OSSERMAN MANIFOLDS AND THE WEYL-SCHOUTEN THEOREM FOR RANK-ONE SYMMETRIC SPACES

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Abstract. A Riemannian manifold is called Osserman (conformally Osserman, respectively), if the
eigenvalues of the Jacobi operator of its curvature tensor (Weyl tensor, respectively) are constant on
the unit tangent sphere at every point. Osserman Conjecture asserts that every Osserman manifold
is either flat or rank-one symmetric. We prove that both the Osserman Conjecture and its conformal
version, the Conformal Osserman Conjecture, are true, modulo a certain assumption on algebraic
curvature tensors in $\mathbb{R}^{16}$. As a consequence, we show that a Riemannian manifold having the same
Weyl tensor as a rank-one symmetric space, is conformally equivalent to it.

1. Introduction

The aim of this paper is twofold. Firstly, we consider Osserman and conformally Osserman manifolds
of dimension 16 (which is the only missing dimension in the proof of the Osserman Conjecture). We
show that both the “genuine” Osserman Conjecture and its conformal version can be reduced to a purely
algebraic question on algebraic curvature tensors in $\mathbb{R}^{16}$. Secondly, we obtain an analogue of the classical
Weyl-Schouten Theorem for rank-one symmetric spaces: a Riemannian manifold of dimension greater
than four having “the same” Weyl tensor as that of a rank-one symmetric space is locally conformally
equivalent to that space.

An algebraic curvature tensor $\mathcal{R}$ on a Euclidean space $\mathbb{R}^n$ is a $(3,1)$ tensor having the same symmetries
as the curvature tensor of a Riemannian manifold. For $X \in \mathbb{R}^n$, the Jacobi operator $\mathcal{R}_X : \mathbb{R}^n \to \mathbb{R}^n$ is
defined by $\mathcal{R}_X Y = \mathcal{R}(X,Y)X$. The Jacobi operator is symmetric and $\mathcal{R}_X X = 0$ for all $X \in \mathbb{R}^n$.

Definition 1. An algebraic curvature tensor $\mathcal{R}$ is called Osserman if the eigenvalues of the Jacobi
operator $\mathcal{R}_X$ do not depend on the choice of a unit vector $X \in \mathbb{R}^n$.

One of the algebraic curvature tensors naturally associated to a Riemannian manifold (apart from
the curvature tensor itself) is the Weyl conformal curvature tensor.

Definition 2. A Riemannian manifold is called (pointwise) Osserman if its curvature tensor at every
point is Osserman. A Riemannian manifold is called conformally Osserman if its Weyl tensor at every
point is Osserman.

It is well-known (and is easy to check directly) that a Riemannian space locally isometric to a
Euclidean space or to a rank-one symmetric space is Osserman. The question of whether the converse
is true is known as the Osserman Conjecture [Oss]:

Osserman Conjecture. Any smooth pointwise Osserman manifold of dimension $n \neq 2,4$ is either flat
or locally rank-one symmetric.

The study of conformally Osserman manifolds was started in [BG1] and then continued in [BG2,
BGNSi, Gil, BGNSt]. Every Osserman manifold is conformally Osserman (which easily follows from
the formula for the Weyl tensor and the fact that every Osserman manifold is Einstein), as also is
every manifold locally conformally equivalent to an Osserman manifold. This motivates the following
conjecture made in [BGNSi]:

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**Conformal Osserman Conjecture.** Any smooth conformally Osserman manifold of dimension $n > 4$ is either conformally flat or locally conformally equivalent to a rank-one symmetric space.

The proof of the Osserman Conjecture for manifolds of dimension not divisible by 4 was given in [Chi], before the conjecture itself was published. The Conformal Osserman Conjecture for manifolds of dimension $n > 6$, not divisible by 4, is proved in [BG1], for manifolds with the structure of a warped product, both conjectures are proved in [BGV].

At present, both the Osserman Conjecture and the Conformal Osserman Conjecture are proved in all the cases, with the only exception when $n = 16$ and one of the eigenvalues of the Jacobi operator has multiplicity 7 or 8 [N1, N2, N3, N4, N5]. The main difficulty in this remaining case lies in the following algebraic fact: it can be shown that in all the other cases, an Osserman algebraic curvature tensor has a **Clifford structure**, so it “looks similar” to the curvature tensor of the complex or the quaternionic projective space (a Clifford structure arises from an orthogonal representation of a Clifford algebra; see Section 2 for details). However, the curvature tensor of the Cayley projective plane (whose Jacobi operator has eigenvalues with multiplicities exactly 7 and 8) is essentially different. This is the only known Osserman curvature tensor without a Clifford structure, which motivates the following algebraic conjecture.

**Conjecture A.** Every Osserman algebraic curvature tensor in $\mathbb{R}^{16}$ whose Jacobi operator has an eigenvalue of multiplicity 7 or of multiplicity 8 either has a Clifford structure or is a linear combination of the constant curvature tensor and the curvature tensor of the Cayley projective plane.

In the latter case, we will say that an algebraic curvature tensor has a **Cayley structure**.

Our main result is the following theorem.

**Theorem 1.** Assuming Conjecture A, both the Osserman Conjecture and the Conformal Osserman Conjecture are true.

As a consequence of Theorem 1, we obtain the following analogue of the Weyl-Schouten Theorem for rank-one symmetric spaces (without assuming Conjecture A):

**Theorem 2.** Suppose that for every point $x$ of a smooth Riemannian manifold $M^n$, $n > 4$, there exists a linear isometry which maps the Weyl tensor of $M^n$ at $x$ on a positive multiple of the Weyl tensor of a rank-one symmetric space $M^6_0$. Then $M^n$ is locally conformally equivalent to $M^6_0$.

Manifolds satisfying the assumption of Theorem 2 can be viewed as conformal analogues of (a subclass of) curvature homogeneous manifolds [TV, Gil].

In dimension four, both theorems are false. For Theorem 2 and the conformal part of Theorem 1, this follows from the fact that a four-dimensional Riemannian manifold is conformally Osserman if and only if it is either self-dual or anti-self-dual [BG2] and that there exist self-dual Kähler metrics on $\mathbb{C}^2$ which are not locally conformally equivalent to locally-symmetric ones [Der]. For the “genuine” Osserman part of Theorem 1, the counterexample is given by the generalized complex space forms [GSV, Corollary 2.7]. However, the Osserman Conjecture is true for four-dimensional **globally** Osserman manifolds, that is, for those whose Jacobi operator has constant eigenvalues on the whole unit tangent bundle [Chi].

The paper is organized as follows. Section 2 gives the algebraic background for the proof of the both theorems: we consider Osserman algebraic curvature tensors on $\mathbb{R}^{16}$, each of which, assuming Conjecture A, has either a Clifford structure (discussed in Sections 2.1,2.2), or a Cayley structure (Section 2.3). The proofs of the both theorems are given in Section 3. We first prove the local version of the conformal part of Theorem 1 using the second Bianchi identity, separately in the Clifford case (Section 3.2) and in the Cayley case (Section 3.3), and then the global version, by showing that the “algebraic type” of the Weyl tensor is the same at all the points of a connected conformally Osserman Riemannian manifold (Section 3.4). The proofs of Theorem 2 and of the “genuine” Osserman part of Theorem 1 easily follow (see the second and the last paragraphs of Section 3, respectively).

The Riemannian manifold $M^n$ is assumed to be smooth (of class $C^\infty$), although both theorems remain valid for class $C^k$, with sufficiently large $k$. 

**Theorem 1.** Assuming Conjecture A, both the Osserman Conjecture and the Conformal Osserman Conjecture are true.
2. OSSERMAN ALGEBRAIC CURVATURE TENSORS IN $\mathbb{R}^{16}$ AND THE CLIFFORD STRUCTURE

Both Theorem 1 and Theorem 2 have to be proved only when $n = 16$ (see Section 1). In this section, we consider all the known Osserman algebraic curvature tensors in $\mathbb{R}^{16}$, namely, the algebraic curvature tensors with a Clifford structure and the algebraic curvature tensors with a Cayley structure.

2.1. Clifford structure. The property of an algebraic curvature tensor $\mathcal{R}$ to be Osserman is quite algebraically restrictive. In the most cases, such a tensor can be obtained by the following construction, suggested in [GSV], which generalizes the curvature tensors of the complex and the quaternionic projective spaces.

Definition 3. A Clifford structure $\text{Cliff}(\nu; J_1, \ldots, J_\nu; \lambda_0, \eta_1, \ldots, \eta_\nu)$ on a Euclidean space $\mathbb{R}^n$ is a set of $\nu \geq 0$ anticommuting almost Hermitian structures $J_i$ and $\nu + 1$ real numbers $\lambda_0, \eta_1, \ldots, \eta_\nu$, with $\eta_i \neq 0$. An algebraic curvature tensor $\mathcal{R}$ on $\mathbb{R}^n$ has a Clifford structure $\text{Cliff}(\nu; J_1, \ldots, J_\nu; \lambda_0, \eta_1, \ldots, \eta_\nu)$ if

$$\mathcal{R}(X, Y)Z = \lambda_0(\langle X, Z \rangle Y - \langle Y, Z \rangle X) + \sum_{i=1}^{\nu} \eta_i(2\langle J_i X, Y \rangle J_i Z + \langle J_i Z, Y \rangle J_i X - \langle J_i Z, X \rangle J_i Y).$$

When this does not create ambiguity, we abbreviate $\text{Cliff}(\nu; J_1, \ldots, J_\nu; \lambda_0, \eta_1, \ldots, \eta_\nu)$ to just $\text{Cliff}(\nu)$.

Remark 1. As it follows from Definition 3, the operators $J_i$ are skew-symmetric, orthogonal and satisfy the equations $\langle J_i X, J_j X \rangle = \delta_{ij} \|X\|^2$ and $J_i J_j + J_j J_i = -2\delta_{ij} \text{id}$, for all $i, j = 1, \ldots, \nu$, and all $X \in \mathbb{R}^n$. This implies that every algebraic curvature tensor with a Clifford structure is Osserman, as by (1) the Jacobi operator has the form $\mathcal{R}_X Y = \lambda_0(\|X\|^2 Y - \langle Y, X \rangle X) + \sum_{i=1}^{\nu} 3\eta_i \langle J_i X, Y \rangle J_i X$, so for a unit vector $X$, the eigenvalues of $\mathcal{R}_X$ are $\lambda_0$ (of multiplicity $n - 1 - \nu$ provided $\nu < n - 1$), 0 and $\lambda_0 + 3\eta_i$, $i = 1, \ldots, \nu$.

From the fact that the $J_i$’s are anticommuting almost Hermitian structures it easily follows that the operators $J_{i_1}, \ldots, J_{i_m}$ with pairwise nonequal $i_j$’s, are skew-symmetric, if $m \equiv 1, 2 \mod (4)$, and are symmetric otherwise.

It turns out that every Osserman algebraic curvature tensor has a Clifford structure in all the dimensions except for $n = 16$, and also in many cases when $n = 16$, as follows from [N3] (Proposition 1 and the second last paragraph of the proof of Theorem 1 and Theorem 2), [N2, Proposition 1] and [N4, Proposition 2.1]. The only known counterexample is an algebraic curvature tensor with a Cayley structure: $\mathcal{R} = a R^C + b R^D$, where $R^C$ and $R^D$ are the curvature tensors of the unit sphere $S^{16}(1)$ and of the Cayley projective plane $\mathbb{O}P^2$, respectively, and $a \neq 0$.

A Clifford structure $\text{Cliff}(\nu)$ on the Euclidean space $\mathbb{R}^n$ turns it into a Clifford module (we refer to [ABS, Part 1], [Hus, Chapter 11] for standard facts on Clifford algebras and Clifford modules). Denote $\text{Cl}(\nu)$ a Clifford algebra on $\nu$ generators $x_1, \ldots, x_\nu$, an associative unital algebra over $\mathbb{R}$ defined by the relations $x_i x_j + x_j x_i = -2\delta_{ij}$ (this condition determines $\text{Cl}(\nu)$ uniquely). The map $\rho : \text{Cl}(\nu) \rightarrow \mathbb{R}^n$ defined on generators by $\rho(x_i) = J_i$ (and $\rho(1) = \text{id}$) is a representation of $\text{Cl}(\nu)$ on $\mathbb{R}^n$. As all the $J_i$’s are orthogonal and skew-symmetric, $\rho$ gives rise to an orthogonal multiplication defined as follows. In the Euclidean space $\mathbb{R}^\nu$, fix an orthonormal basis $e_1, \ldots, e_\nu$. For every $u = \sum_{i=1}^{\nu} u_i e_i \in \mathbb{R}^\nu$ and every $X \in \mathbb{R}^n$, define

$$J_u X = \sum_{i=1}^{\nu} u_i J_i X$$

(when $u = e_i$, we abbreviate $J_{e_i}$ to $J_i$). The map $J : \mathbb{R}^\nu \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ defined by (2) is an orthogonal multiplication: $\|J_u X\|^2 = \|u\|^2 \|X\|^2$ (similarly, we can define an orthogonal multiplication $J : \mathbb{R}^{\nu+1} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ by $J_u X = u_0 X + \sum_{i=1}^{\nu} u_i J_i X$, for $u = \sum_{i=0}^{\nu} u_i e_i \in \mathbb{R}^{\nu+1}$, where $e_0, e_1, \ldots, e_\nu$ is an orthonormal basis for the Euclidean space $\mathbb{R}^{\nu+1}$).

For $X \in \mathbb{R}^n$, introduce the subspaces

$$J X = \text{Span}(J_1 X, \ldots, J_\nu X), \quad \mathcal{I} X = \text{Span}(X, J_1 X, \ldots, J_\nu X).$$

Later we will also use the complexified versions of these subspaces, which we denote $\mathcal{J}_C X$ and $\mathcal{I}_C X$ respectively, for $X \in \mathbb{C}^n$. 
2.2. Algebraic curvature tensors of dimension 16 with a Clifford structure. To find all the algebraic curvature tensors with a Clifford structure in dimension 16, we need to find all the possible ways of turning $\mathbb{R}^{16}$ into a $\text{Cl}(\nu)$-module. A convenient way to describe them is by using the octonions.

In general, the proof of Theorem 1 extensively uses computations in the octonion algebra $\mathbb{O}$ (in particular, the standard identities like $a^* = 2(a, 1) - a$, $(a, b) = (a^*, b^*) = \frac{1}{2}(a^*b + b^*a)$, $a(ab) = a^2b$, $(a, bc) = (b^*a, c) = (ac^*, b)$, $(ab^*)c + (ac^*)b = 2(b, c)a$, $(ab, ac) = (ba, ca) = ||a||^2(b, c)$, for any $a, b, c \in \mathbb{O}$, and similar ones, see e.g., [BG]) and the fact that $\mathbb{O}$ is a division algebra (in particular, any nonzero octonion is invertible: $a^{-1} = ||a||^{-2}a^*$). We will also use the bijectonions $\mathbb{O} \otimes \mathbb{C}$, the algebra over $\mathbb{C}$ with the same multiplication table as that for $\mathbb{O}$. As all the above identities are polynomial, they still hold for bijectonions, with the complex inner product on $\mathbb{C}^8$, the underlying linear space of $\mathbb{O} \otimes \mathbb{C}$.

However, the bioctonion algebra is not a division algebra (and has zero-divisors: $(1 + e_1)(1 - e_1) = 0$).

In the following lemma, (which contains known facts, but will be convenient for us to refer to) we call a representation $\rho$ of $\text{Cl}(\nu)$ in $\mathbb{R}^n$ orthogonal, if all the $\rho(x_i)$ are orthogonal. Representation $\rho_1, \rho_2$ are called equivalent (respectively, orthogonally equivalent), if there exists $T \in \text{GL}(n)$ (respectively, $T \in O(n)$) such that $\rho_2(x) = T\rho_1(x)T^{-1}$, for all $x \in \text{Cl}(\nu)$. For a representation $\rho$, the representation $-\rho$ is defined on the generators of $\text{Cl}(\nu)$ by $(-\rho)(x_i) = -\rho(x_i)$ (induced by the automorphism $\alpha : x_i \rightarrow -x_i$ of $\text{Cl}(\nu)$).

Lemma 1. 1. For any representation of a Clifford algebra $\text{Cl}(\nu)$ in $\mathbb{R}^{16}, \nu \leq 8$. There is exactly one, up to orthogonal equivalence, orthogonal representations $\rho_8$ of $\text{Cl}(8)$ in $\mathbb{R}^{16}$. It can be defined as follows. Identify $\mathbb{R}^{16}$ and $\mathbb{R}^8$ with $\mathbb{O} \oplus \mathbb{O}$ and $\mathbb{O}$ respectively, via linear isometries. Then the orthogonal multiplication (2) defined by $\rho_8$ is given by

\begin{equation}
J_p(a, b) = (bp, -ap^*), \quad \text{for} \quad p \in \mathbb{R}^8 = \mathbb{O}, \quad X = (a, b) \in \mathbb{R}^{16} = \mathbb{O} \oplus \mathbb{O}.
\end{equation}

2. Any orthogonal representation of a Clifford algebra $\text{Cl}(\nu)$, $\nu \geq 4$, in $\mathbb{R}^{16}$ is either a restriction of the representation $\rho_8$ to $\text{Cl}(\nu) \subset \text{Cl}(8)$, or, up to orthogonal equivalence, is $\pm \rho_7$, where $\rho_7$ is the following reducible orthogonal representation of $\text{Cl}(7)$. Identify $\mathbb{R}^{16}$ and $\mathbb{R}^7$ with $\mathbb{O} \oplus \mathbb{O}$ and $\mathbb{O} \perp \mathbb{O}$ respectively, via linear isometries. Then the orthogonal multiplication (2) defined by $\rho_7$ is given by

\begin{equation}
J_p(a, b) = (ap, bp), \quad \text{for} \quad p \in \mathbb{R}^7 = \mathbb{O}', \quad X = (a, b) \in \mathbb{R}^{16} = \mathbb{O} \oplus \mathbb{O}.
\end{equation}

Proof. It is easy to see that if two orthogonal representations are equivalent, then they are orthogonally equivalent. Indeed, suppose that $\rho_1(x_i) = J_i$ and $\rho_2(x_i) = T J_i T^{-1}$, where $T \in \text{GL}(n)$. Then, as both $\rho_1$ and $\rho_2$ are orthogonal, we get $(T^T T) J_i = J_i (T^T T)$. As every $J_i$ commutes with $T^T T$, it also commutes with $S = \sqrt{T^T T}$, the unique symmetric positive definite matrix such that $S^2 = T^T T$. But then $T S^{-1} \in O(n)$ and $\rho_2(x_i) = (T S^{-1}) J_i (T S^{-1})^{-1}$.

By [Hus, Table 6.5], $N_5 = N_6 = N_8 = Z$, $N_7 = Z \oplus Z$, where $N_\nu$ is the free abelian group generated by irreducible representations of $\text{Cl}(\nu)$.

1. The fact that $\nu \leq 8$ follows from [Hus, Theorem 11.8.2]. The orthogonal multiplication defined by (3) is indeed a representation of $\text{Cl}(8)$ in $\mathbb{R}^{16}$ (which follows from the octonion identity $(ap^*)q + (aq^*)p = 2(p, q)a$). As $N_8 = Z$, a representation of $\text{Cl}(8)$ in $\mathbb{R}^{16}$ is unique, up to orthogonal equivalence, hence any orthogonal representation is orthogonally equivalent to the one defined by (3).

2. The restriction of $\rho_8$ to any $\text{Cl}(\nu) \subset \text{Cl}(8)$ defines an orthogonal representation of $\text{Cl}(\nu)$ in $\mathbb{R}^{16}$. As $N_5 = N_6 = Z$, any orthogonal representation of $\text{Cl}(\nu)$, $\nu = 5, 6$, in $\mathbb{R}^{16}$ is equivalent (hence orthogonally equivalent) to it.

For the algebra $\text{Cl}(7)$, the group $N_7 = Z \oplus Z$ is generated by two inequivalent representations in $\mathbb{R}^8$. These are $\pm \sigma$, where on generators, $\sigma(x_i)$ is the right multiplication in $\mathbb{O}$ by the imaginary octonion $x_i$ (note that $\prod_{i=1}^7 \sigma(x_i) = \pm \text{id}$, with $\pm$ replaced by $\mp$ for $-\sigma$). Then there are exactly three (orthogonally) inequivalent (orthogonal) representations of $\text{Cl}(7)$ in $\mathbb{R}^{16}$: $\pm 2\sigma$ and $(-\sigma) \pm \sigma$. As it follows from (4), $\rho_7 = 2\sigma$. Moreover, neither of $\pm \rho_7$ can be a restriction of $\rho_8$ to $\text{Cl}(7)$, as $\prod_{i=1}^7 \rho_8(x_i) = \pm \text{id}$, so $\prod_{i=1}^8 \rho_7(x_i) = \pm \rho_7(x_8)$, which is skew-symmetric, thus contradicting Remark 1. \hfill \Box

Note that an algebraic curvature tensor with a Clifford structure does not change, if we change the signs of (some or all of) the $J_i$’s, so it does not matter, which of $\pm \rho_7$ is defined by (4).

In the proof of Theorem 1 we will use the following Lemma.
Lemma 2.1. Suppose that a Clifford structure on $\mathbb{R}^{16}$ is given by (3). Let $N: \mathbb{R}^{16} \to \mathbb{R}^{8}$ be a quadratic form such that for all $i = 1, \ldots, 8$, the cubic polynomial $\langle N(X), J_i X \rangle$ is divisible by $\|X\|^2$. Then there exist a linear operator $A: \mathbb{R}^{16} \to \mathbb{R}^8$ and vectors $V, U \in \mathbb{R}^{16}$ such that $N(X) = J_A(X)X + \langle V, X \rangle X - U\|X\|^2$.

2. Suppose that a Clifford structure on $\mathbb{R}^{16}$ is given by (4). Let $N = (N_1, N_2): \mathbb{R}^{16} \to \mathbb{R}^{16}$ be a quadratic form, $u$ be a unitary imaginary octonion, and $p \in \mathbb{R}^{16}$ be such that for all $X = (a, b) \in \mathbb{R}^{16}$, $a^* N_1(X) + b^* N_2(X) = (p, X)(\langle a^* b, u \rangle).$ Then there exists $m \in \mathbb{R}^{16}$ such that $N(X) = \|X\|^2(m - \pi_{TX}m) = \|X\|^2 - \langle m, X \rangle X - \sum_{i=1}^7 \langle J_i X, m \rangle J_i X$, where $\pi_{TX}$ is the orthogonal projection to $T_X$.

Proof. 1. For $X = (a, b) \in \mathbb{R}^{16}$, let $N(X) = (N_1(X), N_2(X))$, where $N_1, N_2 : \mathbb{R}^{16} \to \mathbb{R}^8$ are quadratic forms. By the assumption and (3), for any $q \in \mathbb{R}^8$, the cubic polynomial $\langle N(X), J_q X \rangle = \langle N_1, bq \rangle - \langle N_2, aq^* \rangle$ is divisible by $\|X\|^2$, hence so is the polynomial vector $b^* N_1 - N_2 a$. Then there exists a linear operator $L : \mathbb{R}^{16} \to \mathbb{R}^8$ such that $b^* N_1(X) - N_2(X)^* a = (\|a\|^2 + \|b\|^2)L(X)$, which simplifies to

\[
b^* (N_1(X) - bL(X)) = (L(X)a)^* + N_2(X)^* a.
\]

Let for $X = (a, b)$,

\[
N_1(X) - bL(X) = \xi_1(b) + P(a, b) + \phi(a), \quad L(X)a^* + N_2(X)^* = \xi_2(a) + Q(a, b) + \psi(b),
\]

where $\phi, \xi_1, \xi_2, \psi : \mathbb{R}^8 \to \mathbb{R}^8$ are quadratic forms and $P, Q : \mathbb{R}^8 \times \mathbb{R}^8 \to \mathbb{R}^8$ are bilinear forms. Collecting the terms of the same degree in $a$ and $b$ in (5) we get $\xi_1 = \xi_2 = 0$ and

\[
b^* P(a, b) = \psi(b)a, \quad b^* \phi(a) = Q(a, b)a.
\]

Taking $b = 1$ in the second equation we find $\phi(a) = Q(a, 1)a$ and so $Q(a, b)a = b^* (Q(a, 1)a) = 2(b, a, 1)^*(ba)$, which implies $(a^* b^*) Q(a, b) = a^* (2(b, a, 1) - Q(a, b^*) b)$.

For every $b$ from the unit sphere $S^d \subset \mathbb{O}$, we can now apply [N2, Lemma 6], with $Y = a^*, e = b^*$, $L(Y) = Q(a, 1)$ and $F(Y) = 2(b, a, 1) - Q(a, b^*)$. Then $Q(a, 1) = (c(b), a^*)^1 + (d(b), a^*) b^* + ap(b)$, for some maps $c, d, p : S^d \to \mathbb{O}$. As the left-hand side does not depend on $b$, we get $a(p(b_1) - p(b_2)) \in \text{Span}(1, b_1, b_2)$, for all $a \in \mathbb{O}$ and all $b_1, b_2 \in S^d$, so $p(b) = p$, a constant. Taking the real and the imaginary parts of the both sides we obtain that $c(b) = c$ is also a constant and $d(b) = 0$. Then $Q(a, 1) = (c(a), a)^1 + ap$, and therefore $\phi(a) = Q(a, 1)a = \langle m, a \rangle a - \|a\|^2 p$, where $m = c^* + 2p^*$. Then from (6), $Q(a, b) = b^* \phi(a) = (\langle m, a \rangle b^* - (b^* p^*) a^*)a$, so $Q(a, b) = \langle m, a \rangle b^* - (b^* p^*) a^*$. Similarly, from the first equation of (6) we get $\psi(b) = \langle r, b^* \rangle b^* - \|b\|^2 q^*$, $P(a, b) = \langle r, b^* \rangle a - b(qa)$. Then

\[
N_1(X) = bL(X) + \langle r, b^* \rangle a - b(qa) + \langle m, a \rangle a - \|a\|^2 p^*,
\]

\[
N_2(X) = -aL(X)^* + \langle m, a \rangle b - a(p\bar{b}) + \langle r, b^* \rangle b - \|b\|^2 q^*,
\]

which implies $N(X) = J_A(X)X + \langle V, X \rangle X - U\|X\|^2$, for all $X \in \mathbb{R}^{16}$, where $A : \mathbb{R}^{16} \to \mathbb{R}^8$ is a linear operator defined by $A(X) = L(X) + b^* p^* - qa$, and $V = (m, r^*)$, $U = (p^*, q^*) \in \mathbb{R}^{16}$.

2. From the assumption, $N_1((a, 0)) = N_2((0, b)) = 0$, so $N_1((a, b)) = P(a, b) + \xi_1(b), N_2((a, b)) = \xi_2(a) + Q(a, b)$, for some quadratic forms $\xi_1, \xi_2 : \mathbb{R}^8 \to \mathbb{R}^8$ and bilinear forms $P, Q : \mathbb{R}^8 \times \mathbb{R}^8 \to \mathbb{R}^8$. Collecting the terms of degree two in $a$ we get $a^* P(a, b) + b^* \xi_2(a) = (p_1, a)\langle a^* b, u \rangle$, where $p_1 \in \mathbb{R}^8$, $p = (p_1, p_2)$. Substituting $b = 1$ we get $\xi_2(a) = (p_1, a)\langle a^* u - a^* P(a, 1), a^* P(a, 1) \rangle = (p_1, a)(\langle a^* b, u \rangle - \langle a^*, u b^* \rangle)$. Multiplying by $a$ from the left and taking $a \downarrow p_1$ we get $\|a\|^2 (P(a, b) - a(b^* a^* P(a, 1))) = 0$ which can be rewritten as $-2(a^* (b - a, 1)b(a^* P(a, 1))) = \|a\|^2 (b(P(a, 1) - P(b), a)$. Multiplying by $(\langle a, b \rangle - a, 1)^* b^*$ from the left we obtain that all the components of the polynomial vector $\langle a, b \rangle - a, (1)^2 b^* a^* P(a, 1))$ belong to the ideal $I$ generated by $\|a\|^2$ and $(p_1, a)^2$. For a fixed nonzero $b \perp 1$, the quadratic form $\|a, b \rangle - a, (1)^2 b^* \|2$ is not in $I$ (as it is nonzero and vanishes on $\text{Span}(1, b)^\perp$). As the ideal $I$ is prime, it follows that all the components of $a^* P(a, 1)$ belong to $I$. Since $P(a, 1)$ is linear in $a$, we obtain $a^* P(a, 1) = \|a\|^2 c + (p_1, a)\Lambda a$, for some $c_1 \in \mathbb{O}$ and some linear operator $L$ on $\mathbb{O}$. Multiplying this by $a$ from the left we obtain that $a \cdot L a$ is divisible by $\|a\|^2$, so $a \cdot L a = \|a\|^2 c_2$, for some $c_2 \in \mathbb{O}$, so $L a = a^* c_2$, therefore $a^* P(a, 1) = \|a\|^2 c_1 + (p_1, a)^2 c_2$ which implies $P(a, 1) = a c_1 + (p_1, a)^2 c_2$. Then $a^* P(a, b) - b^* (a^* (ac_1 + (p_1, a) c_2)) = (p_1, a)(\langle a^* b, u \rangle - \langle a^*, u b^* \rangle)$, for all $a, b \in \mathbb{O}$, $\|a\|^2 \|b \|^2 c_1 = (p_1, a)(\langle a^* b, u \rangle - \langle a^*, u b^* \rangle + b^* (a^* c_2))$.

Assume $p_1 \neq 0$. Then multiplying by $a$ from the left we obtain that all the components of the polynomial vector $a(\langle a^* b, u \rangle - \langle a^*, u b^* \rangle + b^* (a^* c_2))$ are divisible by $\|a\|^2$. As it is linear in $b$ and
quadratic in \( a \), we obtain \( a(\langle a^*b, u \rangle - \langle a^*, u \rangle b^* + b^*(a^*c_2)) = \|a\|^2L_1b \) for some linear operator \( L_1 \) on \( \mathcal{O} \), so \( \langle a^*b, u \rangle - \langle a^*, u \rangle b^* + b^*(a^*c_2) = a^*L_1b \). Taking \( a = 1 \) we get \( L_1b = \langle b, u \rangle + b^*c_2 \), so \( \langle a^*b, u \rangle - \langle a^*, u \rangle b^* + b^*(a^*c_2) = a^* \langle b, u \rangle + b^*c_2 \), for all \( a, b \in \mathcal{O} \). This implies that for all orthogonal \( a, b \in \mathcal{O} \), \( a(bc_2) \in \text{Span}(a, b, 1) \), so \( bc_2 \in \text{Span}(ab, a, b) \). Thus \( c_2 = 0 \), so \( \langle a^*b, u \rangle - \langle a^*, u \rangle b^* + b^*(a^*c_2) = 0 \), for all \( a, b \in \mathcal{O} \), which leads to a contradiction (take \( b = u, a \perp 1, u \)).

It follows that \( p_1 = 0 \), so \( a^*P(a, b) - |a|^2b^*c_1 = 0 \) which implies \( P(a, b) = a(b^*c_1) \), and then \( \xi_2(a) = (p_1, a)(a^*u - a^*P(a, 1) = -\|a\|^2c_1 \). Similarly, \( Q(a, b) = b(a^*c_3) \) and \( \xi_1(b) = -\|b\|^2c_3 \), for some \( c_3 \in \mathcal{O} \). It follows that \( N(X) = (a(b^*c_1) - \|b\|^2c_3, b(a^*c_3) - |a|^2c_1) \). Then by (4), \( N(X) = \|X\|^2(m - \pi_Xm) = \|X\|^2m - \langle m, X \rangle X - \sum_{i=1}^8(J_iX, m)j_iX \), where \( m = (-c_3, -c_1) \in \mathbb{R}^16 \). \( \square \)

2.3. Algebraic curvature tensors with a Cayley structure. The curvature tensor \( R^o \) of the Cayley projective plane \( \mathcal{O}P^2 \) of the sectional curvature between 1 and 4, is explicitly given in [BG, Eq. 6.12].

Identifying the tangent space \( T_x\mathcal{O}P^2 \) with \( \mathcal{O} \oplus \mathcal{O} \) via a linear isometry, we have for \( X = (x_1, x_2), Y = (y_1, y_2), Z = (z_1, z_2) \in T_x\mathcal{O}P^2 = \mathcal{O} \oplus \mathcal{O} \):

\[
R^o(X, Y)Z = (4\langle x_1, z_1 \rangle y_1 - 4\langle y_1, z_1 \rangle x_1 - \langle z_1y_2 \rangle x_2^2 + \langle z_1x_2 \rangle y_2^2 - \langle x_1y_2 - y_1x_2 \rangle z_2^2, \\
4\langle x_2, z_2 \rangle y_2 - 4\langle y_2, z_2 \rangle x_2 - x_1^2\langle y_1, z_2 \rangle + y_1^2\langle x_1, z_2 \rangle + z_1^2\langle x_1y_2 - y_1x_2 \rangle).
\]

Introducing the symmetric operators \( S_i \in \text{End}(\mathbb{R}^{16}), i = 0, 1, \ldots, 8 \), by \( S_0(x_1, x_2) = (x_1, -x_2), S_i(x_1, x_2) = (e_i^*x_2, x_1^*e_i), i = 1, \ldots, 8, \) where \( \{e_i\} \) is an orthonormal basis for \( \mathcal{O} \), we obtain

\[
R^o(X, Y) = 3X \wedge Y + \sum_{i=0}^8 S_iX \wedge S_iY,
\]

where \( X \wedge Y \) is the skew-symmetric operator defined by \( (X \wedge Y)Z = \langle X, Z \rangle Y - \langle Y, Z \rangle X \).

As it follows from the definition, the operators \( S_i \) are orthogonal and satisfy

\[
S_iS_j + S_jS_i = 2\delta_{ij} \text{id}, \quad 0 \leq i, j \leq 8.
\]

For every \( w \) in the Euclidean space \( \mathbb{R}^9 \), introduce the symmetric operator \( S_w = \sum_{i=0}^8 w_iS_i \). As it follows from (8), the map \( S : \mathbb{R}^9 \times \mathbb{R}^{16} \to \mathbb{R}^{16} \) defined by \( (w, X) \to S_wX \) is an orthogonal multiplication: \( \|S_wX\| = \|w\| \cdot \|X\| \), for all \( w \in \mathbb{R}^9, X \in \mathbb{R}^{16} \). We usually abbreviate \( S_{e_i} \) to \( S_i \).

The operators \( S_i \) define the structure of the Clifford \( \mathcal{C}l^+(9) \)-module on the Euclidean space \( \mathbb{R}^{16} \). Denote \( \mathcal{C}l^+(9) \) the Clifford algebra on nine generators \( x_0, x_1, \ldots, x_8 \), an associative unital algebra over \( \mathbb{R} \) defined by the relations \( x_i x_j + x_j x_i = 2\delta_{ij} \). The map \( \sigma : \mathcal{C}l^+(9) \to \mathbb{R}^{16} \) defined on generators by \( \sigma(x_i) = S_i \) (and \( \sigma(1) = \text{id} \)) is a representation of \( \mathcal{C}l^+(9) \) on \( \mathbb{R}^{16} \). The Clifford algebra \( \mathcal{C}l^+(9) \) is isomorphic to \( \mathbb{R}(16) \oplus \mathbb{R}(16), \) where \( \mathbb{R}(16) \) is the algebra of \( 16 \times 16 \) real matrices [ABS, §4], so \( \sigma \) is surjective. In particular, as by (8) the operator \( \prod_{i=0}^8 S_i \) commutes with all the \( S_i \)'s (hence with all the \( \mathbb{R}(16) \)) and is orthogonal, we have \( \prod_{i=0}^8 S_i = \pm \text{id} \).

As by (8), for every nonzero \( w \in \mathbb{R}^9, S_w^2 = \|w\|^2 \text{id} \) and \( \text{Tr} S_w = 0 \) (multiply (8) by \( S_i \) and take the trace), \( S_w \) has two eigenvalues \( \pm \|w\| \), each of multiplicity 8. Denote \( E_{\|w\|}(S_w) \) the \( \|w\| \)-eigenspace of \( S_w \) and \( E_{\|w\|}(S_w) \) the orthogonal projection of \( \mathbb{R}^{16} \) to \( E_{\|w\|}(S_w) \). Then \( \pi_{E_{\|w\|}(S_w)} = \frac{1}{2}(\|w\|)^{-1}S_w + \text{id} \).

Introduce the subspaces \( \mathcal{L}_k = \text{Span}_{\mathbb{R}}(S_i, \ldots, S_k) \subset \text{End}(\mathbb{R}^{16}), 0 \leq k \leq 9 \) (in particular, \( \mathcal{L}_0 = \mathbb{R} \text{id} \)). On \( \text{End}(\mathbb{R}^{16}) \), introduce the inner product by \( (Q_1, Q_2) = \text{Tr}(Q_1Q_2) \) and the operator \( A \) by

\[
AQ = \sum_{i=0}^8 Q_iS_i, \quad Q \in \text{End}(\mathbb{R}^{16}).
\]

Lemma 3. 1. For \( k = 0, \ldots, 9, \mathcal{L}_{-k} = \mathcal{L}_k \). Moreover, \( \bigoplus_{k=0}^4 \mathcal{L}_k = \text{End}(\mathbb{R}^{16}) \) and \( \text{Sym}(\mathbb{R}^{16}) = \mathcal{L}_0 \oplus \mathcal{L}_4 \). In particular, \( \text{Sym}(\mathbb{R}^{16}) \) is the space of the skew-symmetric endomorphisms of \( \mathbb{R}^{16} \) respectively, and all the direct sums are orthogonal. Moreover, \( \{1, S_1, \ldots, S_k \} \) is an orthonormal basis for \( \mathcal{L}_k, 0 \leq k \leq 4 \).

2. The operator \( A \) is symmetric and does not depend on the choice of an orthonormal basis \( \frac{1}{2}S_i \) for \( \mathcal{L}_1 \). Its eigenspaces are \( \mathcal{L}_k, 0 \leq k \leq 4, \) with the corresponding eigenvalues \( (-1)^k(9 - 2k) \). In particular, \( \text{Skew}(\mathbb{R}^{16}) \) and \( \text{Sym}(\mathbb{R}^{16}) \) are invariant subspaces of \( A \).

3. For every \( w \in \mathbb{R}^9, Q \in \text{End}(\mathbb{R}^{16}), A(Qw) = -A(Q)w + 2SwQ, A(SwQ) = -SwA(Q) + 2QS_w \).

For every \( X, Y \in \mathbb{R}^{16}, A(X \wedge Y) = \sum_{i=0}^8 S_iX \wedge S_iY \).
4. For every $X \in \mathbb{R}^{16}$, $X \in \text{Span}^8_{i=0}(S_i X)$ and $\sum_{i=0}^8 (S_i, X) S_i X = ||X||^2 X$. For every $X, Y \in \mathbb{R}^{16}$, $\sum_{i=0}^8 (2( S_i X, Y) S_i X + (S_i X, X) S_i Y) = ||X||^2 Y + 2( X, Y) X$. For a unit vector $X \in \mathbb{R}^{16}$, $R_X^2 Y = Y$, when $Y \perp \text{Span}^8_{i=0}(S_i X)$ and $R_X^2 Y = \frac{1}{2} Y$, when $Y \in \text{Span}^8_{i=0}(S_i X)$, $Y \perp X$.

5. Let $N : \mathbb{R}^{16} \times \mathbb{R}^8 \rightarrow \mathbb{R}^{16}$ be a bilinear skew-symmetric map such that $\langle X(X, Y), Z \rangle = 0$, for every $w \in \mathbb{R}^9$ and for every $X, Y, Z \in E_{||w||}(S_w)$. Then there exists $q \in \mathbb{R}^{16}$ such that

$$(10) \quad N(X, Y) = (A - \text{id})(X \wedge Y)q = \sum_{i=0}^8 ( (S_i X, q) S_i Y - (S_i Y, q) S_i X - ( (X, q) Y - (Y, q) X).$$

\textbf{Proof.} 1. The fact that $L_{9-k} = L_k$ follows from $\prod_{i=0}^8 (S_i) = \pm \text{id}$. Moreover, all the operators $\frac{1}{2} S_i \ldots S_k$, $i_1 \ldots i_k$, $0 \leq k \leq 4$, are orthonormal. Indeed, from (8), the norm of each of them is 1. The inner product of two different ones is $\frac{1}{16}$ times the trace of some $S' = S_{j_1} \ldots S_{j_p}$, $1 \leq p \leq 8$, which is clearly zero, if $S'$ is skew-symmetric. Otherwise, $p$ or $p = 4$ (by $L_{9-k} = L_k$), and in the both cases, $\text{Tr} S' = 0$, as $S'$ is a product of a symmetric and a skew-symmetric operator: $S_i = S_i \cdot (S_j S_k)$, when $p = 1$, and $S_j S_k S_i S_j \cdot S_i \cdot S_k S_j S_i$, when $p = 4$ (for arbitrary pairwise nonequal $i, j, k, l$). Now, as the $S_i$'s are symmetric, (8) implies that $L_0 + L_1 + L_2 \subset \text{Sym}^{(16)}$ and that $L_0 + L_1 + L_2 \subset \text{Skew}(16)$, with both inclusions being in fact equalities by the dimension count.

2. 3. Directly follow from (8, 9) and the fact that $S_i (X \wedge Y) S_i = S_i X \wedge S_i Y$.

4. For $X = (x_1, x_2) \neq 0$, $x_1, x_2 \in \mathbb{R}$, define $w \in \mathbb{R}^9 = \mathbb{R} \oplus \mathbb{R}$ by $w = ||x||^{-2}((||x_1||^2 - ||x_2||^2)e_0 +2x_1 x_2)$. Then $S_i X = X$ by the definition of the $S_i$'s, so $X \in \text{Span}^8_{i=0}(S_i X)$. As $(S_i X, S_j X) = \delta_{ij} ||X||^2$ by (8), equation $\sum_{i=0}^8 (S_i X, S_i X) = ||X||^2$ follows. The second equation is obtained by polarization. Substituting it to the expression for $R_X^2 Y$ obtained from (7), we get the last statement of the assertion.

5. For every $Z \in \mathbb{R}^{16}$, define $K(Z) \in \text{Skew}(16)$ by $\langle K(Z)X, Y \rangle = \langle N(X, Y), Z \rangle$. As $\pi_{E_{||w||}(S_w)} = \frac{1}{2}(w^{-1}S_w + \text{id})$ we have $\langle \pi_{E_{||w||}(S_w)} \rangle = \langle (w^{-1}S_w + \text{id}) \rangle$ is divisible by $||w||^2$ (the $w$'s are the coordinates of $w$ relative to the orthonormal basis $\{e_0, e_1, \ldots, e_8\}$ for $\mathbb{R}^9$ such that $S_i = S_j$). Then from (8) we obtain that $\langle w \rangle^2 = \langle w \rangle^2 \langle w \rangle^2 \langle K(S_w Z) S_w Z \rangle - \langle \rangle^2 \langle K(S_w Z) S_w Z \rangle = 0$. For every $Z \in \mathbb{R}^9$, define the polynomials $F_{Z,i}(w) = \text{Tr}(K(S_w Z) S_w Z)$. Then $\langle w \rangle^2 = \langle w \rangle^2 \langle F_{Z,i}(w) \rangle - \langle w \rangle^2 \langle F_{Z,i}(w) \rangle$. Let $I$ be the ideal of the polynomial ring $K = \mathbb{R}[w_0, w_1, \ldots, w_8]$ generated by $\langle w \rangle^2$ and let $\pi : K \rightarrow K/I$ be the natural projection. Then $\pi(w)F_{Z,i} - \langle w \rangle^2 F_{Z,i} = 0$, so the 2 × 9-matrix over the ring $K/I$ whose $i$-th row is $(\pi(w_1), \pi(F_{Z,i}))$, $i = 0, 1, \ldots, 8$, has rank at most one. As the ring $K/I$ is a unique factorization domain [Nag], there exist $x, y, u \in K/I$, such that $\pi(w_i) = xu_i$, $\pi(F_{Z,i}) = yu_i$. Since the elements $\pi(w_i) = xu_i$ are coprime, $x$ is invertible, so we can take $x = 1$, hence $\pi(F_{Z,i}) = yu_i$. Lifting this equation up to $K$ we obtain that $\langle w \rangle^2 \langle K(S_w Z) S_w Z \rangle = \langle T S_w Z, w \rangle^2 + \langle G S_w Z, e_i \rangle$. It follows that $\langle w \rangle^2 = \langle T S_w Z, w \rangle^2 + \langle G S_w Z, e_i \rangle$, for all $Z \in \mathbb{R}^{16}$, $w \in \mathbb{R}^9$, $i = 0, 1, \ldots, 8$. Substituting $S_w Z$ as $Z$ we obtain $\langle w \rangle^2 = \langle K(S_w Z) S_w Z \rangle = \langle T S_w Z, w \rangle^2 + \langle G S_w Z, e_i \rangle$. It follows that $\langle w \rangle^2 = \langle T S_w Z, w \rangle^2 + \langle G S_w Z, w \rangle$, where $G S_w Z, w = -S_w Z$. Hence $\langle w \rangle^2 p + \langle S_w Z, G w \rangle^2 = 0$, so $G S_w Z, w = -S_w Z$. Hence $\langle w \rangle^2 = \langle T S_w Z, w \rangle^2 + \langle G S_w Z, e_i \rangle$. In particular, for $w = ej$, $j \neq i$, we get by (8): $\langle T (K(Z) S_j Z) \rangle = \langle (Z, S_j Z) \rangle = \frac{1}{2} \langle j \rangle$, $i < j$.}
Now for any \( K \in \text{Skew}(\mathbb{R}^{16}) \), \( \pi_2(K) = \frac{1}{8}(A + 3\text{id})K \) and \( \pi_3(K) = -\frac{1}{8}(A - 5\text{id})K \), (where \( \pi_i \) is the orthogonal projection to \( L_i \) by assertions 1 and 2, and \( \mathcal{A}(S_wKS_w) = S_wA(K)S_w \) and \( \mathcal{A}(S_wK + KS_w) = -S_wA(K)S_w + 2S_wK + 2KS_w \) by assertion 3, so \( \pi_2(S_wK + KS_w) = S_w\pi_3(K + \pi_3(K)S_w) \) and \( \pi_3(S_wK + KS_w) = S_w\pi_2(K)S_w \). Projecting (11) to \( L_2 \) we then obtain \( S_w\pi_2(K(S_wZ))S_w + ||w||^2\langle\pi_2(K(S_wZ)) + \pi_3(K(Z)) + \pi_3(K(Z)S_w)\rangle = 0 \). As it is shown above, \( \pi_2(K(Z)) = -\frac{3}{2}\pi_2(Z \wedge p) \), so \( -\frac{3}{2}S_w\pi_3((S_wZ)\wedge p + ||w||^2(-\frac{1}{2}\pi_2(S_wZ)\wedge p) + S_w\pi_3(K(Z)) + \pi_3(K(Z)S_w) = 0 \), which (using the fact that \( S_w\pi_2(K)S_w = \pi_2(S_wKS_w) \) and that \( \mathcal{A}(X \wedge Y)S_w = (S_wX) \wedge Y + X \wedge (S_wY) \)) results in \( -9\pi_3(K(Z) - \frac{1}{2}Z \wedge p) = 0 \) by assertion 2.

As \( K(Z) = \pi_2(K(Z)) + \pi_3(K(Z)) \) by assertion 1, it follows that \( K(Z) = \frac{1}{3}(-\pi_2 + \pi_3) \) \( (Z \wedge p) = \frac{1}{16}((-A + 3\text{id}) - (A - 5\text{id})(Z \wedge p) = -\frac{1}{8}(A - \text{id})(Z \wedge p) \). Since \( \langle K(Z)X, Y \rangle = -\frac{1}{16}\langle K(Z), X \wedge Y \rangle \) and \( \mathcal{A} \) is symmetric, \( \langle N(X, Y), Z \rangle = \langle K(Z)X, Y \rangle = \frac{1}{16}\langle A(\text{id})(Z \wedge p), X \wedge Y \rangle = \frac{1}{16}\langle (A(\text{id})(X \wedge Y), Z \wedge p) = \frac{1}{8}\langle (A - \text{id})(X \wedge Y)p, Z \rangle \), so \( N(X, Y) = (A - \text{id})(X \wedge Y)q \), with \( q = \frac{1}{8}p \), as required. \( \square \)

3. CONFORMALLY OSSERMAN MANIFOLDS: PROOF OF THEOREM 1 AND THEOREM 2

In all the cases, except when \( n = 16 \), Theorem 1 is already proved: for the Osserman Conjecture, see [N3, Theorem 2], for the Conformal Osserman Conjecture, see [N5, Theorem 1]. In this section, we will first prove Theorem 1 for conformally Osserman manifold of dimension 16 (assuming Conjecture A) and then deduce from it the proof for “genuine” Osserman manifolds.

Theorem 2 is an easy corollary of Theorem 1, as in Theorem 2 we consider the Riemannian manifolds \( M^n \), for which Conjecture A is already “satisfied” by the assumption: the Weyl tensor of each of them at every point is proportional either to the Weyl tensor of \( S^n = \mathbb{R}^{n+1} / \mathbb{Z} \), or to the Weyl tensor of \( S^{n-1} \). By Theorem 1, \( M^n \) is locally conformally equivalent to a rank-one symmetric space, which is, in fact, \( M^n \), as the Weyl tensors of different rank-one symmetric spaces are different (for instance, because their Jacobi operators have different multiplicities of the eigenvalues).

We start with a brief informal outline of the proof of the conformal part of Theorem 1. Recall that the Weyl tensor of a Riemannian manifold \( M^n \) is defined by

\[
R(X, Y) = \hat{\rho}X \wedge Y - \hat{\rho}Y \wedge X + W(X, Y),
\]

where \( \hat{\rho} = \frac{1}{n-2} \text{Ric} - \frac{n-6}{n-2} \text{scal} \text{id} \), Ric is the Ricci operator, scal is the scalar curvature and \( X \wedge Y \) is the skew-symmetric operator defined by \( (X \wedge Y)Z = (X, Z)Y - (Y, Z)X \). According to Conjecture A, the Weyl tensor has either a Clifford or a Cayley structure. First of all, in Section 3.1 we show that both these structures can be chosen smooth on an open, dense subset \( M' \subset M^{16} \) (see Lemma 4 for the precise statement), so that on every connected component \( M_\alpha \) of \( M' \), the curvature tensor \( R \) of \( M^{16} \) is given by either (13) or (14) (up to a conformal equivalence), with all the operators and the functions involved being locally smooth. Then we establish the local version of the theorem, at every point \( x \in M' \), for each of the two cases separately. This is done by using the differential Bianchi identity and the fact that under a conformal change of the metric, the symmetric tensor field \( \rho \) (which is a linear combination of \( \hat{\rho} \) from (12) and the identity) is a Codazzi tensor, that is, \( (\nabla_X \rho)Y = (\nabla_Y \rho)X \). Using the result of [DS], we show that \( \rho \) must be a constant multiple of the identity, which implies that every connected component \( M_\alpha \subset M' \) is locally conformally equivalent to a symmetric Osserman manifold.

The proof in the Clifford case is given in Section 3.2 (Lemma 5 and Lemma 6), in the Cayley case, in Section 3.3 (Lemma 7). Then, by the result of [GSV, Lemma 2.3], every \( M_\alpha \) is either locally conformally flat or is locally conformal to a rank-one symmetric space. In Section 3.4 we prove the conformal part of Theorem 1 globally, by first showing (using Lemma 8) that \( M \) splits into a disjoint union of a closed subset \( M_0 \), on which the Weyl tensor vanishes, and nonempty open connected subsets \( M_\alpha \), each of which is locally conformal to one of the rank-one symmetric spaces. On every \( M_\alpha \), the conformal factor \( f \) is a well-defined positive smooth function. Assuming that there exists at least one \( M_\alpha \) and that \( M_0 \neq \emptyset \) we show that there exists a point \( x_0 \in M_0 \) on the boundary of a geodesic ball \( B \subset \mathcal{M} \) such that both \( f(x) \) and \( \nabla f(x) \) tend to zero when \( x \to x_0 \), \( x \in B \) (Lemma 9). Then the positive function \( u = f^{7/2} \)
satisfies an elliptic equation in $B$, with $\lim_{x \to x_0, x \in B} u(x) = 0$, hence by the boundary point theorem, the limiting value of the inner derivative of $u$ at $x_0$ must be positive. This contradiction implies that either $M = M_0$ or $M = M_\alpha$, thus proving the conformal part of Theorem 1. The “genuine Osserman” part of Theorem 1 then follows using the result of [Nic].

3.1. Smoothness of the Clifford and of the Cayley structures. Let $M^{16}$ be a connected smooth Riemannian manifold whose Weyl tensor at every point is Osserman. Define a function $N : M^{16} \to \mathbb{N}$ as follows: for $x \in M^{16}$, $N(x)$ is the number of distinct eigenvalues of the operator $W_{X \perp Y}^M$, where $W_X$ is the Jacobi operator associated to the Weyl tensor and $X$ is an arbitrary nonzero vector from $T_x M^{16}$. As the Weyl tensor is Osserman, the function $N(x)$ is well-defined. Moreover, as the set of symmetric operators having no more than $N_0$ distinct eigenvalues is closed in the linear space of symmetric operators on $\mathbb{R}^{15}$, the function $N(x)$ is lower semi-continuous (every subset $\{x : N(x) \leq N_0\}$ is closed in $M^{16}$).

Let $M'$ be the set of points where the function $N(x)$ is continuous (that is, locally constant). It is easy to see that $M'$ is an open and dense (but possibly disconnected) subset of $M^{16}$. The following lemma shows that, assuming Conjecture A, all the “ingredients” of the curvature tensor are locally smooth on every connected component of $M'$.

**Lemma 4.** Let $M^{16}$ be a smooth conformally Osserman Riemannian manifold. Let $M'$ be the (open, dense) subset of all the points of $M^{16}$ at which the number of distinct eigenvalues of the Jacobi operator associated to the Weyl tensor of $M^{16}$ is locally constant.

Assume Conjecture A. Then for every $x \in M'$, there exists a neighborhood $U = U(x)$ with exactly one of the following properties.

(a) There exists $\nu \geq 0$, smooth functions $\eta_1, \ldots, \eta_\nu : U \to \mathbb{R} \setminus \{0\}$, a smooth symmetric linear operator field $\rho$ and smooth anticommuting almost Hermitian structures $J_i$, $i = 1, \ldots, \nu$, on $U$ such that for all $y \in U$ and all $X, Y, Z \in T_y M^{16}$, the curvature tensor of $M^{16}$ has the form

\[
(13) \quad R(X, Y)Z = (X, Z)\rho Y + \langle \rho X, Z \rangle Y - \langle Y, Z \rangle \rho X - \langle \rho Y, Z \rangle X + \sum_{i=1}^\nu \eta_i(2(J_iX, Y)J_iZ + (J_iZ, Y)J_iX - (J_iZ, X)J_iY).
\]

(b) The Riemannian manifold $U$ is conformally equivalent to a Riemannian manifold whose curvature tensor has the form

\[
(14) \quad R(X, Y) = \rho X \wedge Y - \rho Y \wedge X + \varepsilon \sum_{i=0}^8 S_iX \wedge S_iY = \rho X \wedge Y - \rho Y \wedge X + \varepsilon \mathcal{A}(X \wedge Y),
\]

at every point $y \in U$, where $\varepsilon = \pm 1$ and $\rho, S_i, i = 0, \ldots, 8$, are smooth fields of symmetric operators on $U$ satisfying (8).

**Proof.** On every connected component $M_\alpha \subset M'$, the number $N = N(x)$ is a constant, so the operator $W_{X \perp Y}$, where $X$ is a unit vector, has exactly $N$ distinct eigenvalues $\mu_0, \mu_1, \ldots, \mu_{N-1}$, with the multiplicities $m_0, m_1, \ldots, m_{N-1}$. The functions $\mu_i$’s are smooth on $M_\alpha$, and the $m_i$’s are constants, by the smoothness of the characteristic polynomial of $W_{X \perp Y}$. We label them in such a way that $m_0 = \max(m_0, m_1, \ldots, m_{N-1})$.

Clearly, $\sum_{i=0}^{N-1} m_i = 15$ and, as $Tr W_X = 0$, we have $\sum_{i=0}^{N-1} m_i \mu_i = 0$. It follows that if $N = 1$, then $W = 0$, so $M_\alpha$ is conformally flat. Then by (12), the curvature tensor has the form (13), with $\nu = 0$ and a smooth $\rho$.

Suppose $N > 2$. By Conjecture A, $W$ either has a Clifford structure, or a Cayley structure. But in the latter case, the operator $W_{X \perp Y}$ has two distinct eigenvalues (from assertion 4 of Lemma 3). It follows that $W$ has a Clifford structure $\text{Cliff}(\nu)$, at every point of $M_\alpha$ ($\nu$ may a priori depend on $x \in M_\alpha$). By assertion 1 of Lemma 1, $\nu \leq 8$, and by Remark 1, for a unit vector $X$, the eigenvalues of $W_{X \perp Y}$ are $\lambda_0$, of multiplicity $15 - \nu$, and $\lambda_0 + 3\eta_i$, $i = 1, \ldots, \nu$. All of the $\eta_i$’s are nonzero (by Definition 3), some of them can be equal, but not all, as otherwise $N = 2$, so the multiplicity of every eigenvalue $\lambda_0 + 3\eta_i$ is at most $\nu - 1 \leq 15 - \nu$, as $\nu \leq 8$. It follows that the maximal multiplicity is $m_0 = 15 - \nu$ (at $7$), so $\nu = 15 - m_0$, which is a constant on $M_\alpha$. Moreover, $\lambda_0 = \mu_0$ (this is automatically satisfied, unless $\nu = 8$ and $\eta_1 = \cdots = \eta_\nu \neq \eta_\nu$; in the latter case we have two eigenvalues of multiplicity $7$ and we choose the labeling of the $\mu_i$’s so that $\mu_0 = \lambda_0$). The functions $\lambda_0$ and $\lambda_0 + 3\eta_i$ are smooth, as each of them equals
one of the $\mu_i$'s. Moreover, for every smooth unit vector field $X$ on $M_\alpha$ and every $i = 1, \ldots, N - 1$, the $\mu_i$-eigendistribution of $W_{X|X \perp}$ (which must be smooth on $M_\alpha$ and must have a constant dimension $m_i$) is $\text{Span}(J\lambda \eta + 3\eta, J(X,X))$, by Remark 1. By assertion 3 of Lemma 3 of [N5], there exists a neighborhood $U_i(x)$ and smooth anticommuting almost Hermitian structures $J'_i$ (for the $j$'s such that $\lambda_0 + 3\eta_0 = \mu_i$) on $U_i(x)$ such that $\text{Span}(J\lambda \eta + 3\eta, J(X,X)) = \text{Span}(J\lambda \eta + 3\eta_j, J'_i J(X,X))$. Let $W'$ be a (unique) algebraic curvature tensor on $U = \bigcap_{i=1}^N U_i(x)$ with the Clifford structure $\text{Cliff}(\nu; J_1', \ldots, J'_N; \lambda_0, \eta_1, \ldots, \eta_N)$. Then $\nu = 15 - m_0$ is constant and all the $J_i, \eta_i$ and $\lambda_0$ are smooth on $U$. Moreover, for every unit vector field $X$ on $U$, the Jacobi operators $W'_X$ and $W_X$ have the same eigenvalues and eigenvectors by construction, hence $W'_X = W_X$, which implies $W' = W$. Then the curvature tensor on $U$ has the form (13), with the operator $\rho$ given by $\rho = \frac{1}{n \nu} \text{Ric} + \frac{1}{n(n-1)(n-2)} \text{id}$, by (12). As $\lambda_0$ is a smooth function, the operator field $\rho$ is also smooth.

Now consider the case $N = 2$. Again, by Conjecture A, $W$ either has a Clifford structure, or a Cayley structure. In the former case, by Remark 1, for a unit vector $X$ at every point $x \in M_\alpha$, the eigenvalues of $W_{X|X \perp}$ are $\lambda_0$, of multiplicity $15 - \nu$, and $\lambda_0 + 3\eta \neq 0$, of multiplicity $\nu$. In the latter case, there are two eigenvalues, of multiplicities $m_0 = 8$ and $m_1 = 7$, respectively (as it follows from assertion 4 of Lemma 3). In the both cases, $m_0 \geq 8$. It follows that if $m_0 > 8$, the Weyl tensor $W$ has a Clifford structure $\text{Cliff}(m_1)$, at every point $x \in M_\alpha$. Then we can finish the proof as in the case $N > 2$ considered above, as on $M_\alpha$, the functions $\lambda_0 = \mu_0$ and $\lambda_0 + 3\eta = \mu_1$ are smooth and $\text{Span}(J\lambda \eta, J(X,X))$, the $\mu_1$-eigendistribution of $W_{X|X \perp}$, is constant when $m_1 = 7$. Assertion 3 of Lemma 3 of [N5] applies and we obtain the curvature tensor of the form (13) (with $\nu = m_1$ and all the $\eta_i$'s equal) on some neighborhood $U = U(x)$ of an arbitrary point $x \in M_\alpha$.

Suppose now that $N = 2$, and $m_0 = 8, m_1 = 7$. Then at every point $x \in M_\alpha$, the Weyl tensor either has a Clifford structure $\text{Cliff}(7)$, with $\eta_1 = \cdots = \eta_7 = \eta$, or a Clifford structure $\text{Cliff}(8)$, with $\eta_1 = \cdots = \eta_8 = \eta$, or a Cayley structure. Denote $M(7), M(8)$ and $M^{(0)}$ the corresponding subsets of $M_\alpha$, respectively. These three subsets are mutually disjoint. Indeed, if $W = a R^{(5)} + b R^{(6)}, a \neq 0$, would have a Clifford structure, then the same would be true for $R^{(3)} = a^{-1} W - ba^{-1} R^{(5)}$ (as by Definition 3, the set of algebraic curvature tensors with a Clifford structure is invariant under scaling and shifting by a constant curvature tensor). This contradicts the fact that $R^{(3)}$ has no Clifford structure [N3, Remark 1]. Moreover, if $x \in M(7) \cap M(8)$, then for any unit vector $X \in T_x M^{16}$ the operator $W_{X|X \perp}$ has an eigenspace of dimension $7$ spanned by orthonormal vectors $J_1 X, \ldots, J_7 X$ (where the $J_i$ are defined by the Clifford structure $\text{Cliff}(7)$), and the orthogonal eigenspace of dimension $8$ spanned by orthonormal vectors $J_8 X, \ldots, J_{16} X$ (where the $J_i$ are defined by $\text{Cliff}(8)$). Then the fifteen operators $J_1, J'_i$ are anticommuting almost Hermitian structures on $\mathbb{R}^{16}$, which contradicts the fact that $\nu \leq 8$ (assertion 1 of Lemma 1). It follows that $M_\alpha = M(7) \cup M(8) \cup M^{(0)}$. Moreover, each of the three subsets $M(7), M(8), M^{(0)}$ is relatively closed in $M_\alpha$. Indeed, suppose a sequence $(x_n) \subset M(7)$ converges to a point $x^0 \in M_\alpha$. Then the functions $\lambda(x_n)$ and $\lambda(x_n) + \eta(x_n)$ are well-defined (as $M(7), M(8), M^{(0)}$ are disjoint) and bounded (as $\lambda^2(x_n) + (\lambda(x_n) + \eta(x_n))^2 = \mu_0(x_n)^2 + \mu_1(x_n)^2$), so we can assume that both sequences converge by choosing a subsequence; moreover, as all the $J_i(x_n)$ are orthogonal operators, we can choose a subsequence such that all of them converge. Then by continuity, $W(x^0)$ has the form (1), with $\nu = 7$, with the $J_i$'s being anticommuting almost Hermitian structures, and with $\eta \neq 0$, as otherwise $N = 1$, which contradicts the fact that $N = 2$ on $M_\alpha$. Therefore, $x^0 \in M(7)$. The above arguments work for $M(8)$ almost verbatim (by replacing $7$ by $8$), and for $M^{(0)}$, with a slight modification (by replacing the orthogonal operators $J_i$'s by the orthogonal operators $S_i$ from (7)). Hence, as $M_\alpha$ is connected, it coincides with exactly one of the sets $M(7), M(8), M^{(0)}$.

Now, if $M_\alpha = M(7)$ or if $M_\alpha = M(8)$, the proof follows from the same arguments as in the case $N > 2$: we have a Clifford structure for $W$ with a constant $\nu$.

Suppose $M_\alpha = M^{(0)}$. Then by (7, 12), the Weyl tensor on $M_\alpha$ has the form $W(X,Y) = b X \wedge Y + a \sum_{i=0}^8 S_i X \wedge S_i Y$, with $a \neq 0$ (actually, $5b = 3a \neq 0$, as $\text{Tr} W_X = 0$; see (46b)), so by (12), the curvature tensor at every point $x \in M_\alpha$ has the form

$$R(X,Y) = \rho X \wedge Y - \rho Y \wedge X + f \sum_{i=0}^8 S_i X \wedge S_i Y,$$
where $\rho$ is a symmetric operator and $f \neq 0$ (as $N = 2$ on $M_\alpha$). As $S^2_\alpha = \text{id}$ (by (8)) and $\text{Tr} S_i = 0$ (see the proof of assertion 1 of Lemma 3), at every point $x \in M_\alpha$, the Ricci operator, the scalar curvature and the Weyl tensor of $M^{16}$ are given by
\[
\text{Ric} \ X = 14pX + (\text{Tr} \rho - 9f)X, \quad \text{scal} = 30 \text{Tr} \rho - 144f, \quad W(X, Y) = \frac{1}{2} f(3X \wedge Y + 5 \sum_{i=0}^{s} S_i X \wedge S_i Y),
\]
a direct computation using (8) and the fact that the $S_i$’s are symmetric, orthogonal and $\text{Tr} S_i = 0$ gives $|W|^2 = \frac{272}{15} f^2$ (see (46c)). As $f \neq 0$, it follows that $f$ is a smooth function (hence $\rho$ is a smooth symmetric operator, as $\text{scal}$ and $\text{Ric}$ are smooth) on $M_\alpha$. Introduce an algebraic curvature tensor $P$ defined by $P(X, Y) = f^{-1}(R(X, Y) - \rho X \wedge Y + \rho Y \wedge X) = \sum_{i=0}^{s} S_i X \wedge S_i Y = \mathcal{A}(X \wedge Y)$, where the last equation follows from assertion 3 of Lemma 3. As $P$ is smooth, the field $\mathcal{A}$ of the endomorphisms of the bundle $\text{Skew}(M_\alpha)$ of skew-symmetric endomorphisms over $M_\alpha$ is smooth (the fact that $\mathcal{A}$ is an endomorphism of the bundle $\text{Skew}(M_\alpha)$ follows from assertion 2 of Lemma 3).

Then the matrix bundles $\mathcal{A}$, $\mathcal{L}_2(M_\alpha)$ and $\mathcal{L}_3(M_\alpha)$ (assertion 2 of Lemma 3) are also smooth. As the matrix product is smooth, it follows that the subbundle $(\mathcal{L}_2(M_\alpha))^2 = \text{Span}(K_1 K_2, K_1, K_2 \in \mathcal{L}_2(M_\alpha))$ is smooth. By the definition of $\mathcal{L}_2$ and from (8), $(\mathcal{L}_2(M_\alpha))^2 = \mathcal{L}_0(M_\alpha) \oplus \mathcal{L}_2(M_\alpha) \oplus \mathcal{L}_4(M_\alpha)$, with the direct sum being orthogonal relative to the (smooth) inner product in $\text{End}(M_\alpha)$ (assertion 1 of Lemma 3). Since $\mathcal{L}_2(M_\alpha)$ is smooth and $\mathcal{L}_0(M_\alpha)$ is a one-dimensional bundle spanned by the identity operator, the bundle $\mathcal{L}_4(M_\alpha)$ is smooth. Then the bundle $(\mathcal{L}_4(M_\alpha))^2 = \bigoplus_{s=0}^{4} \mathcal{L}_2(M_\alpha) = \bigoplus_{s=0}^{4} \mathcal{L}_s(M_\alpha)$ (the latter equation follows from assertion 1 of Lemma 3) is also smooth. The direct sum on the right-hand side is orthogonal with respect to the smooth inner product and the bundles $\mathcal{L}_s(M_\alpha)$, $s = 0, 2, 3, 4$, are smooth, as it is shown above. Hence $\mathcal{L}_1(M_\alpha)$ is a smooth subbundle of $\text{End}(M_\alpha)$. It follows that on some neighborhood $\mathcal{U}(x)$ of an arbitrary point $x \in M_\alpha$ we can choose nine smooth sections $S_i$, $i = 0, 1, \ldots, 8$, of $\mathcal{L}_1(M_\alpha)$, which are orthogonal and all have norm 4. By assertion 2 of Lemma 3, the operator $\mathcal{A}$ does not change, if we replace $S_i$ by $S_i$, so by assertion 3 of Lemma 3, the curvature tensor (15) remains unchanged if we replace the operators $S_i$ by $S_i$, therefore we can assume $f, \rho$ and the $S_i$’s in (15) to be smooth on $\mathcal{U}$. Under a conformal change of metric $\tilde{g} = h g$, $h > 0$, for a positive smooth function $h = e^{2\phi} : \mathcal{U} \to \mathbb{R}$, the curvature tensor transforms as $\tilde{R}(X, Y) = R(X, Y) - (X \wedge KY + KY \wedge X)$, where $K = H(\phi) - \nabla \phi \otimes \nabla \phi + \frac{1}{2} \|\nabla \phi\|^2 \text{id}$ and $H(\phi)$ is the symmetric operator associated to the Hessian of $\phi$ (see Lemma 8). As $X \wedge Y = \tilde{h} X \wedge Y$ we obtain $\tilde{R}(X, Y) = \tilde{\rho} X \wedge Y - \tilde{\rho} Y \wedge X + f h^{-1} \sum_{i=0}^{s} S_i X \wedge S_i Y$, where $\tilde{\rho} = h^{-1}(\rho - K)$. Taking $h = |f|$ (and dropping the tildes) we obtain (14), with $\varepsilon = \pm 1$ (9).

### 3.2. Clifford case

Let $x \in M'$ and let $\mathcal{U} = \mathcal{U}(x)$ be the neighborhood of $x$ defined in assertion (a) of Lemma 4. By the second Bianchi identity, $(\nabla_{\nabla R})(X, Y) + (\nabla_{\nabla R})(U, X)Y + (\nabla_{\nabla R})(Y, U)Y = 0$. Substituting $R$ from (13) and using the fact that the operators $J_i$’s and their covariant derivatives are skew-symmetric and the operator $\rho$ and its covariant derivatives are symmetric we get:

\[
\begin{align*}
\langle X, Y \rangle (\langle \nabla \rho X \rangle Y - \langle \nabla \rho Y \rangle U) + |Y|^2 (\langle \nabla X \rangle U - \langle \nabla Y \rangle X) + \langle U, Y \rangle (\langle \nabla Y \rangle X - \langle \nabla X \rangle Y) \\
+ \langle \langle \nabla \rho X \rangle U - \langle \nabla \rho Y \rangle Y, X \rangle + \langle \langle \nabla X \rangle Y - \langle \nabla Y \rangle X, U \rangle + \langle \langle \nabla \rho Y \rangle X - \langle \nabla \rho U \rangle Y, Y \rangle \\
+ \sum_{i=1}^{(\nu)} 3 \langle X (\eta_i), J_i U, X \rangle - U (\eta_i) \langle J_i Y, X \rangle J_i Y \\
+ \sum_{i=1}^{(\nu)} 3 \eta_i (2 \langle J_i U, X \rangle J_i Y + \langle J_i Y, X \rangle J_i U - \langle J_i Y, U \rangle J_i X) \\
+ \sum_{i=1}^{(\nu)} 3 \langle X (\eta_i), \langle \nabla_{\nabla J_i} J_i X, Y \rangle + 3 \langle \nabla_{J_i} X, Y \rangle J_i U + 2 \langle \nabla_{J_i} U, X \rangle \rangle J_i Y \\
+ 3 \langle J_i X, Y \rangle \langle \nabla_{\nabla J_i} J_i Y + 3 \langle J_i Y, U \rangle \langle \nabla_{J_i} X \rangle J_i Y + 2 \langle J_i U, X \rangle \langle \nabla_{J_i} Y \rangle \rangle J_i Y \\
+ \langle \langle \nabla_{J_i} Y, X \rangle J_i U + \langle J_i Y, X \rangle \langle \nabla_{J_i} J_i X \rangle U - \langle \langle \nabla_{J_i} Y, J_i U \rangle, X \rangle J_i X - \langle J_i Y, U \rangle \langle \nabla_{J_i} Y, X \rangle X = 0.
\end{align*}
\]

Taking the inner product of (16) with $X$ and assuming $X, Y$ and $U$ to be orthogonal we obtain

\[
\begin{align*}
|X|^2 (\langle Q(Y), U \rangle) + |Y|^2 (\langle Q(X), U \rangle) - \langle Q(Y), Y \rangle \langle U, Y \rangle - \langle Q(X), Y \rangle \langle X, U \rangle \\
+ \sum_{i=1}^{(\nu)} 3 \langle X (\eta_i), J_i U, X \rangle - Y (\eta_i) \langle J_i X, U \rangle - U (\eta_i) \langle J_i Y, X \rangle \rangle J_i Y, X \\
+ \sum_{i=1}^{(\nu)} 3 \eta_i (2 \langle \langle \nabla_{\nabla J_i} X, Y \rangle + \langle \langle \nabla_{J_i} X, Y \rangle U + \langle \langle \nabla_{J_i} U, X \rangle \rangle J_i Y, X \\
- \langle J_i Y, U \rangle \langle \nabla_{J_i} X, Y \rangle - \langle J_i X, U \rangle \langle \nabla_{J_i} Y, X \rangle \rangle) J_i Y, X = 0,
\end{align*}
\]

where $Q = \text{id}$.
where \( Q : \mathbb{R}^{16} \rightarrow \mathbb{R}^{16} \) is the quadratic map defined by
\[
(Q(X), U) = \langle (\nabla_X \rho) U - (\nabla_U \rho) X, X \rangle.
\]

Note that \( \langle Q(X), X \rangle = 0 \).

**Lemma 5.** In the assumptions of Lemma 4, let \( x \in M' \) and let \( U \) be the neighborhood of \( x \) introduced in assertion (a) of Lemma 4. Suppose that \( \nu > 4 \). For every point \( y \in U \), identify \( T_y M^{16} \) with the Euclidean space \( \mathbb{R}^{16} \) via a linear isometry. Then
1. There exist \( m_i, b_{ij} \in \mathbb{R}^{16}, i, j = 1, \ldots, \nu, \) such that for all \( X \in \mathbb{R}^{16} \) and all \( i, j = 1, \ldots, \nu, \)
\[
(19a) \quad Q(X) = 3 \sum_{k=1}^{\nu} \langle m_k, X \rangle J_k X,
\]
\[
(19b) \quad \langle \nabla_X J_i \rangle X = \eta_i^{-1}(\|X\|^2 m_i - \langle m_i, X \rangle X) + \sum_{j=1}^{\nu} \langle b_{ij}, X \rangle J_j X,
\]
\[
(19c) \quad b_{ij} + b_{ji} = \eta_i^{-1} J_i m_i + \eta_j^{-1} J_j m_j.
\]
2. The following equations hold for all \( X, Y, \in \mathbb{R}^{16} \) and all \( i, j = 1, \ldots, \nu, \)
\[
(20a) \quad \nabla \eta_i = 2 J_i m_i,
\]
\[
(20b) \quad \eta_i b_{ij} + b_{ji} \eta_j = 0, \quad i \neq j,
\]
\[
(20c) \quad J_i m_i = \eta_i p, \quad \text{for some } p \in \mathbb{R}^{16}.
\]

**Proof.** 1. Equation (17) is a polynomial equation in \( 48 \) real variables, the coordinates of the vectors \( X, Y, U \). It must still hold if we allow \( X, Y, U \) to be complex and extend the \( J_i \)'s, the \( \nabla X \)'s and \( \langle \cdot, \cdot \rangle \) by complex linearity (bilinearity) to \( \mathbb{C}^{16} \). The complexified inner product \( \langle \cdot, \cdot \rangle \) is a nonsingular quadratic form on \( \mathbb{C}^{16} \) (not a Hermitian inner product on \( \mathbb{C}^{16} \)).

From (17), for any two vectors \( X, Y \in \mathbb{R}^{16} \) with \( Y \perp I_C X \), we get
\[
(21) \quad ||X||^2 Q(Y) + ||Y||^2 Q(X) - \langle Q(X), Y \rangle Y - \langle Q(Y), X \rangle X
\]
\[\quad - \sum_{i=1}^{\nu} 3 \eta_i \langle \langle \nabla_X J_i \rangle X, Y \rangle Y + \langle (\nabla_X J_i) X, Y \rangle X J_i X = 0.\]

By assertion 2 of Lemma 1, the Clifford structure has one of two possible forms. We prove identities (19) separately for each of them.

**Case (a)** The representation of \( \text{Cl}(\nu) \) is a restriction to \( \text{Cl}(\nu) \subset \text{Cl}(8) \) of the representation \( \rho_8 \) of \( \text{Cl}(8) \) given by (3).

By complexification, we can assume that \( J_p X \) is given by equation (3), where \( X = (a, b) \in \mathbb{C}^8 \) and \( a, b, p \in O \otimes C \). Denote \( \mathcal{C} = \{X = (a, b) : \|X\|^2 = 0\} \subset \mathbb{C}^{16} \) the isotropic cone, and for \( X \in \mathbb{C}^{16} \), denote \( T_0 X \mathcal{C} \) the complex linear span of \( X, J_1 X, \ldots, J_8 X \). For \( X \in \mathcal{C} \), the space \( T_0 X \mathcal{C} \) is isotropic: the inner product of any two vectors from \( T_0 X \mathcal{C} \) vanishes. Take \( Y = J_q X \in T_0 X \mathcal{C} \), with some \( q \in O \otimes C \). Then from (21) we obtain:
\[
\sum_{i=1}^{\nu} \eta_i \langle \langle \nabla_X J_i \rangle X, J_q X \rangle J_q X \in T_0 X \mathcal{C}.
\]
Introduce the (\( \mathcal{C} \))-linear operator \( M_X : O \otimes \mathcal{C} \rightarrow O \otimes \mathcal{C} \) by \( M_X(q) = \sum_{i=1}^{\nu} \eta_i \langle \langle \nabla_X J_i \rangle X, J_q X \rangle e_i \). Then, as \( T_0 X \mathcal{C} \) is isotropic, we get \( J_{M_X(q)} J_q X \perp T_0 X \mathcal{C} \), for all \( X \in \mathcal{C} \) and all \( q \in O \otimes \mathcal{C} \). Then by (3), for any \( X = (a, b) \in \mathcal{C} \) and any \( p \in O \otimes \mathcal{C} \) we obtain \( ((-aq^*) M_X(q), -(bp)(M_X(q))^* \perp (bp, -ap^*) \), so
\[
(22) \quad (M_X(q)(bq^*)a = b^*((aq^*)M_X(q)).
\]

The biquaternion equation \( (m(bq^*)a - b^*((aq^*)m) = 0 \) for \( m \in O \otimes \mathcal{C}, \) with \( (a, b, q) \) on the algebraic surface \( \mathcal{S} = \mathcal{C} \times \mathbb{C}^8 \subset \mathbb{C}^{24} \), can be viewed as a system of eight linear equations for \( m \in \mathbb{C}^8 \). Let \( \mathcal{M}(a, b, q) \) be the matrix of this system. As both \( m = q \) and \( m = b^*a \) are solutions to the system, \( \text{rk} \mathcal{M}(a, b, q) \leq 6 \), for all the points \( (a, b, q) \) from a nonempty Zariski open subset of \( \mathcal{S} \). On the other hand, if \( a = q = 1, b \perp 1, \|b\|^2 = -1 \), the equation has the form \( mb^* = b^*m \), which implies that \( m \in \text{Span}_\mathbb{C}(1, b) \), so \( \text{rk} \mathcal{M}(a, b, q) \geq 6 \) for all the points \( (a, b, q) \) from a nonempty Zariski open subset of \( \mathcal{S} \). It follows that for a nonempty Zariski open subset of \( \mathcal{S} \), \( \text{rk} \mathcal{M}(a, b, q) = 6 \) and the solution set is \( \text{Span}_\mathbb{C}(q, b^*a) \). Therefore
from (22), $M_X(q) \in \text{Span}_c(q, b^*a)$, for all $(a, b, q)$ from a nonempty Zariski open subset of $S$, hence $\text{rk}(q, M_X(q), b^*a) \leq 3$, for all $X = (a, b) \in C$, $q \in C^8$. It follows that the linear operators from $C^8$ to $\text{Skew}(C^8)$ defined by $q \mapsto M_X(q) \wedge (b^*a)$ and $q \mapsto q \wedge (b^*a)$ are linearly dependent, for every $X = (a, b) \in C$, so $M_X(q) \wedge (b^*a) = c_X q \wedge (b^*a)$, for $c_X \in C$. Then $(M_X q - c_X q) \wedge (b^*a) = 0$, so $M_X(q) = c_X q + \alpha_X(q) b^*a$, for all $X = (a, b) \in C$ such that $b^*a \neq 0$, where $\alpha_X$ is a linear form on $C^8$. By the definition of $M_X(q)$, $c_X q + \alpha_X(q) b^*a = \sum_{i=1}^{\nu} \eta_i (\langle \nabla_X J_i X, J_J X \rangle e_i)$. Substituting $q = e_j$, $j \leq \nu$, and taking the inner product with $e_k$, $k \leq \nu$, we obtain $\eta_k^{-1} c_X \delta_{jk} + \alpha_X(e_j) \cdot \eta_k^{-1} (b^*a, e_k) + \alpha_X(e_k) \cdot \eta_j^{-1} (b^*a, e_j) = 0$.

As the rank of the $\nu \times \nu$-matrix $A_X$ defined by $(A_X)_{jk} = \alpha_X(e_j) \cdot \eta_k^{-1} (b^*a, e_k) + \alpha_X(e_k) \cdot \eta_j^{-1} (b^*a, e_j)$ is at most two and as $\nu > 4$, we obtain that $c_X = 0$, for all $X = (a, b) \in C$ such that $b^*a \neq 0$, hence $A_X = 0$. Now, if $\nu = 8$, it follows from $A_X = 0$, $b^*a \neq 0$, that $\alpha_X = 0$, so $M_X = 0$. If $\nu < 8$, then, as $c_X = 0$, we have $M_X(q) = \alpha_X(q) b^*a = \sum_{i=1}^{\nu} \eta_i (\langle \nabla_X J_i X, J_J X \rangle e_i)$, so $\alpha_X(q) (b^*a, e_i) = 0$, for all $s > \nu$. Choosing $X = (a, b) \in C$ such that all the components of $b^*a$ are nonzero, we again obtain that $\alpha_X = 0$, so $M_X = 0$. It follows that $M_X(q) = 0$ for all $X$ from a nonempty open subset of $C$, hence for all $X \in C$.

From the definition of $M_X(q)$, it follows that for all $X \in C$ and all $i = 1, \ldots, \nu$, $j = 1, \ldots, 8$, $\langle \langle \nabla_X J_i X, J_J X \rangle, X \rangle$ is defined by $(A_X)_{jk} = \alpha_X(e_j) \cdot \eta_k^{-1} (b^*a, e_k) + \alpha_X(e_k) \cdot \eta_j^{-1} (b^*a, e_j)$ is divisible by $\|X\|^2 \nu \over C$, and hence over $\mathbb{R}$. Then by assertion 1 of Lemma 2, $(\nabla_X J_i X) = \sum_{j=1}^{\nu} \langle b_{ij}, X \rangle J_i J_J X = (V_i, X) X - \|X\|^2 U_i$, for all $i = 1, \ldots, \nu$, where $b_{ij}, V_i, U_i$ are some vectors from $R^{10}$ (note that the summation on the right-hand side is up to $8$, not up to $\nu$, as in (19b)). As $(\langle \nabla_X J_i X, X \rangle, X) = 0$, we have $V_i = U_i$, so

\begin{equation}
(\nabla_X J_i X) = \sum_{j=1}^{\nu} \langle b_{ij}, X \rangle J_i J_J X + \eta_i^{-1} (\|X\|^2 m_i - \langle m_i, X \rangle, X),
\end{equation}

for some $m_i \in R^{10}$. Substituting this into (21), complexifying the resulting expression and taking $X \in C$, $Y = J_X X$, with some $q \in C \in C^8$, the polynomial $\langle \langle X \rangle, J_X X - \langle Q(J_X X), X \rangle \rangle = 0$. As $X$ and $J_X X$ are linearly independent for a nonempty Zariski open subset of $(X, q) \in \mathcal{S} = \mathbb{C} \in C^8$, the polynomial $\langle \langle X \rangle, J_X X \rangle$ vanishes on $\mathcal{S}$, so for all $q \in C \in C^8$, the polynomial $\langle \langle Q(X), J_X X \rangle \rangle$ is divisible by $\|X\|^2$. Then $\langle \langle Q(X), J_X X \rangle \rangle$ is divisible by $\|X\|^2$, for all $i = 1, \ldots, 8$, which by assertion 1 of Lemma 2, implies that $\langle \langle Q(X), Y \rangle \rangle = \|X\|^2 (Y, U)$, for some fixed $U \in R^{10}$, where $Y \perp T^0 X$.

It then follows from (21) and (23) that for all $X, Y \in R^{10}$, with $Y \perp T^0 X$,

\begin{equation}
\|X\|^2 T(Y) + \|Y\|^2 T(X) = 0,
\end{equation}

where the quadratic form $T : R^{10} \rightarrow R^{10}$ is defined by $T(X) = Q(X) - \langle X, U \rangle X - 3 \sum_{i=1}^{\nu} \langle m_i, X \rangle J_i X$. We want to show that $T = 0$. For some $E \in R^{10}$, the quadratic form $t(X) = \langle T(X), E \rangle$ is nonzero. Then from (24), $\|X\|^2 t(Y) + \|Y\|^2 t(X) = 0$, for all $X, Y \in R^{10}$, $Y \perp T^0 X$. If $X = (a, 0)$, $Y = (b, 0)$, $a \perp b$, then $Y \perp T^0 X$ by (3), so $\|a\|^2 t((b, 0)) + \|b\|^2 t((a, 0)) = 0$ which implies $t((a, 0)) = 0$, for all $a \in O$. Similarly, $t((0, b)) = 0$, for all $b \in O$. It follows that $t((a, b)) = \langle L(a, b) \rangle$ for some $L \in \text{End}(O)$. From (3), any $X = (a, b), a, b \neq 0$, and any $Y = (\|b\|^2 bu, \|a\|^2 au^*)$, with $u \perp b^*a$, satisfy $Y \perp T^0 X$. Then from $\|X\|^2 t(Y) + \|Y\|^2 t(X) = 0$ and $t((a, b)) = \langle L(a, b) \rangle$, we obtain $\|u\|^2 L(a, b) = (L(bu), au^*)$, for all $u \perp b^*a$ (the condition $a, b \neq 0$ can be dropped). It follows that $\langle \|u\|^2 L^* b - L(bu) \cdot u \rangle = 0$ for all $a \perp bu$ (where $L^*$ is the operator transposed to $L$), so $\|u\|^2 L^* b - L(bu) \cdot u$ for all $b, u \in O$. Taking the inner product with $bu$ we get $\langle \|u\|^2 L^* b - L(bu) \cdot u \rangle$ for all $b, u \in O$. Taking the inner product with $bu$ we get $\langle \|u\|^2 L^* b - L(bu) \cdot u \rangle = 0$. Taking $u = 1$ we get $L^* = L$, so $Lb \cdot u^* = L(bu)$, for all $b, u \in O$. Substituting $b = 1$ we obtain $L = pu^*$, where $p = L$ is $L$ is $(pb)^* u^* = p(bu)^*$. If $p \neq 0$, this equation, with $b = p$, implies $p^* u = pu^* p$, for all $u \in O$. This contradiction shows that $p = 0$, hence $L = 0$, so $\ell = 0$.

Therefore, the quadratic form $T(X) = Q(X) - \langle X, U \rangle X - 3 \sum_{i=1}^{\nu} \langle m_i, X \rangle J_i X$ vanishes, which implies $Q(X) = \langle X, U \rangle X + 3 \sum_{i=1}^{\nu} \langle m_i, X \rangle J_i X$. Substituting this and (23) into (21), with $Y \perp T^0 X$, we get $U = 0$, which proves (19a).

Now, if $\nu = 8$, then (19b) follows from (23). If $\nu < 8$, choose $X \neq 0, Y = J_X X, s > \nu$. Substituting into (21) and using (23) and (19a) we get $\sum_{i=1}^{\nu} \eta_i (\langle b_{is}, X \rangle J_i J_X X + \langle b_{is}, J_X X \rangle J_J X) = 0$. Taking $X = (a, 0)$ and $X = (0, a)$ and using (3) we get $\langle b_{is}, (a, 0) \rangle = \langle b_{is}, (0, a) \rangle = 0$, so $b_{is} = 0$, for all $1 \leq i \leq \nu < s \leq 8$. This, together with (23), proves (19b).

Equation (19c) follows from (19b) and the fact that $\langle \langle \nabla_X J_i X, J_J X \rangle$ is antisymmetric in $i$ and $j$.  

Case (b) $\nu = 7$ and the representation of $\text{Cl}(7)$ is given by (4).

Let $e_i, i = 1, \ldots, 7$, be a fixed orthonormal basis in $\mathbb{C}' = 1_{\mathbb{C}}$ (or in its complexification), for instance, the one with the multiplication table as in [Bes, Section 3.64].

As in Case (a), by complexification, we can assume that $J_p X$ is given by equation (4), where $X = (a, b) \in \mathbb{C}'^6$ and $a, b, p \in \mathbb{O} \otimes \mathbb{C}, p \perp 1$. We extend $J_p X$ to $\mathbb{O} \otimes \mathbb{C}$ by complex linearity by defining $J_{a,1} X = a X$, for $a \in \mathbb{C}$. Denote $C$ the isotropic cone in $\mathbb{C}'^6$, and for $X \in \mathbb{C}'^6$, denote $\mathcal{L} X$ the complexification of $TX$. Take $Y \in C, q \in \mathbb{O} \otimes \mathbb{C}$. Then $Y = J_q X \in \mathcal{L} X$, so by (21), $\sum_{i=1}^{7} \eta_i ((\nabla X J_i)X, J_q X)J_i J_q X \in \mathcal{L} X$. Introduce the operator $M_X \in \text{End}(\mathbb{O} \otimes \mathbb{C})$ by

$$M_X(q) = \sum_{i=1}^{7} \eta_i ((\nabla X J_i)X, J_q X)e_i. \tag{25}$$

As $\mathcal{L} X$ is isotropic, $J_{M_X(q)} J_q X \perp J_p X$, for all $q, p \in \mathbb{O} \otimes \mathbb{C}$. Then by (4), for any $X \in C, q \in \mathbb{O} \otimes \mathbb{C}$,

$$(aq)^* (a M_X(q)) + (bq)^* (b M_X(q)) = 0.$$ Consider this biociton equation as a linear system for $M \in \text{End}(\mathbb{O} \otimes \mathbb{C})$, with $X = (a, b) \in C$. A direct computation shows that $M_X(q) = q, a^* (bq)$ and $M_X(q) = (q, v), 1, (w, q)a*b$, with arbitrary $v, w \in \mathbb{O} \otimes \mathbb{C}$, are the solutions. When $a = e_2, b = i e_3$, these solutions span a subspace of dimension 18 of $\text{End}(\mathbb{O} \otimes \mathbb{C})$, so for all $X = (a, b)$ from a nonempty Zariski open subset $C_1 \subset C$, the dimension of the solution space is at least 18. On the other hand, a direct computation, with $a = e_2, b = i e_3$ shows that every solution is a linear combination of those above. It follows that the corank of the matrix of the linear system (whose entries are polynomials in the coordinates of $X = (a, b) \in C$) equals 18 for all the points $X = (a, b)$ from a nonempty Zariski open subset $C_2 \subset C$. Then for every $X = (a, b) \in C_3 = C_1 \cap C_2$, the operator $M$ is be a linear combination of the four listed above, that is,

$$M_X(q) = \langle w_X, q \rangle 1 + \langle w_X, q \rangle a*b + \alpha_x q + \beta_X a^*(bq),$$

for all $X = (a, b) \in C_3$, where $v, w : C_3 \to \mathbb{O} \otimes \mathbb{C}, \alpha, \beta : C_3 \to \mathbb{C}$. From (25), $M_X(1) = 0, M_X(q) \perp 1$, so

$$M_X(q) = \text{Im}(\langle w_X, q \rangle a*b - \langle w_X, 1 \rangle a^*(bq) + \alpha_X q), \tag{26}$$

where Im is the operator of taking the imaginary part of a biociton: $\text{Im}(q) = q - \langle q, 1 \rangle 1$. Define the symmetric operator $D$ on $\mathbb{O} \otimes \mathbb{C}$ by $D 1 = 0$, $De_i = \eta_i^{-1} e_i$. From (25) it follows that $(M_X, Dq) = 0$, for all $q \in \mathbb{O} \otimes \mathbb{C}$, so for all $X = (a, b) \in C_3$,

$$\langle w_X, q \rangle (a*b, Dq) = \langle w_X, 1 \rangle (a^*(bq), D) - \langle \alpha_X, \text{Dq} \rangle. \tag{27}$$

Substituting $q = b^* a$ and using the fact that $Dq \perp 1$ we get $(\langle w_X, b^* a \rangle - \alpha_X, (b^* a), D(b^* a)) = 0$. The algebraic function $(\langle b^* a \rangle, D(b^* a))$ is not zero on $C$ (for instance, for $a = i e_2, b = 1$, hence on a nonempty Zariski open subset $C_4 \subset C_3$, we have $\alpha_X = \langle w_X, b^* a \rangle$.

For $x, y \in \mathbb{O} \otimes \mathbb{C}$, define the operators $L_x$ and $x \otimes y$ on $\mathbb{O} \otimes \mathbb{C}$ by $L_x q = x q$ and $(x \otimes y) q = (y, q)x + (x, q)y$. As $L_x^* = L_x$ and $L_a, L_b + L_b, L_a = 2(a, b) \text{id}$, equation (27) can be rewritten as

$$D(a*b) \otimes w_X = \langle w_X, 1 \rangle [D, L_a, L_b] - 2(\text{Im} w_X, b^* a)D. \tag{28}$$

Let $S$ be a symmetric operator commuting with $D$. Multiplying both sides of (28) by $S$ and taking the trace we get $\langle SD(a*b), w_X \rangle = -\text{Im} w_X, b^* a) \text{Tr} SD$. Choosing $S$ in such a way that $SD = \text{Im} w_X, b^* a) = 0$, hence $\langle SD(a*b), w_X \rangle = 0$, for any symmetric $S$ commuting with $D$. Taking $S$ diagonal relative to the basis $e_i$ we obtain $(a*b, e_i) \langle w_X, e_i \rangle = 0$, for all $i = 1, \ldots, 7$. As for a nonempty Zariski open subset of $X = (a, b) \in C$, all the numbers $(a*b, e_i)$ are nonzero, we get $\text{Im} w_X = 0$, that is, $w_X = \gamma x 1$, for all $X$ from a nonempty Zariski open subset of $C$. Then from the above, $\alpha_X = \gamma x (a, b)$ and from (25),

$$M_X(q) = \gamma x ((1, q)a*b - a^*(bq)) + (a, b) q - 2(a, b)(1, q) + (b^* a, q)), \tag{29}$$

for all $X = (a, b)$ from a nonempty Zariski open subset of $C$. It then follows from (25) that for all $i = 1, \ldots, 7$, $(\nabla X J_i)X, J_q X) = \eta_i^{-1} (M_X(q), e_i) = \eta_i^{-1} \gamma x ((1, q)(a^* b, e_i) - (a^* bq, e_i) + (a, b)(q, e_i))$. Introduce the quadratic forms $\Phi_i, \Psi_i : \mathbb{C}'^6 \to \mathbb{C}^8$ by $(\nabla X J_i)X = (\Phi_i(X), \Psi_i(X))$, for $i = 1, \ldots, 7$. Then the above equation and (4) imply $a^* \Phi_i(X) + b^* \Psi_i(X) = \eta_i^{-1} \gamma x ((a^* b, e_i) - b^* (ae_i) + (a, b)e_i)$. It follows that for every fixed $i = 1, \ldots, 7$, the polynomial vectors $a^* \Phi_i(X) + b^* \Psi_i(X)$ and $T_i(X) = (a^* b, e_i) - b^* (ae_i) + (a, b)e_i$ are linearly dependent for all $X = (a, b)$ from a nonempty Zariski open subset
of $C$, that is, for all $X \in C$. Note that $\langle a^*\Phi_i(X) + b^*\Psi_i(X), 1 \rangle = \langle T_i(X), 1 \rangle = 0$ (the first equation follows from $(\nabla_X J_i)X, X = 0$). Then the rank of the $7 \times 2$-matrix $N(X) = (a^*\Phi_i(X) + b^*\Psi_i(X) | T_i(X))$ (whose $j$-th row, $j = 1, \ldots, 7$, is $(a^*\Phi_i(X) + b^*\Psi_i(X), e_j) | \langle T_i(X), e_j \rangle$) is at most one, for all $X \in C$. As $R = \mathbb{C}[X]/(\|X\|^2)$, the coordinate ring of $C$, is a unique factorization domain [Nag], there exist $u_1, u_2 \in R$ and $v$ in the free module $R^7$ such that $\pi(N(X)) = \langle u_1 v | u_2 v \rangle$, where $\pi : \mathbb{C}[X] \to R$ is the natural projection. Let $U_2, V_j \in \mathbb{C}[X]$ be the polynomials of the lowest degree in the cosets $\pi^{-1}u_2$ and $\pi^{-1}v_j$ respectively. Lifting the equation $u_2v_j = \pi(T_i(X), e_j) = \pi(\langle ae_i, be_j \rangle)$, for $j \neq i$, to $\mathbb{C}[X]$ we get $\langle ae_i, be_j \rangle = U_2(X)\langle v_j(X) + \|X\|^2\Xi_j(X), \Xi_j(X)\rangle$, for some $\Xi_j \in \mathbb{C}[X]$. Then $U_2$ and $V_j$ are nonzero (as $\langle ae_i, be_j \rangle$ is not divisible by $\|X\|^2 = \|a\|^2 + \|b\|^2$). Moreover, as the polynomial on the left-hand side is of degree two in $X$, and $\|X\|^2$ is prime in $\mathbb{C}[X]$, the polynomials $U_2$ and $V_j$ are homogeneous, with $\deg U_2 + \deg V_j = 2$ and $\Xi_j$ are constants.

Suppose $\deg U_2 = 2$. Then the $V_j$'s are nonzero constants for all $j \neq i$. It follows that for some nontrivial combination $e'$ of the $e_j$, $j \neq i$, $\langle ae_i, be' \rangle = 0$, for all $a, b \in O$, a contradiction.

Suppose $\deg U_2 = 1$. Then $\deg V_j = 1$, for all $j \neq i$. Taking $a = 0$ we get $\Xi_j = 0$ (as the rank of the quadratic form $U_2((0, b))V_j((0, b))$ in $b$ is at most two), so for some nonzero linear forms $U_2$ and $V_j$, $\langle ae_i, be_j \rangle = U_2(X)\langle v_j(X) + \|X\|^2\Xi_j(X), \Xi_j(X)\rangle$, for some $U_1(X) \in \mathbb{C}[X]$ and $\Xi_j(X) \in \mathbb{C}[X]^8$, with $\langle \Xi_j(X), 1 \rangle = 0$. This implies for some $X \in \mathbb{C}[X]$ and $\Xi_j(X)$ is a unique factorization domain [Nag], there exist $p_i \in \mathbb{R}$ and $\Xi_j(X) \in \mathbb{R}^7$ such that $a^*\Phi_i(X) + b^*\Psi_i(X) = U_1(X)T_i(X) + \|X\|^2\Xi_j(X)$, for some $U_1(X) \in \mathbb{C}[X]$ and $\Xi_j(X) \in \mathbb{C}[X]^8$, with $\langle \Xi_j(X), 1 \rangle = 0$. As the left-hand side is a vector whose components are homogeneous cubic polynomials in $X$, the components of $T_i$ are quadratic forms of $X$, and $\|X\|^2$ is prime in $\mathbb{C}[X]$, then $U_1(X)$ is a linear form and $\Xi_j(X)$ is a linear operator. As both sides are real for $X \in \mathbb{R}^16$, for every $i = 1, \ldots, 7$, there exist $p_i \in \mathbb{R}^16$ and $\Xi_i : \mathbb{R}^16 \to \mathbb{R}^7 \cong \mathbb{O}'$ such that $a^*\Phi_i(X) + b^*\Psi_i(X) = (p_i, X)T_i(X) + \|X\|^2\Xi_i(X) = (p_i, X)((a^*b, e_i) - (a^*\Xi_j, e_i) + (a, b)e_i) + \|X\|^2\Xi_i(X)$, for all $X = (a, b) \in \mathbb{R}^16$. Let $\Phi_i(X) = \Phi_i(X) - a\Xi_jX - \frac{1}{2}(p_i, X)e_i\Xi_j(X) - \|X\|^2\Xi_i(X) = \Psi_i(X) = \Psi_i(X) - b\Xi_jX - \|X\|^2\Xi_i(X)$, for all $X = (a, b) \in \mathbb{R}^16$. Then $a^*\Phi_i(X) + b^*\Psi_i(X) = (p_i, X)(a^*b, e_i)$.

From the Lemma 2.2, with $N_1(X) = \Phi_i(X)$, $N_2(X) = \Phi_i(X)$, $p = p_i$, $u = e_i$, it follows that $\langle \nabla_X J_iX = (\Phi_i(X), \Psi_i(X)) \rangle = (a\Xi_iX + \frac{1}{2}(p_i, X)e_i \Xi_j(X) - \Xi_jX, X - \sum_{j=1}^7(\Xi_iX, e_j)J_iX$, for some $m_i \in \mathbb{R}^16$. As $\langle \nabla_X J_iX, X \rangle = 0$ and $\langle \Xi_jX, 1 \rangle = 0$ we get $\langle p_i, X \rangle(e_i, b^*a) = 0$. Since the polynomial $\langle e_i, b^*a \rangle$ is not divisible by $\|X\|^2$, we obtain $p_i = 0$. Then by (4), $\langle a\Xi_iX + \Xi_jX, b\Xi_iX \rangle = \Xi_iX, X = \sum_{j=1}^7(\Xi_iX, e_j)J_iX$ which implies (19b), with $b_{ij} = J_i(m_i + \Xi_jX)$ and with $m_i$ replaced by $\bar{q}_1^{-1}m_i$. Equation (19c) follows from (19b) and the fact that $\langle \nabla_X J_iX, J_iX \rangle$ is antisymmetric in $i$ and $j$.

We next prove (19a). Substituting (19b) into (21) we obtain

$$\langle Y, Y \rangle^2 - 2\langle Y, X \rangle \langle T(X), Y \rangle - \langle T(X), X \rangle = 0,$$

where the quadratic form $T : \mathbb{R}^16 \to \mathbb{R}^16$ is defined by

$$T(X) = Q(X) - 3 \sum_{i=1}^7 \langle m_i, X \rangle J_iX.$$

For a nonzero $U \in \mathbb{R}^16$, let $X, Y \perp IU$ be linearly independent. Then the eight-dimensional spaces $IX$ and $IY$ are both orthogonal to $U$, so their intersection is nontrivial. But if $J_iX = J_iY$, then $J_{2(u,v)}u - \|u\|^2v = J_{\|v\|^2}v - \|v\|^2u$, so $\dim(IX \cap IY) \geq 2$. It follows that $IX, IY \perp V$, for some nonzero $V \perp U$. Substituting $V$ for $Y$ into (29) and taking the inner product with $U$ we obtain $\|X\|^2\langle T(X), U \rangle = -\|Y\|^2\langle T(V), U \rangle$. Similarly, substituting $V$ for $X$ into (29) and taking the inner product with $U$ we get $\|Y\|^2\langle T(Y), U \rangle = -\|X\|^2\langle T(Y), U \rangle$, so $\|X\|^2\langle T(X), U \rangle = \|Y\|^2\langle T(Y), U \rangle$, for all nonzero $X, Y \perp IU$. It follows that for some function $f : \mathbb{R}^16 \to \mathbb{R}$, which is homogeneous of degree one,

$$\langle T(X), U \rangle = \langle X, f(U) \rangle, \quad \text{for all } X \perp IU.$$ 

Taking the inner product of (29) with $Z \perp IU$ we obtain $\langle T(Y), \|X\|^2Z - \langle X, Z \rangle X + \|Y\|^2\langle T(X), Z \rangle = 0$. By which (31) implies $\langle T(X), Z \rangle = f((X, Z)X) - \|X\|^2Z$, for all $X, Z$ with $\dim(IX \cup IZ) < 16$, that is, with $IX$ and $IZ$ having a nontrivial intersection. In particular, taking $Z = J_iX$ we obtain $\langle T(X), J_iX \rangle = -\|X\|^2f(J_iX)$. Replacing $X$ by $J_iX$ we get $\langle T(J_iX), X \rangle = -\|X\|^2f(J_iX)$, for all $X \in \mathbb{R}^16$. For an arbitrary nonzero $X \in \mathbb{R}^16$, let $U_i, i = 1, \ldots, 8$, be an orthonormal basis for $(IX)^{\perp}$. 
Denoting $\text{Tr} T$ the vector in $\mathbb{R}^{16}$ whose components are the traces of the corresponding components of $T$ and using the fact that $T(X) \perp X$ (which follows from (30) and the fact that $Q(X) \perp X$) we get 
\[ \langle \text{Tr} T, X \rangle = \|X\|^2 \sum_{i=1}^{16} \langle T(J_i X), X \rangle + \sum_{i=1}^{16} \langle T(U_i), X \rangle = -7f(X) + 8f(X) = f(X), \]
by (31). Therefore $f$ is a linear form, $f(X) = \langle l, X \rangle$, for some $l \in \mathbb{R}^{16}$. Then $T(X) = \|X\|^2 \sum_{i=1}^{16} \langle T(J_i X), X \rangle J_i X + \sum_{i=1}^{16} \langle T(U_i), U_i \rangle U_i = -\sum_{i=1}^{16} \langle l, J_i X \rangle J_i X + \|X\|^2 \sum_{i=1}^{16} \langle l, U_i \rangle U_i = \|X\|^2 (\pi_{TX} l - \pi_{TX} X) + \langle l, X \rangle X = \|X\|^2 \sum_{j=1}^{16} \langle l, X \rangle X. \]
Substituting this into (29) and using (31) we get $l = \pi_{TX} X + \pi_{TY} l$, for all $Y \perp X$. Let $l = (l_1, l_2)$. As it follows from (4), for $X = (a, b)$, \[ \pi_{TX} l = \|X\|^2 (a a^* l_1 + b b^* l_2), \]
b(a^* l_1) + \langle b \|^2 l_2), \]
and if $a \neq 0$, the vector $Y = \langle ||b||^2 a q^* - a q||b||^2 q ||b||^2 b \rangle$ satisfies $Y \perp X$, for all $q \in \mathbb{C}$. Then the equation $l = \pi_{TX} X + \pi_{TY} l$ implies \[ \|q\|^2 (a b^* l_2) = (a q) (b q^* l_2), \]
and \[ \|q\|^2 (a^* a l_1) = (b q) (a q^* l_1), \]
for arbitrary $a, b, q, l_2 \in \mathbb{C}$. The first of them implies $l_2 = 0$ (to see that, take $b = q^*$, then the octonions $a, q, l_2$ associate, for every $a, q$, so $l_2 \in \mathbb{C}$; if $l_2 \neq 0$, then the octonions $a, q, b$ a contradiction), the second one can be obtained from the first one by interchanging $a$ and $b$, so it implies $l_1 = 0$. Thus $l = 0$, so $T(X) = 0$, which is equivalent to (19a) by (30).

2. Substitute $X = J_i Y, U \perp X, Y$ into (17) and consider the first term in the second summation. As $\langle J_i Y, Y \rangle = \|Y\|^2 \delta_{ik}$, that term equals $3 \eta_k (2 \langle (\nu \nu U_j X, Y \rangle + \langle (\nu \nu J_i Y, U) \rangle + \langle (\nu \nu J_i Y, X \rangle) \|Y\|^2$. As $J_k$ is orthogonal and skew-symmetric, \[ \langle (\nu \nu J_k X, Y \rangle = \langle (\nu \nu J_k J_i Y, J_i Y \rangle = -\langle J_k (\nu \nu U_j X, J_i Y \rangle = \langle (\nu \nu U_j J_k Y, J_i Y \rangle = 0. \]
Next, \[ \langle (\nu \nu J_k Y, U) \rangle = -\langle (\nu \nu J_k Y, J_i Y \rangle = \langle J_k (\nu \nu J_k Y, J_i Y \rangle = \langle (\nu \nu J_k J_k Y, J_i J_k Y \rangle \]
by (19b). Again by (19b), as $Y = -J_k X, \langle (\nu \nu J_k J_k X, Y \rangle = \langle J_k (\nu \nu J_k X, J_i J_k Y \rangle = \langle \eta_k J_k J_k J_k X, J_i J_k Y \rangle \]
and \[ \|J_k J_k Y \|^2 \delta_{ik} = \delta_{ik} \]
by (19b). Substituting this into (17) and using (19a, 19b) we obtain after simplification:
\[ \|Y\|^2 (2 \langle J_k m_k, U \rangle - \|U\| \eta_k) + \sum_{j=1}^{16} \langle \eta_l b_{kj} + \eta_l b_{j, k}, J_i J_k Y \rangle J_i J_k Y + \langle J_k J_k J_k (J_i J_k Y) \rangle Y = 0. \]
Choose $i \neq j$ such that $k \neq i, j$ and take the eigenvectors of the symmetric orthogonal operator $J_i J_k J_k$ as $U$. For each such $U$, $J_i J_k U = \pm J_i U$, so dim$(J_i J_k U + J_i U) < 2 \nu < 16$, there exists a nonzero $Y \perp J_i U + J_i J_k U$, which implies $U \perp J_i Y + J_i J_k Y$. Substituting such $U$ and $Y$ into (32) we obtain \[ \langle U, 2 \delta_{ik} - \eta_k \rangle = 0. \]
As the eigenvectors of $J_i J_k J_k$ span $\mathbb{R}^{16}$, equation (20a) follows.

Substituting (20a) into (32) we obtain \[ \sum_{j=1}^{16} \langle \eta_k b_{kj} + \eta_l b_{j, k}, J_i J_k Y \rangle J_i J_k Y + \langle J_k J_i J_k (J_i J_k Y) \rangle Y = 0, \]
which implies \[ \sum_{j=1}^{16} \langle \eta_k b_{kj} + \eta_l b_{jk}, J_i J_k Y \rangle J_i J_k Y + \langle J_k J_i J_k (J_i J_k Y) \rangle Y = 0. \]
Equation (20b) now follows from assertion 2 of [N5, Lemma 3].

By (18) and (19a), \[ \langle (\nu \nu J_k) Y - (\nu \nu J_k) X, X \rangle = 3 \sum_{i=1}^{16} \langle m_i, X \rangle J_i J_k X, \]
for all $X, U \in \mathbb{R}^{16}$. Polarizing this equation and using the fact that the covariant derivative of $Y$ is symmetric we obtain \[ \langle (\nu \nu J_k) Y - (\nu \nu J_k) X, (\nu \nu J_k) Y - (\nu \nu J_k) X \rangle - 2 \langle (\nu \nu J_k) Y, X \rangle = 3 \sum_{i=1}^{16} \langle m_i, J_i J_k X \rangle \]
by (19b). Subtracting the same equation, with $Y$ and $U$ interchanged, we get \[ \langle (\nu \nu J_k) Y, (\nu \nu J_k) X \rangle = \sum_{i=1}^{16} \langle m_i, J_i J_k X \rangle \]
by (19b). This, combined with (20c), implies (20c).

To prove (20d), substitute $X \perp Y, U = J_k Y$ into (16). Using (19, 20c) we obtain after simplification:
\[ 3 \langle \nu \nu J_k Y, X \rangle - 3 \eta_k \langle m_k, Y \rangle X + \sum_{i=1}^{16} \langle \eta_k b_{ik} + 2 \delta_{ik} m_k, Y \rangle J_i J_k X \]
mod $(\nu \nu J_k) X$. Subtracting three times the polarized equation (19b) (with $i = k$) and solving for $(\nu \nu J_k) X$ we get
\[ (\nu \nu J_k) X = \sum_{i=1}^{16} \langle \eta_k b_{ik} - \eta_l b_{ik} - 2 \delta_{ik} m_k, Y \rangle J_i J_k X \]
mod $(\nu \nu J_k) X$, for all $X \perp Y$. Choose $s \neq k$ and define the subset $S_{ks} \subset \mathbb{R}^{16} \oplus \mathbb{R}^{16}$ by $S_{ks} = \{ X, Y \} : X, Y \neq 0, X, J_k X, J_s X \perp J_k Y$. It is easy to see that $(X, Y) \in S_{ks} \iff (Y, X) \in S_{ks}$ and that replacing $J_k Y$ by $J_s Y$ in the definition of $S_{ks}$ gives the same set $S_{ks}$. Moreover, the set \( \{ X : (X, Y) \in S_{ks} \} \) (and hence the set \( \{ Y : (X, Y) \in S_{ks} \} \)) spans $\mathbb{R}^{16}$. If $\nu < 8$, this follows from [N1, Lemma 3.2 (4)]. If $\nu = 8$, the Clifford structure is given by (33). Take $X = (a, b)$, with $\|a\| = \|b\| = 1$, and $Y = (bu, au^*)$ for some nonzero $u \in \mathbb{C}$. Then the condition $X \perp J_k Y$ is satisfied and the condition $J_k X \perp J_k Y$ is equivalent to \[ \langle (ac_k') q, y \rangle + \langle (bc_k') q^*, au^* \rangle = 0, \]
for all $q \in \mathbb{C}$, that is, to \( (ac_k') (b) + (au^*) (b) = 0 \). As \( (ac_k') (b) + (au^*) (b) = 2(ac_k', b) - b^* (ac_k') u + 2 (au^*) - b^* (au^*) e_k = 2(ac_k', b) - 2 (ek, u) b^* a + 2 (au^*, b) \), the latter condition is satisfied, if we choose $a, b$ and $u$ in such a way that $b^* a$, $u$ and $e_k$ are orthogonal. Similar arguments for $e_s$ show that for every $X = (a, b)$, with $\|a\| = \|b\| = 1$ and $b^* a \perp e_k, e_s$, there
exists $Y \neq 0$ such that $(X, Y) \in S_k$. In particular, taking $X = (\pm e_i, b)$, with a fixed $e_i \perp e_k, e_s$ and arbitrary unit $b \in \mathbb{O}$ we obtain that the set \{$(X, Y) \in S_k$\} spans $\mathbb{R}^16$.

Now, for $(X, Y) \in S_k$, take the inner product of (33) with $J_kX$. Since $((\nabla Y J_k)X, J_kX)$ is antisymmetric in $k$ and $s$, we get $(3 - \eta_k^{-1})b_{ks} + (3 - \eta_k^{-1})b_{sk} = 0$, for a set of the $Y$'s spanning $\mathbb{R}^16$.

So $(3 - \eta_k^{-1})b_{ks} + (3 - \eta_k^{-1})b_{sk} = 0$, for all $k \neq s$. This and (20b) imply (20d).

Now from (20b, 20d) it follows that $b_{ij} + b_{ji} = 0$ for all $i \neq j$, so by (19c), $\eta^{-1}_i J_i m_i = -\eta^{-1}_j J_j m_j$.

Acting by $J_i J_j$ we obtain that the vector $\eta^{-1}_i J_i m_i$ is the same, for all $i = 1, \ldots, \nu$, which proves (20e). □

**Lemma 6.** In the assumptions of Lemma 4, let $x \in M'$ and let $U$ be the neighborhood of $x$ introduced in assertion (a) of Lemma 4. Then there exists a smooth metric on $U$ conformally equivalent to the original metric whose curvature tensor has the form (13), with $\rho$ a constant multiple of the identity.

**Proof.** If $\nu \leq 4$, the proof follows from [N5, Lemma 7]. Suppose $\nu \geq 4$. Let $f$ be a smooth function on $U$ and let $(\cdot, \cdot)' = e^f(\cdot, \cdot)$. Then $W' = W$, $J_i' = J_i$, $\eta_i' = e^{-f} \eta_i$ and, on functions, $\nabla' = e^{-f} \nabla$, where we use the dash for the objects associated to metric $(\cdot, \cdot)'$. Moreover, the curvature tensor $R'$ still has the form $\rho$ and all the identities of Lemma 5 remain valid.

In the cases considered in Lemma 5, the ratios $\eta_i'/\eta_i$ are constant, as it follows from (20a,20e). In particular, taking $f = \ln|\eta_i|$ we obtain that $\eta_i'$ is a constant, so $\eta_i'$ are constant, $m_i' = 0$ by (20a), so $(\nabla Y J_i')U - (\nabla Y J_i')Y = 0$ by (20c). Dropping the dashes, we obtain that, up to a conformal smooth change of the metric on $U$, the curvature tensor has the form (13), with $\rho$ satisfying the identity $(\nabla X \rho)X = (\nabla X \rho)Y$, for all $X, Y$, that is, with $\rho$ being a symmetric Codazzi tensor.

Then by [DS, Theorem 1], at every point of $U$, for any three eigenspaces $E_\beta, E_\gamma, E_\alpha$ of $\rho$, with $\alpha \notin \{\beta, \gamma\}$, the curvature tensor satisfies $R(X, Y)Z = 0$, for all $X \in E_\beta$, $Y \in E_\gamma$, $Z \in E_\alpha$. It then follows from (13) that

\[
\sum_{i=1}^{\nu} \eta_i (2\langle J_i X, Y \rangle J_i Z + \langle J_i Z, Y \rangle J_i X - \langle J_i Z, X \rangle J_i Y) = 0,
\]

for all $X \in E_\beta$, $Y \in E_\gamma$, $Z \in E_\alpha$, $\alpha \notin \{\beta, \gamma\}$.

Suppose $\rho$ is not a multiple of the identity. Let $E_1, \ldots, E_p$, $p \geq 2$, be the eigenspaces of $\rho$. If $p \geq 2$, denote $E_1' = E_1$, $E_2' = E_2 \oplus \cdots \oplus E_p$. Then by linearity, (34) holds for any $X, Y \in E_\alpha'$, $Z \in E_\beta'$, such that $\{\alpha, \beta\} = \{1, 2\}$. Hence to prove the lemma it suffices to show that (34) leads to a contradiction, in the assumption $p = 2$. For the rest of the proof, suppose that $p = 2$. Denote $\dim E_\alpha = d_\alpha$.

If $\nu < 8$, the claim follows from the proof of [N5, Lemma 7] (see [N5, Remark 4]). Suppose $\nu = 8$, then the Clifford structure is given by (33).

Choosing $Z \in E_\alpha$, $X, Y \in E_\beta$, $\alpha \neq \beta$, and taking the inner product of (34) with $X$ we obtain

\[
\sum_{i=1}^{8} \eta_i \langle J_i X, Y \rangle \langle J_i X, Z \rangle = 0.
\]

It follows that for every $X \in E_\alpha$, the subspaces $E_1$ and $E_2$ are invariant subspaces of the symmetric operator $R'_X \in \text{End}((\mathbb{R}^16)^8)$ defined by $R'_X Y = \sum_{i=1}^{8} \eta_i \langle J_i X, Y \rangle J_i X$. So $R'_X$ commutes with the orthogonal projections $\pi_\beta : \mathbb{R}^16 \to E_\beta$, $\beta = 1, 2$. Then for all $\alpha, \beta = 1, 2$ ($\alpha$ and $\beta$ can be equal), all $X \in E_\alpha$ and all $Y \in \mathbb{R}^16$, $\sum_{i=1}^{8} \eta_i \langle J_i X, \pi_\beta Y \rangle J_i X = \sum_{i=1}^{8} \eta_i \langle J_i X, Y \rangle \pi_\beta J_i X$. Taking $Y = J_i X$ we get that $\pi_\beta J_i X \subset J_i X$, that is, $\pi_\beta J_i X \subset J_i X$, for all $X \in E_\alpha$, $\alpha, \beta = 1, 2$. As $\pi_1 + \pi_2 = \text{id}$, we obtain $J_i X \subset \pi_1 J_i X \oplus \pi_2 J_i X \subset J_i X$, hence

\[
J_i X = \pi_1 J_i X \oplus \pi_2 J_i X,
\]

for all $X \in E_1 \cup E_2$.

As all four functions $f_{\alpha \beta} : E_\alpha \to \mathbb{Z}$, $\alpha, \beta = 1, 2$, defined by $f_{\alpha \beta}(X) = \dim \pi_\beta J_i X$, $X \in E_\alpha$, are lower semi-continuous and $f_{\alpha \beta}(X) + f_{\alpha \beta}(X) = 8$ for all nonzero $X \in E_\alpha$, there exist constants $c_{\alpha \beta}$, with $c_{\alpha \beta} = 8$, such that $\dim \pi_\beta J_i X = c_{\alpha \beta}$, for all $\alpha, \beta = 1, 2$ and all nonzero $X \in E_\alpha$.

Consider two cases.

First assume that there exist no nonzero $Y \in E_\alpha$, $Z \in E_\beta$, $\alpha \neq \beta$, such that $Y \perp J_i Z$. Then it follows from (35) that $\pi_\beta J_i X\subset E_\beta$, for $\alpha \neq \beta$. Then $d_1, d_2 \leq 8$. As $d_1 + d_2 = 16$, we obtain $d_1 = d_2 = 8$ and $J_i X = E_\beta$, for every $X \in E_\alpha$, $\alpha \neq \beta$. Then (34), with $X, Y \in E_\alpha$, $Z \in E_\beta$, $\alpha \neq \beta$, gives $\sum_{i=1}^{8} \eta_i \langle (Z, J_i X) J_i Y - (Z, J_i Y) J_i X \rangle = 0$. Taking the inner product with $J_i X$ we get $\eta_i \langle Z, J_i Y \rangle \langle X \rangle^2 = \sum_{i=1}^{8} \eta_i \langle Z, J_i X \rangle \langle J_i Y, J_i X \rangle$. Taking $Z = J_k X \in E_\beta$ and assuming $X \perp Y$ we
obtain \((\eta_j + \eta_k)(X, J_k J_j Y) = 0, \) for \( k \neq j \). Taking \( X = J_k J_j Y (\in E_\alpha) \) we get \( \eta_j + \eta_k = 0, \) for \( k \neq j, \) so all the \( \eta_i \)'s are zeros, a contradiction with \( \nu = 8 \).

Otherwise, assume that there exist nonzero \( Y \in E_\alpha, Z \in E_\beta, \alpha \neq \beta, \) such that \( Y \perp JZ \). Substituting such \( Y \) and \( Z \) into (34), with \( X \in E_\alpha, \) we obtain 
\[
\sum_{i=1}^{8} \eta_i (2(J_i Y, X)J_i Z + (J_i Z, X) J_i Y) = 0.
\]
Taking \( X \in E_\alpha \) orthogonal to \( \pi_\alpha JY \) (and then to \( \pi_\alpha JZ \)) we obtain \( \pi_\alpha JY = \pi_\alpha JZ \). As the condition \( Y \perp JZ \) is symmetric in \( Y \) and \( Z \), we can interchange \( Y \) and \( Z \) and \( \alpha \) and \( \beta \) to get \( \pi_\beta JY = \pi_\beta JZ \), which by (35) implies that \( JY = JZ \), for any two nonzero vectors \( Y \in E_\alpha, Z \in E_\beta, \alpha \neq \beta, \) such that \( Y \perp JZ \). Now, if for some nonzero \( Y \in E_\alpha \) there exists \( Z \in E_\beta, \alpha \neq \beta, \) such that \( Y \perp JZ \), then by (35), the space \( \pi_\beta JY \) is a proper subspace of \( E_\beta \), so \( c_{\beta} < d_\beta \), which implies that for every nonzero \( X \in E_\alpha \) and any nonzero \( Z \) from the orthogonal complement to \( \pi_\beta JX \) in \( E_\beta \) (which is nontrivial), \( X \perp JZ \), hence \( JX = JZ \), from the above.

Consider an operator \( P = \prod_{i=1}^{8} J_i \). As the \( J_i \)'s are anticommuting almost Hermitian structures, \( P \) is symmetric and orthogonal, and \( Tr P = 0 \), as \( P \) is the product of the symmetric operator \( \prod_{i=1}^{8} J_i \) and the skew-symmetric one, \( J_8 \). So its eigenvalues are \( \pm 1 \), with both eigenspaces \( \mathbb{V}_\pm \) of dimension \( 8 \). As the \( J_i \)'s anticommute, each of them interchanges the eigenspaces of \( P \): \( J_i V_\pm = V_{\pm} \).

From the above, for every unit vector \( X \in E_1 \), there exists a unit vector \( Z \perp X \) such that \( JX = JZ \). Therefore, by (2) there exists \( F : \mathbb{R}^8 \to \mathbb{R}^8 \) such that for all \( u \in \mathbb{R}^8 \), \( JF(u)X = J_0 \langle J, X \rangle u \). As the right-hand side is linear in \( u \), the map \( F \) is also linear: \( F(u) = Au \), for some \( A \in \text{End}(\mathbb{R}^8) \). Moreover, as \( JX \) is an orthogonal multiplication and as \( X \) and \( Z \) are orthonormal, the operator \( A \) is orthogonal and skew-symmetric. Without losing generality, we can assume that an orthonormal basis for \( \mathbb{R}^8 \) is chosen in such a way that \( J_i X = J_{i+4} Z \), for \( i = 1, 2, 3, 4 \), so \( J_1 J_2 J_3 J_4 X = X \), for all \( 1 \leq i \neq j \leq 4 \). As changing an orthonormal basis for \( \mathbb{R}^8 \) may only change the sign of \( P \), it follows that \( X \) is an eigenvector of \( P \), say \( X \in V_+ \). As \( X \) is an arbitrary unit vector from \( E_1 \), by continuity, \( E_1 \subset V_+ \). Similarly, \( E_2 \subset V_- \). But as every \( J_i \) interchanges the \( V_\pm \)'s, we get \( JF_1 = E_2 \), which contradicts the assumption that there exist nonzero \( Y \in E_1 \), \( Z \in E_2 \) with \( Y \perp JZ \).

Hence the Codazzi tensor \( \rho \) is a multiple of the identity. The definition of the Codazzi tensor easily implies that \( \rho \) is a constant multiple of the identity on \( \mathcal{U} \).

By Lemma 6, up to a conformal change of the metric, we can assume that on \( \mathcal{U} \), the curvature tensor has the form (13), with \( \rho \) a constant multiple of the identity. Then (18) implies that \( Q = 0 \), so \( m_i = 0 \) by (19a), \( \nabla \eta_i = 0 \) by (20a) and (\( \nabla_X J_i \)) \( X = \sum_{j=1}^{8} \langle b_{ij}, X \rangle J_j X \) by (19b). Then from (13), (\( \nabla_X R)(X, Y) = 3 \sum_{j=1}^{8} \eta_i (\nabla_X J_i X, Y) J_i X + \langle J_i X, Y \rangle (\nabla_X J_i X) = 3 \sum_{j=1}^{8} \eta_i (b_{ij} + \eta_j b_{ji}, X) \)
\[
\langle J_i X, Y \rangle J_i X = 0, \quad \eta_i b_{ij} + \eta_j b_{ji} = 0 \quad (by \ (20b) \ for \ i \neq j, \text{ and by } (19c) \ for \ i = j, \text{ and by } (19c) \ for \ i = j, \text{ and by } m_i = 0).
\]

It is well-known [Bes, Proposition 2.35] that the equation (\( \nabla_X R)(X, Y) = 0 \) implies \( \nabla R = 0 \), so the metric on \( \mathcal{U} \) is locally symmetric. As \( \rho \) is a multiple of the identity, it follows from (13) that the curvature tensor is Osserman. Then by [GSV, Lemma 2.3], \( \mathcal{U} \) is either flat or is locally isometric to a rank-one symmetric space.

Thus, for every \( x \in M' \) satisfying assertion (a) of Lemma 4, the metric on the neighborhood \( \mathcal{U} = \mathcal{U}(x) \) is either conformally flat or is conformally equivalent to the metric of a rank-one symmetric space.

3.3. Cayley case. In this section, we consider the case when the Weyl tensor has a Cayley structure.

Let \( x \in M' \) and let \( \mathcal{U} = \mathcal{U}(x) \) be neighborhood of \( x \) defined in assertion (b) of Lemma 4. Up to a conformal change of the metric, the curvature tensor on \( \mathcal{U} \) is given by (14). Then
\[
(\nabla_Z R)(X, Y) = (\nabla_Z \rho) X \wedge Y - (\nabla_Z \rho) Y \wedge X + \varepsilon \sum_{i=0}^{8} ((\nabla_Z S_i) X \wedge S_i Y + S_i X \wedge (\nabla_Z S_i) Y),
\]
so the second Bianchi identity has the form
\[
\sigma_{XYZ} (A(Y, X) \wedge Z + \varepsilon \sum_{i=0}^{8} S_i Z \wedge B_i(Y, X)) = 0.
\]
where \( \sigma_{XYZ} \) is the sum taken over the cyclic permutations of \( (X, Y, Z) \), and the skew-symmetric maps \( A, B_w : \mathbb{R}^{16} \times \mathbb{R}^{16} \to \mathbb{R}^{16} \) are defined by
\[
A(Y, X) = (\nabla_Y \rho) X - (\nabla_X \rho) Y, \quad B_w(Y, X) = (\nabla_Y S_w) X - (\nabla_X S_w) Y,
\]
for \( w \in \mathbb{R}^9, \ X, Y \in \mathbb{R}^{16} \), where \( S_w X = \sum_{i=0}^{8} w_i S_i \) is the orthogonal multiplication defined in Section 2.3.
Lemma 7. In the assumptions of Lemma 4, let \( x \in M' \) and let \( U \) be the neighborhood of \( x \) introduced in assertion (b) of Lemma 4. For every point \( y \in U \), identify \( T_y M' \) with the Euclidean space \( \mathbb{R}^{16} \) via a linear isometry. Then

1. There exists a linear map \( N : \mathbb{R}^{16} \to \text{Skew}(\mathbb{R}^{16}) \), \( X \to N_X \), such that \( \nabla_X S_w = [S_w, N_X] \) for all \( w \in \mathbb{R}^9 \), \( X \in \mathbb{R}^{16} \).

2. For every unit vector \( w \in \mathbb{R}^9 \), there exists a linear operator \( L_w : w^* \to E_1(S_w) \) such that for every \( X, Y \in E_1(S_w) \), \( Z \in E_{-1}(S_w) \), and every \( u \in \mathbb{R}^9 \), \( u \perp w \),

\[
\begin{align*}
\pi_{E_1(S_w)} B_w(Y, X) &= 0, \\
\pi_{E_1(S_w)} A(Y, X) &= 0, \\
\pi_{E_1(S_w)} B_u(Y, X) &= (L_w, u)Y - \langle L_w u, Y \rangle X,
\end{align*}
\]

(39)

where \( E_{\pm 1}(S_w) \) are the \( \pm 1 \)-eigenspaces of \( S_w \) and \( \pi_{E_1(S_w)} \) is the orthogonal projection to \( E_1(S_w) \).

3. There exists a bilinear skew-symmetric map \( T : \mathbb{R}^9 \times \mathbb{R}^9 \to \mathbb{R}^{16} \) such that for all \( X, Y \in \mathbb{R}^9 \), \( w \in \mathbb{R}^9 \),

\[
A(Y, X) = 0, \quad B_w(Y, X) = \sum_{i=0}^{8} (T(w, e_i), X)S_i Y - \langle T(w, e_i), Y \rangle S_i X.
\]

4. The tensor \( \rho \) is a constant multiple of the identity on \( U \).

5. \( \nabla_X S_w = -\sum_{i=0}^{8} (T(w, e_i), X)S_i \).

Proof. 1. Denote \( T_i = \nabla_X S_i \). The operators \( T_i \) are symmetric and satisfy \( T_i S_j + T_j S_i + S_i T_j + S_j T_i = 0 \), for all \( i, j = 0, 1, \ldots, 8 \), by (8). In particular, the operators \( S_i T_i \) are skew-symmetric and

\[
\begin{align*}
[S_i, T_j] &= 2T_i, \\
[S_i, S_j T_i] &= T_i + S_j T_i S_i, \quad i \neq j.
\end{align*}
\]

Define \( N : \mathbb{R}^{16} \to \text{Skew}(\mathbb{R}^{16}) \) by \( N_X = \frac{1}{16} (7 \text{id} + A) \sum_{i=0}^{8} S_i T_i J_i \), for \( X \in \mathbb{R}^{16} \). The fact that \( N_X \) is indeed skew-symmetric follows from assertion 2 of Lemma 3, as \( S_i T_j \) are skew-symmetric. Moreover, for any \( i = 0, \ldots, 8 \), from assertion 3 of Lemma 3,

\[ [S_i, A(Q)] = -2[S_i, Q] - A([S_i, Q]) = (-2 \text{id} - A)([S_i, Q]), \]

so

\[ [S_i, N_X] = \frac{1}{16} [S_i, (7 \text{id} + A) \sum_{i=0}^{8} S_j T_j] = \frac{1}{16} (5 \text{id} - A) \sum_{i=0}^{8} [S_i, S_j T_i]. \]

Then by (8) and (40), \([S_i, N_X] = \frac{1}{16}(5 \text{id} - A)(11 \text{id} + A)T_i = -\frac{1}{16}(A^2 + 6A - 55)T_i \). As \( T_i \) is symmetric and \( \text{Tr} T_i = 0 \) (which follows from \( T_i = S_i S_j T_i \) and the fact that \( S_i T_i \) is skew-symmetric), we obtain from assertion 1 of Lemma 3 that \( T_i \in L_1 \oplus L_4 \). Then by assertion 2 of Lemma 3, \((A - \text{id})(A + 7)T_i = 0 \), which implies \([S_i, N_X] = T_i \).

2. By assertion 2 of Lemma 3, the operator \( A \) does not depend on the choice of the orthonormal basis \( \frac{1}{2} S_i \) for \( L_1 \), it follows from (36) that we lose no generality by assuming \( w = e_0 \in \mathbb{R}^9 \). By assertion 1,

\[ \nabla_X S_i = [S_0, N_X], \]

so from (38), \( B_0(Y, X) = [S_0, N_Y]X - [S_0, N_X]Y = (S_0 - \text{id})[N_Y X - N_X Y] = -2\pi_{E_{-1}(S_0)} (N_Y X - N_X Y), \)

so \( \pi_{E_{i}(S_0)} B_0(Y, X) = 0 \). This proves the first identity of (39).

To prove the second one, take \( X, Y, Z \in E_1(S_0) \). Note that by (8), \( S_i X \in E_{-1}(S_0) \) for all \( j \neq 0 \) (and similarly, for \( Y \) and \( Z \)). Then projecting (37) to \( E_1(S_0) \) and using the first identity of (39), we obtain \( \sigma_{XYZ}(\pi_{E_{i}(S_0)} A(Y, X)) \wedge Z = 0 \).

Assuming \( X, Y, Z \) linearly independent and acting by the both sides on a vector \( U \in E_{1}(S_0) \) in \( \text{Span}(X, Y, Z) \), we obtain \( \langle A(Y, X), U \rangle = 0 \), so \( \pi_{E_{i}(S_0)} A(Y, X) \in \text{Span}(X, Y) \) (as \( Z \in E_1(S_0) \) \& \( \text{Span}(X, Y) \) can be chosen arbitrarily). Then for orthonormal vectors \( X, Y \in E_1(S_0), \)

\[
\pi_{E_{i}(S_0)} A(Y, X) = \langle A(Y, X), X \rangle X + \langle A(Y, X), Y \rangle Y,
\]

so the coefficient of \( X \wedge Z \) in \( \sigma_{XYZ}(\pi_{E_{i}(S_0)} A(Y, X)) \wedge Z = 0 \) (with orthonormal \( X, Y, Z \in E_{1}(S_0) \)) gives \( \langle A(Y, X), X \rangle + \langle A(Y, X), Z \rangle = 0 \) (using the fact that \( A(Y, X) \) is antisymmetric in \( X, Y \), by (38)). Then \( \langle A(Y, X), X \rangle = 0 \), hence \( \pi_{E_{i}(S_0)} A(Y, X) = 0 \), for any orthonormal vectors \( X, Y \in E_{1}(S_0) \).

For the remaining two identities, take \( X, Y \in E_1(S_0), \)

\[ Z \in E_{-1}(S_0) \text{ in (37) and project the resulting equation to } E_1(S_0) \cap E_1(S_0). \]

As by (8), \( S_i X, S_i Y \in E_{-1}(S_0), S_i Z \in E_1(S_0), \) for all \( i \geq 1 \), we get

\[
(41) \in \sum_{i=1}^{8} S_i Z \wedge \pi_{E_{i}(S_0)} (B_i(Y, X)) + \pi_{E_{i}(S_0)} \langle A - \varepsilon B_0 \rangle (Z, Y) \wedge X + \pi_{E_{i}(S_0)} \langle A - \varepsilon B_0 \rangle (X, Z) \wedge Y = 0.
\]

Taking the inner product of (41) with \( S_j Z \cap S_k Z \) we find that the expression \( \langle \varepsilon \|Z\|^2 B_j(Y, X) - \langle S_j Z, X \rangle (A - \varepsilon B_0)(Z, Y) - \langle S_j Z, Y \rangle (A - \varepsilon B_0)(X, Z), S_k Z \rangle \) is symmetric in \( j, k \geq 1 \), for all \( X, Y \in E_1(S_0), Z \in E_{-1}(S_0). \)

Fix \( j, k \geq 1, j \neq k \), and take \( Z \in \text{Span}_{a=j,k}(S_a X, S_a Y) \). Then \( S_k \pi_{E_{i}(S_0)} B_j(Y, X) - S_j \pi_{E_{i}(S_0)} B_k(Y, X) \in \text{Span}_{a=j,k}(S_a X, S_a Y) \), so (acting by \( S_j S_k \) on the both sides) \( S_j \pi_{E_{i}(S_0)} B_j(Y, X) + \)
\[ S_k \pi_{E_1(S_0)} B_k(Y, X) \in \text{Span}_{\alpha=j,k}(S_{\alpha}X, S_{\alpha}Y). \] Let \( S_{jk} \subset E_1(S_0) \times E_1(S_0) \) be the set of pairs \((X, Y)\) such that \( X \neq 0, \ Y \notin \text{Span}(X, S_jX, S_kX) \). Then \( S_{jk} \) is open and dense in \( E_1(S_0) \times E_1(S_0) \), and the vectors \( S_jX, S_kX, S_jY, S_kY \) are linearly independent for \((X, Y) \in S_{jk}\). As \( S_j \pi_{E_1(S_0)} B_j(Y, X) + S_k \pi_{E_1(S_0)} B_k(Y, X) \) is skew-symmetric in \( X, Y \) and symmetric in \( j, k \), there exist (rational) functions \( f_{jk}, f_{kj} : S_{jk} \to \mathbb{R} \) such that \( S_j \pi_{E_1(S_0)} B_j(Y, X) + S_k \pi_{E_1(S_0)} B_k(Y, X) = f_{jk}(X, Y) S_jX + f_{kj}(X, Y) S_kX - f_{kj}(Y, X) S_jY - f_{jk}(Y, X) S_kY \), for every \((X, Y) \in S_{jk}\). Taking \( Y' \in E_1(S_0) \setminus \text{Span}(X, S_jX, S_kX, S_jY, S_kY) \) (so that the vectors \( S_jX, S_kX, S_jY, S_kY, S'_jY, S'_kY \) are linearly independent) and replacing \( Y \) by \( aY + bY' \neq 0 \), we obtain from the linearity of the left-hand side that \( f_{jk} \) and \( f_{kj} \) do not depend on the first argument and are linear in the second one. It follows that for some \( u \in \mathbb{R} \), \( \sum_{\alpha=j,k} \pi_{E_1(S_0)} S_{\alpha}X, S_{\alpha}Y \) is linear, \( \langle v_{jk}, Y \rangle S_jX + \langle v_{kj}, Y \rangle S_kX - \langle v_{jk}, X \rangle S_jY - \langle v_{kj}, X \rangle S_kY, \) for all \( X, Y \in E_1(S_0) \). Choose \( i, j, k \) such that \( i, j, k \) are all distinct and add to the above equation the same one with \( j, k \) replaces by \( i, l \). The left-hand side of the resulting equation is symmetric in all four indices \( i, j, k, l \), hence the right-hand side also is. Choosing \( X, Y \in E_1(S_0) \) in such a way that the eight vectors \( S_iX, S_jX, S_kX, S_lX \), \( a = i, j, k, l \), are linearly independent (to do that, take \( X \neq 0 \) and \( Y \notin \text{Span}_{\alpha \neq \{i, j, k, l\}}(X, S_\alpha X) \)), we obtain that \( v_{ijk} = v_{ijl} \), for all \( j, k \geq 1, j \neq k \), with some \( v_{ij} \in E_1(S_0) \). It follows that \( S_j(\pi_{E_1(S_0)} B_j(Y, X) - \langle v_{ij}, Y \rangle X + \langle v_{ij}, X \rangle Y + S_k(\pi_{E_1(S_0)} B_k(Y, X) - \langle v_{ij}, Y \rangle X + \langle v_{ij}, X \rangle Y) = 0 \) for all \( X, Y \in E_1(S_0) \) and all \( j, k \geq 1, j \neq k \), which implies \( \pi_{E_1(S_0)} B_j(Y, X) = \langle v_{ij}, Y \rangle X - \langle v_{ij}, X \rangle Y \). This proves the third identity of (39), if we define the operator \( L_{eq} \) by \( L_{eq}e_j = -v_{ij} \) and extend it by linearity to \( e_0^+ \).

Substituting the third identity of (39), with \( w = e_0 \), to (41) we obtain \( (F(Z)X) \wedge Y = (F(Z)Y) \wedge X \), where the linear operator \( F : E_1(S_0) \to \text{End}(E_1(S_0)) \) is defined by \( (F(Z)X) = \varepsilon \sum_{\alpha=1}^8 (L_{eq}e_{\alpha}, X)S_{\alpha}Z_X \). For every \( u \in \mathbb{R}^9 \), extend the operator \( L_u \) from \( \mathbb{R}^9 \) to \( \mathbb{R}^{16} \) by linearity putting \( L_u w = 0 \), and then define \( L_u : \mathbb{R}^{16} \to \mathbb{R}^{16} \), for all \( u \neq 0 \), by \( L_u = L_u/\|u\| \). The identities (39) then hold for all \( u, w \in \mathbb{R}^9 \), \( u \neq 0 \), if we replace \( E_1(S_0) \) by \( E_{\|w\|}(S_0) = (E_1(S_0)/\|w\|) \). Combining the first and the third identities of (39) we obtain that for all \( u, w \in \mathbb{R}^9 \), \( u \neq 0 \), and all \( X, Y \in E_{\|w\|}(S_0) \),

\[ \pi_{E_1(S_0)} B_u(Y, X) = \langle L_u x, Y \rangle - \langle L_u w, X \rangle X. \]

For every \( u \in \mathbb{R}^9 \), define the quadratic form \( Q_u : \mathbb{R}^{16} \to \mathbb{R}^{16} \) by \( \langle Q_u(Y), X \rangle = \langle B_u(Y, X), Y \rangle \). Taking the inner product of the above equation with \( Y \in E_{\|w\|}(S_0) \) and then integrating by \( Y \) over the unit \( S(w) \subset E_{\|w\|}(S_0) \) we obtain \( \int_{S(w)} Q_u(Y)dY, X = \frac{2\pi}{\omega_7} \langle L_u x, X \rangle \), for all \( X \in E_{\|w\|}(S_0) \), where \( \omega_7 \) is the volume of \( S(w) \). Relative to some orthonormal basis \( \{e_i\} \) for \( \mathbb{R}^{16} \), the \( i \)-th component of \( \int_{S(w)} Q_u(Y)dY \in \mathbb{R}^{16} \) is \( \frac{2\pi}{\omega_7} \text{Tr}(\pi_{E_1(S_0)}(S_{\alpha}u_{\alpha})), \) where \( u_{\alpha} \in \text{Sym}(\mathbb{R}^{16}) \) is the operator associated to the quadratic form \( Y \to \langle Q_u(Y), E_i \rangle \). As \( \pi_{E_1(S_0)} = \sum_{\alpha=1}^8 (\text{id} + \|w\|^{-1}S_{\alpha}u_{\alpha}) \), we obtain \( \int_{S(w)} Q_u(Y)dY = \frac{2\pi}{\omega_7} \langle C_u + \|w\|^{-1}T(u, w), X \rangle \), where a linear operator \( C : \mathbb{R}^9 \to \mathbb{R}^{16} \) and a bilinear map \( T : \mathbb{R}^9 \times \mathbb{R}^9 \to \mathbb{R}^{16} \) are defined by \( \langle C_u, E_i \rangle = \frac{1}{16} \text{Tr}(Q_{u_{\alpha}}), \) and \( \langle T(u, w), E_i \rangle = \frac{1}{16} \text{Tr}(Q_{u_{\alpha}}S_{\alpha}u_{\alpha}), \) for \( 1 \leq i \leq 16 \). It follows that \( \langle C_u + \|w\|^{-1}T(u, w), X \rangle = \langle L_u x, X \rangle \), for all \( X \in E_{\|w\|}(S_0) \), which gives

\[ \pi_{E_{\|w\|}(S_0)}(B_u(Y, X) = \langle C_u + \|w\|^{-1}T(u, w), X \rangle Y + \langle C_u + \|w\|^{-1}T(u, w), Y \rangle X = 0, \]

for all \( X, Y \in E_{\|w\|}(S_0) \). This equation is satisfied, if we substitute \( B'_u(Y, X) = \langle C_u, X \rangle Y - \langle C_u, Y \rangle X + \sum_{i=0}^8 (T(u, e_i), X)S_{\alpha}Y - (T(u, e_i), Y)S_{\alpha}X \) for \( B_u(Y, X) \) (this follows from the fact that the sum on the right-hand side of \( B'_u(Y, X) \) does not depend on the choice of an orthonormal basis \( \{e_i\} \) for \( \mathbb{R}^9 \), so we can take \( e_0 = \|w\|^{-1}w \); then \( S_jX, S_jY \in E_{\|w\|}(S_0) \) for \( i \neq 0 \), by (8)). Therefore, for every \( u \in \mathbb{R}^9 \), the bilinear skew-symmetric map \( B''_u : \mathbb{R}^{16} \times \mathbb{R}^{16} \to \mathbb{R}^{16} \) defined by \( B''_u = B_u - B'_u \) satisfies the hypothesis of assertion 5 of Lemma 3. It follows that for some \( q : \mathbb{R}^9 \to \mathbb{R}^{16} \), \( B''_u(Y, X) = (A - \|w\|^{-1}X \wedge Y \) q(u). As the left-hand side is linear in \( u \), the map \( q \) is a linear, \( q(u) = C'u \), for all \( X, Y \in \mathbb{R}^{16}, u \in \mathbb{R}^9 \),

\[ B_u(Y, X) = \sum_{i=0}^8 S_i(X \wedge Y)(T(u, e_i) + S_{C'u}X) + (X \wedge Y)(C - C'u)u. \]
From the first identity of (39) it now follows that $(T(u, u) + \|u\|C_u, X) = 0$, for all $u \in \mathbb{R}^9$, $X \in E_{\|u\|}(S_u)$. As $\pi_{E_{\|u\|}(S_u)} = \frac{1}{2}(\text{id} + \|u\|^{-1}S_u)$, this implies $\|u\|T(u, u) + S_uCu + (S_uT(u, u) + \|u\|^2Cu) = 0$. Since $\|u\|$ is not a rational function, we obtain $T(u, u) + S_uCu = 0$, for all $u \in \mathbb{R}^9$, so the bilinear map $T' : \mathbb{R}^9 \times \mathbb{R}^9 \to \mathbb{R}^{16}$ defined by $T'(u, w) = T(u, w) + S_uCu$ is skew-symmetric.

By the second identity of (39), the map $A(X, Y)$ satisfies the hypothesis of assertion 5 of Lemma 3, so there exists $q \in \mathbb{R}^{16}$ such that for all $X, Y \in \mathbb{R}^{16}$,

$$A(X, Y) = (A - \text{id})(X \wedge Y)q = \sum_{i=0}^{8}(\langle S_iX, q \rangle S_iY - \langle S_iY, q \rangle S_iX) - ((X, q)Y - (Y, q)X).$$

Substituting (42, 43) into the fourth equation of (39), with $w = e_0$, we obtain $\sum_{i=1}^{8}(S_iq - \varepsilon(T(e_0, e_i) + S_iC'e_0), Y)S_iZ + (2q + \varepsilon(T(e_0, e_i) + (C - 2C')e_0), Z) = \varepsilon S_{2i}Y, Z = \varepsilon(T(e_0, e_i) + (C - 2C')e_0, Z) = \varepsilon Y, L_0e_0 \|Z\|^2$. It follows that the second term on the right-hand side viewed as a polynomial of $Z \in E_{-1}(S_0)$ (with $Y \in E_1(S_0)$, $k > 0$ fixed) is divisible by $\|Z\|^2$. As this term is a product of two linear forms in $Z$, we get $(2q + \varepsilon(T(e_0, e_0) + (C - 2C')e_0, Z) = 0$. From the fact that $T(u, w) = T'(u, w) - S_uCu$, where $T'$ is skew-symmetric, $(q + \varepsilon(C - C')e_0, Z) = 0$, for all $Z \in E_{-1}(S_0)$. As $e_0 \in \mathbb{R}^9$ is an arbitrary unit vector, it follows that $\|u\|q + \varepsilon(C - C')u \in E_{\|u\|}(S_u)$, for all $u \in \mathbb{R}^9$, so $S_i(\langle u\|q + \varepsilon(C - C')u = \|u\|q + \varepsilon(C - C')u) = \|u\|^2q - \varepsilon S_{2i}(C - C')u$. Since $\|u\|$ is not a rational function, we obtain $(C - C')u = \varepsilon S_uq$, for all $u \in \mathbb{R}^9$, so $T(u, w) = S_uC'u = T'(u, w) - S_uCu + S_uC'u = T'(u, w) - \varepsilon S_uC'u = T''(u, w) - \varepsilon(w, u)q$, where $T''(u, w) = T'(u, w) - \varepsilon(S_uu - \langle u, w \rangle id)q$ is skew-symmetric, as $T'$ is skew-symmetric and by (8). Substituting this to (42) we obtain $B_{\beta}(Y, X) = \sum_{i=0}^{8}(\langle T''(u, e_i) - \varepsilon(e_i, u)q, X \rangle Y - \varepsilon(S_uq, X) - \varepsilon(S_uq, Y)X).$ Substituting this expression and (43) into (37) we get after simplification: $\sigma_{XYZ}(2 \sum_{i=0}^{8}(X, q)S_iY \wedge S_iZ - 2(X, q)Y \wedge Z) = 0$, so $(A - \text{id})(\sigma_{XYZ}(X, q)Y \wedge Z) = 0$. From assertion 2 of Lemma 3 it follows that $\sigma_{XYZ}(X, q)Y \wedge Z) = 0$, for all $X, Y, Z \in \mathbb{R}^{16}$, which easily implies $q = 0$. This proves the assertion (with $T''$ denoted by $T$).

4. As it follows from (38) and assertion 3, $\rho$ is a Codazzi tensor. By [DS, Theorem 1], for any two eigenspaces $E_\alpha, E_\beta$ of $\rho$, the exterior product $E_\alpha \wedge E_\beta$ is an invariant subspace of the operator $R$ on the space of bivectors. Suppose $\rho$ is not a multiple of the identity. As in the proof of Lemma 6, by linearity, it suffices to show that the following assumption leads to a contradiction: there exist two orthogonal nontrivial invariant subspaces $E_1, E_2$ of $\rho$ such that $E_1 \wedge E_2$ is an invariant subspace of $R$. By (14), this is equivalent to the fact that every $E_\alpha \wedge E_\beta$ is an invariant subspace of the operator $A$, which is then equivalent to the fact that

$$A(X \wedge Y)Z = \sum_{i=0}^{8}(S_iX \wedge S_iY)Z = 0, \text{ for all } X, Y \in E_\alpha, Z \in E_\beta, \alpha \neq \beta.$$ 

Denote $\dim E_\alpha = d_\alpha > 0$, $\alpha = 1, 2$. By assertion 2 of Lemma 3, the space $\text{Skew}(\mathbb{R}^{16})$ is an invariant subspace of $A$, and the restriction of $A$ to it is a symmetric operator, with eigenvalues $5$ and $-3$, whose corresponding eigenspaces are $E_2$ and $E_3$. As every $E_\alpha \wedge E_\beta$ is an invariant subspace of $A$, we obtain that $E_\alpha \wedge E_\beta = \oplus_{k=2,3}V_{\alpha \beta k}$, where $V_{\alpha \beta k} = \pi_{L_k}(E_\alpha \wedge E_\beta)$. The six subspaces $V_{\alpha \beta k}$, $1 \leq \alpha \leq \beta \leq 2$, $k = 2, 3$, are mutually orthogonal. Moreover, as $\text{Skew}(\mathbb{R}^{16}) = \oplus_{k=2,3}L_k = \oplus_{1 \leq \alpha \leq \beta \leq 2}E_\alpha \wedge E_\beta$, we get $L_k = \oplus_{1 \leq \alpha \leq \beta \leq 2}V_{\alpha \beta k}$ (with all the direct sums above being orthogonal) and $V_{\alpha \beta k} = \pi_{E_\alpha \wedge E_\beta}L_k = (E_\alpha \wedge E_\beta) \cap L_k$, for all $1 \leq \alpha \leq \beta \leq 2$, $k = 2, 3$.

Every nonzero element $K \in V_{\alpha \beta 2}$ has the form $\sum_{j=0}^{8}a_{ij}S_iS_j$, where $a_{ij} = -a_{ji}$. Moreover, as $V_{\alpha \beta 2} \perp (E_\beta \wedge \mathbb{R}^{16})$, $\beta \neq \alpha$, the kernel of every such $K$ contains $E_\beta$. $\beta \neq \alpha$. Choosing an orthonormal basis for $\mathbb{R}^9$, relative to which the skew-symmetric matrix $a_{ij}$ has a canonical form, we get $K = \sum_{i=1}^{4}b_iS_{2i-1}S_{2i}$, with $b_i \neq 0$ (unless all the $b_i$’s are zeros). Then $X \in \text{Ker} K$ if and only if $X = (\sum_{i=2}^{4}c_iS_iS_{2i-1}S_{2i})X$, where $c_i = b_ib_i^{-1}$, $i = 2, 3, 4$. Consider symmetric orthogonal operators $D_i = S_iS_{2i-1}S_{2i} \in L_i$, $i = 2, 3, 4$. By (8), $D_iD_j \in L_4$, $i \neq j$, and $D_2D_3D_4 \in L_1$ (in fact, $D_2D_3D_4 = \pm S_0$). Then by assertion 1 of Lemma 3, $\text{Tr} D_i = \text{Tr} D_jD_i = \text{Tr} D_3D_2D_4 = 0$, $2 \leq i < j \leq 4$. It follows that each of the symmetric orthogonal operators $D_i, D_j, D_3D_2D_4, 2 \leq i < j \leq 4$, has eigenvalues $\pm 1$, both of multiplicity $8$. Furthermore, as $D_i$’s pairwise commute (which again follows from (8)), we can choose an orthonormal basis for $\mathbb{R}^{16}$ relative to which the matrices of the $D_i$’s are diagonal. The $D_i$’s satisfy the above
condition on the multiplicities of eigenvalues if and only if the space $\mathbb{R}^{16}$ splits into the orthogonal sum of two-dimensional subspaces $W(e_2, e_3, e_4)$, $\varepsilon_i = \pm 1$, such that $D_i W(e_2, e_3, e_4) = \varepsilon_i id W(e_2, e_3, e_4)$. From the above, $\text{Ker} \, K$ is the $+1$-eigenspace of the operator $c_2 D_2 + c_3 D_3 + c_4 D_4$. Its eigenvalues are $\lambda = c_2 e_2 + c_3 e_3 + c_4 e_4$, with the corresponding eigenspace $W_{\lambda} = \mathbb{R}(e_2, e_3, e_4)$, where the sum is taken over the set $s_4 = \{\varepsilon_2 e_2, \varepsilon_3 e_3, \varepsilon_4 e_4\}$, with $\dim W_{\lambda} = 2 \# s_4$. Considering the equations $c_2 e_2 + c_3 e_3 + c_4 e_4 = 1$, $\varepsilon_i = \pm 1$, we see that $\dim \text{Ker} \, K$ can be equal to $0, 2, 4, \text{or} 8$, and in the latter case (up to relabeling), $c_2 = \pm 1$, $c_3 = c_4 = 0$, so $K$ is a nonzero multiple of $S_1 S_2 + c_2 S_3 S_4$ and $\text{Ker} \, K$ is the $-2$-eigenspace of the symmetric operator $S_1 S_2 S_3 S_4$. As $\text{Ker} \, K \supset E_{\beta}$, $\dim \text{Ker} \geq d_{\beta}$. It follows from $d_1 + d_2 = 16$ that either one of the spaces $V_{\alpha \alpha}$ is trivial, or $d_1 = d_2 = 8$ and $\text{Ker} \, K = E_{\beta}$, for all nonzero $K \in V_{\alpha \alpha}$ (both for $(\alpha, \beta) = (1, 2)$ and $(\alpha, \beta) = (2, 1)$).

The first possibility leads to a contradiction. Indeed, suppose $V_{112} = 0$. Then $L_2 = V_{122} \oplus V_{222} \subset E_2 \wedge \mathbb{R}^{16}$ which implies that $(KX, Y) = 0$, for all $X \in E_2$ and all $Y \in E_1$, that is, $\langle SX, SY \rangle = 0$, for all $i, j = 0, \ldots, 8$. As for a nonzero $X$, $\dim \text{Span}_{\mathbb{R}}(S(X)) = 9$, it follows that $d_1 = 1$. Then for a nonzero $Z \in E_1$ we can choose $X \in E_2$ such that $Z \perp SX$. Substituting such $X, Z$ and an arbitrary $Y \in E_2$ into (44) we find that $Z \perp SY$. This implies that $E_2$ is an invariant subspace of all the operators $S_i$, hence $E_1$ also is. Then $Z$ is an eigenvector of every $S_i$, which contradicts the fact that the operator $S_i S_j$, $i \neq j$, is orthogonal and skew-symmetric.

Suppose now that $d_1 = d_2 = 8$ and $\text{Ker} \, K = E_{\beta}$, for all nonzero $K \in V_{\alpha \alpha}$ (for $(\alpha, \beta) = (1, 2)$ and $(\alpha, \beta) = (2, 1)$). Choose a nonzero $K \in V_{112}$. As it is shown above, under an appropriate choice of an orthonormal basis for $\mathbb{R}^9$, $K = c(S_1 S_2 + \varepsilon S_3 S_4)$ (for some $\varepsilon = \pm 1$, $c \neq 0$) and $E_2 = \text{Ker} \, K$ is the $\varepsilon$-eigenspace of $S_1 S_2 S_3 S_4$. Then $E_1$ is the $(-\varepsilon)$-eigenspace of $S_1 S_2 S_3 S_4$. As it follows from (8), $S_1 S_2 S_3 S_4 = \tilde{\varepsilon} S_1 S_2 S_3 S_4$, where $\tilde{\varepsilon} = -1$ for $i = 1, \ldots, 4$ and $\tilde{\varepsilon} = 1$ for $i = 5, 6, \ldots, 8$. It follows that for any nonzero $X \in E_2$ and $X, Y \in E_1$, when $1 \leq i \leq 4$ and $S_i X \in E_2$, otherwise. Moreover, as for any nonzero $X \in E_2$, $S_1 S_2 S_4 S_3 = \pm S_1 S_2 S_4$, where $\{i, j, k, l\} = \{1, 2, 3, 4\}$, the dimension of the space $\text{Span}^4_{i=1}(S(X)Y)$, $X \in E_2$, is at most 4, so there exists a nonzero $Y \in E_2$ orthogonal to this subspace. Then $\text{Span}^4_{i=1}(S(X)Y) \perp \text{Span}^4_{i=1}(S(X)Y)$. Substituting such $X$ and $Y$ into (44) and taking $Z = S_1 X \in E_1$ we obtain $\|X \|^2 S_1 Y = 0$, which is a contradiction.

It follows that $\rho$ is a multiple of the identity, at every point $y \in U$. As $\langle \nabla X \rho \rangle(Y) = \langle \nabla Y \rho \rangle(X)$, $\rho$ is in fact a constant multiple of the identity.

5. By assertion 1, there exists a linear map $N : \mathbb{R}^{16} \to \text{Skew}(\mathbb{R}^{16})$ such that $\nabla X S_w = [S_w, N X]$. Then from (38) and assertion 3, $[S_w, N X] = \sum_{i=0}^{8} (\langle T(w, e_i), X \rangle S_i Y - \langle T(w, e_i), Y \rangle S_i X, Y)$. As from (8) $[S_w, \sum_{i=0}^{8} \langle T(w, e_i), X \rangle S_i Y] = \sum_{i=0}^{8} (\langle T(w, e_i), X \rangle S_i, Y)$, for a linear map $N' : \mathbb{R}^{16} \to \text{Skew}(\mathbb{R}^{16})$ defined by $N'_X = N X + \frac{1}{4} \sum_{i=0}^{8} \langle T(w, e_i), X \rangle S_i$, we obtain that $[S_w, N'_X] = [S_w, N X]$, for all $X \in \mathbb{R}^{16}$. Taking a unit vector $w \in \mathbb{R}^{16}$ and $X \in E_{\varepsilon}(S_w)$, ($\varepsilon = \pm 1$), we get $[S_w, N'_X] = (S_w - \varepsilon \text{id}) N'_X = -2 \varepsilon \pi_{E_{\varepsilon}(S_w)} N'_X X$, and similarly, $[S_w, N X] = 2 \varepsilon \pi_{E_{\varepsilon}(S_w)} N X X$, so $\pi_{E_{\varepsilon}(S_w)} N'_X X = \pi_{E_{\varepsilon}(S_w)} N X X$, hence $\langle N'_X, X, Z \rangle = 0$, for all $X \in E_{\varepsilon}(S_w)$, $Y, Z \in E_{-\varepsilon}(S_w)$. As $N'_X$ depends linearly on $X$ and is skew-symmetric, we obtain that $\langle N'_X, X, Z \rangle = 0$, for all $X \in E_{\varepsilon}(S_w)$, $Z \in E_{-\varepsilon}(S_w)$ and all $Y \in \mathbb{R}^{16}$. It follows that $E_{\varepsilon}(S_w)$ and $E_{-\varepsilon}(S_w)$ are invariant subspaces of $N'_X$, so $N'_X$ commutes with $S_w$, for any $w \in \mathbb{R}^{16}$ and any $X \in \mathbb{R}^{16}$ (which, in fact, implies $N' = 0$). Then $\nabla X S_w = [S_w, N X] = [S_w, N X - \frac{1}{4} \sum_{i=0}^{8} \langle T(w, e_i), X \rangle S_i S_j Y] = -\sum_{i=0}^{8} \langle T(w, e_i), X \rangle S_i$, as required. \qed

It follows from assertions 4 and 5 of Lemma 7 and (36) that after a conformal change of metric on $U$, $(\nabla Z R)(X, Y) = \varepsilon \sum_{i,j=0}^{8} (\langle T(w, e_i), Z \rangle S_i X \wedge S_i Y - \langle T(w, e_i), Z \rangle S_i X \wedge S_i Y) = 0$, as $T$ is skew-symmetric. Hence $U$ is a locally symmetric space. Moreover, as $\rho$ is a constant multiple of the identity by assertion 4 of Lemma 7, the curvature tensor (14) is Osserman, so $U$ is locally isometric to a rank-one symmetric space by [GSV, Lemma 2.3] (in fact, to the Cayley projective plane or its noncompact dual, as these are the only two rank-one symmetric spaces of dimension 16 the Jacobi operator of whose curvature tensor has an eigenvalue of multiplicity exactly 8).

Thus, for every $x \in M'$ satisfying assertion (b) of Lemma 4, the metric on the neighborhood $U = U(x)$ is conformally equivalent to the metric of a rank-one symmetric space.
3.4. Proof of Theorem 1. Lemma 4 and the results of Sections 3.2 and 3.3 imply the conformal part of Theorem 1 at the generic points. Namely, every \( x \in M' \) (the latter is an open, dense subset of \( M^{16} \)) has a neighborhood \( U \) which is conformally equivalent to a domain either of a Euclidean space, or of a rank-one symmetric space, that is, of one of the model spaces
\[
\mathbb{R}^{16}, \mathbb{C}P^8, \mathbb{C}H^8, \mathbb{H}P^4, \mathbb{H}H^4, \mathbb{O}P^2, \mathbb{O}H^2,
\]
where we normalize the standard metric \( \hat{g} \) on each of the non-flat spaces above in such a way that the sectional curvature \( K_\sigma \) satisfies \( |K_\sigma| \in [1,4] \).

To prove the conformal part of Theorem 1, we will show that, firstly, the same is true for any \( x \in M^{16} \), and secondly, that the model space to a domain of which \( U \) is conformally equivalent is the same, for all \( x \in M^{16} \). Our proof very closely follows the arguments of [N5] from after Remark 4 to the end of Section 3. We start with the following technical lemma:

**Lemma 8.** Let \( (N^{16}, \langle \cdot, \cdot \rangle) \) be a smooth Riemannian space locally conformally equivalent to one of the \( \mathbb{O}P^2, \mathbb{O}H^2 \), so that \( \hat{g} = f \langle \cdot, \cdot \rangle \), for a positive smooth function \( f = e^{2\phi} : N^{16} \rightarrow \mathbb{R} \). Then the curvature tensor \( R \) and the Weyl tensor \( W \) of \( (N^{16}, \langle \cdot, \cdot \rangle) \) satisfy (with \( \varepsilon = 1 \) for \( \mathbb{O}P^2 \) and \( \varepsilon = -1 \) for \( \mathbb{O}H^2 \)):
\[
\begin{align*}
R(X,Y) &= (X \wedge KY + KX \wedge Y) + \varepsilon f(3X \wedge Y + P(X,Y)), \quad \text{where} \\
P(X,Y) &= A(X \wedge Y) = \sum_{i=0}^{8} S_i X \wedge S_i Y, \quad K = H(\phi) - \nabla \phi \otimes \nabla \phi + \frac{1}{2} \parallel \nabla \phi \parallel^2 \text{id}, \\
W(X,Y) &= W_{O,\varepsilon}(X,Y) = \varepsilon f(\frac{1}{2} X \wedge Y + P(X,Y)), \\
\|W\|^2 &= \frac{32256}{8} f^2, \\
(\nabla_Z W)(X,Y) &= \varepsilon Z f(\frac{1}{2} X \wedge Y + P(X,Y)) + \frac{1}{2} \varepsilon ((P(X,Y), \nabla f \wedge Z) + P((\nabla f \wedge Z)X,Y) + P(X,(\nabla f \wedge Z)Y),
\end{align*}
\]
where \( H(\phi) \) is the symmetric operator associated to the Hessian of \( \phi \), and both \( \nabla \) and the norm are computed relative to \( \langle \cdot, \cdot \rangle \).

**Proof.** By (7), the curvature tensor of \( \mathbb{O}P^2 (\varepsilon = 1) \) and \( \mathbb{O}H^2 (\varepsilon = -1) \) has the form \( \tilde{R}(X,Y) = \varepsilon (3X \wedge Y + \sum_{i=0}^{8} S_i X \wedge S_i Y) \), where \( (X \wedge Y)Z = \hat{g}(X,Z)Y - \hat{g}(Y,Z)X \). Under the conformal change of metric \( \hat{g} = f \langle \cdot, \cdot \rangle = e^{2\phi} \langle \cdot, \cdot \rangle \), the curvature tensor transforms as \( \tilde{R}(X,Y) = R(X,Y) - (X \wedge KY + KX \wedge Y) \).

As \( \hat{g}(X,Y) = f \langle X,Y \rangle = fX \wedge Y \) and the \( S_i \)’s still satisfy (8) and are symmetric for \( \langle \cdot, \cdot \rangle \), equation (46a) follows.

Equation (46b) follows from the definition of the Weyl tensor (12); the norm of \( W \) in (46c) can be computed directly using (8) and the fact that the \( S_i \)'s are symmetric, orthogonal and \( \text{Tr}S_i = 0 \).

Assertion 5 of Lemma 7 is satisfied for \( \hat{g} \), so \( \nabla_Z S_i = \sum_{j=0}^{8} \omega^i_j(Z)S_j \), where \( \nabla \) is the Levi-Civita connection for \( \hat{g} \) and \( \omega^i_j(Z) = -\langle T(e_i,e_j),Z \rangle \), \( \omega^i_i + \omega^j_j = 0 \). As \( \nabla_Z X = \nabla_Z X + Z \phi X + X \phi Z - \langle X,Z \rangle \phi \), we get \( \nabla_Z S_i = \sum_{j=0}^{8} \omega^i_j(Z)S_j + [S_i, \nabla \phi \wedge Z] \), so \( (\nabla_Z P)(X,Y) = P(X,Y), \nabla \phi \wedge Z) = [P((\nabla \phi \wedge Z)X,Y) + P(X,(\nabla \phi \wedge Z)Y) \), which, together with (46d), proves (46d).

From Lemma 4, the results of Sections 3.2 and 3.3 and [N5, Theorem 3], for every point \( x \in M' \), there exists a neighborhood \( U = U(x) \) and a positive smooth function \( f : U \rightarrow \mathbb{R} \) such that the Riemannian space \( (U, f \langle \cdot, \cdot \rangle) \) is isometric to an open subset of one of the model spaces of (45), so at every point \( x \in M' \), the Weyl tensor \( W \) of \( M^{16} \) either vanishes, or has the form \( W_{\nu,\varepsilon} \) given in [N5, Lemma 8, (36b)], with \( n = 16 \) and the corresponding \( \nu \), or has the form \( W_{O,\varepsilon} \) given in (46b).

Here \( W_{\nu,\varepsilon} \) is the Weyl tensor of the corresponding model space \( M_{\nu,\varepsilon} = \mathbb{C}P^8, \mathbb{C}H^8, \mathbb{H}P^4, \mathbb{H}H^4 \), where \( \varepsilon = \pm 1 \) is the sign of the curvature and \( \nu = 1 (\nu = 3) \) for complex (quaternionic) spaces, respectively. The Jacobi operators associated to the different Weyl tensors \( W_{\nu,\varepsilon} \) of [N5, Eq. (36b)] and \( W_{O,\varepsilon} \) of (46b) differ by the multiplicities and the signs of the eigenvalues, so every point \( x \in M' \) has a neighborhood conformally equivalent to a domain of exactly one of the model spaces. Moreover, the function \( f > 0 \) is well-defined when \( W \neq 0 \), as \( \|W\|^2 = C_{\nu,\varepsilon} f^2 \) by [N5, Eq. (36c)] and \( \|W\|^2 = \frac{32256}{8} f^2 \), by (46c).

By continuity, the Weyl tensor \( W \) of \( M^n \) either has the form \( W_{\nu,\varepsilon} \), or the form \( W_{O,\varepsilon} \), or vanishes, at every point \( x \in M^n \) (as \( M' \) is open and dense in \( M^n \), see Lemma 4). Moreover, every point \( x \in M^n \),
at which the Weyl tensor has the form $W_{\nu,\varepsilon}$ or $W_{0,\varepsilon}$, has a neighborhood, at which the Weyl tensor has the same form. Hence $M^n = M_0 \cup \bigcup \alpha M_\alpha$, where $M_0 = \{ x : W(x) = 0 \}$ is closed, and every $M_\alpha$ is a nonempty open connected subset, with $\partial M_\alpha \subset M_0$, such that the Weyl tensor has the same form $W_{\nu,\varepsilon}$ or $W_{0,\varepsilon}$ at every point $x \in M_\alpha$. In particular, $M_\alpha \subset M'$, for every $\alpha$, so that each $M_\alpha$ is locally conformally equivalent to one of the nonflat model spaces (45).

To prove the conformal part of Theorem 1, we need to show that either $M = M_0$ or $M_0 = \emptyset$. Suppose that $M_0 \neq \emptyset$ and that there exists at least one component $M_\alpha$. If $M_\alpha$ is locally conformally equivalent to one of the model spaces $M_{\nu,\varepsilon}$, we get a contradiction following the arguments of [N5] (from after Lemma 8 to the end of Section 3). Suppose $M_\alpha$ is locally conformally equivalent to one of $O(P^2), \Omega H^2$. Let $y \in \partial M_\alpha \subset M_0$ and let $B_\delta(y)$ be a small geodesic ball of $M$ centered at $y$ which is strictly geodesically convex (any two points from $B(y)$ can be connected by a unique geodesic segment lying in $B_\delta(y)$ and that segment realizes the distance between them). Let $x \in B_{\delta/3}(y) \cap M_\alpha$ and let $r = \text{dist}(x, M_\alpha)$. Then the geodesic ball $B = B_r(x)$ lies in $M_\alpha$ and is strictly convex. Moreover, $\partial B$ contains a point $x_0 \in M_0$. Replacing $x$ by the midpoint of the segment $[x_0]$, $x$ and $r/2$, if necessary, we can assume that all the points of $\partial B$, except for $x_0$, lie in $M_\alpha$.

The function $f$ is positive and smooth on $\overline{B} \setminus \{ x_0 \}$ (that is, on an open subset containing $\overline{B} \setminus \{ x_0 \}$, but not containing $x_0$).

**Lemma 9.** When $x \to x_0$, $x \in B$, both $f$ and $\nabla f$ have a finite limit. Moreover, $\lim_{x \to x_0, x \in B} f(x) = 0$.

**Proof.** The fact that $\lim_{x \to x_0, x \in B} f(x) = 0$ follows from (46c) and the fact that $W_{|x_0} = 0$ (as $x_0 \in M_0$).

As the Riemannian space $(B, f(\cdot, \cdot))$ is locally isometric to a rank-one symmetric space $M_0^\Omega$ (where $\varepsilon = \pm 1$ and $M_0^\Omega = O(P^2)$, $M_0^\Omega = \Omega H^2$) and is simply connected, there exists a smooth isometric immersion $\iota : (B, f(\cdot, \cdot)) \to M_0^\Omega$. Since $f$ is smooth on $\overline{B} \setminus \{ x_0 \}$ and $\lim_{x \to x_0, x \in B} f(x) = 0$, the range of $\iota$ is a bounded domain in $M_0^\Omega$. Moreover, as $\lim_{x \to x_0, x \in B} f(x) = 0$, every sequence of points in $B$ converging to $x_0$ in the metric $(\cdot, \cdot)$ is a Cauchy sequence for the metric $f(\cdot, \cdot)$). It follows that there exists a limit $\lim_{x \to x_0, x \in B} f(x) = 0$. Defining for every $x \in B$ the point $S_{x} = \text{Span}_{\iota=0}^{\iota} (S_{i})$ in the Grassmanian $G(9, \text{Sym}(T_{x}M^{16}))$, we obtain that there exists a limit $\lim_{x \to x_0, x \in B} S_{x} = S_{x_0} \subset G(9, \text{Sym}(T_{x_0}M^{16}))$. In particular, if $Z$ is a continuous vector field on $\overline{B}$, then there exists a unit continuous vector field $Y$ on $\overline{B}$ such that $Y \perp \text{Span}_{\iota=0}^{\iota} (S_{i})$ on $B$. For such two vector fields, the function $\theta(Y, Z) = \sum_{j=1}^{16} (\nabla_{E_{j}} W)(E_{j}, Y, Z)$ (where $E_{j}$ is an orthonormal frame on $\overline{B}$) is well-defined and continuous on $\overline{B}$. Using (8, 46d) and assertion 4 of Lemma 3, we obtain by a direct computation that at the points of $B$, $\theta(Y, Z) = \frac{\partial Z}{\partial f} f Z$. As $\theta(Y, Z)$ is continuous on $\overline{B}$, there exists a limit $\lim_{x \to x_0, x \in B} Z f$. Since $Z$ is an arbitrary continuous vector field on $\overline{B}$, $\nabla f$ has a finite limit when $x \to x_0$, $x \in B$. □

As $\lim_{x \to x_0, x \in B} f(x) = 0$ and the $S_{i}$'s are orthogonal, the second term on the right-hand side of (46a) tends to 0 when $x \to x_0$ in $B$. Then the tensor field defined by $(X, Y) \to (X \wedge KY + KX \wedge Y)$ has a finite limit (namely $R_{|x_0}$) when $x \to x_0$ in $B$. It follows that the symmetric operator $K$ has a finite limit at $x_0$. Computing the trace of $K$ and using the fact that $\phi = \frac{1}{2} \ln f$ we get

$$\Delta u = Fu, \quad \text{where } u = f^{1/2}, \quad F = 7 \text{ Tr } K$$

on $B$. Both functions $F$ and $u$ are smooth on $\overline{B} \setminus \{ x_0 \}$ and have a finite limit at $x_0$. Moreover, $\lim_{x \to x_0, x \in B} u(x) = 0$ by Lemma 9 and $u(x) > 0$ for $x \in \overline{B} \setminus \{ x_0 \}$. The domain $B$ is a small geodesic ball, so it satisfies the inner sphere condition (the radii of curvature of the sphere $\partial B$ are uniformly bounded). By the boundary point theorem [Fra, Section 2.3], the inner directional derivative of $u$ at $x_0$ (which exists by Lemma 9, if we define $u(x_0) = 0$ by continuity) is positive. But $\nabla u = \frac{1}{2} f^{5/2} \nabla f$ in $B$, so $\lim_{x \to x_0, x \in B} \nabla u = 0$ by Lemma 9, a contradiction.

This proves the conformal part of Theorem 1.

The “genuine” Osserman part, the Osserman Conjecture (assuming Conjecture A), now easily follows. Indeed, any Osserman manifold $M^n$, $n > 4$, is Einstein, hence by (12) its Weyl tensor is Osserman, hence $M^n$ is locally conformally equivalent to a rank-one symmetric space or to a flat space, as shown above. Then by [Nic, Theorem 4.4], $\nabla W = 0$, so, as $M^n$ is Einstein, it is locally symmetric, and the proof follows from [GSV, Lemma 2.3].
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