Let \( W = |X - Y| \) be the absolute difference between \( X \) and \( Y \) in our coupled space \( (\Omega_{\text{couple}}, \mathcal{F}_{\text{couple}}, \mathbb{P}_{\text{couple}}) \). If \( I_j \) is not \( i \)-bad and was coupled in the second stage, then \( \{ X \in I_j \} \subset \{ W \leq i \} \) for \( i > 1 \). Since on each interval of the form \( [2^n, 2^{n+1}] \), for \( n \geq \log (ic_i) \), we coupled the r.v in such a way that it has no \( i \)-bad intervals, the only mass that may affect the event \( \{ W > i \} \) is \( |g(2^n) - g(2^{n+1})| \). It follows that
\begin{equation}
\mathbb{P}_{\text{couple}}(W > i) \leq \sum_{k = [\log(i)]}^{[\log(ic_i)] - 1} \left| g(2^k) - g(2^{k+1}) \right| + L \left( \frac{ic_i}{4} \right) \left( \frac{ic_i}{4} \right)^{-\alpha} + \left| g \left( \frac{ic_i}{4} \right) \right|.
\end{equation}
We claim now that

\[(4.8) \sum_{k=\log(i)}^{[\log(i)]-1} |g(2^k) - g(2^{k+1})| = o(L(i) i^{-\alpha}).\]

To see that we note that by (4.3) and the UCT, for any $C > \frac{1+2\alpha}{1-2\alpha} and large enough $i$ we have

\[|g(2^k) - g(2^{k+1})| \leq \epsilon^2 \alpha^2 \frac{L(2^{k+1})}{L(2^k)} - 2^{2-\alpha} L\left(\frac{L(2^{k+1})}{L(2^k)}\right),\]

It follows that for large enough $i$ we have

\[\sum_{k=\log(i)}^{[\log(i)]-1} |g(2^k) - g(2^{k+1})| \leq C \epsilon^2 L\left(\frac{L(2^{\log(i)})}{2^{\log(2i)}}\right) - L(\log(i) 2^{-i^{2\alpha}}),\]

and (4.8) is implied. It follows from (4.7) and (4.8) that

\[(4.9) \mathbb{P}_{\text{couple}}(W > i) = o\left(L(i) i^{-\alpha}\right),\]

and it is straightforward to see that (4.9) holds when $i \in \mathbb{R}^+$. 

In the following result we limit ourselves to case where the waiting times of $Y_{1i}$ is such that $\mathbb{P}(W_i > t) \sim c t^{-\alpha}$, where $c$ is some positive constant. This assumption is important for the result.

**Theorem 2.** Let $Y^n_i$ be the CTRW associated with the i.i.d space-time jumps $\{n^{-\frac{1}{2}} W_i, n^{-\frac{1}{2}} J_i\}$ where $\mathbb{P}(W_1 > t) = [\Gamma(1-\alpha)]^{-1} t^{-\alpha} + g_2(t)$ and $g_2(t) = O(t^{-\beta})$ for $\beta > \alpha$ and $J_i \in \mathbb{R}$ has variance 1 and zero mean. Then there exists a CTRW $\tilde{X}^n_i$ associated with the i.i.d space-time jumps $\{n^{-\frac{1}{2}} J_i, n^{-1} U_i\}$ s.t $U_i$ has finite mean, a sequence of inverse subordinators $E^n_{j_i}$ so that for every $c < \xi_0$,

\[\rho_{j_i}\left(Y^n_i, \tilde{X}^n_{E^n_{j_i}}\right) < C n^{-c},\]

where $\xi_0 = \min\left\{\frac{\alpha}{\alpha+1}, \frac{\beta-\alpha}{2\alpha+1}\right\}$.

**Proof.**

**Step 1** First consider for every $n$ the sequence $\{a_n W_i\}_{i=1}^{\infty}$. If $\mathbb{P}(W_1 > t) = L(t) t^{-\alpha}$, by Lemma 5 we can approximate $W_1$ by a distribution $U \in \mathcal{G}_{\epsilon}$ s.t. $\mathbb{P}(U > t) \sim L(t) t^{-\alpha}$. Let $\{U_i\}$ be a sequence of i.i.d r.v’s s.t $U_1 \sim U$. Define $X^n_i$ to be the CTRW associated with the space time jumps $\{n^{-\frac{1}{2}} J_i, n^{-\frac{1}{2}} U_i\}$. We wish to construct a set $A \in \Omega_{\text{couple}}$ of probability larger than $1 - \epsilon$ on which we can bound the distance $(d_{j_i})$ between two trajectories of the processes $X^n_i$ and $Y^n_i$. In order to use Lemma 8 we must limit our discussion to finite number of jumps by time $T$. We shall use the fact that for every coupling $p_{X,Y}$ of some r.v.s $X$ and $Y$, if $p_X(X \in A) < \frac{\epsilon}{4}$ and $p_Y(Y \in B) < \frac{\epsilon}{4}$ then
\[ p_{X,Y} \left( |X - Y| \leq |\{X \in A\} \cup \{Y \in B\} \right) > \epsilon \right) \leq \frac{\epsilon}{2}, \]

for any coupling \( p_{X,Y} \) of \( p_X \) and \( p_Y \). And so, if we show that

\[ p_{X,Y} \left( |X - Y| \leq |\{X \in A\} \cap \{Y \in B\} \right) > \epsilon \right) \leq \frac{\epsilon}{2}, \]

we see that \( p_{X,Y} \left( |X - Y| > \epsilon \right) = p_{X,Y} \left( |X - Y| \leq |\{X \in A\} \cap \{Y \in B\} \right) + |X - Y| \leq |\{X \in A\} \cap \{Y \in B\} \right) > \epsilon \leq \frac{\epsilon}{2} \).

Suppose there exists a sequence \( M_1 (n) \to \infty \) s.t. for large enough \( n \)

\[ \mathbb{P} \left( \sum_{i=1}^{M_1(n)} a_n W_i \leq T \right) \leq \frac{\epsilon}{8}, \]

\[ \mathbb{P} \left( \sum_{i=1}^{M_1(n)} a_n U_i \leq T \right) \leq \frac{\epsilon}{8}. \]

Next assume there exists a sequence \( M_2 (n) \to \infty \) s.t. for large enough \( n \)

\[ \mathbb{P} \left( \sum_{i=1}^{M_2(n)} a_n W_i \leq \frac{\epsilon}{2} \right) < \frac{\epsilon}{8}, \]

\[ \mathbb{P} \left( \sum_{i=1}^{M_2(n)} a_n U_i \leq \frac{\epsilon}{2} \right) < \frac{\epsilon}{8}. \]

Moreover, assume that for large enough \( n \)

\[ \mathbb{P} \left( S_{\{0,1,\ldots,M_2(n)\}} > \frac{\epsilon}{4} \right) < \frac{\epsilon}{8}, \]

where for a set \( A \subset \mathbb{Z}_+ \), with \( j_i = \inf A \),

\[ S_A = \sup_{i \in A} \left| j_i \sum_{j=j_i}^{i} n^{-\frac{1}{2}} J_j \right|. \]

Define the random sets

\[ B_n^Y [a,b] = \left\{ j : \sum_{i=1}^{j} a_n W_i \in [a,b] \right\}, \]

\[ B_n^X [a,b] = \left\{ j : \sum_{i=1}^{j} a_n U_i \in [a,b] \right\}, \]

that is, \( B_n^Y [a,b] \) is the set of the indices of the jumps that occurred in the time interval \( [a,b] \). Also define \( i_0 = \inf B_n^Y [a,b] \) and \( i_0^X = \inf B_n^X [a,b] \). By Lemma \( 8 \) we know that we can construct a probability space \( (\Omega_{\text{couple}}, \mathcal{F}_{\text{couple}}, \mathbb{P}_{\text{couple}}) \) on which one can find the sequence \( \{W_i\} \) and \( \{U_i\} \) s.t. for large enough \( n \) we have

\[ \mathbb{P}_{\text{couple}} \left( \sum_{i=1}^{M_1(n)} a_n |W_i - U_i| > \frac{\epsilon}{2} \right) < \frac{\epsilon}{4}. \]
It is implied that for $N_0$ large enough, for every $n > N_0$, one can find a set $A_n \in \mathcal{F}_{\text{couple}}$ s.t $\mathbb{P}_{\text{couple}} (A_n) > 1 - \epsilon$ and conditioned on $A_n$ we have

$$
\mathbb{P}_{\text{couple}} \left( |B_n^X[0,T]| > M_1(n) \middle| A_n \right) = 0
$$

$$
\mathbb{P}_{\text{couple}} \left( \left| B_n^X[T - \frac{\epsilon}{2}, T] \right| > M_2(n) \middle| A_n \right) = 0
$$

$$
\mathbb{P}_{\text{couple}} \left( \left| \sum_{J \in F_n} J \right| > \frac{\epsilon}{4} \middle| A_n \right) = 0
$$

(4.13)

$$
\mathbb{P}_{\text{couple}} \left( \sum_{i=1}^{M(n)} a_n \left| W_i - U_i \right| > \frac{\epsilon}{2} \middle| A_n \right) = 0,
$$

where the first three equations in (4.13) are true for the sets $B_n^X[0,T]$ and $B_n^X[T - r^{-1}, T]$ as well.

**Step 2** Let $M \in \mathbb{Z}_+$ and $d_i^M : (\mathbb{R} \times \mathbb{R}_+)^M \to \mathbb{R}_+$ be the metric on vectors of real numbers defined by

$$
d_i^M \left( \{a_n^1, a_n^2\}, \{b_m^1, b_m^2\} \right) = \sum_{n=1}^{M} |a_n^1 - b_m^1| + |a_n^2 - b_m^2|.
$$

Consider the set $\mathcal{A} = \left\{ (J_i, W_i) \in (\mathbb{R} \times \mathbb{R}_+)^M : \sum_{i=1}^{M} W_i \leq T \right\}$ equipped with $d_i^M$, i.e.

$$
d_i^M \left( (J_i^1, W_i^1) , (J_i^2, W_i^2) \right) = \sum_{i=1}^{M} |J_i^1 - J_i^2| + |W_i^1 - W_i^2|.
$$

Define the mapping $\mathcal{T} : \mathcal{A} \to \mathbb{D}[0,T]$ by

$$
(J_i, W_i)_{i=1}^{M} \mapsto f_t = \sum_{i=1}^{M} J_i \left\{ \sum_{i=1}^{M} W_i \leq t \right\}.
$$

We claim that $\mathcal{T} : (\mathcal{A}, d_i^M) \to (\mathbb{D}[0,T], d_{J_t})$ is a contraction. To see that, let $(J_i^1, W_i^1), (J_i^2, W_i^2) \in (\mathbb{R} \times \mathbb{R}_+)^M$, and define

$$
\lambda_i = \begin{cases} 
\frac{t W_i^2}{W_i} & 0 \leq t < W_i^1 \\
(t - W_i^1) \frac{W_i^2}{W_i} + W_i^2 & W_i^1 \leq t < W_i^1 + W_i^2 \\
\vdots & \vdots \\
(t - \sum_{i=1}^{M-1} W_i^1) \frac{W_i^2}{W_i} + \sum_{i=1}^{M-1} W_i^1 & \sum_{i=1}^{M-1} W_i^1 \leq t \leq T 
\end{cases}
$$

Note that

$$
\| \mathcal{T} \left( (J_i^1, W_i^1) \right) (\lambda_i) - \mathcal{T} \left( (J_i^2, W_i^2) \right) (t) \| \leq \sup_{t} |J_i^1 - J_i^2| \leq \sum_{i=1}^{M} |J_i^1 - J_i^2|,
$$

since the regeneration points of $\mathcal{T} \left( (J_i^2, W_i^2) \right) (t)$ and $\mathcal{T} \left( (J_i^1, W_i^1) \right) (\lambda_i)$ are the same. Next note that since $\lambda_i$ is piece-wise linear

$$
\| \lambda_i - t \| \leq \sup_{t_i} |\lambda_i - t_i|,
$$
where \( t_i \in \left\{ \sum_{j=1}^{i} W_j^1 : 1 \leq i \leq M \right\} \). Or equivalently,

\[
\| \lambda_t - t \| = \sup_{1 \leq i \leq M} \left| \sum_{j=1}^{i} W_j^2 - \sum_{j=1}^{i} W_j^1 \right| \leq \sum_{i=1}^{M} |W_i^1 - W_i^2| .
\]

It follows that

\[
d_{J_1} (T [ (J_1^1, W_1^1) ] (t) - T [ (J_2^2, W_2^2) ] (t) ) \\
\leq \| \lambda_t - t \| \wedge \| T [ (J_1^1, W_1^1) ] (\lambda_t) - T [ (J_2^2, W_2^2) ] (t) \| \\
\] (4.14)

so that \( T \) is indeed a contraction. Next, let \( x^n_\omega \) and \( y^n_\omega \) be two realizations of \( X^n_\omega \) and \( Y^n_\omega \) respectively on the set \( A^n \). Suppose w.l.o.g that \( x^n_\omega \) has at least the same number of jumps by time \( T - \frac{\epsilon}{2} \) as \( y^n_\omega \), that is

\[
\left| B^n_Y [0, T - \frac{\epsilon}{2}] \right| \leq \left| B^n_X [0, T - \frac{\epsilon}{2}] \right| .
\]

By (4.13) we have

\[
\sum_{i=1}^{M_{(n)}} a_n |W_i - U_i| < \frac{\epsilon}{2} .
\]

which implies that one can find \( J_{diff} := |B^n_X [0, T - \frac{\epsilon}{2}]| - |B^n_Y [0, T - \frac{\epsilon}{2}]| \) jumps of \( y^n_\omega \) in the interval \( [T - \frac{\epsilon}{2}, T] \). Let \( \tilde{x}_t \in \mathbb{D}[0, T] \) s.t

\[
\tilde{x}_t = x_t - \sum_{i \in B^n_X [T - \frac{\epsilon}{2}, t]} n^{-\frac{1}{2}} J_i ,
\]

where if \( t < T - \frac{\epsilon}{2} \) the summation vanishes. In words, \( \tilde{x}_t \) equals to \( x_t \) up to time \( T - \frac{\epsilon}{2} \) and equals \( x_t (T - \frac{\epsilon}{2}) \) on the interval \( [T - \frac{\epsilon}{2}, T] \). Next we define the time \( T_{diff} = \inf \{ t : |B^n_Y [T - \frac{\epsilon}{2}, t] - J_{diff} | \geq J_{diff} \} \) and

\[
\tilde{y}_t = y_{T_{diff} \wedge t} .
\]

\( \tilde{y}_t \) stands for the function that equals \( y_t \) up to the point where it has jumped the same number of jumps as \( \tilde{x}_t \). Next note that on \( A^n \),

\[
d_{J_1} (x_t, y_t) \leq d_{J_1} (\tilde{x}_t, \tilde{y}_t) + d_{J_1} (x_t - \tilde{x}_t, y_t - \tilde{y}_t) \\
\leq d_{J_1}^2 ((J_1^n(\omega), a_n W_i(\omega)), (J_1^n(\omega), a_n U_i(\omega))) \\
+ \sum_{B^n_X [T - \frac{\epsilon}{2}, T]} + \sum_{B^n_Y [T - \frac{\epsilon}{2}, T]} \\
< \epsilon ,
\]

(4.15)

where \( \omega \in \Omega_{couple} \) is such that

\[
T (J_1^n(\omega), a_n W_i(\omega)) = x_t \\
T (J_1^n(\omega), a_n U_i(\omega)) = y_t \\
\left| B^n_X [0, T - \frac{\epsilon}{2}] \right| = M .
\]

Inequality (4.15) follows from (4.13) and (4.14). We have showed that there exists a coupling s.t

\[
P_{couple} (d_{J_1} (X^n_\omega, Y^n_\omega) > \epsilon) < \epsilon ,
\]
or that
\[ \rho \mathcal{J}_1(X^n_t, X^n_t) < \epsilon. \]

Step 3 Let \( W \) be a r.v s.t \( W \sim W_1 \). In order to approximate \( W \) by elements in \( \Omega_s^* \) we follow the recipe in Lemma 5. We first introduce \( W_m = W_{1[0,m]} \) and
\[ \mu^m_i = \mathbb{E}((W_m)^i), \]
the \( i \)'th moment of \( W_m \). We proceed to defining the symbol
\[ \psi(s) = -s - (\mu^m_1)^{-1} \int_0^\infty (e^{-sy} - 1) \frac{\alpha y^{\alpha-1}}{\Gamma(1-\alpha)} dy. \]

Note that \( \psi(s) \) is the symbol of the subordinator \( t + D_{1/\mu^m_1} \), where \( D_t \) is the standard stable subordinator of index \( 0 < \alpha < 1 \) whose LT is \( \mathbb{E}(e^{-sD_t}) = e^{-ts^\alpha} \). Note that we somewhat deviate form the recipe in Lemma 5, where we would use the symbol
\[ \psi'(s) = -s - (\mu^m_1)^{-1} \int_0^\infty (e^{-sy} - 1) f(dy), \]
where \( f(dy) \) is the distribution of \( W \). However, since the purpose of the \( f(dy) \) in Equation (4.16) is to obtain a regularity of \( s^\alpha L(s^{-1}) \) around zero for \( \psi(s) \), it is clear that in this case a stable subordinator would do the job. An expression for the tail of a stable subordinator at time \( t > 0 \) can be found in [15, Eq. 2.4.3] to be (with some algebraic manipulations)
\[ F^D_t(x) = \sum_{n=1}^\infty (-1)^{n-1} \frac{x^{-\alpha n t^n}}{\Gamma(1-\alpha n) n!} x > 0, t > 0. \]

Let \( h_m(dy) = \Phi_{\psi'}(f_m) \), where \( f_m(dy) = \mathbb{P}(W_m \in dy) \). We have for \( x > m \)
\[ h_m(x) = \mathbb{P}(W_m + D_{\nu_m/\mu^m_1} > x) \]
\[ = \int_0^\infty F^D_{\nu_m/\mu^m_1}(x-y) f_m(dy). \]

Moreover, we see that for \( x > m \)
\[ \int_0^\infty F^D_{\nu_m/\mu^m_1}(x) f_m(dy) \leq h_m(x) \leq \int_0^\infty F^D_{\nu_m/\mu^m_1}(x-m) f_m(dy), \]
which, by the analyticity of \( F^D_t \) and the compact support of \( f_m(dy) \) shows that
\[ \frac{x^{-\alpha}}{\Gamma(1-\alpha)} - \frac{x^{-2\alpha}}{\Gamma(1-2\alpha)} \frac{\mu^m_2}{(\mu^m_1)^2} + O(x^{-3\alpha}) \leq h_m(x) \]
\[ \leq \frac{(x-m)^{-\alpha}}{\Gamma(1-\alpha)} - \frac{(x-m)^{-2\alpha}}{\Gamma(1-2\alpha)} \frac{\mu^m_2}{(\mu^m_1)^2} + O(x^{-3\alpha}), \]
or that
\[ h_m(x) = \frac{x^{-\alpha}}{\Gamma(1-\alpha)} + g_1(x), \]
where \( g_1 \sim x^{-2\alpha} \). Let \( U^m = \Phi_{g_1}(W_1^m) \), so that \( \mathbb{P}(U^m > x) = \bar{h}_m(x) = \frac{1}{\Gamma(1-\alpha)} + g_1(x) \). Now, since \( \mathbb{P}(W \geq x) = \frac{1}{\Gamma(1-\alpha)} + g_2(x) \) where \( g_2(x) = O(x^{-\beta}) \) then \( \bar{h}(x) = \mathbb{P}(W \geq x) + g_3(x) \) where

\[
g_3(x) = O(x^{-\gamma}),
\]

with \( \gamma = \min\{2\alpha, \beta\} \). We assume our probability space has two sequences of i.i.d r.v.s \( \{U_i\} \) and \( \{W_i\} \) where \( U_1 \sim U^m \) and \( W_1 \sim W \). Applying the coupling in Lemma 8 with \( Y = U^m \) and \( X = W \) and substituting \( g_3(x) \) in place of \( g(x) \), making the same calculation down to (4.5) (note that here \( L(t) = \frac{1}{\Gamma(1-\alpha)} \) we see that \( i \)-bad \( I_j \)'s can be found for \( j > Ci^{\frac{\gamma}{\alpha+\beta}} \). Using this in (4.7), denoting \( Z = |X - Y| \), we see that the summation on the r.h.s contributes to \( \mathbb{P}_{\text{couple}}(Z > t) \) at most \( O(t^{-\gamma}) \), the second term contributes \( O\left(t^{-\frac{\alpha(\gamma+1)}{\alpha+\beta}}\right) \) whereas the last term gives not more than \( O\left(t^{-\gamma\left(\frac{\alpha+1}{\alpha+\beta}\right)}\right) \). We conclude that

\[
\mathbb{P}_{\text{couple}}(Z > t) = O\left(t^{-\xi_0}\right),
\]

where \( \xi_0 = \frac{\alpha(\gamma+1)}{\alpha+\beta} \). Let \( c(\xi) = \frac{\xi - \alpha}{3\xi + \alpha} \) and note that \( c(\xi) \) is strictly increasing on \([0, \infty)\). Fix \( \xi \in [0, \xi_0) \) and write \( c := c(\xi) \). Let \( c' = \frac{\alpha}{\xi} \). \( M_1(n) = c'n \log n \), \( M_2(n) = c'n^{1-\alpha c'} \log n \) in (4.13), by Chernoff’s bound we have

\[
\mathbb{P}_{\text{couple}} \left( \sum_{i=1}^{M_1(n)} n^{-\frac{1}{2}} W_i \leq T \right) \leq e^{sT} (1 - n^{-1}s\alpha + o(n^{-1}s\alpha))'n \log n
\]

(4.19)

\[
\mathbb{P}_{\text{couple}} \left( \sum_{i=1}^{M_2(n)} n^{-\frac{1}{2}} W_i \leq n^{-c'} \right) \leq e^s \left( 1 - n^{-c'\alpha-1}s\alpha + o(n^{-c'\alpha-1}s\alpha) \right)'n^{1-\alpha c'} \log n.
\]

(4.20)

taking \( s = 1 \) in (4.19) and in (4.20) we see that for large enough \( n \) we have

\[
\mathbb{P}_{\text{couple}} \left( \sum_{i=1}^{M_1(n)} n^{-\frac{1}{2}} W_i \leq T \right) \leq Cn^{-c'}
\]

\[
\mathbb{P}_{\text{couple}} \left( \sum_{i=1}^{M_2(n)} n^{-\frac{1}{2}} W_i \leq n^{-c'} \right) \leq Cn^{-c'}.
\]

If \( \{Z_i\} \) are i.i.d r.v.s s.t \( Z_1 \sim Z \), by [12, Eq. 1.1] (with \( t = 1 \)) we have

\[
\mathbb{P}_{\text{couple}} \left( n^{-\frac{1}{2}} \sum_{i=1}^{M_1(n)} Z_i > n^{-c'} \right) \leq M_1(n) \mathbb{P}_{\text{couple}} \left( Z > n^{-c'} \right)
\]

\[
+ \exp \left( 1 - \frac{A(n)}{n^{\frac{1}{2} - c'}} \log \left( \frac{n^{\frac{1}{2} - c'}}{A(n)} \right) \right),
\]

whenever

\[
\frac{n^{\frac{1}{2} - c'}}{A(n)} > 1,
\]

(4.21)
where
\[ A(n) = c'n \log n \int_0^{\frac{1}{n^\frac{1}{\alpha}-\epsilon}} y \mathbb{P}_{\text{couple}}(Z \leq dy). \]

To see that \((4.21)\) indeed holds, use the fact that \(\mathbb{P}_{\text{couple}}(Z > y) = O(y^{-\xi_0})\) and therefore that \(A(n) = O\left(cn \log (n) \frac{1}{n^{\frac{1}{\alpha}-\epsilon}}(1-\epsilon)\right)\), and

\[ -c = \left(c' - \frac{1}{\alpha}\right) \xi + 1, \]
\[ > \left(c' - \frac{1}{\alpha}\right) \xi_0 + 1, \]

to conclude that for large enough \(n\)

\[ \frac{n^{\xi} \left(\frac{1}{n^\frac{1}{\alpha}-\epsilon}\right)}{A(n)} \sim \frac{1}{[cn \log n] n^{-\xi_0} \left(\frac{1}{n^\frac{1}{\alpha}-\epsilon}\right)} > 1, \]

and \(\frac{A(n)}{n^{\xi} \left(\frac{1}{n^\frac{1}{\alpha}-\epsilon}\right)} = O(n^{-c})\). Using Markov’s inequality we obtain for \(\xi < \xi_0\)

\[ \mathbb{P}_{\text{couple}}(n^{-\frac{1}{\alpha}} \sum_{i=1}^{M_1(n)} Z > n^{-c}) \leq n^{(c' - \frac{1}{\alpha})\xi} n e^c \log n \mathbb{E}(Z^\xi) \]
\[ + o(n^{-c}). \]

By \((4.22)\) we see that \((4.23)\) is bounded by \(Cn^{-c} \log n\). By Doob’s inequality we see that

\[ \mathbb{P}_{\text{couple}}(\mathcal{N} \{1, \ldots, M_2(n)\}, Y_{n,t} > n^{-c}) \leq M_2(n) n^{2c} n^{-1} \mathbb{E}(J_1^2) \]
\[ \leq n^{-\alpha c' + 2c} \log n \mathbb{E}(J_1^2) \]
\[ = n^{-c} \log n \mathbb{E}(J_1^2). \]

Hence, we have verified all the all inequalities in \((4.13)\) with the bound \(Cn^{-c}\) for \(c < c(\xi_0)\). Following the arguments in Step 2 we see that if \(X^n_t\) is the CTRW associated with the space-time jumps \(\{n^{-\frac{1}{\alpha}} J_i, n^{-\frac{1}{\alpha}} U_i\}\) then

\[ \rho_{J_1}(Y^n_t, X^n_t) < Cn^{-c}. \]

Finally, we note that since \(U_1 \in \mathcal{L}_a^{\alpha}\), by Proposition 3 we may represent \(X^n_t\) as \(\tilde{X}_{E^n_t}\), where \(\tilde{X}_{E^n_t}\) is the CTRW associated with the space-time jumps \(\{n^{-\frac{1}{\alpha}} J_i, n^{-1} \tilde{U}_i\}\), where \(\tilde{U}_i \sim U^{\alpha}\) (and so has finite mean) and \(E^n_t\) is a sequence of inverse-subordinators of the subordinators \(D^n_t = t + D_{t/\mu_1^n}\)

where \(E(U^{\alpha}) = \mu_1^n\).

\[ \square \]

**Remark 6.** Note that the choice of the skew of the tail of \(W_1\) need not be \([\Gamma(1-\alpha)]^{-1}\), i.e. one can take \(W \text{ s.t } \mathbb{P}(W_1 > t) \sim Ct^{-\alpha}\) for any \(C > 0\). We then approximate \(W_1\) by r.v’s in \(\mathcal{L}_a^{\alpha(1-\alpha)}\).
Working along the same lines of Theorem 2 it is not hard to see that if \( Y^n_t \) is as in Theorem 2 but with i.i.d spatial jumps \( \{ n^{-1}J_i \} \) where \( \mathbb{E} ( J_1 ) = 1 \) (and therefore \( Y^n_t \overset{d}{=} E_t \)) then there exists a time-changed CTRW \( \hat{X}^{n}_{E^n_t} \) s.t

\[
\rho_{J_1} \left( Y^n_t, \hat{X}^{n}_{E^n_t} \right) < C n^{-c},
\]

for any \( c < \frac{\xi_0 - \alpha}{\alpha ( \xi_0 + 1 )} \). Note that one cannot expect for a rate of convergence in (4.24) better then \( O \left( n^{-\frac{1}{\alpha}} \right) \) as the scaling is of \( n^{-\frac{1}{\alpha}} \) (unless \( U_1 \sim W_1 \)). Nevertheless, is it possible to get arbitrarily close to \( O \left( n^{-\frac{1}{\alpha}} \right) \)? Suppose we somehow manage to find a subordinator \( D'_t \) s.t

\[
P \left( D'_t W_m > t \right) - P \left( W_1 > t \right) = O \left( t^{-\gamma} \right),
\]

for \( \gamma > 2\alpha \). Then as \( \gamma \to \infty \) we see that (4.24) holds for every \( c < \frac{1}{\alpha} \). Unfortunately, we could not find a way to improve \( \gamma \) beyond \( 2\alpha \). Another point worth mentioning is that the constant controlling \( g_3 \) in (4.18) grows as we better approximate \( W \) by \( W_m \). This can be seen from the term \( \mu^2_2/(\mu_1^n)^2 \) in (4.17). Indeed, by the regular variation of the tail of \( W \) we see that \( \mu^2_2/(\mu_1^n)^2 \sim m^\alpha \) as \( m \to \infty \). An interesting question in that regard is whether there exists a better choice of \( W_m \) (possibly where \( W_m \) has infinite slowly varying mean) so that this undesirable phenomenon be avoided?

4.1. Example: Pareto Distribution. We would like to consider an example in which we use Theorem 2. Let \( Y^n_t \) be the CTRW associated with the i.i.d space-time jumps \( \left( n^{-\frac{1}{\alpha}} J_1, n^{-\frac{1}{\alpha}} W_i \right) \), where \( J_1 \) has finite second moment and zero mean and \( W_1 \) has the so-called Pareto distribution \( f(dy) \), i.e.

\[
\hat{f} ( t ) = \mathbb{P} ( W_1 \geq t ) = \begin{cases} 
\frac{t^{-\alpha}}{\Gamma ( 1 - \alpha )} & t > \Gamma ( 1 - \alpha )^{-1/\alpha} \\
1 & t \leq \Gamma ( 1 - \alpha )^{-1/\alpha},
\end{cases}
\]

where \( 0 < \alpha < 1 \). Using Theorem 2, we have \( \beta = \infty \) and therefore \( \xi_0 = \frac{\alpha}{\alpha + 4} \). It follows that

\[
\rho_{J_1} \left( Y^n_t, \hat{X}^{n}_{E^n_t} \right) < C n^{-c},
\]

with \( c < \frac{\alpha}{\xi_0 + 4} \), where \( \hat{X}^{n}_{E^n_t} \) is the process constructed in Theorem 2.

Acknowledgement. The author would like to thank Eli Barkai and Ofer Zeitouni for helpful discussions on different aspects of the problem. The author would also like to thank Mark Meerschaert and Peter Straka for reading the manuscript and offering their insightful remarks.
REFERENCES

[1] Gérard Ben Arous, Manuel Cabezas, Jiří Černý, Roman Royfman, et al. Randomly trapped random walks. The Annals of Probability, 43(5):2405–2457, 2015.

[2] Gérard Ben Arous and Jiří Černý. Scaling limit for trap models on. The Annals of Probability, pages 2356–2384, 2007.

[3] Eli Barkai, Ralf Metzler, and Joseph Klafter. From continuous time random walks to the fractional fokker-planck equation. Physical Review E, 61(1):132, 2000.

[4] Peter Becker-Kern, Mark M Meerschaert, and Hans-Peter Scheffler. Limit theorems for coupled continuous time random walks. Annals of Probability, pages 730–756, 2004.

[5] Nicholas H Bingham, Charles M Goldie, and Jef L Teugels. Regular variation, volume 27. Cambridge university press, 1989.

[6] Luiz Renato Gonçalves Fontes, Marco Isopi, and CM Newman. Random walks with strongly inhomogeneous rates and singular diffusions: convergence, localization and aging in one dimension. Annals of probability, pages 579–604, 2002.

[7] Vassili N Kolokoltsov. Markov processes, semigroups, and generators, volume 38. Walter de Gruyter, 2011.

[8] Mark Meerschaert, Erkan Nane, and P Vellaisamy. The fractional poisson process and the inverse stable subordinator. Electron. J. Probab, 16(59):1600–1620, 2011.

[9] Mark Meerschaert and Hans-Peter Scheffler. Triangular array limits for continuous time random walks. Stochastic processes and their applications, 118(9):1606–1633, 2008.

[10] Mark Meerschaert, Hans-Peter Scheffler, et al. Limit theorems for continuous-time random walks with infinite mean waiting times. Journal of applied probability, 41(3):623–638, 2004.

[11] Elliott W Montroll and George H Weiss. Random walks on lattices. ii. Journal of Mathematical Physics, 6(2):167–181, 1965.

[12] Sergey V Nagaev. Large deviations of sums of independent random variables. The Annals of Probability, pages 745–789, 1979.

[13] René L Schilling, Renming Song, and Zoran Vondracek. Bernstein functions: theory and applications, volume 37. Walter de Gruyter, 2012.

[14] Peter Straka and Bruce Ian Henry. Lagging and leading coupled continuous time random walks, renewal times and their joint limits. Stochastic Processes and their Applications, 121(2):324–336, 2011.

[15] Vladimir M Zolotarev. One-dimensional stable distributions, volume 65. American Mathematical Soc., 1986.