Hard congestion limit of the dissipative Aw–Rascle system

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Abstract

In this study, we analyse the famous Aw–Rascle system in which the difference between the actual and the desired velocities (the offset function) is a gradient of a singular function of the density. This leads to a dissipation in the momentum equation which vanishes when the density is zero. The resulting system of PDEs can be used to model traffic or suspension flows in one dimension with the maximal packing constraint taken into account. After proving the global existence of smooth solutions, we study the so-called ‘hard congestion limit’, and show the convergence of a subsequence of solutions towards a weak solution of a hybrid free-congested system. This is also illustrated numerically using a numerical scheme proposed for the model studied. In the context of suspension flows, this limit can be seen as the transition from a suspension regime, driven by lubrication forces, towards a granular regime, driven by the contacts between the grains.

Keywords: Aw–Rascle system, suspension flows, maximal packing, weak solutions  
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1. Introduction

The purpose of this work is to study a singular limit $\varepsilon \to 0$ for the following generalization of the Aw and Rascle [3] and Zhang [36] system

\begin{align}
\partial_t \rho + \partial_x (\rho u) &= 0, \\
\partial_t (\rho w) + \partial_x (\rho u w) &= 0,
\end{align}

on one-dimensional periodic domain $\Omega = T$. The unknowns of the system are the density $\rho$ and the velocity of motion $u$. The quantity $w$ denotes the desired velocity of motion and it differs from the actual velocity $u$ by the offset function. This function describes the cost of moving in certain direction and it depends on the congestion of the flow. In our case the offset function is equal to the gradient of $p(\rho)$, more precisely:

\begin{equation}
w = u + \partial_x p(\rho),
\end{equation}

where

\begin{equation}
p(\rho) = \frac{\varepsilon F(\rho)}{(1 - \rho)^\beta}, \quad \text{with } F(s) \to 0, \quad \beta > 1.
\end{equation}

This singular function plays formally the role of a barrier by preventing the density to exceed the maximal fixed threshold $\bar{\rho} \equiv 1$. The motivation to study this model and its asymptotic limit $\varepsilon \to 0$ comes mainly from two areas of applications:

**The Aw–Rascle model for traffic [3].** The system \((1)\) with scalar offset function, i.e. with $w = u + \rho^\gamma$ for $\gamma > 1$, usually known as the Aw–Rascle system (or sometimes ARZ model) has been derived from the Follow the Leader (FTL) microscopic model of one lane vehicular traffic in \([2]\). The drawback of that model is that the offset function $\rho^\gamma$, does not preserve the maximal density constraint, i.e. solutions satisfy the maximal density constraint $\rho^0 \leq \bar{\rho}$ initially but evolve in finite time to a state which violates this constraint. Moreover, the velocity offset should be very small unless the density $\rho$ is very close to the maximal value, $\bar{\rho} = 1$. Indeed, the drivers do not reduce their speed significantly if the traffic is not congested enough. To incorporate these features the authors of \([6]\) proposed to work with the asymptotic limit ($\varepsilon \to 0$) of \((1)\) with $w = u + p(\rho)$, and a singular scalar offset function $p$ given by \((3)\). The singular Aw–Rascle system and its asymptotic limit $\varepsilon \to 0$ has been studied numerically in \([8]\), and derived from a FTL approximation in \([7]\). To be able to use this model in the multi-dimensional setting, where velocity and offset function should have the same physical dimension, a possible way would be to take for the offset a gradient rather than a scalar function (see the recent paper \([5]\) for a proposition and analysis of a multi-d extension of the classical Aw–Rascle model). The use of a gradient can be interpreted as ability of the driver to relax their velocity to an average of the speed of the front and the rear vehicles, weighted according to the local density. So, unlike in the classical Aw–Rascle model, both front and rear interactions would have to be incorporated at the level of the particle model. This seems to be a reasonable assumption for interactions between vehicles that can change lanes and overtake each others.

**The lubrication model.** Equations \((1)\)–\((3)\) appear also in modeling of suspension flows, i.e. flows of grains suspended in a viscous fluid. To explain this context better, note that system
(1) with \( w_\varepsilon \) given by (2) can be rewritten (formally) as the pressureless compressible Navier–Stokes equations with density dependent viscosity coefficient

\[
\begin{align}
\partial_t \rho_\varepsilon + \partial_x (\rho_\varepsilon u_\varepsilon) &= 0, \\
\partial_t (\rho_\varepsilon u_\varepsilon) + \partial_x (\rho_\varepsilon u_\varepsilon^2) - \partial_x (\lambda_\varepsilon (\rho_\varepsilon) \partial_x u_\varepsilon) &= 0, \\
\end{align}
\]

where

\[
\lambda_\varepsilon (\rho_\varepsilon) = \rho_\varepsilon^2 p'_\varepsilon (\rho). 
\]

In system (4) the singular diffusion coefficient \( \lambda_\varepsilon (\rho_\varepsilon) \) represents the repulsive lubrication forces and \( \varepsilon \) is linked to the viscosity of the interstitial fluid. This system has been derived from a microscopic approximation in [23]. The limit \( \varepsilon \to 0 \) models the transition between the suspension regime, dominated by the lubrication forces, towards the granular regime dictated by the contacts between the solid grains.

Formally, performing the limit \( \varepsilon \to 0 \) in (1) (or equivalently in (4)), we expect to get the solution \((\rho, u)\) of the compressible pressureless Euler system, at least when \( \rho < 1 \). In the region where \( \rho = 1 \) we expect that the singularity of the offset function (3) (equivalently the singularity of the viscosity coefficient (5)) will prevail giving rise to additional forcing term. The limiting equations then read

\[
\begin{align}
\partial_t \rho + \partial_x (\rho u) &= 0, \\
\partial_t (\rho u) + \partial_x (\rho u^2) + \partial_x P &= 0, \\
0 \leq \rho \leq 1, \quad (1 - \rho) \pi &= 0, \quad \pi \geq 0, \\
\end{align}
\]

where \( \pi \) is the additional unknown obtained as a limit of certain singular function of \( \rho_\varepsilon \), that will be specified later on. This limiting system has been derived formally before in the papers of Lefebvre–Lepot and Maury [23] and then the Lagrangian solutions based on an explicit formula using the monotone rearrangement associated to the density were constructed by Perrin and Westdickenberg [32]. As explained in this latter work, system (6) is related to the constrained Euler equations

\[
\begin{align}
\partial_t \rho + \partial_x (\rho u) &= 0, \\
\partial_t (\rho u) + \partial_x (\rho u^2) + \partial_x P &= 0, \\
\rho \leq 1, \quad (1 - \rho) P &= 0, \quad P \geq 0, \\
\end{align}
\]

studied for instance by Berthelin in [4] or Preux and Maury [26], by splitting the momentum equation (6b) as follows:

\[
\begin{align}
\partial_t (\rho u) + \partial_x (\rho u^2) + \partial_x P &= 0, \\
\partial_t \pi + u \partial_x \pi &= P. \\
\end{align}
\]

Consequently, solutions to system (6) are particular solutions to (7). In [4] the constrained Euler equations are obtained through the sticky blocks approximation, while in [1, 9, 18], the constrained Euler system is approximated by the compressible Euler equations with singular pressure \( P_\varepsilon = P_\varepsilon (\rho_\varepsilon) \).

Similar asymptotic limit passage \( \varepsilon \to 0 \) was analysed in the multi-dimensional setting by Bresch, Necasova and Perrin [14] in the case of heterogeneous fluids flows described by compressible Birkmann equations with singular pressure and bulk viscosity coefficient. The full
compressible Navier–Stokes system with exponentially singular viscosity coefficients and pressure was considered by Perrin in [29, 30]. The asymptotic limit when \( \epsilon \to 0 \) in the singular pressure term that leads to the two-phase compressible/incompressible Navier–Stokes equations was considered even earlier in the context of crowd dynamics, see Bresch, Perrin and Zatorska [15], Perrin and Zatorska [33], Degond, Minakowski and Zatorska [20], Degond et al [19]. Moreover, an interested reader can also consult Lions and Masmoudi [25] and Vauchelet and Zatorska [35] for different approximation of the two-phase system. For an overview of results and discussion of models described by the free/congested two-phase flows we refer to [31].

Our paper contains two main results: the existence of strong solutions to (1) at \( \epsilon \) fixed, and the convergence as \( \epsilon \to 0 \) of the solutions towards a solution of the limit system (6). Let us be a little bit more precise about the framework and the difficulties associated to system (1).

To study (1) at \( \epsilon \) fixed, we take advantage of the reformulation (4) as (pressureless) Navier–Stokes equations with a density dependent viscosity. It is now well-known that, in addition to the classical energy estimate which provides a control of \( \sqrt{\lambda_{\epsilon}} \partial_{\epsilon} u_{\epsilon} \), the BD entropy (BD for Bresch and Desjardins [13]) yields a control of a gradient of some density function. We show in this paper, that this estimate is precisely the key ingredient to ensure the maximal density constraint, namely we will show that \( \| \rho_{\epsilon} \|_{L^\infty} \leq C_\epsilon \) for some constant \( C_\epsilon < 1 \) tending to 1 as \( \epsilon \to 0 \).

As \( \epsilon \to 0 \), the main issue which is common to the analysis of Navier–Stokes equations with (degenerate close to vacuum) density dependent viscosities, is the fact that, \textit{a priori}, we do not have any uniform control in \( L^1 \) of \( \partial_{\epsilon} u_{\epsilon} \). Therefore, the identification of the limit of the nonlinear convective term \( \rho_{\epsilon} u_{\epsilon}^2 \) is not direct. An important difference and difficulty in comparison with [30] or other studies on Navier–Stokes equations with degenerate viscosities, is that the viscosity vanishes, namely the viscosity \( \lambda_{\epsilon}(\rho) \) goes to 0 as \( \epsilon \to 0 \) for any \( \rho < 1 \). As a result, the control of the gradient of \( \rho_{\epsilon} \) provided by the BD entropy is not uniform with respect to \( \epsilon \). This prevents us from using the Mellet–Vasseur type of estimates [28] to pass to the limit in the convective term. An alternative approach was provided by Boudin in his work [12] devoted to the the vanishing viscosity limit for pressureless gases. He studied system (4) in which the singular viscosity term \( \partial_{\epsilon}(\lambda_{\epsilon} \partial_{\epsilon} u_{\epsilon}) \) is replaced by the non-singular term \( \epsilon \partial_{\epsilon}^2 u_{\epsilon} \).

The key ingredient of [12] is the concept of \textit{duality solutions} for the limit pressureless gas equations. In this framework, introduced by Bouchut and James in [10, 11], it is particularly important to ensure a \textit{one-sided Lipschitz condition}, or the Oleinik entropy condition, on the velocity field. It is related to the \textit{compressive property} of the dynamics which turns out to be useful also in other ‘compressive systems’ such as aggregation equations (see for instance the works of James and Vauchelet [21, 22]). Note that Berthelin in [4] derived the estimate \( \partial_{\epsilon} u \leq 1/t \) for solutions of the constrained Euler equations obtained through the sticky blocks approximation. Building upon the recent developments of Constantin \textit{et al} [17] (see also [16]) around the regular solutions for the Navier–Stokes equations, we derive the \( \epsilon \)-uniform one-sided Lipschitz condition \( \partial_{\epsilon} u_{\epsilon} \leq C \) on the approximate solution. This estimate requires no vacuum at the level of fixed \( \epsilon \). The \( \epsilon \)-dependent lower bound on the density \( \rho_{\epsilon} \) is derived by imposing a specific behavior of \( \epsilon \)-dependent viscosity close to vacuum (see (8)-(9) below).

The outline of this paper is to first show that for \( \epsilon \) fixed, \((\rho_{\epsilon}, u_{\epsilon})\) is a regular solution to (4). We prove the existence of regular solutions to this system in section 3 following the approach of Constantin \textit{et al} [17]. Then, we derive estimates uniform with respect to \( \epsilon \), including the one-sided Lipschitz condition, in section 4. We justify the limit passage \( \epsilon \to 0 \) in section 5 and show that the limit is solution to the constrained Euler equations. Finally, we propose in section 6 a numerical scheme to illustrate the behavior of the solutions to (4) as \( \epsilon \to 0 \).
the convenience of the reader we included in the Appendix all details of technical estimates of higher regularity of solutions for \( \varepsilon \) fixed.

2. Main results

Our first main result concerns the existence of strong solutions to system (1) for specific constitutive laws. With a slight abuse of notation, we re-define the functions \( p_\varepsilon, w_\varepsilon, \) and \( \lambda_\varepsilon(\rho_\varepsilon) \), from the introduction.

We consider system (1) with \( w_\varepsilon \) given by

\[
w_\varepsilon = u_\varepsilon + \partial_t p_\varepsilon(\rho_\varepsilon) + \partial_t \varphi_\varepsilon(\rho_\varepsilon),
\]

where

\[
p_\varepsilon(\rho_\varepsilon) = \varepsilon \frac{\rho_\varepsilon^2}{(1 - \rho_\varepsilon)^2}, \quad \varphi_\varepsilon(\rho_\varepsilon) = \frac{\varepsilon}{\alpha - 1} \rho_\varepsilon^{\alpha-1}, \quad \gamma > 0, \quad \beta > 1, \quad \alpha \in \left(0, \frac{1}{2}\right).
\]

It means that in (3) we consider \( F(\rho) = \rho^\gamma + (\alpha - 1)^{-1}(1 - \rho)^{\beta} \rho^{\alpha-1} \) instead of \( F(\rho) = \rho^\gamma \), for example. Note however, that due to additional approximation \( \varphi_\varepsilon \) we have \( \lim_{\varepsilon \to 0} F(s) = -\infty \) and not \( \lim_{\varepsilon \to 0} F(s) = 0 \).

The approximation of (4) can be written in the following form

\[
\begin{align*}
\partial_t \rho_\varepsilon + \partial_x (\rho_\varepsilon u_\varepsilon) &= 0, \\
\partial_t (\rho_\varepsilon u_\varepsilon) + \partial_x \left( \rho_\varepsilon u_\varepsilon^2 \right) - \partial_x \left( \lambda_\varepsilon(\rho_\varepsilon) \partial_x u_\varepsilon \right) &= 0,
\end{align*}
\]

with \( \lambda_\varepsilon \) re-defined as

\[
\lambda_\varepsilon(\rho) = \rho^2 \lambda_\varepsilon(\rho) + \rho^2 \varphi_\varepsilon(\rho) = \rho^2 \lambda_\varepsilon(\rho) + \varepsilon \rho^\alpha.
\]

Given the values of the parameters, \( \alpha, \beta, \gamma \), we observe that the behavior of \( \lambda_\varepsilon \) is dictated by \( \rho^2 \varphi_\varepsilon(\rho) \) close to vacuum, and it is dictated by the singularity of \( \rho^2 \lambda_\varepsilon(\rho) \) close to the maximal density constraint \( \rho \equiv 1 \).

We supplement this system with the following set of initial conditions

\[
\rho_\varepsilon|_{t=0} = \rho_\varepsilon^0, \quad u_\varepsilon|_{t=0} = u_\varepsilon^0.
\]

The existence of unique global smooth solution to the approximate problem (10) at \( \varepsilon > 0 \) fixed is stated in the theorem below.

**Theorem 2.1.** Let \( \varepsilon > 0 \) be fixed, and let \( p_\varepsilon, \varphi_\varepsilon \) be given by (9). Assume that the initial data (11) satisfy \( \rho_\varepsilon^0, u_\varepsilon^0 \in H^2(\Omega), \) with \( 0 < \rho_\varepsilon^0 < 1 \).

Then, for all \( T > 0 \), there exists a unique global solution \( (\rho_\varepsilon, u_\varepsilon) \) to system (10) such that \( 0 < \rho_\varepsilon(t,x) < 1 \) for all \( t \in [0,T], x \in \Omega, \) and

\[
\rho_\varepsilon \in C\left([0,T];H^3(\Omega)\right), \quad u_\varepsilon \in C\left([0,T];L^2\left(0,T;H^4(\Omega)\right)\right) \cap L^2(0,T;H^4(\Omega)).
\]

We further show that solutions \( \rho_\varepsilon, u_\varepsilon \) from the class (12) satisfy some uniform in \( \varepsilon \) estimates that allow us to justify the asymptotic limit \( \varepsilon \to 0 \) in the weak sense. To this end, we rewrite system (10) as follows

\[
\begin{align*}
\partial_t \rho_\varepsilon + \partial_x (\rho_\varepsilon u_\varepsilon) &= 0, \\
\partial_t (\rho_\varepsilon u_\varepsilon) + \partial_x \left( \rho_\varepsilon u_\varepsilon^2 \right) - \partial_x \left( \lambda_\varepsilon(\rho_\varepsilon) \partial_x u_\varepsilon \right) &= 0,
\end{align*}
\]
\[
\begin{align*}
\frac{\partial}{\partial t} (\rho \rho_x + \partial_x (\rho_x u_x)) &= 0, \\
\frac{\partial}{\partial t} (\rho \rho_x u_x + \partial_x \pi_x (\rho_x)) + \partial_x ((\rho_x u_x + \partial_x \pi_x (\rho_x)) u_x) &= 0,
\end{align*}
\]

(13a) (13b)

where we denoted
\[
\pi_x^\varepsilon (\rho_x) = \rho_x p_x^\varepsilon (\rho_x) + \rho_x \varphi_x^\varepsilon (\rho_x).
\]

(14)

We show that for \( \varepsilon \to 0 \) the solutions of (13) converge to an entropy weak solution of (6) with the unknowns \( \rho, u, \pi \) in the sense specified in the following result.

**Theorem 2.2.** Let assumptions from theorem 2.1 be satisfied, and moreover let
\[
0 < \rho_x^0 (x) \leq 1 - C_0 \varepsilon \frac{1}{\varepsilon} \quad \forall x \in T,
\]
\[
\max (\text{ess sup}) (\lambda_x (\rho_x^0) \partial_x u_x^0), 0) \leq C_1 \varepsilon \frac{1}{\varepsilon},
\]
\[
\| \sqrt{\rho_x^0} u_x^0 \|_{L^2} \leq C,
\]
\[
\| \partial_x \pi_x (\rho_x^0) \|_{L^2} + \| \sqrt{\rho_x^0} \partial_x \varphi (\rho_x^0) \|_{L^2} \leq C,
\]
\[
0 < M^\varepsilon \leq M^0 = \int_T \rho_x^0 \, dx \leq M^0 < |T|,
\]

for some \( C_0, C_1, C, M^0, M^0 > 0 \) independent of \( \varepsilon \). Then:

1. The solution \( (\rho_x, u_x) \) given by theorem 2.1 satisfies the following uniform estimates
\[
C \varepsilon \frac{1}{\varepsilon} \leq \rho_x (t, x) \leq 1 - C \varepsilon \frac{1}{\varepsilon} \quad \forall (t, x) \in [0, T] \times T,
\]
\[
\| \pi_x \|_{L^{\infty} H} \leq C,
\]

(20) (21)

and the one-sided Lipschitz condition
\[
\partial_x u_x (t, x) \leq C \quad \forall (t, x) \in [0, T] \times T,
\]

(22)

for some \( C > 0 \), independent of \( \varepsilon \). Moreover, the following inequality holds for any \( S \in C^1 (\mathbb{R}) \) convex:
\[
\partial_t (\rho_x S (u_x)) + \partial_x (\rho_x u_x S (u_x)) - \partial_x (\lambda_x (\rho_x) S (u_x)) \leq 0, \quad \forall (t, x) \in (0, T) \times T.
\]

2. Let in addition
\[
\rho_x^0 \to \rho^0 \text{ weakly in } L^2 (T),
\]
\[
\rho_x^0 u_x^0 \to \rho^0 u^0 \text{ weakly in } L^2 (T),
\]
\[
\partial_x \pi_x (\rho_x^0) \to \partial_x \pi^0 \text{ weakly in } L^2 (T).
\]

(23)

Then there exists a subsequence \( (\rho_{x_n}, u_{x_n}, \pi_x (\rho_{x_n})) \) of solutions to (13)–(14) with initial datum \( (\rho_{x_n}^0, u_{x_n}^0, \pi_x (\rho_{x_n}^0)) \), which converges to \( (\rho, u, \pi) \) a weak solution of (6) with initial datum \( (\rho^0, u^0, \pi^0) \).
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More precisely we have $0 \leq \rho \leq 1$ a.e. and the following convergences hold:

$$
\rho_\varepsilon \rightarrow \rho \text{ weakly-* in } L^\infty ((0,T) \times \mathbb{T}),
$$

$$
\pi_\varepsilon (\rho_\varepsilon) \rightarrow \pi \text{ weakly-* in } L^\infty (0,T; H^1 (\mathbb{T})),
$$

$$
u_\varepsilon \rightarrow \nu \text{ weakly-* in } L^\infty ((0,T) \times \mathbb{T}).
$$

Moreover, for the limiting system, the following (entropy) conditions hold:

- **one-sided Lipschitz condition**

  $$
  \partial_t u \leq C \text{ in } \mathcal{D}',
  $$
  
- **entropy inequality:**

  $$
  \partial_t (\rho S (u)) + \partial_x (\rho u S (u)) - \partial_x \Lambda_S \leq 0 \text{ in } \mathcal{D}',
  $$

  for any convex $S \in C^1 (\mathbb{R})$, and $\Lambda_S \in \mathcal{M}((0,T) \times \mathbb{T})$ satisfying $|\Lambda_S| \leq \text{Lip}_S |\Lambda|$ where $-\Lambda = -\lambda_\varepsilon (\rho) \partial_x u \in \mathcal{M}^+ ((0,T) \times \mathbb{T})$.

**Remark 2.3.** Let us first comment on the choice of initial datum:

- The assumptions for the initial velocity include the upper bound for $(\lambda_\varepsilon (\rho_\varepsilon^0) \partial_x u_\varepsilon^0)_+$ to deduce the one-sided Lipschitz condition discussed in the introduction. This control amounts to take small values of $(\partial_x u_\varepsilon^0)_+$, except in the low density regions.

- The initial condition (19) implies that the limit system (for $\varepsilon = 0$) cannot be fully congested. This condition is required to control the singular potential $\pi_\varepsilon (\rho_\varepsilon)$ in section 4.1. Analogous constraint has been imposed to study the asymptotic limit of the compressible Navier–Stokes equations with singular pressure, see [33, 34].

**Remark 2.4.** The choice of the approximate function $\varphi_\varepsilon$ provides uniform lower bound of the density. It was shown in [28] that, for viscosity coefficient proportional to $\rho^\alpha$ with $\alpha \in [0,1/2]$, the weak solutions to compressible Navier–Stokes equations have density bounded away from zero by a constant. We use this property at the level of $\varepsilon$ being fixed in order to derive the one-sided Lipschitz condition, but we also show that in the limit passage $\varepsilon \rightarrow 0$, $\varphi_\varepsilon$ converges to zero strongly. We also remark that the lower bound for the density could be obtained without the use of $\varphi_\varepsilon$, by assuming more regularity on the initial data uniformly w.r.t. $\varepsilon$, see [27].

**Remark 2.5.** The main difficulty in studying the $\varepsilon \rightarrow 0$ limit passage is to justify convergence of the nonlinear terms. In particular, to pass to the limit in the convective term $\rho_\varepsilon u_\varepsilon$ one needs compactness of the velocity sequence w.r.t. space-variable. For compressible Navier–Stokes equations with constant viscosity coefficient this sort of information is deduced directly from the a-priori estimates. When the viscosity coefficient is degenerate, one can compensate lack of the a-priori estimate by higher regularity of the density via the so-called Bresch-Desjardins estimate. Here, the compactness w.r.t. space follows directly from the one-sided Lipschitz condition, which is possible to deduce because the system is pressureless.

**Remark 2.6 (more general singular functions $p_\varepsilon$).** The specific form of the function $p_\varepsilon$ (which blows up close to 1 like a power law) is used in the paper to exhibit the small scales associated to the singular limit $\varepsilon \rightarrow 0$ (see in particular lemma 3.3). Nevertheless, we expect similar results for more general (monotone) hard-sphere potentials. All the estimates will then depend on the specific balance between the parameter $\varepsilon$ and the type of the singularity close to 1 encoded in the function $p_\varepsilon$. 

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3. Proof of theorem 2.1

The first step of the proof is to construct local in time unique regular solution to (10). Thanks to the presence of the approximation term $\varphi_\varepsilon$ this can be done following the iterative scheme, described for example in appendix B of Constantin et al [17]. The extension of this solution to the global in time solution requires some uniform (with respect to time) estimates that are presented below.

3.1. Basic energy estimates

In this section we assume that $\rho_\varepsilon, u_\varepsilon$ are regular solutions to (10) in the time interval $[0, T]$ in the class (12), and that $\rho_\varepsilon$ is non-negative. For such solutions we first obtain straight from the continuity equation (10a) that

$$\|\rho_\varepsilon\|_{L^1_t} = \|\rho_0\|_{L^1},$$

for all $t \in [0, T]$. Multiplying the momentum equation (10b) by $u_\varepsilon$ and integrating by parts we obtain the classical energy estimate.

Lemma 3.1. For a regular solutions of system (10), we have

$$\|\sqrt{\rho_\varepsilon} u_\varepsilon\|_{L^\infty_t L^2}^2 + \|\sqrt{\lambda_\varepsilon (\rho_\varepsilon)} \partial_x u_\varepsilon\|_{L^2_t L^2}^2 \leq C E_{0,\varepsilon},$$

with

$$E_{0,\varepsilon} := \|\sqrt{\rho_0^0} u_0^0\|_{L^2}^2.$$ 

The next energy estimate is an analogue of Bresch–Desjardins entropy for the compressible Navier–Stokes equations. We first introduce the quantity

$$H'_\varepsilon (\rho_\varepsilon) = p_\varepsilon (\rho_\varepsilon) + \varphi_\varepsilon (\rho_\varepsilon).$$

Lemma 3.2. For a regular solution of system (10), we have

$$\|\sqrt{\rho_\varepsilon} w_\varepsilon\|_{L^\infty_t L^2}^2 + \|H_\varepsilon (\rho_\varepsilon)\|_{L^\infty_t L^2}^2 + \|\sqrt{\rho_\varepsilon^0} \partial_x (p_\varepsilon (\rho_\varepsilon) + \varphi_\varepsilon (\rho_\varepsilon))\|_{L^2_t L^2}^2 \leq C E_{1,\varepsilon} (1 + T),$$

with

$$E_{1,\varepsilon} := \|\sqrt{\rho_0^0} w_0^0\|_{L^2}^2 + \|H_\varepsilon (\rho_0^0)\|_{L^2}^2.$$ 

Proof. Recall that at the level of regular solutions, the system (10) can be reformulated as (1) with $w_\varepsilon$ given by (8). Therefore, multiplying the equation (1b) by $w_\varepsilon$ and integrating by parts we easily obtain

$$\|\sqrt{\rho_\varepsilon^0} w_\varepsilon\|_{L^\infty_t L^2} \leq \|\sqrt{\rho_0^0} w_0^0\|_{L^2}.$$ 

Using again formula (8) to substitute for $u_\varepsilon$ in (10a) we obtain a porous medium equation for $\rho_\varepsilon$, namely:

$$\partial_t \rho_\varepsilon + \partial_x (\rho_\varepsilon w_\varepsilon) - \partial_x (\rho_\varepsilon \partial_x (p_\varepsilon (\rho_\varepsilon) + \varphi_\varepsilon (\rho_\varepsilon))) = 0.$$
Multiplying this equation by $H'_\varepsilon(\rho_\varepsilon)$ and integrating over space and time, we get

$$\sup_{t \in (0,T)} \int_T H_\varepsilon(\rho_\varepsilon) \, dx + \|\sqrt{\rho_\varepsilon} \partial_x (H'_\varepsilon(\rho_\varepsilon))\|_{L^2_x}^2 \leq \int_T H_\varepsilon(\rho_\varepsilon^0) \, dx + C \|\sqrt{\rho_\varepsilon^0} w_\varepsilon\|_{L^2_x}^2,$$

$$\leq \int_T H_\varepsilon(\rho_\varepsilon^0) \, dx + CT \|\sqrt{\rho_\varepsilon^0} w_\varepsilon\|_{L^2_x}^2.$$

Note that $H_\varepsilon$ consists of two components: one positive but singular $H'_\varepsilon(\rho) = \int_0^\varepsilon \varphi_\varepsilon(r) \, dr$, and a non-singular negative one $H''_\varepsilon(\rho) = \int_0^\varepsilon \varphi_\varepsilon(r) \, dr$ since $\alpha < 1$ (see (9)). However we can absorb this negative contribution in $H'_\varepsilon(\rho)$ as follows. Let us first observe that

$$H'_\varepsilon(\rho) \sim \frac{\varepsilon}{(1-\rho)^{\frac{\alpha}{\alpha-1}},}$$

and therefore, for any $C_1, C_2 > 0$ (independent of $\varepsilon$) there exist $C'_1, C'_2$ (also independent of $\varepsilon$) such that

$$H'_\varepsilon(\rho_\varepsilon) \{\rho_\varepsilon \geq 1-C'_1\varepsilon^{-\frac{1}{\alpha-1}}\} \geq C_1 \rho_\varepsilon^\beta \{\rho_\varepsilon \geq 1-C'_1\varepsilon^{-\frac{1}{\alpha-1}}\};$$

$$\int_T H'_\varepsilon(\rho_\varepsilon) \{\rho_\varepsilon \leq 1-C'_2\varepsilon^{-\frac{1}{\alpha-1}} \leq \rho_\varepsilon \leq 1-C'_2\varepsilon^{-\frac{1}{\alpha-1}}\} \, dx \geq C_2 \varepsilon.$$

Hence, splitting the integral of $H''_\varepsilon$ into two parts, we get

$$\int_T |H''_\varepsilon(\rho_\varepsilon)| \, dx = \int_T \frac{\varepsilon}{(1-\rho_\varepsilon)} \rho_\varepsilon^\beta \{\rho_\varepsilon \leq 1-C'_1\varepsilon^{-\frac{1}{\alpha-1}}\} \, dx + \int_T \frac{\varepsilon}{(1-\rho_\varepsilon)} \rho_\varepsilon^\beta \{\rho_\varepsilon \geq 1-C'_1\varepsilon^{-\frac{1}{\alpha-1}}\} \, dx$$

$$\leq \frac{\varepsilon}{(1-\rho_\varepsilon)} |T| + \int_T |H'_\varepsilon(\rho_\varepsilon)| \{\rho_\varepsilon \geq 1-C'_1\varepsilon^{-\frac{1}{\alpha-1}}\} \, dx$$

$$\leq \int_T |H''_\varepsilon(\rho_\varepsilon)| \{\rho_\varepsilon \leq 1-C'_2\varepsilon^{-\frac{1}{\alpha-1}} \leq \rho_\varepsilon \leq 1-C'_2\varepsilon^{-\frac{1}{\alpha-1}}\} \, dx + \int_T |H'_\varepsilon(\rho_\varepsilon)| \{\rho_\varepsilon \geq 1-C'_2\varepsilon^{-\frac{1}{\alpha-1}}\} \, dx$$

for sufficiently large $C_1', C_2' > 0$, independent of $\varepsilon$.

This gives the result of the lemma. \(\square\)

### 3.2. The upper and lower bounds on the density

The purpose of this section is to prove that $\rho_\varepsilon$ is uniformly (in time) bounded from above by $\overline{\rho}_\varepsilon$ and from below by $\underline{\rho}_\varepsilon$. This will be done in the two lemmas below.

**Lemma 3.3 (upper bound on the density).** Let $T > 0$, and let $(\rho_\varepsilon, u_\varepsilon)$ be a solution to (10) in the class (12) satisfying the energy estimates (27) and (29). Assume moreover that initially $E_{0,\varepsilon}$ and $E_{1,\varepsilon}$, defined in lemmas 3.1 and 3.2, are bounded uniformly with respect to $\varepsilon$. Then there exists a positive constant $C$ independent of $\varepsilon$ and $T$ such that

$$\rho_\varepsilon(t,x) \leq 1 - C \left(\frac{\varepsilon}{1 + \sqrt{T}}\right)^{\frac{1}{\alpha-1}} := \overline{\rho}_\varepsilon \quad \forall \, t \in [0,T], \, x \in \mathbb{T}. \quad (31)$$
**Proof.** First of all, the $L_{t}^\infty L_{x}^{1}$ bound (29) on $H_{x}^{s}$ provides the upper bound

$$\|\rho_{\varepsilon}\|_{L_{t}^{\infty}L_{x}^{1}} \leq 1.$$ 

One can actually derive a more precise upper bound on the density thanks to the previous estimates. First the Nash inequality provides

$$\|H_{x}^{s}(\rho_{\varepsilon}(t, \cdot))\|_{L_{x}^{2}} \leq C \|H_{x}^{s}(\rho_{\varepsilon}(t, \cdot))\|_{L_{x}^{2}}^{2/3} \|\partial_{s}H_{x}^{s}(\rho_{\varepsilon}(t, \cdot))\|_{L_{x}^{2}}^{1/3} + C \|H_{x}^{s}(\rho_{\varepsilon}(t, \cdot))\|_{L_{x}^{1}},$$

and we have by Sobolev inequality that

$$\|H_{x}^{s}(\rho_{\varepsilon}(t, \cdot))\|_{L_{x}^{2}} \leq \|H_{x}^{s}(\rho_{\varepsilon}(t, \cdot))\|_{L_{x}^{2}}^{1/2} \|\partial_{s}H_{x}^{s}(\rho_{\varepsilon}(t, \cdot))\|_{L_{x}^{2}}^{1/2} + \|H_{x}^{s}(\rho_{\varepsilon}(t, \cdot))\|_{L_{x}^{2}}\left( \|\partial_{s}H_{x}^{s}(\rho_{\varepsilon}(t, \cdot))\|_{L_{x}^{2}}^{2/3} + \|L_{x}^{s}(\rho_{\varepsilon}(t, \cdot))\|_{L_{x}^{1}} \right).$$

Now, we observe that due to $p_{\varepsilon}(\rho_{\varepsilon}), \varphi_{\varepsilon}(\rho_{\varepsilon}) > 0$, the estimate (29) provides the bound

$$\|\sqrt{p_{\varepsilon}}p_{\varepsilon}(\rho_{\varepsilon})\partial_{x}\rho_{\varepsilon}\|_{L_{t}^{\infty}L_{x}^{2}} \leq C(E_{1,\varepsilon}(1 + T) + E_{0,\varepsilon}),$$

and on the other hand

$$\|\sqrt{p_{\varepsilon}}p_{\varepsilon}(\rho_{\varepsilon})\partial_{x}\rho_{\varepsilon}\|_{L_{t}^{\infty}L_{x}^{2}}^{2} = \left| \sqrt{p_{\varepsilon}}p_{\varepsilon}(\rho_{\varepsilon})\partial_{x}\rho_{\varepsilon} \right|_{L_{t}^{\infty}L_{x}^{2}}^{2}.$$ 

Since

$$p_{\varepsilon}(\rho_{\varepsilon}) = \varepsilon \frac{\rho_{\varepsilon}^{2} - 1}{(1 - \rho_{\varepsilon})^{\beta+1}} \left[ \gamma (1 - \rho_{\varepsilon}) + \beta \rho_{\varepsilon} \right],$$

we deduce that

$$\sqrt{p_{\varepsilon}}p_{\varepsilon}(\rho_{\varepsilon}) = \frac{\left[ \gamma (1 - \rho_{\varepsilon}) + \beta \rho_{\varepsilon} \right] \rho_{\varepsilon}^{1/2}}{1 - \rho_{\varepsilon}}$$

$$= \left( \gamma + \beta \frac{\rho_{\varepsilon}}{1 - \rho_{\varepsilon}} \right) \rho_{\varepsilon}^{1/2} \\
\geq \gamma \|\rho_{\varepsilon}\|_{L_{t}^{\infty}L_{x}^{2}}^{1/2}.$$ 

Therefore

$$\|\partial_{s}H_{x}^{s}(\rho_{\varepsilon})\|_{L_{t}^{\infty}L_{x}^{2}} \leq \gamma^{-1} \|\rho_{\varepsilon}\|_{L_{t}^{\infty}L_{x}^{2}}^{1/2} \|\sqrt{p_{\varepsilon}}p_{\varepsilon}(\rho_{\varepsilon})\partial_{x}\rho_{\varepsilon}\|_{L_{t}^{\infty}L_{x}^{2}}$$

$$\leq C \|\rho_{\varepsilon}\|_{L_{t}^{\infty}L_{x}^{2}}^{1/2} \sqrt{(E_{1,\varepsilon}(1 + T) + E_{0,\varepsilon})}$$

$$\leq C \sqrt{(E_{1,\varepsilon}(1 + T) + E_{0,\varepsilon}),}$$

and so, coming back to (32):

$$\|H_{x}^{s}(\rho_{\varepsilon})\|_{L_{t}^{\infty}L_{x}^{2}} \leq C \sqrt{(E_{1,\varepsilon}(1 + T) + E_{0,\varepsilon}).}$$

(34)
Finally, it is easy to show that this latter bound yields
\[
\|\rho_\varepsilon\|_{L^\infty_t L^1_x} \leq 1 - C \left( \frac{\varepsilon}{1 + \sqrt{T}} \right)^{\frac{1}{\alpha}},
\]
for some constant \( C \) independent of \( \varepsilon \) and \( T \), provided that \( E_{1,\varepsilon} \) and \( E_{0,\varepsilon} \) are bounded uniformly w.r.t. \( \varepsilon \).

\[\tag{35}\]

**Remark 3.4.** This upper bound ensures that \( \rho_\varepsilon \) never reaches 1 at \( \varepsilon > 0 \) fixed in finite time, and so \( p_\varepsilon (\rho_\varepsilon) \) remains bounded by a constant which depends a-priori on \( \varepsilon \). Indeed, we have
\[
p_\varepsilon (\rho_\varepsilon (t,x)) \leq \frac{\varepsilon}{(1 - \|\rho_\varepsilon\|_{\infty})^\beta} \leq C \left( 1 + \sqrt{T} \right)^{\frac{\beta}{1 - \frac{1}{2}}} \varepsilon^{-\frac{1}{1 - \frac{1}{2}}} \quad \forall \ t \in [0, T], \ x \in \mathbb{T}.
\]

Note that the same type of estimate holds for the potential \( \pi_\varepsilon (\rho_\varepsilon) \) defined in (14).

In the next step we progress with the lower bound on the density. This is the only place where the artificial approximation term \( \varphi_\varepsilon \) matters.

**Lemma 3.5 (lower bound on the density).** Let the assumptions of Lemma 3.3 be satisfied. Then for a constant \( C > 0 \) independent of \( \varepsilon \) and \( T \) we have:
\[
\left\| \frac{1}{\rho_\varepsilon} \right\|_{L^\infty_t L^1_x} \leq C \varepsilon^{-\frac{1}{2 - \alpha}} (1 + T)^{\frac{1}{2 - \alpha}} =: \frac{1}{\bar{p}_\varepsilon}.
\]

\[\tag{36}\]

**Proof.** The estimate (29) ensures that \( \sqrt{\rho_\varepsilon} \partial_x \varphi_\varepsilon (\rho_\varepsilon) \) is controlled uniformly w.r.t. \( \varepsilon \) in \( L^\infty_t L^2_x \), and so, there exists \( C > 0 \) independent of \( \varepsilon \) and \( T \) such that:
\[
\| \partial_x \left( \varepsilon \rho_\varepsilon^{\alpha - \frac{1}{2}} \right) \|_{L^2_t L^\infty_x} \leq C (E_{1,\varepsilon} (1 + T) + E_{0,\varepsilon}).
\]

\[\tag{37}\]

On the other hand, we have by conservation of mass
\[
\int_\mathbb{T} \rho_\varepsilon (t,x) \, dx = \int_\mathbb{T} \rho_\varepsilon^0 (x) \, dx = M^\varepsilon_0,
\]
and by hypothesis (19) we ensure that
\[
0 < M^\varepsilon_0 \leq M^0_0 < |\mathbb{T}|,
\]
for some \( M^0_0 \) independent of \( \varepsilon \). Therefore, for any time \( t \), there exists some \( \bar{x}(t) = \bar{x}(t, \varepsilon) \in \mathbb{T} \) such that
\[
\rho_\varepsilon (t, \bar{x}(t)) \geq \frac{M^0_0}{|\mathbb{T}|}.
\]
For all \( t \in (0, T) \) and \( x \in \mathbb{T} \) we can write
\[
\varepsilon (\rho_\varepsilon (t,x))^{\alpha - 1/2} - \varepsilon (\rho_\varepsilon (t, \bar{x}(t)))^{\alpha - 1/2} = \int_{\bar{x}(t)}^x \varepsilon \partial_x (\rho_\varepsilon (t,x))^{\alpha - 1/2},
\]

\[\tag{38}\]
so that
\[
\varepsilon (\rho_\varepsilon(t,x))^{\alpha-1/2} \leq \varepsilon (\rho_\varepsilon(t,\bar{x}(t)))^{\alpha-1/2} + |x - \bar{x}(t)|^{1/2} \| \partial_x \left( \varepsilon \rho_\varepsilon^{\alpha-1/2} \right) \| L^2_x.
\]

Hence, \(\varepsilon (\rho_\varepsilon(t,x))^{\alpha-1/2} \leq C \sqrt{1 + T}\) and finally
\[
\rho_\varepsilon^{-1}(t,x) \leq C \varepsilon^{-1/2 - \alpha} (1 + T)^{1/2} \quad \forall t > 0, x \in T.
\]

**Remark 3.6.** For the sake of simplicity we sometimes estimate \(\rho_\varepsilon\) as follows
\[
\rho_\varepsilon^{-1} = C \varepsilon^{-1/2 - \alpha} (1 + T)^{1/2} \leq C(T) \varepsilon^{-2}.
\]

**3.3. Further regularity estimates**

**3.3.1. Control of the singular diffusion.** In the next step we provide the estimates of
\[
V_\varepsilon := \lambda_\varepsilon (\rho_\varepsilon) \partial_x u_\varepsilon,
\]
following the reasoning of Constantin *et al* [17] (\(V_\varepsilon\) corresponds to what is called *active potential* in [17]).

**Lemma 3.7.** The variable \(V_\varepsilon\) satisfies
\[
\partial_t V_\varepsilon + \left( u_\varepsilon + \frac{\lambda_\varepsilon (\rho_\varepsilon)}{\rho_\varepsilon^2} \partial_x \rho_\varepsilon \right) \partial_x V_\varepsilon - \frac{\lambda_\varepsilon (\rho_\varepsilon)}{\rho_\varepsilon} \partial_x^2 V_\varepsilon = - \frac{(\lambda_\varepsilon' (\rho_\varepsilon) \rho_\varepsilon + \lambda_\varepsilon (\rho_\varepsilon))}{(\lambda_\varepsilon (\rho_\varepsilon))^2} V_\varepsilon^2. \tag{39}
\]

**Proof.** Dividing (40) by \(\rho_\varepsilon > 0\) and using the continuity equation we get
\[
\partial_t u_\varepsilon + u_\varepsilon \partial_x u_\varepsilon - \frac{1}{\rho_\varepsilon} \partial_x V_\varepsilon = 0.
\]

Next, taking the space derivative we obtain
\[
\partial_t \partial_x u_\varepsilon + \partial_x (u_\varepsilon \partial_x u_\varepsilon) - \partial_x \left( \frac{1}{\rho_\varepsilon} \partial_x V_\varepsilon \right) = 0.
\]

On the other hand, multiplying (40a) by \(\lambda_\varepsilon'(\rho_\varepsilon)\), we get
\[
\partial_t \lambda_\varepsilon (\rho_\varepsilon) + \partial_x \lambda_\varepsilon (\rho_\varepsilon) u_\varepsilon + \lambda_\varepsilon'(\rho_\varepsilon) \rho_\varepsilon \partial_x u_\varepsilon = 0.
\]
Hence
\[ \partial_t V = \lambda (\rho \varepsilon) \partial_t \partial_u \varepsilon + \partial_t u \varepsilon \partial_k \lambda \rho \varepsilon \]
\[ = - \lambda (\rho \varepsilon) \partial_t u \varepsilon \partial_k \lambda \rho \varepsilon + \lambda (\rho \varepsilon) \partial_t \left( \frac{1}{\rho \varepsilon} \partial_k V \right) - \partial_t \lambda (\rho \varepsilon) u \varepsilon \partial_k \lambda \rho \varepsilon - \lambda' (\rho \varepsilon) \rho \varepsilon (\partial_k u \varepsilon)^2 \]
\[ = - \partial_t (u \varepsilon V) + \partial_t \left( \frac{\lambda (\rho \varepsilon)}{\rho \varepsilon} \partial_k V \right) - \lambda' (\rho \varepsilon) \rho \varepsilon \left( \frac{V}{\lambda (\rho \varepsilon)} \right)^2 - \partial_t \lambda (\rho \varepsilon) \partial_k V \]
\[ = - u \varepsilon \partial_k V - \frac{1}{\rho \varepsilon} V^2 \partial_t + \partial_t \left( \frac{\lambda (\rho \varepsilon)}{\rho \varepsilon} \partial_k V \right) - \lambda' (\rho \varepsilon) \rho \varepsilon \left( \frac{V}{\lambda (\rho \varepsilon)} \right)^2 - \partial_t \lambda (\rho \varepsilon) \partial_k V, \]
(40)
from which (39) follows.

**Lemma 3.8.** We have
\[ ||V||_{L^2_t L^2}^2 + \varepsilon ||\partial_t V||_{L^2_t L^2}^2 \leq C ||V(0)||_{L^2}^4 \text{ exp} \left[ T \left( \varepsilon^{-3} ||R||_{L^\infty}^4 + \varepsilon^{-1} \right) \right] \]
where
\[ ||R||_{L^\infty} := ||\partial_t \lambda \rho \varepsilon ||_{L^\infty} \leq \frac{C}{\sqrt{T}} \left( 1 + \sqrt{T} \right). \]

**Proof.** We multiply (39) by $V$ and integrate w.r.t. space to get
\[ \frac{d}{dt} \int_{\mathbb{T}^2} \frac{V^2}{2} \, dx + \int_{\mathbb{T}} \frac{\lambda (\rho \varepsilon)}{\rho \varepsilon} (\partial_k V)^2 \, dx \]
\[ = - \int_{\mathbb{T}} \partial_t \left( \frac{\lambda (\rho \varepsilon)}{\rho \varepsilon} \right) V^2 \partial_k V \, dx - \int_{\mathbb{T}} \lambda (\rho \varepsilon) \partial_k \rho \varepsilon \partial_k V \, dx - \int_{\mathbb{T}} u \varepsilon \partial_k V \, dx \]
\[ = - \int_{\mathbb{T}} \partial_t \lambda (\rho \varepsilon) \rho \varepsilon \partial_k \rho \varepsilon \partial_k V \, dx - \int_{\mathbb{T}} u \varepsilon \partial_k V \, dx \]
\[ = - \int_{\mathbb{T}} \partial_t \lambda (\rho \varepsilon) \rho \varepsilon \partial_k \rho \varepsilon \partial_k V \, dx - \int_{\mathbb{T}} u \varepsilon \partial_k V \, dx \]
\[ = I_1 + I_2 + I_3. \]

Observe that
\[ \frac{\lambda (\rho \varepsilon)}{\rho \varepsilon} = \rho_p p' (\rho \varepsilon) + \varepsilon \rho_p^{-1} \geq \varepsilon. \]
(42)

We will use this estimate to control each of the terms $I_1, I_2, I_3$ separately below.

**Control of $I_1$.** We have
\[ |I_1| \leq \int_{\mathbb{T}} \frac{\partial_t \lambda (\rho \varepsilon)}{\rho \varepsilon} \partial_k V \, dx \leq \||R||_{L^2} ||\partial_t V||_{L^2} \||V||_{L^\infty}. \]
Let us estimate $\|R\|_{L^\infty_2 L^2}$ To this purpose we split $\lambda_\varepsilon(\rho_\varepsilon)$ to the singular and the non-singular parts denoted by

$$\lambda_\varepsilon^s = \rho_\varepsilon^2 p'_s(\rho_\varepsilon), \quad \lambda_\varepsilon^n = \varepsilon \rho_\varepsilon. $$

For the singular part we have

$$
\left\| \frac{\partial_\varepsilon \lambda_\varepsilon^s}{\rho_\varepsilon} \right\|_{L^\infty_2 L^2} \leq \left\| \frac{2 \rho_\varepsilon p'_s(\rho_\varepsilon) + \rho_\varepsilon^3 p''_s(\rho_\varepsilon)}{\rho_\varepsilon} \right\|_{L^\infty_2 L^2} \leq 2 \sqrt{\rho_\varepsilon} + \rho_\varepsilon^{3/2} p''_s(\rho_\varepsilon) \left\| \frac{p'_s(\rho_\varepsilon)}{\rho_\varepsilon} \right\|_{L^\infty} \leq C \left( \frac{1}{\sqrt{\rho_\varepsilon}} + \frac{1}{\sqrt{T}(1 - \rho_\varepsilon)} \right) \| \sqrt{\rho_\varepsilon} p'_s(\rho_\varepsilon) \partial_\varepsilon \rho_\varepsilon \|_{L^\infty_2 L^2},
$$

and for the non-singular part

$$
\left\| \frac{\partial_\varepsilon \lambda_\varepsilon^n}{\rho_\varepsilon} \right\|_{L^\infty_2 L^2} = \| \varepsilon \rho_\varepsilon^{\alpha - 3/2} \partial_\varepsilon \rho_\varepsilon \|_{L^\infty_2 L^2} \leq C \rho_\varepsilon^{-1/2} \| \varepsilon \rho_\varepsilon^{\alpha - 3/2} \partial_\varepsilon \rho_\varepsilon \|_{L^\infty_2 L^2}.
$$

Therefore, using estimate (29), we get

$$
\| R \|_{L^\infty_2 L^2} \leq C \frac{1 + \sqrt{T}}{\sqrt{T}(1 - \rho_\varepsilon)}. \tag{43}
$$

Coming back to the control of $I_1$, we have:

$$
|I_1| \leq \| V_\varepsilon \|_{L^\infty_2} \| \partial_\varepsilon V_\varepsilon \|_{L^2} \| R \|_{L^2} \leq C \| \partial_\varepsilon V_\varepsilon \|_{L^2} \left( \| \partial_\varepsilon V_\varepsilon \|^{1/2}_{L^2} \| V_\varepsilon \|^{1/2}_{L^2} + \| V_\varepsilon \|_{L^2} \right) \| R \|_{L^2} \leq C \varepsilon \| \partial_\varepsilon V_\varepsilon \|_{L^2} + C \left( \varepsilon^{-3} \| V_\varepsilon \|_{L^2}^2 \| R \|_{L^2}^2 + \varepsilon^{-1} \| V_\varepsilon \|_{L^2} \| R \|_{L^2}^2 \right). \tag{44}
$$

Control of $I_2$. Using the Hölder and the Young inequalities we obtain

$$
|I_2| \leq \int_T \left| u_\varepsilon \| \partial_\varepsilon V_\varepsilon \|_{L^2} \right| \| x_\varepsilon \| \left| x_\varepsilon \| \partial_\varepsilon V_\varepsilon \|_{L^2} \| u_\varepsilon \|_{L^\infty_2} \leq \| \partial_\varepsilon V_\varepsilon \|_{L^2} \| V_\varepsilon \|_{L^2} \| u_\varepsilon \|_{L^\infty_2} \leq C \varepsilon \| \partial_\varepsilon V_\varepsilon \|_{L^2}^2 + C \varepsilon^{-1} \| V_\varepsilon \|_{L^2}^2 \| u_\varepsilon \|_{L^2}^2,
$$

with

$$
\| u_\varepsilon \|_{L^2}^2 \leq \| u_\varepsilon \|_{L^2}^2 + \left( \frac{1}{\lambda_\varepsilon(\rho_\varepsilon)} \right)^{1/2} L_\infty \| V_\varepsilon \|_{L^2} \| u_\varepsilon \|_{L^2} \leq \rho_\varepsilon^{-1/2} \| \sqrt{\rho_\varepsilon} u_\varepsilon \|_{L^2} \| e^{-2} \rho_\varepsilon^{-2\alpha} \| V_\varepsilon \|_{L^2} \leq \rho_\varepsilon^{-1/2} \| \sqrt{\rho_\varepsilon} u_\varepsilon \|_{L^2} \| e^{-2} \rho_\varepsilon^{-2\alpha} \| V_\varepsilon \|_{L^2},
$$

and

$$
\| u_\varepsilon \|_{L^\infty_2} \| u_\varepsilon \|_{L^2} \| V_\varepsilon \|_{L^2} \| u_\varepsilon \|_{L^\infty_2} \leq \rho_\varepsilon^{-1/2} \| \sqrt{\rho_\varepsilon} u_\varepsilon \|_{L^2} \| e^{-2} \rho_\varepsilon^{-2\alpha} \| V_\varepsilon \|_{L^2},
$$

and
hence
\[ |I_2| \leq C \| \partial_t V_\varepsilon \|_{L^2_t}^2 + C \varepsilon_{-1} \rho_{-1/2} \| V_\varepsilon \|_{L^2_t}^2 \sqrt{\rho_{\varepsilon} u_{\varepsilon}} \|_{L^2_t}^2 + C \varepsilon_{-3} \rho_{-\alpha} \| V_\varepsilon \|_{L^2_t}^2. \]  
(45)

Control of $I_3$. We apply the Nash inequality and the Hölder inequality to get
\[
|I_3| \leq \int \left| \frac{\lambda'_\varepsilon (\rho_\varepsilon \varepsilon + \lambda_{\varepsilon} (\rho_\varepsilon))}{(\lambda_{\varepsilon} (\rho_\varepsilon))^2} \right| \| V_\varepsilon \|_3^3 \, dx
\leq \left\| \frac{\lambda'_\varepsilon (\rho_\varepsilon \varepsilon + \lambda_{\varepsilon} (\rho_\varepsilon))}{(\lambda_{\varepsilon} (\rho_\varepsilon))^2} \right\|_{L^\infty} \left( \| \partial_t V_\varepsilon \|_{L^2_t}^{1/2} \| V_\varepsilon \|_{L^2_t}^{5/2} + \| V_\varepsilon \|_{L^3_t}^3 \right)
\leq C \frac{\rho_{-3}^2}{(1 - \rho_{-3})^{3/2}} \left( \| \partial_t V_\varepsilon \|_{L^2_t}^{1/2} \| V_\varepsilon \|_{L^2_t}^{5/2} + \| V_\varepsilon \|_{L^3_t}^3 \right)
\leq C \frac{\rho_{-3}^2}{(1 - \rho_{-3})^{3/2}} \| V_\varepsilon \|_{L^2_t}^3 + C \frac{\rho_{-3}^{2\alpha} ((1 - \rho_{-3})^{\beta + 2})^{4/3} \| V_\varepsilon \|_{L^2_t}^{5/2 \times 4/3} + \rho_{-3}^{2\alpha} ((1 - \rho_{-3})^{\beta + 2})^{4/3} \| V_\varepsilon \|_{L^2_t}^3}{(1 - \rho_{-3})^{3/2}}.
\]

Putting everything together, we get
\[
\frac{d}{dt} \| V_\varepsilon \|_{L^2_t}^2 + C \| \partial_t V_\varepsilon \|_{L^2_t}^2 \leq C \left( e^{-3} \| V_\varepsilon \|_{L^2_t}^2 \| R \|_{L^2_t}^2 + e^{-1} \| V_\varepsilon \|_{L^2_t}^2 \| R \|_{L^2_t}^2 + e^{-1} \rho_{-1/2} \| V_\varepsilon \|_{L^2_t}^2 \sqrt{\rho_{\varepsilon} u_{\varepsilon}} \|_{L^2_t}^2 \right)
\leq C \left( e^{-3} \| R \|_{L^2_t}^4 + e^{-1} \rho_{-1/2} \sqrt{\rho_{\varepsilon} u_{\varepsilon}} \|_{L^2_t}^2 \right) + \frac{\rho_{-3}^{2\alpha} ((1 - \rho_{-3})^{\beta + 2})^{4/3} \| V_\varepsilon \|_{L^2_t}^3}{(1 - \rho_{-3})^{3/2}}.
\]

and (41) follows by Gronwall’s inequality. \( \square \)

From the above result we infer the estimates on $\partial_t u_\varepsilon$ (which are not uniform with respect to $\varepsilon$).

**Lemma 3.9.** We have:

(i) $\partial_t u_\varepsilon$ is bounded in $L^2_t L^2_x$ with the estimate
\[
\| \partial_t u_\varepsilon \|_{L^2_t L^2_x} \leq \frac{1}{\sqrt{\lambda_{\varepsilon} (\rho_\varepsilon)}} \| \lambda_{\varepsilon} (\rho_\varepsilon) \partial_t u_\varepsilon \|_{L^2_t L^2_x} \leq C \varepsilon^{-1/2} \rho_{-\alpha/2} \lambda_{\varepsilon}^{1/2} \cdot \]  
(46)

(ii) $\partial_t u_\varepsilon$ is bounded in $L^\infty_t L^2_x$ with the estimate
\[
\| \partial_t u_\varepsilon \|_{L^\infty_t L^2_x} \leq \frac{1}{\lambda_{\varepsilon} (\rho_\varepsilon)} \| V_\varepsilon \|_{L^\infty_t L^2_x}.
\]  
(47)
Part (We begin with a formal derivation of higher regularity estimates. For $l \geq 1$ regularity estimates, we assume that the functions $V(x)$ are sufficiently smooth. We introduce the commutator notation

\[ [D,f] = D(f(x)) - f(x)D(x) \]

which is any differential operator and $f, g$ are sufficiently smooth functions. Using this notation we rewrite the above equation as:

\[ \frac{1}{2} \frac{d}{dt} \| \partial_t^m \rho_c \|_{L^2}^2 = - \int_T \partial_t^m \rho_c (\partial_t^m (u_c \partial_t \rho_c) + \partial_t^m (\rho_c \partial_t u_c)) \, dx. \]

\[ \| \partial_t^2 u_c \|_{L^2} \leq C \left\| \frac{1}{\lambda_c (\rho_c)} \right\|_{L^\infty_t} \| \partial_t V_c \|_{L^2_t}^2 + C \left\| \frac{1}{\lambda_c (\rho_c)} \right\|_{L^\infty_t} \| \partial_t \lambda_c (\rho_c) \|_{L^\infty_t} \left( \| \partial_t \lambda_c (\rho_c) \|_{L^2_t}^2 + 1 \right) \| \partial_t u_c \|_{L^2_t}. \]

\[ (48) \]

**Proof.** Part (i) follows immediately from (27). Part (ii) results from the definition of $V_c$ and the control of $V_c$ in $L^\infty_t L^2_x$ is provided by estimate (41). And finally, part (iii) follows by differentiating $V_c$, we get $\partial_t V_c = \partial_t \lambda_c \partial_t u_c + \lambda_c \partial_t^2 u_c$, so that

\[ \| \partial_t^2 u_c \|_{L^2_t} \leq \left\| \frac{1}{\lambda_c (\rho_c)} \right\|_{L^\infty_t} (\| \partial_t V_c \|_{L^2_t} + \| \partial_t \lambda_c \|_{L^\infty_t} \| \partial_t u_c \|_{L^2_t}) \]

\[ \leq C \left\| \frac{1}{\lambda_c (\rho_c)} \right\|_{L^\infty_t} (\| \partial_t V_c \|_{L^2_t} + \| \partial_t \lambda_c \|_{L^\infty_t} (\| \partial_t u_c \|_{L^2_t} \| \partial_t^2 u_c \|_{L^2_t} + \| \partial_t u_c \|_{L^2_t})). \]

\[ \square \]

3.3.2. The higher order regularity estimates. We begin with a formal derivation of higher regularity estimates, for which we assume that the functions $\rho_c, u_c$ are sufficiently smooth.

**Lemma 3.10.** Let $m \geq 2$ and let $\rho_c, u_c \in C^1([0,T]; C^m(\mathbb{T}))$, $u_c \in C([0,T]; C^{m+1}(\mathbb{T}))$ satisfy (10a), then we have

\[ \frac{1}{2} \frac{d}{dt} \| \partial_t^m \rho_c \|_{L^2_t}^2 \leq C \left( \| \partial_t^m u_c \|_{L^2_t} \| \partial_t^m \rho_c \|_{L^2_t} + \| \rho_c \|_{L^\infty_t} \| \partial_t^m \rho_c \|_{L^2_t} \| \partial_t^{m+1} u_c \|_{L^2_t} \right). \]

\[ (49) \]

Moreover, for $l \geq 1$ and $\rho_c \in C([0,T]; C^{l+1}(\mathbb{T}))$, $u_c \in C([0,T]; C^l(\mathbb{T}))$, $V_c \in C^1([0,T]; C^{l+2}(\mathbb{T}^d))$ satisfying (39), we have

\[ \frac{1}{2} \frac{d}{dt} \int_T \| \partial_t^l V_c \|_{L^2_t}^2 \, dx = - \int_T \partial_t^l \left( \left( u_c + \frac{\lambda_c (\rho_c)}{\rho_c^2} \partial_t \rho_c \right) \partial_t V_c \right) \partial_t^l V_c \, dx \]

\[ + \int_T \partial_t^l \left( \frac{\lambda_c (\rho_c)}{\rho_c} \partial_t^2 V_c \right) \partial_t^l V_c \, dx \]

\[ - \int_T \partial_t^l \left( \frac{\lambda_c (\rho_c)}{\rho_c} \partial_t u_c + \frac{\lambda_c (\rho_c)}{\rho_c^2} \right) V_c^2 \partial_t^l V_c \, dx. \]

\[ (50) \]

**Proof.** First, differentiating (10a) $m$ times with respect to $x$ and multiplying by $\partial_t^m \rho_c$, we deduce

\[ \frac{1}{2} \frac{d}{dt} \int_T \| \partial_t^m \rho_c \|_{L^2_t}^2 \, dx = - \int_T \partial_t^m \rho_c (\partial_t^m (u_c \partial_t \rho_c) + \partial_t^m (\rho_c \partial_t u_c)) \, dx. \]
\[ \frac{1}{2} \frac{d}{dt} \int \left[ \frac{\partial_t \rho_c}{\rho_c} \right]^2 \, dx = - \int \frac{\partial_t \rho_c}{\rho_c} \left[ \left[ \frac{\partial_t \rho}{\rho} \right] \partial_t \rho_c \right] \, dx - \int \frac{\partial_t \rho_c}{\rho_c} \partial_t \rho_c \, dx + \int \partial_t^2 \rho_c u_c \, dx + \int \partial_t^2 \rho_c \partial_t^2 \rho_c \, dx + \int \partial_t \rho_c \partial_t^2 u_c \, dx. \] (51)

On the periodic domain, for any \( g \in W^{n,2}(\mathbb{T}) \) with \( n \geq 3 \), we have
\[ \| \partial_t g \|_{L^\infty} \leq C \| \partial_t^2 g \|_{L^2} \leq C(n) \| \partial_t^3 g \|_{L^2}, \] (52)
see f.i. [17, page 12]. Inequality (52) and the Kato–Ponce theory yield
\[ \| \left[ \frac{\partial_t \rho_c \rho_c}{\rho_c} \right] \partial_t \rho_c \|_{L^2} \leq C \left( \| \partial_t^3 \rho_c \|_{L^2} \right) \| \partial_t \rho_c \|_{L^\infty} + \| \partial_t \rho_c \|_{L^2} \| \partial_t \rho_c \|_{L^2} \right), \] and similarly
\[ \| \left[ \frac{\partial_t \rho_c \rho_c}{\rho_c} \right] \partial_t \rho_c \|_{L^2} \leq C \left( \| \partial_t^3 \rho_c \|_{L^2} \right) \| \partial_t \rho_c \|_{L^\infty} + \| \partial_t \rho_c \|_{L^2} \| \partial_t \rho_c \|_{L^2} \right). \]

Going back to (51), we notice that
\[ - \int \frac{\partial_t \rho_c}{\rho_c} \partial_t \rho_c \, dx = \frac{1}{2} \int \partial_t \rho_c \, dx. \]
This implies
\[ \int \partial_t \rho_c \, dx \leq C \| \partial_t \rho_c \|_{L^\infty} \| \partial_t^2 \rho_c \|_{L^2}. \]
Combining all these estimates and plugging into (51) we finally obtain (49). Next, differentiating (39) \( l \) times with respect to space and multiplying by \( \partial_t^l V_c \) we obtain (50).

We already have the estimate of \( V_c \) in \( L^\infty L^2 \) and \( \partial_t V_c \) in \( L^2 L^2 \) from (41). Also, lemma 3.3 and estimate (37) give
\[ \| \partial_t \rho_c \|_{L^\infty L^2} = \left( \alpha - \frac{1}{2} \right) \left\| \rho_c \right\|_{L^\infty L^2} + \frac{1}{\varepsilon} \left( \varepsilon \| \rho_c \|_{L^\infty L^2} \right) \leq C \left( \rho_c, \alpha, T, E_0, E_1 \right). \] (53)
Using the formulas from lemma 3.10, we want to derive the estimates for two further orders of regularity:

- First, the case corresponding to \( m = 2 \) and \( l = 1 \) in (49) and (50), respectively.
- Then, the case corresponding to \( m = 3 \) and \( l = 2 \) in (49) and (50), respectively.

The obtained result is summarised in the statement below.

**Proposition 3.11.** Let \( \varepsilon > 0 \) be fixed and \( (\rho_c, u_c) \) be a regular solution of system (10) with
\[ E_{3,\varepsilon} = E_{1,\varepsilon} + \| \partial_t^2 u_c \|_{L^2} + \| \partial_t^2 \rho_c \|_{L^2} + \| \partial_t^2 u_c \|_{L^2} + \| \partial_t \rho_c \|_{L^2}. \] (54)
Then we have
\[ \| \partial_t^3 \rho_c \|_{L^\infty L^2} + \| \partial_t^3 V_c \|_{L^\infty L^2} + \| \partial_t^3 u_c \|_{L^\infty L^2} + \| \partial_t \rho_c \|_{L^\infty L^2} \leq C \left( \varepsilon, E_{3,\varepsilon}, \rho_c, \| \rho_c \|, T \right). \] (55)
Proposition 3.11 is an analogue of [17, lemma 4.3] and it is based on the lemma A.1 which in turns is an analogue of [17, lemma 4.2]. For completeness we include the proofs of both results, see the appendix.

4. Estimates uniform in $\varepsilon$

In this section we first recall and derive additional uniform w.r.t. $\varepsilon$ estimates which eventually will allow us to let $\varepsilon \to 0$. The two key estimates of this section are: the one-sided Lipschitz condition on $u_\varepsilon$, and the control of the singular potential $\pi_\varepsilon$.

4.1. Estimates based on the energy bounds

First note that the lemmas 3.1 and 3.2 are already uniform with respect to $\varepsilon$, and so we have the following uniform bounds

**Proposition 4.1.** Let the initial data $\rho_0^\varepsilon, u_0^\varepsilon$ be such that

$$E_{0,\varepsilon} + E_{1,\varepsilon} \leq C \quad (56)$$

for some $C$ independent of $\varepsilon$. Then $\rho_\varepsilon, u_\varepsilon$ enjoy the following bounds:

$$\|\sqrt{\rho_\varepsilon} u_\varepsilon\|_{L^\infty_t L^2_x} \leq C,$$

$$\|\sqrt{\rho_\varepsilon} w_\varepsilon\|_{L^\infty_t L^2_x} \leq C,$$

$$\|\sqrt{\lambda_\varepsilon (\rho_\varepsilon)} \partial_x u_\varepsilon\|_{L^2_t L^2_x} \leq C,$$

$$\|\rho_\varepsilon\|_{L^\infty_t L^\infty_x} \leq 1.$$

(57)

4.2. One-sided Lipschitz condition on $\partial_x u_\varepsilon$

The purpose of this subsection is to prove that $u_\varepsilon$ satisfies a one-sided Lipschitz condition, which will yield a control of the full norm of the $\partial_x u_\varepsilon$.

**Proposition 4.2.** Let $\rho_\varepsilon, u_\varepsilon$ be solution to system (10), and set

$$A_\varepsilon := \max (\text{ess sup} (\lambda_\varepsilon (\rho_\varepsilon^0) \partial_x u_\varepsilon^0), 0).$$

Then

$$V_\varepsilon = \lambda_\varepsilon (\rho_\varepsilon) \partial_x u_\varepsilon \leq A_\varepsilon.$$

(58)

In particular:

- If $A_\varepsilon \to 0$ as $\varepsilon \to 0$, then

$$\lambda_\varepsilon (\rho_\varepsilon) \partial_x u_\varepsilon \to 0 \quad \text{as} \quad \varepsilon \to 0; \quad (59)$$

- If $A_\varepsilon \leq \lambda_\varepsilon (\rho_\varepsilon) \leq C \varepsilon^{-\alpha}$, for some $C$ independent of $\varepsilon$, then

$$\partial_x u_\varepsilon \leq C. \quad (60)$$
**Proof.** The starting point is derivation of equation for
\[ \tilde{V}_e = (K_e A_e t + 1) V_e. \]
Similarly to proof of proposition 3.7 we can show that
\[
\partial_t \tilde{V}_e + \partial_x \left( u_e \tilde{V}_e \right) - \partial_t \left( \frac{\lambda_e (\rho_e)}{\rho_e} \partial_x \tilde{V}_e \right)
= \frac{K_e A_e}{K_e A_e t + 1} \tilde{V}_e - \left( \frac{\lambda_e (\rho_e)}{\lambda_e (\rho_e)} \right)^2 - \frac{\partial_t \lambda_e (\rho_e)}{\rho_e} \partial_x \tilde{V}_e, \tag{61}
\]
which for \( \varepsilon \) fixed holds pointwisely. We now derive the renormalised equation for \( \tilde{V}_e \). To this purpose we multiply (61) by \( S'(\tilde{V}_e) \), where \( S \) is smooth, increasing and convex function, we obtain
\[
\partial_t S(\tilde{V}_e) + \partial_x \left( u_e S(\tilde{V}_e) \right) - \partial_t \left( \frac{\lambda_e (\rho_e)}{\rho_e} S'(\tilde{V}_e) \partial_x \tilde{V}_e \right)
= \left( S(\tilde{V}_e) - S'(\tilde{V}_e) \tilde{V}_e \right) \partial_x u_e - S''(\tilde{V}_e) \frac{\lambda_e (\rho_e)}{\rho_e} \left( \partial_x \tilde{V}_e \right)^2
\]
\[
+ \frac{K_e A_e}{K_e A_e t + 1} \tilde{V}_e - \left( \frac{\lambda_e (\rho_e)}{\lambda_e (\rho_e)} \right)^2 - \frac{\partial_t \lambda_e (\rho_e)}{\rho_e} \partial_x S(\tilde{V}_e).
\]
We set \( S(y) = F_\eta(y) \) where \( F_\eta, \eta > 0 \), is a regularization of \( \cdot - A_e \) + :
\[
F_\eta(y) = \begin{cases} 
0 & \text{if } y - A_e \leq \eta \\
\frac{y - A_e - \eta}{2} + \frac{\eta}{2} \sin \left( \frac{\pi}{\eta} \frac{y - A_e}{2} \right) & \text{if } \eta \leq y - A_e \leq 2\eta \\
y - A_e - \frac{3\eta}{2} & \text{if } y - A_e \geq 2\eta
\end{cases}
\]
For \( \eta > 0 \) fixed, \( F_\eta'' \geq 0, 0 \leq \eta \leq 1 \), and
\[
|F_\eta(y) - (y - A_e) F_\eta''(y)| \leq \left( \frac{3}{2} + \frac{1}{2\pi} \right) \eta = \kappa \eta.
\]
Note that for such choice of \( S \) the second and the fourth terms on the rhs of (62) are non-positive. Therefore, integrating (62) in space, we then get:
\[
\frac{d}{dt} \int_T F_\eta(\tilde{V}_e(t)) \, dx \leq \int_T \left| F_\eta(\tilde{V}_e) - F_\eta'(\tilde{V}_e) (\tilde{V}_e - A_e) \right| \partial_x u_e \, dx
\]
\[
+ \int_T \frac{A_e}{K_e A_e t + 1} F_\eta'(\tilde{V}_e) \tilde{V}_e \left( K_e - \frac{1}{\lambda_e (\rho_e)} \right) \, dx
\]
\[
+ \int_T \left| \partial_t \left( \frac{\partial_x \lambda_e (\rho_e)}{\rho_e} \right) \right| F_\eta(\tilde{V}_e) \, dx.
\]
The first term on the rhs can be controlled using (64) and (57), and the $\varepsilon$-dependent bound from below for $\rho_\varepsilon$ (36), we obtain
\begin{equation}
\int_0^T \int_T \left| F_\eta \left( \tilde{V}_\varepsilon \right) - F_\eta \left( \tilde{V}_\varepsilon - A_\varepsilon \right) \right| \left| \partial_\varepsilon u_\varepsilon \right| \text{d}x \text{d}t \leq \kappa \eta \int_0^T \int_T \left| \partial_\varepsilon u_\varepsilon \right| \text{d}x \text{d}t \leq \kappa C(T) \eta \varepsilon^{-\alpha'}
\end{equation}

with some $\alpha' > 0$. For the second term of the right-hand side of (65), we notice that
\begin{equation}
\int_T \frac{A_\varepsilon}{K_\varepsilon A_\varepsilon t + 1} F_\eta' \left( \tilde{V}_\varepsilon \right) \tilde{V}_\varepsilon \left( K_\varepsilon - \frac{1}{\lambda_\varepsilon (\rho_\varepsilon)} \right) \text{d}x
\end{equation}

Now, we recall that $\rho_\varepsilon \leq 1 - C_1 \varepsilon^{\frac{1}{m+1}}$, with $C_1 > 0$ independent of $\varepsilon$, so that $\lambda_\varepsilon (\rho_\varepsilon) \leq C_2 \varepsilon^{-\frac{1}{m+1}}$. Hence, taking
\begin{equation}
K_\varepsilon \leq C_2^{-1} \varepsilon \frac{1}{m+1}
\end{equation}

we observe that the integral (67) is non-positive so that, by Gronwall’s inequality,
\begin{equation}
\int_T F_\eta' \left( \tilde{V}_\varepsilon (t) \right) \leq \left[ \int_T F_\eta' \left( \tilde{V}_\varepsilon (0) \right) + C_\eta \int_T \left| \partial_\varepsilon u_\varepsilon \right| \right] \exp \left( \left\| \partial_\varepsilon \left( \frac{\partial_\varepsilon \lambda_\varepsilon (\rho_\varepsilon)}{\rho_\varepsilon} \right) \right\|_{L^1} \right)
\end{equation}

Passing to the limit $\eta \to 0$, and observing that initially $(\tilde{V}_\varepsilon (0) - A_\varepsilon)_+ = (V_\varepsilon (0) - A_\varepsilon)_+ = 0$, we conclude that
\begin{equation}
\int_T \left( \tilde{V}_\varepsilon (t) - A_\varepsilon \right)_+ \text{d}x \leq 0.
\end{equation}

This implies that
\begin{equation}
V_\varepsilon (t,x) \leq \frac{A_\varepsilon}{K_\varepsilon A_\varepsilon t + 1} \leq A_\varepsilon.
\end{equation}

This concludes the proof of (58). The implication (59) is straightforward, and (60) follows from (58) and the lower bound of the density (36).

**Remark 4.3.** Note that we have from (69) an Oleinik entropy condition at $\varepsilon > 0$ fixed, namely a decreasing in time: $\left( \lambda_\varepsilon \partial_\varepsilon u_\varepsilon \right)_+ \leq (K_\varepsilon t)^{-1}$. Unfortunately, due to the condition on $K_\varepsilon$ (68), this estimate degenerates as $\varepsilon \to 0$.

As a consequence of proposition 4.2 and due to the periodicity of the domain, we can control the whole norm of the velocity gradient.

**Corollary 4.4.** We have
\begin{equation}
\left\| \partial_\varepsilon u_\varepsilon \right\|_{L^\infty T^1} \leq C
\end{equation}

for a constant $C$ independent of $\varepsilon$. As a consequence of it, we deduce
\begin{equation}
\left\| u_\varepsilon \right\|_{L^\infty T} \leq C.
\end{equation}
Proof. Let us denote $D_\varepsilon := (\partial_x u_\varepsilon)_+$. We have, for any $t$

$$
\int_T |\partial_x u_\varepsilon(t, x)| \, dx = \int_T (2D_\varepsilon(t, x) - \partial_x u_\varepsilon(t, x)) \, dx = \int_T 2D_\varepsilon(t, x) \, dx \\
\text{one-sided Lip. (60)} \leq 2\bar{C}.
$$

Taking the supremum w.r.t. $t$ we finally obtain the uniform control of $\|\partial_x u_\varepsilon\|_{L_1^\infty}$.

As a consequence of estimates (57) and nonnegativity of the density, one obtains

$$
\|u_\varepsilon\|_{L_1^\infty} \leq C,
$$

applying the generalized Poincaré inequality (see proposition A.2 in appendix). Finally from the Sobolev embedding, we obtain the $L_1^\infty$ bound.

### 4.3. Improved potential estimate

Note that so far we are lacking the uniform bound on the singular part of the potential $\pi_\varepsilon(\rho_\varepsilon)$ (defined in (14), recall also remark 3.4 and estimate (35)). This is the purpose of the next lemma.

**Lemma 4.5.** Let the conditions of the previous proposition be satisfied and assume furthermore the condition (19). We have then

$$
\|\pi_\varepsilon(\rho_\varepsilon)\|_{L_1^\infty} + \|\partial_x \pi_\varepsilon(\rho_\varepsilon)\|_{L_2^\infty} \leq C,
$$

for a positive constant $C$ independent of $\varepsilon$.

**Proof.** We start with the control of the gradient $\partial_x \pi_\varepsilon(\rho_\varepsilon)$ which easily derived from estimate (57)

$$
\|\partial_x \pi_\varepsilon(\rho_\varepsilon)\|_{L_2^\infty} \leq \|\sqrt{\rho_\varepsilon}\|_{L_1^\infty} \|\sqrt{\rho_\varepsilon} \partial_x \rho_\varepsilon(\rho_\varepsilon) + \varphi_\varepsilon(\rho_\varepsilon)\|_{L_2^\infty} \leq \|\sqrt{\rho_\varepsilon}\|_{L_1^\infty} \left(\|\sqrt{\rho_\varepsilon} w_\varepsilon\|_{L_2^\infty} + \|\sqrt{\rho_\varepsilon} u_\varepsilon\|_{L_2^\infty}\right) \leq C. \tag{73}
$$

Assume that $T$ can be identified with $[0, 1]$ supplemented with periodic boundary conditions. We first multiply equation (10b) by

$$
\psi(t, x) = \int_0^x (\rho_\varepsilon(t, y) - \langle \rho_\varepsilon \rangle) \, dy,
$$

where

$$
\langle \rho_\varepsilon \rangle := \frac{M_\varepsilon^0}{|T|}.
$$
Note that $\psi$ is periodic in space. After time and space integration, we get:

$$
\int_0^t \int_\Omega \rho_e^2 p_e' (\rho_e) \partial_t u_e (\rho_e - \langle \rho_e \rangle) \, dx \, dt' + \int_0^t \int_\Omega \rho_e^2 \varphi_e' (\rho_e) \partial_x u_e (\rho_e - \langle \rho_e \rangle) \, dx \, dt' \\
= \int_0^t \int_\Omega \rho_e u_e^2 (\rho_e - \langle \rho_e \rangle) \, dx \, dt' + \int_0^t \int_\Omega (\rho_e u_e) \partial_t \psi \, dx \, dt' \\
+ \int_\Omega (\rho_e u_e \psi) (0, x) \, dx - \int_\Omega (\rho_e u_e \psi) (t, x) \, dx.
$$

(74)

From the boundedness of the initial conditions and due to (57), we get that the last two terms are uniformly bounded. By the same token, noticing that

$$
\partial_t \psi = \int_0^x \partial_t \rho_e \, dy = - \int_0^x \partial_x (\rho_e u_e) \, dy = \rho_e u_e (t, 0) - \rho_e u_e (t, x),
$$

and using in addition the uniform bound (71), the third term on the rhs of (74) is bounded as well. The first of the rhs of (74) is bounded thanks to (57). Ultimately, we infer that

$$
\left| \int_0^t \int_\Omega \rho_e^2 p_e' (\rho_e) \partial_x u_e (\rho_e - \langle \rho_e \rangle) \, dx \, dt' + \int_0^t \int_\Omega \rho_e^2 \varphi_e' (\rho_e) \partial_x u_e (\rho_e - \langle \rho_e \rangle) \, dx \, dt' \right| \leq C. \quad (75)
$$

Now, for $b(\rho) := (\rho - \langle \rho_e^0 \rangle) \pi_e (\rho)$, we have

$$
\partial_t b (\rho_e) + \partial_x (b (\rho_e) \, u_e) = - (b' (\rho_e) \rho_e - b (\rho_e)) \partial_t u_e,
$$

that is

$$
\partial_t \left[ (\rho_e - \langle \rho_e^0 \rangle) \pi_e (\rho_e) \right] + \partial_x \left[ (\rho_e - \langle \rho_e^0 \rangle) \pi_e (\rho_e) \, u_e \right] \\
= - \left[ \rho_e \, (\rho_e - \langle \rho_e^0 \rangle) \pi_e (\rho_e) \right] + \langle \rho_e^0 \rangle \pi_e (\rho_e) \partial_t u_e.
$$

Integrating over space and time we have

$$
\int_\Omega (\rho_e - \langle \rho_e^0 \rangle) \pi_e (\rho_e) (t) \, dx = \int_\Omega (\rho_e^0 - \langle \rho_e^0 \rangle) \pi_e (\rho_e^0) \, dx \\
- \int_0^t \int_\Omega \rho_e \, (\rho_e - \langle \rho_e^0 \rangle) \pi_e (\rho_e) \, \partial_t u_e \, dx \, dt' \\
- \int_0^t \int_\Omega \langle \rho_e^0 \rangle \pi_e (\rho_e) \, \partial_t u_e \, dx \, dt' = I_1 + I_2 + I_3.
$$

The hypothesis (15) on the initial data yields directly the control of the first integral. Indeed, since we assumed $0 < \rho_e^0 \leq 1 - C \varepsilon$, we have

$$
\pi_e (\rho_e^0) \leq C \frac{\varepsilon}{(1 - \sup_x \rho_e^0)^3} + C \varepsilon \left( \sup_x \rho_e^0 \right)^{\alpha} \leq C,
$$

so that $|I_1| \leq C$. 

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For the second integral, we recall that
\[
π'_ε(ρ_ε) = ρ_ε p'_ε(ρ_ε) + ρ_ε φ'_ε(ρ_ε).
\]
Hence, estimate (75) yields
\[
|I_2| ≤ \left| \int_0^t \int_T \rho_ε^2 p'_ε(ρ_ε) \partial_ε u_ε (ρ_ε − ⟨ ρ^0_ε ⟩) \, dx \, dt' + \int_0^t \int_T \rho_ε^2 φ'_ε(ρ_ε) \partial_ε u_ε (ρ_ε − ⟨ ρ^0_ε ⟩) \, dx \, dt' \right| ≤ C.
\]

Finally, performing an integration by parts, we deduce from estimates (71) and (73)
\[
|I_3| = \left| \int_0^t \int_T (ρ^0_ε) \partial_ε π_ε(ρ_ε) u_ε \, dx \, dt' \right| ≤ (ρ^0_ε) \| u_ε \|_{L^∞_t} \| \partial_ε π_ε(ρ_ε) \|_{L^1_x} \leq C \| u_ε \|_{L^∞_t} \| \partial_ε π_ε(ρ_ε) \|_{L^∞_t} \leq C.
\]
Therefore, we have
\[
\left| \int_T (ρ_ε − ⟨ ρ^0_ε ⟩) π_ε(ρ_ε)) \, dx \right| ≤ C.
\] (76)

From the hypotheses (15)–(19), we ensure that
\[
⟨ ρ^0_ε ⟩ ≤ \hat{ρ} = \frac{M^0}{|T|} < 1,
\] (77)
and we define \( ρ^m_ε = \frac{1 + ⟨ ρ^0_ε ⟩}{2} ≤ \frac{1 + \hat{ρ}}{2} \). In order to estimate
\[
\int_T π_ε(ρ_ε) \, dx,
\]
we split the integral as
\[
\int_T π_ε(ρ_ε) \, dx = \int_T π_ε(ρ_ε) \mathbf{1}_{ρ_ε < ρ^m_ε} \, dx + \int_T π_ε(ρ_ε) \mathbf{1}_{ρ_ε ≥ ρ^m_ε} \, dx.
\]

When \( ρ_ε < ρ^m_ε \), the term \( π_ε(ρ_ε) \) remains far from the singularity uniformly with respect to \( ε \) and therefore
\[
\int_T π_ε(ρ_ε) \mathbf{1}_{ρ_ε < ρ^m_ε} \, dx ≤ C.
\]

Since, \( ρ^m_ε ≤ \frac{1 + \hat{ρ}}{2} \), we have
\[
C ≥ \left| \int_T (ρ_ε − ⟨ ρ^0_ε ⟩) π_ε(ρ_ε)) \, dx \right| ≥ \frac{1 - \hat{ρ}}{2} \int_T π_ε(ρ_ε) \mathbf{1}_{ρ_ε ≥ ρ^m_ε} \, dx.
\]

Therefore, we obtain our desired result
\[
\| π_ε(ρ_ε) \|_{L^∞_t L^1_x} ≤ C.
\]
Corollary 4.6. We have
\[ \| \lambda(\rho) \partial_t u \|_{L^1_t} \leq C \]  \hspace{1cm} (78)
for a constant \( C \) independent of \( \varepsilon \).

Proof. Note that for \( \varepsilon \) fixed, \( s_\varepsilon = \rho_\varepsilon p_\varepsilon(\rho_\varepsilon) \) satisfies the following equation
\[ \partial_t s_\varepsilon(\rho_\varepsilon) + \partial_x (s_\varepsilon(\rho_\varepsilon) u_\varepsilon) = -\rho_\varepsilon^2 p_\varepsilon'(\rho_\varepsilon) \partial_x u_\varepsilon. \]  \hspace{1cm} (79)

Now, considering the cases of \( \rho_\varepsilon \) far away from 1 and close to 1 separately, and observing that \( s_\varepsilon(\rho) \) and \( \pi_\varepsilon(\rho) \) have the same singularity when \( \rho \) is close to 1, with the help of estimate (72), we obtain
\[ \| s_\varepsilon(\rho_\varepsilon) \|_{L^\infty_t L^1_x} \leq C. \]

Integrating equation (79), and using the control of the initial density (15) we deduce
\[ \int_0^t \int_T \rho_\varepsilon^2 p_\varepsilon'(\rho_\varepsilon) \partial_x u_\varepsilon \, dx \, dt \leq C. \]  \hspace{1cm} (80)

Next, we proceed exactly in the same way as in the proof of corollary 4.4 for \( V_\varepsilon = \lambda_\varepsilon(\rho_\varepsilon) \partial_t u_\varepsilon \) by combining the control (80) of \( \int_0^t \int_T V_\varepsilon \, dx \, dt \) with the control of the positive part given by (58).

5. Passage to the limit \( \varepsilon \to 0 \)

The purpose of this section is to prove theorem 2.2. We will show that when \( \varepsilon \to 0 \) the sequence of solutions \( (\rho_\varepsilon, u_\varepsilon, \pi_\varepsilon(\rho_\varepsilon)) \) converges to \( (\rho, u, \pi) \), the distributional solution of (6).

Proof of theorem 2.2. Thanks to the uniform bounds from the previous section, there exist \( \rho \in [0, 1] \), \( u \), and \( \pi \geq 0 \) such that
\[ \rho_\varepsilon \rightharpoonup \rho \text{ weakly-* in } L^\infty((0, T) \times \mathbb{T}), \]
\[ u_\varepsilon \rightharpoonup u \text{ weakly-* in } L^\infty((0, T) \times \mathbb{T}), \]
\[ \pi_\varepsilon(\rho_\varepsilon) \rightharpoonup \pi \text{ weakly-* in } L^\infty(0, T; H^1(\mathbb{T})), \]
up to a selection of a subsequence.

We can immediately justify that
\[ (1 - \rho_\varepsilon) \pi_\varepsilon(\rho_\varepsilon) \to 0 \text{ strongly in } L^q((0, T) \times \mathbb{T}), \quad q > 1, \]  \hspace{1cm} (81)
and that the approximate viscosity term converges to 0 strongly, i.e.
\[ \rho_\varepsilon \varphi_\varepsilon(\rho_\varepsilon) \to 0 \text{ strongly in } L^\infty((0, T) \times \mathbb{T}). \]

To pass to the limit in the nonlinear terms we first use the continuity and momentum equations of system (10) to deduce that for any \( p < \infty \) we have
\[ \| \partial_\rho \rho_\varepsilon \|_{L^\infty W^{-1, p}} + \| \partial_t (\rho_\varepsilon u_\varepsilon) \|_{L^\infty W^{-1, 2}} \leq C, \]  \hspace{1cm} (82)
where to estimate the time derivative of momentum, we use that \( \lambda_\varepsilon(\rho_\varepsilon) \approx \pi_\varepsilon(\rho_\varepsilon)^{1 + \frac{1}{2}} \), along with uniform estimates (57) and (72).

Combining the control of \( \partial_t \rho_\varepsilon \) with the control of \( \partial_t \pi_\varepsilon(\rho_\varepsilon) \) we can apply the standard compensated compactness argument (see, lemma 5.1 from [24]) to justify that

\[
(1 - \rho_\varepsilon) \pi_\varepsilon(\rho_\varepsilon) \to (1 - \rho) \pi \quad \text{in } \mathcal{D}',
\]

and so, from (81), we deduce that \((1 - \rho)\pi = 0\) a.e. in \((0, T) \times \mathbb{T}\) with \(1 - \rho \geq 0, \pi \geq 0\).

Similarly, combining the control of gradient of velocity (70) with the uniform estimates for the time derivatives (82) we can justify that

\[
\rho_\varepsilon u_\varepsilon \to \rho u \quad \text{and} \quad \rho_\varepsilon u_\varepsilon^2 \to \rho u^2
\]

in the sense of distributions.

Finally, we can use the equation for \( \partial_t \partial_t \pi_\varepsilon(\rho_\varepsilon) \)

\[
\partial_t \partial_t \pi_\varepsilon(\rho_\varepsilon) + \partial_t (u_\varepsilon \partial_t \pi_\varepsilon) = -\partial_t (\lambda_\varepsilon(\rho_\varepsilon) \partial_t u_\varepsilon)
\]

to deduce that

\[
\partial_t \pi_\varepsilon(\rho_\varepsilon) \in L^2_t W^{-1,1}_x,
\]

so, repeating the previous argument we can justify that also

\[
u_\varepsilon \partial_t \pi_\varepsilon(\rho_\varepsilon) \to u \partial_t \pi,
\]

in the sense of the distributions. With this argument, we can justify completely the passage to the limit in the equations leading to system (6).

The last part is to verify the entropy conditions for the limiting system. First, it is clear that the one-sided Lipschitz estimate holds on the limit velocity \( u \):

\[
\partial_s u \leq C \quad \text{in } \mathcal{D}'.
\]

Next, we write that for fixed \( \varepsilon \), smooth function \( S \):

\[
\partial_t (\rho_\varepsilon S(\varepsilon_\varepsilon)) + \partial_t (\rho_\varepsilon u_\varepsilon S(\varepsilon_\varepsilon)) - \partial_t (S'(u_\varepsilon) \lambda_\varepsilon(\rho_\varepsilon) \partial_t u_\varepsilon) = S''(u_\varepsilon) \lambda_\varepsilon(\rho_\varepsilon)(\partial_t u_\varepsilon)^2,
\]

hence, for convex function \( S \):

\[
\partial_t (\rho_\varepsilon S(\varepsilon_\varepsilon)) + \partial_t (\rho_\varepsilon u_\varepsilon S(\varepsilon_\varepsilon)) - \partial_t (S'(u_\varepsilon) \lambda_\varepsilon(\rho_\varepsilon) \partial_t u_\varepsilon) \leq 0.
\]

As previously, we pass to the limit in the sense of distribution in the first two nonlinear terms thanks to compensated compactness arguments. Next, since \((\lambda_\varepsilon(\rho_\varepsilon) \partial_t u_\varepsilon)\varepsilon_\varepsilon\) is bounded in \(L^1_{t,x}\) from (78), it converges to some \( \Lambda \in M((0, T) \times \mathbb{T}) \). Recall that \( (u_\varepsilon)\varepsilon_\varepsilon\) is bounded in \(L^\infty_{t,x}\), so \((S'(u_\varepsilon) \lambda_\varepsilon(\rho_\varepsilon) \partial_t u_\varepsilon)\varepsilon_\varepsilon\) is bounded in \(L^1_{t,x}\) and converges to some \( \Lambda_\varepsilon \in M((0, T) \times \mathbb{T}) \), where \( |\Lambda_\varepsilon| \leq \text{Lip}_\varepsilon |\Lambda| \). Finally, we have proven that:

\[
\partial_t (\rho S(u)) + \partial_t (\rho u S(u)) - \partial_t \Lambda_\varepsilon \leq 0.
\]
In addition, from (59) we have for all $t \in [0, T]$, $x \in \mathbb{T}$
\[
\lambda_\varepsilon (\rho_\varepsilon (t,x)) (\partial_x u_\varepsilon (t,x))_+ \to 0 \quad \text{as } \varepsilon \to 0,
\]
and therefore
\[
\Lambda \leq 0. \tag{86}
\]

The proof of the theorem 2.2 is therefore complete. \hfill \Box

6. Numerical approximations and illustrations

In this section, we first propose a discretization of model (1)–(3) (dropping the viscosity $\varphi_\varepsilon$) and then performs numerical simulations to illustrate the behavior of the solutions in the limit $\varepsilon \to 0$. In particular, we compare with the dynamics the Euler equation with density constraint (7).

6.1. Numerical scheme

We consider the following formulation of the system:
\[
\partial_t \rho + \partial_x (\rho w) - \partial_x (\rho \partial_x p_\varepsilon (\rho)) = 0,
\]
\[
\partial_t (\rho w) + \partial_x (\rho w^2) - \partial_x (\rho w \partial_x p_\varepsilon (\rho)) = 0,
\]
and look for a numerical solution on $[0, T] \times [0, L]$. The density equation involves a non-linear diffusion operator, with unbounded diffusion coefficient $\rho p_\varepsilon (\rho)$. Therefore implicit schemes are required to avoid too stringent stability condition (in particular with respect to $\varepsilon$).

6.1.1. Time semi-discretization. We denote by $(\rho^n, w^n)$ the quantities at the discrete time $t^n = n \Delta t$, with $n \in \mathbb{N}$ and where $\Delta t > 0$ denotes the time step. We consider the following time semi-discretization scheme:
\[
\frac{\rho^{n+1} - \rho^n}{\Delta t} + \partial_x (\rho w^n) - \partial_x (\rho \partial_x p_\varepsilon (\rho^{n+1})) = 0, \tag{87}
\]
\[
\frac{(\rho w)^{n+1} - (\rho w)^n}{\Delta t} + \partial_x ((\rho w)^n) - \partial_x (\rho \partial_x p_\varepsilon (\rho^{n+1})) = 0. \tag{88}
\]

In order to prevent the density from exceeding the maximal density equal to 1, the first equation is solved in singular pressure variable $\rho^{n+1}$:
\[
\frac{\rho_\varepsilon (\rho^{n+1}) - \rho_\varepsilon (\rho^n)}{\Delta t} + \partial_x (\rho w^n) - \partial_x (\rho \partial_x \rho_\varepsilon (\rho^{n+1})) = 0, \tag{89}
\]
where $\rho_\varepsilon (\rho)$ is the inverse function of $\rho_\varepsilon (\rho)$. In practice, at the $n$th iteration, we first solve the elliptic equation (89) to get $\rho^{n+1}$ and deduce $\rho^{n+1} = \rho_\varepsilon (\rho^{n+1})$ which is less than 1 due to the singularity of the pressure law. Finally $w^{n+1}$ is obtained from (88).
6.1.2. Full discretization. Let us consider the \((p^\rho_n, w^\rho_n)\) on a uniform mesh \(x_i = (j + 1/2)\Delta x\), with \(\Delta x > 0\) and \(j \in \{0, \ldots, J-1\}\), with \(J\Delta x = L\). We use finite-volume discretization formula for the advection terms and finite difference discretizations for the elliptic operators:

\[
\rho_C \left( \frac{p^\rho_{n+1} - p^\rho_n}{\Delta t} \right) + \frac{F^n_{\rho,j+\frac{1}{2}} - F^n_{\rho,j-\frac{1}{2}}}{\Delta x} = \frac{\left( \rho^\rho_{j+1} + \rho^\rho_j \right) \left( p^\rho_{n+1} - p^\rho_{n+1} \right) - \left( \rho^\rho_n + \rho^\rho_{j-1} \right) \left( p^\rho_{n} - p^\rho_{j-1} \right)}{2\Delta x^2} = 0,
\]

(90)

\[
\rho_F \left( \frac{\rho w_{n+1}^{j+1} - \rho w_n^j}{\Delta t} \right) + \frac{F^n_{\rho w,j+\frac{1}{2}} - F^n_{\rho w,j-\frac{1}{2}}}{\Delta x} = \frac{\left( \rho w_{j+1}^{n+1} + (\rho w)^n_{j+1} \right) \left( p^\rho_{j+1} - p^\rho_{j+1} \right) - \left( \rho w_n^{n+1} + (\rho w)_n^j \right) \left( p^\rho_{n} - p^\rho_{j-1} \right)}{2\Delta x^2} = 0,
\]

(91)

where the fluxes are Lax–Friedrichs fluxes given by:

\[
F^n_{\rho,j+\frac{1}{2}} = \frac{(\rho w)^n_{j+1} + (\rho w)^n_{j+1}}{2} - \frac{1}{2} C^n \left( p^\rho_{j+1} - p^\rho_j \right),
\]

\[
F^n_{\rho w,j+\frac{1}{2}} = \frac{(\rho w^2)^n_{j+1} + (\rho w^2)^n_{j+1}}{2} - \frac{1}{2} C^n \left( (\rho w)^n_{j+1} - (\rho w)^n_j \right),
\]

where \(C^n = \max_i \{|w^n_i|\} = \max_{\rho} \{|w^\rho - (\rho^\rho_{j+1}) - (\rho^\rho_{j-1})]/(2\Delta x)|\}\) is an approximation of the maximal effective velocity. Gathering all the explicit terms on the right-hand side, equation (90) can be written as follows:

\[
\rho_C \left( \frac{p^\rho_{n+1} - p^\rho_n}{\Delta t} \right) = \rho_C \left( \frac{p^\rho_{n} - p^\rho_{j-1}}{\Delta t} \right) + \frac{F^n_{\rho,j+\frac{1}{2}} - F^n_{\rho,j-\frac{1}{2}}}{\Delta x}.
\]

(92)

This equation can be complemented by periodic or Dirichlet boundary conditions according to the test case. This is a non-linear equation solved using Newton’s algorithm. To summarize, at the \(n\)-th iteration, we first compute \((p^\rho_{n+1})_j\) with equation (92) and then we compute \((w^\rho_{n+1})_j\) with equation (91). Note that the time step should satisfy the CFL stability condition: \(C^n \Delta t \leq \Delta x\), for all \(n \in \mathbb{N}\).

6.2. Numerical results

Here, we propose two numerical tests to illustrate the behavior of the numerical solutions as \(\varepsilon \to 0\). We compare the results with numerical simulations of the Euler system with maximal density constraints. These latter are obtained with the numerical scheme proposed in [32] based on a Lagrangian formulation.

6.2.1. Test case 1: compressive initial condition. We first consider the following initial condition:

\[
\rho^0(x) = 0.5, \quad w^0(x) = 0.5 \sin(2\pi x),
\]

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Figure 1. Solution at time $T = 0.3$ with compressive initial condition for different $\varepsilon$. For $\varepsilon > 0$: numerical scheme presented in section 6.1 with $\Delta t = 0.001$, $J = 150$. For $\varepsilon = 0$: Lagrangian scheme \cite{32} for the congested Euler system with 400 discretization points.

on the domain $[0, 1]$ with periodic boundary conditions. The compressive desired velocity $w^0$, which is positive for $x < 0.5$ and negative for $x > 0.5$, induces an increase of density in the middle of the domain. The numerical parameters are as follows: $\Delta t = 0.001$ and $J = 150$ and we consider $\gamma = \beta = 2$ in the definition (1) of $p_\varepsilon$. For the congested Euler system, we use 400 discretization points.

Figure 1 shows the solution at time $T = 0.3$ for different values of $\varepsilon$: we plot the density (left), the velocity $u = w - \partial_x p_\varepsilon(\rho)$ and the adhesion potential $-\pi_\varepsilon(\rho)$ defined in (14). Curves labeled by ’$\varepsilon = 0’$ correspond to the numerical solutions of the constrained Euler system (7). As expected, the approximated solutions tend to the solution to the congested Euler system as $\varepsilon \to 0$. In particular, at the limit, the solution solution presents a discontinuity between a congested phase with constant velocity (incompressibility condition $\partial_x u = 0$) and a free phase.

6.2.2. Test case 2: dilatant initial condition. We then perform a dilatant test-case in the congested regime. We consider the following initial data:

$$
\rho^0_\varepsilon(x) = 1 - 2 \varepsilon x, \quad w^0_\varepsilon(x) = \tanh((x - 0.5)/\sigma),
$$

with $\sigma = 0.05$, on the domain $\mathbb{R}$. The initial density has been chosen such that $p_\varepsilon(\rho^0_\varepsilon)$ tends to a positive value as $\varepsilon \to 0$: this is the critical scale to approach a congested regime. In practice, we use Neumann boundary conditions for the approximated system with $J = 200$ points and free boundary conditions for the congested Euler model with 200 discretization points on the interval.

On figure 2, we compare the solutions at time $T = 0.03$ for different $\varepsilon$. We still observe that the solutions tends to the congested Euler dynamics as $\varepsilon \to 0$. We note that the asymptotic adhesion potential equals zero: indeed, due to desired increasing velocity profile, there is no congested force. In consequence, the desired velocity is constant in time.
Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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Appendix

Here our goal is to prove proposition 3.11. To do so, we first state the following lemma, which provides an estimate of \( \rho_\varepsilon \) and \( V_\varepsilon \) for the case \( m = 2 \) in (49) and \( l = 1 \) in (50). Then using this lemma we derive the higher order regularity estimates for \( \rho_\varepsilon \) and \( V_\varepsilon \) and conclude the proof of proposition 3.11.

**Lemma A.1.** Let \( \varepsilon > 0 \) be fixed and \( (\rho_\varepsilon, u_\varepsilon) \) be a regular solution of system (10) with

\[
E_{2,\varepsilon} = E_{1,\varepsilon} + \| \partial_\varepsilon^2 u_\varepsilon^0 \|_{L^2_t} + \| \partial_\varepsilon^2 \rho_\varepsilon^0 \|_{L^2_t}.
\]

Then we have

\[
\| \partial_\varepsilon^3 \rho_\varepsilon \|_{L^\infty_t L^2_x} + \| \partial_\varepsilon V_\varepsilon \|_{L^\infty_t L^2_x} + \| \partial_\varepsilon^2 V_\varepsilon \|_{L^2_t L^2_x} + \| \partial_\varepsilon^2 u_\varepsilon \|_{L^\infty_t L^2_x} + \| \partial_\varepsilon u_\varepsilon \|_{L^2_t L^2_x} \leq C \left( \varepsilon, E_{1,\varepsilon}, \rho_\varepsilon, \varepsilon T \right).
\]

**Proof.** Here we recall our estimate of \( \partial_\varepsilon \rho_\varepsilon \) in \( L^\infty_t L^2_x \):

\[
\| \partial_\varepsilon \rho_\varepsilon \|_{L^\infty_t L^2_x} = \left( \alpha - \frac{1}{2} \right)^{-1} \| \rho_\varepsilon^{\frac{1}{2} \alpha} \|_{L^\infty_t L^2_x} \leq C \left( \varepsilon, \alpha, T, E_0, E_1 \right).
\]
We consider the case that corresponds to \( m = 2 \) in (49). Therefore, we have

\[
\frac{1}{2} \frac{d}{dt} \| \partial_t^2 \rho \|_{L_2}^2 \leq C \left( \| \partial_t^2 u \|_{L_2} \| \partial_t^2 \rho \|_{L_2}^2 + \| \partial_t^2 \rho \|_{L_2} \| \partial_t^2 u \|_{L_2} \right).
\]

On the other hand, integrating by parts with \( l = 1 \) in (50) gives us

\[
\frac{1}{2} \frac{d}{dt} \int_T |\partial_t V| \, dx + \int_T \frac{\lambda_c(\rho_e)}{\rho_e} |\partial_t^2 V| \, dx
\]

\[
= \int_T \left( u_e + \frac{\lambda_c(\rho_e)}{\rho_e} \partial_t \rho \right) \partial_t V \partial_t^2 V + \int_T \frac{(\lambda_c' \rho_e) \rho_e + \lambda_c(\rho_e))}{(\lambda_c(\rho_e))^2} \partial_t^2 V \, dx.
\]

From the observation

\[
\frac{\lambda_c(\rho_e)}{\rho_e} = \rho_e \rho_e' \rho_e + \varepsilon \rho^{\alpha - 1} \geq \varepsilon
\]

we deduce that

\[
\frac{1}{2} \frac{d}{dt} \int_T |\partial_t V| \, dx + \varepsilon \int_T |\partial_t^2 V| \, dx
\]

\[
\leq \int_T \left( u_e + \frac{\lambda_c(\rho_e)}{\rho_e} \partial_t \rho \right) \partial_t V \partial_t^2 V + \frac{(\lambda_c' \rho_e + \lambda_c(\rho_e))}{(\lambda_c(\rho_e))^2} \partial_t^2 V \, dx + \int_T (\lambda_c'(\rho_e) \rho_e + \lambda_c(\rho_e)) \partial_t \rho \partial_t V \partial_t^2 V \, dx := \sum_{j=1}^3 J_i.
\]

**Control of \( J_1 \)**. We proceed similarly as in the case of \( I_2 \) of lemma 3.8. We have

\[
|J_1| \leq \|u_e\|_{L_\infty} \|\partial_t V\|_{L_2} \|\partial_t^2 V\|_{L_2} \leq \frac{\varepsilon}{16} \|\partial_t^2 V\|_{L_2}^2 + \frac{4 \varepsilon}{\varepsilon} \|u_e\|_{L_\infty}^2 \|\partial_t V\|_{L_2}^2.
\]

We recall

\[
\|u_e\|_{L_\infty} \leq \rho^{-1/2} \|\partial_t u_e\|_{L_2} + \varepsilon^{-2} \rho^{-2\alpha} \|V_e\|_{L_2}^2
\]

to conclude

\[
|J_1| \leq \frac{\varepsilon}{16} \|\partial_t^2 V\|_{L_2}^2 + \frac{4 \varepsilon^2}{\varepsilon} \|\partial_t^2 V\|_{L_2}^2 + \varepsilon^{-3} \rho^{-2\alpha} \|V_e\|_{L_2}^2.
\]

**Control of \( J_2 \)**. For this term, we observe

\[
|J_2| \leq C \left( \rho_e \rho_e' \rho_e \right) \|\partial_t \rho \|_{L_\infty} \|\partial_t V\|_{L_2} \|\partial_t^2 V\|_{L_2}
\]

\[
\leq \frac{\varepsilon}{16} \|\partial_t^2 V\|_{L_2}^2 + C \left( \rho_e \rho_e' \rho_e \right) \|\partial_t \rho \|_{L_\infty} \|\partial_t^2 V\|_{L_2}^2,
\]

from which we deduce

\[
|J_2| \leq \frac{\varepsilon}{16} \|\partial_t^2 V\|_{L_2}^2 + C \left( \rho_e \rho_e' \rho_e \right) \|\partial_t \rho \|_{L_\infty} \|\partial_t^2 V\|_{L_2}^2.
\]
Control of $J_3$. The bound for $\left\| \frac{(\lambda'_{\varepsilon}(\rho_\varepsilon)\rho_\varepsilon + \lambda_{\varepsilon}(\rho_\varepsilon))}{(\lambda_{\varepsilon}(\rho_\varepsilon))^2} \right\|_{L^\infty}$ implies

$$|J_3| \leq \frac{\varepsilon}{16} \| \partial_2^2 V_\varepsilon \|^2_{L^2_T} + C \left( \varepsilon, \bar{\rho}_\varepsilon, \bar{\rho}_\varepsilon' \right) \int_T |V_\varepsilon|^4 dx.$$

Moreover, using the Nash inequality we obtain

$$\| V_\varepsilon \|^4_{L^4_T} \leq C \left( \| V_\varepsilon \|^3_{L^2_T} \| \partial_t V_\varepsilon \|_{L^2_T} + \| V_\varepsilon \|^4_{L^2_T} \right).$$

Thus we deduce

$$|J_3| \leq \frac{\varepsilon}{16} \| \partial_2^2 V_\varepsilon \|^2_{L^2_T} + C \left( \varepsilon, \bar{\rho}_\varepsilon, \bar{\rho}_\varepsilon' \right) \left( \| V_\varepsilon \|^3_{L^2_T} \| \partial_t V_\varepsilon \|_{L^2_T} + \| V_\varepsilon \|^4_{L^2_T} \right).$$

Finally combining (95)–(97), we get

$$\frac{1}{2} \frac{d}{dt} \int_T |\partial_t V_\varepsilon|^2 dx + \frac{13}{16} \varepsilon \int_T |\partial_2^2 V_\varepsilon|^2 dx \leq 4 \left( \bar{\rho}_\varepsilon^{-1/2} e^{-1} \| \sqrt{\rho_\varepsilon} u_\varepsilon \|^2_{L^2_T} + \varepsilon^{-3} \bar{\rho}_\varepsilon^{-2\alpha} \| V_\varepsilon \|^2_{L^2_T} \right) \| \partial_t V_\varepsilon \|^2_{L^2_T} + C \left( \varepsilon, \bar{\rho}_\varepsilon, \bar{\rho}_\varepsilon' \right) \left( \| V_\varepsilon \|^4_{L^2_T} \| \partial_t V_\varepsilon \|_{L^2_T} + \| V_\varepsilon \|^4_{L^2_T} \right).$$

Now we want to derive an expression for $\partial_3^4 u_\varepsilon$ in terms of $V_\varepsilon$ and its derivatives. A direct calculation gives us

$$\partial_3^4 V_\varepsilon = 2 \partial_t (\lambda_{\varepsilon}(\rho_\varepsilon)) \partial_2^2 u_\varepsilon + \partial_3^2 (\lambda_{\varepsilon}(\rho_\varepsilon)) \partial_3 u_\varepsilon + \lambda_{\varepsilon}(\rho_\varepsilon) \partial_3^4 u_\varepsilon.$$

Using the relation between $\partial_t V_\varepsilon$ and $\partial_t u_\varepsilon$, we have

$$\partial_3^4 u_\varepsilon = \frac{1}{\lambda_{\varepsilon}(\rho_\varepsilon)} \partial_3^2 V_\varepsilon - \frac{2}{\lambda_{\varepsilon}(\rho_\varepsilon)} \partial_3 \lambda_{\varepsilon}(\rho_\varepsilon) \partial_3 V_\varepsilon + \lambda_{\varepsilon}(\rho_\varepsilon)^2 \left( \frac{\partial_3 \lambda_{\varepsilon}(\rho_\varepsilon)}{\lambda_{\varepsilon}(\rho_\varepsilon)} \right)^2 V_\varepsilon$$

$$- \left( \frac{\lambda'_{\varepsilon}(\rho_\varepsilon) \| \partial_3 u_\varepsilon \|^2 + \lambda''_{\varepsilon}(\rho_\varepsilon) \partial_3^2 \rho_\varepsilon}{\lambda_{\varepsilon}(\rho_\varepsilon)^2} \right) V_\varepsilon := \sum_{i=1}^4 B_i.$$

Denoting $\bar{A} = \frac{\partial_3 \lambda_{\varepsilon}(\rho_\varepsilon)}{\lambda_{\varepsilon}(\rho_\varepsilon)}$ we observe that

$$\| \bar{A} \|^4_{L^\infty_T} \leq C \left( \bar{\rho}_\varepsilon, \bar{\rho}_\varepsilon' \right) \| \partial_3 \rho_\varepsilon \|^4_{L^\infty_T} \leq C \left( \bar{\rho}_\varepsilon, \bar{\rho}_\varepsilon' \right) \| \partial_3^2 \rho_\varepsilon \|^4_{L^2_T}.$$
Using the above estimate, we obtain the following bounds:

\[
\begin{align*}
\|B_1\|_{L^2_t} & \leq \left\| \frac{1}{\lambda_x(\rho_e)} \right\|_{L^2_t} \|\partial^2_t V_e\|_{L^2_t} \leq C \left( \rho_e, \rho_e \right) \|\partial^2_t V_e\|_{L^2_t}; \\
\|B_2\|_{L^2_t} & \leq C \|\tilde{A}\|_{L^\infty_t} \|\partial_t V_e\|_{L^2_t} \leq C \left( \rho_e, \rho_e \right) \|\partial^2_t \rho_e\|_{L^2_t} \|\partial_t V_e\|_{L^2_t}; \\
\|B_3\|_{L^2_t} & \leq C \|\tilde{A}\|_{L^\infty_t} \|\tilde{A}\|_{L^2_t} \|\partial_t V_e\|_{L^2_t} \leq C \left( \rho_e, \rho_e \right) \|\partial_t \rho_e\|_{L^2_t} \|\partial_t \rho_e\|_{L^2_t} \|V_e\|_{H^1_t}; \\
\|B_4\|_{L^2_t} & \leq C \left( \rho_e, \rho_e \right) \|\partial_t \rho_e\|_{L^2_t} \|\partial_t V_e\|_{L^2_t} \|V_e\|_{H^1_t} + C \left( \rho_e, \rho_e \right) \|\partial^2_t \rho_e\|_{L^2_t} \|V_e\|_{H^1_t} \\
& \leq C \left( \rho_e, \rho_e \right) \left( \|\partial^2_t \rho_e\|_{L^2_t} \|\partial_t \rho_e\|_{L^2_t} \|V_e\|_{H^1_t} + \|\partial^2_t \rho_e\|_{L^2_t} \|V_e\|_{H^1_t} \right).
\end{align*}
\]

Therefore, we have

\[
\|\partial^3_t u_e\|_{L^2_t} \leq C \left( \rho_e, \rho_e \right) \left( \|\partial^3_t V_e\|_{L^2_t} + \left( \|\partial_t \rho_e\|_{L^2_t} \|V_e\|_{H^1_t} + \|V_e\|_{H^1_t} \right) \|\partial^2_t \rho_e\|_{L^2_t} \right). \tag{99}
\]

We recall

\[
\frac{1}{2} \frac{d}{dt} \|\partial^2_t \rho_e\|_{L^2_t}^2 \leq C \left( \rho_e, \rho_e \right) \left( \|\partial^2_t u_e\|_{L^2_t} \|\partial^2_t \rho_e\|_{L^2_t} + \|\partial^2_t \rho_e\|_{L^2_t} \|\partial^3_t u_e\|_{L^2_t} \right)
\]

and substitute (99) in the estimate for \(\partial^2_t \rho_e\), we get

\[
\frac{1}{2} \frac{d}{dt} \|\partial^2_t \rho_e\|_{L^2_t}^2 \leq C \left( \rho_e, \rho_e \right) \left( \|\partial^2_t u_e\|_{L^2_t} \|\partial^2_t \rho_e\|_{L^2_t} \\
+ C \left( \rho_e, \rho_e \right) \left( \|\partial^2_t u_e\|_{L^2_t} \|V_e\|_{H^1_t} + \|\partial_t \rho_e\|_{L^2_t} \|V_e\|_{H^1_t} + \|\partial \rho_e\|_{L^2_t} \|V_e\|_{H^1_t} \right) \|\partial^2_t \rho_e\|_{L^2_t}^2 \right).
\]

Now we add the above estimate with (98) to obtain

\[
\frac{1}{2} \frac{d}{dt} \|\partial^2_t \rho_e\|_{L^2_t}^2 + \frac{1}{2} \frac{d}{dt} \|\partial_t V_e\|_{L^2_t}^2 + \frac{13}{16} \|\partial^2_t V_e\|_{L^2_t}^2 \leq C \left( \rho_e, \rho_e \right) \left( \|\partial^2_t V_e\|_{L^2_t} \|\partial^2_t \rho_e\|_{L^2_t} \\
+ C \left( \rho_e, \rho_e \right) \left( \|\partial^2_t u_e\|_{L^2_t} \|V_e\|_{H^1_t} + \|\partial_t \rho_e\|_{L^2_t} \|V_e\|_{H^1_t} + \|\partial \rho_e\|_{L^2_t} \|V_e\|_{H^1_t} \right) \|\partial^2_t \rho_e\|_{L^2_t}^2 \right)
\]

\[
+ 4 \left( \rho_e^{-1/2} e^{-1} \right) \|\sqrt{\rho_e} u_e\|_{L^2_t} + e^{-3} \rho_e^{-2\alpha} \|V_e\|_{L^2_t}^2 \left[ \|\partial \rho_e\|_{L^2_t} \right] \|\partial^2_t \rho_e\|_{L^2_t}^2 \\
+ C \left( \rho_e, \rho_e \right) \left( \|V_e\|_{L^2_t}^2 \|\partial_t \rho_e\|_{L^2_t} + \|V_e\|_{L^2_t}^4 \right).
\]

Using the Young’s inequality we deduce

\[
\|\partial^2_t V_e\|_{L^2_t} \leq C \left( \rho_e, \rho_e \right) \left( \|\partial^2_t \rho_e\|_{L^2_t} \right) \leq \frac{\epsilon}{16} \|\partial^2_t V_e\|_{L^2_t}^2 + \frac{4}{\epsilon} \|\partial^2_t \rho_e\|_{L^2_t}^2.
\]

Similarly, we derive the following inequality:

\[
\frac{1}{2} \frac{d}{dt} \|\partial^2_t \rho_e\|_{L^2_t}^2 + \frac{3}{4} \|\partial^2_t V_e\|_{L^2_t}^2 \leq F_1(t) \|\partial^2_t \rho_e\|_{L^2_t}^2 + F_2(t) \|\partial_t V_e\|_{L^2_t}^2 + G(t), \tag{100}
\]
where

\[ F_1 (t) = C \left( \varepsilon, \rho_c, \rho_e \right) \left( \| \partial^2_t u_e \|_{L^2} + \| V_e \|_{H^1} + \| \partial_x \rho_c \|_{L^2} \| V_e \|_{H^1} + \| \partial_x V_e \|_{L^2}^2 + 1 \right) \]

\[ F_2 (t) = 4 \left( \rho_c^{-1/2} e^{-1} \| \sqrt{\rho_c} u_e \|_{L^2}^2 + e^{-3} \rho_c^{-2} \| V_e \|_{L^2}^2 \right) \]

and

\[ G (t) = C \left( \varepsilon, \rho_c, \rho_e \right) \left( \| V_e \|_{L^2}^2 \| \partial_x V_e \|_{L^2} + \| V_e \|_{L^2}^4 \right) . \]

From our earlier estimates on the time interval \((0, T)\) described in Lemma 3.8, we have

\[ \| F_1 \|_{L^1} \leq C \left( \varepsilon, \rho_c, \rho_e, T \right) \left( \| \partial^2_t u_e \|_{L^2} + \| V_e \|_{L^2} + \| \partial_x \rho_c \|_{L^2} \| V_e \|_{L^2} + \| \partial_x V_e \|_{L^2}^2 + T \right), \]

\[ \| F_2 \|_{L^1} \leq C \left( \varepsilon, \rho_c, \rho_e, T \right) \left( \rho_c^{-1/2} e^{-1} \| \sqrt{\rho_c} u_e \|_{L^2}^2 + e^{-3} \rho_c^{-2} \| V_e \|_{L^2}^2 \right), \]

\[ \| G \|_{L^1} \leq C \left( \varepsilon, \rho_c, \rho_e, T \right) \left( \| V_e \|_{L^2}^3 \| \partial_x V_e \|_{L^2} + \| V_e \|_{L^2}^4 \right), \]

and consequently

\[ \| F_1 \|_{L^1} + \| F_2 \|_{L^1} + \| G \|_{L^1} \leq C \left( \varepsilon, \rho_c, \rho_e, E_{1, \varepsilon}, T \right) . \]

Integrating equation (100) with respect to time, and using the additional hypothesis on initial data (93) and Grönwall’s inequality, we conclude

\[ \| \partial^2_t \rho_c \|_{L^\infty L^2} + \| \partial_x V_e \|_{L^2 L^2}^2 + \frac{\varepsilon}{2} \| \partial^2_t V_e \|_{L^2 L^2}^2 \leq C \left( \varepsilon, E_{2, \varepsilon}, \rho_c, T \right) . \]

From this we deduce the following estimates:

The \( L^\infty L^2 \) estimate for \( \partial^2_t u_e \).

\[ \| \partial^2_t u_e \|_{L^\infty L^2} \leq \frac{1}{\lambda_c (\rho_e)} \| \partial_x V_e \|_{L^2} + \lambda_c (\rho_e) \| V_e \|_{L^2} + \| \partial_x \rho_c \|_{L^2} \| \partial_x u_e \|_{L^2} \| V_e \|_{L^2} \leq C \left( \varepsilon, E_{2, \varepsilon}, \rho_c, T \right) . \]

The \( L^2 L^2 \) estimate for \( \partial^3_x u_e \). We write

\[ \partial^3_x u_e = \frac{1}{\lambda_c (\rho_e)} \partial^2_t V_e - \frac{2 \partial_x \lambda_c (\rho_e)}{\lambda_c (\rho_e)} \partial_x V_e + 2 \left( \frac{\partial_x \lambda_c (\rho_e)}{\lambda_c (\rho_e)} \right)^2 V_e \]

\[ - \left( \frac{\lambda'_c (\rho_e) | \partial_x \rho_c |^2 + \lambda''_c (\rho_e) \partial^2_x \rho_c}{\lambda_c (\rho_e)^2} \right) V_e , \]
and from (99), we get
\[
\| \partial_t^3 u_e \|_{L^2_l} \leq C \left( \rho_e, \rho_e^2 \right) \left( \| \partial_t^2 V_e \|_{L^2_l} + \left( \| \partial_t \rho_e \|_{L^2_l} + \| \partial_t V_e \|_{L^2_l} \right) \right)^2 
+ \| \partial_t^2 \rho_e \|_{L^2_l}^2 + \| \partial_t V_e \|_{L^2_l} \| \partial_t^2 \rho_e \|_{L^2_l} 
+ C \left( \rho_e, \rho_e^2 \right) \left( \| \partial_t \rho_e \|_{L^2_l} + \| \partial_t V_e \|_{L^2_l} \right) \| \partial_t^2 u_e \|_{L^2_l} 
\]

The proof of the lemma is complete.

Having the above lemma at hand, we next prove the regularity estimates from proposition 3.11 with the initial data satisfying (54).

**Proof of proposition 3.11.** In order to prove the proposition, at first we notice that this corresponds to the case \( m = 3 \) and \( l = 2 \) in (51) and (50), respectively.

For \( m = 3 \) in (49) we obtain
\[
\frac{1}{2} \partial_t \| \partial_t^3 \rho_e \|_{L^2_l}^2 \leq C \left( \| \partial_t^3 u_e \|_{L^2_l} \| \partial_t^3 \rho_e \|_{L^2_l}^2 + \| \partial_t^2 \rho_e \|_{L^2_l} \| \partial_t^4 u_e \|_{L^2_l} \right). 
\]

Similarly, \( l = 2 \) in (50) gives us
\[
\frac{1}{2} \partial_t \| \partial_t^2 V_e \|_{L^2_l}^2 = - \int \partial_t^2 \left( \left( u_e + \frac{\lambda_e}{\rho_e} \partial_t \rho_e \right) \partial_t V_e \right) \partial_t^2 V_e \, dx 
+ \int \partial_t^2 \left( \frac{\lambda_e}{\rho_e} \partial_t^2 V_e \right) \partial_t^2 V_e \, dx 
- \int \partial_t^2 \left( \frac{\lambda_e}{\rho_e} \partial_t \rho_e + \frac{\lambda_e}{\rho_e} \right) \partial_t^2 V_e \, dx. 
\]

Applying integration by parts for the terms on the right side of the above equation, followed by an adjustment of the terms, we get
\[
\frac{1}{2} \partial_t \| \partial_t^2 V_e \|_{L^2_l}^2 + \int \frac{\lambda_e}{\rho_e} \| \partial_t^3 V_e \|_{L^2_l}^2 
= \int u_e \partial_t^2 V_e \partial_t V_e + \int \partial_t u_e \partial_t V_e \partial_t^2 V_e \, dx + \int \frac{\lambda_e}{\rho_e} \partial_t^2 \rho_e \partial_t V_e \partial_t^3 V_e \, dx 
+ \int \frac{\rho_e \lambda_e}{\rho_e} \partial_t V_e \partial_t^2 \rho_e \partial_t^3 V_e \, dx 
+ \int \partial_t \left( \frac{\lambda_e}{\rho_e} \partial_t \rho_e + \frac{\lambda_e}{\rho_e} \right) \partial_t^2 V_e \partial_t^2 \rho_e \partial_t^3 V_e \, dx 
+ \int \partial_t \left( \frac{\lambda_e}{\rho_e} \partial_t \rho_e + \frac{\lambda_e}{\rho_e} \right) \partial_t V_e \partial_t^2 \rho_e \partial_t^3 V_e \, dx 
+ 2 \int \left( \frac{\lambda_e}{\rho_e} \partial_t \rho_e + \frac{\lambda_e}{\rho_e} \right) \partial_t^2 \rho_e \partial_t V_e \partial_t^3 V_e \, dx := \sum_{i=1}^{6} K_i. 
\]

We use lemma A.1 to estimate terms \( K_i \) for \( i = 1, \ldots, 6 \). First, we recall the inequality
\[
\frac{\lambda_e}{\rho_e} \geq \varepsilon 
\]

We conclude
\[
\int \frac{\lambda_e}{\rho_e} \| \partial_t^3 V_e \|_{L^2_l}^2 \, dx \geq \varepsilon \int \| \partial_t^2 V_e \|_{L^2_l}^2 \, dx. 
\]
Control of $K_1$. Here we have

$$|K_1| \leq \||u_c||_{L^\infty}\||\partial_1^2 V_c||_{L^2}||\partial_1^3 V_c||_{L^2} \leq \frac{\epsilon}{16}||\partial_1^3 V_c||_{L^2}^2 + 4\frac{\epsilon}{\epsilon}||u_c||_{L^\infty}^2||\partial_1^3 V_c||_{L^2}^2.$$

Proceeding similarly as in (95), we obtain

$$|K_1| \leq \frac{\epsilon}{16}||\partial_1^3 V_c||_{L^2}^2 + 4\left(\frac{\rho_c}{\epsilon}^{-1/2}e^{-1}\||\sqrt{\rho_c}u_c||_{L^2}^2 + \epsilon^{-3}\frac{\rho_c}{\epsilon}^{-2\alpha}||V_c||_{L^2}^2\right)||\partial_1^3 V_c||_{L^2}^2. \quad (103)$$

Control of $K_2$. Also, for this term we use Young’s inequality and the inequality (52) to get

$$|K_2| \leq \||\partial_1 u_c||_{L^2}||\partial_1 V_c||_{L^\infty}||\partial_1^3 V_c||_{L^2} \leq \frac{\epsilon}{16}||\partial_1^3 V_c||_{L^2}^2 + \frac{8}{\epsilon}||\partial_1 u_c||_{L^2}^2||\partial_1^3 V_c||_{L^2}^2.$$

Hence, we have

$$|K_2| \leq \frac{\epsilon}{16}||\partial_1^3 V_c||_{L^2}^2 + C\left(\epsilon, \rho_c, \rho_c\right)\left(\||\partial_1 u_c||_{L^2}^2||\partial_1^2 V_c||_{L^2}^2 + \||\partial_1 u_c||_{L^2}^2\||\partial_1^3 V_c||_{L^2}^2\right). \quad (104)$$

Control of $K_3$. We note that

$$|K_3| \leq \left|\frac{\lambda_c\left(\rho_c\right)}{\rho_c^2}\right|_{L^\infty}||\partial_1^2 \rho_c||_{L^\infty}||\partial_1 V_c||_{L^2}||\partial_1^3 V_c||_{L^2} \leq \frac{\epsilon}{16}||\partial_1^3 V_c||_{L^2}^2 + \frac{C}{\epsilon}||\partial_1 u_c||_{L^2}^2||\partial_1^3 V_c||_{L^2}^2. \quad (105)$$

Control of $K_4$. We start with the following estimate:

$$|K_4| \leq \left|\frac{\rho_c\lambda_c'\left(\rho_c\right) - \lambda_c\left(\rho_c\right)}{\lambda_c^2\left(\rho_c\right)}\right|_{L^\infty}||\partial_1 u_c||_{L^2}^2||\partial_1 V_c||_{L^2}||\partial_1^3 V_c||_{L^2}.$$

Additionally, we apply Young’s inequality to get

$$|K_4| \leq \frac{\epsilon}{16}||\partial_1^3 V_c||_{L^2}^2 + \frac{C}{\epsilon}||\partial_1 u_c||_{L^2}^2||\partial_1^3 V_c||_{L^2}^2. \quad (106)$$

Control of $K_5$. A direct calculation gives us the following identity:

$$\partial_1\left(\frac{\lambda_c'\left(\rho_c\right)\rho_c + \lambda_c\left(\rho_c\right)}{\lambda_c\left(\rho_c\right)^2}\right) = \left[\frac{1}{\lambda_c\left(\rho_c\right)^2}(\lambda_c''\left(\rho_c\right)\rho_c + 2\lambda_c'\left(\rho_c\right)) - \frac{\lambda_c'\left(\rho_c\right)}{\lambda_c\left(\rho_c\right)^2}(\lambda_c'\left(\rho_c\right)\rho_c + \lambda_c\left(\rho_c\right))\right] \partial_1 \rho_c.$$
followed by
\[ |K_3| \leq C \left( \bar{\rho}_c, \rho_c \right) \| \partial_c \rho_c \|_{L^2} \| V_c \|_{L^\infty}^2 \| \partial^4_c V_c \|_{L^2}. \]

By using the Young inequality, we can further deduce
\[ |K_3| \leq \frac{\varepsilon}{16} \| \partial^3_c V_c \|_{L^2}^2 + C \left( \varepsilon, \bar{\rho}_c, \rho_c \right) \| \partial_c \rho_c \|_{L^2}^2 \| V_c \|_{L^4}^4. \] (107)

**Control for K_δ.** Here we observe that
\[
|K_δ| \leq \left\| \frac{\left( \lambda'_c(\rho_c) \right) \rho_c + \lambda_c(\rho_c)}{\left( \lambda_c(\rho_c) \right)^2} \right\|_{L^\infty} \| \partial_c V_c \|_{L^2} \| V_c \|_{L^\infty} \| \partial^3_c V_c \|_{L^2}.
\]

This implies
\[ |K_δ| \leq \frac{\varepsilon}{16} \| \partial^3_c V_c \|_{L^2}^2 + C \left( \varepsilon, \bar{\rho}_c, \rho_c \right) \| V_c \|_{L^4}^4. \] (108)

Therefore, adding inequalities (103)–(108), we have
\[
\frac{1}{2} \frac{d}{dt} \int_T |\partial^2_c V_c|^2 \, dx + \frac{5}{8} \varepsilon \int_T |\partial_c \rho_c|^2 \, dx \leq C \left( \varepsilon, \bar{\rho}_c, \rho_c \right) \| \partial_c V_c \|_{L^2}^2 \| \partial^3_c \rho_c \|_{L^2}^2
+ C \left( \varepsilon, \bar{\rho}_c, \rho_c \right) \left( \left\| \left( \lambda'_c(\rho_c) \right) \rho_c + \lambda_c(\rho_c) \right\|_{L^\infty} \| \partial_c V_c \|_{L^\infty} \| \partial^3_c V_c \|_{L^2}^2 \right)
+ C \left( \varepsilon, \bar{\rho}_c, \rho_c \right) \left( \| \partial_c V_c \|_{L^2} \| \partial^2_c \rho_c \|_{L^2}^2 + \left( 1 + \| \partial_c \rho_c \|_{L^2}^2 \right) \| V_c \|_{L^4}^4 \right). \] (109)

Next, we would like to estimate \( \| \partial^4_c u_c \|_{L^2} \). A direct computation leads to the following identity:
\[
\partial^4_c u_c = \frac{1}{\lambda_c(\rho_c)} \partial^3_c V_c - \left( \frac{2 \lambda_c'(\rho_c)}{\lambda_c(\rho_c)} \right) \partial_c \rho_c \partial^3_c V_c
+ \left( \frac{2 \lambda_c(\rho_c)}{\lambda_c(\rho_c)^2} - \left( \frac{\lambda_c'(\rho_c)}{\lambda_c(\rho_c)^2} \right) \right) \partial^3_c \rho_c \partial_c V_c
+ \left( \frac{2 \lambda_c(\rho_c)}{\lambda_c(\rho_c)^2} - \left( \frac{\lambda_c'(\rho_c)}{\lambda_c(\rho_c)^2} \right) \right) \partial^3_c \rho_c \partial_c V_c
+ \left( \frac{2 \lambda_c(\rho_c)}{\lambda_c(\rho_c)^2} - \left( \frac{\lambda_c'(\rho_c)}{\lambda_c(\rho_c)^2} \right) \right) \partial^3_c \rho_c \partial_c V_c.
\]
We rewrite the above expression as

\begin{align*}
\partial_t^3 u(x) &= R_1 (\lambda (\rho_x), \lambda' (\rho_x), \lambda'' (\rho_x)) \partial_t^3 V(x) + R_2 (\lambda (\rho_x), \lambda' (\rho_x), \lambda'' (\rho_x)) \partial_t \rho_x \partial_t^2 V(x) \\
&\quad + R_3 (\lambda (\rho_x), \lambda' (\rho_x), \lambda'' (\rho_x)) \partial_t \rho_x \partial_t V(x) + R_4 (\lambda (\rho_x), \lambda' (\rho_x), \lambda'' (\rho_x)) \partial_t^2 \rho_x \partial_t V(x) \\
&\quad + R_5 (\lambda (\rho_x), \lambda' (\rho_x), \lambda'' (\rho_x)) \partial_t \rho_x \partial_t^2 V(x) + R_6 (\lambda (\rho_x), \lambda' (\rho_x), \lambda'' (\rho_x)) (\partial_t \rho_x)^3 V(x) \\
&\quad + R_7 (\lambda (\rho_x), \lambda' (\rho_x), \lambda'' (\rho_x)) \partial_t^3 \rho_x V(x) := \sum_{i=1}^{7} D_i,
\end{align*}

where for each $i = 1, \ldots, 7$ we have

\[
\| R_i (\lambda (\rho_x), \lambda' (\rho_x), \lambda'' (\rho_x)) \|_{L^\infty} \leq C \left( \rho_x, \rho_x \right).
\]

Therefore, we get the following estimates:

\[
\begin{align*}
\| D_1 \|_{L^2} &\leq C \left( \rho_x, \rho_x \right) \| \partial_t^3 V(x) \|_{L^2} \\
\| D_2 \|_{L^2} &\leq C \left( \rho_x, \rho_x \right) \| \partial_t^3 V(x) \|_{L^2} \| \partial_t \rho_x \|_{L^\infty} \leq C \left( \rho_x, \rho_x \right) \| \partial_t^3 V(x) \|_{L^2} \| \partial_t \rho_x \|_{L^2} \\
\| D_3 \|_{L^2} &\leq C \left( \rho_x, \rho_x \right) \| \partial_t V(x) \|_{L^2} \| \partial_t \rho_x \|_{L^2} \leq C \left( \rho_x, \rho_x \right) \| \partial_t V(x) \|_{L^2} \| \partial_t \rho_x \|_{L^2} \\
\| D_4 \|_{L^2} &\leq C \left( \rho_x, \rho_x \right) \| \partial_t \rho_x \|_{L^2} \| \partial_t \rho_x \|_{L^2} \leq C \left( \rho_x, \rho_x \right) \| \partial_t \rho_x \|_{L^2} \| \partial_t \rho_x \|_{L^2} \\
\| D_5 \|_{L^2} &\leq C \left( \rho_x, \rho_x \right) \| \partial_t V(x) \|_{L^2} \| \partial_t \rho_x \|_{L^2} \leq C \left( \rho_x, \rho_x \right) \| \partial_t V(x) \|_{L^2} \| \partial_t \rho_x \|_{L^2} \\
\| D_6 \|_{L^2} &\leq C \left( \rho_x, \rho_x \right) \| \partial_t \rho_x \|_{L^2} \| \partial_t \rho_x \|_{L^2} \leq C \left( \rho_x, \rho_x \right) \| \partial_t \rho_x \|_{L^2} \| \partial_t \rho_x \|_{L^2} \\
\| D_7 \|_{L^2} &\leq C \left( \rho_x, \rho_x \right) \| \partial_t \rho_x \|_{L^2} \| \partial_t \rho_x \|_{L^2} \leq C \left( \rho_x, \rho_x \right) \| \partial_t \rho_x \|_{L^2} \| \partial_t \rho_x \|_{L^2}.
\end{align*}
\]

Now, going back to (101) and plugging the above estimate in it, we obtain

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \| \partial_t^3 \rho_x \|_{L^2}^2 &\leq C \left( \rho_x, \rho_x \right) \| \partial_t^3 V(x) \|_{L^2} \| \partial_t^3 \rho_x \|_{L^2} \\
&\quad + C \left( \rho_x, \rho_x \right) \left( \| \partial_t \rho_x \|_{H^1} + \| \partial_t^3 u(x) \|_{L^2} \right) \| \partial_t^3 \rho_x \|_{L^2} \\
&\quad + C \left( \rho_x, \rho_x \right) \left( \| \partial_t \rho_x \|_{H^1} \| \partial_t \rho_x \|_{L^2} \| \partial_t^2 V(x) \|_{L^2} \right) \\
&\quad + C \left( \rho_x, \rho_x \right) \left( \| \partial_t \rho_x \|_{H^1} \left( \| \partial_t \rho_x \|_{H^1} + \| \rho_x \|_{L^2} + \| \rho_x \|_{H^1} \right) \right) \| \partial_t^3 \rho_x \|_{L^2}.
\end{align*}
\]

Now we add the above inequality with (109) and use the following inequality

\[
C \left( \rho_x, \rho_x \right) \| \partial_t^3 V(x) \|_{L^2} \| \partial_t^3 \rho_x \|_{L^2} \leq \frac{\epsilon}{8} \| \partial_t^3 V(x) \|_{L^2}^2 + C \left( \epsilon, \rho_x, \rho_x \right) \| \partial_t^3 \rho_x \|_{L^2}^2
\]

to deduce

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \| \partial_t^3 \rho_x \|_{L^2}^2 + \frac{1}{2} \frac{d}{dt} \| \partial_t^3 \rho_x \|_{L^2}^2 + \frac{1}{2} \frac{d}{dt} \| \partial_t^3 V(x) \|_{L^2}^2 &\leq \tilde{F}_1 \left( t \right) \| \partial_t^3 \rho_x \|_{L^2}^2 + \tilde{F}_2 \left( t \right) \| \partial_t^3 \rho_x \|_{L^2}^2 + \tilde{G} \left( t \right)
\end{align*}
\]

(110)
\[ \tilde{F}_1(t) = C(\varepsilon, \bar{\rho}_\varepsilon, \bar{\rho}_\varepsilon) \left( \| V_\varepsilon \|_{L^2} + \| \partial_3^2 u_\varepsilon \|_{L^2} + \| \partial_4 u_\varepsilon \|_{L^2} + \| \partial_5^2 V_\varepsilon \|_{L^2}^2 + \| V_\varepsilon \|_{H^1}^2 + \| \rho_\varepsilon \|_{H^1}^2 + 1 \right), \]

\[ \tilde{F}_2(t) = C(\varepsilon, \bar{\rho}_\varepsilon, \bar{\rho}_\varepsilon) \left( \left( \frac{\rho_\varepsilon}{2} \right)^{1/2} \epsilon^{-1} \| \sqrt{\rho_\varepsilon} u_\varepsilon \|_{L^2}^2 + \epsilon^{-3} \frac{\rho_\varepsilon}{2} \epsilon^{-2} \| V_\varepsilon \|_{L^2}^2 \right) + \| \partial_3^2 u_\varepsilon \|_{L^2}^2 \right) + \| \rho_\varepsilon \|_{H^2}^2, \]

and

\[ \tilde{G}(t) = C(\varepsilon, \bar{\rho}_\varepsilon, \bar{\rho}_\varepsilon) \left( \| \partial_3^2 u_\varepsilon \|_{L^2}^2 + \| \partial_4^2 u_\varepsilon \|_{L^2}^2 + \left( \sum_{k=1}^{4} \| \rho_\varepsilon \|_{H^2}^k \right) \left( \sum_{k=1}^{4} \| V_\varepsilon \|_{H^1}^k \right) \right). \]

From lemmas 3.8 and A.1, we have

\[ \| \tilde{F}_1 \|_{L_t^1} + \| \tilde{F}_2 \|_{L_t^1} + \| \tilde{G} \|_{L_t^1} \leq C \left( \varepsilon, \bar{\rho}_\varepsilon, \bar{\rho}_\varepsilon, E_2, T \right). \]

Now, we introduce an additional hypothesis

\[ \| \partial_3^2 u_\varepsilon \|_{L_t^2} + \| \partial_4^2 V_\varepsilon \|_{L_t^2} < \infty. \]

Again we use Grönwall’s inequality to conclude

\[ \| \partial_3^2 u_\varepsilon \|_{L_t^\infty} + \| \partial_4^2 V_\varepsilon \|_{L_t^\infty} \leq C \left( \varepsilon, E_3, \bar{\rho}_\varepsilon, \bar{\rho}_\varepsilon, T \right), \]

where

\[ E_3 = E_2 + \| \partial_3^2 \rho_\varepsilon \|_{L_t^2} + \| \partial_4^2 V_\varepsilon \|_{L_t^2}. \]

We proceed analogously as in the proof of lemma A.1 to obtain the \( L_t^\infty L_t^2 \) estimate of \( \partial_3^2 u_\varepsilon \) and the \( L_t^2 L_t^2 \) estimate of \( \partial_4^2 u_\varepsilon. \)

Next, we state and prove a generalized Poincaré inequality:

**Proposition A.2.** Let \( r \) be non-negative function such that

\[ 0 < M_0 \leq \int_T r \, dx < \infty, \quad r \in L_t^\infty(\mathbb{T}). \]  

(111)

Then, there exists a positive constant \( C \) (depending on \( M_0 > 0 \)) such that the following inequality holds

\[ \| u \|_{L_t^2} \leq C \left( \| \partial_3 u \|_{L_t^2} + \int_T r |u| \, dx \right), \]

(112)

for any \( u \in W^{1,1}(\mathbb{T}). \)
Proof. We prove the statement by methods of contradiction. Suppose (112) is not true, then there exists a sequence $\{u_n\}_{n \in \mathbb{N}}$ and $\{r_n\}_{n \in \mathbb{N}}$ such that

$$
\|u_n\|_{L^1_x} = 1, \quad \|\partial_x u_n\|_{L^1_x} + \int_T r_n |u_n| \, dx \leq \frac{1}{n}
$$

and

$$r_n \rightharpoonup^* r \text{ weakly-}^* \text{ in } L^\infty_x.$$

Therefore, we have

$$
\|u_n\|_{W^{1,1}_x} \leq 2.
$$

As a consequence of compact embedding of $W^{1,1}_x$ in $L^1_x$, we obtain

$$u_n \rightharpoonup u \text{ strongly in } L^1_x.$$

Next, the bound $\|\partial_x u_n\|_{L^1_x} \leq \frac{1}{n}$ yields

$$\partial_x u_n \rightharpoonup 0 \text{ strongly in } L^1_x.$$

The above two statements imply

$$u_n \rightharpoonup u \text{ strongly in } W^{1,1}_x \text{ and } \partial_x u = 0 \text{ a.e..}$$

Now, note that the weak-$^*$ convergence of $r_n$ in $L^\infty_x$ and strong convergence of $u_n$ in $L^1_x$ helps us to deduce

$$\int_T r \, dx = 0,$$

that contradicts the hypothesis (111).

References

[1] Ancona F, Bianchini R and Perrin C 2023 Hard-congestion limit of the p-system in the BV setting ESAIM: Proc. Surv. 72 41–63
[2] Aw A, Klar A, Rascle M and Materne T 2002 Derivation of continuum traffic flow models from microscopic follow-the-leader models SIAM J. Appl. Math. 63 259–78
[3] Aw A and Rascle M 2000 Resurrection of second order models of traffic flow SIAM J. Appl. Math. 60 916–38
[4] Berthelin F 2002 Existence and weak stability for a pressureless model with unilateral constraint Math. Models Methods Appl. Sci. 12 249–72
[5] Berthelin F 2022 Existence result for a two-dimensional system of conservation laws with unilateral constraints Nonlinear Anal. 232 113248
[6] Berthelin F, Degond P, Delitala M and Rascle M 2008 A model for the formation and evolution of traffic jams Arch. Ration. Mech. Anal. 187 185–220
[7] Berthelin F and Goatin P 2017 Particle approximation of a constrained model for traffic flow Nonlinear Differ. Equ. Appl. 24 1–16
[8] Berthelin F, Goudon T, Polizzi B and Ribot M 2017 Asymptotic problems and numerical schemes for traffic flows with unilateral constraints describing the formation of jams Netw. Heterog. Media. 12 591–617
[9] Bianchini R and Perrin C 2021 Soft congestion approximation to the one-dimensional constrained Euler equations *Nonlinearity*, **34** 6901–29
[10] Bouchut F and James F 1998 One-dimensional transport equations with discontinuous coefficients *Nonlinear Anal.*, **32** 891
[11] Bouchut F and James F 1999 Duality solutions for pressureless gases, monotone scalar conservation laws and uniqueness *Commun. PDE*, **24** 2173–89
[12] Boudin L 2000 A solution with bounded expansion rate to the model of viscous pressureless gases *SIAM J. Math. Anal.*, **32** 172–93
[13] Bresch D and Desjardins B 2006 On the construction of approximate solutions for the 2D viscous shallow water model and for compressible Navier–Stokes models *J. Math. Pures Appl.*, **86** 362–8
[14] Bresch D, Nečasová Š and Perrin C 2019 Compression effects in heterogeneous media *J. L’École Polytech. Math.*, **6** 433–67
[15] Bresch D, Perrin C and Zatorska E 2014 Singular limit of a Navier-Stokes system leading to a free/congested zones two-phase model *C. R. Math. Acad. Sci. Paris*, **352** 685–90
[16] Burtea C and Haspot B 2020 New effective pressure and existence of global strong solution for compressible Navier-Stokes equations with general viscosity coefficient in one dimension *Nonlinearity*, **33** 2077–105
[17] Constantin P, Drivas T D, Nguyen H Q and Pasqualotto F 2020 Compressible fluids and active potentials *Ann. Inst. Henri Poincare C.*, **37** 145–80
[18] Degond P, Hua J and Navoret I. 2011 Numerical simulations of the euler system with congestion constraint *J. Comput. Phys.*, **230** 8057–88
[19] Degond P, Minakowski P, Navoret L and Zatorska E 2018 Finite volume approximations of the Euler system with variable congestion *Comput. Fluids*, **169** 23–39
[20] Degond P, Minakowski P and Zatorska E 2018 Transport of congestion in two-phase compressible/incompressible flows *Nonlinear Anal. Real World Appl.*, **42** 485–510
[21] James F and Vauchelet N 2013 Chemotaxis: from kinetic equations to aggregate dynamics *Nonlinear Differ. Equ. Appl.*, **20** 101–27
[22] James F and Vauchelet N 2016 Equivalence between duality and gradient flow solutions for one-dimensional aggregation equations *Discrete Contin. Dyn. Syst. A*, **36** 1355–82
[23] Lefebvre-Lepot A and Maury B 2011 Micro-macro modelling of an array of spheres interacting through lubrication forces *Adv. Math. Sci. Appl.*, **21** 535–57 (available at: https://mcm-www.jwu.ac.jp/~aikit/AMSA/current.html#fh5co-tab-feature-vertical13)
[24] Lions P-L. 1996 *Mathematical Topics in Fluid Mechanics: Volume 2: Compressible Models* vol 2 (Oxford University Press on Demand)
[25] Lions P-L and Masmoudi N 1999 On a free boundary barotropic model *Ann. Inst. Henri Poincare C*, **16** 373–410
[26] Maury B and Preux A 2017 Pressureless euler equations with maximal density constraint: a time-splitting scheme *Topological Optimization and Optimal Transport: In the Applied Sciences* vol 17 p 333
[27] Mehmood M A 2023 Hard congestion limit of the dissipative aw–rascle system with a polynomial offset function (in preparation)
[28] Mellet A and Vasseur A 2007/08 Existence and uniqueness of global strong solutions for one-dimensional compressible Navier-Stokes equations *SIAM J. Math. Anal.*, **39** 1344–65
[29] Perrin C 201609 Pressure-dependent viscosity model for granular media obtained from compressible Navier–Stokes equations *Appl. Math. Res. Express*, **2016** 289–333
[30] Perrin C 2017 Modelling of phase transitions in granular flows *Nonlinear Anal. Real World Appl.*, **32** 1–34
[31] Perrin C 2018 An overview on congestion phenomena in fluid equations *J. EDP*, **1**–34
[32] Perrin C and Westdickenberg M 2018 One-dimensional granular system with memory effects *SIAM J. Math. Anal.*, **50** 5921–46
[33] Perrin C and Zatorska E 2015 Free/congested two-phase model from weak solutions to multidimensional compressible Navier-Stokes equations *Commun. PDE*, **40** 1558–89
[34] Pokorný M, Wróblewska-Kaminśka A and Zatorska E 2022 Two-phase compressible/incompressible navier–stokes system with inflow-outflow boundary conditions *J. Math. Fluid Mech.*, **24** 87
[35] Vauchelet N and Zatorska E 2017 Incompressible limit of the Navier-Stokes model with a growth offset function (in preparation)
[36] Zhang H M 2002 A non-equilibrium traffic model devoid of gas-like behavior *Transp. Res. B*, **36** 275–90