Corrigendum: Distillability and PPT entanglement of low-rank quantum states

2011 J. Phys. A: Math. Theor. 44 285303

Lin Chen\(^1,3\) and Dragomir \(\check{Z}\)-Đoković\(^2\)

\(^1\)Centre for Quantum Technologies, National University of Singapore, 3 Science Drive 2, Singapore 117542
\(^2\)Department of Pure Mathematics and Institute for Quantum Computing, University of Waterloo, Waterloo, Ontario, N2L 3G1, Canada

E-mail: cqtcl@nus.edu.sg and djokovic@uwaterloo.ca

Received 7 December 2011
Published 19 January 2012
Online at stacks.iop.org/JPhysA/45/059501

There is an error in the proof of theorem 8 of [1]. It is easy to fix it and the theorem remains valid. The error is at the end of the proof. Namely, the blocks \(C_i\) are \(R \times N\) matrices with \(R \geq N\). In the case \(R > N\), the formula for the determinant of \(\langle x|\rho|x\rangle\) given at the bottom of page 9 does not make sense.

To correct the error, we need to prove the claim about the coefficient of \(t_1^{2r_1}t_2^{2r_2}\ldots t_k^{2r_k}\) in the expansion of \(\det(\langle x|\rho|x\rangle)\). For convenience, let \(L\) be the \(N \times N\) matrix

\[ L := \langle x|\rho|x\rangle = \sum_{i,j=1}^k t_i t_j C_i^\dagger C_j, \]

whose entries are quadratic forms in the real variables \(t_1, \ldots, t_k\). We also set \(P := \det L\), and assume that it is fully expanded. We proceed in \(k\) steps.

We begin the first step by forming the partition \(L = \begin{pmatrix} L_1 & L_2 \\ L_3 & L_4 \end{pmatrix}\), where the block \(L_1\) is square of order \(r_1\). Since \(C_1 = I_{r_1} \oplus 0\) and all blocks \(C_2 = 0\), the variable \(t_1\) does not occur in the blocks \(L_2\) and \(L_4\). Hence, if we collect all terms in \(P\) which are divisible by \(t_1^{2r_1}\), we obtain the polynomial \(t_1^{2r_1}P_1\), where \(P_1 := \det L_4\).

In the second step, we repeat this argument with the matrix \(L_4 = \sum_{i,j=2}^{k} t_j t_j C_{i4}^\dagger C_{j4}\) instead of \(L\). We partition it as \(L_4 = \begin{pmatrix} L_{41} & L_{42} \\ L_{43} & L_{44} \end{pmatrix}\), where the block \(L_{41}\) is square of order \(r_2\). Since \(C_{41} = I_{r_2} \oplus 0\) and all blocks \(C_{42} = 0\), the variable \(t_2\) does not occur in the blocks \(L_{42}\) and \(L_{44}\). Hence, if we collect all terms in \(P_1\) which are divisible by \(t_2^{2r_2}\), we obtain the polynomial \(t_2^{2r_2}P_2\), where \(P_2 := \det L_{44}\).

The remaining \(k-2\) steps are similar. At the end, we obtain that the coefficient of \(t_1^{2r_1}t_2^{2r_2}\ldots t_k^{2r_k}\) in \(P\) is indeed 1.

3 Author to whom any correspondence should be addressed.
Reference

[1] Chen L and Doković D Ž 2011 Distillability and PPT entanglement of low rank quantum states J. Phys. A: Math. Theor. 44 285303
Distillability and PPT entanglement of low-rank quantum states

Lin Chen$^{1,3}$ and Dragomir Ž. Đoković$^2$

$^1$ Centre for Quantum Technologies, National University of Singapore, 3 Science Drive 2, Singapore 117542
$^2$ Department of Pure Mathematics and Institute for Quantum Computing, University of Waterloo, Waterloo, Ontario, N2L 3G1, Canada

E-mail: cqtcl@nus.edu.sg and djokovic@uwaterloo.ca

Received 25 February 2011, in final form 18 May 2011
Published 14 June 2011
Online at stacks.iop.org/JPhysA/44/285303

Abstract

The bipartite quantum states $\rho$, with rank strictly smaller than the maximum of the ranks of the reduced states $\rho_A$ and $\rho_B$, are distillable by local operations and classical communication (Horodecki P, Smolin J A, Terhal B M and Thapliyal A V 2003 Theor. Comput. Sci. 292 589–96; 1999 arXiv:quant-ph/9910122). Our first main result is that this is also true for NPT states with rank equal to this maximum. (A state is PPT if the partial transpose of its density matrix is positive semidefinite, and otherwise it is NPT.) This was conjectured first in 1999 in the special case when the ranks of $\rho_A$ and $\rho_B$ are equal (see (Horodecki P, Smolin J A, Terhal B M and Thapliyal A V 2003 Theor. Comput. Sci. 292 589–96; 1999 arXiv:quant-ph/9910122). Our second main result provides a complete solution of the separability problem for bipartite states of rank 4. Namely, we show that such a state is separable if and only if it is PPT and its range contains at least one product state. We also prove that the so-called checkerboard states are distillable if and only if they are NPT.

PACS numbers: 03.65.Ud, 03.67.Mn, 03.65.-a, 03.67.Hk

1. Introduction

Pure quantum entanglement plays the essential role in various quantum information tasks, such as GHZ states in quantum teleportation [5] and graph states in quantum computation [11]. However, pure entangled states are always coupled with the environment due to the unavoidable decoherence. As a result, they become mixed states which usually cannot be used directly, or become quite useless for quantum information tasks [46]. Hence, extracting pure entanglement from mixed states is a basic task in quantum information. Formally, this task is called entanglement distillation and the entanglement...
measure quantifying the asymptotically obtainable pure entanglement is called distillable entanglement; the states from which we can obtain pure entanglement are distillable [9]. Apart from the mentioned physical applications, the distillable entanglement is the lower bound of many well-known entanglement measures such as the entanglement cost, entanglement of formation [9] and the squashed entanglement [7, 18]. An upper bound for distillable entanglement is provided by the distillable key [17, 35], and recently the gap between them has been experimentally observed [20]. Much effort has been devoted to the investigation of entanglement distillation; for a review see [38].

For a state $\rho$ acting on $\mathcal{H}_A \otimes \mathcal{H}_B$, the partial transpose, computed in an orthonormal (o.n.) basis $\{|a_i\rangle\}$ of system $A$, is defined by $\rho^T := \sum_{ij} |a_i\rangle\langle a_j| \otimes \langle a_j| \rho |a_i\rangle$. Mathematically, we say that $\rho$ is 1-distillable if there exists a pure bipartite state $|\psi\rangle$ of Schmidt rank 2 such that $\langle \psi | \rho^T | \psi \rangle < 0$ [21]. More generally, we say that $\rho$ is $n$-distillable if the state $\rho^\otimes n$ is 1-distillable. Finally, we say that $\rho$ is distillable under local operations and classical communication (LOCC) if it is $n$-distillable for some $n \geq 1$.

It is a famous open question whether all bipartite NPT states, i.e. the states $\rho$ such that $\rho^T$ has at least one negative eigenvalue, are distillable. Although many papers deal with the problem of distillation of pure entanglement from NPT states [3, 15, 38, 44, 50], the complete solution is still unknown. In this paper, we will solve an open question, which is an important special case of the distillation problem.

It follows easily from the definition of distillability given above that bipartite PPT states, i.e. states with positive semidefinite partial transpose, are not distillable by LOCC [30, 32]. In recent years, PPT entangled states have been extensively studied in connection with the phenomena of entanglement activation and universal usefulness [33, 42], the distillable key [35], the symmetry permutations [49] and entanglement witnesses [40], both in theory and experiment [1].

Therefore, it is an important and basic question to decide whether a given PPT state is separable [38]. The famous partial transpose criterion says that the separable states are PPT [45]. Furthermore, Horodecki et al showed that this is necessary and sufficient for states in $M \otimes N$ quantum systems with $MN \leq 6$ [29]. However, the problem becomes difficult for any bigger dimension. For instance, the first examples of PPT entangled states acting on $2 \otimes 4$ and $3 \otimes 3$ were discovered in 1997 [30], but it is still an open problem to decide which PPT states in these systems are separable. The situation is not much better if we consider the separability problem for bipartite states of fixed rank, where the cases of ranks 4 and 5 remained unresolved. We have discovered a simple criterion for separability of states of rank 4.

Throughout the paper the spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ are finite dimensional. We will ignore the normalization condition for states since it does not affect the process of entanglement distillation. Thus, unless stated otherwise, we assume that the states are non-normalized, and we set $M = \text{Dim } \mathcal{H}_A$ and $N = \text{Dim } \mathcal{H}_B$. As our definition of an ‘$M \times N$ state’ can be easily misinterpreted, we state it formally.

**Definition 1.** A bipartite state $\rho$ is an $M \times N$ state if the reduced states $\rho_A = \text{Tr}_B(\rho)$ and $\rho_B = \text{Tr}_A(\rho)$ have ranks $M$ and $N$, respectively.

Our first main result, theorem 10, asserts that $M \times N$ rank-$N$ NPT states are distillable under LOCC. The stronger statement that these states are in fact 1-distillable under LOCC is an immediate consequence of the proof. The special case $M = N$ was proposed in 1999 as a conjecture [37], conjecture 1. As pointed out in [37], it follows from this special case that all rank-3 entangled states are distillable. Hence, we recover the main result of [14]. We give the details in sections 2 and 3.
Our second main result, theorem 22, solves the separability problem for bipartite states of rank 4. Namely, it asserts that a bipartite state of rank 4 is separable if and only if it is PPT and its range contains at least one product state. This is a numerically operational criterion, e.g. see the method introduced in [36]. Therefore, $3 \times 3$ PPT states of rank 4 constitute the first class of PPT states of fixed dimension and rank for which the separability problem can be operationally decided. We also discuss the analytical method based on the Plücker coordinates and the Grassmann variety. We give the details in sections 4, 5, and the appendix.

We also apply our results to characterize the quantum correlations inside a tripartite pure state $|\psi\rangle$. In section 6, theorem 25, we show that the reduced density operators $\rho_{AB}$ and $\rho_{AC}$ of $\rho = |\psi\rangle\langle\psi|$ are undistillable if and only if they are separable. In other words, there is no tripartite pure state with two undistillable entangled reduced density operators. We also discuss the relation between distillability and quantum discord [13], where the latter is another kind of quantum correlation.

To investigate the states beyond $M \times N$ states of rank $N$, we study the family of checkerboard states. This family consists of two-qutrit states [23], generically of rank 4, which generalizes the family constructed by Bruß and Peres [12]. In section 7, we prove that the NPT checkerboard states are 1-distillable. In section 8, we analyze further the full-rank properties (defined below) used in the proof of our first main result. We conclude and discuss our results in section 9.

For convenience and reference, we list some known results on distillability of bipartite NPT states under LOCC.

(A) The $2 \times N$ NPT states are distillable [6, 21, 31].
(B) The states violating the reduction criterion are distillable [28].
(C) The $M \times N$ states of rank less than $M$ or $N$ are distillable [37].
(D) The NPT states of rank at most 3 are distillable [14].
(E) All NPT states can be converted by LOCC into NPT Werner states [19, 21, 52].

The states mentioned in (A–D) are in fact 1-distillable. For the states in result (A), it suffices to observe that generic pure states in a $2 \otimes N$ space have Schmidt rank 2. For (B) see [15] or the next section. The 1-distillability in result (C) follows from that of (B) and the proof of [37], theorem 1. The assertion for states in result (D) can be deduced from those of (A) and (B) [14].

In view of result (E), one might hope to solve the problem of entanglement distillation by distilling the NPT Werner states. Unfortunately, although this problem has been studied extensively in the past decade, it still remains an open and apparently very hard problem [44]. It is thus of high importance to address the non-Werner states and propose useful tools for their distillation. In particular, one can see that result (B), namely reduction criterion, plays a quite important role for distillation of generic entangled states. For example, we have argued that other three main results (A), (C) and (D) are partially or totally derivable from the reduction criterion.

Let us introduce the right full-rank property (RFRP) via the Hermitian observable $\text{Tr}_A(|x\rangle\langle x|\rho) = \langle x|\rho|x\rangle$ where $|x\rangle \in \mathcal{H}_A$. We say that $\rho$ has the RFRP if the operator $\langle x|\rho|x\rangle$ is invertible (i.e. has full rank) for some $|x\rangle \in \mathcal{H}_A$. One defines similarly the left full-rank property (LFRP) by using the Hermitian observable $\text{Tr}_B(|y\rangle\langle y|\rho) = \langle y|\rho|y\rangle$ with $|y\rangle \in \mathcal{H}_B$. Every result about RFRP has its analog for LFRP, and we work mostly with the former.

We point out that RFRP has appeared previously in [36] where it was shown that all $M \times N$ ($M \leq N$) PPT states of rank $N$ have RFRP (see the proof of theorem 1 in that paper).
We prove in theorem 8 that bipartite states violating LFRP or RFRP are distillable under LOCC. We refer to this result as the full-rank criterion. It is a crucial ingredient in the proof of theorem 10. Let us also mention that theorems 8 and 10 remain valid if we replace the word ‘distillable’ with ‘1-distillable’ (see the next section).

The questions of separability and distillability for the $M \times N$ states $\rho$ of rank $R \leq N$, with $M \leq N$, have a very simple answer: they are separable if and only if they are PPT, and they are distillable if and only if they are NPT. The result on separability is proved in [36], and the one on distillability in [37], theorem 1, when $R < N$, while our first main result handles the case $R = N$.

As a transition between the first and second main results, we introduce the concepts of reducible and irreducible quantum states. The sum $\rho$ of bipartite states $\rho_i$ is $B$-direct if $R(\rho_B)$ is a direct sum of $R((\rho_i)_B)$. A bipartite state $\rho$ is reducible if it is a $B$-direct sum of two states and otherwise $\rho$ is irreducible. In the language of quantum state transformation, a reducible state is stochastic-LOCC (SLOCC) equivalent to the sum of $\rho_i$ whose reduced density operators $(\rho_i)_B$ are pairwise orthogonal [22]. We formally state this fact in proposition 15. More properties of reducible states are also introduced, e.g., to study the distillability in propositions 17, 18 and PPT, separability in corollary 16.

Based on the reducibility, we can easily decide the separability and distillability of the bipartite state $\rho$ of rank 4 when there is some $|x\rangle \in \mathcal{H}_A$ such that the operator $\langle x|\rho|x\rangle$ is non-invertible. This is demonstrated in lemma 20. We solve the same problem when there is a product state in the range of $\rho$ in proposition 21. In terms of these results we reach our second main result. That is, a bipartite state of rank 4 is separable if and only if it is PPT and there is a product state in its range. This is illustrated in theorem 22 and is numerically operational. We also propose an analytical method in terms of the Grassmann variety to implement our result. In the case of two-dimensional subspaces of a $2 \otimes 3$ space we give a simple equation which is satisfied if and only if the subspace contains a product state. Similarly, in the appendix we give a single equation test for the existence of a product state in the three-dimensional subspace of a $2 \otimes 4$ space. Such an equation also exists in the case of four-dimensional subspaces of a $3 \otimes 3$ space. We were not able to compute it. The method is totally analytical and does not rely on numerical estimation. The reducibility also constitutes the technical basis for the distillability of checkerboard states, as proved in theorem 28.

We also apply the main results to derive a few physical facts. For example, we prove in theorem 25 that a tripartite pure state cannot have two undistillable entangled reduced density operators. This is realized by the fact that the two reduced density operators have a maximal local rank equal to its rank, as well as the first main result and proposition 3.

Throughout the paper we write $I_k$ resp. $0_k$ for the identity resp. zero $k \times k$ matrix. The inequality $H \geq 0$ means that $H$ is a positive semidefinite Hermitian operator or matrix. Similarly, $H > 0$ means that $H$ is positive definite. We denote by $\mathcal{R}(\rho)$ the range space of an operator $\rho$.

We write $[X, Y]$ for the commutator $XY - YX$. We recall that if $X$ is normal and $Y$ arbitrary, then $[X, Y] = 0$ implies $[X^\dagger, Y] = 0$. This follows from the fact that, for a normal operator $X$, there exists a linear polynomial $f(t)$ such that $X^\dagger = f(X)$.

2. Preliminary facts

Before proceeding with the proofs of our results, let us recall some additional facts. First, by using an o.n. basis $\{|u_i\rangle\}$ of $\mathcal{H}_A$, we can write any state as $\rho = \sum_{i,j=1}^{M} |u_i\rangle\langle u_j| \otimes \sigma_{ij}$. Hence, $\rho$ is represented by its matrix
\[
\rho = \begin{pmatrix}
\sigma_{11} & \sigma_{12} & \cdots & \sigma_{1M} \\
\sigma_{21} & \sigma_{22} & \cdots & \sigma_{2M} \\
\vdots & \vdots & \ddots & \vdots \\
\sigma_{M1} & \sigma_{M2} & \cdots & \sigma_{MM}
\end{pmatrix}.
\] (1)

Usually we assume that \{\{|u_i\}\} is the computational basis \{|i\}\).

Second, the RFRP (defined in the introduction) was an essential tool used in [36] and will also play an important role in this paper. Clearly, RFRP is equivalent to the following assertion. One can choose an o.n. basis \{\{|u_i\}\} of \(\mathcal{H}_A\) such that some diagonal block \(\sigma_{ii}\) in the above matrix has rank \(N\).

The condition that \(\text{rank}(\langle x|\rho|x\rangle) = N\) is equivalent to \(\ker(\langle x|\rho|x\rangle) = 0\). It follows that \(\rho\) violates RFRP if and only if for each state \(|x\rangle\in\mathcal{H}_A\), there exists a state \(|y\rangle\in\mathcal{H}_B\) such that \(|x\rangle\otimes|y\rangle\in\ker\rho\).

\textbf{Example 2.} We use an argument from the proof of [36], lemma 3, to show that all \(M \times N\) separable states \(\rho\) have RFRP. We can write \(\rho\) as a finite sum of non-normalized product states.

\textbf{Proposition 3.} If \(\rho\) is an \(M \times N\) PPT state with \(M \leq N\) and rank \(N\), then \(\rho\) is separable and can be written as a sum of \(N\) product states

\[\rho = \sum_{i=1}^{N} |a_i, b_i\rangle\langle a_i, b_i|.
\]

If \(A\) and \(B\) are linear operators on \(\mathcal{H}_A\) and \(\mathcal{H}_B\), respectively, then we refer to \(A \otimes B\) as a local operator. The abbreviation ILO will refer to invertible local operators, i.e. to operators \(A \otimes B\) with both \(A\) and \(B\) invertible. It is easy to see that RFRP is invariant under ILOs, i.e. if \(A \otimes B\) is an ILO and the state \(\rho\) has RFRP, then the transformed state \((A \otimes B)^\dagger \rho (A \otimes B)\) also has RFRP. Moreover, if \(\rho\) violates RFRP, then so does \((A \otimes B)^\dagger \rho (A \otimes B)\) even if \(A \otimes B\) is not invertible.

Third, there is a simple way to prove distillability of \(\rho\) which can be applied in many cases. For that purpose observe that if \(\rho\) is given by its matrix (1), then

\[
\rho^\Gamma = \begin{pmatrix}
\sigma_{11} & \sigma_{12} & \cdots & \sigma_{1M} \\
\sigma_{21} & \sigma_{22} & \cdots & \sigma_{2M} \\
\vdots & \vdots & \ddots & \vdots \\
\sigma_{M1} & \sigma_{M2} & \cdots & \sigma_{MM}
\end{pmatrix}.
\]

Since \(\rho\) is a Hermitian matrix, so is \(\rho^\Gamma\). Let \(X = \begin{pmatrix} a & b^* \\ b & c \end{pmatrix}\) be a principal 2 \(\times\) 2 submatrix of \(\rho^\Gamma\). Thus \(a\) and \(c\) are the diagonal entries of \(\rho^\Gamma\).

\textbf{Lemma 4.} If \(ac = 0\) while \(b \neq 0\), then \(\rho\) is distillable.

\textbf{Proof.} Since \(X\) has a negative eigenvalue and the diagonal blocks \(\sigma_{ii} \geq 0\), the diagonal entries \(a\) and \(c\) must belong to different diagonal blocks, say \(\sigma_{kk}\) and \(\sigma_{ll}\). Let \(P\) be the orthogonal projector onto the two-dimensional subspace of \(\mathcal{H}_A\) spanned by \(|k\rangle\) and \(|l\rangle\). Then the projected
state $\rho' := (P \otimes I_N)\rho(P \otimes I_N)$ is an NPT state acting on a $2 \otimes N$ system. Hence, $\rho'$ is distillable by result (A). Consequently, $\rho$ is also distillable.

Whenever the distillability of $\rho$ can be proved by direct application of this simple lemma, we say that the state $\rho$ is \textit{trivially distillable}. For example, by using this concept we generalize theorem 2 of [37].

\textbf{Theorem 5.} Let $\rho = |\psi\rangle\langle\psi| + \sigma$ be an $M \times N$ state where $\sigma$ is any state and $|\psi\rangle$ is a pure entangled state. If $r := \text{rank}(\sigma_A) < M$, then $\rho$ is distillable.

\textbf{Proof.} We may assume that $\{|1\rangle, \ldots, |r\rangle\}$ is a basis of $\mathcal{R}(\sigma_A)$. We can write $|\psi\rangle = \sum_{i=1}^{M} |i, \psi_i\rangle$ where $|\psi_i\rangle \in \mathcal{H}_B$. Since $r < M$, we must have $|\psi_M\rangle \neq 0$. Since $|\psi\rangle$ is entangled, at least one of the $|\psi_k\rangle$, $k < M$, is not parallel to $|\psi_M\rangle$. Let us fix such index $k$. Let $V$ be an ILO on $\mathcal{H}_B$ such that $V|\psi_k\rangle = |1\rangle$ and $V|\psi_M\rangle = |2\rangle$. If $P = |k\rangle\langle k| + |M\rangle\langle M|$, then the state $\rho' := (P \otimes \rho)P \otimes V^\dagger$ acts on a $2 \otimes N$ system.

We claim that $\rho'$ is trivially distillable. Since this is the first time that we are applying lemma 4, we give all the details. Since $P \otimes V|\psi\rangle = |k, 1\rangle + |M, 2\rangle$, we have

$$P \otimes V|\psi\rangle\langle\psi|P \otimes V^\dagger = [k, 1]|k, 1\rangle + [M, 2]|M, 2\rangle + [k, 1]|M, 2\rangle + [M, 2]|k, 1\rangle.$$  

$$P \otimes V|\psi\rangle\langle\psi|P \otimes V^\dagger = [k, 1]|k, 1\rangle + [M, 2]|M, 2\rangle + [M, 1]|k, 2\rangle + [k, 2]|M, 1\rangle.$$  

On the other hand, since $\sigma(|M\rangle\langle M| \otimes V)^\dagger = 0$, the $2N \times 2N$ matrix of $(P \otimes V)\sigma(P \otimes V)^\dagger$ has its nonzero entries all in the $N \times N$ submatrix contained in the first $N$ rows and columns. Hence, the $2 \times 2$ principal submatrix of $(\rho')^\dagger$ in rows 2 and $N + 1$ has the form $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, which proves our claim. It follows from this claim that $\rho$ is also distillable. \hfill $\square$

Theorem 2 of [37] is obtained by taking $\sigma$ to be a pure product state.

In the following proposition we represent $M \times N$ states in a convenient matrix form which will be used in our proofs. We include a result proved in [36] for the states which are also PPT.

\textbf{Proposition 6.} Let $\rho$ be an $M \times N$ state of rank $R$.

(i) $\rho$ can be written as

$$\rho = \sum_{i, j} |i\rangle|j\rangle \otimes C_j^\dagger C_j = (C_1, \ldots, C_M)^\dagger \cdot (C_1, \ldots, C_M),$$  

where $C_i$ are $R \times N$ matrices such that $\rho_B = \sum_i C_i^\dagger C_i > 0$.

(ii) If $R = N$ and $\rho$ has RFRP, then there exists an invertible local operator $A \otimes B$ such that

$$\rho' := (A \otimes B)^\dagger \rho A \otimes B$$

$$= (C_1, \ldots, C_{M-1}, I_N)^\dagger \cdot (C_1, \ldots, C_{M-1}, I_N).$$

Moreover, $\rho'$ is PPT if and only if the $C_i$ are pairwise commuting normal matrices.

\textbf{Proof.} The assertion (i) follows from the spectral decomposition theorem. Indeed, by that theorem we have

$$\rho = \sum_{i=1}^{R} |\psi_i\rangle\langle\psi_i|,$$

$$|\psi_i\rangle = \sum_{j=1}^{M} |j, \psi_{ij}\rangle,$$  

and

$$\rho = \sum_{i,j} |i,j\rangle \otimes C_j^\dagger C_j,$$  

where $C_i$ are $R \times N$ matrices such that $\rho_B = \sum_i C_i^\dagger C_i > 0$. Hence, $\rho$ is distillable by result (A). Consequently, $\rho$ is also distillable.

Theorem 5 of [37] is obtained by taking $\sigma$ to be a pure product state.

In the following proposition we represent $M \times N$ states in a convenient matrix form which will be used in our proofs. We include a result proved in [36] for the states which are also PPT.

\textbf{Proposition 6.} Let $\rho$ be an $M \times N$ state of rank $R$.

(i) $\rho$ can be written as

$$\rho = \sum_{i, j} |i\rangle|j\rangle \otimes C_j^\dagger C_j = (C_1, \ldots, C_M)^\dagger \cdot (C_1, \ldots, C_M),$$  

where $C_i$ are $R \times N$ matrices such that $\rho_B = \sum_i C_i^\dagger C_i > 0$.

(ii) If $R = N$ and $\rho$ has RFRP, then there exists an invertible local operator $A \otimes B$ such that

$$\rho' := (A \otimes B)^\dagger \rho A \otimes B$$

$$= (C_1, \ldots, C_{M-1}, I_N)^\dagger \cdot (C_1, \ldots, C_{M-1}, I_N).$$

Moreover, $\rho'$ is PPT if and only if the $C_i$ are pairwise commuting normal matrices.

\textbf{Proof.} The assertion (i) follows from the spectral decomposition theorem. Indeed, by that theorem we have

$$\rho = \sum_{i=1}^{R} |\psi_i\rangle\langle\psi_i|,$$

$$|\psi_i\rangle = \sum_{j=1}^{M} |j, \psi_{ij}\rangle,$$  

and
where $|\psi_i\rangle$ are non-normalized pure states. Since these states span the range of $\rho$, they must be linearly independent. However, they are not uniquely determined by $\rho$. The matrices $C_j$ can be chosen as follows:
\[
C_j = (|\psi_{1j}\rangle, \ldots, |\psi_{Rj}\rangle)^\dagger, \quad j = 1, \ldots, M. \tag{4}
\]
In other words, $\langle \psi_{ij} |$ is the $i$th row of $C_j$. Since $\rho_B \geq 0$ and $\text{rank } \rho_B = M$, we have $\rho_B > 0$.

For the first assertion of (ii) we may assume that $C_M$ is invertible and then apply the local operator $I_A \otimes C_M^{-1}$. It is shown in [36] that the PPT condition implies that the $C_i$s are pairwise commuting normal matrices. The converse is straightforward. □

The next example shows that there exist $M \times N$ NPT states of rank $N$ which violate RFRP.

Example 7. For the (non-normalized) $3 \times 3$ rank-3 antisymmetric state
\[
\rho_{\text{as}} = \sum_{1 \leq i < j \leq 3} (|ij\rangle - |ji\rangle)(\langle ij| - \langle ji|), \tag{5}
\]
the operator $\langle x | \rho_{\text{as}} | x \rangle$ has rank 2 for all nonzero vectors $|x\rangle_A = \sum \xi_i |i\rangle \in \mathcal{H}_A$. This can be verified as follows. We first compute the matrices $C_i$. In this example we can choose $|\psi_i\rangle = |jk\rangle - |kj\rangle$ where $(i, j, k)$ is a cyclic permutation of $(1, 2, 3)$. Then we find that
\[
C_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad C_3 = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{6}
\]
Thus $\langle x | \rho_{\text{as}} | x \rangle = X^\dagger X$ where $X = \sum \xi_i C_i$. Since $X$ is antisymmetric, its rank must be even, and so both $X$ and $\langle x | \rho_{\text{as}} | x \rangle$ have rank 2.

For a more general class of examples violating RFRP see section 8. The problem of characterizing the states violating RFRP remains open.

Let us make two important observations. First, there is a stronger version of result (B) from the introduction, due to Clarisse [15]. Namely, he has shown that a bipartite state $\rho$ which violates the reduction criterion is in fact 1-distillable [21]. This is much stronger than being merely distillable.

Second, assume that a bipartite state $\sigma$ is obtained from $\rho$ by applying a local operator $\sigma = (A \otimes B)^\dagger \rho (A \otimes B)$, where $A \otimes B$ may be singular. If $\sigma$ is 1-distillable, then $\rho$ is also 1-distillable. Indeed if the Hermitian operator $D$ is a 1-distillation witness detecting $\sigma$, i.e. $\text{Tr}(\sigma D) < 0$ while $\text{Tr}(\sigma' D) \geq 0$ for all 1-undistillable states $\sigma'$, then $(A \otimes B) D (A \otimes B)^\dagger$ is a 1-distillation witness detecting $\rho$.

One can easily verify that in all cases that arise in our proofs, the above two observations guarantee that the state $\rho$ from which we start is not only distillable but also 1-distillable. Hence, theorems 8 and 10 remain valid when we replace the word ‘distillable’ with ‘1-distillable’.

3. Distillability of $M \times N$ NPT states of rank $N$

In this section, we prove our main result, theorem 10. However, our first objective is to show that the states which violate LFRP or RFRP are distillable. Since PPT states are not distillable, it follows that all PPT states must possess both LFRP and RFRP.

For convenience, we denote by $X[k]$ the submatrix of the matrix $X$ consisting of the last $k$ columns. If a matrix $X$ factorizes as $X = YZ$, then we say that $Z$ is a right factor of $X$. 

7
Theorem 8. Bipartite states which violate LFRP or RFRP are distillable.

Proof. It suffices to prove that any $M \times N$ state $\rho$ which violates RFRP is distillable. Let $R$ be the rank of $\rho$. If $R < N$, then $\rho$ is distillable by result (C). From now on we assume that $R \geq N$. By proposition 6 we have $\rho = (C_1, \ldots, C_M)^\dagger \cdot (C_1, \ldots, C_M)$ where each matrix $C_i$ is of size $R \times N$, and $\rho_B = \sum_i C_i^\dagger C_i > 0$.

We can replace $(C_1, \ldots, C_M)$ with $U(C_1, \ldots, C_M)$ where $U$ is a unitary matrix, without changing $\rho$. The effect of an invertible local transformation $\rho \rightarrow (I \otimes B)^\dagger \rho (I \otimes B)$ is to replace each $C_i$ by $C_iB$. Recall that RFRP is preserved by these local transformations. In order to prove the theorem we can apply these kinds of transformations as many times as needed.

Since $\rho_A$ is invertible, each $C_i \neq 0$. Let $r_i$ be the rank of $C_i$. By multiplying $(C_1, \ldots, C_M)$ by a unitary matrix $U_i$ on the left-hand side, we may assume that the last $R - r_i$ rows of $C_i$ are zero. We choose an invertible matrix $B_i$ such that $C_iB_i = I_{r_i} \oplus 0$. By multiplying all $C_is$ by $B_i$ on the right-hand side, we may assume that $C_i = I_{r_i} \oplus 0$. Since $\rho$ violates RFRP, we have $r_i < N$.

For a sequence of indexes $1 \leq i_1 < i_2 < \cdots < i_k \leq M$ we denote by $\rho_{i_1,\ldots,i_k}$ the corresponding principal submatrix of $\rho$ of size $kN \times kN$, i.e.

$$\rho_{i_1,\ldots,i_k} = (C_{i_1}, \ldots, C_{i_k})^\dagger \cdot (C_{i_1}, \ldots, C_{i_k}).$$

The corresponding principal submatrix of $\rho^\Gamma$ will be denoted by $\rho_{i_1,\ldots,i_k}^\Gamma$. For instance, for $i > 1$ we have

$$\rho_{1,i} = (C_1, C_i)^\dagger \cdot (C_1, C_i) = \begin{pmatrix} C_1^\dagger C_1 & C_1^\dagger C_i \\ C_i^\dagger C_1 & C_i^\dagger C_i \end{pmatrix}$$

and

$$\rho_{1,i}^\Gamma = \begin{pmatrix} C_1^\dagger C_1 & C_i^\dagger C_1 \\ C_i^\dagger C_1 & C_i^\dagger C_i \end{pmatrix}.$$

We split each $C_i$ into four blocks $C_i = \begin{pmatrix} C_{i_1} & C_{i_2} \\ C_{i_3} & C_{i_4} \end{pmatrix}$ with $C_{i_1}$ square of size $r_i$. Since $C_1 = I_{r_1} \oplus 0$, we have

$$\rho_{1,i} = \begin{pmatrix} I_{r_i} & 0 & C_{i_1} & C_{i_2} \\ 0 & 0 & 0 & 0 \\ C_{i_1}^\dagger & 0 & * & * \\ C_{i_2}^\dagger & 0 & * & * \end{pmatrix}, \quad i > 1,$$

where the asterisk stands for an unspecified block. If some $C_{i_2} \neq 0$, then $\rho$ is trivially distillable. Thus, we may assume that all $C_{i_2} = 0$.

Since $\rho_B$ has rank $N$, its submatrix $\rho_B[N - r_i]$ must have rank $N - r_i$. Since $C_{i_2} = 0$, $C_{i_4}$ is a right factor of the submatrix $(C_i^\dagger C_i)[N - r_i]$. Therefore, $C_{i_4} \neq 0$ for at least one index $i > 1$. By permuting the $C_i$s with $i > 1$, we may assume that $C_{i_4} \neq 0$. Let $r_2$ be its rank. By multiplying $(C_1, \ldots, C_M)$ by a unitary matrix $I_{r_1} \oplus U_2$ on the left-hand side, we may assume that the last $R - r_2$ rows of $C_{i_4}$ are zero. Let $B_2$ be an invertible matrix such that $C_{i_4}B_2 = I_{r_2} \oplus 0$. By multiplying each $C_i$ by $I_{r_1} \oplus B_2$ on the right-hand side, we may assume that $C_{i_4} = I_{r_2} \oplus 0$. Note that these operations performed on $(C_1, \ldots, C_M)$ do not alter $C_1$ and that the equalities $C_{i_2} = 0$ remain valid.
Assume that $r_1 + r_2 < N$. We split each $C_{ij}$ into four blocks $C_{ij} = \left( \begin{array}{cc} C_{i1} & C_{i2} \\ C_{i3} & C_{i4} \end{array} \right)$, with $C_{i4}$ square of size $r_2$. For $i > 2$ we have

$$
\rho_{2,i} = \left( \begin{array}{ccc} C_{i2}^\dagger & C_{i3}^\dagger & C_{i4}^\dagger \\ C_{i2} & C_{i3} & C_{i4} \\ C_{i2}^\dagger & C_{i3}^\dagger & C_{i4}^\dagger \end{array} \right).
$$

We extract from it the principal submatrix

$$
\left( \begin{array}{cc} C_{i2}^\dagger & C_{i3}^\dagger \\ C_{i2} & C_{i3} \end{array} \right) = \left( \begin{array}{ccc} I_{r_2} & 0 & C_{i41} \\ 0 & 0 & 0 \\ C_{i41} & 0 & * \\ 0 & * & * \end{array} \right).
$$

If some $C_{i42} \neq 0$, then $\rho$ is trivially distillable. Thus, we may assume that all $C_{i42} = 0$.

Since $\rho_B$ has rank $N$, the matrix $\rho_B[N - r_1 - r_2]$ must have rank $N - r_1 - r_2$. Since $C_{i2} = 0$ and $C_{i42} = 0$, we have

$$
C_i = \left( \begin{array}{ccc} C_{i1} & 0 & 0 \\ * & C_{i41} & 0 \\ * & C_{i43} & C_{i44} \end{array} \right).
$$

Consequently, $C_{i44}$ is a right factor of the submatrix $(C_i C_i)[N - r_1 - r_2]$. Therefore, $C_{i44} \neq 0$ for at least one index $i > 2$. By permuting the $C_i$s with $i > 2$, we may assume that $C_{344} \neq 0$. Let $r_3$ be its rank. By multiplying $(C_1, C_2, \ldots, C_M)$ by a unitary matrix $U_{r_1 + r_2} \oplus U_1$ on the left-hand side, we may assume that the last $R - r_1 - r_2 - r_3$ rows of $C_{344}$ are zero. Let $B_3$ be an invertible matrix such that $C_{344} B_3 = I_{r_3} \oplus 0$. By multiplying each $C_i$ by $I_{r_1 + r_2} \oplus B_3$ on the right-hand side, we may assume that $C_{344} = I_{r_3} \oplus 0$. Note that these operations performed on $(C_1, C_2, \ldots, C_M)$ alter neither $C_1$ nor $C_{24}$ and that the equalities $C_{i2} = 0$ and $C_{i42} = 0$ remain valid.

If $r_1 + r_2 + r_3 < N$ we can continue by splitting each $C_{i44}$ into four blocks, etc. This process terminates as soon as $\rho$ has been shown to be trivially distillable. Otherwise it can be continued as long as $r_1 + \cdots + r_k < N$ and $k < M$. However, note that if $k$ becomes equal to $M$, then we must have $r_1 + \cdots + r_M = N$ because rank $\rho_B = N$.

We claim that the process must terminate while the inequality $r_1 + \cdots + r_k < N$ is still valid. Indeed assume that we reach a point where $r_1 + \cdots + r_k = N$. Since $k \leq M$ we can set $|x\rangle = t_1 |1\rangle + \cdots + t_k |k\rangle \in \mathcal{H}_M$, where $t_1, \ldots, t_k$ are real parameters. Then we have

$$
\langle x | \rho | x \rangle = \sum_{i,j=1}^{k} t_i t_j C_i^\dagger C_j = \left( \sum_{i=1}^{k} t_i C_i \right)^\dagger \left( \sum_{i=1}^{k} t_i C_i \right).
$$

It follows that

$$
\det(\langle x | \rho | x \rangle) = \left| \det \left( \sum_{i=1}^{k} t_i C_i \right) \right|^2.
$$

The determinant on the left-hand side is a polynomial in the parameters $t_1, \ldots, t_k$. As $C_{i2} = 0$, $C_{i42} = 0, \ldots$ for all $i$, the coefficient of $t_1^{2r_1} t_2^{2r_2} \cdots t_k^{2r_k}$ in this determinant is equal to 1. Hence,
we can choose the values of the parameters \( t_i \) so that the operator \( \langle x | \rho | x \rangle \) becomes invertible. This contradicts the hypothesis that \( \rho \) violates RFRP, and completes the proof. \( \Box \)

Let us also record the observation we made earlier.

**Corollary 9.** All bipartite PPT states possess LFRP and RFRP.

For \( M \times N \) states with \( M \leq N \) and rank \( N \) this was proved in [36].

By using theorem 8 we can now prove our main result.

**Theorem 10.** The \( M \times N \) PPT states of rank \( N \) are distillable under LOCC.

**Proof.** By result (C) we may assume that \( M \leq N \). In view of theorem 8, it suffices to prove the assertion for \( M \times N \) PPT states \( \rho \) of rank \( N \) which have RFRP. Since \( \rho \) is PPT, we must have \( M \geq 2 \). By proposition 6 we can assume that \( \rho = (C_1, \ldots, C_{M-1}, I_N) \cdot (C_1, \ldots, C_{M-1}, I_N) \) where \( C_i \) are \( N \times N \) matrices. Let \( \rho_i = (P_i \otimes I) \rho (P_i \otimes I) \) where \( P_i = |i\rangle \langle i| + |M|/|M|, i < M \).

If some \( \rho_i \) is PPT, then \( \rho_i \) and \( \rho \) is distillable by result (A). Thus, we may assume that all \( \rho_i \) states are PPT. By applying proposition 6 (ii) to \( \rho_i \), we deduce that \( C_i \) is a normal matrix. Since \( \rho \) is PPT, the same proposition implies that there exist \( i, j \) such that \( |C_i, C_j| \neq 0 \). In particular, \( M \geq 3 \). We may assume that \( i = 1 \) and \( j = 2 \).

Let \( \rho' = (V \otimes I_N) \rho (V \otimes I_N) \) where \( V = (x | 1 \rangle + |2\rangle)/2 + |M|/|M| \) and \( x \) is a complex parameter. Obviously we have

\[
(C_1, \ldots, C_{M-1}, I_N) (V \otimes I_N) = (0, X, 0, \ldots, 0, I_N),
\]

where \( X = xC_1 + C_2 \). Since the range of \( \rho_0' \) is contained in the subspace spanned by \( |2\rangle \) and \( |M\rangle \), we can view \( \rho' \) as a state acting on a \( 2 \otimes N \) space. Then, its density matrix is

\[
\rho' = (X, I_N) \cdot (X, I_N)
\]

and we have

\[
\rho_0' = \begin{pmatrix}
\text{Tr}(X^\dagger X) & \text{Tr}(X^\dagger) \\
\text{Tr}(X) & N
\end{pmatrix}.
\]

Hence, \( \det(\rho_0') = N \text{Tr}(X^\dagger X) - |\text{Tr}(X)|^2 \). Since \( [C_1, C_2] \neq 0 \), the matrices \( X \) and \( I_N \) are linearly independent. Consequently, \( \det(\rho_0') > 0 \) by the Cauchy–Schwarz inequality. Since \( \rho_0' = I_N + X^\dagger X > 0 \), we conclude that \( \rho' \) is a \( 2 \times N \) state for all \( x \). Evidently its rank is \( N \).

Assume that \( \rho' \) is PPT for all values of \( x \). By proposition 6, the matrix \( X = xC_1 + C_2 \) is normal. Since \( x \) is arbitrary and both \( C_1 \) and \( C_2 \) are normal, we deduce that \( x[C_1, C_2] = x^\ast [C_1^\dagger, C_2] \). By setting first \( x = 1 \) and then that \( x \) is the imaginary unit, we deduce that \( [C_1, C_2] = 0 \). Consequently, also \( [C_1, C_2] = 0 \) because \( C_2 \) is normal, and we have a contradiction. We conclude that \( \rho' \) must be PPT for at least one value of \( x \). Such \( \rho' \) is distillable by result (A) and, consequently, \( \rho \) is also distillable. This completes the proof. \( \Box \)

This theorem gives a new family of bipartite distillable states under LOCC. In particular, for \( M = N \) we obtain a positive answer to an open problem proposed in [37].

### 4. Reducible and irreducible states

In the previous sections we have shown that the states violating LFRP or RFRP are distillable and discussed several applications of this result. In this and subsequent sections we would like to explore further the structure of states having both full-rank properties. This problem is important for two main reasons. First, most bipartite quantum states have LFRP and RFRP, i.e. this is a generic property of the state space (see section 8). Second, by theorem 10, the distillability problem for bipartite states reduces to the case of \( M \times N \) states of rank bigger
than $\max(M, N)$. We introduce some new concepts such as reducible states, and use them to construct new classes of distillable states. They complement theorems 8 and 10.

We begin with the definition of reducible and irreducible states.

**Definition 11.** The sum $\rho$ of bipartite states $\rho_i$ is $B$-direct if $R(\rho_B) = \text{a direct sum of } R((\rho_i)_B)$. A bipartite state $\rho$ is reducible if it is a $B$-direct sum of two states and otherwise $\rho$ is irreducible. We denote them by $\rho_{re}$ and $\rho_{ir}$, respectively.

It is clear that if $A \otimes B$ is an arbitrary ILO, then $\rho$ is reducible if and only if $(A \otimes B)\rho(A \otimes B)^d$ is reducible.

We observe that in the case $N = 3$ the reducible NPT states are distillable.

**Lemma 12.** Any $M \times 3$ reducible NPT state is distillable.

**Proof.** Such a state $\rho$ is a $B$-direct sum of two states with $B$-local ranks 1 and 2. Since the one with $B$-local rank 1 is separable, the other one must be NPT and so it is distillable by result (A). Consequently, $\rho$ is distillable too. \(\square\)

The following lemma is obvious.

**Lemma 13.** Any bipartite state is a finite $B$-direct sum of irreducible states.

We point out that this decomposition into irreducibles is not unique in general. For instance any $1 \times 2$ state $\rho$ has infinitely many $B$-direct decompositions $\rho = \rho_1 + \rho_2$, where $\rho_1$ and $\rho_2$ are product states. Here is another more interesting example with entangled irreducible summands.

**Example 14.** Let $\rho$ be the $4 \times 4$ (non-normalized) state which is by definition the $B$-direct sum $\rho = \rho_1 + \rho_2$ of two irreducible states $\rho_1 = 2|\phi_1\rangle\langle\phi_1|$ and $\rho_2 = 2|\phi_2\rangle\langle\phi_2|$ where $\phi_1 = |11\rangle + |22\rangle$ and $\phi_2 = |13\rangle + |24\rangle$. It admits another such decomposition $\rho = \rho_1 + \rho_2'$, where $\rho_1' = |\phi_1'\rangle\langle\phi_1'|$ and $\rho_2' = |\phi_2'\rangle\langle\phi_2'|$ with $\phi_1' = |11\rangle + |13\rangle + |22\rangle + |24\rangle$ and $\phi_2' = |11\rangle - |13\rangle + |22\rangle - |24\rangle$.

The reducible states have the following important property.

**Proposition 15.** Let $\rho = \rho_1 + \rho_2$ (a $B$-direct sum) be a reducible $M \times N$ state. Then, there exists a Hermitian operator $V > 0$ on $\mathcal{H}_B$ such that the states $\rho_i' = (I \otimes V)\rho_i(I \otimes V)$ ($i = 1, 2$) have orthogonal ranges, i.e. $(\rho_i')_B(\rho_i')_B = 0$.

**Proof.** For convenience, set $\sigma_i = (\rho_i)_B$. Since $\mathcal{H}_B = R(\sigma_1) \oplus R(\sigma_2)$, we have $\sigma := \sigma_1 + \sigma_2 > 0$. We set $V = \sigma^{-1/2}$ and $\sigma_i' = V \sigma_i V$. Since $\sigma_1' + \sigma_2' = I_N$ and rank$(I_N - \sigma_i') = \text{rank}(\sigma_i') = N - \text{rank}(\sigma_i')$, it follows that $\sigma_i'$ is an orthogonal projector. Hence $\sigma_1'\sigma_2' = \sigma_1'(I_N - \sigma_1') = 0$. For $\rho_i' = (I \otimes V)\rho_i(I \otimes V)$, we have $(\rho_i')_B V = V \sigma_i V = \sigma_i'$ and so $(\rho_i')_B(\rho_i')_B = \sigma_i'\sigma_i' = 0$. \(\square\)

**Corollary 16.** If $\rho = \sum_i \rho_i$ is a $B$-direct sum, then $\rho$ is separable (PPT) if and only if every $\rho_i$ is separable (PPT). In particular, the reducible $3 \times 3$ state $\rho$ is separable if and only if it is PPT.

For convenience, we refer to the following states [34], section 4, as label states

$$\rho_{ia} := \sum_i p_i|a_i, b_i\rangle\langle a_i, b_i|_{A_iB_i} \otimes |\psi_i\rangle\langle\psi_i|_{A_iB_i},$$
where the product states $|a_i, b_i\rangle$ are distinguishable via LOCC. In particular, when $|a_i, b_i\rangle = |1, i\rangle$, the state $\rho_{ia}$ becomes a reducible state. It is also known that three fundamental entanglement measures, i.e. the distillable entanglement $E_d$, the entanglement cost $E_c$, and the entanglement of formation $E_f$ are all equal for such states \[34\]. Indeed, because $|a_i, b_i\rangle$ are distinguishable via LOCC, by measurements Alice and Bob can get each pure state $|\psi_i\rangle$ with probability $p_i$. Therefore, $E_d(\rho_{ia}) = \sum_i p_i S(\text{Tr}_A |\psi_i\rangle\langle\psi_i|)$, where $S(\rho)$ is the von Neumann entropy. On the other hand, by the definition of $E_c$ and $E_f$ \[9\], we have the inequalities

$$E_d(\rho_{ia}) \leq E_c(\rho_{ia}) \leq E_f(\rho_{ia}) \leq \sum_i p_i S(\text{Tr}_A |\psi_i\rangle\langle\psi_i|).$$

So the three entanglement measures coincide.

One can prove a similar fact for reducible states $\rho = \sum_i p_i \rho_i$, provided that $(\rho_i)_B(\rho_j)_B = 0$ for $i \neq j$. We assume here that the states $\rho_i$ and $\rho_{ia}$ are all normalized, $p_i > 0$, $\sum_i p_i = 1$, and the sum is B-direct. Thus, Alice and Bob can get each state $\rho_i$ with probability $p_i$. We have

$$E_d(\rho_{re}) = \sum_i p_i E_d(\rho_i) \leq E_c(\rho_{re}) \leq E_f(\rho_{re}) \leq \sum_i p_i E_f(\rho_i). \quad (7)$$

Evidently, the three entanglement measures coincide if and only if $E_d(\rho_i) = E_f(\rho_i)$ for all $i$, e.g. this is satisfied for the label states\[4\]. Thus we have the following result.

**Proposition 17.** If $\rho = \sum_i p_i$ is a B-direct sum, then $\rho$ is distillable if and only if at least one of the $\rho_i$ is distillable.

**Proof.** By proposition 15, we may assume that the states $\rho_i$ are distinguishable by LOCC. From the equality in equation (7) we see that the inequality $E_d(\rho_{re}) > 0$ holds if and only if $E_d(\rho_i) > 0$ for at least one index $i$. \(\square\)

Together with lemma 13, this proposition shows that the question of deciding whether an arbitrary bipartite state is distillable reduces to the case of irreducible states. However, the distillability problem for irreducible states is hard. We tackle a special case in the following proposition.

**Proposition 18.** Let $\rho$ be an irreducible $M \times N$ state such that $\mathcal{H}_A' \otimes |b\rangle \subseteq \ker \rho$ for some $(M - 1)$-dimensional subspace $\mathcal{H}_A' \subseteq \mathcal{H}_A$ and some state $|b\rangle \in \mathcal{H}_B$. Then, $\rho$ is distillable.

**Proof.** We may assume that $\mathcal{H}_A'$ is spanned by the basis vectors $|i\rangle_A$, $i > 1$, and that $|b\rangle = |1\rangle_B$. By proposition 6, we have $\rho = (C_1, \ldots, C_M)(C_1, \ldots, C_M)$, where the $C_i$ are $R \times N$ matrices and $R = \text{rank} \rho$. Moreover, the first columns of the $C_i$ are 0 for $i > 1$. Since $\rho_{1b}$ is invertible, the first column of $C_1$ is not 0. By multiplying $(C_1, \ldots, C_M)$ by a unitary matrix on the left-hand side, we may assume that only the first component of the first column of $C_1$ is nonzero. Clearly, we can also assume that the same is true for the first row of $C_1$. Since $\rho$ is irreducible, at least one of the first rows of the $C_i$, $i > 1$, must be nonzero. It follows that $\rho$ is trivially distillable. \(\square\)

In view of lemma 12, we have the following corollary.

**Corollary 19.** If $N = 3$, we can remove the irreducibility hypothesis from proposition 18.

It is easy to construct examples of bipartite states having LFRP and RFRP which can be shown to be distillable by proposition 18. Clearly, theorem 8 cannot be applied to such states.

\[4\] However, the equality does not hold for most of the known mixed bipartite states. The reason is that, according to the computable results, the distillable entanglement is smaller than the entanglement cost. It is also an important open problem to decide whether the equality holds only for the label states (including the pure states).

\[12\]
5. Separability criterion for bipartite states of rank 4

The problem of deciding whether a state is separable has been shown to be NP-hard [25], and hence the progress in solving instances of this problem is significant for both quantum information theory and computer science. Most of the recent contributions focus on the numerical methods with an objective to improve the efficiency in some special cases [27].

On the other hand, the PPT condition, a celebrated decision criterion, is a necessary condition for separability of arbitrary $M \times N$ states. It is sufficient only if $MN \leq 6$. It is also sufficient for some other classes of bipartite states. For instance, this is the case when the rank, $R$, of $\rho$ satisfies the inequality $R \leq \max(M, N)$. However, note that if $R < \max(M, N)$, then $\rho$ is NPT by result (C). Thus, the PPT condition forces the inequality $\max(M, N) < R$. If $\max(M, N) = R$, then the above result applies and it remains to consider only the case $\max(M, N) < R$. We infer that if $R \leq 3$, then $\rho$ is separable if and only if it is PPT, i.e. there are no PPT entangled states with $R \leq 3$. However, when $R = 4$ and $M = N = 3$ such states do exist and the problem arises to decide whether a given $3 \times 3$ PPT state of rank 4 is separable. In this section, we provide a simple answer to this question, and thereby obtain a criterion for separability of arbitrary bipartite states of rank 4.

We start with an easy observation.

**Lemma 20.** Let $\rho$ be a $3 \times N$ state such that, for some $|a\rangle \in \mathcal{H}_A$, rank$\langle a | \rho | a \rangle = 1$. If $\rho$ is NPT, then it is distillable. If $\rho$ is PPT and $N = 3$, then $\rho$ is separable.

**Proof.** We prove both assertions at the same time. We may assume that $|a\rangle = |1\rangle$. Consequently, we have $\rho = (C_1, C_2, C_3)^\dagger \cdot (C_1, C_2, C_3)$, where the blocks $C_i$ are $R \times N$, $R$ is the rank of $\rho$ and $C_1$ has rank 1. After multiplying $(C_1, C_2, C_3)$ by a unitary matrix on the left-hand side and multiplying each $C_i$ by the same invertible matrix on the right-hand side, we may assume that

$$C_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} x & u \\ C_{21} & C_{22} \end{pmatrix}, \quad C_3 = \begin{pmatrix} y & v \\ C_{31} & C_{32} \end{pmatrix},$$

where the first blocks are $1 \times 1$. If $u \neq 0$ or $v \neq 0$, then $\rho$ is trivially distillable and so is NPT. Thus, we may assume that $u = v = 0$. By applying an ILO on $\mathcal{H}_A$, we may also assume that $x = y = 0$. After switching the parties $A$ and $B$, $\rho$ becomes reducible. Hence, if $\rho$ is NPT it is distillable by lemma 12, and otherwise it is separable by corollary 16.

The following proposition is the crucial step in the proof of our separability criterion.

**Proposition 21.** Let $\rho$ be a $3 \times 3$ state of rank 4 containing at least one product state in its range. Then $\rho$ is either distillable or separable. (In the former case it is NPT and in the latter PPT.) Equivalently, $\rho$ cannot be PPT and entangled.

**Proof.** If $\rho$ is reducible, then the assertion follows from lemma 12 and corollary 16. Hence, we may assume that $\rho$ is irreducible.

We can write $\rho$ as $\rho = \sum_{i=1}^3 |\psi_i\rangle \langle \psi_i|$, where $|\psi_1\rangle$ is a product state. This gives the factorization $\rho = (C_1, C_2, C_3)^\dagger \cdot (C_1, C_2, C_3)$ with the blocks $C_i$ of size $4 \times 3$. By applying an ILO, we may assume that

$$C_1 = \begin{pmatrix} 1 & 0 \\ C_{11} & C_{12} \end{pmatrix}, \quad C_2 = \begin{pmatrix} 0 & 0 \\ C_{21} & C_{22} \end{pmatrix}, \quad C_3 = \begin{pmatrix} 0 & 0 \\ C_{31} & C_{32} \end{pmatrix},$$

where the blocks $C_{ij}$ are of size $3 \times 2$. If the projected state $\rho' := (C_{12}, C_{22}, C_{32})^\dagger \cdot (C_{12}, C_{22}, C_{32})$ is entangled, then both $\rho'$ and $\rho$ are distillable. So, we may assume that $\rho'$ is
separable. Since a separable state of rank 3 is a sum of three product states (see proposition 3), we may assume that \( C_2 = D_i C_i \), \( i = 1, 2, 3 \), where \( C \) is a 3 \( \times \) 2 matrix and the \( D_i \) are diagonal matrices. If \( D_2 \) and \( D_3 \) are linearly dependent, then we can assume that one of them is 0 and so lemma 20 implies that \( \rho \) is distillable or separable. Thus, we may assume that \( D_2 \) and \( D_3 \) are linearly independent. They remain independent after removing one of the rows in each of them, say the first row. By using an ILO on system A, we may assume that the diagonal entries of \( D_1 \), \( D_2 \) and \( D_3 \) are \( (d_1, 0, 0) \), \( (d_2, 1, 0) \) and \( (d_3, 0, 1) \), respectively. The matrices \( C_i \) now have the form

\[
C_1 = \begin{pmatrix} 1 & 0 & 0 \\ u_1 & d_1 \alpha & d_1 \beta \\ u_2 & 0 & 0 \\ u_3 & 0 & 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 0 & 0 & 0 \\ v_1 & d_2 \alpha & d_2 \beta \\ v_2 & \gamma & \delta \\ v_3 & 0 & 0 \end{pmatrix}, \quad C_3 = \begin{pmatrix} 0 & 0 & 0 \\ w_1 & d_3 \alpha & d_3 \beta \\ w_2 & 0 & 0 \\ w_3 & \epsilon & \zeta \end{pmatrix}.
\]

If \( d_1 = 0 \) or \( \alpha = \beta = 0 \), we can again use lemma 20. Thus, by using an ILO on system B, we may assume that \( d_1 = \alpha = 1 \) and \( \beta = 0 \), as well as that \( u_1 = 0 \). One of \( \delta \) and \( \zeta \) must be nonzero, and so we may assume that, say, \( \delta = 1 \), as well as \( v_2 = \gamma = 0 \).

For convenience, denote by \( \rho_{ij} \) the projected state \((C_i, C_j)\) for \( i \leq j \leq 3 \). If \( u_2 \neq 0 \), then \( \rho_{12} \) is trivially distillable, and so we may assume that \( u_2 = 0 \). Now we have

\[
C_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ u_3 & 0 & 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 0 & 0 & 0 \\ v_1 & d_2 & 0 \\ 0 & 0 & 1 \\ v_3 & 0 & 0 \end{pmatrix}, \quad C_3 = \begin{pmatrix} 0 & 0 & 0 \\ w_1 & d_3 & 0 \\ w_2 & 0 & 0 \\ w_3 & \epsilon & \zeta \end{pmatrix}.
\]

By subtracting from \( C_2 \) and \( C_3 \) suitable multiples of \( C_1 \), we may assume that

\[
C_2 = \begin{pmatrix} -d_2 & 0 & 0 \\ v_1 & 0 & 0 \\ 0 & 0 & 1 \\ v_3' & 0 & 0 \end{pmatrix}, \quad C_3 = \begin{pmatrix} -d_1 & 0 & 0 \\ w_1 & 0 & 0 \\ w_2 & 0 & 0 \\ w_3' & \epsilon & \zeta \end{pmatrix}.
\]

If \( v_1 \neq 0 \), then \( \rho_{12} \) is trivially distillable, so we may assume that \( v_1 = 0 \).

Assume that \( \zeta = 0 \). Since \( \rho \) is irreducible, we must have \( u_2 \neq 0 \) and so \( \rho_{23} \) is trivially distillable. Assume now that \( \zeta \neq 0 \). If \( u_3 \neq 0 \), then \( \rho_{13} \) is trivially distillable, so we may assume that \( u_3 = 0 \). If \( w_1 \neq 0 \), then it is easy to see that the state \( \rho_{13} \) is distillable. Indeed, for that purpose we may assume that \( \epsilon = 0 \) and then \( \rho_{13} \) becomes trivially distillable. Thus, we may assume that also \( w_1 = 0 \), and so

\[
C_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} -d_2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ v_3' & 0 & 0 \end{pmatrix}, \quad C_3 = \begin{pmatrix} -d_1 & 0 & 0 \\ w_2 & 0 & 0 \\ w_3' & \epsilon & \zeta \end{pmatrix}.
\]

Since \( \rho \) is irreducible, we must have \( \epsilon \neq 0 \). If \( v_3' \neq 0 \), then \( \rho_{23} \) is trivially distillable. Thus, we assume that \( v_3' = 0 \). If \( u_2 = 0 \), then \( \rho \) is separable. Thus, we assume that \( u_2 \neq 0 \). Then, we examine the projected state \( \rho_{23} \). As \( \epsilon \neq 0 \), we can kill \( w_3' \) and \( \zeta \) in \( C_3 \) to obtain a trivially distillable state. This completes the proof.

Note that the above proof actually shows that the NPT states treated in this proposition are in fact 1-distillable.

We now present our separability criterion for bipartite states of rank 4.
Theorem 22. A bipartite state of rank 4 is separable if and only if it is PPT and its range contains at least one product state.

Proof. The conditions are obviously necessary. To prove the sufficiency, let $\rho$ be an $M \times N$ bipartite state of rank 4. By result (C), we must have $M, N \leq 4$. If $\max(M, N) = 4$, then proposition 3 shows that $\rho$ is separable. In view of the Peres–Horodecki criterion, it remains to consider the case $M = N = 3$. Hence, we can invoke proposition 21 to complete the proof. □

As an open question we ask whether proposition 21 can be extended to some other classes of PPT states, such as $2 \times 4$ PPT entangled states of rank 5. Such states do exist [30] (see also [2], equation (6)).

Hence, to decide whether a $3 \times 3$ PPT state $\rho$ of rank 4 is separable, one just has to check whether there is a product state in the range of $\rho$. This is numerically operational because the dimension is low. For example, one can refer to the discussion in [36], section 4.

Here, we present the analytical approach to this problem. We only need to consider the linear combination of four linearly independent $3 \times 3$ matrices $A = [a_{ij}]$, $B = [b_{ij}]$, $C = [c_{ij}]$, $D = [d_{ij}]$. The problem is to decide whether the variables $w, x, y, z$ can be chosen so that the matrix

$$E = wA + xB + yC + zD$$

has rank 1. This is equivalent to the requirement that each $2 \times 2$ minor of $E$ vanishes.

Mathematically, we have a set of nine equations

$$(waij + xbij + ycij + zdij)(wakl + xbkl + yckl + zdkl) - (wail + xbil + ycil + zdil)(wakj + xbkj + yckj + zdkj) = 0,$$

with $i < k$ and $j < l$. Thus, rank $E = 1$ if and only if these equations have a nonzero solution for $w, x, y, z$.

It is instructive to take a look at the following example which requires more algebraic background.

Example 23. Let us consider the simpler problem of deciding whether a two-dimensional subspace $V$ of a $2 \otimes 3$ system contains a product vector. Let $\{|a\rangle, |b\rangle\}$ be an arbitrary basis of $V$. Thus

$$|a\rangle = \sum_{i,j} a_{ij}|i, j\rangle, \quad |b\rangle = \sum_{i,j} b_{ij}|i, j\rangle,$$

where $i = 1, 2$ and $j = 1, 2, 3$. Let us form the $2 \times 6$ matrix from the components of these two states:

$$P = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{21} & a_{22} & a_{23} \\ b_{11} & b_{12} & b_{13} & b_{21} & b_{22} & b_{23} \end{pmatrix}.$$ 

Denote by $P^{ij}, i < j$, the $2 \times 2$ submatrix of $P$ made up from the $i$th and $j$th columns. The Plücker coordinates of $V$ are the 15 determinants $p_{ij} = \det P^{ij}$. If we change the basis, the Plücker coordinates of $V$ will be changed only by an overall factor. We point out that the Plücker coordinates are algebraically dependent. For instance, we have $p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} = 0$. A two-dimensional subspace $V$ can be viewed as a point of the so-called Grassmann variety (or Grassmannian) $G_{2,6}$. The set of the points $V$ which contain a product vector form a closed subvariety of this Grassmannian. By using the known facts about the incidence varieties as presented in [26], one can show that this subvariety is in fact a hypersurface, i.e. it is given
by just one algebraic equation in the Plücker coordinates. This equation can be computed explicitly; it is given by a homogeneous polynomial of degree 3:

$$2p_{12} p_{34} p_{56} + p_{12} p_{26} p_{46} + p_{13} p_{15} p_{56} + p_{23} p_{24} p_{46} + p_{13} p_{35} p_{45}$$

$$= p_{13} p_{25} p_{46} - p_{15} p_{24} p_{56} - p_{12} p_{35} p_{46} - p_{12} p_{16} p_{56} - p_{23} p_{34} p_{45} = 0.$$  

Thus, $V$ contains a product vector if and only if its Plücker coordinates satisfy this equation. The necessity can be easily verified by setting $a_{ij} = a_i a_j$, computing the $p_{ij}$ and then verifying that the above equation is identically satisfied. For sufficiency, due to the fact that the $p_{ij}$ are algebraically dependent, one must verify that the left-hand side of the equation is not identically zero. For instance, if we set $a_{11} = a_{22} = b_{12} = b_{23} = 1$ and all other $a_{ij}$ and $b_{ij}$ equal to 0, then the left-hand side of the equation is equal to $-1$.

A similar polynomial equation exists for three-dimensional subspaces $V$ of a $2 \otimes 4$ system. The polynomial is again homogeneous, but now has degree 4, and it is an integer linear combination of 149 monomials in the Plücker coordinates $p_{ijk}$, $1 \leq i < j < k \leq 8$. This equation is given in the appendix. In the case of four-dimensional subspaces of a $3 \otimes 3$ system again there is such an equation, but so far we were not able to compute it explicitly.

We note that proposition 21 is in agreement with a conjecture proposed in [41], where the authors ask whether all $3 \times 3$ PPT entangled states of rank 4 are equivalent via stochastic LOCC to the PPT entangled states arising from the unextendible product basis (UPB) [8], i.e. the states $V_A \otimes V_B (I_3 - \sum_{i=1}^3 |a_i, b_i\rangle\langle a_i, b_i|) V_A^\dagger \otimes V_B^\dagger$ where $V_A, V_B$ are invertible and $|a_i, b_i\rangle$ are the five members of a UPB. Indeed, proposition 21 shows that $R(\rho)$ contains no product state, and so provides evidence in support of the conjecture.

6. Some applications

In this section, we apply theorem 10 to study the quantum correlations inside tripartite pure states. We also discuss its meaning in terms of quantum discord [43]. We denote by $d_A, d_B, d_C$ the dimensions of the Hilbert spaces $\mathcal{H}_A, \mathcal{H}_B, \mathcal{H}_C$, respectively.

As a bipartite state is always a reduced density operator of a tripartite pure state, say $\rho = |\psi\rangle \langle \psi|$, we can use theorem 10 to characterize the quantum correlation by means of distillable entanglement. For example, we ask when are $\rho_{AB}$ and $\rho_{AC}$ simultaneously undistillable or bound entangled and which states $|\psi\rangle$ satisfy such a property. The question was first studied in 1999 by Thapliyal [48], who showed that an $N$-partite pure state $|\psi\rangle$ has fully $(N - 1)$-partite separable states if and only if $|\psi\rangle$ is a GHZ state up to local unitary. Based on theorem 10, we completely solve this problem for the tripartite system. We need a lemma to prove our theorem.

**Lemma 24.** For a tripartite pure state $\rho = |\psi\rangle \langle \psi|$, the bipartite reduced density operators $\rho_{AB}$ and $\rho_{AC}$ are PPT if and only if $|\psi\rangle = \sum_{i=1}^d |a_i\rangle |i\rangle$ up to local unitary operations.

**Proof.** The sufficiency is obvious, e.g. the reduced state $\rho_{AB} = \text{Tr}_C \rho = \sum_i |a_i\rangle \langle a_i |$ is separable. Let us show the necessity. Suppose $\rho_{AB}^F \geq 0$ and $\rho_{AC}^F \geq 0$. Recall that $d_B = \text{rank} \rho_{AC}$ and $d_C = \text{rank} \rho_{AB}$. If follows from result (C) that $d_B \geq \max(d_A, d_C)$ and $d_C \geq \max(d_A, d_B)$. So we have $d := d_B = d_C \geq d_A$. According to proposition 3, the states $\rho_{AB}$ and $\rho_{AC}$ are separable and $\rho_{AB} = \sum_{i=1}^d |a_i, b_i\rangle \langle a_i, b_i|$, where the states $|b_i\rangle$ span $\mathcal{H}_B$. So, $|\psi\rangle = \sum_{i=1}^d |a_i, b_i, i\rangle$ up to local unitary operations.

We choose a subset $\{ |g_1\rangle, \ldots, |g_m\rangle \}$ of $\{ |a_1\rangle, \ldots, |a_d\rangle \}$ such that (i) $|g_i\rangle$ and $|g_j\rangle$ are not parallel if $i \neq j$, and (ii) each $|a_i\rangle$ is parallel to some $|g_j\rangle$. For each $k \in \{1, \ldots, m\}$ let $S_k$ be
the set of all $i$ such that $|a_i\rangle$ is parallel to $|g_k\rangle$. Thus, the sets $S_1, \ldots, S_m$ form a partition of the set $\{1, \ldots, d\}$.

The state $\rho_{AC}$ is also separable and has rank $d$. It follows from proposition 3 again that $\rho_{AC} = \sum_{i=1}^d |e_i, f_i\rangle\langle e_i, f_i|$. Since the product states $|a_i, i\rangle$ span $\mathcal{R}(\rho_{AC})$, there is an invertible matrix $[c_{ij}]$ of order $d$ such that

$$|e_i, f_i\rangle = \sum_{j=1}^d c_{ij} |a_j, j\rangle.$$  

Since the $|j\rangle_C$ form an o.n. basis of $\mathcal{H}_C$, this equation implies that whenever coefficients $c_{ij}$ and $c_{il}$ are nonzero, $|a_j\rangle$ and $|a_l\rangle$ must be parallel. In other words, there is a $k$ such that $|e_i, f_i\rangle$ is a linear combination of $|g_k, j\rangle$ with $j \in S_k$. For each $k \in \{1, \ldots, m\}$, let $\mathcal{H}_k$ be the subspace of $\mathcal{H}_C$ spanned by the $|j\rangle$ with $j \in S_k$, and let $F_k$ be the set of all $i$ such that $|e_i, f_i\rangle$ is a linear combination of $|g_k, j\rangle$ with $j \in S_k$. Clearly, the sets $F_1, \ldots, F_m$ also form a partition of $\{1, \ldots, d\}$. It is easy to see that, for each $k$, the sets $F_k$ and $S_k$ have the same cardinality, say $d_k$.

By using the spectral decomposition, we can rewrite the state $\sigma_k = \sum_{i \in F_k} |e_i, f_i\rangle\langle e_i, f_i|$ as $\sigma_k = \sum_{i=1}^{d_k} |g_k, f_{k,i}\rangle\langle g_k, f_{k,i}|$, where the states $|g_k, f_{k,i}\rangle$, $1 \leq i \leq d_k$, form an orthogonal basis of $\mathcal{H}_k$. By taking the sum of all $\sigma_k$s, and renaming all product states $|g_k, f_{k,i}\rangle$ as $|e'_j, f'_j\rangle$, we obtain

$$\rho_{AC} = \sum_{j=1}^d |e'_j, f'_j\rangle\langle e'_j, f'_j|,$$

where the states $|f'_j\rangle$ are pairwise orthogonal. The purification of this state, namely $|\psi\rangle$, reads $|\psi\rangle = \sum_{i=1}^d |e'_i\rangle|ii\rangle$, up to local unitary operations. This proves the necessity. \hfill \square

**Theorem 25.** For a tripartite pure state $\rho = |\psi\rangle\langle\psi|$, the reduced states $\rho_{AB}$ and $\rho_{AC}$ are undistillable if and only if $|\psi\rangle = \sum_{i=1}^d |a_i\rangle|ii\rangle$ up to local unitary operations.

**Proof.** The sufficiency readily follows from the separability of $\rho_{AB}$ and $\rho_{AC}$. Let us show the necessity. As in the above proof, we have $d_B = d_C = d_A$. By theorem 10, the states $\rho_{AB}$ and $\rho_{AC}$ are PPT. Then, the assertion follows from lemma 24. \hfill \square

**Corollary 26.** For a tripartite state $|\psi\rangle$, all three bipartite reduced states are undistillable if and only if $|\psi\rangle$ is a generalized GHZ state, i.e. $|\psi\rangle = \sum_{i=1}^d a_i |ii\rangle$ up to local unitary operations.

Theorem 25 reveals a new constraint for the distillable entanglement. When a tripartite pure state has two undistillable reduced density operators, then they have to be separable. In other words, when a tripartite pure state has a bound entangled reduced state, such as a PPT entangled state, then the other two reduced states must be NPT and distillable. So, regardless of whether bound entanglement is PPT or NPT as conjectured in [21], it is not a generic property shared by two parties of a tripartite pure state. The question whether a mixed tripartite state may have two bound entangled reduced states is still an open problem.

It is interesting that there is no counterpart to corollary 26 for mixed states. For example, we consider the three-qubit PPT entangled state $\rho = I - \sum_{i=1}^d |\psi_i\rangle\langle\psi_i|$, where the normalized
and $|a\rangle, |a\perp\rangle, |b\rangle, |b\perp\rangle$ and $|c\rangle, |c\perp\rangle$ are orthonormal, respectively. All three bipartitions of this state, i.e. $\rho_{A:BC}, \rho_{B:AC}, \rho_{C:AB}$, are separable. Hence, all three bipartite reduced states are separable too. However, the state $\rho$ is PPT entangled and we cannot distill any (bipartite) pure entanglement for quantum information tasks. This is essentially different from corollary 26. From sufficiently many copies of the generalized GHZ state, one can asymptotically generate a standard GHZ state $|000\rangle + |111\rangle$ based on the BBPS protocol [4]. Therefore, besides the distillable entanglement, we need additional parameters to characterize the quantum correlation inside mixed tripartite states.

On the other hand, theorem 25 can be better understood via quantum discord [43], which measures the bipartite quantum correlation beyond entanglement. Indeed, quantum discord is larger than zero for many separable states, and it is equal to zero iff the separable state is diagonal in one of the systems, e.g. $\sum_i \rho_i \otimes |i\rangle\langle i|$. In this state any measurement on system A will lead to pairwise commuting Hermitian operators on system B, which is a basic property of classical mechanics. In this sense, system B is classical and it has no quantum features. Furthermore, the converse statement is also true. That is, the state has the form $\sum_i \rho_i \otimes |i\rangle\langle i|$ if system B is classical [13]. For simplicity, we say that the state $\rho_{AB}$ is classical when system B is classical. By using theorem 25 and the definition of quantum discord, we have

**Lemma 27.** For a tripartite pure state $|\psi\rangle$, the reduced states $\rho_{AB}$ and $\rho_{AC}$ are undistillable if and only if they have zero quantum discord. In particular, $|\psi\rangle$ is a generalized GHZ state if and only if all reduced states have zero quantum discord.

Lemma 27 shows that the collective undistillability of reduced states in a tripartite pure state will lead to the disappearance of quantum correlation among them. It is unknown whether this is also true for mixed tripartite states.

To grasp the meaning of this result more intuitively, we consider a practical case in which Alice and Bob are correlated in a state $\rho_{AB}$, and there is a third party, Charlie, from the environment. Suppose they are in a pure state $|\psi\rangle$ and share sufficiently many copies of $|\psi\rangle$.

First, when at most one system is classical, say Alice, we can distill at least two bipartite reduced states by theorem 25 and lemma 27. Second, when two systems are classical, say Bob and Charlie, then there may be either one or two distillable bipartite reduced states. The former tripartite state is just $|\psi\rangle = \sum_{i=1}^d |a_i\rangle|i\rangle$ of lemma 27, while the latter corresponds to the purification of a fully classical state of the form $\sum_{ij} a_{ij} |ij\rangle\langle ij|$, up to local unitary [13]. In other words, the state $\sum_{ij} \sqrt{a_{ij}} |\psi_{ij}\rangle |ij\rangle$ has classical systems $B, C$ in the same reduced state. Meanwhile, we have distillable $\rho_{AB}$ and $\rho_{AC}$ when $d_A > \max(d_B, d_C)$ in view of result (C). Finally, when all three systems are classical, the state $|\psi\rangle$ is a generalized GHZ state. One cannot distill entanglement between any two parties. This indicates that the less classical systems there are in a composite system, the more distillable reduced states there are. This provides a connection between quantum discord and distillable entanglement, where the former is viewed as a kind of quantum correlation rather than entanglement measure.

5 A connection between quantum discord and the entanglement of formation, which is another important entanglement measure, has been found by several authors, see, e.g., arXiv:1007.1814v1 [quant-ph] and arXiv:1006.4727v2 [quant-ph].
7. Distillation of some $3 \times 3$ states of rank 4

So far we have identified some distillable entangled states, e.g. theorem 10 shows that $M \times N$ NPT states of rank $N$ are 1-distillable. A natural question then arises: can we construct some more complex distillable states with specified dimension and rank? In this section, we focus on the simplest nontrivial case, namely $3 \times 3$ states of rank 4. In particular, we will show that the checkerboard states [23], which generalize the celebrated Bruß–Peres family, are distillable if and only if they are NPT. Some of the results are also extendible to higher dimensions.

We now examine the so-called checkerboard states, i.e. the states $\rho$ acting on a $3 \otimes 3$ system and defined by equation (3) with $M = N = 3$, $R = 4$, and the $|\psi_i\rangle$ given by

$$
|\psi_1\rangle = |1\rangle(a|1\rangle + d|3\rangle) + |2\rangle(c|2\rangle) + |3\rangle(b|1\rangle + e|3\rangle),
$$

$$
|\psi_2\rangle = |1\rangle(g|2\rangle) + |2\rangle(f|1\rangle + i|3\rangle) + |3\rangle(h|2\rangle),
$$

$$
|\psi_3\rangle = |1\rangle(j|1\rangle + m|3\rangle) + |2\rangle(l|2\rangle) + |3\rangle(k|1\rangle + n|3\rangle),
$$

$$
|\psi_4\rangle = |1\rangle(q|2\rangle) + |2\rangle(p|1\rangle + s|3\rangle) + |3\rangle(r|2\rangle).
$$

The parameters $a, b, \ldots, s$ are arbitrary complex numbers. Each pure state $|\psi_i\rangle$ is written above in the form $|\psi_i\rangle = \sum_j \langle i | \psi_j \rangle$. By using these $|\psi_j\rangle$ and equation (4) we have $\rho = (C_1, C_2, C_3)^\dagger \cdot (C_1, C_2, C_3)$, where the complex conjugates of the blocks $C_i$ are given by

$$
C_1^* = \begin{pmatrix}
a & 0 & d \\
0 & g & 0 \\
j & 0 & m
\end{pmatrix},
$$

$$
C_2^* = \begin{pmatrix}
0 & c & 0 \\
f & 0 & i \\
0 & l & 0
\end{pmatrix},
$$

$$
C_3^* = \begin{pmatrix}
b & 0 & e \\
0 & h & 0 \\
k & 0 & n
\end{pmatrix}.
$$

Generically, these states $\rho$ have rank $4$ and $R(\rho)$ contains no product state (see [23]) and, consequently, $\rho$ is entangled. One can find in the same paper two concrete examples of NPT checkerboard states $\rho$ which are distillable. In the next theorem, we prove that this feature is shared by all NPT checkerboard states.

**Theorem 28.** All NPT checkerboard states are distillable.

**Proof.** Let $\rho$ be an NPT checkerboard state. We have to show that $\rho$ is distillable. This is certainly the case if one of the local ranks of $\rho$ is less than 3. Hence, we may assume that $\rho$ is a $3 \times 3$ state. We divide the proof into two steps. First, we eliminate all but five parameters from the matrices $C_i^*$. Second, we analytically investigate the partial transpose of various $2 \times 2$ and $2 \times 3$ projected states of $\rho$. If any of them is entangled, then $\rho$ is distillable and so we can dismiss such cases.

Let us carry out the first step. By an argument similar to the one in the beginning of the proof of proposition 21, we can assume that $b = e = j = m = 0$. By using an ILO, we can also assume that $a = g = 1$ and $d = q = 0$. Thus, we have

$$
C_1^* = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix},
$$

$$
C_2^* = \begin{pmatrix}
0 & c & 0 \\
f & 0 & i \\
0 & l & 0
\end{pmatrix},
$$

$$
C_3^* = \begin{pmatrix}
0 & 0 & 0 \\
0 & h & 0 \\
k & 0 & n
\end{pmatrix}.
$$

If $i \neq 0$, then $\rho$ is trivially distillable, and so we assume that $i = 0$. If $s = 0$, then $\rho$ is distillable by corollary 19, so we may assume that $s \neq 0$. Similarly, we may assume that $n \neq 0$. By multiplying $C_2^*$ by $1/f$, we may assume that $f = 1$. By multiplying the third
columns of $C^*_i$ by 1/s, we may also assume that $s = 1$. By multiplying $C^*_s$ by 1/n, we may assume that $n = 1$. By using an ILO we can also assume that $p = 0$. We now have

$$
C^*_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C^*_2 = \begin{pmatrix} 0 & c & 0 \\ 1 & 0 & 0 \\ 0 & l & 0 \end{pmatrix}, \quad C^*_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & h & 0 \\ k & 0 & 1 \end{pmatrix}. 
$$

This completes the first step.

Now we carry out the second step. Let $\sigma = (V \otimes I) \rho (V^\dagger \otimes I)$, where $V^\dagger = (x|1 + |3\rangle) (|1 + |2\rangle |2\rangle$ and $x$ is a complex parameter. Assume that $\sigma$ is PPT for all $x$. After permuting simultaneously the rows and columns of $\sigma^\dagger$, we obtain a direct sum $A \oplus B$ of two $3 \times 3$ Hermitian matrices. Since $\sigma$ is PPT, we must have

$$
det A = |x|^2 (|c|^2 + |l|^2 - |r|^2) - |x + h^* - kr^*|^2 \geq 0, \\
det B = |x + h^*|^2 - |cx + lk^*|^2 + |r|^2 - |l|^2 \geq 0.
$$

By setting $x = 0$, the first inequality gives $h = rk^*$. After eliminating $h$, the above inequalities imply that $|c| = 1$ and $|l| = |r|$, and finally that $l = cr^*k/k^*$ if $k \neq 0$. By using these equalities, a computation shows that $\rho$ is PPT which is a contradiction. Hence, $\sigma$ must be NPT for some $x$, and so this particular $\sigma$ and $\rho$ are distillable. One can similarly and more easily handle the case $k = 0$. This completes the proof. \hfill \Box

8. More on full-rank properties

Theorem 8 gives a new theoretical tool, the full-rank criterion, for the detection of bipartite states distillable under LOCC. This criterion is similar to the well-known reduction criterion [28], for both criteria ensure the distillability of the states which violate them. It is easy to show that these two criteria are incomparable. First, the distillable $3 \times 3$ antisymmetric Werner state $\rho_{\text{W}}$ is detected by the full-rank criterion but not by reduction criterion. Second, there are 1-distillable states $\rho$ which can be detected by the reduction criterion but not by the full-rank criterion. An example is the 2 $\times$ 2 entangled state $\rho = |\psi_1\rangle |\psi_1\rangle + |\psi_2\rangle |\psi_2\rangle$, where $|\psi_1\rangle = \sqrt{p/2} (00 + |11\rangle)$ and $|\psi_2\rangle = \sqrt{(1-p)/2} (00 - |11\rangle)$ with $0 < p < 1$ and $p \neq 1/2$. It follows from proposition 30 below that all 2 $\times$ 2 states have both LFRP and RFRP. Third, some distillable states are detected by both criteria such as $M \times N$ states of rank $< N$, in particular the pure entangled states.

Since the states violating either criterion are distillable, they must be NPT. In other words, the PPT states satisfy both the reduction criterion and the full-rank criterion.

Next, we consider the states with RFRP. Let $\rho$ be an $M \times N$ state of rank $R$ written as in equation (2) where the $C_i$ are $R \times N$ matrices. The matrix of $\rho$ is the block matrix $(C_i | C_j)$, $i, j = 1, \ldots, M$, where each block is an $N \times N$ matrix. If rank($C_i$) $< N$, then the rank of the submatrix $(C_i | C_j | \ldots | C_j | M)$ is $< N$. Hence, if rank($C_i$) $< N$ for all $i$, then $R \leq M(N - 1)$. We conclude that if the rank of $\rho$ exceeds $M(N - 1)$, then $\rho$ must have RFRP (in fact one of the $C_i$s must have rank $N$).

On the other hand, if $R < N$, then $\rho$ violates RFRP. This follows from the fact that rank$(\langle x | \rho | x \rangle) \leq R$ for all $|x\rangle \in \mathcal{H}_A$. Moreover, the $N \times N$ states $\rho$ in the antisymmetric space violate RFRP and some of them have rank as large as $N(N - 1)/2$. To prove this, note that the states $|\pi_{ij}\rangle = |i\rangle - |j\rangle$, $1 \leq i < j \leq N$, form a basis of the antisymmetric space $\mathcal{A}$. We have $\rho = \sum_k |\psi_k\rangle |\psi_k\rangle$, where $|\psi_k\rangle \in \mathcal{A}$ are non-normalized states. Let $|x\rangle \in \mathcal{H}_A$ be a nonzero vector. For $i < j$, we have

$$
\langle \pi_{ij} | x, x \rangle = (i, j|x, x\rangle - (j, i|x, x\rangle = 0.
$$
Since each $|\psi_k\rangle$ is a linear combination of the $|\pi_{ij}\rangle$, it follows that $|x, x\rangle \in \ker \rho$. Consequently, $\rho$ violates RFRP.

For convenience, we collect the above results in a proposition.

**Proposition 29.** Let $R$ be the maximum rank of $M \times N$ states which violate RFRP. Then $R \leq M(N - 1)$ and, if $M = N$, $R \geq N(N - 1)/2$.

How can we verify whether $\rho$ has RFRP? To answer this question, let us write $\rho$ as in equation (2), where the $C_i$ are $R \times N$ matrices and $R = \text{rank}(\rho)$. For $|x\rangle_A = \sum_k \xi_k |k\rangle$, we have

$$\langle x | \rho | x \rangle = \left( \sum_k \xi_k C_k \right)^\dagger \left( \sum_k \xi_k C_k \right),$$

and so $\text{rank}(\langle x | \rho | x \rangle) = \text{rank}(\sum_k \xi_k C_k)$. For small $N$ the answer to our question can be obtained simply by computing all $N \times N$ minors of the matrix $\sum_k \xi_k C_k$.

In general, $\rho$ violates RFRP if and only if the space of $R \times N$ matrices spanned by $C_1, \ldots, C_M$ contains no matrix of rank $N$. This is certainly the case if $R < N$. The above problem is related to the still open problem of Edmonds [24], from theoretical computer science, which asks to decide whether a given linear subspace of complex $M \times N$ matrices contains a matrix of rank $M$.

A similar test is valid for the LFRP. For $|y\rangle_B = \sum_l \eta_l |l\rangle$, we have $\langle y | \rho | y \rangle = (\langle y | C_i^\dagger C_j | y \rangle)$, where the right-hand member is an $M \times M$ matrix with the indicated entries. It follows that the rank of $\langle y | \rho | y \rangle$ is equal to the rank of the $R \times M$ matrix $(C_1 |y\rangle, \ldots, C_M |y\rangle)$. Let $K_i$, $i = 1, \ldots, N$, be the $R \times M$ matrix defined as follows: the $j$th column of $K_i$ is the $i$th column of $C_j$. Then, $\rho$ violates LFRP if and only if the space of $R \times M$ matrices spanned by $K_1, \ldots, K_N$ contains no matrix of rank $M$.

It is not hard to construct examples of states which possess LFRP but violate RFRP. In the next proposition we consider the RFRP for $M \times 2$ states.

**Proposition 30.** All $M \times 2$ states have RFRP. Equivalently, all $2 \times N$ states have LFRP.

**Proof.** The equivalence is clear since we can interchange the parties A and B. We prove only the first assertion.

The proof is by contradiction. Let us assume that there exists an $M \times 2$ state $\rho$ which violates RFRP. Then the rank, $R$, of $\rho$ must be at least 2. We have

$$\rho = \sum_{i,j=1}^M |i\rangle \langle j| \otimes C_i^\dagger C_j,$$

where $C_i$ are $R \times 2$ matrices. As $\text{rank}(\rho_{ij}) = M$, $C_i \neq 0$ for each index $i$. Since $\rho$ violates RFRP, the matrix $\sum_j \xi_j C_i$ must have rank less than 2 for arbitrary choice of complex numbers $\xi_j$. It follows that each $C_i$ must have rank 1 and so $C_i = |v_i\rangle \langle \phi_i|$, where $|\phi_i\rangle = \alpha_i |1\rangle + \beta_i |2\rangle \in \mathcal{H}_B$ and $|v_i\rangle$ are nonzero vectors.

We have $C_i^\dagger C_j = \langle v_i | v_j \rangle |\phi_i\rangle |\phi_j\rangle$. Hence,

$$\rho_B = \sum_i C_i^\dagger C_i = \sum_i \|v_i\|^2 \left( |\alpha_i|^2 \alpha_i^\dagger \beta_i + |\beta_i|^2 \alpha_i \beta_i^\dagger \right).$$
By using the Lagrange identity, we find that
\[
\det(\rho_B) = \left( \sum_i |\alpha_i|^2 v_i \right) \left( \sum_j |\beta_j|^2 v_j \right) - \left( \sum_i |\alpha_i \beta_j^*|^2 v_i \right)^2 \\
= \sum_{i<j} \|v_i\|^2 \|v_j\|^2 |\alpha_i \beta_j - \alpha_j \beta_i|^2.
\]
Since all \(|v_i| \neq 0\) and \(\det(\rho_B) > 0\), there exists a pair of indexes \(i, j\) such that \(\alpha_i \beta_j - \alpha_j \beta_i \neq 0\). Without any loss of generality we may assume that \(i = 1\) and \(j = 2\). We know that the \(R \times 2\) matrix
\[
\begin{bmatrix}
\xi_1 C_1 + \xi_2 C_2 = \xi_1 |v_1| (|\phi_1| + |\phi_2|) + \xi_2 |v_2| (|\phi_1| + |\phi_2|) \\
= (\xi_1 \alpha_1^* |v_1| + \xi_2 \alpha_2^* |v_2|, \xi_1 \beta_1^* |v_1| + \xi_2 \beta_2^* |v_2|)
\end{bmatrix}
\]
has rank less than 2 for arbitrary complex numbers \(\xi_1\) and \(\xi_2\). Thus, all of its \(2 \times 2\) minors must vanish:
\[
0 = \begin{vmatrix}
\xi_1 \alpha_1^* v_{1,k} + \xi_2 \alpha_2^* v_{2,k} & \xi_1 \beta_1^* v_{1,k} + \xi_2 \beta_2^* v_{2,k} \\
\xi_1 \alpha_1^* v_{1,l} + \xi_2 \alpha_2^* v_{2,l} & \xi_1 \beta_1^* v_{1,l} + \xi_2 \beta_2^* v_{2,l}
\end{vmatrix},
\]
where \(1 \leq k < l \leq M\) and \(v_{i,k}\) denotes the \(k\)th component of \(|v_i|\). Since \(\xi_1\) and \(\xi_2\) are arbitrary and \(\alpha_1 \beta_2 - \alpha_2 \beta_1 \neq 0\), we have \(v_{1,k} v_{2,l} - v_{2,k} v_{1,l} = 0\) for all \(k < l\). Hence, the vectors \(|v_1|\) and \(|v_2|\) are linearly dependent. Without any loss of generality we may assume that \(|v_1| = |v_2|\).

Recall from equations (3) and (4) that we can write
\[
\rho = \sum_{i=1}^R |\psi_i\rangle \langle \psi_i|,
\]
where \(|\psi_i\rangle = \sum_j |j\rangle \otimes |\psi_{ij}\rangle\) and \(|\psi_{ij}\rangle\) is the \(i\)th row of \(C_j\), i.e.
\[
|\psi_{ij}\rangle = v_{ij}^\dagger (\alpha_j |1\rangle + \beta_j |2\rangle) = v_{ij}^\dagger |\phi_j\rangle.
\]
Since \(|v_1| = |v_2|\), we obtain that
\[
|\psi_{ij}\rangle = v_{ij}^\dagger (|1, \phi_1\rangle + |2, \phi_2\rangle) + \sum_{j=2}^R v_{ij}^\dagger |j, \phi_j\rangle, \quad i = 1, \ldots, R.
\]
We conclude that \(\mathcal{R}(\rho)\) is contained in the subspace spanned by the vectors \(|1, \phi_1\rangle + |2, \phi_2\rangle\) and \(|j, \phi_j\rangle\) for \(j = 3, \ldots, R\). This contradicts the hypothesis that \(\text{rank}(\rho) = R\), and completes the proof. \(\square\)

In particular, all \(2 \times 2\) states have LFRP and RFRP. While all \(2 \times N\) states have LFRP, even the simplest non-trivial case, i.e. a \(2 \times 3\) state, may violate RFRP. Indeed, one can easily verify that the \(2 \times 3\) state \(\rho = (|11\rangle + |22\rangle) (|11\rangle + |22\rangle) + |23\rangle |23\rangle + |13\rangle |13\rangle\) violates RFRP.

Hence, for any \(2 \times N\) state \(\rho\) of rank \(R \leq N\) there exists a state \(|\chi\rangle \in \mathcal{H}_A\) such that \(\text{rank}(\langle \chi | \rho | \chi \rangle < N\). However, this is not true for \(R > N\).

Finally, we have seen that the antisymmetric state \(\rho_{as}\) given by equation (5) violates RFRP. However, as one can easily verify, its two-copy version \(\rho_{as} \otimes \rho_{as}\) has RFRP.

9. Conclusions

Following [36], we have introduced two full-rank properties, LFRP and RFRP, and proved that the bipartite state violating at least one of them is 1-distillable. We refer to this result as
the full-rank criterion for distillability. By using it, we obtained our first main result which asserts that the $M \times N$ NPT states of rank $N$ are 1-distillable. In particular, this result provides the affirmative solution to an open problem first proposed in 1999. This result leads to a new characterization of the distillable entanglement; namely, a tripartite pure state $|\psi\rangle$ cannot have two undistillable entangled bipartite reduced states. We also derived an explicit expression for tripartite pure states having two undistillable bipartite reduced states. Both of these states turn out to be separable. On the other hand, we define reducible and irreducible states and use them to distill some entangled states possessing both LFRP and RFRP. The most important result in this direction is that the checkerboard states [23] are distillable under LOCC if and only if they are NPT.

We now list the most interesting results proved in this paper. We just continue the enumeration of results (A–E) from the introduction.

(F) Bipartite $M \times N$ NPT states of rank $\max(M, N)$ are 1-distillable.
(G) A bipartite rank-4 state is separable if and only if it is PPT and its range contains at least one product state.
(H) If a bipartite state violates at least one of the two full-rank conditions, then it is 1-distillable.
(I) For a tripartite pure state $|\psi\rangle$, all three bipartite reduced states are undistillable if and only if $|\psi\rangle$ is a generalized GHZ state.
(J) The NPT checkerboard states (acting on a $3 \otimes 3$ system) are 1-distillable.

For convenience, let us summarize what is known and what remains to be done to finish off the coarse classification of rank-4 states $\rho$ in a $3 \otimes 3$ system. There are two cases:

1. $R(\rho)$ contains a product state;
2. $R(\rho)$ contains no product state.

In case (1), we know that if $\rho$ is PPT, then it is separable, and otherwise it is 1-distillable. In case (2), $\rho$ is always entangled and may be PPT or NPT. For the PPT subcase, there is a very precise conjecture [41] describing how these PPT entangled states can be generated from UPBs. For the NPT subcase, we conjecture that these states are distillable, and we know that this is true for the checkerboard states (see theorem 2).

There are many related open problems and conjectures for further study that originate from this paper. First, we have seen from proposition 18 that an irreducible bipartite state $\rho$, for which there exists a nonzero vector $|x\rangle$ such that $C_i|x\rangle = 0$ for all but one of the indexes $i$, is distillable. (Note that if $C_i|x\rangle = 0$, then $|i, x\rangle \in \ker(\rho_i)$.) Thus, it is natural to ask whether we can relax the above condition on the kernels of the $C_i$s. For example, one can conjecture that any irreducible $M \times N$ NPT state $\rho$, with one of the $C_i$ of deficient rank, is distillable? Since Werner states have full rank, all of its blocks $C_i$ also have full rank, and so they do not contradict the conjecture. Because of the rank condition on the $C_i$, the question evidently depends on the analysis of product states in the kernel of $\rho$. For example, it is a well-known fact that in an $M \otimes N$ system every subspace $V \subseteq \mathcal{H}_A \otimes \mathcal{H}_B$ with $\dim V > (M-1)(N-1)$ contains a product vector. To prove this, we can just apply [26], proposition 11.4, to the Segre variety consisting of all product vectors.

If the above conjecture turns out to be true, it would follow that every bipartite $M \times N$ (for $M \leq N$) NPT state $\rho$ of rank 4 is distillable. Indeed, if $M < 3$, then $\rho$ is distillable by result (A). If $N > 4$, then $\rho$ is distillable by result (C). Thus, we may assume that $M = N = 3$. As $\ker(\rho)$ has dimension 5, it contains a product state. By the conjecture, $\rho$ is distillable.

To state the second conjecture, let $\rho$ be a bipartite NPT state acting on some $M \otimes N$ system. We can view $\rho^\otimes_k$ as a bipartite state acting on a $M^k \otimes N^k$ system. Then, the conjecture claims that for some $k \geq 1$, the state $\rho^\otimes_k$ can be locally transformed into an $m \times n$ NPT state of rank $\max(m, n) + 1$. 23
If this conjecture is true, then we can project some tensor power of an entangled 1-undistillable Werner state \( \rho_w \) onto an \( m \times n \) \( (m \leq n) \) NPT state \( \sigma \) of rank \( n + 1 \). Because it is widely believed that \( \rho_w \) is undistillable, we can thus conjecture that there exists an undistillable \( M \times N \) NPT state of rank \( N + 1 \). This would then imply that the value \( N \) for the rank of the states in theorem 10 is maximal.

The third question is about the irreducible states. From lemma 13 we know that the distillation problem relies on further investigation of irreducible states. For example we ask: can the tensor product of two irreducible states be a reducible state? The existence of such a phenomenon would become a sort of activation of reducibility, which is akin to the activation of PPT bound entanglement [33].

We also have seen that all \( M \times N \) NPT states of rank \( N \) are 1-distillable. In fact, researchers have previously shown that such states cannot be PPT entangled. However, it is also well known that there are PPT entangled \( M \times N \) states with rank bigger than \( N \), and meanwhile, there exist 1-undistillable but \( n \)-distillable NPT states such as the Watrous state [51]. Is this an essential difference between the above two families of states? Does the existence of PPT entanglement imply that of 1-undistillable NPT states?

Another interesting problem is to decide which PPT states are entangled. For example, what is the relationship between the existence of product states in the range and entanglement of PPT states? Can we conjecture that if there are more known product states in the range of a PPT state, then it becomes easier to decide whether it is entangled?

Acknowledgments

We thank the two anonymous referees for their valuable suggestions. Consequently, the paper has been thoroughly reorganized and considerably improved. We thank Andreas Winter for pointing out that the 3 \( \times \) 3 antisymmetric state violates RFRP. We also thank L Skowronek for his email [47] which led to a minor correction in the original version of the paper. Part of this work was done when LC was visiting Professor Heng Fan at the Institute of Physics, CAS, Beijing, China, and Professor Runyao Duan at the University of Technology, Sydney, Australia. The CQT is funded by the Singapore MoE and the NRF as part of the Research Centres of Excellence programme. The second author was supported in part by an NSERC Discovery Grant.

Appendix

We give here the equation which is necessary and sufficient for a three-dimensional subspace \( V \) of a \( 2 \otimes 4 \) system to contain at least one product state. Let \( \{|a\rangle, |b\rangle, |c\rangle\} \) be an arbitrary basis of \( V \). Thus,

\[
|a\rangle = \sum_{i=1}^{2} \sum_{j=1}^{4} a_{ij} |i, j\rangle ,
\]

and similarly for \( |b\rangle \) and \( |c\rangle \). Let us form the \( 3 \times 8 \) matrix from the components of these three states:

\[
P = \begin{pmatrix}
a_{i1} & a_{i2} & a_{i3} & a_{i4} & a_{21} & a_{22} & a_{23} & a_{24} \\
b_{i1} & b_{i2} & b_{i3} & b_{i4} & b_{21} & b_{22} & b_{23} & b_{24} \\
c_{i1} & c_{i2} & c_{i3} & c_{i4} & c_{21} & c_{22} & c_{23} & c_{24}
\end{pmatrix} ,
\]

Denote by \( p_{ijk} \), \( 1 \leq i < j < k \leq 8 \), the \( 3 \times 3 \) submatrix of \( P \) made up from the \( i \)th, \( j \)th and \( k \)th columns. The Plücker coordinates of \( V \) are the 56 determinants \( p_{ijk} = \det p_{ijk} \). If we
change the basis, the Plücker coordinates of \( V \) will be changed only by an overall factor. The equation we alluded to is the following:

\[
p_{123} F_1 + p_{124} F_2 + p_{134} F_3 + p_{234} F_4 = 0,
\]

where

\[
F_1 = 3p_{124} p_{578} p_{678} - 3p_{125} p_{478} p_{678} + p_{126} p_{478} p_{578} + 2p_{127} p_{458} p_{678} - p_{127} p_{468} p_{578}
+ p_{128} p_{178} p_{678} - p_{128} p_{378} p_{578} - 2p_{128} p_{158} p_{678} + p_{128} p_{368} p_{578} - 3p_{134} p_{568} p_{678}
+ 3p_{135} p_{468} p_{678} - p_{136} p_{458} p_{678} - p_{136} p_{468} p_{578} + p_{137} p_{468} p_{568} - p_{138} p_{168} p_{678}
+ 3p_{138} p_{258} p_{678} + p_{138} p_{268} p_{578} - p_{138} p_{368} p_{568} + 2p_{145} p_{278} p_{678} - p_{145} p_{368} p_{578}
- 2p_{145} p_{367} p_{678} - 2p_{146} p_{178} p_{678} + 2p_{146} p_{278} p_{578} + 4p_{146} p_{358} p_{678} - 2p_{146} p_{368} p_{578}
- 2p_{146} p_{357} p_{678} + 2p_{146} p_{467} p_{578} + p_{147} p_{168} p_{678} - p_{147} p_{258} p_{678} - p_{147} p_{268} p_{578}
+ p_{147} p_{368} p_{568} - p_{147} p_{467} p_{568} + 3p_{148} p_{368} p_{578} - 2p_{148} p_{458} p_{678} - 2p_{148} p_{468} p_{578}
+ 2p_{148} p_{248} p_{578} - 2p_{148} p_{258} p_{568} - 2p_{148} p_{258} p_{578} + 3p_{148} p_{358} p_{568} - 2p_{148} p_{258} p_{578}
- 2p_{148} p_{358} p_{678} + 4p_{148} p_{368} p_{578} + 3p_{148} p_{458} p_{678} - 2p_{148} p_{468} p_{578} - p_{148} p_{348} p_{678}
+ p_{245} p_{258} p_{578} + p_{245} p_{258} p_{568} - p_{245} p_{345} p_{678} + p_{246} p_{358} p_{568} + p_{246} p_{356} p_{568} + p_{246} p_{456} p_{578} - p_{246} p_{457} p_{568},
\]

\[
F_2 = p_{125} p_{578} p_{678} - p_{127} p_{178} p_{678} + p_{127} p_{278} p_{578} + 2p_{127} p_{358} p_{678} - p_{127} p_{368} p_{578}
- 2p_{127} p_{357} p_{678} + p_{127} p_{467} p_{578} + 3p_{135} p_{567} p_{678} - p_{135} p_{278} p_{678} - p_{135} p_{368} p_{578}
- 2p_{135} p_{467} p_{578} + p_{136} p_{178} p_{678} - p_{136} p_{278} p_{578} - 2p_{136} p_{358} p_{678} + p_{136} p_{368} p_{578}
+ 4p_{136} p_{458} p_{578} - p_{137} p_{168} p_{678} - p_{137} p_{258} p_{678} - p_{137} p_{258} p_{578} - p_{137} p_{268} p_{578}
+ p_{137} p_{368} p_{568} - p_{137} p_{467} p_{678} + 2p_{145} p_{358} p_{678} - p_{146} p_{357} p_{678} - p_{147} p_{356} p_{678}
+ p_{147} p_{256} p_{678} + p_{147} p_{267} p_{578} - p_{147} p_{367} p_{568} + p_{147} p_{467} p_{568} - 3p_{234} p_{457} p_{567},
\]

\[
F_3 = p_{135} p_{268} p_{678} + p_{136} p_{168} p_{678} - p_{136} p_{258} p_{678} + p_{136} p_{268} p_{578} - p_{136} p_{368} p_{568}
- 2p_{136} p_{456} p_{578} + p_{136} p_{456} p_{568} - p_{145} p_{267} p_{578} + p_{146} p_{167} p_{678} - p_{146} p_{257} p_{678}
- p_{146} p_{267} p_{578} + p_{146} p_{356} p_{568} + p_{146} p_{367} p_{678} - p_{146} p_{456} p_{578}
- p_{235} p_{268} p_{578} + p_{235} p_{368} p_{568} - 2p_{235} p_{456} p_{578} - p_{235} p_{467} p_{568} - p_{236} p_{268} p_{578}
+ p_{236} p_{358} p_{568} + 2p_{236} p_{456} p_{578} - p_{236} p_{457} p_{568} + 2p_{236} p_{457} p_{578}
- p_{245} p_{356} p_{568} + p_{245} p_{356} p_{578} + p_{245} p_{357} p_{578} + p_{245} p_{357} p_{678} - p_{246} p_{356} p_{678}
- p_{246} p_{356} p_{678} + p_{246} p_{357} p_{678} - 2p_{246} p_{457} p_{678} - p_{246} p_{458} p_{578} + p_{246} p_{458} p_{568} - p_{246} p_{458} p_{568},
\]

\[
F_4 = p_{235} p_{258} p_{578} - p_{235} p_{358} p_{568} - 2p_{235} p_{456} p_{578} + 2p_{235} p_{457} p_{568} - p_{245} p_{257} p_{578}
+ p_{245} p_{356} p_{578} - p_{245} p_{457} p_{568} - p_{245} p_{457} p_{578} - p_{245} p_{456} p_{578} + p_{245} p_{456} p_{568},
\]

Although this equation looks complicated, it can be easily programmed on a computer and used to decide whether \( V \) contains a product state. The main point is that we do not need to solve numerically any algebraic equations.
References

[1] Amselem E and Bourennane M 2009 Nat. Phys. 5 748
[2] Augusiak R, Grabowski J, Kus M and Lewenstein M 2009 arXiv:0907.4979 [quant-ph]
[3] Bandyopadhyay S and Roychowdhury V 2003 Phys. Rev. A 68 022319
[4] Bennett C H, Bernstein H J, Popescu S and Schumacher B 1996 Phys. Rev. A 53 2046
[5] Bennett C H, Brassard G, Crepeau C, Jozsa R, Peres A and Wootters W 1993 Phys. Rev. Lett. 70 1895
[6] Bennett C H, Brassard G, Popescu S, Schumacher B, Smolin J and Wootters W 1996 Phys. Rev. Lett. 76 722
[7] Brandao F, Christandl M and Yard J 2010 arXiv:1010.1750 [quant-ph]
[8] Bennett C H, DiVincenzo D P, Mor T, Shor P W, Smolin J A and Terhal B M 1999 Phys. Rev. Lett. 82 5385
[9] Bennett C H, DiVincenzo D, Smolin J and Wootters W 1996 Phys. Rev. A 54 3824
[10] Bravyi S 2004 Quantum Inf. Process. 3 309
[11] Briegel H J and Raussendorf R 2001 Phys. Rev. Lett. 86 910
[12] Bruß D and Peres A 2000 Phys. Rev. A 61 030301
[13] Chen L, Chitambar E, Modi K and Vacanti G 2010 Phys. Rev. A 83 020101
[14] Chen L and Chen Y-X 2008 Phys. Rev. A 78 022318
[15] Clarisse L 2005 Phys. Rev. A 71 032332
[16] Coffman V, Kundu J and Wootters W K 2000 Phys. Rev. A 61 052306
[17] Christandl M, Schuch N and Winter A 2010 Phys. Rev. Lett. 104 240405
[18] Christandl M and Winter A 2004 J. Math. Phys. 45 829
[19] Dur W, Cirac J I, Lewenstein M and Bruß D 2000 Phys. Rev. A 61 062313
[20] Dobek K, Karpinski M, Demkowicz-Dobrzanski R, Banaszek K and Horodecki P 2011 Phys. Rev. Lett. 106 030501
[21] DiVincenzo D P, Shor P W, Smolin J A, Terhal B M and Thapliyal A V 2000 Phys. Rev. A 61 062312
[22] Dur W, Vidal G and Cirac J I 2000 Phys. Rev. A 62 062314
[23] Edmonds J 1967 1967 J. Res. Natl Bur. Stand. B 71 241
[24] Gurvits L 2003 Proc. 35th ACM Symp. on Theory of Computing (New York: ACM Press) p 10
[25] Harrow A and Montanaro A 2010 arXiv:1001.0017 [quant-ph]
[26] Horodecki M and Horodecki P 1999 Phys. Rev. A 59 4206
[27] Horodecki M, Horodecki P and Horodecki R 1996 Phys. Lett. A 223 1
[28] Horodecki P 1997 Phys. Lett. A 232 333
[29] Horodecki M, Horodecki P and Horodecki R 1997 Phys. Rev. Lett. 78 574
[30] Horodecki P, Horodecki M and Horodecki R 1998 Phys. Rev. Lett. 80 5239
[31] Horodecki P, Horodecki M and Horodecki R 1999 Phys. Rev. Lett. 82 1056
[32] Horodecki P, Horodecki M and Horodecki R 1998 Acta Phys. Slovaca 48 141
[33] Horodecki P, Horodecki R and Horodecki M 1998 arXiv:quant-ph/9805072
[34] Horodecki P, Horodecki K, Horodecki M, Horodecki P and Oppenheim J 2002 Phys. Rev. Lett. 89 160502
[35] Horodecki P, Lewenstein M, Vidal G and Cirac J I 2000 Phys. Rev. A 62 032310
[36] Horodecki P, Smolin J A, Terhal B M and Thapliyal A V 2003 Theor. Comput. Sci. 292 589
[37] Horodecki P, Smolin J A, Terhal B M and Thapliyal A V 1999 arXiv:quant-ph/9910122
[38] Horodecki R, Horodecki P, Horodecki M and Horodecki K 2009 Rev. Mod. Phys. 81 865
[39] Kraus B, Lewenstein M and Cirac J I 2002 Phys. Rev. A 65 042327
[40] Lewenstein M, Kraus B, Cirac J I and Horodecki P 2000 Phys. Rev. A 62 052310
[41] Leinaas J M, Myrheim J and Sollid P 2010 Phys. Rev. A 81 062330
[42] Masanes L 2006 Phys. Rev. Lett. 96 150501
[43] Oliulver I and Zurek W H 2002 Phys. Rev. Lett. 88 017901
[44] Pankowski L, Piani M, Horodecki M and Horodecki P 2010 IEEE Trans. Inf. Theory 56 4085
[45] Peres A 1996 Phys. Rev. Lett. 77 1413
[46] Siomau M and Fritzsche S 2011 arXiv:1101.3275 [quant-ph]
[47] Toth G and Guhne O 2009 Phys. Rev. Lett. 102 170503
[48] Vianna R O and Doherty A C 2006 Phys. Rev. A 74 052306
[49] Watrous J 2004 Phys. Rev. Lett. 92 010502
[50] Werner R F 1989 Phys. Rev. A 40 4277