Abstract. We introduce and study skew product Smale endomorphisms over finitely irreducible shifts with countable alphabets. This case is different from the one with finite alphabets and we develop new methods. In the conformal context we prove that almost all conditional measures of equilibrium states of summable Hölder continuous potentials are exact dimensional and their dimension is equal to the ratio of (global) entropy and Lyapunov exponent. We show that the exact dimensionality of conditional measures on fibers implies global exact dimensionality of the original measure. We then study equilibrium states for skew products over expanding Markov–Rényi transformations and settle the question of exact dimensionality of such measures. We apply our results to skew products over the continued fraction transformation. This allows us to extend and improve the Doeblin–Lenstra conjecture on Diophantine approximation coefficients to a larger class of measures and irrational numbers.

Key words: skew product endomorphisms, exact dimensional measures, conditional measures, natural extensions, continued fractions

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1. Introduction

We introduce and explore skew product Smale endomorphisms modeled on subshifts of finite type with countable alphabets. We study the thermodynamic formalism for skew product Smale endomorphisms over countable-to-1 maps, in particular natural extensions of countable-to-1 endomorphisms, such as EMR (expanding Markov–Rényi) maps, Gauss maps, etc. The case of countable alphabet is different from the one with finite alphabet, since limit sets are usually non-compact and new methods are needed to prove exact dimensionality of measures. Also, our notion of Smale space is different from, although inspired by, the respective notion from [41].

One of our objectives is to develop the thermodynamic formalism of such dynamical systems. In order to do this, we first recall in §2 the foundations of thermodynamics formalism of one-sided subshifts of finite type modeled on a countable (either finite or infinite) alphabet, from [15, 16]. Passing on to the two-sided shifts in §3, we provide a thermodynamic formalism of Hölder continuous summable potentials with respect to two-sided subshifts of finite type. This also includes a characterization of Gibbs states in terms of conditional measures, which has no counterpart for one-sided shifts.

We then define in §4 skew product Smale endomorphisms, modeled on countable alphabet subshifts of finite type, and specify their several significant subclasses. We define them in §5 and study them in §6. In Theorem 6.2, we show that projections of almost everywhere conditional measures of equilibrium states of summable Hölder continuous potentials are exact dimensional and their dimension is the ratio of global entropy and Lyapunov exponent. In the proof we develop new methods suited for the countable alphabet case.

We prove in Theorem 7.3 a version of Bowen’s formula giving the Hausdorff dimension of each fiber as the zero of a pressure function; we deal also with the case when the pressure function has no zero. Exact dimensional measures have a long history, being studied in many settings, for example [2, 8, 9, 16, 22, 30, 32, 34, 35, 48], to name a few.

Then, in §8, we pass to general skew products over countable-to-1 endomorphisms. For endomorphisms, the study of Hausdorff dimension is in general different than for invertible systems and specific phenomena appear (see for example [9, 21, 22, 25–29, 40]). We prove, under a condition of $\mu$-injectivity for the coding of the base map, the exact dimensionality of conditional measures of equilibrium measures in stable manifold fibers, building on [22].
We consider general skew product endomorphisms $F : X \times Y \to X \times Y$ of the form

$$F(x, y) = (f(x), g(x, y))$$

over countable-to-1 endomorphisms $f : X \to X$, where $X$ is a metric space (not only $E^1_+\mathbb{A}$) and $Y \subset \mathbb{R}^d$. Then $f$ is coded by a symbol space with countably many symbols and we find in Theorem 8.4 a closed formula for pointwise dimensions of conditional measures on fibers of $F$. Then, in Theorem 8.6, we prove that, if the conditional measures of an equilibrium measure $\mu_\phi$ on fibers are exact dimensional, and if the projection of $\mu_\phi$ in the base is also exact dimensional, then $\mu_\phi$ is exact dimensional globally.

Next, we study several main classes of skew product endomorphisms over countable-to-1 maps, in particular natural extensions (inverse limits); for dynamics on inverse limits, see for example [20–22, 24, 25, 27, 40]. In §9, we study EMR maps $f : I \to I$ and conformal Smale skew product endomorphisms $F : I \times Y \to I \times Y$ over $f$. In Theorem 9.3, we prove exact dimensionality of conditional measures on fibers of $F$. In particular, we consider the continued fraction Gauss map

$$G(x) = \left\{ \frac{1}{x} \right\}, \quad x \in (0, 1],$$

coded by a countable alphabet; and the Manneville–Pomeau maps

$$f_\alpha(x) = x + x^{1+\alpha} \pmod{1}, \quad x \in [0, 1],$$

$\alpha > 0$. In Theorem 9.6, we prove that a class of equilibrium measures are exact dimensional globally on $I \times Y$.

In §10, we apply our results to Diophantine approximation of irrational numbers $x$ and we generalize the Doeblin–Lenstra conjecture about the approximation coefficients $\Theta_n(x)$ in continued fraction representation to equilibrium measures $\mu_s$ of potentials

$$-s \log |G| : [0, 1] \to \mathbb{R}, \quad s > 1/2,$$

where we recall that $G$ is the Gauss map $G(x) = \{1/x\}, x \in (0, 1]$. If the continued fraction representation of an irrational number $x \in [0, 1)$ is

$$x = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \ldots}}} = [a_1, a_2, \ldots],$$

with $a_i \geq 1$ being an integer for all $i \geq 1$, and, if $p_n(x)/q_n(x) = [a_1, \ldots, a_n] \in \mathbb{Q}$, $n \geq 1$, then the approximation coefficients (see [12]) are

$$\Theta_n(x) := q_n(x)^2 \cdot \left| x - \frac{p_n(x)}{q_n(x)} \right|, \quad n \geq 1.$$

The original Doeblin–Lenstra conjecture (see for example [3, 12]) gives information about the frequency of having consecutive pairs $(\Theta_k(x), \Theta_{k-1}(x))$ in some prescribed set, and it involves the lift of the Gauss measure $\mu_G$ to the natural extension space $[0, 1)^2$ of the continued fraction transformation; thus, it is valid for Lebesgue-a.e. (almost every) $x \in [0, 1)$. By contrast, in our case we take the numbers $x$ from the complement of this set. The natural extension $([0, 1)^2, \tilde{G})$ of $G$ is a skew product which falls into our
setting. Hence, we can apply in Theorem 10.1 the results obtained above. Using the exact dimensionality of \( \hat{\mu}_s \) on the natural extension, we also make the Doeblin–Lenstra conjecture more precise. Namely, in Theorem 10.2, we estimate the asymptotic frequency of having the pairs \((\Theta_k(x), \Theta_{k-1}(x))\) \(r\)-close to pairs \((z, z')\), for \(1 \leq k \leq n\), if \(n\) becomes larger and larger, for all irrational numbers \(x\) from a measurable set \(\Lambda_s \subset [0, 1]\) with \(\mu_s(\Lambda_s) = 1\) and \(HD(\Lambda_s) > 0\). We emphasize that the Lebesgue measure of \(\Lambda_s\) is zero, so it is not covered by the original Doeblin–Lenstra conjecture. We provide its exact value in formula (2) of Theorem 10.2.

Our Smale skew product endomorphisms are related also to the notion of chains with complete connections, introduced in [31] and studied for instance in [7, 11, 31]. Our endomorphisms are related also to iterated function systems with place-dependent probabilities (see [1]), which are systems of contractions \(S = \{\phi_i : X \to X\}_{i \in I}\) with limit set \(J_S\), where instead of the classical self-similar measure associated to a probability vector \((p_i, i \in I)\), one considers an invariant measure \(\mu\) on \(J_S\) associated to a probabilistic vector composed of variable weights \(p_i : X \to [0, 1]\) for \(i \in I\).

Many authors studied various related aspects of thermodynamic formalism and its relations to dimension theory, for example [2, 4, 5, 9, 10, 13, 14, 16–21, 23–28, 30, 32–37, 39, 41, 42, 44, 46, 47], and this list is far from complete.

2. One-sided thermodynamic formalism

In this section we collect some fundamental ergodic (thermodynamic formalism) results concerning one-sided symbolic dynamics. All of them can be found with proofs in [15, 16]. Let \(E\) be a countable set and let \(A : E \times E \to \{0, 1\}\) be a matrix. A finite or countable infinite tuple \(\omega\) of elements of \(E\) is called \(A\)-admissible if and only if \(A_{ab} = 1\) for any two consecutive elements \(a, b\) of \(E\).

The matrix \(A\) is said to be finitely irreducible if there exists a finite set \(F\) of finite \(A\)-admissible words so that for any two elements \(a, b\) of \(E\) there exists \(\gamma \in F\) such that the word \(a\gamma b\) is \(A\)-admissible. In the following, the incidence matrix \(A\) is assumed to be finitely irreducible. Given \(\beta > 0\), define the metric \(d_\beta\) on \(E^N\) by

\[
d_\beta((\omega_n)_0^\infty, (\tau_n)_0^\infty) = \exp(-\beta \max\{n \geq 0 : (0 \leq k \leq n) \Rightarrow \omega_k = \tau_k\})
\]

with the standard convention that \(e^{-\infty} = 0\). Note that all the metrics \(d_\beta, \beta > 0\), on \(E^N\) are Hölder continuously equivalent and they induce the product topology on \(E^N\). Let

\[
E_A^+ = \{(\omega_n)_0^\infty : \forall n \in \mathbb{N} \ A_{\omega_n \omega_{n+1}} = 1\}.
\]

Then \(E_A^+\) is a closed subset of \(E^N\) and we endow it with the topology and metrics \(d_\beta\) inherited from \(E^N\). The shift map \(\sigma : E^Z \to E^Z\) is defined by the formula \(\sigma((\omega_n)_0^\infty) = ((\omega_n)_0^\infty)\), \(\sigma(E_A^+) \subset E_A^+\), and \(\sigma : E_A^+ \to E_A^+\) is continuous. For every finite word \(\omega = \omega_0 \omega_1 \ldots \omega_{n-1}\), put \(|\omega| = n\), the length of \(\omega\), and \(|\omega| = \tau \in E_A^+ : \forall (0 \leq j \leq n-1) : \tau_j = \omega_j\}\), the cylinder generated by \(\omega\). Let \(\psi : E_A^+ \to \mathbb{R}\) be continuous; then the topological pressure \(P(\psi)\) is

\[
P(\psi) = \lim_{n \to \infty} \frac{1}{n} \log \sum_{|\omega| = n} \exp(\sup(S_n \psi|_{|\omega|}))
\]

and the limit exists, as the sequence \(\log \sum_{|\omega| = n} \exp(\sup(S_n \psi|_{|\omega|}))\), \(n \in \mathbb{N}\), is subadditive. The following theorem, a weaker version of the variational principle, was proved in [16].
Theorem 2.1. If $\psi : E_A^+ \to \mathbb{R}$ is a continuous function and $\mu$ is a $\sigma$-invariant Borel probability measure on $E_A^+$ such that $\int \psi \, d\mu > -\infty$, then $h_\mu(\sigma) + \int_{E_A^+} \psi \, d\mu \leq P(\psi)$.

We say that the function $\psi : E_A^+ \to \mathbb{R}$ is summable if and only if

$$\sum_{e \in E} \exp(\sup(e)) < +\infty.$$ 

A shift-invariant Borel probability measure $\mu$ on $E_A^+$ is called a Gibbs state of $\psi$ provided that there are a constant $C \geq 1$ and $P \in \mathbb{R}$ such that

$$C^{-1} \leq \frac{\mu([\omega])}{\exp(S_n\psi(\tau) - Pn)} \leq C \quad (2.1)$$

for all $n \geq 1$, all admissible words $\omega$ of length $n$, and all $\tau \in [\omega]$. It clearly follows from (2.1) that if $\psi$ admits a Gibbs state, then $P = P(\psi)$.

Definition 2.2. A function $g : E_A^+ \to \mathbb{C}$ is called Hölder continuous if it is Hölder continuous with respect to one, equivalently all, metrics $d_\beta$. Then there exists $\beta > 0$ such that $g$ is Lipschitz continuous with respect to $d_\beta$. The corresponding Lipschitz constant is $L_\beta(g)$.

The proofs of the following three results can be found for instance in [16] and [15]. Regarding the result of Theorem 2.3, it was shown in [43] that if the shift is topologically mixing, then finite irreducibility is also necessary. For Theorem 2.4, see [42].

Theorem 2.3. For every Hölder continuous summable potential $\psi : E_A^+ \to \mathbb{R}$, there exists a unique Gibbs state $\mu_\psi$ on $E_A^+$. The measure $\mu_\psi$ is ergodic.

Theorem 2.4. Suppose that $\psi : E_A^+ \to \mathbb{R}$ is a Hölder continuous potential. Then, denoting by $P_F(\psi)$ the topological pressure of $\psi|_{F^+}$ with respect to the shift map $\sigma : F^+_A \to F^+_A$, we have $P(\psi) = \sup\{P_F(\psi)\}$, where the supremum is taken over all finite subsets $F$ of $E$; equivalently over all finite subsets $F$ of $E$ such that the matrix $A|_{F \times F}$ is irreducible.

Theorem 2.5. (Variational principle for one-sided shifts) Suppose that $\psi : E_A^+ \to \mathbb{R}$ is a Hölder continuous summable potential. Then

$$\sup \left\{ h_\mu(\sigma) + \int_{E_A^+} \psi \, d\mu, \mu \circ \sigma^{-1} = \mu \text{ and } \int \psi \, d\mu > -\infty \right\} = P(\psi) = h_{\mu_\psi}(\sigma) + \int_{E_A^+} \psi \, d\mu_\psi$$

and $\mu_\psi$ is the only measure at which this supremum is attained.

Any measure that realizes the supremum in the above variational principle is called an equilibrium state for $\psi$. Then Theorem 2.5 can be reformulated as follows.

Theorem 2.6. If $\psi : E_A^+ \to \mathbb{R}$ is a Hölder continuous summable potential, then the Gibbs state $\mu_\psi$ is a unique equilibrium state for $\psi$.

Also, due to the irreducibility of the incidence matrix $A$, we have the following proposition.
3. Two-sided thermodynamic formalism

As in the previous section let $E$ be a countable set and let $A : E \times E \to \{0, 1\}$ be a finitely irreducible matrix. Given $\beta > 0$, we define the metric $d_\beta$ on $E^\mathbb{Z}$ by

$$d_\beta((\omega)_{-\infty}^\infty, (\tau)_{-\infty}^\infty) = \exp(-\beta \max\{n \geq 0 : \forall k \in \mathbb{Z} \mid |k| \leq n \Rightarrow \omega_k = \tau_k\})$$

with the standard convention that $e^{-\infty} = 0$. Note that all the metrics $d_\beta$, $\beta > 0$, on $E^\mathbb{Z}$ are Hölder continuously equivalent and they induce the product topology on $E^\mathbb{Z}$. We set

$$E_A = \{(\omega)_{-\infty}^\infty : \forall n \in \mathbb{Z} \ A_{\omega_n\omega_{n+1}} = 1\}.$$  

Obviously, $E_A$ is a closed subset of $E^\mathbb{Z}$ and we endow it with the topology and metrics $d_\beta$ inherited from $E^\mathbb{Z}$. The two-sided shift map $\sigma : E^\mathbb{Z} \to E^\mathbb{Z}$ is defined as $\sigma((\omega)_{-\infty}^\infty) = ((\omega_{n+1})_{-\infty}^\infty)$. Clearly, $\sigma(E_A) = E_A$ and $\sigma : E_A \to E_A$ is a homeomorphism.

**Definition 3.1.** A function $g : E_A \to \mathbb{C}$ is said to be Hölder continuous provided that it is Hölder continuous with respect to one, equivalently all, metrics $d_\beta$. Then there exists at least one (in fact an open segment) parameter $\beta > 0$ such that $g$ is Lipschitz continuous with respect to $d_\beta$. The corresponding Lipschitz constant is denoted by $L_\beta(g)$.

For every $\omega \in E_A$ and all $-\infty \leq m \leq n \leq \infty$, we set $\omega|m^n = \omega_m\omega_{m+1} \ldots \omega_n$.

Let $E_A^*$ be the set of all $A$-admissible finite words. For $\tau \in E^*$, $\tau = \tau_m\tau_{m+1} \ldots \tau_n$, we set

$$[\tau]_m^n = \{\omega \in E_A : \omega|m^n = \tau\}$$

and call $[\tau]_m^n$ the cylinder generated by $\tau$ of size from $m$ to $n$. The family of all cylinders of size from $m$ to $n$ will be denoted by $C_m^n$. If $m = 0$, we simply write $[\tau]$ for $[\tau]_m^n$.

Let $\psi : E_A \to \mathbb{R}$ be a continuous function. The topological pressure $P(\psi)$ is defined by

$$P(\psi) := \lim_{n \to \infty} \frac{1}{n} \log \sum_{\omega \in C_0^{n-1}} \exp(\sup(S_n \psi|_{[\omega]}))$$

(3.1)

and the limit exists due to the same subadditivity argument. Similarly as in Theorem 2.1, we immediately obtain the following theorem.

**Theorem 3.2.** If $\psi : E_A \to \mathbb{R}$ is a continuous function and $\mu$ is a $\sigma$-invariant Borel probability measure on $E_A$ such that $\int \psi \, d\mu > -\infty$, then

$$h_\mu(\sigma) + \int_{E_A} \psi \, d\mu \leq P(\psi).$$

A shift-invariant Borel probability measure $\mu$ on $E_A$ is called a *Gibbs state* of $\psi$ if there are a constant $C \geq 1$ and $P \in \mathbb{R}$ such that

$$C^{-1} \leq \frac{\mu([\omega]_{0}^{n-1})}{\exp(S_n \psi(\omega) - Pn)} \leq C$$

(3.2)
The function \( \omega \) and, for \( 0 \leq \beta \leq 1 \), we get
\[
\psi_2 - \psi_1 = u - u \circ \sigma.
\]
Any function of the form \( u - u \circ \sigma \) is called a cohomological function. A function \( \psi : E_A \to \mathbb{R} \) is called cohomologous to a constant, say \( b \in \mathbb{R} \), provided that \( \psi - b \) is a cohomological function. Notice that any two functions on \( E_A \), cohomologous in \( C(E_A) \), the class of all real-valued bounded functions on \( E_A \), have the same topological pressure and the same set of Gibbs measures.

A function \( \psi : E_A \to \mathbb{R} \) is called past-independent if for every \( \tau \in C_0^\infty \) and for all \( \omega, \rho \in [\tau] \), we have \( \phi(\omega) = \phi(\tau) \).

To apply the previous section we need the following lemma.

**Lemma 3.3.** Every Hölder continuous function \( \psi : E_A \to \mathbb{R} \) is cohomologous to a past-independent Hölder continuous function \( \psi^+ : E_A \to \mathbb{R} \) in the class \( H_B \) of all bounded Hölder continuous functions.

**Proof.** The proof is essentially the same as in [4, Lemma 1.6, p. 11]. For every letter \( e \in E \), fix some infinite word \( \overline{e} \in E_A(\infty, -1) \) such that \( A_{\overline{e}1} = 1 \). Then for \( \omega \in E_A \), fix as before an infinite word \( \overline{\omega} \in E_A(\infty, -1) \) and put \( \overline{\omega} = \overline{\omega}|\infty^\omega \). Hence, \( \overline{\omega} \) is the same as \( \omega \) starting from index 0, but has fixed elements with negative indices. Now set
\[
\psi(\omega) = \sum_{j=0}^\infty (\psi(\sigma^j(\omega)) - \psi(\sigma^j(\overline{\omega}))).
\]
Fix \( \beta > 0 \) so that \( \psi \) is Lipschitz continuous with respect to \( d_\beta \). For \( j \geq 0 \), \( [\sigma^j(\omega)]_{\infty} = [\sigma^j(\overline{\omega})]_{\infty} \), so \( d_\beta(\sigma^j(\omega), \sigma^j(\overline{\omega})) \leq e^{-\beta j} \), and thus
\[
|\psi(\sigma^j(\omega)) - \psi(\sigma^j(\overline{\omega}))| \leq L_\beta(\psi)e^{-\beta j}.
\]
(3.3)
The function \( u \) is well defined and continuous. If \( d_\beta(\omega, \tau) = e^{-\beta n} \), then \( [\omega]_{n\in\mathbb{N}} = [\tau]_{n\in\mathbb{N}} \)
and, for \( 0 \leq j \leq n \),
\[
|\psi(\sigma^j(\omega)) - \psi(\sigma^j(\tau))| \leq L_\beta(\psi)d_\beta(\sigma^j(\omega), \sigma^j(\tau)) \leq L_\beta(\psi)e^{-\beta(n-j)}
\]
and
\[
|\psi(\sigma^j(\overline{\omega})) - \psi(\sigma^j(\overline{\tau}))| \leq L_\beta(\psi)d_\beta(\sigma^j(\overline{\omega}), \sigma^j(\overline{\tau})) \leq L_\beta(\psi)e^{-\beta(n-j)}.
\]
Thus, using also (3.3), we get
\[
|u(\omega) - u(\tau)| \leq 2L_\beta(\psi)\sum_{j=0}^{E(n/2)} e^{-\beta(n-j)} + 2L_\beta(\psi)\sum_{j>E(n/2)} e^{-\beta j}
\]
\[
\leq 4L_\beta(\psi)(1 - e^{-\beta})(1 - e^{-\beta(n/2)}).
\]
So, \( u : E_A \to \mathbb{R} \) is Lipschitz continuous with respect to the metric \( d_{\beta/2} \) and, by (3.3), it is bounded. So, \( u \in H_{\beta/2} \). Hence, \( \psi^+ = \psi - u + u \circ \sigma \) is Lipschitz continuous with respect to \( d_{\beta/2} \). It follows that \( \psi^+ \) is past-independent. \( \square \)
In the setting of the above lemma, let \( \psi^+ \) be the factorization of \( \psi^+ \) on \( E^+_A \), i.e.
\[
\psi^+ = \overline{\psi}^+ \circ \pi_0,
\]
where \( \pi_0 : E_A \to E^+_A \), \( \pi_0(\omega) = (\omega_0 \omega_1 \ldots) \), \( \omega \in E_A \). As a consequence of this lemma, we get the following lemma.

**Lemma 3.4.** If \( \psi : E_A \to \mathbb{R} \) is a Hölder continuous potential, then \( \mathcal{P}(\psi) = \mathcal{P}(\psi^+) \), where, we recall, the former pressure is taken with respect to the two-sided shift \( \sigma : E_A \to E_A \) while the latter one is taken with respect to the one-sided shift \( \sigma^+ : E^+_A \to E^+_A \).

Then, from this lemma and Theorem 2.4, we get the following theorem.

**Theorem 3.5.** Suppose that \( \psi : E_A \to \mathbb{R} \) is a Hölder continuous potential. Then, denoting by \( \mathcal{P}_F(\psi) \) the topological pressure of \( \psi|_F \) with respect to \( \sigma : F_A \to F_A \), we have that
\[
\mathcal{P}(\psi) = \sup \{ \mathcal{P}_F(\psi) \},
\]
where the supremum is taken over all finite subsets \( F \) of \( E \); equivalently over all finite subsets \( F \) of \( E \) such that the matrix \( A|_{F \times F} \) is irreducible.

We call the function \( \psi : E_A \to \mathbb{R} \) summable if and only if
\[
\sum_{e \in E} \exp(\sup(\psi|_e)) < \infty.
\]
As in the case of one-sided shift, we have the following proposition.

**Proposition 3.6.** A Hölder continuous function \( \psi : E_A \to \mathbb{R} \) is summable if and only if \( \mathcal{P}(\psi) < \infty \).

From Lemma 3.3 (the coboundary appearing there is bounded), we get the following lemma.

**Lemma 3.7.** Every Hölder continuous summable function \( \psi : E_A \to \mathbb{R} \) is cohomologous to a past-independent Hölder continuous summable function \( \psi^+ : E_A \to \mathbb{R} \) in the class \( H_B \) of all bounded Hölder continuous functions.

**Theorem 3.8.** For every Hölder continuous summable potential \( \psi : E_A \to \mathbb{R} \), there exists a unique Gibbs state \( \mu_\psi \) on \( E_A \). The measure \( \mu_\psi \) is ergodic.

**Proof.** Let \( \psi^+ \) be the past-independent Hölder continuous summable potential ascribed to \( \psi \) in Lemma 3.7. It follows from Theorem 2.3 that there exists a unique measure \( \mu^+_\psi \) on \( E^+_A \) for which (3.2) is satisfied. Also, \( \mu^+_\psi \) is ergodic. Since \( \mu^+_\psi \) is invariant, the formula
\[
\mu_\psi([\omega]|_m) = \mu^+_\psi(\sigma^m([\omega]|_m)) = \mu^+_\psi([\eta]|_0^{n-m}),
\]
with \( \eta_0 = \omega_m, \ldots, \eta_{n-m} = \omega_n \), for \( |\omega| = n - m + 1 \), gives rise to a shift-invariant measure \( \mu_\psi \) on \( E_A \), for which (3.2) holds. Thus, \( \mu_\psi \) is a Gibbs state for \( \psi \) and is ergodic. Passing to the uniqueness, if \( \mu \) is a Gibbs state for \( \psi \), then from its invariance and (3.2), for all \( n \geq 0 \) and all \( \omega \in E_A \), we have
\[
C^{-1} \leq \frac{\mu([\omega]|_{n-n})}{\exp(S_{2n+1} \psi(\sigma^{-n}(\omega)) - \mathcal{P}(\psi)n)} \leq C.
\]
Any Gibbs states of $\psi$ are equivalent and, as one of them is ergodic, uniqueness follows.

**THEOREM 3.9.** (Variational principle for two-sided shifts) Suppose that $\psi : E_A \to \mathbb{R}$ is a Hölder continuous summable potential. Then

$$
\sup \left\{ h_\mu(\sigma) + \int_{E_A} \psi \, d\mu : \mu \circ \sigma^{-1} = \mu \text{ and } \int \psi \, d\mu > -\infty \right\} = P(\psi) = h_{\mu_\psi}(\sigma) + \int_{E_A} \psi \, d\mu
$$

and $\mu_\psi$ is the only measure at which this supremum is taken on.

*Proof.* We replace $\psi$ by the past-independent Hölder continuous summable potential $\psi^+$ resulting from Lemma 3.7. Since the dynamical system $(\sigma, E_A)$ is canonically isomorphic to the natural extension of $(\sigma, E_A^+)$, the map $\mu \mapsto \mu \circ \pi_0^{-1}$ gives a bijection between the space $M_\sigma(E_A)$ of $\sigma$-invariant probabilities on $E_A$ and the space $M_\sigma(E_A^+)$ of $\sigma$-invariant probabilities on $E_A^+$, which preserves entropies. Since $P(\psi) = P(\psi^+)$ by Lemma 3.4, and since for every $\mu \in M_\sigma(E_A)$, $\int_{E_A^+} \psi^+ \, d\mu \circ \pi_0^{-1} = \int_{E_A} \psi^+ \circ \pi_0 \, d\mu = \int_{E_A} \psi^+ \, d\mu$, we are done, due to Theorem 2.5. □

Any measure that realizes the supremum value in the above variational principle is called an equilibrium state for $\psi$. Then Theorem 3.9 can be reformulated as follows.

**THEOREM 3.10.** If $\psi : E_A \to \mathbb{R}$ is a Hölder continuous summable potential, then the Gibbs state $\mu_\psi$ is a unique equilibrium state for $\psi$.

We will need however more characterizations of Gibbs states. Define the partition

$$
\mathcal{P}^- = \{ [\omega]_0^\infty : \omega \in E_A \} = \{ [\omega] : \omega \in E_A^+ \}.
$$

The partition $\mathcal{P}^-$ is a measurable partition of $E_A$ and two elements $\alpha, \beta \in E_A$ belong to the same element of this partition if and only if $\alpha|_0^\infty = \beta|_0^\infty$. If $\mu$ is a Borel probability measure on $E_A$, we let

$$
\{ \overline{\mu}^\tau : \tau \in E_A \}
$$

be a canonical system of conditional measures induced by partition $\mathcal{P}^-$ and measure $\mu$ (see Rokhlin [38]). Each $\overline{\mu}^\tau$ is a Borel probability measure on $[\tau|_0^\infty]$ and we will frequently write $\overline{\mu}^\omega$, $\omega \in E_A^+$, to denote the corresponding conditional measure on $[\omega]$. Denote by

$$
\pi_0 : E_A \to E_A^+, \quad \pi_0(\tau) = \tau|_0^\infty, \quad \tau \in E_A.
$$

the canonical projection to $E_A^+$. The system $\{ \overline{\mu}^\omega : \omega \in E_A^+ \}$ is determined by the fact that

$$
\int_{E_A} g \, d\mu = \int_{E_A^+} \int_{[\omega]} g \, d\overline{\mu}^\omega \, d(\mu \circ \pi_0^{-1})(\omega)
$$

for every measurable function $g \in L^1(\mu)$ [38]. It is evident from this characterization that if we change such a system on a set of zero $\mu \circ \pi_0^{-1}$-measure, then we also obtain a system
of conditional measures. The canonical system of conditional measures induced by \( \mu \) is uniquely defined up to a set of zero \( \mu \circ \pi^{-1}_0 \)-measure. We say that a collection

\[
\{ \overline{\mu}^\omega : \omega \in E_A^+ \}
\]

defines a global system of conditional measures of \( \mu \) if this is indeed a system of conditional measures of \( \mu \) and a measure \( \overline{\mu}^\omega \) is defined for every \( \omega \in E_A^+ \), rather than only on a set of full \( \mu \circ \pi^{-1}_0 \)-measure. The first characterization of Gibbs states is the following theorem.

**Theorem 3.11.** Suppose that \( \psi : E_A \to \mathbb{R} \) is a Hölder continuous summable potential. Let \( \mu \) be a Borel probability shift-invariant measure on \( E_A \). Then \( \mu = \mu_\psi \), the unique Gibbs state for \( \psi \), if and only if there exists \( D \geq 1 \) such that

\[
D^{-1} \leq \frac{\overline{\mu}^\omega([\tau \omega|_n^\infty])}{\exp(S_n \psi(\rho) - P(\psi)n)} \leq D
\]

for every \( n \geq 1 \), \( \mu \circ \pi^{-1}_0 \)-a.e. \( \omega \in E_A^+ \), \( \overline{\mu}^\omega \)-a.e. \( \tau \omega \in E_A(-n, \infty) \) with \( A_{\tau^{-1} \omega_0} = 1 \), and \( \rho \in [\tau \omega|_0^\infty] \). Also, there exists a global system of conditional measures of \( \mu_\psi \) such that

\[
D^{-1} \leq \frac{\overline{\mu}_\psi^\omega([\tau \omega|_n^\infty])}{\exp(S_n \psi(\rho) - P(\psi)n)} \leq D
\]

for every \( \omega \in E_A^+ \), \( n \geq 1 \), \( \tau \in E_A(-n, -1) \) with \( A_{\tau^{-1} \omega_0} = 1 \), and every \( \rho \in [\tau \omega|_0^\infty] \).

**Proof.** Suppose that (3.4) holds. For every \( \omega \in E_A \) (note that here indeed ‘for every’, although (3.4) is assumed to hold only for \( \mu \circ \pi^{-1}_0 \)-a.e. \( \omega \in E_A^+ \)) and every \( n \geq 1 \), if \( \omega|_0^{n-1}|_{-n} \) denotes the finite word \( \eta_{-n} \ldots \eta_{-1} \) with \( \eta_{-n} = \omega_0, \ldots, \eta_{-1} = \omega_{n-1} \), then we obtain

\[
\mu([\omega|_0^{n-1}]) = \mu(\sigma^n([\omega|_0^{n-1}])) = \mu([\omega|_0^{n-1}|_{-n}]) = \int_{E_A^+} \overline{\mu}^\tau([\omega|_0^{n-1}|_{-n}]) d\mu \circ \pi^{-1}_0(\tau)
\]

\[
= \int_{E_A^+} \overline{\mu}^\tau([\omega|_0^{n-1}|_{-n}]) d\mu \circ \pi^{-1}_0(\tau)
\]

\[
\geq \exp(S_n \psi(\omega) - P(\psi)n) \sum_{e \in E : A_{\omega_{n-1} e = 1}} \mu([e]).
\]

Consequently,

\[
\mu([\omega|_0^{n-1}]) \geq \exp(S_n \psi(\omega) - P(\psi)n).
\]

In order to prove the opposite inequality, notice that because of finite irreducibility of the matrix \( A \) there exists a finite set \( F \subset E \) such that for every \( a \in E \) there exists \( b \in F \) such that \( A_{ab} = 1 \). Since \( \mu \) is a non-zero measure, there exists \( c \in E \) such that \( \mu([c]) > 0 \). Invoking finite irreducibility of the matrix \( A \) again, we see that for every \( e \in E \) there exists a finite word \( \alpha \) such that \( e\alpha c \) is \( A \)-admissible. Put \( k = |e\alpha| \). It then follows from (3.6) that

\[
\mu([e]) \geq \mu([e\alpha]) \geq \exp(S_k \psi(\rho) - P(\psi)k) \mu([c]) > 0
\]

for every \( \rho \in [e\alpha] \). Hence, \( T = \min \{ \mu([e]) : e \in F \} > 0 \). Continuing (3.6), we see that \( \mu([\omega|_0^{n-1}]) \geq T \exp(S_n \psi(\omega) - P(\psi)n) \). Combining this with (3.7), we see that \( \mu \) is a Gibbs state for \( \psi \) and the first assertion of the theorem is established.
Now, to complete the proof, we need to define a global system of conditional measures of \( \mu_\psi \) such that (3.5) holds for every \( \omega \in E_A^+ \), \( n \geq 1 \), \( \tau \in E_A(\omega, -1) \) with \( A_{\tau^{-1} \omega_0} = 1 \), and every \( \rho \in \sigma^{-n}([\tau\omega]_{\infty n}) = [\tau\omega]_{0}^{\infty} \). Indeed, let \( L : \ell_\infty \rightarrow \ell_\infty \) be a Banach limit. Note that

\[
\frac{\mu_\psi([\tau\omega]_{n}^{k-1})}{\mu_\psi(\omega)_{0}^{k-1})} = \frac{\mu_\psi([\tau\omega]_{0}^{n+k-1})}{\mu_\psi(\omega)_{0}^{k-1})} \times \frac{\exp(S_{n+k}(\rho) - P(\psi)(n+k))}{\exp(S_k(\sigma^n(\rho) - P(\psi))k)}
\]

belonging to \( \ell_\infty \) (comparability constants from Gibbs property of \( \mu_\psi \)). So, the sequence

\[
\left(\frac{\mu_\psi([\tau\omega]_{n}^{k-1})}{\mu_\psi(\omega)_{0}^{k-1})}\right)_{k=1}^{\infty}
\]

belongs to \( \ell_\infty \). We can then define

\[
\overline{\phi}_\psi([\tau\omega]_{\infty n}) := L\left(\left(\frac{\mu_\psi([\tau\omega]_{n}^{k-1})}{\mu_\psi(\omega)_{0}^{k-1})}\right)_{k=1}^{\infty}\right).
\]

For every \( g : [\omega] \rightarrow \mathbb{R} \) and a linear combination \( \sum_{j=1}^{s} a_j \mathbb{1}_{[\tau_0^{-1}]} \), the sequence

\[
\frac{\mu_\psi(\sum_{j=1}^{s} a_j \mathbb{1}_{[\tau_0^{-1}]})}{\mu_\psi(\omega)_{0}^{k-1})} \simeq \mu_\psi\left(\sum_{j=1}^{s} a_j \mathbb{1}_{[\tau_0^{-1}]}ight)
\]

with the same comparability constants as above, belongs to \( \ell_\infty \). We can then define

\[
\overline{\phi}_\psi\left(\sum_{j=1}^{s} a_j \mathbb{1}_{[\tau_0^{-1}]}ight) := L\left(\left(\frac{\mu_\psi(\sum_{j=1}^{s} a_j \mathbb{1}_{[\tau_0^{-1}]})}{\mu_\psi(\omega)_{0}^{k-1})}\right)_{k=1}^{\infty}\right).
\]

So, we have defined a function \( \overline{\phi}_\psi \) from the vector space \( \mathcal{V} \) of all linear combinations as above to the set of real numbers. Since the Banach limit is a positive linear operator, so is the function \( \overline{\phi}_\psi : \mathcal{V} \rightarrow \mathbb{R} \). Furthermore, because of monotonicity of Banach limits, and because of (3.9), \( \overline{\phi}_\psi(g) \leq 0 \) whenever \( (g_n)_{n=1}^{\infty} \) is a monotone decreasing sequence of functions in \( \mathcal{V} \) converging pointwise to 0. Therefore, the Daniell–Stone theorem gives a unique Borel probability measure on \([\omega]\), whose restriction to \( \mathcal{V} \) coincides with \( \overline{\phi}_\psi \). We keep the same symbol \( \overline{\phi}_\psi \) for this extension. Now it follows from the martingale convergence theorem that, for \( \mu_\psi \circ \pi_0^{-1} \)-a.e. \( \omega \in E_A^+ \) and every \( \tau \in E_A(\omega, -1) \) with \( A_{\tau^{-1} \omega_0} = 1 \), the limit

\[
\lim_{k \rightarrow \infty} \frac{\mu_\psi([\tau\omega]_{n}^{k-1})}{\mu_\psi(\omega)_{0}^{k-1})}
\]

exists and equals the conditional measure of \( \mu_\psi \) on \([\omega]\). By properties of Banach limits,

\[
\frac{\mu_\psi([\tau\omega]_{n}^{k-1})}{\mu_\psi(\omega)_{0}^{k-1})} = \lim_{k \rightarrow \infty} \frac{\mu_\psi([\tau\omega]_{n}^{k-1})}{\mu_\psi(\omega)_{0}^{k-1})}
\]

and thus the collection \( \{\overline{\phi}_\psi : \omega \in E_A^+\} \) is indeed a global system of conditional measures of \( \mu_\psi \). Using also (3.8), this completes the proof. \( \square \)
Similarly, let
\[ \mathcal{P}_+ = \{[\omega]_{-\infty}^{-1} : \omega \in E_A \} \]
and, given a Borel probability measure \( \mu \) on \( E_A \), let \( \{\mu^+\omega : \omega \in E_A\} \) be the corresponding canonical system of conditional measures. As in Theorem 3.11, we prove the following theorem.

**Theorem 3.12.** Suppose that \( \psi : E_A \to \mathbb{R} \) is a Hölder continuous summable potential. Let \( \mu \) be a Borel probability shift-invariant measure on \( E_A \). Then \( \mu \) is equal to the unique Gibbs state \( \mu_\psi \) of \( \psi \) if and only if there exists \( D \geq 1 \) such that for all \( \omega \in E_A(-\infty, -1) \), \( n \geq 1 \), \( \tau \in E_A(0, n-1) \) with \( A_{\omega-1}\tau_0 = 1 \), and \( \rho \in [\omega\tau]_{-\infty}^{n-1} \), the conditional measures \( \mu^+\omega \) satisfy
\[ D^{-1} \leq \frac{\mu^+\omega([\omega\tau]_{-\infty}^{n-1})}{\exp(S_n\psi(\rho) - P(\psi)n)} \leq D. \] (3.10)

4. **Skew product Smale spaces of countable type**

We keep the notation from the previous two sections.

**Definition 4.1.** Let \((Y, d)\) be a complete bounded metric space and take for every \( \omega \in E_A^+ \) an arbitrary set \( Y_\omega \subset Y \) and a continuous injective map \( T_\omega : Y_\omega \to Y_{\sigma(\omega)} \). Define
\[ \hat{Y} := \bigcup_{\omega \in E_A^+} \{\omega\} \times Y_\omega \subset E_A^+ \times Y. \]

Define the map \( T : \hat{Y} \to \hat{Y} \) by \( (T(\omega), y) = (\sigma(\omega), T_\omega(y)) \). The pair \((\hat{Y}, T : \hat{Y} \to \hat{Y})\) is called a skew product Smale endomorphism if there exists \( \lambda > 1 \) such that \( T \) is fiberwise uniformly contracting, i.e. for all \( \omega \in E_A^+ \) and all \( y_1, y_2 \in Y_\omega \),
\[ d(T_\omega(y_2), T_\omega(y_1)) \leq \lambda^{-1}d(y_2, y_1). \] (4.1)

Note that for every \( \tau \in E_A(-n, \infty) \), the composition \( T_\tau^n = T_{\tau|_{\tau_n}} \circ T_{\tau|_{\tau_2}} \circ \ldots \circ T_{\tau|_{\tau_{n+1}}} : Y_{\tau_n} \to Y_{\tau_0} \) is well defined. Therefore, for every \( \tau \in E_A \), we can define the map
\[ T^n := T^n_{\tau|_{\tau_n}} := T_{\tau|_{\tau_n}} \circ T_{\tau|_{\tau_{n+1}}} \circ \ldots \circ T_{\tau|_{\tau_{n+1}}} : Y_{\tau|_{\tau_n}} \to Y_{\tau|_{\tau_0}}. \]

Then the sequence \((T^n_\tau(Y_{\tau|_{\tau_n}}))_{n=0}^\infty\) consists of descending sets and
\[ \text{diam}(T^n_\tau(Y_{\tau|_{\tau_n}})) \leq \lambda^{-n} \text{diam}(Y). \] (4.2)

The same is then true for the closures of these sets, i.e. we have that the sequence \((\overline{T^n_\tau(Y_{\tau|_{\tau_n}})})_{n=0}^\infty\) consists of closed descending sets and \( \text{diam}(\overline{T^n_\tau(Y_{\tau|_{\tau_n}}})) \leq \lambda^{-n} \text{diam}(Y) \).

Since the metric space \((Y, d)\) is complete, we conclude that its intersection
\[ \bigcap_{n=1}^{\infty} \overline{T^n_\tau(Y_{\tau|_{\tau_n}})} \]
is a singleton. Denote its only element by \( \hat{\pi}_2(\tau) \). So, we have defined the map
\[ \hat{\pi}_2 : E_A \to Y \]
and next define the map \( \hat{\pi} : E_A \to E_A^+ \times Y \) by the formula
\[
\hat{\pi}(\tau) = (\tau|_0^\infty, \hat{\pi}_2(\tau))
\]
and the truncation to the elements of non-negative indices by
\[
\pi_0 : E_A \to E_A^+, \quad \pi_0(\tau) = \tau|_0^\infty.
\]
In the notation for \( \pi_0 \) we drop the hat symbol, as this projection is in fact independent of the skew product on \( \hat{Y} \). For all \( \omega \in E_A^+ \), define the \( \hat{\pi}_2 \)-projection of the cylinder \( \{\omega\} \subset E_A \),
\[
J_\omega := \hat{\pi}_2(\{\omega\}) \in Y,
\]
and call these sets the stable Smale fibers of the system \( T \). The global invariant set is
\[
J := \hat{\pi}(E_A) = \bigcup_{\omega \in E_A^+} \{\omega\} \times J_\omega \subset E_A^+ \times Y,
\]
called the Smale space (or the fibered limit set) induced by the Smale presystem \( T \).

For each \( \tau \in E_A \), we have \( \hat{\pi}_2(\tau) \in \hat{Y}|_0^\infty \), so \( J_\omega \subset \hat{Y}_\omega \) for every \( \omega \in E_A^+ \). Since all \( T_{\omega} : Y_\omega \to Y_{\sigma(\omega)} \) are Lipschitz continuous with Lipschitz constant \( \lambda^{-1} \), they extend uniquely to Lipschitz continuous maps from \( \hat{Y}_\omega \) to \( \hat{Y}_{\sigma(\omega)} \) with Lipschitz constant \( \lambda^{-1} \).

**Proposition 4.2.** For every \( \omega \in E_A^+ \), we have that
\[
T_{\omega}(J_\omega) \subset J_{\sigma(\omega)},
\]
\[
\bigcup_{e \in E, A_{\omega e} = 1} T_{e\omega}(J_{e\omega}) = J_\omega, \quad \text{and}
\]
\[
T \circ \hat{\pi} = \hat{\pi} \circ \sigma.
\]

**Proof.** Let \( y \in J_\omega \); then there exists \( \tau \in E_A(-\infty, -1) \) such that \( A_{\tau, -1 e\omega} = 1 \) and \( y = \hat{\pi}_2(\tau \omega) \). Then
\[
\{T_{\omega}(y)\} = T_{\omega} \left( \bigcap_{n=1}^{\infty} T_{\tau \omega}(Y|_{1-n\omega}^{-1}) \right) \subset \bigcap_{n=1}^{\infty} T_{\tau \omega}(T_{\tau \omega}(Y|_{1-n\omega}^{-1})) \subset \bigcap_{n=1}^{\infty} T_{\omega}(T_{\tau \omega}(Y|_{1-n\omega}^{-1}))
\]
\[
= \bigcap_{n=1}^{\infty} \left( T_{\tau \omega}(Y|_{1-n\omega}^{-1}) \right) = \bigcap_{n=1}^{\infty} \left( T_{\tau \omega}(Y|_{1-n\omega}^{-1}) \right)
\]
\[
= \hat{\pi}_2(\tau|_{-\infty}^{-1} A_{\omega \sigma(\omega)}(\sigma(\omega))) \subset J_{\sigma(\omega)}.
\]
Thus, \( T_{\omega}(J_\omega) \subset J_{\sigma(\omega)} \), meaning that (4.4) holds, and, as \( \{T_{\omega}(y)\} \) and \( \{\hat{\pi}_2(\tau|_{-\infty}^{-1} A_{\omega \sigma(\omega)}(\sigma(\omega)))\} \), the respective sides of (4.7), are singletons, we therefore get
\[
T_{\omega} \hat{\pi}_2(\tau \omega) = \hat{\pi}_2 \circ \sigma(\tau \omega),
\]
meaning that (4.6) holds. The inclusion \( \bigcup_{e \in E, A_{\omega e} = 1} T_{e\omega}(J_{e\omega}) \subset J_\omega \) holds because of (4.4). In order to prove the opposite one, let \( z \in J_\omega \). Then \( z = \hat{\pi}(\gamma \omega) \) with some \( \gamma \in E_A(-\infty, -1) \), where \( A_{\gamma, -1 e\omega} = 1 \). Formula (4.8) then yields
\[
z = \hat{\pi}_2 \circ \sigma(\gamma|_{-\infty}^{-2} Y|_{-1}^{0} \omega) = T_{\gamma, -1 e\omega} \circ \hat{\pi}_2(\gamma|_{-\infty}^{-2} Y|_{-1}^{0} \omega) \in T_{\gamma, -1 e\omega}(J_{\gamma, -1 e\omega}).
\]
So, \( J_\omega \subset \bigcup_{e \in E, A_{\omega e} = 1} T_{e\omega}(J_{e\omega}) \). \( \square \)
Similarly, we obtain \( J_\omega = \bigcup_{\tau \in E_A^+} T_{\tau \omega}(J_{\tau \omega}) \), for all \( \omega \in E_A^+ \) and \( n > 0 \). By formula (4.4), we have \( T(J) \subset J \), so consider the system
\[
T : J \to J,
\]
which we call the skew product Smale endomorphism generated by the Smale system \( T : \hat{Y} \to \hat{Y} \). By (4.5), we have the following observation.

**Observation 4.3.** The map \( T : J \to J \) is surjective.

**Observation 4.4.** If \( T : \hat{Y} \to \hat{Y} \) is a skew product Smale system, then the following statements are equivalent:
(a) for every \( \xi \in J \), the fiber \( \hat{\pi}^{-1}(\xi) \subset E_A \) is compact;
(b) for every \( y \in Y \), the fiber \( \hat{\pi}_2^{-1}(y) \subset E_A \) is compact;
(c) for every \( \xi = (\omega, y) \in J \), the set \( \{e \in E : A_{\omega y} = 1 \text{ and } y \in T_{\omega y}(J_{\omega y})\} \) is finite.

If any of these conditions is satisfied, we call the Smale system \( T : J \to J \) of compact type.

**Remark 4.5.** In item (a) of Observation 4.4, one can replace \( J \) by \( \hat{Y} \).

**Observation 4.6.** If for every \( y \in Y \) the set \( \{e \in E : A_{\omega y} = 1 \text{ and } y \in T_{\omega y}(J_{\omega y})\} \) is finite for every \( \omega \in E_A^+ \), then \( T : J \to J \) is of compact type.

From now on we assume that \( T : \hat{Y} \to \hat{Y} \) is a skew product Smale system of compact type.

If for every \( \xi \in \hat{Y} \) (or in \( J \)) the fiber \( \hat{\pi}^{-1}(\xi) \subset E_A \) is finite, we call the skew product Smale system \( T \) of finite type.

**Observation 4.7.** If the skew product Smale system \( T : \hat{Y} \to \hat{Y} \) is of finite type, then it is also of compact type.

The Smale system \( T : \hat{Y} \to \hat{Y} \) is called of bijective type if, for every \( \xi \in J \), the fiber \( \hat{\pi}^{-1}(\xi) \) is a singleton. Equivalently, the map \( \hat{\pi} : E_A \to J \) is injective; then also \( T : J \to J \) is bijective. A Smale skew product of bijective type is clearly of finite type and thus of compact type.

**Definition 4.8.** We call a Smale endomorphism continuous if the global map \( T : J \to J \) is continuous with respect to the relative topology inherited from \( E_A^+ \times Y \).

**Lemma 4.9.** For every \( n \geq 1 \) and every \( \tau \in E_A(-n, \infty) \), we have that
\[
\hat{\pi}_2([\tau]) = T^n_\tau(J_\tau) \text{ and equivalently for every } \tau \in E_A, \quad \hat{\pi}_2([\tau|_{-n}^\infty]) = T^n_\tau(J_{\tau|_{-n}^\infty}).
\]

**Proof.** From (4.6), we get \( T^n_\tau(J_{\tau|_{-n}}) = T^n_\tau \circ \hat{\pi}_2([\tau|_{-n}^\infty]) = \hat{\pi}_2 \circ \sigma^n([\tau|_{-n}^\infty]) = \hat{\pi}_2([\tau|_{-n}^\infty]) \).

As a consequence of (4.2), we get the following observation.

**Observation 4.10.** For every \( \omega \in E_A \), the map \([\omega|_{0}^\infty] \ni \tau \mapsto \hat{\pi}_2(\tau) \in J_{\omega|_{0}^\infty} \subset Y \) is Lipschitz continuous if \( E_A \) is endowed with the metric \( d_{k-1} \).
Note that for every \( \tau \in E_A^n, n \geq 1 \), we have \( \hat{\pi}([\tau]) = \bigcup_{\omega \in [\tau]}{\omega} \times J_\omega \).

Let \( M(E_A) \) be the topological space of Borel probability measures on \( E_A \) with the topology of weak convergence and \( M_\sigma(E_A) \) be its closed subspace consisting of \( \sigma \)-invariant measures. Likewise, let \( M(J) \) be the space of Borel probability measures on \( J \) with the topology of weak convergence and let \( M_T(J) \) be its closed subspace of \( T \)-invariant measures. The following fact is well known and easy to prove.

**Lemma 4.11.** Let \( W \) and \( Z \) be Polish spaces. Let \( \mu \) be a Borel probability measure on \( Z \), let \( \hat{\mu} \) be its completion, and denote by \( \hat{B}_\mu \) the complete \( \sigma \)-algebra of all \( \hat{\mu} \)-measurable subsets of \( Z \). Let \( f : W \to Z \) be a Borel measurable surjection and let \( g : W \to \mathbb{R} \) be a Borel measurable function. Define the functions \( g_\ast, g^* : Z \to \mathbb{R} \) respectively by

\[
g_\ast(z) := \inf\{g(w) : w \in f^{-1}(z)\} \quad \text{and} \quad g^*(z) := \sup\{g(w) : w \in f^{-1}(z)\}.
\]

Then these two functions are measurable with respect to the \( \sigma \)-algebra \( \hat{B}_\mu \). If in addition the map \( f : W \to Z \) is locally one-to-one, then both \( g_\ast \) and \( g^* : Z \to \mathbb{R} \) are Borel measurable.

Now we prove the following theorem.

**Theorem 4.12.** If \( T : J \to J \) is a continuous skew product Smale endomorphism of compact type, then the map \( M_\sigma(E_A) \ni \mu \longmapsto \mu \circ \hat{\pi}^{-1} \in M_T(J) \) is surjective.

**Proof.** Fix \( \mu \in M_T(J) \). Let \( B_b(E_A) \) and \( B_b(J) \) be the vector spaces of all bounded Borel measurable real-valued functions defined respectively on \( E_A \) and \( J \). Let \( B_b^+(E_A) \) and \( B_b^+(J) \) be the respective convex cones consisting of non-negative functions. Let

\[
\hat{B}_b(E_A) := \{g \circ \hat{\pi} : g \in B_b(J)\}.
\]

Clearly, \( \hat{B}_b(E_A) \) is a vector subspace of \( B_b(E_A) \) and, as \( \hat{\pi} : E_A \to J \) is a surjection, for each \( h \in \hat{B}_b(E_A) \) there exists a unique \( g \in B_b(J) \) such that \( h = g \circ \hat{\pi} \). Thus, treating, via integration, \( \mu \) as a linear functional from \( B_b(J) \) to \( \mathbb{R} \), the formula

\[
\hat{B}_b(E_A) \ni g \circ \hat{\pi} \longmapsto \hat{\mu}(g \circ \hat{\pi}) := \mu(g) \in \mathbb{R}
\]

defines a positive linear functional from \( \hat{B}_b(E_A) \) to \( \mathbb{R} \). By Lemma 4.11 applied to \( \hat{\pi} : E_A \to \mathbb{R} \), for every \( h \in B_b(E_A) \), the function \( h_\ast \circ \hat{\pi} : E_A \to \mathbb{R} \) belongs to \( \hat{B}_b(E_A) \). Also, \( h - h_\ast \circ \hat{\pi} \geq 0 \); thus, \( h - h_\ast \circ \hat{\pi} \in B_b^+(E_A) \). Riesz’s theorem produces then a positive linear functional \( \mu^* : B_b(E_A) \to \mathbb{R} \) such that \( \mu^*(h) = \hat{\mu}(h) \) for every \( h \in \hat{B}_b(E_A) \). But \( \mu^* \) restricted to the space \( C_b(E_A) \) of bounded continuous functions on \( E_A \) remains linear and positive.

**Claim 1.** If \( (g_n)_{n=1}^\infty \) is a monotone decreasing sequence of non-negative functions in \( C_b(E_A) \) converging pointwise in \( E_A \) to the function identically equal to zero, then \( \lim_{n \to \infty} \mu^*(g_n) \) exists and is equal to zero.

**Proof.** Clearly, \( (g_n^+)_{n=1}^\infty \) is a monotone decreasing sequence of non-negative bounded functions that, by Lemma 4.11, belong to \( B(J) \) and thus to \( B_b^+(J) \). Fix \( y \in J \). Since our map \( T : J \to J \) is of compact type, the set \( \hat{\pi}^{-1}(y) \subset E_A \) is compact. Therefore, Dini’s theorem applies to let us conclude that the sequence \( (g_n|_{\hat{\pi}^{-1}(y)})_{n=1}^\infty \) converges uniformly to
zero. Since all these functions are non-negative, this just means that the sequence \((g_n^*)_{n=1}^{\infty}\) converges to zero. In conclusion, \((g_n^*)_{n=1}^{\infty}\) is a monotone decreasing sequence of functions in \(B_+(J)\) converging pointwise to zero. Therefore, as also \(g_n \leq g_n^* \circ \hat{\pi}\), we get
\[
0 \leq \lim_{n \to \infty} \mu^*(g_n) \leq \lim_{n \to \infty} \mu^*(g_n \circ \hat{\pi}) = \lim_{n \to \infty} \hat{\mu}(g_n^* \circ \hat{\pi}) = \lim_{n \to \infty} \mu(g_n^*) = 0.
\]
So, \(\lim_{n \to \infty} \mu^*(g_n)\) exists and is equal to zero. \(\square\)

Having Claim 1, the Daniell–Stone representation theorem implies that \(\mu^*\) extends uniquely from \(C_b(E_A)\) to an element of \(M(E_A)\). We denote it also by \(\mu^*\).

**Claim 2.** For every \(\varepsilon > 0\), there exists \(K_\varepsilon\), a compact subset of \(E_A\), such that
\[
\hat{\pi}^{-1}(\hat{\pi}(K_\varepsilon)) = K_\varepsilon \text{ and } \mu(\hat{\pi}(K_\varepsilon)) \geq 1 - \varepsilon/2.
\]

**Proof.** Fix \(k \in \mathbb{Z}\) and let \(p_k : E^+ \to E\) be the projection on the \(k\)th coordinate, i.e. \(p_k((\gamma_n)_{n=-\infty}^{\infty}) = \gamma_k\). Fix \(\varepsilon > 0\). We assume without loss of generality that \(E = \{1, 2, \ldots\}\). Since \(T : J \to J\) is of compact type, each set \(\hat{\pi}^{-1}(y) \subset E_A\), \(y \in J\), is compact and thus the function \(p_k^* : J \to \mathbb{R}\), defined in Lemma 4.11, takes values in \(\mathbb{R}\) and is Borel measurable. Thus, \(p_k^* \circ \hat{\pi} : E_A \to \mathbb{N}\) is Borel measurable and there exists \(n_k \geq 1\) so that
\[
\mu((p_k^*)^{-1}([n_k + 1, \infty))) < 2^{-|k|-4} \varepsilon. \tag{4.9}
\]
Since \(\mu\) is inner regular, by Lusin’s theorem, Borel measurability of the function \(p_k^* : J \to \mathbb{N}\) yields the existence of closed subsets \(J_k \subset J\) such that
\[
\mu(J_k) \geq 1 - \varepsilon 2^{-|k|-4} \quad \text{and} \quad p_k^*|_{J_k} : J_k \to \mathbb{N} \text{ is continuous.}
\]

Define
\[
J_\infty := \bigcap_{k \in \mathbb{Z}} J_k.
\]
Then \(J_\infty\) is a closed subset of \(J\),
\[
\mu(J_\infty) \geq 1 - \frac{\varepsilon}{4}, \tag{4.10}
\]
and each map \(p_k^*|_{J_\infty} : J_\infty \to \mathbb{N}\) is continuous. Define also
\[
K_\varepsilon := \bigcap_{k \in \mathbb{Z}} (p_k^*|_{J_\infty} \circ \hat{\pi}|_{\hat{\pi}^{-1}(J_\infty)})^{-1}([n_k + 1, \infty]).
\]
By the definition of the maps \(p_k^*\), we have that
\[
\hat{\pi}^{-1}(\hat{\pi}(K_\varepsilon)) = K_\varepsilon \quad \text{and} \quad \hat{\pi}(K_\varepsilon) = J_\infty \cap \bigcap_{k \in \mathbb{Z}} (p_k^*)^{-1}([n_k + 1, \infty]). \tag{4.11}
\]
Therefore, using (4.10) and (4.9), we get
\[
\mu(J \setminus \hat{\pi}(K_\varepsilon)) \leq \mu(J \setminus J_\infty) + \sum_{k \in \mathbb{Z}} \mu((p_k^*)^{-1}([n_k + 1, \infty))) \leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}. \tag{4.12}
\]
Since \(p_k^*|_{J_\infty}\), \(k \in \mathbb{Z}\), are continuous, \(K_\varepsilon\) is closed in \(E_A\). Also, \(K_\varepsilon \subset \bigcap_{k \in \mathbb{Z}} [1, n_k]\) and this product is compact, so \(K_\varepsilon\) is compact. Using (4.11) and (4.12) completes the proof. \(\square\)
Using that \( \mu \) is \( T \)-invariant, and Urysohn’s approximation method, we show the following claim.

**Claim 3.** If \( \varepsilon > 0 \) and \( K_\varepsilon \subset E_A \) is the compact set produced in Claim 2, then \( \mu^* \circ \sigma^{-j} \geq 1 - \varepsilon \) for all integers \( j \geq 0 \).

**Proof.** Fix \( \varepsilon > 0 \) arbitrary. Fix an integer \( j \geq 0 \). Since measure \( \mu^* \circ \sigma^{-j} \circ \hat{\pi}^{-1} \) is outer regular and \( \hat{\pi}(K_\varepsilon) \) is a Borel (since compact) set, there exists an open set \( U \subset J \) such that

\[
\hat{\pi}(K_\varepsilon) \subset U \quad \text{and} \quad \mu^* \circ \sigma^{-j} \circ \hat{\pi}^{-1}(U \setminus \hat{\pi}(K_\varepsilon)) \leq \varepsilon / 2.
\]

Now Urysohn’s lemma produces a continuous function \( u : J \to [0, 1] \) such that \( u(\hat{\pi}(K_\varepsilon)) = 1 \) and \( u(E_A \setminus U) \subset [0] \). Then, by our construction of \( \mu^* \) and by Claim 2,

\[
\mu^* \circ \sigma^{-j}(K_\varepsilon) = \mu^* \circ \sigma^{-j} \circ \hat{\pi}^{-1}(\hat{\pi}(K_\varepsilon)) \geq \mu^* \circ \sigma^{-j} \circ \hat{\pi}^{-1}(U) - \frac{\varepsilon}{2} = \mu^*(1_U \circ \hat{\pi} \circ \sigma^j) - \frac{\varepsilon}{2} = \mu^*(1_U \circ T^j) - \frac{\varepsilon}{2} \geq \mu^*(u \circ T^j \circ \hat{\pi}) - \frac{\varepsilon}{2} = \mu(u) - \frac{\varepsilon}{2} \geq \mu(\hat{\pi}(K_\varepsilon)) - \frac{\varepsilon}{2} \geq 1 - \varepsilon.
\]

Now, for every \( n \geq 1 \), set

\[
\mu_n^* := \frac{1}{n} \sum_{j=0}^{n-1} \mu^* \circ \sigma^{-j}.
\]

It directly follows from Claim 3 that

\[
\mu_n^*(K_\varepsilon) \geq 1 - \varepsilon
\]

for every \( \varepsilon > 0 \) and all \( n \geq 1 \). Also, since, by Claim 2, each set \( K_\varepsilon \) is compact, the sequence of measures \( (\mu_n^*)_{n=1}^\infty \) is tight with respect to the weak topology on \( M_\sigma(E_A) \). There thus exists \( (n_k)_{k=1}^\infty \), an increasing sequence of positive integers, such that \( (\mu_{n_k}^*)_{k=1}^\infty \) converges weakly; denote its limit by \( \nu \in M(E_A) \). A standard argument shows that \( \nu \in M_\sigma(E_A) \). By the definitions of \( \hat{\mu} \) and \( \mu^* \), we get for every \( g \in C_h(E_A) \) and every \( n \geq 1 \) that

\[
\mu_n^* \circ \hat{\pi}^{-1}(g) = \mu_n^*(g \circ \hat{\pi}) = \frac{1}{n} \sum_{j=0}^{n-1} \mu^* \circ \sigma^{-j} \circ (g \circ \hat{\pi}) = \frac{1}{n} \sum_{j=0}^{n-1} \mu^*(g \circ \hat{\pi} \circ \sigma^j)
\]

\[
= \frac{1}{n} \sum_{j=0}^{n-1} \mu^*(g \circ T^j \circ \hat{\pi}) = \frac{1}{n} \sum_{j=0}^{n-1} \hat{\mu}(g \circ T^j \circ \hat{\pi})
\]

\[
= \frac{1}{n} \sum_{j=0}^{n-1} \mu(g \circ T^j) = \frac{1}{n} \sum_{j=0}^{n-1} \mu(g) = \mu(g).
\]

So, \( \mu_n^* \circ \hat{\pi}^{-1} = \mu \) for all \( n \geq 1 \); thus,

\[
\nu \circ \hat{\pi}^{-1} = \lim_{k \to \infty} \mu_{n_k}^* \circ \hat{\pi}^{-1} = \lim_{k \to \infty} \mu_{n_k}^* \circ \hat{\pi}^{-1} = \mu.
\]
Observation 4.13. If $T$ is a Smale endomorphism and $\mu \in M_\sigma(E_A)$, then
\[
h_{\mu \circ \hat{\pi}^{-1}}(T) = h_\mu(\sigma).
\]

Proof. We have two standard inequalities $h_{\mu \circ \hat{\pi}^{-1}}(T) \leq h_\mu(\sigma)$ and $h_{\mu \circ \hat{\pi}^{-1} \circ \pi_0^{-1}}(\sigma) \leq h_{\mu \circ \hat{\pi}^{-1}}(T)$. But $\pi_0 : E_A \to E_A^+$, $\pi_0(\tau) = \tau|_0^\infty$ is the canonical projection from $E_A$ to $E_A^+$. So, the measure $\mu \in M_\sigma(E_A)$ is Rokhlin’s natural extension of the measure $\mu \circ \hat{\pi}^{-1} \circ \pi_0^{-1} \in M_\sigma(E_A^+)$. Hence, $h_{\mu \circ \hat{\pi}^{-1} \circ \pi_0^{-1}}(\sigma) = h_\mu(\sigma)$. So, from the above inequalities, $h_{\mu \circ \hat{\pi}^{-1}}(T) = h_\mu(\sigma)$.

Now we define the topological pressure of continuous real-valued functions on $J$ with respect to the dynamical system $T : J \to J$. Since the space $J$ is not compact, there is no canonical candidate for such definition and we choose the one which will turn out to behave well on the theoretical level (variational principle) and serves well for practical purposes (Bowen’s formula). For every finite admissible word $\omega \in E_A^+$, let
\[
[\omega]_T =: \hat{\pi}_2([\omega]) \subset J.
\]
If $\psi : J \to \mathbb{R}$ is a continuous function, we define
\[
P(\psi) = P_T(\psi) := \lim_{n \to \infty} \frac{1}{n} \log \sum_{\omega \in C^{n-1}} \exp(\sup(S_n \psi|[\omega]|_T)),
\]
where $S_n \psi = \sum_{j=0}^{n-1} \psi \circ T^j$, $n \geq 1$. The limit above exists since the sequence $\mathbb{N} \ni n \mapsto \log \sum_{\omega \in C^{n-1}} \exp(\sup(S_n \psi|[\omega]|_T))$ is subadditive. We call $P_T(\psi)$ the topological pressure of the potential $\psi : J \to \mathbb{R}$ with respect to the dynamical system $T : J \to J$. As an immediate consequence of this definition and Definition 3.1, we get the following observation.

Observation 4.14. If $\psi : J \to \mathbb{R}$ is a continuous function, then
\[
P_T(\psi) = P_\sigma(\psi \circ \hat{\pi}).
\]

The following theorem follows immediately from Theorem 3.9, Observation 4.14, Theorem 4.12, and Observation 4.13, and we will provide such proof.

Theorem 4.15. If $\psi : J \to \mathbb{R}$ is a continuous function, and $\mu \in M_T(J)$ is such that $\psi \in L^1(J, \mu)$ and $\int J \psi \ d\mu > -\infty$, then $h_\mu(T) + \int J \psi \ d\mu \leq P_T(\psi)$.

Proof. By Theorem 4.12, there exists $\nu \in M_\sigma(E_A)$ such that $\nu \circ \hat{\pi}^{-1} = \mu$. The other theorems listed immediately above give
\[
h_\mu(T) + \int J \psi \ d\mu = h_{\nu \circ \hat{\pi}^{-1}}(T) + \int J \psi \ d(\nu \circ \hat{\pi}^{-1})
\]
\[
= h_\nu(\sigma) + \int_{E_A} \psi \circ \hat{\pi} \ d\hat{\nu} \leq P_\sigma(\psi \circ \hat{\pi}) = P_T(\psi).
\]

We have the following two definitions.
Definition 4.16. The measure $\mu \in M_T(J)$ is called an equilibrium state of the continuous potential $\psi : \hat{Y} \to \mathbb{R}$ if $\int \psi \, d\mu > -\infty$ and
\[ h_\mu(T) + \int_J \psi \, d\mu = P_T(\psi). \]

Definition 4.17. The potential $\psi : J \to \mathbb{R}$ is called summable if
\[ \sum_{e \in E} \exp(\sup(\psi|_{[e]_T})) < \infty. \]

Observation 4.18. The potential $\psi : J \to \mathbb{R}$ is summable if and only if $\psi \circ \hat{\pi} : E_A \to \mathbb{R}$ is summable.

Definition 4.19. We call a continuous skew product Smale endomorphism $T : \hat{Y} \to \hat{Y}$ Hölder if the projection $\hat{\pi} : E_A \to J$ is Hölder continuous.

We now establish an important property of Hölder skew product Smale endomorphisms of compact type and then will describe a general construction of such endomorphisms.

Theorem 4.20. If $T : J \to J$ is a Hölder skew product Smale endomorphism of compact type and $\psi : J \to \mathbb{R}$ is a Hölder summable potential, then $\psi$ admits a unique equilibrium state, denoted by $\mu_\psi$. In addition, $\mu_\psi = \mu_{\psi \circ \hat{\pi}} \circ \hat{\pi}^{-1}$, where $\mu_{\psi \circ \hat{\pi}}$ is the unique equilibrium state of $\psi \circ \hat{\pi} : E_A \to \mathbb{R}$ with respect to $\sigma : E_A \to E_A$.

Proof. The potential $\psi \circ \hat{\pi} : E_A \to \mathbb{R}$ is a summable Hölder continuous potential, so it has a unique equilibrium state $\mu_{\psi \circ \hat{\pi}}$ by Theorem 2.6. By Observations 4.14 and 4.4, we have
\begin{align*}
    h_T(\mu_{\psi \circ \hat{\pi}} \circ \hat{\pi}^{-1}) + \int_J \psi \, d(\mu_{\psi \circ \hat{\pi}} \circ \hat{\pi}^{-1}) &= h_\sigma(\mu_{\psi \circ \hat{\pi}}) + \int_{E_A} \psi \circ \hat{\pi} \, d(\mu_{\psi \circ \hat{\pi}}) \\
    &= P_\sigma(\psi \circ \hat{\pi}) \\
    &= P_T(\psi).
\end{align*}

We have to show that, if $\mu$ is an equilibrium measure of $\psi$, then $\mu = \mu_{\psi \circ \hat{\pi}} \circ \hat{\pi}^{-1}$. In this case, from Theorem 4.12, we get $\mu = \nu \circ \hat{\pi}^{-1}$ for some $\nu \in M_\sigma(E_A)$. Then, by Observation 4.14,
\begin{align*}
    h_\nu(\sigma) + \int_{E_A} \psi \circ \hat{\pi} \, d\nu &= h_{\nu \circ \hat{\pi}^{-1}}(T) + \int_J \psi \, d(\nu \circ \hat{\pi}^{-1}) = h_\mu(T) + \int_J \psi \, d\mu \\
    &= P_T(\psi) \\
    &= P_\sigma(\psi \circ \hat{\pi}).
\end{align*}

Hence, $\nu$ is an equilibrium state of $\psi \circ \hat{\pi} : E_A \to \mathbb{R}$ and $\nu = \mu_{\psi \circ \hat{\pi}}$ (see Theorem 2.6). □

Now we provide the promised construction of Hölder Smale skew product endomorphisms. Start with $(Y, d)$, a complete bounded metric space, and assume that we are given for every $\omega \in E_A^+$ a continuous closed injective map $T_\omega : Y \to Y$ satisfying the following conditions:
\[ d(T_\omega(y_2), T_\omega(y_1)) \leq \lambda^{-1} d(y_2, y_1). \]  

(4.13)
for all \( y_1, y_2 \in Y \) and some \( \lambda > 1 \) independent of \( \omega \),

\[
d_{\infty}(T_\beta, T_\alpha) := \sup\{d(T_\beta(\xi), T_\alpha(\xi)) : \xi \in Y\} \leq Cd_\kappa(\beta, \alpha)
\]

(4.14)

with some constants \( C \in (0, \infty), \kappa > 0 \), and all \( \alpha, \beta \in E_A^+ \). Then

\[
\hat{Y} = E_A^+ \times Y,
\]

and we call \( T : \hat{Y} \to \hat{Y} \) a skew product Smale system of global character. We may assume without loss of generality that

\[
\kappa \leq \frac{1}{2} \log \lambda.
\]

(4.15)

**Theorem 4.21.** Each skew product Smale system of global character is Hölder.

**Proof.** Let \( T : E_A^+ \times Y \to E_A^+ \times Y \) be such a skew product Smale system. We first show that \( T : E_A^+ \times Y \to E_A^+ \times Y \) is continuous. It is enough to show that \( p_2 \circ T : E_A^+ \times Y \to Y \) is continuous, with \( p_2 \) the projection to the second coordinate. For all \( \alpha, \beta \in E_\omega^+ \) and \( z, w \in Y \),

\[
d(p_2 \circ T(\alpha, z), p_2 \circ T(\beta, w)) = d(T_\alpha(z), T_\beta(w))
\]

\[
\leq d(T_\alpha(z), T_\beta(z)) + d(T_\beta(z), T_\beta(w))
\]

\[
\leq d_{\infty}(T_\alpha, T_\beta) + \lambda^{-1} d(z, w)
\]

\[
\leq Cd_\kappa(\alpha, \beta) + \lambda^{-1} d(z, w)
\]

and continuity of the map \( p_2 \circ T : E_A^+ \times Y \to Y \) is proved. So, the continuity of \( T : E_A^+ \times Y \to E_A^+ \times Y \) is proved and thus \( T : J \to J \) is continuous too. We now show that \( T : J \to J \) is Hölder. So, fix an integer \( n \geq 0 \), two words \( \alpha, \beta \in E_A \), and \( \xi \in Y \). We then have

\[
d(T_{\alpha}^{n+1}(\xi), T_{\beta}^{n+1}(\xi)) = d(T_\alpha^n(T_\alpha^{(\infty)}(\xi)), T_\beta^n(T_\beta^{(\infty)}(\xi)))
\]

\[
\leq d(T_\alpha^n(T_\alpha^{(\infty)}(\xi)), T_\beta^n(T_\beta^{(\infty)}(\xi)))
\]

\[
+ d(T_\beta^n(T_\beta^{(\infty)}(\xi)), T_\beta^n(T_\beta^{(\infty)}(\xi)))
\]

\[
\leq \lambda^{-n} d(T_\alpha^{[\infty]}(\xi), T_\beta^{[\infty]}(\xi)) + d_{\infty}(T_\alpha^n, T_\beta^n)
\]

\[
\leq \lambda^{-n} Cd_\kappa(\alpha^{[\infty]}, \beta^{[\infty]}) + d_{\infty}(T_\alpha^n, T_\beta^n).
\]

(4.16)

Let \( p \geq -1 \) be uniquely determined by the property that

\[
d_\kappa(\alpha, \beta) = e^{-kp}.
\]

(4.17)

Consider two cases. First assume that \( d_\kappa(\alpha, \beta) \geq e^{-kn} \). Then, using also (4.15), we get

\[
\lambda^{-n} d_\kappa(\alpha^{[\infty]}, \beta^{[\infty]}) \leq e^{-2kn} \leq e^{-kn} d_\kappa(\alpha, \beta).
\]

(4.18)

So, assume that

\[
d_\kappa(\alpha, \beta) < e^{-kn}.
\]

(4.19)

Then \( n < p \), so \( n + 1 \leq p \), whence

\[
d_\kappa(\alpha^{[\infty]}, \beta^{[\infty]}) = e^{-k(n+2)} e^{-kp} = e^{-k(n+2)} d_\kappa(\alpha, \beta) \leq e^{-kn} d_\kappa(\alpha, \beta).
\]

Hence,

\[
\lambda^{-n} d_\kappa(\alpha^{[\infty]}, \beta^{[\infty]}) \leq e^{-kn} d_\kappa(\alpha, \beta).
\]
Inserting this and (4.18) into (4.16) in either case yields
\[ d(T^{n+1}_\alpha(\xi), T^{n+1}_\beta(\xi)) \leq d_\infty(T^n\alpha, T^n\beta) + C e^{-\kappa n} d_\kappa(\alpha, \beta). \]

Taking the supremum over all \( \xi \in Y \), we get \( d_\infty(T^{n+1}_\alpha, T^{n+1}_\beta) \leq d_\infty(T^n\alpha, T^n\beta) + C e^{-\kappa n} d_\kappa(\alpha, \beta) \). Thus, by induction,
\[ d_\infty(T^n_\alpha, T^n_\beta) \leq C d_\kappa(\alpha, \beta) \sum_{j=0}^{n-1} e^{-\kappa j} \leq C d_\kappa(\alpha, \beta) \sum_{j=0}^{\infty} e^{-\kappa j} = C(1 - e^{-\kappa})^{-1} d_\kappa(\alpha, \beta) \]

for all \( \alpha, \beta \in E_A \) and all integers \( n \geq 0 \). Recall that the integer \( p \geq -1 \) is determined by (4.17). Assume that \( p \geq 0 \). Then, using (4.20), (4.19), and (4.2), we get
\[ d(\hat{\pi}_2(\alpha), (\hat{\pi}_2(\alpha)) \leq \text{diam}(T^p_\alpha(Y)) + \text{diam}(T^p_\beta(Y)) + d_\infty(T^p_\alpha, T^p_\beta) \]
\[ = \lambda^{-p} \text{diam}(Y) + \lambda^{-p} \text{diam}(Y) + \frac{C}{1 - e^{-\kappa}} d_\kappa(\alpha, \beta) \]
\[ \leq 2 \text{diam}(Y) d_\kappa(\log \lambda)/\kappa(\alpha, \beta) + \frac{C}{1 - e^{-\kappa}} d_\kappa(\alpha, \beta). \]

Thus, \( \hat{\pi}_2 : E_A \to Y \) is Hölder continuous, so \( \pi : E_A \to Y \) is Hölder continuous.

\[ \square \]

5. **Conformal skew product Smale endomorphisms**

In this section we keep the setting of skew product Smale endomorphisms. However, we assume more about the spaces \( Y_\omega, \omega \in E_A^* \), and the fiber maps \( T_\omega : Y_\omega \to Y_{\sigma(\omega)} \), namely the following.
(a) \( Y_\omega \) is a closed bounded subset of \( \mathbb{R}^d \), with some \( d \geq 1 \) such that \( \text{Int}(Y_\omega) = Y_\omega \).
(b) Each map \( T_\omega : Y_\omega \to Y_{\sigma(\omega)} \) extends to a \( C^1 \) conformal embedding from \( Y_\omega^* \) to \( Y_{\sigma(\omega)}^* \), where \( Y_\omega^* \) is a bounded connected open subset of \( \mathbb{R}^d \) containing \( Y_\omega \). The same symbol \( T_\omega \) denotes this extension and we assume that \( T_\omega : Y_\omega^* \to Y_{\sigma(\omega)}^* \) satisfies condition (c):
(c) Formula (4.1) holds for all \( y_1, y_2 \in Y_\omega^* \), perhaps with some smaller constant \( \lambda > 1 \).
(d) (Bounded distortion property 1) There exist constants \( \alpha > 0 \) and \( H > 0 \) such that for all \( y, z \in Y_\omega^* \), we have that
\[ |\log |T'_\omega(y)|| - \log |T'_\omega(z)||| \leq H||y - z||^\alpha. \]
(e) The function \( E_A \ni \tau \mapsto \log |T'_\omega(\hat{\pi}_2(\omega))| \) \( \in \mathbb{R} \) is Hölder continuous.
(f) (Open set condition) For every \( \omega \in E_A^+ \) and for all \( a, b \in E \) with \( A_\omega a_\omega = A_\omega b_\omega = 1 \) and \( a \neq b \), we have
\[ T_{\omega a_\omega}(\text{Int}(Y_{\omega a_\omega})) \cap T_{\omega b_\omega}(\text{Int}(Y_{\omega b_\omega})) = \emptyset. \]
(g) (Strong open set condition) There exists a measurable function \( \delta : E_A^+ \to (0, \infty) \) such that for every \( \omega \in E_A^+ \),
\[ J_\omega \cap (Y_\omega \setminus \overline{B(Y_\omega, \delta(\omega))}) \neq \emptyset. \]

Any skew product Smale endomorphism satisfying conditions (a)–(g) will be called in the following a **conformal skew product Smale endomorphism**.

A standard calculation based on (c), (d), and (e) yields the following.
(BDP2) (Bounded distortion property 2) For some constant $H$, we have that

$$|\log |(T^n_\tau)'(y)| - \log |(T^n_\tau)'(z)|| \leq H ||y - z||^\alpha$$

for all $\tau \in E_A$, $y, z \in Y^*_{\tau|_{-n}}$, and all $n > 0$.

An immediate consequence of (BDP2) is the following version.

(BDP3) (Bounded distortion property 3) For all $\tau \in E_A$, all $n \geq 0$, and all $y, z \in Y^*_{\tau|_{-n}}$, if $K := \exp(H \text{diam}^\alpha(Y))$, then we have that

$$K^{-1} \leq |(T^n_\tau)'(y)| \leq K.$$

Remark 5.1. Bounded distortion property 1, (d), is always satisfied if $d \geq 2$. If $d = 2$, this is due to the Koebe distortion theorem since each conformal map in $\mathbb{C}$ is either holomorphic or antiholomorphic. If $d \geq 3$, this follows from the Liouville representation theorem saying that conformal maps in $\mathbb{R}^d$, $d \geq 3$, are either Möbius transformations or similarities.

Recall also that we say that a conformal skew product Smale endomorphism is Hölder if the condition of Hölder continuity for $\hat{\pi} : E_A \to J$ is satisfied; see Definition 4.19.

As an immediate consequence of the open set condition (f), we get the following lemma.

**Lemma 5.2.** Let $T : \hat{Y} \to \hat{Y}$ be a conformal skew product Smale endomorphism. If $n \geq 1$, $\alpha, \beta \in E_A(-n, \infty)$, $\alpha|_0^\infty = \beta|_0^\infty$, and $\alpha|_{-n}^{-1} = \beta|_{-n}^{-1}$, then

$$T^n_\alpha(\text{Int}(Y_\alpha)) \cap T^n_\beta(\text{Int}(Y_\beta)) = \emptyset.$$  

In fact, we have more:

$$T^n_\alpha(\text{Int}(Y_\alpha)) \cap T^n_\beta(\text{Int}(Y_\beta)) = \emptyset = T^n_\alpha(Y_\alpha) \cap T^n_\beta(\text{Int}(Y_\beta)).$$

**Lemma 5.3.** Let $T : \hat{Y} \to \hat{Y}$ be a conformal skew product Smale endomorphism. If $n \geq 1$ and $\tau \in E_A(-n, \infty)$, then

$$\hat{\pi}_2^{-1}(T^n_\tau(\text{Int}(Y_\tau))) \subset [\tau].$$

**Proof.** Let $\gamma \in \hat{\pi}_2^{-1}(T^n_\tau(\text{Int}(Y_\tau)))$ and hence $\gamma|_0^\infty = \tau|_0^\infty$ and

$$\hat{\pi}_2(\gamma) \in T^n_\tau(\text{Int}(Y_\tau)) \subset Y_{\tau|_0^\infty}.$$  

Also, $\hat{\pi}_2(\gamma) \in T^n_{\gamma|_{-n}^\infty}(Y_{\gamma|_{-n}^\infty})$. From Lemma 5.2, it follows that $\gamma|_{-n}^0 = \tau$, so $\gamma \in [\tau]$. □

We will also use the following.

(h) (Uniform geometry condition) There exists $R > 0$ so that for all $\omega \in E_A^+$, there exists $\xi_\omega \in Y_\omega$ with

$$B(\xi_\omega, R) \subset Y_\omega.$$  

The primary significance of the uniform geometry condition (h) lies in the following lemma.
Lemma 5.4. If \( T : \hat{Y} \to \hat{Y} \) is a Hölder conformal skew product Smale endomorphism satisfying the uniform geometry condition \((h)\), then for every \( \gamma \geq 1 \) there exists \( \Gamma_\gamma > 0 \) such that if \( \mathcal{F} \subset E^+_A(\infty, -1) \) is a collection of mutually incomparable \((finite)\) words, so that \( A_{\tau_{-100}} = 1 \) for some \( \omega \in E^+_A \) and all \( \tau \in \mathcal{F} \), and so that for some \( \xi \in Y_\omega \),
\[
T_{\tau_0}^{|\tau|}(Y_{\tau_0}) \cap B(\xi, r) \neq \emptyset \quad \text{with} \quad \gamma^{-1}r \leq \text{diam}(T_{\tau_0}^{|\tau|}(Y_{\tau_0})) \leq yr,
\]
then the cardinality of \( \mathcal{F} \) is bounded above by \( \Gamma_\gamma \).

Proof. The family \( \{T_{\tau_0}^{|\tau|}(\text{Int}(Y_{\omega_T})) : \tau \in \mathcal{F}\} \) consists of mutually disjoint sets in \( Y_\omega \). We get
\[
T_{\tau_0}^{|\tau|}(\text{Int}(Y_{\tau_0})) \supset T_{\tau_0}^{|\tau|}(B(\xi_{\tau_0}, R))
\]
\[
\supset B(T_{\tau_0}^{|\tau|}(\xi_{\tau_0}, K^{-1}R|(T_{\tau_0}^{|\tau|})'(\xi_{\tau_0}))
\]
\[
\supset B(T_{\tau_0}(\xi_{\tau_0}), K^{-2}r\gamma^{-1}r)
\]
from the uniform geometry condition. Also, \( T_{\tau_0}^{|\tau|}(\text{Int}(Y_{\omega_T})) \subset B(\xi, (1 + \gamma)r) \).

\[ \square \]

6. Volume lemmas

We keep the setting of §5, with \( T : \hat{Y} \to \hat{Y} \) a conformal skew product Smale endomorphism, i.e. satisfying conditions \((a)\)–\((g)\) of §5.

Definition 6.1. We say in general that a measure \( \mu \) is exact dimensional on a space \( X \) if its pointwise dimension \( d_\mu(x) \) at \( x \), defined by the formula
\[
d_\mu(x) = \lim_{r \to 0} \frac{\log \mu(B(x, r))}{\log r},
\]
exists for \( \mu \)-a.e. \( x \in X \) and \( d_\mu(\cdot) \) is constant \( \mu \)-almost everywhere (see [32]).

If \( \mu \) is a Borel probability \( \sigma \)-invariant measure on \( E_A \), then by \( \chi_\mu(\sigma) \) we denote its Lyapunov exponent, defined by the formula
\[
\chi_\mu(\sigma) := -\int_{E_A} \log |T'_{\tau_0}(\eta_2(\tau))| \, d\mu(\tau) = -\int_{E_A^+} \int_{[\omega]} \log |T'_{\omega}(\eta_2(\tau))| \, d\mu^\omega(\tau) \, dm(\omega),
\]
where \( m = \mu \circ \pi_0^{-1} = \pi_1 \circ \mu \) is the canonical projection of \( \mu \) onto \( E^+_A \).

We prove now the exact dimensionality of projections of conditional measures for equilibrium states onto the fibers.

Theorem 6.2. (Exact dimensionality of conditional measures in fibers) Let \( T : \hat{Y} \to \hat{Y} \) be a Hölder conformal skew product Smale endomorphism and let \( \psi : E_A \to \mathbb{R} \) be a Hölder continuous summable potential. Then for the projection \( \hat{\pi}_2 \circ \hat{\pi}_1^{-1} \) of the conditional measure to the fiber \( J_\omega \), we have
\[
\text{HD}(\hat{\mu}^\omega_\psi \circ \hat{\pi}_2^{-1}) = \frac{h_{\mu_\psi}(\sigma)}{\chi_{\mu_\psi}(\sigma)} = \frac{P_\sigma(\psi) - \int \psi \, d\mu_\psi}{\chi_{\mu_\psi}(\sigma)}
\]
for \( m_\psi \)-a.e. \( \omega \in E^+_A \), where \( m_\psi = \mu_\psi \circ \pi_0^{-1} \). Moreover, for \( m_\psi \)-a.e. \( \omega \in E^+_A \) the measure \( \hat{\mu}^\omega_\psi \circ \hat{\pi}_2^{-1} \) is exact dimensional and its pointwise dimension is given by
\[
\lim_{r \to 0} \frac{\log \hat{\mu}^\omega_\psi \circ \hat{\pi}_2^{-1}(B, r)}{\log r} = \frac{h_{\mu_\psi}(\sigma)}{\chi_{\mu_\psi}(\sigma)} \quad (6.1)
\]
for \( m_\psi \)-a.e. \( \omega \in E^+_A \) and \( \hat{\mu}^\omega_\psi \circ \hat{\pi}_2^{-1} \)-a.e. \( z \in J_\omega \).
To prove the opposite inequality, note that the set $E_A \ni E_A \subset E_A$ such that $\mu_\psi(E_A, \psi) = 1$,

$$\lim_{n \to \infty} \frac{1}{n} \log |(T^n_\tau')(\hat{\tau}_2(\sigma^{-n}(\tau)))| = -\chi_{\mu_\psi}(\sigma)$$

and

$$\lim_{n \to \infty} \frac{1}{n} S_n \psi(\sigma^{-n}(\tau)) = \int_{E_A} \psi \ d\mu_\psi$$

(6.2)

for every $\tau \in E_A, \psi$. For arbitrary $\omega \in E_A^+$, denote now

$$v_\omega := \mu_\psi \circ \hat{\tau}_2^{-1},$$

which is a Borel probability measure on $J_\omega$. Fix $\tau \in E_A, \psi$. Fix a radius

$$r \in (0, \text{diam}(Y_{p_2(\tau)}/2)).$$

Let $z = \hat{\tau}_2(\tau)$ and consider the least integer $n = n(z, r) \geq 0$ so that

$$T^n_\tau(Y_{|\infty_n}) \subset B(z, r).$$

If $r > 0$ is small enough (depending on $\tau$), then $n \geq 1$ and $T^{n-1}_\tau(Y_{|\infty_n}) \not\subset B(z, r)$. Since $z \in T^{n-1}_\tau(Y_{|\infty_n})$, this implies that

$$\text{diam}(T^{n-1}_\tau(Y_{|\infty_n})) \geq r.$$  

(6.3)

Write $\omega := \tau|\infty_n$. It follows from (6.3), Lemma 4.9, and Theorem 3.11 that

$$v_\omega(B(z, r)) \geq v_\omega(\hat{\tau}_2([\tau|\infty_n])) = \mu_\psi \circ \hat{\tau}_2^{-1}(\hat{\tau}_2([\tau|\infty_n])) \geq \mu_\psi([\tau|\infty_n])$$

$$\geq D^{-1} \exp(S_n \psi(\sigma^{-n}(\tau)) - P_\sigma(\psi) n).$$

(6.4)

By taking logarithms and using (6.4), this gives that

$$\frac{\log v_\omega(B(z, r))}{\log r} \leq -\log D + S_n \psi(\sigma^{-n}(\tau)) - P_\sigma(\psi) n.$$  

So, applying (BDP3), we get that

$$\frac{\log v_\omega(B(z, r))}{\log r} \leq -\log D + S_n \psi(\sigma^{-n}(\tau)) - P_\sigma(\psi) n$$

$$\log K + \log(\text{diam}(Y_{|\infty_n})) + \log |(T^{n-1}_\tau)'(\hat{\tau}_2(\sigma^{-n}(\tau)))|,$$

so by dividing both numerator and denominator by $n$, and using that $\text{diam}(Y_{|\infty_n}) = \text{diam}(Y)$ and (6.2), this yields

$$\lim_{r \to 0} \frac{\log v_\omega(B(z, r))}{\log r} \leq \lim_{n \to \infty} (1/n) S_n \psi(\sigma^{-n}(\tau)) - P_\sigma(\psi)$$

$$\lim_{n \to \infty} (1/n) \log |(T^{n-1}_\tau)'(\hat{\tau}_2(\sigma^{-n}(\tau)))|$$

$$= P_\sigma(\psi) - \int \psi \ d\mu_\psi$$

$$\chi_{\mu_\psi}(\sigma).$$

(6.6)

To prove the opposite inequality, note that the set $\hat{\tau}_2^{-1}(J_\omega \backslash B(Y_{\omega}^c, \delta(\omega)))$ is open in $[\omega] \subset E_A$, it is not empty by (g), and thus

$$\mu_\psi(\hat{\tau}_2^{-1}(J_\omega \backslash B(Y_{\omega}^c, \delta(\omega)))) > 0$$
for every \( \omega \in E_A^+ \). Consequently, \( \mu_\psi(Z) > 0 \), where

\[
Z := \bigcup_{\omega \in E_A^+} \hat{\pi}^{-1}_2(J_\omega \setminus \overline{B}(Y^c_\omega, \delta(\omega))).
\]

Since \( \delta : E_A^+ \to (0, \infty) \) is measurable, there exists \( R > 0 \) such that \( \mu_\psi(Z_R) > 0 \), where

\[
Z_R := \bigcup_{\omega \in E_A^+} \hat{\pi}^{-1}_2(J_\omega \setminus \overline{B}(Y^c_\omega, R)).
\]

Consider the set \( N(\tau) := \{ k \geq 0 : \sigma^{-k}(\tau) \in Z_R \} \). Represent this set \( N(\tau) \) as a strictly increasing sequence \( (k_n(\tau))_{n=1}^\infty \). By Birkhoff’s ergodic theorem, there is a measurable set \( \tilde{E}_A, \psi \subset E_A, \psi \) with \( \mu_\psi(\tilde{E}_A, \psi) = 1 \) and, for every \( \tau' \in \tilde{E}_A, \psi \),

\[
\lim_{n \to \infty} \frac{\text{Card}\{0 \leq i \leq n, \sigma^{-i}(\tau') \in Z_R\}}{n} = \mu_\psi(Z_R).
\]

Now we put \( k_n(\tau) \geq n \), instead of \( n \) above, and notice that

\[
\text{Card}\{0 \leq i \leq k_n(\tau), \sigma^{-i}(\tau') \in Z_R\} = n.
\]

Therefore, as \( \mu_\psi(Z_R) > 0 \), we obtain for every \( \tau \in \tilde{E}_A, \psi \) and any \( n \) large, that

\[
\lim_{n \to \infty} \frac{k_n(\tau)}{n} = \frac{1}{\mu_\psi(Z_R)}.
\]

Hence, for every \( \tau \in \tilde{E}_A, \psi \),

\[
\lim_{n \to \infty} \frac{k_{n+1}(\tau)}{k_n(\tau)} = 1. \tag{6.7}
\]

Fix \( \tau \in \tilde{E}_A, \psi \), \( \omega = \tau|_0^\infty \), and let the largest \( n = n(\tau, r) \geq 1 \) be such that with \( j \geq 1 \),

\[
K^{-1} \cdot |(T^{k_n}_\tau)(\hat{\pi}_2(\sigma^{-k_n}(\tau)))| R \geq r. \tag{6.8}
\]

Then

\[
K^{-1} \cdot |(T^{k_{n+1}}_\tau)(\hat{\pi}_2(\sigma^{-k_{n+1}}(\tau)))| R < r. \tag{6.9}
\]

It follows from (6.8) and (BDP3) that \( B(z, r) \subset T^{k_n}_\tau(B(\hat{\pi}_2(\sigma^{-k_n}(\tau)), R)) \subset T^{k_n}_\tau(\text{Int}(Y^c|_{-k_n} \tau)) \). Hence, invoking also Lemma 5.3 and Theorem 3.11, we infer that

\[
v_\omega(B(z, r)) \leq \overline{\mu}_\psi^o(\tau|_{-k_n}^\infty \tau) \leq D \exp(S_{k_n} \psi(\sigma^{-k_n}(\tau)) - P_\sigma(\psi)k_n).
\]

By taking logarithms and using (6.9), this gives

\[
\log \frac{v_\omega(B(z, r))}{\log r} \geq \frac{\log D + S_{k_n} \psi(\sigma^{-k_n}(\tau)) - P_\sigma(\psi)k_n}{-\log K + \log |(T^{k_n+1}_\tau)(\hat{\pi}_2(\sigma^{-k_{n+1}}(\tau)))|}.
\]

Dividing both numerator and denominator above by \( k_n \), and using (6.2) and (6.7), yields

\[
\lim_{r \to 0} \frac{\log \frac{v_\omega(B(z, r))}{\log r}}{\log r} \geq \lim_{n \to \infty} \frac{1}{k_n} S_{k_n} \psi(\sigma^{-k_n}(\tau)) - P_\sigma(\psi)
\]

\[
= \frac{P_\sigma(\psi) - \int \psi d\mu_\psi}{\chi_{\mu_\psi}(\sigma)}.
\]

From (6.6), it follows that (6.1) holds for all \( \tau \in \tilde{E}_A, \psi \). \( \square \)
If \( \mu \) is now a Borel probability \( T \)-invariant measure on the fibered limit set \( J \), then by \( \chi_\mu(T) \) we again denote its Lyapunov exponent, defined by

\[
\chi_\mu(T) := -\int_J \log |T'_\omega(z)| \, d\mu(\omega, z) = -\int_{E_A^+} \int_{J_\omega} \log |T'_\omega(z)| \, d\mu^\omega(z) \, dm(\omega),
\]

where \( m = \mu \circ \pi_0^{-1} \) is the projection of \( \mu \) onto \( E_A^+ \) and \( (\mu^\omega)_{\omega \in E_A^+} \) is the canonical system of conditional measures of \( \mu \) for the measurable partition \( \{ \omega \times J_\omega \}_{\omega \in E_A^+} \).

Now we prove the following corollary.

**Corollary 6.3.** Let \( T : \hat{Y} \to \hat{Y} \) be a Hölder conformal Smale endomorphism of compact type. Let \( \psi : J \to \mathbb{R} \) be a Hölder continuous summable potential. Then

\[
\text{HD}(\mu^\psi) = \frac{\chi_\mu^\psi(T)}{\chi_\mu^\psi(T)} = \frac{P_T(\psi) - \int \psi \, d\mu^\psi}{\chi_\mu^\psi(T)}
\]

for \( \mu_\psi \)-a.e. \( \omega \in E_A^+ \), where \( \mu_\psi = \mu_\psi \circ \hat{\pi}^{-1} \). Moreover, for \( \mu_\psi \)-a.e. \( \omega \in E_A^+ \), the measure \( \mu^\psi \) is exact dimensional and, for \( \mu_\psi \)-a.e. \( \omega \in E_A^+ \) and \( \mu^\psi \)-a.e. \( z \in J_\omega \),

\[
\lim_{r \to 0} \frac{\log \mu^\psi(B(z, r))}{\log r} = \frac{\chi_\mu^\psi(T)}{\chi_\mu^\psi(T)}.
\]

**Proof.** Let \( \hat{\psi} := \psi \circ \hat{\pi} : E_A \to \mathbb{R} \). By Theorem 4.20, \( \mu_\hat{\psi} = \mu_\psi \circ \hat{\pi}^{-1} \) is the unique equilibrium state of the potential \( \hat{\psi} \) and the shift map \( \sigma : E_A \to E_A \). By Observation 4.14, \( P_T(\hat{\psi}) = P_\sigma(\hat{\psi}) \) and, by Observation 4.13, \( h_\mu^\psi(T) = h_\mu^\psi(\sigma) \). Since in addition \( \chi_\mu^\psi(T) = \chi_\mu^\psi(\sigma) \), the proof follows immediately from Theorem 6.2 applied to \( \hat{\psi} : E_A \to \mathbb{R} \). \( \square \)

The uniform geometry condition (h) was not required in this section; it will be used in the next one.

7. **Bowen’s formula**

We keep the setting of §§5 and 6, so \( T : \hat{Y} \to \hat{Y} \) is a conformal skew product Smale endomorphism, i.e. satisfies conditions (a)–(g) of §5. We however emphasize that in §7, condition (h), i.e. the uniform geometry condition, is assumed.

For every \( t \geq 0 \), let \( \psi_t : J \to \mathbb{R} \) be the function

\[
\psi_t(\omega, y) = -t \log |T'_\omega(y)|.
\]

Define \( \mathcal{F}(T) \) to be the set of parameters \( t \geq 0 \) for which the potential \( \psi_t \) is summable, i.e.

\[
\sum_{\sigma \in E} \exp(\sup(\psi_t[\sigma])) < \infty.
\]

This means that

\[
\sum_{\sigma \in E} \sup[|T_{e\sigma}|_\infty : \sigma \in E_A(1, \infty), \ A_{e\sigma} = 1] < \infty.
\]

For every \( t \geq 0 \), let

\[
P(t) := P_T(\psi_t)
\]

and call \( P(t) \) the topological pressure of the parameter \( t \). From Proposition 3.6, we have \( \mathcal{F}(T) = \{ t \geq 0 : P(t) < \infty \} \). We record the following basic properties of this pressure.
Proposition 7.1. The pressure function $t \mapsto P(t), \ t \in [0, \infty)$ has the following properties:

(a) $P$ is monotone decreasing;
(b) $P|_F(T)$ is strictly decreasing;
(c) $P|_F(T)$ is convex, real-analytic, and Lipschitz continuous.

Proof. All these statements except real analyticity follow easily from the definitions, plus, due to Lemma 3.4 and Observation 4.14, from their one-sided shift counterparts. □

Now we can define two significant numbers associated with the Smale endomorphism $T$:

$$\theta_T := \inf \{ t \geq 0 : P(t) < \infty \} \quad \text{and} \quad B_T := \inf \{ t \geq 0 : P(t) \leq 0 \}.$$

The number $B_T$ is called Bowen’s parameter of the system $T$. Clearly, $\theta_T \leq B_T$.

Theorem 7.2. If $T : \hat{Y} \to \hat{Y}$ is a Hölder conformal skew product Smale endomorphism satisfying the uniform geometry condition (h), then, for every $\omega \in E_A^+$,

$$\text{HD}(J_\omega) = B_T.$$ 

Theorem 7.3. If $T : \hat{Y} \to \hat{Y}$ is a Hölder conformal skew product Smale endomorphism satisfying the uniform geometry condition (h) and the alphabet $E$ is finite, then $\bar{p}_{B_T}^\omega$ is an Ahlfors regular measure on $J_\omega$ for every $\omega \in E_A^+$. In particular, for every $\omega \in E_A^+$,

$$\text{HD}(J_\omega) = B_T.$$ 

Proof. Put $h := B_T$. Fix $\omega \in E_A^+$ and $z = \hat{\pi}_2(\tau) \in J_\omega$ arbitrary. Let $n = n(z, r)$ be given by (6.3) and denote $\nu_\omega := \bar{p}_{B_T}^\omega \circ \hat{\pi}_2^{-1}$. The formula (6.5) gives, for $\psi = \psi_h$,

$$\nu_\omega(B(z, r)) \geq D^{-1} \exp(S_n \psi(\sigma^{-n}(\tau))) = D^{-1}|(T^n_T)'(\hat{\pi}_2(\sigma^{-n}(\tau)))|^h. \quad (7.1)$$
Now, since \( E_A \) is compact (as \( E \) is finite) and since \( E_A \ni \tau \mapsto |T'_\tau(\hat{\pi}_2(\tau))| \in (0, \infty) \) is continuous, we conclude that there exists a constant \( M \in (0, \infty) \) such that

\[
M^{-1} \leq \inf\{|T'_\tau(\hat{\pi}_2(\tau))| : \tau \in E_A\} \leq \sup\{|T'_\tau(\hat{\pi}_2(\tau))| : \tau \in E_A\} \leq M. \tag{7.2}
\]

Having this and inserting (6.4) into (7.1), we get

\[
\nu_\omega(B(z, r)) \geq (DM^h)^{-1} r^h. \tag{7.3}
\]

To prove an appropriate inequality in the opposite direction, let

\[
\mathcal{F}(z, r) := \{ \tau \in E^*_A(-\infty, -1) : T'_{\tau\omega}(Y_{\tau\omega}) \cap B(z, r/2) \neq \emptyset, \quad \text{diam}(T'_{\tau\omega}(Y_{\tau\omega})) \leq r/2 \text{ and diam}(T'_{\tau\omega}(Y_{\tau\omega})) > r/2 \}.
\]

Thus, \( \mathcal{F}(z, r) \) consists of mutually incomparable elements of \( E^*_A(-\infty, -1) \), so, using (7.2) and (BDP3), we get for every \( \tau \in \mathcal{F}(z, r) \) with \( n := |\tau| \) that

\[
\text{diam}(T^n_{\tau\omega}(Y_{\tau\omega})) = \text{diam}(T^{n-1}_{\tau\omega}(T_{\tau\omega}(Y_{\tau\omega}))) \geq K^{-1}\|T^{n-1}_{\tau\omega}(Y_{\tau\omega})\| \text{diam}(T_{\tau\omega}(Y_{\tau\omega})) \\
\geq K^{-2}\|T^{n-1}_{\tau\omega}(Y_{\tau\omega})\| \|T'_{\tau\omega}\| \text{diam}(Y_{\tau\omega}) \\
\geq 2K^{-2}M^{-1}R\|T^{n-1}_{\tau\omega}(Y_{\tau\omega})\| \|T'_{\tau\omega}\| \text{diam}(Y_{\tau\omega}) \\
\geq 2K^{-3}M^{-1}R \text{ diam}(Y)^{-1} \text{diam}(T^{n-1}_{\tau\omega}(Y_{\tau\omega})) \\
\geq K^{-3}M^{-1}R \text{ diam}(Y)^{-1} \text{diam}(u).
\]

Thus, Lemma 5.4 applies with the radius equal to \( r/2 \), given that \#\( \mathcal{F}(z, r) \) \( \leq \gamma \), where \( \gamma := \max\{1, 2K^3MR^{-1}\text{diam}(Y)\} \). Since also \( \hat{\pi}_2^{-1}(B(z, r)) \subset \bigcup_{\tau \in \mathcal{F}(z, r)} [\tau\omega] \), we therefore get

\[
\nu_\omega(B(z, r)) \leq \hat{\pi}_h^\omega \circ \hat{\pi}_2^{-1}\left( \bigcup_{\tau \in \mathcal{F}(z, r)} [\tau\omega] \right) \leq \sum_{\tau \in \mathcal{F}(z, r)} \hat{\pi}_h^\omega \circ \hat{\pi}_2^{-1}(\{\tau\omega\}) \leq K^h \sum_{\tau \in \mathcal{F}(z, r)} \text{diam}^h(T_{\tau\omega}(Y_{\tau\omega})) \leq 2^h K^h \#\mathcal{F}(z, r)^h,
\]

and along with (7.3) this shows that \( \nu_\omega \) is Ahlfors regular with exponent \( h = B_T \).

\[\square\]

Proof of Theorem 7.2. Fix \( t > B_T \) arbitrary; then \( P(t) < 0 \), so for every integer \( n \geq 1 \) large and \( \omega \in E^+_A \), we have

\[
\sum_{\tau \in \mathcal{E}_A(-n, -1), A_{\tau\omega_0} = 1} \|T^n_{\tau\omega}\| \leq \exp\left(\frac{1}{2} P(t)n\right).
\]

Thus, by (BDP2),

\[
\sum_{\tau \in \mathcal{E}_A(-n, -1), A_{\tau\omega_0} = 1} \text{diam}^h(T^n_{\tau\omega}(Y_{\tau\omega})) \leq K^h \exp\left(\frac{1}{2} P(t)n\right). \tag{7.4}
\]
We have $P(t) < 0$, $\{ T^n_{\tau}\omega}(Y_{\tau}) : \tau \in E^n_A(-n, -1), A_{-t_{\lambda\omega}} = 1 \}$ is a cover of $J_{\omega}$, and diameters of its sets converge to zero; so, from (7.4), $H_f(J_{\omega}) = 0$. Thus, $HD(J_{\omega}) \leq t$ and

$$HD(J_{\omega}) \leq B_T.$$  

For the opposite inequality, fix $0 \leq t < B_T$. Then $P(t) > 0$ and, from Theorem 3.5, $P_F(t) > 0$ for a finite set $F \subset E$ such that $A|_{F \times F}$ is irreducible. Theorem 7.3 gives $HD(J_{\omega}(F)) > t$ for all $\omega \in E^+_A$. But $J_{\omega}(F) \subset J_{\omega}$ and hence $HD(J_{\omega}) \geq t$. As $t < B_T$ is arbitrary, we get

$$HD(J_{\omega}) \geq B_T.$$

8. **General skew products over countable-to-1 endomorphisms**

We want to enlarge the class of endomorphisms with exact dimensionality of conditional measures on fibers. For general thermodynamic formalism of endomorphisms related to our approach, one can see [21–23, 26, 28, 29, 41], etc. Our results on exact dimensionality of conditional measures in fibers extend a result on exact dimensionality of conditional measures on stable manifolds of hyperbolic endomorphisms from [22]. We will apply the results obtained in previous sections to skew products over countable-to-1 endomorphisms. This includes EMR maps, continued fraction transformations, etc.

First, we prove a result about skew products whose base transformations are modeled by one-sided shifts on a countable alphabet. Assume that we have a skew product $F : X \times Y \to X \times Y$, where $X$ and $Y$ are complete bounded metric spaces, $Y \subset \mathbb{R}^d$ for some $d \geq 1$, and

$$F(x, y) = (f(x), g(x, y)),$$

where the map

$$Y \ni y \mapsto g(x, y)$$

is injective and continuous for every $y \in Y$. Denote the map $Y \ni y \mapsto g(x, y)$ also by $g_x(y)$. Assume that $f : X \to X$ is at most countable-to-1, and its dynamics is modeled by a one-sided Markov shift on a countable alphabet $E$ with the matrix $A$ finitely irreducible, i.e. there exists a surjective Hölder continuous map, called *coding*,

$$p : E^+_A \to X$$

such that $p \circ \sigma = f \circ p$.

Assume that conditions (a)–(g) from §5 are satisfied for $T_\omega : Y_{\omega} \to Y_{\sigma\omega}$ given by $T_\omega := g_{p(\omega)}, \omega \in E^+_A$. Then we call $F : X \times Y \to X \times Y$ a *generalized conformal skew product Smale endomorphism*.

Given the skew product $F$ as above, we can also form a skew product endomorphism in the following way: define for every $\omega \in E^+_A$, the fiber map $\hat{F}_\omega : Y \to Y$ by

$$\hat{F}_\omega(y) = g(p(\omega), y).$$

The system $(\hat{Y}, \hat{F})$ is called the *symbolic lift* of $F$. If $\hat{Y} = E^+_A \times Y$, we obtain a conformal skew product Smale endomorphism $\hat{F} : \hat{Y} \to \hat{Y}$ given by

$$\hat{F}(\omega, y) = (\sigma(\omega), \hat{F}_\omega(y)).$$  

(8.1)
As in the beginning of §4, we study the structure of fibers $J_{\omega}$, $\omega \in E^+_A$ and later of the sets $J_x$, $x \in X$. From definition, $J_{\omega} = \hat{\pi}_2([\omega])$ and it is the set of points of type

$$\bigcap_{n \geq 1} \hat{F}_{\tau_{-1}\omega} \circ \hat{F}_{\tau_{-2}\omega} \circ \ldots \circ \hat{F}_{\tau_{-n}\ldots \tau_{-1}\omega}(Y).$$

Let us call an \textit{n-prehistory} of the point $x$ with respect to the system $(f, X)$, any finite sequence of points in $X$: $(x, x_{-1}, x_{-2}, \ldots, x_{-n}) \in X^{n+1}$, where $f(x_{-1}) = x$, $f(x_{-2}) = x_{-1}$, $\ldots$, $f(x_{-n}) = x_{-n+1}$. Call a \textit{complete prehistory} (or simply a \textit{prehistory}) of $x$ with respect to $(f, X)$, any infinite sequence of consecutive preimages in $X$, i.e. $\hat{x} = (x, x_{-1}, x_{-2}, \ldots)$, where $f(x_{i}) = x_{i+1}, i \geq -1$. The space of complete prehistories is denoted by $\hat{X}$ and is called the \textit{natural extension} (or \textit{inverse limit}) of $(f, X)$. We have a bijection $\hat{f}: \hat{X} \to \hat{X}$,

$$\hat{f}(\hat{x}) = (f(x), x, x_{-1}, \ldots).$$

In this paper, we use the terms inverse limit and natural extension interchangeably, without having necessarily a fixed invariant measure defined on the space $X$.

We consider on $\hat{X}$ the canonical metric, which induces the topology equivalent to the one inherited from the product topology on $X^N$. Then $\hat{f}$ becomes a homeomorphism. For more on dynamics of endomorphisms and inverse limits, one can see for example [6, 9, 21–24, 26–28, 41].

In the above notation, we have $f(p(\tau_{-1}\omega)) = p(\omega) = x$ and, for all the prehistories of $x$, $\hat{x} = (x, x_{-1}, x_{-2}, \ldots) \in \hat{X}$, consider the set $J_x$ of points of type

$$\bigcap_{n \geq 1} g_{x_{-1}} \circ g_{x_{-2}} \circ \ldots \circ g_{x_{-n}}(Y).$$

Notice that, if $\hat{\eta} = (\eta_0, \eta_1, \ldots)$ is another sequence in $E^+_A$ such that $p(\hat{\eta}) = x$, then for any $\eta_{-1}$ so that $\eta_{-1}\hat{\eta} \in E^+_A$, we have $p(\eta_{-1}\hat{\eta}) = x'_{-1}$, where $x'_{-1}$ is some 1-preimage (i.e. preimage of order 1) of $x$. Hence, from the definitions and the discussion above, we see that

$$J_x = \bigcup_{\omega \in E^+_A, p(\omega) = x} J_{\omega}. \quad (8.2)$$

Let us denote the respective fibered limit sets for $T$ and $F$ by

$$J = \bigcup_{\omega \in E^+_A} \{\omega\} \times J_{\omega} \subset E^+_A \times Y \quad \text{and} \quad J(X) := \bigcup_{x \in X} \{x\} \times J_x \subset X \times Y. \quad (8.3)$$

So, $\hat{F}(J) = J$ and $F(J(X)) = J(X)$. The H"older continuous projection $p_J : J \to J(X)$ is

$$p_J(\omega, y) = (p(\omega), y)$$

and we obtain $F \circ p_J = p_J \circ \hat{F}$. In the following, $\hat{\pi}_2 : E_A \to Y$ and $\hat{\pi} : E_A \to E^+_A \times Y$ are the maps defined in §4 and

$$\hat{\pi}(\tau) = (\tau|_0^\infty, \hat{\pi}_2(\tau)).$$

Now it is important to know if enough points $x \in X$ have unique coding sequences in $E^+_A$. 


Definition 8.1. Let $F : X \times Y \to X \times Y$ be a generalized conformal skew product Smale endomorphism. Let $\mu$ be a Borel probability measure $X$. We then say that the coding $p : E_A^+ \to X$ is $\mu$-injective if there exists a $\mu$-measurable set $G \subset X$ with $\mu(G) = 1$ such that for every point $x \in G$, the set $p^{-1}(x)$ is a singleton in $E_A^+$.

Denote such a set $G$ by $G_\mu$ and for $x \in G_\mu$ the only element of $p^{-1}(x)$ by $\omega(x)$.

Proposition 8.2. If the coding $p : E_A^+ \to X$ is $\mu$-injective, then for every $x \in G_\mu$, we have

$$J_x = J_{\omega(x)}.$$

Proof. Take $x \in G_\mu$ and let $x_{-1} \in X$ be an $f$-preimage of $x$, i.e. $f(x_{-1}) = x$. Since $p : E_A^+ \to X$ is surjective, there exists $\eta \in E_A^+\hat{\ }$ such that $p(\eta) = x_{-1}$. But this implies that $f(x_{-1}) = f \circ p(\eta) = p \circ \sigma(\eta) = x$. Then, from the uniqueness of the coding sequence for $x$, it follows that $\sigma(\eta) = \omega(x)$, whence $x_{-1} = p(\omega_{-1}(x))$ for some $\omega_{-1} \in E$. Since $J_x = \bigcap_{n \geq 1} g_{x_{-1}} \circ g_{x_{-2}} \circ \ldots \circ g_{x_{-n}}(Y)$, it follows that $J_x = J_{\omega(x)}$. \qed

In the following, we work only with $\mu$-injective codings, and the measure $\mu$ will be clear from the context. Also, given a metric space $X$ with a coding $p : E_A^+ \to X$ and a potential $\phi : X \to \mathbb{R}$, we say that $\phi$ is Hölder continuous if $\phi \circ p$ is Hölder continuous.

Now consider a potential $\phi : J(X) \to \mathbb{R}$ such that the potential

$$\hat{\phi} := \phi \circ p_J \circ \hat{\tau} : E_A \to \mathbb{R}$$

is Hölder continuous and summable. For example, $\hat{\phi}$ is Hölder continuous if $\phi : J(X) \to \mathbb{R}$ is itself Hölder continuous. This case will be quite frequent in examples below. Define now

$$\mu_\phi := \mu_{\hat{\phi}} \circ (p_J \circ \hat{\tau})^{-1}$$

and call it the equilibrium measure of $\phi$ on $J(X)$ with respect to the skew product $F$.

Now let us consider the partition $\xi'$ of $J(X)$ into the fiber sets $\{x\} \times J_x$, $x \in X$, and the conditional measures $\mu^x_\phi$ associated to $\mu_\phi$ with respect to the measurable partition $\xi'$ (see [38]). Recall that for each $\omega \in E_A^+$, we have $\hat{\tau}_2(\{\omega\}) = J_\omega$.

Denote by $p_1 : X \times Y \to X$ the canonical projection onto the first coordinate, i.e.

$$p_1(x, y) = x.$$ 

Theorem 8.3. Let $F : X \times Y \to X \times Y$ be a generalized conformal skew product Smale endomorphism. Let $\phi : J(X) \to \mathbb{R}$ be a potential such that $\hat{\phi} = \phi \circ p_J \circ \hat{\tau} : E_A \to \mathbb{R}$ is a Hölder continuous summable potential on $E_A$. Assume that the coding $p : E_A^+ \to X$ is $\mu_\phi \circ p_1^{-1}$-injective and denote the corresponding set $G_{\mu_\phi} \subset X$ by $G_\phi$. Then:

1. $J_x = J_{\omega(x)}$ for every $x \in G_\phi$;
2. with $\bar{\mu}_\phi^\omega$, $\omega \in E_A^+$ the conditional measures of $\mu_{\hat{\phi}}$, we have for $\mu_\phi \circ p_1^{-1}$-a.e. $x \in G_\phi$,

$$\mu^x_\phi = \bar{\mu}_{\phi}^{\omega(x)} \circ (p_J \circ \hat{\tau})^{-1}$$

or, equivalently, if $\mu^x_\phi$ and $\bar{\mu}_{\phi}^{\omega(x)}$ are viewed on $J_x$ and $E_A^-$, then $\mu^x_\phi = \bar{\mu}_{\phi}^{\omega(x)} \circ \hat{\tau}_2^{-1}$. 

Proof. Part (1) is just Proposition 8.2. We thus deal with part (2) only. By the definition of conditional measures, we have for every \( \mu_\phi \)-integrable function \( H : J(X) \to \mathbb{R} \) that

\[
\int_{J(X)} H \, d\mu_\phi = \int_{E_\Lambda} H \circ p_J \circ \hat{\pi} \, d\tilde{\mu}_\phi = \int_{E_\Lambda^+} \int_{[\omega]} H \circ p_J \circ \hat{\pi} \, d\tilde{\mu}_\phi \circ \pi_1^{-1}(\omega)
\]

and

\[
\int_{J(X)} H \, d\mu_\phi = \int_X \int_{\{x\} \times J_\varepsilon} H \, d\mu_\phi \circ p_1^{-1}(x).
\]

But, from the definitions of various projections,

\[
\mu_\phi \circ p_1^{-1} = \mu_\phi \circ (p_J \circ \hat{\pi})^{-1} \circ p_1^{-1} = \mu_\phi \circ (p_1 \circ p_J \circ \hat{\pi})^{-1}
\]

\[
= \mu_\phi \circ (p \circ \pi_1)^{-1} = \mu_\phi \circ \pi_1^{-1} \circ p^{-1}.
\]

Therefore, remembering also that \( \mu_\phi \circ p_1^{-1}(G_\phi) = 1 \), we get that

\[
\int_{E_\Lambda^+} \int_{[\omega]} H \circ p_J \circ \hat{\pi} \, d\tilde{\mu}_\phi \circ \pi_1^{-1}(\omega)
\]

\[
= \int_{E_\Lambda^+} \int_{\{p(\omega)\} \times J_{p(\omega)}} H \, d\tilde{\mu}_\phi \circ (p_J \circ \hat{\pi})^{-1} \, d\mu_\phi \circ \pi_1^{-1}(\omega)
\]

\[
= \int_{G_\phi} \int_{\{x\} \times J_\varepsilon} H \, d\tilde{\mu}_\phi \circ (p_J \circ \hat{\pi})^{-1} \, d\mu_\phi \circ \pi_1^{-1} \circ p^{-1}(x)
\]

\[
= \int_{G_\phi} \int_{\{x\} \times J_\varepsilon} H \, d\tilde{\mu}_\phi \circ (p_J \circ \hat{\pi})^{-1} \, d\mu_\phi \circ p_1^{-1}(x).
\]

Hence this, together with (8.5) and (8.6), gives

\[
\int_{G_\phi} \int_{\{x\} \times J_\varepsilon} H \, d\mu_\phi \circ p_1^{-1}(x) = \int_{G_\phi} \int_{\{x\} \times J_\varepsilon} H \, d\tilde{\mu}_\phi \circ (p_J \circ \hat{\pi})^{-1} \, d\mu_\phi \circ p_1^{-1}(x).
\]

Thus, the uniqueness of the system of Rokhlin’s canonical conditional measures yields \( \mu_\phi^x = \mu_\phi^{\omega(x)} \circ (p_J \circ \hat{\pi})^{-1} \) for \( \mu_\phi \circ p_1^{-1} \)-a.e. \( x \in G_\phi \). This means that the first part of (2) is established. But \( p_J \circ \hat{\pi} = (p \circ \pi_1) \times \hat{\pi}_2 \) and thus \( p_J \circ \hat{\pi} |_{[\omega(x)]=\{x\} \times \hat{\pi}_2} = [\omega(x)] \). □

Define the Lyapunov exponent for an \( F \)-invariant measure \( \mu \) on \( J(X) = \bigcup_{x \in X} \{x\} \times J_\varepsilon \) by

\[
\chi_\mu(F) = -\int_{J(X)} \log |g_\mu^x(y)| \, d\mu(x, y).
\]

In conclusion, from Theorem 8.3, Theorem 6.2, and definition (8.4), we obtain the following result for skew product endomorphisms over countable-to-1 maps \( f : X \to X \).

**Theorem 8.4.** Let \( F : X \times Y \to X \times Y \) be a generalized conformal skew product Smale endomorphism. Let \( \phi : J(X) \to \mathbb{R} \) be a potential such that

\[
\hat{\phi} := \phi \circ p_J \circ \hat{\pi} : E_\Lambda \to \mathbb{R}
\]

is Hölder continuous summable. Assume that the coding \( p : E_\Lambda^+ \to X \) is \( \mu_\phi \circ p_1^{-1} \)-injective.
Then, for $\mu_\phi \circ p_1^{-1}$-a.e. $x \in X$, the conditional measure $\mu^x_\phi$ is exact dimensional on $J_x$ and
\[
\lim_{r \to 0} \frac{\log \mu^x_\phi(B(y, r))}{\log r} = h_{\mu_\phi}(F) / \chi_{\mu_\phi}(F) = \text{HD}(\mu^x_\phi),
\]
for $\mu_\phi$-a.e. $y \in J_x$; hence, equivalently, for $\mu_\phi$-a.e. $(x, y) \in J(X)$.

As an immediate consequence of this theorem, we get the following corollary.

**Corollary 8.5.** Let $F : X \times Y \to X \times Y$ be a generalized conformal skew product Smale endomorphism. Let $\phi : J(X) \to \mathbb{R}$ be a Hölder continuous potential such that
\[
\sum_{e \in E} \exp(\sup(\phi|_{\pi^{-1}([e]) \times Y})) < \infty.
\]
Assume that the coding $p : E^+_A \to X$ is $\mu_\phi \circ p_1^{-1}$-injective. Then, for $\mu_\phi \circ p_1^{-1}$-a.e. $x \in X$, the conditional measure $\mu^x_\phi$ is exact dimensional on $J_x$ and, for $\mu_\phi$-a.e. $y \in J_x$,
\[
\lim_{r \to 0} \frac{\log \mu^x_\phi(B(y, r))}{\log r} = h_{\mu_\phi}(F) / \chi_{\mu_\phi}(F) = \text{HD}(\mu^x_\phi).
\]

By using Theorem 8.4, we will prove exact dimensionality of conditional measures of equilibrium states on fibers for many types of skew products.

First, we prove a result about *global* exact dimensionality of measures on $J(X)$.

**Theorem 8.6.** Let $F : X \times Y \to X \times Y$ be a generalized conformal skew product Smale endomorphism. Assume that $X \subset \mathbb{R}^d$ with some integer $d \geq 1$. Let $\mu$ be a Borel probability $F$-invariant measure on $J(X)$, and $(\mu^x)_{x \in X}$ be Rokhlin’s canonical system of conditional measures of $\mu$, with respect to the partition $\{[x] \times J_x\}_{x \in X}$. Assume that:

(a) there exists $\alpha > 0$ such that for $\mu \circ p_1^{-1}$-a.e. $x \in X$, the conditional measure $\mu^x$ is exact dimensional and $\text{HD}(\mu^x) = \alpha$;

(b) the measure $\mu \circ p_1^{-1}$ is exact dimensional on $X$.

Then the measure $\mu$ is exact dimensional on $J(X)$ and, for $\mu$-a.e. $(x, y) \in J(X)$,
\[
\text{HD}(\mu) = \lim_{r \to 0} \frac{\log \mu(B((x, y), r))}{\log r} = \alpha + \text{HD}(\mu \circ p_1^{-1}).
\]

**Proof.** Denote the canonical projection to the first coordinate by $p_1 : X \times Y \to X$. Let then $\nu := \mu \circ p_1^{-1}$. Denote the Hausdorff dimension $\text{HD}(\nu)$ by $\gamma$. From the exact dimensionality of the conditional measures of $\mu$, we know that for $\nu$-a.e. $x \in X$ and for $\mu^x$-a.e. $y \in Y$,
\[
\lim_{r \to 0} \frac{\log \mu^x(B(y, r))}{\log r} = \alpha.
\]

Then, for any $\varepsilon \in (0, \alpha)$ and any integer $n \geq 1$, consider the following Borel set in $X \times Y$:
\[
A(n, \varepsilon) := \left\{ z = (x, y) \in X \times Y : \alpha - \varepsilon < \frac{\log \mu^x(B(y, r))}{\log r} < \alpha + \varepsilon \ \forall r \in (0, 1/n) \right\}.
\]

From the definition, it is clear that $A(n, \varepsilon) \subset A(n + 1, \varepsilon)$ for all $n \geq 1$. Moreover, setting
\[
X'_Y := \bigcap_{\varepsilon > 0} \bigcup_{n=1}^{\infty} A(n, \varepsilon),
\]
it follows from the exact dimensionality of almost all the conditional measures of $\mu$ and from the equality of their pointwise dimensions that $\mu(X_\gamma) = 1$. For $\varepsilon > 0$ and $n \geq 1$, consider also the following Borel subset of $X$:

$$D(n, \varepsilon) := \left\{ x \in X : \gamma - \varepsilon < \frac{\log \nu(B(x, r))}{\log r} < \gamma + \varepsilon \forall r \in (0, 1/n) \right\}.$$ 

We know that $D(n, \varepsilon) \subset D(n + 1, \varepsilon)$ for all $n \geq 1$ and, from the exact dimensionality of $\nu$, we obtain that for every $\varepsilon > 0$,

$$\nu(\bigcup_{n=1}^{\infty} D(n, \varepsilon)) = 1.$$ For $\varepsilon > 0$ and an integer $n \geq 1$, let us denote now

$$E(n, \varepsilon) := A(n, \varepsilon) \cap p^{-1}_1(D(n, \varepsilon)).$$

Clearly, from the above, we have that for any $\varepsilon > 0$,

$$\lim_{n \to \infty} \mu(E(n, \varepsilon)) = 1. \tag{8.9}$$

Using the definition of conditional measures similarly as in [22, 24], and the definitions of $A(n, \varepsilon)$ and $D(n, \varepsilon)$, we have that, for any $z \in E(n, \varepsilon)$, $x = \pi_1(z)$ and any $n \geq 1$, $\varepsilon > 0$, $0 < r < 1/n$,

$$\mu(E(n, \varepsilon) \cap B(z, r)) = \int_{D(n, \varepsilon) \cap B(x, r)} \mu^\gamma(B(z, r) \cap ([y] \times Y) \cap A(n, \varepsilon)) \, dv(y)
\leq \int_{D(n, \varepsilon) \cap B(x, r)} r^{\alpha - \varepsilon} \, dv(y) = r^{\alpha - \varepsilon} \nu(D(n, \varepsilon) \cap B(x, r))
\leq r^{\alpha + \gamma - 2\varepsilon}. \tag{8.10}$$

Since $\mu(E(n, \varepsilon)) > 0$ for all $n \geq 1$ large enough, it follows from the Borel density lemma and the Lebesgue density theorem that, for $\mu$-a.e. $z \in E(n, \varepsilon)$, we have that

$$\lim_{r \to 0} \frac{\mu(B(z, r) \cap E(n, \varepsilon))}{\mu(B(z, r))} = 1.$$ 

Thus, for any $\theta > 1$ arbitrary, there exists a subset $E(n, \varepsilon, \theta)$ of $E(n, \varepsilon)$ such that

$$\mu(E(n, \varepsilon, \theta)) = \mu(E(n, \varepsilon))$$

and, for all $z \in E(n, \varepsilon, \theta)$, there exists $r(z, \theta) > 0$ so that for $0 < r < \inf\{r(z, \theta), 1/n\}$, we have from (8.10),

$$\mu(B(z, r)) \leq \theta \mu(E(n, \varepsilon) \cap B(z, r)) \leq \theta \cdot r^{\alpha + \gamma - 2\varepsilon}.$$ 

Thus, for any point $z \in E(n, \varepsilon, \theta)$, we obtain

$$\lim_{r \to 0} \frac{\log \mu(B(z, r))}{\log r} \geq \alpha + \gamma - 2\varepsilon.$$ 

Now, since $\mu(E(n, \varepsilon, \theta)) = \mu(E(n, \varepsilon))$, it follows from (8.9) that $\mu(\bigcup_{n=1}^{\infty} E(n, \varepsilon, \theta)) = 1$. Hence,

$$\mu(\bigcap_{\varepsilon > 0} \bigcup_{\theta > 1} \bigcap_{n=1}^{\infty} E(n, \varepsilon, \theta)) = 1.$$
and, for every \( z \in \bigcap_{\varepsilon > 0} \bigcap_{\theta > 1} \bigcup_n E(n, \varepsilon, \theta) \), we have the inequality
\[
\lim_{r \to 0} \frac{\log \mu(B(z, r))}{\log r} \geq \alpha + \gamma.
\]
Conversely, from the exact dimensionality of \( \nu \) and of the conditional measures of \( \mu \), and by taking \( x = \pi_1(z) \), we have that for all \( r \in (0, 1/n) \),
\[
\mu(B(z, r) \cap E(n, \varepsilon)) = \int_{D(n,\varepsilon) \cap B(x,r)} \mu^y(B(z, r) \cap A(n, \varepsilon) \cap \{y\} \times Y) \, d\nu(y)
\geq r^{\alpha + \gamma + 2\varepsilon}.
\]
Thus,
\[
\mu(B(z, r)) \geq \mu(B(z, r) \cap E(n, \varepsilon)) \geq r^{\alpha + \gamma + 2\varepsilon}
\]
for all \( z \in E(n, \varepsilon) \) and \( r \in (0, 1/n) \). Making use of (8.9), we deduce that \( \mu \) is exact dimensional and thus for \( \mu \)-a.e. \( z \in X \times Y \) we obtain the conclusion that
\[
\lim_{r \to 0} \frac{\log \mu(B(z, r))}{\log r} = \alpha + \gamma.
\]

9. Skew products over EMR endomorphisms

We now consider EMR (expanding Markov–Rényi) maps on the interval, and we construct skew product endomorphisms over these maps which contract in fibers. This EMR class contains important examples of endomorphisms coded by a shift space with countable alphabet, such as the continued fraction Gauss transformation. The Manneville–Pomeau map, having an indifferent fixed point (parabolic point), is not EMR, but one can associate to it a countable uniformly hyperbolic system by inducing via the first return map. Let us first give the definition of EMR maps from [34].

**Definition 9.1.** Let \( I \) be a closed bounded interval in \( \mathbb{R} \) and assume that \( I = \bigcup_{n=0}^{\infty} I_n \), where \( I_n, n \geq 0 \), are closed intervals with mutually disjoint interiors. A map \( f : I \to I \) is called EMR if:
(a) for every \( n \geq 0 \), \( f \mid_{\text{Int}(I_n)} \) is a \( C^2 \) map;
(b) there exists an iterate of \( f \) which is uniformly expanding, i.e. there exist \( \lambda > 1 \) and a positive integer \( m \) such that
\[
|(f^m)'(x)| \geq \lambda
\]
for all \( x \in \bigcap_{k=0}^{m} f^{-k}(\bigcup_{n \geq 0} \text{Int}(I_n)) \);
(c) the map \( f \) is full Markov, i.e. \( f(I_n) = I \) for every integer \( n \geq 0 \);
(d) the map \( f \) satisfies Rényi’s condition, i.e.
\[
\sup \left\{ \frac{|f^n(x)|}{|f'(y)| \cdot |f'(z)|} : n \in \mathbb{N} \text{ and } x, y, z \in I_n \right\} < +\infty.
\]
For an EMR map \( f : I \to I \), there is a coding with a shift space on countably many symbols
\[
p : \mathbb{N}^\mathbb{N} \to I.
\]
Every point $x$ uniquely determined by the condition that
\[
\{ p(\omega) \} = \bigcap_{n=0}^{\infty} f^{-n}(I_{\omega_n}).
\]

Every point $x$ which never hits the boundary of any interval $I_n$ under an iterate of $f$ has a unique such coding, i.e. there exists a unique $\omega \in \mathbb{N}^\mathbb{N}$ with $p(\omega) = x$. Thus, $p : E_1^+ \to X$ is injective outside a countable set.

Two significant classes of examples used in the following are: the continued fraction map as a class in itself and a class derived from Manneville–Pomeau maps. The continued fraction (Gauss) map is $G : [0, 1] \to [0, 1]$,
\[
G(x) := \left\{ \frac{1}{x} \right\} = \frac{1}{x} - \left[ \frac{1}{x} \right] \quad \text{for } x \neq 0 \quad \text{and} \quad f_1(0) = 0.
\]

The Manneville–Pomeau maps $f_{\alpha, \alpha} : [0, 1] \to [0, 1]$ are defined by
\[
f_{\alpha}(x) = x + x^{1+\alpha} \pmod{1},
\]
where $\alpha > 0$. Notice that $f_{\alpha}$ has an indifferent fixed point at 0, i.e. $f_{\alpha}(0) = 0$ and $f_2(0) = 1$. So, $f_2$ is not an EMR map. It was shown, and it is easy to see, that some induced (first return) map of $f_{\alpha}$ is EMR. Indeed, let $c$ be the unique solution on $[0, 1]$ to the equation
\[
c_\alpha + c_\alpha^\alpha = 1.
\]

Obviously, both restrictions
\[
f_{\alpha} \mid_{[0, c_\alpha]} \quad \text{and} \quad f_{\alpha} \mid_{[c_\alpha, 1]}
\]
are injective and $f_{\alpha}([0, c]) = [0, 1] = f_{\alpha}([c, 1])$. The first return (induced) map of $f_{\alpha}$ from $[c_\alpha, 1]$ onto $[c_\alpha, 1]$ is expanding, i.e. it satisfies condition (b) of Definition 9.1. The reason is that $f'(x) > 1$ for every $x \in (0, 1]$ and, if $x \in (c, 1)$, then it will take a sufficiently long time for $x$ to enter $[c, 1]$ again under forward iteration of $f_{\alpha}$, so that the derivative of this iterate evaluated at $x$ will be larger than some constant larger that 1. In fact, denoting the interval $[c_\alpha, 1]$ by $I_0$, it is very easy to picture the first return map
\[
f_{\alpha, I_0} : I_0 \to I_0.
\]

Indeed, define a decreasing sequence $(a_n)_{n=0}^\infty$ inductively as follows: $a_0 := 1$ and, if $a_n \in I_0$, $n \geq 0$, has been defined, then $a_{n+1}$ is defined to be the only point in $I_0$ such that
\[
f(a_{n+1}) = a_n.
\]

For every $n \geq 1$, let $I_n := [a_n, a_{n-1}]$. Then $f_2^n(I_{n+1}) = I_0$. For all $n > 0$, so the induced map (first return time) to $I_0$ is
\[
f_{2, I_0}(x) = f_2^n(x), \quad x \in I_{n+1}, \quad n \geq 0. \tag{9.1}
\]

The first statement of the following proposition follows from [44], while the second one follows from [45].

**Proposition 9.2.** Both the Gauss map $G : [0, 1] \to [0, 1]$ and the induced Manneville–Pomeau maps $f_{\alpha, I_0} : I_0 \to I_0$ are EMR.
Consider now a general EMR map $f: I \to I$ and a skew product $F: I \times Y \to I \times Y$, where $Y \subset \mathbb{R}^d$ is a bounded open set, with $F(x, y) = (f(x), g(x, y))$. Recall that the symbolic lift of $F$ is the map $\tilde{F}: \mathbb{N}^N \times Y \to \mathbb{N}^N \times Y$ given by the formula

$$\tilde{F}(\omega, y) = (\sigma(\omega), g(p(\omega), y))$$

for all $(\omega, y) \in \mathbb{N}^N \times Y$. If the symbolic lift $\tilde{F}$ is a Hölder conformal skew product Smale endomorphism, then we say by extension that $F$ is a Hölder conformal skew product endomorphism over $f$.

Recall now the observation after Definition 9.1 that the coding $\pi$ of an EMR map is injective outside a countable set; and, from (8.3), the fibered limit set of $F$ is

$$J(I) = \bigcup_{x \in I} \{x\} \times J_x.$$  

Let $\phi: J(I) \to \mathbb{R}$ be a Hölder continuous summable potential on $J(I)$ and let $\mu_\phi$ be its equilibrium measure defined in (8.4). Then $p$ can be easily shown to be $\mu_\phi \circ p_1^{-1}$-injective (as $\mu_\phi$ is invariant, ergodic, and has full topological support). So, as an immediate consequence of from Theorem 8.4, we get the following theorem.

**Theorem 9.3.** (Exact dimensionality of conditional measures for EMR maps) Let $f: I \to I$ be an EMR map. Let $Y \subset \mathbb{R}^d$ be an open bounded set and let $F: I \times Y \to I \times Y$ be a Hölder conformal skew product endomorphism over $f$. Let $\phi: J(I) \to \mathbb{R}$ be such that

$$\hat{\phi} := \phi \circ p_1 \circ \hat{\pi}: \mathbb{N}^\mathbb{Z} \to \mathbb{R}$$

is Hölder continuous summable, where summability can be expressed as

$$\sum_{e \in \mathbb{N}} \exp(\sup(\phi|_{p([e]) \times Y})) < +\infty.$$  

Then, with $\mu_\phi$ being the equilibrium measure of $\phi$ (see also (8.4)), we have for $\mu_\phi \circ p_1^{-1}$-a.e. $x \in I$ that the conditional measure $\mu_\phi^x$ is exact dimensional on $J_x$ and

$$\lim_{r \to 0} \frac{\log \mu_\phi^x(B(y, r))}{\log r} = h_{\mu_\phi}(F) = \chi_{\mu_\phi}(F) = \text{HD}(\mu_\phi^x)$$

for $\mu_\phi^x$-a.e. $y \in J_x$; equivalently, for $\mu_\phi$-a.e. $(x, y) \in J(I)$.

As an immediate consequence of this theorem, we get the following corollary.

**Corollary 9.4.** (Exact dimensionality of conditional measures for EMR maps, II) Let $f: I \to I$ be an EMR map. Let $Y \subset \mathbb{R}^d$ be an open bounded set and let $F: I \times Y \to I \times Y$ be a Hölder conformal skew product endomorphism over $f$. Let $\phi: I \to \mathbb{R}$ be a potential. Set

$$\psi := \phi \circ p_1: I \times Y \to \mathbb{R}.$$  

Assume that $\hat{\psi} = \phi \circ p_1 \circ p_1 \circ \hat{\pi}: \mathbb{N}^\mathbb{Z} \to \mathbb{R}$ is Hölder continuous summable, where summability can be expressed as $\sum_{e \in \mathbb{N}} \exp(\sup(\phi|_{p_e})) < +\infty$. Then:
(a) if $\mu_\psi$ is the equilibrium measure of $\psi$, and $\mu_\phi = \mu_\psi \circ p_1^{-1}$, then for $\mu_\phi$-a.e. $x \in I$, the conditional measure $\mu_\psi^x$ is exact dimensional on $J_x$ and

$$\lim_{r \to 0} \frac{\log \mu_\psi^x(B(y, r))}{\log r} = \frac{h_{\mu_\phi}(F)}{\chi_{\mu_\phi}(F)}$$

for $\mu_\psi^x$-a.e. $y \in J_x$; hence, equivalently for $\mu_\psi$-a.e. $(x, y) \in J(I)$;

(b) if $\mu_\phi$ is exact dimensional on $I$, then $\mu_\psi$ is exact dimensional on $I \times Y$.

**Proof.** Item (a) is an immediate consequence of Theorem 9.3, while part (b) is an immediate consequence of Theorem 8.6. \qed

We denote the class of potentials considered in Corollary 9.4 by $\mathcal{W}$. As an immediate consequence, we get the following corollary.

**Corollary 9.5.** Let $f$ be either the Gauss map $G$ or the induced map $f_{a, I_0}$ of some Manneville–Pomeau map $f_a$. Consider an open bounded set $Y \subset \mathbb{R}^d$, and $F : I \times Y \longrightarrow I \times Y$, a Hölder conformal skew product endomorphism over $f$. Let $\phi : I \longrightarrow \mathbb{R}$ be a potential from $\mathcal{W}$ and set $\psi := \phi \circ p_1 : I \times Y \longrightarrow \mathbb{R}$. Then:

(a) if $\mu_\psi$ is the equilibrium measure of $\psi$, and $\mu_\phi := \mu_\psi \circ p_1^{-1}$, then for $\mu_\phi$-a.e. $x \in I$, the conditional measure $\mu_\phi^x$ is exact dimensional on $J_x$ and

$$\lim_{r \to 0} \frac{\log \mu_\phi^x(B(y, r))}{\log r} = \frac{h_{\mu_\phi}(F)}{\chi_{\mu_\phi}(F)}$$

for $\mu_\phi^x$-a.e. $y \in J_x$; hence, equivalently for $\mu_\phi$-a.e. $(x, y) \in J(I)$;

(b) if $\mu_\phi$ is exact dimensional on $I$, then $\mu_\psi$ is exact dimensional on $I \times Y$.

In [16, 34, 39], and others, there was developed and studied the multifractal analysis of equilibrium states for a class of Hölder continuous summable potentials. As almost always, multifractal analysis requires the use of an auxiliary potential and the quantity $T(q)$ is commonly referred to as a temperature. We now want to discuss exact dimensionality of equilibrium states for a class of Hölder continuous summable potentials. These fit to our setting of this section.

If $\phi \in \mathcal{W}$, then for every $q \geq 1$ and $t > \theta := 1/2$ if $f = G$, or $t > \theta := \alpha/(\alpha + 1)$ if $f = f_{a, I_0}$, the potential

$$\phi_{q, t} := -t \log |f'| + q(\phi - P(\phi)) : I \longrightarrow \mathbb{R}$$

also belongs to $\mathcal{W}$. Since $\lim_{t \to 0} P(\phi_{q, t}) = +\infty$ and $\lim_{t \to +\infty} P(\phi_{q, t}) = -\infty$, there exists a unique number $T(q) > \theta$ such that

$$P(\phi_{q, T(q)}) = 0.$$

We see that $P(\phi_{1,0}) = 0$, so $T(1) = 0$.

Consider a skew product map

$$F : I \times Y \longrightarrow I \times Y$$

over $G$ or over $f_{a, I_0}$ as in Corollary 9.5 and let $\phi \in \mathcal{W}_0$. If, as usually, $p_1 : I \times Y \to I$ is the projection on the first coordinate, define the potentials on $I \times Y$,

$$\psi_{q, t} := \phi_{q, t} \circ p_1 : I \times Y \longrightarrow \mathbb{R} \quad \text{and} \quad \psi_q := \psi_{q, T(q)} : I \times Y \longrightarrow \mathbb{R}.$$
For every $q \geq 1$, let $\mu_{\psi_q}$, the equilibrium measure of $\psi_q$, be defined by formula (8.4). Let

$$\mu_q := \mu_{\psi_q} \circ p_1^{-1}.$$  

The measure $\mu_q$ is called the equilibrium state (measure) for the potential $\phi_{q, T(q)}$ on $I$.

We shall prove the following theorem.

**Theorem 9.6.** In the above setting, if $F : I \times Y \to I \times Y$ is either a Hölder conformal skew product endomorphism over the continued fraction transformation $G$ or the induced map $f_{\alpha, I_0}$, $\alpha > 0$, of the Manneville–Pomeau map $f_{\alpha}$, and if $\phi \in \mathcal{W}$, then for every $q \geq 1$, the measure $\mu_{\psi_q}$ is exact dimensional on $I \times Y$.

**Proof.** From [34, Theorems 1 and 2 and Proposition 3] (see also [16] and [31]) for much more general treatment), we obtain the exact dimensionality of the measures $\mu_{\phi_q}$ on $I$. Therefore, our theorem follows directly from Corollary 9.5. □

10. **Diophantine approximants and the Doeblin–Lenstra conjecture**

We want to apply the results about skew products to certain properties of Diophantine approximants, making the conjecture of Doeblin and Lenstra more general and precise. Recall that the continued fraction (Gauss) transformation is given by

$$G(x) = \begin{cases} \frac{1}{x} - n & \text{if } x \in \left(\frac{1}{n + 1}, \frac{1}{n}\right) \\ 0 & \text{otherwise} \end{cases}$$

(10.1)

and $G(x) = 0$ otherwise. Then the corresponding coding map

$$\pi_G : \mathbb{N}^\mathbb{N} \to [0, 1]$$

is given by the formula

$$\pi_G(\omega) = [\omega_1, \omega_2, \ldots] = \frac{1}{\omega_1 + \frac{1}{\omega_2 + \frac{1}{\ldots + \frac{1}{\omega_n + \ldots}$$

and it is a bijection between $\mathbb{N}^\mathbb{N}$ and the set of irrational numbers in $[0, 1]$.

If we truncate the representation $[\omega_1, \omega_2, \ldots]$ at an integer $n \geq 1$, then we obtain a rational number $p_n/q_n$, called the $n$th convergent of $x := \pi_G(\omega)$, where $p_n, q_n \geq 1$ are relatively prime integers and

$$p_n = [\omega_1, \ldots, \omega_n].$$

If needed, we shall also denote $p_n$ and $q_n$ respectively by $p_n(\omega)$ and $q_n(\omega)$ or also by $p_n(x)$ and $q_n(x)$, in order to indicate their dependence on $\omega$ and $x$. We will also sometimes write $\omega_n(x)$ for $\omega_n$. Let us now introduce (see for example [12]) the numbers

$$\Theta_n := \left| x - \frac{p_n}{q_n} \right| \cdot q_n^2, \quad n \geq 1.$$  

These numbers $\Theta_n$ also depend on $\omega$ or (equivalently) on $x$ and will also be denoted by $\Theta_n(\omega)$ or $\Theta_n(x)$. Denote:

$$T_n = T_n(\omega) := \pi_G(\sigma^n(\omega)) = [\omega_{n+1}, \omega_{n+2}, \ldots], \quad n \geq 1,$$

and
\[ V_n = V_n(\omega) := [\omega_n, \ldots, \omega_1], \quad n \geq 1. \]

We will also denote them respectively by \( T_n(x) \) and \( V_n(x) \). We see that the number \( T_n(x) \) represents the future of \( x \) while the number \( V_n(x) \) represents the past of \( x \). It follows directly from the definitions that for every integer \( n \geq 1 \), we have that

\[ V_n = \frac{q_{n-1}}{q_n}, \quad \Theta_{n-1} = \frac{V_n}{1 + T_n V_n}, \quad \text{and} \quad \Theta_n = \frac{T_n}{1 + T_n V_n}. \quad (10.2) \]

We will use again natural extension systems (see for example [21, 24, 26, 40]), which belong, in certain cases, to our class of skew products. The natural extension \( \tilde{G} = G : [0, 1] \times [0, 1] \rightarrow [0, 1] \times [0, 1] \) of the Gauss map \( G = G : [0, 1] \rightarrow (0, 1) \) is such an example and is given by the formula

\[ \tilde{G}(x, y) = \left( T_1(x), \frac{1}{\omega_1(x) + y} \right). \]

It follows from this that

\[ \tilde{G}(x, 0) = \left( T_1(x), \frac{1}{\omega_1(x)} \right) \quad \text{and} \quad \tilde{G}^2(x, 0) = \left( T_2(x), \frac{1}{\omega_2(x) + 1/\omega_1(x)} \right). \]

By induction, we obtain for every \( n \geq 1 \) that

\[ \tilde{G}^n(x, 0) = (T_n(x), [\omega_n(x), \ldots, \omega_1(x)]) = (T_n(x), V_n(x)). \]

The approximation coefficients \( \Theta_n \) were the object of an important conjecture originally stated by Doeblin and reformulated in the 1980s by Lenstra (see [12]), namely that for Lebesgue-a.e. \( x \in [0, 1) \) the frequency of appearances of \( \Theta_n(x) \) in the interval \([0, t] \), \( t \in [0, 1) \), is given by the function \( F : [0, 1] \rightarrow [0, 1] \) given by

\[ F(t) = \begin{cases} \frac{t}{\log 2} & \text{if } t \in [0, 1/2), \\ \frac{1}{\log 2}(1 - t + \log 2t) & \text{if } t \in [1/2, 1]. \end{cases} \]

More precisely, the \textit{Doeblin–Lenstra conjecture} says that for Lebesgue-a.e. \( x \in [0, 1] \) and all \( t \in [0, 1] \),

\[ \lim_{n \to \infty} \frac{\#\{1 \leq k \leq n : \Theta_k(x) \leq t\}}{n} = F(t), \]

i.e. the above limit exists and is equal to the above function \( F(t) \).

This conjecture was solved by Bosma, Jager and Wiedijk in the 1980s; see [3]. In the proof, they needed fundamentally the \textit{natural extension} \([0, 1] \times [0, 1], \tilde{G}, \tilde{\mu}_G \) of the continued fraction dynamical system \( \tilde{G} \) with the classical Gauss measure \( \mu_G \) defined by

\[ \mu_G(A) = \frac{1}{\log 2} \int_A \frac{1}{1 + x} \, dx \]

for every Borel set \( A \subset \mathbb{R} \). Indeed, in the expression of \( \Theta_n \), we have both the future \( T_n \) as well as the past \( V_n \); thus, the natural extension is the right construction in this case.

Let us now apply our results on skew products for the natural extension \( \tilde{G} \) to the lifts of certain invariant measures, in fact some equilibrium states. We recall that the potentials \( \phi_s : I \rightarrow \mathbb{R} \), given by the formula

\[ \phi_s(x) = -s \log |G'(x)|, \quad x \in [0, 1), \]
Then let \( y \) such that for every \( x \in [0, 1] \times [0, 1] \) and \( c < 1 \) in Corollary 9.4 and defined by formula (8.4). Furthermore, let 

\[
\hat{\mu}_s = \mu_{\psi_s}
\]

be the equilibrium measure of \( \phi_s \) considered in the same corollary. From this corollary, we know that \( \hat{\mu}_s \) is exact dimensional on \( [0, 1] \times [0, 1] \). Our purpose is now to describe the asymptotic frequencies with which \( \Theta_n(x) \) come close to arbitrary values, when \( x \) is \( \mu_s \)-generic, instead of \( x \) in a set of full Lebesgue measure as in the original Doeblin–Lenstra conjecture. We know from this corollary that the measure \( \hat{\mu}_s \) is exact dimensional on \( [0, 1] \times [0, 1] \) and we have a formula for its Hausdorff dimension.

Our purpose now is to describe asymptotic frequencies with which the approximation coefficients \( \Theta_n(x) \) of \( x \in [0, 1] \) become close to certain given arbitrary values, when \( x \) is Lebesgue non-generic (i.e. \( x \) belongs to a set of Lebesgue measure zero). In fact, these \( x \) will be generic for equilibrium measures \( \mu_s \), which except for \( s = 1 \) are singular with respect to the Lebesgue measure.

First let us prove the following result about the asymptotic frequency of appearance of \((T_n(x), V_n(x))\) in all squares of \( \mathbb{R}^2 \), with respect to the measure \( \hat{\mu}_s \).

**Theorem 10.1.** If \( s > 1/2 \), then for \( \mu_s \)-a.e. \( x \in [0, 1] \) and for all four real numbers \( a < b \) and \( c < d \), we have that

\[
\lim_{n \to \infty} \frac{\# \{ k \in \{ 0, 1, \ldots, n-1 \} : (T_k(x), V_k(x)) \in (a, b) \times (c, d) \}}{n} = \hat{\mu}_s((a, b) \times (c, d)).
\]

**Proof.** Denote \( A = (a, b) \times (c, d) \) and for every \( \varepsilon > 0 \) let

\[
A(\varepsilon) := (a, b) \times (c - \varepsilon, d + \varepsilon) \quad \text{and} \quad A(-\varepsilon) := (a, b) \times (c + \varepsilon, d - \varepsilon).
\]

Then

\[
A(-\varepsilon) \subset A \subset A(\varepsilon).
\]

Let \( x \in [0, 1] \setminus \mathbb{Q} \). Then \( x = [\omega_1(x), \omega_2(x), \ldots] \). Hence, there exists an integer \( n_\varepsilon \geq 1 \) such that for every \( y \in [0, 1] \) and every integer \( n \geq n_\varepsilon \), we have that

\[
|\{ \omega_n(x), \omega_{n-1}(x), \ldots, \omega_1(x) + y \} - \{ \omega_n(x), \omega_{n-1}(x), \ldots, \omega_1(x) \} | < \varepsilon.
\]

Thus, if \( \tilde{G}^n(x, y) = (T^n(x), [\omega_n(x), \omega_{n-1}(x), \ldots, \omega_1(x) + y] \in A(-\varepsilon) \), then \((T_n(x), V_n(x)) \in A \) and, if \((T_n(x), V_n(x)) \in A \), then \( \tilde{G}^n(x, y) \in A(\varepsilon) \). Therefore for every \( x \in [0, 1] \setminus \mathbb{Q} \) and every \( y \in [0, 1] \), we obtain that

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{1}_{A(-\varepsilon)}(\tilde{G}^k(x, y)) \leq \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{1}_A(\tilde{G}^k(x, 0)) \leq \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{1}_A(\tilde{G}^k(x, 0)) \leq \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{1}_{A(\varepsilon)}(\tilde{G}^k(x, y)). \tag{10.3}
\]
Since the equilibrium measure $\hat{\mu}_s$ is ergodic on $[0, 1]^2$ with respect to the map $\tilde{G}$ and since $\hat{\mu}_s$ projects on $\mu_s$, the equilibrium state of the potential $\phi_s$, it follows from Birkhoff’s ergodic theorem that for $\mu_s$-a.e. $x \in [0, 1]$ there exist $y_1 \in [0, 1]$ and $y_2 \in [0, 1]$ such that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{1}_{A(\varepsilon)}(\tilde{G}^k(x, y)) = \hat{\mu}_s(A(\varepsilon))$$

and

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{1}_{A(\varepsilon)}(\tilde{G}^k(x, 0)) = \hat{\mu}_s(A(\varepsilon)).$$

Along with (10.3), these yield

$$\hat{\mu}_s(A(\varepsilon)) \leq \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{1}_{A}(\tilde{G}^k(x, 0)) \leq \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{1}_{A}(\tilde{G}^k(x, 0)) \leq \hat{\mu}_s(A(\varepsilon)).$$

(10.4)

Noting that $\hat{\mu}_s$ does not charge the boundary of $A$ and letting, in the above inequality, $\varepsilon > 0$ to 0 over a (countable) sequence, we obtain that for $\mu_s$-a.e. $x \in [0, 1]$, 

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{1}_{A}(T_k(x), V_k(x)) = \hat{\mu}_s(A).$$

We prove now that for $x \in \Lambda_s$ (recall that $\Lambda_s$ has zero Lebesgue measure, but $\mu_s$-measure equal to 1), the approximation coefficients $\Theta_n(x), \Theta_{n-1}(x)$ behave very erratically. The following theorem says that for irrational numbers $x \in \Lambda_s$, the behavior of the consecutive numbers $\Theta_k(x), \Theta_{k-1}(x)$ is chaotic, and we can estimate the asymptotic frequency that $\Theta_k(x)$ is $r$-close to some $z$, while $\Theta_{k-1}(x)$ is $r$-close to some $z'$. This asymptotic frequency is comparable to $r^{\delta(\hat{\mu}_s)}$, regardless of the point $x \in \Lambda_s$ or the numbers $z, z'$ chosen.

**Theorem 10.2.** For every $s > 1/2$, there exists a Borel set $\Lambda_s \subset [0, 1] \setminus \mathbb{Q}$ with $\mu_s(\Lambda_s) = 1$ and with the following properties:

1. $\text{HD}(\Lambda_s) = h_{\mu_s}(G)/\chi_{\mu_s}$;
2. for every $x \in \Lambda_s$, we have that $\lim_{n \to \infty} (1/n) \log |x - p_n(x)/q_n(x)| = \chi_{\mu_s}$;
3. for every $x \in \Lambda_s$ and $\hat{\mu}_s$-a.e. $(z, z') \in [0, 1)^2$, we have that

$$\lim_{r \to 0} \lim_{n \to \infty} \frac{\{0 \leq k \leq n - 1 : (\Theta_k(x), \Theta_{k-1}(x)) \in B(\frac{z}{1+zz'}, r) \times B(\frac{z'}{1+zz'}, r)\}}{n} = \text{HD}(\hat{\mu}_s);$$

4. $\chi_{\mu_s,s>1/2} = [\chi_{\mu_{1/2}}, +\infty)$;
5. $\text{HD}(\hat{\mu}_s) = \frac{h_{\mu_s}(G)}{\chi_{\mu_s}} + \frac{h_{\hat{\mu}_s}(\tilde{G})}{\int_{[0,1]^2} \log(\omega_1(x) + y) \, d\hat{\mu}_s(x, y)}$.

**Proof.** Since $\text{HD}(\mu_s) = h_{\mu_s}(G)/\chi_{\mu_s}$, there exists a Borel set $\Lambda^*_s \subset I \setminus \mathbb{Q}$ such that:

1. $\mu_s(\Lambda^*_s) = 1$;
(b) $\text{HD}(\Lambda^*_s) = h_{\mu_s}(G)/\chi_{\mu_s}$;

(c) each Borel subset of $\Lambda_s^*$ with full measure $\mu_s$ has Hausdorff dimension equal to $h_{\mu_s}(G)/\chi_{\mu_s}$.

Define now $\Lambda_s$ to be the set of all points $x$ in $\Lambda_s^*$ for which (c) above holds and for which the assertion of Theorem 10.1 holds. By this theorem, by a result from [34], and by the properties (a), (b), and (c), it follows that the set $\Lambda_s$ satisfies the conditions (1) and (2) above, and moreover that $\mu_s(\Lambda_s) = 1$.

Now let us show (3). So, fix $x \in \Lambda_s$ and let arbitrary $z, z' \in [0, 1)$ be given. Because of formulas (10.2), there exists some constant $C \geq 1$ such that for all radii $r > 0$ we have the following two implications:

1. if, for some integer $k \geq 1$, $(T_k(x), V_k(x)) \in B(z, r) \times B(z', r)$, then
   $$(\Theta_k(x), \Theta_{k-1}(x)) \in B\left(\frac{z}{1 + zz'}, Cr\right) \times B\left(\frac{z'}{1 + zz'}, Cr\right);$$

2. if $(\Theta_k(x), \Theta_{k-1}(x)) \in B(z/(1 + zz'), r) \times B(z'/(1 + zz'), r)$, then $(T_k, V_k) \in B(z, Cr) \times B(z', Cr)$.

It therefore follows from Theorem 10.1 that

$$\hat{\mu}_s(B(z, C^{-1}r) \times B(z', C^{-1}r))$$

$$\leq \lim_{n \to \infty} \#\{0 \leq k \leq n - 1 : (\Theta_k(x), \Theta_{k-1}(x)) \in B(z/(1 + zz'), r) \times B(z'/(1 + zz'), r)\}$$

$$\leq \hat{\mu}_s(B(z, Cr) \times B(z', Cr)).$$

(10.5)

Since by Corollary 9.4 the measure $\hat{\mu}_s$ is exact dimensional, we have that

$$\lim_{r \to 0} \frac{\log \hat{\mu}_s(B(z, C^{-1}r) \times B(z', C^{-1}r))}{\log r} = \text{HD}(\hat{\mu}_s)$$

$$= \lim_{r \to 0} \frac{\log \hat{\mu}_s(B(z, Cr) \times B(z', Cr))}{\log r}. $$

Along with (10.5), this finishes the proof of (3). By [16, 34], the function $(1/2, +\infty) \ni s \mapsto \chi_{\mu_s}$ is strictly increasing and $\lim_{s \to \infty} \chi_{\mu_s} = \infty$. Hence, (4) follows.

Next we compute the Lyapunov exponent of the contraction $y \mapsto 1/(\omega_1(x) + y)$ in the fiber over $x$, which is equal to $2 \int_{[0, 1]^2} \log(\omega_1(x) + y) \, d\hat{\mu}_s(x, y)$, and we have that the entropy of the natural extension is $h_{\hat{\mu}_s}(\widetilde{G})$ (see also [24, 26]). Hence, we use Theorems 8.4 and 9.6 to obtain the Hausdorff dimension of the lift measure $\hat{\mu}_s$ on $[0, 1) \times [0, 1)$, thus proving (5). In conclusion, all the statements of the theorem are proved. \qed

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