Convex resource theory of non-Markovianity

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Abstract

We establish a convex resource theory of non-Markovianity inducing information backflow under the constraint of small time intervals within the temporal evolution. We identify the free operations and a generalized bona-fide measure of non-Markovian information backflow. The framework satisfies the basic properties of a consistent resource theory. The proposed resource quantifier is lower bounded by the optimization free Rivas–Huelga–Plenio (RHP) measure of non-Markovianity. We next define the robustness of non-Markovianity and show that it can directly linked with the RHP measure of non-Markovianity through a lower bound. This enables a physical interpretation of the RHP measure. We further relate robustness of non-Markovianity with the quantum capacity of dephasing channels.

Keywords: open quantum system, resource theory, non-Markovianity, robustness

1. Introduction

Control and manipulation of characteristic quantum traits of any physical system are more often than not hindered by decoherence resulting from unavoidable coupling with noisy environments. Theory suggests that as a result of decoherence, the system monotonically relaxes to thermal equilibrium, or generally, to a non-equilibrium steady state [1–4]. The one-way information flow characterized by the monotonic relaxation towards the stationary states is a direct
consequence of the Born–Markov approximation [4], which is valid for very large stationary environments, leading to complete positive (CP)-divisibility of the dynamics [5–7]. However, beyond Born–Markov limit, the CP-divisibility breaks down [8], triggering non-Markovian (NM) backflow of information [9–24].

Recently, it has been shown that NM information backflow acts as a resource in various quantum mechanical tasks. For example, it has been shown that NM allows perfect teleportation with mixed states [25], efficient entanglement distribution [26], improvement of capacity for long quantum channels [27] and efficient work extraction from an Otto cycle [28]. The exploitation of information backflow as a resource has also been considered in entangling operations and quantum metrology [29–33]. For all of these mentioned cases, the accomplishment of the tasks has been done by harnessing information backflow, which can be understood as resource inter-conversion. NM can thus be inter-converted via information backflow, into other resources like entanglement, coherent information, extractable work etc. It can also be exploited for efficient quantum control [34]. Information theories can be viewed as examples of resource inter-conversion [35]. Since NM can be converted into other resources, the necessity of a resource theory (RT) of non-Markovianity (RTNM) is strongly established.

The construction of resource theories in connection with various quantum phenomena such as entanglement [36, 37], coherence [38], reference frame and asymmetry [39], nonlocality [40], non-Gaussianity [41], and thermodynamics [42, 43], has flourished in recent years. RT can be broadly classified into two categories: static and dynamic [44]. Resource theories of entanglement and coherence fall under the first category. Recently the idea of dynamical resource theories has evolved in a series of works [45–48]. Since dynamics of a quantum system is governed by completely positive trace preserving (CPTP) maps, to identify a dynamical resource one needs to characterize CPTP maps which cannot generate the corresponding resource. The well known Choi–Jamilkowski isomorphism can be a useful tool here. NM being a property of a dynamics, can also be regarded as dynamical resource. However, NM is a property of time parametrized family of CPTP maps or dynamical maps implying that here one needs to consider maps depending on time. In other dynamical resource theories, resourceless operations do not change in time. But in the case of NM dynamics the resource can be generated anywhere within the dynamics, and hence, the construction of RTNM is different from other resource theories.

Previously, there has been some attempts to construct RTNMs based on various approaches [49–51]. A formalism based on a tripartite scenario was employed in [49] where the classical notion of non-Markovianity emerges as a special case of the approach. It may be noted that the phenomenon of NM is not restricted to the domain of quantum theory [52]. In another resource theoretic approach to address NM, genuine quantum memory has been proposed as a resource in [50] where a game played between two players to certify quantum memory has been devised. Further, a RT of non-Markovianity has been formulated recently via the process tensor formalism [51], where the concept of super-processes has been defined which transforms the processes and characterizes multi-time processes in a client-server scenario. Additionally, NM monotones have also been proposed in terms of quantum mutual information and quantum coherence [53, 54], though without a formal resource theoretic framework.

In the present work we employ the Choi–Jamilkowski isomorphism to construct a convex RT of NM by identifying free and resourceful operations. Our RTNM encapsulates information backflow caused by indivisibility of quantum channels as the central feature. Note that though divisible operations do not exhaust all Markovian operations [55, 56], CP-divisible NM operations cannot generate information backflow. Hence, it is legitimate to consider only indivisible operations as resourceful operations. Our RTNM is valid for arbitrary finite dimensional single or any-partite systems, satisfying all the basic ingredients of a RT [57]. In our RT
framework, we define the robustness of NM and relate it to the RHP measure. Further, robustness defined by us, has an explicit physical implication on the quantum capacity of dephasing channels.

Our approach is thus quite different from the above-mentioned RTNMs [49–51]. Our formalism closely parallels other information theoretic resource theories such as of entanglement and non-locality in the sense of providing a way to define the concept of robustness of NM, as well as to construct a theory of NM witnesses [58, 59]. A clear difference between our approach and the resource theories of entanglement though lies on the significance of free and resourceful operations, as explained in details in the next section.

Our construction of RTNM is restricted to the operations having Lindblad type generator

\[
\dot{\rho}(t) = \mathcal{L}(\rho(t)) = \sum_{k=1}^{n \leq d_k^2} \Gamma_k(t) \left( L_k \rho(t)L_k^\dagger - \frac{1}{2} L_k^\dagger L_k \rho(t) - \frac{1}{2} \rho(t)L_k^\dagger L_k \right),
\]

where \( \Gamma_k(t) \)s are the Lindblad coefficients, \( L_k \)s are the Lindblad operators and \( d_k \) is the dimension of the system. For divisible evolutions \( \Gamma_k(t) \geq 0, \forall k, t \). The paper is organized as follows.

In the next section we formally construct our RTNM and define a measure of NM. We show how it may be related to the RHP measure [5] of NM. In section 3 we introduce the concept of robustness of NM. We further demonstrate an application of our RTNM in quantum channel capacity improvement for dephasing channels. Section 4 is reserved for a summary of our results and some concluding remarks.

2. Resource theory of non-Markovianity

There are many types of physical phenomena in quantum mechanics, defined directly at the level of quantum states by imposing constraints over the physical operations. To construct a RT concerning one of such phenomena, one usually needs to identify the states containing the signature of such resources; or conversely, one needs to find the states not containing such resources, defined as the ‘free states’. An important component of any quantum RT is the set of constrained quantum operations under which the resource cannot increase. These operations are called the ‘free operations’. In case of NM, one needs to keep in mind a crucial fact that non-Markovianity is a property of quantum processes, rather than that of states. Hence, in this respect the RT of NM is fundamentally different from the existing resource theories of entanglement, coherence or thermodynamics. Here, identification of free states is not relevant.

Rather, it is important to identify the resourceless operations, with NM being a dynamical resource. Automatically, these would identify the operations which can be considered as resource. From now onwards the term ‘free operations’ will be used to imply resourceless operations in the context of non-Markovianity. (Here it is important not to get confused with the term ‘free operations’ usually used in the RT of entanglement or thermodynamics). Further, we need a resource quantifier, and for this, it is technically convenient if the RT structure is convex. Quantum RTs such as entanglement, coherence, asymmetry and athermality possess such structure. In the following we construct these basic components for the RTNM and prove that it satisfies all the requirements of a convex RT under a particular constraint.

Free operations. A one parameter family of CPTP maps \( \{ \Lambda_t; t \geq 0 \} \) acting on state space is known as dynamical map. A dynamical map is said to be divisible if

\[
\Lambda_{t_2} = \Lambda(t_2, t_1)\Lambda_{t_1} \quad \text{where} \quad t_2 > t_1,
\]

3
with \( \Lambda(t_2, t_1) \) being a CPTP map for all \( t_1, t_2 \). Here \( \Lambda(t_2, t_1) \) is two parameter family of maps acting on the state space. \( \Lambda(t_2, t_1) \) is known as propagator of the dynamics and it characterizes the nature of the dynamics under consideration. A dynamical map is defined as CP divisible if the corresponding propagator is a CP trace preserving map. We define the CP divisible operations as the resourceless or free operations. They are expressed as

\[
\Lambda^\mathcal{M}(t_2, t_1) \equiv T \exp \left( \int_{t_1}^{t_2} \mathcal{L}_d dt \right),
\]

where \( \mathcal{L}_d(\cdot) = \sum \Gamma_k(t) \left( L_k(\cdot) L_k^\dagger - \{ L_k^\dagger L_k, \cdot \} \right) \) with \( \Gamma_k(t) \geq 0, \forall k, t \) and \( T \) stands for the time ordering product. It is important to mention here that in general CP divisible operations do not form a convex set [60, 61]. To have a convex RT we put a further constraint on the definition of resourceless operations, given by the small time interval approximation.

In order to deal with the operations, one has to work with positive maps acting on the space of bounded operators on some Hilbert space. Though it is often hard to work with such maps, the famous Choi–Jamilkowski isomorphism [62, 63] plays the crucial role here. The Choi–Jamilkowski isomorphism translates everything to the language of operators from the language of maps. We shall use the Choi operator corresponding to a dynamical map for detailed study. Considering the dynamical evolution between time intervals \( t_1 \) and \( t_2 \) where \( t_2 > t_1 \), and denoting \( t_2 = t + \epsilon \) and \( t_1 = t \) for some positive \( \epsilon \), we can define

\[
\mathcal{C}^\mathcal{M}(t + \epsilon, t) = I \otimes \Lambda^\mathcal{M}(t + \epsilon, t)(|\psi\rangle \langle \psi|) \quad \text{with } \epsilon > 0,
\]

where \( |\psi\rangle \) is the maximally entangled state of \( d \times d \) dimension, for a \( d \) dimensional system. It is to be observed that for a divisible dynamics, the Choi operator corresponding to the operation is a valid density matrix and hence a quantum state.

Note here that the RHP measure of NM [8] is given by

\[
g^\mathcal{N}(t) = \lim_{\epsilon \to 0^+} \frac{\| \mathcal{C}^\mathcal{N}(t + \epsilon, t) - I \|_1 - 1}{\epsilon},
\]

where \( \| \cdot \|_1 \) is the trace norm and \( \mathcal{C}^\mathcal{N}(t + \epsilon, t) = I \otimes \Lambda^\mathcal{N}(t + \epsilon, t)(|\psi\rangle \langle \psi|) \) is the Choi operator corresponding to any operation \( \Lambda^\mathcal{N} \). For a divisible evolution \( \Lambda^\mathcal{M} \), \( g^\mathcal{M}(t) \) is always zero since \( \mathcal{C}^\mathcal{M}(t + \epsilon, t) \) is a valid state and \( \| \mathcal{C}^\mathcal{M}(t + \epsilon, t) \|_1 = 1 \). We restrict ourselves within finite dimensional systems. We have considered implicitly the trace of Choi operator corresponding to a free operation to be unity. This is because of the fact that the operation is always trace preserving. Therefore, for divisible operations, the output is always a valid state. So both the trace and trace norm of such outputs will always be unity. On the other hand, indivisible operations will give rise to negative eigenvalues of the output. For those cases the trace and hermiticity preserving features of the map ensure that the trace norm of the output will always exceed unity.

As mentioned in the introduction, we only consider those operations having Lindblad type generators as \( \Lambda(t + \epsilon, t) \equiv T \exp \left( \int_t^{t+\epsilon} \mathcal{L} dt \right) \). Moreover, if we consider

\[
\epsilon \Gamma_k(t) \ll 1 (\forall k),
\]

which we call as the small time interval approximation, we can take the operation as \( \Lambda(t + \epsilon, t) \equiv I + \epsilon \mathcal{L} \) by considering only up to 1st order terms in the Taylor series expansion. Thus we collect Choi operators corresponding to all such families of dynamical maps.
and define the set of free Choi operators as

$$
\mathcal{F} = \left\{ \mathcal{C}_\alpha^M(t + \epsilon, t) : \alpha \in J \quad \text{and} \quad \| \mathcal{C}_\alpha^M(t + \epsilon, t) \|_1 = 1 \quad \forall \ t, \epsilon, \alpha \right\}, \quad (7)
$$

with the condition $\epsilon \Gamma_k(t) \ll 1(\forall k)$. Here $J$ is any index set. Note that in general the Lindblad operators do not commute at different times, requiring a time ordering operation while defining the set $\mathcal{F}$. However, in our present small time interval approximation, no time ordering is required since no higher order terms are involved.

Based on the previous definitions, we now prove the following propositions stating particularly essential properties. Before proceeding to the proofs one should keep in mind the following facts again. The properties discussed here are the features of resourceless dynamical maps, viz, divisible maps. Since we have the Choi–Jamilkowski isomorphism [62, 63] in hand, we shall prove the propositions at the level of corresponding Choi operators. Note that we will be defining a resource which is the sole property of the map, not the state. Therefore, in the physical picture, there is no concept of free or resourceful states here.

**Proposition 1.** In the limit of $\epsilon \Gamma_k(t) \ll 1(\forall k)$, the set of free Choi operators form a convex set.

**Proof.** To prove the convexity of a set, it is enough to show that the convex combination of two elements of the set also belongs to the same set. Let us consider two Markovian operations $\Lambda_{M(1)}^M$ and $\Lambda_{M(2)}^M$. Both of them belong to the two parameter family of dynamical maps. Without loss of generality, let us assume that both the dynamics have occurred within the time $t$ to $t + \epsilon$ for some positive $\epsilon$. According to our assumption both the dynamics have Lindblad type generators, i.e.,

$$
\Lambda_{M(1)}^M(t + \epsilon, t) = \mathcal{T} \exp \left( \int_{t}^{t + \epsilon} \mathcal{L}_{1}^{(1)}(t') dt' \right),
$$

$$
\Lambda_{M(2)}^M(t + \epsilon, t) = \mathcal{T} \exp \left( \int_{t}^{t + \epsilon} \mathcal{L}_{1}^{(2)}(t') dt' \right).
$$

Here $\mathcal{L}_{1}^{(1)}$ and $\mathcal{L}_{1}^{(2)}$ are Lindblad type generators with positive coefficients. In the limit of $\epsilon \Gamma_k(t) \ll 1(\forall k)$, we can expand the exponentials and neglect the 2nd and higher order terms (time ordering not required). Therefore, we have

$$
\Lambda_{M(1)}^M(t + \epsilon, t) = \mathcal{I} + \epsilon \mathcal{L}_{1}^{(1)}, \quad \Lambda_{M(2)}^M(t + \epsilon, t) = \mathcal{I} + \epsilon \mathcal{L}_{1}^{(2)}. \quad (8)
$$

We define another map

$$
\Lambda^M(t + \epsilon, t) = p\Lambda_{M(1)}^M(t + \epsilon, t) + (1 - p)\Lambda_{M(2)}^M(t + \epsilon, t),
$$

$$
= \mathcal{I} + \epsilon [p\mathcal{L}_{1}^{(1)} + (1 - p)\mathcal{L}_{1}^{(2)}] = \mathcal{I} + \epsilon \mathcal{L}, \quad (9)
$$

with $\mathcal{L} = p\mathcal{L}_{1}^{(1)} + (1 - p)\mathcal{L}_{1}^{(2)}$ and $0 \leq p \leq 1$. Clearly, $\mathcal{L}$ is also a Lindblad type generator with positive coefficients, which proves $\Lambda^M(t + \epsilon, t)$ also belongs to the set of divisible Markovian maps. Then, if the Choi operators corresponding to the operations $\Lambda_{M(1,2)}^M$ are denoted by $\mathcal{C}_{\alpha(1,2)}^M$, then their convex combination $\sigma = p\mathcal{C}_{\alpha(1)}^M + (1 - p)\mathcal{C}_{\alpha(2)}^M$, can be represented as

$$
\sigma = \mathcal{I} \otimes \Lambda^M(t + \epsilon, t)|\psi\rangle\langle\psi|.
$$

Since $\Lambda^M(t + \epsilon, t)$ is a divisible map, we have $\sigma \in \mathcal{F}$. □
In a generic RT one always expects some natural features of free states. Now let us prove the following propositions where we show that free Choi operators in the context of RT of non-Markovianity also possess these properties.

**Proposition 2A.** Resourceful Choi operators cannot be generated under tensor product, partial trace and permutations of spatially separated subsystems.

**Proof.** Let us consider two arbitrary Choi operators $\rho^M, \sigma^M \in \mathbb{F}$ (here we drop the index $\alpha$ of the Choi operators for brevity). Therefore $\|\rho^M\|_1 = \|\sigma^M\|_1 = 1$. From the properties of trace norm [64] we have, $\|\rho^M \otimes \sigma^M\|_1 = \|\rho^M\|_1 \|\sigma^M\|_1 = 1$. We know that for a NM resourceful Choi operator, the trace norm must be strictly greater than 1 in some intermediate region. Therefore $\rho^M \otimes \sigma^M$ is not a resourceful Choi operator. This mathematical property has a nice physical interpretation. It states that even if there are two parallel processes running simultaneously and both of them are resourceless in nature, then the resulting process is again resourceless.

Similarly we can prove $\sigma^M \otimes \rho^M$ is also not resourceful.

To show partial trace cannot generate the resource, let $\rho^M_{\alpha} \in \mathbb{F}$ be an arbitrary bipartite Choi operator. If we take partial trace with respect to subsystem B, the reduced subsystem becomes $\rho^M_A = \text{Tr}_B[\rho^M_{\alpha}]$. Now, $\|\rho^M_A\|_1 = \text{Tr}[\sqrt{\rho^M_A \rho^M_A}] = \text{Tr}[\rho^M_A]$, as $\rho^M_A$ is a positive operator. Hence, we have

$$\rho^M_A = \sum_{j=1}^{d_B} (\mathbb{I}_A \otimes \langle b_j |) \rho^M_{\alpha} (\mathbb{I}_A \otimes | b_j \rangle),$$

(10)

where $d_B$ is the dimension of subsystem B having orthonormal basis $\{|b_i\}\}$. Therefore,

$$\text{Tr}[\rho^M_A] = \text{Tr} \left[ \rho^M_{\alpha} \left( \mathbb{I}_A \otimes \sum_{j=1}^{d_B} | b_j \rangle \langle b_j | \right) \right]$$

$$= \text{Tr}[\rho^M_{\alpha} (\mathbb{I}_A \otimes \mathbb{I}_B)]$$

$$= \text{Tr}[\rho^M_{\alpha}] = 1.$$  

So, we prove that $\rho^M_A$ is not a resourceful Choi operator. □

Here it may be re-emphasized that the picture which we construct is in terms of the Choi operators corresponding to the dynamical maps. Here in the previous proposition, when we prove that the tensor product of two free Choi operators is also free, we basically consider two different divisible processes acting separately on two separate states. Therefore, these two processes have absolutely no connection to each other, and hence, the total process is also divisible, if each of them are.

**Proposition 2B.** The set of free Choi operators $\mathbb{F}$ is a compact set.

**Proof.** A subset in an Euclidean space is compact if and only if it is bounded and closed (contains all its limit points). Boundedness of $\mathbb{F}$ in trace norm is clear from its definition and closedness of $\mathbb{F}$ follows from the continuity of trace norm. Hence $\mathbb{F}$ is compact. □

**Proposition 2C.** Free operations cannot generate a resourceful Choi operator.
We have considered divisible operations as free operations. Application of a free operation on another free operation basically means concatenation of two divisible operations which gives rise to another single divisible operation. Therefore, the Choi operator corresponding to the resulting operation cannot be resourceful.

These propositions have the following implications. One can define free Choi operators in an arbitrary finite dimension. The tensor product of two free Choi operators and the reduced state of a free Choi operator cannot be resourceful [57], and free operations cannot generate the resource.

Measure of non-Markovianity. A proper measure of non-Markovianity can be constructed by the minimum contractive distance between a NM and the set of all Markovian operations. The Choi–Jamilkowski isomorphism [62, 63] reduces this problem to finding the minimum distance between the corresponding Choi operators. A measure of NM can be defined as

\[ D_{T}(t) = \lim_{\epsilon \to 0} \frac{M_{T}(t + \epsilon, t) - 0}{\epsilon}, \]

where \( M_{T}(t + \epsilon, t) = \inf_{C_{N}(t + \epsilon, t)} D(C_{N}(t + \epsilon, t) | C_{M}(t + \epsilon, t)), \) where \( D(\cdot | \cdot) \) is any metric contractive under CPTP maps. \( M_{T}(C_{N}(t + \epsilon, t)) \) can only be a non-zero positive quantity in the time span \( \epsilon \), where CP breaks down. The optimization is done over the free Choi operators \( C_{M}(t + \epsilon, t) \). This is extremely difficult to calculate because the free operations \( \Lambda^{M} \) do not form a convex set [60, 61]. We overcome this difficulty by virtue of the following arguments and proposition.

It can be shown that in the limit of \( \epsilon \to 0 \), the set of all Choi operators \( A \) is also a convex set. Using proposition 1, we define the following measure of NM, taking the right derivative of \( M_{T}(t + \epsilon, t) \) in the CP breaking region as

\[ D_{T}(t) = \lim_{\epsilon \to 0} \frac{M_{T}(t + \epsilon, t) - 0}{\epsilon}, \]

where \( M_{T}(t + \epsilon, t) = \inf_{C_{N}(t + \epsilon, t)} \| |C_{N}(t + \epsilon, t) - C_{M}(t + \epsilon, t)|\|_{1}, \) with \( \| \cdot \|_{1} \) is the trace norm. The minimum distance \( M \) at the previous time \( t \) is taken to be zero, since divisibility is not broken before \( t \), and the Choi operator at that time belongs to the set of free Choi operators. For any quantum evolution, we always have \( D_{T}(t) \geq 0 \). The equality holds for divisible Markovian evolutions. The optimization involved in evaluating \( D_{T}(t) \) is now easier because the set of free Choi operators now forms a convex set. Moreover, by virtue of propositions 2B and 1, we know that the set of free Choi operators \( F \) in the limit \( \epsilon \to 0 \) is convex and compact. We can thus apply the Krein–Milman theorem [65] to state that \( F \) is the convex hull of its extreme points. Hereafter, for brevity we use the short hand notation \( C_{N}(t + \epsilon, t) = C_{N}(t) \) and \( C_{M}(t + \epsilon, t) = C_{M}(t) \) (which refer, as usual to two parameter families of Choi operators).

In quantum theory, a resource measure [44] is said to be a bona fide measure if it is faithful, convex and a monotone under the free operation corresponding to that RT. In the present context, faithfulness of a measure means that the resource quantifier will give non-zero value if a resourceful operation is considered. Since we are discussing a convex RT, convexity of the resource measure follows naturally. Finally, monotonicity under free operations is the next important property of a good resource quantifier. For the RT of non-Markovianity, monotonicity stems from the fact that concatenating one or more free operations to the original operation cannot increase the resource, and hence, the resource quantifier should be non increasing under free operations.

**Proposition 3.** \( D_{T}(t) \) is a bona fide measure of NM.

**Proof.** To prove \( D_{T}(t) \) is a bona fide measure, we have to prove it is faithful, convex and a monotone under the free operations.

The measure is faithful if \( D_{T}(t) \geq 0 \) and \( D_{T}(t) = 0 \) if \( C_{M}(t) \in F \). Clearly \( D_{T}(C_{O}(t)) \geq 0 \) \( \forall C_{O}(t) \) from the definition. We now show that \( D_{T}(C_{M}(t)) = 0 \) if \( C_{M}(t) \in F \). Here the part is
obvious from the definition. To prove the only if part, let $\mathcal{D}_T(C^N(t)) = 0$ for some $C^N(t)$. Then $\mathcal{D}_T(C^N(t)) = 0 \Rightarrow \lim_{\epsilon \to 0} \frac{\mathcal{D}_T(C^N(t) + \epsilon)}{\epsilon} = 0 \Rightarrow M_T(C^N(t)) = 0$, since $\epsilon$ is a finite positive number. Therefore $M_T(C^N(t)) = \inf_{C^M \in \mathcal{M}} \mathcal{D}_T(C^N(t)|C^M(t)) = 0$. This implies $C^N(t) \in C(F) = F$ since $F$ is a closed set. Therefore $\mathcal{D}_T(t)$ is faithful.

To prove the convexity of $\mathcal{D}_T(t)$, we consider $C^N_1(t)$ and $C^N_2(t)$ be two Choi operators. By the convexity property of the set of all Choi operators $\mathcal{A}$, we know $C^N_{\lambda}(t) = \rho C^N(t) + (1 - \rho)C^N_2(t$) is also a Choi operator. Therefore, by virtue of the triangle inequality, we have $||C^N_{1}(t) - C^N(t) - C^M(t)||_1 \leq ||C^N_1(t) - C^N(t)||_1 + (1 - \rho)||C^N_2(t) - C^M(t)||_1$, for all $C^M(t)$. Consequently, we get $\inf_{C^M(t)} ||C^N_1(t) - C^N(t)||_1 \leq \inf_{C^M(t)} ||C^N_1(t) - C^N(t)||_1 + (1 - \rho)\inf_{C^M(t)} ||C^M_2(t) - C^M(t)||_1$. This in turn proves the convexity relation $\mathcal{D}_T(pC^N_1(t) + (1 - \rho)C^M(t)) \leq p\mathcal{D}_T(C^N(t)) + (1 - \rho)\mathcal{D}_T(C^M(t))$.

To prove the monotonicity of $\mathcal{D}_T(t)$, we consider a divisible free operation: $\rho(t_2) = I \otimes \Lambda(t_2, t_1)(\rho(t_1)) = \text{Tr}_E [V(t_2, t_1) \otimes \sigma_E V^\dagger(t_2, t_1)]$, where $V(t)$ is a global unitary acting on the composite system-environment Hilbert space and $\sigma_E$ is the initial state of the environment. Therefore, we have $||C^N(t + \Delta) - C^M(t + \Delta)||_1 = ||\text{Tr}_E [V(t + \Delta, t)(C^N(t) - C^M(t)) \otimes \sigma_E V(t + \Delta, t)]||_1$. Using the trace norm inequality $||\text{Tr}_E[A_{ab}]||_1 \leq ||[A_{ab}]||_1$, for any bounded operator $A_{ab}$, and preservation of trace norm under unitary rotation, we thus have $||C^N(t) - C^M(t)||_1 = ||[C^N(t) - C^M(t)] \otimes \sigma_E V(t + \Delta, t)||_1 \leq ||\text{Tr}_E [V(t + \Delta, t)(C^N(t) - C^M(t)) \otimes \sigma_E V(t + \Delta, t)]||_1$. Therefore we have $||C^N(t + \Delta) - C^M(t + \Delta)||_1 \leq ||C^N(t) - C^M(t)||_1$.

Now $M_T(t, t) = \inf_{C^M(t)} ||C^N(t) - C^M(t)||_1 = ||C^N(t) - C^M(t)||_1$, with $C^M(t)$ being the free Choi operator from which the distance is minimum. Using the fact $I \otimes \Lambda(t_1 + \Delta, t)(C^M(t)) \in F$, we have $\inf_{\epsilon \geq \Delta \otimes \Lambda(t_1 + \Delta, t)(C^N(t)) - I \otimes \Lambda(t_1 + \Delta, t)(C^M(t))} ||I - \Delta \otimes \Lambda(t_1 + \Delta, t)(C^N(t)) - I \otimes \Lambda(t_1 + \Delta, t)(C^M(t))||_1 \leq ||\Delta \otimes \Lambda(t_1 + \Delta, t)(C^N(t)) - I \otimes \Lambda(t_1 + \Delta, t)(C^M(t))||_1 \leq ||C^N(t) - C^M(t)||_1$. It is evident from equation (11) that $\mathcal{D}_T(t) \leq \mathcal{D}_T(t)$, proving the monotonicity of $\mathcal{D}_T(t)$ under divisible operations. This completes the proof of the proposition.

We further reduce the complexity of calculating the measure of NM, by constructing a lower bound of $\mathcal{D}_T(t)$ in the following theorem.

**Theorem 1.** Let $\Lambda^N$ be a map corresponding to some operation $\mathcal{N}$ and $g^N(t)$ be the RHP measure, then $\mathcal{D}_T(t)$ is bounded below by $g^N(t)$, i.e. $\mathcal{D}_T(t) \geq g^N(t)$.

**Proof.** We have the expression of our NM measure

$$\mathcal{D}_T(t) = \lim_{\epsilon \to 0^+} \inf_{C^M} ||C^N(t) - C^M(t)||_1 / \epsilon.$$  

Using the reverse triangle inequality: $||A - B||_1 \geq ||A||_1 - ||B||_1$ with $||C^N(t)||_1 \geq 1$ and $||C^M(t)||_1 = 1 \forall C^M$, we have

$$\mathcal{D}_T(t) \geq \lim_{\epsilon \to 0^+} \frac{||C^N(t)||_1 - 1}{\epsilon} = g^N(t) \geq .$$

Interestingly, $\mathcal{D}_T(t)$ is lower bounded by $g^N(t)$, which is optimization free and easier to calculate. Note that $g^N(t)$ is the time derivative of the trace norm of the Choi operator [8]. When the divisibility breaks down, the norm of the corresponding Choi operator is strictly
greater than 1. In those regions we have \( g^N(t) > 0 \), showing that the RHP measure is a witness of CP-indivisibility, whereas \( D_H(t) \) is the time derivative of the minimum distance between the Choi operator corresponding to a specific evolution and all possible free Choi operators.

**Corollary.** The RHP measure \( g^N(t) \) is also a bona fide measure of NM.

**Proof.** The faithfulness of \( g^N(t) \) follows from its original definition \([8]\). The convexity of \( g^N(t) \) follows from the convexity of \( \|C_N(t)\|_1 \), as proved in proposition 3. The monotonicity of \( g^N(t) \) follows from the trace inequality \( \|\text{Tr}_b[A_{ab}]\|_1 \leq \|A_{ab}\|_1 \), for any bounded operator \( A_{ab} \), as done in proposition 3. \( \square \)

### 3. Robustness of non-Markovianity

We now propose the concept of robustness of NM (RONM), on similar lines of the robustness of the resource theories of entanglement [66–68], coherence [69] and asymmetry [70]. In accordance with the definitions of robustness for other quantum resources, we define RONM as the minimum amount of noise (Markovian or NM) needed to be added to a NM evolution to make the resulting evolution Markovian. Hence, the formal definition of RONM follows as

\[
\mathcal{R}_{\mathcal{A}}(C^N(t)) = \inf_{s} \left\{ s \geq 0 : \frac{C^N(t) + s\tau^N(t)}{1 + s} = \delta^M(t) \in \mathbb{F} \right\}, \quad (12)
\]

where \( \tau^N(t) \) is an arbitrary element from the set of all Choi operators \( \mathcal{A} \). After achieving the minimization for Choi operators \( \tau^N(t) \) and \( \delta^{M(t)} \), we write

\[
C^N(t) = [1 + \mathcal{R}_\mathcal{A}(C^N(t))]|\delta^M(t) - \mathcal{R}_\mathcal{A}(C^N(t))\tau^N(t)|. \quad (13)
\]

It may be noted here that in a recent paper \([71]\) another definition of robustness of NM has also been proposed. However, the authors have considered only the divisible operations over which the optimization is based on. It is important to mention here the fact that convex mixing of a NM noise with another NM dynamics can make it Markovian \([72]\). Recently the same has been demonstrated experimentally \([73]\). Therefore the definition provided in \([71]\) may of course be suitable for certain situations but it lacks generality, since it excludes a considerable portion of noise in the definition.

We now present the proof of the following proposition where we establish that \( \mathcal{R}_{\mathcal{A}}(C^N(t)) \) is also a very useful measure of NM.

**Proposition 4.** \( \mathcal{R}_{\mathcal{A}}(C^N(t)) \) is a faithful, convex measure of NM and lower bounded by \( \frac{1}{2} \int_0^1 \delta^N(t')dt' \).

**Proof.** The faithfulness of RONM follows from the definition as

\[
\mathcal{R}_{\mathcal{A}}(C^Q(t)) \geq 0 \quad \text{and} \quad \mathcal{R}_{\mathcal{A}}(C^Q(t)) = 0 \Leftrightarrow C^Q(t) \in \mathbb{F}.
\]

To prove the convexity of \( \mathcal{R}_{\mathcal{A}}(C^N(t)) \), let us consider two arbitrary Choi operators \( C^N_1(t) \) and \( C^N_2(t) \), expressed as the pseudo-mixture \( C^N_i(t) = [1 + \mathcal{R}_{\mathcal{A}}(C^N_i(t))]|\delta^M_i(t) - \mathcal{R}_{\mathcal{A}}(C^N_i(t))\tau^N_i(t)| \) (for \( i = 1, 2 \)). The convex structure of \( \mathcal{A} \) ensures that the convex combination of these two Choi operators will also be another Choi operator. Now considering the convex decomposition \( C^N(t) = pC^N_1(t) + (1 - p)C^N_2(t) \) (with \( 0 \leq p \leq 1 \)) and utilizing the pseudo-mixtures written above, the following pseudo-mixture \( C^N(t) = [1 + s]\delta^M(t) - s(C^N(t))\tau^N(t) \) can be written with \( \delta^M(t) = [p(1 + \mathcal{R}_{\mathcal{A}}(C^N_1(t)))|\delta^M_1(t) - \mathcal{R}_{\mathcal{A}}(C^N_1(t))\tau^N_1(t)|/(1 + s) + (1 - p)(1 + \mathcal{R}_{\mathcal{A}}(C^N_2(t)))|\delta^M_2(t) - \mathcal{R}_{\mathcal{A}}(C^N_2(t))\tau^N_2(t)|/(1 + s) \)
and \( \tau^N(t) = \{ p \mathcal{R}_N(\mathcal{C}^N_1(t)) \tau^N_1(t) + (1 - p) \mathcal{R}_N(\mathcal{C}^N_2(t)) \tau^N_2(t) \} / s \). The trace preservation property of any arbitrary quantum operation guarantees the hermiticity of Choi operator with unit trace. Therefore, from the normalization condition, we get \( s = p \mathcal{R}_N(\mathcal{C}^N_1(t)) + (1 - p) \mathcal{R}_N(\mathcal{C}^N_2(t)) \). Now, from the definition of RONM, we have \( \mathcal{R}_N(\mathcal{C}^N(t)) \leq s \). Thus, the convexity of \( \mathcal{R}_N(\mathcal{C}^N(t)) \) is proved.

To find the lower bound of \( \mathcal{R}_N(\mathcal{C}^N(t)) \), we use a result from entanglement theory [74]. As we have seen from equation (13), the Choi operator \( \mathcal{C}^N(t) \) can be written in the pseudo mixture \( \mathcal{C}^N(t) = c_+ \delta^{N+} - c_- \tau^{N-} \), with \( c_+ = 1 + \frac{1}{2} \mathcal{R}_N(\mathcal{C}^N(t)) \) and \( c_- = \mathcal{R}_N(\mathcal{C}^N(t)) \). Now from [74] we know, that for the Harmitian matrix \( \mathcal{C}^N(t) \), there exist a minimal decomposition \( \mathcal{C}^N(t) = a_+ \rho^+ - a_- \rho^- \), such that \( ||\mathcal{C}^N(t)||_1 = a_+ + a_- \) is minimum and \( \rho^+, \rho^- \) has disjoint support.

Therefore, if \( \mathcal{P} \) be the projector onto the negative eigenvalue subspace of \( \mathcal{C}^N(t) \), then \( a_- = -\text{Tr}[\mathcal{P} \mathcal{C}^N(t)] \) is the sum of absolute values of negative eigenvalues [74]. Let us consider the spectral decomposition \( \mathcal{C}^N(t) = \sum l |\lambda^N_+ l\rangle \langle \lambda^N_+ l| + \sum k |\lambda^N_- k\rangle \langle \lambda^N_- k| \) with \( \lambda^N_+ l \)'s and \( \lambda^N_- k \)'s being the positive and negative eigenvalues respectively. Therefore, the trace preservation condition yields \( \sum l \lambda^N_+ l = 1 + \sum l |\lambda^N_+ l| \). Thus, we have \( ||\mathcal{C}^N(t)||_1 = \sum l \lambda^N_+ l + \sum k |\lambda^N_- k| = 1 + 2 \sum k |\lambda^N_- k| \). Since from [74] we know \( \mathcal{R}_N(\mathcal{C}^N(t)) = c_- \geq a_- = \sum l |\lambda^N_+ l| \), we get the following inequality

\[
\mathcal{R}_N(\mathcal{C}^N(t)) \geq \frac{||\mathcal{C}^N(t)||_1 - 1}{2}.
\]

Now because of the fact that \( g^N \) is the right derivative of the trace norm of Choi operator, we have

\[
\mathcal{R}_N(\mathcal{C}^N(t)) \geq \frac{1}{2} \int_0^t g^N(t') dt' = \frac{1}{2} \mathcal{N}_R(t).
\]

This proves the proposition. \( \square \)

Here we see that the RONM is lower bounded by the integral of RHP measure \( g^N(t) \). This provides a physical interpretation of the RHP measure. RONM physically means the endurance of a NM operation under mixing with arbitrary noise. The normalized measure of NM [8] can therefore be found to satisfy the following inequality

\[
\mathcal{T}(t) = \frac{\mathcal{N}_R(t)}{1 + \mathcal{N}_R(t)} \leq \frac{2 \mathcal{R}_N(\mathcal{C}^N(t))}{1 + 2 \mathcal{R}_N(\mathcal{C}^N(t))}.
\]

Finally, as an application of RTNM, we consider the NM induced improvement of quantum capacity for dephasing channels. It has been shown earlier [27, 75], that the quantum capacity of a dephasing channel can be represented by \( \mathcal{Q}_D(t) = 1 - H_2(\frac{1 + e^{2 \mathcal{R}_N^R(t) - G_M(t)}}{2}) \), where \( H_2(\cdot) \) is the binary Shannon entropy and \( G(t) = \frac{1}{2} \int_{\Gamma(t)} \Gamma(t') dt' \), for a qubit dephasing channel \( \dot{\rho}(t) = e^{-i \Gamma(t) \sigma_z} \rho(t) e^{i \Gamma(t) \sigma_z} - \rho(t) \). For a NM dephasing channel, the quantum capacity satisfies the following inequality

\[
\mathcal{Q}_D(t) \leq 1 - H_2 \left( \frac{1 + e^{2 \mathcal{R}_N^R(t) - G_M(t)}}{2} \right).
\]

where \( G_M(t) = \frac{1}{2} \int_{\Gamma(t) \geq 0} \Gamma(t') dt' \), and \( \mathcal{R}_N^R(t) \) is the robustness of the NM dephasing channel. Since \( G_M(t) - N^P_M(t) = G(t) \geq 0 \) with \( N^P_M(t) = \frac{1}{2} \int_{\Gamma(t) \geq 0} \Gamma(t') dt' \), the total exponential term on the right-hand side \( e^{-G_M(t)} \leq 1 \). However, if the evolution is sufficiently indivisible, so that \( G(t) \rightarrow 0 \), even for a sufficiently long time evolution, the capacity of the channel improves tremendously. This is indeed possible in practical situations, where the dephasing coefficient takes the form \( \Gamma(t) \sim \tan(t) \) [8]. For these evolutions, we have \( G(t) = 1 - \ln(\cos(t)) \), which can vanish even
for sufficiently large instances of time, and hence, $Q_D(t) = 1$ for such situations. Therefore, near perfect communication can be achieved through lossy quantum channels, using NM as a resource.

4. Conclusions

In this work, we have constructed a convex RTNM under the constraint of small time intervals, which satisfies all the properties of RT. We have defined the divisible operations as the free operations and constructed a bona fide measure of NM. In a recent work [56], a measure of NM in terms of minimum quasi-distances has been proposed, where the minimization is done over all Markov processes. Due to the non-convexity of Markov processes, this optimization is extremely difficult to compute in practice. On the other hand, calculating our proposed measure $D_T(t)$ is much easier, since the free Choi operators ($C_M(t)$) form a convex set in a sufficiently small time interval. Moreover, $D_T(t)$ is lower bounded by the optimization free RHP measure of NM.

We have further constructed robustness of NM, which is the degree of endurance of any NM operation under the mixing with noise. We have shown that it is faithful and convex, and has also possesses an easily computable optimization free lower bound (14). Moreover, we have directly connected RONM with the RHP measure through this lower bound, presenting an operational interpretation of the RHP measure which itself is shown to be a bona fide measure in our RT framework. Our formalism encompasses three different NM measures, viz $D_T(t)$, RHP measure and RONM, within a single framework.

We have presented a clear physical application of indivisible NM operations in enhancing the quantum capacity of dephasing channels. In cases where the dynamical map is dephasing in nature, it may thus be possible to infer the robustness of non-Markovianity. This may play an important role in actual information processing using non-Markovianity as resource, since one would be able to relate channel capacities with a quantitative measure of non-Markovianity. Further possibilities of applications of our RT of non-Markovianity in noisy quantum communications are thereby motivated. It may also be interesting to study the scenarios where the non-Markovianity is detrimental, for example in dynamical decoupling [76] or in the emergence of quantum darwinism [77].

Before concluding, it may be noted that though our study is closely related to the RHP non-Markovianity, it is known that if the dynamics is Markovian in the RHP sense, it is also Markovian in the Breuer–Laine–Piilo (BLP) [6] sense, and equivalence of the RHP and BLP measures has been proposed [78]. BLP NM operations are contained in the set of RHP NM operations. Nonetheless, there can be such operations which are not divisible, but do not show information backflow, for example, some eternal NM operations [79]. Our present analysis may be extended based on the BLP perspective as well, opening an interesting avenue for further exploration.

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