Gauging Cosets

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Abstract

We show how to gauge the set of raising and lowering generators of an arbitrary Lie algebra. We consider \(SU(N)\) as an example. The nilpotency of the BRST charge requires constraints on the ghosts associated to the raising and lowering generators. To remove these constraints we add further ghosts and we need a second BRST charge to obtain nontrivial cohomology. The second BRST operator yields a group theoretical explanation of the grading encountered in the covariant quantization of superstrings.
1. Introduction

Nonlinear sigma models based on group or coset manifolds allow one to construct interacting string models with nontrivial backgrounds. Three classes of models have been obtained in this way: i) standard WZNW models on group manifolds (this construction is possible for any compact or noncompact group) \[1\]. ii) WZNW models on particular coset manifolds (this construction seems only possible if the structure constants with 3 coset indices are invariant with respect to the subgroup, see for example \[2\]) iii) gauged WZNW models \[3\]. In the latter case the gauging is performed by introducing a set of gauge fields \(A_z\) and \(A_{\bar{z}}\) coupled to the currents of a subgroup. The gauge fixing \(A_{\bar{z}} = 0\) leads to a second WZNW model with \(h\)-currents and ghost currents. Gauging once again this BRST invariant action yields the sum of gauge, \(h\) and ghost currents which forms the starting point for the construction of the final BRST charge \[4\]. This BRST charge implements the constraints at the level of the Fock space and selects the physical subspace of the theory.

In the present paper we present a new method to obtain interesting nonlinear sigma models by imposing (gauging) the constraints related to coset generators and not those of a subgroup. This is inspired by recent developments in superstring theory. To construct a quantized superstring one may begin with the set of second class constraints \(d_{\sigma} \) (the conjugate momenta of the fermionic coordinates \(\theta^\alpha\)) as starting point. A BRST charge \(Q = \oint \lambda^\alpha d_{\sigma} \) is constructed and it is made nilpotent by imposing suitable quadratic constraints on the 16 complex commuting ghosts \(\lambda^\alpha\). The operators \(d_{\sigma}\) generate a particular non-semisimple superalgebra, with further generators \(\Pi_{zm}\) and \(\partial_z \theta^\alpha\) which form a sub-superalgebra. In one (noncovariant) approach one imposes the constraint \(Q|_{\text{phys}} = 0\) on the physical states. \[5\]. So in this approach one gauges \(d_{\sigma}\). However, since the ghost fields are constrained, gauging \(d_{\sigma}\) does not imply that all corresponding conjugate variables \(\theta^\alpha\)
are removed by the cohomology. Rather, the dependence of the vertex operators on $\theta^\alpha$ is only restricted by the field equations.

In another (covariant) approach one introduces by hand a ghost doublet $(b, c_z)$ in order to obtain a nilpotent BRST charge, and one imposes all three constraints, but then one restricts the carrier space by imposing a “grading condition” in order that not all $x^m$ and $\theta^\alpha$ are removed from the cohomology. With this grading condition one obtains the same spectrum as from the noncovariant approach, so one has undone the gauging of $\Pi_{zm}$ and $\partial_z \theta$ in some sense.

Although imposing the grading condition by hand yields to correct cohomology, we have suspected for a long time that there exists another charge whose vanishing on physical states achieves the same purpose. As a first step in this direction we have recently removed the ghost pair $(b, c_z)$ (and also another ghost pair $(\omega, \eta_z)$ which we also introduced by hand to have vanishing central charge) by “gauging” the WZNW model based on superalgebra of $d_{z\alpha}, \Pi_{zm}$ and $\partial_z \theta^\alpha$. The procedure of gauging leads to an extra set of currents $d_{z\alpha}^{(h)}, \Pi_{zm}^{(h)}$ and $\partial_z \theta^{(h)\alpha}$ with opposite sign for the double poles, and as a consequence it has automatically a nilpotent BRST charge and a vanishing central charge. So in this approach there are no longer any ghosts added by hand. The second step is then to find the charge which takes over the role of the grading.

In this article we give a general construction of such a charge. It turns out to be a second BRST charge which anticommutes with the usual BRST charge and which arises naturally in the process of “gauging coset generators”. We shall consider a general simple Lie algebra, instead of the nonsemisimple Lie algebra which appears in the string model. The two main ideas on which our approach is based are, on the one hand, the structure of Lie (or affine Lie) algebras on the Cartan-Weyl basis, and, on the other hand, the BRST approach to second class constraints.

It is well-known how to gauge a subgroup $H$ of a Lie group $G$: one decomposes the generators into coset generators $K_{\alpha}$ and subgroup generators $H_i$ and one constructs the BRST charge $Q = c^i H_i + \frac{1}{2} b_i f^i_{jk} c^k c^j$ where $c^i, b_i$ are the ghosts and the antighosts, respectively, and $f^i_{jk}$ the structure constants of $H$. The $H_i$ are then first class constraints which annihilate physical states: $H_i |_{phys} = 0$ (which becomes $Q |_{phys} = 0$ in the BRST approach). Often an explicit representation of $K_{\alpha}$ and $H_i$ in terms of differential operators or conformal field theory is given.

The main difference between gauging a subgroup and gauging a coset boils down to the fact that the generators of the subgroup form a closed algebra of constraints whereas
the generators of the coset do not. From a Hamiltonian point of view this means that
the constraints imposed by $K_\alpha$ are second class constraints. Namely, if $K_\alpha$ is a first class
constraint, it annihilates physical states $K_\alpha|\text{phys}\rangle = 0$, but then also $[K_\alpha, K_\beta]|\text{phys}\rangle = 0$.
The set of generators $[K_\alpha, K_\beta]$ in general closes on the generators $H_i$ (and only $H_i$ if
one has a symmetric coset decomposition). If $H_i$ is not to be a constraint, $H_i|\text{phys}\rangle$
should not vanish for all physical states, but then $[K_\alpha, K_\beta]|\text{phys}\rangle$ should be non-vanishing.
Thus gauging coset generators leads to second class constraints. Using the Dirac brackets
to implement the constraints and eliminating the variables associated with the second
class constraints, in general reduces the isometry group of target space and looses the
manifest symmetry of the theory. (In the case of superstrings, separating the second class
fermionic constraints from the first class constraints, one necessarily breaks the super-
Poincaré covariance of the Green-Schwarz sigma model.) There is, however, a way to
preserve all symmetries, namely by using BRST methods. This paper presents a covariant
BRST approach to the gauging of coset generators.

The basic idea is to start with a BRST charge $Q_{K,0} = \xi^\alpha K_\alpha$ instead of the coset
generators $K_\alpha$ by themselves. The nilpotency of $Q_{K,0}$ requires that $\xi^\alpha \xi^\beta [K_\alpha, K_\beta]$ vanishes,
and this can be achieved by imposing suitable constraints on the ghosts $\xi$. Next we relax
these constraints, which requires the introduction of further ghosts, but in order that the
cohomology does not depend on the extra ghosts, we need a second BRST charge. As we
already mentioned, this is the main idea upon which a new approach to the quantization
of the superstring is based. In that particular example, $K_\alpha = d_\alpha$ (cf. [5]) are the conjugate
momenta of the spacetime fermionic coordinates $\theta^\alpha$, but in this article we want to develop
a general formalism applicable to any physical system.

The content of this paper is as follows: in sec. 2, we derive the constraints on the
ghost fields and we define the constrained cohomology. In sec. 3 we remove the constraints
by adding new ghost fields while keeping the extended BRST charge $Q_K$ nilpotent. At the
same time we introduce a new nilpotent BRST operator $Q_C$ whose role is to remove the
new ghosts from physical observables. Without $Q_C$ there is no cohomology for $Q_K$ but
with $Q_C$ the pair $(Q_K, Q_C)$ leads to nontrivial “relative cohomology”. We mention that
the idea of using a second BRST charge has been proved useful in string theory [8], in a
6 dimensional supersymmetric formulation of superstrings on a Calabi-Yau manifold [3],
in topological field theory [10], in string field theory [11], and very recently in a string-
inspired formulation of harmonic superspace [12]. In section 4 we find as an unexpected
bonus the solution to a problem that has kept us thinking for a long time. In our work on
the covariant quantum superstring we found the need to introduce a new quantum number for the ghosts, called grading, whose role was to restrict the vertex operators such that nontrivial cohomology resulted. We now propose that this grading has a group theoretical meaning which is intimately linked to the second BRST charge. In sec. 5, we construct the currents of a conformal field theory obtained by gauging the coset generators of $SU(N)$, and extend the discussion from the affine Lie algebra to the Virasoro algebra and beyond. In sec. 6 we mention possible applications of the formalism developed in this article. In an appendix we illustrate our method by explicitly working through the example of $SU(3)$.

2. Gauging the coset: the first BRST charge $Q_K$

Consider a simple Lie algebra decomposed into the Cartan-Weyl basis of raising operators $E_\alpha$ associated with positive roots, lowering operators $E_{-\alpha}$ associated with negative roots, and Cartan generators $H_i$

$$[E_\alpha, E_{-\alpha}] = \alpha^i H_i, \quad [H_i, E_{\pm\alpha}] = \pm \alpha_i E_{\pm\alpha},$$

$$[E_\alpha, E_\beta] = N_{\alpha,\beta} E_{\alpha+\beta} \quad \text{if} \quad \alpha + \beta \neq 0, \quad [H_i, H_j] = 0. \quad (2.1)$$

We choose the phase convention $N_{\alpha,\beta} = -N_{-\alpha,-\beta}$, and $E_{-\alpha} = (E_\alpha)^\dagger$. The index $i$ of $\alpha_i$ has been lowered with the Killing-Cartan metric according to $g_{ij} \alpha^j = g_{\alpha,-\alpha} \alpha_i$. (Often one normalizes the $E_\alpha$ such that $g_{\alpha,-\alpha} = 1$, but we shall not require this). We begin with the BRST charge

$$Q_{K,0} = \sum_{\alpha \in \Delta} \xi^\alpha E_\alpha - \frac{1}{2} \sum_{\alpha,\beta \in \Delta} \beta_{\alpha+\beta} N_{\alpha,\beta} \xi^\alpha \xi^\beta \quad (2.2)$$

where the sum is over all roots (we denote by $\Delta$ the set of all roots and $\Delta_+$ the set of positive roots). The $\xi^\alpha$ are anticommuting ghosts and $\beta_\alpha$ are the corresponding antighosts. One may view this as an expression in quantum mechanics with brackets $\{\xi^\alpha, \beta_\beta\} = \delta^\alpha_\beta$ or an expression of the holomorphic sector in string theory with operator product expansion.

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3 For the super-Poincaré group in 10 dimensions generated by $Q_\alpha$ and $P_m$, the Killing-Cartan metric is zero. One can modify the Lie algebra by adding a new central charge, and in this way one obtains a non-degenerate metric [13]. We propose that in such cases one uses this metric to lower the index of $\alpha^i$. 
\( \xi^\alpha(z) \beta_\beta(w) \sim \delta_\beta^\alpha \frac{1}{z - w} \). (In the latter case one should add an integration \( \oint dz \) to the definition of \( Q_{K,0} \)). Nilpotency of \( Q_{K,0} \) requires the following constraint:

\[
\sum_{\alpha \in \Delta_+} \alpha^i \xi^\alpha \xi^{-\alpha} = 0 ,
\]

where the index \( i \) labels the Cartan generators. For example, if the group is \( SU(2) \), the two ghosts \( \xi^\pm \) correspond to the raising and lowering generators \( E_\pm \) and the constraint (2.3) becomes \( \xi^+ \xi^- = 0 \) of which \( \xi^- = \xi^+ \) is a solution. There are thus solutions in general, but they break the \( H \)-invariance (in the case of the ten-dimensional superstring any solution of the constraints \( \lambda \gamma^m \lambda = 0 \) breaks the manifest Lorentz covariance). The constraints in (2.3) clearly commute with each other and are invariant under the BRST transformations generated by (2.2) (see below). Hence they are first class constraints on the ghost fields. They generate gauge transformations on the antighosts \( \beta_\alpha \)

\[
\delta_\epsilon \beta_\alpha = \left[ \epsilon_i \sum_{\beta \in \Delta_+} \beta^i \xi^\beta \xi^{-\beta}, \beta_\alpha \right] = -\epsilon_i \alpha^i \xi^{-\alpha} \quad (2.4)
\]

where \( \epsilon^i \) are infinitesimal local parameters (one for each generator of the Cartan subalgebra).

In order that the constraints (2.3) are compatible with the BRST symmetry, they should be invariant under it. One can check by using the Jacobi identity

\[
\left[ [E_\alpha, E_\beta], E_{-\alpha-\beta} \right] + \left[ [E_\beta, E_{-\alpha-\beta}], E_\alpha \right] + \left[ [E_{-\alpha-\beta}, E_\alpha], E_\beta \right] = 0
\]

that this is the case

\[
[Q_{K,0}, \delta_\epsilon] = \left[ \sum_{\alpha \in \Delta} \xi^\alpha E_\alpha - \frac{1}{2} \sum_{\alpha, \beta \in \Delta} \beta_{\alpha+\beta} N_{\alpha,\beta} \xi^\alpha \xi^\beta, \sum_{i} \epsilon_i \sum_{\alpha \in \Delta_+} \alpha^i \xi^\alpha \xi^{-\alpha} \right] = 0 . \quad (2.5)
\]

The space \( \mathcal{M} \) on which the Lie algebra acts is the group manifold \( \mathcal{G} \) parametrized by a set of coordinates; in that case the generators \( E_\alpha \) can be represented by differential

4 In the case of superstrings the generators of the coset are represented by covariant derivatives \( d_z^\alpha \) and the proper maximal subgroup is generated by the translations \( \Pi^m_z \) and the fermionic translations \( \partial_\beta \theta^\alpha \). The constraints (2.3) correspond to \( \lambda^\alpha \gamma^m \lambda = 0 \) for the commuting spinors \( \lambda^\alpha \), and such \( \lambda^\alpha \) are called pure spinors.

5 Notice that for Grassmann variables one has \( \xi^+ \equiv \delta(\xi^+) \), the Dirac delta function with respect to the Berezin integration, \( \int d\xi^+ \delta(\xi^+) f(\xi^+) = \int d\xi^+ f(\xi^+) = f(0) \). For \( SU(3) \), one has the constraints

\[
\xi^1 \xi^{-1} + \frac{1}{2} \xi^2 \xi^{-2} + \frac{1}{2} \xi^3 \xi^{-3} = 0 \quad \text{and} \quad \xi^2 \xi^{-2} - \xi^3 \xi^{-3} = 0,
\]

which can be solved by setting \( \xi^{\pm 1} = \pm \xi^{\pm 2} = \pm \xi^{\pm 3} \) or by the minimal solution \( \xi^\alpha = \xi^{-\alpha} \) for all \( \alpha \),

\[5\]
operators \( D_\alpha \) acting on functions defined on \( G \). We extend \( M \) to the space \( \hat{M} \) which contains the ghosts \( \xi^\alpha \). Then, we impose the constraints (2.3) to define the reduced functional space \( \hat{M}' \) on which we compute the cohomology \( H(Q_{K,0}, \hat{M}'). \)

The space \( \hat{M}' \) decomposes into subspaces \( \hat{M}'(n) \) with ghost number \( n \). Consider the sector with ghost number one, containing the following functions

\[
\Phi^{(1)} = \sum_{\alpha \in \Delta} \xi^\alpha A_\alpha(M). \tag{2.6}
\]

where \( A_\alpha(M) \) is a function on the group manifold \( M \). Acting with \( Q_{K,0} \) in (2.2) on \( \Phi^{(1)} \) while imposing the constraint (2.3) leads to restrictions on the fields \( A_\alpha \)

\[
[Q_{K,0}, \Phi^{(1)}] = \frac{1}{2} \sum_{\alpha, \beta \in \Delta} \xi^\alpha \xi^\beta \left( D_\alpha A_\beta - D_\beta A_\alpha - N_{\alpha, \beta} A_{\alpha + \beta} \right) = 0. \tag{2.7}
\]

The general solution of this equation is

\[
F_{[\alpha, \beta]} \equiv \left( D_\alpha A_\beta - D_\beta A_\alpha - N_{\alpha, \beta} A_{\alpha + \beta} \right) = \delta_{\alpha + \beta, 0} \sum_i \alpha^i W_i. \tag{2.8}
\]

The left hand side can be viewed as the curvature \( F_{[\alpha, \beta]} \) of the group manifold along all roots. We have \( F_{\alpha,-\alpha} = \sum_i \alpha^i W_i \) for each root \( \alpha \in \Delta_+ \). Because the curvatures are non-vanishing when \( \alpha = -\beta \), the fields \( A_\alpha \) are nontrivial (not pure gauge). By acting with \( E_\gamma + A_\gamma \) on \( F_{\alpha, \beta} \) and summing over all cyclic permutations of \( \alpha, \beta, \gamma \), the Bianchi identities yield constraints on \( A_\alpha \) and \( W^i \), but we do not analyze these issues here further. One could completely gauge the group \( G \) by adding the Cartan generators multiplied by the corresponding ghosts to the BRST charge (see below). In that case, all curvatures vanish, implying that there are no propagating physical degrees of freedom.

There are solutions of (2.8) which are given by the BRST exact elements of \( \hat{M}' \). For example at ghost number one we may pick a function \( \Omega \in \hat{M}'(0) \) with ghost number zero and acting with \( Q_{K,0} \) on it, one obtains the gauge transformation

\[
\delta \Phi^{(1)} = \{Q_{K,0}, \Omega\} = \sum_\alpha \xi^\alpha D_\alpha \Omega. \tag{2.9}
\]

\textsuperscript{6} A rather simple example is the \( SU(2) \) case. In that case \( \Phi^{(1)} = \xi^+ A_+ + \xi^- A_- \) and \( \{Q_{K,0}, \Phi^{(1)}\} = \xi^+ \xi^- (D_+ A_- - D_- A_+) = 0 \) due to the constraints \( \xi^+ \xi^- = 0 \). This implies that any function \( A_{\pm} \) such that \( A_{\pm} \neq D_{\pm} \Omega \) (where \( \Omega \in M(0) \)) belongs to the cohomology \( H^{(1)}(Q_{K,0}|\hat{M}') \). In addition, it is obvious that \( \hat{M}'(n) = \emptyset \) for \( n \geq 2 \). Finally, we have \( \Phi^{(0)} = A \) where \( A \in M \).

Acting with \( Q_{K,0} \) one finds \( \xi^+ D_+ A + \xi^- D_+ A = 0 \) and using the solution \( \xi^+ = \xi^- \), it follows that \( (D_+ - D_-) A = 0 \) which has nontrivial solutions. So both \( H^{(0)}(Q_{K,0}|\hat{M}') \) and \( H^{(1)}(Q|\hat{M}') \) are not empty.
This gives the gauge transformations $\delta A_\alpha = D_\alpha \Omega$. By inserting this gauge transformation into (2.8) one gets $\delta W_i = D_i \Omega$ where $D_i$ is the differential operator acting on the Cartan coordinates of the group manifold. The solution of (2.8) is then given by $A_\alpha = D_\alpha \Omega$ and $W_i = D_i \Omega$. By assuming that the group manifold is independent of those coordinates we obtain that $W_i$ are invariant under the gauge transformation.

For later use, we note that the Casimir operator $C_2$ is given by

$$C_2 = \sum_{\alpha \in \Delta^+} g_{\alpha,-\alpha} (E_\alpha E_{-\alpha} + E_{-\alpha} E_\alpha) + \sum_{i,j} g_{ij} H_i H_j$$  \hspace{1cm} (2.10)$$

and it is BRST invariant, $[Q_{K,0}, C_2] = 0$.

In the next section we relax the constraints, but before moving on we would like to point out that the approach to constrained systems of this section is already a generalization of the pure spinor formalism [3] for the superstring to a wider class of models. From this point of view Berkovits’ pure spinor formulation corresponds to the gauging of the coset currents of a particular WZNW model [7].

3. No constraints: the second BRST charge $Q_C$

Working with constrained fields is not very practical and therefore it is desirable to remove the constraints. The most straightforward way to remove them is to implement these constraints at the level of the BRST cohomology, by adding new ghosts which are Lagrange multipliers. By requiring nilpotency of the BRST charge, further terms in the BRST charge $Q_K$ can be determined. However, this procedure renders the cohomology empty (except for a few non-propagating degrees of freedom at zero momentum [14]). Therefore, we develop a method which does recover the correct cohomology. First we construct the full BRST charge by introducing new ghosts. Then we construct a second BRST charge $Q_C$ whose role is to remove the new ghosts and to yield the same nontrivial cohomology as we started with.

The constraints (2.3) we add two further terms to the BRST charge $Q_{K,0}$

$$Q_{K,-1} = - \sum_{\alpha \in \Delta^+} \bar{\eta}_i \alpha^i \xi^\alpha \xi^{-\alpha}, \hspace{1cm} Q'_{K,1} = \sum_i \eta_i H_i,$$  \hspace{1cm} (3.1)$$

A similar analysis for the superparticle has been pursued in [15].
The anticommuting ghosts $\eta^i$ and $\bar{\eta}_j$ satisfy the bracket \{\eta^i, \bar{\eta}_j\} = \delta^i_j$ or $\eta^i(z)\bar{\eta}_j(w) \sim \delta^i_j(z - w)^{-1}$. At this stage, $(Q_{K,-1} + Q_{K,0} + Q'_{K,1})^2$ contains only terms of the form $\sum \alpha_i \eta^i \xi^\alpha E_\alpha$, and they can be canceled by adding the usual three ghost term

$$Q''_{K,1} = \sum_{i, \alpha} \alpha_i \eta^i \beta \xi^\alpha. \quad (3.2)$$

The BRST charge can be decomposed into terms $Q_{K,n}$ with different grading $n$ if one assigns the following grading to the ghosts and antighosts

$$gr(\xi^\alpha) = 0, \quad gr(\eta^i) = 1, \quad gr(\beta \alpha) = 0 \quad gr(\bar{\eta}_i) = -1. \quad (3.3)$$

The nilpotency of the BRST charge $Q_K$ and the existence of this grading lead to a filtration of the nilpotency relations

$$Q^2_{K,-1} = 0, \quad \{Q_{K,-1}, Q_{K,0}\} = 0, \quad \{Q_{K,0}, Q_{K,0}\} + 2 \{Q_{K,1}, Q_{K,-1}\} = 0, \quad (3.4)$$

$$\{Q_{K,0}, Q_{K,1}\} = 0, \quad Q^2_{K,1} = 0.$$

The second equation implies the invariance of the constraints (2.3). The third equation tells us that the charge $Q_{K,0}$ is nilpotent up to the constraints (2.3). The fourth relation holds since it is proportional to the sum of $(\alpha + \beta)_i$, $-\alpha_i$ and $-\beta_i$. The last equation expresses the simple fact that the Cartan generators are abelian ($\eta^i \eta^j H_i H_j = 0$).

The new BRST operator $Q_K$ has trivial cohomology. (We will demonstrate this later in a model of conformal field theory where the BRST operator can be obtained from a $G/G$ gauged WZNW model.) This can be understood as follows: the BRST charge $Q_K$ contains the operators $E_\alpha, E_{-\alpha}$ and $H_i$ for the roots and Cartan subalgebra. Therefore, the BRST closed functions of $\hat{M}_1$ (i.e. the unconstrained space with all ghosts) are given by

$$\Phi^{(1)} = \sum_{\alpha} \xi^\alpha A_\alpha + \sum_i \eta^i A_i. \quad (3.5)$$

By defining the curvatures of the fields of $A_\alpha$ and $A_i$ as usual

$$F_{[\alpha, \beta]} = D_{[\alpha} A_{\beta]} - N_{\alpha, \beta} A_{\alpha+\beta} \quad \text{if} \quad \alpha + \beta \neq 0 \quad F_{\alpha, -\alpha} = D_{[\alpha} A_{-\alpha]} - \alpha^i A_i,$$

$$F_{\pm \alpha, i} = D_i A_{\pm \alpha} - D_{\pm \alpha} A_i + \alpha_i A_\alpha, \quad F_{i, j} = D_{[i} A_{j]}, \quad (3.6)$$

the equation $\{Q_K, \Phi^{(1)}\} = 0$ implies that all curvatures vanish. Therefore, we can start solving this system by observing that $A_i = D_i \Omega$ solves locally $F_{i, j} = 0$, and inserting this
result in \( F_{\pm \alpha, i} = 0 \), one gets \( A_{\pm \alpha} = D_{\pm \alpha} \Omega \). So, the general solution of (3.6) are pure
gauge connections which corresponds to BRST exact \( \Phi^{(1)} = \{ Q_K, \Omega \} \). The same results
hold for other ghost numbers. In order not to remove the nontrivial cohomological classes
of \( \hat{\mathcal{M}}' \), we have to establish a new definition of physical observables. To this purpose we
introduce a second BRST charge \( Q_C \). This requires to extend again the set of ghost fields.

The additional ghosts have the role to remove the ghosts \( \eta^i \) and \( \bar{\eta}_i \) from the cohomology. We add a quartet (two new doublets) of fields. The first doublet contains a pair of
anticommuting fields \( \eta'^i \) and \( \bar{\eta}'_i \) with brackets \( \{ \eta'^i, \bar{\eta}'_j \} = \delta^i_j \) and with the same quantum
number as \( \eta^i \) and \( \bar{\eta}_i \), where the index \( i \) runs over the Cartan subalgebra. In addition, we
introduce two commuting fields \( \phi^i \) and \( \bar{\phi}_i \) with brackets \( [\phi^i, \bar{\phi}_j] = \delta^i_j \). The new ghosts have
the following ghost and grading numbers, respectively

\[
\eta'^i \ (1, 1), \quad \bar{\eta}'_i \ (-1, -1), \quad \phi^i \ (0, 0), \quad \bar{\phi}_i \ (0, 0). \tag{3.7}
\]

We follow now the procedure of [10]. The \((\phi, \bar{\phi})\) form with the \((\eta, \bar{\eta})\) a quartet of \( Q_C \), but the same \((\phi, \bar{\phi})\) form with \((\eta', \bar{\eta}')\) another quartet of \( Q_K \). In this way, we remove all
six ghosts \( \eta, \bar{\eta}, \eta', \bar{\eta}', \phi, \bar{\phi} \) from the cohomology. The new BRST charge \( Q_C \) is given by

\[
Q_C = \sum_{\alpha \in \Delta_+} \eta'^i \alpha^i \xi^\alpha \xi^{-\alpha} + \sum_i \bar{\phi}_i \eta^i. \tag{3.8}
\]

It is obviously nilpotent. The second term removes \( \bar{\eta}' \) (and its conjugate \( \eta_i \) can be set to
zero) from the space \( \hat{\mathcal{M}}' \). The first term is needed in order that the second BRST charge
commutes with the original charge \( Q_K \). However, we also have to add a new piece to \( Q_K \)

\[
Q_K \to Q_K + \sum_i \bar{\phi}_i \eta'^i. \tag{3.9}
\]

This extended \( Q_K \) is clearly also nilpotent.

The new term in \( Q_K \) removes the variables \( \eta', \bar{\eta}', \phi \) and \( \bar{\phi} \) from the cohomology of \( Q_K \).
Although we keep all ghosts in our covariant approach, note that if we would remove all of
them, we would have to impose by hand the original constraints in (2.3). The argument

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8 Since \( Q_C \bar{\eta}_i = \bar{\phi}_i \) and \( Q_C \phi^i = -\eta^i \), any function \( F(\eta, \phi, \eta', \ldots) \) which is annihilated by \( Q_C \) is
independent of \( \phi \) and \( \eta \) up to terms which are \( Q_C \) exact. In fact, there is an homotopy operator
\( K_C \), satisfying \( \{ K_C, Q_C \} = N_{\phi, \phi} + N_{\eta, \bar{\eta}} \) and given by \( K_C = \phi^i \bar{\eta}_i \). Then any state with \( \phi \) and \( \eta \)
dependence which is \( Q_C \)-closed is also \( Q_C \)-exact. Similarly, \( Q_K \bar{\eta}'_i = \bar{\phi}_i \) and \( Q_K \phi^i = -\eta'^i \), and
\( Q_K \) removes the \( \phi_i \) dependence while terms depending on \( \eta'_i \) are \( Q_K \)-exact.
is the same as in the case of the $A_0 = 0$ gauge in QED, where one has to impose by hand its missing field equation, the Gauss constraint. Notice that the second BRST charge has only terms with grading number $-1$ and $+1$.

The two BRST charges anticommute

$$Q_K^2 = 0, \quad \{Q_K, Q_C\} = 0, \quad Q_C^2 = 0. \quad (3.10)$$

To prove that the terms with $N_{\alpha,\beta}$ cancel in $\{Q_K, Q_C\}$ one may use the Jacobi identities with $E_\alpha, E_\beta$ and $E_{-\alpha-\beta}$. The terms proportional to $\bar{\eta}'\eta$ cancel because they occur in pairs with opposite signs. In addition, it is easy to see that the first BRST charge $Q_K$ can be written in the following way

$$Q_K = \sum_\alpha \xi^\alpha E_\alpha - \frac{1}{2} \sum_{\alpha,\beta} \beta_{\alpha+\beta} N_{\alpha,\beta} \xi^\alpha \xi^\beta + \left[ Q_C, \sum_i \left[ \bar{\eta}_i \eta_i' - \phi_i (H_i + \sum_\alpha \alpha_i \beta_\alpha \xi^\alpha) \right] \right], \quad (3.11)$$

which shows that the additional terms are indeed trivial with respect to the second BRST charge. Furthermore, $Q_C$ is $Q_K$-exact, namely

$$Q_C = [Q_K, \bar{\eta}_i \eta_i']. \quad (3.12)$$

The most general vertex with ghost number one can be written as follows

$$\Phi^{(1)} = \sum_\alpha \xi^\alpha A_\alpha + \{Q_C, \sum_i \phi^i W_i\}. \quad (3.13)$$

The second term is $Q_C$-trivial and it is needed in order that $\Phi^{(1)}$ is in the cohomology of $Q_K$. Physical states are identified with $H(Q_K|H(Q_C, \hat{M}))$: they are $Q_C$-closed, and $Q_K$-closed modulo $Q_C$-exact terms. To work this out in more detail note that the $Q_K$ cohomology can be written as follows

$$\{Q_K, \Phi^{(1)}\} = \left\{ Q_{K,0} + [Q_C, X], \sum_\alpha \xi^\alpha A_\alpha + [Q_C, \sum_i \phi^i W_i] \right\} = \quad (3.14)$$

$$\left\{ Q_{K,0}, \sum_\alpha \xi^\alpha A_\alpha \right\} + \left\{ Q_C, [X, \sum_\alpha \xi^\alpha A_\alpha] - [Q_{K,0}, \sum_i \phi^i W_i] + [X, [Q_C, \sum_i \phi^i W_i]] \right\} =$$

$$\left\{ Q_{K,0}, \sum_\alpha \xi^\alpha A_\alpha \right\} + \{Q_C, Z\}.$$

where $X$ is the $Q_C$-exact term in (3.11) and $Z$ is the $Q_C$-exact term in (3.13). We used $\{Q_C, \sum_\alpha \xi^\alpha A_\alpha\} = 0$. This shows that we have obtained the correct cohomology: the $Q_K$-closed vertex operators constructed from all ghosts given modulo the $Q_C$-exact terms coincide with the vertex operators which are constructed from $\xi^\alpha$, and which are $Q_{K,0}$-closed modulo the constraints in (2.3).
4. An interpretation of the grading

In our work on the covariant quantization of the superstring we were forced to exclude certain terms from the massless vertex operators in order to obtain a nontrivial cohomology. We achieved this by assigning a grading to the various ghosts which appear in our work, and requiring that vertex operators contain only terms with nonnegative overall grading. The deep meaning of this grading has eluded us up till now although we have shown that it is related to homological perturbation theory \[6,16\]. We now present an interpretation of this grading condition.

In the previous section, we used a second BRST charge to select the physical subspace. This suggests that there exists another quantum number for the ghosts and antighosts besides the ghost number. This is the quantum number given in (3.3) and (3.7). However, the notion of a grading was extracted from the BRST approach, and we should explore whether the grading has a meaning independently of this construction, in particular whether there is a property of Lie algebras which leads to the concept of this grading.

Let us return to the Lie algebra on the Cartan-Weyl basis

\[
[H_i, H_j] = 0, \quad [E_{\alpha}, E_{-\alpha}] = \alpha^i H_i, \\
[H_i, E_{\pm\alpha}] = \pm\alpha_i E_{\pm\alpha}, \quad [E_{\alpha}, E_{\beta}] = N_{\alpha,\beta} E_{\alpha+\beta}.
\]

These commutators are preserved by the following transformations

\[
H_i \rightarrow \lambda H_i, \quad E_{\alpha} \rightarrow E_{\alpha} \quad \alpha_i \rightarrow \lambda \alpha_i
\]

where \(\lambda \neq 0\). As a consequence \(\alpha^i \rightarrow \lambda^{-1} \alpha^i\). We identify the grading with the power of \(\lambda\) in this automorphism. Thus we assign a grading +1 to each Cartan generator. The transformation rule for the roots \(\alpha_i\) (for each component we assume the same dilatation) is a consequence of the dilatation of the Cartan generators.

By viewing \(E_{\alpha}\) and \(H_i\) as constraints on the physical states: \(E_{\alpha}|\psi\rangle = 0\) and \(H_i|\psi\rangle = 0\) the contraction \(\lambda \rightarrow 0\) reduces the set of the constraints to those implemented by the lowering and raising operators \(E_{\alpha}\) with \(\alpha \in \Delta\), but they become second class constraints. In fact the r.h.s. of \([E_{\alpha}, E_{-\alpha}] = \alpha^i H_i\) has a finite limiting value. This implies that we need to implement a quantization procedure which deals with second class constraints.

Applying these considerations to the superstring, we are led to the following gradings: \(\lambda^\alpha\) has grading zero, and \(\xi^m\) and \(\chi_\alpha\) have grading one. This is not the grading proposed in
but remarkably, it gives the same suppression of terms in the vertex operator and thus
the same spectrum. (More specifically: the terms $b\lambda^\alpha \lambda^\beta$ and $b\lambda^\alpha \xi^m$ which had grading
$-4 + 1 + 1$ and $-4 + 1 + 2$, respectively, were rejected. The new grading rejects only the
term $b\lambda^\alpha \lambda^\beta$).

At the level of the BRST charge (assigning zero grading to any ghosts and antighosts)
we see that by rescaling the Cartan generators (and the roots $\alpha_i$) the terms $Q'_{K,1}$ and $Q''_{K,1}$
vanish. However, $Q_{K,-1}$ explodes if we do not require that $\sum_{\alpha \in \Delta} \alpha^i \xi^\alpha \xi^{-\alpha} = 0$, namely
if we do not require that the constraints (2.3) are satisfied. At this point it is clear that
we can reabsorb the rescaling (4.2) into the ghost fields by assigning a suitable grading or,
equivalently, by rescaling them. In this way the BRST charge $Q_K$ is decomposed into the
pieces $Q_{K,-1} + Q_{K,0} + Q_{K,1}$ and the second BRST charge $Q_C$ into $Q_C = Q_{C,-1} + Q_{C,1}$.

To conclude, the grading is a property of Lie algebras which we transfer to the ghost
fields.

5. An example: $SU(N)$ conformal field theory

We now present an example where the general formalism developed in the previous
sections is worked out explicitly. The example is the conformal field theory associated to
the affine Lie algebra of currents based on $SU(N)$. The generators are represented by
holomorphic currents $E_\alpha(z), H_i(z)$. We neglect the anti-holomorphic sector of the theory.
There are double poles in the OPE’s of these currents

$$E_\alpha(z)E_{-\alpha}(w) \sim \frac{\alpha^i H_i(w)}{z-w} + \frac{k}{(z-w)^2},$$

$$E_\alpha(z)E_\beta(w) \sim N_{\alpha, \beta} \frac{E_{\alpha+\beta}}{(z-w)}, \quad \text{if } \alpha + \beta \neq 0$$

$$H_i(z)E_\alpha(w) \sim \frac{\alpha_i E_\alpha(w)}{(z-w)}, \quad H_i(z)H_j(w) \sim \frac{k g_{ij}}{2(z-w)^2}.$$
WZNW model with level $-2N - k$ (where $N$ is the dimension of the gauge group $SU(N)$) and those fields are associated to the gauge fields needed to impose the constraints.\footnote{In our earlier work, we used the formalism with a ghost pair $(c_z, b)$, satisfying $c_z(z)b(w) \sim \frac{1}{z-w}$, to remove the anomaly in the BRST charge, but recently we have dropped the ghosts $(c_z, b)$ in favor of the auxiliary $h$-currents \cite{7}.}

The BRST current $j_{K,0}$ corresponding to $Q_{K,0}$ introduced in section 2 is given by

$$j_{K,0}(z) = \sum_{\alpha \in \Delta} \xi^\alpha \left( E_\alpha(z) + E^{(h)}_\alpha(z) \right) - \frac{1}{2} \sum_{\alpha, \beta \in \Delta} N_{\alpha,\beta} \beta_\alpha \beta_\beta \xi^\alpha \xi^\beta$$

(5.2)

which is nilpotent up to the constraints (2.3). The combination of currents $(E_\alpha + E^{(h)}_\alpha + \sum_\beta N_{\alpha,\beta} \beta_\alpha \beta_\beta \xi^\beta)$ does not have double poles and it can be used to construct the left-moving sector of the BRST charge $Q_{K,1} = \oint dz j_{K,0}(z)$. All single poles of the combination $(E_\alpha + E^{(h)}_\alpha + \sum_\beta N_{\alpha,\beta} \beta_\alpha \beta_\beta \xi^\beta)$ are cancelled except those proportional to the Cartan generators.

The constraints (2.3) generate gauge transformations of the antighosts and therefore any conformal field theory operator should be compatible with those transformations.

The definition of physical states is given as in the previous section (cf. eq. (2.7)) in the constrained cohomology of $Q_{K,0}$. Using the Sugawara construction, the energy-momentum tensor is given by

$$T_{zz} = \frac{1}{2(k + N)} \sum_{\alpha \in \Delta} \frac{|\alpha|^2}{2} (E_{+\alpha} E_{-\alpha} + E_{-\alpha} E_{+\alpha}) - \frac{1}{2(k + N)} \sum_{\alpha \in \Delta} \frac{|\alpha|^2}{2} (E^{(h)}_{+\alpha} E^{(h)}_{-\alpha} + E^{(h)}_{-\alpha} E^{(h)}_{+\alpha})$$

$$+ \sum_i (H_i H_i + H_i^{(h)} H_i^{(h)}) - \sum_{\alpha \in \Delta} \beta_\alpha \partial_z \xi^\alpha.$$

(5.3)

The last term needs some explanation. The ghost fields $\xi^\alpha$ are constrained and, as a consequence, the antighosts transform under gauge transformation generated by the constraint (2.3). This means that the constraint eliminates some of the ghosts $\xi^\alpha$ (for example, for $SU(2)$ there is only one independent ghost field $\xi^+ = \xi^-$) and thus $T_{zz}$ (as well as the Lagrangian) depends only on certain combinations of the antighost fields (for $SU(2)$, the ghost dependent term is given by $\beta_+ \partial \xi^+ + \beta_- \partial \xi^- = (\beta_+ + \beta_-) \partial \xi^+ = \tilde{\beta}_+ \partial \xi^+$ where $\tilde{\beta}_+ = \beta_+ + \beta_-$ is the combination gauge invariant under $\delta_\epsilon \beta_\pm = \epsilon \xi^\pm$ and...
\[ \delta_\epsilon \beta_+ = -\epsilon \xi^- = -\epsilon \xi^+ . \] The tensor \( T_{zz} \) is invariant under BRST transformations, 
[\[ Q_{K,1}, T_{zz} \] = 0. \]

The total conformal charge of the system is
\[
c_{SU(N)} = \frac{k(N^2 - 1)}{k + N} - \frac{(-2N - k)(N^2 - 1)}{k + N} - 2[N(N - 1) - (N - 1)] = 4(N - 1) \tag{5.4}
\]
where the last term is due to the ghosts and antighosts associated with the \( N(N - 1) \) roots minus the number of the constraints \( N - 1 \) (see (2.3)). The factor \(-2\) comes from the conformal weight \((0, 1)\) and statistics of the pairs \((\xi^\alpha, \beta_\alpha)\). The total conformal charge is always positive and it does not depend on the level of the WZW action. Notice that without the constraints and with the ghosts associated to the Cartan generators, the last term in (5.4) would be
\[
-2[N(N-1) + (N-1)] = -2(N^2-1) \quad \text{and it cancels exactly the first two terms in } c_{SU(N)}. \quad \text{This coincides with the topological model } G/G. \quad \text{(The total central charge vanishes because } T \text{ is the energy-momentum tensor for a twisted superconformal algebra.)}
\]
The total central charge is positive (the theory is unitary) and it can vanish only if \( N = 1 \) which is a trivial case. However, another way to make the central charge vanish (except for example by adding suitable ghosts for reparametrizations) is to add a fermionic counterpart to the generators \( E_\alpha \) and \( H_i \). So, for the superalgebra \( SU(M|N) \), we have \( \{E_\alpha, H_i\} \) and \( \{E'_{\alpha'}, H'_{i'}\} \) for the subgroup \( SU(M) \times SU(N) \). In addition we have \( 2M \times N \) fermionic generators \( Q_{ab'} \) where \( a = 1, \ldots, M \) and \( b' = 1, \ldots N \). In that case we have to decide which coset we need to gauge and therefore we have to introduce bosonic ghosts.

Given a BRST current \( j_{K,z} \) and the energy-momentum tensor \( T_{zz} \), there is an additional operator worth the be mentioned: the ghost current. In the present case it is given by
\[
J^h_z = \sum_{\alpha \in \Delta} \beta_\alpha \xi^\alpha \quad \text{which is invariant under the gauge transformations } \delta_\epsilon \text{ generated by the first class constraints (2.3). In order to compute the coefficient of the double pole of } J(z).J(w) \text{ first solve the constraint (2.3), then one can choose a gauge for the antighosts.} \tag{10}
\]
The next step is to construct the second BRST charge \( Q_C \) for this conformal field theory. As a consequence we have to modify the BRST charge \( Q_K \). As in the previous section we introduce the fields \( \eta^i, \bar{\eta}_i \) to remove the ghost constraints (2.3) and to implement the constraints associated to the Cartan generators. (Notice that the enlargement of the \( \eta^i, \bar{\eta}_i \) in the \( SU(2) \) case, one has \( \xi^+ = \xi^- \) and \( J = \beta_+ \xi^+ + \beta_- \xi^- = (\beta_+ + \beta_-) \xi^+ = \tilde{\beta}_+ \xi^+ \). The last expression involves only free fields and we can compute the coefficient straightforwardly \( J(z).J(w) \sim (z - w)^{-2} \).
set of the constraints to the complete algebra leads to vanishing cohomology unless an addition constraint is added.) The new BRST current is

\[ j_K(z) = j_K,0(z) + \sum_{i,\alpha \in \Delta_+} \bar{\eta}_i \alpha^i \xi^{+\alpha} \xi^{-\alpha} + \sum_i [\eta^i (H_i + H_i^{(h)}) + \sum_{\alpha \in \Delta} \eta^i \alpha \beta_\alpha \xi^\alpha] \] (5.5)

and the new energy-momentum tensor is modified into

\[ T_{zz} \to T_{zz} + \bar{\eta}_i \partial \eta^i. \]

By adding the new fields and by modifying the energy-momentum tensor we find that the total central charge of the new \( T \) vanishes. This is due to the topological nature of the model under the analysis (see for example [14] for a complete analysis of the BRST cohomology for \( G/G \) and \( G/H \) models).

In fact, it is easy to see that the ghost introduced can be viewed as twisted fermions on the worldsheet

\[ \psi_\alpha = \frac{\xi_\alpha + \beta_\alpha}{2}, \quad \bar{\psi}_\alpha = \frac{\xi_\alpha - \beta_\alpha}{2}, \] (5.6)

\[ \bar{\psi}^i = \frac{\eta^i + \bar{\eta}^i}{2}, \quad \psi^i = \frac{\eta^i - \bar{\eta}^i}{2}, \]

and the BRST symmetry as a twisted supersymmetry on the worldsheet. To compute the total central charge it is sufficient to compute the anomaly in the ghost current

\[ J^{gh}(z) = - \sum_{\alpha} \xi_\alpha \beta^\alpha - \sum_i \eta^i \bar{\eta}_i. \] (5.7)

Since the ghosts fields are free fields, the coefficient of the double pole of \( J^{gh}(z)J^{gh}(w) \) is \( N^2 - 1 \), namely the dimension of the Lie group \( SU(N) \). For the supergroup \( SU(M|N) \) one has \( N^2 + M^2 - 2MN - 2 \).

Following the previous sections, we have to define a new BRST charge which leads to the correct cohomology of the theory. For that purpose we follow the previous section and we add the topological quartet formed by the commuting fields \((\phi^i, \bar{\phi}_i)\) and by the anticommuting ghosts \((\eta^i, \bar{\eta}_i)\). They are needed to remove the ghosts \( \eta^i, \bar{\eta}_i \) added in (5.3). The introduction of new fields might modify the central charge, but introducing topological quartets the total central charge remains zero. Nevertheless the coefficient of the double pole in \( J^{gh}(z)J^{gh}(w) \), where

\[ J^{gh}(z) = - \sum_{\alpha \in \Delta} \xi_\alpha \beta^\alpha - \sum_i \eta^i \bar{\eta}_i - \sum_i \eta^i \bar{\eta}^i. \] (5.8)
changes from \(N^2 - 1\) to \(N^2 + N - 2\).

The new BRST charge is given by
\[
j_C = \sum_{i, \alpha \in \Delta^+} \bar{\eta}_i^i \alpha^i \xi^\alpha \xi^{-\alpha} + \sum_i \bar{\phi}_i \eta^i, \tag{5.9}\]
and \(j_K\) has to modified as follows
\[
j_K \rightarrow j_K + \sum_i \bar{\phi}_i \eta^i; \tag{5.10}\]
Both currents are nilpotent and they anticommute \(j_K(z)j_C(w) \sim 0\). In the present framework, we can establish a new conserved current
\[
J^{gr} = -\sum_i \eta^i \bar{\eta}_i - \sum_i \eta'^i \bar{\eta}'_i; \tag{5.11}\]
which corresponds to the assignment in (3.3). Notice that the second BRST current \(j_C\) contains only pieces with grading \(-1\) and \(+1\). We can clearly make any linear combination of the current \(J^{gh}\) and \(J^{gr}\). The coefficient of the double poles of the second charge is \(2(N - 1)\). Notice that the construction achieved so far resembles the \(N = 4\) embedding of the RNS superstrings provided in [8]. In particular, for the \(N=4\) formulation of the superstring, the two BRST charges \(Q_1\) and \(Q_2\) implement the superdiffeomorphisms at the quantum level and restrict the Fock subspace to the small Hilbert space. The two currents \(J_1\) and \(J_2\) are identified with ghost and picture number. Both BRST charges have ghost number one, but while the second has picture \(-1\), the first one is a sum of terms with definite picture. This reproduces the structure outlined above. The ghost number and the picture number have to be identified with \(J^{gh}\) and \(J^{gr}\), and the two BRST charges with \(Q_K\) and \(Q_C\). We can push the analogy even further: the motivation to introduce a second BRST charge in the RNS context is the enlargement of the functional space from the small Hilbert space (without the zero mode of \(\xi\)) to the large Hilbert space (with \(\xi_0\)). The second BRST charge restricts again the space, but then one can work covariantly (namely with all the modes of the field \(\xi\)). The motivation to introduce the second BRST charge \(Q_C\) in the present framework is the enlargement of the functional space to a space without constraints (2.3).

As outlined in the previous section, one can construct the vertex operators and study the spectrum. As has been already shown at the massless level, this model has nontrivial solutions to the equations of motion. The conformal field theory approach should however lead to the analysis of the complete tower of states. A detailed study for the superstring will be presented elsewhere.
6. Conclusions and outlook

In this article we have shown how to gauge the set of raising and lowering operators for an arbitrary Lie algebra in a covariant way. One needs to introduce more ghosts, and then remove their effects by a second BRST charge. It would be interesting to consider other cosets, for example the subset of all raising operators which plays a role in the derivation of harmonic superspace from pure spinors [12].

To complete the analysis of the conformal field theory of the previous section, we should repeat the analysis given in [17] and [18], leading to a Kazama algebra [19] and, after adding a Koszul quartet (a topological gravity quartet), we should obtain an $N = 2$ superconformal algebra [7]. Using the ghosts of topological gravity one can construct a new BRST charge to implement the reparametrization invariance as has been discussed in [20], but in addition we expect to need another BRST charge to recover the observables of the gravity sector. For this purpose we intend to use the formalism developed in this paper.

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Appendix A. $SU(3)$ as an example

We apply our results to $SU(3)$ as an example. To parametrize the compact basis we take the usual set of matrices with the normalization $\text{tr}(\lambda_a\lambda_b) = 2\delta_{ab}$, so the first Gell-Mann matrix is given by

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$ (A.1)

On the Cartan-Weyl basis, the raising generators are $E_I = \frac{i}{4}(\lambda_1 + i\lambda_2)$, $E_{II} = \frac{i}{4}(\lambda_4 + i\lambda_5)$, and $E_{III} = \frac{i}{4}(\lambda_6 - i\lambda_7)$, while $E_{-I} = E_I^\dagger = \frac{i}{4}(\lambda_1 - i\lambda_2)$, . . . The Cartan generators
are the hermitian matrices $H_\tau = \frac{1}{2}\lambda_3$ and $H_Y = \frac{1}{2}\lambda_8$. The commutation relations which determine $N_{\alpha,\beta}$ read

\[
\begin{align*}
[E_{-III}, E_{-II}] &= -\frac{1}{2}E_{-I}, & [E_{II}, E_{III}] &= -\frac{1}{2}E_I, \\
[E_{III}, E_{-I}] &= -\frac{1}{2}E_{-II}, & [E_{I}, E_{-III}] &= -\frac{1}{2}E_{II}, \\
[E_{I}, E_{-II}] &= \frac{1}{2}E_{III}, & [E_{II}, E_{-I}] &= \frac{1}{2}E_{-III},
\end{align*}
\]  

(A.2)

One easily derives

\[
\begin{align*}
[E_{I}, E_{-I}] &= \frac{1}{2}H_T, & [E_{II}, E_{-II}] &= \frac{1}{2}H_T + \frac{\sqrt{3}}{4}H_Y, \\
[E_{III}, E_{-III}] &= \frac{1}{4}H_T - \frac{\sqrt{3}}{4}H_Y, & [H_T, E_I] &= E_I, \\
[H_Y, E_I] &= 0, & [H_T, E_{II}] &= \frac{1}{2}E_{III}, \\
[H_Y, E_{II}] &= \frac{\sqrt{3}}{2} E_{III}, & \ldots
\end{align*}
\]  

(A.3)

The normalization $N_{\alpha,\beta} = -N_{-\alpha,-\beta}$ is satisfied. The roots are $(\pm 1,0)$ and $(\pm 1/2, \pm \sqrt{3}/2)$. The Cartan-Killing metric $g_{AB} = f_{AP}^Q f_{BQ}^P$ is given by $g_{ij} = 3\delta_{ij}$, and $g_{\alpha,-\alpha} = 3/2$ for each root. The usual relation $g_{\alpha,-\alpha}\alpha_i = g_{ij}\alpha^i$ is clearly satisfied. We shall occasionally need $g^{ij}\beta_j = g^{\beta,-\beta}\beta^i$. (We do not rescale $E_{\alpha}$ such that $g_{\alpha,-\alpha} = 1$; hence, the indices of $\alpha^i$ and $H^i$ are lowered by the matrix $g_{ij}g^{\alpha,-\alpha} = 2g_{ij}/3$, but we shall never have occasion to lower indices).

The constraints in (2.3) become

\[
\begin{align*}
C^T &= \frac{1}{2}\xi^I\xi_{-I} + \frac{1}{4}\xi^{II}\xi_{-II} + \frac{1}{4}\xi^{III}\xi_{-III} = 0, & C^Y &= \frac{\sqrt{3}}{4}(\xi^{II}\xi_{-II} - \xi^{III}\xi_{-III}) = 0. \\
\end{align*}
\]  

(A.4)

The BRST charge $Q_{K,0}$ in (2.2) becomes

\[
Q_{K,0} = \left(\xi^I E_I + \ldots + \xi_{-III}E_{-III}\right) 
\]  

(A.5)

\[
+ \frac{1}{2}\left(\beta_1\xi^{II}\xi^{III} + \beta_{II}\xi^I\xi_{-III} - \beta_{III}\xi^I\xi_{-II} - \beta_{-I}\xi^{II}\xi_{-III} - \beta_{-II}\xi^{III}\xi_{-II} + \beta_{-III}\xi^{III}\xi_{-II}\right)
\]

The constraints commute with $Q_{K,0}$, as one may check by explicit computation.

The quadratic Casimir operator is given by

\[
C_2 = \frac{2}{3}\left(E_I E_{-I} + E_{II} E_{-II} + E_{III} E_{-III} + E_{-I} E_I + E_{-II} E_{II} + E_{-III} E_{III}\right) 
\]  

(A.6)
\[ \frac{1}{3} (H_T H_T + H_Y H_Y) . \]

The square of \( Q_{K,0} \) contains only constraints

\[ (Q_{K,0})^2 = C^T H_T + C^Y H_Y . \]  

(A.7)

Adding

\[
Q_{K,-1} = - \bar{\eta}^T C^T - \bar{\eta} Y C^Y \\
Q'_{K,1} = \eta^T H_T + \eta Y H_Y \bar{\phi} T \eta^T + \bar{\phi} Y \eta^Y , \\
Q''_{K,1} = \eta^T \left( \frac{1}{2} \beta_I \xi^I + \frac{1}{2} \beta_{II} \xi^{II} + \frac{1}{2} \beta_{III} \xi^{III} - \frac{1}{2} \beta_{-I} \xi^{-I} - \frac{1}{2} \beta_{-II} \xi^{-II} - \frac{1}{2} \beta_{-III} \xi^{-III} \right) + \\
+ \frac{\sqrt{3}}{2} \eta^Y \left( \beta_{II} \xi^{II} - \beta_{III} \xi^{III} - \beta_{-II} \xi^{-II} + \beta_{-III} \xi^{-III} \right) 
\]

(A.8)

we find the nilpotent BRST operator \( Q_K \).

The second BRST operator \( Q_C \) is given by

\[
Q_C = \eta^T \left( \frac{1}{2} \xi^I \xi^{-I} + \frac{1}{4} \xi^{II} \xi^{-II} + \frac{1}{4} \xi^{III} \xi^{-III} \right) + \\
+ \frac{\sqrt{3}}{4} \eta^Y \left( \xi^{II} \xi^{-II} - \xi^{III} \xi^{-III} \right) + \bar{\phi} T \eta^T + \bar{\phi} Y \eta^Y , 
\]

(A.9)

and one may verify by direct computation that indeed anticommutes with \( Q_K \).

Appendix B. The Haar measure for \( SU(2)/U(1) \) from the BRST cohomology

In this appendix we present an application of the formalism presented in the previous sections. We consider the group \( SU(2) \) and we parametrize the matrix of the coset \( SU(2)/U(1) \) with a single complex vector \( p_i \) with \( i = 1, 2 \). We assume that \( p_i \) is normalized to unity and a given \( u \in SU(2)/U(1) \) can be written as \( u = (p_i, \epsilon_{ij} \bar{p}^j) \).

Associated to the generators \( E_\pm \) and \( H \), we introduce the following differential operators

\[
D_+ = p_i \epsilon^{ij} \partial_{\bar{p}^j} , \quad D_- = \bar{p}^i \epsilon_{ij} \partial_{p_j} , \quad [D_+, D_-] = D_0 = p_i \partial_i - \bar{p}^i \partial_{\bar{p}^i} \quad (B.1)
\]

\[
[D_0, D_+] = D_+ , \quad [D_0, D_-] = -D_- ,
\]

19
and the BRST charge

\[ Q_{K,0} = \xi^+ D_+ + \xi^- D_- \]  

which is nilpotent if \( \xi^+\xi^- = 0 \). Acting with \( Q_{K,0} \) on the vector \( p_i \) and its conjugate \( \bar{p}^i \) (they are treated as independent), \((B.2)\) leads to

\[
\{ Q_{K,0}, \bar{p}^i \} = \xi^+ p_k \epsilon^{ki}, \quad \{ Q_{K,0}, \xi^+ \} = 0, \quad \{ Q_{K,0}, \xi^- \} = 0.
\] (B.3)

Let us introduce the homogenous forms

\[
\omega^+ = \bar{p}^i \epsilon_{ij} d\bar{p}^j, \quad \omega^- = p_i \epsilon^{ij} dp_j.
\] (B.4)

They are dual of \( D_+ \) and \( D_- \) in the sense that \( \langle D_\pm, \omega^\pm \rangle = 1 \) and \( \langle D_\pm, \omega^\mp \rangle = 0 \) when

\[
\langle \partial_{p_i}, dp_j \rangle = \delta^i_j, \quad \langle \partial_{\bar{p}_i}, d\bar{p}_j \rangle = \delta^i_j, \quad \text{etc.}
\]

Their BRST variations are given by

\[
\{ Q_{K,0}, \omega^+ \} = -2\xi^+ p_i dp^i + d\xi^+, \quad \{ Q_{K,0}, \omega^- \} = 2\xi^- \bar{p}_i dp_i + d\xi^-,
\] (B.5)

where we used that \( Q_{K,0} \) and the exterior derivative \( d \) anticommute, and \( d\xi^+ \) and \( d\xi^- \) have to be considered as the worldsheet derivatives of the ghost fields.

With some algebra, it is easy to show that

\[
\{ Q_{K,0}, \omega^+ \omega^- \} = d\left( \xi^+ \omega^- + \omega^+ \xi^- \right),
\] (B.6)

Then, computing the \( Q_{K,0} \) variation of \( \left( \xi^+ \omega^- + \omega^+ \xi^- \right) \), one gets

\[
\left\{ Q_{K,0}, \left( \xi^+ \omega^- + \omega^+ \xi^- \right) \right\} = \xi^+ \xi^- (-4\bar{p}^k dp_k) - d(\xi^+ \xi^-)
\] (B.7)

which is zero because of the constraints \( \xi^+\xi^- = 0 \). This shows that \( \omega^+\omega^- \) belongs to the cohomology of \( Q_{K,0} \) modulo \( d \)-exact terms, and satisfies the descent equations. By using the parametrization \( z = \frac{p_1}{p_2} \) and \( \bar{z} = \frac{\bar{p}^1}{p^2} \), a simple exercise shows that

\[
\omega^+\omega^- = \frac{dz d\bar{z}}{(1 + |z|^2)^2}.
\] (B.8)

This is the Haar measure of the coset \( SU(2)/U(1) \).
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