ELKO Spinor Fields: Lagrangians for Gravity derived from Supergravity

Roldão da Rocha

Centro de Matemática, Computação e Cognição, 
Universidade Federal do ABC, 09210-170, Santo André, SP, Brazil

J. M. Hoff da Silva

Instituto de Física Teórica, Universidade Estadual Paulista, 
Rua Pamplona 145 01405-900 São Paulo, SP, Brazil

Dual-helicity eigenspinors of the charge conjugation operator (ELKO spinor fields) belong — together with Majorana spinor fields — to a wider class of spinor fields, the so-called flagpole spinor fields, corresponding to the class-(5), according to Lounesto spinor field classification based on the relations and values taken by their associated bilinear covariants. There exists only six such disjoint classes: the first three corresponding to Dirac spinor fields, and the other three respectively corresponding to flagpole, flag-dipole and Weyl spinor fields. Using the mapping from ELKO spinor fields to the three classes Dirac spinor fields, it is shown that the Einstein-Hilbert, the Einstein-Palatini, and the Holst actions can be derived from the Quadratic Spinor Lagrangian (QSL), as the prime Lagrangian for supergravity. The Holst action is related to the Ashtekar’s quantum gravity formulation. To each one of these classes, there corresponds a unique kind of action for a covariant gravity theory. Furthermore we consider the necessary and sufficient conditions to map Dirac spinor fields (DSFs) to ELKO, in order to naturally extend the Standard Model to spinor fields possessing mass dimension one. As ELKO is a prime candidate to describe dark matter and can be obtained from the DSFs, via a mapping explicitly constructed that does not preserve spinor field classes, we prove that — in particular — the Einstein-Hilbert, Einstein-Palatini, and Holst actions can be derived from the QSL, as a fundamental Lagrangian for supergravity, via ELKO spinor fields. The geometric meaning of the mass dimension-transmuting operator — leading ELKO Lagrangian into the Dirac Lagrangian — is also pointed out, together with its relationship to the instanton Hopf fibration.

PACS numbers: 03.65.Pm, 04.50.+h, 11.25.-w, 98.80.Jk

I. INTRODUCTION

Spinor fields can be classified according to the values assumed by their respective bilinear covariants. There are only six classes of spinor fields [1, 2, 3]: three of them are related to the three non-equivalent classes of Dirac spinor fields (DSFs), and the others are constituted respectively by the so-called flag-dipole, flagpole and Weyl spinor fields [1, 2, 3]. Majorana and ELKO (Eigenspinoren des Ladungskonjugationsoperators, or dual-helicity eigenspinors of the charge conjugation operator) spinor fields are special subclasses of flagpole spinor fields [4].

ELKO spinor fields are unexpected spin one-half — presenting mass dimension 1 — matter fields, which belong to a non-standard Wigner class [30, 31], and are obtained from a complete set of dual-helicity eigenspinors of the charge conjugation operator. Due to the unusual mass dimension, ELKO spinor fields interact in few possibilities with the Standard Model particles, which instigates it to be a prime candidate to describe dark matter1. Indeed, the new matter fields — constructed via ELKO [34] — are dark with respect to the matter and gauge fields of the Standard Model (SM), interacting only with gravity and the Higgs boson [30, 31, 33, 34, 40]. Moreover, it is essential to try to incorporate ELKO spinor fields in some extension of the SM, identifying new fields to dark matter and suggesting how the dark matter sector Lagrangian density arises from a mass dimension-transmuting symmetry. We additionally have already considered the possibility of incorporating the dynamics of ELKO spinor fields, extending the SM in order to accomplish the dynamical, as well the not less fundamental, algebraic, topological and geometric properties, associated with ELKO. In [35] the underlying equivalence between Dirac spinor fields (DSFs) and ELKO was analyzed and investigated and the conditions under which the DSFs can be led to an ELKO were constructed,

1 Other motivations for the ELKO to be a prime candidate to describe dark matter can be seen in, e.g., [30, 31, 34].
since they are inherently distinct and represent disjoint classes in Lounesto spinor field classification. Any invertible map that takes Dirac particles and leads to ELKO is also capable to make mass dimension transmutations, since DSFs present mass dimension three-halves, instead of mass dimension one associated with ELKO. In this previous paper the initial pre-requisites to construct a natural extension of the Standard Model (SM) in order to incorporate ELKO were provided, and consequently a possible description of dark matter in this context.

By using one specific class of DSF — seen as an equivalence class of Dirac spinor fields — and imposing a condition of constant spinor field, it has already been shown that the Einstein-Hilbert Lagrangian of General Relativity (GR), as well as the Lagrangian of its teleparallel equivalent (GR∥), can be recast as a quadratic spinor Lagrangian (QSL) and charge conjugation operators. Any invertible operator equals +1 and -1, respectively for DSFs and ELKO. The Holst action is shown to be equivalent to the Ashtekar formulation of Quantum Gravity.

In a previous paper, the equivalence between the underlying algebraic structure of the DSFs and the corresponding gravity theory actions were established, and one of the main aims of the present paper is to obtain the Lagrangians of some of the current theories for gravity and quantum gravity exclusively using ELKO spinor fields. This equivalence enables us to better characterize and understand the nature of the spinor field that constitutes the QSL. ELKO spinor fields can be led to DSFs, and the Einstein-Hilbert, the Einstein-Palatini and the Holst actions, respectively, can be derived from a QSL, when we consider ELKO spinor fields. We begin by showing first that the spinor-valued 1-form field entering the QSL has necessarily to be constructed by a tensor product between a Dirac spinor field and a Clifford algebra-valued 1-form: no other spinor fields can lead either to the Holst action, or to the particular cases of Einstein-Hilbert and Einstein-Palatini actions. These three gravitational actions correspond respectively to a class-(2), class-(3), and class-(1) DSFs, which present complete correspondence to ELKO spinor fields. ELKO spinor fields that are mapped into classes-(2) and -(3) of DSFs together give the Einstein-Hilbert and Einstein-Palatini actions, and the ELKO spinor fields that are mapped into class-(1) DSF gives alone the complete Holst action, which shows up also in the proof of gravitational theory as a SUSY gauge theory. Furthermore, we assume a more general approach, where the ELKO spinor field is not a constant spinor field anymore. As a consequence, the boundary term of the QSL will have many additional terms that can be related to some physical identities, and may unravel additional properties.

The paper is organized as follows: after presenting some algebraic preliminaries in Section II, we briefly introduce in Section III the ELKO spinor fields as well as we recall the conditions under which a DSF can be mapped into an ELKO spinor field. We also point out the geometric meaning of the mass dimension-transmuting symmetry between ELKO and Dirac spinor fields. In Section IV we investigate the QSL, and in Section V after briefly presenting the Lounesto spinor field classification, as well as some important features of each spinor field class, we show that Einstein-Hilbert, Einstein-Palatini, and Holst actions can be derived from a QSL provided we do not restrict ourselves to the case of a class-(2) DSF, also deriving such Lagrangians for gravity, via ELKO spinor fields. In the last Section all the results obtained are discussed.

II. PRELIMINARIES

This section is devoted to briefly introduce the mathematical pre-requisites to completely recall the definition of Clifford algebra-valued differential forms on a manifold $M$. For more details, see, e.g., [6, 12].
We denote by $M = (M, g, \nabla, \tau_g, 1)$ the spacetime structure: $M$ denotes a 4-dimensional manifold, $g \in \text{sec} T^*_M$ is the metric associated with the cotangent bundle, $\nabla$ is the Levi-Civita connection of $g$, $\tau_g \in \text{sec} \Lambda^1(T^*M)$ defines a spacetime orientation and $\dag$ refers to an equivalence class of timelike 1-form fields defining a time orientation. By $F(M)$ we mean the (principal) bundle of frames, by $P_{\SO^r_1}(M)$ the orthonormal frame bundle, and $P_{\SO^s_1}(M)$ denotes the orthonormal coframe bundle. We consider $M$ a spin manifold, and then there exists $P_{\Spin^r_1}(M)$ and $P_{\Spin^s_1}(M)$ which are respectively the spin frame and the spin coframe bundles. We denote by $s : P_{\Spin^r_1}(M) \rightarrow P_{\SO^r_1}(M)$ the fundamental mapping present in the definition of $P_{\Spin^r_1}(M)$. A spin structure on $M$ consists of a principal fiber bundle $\pi_s : P_{\Spin^r_1}(M) \rightarrow M$, with group $\Spin^r_1$, and the map

$$s : P_{\Spin^r_1}(M) \rightarrow P_{\SO^r_1}(M)$$

satisfying the following conditions:

(i) $\pi(s(p)) = \pi_s(p)$, $\forall p \in P_{\Spin^r_1}(M)$; $\pi$ is the projection map of the bundle $P_{\SO^r_1}(M).

(ii) $s(p\phi) = s(p)\text{Ad}_\phi$, $\forall p \in P_{\Spin^r_1}(M)$ and $\text{Ad} : \Spin^r_1 \rightarrow \text{Aut}(C\ell_{1,3})$, $\text{Ad}_\phi : C\ell_{1,3} \ni \Xi \mapsto \phi\Xi\phi^{-1} \in C\ell_{1,3}$.

We recall now that sections of $P_{\Spin^r_1}(M)$ are orthonormal coframes, and that sections of $P_{\Spin^s_1}(M)$ are also orthonormal coframes such that two coframes differing by a $2\pi$ rotation are distinct and two coframes differing by a $4\pi$ rotation are identified. Next we introduce the Clifford bundle of differential forms $C\ell(M, g)$, which is a vector bundle associated with $P_{\Spin^r_1}(M)$. Their sections are sums of non-homogeneous differential forms, which will be called Clifford fields. We recall that $C\ell(M, g) = P_{\SO^r_1} \times_{\text{Ad}} C\ell_{1,3}$, where $C\ell_{1,3} \simeq M(2, \mathbb{R})$ is the spacetime algebra. Details of the bundle structure are as follows $[12, 14, 15]$

(1) Let $\pi : C\ell(M, g) \rightarrow M$ be the canonical projection of $C\ell(M, g)$ and let $\{ U_\alpha \}$ be an open covering of $M$. There are trivialization mappings $\psi_i : \pi^{-1}(U_i) \rightarrow U_i \times C\ell_{1,3}$ of the form $\psi_i(p) = (\pi_c(p), \psi_{i,x}(p)) = (x, \psi_{i,x}(p))$. If $x \in U_i \cap U_j$ and $p \in \pi^{-1}(x)$, then

$$\psi_{i,x}(p) = h_{ij}(x)\psi_{j,x}(p)$$

for $h_{ij}(x) \in \text{Aut}(C\ell_{1,3})$, where $h_{ij} : U_i \cap U_j \rightarrow \text{Aut}(C\ell_{1,3})$ are the transition mappings of $C\ell(M, g)$. We recall that every automorphism of $C\ell_{1,3}$ is inner. Then,

$$h_{ij}(x)\psi_{j,x}(p) = a_{ij}(x)\psi_{i,x}(p)a_{ij}^{-1}(x)$$

for some $a_{ij}(x) \in C\ell_{1,3}$, the group of invertible elements of $C\ell_{1,3}$.

(2) As it is well known, the group $\SO^r_1$ has a natural extension in the Clifford algebra $C\ell_{1,3}$. Indeed, we know that $C\ell_{1,3}$ is the group of invertible elements of $C\ell_{1,3}$ acts naturally on $C\ell_{1,3}$ as an algebra automorphism through its adjoint representation. A set of lifts of the transition functions of $C\ell(M, g)$ is a set of elements $\{ a_{ij} \} \subset C\ell_{1,3}^*$ such that, if $^5$

$$\text{Ad} : \phi \mapsto \text{Ad}_\phi$$

$$\text{Ad}_\phi(\Xi) = \phi\Xi\phi^{-1}, \forall \Xi \in C\ell_{1,3},$$

then $\text{Ad}_{a_{ij}} = h_{ij}$ in all intersections.

(3) Also $\sigma = \text{Ad}|_{\Spin^r_1}$ defines a group homeomorphism $\sigma : \Spin^r_1 \rightarrow \SO^r_1$ which is onto with kernel $\mathbb{Z}_2$. We have that $\text{Ad}_{-1} = \text{identity}$, and so $\text{Ad} : \Spin^r_1 \rightarrow \text{Aut}(C\ell_{1,3})$ descends to a representation of $\SO^r_1$. Let us call $\text{Ad}'$ this representation, i.e., $\text{Ad}' : \SO^r_1 \rightarrow \text{Aut}(C\ell_{1,3})$. Then we can write $\text{Ad}'(\phi)\Xi = \text{Ad}_\phi\Xi = \phi\Xi\phi^{-1}$.

(4) It is clear that the structure group of the Clifford bundle $C\ell(M, g)$ is reducible from $\text{Aut}(C\ell_{1,3})$ to $\SO^r_1$. The transition maps of the principal bundle of oriented Lorentz cotetrad $P_{\SO^r_1}(M)$ can thus be (through $\text{Ad}'$) taken as transition maps for the Clifford bundle. We then have $[10]$

$$C\ell(M, g) = P_{\SO^r_1} \times_{\text{Ad}'} C\ell_{1,3},$$

i.e., the Clifford bundle is a vector bundle associated with the principal bundle $P_{\SO^r_1}(M)$ of orthonormal Lorentz coframes.

---

$^5$ Recall that $\Spin^r_1 = \{ \phi \in C\ell_{1,3}^* : \phi^2 = 1 \} \simeq \text{SL}(2, \mathbb{C})$ is the universal covering group of the restricted Lorentz group $\SO^r_1$. Notice that $C\ell_{1,3} \simeq C\ell_{3,0} \simeq M(2, \mathbb{C})$, the even subalgebra of $C\ell_{1,3}$ is the Pauli algebra.
Recall that $\mathcal{G}(T^*_x M, g_x)$ is also a vector space over $\mathbb{R}$ which is isomorphic to the exterior algebra $\Lambda(T^*_x M)$ of the cotangent space and $\Lambda(T^*_x M) = \otimes_{\ell=0}^{k} \Lambda^k(T^*_x M)$, where $\Lambda^k(T^*_x M)$ is the $(\frac{k}{2})$-dimensional space of $k$-forms over a point $x$ on $M$. There is a natural embedding $\Lambda(T^* M) \hookrightarrow \mathcal{C}(M, g)$ \ref{12} and sections of $\mathcal{G}(M, g)$ — Clifford fields — can be represented as a sum of non-homogeneous differential forms. Let $\{e_a\} \in \mathbf{PSO}_{1,3}(M)$ (the orthonormal frame bundle) be a tetrad basis for $TU \subset TM$ (given an open set $U \subset M$). Moreover, let $\{\vartheta^a\} \in \mathbf{PSO}_{1,3}(M)$. Then, for each $a = 0, 1, 2, 3$, $\vartheta^a \in \sec \Lambda^1(T^* M) \hookrightarrow \sec \mathcal{G}(M, g)$. We recall next the crucial result \ref{12,16} that in a spin manifold we have:

$$\mathcal{C}(M, \eta) = P_{\text{Spin}^c_{1,3}}(M) \times_{\text{Ad}} \mathcal{C}I_{1,3}. \tag{6}$$

Spinor fields are sections of vector bundles associated with the principal bundle of spinor coframes. The well known Dirac spinor fields are sections of the bundle

$$S_c(M, \eta) = P_{\text{Spin}^c_{1,3}}(M) \times_\mu \mathbb{C}^4, \tag{7}$$

with $\mu_c$ the $D^{(1/2,0)} \oplus D^{(0,1/2)}$ representation of $\text{Spin}^c_{1,3} \cong \text{SL}(2, \mathbb{C})$ in $\text{End}(\mathbb{C}^4)$ \ref{17}.

The orthonormal coframe field $\{\vartheta^a\} \in \sec \Lambda^1(T^* M)$ can be related to the metric $g$ by $g = \eta_{a\bar{b}} \vartheta^a \otimes \vartheta^{\bar{b}}$, with $(\eta_{a\bar{b}}) = \text{diag}(1, -1, -1, -1)$. The Clifford product will be denoted by juxtaposition. Given two arbitrary (in general non-homogeneous) form fields $\xi, \zeta \in \sec \Lambda(T^* M)$, the dual Hodge operator $\ast : \sec \Lambda^p(T^* M) \rightarrow \sec \Lambda^{4-p}(T^* M)$ is defined explicitly by $\xi \wedge \ast \zeta = G(\xi, \eta)$, where $G : \sec \Lambda(T^* M) \times \sec \Lambda(T^* M) \rightarrow \mathbb{R}$ denotes the metric extended to the space of form fields.

The coframe field $\{\vartheta^a\}$ and the metric-compatible connection 1-form $\omega^{ab}$ are potentials for the curvature and the torsion, expressed respectively by the structure equations

$$\Omega^a_b = d\vartheta^a + \omega^{a}_{\bar{b}} \wedge \vartheta^b \in \sec \Lambda^2(T^* M) \quad \text{and} \quad \Theta^a = d\vartheta^a + \omega^{a}_{\bar{b}} \wedge \vartheta^{\bar{b}} \in \sec \Lambda^2(T^* M). \tag{8}$$

The connection coefficients are implicitly given by $\omega^{ab} = \omega_{abc} \vartheta^c$, and the torsion can be decomposed in its irreducible components under the global Lorentz group as \ref{18}

$$\Theta^a = (1) \Theta^a + (2) \Theta^a + (3) \Theta^a \tag{9}$$

where

$$(2) \Theta^a = \frac{1}{3} \vartheta^a \wedge (\vartheta^b \wedge \Theta_b), \quad (3) \Theta^a = - \frac{1}{3} \ast (\vartheta^a \wedge \Theta), \quad (1) \Theta^a = \Theta^a - (2) \Theta^a - (3) \Theta^a, \tag{10}$$

with $a = \ast(\Theta_b \wedge \vartheta^b)$ denoting the axial 1-form associated with the axial torsion $(3) \Theta^a$. The term $\ast a$ is the well known translational Chern-Simmons 3-form field \ref{13,21,22}, whose total derivative $d \ast a$ is the Nieh-Yan 4-form field \ref{19,20,22}.

Clifford algebra-valued differential forms (on Minkowski spacetime) are elements of $\sec \Lambda(T^* M) \otimes \mathcal{C}I_{1,3}$. In particular, Eqs. \ref{3} are written as

$$\Omega = d\omega + \omega \wedge \omega \quad \text{and} \quad \Theta = d\vartheta + \vartheta \wedge \vartheta + \vartheta \wedge \omega, \tag{11}$$

where

$$\vartheta = \vartheta^a \otimes \gamma_a, \quad \omega = \frac{1}{4} \omega^{ab} \otimes \gamma_{ab}, \quad \Theta = \Theta^a \otimes \gamma_a, \quad \Omega = \frac{1}{4} \Omega^{ab} \otimes \gamma_{ab}, \tag{12}$$

with $\gamma_{ab} = \frac{1}{2}(\gamma_a \gamma_b - \gamma_b \gamma_a)$. All operations in the exterior algebra of differential forms are trivially induced on the space of Clifford-valued differential forms. In particular, given $\vartheta^a \in \Lambda(V)$, the total derivative $d(\vartheta^a \otimes \gamma_a)$ is given by $d(\vartheta^a) \otimes \gamma_a$ and, given a $p$-form field basis $\{\vartheta^f\}$ and a Clifford algebra basis $\{\gamma_f = \gamma_a \gamma_b \gamma_c \ldots\}$, the exterior product between two elements $\Phi = \Phi^f \otimes \gamma_f$ and $\Gamma = \Gamma^f \otimes \gamma_f$ of $\sec \Lambda(T^* M) \otimes \mathcal{C}I_{1,3}$ is given by $\ref{13,15}$

$$\Phi \wedge \Gamma = (\Phi^f \otimes \gamma_f) \wedge (\Gamma^f \otimes \gamma_f) = (\Phi^f \wedge \Gamma^f) \otimes \gamma_f \gamma_f. \tag{13}$$
III. ELKO SPINOR FIELDS

In this Section the formal properties of ELKO spinor fields are briefly revised \[30, 31, 32\] and the map between ELKO spinor fields and DSFs recalled. An ELKO, denoted by \(\Psi\), corresponding to a plane wave with momentum \(p = (p^0, \mathbf{p})\) can be written, without loss of generality, as \(\Psi(p) = \lambda(p)e^{-ip\cdot x}\) (or \(\Psi(p) = \lambda(p)e^{ip\cdot x}\)) where

\[
\lambda(p) = \begin{pmatrix} \Phi L(p) \\ \phi L(p) \end{pmatrix},
\]

\(\phi_L(p)\) denotes a left-handed Weyl spinor, and given the rotation generators denoted by \(\hat{J}\), the Wigner’s spin-1/2 time reversal operator \(\Phi\) satisfies \(\Phi\hat{J}\Phi^{-1} = -\hat{J}^\ast\). Hereon, as in \[30\], the Weyl representation of \(\gamma^\mu\) is used, i.e.,

\[
\gamma_0 = \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad -\gamma_k = \gamma^k = \begin{pmatrix} 0 & -\sigma_k \\ \sigma_k & 0 \end{pmatrix}, \quad \gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3 = -i\gamma^{0123} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

(15)

where

\[
\mathbb{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbb{O} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

(16)

ELKO spinor fields are eigenspinors of the charge conjugation operator \(C\), i.e., \(C\lambda(p) = \pm \lambda(p)\), for \(C = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}\). The operator \(K\) is responsible for the \(C\)-conjugation of spinor fields appearing on the right. The plus sign stands for self-conjugate spinors, \(\lambda^S(p)\), while the minus yields anti self-conjugate spinors, \(\lambda^A(p)\). Explicitly, the complete form of ELKO spinor fields can be found by solving the equation of helicity \((\sigma \cdot \mathbf{p})\phi^\pm(0) = \pm \phi^\pm(0)\) in the rest frame and subsequently performing a boost, in order to recover the result for any \(p\) \[30\]. Note that the helicity of \(i\Phi [\phi_L(p)]\ast\) is opposed to that of \(\phi_L(p)\), since \((\sigma \cdot \hat{p})\Phi [\phi_L^\pm(0)]\ast = \mp \Phi [\phi_L^\pm(0)]\ast\). Here \(\hat{p} := \mathbf{p}/||\mathbf{p}||\) = \((\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)\).

The four spinor fields are given by

\[
\lambda^{S/A}_{\{\mp, \pm\}}(p) = \sqrt{\frac{E + m}{2m}} \left(1 \mp \frac{\mathbf{p}}{E + m} \right) \lambda^{S/A}_{\{\mp, \pm\}}(0),
\]

(17)

where \(\lambda_{\{\mp, \pm\}}(0) = \begin{pmatrix} \pm i\Theta[\phi^\pm(0)]\ast \\ \phi^\mp(0) \end{pmatrix}\).

(18)

The phases are adopted so that

\[
\phi^+(0) = \sqrt{m} \begin{pmatrix} \cos(\theta/2)e^{-i\phi/2} \\ \sin(\theta/2)e^{i\phi/2} \end{pmatrix}, \quad \phi^-(0) = \sqrt{m} \begin{pmatrix} -\sin(\theta/2)e^{-i\phi/2} \\ \cos(\theta/2)e^{i\phi/2} \end{pmatrix},
\]

(19)

at rest, and since \(\Theta[\phi^\pm(0)]\ast\) and \(\phi^\mp(0)\) present opposite helicities, ELKO cannot be an eigenspinor field of the helicity operator, and indeed carries both helicities. In order to guarantee an invariant real norm, as well as positive definite norm for two ELKO spinor fields, and negative definite norm for the other two, the ELKO dual is given by \[30\]

\[
\lambda^{\mp/S \ast}_{\{\mp, \pm\}}(p) = \pm i \left[\lambda^{S/A}_{\{\mp, \pm\}}(p)\right]^{\ast} \gamma^0.
\]

(20)

It is useful to choose \(i\Theta = \sigma_2\), as in \[30\], in such a way that it is possible to express

\[
\lambda(p) = \begin{pmatrix} \sigma_2\Phi L(p) \\ \phi L(p) \end{pmatrix},
\]

(21)

Now, any flagpole spinor field is an eigenspinor field of the charge conjugation operator \[1, 2\], here represented by \(C\psi = -\gamma^2\psi\ast\). Indeed

\[
-\gamma^2 \lambda\ast = \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} \begin{pmatrix} (\sigma_2\phi)^\ast \\ \phi^\ast \end{pmatrix} = \begin{pmatrix} \sigma_2\phi^\ast \\ -\sigma_2\phi^\ast \end{pmatrix} = \lambda.
\]
Let us make a brief recall of which are the conditions a Dirac spinor field must obey to be led to an ELKO. In [39] there has been proved that not all DSFs can be led to ELKO, but only a subset of the three classes — under Lounesto classification — of DSFs restricted to some conditions. Explicitly, by taking a DSF

\[ \psi(p) = \left( \frac{\phi_R(p)}{\phi_L(p)} \right) = \left( \sigma_2 \phi_L^*(p) \right), \]

and taking into account that \( \phi_R(p) = \chi \phi_L(p) \), where \( \chi = \frac{E+\sigma \cdot p}{m} \) and \( \kappa \psi = \psi^* \), and denoting the 4-component DSF by \( \psi = (\psi_1, \psi_2, \psi_3, \psi_4)^T \), \( \psi_r \in \mathbb{C}, r = 1, \ldots, 4 \), we have the simultaneous conditions a DSF must obey in order for it to be led to an ELKO [39]:

\[\begin{align*}
0 &= \Re(\psi_1^* \psi_3) = \Re(\psi_2^* \psi_4) \\
0 &= \Re(\psi_2^* \psi_3) + \Re(\psi_1^* \psi_4) \\
0 &= \Im(\psi_1^* \psi_4) - \Im(\psi_2^* \psi_3) - 2\Im(\psi_3^* \psi_4) - 2\Im(\psi_1^* \psi_2).
\end{align*}\]  

In what follows we obtain the extra necessary and sufficient conditions for each class of DSFs.

As additional conditions on class-(2) Dirac spinor fields, we also have:

\[ \Re(\psi_1^* \psi_4) + \Im(\psi_2^* \psi_3) = 0. \]  

For the class-(3) of spinor fields, the additional condition was obtained in [39]:

\[ \Im(\psi_1^* \psi_4) - \Im(\psi_2^* \psi_3) - 2\Im(\psi_1^* \psi_2) = 0. \]

Class-(1) DSFs must obey all the conditions given by Eqs. (23), (24), and (25). Note that if one relaxes the condition given by Eq. (24) or Eq. (25), DSFs of types-(3) and -(2) are respectively obtained.

Using the decomposition \( \psi_j = \psi_{ja} + i \psi_{jb} \) (where \( \psi_{ja} = \Re(\psi_j) \) and \( \psi_{jb} = \Im(\psi_j) \)) it follows that \( \Re(\psi_i^* \psi_j) = \psi_{ia} \psi_{ja} + \psi_{ib} \psi_{jb} \) and \( \Im(\psi_i^* \psi_j) = \psi_{ia} \psi_{ja} - \psi_{ib} \psi_{jb} \) for \( i, j = 1, \ldots, 4 \). So, in components, the conditions in common for all types of DSFs are

\[\begin{align*}
\psi_{1a} \psi_{3a} + \psi_{1b} \psi_{3b} &= 0, \\
\psi_{2a} \psi_{4a} + \psi_{2b} \psi_{4b} &= 0.
\end{align*}\]

and the additional conditions for each case are summarized in Table I below.

| Class | Additional conditions |
|-------|-----------------------|
| (1)   | \( \psi_{2a} (\psi_{3a} - \psi_{3b}) + \psi_{2b} (\psi_{1a} + \psi_{1b}) = 0 = \psi_{1a} \psi_{4a} - \psi_{1b} \psi_{4b} \) |
| (2)   | \( \psi_{1a} \psi_{3a} - \psi_{3b} \psi_{4a} = 0 = \psi_{2a} \psi_{3a} + \psi_{2b} \psi_{3b} + \psi_{1a} \psi_{4a} + \psi_{1b} \psi_{4b} \) |
| (3)   | \( \psi_{2a} (\psi_{3a} - \psi_{3b}) + \psi_{2b} (\psi_{3a} + \psi_{3b}) = 0 \) and \( (\psi_{1a} \psi_{3b} - \psi_{1b} \psi_{3a}) - (\psi_{2a} \psi_{3b} - \psi_{2b} \psi_{3a}) - 2(\psi_{2a} \psi_{4a} - \psi_{2b} \psi_{4b}) - 2(\psi_{1a} \psi_{2b} - \psi_{1b} \psi_{2a}) = 0 \) |

TABLE I: Additional conditions, in components, for class (1), (2) and (3) Dirac spinor fields.

The explicit mappings obtained above present the same form of the instanton Hopf fibration map \( S^3 \rightarrow S^7 \rightarrow S^4 \) mapping obtained in [37], and can be interpreted as the geometric meaning of the mass dimension-transmuting operator obtained in [31], where we obtained mapping between ELKO and Dirac spinor fields. Some results involving the instanton Hopf fibration can also be seen in this context, e.g., in [38]. In [40] it was shown the reason why the above conditions prevent the Hopf fibration to be described by ELKO spinor fields.

**IV. THE QUADRATIC SPINOR LAGRANGIAN**

It is well known that given a spinor-valued 1-form field \( \Psi \), the quadratic spinor Lagrangian (QSL) is given by

\[ L_{\Psi} = 2 D \bar{\Psi} \wedge \gamma_5 D \Psi = 2 \bar{\Psi} \wedge \Omega_{55} \wedge \Psi + d[(D \bar{\Psi}) \wedge \gamma_5 \Psi + \bar{\Psi} \wedge \gamma_5 D \Psi], \]  

(28)
where \( D \Psi = d \Psi + \omega \wedge \Psi \) and \( D \bar{\Psi} = d \bar{\Psi} + \bar{\Psi} \wedge \omega \). Now, choose the ansatz \( \Psi = \psi \otimes \vartheta \),

\[
\psi = \psi \otimes \vartheta ,
\]

where \( \vartheta \) denotes the orthonormal frame 1-form \( \vartheta = \vartheta^a \otimes \gamma_a = h^a_{\mu} dx^\mu \otimes \gamma_a \) and \( \psi \) is a spinor field. The action of the spinor covariant exterior derivative \( D \), mapping a spinor-valued 1-form field \( \Psi \) into a spinor-valued 2-form field \( D \Psi \) is explicitly given by

\[
D \Psi = \vartheta^a \wedge [\partial^{(s)} \psi \otimes \vartheta + \psi \otimes (\nabla_{e_a} + (e_a \wedge \Theta^c) \wedge e_c) \vartheta],
\]

where the spin-Dirac operator \( \partial^{(s)} \) acting on spinor fields \( \psi \) and the covariant derivative \( \nabla_{e_a} \) acting on Clifford-valued 1-form fields are given respectively by

\[
\partial^{(s)} \psi = \partial_a \psi + \frac{1}{2} \omega_a \psi, \quad \nabla_{e_a} \vartheta = \partial_a \vartheta + \frac{1}{2} [\omega_a, \vartheta],
\]

where \( \omega_a = \omega_{abc}(e_b \otimes \vartheta^c) \).

It is important to remark that the ansatz given by Eq.\( (29) \) arises in different contexts: in \( [8] \) \( \psi \) is a Dirac spinor field used to prove the equivalence between QSL and the Lagrangians describing General Relativity (GR), its teleparallel equivalent GR\(_T\), and the Møller Lagrangian; in \( [10] \) \( \psi \) is an auxiliary Majorana spinor used to prove that gravitation can be described as a SUSY gauge theory; in \( [25, 26] \) \( \psi \) is an anticommuting Majorana spinor described by Grassmann superspace coordinates, which generates the spinor supersymmetric conserved current. The QSL was first proposed in \( [13] \) in the proof of positive energy theorem.

Up to our knowledge, there are no identities like the spinor-curvature identities that yield the term linear in curvature which reduces to the scalar curvature \( [10] \). One of the spinor-curvature identities related which this issue is given by

\[
2 D(\bar{\psi} \xi) \wedge D(\zeta \psi) = 2(-1)\xi \bar{\psi} \xi \wedge \Omega \wedge (\zeta \psi) + d[\bar{\psi} \xi \wedge D(\zeta \psi) - (-1)\bar{\psi} D(\zeta \psi) \wedge \zeta \psi],
\]

where now \( \xi \in \sec \Lambda p(T^* M) \otimes C\ell_{1,3} \) and \( \zeta \in \sec \Lambda (T^* M) \otimes C\ell_{1,3} \). The scalar curvature appears in a natural way in the case where \( \Psi \) in QSL is a spinor-valued 1-form field, as suggested in Eq.\( (29) \).

Substituting the ansatz \( (29) \) in the QSL (Eq.\( (28) \)), it follows

\[
\mathcal{L}_\Psi = \mathcal{L}(\psi, \vartheta, \omega) = 2 D(\bar{\psi} \partial) \gamma_5 \wedge D(\partial \psi) = -\bar{\psi} \psi \Omega_{ab} \wedge \star(\partial^a \wedge \partial^b) + \bar{\psi} \gamma_5 \psi \Omega_{ab} \wedge \partial^a \wedge \partial^b + d[D(\bar{\psi} \partial) \gamma_5 \psi \vartheta + \bar{\psi} \vartheta \gamma_5 D(\partial \psi)],
\]

and it is easy to see that when the spinor field satisfies the normalization conditions

\[
\bar{\psi} \psi = 1, \quad \bar{\psi} \gamma_5 \psi = 0,
\]

the original QSL can be written as

\[
\mathcal{L}_\Psi = -\Omega_{ab} \wedge \star(\partial^a \wedge \partial^b) + d[D(\bar{\psi} \partial) \wedge \gamma_5 \psi \vartheta + \bar{\psi} \vartheta \wedge \gamma_5 D(\partial \psi)]
\]

In this way, the DSF \( \psi \) enters in the QSL only at the boundary and does not appear in the equations of motion. In fact, up to the boundary term, the Lagrangian is given by

\[
\mathcal{L}_\Psi = -\Omega_{ab} \wedge \star(\partial^a \wedge \partial^b),
\]

which is the Einstein-Hilbert Lagrangian. Eq.\( (34) \) shows that the action \( S_\Psi = \int \mathcal{L}_\Psi \) does not depend locally on the Dirac spinor field \( \psi \).

Tung and Nester \( [8] \) asserted that a change on the spinor field will leave the action \( S_\Psi \) unchanged, and then the spinor field has a six-parameter — four complex parameters constrained by Eqs.\( (32) \) — local gauge invariance. The theory also presents a Lorentz gauge freedom related to the transformations of the orthonormal frame field. They prove that the spinor field gauge freedom induces a Lorentz transformation on the orthonormal frame field, and the boundary term has only one physically independent degree of freedom \( [8] \). They also admit a suitable choice fixing one of the two Lorentz gauges by tying the DSF to the orthonormal coframe field together. So, the spinor gauge freedom related to the six parameter DSF \( \psi \) is (2 to 1) equivalent to the Lorentz transformations for the associated orthonormal frame. The choice \( d\psi = 0 \) clearly implies that \( \psi \) is a constant spinor. However, other choices are possible where the spinor field \( \psi \) is not constant anymore — \( d\psi \neq 0 \).
V. QSL AS THE FUNDAMENT OF GRAVITY VIA THE CLASSIFICATION OF ELKO SPINOR FIELDS

Classical spinor fields\(^6\) carrying a \(D(1,2,0) \oplus D(0,1/2)\), or \(D(1/2,0)\), or \(D(0,1/2)\) representation of \(\text{SL}(2, \mathbb{C}) \simeq \text{Spin}^e_{1,3}\) are sections of the vector bundle

\[
P_{\text{Spin}^e_{1,3}}(M) \times_\rho \mathbb{C}^4,
\]

where \(\rho\) stands for the \(D(1,2,0) \oplus D(0,1/2)\) (or \(D(1/2,0)\) or \(D(0,1/2)\)) representation of \(\text{Spin}^e_{1,3}\) in \(\mathbb{C}^4\). Other important spinor fields, like Weyl spinor fields, are obtained by imposing some constraints on the sections of \(P_{\text{Spin}^e_{1,3}}(M) \times_\rho \mathbb{C}^4\). See, e.g., 12 for details. Given a spinor field \(\psi \in \sec P_{\text{Spin}^e_{1,3}}(M) \times_\rho \mathbb{C}^4\) the following sections of \(\Lambda(T^* M) = \oplus_{\nu=0}^4 \Lambda^\nu(T^* M) \hookrightarrow C\ell(M, g)\) [12]

\[
\begin{align*}
\sigma &= \bar{\psi} \psi, \quad J = J_\mu \partial^\mu = \bar{\psi} \gamma_\mu \psi \partial^\mu, \quad S = S_{\mu \nu} \partial^\mu \partial^\nu = \frac{1}{2} \bar{\psi} \gamma_0 i \gamma_\mu \gamma_\nu \psi \partial^\mu \wedge \partial^\nu, \\
K &= \bar{\psi} \gamma_{0123} \gamma_\mu \psi \partial^\mu, \quad \chi = -\bar{\psi} \gamma_{0123} \psi,
\end{align*}
\]

with \(\sigma, \chi \in \sec \Lambda^0(T^* M), J, K \in \sec \Lambda^1(T^* M)\) and \(S \in \sec \Lambda^2(T^* M) \hookrightarrow C\ell(M, g)\). In the formul\ae\ appearing in Eq. 37, the set \(\{\gamma_\mu\}\) can be thought of as being the Dirac matrices, but we prefer not to make reference to any kind of representation, in order to preserve the algebraic character of the theory. When required, it is possible to use any suitable representation. Also, \(\{1_4, \gamma_\mu, \gamma_\mu \gamma_\nu, \gamma_\mu \gamma_\nu \gamma_\rho, \gamma_0 \gamma_1 \gamma_2 \gamma_3\}\) is a basis for \(C\ell(M, g), \mu < \nu < \rho\), and \(1_4 \in \mathbb{C}^4\) is the identity matrix.

Lounesto spinor field classification — representation independent — is given by the following spinor field classes 1, 2, where in the first three classes it is implicit that \(J, K, S \neq 0\):

1. \(\sigma \neq 0, \; \chi \neq 0\).
2. \(\sigma \neq 0, \; \chi = 0\).
3. \(\sigma = 0, \; \chi \neq 0\).
4. \(\sigma = 0, \; \chi = 0, \; K \neq 0, \; S \neq 0\).
5. \(\sigma = 0, \; \chi = 0, \; K = 0, \; S \neq 0\).
6. \(\sigma = 0, \; \chi = 0, \; K \neq 0, \; S = 0\).

The current density \(J\) is always non-zero. Classes (1), (2), and (3) are called Dirac spinor fields for spin-1/2 particles, and classes (4), (5), and (6) are called, respectively, flag-dipole, flagpole and Weyl spinor fields. Majorana and ELKO spinor fields 4, 30, 31 are a particular case of a class-(5) spinor field. It is worthwhile to point out a peculiar feature of spinor fields of class (4), (5), and (6): although \(J\) is always non-zero, we have \(J^2 = -K^2 = 0\). Although the choices given by Eq. (33) is restricted to class-(2) DSFs, we can explore other choices for values of \(\sigma = \bar{\psi} \psi\) and \(\chi = \bar{\psi} \gamma_5 \psi\), and also investigate the QSL from the point of view of classes (1) and (3) spinor fields.

Now, if instead of class-(2) we consider the class-(3) DSF, in which case the spinor field satisfies the normalization conditions

\[
\sigma = \bar{\psi} \psi = 0, \quad \chi = \bar{\psi} \gamma_5 \psi = 1,
\]

then the original QSL can be written as

\[
\mathcal{L}_\psi = -\Omega_{ab} \wedge (\partial^a \wedge \partial^b) + d[D(\bar{\psi} \partial) \wedge \gamma_5 \psi \partial + \bar{\psi} \partial \wedge \gamma_5 D(\partial \psi)].
\]

The class-(1) DSF \(\psi\) enters the QSL only at the boundary, and consequently it does not appear in the equations of motion. Up to the boundary term, therefore, the Lagrangian is given by the Einstein-Palatini Lagrangian

\[
\mathcal{L}_\psi = -\Omega_{ab} \wedge (\partial^a \wedge \partial^b).
\]

\(^6\) As is well known, quantum spinor fields are operator valued distributions. It is not necessary to introduce quantum fields in order to know the algebraic classification of ELKO spinor fields.
It is immediate to see that, by considering a class-(1) DSF, characterized by the conditions \( \sigma \neq 0 \) and \( \chi \neq 0 \), the most general Holst action, given by

\[
S^\Psi_{\psi} = \bar{\psi} \psi \int \Omega_{ab} \wedge \ast (\vartheta^a \wedge \vartheta^b) + \bar{\psi} \gamma_5 \psi \int \Omega_{ab} \wedge (\vartheta^a \wedge \vartheta^b),
\]

follows naturally \[45\]. In fact, this action comes from the QSL associated with a class-(1) DSF map from Dirac to ELKO spinor fields shown in Sec.(III) is a one-to-one correspondence to the instanton Hopf fibration \[4, 5, 6, 7, 39\]. It has been shown in \[4\] that ELKO is a type-(5), flagpole spinor field. In addition, type-(1) DSFs — of the current theories for gravity, from the ELKO spinor fields viewpoint, based also in the previous results in most general Holst action, given by Eqs.(23, 24, 25) are exactly in correspondence to the instanton Hopf fibration equation, in the Clifford algebra arena, as shown in \[37\]. It would suggest the reason why the ELKO spinor fields satisfy the Klein-Gordon equation, instead of the Dirac equation. Physically, as ELKO presents mass dimension one \[30, 31, 32\], while any other type of arena, as shown in \[37\]. It would suggest the reason why the ELKO spinor fields satisfy the Klein-Gordon equation, instead of the Dirac equation. Physically, as ELKO presents mass dimension one \[30, 31, 32\], while any other type of spin-1/2 spinor field present mass dimension 3/2, the conditions obtained in Sec.(III) might introduce the geometric explanation for this physical open problem.

VI. CONCLUDING REMARKS

The main purpose of this paper is to investigate and discuss the QSL as the fundamental Lagrangian for some of the current theories for gravity, from the ELKO spinor fields viewpoint, based also in the previous results in 4, 5, 6, 7, 39. It has been shown in 4 that ELKO is a type-(5), flagpole spinor field. In addition, type-(1) DSFs — under Lounesto spinor field classification — present seven degrees of freedom, and it can be shown that the mapping from Dirac to ELKO spinor fields shown in Sec.(III) is a one-to-one correspondence to the instanton Hopf fibration map \( S^3 \rightarrow S^7 \rightarrow S^4 \) [37]. The conditions that the Dirac spinor fields must satisfy in order to be led to ELKO, explicitly given by Eqs. (23, 24, 25) are exactly in correspondence to the instanton Hopf fibration equation, in the Clifford algebra arena, as shown in [37]. It would suggest the reason why the ELKO spinor fields satisfy the Klein-Gordon equation, instead of the Dirac equation. Physically, as ELKO presents mass dimension one \[30, 31, 32\], while any other type of spin-1/2 spinor field present mass dimension 3/2, the conditions obtained in Sec.(III) might introduce the geometric explanation for this physical open problem.

QSL makes use of a general auxiliary spin-3/2 field that can be expressed as the tensor product between an auxiliary spinor field \( \psi \) and a Clifford-valued 1-form \( \theta \). This auxiliary spinor field \( \psi \) was first introduced by Witten as a convenient tool in the proof of the positive-energy theorem of Einstein gravity 43. When the QSL is required to yield Einstein-Hilbert, Einstein-Palatini, and Holst actions, it follows naturally that the auxiliary spinor-valued 1-form field composing the QSL takes the form of an ELKO, when we take into account the mapping in Eqs.(23, 24, 25). Any other choice of spinor field leads, up to a boundary term, to a null QSL 45. In the light of Sec.(III), the spinor-valued 1-form field of the QSL must necessarily be constituted by a tensor product between an ELKO spinor field and a Clifford algebra-valued 1-form.

Einstein-Hilbert, Einstein-Palatini, and Holst actions correspond respectively to the mapping between ELKO spinor fields and DSFs of class-(2) (given by Eqs. (23, 24)), class-(3) (given by Eqs. (25, 26)), and class-(1) (given by Eq. (24, 25, 23)) DSFs. And, as ELKO spinor fields can be obtained from the DSFs, via a mapping explicitly constructed that does not preserve spinor field classes, under Lounesto classification 39, we conclude that — in particular — the Einstein-Hilbert, Einstein-Palatini, and Holst actions can be derived from the QSL, as a fundamental Lagrangian for supergravity, only using ELKO spinor fields7.

Although the choice \( d\psi = 0 \), and the normalization conditions \( \sigma = \bar{\psi}\psi = 1 \) and \( \chi = \bar{\psi} \gamma_5 \psi = 0 \) — corresponding to a class-(2) Dirac spinor field — gives the best option to prove the equivalence between the QSL and the Lagrangians associated with general relativity and teleparallel gravity, they are restrictive if we are interested in more general analyses. Also, the ELKO spinor field mapping — Eqs. (23, 24, 25) — into classes (2) and (3) of DSFs can be chosen to give the complete QSL Holst action, each one corresponding respectively to one of its pieces \( \bar{\psi} \psi \int \Omega_{ab} \wedge \ast (\vartheta^a \wedge \vartheta^b) \) or \( \bar{\psi} \gamma_5 \psi \int \Omega_{ab} \wedge (\vartheta^a \wedge \vartheta^b) \). Furthermore, the ELKO spinor field mapping (Eqs. (23)) into class-(1) Dirac spinor field gives alone the complete Holst action, since in this case \( \sigma = \bar{\psi}\psi \neq 0 \) and \( \chi = \bar{\psi} \gamma_5 \psi \neq 0 \).

VII. ACKNOWLEDGMENT

The authors are very grateful to Prof. Dharamvir Ahluwalia for pointing out elucidating and enlightening viewpoints. Roldão da Rocha thanks to Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP) (2008/06483-5).

7 In [40] the super-Poincare algebra was obtained in the context of a 3-dimensional Euclidean space.
and J. M. Hoff da Silva thanks to CAPES-Brazil for financial support.

[1] P. Lourenço, Clifford Algebras, Relativity and Quantum Mechanics, in P. Letelier and W. A. Rodrigues, Jr. (eds.), Gravitation: the Spacetime Structure, Proc. of the 8th Latin American Symposium on Relativity and Gravitation, Águas de Lindóia, Brazil, 25-30 July 1993, World-Scientific, London 1993.

[2] P. Lourenço, Clifford Algebras and Spinors, 2nd ed., p. 152-173, Cambridge Univ. Press, Cambridge 2002.

[3] H. T. Noh and Relativistic, Antiparallel Spinors and Q-Cartan Motions in Phase Space, Found. Phys. 16 (1986) 708-709.

[4] R. da Rocha and W. A. Rodrigues, Jr., Where are ELKO Spinor Fields in Lourenço Spinor Field Classification?, Mod. Phys. Lett. A21 (2006) 65-74 [arXiv:math-ph/0506075v3].

[5] R. da Rocha and W. A. Rodrigues, Jr., The Einstein-Hilbert Lagrangian density in a 2-dimensional spacetime is an exact differential, Mod. Phys. Lett. A21 (2006) 1519-1527 [arXiv:math-ph/0512168v7].

[6] W. A. Rodrigues, Jr., R. da Rocha, and J. Vaz, Jr., Hidden Consequence of Active Local Lorentz Invariance, Int. J. Geom. Meth. Mod. Phys. 2 (2005) 365-357 [arXiv:math-ph/0501064v6].

[7] R. da Rocha and W. A. Rodrigues, Jr., The Dirac-Hestenes Equation for Spherical Symmetric Potentials in the Spherical and Cartesian Gauges, Int. J. Mod. Phys. A21 (2006) 4071-4082 [arXiv:math-ph/0601018v2].

[8] J. M. Nester and R. S. Tung, A Quadratic Spinor Lagrangian for General Relativity, Gen. Rel. Grav. 27 (1995) 115-119 [arXiv:gr-qc/9407004].

[9] R. S. Tung and T. Jacobson, Spinor One-forms as Gravitational Potentials, Class. Quantum Grav. 12 (1995) L51-55 [arXiv:gr-qc/9502037].

[10] J. M. Nester, R. S. Tung, and V. V. Zhytnikov, Some Spinor-Curvature Identities, Class. Quantum Grav. 11 (1994) 983-987 [arXiv:gr-qc/9403026].

[11] R. S. Tung, Gravitation as a supersymmetric gauge theory, Phys. Lett. A 264 (2000) 341-345 [arXiv:gr-qc/9904008].

[12] R. A. Mosna and W. A. Rodrigues, Jr., The bundles of algebraic and Dirac-Hestenes spinor fields, J. Math. Phys. 45 (2004) 2945-2988 [arXiv:math-ph/0212035].

[13] A. Dimakis and F. Müller-Hoissen, Clifford calculus with applications to classical field theories, Class. Quantum Grav. 8 (1991) 2093-2132.

[14] A. Dimakis and F. Müller-Hoissen, On a gauge condition for orthonormal three-frames, Phys. Lett. 142 A (1989) 73-74.

[15] F. Estabrook, Lagrangians for Ricci-flat geometries, Class. Quantum Grav. 8 (1991) L151-154.

[16] H. B. Lawson, Jr. and M. L. Michelson, Spin Geometry, Princeton University Press, Princeton, 1989.

[17] Y. Choquet-Bruhat, C. DeWitt-Morette, and M. Dillard-Bleick, Identities, World Spinors, and Breaking of Dilation Invariance, Phys. Rev. B25 (1985) 1-171 [arXiv:gr-qc/9402012].

[18] O. Chandia and J. Zanelli, Topological invariants, instantons and chiral anomaly on spaces with torsion, Phys. Rev. D55 (1997) 7580-7585 [arXiv:hep-th/9702025].

[19] R. Troncoso and J. Zanelli, Gauge Supergravities for all Odd Dimensions Int. J. Theor. Phys. 38 (1999) 1181-1206 [arXiv:hep-th/9807029].

[20] Y. N. Obukhov, E. W. Mielke, J. Budzcies, and F. W. Hehl, On the chiral anomaly in non-Riemannian spacetimes, Found. Phys. 27 (1997) 1221-1236 [arXiv:gr-qc/9702011].

[21] F. W. Hehl, J. D. McCrea, E. W. Mielke, and Y. Ne’eman, Metric-Affine Gauge Theory of Gravity: Field Equations, Noether Identities, World Spinors, and Breaking of Dilution Invariance, Phys. Rev. B25 (1985) 1-171 [arXiv:gr-qc/9402012].

[22] O. Chandia and J. Zanelli, Supersymmetric Particle in a Spacetime with Torsion and the Index Theorem, Phys. Rev. D58 (1998) 045014 [arXiv:hep-th/9803034].

[23] I. Bars and W. A. MacDowell, Spinor theory of general relativity without elementary gravitons, Phys. Lett. B71 (1977) 111-114.

[24] I. Bars and W. A. MacDowell, A spin-3/2 theory of gravitation, Gen. Rel. Grav. 19 (1979) 205-209.

[25] P. Chinea, A Clifford algebra approach to general relativity, Gen. Rel. Grav. 21 (1986) 21.

[26] J. P. Crawford, On the Algebra of Dirac Bispinor Densities: Factorization and Inversion Theorems, J. Math. Phys. 26 (1985) 1429-1441.

[27] P. R. Holland, Minimal Ideals and Clifford Algebras in the Phase Space Representation of spin-1/2 Fields, p. 273-283 in Chisholm J S R and Common A R (eds.), Proceedings of the Workshop on Clifford Algebras and their Applications in Mathematical Physics (Canterbury 1985), Reidel, Dordrecht 1986.

[28] D. V. Ahluwalia-Khalilova and D. Grumiller, Spin Half Fermions with Mass Dimension One: Theory, Phenomenology, and Dark Matter, J. Cosmol. Astropart. Phys. JCAP (2005) 012 [arXiv:hep-th/0412080].

[29] D. V. Ahluwalia-Khalilova and D. Grumiller, Dark Matter: A spin one half fermion field with mass dimension two, Phys. Rev. D 72 (2005) 067701 [arXiv:hep-th/0410192].

[30] D. V. Ahluwalia-Khalilova, Dark Matter, and its darkness, Int. J. Mod. Phys. D 15 (2006) 2267-2278 [arXiv:hep-th/0603545v3].

[31] D. V. Ahluwalia-Khalilova, Theory of neutral particles: McLennan-Case construct for neutrino, its generalization, and a new wave equation, Int. J. Mod. Phys. A11 (1996) 1855-1874 [arXiv:hep-th/9409154v2].

[32] D. V. Ahluwalia, Cheng-Yang Lee, D. Schritt, T. F. Watson, Dark matter and dark gauge fields, [arXiv:0712.4190v2 [hep-ph]]. Local fermionic dark matter with mass dimension one, [arXiv:0804.1854v3 [hep-th]].

[33] C. G. Boehmer, The Einstein-Elko system – Can dark matter drive inflation?, Annalen Phys. 16 (2007) 325-341 [arXiv:gr-qc/0701087v1]; The Einstein-Cartan-Elko system, Annalen Phys. 16 (2007) 38-44 [arXiv:gr-qc/0607088v1]; Dark spinor inflation – theory primer and dynamics, Phys. Rev. D 77 (2008) 123533 [arXiv:0804.0616v1 [astro-ph]].

[34] R. da Rocha, The super-Poincare algebra via pure spinors and the interaction principle in 3D Euclidean space, Braz. J. Phys. 35 4B (2005) 1138-1139.
[37] J. Vaz, Jr., Construction of Monopoles and Instantons by using Spinors and the Inversion Theorem, in *Clifford Algebras and their Applications in Mathematical Physics*, V. Dietrich et al. (eds.), pp. 401-421, Kluwer, Dordrecht 1998.

[38] R. da Rocha and J. Vaz, Jr., Clifford algebra-parametrized octonions and generalizations, *J. Algebra* 301 (2006) 459-473 [arXiv:math-ph/0603053v1].

[39] R. da Rocha and J. M. Hoff da Silva, From Dirac spinor fields to eigenspinoren des ladungskonjugationsoperators, *J. Math. Phys.* 48 (2007) 123517 [arXiv:0711.1103v1 [math-ph]].

[40] R. da Rocha and J. M. Hoff da Silva, ELKO, flagpole and flag-dipole spinor fields, and the instanton Hopf fibration, to appear in *Adv. Appl. Clifford Alg.* (2009) [arXiv:0811.2717v1 [math-ph]].

[41] C. H. Chou, R. S. Tung, and H. L. Yu, Origin of the Immirzi Parameter, *Phys. Rev.* D 72 (2005) 064016 [arXiv:gr-qc/0509028].

[42] S. Holst, Barbero's Hamiltonian derived from a generalized Hilbert-Palatini action, *Phys. Rev.* D 53 (1996) 5966-5969 [arXiv:gr-qc/9511026].

[43] E. Witten, A new proof of the positive energy theorem, *Commun. Math. Phys.* 80 (1981) 381-402.

[44] W. Bardeen and B. Zumino, Consistent and Covariant Anomalies in Gauge and Gravitational Theories, *Nucl. Phys.* B244 (1984) 421.

[45] R. da Rocha R and J. G. Pereira, The quadratic spinor Lagrangian, axial torsion current, and generalizations, *Int. J. Mod. Phys.* D 16 (2007) 1653-1667 [arXiv:gr-qc/0703076v1].