Fusion algebra of critical percolation

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Abstract. We present an explicit conjecture for the chiral fusion algebra of critical percolation considering Virasoro representations with no enlarged or extended symmetry algebra. The representations that we take to generate fusion are countably infinite in number. The ensuing fusion rules are quasirational in the sense that the fusion of a finite number of these representations decomposes into a finite direct sum of these representations. The fusion rules are commutative, associative and exhibit an $s\ell(2)$ structure. They involve representations which we call Kac representations of which some are reducible yet indecomposable representations of rank 1. In particular, the identity of the fusion algebra is a reducible yet indecomposable Kac representation of rank 1. We make detailed comparisons of our fusion rules with the recent results of Eberle–Flohr and Read–Saleur. Notably, in agreement with Eberle–Flohr, we find the appearance of indecomposable representations of rank 3. Our fusion rules are supported by extensive numerical studies of an integrable lattice model of critical percolation. Details of our lattice findings and numerical results will be presented elsewhere.

Keywords: conformal field theory, percolation problems (theory)

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1. Introduction

Percolation [1]–[3] has its origins in the paper [4] by Broadbent and Hammersley from 1957. Despite its relatively simple description, the subtleties and richness of percolation continue to hold much interest and even surprises after 50 years. One exciting recent development is the demonstration [5,6] that the continuum scaling limit of percolation on the lattice yields a conformally invariant measure in the plane with connections to stochastic Loewner evolution [7]–[12]. This is achieved by considering discrete analytic functions on the lattice. Another intriguing development is the unexpected connection [13]–[16] between the ground state of percolation, viewed as a stochastic process, and fully packed loop refinements of enumerations of symmetry classes of alternating sign matrices.

Percolation as a conformal field theory (CFT) has some novel aspects being a non-rational and non-unitary theory with a countably infinite number of scaling fields. Most importantly, as argued in [17]–[19] for example, it is a logarithmic CFT with the consequence that it admits indecomposable representations of the Virasoro algebra [20]. The first systematic study of logarithmic CFT appeared in [21]. Logarithmic CFTs are currently the subject of intensive investigation, see [22]–[39] and references therein. There is of course a long history of studying percolation as the continuum scaling limit of lattice models [40]–[42]. Here, however, it is convenient to regard critical percolation as a member of the family $\mathcal{LM}(p,p')$ of logarithmic CFTs defined as the continuum scaling limit of integrable lattice models [43]. The first two members $\mathcal{LM}(1,2)$ and $\mathcal{LM}(2,3)$ correspond to critical dense polymers and critical percolation (bond percolation on the square lattice), respectively. This solvable model of critical dense polymers was considered in [44].

In this paper, we are interested in the fusion algebra of $\mathcal{LM}(2,3)$ and we present an explicit conjecture for the fusion rules generated from two fundamental representations, here denoted $(2,1)$ and $(1,2)$. The identity of this fundamental fusion algebra is denoted $\mathcal{LM}(2,3)$. The fundamental fusion algebra is denoted

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(1,1) and is a reducible yet indecomposable representation of rank 1. Our fusion rules are supported by extensive numerical studies of our integrable lattice model of critical percolation. Details of our lattice findings and numerical results will be presented elsewhere.

It appears natural to suspect that the so-called augmented $c_{p,p'}$ models [45] are equivalent to our logarithmic minimal models $LM(p, p')$. In particular, we believe that the augmented $c_{2,3}$ model is equivalent to critical percolation $LM(2, 3)$. Much is known [46] about the fusion algebras of the augmented $c_{p,p'}$ models with $p = 1$ while much less is known about the fusion algebras of these models for $p > 1$. For critical percolation, the most complete information on fusion comes from Eberle and Flohr [45] who systematically applied the Nahm algorithm [46,47] to obtain fusions level by level. A careful comparison shows that our fusion rules are compatible with their results [45]. In particular, we confirm their observation of indecomposable representations of rank 3. We also make a detailed comparison of our fusion rules with the results of [48] which we find correspond to a subalgebra of our fusion algebra of critical percolation.

1.1. Kac representations

Critical percolation $LM(2, 3)$ has central charge $c = 0$ and conformal weights

$$\Delta_{r,s} = \frac{(3r - 2s)^2 - 1}{24}, \quad r, s \in \mathbb{N}. \tag{1.1}$$

The set of distinct conformal weights is $\{\Delta_{k,1}, \Delta_{k+1,2}, \Delta_{k+1,3}; k \in \mathbb{N}\} = \{\Delta_{1,k+1}, \Delta_{2,k+2}; k \in \mathbb{N}\}$.

From the lattice, a Kac representation $(r, s)$ arises for every pair of integer Kac labels $r, s$ in the first quadrant of the infinitely extended Kac table, see figure 1. This relaxes the constraint $r = 1, 2$ considered in [43]. The lattice description of the full set of Kac representations will be discussed in detail elsewhere. The conformal character of the Kac representation $(r, s)$ is given by

$$\chi_{r,s}(q) = \frac{q^{1/24+\Delta_{r,s}}}{\eta(q)} (1 - q^{rs}) \tag{1.2}$$

where the Dedekind eta function is defined by

$$\eta(q) = q^{1/24} \prod_{m=1}^{\infty} (1 - q^m). \tag{1.3}$$

We will denote the character of the irreducible Virasoro representation of conformal weight $\Delta_{r,s}$ by $\text{ch}_{r,s}(q)$. These irreducible characters [49] read

$$\text{ch}_{2k-1,a}(q) = K_{12,6k-3-2a;k}(q) - K_{12,6k-3+2a;k}(q), \quad a = 1, 2$$

$$\text{ch}_{2k+1,3}(q) = \frac{1}{\eta(q)} (q^{3(2k-1)^2/8} - q^{3(2k+1)^2/8})$$

$$\text{ch}_{2k,b}(q) = \frac{1}{\eta(q)} (q^{(3k-b)^2/6} - q^{(3k+b)^2/6}), \quad b = 1, 2, 3 \tag{1.4}$$
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Figure 1. Extended Kac table of critical percolation $\mathcal{LM}(2,3)$ showing the conformal weights $\Delta_{r,s}$ of the Kac representations $(r,s)$. Except for the identifications $(2k,3k') = (2k',3k)$, the entries relate to distinct Kac representations even if the conformal weights coincide. This is unlike the irreducible representations which are uniquely characterized by their conformal weight. The periodicity of conformal weights $\Delta_{r,s} = \Delta_{r+2s+3}$ is made manifest by shading the rows and columns with $r \equiv 0 \pmod{2}$ or $s \equiv 0 \pmod{3}$. The Kac representations which happen to be irreducible representations are marked with a red shaded quadrant in the top right corner. These do not exhaust the distinct values of the conformal weights. For example, the irreducible representation with $\Delta_{1,1} = 0$ does not arise as a Kac representation. By contrast, the Kac table of the associated rational (minimal) model consisting of the shaded $1 \times 2$ grid in the lower left corner is trivial and contains only the operator corresponding to the irreducible representation with $\Delta = 0$.

where $k \in \mathbb{N}$ while $K_{n,\nu,k}(q)$ is defined as

$$K_{n,\nu,k}(q) = \frac{1}{\eta(q)} \sum_{j \in \mathbb{Z} \setminus \{1, \ldots, k-1\}} q^{(nj-\nu)^2/2n}. \quad (1.5)$$

It follows that for $k = 1$, the first expression in (1.4) reduces to the well-known irreducible character

$$\text{ch}_{1,a}(q) = \frac{1}{\eta(q)} \sum_{j \in \mathbb{Z}} (q^{(12j-1)^2/24} - q^{(12j+5)^2/24}) = 1, \quad a = 1, 2. \quad (1.6)$$

A priori, a Kac representation can be either irreducible or reducible. In the latter case, it could be fully reducible (in which case it would be a direct sum of irreducible representations) or its direct-sum decomposition could involve at least one reducible but indecomposable representation of rank 1 (possibly in addition to some reducible but indecomposable representation of rank 1).
irreducible representations. We will only characterize the Kac representations appearing in the fusion algebras to be discussed in the present work. Among these are the Kac representations \( \{ (2k, 1), (2k, 2), (2k, 3), (1, 3k), (2, 3k); k \in \mathbb{N} \} \). Since their characters all correspond to irreducible Virasoro characters, these Kac representations must themselves be irreducible. They constitute an exhaustive list of irreducible Kac representations. Two Kac representations are naturally identified if they have identical conformal weights and are both irreducible. The relations

\[
(2k, 3) = (2, 3k)
\]

are the only such identifications. More general relations are considered in (2.13) and (2.14). Here we merely point out that two Kac characters (1.2) are equal \( \chi_{r,s}(q) = \chi_{r',s'}(q) \) if and only if \( (r', s') = (r, s) \) or \( (r', s') = (2s/3, 3r/2) \). That is, the only equalities between Kac characters are of the form \( \chi_{2k, 3k'}(q) = \chi_{2k', 3k}(q) \). According to (2.14), a similar equality applies to the Kac representations themselves: \( (2k, 3k') = (2k', 3k) \).

The only reducible Kac representations entering the fundamental fusion algebra to be discussed below are \( (1, 1) \) and \( (1, 2) \) and they are both indecomposable representations of rank 1, cf section 2.6. The indecomposable representations of higher rank appearing in the fusion algebra may be described in terms of Kac representations and their characters. We therefore list the decompositions of the relevant Kac characters in terms of irreducible characters

\[
\begin{align*}
\chi_{2k-1,b}(q) &= \text{ch}_{2k-1,b}(q) + (1 - \delta_{b,3}\delta_{k,1})\text{ch}_{2k+1,b}(q), & b = 1, 2, 3 \\
\chi_{a,3k-6}(q) &= \text{ch}_{a,3k-6}(q) + (1 - \delta_{a,2}\delta_{k,1})\text{ch}_{a,3k+b}(q), & a, b = 1, 2 \\
\chi_{3,3k+b}(q) &= \text{ch}_{1,3k-3+b}(q) + \text{ch}_{1,3k+b}(q) + \text{ch}_{1,3k+3-b}(q) + \text{ch}_{1,3k+6-b}(q) + \text{ch}_{1,3k+9-b}(q), & b = 1, 2
\end{align*}
\]

where \( k \in \mathbb{N} \). The decomposition in the general case is discussed in the appendix of [43].

2. Fusion algebras

The fundamental fusion algebra \( \langle (2, 1), (1, 2) \rangle \) is defined as the fusion algebra generated by the fundamental representations \( (2, 1) \) and \( (1, 2) \). We find that closure of this fusion algebra requires the inclusion of a variety of other representations

\[
\langle (2, 1), (1, 2) \rangle = \langle (1, 1), (1, 2), (2k, a), (1, 3k), (2k, 3), R_{2k,a}^{1,0}, R_{2k,3}^{1,0}, R_{a,3k}^{0,b}, R_{2k,3}^{1,b}; \\
& a, b = 1, 2; k \in \mathbb{N} \rangle
\]

(2.1)
to be discussed next.

2.1. Indecomposable representations of rank 2 or 3

For \( k \in \mathbb{N} \), the representations denoted by \( R_{2k,1}^{1,0}, R_{2k,2}^{1,0}, R_{2k,3}^{1,0}, R_{1,3k}^{0,1}, R_{1,3k}^{0,1}, R_{2,3k}^{0,2}, R_{2,3k}^{0,1} \) and \( R_{2,3k}^{0,2} \) are indecomposable representations of rank 2, while \( R_{2k,3}^{1,1} \) and \( R_{2k,3}^{1,2} \) are indecomposable

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representations of rank 3. Their characters read

\[
\chi[\mathcal{R}_{2k,0}^{1,0}(q)] = \chi_{2k-1, b}(q) + \chi_{2k+1, b}(q) \\
= (1 - \delta_{b,3}\delta_{k,1})\chi_{2k-1, b}(q) + 2\chi_{2k+1, b}(q) + \chi_{2k+3, b}(q), \quad b = 1, 2, 3 \\
\chi[\mathcal{R}_{a,3k}^{0, b}(q)] = \chi_{a, 3k-b}(q) + \chi_{a, 3k+b}(q) \\
= (1 - \delta_{a, 2}\delta_{k, 1})\chi_{a, 3k-b}(q) + 2\chi_{a, 3k+b}(q) + \chi_{a, (k+2)-b}(q), \quad a, b = 1, 2 \\
\chi[\mathcal{R}_{2k,3}^{1, b}(q)] = \chi_{2k-1, 3-b}(q) + \chi_{2k-1, 3+b}(q) + \chi_{2k-3+b}(q) + \chi_{2k+3+b}(q) \\
= (1 - \delta_{k, 1})\chi_{1, 3k-3-b}(q) + 2(1 - \delta_{k, 1})\chi_{1, 3k-3+b}(q) + 2\chi_{1, 3k-b}(q) \\
+ 4\chi_{1, 3k+b}(q) + (2 - \delta_{k, 1})\chi_{1, 3k+3-b}(q) + 2\chi_{1, 3k+3+b}(q) \\
+ 2\chi_{1, 3k+6-b}(q) + \chi_{1, 3k+9-b}(q), \quad b = 1, 2
\]

indicating that one may consider the indecomposable representations as ‘indecomposable combinations’ of Kac representations. The participating Kac representations are of course the ones whose characters appear in (2.2). In the case of the indecomposable representation \(\mathcal{R}_{2k,b}^{1,0}\) (or \(\mathcal{R}_{a,3k}^{0,b}\)) of rank 2, our lattice analysis indicates that a Jordan cell is formed between every state in \(\chi_{2k+1,b}(q)\) (or \(\chi_{a,3k+b}(q)\)) and its partner state in the second copy of \(\chi_{2k+1,b}(q)\) (or \(\chi_{a,3k+b}(q)\)), and nowhere else. In the case of the indecomposable representation \(\mathcal{R}_{2k,3}^{1, b}\) of rank 3, our lattice analysis indicates that for every quartet of matching states in the four copies of \(\chi_{1,3k+b}(q)\), a rank-3 Jordan cell is formed along with a single state. It likewise appears that a Jordan cell of rank 2 is formed between every pair of matching states in the irreducible components with multiplicity 2.

The notation \(\mathcal{R}_{r,s}^{a,b}\) is meant to reflect simple properties of the higher-rank indecomposable representations. The pair of lower indices thus refers to a ‘symmetry point’ in the Kac table around which an indecomposable combination of Kac representations are located. The pair of upper indices indicates the distribution of these representations of which there are either two (if \(a = 0\) or \(b = 0\)) or four (if \(a, b \neq 0\)). Their locations correspond to endpoints or corners, respectively, of a line segment or a rectangle with centre at \((r, s)\). This structure is encoded neatly in the character expressions (2.2).

It follows from the lattice that the fundamental fusion algebra may be described by separating the representations into a horizontal and a vertical part. Before discussing implications of this, we examine the two directions individually, and introduce some abbreviations. To compactify the fusion rules, we use the notation

\[
(r, -s) \equiv (-r, s), \quad \mathcal{R}_{r,s}^{a,b} \equiv \mathcal{R}_{-r,-s}^{a,b} \equiv -\mathcal{R}_{r,s}^{a,b}
\]

implying, in particular, that \((0, 0) \equiv (r, 0) \equiv \mathcal{R}_{0,0}^{a,b} \equiv \mathcal{R}_{r,0}^{a,b} \equiv 0\), and define the Kronecker delta combinations

\[
\delta_{j,k\{k'}^{(2)} = 2 - \delta_{j,k-k'} - \delta_{j,k+k'} \\
\delta_{j,k\{k'}^{(4)} = 4 - 3\delta_{j,k-k'} - 2\delta_{j,k+k'} - \delta_{j,k+k'+1} - \delta_{j,k-k'+1} - 2\delta_{j,k+k'} - 3\delta_{j,k+k'+1} \\
\delta_{j,k\{k'}^{(8)} = 8 - 7\delta_{j,k-k'} - 6\delta_{j,k+k'} - 4\delta_{j,k-k'+1} - 2\delta_{j,k-k'+1} - \delta_{j,k-k'+1} - 2\delta_{j,k+k'} - 6\delta_{j,k+k'+1} - 7\delta_{j,k+k'+1}.
\]

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2.2. Horizontal fusion algebra

The horizontal fusion algebra $\langle (2, 1) \rangle$ is defined as the fusion algebra generated by the fundamental representation $(2, 1)$. We find that closure of this fusion algebra requires the inclusion of the Kac representations $(2k, 1)$ and the rank-2 indecomposable representations $\mathcal{R}^{1,0}_{2k,1}$

$$\langle (2, 1) \rangle = \langle (2k, 1), \mathcal{R}^{1,0}_{2k,1}; k \in \mathbb{N} \rangle. \quad (2.5)$$

We conjecture that the fusion algebra $\langle (2, 1) \rangle$ reads

$$(2k, 1) \otimes (2k', 1) = \bigoplus_{j = |k-k'|+1}^{k+k'-1} \mathcal{R}^{1,0}_{2j,1}, \quad (2.6)$$

$$\mathcal{R}^{1,0}_{2k,1} \otimes \mathcal{R}^{1,0}_{2k',1} = \bigoplus_{j = |k-k'|}^{k+k'} \delta_{j,k,k'}^{(2)} \mathcal{R}^{1,0}_{2j,1}. \quad (2.7)$$

This fusion algebra does not contain an identity.

2.3. Vertical fusion algebra

The vertical fusion algebra $\langle (1, 2) \rangle$ is defined as the fusion algebra generated by the fundamental representation $(1, 2)$. We find that closure of this fusion algebra requires the inclusion of the Kac representations $(1, 1)$ and $(1, 3k)$ and the rank-2 indecomposable representations $\mathcal{R}^{0,b}_{2k,1}$

$$\langle (1, 2) \rangle = \langle (1, 1), (1, 2), (1, 3k), \mathcal{R}^{0,b}_{1,3k}; b = 1, 2; k \in \mathbb{N} \rangle. \quad (2.8)$$

Letting $X$ denote any of these representations, we conjecture that the fusion algebra $\langle (1, 2) \rangle$ reads

$$(1, 1) \otimes X = X$$

$$(1, 2) \otimes (1, 2) = (1, 1) \oplus (1, 3)$$

$$(1, 2) \otimes (1, 3k) = \mathcal{R}^{0,1}_{1,3k}$$

$$(1, 2) \otimes \mathcal{R}^{0,1}_{1,3k} = \mathcal{R}^{0,2}_{1,3k} \oplus 2(1, 3k)$$

$$(1, 2) \otimes \mathcal{R}^{0,2}_{1,3k} = \mathcal{R}^{0,1}_{1,3k} \oplus (1, 3(3k-1)) \oplus (1, 3(k+1))$$

$$(1, 3k) \otimes (1, 3k') = \bigoplus_{j = |k-k'|+1}^{k+k'-1} (\mathcal{R}^{0,2}_{1,3j} \oplus (1, 3j))$$

$$(1, 3k) \otimes \mathcal{R}^{0,1}_{1,3k'} = \left( \bigoplus_{j = |k-k'|+1}^{k+k'-1} 2\mathcal{R}^{0,1}_{1,3j} \right) \oplus \left( \bigoplus_{j = |k-k'|}^{k+k'} \delta_{j,k,k'}^{(2)} (1, 3j) \right)$$

$$(1, 3k) \otimes \mathcal{R}^{0,2}_{1,3k'} = \left( \bigoplus_{j = |k-k'|}^{k+k'} \delta_{j,k,k'}^{(2)} \mathcal{R}^{0,1}_{1,3j} \right) \oplus \left( \bigoplus_{j = |k-k'|+1}^{k+k'-1} 2(1, 3j) \right)$$

$$\delta_{j,k,k'}^{(2)} = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{otherwise} \end{cases}$$

$$\mathcal{R}^{0,b}_{1,3k} = \bigoplus_{j = |k-k'|}^{k+k'} \mathcal{R}^{0,b}_{1,3j}$$

$$\mathcal{R}^{0,b}_{1,3k} \otimes \mathcal{R}^{0,b}_{1,3k'} = \bigoplus_{j = |k-k'|}^{k+k'} \mathcal{R}^{0,b}_{1,3j} \otimes \mathcal{R}^{0,b}_{1,3j'}$$

$$\mathcal{R}^{0,b}_{1,3k} \otimes X = X$$

$$\mathcal{R}^{0,b}_{1,3k} \otimes \mathcal{R}^{0,b}_{1,3k'} = \mathcal{R}^{0,b}_{1,3k} \otimes \mathcal{R}^{0,b}_{1,3k'}$$

$$\mathcal{R}^{0,b}_{1,3k} \otimes \mathcal{R}^{0,b}_{1,3k'} = \mathcal{R}^{0,b}_{1,3k} \otimes \mathcal{R}^{0,b}_{1,3k'}$$

$$\mathcal{R}^{0,b}_{1,3k} \otimes \mathcal{R}^{0,b}_{1,3k'} = \mathcal{R}^{0,b}_{1,3k} \otimes \mathcal{R}^{0,b}_{1,3k'}$$
Kac representations, this implies that the representations may be separated into a horizontal and a vertical part. For the both associative and commutative. As already announced, it also follows from the lattice description that the fundamental fusion algebra of critical percolation discussed by Read and Saleur in \[48\]. To appreciate this, we provide a dictionary for translating the representations generating the subalgebra (2.9) into the notation used in \[48\].

\[
\mathcal{R}_{1,3}^{0,1} \otimes \mathcal{R}_{1,3}^{0,1} = \left( \bigoplus_{j=|k-k'|}^{k+k'} \delta_{j,k,k'}^{(2)} \mathcal{R}_{1,3}^{0,1} \right) \oplus \left( \bigoplus_{j=|k-k'|+1}^{k+k'-1} \delta_{j,k,k'}^{(2)} \mathcal{R}_{1,3}^{0,2} \right) \oplus \left( \bigoplus_{j=|k-k'|}^{k+k'} \delta_{j,k,k'}^{(2)} \mathcal{R}_{1,3}^{0,2} \right) \oplus \left( \bigoplus_{j=|k-k'|+1}^{k+k'-1} \delta_{j,k,k'}^{(2)} \mathcal{R}_{1,3}^{0,2} \right)
\]

(2.8)

It is noted that for \( j = |k-k'| - 1 \) (mod 2), as in \( \mathcal{R}_{1,3}^{0,2} \otimes \mathcal{R}_{1,3}^{0,2} \), the fusion multiplicity \( \delta_{j,k,k'}^{(4)} \) reduces to \( 4 - 3 \delta_{j,k-k'|-1} - \delta_{j,k-k'|+1} - \delta_{j,k+k'-1} - 3 \delta_{j,k+k'+1} \). The representation (1,1) is the identity of this vertical fusion algebra.

### 2.4. Comparison with Read and Saleur

It is verified that

\[
\langle (1,1), (1, 6k - 3) \rangle, \mathcal{R}_{1,6k}^{0,1}, \mathcal{R}_{1,6k-3}^{0,2} ; k \in \mathbb{N}
\]

(2.9)

is a subalgebra of the vertical fusion algebra. It corresponds to the fusion algebra of critical percolation discussed by Read and Saleur in \[48\]. To appreciate this, we provide a dictionary for translating the representations generating the subalgebra (2.9) into the notation used in \[48\].

\[
(1, 1) \leftrightarrow \mathcal{R}_0 \quad (1, 2j + 1) \leftrightarrow \mathcal{R}_j, \quad j \equiv 1 \text{ (mod 3)}
\]

\[
\mathcal{R}_{1,2j-1}^{0,2} \leftrightarrow \mathcal{R}_j, \quad j \equiv 2 \text{ (mod 3)}
\]

\[
\mathcal{R}_{1,2j}^{0,1} \leftrightarrow \mathcal{R}_j, \quad j \equiv 0 \text{ (mod 3)}
\]

(2.10)

where \( j \in \mathbb{N} \). We find that their fusion algebra is in agreement with the subalgebra (2.9) of the vertical fusion algebra \( \langle (1, 2) \rangle \) which itself is a subalgebra of the fundamental fusion algebra \( \langle (2, 1), (1, 2) \rangle \) of critical percolation.

### 2.5. Fundamental fusion algebra

It follows from the lattice description that the fundamental fusion algebra \( \langle (2, 1), (1, 2) \rangle \) is both associative and commutative. As already announced, it also follows from the lattice that the representations may be separated into a horizontal and a vertical part. For the Kac representations, this implies

\[
(r, s) = (r, 1) \otimes (1, s).
\]

(2.11)
For the purposes of examining the fundamental fusion algebra, we introduce the representations

\[(2k, 3k') = (2k, 1) \otimes (1, 3k'), \quad \mathcal{R}^{1,0}_{2k,3k'} = \mathcal{R}^{1,0}_{2k,1} \otimes (1, 3k') \]
\[\mathcal{R}^{0,b}_{2k,3k'} = (2k, 1) \otimes \mathcal{R}^{0,b}_{1,3k'}, \quad \mathcal{R}^{1,b}_{2k,3k'} = \mathcal{R}^{1,0}_{2k,1} \otimes \mathcal{R}^{0,b}_{1,3k'} \]

(2.12)

thus defined as the result of certain simple fusions of ‘a horizontal and a vertical representation’. As we will show elsewhere, these representations may be decomposed in terms of the representations listed in (2.1)

\[(2k, 3k') = \bigoplus_{j=|k-k'|+1, \text{by } 2}^{k+k'-1} (2j, 3), \quad (2.13)\]
\[\mathcal{R}^{1,0}_{2k,3k'} = \bigoplus_{j=|k-k'|+1, \text{by } 2}^{k+k'-1} \mathcal{R}^{1,0}_{2j,3}, \quad \mathcal{R}^{0,b}_{2k,3k'} = \bigoplus_{j=|k-k'|+1, \text{by } 2}^{k+k'-1} \mathcal{R}^{0,b}_{2j,3} \]

(2.13)

with

\[(2k, 3k') = (2k', 3k), \quad \mathcal{R}^{1,0}_{2k,3k'} = \mathcal{R}^{1,0}_{2k',3k}, \quad \mathcal{R}^{0,b}_{2k,3k'} = \mathcal{R}^{0,b}_{2k',3k}, \quad \mathcal{R}^{1,b}_{2k,3k'} = \mathcal{R}^{1,b}_{2k',3k} \]

(2.14)

as special identifications extending the set (1.7). The fundamental fusion algebra is now obtained by simply applying (2.12) and (2.13) to the fusion of a pair of representations in (2.1). We illustrate this with a general but somewhat formal evaluation where we let

\[A_{r,s} = \bar{a}_{r,1} \otimes a_{1,s}, \quad B_{r',s'} = \bar{b}_{r',1} \otimes b_{1,s'}, \quad \bar{a}_{r,1} \otimes \bar{b}_{r',1} = \bigoplus_{s''} c_{s''} \]

and \(a_{1,s} \otimes b_{1,s'} = \bigoplus_{s''} c_{s''}.\) Our fusion prescription now yields

\[A_{r,s} \otimes B_{r',s'} = \left( \bar{a}_{r,1} \otimes a_{1,s} \right) \otimes \left( \bar{b}_{r',1} \otimes b_{1,s'} \right) = \left( \bar{a}_{r,1} \otimes \bar{b}_{r',1} \right) \otimes \left( a_{1,s} \otimes b_{1,s'} \right) \]

\[= \left( \bigoplus_{s''} \bar{c}_{r',1} \right) \otimes \left( \bigoplus_{s''} c_{s''} \right) = \bigoplus_{s'''} \bar{C}_{s'''} \]

(2.15)

where \(C_{s'''} = \bar{c}_{s'''} \otimes c_{1,s''}.\) Using this, the fundamental fusion algebra \(\langle (2, 1), (1, 2) \rangle\) follows straightforwardly from the fusion algebras \(\langle (2, 1) \rangle\) and \(\langle (1, 2) \rangle\) together with (2.12) and (2.13). In particular, it follows readily that the Kac representation \((1, 1)\) is the identity of the fundamental fusion algebra \(\langle (2, 1), (1, 2) \rangle\).

In this brief communication, we will only apply this fusion prescription explicitly to the fusion of the two rank-2 indecomposable representations \(\mathcal{R}^{1,0}_{2k,2} \otimes \mathcal{R}^{0,2}_{2,3k'}\)

\[\mathcal{R}^{1,0}_{2k,2} \otimes \mathcal{R}^{0,2}_{2,3k'} = \left( \mathcal{R}^{1,0}_{2k,1} \otimes (1, 2) \right) \otimes \left( (2, 1) \otimes \mathcal{R}^{0,2}_{1,3k'} \right) \]
\[= \left( \mathcal{R}^{1,0}_{2k,1} \otimes (2, 1) \right) \otimes \left( (2k-1, 1) \oplus 2(2k, 1) \oplus (2k+1, 1) \right) \]
\[\otimes \left( \mathcal{R}^{0,1}_{1,3k'} \oplus (1, 3(k'-1)) \oplus (1, 3(k'+1)) \right) \]
\[= \left( \bigoplus_{j=|k-k'|}^{k+k'} \delta^{(2)}_{j,|k,k'|} \mathcal{R}^{0,1}_{2,3} \right) \otimes \left( \bigoplus_{j=|k-k'|}^{k+k'+1} \delta^{(4)}_{j,|k,k'|} (2j, 3) \right) \]

(2.16)
and to the fusion of two rank-3 indecomposable representations
\[
\mathcal{R}^{1,1}_{2k,3} \otimes \mathcal{R}^{1,1}_{2k',3} = \left( \mathcal{R}^{1,0}_{2k,1} \otimes \mathcal{R}^{1,1}_{1,3} \right) \otimes \left( \mathcal{R}^{1,0}_{2k',1} \otimes \mathcal{R}^{1,0}_{1,3} \right) = \left( \mathcal{R}^{1,0}_{2k,1} \otimes \mathcal{R}^{1,0}_{2k',1} \right) \otimes \left( \mathcal{R}^{0,1}_{1,3} \otimes \mathcal{R}^{0,1}_{1,3} \right)
\]
\[
= \left( \bigoplus_{j=[k-k']} \delta_{j, \{k,k'\}}^{(2)} \mathcal{R}^{1,0}_{2j,1} \right) \otimes \left( \mathcal{R}^{0,1}_{1,6} \oplus 2 \mathcal{R}^{0,2}_{1,3} \oplus 4(1,3) \right)
\]
\[
= \left( \bigoplus_{j=[k-k']} \delta_{j, \{k,k'\}}^{(4)} \mathcal{R}^{1,1}_{2j,3} \right) \oplus \left( \bigoplus_{j=[k-k']} \delta_{j, \{k,k'\}}^{(2)} \left( 2 \mathcal{R}^{1,2}_{2j,3} \oplus 4 \mathcal{R}^{1,0}_{2j,3} \right) \right)
\]
(2.17)

and likewise
\[
\mathcal{R}^{1,2}_{2k,3} \otimes \mathcal{R}^{1,2}_{2k',3} = \left( \bigoplus_{j=[k-k']} \delta_{j, \{k,k'\}}^{(2)} \mathcal{R}^{1,1}_{2j,3} \right) \oplus \left( \bigoplus_{j=[k-k']} \delta_{j, \{k,k'\}}^{(4)} \left( \mathcal{R}^{1,2}_{2j,3} \oplus 2 \mathcal{R}^{1,0}_{2j,3} \right) \right)
\]
\[
= \left( \bigoplus_{j=[k-k']} \delta_{j, \{k,k'\}}^{(4)} \mathcal{R}^{1,2}_{2j,3} \right) \oplus \left( \bigoplus_{j=[k-k']} \delta_{j, \{k,k'\}}^{(2)} \left( 2 \mathcal{R}^{1,2}_{2j,3} \right) \right)
\]
(2.18)

Several subalgebras of the fundamental fusion algebra are easily identified. An interesting example is the one generated by the set of rank-3 indecomposable representations and the rank-2 indecomposable representations \(\mathcal{R}^{1,0}_{2k,3}\). Two other noteworthy subalgebras are the ones generated by all the representations in (2.1) except \((1,2)\) or \((1,1)\) and \((1,2)\).

We wish to point out that, at the level of Kac characters, the horizontal, vertical and fundamental fusion algebras are all compatible with the \(\mathfrak{sl}(2)\) structure
\[
\phi_n \otimes \phi_{n'} = \bigoplus_{m=|n-n'|+1, \text{ by } 2}^{n+n'-1} \phi_m.
\]
(2.19)

This is straightforward to establish for the horizontal and vertical fusion algebras as illustrated by the fusion \(\mathcal{R}^{1,0}_{2k,1} \otimes \mathcal{R}^{1,0}_{2k',1}\) where (2.19) yields
\[
\chi[\mathcal{R}^{1,0}_{2k,1} \otimes \mathcal{R}^{1,0}_{2k',1}](q) = \left( \chi_{2k-1,1}(q) + \chi_{2k+1,1}(q) \right) \otimes \left( \chi_{2k'-1,1}(q) + \chi_{2k'+1,1}(q) \right)
\]
\[
= \sum_{j=[2k-2k'+2]+1, \text{ by } 2}^{2(k+k')-3} \chi_{j,1}(q) + \sum_{j=[2k-2k'+2]+1, \text{ by } 2}^{2(k+k')-1} \chi_{j,1}(q)
\]
\[
+ \sum_{j=[2k-2k'+2]+1, \text{ by } 2}^{2(k+k')-1} \chi_{j,1}(q) + \sum_{j=[2k-2k'+2]+1, \text{ by } 2}^{2(k+k')+1} \chi_{j,1}(q)
\]
\[
= \sum_{j=[k-k']}^{k+k'} \delta_{j, \{k,k'\}}^{(2)} \left( \chi_{2j-1,1}(q) + \chi_{2j+1,1}(q) \right)
\]
(2.20)
while

\[ \chi[R_{2k,1}^{1,0} \otimes R_{2k',1}^{1,0}](q) = \sum_{j = |k - k'|}^{k + k'} \delta_{j,k,k'}^{(2)} \chi[R_{2j,1}^{1,0}](q) \]

\[ = \sum_{j = |k - k'|}^{k + k'} \delta_{j,k,k'}^{(2)} \left( \chi_{2j-1,1}(q) + \chi_{2j+1,1}(q) \right). \tag{2.21} \]

The separation into a horizontal and a vertical part (2.11) and (2.12) then implies that the characters of the fundamental fusion algebra exhibit two independent \(s\ell(2)\) structures as in (2.19)—one in each direction. This is clearly reminiscent of the fusion algebras of rational (minimal) models where the \(s\ell(2)\) structures are carried by the (characters of the) irreducible representations. Here, on the other hand, the \(s\ell(2)\) structures are tied to the Kac representations but, due to the higher-rank indecomposable nature of some other representations, only at the level of their characters.

2.6. Comparison with Eberle and Flohr

To facilitate a comparison with [45] by Eberle and Flohr, we provide a partial dictionary relating our notation to the one used in [45]. In the orders specified, the translation reads

\begin{align*}
\{(2k, b), (1, 3k)\} &\longrightarrow \{\mathcal{V}(\Delta_{2k,b}), \mathcal{V}(\Delta_{1,3k})\}, \quad b = 1, 2, 3; \quad k \in \mathbb{N} \\
\{(1, 1), (1, 2)\} &\longrightarrow \{R^{(1)}(0)_2, R^{(1)}(0)_1\} \\
\{R_{2,1}^{1,0}, R_{6,1}^{1,0}, R_{8,1}^{1,0}\} &\longrightarrow \{R^{(2)}(0, 2)_7, R^{(2)}(2, 7), R^{(2)}(7, 15), R^{(2)}(15, 26)\} \\
\{R_{2,2}^{1,0}, R_{6,2}^{1,0}, R_{8,2}^{1,0}\} &\longrightarrow \{R^{(2)}(0, 1)_5, R^{(2)}(1, 5), R^{(2)}(5, 12), R^{(2)}(12, 22)\} \\
\{R_{2,3}^{1,0}, R_{4,3}^{1,0}, R_{6,3}^{1,0}, R_{8,3}^{1,0}\} &\longrightarrow \{R^{(2)}(1/3, 1/3), R^{(2)}(1/3, 10/3), R^{(2)}(10/3, 28/3), \ R^{(2)}(12/3, 55/3)\} \\
\{R_{1,3}^{1,1}, R_{1,3}^{0,1}, R_{1,9}^{0,1}, R_{1,12}^{0,1}\} &\longrightarrow \{R^{(2)}(0, 1)_7, R^{(2)}(2, 5), R^{(2)}(7, 12), R^{(2)}(15, 22)\} \\
\{R_{2,3}^{0,1}, R_{2,6}^{0,1}, R_{2,12}^{0,1}\} &\longrightarrow \{R^{(2)}(1/8, 1/8), R^{(2)}(5/8, 21/8), R^{(2)}(33/8, 65/8), \ R^{(2)}(85/8, 133/8)\} \\
\{R_{0,1}^{0,1}, R_{0,2}^{0,2}, R_{1,12}^{0,2}\} &\longrightarrow \{R^{(2)}(0, 2)_5, R^{(2)}(1, 7), R^{(2)}(5, 15), R^{(2)}(12, 26)\} \\
\{R_{2,2}^{0,2}, R_{2,6}^{0,2}, R_{2,12}^{0,2}\} &\longrightarrow \{R^{(2)}(5/8, 5/8), R^{(2)}(1/8, 33/8), R^{(2)}(21/8, 85/8), \ R^{(2)}(65/8, 161/8)\} \\
\{R_{2,3}^{1,1}, R_{6,3}^{1,1}, R_{8,3}^{1,1}\} &\longrightarrow \{R^{(3)}(0, 0, 1, 1), R^{(3)}(0, 1, 2, 5), R^{(3)}(2, 5, 7, 12), \ R^{(3)}(7, 12, 15, 22)\} \\
\{R_{2,3}^{1,2}, R_{6,3}^{1,2}, R_{8,3}^{1,2}\} &\longrightarrow \{R^{(3)}(0, 0, 2, 2), R^{(3)}(0, 1, 2, 7), R^{(3)}(1, 5, 7, 15), \ R^{(3)}(5, 12, 15, 26)\}.
\end{align*}
The only three fusions of rank-3 indecomposable representations considered in [45] correspond to

\[
\begin{align*}
\mathcal{R}_{2,3}^{1,1} \otimes \mathcal{R}_{2,3}^{1,1} &= \mathcal{R}_{2,3}^{1,1} \oplus 2\mathcal{R}_{4,3}^{1,1} \oplus \mathcal{R}_{6,3}^{1,1} \oplus 4\mathcal{R}_{2,3}^{1,2} \oplus 2\mathcal{R}_{4,3}^{1,2} \oplus 8\mathcal{R}_{2,3}^{1,0} \oplus 4\mathcal{R}_{4,3}^{1,0} \\
\mathcal{R}_{2,3}^{1,1} \otimes \mathcal{R}_{2,3}^{1,2} &= 4\mathcal{R}_{2,3}^{1,1} \oplus 2\mathcal{R}_{4,3}^{1,1} \oplus \mathcal{R}_{6,3}^{1,1} \oplus 2\mathcal{R}_{4,3}^{1,2} \oplus 2\mathcal{R}_{6,3}^{1,0} \oplus 4\mathcal{R}_{4,3}^{1,0} \oplus 2\mathcal{R}_{6,3}^{1,0} \\
\mathcal{R}_{2,3}^{1,2} \otimes \mathcal{R}_{2,3}^{1,2} &= \mathcal{R}_{2,3}^{1,3} \oplus 2\mathcal{R}_{4,3}^{1,1} \oplus \mathcal{R}_{6,3}^{1,1} \oplus 4\mathcal{R}_{4,3}^{1,2} \oplus 2\mathcal{R}_{2,3}^{1,0} \oplus 2\mathcal{R}_{2,3}^{1,0} \\
&\quad \oplus 2\mathcal{R}_{4,3}^{1,0} \oplus 2\mathcal{R}_{6,3}^{1,0} \oplus \mathcal{R}_{8,3}^{1,0}.
\end{align*}
\]

Likewise, the only fusion of the type (2.16) considered in [45] corresponds to

\[
\mathcal{R}_{2,2}^{1,0} \otimes \mathcal{R}_{2,3}^{0,2} = 2\mathcal{R}_{2,3}^{0,1} \oplus \mathcal{R}_{4,3}^{0,1} \oplus (2, 3) \oplus 2(4, 3) \oplus (6, 3).
\]

We find that our fusion rules reduce to the many examples examined by Eberle and Flohr [45]. This confirms their observation that indecomposable representations of rank 3 are required. Our results also demonstrate that the fusion algebra closes without the introduction of indecomposable representations of higher rank than 3.

Eberle and Flohr also presented an algorithm [45] for computing fusion products in the augmented \(c_{p,p'}\) models, in particular in the augmented \(c_{2,3} = 0\) model. Their algorithm is rooted in the many explicit examples examined in their paper and yields fusion rules which are both commutative and associative. Considering the affirmative comparison of our fusion rules with their examples, we believe that their algorithm for the augmented \(c_{2,3}\) model yields results equivalent to our explicit fusion rules for critical percolation \(\mathcal{L}M(2, 3)\).

### 2.7. Kac representations revisited

As already indicated and also discussed in [45], the two representations \((1, 1)\) and \((1, 2)\) (there denoted \(\mathcal{R}^{(1)}(0)_{2}\) and \(\mathcal{R}^{(1)}(0)_{1}\), respectively) are not fully reducible. We quote Eberle and Flohr:

On the other hand, the representations \(\mathcal{R}^{(2)}(0, 1)_{5}\) and \(\mathcal{R}^{(2)}(0, 1)_{7}\) contain a state with weight 0 which generates a subrepresentation \(\mathcal{R}^{(1)}(0)_{1}\). This subrepresentation is indecomposable but neither is it irreducible nor does it exhibit any higher-rank behaviour. It only exists as a subrepresentation as it needs the embedding into the rank 2 representation in order not to have null-vectors at both levels 1 and 2. But, nevertheless, being a subrepresentation of a representation in the spectrum it has to be included into the spectrum, too.

This is corroborated by our findings. From the lattice, the two representations \((1, 1)\) and \((1, 2)\) arise in the conformal scaling limit from very simple and natural boundary conditions. This supports our assertion that these Kac representations are indeed physical. Furthermore, since one is immediately faced with problems when attempting to include their irreducible components

\[
(1, 1) : \{\mathcal{V}(0), \mathcal{V}(2)\}, \quad (1, 2) : \{\mathcal{V}(0), \mathcal{V}(1)\}
\]

in the fusion algebra, we advocate to consider fusion algebras of critical percolation generated from Kac representations and indecomposable representations of higher rank. The only irreducible representations appearing in these fusion algebras are therefore
themselves Kac representations, that is, they belong to the set of irreducible Kac representations \{ (2k, 1), (2k, 2), (2k, 3) = (2, 3k), (1, 3k) \}. Natural extensions of the horizontal, vertical and fundamental fusion algebras involve all the associated Kac representations and read

\[
\langle (2, 1), (3, 1) \rangle, \quad \langle (1, 2), (1, 4) \rangle, \quad \langle (2, 1), (3, 1), (1, 2), (1, 4) \rangle
\]

respectively. They will be addressed elsewhere. Further evidence in support of the relevance of Kac representations in logarithmic CFT may be found in [50] where quotient modules with characters (1.2) are found to arise naturally in the limit of certain sequences of minimal models.

3. Conclusion

We have presented explicit general conjectures for the chiral fusion algebras of critical percolation, and we have exhibited dictionaries to facilitate comparison of our results with the particular results of Eberle and Flohr [45] and Read and Saleur [48]. Importantly, we observe the appearance of rank-3 indecomposable representations in agreement with Eberle and Flohr. Our fundamental fusion algebra is built from independent horizontal and vertical algebras that, at the level of characters, respect an underlying \( sl(2) \) structure. The identity \((1, 1)\) of this fundamental fusion algebra is a reducible yet indecomposable Kac representation of rank 1.

Our reported fusion rules are supported by extensive numerical investigations of an integrable lattice model of critical percolation. These lattice results will be presented elsewhere. We also hope to discuss elsewhere the full fusion algebra encompassing all of the Kac representations as well as extensions to general logarithmic minimal models.

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