LARGE DEVIATIONS OF EMPIRICAL MEASURES OF ZEROS ON RIEMANN SURFACES

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Abstract. We determine an LDP (large deviations principle) for the empirical measure
\[ \tilde{Z}_s := \frac{1}{N} \sum_{\zeta : s(\zeta) = 0} \delta_\zeta, \quad (N := \#\{\zeta : s(\zeta) = 0\}) \]
of zeros of random holomorphic sections \( s \) of random line bundles \( L \to X \) over a Riemann surface \( X \) of genus \( g \geq 1 \). In a previous article [ZZ], O. Zeitouni and the author proved such an LDP in the \( g = 0 \) case of \( \mathbb{CP}^1 \) using an explicit formula for the JPC (joint probability current) of zeros of Gaussian random polynomials. The main purpose of this article is to define Gaussian type measures on the “vortex moduli space” of all holomorphic sections of all line bundles of degree \( N \) and to calculate its JPC as a volume form on the configuration space \( X^{(N)} \) of \( N \) points of \( X \). The calculation involves the higher genus analogues of Vandermonde determinants, the prime form and bosonization.

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In a recent article, O. Zeitouni and the author [ZZ] proved an LDP (large deviations principle) for the empirical measure
\[ \tilde{Z}_s := d\mu_\zeta := \frac{1}{N} \sum_{\zeta : s(\zeta) = 0} \delta_\zeta, \quad N := \#\{\zeta : s(\zeta) = 0\} \]
of zeros of Gaussian random polynomials \( s \) of degree \( N \) in one complex variable (where \( \delta_\zeta \) is the Dirac point measure at \( \zeta \).) The purpose of this continuation is to generalize the LDP to holomorphic sections of line bundles \( L \to X \) of degree \( N \) over a compact Riemann surface \( X \) of genus \( g \geq 1 \). Roughly speaking, the LDP determines the asymptotic probability that a configuration \( \{P_1, \ldots, P_N\} \) of \( N \) points is the zero set of a random holomorphic section \( s \) of

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a line bundle $L$ of degree $N$ as $N \to \infty$. The essentially new aspect of higher genus Riemann surfaces is that $L$ is not unique but rather varies over the $g$-dimensional Picard variety $\text{Pic}^N$ of holomorphic line bundles of degree $N$. As recalled in \S1, the space $\text{Pic}^N$ of line bundles of degree $N$ is a compact complex torus of dimension $g$. The space of all holomorphic sections of all line bundles of degree $N$ is therefore the total space

$$E^N := \bigcup_{\xi \in \text{Pic}^N} H^0(X, \xi)$$

of the complex holomorphic vector bundle (the Picard bundle),

$$\pi_N : E^N \to \text{Pic}^N$$

whose fiber $E^N_\xi$ over $\xi$ is the space $H^0(X, \xi)$ of holomorphic sections of $\xi$. Since $E_N$ is a vector bundle rather than a vector space, it does not carry a Gaussian measure as in the genus zero case of \cite{Zam}. But it does carry closely related types of Gaussian-like measures introduced in Definitions \ref{def:2}--\ref{def:4}. Since we are interested in zeros, it is natural to identify sections which differ by a constant multiple, i.e. to projectivize $H^0(X, \xi)$ to $\mathbb{P}H^0(X, \xi)$, and to push forward the Gaussian type measures to Fubini-Study type probability measures on the $\mathbb{CP}^{N-g}$ bundle

$$\mathbb{P}E^N := \bigcup_{\xi \in \text{Pic}^N} \mathbb{P}H^0(X, \xi) \to \text{Pic}^N.$$  

We endow the total space with Fubini-Study volume forms along the fibers and Haar measure along the base. The resulting probability measure is called the \textit{Fubini-Study-fiber ensemble} (Definition \ref{def:2}). In addition we will define a Fubini-Study fiber ensemble over the configuration space $X^{(g)}$ of $g$ points (Definition \ref{def:3}) and a more linear \textit{projective linear ensemble} by embedding $\mathbb{P}E^N$ in a higher dimensional projective space (Definition \ref{def:4}).

The projectivized Picard bundle $\mathbb{P}E^N$ is analytically equivalent to the configuration space

$$X^{(N)} = \text{Sym}^N X := \underbrace{X \times \cdots \times X}_{N} / S_N$$

of $N$ points of $X$ under the ‘zero set’ or divisor map

$$D : \mathbb{P}E_N \to X^{(N)}, \quad D(\xi, s) = \zeta_1 + \cdots + \zeta_N, \quad Z_s = \{\zeta_j\}.$$ 

Here, $S_N$ is the symmetric group on $N$ letters. This analytic equivalence explains why it is natural to view the line bundle as well as the section as a random variable. Any configuration $\{\zeta_1, \ldots, \zeta_N\}$ is the zero set of some section of some line bundle of degree $N$, but the possible zero sets of sections $s \in H^0(X, L)$ of a fixed $L \in \text{Pic}^N$ lie on a codimension $g$ submanifold of to $X^{(N)}$. (As will be recalled in \S1 the submanifold is a fiber of the Abel-Jacobi map $A_N : X^{(N)} \to \text{Jac}(X)$.) $\mathbb{P}E_N$ is also the moduli space of abelian vortices of Yang-Mill-Higgs fields of vortex number $N$ (\cite{Sam, MN}), and is therefore also called the vortex moduli space.

A Gaussian type probability measure on $E_N$ weights a section in terms of its $L^2$ norm with respect to an inner product, or equivalently in terms of its coefficients relative to an orthonormal basis. Under $D$, it induces a probability measure $\tilde{K}^N$ on $X^{(N)}$, which is called the \textit{joint probability distribution} JPD of zeros of the random sections $s \in \mathbb{P}E_N$. We also refer to it as the JPC (joint probability current) since it is naturally a (possibly singular) volume form on $X^{(N)}$. Its integral over an open set $U \subset X^{(N)}$ is the probability that the zero set of the random section lies in $U$. The first objective of this article is to calculate it explicitly
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for the Gaussian-type probability measures on $\mathcal{E}_N$ mentioned above in terms of products of Green’s functions and prime forms. This calculation is much more involved than in the genus zero case, and is based on bosonization formulae on higher genus Riemann surfaces in [ABMNV, VV, F, DP, W].

We then use the explicit formula for $\vec{K}^N$ to obtain a rate function for the large deviations principle, which (roughly speaking) gives the asymptotic probability as $N \to \infty$ that a configuration of $N$ points is the zero sets of a random section. Since the spaces $X^{(N)}$ change with $N$, we encode a configuration by its empirical measure,

$$\{\zeta_1, \ldots, \zeta_N\} \to \mu_\zeta := \frac{1}{N} \sum_{j=1}^N \delta_{\zeta_j} \in \mathcal{M}(X).$$

We denote the map from configurations to empirical measures by

$$\mu : X^{(N)} \to \mathcal{M}(X),$$

where $\mathcal{M}(K)$ denotes the convex set of probability measures on a set $K$. Under the map $\mu$ the JPD (or JPC) pushes forward to a probability measure

$$\text{Prob}_N = \mu_* \vec{K}^N$$
on $\mathcal{M}(X)$.

Our main result is the analogue in higher genus of the LDP in genus zero. To state the result, we need some further notation. As in [ZZ], the input for our probability measures on sections is a pair $(\omega, \nu)$ where $\omega$ is a real $(1, 1)$ form and where $\nu$ is a probability measure on $X$ satisfying the two rather small technical hypotheses (37) and (39) carried over from [ZZ]. We denote by $G_\omega$ the Green’s function of $X$ with respect to $\omega$, and by

$$U^\mu_\omega(z) = \int_X G_\omega(z, w)d\mu(w)$$

the Green’s potential of a probability measure $\mu \in \mathcal{M}(X)$ (see §4.1 for background). We define the Green’s energy of $\mu \in \mathcal{M}(X)$ by

$$\mathcal{E}_\omega(\mu) = \int_{X \times X} G_\omega(z, w)d\mu(z)d\mu(w).$$

As in [ZZ], we have:

**Definition 1.** The LD rate functional is defined by,

$$I^{\omega,K}(\mu) = -\frac{1}{2} \mathcal{E}_\omega(\mu) + \sup_K U^\mu_\omega, \quad \mu \in \mathcal{M}(X)$$

We also let

$$E_0(\omega) = \inf_{\mu \in \mathcal{M}(X)} I^{\omega,K}(\mu), \quad \bar{I}^{\omega,K} = I^{\omega,K} - E_0(\omega).$$

It is proved in [ZZ] that the infimum $\inf_{\mu \in \mathcal{M}(X)} I^{\omega,K}(\mu)$ is achieved at the unique Green’s equilibrium measure $\nu_{\omega,K}$ with respect to $(\omega, K)$, and $E_0(\omega) = \frac{1}{2} \log \text{Cap}_\omega(K)$, where $\text{Cap}_\omega(K)$ is the Green’s capacity. By the Green’s equilibrium measure we mean the minimizer of $-\mathcal{E}_\omega$ on $\mathcal{M}(K)$. 

**Theorem 1.** Assume that $d\nu \in \mathcal{M}(X)$ satisfies the Bernstein-Markov property (37) and that its support is nowhere thin (39). Then, for the Fubini-Study-fiber ensembles (see Definitions 2-3), $\tilde{I}_{\omega,K}$ of (13) is a convex rate function and the sequence of probability measures $\{\text{Prob}_N\}$ on $\mathcal{M}(X)$ defined by (9) satisfies a large deviations principle with speed $N^2$ and rate function $\tilde{I}_{\omega,K}$, whose unique minimizer $\nu_{h,K} \in \mathcal{M}(X)$ is the Green’s equilibrium measure of $K$ with respect to $\omega$.

Roughly speaking an LDP with speed $N^2$ and rate function $I$ states that, for any Borel subset $E \subset \mathcal{M}(X)$,

$$\frac{1}{N^2} \log \text{Prob}_N\{\sigma \in \mathcal{M} : \sigma \in E\} \to -\inf_{\sigma \in E} I(\sigma).$$

The rate function is the same as in the genus zero case in [ZZ]. The analytical problems involved in proving the LDP from the formula for the JPC are similar to those in the genus zero case of [ZZ]. The principal new feature in this work is the derivation of an explicit formula for the JPC for our ensembles.

Roughly speaking, the existence of the LDP is due to the explicit relation between two structures on $X^{(N)}$:

- **The fiber bundle structure coming from the Abel-Jacobi fibrations** $X^{(N)} \to \text{Jac}(X)$ (2)-(4). This gives rise to Fubini-Study metrics along the fibers, which are the source of the probability measures we define, based on the idea that the term ‘random section’ should refer to random coefficients relative to a fixed basis.

- **The product structure of $X^{(N)}$**: Although it is the quotient of the Cartesian product $X^N$ by the symmetric group $S_N$, the order $N!$ of $S_N$ is negligible when taking the $\frac{1}{N^2} \log$ limit, so we may think of $X^{(N)}$ as essentially a product. As such, it carries the exterior product measures $\pi_1^*\omega \boxtimes \cdots \boxtimes \pi_N^*\omega$ where $\omega$ is any Kähler form on $X$ and $\pi_j : X^N \to X$ is the projection onto the $j$th factor. In practice, we use the local coordinate volume form $\prod_{j=1}^N d\zeta_j \wedge d\overline{\zeta}_j$ where $\zeta_j$ is the local uniformizing coordinate on $X$. The main step in the proof of Theorem 1 is to express the probability measures on $\mathbb{P}E_N$ defined by the fiber bundle structure in terms of product measures. Estimates of the product measure in Lemma 18 are then crucial to the proof of Theorem 1.

The relation between the fiber bundle structure and the product structure of $X^{(N)}$ permeates the proof of Theorem 1 sometimes in an implicit way. For instance, our use of bosonization in calculating the JPC is essentially to relate these structures.

### 0.1. The Projective Linear Ensemble and the Fubini-Study-fiber ensemble

We now define the basic probability measures on $\mathbb{P}E_N$ of this article. As discussed in [ZZ, SZ] (and elsewhere), Gaussian (or Fubini-Study) measures on a vector space $V$ (or projective space $\mathbb{P}V$) correspond to a choice of Hermitian inner product on $V$. When $V = H^0(X, \xi)$, inner products may be defined by choosing a Hermitian metric $h$ on $\xi$ and a probability measure $\nu$ on $X$. The data $(h, \nu)$ induces the inner product

$$||s||_{G(h,\nu)}^2 := \int_X |s(z)|^2_h \, d\nu(z).$$

(14)

The Hermitian inner product $G(h, \nu)$ in turn induces a complex Gaussian measure $\gamma(h, \nu)$ on $H^0(X, \xi)$: If $\{S_j\}$ is an orthonormal basis for $G(h, \nu)$, then the Gaussian measure is given
in coordinates with respect to this basis by,

\[ d\gamma(s) := \frac{1}{\pi^m} e^{-|c|^2} dc, \quad s = \sum_{j=1}^{d} c_j S_j, \quad c = (c_1, \ldots, c_d) \in \mathbb{C}^d. \tag{15} \]

Here, \( d = \dim_{\mathbb{C}} H^0(X, \xi) \). As in [ZZ] we assume throughout that \( \nu \) is a Bernstein-Markov measure whose support is non-thin at all of its points (see [1.10] for background).

Since we are studying zeros, it is natural to push a Gaussian measure \( \gamma_{G(h,\nu)} \) on \( H^0(X, \xi) \) under the natural projection \( H^0(X, \xi)^* \to \mathbb{P}H^0(X, \xi) \) to obtain a Fubini-Study measure \( dF_{SH} \) induced by \((h, \nu)\). In fact, our important maps factor through the projective space, so it simplifies things to use Fubini-Study volume forms from the start. We refer to [ZZ], §3.2, for further discussion of this step.

The fact that \( E_N \) is a vector bundle rather than a vector space complicates this picture in two ways: first, we need to define a family \( G_N(\xi) \) of Hermitian inner products on the spaces \( H^0(X, \xi) \) as \( \xi \) varies over \( \text{Pic}^N \). If we are given a family \( G_N(\xi) \) of Hermitian metrics on \( H^0(X, \xi), \xi \in \text{Pic}^N \), we denote by \( \gamma_{G_N(\xi)} \) the associated family of Gaussian measures on \( H^0(X, \xi) \) and by \( dV_{FS}^{G_N(\xi)} \) the associated family of Fubini-Study measures on \( \mathbb{P}H^0(X, \xi) \) (the pushforwards of the \( \gamma_{G_N(\xi)} \)). Second, Gaussian measures along the fibers of \( E_N \) do not define a probability measure on \( E_N \); we also need a probability measure on the ‘base’, \( \text{Pic}^N \). For expository simplicity, we assume that the base measure is normalized Haar measure \( d\theta \) on \( \text{Pic}^N \approx \text{Jac}(X) \).

**Definition 2.** Given a family of Hermitian inner products \( G_N(\xi) \) for \( \xi \in \text{Pic}^N \), and a volume form \( dc \) on \( \text{Pic}^N \), the associated Fubini-Study-fiber measure \( d\tau_{FSH} \) on \( \mathbb{P}E_N \) is the fiber-bundle product measure,

\[ \int_{\mathbb{P}E_N} F(s) d\tau_{FSH}^{G_N(\xi)} := \int_{\text{Pic}^N} \left\{ \int_{H^0(X, \xi)} F(s) dV_{FS}^{G_N(\xi)} \right\} d\sigma(\xi), \tag{16} \]

where \( dV_{FS}^{G_N(\xi)} \) are the fiber Fubini-Study volume forms defined by the Hermitian inner products on \( H^0(X, \xi) \) induced by \( G_N(\xi) \). We also denote by \( \hat{\tau}_N \) the analogous construction for Gaussian-fiber measures. In the special case where \( d\sigma \) is normalized Haar measure, we call \( d\tau_{FSH} \) Fubini-Study-Haar measure.

In genus zero, where the only line bundle of degree \( N \) is \( \mathcal{O}(N) \), the inner products \( G_N(\omega, \nu) \) on \( H^0(\mathbb{C}^P^1, \mathcal{O}(N)) \) were those induced from tensor powers \( h^N \) of a fixed Hermitian metric \( h \) with curvature form \( \omega \) on \( \mathcal{O}(1) \) and from \( \nu \). In higher genus, we would like to induce a family of inner products \( G_N(\omega, \nu) \) on all \( H^0(X, \xi) \) from the data \((\omega, \nu)\). It is a slight but useful change in viewpoint to regard the basic data as \((\omega, \nu)\) rather than \((h, \nu)\) since \( h \) is determined by \( \omega \) only up to a constant.

The data \((\omega, \nu)\) can be used to determine a family \( G_N(\omega, \nu) \) of Hermitian inner products on \( H^0(X, \xi) \) in at least two natural ways. The first is to choose a family of Hermitian metrics \( h_N(\xi) \) on the family \( \xi \) such that the curvature \((1,1)\) forms of the \( h_N(\xi) \) equal \( \omega \). We will refer to such a family as \( \omega \)-admissible. Unfortunately, such a family is only determined up to a function on \( \text{Pic}^N \). The function may be fixed up to an overall constant using the so-called Faltings’ metric on the determinant line bundle \( \bigwedge^{\text{top}} H^0(X, \xi) \). This is a natural and
attractive approach, but requires a certain amount of background to define. In the end, the overall constant (or function) turns out to be irrelevant to the LDP.

There is a second approach which is less obvious but is simpler and asymptotically equivalent as $N \to \infty$. It arises from an auxiliary Hermitian line bundle $\mathcal{L}_{N+g} \to X$ of degree $N + g$, whose space $H^0(X, \mathcal{L}_{N+g})$ of holomorphic sections we call the large vector space (see [2.1] for background). Since $\dim H^0(X, \mathcal{L}_{N+g}) = N + 1$ and $X^{(N)} = \dim \mathbb{P}H^0(X, \mathcal{L}_{N+g})$, we regard $\mathbb{P}H^0(X, \mathcal{L}_{N+g})$ as a kind of linear model for $X^{(N)}$. As discussed below, there exists a finite branched holomorphic cover $X^{(N)} \to \mathbb{P}H^0(X, \mathcal{L}_{N+g})$ (away from a certain Wirtinger subvariety) which restricts on each fiber of the Abel-Jacobi map to an embedding of projective spaces. The choice of $\mathcal{L}_{N+g}$ is arbitrary but may be uniquely fixed by picking a base point $P_0$ and defining

$$\mathcal{L}_{N+g} = \mathcal{O}((N + g)P_0)$$

to be the line bundle of the divisor $(N + g)P_0$ (see [11] for the notation). We recall (from [Gu2], p. 107) that for each $\xi \in \text{Pic}^N$, there is an embedding

$$\iota_\xi: \mathcal{L}_{N+g} \to H^0(X, \mathcal{L}_{N+g}) : \mathcal{D}(s) \geq A_{\mathcal{L}_{N+g}}(\xi)$$

where $A_{\mathcal{L}_{N+g}}: \text{Pic}^{N+g} \to X^{(g)}$ is the Abel-Jacobi map associated to $\mathcal{L}_{N+g}$ (see [11]).

We use this to show that $\mathcal{E}_N$ is analytically equivalent (away from a certain Wirtinger subvariety) to the holomorphic vector bundle $\tilde{\mathcal{E}}_N \to X^{(g)}$ with fiber

$$\tilde{\mathcal{E}}_{P_1, \ldots, P_g} := \{ s \in H^0(X, \mathcal{L}_{N+g}) : \mathcal{D}(s) \geq P_1 + \cdots + P_g \}. \quad (18)$$

To be more precise, the Abel-Jacobi map $X^{(g)} \to \text{Jac}(X) \simeq \text{Pic}^N$ is a branched cover, with branch locus along a Wirtinger subvariety (cf. [2.3]). It turns out to be more convenient to use $\tilde{\mathcal{E}}_N$ rather than $\mathcal{E}_N$ since there is a simpler map from $\mathbb{P}\tilde{\mathcal{E}}_N \to \mathbb{P}H^0(X, \mathcal{L}_{N+g})$. From a probability point of view, this model is equivalent to that of Definition [2] but for emphasis we give it a separate definition.

**Definition 3.** Given a family of Hermitian inner products $G_N(\xi)$ for $\xi \in \text{Pic}^N$, and a volume form $d\sigma$ on $X^{(g)}$, the associated Fubini-Study-$X^{(g)}$ measure $d\tau_{N}^{FS}$ on $\mathbb{P}\tilde{\mathcal{E}}_N$ is the fiber-bundle product measure,

$$\int_{\mathbb{P}\tilde{\mathcal{E}}_N} F(s)d\tau_{N}^{FS}(s) := \int_{X^{(g)}} \{ \int_{\mathbb{P}H^0(X, \xi)} F(s)dV_{G_N(\xi)}^{FS} \} d\sigma(P_1 + \cdots + P_g), \quad (19)$$

where $dV_{G_N(\xi)}^{FS}$ are the fiber Fubini-Study volume forms defined by the Hermitian inner products on $H^0(X, \xi)$ induced by $G_N(\xi)$. We also denote by $\tilde{\gamma}_N$ the analogous construction for Gaussian-Haar measures.

We now use this set-up to define a family $G_N(\omega, \nu)$ of Hermitian inner products on $H^0(X, \xi)$. We first choose a Hermitian metric $h_0$ on $\mathcal{O}(P_0)$ with curvature $(1,1)$-form $\omega$. This induces Hermitian metrics $h_{N+g}$ on all the line bundles $\mathcal{L}_{N+g}$. Together with the positive measure $\nu$ on $X$, we obtain the inner product $G_{N+g}(h, \nu)$ on $H^0(X, \mathcal{L}_{N+g})$ and by restriction, on the subspaces $\iota_\xi: \mathcal{L}_{N+g} \to H^0(X, \xi)$. Since the embeddings are isomorphisms, we obtain inner products on $H^0(X, \xi)$ as $\xi$ varies over $\text{Pic}^N$ (see [2]). We refer to this family of inner products as an $\omega$-admissible family of Hermitian inner products (see [2, 3]). We then endow $H^0(X, \mathcal{L}_{N+g})$ and the fibers $H^0(X, \xi), \xi \in \text{Pic}^N$ with the Gaussian measures $\gamma_{G_{N+g}(h, \nu)}$. 
resp. \( \gamma_{G_N(h,\nu)} \), associated to the inner products. The Gaussian measures on \( H^0(X,\xi) \) are (up to identification by the embedding) the conditional Gaussian measures of \( \gamma_{G_{N+g}(h,\nu)} \).

We then projectivize to obtain Fubini-Study metrics associated to the inner products on the fibers \( \mathbb{P}H^0(X,\xi) \) and on \( \mathbb{P}(X,\mathcal{L}_{N+g}) \). The induced maps (cf. (17)),

\[
\begin{align*}
\psi_{\mathcal{L}_{N+g}} : \mathbb{P}\tilde{\mathcal{E}}_N &\to \mathbb{P}H^0(X,\mathcal{L}_{N+g}), \\
\psi_{\mathcal{L}_{N+g}}[s] &= [\iota_\xi,\mathcal{L}_{N+g}(s)], \text{ where } s \in H^0(X,\xi)
\end{align*}
\]

are by definition isometric along the subspaces \( \mathbb{P}H^0(X,\xi) \). In Proposition 3, we show that this map is a branched covering map of degree \( (N_g) \) away from the exceptional (Wirtinger) locus. Further \( \psi_{\mathcal{L}_{N+g}} \) pulls back the Hermitian hyperplane bundle of \( \mathbb{P}H^0(X,\mathcal{L}_{N+g}) \) (with its Fubini-Study metric) to the natural hyperplane bundle over \( \mathbb{P}\tilde{\mathcal{E}}_N \) (with the Hermitian metric above).

We use this construction to obtain our third ensemble:

**Definition 4.** We define the projective linear ensemble \((\mathbb{P}\mathcal{E}_N,d\lambda_{PL})\) by

\[
d\lambda_{PL} = \frac{1}{(N_g)!}\psi_{\mathcal{L}_{N+g}}^*dV_{G_{N+g}(h,\nu)}^{FS}
\]

where \( dV_{G_{N+g}(h,\nu)}^{FS} \) is the Fubini-Study volume form on \( \mathbb{P}H^0(X,\mathcal{L}_{N+g}) \) and \( \psi_{\mathcal{L}_{N+g}} : \mathbb{P}\tilde{\mathcal{E}}_N \to \mathbb{P}H^0(X,\mathcal{L}_{N+g}) \) is the map (20).

The Fubini-Study-Haar measure and the projective linear measure are closely related since they are top exterior powers of differential forms which agree along the fibers of the projection \( \mathcal{P}_N \to \text{Pic}^N \) to a base of fixed dimension \( g \). They are compared in detail in §6. A technical complication in this approach is that the pulled-back Fubini-Study probability measure is only a semi-positive volume form on \( X^{(N)} \) which vanishes along the branch locus.

0.2. Joint probability current. Given a probability measure \( \tau_N \) on \( \mathcal{E}^N \) (or \( \mathbb{P}\mathcal{E}^N \)) the corresponding JPC is defined as follows:

**Definition 5.**

\[
\tilde{K}^N(\zeta_1,\ldots,\zeta_N) := \mathcal{D}_*\tau_N \in \mathcal{M}(X^{(N)})
\]

of the measure \( \tau_N \) under the divisor map \( \mathcal{D} \).

Since we are dealing with a number of measures \( \tau_N \), we often subscript \( \tilde{K}^N \) to indicate the associated measure \( \tau_N \).

The main task of this article is to express the JPC of the ensembles in the previous section in terms of empirical measures of zeros (1), and to extract an approximate rate function from it. Less formally, we need to express the JPC in terms of ‘zeros coordinates’, i.e. the natural coordinates \( \{\zeta_1,\ldots,\zeta_N\} \) of \( X^{(N)} \). Elementary symmetric functions of the \( \zeta_j \) define local coordinates on \( X^{(N)} \). What makes the expression difficult is that the underlying probability measures on \( H^0(X,\xi) \) and \( \mathcal{E}^N \) are expressed in terms of coefficients relative to a basis of sections. Thus we need to ‘change variables’ from coefficients to zeros. As mentioned above, we are essentially changing from variables adapted to the Abel-Jacobi fibration structure of \( X^{(N)} \) to those adapted to its product structure.
In the genus zero case, the inverse map from zeros to coefficients is given by the Newton-Vieta formula,
\[
\prod_{j=1}^{N} (z - \zeta_j) = \sum_{k=0}^{N} (-1)^k e_{N-k}(\zeta_1, \ldots, \zeta_N) \; z^k,
\]
where \( e_j = \sum_{1 \leq p_1 < \cdots < p_j \leq N} z_{p_1} \cdots z_{p_j} \) are the elementary symmetric functions. The Jacobian of the map from coefficients to zeros is thus \(|\Delta(\zeta)|^2\) where
\[
\Delta(\zeta) = \det(\zeta_j^k) = \prod_{j<k} (\zeta_j - \zeta_k)
\]
is the Vandermonde determinant. In effect, we must generalize both formulae to higher genus. Generalizing the Vieta formula is the subject of §3.1. Generalizing the Vandermonde determinant formula is the subject of §3.3.

In particular, we need the analogue of \( z - w \) in higher genus. As is well-known, the analogue is the ‘prime form’ \( E(z, w) \) of \( X \). It is a holomorphic section of a certain line bundle over \( X \times X \) with divisor equal to \( D \), the diagonal in \( X \times X \). \( E(z, w) \) has a well-known expression in terms of theta-functions, but for our purposes it may be characterized as follows: the space \( H^0(X \times X, \mathcal{O}(D)) \) of holomorphic sections of \( \mathcal{O}(D) \) is one-dimensional \( \mathbb{R} \). The bundle of which \( E(z, w) \) is a section is isomorphic to \( \mathcal{O}(D) \to X \times X \). Hence to specify \( E(z, w) \) we need to specify an element of \( H^0(X \times X, \mathcal{O}(D)) \). Roughly speaking, we do this by specifying that \( E(z, w) \sim z - w \) near the diagonal. This depends on a choice of local coordinates, which we define by uniformizing, i.e. by expressing \( X = \tilde{X}/\Gamma \) where \( \tilde{X} \) is the universal holomorphic cover and \( \Gamma \) is the deck transformation group. We then select a fundamental domain \( \mathcal{F} \) for \( \Gamma \) in \( \tilde{X} \) and use the global coordinates on \( \tilde{X} \) as local coordinates on \( X \). For more details, we refer to §1.8.

As above, we denote by \( \mathcal{O}(P) \) the point line bundle, i.e. the line bundle associated to the divisor \( \{ P \} \). For \( g \geq 1 \), the space \( H^0(X, \mathcal{O}(P)) \) is one (complex) dimensional, and may be identified as the pullback under the embedding \( \iota_P : X \to X \times X, \iota_P(z) = (z, P) \) of the line bundle \( \mathcal{O}(D) \to X \times X \). The section \( 1_{\mathcal{O}(P)}(z) \) is defined to be \( \iota_P^* E(z, \cdot) = E(z, P) \). Using products of the prime form, we define ‘canonical sections’ \( S_{\zeta_1, \ldots, \zeta_N} \) of high degree line bundles with prescribed zeros (see §2). We also need to define a certain Bergman kernel for the orthogonal projection onto \( H^0(X, \mathcal{L}_{N+g}) \) with respect to the inner product \( G_{N+g}(h_0, \nu) \). It is not quite the standard one, but rather projects onto the codimension one subspace \( H^0_{P_0}(X, \mathcal{L}_{N+g}) \) of sections vanishing at the base point \( P_0 \). This kind of Bergman kernel is studied in [SZZ] and is called there the conditional Szegö kernel. Thus, we put
\[
B_N = \text{the Bergman kernel on } H^0_{P_0}(X, \mathcal{L}_{N+g}) \text{ w.r.t. the inner product } G_{N+g}(h_0, \nu).
\]
As will be shown below, \( B_N \) is almost the same as the Bergman kernel for a slightly modified line bundle \( E_{N+g-1} \) of degree \( N + g - 1 \) (see §54). Their determinants differ by products of \( 1_{P_0} \).

Before stating the result, we emphasize a further notational convention used throughout: when discussing sections of a line bundle \( L \), we fix a local frame \( e_L \) and express a section by \( s = fe_L \) where \( f \) is a locally defined function. As discussed in §1.11, the orthogonal projection onto \( H^0(X, L) \) is a section of \( L \otimes L^{-1} \) (the Szegö kernel). When we express it in
terms of $e_L \otimes \bar{e}_L$, the local function is the Bergman kernel $B_L$ above. In a similar way, we use $E(z, P)$ to be the local expression for $1_{\mathcal{O}(P)}(z)$ relative to the canonical frame for $\mathcal{O}(P)$. The former is standard, but the latter is not a standard convention.

The following Theorem expresses the JPC in terms of coordinates on $X(N)$ or more precisely its further lift to $X^N$. The coordinates are products of uniformizing coordinates on the factors. As mentioned above, and as will be clarified during the proof, the local expressions are understood to be given on the complement of the exceptional Wirtinger loci.

**Theorem 2.** (I) The JPC of zeros in the projective linear Fubini-Study probability space $\mathbb{P}E_N$ with measure in Definition 4 is given in local coordinates by

$$
\hat{K}_{PL}^N(\zeta_1, \ldots, \zeta_N) = \frac{1}{Z_N(h)} \frac{F_N(\zeta_1, \ldots, \zeta_N, P_0) \prod_{j=1}^{N-1} \prod_{j \neq k}^N E(\zeta_j, \zeta_k) \prod_{j=1}^N d\zeta_j \wedge d\bar{\zeta}_j}{\det(B_N(\zeta_j, \zeta_k))_{j,k=1}^N} \prod_{j=1}^N d\zeta_j \wedge d\bar{\zeta}_j
$$

$$
\times \left( \int_X \left| \prod_{j=1}^N E(P_j, z) \right|^2_{h_g} \cdot \left| \prod_{j=1}^N E(\zeta_j, z) \right|^2_{h_N} dv(z) \right)^{-N-1},
$$

where $P_1 + \cdots + P_g = A_{E_{N+g}}(\zeta_1 + \cdots + \zeta_N)$. (defined in (72)) and $Z_N(h)$ is a normalizing constant so that $\hat{K}^N$ has mass one. Here, $F_N(\zeta_1, \ldots, \zeta_N, P_0)$ is defined in (64).

Furthermore,

$$
(II) \quad \hat{K}_{PL}^N(\zeta_1, \ldots, \zeta_N) = \frac{1}{Z_N(h)} \frac{\exp(\sum_{\zeta_j} G(\zeta_j, \zeta_j) \prod_{j=1}^N |\rho_\omega(\zeta_j)|^2 d^2 \zeta_j) \prod_{j=1}^N d\mu_{\zeta_j}}{1_{\mathbb{CP}^1} \prod_{j=1}^N |(z - \zeta_j)|^2 e^{-N\varphi(z)} dv(z)} \frac{F_N(\zeta_1, \ldots, \zeta_N)}{N+1},
$$

where $d\mu_{\zeta_j} = \sum_{j=1}^g \delta_{P_j}(\zeta_j)$, where $\rho_\omega$ is defined in (70), and where $F_N$ is defined in (73).

Note that $\prod_{j=1}^N d^2 \zeta_j$ is not well-defined on $X(N)$ but only on the Cartesian product $X^N$; however, $K_{PL}^N$ is symmetric under permutations and descend to the quotient. We say more about this in (122) (see [A] for background on holomorphic forms on symmetric powers). To explain the first formula, we observe that the numerator and denominator involve Hermitian inner products of sections. When expressed in terms of the same local frame, the frames and metric factors cancel. This will be explained in a Remark after the statement of Proposition 5. The main difference between expressions (I) and (II) is that the difficult Bergman determinant $\det(B_N(\zeta_j, \zeta_k))_{j,k=1}^N$ in (I) has been expressed in terms of products of norms of the Green’s function (or the prime form). It is an instance of the theme mentioned above of relating the fiber bundle structure of $X(N)$ to its product structure. The identity relating the Bergman determinant and products of the prime form is known as the bosonization formula on Riemann surfaces [ABMN, [V]]. This complicated identity brings in other special factors which we call $F_N$. When we take $\frac{1}{N}$ log this factor will evaporate in the limit.

Since $\hat{K}^N$ is the pull-back of a smooth form under a smooth map (defined on the complement of a hypersurface), the zeros of the denominator must be cancelled by those of the numerator. This is discussed after the proof in (376).

The corresponding formula in the genus zero case $\mathbb{CP}^1$ is given in Proposition 3 of [ZZ]:

$$
\hat{K}^N(\zeta_1, \ldots, \zeta_N) = \frac{1}{Z_N(h)} \frac{\prod_{j=1}^N |\Delta(\zeta_1, \ldots, \zeta_N)|^2 d^2 \zeta_1 \cdots d^2 \zeta_N}{\left( \int_{\mathbb{CP}^1} \prod_{j=1}^N |(z - \zeta_j)|^2 e^{-N\varphi(z)} dv(z) \right)^{N+1}}.
$$
Here, we abbreviate \( d^2 \zeta = d\zeta \wedge d\bar{\zeta} \). Thus, the expression

\[
\Delta_j(\zeta_1, \ldots, \zeta_N) := \frac{\prod_{j=1}^g \prod_{k=1}^N E(P_j, \zeta_k) \cdot \prod_{j,k \neq j} E(\zeta_j, \zeta_k)}{\sqrt{\det (B_N(\zeta_j, \zeta_k))_{j,k=1}^n}}
\]

is a higher genus generalization of the Vandermonde determinant in the sense that it is the Jacobian of the change of variables from coefficients to zeros. The same Jacobian arises in bosonization formulae (see e.g. (4.15) in [ABMNV]).

For the purposes of this article, Theorem 2 is mainly useful to obtain a similar formula for the JPC \( \tilde{K}_{FSH}^N \) of the Fubini-Study-fiber ensembles of Definitions 2 and 3. These are the ensembles for which we prove Theorem 1. We obtain formulae for \( \tilde{K}_{FSH}^N \) by relating them explicitly to \( \tilde{K}_{PL}^N \) as given in Theorem 2. The formula for \( \tilde{K}_{FSH}^N \) is too complicated to state in the introduction, and requires many considerations from Abel-Jacobi theory. It is stated and proved in Theorem 6. We summarize it as follows:

**Theorem 3.** The JPC of zeros in the Fubini-Study-fiber probability spaces \( \mathbb{P} \mathcal{E}_N \) (resp. \( \mathbb{P} \tilde{\mathcal{E}}_N \)) with measure in Definition 2 (resp. Definition 3) pulls back to \( X^N \) as the form,

\[
\tilde{K}_{FSH}^N(\zeta_1, \ldots, \zeta_N) = \frac{J_N(\zeta_1, \ldots, \zeta_N)}{Z_N(\omega)} \exp \left( \frac{1}{2} \sum_{i \neq j} G_\omega(\zeta_i, \zeta_j) \right) \prod_{j=1}^N d^2 \zeta_j \left( \int_X e^{\int_X G_\omega(z,w) d\mu_\zeta(w)} e^{\int_X G_\omega(z,w) d\mu_{P(\zeta)}(w)} d\nu(z) \right)^{N+1},
\]

where \( J_N \) is defined in Theorem 6.

The main idea is that the volume forms \( \tilde{K}_{PL}^N \) and \( \tilde{K}_{FSH}^N \) on \( \mathbb{P} \mathcal{E}_N \) only differ in ‘horizontal’ directions with respect to connections on \( X^{(N)} \rightarrow \text{Jac}(X) \) resp. \( \tilde{X}^{(N)} \rightarrow X^{(g)} \), which has a fixed dimension \( g \). Hence, they are equivalent up to small errors as \( N \rightarrow \infty \).

### 0.3. Approximate rate function

We use Theorems 2 and 3 to derive an approximate rate functional for the large deviations principle. We need to introduce the following functionals:

**Definition 6.** Let \( \zeta \in X^{(N)} \) and let \( \mu_\zeta \) be as in (7). Also let \( \mu_{P(\zeta)} \) be the sum of point masses at the \( g \) points in the image of \( \zeta \) under the Abel-Jacobi map. Let \( D = \{(z, z) : z \in X\} \) be the diagonal. Put:

\[
\begin{align*}
\mathcal{E}_N^\omega(\mu_\zeta) &= \int_{X \times X \setminus D} G_\omega(z, w) d\mu_\zeta(z) d\mu_\zeta(w), \\
J_N^{\omega, \nu}(\mu_\zeta) &= \log \| e^{\int_U G_\omega(z, w) d\mu_{P(\zeta)}(w)} \|_{L^1(\nu)}
\end{align*}
\]

Then put

\[
I_N^{\omega, \nu}(\mu_\zeta) = -\frac{1}{2} \mathcal{E}_N^\omega(\mu_\zeta) + \frac{N + 1}{N} J_N^{\omega, \nu}(\mu_\zeta).
\]

We then have,

**Proposition 1.** Write \( \tilde{K}_{FSH}^N(\zeta_1, \ldots, \zeta_N) = D_N^N(\zeta_1, \ldots, \zeta_N) \prod_{j=1}^N d^2 \zeta_j \), and similarly define \( D_{FSH}^N \). Then,

\[
D_N^N(\zeta_1, \ldots, \zeta_N) = \frac{1}{Z_N(h)} e^{-N^2(I_N^{\omega, \nu}(\mu_\zeta) + O(1))}.
\]
We observe that \( D^N \) is precisely the coefficient relating \( \tilde{K}^N \) (defined using the fiber bundle structure of \( P\mathcal{E}_N \)) to the local Lebesgue product measure on \( X^{(N)} \). We could use any product measure \( \omega_0 \otimes \cdots \otimes \omega_0 \) here, but it is sufficient to use \( \prod_j \Delta^2 \zeta_j \) in local uniformizing coordinates. The proof of Theorem 1 from Proposition 1 uses almost the same analysis as in [ZZ]. Hence the emphasis of this article is on the proof of Theorems 2 and 3 and of Proposition 1.

Finally, we thank R. Wentworth and S. Wolpert for helpful conversations on the prime form and on bosonization formulae.

1. Background

The main purpose of this section is to review the Abel-Jacobi theory we need. To the extent possible, we follow the notation of [ACGH, ABMNV, Gu1, Gu2]. We also briefly review some definitions and notation regarding Gaussian and Fubini-Study random holomorphic sections. We use throughout the same notation and terminology as in [ZZ], and when the definitions are essentially the same as in genus zero, we refer the reader to that article.

We denote by \( X \) a compact smooth Riemann surface of genus \( g \). Throughout we fix a base point \( P_0 \in X \). By the uniformization theorem, it may be expressed as \( \tilde{X} \setminus \Gamma \) where \( \tilde{X} \) is the universal conformal cover (= \( \mathbb{CP}^1 \) if \( g = 0 \), = \( \mathbb{C} \) if \( g = 1 \) and = \( \mathcal{H} \) (the upper half plane) if \( g \geq 2 \)). Here, \( \Gamma \) is the deck transformation group or fundamental group, realized as conformal automorphisms of \( \tilde{X} \). We also denote by \( \pi_1(X, P) \) generators of \( \Gamma \) defined as in [Gu2] (page 4). They depend on a choice of base point \( P_0 \) and a marking of \( X \); we refer to [Gu2] for the background.

We further denote by \( \omega_1, \ldots, \omega_g \) a basis for the holomorphic differential one forms of \( X \). We assume the \( A, B \) cycles and basis are canonical, i.e. \( \int_{A_i} \omega_j = \delta_{ij} \). For any holomorphic differential, we denote by \( w(z) = \int_{z_0}^z \omega \) its Abelian integral, a holomorphic function on \( \tilde{X} \) with \( w(z_0) = 0 \). In the case of the basis differentials \( \omega_k \) the Abelian integrals are denoted \( w_k \).

We denote line bundles by \( L, \xi \) or by divisors of holomorphic sections \( s \in H^0(X, \xi) \). The Chern class of \( L \) is denoted as usual by \( c_1(L) \). We also put \( h^0(X, L) = \dim H^0(X, L) \).

1.1. Point line bundles over Riemann surfaces of genus \( g > 0 \). Given \( P \in X \), there exists a unique holomorphic line bundle \( \mathcal{O}(P) \) with the properties that \( \dim H^0(X, \mathcal{O}(P)) = 1 \) and \( c_1(\mathcal{O}(P)) = 1 \) and so that each non-zero section vanishes at (and only at) the point \( P \). These are the line bundles defined by a point divisor \( \{ P \} \), defined by an atlas of two charts, \( U_0 = X \setminus \{ P \} \) and \( U_1 \) a small disc around \( P \), and the transition function \( g_{01} = z - P \).

The line bundle \( \mathcal{O}(P) \) has a canonical section \( 1_{\mathcal{O}(P)} \) corresponding to the meromorphic function 1 under the correspondence between meromorphic functions and holomorphic sections of \( H^0(X, \mathcal{O}(P)) \). It is the section locally represented by 1 on \( X \setminus \{ P \} \). Our notation follows [ABMNV]; the same section is called the ‘constant section’ and is denote by 1 in [Fal].

These sections are canonical once we fix the coordinates and atlas, but that choice itself is non-canonical. The choices can be made consistently as \( P \) varies by working on \( X \times X \). As in the introduction, we define local coordinates by uniformizing. We let \( \{ U_{0, \alpha}, z_0 \} \) denote the corresponding atlas of \( X \). We then define an atlas of local trivializations of \( \mathcal{O}(D) \) taking the cover \( \{ U_0 \times U_0, U_0 \} \) of \( X \times X \) where \( U_0 = X \times X \setminus D \). Then the local holomorphic functions \( f_\alpha(Q, P) = z_\alpha(P) - z_\alpha(Q) \) are local defining functions of \( D \) in \( U_\alpha \times U_\alpha \). Hence their ratios
degeneracies of \( p \) \( X \) where \( \Delta \) is the Vandermonde determinant. As this shows, any smooth volume form on \( X \) therefore descends to \( p \) in terms of local coordinate volume form.

We briefly review this fact and the associated local coordinates on \( X \). We also recall that \( O(\mathcal{O}(D)) \) is the quotient of the Cartesian product \( \mathcal{O}(\mathcal{D}) \) by the action of the symmetric group \( S_r \). The action has non-trivial isotropy along the large diagonals \( \zeta_j = \zeta_k \) \( j \neq k \). However, \( X^{(r)} \) is a complex analytic manifold of dimension \( r \) and the natural projection

\[ p_r : X^r \rightarrow X^{(r)} \]

is a complex analytic \( r! \) sheeted branched covering map with branch locus on the large diagonals. We briefly review this fact and the associated local coordinates on \( X^{(r)} \); the degeneracies of \( p_r \) will be important later in explaining cancellations of zeros and poles in \( K^N_{\mathcal{F}} \). We refer to ([Gu2], Theorem 9; or [GH], p. 236).

Let \( D = p_1 + \cdots + p_r \in X^{(r)} \). Let \( U_i \) be a neighborhood of \( p_i \) and \( z_i \) a local coordinate in \( U_i \). We assume that \( U_i \neq U_j \) if \( p_i \neq p_j \) and \( U_i = U_j \), \( z_i = z_j \) if \( p_i = p_j \). Let \( \sigma_1, \ldots, \sigma_r \) be the elementary symmetric functions. Then

\[ q_1 + \cdots + q_r \rightarrow (\sigma_1 \{ z_i(q_i) \}, \ldots, \sigma_r \{ z_i(q_i) \}) \]

is a local coordinate system on \( p_r(U_1 \times \cdots \times U_r) \) defining \( X^{(r)} \) as a complex manifold. Away from the large diagonals, \( p_r \) is a covering map and we can use \( (z_1(p_1), \ldots, z_r(p_r)) \) as local coordinates.

A volume form on \( X^{(r)} \) pulls back under \( p_r \) to a smooth form on \( X^r \) which may be written in terms of local coordinate volume form

\[ d\sigma_1 \wedge \cdots \wedge d\sigma_r = \Delta(\zeta)d\zeta_1 \wedge \cdots \wedge d\zeta_r, \]

where \( \Delta \) is the Vandermonde determinant. As this shows, any smooth volume form on \( X^{(r)} \) lifts to a semi-volume form with zeros on the branch locus of \( p_r \). For more on holomorphic forms on \( X^{(r)} \) we refer to [A].

We also recall that \( X^{(g)} \) is a projective Kähler manifold. To see this, let \( L \rightarrow X \) be an ample line bundle and let \( \tilde{L} = L_{z_1} \boxtimes \cdots \boxtimes L_{z_g} \) be the exterior tensor bundle on \( X^r \). Then \( \boxtimes_{\tau \in S_r} \tau^* \tilde{L} \) defines an ample line bundle on \( X^r \) which is invariant under permutations and therefore descends to \( X^{(r)} \). Positively curved metrics on this bundle give a supply of Kähler forms on \( X^{(r)} \) which can be used to specify the volume forms on \( X^{(g)} \) in Definition 3.

1.3. Products of point line bundles and canonical sections. Point bundles can be used to generate all other line bundles. We denote by \( \mathcal{O}(P_1 + \cdots + P_n) = \mathcal{O}(P_1) \otimes \cdots \otimes \mathcal{O}(P_n) \) the tensor product of the point bundles. In general, given two divisors \( D, D' \), \( \mathcal{O}(D)\mathcal{O}(D') = \mathcal{O}(D + D') \).

Products of point line bundles have a canonical section, depending only on the local trivializing charts and coordinates used to define \( \mathcal{O}(P) \).

**Definition 7.** The canonical section of \( \mathcal{O}(\zeta_1 + \cdots + \zeta_N) \) is defined by

\[ 1_{\zeta_1 + \cdots + \zeta_N}(z) := \prod_{j=1}^N 1_{\mathcal{O}(\zeta_j)}(z). \]
By Riemann-Roch, for \( N \geq 2g - 1 \) we have
\[
\dim H^0(X, \mathcal{O}(P_1 + \cdots + P_N)) = N + 1 - g.
\]
This is the source of the difference between \( g = 0 \) and \( g > 0 \): the number of zeros of sections is greater by \( g \) than the dimension of the projective space of sections. Indeed, the configurations of zeros of sections \( s \in H^0(X, \xi) \) form the fiber of the Abel-Jacobi map \( A_N : X^{(N)} \to \text{Jac}(X) \). This is the next object to review.

1.4. Jacobian and Picard varieties. The Jacobian variety \( \text{Jac}(X) \) may be identified with the compact complex \( g \)-dimensional torus \( \mathbb{C}^g / \mathcal{P} \) of unitary characters \( \chi \) of the fundamental group \( \pi_1(X) \) or equivalently of \( H^1(X, \mathbb{Z}) \). Here, \( \mathcal{P} \) is the period lattice of \( X \).

\[ \text{Pic}^0 = \text{Div}_0 / \text{PDiv} \] is the space of divisor classes of degree zero, i.e. divisors of degree 0 modulo principal divisors (divisors of meromorphic functions). We denote the divisor class of \( D \) by \( [D] \). Thus, \( \text{Pic}^0 \) is a compact complex torus of dimension \( g \). For general \( r \in \mathbb{Z} \), \( \text{Pic}^r \) is the space of holomorphic line bundles of Chern class \( r \). \( \text{Pic}^r \) is a homogeneous space for \( \text{Pic}^0 \) under tensor product. If we fix a base point \( P_0 \), we obtain an identification \( \text{Pic}^r(X) \simeq \text{Pic}^0(X) \) by
\[ \xi \in \text{Pic}^r(X) \to \xi \mathcal{O}(-rP_0). \]

Abel’s theorem states that the image is 0 if \( \sum_{j=1}^r \zeta_j - rP_0 \) is a principal divisor (the divisor of a meromorphic function).

When \( r = g \) the Abelian sums map is an analytic isomorphism away from a (Wirtinger) hypersurface. It may be re-formulated in terms of Picard varieties as follows: The map \( P_1 + \cdots + P_g \to \mathcal{O}(gP_0 - (P_1 + \cdots + P_g)) \) is an analytic diffeomorphism
\[
A_g : X^{(g)} \simeq \text{Pic}^0(X)
\]
outside of a codimension one subvariety. That is, any \( \xi \in Pic^0(C) \) may be represented in the form

\[
\xi = \mathcal{O}(P_1 + \cdots + P_g)\mathcal{O}(gP_0)^{-1},
\]

for some \( P_1 + \cdots + P_g \). The representation is unique when \( \dim H^0(X, \mathcal{O}(P_1 + \cdots + P_g)) = 1 \), and this holds for generic \( P_1 + \cdots + P_g \) (\[Giu\] (pages 116-120)).

The exceptional points lie on Wirtinger varieties, which play an important role in the formulae for the JPC. There are several related definitions accordingly as we regard them as subsets of \( \text{Jac}(X) \) or \( \text{Pic}^N(X) \) or \( X^{(g)} \). The Wirtinger varieties \( W_r \subset \text{Jac}(X) \) are defined by \( W_r = W_1 + \cdots + W_1 \subset \text{Jac}(X) \) where \( W_1 \) is the image of \( X \) under \( A_1 \). They depend on the choice of base point \( P_0 \), and one has \( \dim W_r = r \) for \( 1 \leq r \leq g \) and \( W_g = \text{Jac}(X) \). We also recall \textit{Jacobi’s inversion formula}: The Abelian sums map \( A_g : X^{(g)} \to \text{Jac} X \) is a surjective holomorphic map with a codimension one singular set of effective divisors \( P_1 + \cdots + P_g \) for which \( \dim H^0(X, \mathcal{O}(P_1 + \cdots + P_g)) = 2 \); equivalently, there exists a different element \( Q_1 + \cdots + Q_g \) for which \( \mathcal{O}(P_1 + \cdots + P_g) = \mathcal{O}(Q_1 + \cdots + Q_g) \). The image of this set under \( A_g \) is \( W_1 \), which has dimension \( g - 2 \) by the Brill-Noether formula (see \[ACGH\]). Viewing \( W_g \subset \text{Pic}^g(X) \) as the subset with \( \dim H^0(X, \xi) = 2 \), one has a map \( X \times W_g \to W_{g-1} \) with \( (P_0, \xi) \to \xi A_{\mathcal{O}(P_0)} \), whose sections correspond to the sections of \( \xi \) vanishing at \( P_0 \). Fixing the base point, this map embeds \( W_1 \subset W_{g-1} \) as a codimension one subset.

Fix a line bundle \( \mathcal{L}_{N+g} \) of degree \( g + N \). Let \( \zeta_1 + \cdots + \zeta_N \in X^{(N)} \). Then there exists a point \( P_1 + \cdots + P_g \) so that \( \mathcal{O}(\zeta_1) \otimes \cdots \otimes \mathcal{O}(\zeta_N) \simeq \mathcal{L}_{N+g} \mathcal{O}(P_1 + \cdots + P_g)^{-1} \). The point is unique when \( \mathcal{L}_{N+g} \otimes \mathcal{O}(- \zeta_1 - \cdots - \zeta_N) \notin W_1 \), i.e. lies outside of the codimension one Wirtinger subvariety \( W_1 \) of line bundles \( \xi \) of degree \( g \) with \( \dim H^0(X, \xi) \geq 2 \).

We take \( \mathcal{L}_{N+g} \) to be \( (N + g)P_0 \) throughout, and further define

\[
X_{N+g}^{(N)} = \{ \zeta_1 + \cdots + \zeta_N \in X^{(N)} : (N + g)P_0 - \zeta_1 + \cdots + \zeta_N \in W_1 \},
\]

and

\[
A_{\mathcal{L}_{N+g}} : X^{(N)} \setminus X_{N+g}^{(N)} \to X^{(g)}, \quad A_{\mathcal{L}_{N+g}}(\zeta_1 + \cdots + \zeta_N) = P_1 + \cdots + P_g.
\]

where as above \( (N + g)P_0 - (\zeta_1 + \cdots + \zeta_N) = P_1 + \cdots + P_g \), for a unique \( P_1 + \cdots + P_g \).

In the notation of \[ACGH\], \( X^{(N)}_{N+g} \) is the inverse image of \( W_1 \) under the Abel sums maps \( X^{(N)} \to \text{Jac}(X) \).

**Definition 8.** Let \( \xi \in \text{Pic}^N(X) \) and let \( s \in \mathbb{P}H^0(X, \xi) \). If \( \mathcal{O}((N + g)P_0 - \mathcal{D}(s)) \notin W_1 \) we define the canonical factor associated to \( [s] \) to be the factor \( \prod_{j=1}^g \mathbf{1}_{\mathcal{O}(P_j)} \) where \( A_{\mathcal{L}_{N+g}}(\mathcal{D}(s)) = P_1 + \cdots + P_g \).

1.6. Picard bundles, vortex moduli space and Poincaré bundles. In the introduction, we defined the Picard bundles \[3\] and their projectivizations \( \mathbb{P}\mathcal{E}_N \), the vortex moduli spaces. It is well-known that \( \mathbb{P}\mathcal{E}_N \) is analytically equivalent to \( X^{(N)} \) for \( N \geq 2g - 1 \). For the proof we refer to \[Gu2\], Corollary 2 to Theorem 10; and Corollary 2 to Theorem 15. See also \[ACGH\] and also \[Mat, Sch\] for the original proofs.

In \[FL\] it is proved that the Picard bundles are negative complex vector bundles, i.e. that \( \mathcal{E}_N^* \) is an ample vector bundle. It is equivalent that \( \mathcal{O}_{\mathbb{P}(1)} \to \mathbb{P}(\mathcal{E}_N) \) is an ample line bundle. See also \[ACGH\], Ch. VII.
1.7. **Theta functions, theta divisor and Riemann’s vector** $\Delta$. Riemann’s theta function $\theta$ is a section of a certain line bundle $\Theta_{\alpha} \to \text{Jac}(X)$. Its divisor is also denoted by $\Theta$. We refer the reader to [DP, F, Gu1, Gu2, ABMNV, VV] for background. It is also customary to denote by $\Theta$ the set of line bundles $\xi \in \text{Pic}^g(X)$ such that $h^0(X, \xi) = 1$, i.e. which have a global section (as in [Fal, ABMNV]).

As above, we denote by $W_{g-1} \subset \text{Pic}^{g-1}(X)$ the subvariety of effective divisors of degree $g - 1$. Riemann’s vanishing theorem states that there exists a point $\Delta \in \text{Pic}^{g-1}(X)$ so that $\Theta = W_{g-1} + \Delta$ (i.e. it is the translate of $W_{g-1}$ by this point of the torus). Thus, $\theta(\sum_{j=1}^{g-1} P_j - \Delta) \equiv 0$ and $\theta(z) = 0$ if and only if $z = \sum_{j=1}^{g-1} P_j - \Delta$ for some $P_j$. We also denote by $\theta[\alpha]$ the theta function with characteristic $\alpha$ (a choice of spin structure).

The ‘map’ $\zeta_1 + \cdots + \zeta_N \to P_1 + \cdots + P_g$ in (32) is the composition of the Abel sums map $X^{(N)} \to \text{Jac}(X)$ with the Jacobi inversion ‘map’ $\text{Jac}(X) \to X^{(g)}$. The latter may be described as follows ([ACGH], p. 28): Let $A_1 : X \to \text{Jac}(X)$ be the Abel embedding. Then the inversion map $\psi : X^{(g)} \to \text{Jac}(X)$ is given by $\psi(\lambda) = A_1^*(\Theta_{\lambda+\Delta})$ as long as $A_1(X)$ is not contained in the translate $\Theta_{\lambda}$ of $\Theta$ by $\lambda$. Here, $A_1^*(\Theta_{\lambda+\Delta})$ is the effective divisor of degree $g$ on $X$. Thus, we have

$$P_1 + \cdots + P_g = A_1^*(\Theta_{\lambda_1+\cdots+\lambda_N+\Delta}),$$

where $A_N$ is (as above) the Abelian sums map with basepoint $P_0$.

1.8. **Canonical section of a point line bundle and the prime form.** The canonical sections $1_{O(P)}(z)$ are defined rather abstractly. After making identifications, they may be more concretely expressed in terms of $\theta$-functions. The prime form is a holomorphic differential form on $X$ of type $(-1/2, -1/2)$ of the form $\frac{z-w}{\sqrt{dz}\sqrt{dw}}$. The degree of the canonical bundle is $2g - 2$, and that of the square root spin bundles is $g - 1$. So $\frac{z-w}{\sqrt{dz}\sqrt{dw}}$ has a zero at $z = w$ plus $g - 1$ other zeros at points independent of $w$. To remove the extra zeros, one defines the prime form by (cf. [F], Definition 2.1)

$$E(z, w) = \frac{\theta[\alpha](w-z)}{\sqrt{\omega_\alpha(z)\sqrt{\omega_\alpha(w)}}}$$

where $\alpha$ is an odd spin structure (i.e. a choice of square root of the canonical bundle $K_X$) and $\sqrt{\omega_\alpha}$ is a certain holomorphic section of the corresponding spin bundle. This prime form vanishes only when $z = w$. To tie this discussion together with the abstract one in §1.4 we have (see e.g. [DP], (6.54)):

**Proposition 2.** Let $i_w : X \to X \times X$ be the map $i_w(z) = (z, w)$, let $\mu(z, w) = A_1(z - w)$, and let $\Theta$ be the theta-divisor. Then each fixed $w$,

- $i_w^*\mu^*\Theta \otimes K^{1/2}$ is a line bundle of degree $g - (g - 1) = 1$;
- $i_w^*\mu^*\Theta \otimes K^{-1/2} = \zeta_w$
- $1_w(z) = E(z, w) \in H^0(X, i_w^*\mu^*\Theta \otimes K^{-1/2})$

It is not important in this article whether we use the explicit formula in terms of theta functions or the more abstract definition in terms of point line bundles. We will be taking the Hermitian norms of these sections and construct the metrics so that the isomorphism from the point line bundle setting to the theta function setting is isometric. We denote a Hermitian metric on $O(P)$ by $h_P$ so that the norm of $1_{O(P)}(z)$ is $\|1_{O(P)}(z)\|_{h_P(z)}$. It is
equivalent to define a Hermitian metric on $O(D) \to X \times X$ and pull it back under $\iota_P$, so that $||1_{O(P)}(z)||_{h_P} = ||E(z, P)||_{h(z)\otimes h(P)}$.

1.9. **Hermitian metrics and Chern classes.** Let $h$ be a smooth Hermitian metric on a holomorphic line bundle $L \to X$. Its Chern form is defined by

$$c_1(h) = \omega_h := -\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log ||e_L||_h^2,$$  \hspace{1cm} (35)

where $e_L$ denotes a local holomorphic frame (= nonvanishing section) of $L$ over an open set $U \subset M$, and $||e_L||_h = h(e_L, e_L)^{1/2}$ denotes the $h$-norm of $e_L$. We say that $h$ is positive if the (real) 2-form $\omega_h$ is a positive $(1, 1)$ form, i.e. defines a Kähler metric.

For any smooth Hermitian metric $h$ and local frame $e_L$ for $L$, we write $||e_L||_h^2 = e^{-\varphi}$ (or, $h = e^{-\varphi}$), and

$$\omega_h = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \varphi = dd^c \varphi.$$  \hspace{1cm} (36)

We refer to $\varphi = -\log ||e_L||_h^2$ as the potential of $\omega_h$ in $U$, or as the Kähler potential when $\omega_h$ is a Kähler form. As in [ZZ], we are interested in general smooth metrics, not only those where $\omega_h$ is positive. The metric $h$ induces Hermitian metrics $h^N$ on $L^N = L \otimes \cdots \otimes L$ given by $||s \otimes \cdots \otimes s||_h^N = ||s||_h^N$. The $N$-dependent factor $e^{-N\varphi}$ is then the local expression of $h^N$ in the local frame $e^N$.

In the reverse direction, suppose that we are given a smooth $(1, 1)$ form $\omega$ with $\int_X \omega = 1$ and a line bundle $L$ of degree one. Then there exists a Hermitian metric $h$ on $L$, unique up to a multiplicative constant, with $\omega_h = \omega$. To see this, let $h_*$ be any Hermitian metric on $L$. Then $\int_X (\omega - \omega_*) = 0$ so there exists a $\varphi_0 \in C^\infty(X)$ orthogonal to the constant functions so that $\omega - \omega_* = dd^c \varphi_0$. It is unique up to an additive constant but can be normalized to have integral zero with respect to $\omega_*$. Then $h = e^{-\varphi_0}h_*$. We adopt the following terminology from [Fall] and elsewhere: Given a real $(1, 1)$ form $\omega_0 \in H^2(X, \mathbb{Z})$, a Hermitian metric on a line bundle $\xi \to X$ is called $\omega_0$-admissible if its curvature $(1, 1)$ form (36) equals $\omega_0$.

1.10. **Hermitian inner products on $H^0(X, \xi)$.** As mentioned in the introduction, and as reviewed below, our Gaussian and Fubini-Study measures are induced by a choice of data $(\omega_0, \nu)$ where $\omega_0 \in H^2(X, \mathbb{Z})$ has $\int \omega_0 = 1$ and where $\nu$ is a probability measure on $X$ satisfying the following two technical assumptions: First is the weighted Bernstein-Markov condition (see [B] [ZZ] or [BE], Definition 4.3 and references):

For all $\epsilon > 0$ there exists $C_\epsilon > 0$ so that

$$\sup_K ||s(z)||_{h^N} \leq C_\epsilon e^{\epsilon N} ||s||_{G_N(h, \nu)}, \quad s \in H_0(\mathbb{CP}^1, O(N)).$$  \hspace{1cm} (37)

Here, and throughout this article, we write

$$K = \text{supp } \nu.$$  \hspace{1cm} (38)

Second is the assumption that

$K$ is non-thin at all of its points.  \hspace{1cm} (39)

We refer to [ZZ] for further discussion of these rather mild assumptions.
We only assume that $h$ is a $C^\infty$ metric as in [SZ, ZZ, B, Ber1]. In the local frame any holomorphic section may be written $s = f e$ where $f \in \mathcal{O}(U)$ is a local holomorphic function. The inner product (14) then takes the form,

$$||s||_{G(h, \nu)} = \int_{\mathbb{C}} |f(z)|^2 e^{-N \varphi} d\nu(z).$$

(40)

1.11. **Szegö projectors and Bergman kernels.** Let $(L, h)$ be any Hermitian holomorphic line bundle, let $\nu$ be a probability measure, and let $G = G(h, \nu)$ be the Hermitian inner product (14)-(11). We denote by $\Pi_G : L^2(X, L) \to H^0(X, L)$ the orthogonal projection with respect to $G$. It is a section of $L \otimes L \to X \times X$. If we choose a local frame $e_L$ for $L$, the it may be expressed as $\Pi_G = B_G(z, w) e_L \otimes e^*_L$. The coefficient is the Bergman kernel function relative to a frame $e_L$. If $S_j$ is an orthonormal basis of $H^0(X, L)$ with respect to $G$, then locally $S_j = f_j e_L$ where $f_j$ are local holomorphic functions and $B_G(z, w) = \sum_{j=1}^d f_j(z) \overline{f_j}(w)$ where $d = \dim H^0(X, L)$. Thus, if $s = f e_L$ is any section,

$$\Pi_G s(z) = \left( \int_X B_G(z, w) f(w) e^{-\varphi} d\nu \right) e_L(z), \quad (e^{-\varphi} = |e_L|^2_h)$$

1.12. **Coherent states.** Given the inner product $G = G(h, \nu)$ on $H^0(X, L)$ and a point $P \in X$, the associated coherent state is defined by

$$\Phi^P_G := \Pi_G(\cdot, P).$$

(41)

Its important property is that

$$s(P) = \langle s, \Phi^P_G \rangle_G, \quad s \in H^0(X, L).$$

(42)

Thus, $\Phi^P_G$ represents the evaluation functional at $P$. It is also useful to a scalar valued evaluation functional by picking a non-zero $v_P \in L_P$ and tensoring $\Phi^P_G \otimes v^*_P$, i.e. producing $\langle s(P), v_P \rangle_G$.

1.13. **Fubini-Study volume form.** Let $Z \in \mathbb{C}^{d+1}$ and let $||Z||^2 = \sum_{j=1}^{d+1} |Z_j|^2$. In the open dense chart $Z_0 \neq 0$, and in affine coordinates $w_j = \frac{Z_j}{Z_0}$, the Fubini-Study volume form is given by,

$$dVol_I = \frac{\prod_i dw_i \wedge d\bar{w}_i}{(1 + ||W||^2)^{d+1}}.$$

(43)

We sometimes work with a basis which is not orthonormal for the inner product, and therefore need a more general formula where the inner product $||Z||^2$ is replaced by any Hermitian inner product $||AZ||^2$ on $\mathbb{C}^{d+1}$ where $A \in GL(d+1, \mathbb{C})$. The Fubini-Study metric for this inner product is $\partial \bar{\partial} \log ||AZ||^2$. On the affine hyperplane $\{(w, 1)\}$, the matrix $A = \begin{pmatrix} A & \bar{b} \\ \bar{c} & d \end{pmatrix}$ ‘acts’ by the projective linear map $A_P \cdot W = \frac{AW + \bar{b}}{\bar{c}W + d}$; ‘acts’ is in quotes because the group does not preserve this hyperplane. A general inner product on $\mathbb{C}^{d+1}$ may be expressed in the form $||AZ||^2$ for some $A \in GL(N+1, \mathbb{C})$. The associated Fubini-Study metric
is $\frac{i}{2\pi} \partial \bar{\partial} \log ||AZ||^2 = A^* \frac{i}{2\pi} \partial \bar{\partial} \log ||Z||^2$. Hence, in homogeneous coordinates, the induced volume form in the affine chart is

$$dVol_A = \left( (w, 1)^* A^* \frac{i}{2\pi} \partial \bar{\partial} \log ||Z||^2 \right)^d = A_F^d \prod_i dw_i \wedge d\bar{w}_i \left( 1 + ||W||^2 \right)^{d+1}. \quad (44)$$

As in $[ZZ]$, under the natural projection $\pi : \mathbb{C}^{d+1} - \{0\} \to \mathbb{CP}^d$,

$$\pi^* dVol = |\det A|^2 |(AZ)_0|^2 \cdot \left( \frac{\partial}{\partial Z_0} \wedge \frac{\partial}{\partial \bar{Z}_0} \right)^{-1} \left( \prod_{i=1}^d dZ_i \wedge d\bar{Z}_i \right), \quad (45)$$

where $\left( \frac{\partial}{\partial Z_0} \wedge \frac{\partial}{\partial \bar{Z}_0} \right)^{-1} \left( \prod_{i=1}^d dZ_i \wedge d\bar{Z}_i \right)$ is the coefficient of $dZ_0 \wedge d\bar{Z}_0$ in the form $d(A^*Z)_0 \wedge d(A^*\bar{Z})_0$.

2. Inner Products Induced by the Canonical Embedding

In this section, we provide background and definitions for the objects introduced in [0.1]. In particular, we define the large vector space $H^0(X, L_{N+g})$ and the embeddings of the spaces $H^0(X, \xi)$ with $\xi \in \text{Pic}^N(X)$ into it.

2.1. The large vector space $H^0(X, L_{N+g})$. Given a line bundle $L$ and a divisor $D = \sum_i a_i P_i$, let $\mathcal{L}(L; D)$ denote the vector space of meromorphic sections $s$ satisfying $D + \mathcal{D}(s) \geq 0$. Let $s_0$ be a global holomorphic section of $[D]$ with $\mathcal{D}(s_0) = D$. We recall that if $D$ is an effective divisor and $s_0$ be a section of $H^0(X, [D])$ with $\mathcal{D}(s_0) = [D]$, then multiplication by $s_0$ gives an identification $H^0(X, \mathcal{O}(L \otimes [-D])) \cong \mathcal{L}(L; -D)$.

We make constant use of the following case (see [Gu2], page 107): Let $\mathcal{L}_{N+g} = \mathcal{O}((N + g)P_0) \in \text{Pic}^{N+g}$, let $\zeta = \zeta_1 + \cdots + \zeta_N \in X^{(N)} \setminus X^{(N)}_{N+g}$, i.e., assume that $\mathcal{O}((N + g)P_0)\mathcal{O}(-\zeta_1 + \cdots + \zeta_N) \notin W^1_g$, and let $P_1 + \cdots + P_g = A_{\mathcal{L}_{N+g}}(\zeta_1 + \cdots + \zeta_N)$. Then multiplication by $\prod_{j=1}^g 1_{\mathcal{O}(P_j)}$ defines the isomorphism $[17]$

$$H^0(X, \mathcal{O}(\zeta_1 + \cdots + \zeta_N)) \cong H^0(X, \mathcal{O}((N + g)P_0 - (P_1 + \cdots + P_g)))$$

$$\cong \{ s \in H^0(X, \mathcal{O}((N + g)P_0)) : \mathcal{D}(s) \geq P_1 + \cdots + P_g \}. \quad (46)$$

We now take the product of the sections defined in Definitions [7.8] to produce canonical sections of $H^0(X, \mathcal{L}_{N+g})$.

DEFINITION 9. Assume that $\zeta = \zeta_1 + \cdots + \zeta_N \in X^{(N)} \setminus X^{(N)}_{N+g}$. Then we define the section $S_{\zeta_1, \ldots, \zeta_N} \in H^0(X, \mathcal{L}_{N+g})$ by,

$$S_{\zeta_1, \ldots, \zeta_N}(z) := \prod_{j=1}^g 1_{\mathcal{O}(P_j)}(z) \prod_{j=1}^N 1_{\mathcal{O}(\zeta_j)}(z),$$

where $1_{\mathcal{O}(P_j)}(z)$ and $1_{\mathcal{O}(\zeta_j)}(z)$.
2.2. The vector bundle $\tilde{E}_N$. Let $L_{N+g}$ be a line bundle of degree $N + g$, which as always we take to be $O((N + g)P_0)$. We then define a vector bundle over $X^{(g)}$:

**Definition 10.** $\tilde{E}_N \to X^{(g)}$ is the vector bundle with fiber,

$$\tilde{E}_{P_1 + \cdots + P_g} := H^0(X, L_{N+g}O(-(P_1 + \cdots + P_g))).$$  \hspace{1cm} (47)

We further define the extended configuration space

$$\tilde{X}^{(N)} := \mathbb{P}\tilde{E}_N = \{([s], P_1 + \cdots + P_g) : \mathcal{D}(s) + P_1 + \cdots + P_g = (N + g)P_0 \}$$

$$= \{((\zeta_1 + \cdots + \zeta_N, P_1 + \cdots + P_g) : \zeta_1 + \cdots + \zeta_N + P_1 + \cdots + P_g = (N + g)P_0 \}.$$

Here as usual, equality means equality of divisor classes. There is a natural map $\tilde{X}^{(N)} \to X^{(N)}$ which is an analytic isomorphism away from $X^{(N)}$, i.e. on the locus where there exists a unique $P_0 + \cdots + P_g$ so that $\zeta_1 + \cdots + \zeta_N + P_1 + \cdots + P_g = (N + g)P_0$. In effect, we add extra points to the configuration space when the representation is not unique. The purpose for doing this is to analyze the behavior of volume forms along the bad set.

There exists a natural diagram of maps

$$\begin{array}{ccc}
\mathbb{P}\tilde{E}_N & \xrightarrow{\rho} & \mathbb{P}E_N = X^{(N)} \\
\downarrow \pi & & \downarrow \rho \\
X^{(g)} & \xrightarrow{\xi} & \text{Pic}^N \simeq \text{Jac}(X),
\end{array}$$

where the bottom arrow is the map $P_1 + \cdots + P_g \to L \otimes O(-(P_1 + \cdots + P_g))$ and the top arrow is the induced identification of sections. We need the diagram of inverse maps and run into the usual Jacobi inversion problem, i.e. that the the maps are not invertible along the the Wirtinger subvarieties. But they are analytic isomorphisms away from these subvarieties:

$$\begin{array}{ccc}
X^{(N)} \setminus X^{(N)}_{N+g} & \simeq & \mathbb{P}E_N \setminus W^{N}_{(N+g)P_0} \\
\downarrow \pi & \xrightarrow{\xi} & \downarrow \rho \\
\text{Pic}^N & \xrightarrow{\xi} & X^{(g)}
\end{array} \hspace{1cm} (49)$$

The bottom map is singular (degenerate) along the hypersurface of $W^1_g \subset X^{(g)}$ where $\dim H^0(X, O(P_1 + \cdots + P_g)) = 2$. The image under $\pi$ of this set is the set of line bundles $\xi \in \text{Pic}^N$ so that $L_{N+g}^{-1} \xi \in W^1_g$. So the singular sets of the maps are compatible with the diagram.

A natural question is to compare $\mathbb{P}H^0(X, L_{N+g}), \mathbb{P}E_N$ and and $\mathbb{P}\tilde{E}$. As in (16), there is a natural identification

$$\tilde{E}_{P_1 + \cdots + P_g} := \{s \in H^0(X, L_{N+g}) : \mathcal{D}(s) \geq P_1 + \cdots + P_g \},$$  \hspace{1cm} (50)

defined by multiplication by $\sigma_L = \prod_{j=1}^g 1_{O(P_j)}$. Since we understand the relation between $\mathbb{P}E_N$ and and $\mathbb{P}\tilde{E}$, the main point is the following

**Proposition 3.** The fiberwise canonical map $\sigma_L : \mathbb{P}\tilde{E}_N \to \mathbb{P}H^0(X, L_{N+g})$,

$$\sigma_L(P_1 + \cdots + P_g, s) = \prod_{j=1}^g 1_{O(P_j)} s, \quad P_1 + \cdots + P_g = \rho(s)$$
is an analytic branched \((N+g)\)-fold cover taking the fiber of \(\mathbb{P}\hat{\mathcal{E}}_N\) over \(P_1 + \cdots + P_g \in X^{(g)}\) to the subspace
\[
\mathbb{P}\{S \in H^0(X, \mathcal{O}(\mathcal{L}_{N+g}) : \mathcal{D}(S) \geq P_1 + \cdots + P_g \} \subseteq \mathbb{P}H^0(X, \mathcal{L}_{N+g}).
\]
The branch locus \(\mathcal{B}^{(N)}_{N+g} \subset \hat{X}^{(N)}\) consists of \(\{ (\zeta_1 + \cdots + \zeta_N, P_1 + \cdots + P_g) \}\) with multiple points.

**Proof.** We first show that \(D\sigma_\mathcal{L}\) is an isomorphism on the open dense subset \(\hat{X}^{(N)} \setminus \mathcal{B}^{(N)}_{N+g}\). Since a holomorphic map of complex manifolds of the same dimension is surjective if its differential is surjective at one point, this will prove that \(\sigma_\mathcal{L}\) is surjective and that it is a branched covering map.

Let \(s_t\) be a curve in \(\hat{\mathcal{E}}_N\), and let \(\sigma_\mathcal{L}(s_t)\) be the image curve in \(H^0(X, \mathcal{L}_{N+g})\). Then
\[
D\sigma_\mathcal{L}(s) = \frac{d}{dt}_{t=0}s(t)\sigma_\mathcal{L}(t) = \hat{s}_0 \prod_{j=1}^g \mathbf{1}_{\mathcal{O}(P_j)} + s_0 \frac{d}{dt}_{t=0} \prod_{j=1}^g \mathbf{1}_{\mathcal{O}(P_j)}.
\]
If the left side is zero, then
\[
\frac{\hat{s}}{s} = -\sum_{j=1}^g \frac{\mathbf{1}_{\mathcal{O}(P_j)}}{\mathbf{1}_{\mathcal{O}(P_j)}}.
\]
Since \(s = C \prod_{j=1}^N \mathbf{1}_{\mathcal{O}(\zeta_j)}\), this implies
\[
\sum_{k=1}^N \frac{\mathbf{1}_{\mathcal{O}(\zeta_k)}}{\mathbf{1}_{\mathcal{O}(\zeta_k)}} + \sum_{j=1}^g \frac{\mathbf{1}_{\mathcal{O}(P_j)}}{\mathbf{1}_{\mathcal{O}(P_j)}} = 0.
\]
If the \(\{\zeta_k, P_j\}\) are all distinct then this equation cannot hold since the poles on the left side occur at some of the \(\{\zeta_k\}\) and on the right at some of the \(\{P_j\}\). Hence \(D\sigma_\mathcal{L}\) is injective (and therefore an isomorphism) away from the diagonal.

Further, it is \((N+g)\) to 1 on the same set. Indeed, given \([S] \in \mathbb{P}H^0(X, \mathcal{L}_{N+g})\), we split up its \(N+g\) zeros into two groups, one of \(N\) points \(\zeta_1 + \cdots + \zeta_N\) and one of \(g\) points \(P_1 + \cdots + P_g\). There does not appear to be any preferred way to do this, so we consider all possible ways. Since \(s\) is a section, \(\prod_{j=1}^N \mathcal{O}(\zeta_j) \otimes \prod_{j=1}^g \mathcal{O}(P_j) = \mathcal{L}_{N+g}\), it follows that \(A_{\mathcal{L}_{N+g}}(\zeta_1 + \cdots + \zeta_N) = P_1 + \cdots + P_g\). Hence \([S]\) is the image of the section \(\prod_{j=1}^N \mathbf{1}_{\mathcal{O}(P_j)}\) under the canonical map. When the points \(\{\zeta_1, \ldots, \zeta_N, P_1, \ldots, P_g\}\) are distinct, there are \((N+g)\) to split up the zeros into a subset of \(N\) elements and a subset of \(g\) elements. Hence the line through \(s\) is the image of \((N+g)\) sections, and it is clear that these are the only ways that \(S\) is the image of an element. On the hypersurfaces where \(\{\zeta_j, P_k\}\) has multiple points, there are fewer ways to split up the set of zeros; this is the branch locus.

\[\square\]

Proposition 3 indicates why it is simpler to work with \(\mathbb{P}\hat{\mathcal{E}}_N\) than \(\mathbb{P}\mathcal{E}_N\). In the latter case we would need to puncture out \(X^{(N)}_{N+g}\) in order to defined the map to \(\mathbb{P}H^0(X, \mathcal{L}_{N+g})\), since \(P_1 + \cdots + P_g\) would not be uniquely defined on the Wirtinger subvariety. In the case of \(\mathbb{P}\hat{\mathcal{E}}\), the the line bundle is duplicated at points \(P_1 + \cdots + P_g, P'_1 + \cdots + P'_g\) so that \(\mathcal{L}_{N+g} \otimes \mathcal{O}(-(P_1 + \cdots + P_g)) \simeq \mathcal{L}_{N+g} \otimes \mathcal{O}(-(P'_1 + \cdots + P'_g))\). This redundancy makes it possible to define the map to \(\mathbb{P}\hat{\mathcal{E}}\) at all points of \(\mathbb{P}\hat{\mathcal{E}}_N\).
2.3. \(\omega\)-admissible family of Hermitian metrics on \(\tilde{E}_N\). Having reviewed the relevant Abel-Jacobi theory, we now return to the question of defining Hermitian metrics on the vector bundle, \(\tilde{E}_N\), i.e. a smooth family of Hermitian inner products on the spaces \(H^0(X, \xi)\) for \(\xi \in \text{Pic}^N\). As mentioned in the introduction, we define them by fixing a Hermitian metric \(h_0\) on \(O(P_0)\) and the associated Hermitian metrics \(h_0^{N+g}\) on \(O((N+g)P_0) = L_{N+g}\) with Chern form \(\omega\). Together with the Bernstein-Markov measure \(\nu\) we obtain an inner product \(G^N_{N+g}(h, \nu)\) as in (14). We then define inner products \(G_N(h, \nu, L)\) on \(H^0(X, L)\) by specifying that the maps \(\sigma_L\) are isometric. That is, we restrict \(G^N_{N+g}(h, \nu)\) to each embedded subspace and thus induce an inner product on each \(H^0(X, L)\). We refer to the family of such inner products as the \(\omega\)-admissible Hermitian metrics. The Gaussian measures induced Fubini-Study measures on the associated projective spaces of sections.

2.4. \(\omega_0\)-admissible metrics and admissible Hermitian inner products. As mentioned in the introduction, there is another natural way to define a family of \(\omega_0\)-admissible Hermitian inner products on the line bundles \(\xi \in \text{Pic}^N(X)\) and an associated family of \(\omega_0\)-admissible Hermitian inner products on the spaces \(H^0(X, \xi)\). Namely, we equip the line bundles \(\xi\) with \(\omega_0\) admissible metrics and then use (14) to define associated Hermitian inner products. This approach involves the complication that the admissible line bundle metrics are only unique up to a constant, and therefore the family of metrics as \(\xi\) ranges over \(\text{Pic}^N\) is only unique up to a function on \(\text{Pic}^N\). This constant can be fixed up to an overall constant \(C_N\) by using the Faltings metric on the associated determinant line bundle of \(E_N \to \text{Pic}^N\), i.e. a Hermitian metric on \(\bigwedge^\text{top} E_N\).

A further complication is that the Hermitian inner products on \(H^0(X, \xi)\) differ from the inner products defined by the canonical embedding by the factor of \(\sigma_L = \prod_{j=1}^g 1_{O(P_j)}(z)\). That is, the canonical embeddings would not be isometric if we used \(\omega_0\)-admissible metrics to define admissible Hermitian inner products on \(H^0(X, \xi)\). To deal with this complication, one would need to express to directly evaluate the Fubini-Study-Haar ensemble with these inner products in terms of zeros coordinates.

Although the approach in terms of \(\omega_0\)-admissible Hermitian metrics and inner products seems very natural and attractive, we opt for the large vector space (and projective linear ensemble) for simplicity of exposition.

3. Proof of Theorem 2 (I)

In this section, we prove the first formula for the JPC for the projective linear ensemble in terms of the prime form and Bergman kernel determinant. Starting from the general formula for the Fubini-Study volume form with respect to an inner product on the large projective space \(PH^0(X, L_N)\) (§1.13), we pull back this volume form by a generalized Newton-Vieta map from zeros to sections (§3.3) to obtain a volume form on \(X^{(N)}\). We express it in the natural configuration space coordinates (zeros coordinates) as the quotient of a ‘Vandermonde’ and an \(L^2\) factor. In Proposition 5 we express the Vandermonde determinant in terms of the prime form. Finally we write the denominator in terms of the prime form to complete the proof.
We apply formula (143) to the projective space \( \mathbb{P}^d \), where as above \( \mathcal{L}_{N+g} \) is a line bundle of degree \( N + g \), endowed with the inner product (144) with \( h \) an admissible metric. To obtain an identification with \( \mathbb{C}^d \), we need introduce a basis of \( H^0(X, \mathcal{L}_{N+g}) \).

3.1. Coordinates relative to a basis of \( H^0(X, \mathcal{L}_{N+g}) \). We now define a special orthonormal basis of \( H^0(X, \mathcal{L}_{N+g}) \) and (at the same time) an affine chart \( \mathbb{C}^{N+g-1} \) for \( \mathbb{P}^d \).

We follow an analogy with the genus zero case, and we begin by explaining that case in a form suitable for generalization. When \( X = \mathbb{P}^1 \), we often use the basis \( \{ z^j \} \) on the right side of (22). This basis of polynomials represents local holomorphic coefficients relative to the frame \( e^N(z) \) of \( \mathcal{O}(N) \rightarrow \mathbb{C}^1 \) corresponding to the homogeneous polynomial \( z_N^j \) in coordinates \( (z_0, z_1) \) on \( \mathbb{C}^2 \). In these coordinates \( z = \frac{z_1}{z_0} \) and \( z^j e^N = \frac{z_1^j}{z_0^{N-j}} \). The section \( z_1^j \) is distinguished in this basis because the coefficient of \( z_N \) of the product \( \prod_{j=1}^{N} (z - \zeta_j) = z^N + \cdots \) always equals one. Hence an affine chart for \( \mathbb{P}^d(X, \mathcal{O}(N)) \) is given by the affine space of monic polynomials of this form; the associated affine coordinates \( w_j \) are the coefficients relative to \( 1, \ldots, z^{N-1} \).

The element \( z_N \) has a natural generalization to the line bundle \( \mathcal{L}_N \): Namely, it is the coherent state \( \Phi^\infty_{h, \mathbb{P}^1} \) for the Fubini-Study inner product centered at the point \( \infty \in \mathbb{C}^1 \) (see (112)). Indeed, for any holomorphic section (polynomial) \( s \), \( \langle s, \Phi^\infty_{h, \mathbb{P}^1} \rangle = s(\infty) \), while \( \langle s, z_N \rangle = a_N \), where \( s = \sum_{j=0}^{N} a_j z^j \). So we need to see that \( a_N = s(\infty) \). But in homogeneous coordinates, \( z_0 = 0 \) defines \( \infty \), so all monomials \( z^j e^N = \frac{z_1^j}{z_0^{N-j}} \) with \( j \neq N \) vanish at \( \infty \).

Given the inner product \( G_N(h, \nu) \) on \( H^0(X, \mathcal{L}_{N+g}) \), the Szegö projector \( \Pi_{\mathcal{L}_{N+g}} \) is defined to be the orthogonal projection from all \( L^2 \) sections of \( \mathcal{L}_{N+g} \) onto the space \( H^0(X, \mathcal{L}_{N+g}) \) with respect to \( G_N(h, \nu) \). For simplicity, we do not include the data \( (h, \nu) \) in the notation for \( \Pi_{\mathcal{L}_{N+g}} \). Given a point \( P \in X \), the associated coherent state is defined as in (112) i.e.

\[
\Phi^P_{N+g} := \Pi_{\mathcal{L}_{N+g}} (\cdot, P). \tag{51}
\]

Its important property is that

\[
s(P) = \langle s, \Phi^P_{N+g} \rangle_{G_N(h, \nu)}, \quad s \in H^0(X, \mathcal{L}_{N+g}). \tag{52}
\]

Strictly speaking, \( \Phi^P(z) \) is a section with values in \( \mathcal{L}_P \), and we need to tensor with a covector in \( \mathcal{L}_P^* \) to cancel this factor.

To generalize \( z_N \) we therefore pick a base point \( P_0 \) and use the element

\[
\hat{\psi}_0 := \frac{\Phi^P_{0, N+g}}{||\Phi^P_{0, N+g}||}
\]
as a distinguished basis element of \( H^0(X, \mathcal{L}_{N+g}) \). We also use it as a local frame for \( \mathcal{L}_{N+g} \rightarrow X \). We then pick a \( G_{N+g}(\omega, \nu) \)-orthonormal basis \( \{ \hat{\psi}_j \} \) for \( H^0_{P_0}(X, \mathcal{L}_{N+g}) \cong H^0(X, E_{N+g-1}) \), and write them locally as \( \hat{\psi}_j = \hat{\psi}_j \Phi^P_{0, N+g} \), where \( \hat{\psi}_j \) are local coefficient functions in the frame (An advantage of working with the large vector space is that we can fix a convenient basis for it.)

We then have the orthogonal decomposition,

\[
H^0(X, \mathcal{L}_{N+g}) = H^0_{P_0}(X, \mathcal{L}_{N+g}) \oplus \mathbb{C} \Phi^P_{0, N+g}, \quad \text{where} \quad H^0_{P_0}(X, \mathcal{L}_{N+g}) = \{ s : s(P_0) = 0 \}. \tag{53}
\]

We further define the auxiliary line bundle

\[
E_{N+g-1} = \mathcal{L}_{N+g} \otimes \mathcal{O}(-P_0) \tag{54}
\]
Thus, we define the isomorphism
\[ \otimes 1_{P_0} : H^0(X, E_{N+g-1}) \to H^0_{P_0}(X, \mathcal{L}_{N+g}). \] (55)

Equipped with the admissible metrics, this isomorphism is an isometry. We only introduce \( E_{N+g-1} \) to quote relevant facts from Abel-Jacobi theory from the literature, and to be able to speak of Bergman kernels rather than the conditional Bergman kernels for \( H^0_{P_0}(X, \mathcal{L}_{N+g}) \); the latter are the principal objects.

We let \( Z_j \) denote coordinates with respect to \( \{ \psi_j \} \). We view \( H^0_{P_0} = \{ Z : Z_0 = 0 \} \) as the ‘hyperplane at infinity’ and define the affine coordinates \( w \) on \( \mathbb{P}H^0(X, \mathcal{L}_{N+g}) \) by \( w_j = \frac{Z_j}{Z_0} \). Thus, the projective coordinates are ratios of the coefficient functions \( f_j \) of the section in the frame.

**Definition 11.** We then put
\[ \mathcal{E}_{N+1-j}(\zeta_1, \ldots, \zeta_N) = w_j(S_\zeta), \quad (j = 0, \ldots, N). \] (56)

Thus,
\[ \tilde{S}_{\zeta_1, \ldots, \zeta_N} := ||\Phi^{P_0}_{N+g}|| \frac{S_{\zeta_1, \ldots, \zeta_N}}{\langle S_{\zeta_1, \ldots, \zeta_N}, \Phi^{P_0}_{N+g} \rangle} := \sum_{j=0}^{N} \mathcal{E}_{N-j}(\zeta_1, \ldots, \zeta_N) \psi_j(z). \] (57)

By definition,
\[ \mathcal{E}_0(\zeta_1, \ldots, \zeta_N) = 1, \] (58)

generalizing the affine space of monic polynomials in the basis \( \{ z^j \} \). In that case, \( \langle S_{\zeta_1, \ldots, \zeta_N}, z^N \rangle = 1 \).

Since \( S_{\zeta_1, \ldots, \zeta_N} \) is not well-defined if \( \zeta_1 + \cdots + \zeta_N \in X^{(N)}_{N+g} \), we regard it as defined only on the complement. Alternatively, it is a well-defined map from extended configuration space \( \tilde{X}^{(N)} \).

3.2. **Pull back of the Fubini-Study volume form on** \( \mathbb{P}H^0(X, \mathcal{L}_{N+g}) \). The first step in the proof of Theorem 2 (I) is the following preliminary version of the formula for the JPC \( \tilde{K}^N_{PL} \) of the projective linear ensemble:

**Proposition 4.** The pullback to \( X^{(N)} \setminus X^{(N)}_{N+g} \) under \( \psi_\mathcal{C} \) (or to \( \tilde{X}^{(N)} \) under \( \sigma_\mathcal{C} \)) of the Fubini-Study volume form on \( \mathbb{P}H^0(X, \mathcal{L}_{N+g}) \) with respect to the inner product \( G_{N+g}(\omega, \nu) \) is given by,
\[ \tilde{K}^N_{PL} = ||\Phi^{P_0}_{N+g}||^{-2(N+1)} ||\Phi^{P_0}_{N+g}||^{2N+2} \prod_{j=1}^{N} \frac{d\mathcal{E}_j \wedge d\tilde{\mathcal{E}}_j}{||S_{\zeta_1, \ldots, \zeta_N}||^{2(N+1)} L^2(G_{N+g}(\omega, \nu))}. \] (59)

**Proof.** We evaluate the Fubini-Study volume form \( (45) \) on \( \mathbb{P}H^0(X, \mathcal{L}_{N+g}) \) with respect to the inner product \( G_{N+g}(\omega, \nu) \). The homogeneous coordinates \( w \) on the chart where \( Z_0 = \langle S_{\zeta_1, \ldots, \zeta_N}, \Phi^{P_0}_{N+g} \rangle \neq 0 \) are defined in \( (56) \), or equivalently, \( Z_j = \langle S_{\zeta_1, \ldots, \zeta_N}, \frac{\Phi^{P_0}_{N+g}}{||\Phi^{P_0}_{N+g}||} \rangle \mathcal{E}_{N+1-j}(\zeta_1, \ldots, \zeta_N) \), and
\[ S_{\zeta_1, \ldots, \zeta_N} = \prod_{j=1}^{g} 1_{\mathcal{O}(P_j)}(z) \cdot \prod_{j=1}^{N} 1_{\mathcal{O}(\zeta_j)}(z), = \sum_{k=0}^{N} Z_k \psi_k. \]
We then pull back the form (45) under the section \((w, 1) = \|\Phi_{N+g}^{P_0}\|^2 \frac{S_{\zeta_1, \ldots, \zeta_N}}{\langle S_{\zeta_1, \ldots, \zeta_N}, \Phi_{N+g}^{P_0} \rangle}\). Since we have picked an orthonormal basis for \(H^0(X, L_{N+g})\), the matrix \(A\) in (45) is the identity matrix. The formula for the denominator follows from

\[
(1 + \|W\|^2)^{N+1} = \|\Phi_{N+g}^{P_0}\|^{2(N+1)} \|\langle S_{\zeta_1, \ldots, \zeta_N}, \Phi_{N+g}^{P_0} \rangle\|^{-2(N+1)} \|S_{\zeta}\|^{2(N+1)}.
\]

3.3. Vandermonde in higher genus. The next step is to simplify the form in the numerator in Proposition 4. By the higher genus Vandermonde determinant we mean Jacobian determinant \(\det \frac{\partial \bar{\psi}}{\partial \zeta}\) defined by

\[
d\mathcal{E}_1 \wedge \cdots \wedge d\mathcal{E}_N = \det \left( \frac{\partial \bar{\psi}}{\partial \zeta} \right) \prod_{j=1}^{N} d\zeta_j.
\]

(60)

Here, we assume that \(\zeta_1, \ldots, \zeta_N\) does not lie in the branch locus of \(p_N : X^N \to X^{(N)}\) so that we can use \(\zeta_j\) as local coordinates (see 11.2). That is, we fix a trivializing chart \(U\) for \(\mathcal{O}(P_0)\) centered at \(P_0\), and a trivializing frame \(e\) for \(\mathcal{O}(P_0)\). We let \(\zeta\) denote a local holomorphic coordinate in \(U\) which vanishes at \(P_0\) and we denote the associated coordinates on \(U^{(N)}\) by \(\{\zeta_1, \ldots, \zeta_N\}\). We express each section of \(L_{N+g}\) as a local holomorphic function times this frame. For simplicity, we use the same notation for sections and their local holomorphic functions relative to this frame. We now prove the formula alluded to in (26).

**Proposition 5.** Let \(\mathcal{E}_{N-j}(\zeta_1, \ldots, \zeta_N)\) be defined by (57). Then

\[
\det \left( \frac{\partial \bar{\psi}_{N-j}(\zeta)}{\partial \zeta_k} \right)_{j, k=1}^{N} = \|\Phi_{N+g}^{P_0}\|^N \prod_{j=1}^{N} \prod_{j=1}^{g} 1_{\mathcal{O}(P_j)}(\zeta_j) \det \left( \bar{\psi}_{N-j}(\zeta) \right) \prod_{j=1}^{N} \prod_{j \neq k} 1_{\mathcal{O}(\zeta_j)}(\zeta_k).
\]

The determinant omits \(j = 0\), in which case \(\mathcal{E}_N \equiv 1\) and the derivatives vanish. We also observe that when taking the norm square of this expression, the factor \(\|\langle S_{\zeta_1, \ldots, \zeta_N}, \Phi_{N+g}^{P_0} \rangle\|^{2N+2}\) in Lemma 4 cancels all but two powers in the norm-square of \(\langle S_{\zeta_1, \ldots, \zeta_N}, \Phi_{N+g}^{P_0} \rangle\) in the denominator of (5).

**Remark:** The Slater determinant \(\det \left( \bar{\psi}_{N-j}(\zeta_j) \right)\) is a section of the highest power of the exterior tensor product

\[
\pi_1^* L_{N+g} \otimes \cdots \otimes \pi_N^* L_{N+g} \to X^N
\]

where \(\pi_j : X^N \to X\) is the projection to the \(j\)th factor. On the other hand,

\[
\left( \prod_{j=1}^{g} 1_{\mathcal{O}(P_j)}(\zeta_j) \cdot \prod_{j \neq k} 1_{\mathcal{O}(\zeta_j)}(\zeta_k) \right) d\zeta_k = \pi_k^* \partial \left( \prod_{j=1}^{g} 1_{\mathcal{O}(P_j)} \cdot \prod_{j=1}^{N} \prod_{j \neq k} 1_{\mathcal{O}(\zeta_j)} \right)(\zeta_k) \in \pi_k^* L_{N+g} \otimes K_X.
\]

Taking the exterior tensor product \(\prod_{k=1}^{N}\) of these \((1, 0)\) forms and taking the ratio with the Slater determinant produces a well-defined \((N, 0)\) form on \(X^{(N)}\), i.e. a section of \(\pi_1^* K_X \otimes \cdots \otimes \pi_N^* K_X \to X^N\). Thus, as mentioned after the statement of Theorem 2, the ratio is well defined without the choice of a Hermitian metric. Note also that we only obtain special sections since \(\bar{\psi}_j \in H^0_{P_0}(X, L_{N+g})\).

The proof of Proposition 5 consists of two Lemmas.
LEMMA 1. We have the following identity on determinants of $N \times N$ matrices,

$$\det \left( \sum_{n=1}^{N} \frac{\partial \varepsilon_{N,n}}{\partial \zeta_i} \psi_n(\zeta_1), \ldots, \sum_{n=1}^{N} \frac{\partial \varepsilon_{N,n}}{\partial \zeta_i} \psi_n(\zeta_N) \right) = \det \left( \sum_{n=1}^{N} \frac{\partial \varepsilon_{N,n}}{\partial \zeta_j} \psi_n(\zeta_1), \ldots, \sum_{n=1}^{N} \frac{\partial \varepsilon_{N,n}}{\partial \zeta_j} \psi_n(\zeta_N) \right)$$

As above, the sums omit $n = 0$ so that $\varepsilon_0 = 1$.

Proof. For each $n$ we consider the row vector $\Psi_n(\zeta) := [\psi_n(\zeta_1), \ldots, \psi_n(\zeta_N)]$. Then the $j$th row of our matrix is $\sum_{n=1}^{N} \frac{\partial \varepsilon_{N,n}}{\partial \zeta_j} \Psi_n(\zeta)$, so we are calculating

$$\sum_{n=1}^{N} \frac{\partial \varepsilon_{N,n}}{\partial \zeta_1} \Psi_n(\zeta) \wedge \cdots \wedge \sum_{n=1}^{N} \frac{\partial \varepsilon_{N,n}}{\partial \zeta_N} \Psi_n(\zeta).$$

Clearly this gives the sum

$$\left( \sum_{\sigma \in \Sigma_N} \epsilon(\sigma) \frac{\partial \varepsilon_1}{\partial \zeta_{\sigma(1)}} \cdots \frac{\partial \varepsilon_N}{\partial \zeta_{\sigma(N)}} \right) \Psi_1(\zeta) \wedge \Psi_2(\zeta) \wedge \cdots \wedge \Psi_N(\zeta)$$

stated in the Proposition.

We now calculate the left side in Lemma 1 in a different way:

LEMMA 2. With the above notation and conventions, we have

$$\det \left( \sum_{n=1}^{N} \frac{\partial \varepsilon_{N,n}}{\partial \zeta_1} \psi_n(\zeta_1), \ldots, \sum_{n=1}^{N} \frac{\partial \varepsilon_{N,n}}{\partial \zeta_1} \psi_n(\zeta_N) \right)$$

$$= ||\Phi_{P_{hN}}^P||^N \langle S_{\zeta_1, \ldots, \zeta_N}, \Phi_{P_{hN}}^P \rangle^{-N} \prod_{k=1}^{N} \left( \prod_{j=1}^{g} 1_{\mathcal{O}(P_j)}(\zeta_k) \cdot \prod_{j,k \neq j} 1_{\mathcal{O}(\zeta_j)}(\zeta_k) \right).$$

Proof. By definition of $S_{\zeta_1, \ldots, \zeta_N}$ (Definition 9) and by (57), we have

$$||\Phi_{P_{hN}}^P|| \langle S_{\zeta_1, \ldots, \zeta_N}, \Phi_{P_{hN}}^P \rangle^{-1} \prod_{j=1}^{g} 1_{\mathcal{O}(P_j)}(\zeta_k) \cdot \prod_{j=1}^{N} 1_{\mathcal{O}(\zeta_j)}(z) = \sum_{j=0}^{N} \varepsilon_{N-j}(\zeta_1, \ldots, \zeta_r) \psi_j(z).$$

Recall that we are working in the chart where $\langle S_{\zeta_1, \ldots, \zeta_N}, \Phi_{P_{hN}}^P \rangle \neq 0$. If we differentiate this factor or $\prod_{j=1}^{g} 1_{\mathcal{O}(P_j)}$ and set $z = \zeta_k$, the second factor vanishes. So we only need to differentiate the factor $\prod_{j=1}^{N} 1_{\mathcal{O}(\zeta_j)}(z)$ and multiply it by the other two factors with $z = \zeta_k$.

The factor $1_{\mathcal{O}(\zeta_j)}(\zeta_k)$ vanishes if $j = k$, so differentiation in $\zeta_n$ (in the local coordinate) must remove this factor to get a non-zero result on the left side when we set $z = \zeta_k$. Using that $1_{P}(Q) \sim P - Q$ near the diagonal, we get

$$\frac{\partial}{\partial \zeta_n} \prod_{j=1}^{N} 1_{\mathcal{O}(\zeta_j)}(z) |_{z=\zeta_k} = \prod_{j \neq n} 1_{\mathcal{O}(\zeta_j)}(\zeta_n)$$

$$= \delta_{nk} \prod_{j \neq n} 1_{\mathcal{O}(\zeta_j)}(\zeta_k).$$

Here we differentiate the local expression for $1_{\mathcal{O}(P)}(z)$ as if it were a function rather than a section. To be more precise, we implicitly use the Chern connection for the admissible metric.
on \(O(1)\). But the connection term vanishes when we evaluate at \(z = \zeta_k\), so the covariant derivative produces the same result as the local derivative.

It follows that
\[
\det \left( \frac{\partial}{\partial \zeta_j} S_{\zeta_1, \ldots, \zeta_N}(z) \big|_{z=\zeta_k} \right) = \prod_{k=1}^{N-1} \prod_{j=1}^{g} 1_{O(P_j)}(\zeta_k) \left( \prod_{j:j \neq k} 1_{O(\zeta_j)}(\zeta_k) \right).
\]

Multiplying by the other factors completes the proof of the Lemma.

Remark:

Further, the Slater determinant \(\det(\psi_j(z)) = \Delta(\zeta)\). So the two factors in Proposition 5 cancel to leave \(\Delta(\zeta)\).

Combining the two Lemmas we obtain Proposition \(\Box\). 

3.4. Slater determinant and Bergman determinant. To complete the proof of Theorem 2 (I), we relate the Slater determinants \(\det(\psi_j(z))_{j,n=1}^{N} \) in Lemma \(\Box\) to Bergman determinants. The following Lemma is a general fact for any Hilbert space (it also applies to \(H^0_p(X, L_{N+g})\)).

**Lemma 3.** Let \(E\) be a line bundle of degree \(n + g - 1 \geq 2g - 1\), and let \(G\) be an inner product on \(H^0(X, E)\). Let \(\psi_j = \{f_j e_E\}_{j=1}^{n}\) be a basis for \(H^0(X, E)\) Then any \((\zeta_1, \ldots, \zeta_n) \in X(n)\),
\[
\frac{\left| \det (f_j(\zeta_k))_{j,k=1}^{n} \right|^2}{\det (\langle \psi_j, \psi_k \rangle_G)} = \det (B_G(\zeta_j, \zeta_k))_{j,k=1}^{n}.
\]

If we include the frame \(e_E\) we would have the additional factor of \(\prod_{j,k=1}^{n} e_E(\zeta_j) \otimes e_E(\zeta_k)^*\) on both sides. If we then contract with the Hermitian metric \(h = e^{-\varphi}\) we would multiply both sides by \(e^{-\sum_{j=1}^{n} \varphi(\zeta_j)}\).

**Proof.** This follows from the general formula that if \(\{v_j\}\) is a basis of an inner product space \(V\), then
\[
|v_1 \wedge \cdots \wedge v_n|^2 = \det (\langle v_j, v_k \rangle). \tag{62}
\]
We consider a basis of coherent states $\Phi^G_k$ for $H^0(X, E)$ ($k = 1, \ldots, n$) with respect to $G$. Then $\psi_j(\zeta_k) = \langle \psi_j, \Phi^G_k \rangle_G$. We define the matrices,

$$M := (\langle \Phi^G_k, \Phi^G_\ell \rangle) = (B_G(\zeta_j, \zeta_\ell), \quad A = (A^k_j), \quad \psi_j = \sum A^k_j \Phi^G_k.$$

Then

$$\det (\langle \psi_j, \psi_k \rangle_G) = \det A^*A \det M = |\det A|^2 \det M.$$ 

Also,

$$\langle \psi_j, \Phi^G_k \rangle_G = AM.$$ 

It follows that

$$\frac{|\det (\langle \psi_j, \Phi^G_k \rangle_G)|^2}{\det (\langle \psi_j, \psi_k \rangle_G)} = \frac{|\det A|^2 |\det M|^2}{|\det A|^2 |\det M|} = \det M.$$

We now consider the determinant $\det (\psi_j(\zeta_k))_{j,n=1}^{N}$ in Lemma 1. Note that this matrix omits the column $n = 0$. Hence, the relevant Bergman kernel is the conditional Bergman kernel for $H^0_{P_0}(X, L_{N+g})$ with respect to the admissible metric and measure $d\nu$ (see (1.1)). Since this is not a standard object, we use the isomorphism of this conditional space with $H^0(X, E_{N+g-1})$ with the induced inner product. Then we can use the Bergman kernel for this space of sections. On the other hand, its basis $\psi_j^E$ of sections differ from those $\psi_j$ of $L_{N+g}$ by the factor $1_{\Omega(P_0)}$, which will show up in its Slater determinant.

**Corollary 4.** Let $\{\psi_j\}$ be the orthonormal basis of $H^0_{P_0}(X, L_{N+g})$ and let $B_N$ be the conditional Bergman kernel for the admissible inner product. Let $G$ be the isometric inner product on $H^0(X, E_{N+g-1}) \simeq H^0_{P_0}(X, L_{N+g})$ and let $B_{E_{N+g-1}}$ be its Bergman kernel. Then, the Slater determinant in the denominator of Proposition 3 is given by,

$$\left| \det (\psi_j(\zeta_k))_{j,k=0}^{N-1} \right|^2 = \left| \prod_{k=1}^{N} 1_{P_0}(\zeta_k) \right|^2 \left| \det (\psi_j^E(\zeta_k))_{j,k=0}^{N-1} \right|^2$$

$$= \left| \prod_{k=1}^{N} 1_{P_0}(\zeta_k) \right|^2 \det (B_{E_{N+g-1}}(\zeta_j, \zeta_k))_{j,k=1}^{N} \det (\langle \psi_j, \psi_k \rangle_G).$$

3.5. **Projective linear ensemble: Proof of I of Theorem 2.** Put:

$$F_N(\zeta_1, \ldots, \zeta_N, P_0) = \frac{||\Phi_{N}^P||^{-2}}{||S_{\zeta_1, \ldots, \zeta_N, \Phi_{N}^P}||^2}. \quad (64)$$

We now state I of Theorem 2 in a more precise way and complete the proof:

**Theorem 4.** The pull-back under $\psi_L : X^{(N)} \times X^{(N)}_{N+g} \rightarrow \mathbb{P}H^0(X, L_{N+g})$ (resp. $\sigma_L : X^{(N)} \rightarrow \mathbb{P}H^0(X, L_{N+g})$) of the Fubini-Study volume form, is given in uniformizing coordinates on the cover $X^N$ by,

$$\left(\frac{1}{Z_N(h)} \frac{\mathcal{F}_N(\zeta_1, \ldots, \zeta_N, P_0)}{\det(B_N(\zeta_j, \zeta_\ell))}, \Pi^N_{j=k+1} E(\zeta_j, \zeta_\ell) \Pi^N_{j=1} E(\zeta_j, \zeta_\ell) \right)^2 \prod_{j=1}^{N} d\zeta_j \wedge d\tilde{\zeta}_j$$

$$\times \left( \int_X \left| \Pi_{j=1}^g E(P_j, z) \right|^2_{h_g} \cdot \left| \Pi_{j=1}^{N} E(\zeta_j, z) \right|^2_{h_N} d\nu(z) \right)^{-N-1},$$
where \( P_1 + \cdots + P_g = A \mathcal{L}_{N+g}(\zeta_1 + \cdots + \zeta_N) \). (defined in \( \mathbb{I}_g \)) and \( Z_N(h) \) is a normalizing constant so that \( \tilde{K}^N \) has mass one. It extends to a smooth form on extended configuration space \( \tilde{X}^{(N)} \).

The proof follows Proposition 4, Proposition 5 and Corollary 4. That is, we first use

\[
\tilde{K}^N_{PL} = \left| \Phi_{h^N}^P \right|^{-2(N+1)} \left| \left\langle S_{\zeta_1, \ldots, \zeta_N}, \Phi_{h^N}^P \right\rangle \right|^{2N+2} \frac{\prod_{j=1}^N d\mathcal{E}_j \wedge d\bar{\mathcal{E}}_j}{\prod_{j=1}^g 1_{\mathcal{O}(P_j)}(z) \prod_{j=1}^N 1_{\mathcal{O}(\zeta_j)}||2(N+1)||_2(G_{N+g}(\omega, \nu))} (65)
\]

Second we use

\[
det \left( \left( \frac{\partial \zeta_{N+j}}{\partial \zeta_k} \right)_{j,k=1}^N \right) = \left| \Phi_{h^N}^P \right|^N \frac{\prod_{k=1}^N \prod_{j=1}^g 1_{\mathcal{O}(P_j)}(\zeta_k) \left\langle S_{\zeta_1, \ldots, \zeta_N}, \Phi_{h^N}^P \right\rangle^N \det \left( \psi_n(\zeta_j) \right) \left( \prod_{k=1}^N \prod_{j, k \neq j} 1_{\mathcal{O}(\zeta_j)}(\zeta_k) \right)}{\prod_{j=1}^g 1_{\mathcal{O}(P_j)}(z) \prod_{j=1}^N 1_{\mathcal{O}(\zeta_j)}}.
\]

Here it is assumed that \( \{ \psi_n \} \) is orthonormal. Finally, we take the norm square and simplify.

It is a smooth form on extended configuration space since it is the pullback of a smooth form under a smooth map.

3.6. Analysis of zeros. Since \( \tilde{K}^N_{PL} \) is a smooth form, the zeros of the denominator must be cancelled by the zeros of the numerator in the expression (I) of Theorem 2. We now verify this as a check on the calculation. We begin with the Bergman determinant:

**Lemma 5.** The zero set of \( \det \left( B_{E_{N+g-1}}(\zeta_j, \zeta_k) \right)_{j,k=1}^N \) on \( X^{(N)} \) consists of the ‘diagonals’ \( \zeta_j = \zeta_k \), together with the \( \zeta_1 + \cdots + \zeta_N \) such that \( [E_{N+g-1}] - (\zeta_1 + \zeta_2 + \cdots + \zeta_N) \in \Theta = W_{g-1}^1 \), i.e. has a one-dimensional space of holomorphic sections. Multiplication by \( 1_{P_0} \) maps this subspace to a one-dimensional subspace of \( H^0(P_0, \mathcal{L}_{N+g}) \) vanishing at \( P_0, \zeta_1, \ldots, \zeta_N \).

We will denote this set by \( X^{(N)}_{N+g-1} \), analogously to \( X^{(N)}_{N+g} \). Thus we now have three ‘bad’ sets:

- The branch locus \( B_{N+g}^{(N)} \) of the map \( \mathbb{P}\mathcal{E}_N \to \mathbb{P}H^0(X, \mathcal{L}_{N+g}) \).
- \( X^{(N)}_{N+g} \), the set where the representation \( \mathcal{L} - \zeta_1 + \cdots + \zeta_N = P_1 + \cdots + P_g \) fails to be unique.
- \( X^{(N)}_{N+g-1} \) where \( E_{N+g-1} - \sum_{j=1}^N \zeta_j = Q_1 + \cdots + Q_{g-1} \), or equivalently \( \mathcal{L} - \zeta_1 + \cdots + \zeta_N = P_0 + Q_1 + \cdots + Q_{g-1} \). The ‘extra zeros’ of the Bergman determinant are in \( X^{(N)}_{N+g-1} \).

**Proof.** Let \( E \) be a line bundle of degree \( N + g - 1 \) and recall the Abel type maps,

\[
A_{E,g-1} : X^{(N)} \to \text{Pic}^{g-1}(X), \ A_{E,g-1}(P_1 + \cdots + P_N) := \mathcal{O}(E)(\mathcal{O}(P_1 + \cdots + P_N))^{-1}. \quad (66)
\]

As mentioned above, in \( \text{Pic}^{g-1} \) there is the divisor \( \Theta \) of bundles which have global holomorphic sections. Given a base point it can be identified with the Wirtinger variety \( W_{g-1}^1 \subset \text{Jac}(X) \) under the further tensor product by \( \mathcal{O}(-g+1)P_0 \). If \( \zeta_1 + \cdots + \zeta_N \not\in A_{E,g-1}^1(\Theta) \) then the map \( s \in H^0(X, E) \to \bigoplus_{j=1}^N s(\zeta_j) \) defines an isomorphism

\[
H^0(X, E) \simeq \bigoplus_{j=1}^N E[\zeta_j]. \quad (67)
\]

Indeed, the map fails to be an isomorphism if and only if there exists a non-zero section \( s \) such that \( \mathcal{D}(s) \geq \zeta_1 + \cdots + \zeta_N \). Then \( \prod_{j=1}^N 1_{\mathcal{O}(\zeta_j)} \in H^0(X, \mathcal{O}(E)(\mathcal{O}(\zeta_1 + \cdots + \zeta_N))^{-1}) \) is a non-zero
section and $O(E)(O(\zeta_1+\cdots+\zeta_N))^{-1}) \in \Theta$. Conversely, if $H^0(X,O(E)(O(\zeta_1+\cdots+\zeta_N))^{-1})$ has a non-trivial section $e_{g-1}$ then $e_{g-1} \prod_{j=1}^{N} 1_{O(\zeta_j)} \in H^0(X,E_{N+g-1})$ is a section vanishing at all $\zeta_j$.

Now let $E = E_{N+g-1} = (N+g-1)P_0$ and let $\{\Phi^j_G\}_{j=1}^N$ be the set of coherent states in $H^0(X,E_{N+g-1}) \simeq H^0_{pb}(X,L_{N+g})$ centered at the points $\{\zeta_j\}$. Then the Gram matrix of inner products of these coherent states is

$$((\Phi^j_G, \Phi^k_G)_G) = (B_G(\zeta_j, \zeta_k)).$$

It follows that the zeros of the Bergman determinant are the points $\{\zeta_1, \ldots, \zeta_N\}$ such that the coherent states $\Phi^j_G$ fail to be linearly independent or equivalently such that the evaluation map in $\{\Phi^j_G\}$ fails to be an isomorphism, and so $\{\zeta_1, \ldots, \zeta_N\} \in A^{-1}_{E_{N+g-1},G}(\Theta)$.

In this case, $H^0(X,E_{N+g-1}) \simeq H^0_{pb}(X,L_{N+g})$ is one-dimensional, hence there exists a one-dimensional space of $s \in H^0(X,E_{N+g-1})$ vanishing at all $\zeta_j$. The last statement is then obvious.

\section*{Lemma 6.}

As functions on extended configuration space $\tilde{X}^{(N)}$, the zeros of $\det (B_{E_{N+g-1}}(\zeta_j, \zeta_k))$ are cancelled by those of the numerator

$$\prod_{j=1}^{N} \prod_{j \neq k}^{g} 1_{O(\zeta_j)}(\zeta_k) \prod_{k=1}^{N} 1_{O(\zeta_j)}(\zeta_k)^2.$$

\textbf{Proof.} Given $\zeta_2, \ldots, \zeta_N$, $\det (B_{E_{N+g-1}}(\zeta_j, \zeta_k))_{j=1}^{N}$ vanishes at $\zeta_1 = \zeta_2, \ldots, \zeta_N$ and at $g$ further points $\zeta_1 + Q_1 + \cdots + Q_{g-1}$ so that $[E_{N+g-1}] = \sum_{j=2}^{N} \zeta_j + \zeta_1 + Q_1 + \cdots + Q_{g-1}$. This follows from Lemma 5. Vanishing of the Bergman determinant is equivalent to vanishing of the Slater determinant $(S_k(\zeta_j)$ for some (hence any) basis $\{S_k\}$ of $H^0(X,E_{N+g-1})$. Suppose there exists a section $S$ vanishing at $\zeta_2 \cdot \cdot \cdot + \zeta_N + \zeta + Q_1 + \cdots + Q_{g-1}$. Putting $S_1 = S$ shows that the Slater determinant vanishes at any $N$-element subset of this configuration.

Viewed as a function of $\zeta_1$ (or any other index), the product

$$\left( \prod_{k=1}^{N} \prod_{j \neq k}^{g} 1_{O(\zeta_j)}(\zeta_k) \prod_{k=1}^{N} 1_{O(\zeta_j)}(\zeta_k) \right)$$

also vanishes at any $N$-element subset of this configuration. Indeed, adding $P_0$ to a configuration from $X_{N+g-1}^{(N)}$ produces a special type of configuration from $X_{N+g}^{(N)}$ with $Q_1 + \cdots + Q_{g-1} + P_0 = P_1 + \cdots + P_g$. This representation is unique since $\dim H^0(X,O(Q_1 + \cdots + Q_{g-1})) = 1$. It follows that $[L_{N+g}] = \sum_{j=1}^{N} \zeta_j + P_0 + Q_1 + \cdots + Q_{g-1}$. \hfill \Box

\textbf{Remark:} The factor $\left( \prod_{k=1}^{N} \prod_{j \neq k}^{g} 1_{O(\zeta_j)}(\zeta_k) \right)$ cancels all of the ‘coincidence zeros’ and indeed it has two factors of $\zeta_j - \zeta_k$ for each $j \neq k$ while the Bergman determinant has one. This effectively leaves a product $\prod_{j<k} 1_{O(\zeta_j)}(\zeta_k)$. The factor $\prod_{k=1}^{N} \prod_{j=1}^{g} 1_{O(\zeta_j)}(\zeta_k)$ always vanishes when $\zeta_k = P_j$ for some $j, k$, including when $\zeta_1 + \cdots + \zeta_N \notin X_{N+g-1}^{(N)}$, i.e. when $(N+g)P_0 - \sum_{j=1}^{N} \zeta_j = P_1 + \cdots + P_g$ but there does not exist $Q_1 + \cdots + Q_{g-1}$ such that $(N+g)P_0 - \sum_{j=1}^{N} \zeta_j = P_0 + Q_1 + \cdots + Q_{g-1}$, i.e. when
no $P_j$ equals $P_0$. In this sense it has ‘more zeros’ than necessary to cancel the zeros of the Bergman determinant. One may view the extra zeros as arising from the degeneracy of the map $D\psi_{N+g}$ in Proposition 3.

4. JPC and Green’s functions: Proof of II of Theorem 2

In this section, we rewrite the formula for the JPC of the projective linear ensemble in terms of Green’s functions. In particular, we rewrite the Bergman (or Slater) determinant as a product of values of the Green’s function. This step is crucial to obtain the large deviations rate functional. The product formula for the Slater/Bergman determinant follows from the bosonization formula of [ABMNV, VV, Fal, F].

To state the results, we need some further notation. We put

$$\frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right),$$

and

$$\partial f = \frac{\partial f}{\partial z} dz; \quad \bar{\partial} f = \frac{\partial f}{\partial \bar{z}} d\bar{z}.$$ 

Thus, $dd^c f = \frac{i}{8\pi} \Delta f dz \wedge d\bar{z}$, where $\Delta$ is the standard Euclidean Laplacian, and

$$\Delta \left( \frac{1}{2\pi} \log |z| \right) = \delta_0 \iff dd^c (2 \log |z|) = \delta_0 dx \wedge dy. \quad (68)$$

4.1. Green’s function with respect to a Hermitian metric. Given a real $(1, 1)$ form $\omega$, we define the Green’s function $G_\omega$ relative to $\omega$ to be the unique solution $G_\omega(z, \cdot) \in \mathcal{D}'(X)$ of

$$\begin{cases} 
(i) & dd^c_\omega G_\omega(z, w) = \delta_z(w) - \omega_w, \\
(ii) & G_\omega(z, w) = G_\omega(w, z), \\
(iii) & \int_X G_\omega(z, w) \omega_w = 0, 
\end{cases} \quad (69)$$

where the equality in the top line is in the sense of $(1, 1)$ forms. We refer to [ZZ] for background (see the proof of Proposition 10); uniqueness follows from condition (iii).

We note that in [ZZ] the Green’s function was denoted $G_h$ with respect to a Hermitian metric $h$ on $\mathcal{O}(1)$. But the Green’s function depends only on the curvature $(1, 1)$ form of $h$, so we denote it here by $G_\omega$.

The Green’s potential of a measure in higher genus is defined precisely as in genus zero in [ZZ]. The Green’s potential of a measure is uniquely characterized as the solution of

$$\begin{cases} 
dd^c U^\mu_\omega = \mu - \omega; \\
\int_X U^\mu_\omega \omega = 0. 
\end{cases}$$

As in the genus zero case of [ZZ], the Green’s function may be expressed in terms of local holomorphic coordinates and a local potential $\varphi$ for $\omega$ (i.e. $\omega = dd^c \varphi$). A more invariant expression for $G_\omega$ is in terms of the prime form (see §1.1 for background).

**Proposition 7.** Let $(X, \omega)$ be a compact Riemann surface equipped a $(1, 1)$ form $\omega$. Let $\mathcal{O}(w)$ be the point line bundle and let $1_{\mathcal{O}(w)}(z)$ be its canonical section. Also, let $h_w$ be an
\(\omega\)-admissible metric on \(\mathcal{O}(w)\). Then the Green’s function \(G_\omega\) is given by
\[
G_\omega(z, w) = \log \|1_{\mathcal{O}(w)}(z)\|_{h_{(\omega,w)}}^2 - \frac{1}{X_\omega} \int_X \log \|1_{\mathcal{O}(w)}(z)\|_{h_w(z)}^2 \omega_z = \log \|E(z, w)\|_{h(z)\Omega(h(w))}.
\]

**Proof.** We apply \(i\bar{\partial} = 4d\delta\) in the \(z\) variable to both sides of the formula. On the left, we obtain
\[
\frac{\delta_w(z)}{\omega} - \frac{1}{4\lambda} \omega,
\]
where \(A = \int_X \omega\). On the right side, we write \(1_{\mathcal{O}(w)} = (z - w)e\) relative to a local holomorphic function \(e\) of \(\mathcal{O}(w)\) near \(w\). Then in the \(z\) variable,
\[
\partial_c \log \|1_{\mathcal{O}(w)}(z)\|_{h_{(\omega,w)}}^2 = \delta_w(z) - \partial^c \|e(z)\|_{h_{(\omega,w)}}^2 = \delta_w(z) - \omega,
\]
since \(\varphi_w(z) = \log \|e(z)\|_{h_{(\omega,w)}}^2\) is a potential for \(\omega\). Note that \(\omega_{h(z,w,w)} = \frac{1}{4\lambda} \omega\), since \(\omega_{h(z,w,w)}\) is the curvature of a line bundle of Chern class one and is harmonic with respect to \(\omega\). Hence after taking \(\partial^c\) the integral vanishes and we get \(\delta_w(z) - \frac{1}{4\lambda} \omega\) on the right side.

It follows that
\[
d\delta_z (G_\omega(z, w) - \log \|1_{\mathcal{O}(w)}(z)\|_{h_{(\omega,w)}}^2) = 0
\]
and since both are globally well-defined their difference must be a constant, possibly depending on \(w\). The constant is determined by the condition that \(\int_X G(z, w) \omega_w(z) = 0\) for all \(w\).

**Remark:** (i) It is not obvious that \(G(z, w) = G(w, z)\) from this formula, but this must be the case since the Green’s function satisfying (i) and (iii) is unique.

(ii) The admissible Hermitian metric is only defined up to a multiplicative constant. However, the right side of the formula is independent of the choice of the constant.

(iii) Since the genus \(g \geq 1\), we may also express \(X = \tilde{X}/\Gamma\) where \(\tilde{X}\) is the universal holomorphic cover. Then \(G\) is a \(\Gamma\)-invariant solution of (i) - (ii) on \(\tilde{X}\), and (iii) holds for the integral over a fundamental domain \(\mathcal{F}\) for \(\Gamma\). Indeed, \(\partial^c (G_\omega - 2 \log |z - w|) = \omega\) where derivatives are taken in either the \(z\) or \(w\) variable, and so \(G_\omega - 2 \log |z - w|\) is a smooth potential for \(\omega\). If we subtract \(\varphi(z) + \varphi(w)\), then \(\partial^c\) of the result equals zero in each variable. Hence the difference must be a constant and the constant is determined by (iii).

### 4.2. Mean value of Green’s function.

Henceforth, we define
\[
\rho_\omega(w) := \frac{1}{\mathcal{F}X} \int_X \log \|1_{\mathcal{O}(w)}(z)\|_{h_w(z)}^2 \omega_z.
\]

Let us relate this to the analogous ‘constant’ when \(g = 0\). In [ZZ], we defined \(E(h) := (\int_{\mathbb{C}P^1} \varphi(z) \omega_h + 4\pi \rho_\varphi(\infty))\), where \(\rho_\varphi\) was a certain Robin constant. In Lemma 8 of [ZZ] we showed that in each trivializing affine chart of \(\mathcal{O}(1) \to \mathbb{C}P^1\), and relative to the frame \(e(z)\) over the affine chart \(\mathbb{C}\) in which \(h = e^{-\varphi}\) and \(\omega_h = d\delta^c \varphi\), the Green’s function has the local expression, \(G_j(z, w) = 2 \log |z - w| - \varphi_j(z) - \varphi_j(w) + E(h)\), and \(\int_{\mathbb{C}} G_j(z, w) d\delta^c \varphi_j = 0\). In particular, we showed that \(\int_{\mathbb{C}P^1} \log |z - w|^2 h_{(\omega_w,\omega_h)} = \text{a constant in } z\), equal to \(E(h)\).

In higher genus, we also have \(\log \|1_{\mathcal{O}(w)(z)}\|_{h_w(z)}^2 = \log |z - w| - \varphi(z) - \varphi(w)\) relative to the canonical frame of \(\mathcal{O}(w)\). We could use Green’s formula on a fundamental domain \(\mathcal{F}\) to simplify the formula for \(\rho_\omega(w)\) and also express it in terms of the potential. But we do not need the expression, so we leave the details to the reader.
4.3. Norms of canonical sections and Green’s functions. We now tie together the Vieta maps with the Green’s potentials of the discrete measures \( \mu = \mu_\zeta \).

**Lemma 8.** Let \( h \) be an \( \omega \)-admissible metric. Then with \( \rho_\omega \) as in \((70)\),
- \((i)\) \( \frac{1}{N} \log ||1_{\zeta_1+\ldots+\zeta_N}(z)||_h^N - \frac{1}{N} \int_X \log ||1_{\zeta_1+\ldots+\zeta_N}||_h^2 \omega = U^\mu_\omega(z) \). Hence,

\[
||1_{\zeta_1+\ldots+\zeta_N}(z)||_h^N e^{-\frac{1}{N} \int_X \log ||1_{\zeta_1+\ldots+\zeta_N}||_h^2 \omega} = e^{U^\mu_\omega}.
\]
- \((ii)\) \( \int_X \log ||1_{\zeta_1+\ldots+\zeta_N}(w)||_h^2 \omega = \sum_{j=1}^N \rho_\omega(\zeta_j) \).
- \((iii)\) \( \frac{1}{N} \log ||S_\zeta(z)||_h^N - \frac{1}{N} \int_X \log ||S_\zeta||_h^2 \omega = U^\mu_\omega(z) + \frac{1}{N} U^\mu_{\omega(G)}(z) \). Hence,

\[
||S_\zeta(z)||_h^N e^{-\frac{1}{N} \int_X \log ||S_\zeta||_h^2 \omega} = e^{U^\mu_\omega} + \frac{1}{N} U^\mu_{\omega(G)}.
\]
- \((iv)\) \( \int_X \log ||S_\zeta(w)||_h^2 \omega h = \sum_{j=1}^N \rho_\omega(\zeta_j) + \sum_{j=1}^g \rho_\omega(P_j) \).

**Proof.** Since \( \log ||1_{\zeta_1+\ldots+\zeta_N}||_h^2 = \sum_{j=1}^N \log ||1_{\zeta_j}||_h^2 \), it follows from Proposition \((7)\) that

\[
d\mu_\zeta = dd^c \frac{1}{N} \log ||1_{\zeta_1+\ldots+\zeta_N}||_h^2 + \omega,
\]
and that

\[
d\mu_\zeta + \frac{1}{N} d\mu_{P(\zeta)} = dd^c \frac{1}{N} \log ||1_{\zeta_1+\ldots+\zeta_N}1_{P_1+\ldots+P_g}||_h^2 + (1 + \frac{g}{N}) \omega,
\]
it follows by integration against \( G_\omega(z, w) \) that

\[
U^\mu_\omega(z) := \int_X G_\omega(z, w) d\mu_\zeta(w) = \int_X G_\omega(z, w) \left(dd^c \frac{1}{N} \log ||1_{\zeta_1+\ldots+\zeta_N}||_h^2 + \omega\right)
= \int_X G_\omega(z, w) \left(dd^c \frac{1}{N} \log ||1_{\zeta_1+\ldots+\zeta_N}||_h^2\right)
= \frac{1}{N} \log ||1_{\zeta_1+\ldots+\zeta_N}||_h^2(z) - \frac{1}{N} \int_X \log ||1_{\zeta_1+\ldots+\zeta_N}||_h^2(z) \omega.
\]

\(\square\)

4.4. **Bosonization formulae.** Bosonization formulae equate the determinantal correlation functions of a free fermionic field theory in two dimensions to the symmetric correlation functions of a corresponding bosonic field theory. For our purposes, the important point is that the bosonization formulae express Slater or Bergman determinants in terms of products of prime forms (or exponentials of Green’s functions). We give further background in the Appendix \((8)\).

With some adjustments to make the notation consistent with this article, formula (5.4) of \(\text{ABMNV} \) states the following:

**Theorem 5.** Let \( L \to X \) be the line bundle of degree \( d \) of the divisor \( \sum_{j=1}^d a_j \). Let \( \{\psi_j\}_{j=1}^p \) be a basis of \( H^0(X, L) \) where \( p = \dim H^0(X, L) = d + 1 - g \). Then there exists a constant \( A_N(\omega, g) \) such that
\[
\left| \det \left( \begin{array}{ccc}
\psi_1(\zeta_1) & \cdots & \psi_p(\zeta_1) \\
\psi_1(\zeta_p) & \cdots & \psi_p(\zeta_p)
\end{array} \right) \right|^2 = A_N(g, \omega) \prod_{i<j} |E(\zeta_i, \zeta_j)|^2
\]

(71)

\[
\left| \det(\langle \psi_i, \psi_j \rangle) \right|^2 \mathcal{N}(L \otimes \mathcal{O}(\sum_{j=1}^p \zeta_j) \otimes \mathcal{L}_A^{-1})
\]

The constant \(A(g, \omega)\) is a quite complicated constant, independent of \(\{\zeta_j\}\) and the choice of \(\{\psi_k\}\), involving a ratio of determinants of two Laplacians multiplied by some extra factors involving \(\theta\)-functions, a metric \(g\) on \(X\) and spin structures. Also, \(\mathcal{L}_A\) is a spin structure (a bundle of degree \(g-1\)) depending on a choice of homology basis \(\mathcal{A}\) of \(X\), and \(\mathcal{N}(L \otimes \mathcal{O}(\sum_{j=1}^p \zeta_j) \otimes \mathcal{L}_A^{-1})\) is defined by

\[
\mathcal{N}_A(z) = e^{-4\pi i(Y_{Y,y})} |\theta(z)|^2,
\]

(72)

where \(z\) is the divisor class \(z = L \otimes \mathcal{O}(\sum_{j=1}^p \zeta_j) \otimes \mathcal{L}_A^{-1}\). In addition, \(\tau\) is the period matrix and \(Y\) is the matrix of inner products of the holomorphic one forms (we refer to [ABMNV] for further details).

The \(\theta\)-factor is quite important since it cancels the extra zeros. The rest of the details of the formula are mostly irrelevant for our purposes, since they are of lower logarithmic order than the LDP rate function.

We apply the bosonization formula to the line bundle \(E_{N+g-1}\) of (54) of degree \(d = N + g - 1\). Hence \(p = N\) and the relevant theta function is

\[
\theta((N + g - 1)P_0 - \sum_{j=1}^N \zeta_j - \Delta).
\]

Here we suppress \(\tau\) since the complex structure is fixed in our discussion.

#### 4.5. Completion of the proof of II of Theorem [2]"

To state the next result, we need some further notation. Motivated by formula I of Theorem [2] and by the detailed bosonization formula (see §4.1, Theorem [5] and the Appendix §8), we put

\[
F_N(\zeta_1, \ldots, \zeta_N) : = A(N, g)e^{-(N+1)\sum_{j=1}^g \rho_\omega(P_j(\zeta))} \mathcal{F}_N(\zeta_1, \ldots, \zeta_N, P_0) \left| \prod_{k=1}^N 1_{P_k(\zeta_k)} \right|^{-2}
\]

\[
\left| \left( \prod_{k=1}^N \prod_{j=1}^g E(P_j, \zeta_k) \right) \right|^2 \left| \left( \theta((N + g - 1)P_0 - \sum_{i=1}^N \zeta_i - \Delta) \right) \right|^{-2},
\]

(73)

where \(\mathcal{F}_N\) is defined in (64). Here we use that \(| \det(\langle \psi_i, \psi_j \rangle) |^2 = 1\). We view \(\mathcal{F}_N\) as defined on \(X^{(N)} \setminus X_{N+g}^{(N)}\) or on extended configuration space \(\tilde{X}^{(N)}\). In the latter case we write \(F(\zeta_1 + \cdots + \zeta_N, P_1 + \cdots + P_g)\).

**Lemma 9.** Let \(h = e^{-\varphi}\) be a smooth Hermitian metric on \(\mathcal{O}(P_0)\), and let \(\omega_h, G_\omega\) be as above. Then, the joint probability current for the projective linear ensemble is given by:

\[
\mathcal{K}_N(\zeta_1, \ldots, \zeta_N) = \frac{1}{Z_N(\omega)} \exp \left( \sum_{\zeta \neq \zeta_j} G_\omega(\zeta, \zeta_j) \right) 
\]

\[
\cdot F_N(\zeta_1, \ldots, \zeta_N) e^{-2N \int \rho_\omega(w) d\mu_\zeta(w)},
\]
where $\rho_\omega$ is defined in (70), where $F_N$ is defined in (73), and where $Z_N$ is a normalizing constant (see §8).

Proof. We first rewrite the numerator
\[
\prod_{k=1}^{N} \left| \prod_{j=1}^{g} E(P_j, \zeta_k) \cdot \prod_{j:k \neq j} E(\zeta_j, \zeta_k) \right|^2 / \det \left( B_N(\zeta_j, \zeta_k) \right)_{j,k=1}^{N}
\] (74)
of the expression I in Theorem 2 in terms of the Green’s function. It is a product of three factors:

• (i) $\prod_{k=1}^{N} \left| \prod_{j:k \neq j} E(\zeta_j, \zeta_k) \right|$;

• (ii) $\prod_{k=1}^{N} \left| \prod_{j=1}^{g} E(P_j, \zeta_k) \right|^2$;

• (iii) $(\det (B_N(\zeta_j, \zeta_k)))_{j,k=1}^{N}^{-1}$.

By Proposition 7 and Lemma 8, (i) equals
\[
\left| \prod_{j:k \neq j=1}^{N} 1_{\mathcal{O}(\zeta_j)}(\zeta_k) \right|^2 = \exp \left( \sum_{i<j}^{N} G^\omega(\zeta_i, \zeta_j) \right) \exp \left( -(N - 1) \sum_{j=1}^{N} \rho_\omega(\zeta_j) \right).
\] (75)

Hence,
\[
2 \sum_{i<j}^{N} \log \left| 1_{\mathcal{O}(\zeta_j)}(\zeta_k) \right|^2 = \sum_{i<j}^{N} G_h(\zeta_i, \zeta_j) + (N - 1) \sum_{j=1}^{N} \rho_\omega(\zeta_j)
\]
\[
= N^2 \int_{X \times X \setminus \Delta} G_h(z, w)d\mu_\zeta(z)d\mu_\zeta(w) - N(N - 1) \int \rho_\omega(w)d\mu_\zeta(w).
\] (76)

To express (iii) in terms of Green’s functions, we use the bosonization formula (4.4) for the Bergman determinant or of the Hermitian line bundle $E = E_{N+g-1}$. It gives:
\[
\left| \det \left( \begin{array}{ccc}
\psi^E_1(\zeta_1) & \cdots & \psi^E_N(\zeta_1) \\
\psi^E_1(\zeta_N) & \cdots & \psi^E_N(\zeta_N)
\end{array} \right) \right|^2 = A_N(g, \omega) \prod_{i<j} \left| E(\zeta_i, \zeta_j) \right|^2
\]
\[
\cdot \left| \det \left( \psi^E_i, \psi^E_j \right) \right|^2 \mathcal{N}(E_{N+g-1} \otimes \mathcal{O}( - \sum_{j=1}^{N} \zeta_j ) \otimes L_A^{-1})
\] (77)

Here, by (72), up to an $N$-independent positive constant,
\[
\mathcal{N}(E_{N+g-1} \otimes \mathcal{O}( - \sum_{j=1}^{N} \zeta_j ) \otimes L_A^{-1}) = ||\theta((N + g - 1)P_0 - \sum_{i=1}^{N} \zeta_i - \Delta)||^2.
\]

Since
\[
\left| \det (\psi_n(\zeta_j)) \right|^2 = \left| \prod_{k=1}^{N} 1_{\mathcal{P}_b}(\zeta_k) \right|^2 \det \left( B_{E_{N+g-1}}(\zeta_j, \zeta_k) \right)_{j,k=1}^{N},
\]
we need to multiply the product side of the bosonization formula by $\left| \prod_{k=1}^{N} 1_{\mathcal{P}_b}(\zeta_k) \right|^2$.

Again using Proposition 7 and Lemma 8, we then convert the denominator
Proof. We first note that the poles of $g$ with a one-dimensional space of holomorphic sections vanishing at $g$ in Lemma 6. Hence it suffices to prove the second statement, which comes down to the equivalent statement.

4.6. Analysis of zeros. We now update §3.6 to take into account the above results using bosonization. We first re-consider Lemma 6 in the light of the bosonization formula.

Lemma 10. $F_N$ is holomorphic on $\tilde{X}^{(N)}$. That is,

$$\left(\prod_{k=1}^{N} \prod_{j=1}^{g} E(P_j, \zeta_k)\right) \left(\theta((N + g - 1)P_0 - \sum_{i=1}^{N} \zeta_i - \Delta)\right)^{-1}$$

has no poles. Moreover, the same is true on extended configuration space $\tilde{X}^{(N)}$.

Proof. We first note that the poles of $\left|\prod_{k=1}^{N} \prod_{j=1}^{g} E(P_j, \zeta_k)\right|^{-2}$ are cancelled by the zeros of $\mathcal{F}_N$. Hence it suffices to prove the second statement, which comes down to the equivalent statement in Lemma 6.

By Riemann’s vanishing theorem, the zeros of $\theta((N + g - 1)P_0 - \sum_{i=1}^{N} \zeta_i - \Delta)$ occur at $\{\zeta_1, \ldots, \zeta_N\}$ such that $[(N + g - 1)P_0 - \sum_{i=1}^{N} \zeta_i] = Q_1 + \cdots + Q_{g-1}$ is the divisor class of a line bundle of degree $g - 1$ with a one-dimensional space of holomorphic sections. Equivalently, $[(N + g)P_0 - \sum_{i=1}^{N} \zeta_i] = P_0 + Q_1 + \cdots + Q_{g-1}$ is the divisor class of a line bundle of degree $g$ with a one-dimensional space of holomorphic sections vanishing at $P_0$. But this implies that $P_0 + Q_1 + \cdots + Q_{g-1} = P_1 + \cdots + P_g$ when the representation is unique, so that $\left(\prod_{k=1}^{N} \prod_{j=1}^{g} E(P_j, \zeta_k)\right)$ vanishes when some $\zeta_k$ equals some $Q_j$. 

\[\square\]
The same formula is valid on $\tilde{X}^{(N)}$.

We sum up in the

**Proposition 6.** There exists a constant $B(N, g)$ such that, as holomorphic sections on $\tilde{X}^{(N)}$,

$$F_N(\zeta_1, \ldots, \zeta_N, P_1 + \cdots + P_g) : = B(N, g)e^{-(N+1)\sum_{j=1}^{g} \rho_\omega(P_j(\zeta))} \left| \prod_{k=1}^{N} \prod_{j=1}^{g} E(P_j, \zeta_k) \right|^2 
\cdot ||\theta(P_1 + \cdots + P_g - P_0 - \Delta)||^{-2} \prod_{j=1}^{g} |E(P_j, P_0)|^2 \left| \Phi_{h_N}^{P_0} \right|^{-2}$$

We have used Lemma 3.6 together with the cancellation,

$$F_N(\zeta_1, \ldots, \zeta_N) \prod_{j=1}^{N} E(P_0, \zeta_j)|^{-2} = \left| \Phi_{h_N}^{P_0} \right|^{-2} \prod_{j=1}^{g} |E(P_j, P_0)|^2.$$
5.2. **Formule for** $d\tau_N^{FSH}$ **and** $d\lambda_{PL}$. The Hermitian inner products $G_N(h, \nu)$ of (13) induce a Hermitian metric on $Z_N$, corresponding to the choice of $\omega_0$-admissible metrics on $\xi \in \text{Pic}^N(X)$ (§2.4). That is, at a point $\xi \in X^{(N)}$, the norm of the vector $1_{\mathcal{O}(\xi)} \in Z_\xi$ is $\|1_{\mathcal{O}(\xi)}\|_{G_N(h, \nu)}$. The curvature (1, 1) form of the Chern connection is given by the (1, 1) form,

$$\omega_Z(\xi) = \frac{i}{2} \partial \bar{\partial} \log \|1_{\mathcal{O}(\xi)}\|_{G_N(h, \nu)} \text{ on } X^{(N)},$$

(81)

where $\partial \bar{\partial}$ is the operator on $X^{(N)}$, with Kähler potential given by the pluri-subharmonic function $\log \|1_{\mathcal{O}(\xi)}\|_{G_N(h, \nu)}$. This choice is most natural geometrically, but as in §2.4 we opt for the somewhat more complicated Hermitian metric on $Z_N$ in Definition 9 i.e. the Hermitian metric $\|S_{\xi_1, ..., \xi_N}\|_{G_{N+g}(h, \nu)} = \|1_{\mathcal{O}(\xi)}1_{\mathcal{O}(P(\xi))}\|_{G_{N+g}(h, \nu)}$. Its curvature (1, 1) form is given by,

$$\tilde{\omega}_Z(\xi) = \frac{i}{2} \partial \bar{\partial} \log \|S_{\xi_1, ..., \xi_N}\|_{G_{N+g}(h, \nu)}.$$  

(82)

Further, if we equip $\Theta \to \text{Jac}(X)$ with its standard Hermitian metric $h_\Theta$ with curvature the Euclidean (1, 1) form $\omega_\Theta := \sum dz_i \otimes \bar{dz}_i$ on $\text{Jac}(X)$, then $\omega_Z + A_N^* \omega_\Theta$ is the curvature form of $Z_N \otimes A_N^* \omega_\Theta$ (and similarly for the $\tilde{\omega}_Z$ version).

**Proposition 7.** We have,

- **The Fubini-Study-Haar probability measure** (Definition 2) is given by

$$d\tau_N^{FSH} = \tilde{\omega}_Z^{N-g} \wedge A_N^* \omega_\Theta.$$  

(83)

- **The probability measure of the Fubini-Study-X**(g)

$$d\tau_N^{FSX(g)} = \tilde{\omega}_Z^{N-g} \wedge \rho_N \text{d}\sigma,$$  

(84)

where \( \rho_N \) is the curvature form of $Z_N \otimes A_N^* \omega_\Theta$ (c.f. (49))

- **The projective linear ensemble probability measure** $d\lambda_{PL}$ (Definition 4) is given by $\tilde{\omega}_Z$.

**Proof.** The Proposition follows easily from the following

**Lemma 11.**

- $\tilde{\omega}_Z = \psi_{N+g}^* \omega_{FS,N+g}$, where $\omega_{FS,N+g}$ is the Fubini-Study metric on $\mathbb{P}H^0(X, \mathcal{L}_{N+g})$ corresponding to the choice of Hermitian inner product $G_{N+g}(h, \nu)$. It is a semi-positive (1, 1) form.

- $\tilde{\omega}_Z \wedge \omega_\Theta$ is a strictly positive (1, 1) form on $X^{(N)}$.

We consider the following diagram.

$$
\begin{array}{ccc}
Z_N & \xrightarrow{\sigma_L^*} & \sigma_L^* \mathcal{O}(1) \\
\pi \downarrow & \downarrow \rho & \downarrow \pi \\
X^{(N)}, \tilde{\omega}_Z & \xrightarrow{\tilde{\psi}_{N+g}} & (\mathbb{P}^N, \sigma_L^* \omega_{FS,N+g}) \\
A_N \downarrow & \downarrow \text{Definition } (10) & (\mathbb{P}H^0(X, \mathcal{L}_{N+g}), \omega_{FS}) \\
\text{Jac}(X) & \xrightarrow{\sim} & X^{(g)}
\end{array}
$$

(85)

where \( \sim \) is the identification, and $\sigma_L$ is the branched covering map, discussed in §2.2.
The inner product $G_{N+g}(h, \nu)$ on $H^0(X, \mathcal{L}_{N+g})$ induces a Fubini-Study Hermitian metric $G_{N+g}(h, \nu)$ on $\mathcal{O}(1) \to \mathbb{P}H^0(X, \mathcal{L}_{N+g})$ and the Fubini-Study form $\omega_{FS,N+g}$ on $\mathbb{P}H^0(X, \mathcal{L}_{N+g})$ is its curvature $(1, 1)$ form. Under the holomorphic map $\psi_{L_{N+g}}$ and its lift to $\mathcal{O}(1)$, the Hermitian metric and curvature form pull back to a Hermitian metric and its curvature form on $\psi_{L_{N+g}}^* \mathcal{O}(1)$. The pulled back Hermitian metric is the natural Hermitian metric on $\mathcal{O}(1) \to \mathbb{P}E_N$ defined by the Hermitian inner products. Since $\psi_{L_{N+g}}$ is holomorphic, $\psi_{L_{N+g}}^* \omega_{FS,N+g} = \omega_{FE_N}$, where the latter is $\tilde{\omega}_g$ transported to $\mathbb{P}E_N$.

It follows that $\tilde{\omega}_g$ is semi-positive and strictly positive away from the singular set of $\psi_{L_{N+g}}$ determined in Proposition 3. It is also strictly positive along the fibers of $\mathbb{P}E_N \to X^g$. Since $A_N^g \omega_{\Theta}$ is semi-positive and strictly positive along the ‘horizontal’ directions transverse to the fibers, the sum of the two is strictly positive.

By the definitions (2) and (3),

$$
\begin{align*}
\psi_{L_{N+g}}^* (\tilde{\omega}_{FS,N+g}^{N-g}) &\wedge A_{N}^g d\omega_{\Theta} = \tilde{\omega}_Z^{N-g} \wedge A_{N}^g \omega_{\Theta}, \\
\psi_{L_{N+g}}^* (\omega_{FS,N+g}^N) &\wedge A_{N}^g \omega_{\Theta} = \tilde{\omega}_Z^N.
\end{align*}
$$

Since $\psi_{L_{N+g}}$ is singular along the Wirtinger subvariety of $\mathbb{P}E_N$, or equivalently along $X^N_{N+g}$, it is helpful to factor the pullbacks as follows:

**Corollary 12.**
- The JPC for the Fubini-Study-Haar ensemble is given by $K_N = D_s \left( \int_{\mathcal{O}} \omega_{FS,N+g}^{N-g} \right) \wedge A_{N}^g \omega_{\Theta}$,
- The JPC for the projective linear ensemble is given by $K_N = D_s \left( \int_{\mathcal{O}} \omega_{FS,N+g}^N \right)$

6. **Proof of Theorem 3**

In Proposition 7 we describe the two probability measures as volume forms on $\mathbb{P}E_N$ or equivalently on $X^{(N)}$. We now use this relation to give an explicit formula for $K_{FSH}^N$:

**Theorem 6.** We have

$$
K_{FSH}^N(\zeta_1, \ldots, \zeta_N) = \mathcal{J}_N(\zeta_1, \ldots, \zeta_N) \frac{1}{Z_N(\omega)} \frac{\exp \left( \frac{i}{2} \sum_{i \neq j} G(\zeta_i, \zeta_j) \right) \prod_{j=1}^N d^2 \zeta_j}{(\int_X e^{N \int_X G(z, w) d\mu(z) w e^{X G(z, w) d\mu(\zeta) w} d\nu(z))^{N+1}},
$$

where (for a certain constant $B(N, g)$),

$$
\mathcal{J}_N(\zeta_1, \ldots, \zeta_N) = B(N, g) e^{-(N+1) \sum_{j=1}^g P_j(\zeta)} ||\Phi_{h_N}^P||^{-2} \prod_{j=1}^g |P_j - P_0 - \Delta||^{-2} \prod_{j=1}^g |E(P_j, P_0)|^2 (\det \left( \Phi_{N+g}^P, \Phi_{N+g}^P \right))
$$

$$
\left| \prod_{j \neq n} E(P_n, P_j) \right|^{-2} |\det DA_g|^2,
$$

where $DA_g$ is the derivative of the Abel map.


Here,
\[ ||\theta(P_1 + \cdots + P_g - P_0 - \Delta)||^{-2} \prod_{j=1}^{g} |E(P_j, P_0)|^2 |\det DA_g|^2 \]
is a smooth positive function, and
\[
\det \left( \Phi_{N+g}^j(P_k) \right) \prod_{j \neq n} E(P_n, P_j)^{-2}.
\]
is a smooth nowhere vanishing function on \(X'^{(g)}\) times \(|\prod_{j<n} E(P_j, P_n)|^{-2}\).

**Proof.** It suffices to calculate the ratio of the volume forms, i.e. the Jacobian
\[
J_N := \frac{\tilde{\omega}_N}{\tilde{\omega}_N^{-g} \wedge A_N^* \omega^g_{\Theta}}.
\]
The discussion is similar for the ensembles of Definition 3, with forms pulled back from \(X'^{(g)}\). In fact, the proof involves passing back and forth between \(\bar{X}'(N)\) and \(X'^{(N)}\) and between \(\text{Jac}(X)\) and \(X'^{(g)}\).

We do this by evaluating numerator and denominator on frames for the vertical bundle \(V \mathbb{P} \mathcal{E}_N\) and for the horizontal space \(H \mathbb{P} \mathcal{E}_N\) defined by the connection:
\[
H_{(L,[k])} \mathbb{P} \mathcal{E}_N := V_{(L,[k])}^+ \text{ with respect to the metric } \tilde{\omega}_N + A_N^* \omega_{\Theta}.
\]
In fact, the same connection is defined by the condition that the horizontal space is \(\tilde{\omega}_N\)-orthogonal to the vertical space. Indeed, \(A_N^* \omega_{\Theta}\) is a ‘horizontal’ \((1,1)\) form, i.e. \(A_N^* \omega_{\Theta}(V, W) = 0\) for all \(W \in T_{(L,[k])} \mathbb{P} \mathcal{E}_N\) if \(V\) is vertical. It follows that \(\tilde{\omega}_N(V, H) = (\tilde{\omega}_N + A_N^* \omega_{\Theta})(H, V)\). The difference in the metrics lies in the fact that \((\tilde{\omega}_N + A_N^* \omega_{\Theta})\) is always non-degenerate in the horizontal subspace.

6.1. **Calculation of the Jacobian** \(J_N\). We let \(V_1, \ldots, V_N\) be a vertical orthonormal frame for \(V\) with respect to \(\tilde{\omega}_N + A_N^* \omega_{\Theta}\), or equivalently, \(\tilde{\omega}_N\); this makes sense, since they are the same metric along the fibers.

**Lemma 13.** The Jacobian \((88)\) is given by
\[
J_N = \frac{\det \langle H_i, \tilde{H}_j \rangle_{\tilde{\omega}_N}}{\det (A_N^* \omega^g_{\Theta}(H_i, H_j))}
\]

**Proof.** For \(\tilde{\omega}_N\), the Gram matrix of inner products of the elements of the frame is the \(N \times N\) matrix
\[
G_N := \begin{pmatrix}
\langle V_i, V_j \rangle & \langle V_i, H_j \rangle \\
\langle V_i, H_j \rangle & \langle H_i, H_j \rangle
\end{pmatrix} = \begin{pmatrix}
I_{(N-g) \times (N-g)} & 0_{(N-g) \times g} \\
0_{g \times (N-g)} & I_{g \times g}
\end{pmatrix}.
\]
Here, \(\langle , \rangle\) is short for the Hermitian semi-metric defined by \(\tilde{\omega}_N\). Then,
\[
\tilde{\omega}_N^N(V_1, \bar{V}_1, \ldots, V_{N-g}, \bar{V}_{N-g}, H_1, \bar{H}_1, \ldots, H_g, \bar{H}_g) = \det G_N.
\]
In the case of \( \tilde{\omega}^{N-g}_{Z} \land A_{N}^{*} \omega^{g}_{\Theta} \), the volume of the frame is the determinant of the matrix
\[
\begin{pmatrix}
\langle V_{i}, V_{j} \rangle & 0 \\
0 & A_{N}^{*} \omega^{g}_{\Theta}(H_{i}, \bar{H}_{j})
\end{pmatrix} = \begin{pmatrix}
I_{(N-g) \times (N-g)} & 0_{(N-g) \times g} \\
0_{g \times (N-g)} & A_{N}^{*} \omega^{g}_{\Theta}(H_{i}, \bar{H}_{j})
\end{pmatrix}.
\]

We now evaluate each horizontal block determinant by carrying the connection over to \( X^{(N)} \) under the identification with \( \mathbb{P}E_{N} \) and horizontally lift a curve from \( \text{Jac}(X) \) to \( X^{(N)} \) starting at a point \( \tilde{\zeta} = \zeta_{1} + \cdots + \zeta_{N} \). The point \( \tilde{\zeta} \) corresponds to a line of sections \([s] \in \mathbb{P}E_{N} \).

We fix the standard basis of holomorphic vector fields \( \tilde{H}_{i} \) in \( \mathbb{C}^{N+1} \). We let \( H_{j} \) be the horizontal lift of \( \frac{\partial}{\partial \bar{z}_{j}} \). We denote the integral curves by \( x_{j}(t), y_{j}(t) \). We want to calculate the matrix of inner products of their tangent vectors with \( \tilde{\zeta} \) under the identification with \( \mathbb{P}H^{0}(X, L_{N+g}) \). This complement may be identified with \( \mathbb{P}H^{0}(X, L_{N+g}) \) under \( \sigma_{L} \). We thus need to consider the associated curves \( \tilde{P}^{k}(t) = P_{1}^{k}(t) + \cdots + P_{g}^{k}(t) \) in \( X^{g} \) which are the images of \( \zeta_{k}(t) = \zeta_{k} + \cdots + \zeta_{N} \) under \( A_{\mathbb{C}^{N+1}} \). The map is only well-defined away from \( X^{N+g} \). (Here, it would have been preferable to work on \( \tilde{X}^{(N)} \), but then the connection would degenerate).

The curves \( P_{1}^{k}(t) + \cdots + P_{g}^{k}(t) \) in \( X^{g} \) are the same as the images of the \( x_{k}(t), y_{k}(t) \) under the inverse Abelian sums map \( A_{g}^{-1} : \text{Jac} \rightarrow X^{(g)} \). The inverse map is not well-defined on \( W_{1}^{+} \) (\((\ref{inv})\)). We are calculating volume forms so it is sufficient to work on the complement of this set, but the degeneracy set of \( A_{g} \) makes itself felt in the zeros and singularities of the forms.

Since \( \tilde{\omega}_{Z_{N}} = \psi_{L_{N+g}}^{*} \omega_{FS,N,g} \) (Lemma \((\ref{inv})\)), or \( \sigma_{L}^{*} \omega_{FS,N,g} \) on \( \tilde{X}^{(N)} \), the horizontal space \( H_{(\{s\}, P_{1} + \cdots + P_{g})}(\mathbb{P}E_{N}) \) maps under \( D\sigma_{L} \) to the orthogonal complement of the tangent space to the canonically embedded image of \( \mathbb{P}H^{0}(X, L) \). This complement may be identified with the orthogonal complement in \( H^{0}(X, L_{N+g}) \) to the subspace \( \{ s \in H^{0}(X, L_{N+g}) : D(s) \geq P_{1} + \cdots + P_{g} \} \) with respect to the inner product \( G_{N+g}(h, \nu) \). Indeed, the natural projection \( \pi : \mathbb{C}^{d+1} - \{ 0 \} \rightarrow \mathbb{C}^{d} \) is a Riemannian submersion when the spaces are equipped with compatible Euclidean, resp. Fubini-Study, metrics. It is a principal \( \mathbb{C}^{*} \) bundle and carries a natural (Hopf) connection.

We denote by \([s_{k}(t)]\) the horizontal curve in \( \mathbb{P}E_{N} \) whose initial tangent vector is \( H_{k} \). Under \( \sigma_{L} \) it goes to a curve in \( \mathbb{P}H^{0}(X, L_{N+g}) \). We use the well-known identification of the projective space \( \mathbb{P}H \) associated to a Hilbert space with the set of rank one Hermitian orthogonal projections in the space of Hermitian operators on \( H \). Thus, to \([s_{k}(t)]\) we associate the curve \( \pi_{k}(t) = \frac{s_{k}(t) \circ \pi_{m}^{*}(t)}{||s_{k}(t)||_{L^{2}}} \) of projections. Then the Fubini-Study inner product of tangent vectors is given by \( \langle \tilde{n}_{k}, \tilde{n}_{m} \rangle = Tr \tilde{n}_{k} \circ \tilde{n}_{m}^{*} \).

We then put
\[
s_{k}(t) = \prod_{j=1}^{g} E(\cdot, P_{j}^{k}(t)) \prod_{j=1}^{N} E(\cdot, \zeta_{j}^{k}(t)), \quad S_{k}(t) = e^{i\theta_{k}(t)} \frac{s_{k}(t)}{||s_{k}(t)||}.
\]
where the phase $e^{iθ_k(t)}$ is chosen so that $⟨\dot{S}_k(t), S_k(t)⟩ = 0$. With this phase condition the map $ψ_t → |ψ_t⟩⟨ψ_t|$ is an isometry, i.e.

$$Tr{\hat{π}_k}(0){\hat{π}_m}(0) = ⟨\dot{S}_k(0), \dot{S}_m(0)⟩.$$  \hfill (90)

Horizontality is the condition that $\dot{S}_k(0) \perp H^0(X, O(ζ_1 + ⋯ + ζ_N))$. As long as the $P_j$ are distinct, the coherent states $\{Φ_P^j\}_{N+g}$ form a basis of the ortho-complement. Hence, there exists a matrix $C = (C_j^k)$ of coefficients so that

$$\frac{d}{dt}|_{t=0} S_k(t) = \sum_{j=1}^{g} C_j^k Φ_P^j_{N+g}.$$  \hfill (91)

**Remark:** Abbreviate $L = L_{N+g}$. Since $Φ_P^j_{N+g}(z) ∈ L_z ⊗ L^*_{P_j}$, the left side of (91) is a section of $L$ while the right side takes values in $L ⊗ L^*$. Hence, $C_j^k$ must take values in $L_{P_j}$ and the product $C_j^k Φ_P^j_{N+g}$ implicitly involves a contraction. It will be seen below that these extra factors cancel, so rather than introduce new notation we will keep track of them in a series of remarks.

The next step is to calculate the matrix $C$. To this end, we introduce the matrix

$$M_P = M = (M_{ij}), \quad \text{with } M_{ij} := ⟨Φ_P^i_{N+g}, Φ_P^j_{N+g}⟩ = det(Φ_P^j_{N+g}(P_k)).$$

**Remark:** Since $Φ_P^j_{N+g}(P_k) = Π_{N+g}(P_k, P_j)$, this is matrix is also $L ⊗ L^*$-valued. As discussed in [1], we could write it as $B_{N+g}(P_k, P_j)$ times a frame for $L ⊗ L^*$. As noted above, the frames will cancel later.

We further define the matrix $Q = Q_{s, \bar{s}}$ by

$$(Q_{nk}) = \frac{d}{dt}|_{t=0} S_k(t)(P_n).$$  \hfill (92)

**Remark:** This matrix is $L$-valued. Since the curve has been lifted to $H^0(X, L_{N+g})$, the tangent vectors may be identified with elements of this vector space. Hence $Q_{nk} ∈ L_{N+g}(P_n)$.

It is clear that the $\frac{d}{dt}|_{t=0}$ derivative only produces a non-zero term when the factor $E(P_n, P_n^k(t))$ is differentiated. Thus, we have

$$Q_{kn} = \left(\prod_{j ≠ n} E(P_n, P_j) \prod_{j=1}^{N} E(P_n, ζ_j) \frac{d}{dt}|_{t=0} E(P_n, P_n^k(t))\right)\left|| \left(\prod_{j} E(\cdot, P_j) \prod_{j=1}^{N} E(\cdot, ζ_j)\right) \right||_{L^2}e^{iθ_k(t)}.$$  \hfill (93)

Since $E$ is a section of a line bundle, it should require a connection to differentiate it in the second component. However, since $E(z, w)$ vanishes on the diagonal, all connections give the same result, i.e. the derivative of the coefficient $P_n - P_n^k(t)$ relative to a frame. Since the derivative does not touch the frame, it produces another holomorphic section of the same
line bundle. We then identify $d\tau t_0 E(P_n, P^k_n(t))$ with the scalar $-\dot{\hat{P}}^k_n(0)$, the coordinate of the $n$-component of the $k$th curve in our choice of local coordinates. We thus have,

$$
\det Q_{s, F} = \left( \prod_{\ell=1}^N \prod_{j=1}^N E(P_\ell, \zeta_j) \right) \left( \prod_{j\neq n} E(P_n, P_j) \right) \left( \prod_j E(\cdot, P_j) \prod_{j=1}^N E(\cdot, \zeta_j) \right) \left( \det(-\dot{\hat{P}}^k_n(0)) \right).
$$

To put the last determinant in an invariant form, we note that $(-\dot{\hat{P}}^k_n(0))$ is the matrix of $DA^{-1}_g$ relative to the standard basis $\partial_z j$ of $\text{Jac}(X)$ and the coordinate basis of $X^{(g)}$.

We now claim that

**Lemma 14.** Let $H_j$ be the horizontal lifts of the coordinate vector fields $\frac{\partial}{\partial z_j}$ on $\text{Jac}(X)$ to $H([s], P_1 + \cdots + P_g)\mathbb{P}E_N$. Then

$$
\det(\langle H_i, H_j \rangle_{\mathbb{P}E_N}) = \det(M^{-1})|\det Q|^2,
$$

and therefore,

$$
J_N = (\det M^{-1})|\det Q|^2.
$$

**Remark:** We observe that both $\det M$ and $|\det Q|^2$ take values in $\mathcal{L} \otimes \mathcal{L}^*$. Hence if we expressed them relative to a frame $e_\mathcal{L}$ for $\mathcal{L}$, the frames would cancel in the quotient. If we we write a section $\dot{S}_k(P_n) = f_k(P_n)e_\mathcal{L}(P_n)$ then the ratio leaves a Slater determinant $\parallel \det f_k(P_n) \parallel^2$ in the numerator and the Bergman determinant $B_{N+g}(P_k, P_j)$ in the denominator. Thus the ratio is a positive scalar function.

We now prove the Lemma:

**Proof.** In view of Lemma 13, the second statement follows from the first. We now prove the first statement. Taking inner products and using (91), we have

$$
(\langle H_i, H_j \rangle_{\mathbb{P}E_N}) = CMC^*
$$

and

$$
\det(\langle H_i, H_j \rangle_{\mathbb{P}E_N}) = \det \left( \Phi^{F_i}_{N+g}, \Phi^{F_j}_{N+g} \right) \det(C^*C) = \det M \det(C^*C).
$$

(96)

It also follows by setting $z = P_n$ in (99) that

$$
Q_{kn} = \sum_{j=1}^g C^{k}_{j} \Phi^{P_j}_{N+g}(P_n).
$$

(97)

Thus, we have

$$
Q = CM \iff C = QM^{-1}.
$$

(98)

Here, we are assuming that the $P_j$ are distinct, hence that $M$ is invertible (Lemma 101).

It follows that

$$
\det(\langle H_i, H_j \rangle) = \det(Q^*Q) \det M^{-1}.
$$

(99)

\(\square\)
6.2. Configuration spaces and lifts of $\tilde{R}_{FSH}^N$. Using that

$$\tilde{R}_{FSH}^N(\zeta_1, \ldots, \zeta_N) \equiv \tilde{\omega}_{Z}^{-g} \land A_N^* \omega_0 = J_N^{-1} \tilde{\omega}_{Z} = J_N^{-1} \tilde{\omega}_{Z}^N,$$

it follows from Lemma [14] that the lift of $\tilde{R}_{FSH}^N$ to the Cartesian product $X^N$ is given by

$$\tilde{R}_{FSH}^N(\zeta_1, \ldots, \zeta_N) = \mathcal{J}_N(\zeta_1, \ldots, \zeta_N) \frac{1}{Z_N(\omega)} \frac{\exp \left( \frac{1}{2} \sum_{i \neq j} G_{\omega}(\zeta_i, \zeta_j) \right) \prod_{j=1}^N d^2 \zeta_j}{(\int_X e^N \int_X G_{\omega}(z, w) d\zeta(w) e^N \int_X G_{\omega}(z, w) d\mu_{p(\zeta)}(w) d\nu(z))^{N+1}},$$

where (for a constant $B(N, g)$),

$$\mathcal{J}_N(\zeta_1, \ldots, \zeta_N) = \cdot F_N(\zeta_1, \ldots, \zeta_N) e^{-2N \int \rho_\omega(w) d\zeta(w) (\det M) \cdot \det M}^2$$

$$= B(N, g) e^{-(N+1) \sum_{j=1}^N \rho_\omega(P_j) (\det M)} \left| \Phi_{P_0}^N \right|^{-2} \left| \prod_{k=1}^N \prod_{j=1}^g E(P_j, \zeta_k) \right|^2$$

$$\cdot \left| \theta(P_1 + \cdots + P_g - P_0 - \Delta) \right|^{-2} \prod_{j=1}^g |E(P_j, P_0)|^2 \left( \Phi_{P_0}^N \right)$$

$$\left| \prod_{j \neq n} E(P_n, P_j) \right|^{-2} | \det DA_g |^2$$

We next claim that

$$\left| \theta(P_1 + \cdots + P_g - P_0 - \Delta) \right|^{-2} \prod_{j=1}^g |E(P_j, P_0)|^2 | \det DA_g |^2$$

is a smooth positive function. To see this, we recall that $\theta(P_1 + \cdots + P_g - P_0 - \Delta) = 0$ whenever there exists $Q_1 + \cdots + Q_{g-1} \in X^{(g-1)}$ so that $P_1 + \cdots + P_g - P_0 = Q_1 + \cdots + Q_{g-1}$. This can occur if some $P_j = P_0$ and the $Q_j$ are the remaining $P_k$'s, and such poles are cancelled by $\prod_{j=1}^g |E(P_j, P_0)|^2$. It can also happen at other points on the Wirtinger variety $W^1_g$ where the representation fails to be unique, and these poles are cancelled by $| \det DA_g |^2$.

We further claim that the factor

$$\det \left( \Phi_{P_0}^N \right) \left| \prod_{j \neq n} E(P_n, P_j) \right|^{-2}.$$

is a smooth nowhere vanishing function on $X^{(g)}$ times $| \prod_{j \leq n} E(P_j, P_n) |^{-2}$. Thus, the $\det M$ factor cancels ‘half’ the poles of the second factor.

To explain this, we note that $\det((H_i, H_j)_{\bar{\omega}_Z}^N) = 0$ if and only if $([s], P_1 + \cdots + P_g) \in B_{N+g}$, the branch locus of the map $\sigma_L$. Indeed, $\{H_1, \ldots, H_g\}$ is a basis of the horizontal space at all points, so the determinant can only vanish when $\bar{\omega}_Z$ degenerates. Since it is the pullback
of a non-degenerate form under \( \sigma_L \), it only degenerates on the branch locus. We recall that this is the locus where \( \zeta_1 + \cdots + \zeta_N + P_1 + \cdots + P_g \) has multiplicity (i.e. at least two terms coincide). We observe that also

\[
\det \bar{M}_{\bar{P}} = 0 \iff \exists j \neq k : \ P_j = P_k.
\]  

Indeed, \( \det M = 0 \) if and only if the map

\[
\{ s \in H^0(X, \mathcal{L}_{N+g}) : \mathcal{D}(s) \geq P_1 + \cdots + P_g \} \rightarrow \mathcal{L}_{N+g}[P_1] \oplus \cdots \oplus \mathcal{L}_{N+g}[P_g]
\]
on the given ortho-complement, sending \( s \rightarrow (s(P_1), \ldots, s(P_g)) \), has a kernel. The kernel is trivial if the \( P_j \) are distinct, since \( s(P_1) = \cdots = s(P_g) = 0 \) implies that \( s \) lies both in the given subspace and its orthocomplement. When there are multiplicities, then there is a non-trivial kernel; one needs to supplement the map with the derivatives of \( s \) at the multiple points.

This completes the proof of Theorem 6.

\[ \square \]

7. LDP for the projective linear ensemble: Proof of Theorem 1

In this section we prove Theorem 1 for the projective linear ensemble. More precisely, we reduce the proof to the results of [ZZ].

For the sake of completeness, we recall the definition of the LDP: if \( B(\sigma, \delta) \) denotes the ball of radius \( \delta \) around \( \sigma \in \mathcal{M}(\mathbb{CP}^1) \) in the Wasserstein metric, and \( \overline{B(\sigma, \delta)} \) denote its interior (respectively, its closure), then

\[
- \inf_{\mu \in \overline{B(\sigma, \delta)}} \tilde{I}^{\omega,K}(\mu) \leq \liminf_{N \to \infty} \frac{1}{N^2} \log \text{Prob}_N(B(\sigma, \delta))
\]

\[
\leq \limsup_{N \to \infty} \frac{1}{N^2} \log \text{Prob}_N(B(\sigma, \delta)) \leq - \inf_{\mu \in \overline{B(\sigma, \delta)}} \tilde{I}^{\omega,K}(\mu).
\]  

(102)

Once we have found the approximate rate functional, and have expressed it in terms of Green’s functions, we can take its limit precisely as in [ZZ] and obtain the LDP.

For any ensemble, we express the lift of \( \bar{K}^N \) to the Cartesian product as

\[
\bar{K}^N(\zeta_1, \ldots, \zeta_N) = D_N(\zeta_1, \ldots, \zeta_N) \prod_{j=1}^N d^2\zeta_j.
\]  

(103)

7.1. An approximate rate function: Proof of Proposition 1. The Proof of Proposition 1 is similar to that of Lemma 18 of [ZZ], the principal new feature being the coefficient function \( F_N \).

We express the JPC of Theorem 2 in terms of the empirical measures \( \mu_\zeta \). The following Lemma proves Proposition 1

**Lemma 15.** We have

\[
\bar{K}_N(\zeta_1, \ldots, \zeta_N) = \frac{1}{Z_N(h)} e^{-N^2\left(-\frac{1}{2} \xi_2(\mu_\zeta) + \frac{N+1}{2} \tau^+ N(\mu_\zeta) \right)} \kappa_N,
\]

where (cf. Proposition 6),

\[
\kappa_N = F_N(\zeta_1, \ldots, \zeta_N) \prod_j d^2\zeta_j.
\]
Proof. We first observe that the main factor in (II) of Theorem 2 can be rewritten in terms of the empirical measure to get,

$$\exp \left( \frac{1}{2} \sum_{i \neq j} G_\omega(\zeta_i, \zeta_j) \right) \left( \int_X e^{N} f_x G_\omega(z, w) d\mu_\zeta(w) e^{f_x G_\omega(z, w) d\mu_P(z)} d\nu(z) \right)^{\frac{N+1}{2}} = e^{-N^2 I^\omega,\nu_N(\mu_\zeta)}.$$ 

Indeed, by taking the right side as the definition of $I^\omega,\nu_N$, we get

$$I^\omega,\nu_N = -\frac{1}{N^2} \sum_{i \neq j} \frac{1}{2} G_\omega(\zeta_i, \zeta_j) + \frac{N+1}{N^2} \log \left( \int_X e^{N} f_x G_\omega(z, w) d\mu_\zeta(w) e^{f_x G_\omega(z, w) d\mu_P(z)} d\nu(z) \right)$$

$$= -\frac{1}{N^2} \int_{X \times X \setminus D} G_\omega(z, w) d\mu_\zeta(z) d\mu_\zeta(w) + \frac{N+1}{N^2} \log \left( \int_X e^{N} f_x G_\omega(z, w) d\mu_\zeta(w) e^{f_x G_\omega(z, w) d\mu_P(z)} d\nu(z) \right)$$

$$= -\frac{1}{N^2} \left( -\frac{1}{2} E^\omega_N(\mu_\zeta) + \frac{N(N+1)}{N^2} J^\omega_N(\mu_\zeta) \right),$$

as one sees by comparing with Definition 0. We then combine the rest of the factors into $\kappa_N$.\hfill \square

7.2. Properties of the rate function. The properties of the rate function in higher genus are similar to those proved in [ZZ] Section 6 in genus zero, and the proofs are the same, so we state them rapidly and refer to [ZZ] for the proof.

Proposition 16. (see [ZZ], Proposition 24) The function $I_1$ of (12) has the following properties:

1. It is a lower-semicontinuous functional.
2. It is strictly convex.
3. Its unique minimizer is the equilibrium measure $\nu_{h,K}$.
4. Its minimum value equals $\frac{1}{2} \log \text{Cap}_h(K)$.

Set

$$E_0(h) = \inf_{\mu \in M(X)} I^{\omega,K}_N(\mu), \quad \bar{I}_{1,K} = I^{\omega,K} - E_0(h).$$

The infimum $\inf_{\mu \in M(X)} I^{\omega,K}(\mu)$ is achieved at the Green’s equilibrium measure $\nu_{\omega,K}$ with respect to $(\omega, K)$, and $E_0(h) = \frac{1}{2} \log \text{Cap}_h(K)$, where (as above) $\text{Cap}_h(K)$ is the Green’s capacity with respect to $\omega$. For background we refer to [ZZ] and its references.

7.3. Completion of proof of Theorem 1. Given Theorem 3 Lemma 15 and the arguments of [ZZ], the main remaining complication in proving Theorem 1 is to deal with the singular factor $\left| \prod_{j<n} E(P_n, P_j) \right|^{-2}$ described in the statement of Theorem 6. Of course it must be cancelled by the other factors since the form is smooth, but we need to make the necessary estimates to take the limit as $N \to \infty$. We recall that this factor arises since the pullback of the Fubini-Study volume form on $\mathbb{P}H^0(X, \mathcal{L}_{N+g})$ to $\mathbb{P}E_N$ has degeneracies on the branch locus, while those of the Fubini-Study-fiber volume forms do not.

The cancellation of this factor comes from the fact (discussed in [12]) that $\tilde{K}_{FSH}^N$ is a smooth non-degenerate form on $\mathbb{P}E_N \simeq X^{(N)}$ which is bundle-like and which contains the lift of a factor on $X^{(g)}$. This factor can be expressed in coordinates $P_1 + \cdots + P_g$ in the image of the Abel map and its lift to $X^g$ contains a Vandermonde type factor $\left| \prod_{j<n} E(P_n, P_j) \right|^{2}$.
cancelling the singular one above. To determine the ratio, we need to change coordinates again from $\zeta_1 + \cdots + \zeta_N$ to $P_1 + \cdots + P_g$ and $N - g$ remaining coordinates along the fiber, $\mathbb{P}H^0(X, \mathcal{O}((N+g)P_0 - (P_1 + \cdots + P_g))$. To prove Theorem 6 i.e. to study empirical measures, we need the remaining $N - g$ coordinates to be divisor coordinates rather than coefficients relative to a basis. Indeed, this was the main reason for introducing $L_{N+g}$ in the first place.

We therefore seek an $X^{N-g}$-valued fiber coordinate $\eta_1 + \cdots + \eta_{N-g}$ which are coordinates of $N - g$ zeros of sections in the fiber $\mathbb{P}H^0(X, \mathcal{O}((N+g)P_0 - (P_1 + \cdots + P_g))$. Since $\dim \mathbb{P}H^0(X, \mathcal{O}((N+g)P_0 - (P_1 + \cdots + P_g)) = N - g$, a section is specified up to scalar multiples by $N - g$ zeros. Equating $\mathbb{P}\mathcal{E}_N = X^{(N)}$, it is equivalent to define (almost everywhere) an analytic function

$$\tilde{\zeta}(\vec{\eta}, \vec{P}) \in X^{(N)} : A_N(\zeta) = P_1 + \cdots + P_g$$

whose image is an open dense (indeed, Zariski open) subset of $X^{(N)}$. There are of course $\binom{N}{g}$ ways to select a subset of $N - g$ zeros from $\zeta_1 + \cdots + \zeta_N$, and what we are claiming is that there exists a well-defined analytic branch of the correspondence $X^{(N)} \to X^{(N-g)} \times X^{(g)}$. whose graph is given by

$$\{(\zeta_1 + \cdots + \zeta_N; \zeta_{j_1} + \cdots + \zeta_{j_{N-g}} : P_1 + \cdots + P_g) \subset X^{(N)} \times (X^{(N-g)} \times X^{(g)})$$

where as usual $P_1 + \cdots + P_g = A_N(\zeta_1 + \cdots + \zeta_N)$. Existence of such a branch follows from the fact that correspondence is a covering map on the complement of the branch locus, and the complement is a Zariski open set (see also [Mat]).

We recall that the expression for $\tilde{K}_{FSH}^N$ in Theorem 6 is for the pull back of this form to $X^N$ in the local coordinates $\zeta_1, \ldots, \zeta_N$. We now use the local coordinates $\eta_1, \eta_2, \ldots, \eta_{N-g}, P_1, \ldots, P_g$ instead. By definition of a Fubini-Study-fiber bundle probability measure, $\tilde{K}_{FSH}^N$ has the wedge product of a form $d\sigma$ lifted from $X^{(g)}$ and a smooth Fubini-Study volume form along the fibers. The former is a smooth positive multiple of $|\prod_{j \neq k=1}^g E(P_j, P_k)|^2 \prod_{j=1}^g |dP_j \wedge d\bar{P}_j|$ and a smooth form in $\eta$. It follows that the factor $|\prod_{j<n} E(P_n, P_j)|^2$ in the formula of Theorem 6 is cancelled by the same Vandermonde factor arising from the expression of $\tilde{K}_{FSH}^N$ in the coordinates $(\vec{\eta}, \vec{P})$. This changes the definition of $\tilde{K}_N$ in Lemma 15 and Theorem 6 to

$$\tilde{K}_N = \tilde{J}_N \prod_{j=1}^g dP_j \wedge d\bar{P}_j \prod_{j=1}^{N-g} d\eta_j \wedge d\bar{\eta}_j,$$

with

$$\tilde{J}_N(\eta_1, \ldots, \eta_{N-g}, P_1 \ldots, P_g) = B(N,g) e^{-(N+1)\sum_{j=1}^g \rho_{\sigma}(P_j)} ||\Phi_{1/\Delta}^P||^{-2}$$

$$\cdot ||\theta(P_1 + \cdots + P_g - P_0 - \Delta)||^{-2} \prod_{j=1}^g |E(P_j, P_0)|^2$$

$$\quad (\det \left( \Phi_{N+g}^P \Phi_{N+g}^\ast \right) \left( \prod_{j<n} E(P_n, P_j) \right)^{-2} |\det DA_g|^2.$$ 

(105, 106)

7.3.1. LD upper bound. We first sketch the proof of the upper bound part of the large deviation principle from [ZZ]:

$$\sum_{j=1}^g dP_j \wedge d\bar{P}_j \prod_{j=1}^{N-g} d\eta_j \wedge d\bar{\eta}_j,$$
Lemma 17.
\[ \lim_{\delta \downarrow 0} \limsup_{N} \frac{1}{N^2} \log \Prob_N(B(\sigma, \delta)) \leq -\tilde{I}_N^\omega K(\sigma). \] (107)

Proof. The first step is to prove the following: let \( \epsilon > 0 \) and let \( K = \text{supp}\nu \). If \( \nu \) satisfies the Bernstein-Markov condition \( (37) \), then there exists a \( N_0 = N_0(\epsilon) \) such that for all \( N > N_0 \) and all \( \mu_\xi \in \mathcal{M}(X) \), (with \( \rho_\omega \) as in \( (70) \))
\[
\log \| e^{U_\omega^{\mu_\xi}} e^\left( \frac{1}{N} U_{\mu^\rho}^{\xi}(\omega) \right) \|_{L^N(\nu)} \geq \sup_{z \in K} (U_\omega^{\mu_\xi}) - \epsilon .
\]

This is similar to Lemma 30 of \([ZZ]\), with two modifications. First, there is the new factor \( \frac{1}{N} U_{\mu^\rho}^{\xi}(\omega) \) (which is absorbed in the \( \epsilon \)). Second, the Bernstein-Markov assumption is now a uniform estimate comparing \( L^2 \) norms and sup norms of sections \( s \in H^0(C, \xi) \) as \( \xi \) varies over \( \text{Pic}^N \):
\[
\sup_{z \in K} |s(z)|_{h_\xi} \leq C \epsilon^{-N} \left( \int_K |s(z)|^2 \, d\nu(z) \right)^{1/2} , \quad \forall \xi \in \text{Pic}^N , \ s \in H^0(X, \xi) .
\] (108)

Here, \( h_\xi(\omega) \) is the admissible metric on the line bundle \( \xi(\xi) \) where \( s_\xi \in H^0(X, \xi(\xi)) \). By Lemma \( 8 \) we may write,
\[
|s_\xi(z)|_{h_\xi}^2 = e^{N(U_\omega^{\mu_\xi}(z) - \frac{1}{N} U_{\mu^\rho}^{\xi}(\omega))} , \quad \forall \xi \in X^{(N)} .
\]

Hence,
\[
\| e^{U_\omega^{\mu_\xi}} e^\left( \frac{1}{N} U_{\mu^\rho}^{\xi}(\omega) \right) \|_{L^N(\nu)} = \left( \int_K |s_\xi(z)|_{h_\xi(\omega)}^2 \, d\nu(z) \right)^{1/N} \geq \left( C \epsilon^{-N} \sup_{z \in K} |s_\xi(z)|_{h_\xi(\omega)}^2 \right)^{1/N} ,
\]

and
\[
\log \| e^{U_\omega^{\mu_\xi}} e^\left( \frac{1}{N} U_{\mu^\rho}^{\xi}(\omega) \right) \|_{L^N(\nu)} \geq \sup_{z \in K} U_\omega^{\mu_\xi} - \epsilon + \frac{1}{N} \log C \epsilon ,
\]
for all \( \epsilon > 0 \).

Write
\[
\Theta_N = -\frac{1}{N^2} \log \hat{Z}_N(h) .
\] (109)

In §1A we show that (as in \([ZZ]\), \( \Theta_N \to_{N \to \infty} \log \text{Cap}_\omega(K) \).

By Lemma \( 9 \) and Lemma \( 15 \)
\[
\frac{1}{N^2} \log \Prob_N(B(\sigma, \delta)) = \frac{1}{N^2} \log \int_{\xi \in X^{(N)}, \mu_\xi \in B(\sigma, \delta)} e^{-N^2(I_N(\mu_\xi) - \bar{\kappa}_N)} \Theta_N ,
\] (110)

where \( \bar{\kappa}_N \) is the smooth, non-negative \( (N, N) \) form defined in \( (105) \), and \( I_N \) is the approximate rate function.

Fix \( M \in \mathbb{R} \) and let \( G_\omega^M = G_\omega \vee (-M) \) be the truncated Green function and let \( \mathcal{E}_\omega^M \) be the Green’s energy associated to the truncated Green’s function. As in \([ZZ]\), \( G_\omega^M \) is continuous on \( X \times X \) and
\[
- \frac{1}{N^2} \sum_{i < j} G_\omega(\xi_i, \xi_j) \geq \mathcal{E}_\omega^M(\mu_\xi) - \frac{C(M)}{N} ,
\]
where the constant \( C(M) \) does not depend on \( \xi \).
It follows that, for any $\epsilon > 0$ and all $N > N_0(\epsilon)$,

$$\frac{1}{N^2} \log \Prob_N(B(\sigma, \delta)) \leq \frac{1}{N^2} \log \int_{\xi \in X^{(N)}} e^{\frac{1}{2} \log M(\mu_\xi) - N^2 J^K(\mu_\xi)} k_N$$

$$+ \left( \Theta_N + \frac{C'(M)}{N} + \epsilon \right),$$

for some constant $C'(M)$.

It follows that

$$\limsup_N \frac{1}{N^2} \log \Prob_N(B(\sigma, \delta)) \leq \limsup_{N \to \infty} \Theta_N$$

$$+ \limsup_{\delta \to 0} \sup_{\mu \in B(\sigma, \delta)} \left( -\frac{1}{2} \log M(\sigma) + J^K(\sigma) \right)$$

$$\leq \limsup_N \frac{1}{N^2} \log \int_{X^{(N)}} k_N.$$

We now claim:

**Lemma 18.** Let $\tilde{k}_N$ be the smooth $(N, N)$ form defined in (105). Then

$$\frac{1}{N^2} \left| \log \int_{X^{(N)}} \tilde{k}_N \right| = O\left( \frac{\log N}{N} \right).$$

*Proof.* Omitting the constant $B(N, g) \left( \det(\nabla_k P_n) \right), \| \Phi_{N+g} \|^2$ (which may be absorbed into the overall normalizing constant $Z_N$), the statement comes down to showing that

$$\frac{1}{N^2} \left| \log \int_{X^{(N)}} e^{-\left( N + 1 \right) \sum_{j=1}^g \rho_{\omega}(P_j(\xi))} \right|$$

$$\| \theta(P_1 + \cdots + P_g - P_0 - \Delta) \|^{-2} \prod_{j=1}^g \left| E(P_j, P_0) \right|^2 \left| \det A_g \right|^2$$

$$(\det \left( \Phi_{N+g}^P, \Phi_{N+g}^P \right)^* \left( \Pi_{j<n} E(P_n, P_j) \right) \right)^2 \prod_{j=1}^{N-g} \eta_j \prod_{j=1}^{g} dP_j = O\left( \frac{\log N}{N} \right).$$

We observe that the integrand is a smooth function only of $P_1 + \cdots + P_g \in X^{(g)}$ and only the factor $e^{-\left( N + 1 \right) \sum_{j=1}^g \rho_{\omega}(P_j(\xi))} \left( \det \left( \Phi_{N+g}^P, \Phi_{N+g}^P \right) \right)$ depends on $N$.

We first integrate out the $\eta_j$ variables and obtain the Lebesgue volume of $X^{(N-g)}$ in the $\eta_j$ coordinates. It lifts back to $X^{N-g}$ as a product measure and therefore (as in [ZZ]), we have

$$\frac{1}{N^2} \left| \log \int_{X^{(N-g)}} \prod_{j=1}^{N-g} \eta_j \right| = O\left( \frac{\log N}{N} \right).$$

This reduces us to studying the remaining integral over $X^{(g)}$. Since the factor $\| \theta(P_1 + \cdots + P_g - P_0 - \Delta) \|^{-2} \prod_{j=1}^g \left| E(P_j, P_0) \right|^2 \left| \det A_g \right|^2$ is smooth, positive function on $X^{(g)}$ which is independent of $N$, it is bounded above by a constant $C_g > 0$ and below by another constant $c_g > 0$. Hence it may be removed from the integral at the cost of a remainder $O\left( \frac{1}{N^2} \right)$. Also, the factor $e^{-\left( N + 1 \right) \sum_{j=1}^g \rho_{\omega}(P_j(\xi))}$ may be removed at the cost of a remainder $O\left( \frac{1}{N} \right)$. Consequently, it suffices to show that

$$\frac{1}{N^2} \left| \log \int_{X^{(g)}} \left( \det \left( \Phi_{N+g}^P, \Phi_{N+g}^P \right) \right)^* \left( \Pi_{j<n} E(P_n, P_j) \right) \right|^2 \prod_{j} dP_j = O\left( \frac{\log N}{N} \right).$$
Denote by \( X^g(N) \) the ‘well-separated’ set of \((P_1, \ldots, P_g)\) so that \( d(P_j, P_k) \geq \frac{\log N}{\sqrt{N}} \) for all \( j \neq k \), and then put
\[
\int_{X^g} = \int_{X^g(N)} + \int_{X^g \backslash X^g(N)}.
\]

On \( X^g \backslash X^g(N) \), the off-diagonal elements are of order \( N^{-p} \) for any desired \( p > 0 \). The diagonal elements are of order \( N \). Hence
\[
\det \left( \langle P_{N+g}^j, P_{N+g}^k \rangle \right) \geq N^g - O(N^{-p}), \quad \text{on } X^g \backslash X^g(N).
\]
Since also \( \left| \left( \prod_{j<n} E(P_n, P_j) \right) \right|^{-2} \geq \epsilon_0 > 0 \) for some constant \( \epsilon_0 \) depending independent of \( N \), the integrand of the \( X^g \backslash X^g(N) \) integral is bounded below by \( \epsilon_0 N^g \) and therefore the integral is bounded below by a constant (depending only on the genus) times \( N^g \). Since the integrand is positive, the addition of the integral over \( X^g(N) \) only increases the quantity and therefore,
\[
\frac{1}{N^2} \log \int_{X^g} \left( \det \left( \langle P_{N+g}^j, P_{N+g}^k \rangle \right) \right)^* \left| \left( \prod_{j<n} E(P_n, P_j) \right) \right|^{-2} \prod_j d^2P_j \geq C_g \frac{\log N}{N}.
\]
This proves the lower bound half of the desired estimate.

We then prove the upper bound. The main contribution comes from the integral over \( X^g(N) \). On \( X^g \backslash X^g(N) \), \( |E(P_j, P_k)| \geq \frac{\log N}{\sqrt{N}} \) and the product \( \left| \left( \prod_{j<n} E(P_n, P_j) \right) \right| \) is bounded below by \( (\frac{\log N}{\sqrt{N}})^g \). Also by the Hadamard inequality, and the fact that \( \|P^g_{N+g}\|_{L^\infty} \leq N \), \( |(\det \left( \langle P_{N+g}^j, P_{N+g}^k \rangle \right)^*| \) is bounded above by \( N^g \). Hence, \( \frac{1}{N^2} \log \) of the integral over 
\( X^g \backslash X^g(N) \) is \( O(\frac{\log N}{N}) \).

Thus it suffices to give an upper bound for the integral over \( X^g(N) \). By a slight extension of the same argument, we decompose the remaining set \( \{ \bar{P} \in X^g : \exists \{j : d(P_j, P_j) < \frac{\log N}{\sqrt{N}} \} \} \) into sets where there are \( r \) clusters of points each within \( < \frac{\log N}{\sqrt{N}} \) of each other and such that points of each cluster are \( \geq \frac{\log N}{\sqrt{N}} \) apart for distinct clusters. If each cluster contains just one point then we are back to the case with \( \geq \frac{\log N}{\sqrt{N}} \) separated points. In each cluster, and with \( j \neq k \) we write \( \Phi^j(z) - \Phi^k = (P_j - P_k) F_{N}(P_j, P_k) \). We then multiply by \( (E(P_j, P_k))^{-1} \). There are two entries for each \( (P_j, P_k) \) and so the cancellation leaves the smooth matrix function \( \det(F(P_j, P_k)) \) and we need an upper bound for \( \frac{1}{N^2} \log \int_{X^g(N)} \det(F(P_j, P_k)) \). Each column involves at most one derivative of \( \Phi^j_{N+g} \), whose norm is then at most \( N^2 \). By the Hadamard determinant inequality, \( \frac{1}{N^2} \log \int_{X^g(N)} \det(F(P_j, P_k)) \prod d^2P_j \) is of order at most \( \frac{\log N}{N} \).

We now complete the proof of the upper bound: As in [ZZ], \( \mathcal{E}^M_\omega(\sigma) \) is continuous and \( J^K_\omega(\sigma) \) is lower semi-continuous with respect to weak convergence. It follows that
\[
\lim \limsup_{\delta, N} \frac{1}{N^2} \log \text{Prob}_N(B(\sigma, \delta)) \leq \lim_{N \to \infty} \Theta_N + \frac{1}{2} \mathcal{E}_h^M(\sigma) - J_h^K(\sigma) + \epsilon.
\]
Since \( \mathcal{E}^M_\omega(\sigma) \to \mathcal{E}_\omega(\sigma) \) as \( M \to \infty \) by monotone convergence, and since \( \epsilon \) is arbitrary, we obtain [107]. \( \square \)
To complete the proof of Theorem 1, we also need to prove the lower bound:

**Lemma 19.**

\[
\lim_{\delta \downarrow 0} \liminf_N \frac{1}{N^2} \log \mathbf{P}_N(B(\sigma, \delta)) \geq -\tilde{I}_{\omega,K}(\sigma). \tag{111}
\]

Exactly as in [ZZ] (Lemma 31), it suffices to prove (111) when \(\sigma = f_{\omega} \in M(X)\) with \(f\) a strictly positive and continuous function on \(X\).

**Proof.** We closely follow the proof of the LDP lower bound in [ZZ] until the last step where we apply Lemma 18 rather than the product measure argument in [ZZ]. Under the assumption that \(\sigma = f_{\omega}\), we can construct a sequence of discrete probability measures

\[
d\sigma_N = \frac{1}{N} \sum_{j=1}^{N} \delta_{Z_j} \in B(\sigma, \delta)
\]

with the following properties:
1. \(\sigma_N \in B(\sigma, \delta/2)\) for all \(N\) large;
2. \(d(Z_i, Z_j) \geq C(\sigma, \delta)/\sqrt{N}\) for \(i \neq j\).

Define

\[
D_N^\eta = \{\zeta \in X^N : d(\zeta_j, Z_j) \leq \eta \cdot \frac{N}{N}, \ j = 1, \ldots, N\}.
\]

Then, for \(\eta\) small enough and all \(N\) large, all \(\zeta \in D_N^\eta\) satisfy that \(\mu_\zeta \in B(\sigma, \delta)\). Since \(D_N^\eta \subset B(\sigma, \delta)\),

\[
\mathbf{P}_N(B(\sigma, \delta)) \geq \int_{D_N^\eta} e^{-N^2 I_N(\mu_\zeta)} \tilde{\kappa}_N + \Theta_N, \tag{112}
\]

where \(I_N = I_N^{\omega,\mu}\) is the approximate rate function. Following the Green’s function estimates up to (62) of [ZZ], we get that for any \(\epsilon' > 0\) and all \(N\) large enough,

\[
\mathbf{P}_N(B(\sigma, \delta)) \geq e^{-N^2 I^{\omega,K}(\sigma)-3\epsilon'N^2} \int_{D_N^\eta} \tilde{\kappa}_N \tag{113}
\]

By Lemma 18, we have the lower bound (for some \(C \geq 0\))

\[
\frac{1}{N^2} \log \int_{X^{(N)}} \tilde{\kappa}_N \geq -C \log \frac{N}{N},
\]

and together with (113) it implies the desired lower bound (111).

\[\square\]

7.4. **The normalizing constant:** Proof of Lemma 20. To complete the proof of Theorem 1 we need to determine the logarithmic asymptotics of the normalizing constant \(\tilde{Z}_N(\omega)\), or equivalently of \(\Theta_N\) (109). In fact, in the course of the proof we also introduced a constant \(B(N, g)\) and in the proof it was also absorbed into \(\Theta_N\). We now denote the overall constant by \(\tilde{Z}_N(\omega)\). We have proved the LDP for the measure multiplied by this constant. We then determine \(\Theta_N\) from the fact that \(\mathbf{P}_N\) is a probability measure. As in [ZZ], Lemma 4 (see Section 7.4):
**Lemma 20.** We have,

\[-\lim_{N \to \infty} \frac{1}{N^2} \Theta_N = \lim_{N \to \infty} \frac{1}{N^2} \log \hat{Z}_N(\omega) = \frac{1}{2} \log \text{Cap}_h(K).\]

For the sake of completeness, we include the proof from [ZZ]:

**Proof.** By Proposition [10] and the proof of the large deviations upper bound in Lemma [107],

\[
0 = \lim_{N \to \infty} \frac{1}{N^2} \log \text{Prob}_N(\mathcal{M}(X))
\]

\[
\leq \limsup_{N \to \infty} \frac{1}{N^2} \log \hat{Z}_N(h) - \inf_{\mu \in \mathcal{M}(X)} \log h \Gamma(\omega, h)(\mu)
\]

\[
= \limsup_{N \to \infty} \frac{1}{N^2} \log \hat{Z}_N(h) - \log h \Gamma(\omega, h)
\]

\[
= \limsup_{N \to \infty} \frac{1}{N^2} \log \hat{Z}_N(h) - \frac{1}{2} \log \text{Cap}_h(K).
\]

A similar argument using the large deviations lower bound shows the reverse inequality for \(\liminf_{N \to \infty} \frac{1}{N^2} \log \hat{Z}_N\).

\[
\square
\]

8. **Appendix on determinants and bosonization**

Up to the constant factor, the bosonization we quote in Lemma [5] is relatively simple to prove (see Fay [F]). Following Fay’s presentation, we describe line bundles and their sections by their automorphy factors. For genus one, \(X\) is expressed as \(\mathbb{C} \setminus \Gamma\) while for \(g \geq 2\), \(X = \mathbb{H} \setminus \Gamma\) with \(\Gamma \subset SL(2, \mathbb{R})\) and \(\mathbb{H}\) the upper half plane. In either case, \(L\) is defined by a factor of automorphy \(\varphi_\gamma(z)\),

\[
\varphi_\gamma(z) = \chi_\gamma \frac{t(\gamma z)}{t(z)} \prod_{i=1}^d \frac{E(\gamma z, a_i)}{E(z, a_i)}, \quad (\gamma \in \Gamma),
\]

where \(t(z)\) is a nowhere vanishing holomorphic function on \(\mathbb{H}\). We also put

\[
\sigma(p) = \exp \left( - \sum_{k=1}^g \int_{A_k} v_k(x) \log E(x, p) \right), \quad \sigma(p, p_1) = \frac{\sigma(p)}{\sigma(p_1)}.
\]

By Proposition 1.2 of [F], \(\sigma(p)\) is a nowhere vanishing holomorphic automorphic function on \(\mathbb{H}\) with automorphic factors given in [F] (1.12). Finally, let \(\Delta\) denote the vector of Riemann constants and let \(\theta[\chi]\) denote the theta function with characteristic \(\chi\).

Theorem 1.3 of [F] states the following: Let \(L\) be the line bundle \(\chi \otimes D\) of degree \(d \geq 1\) with \(D = \sum_{i=1}^N a_i\) and with \(\chi\) a unitary character. Then there exists a constant \(f_L\) depending only on the marking of the Riemann surface so that, for any basis \(\{\omega_j\}\) of \(H^0(L)\) and any points \(x_1, \ldots, x_{d+1-g} \in X\), so that

\[
\det (\omega_j(x_k))_{j,k=1}^{d+1-g} = f_L^{-1} \theta[\chi] \left( \sum_{i=1}^d a_i - \sum_{i=1}^{d+1-g} x_i - \Delta \right) \prod_{i<j} E(x_i, x_j) \prod_{i=1}^{d+1-g} t(x_i) \prod_{i=1}^{d+1-g} \sigma(x_i, z_0), \quad (114)
\]

Let us sketch the proof of (114). The main point is to show that \(f_L\) is a constant depending only on the marking. The first point is that \(f_L\) is a meromorphic function of \(x_i \in X\). As noted above, the Slater determinant \(\det (\omega_j(x_k))\) is a section of \(\pi_1^1 L \otimes \cdots \otimes \pi_1^{d+1-g} L \to X^{(d+1-g)}\).
With Fay’s definition of the prime form $E$, the right side is also a section of this bundle. To verify the formula is to prove that $f_L$ has no zeros or poles.

For generic $x_2, \ldots, x_{d+1-g}$, $\det(\omega_j(x_k))$ vanishes at $x_1 = x_2, \ldots, x_{d+1-g}$ and at $g$ further points $\xi$ such that $x_2 + \cdots + x_{d+1-g} + \xi = [L] \in \text{Pic}^d(X)$. Also, $h^1(L \otimes (-\sum_{j=2}^{d+1-g} x_j)) = 0$ for generic $x_2, \ldots, x_{d+1-g}$. By Riemann’s theorem (4.17), the zeros of $\theta[\chi](\sum_{i=1}^{d} a_i - \sum_{i=1}^{d+1-g} x_i - \Delta)$ in $x_1$ occur at $g$ points $\eta$ for which

$$[L] - \sum_{i=2}^{d+1-g} x_i = \eta.$$  

It follows that $\xi = \eta$ and that $f_L$ has no zeros or poles in $x_1$. Similarly, $f_L$ is constant in all of the variables $x_j$. Hence it is constant, completing the proof.

The missing detail in this formula is an explicit formula for $f_L$, which in our problem depends on $N$. It is possible that this factor is cancelled by the normalizing factor $Z_N(h)$ in Theorem 2. But it is useful to recall the explicit formula for $f_L$.

8.1. Bosonization formulae. In view of the number of complicated invariants, it is useful to compare this to the original bosonization formulae of [ABMNV] (see also [VV] and [Fal]). A special case of (4.15) of [ABMNV] (in the notation of that article) is the formula,

$$\frac{\det' \partial\bar{\partial} \psi_j}{\det(\psi_j, \psi_j)} \cdot \frac{\det' \partial\bar{\partial} \bar{\psi}_j}{\det(\bar{\psi}_j, \bar{\psi}_j)} \cdot \det \begin{pmatrix} \psi_1(P_1) & \cdots & \psi_p(P_1) \\ \psi_1(P_p) & \cdots & \psi_p(P_p) \end{pmatrix}^2 = \left( \frac{\det' \partial\bar{\partial} \bar{\psi}_j}{\det(\bar{\psi}_j, \bar{\psi}_j)} \right)^{-\frac{1}{2}} \cdot N(z) \cdot \prod_{i,j=1}^p G(P_i, P_j).$$  

(115)

In this formula, a Riemannian metric is given on $X$, $\bar{\partial} \partial$ is its Laplacian, $A_X$ is the area of $X$, and $G$ is its ‘regulated coincident’ Green’s function. Further $\mathcal{L}_b$ is a holomorphic line bundle of degree $2\lambda(g - 1)$, $(\cdot, \cdot)$ is an inner product on $H^0(X, \mathcal{L}_b)$ and $\{\psi_j\}$ is a basis of $H^0(X, \mathcal{L}_b)$. Also, $\partial_{\mathcal{L}_b} : C^\infty(X, \mathcal{L}_b) \to C^\infty(X, \mathcal{L}_b \otimes \bar{K}_X)$ is the natural $\bar{\partial}$ operator and $\bar{\partial}_{\mathcal{L}_b}$ is its adjoint in the inner product induced by the Hermitian metric on $\mathcal{L}_b$ and the Riemannian metric on $X$. Also, $G(P_i, P_j) = E(P_i, P_j)$ in our notation, $(iY)^{-1}$ is the period matrix, and the factor $\det(iY)^{-1}N$ is the spin $\frac{1}{2}$ determinant. The rest of the notation is defined in (4.14).

It follows that the constant $A_N(g, \omega)$ defined there is a ratio twisted Laplace determinants and some non-vanishing factors independent of $N$. The proof that the logarithms of Laplace determinants depend only linearly on $N$ is given in [BV].

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