EXPLICIT DESCRIPTION OF JUMPING PHENOMENA ON MODULI SPACES OF PARABOLIC CONNECTIONS AND HILBERT SCHEMES OF POINTS ON SURFACES

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Abstract. In this paper, we investigate the apparent singularities and the dual parameters of rank 2 parabolic connections on \( \mathbb{P}^1 \) and rank 2 (parabolic) Higgs bundle on \( \mathbb{P}^1 \). Then we obtain explicit descriptions of Zariski open sets of the moduli space of the parabolic connections and the moduli space of the Higgs bundles. For \( n = 5 \), we can give global descriptions of the moduli spaces in detail.

1. Introduction

The purpose of this paper is to give explicit descriptions of the moduli space of rank 2 (parabolic) Higgs bundles on \( \mathbb{P}^1 \) and the moduli space of rank 2 parabolic connections on \( \mathbb{P}^1 \) by the apparent singularities and their dual parameters. It is well-known that the apparent singularities and their dual parameters are coordinates on Zariski open sets of these moduli spaces. Historically, Okamoto [11] described Hamiltonians of the Garnier systems by the apparent singularities and their duals, for the Garnier systems are obtained by the isomonodromic deformations of rank 2 parabolic connections on \( \mathbb{P}^1 \). The apparent singularities and their duals are introduced as coordinates for a Zariski open set of Okamoto’s space of initial conditions, which are nothing but the moduli space of rank 2 parabolic connections on \( \mathbb{P}^1 \). Arinkin–Lysenko [1] and Oblezin [12] studied the moduli space of rank 2 parabolic connections on \( \mathbb{P}^1 \) more systematically, and they also introduced the apparent singularities and their duals as coordinates for Zariski open sets of moduli spaces. Oblezin also showed that the moduli space of rank 2 parabolic connections on \( \mathbb{P}^1 \) is birational to the Hilbert scheme of points on the blowing up of the total space of a certain line bundle on \( \mathbb{P}^1 \). Dubrovin and Mazzocco discussed the apparent singularities and their duals for higher rank cases on \( \mathbb{P}^1 \) in detail [4].

For the moduli spaces of parabolic connections on arbitrary genus curves, Inaba–Iwasaki–Saito [6] and Inaba [5] established the existence of good moduli spaces of stable parabolic connections, and it is interesting to describe their geometric structures. Saito and Szabo are developing a systematic treatment of apparent singularities and their duals for general parabolic connections on higher genus curves [13]. One can show the similar geometric description of the moduli spaces of parabolic Higgs bundles. The main purpose of this paper is to give more explicit description of the total space of the moduli spaces of rank 2 parabolic connections on \( \mathbb{P}^1 \). For the purpose, we need treat the following particular cases. The first case is that the apparent singularities approaches to the regular singularities of Higgs fields or connections (This case is already treated in [12, Section 3.7]). The second case is that the apparent singularities have multiplicities. The third case is that the type of the underlying bundle is jumping. The jumping phenomenon happens to \( n \)-regular singularities cases where \( n \geq 5 \). For the first and second cases, we can give an explicit description of families for the \( n \)-point regular singularities case. For the third case, we give an explicit description of jumping families parameterized by the apparent singularities and their duals for the 5-point regular singularities case. Oblezin [12] considered the stratification of the moduli spaces of rank 2 parabolic connections on \( \mathbb{P}^1 \) associated to the bundle type of underlying bundles, and gave geometric descriptions of each strata of the moduli spaces, separately. On the other hand, in this paper, we try to give a global geometric description of the moduli spaces including the jumping phenomena.
phenomena of bundle type. As the result, we obtain a global description of the moduli spaces for the 5-point regular singularities case, and give an explicit description of universal families of Higgs bundles and connections.

Fix points \( t_1, \ldots, t_n \in \mathbb{P}^1 \) (\( t_i \neq t_j \)), and set \( D = t_1 + \cdots + t_n \). We consider pairs \((E, \nabla)\) where \( E \) is a rank 2 vector bundle on \( \mathbb{P}^1 \) and \( \nabla : E \to E \otimes \Omega^1_{\mathbb{P}^1}(D) \) a connection having simple poles supported on \( D \). At each pole, we have two residual eigenvalues \( \{ \lambda^+, \lambda^- \} \) of \( \nabla \), \( i = 1, \ldots, n \); they satisfy Fuchs relation \( \sum_i (\lambda^+ + \lambda^-) + d = 0 \) where \( d = \deg(E) \). Moreover, we introduce parabolic structures \( I = \{ t_i \}_{1 \leq i \leq n} \) such that \( I_i \) is a one dimensional subspace of \( E_{t_i} \), which corresponds to an eigenspace of the residue of \( \nabla \) at \( t_i \) with the eigenvalue \( \lambda^+ \). Note that when \( \lambda^+ \neq \lambda^- \), the parabolic structure \( I \) is determined uniquely by the connection \((E, \nabla)\). Fixing a spectral data \( \xi = (\xi^+) \) with integral sum \(-d\) and introducing the weight \( \nu \), we can construct the moduli space \( M^w_{t, \xi}(gl_2) \) of \( w \)-stable \( \xi \)-parabolic connections \((E, \nabla, I)\) by Geometric Invariant Theory and the moduli space \( M^w_{t, \xi}(gl_2) \) turns to be a smooth irreducible quasi-projective variety of dimension \( 2n - 6 \) for generic weight \( \nu \) (see [3]). Note that, when

\[
\sum_{i=1}^n \xi_1^i \notin \mathbb{Z}
\]

for any \( \{ \epsilon_i \} \in \{ +, - \} \), every parabolic connection \((E, \nabla, I)\) is irreducible, hence stable. Therefore the moduli space \( M^w_{t, \xi}(gl_2) \) does not depend on the choice of the weight \( \nu \) in such cases. It is known that the moduli spaces coincide with the spaces of initial conditions for Garnier systems, and the case \( n = 4 \) corresponding to the Painlevé VI equation, for such differential equations are nothing but isomonodromic deformations for linear connections. Next, we fix \( \xi = (\xi^+)_{1 \leq i \leq n} \) where \( \sum (\xi_1^+ + \xi_-^+) = 0 \). In the same way as above, we can define \( w \)-stable \( \xi \)-parabolic Higgs bundle \((E, \Phi, I)\). Here \( E \) is a rank 2 vector bundle on \( \mathbb{P}^1 \), \( \Phi : E \to E \otimes \Omega^1_{\mathbb{P}^1}(D) \) is an \( sl_n \)-morphism, and \( I \) is the parabolic structure. At each point \( t_i \), residual eigenvalues of \( \Phi \) are \( \{ \xi_1^+, \xi_-^+ \} \) of \( \Phi \). Let \( M_{H, t, \xi}^w(gl_2) \) be the moduli space of \( w \)-stable \( \xi \)-parabolic Higgs bundles.

By suitable transformations, we may assume that \( d = \deg(E) = -1 \) and \( \xi \) can be normalized as follows. For connection cases, we can put

\[
\begin{align*}
\xi_1^+ &= \nu_i, \\
\xi_-^+ &= -\nu_i, \\
\xi_- &= 1 - \nu_n,
\end{align*}
\]

and for Higgs cases, we can put \( \xi_1^+ = \nu_i \) and \( \xi_-^+ = -\nu_i \) (\( i = 1, \ldots, n \)), for some \( \nu = (\nu_1, \ldots, \nu_n) \in \mathbb{C}^n \). Let \( M \) and \( M_H \) be the moduli space of \( \nu \)-sl2-parabolic connections and the moduli space of \( \nu \)-sl2-parabolic Higgs bundles, respectively. By natural isomorphisms, we have \( M \cong M_{H, t, \xi}^w(gl_2) \) and \( M_H \cong M_{H, t, \xi}^w(gl_2) \). (Note that the moduli space \( M \) is noting but the moduli space of (modified) \((\nu_1, \ldots, \nu_n)\)-bundles on \( \mathbb{P}^1 \) treated in [11] and [12].)

For the moduli space \( M_H \), we obtain the following results. First, we consider the Zariski open set \( M_0^H \), which is the locus where the type of the underlying bundle is \( \mathcal{O} \oplus \mathcal{O}(-1) \). Let \( K'_n \) be some Zariski open set of some blowing-up of the Hirzebruch surface of degree \( n - 2 \). (See Figure 1.) By the explicit computation of the apparent singularities and their dual parameters, we have the following

**Theorem 1.1 (Theorem [3.1].)** By the apparent singularities and dual parameters, we have a map

\[
M_0^H \to \text{Hilb}^{n-3}(K'_n),
\]

and this map is injective. Moreover, we can give an explicit description of the universal family \((E^{(0)}, \Phi^{(0)}) \to M_0^H \times \mathbb{P}^1 \).

Suppose \( n = 5 \). We consider the total moduli space \( M_H \), which also includes the jumping locus. The type of the underlying bundle of members are \( \mathcal{O} \oplus \mathcal{O}(-1) \) (generic) or \( \mathcal{O}(1) \oplus \mathcal{O}(-2) \) (jumping locus). Let \( \tilde{M}_H \) be the moduli space of \( \nu \)-sl2-parabolic Higgs bundles with a cyclic vector \( \sigma \in H^0(\mathbb{P}^1, E) \). The moduli
space $\tilde{M}_H$ is the blowing-up of $M_H$ along the jumping locus. We take some blowing-up of $\text{Hilb}^2(K_H')$, denoted by $\tilde{\text{Hilb}}^2(K_H')$. Then we have the map $\tilde{M}_H \to \tilde{\text{Hilb}}^2(K_H')$.

**Theorem 1.2** (Theorem 3.2 and Section 4.1). Suppose that $n = 5$. The map $\tilde{M}_H \to \tilde{\text{Hilb}}^2(K_H')$ is injective. Moreover, we can give an explicit description of the universal family $(\tilde{E}, \tilde{\Phi}, \tilde{\sigma}) \to \tilde{M}_H \times \mathbb{P}^1$.

For the moduli space $M$ of connections, which is isomorphic to $M_W^{2\iota}(gl_2)$, we have the following results. First, we consider the Zariski open set $M_0$ which is the locus where the type of the underlying bundle is $O \oplus O(-1)$. Let $\tilde{K}_n'$ be some Zariski open set of some blowing-up of the Hirzebruch surface of degree $n - 2$. By the same argument as in the Higgs case, we have the following

**Theorem 1.3** (Theorem 5.2). By the apparent singularities and dual parameters, we have a map

$$M^0 \to \text{Hilb}^{n-3}(\tilde{K}_n'),$$

and this map is injective. Moreover, we can give an explicit description of the universal family $(\tilde{E}^{(0)}, \tilde{\nu}^{(0)}) \to M^0 \times \mathbb{P}^1$.

Parts of Theorem 1.1 and Theorem 1.3 are already contained in [11, 17, 12, 116]. For $n = 4$, the results are discussed in [11] and [7]. Oblin [12] gives a map from $M_H^0$ (resp. $M^0$) to a $(n - 3)$-th symmetric product of $K_n'$ (resp. $\tilde{K}_n'$), which is an isomorphism on a certain open set. For $n = 5$, the injectivities are discussed in [116].

Suppose $n = 5$. We consider the moduli space $M$, which includes the jumping locus. The type of the underlying bundle of members of the jumping locus is $O(1) \oplus O(-2)$. Let $\tilde{M}$ be the moduli space of $\nu$-$sl_2$-parabolic connections with a cyclic vector $\sigma \in H^0(\mathbb{P}^1, E)$. The moduli space $\tilde{M}$ is the blowing-up of $M$ along the jumping locus. Let $C_\infty$ be the $\infty$-section of the Hirzebruch surface of degree $n - 2$.

**Theorem 1.4** (Theorem 5.5). Let $\phi: \tilde{M} \to \text{Hilb}^2(\tilde{K}_n' \cup C_\infty)$ be the birational map constructed by the apparent singularities and the dual parameters. By taking some sequence of blowing-ups $\tilde{\text{Hilb}}^2(\tilde{K}_n' \cup C_\infty) \to \text{Hilb}^2(\tilde{K}_n' \cup C_\infty)$, we have the injective map $\phi: \tilde{M} \to \text{Hilb}^2(\tilde{K}_n' \cup C_\infty)$ for $\phi$. The moduli space $\tilde{M}$ is biregular to its image $\tilde{\phi}(\tilde{M}) \subset \tilde{\text{Hilb}}^2(\tilde{K}_n' \cup C_\infty)$.

The organization of this paper is as follows. In Section 2, we introduce definitions and notations which are necessary in this paper. In Section 3, [3.1] we consider the moduli spaces of $\nu$-$sl_2$-parabolic Higgs bundles with bundle type $O \oplus O(-1)$. We show Theorem 1.1 (Theorem 3.1) by explicit calculations of apparent singularities. In [3.2], we consider the moduli spaces of $\nu$-$sl_2$-parabolic Higgs bundles with a cyclic vector for $n = 5$. We show the injectivity of the map in Theorem 1.2 (Theorem 3.2) by explicit calculations of apparent singularities and spectral curves. In Section 4, we construct an explicit jumping families of $\nu$-$sl_2$-parabolic Higgs bundles by the lower and upper modifications. In particular, in 4.1, we give an explicit description of the universal family of $\tilde{M}_H$. In Section 5, we consider the moduli spaces of $\nu$-$sl_2$-parabolic connections. In [5.1], we show Theorem 1.3 (Theorem 5.2) by the same way as in the Higgs case. In [5.2], we construct an explicit jumping family of connections for $n = 5$, and in [5.3], we analyze the behavior of the apparent singularities and their duals when the parameters of the jumping family approach to the jumping locus. Finally, we obtain Theorem 1.4 (Theorem 5.5).

2. Preliminaries

In this section, first, we define $\nu$-$sl_2$-parabolic connections and $\nu$-$sl_2$-parabolic Higgs bundles, and recall the well-known facts of the connections and Higgs bundles. In 2.2, we describe some blowing-ups of the Hirzebruch surface $\Sigma_{n-2}$, which are target spaces of the map defined by the apparent singularities and their duals. In 2.3, we discuss descriptions of Higgs fields. Since we consider Higgs bundles on $\mathbb{P}^1$, ...
the underlying vector bundles split into the direct sum of line bundles. Then we can describe the Higgs fields explicitly. By the automorphisms of vector bundles, we can normalize the Higgs fields to reduce the number of parameters. In 2.4, we discuss the apparent singularities and the dual parameters of Higgs bundles, which give a map from the moduli space of Higgs bundles (with a cyclic vector) to the symmetric product of the total space of $\Omega^1_{\nu}(D)$. In 2.5, we discuss the transformations called the lower modification and the upper modification, and we use the transformations for a construction of a universal family of the moduli space of Higgs bundles (with a cyclic vector). The contents of 2.1, 2.2, and 2.5 basically follow the expositions from [1] and [12].

### 2.1. $\mathfrak{sl}_2$-connections and $\mathfrak{sl}_2$-Higgs bundles.

We introduce $\mathfrak{sl}_2$-parabolic connections and $\mathfrak{sl}_2$-parabolic Higgs bundles, and we consider relations between the moduli space $\mathcal{M}^{\mu\nu}_{\nu}(\mathfrak{g}\mathfrak{l}_2)$ and these moduli spaces.

Fix complex numbers $\nu_1, \ldots, \nu_n \in \mathbb{C}$. Suppose that $\nu_1 \cdots \nu_n \neq 0$ and

$$\sum_{i=1}^{n} \epsilon_i \nu_i \notin \mathbb{Z}$$

for any $(\epsilon_i), \epsilon_i \in \{1, -1\}$.

**Definition 2.1.** A $\nu$-$\mathfrak{sl}_2$-parabolic connection on $\mathbb{P}^1$ is a triplet $(E, \nabla, \varphi)$ such that

1. $E$ is a rank 2 vector bundle on $\mathbb{P}^1$,
2. $\nabla: E \to E \otimes \Omega^1_{\nu}(D)$ is a connection,
3. $\varphi: \bigwedge^2 E \cong \mathcal{O}_{\mathbb{P}^1}(-1)$ is a horizontal isomorphism,
4. the residue $\text{res}_{t_i}(\nabla)$ of the connection $\nabla$ at $t_i$ has eigenvalues $\nu_i^\pm$, $1 \leq i \leq n$.

Here we put $\nu_i^\pm := \pm \nu_i$ ($i = 1, \ldots, n - 1$), $\nu_n^+ := \nu_n$, $\nu_n^- := 1 - \nu_n$.

Denote by $\mathcal{M}$ the moduli stack of $\nu$-$\mathfrak{sl}_2$-parabolic connections on $\mathbb{P}^1$, and by $M$ its coarse moduli space. Let $\nabla': \mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}_{\mathbb{P}^1} \otimes \Omega^1_{\nu}(D)$ be the connection defined by

$$f \mapsto df + \left( \frac{1}{2} \sum_{i=1}^{n-1} (-\xi_i^+ - \xi_i^-) \frac{dz}{z-t_i} + \frac{1}{2} \sum_{i=1}^{n-1} (\xi_i^+ + \xi_i^-) \frac{dz}{z-t_i} \right) f.$$ 

Suppose that the condition (1) holds and $\xi_i^+ \neq \xi_i^-$ for $i = 1, \ldots, n$. Then we have an isomorphism

$$M^{\mu\nu}_{\nu}(\mathfrak{g}\mathfrak{l}_2) \to M = M_{\nu},$$

$$(E, \nabla, \varphi) \mapsto ((E, \nabla) \otimes (\mathcal{O}_{\mathbb{P}^1}, \nabla'), \varphi),$$

where $\nu = (\nu_1, \ldots, \nu_n)$, $\nu_i = (\xi_i^+ - \xi_i^-)/2$ ($i = 1, \ldots, n - 1$), $\nu_n = \xi_n^+ + \sum_{i=1}^{n-1} (\xi_i^+ - \xi_i^-)/2$.

**Definition 2.2.** A $\nu$-$\mathfrak{sl}_2$-parabolic Higgs bundle on $\mathbb{P}^1$ is a triplet $(E, \Phi, \varphi)$ such that

1. $E$ is a rank 2 vector bundle on $\mathbb{P}^1$,
2. $\Phi: E \to E \otimes \Omega^1_{\nu}(D)$ is an $\mathcal{O}_{\mathbb{P}^1}$-morphism,
3. $\varphi: \bigwedge^2 E \cong \mathcal{O}_{\mathbb{P}^1}(-1)$ is an isomorphism and $\text{tr}(\Phi) = 0$,
4. the residue $\text{res}_{t_i}(\Phi)$ of the connection $\Phi$ at $t_i$ has eigenvalues $\pm \nu_i$, $1 \leq i \leq n$.

Denote by $\mathcal{M}_H$ the moduli stack of $\nu$-$\mathfrak{sl}_2$-parabolic Higgs bundles on $\mathbb{P}^1$, and by $M_H$ its coarse moduli space. We have a stratification of $M_H$ as follows. By the irreducibility of $(E, \Phi, \varphi) \in M_H$, we have the following

**Proposition 2.3.** For $(E, \Phi, \varphi) \in M_H$, we have

$$E \cong \mathcal{O}(k) \oplus \mathcal{O}(-k - 1)$$

where $0 \leq k \leq \left\lfloor \frac{n-3}{2} \right\rfloor$. 


Let $M^0_H$ be the subvariety of $M_H$ where $E \cong \mathcal{O}(k) \oplus \mathcal{O}(-k-1)$. Then

$$M_H = M^0_H \cup \cdots \cup M^{(n-3)/2}_H.$$ 

Note that the stratum $M^0_H$ is a Zariski open dense of $M_H$.

Moreover, we introduce $\nu$-\textit{sl}$\mathfrak{2}$-parabolic connection on $\mathbb{P}^1$ with a cyclic vector and $\nu$-\textit{sl}$\mathfrak{2}$-parabolic Higgs bundle on $\mathbb{P}^1$ with a cyclic vector.

**Definition 2.4.** A $\nu$-\textit{sl}$\mathfrak{2}$-parabolic connection on $\mathbb{P}^1$ with a cyclic vector is a tuple $(E, \nabla, \phi, [\sigma])$ such that

1. $E$ is a rank 2 vector bundle on $\mathbb{P}^1$,
2. $\nabla: E \to E \otimes \Omega^1_{\mathbb{P}^1}(D)$ is a connection,
3. $\phi: \bigwedge^1 E \cong \mathcal{O}_{\mathbb{P}^1}(-1)$ is a horizontal isomorphism,
4. the residue $\text{res}_{t_i}(\nabla)$ of the connection $\nabla$ at $t_i$ has eigenvalues $\nu^\pm_i$, $1 \leq i \leq n$.
5. $[\sigma] \subset H^0(\mathbb{P}^1, E)$ is a 1-dimensional subspace generated by a nonzero section $\sigma \in H^0(\mathbb{P}^1, E)$.

Denote by $\hat{M}$ the moduli stack of $\nu$-\textit{sl}$\mathfrak{2}$-parabolic connections on $\mathbb{P}^1$ with a cyclic vector, and by $\hat{M}$ its coarse moduli space. For description of the moduli spaces $\hat{M}$, we denote by $\hat{M}$ the coarse moduli space. For $n = 5$, the moduli space $\hat{M}$ is the blowing-up of $M_H$ along $M^0_H$.

**Definition 2.5.** A $\nu$-\textit{sl}$\mathfrak{2}$-parabolic Higgs bundle on $\mathbb{P}^1$ with a cyclic vector is a tuple $(E, \Phi, \phi, [\sigma])$ such that

1. $E$ is a rank 2 vector bundle on $\mathbb{P}^1$,
2. $\Phi: E \to E \otimes \Omega^1_{\mathbb{P}^1}(D)$ is an $\mathcal{O}_{\mathbb{P}^1}$-morphism,
3. $\phi: \bigwedge^1 E \cong \mathcal{O}_{\mathbb{P}^1}(-1)$ is an isomorphism and $\text{tr}(\Phi) = 0$,
4. the residue $\text{res}_{t_i}(\nabla)$ of the connection $\nabla$ at $t_i$ has eigenvalues $\pm \nu_i$, $1 \leq i \leq n$.
5. $[\sigma] \subset H^0(\mathbb{P}^1, E)$ is a 1-dimensional subspace generated by a nonzero section $\sigma \in H^0(\mathbb{P}^1, E)$.

Denote by $\hat{M}_H$ the moduli stack of $\nu$-\textit{sl}$\mathfrak{2}$-parabolic Higgs bundles on $\mathbb{P}^1$ with a cyclic vector, and by $\hat{M}_H$ its coarse moduli space. For $n = 5$, the moduli space $\hat{M}_H$ is the blowing-up of $M_H$ along $M^0_H$.

### 2.2. Hirzebruch surfaces and the blowing-ups.

For description of the moduli spaces $M$ and $M_H$, we introduce some blowing-ups of the Hirzebruch surface $\Sigma_{n-2}$. Put $L := \Omega^1_{\mathbb{P}^1}(D)$. Let $\mathcal{L}$ be the total space of the line bundle $L$. Note that $\mathcal{L} = \Sigma_{n-2} \setminus C_\infty$ where $C_\infty$ is the infinity section $(C_\infty)^2 = -(n-2)$.

First, we construct a blowing-up of the Hirzebruch surface $\Sigma_{n-2}$ corresponding to $M_H$. Let $\pi: \mathcal{L} \to \mathbb{P}^1$ be the projection and let $\tau_i: \pi^{-1}(t_i) \xrightarrow{\cong} \mathbb{C}$ be the residue map. Put $\nu_i^+ := \nu_i, \nu_i^- := -\nu_i$ for $i = 1, \ldots, n$, and $\nu_i^\pm := \tau_i^{-1}(\nu_i^\pm)$. Set

$$K'_n := \left( \text{Bl}_{\nu_i^\pm \mathcal{L}} \right) \setminus \left( \bigcup_i F_i \right)$$

where $\text{Bl}_{\nu_i^\pm \mathcal{L}}$ is the blowing-up of $\mathcal{L}$ at $\nu_i^\pm$ for $i = 1, \ldots, n$, and $F_i$ are the proper pre-images of the fiber $F_i$ for $i = 1, \ldots, n$. We denote by $K_n$ the image of $K'_n$ by the projection $K'_n \to \mathcal{L}$ (see Figure 1).

Second, we construct a blowing-up of the Hirzebruch surface $\Sigma_{n-2}$ corresponding to $M$. Let $\pi: \mathcal{L} \to \mathbb{P}^1$ be the projection and let $\tau_i: \pi^{-1}(t_i) \xrightarrow{\cong} \mathbb{C}$ be the residue map. Set

$$K'_n := \left( \text{Bl}_{\nu_i^\pm \mathcal{L}} \right) \setminus \left( \bigcup_i F_i \right)$$

where $\nu_i^\pm := \tau_i^{-1}(\nu_i^\pm)$. Here, $\nu_i^+ := \nu_i, \nu_i^- := -\nu_i$ for $i = 1, \ldots, n-1$ and $\nu_n^+ := \nu_n, \nu_n^- := 1 - \nu_n$. We denote by $K_n$ the image of $K'_n$ by the projection $K'_n \to \mathcal{L}$. 


2.3. Description of Higgs fields. Put \( U_0 := \mathbb{P}^1 \setminus \{ \infty \} \), \( U_\infty := \mathbb{P}^1 \setminus \{ 0 \} \). Let \( z \) and \( w \) be the coordinates on \( U_0 \) and \( U_\infty \), respectively. Put
\[
\omega_z := \frac{dz}{z(z-1)(z-x_1)\cdots(z-x_{n-3})} \quad \text{and} \quad R_k := \begin{pmatrix} z^k & 0 \\ 0 & \frac{1}{z^{k+1}} \end{pmatrix}, \quad 0 \leq k \leq \left\lfloor \frac{n-3}{2} \right\rfloor.
\]
We consider an explicit description of the Higgs field of \((E, \Phi) \in M_H\). Suppose that \( E \cong \mathcal{O}(k) \oplus \mathcal{O}(-k-1) \) where \( 0 \leq k \leq [(n-3)/2] \). We can describe the Higgs field \( \Phi \) as follows:
\[
\Phi = \begin{cases} A^k_z \otimes \omega_z & \text{on } U_0 \\ R_k^{-1}(A^k_z \otimes \omega_z)R_k & \text{on } U_\infty \end{cases}
\quad \text{where} \quad A^k_z := \begin{pmatrix} f_{11}^{(n-2)}(z) & f_{12}^{(n+2k-1)}(z) \\ f_{21}^{(n-2k-3)}(z) & -f_{11}^{(n-2)}(z) \end{pmatrix}
\]
where \( f_{ij}^{(l)}(z) \) is a polynomial in \( z \) of degree at most \( l \). By the irreducibility, we have \( f_{21}^{(n-2k-3)}(z) \neq 0 \).

We consider automorphisms of the vector bundle \( E \cong \mathcal{O}(k) \oplus \mathcal{O}(-k-1) \). Any element \( P \in \text{Hom}_{\mathcal{O}(k)}(E, E) \cong H^0(\mathbb{P}^1, \mathcal{O} \oplus \mathcal{O}(2k+1) \oplus \mathcal{O}) \) is described as follows:
\[
P_{U_0} = \begin{pmatrix} s & p^{(2k+1)}(z) \\ 0 & t \end{pmatrix} \quad \text{on } U_0, \quad P_{U_\infty} = \begin{pmatrix} s & w^{2k+1}p^{(2k+1)}(1/w) \\ 0 & t \end{pmatrix} \quad \text{on } U_\infty
\]
where \( s, t \in \mathbb{C} \) and \( p^{(2k+1)}(z) \) is a polynomial in \( z \) of degree at most \( 2k+1 \). If \( st \neq 0 \), then \( P \in \text{Aut}(E) \). We take \( P \in \text{Aut}(E) \). Then we have
\[
\text{the (1,1)-entry of } P_{U_0}^{-1}A_zP_{U_0} = \frac{tf_{11}^{(n-2)}(z) - pf_{21}^{(2k+1)}(z) f_{21}^{(n-2k-3)}}{t}
\]
\[
\text{the (2,1)-entry of } P_{U_0}^{-1}A_zP_{U_0} = \frac{sf_{21}^{(n-2k-3)}(z)}{t}.
\]

We consider simple descriptions of Higgs fields by the automorphisms of \( E \). First, we consider the \((2,1)\)-entry. Let \( \{[s_1 : 1], \ldots, [s_t : 1], [1 : q_{t+1}], \ldots, [1 : q_{n-2k-3}]\} \) be the zeros of \( f_{21}^{(n-2k-3)}(z) \) on \( \mathbb{P}^1 \) where \( 0 \leq i \leq n-2k-3 \). By the automorphisms of \( E \), we can put
\[
f_{21}^{(n-2k-3)}(z) := (s_1 z - 1) \cdots (s_t z - 1)(z - q_{t+1}) \cdots (z - q_{n-2k-3}).
\]
Second, we consider the (1,1)-entry. We assume that the coefficient of \( z^l \) in the polynomial \( f_{21}^{(n-2k-3)}(z) \) is nonzero for some \( l \) \((0 \leq l \leq n-2k-3)\). By the automorphisms of \( E \), we can put
\[
f_{11}^{(n-2)}(z) := a_{n-2}z^{n-2} + \cdots + a_{t+2k+2}z^{t+2k+2} + a_{t-1}z^{t-1} + \cdots + a_0.
\]
In particular, if \( i = 0 \), that is, \( f_{21}^{(n-2k-3)}(z) := (z - q_1) \cdots (z - q_{n-2k-3}) \), then the coefficient of \( z^{n-2k-3} \) in the polynomial \( f_{21}^{(n-2k-3)}(z) \) is nonzero. In this case, we can put
\[
 f_{11}^{(n-2)}(z) := a_{n-2k-4}z^{n-2k-4} + \ldots + a_0.
\]

### 2.4. Apparent singularities and the dual parameters

We recall the apparent singularities and the dual parameters introduced by Saito-Szabo \[13\]. Let \((E, \Phi, \varphi) \in M_H\). If \( E \cong \mathcal{O} \oplus \mathcal{O}(-1) \), then the apparent singularities and the dual parameters coincide with the geometric Darboux coordinates due to Oblezin \[12\] Section 3, which gives a geometric interpretation of the Sklyanin formulas from \[15\]. We fix a section \( \sigma \in H^0(\mathbb{P}^1, E) \). For the section \( \sigma \), we define the following composition
\[
 \mathcal{O}_{p_1} \xrightarrow{\sigma} E \xrightarrow{\Phi} E \otimes L \longrightarrow (E / \mathcal{O}_{p_1}) \otimes L.
\]

The composition \( \mathcal{O}_{p_1} \to (E / \mathcal{O}_{p_1}) \otimes L \) is injective. Then we can define a subsheaf \( F^0 \subset E \) such that \( \mathcal{O}_{p_1} \to (F^0 / \mathcal{O}_{p_1}) \otimes L \) is isomorphic. By the isomorphism \( F^0 / \mathcal{O}_{p_1} \cong L^{-1} \), we have \( F^0 \cong \mathcal{O}_{p_1} \otimes L^{-1} \).

Therefore, we have the following exact sequence.
\[
 0 \longrightarrow \mathcal{O}_{p_1} \otimes L^{-1} \longrightarrow E \longrightarrow T_A \longrightarrow 0
\]
where \( T_A \) is a torsion sheaf. By the Riemann-Roch theorem, we have that the torsion sheaf \( T_A \) is length \( n - 3 \). The exact sequence \((10)\) is called a Frobenius-Hecke sheaf originally introduced by Drinfeld (see \[3\] and \[12\] Section 3.3).

**Definition 2.6.** For \((E, \Phi, \varphi) \in M_H\) and a nonzero section \( \sigma \in H^0(\mathbb{P}^1, E) \), we call the support of \( T_A \) apparent singular points of a \( \nu \)-\( \mathfrak{s}_2 \)-parabolic Higgs bundle with a cyclic vector \((E, \Phi, \varphi, [\sigma])\).

Next, we define dual parameters of \((E, \Phi, \varphi, [\sigma])\). Let \( C_{s} \) be the spectral curve of \((E, \Phi, \varphi)\). Let \( G \) be a torsion free sheaf of rank 1 on \( C_{s} \) corresponding to \((E, \Phi, \varphi)\), which satisfies \( E = \pi_G \). Since \( H^0(C_{s}, G) \cong H^0(\mathbb{P}^1, E) \), for a section \( \sigma \in H^0(\mathbb{P}^1, E) \), we have the short exact sequence
\[
 0 \longrightarrow \mathcal{O}_{C_s} \xrightarrow{\sigma} G \longrightarrow T_B \longrightarrow 0
\]
where \( T_B \) is a torsion sheaf on \( C_s \) of length \( n - 3 \). We take the direct image of the short exact sequence. Since \( \pi_*(\mathcal{O}_{C_s}) = \mathcal{O}_{p_1} \otimes L^{-1} \) and \( \pi_*G = E \), we have
\[
 0 \longrightarrow \mathcal{O}_{p_1} \otimes L^{-1} \xrightarrow{\pi_*\sigma} E \longrightarrow \pi_*(T_B) \longrightarrow 0.
\]

We may show that this short exact sequence coincides with the short exact sequence \((10)\). In particular, we have \( \pi_*(T_B) = T_A \), whose support is the apparent singularities of \((E, \Phi, \varphi, [\sigma])\). Set \( \text{Supp}(T_B) = \{\tilde{p}_1, \ldots, \tilde{p}_{n-3}\} \), where \( \tilde{p}_i \) is a point on \( C_s \). Put \( \tilde{p}_i = (q_i, p_i) \) where \( q_i = \pi(\tilde{p}_i) \), which is an apparent singularity, and \( p_i \in L_{q_i} \).

**Definition 2.7.** For \((E, \Phi, \varphi) \in M_H\) and a nonzero section \( \sigma \in H^0(\mathbb{P}^1, E) \), we call \( \{p_1, \ldots, p_{n-3}\} \) dual parameters of a \( \nu \)-\( \mathfrak{s}_2 \)-parabolic Higgs bundle with a cyclic vector \((E, \Phi, \varphi, [\sigma])\).

We consider the apparent singularities and the dual parameters of \((E, \Phi, \varphi, [\sigma])\) where \( E \cong \mathcal{O} \oplus \mathcal{O}(-1) \). In this case, the Higgs field \( \Phi \) is described as follows:
\[
 \Phi = \begin{cases} 
 A^0_z \otimes \omega_z & \text{on } U_0, \\
 R_0^{-1}(A^0_z \otimes \omega_z)R_0 & \text{on } U_\infty 
\end{cases}
\]
where \( A^0_z := \begin{pmatrix} f_{11}^{(n-2)}(z) & f_{12}^{(n-1)}(z) \\ f_{21}^{(n-3)}(z) & f_{21}^{(n-2)}(z) \end{pmatrix} \).

The apparent singularities are the zeros of \( f_{21}^{(n-3)}(z) \) on \( \mathbb{P}^1 \), denoted by \( \{q_1, \ldots, q_{n-3}\} \). The dual parameters are \( \{p_1, \ldots, p_{n-3}\} \) where we put \( p_i := f_{11}^{(n-2)}(q_i) \). Then, for \((\mathcal{O} \oplus \mathcal{O}(-1), \Phi, \varphi)\), we have
\[
 \{(q_1, p_1), \ldots, (q_{n-3}, p_{n-3})\} \in \text{Sym}^{n-3}(\mathbb{K}_n).
\]

Next, we consider the case \( E \cong \mathcal{O}(k) \oplus \mathcal{O}(-k - 1) \) where \( k > 0 \). In this case, the Higgs field \( \Phi \) is described as follows:
\[
 \Phi = \begin{cases} 
 A^k_z \otimes \omega_z & \text{on } U_0, \\
 R_0^{-1}(A^k_z \otimes \omega_z)R_0 & \text{on } U_\infty 
\end{cases}
\]
where \( A^k_z := \begin{pmatrix} f_{11}^{(n-2)}(z) & f_{12}^{(n+k-1)}(z) \\ f_{21}^{(n-2k-3)}(z) & f_{21}^{(n-2k-4)}(z) \end{pmatrix} \).
Let \( \sigma \in H^0(\mathbb{P}^1, E) \cong H^0(\mathbb{P}^1, \mathcal{O}(k)) \) be a section of \( E \), and let \( \{q_1, \ldots, q_k\} \in \text{Sym}^{k}(\mathbb{P}^1) \) be the zeros of the section \( \sigma \). We denote by \( \{q_{2k+1}, \ldots, q_{n-3}\} \in \text{Sym}^{n-2k-3}(\mathbb{P}^1) \) the zeros of \( f_{21}^{(n-2k-3)}(z) \) on \( \mathbb{P}^1 \). The apparent singularities of \((E, \Phi, \varphi, [\sigma])\) are the following
\[
\{2q_1, \ldots, 2q_k, q_{2k+1}, \ldots, q_{n-3}\} \in \text{Sym}^{n-3}(\mathbb{P}^1).
\]
We compute the dual parameters of \((E, \Phi, \varphi, [\sigma])\). We take \( \sigma \in H^0(C_1, G) \cong H^0(\mathbb{P}^1, \pi_1 G) \) corresponding to \( \sigma \in H^0(\mathbb{P}^1, E) \). The section \( \sigma \in H^0(C_1, G) \) has the following zeros
\[
\{(q_1, p_1), (q_1, -p_1), \ldots, (q_k, p_k), (q_k, -p_k)\}
\]
where \( \det(p_iI - \Phi|_{q_i}) = 0 \) for \( i = 1, \ldots, k \). The dual parameters are \( \{p_1, -p_1, \ldots, p_k, -p_k, p_{2k+1}, \ldots, p_{n-3}\} \) where we put \( p_i := f_{11}^{(n-2)}(q_i) \) for \( i = 2k+1, \ldots, n-3 \). Then, for \((E, \Phi, \varphi, [\sigma])\), we have
\[
\{(q_1, p_1), (q_1, -p_1), \ldots, (q_k, p_k), (q_k, -p_k), (q_{2k+1}, p_{2k+1}), \ldots, (q_{n-3}, p_{n-3})\} \in \text{Sym}^{n-3}((\mathcal{K}_1)\).
\]

2.5. Lower and upper modifications. In this subsection, following [12] Section 2], we describe the lower and the upper modifications. Let \( E \) be an algebraic vector bundle on \( \mathbb{P}^1 \) of rank 2 and of degree \( d \). Fix a point \( a \in \mathbb{P}^1 \). Let \( l \subset E|_a \) be a 1-dimensional subspace.

**Definition 2.8.** We call
\[
(a,l)^{\text{low}}(E) := \{ s \in E \mid s(a) \in l \}, \quad (a,l)^{\text{up}}(E) := (a,l)^{\text{low}}(E) \otimes \mathcal{O}(a)
\]
the lower and the upper modifications of \( E \), respectively.

*The lower and the upper modifications* provide the following exact sequences
\[
0 \rightarrow (a,l)^{\text{low}}(E) \rightarrow E \rightarrow E|_a/l \rightarrow 0,
\]
\[
0 \rightarrow E \rightarrow (a,l)^{\text{up}}(E) \rightarrow l \otimes \mathcal{O}(a) \rightarrow 0,
\]
respectively. In other words, we change our bundle rescaling the basis of sections in the neighborhood of a point \( a \) as follows. Given a local decomposition \( V = l \oplus l' \) of \( E \cong V \otimes \mathcal{O} \), we put the local basis \( \{s_1(z), s_2(z)\} \) with \( l \otimes \mathcal{O} \cong \langle s_1(z) \rangle \) and \( l' \otimes \mathcal{O} \cong \langle s_2(z) \rangle \). Then the basis of the lower modification \((a,l)^{\text{low}}\) of the bundle is generated by the sections \( \{s_1(z), (z-x)s_2(z)\} \), and of the upper one \((a,l)^{\text{up}}\) by \( \{(z-x)^{-1}s_1(z), s_2(z)\} \). Consequently, in the punctured neighborhood, we may represent the action of the modifications by the following gluing matrices
\[
(a,l)^{\text{low}} = \begin{pmatrix} 1 & 0 \\ 0 & (z-a) \end{pmatrix}, \quad (a,l)^{\text{up}} = \begin{pmatrix} (z-a)^{-1} & 0 \\ 0 & 1 \end{pmatrix}.
\]

3. Geometric description of the moduli spaces

Suppose that \( \nu \) satisfies the condition [3] and \( \nu_1 \cdots \nu_n \neq 0 \). We put
\[
(t_1, t_2, t_n) := (0, 1, \infty),
\]
\[
(\nu_1^+, \ldots, \nu_n^+) := (\pm \nu_1, \pm \nu_2, \ldots, \pm \nu_n), \quad \text{and}
\]
\[
\hat{\nu}_i := \nu_i(t_i - t_1) \cdots (t_i - t_{i-1})(t_i - t_{i+1}) \cdots (t_i - t_{n-1}) \text{ for } i = 1, \ldots, n - 1.
\]

First, we consider the apparent singularities and the dual parameters of members of \( M^0_H \) for \( n \geq 4 \). Then we have Theorem 1.1 (Theorem 3.1). Second, we assume \( n = 5 \). We consider the apparent singularities and the dual parameters of members of \( \tilde{M}_H \). Then we have the first assertion of Theorem 1.2 (Theorem 3.2).
3.1. Geometric description of $M^0_H$ for $n \geq 4$. Let $(O \oplus O(-1), \Phi, \varphi) \in M^0_H$, and $K'_n$ be the Zariski open set of the blowing-up of Hirzebruch surface of degree $n - 2$ defined in \[2\] and $K_n$ be the contraction $K'_n \rightarrow K_n$. Since $\dim H^0(\mathbb{P}^1, O \oplus O(-1)) = 1$, sections are determined uniquely up to constant. Then the apparent singularities and the dual parameters are determined by $(E, \Phi, \varphi)$. Let $\{(q_1, p_1), \ldots, (q_{n-3}, p_{n-3})\} \in \text{Sym}^{n-3}(K_n)$ be the apparent singularities and the dual parameters of $(E, \Phi, \varphi)$. We consider the map

\[
M^0_H \rightarrow \text{Sym}^{n-3}(K_n)
\]
\[
(E, \Phi, \varphi) \mapsto \{(q_1, p_1), \ldots, (q_{n-3}, p_{n-3})\},
\]

which is essentially constructed in \[12\] Section 3]. Since this map is not injective, we consider the composite of the Hilbert-Chow morphism and the blowing-up

\[
\text{Hilb}^{n-3}(K'_n) \rightarrow \text{Sym}^{n-3}(K'_n) \rightarrow \text{Sym}^{n-3}(K_n).
\]

Then we have the following

**Theorem 3.1.** The map \[11\] is extended to

\[
M^0_H \rightarrow \text{Hilb}^{n-3}(K'_n).
\]

The map is injective. Moreover, we can give an explicit description of the universal family $(E^{(0)}, \Phi^{(0)}) \rightarrow M^0_H \times \mathbb{P}^1$. 

The image of the map $M^0_H \rightarrow \text{Hilb}^{n-3}(K'_n)$ is described as follows. The image of the map $M^0_H \rightarrow \text{Sym}^{n-3}(K'_n)$ is

\[
\{ x = \{n_1 \hat{p}_1, \ldots, n_r \hat{p}_r\} \in \text{Sym}^{n-3}(K'_n) \mid \pi(\hat{p}_i) \neq \pi(\hat{p}_j) \text{ for } i \neq j \}
\]

where $n_j$ are integers such that $n_1 + \cdots + n_r = n - 3$, $n_j \geq 1$ ($j = 1, \ldots, r$) and $\pi$ is the projection $K'_n \rightarrow \mathbb{P}^1$. For $x = \{n_1 \hat{p}_1, \ldots, n_r \hat{p}_r\}$, let $M^0_H$ be the fiber of $x$ under $M^0_H \rightarrow \text{Sym}^{n-3}(K'_n)$. Then $M^0_H \cong \mathbb{C}^{n-1} \times \cdots \times \mathbb{C}^{n-1}$ where $\mathbb{C}^{n-1} \times \cdots \times \mathbb{C}^{n-1}$ is an affine open set of the fiber of $x$ under the Hilbert-Chow morphism.

The explicit description of the family is as follows. For simplicity, we assume that $\{q_1, \ldots, q_{n-3}\} \subset \mathbb{P}^1 \setminus \{\infty\}$. If the apparent singularities are distinct, then the explicit description is the following

\[
A_z \otimes \omega_z \ 	ext{on } U_0, \quad A_z = \begin{pmatrix} a_{n-4} z^{n-4} + \cdots + a_0 & f^{(n-1)}_{12}(z) \\ (z-q_1) \cdots (z-q_{n-3}) & -(a_{n-4} z^{n-4} + \cdots + a_0) \end{pmatrix}
\]

where

\[
a_i = \sum_{k=1}^{n-3} \sum_{j=1}^{n-3} u_{i+1,j} l_{jk} p_k, \quad R_0 = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{z} \end{pmatrix}
\]

Here $(q_1, p_1), \ldots, (q_{n-3}, p_{n-3})$ are pairs of the apparent singularities and their duals, the element $l_{ij}$ is defined by the relations $l_{ij} = 0$ for $i < j$, $l_{11} = 1$ and $l_{ij} = \prod_{k=1, k \neq j}^{n} (q_j - q_k)$ otherwise, and the element $u_{ij}$ is defined by the relations $u_{ii} = 1$, $u_{1i} = 0$ and $u_{ij} = u_{i-1,j-1} - u_{i,j-1} q_j - 1$ otherwise, that is, the matrix $(u_{i,j} l_{jk})_{ik}$ is the inverse of the Vandermonde matrix. Here we set $u_{00} = 0$. We omit the description of $f^{(n-1)}_{12}(z)$, which is a polynomial in $z$ of degree $n - 1$, since the description is lengthened and is not necessary below. Next, we consider the case where the apparent singularities and their duals have multiplicities. Let $n_j$ ($j = 1, \ldots, r$) be integers such that $n_1 + \cdots + n_r = n - 3$, $n_j \geq 1$. For each $j$ ($j = 1, \ldots, r$), we assume that $q_i \in \mathbb{P}^1 \setminus \{\infty\}$ and $p_i = y_j \in \mathbb{C}$ ($i = \sum_{k=1}^{j} n_k + 1, \ldots, \sum_{k=1}^{j} n_k$), that is, the pair of the apparent singularities and their duals is $\{n_1(x_1, y_1), \ldots, n_r(x_r, y_r)\}$. In this case, for each $j$ ($j = 1, \ldots, r$) we substitute

\[
p_i = y_j + (q_i - x_j)(\lambda_0^{(j)} + \lambda_1^{(j)} (q_i - x_j) + \cdots + \lambda_{n_j-2}^{(j)} (q_i - x_j)^{n_j-2}), \quad i = \sum_{k=1}^{j-1} n_k + 1, \ldots, j \sum_{k=1}^{j} n_k,
\]

for the coefficient \[14\]. Here $\lambda_0^{(j)}, \ldots, \lambda_{n_j-2}^{(j)}$ are coordinates of the affine open set $\mathbb{C}^{n_j-1}$ of the fiber of the Hilbert-Chow morphism. Then we can define the coefficients $a_i$ for this case.
Proof of Theorem 3.1. Put

$$M_H^{00} := \{(E, \Phi, \varphi) \in M_H^0 \mid \{q_1, \ldots, q_{n-3}\} : \text{the apparent singularities of } (E, \Phi, \varphi) \}$$

and

$$M_H^{000} := \{(E, \Phi, \varphi) \in M_H^0 \mid \{q_1, \ldots, q_{n-3}\} : \text{the apparent singularities of } (E, \Phi, \varphi) \}.\$$

Step 1. In this step, we show that the restriction $M_H^{000} \to \text{Sym}^{n-3}(\mathcal{K}_n)$ is injective. The image of this restriction is the following

$$\text{Image}(M_H^{000}) := \left\{(q_1, p_1, \ldots, (q_{n-3}, p_{n-3})) \mid \begin{array}{c} q_i \neq q_j \ (i \neq j) \\
q_i \notin \{t_1, \ldots, t_n\} \ (i = 1, \ldots, n-3) \end{array}\right\} \subset \text{Sym}^{n-3}(\mathcal{K}_n).$$

Let

$$\{(s_1 : 1), u_1, \ldots, (s_i : 1, u_i), (1 : q_{i+1}, p_{i+1}), \ldots, (1 : q_{n-3}, q_{n-3})\}\) be an element of $\text{Image}(M_H^{000})$ where $0 \leq i \leq n-3$. We show that the entries $f_{21}^{(n-3)}(z)$, $f_{11}^{(n-2)}(z)$, and $f_{12}^{(n-1)}(z)$ of the description [3] are determined by the element [15] up to automorphisms of $\mathcal{O} \oplus \mathcal{O}(-1)$. First, we consider the entry $f_{21}^{(n-3)}(z)$. By the definition of the apparent singularities and an automorphism of $\mathcal{O} \oplus \mathcal{O}(-1)$, we can put

$$f_{21}^{(n-3)}(z) = (s_1 z - 1) \cdots (s_i z - 1) (z - q_{i+1}) \cdots (z - q_{n-3}).$$

Second, we consider the entry $f_{11}^{(n-2)}(z)$. Since $s_1 \cdots s_i \neq 0$, the coefficient $z^{n-3}$ in $f_{21}^{(n-3)}(z)$ is nonzero. Then, by the automorphism of $\mathcal{O} \oplus \mathcal{O}(-1)$, we can put $f_{11}^{(n-2)}(z) = a_n z^{n-4} + \cdots + a_0$ as in the description [9]. By the definition of the dual parameters, we have that $u_j = s_i z f_{11}^{(n-2)}(1/s_j)$ for $1 \leq j \leq i$ and $p_i = f_{11}^{(n-2)}(q_j)$ for $i + 1 \leq j \leq n - 3$. Then we have the following system

$$\begin{pmatrix} u_1 \\ \vdots \\ p_{i+1} \\ \vdots \\ p_{n-3} \end{pmatrix} = \begin{pmatrix} 1 & \cdots & s_1^{n-3} & s_1^{n-4} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \cdots & s_i^{n-3} & s_i^{n-4} \\ p_{i+1} & \cdots & q_{i+1} & 1 \\ \vdots & \vdots & \vdots & \vdots \\ p_{n-3} & \cdots & q_{n-3} & 1 \end{pmatrix} \begin{pmatrix} a_{n-4} \\ \vdots \\ a_1 \\ a_0 \end{pmatrix}. \$$

We can determine the coefficients $a_{n-4}, \ldots, a_0$ by an element [15].

Third, we consider the entry $f_{12}^{(n-1)}(z)$. We put $f_{12}^{(n-1)}(z) := b_{n-1} z^{n-1} + b_{n-2} z^{n-2} + \cdots + b_0$. We solve the equations

$$\text{det}(\text{res}_i \Phi) = -\nu_i^2, \quad i = 1, \ldots, n.$$  

Then we have

$$f_{12}^{(n-1)}(t_i) = -f_{12}^{(n-2)}(t_i)^2 + \nu_i^2 f_{21}^{(n-3)}(t_i) \quad \text{(for } i = 1, \ldots, n-1), \text{ and } b_0 = \frac{-(w^{n-2} f_{11}^{(n-2)}(1/w))^2_{w=0} + \nu_n^2}{(w^{n-3} f_{21}^{(n-3)}(1/w))_{w=0}}. \$$

Since $f_{11}^{(n-2)}$ and $f_{21}^{(n-3)}$ are determined by the element [15], we can determine the coefficients $b_0, \ldots, b_{n-1}$. As the result, we obtain that the map $M_H^{000} \to \text{Sym}^{n-3}(\mathcal{K}_n)$ is injective.

Step 2. In this step, we extend the map $M_H^{000} \to \text{Sym}^{n-3}(\mathcal{K}_n)$ to $M_H^{00} \to \text{Sym}^{n-3}(\mathcal{K}_n')$, and we show that the extended map $M_H^{00} \to \text{Sym}^{n-3}(\mathcal{K}_n')$ is injective. For the element [15], we can put

$$f_{11}^{(n-2)}(z) := p_j + (z - q_j) \tilde{f}(z), \text{ and } f_{21}^{(n-3)}(z) := (s_1 z - 1) \cdots (s_i z - 1) (z - q_{i+1}) \cdots (z - q_{n-3})$$
where \( \tilde{f}_{11}(z) \) is a polynomial of degree at most \( n - 3 \) in \( z \). The polynomial \( \tilde{f}_{11}(z) \) is determined by the element \([15]\) up to automorphisms of \( \mathcal{O} \oplus \mathcal{O}(-1) \). By the condition \([16]\), we have

\[
f_{12}^{(n-1)}(t_i) = \frac{-(t_i - q_j)\tilde{f}_{11}(t_i)^2 + \nu \tilde{f}_{11}(t_i)^2}{(t_i - q_1) \cdots (t_i - q_{n-3})} \]

\[
= \frac{-(t_i - q_j)(\tilde{f}_{11}(t_i)^2 + 2p_j \tilde{f}_{11}(t_i)) - p_j^2 + \nu^2}{(t_i - q_1) \cdots (t_i - q_{n-3})}.
\]

We consider the blowing-up \( \mathcal{K}'_n \to \mathcal{K}_n \). Let \( \epsilon \in \{+,-\} \). We define the blowing-up parameters \( \nu_j \epsilon \) at \( (t_i, \epsilon \nu_j) \in \mathcal{K}_n \) as \( p_j - \epsilon \nu_j = \nu_j \epsilon (q_j - t_i) \). We substitute \( p_j = \nu_j \epsilon (q_j - t_i) + \epsilon \nu_j \) for the formula \([17]\). Then we have

\[
f_{12}^{(n-1)}(t_i) = \frac{-(t_i - q_j)(\tilde{f}_{11}(t_i)^2 + 2p_j \tilde{f}_{11}(t_i)) - (\nu_j \epsilon(q_j - t_i))^2 - \epsilon 2\nu_j \nu_j \epsilon (q_j - t_i)}{(t_i - q_1) \cdots (t_i - q_{n-3})}
\]

\[
= \frac{-(\tilde{f}_{11}(t_i)^2 + 2p_j \tilde{f}_{11}(t_i)) - (\nu_j \epsilon)^2(q_j - t_i) + \epsilon 2\nu_j \nu_j \epsilon (q_j - t_i)}{(t_i - q_1) \cdots (t_i - q_{j-1})(t_i - q_{j+1}) \cdots (t_i - q_{n-3})}.
\]

We consider the behavior of \( f_{12}^{(n-1)}(t_i) \) as \( q_j \to t_i \). The limit \( \lim_{q_j \to t_i} f_{12}^{(n-1)}(t_i) \) is convergence, and the convergence value is determined by the apparent singularities, the dual parameters, and the blowing-up parameters \( \nu_j \epsilon \). By the same argument as in Step 1, we can determine the all coefficients of \( f_{12}^{(n-1)}(z) \).

Then we obtain the map \( M_H^{00} \to \text{Sym}^{n-3}(\mathcal{K}'_n) \), and this map is injective.

**Step 3.** In this step, we extend the map \( M_H^{00} \to \text{Sym}^{n-3}(\mathcal{K}'_n) \) to \( M_H^0 \to \text{Hilb}^{n-3}(\mathcal{K}'_n) \), and we show that the extended map \( M_H^0 \to \text{Hilb}^{n-3}(\mathcal{K}'_n) \) is injective. Let

\[
\{(s_1 : 1, u_1), \ldots, (s_{n_0} : 1, u_{n_0}), (1 : q_1, p_1^1), \ldots, (1 : q_{n_1}^{1}, p_1^{n_1}), \ldots, (1 : q_{r}^{1}, p_1^{r}), \ldots, (1 : q_{n_r}^{r}, p_r^{n_r})\}
\]

be an element of \( \text{Image}(M_H^{000}) \) where \( n_0 + \cdots + n_r = n - 3 \), \( n_j \geq 1 \) \( j = 1, \ldots, r \). By the element \([19]\), we can determine the entries \( f_{21}^{(n-3)}(z) \), \( f_{11}^{(n-2)}(z) \), and \( f_{12}^{(n-1)}(z) \) of the description \([5]\) (Step 1). Let \( \tilde{x} \) be a point of \( \text{Sym}^{n-3}(\mathcal{K}'_n) \), denoted by

\[
\tilde{x} = \{n_0([0 : 1], y_0, a_0^{i, \pm}), n_1([1 : x_1], y_1, a_1^{i, \pm}), \ldots, n_r([1 : x_r], y_r, a_r^{i, \pm})\} \in \text{Sym}^{n-3}(\mathcal{K}'_n)
\]

where \( y_l + \epsilon \nu_l = a_l^{i, \pm}(x_l - t_i) \) for \( l = 0, \ldots, r \) and \( \epsilon \in \{+,-\} \). We consider the behavior of the entries as \( s_j \to 0 \) \( (j = 1, \ldots, n_0) \), \( q_k^l \to x_l \) and \( p_k^l \to y_l \) \( (k = 1, \ldots, n_l \text{ and } l = 1, \ldots, r) \) where \( x_l \neq x_{l_1} \) \((l_1 \neq l) \). Let \( \mathcal{I} \) be an ideal contained in the fiber of \( \tilde{x} \) by the Hilbert-Chow morphism \( \text{Hilb}^{n-3}(\mathcal{K}'_n) \to \text{Sym}^{n-3}(\mathcal{K}'_n) \).

We show that the entries \( f_{11}, f_{12}, f_{21} \) are determined by the ideal \( \mathcal{I} \) up to automorphisms.

On a neighborhood of \( \mathcal{I} \), the Hilbert scheme of points \( \text{Hilb}^{n-3}(\mathcal{K}'_n) \) is isomorphic to \( \text{Hilb}^{n_0}((\mathcal{K}'_n) \times \cdots \times \text{Hilb}^{n_r}(\mathcal{K}'_n)) \). We denote by \( (\mathcal{I}_{x_0, y_0}, \ldots, \mathcal{I}_{x_r, y_r}) \) the image of \( \mathcal{I} \). We put \( \mathcal{I}_{x_i, y_i} = ((q - x_i)^{n_i}, f_{x_i, y_i}(q, p)) \) where \( f_{x_i, y_i}(q, p) := (p - y_i) - (q - x_i)(\lambda_0^{(i)} + \lambda_1^{(i)}(q - x_j) + \cdots + \lambda_{n_j-2}^{(i)}(q - x_i)^{n_j-2}) \) as in the proof of \([10]\) Theorem 1.13).

We consider the entry \( f_{11}^{(n-2)}(z) \). For a neighborhood of the ideal \( \mathcal{I} \), we can assume that the coefficient of \( z^{n-3} \) in \( f_{21}^{(n-3)}(z) \) is nonzero. Then we can normalize \( f_{11}^{(n-2)}(z) \) as

\[
f_{11}^{(n-2)}(z) := a_{n-2}z^{n-2} + \cdots + a_{n-n_0-1}z^{n-n_0-1} + a_{n-n_0-4}z^{n-n_0-4} + \cdots + a_0
\]

by automorphisms of \( \mathcal{O} \oplus \mathcal{O}(-1) \). By the definition of dual parameters, we have the following system

\[
\begin{pmatrix}
    u_{1} \\
    \vdots \\
    u_{n_0} \\
    p_1^1 \\
    \vdots \\
    p_r^r
\end{pmatrix}
= \begin{pmatrix}
    1 & s_1 & \cdots & s_{n_0}^{n_0-1} & s_{n_0}^{n_0+2} & \cdots & s_{n_0}^{n_0-2} \\
    \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\
    1 & (q_1)^{n_1-2} & \cdots & (q_1)^{n_0-n_0-1} & (q_1)^{n_0-n_0-2} & \cdots & 1 \\
    \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\
    1 & (q_r)^{n_r-2} & \cdots & (q_r)^{n_0-n_0-1} & (q_r)^{n_0-n_0-4} & \cdots & 1
\end{pmatrix}
\begin{pmatrix}
    a_{n-2} \\
    \vdots \\
    a_{n-n_0-1} \\
    a_{n-n_0-4} \\
    \vdots \\
    a_0
\end{pmatrix}
\]
We substitute

\[ u_k = y_0 + s_k (\lambda_0^{(k)} + \lambda_1^{(k)} s_k + \cdots + \lambda_{n_k-2}^{(k)} s_k^{n_k-2}) , \quad k = 1, \ldots, n_0, \]

and

\[ p_k^j = y_l + (q_k^j - x_l) (\lambda_0^{(j)} \lambda_1^{(j)} (q_k^j - x_l) + \cdots + \lambda_{n_l-2}^{(j)} (q_k^j - x_l)^{n_l-2}), \quad k = 1, \ldots, n_l, \]

where \( j = 1, \ldots, r \), in the system \( \{ \} \). Then we can show that the coefficients of the polynomial \( \{ \} \) are defined when \( s_k = 0 \) and \( q_k^j = x_l \). Moreover, these coefficients are determined by the apparent singularities, the dual parameters, and the parameters \( \lambda_0^{(i)}, \lambda_1^{(i)}, \ldots, \lambda_{n_i-2}^{(i)} \).

We consider the entry \( f_{12}^{(n-1)}(z) \). If \( x_l \notin \{ t_1, \ldots, t_n \} \), then we have the value \( f_{12}^{(n-1)}(t_i) \in \mathbb{C} \) as \( q_k^j \to x_l \) for any \( k = 1, \ldots, n_i \). By the values \( f_{12}^{(n-1)}(t_1), \ldots, f_{12}^{(n-1)}(t_n) \), we can determine all coefficients of \( f_{12}^{(n-1)}(z) \). Next, we consider the behavior of the value \( f_{12}^{(n-1)}(t_i) \) as \( x_l \to t_i \). Let \( \epsilon \in \{ +, - \} \). We consider the ideal \( \mathcal{I}_{x_l, a_{i',i}}^{n_l} \), \( f_{x_l, a_{i', i}}^{n_l} \), where

\[ \mathcal{I}_{x_l, a_{i',i}}^{n_l} = ((q - t_i)^{n_l}, f_{x_l, a_{i', i}}^{n_l}(q, v^{i',e})) \]

For the ideal \( \mathcal{I}_{x_l, a_{i',i}}^{n_l} \), we can describe the dual parameter \( p_k^j \) \( (k = 1, \ldots, n_l) \) as follows:

\[ p_k^j = \epsilon \nu_i + v_{k, l}(q_k^j - t_i) = \epsilon \nu_i + (a_{i', e} + (q_k^j - x_l)(\lambda_0^{(j')} + \lambda_1^{(j')} (q_k^j - x_l) + \cdots + \lambda_{n_{l_i}-2}^{(j')} (q_k^j - x_l)^{n_{l_i}-2}))(q_k^j - t_i); \]

where \( \hat{\lambda}_j = \lambda_{j-1}^{(j')} + \lambda_{j-1}^{(j-i)}(x_l - t_i) \) for \( j = 1, \ldots, n_i - 1 \). Here we put \( \lambda_{-1'} : = a_{1', e} \). For the ideal \( \mathcal{I}_{x_l, a_{i',i}}^{n_l} \), we put

\[ f_{12}^{(n-2)}(z) : = \epsilon \nu_i + a_{i', e}(x_l - t_i) + \hat{\lambda}_1(z - x_l) + \hat{\lambda}_2(z - x_l)^2 + \cdots + \hat{\lambda}_{n_{l_i}-1}(z - x_l)^{n_{l_i}-1} + (z - x_l)^{n_{l_i}-1} \]

We substitute \( z = t_i \) for \( f_{12}^{(n-2)}(z) \). Then we have \( f_{12}^{(n-2)}(t_i) = \epsilon \nu_i + a_{i', e}(x_l - t_i)^{n_{l_i}}(f_{11}(t_i) - \lambda_{n_{l_i}-2}) \). We consider the value of \( f_{12}^{(n-1)}(t_i) \) as follows:

\[ f_{12}^{(n-1)}(t_i) = \frac{-f_{11}^{(n-1)}(t_i)^2 + \nu_i^2}{(t_i - x_l)^{n_l}} = \frac{-f_{11}^{(n-1)}(t_i)^2 + \nu_i^2}{(t_i - x_l)^{n_l}} = \frac{\alpha + \beta(t_i - x_l)^{n_l}}{(t_i - q_{n_{l_i}+1})(t_i - q_{n_{l_i}})} \]

where \( \alpha := \beta \nu_i (f_{11}(t_i) - \lambda_{n_{l_i}+1}) \) and \( \beta := (f_{11}(t_i) - \lambda_{n_{l_i}-2})^2 \). Then, we have the finite value \( f_{12}^{(n-1)}(t_i) \) as \( x_l \to t_i \). By the values \( f_{12}^{(n-1)}(t_1), \ldots, f_{12}^{(n-1)}(t_n) \) as \( x_l \to t_i \), we can determine all coefficients of \( f_{12}^{(n-1)}(z) \).

We obtain an extended map \( M_{n}^{0_n} \to \text{Hilb}^{n-3}(\mathcal{K}_n) \) by the assignment of \( \mathcal{I}_{x_l, a_{i',i}}^{n_l} = ((q - t_i)^{n_l}, f_{x_l, a_{i', i}}^{n_l}(q, v^{i',e})) \) to the matrix \( \begin{pmatrix} f_{11}^{(n-2)} & f_{12}^{(n-1)} \\ f_{21}^{(n-2)} & f_{22}^{(n-1)} \end{pmatrix} \). This procedure is confirmed by the implication that the existence of \( \lim_{x_l \to t_i} f_{12}^{(n-1)}(t_i) \) determines the data \( f_{x_l, a_{i', i}}^{n_l}(q, v^{i',e}) \) from the argument above. The extended map is injective.

\[ \square \]

3.2. Geometric description of \( \widehat{M}_H \) for \( n = 5 \). Suppose that \( n = 5 \). By the apparent singularities and the dual parameters of \( (E, \Phi, \varphi, [\sigma]) \in \widehat{M}_H \), we have the following map

\[ \widehat{M}_H \to \text{Sym}^2(\mathcal{K}_5) \]

\[ (E, \Phi, \varphi, [\sigma]) \mapsto \{(q_1, p_1), (q_2, p_2)\} \]

where \( \{q_1, q_2\} \) are apparent singularities and \( \{p_1, p_2\} \) are their dual parameter where \( p_i \) corresponds to \( q_i \) for \( i = 1, 2 \). There exists a stratification \( \widehat{M}_H = \widehat{M}_H^0 \cup \widehat{M}_H^1 \) where \( \widehat{M}_H^0 \) is the locus such that
Then we have the natural extended map \( \widetilde{\zeta} \) where \( \widetilde{\zeta} \). Since \( (q_1, p_1), (q_2, p_2) \) are the pairs of the apparent singularities and the dual parameters. We define the map \( \widetilde{\mathcal{M}}_H \) passes through the points \( (q_1, p_1), (q_2, p_2) \) by \( \widetilde{\mathcal{M}}_H \) := \{ (E, \Phi, \varphi, [\sigma]) \in \mathcal{M}_H \mid \lambda_+ = \infty \} \in \text{Hilb}^2(\mathcal{K}_5') \)

where \((q, p), (q, -p)\) are the pairs of the apparent singularities and the dual parameters, and \(\lambda_+\) is the parameter of the fiber of the Hilbert-Chow morphism, that is, \(p_2 - p_1 = \lambda_+(q_2 - q_1)\).

Next, we extend the map \( \mathcal{M}_H \rightarrow \text{Hilb}^2(\mathcal{K}_5') \). For \((E, \Phi, \varphi, [\sigma]) \in \mathcal{M}_H\), we describe the Higgs field \( \Phi \) as follows:

\[
\Phi = \begin{cases}
A_1^1 \otimes \omega_z & \text{on } U_0 \\
R_0^{-1}(A_1^1 \otimes \omega_z)R_0 & \text{on } U_\infty
\end{cases}
\]

where \( A_1^1 := \left( \begin{array}{cc} f^{(3)}_{11}(z) & f^{(6)}_{12}(z) \\ f^{(6)}_{21}(z) & -f^{(3)}_{11}(z) \end{array} \right) \).

By the automorphism of \( E \cong \mathcal{O}(1) \oplus \mathcal{O}(-2) \), we can normalize \( A_2^1 \) as follows:

\[
A_2^1 = \begin{pmatrix} 0 & f^{(6)}_{12}(z) \\ 1 & 0 \end{pmatrix}
\]

where we put \( f^{(6)}_{12}(z) := b_0 + b_1 z + \cdots + b_6 z^6 \). The spectral curve \( C_s \subset \mathcal{K}_5 \) is defined by \( \eta^2 - f^{(6)}_{12}(z) = 0 \). The curve \( C_s \) passes through the points \( (t_1, \hat{v}_1), (t_1, -\hat{v}_1), \ldots, (t_n, \hat{v}_n), (t_n, -\hat{v}_n), (q, p), (q, -p) \). Here, \((q, p), (q, -p)\) are the pairs of the apparent singularities and the dual parameters. We define the map \( \mathcal{M}_H \rightarrow \text{Hilb}^2(\mathcal{K}_5') \) by

\[
\mathcal{M}_H \ni (E, \Phi, \varphi, [\sigma]) \rightarrow \{ (t_i, \hat{v}_i, v), (t_i, -\hat{v}_i, -v) \} \in \text{Hilb}^2(\mathcal{K}_5')
\]

where \( \{(t_i, \hat{v}_i), (t_i, -\hat{v}_i)\} \) are the apparent singularities and the dual parameters of \((E, \Phi, \varphi, [\sigma])\), and

\[
v = \lim_{q \to t_i} p - \hat{v}_i = \frac{1}{2 \hat{v}_i} \frac{d}{dz} f^{(6)}_{12}(t_i).
\]

Then we have the natural extended map \( \mathcal{M}_H \rightarrow \text{Hilb}^2(\mathcal{K}_5') \).

Let \( \text{Hilb}^2(\mathcal{K}_5') \) be the blowing-up of \( \text{Hilb}^2(\mathcal{K}_5') \) along \( Z \). We show that the spectral curves are determined by the point of \( \text{Hilb}^2(\mathcal{K}_5') \). Let \( \breve{x} := \{ (q_1, p_1, v_1^{\pm}), (q_2, p_2, v_2^{\pm}) \} \), \( \lambda_+, \lambda_- \) be a point of \( \text{Hilb}^2(\mathcal{K}_5') \) where \( p_2 - p_1 = \lambda_+(q_2 - q_1) \) and \( p_2 + p_1 = \lambda_-(q_2 - q_1) \). First, any spectral curves pass through the points \( (t_i, \hat{v}_i) \) and \( (t_i, -\hat{v}_i) \) for \( i = 1, \ldots, n \), that is, the polynomial \( f^{(6)}_{12}(z) \) satisfies the condition \( \hat{v}_i^2 - f^{(6)}_{12}(t_i) = 0 \). By the equations, we can determine the coefficients \( b_1, b_2, b_3, b_6 \) by \( b_4, b_5 \). Second, the spectral curves pass through the points \( \{ (q_1, p_1), (q_2, p_2) \} \), that is, \( p_1^2 - f^{(6)}_{12}(q_1) = 0 \) for \( i = 1, 2 \). If \( q_1 \neq q_2 \), then the coefficients \( b_4 \) and \( b_5 \) are determined by \( (q_1, p_1) \) and \( (q_2, p_2) \), that is, the spectral curve is determined by
the apparent singularities and the dual parameters. We consider the behavior of the spectral curve as \( q_2 \to q_1 \) and \( p_2 \to -p_1 \). We consider the following equations

\[
\begin{cases}
p_1^2 - \tilde{f}_1''(q_1) = 0 \\(-p_1 + \lambda_-(q_2 - q_1))^2 - \tilde{f}_1''(q_2) = 0.
\end{cases}
\]

as \( q_2 \to q_1 \) and \( p_2 \to -p_1 \). When \( q_1, q_2 \notin \{t_1, \ldots, t_5\} \), we can determine the coefficients \( b_4 \) and \( b_5 \) by \( q_1, p_1 \) and \( \lambda_- \). We consider the case \( q_1 = q_2 = t_i \) for some \( i \). Let \( \lambda^+_i \) be the parameter such that \( v^{i,+}_1 + v^{i,+}_2 = \lambda^+_i(q_2 - q_1) \), which is a blowing-up parameter of \( \tilde{\text{Hilb}}^2(K'_0) \to \text{Hilb}^2(K'_0) \). When we take \( q_1 \to t_i \) and \( q_2 \to t_i \), we have the following equations

\[
\begin{cases}
v^{i,+}_1 = \frac{1}{2 \sigma_i^2} \frac{d^2}{dz^2} \tilde{f}_1''(t_i) \\\lambda_- = \frac{1}{2 \sigma_i^2} \left( \frac{d^2}{dz^2} \tilde{f}_1''(t_i) - \frac{1}{2 \sigma_i^2} \left( \frac{d}{dz} \tilde{f}_1''(t_i) \right)^2 \right).
\end{cases}
\]

By these equations, we can determine the coefficients \( b_4 \) and \( b_5 \) by \( q_1, v^{i,+}_1 \) and \( \lambda_- \). Therefore, spectral curves are determined by the point of \( \tilde{\text{Hilb}}^2(K'_0) \). If the underlying vector bundles of Higgs bundles are \( \mathcal{O}(1) \oplus \mathcal{O}(-2) \), then Higgs fields are determined by points of \( \tilde{\text{Hilb}}^2(K'_0) \) by the normalization [26]. On the other hand, cyclic vectors of bundle type \( \mathcal{O}(1) \oplus \mathcal{O}(-2) \) are determined by the apparent singularities \( q_1 = q_2 \). Then we have that the restriction map \( \hat{\mathcal{M}}_H \to \text{Hilb}^2(K'_0) \) is injective. Finally, we obtain that the restriction map \( \hat{\mathcal{M}}_H \to \text{Hilb}^2(K'_0) \) is injective.

We can describe the image of the map [25] as follows. We define parameters \((v^{j,\pm})_{1 \leq j \leq 5}\) by

\[p - v^{j,\pm}_j(t_j - t_k_1)(t_j - t_k_2)(t_j - t_k_3) = v^{j,\pm}(q - t_j)\]

The parameters \((v^{i,\pm})_{1 \leq i \leq 5}\) are blowing-up parameters of \( \text{Bl}_{v^{i,\pm}} L \to L \). The moduli space \( \hat{\mathcal{M}}_H \) is stratified as \( \hat{\mathcal{M}}_H = \hat{\mathcal{M}}_H^0 \cup \hat{\mathcal{M}}_H^1 \) where \( \hat{\mathcal{M}}_H^1 \) is the locus such that \((E, \Phi, \varphi, \sigma) \in \hat{\mathcal{M}}_H^1 \) satisfies \( E \cong \mathcal{O}(i) \oplus \mathcal{O}(-i - 1) \). Then the images of \( \hat{\mathcal{M}}_H^0 \) and \( \hat{\mathcal{M}}_H^1 \) in \( \tilde{\text{Hilb}}^2(K'_0) \) are the following

\[
\hat{\mathcal{M}}_H^0 \cong \left\{ \left( ((q_1, p_1, v^{i,\pm}_1), (q_2, p_2, v^{i,\pm}_2)) \right) \mid q_1 \neq q_2, v^{i,\pm}_j \in \mathbb{C} \right\} \cup \\
\left\{ \left( ((q_1, p_1, v^{i,\pm}_1), (q_2, p_2, v^{i,\pm}_2)), \lambda_+ \right) \mid q_1 - q_2 = p_1 - p_2 = 0, \lambda_+ \in \mathbb{C} \right\}
\]

\[
\hat{\mathcal{M}}_H^1 \cong \left\{ \left( ((q_1, p_1, v^{i,\pm}_1), (q_2, p_2, v^{i,\pm}_2)), \lambda_+, \lambda_- \right) \mid q_1 - q_2 \neq t_j \text{ for any } j = 1, \ldots, 5, \right. \\
q_1 - q_2 = p_1 + p_2 = 0, \lambda_+ = \infty, \lambda_- \in \mathbb{C} \right. \cup \\
\left. \left\{ \left( ((q_1, p_1, v^{i,\pm}_1), (q_2, p_2, v^{i,\pm}_2)), \lambda_+, \lambda_- \right) \mid q_1 = q_2 = t_j, \right. \\
p_1 - p_2 = v^{j,\pm}_j(t_j - t_k_1)(t_j - t_k_2)(t_j - t_k_3), \lambda_+ = \infty, \lambda_- = \mathbb{C} \right. \cup \\
\left. \left\{ \left( ((q_1, p_1, v^{i,\pm}_1), (q_2, p_2, v^{i,\pm}_2)), \lambda_+, \lambda_- \right) \mid q_1 = q_2 = t_j, \right. \\
p_1 = p_2 = v^{j,\pm}_j(t_j - t_k_1)(t_j - t_k_2)(t_j - t_k_3), \lambda_+ = \infty, \lambda_- = \mathbb{C} \right. \cup \\
\left. \left\{ \left( ((q_1, p_1, v^{i,\pm}_1), (q_2, p_2, v^{i,\pm}_2)), \lambda_+, \lambda_- \right) \mid q_1 = q_2 = t_j, \right. \\
p_1 = p_2 = v^{j,\pm}_j(t_j - t_k_1)(t_j - t_k_2)(t_j - t_k_3), \lambda_+ = \infty, \lambda_- = \mathbb{C} \right. \cup \\
\left. \left\{ \left( ((q_1, p_1, v^{i,\pm}_1), (q_2, p_2, v^{i,\pm}_2)), \lambda_+, \lambda_- \right) \mid q_1 = q_2 = t_j, \right. \\
p_1 = p_2 = v^{j,\pm}_j(t_j - t_k_1)(t_j - t_k_2)(t_j - t_k_3), \lambda_+ = \infty, \lambda_- = \mathbb{C} \right.
\}
\]

Here, \( \lambda_\pm \) and \( \lambda_\pm' \) satisfy the following relations

\[p_1 - p_2 = \lambda_+(q_1 - q_2), \quad v^{i,\pm}_1 - v^{i,\pm}_2 = \lambda^{i}_+(q_1 - q_2)\]

\[p_1 + p_2 = \lambda_-(q_1 - q_2), \quad v^{i,\pm}_1 + v^{i,\pm}_2 = \lambda^{i}_-(q_1 - q_2)\]

**Remark 3.3.** We consider the map

\[\hat{\mathcal{M}}_H^1 \to \mathbb{P}^1(E, \Phi, \varphi, \sigma) \to q_1\]
given by the description \([28]\) and the natural projection. Any fiber of this map is \(C^2\), which is isomorphic to \(M_H^1\). Note that \(q_1\) is a coordinate of \(\mathbb{P}H^0(\mathbb{P}, E)\), which is a blowing-up parameter of \(\hat{M}_H \rightarrow M_H\).

4. Jumping families for Higgs bundles

In this section, we give an explicit description of the universal family of the moduli space \(\hat{M}_H\) for \(n = 5\). For the purpose, we need give a description of jumping family, which is a family of Higgs fields such that for generic parameters, the underlying vector bundles are \(\mathcal{O} \oplus \mathcal{O}(-1)\) and for special parameters, the underlying vector bundles are \(\mathcal{O}(1) \oplus \mathcal{O}(-2)\). Descriptions of jumping families are given by the lower and upper modifications. In \([4, 2]\) we apply the description of jumping families to the case \(n \geq 4\). Then we obtain explicit descriptions of jumping families for the case \(n \geq 4\).

4.1. Jumping family for \(n = 5\). Suppose that \(n = 5\). We consider the following covering of \(\hat{M}_H\):

\[
\begin{align*}
V_0 &:= \hat{M}_H^1 \subset \hat{M}_H, \\
V_1 &:= \left\{ (E, \Phi, \varphi) \in M_H^{00} \mid \{ \tilde{p}_1, \tilde{p}_2 \} \not\subset \text{Sym}^2[\text{branch points of } C_s] \right\} \cup \hat{M}_H^1 \subset \hat{M}_H,
\end{align*}
\]

where \(C_s\) is the spectral curve of \((E, \Phi, \varphi)\) and \(\tilde{p}_1 = (q_1, p_1) \in C_s\) is an apparent singularity and its dual of \((E, \Phi, \varphi)\). Since we consider sL2-Higgs bundles, \(p_1 = 0\) implies that \(\tilde{p}_1 = (q_1, p_1) \in C_s\) is a branch point of \(C_s\). By Theorem \([3, 1]\) we have an explicit description of the universal family \((E_{V_1}, \Phi_{V_1}[\sigma_{V_1}])\) on \(V_1 \times \mathbb{P}^1\).

Now we give an explicit description of the universal family \((E_{V_1}, \Phi_{V_1}[\sigma_{V_1}])\) on \(V_1 \times \mathbb{P}^1\). Then we have an explicit description of the universal family \((\hat{E}, \hat{\Phi}, [\hat{\sigma}])\) on \(\hat{M}_H \times \mathbb{P}^1\):

\[
\begin{align*}
(V_1 \times \mathbb{P}^1) &\subset \hat{M}_H \times \mathbb{P}^1 \rightarrow \hat{M}_H \times \mathbb{P}^1 \rightarrow V_0 \times \mathbb{P}^1.
\end{align*}
\]

Let

\[
\Phi_{V_0} = \begin{cases}
A_z^0 \otimes \omega_z & \text{on } U_0, \\
R_0^{-1}(A_z^0 \otimes \omega_z)R_0 & \text{on } U_\infty,
\end{cases}
\]

where \(A_z^0 = \begin{pmatrix} a_1z + a_0 & f_1^4(z) \\ (z - q_1)(z - q_2) & -(a_1z + a_0) \end{pmatrix}\), be the family on \(V_0\) obtained by Theorem \([3, 1]\) for \(n = 5\). Here we set

\[
a_1 := \frac{p_1 - p_2}{q_1 - q_2}, \quad a_0 := -\frac{p_1q_2 - p_2q_1}{q_1 - q_2}, \quad R_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1/z \end{pmatrix}
\]

and we assume that \(q_1, q_2 \neq \infty\) for simplicity. Set

\[
X := \left\{ ((q_1, p_1), (q_2, p_2), \lambda) \in (K^*_0)^2 \times \mathbb{C} \mid \begin{array}{c}
p_2 - p_1 = \lambda(q_2 - q_1), \\
p_1 \neq 0, \text{ and } q_2 - q_1 \neq 0
\end{array} \right\}
\]

and

\[
\bar{X} := X \cup \left\{ ((q_1, p_1), (q_2, p_2), \lambda) \in (K^*_0)^2 \times \mathbb{C} \mid p_2 - p_1 = q_2 - q_1 = 0 \right\}.
\]

Let \(P_1, P_2,\) and \(P_3\) be the following matrices

\[
P_1 := \begin{pmatrix} 1 & 0 \\ 0 & z - q_1 \end{pmatrix}, \quad P_2 := \begin{pmatrix} 1 & 0 \\ \frac{z - q_1}{2p_1} & 1 \end{pmatrix}, \quad P_3 := \begin{pmatrix} z - q_1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

Proposition 4.1. We define a family of \(v\)-sL2-parabolic Higgs bundle \((E_X, \Phi_X, \varphi_X, [\sigma_X])\) with a cyclic vector on \(X \times \mathbb{P}^1\) as

\[
\Phi_X = \begin{cases}
(P_1P_2P_3)^{-1}(A_z^0 \otimes \omega_z)P_1P_2P_3 & \text{on } U_0, \\
R_0^{-1}(A_z^0 \otimes \omega_z)R_0 & \text{on } U_\infty^0
\end{cases}
\]

where \(U_\infty^0 = \text{Spec } \mathbb{C}[w, 1/(q_1w - 1)]\). Here \(\sigma_X\) is the element of \(H^0(\mathbb{P}^1, E_X)\) such that the zero of \(\sigma_X\) is \(q_1\) when \(E_X \cong \mathcal{O}(1) \oplus \mathcal{O}(-2)\). We can extend this family on \(X \times \mathbb{P}^1\) to the family on \(\hat{X} \times \mathbb{P}^1\), naturally. For
the extended family, we have the following. If \( q_1 \neq q_2 \), then the underlying vector bundle is \( \mathcal{O} \oplus \mathcal{O}(-1) \). If \( q_1 = q_2 \), then the underlying vector bundle is \( \mathcal{O}(1) \oplus \mathcal{O}(-2) \).

**Proof.** We describe the construction of a family of \( \nu \)-\( \mathfrak{sl}_2 \)-parabolic connections with a cyclic vector on \( X \times \mathbb{P}^1 \) so that this family satisfies the assertions of the proposition. As the result, we obtain the family which has the description \( (33) \). Then this proposition follows from this construction.

By the natural map \( X \to \mathcal{V}_0 \), the family \( \Phi_{\mathcal{V}_0} \) on \( \mathcal{V}_0 \times \mathbb{P}^1 \) induces the family on \( X \times \mathbb{P}^1 \), denoted by \( \Phi_X^0 \). We consider the lower and upper modifications \( (q_1, l_{p_1})^{\text{up}} \circ (q_1, l_{p_1})^{\text{low}}(\Phi_X^0) \), denoted by \( \Phi_X \), where \( l_{p_1} \) is a one dimensional subspace of \( \mathcal{E}|_{q_1} \) which corresponds to the eigenspace of the residue of \( \Phi_X^0 \) at \( q_1 \) with the eigenvalue \( p_1 \). Explicitly, the modifications are described as follows. We consider the following diagram

\[
\begin{array}{ccc}
U_0 \times \mathbb{C}^2 & \xleftarrow{R_0} & U_\infty^1 \times \mathbb{C}^2 \\
\downarrow{P_1} & \leftarrow & \downarrow{T(q_2)} \\
U_0 \times \mathbb{C}^2 & \xrightarrow{P_2 \circ P_3} & U_0 \times \mathbb{C}^2 \\
\end{array}
\]

where

\[ T := \begin{pmatrix} z - q_1 & 0 \\ \frac{2q_1 - q_2}{2p_1} & \frac{1}{z - q_1} \end{pmatrix}. \]

In particular,

\[ T^0_{q_1} = \lim_{q_2 \to q_1} T = \begin{pmatrix} z - q_1 & 0 \\ 0 & \frac{1}{z - q_1} \end{pmatrix}. \]

Here, the transformation \( P_1 \) implies the lower modification \( (q_1, l_{p_1})^{\text{low}} \) and the transformation \( P_2 \circ P_3 \) implies the upper modification \( (q_1, l_{p_1})^{\text{up}} \). Namely, we describe \( \Phi_X \) as

\[ \Phi_X = (q_1, l_{p_1})^{\text{up}} \circ (q_1, l_{p_1})^{\text{low}}(\Phi_X) \]

\[ = \begin{cases} (P_1 P_2 P_3)^{-1}(A^0_1 \otimes \omega_2)P_1 P_2 P_3 & \text{on } U_0 \\ R_0^{-1}(A^0_2 \otimes \omega_2)R_0 & \text{on } U_\infty^1. \end{cases} \]

Taking the limit \( q_2 \to q_1 \), we have Higgs bundles of bundle type \( \mathcal{O}(1) \oplus \mathcal{O}(-2) \). Then we have the description of the family of the Higgs fields \( (33) \), and the family satisfies the assertion of the proposition. By the transition function \( (34) \), the zero of cyclic vectors is \( q_1 \) when \( E_X \cong \mathcal{O}(1) \oplus \mathcal{O}(-2) \). □

By this proposition, we have a map \( \hat{X} \to \mathcal{V}_2 \subset \hat{\mathcal{M}}_H \).

Next we compute the apparent singularities and the dual parameters of \( \Phi_X \) when \( q_1 \neq q_2 \). Put

\[ Q_1 := \begin{pmatrix} z - q_1 & 2p_1 \\ -\frac{q_1 - q_2}{2p_1} & 0 \end{pmatrix}, \quad Q_2 := \begin{pmatrix} 1 & -\frac{2p_1}{q_1 - q_2} \\ 0 & 1 \end{pmatrix}. \]

Then we have the following diagram

\[
\begin{array}{ccc}
U_0 \times \mathbb{C}^2 & \xrightarrow{T(q_2)} & U_\infty^1 \times \mathbb{C}^2 \\
\downarrow{Q_1} & \leftarrow & \downarrow{Q_2} \\
U_0 \times \mathbb{C}^2 & \xleftarrow{R_0} & U_\infty^2 \times \mathbb{C}^2. \\
\end{array}
\]

By the transformation by \( Q_1 \) and \( Q_2 \), we can describe \( \Phi_X \) as in the description \( (33) \). The apparent singularities and the dual parameters of \( \Phi_X \) is \{\((q_1, -p_1), (q_2, p_2)\)\} (Figure 2). Then we have a map
Here, the entries of $A$. Moreover, we can extend this map to $\tilde{X} \to \text{Hilb}^2(K'_5)$ by

$$\tilde{X} \setminus X \ni ((q_1, p_1), (q_1, p_1), \lambda) \mapsto ((q_1, -p_1), (q_1, p_1), \infty, \lambda) \in \text{Hilb}^2(K'_5)$$

where $(\infty, \lambda)$ are values of a parameter of pre-images of the Hilbert-Chow morphism and of a blowing-up parameter of $\text{Hilb}^2(K'_5) \to \text{Hilb}^2(K'_5)$. We obtain the commutative diagram

$$\begin{array}{ccl}
\tilde{X} & \longrightarrow & \text{Hilb}^2(K'_5) \\
\downarrow & & \\
\mathcal{V}_1 & \longrightarrow & \text{Hilb}^2(K'_5).
\end{array}$$

By this diagram, the explicit description of the family (33) induces an explicit description of the universal family on $\mathcal{V}_1 \times \mathbb{P}^1$ which is parametrized by the apparent singularities and their duals.

**Figure 2.** The apparent singularities and the dual parameters

### 4.2. Jumping families for $n \geq 5$

In this section, we give an explicit description of jumping families for $n \geq 5$ as in the previous section.

Let

$$\Phi_0 = \begin{cases} 
A_0^0 \otimes \omega_z & \text{on } U_0 \\
R_0^{-1}(A_0^0 \otimes \omega_w)R_0 & \text{on } U_\infty
\end{cases},$$

where

$$A_0^0 = \begin{pmatrix}
0 & f_1^{(n-2)}(z) & f_1^{(n-1)}(z) \\
(z-q_1) & 0 & -(z-q_1) \\
(z-q_1) & -(z-q_1) & 0
\end{pmatrix},$$

be the family on $M_H^n$ obtained by Theorem 3.1. We assume that $q_1, \ldots, q_{n-3} \neq \infty$ for simplicity.

First, we construct a family having Higgs bundles of bundle type $\mathcal{O}(1) \oplus \mathcal{O}(-2)$ from the family $\Phi_0$ by lower and upper modifications as in 4.1. Fix $(q_1, p_1) \in K'_n$ and assume that $q_1 \neq q_2$. Put

$$\Phi_1 := (q_1, t_{p_1})^{\text{up}} \circ (q_1, t_{p_1})^{\text{low}}(\Phi_0).$$

If we take the limit $q_2 \to q_1$ of $\Phi_1$, then we have Higgs bundles of bundle type $\mathcal{O}(1) \oplus \mathcal{O}(-2)$:

$$\lim_{q_2 \to q_1} \Phi_1 = \begin{cases}
A_1^1 \otimes \omega_z & \text{on } U_0 \\
(R_1^{q_2})^{-1}(A_1^1 \otimes \omega_w)R_1^{q_2} & \text{on } U_\infty
\end{cases},$$

where

$$A_1^1 = \begin{pmatrix}
0 & f_1^{(n-2)}(z) & f_1^{(n+1)}(z) \\
(z-q_1) & 0 & -(z-q_1) \\
(z-q_1) & -(z-q_1) & 0
\end{pmatrix},$$

and

$$R_1^{q_2} = \begin{pmatrix}
z-q_1 & 0 \\
0 & 1 \\
\frac{1}{z(q_1-q_1)}
\end{pmatrix}.$$

Here, the entries of $A_1^1$ satisfy the following equations:

$$\begin{aligned}
p_i - f_1^{(n-2)}(q_i) &= 0 \quad \text{for } i = 3, \ldots, n-3, \\
p_i^2 - \det(A_1^1)_{z=q_i} &= 0, \\
\det(\text{res}_{z=q_i} \Phi_1) - \nu_i^2 &= 0 \quad \text{for } i = 1, \ldots, n.
\end{aligned}$$

Note that $q_1$ is the zero of the corresponding cyclic vector $\sigma \in H^0(\mathbb{P}^1, E)$. 


Next, we construct a family having Higgs bundles of bundle type \( O(2) \oplus O(-3) \) from the family \( \Phi_1 \). Here, we assume that \( q_1, \ldots, q_{n-3} \neq \infty \) for simplicity. For the Higgs field \( (29) \), fix \( (q_3, p_3) \in K_n \) and assume that \( q_3 \neq q_4 \). Put \( \Phi_2 := (q_3, l_{p_3})^{up} \circ (q_3, l_{p_3})^{low} \left( \lim_{q_2 \to q_1} \Phi_1 \right) \).

If we take the limit \( q_4 \to q_3 \) of \( \Phi_2 \), then we have Higgs bundles of bundle type \( O(2) \oplus O(-3) \):

\[
\lim_{q_2 \to q_1} \Phi_2 = \begin{cases} A_2^2 \otimes \omega_z & \text{on } U_0 \\ (R_2^{q_1})^{-1}(A_2^2 \otimes \omega_z) & \text{on } U_0^{q_1} \end{cases}
\]

where \( A_2^2 = \begin{pmatrix} -f_{11}^{(n-2)}(z) & f_{12}^{(n-3)}(z) \\ (z-q_5) \cdots (z-q_{n-3}) & -f_{11}^{(n-2)}(z) \end{pmatrix} \)

and

\[
R_2^{q_1} = \begin{pmatrix} (z-q_1)(z-q_3) & 0 \\ 0 & z/(z-q_3) \end{pmatrix}.
\]

Here, the entries of \( A_2^2 \) satisfy the following equations:

\[
\begin{align*}
p_i - f_{11}^{(n-2)}(q_i) &= 0 & \text{for } i = 5, \ldots, n-3, \\
p_i^2 - \det(A_2^2)_{i=1} &= 0 & \text{for } i = 1, 3, \\
\det \left( \text{res}_{z=q_i} \Phi_2 - \nu^i \right) &= 0 & \text{for } i = 1, \ldots, n.
\end{align*}
\]

Note that \( q_1, q_3 \) is the zeros of the corresponding cyclic vector \( \sigma \in H^0(\mathbb{P}^1, E) \). We continue this process. Then we have family having Higgs bundles of bundle type \( O(k) \oplus O(-k-1) \) for \( k = 1, \ldots, [(n-3)/2] \).

5. Geometric description for connection cases

Suppose that \( \nu \) satisfies the condition \( (3) \) and \( \nu_1 \cdots \nu_n \neq 0 \). We put

\[
(t_1, \ldots, t_n) := (0, 1, x_1, \ldots, x_{n-3}, \infty),
\]

\[
(\nu_1^+, \ldots, \nu_{n-1,3}^+, \nu_{n}^-) := (\pm \nu_0, \pm \nu_1, \ldots, \pm \nu_{n-1}, \nu_n, 1-\nu_n),
\]

\[
\nu_i := \nu_i(t_i - t_1) \cdots (t_i - t_{i-1})(t_i - t_{i+1}) \cdots (t_i - t_{n-1}) \text{ for } i = 1, \ldots, n-1.
\]

Let \( M^k \) (resp. \( \widehat{M}^k \)) be the subvariety of \( M \) (resp. \( \widehat{M} \)) where \( E \equiv O(k) \oplus O(-k-1) \). First, we define the apparent singularities and the dual parameters of \( (E, \nabla, \varphi) \in M^0 \), and we compute the apparent singularities and the dual parameters for \( n \geq 4 \). Then we have Theorem 1.3 (Theorem 5.2). Second, we assume \( n = 5 \). We construct a jumping family. Third, we compute the apparent singularities of this jumping family on the locus of bundle type \( O \oplus O(-1) \), and we analyze the behavior of the apparent singularities of the jumping families when the parameter closes to the jumping locus. Then we obtain a map from \( \widehat{M} \) to the Hilbert scheme of points on some surface. Moreover, we take some sequence of blowing-ups of the Hilbert scheme. Then we obtain an injective map from \( \widehat{M} \) to the blowing-ups.

5.1. Geometric description of \( M^0 \) for \( n \geq 4 \). Let \( (E, \nabla, \varphi) \in M \). We can define the apparent singularities of \( (E, \nabla, \varphi) \in M \) as follows. We fix a section \( \sigma \in H^0(\mathbb{P}^1, E) \). For the section \( \sigma \), we define the following composition

\[
O_{p_1} \xrightarrow{\sigma} E \xrightarrow{\nabla} E \otimes L \rightarrow (E/O_{p_1}) \otimes L.
\]

The composition \( O_{p_1} \rightarrow (E/O_{p_1}) \otimes L \) is an \( O_{p_1} \)-morphism, which is injective. Then we can define a subsheaf \( F^0 \subset E \) such that \( O_{p_1} \rightarrow (F^0/O_{p_1}) \otimes L \) is an isomorphism. By the isomorphism \( F^0/O_{p_1} \cong L^{-1} \), we have \( F^0 \cong O_{p_1} \oplus L^{-1} \). Therefore, we have the following exact sequence

\[
0 \rightarrow O_{p_1} \otimes L^{-1} \rightarrow E \rightarrow T_A \rightarrow 0
\]

where \( T_A \) is a torsion sheaf. By the Riemann-Roch theorem, we have that the torsion sheaf \( T_A \) is length \( n-3 \).

**Definition 5.1.** For \( (E, \nabla, \varphi) \in M \) and a nonzero section \( \sigma \in H^0(\mathbb{P}^1, E) \), we call the support of \( T_A \) apparent singularities of a \( \nu \)-parabolic connection with a cyclic vector \( (E, \nabla, \varphi, [\sigma]) \).
For $(E, \nabla, \varphi) \in M^0$, we define dual parameters as follows. Since $E \cong \mathcal{O} \oplus \mathcal{O}(-1)$, we can denote the connection $\nabla$ by

$$
\nabla = \begin{cases} 
d + A_2^0 \otimes \omega_2, & \text{on } U_0 \\
g + R_0^{-1}dR_0 + R_0^{-1}(A^0 \otimes \omega_2)R_0, & \text{on } U_\infty 
\end{cases}
$$

where $A_2^0 := \begin{pmatrix} f^{(n-2)}_1(z) & f^{(n-2)}_2(z) \\ f^{(n-3)}_1(z) & -f^{(n-2)}_1(z) \end{pmatrix}$.

Note that the zeros of the polynomial $f^{(n-2)}_2(z)$ are the apparent singularities of $(E, \nabla, \varphi)$. We denote by $\{q_1, \ldots, q_{n-3}\}$ the apparent singularities. We put $p_i := f^{(n-2)}_1(q_i) \in L_{q_i}$. We call $\{p_1, \ldots, p_{n-3}\}$ the dual parameters of $(E, \nabla, \varphi) \in M^0$. The definition of the apparent singularities and the dual parameters is already given by Oblezin [12, Section 3].

Let $\tilde{K}'_n$ be the Zariski open set of the blowing-up of Hirzebruch surface of degree $n - 2$ defined in 2.2 and $\tilde{K}_n$ be the contraction $\tilde{K}'_n \to \tilde{K}_n$. Then we can define the following map

$$
M^0 \longrightarrow \text{Sym}^{n-3}(\tilde{K}_n)
$$

\begin{equation}
(E, \nabla, \varphi) \mapsto \{(q_1, p_1), \ldots, (q_{n-3}, p_{n-3})\},
\end{equation}

which is already constructed in [12, Section 3]. We consider the composite of the Hilbert-Chow morphism and the blowing-up

$$
\text{Hilb}^{n-3}(\tilde{K}'_n) \longrightarrow \text{Sym}^{n-3}(\tilde{K}'_n) \longrightarrow \text{Sym}^{n-3}(\tilde{K}_n),
$$

where $\tilde{K}'_n \to \tilde{K}_n$ is the blowing up defined in 2.2. By the same argument as in the proof of Theorem 3.1 we obtain the following

**Theorem 5.2.** We can extend the map \((41)\) to

$$
M^0 \longrightarrow \text{Hilb}^{n-3}(\tilde{K}'_n).
$$

This map is injective. Moreover, we can give an explicit description of the universal family $(\hat{E}^{(0)}, \hat{\varphi}^{(0)}) \to M^0 \times \mathbb{P}^1$.

### 5.2. Jumping family for $n = 5$.

Suppose that $n = 5$. In this section, we give an explicit description of a jumping family of connections.

Let $K'_n$ be the Zariski open set of the blowing-up of Hirzebruch surface of degree $n - 2$ corresponding to the moduli space of parabolic Higgs bundles (defined in 2.2), and $K_n$ be the contraction $K'_n \to K_n$. Fix $(\epsilon_0, e_1) \in \mathbb{C}^2$. Set

\begin{equation}
X := \{(q_1, p_1), (q_2, p_2), \lambda) \in (K'_n)^2 \times \mathbb{C} \mid p_2 - p_1 = \lambda(q_2 - q_1), p_1 \neq 0, \text{and } q_2 - q_1 \neq 0 \}
\end{equation}

and

\begin{equation}
\hat{X} := X \cup \{(q_1, p_1), (q_2, p_2), \lambda) \in (K'_n)^2 \times \mathbb{C} \mid p_2 - p_1 = q_2 - q_1 = 0 \},
\end{equation}

which are defined in 4.1 and let $Q_1, Q_2$ and $R_0$ be the following matrices

$$
Q_1 := \begin{pmatrix} -q_1 & 2p_1 \\ -q_1 - q_2 & 2p_1 \\ \frac{q_1 - q_2}{w} & 0 \end{pmatrix}, \quad Q_2 := \begin{pmatrix} 1 & 2p_1 w^2 \\ 0 & (q_1 - q_2)w + q_2 - 1 \end{pmatrix}, \quad R_0 := \begin{pmatrix} 1 & 0 \\ 0 & 1/z \end{pmatrix},
$$

respectively. ($Q_1$ and $Q_2$ were defined in 4.1).

**Proposition 5.3.** We define a family of $\nu$-sl2-parabolic connections $(E_X, \nabla_X, \varphi_X, [\sigma_X])$ with a cyclic vector on $X \times \mathbb{P}^1$ as

\begin{equation}
\nabla_X = \begin{cases} 
d + \bar{A}_0 dQ_1^{-1} + Q_1 \tilde{A}_0 Q_1^{-1}, & \text{on } U_0 = \text{Spec } \mathbb{C}[z] \\
d + \bar{Q}_2 dQ_2^{-1} + Q_2 \left(R_0 dR_0 + R_0^{-1} \tilde{A}_0 R_0 \right)Q_2^{-1}, & \text{on } U_{\tilde{Q}_2} = \text{Spec } \mathbb{C}[w, 1/(q_1 w - 1)]
\end{cases}
\end{equation}

Here $\tilde{A}_0$ is defined below, and $\sigma_X$ is the element of $H^0(\mathbb{P}^1, E_X)$ such that the zero of $\sigma_X$ is $q_1$ when $E_X \cong \mathcal{O}(1) \oplus \mathcal{O}(-2)$. We can extend this family on $X \times \mathbb{P}^1$ to the family on $\hat{X} \times \mathbb{P}^1$, naturally. For the extended family, we have the following. If $q_1 \neq q_2$, then the underlying vector bundle is $\mathcal{O} \oplus \mathcal{O}(-1)$. If $q_1 = q_2$, then the underlying vector bundle is $\mathcal{O}(1) \oplus \mathcal{O}(-2)$. 

We define a family of connection matrices $\tilde{A}_{U_0}$ on $X \times U_0$ as
\begin{equation}
\tilde{A}_{U_0} = \begin{pmatrix} F_{11}(z) & F_{12}(z) \\ F_{21}(z) & -F_{11}(z) \end{pmatrix} \frac{dz}{z(z-1)(z-x_1)(z-x_2)}
\end{equation}
where
\begin{align*}
F_{11}(z) &:= \frac{2p_1(z-q_1)}{q_2-q_1} + (z-q_1)\lambda + \frac{1}{2} \left( (1-2\nu_5 e_0(q_2-q_1))z^3 + (-2\nu_5 e_1(q_2-q_1) + q_1 - x_1 - x_2 - 1)z^2 \\
&\quad + (2p_1 - \nu_5 q_1^2(q_2-q_1))e_0 + 2\nu_5 q_1(q_2-q_1)e_1 - \frac{\nu_5(q_2-q_1)}{p_1}q_1(q_1-1)(q_1-x_1)(q_1-x_2) \\
&\quad + q_1^2 - (1+x_1+x_2)q_1 + x_1 + x_2 + x_1x_2 \right)z - 2p_1(1+e_1+e_0q_1) + (q_1-1)(1-q_1-x_1)(q_1-x_2), \\
F_{21}(z) &:= (1+e_0(q_1-q_2))z^2 - (q_1 + q_2 + e_1(q_2-q_1))z \\
&\quad + q_1q_2 + \frac{(q_1-q_2)q_1}{2p_1}(-2e_1p_1 - 2e_0q_1 + (q_1-1)(1-q_1-x_1)(q_1-x_2)).
\end{align*}
We omit the description of $F_{12}(z)$, since the description is lengthened and is not necessary for computation of the apparent singularities and their duals.

By this proposition, we have a map $\tilde{X} \to \tilde{M}$. On the other hand, we have a map $X \to \text{Hilb}^{n-3}(\tilde{K}_n)$ defined by $((q_1, p_1), (q_2, p_2), \lambda) \mapsto \{(q_1', p_1'), (q_2', p_2')\}, \lambda$ where $q_1', q_2'$ are the zero of $F_{21}(z)$, and $p_1', p_2'$ are their duals, that is, $p_i = F_{11}(q_i')$. Then we have the following diagram
\begin{equation}
\begin{tikzcd}
\tilde{X} \ar{r} & \tilde{M} \ar{r} & \text{Hilb}^2(\tilde{K}_n).
\end{tikzcd}
\end{equation}

**Proof of Proposition 5.3** We describe the construction of a family of $\nu$-$\mathfrak{sl}_2$-parabolic connections with a cyclic vector on $X \times \mathbb{P}^1$ so that this family satisfies the assertions of the proposition. As the result, we obtain the family which has the description (44). Then this proposition follows from this construction.

Let $A_0^0 \otimes \omega_z$ be the Higgs field (29) on $X \times U_0$ defined in (4.1). Let $\Phi_X$ be the Higgs field on $X \times \mathbb{P}^1$ defined by
\begin{equation}
P^{-1}(A_0^0 \otimes \omega_z)P \text{ on } U_0, \quad \text{and} \quad R_0^{-1}(A_0^0 \otimes \omega_z)R_0 \text{ on } U_\infty
\end{equation}
where the eigenvalues of the residue matrices are
\begin{align*}
\begin{array}{|c|c|c|c|c|}
\hline
\text{res}_0 \Phi_X & \text{res}_1 \Phi_X & \text{res}_{i_1} \Phi_X & \text{res}_{i_2} \Phi_X & \text{res}_{\infty} \Phi_X \\
\hline
\nu'_1, -\nu'_1 & \nu'_2, -\nu'_2 & \nu'_3, -\nu'_3 & \nu'_4, -\nu'_4 & \nu'_5, -\nu'_5 \\
\hline
\end{array}
\end{align*}
Here we put $P := P_1P_2P_3$ where $P_1, P_2,$ and $P_3$ are the matrices (32). Note that the underlying vector bundles have bundle type $\mathcal{O}(1) \oplus \mathcal{O}(-2)$ when $q_1 = q_2$. When $q_1 \neq q_2$, by $Q_1$ and $Q_2$ which are the matrices (36), we denote the Higgs field $\Phi_X$ by
\begin{equation}
Q_1^{-1}P^{-1}(A_0^0 \otimes \omega_z)PQ_1 \text{ on } U_0, \quad \text{and} \quad Q_2^{-1}R_0^{-1}(A_0^0 \otimes \omega_z)R_0Q_2 \text{ on } U_\infty
\end{equation}
as in the description (3).

Now we construct a family of initial connections $\nabla_0$ on $X \times \mathbb{P}^1$ and we determine $\nu'_1, \ldots, \nu'_5$ so that $\nabla_0 + \Phi_X$ is the desired family $\nabla_X$. We put
\begin{equation}
T_\infty = \begin{pmatrix} 1 & \nu'_5 \\ 0 & 1 \end{pmatrix}.
\end{equation}
Then we have
\begin{align*}
T_\infty^{-1} \text{res}_n \Phi \nu_0 T_\infty & = T_\infty^{-1} \begin{pmatrix} 0 & -\nu'^2_5 \\ -1 & 0 \end{pmatrix} T_\infty = \begin{pmatrix} \nu'_5 & 0 \\ -1 & -\nu'_5 \end{pmatrix}.
\end{align*}
Let \( \nabla_0 : \mathcal{O} \oplus \mathcal{O}(-1) \to (\mathcal{O} \oplus \mathcal{O}(-1)) \otimes \Omega^1_{\mathcal{O}}(D) \) be the connection defined by

\[
\begin{align*}
&\begin{cases}
  d + R_{0,w}^{-1} dR_{0,w} + R_{0,w}^{-1} T_{\infty}(B_w \otimes \omega_w)T_{\infty}^{-1}R_{0,w} & \text{on } U_0 \\
  d + T_{\infty}(B_w \otimes \omega_w)T_{\infty}^{-1} & \text{on } U_\infty
\end{cases}
\end{align*}
\]

where we put \( R_{0,w} := \begin{pmatrix} 1 & 0 \\ 0 & 1/w \end{pmatrix} \) and

\[
B_w := \begin{pmatrix} f_3 w^3 + f_2 w^2 + f_1 w + f_0 \\ (q_1 - q_2)(e_2 w^2 + e_1 w + e_0) \end{pmatrix} (w-1)(x_1 w-1)(x_2 w-1) - (f_3 w^3 + f_2 w^2 + f_1 w + f_0)
\]

where \( f_0, \ldots, f_3, d_1, \ldots, d_4, e_0, e_1, \) and \( e_2 \) are parameters. When \( q_1 \neq q_2 \), we define the connection \( \nabla_0 + \Phi_X \) as

\[
\begin{align*}
&\begin{cases}
  d + R_{0,w}^{-1} dR_{0,w} + R_{0,w}^{-1} T_{\infty}(B_w \otimes \omega_w)T_{\infty}^{-1}R_{0,w} + Q_1^{-1} P^{-1}(A^0 \otimes \omega_z)PQ_1 & \text{on } U_0 \\
  d + T_{\infty}(B_w \otimes \omega_w)T_{\infty}^{-1} + Q_2^{-1} R_0^{-1}(A^0 \otimes \omega_z)R_0Q_2 & \text{on } U_\infty.
\end{cases}
\end{align*}
\]

Then the eigenvalues of the residue matrix of \( \nabla_0 + \Phi_X \) at \( \infty \) is the following:

| \text{res}_\infty \Phi_X | \text{res}_\infty (\nabla_0 + \Phi_X) |
|-----------------------|---------------------|
| \nu'_5, -\nu'_5    | -f_0 + \nu'_5, 1 - (-f_0 + \nu'_5). |

We put \( (\nabla_0)U_0 = R_{0,w}^{-1} dR_{0,w} + R_{0,w}^{-1} T_{\infty}(B_w \otimes \omega_w)T_{\infty}^{-1}R_{0,w} \). First, we determine the parameters of \( B_w \) so that the limit \( \lim_{q_2 \to q_1} (Q_1 d(Q_1^{-1}) + Q_1(\nabla_0)U_0 Q_1^{-1}) \) is convergence. We claim that if we determine the parameters \( f_0, f_1, f_2, d_1, d_2, e_1, e_3 \) such that the following polynomial

\[
(2 f_0 - 1) z^4 + (1 + 2 f_1 - 2 f_0 q_1 + x_1 + x_2) z^3 + (2 f_2 + 2 e_0 p_1 - 2 f_1 q_1 - x_1 - x_2 - x_1 x_2) z^2
+ (2 f_3 + 2 e_1 p_1 - 2 f_2 q_1 + x_1 x_2) z + 2(e_2 p_1 - f_3 q_1)
\]

is identically zero, then the limit is convergence. We solve the simultaneous linear equations as follows:

\[
\begin{align*}
f_0 &= \frac{1}{2}, \quad f_1 = \frac{1}{2}(q_1 - x_1 - x_2 - 1), \quad f_2 = \frac{1}{2}(-2 e_0 p_1 + q_1^2 + x_1 + x_2 + x_1 x_2 - q_1(1 + x_1 + x_2)), \\
f_3 &= \frac{1}{2}(-2 e_1 p_1 - 2 e_0 p_1 q_1 + (q_1 - 1)(q_1 - x_1)(q_1 - x_2)), \quad \text{and} \\
e_2 &= \frac{q_1}{2 p_1}(-2 e_1 p_1 - 2 e_0 p_1 q_1 + (q_1 - 1)(q_1 - x_1)(q_1 - x_2)).
\end{align*}
\]

Then we can define the limit \( \lim_{q_2 \to q_1} (\nabla_0 + \Phi_X) \) as

\[
\begin{align*}
\lim_{q_2 \to q_1} (d + Q_1 d(Q_1^{-1}) + Q_1(\nabla_0)U_0 Q_1^{-1} + P(q_2)^{-1}(A^0 \otimes \omega_z)P(q_2)) & \text{ on } U_0 \\
\lim_{q_2 \to q_1} (d + Q_2 d(Q_2^{-1}) + Q_2 T_{\infty}(B_w \otimes \omega_w)T_{\infty}^{-1}Q_2^{-1} + R_0^{-1}(A^0 \otimes \omega_z)R_0) & \text{ on } U_\infty,
\end{align*}
\]

which has bundle type \( \mathcal{O}(1) \oplus \mathcal{O}(-2) \).

Secondly, we determine the remainder parameters of \( B_w \) so that the residue matrix of \( \nabla_0 + \Phi_X \) at \( t_i \) has the eigenvalues \( (\nu_i, -\nu_i) \) (resp. \( (\nu_i, 1 - \nu_i) \)) for \( i = 1, \ldots, 4 \) (resp. \( i = 5 \)). We take eigenvectors of the residue matrices of \( \Phi_X \) at \( t_i \) as follows:

\[
\begin{align*}
&\begin{pmatrix} \nu'^*_i \nu'^*_i \end{pmatrix} = \begin{pmatrix} f_i^+(q_1, p_1, \lambda) \end{pmatrix}, \quad \begin{pmatrix} (q_1 - q_2)(q_1 - t_i)(q_2 - t_i) \end{pmatrix} \quad \text{associated to } \nu'_i \\
&\begin{pmatrix} \nu'^*_i \nu'^*_i \end{pmatrix} = \begin{pmatrix} f_i^-(q_1, p_1, \lambda) \end{pmatrix}, \quad \begin{pmatrix} (q_1 - q_2)(q_1 - t_i)(q_2 - t_i) \end{pmatrix} \quad \text{associated to } -\nu'_i.
\end{align*}
\]

Here, we put \( f_i^+(q_1, p_1, \lambda) = p_1(q_2 - t_i) + p_2(q_1 - t_i) - \epsilon \nu'_i(q_1 - q_2)(t_i - t_j)(t_k - t_i)(t_i - t_k) \) where \( \epsilon \in \{+, -\} \).

Set

\[
T^*_t = \begin{pmatrix} 1 & f_t^+(q_1, p_1, \lambda) \\ 0 & (q_1 - q_2)(q_1 - t_i)(q_2 - t_i) \end{pmatrix} \quad i = 1, \ldots, 4, \quad \epsilon \in \{+, -\}.
\]

Then we have

\[
(T^*_t)^{-1} (\text{res}_t(Q_1^{-1} P^{-1}(A^0 \otimes \omega_z)PQ_1)) T^*_t = \begin{pmatrix} -\epsilon \nu'_i & * \\ 0 & \epsilon \nu'_i \end{pmatrix}.
\]
We fix a tuple $\epsilon = (\epsilon_1, \ldots, \epsilon_4)$ where $\epsilon_i \in \{+, -\}$. For the tuple $\epsilon$, we solve the following four linear equations
\[
((T_{i*}^\epsilon)^{-1}(\text{res}_t(R_0^{-1}dR_{0,w} + R_0^{-1}T_{\infty}(B_w \otimes \omega_w)T_{\infty}^{-1}R_{0,w}))T_{i*}^\epsilon))_{21} = 0 \quad i = 1, \ldots, 4
\]
where $A_{ij}$ is the $(i, j)$-entry of a $(2 \times 2)$-matrix $A$. Explicitly, we can describe these equations as follows:
\[
(q_1 - q_2)(q_1 - t_i)^2(q_2 - t_i)^2(t_i^2 d_1 + t_i^2 d_2 + t_id_3 + d_4) + 2(q_1 - t_j)(q_2 - t_j)g_i(q_1, p_1, \lambda)(t_i^2 f_0 + t_i^2 f_1 + t_if_2 + f_3)
\]
\[
- (g_i^*(q_1, p_1, \lambda))^2(t_i^2 e_0 + t_ie_1 + e_2) = 0 \quad i = 1, \ldots, 4
\]
where we put
\[
g_i^*(q_1, p_1, \lambda) := p_1(q_2 - t_i) + p_2(q_1 - t_i) - \nu_2^* t_i(q_1 - q_2)(q_1 - t_i)(q_2 - t_i) + \epsilon_i \nu_1^*(q_1 - q_2)(t_i - t_j)(t_i - t_k)(t_i - t_l)
\]
\[
= p_1(q_1 - t_i) + p_1(q_2 - t_i) - (q_1 - q_2)((q_1 - t_i)\lambda + \nu_2^* t_i(q_1 - t_i)(q_2 - t_i) - \epsilon_i \nu_1^*(t_i - t_j)(t_i - t_k)(t_i - t_l)).
\]
Then the solution of the equations (52) is the following
\[
\begin{pmatrix}
    d_1 \\
    d_2 \\
    d_3 \\
    d_4
\end{pmatrix} = 
\begin{pmatrix}
    0 & 0 & 0 & 1 \\
    1 & 1 & 1 & 1 \\
    x_1^2 & x_1^2 & x_1 & 1 \\
    x_2^2 & x_2^2 & x_2 & 1
\end{pmatrix}^{-1}
\begin{pmatrix}
    h_{\epsilon_1}^*(q_1, p_1, \lambda) \\
    h_{\epsilon_2}^*(q_1, p_1, \lambda) \\
    h_{\epsilon_3}^*(q_1, p_1, \lambda) \\
    h_{\epsilon_4}^*(q_1, p_1, \lambda)
\end{pmatrix}
\]
where
\[
h_{\epsilon_1}^*(q_1, p_1, \lambda) = \frac{1}{2}(q_1 - t_i)(-2\epsilon_1 p_1 - 2\epsilon_0 p_1(q_1 + t_i) + (q_1 - t_j)(q_1 - t_k)(q_1 - t_l))
\]
\[
- 2(q_1 - t_i)((q_1 - t_i)\lambda + \nu_2^* t_i(q_1 - t_j)(q_2 - t_i) - \epsilon_i \nu_1^*(t_i - t_j)(t_i - t_k)(t_i - t_l))
\]
\[
+ \frac{q_1 - q_2}{p_1}((q_1 - t_i)\lambda + \nu_2^* t_i(q_1 - t_i)(q_2 - t_i) - \epsilon_i \nu_1^*(t_i - t_j)(t_i - t_k)(t_i - t_l))^2.
\]
We consider the case $q_1 = t_i$ for some $i$. For fixed $\epsilon = (\epsilon_1, \ldots, \epsilon_5)$, $\epsilon_i \in \{+, -, 0\}$, the domain of the function
\[
h_{\epsilon_i}^*(q_1, p_1, \lambda) := \frac{h_{\epsilon_i}^*(q_1, p_1, \lambda)}{(q_1 - t_i)^2(q_2 - t_i)^2}
\]
is extended to $q_1 = t_i$, $p_1 = \epsilon_i \nu_i$ and $\lambda = \nu_i^* \epsilon_i$. Here, $\nu_i^* \epsilon_i$ is the blowing-up parameter of $K_{\pi_i^*} \to K_{\pi}$ at $(t_i, \epsilon_i \nu_i)$. Therefore we can extend the family $\nabla_{\epsilon_0}$ to $q_1 = t_i$, $p_1 = \epsilon_i \nu_i$ and $\lambda = \nu_i^* \epsilon_i$ when we substitute the solution $d_1, \ldots, d_4$ associated to $\epsilon$.

We compute the eigenvalue of the residues of $\nabla_{\epsilon_0}$ at $t_i$ for $i = 1, \ldots, 5$. We put
\[
\alpha_{\epsilon_i}^i(q_1, q_2, p_1, \lambda) := (T_{i*}^\epsilon)^{-1}(\text{res}_t(R_0^{-1}dR_{0,w} + R_0^{-1}T_{\infty}(B_w \otimes \omega_w)T_{\infty}^{-1}R_{0,w}))T_{i*}^\epsilon)_{22}
\]
(54)
\[
= \frac{\beta_{\epsilon_1}^1(q_1, p_1, \lambda)\beta_{\epsilon_2}^2(q_1, q_1, p_1, \lambda)}{2p_1(q_2 - t_i)(t_i - t_j)(t_i - t_k)(t_i - t_l)} \quad i = 1, \ldots, 4,
\]
where
\[
\beta_{\epsilon_1}^1(q_1, p_1, \lambda) := -2\epsilon_1 p_1 - 2\epsilon_0 p_1(q_1 + t_i) + (q_1 - t_j)(q_1 - t_k)(q_1 - t_l),
\]
\[
\beta_{\epsilon_2}^2(q_1, q_1, p_1, \lambda) := (p_1(q_1 - t_i) - (q_1 - q_2)((q_1 - t_i)\lambda + \nu_2^* t_i(q_1 - t_i)(q_2 - t_i) - \epsilon_i \nu_1^*(t_i - t_j)(t_i - t_k)(t_i - t_l)).
\]
Then we have
\[
((T_{i*}^\epsilon)^{-1}(\text{res}_t((\nabla_{\epsilon_0})_{t_i}))T_{i*}^\epsilon)_{22}
\]
for $i = 1, \ldots, 4$. Here we put $((\nabla_{\epsilon_0})_{t_i}) = R_0^{-1}dR_{0,w} + R_0^{-1}T_{\infty}(B_w \otimes \omega_w)T_{\infty}^{-1}R_{0,w}.$

We define the family of connections $\nabla_X$ as $\nabla_0 + \Phi_X$. The limit $\lim_{t_i \to q_1} \nabla_X$ has bundle type $O(1) \oplus O(-2)$. The eigenvalues of the residues of $\nabla_X$ at $0, 1, x_1, x_2$, and $\infty$ are the following

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
\text{res}_0 & \nabla_X & \text{res}_1 & \nabla_X & \text{res}_2 & \nabla_X & \text{res}_\infty & \nabla_X \\
\hline
-\alpha_{\epsilon_1}^1 + \epsilon_i \nu_1^* & -\alpha_{\epsilon_2}^2 + \epsilon_2 \nu_2^* & -\alpha_{\epsilon_3}^3 + \epsilon_3 \nu_3^* & -\alpha_{\epsilon_4}^4 + \epsilon_4 \nu_4^* & -\frac{1}{2} + \nu_5^* \\
\alpha_{\epsilon_1}^1 - \epsilon_1 \nu_1^* & \alpha_{\epsilon_2}^2 - \epsilon_2 \nu_2^* & \alpha_{\epsilon_3}^3 - \epsilon_3 \nu_3^* & \alpha_{\epsilon_4}^4 - \epsilon_4 \nu_4^* & \frac{1}{2} - \nu_5^* \\
\hline
\end{array}
\]
where we put $\alpha_i^\ast := \alpha_i^\ast(q_2, p_i, \lambda)$. For the fixed tuple $(\nu_1, \ldots, \nu_5)$, we determine the eigenvalues $(\nu_1', \ldots, \nu_5')$ of Higgs field $\Phi_X$ as follows. We solve the following equations for $\nu_i'$:

\begin{equation}
-\frac{1}{2} + \nu_i' = \nu_5, \quad \text{and} \quad -\alpha_i^\ast + \epsilon_i \nu_i' = \nu_i \quad \text{for} \quad i = 1, \ldots, 4.
\end{equation}

Then $\nu_i'$ is determined by $\nu_i$ and $\nu_5$ (when at least $q_2$ is close to $q_1$ enough). Then the eigenvalues of residues of the family $\nu_X$ at $t_i$ ($i = 1, \ldots, 5$) are $\nu_i^\pm$. Let $\tilde{A}_U$ be the connection matrix

\begin{equation}
R_{0, w}^{-1} dR_{0, w} + R_{0, w}^{-1} T_{\infty} (B_w \otimes \omega_w) T_{\infty}^{-1} R_{0, w} + Q_1 P^{-1}(A_w \otimes \omega_x) PQ_1
\end{equation}

(see (51)). Then, we obtain the family [44], which satisfies the assertion of Proposition 5.3 by the construction.

\begin{remark}
The connection $\nabla_X$ is parametrized by $e_0$ and $e_1$. We can take $e_0, e_1 \in \mathbb{C}$ freely.
\end{remark}

5.3. The apparent singularities and the dual parameters of $\nabla_X$. Let $C_{\infty}$ be the $\infty$-section of the Hirzebruch surface of $n - 2$. Let $\phi: \tilde{M} \to \text{Hilb}^2(\tilde{K}_5^0 \cup C_{\infty})$ be the birational map constructed by the apparent singularities and the dual parameters. For $n = 5$, by analyzing the behavior of the apparent singularities and their duals of $\nabla_X$ as $q_2 \to q_1$, we obtain the following

**Theorem 5.5.** By taking a sequence of blowing-ups $\text{Hilb}^2(\tilde{K}_5^0 \cup C_{\infty}) \to \text{Hilb}^2(\tilde{K}_5^0 \cup C_{\infty})$, we have the injective map $\tilde{M} \to \text{Hilb}^2(\tilde{K}_5^0 \cup C_{\infty})$. The moduli space $\tilde{M}$ is birational to its image $\text{Image} \tilde{M} \subset \text{Hilb}^2(\tilde{K}_5^0 \cup C_{\infty})$.

**Proof.** First, we construct a map $\tilde{M} \to \text{Hilb}^2(\tilde{K}_5^0 \cup C_{\infty})$ by the birational map $\phi$. The apparent singularities of $\nabla_X$ is the zero of the polynomial

\begin{equation}
F_{21}(z) = (1 + e_0(q_1 - q_2))z^2 - (q_1 + q_2 + e_1(q_2 - q_1))z
\end{equation}

\begin{equation}
+ q_1 q_2 + \frac{(q_1 - q_2)q_1}{2p_1} (-2e_1 p_1 - 2e_0 q_1 + (q_1 - 1)(q_1 - x_1)(q_1 - x_2)).
\end{equation}

Let $q_1'$ and $q_2'$ be the solution of the equation $F_{21}(z) = 0$:

\begin{equation}
q_1' = q_1 + a_1(q_2 - q_1) - \frac{\sqrt{a_3(q_2 - q_1)}}{a_2},
\end{equation}

\begin{equation}
q_2' = q_1 + a_1(q_2 - q_1) + \frac{\sqrt{a_3(q_2 - q_1)}}{a_2}
\end{equation}

where we set

\begin{equation}
a_1 := \frac{1 + e_1 + 2e_0 q_1}{2 - 2e_0(q_2 - q_1)},
\end{equation}

\begin{equation}
a_2 := 2p_1(-1 + e_0(q_2 - q_1)),
\end{equation}

\begin{equation}
a_3 := p_1(-1 + e_1 + 2e_0 q_1)^2(q_1 - q_2) + 2q_1(q_1 - 1)(1 + e_0(q_1 - q_2))(q_1 - x_1)(q_1 - x_2)).
\end{equation}

Note that $\lim_{q_2' \to q_1} q_1' = \lim_{q_2' \to q_1} q_2' = q_1$. The dual parameter of $q_i'$ is $p_i' = F_{11}(q_i')$ for $i = 1, 2$. If we consider $\lim_{q_2' \to q_1} F_{11}(q_1')$, then the limit diverges (Figure 3). Let $s$ be the parameter such that $\tilde{q}_i' - \tilde{q}_i' = s(q_2' - q_1)$ where $\tilde{q}_i' = 1/p_i$ for $i = 1, 2$. By explicit computation, we have

\begin{equation}
\lim_{q_2' \to q_1} s = \frac{1}{q_1(q_1 - 1)(q_1 - x_1)(q_1 - x_2)}.
\end{equation}

Then we can extend the map $X \to \text{Hilb}^2(\tilde{K}_5^0)$ to $\tilde{X} \to \text{Hilb}^2(\tilde{K}_5^0 \cup C_{\infty})$ by

\begin{equation}
\tilde{X} \setminus X \ni ((q_1, p_1), (q_1, p_1), \lambda) \mapsto \{(q_1, \infty), (q_1, \infty), \frac{1}{q_1(q_1 - 1)(q_1 - x_1)(q_1 - x_2)} \} \in \text{Hilb}^2(\tilde{K}_5^0 \cup C_{\infty}).
\end{equation}

We can define a map $\phi: \tilde{M} \to \text{Hilb}^2(\tilde{K}_5^0 \cup C_{\infty})$ so that $\phi|_{\tilde{M}_0}$ is the map in Theorem 5.5 and the diagram

\begin{equation}
\begin{array}{ccc}
\tilde{X} & \to & \text{Hilb}^2(\tilde{K}_5^0 \cup C_{\infty}) \\
\downarrow \phi & & \\
\tilde{M} & \to & \text{Hilb}^2(\tilde{K}_5^0 \cup C_{\infty}).
\end{array}
\end{equation}

is commutative.
First, we take the blowing-up of \( \text{Hilb}^2(\mathcal{K}_5' \cup C_\infty) \) of \( \text{Hilb}^2(\mathcal{K}_5' \cup C_\infty) \) such that

\[
\xymatrix{
\text{Hilb}^2(\mathcal{K}_5' \cup C_\infty) \ar[r]^-{\phi} \ar[d] & \tilde{\mathcal{M}} \ar[d] \\
\text{Hilb}^2(\mathcal{K}_5' \cup C_\infty)
}
\]

where \( \phi \) is injective. Set \( \bar{p}'_1 := \frac{1}{p_1} \). Let \( ((q'_1, \bar{p}'_1), (q'_2, \bar{p}'_2)) \) be coordinates on \((\mathcal{K}_5' \cup C_\infty) \times (\mathcal{K}_5' \cup C_\infty) \setminus (C_0 \cup \pi^{-1}(\infty)) \times (C_0 \cup \pi^{-1}(\infty)) \).

First, we take the blowing-up of \((\mathcal{K}_5' \cup C_\infty) \times (\mathcal{K}_5' \cup C_\infty) \) along the ideal \( \mathcal{I}_1 = (q'_2 - q'_1, \bar{p}'_2 - \bar{p}'_1) \), denoted by \( \text{Bl}_{1} \). Note that \( \text{Hilb}^2(\mathcal{K}_5' \cup C_\infty) = \text{Bl}_{1}/\mathcal{S}_2 \). We defined the blowing-up parameter \( s \) as \( \bar{p}'_2 - \bar{p}'_1 = s(q'_2 - q'_1) \), and we obtained

\[
\lim_{q_2 \to q_1} s = \frac{1}{q_1(q_1 - 1)(q_1 - x_1)(q_1 - x_2)}.
\]

Second, we take the blowing-up of \( \text{Bl}_{1} \) along the ideal

\[
\mathcal{I}_2 = \left( q'_2 - q'_1, \bar{p}'_2 - \bar{p}'_1, s - \frac{1}{q(q - 1)(q - x_1)(q - x_2)}, \bar{p}'_2 + \bar{p}'_1 \right)
\]

where \( q := \frac{q'_1 + q'_2}{2} \),

denoted by \( \text{Bl}_{2} \). We define the blowing-up parameters \( t_1, t_2 \) as

\[
s = \frac{1}{q(q - 1)(q - x_1)(q - x_2)} = t_1(q'_2 - q'_1), \quad \bar{p}'_2 + \bar{p}'_1 = t_2(q'_2 - q'_1).
\]

By explicit calculations, we have \( \lim_{q_2 \to q_1} t_1 = \lim_{q_2 \to q_1} t_2 = 0 \). Third, we take the blowing-up of \( \text{Bl}_{2} \) along the ideal

\[
\mathcal{I}_3 = \left( q'_2 - q'_1, \bar{p}'_2 - \bar{p}'_1, s - \frac{1}{q(q - 1)(q - x_1)(q - x_2)}, \bar{p}'_2 + \bar{p}'_1, t_1, t_2 \right),
\]

denoted by \( \text{Bl}_{3} \). We define the blowing-up parameters \( u_1, u_2 \) as \( t_1 = u_1(q'_2 - q'_1), \quad t_2 = u_2(q'_2 - q'_1) \).

By explicit calculations, we have

\[
\begin{align*}
\lim_{q_2 \to q_1} u_1 &= \frac{-\lambda}{4q_1^2(q_1 - 1)^2(q_1 - x_1)^2(q_1 - x_2)^2} + \frac{u^{(1)}_1(q_1)p_1}{4q_1^2(q_1 - 1)^3(q_1 - x_1)^3(q_1 - x_2)^3} \\
&\quad + \frac{u^{(0)}_1(q_1)}{16q_1^4(q_1 - 1)^3(q_1 - x_1)^3(q_1 - x_2)^3}, \\
\lim_{q_2 \to q_1} u_2 &= \frac{-4q_1^3 + 3q_1^2(1 + x_1 + x_2) - 2q_1(x_1 + x_2 + x_1x_2) + x_1x_2}{4q_1^2(q_1 - 1)^2(q_1 - x_1)^2(q_1 - x_2)^2}
\end{align*}
\]
where \( u_{1}(q_1) \) and \( u_{0}(q_1) \) are polynomials in \( q_1 \), which are independent of \( \lambda \) and \( p_1 \). Fourth, we take the blowing-up of \( \text{Bl}_4 \) along the ideal

\[
\mathcal{I}_4 = \left( q'_2 - q'_1, p'_2 - p'_1, s - \frac{1}{q(q-1)(q-x_1)(q-x_2)}, p'_2 + t'_1, t'_2, \right. \\
\left. u_2 - \frac{-4q^3 + 3q^2(1+x_1+x_2) - 2q(x_1 + x_2 + x_1x_2) + x_1x_2}{4q^2(q-1)^2(q-x_1)^2(q-x_2)^2}, \right).
\]

denoted by \( \text{Bl}_4 \). We put blowing-up parameters \( v \) as follows:

\[
u_2 - \frac{-4q^3 + 3q^2(1+x_1+x_2) - 2q(x_1 + x_2 + x_1x_2) + x_1x_2}{4q^2(q-1)^2(q-x_1)^2(q-x_2)^2} = v(q'_2 - q'_1).
\]

By explicit calculations, we have \( \lim_{q_2 \to q_1} v = 0 \). Finally, we take the blowing-up of \( \text{Bl}_4 \) along the ideal

\[
\mathcal{I}_5 = \left( q'_2 - q'_1, p'_2 - p'_1, s - \frac{1}{q(q-1)(q-x_1)(q-x_2)}, p'_2 + t'_1, t'_2, \right. \\
\left. u_2 - \frac{-4q^3 + 3q^2(1+x_1+x_2) - 2q(x_1 + x_2 + x_1x_2) + x_1x_2}{4q^2(q-1)^2(q-x_1)^2(q-x_2)^2}, v, \right).
\]

denoted by \( \text{Bl}_5 \). We define the blowing-up parameter \( w \) as \( v = w(q'_2 - q'_1) \). By explicit calculations, we have

\[
\lim_{q_2 \to q_1} w = \frac{\lambda}{8q_1^3(q_1-1)^3(q_1-x_1)^2(q_1-x_2)^2} \left( 4q_1^3 - 3q_1^2(1+x_1+x_2) + 2q_1(x_1 + x_2 + x_1x_2) - x_1x_2 \right)
\]

\[
+ \frac{-w^{(1)}(q_1)p_1}{8q_1^3(q_1-1)^3(q_1-x_1)^2(q_1-x_2)^2} + \frac{w^{(0)}(q_1)}{64q_1^3(q_1-1)^4(q_1-x_1)^4(q_1-x_2)^2}
\]

where \( w^{(1)}(q_1) \) and \( w^{(0)}(q_1) \) are polynomials in \( q_1 \), which are independent of \( \lambda \) and \( p_1 \). We define the blowing-ups \( \tilde{\text{Hilb}}^2(\tilde{K}_5) \) of \( \text{Hilb}^2(\tilde{K}_5) \) as \( \tilde{\text{Hilb}}^2(\tilde{K}_5) = \text{Bl}_5/\mathfrak{S}_2 \). Here, \( \mathfrak{S}_2 \) is the symmetric group, which acts on \( \text{Bl}_5 \) naturally.

We consider the family \( \lim_{q_2 \to q_1} \nabla_X \), which has bundle type \( \mathcal{O}(1) \oplus \mathcal{O}(-2) \). The family is parameterized by \( (\lambda, p_1, q_1) \in \mathbb{C}^3 \). We fix the parameter \( q_1 \). We can regard the parameters \( (\lambda, p_1) \) as coordinates of \( M^1 \), which is isomorphic to \( \mathbb{C}^2 \). The family \( \lim_{q_2 \to q_1} \nabla_X \) where \( q_1 \) is fixed is a universal family of \( M^1 \).

The theorem follows from (57) and (58). \( \square \)

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