MOTIVIC HAAR MEASURE ON REDUCTIVE GROUPS

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Abstract. We define a motivic analogue of the Haar measure for groups of the form $G(k((t)))$, where $k$ is an algebraically closed field of characteristic zero, and $G$ is a reductive algebraic group defined over $k$. A classical Haar measure on such groups does not exist since they are not locally compact. We use the theory of motivic integration introduced by M. Kontsevich to define an additive function on a certain natural Boolean algebra of subsets of $G(k((t)))$. This function takes values in the so-called dimensional completion of the Grothendieck ring of the category of varieties over the base field. It is invariant under translations by all elements of $G(k((t)))$, and therefore we call it a motivic analogue of Haar measure. We give an explicit construction of the motivic Haar measure, and then prove that the result is independent of all the choices that are made in the process, even though we have no general uniqueness statement.

0. Introduction

In this paper we define a version of Haar measure on groups that arise when taking the set of points of an algebraic group over a “large” local field. For an algebraic group $G$ defined over an algebraically closed field $k$ of characteristic zero, we consider the set of its points $G(F)$ over the field $F = k((t))$ of Laurent series with coefficients in $k$. Since $F$ is a local field, it can be expected that $G(F)$ would be in many ways analogous to a $p$-adic group. However, there is no hope for a Haar measure on $G(F)$ in the usual sense, since, unlike the $p$-adic situation, the set $G(F)$ is not locally compact. Our objective is to define a “variety-valued” invariant measure on $G(F)$ in the case when $G$ is reductive, and give an explicit formula for such a measure. We are able to do this by means of the theory of motivic integration introduced by M. Kontsevich, [13].

In the original theory of motivic integration, the motivic measures live on arc spaces of (smooth) varieties and take values in a certain completion of the Grothendieck ring of the category of all algebraic varieties over $k$. The arc spaces are defined as follows. For an algebraic variety $X$ over $k$, the space of formal arcs on $X$ is denoted by $\mathcal{L}(X)$. It is the inverse limit

$$\lim_{\leftarrow} \mathcal{L}_n(X)$$

in the category of $k$-schemes of the schemes $\mathcal{L}_n(X)$ representing the functors defined on the category of $k$-algebras by

$$R \mapsto \text{Mor}_{k}\text{-schemes}(\text{Spec } R[t]/t^{n+1}R[t], X).$$
The set of $k$-rational points of $\mathcal{L}(X)$ can be identified with the set of points of $X$ over $k[[t]]$, that is,

$$\text{Mor}_{k\text{-schemes}}(\text{Spec} k[[t]], X).$$

There are canonical morphisms $\pi_n : \mathcal{L}(X) \to \mathcal{L}_n(X)$ – on the set of points, they correspond to truncation of arcs. In particular, when $n = 0$, we get the the natural projection $\pi_X : \mathcal{L}(X) \to X$. The canonical motivic measure is an additive function (whose values are, roughly speaking, equivalence classes of $k$-varieties) on a certain algebra of subsets of the space $\mathcal{L}(X)$ (see [3]). In the case when $X$ is a smooth variety over $k$, this function assigns to the sets of the form $\pi_X^{-1}(\pi_X(A))$ with $A$ a subvariety of $X$ the equivalence class of $A$. Loosely speaking, the canonical motivic measure “projects under $\pi_X$ to the tautological measure on $X$” (see sections 1.2, 1.3). Such a normalization makes the motivic measure on $\mathcal{L}(X)$ unique, hence the term “canonical”.

For an algebraic group $G$, uniqueness implies that the canonical motivic measure on $\mathcal{L}(G)$ is automatically invariant under translations by the elements of $\mathcal{L}(G)$. We observe that by definition of an arc space, the set of $k$-points of $\mathcal{L}(G)$ is in bijection with the set of $k[[t]]$-points of $G$, that is, with the set of integral points in $G(F)$ (In the $p$-adic analogy, $\mathcal{L}(G)$ corresponds to a maximal compact subgroup inside a $p$-adic group). Our task is to extend the motivic measure beyond the integral points of $G(F)$ in such a way that it would be invariant under the translations by all elements of $G(F)$.

For our construction, the arc spaces will not quite suffice because $G(F)$ is not in bijection with the set of $k$-points of any arc space. We will need a slightly more general setup, described in the Bourbaki talk by E. Looijenga [15], and also the language of ind-schemes, needed to handle objects that are “bigger” than arc spaces. We review all the necessary definitions and theorems in the next section. In Section 2, we first extend the motivic measure on $\mathcal{L}(\mathbb{A}^n)$ to the ind-scheme over $k$ whose set of $k$-points coincides with the $F$-points of $\mathbb{A}^n$. We then transport the motivic measure from the affine space to a full measure subset of $G(F)$ (namely, the big cell), using the translation-invariant differential form on $G$.

Acknowledgement. I am deeply grateful to my advisor T. C. Hales for suggesting this project and guiding me through it, and to F. Loeser, A.-M. Aubert, A. Bravo, Ju-Lee Kim, J. Korman, A. Kuronya, E. Lawes, N. Ramachandran, M. Roth and V. Vologodsky for helpful conversations and suggestions.

1. Preliminaries

1.1. The space of sections. Almost everything in the following three subsections is quoted from [15].

We reserve the symbol $\mathbb{D}$ for $\text{Spec} k[[t]]$. The term $\mathbb{D}$-variety will mean a separated reduced scheme that is flat and of finite type over $\mathbb{D}$ and whose
closed fiber is reduced. For a \( \mathcal{D} \)-variety, \( \mathcal{X}/\mathcal{D} \), with closed fiber \( X \), we consider the set \( \mathcal{X}_n \) of sections of its structure morphism up to order \( n \). By sections up to order \( n \) we mean morphisms over \( \mathcal{D} \) from \( \text{Spec} \, k[t]/(t^{n+1}) \) to \( \mathcal{X} \) which make the following diagram commute

\[
\begin{array}{ccc}
\text{Spec} \, k[t]/(t^{n+1}) & \xrightarrow{\cdot} & \mathcal{D}, \\
\downarrow & & \downarrow \\
\mathcal{X} & \xrightarrow{\text{structure morphism}} & \mathcal{X}_n
\end{array}
\]

where the vertical arrow is the structure morphism of \( \mathcal{X} \).

The set \( \mathcal{X}_n \) is the set of closed points of a \( k \)-variety (which we will also denote by the same symbol \( \mathcal{X}_n \)), [9], Section 4.2. Naturally, \( \mathcal{X}_0 = X \). The set \( \mathcal{X}_\infty \) of sections of the structure morphism \( \mathcal{X} \to \mathcal{D} \) is the projective limit of \( \mathcal{X}_n \)'s, and therefore it is a set of closed points of a provariety over \( k \) (by definition, a provariety is a projective limit of a system of varieties; it is a scheme over \( k \), which in our case is not of finite type). If \( \mathcal{X}/\mathcal{D} \) is of the form \( \mathcal{X} \times \mathcal{D} \to \mathcal{D} \), with \( X \) – a \( k \)-variety, then we get the arc spaces described in the introduction: \( \mathcal{X}_n = L_n(X) \) and \( \mathcal{X}_\infty = L(X) \).

As in the case of arc spaces, we have projection morphisms \( \pi_m : \mathcal{X}_m \to \mathcal{X}_n \) and \( \pi_n : \mathcal{X} \to \mathcal{X}_n \) for all \( m \geq n \). (When \( n = 0 \), we shall write \( \pi_X \) and \( \pi_X^0 \) instead of \( \pi_0 \), \( \pi_0^0 \).) A fiber of \( \pi_m+1 \) lies in an affine space over the Zariski tangent space of the base point.

Recall that a constructible subset of a variety \( V \) is a finite disjoint union of (Zariski) locally closed subvarieties of \( V \).

**Definition 1.** A set \( A \subset \mathcal{X}_\infty \) is called weakly stable at level \( n \), if it is a union of fibers of \( \pi_n : \mathcal{X} \to \mathcal{X}_n \), and \( \pi_n(A) \) is constructible. A subset \( A \subset \mathcal{X}_\infty \) is called stable at level \( n \), if it is weakly stable at level \( n \) and for all \( m \geq n \), \( \pi_{m+1}(A) \to \pi_m(A) \) is a piecewise trivial fibration over \( \pi_m(A) \) with fiber \( \mathbb{k}_d^2 \), where \( d = \dim \mathcal{X}_0 \). (For a definition of piecewise trivial fibration, see [6], p.6.) A set is called (weakly) stable if it is (weakly) stable at some level \( n \).

**Remark 2.** It immediately follows from the definition that a set which is stable at level \( n \) is also stable at level \( m \) for all \( m \geq n \). If \( \mathcal{X}/\mathcal{D} \) is smooth and of pure dimension, a weakly stable set is automatically stable (for smooth \( \mathcal{X} \), a fiber of the projection from \( \mathcal{X}_{n+1} \) to \( \mathcal{X}_n \) is an affine space of dimension \( d = \dim \mathcal{X} \) over the tangent space of the base point, [15], p.4). It is also worth mentioning that it is not obvious and not always true that \( \mathcal{L}(X) \) is stable at level 0. The fact that it is stable at some level is a theorem (see e.g. [15], Proposition 3.1). When \( X \) is smooth, it follows from the proof of Proposition 3.1, [15] that \( \mathcal{L}(X) \) is actually stable at level 0.

1.2. The ring \( \hat{\mathcal{M}} \). Now let us describe the ring \( \hat{\mathcal{M}} \) where the measure will take values. Let \( \mathcal{V}_k \) denote the category of all varieties over \( k \), and let \( K_0(\mathcal{V}_k) \) be the Grothendieck ring of this category. Let \( \mathbb{L} = [\mathbb{A}^1] \) denote the isomorphism class of the affine line – an element in \( K_0(\mathcal{V}_k) \). The notation
comes from its motivic interpretation: it corresponds to the so-called Lefschetz motive under the map from $K_0(V_k)$ to the ring of Chow motives [16]. Consider the localization of $K_0(V_k)$ at $L$: $\mathcal{M} = K_0(V_k)[L^{-1}]$. In order to get a measure on an interesting algebra of subsets of $\mathcal{X}_\infty$, we need to complete the ring $\mathcal{M}$. Given $m \in \mathbb{Z}$, let $F_m \mathcal{M}$ be the subgroup of $\mathcal{M}$ generated by the elements of the form $[Z]L^{-r}$ with $\dim Z \leq m + r$. This is a filtration of $\mathcal{M}$ as a ring: $F_m \mathcal{M} \cdot F_n \mathcal{M} \subset F_{m+n} \mathcal{M}$. This filtration is called the dimensional filtration. Denote by $\hat{\mathcal{M}}$ the separated completion of $\mathcal{M}$ with respect to this filtration, i.e.

$$\hat{\mathcal{M}} = \lim_{\leftarrow} \mathcal{M}/F_m \mathcal{M}.$$ 

This is called the dimensional completion. Our motivic measure will be $\hat{\mathcal{M}}$-valued.

**Remark 3.** A recent work of F. Loeser and J. Sebag [14] suggests that it should be possible to define a motivic measure that would take values in the ring $\mathcal{M}$, without the completion. However, we will not pursue this idea here.

1.3. A measure on the space of sections. Let $A$ be a subset of $\mathcal{X}_\infty$ which is stable at level $n$. Observe that by definition of stability, the number $(\dim \pi_m(A) - md)$ is independent of the choice of $m \geq n$ (here $d$ is the dimension of the closed fiber $X$ of $\mathcal{X}$). We call this number the virtual dimension $\dim A$ of $A$. The class $[\pi_m(A)]L^{-md} \in \mathcal{M}$ also does not depend on $m$; we denote it by $\tilde{\mu}_X(A)$. The collection of stable subsets of $\mathcal{X}_\infty$ is a Boolean ring (i.e., is closed under finite union and difference), on which $\tilde{\mu}_X$ defines a finite additive measure.

Let $\mu_X$ be the composition of $\tilde{\mu}_X$ and the completion map $\mathcal{M} \to \hat{\mathcal{M}}$. We call it the motivic measure on $\mathcal{X}$. A subset $A \subset \mathcal{X}_\infty$ is called measurable if for every (negative) integer $m$ there exists a stable subset $A_m \subset \mathcal{X}_\infty$ and a sequence $(C_i \subset \mathcal{X}_\infty)_{i=0}^{\infty}$ of stable subsets such that the symmetric difference $A \Delta A_m$ is contained in $\bigcup_{i \in \mathbb{N}} C_i$ with $\dim C_i < m$ for all $i$ and $\dim C_i \to -\infty$ as $i \to \infty$.

Now we cite the key proposition, which is a generalization of Denef and Loeser’s theorem [4].

**Proposition 4.** ([13], Proposition 2.2) The measurable subsets of $\mathcal{X}_\infty$ make up a Boolean subring and $\mu_X$ extends to a measure on this ring by

$$\mu_X(A) := \lim_{m \to -\infty} \mu_X(A_m).$$

In particular, the above limit exists in $\hat{\mathcal{M}}$ and its value depends only on $A$.

**Remark 5.** Notice that this definition of the measure differs from the one in [3] and [4] by a factor of $L^d$ (with our normalization, the projection of the motivic measure under $\pi_X$ is the “tautological” measure on $X$, as it was described in the introduction).
1.4. The transformation rule. The following crucial results from [15] show that the additive function of sets $\mu_X$ possesses the properties expected of a measure in the classical sense.

**Proposition 6.** ([15], Proposition 3.1) For a $\mathbb{D}$-variety $X/\mathbb{D}$ of pure relative dimension over $\mathbb{D}$, the preimage of any constructible subset under $\pi_n : X_n \to X$ is measurable. In particular, $X_\infty$ is measurable. If $\mathcal{Y} \subset X$ is nowhere dense, then $\mathcal{Y}_\infty$ is of measure zero.

For $X/\mathbb{D}$ of pure relative dimension we have a notion of an integrable function $\Phi : X_\infty \to \mathcal{M}$. This requires the fibers of $\Phi$ to be measurable and the sum $\sum a \mu_X(\Phi^{-1}(a))a$ ($a \in \mathcal{M})$ to converge, i.e., at most countably many nonzero terms $(\mu_X(\Phi^{-1}(a_i))a_i)_{i \in \mathbb{N}}$ are allowed, and the condition $\mu_X(\Phi^{-1}(a_i))a_i \in F_{m_i} \mathcal{M}$ with $\lim_{i \to \infty} m_i = -\infty$ is required to hold. The motivic integral of $\Phi$ is then by definition the value of this series:

$$\int \Phi \, d\mu_X = \sum_i \mu_X(\Phi^{-1}(a_i))a_i.$$ 

An integrable function of particular interest arises from an ideal, $\mathcal{I}$, in the structure sheaf, $\mathcal{O}_X$, of $X$. Such an ideal defines a function $\text{ord}_\mathcal{I} : X_\infty \to \mathbb{N} \cup \{\infty\}$ by assigning to $\gamma \in X_\infty$ the multiplicity of $\gamma^* \mathcal{I}$ as follows. Let $\gamma(o)$ denote the “constant term of $\gamma$”, that is, the image of the closed point, $o$, of $\mathbb{D}$ in the closed fiber of $X$. The map $\gamma^*$ is the map of rings $\mathcal{O}_{X, \gamma(o)} \to k[[t]]$ that induces $\gamma$. Then $\gamma^*$ applied to $\mathcal{I}$ means the base change of $\mathcal{I}$ to $k[[t]]$.

That is, $\gamma^* \mathcal{I}$ is a sheaf on $\text{Spec} k[[t]]$, whose stalk over the closed point is the $k[[t]]$-module $k[[t]] \otimes_{\mathcal{O}_{X, \gamma(o)}} M$, where $M$ is the $\mathcal{O}_{X, \gamma(o)}$-module that corresponds, in the world of rings, to the stalk of $\mathcal{I}$ at $\gamma(o)$, and $k[[t]]$ is an $\mathcal{O}_{X, \gamma(o)}$-module via the map $\gamma^*$. An example of the function $\text{ord}_\mathcal{I}$ when $X = \mathcal{L}(X)$, and $\mathcal{I}$ is the sheaf corresponding to a divisor, $D$, is considered in detail in Section 2.2 of [3] (where $\gamma^* \mathcal{I}$ is denoted $\gamma \cdot D$). The condition $\text{ord}_\mathcal{I} \gamma = n$ only depends on the $n$-jet of $\gamma$, and it defines a constructible subset $C_n \subset X_n$. It turns out that the set defined by $\text{ord}_\mathcal{I} \gamma = \infty$ is of measure zero, and the function $L^{-\text{ord}_\mathcal{I}}$ is integrable.

We can now state the theorem that is key for all applications – the transformation rule. Let $H : \mathcal{Y} \to X$ be a morphism of $\mathbb{D}$-varieties of pure relative dimension $d$. We define the Jacobian ideal $J_H \subset \mathcal{O}_\mathcal{Y}$ of $H$ as the 0-th Fitting ideal of the sheaf of relative differentials $\Omega_{\mathcal{Y}/X}$ (for definitions, see [10], II.8.9.2 and [3], Sections 16.1, 20.2).

**Theorem 7.** ([15], Theorem 3.2) Let $H : \mathcal{Y} \to X$ be a $\mathbb{D}$-morphism of pure dimensional $\mathbb{D}$-varieties with $\mathcal{Y}/\mathbb{D}$ smooth. If $A$ is a measurable subset of $\mathcal{Y}_\infty$ with $H|_A$ injective, then $HA$ is measurable and $\mu_X(HA) = \int_A L^{-\text{ord}_{J_H}} d\mu_\mathcal{Y}$.

**Example 8.** Suppose $H : \mathcal{L}(Y) \to \mathcal{L}(X)$ is induced by an isomorphism $h : Y \to X$. Then $H$ preserves the measure: $\mu_{\mathcal{L}(X)}(HA) = \mu_{\mathcal{L}(Y)}(A)$ for any measurable subset $A \subset \mathcal{L}(Y)$.
Proof. An isomorphism of algebraic varieties induces an isomorphism on their tangent bundles. Hence, $\mathcal{J}_H$ is trivial (i.e. it is the ideal sheaf that coincides with the structure sheaf of $\mathcal{L}(Y)$). The function $L^{-\text{ord}_{\mathcal{J}_H}}$ is identically equal to 1 on $\mathcal{L}(Y)$ in this case.

We will need to use the transformation rule in a slightly more general situation, when $\mathcal{Y}$ is not smooth over $\mathcal{D}$ but is allowed to have a singularity in the closed fiber. In this case, however, the set $A$ will be assumed to be away from the singularity.

For a $\mathcal{D}$-variety $\mathcal{X}$ of pure relative dimension $d$, we denote by $\mathcal{J}(\mathcal{X}/\mathcal{D})$ the $d$-th Fitting ideal of $\Omega_{\mathcal{X}/\mathcal{D}}$. It defines the locus where $\mathcal{X}$ fails to be smooth over $\mathcal{D}$, see [15], Section 9.

**Lemma 9.** Let $H : \mathcal{Y} \to \mathcal{X}$ be a $\mathcal{D}$-morphism of pure dimensional $\mathcal{D}$-varieties; assume that the generic fiber of $\mathcal{Y}$ is smooth. Let $A$ be a measurable subset of $\mathcal{Y}_\infty$ with $H|_A$ injective and such that for all $\gamma \in A$, $\gamma(o)$ is in the regular locus of $\mathcal{Y}$. Then the transformation rule holds for the set $A$: $\mu_\mathcal{X}(HA) = \int_A L^{-\text{ord}_{\mathcal{J}_H}} d\mu_\mathcal{Y}$.

**Proof.** We follow the proof of the transformation rule in [15]. The proof rests on the Key Lemma 9.2, and that is where the assumption that $\mathcal{Y}$ is smooth appears first. Here is the statement of Lemma 9.2, [15]:

Suppose $\mathcal{Y}/\mathcal{D}$ is smooth and let $A \subset \mathcal{Y}_\infty$ be a stable subset of level $l$. Assume that $H|_A$ is injective and that $\text{ord}_{\mathcal{J}_H}|_A$ is constant equal to $e < \infty$. Then for $n \geq \sup\{2e, l + e, \text{ord}_{\mathcal{J}(\mathcal{X}/\mathcal{D})}|_{HA}\}$, $H_n : \pi_n A \to H_n \pi_n A$ has the structure of affine-linear bundle of dimension $e$. (Here $H_n$ is the truncation of the map $H$, that is, the map induced by $H$ on $\mathcal{Y}_n$.)

We claim that the same statement holds if the assumption that $\mathcal{Y}$ is smooth is replaced by the weaker assumption from the statement of our lemma.

There are two implications of smoothness of $\mathcal{Y}$ that are used in the proof of Lemma 9.2. The first one is that for all points $\gamma \in A$, the $\mathcal{O}$-module $\gamma^* \Omega_{\mathcal{Y}/\mathcal{D}}$ is torsion-free, where $\mathcal{O} = k[[t]]$ (recall the definition of $\gamma^*$ applied to an ideal sheaf – it is basically the base change to $k[[t]]$ using the map of rings $\gamma^*$). For this statement to hold for all $\gamma \in A$, it is not necessary for $\mathcal{Y}$ to be smooth over $\mathcal{D}$. It is sufficient that $\gamma(o)$ is in $\mathcal{Y}_{\text{reg}}$ and the generic fiber of $\mathcal{Y}$ is smooth. We show this by computing the $d$th Fitting ideal of the $k[[t]]$-module $\gamma^* \Omega_{\mathcal{Y}/\mathcal{D}}$ in the same way as as it is done in [15], Section 9. Recall that $\mathcal{J}(\mathcal{Y}/\mathcal{D})$ stands for the $d$-th Fitting ideal of $\Omega_{\mathcal{Y}/\mathcal{D}}$, where $d$ is the relative dimension of $\mathcal{Y}$. Since Fitting ideals commute with base change, $\gamma^*(\mathcal{J}(\mathcal{Y}/\mathcal{D})) = \text{Fitt}_d(\gamma^* \Omega_{\mathcal{Y}/\mathcal{D}})$. The latter Fitting ideal measures the length of torsion of $\gamma^* \Omega_{\mathcal{Y}/\mathcal{D}}$: if a $k[[t]]$-module of rank $d$ has torsion of length $e$, its $d$th Fitting ideal is $(t^e)$. It remains to observe that the order with respect to $t$ of the ideal $\gamma^*(\mathcal{J}(\mathcal{Y}/\mathcal{D}))$ is the multiplicity of $\gamma$ along the locus defined by $\mathcal{J}(\mathcal{Y}/\mathcal{D})$, that is, the singular locus of $\mathcal{Y}$ (see [15], Section 9). By assumption, $\gamma$ maps $\mathcal{D}$ to the regular part of $\mathcal{Y}$, thus $\text{ord}_t(\gamma^*(\mathcal{J}(\mathcal{Y}/\mathcal{D}))$ is equal to 0.
1.6. In [2], the k-algebras defined by GL group" confusion between the functor and the set of set Mor( X, S) change the notation and denote the functors defined above by GL S, more can be said. The following definitions are quoted from [4].

Let k, as above, be an algebraically closed field of characteristic 0. By definition, a k-space (resp., k-group) is a functor from the category of k-algebras to the category of sets (resp., of groups) which is a sheaf for the faithfully flat topology (see [2] for the details of the definition). The category of schemes can be viewed as a full subcategory in the category of k-spaces. Direct limits exist in the category of k-spaces; we’ll say that a k-space (resp., a k-group) is an ind-scheme (resp., ind-group) if it is the direct limit of a directed system of schemes. Note that an ind-group is not necessarily a limit of a directed system of algebraic groups. Let (Xα)α∈I be a directed system of schemes, X its limit in the category of k-spaces, and S a scheme. The set Mor(S, X) of morphisms of S into X is the direct limit of the sets Mor(Mor(S, Xα), and the set Mor(X, S) is the inverse limit of the sets Mor(Xα, S).

1.5. k-spaces. Let G be a linear algebraic group. As noted in the introduction, the set of k-points of Σ(G) is in bijection with G(k[[t]]). With the use of the framework of k-spaces [4], more can be said. The following definitions are quoted from [4].

Let k, as above, be an algebraically closed field of characteristic 0. By definition, a k-space (resp., k-group) is a functor from the category of k-algebras to the category of sets (resp., of groups) which is a sheaf for the faithfully flat topology (see [2] for the details of the definition). The category of schemes can be viewed as a full subcategory in the category of k-spaces. Direct limits exist in the category of k-spaces; we’ll say that a k-space (resp., a k-group) is an ind-scheme (resp., ind-group) if it is the direct limit of a directed system of schemes. Note that an ind-group is not necessarily a limit of a directed system of algebraic groups. Let (Xα)α∈I be a directed system of schemes, X its limit in the category of k-spaces, and S a scheme. The set Mor(S, X) of morphisms of S into X is the direct limit of the sets Mor(Mor(S, Xα), and the set Mor(X, S) is the inverse limit of the sets Mor(Xα, S).

1.6. In [4], the k-group GLr(k((t)))) is the functor on the category of k-algebras defined by R → GLr(R((t))), and the “maximal compact subgroup” GLr(k[[t]]) is the subfunctor R → GLr(R[[t]]). In order to avoid confusion between the functor and the set of k((t))-points of GLr, we will change the notation and denote the functors defined above by GLr((t)) and GLr[[t]], respectively.

There is a filtration of the k-group GLr((t)) by the subfunctors GLr(N), where GLr(N)(R) is the set of matrices A(t) in GLr(R((t))) such that both A(t) and A(t)−1 have no poles of order greater than N, that is, all their entries can be written as ∑∞i=−N aiti with ai ∈ R.

The construction of the previous paragraph applies to any affine variety. Indeed, let X = Spec R[x1, . . . , xd]/I. For a k-algebra R, define X(N)(R) to be the set of elements of k d(R) satisfying the equations in I and having poles of order not greater than N in the sense defined above. By X((t)) we will denote the direct limit of X(N); naturally, X((t)) is a subfunctor of k d((t)).

Theorem 1.2 of [4] states that the k-group GLr[[t]] (GLr(k[[t]]) in the notation of the authors) is represented by an affine group scheme and that (GLr(N))N≥0 are represented by schemes, making the the k-group GLr((t)) an ind-group. The proof uses only the fact that GLr is an affine variety: to show that GLr(N) is represented by a scheme, one needs to think of GLr as
the closed subset of the affine space $M_r \times M_r$ ($M_r$ being the space of all $r \times r$-matrices) defined by the equation $AB = Id$. The equation $AB = Id$ (which is, in fact, the system of $r^2$ equations in $r^4$ variables) can be substituted with any finite number of polynomial equations in $d$ variables, and the proof will carry over to any closed subvariety of $\mathbb{A}^d$. Thus if $X$ is closed in $\mathbb{A}^d$, the $k$-space $X((t))$ is represented by the ind-scheme that is the direct limit of schemes representing the functors $X^{(N)}$. We will denote these schemes by the same symbol $X^{(N)}$. The affine space $\mathbb{A}^d((t))$ itself and its filtration by $(\mathbb{A}^d)^{(N)}$ are discussed in detail in the next section.

In the case $X = G$ — a reductive algebraic group, $G((t))$ is an ind-group.

All of the above is summarized in the following proposition; we omit its rigorous proof.

**Proposition 10.** Let $G$ be a reductive algebraic group. Then $\mathcal{L}(G)$ is embedded in the ind-group $G((t))$, and $G((t))$ is a direct limit of affine schemes $(G^{(N)})_{N \geq 0}$ in the category of $k$-spaces, with $G^{(0)} = \mathcal{L}(G)$ representing $G[[t]]$.

1.7. **The space $\mathbb{A}^d((t))$.** We first focus our attention on affine space since we used it above to define $X((t))$ for $X$ an affine variety, and all the subsequent constructions will also be based upon it.

1.7.1. We begin with the arc space of the affine line $\mathcal{L}(\mathbb{A}^1)$.

By definition, $\mathcal{L}_n(\mathbb{A}^1)$ represents the functor

$$R \to \text{Mor}(\text{Spec } R[t]/t^{n+1}R[t], \mathbb{A}^1) = \text{Mor}(k[x], R[t]/t^{n+1}R[t])$$

$$\cong R[t]/t^{n+1}R[t] \cong R^{n+1}.$$  

Hence, $\mathcal{L}_n(\mathbb{A}^1) \cong \mathbb{A}^{n+1}$, and the natural projection $\mathcal{L}_{n+1}(\mathbb{A}^1) \rightarrow \mathcal{L}_n(\mathbb{A}^1)$ corresponds to the map $R[t]/t^{n+2}R[t] \rightarrow R[t]/t^{n+1}R[t]$ that takes $P \in R[t]/t^{n+2}R[t]$ to $(P \mod t^{n+1})$, which, in turn, corresponds to the map $(T_0, \ldots, T_{n+1}) \rightarrow (T_0, \ldots, T_n)$ from $\mathbb{A}^{n+2}$ to $\mathbb{A}^{n+1}$. We conclude that the inverse limit of the system $\mathcal{L}_n(\mathbb{A}^1)$ coincides with the inverse limit of the spaces $\mathbb{A}^n$ with natural projections. The latter is the scheme $\mathbb{A}^\infty = \text{Spec } k[T_1, T_2, \ldots]$ (see e.g. [2] and references therein for a detailed treatment of $\mathbb{A}^\infty$, but note that all we will use here is its existence as a $k$-scheme).

1.7.2. We can also consider $\mathbb{A}^1$ with its additive group structure, that is, the group $\mathcal{G}_a$. Let $\mathcal{G}_a^{(N)}$ be the functor

$$R \to \{\text{elements of } R((t)) \text{ with poles of order } \leq N\}.$$  

An element of $R((t))$ with poles of order not greater than $N$ is nothing but a sequence of coefficients $(a_{-N}, \ldots, a_0, a_1, \ldots)$, where $a_i \in R$, $i = 1, 2, \ldots$; thus

$$\mathcal{G}_a^{(N)} \cong \text{Spec } k[T_{-N}, \ldots, T_0, \ldots] \cong \text{Spec } k[T_0, T_1, \ldots] = \mathcal{G}_a^{(0)} = \mathcal{L}(\mathcal{G}_a).$$  

An analogous argument works for $\mathbb{A}^d((t))$ with $d \in \mathbb{N}$. In particular, $(\mathbb{A}^d)^{(N)}$ is isomorphic over $k$ to $\mathcal{L}(\mathbb{A}^d)$ for all $N \in \mathbb{N}$. Denote this isomorphism by $S_N$.  


1.7.3. Recall the notations: \( F = k((t)) \), \( \mathbb{D} = \text{Spec} \ k[[t]] \). If \( R \) is a \( k \)-algebra, by \( R \)-points of a \( k \)-space we will simply mean the set which is an image of \( R \) (recall that a \( k \)-space is a functor from \( k \)-algebras to sets). In all that follows we will be mostly concerned with the set of \( k \)-points of \( \mathbb{A}^d((t)) \), because this set is in bijection with \( \mathbb{A}^d(F) \).

So far, we have described one way of thinking of \( \mathbb{A}^d((t))(k) \): a union of the sets of \( k \)-points of the schemes over \( k \) forming the directed system \( (\mathbb{A}^d)^{(N)} \). Each isomorphism \( S_N \) between \( (\mathbb{A}^d)^{(N)} \) and \( (\mathbb{A}^d)^{(0)} = \mathcal{L}(\mathbb{A}^d) \) induces a bijection on the sets of their \( k \)-points, shifting the indices of a power series corresponding to a given point by \( N \) to the right. We recall that \( \mathcal{L}(\mathbb{A}^d) = (\text{Spec} \ k[T_0, \ldots , T_n, \ldots ])^d \). Now observe that the set of \( k \)-points of \( \mathcal{L}(\mathbb{A}^d) \) is in natural bijection with the set of \( k[[t]] \)-points of the affine space \( \mathbb{A}^d \) as a scheme over \( k[[t]] \), that is, of \( \text{Spec} \ k[[t]][x_1, \ldots , x_d] \). This gives another, sometimes more convenient, way of looking at \( k \)-points of \( \mathbb{A}^d((t)) \).

Fix a positive integer \( N \) and consider the \( k[[t]] \)-morphism \( S_N \) from \( \text{Spec} \ k[[t]][u_1, \ldots , u_d] \) to \( \text{Spec} \ k[[t]][x_1, \ldots , x_d] \) (i.e., to itself), induced by the map of rings \( x_i \mapsto t_N u_i \), \( i = 1 \ldots d \). On \( k[[t]] \)-points (which are again viewed as \( d \)-tuples of power series with coefficients in \( k \)), this map induces multiplication by \( t^N \), that is, a shift of all indices to the right by \( N \). Observe that, even though it is not an injective map of \( k[[t]] \)-schemes, on \( k[[t]] \)-points it is an injection. Thus, if we take two copies of \( \mathbb{A}^d \) over \( k[[t]] \) and the morphism \( S_N \) between them, we can identify the set of \( k[[t]] \)-points of the image of \( S_N \) with \( k \)-points of \( \mathcal{L}(\mathbb{A}^d) \), and then the set of \( k[[t]] \)-points of the source copy of \( \mathbb{A}^d \) will be naturally identified with the set of \( k \)-points of \( (\mathbb{A}^d)^{(N)} \). This is an alternative description of the map induced on \( k \)-points of \( (\mathbb{A}^d)^{(N)} \) by the isomorphism \( S_N : (\mathbb{A}^d)^{(N)} \to \mathcal{L}(\mathbb{A}^d) \).

1.8. **Morphisms.** By definition, \( G((t)) \) is a \( k \)-space, that is, a functor. Then a morphism between two such objects is a morphism of functors (a natural transformation). However, we can use the fact that \( G((t)) \) is represented by an ind-scheme. By a morphism between two affine ind-schemes \( X = \lim X_i \) and \( Y = \lim Y_i \) we shall mean a map of sets \( \phi : X \to Y \) such that each \( \phi(X_i) \) is contained in some \( Y_j \), and the induced map \( X_i \to Y_j \) is a morphism of schemes.

Let \( G \) be an algebraic group. Then we can define an action of \( G(F) \) (the group of \( k((t)) \)-points of \( G \)) on the ind-group \( G((t)) \) by left or right translations in the same way as it is done for group schemes, see e.g. Section 4.2 of \( [3] \).

2. **A construction of the motivic measure on \( G((t)) \)**

We begin with a construction of an additively invariant motivic measure on the affine space \( \mathbb{A}^d((t)) \). Then we use the structure theory of \( G \) to reduce the problem of constructing a measure on \( G((t)) \) to the construction on \( \mathbb{A}^d((t)) \).
2.1. Haar measure on the affine space. The algebra of measurable subsets of the space $\mathfrak{L}(X)$ was defined in Appendix in [3] for any variety $X$. In particular, we have an algebra of measurable sets in $\mathfrak{L}(A^d)$. However, notice that in [3], the expression “a subset of a scheme” means a subset of the underlying topological space, whereas for us (as well as in [13]) a subset of $\mathfrak{L}(X)$ is a subset of the set of closed points of $X$ contained in $(\tilde{d})$.

Definition 11. We call a subset of $X$ of the schemes $A^d$ the algebra of sets defined in [6] with the set of closed points of $A$ contained in $(\tilde{d})$. Let $(\tilde{a})$ be a measurable subset of $\mathfrak{L}(X)$ by taking the intersection of all elements of the algebra of sets defined in [3] with the set of closed points of $\mathfrak{L}(X)$. In general, by a subset of an ind-scheme $X$ which is a direct limit of $k$-schemes $X^{(N)}$ we shall mean an increasing union of subsets of the sets of closed points of the schemes $X^{(N)}$.

Example 12. Let $\mathcal{X} = \mathfrak{L}([\mathbb{G}_a])$, and denote the corresponding motivic measure (from Proposition 11) by $\mu_\mathcal{X}$. Consider a decreasing filtration of $[\mathbb{G}_a]$ by the subsets $t^n[[t]]$, $n = 0, 1, \ldots$. Denote the corresponding algebraic subsets of $\mathfrak{L}([\mathbb{G}_a])$ by $B_n$, so that the set of $k$-points of $B_n$ is $t^n[[t]]$. Let us calculate their volumes. The set $B_n$ $(n \in \mathbb{N})$ is precisely the fiber of $\mathfrak{L}([\mathbb{G}_a])$ over the point $0_{n-1} = (0, \ldots, 0) \in \mathfrak{L}_{n-1}([\mathbb{G}_a])$. Hence, by definition,

$$\tilde{\mu}_\mathcal{X}(B_n) = \mathbb{L}^{-n+1}([\pi_{n-1}(B_n)]) = \mathbb{L}^{-n+1}([\text{pt}])$$

$$= \mathbb{L}^{-n+1}1 = \mathbb{L}^{-n+1}.$$

The total volume $\tilde{\mu}_\mathcal{X}(\mathfrak{L}([\mathbb{G}_a]))$ is by definition $[\mathbb{A}^1]\mathbb{L}^0 = \mathbb{L}$, so we have

$$\tilde{\mu}_\mathcal{X}(B_n) = \mathbb{L}^{-n}\tilde{\mu}_\mathcal{X}(\mathfrak{L}([\mathbb{G}_a])).$$

2.1.1. Bounded measurable subsets form an algebra of sets (closed only under finite unions, though). In order to define a measure on this algebra, we need to calculate the volumes of some special subsets of $\mathfrak{L}(A^d)$. We do it in the case $d = 1$ first.

Example 12. Let $\mathcal{X} = \mathfrak{L}([\mathbb{G}_a])$, and denote the corresponding motivic measure (from Proposition 11) by $\mu_\mathcal{X}$. Consider a decreasing filtration of $[\mathbb{G}_a]$ by the subsets $t^n[[t]]$, $n = 0, 1, \ldots$. Denote the corresponding algebraic subsets of $\mathfrak{L}([\mathbb{G}_a])$ by $B_n$, so that the set of $k$-points of $B_n$ is $t^n[[t]]$. Let us calculate their volumes. The set $B_n$ $(n \in \mathbb{N})$ is precisely the fiber of $\mathfrak{L}([\mathbb{G}_a])$ over the point $0_{n-1} = (0, \ldots, 0) \in \mathfrak{L}_{n-1}([\mathbb{G}_a])$. Hence, by definition,

$$\tilde{\mu}_\mathcal{X}(B_n) = \mathbb{L}^{-n+1}([\pi_{n-1}(B_n)]) = \mathbb{L}^{-n+1}([\text{pt}])$$

$$= \mathbb{L}^{-n+1}1 = \mathbb{L}^{-n+1}.$$

The total volume $\tilde{\mu}_\mathcal{X}(\mathfrak{L}([\mathbb{G}_a]))$ is by definition $[\mathbb{A}^1]\mathbb{L}^0 = \mathbb{L}$, so we have

$$\tilde{\mu}_\mathcal{X}(B_n) = \mathbb{L}^{-n}\tilde{\mu}_\mathcal{X}(\mathfrak{L}([\mathbb{G}_a])).$$

2.1.2. Now we can define a motivic measure on $\mathfrak{L}(\mathbb{G}_a((t)))$. We keep the notation of the previous example. Let $A$ be a measurable subset of $\mathfrak{L}(\mathbb{G}_a^{(N)})$, i.e., its image $B = S_N(A)$ in $\mathfrak{L}(\mathbb{G}_a^{(0)}) = \mathfrak{L}(\mathbb{A}^1)$ is measurable. Then we set

$$\mu_\mathcal{X}(A) = \mathbb{L}^N\tilde{\mu}_\mathcal{X}(B).$$

On the level of rings, the inclusion $\mathfrak{G}_a^{(N-1)} \hookrightarrow \mathfrak{G}_a^{(N)}$ corresponds to the map induced by $T_{-N} \mapsto 0$ from $k[T_{-N}, T_{-N+1}, \ldots]$ to $k[T_{-N+1}, \ldots]$. The map $S_N$ identifies the scheme $\mathfrak{G}_a^{(N)}$ with $\mathfrak{L}(\mathfrak{G}_a)$, and therefore the image of its subset $\mathfrak{G}_a^{(N-1)}$ maps isomorphically onto the fiber of $\mathfrak{L}(\mathfrak{G}_a)$ over 0, that
is, the set $B_1$. Similarly, for $M < N$, $S_N \left( \mathcal{G}_a^{(N)} \right) = B_{N-M}$. Then the relation \( \text{[1]} \) guarantees that the volume $\mu_a(\mathcal{G}_a^{(N)})$ is well defined. A similar calculation applied to an arbitrary measurable subset of $\mathcal{G}_a^{(N)}$ would show that the measure $\mu_a$ is well defined.

**Remark 13.** It is possible to arrive at the same conclusions without writing down the sets $B_n$ and their volumes explicitly, but by using the transformation rule and the following lemma.

**Lemma 14.** The order of Jacobian $\text{ord}_t J_{\tilde{S}_N}(\gamma)$ of the map $\tilde{S}_N : \mathcal{L}(\mathbb{A}^d) \to \mathcal{L}(\mathbb{A}^d)$ is equal to $Nd$ for all $\gamma \in \mathcal{L}(\mathbb{A}^d)$.

**Proof.** As in \[1.7.3\], we think of the closed points of $\mathcal{L}(\mathbb{A}^d)$ as sections of the structure morphism of the scheme Spec $k[[t]][x_1, \ldots, x_d]$ over $k[[t]]$. We have the map $\tilde{S}_N : \text{Spec } B \to \text{Spec } A$, where $A = k[[t]][x_1, \ldots, x_d]$, $B = k[[t]][u_1, \ldots, u_d]$, $x_i \mapsto t^N u_i$. There is an exact sequence of modules of differentials (\[8\], Section 16.1):

\[
\Omega_{A/k[[t]]} \otimes_A B \longrightarrow \Omega_{B/k[[t]]} \longrightarrow \Omega_{B/A} \longrightarrow 0.
\]

We see that $\Omega_{B/A}$ is a torsion $B$-module, and the above exact sequence is its free presentation. Hence $\text{Fitt}_0(\Omega_{B/A}) = (\det(t^N \text{Id})) = (t^{Nd}) \subset B$ by \[8\], Section 20.2. Therefore the Jacobian ideal of the map $\tilde{S}_N$ is the ideal sheaf $(t^{Nd})$ on Spec $B$. Let $\gamma : \text{Spec } k[[t]] \to \text{Spec } k[[t]][u_1, \ldots, u_d]$ be a section. The stalk of $\mathcal{J}_{\tilde{S}_N}$ at every point is the ideal $(t^{Nd})$ in the local ring of that point, i.e., it is an ideal of $k[[t]]$ embedded into the local ring of the point.

Any section $\gamma$ fixes $k[[t]]$ by definition, so the pullback of $\mathcal{J}_{\tilde{S}_N}$ to $k[[t]]$ by $\gamma$ is the ideal $(t^{Nd})$ itself. Thus $\text{ord}_t \mathcal{J}_{\tilde{S}_N}(\gamma) = Nd$. \(\square\)

2.1.3. Recall the notation: $\tilde{\mu}_a$ is the canonical measure on $\mathcal{L}(\mathbb{A}^d)$ (see Proposition \[3\]).

**Definition 15.** Let $A \subset (\mathbb{A}^d)^{(N)}$ be a bounded measurable subset. Then define

\[
\mu_a(A) = \mathbb{L}^{Nd} \tilde{\mu}_a(S_N(A)).
\]

**Lemma 16.** The measure $\mu_a$ is well defined and additively invariant.

**Proof.** The first statement is proved exactly the same way as in \[2.1.2\]. The invariance follows from the transformation rule, but it is also easy to check this statement by hand, using the explicit definition of the measure $\mu_a$ and the fact that translations are isomorphisms. \(\square\)

**Remark 17.** By invariance here we mean that the translates of bounded measurable subsets are again bounded measurable, of the same measure.

It is now possible to define the full algebra of measurable sets in $\mathbb{A}^d((t))$. 

 HAAR MEASURE
**Definition 18.** We call a subset \( B \subset \mathbb{A}^d((t)) \) measurable if it can be represented as a disjoint countable union of bounded measurable subsets \( B = \bigcup_{n \in \mathbb{N}} B_n \), such that the series of their measures \( \sum_{n=1}^{\infty} \mu_n(B_n) \) converges in the ring \( \mathcal{M} \). The measure of \( B \) is defined as \( \mu_n(B) = \sum_{n=1}^{\infty} \mu_n(B_n) \).

The proof that \( \mu_n(B) \) does not depend on a particular collection \( B_n \) mimics standard measure theory, with the use of a norm on \( \mathcal{M} \) introduced in the Appendix in [6]. It is easy to see that the measure \( \mu_n \) extended to the \( \sigma \)-algebra of measurable sets is still translation-invariant.

2.2. **Notation.** Let \( X \) be an affine variety, \( X((t)) \) – the ind-scheme defined as in [1.6], and \( U \) – a Zariski open subset of \( X \) with \( Z = X \setminus U \) closed. Then \( Z((t)) \) is a subfunctor of \( X((t)) \). By \( \mathcal{C}_X(U) \) we will denote the ind-scheme which is the direct limit of the schemes \( X^{(N)} \setminus Z^{(N)} \) – that is, the “complement of \( Z((t)) \) in \( X((t)) \)”. We shall denote by \( \mathcal{C}_X^0(U) \) the complement of \( Z[[t]] \) in \( X[[t]] \). Notice that \( \mathcal{C}_X^0(U) \) is not the same as \( U[[t]] \) – in general, it is much larger. By the construction, there is an inclusion morphism of ind-schemes \( \mathcal{C}_X(U) \hookrightarrow X((t)) \). Later we will slightly abuse the terminology by thinking of \( \mathcal{C}_X(U) \) as a measurable subset of \( X((t)) \), meaning that the set of closed points of \( \mathcal{C}_X(U) \) can be thought of as a subset of the set of closed points of \( X((t)) \).

Example 19. \( X = \mathbb{A}^1 \), \( Z = \{0\} \), \( U = X \setminus Z \). Then \( \mathcal{L}(U) \) is the set of \( B_1 \) from Example 12, that is, the fiber of \( \pi_X \) over \( 0 \in \mathcal{L}_0(X) \), so its motivic volume is different from the volume of \( X \). However, \( \mathcal{C}_X^0(U) \) is the complement of \( \mathcal{L}(Z) \) in \( \mathcal{L}(\mathbb{A}^1) \), that is, a complement of a single point, so the motivic volume of \( \mathcal{C}_X^0(U) \) coincides with the motivic volume of \( \mathbb{A}^1 \).

In this example, \( \mathcal{C}_X(U) = U((t)) \) is the functor that assigns to every ring \( R \) the set of Laurent series with coefficients in \( R \) such that at least one of the coefficients is a unit in \( R \). Also, notice that \( U((t)) \cap \mathcal{L}(X) = \mathcal{C}_X^0(U) \).

2.3. Once and for all, we choose the standard coordinates \( x_1, \ldots, x_d \) on \( \mathbb{A}^d \). Let \( \omega \) be a top degree differential form \( \omega = gdx_1 \wedge \cdots \wedge dx_d \) defined on a Zariski open subset \( U \subset \mathbb{A}^d \), where \( g \) is a regular function on \( U \). Then define the measure \( \mu_{|\omega|} \) on \( \mathcal{C}_{\mathbb{A}^d}(U) \) by

\[
\mu_{|\omega|}(A) = \int_A \mathbb{L}^{-\text{ord}_t(g \circ \gamma)} d\mu_\gamma(\gamma),
\]

where \( \mu_\gamma \) is the motivic measure on \( \mathbb{A}^d((t)) \), \( A \) is a bounded measurable set contained in \( \mathcal{C}_{\mathbb{A}^d}(U) \); \( \text{ord}_t(g \circ \gamma) \) for \( \gamma \in (\mathbb{A}^d)^{(N)} \) is the order of vanishing of the formal power series \( g(\gamma) \) at \( t = 0 \) (if the series has a pole at \( t = 0 \), the order is negative).

In this notation, the measure on \( \mathbb{A}^d((t)) \) defined in 2.1.3 is the one that corresponds to the form \( dx_1 \wedge \cdots \wedge dx_d \).
By definition of the measure $\mu_\alpha$, the integral in (3) can be written as
\begin{equation}
\mu_{|\omega|}(A) = \int_A \mathbb{L}^{-\ord_t(g\gamma)} d\mu_\alpha(\gamma) = \int_{S_N(A)} \mathbb{L}^{-\ord_t(\tilde{g}\gamma)+Nd} d\mu_\alpha(\gamma)
\end{equation}
for any $N \geq 0$, where $\tilde{g}(t^N x_1, \ldots, t^N x_d) = g(x_1, \ldots, x_d)$. In particular, since for a bounded set $A$ the number $N$ can be chosen big enough to ensure $S_N(A) \subset \mathbb{L}(\mathbb{A}^d)$, the motivic integral in the right-hand side of (3) exists (see [9]), and therefore the integral in (3) is also defined (we can use the right-hand side of (4) as its definition).

2.4. A coordinate system on the big cell. Let $G$ be a connected reductive algebraic group defined over $k$. Let $T \subset G$ be a maximal torus (recall that the field $k$ is assumed algebraically closed, so $T$ is automatically split), $m$ – its dimension, $\Delta$ – a choice of simple roots of the Lie algebra of $G$, $n$ – the cardinality of $\Delta$, $B \supset T$ – the Borel subgroup corresponding to $\Delta$, $U$ – its unipotent radical, $B^-$ – the opposite Borel subgroup with respect to $T$, $U^-$ – its unipotent radical. Then ([11], p.174), the product morphism is an isomorphism of algebraic varieties
\[ U^- \times T \times U \to \Omega', \]
where $\Omega' \subset G$ is a Zariski open subset (a big cell). For our purposes, it is more convenient to consider its conjugate, the set $\Omega = U^- \times U \times T$. The unipotent subgroup $U$ (respectively, $U^-$) is isomorphic to a cartesian product of root subgroups $U_\alpha$ corresponding to positive (respectively, negative) roots. Choose a generator for each $U_\alpha$, and denote it by $x'_\alpha$ if $\alpha$ is positive, and by $y'_\alpha$ if $\alpha$ is negative. Each $U_\alpha$ can be identified with a one-dimensional subspace $g_\alpha$ in the Lie algebra of $G$. Denote by $x_\alpha$ (resp., $y_\alpha$) the generator of $g_\alpha$ that corresponds to $x'_\alpha$ (resp., $y'_\alpha$) under this isomorphism. This defines a coordinate system on $U^- \times U$. Next, choose a coordinate system $s_1, \ldots, s_m$ on $T$ by representing it as a product of $m$ copies of $\mathbb{G}_m$ and choosing a coordinate $s_j$ on each of them. Hence we have defined a coordinate map $i : \Omega \to \mathbb{A}^d$, $d = 2n + m$. It is defined over $k$. The image of this map is the Zariski open subset of $\mathbb{A}^d$ defined by $s_1 \cdot \ldots \cdot s_m \neq 0$.

2.5. Let $\Omega$ be the big cell of $G$, as in the previous subsection. Denote by $Z$ the complement of $\Omega$ in $G$ – a constructible subset which is a union of a finite number of closed subvarieties of $G$ defined over $k$. Recall from [14] that the set of $F$-points of $G$ can be identified with the set of $k$-points of the ind-group $G((t))$, which is a direct limit of the system $(G^{(N)})_{N \geq 0}$. Under this bijection the set $\Omega(F)$ is identified with the set of $k$-points of $\mathfrak{C}_G(\Omega)$.

We recall from [2.2] that by definition $G((t)) = \mathfrak{C}_G(\Omega) \cup Z((t))$. Observe that the map $i$ from the previous subsection extends to a map from $\mathfrak{C}_G(\Omega)$ to $\mathbb{A}^d((t))$; it is still a map over $k$, and we will denote it by the same letter $i$. 
Let $\omega$ be a 1-form on $\Omega$ that is defined by the following expression in the coordinates $(x, y, s)$ defined in 2.4:

$$\omega = dx_1 \wedge \cdots \wedge dx_n \wedge dy_1 \wedge \cdots \wedge dy_n \wedge \frac{ds_1}{s_1} \wedge \cdots \wedge \frac{ds_m}{s_m} = dx \wedge dy \wedge ds.$$  

(5)

Lemma 20. The form $\omega$ is invariant under left and right translations on $G$.

We omit the proof.

Recall that by a subset of the ind-scheme $G((t))$ we mean a union of subsets of closed points of the schemes $G^{(N)}$. Now we are ready to define a motivic measure on $G((t))$.

Definition 21. Let $B$ be a subset of $G((t))$. We say that $B$ is $\Omega$-measurable if $B$ can be represented as a (disjoint) union $B = C \cup A$, where $C \subset Z((t))$ and $A$ is a measurable subset of $\mathcal{C}_G(\Omega)$. Here we say that a subset $A$ of $\mathcal{C}_G(\Omega)$ is measurable if its image $i(A)$ is a measurable subset of $\mathbb{A}^d((t))$. For $B = C \cup A$ measurable, set

$$\mu_\Omega(B) = \mu_{|(i^{-1})^*(\omega)|}(i(A)).$$

(6)

We call a measurable subset bounded, if it is contained in $\mathcal{C}_G(\Omega)$ and its image under the map $i$ is a bounded measurable subset of $\mathbb{A}^d((t))$.

Proposition 22. Let $g$ be an element of $G(F)$, and let $A$ be a bounded $\Omega$-measurable set in $\mathcal{C}_G(\Omega)$ such that $g^{-1}A$ is also contained in $\mathcal{C}_G(\Omega)$ and bounded. Then $\mu_\Omega(A) = \mu_\Omega(g^{-1}A)$.

Proof. Let us denote by $L_g$ the left translation by $g$ viewed as an automorphism of $G$ defined over the field $F$. On the open subset $\Omega \cap g^{-1}\Omega$ it can be represented as a rational map in the coordinates $x, y, s$ that were defined in 2.4. We denote this map by $h(x, y, s)$, and its Jacobian matrix by $J$. More precisely, $h$ is a birational map from $\mathbb{A}^d$ to $\mathbb{A}^d$ over $F$ defined by the formula $h(x, y, s) = i(L_g(i^{-1}(x, y, s)))$. Thus $J$ is an $F$-valued regular function on $\Omega$, and by Lemma 20 we have

$$p(h(x, y, s)) \cdot \det J \cdot dx \wedge dy \wedge ds = p(x, y, s)dx \wedge dy \wedge ds,$$

(7)

where $p(x, y, s) = 1/s_1 \ldots s_m$. Now the goal is to represent the restriction of the map $L_g$ to the given set $g^{-1}A$ as a restriction of a $k[[t]]$-morphism of $\mathbb{D}$-varieties, so that the transformation rule for motivic measures can be applied to it.

The sets $A$ and $g^{-1}A$ are both contained in $\mathcal{C}_G(\Omega)$ and are bounded by assumption. By definition, this means that $i(A)$ is a measurable subset of $(\mathbb{A}^d)^{(N)}$ for some $N \geq 0$, and that $i(g^{-1}A)$ is defined and is contained in $(\mathbb{A}^d)^{(M)}$ for some $M \geq 0$. We choose both integers $M$, $N$ to be minimal possible. Also, we can assume without loss of generality that $A$ is stable.
We will need the expression $h(t^{-M}x, t^{-M}y, t^{-M}s)$. We write it in the form

$$h(t^{-M}x, t^{-M}y, t^{-M}s) = \left( \tilde{f}_1(x, y, s), \ldots, \tilde{f}_d(x, y, s) \right),$$

where $\tilde{f}_i, i = 1, \ldots, d$ and $\Delta$ are in $k[[t]][x, y, s]$, and $\gcd(\tilde{f}_1, \ldots, \tilde{f}_d, \Delta) = 1$.

Let us break up the set $A$ according to the order of vanishing of $\Delta$ on $S_M(i(g^{-1}A))$:

$$A = \bigcup_{e \geq 0} A_e,$$
$$A_0 = \{ \gamma \in A \mid \text{ord}_t \Delta(S_M \circ i \circ g^{-1}\gamma) \leq 0 \};$$
$$A_e = \{ \gamma \in A \mid \text{ord}_t \Delta(S_M \circ i \circ g^{-1}\gamma) = e \} \text{ for } e \geq 1.$$

Now we are ready to construct, for each $e = 0, 1, \ldots$, a scheme $\mathcal{X}_e$ over $\mathbb{D}$ and two $\mathbb{D}$-morphisms $H_1$ and $H_2$ from $\mathcal{X}_e$ to $\mathbb{A}^d[[t]]$, such that the the following conditions hold:

(i) There exists a measurable subset $B$ of $(\mathcal{X}_e)_{\infty}$ such that $H_1$ induces a bijection between $B$ and $S_M(i(g^{-1}A_e))$.

(ii) The morphism $H_2$ induces a bijection between $B$ and $S_e(i(A_e))$.

(iii) The following diagram (of maps of sets) commutes:

$$\begin{array}{cccc}
S_M(i(g^{-1}A_e)) & \xrightarrow{H_1} & B & \xrightarrow{H_2} & S_e(i(A)) \\
\downarrow S_M & & \downarrow h & & \downarrow S_e \\
i(g^{-1}A_e) & & i(A_e) & & \end{array}$$

Define the scheme $\mathcal{X}_e$ to be

$$\mathcal{X}_e = \text{Spec } k[[t]][x_1, \ldots, x_n, y_1, \ldots, y_n, s_1, \ldots, s_m, z]/(z\Delta - t^e).$$

Let $H_1 : \mathcal{X}_e \rightarrow \mathbb{A}^d[[t]] = \text{Spec } k[[t]][u_1, \ldots, u_d]$ be the morphism of schemes induced by the identity map on the first $d$ variables:

$$u_i \mapsto x_i, 1 \leq i \leq n; u_i \mapsto y_{i-n}, n + 1 \leq i \leq 2n; u_i \mapsto s_{i-2n}, 2n < i \leq d.$$

When $e = 0$, the map $H_1$ is nothing but the inclusion morphism of the open subset of $\mathbb{A}^d[[t]]$ defined by $\Delta \neq 0$ into $\mathbb{A}^d[[t]]$.

Let $H_2 : \mathcal{X}_e \rightarrow \mathbb{A}^d[[t]] = \text{Spec } k[[t]][u_1, \ldots, u_d]$ be the morphism defined by

$$u_i \mapsto z\tilde{f}_i(x, y, s), \quad i = 1, \ldots, d.$$
Naturally, $H_1$ induces a bijection on $k[[t]]$-points. Let $B \subset (X_e)_\infty$ be the preimage of the set $S_M(i(g^{-1}A_e))$ under this bijection. Then it immediately follows from the definition of $H_2$ that the property (iii) holds (recall that $S_M$ is a bijection between the sets $i(g^{-1}A_e)$ and $S_M(i(g^{-1}A_e))$; $h(t^{-M}x, t^{-M}y, t^{-M}s) = \left(\frac{k}{x}, \frac{k}{y}\right)$, and $z = \frac{e}{M}$). Since the map $h$ is a coordinate expression of a translation by a group element, it is a bijection; thus the commutativity of the diagram implies that $H_2$ induces a bijection between the set $B$ and the set $S_e(i(A_e))$.

In the case $e = 0$ the scheme $X_e$ is smooth over $\mathbb{D}$. For $e > 0$, $X_e$ has smooth generic fiber, and the singular locus in its closed fiber is defined by the equations $\Delta(x, y, s) = z = 0$. We observe that the $z$-coordinate of $\gamma(o)$ (the image of the closed point of $\mathbb{D}$) is not equal to zero for any element $\gamma$ of the set $B$ since $\Delta$ is assumed to vanish exactly up to order $e$ on the image of $\gamma$ in $S_M(i(g^{-1}A_e))$. That is, $\gamma(o)$ does not lie in the singular locus of the closed fiber of $X_e$. Since $A$ is assumed to be stable, the set $B$ is also stable: the condition $\text{ord}_\Delta(\gamma) = e$ depends only on the $e$-jet of $\gamma$.

Indeed, the stability of $A$ implies the stability of all the sets $A_e$. The only formal difference between $B$ and $A_e$ is that the points in $B$ have an extra coordinate $z = z_0 + z_1t + \cdots + z_n t^n + \ldots$, and satisfy an extra equation $z\Delta(x, y, s) = t^e$. By our assumption on $A_e$ and by the definition of $B$, the order of $\Delta/t^e$ is equal to 0. Hence, if $n > e$, each equation in $z_{n+1}$ of the form $(z_0 + \cdots + z_{n+1} t^{n+1} + \ldots)\Delta(x, y, s) = t^e + t^{n+1}g(t)$, $g(t) \in k[[t]]$ with fixed $x, y, s$ and fixed $z_0, \ldots, z_n$ has a unique solution. Therefore, the set $B$ is stable at level $e$ or the level of $A$, whichever is greater.

It follows now from Lemma 9 that the transformation rule can be applied to the restriction of the morphisms $H_1$ and $H_2$ to the set $B$. Let us denote the motivic measure on $X_e$ that was defined in Section 1.3 by $d\mu_e$, and the motivic measure on $k[[t]]$ by $d\mu_a$, as before. By the transformation rule, we have

$$d\mu_a|_{S_M(i(g^{-1}A_e))} = \mathbb{L}^{-\text{ord}J_H1}d\mu_e|_B,$$

$$d\mu_a|_{S_e(i(A_e))} = \mathbb{L}^{-\text{ord}J_H2}d\mu_e|_B.$$  

It remains to calculate $J_{H_1}$ and $J_{H_2}$. We start with the Jacobian of $H_1$. Let $R_1$ be the ring $R_1 = k[[t]][u_1, \ldots, u_d]$, and let $R_2 = k[[t]][x_1, \ldots, x_n, y_1, \ldots, y_n, s_1, \ldots, s_m, z]/(z\Delta - t^e)$. By definition, $J_{H_1}$ is the 0-th Fitting ideal of the module $\Omega_{R_2/R_1}$, where the map $R_1 \to R_2$ is given by the formula (9). We have the exact sequence

$$0 \to \Omega_{R_1/k[[t]]} \otimes_{R_1} R_2 \to \Omega_{R_2/k[[t]]} \to \Omega_{R_2/R_1}$$

Hence, $\Omega_{R_2/R_1}$ is in this case a torsion $R_2$-module isomorphic to $R_2[\sigma]/\sigma\Delta$. Its 0th Fitting ideal is $(\Delta)$. Notice that by the remark on definition of the set $B$ earlier in this proof, $\text{ord}_\Delta(\gamma^*\Delta) = e$ for all $\gamma \in B$.

Let us now calculate the Jacobian of $H_2$. The rings $R_1$ and $R_2$ remain the same, but the map $R_1 \to R_2$ is given by the formula (10) now. Then $\Omega_{R_2/R_1}$ is the $R_2$-module generated over $R_2$ by the formal symbols
dx_1, \ldots, dx_n, dy_1, \ldots, dy_m, ds_1, \ldots, ds_m, dz$ with the relations obtained by setting to zero the formal derivatives of the polynomials $z \Delta$ and $z \tilde{f}_i(x, y, s)$, $i = 1, \ldots, d$. Hence, by definition of the Fitting ideal, the 0th Fitting ideal of this module is generated by the following $(d + 1) \times (d + 1)$-determinant:

$$
\begin{vmatrix}
\frac{\partial f_1}{\partial x_1} & \ldots & \frac{\partial f_1}{\partial x_d} & \frac{\partial f_1}{\partial s_1} & \frac{\partial f_1}{\partial s_m} \\
\frac{\partial f_1}{\partial x_2} & \ldots & \frac{\partial f_1}{\partial x_d} & \frac{\partial f_1}{\partial s_1} & \frac{\partial f_1}{\partial s_m} \\
\vdots & \cdots & \vdots & \vdots & \vdots \\
\frac{\partial f_1}{\partial s_1} & \ldots & \frac{\partial f_1}{\partial s_m} & 0 & 0 \\
\frac{\partial f_1}{s_d} & \ldots & \frac{\partial f_1}{s_m} & \Delta & \Delta \\
\end{vmatrix} = z^d
$$

By the formula (8), the latter determinant is equal to $z^d \Delta^d (t^{-Md} \det J) \Delta$, where $\det J$ is the Jacobian determinant of the map $h$ that was defined in the beginning of the proof (we are using the equality $\frac{\partial f}{\partial x} - \frac{\partial f}{\partial x} \Delta = \Delta (\frac{\partial f/\Delta}{\partial x})$). Finally, we see that the Jacobian ideal of the map $H_2$ is the ideal $(t^{(e-M)d} \det J \Delta)$.

Let $\tilde{p}(t^M x, t^M y, t^M z) = p(x, y, z) = 1/s_1 \ldots s_m = \tilde{p}(t^e x, t^e y, t^e z)$. With these notations, by (9), (11), and (12), get:

$$
\mu_{\Omega}(g^{-1} A_e) = \mathbb{L}^M \int_{S_M(i(\mathcal{A}_e))} \mathbb{L}^{-\text{ord}_t \tilde{p}^e} d\mu_s(\gamma) = \mathbb{L}^M \int_B \mathbb{L}^{-\text{ord}_t \tilde{p}^e H_1(\gamma) - \text{ord}_t \mathcal{J}_{H_1}(\gamma)} d\mu_s(\gamma);
$$

$$
\mu_{\Omega}(A_e) = \mathbb{L}^{ed} \int_B \mathbb{L}^{-\text{ord}_t \tilde{p}^e H_2(\gamma) - \text{ord}_t \mathcal{J}_{H_2}(\gamma)} d\mu_s(\gamma).
$$

It remains to compare the subintegral expressions. We need to show that

$$
M - (\text{ord}_t \tilde{p} \circ H_1(\gamma) + \text{ord}_t \mathcal{J}_{H_1}(\gamma)) = e - (\text{ord}_t \tilde{p} \circ H_2(\gamma) + \text{ord}_t \mathcal{J}_{H_2}(\gamma))
$$

for $\gamma \in B$. This equality immediately follows from (7) and the formulas for $\mathcal{J}_{H_1}$ and $\mathcal{J}_{H_2}$.

We have shown that $\mu_{\Omega}(A_e) = \mu_{\Omega}(g^{-1} A_e)$ for $e = 0, 1, \ldots$. Hence, by the additivity of the measure, $\mu_{\Omega}(A) = \mu_{\Omega}(g^{-1} A)$.

**Theorem 23.** The measure $\mu_{\Omega}$ is translation-invariant (both on the left and on the right).

**Proof.** We will prove left-invariance; right-invariance is proved identically. Let $A$ be an $\Omega$-measurable subset of $G(t)$, and $g \in G(F)$. We need to show that $\mu_{\Omega}(A) = \mu_{\Omega}(g^{-1} A)$. We can assume that $A$ is bounded $\Omega$-measurable without loss of generality, since any unbounded $\Omega$-measurable set by definition can be represented as a countable disjoint union of bounded $\Omega$-measurable sets.
Let us break up the set \( g^{-1}A \) according to the maximal order of pole of the coordinates of its points: \( g^{-1}A = \bigcup_{n=0}^\infty B_n \cup B_\infty \), where

\[
B_0 = g^{-1}A \cap \mathcal{L}(G), \\
B_n = \{ \gamma \in g^{-1}A \mid \gamma \in \mathcal{C}_G(\Omega) ; i(\gamma) \in (\mathbb{A}^d)^{(N)} \setminus (\mathbb{A}^d)^{(N-1)} \}, n \geq 1, \\
B_\infty = \{ \gamma \in g^{-1}A \mid \gamma \notin \mathcal{C}_G(\Omega) \}.
\]

Then \( \mu_\Omega(gB_n) = \mu_\Omega(B_n) \) for \( n \geq 0 \) by Proposition 22, \( \mu_\Omega(B_\infty) = 0 \) by definition. It remains to show that \( \mu_\Omega(gB_\infty) = 0 \): then we will have

\[
\mu_\Omega(g^{-1}A) = \sum_{n=1}^\infty \mu_\Omega(B_n) + \mu_\Omega(B_\infty) = \sum_{n=1}^\infty \mu_\Omega(gB_n) + \mu_\Omega(gB_\infty) = \mu_\Omega(A).
\]

The set \( gB_\infty \) is contained in the set \( E = gZ([t]) \cap \mathcal{C}_G(\Omega) \), so it suffices to show that the set \( E \) has measure 0. We can represent it as a disjoint union of bounded subsets of \( \mathcal{C}_G(\Omega) \): \( E = \bigcup_{N=0}^\infty E_N \) with \( E_0 = E \cap \mathcal{L}(G) \) and \( E_N = E \cap (\Omega(N) \setminus \Omega(N-1)) \) for \( N \geq 1 \). It remains to observe that \( S_N(i(E_N)) \) is well defined and it is a locally closed subscheme of \( \mathcal{L}(\mathbb{A}^d) \). Its relative dimension over \( k[[t]] \) is less than \( d \), and therefore by definition of the measure on the affine space we have \( \mu_\Omega(S_N(i(E_N))) = 0 \). This implies \( \mu_\Omega(E_N) = 0 \) for all \( N \geq 1 \); hence \( \mu_\Omega(E) = 0 \).

**Corollary 24.** The algebra of \( \Omega \)-measurable sets and the measure \( \mu_\Omega \) itself do not depend on the choice of the torus \( T \) or the set of positive roots (that is, \( \Omega \) can be dropped from the notation).

**Proof.** Follows from the theorem and the fact that all the big cells are conjugate in \( G \) over \( k \) (recall that we are assuming \( k \) to be algebraically closed).

2.6. As stated in the introduction, the goal was to define a motivic measure on \( G((t)) \) that would extend the canonical motivic measure on \( \mathcal{L}(G) \). The following theorem shows that we have achieved it.

**Theorem 25.** Let \( \Omega \) be any big cell in the group \( G \). Then \( \mathcal{L}(G) \) is \( \Omega \)-measurable, and the restriction of \( \mu_\Omega \) to \( \mathcal{L}(G) \) coincides with the canonical motivic measure on \( \mathcal{L}(G) \).

**Proof.** Let us denote the canonical motivic measure on \( \mathcal{L}(G) \) by \( \mu_G \). Denote the complement of \( \Omega \) in \( G \) by \( Z \), as before. First, notice that \( \mu_G(\mathcal{L}(Z)) = 0 \) by the axioms of the canonical measure; \( \mathcal{L}(G) = \mathcal{L}(Z) \cup (\mathcal{C}_G(\Omega) \cap \mathcal{L}(G)) \), and \( \mu_\Omega(Z) = 0 \) by definition of \( \mu_\Omega \). Therefore, we only need to show that the restrictions of \( \mu_\Omega \) and \( \mu_G \) to \( \mathcal{C}_G(\Omega) \cap \mathcal{L}(G) \) coincide (and are defined on the same algebra of sets).

Consider the multiplication map \( U^{- \times} U \times T \to G \) over \( k \). This map is an isomorphism between \( \mathbb{A}^n \times \mathbb{A}^n \times \mathbb{G}_m \) and \( \Omega \) over \( k \). It induces an isomorphism (over \( k[[t]] \)) of the arc spaces: \( \mathcal{L}(\mathbb{A}^n) \times \mathcal{L}(\mathbb{A}^n) \times \mathcal{L}(\mathbb{G}_m) \to \mathcal{L}(\Omega) \). If we apply the transformation rule to this isomorphism, we immediately obtain that the
restrictions of \( \mu_\Omega \) and \( \mu_G \) to \( \Sigma(\Omega) \) coincide, by Example 8 and the observation that \( \text{ord}_t(s_1 \ldots s_m) = 1 \) on \( \mathbb{C}^m[[t]] \).

Since \( \Sigma(\Omega) \) is a smaller set than \( \Sigma_G(\Omega) \cap \Sigma(G) \), the equality between the two measures restricted to \( \Sigma(\Omega) \) is not enough. However, we claim that a finite number of translates of \( \Sigma(\Omega) \) cover the whole arc space of \( G \), and then the theorem follows immediately.

The claim can be proved, for example, as follows. At first consider the situation over \( k \). All possible big cells cover the group \( G(k) \) (recall that \( k \) is assumed algebraically closed, and hence even Borel subgroups cover \( G(k) \)). Since \( G \) is quasicompact in Zariski topology, and the big cells are Zariski open, there exists a finite subcover by some big cells \( \Omega_1(k) \ldots \Omega_n(k) \). The arc space \( \Sigma(G) \) itself is stable at level 0 since \( G \) is a smooth variety, and so are \( \Sigma(\Omega_1) \ldots \Sigma(\Omega_n) \), by Remark 3. In particular, \( \Sigma(G) = \pi_0^{-1}(G) \), \( \Sigma(\Omega_i) = \pi_0^{-1}(\Omega_i) \), \( i = 1, \ldots, n \). It follows that \( \bigcup_{i=1}^n \Sigma(\Omega_i) = \Sigma(G) \). Hence, any \( \mu_G \)-measurable subset \( A \) of \( \Sigma(G) \) can be broken up into a disjoint union \( A = \bigcup_{i=1}^n A_i \) with \( A_i \subset \Sigma(\Omega_i) - \Omega_i \)-measurable. By Corollary 24, any \( \Omega_i \)-measurable set is also \( \Omega \)-measurable, and \( \mu_\Omega(A_i) = \mu_G(A_i) \) for any \( i = 1, \ldots, n \). On the other hand, we have shown in the beginning of this proof that \( \mu_\Omega(A_i) = \mu_G(A_i) \). Hence, \( \mu_\Omega(A) = \sum_{i=1}^n \mu_G(A_i) = \mu_G(A) \).

**Remark 26.** 1. It is possible to construct explicitly the finite number of translates of the given big cell \( \Omega \) that cover \( \Sigma(G) \). It can be done by means of Bruhat decomposition and the following statement (\( \clubsuit \), Section 2.1, p.43): if \( w, s \) are elements of the Weyl group of \( G \) satisfying \( l(s) = 1 \) and \( l(sw) = l(w) + 1 \), and \( n \in G \) is a representative of \( s \), then \( nBwB \) is contained in \( B(sw)B \) (here \( B \) is a fixed Borel subgroup, and \( l(w) \) stands for length of \( w \)).

2. The statement of the last theorem can be proved directly by a Jacobian calculation in a way similar to the proof of Proposition 22. Namely, after having established the equality of the two measures on \( \Sigma(\Omega) \), we could subdivide the remaining part of \( \Sigma(G) \cap \Omega((t)) \) into a disjoint union of subsets according to the order of pole of \( \Omega \)-coordinates of its elements, and then repeat the procedure described in Proposition 22: construct an auxiliary \( \mathbb{D} \)-variety corresponding to each piece with a given order of pole and a \( k[[t]] \)-morphism from it to \( \Sigma(G) \) which corresponds to the natural inclusion of the big cell into \( G \). A complicated calculation shows that the Jacobian ideal of this morphism coincides with the principal ideal generated by \( (s_1 \ldots s_m) \) (recall that \( s_1, \ldots, s_m \) are the coordinates of the torus component of the given element of the big cell). Then the statement follows from the Jacobian transformation rule applied to this morphism.

**2.7. Concluding remarks.** Finally, I would like to mention briefly a few closely related questions which have not been discussed so far, and which I hope to return to in the future.

**2.7.1. Uniqueness.** The classical Haar measure is unique up to a scalar multiple. The canonical motivic measure on \( \Sigma(G) \) is unique because it is normalized in such a way that it projects to the tautological measure on the
variety \( G \). Our construction of the motivic measure on \( G((t)) \) gives an answer that does not depend on the choice of the big cell (Corollary 24) and coincides with the unique motivic measure on \( \mathfrak{L}(G) \). However, at the moment I have no proof (and not even a precise formulation) of a general uniqueness statement.

2.7.2. The assumptions that the ground field \( k \) is algebraically closed and has characteristic 0 were adopted because we followed the exposition of [15] where these assumptions were made. However, it should be possible to extend our result without any difficulty to the case when \( k \) is not algebraically closed but the group \( G \) is assumed split over \( F \). It would also be interesting to construct a motivic Haar measure for reductive groups that are not split over \( F \).

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