A recursive approach for the enumeration of the homomorphisms from a poset $P$ to the chain $C_3$

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Abstract
Let $\mathcal{H}(P, C_3)$ be the set of order homomorphisms from a poset $P$ to the chain $C_3 = 1 < 2 < 3$. We develop a recursive approach for the calculation of the cardinality of $\mathcal{H}(P, C_3)$, and we apply it on several types of posets, including $P = C_3 \times C_3 \times C_k$ and $P = \mathcal{H}(C_k, C_3)$; for the latter poset $P$, we derive a direct formula for $\# \mathcal{H}(P, C_3)$.

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1 Introduction
Let $C_n$ be the set $\{1, \ldots, n\}$ equipped with the natural order. The number of homomorphisms from a poset $P$ to $C_n$ is the value at $x = n$ of the order polynomial $\Omega_P(x)$ of $P$, introduced by Stanley in the early seventies [14, 15, 16]. Due to [14, Theorem 2], the order polynomial can be written as

$$\Omega_P(x) = \sum_{d=0}^{\#P-1} w_P(d) \left( \frac{\#P + x - 1 - d}{\#P} \right),$$

where $w_P(d)$ is the number of linear extensions of $P$ (regarded as permutations of a natural labeling of $P$) with exactly $d$ descents.

The research about the order polynomial focuses on relating it to other structures and polynomials. The following incomplete list of topics and references highlights the variety of subjects. The connections between the order polynomial, the chromatic polynomial in graph theory, and the Erhart polynomial of an order polytope have already been seen by Stanley [14, 17]. Edelman and Klingsberg [7] use the lattice of sub-posets of a poset in order to prove identities for the order polynomial, and Wagner [20] asks for the zeros of another polynomial related to it. Hamaker et al. [8] and Browning et al. [3] study posets with identical order polynomial, and Jochemko [10] connects the concepts of the
order polynomial and Pólya’s enumeration theorem. The general formula presented by Thomas [19] expresses $\Omega_P(n)$ by means of coefficients related to the order polytope of $P$; just as [1], Thomas’ formula provides structural insight, but is not intended to be a tool for practical calculation.

Direct attempts to really calculate the order polynomial or its values are restricted in most cases to examples and exercises [1, Abschnitt III.4], [18, Section 3.15, in part. Ex. 3.62, 3.66]. For posets with an appropriate structure, the order polynomial has a product formula and is thus (at least in principle) easier to calculate and to evaluate; a survey with references is presented by Hopkins [9]. The reason for the reserve to calculate $\Omega_P(x)$ or its values is the complexity of the task: Brightwell and Winkler [2] showed that the calculation of $\Omega_P(#P)$ (the number of linear extensions of $P$) is #P-complete.

This number of linear extensions is according to [18, p. 258] probably the single most useful number for measuring the “complexity” of a poset. An own branch of mathematics has developed around it, and presumably every reader has already been in touch with it. However, there is also some interest to know the values $\Omega_P(n)$ for small integers $n$. $\Omega_P(1) = 1$ is trivial, and with $D(P)$ being the down-set lattice of $P$, the equation $\Omega_P(2) = \#D(P)$ belongs to the basics of order theory, and the enumeration of $D(P)$ is a standard task which often can be done manually. The numbers $h(P) \equiv \Omega_P(3) = \#H(P,C_3)$ are considerably more complicated to determine, and we present in this article a recursive method for their effective calculation.

After recalling common terms and notation in Section 2, we develop our approach in Section 3. Our first starting point is the well-known isomorphism $H(P,C_3) \simeq H(C_2,D(P))$ yielding

$$h(P) = \sum_{D \in D(P)} \#_{D(P)} D,$$

a summation formula widely used, e.g., in the computation of Dedekind numbers [4]. The second starting point is the flexible concept of the generalized vertical sum of posets introduced by the author and Erné [5, 6]. For a generalized vertical sum $R$ of two posets $P$ and $Q$, it has been shown [6] that the down-set lattice $D(R)$ of $R$ is the disjoint union of certain sets $J_T(R)$, with $T$ running through $D(Q)$; we recall this result in Theorem 1. With

$$a_T(R) \equiv \sum_{D \in J_T(R)} \#_{D(R)} D \quad \text{for all } T \in D(Q),$$

we thus get

$$h(R) = \sum_{T \in D(Q)} a_T(R).$$

We now postulate without loss of generality that the posets $P$ and $Q$ are linked in $R$ by two sub-posets $S^- \subseteq P$ and $S^+ \subseteq Q$ and a mapping $\sigma$ from $D(S^+)$ to subsets of $S^-$ which is compatible with the structure of $R$. The sub-posets $S^+$ and $S^-$ give raise to several interrelated generalized vertical sums, and their respective sets $J_T(R')$ are linked in Lemma 1 by isomorphisms. Based on these results, a recursive formula for the coefficients $a_T(R)$ with $T \in D(S^+)$ is derived in Theorem 2. It refers to proper sub-posets of $R$ only and offers thus
the possibility to calculate \( h(R) \) recursively. The approach is very flexible and can be set up in different ways for a given poset \( R \). It even gives raise to new ways to calculate Dedekind numbers, because \( h(C_k^4) \) is the \((k+1)\)th Dedekind number.

For posets \( R \) fitting well to the structure of the recursion, \( h(R) \) can be calculated manually. We do so in Section 4.1. In Section 4.1 we work with posets \( R = W \times C_k \), and we calculate \( h(W \times C_k) \) for several posets \( W \), including the chain \( C_n \), the poset \( \Lambda \) with \( \Lambda \)-shaped diagram, the diamond \( C_2 \times C_2 \), and the Noughts and Crosses grid \( C_3 \times C_3 \). In Section 4.2 we treat the posets \( R = H(C_2, C_3) \cong H(C_{k-1}, C_3) \) and derive a closed formula for \( h(R) \).

Besides of its mathematical interest, the number \( h(R) \) has some relevance in other areas of science, too, e.g., for ensemble based systems in machine learning [13], also known as multiple classifier systems, committee of classifiers, or mixture of experts. Assume that \( k \) experts (humans, robots, recognition systems, ...) have the task to rank objects on a scale with \( v \) signed to \( s \). Because a better ranking of the experts cannot downgrade the summary value, the possible summary rules are the elements of \( \mathcal{H}(C_k^4, C_3) \), and fundamental questions ask for their number, classification etc.

For \( v = 2 \), the summary rules are the monotone Boolean functions [11, 12], and the figures \#(\mathcal{H}(C_k^4, C_3)) \) are the Dedekind numbers [4] known up to \( k = 8 \). For \( v = 3 \), we deal with monotone ternary functions. The number \( h(C_3^2) = 175 \) can still be determined with paper and pencil, but already the calculation of \( h(C_3^2) = 211250 \) done in Section 4.1 is out of reach of manual calculation. Also Section 4.2 has a connection to summary rules, because \( h(\mathcal{H}(C_2, C_3)) \) is the number of symmetric summary rules.

## 2 Basics and Notation

We are working with finite posets, thus ordered pairs \( P = (X, \leq_P) \) consisting of a finite set \( X \) (the carrier of \( P \)) and a partial order relation \( \leq_P \) on \( X \), i.e., a reflexive, antisymmetric, and transitive subset of \( X \times X \). Due to reflexivity, the diagonal \( \Delta_X \equiv \{(x,x) \mid x \in X \} \) is always a subset of \( \leq_P \). As usual, we write \( x \leq_P y \) for \((x,y) \in \leq_P\).

We say that \( y \in P \) is covered by \( x \in P \), iff \( y \neq x \) and \( y \leq_P x \) without any additional point between them: \( y \leq_P z \leq_P x \Rightarrow z \in \{x,y\} \) for all \( z \in P \).

Two elementary posets can be defined on any set \( X \): The antichain \( (X,\Delta_X) \) and the chain which is up to isomorphism characterized by \( x \leq_P y \) or \( y \leq_P x \) for all \( x,y \in X \). For a finite set \( X \) with \( k \equiv \#X \), we write \( A_k \) for the antichain on \( X \) and \( C_k \) for the chain on \( X \). In what follows, \( C_k \) is always the set \{1, \ldots, k\} equipped with the natural order.

A poset \( Q = (Y,\leq_Q) \) is called a sub-poset of \( P \) iff \( Y \subseteq X \) and \( \leq_Q \subseteq \leq_P \), and for \( Y \subseteq X \), the induced poset \( P|_Y \) is defined as \((Y,\leq_P \cap (Y \times Y))\); however, we write \( P \setminus Y \) instead of \( P|_Y \).

Given two posets \( P = (X,\leq_P) \) and \( Q = (Y,\leq_Q) \), we can construct new posets with them. \( P \times Q \) is the poset with carrier \( X \times Y \) and the component-wise defined partial order relation. If \( X \) and \( Y \) are disjoint, the direct sum \( P + Q \)
and the ordinal sum $P \oplus Q$ are posets on $X \cup Y$ with partial order relations

\[
\leq_{P+Q} \equiv \leq_P \cup \leq_Q,
\leq_{P\oplus Q} \equiv \leq_P \cup \leq_Q \cup (X \times Y).
\]

The generalized vertical sums have been introduced by the author and Erné \[5, 6\] as structures "in-between" direct sums and ordinal sums:

**Definition 1** \([5, 6]\). Let \(P = (X, \leq_P), Q = (Y, \leq_Q)\) be posets with disjoint carriers \(X\) and \(Y\). A poset \(R = (X \cup Y, \leq_R)\) on \(X \cup Y\) is called a generalized vertical sum of \(P\) and \(Q\) iff

\[
\leq_P \cup \leq_Q \subseteq \leq_R \subseteq \leq_P \cup \leq_Q \cup (X \times Y).
\]

We call \(P\) the lower part and \(Q\) the upper part of \(R\); “generalized vertical sum” is abbreviated as “g.v.s.” in what follows.

Down-sets are one of the fundamental concepts in order theory. Given a poset \(P = (X, \leq_P)\), a subset \(D \subseteq X\) is called a down-set or order ideal of \(P\) iff \(x \in D\) holds for every \(x \in X\) for which a \(y \in D\) exists with \(x \leq_P y\). For \(B \subseteq X\) and \(x \in X\), we define the down-sets created by \(B\) and \(x\) in \(P\) by

\[
\downarrow_P B \equiv \{ a \in X \mid a \leq_P b \text{ for a } b \in B \},
\downarrow_P x \equiv \downarrow_P \{ x \}.
\]

The set of down-sets of \(P\) is denoted by \(\mathcal{D}(P)\). Together with set inclusion, \(\mathcal{D}(P)\) is a partial order (even a lattice). For a down-set \(D \in \mathcal{D}(P)\), the symbol \(\downarrow_{\mathcal{D}(P)} D\) thus indicates the down-set created by \(D\) in \(\mathcal{D}(D)\):

\[
\mathcal{E}_D(P) \equiv \downarrow_{\mathcal{D}(P)} D = \{ E \in \mathcal{D}(P) \mid E \subseteq D \}.
\]

Up-sets are the duals of downsets: a subset \(U \subseteq X\) is called an up-set or order filter of \(P\) iff \(x \in U\) holds for every \(x \in X\) for which a \(y \in U\) exists with \(y \leq_P x\).

In order to make the line of thought more conclusive, we frequently identify a down-set of a poset with the poset induced by it, e.g., by calling \(P\) a down-set of \(P\).

A mapping \(\xi : X \to Y\) is called an (order) homomorphism from a poset \(P = (X, \leq_P)\) to a poset \(Q = (Y, \leq_Q)\) iff \(x \leq_P y\) implies \(\xi(x) \leq_Q \xi(y)\) for all \(x, y \in X\). The set of order homomorphisms from \(P\) to \(Q\) is denoted by \(\mathcal{H}(P,Q)\). We make \(\mathcal{H}(P,Q)\) being a poset by equipping it with the usual point-wise partial order \(\leq_{\mathcal{H}(P,Q)}\) defined by

\[
\xi \leq_{\mathcal{H}(P,Q)} \zeta \equiv \xi(x) \leq_Q \zeta(x) \text{ for all } x \in X.
\]

“\(\simeq\)” indicates isomorphism of posets.

From set theory, we use the following symbols:

\[
\emptyset \equiv \emptyset, \\
n \equiv \{1, \ldots, n\} \text{ for every } n \in \mathbb{N}, \\
\{n\} \equiv \{1, \ldots, n\} \text{ for every } n \in \mathbb{N}_0,
\]

and for every set \(X\), the symbol \(\mathcal{P}(X)\) denotes the power set of \(X\).
3 The recursion

For the determination of the cardinality of \( \mathcal{H}(R, C_3) \), we start with the general isomorphism [4, p. 4]
\[
\mathcal{H}(R, \mathcal{D}(Q)) \simeq \mathcal{H}(Q, \mathcal{D}(R)),
\]
yielding
\[
\mathcal{H}(R, C_3) \simeq \mathcal{H}(C_2, \mathcal{D}(R)).
\]
We can thus determine \( \#\mathcal{H}(R, C_3) \) by means of the summation formula
\[
h(R) = \sum_{D \in \mathcal{D}(R)} \# \downarrow_{\mathcal{D}(R)} D = \sum_{D \in \mathcal{D}(R)} \# \mathcal{E}_D(R).
\]
This formula is well-known and has widely been used in the computation of Dedekind numbers [4].

Fundamental for our approach is the following theorem describing the down-set lattice of a generalized vertical sum:

**Theorem 1** ([6], Theorem 3.3). Let \( P = (X, \leq_P), Q = (Y, \leq_Q) \) be posets with disjoint carriers \( X \) and \( Y \) and let \( R = (X \cup Y, \leq_R) \) be a g.v.s. with lower part \( P \) and upper part \( Q \). Then the down-set lattice of \( R \) is given by the following disjoint union:
\[
\mathcal{D}(R) = \bigcup_{T \in \mathcal{D}(Q)} \{ D \cup \downarrow_R T \mid D \in \mathcal{D}(P \downarrow_R T) \}.
\]
In what follows, the symbols \( P, Q, R \) etc. are used as in this theorem.

Theorem 1 provides a flexible tool to investigate posets and their down-set lattices systematically, because for every down-set \( D \in \mathcal{D}(R) \), the poset \( R \) is a g.v.s. of \( R|_D \) and \( R \setminus D \). With \( D \) being the antichain of the minimal points of \( R \), this approach has been used in [5, 6] for the enumeration of down-sets of posets and for the enumeration of posets with a certain characteristic.

We define for every \( T \in \mathcal{D}(Q) \)
\[
\mathcal{J}_T(R) \equiv \{ D \in \mathcal{D}(R) \mid D \cap Y = T \}.
\]
Due to \( Q = R|_Y \), the sets \( \mathcal{J}_T(R), T \in \mathcal{D}(Q) \), form a partition of \( \mathcal{D}(R) \). Therefore,
\[
h(R) = \sum_{T \in \mathcal{D}(Q)} a_T(R),
\]
where
\[
a_T(R) \equiv \sum_{D \in \mathcal{J}_T(R)} \# \mathcal{E}_D(R) \quad \text{for all} \ T \in \mathcal{D}(Q).
\]
In the case of \( X \subseteq \downarrow_R Y \), we have \( \mathcal{J}_Y(R) = \{ X \cup Y \} \), hence
\[
a_Y(R) = \# \mathcal{D}(R).
\]

**Definition 2.** In what follows, \( B^- \subseteq X \) is a fixed up-set of \( P \) and \( B^+ \subseteq Y \) is a fixed down-set of \( Q \). We set \( S^- \equiv P|_{B^-}, S^+ \equiv Q|_{B^+} \), and we assume that \( \sigma : \mathcal{D}(S^+) \to \mathcal{P}(B^-) \) is a mapping with
\[
\forall T \in \mathcal{D}(S^+) : \sigma(T) \subseteq X \cap \downarrow_R T \subseteq \downarrow_P \sigma(T).
\]
Because $B^-$ is an up-set of $P$, the poset $P$ is a g.v.s. of $P \setminus B^-$ and $S^-$. Similarly, $Q$ is a g.v.s. of $S^+$ and $Q \setminus B^+$. For later use, we note that the first inclusion in (6) enforces
\[ \sigma(\emptyset) = \emptyset. \] (7)

Firstly, we realize that the assumptions in Definition 2 are not restrictive. For a given poset $R$, let $U$ be an up-set different from $R$ and $\emptyset$. We define $P \equiv R \setminus U$ and $Q \equiv R \upharpoonright U$. We select for $B^+$ any non-empty down-set of $Q$ and $B^-$ as the up-set of $P$ created by the points of $P$ which are covered by points of $B^+$ in $R$. ($B^-$ can be empty; we come back to this case at the end of the section.) With the mapping
\[ \sigma : D(S^+) \to \mathcal{P}(B^-), \]
\[ T \mapsto B^- \cap \downarrow R T, \]
all requirements in Definition 2 are fulfilled. However, such a schematic choice of $S^+$, $S^-$, and $\sigma$ may be unfavorable. For the effective calculation of $h(R)$, they should be selected in such a way, that they match the structure of the recursion, as discussed at the beginning of Section 4.

The isomorphisms in the following lemma are the key for the recursive approach; they are illustrated in the Figures 2 and 5 in Section 4.

**Lemma 1.** For every $T \in D(S^+)$, the mapping
\[ \tau_T : J_T(R) \to \bigcup_{U \in D(S^-) \atop \sigma(T) \leq U} J_U(P), \] (8)
\[ D \mapsto D \setminus T \]
is an isomorphism with inverse $D' \mapsto D' \cup T$; in particular
\[ J_\emptyset(R) \simeq D(P). \] (9)
Furthermore, for every $N \in D(Q)$, the mapping
\[ \beta_N : \{ D \in D(R) \mid N \subseteq D \} \to D(R \setminus \downarrow R N), \] (10)
\[ D \mapsto D \setminus \downarrow R N \]
is an isomorphism with inverse $D' \mapsto D' \cup \downarrow R N$. In particular, for $T \in D(Q)$ with $N \subseteq T$,
\[ J_T(R) \simeq J_{T \setminus N}(R \setminus \downarrow R N) \] (11)

**Proof.** [8]: Let $T \in D(S^+)$. Because $B^+$ is a down-set of $Q$, we have $D(S^+) \subseteq D(Q)$, and $J_T(R)$ is well-defined. And because $P$ is a g.v.s. of $P \setminus B^-$ and $S^-$, also $J_U(P)$ is well-defined for every $U \in D(-)$. Let $D \in J_T(R)$. Because $P$ is the lower part of $R$, we have $D \setminus T \in D(P)$, and due to the first inclusion in (6), we even have $\sigma(T) \subseteq D \setminus T$, thus $\sigma(T) \subseteq (D \setminus T) \cap B^-$. The sets $J_U(P)$, $V \in D(S^-)$, form a partition of $D(P)$; therefore, the set $D \setminus T$ belongs to the union on the right of (8), and the mapping $\tau_T$ is well-defined.

Let $D'$ belong to the union on the right of (8). By case discrimination, we show that $D \equiv D' \cup T$ is a down-set of $R$. Let $x \in D$ and $y \in R$ with $y \leq R x$:
\[ x \in D', y \in P: \] \[ y \in D. \]
\[ x \in D', y \in Q: \text{impossible due to } \leq_R \cap (Y \times X) = \emptyset. \]
\[ x \in T, y \in P: \text{The second inclusion in (6) delivers} \]
\[ y \in \downarrow_P \sigma(T) \subseteq D'. \]
\[ x \in T, y \in Q: T \in \mathcal{D}(S^+) \subseteq \mathcal{D}(Q) \text{ yields } y \in D. \]

Therefore, \( D \) is a down-set of \( R \), and \( D \in \mathcal{J}_T(R) \) follows. We conclude that \( \tau_T \)
has the inverse \( D' \mapsto D' \cup T \). Isomorphism follows, because \( \tau_T \) and its inverse are both homomorphisms with respect to “\( \subseteq \)”.

\[ \theta : \text{Follows with } [3] \text{ and } [7], \text{ because the sets } \mathcal{J}_U(P), U \in \mathcal{D}(S^-), \text{ form a partition of } \mathcal{D}(P). \]

\[ \varphi : \text{It is easily seen that } D \setminus \downarrow_R M \text{ is indeed a down-set of } R \setminus \downarrow_R M \text{ for every } D \in \mathcal{D}(R) \text{ and every subset } M \subseteq X \cup Y. \]

Let \( N \in \mathcal{D}(Q) \). We show that the indicated inverse of \( \beta_N \) is well-defined.

Let \( P' \equiv P \setminus \downarrow_R N, Q' \equiv Q \setminus N, R' \equiv R \setminus \downarrow_R N \). The set \( X \setminus \downarrow_R N \) is a down-set of \( R' \) and \( Y \setminus N \) is an up-set of \( R' \); the poset \( R' \) is thus a g.v.s. of \( P' \) and \( Q' \). Let \( D \in \mathcal{D}(R') \). According to [3], there exists a unique \( T' \in \mathcal{D}(Q') \) and a unique \( D' \in \mathcal{D}(P' \setminus \downarrow_R T') \) with \( D = D' \cup \downarrow_R T' \). We have \( N \cup T \in \mathcal{D}(Q) \) and additionally \( P' \setminus \downarrow_R T' = P \setminus \downarrow_R (N \cup T) \), and [4] delivers that indeed \( D \cup \downarrow_R N = D' \cup \downarrow_R (N \cup T) \) is a down-set of \( R \) containing \( N \).

Isomorphism follows again because \( \beta_N \) and its inverse are both homomorphisms with respect to “\( \subseteq \)”.

[11] is a direct consequence, because of \( T \setminus N \in \mathcal{D}(Q') \).

\[ \square \]

From the definitions, it is directly understandable that the coefficient \( a_T(R) \)
is not affected by \( Q \setminus T \). The proof of this useful observation is technical:

**Lemma 2.** Let \( Y' \in \mathcal{D}(Q) \). With \( Q' \equiv Q|_{Y'}, R' \equiv R|_{X \cup Y'}, \) we have \( a_T(R') = a_T(R) \) for every \( T \in \mathcal{D}(Q') \).

**Proof.** Because of \( Y' \in \mathcal{D}(Q) \) and \( X \cup Y' \in \mathcal{D}(R) \), we have \( \mathcal{D}(Q') \subseteq \mathcal{D}(Q) \) and \( \mathcal{D}(R') \subseteq \mathcal{D}(R) \). Moreover, \( \mathcal{D}(Q') \) is a down-set of \( \mathcal{D}(Q) \), and \( R' \) is a g.v.s. of \( P \) and \( Q' \).

Let \( T \in \mathcal{D}(Q') \). The equation \( a_T(R') = a_T(R) \) will be a direct consequence of

\[ \mathcal{J}_T(R') = \mathcal{J}_T(R), \quad \text{and } \quad \mathcal{E}_D(R') = \mathcal{E}_D(R) \quad \text{for all } D \in \mathcal{J}_T(R'). \]

Because of \( T \subseteq Y' \), we have \( \leq_R \cap (X \times T) = \leq_R \cap (X \times T) \), hence \( X \cap \downarrow_R T = X \cap \downarrow_R T \). Because additionally \( Y' \cap \downarrow_R T = T = Y \cap \downarrow_R T \),
\[ \downarrow_R T = \downarrow_R T, \]
and applying [4] twice yields [12]:

\[ \mathcal{J}_T(R') = \{ D \cup \downarrow_R T \mid D \in \mathcal{D}(P \setminus \downarrow_R T) \} \]
\[ = \{ D \cup \downarrow_R T \mid D \in \mathcal{D}(P \setminus \downarrow_R T) \} = \mathcal{J}_T(R). \]

\[ 7 \]
Now let $D \in \mathcal{D}(Q')$. $\downarrow_{\mathcal{D}(Q)} D$ is a down-set of $\mathcal{D}(Q)$ with $D \in \mathcal{D}(Q')$, and because $\mathcal{D}(Q')$ is a down-set of $\mathcal{D}(Q)$, we have $\downarrow_{\mathcal{D}(Q)} D \subseteq \mathcal{D}(Q')$, hence $\downarrow_{\mathcal{D}(Q)} D \subseteq \downarrow_{\mathcal{D}(Q')} D$. We conclude $\downarrow_{\mathcal{D}(Q)} D = \downarrow_{\mathcal{D}(Q')} D$, because $Q'$ is a sub-poset of $Q$. All together, application of (14) yields

$$\downarrow_{\mathcal{D}(Q)} O = \downarrow_{\mathcal{D}(Q')} O \quad \text{for all} \quad O \in \downarrow_{\mathcal{D}(Q')} D = \downarrow_{\mathcal{D}(Q)} D,$$

implying $\downarrow_{\mathcal{D}(Q')} D = \downarrow_{\mathcal{D}(Q)} D$. Now (13) results, because due to (4),

$$\mathcal{E}_D(D') = \bigcup_{O \in \downarrow_{\mathcal{D}(Q')} D} \{ F \cup \downarrow_{\mathcal{D}(Q')} O \mid F \in \mathcal{D}(P \setminus \downarrow_{\mathcal{D}(Q')} O), \ F \subseteq X \cap \downarrow_{\mathcal{D}(Q')} D \},$$

$$\mathcal{E}_D(D) = \bigcup_{O \in \downarrow_{\mathcal{D}(Q)} D} \{ F \cup \downarrow_{\mathcal{D}(Q)} O \mid F \in \mathcal{D}(P \setminus \downarrow_{\mathcal{D}(Q)} O), \ F \subseteq X \cap \downarrow_{\mathcal{D}(Q)} D \}.$$

In the following theorem, it is described how the coefficients $a_T(R)$ with $T \in \mathcal{D}(S^+)$ can be determined recursively:

**Theorem 2.** Let $M$ be the set of the minimal points of $Q$ and let $P^+ \equiv R|_X \cup B^+$. Then, for all $T \in \mathcal{D}(S^+)$

$$a_T(R) = \sum_{U \in \mathcal{D}(S^+)} a_U(P) + \sum_{\mu = 1}^{\#(M \cap T)} (-1)^{\mu - 1} \sum_{N \subseteq \mathcal{M} \setminus T} a_{T \setminus N}(P^+ \setminus \downarrow_{P^+} N). \quad (15)$$

In particular,

$$a_{\emptyset}(R) = h(P). \quad (16)$$

**Proof.** We prove the following two equations which immediately yield (15):

$$\sum_{D \in \mathcal{J}_R(T)} \#(\mathcal{E}_D(R) \cap \mathcal{J}_\emptyset(R)) = \sum_{U \in \mathcal{D}(S^+)} a_U(P), \quad (17)$$

$$\sum_{D \in \mathcal{J}_R(T) \setminus \mathcal{J}_\emptyset(R)} \#(\mathcal{E}_D(R) \setminus \mathcal{J}_\emptyset(R)) = \sum_{\mu = 1}^{\#(M \setminus T)} (-1)^{\mu - 1} \sum_{N \subseteq \mathcal{M} \setminus T} a_{T \setminus N}(P^+ \setminus \downarrow_{P^+} N). \quad (18)$$

We start with the proof of $\mathcal{E}_D(R) \cap \mathcal{J}_\emptyset(R) = \mathcal{E}_{\tau_T(D)}(P)$. Let $E \in \mathcal{E}_D(R) \cap \mathcal{J}_\emptyset(R)$. $E \in \mathcal{E}_D(R)$ and $E \cap Y = \emptyset$ means $E \in \mathcal{D}(P)$, and $E \subseteq D$ additionally yields $E = E \setminus T \subseteq D \setminus T = \tau_T(D)$. According to Lemma [1], $\tau_T(D)$ is a down-set of $P$, and $E \in \mathcal{E}_{\tau_T(D)}(P)$ follows. On the other hand, $E' \in \mathcal{E}_{\tau_T(D)}(P)$ yields $E' \in \mathcal{E}_D(R)$ with $E' \subseteq \tau_T(D) = D \setminus T \subseteq D$ and $E' \cap Y = \emptyset$, hence $E' \in \mathcal{E}_D(R) \cap \mathcal{J}_\emptyset(R)$.

Now Lemma [4] yields

$$\sum_{D \in \mathcal{J}_R(T)} \#(\mathcal{E}_D(R) \cap \mathcal{J}_\emptyset(R)) = \sum_{D \in \mathcal{J}_R(T)} \#\mathcal{E}_{\tau_T(D)}(P)$$

$$\sum_{U \in \mathcal{D}(S^+)} \sum_{D' \in \mathcal{J}_R(T)} \#\mathcal{E}_{D'}(P) = \sum_{U \in \mathcal{D}(S^+)} a_U(P).$$

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For \( T = \emptyset \), the right side of the formula is zero, and also the left side is zero due to \( D \in J_0(R) \Rightarrow \mathcal{E}_D(R) \subseteq J_0(R) \).

For \( T \neq \emptyset \), we have \( M \cap T \neq \emptyset \). Let \( \emptyset \neq N \subseteq M \cap T \). Then, for \( D \in J_T(R) \),

\[
\mathcal{E}_N^D(R) \equiv \{ E \in \mathcal{E}_D(R) \mid N \subseteq E \} = \{ E \in D(R) \mid N \subseteq E \subseteq D \} = \{ E \in D(R \cup_R N) \mid E \subseteq D \cup_R N \} = \mathcal{E}_D(R \cup_R N).
\]

Now \([11]\) yields with \( R' \equiv R \cup_R N \),

\[
\sum_{D \in J_T(R)} \# E_N^D(R) = \sum_{D' \in J_T(N \cap R')} \# \mathcal{E}_{D'}(R') = a_{T \cap N}(R').
\]

For every \( D \in J_T(R) \), the set \( \mathcal{E}_D(R) \setminus J_0(R) \) contains exactly the sets \( E \in \mathcal{E}_D(R) \) with \( m \in E \) for an \( m \in M \cap T \). Therefore,

\[
\sum_{D \in J_T(R)} \# (\mathcal{E}_D(R) \setminus J_0(R)) = \sum_{D' \in J_T(N \cap R')} \# \mathcal{E}_{D'}(R') \equiv \sum_{\mu=1}^{\#(M \cap T)} (-1)^{\mu-1} \sum_{N \subseteq M \cap T \atop \# N = \mu} \# \mathcal{E}_N^D(R) = \sum_{\mu=1}^{\#(M \cap T)} (-1)^{\mu-1} \sum_{N \subseteq M \cap T \atop \# N = \mu} a_{T \cap N}(R'),
\]

and \([18]\) follows, because Lemma \([2]\) delivers \( a_{T \cap N}(R') = a_{T \cap N}(P \setminus \downarrow_{P^+} N) \) for \( T \in \mathcal{D}(S^+) \).

\([16]\): For \( T = \emptyset \), the double sum on the right of \([15]\) is zero, hence

\[
a_0(R) \equiv \sum_{U \in \mathcal{D}(S^-)} a_U(P) = h(P),
\]

because \( P \) is a g.v.s. of \( P \setminus B^- \) and \( S^- \).

In Section \([4]\) we frequently work with structures as in the following corollary:

**Corollary 1.** Assume

\[
Q \text{ has a minimum point } \bot,
\]

\[
\sigma : \mathcal{D}(S^+) \to \mathcal{D}(S^-) \text{ is an isomorphism}.
\]

Then, setting \( P^+ \equiv R|_{X \setminus B^+} \) again,

\[
h(R) = h(P^+ \setminus \downarrow_{P^+} \bot) + \sum_{D \in \mathcal{D}(Q) \setminus \mathcal{D}(S^+)} a_D(R)
\]

\[
+ \sum_{T \in \mathcal{D}(S^-)} \left( \# \downarrow_{\mathcal{D}(S^-) T} \right) \cdot a_T(P).
\]
Proof. Observing that the double-sum on the right of (15) is zero for \( T = \emptyset \), we get

\[
\sum_{T \in \mathcal{D}(S^+)} a_T(R) = \sum_{T \in \mathcal{D}(S^+)} \left( \sum_{U \in \mathcal{D}(S^-)} a_U(P) + \sum_{\emptyset \neq N \subseteq \{ \bot \} \cap \mathcal{T}} a_{T \setminus N}(P^+ \downarrow_{P^+} N) \right)
\]

\[
= \sum_{T \in \mathcal{D}(S^-)} \left( \# \mathcal{D}(S^-) - T \right) \cdot a_T(P) + \sum_{T \in \mathcal{D}(S^+ \setminus \{ \emptyset \})} a_{T \setminus \{ \bot \}}(P^+ \downarrow_{P^+} \bot).
\]

Because \( P^+ \downarrow_{P^+} \bot \) is a g.v.s. of \( P \downarrow_{P^+} \bot \) and \( S^+ \setminus \{ \bot \} \), the right sum is

\[
\sum_{T \in \mathcal{D}(S^+ \setminus \{ \emptyset \})} a_{T \setminus \{ \bot \}}(P^+ \downarrow_{P^+} \bot) = \sum_{T \in \mathcal{D}(S^+ \setminus \{ \bot \})} a_T(P^+ \downarrow_{P^+} \bot)
\]

\[
= h(P^+ \downarrow_{P^+} \bot).
\]

However, Formula (19) is of limited value, because for the calculation of the \( a \)-coefficients, we still need Formula (15) from Theorem 2.

We have mentioned after Definition 2 that we can run the recursive approach always with \( B^+ \) being any non-empty down-set of \( Q \) and \( B^- \) being the up-set of \( P \) created by the points of \( P \) covered by points of \( B^- \) in \( R \). For these choices, \( B^- \) may be empty. (An example is the poset \( \bar{R} \) in Figure 4, take the two large Lambdas as \( P \) and the remaining part as \( Q \), and define \( B^- \) as the singleton containing the minimum point of \( Q \) only.) In this case, \( S^- \) is the empty poset, \( \sigma(T) = \emptyset \) for all \( T \in \mathcal{D}(S^+) = \mathcal{P}(B^+) \), and the isomorphism \( \tau_T \) in (8) reduces to

\[
\tau_T : \mathcal{J}_T(R) \rightarrow \mathcal{J}_0(P) \cong \mathcal{D}(P),
\]

\[
D \mapsto D \setminus T.
\]

Indeed, if no point of \( P \) is covered by a point of \( B^+ \), then \( R|_{X \cup B^+} = P + S^+ \), hence \( \mathcal{J}_T(R) \simeq \mathcal{D}(P) \times \{ T \} \simeq \mathcal{D}(P) \) for every \( T \in \mathcal{D}(S^+) = \mathcal{P}(B^+) \).

\section{Application}

According to Theorem 2, we need for the calculation of the coefficients \( a_T(R) \) with \( T \in \mathcal{D}(S^+) \) the values of \( a_U(P) \) with \( U \in \mathcal{D}(S^-) \) and the coefficients \( a_{T \setminus N}(P^+ \downarrow_{P^+} N) \) for all non-empty sets \( N \) of minimal points of \( S^+ \). Additionally, we need the coefficients \( a_D(R) \) with \( D \in \mathcal{D}(Q) \setminus \mathcal{D}(S^+) \) for the final calculation of \( h(R) \). For a given poset \( R \), the choice of \( P, Q, \) and \( S^+ \) shifts the balance between these three types of calculation:

- A large sub-poset \( S^+ \) of \( Q \) reduces the number of down-sets \( D \in \mathcal{D}(Q) \setminus \mathcal{D}(S^+) \) for which \( a_D(R) \) has to be calculated separately (even to zero for \( S^+ = Q \)), but makes the calculation of the coefficients \( a_{T \setminus N}(P^+ \downarrow_{P^+} N) \) in the second sum in (15) more demanding.
• A small sub-poset $P$ of $R$ makes the determination of the coefficients $a_U(P)$ with $U \in \mathcal{D}(S^-)$ easier in the first sum in (15), but puts a larger burden on at least one of the two other types of calculation.

• The number of minimal points of $S^+$ affects exponentially the number of terms in the second sum in (15).

Our approach fits thus best to posets $R$ with the following properties:

• $P$ and $P^+$ have a simple structure or are closely related to $R$;

• $a_D(R)$ can easily be calculated for all $D \in \mathcal{D}(Q) \setminus \mathcal{D}(S^+)$;

• the number of minimal points of $S^+$ is small.

For such posets, the calculation of $h(R)$ is possible with ordinary table calculation, as we will see in this section. We work with posets $R = W \times C_k$ for different posets $W$ in Section 4.1 (including $W = C_3 \times C_3$), and with $R = \mathcal{H}(C_2, C_k) \simeq \mathcal{H}(C_{k-1}, C_3)$ in Section 4.2. The posets $S^+$ and $S^-$ are always isomorphic, and the mapping $\sigma$ fulfilling (6) is induced by the respective isomorphism $\iota : S^+ \to S^-$:

$$\forall T \in \mathcal{D}(S^+) : \quad \sigma(T) \equiv \iota[T].$$

In Section 4.1, we have $S^+ = Q$ in all cases, but in Section 4.2, we will see that a different choice of $S^+$ can even be beneficial.

For almost all posets $R$ in this section, we have $X \subseteq \downarrow R Y$. (The exception is the poset in the lower part of Figure 4). For these, equation (5) delivers $a_Q(R) = \# \mathcal{D}(R) = \# \mathcal{H}(R, C_2)$, and as a by-product, we get the number of surjective homomorphisms $R \to C_3$ via

$$h(R) = 3 \cdot a_Q(R) + 3. \quad (20)$$

Due to the general isomorphism $\mathcal{H}(P_1, \mathcal{H}(P_2, P_3)) \simeq \mathcal{H}(P_1 \times P_2, P_3)$ and $\mathcal{H}(C_2, C_2) \simeq C_3$, we have $h(R) = \# \mathcal{D}(R \times C_2)$ for every poset $R$. In particular, the recursive approach gives raise to new ways to calculate Dedekind numbers, because the $(k+1)^{\text{th}}$ Dedekind number is $\# \mathcal{D}(C_k^{k+1}) = h(C_k^k)$.

### 4.1 $R = W \times C_k$

For every poset $W$, the product $W \times C_k$ fits into the frame of Definition 2 via

- $P \equiv W \times C_{k-1}$
- $Q \equiv W \times \{k\}$,
- $S^- \equiv W \times \{k-1\}$,
- $S^+ \equiv W \times \{k\}$,
- $\iota(w, k) \equiv (w, k-1)$
- $P^+ = W \times C_k$.

In order to avoid unnecessary formalism, we identify $S^-$ and $S^+$ with $W$. Due to (6), we always have $a_W(W \times C_k) = \# \mathcal{D}(W \times C_k)$.

In the following sections, we determine $h(C_n \times C_k)$, $h(\Lambda \times C_k)$, $h(\diamond \times C_k)$, and $h(C_3 \times C_3 \times C_k)$, where $\Lambda \equiv A_2 \oplus A_1$ is the poset with $\Lambda$-shaped diagram and $\diamond \equiv C_2 \times C_2$ is the diamond. As a by-product of the calculation of $h(C_3 \times C_3 \times C_k)$, we get $h(W \times C_k)$ for eight additional posets $W$. 

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4.1.1 $C_n \times C_k$

In order to simplify notation, we write $a_j(C_n \times C_k)$ instead of $a_j(C_n \times C_k)$ for all $j \in \mathbb{N}_0$. For $k = 1$, $a_j(C_n) = j + 1$ and $h(C_n) = \frac{(n+1)(n+2)}{2}$ are trivial.

**Theorem 3.** For $k \geq 2$,

$$a_0(C_n \times C_k) = h(C_n \times C_{k-1}),$$

\[\forall j \in \mathbb{N} : a_j(C_n \times C_k) = a_{j-1}(C_{n-1} \times C_k) + \sum_{i=j}^{n} a_i(C_n \times C_{k-1}),\]

hence $h(C_n \times C_k) = h(C_{n-1} \times C_k) + \sum_{i=0}^{n} (i+1) \cdot a_i(C_n \times C_{k-1})$.

**Proof.** The first equation is due to (9). Let $j \in \mathbb{N}$. The symbols in the left sum of (15) become

$T = j,$

$\{ U \in \mathcal{D}(S^-) \mid i[T] \subseteq U \} = \{ i \mid j \leq i \leq n \}.$

The right sum reduces to the single term $a_{T\\setminus\downarrow \downarrow} (P^+ \setminus \downarrow P^- \downarrow)$ with

$\downarrow = 1,$

$T \setminus \{ \downarrow \} \simeq C_{j-1},$

$P^+ \setminus \downarrow P^- \downarrow \simeq C_{n-1} \times C_k,$

and the formula for $a_j(C_n \times C_k)$ with $j \in \mathbb{N}$ follows. The last formula is (19).

The coefficients $h(C_n \times C_k)$ and $a_j(C_n \times C_k)$ are shown in Figure 7 in the Appendix for $n \in \mathbb{N}$ and $k \in \mathbb{N}$.

4.1.2 $\Lambda \times C_k$ and $\diamondsuit \times C_k$

The posets $\Lambda \times C_3$ and $\diamondsuit \times C_3$ are shown in Figure 1. We start with $\Lambda$. Even if one of our results is a closed formula for $h(\Lambda \times C_k)$, we remain interested in the coefficients $a_T(\Lambda \times C_k)$ and derive formulas for them, because we need them for $\diamondsuit$ and $C_3 \times C_3$.

We denote with $\top$ the maximum point of $\Lambda$ and with $\ell, r$ the minimal points. In $\mathcal{D}(\Lambda \times C_k)$, the set of down-sets $D$ with $(\top, j) \in D$ and $(\top, j + 1) \notin D$ is isomorphic to $C_{k+1-j} \times C_{k+1-j}$. All together, $\mathcal{D}(\Lambda \times C_k)$ looks like a step pyramid, as shown in Figure 2 for $\Lambda \times C_3$. We denote with $f$ the “floor” of the pyramid, starting with $f = 0$ for the ground floor and ending with $f = k$ for the “antenna” $\Lambda \times C_k$ on top.

The set $\mathcal{J}_\Lambda(\Lambda \times C_k)$ contains the antenna only, and $\mathcal{J}_{\{\ell, r\}}(\Lambda \times C_k)$ consists of the points of the “backbone” of the pyramid marked in the second drawing in the top row of Figure 2.

$$\mathcal{J}_{\{\ell, r\}}(\Lambda \times C_k) = \{ D_f \mid f \in k-1 \},$$

with $D_f \equiv (\Lambda \times C_k) \setminus \{ (\top, j) \mid f + 1 \leq j \leq k \}$
Figure 1: The posets $\Lambda \times C_3$ and $\diamondsuit \times C_3$.

Figure 2: Top row, from left to right: $\mathcal{D}(\Lambda \times C_3)$, $\mathcal{J}_{\{\ell,r\}}(\Lambda \times C_3)$, $\mathcal{J}_{\{\ell\}}(\Lambda \times C_3)$, and $\mathcal{J}_\emptyset(\Lambda \times C_3) \simeq \mathcal{D}(\Lambda \times C_2)$. Bottom row: illustration of the isomorphisms $\beta_{\{\ell,r\}}$ and $\beta_{\{\ell\}}$ from Lemma 1; explanations in text.
Lemma 1, upper part of Figure 2). Finally, due to the isomorphisms \( \beta \) of the pyramid \( a \times 2 \) times to as shown in the lower part of Figure 2.

The missing numbers \( a \) (16), and summing up \( a \) of the points belonging to the floors 0 ... \( f \) without the backbone, and \( J_0(\Lambda \times C_k) \approx \mathcal{D}(\Lambda \times C_{k-1}) \) contains the inner points and the front points of the pyramid, as shown in Figure 2.

Finally, it is easily seen that the contribution of the \( f \)th floor is \( (k + 1 - f)^2 \), hence

\[
a(\Lambda \times C_k) = \sum_{f=0}^{k} (k + 1 - f)^2 = \frac{(k+1)(k+2)(2k+3)}{6}.
\]

For the calculation of \( a(\ell, r) \), we realize that \( E_D(\Lambda \times C_k) \) consists of the points belonging to the floors 0 ... \( f \). Each floor contributes thus \( (k - f) \)-times to \( a(\ell, r) \), hence

\[
a(\ell, r) = \sum_{f=0}^{k-1} (k - f)(k + 1 - f)^2 = \frac{k(k+1)(k+2)(3k+5)}{12}.
\]

Finally, it is easily seen that the contribution of the \( f \)th floor to the number \( a(\Lambda \times C_k) \) is

\[
F_f(k) = \sum_{i=1}^{k-f} \sum_{j=1}^{k-f} (i + \phi)(j + \phi)
\]

\[
eq \left( f(f+2) + 3(k+2)^2 \right) \cdot \frac{(f+1)(k+1 - f)^2}{12},
\]

and summing up \( F_f(k) \) from 0 to \( k \) delivers \( a(\Lambda \times C_k) \). In particular, due to [16],

\[
h(\Lambda \times C_k) = \sum_{f=0}^{k+1} F_f(k+1).
\]

The missing numbers \( a(\ell) \) of \( (\Lambda \times C_k) \) are given by

\[
\frac{1}{2} \left( h(\Lambda \times C_k) - a(\Lambda \times C_k) - a(\ell, r) \right).
\]
We come to $\diamondsuit = C_2 \times C_2$. We denote with $\top$ and $\bot$ the maximum and minimum point of $\diamondsuit$ and with $\ell, r$ the remaining points. Due to

$$(\diamondsuit \times C_k) \downarrow \diamondsuit \times C_k \downarrow \simeq \Lambda \times C_k,$$

Theorem 2 yields

$$a_\emptyset(\diamondsuit \times C_k) = h(\diamondsuit \times C_{k-1}),$$
$$a_{\bot}(\diamondsuit \times C_k) = h(\diamondsuit \times C_{k-1}) - a_\emptyset(\diamondsuit \times C_{k-1}) + h(\Lambda \times C_k),$$
$$a_{\bot, \ell}(\diamondsuit \times C_k) = a_{\bot, \ell}(\diamondsuit \times C_{k-1}) + a_{\bot, \ell, r}(\diamondsuit \times C_{k-1}) + a_\emptyset(\diamondsuit \times C_{k-1}) + a_\ell(\Lambda \times C_k),$$
$$a_{\bot, r}(\diamondsuit \times C_k) = a_{\bot, r}(\diamondsuit \times C_{k-1}),$$
$$a_{\bot, \ell, r}(\diamondsuit \times C_k) = a_{\bot, \ell, r}(\diamondsuit \times C_{k-1}) + a_\emptyset(\diamondsuit \times C_{k-1}) + a_{\ell, r}(\Lambda \times C_k),$$
$$a_\emptyset(\diamondsuit \times C_k) = a_\emptyset(\diamondsuit \times C_{k-1}) + a_\emptyset(\Lambda \times C_k).$$

For $k \in 5$, the values of $h(\Lambda \times C_k), a_T(\Lambda \times C_k), h(\diamondsuit \times C_k)$, and $a_T(\diamondsuit \times C_k)$ are shown in Figure 8 in the Appendix. We have $a_\Lambda(\Lambda \times C_k) = \#D(\Lambda \times C_k)$ and $a_\emptyset(\diamondsuit \times C_k) = \#D(\diamondsuit \times C_k)$. 

4.1.3 $C_3 \times C_3 \times C_k$

Due to the direct calculation of the coefficients $a_T(\Lambda \times C_k)$, only a single recursive step was required in the calculation of the coefficients $a_T(\diamondsuit \times C_k)$. The case $C_3 \times C_3 \times C_k$ is more demanding. We have to step recursively through the posets in the diagram in Figure 3. An arrow upwards from a poset $V$ to a poset $W$ in this transitive diagram indicates that the $a$-coefficients of $V \times C_k$ are required for the calculation of the $a$-coefficients of $W \times C_k$. In the figure, also the values of $h(W \times C_k)$ and $\#D(W \times C_k)$ are shown for $k \in 5$. For the chains and $\Lambda, \diamondsuit$, see Figures 7 and 8 in the Appendix.

At the end of the introduction, we mentioned machine learning and ensemble based systems as application of monotone ternary functions. Here, the surjective homomorphisms from $C_3^k$ to $C_3$ are of particular interest. We calculate their number with formula (20). Additional regularity is introduced by demanding that a summary rule $s$ has to respect an unanimous decision of the experts, i.e., $s(i, \ldots, i) = i$ for all $i \in 3$. (Because we work with monotone summary rules only, this postulate is equivalent to “$\min r \leq s(r) \leq \max r$ for all $r \in C_3^k$.”) The numbers of these homomorphisms are shown in the following table for $k = 1, 2, 3$. 

| $k$ | $h(C_3^k)$ | surjective | $\xi(i, \ldots, i) = i$ |
|-----|-------------|------------|-------------------------|
| 1   | 10          | 1          | 1                       |
| 2   | 175         | 118        | 64                      |
| 3   | 211250      | 208313     | 116211                  |

For $k = 3$, the number of the homomorphisms $\xi \in H(C_3^3, C_3)$ with $\xi(i, i, i) = i$ for all $i \in 3$ has been calculated as follows. With $R$ being the poset shown in the lower part of Figure 3, $h(R) = 46540$ is the number of homomorphisms $\xi \in H(C_3^3, C_3)$ with $\xi(1, 1, 1) = \xi(2, 2, 2) = 1$ and $\xi(3, 3, 3) = 3$. (The number has been calculated by applying the recursive method on the dual of $R$.) From
Figure 3: The posets $W$ required for the calculation of $h(C_3 \times C_3 \times C_k)$, and the numbers $h(W \times C_k)$ (top table) and $\#D(W \times C_k)$ (bottom table) for $k \in \{5\}$. For the chains, $\Lambda$, and $\diamondsuit$, see Figures 7 and 8 in the Appendix.
Figure 4: The poset $C_3^3$ and the poset $R$ required for the enumeration of the homomorphisms $\xi \in \mathcal{H}(C_3^3, C_3)$ with $\xi(i, \ldots, i) = i$ for all $i \in \mathbb{3}$.

these, $489 = \#\mathcal{D}(R) = \#\mathcal{H}(R, C_2)$ (calculated via (4)) have 2 not in their image; therefore, we have 46051 surjective homomorphisms from $C_3^3$ to $C_3$ with $\xi(1,1,1) = \xi(2,2,2) = 1$ and $\xi(3,3,3) = 3$. This is also the number of surjective homomorphisms from $C_3^3$ to $C_3$ with $\xi(1,1,1) = 1$ and $\xi(2,2,2) = \xi(3,3,3) = 3$, and 116211 results as number of homomorphisms $\xi \in \mathcal{H}(C_3^3, C_3)$ with $\xi(i, i, i) = i$ for all $i \in \mathbb{3}$.

4.2 \( R = \mathcal{H}(C_2, C_k) \simeq \mathcal{H}(C_{k-1}, C_3) \)

As pointed out in the introduction, $h(C_3^k)$ is the number of the different ways how ternary rankings of an object by $k$ experts can be summarized by monotone summary rules in ensemble based systems. For several important summary rules like the different versions of majority voting [13], the resulting summary does not depend on who of the experts gave which rank; here, the summary rule has to be a symmetric function. In this case, we can order the rankings of the experts in non-decreasing order which exchanges the domain $C_k^3$ of the summary rules against the poset $\mathcal{H}(C_k, C_3)$. We are thus dealing with the homomorphisms contained in $\mathcal{H}(\mathcal{H}(C_2, C_k), C_3)$. Because of

$$h(\mathcal{H}(C_{k-1}, C_3)) \equiv \mathcal{H}(C_2, C_k),$$ \hspace{1cm} (21)

the enumeration of the symmetric summary rules is equivalent to the enumeration of the homomorphisms contained in $\mathcal{H}(\mathcal{H}(C_2, C_k), C_3)$. In this section we prove

$$h(\mathcal{H}(C_2, C_k)) = \sum_{i=0}^{k} \left[ \binom{k+i}{k} - \binom{k+i}{k+1} \right] 2^{k-i}. \hspace{1cm} (22)$$

It looks like that this number is simply $\binom{2k+1}{k+1}$, but we did not make an effort to prove this equality.
Figure 5: The posets $\mathcal{H}(C_2, C_4)$ and $\mathcal{H}(C_2, C_5)$ together with their down-set lattices and the respective sets $\mathcal{J}_j(4)$ and $\mathcal{J}_j(5)$. 
4.2.1 Recursion

We represent a homomorphism $\xi \in \mathcal{H}(C_2, C_k)$ by the pair $(\xi(1), \xi(2))$, as shown in Figure 5. For $k \geq 2$, the set $\mathcal{H}(C_2, C_k)$ fits in the frame of Definition 2 in the following way:

$$
P \equiv \mathcal{H}(C_2, C_{k-1}),
Q \equiv C_k \times \{k\},
S^- \equiv C_{k-1} \times \{k-1\},
S^+ \equiv C_{k-1} \times \{k\},
\ell(j, k) \equiv (j, k-1) \text{ for all } j \leq k-1,
P^+ = \mathcal{H}(C_2, C_k) \setminus \{(k, k)\}.
$$

$Q$ is thus the top-diagonal in the Hasse diagram of $\mathcal{H}(C_2, C_k)$. Differently from Section 4.1 we have $S^+ \neq Q$ and $P^+ \neq R$ now.

Let $k \geq 2$ be fixed. In order to unburden the notation, we use short-terms also in this section:

$$
\mathcal{J}_j(k) \equiv \mathcal{J}_{C_j \times \{k\}}(\mathcal{H}(C_2, C_k)),
$$

and

$$
a_j(k) \equiv a_{C_j \times \{k\}}(\mathcal{H}(C_2, C_k))
$$

for every $j \in \mathbb{N}$, hence

$$
\mathcal{J}_z(k) = \{D \in \mathcal{D}(\mathcal{H}(C_2, C_k)) \mid (1, k) \notin D\},
\forall j \leq k-1 : \mathcal{J}_j(k) = \{D \in \mathcal{D}(\mathcal{H}(C_2, C_k)) \mid (j, k) \in D, (j + 1, k) \notin D\},
\mathcal{J}_k(k) = \{\mathcal{H}(C_2, C_k)\}.
$$

Figure 5 shows $\mathcal{H}(C_2, C_k)$ and $\mathcal{D}(\mathcal{H}(C_2, C_k))$ for $k = 4$ and $k = 5$ together with the respective sets $\mathcal{J}_j(4)$ and $\mathcal{J}_j(5)$. The isomorphisms $\tau_T$ from Lemma 4 are clearly visible. For the illustration of the isomorphism $\beta_N$ in the lemma, observe that with $N \equiv \{(1, k)\}$, the set $\downarrow \mathcal{H}(C_2, C_k)N$ is the bottom diagonal $(1, 1), \ldots, (1, k) \in \mathcal{H}(C_2, C_k)$, hence $\mathcal{H}(C_2, C_k) \setminus \downarrow \mathcal{H}(C_2, C_k)N \simeq \mathcal{H}(C_2, C_{k-1})$, and formula (10) becomes

$$
\bigcup_{j=1}^{k} \mathcal{J}_j(k) \simeq \mathcal{D}(\mathcal{H}(C_2, C_{k-1})),
$$

(23)

as confirmed by Figure 5 for $k = 5$.

Theorem 4. We have

$$
\mathcal{J}_0(k) \simeq \mathcal{D}(\mathcal{H}(C_2, C_{k-1})) \simeq \mathcal{D}(\mathcal{H}(C_2, C_k)) \setminus \mathcal{J}_0(k),
$$

(24)

hence

$$
\# \mathcal{D}(\mathcal{H}(C_2, C_k)) = 2^k.
$$

(25)

Furthermore, $a_1(0) = 1, a_1(1) = 2$, and for every $k \geq 2$,

$$
a_0(k) = a_1(k) = h(\mathcal{H}(C_2, C_{k-1})),
\forall j \leq k-1 : a_j(k) = \sum_{i=j-1}^{k-1} a_i(k-1),
a_k(k) = 2^k.
$$

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Proof. The first isomorphism in (24) is (9), and the second one is (23). With this result, (25) follows due to \( \#D(H(C_2, C_1)) = 2 \), and (25) yields \( a_k(k) = 2^k \) because of \( J_k(k) = H(C_2, C_k) \).

\( a_0(1) = 1, a_1(1) = 2 \) is easily seen. For \( k \geq 2 \), \( a_0(k) = h(H(C_2, C_{k-1})) \) is due to (10). Let \( j \in k-1 \). The right sum in (15) reduces to the term \( a_{T\setminus\{\perp\}}(P^+ \downarrow_{P^+} \perp) \) with

\[
\perp = (1, k), \\
T = C_j \times \{k\}, \\
T \setminus \{\perp\} \simeq C_{j-1} \times \{k-1\}, \\
P^+ \downarrow_{P^+} \perp = P^+ \setminus \{(1) \times C_k\} \\
\simeq H(C_2, C_{k-1}) \setminus \{(k-1, k-1)\},
\]

and because of \( j-1 < k-1 \), Lemma 2 yields

\[ a_{T\setminus\{\perp\}}(P^+ \downarrow_{P^+} \perp) = a_{j-1}(k-1). \]

All together, (15) delivers \( a_k(j) = \sum_{i=j-1}^{k-1} a_{k-1}(i) \); in particular, \( a_1(k) = h(H(C_2, C_{k-1})) \).

The numbers \( h(H(C_2, C_k)) \) and \( a_j(k) \) for \( k \in 10 \) are shown in Figure 9 in the Appendix. We have \( h(H(C_2, C_k)) = \binom{2k+1}{k} \) for all \( k \in 10 \).

4.2.2 A polynomial approach

With our choice of \( S^+ \), the coefficient \( a_k(k) \) was taken out of the recursion and turned out to be \( 2^k \). In the following definition, we use it to introduce polynomials:

**Definition 3.** For \( k \in \mathbb{N} \), \( j \in k_0 \), we define polynomials \( q_j^{(k)}(x) \) by setting \( q_0^{(1)}(x) \equiv 1 \), \( q_1^{(1)}(x) \equiv x \), and, for every \( k \geq 2 \),

\[
q_0^{(k)}(x) \equiv q_1^{(k)}(x), \\
j \in k-1: \quad q_j^{(k)}(x) \equiv \sum_{i=j+1}^{k-1} q_i^{(k-1)}(x), \\
q_k^{(k)}(x) \equiv x^k.
\]

For \( j \in k-1_0 \), the degree of \( q_j^{(k)}(x) \) is \( k-1 \), and the comparison with Theorem 4 shows \( q_j^{(k+1)}(2) = h(H(C_2, C_k)) \) for all \( k \in \mathbb{N} \).

**Lemma 3.** Let \( G \) be the directed graph shown in Figure 7 with vertex set

\[
V \equiv \{(k, j) \mid k \in \mathbb{N}_0, j \in k_0\}.
\]
For every \((k, j) \in V\), \(i \in \mathbb{N}_0\), let \(\pi_j^{(k)}(i)\) denote the number of paths in \(G\) starting in \((i, i)\) and ending in \((k, j)\) (with \(\pi_k^{(k)}(k) \equiv 1\)). Then, for all \(k \in \mathbb{N}_0\),

\[
q_j^{(k)}(x) = \sum_{i=0}^{k-1} \pi_j^{(k)}(i) \cdot x^i \quad \text{for all } j \in \mathbb{N}_0; \quad (26)
\]

and \(q_k^{(k)}(x) = \sum_{i=0}^{k-1} \pi_k^{(k)}(i) \cdot x^i\).

Proof. The equation for \(q_k^{(k)}(x)\) follows because of \(\pi_k^{(k)}(i) = \delta_{ki}\) (Kronecker Delta).

Equality (26) holds for \(k = 0\) due to \(q_0^{(1)}(x) = 1 = \pi_0^{(1)}(0) \cdot x^0\). Assume that (26) has been proven for \(k-1 \in \mathbb{N}_0\), and let \(j \in \mathbb{N}_0\). According to the definition of \(q_j^{(k)}(x)\) and the induction hypothesis, the coefficient of \(x^i\) in \(q_j^{(k)}(x)\) is

\[
\sum_{\ell=j-1}^{k-1} \pi_{\ell}^{(k-1)}(i). \quad (27)
\]

Every path from \((i, i)\) to \((k, j)\) has to run over one of the vertices \((k-1, \ell)\) with \(j-1 \leq \ell \leq k\). For \(j-1 \leq \ell \leq k\), let \(\mathcal{P}(\ell)\) be the set of paths in \(G\) from \((i, i)\) to \((k-1, \ell)\). Each path in \(\mathcal{P}(\ell)\) can be extended to a path from \((i, i)\) to \((k, j)\) by adding a step diagonally right upwards to \((k, \ell + 1)\) followed by \(\ell + 1 - j\) steps downwards. Defining for all \(j-1 \leq \ell \leq k\),

\[
\mathcal{P}'(\ell) \equiv \{ p(k, \ell + 1)(k, \ell) \cdots (k, j) \mid p \in \mathcal{P}(\ell) \},
\]

the sets \(\mathcal{P}'(\ell)\) are pairwise disjoint with \(\#\mathcal{P}'(\ell) = \#\mathcal{P}(\ell) = \pi_{\ell}^{(k-1)}(i)\). The set \(\bigcup_{\ell=j-1}^{k-1} \mathcal{P}'(\ell)\) contains the paths from \((i, i)\) to \((k, j)\) running over \((k, j+1)\), and the set \(\mathcal{P}'(j-1)\) contains the paths from \((i, i)\) to \((k, j)\) running over \((k-1, j-1)\), and (27) is the number of paths from \((i, i)\) to \((k, j)\).

The polynomial \(q_0^{(k)}(x)\) remains. Because of \(k > 0\), every path from \((i, i)\) to \((k, 0)\) must run over \((k, 1)\). The number \(\pi_0^{(k)}(i) = \pi_1^{(k)}(i)\) is thus the number of paths in \(G\) from \((i, i)\) to \((k, 0)\).
Equation (22) is a direct consequence of the following theorem:

**Theorem 5.** For all $k \in \mathbb{N}_0$,

$$q_0^{(k+1)}(x) = \sum_{i=0}^{k} \left[ \binom{k+i}{k} - \binom{k+i}{k+1} \right] x^{k-i}. \quad (28)$$

**Proof.** Let $k \in \mathbb{N}_0$. According to Lemma 3, we have

$$q_0^{(k+1)}(x) = \sum_{i=0}^{k} \pi_0^{(k+1)}(i) \cdot x^i,$$

where $\pi_0^{(k+1)}(i)$ is the number of paths in $G$ from $(i, i)$ to $(k+1, 0)$. It is also the number of paths in $G$ from $(i+1, i)$ to $(k+1, 0)$. Each of these paths we can step backwards from $(k+1, 0)$ to $(i+1, i)$. Such a reversed path can be described by an unique sequence of $k$ letters $U$ (for steps upwards in $G$) and $k-i$ letters $D$ (for diagonal steps downwards) in which the $\ell^{th}$ occurrence of $D$ is preceded by at least $\ell$ occurrences of $U$. Let $u_1, \ldots, u_k$ be the indexes of the $U$-letters in such a sequence, and $d_1, \ldots, d_{k-i}$ the indexes of the $D$-letters. Writing $u_1, \ldots, u_k$ into the upper row of a Ferrers diagram of type $[k, k-i]$ and $d_1, \ldots, d_{k-i}$ into the lower one yields a standard Young-tableau of type $[k, k-i]$. This mapping between reversed $(i+1, i)$-$(k+1, 0)$-paths in $G$ and standard Young-tableaux of type $[k, k-i]$ is bijective, and application of the hook-length-formula of Frame, Robinson, and Thrall yields

$$\pi_0^{(k+1)}(i) = \frac{(2k-i)!}{(k+1)! \cdot (k-i)!} = \binom{2k-i}{k} - \binom{2k-i}{k+1}.$$

The polynomial coefficients $\binom{k+i}{k} - \binom{k+i}{k+1}$ are shown in Figure 10 in the Appendix for $k \in \mathbb{N}_0$ and $i \in \mathbb{N}_0$.

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5 Appendix

| $n$ | $N_{C_n}$ | 0 | 1 |
|-----|-----------|---|---|
| 1   | 14        | 1 | 2 |
| 2   | 84        | 1 | 2 |
| 3   | 330       | 1 | 2 |
| 4   | 1001      | 1 | 2 |
| 5   | 2548      | 1 | 2 |
| 6   | 5712      | 1 | 2 |
| 7   | 11628     | 1 | 2 |
| 8   | 21945     | 1 | 2 |
| 9   | 38962     | 1 | 2 |
| 10  | 65780     | 1 | 2 |

| $k$ | $N_{C_k}$ | 0 | 1 | 2 | 3 |
|-----|-----------|---|---|---|---|
| 1   | 10        | 1 | 2 | 3 | 4 |
| 2   | 30        | 1 | 2 | 3 | 4 |
| 3   | 105       | 1 | 2 | 3 | 4 |
| 4   | 205       | 1 | 2 | 3 | 4 |
| 5   | 105       | 1 | 2 | 3 | 4 |

| $n$ | $N_{\Lambda \times C_n}$ | $\emptyset$ | $\{\ell\}$ | $\{r\}$ | $\{\ell, r\}$ | $\Lambda$ |
|-----|--------------------------|------------|-----------|--------|-------------|-------|
| 1   | 14                       | 1          | 2         | 4      | 5           | 6     |
| 2   | 84                       | 1          | 2         | 4      | 5           | 6     |
| 3   | 330                      | 1          | 2         | 4      | 5           | 6     |
| 4   | 1001                     | 1          | 2         | 4      | 5           | 6     |
| 5   | 2548                     | 1          | 2         | 4      | 5           | 6     |
| 6   | 5712                     | 1          | 2         | 4      | 5           | 6     |
| 7   | 11628                    | 1          | 2         | 4      | 5           | 6     |
| 8   | 21945                    | 1          | 2         | 4      | 5           | 6     |
| 9   | 38962                    | 1          | 2         | 4      | 5           | 6     |
| 10  | 65780                    | 1          | 2         | 4      | 5           | 6     |

| $n$ | $N_{\Lambda \times C_n}$ | $\emptyset$ | $\{\ell\}$ | $\{r\}$ | $\{\ell, r\}$ | $\Lambda$ |
|-----|--------------------------|------------|-----------|--------|-------------|-------|
| 1   | 10                       | 1          | 2         | 3      | 5           | 6     |
| 2   | 60                       | 1          | 2         | 3      | 5           | 6     |
| 3   | 105                      | 1          | 2         | 3      | 5           | 6     |
| 4   | 205                      | 1          | 2         | 3      | 5           | 6     |
| 5   | 105                      | 1          | 2         | 3      | 5           | 6     |
| 6   | 196                      | 1          | 2         | 3      | 5           | 6     |

| $n$ | $N_{\Omega \times C_n}$ | $\emptyset$ | $\{\ell\}$ | $\{r\}$ | $\{\ell, r\}$ | $\Omega$ |
|-----|--------------------------|------------|-----------|--------|-------------|-------|
| 1   | 14                       | 1          | 2         | 4      | 5           | 6     |
| 2   | 84                       | 1          | 2         | 4      | 5           | 6     |
| 3   | 330                      | 1          | 2         | 4      | 5           | 6     |
| 4   | 1001                     | 1          | 2         | 4      | 5           | 6     |
| 5   | 2548                     | 1          | 2         | 4      | 5           | 6     |
| 6   | 5712                     | 1          | 2         | 4      | 5           | 6     |
| 7   | 11628                    | 1          | 2         | 4      | 5           | 6     |
| 8   | 21945                    | 1          | 2         | 4      | 5           | 6     |
| 9   | 38962                    | 1          | 2         | 4      | 5           | 6     |
| 10  | 65780                    | 1          | 2         | 4      | 5           | 6     |

Figure 7: The coefficients $h(C_n \times C_k)$ and $a_j(C_n \times C_k)$ for $n \in \mathbb{N}_4$ and $k \in \mathbb{N}_5$.

| $n$ | $N_{\Omega \times C_n}$ | $\emptyset$ | $\{\ell\}$ | $\{r\}$ | $\{\ell, r\}$ | $\Omega$ |
|-----|--------------------------|------------|-----------|--------|-------------|-------|
| 1   | 14                       | 1          | 2         | 4      | 5           | 6     |
| 2   | 84                       | 1          | 2         | 4      | 5           | 6     |
| 3   | 330                      | 1          | 2         | 4      | 5           | 6     |
| 4   | 1001                     | 1          | 2         | 4      | 5           | 6     |
| 5   | 2548                     | 1          | 2         | 4      | 5           | 6     |
| 6   | 5712                     | 1          | 2         | 4      | 5           | 6     |
| 7   | 11628                    | 1          | 2         | 4      | 5           | 6     |
| 8   | 21945                    | 1          | 2         | 4      | 5           | 6     |
| 9   | 38962                    | 1          | 2         | 4      | 5           | 6     |
| 10  | 65780                    | 1          | 2         | 4      | 5           | 6     |

Figure 8: $h(\Lambda \times C_k)$, $a_T(\Lambda \times C_k)$, $h(\Diamond \times C_k)$, and $a_T(\Diamond \times C_k)$ for $k \in \mathbb{N}_{10}$. We have $a_T(\Lambda \times C_k) = \#D(\Lambda \times C_k)$ and $a_T(\Diamond \times C_k) = \#D(\Diamond \times C_k)$.
Figure 9: The numbers $b(H(C_2,C_k))$ and $a_j(k)$ for $k \in 10$.

Figure 10: The polynomial coefficients $\binom{k+i}{k} - \binom{k+i}{k+1}$ for $k \in 10$.

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