FACTORIZATIONS AND INARIANT SUBSPACES
FOR WEIGHTED SCHUR CLASSES

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Abstract. We study factorizations of operator valued functions of weighted
Schur classes over multiply-connected domains. There is a correspondence
between functions from weighted Schur classes and so-called “conservative
curved” systems introduced in the paper. We show that the fundamental
relationship between invariant subspaces of the main operator of a conserva-
tive system and factorizations of the corresponding operator valued function of
Schur class, which is well known in the case of the unit disk, can be extended
to our case. We develop new notions and constructions and discuss changes
that should be made to the standard theory to obtain desired generalization.

0. Introduction
It is well known [1,2] that there is an one-to-one correspondence between (simple)
unitary colligations
\[ \mathfrak{A} = \begin{pmatrix} T & N \\ M & L \end{pmatrix} \in \mathcal{L}(H \oplus \mathcal{H}, H \oplus \mathcal{M}), \quad \mathfrak{A}^* \mathfrak{A} = I, \quad \mathfrak{A} \mathfrak{A}^* = I \]
and operator valued functions \( \Theta(z) \) of the Schur class
\[ S = \{ \Theta \in H^\infty(\mathbb{D}, \mathcal{L}(\mathcal{H}, \mathcal{M})) : ||\Theta||_\infty \leq 1 \}. \]
Here \( H, \mathcal{H}, \mathcal{M} \) are separable Hilbert spaces and \( \mathcal{L}(\mathcal{H}, \mathcal{M}) \) is the space of all bounded
linear operators acting from \( \mathcal{H} \) to \( \mathcal{M} \). The mapping defined by the formula
\( \Theta(z) = L^* + zN^*(I - zT^*)^{-1}M^*, \quad |z| < 1 \) is one of the directions of the above mentioned
correspondence. The operator valued function \( \Theta(z) \) is called the characteristic
function of the unitary colligation \( \mathfrak{A} \) and its property \( ||\Theta||_\infty \leq 1 \) is the consequence
of the unitarity property of the colligation \( \mathfrak{A} \).

The reverse direction of the correspondence is realized via functional model [1,2],
whose essential ingredients are Hardy spaces \( H^2 \) and \( H^2_\mathbb{D} \) (see [3]). These two sides
of the theory (unitary colligations and Schur class functions) are equipollent:
both have simple, clear and independent descriptions and we can easily change a point
of view from unitary colligations to Schur class functions and back. This context
gives a nice opportunity to connect operator theory and function theory in a very
deep and fruitful manner [4].

One of the cornerstones of the theory is the link (see [1,2]) between factorizations
of characteristic function \( \Theta(z) \) and invariant subspaces of operator \( T \), which goes
back to [5] and [6]. The most simple way to explain this connection is to look at it

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from the point of view of systems theory and to employ the well-known correspondence between unitary colligations \( \mathfrak{A} \) and conservative linear discrete time-invariant systems \( \Sigma = (T, M, N, L; H, \mathfrak{N}, \mathfrak{M}) \) (see, e.g., [1])
\[
\begin{align*}
    x(n + 1) &= Tx(n) + Nu(n), \quad x(n) \in H, \quad u(n) \in \mathfrak{N}, \\
    y(n) &= Mx(n) + Lu(n), \quad y(n) \in \mathfrak{M}, \quad n \geq 0.
\end{align*}
\]
The conservative property (the property of energy balance) of \( \Sigma \) corresponds to the unitarity property of the colligation \( \mathfrak{A} \). If we send the sequence \( u(n) \) into the system \( \Sigma \) with the initial state \( x(0) = 0 \), we get the identity \( \tilde{y}(z) = S(z)\tilde{u}(z) \), where \( \tilde{u}(z) = \sum_{n=0}^{\infty} z^n u(n) \), \( \tilde{y}(z) = \sum_{n=0}^{\infty} z^n y(n) \), and \( S(z) = L + zM(I - zT)^{-1}N \).

Note that the transfer function \( S(z) \) of the system \( \Sigma \) is equal to \( \Theta^\ast(z) \), where \( \Theta^\ast(z) := \Theta^\prime(z) \) is the dual function to the characteristic function \( \Theta(z) \) of the unitary colligation \( \mathfrak{A} \).

Sending the output of a system \( \Sigma_2 = (T_2, M_2, N_2, L_2; H_2, \mathfrak{N}, \mathfrak{L}) \) into the input of a system \( \Sigma_1 = (T_1, M_1, N_1, L_1; H_1, \mathfrak{L}, \mathfrak{M}) \), we obtain the cascade system \( \Sigma_{21} := \Sigma_2 \cdot \Sigma_1 = (T_1 \circ T_2, M_1 \circ M_2, N_1 \circ N_2, L_1 \circ L_2; H_1 \circ H_2, \mathfrak{L}, \mathfrak{M}) \). It is clear that the transfer function \( S_{21}(z) \) of the system \( \Sigma_{21} \) is the product of the transfer functions of systems \( \Sigma_1, \Sigma_2 \) and it is easily shown that
\[
\Sigma_{21} = \left(\begin{array}{c}
T_1 \\
0
\end{array}\right) \left(\begin{array}{c}
N_1 M_2 \\
T_2
\end{array}\right), \quad \left(\begin{array}{c}
(M_1, L_1 M_2) \\
(N_1 L_2, N_2)
\end{array}\right), \quad \left(\begin{array}{c}
L_2 L_1
\end{array}\right),
\]
where \( H_{21} = H_1 \oplus H_2 \). The subspace \( H_1 \) is invariant under the operator \( T_{21} \) and therefore, if we fix the characteristic function \( \Theta_{21}(z) \), one may hope to study invariant subspaces of the operator \( T_{21} \) using this approach. Unfortunately, there are some pitfalls for this: the operator \( T_{21} \) can vary when we run over all factorizations of \( \Theta_{21}(z) \). More precisely, the variable part is the unitary component \( T_{21u} \) from the decomposition \( T_{21} = T_{21u} \oplus T_{21l} \) into completely non-unitary and unitary parts [1]. In this connection, recall that any conservative system \( \Sigma \) can be uniquely represented in the form \( \Sigma = \Sigma_s \oplus \Sigma_u \), where \( H_s = H_c \ominus H_u = H \ominus H_s \), \( H_c = \vee_{n \geq 0} T^n N(\mathfrak{M}) \), \( H_u = \vee_{n \geq 0} (T^{n+})^a M^a(\mathfrak{L}) \). Here \( \Sigma_s \) and \( \Sigma_u \) are the simple and “unitary” parts of the system \( \Sigma \), respectively. A system \( \Sigma \) is called simple if \( H = H_s \). A system \( (T, 0, 0, 0; H, \{0\}) \) is called “purely unitary” system if the operator \( T \) is unitary.

B. Sz.-Nagy and C. Foiaş established the following criterion (see [1, 2]): the product of consecutive systems \( \Sigma_{21} = \Sigma_2 \cdot \Sigma_1 \) is simple if and only if the corresponding factorization \( \Theta_{21}(z) = \Theta_2(z)\Theta_1(z) \) is regular. The product \( \Theta_{21}(z) = \Theta_2(z)\Theta_1(z) \) of Schur class functions is called regular [2] if
\[
\text{Ran}(I - \Theta_2(z)\Theta_2(z))^{1/2} \cap \text{Ran}(I - \Theta_1(z)\Theta_1(z))^{1/2} = \{0\}, \quad \text{a.e.} \ z \in \mathbb{T}.
\]
This definition of regularity is equivalent to standard one from [1].

Moreover, B. Sz.-Nagy and C. Foiaş described (Theorems VII.1.1 and VII.4.3 in [1]) an order preserving one-to-one correspondence between regular factorizations of a characteristic function and invariant subspaces of the corresponding model operator, where the order relation for invariant subspaces is the ordinary inclusion and for factorization the order relation is \( \Theta_2 \Theta_1 \prec \Theta_2' \Theta_1' \), where we write \( \Theta_2 \Theta_1 \prec \Theta_2' \Theta_1' \) if there exists \( \theta \in S \) such that \( \Theta_2 = \Theta_2' \theta \) and \( \Theta_1 = \theta \Theta_1 \). Extension of this correspondence between factorizations and invariant subspaces to the case of weighted Schur classes is the main aim of the present paper.
We shall consider operator valued functions (or rather, sets of operator valued functions) of weighted Schur classes \( S_\Xi \) :

\[
S_\Xi := \{ (\Theta^+, \Xi_+, \Xi_-) : \Theta^+ \in H^\infty(G_+; L(\mathfrak{N}_+, \mathfrak{N}_-)),
\forall \zeta \in C, \forall n \in \mathfrak{N}_+ \mid \theta^+(\zeta)n\|_{-\zeta} \leq \|n\|_{+\zeta}\},
\]

(Cfn)

where \( \mathfrak{N}_\pm \) are separable Hilbert spaces; \( G_+ \) is a finite-connected domain of the complex plane \( \mathbb{C} \) bounded by a rectifiable Carleson curve \( C \), \( G_- = \mathbb{C} \setminus \text{clos}G_+ \) and \( \infty \in G_- \); \( \Xi_\pm \) are operator valued weights such that \( \Xi_\pm, \Xi^{-1}_\pm \in L^\infty(C; L(\mathfrak{N}_\pm)) \), \( \Xi_\pm(\zeta) \geq 0 \), \( \zeta \in C \), and \( \|n\|_{\pm\zeta} := (\Xi_\pm(\zeta)n, n)^{1/2}, n \in \mathfrak{N}_\pm \). We shall also use the parallel notation \( \Theta \in \text{Cfn} \) whenever \( \Theta \in S_\Xi \).

First, we recall the construction of free functional model of Sz.-Nagy-Foiaş type (see \[8, 14, 15\]). Let \( \Pi = (\pi_+, \pi_-) \) be a pair of operators \( \pi_\pm \in L(L^2(C, \mathfrak{N}_\pm), H) \) such that

\[
\begin{align*}
(i)_1 & \quad (\pi^+_\pm \pi_\pm) z = z(\pi^+_\pm \pi_\pm); \quad (i)_2 \quad \pi^+_\pm \pi_\pm >> 0; \\
(ii)_1 & \quad (\pi^-_\pm \pi_\pm) z = z(\pi^-_\pm \pi_\pm); \quad (ii)_2 \quad P_-(\pi^-_\pm \pi_\pm) P_+ = 0; \quad (\text{Mod}) \\
(iii) & \quad \text{Ran} \pi_+ \vee \text{Ran} \pi_- = H,
\end{align*}
\]

where \( \mathfrak{N}_\pm, H \) are separable Hilbert spaces; the notation \( \Lambda >> 0 \) means that \( \exists c > 0 \) such that \( \forall u \langle Au, u \rangle \geq c \langle u, u \rangle \); the (nonorthogonal) projections \( P_\pm \) are uniquely determined by conditions \( \text{Ran} P_\pm = E^2(G_\pm, \mathfrak{N}_\pm) \) and \( \text{Ker} P_\pm = E^2(G_\mp, \mathfrak{N}_\mp) \); the spaces \( E^2(G_\pm, \mathfrak{N}_\pm) \) are Smirnov spaces \( \mathfrak{K} \) of vector valued functions with values in \( \mathfrak{N}_\pm \) (since the curve \( C \) is a Carleson curve, such projections exist); the operators \( \pi^\pm_\pm \) are adjoint to \( \pi_\pm \) if we regard \( \pi_\pm : L^2(C, \mathfrak{N}_\pm) \rightarrow H \) as operators acting from weighted spaces \( L^2 \) with operator valued weights \( \Xi_\pm = \pi^\pm_\pm \pi_\pm \). In this interpretation \( \pi_\pm \) are isometries.

Note that, in our model, we strive to retain analyticity in both the domains \( G_+ \) and \( G_- \) with the aim to reserve possibility to exploit techniques typical for boundary-value problems (singular integral operators, the Riemann-Hilbert problem, the stationary scattering theory, including the smooth methods of T.Kato). Thus we will use both the Smirnov spaces \( E^2(G_\pm) \), which are analogues of the Hardy spaces \( H^2 \) and \( H^2 \). The requirement of analyticity in both the domains conflicts with orthogonality: in general, the decomposition \( L^2(C) = E^2(G_+)+E^2(G_-) \) is not orthogonal. Note that the combination “analyticity only in \( G_+ \) and orthogonality” is a mainstream of development in the multiply-connected case starting from \[10\]. In this paper we sacrifice the orthogonality and therefore at this point we fork with traditional way of generalization of Sz.-Nagy-Foiaş theory \[11, 12, 13\]. Nevertheless, our requirements are also substantial and descend from applications (see \[8, 14, 15\]): in \[8\] we studied the duality of spectral components for trace class perturbations of a normal operator with spectrum on a curve; the functional model from \[14\] goes back to the paper \[15\], which is devoted to spectral analysis of linear neutral functional differential equations.

The operator \( \pi^-_\pm \pi_+ \) can be regarded as an analytic operator valued function \( (\pi^-_\pm \pi_+)(z), z \in G_+ \). In this connection, we shall say that the set of operator valued functions

\[
\Theta = (\pi^-_\pm \pi_+, \pi^+_\pm \pi_+, \pi^-_\pm \pi_-) \in S_\Xi.
\]

(MtoC)

is the characteristic function for a model \( \Pi \). Note also that the relation (MtoC) defines the transformation \( \Theta = F_{\text{cm}}(\Pi) \). Conversely, for a given \( \Theta \in S_\Xi \), it is possible to construct (up to unitary equivalence) a functional model \( \Pi \in \text{Mod} \)
such that \( \Theta = (\pi_+^+ \pi_+, \pi_+^+ \pi_+, \pi_-^+ \pi_-) \), i.e., there exists the inverse transformation \( F_{mc} : F_{cm}^{-1} \) (see Prop.1).

At this moment we should look for a suitable generalization of conservative systems (=unitary colligations). We define curved conservative systems in terms of the functional model. Let \( \Pi \in \text{Mod} \). Define the model system \( \widehat{\Sigma} = F_{sm}(\Pi) := (\widehat{T}, \widehat{M}, \widehat{N}, \widehat{\Theta}_u, \widehat{\Xi}; \mathcal{K}_\Theta, \mathcal{R}_+ , \mathcal{R}_-) \), where

\[
\widehat{T} \in \mathcal{L}(\mathcal{K}_\Theta), \quad \widehat{T} f := U f - \pi_+ \widehat{M} f ;
\]

\[
\widehat{M} \in \mathcal{L}(\mathcal{K}_\Theta, \mathcal{R}_+), \quad \widehat{M} f := \frac{1}{2\pi i} \int_C (\pi_+ f)(z) dz ;
\]

\[
\widehat{N} \in \mathcal{L}(\mathcal{R}_-, \mathcal{K}_\Theta), \quad \widehat{N} n := P_{\Theta u} \pi_- n ;
\]

\( \widehat{\Theta}_u \) is the unitary ”part” of \( \widehat{\Theta} = (\pi_+^+ \pi_+, \widehat{\Xi}) \);

\[
\widehat{\Xi} := (\pi_+^+ \pi_+, \pi_-^+ \pi_-) ;
\]

\( f \in \mathcal{K}_\Theta := \text{Ran} \ P_{\Theta u} , \ P_{\Theta u} := (I - \pi_+ P_+ \pi_+^+)(I - \pi_- P_- \pi_-^+), \ n \in \mathcal{R}_-, \) and the normal operator \( U \) with absolutely continuous spectrum lying on \( C \) is uniquely determined by conditions \( U \pi_{\pm} = \pi_{\pm} z \). In the sequel, we shall refer the operator \( \widehat{T} \) as the model operator. The unitary ”part” \( \Theta_u \) is determined by the unitary constant part \( \Theta^0 \) from pure-unitary decomposition \([1,2]\) of Schur class function \( \Theta^0(w) = \Theta^0_u(w) \oplus \Theta^0_\ell \), \( w \in D \), where \( \Theta^0(w) \) is the lift of the (multiple valued character-automorphic) operator valued function \( (\chi_- \Theta^+ \chi_+^{-1} )(z) \) to the universal cover space \([10] \); \( \chi_\pm \) are outer (character-automorphic) operator valued functions such that \( \chi_+^+ \chi_- = \Xi_{\pm} \).

Note also that the formulas (MtoS) define the transformation \( \widehat{\Sigma} = F_{sm}(\Pi) \).

A coupling of operators and Hilbert spaces \( \Sigma = (T, M, N, \Theta_u, \Xi; H, \mathcal{R}, \mathcal{M}) \) is called a conservative curved system if there exists a functional model \( \Pi \) with \( \mathcal{R}_+ = \mathcal{R} \) and \( \mathcal{R}_- = \mathcal{M} \), a Hilbert space \( \mathcal{K}_u \), a normal operator \( \widehat{T}_u \in \mathcal{L}(\mathcal{K}_u) \), and an operator \( X \in \mathcal{L}(H, \mathcal{K}_\Theta \oplus \mathcal{K}_u \oplus H) \) such that \( \sigma(\widehat{T}_u) \subset C, X^{-1} \in \mathcal{L}(\mathcal{K}_\Theta \oplus \mathcal{K}_u \oplus H) \), and

\[
\Sigma = (T, M, N, \Theta_u, \Xi; H, \mathcal{R}, \mathcal{M}) \sim (\widehat{\Sigma} \oplus \widehat{\Sigma}_u), \quad \text{(Sys)}
\]

where \( \widehat{\Sigma} = F_{sm}(\Pi) \) and \( \widehat{\Sigma}_u = (\widehat{T}_u, 0, 0, 0; \mathcal{K}_u, \{0\}, \{0\}) \). We write \( \Sigma_1 \sim \Sigma_2 \) if

\[
XT_1 = T_2 X, \quad M_1 = M_2 X, \quad N_1 X = N_2, \quad \Theta_{1u} = \Theta_{2u}, \quad \Xi_1 = \Xi_2 .
\]

The spaces \( \mathcal{K}_\Theta \) and \( \mathcal{K}_u \) play roles of the simple and “unitary” subspaces of the system \( \widehat{\Sigma} \oplus \widehat{\Sigma}_u \), respectively. A curved conservative system \( \Sigma \) is called simple if

\[
\rho(T) \cap G_+ \neq \emptyset \quad \text{and} \quad \bigcap_{z \in \rho(T)} \text{Ker} M(T - z)^{-1} = \{0\} .
\]

In the case of unitary colligations this definition is equivalent to standard one \([2]\) whenever \( \rho(T) \cap D \neq \emptyset \). Note that there appear some troubles if we attempt to extend the standard definition (simple subspace = controllable subspace \( \vee \) observable subspace) straightforwardly.

In the case when \( G_+ = D \) and \( \Xi_{\pm} = I \), for a conservative curved system \( \Sigma = (T, M, N, \Theta_u, \Xi; H, \mathcal{R}, \mathcal{M}) \), we can consider the block-matrix \( \mathfrak{A} = \begin{pmatrix} T & N \\ M & L \end{pmatrix} \), where \( L = \Theta^+(0)^* \). It is readily shown that \( \mathfrak{A} \) is a unitary colligation and \( \Theta^+(z) = L^* + z N^+(I - z T^*)^{-1} M^* \), \( |z| < 1 \), i.e. \( \Sigma \) is a conservative system and \( \Theta^+ \) is the Sz.-Nagy-Foiaş characteristic function. In the case of simple-connected
domains we lose the property of unitarity for the matrix $\mathfrak{A}$ but we can regard a system $\Sigma = (T, M, N)$ as the result of certain transformation (deformation) of a unitary colligation (=conservative system) $\mathfrak{A}$ $\mathfrak{B}$. Another reason to call our systems “curved conservative” is the fact that the characteristic function of such a system is a weighted Schur function.

Thus, we have defined the notion of conservative curved system. Note that linear similarity (instead of unitary equivalence for unitary colligations) is a natural kind of equivalence for conservative curved systems and duality is a substitute for orthogonality.

The following diagram shows relationships between models, characteristic functions, and conservative curved systems

$\text{Cfn} \xrightarrow{\mathcal{F}_{mc}} \text{Mod} \xrightarrow{\mathcal{F}_{sm}} \text{Sys}$.

As we can now see, characteristic functions and conservative curved systems are not on equal terms: first of them play a leading role because the definition of conservative curved system depends on the functional model, which, in turn, is uniquely determined by the characteristic function. But, surprisingly, the conservative curved systems is a comparatively autonomous notion (i.e., though we define such systems in terms of the functional model, many properties and operations with conservative curved systems can be formulated intrinsically and do not refer explicitly to the functional model) and one of the aims of this paper is to “measure” a degree of this autonomy with the point of view of the correspondence “factorizations of characteristic function $\leftrightarrow$ invariant subspaces”.

If we are going to follow the way described above for conservative systems, we need to introduce transfer functions. For a curved conservative system $\Sigma$, we define the transfer function

$\Upsilon = (\Upsilon(z), \Theta_u, \Xi)$, where $\Upsilon(z) := M(T - z)^{-1}N$.

The formula (StoT) defines also the transformation $\Upsilon = \mathcal{F}_{ts}(\Sigma)$. Then, using the functional model, the transformation $\mathcal{F}_{tc} = \mathcal{F}_{ts} \circ \mathcal{F}_{sc}$ can be computed as

$\Upsilon(z) = \begin{cases} 
\Theta_+(z) - \Theta^+(z)^{-1}, & z \in G_+ \cap \rho(T); \\
-\Theta_-(z), & z \in G_. 
\end{cases}$

In this connection, note that the spectrum of a model operator coincides with the spectrum of a characteristic function, i.e., $z \in G_+ \cap \rho(T) \iff \exists \Theta^+(z)^{-1}$. The operator valued functions $\Theta_\pm(z)$ are defined by the formulas

$\Theta_\pm(z) n := (P_\pm \Theta_\pm n)(z)$, $z \in G_\pm$, $n \in \mathfrak{N}_\pm$;

$\Theta^-(\zeta) := (\pi_+^\dagger \pi_-)(\zeta) = \Xi_+(\zeta)^{-1} \Theta^+(\zeta) \Xi_-(\zeta)$, $\zeta \in \mathbb{C}$.

In the case when $G_+ = \mathbb{D}$ and $\Xi_\pm \equiv I$ we get $\Theta^-(\zeta) = \Theta^+(1/\zeta)^* \cdot |\zeta| = 1$ and therefore, $\Theta^+(z) = \Theta^+(0)^*$, $|z| < 1$; $\Theta^-(z) = \Theta^+(1/\bar{z})^* - \Theta^+(0)^*$, $|z| > 1$.

Thus we arrive at the complete diagram

$\text{Mod} \xrightarrow{\mathcal{F}_{mc}} \text{Cfn} \xrightarrow{\mathcal{F}_{sm}} \text{Sys} \xrightarrow{\mathcal{F}_{tc}} \text{Tfn}$.
Unfortunately, we have obtained almost nothing for our purpose: to study the correspondence “factorizations ↔ invariant subspaces”. The main difficulty is to invert the arrows $F_{tc}$ and $F_{ts}$. In the case of the unit circle the transfer function can be calculated as $\Upsilon(z) = \Theta^+(0)^* - \Theta^+(1/z)^*$, $|z| > 1$ and, conversely, one can easily recover the characteristic function $\Theta^+(z)$ from the transfer function $\Upsilon(z)$ (see [17] for this case and for the case of simple connected domains). But, in general, especially for multiply-connected domains, the latter is a considerable problem. Note that the condition $\Upsilon(z) = M(T-z)^{-1}N \in N(G_+ \cup G_-)$ (that is, the transfer function $\Upsilon(z)$ is an operator valued function of Nevanlinna class: $\Upsilon(z) = 1/\delta(z) \Omega(z)$, where $\delta \in H^\infty(G_+ \cup G_-)$ and $\Omega \in H^\infty(G_+ \cup G_-, \mathcal{L}(\mathfrak{R}, \mathfrak{R}_+))$ is sufficient for uniqueness of characteristic function and there is a procedure recovering the characteristic function from a given transfer function. Moreover, under this assumption it is possible to give intrinsic description for conservative curved systems. Note that we reap the benefit of functional model when we are able to determine that some set of operators $(T, M, N)$ is a conservative curved system [8, 14, 9]. The author plans to address these problems elsewhere.

Thus we distinguish notions of characteristic and transfer function and there are no simple enough (and suitable in the study of factorizations) relationships between them. These circumstances dictate that we have to use only the partial diagram (dgr) and to ignore other objects and transformations related to transfer functions from the complete diagram (Dgr). Note also that we study the correspondence “factorizations of characteristic function ↔ invariant subspaces of operator $T$” in contrast to the correspondence studied in [7]: “factorizations of transfer function ↔ invariant subspaces”. At this point we fork with [7].

The paper is organized as follows. In Section 1 we deal with the fragment $\text{Cfn} \xrightarrow{\mathcal{F}_{mc}} \text{Mod}$: in the context of the functional model we develop the constructions corresponding to factorizations of characteristic functions. If we restrict ourselves to regular factorizations, we can keep on to exploit the functional model $\text{Mod}$. But to handle arbitrary factorizations and to obtain a pertinent definition of the product of conservative curved systems we need some generalization of $\text{Mod}$. Moreover, the order relation $\Theta_2 \Theta_1 < \Theta_2' \Theta_1'$ implies the factorizations like $\Theta_2 \Theta_1$ and therefore we need a functional model suited to handle factorizations of characteristic function with three or more multipliers. With this aim we introduce the notion of n-model $\text{Mod}_n$ and extend the transformations $\mathcal{F}_{mc}$ and $\mathcal{F}_{cm}$ to this context. In the rest part of the section we study geometric properties of n-models in depth and do this mainly because they form a solid foundation for our definition of the product of curved conservative systems in the next section.

At this moment it is unclear how to define the product of conservative curved systems. As a first approximation we can consider the following construction. Let $\Sigma_1 \sim \hat{\Sigma}_1 = \mathcal{F}_{sm}(\mathcal{F}_{mc}(\Theta_1))$ and $\Sigma_2 \sim \hat{\Sigma}_2 = \mathcal{F}_{sm}(\mathcal{F}_{mc}(\Theta_2))$. Then the candidate for their product is $\hat{\Sigma}_{21} = \mathcal{F}_{sm}(\mathcal{F}_{mc}(\Theta_2 \Theta_1))$, where $\mathcal{F}_{mc}(\Theta_2 \Theta_1)$ is 3-model corresponding to the factorization $\Theta_{21} = \Theta_2 \cdot \Theta_1$. Our aim is to define the product $\Sigma_2 \cdot \Sigma_1$ by explicit formulas without referring to the functional model. In Section 2 we suggest such a definition and study basic properties of it. The main one among those properties is the property that the product of systems $\Sigma_2 \cdot \Sigma_1$ is a conservative curved system too (Theorem A). The geometry properties of n-model established in Section 1 play crucial role in our reasoning.
In Section 3 we establish the correspondence between two notions of regularity. The first of them is the regularity of the product of conservative curved systems $\Sigma_2 \cdot \Sigma_1$, the second one is the notion of regular factorization of operator valued functions $[1, 2]$, which we extend to the weighted Schur classes. We obtain this correspondence indirectly: introduce the notion of regularity for models and establish separately the correspondences $\text{Cfn} \leftrightarrow \text{Mod}$ and $\text{Sys} \leftrightarrow \text{Mod}$.

In Section 4 we study the transformation $F_{ic}$ defined therein, which takes a factorization $\Theta_2 \Theta_1$ of characteristic function to the invariant subspace of the model operator of the system $\hat{\Sigma}_{21} = F_{ic}(F_{mc}(\Theta_2 \Theta_1))$. We show that this mapping is surjective. Combining this property of $F_{ic}$ with the criterion of regularity from Section 3, we establish the main result of the paper: there is an order preserving one-to-one correspondence between regular factorizations of a characteristic function and invariant subspaces of the resolvent of the corresponding model operator. In conclusion we translate results obtained for model operators into the language of conservative curved systems.

Note that the multiply connected domain specific appears essentially only in the proof of Prop. 4.1. So, at first a reader can study the paper assuming that one-to-one correspondence between regular factorization of a characteristic function and invariant subspaces of the resolvent of the corresponding model operator.

In conclusion we translate results obtained for model operators into the language of conservative curved systems.

Note that the multiply connected domain specific appears essentially only in the proof of Prop. 4.1. So, at first a reader can study the paper assuming that one-to-one correspondence between regular factorization of a characteristic function and invariant subspaces of the resolvent of the corresponding model operator.

1. Geometric properties of n-model

We start with the definition of an $n$-characteristic function, which formalizes products of weighted Schur class functions like the following $\theta_n \cdot \ldots \cdot \theta_2 \theta_1$: in fact, we merely rearrange them $\Theta_{ij} := \theta_{i-1} \cdot \ldots \cdot \theta_j$.

Definition. Let $\Xi_k, k = 1, n$ be operator valued weights such that $\Xi_k, \Xi_k^{-1} \in L_{\infty}(C, L(H_k))$, $\Xi_k(\zeta) \geq 0, \zeta \in C$. A set of analytic in $G_+$ operator valued functions $\Theta = \{\Theta_{ij} : i \geq j\}$ is called an $n$-characteristic function if $\Theta_{ij} \in S_\Xi$ with weights $\Xi = (\Xi_i, \Xi_j)$ and $\forall i \geq j \geq k \Theta_{ik} = \Theta_{ij} \Theta_{jk}$.

We assume that $\Theta_{kk} := I$ and denote by $\text{Cfn}_n$ the class of all $n$-characteristic functions. In the sequel, we shall usually identify $3$-characteristic function with the factorization of Schur class function $\theta = \theta_2 \cdot \theta_1$, where $\theta = (\Theta_{31}, \Xi_1, \Xi_3), \theta_1 = (\Theta_{21}, \Xi_1, \Xi_2), \theta_2 = (\Theta_{32}, \Xi_2, \Xi_3)$. It is clear how to define the product of $n$-characteristic functions $\Theta = \Theta'' \cdot \Theta'$: assuming that $\Xi''_i = \Xi''_j$, we need only to renumber multipliers, for instance, $\Theta_{ij} = \Theta''_{i-n'+1,i'} \Theta'_{i'}$, $i \geq n' \geq j$.

In the context of functional models a corresponding notion is the notion of $n$-functional model.

Definition. An $n$-tuple $\Pi = (\pi_1, \ldots, \pi_n)$ of operators $\pi_k \in L(L^2(C, H_k), \mathcal{H})$ such that

(i) $\forall k (\pi_k^2 \pi_k^*) z = z(\pi_k^* \pi_k); \quad \pi_k^2 \pi_k \geq 0$;
(ii) $\forall j \geq k (\pi_j^2 \pi_k^*) z = z(\pi_j^* \pi_k); \quad P_-(\pi_j^2 \pi_k^*) P_+ = 0$;
(iii) $\forall i \geq j \geq k \pi_i^2 \pi_k = \pi_i^2 \pi_j \pi_j^* \pi_k$;
(iv) $\mathcal{H}_{\pi_{i \vee \cdots \vee i_k}} = \mathcal{H}$

is called an $n$-model.
Here $H_{\pi_n,\cdots,\pi_1} := \bigvee_{k=1}^n \text{Ran} \pi_k$. The definition is an extension of the definition (Mod): namely, $\text{Mod} = \text{Mod}_2$. It is readily seen that $\Theta = \{\pi_i^\dagger \pi_j\}_{i<j}$ is an $n$-characteristic function with weights $\Xi_k = \pi_k^* \pi_k$ and therefore we have defined the transformation $F_{cm} : \text{Mod}_n \to \text{Cfm}_n$. The existence of the “inverse” transformation $F_{mc}$ follows from

**Proposition 1.1.** Suppose $\Theta \in \text{Cfm}_n$. Then $\exists \Pi \in \text{Mod}_n$ such that $\Theta = F_{cm}(\Pi)$. If also $\Theta = F_{cm}(\Pi')$, then there exists an unitary operator $X : H_{\pi_n,\cdots,\pi_1} \to H'_{\pi_n,\cdots,\pi_1}$ such that $\pi_k^\dagger X \pi_k = X \pi_k$.

**Proof.** We put $H = \oplus_{k=1}^n H_k^\Delta$, where $H_k^\Delta = \text{col} \Delta_{kk+1}k^2(C,\mathcal{G}_k)$, $k = 1, n-1$, $H_n^\Delta = L^2(C,\mathcal{G}_n)$, $\Delta_{kk+1} := (I - \Theta_{k+1k})^{1/2}$, and $\Theta_{k+1k}$ is adjoint to the operator $\Theta_k: L^2(C,\Xi_k) \to L^2(C,\Xi_{k+1})$. Let $\nu_k$, $k = 1, n$ be the operators of embedding of $H_k^\Delta$ into $H$ and

$$\pi_n := \nu_n, \quad \pi_k := \nu_{k+1}\pi_{k+1} + \nu_k\Delta_{kk+1}, \quad k = 1, n-1.$$ 

It can easily be calculated that

$$\pi_k = \nu_k\Theta_nk + \nu_{n-1}\Delta_{n-1n-1}\Theta_{n-1k} + \ldots + \nu_{j}\Delta_{jj+1}\Theta_{jk} + \ldots + \nu_k\Delta_{kk+1}. $$

From this identity we get $\pi_i^\dagger \pi_j = \Theta_{ij}$, $i \geq j$.

The existence and unitary property of $X$ follows from the identity

$$\sum_{i,j=1}^n (\pi_i^\dagger u_i, u_j) L^2(C,\Xi_i) = \sum_{i,j=1}^n (\pi_i^\dagger u_i, u_j) L^2(C,\Xi_i) = ||\pi_i^\dagger u_i + \ldots + \pi_n^\dagger u_n||^2. \quad \square$$

The construction of Prop. 1.1 is simplified if all functions $\Theta_{ij}$ are two-sided $\Xi$-inner. In this case $H = L^2(C)$ and $\pi_k = \Theta_{nk}$.

We can consider an equivalence relation $\sim$ in $\text{Mod}_n : \Pi \sim \Pi'$ if there exists an unitary operator $X : H_{\pi_n,\cdots,\pi_1} \to H'_{\pi_n,\cdots,\pi_1}$ such that $\pi_k^\dagger X \pi_k = X \pi_k$. It is clear that the transformation $F_{cm}$ induces a transformation $F_{cm} : \text{Mod}_n \to \text{Cfm}_n$ such that $F_{cm}(\Pi') = F_{cm}(\Pi)$, $\Pi' \in \text{Cfm}_n$. By Prop. 1.1 there exists the inverse transformation $F_{mc} : \text{Cfm}_n \to \text{Mod}_n$. But, in the sequel, we shall usually ignore this equivalence relation and use merely the transformations $F_{cm}$ and $F_{mc}$.

The product of $n$-models $\Pi', \Pi''$ with the only restriction $\pi'_n \pi''_n = \pi''_n \pi'_n$ is defined (up to unitary equivalence) as $\Pi = \Pi' \cdot \Pi'' := F_{mc}(F_{cm}(\Pi')) \cdot F_{mc}(\Pi'')$.

Using the construction of Prop. 1.1 we can uniquely determine the normal operator $U = X_2 X^{-1} \in \mathcal{L}(H_{\pi_n,\cdots,\pi_1})$ with absolutely continuous spectrum $\sigma(U) \subset C$ such that $U \pi_k = \pi_k z$, where $X : H_{\pi_n,\cdots,\pi_1} \to H_{\pi_n,\cdots,\pi_1}$ is an unitary operator such that $\pi_k = X \pi_k$; the operators $\tilde{\pi}_k$ are constructed for $n$-characteristic function $\Theta = F_{cm}(\Pi)$ as in Prop. 1.1.

Taking into account the existence of a such operator $U$, note that $F_{sc}(\Theta) = F_{sc}(\Theta_1) \oplus \tilde{\Sigma}_u$, where the system $\tilde{\Sigma}_u = (\tilde{T}_u,0,0,0)$ is a “purely normal” system with the normal operator $\tilde{T}_u = U | (H_{\pi_n,\cdots,\pi_1} \ominus H_{\pi_n,\cdots,\pi_1})$, $\sigma(\tilde{T}_u) \subset C$.

Let $\Pi \in \text{Mod}_n$. Now we define our building bricks: orthoprojections $P_{\pi_n,\cdots,\pi_1}$ onto $H_{\pi_n,\cdots,\pi_1}$ and projections $q_{i\pm} := \pi_i P_{\pi_n,\cdots,\pi_1}^\dagger$.

**Lemma 1.2.** For $i \geq j \geq k \geq l \geq m$

1. $q_{i} - q_{j} = 0$; 2. $q_{i} + q_{l} = \pi_{i} \pi_{l} = P_{\pi_i}$; 3. $P_{\pi_j,\cdots,\pi_1}(I - \pi_k \pi_k^\dagger)P_{\pi_j,\cdots,\pi_1} = 0$; 4. $P_{\pi_j,\cdots,\pi_1}(I - \pi_k \pi_k^\dagger)P_{\pi_j,\cdots,\pi_1} = 0$. 


Proof. Statement 1) is a direct consequence of (ii) in \((\text{Mod}_n)\). Statement 2) is obvious. Statement 3) is equivalent to the relation 
\[ \forall f, g \in \mathcal{H}, \quad ((I - \pi_k \pi_k^\dagger)\mathcal{P}_{\pi_i \vee \cdots \vee \pi_m} f, \mathcal{P}_{\pi_i \vee \cdots \vee \pi_j} g) = 0. \]
The latter can be rewritten in the form 
\[ ((I - \pi_k \pi_k^\dagger)\pi_{i'} u, \pi_{i'} v) = 0, \quad j \leq i' \leq i, \quad m \leq i' \leq l \]
and is true because of (iii) in \((\text{Mod}_n)\). Statement 4) can be obtained from Statement 3) by conjugation. \( \square \)

We also define the projections 
\[ P_{(ij)} := \mathcal{P}_{\pi_i \vee \cdots \vee \pi_j} (I - q_j^+)(I - q_i^-), \quad i \geq j. \]

It is easily shown that \( P_{(ij)} = 0 \) and 
\[ P_{(ij)} = (I - q_j^+)(I - q_j^-) = (I - q_i^-)P_{\pi_i \vee \cdots \vee \pi_j}. \]

Indeed, \( P_{\pi_i \vee \cdots \vee \pi_j} - P_{\pi_i} \) is orthoprojection onto \( \mathcal{H}_{\pi_i \vee \cdots \vee \pi_j} \otimes \mathcal{H}_{\pi_i} \), \( P_{\pi_i} q_{i\pm} = q_{i\pm} P_{\pi_i} \) and \((P_{\pi_i \vee \cdots \vee \pi_j} - P_{\pi_i}) q_{i\pm} = q_{i\pm} (P_{\pi_i \vee \cdots \vee \pi_j} - P_{\pi_i}) = 0 \). The same is hold for \( \pi_j \).

Then, 
\[
P_{(ij)}^2 = P_{\pi_i \vee \cdots \vee \pi_j} (I - q_j^+)(I - q_j^-)(I - q_i^-)P_{\pi_i \vee \cdots \vee \pi_j} \\
= P_{\pi_i \vee \cdots \vee \pi_j} [(I - q_j^+)(I - q_j^-)(I - q_i^-) - (I - q_j^+)(I - q_i^-)]P_{\pi_i \vee \cdots \vee \pi_j} \\
= P_{\pi_i \vee \cdots \vee \pi_j} (I - q_j^+)(I - q_i^-) = P_{(ij)}. 
\]

Note also that \( P_{(ij)} = \Theta_{\pi j} P_{\pi_j} \Theta_{\pi_i}^{-1} P_{\pi_i} \Theta_{\pi_j}^{-1} \) whenever all functions \( \Theta_{\pi j} \) are two-sided \( \Xi \)-inner (recall that then we can choose \( \pi_k = \Theta_{nk} \)).

**Lemma 1.3.** For \( i \geq j \geq k \geq l \), one has 

1) \( P_{(ij)} q_{k+} = 0; \) \quad 2) \( q_{i-} P_{(jk)} = 0; \) \quad 3) \( P_{(ij)} P_{(kl)} = 0; \)

4) \( P_{(ik)} P_{(jk)} = P_{(jk)}; \) \quad 5) \( P_{(ij)} P_{(ik)} = P_{(ij)}; \) \quad 6) \( P_{(jk)} P_{(ij)} = 0. \)

**Proof.** Using Lemma 1.2 we have 

1) 
\[
P_{(ij)} q_{k+} = P_{\pi_i \vee \cdots \vee \pi_j} (I - q_j^+)(I - q_i^-)q_{k+} = P_{\pi_i \vee \cdots \vee \pi_j} (I - q_j^+) q_{k+} \\
= P_{\pi_i \vee \cdots \vee \pi_j} [(I - \pi_j \pi_j^\dagger) + q_j] q_{k+} = P_{\pi_i \vee \cdots \vee \pi_j} (I - \pi_j \pi_j^\dagger) \pi_k P_{\pi_j^\dagger} = 0; 
\]

2) 
\[
q_{i-} P_{(jk)} = q_{i-} (I - q_k^-) P_{\pi_j \vee \cdots \vee \pi_k} q_{i-} (I - q_j^-) P_{\pi_j \vee \cdots \vee \pi_k} \\
= q_{i-} [(I - \pi_j \pi_j^\dagger) + q_j] P_{\pi_j \vee \cdots \vee \pi_k} = q_{i-} P_{\pi_i \pi_i^\dagger} (I - \pi_j \pi_j^\dagger) P_{\pi_j \vee \cdots \vee \pi_k} = 0; 
\]

3) 
\[
P_{(ij)} P_{(kl)} = P_{(ij)} (I - q_l^+)(I - q_k^-) P_{\pi_j \vee \cdots \vee \pi_k} = P_{(ij)} (I - q_k^-) P_{\pi_k \vee \cdots \vee \pi_j} \\
= P_{(ij)} [(I - \pi_k \pi_k^\dagger) + q_k] P_{\pi_k \vee \cdots \vee \pi_j} \\
= (I - q_j^+)(I - q_i^-) P_{\pi_i \vee \cdots \vee \pi_j} (I - \pi_k \pi_k^\dagger) P_{\pi_k \vee \cdots \vee \pi_j} = 0; 
\]

4) 
\[
P_{(ik)} P_{(jk)} = P_{\pi_i \vee \cdots \vee \pi_k} (I - q_k^-)(I - q_i^-) P_{(jk)} = P_{\pi_i \vee \cdots \vee \pi_k} (I - q_k^+). \]
5) \[ P_{(ij)}P_{(ik)} = P_{(ij)}(I - q_{k+})(I - q_k)P_{\pi_1 \ldots \pi_k} = P_{(ij)}(I - q_{i-})P_{\pi_1 \ldots \pi_k} = P_{(ij)}; \]

6) \[ P_{(jk)}P_{(ij)} = P_{\pi_j \ldots \pi_k}(I - q_{k+})(I - q_{j-})(I - q_j) = P_{\pi_j \ldots \pi_k}(I - q_{j+})P_{\pi_1 \ldots \pi_j} \]

\[ = (I - q_{k+})P_{\pi_j \ldots \pi_k}(I - q_{j+})P_{\pi_1 \ldots \pi_j}(I - q_i) = 0. \]

We also define the subspaces

[\mathcal{K}_{(ij)} := \text{Ran} P_{(ij)}, \quad \mathcal{H}_{ij} := \mathcal{H}_{\pi_1 \ldots \pi_j}, \quad \mathcal{H}_{ij+} := \mathcal{H}_{ij} \cap \text{Ker} q_{i-}, \quad \mathcal{D}_{j+} := \text{Ran} q_{j+}.]

It is easy to prove that \( \mathcal{H}_{ij+} \cap \text{Ker} P_{(ij)} = \mathcal{D}_{j+} \). Indeed, let \( f \in \mathcal{H}_{ij+} \cap \text{Ker} P_{(ij)} \). Then

\[ f = (I - P_{(ij)})f = f - (I - q_{j+})(I - q_{i-})P_{\pi_1 \ldots \pi_j}f \]

\[ = f - (I - q_{j+})(I - q_{i-})f = f - (I - q_{j+})f = q_{j+}f \in \mathcal{D}_{j+}. \]

Conversely, let \( f \in \mathcal{D}_{j+} \). Then \( f = q_{j+}f \in \mathcal{H}_{ij} \) and therefore \( q_{j-}f = q_{i-}q_{j+}f = 0 \), that is, \( f \in \mathcal{H}_{ij+} \). Hence we have

\[ P_{(ij)}f = (I - q_{j+})(I - q_{i-})P_{\pi_1 \ldots \pi_j}f = (I - q_{j+})(I - q_{i-})f = (I - q_{j+})f = 0 \]

and \( f \in \mathcal{H}_{ij+} \cap \text{Ker} P_{(ij)} \).

Translating the assertions of the above lemmas into the language of geometry, we obtain

\[ \mathcal{K}_{(ij)} \subset \mathcal{H}_{ij+}, \quad \mathcal{K}_{(jk)} \subset \mathcal{K}_{(ik)}, \quad \mathcal{H}_{jk+} \subset \mathcal{H}_{i+}, \quad i \geq j \geq k \geq l. \]

Indeed, let \( f \in \mathcal{K}_{(ij)} \). Then \( f = P_{(ij)}f \in \mathcal{H}_{ij} \) and \( g_{i-}f = q_{i-}(I - q_{j-})(I - q_{i-})f = g_{i-}(I - q_{j-})f = 0 \Rightarrow f \in \mathcal{K}_{(ii)}. \) The inclusion \( \mathcal{K}_{(ijk)} \subset \mathcal{K}_{(ik)} \) is a straightforward consequence of Lemma 1.3.4. Let \( f \in \mathcal{H}_{jk+} \). Then

\[ g_{i-}f = g_{i-}(I - q_{j-})P_{\pi_1 \ldots \pi_k}f = g_{i-}(I - q_{j+})f = 0, \]

and therefore \( f \in \mathcal{H}_{i+}. \)

Let \( 1 = m_1 \leq \ldots \leq m_i \leq \ldots \leq m_N = n \). We define the operators

\[ P_{[m, m_j]} := P_{[m_{j+1}, m_j]}(I - P_{[m_{j+2}, m_{j+1}]}) \ldots (I - P_{[m_{m_i-1}, m_{m_i-1}]}) \]

\[ + P_{[m_{j+2}, m_{j+1}]}(I - P_{[m_{j+3}, m_{j+2}]}) \ldots (I - P_{[m_{m_i-1}, m_{m_i-1}]}) \]

\[ + \ldots + P_{[m_{m_i-2}, m_{m_i-1}]}(I - P_{[m_{m_i-1}, m_{m_i-1}]}), \quad i \geq j. \]

Note that our notation is ambiguous: the projection \( P_{[m, m_j]} \) depends on the whole chain \( m_j \leq \ldots \leq m_i \) but not only on two numbers \( m_j \) and \( m_i \). The following properties of operators \( P_{[m, m_j]} \) are straightforward consequences of Lemma 1.3.

**Proposition 1.4.** For \( i \geq j \geq k \geq l \), \( 1) \) \( P_{[m, m_j]}q_{m+k+} = 0; \) \( 2) q_{m_i}P_{[m, m_i]} = 0; \) \( 3) P_{[m, m_j]}P_{[m_k, m_i]} = 0. \)
Further, since $I - P_{[m,m_1]} = (I - P_{[m_1,m_1+1]})(I - P_{[m_1+2,m_{j+1}]} ) \ldots (I - P_{(m,m_{i-1})})$, we get the following recursion relation
\[ P_{[m,m_j]} = P_{[m,m_k]}(I - P_{[m_1,m_j]}), \quad i \geq j \geq k. \]
Since $P_{[m_j+1,m_j]} = P_{[m_1+1,m_j]}$, we obtain by induction that the operator $P_{[m,m_j]}$ is a projection and
\[ \mathcal{K}_{[m,m_j]} = \mathcal{K}_{(m,m_{i-1})} + \ldots + \mathcal{K}_{(m_j+1,m_j)}, \]
where $\mathcal{K}_{[m,m_j]} := \text{Ran} P_{[m,m_j]}$. We use the notation $H = H' + H''$ if there exists a projection $P'$ such that $H' = \text{Ran} P'$, $H'' = \text{Ker} P'$. Besides we have
\[ \mathcal{K}_{[m,m_j]} \subset \mathcal{H}_{m,m_j} , \quad \mathcal{D}_{m_j} \subset \text{Ker} P_{[m,m_j]} . \]
The first inclusion follows straightforwardly from Prop. 1.4(2). The second one is a consequence of Prop. 1.4(1).

Though we systematically strive to deal only with nonorthogonal projections $q_{i+}$, sometimes we have to employ their orthogonal counterparts $q^\prime_{i+}$. By [10], there exists an isometries $\pi^i_1 \in \mathcal{L}(L^2(C, \mathcal{G}_i), \mathcal{H})$ such that $\text{Ran} \pi^i_1 = \text{Ran} \pi_i$ and $\text{Ran} q_{i+} = \pi^i_1 E^i_0(G_+, \mathcal{G}_i)$, where $E^i_0(G_+, \mathcal{G}_i)$ is the Smirnov space of character-automorphic functions (see [10][11][12] for the definition). Let $q^\prime_{i+} = \pi^i_1 P^\prime - \pi^i_1 \pi^i_2 P^\prime_+$. Then $\text{Ran} q^\prime_{i+} = \text{Ran} q_{i+}$, $P^\prime = \pi^i_1 \pi^i_2 P^\prime_+$, and $P^\prime_+ \pi^i_1 \pi^i_2 P^\prime_+ = 0$. Define also the projections $q^\prime_{i-} = \pi^i_1 P^\prime - \pi^i_1 \pi^i_2$. Then we have
\[ q_{i-} q^\prime_{j+} = 0 \quad \text{and} \quad q^\prime_{i-} q_{j+} = 0, \quad i \geq j. \]
Indeed,
\[ q_{i-} q^\prime_{j+} = \pi^i_1 P^\prime - \pi^i_1 \pi^i_2 P^\prime_+ \pi^i_2 \pi^i_1 = \pi^i_1 P^\prime - \pi^i_1 \pi^i_2 P^\prime_+ \pi^i_1 \pi^i_2 P^\prime_+ = 0 + 0 = 0. \]

By the same reason, $q^\prime_{i-} q_{j+} = 0$. Using these identities and repeating mutatis mutandis proof of Lemma 1.4, we obtain
\[ P^\prime_{(ij)} q_{k+} = P_{(ij)} q^\prime_{k+} = 0; \quad q^\prime_{i-} P_{(jk)} = q_{i-} P^\prime_{(jk)} = 0; \]
\[ P^\prime_{(ij)} P_{(kl)} = P_{(ij)} P^\prime_{(kl)} = 0, \quad i \geq j \geq k \geq l. \]
Then, evidently,
\[ P_{[m,m_j]} P^\prime_{[m,m_k]} = P_{[m,m_j]} P^\prime_{[m,m_k]} = P^\prime_{[m,m_j]} P^\prime_{[m,m_k]} = 0. \]
Since $\text{Ran} q^\prime_{i+} = \text{Ran} q_{i+}$, we have $\mathcal{D}^\prime_{i+} = \mathcal{D}_{i+}$. Evidently, $\mathcal{H}^\prime_{ij} = \mathcal{H}_{ij}$. Further, let $f \in \mathcal{H}_{ij+} = \mathcal{H}_{ij} \cap \text{Ker} q_{i-}$. Then $q_{i-} f = 0$, that is $\pi^i_1 \pi^i_2 f = q_{i+} f$, and
\[ q^\prime_{i-} f = \pi^i_1 \pi^i_2 f - q_{i+} f = \pi^i_1 \pi^i_2 f - q_{i+} f = q_{i+} f - q_{i+} f \in \mathcal{D}^\prime_{i+} = \mathcal{D}^\prime_{i+}. \]
Therefore, $q^\prime_{i-} f = q_{i+}^2 f = q_{i+}^2 q_{i+} g = 0$ and $\mathcal{H}_{ij+} \subset \mathcal{H}^\prime_{ij+}$. For the same reason, $\mathcal{H}^\prime_{ij+} \subset \mathcal{H}_{ij+}$. Thus, we have
\[ \mathcal{D}^\prime_{i+} = \mathcal{D}_{i+}, \quad \mathcal{H}^\prime_{ij} = \mathcal{H}_{ij}, \quad \mathcal{H}^\prime_{ij+} = \mathcal{H}_{ij+} \]
and therefore
\[ \mathcal{K}^\prime_{[m,m_j]} \subset \mathcal{H}_{m,m_j} + \mathcal{D}_{m_j} \subset \text{Ker} P_{[m,m_j]} . \]
The following Proposition affirms a more delicate property of projections $P_{[m,m_j]}$. \

Proposition 1.5. One has $\mathcal{H}_{m,m_j^+} \cap \text{Ker } P_{[m,m_j]} = \mathcal{D}_{m_j^+}$, $i \geq j$.

But beforehand we need to prove the following elementary lemmas, which are of interest in their own right.

**Lemma (i).** Suppose $M, N_+, N_-$ are subspaces of a Hilbert space and $N_+ \perp N_-$. Then $(N \lor M) \oplus N_- = ((N \lor M) \oplus N) \oplus N_+$, where $N = N_+ \oplus N_-$.

**Proof.** Let $f \in (N \lor M) \oplus N_-$. Then $f \in (N \lor M), f \perp N_-$. We have $f = f_N + f_{N_-}$, where $f_N \in N$, $f_{N_-} \in N_-$. Then $f_N = f - f_{N} \perp N_-$ and $f_N \in N_+$. Hence, $f = f_N + f_N \in ((N \lor M) \oplus N) \oplus N_+$.

Conversely, let $f \in N_+$. Then $f \in N$, $f \perp N_-$ and therefore $f \in (N \lor M) \oplus N_-$. Hence, $((N \lor M) \oplus N) \oplus N_+ \subset (N \lor M) \oplus N_-$. □

**Lemma (ii).** Suppose $P \in \mathcal{L}(\mathcal{H})$ is an orthogonal projection; $\mathcal{D}_+, \mathcal{H}_+$ are subspaces of $\mathcal{H}$ such that $\mathcal{D}_+ \subset \mathcal{H}_+$, $\mathcal{K} = \text{Ran } P \subset \mathcal{H}_+$ and $\mathcal{D}_+ \subset \text{Ker } P$. Then the following conditions are equivalent:

1) $\mathcal{H}_+ \cap \text{Ker } P = \mathcal{D}_+$; 2) $\mathcal{H}_+ = \mathcal{K} + \mathcal{D}_+$; 3) Ker$(P|\mathcal{H}_+) = \mathcal{D}_+$.

**Proof.** 1) $\Rightarrow$ 2) Let $f \in \mathcal{H}_+$. Then $f = f_1 + f_2$, where $f_1 = Pf \in \mathcal{K}$ and $f_2 = (I - P)f \in \text{Ker } P$. Since $\mathcal{K} \subset \mathcal{H}_+$, we get $f_2 = f - Pf \in \mathcal{H}_+$ and therefore $f_2 \in \mathcal{H}_+ \cap \text{Ker } P = \mathcal{D}_+$.

2) $\Rightarrow$ 3) It is clear that $\mathcal{D}_+ \subset \text{Ker}(P|\mathcal{H}_+)$. Let $f \in \text{Ker}(P|\mathcal{H}_+) \subset \mathcal{H}_+$. Then $0 = Pf = f_1 + f_2$, where $f_1 \in \mathcal{K}$ and $f_2 \in \mathcal{D}_+$. Then $0 = Pf = (f_1 + f_2) = f_2$ and therefore $f = f_2 \in \mathcal{D}_+$.

3) $\Rightarrow$ 1) It is clear that $\mathcal{D}_+ \subset \mathcal{H}_+ \cap \text{Ker } P$. Let $f \in \mathcal{H}_+ \cap \text{Ker } P$. Then $f \in \text{Ker} (P|\mathcal{H}_+) = \mathcal{D}_+$. □

**Lemma (iii).** Suppose $P_1$ and $P_2$ are projections such that Ker $P_1 = \text{Ker } P_2$. Then $P_1 P_2 = P_1$ and $P_2 P_1 = P_2$.

**Proof.** Since $\text{Ran } (I - P_2) = \text{Ker } P_2$, we get $P_1 (I - P_2) = 0$. Hence, $P_1 P_2 = P_1$. □

**Corollary.** Suppose $P_1, P_2 \in \mathcal{L}(\mathcal{H})$ are projections; $\mathcal{D}_+, \mathcal{H}_+$ are subspaces of $\mathcal{H}$ such that $\mathcal{D}_+ \subset \mathcal{H}_+$, $\text{Ran } P_1 \subset \mathcal{H}_+$, $\text{Ran } P_2 \subset \mathcal{H}_+$, $\mathcal{D}_+ \subset \text{Ker } P_1$, $\mathcal{D}_+ \subset \text{Ker } P_2$ and Ker$(P_1|\mathcal{H}_+) = \text{Ker} (P_2|\mathcal{H}_+) = \mathcal{D}_+$. Then $P_1 P_2 P_1 = P_1$ and $P_2 P_1 P_2 = P_2$.

**Proof.** It is clear that $P_1|\mathcal{H}_+, P_2|\mathcal{H}_+$ are projections. By Lemma (iii), we have $\text{(P_1|\mathcal{H}_+)} (P_2|\mathcal{H}_+) = P_1|\mathcal{H}_+$. Then $P_1 P_2 P_1 f = P_1 P_2 (P_1 f) = P_1 (P_1 f) = P_1 f$. □

**Proof (of Proposition 1.5).** First, we prove our assertion in the orthogonal context. Consider orthogonal projections

$$P'_{(ij)} = P_{\pi_1 \lor \ldots \lor \pi_j} (I - q_{j_+}') (I - q_{i_-}') ,$$

Since operators $q_{j_+}', q_{i_-}'$ are selfadjoint, we have $(q_{j_+}' q_{i_-}')_* = q_{i_-}' q_{j_+}' = q_{i_-} q_{j_+} = 0$

and hence

$$P'_{(ij)} = P_{\pi_1 \lor \ldots \lor \pi_j} (I - q_{j_+}' - q_{i_-}') = P_{\pi_1 \lor \ldots \lor \pi_j} - q_{j_+}' - q_{i_-} ,$$
Define subspaces \( \mathcal{N}_k \pm := \text{Ran} q_{k \pm} \), \( N_k := N_{k+} \oplus N_{k-} = \text{Ran} \pi'_k \), \( k = \overline{1,n} \). Then we have \( P_{\pi'_k,\ldots,\pi'_j} = q_{k'-} + P'_{(ij)} + q_{j'}^{j'} + \) and
\[
\mathcal{H}_{ij} = N_i - \mathcal{K}'_{(ij)} \oplus N_{j+}, \quad \mathcal{H}_{ij+} = \mathcal{K}'_{(ij)} \oplus N_{j+}, \quad \mathcal{D}_{j+} = N_{j+}.
\]
In particular, we get \( N_k \cap N_{k+1} = N_{k+1-} \oplus \mathcal{K}'_{(k+1,k)} \oplus N_{k+} \) and therefore \( N_{k+} \oplus \mathcal{K}'_{(k+1,k)} = (N_k \cap N_{k+1}) \oplus N_{k+1-} \). Applying the former identity and Lemma \((vi)\)
\[i - j\text{ times, we have}
\]
\[
N_{j+} \oplus \mathcal{K}'_{(j+1,j)} \oplus \mathcal{K}'_{(j+2,j+1)} \oplus \cdots \oplus \mathcal{K}'_{(i,i-1)} \oplus N_i
\]
\[= [(N_j \cap N_{j+1}) \oplus N_{j+1-}] \oplus \mathcal{K}'_{(j+2,j+1)} \oplus \cdots \oplus \mathcal{K}'_{(i,i-1)} \oplus N_i
\]
\[= [(N_j \cap N_{j+1}) \oplus N_{j+1-}] \oplus \mathcal{K}'_{(j+2,j+1)} \oplus \cdots \oplus \mathcal{K}'_{(i,i-1)} \oplus N_i
\]
\[= \cdots = [(N_j \cap N_{j+1}) \oplus N_{j+1-}] \oplus \mathcal{K}'_{(j+2,j+1)} \oplus \cdots \oplus \mathcal{K}'_{(i,i-1)} \oplus N_i
\]
\[= [N_{j+1} \cap N_{j+2}] \oplus \cdots \oplus [N_{i-1} \cap N_i] \oplus N_i.
\]
On the other hand, we have already shown \( \mathcal{H}_{ij} = N_{i-} \oplus \mathcal{K}'_{(ij)} \oplus N_{j+} \). Therefore,
\[
\mathcal{K}'_{(ij)} = \mathcal{K}'_{(ii-1)} \oplus \cdots \oplus \mathcal{K}'_{(j+1,j)}.
\]
Then we have \( \mathcal{K}'_{[m,m_j]} = \mathcal{K}'_{[m,m_j]} \) and \( \mathcal{K}'_{[m,m_k]} = \mathcal{K}'_{[m,m_j]} \oplus \mathcal{K}'_{[m,m_k]} \). It is easy to check that \( \mathcal{H}_{m,m_j} \cap \text{Ker} P_{[m,m_j]} = D_{m_j+} \).

For the nonorthogonal case, we shall use induction. In fact, we have already shown that
\[
\mathcal{H}_{m_{j+1},m_j} \cap \text{Ker} P_{[m_{j+1},m_j]} = D_{m_j+} \cap \text{Ker} P_{[m_{j+1},m_j]} = D_{m_j+}, \quad j = \overline{1,n}.
\]
Let \( i \geq j \geq k \). Assume that \( \mathcal{H}_{m,m_j} \cap \text{Ker} P_{[m,m_j]} = D_{m_j+} \) and \( \mathcal{H}_{m,m_k} \cap \text{Ker} P_{[m,m_k]} = D_{m_k+} \). Let \( f \in \mathcal{H}_{m,m_j} \cap \text{Ker} P_{[m,m_k]} \). Using the recursion relation
\[
P_{[m,m_k]} = P_{[m,m_k]}[I - P_{[m,m_j]}] + P_{[m,m_j]}
\]
and properties of projection \( P_{[m,j]} \), we have
\[
P_{[m,m_k]}f = P_{[m,m_k]}(I - P_{[m,m_j]}) + P_{[m,m_j]}f = P_{[m,m_j]}f = 0.
\]
Then, since \( P_{[m,m_j]}f = 0 \), we also have \( P_{[m,m_k]}f = 0 \). On the other hand, the vector \( f \) can be decomposed \( f = f'_{ij} + f'_{jk} + g \), where \( f'_{ij} \in \mathcal{K}'_{[m,m_j]} \), \( f'_{ij} \in \mathcal{K}'_{[m,m_j]} \) and \( g \in D_{m_j+} \). Since \( P_{[m,m_j]}f_{[m,m_k]} = 0 \), we have
\[
0 = P_{[m,m_j]}f = P_{[m,m_j]}(f'_{ij} + f'_{jk} + g) = P_{[m,m_j]}f'_{ij}.
\]
By Lemma \((iii)\), \( \text{Ker}(P_{[m,m_j]} \mathcal{H}_{m,m_j}) = D_{m_j+} \). Then, by Corollary of Lemma \((iii)\), we obtain \( 0 = P_{[m,m_j]}f_{ij} = P_{[m,m_j]}f'_{ij} = f'_{ij} \). Further, \( 0 = P_{[m,m_k]}f = P_{[m,m_k]}(f'_{jk} + g) = P_{[m,m_k]}f'_{jk} \). As above, we get \( 0 = f'_{jk} \). Thus, we have \( f = g \in D_{m_j+} \) and therefore \( \mathcal{H}_{m,m_j} \cap \text{Ker} P_{[m,m_k]} \subset D_{m_j+} \). The inverse inclusion is obvious. \( \square \)
Remark. Since \( P_{[31]} = P_{[32]} + P_{[21]}(I - P_{[32]}) = P_{[32]} + P_{[21]} \), by Prop. 15 and Corollary of Lemma \([22]\) we obtain the following identities

\[
(P_{[32]} + P_{[21]})(P_{[31]}(P_{[32]} + P_{[21]})) = P_{[32]} + P_{[21]}
\]

and

\[
P_{[31]}(P_{[32]} + P_{[21]})(P_{[31]} = P_{[31]}.\]

This means that

\[
(P_{[31]}|K_{[31]})^{-1} = (P_{[21]} + P_{[32]})|K_{[31]}, \quad ((P_{[21]} + P_{[32]})|K_{[31]})^{-1} = P_{[31]}|K_{[31]}.
\]

Example. Let \( w = \varphi(z) = z + \varepsilon z^2, \ |\varepsilon| < 1/2, \ G_+ = \varphi(\mathbb{D}), \ C = \varphi(\mathbb{T}). \) We put

\[
\theta(w) = \frac{2w}{1 + \sqrt{1 + 4\varepsilon w}}, \quad w \in G_+, \quad \Theta_{ij}(w) = \theta(w)^{i-j}, \quad 1 \leq j \leq i \leq n,
\]

and \( \Xi_{ij}(w) = 1, w \in C. \) It can easily be checked that \( |\theta(w)| = 1, \ w \in C. \) Then \( P_{ij} = P_{ij}^{(n)} = \theta^{n-j}P_{\theta^{i-j-1}}\theta^{i-n}. \) For the functions

\[
f_{ij}^{(n)}(w) = \theta(w)^{n-j}w^{-k}, \quad k = 1, i-j,
\]

we have \( f_{ij}^{(n)} \in K_{ij}^{(n)} = \text{Ran} P_{ij}^{(n)}. \) By \([24]\), \( K_{ij}^{(n)}(\varepsilon) = P_{ij}^{(n)}(\varepsilon) K_{ij}^{(n)}(0) \) and \( K_{ij}^{(n)}(0) = P_{ij}^{(n)}(0) K_{ij}^{(n)}(\varepsilon). \) Since \( \dim K_{ij}^{(n)}(0) = i-j, \) the functions \( f_{ij}^{(n)}(w) \) form a basis of the subspace \( K_{ij}^{(n)} \). Note also that \( P_{ij}^{(n)} = \theta_{i-j+1}^{n-i}P_{i-j+1}^{(i-j+1)}\theta^{i-n} \) and therefore \( K_{ij}^{(n)} = \theta_{i-j+1}^{n-i}K_{i-j+1}^{(i-j+1)}. \)

Consider particular cases. In the case of \( n = 3 \) we have

\[
f_{1}^{31} = \theta^2/w, \quad f_{2}^{31} = \theta^2/w^2 \quad \text{and} \quad f_{1}^{21} = \theta^2/w, \quad f_{1}^{32} = \theta/w.
\]

Hence, \( K_{[21]} \subset K_{[31]}, \ K_{[32]} \not\subset K_{[31]} \) and \( K_{[32]} + K_{[21]} \neq K_{[31]} \).

In the case of \( n = 5 \) it can be calculated that

\[
P_{[21]}f_{1}^{53} = -\varepsilon^2 f_{1}^{21} \quad \text{and} \quad P_{[21]}f_{2}^{53} = 2\varepsilon^3 f_{1}^{21}.
\]

Therefore, \( P_{[21]}(P_{[21]}(P_{[32]} \neq 0. \) Our calculations are based on the formula

\[
P_{(k)l}f_{ij}^{p} = \theta_{i-j}^{n-l}w^{-j}P_{u_{i-j-1}(w)} - 2^{l-k}\theta^{n-l}P_{w^{-j-k}u_{k-\ell-1}(w)} - \frac{w^{-j-k}u_{k-\ell-1}(w)}{1 + \sqrt{1 + 4\varepsilon w})^{l-k},
\]

where \( 1 \leq i, j, k, l \leq n, \ p = 1, i-j, \ i \geq j, \ k \geq l, \) and

\[
u_{q,r}(w) := 2^{-r}w^{-q}P_{w^{q}}(1 + \sqrt{1 + 4\varepsilon w})^{r}.
\]

It can easily be checked that \( u_{q,r}(w) \equiv 0, \ q \geq 0. \) For \( q < 0, \) we use the Residue Theorem calculating \( P_{+}w^{q}(1 + \sqrt{1 + 4\varepsilon w})^{r} \) and interpreting the projection \( P_{+} \) as the boundary values of the Cauchy integral operator. In particular, we get

\[
u_{-1,r}(w) = 1; \quad \nu_{-2,r}(w) = 1 + r\varepsilon w; \quad \nu_{-3,r}(w) = \frac{1}{2}(2 + 2r\varepsilon w + (r - 3)\varepsilon^2 w^2);
\]

\[
u_{-4,r}(w) = \frac{1}{6}(2 + 6r\varepsilon w + 3r(r - 3)\varepsilon^2 w^2 + 3r(r - 4)(r - 5)\varepsilon^4 w^3).
\]
2. Product of Conservative Curved Systems

**Definition.** Let \( \Sigma_k = (T_k, M_k, N_k, \Theta_k, \Xi_k; H_k, \mathfrak{N}_{k+}, \mathfrak{N}_{k-}) \), \( k = 1, 2 \) be conservative curved systems, \( G_{1+} = G_{2+} \), \( \mathfrak{N}_{1+} = \mathfrak{N}_{2+} \) and \( \Xi_{1-} = \Xi_{2+} \). We define the product of them as

\[
\Sigma_{21} = \Sigma_2 \cdot \Sigma_1 := (T_{21}, M_{21}, N_{21}, \Theta_{21}; H_{21}, \mathfrak{N}_{1+}, \mathfrak{N}_{2-})
\]

with \( \Theta_{21} = \Theta_{2u} \Theta_{1u} \), \( \Xi_{21} = (\Xi_{1+}, \Xi_{2-}) \), \( H_{21} = H_1 \oplus H_2 \),

\[
T_{21} = \begin{pmatrix} T_1 & N_1 M_2 \\ 0 & T_2 \end{pmatrix}, \quad M_{21} = (M_1, M_{21}^*) , \quad N_{21} = \begin{pmatrix} M_{21}^* \\ N_2 \end{pmatrix} , \tag{Prod}
\]

\[
M_{21}^* f_2 = -\frac{1}{2\pi i} \int_{C} \Theta_1^{-} (\zeta) [M_2 (T_2 - \cdot)^{-1} f_2]^{-} (\zeta) d\zeta , \quad f_2 \in H_2 ,
\]

\[
M_{11}^* f_1 = -\frac{1}{2\pi i} \int_{\Xi_2} \Theta_1^{-} (\zeta) [N_1 (T_1^* - \cdot)^{-1} f_1]^{-} (\zeta) d\zeta , \quad f_1 \in H_1 ,
\]

where \([M_2 (T_2 - \cdot)^{-1} f_2]^{-} \) and \([N_1 (T_1^* - \cdot)^{-1} f_1]^{-} \) are the boundary limits of \( M_2 (T_2 - z)^{-1} f_2 \) and \( N_1 (T_1^* - z)^{-1} f_1 \) from the domains \( G_+ \) and \( G_- := \{ \tilde{z} : z \in G_- \} \), respectively; \( \Theta_2^* = \Theta_2^\sim \) (see (CtoT) for the definition of \( \Theta^- \)).

Note that we can consider the product \( \Sigma_2 \cdot \Sigma_1 \) without the assumption that \( \Sigma_1, \Sigma_2 \) are conservative curved systems. We need only to assume additionally that \( \forall f_2 \in H_2 : M_2 (T_2 - z)^{-1} f_2 \in E_2 (G_-) \) and \( \forall f_1 \in H_1 : N_1 (T_1^* - z)^{-1} f_1 \in E_2 (G_-) \). For conservative curved systems these assumptions are always satisfied (it follows from the definition of conservative curved system).

We start to justify the definition with the observation that in case of unitary colligations we get the standard algebraic definition \[2\]: \( M_{21}^* = \Theta_1^+ (0)^* M_2 \) and \( N_{21}^* := M_{21}^* = N_1 \Theta_2^+ (0)^* \) (see the Introduction). Indeed, since in this case \( \Theta_1^+ \equiv \Theta_2^+ (0)^* = L \) and \( M_2 (T_2 - z)^{-1} f_2 \in E_2 (G_-) \), we obtain

\[
M_{21}^* f_2 = -\frac{1}{2\pi i} \int_{C} \Theta_1^{-} (\zeta) [M_2 (T_2 - \cdot)^{-1} f_2]^{-} (\zeta) d\zeta = -\frac{1}{2\pi i} \int_{C} \Theta_1^{-} (\zeta) [M_2 (T_2 - \cdot)^{-1} f_2]^{-} (\zeta) d\zeta = L_1 M_2 .
\]

By a similar computation, we get \( M_{21}^* = N_1 L_2 \). Besides, we have

**Proposition 2.1.** 1) \( \Sigma_1 \sim \Sigma_1^*, \Sigma_2 \sim \Sigma_2 ^* \Rightarrow \Sigma_2 \cdot \Sigma_1 \sim \Sigma_2^* \cdot \Sigma_1^* \); 2) \( (\Sigma_2 \cdot \Sigma_1)^* = \Sigma_1^* \cdot \Sigma_2^* \).

Here \( \Sigma^* := (T^*, N^*, \Theta^*, \Xi^*) \), \( \Xi_{\pm} = \Xi_\pm^{-1} \).

**Proof.** 1) Let \( \Sigma_1 \cong \Sigma_1^*, \Sigma_2 \cong \Sigma_2^* \). Then

\[
M_{21}^* f_2 = -\frac{1}{2\pi i} \int_{C} \Theta_1^{-} (\zeta) [M_2 (T_2 - \cdot)^{-1} f_2]^{-} (\zeta) d\zeta
\]

\[
= -\frac{1}{2\pi i} \int_{C} \Theta_1^{-} (\zeta) [M_2^* (T_2^* - \cdot)^{-1} X_2 f_2]^{-} (\zeta) d\zeta = M_{21}^* X_2 f_2 .
\]
Hence, $M_{21}^2 = M_{21}^{21'} X_2$. Similarly, $X_1 N_{11}^{21} = N_{11}^{21'}$. Define $X_{21} := \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix}$.

Then, we get

$$T_{21} X_{21} = \begin{pmatrix} T_1' & N_1'X_2 \\ 0 & T_2' \end{pmatrix} \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} = \begin{pmatrix} T_1'X_1 & N_1'X_2 \\ 0 & T_2'X_2 \end{pmatrix} = X_{21} T_{21};$$

$$M_{21} X_{21} = \begin{pmatrix} M_1' & M_2^{21'} \\ 0 & X_2 \end{pmatrix} = \begin{pmatrix} M_1' & M_2^{21'} \end{pmatrix} = M_{21};$$

$$N_{21} = \begin{pmatrix} N_{11}^{21'} \\ N_{21} \end{pmatrix} = \begin{pmatrix} X_1 N_{11}^{21'} \\ X_2 N_{21} \end{pmatrix} = \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} \begin{pmatrix} N_{11}^{21} \\ N_{21} \end{pmatrix} = X_{21} N_{21};$$

and thus $\Sigma_2 \cdot \Sigma_1 = \Sigma_{21}^2 \cdot \Sigma_2 \cdot \Sigma_1^2$.

2) Let $\Sigma_{21*} = (\Sigma_2 \cdot \Sigma_1)^*$. By straightforward calculation, we get

$$T_{21} = \begin{pmatrix} T_1^* & 0 \\ M_2 N_1^* & T_2^* \end{pmatrix}; \quad M_{21} = (N_{11}^{21*}, N_{21}^*); \quad N_{21} = \begin{pmatrix} N_{11}^{21*} \\ M_{21} \end{pmatrix}.$$

On the other hand, let $\Sigma_{21}^* = \Sigma_1^* \cdot \Sigma_2^*$. Then we get

$$T_{21} = \begin{pmatrix} T_1 & 0 \\ N_2 M_1 & T_2 \end{pmatrix} = \begin{pmatrix} T_1 & 0 \\ M_2 N_1 & T_2 \end{pmatrix} = T_{21}.$$

Since

$$M_{21}^{21'} f_1 = -\frac{1}{2\pi i} \int_C \Theta_{21}(\zeta) [M_{21}(T_{21} - \cdot)^{-1} f_1] \cdot (\zeta d\zeta$$

$$= -\frac{1}{2\pi i} \int_C \Theta_{21}(\zeta) [N_{21}^* (T_{21} - \cdot)^{-1} f_1] \cdot (\zeta d\zeta = N_{21}^{21*} f_1, \quad f_1 \in H_1,$$

we have

$$M_{21} = (M_{21}^{21'}, M_{21}^*) = (N_{21}^{21*}, N_{21}^*) = M_{21}.$$

Since

$$N_{21}^{21} f_2 = -\frac{1}{2\pi i} \int_C \Theta_{21}(\zeta) [N_{21} (T_{21} - \cdot)^{-1} f_2] \cdot (\zeta d\zeta$$

$$= -\frac{1}{2\pi i} \int_C \Theta_{21}(\zeta) [M_{21} (T_{21} - \cdot)^{-1} f_2] \cdot (\zeta d\zeta = M_{21}^{21} f_2, \quad f_2 \in H_2,$$

we have

$$N_{21} = \begin{pmatrix} N_{21}^{21} \\ N_{21}^* \end{pmatrix} = \begin{pmatrix} M_{21}^{21*} \\ M_{21}^* \end{pmatrix} = N_{21}.$$

Thus we get $\Sigma_1 \cdot \Sigma_2 = \Sigma_{21}^* = \Sigma_{21*} = (\Sigma_2 \cdot \Sigma_1)^*$. $\square$
Further, we shall say that a triple of operators \((T, M, N)\) is a realization of a
transfer function \(\Upsilon = F_{\text{tc}}(\Theta)\) if \(\Upsilon(z) = M(T - z)^{-1}N\).

**Proposition 2.2.** Suppose that triples \((T_1, M_1, N_1)\) and \((T_2, M_2, N_2)\) are realizations of transfer functions \(\Upsilon_1 = F_{\text{tc}}(\Theta_1)\) and \(\Upsilon_2 = F_{\text{tc}}(\Theta_2)\), respectively. Suppose also that \(\forall f_1 \in H_1 : N_1(T_1^*-z)^{-1}f_1 \in E^2(G_-)\) and \(\forall f_2 \in H_2 : M_2(T_2-z)^{-1}f_2 \in E^2(G_-)\). Then the triple \((T_21, M_21, N_21)\) defined by (Prod) is a realization of the
transfer function \(\Upsilon_21 = F_{\text{tc}}(\Theta_2\Theta_1)\).

**Proof.** For the sake of simplicity, consider the case \(\Theta_{k-} \in H^\infty(G_-, L(\mathfrak{h}_k, \mathfrak{h}_{k+}))\), \(k = 1, 2\) (in the general case we need to use expressions like \((M_{21}(T_21-\lambda)^{-1}N_{21}n, m)\)). It can easily be shown that

\[
(T_{21} - \lambda)^{-1} = \begin{pmatrix}
(T_1 - \lambda)^{-1} & -(T_1 - \lambda)^{-1}N_1M_2(T_2 - \lambda)^{-1} \\
0 & (T_2 - \lambda)^{-1}
\end{pmatrix}.
\]

Then, by straightforward computation, we obtain

\[
M_{21}(T_2 - \lambda)^{-1}N_{21} = M_1(T_1 - \lambda)^{-1}N_1^21 - \Upsilon_1(\lambda)\Upsilon_2(\lambda) + M_2^21(T_2 - \lambda)^{-1}N_2.
\]

Here, we have

\[
M_2^21(T_2 - \lambda)^{-1}N_2 =
\]

\[
= -\frac{1}{2\pi i} \int_{C} \Theta_1(\zeta) \left[ M_2(T_2 - \cdot)^{-1}(T_2 - \lambda)^{-1}N_2 \right](\zeta) d\zeta
\]

\[
= -\frac{1}{2\pi i} \int_{C} \frac{\Theta_1(\zeta)}{\zeta - \lambda} \left[ M_2(T_2 - \cdot)^{-1}N_2 - M_2(T_2 - \lambda)^{-1}N_2 \right](\zeta) d\zeta
\]

\[
= -\frac{1}{2\pi i} \int_{C} \frac{\Theta_1(\zeta)(\Upsilon_2(\zeta) - \Upsilon_2(\lambda))}{\zeta - \lambda} d\zeta
\]

\[
= -\frac{1}{2\pi i} \int_{C} \frac{\Theta_1(\zeta)(\Upsilon_2(\zeta) - \Upsilon_2(\lambda))}{\zeta - \lambda} d\zeta + \frac{1}{2\pi i} \int_{C} \frac{\Theta_1(\zeta)\Upsilon_2(\lambda)}{\zeta - \lambda} d\zeta
\]

\[
= \frac{1}{2\pi i} \int_{C} \frac{\Theta_1(\zeta)\Theta_2(\zeta)}{\zeta - \lambda} d\zeta + \left\{ \Theta_1^+(\lambda)(\Theta_2^+(\lambda) - \Theta_2^-(\lambda)^{-1}), \lambda \in G_+ \right\}.
\]
Similarly, we have

\[ N_{21}^* (T_1^* - \lambda)^{-1} M_1^* = \]

\[ = \frac{1}{2\pi i} \int_C \frac{\Theta_{2+}(\zeta)\Theta_{1-}(\zeta)}{\zeta - \lambda} d\zeta + \left\{ \begin{array}{ll}
\Theta_{2+}(\lambda)(\Theta_{1+}(\lambda) - \Theta_{1+}^*(\lambda)^{-1}), & \lambda \in G_+ \\
0, & \lambda \in G_-
\end{array} \right. \]

\[ = \frac{1}{2\pi i} \int_C \frac{\Theta_{2+}(\zeta)\Theta_{1-}(\zeta)}{\zeta - \lambda} d\zeta + \left\{ \begin{array}{ll}
\Theta_{2+}(\lambda)(\Theta_{1+}(\lambda) - \Theta_{1+}^*(\lambda)^{-1}), & \lambda \in G_+ \\
0, & \lambda \in G_-
\end{array} \right. \]

Hence,

\[ M_1 (T_1 - \lambda)^{-1} N_{21}^* = \]

\[ = \frac{1}{2\pi i} \int_C \frac{\Theta_{1-}^*(\zeta)\Theta_{2+}^*(\zeta)}{\zeta - \lambda} d\zeta + \left\{ \begin{array}{ll}
(\Theta_{1+}\lambda)^* - (\Theta_{1+}(\lambda)^*)^{-1}) \Theta_{2+}^*(\lambda)^*, & \lambda \in G_+ \\
0, & \lambda \in G_-
\end{array} \right. \]

\[ = \frac{1}{2\pi i} \int_C \frac{\Theta_{1-}^*(\zeta)\Theta_{2+}^*(\zeta)}{\zeta - \lambda} d\zeta + \left\{ \begin{array}{ll}
(\Theta_{1+}(\lambda) - \Theta_{1+}(\lambda)^{-1}) \Theta_{2+}^*(\lambda), & \lambda \in G_+ \\
0, & \lambda \in G_-
\end{array} \right. \]

Consider the case when \( \lambda \in G_- \). Then

\[ M_{21} (T_{21} - \lambda)^{-1} N_{21} = \]

\[ = \frac{1}{2\pi i} \int_C \frac{\Theta_{1-}^*(\zeta)\Theta_{2+}^*(\zeta)}{\zeta - \lambda} d\zeta - \Theta_{1-}(\lambda)\Theta_{2-}^*(\lambda) + \frac{1}{2\pi i} \int_C \frac{\Theta_{1+}^*(\zeta)\Theta_{2-}^*(\zeta)}{\zeta - \lambda} d\zeta + 0 \]

\[ = \frac{1}{2\pi i} \int_C \frac{\Theta_{1-}(\zeta)\Theta_{2+}(\zeta)}{\zeta - \lambda} d\zeta + \frac{1}{2\pi i} \int_C \frac{\Theta_{1+}(\zeta)\Theta_{2-}(\zeta)}{\zeta - \lambda} d\zeta \]

\[ + \frac{1}{2\pi i} \int_C \frac{\Theta_{1+}(\zeta)\Theta_{2-}(\zeta)}{\zeta - \lambda} d\zeta + \frac{1}{2\pi i} \int_C \frac{\Theta_{1+}(\zeta)\Theta_{2+}(\zeta)}{\zeta - \lambda} d\zeta \]

\[ = \frac{1}{2\pi i} \int_C \frac{(\Theta_{1+}(\zeta) + \Theta_{1-}(\zeta))(\Theta_{2+}(\zeta) + \Theta_{2-}(\zeta))}{\zeta - \lambda} d\zeta \]

\[ = \frac{1}{2\pi i} \int_C \frac{\Theta_{1+}(\zeta)\Theta_{2+}(\zeta)}{\zeta - \lambda} d\zeta = \frac{1}{2\pi i} \int_C \frac{\Theta_{21}(\zeta)}{\zeta - \lambda} d\zeta = -\Theta_{21}(\lambda). \]
Consider the case when \( \lambda \in G_+ \). Then

\[
M_{21}(T_{21} - \lambda)^{-1}N_{21} =
\]

\[
= \frac{1}{2\pi i} \int_{C} \frac{\Theta^{-1}_{1-}(\zeta)\Theta^{-1}_{2+}(\zeta)}{\zeta - \lambda} \, d\zeta + (\Theta_{1+}^{-1}(\lambda) - \Theta_{1}^{+}(\lambda)^{-1})\Theta_{2+}^{-1}(\lambda)
\]

\[
- (\Theta_{1+}^{-1}(\lambda) - \Theta_{1}^{+}(\lambda)^{-1})(\Theta_{2+}^{-1}(\lambda) - \Theta_{2}^{+}(\lambda)^{-1})
\]

\[
+ \frac{1}{2\pi i} \int_{C} \frac{\Theta_{1-}^{-1}(\zeta)\Theta_{2+}^{-1}(\zeta)}{\zeta - \lambda} \, d\zeta + \Theta_{1+}(\lambda)(\Theta_{2+}^{-1}(\lambda) - \Theta_{2}^{+}(\lambda)^{-1})
\]

\[
= \frac{1}{2\pi i} \int_{C} \frac{\Theta_{1-}^{-1}(\zeta)\Theta_{2+}^{-1}(\zeta) + \Theta_{1+}^{-1}(\zeta)\Theta_{2+}^{-1}(\zeta)}{\zeta - \lambda} \, d\zeta + \Theta_{1+}(\lambda)(\Theta_{2+}^{-1}(\lambda) - \Theta_{2}^{+}(\lambda)^{-1})^{-1}
\]

\[
= \frac{1}{2\pi i} \int_{C} \frac{\Theta_{1-}^{-1}(\zeta) - \Theta_{1+}^{-1}(\zeta)\Theta_{2+}^{-1}(\zeta) + (\Theta_{1-}^{-1}(\zeta) - \Theta_{1+}^{-1}(\zeta))\Theta_{2+}^{-1}(\zeta)}{\zeta - \lambda} \, d\zeta
\]

\[
+ \Theta_{1+}(\lambda)\Theta_{2+}^{-1}(\lambda) - \Theta_{1}^{+}(\lambda)^{-1}\Theta_{2}^{+}(\lambda)^{-1}
\]

\[
= \frac{1}{2\pi i} \int_{C} \frac{\Theta_{1-}^{-1}(\zeta)\Theta_{2+}^{-1}(\zeta)}{\zeta - \lambda} \, d\zeta - \Theta_{1+}^{-1}(\lambda)\Theta_{2+}^{-1}(\lambda) + \Theta_{1+}^{-1}(\lambda)(\Theta_{2+}^{-1}(\lambda) - \Theta_{2}^{+}(\lambda)^{-1})^{-1}
\]

\[
= \frac{1}{2\pi i} \int_{C} \frac{\Theta_{2+}^{-1}(\zeta)}{\zeta - \lambda} \, d\zeta - \Theta_{2+}^{-1}(\lambda)^{-1} = \Theta_{21}(\lambda) - \Theta_{21}(\lambda)^{-1}.
\]

Thus, for \( \lambda \in (G_+ \cup G_-) \cap \rho(T_{21}) \), we obtain

\[
M_{21}(T_{21} - \lambda)^{-1}N_{21} = \Upsilon_{21}(\lambda) = \begin{cases} \Theta_{21+}^{-1}(\lambda) - \Theta_{21+}^{+}(\lambda)^{-1}, & \lambda \in G_+ \cap \rho(T_{21}) \\ -\Theta_{21-}^{-1}(\lambda), & \lambda \in G_- \end{cases}
\]

That is \( \Upsilon_{21} = F_{ic}(\Theta_2\Theta_1) \). \( \square \)

Thus we have obtained important properties of product of systems. But the main question whether the product \( \Sigma_2 \cdot \Sigma_1 \) of conservative curved systems \( \Sigma_1, \Sigma_2 \) is a conservative curved system too leaves unexplained. The following Proposition answers this question. It also answers a question about author's motivation of the definition (Prod): in fact, the connection between the product of systems and the product of models established in the Proposition sheds genuine light on our definition (Prod).

**Proposition 2.3.** Suppose \( \Pi_1, \Pi_2 \in \text{Mod} \), \( \Pi = \Pi_2 \cdot \Pi_1 \), \( \Sigma_1 = F_{sm}(\Pi_1) \), \( \Sigma_2 = F_{sm}(\Pi_2) \), \( \Sigma_{21} = \Sigma_2 \cdot \Sigma_1 \), and \( \tilde{\Sigma} = F_{sm}(\Pi) \). Then \( \Sigma_{21} \sim \tilde{\Sigma} \).

We hope that it will cause no confusion if we use the same symbol \( F_{ms} \) for the transformations \( F_{ms} : \text{Mod} \to \text{Sys} \) and \( F_{ms} : \text{Mod}_n \to \text{Sys} : \) the latter one is defined by (MtoS) as well (with \( \pi_+ = \pi_1 \) and \( \pi_- = \pi_n \)).

**Proof.** Let

\[
\Sigma_1 = (T_1, M_1, N_1), \quad \Sigma_2 = (T_2, M_2, N_2), \quad \Sigma_{21} = (T_{21}, M_{21}, N_{21})
\]
Let also $\Pi = (\pi_1, \pi_2, \pi_3), \bar{\Sigma} = (\bar{T}, \bar{M}, \bar{N}) = \mathcal{F}_{sm}(\Pi)$ and
\[ \hat{\Sigma}_1 = (\hat{T}_1, \hat{M}_1, \hat{N}_1) = \mathcal{F}_{sm}(\pi_1, \pi_2), \quad \hat{\Sigma}_2 = (\hat{T}_2, \hat{M}_2, \hat{N}_2) = \mathcal{F}_{sm}(\pi_2, \pi_3). \]

It is obvious that the systems $\hat{\Sigma}_k$ and $\Sigma_k, k = 1, 2$ are unitarily equivalent.

Since there are no simple and convenient expressions for operators $\hat{T}^*, \hat{M}^*, \hat{N}^*$ in terms of the model $\Pi$, we need to employ the dual model $\bar{\Sigma} = (\pi_{\Sigma}, \pi_{\bar{\Sigma}})$. In $[8, 9]$, the author noticed that it was convenient to use the pair of operators $W, W^* \in \mathcal{L}(C, \mathfrak{N})$ for a model and the dual one simultaneously. We extend this construction to 3-models. By $[8, 9]$, there exist operators $W_k, W_{ak} \in \mathcal{L}(H, \mathfrak{N}), k = 1, 2$ such that $W_{ak}W_k = I$, $W_kW_{ak} = P_k$, and
\[
\begin{align*}
\hat{T}_1W_k &= W_kT_k, \\
\hat{M}_1W_k &= M_k, \\
\hat{N}_1W_k &= N_k.
\end{align*}
\]

Define $W_{21} := (W_1, W_2)$ and $W_{a21} := (W_{a1}, W_{a2})$. By Lemma $[8,9]$, $P_{(21)}P_{(32)} = P_{(21)}P_{(32)} = 0$. This implies
\[
W_{a21}W_{21} = \text{diag}(I, I) \quad \text{and} \quad W_{21}W^*_{a21} = P_{(21)} + P_{(32)}.
\]

We put
\[
\begin{align*}
\hat{T}_{21} &= W_{21}T_{21}W^*_{a21}, \\
\hat{M}_{21} &= M_{21}W^*_{a21}, \\
\hat{N}_{21} &= N_{21}W^*_{a21},
\end{align*}
\]
and (see Remark after Prop. $[1,2]$)
\[
\begin{align*}
\hat{T}_{21} &= (P_{(21)} + P_{(32)}) \hat{T}_{P(31)}, \\
\hat{M}_{21} &= \hat{M}_{P(31)}, \quad \hat{N}_{21} = (P_{(21)} + P_{(32)}) \hat{N}, \\
\hat{T}_{a21} &= (P_{(21)} + P^*_{(32)}) \hat{T}_{aP(31)}, \\
\hat{M}_{a21} &= \hat{M}_{aP(31)}, \quad \hat{N}_{a21} = (P_{(21)} + P^*_{(32)}) \hat{N}.
\end{align*}
\]

Our aim is to show that $(\hat{T}_{21}, \hat{M}_{21}, \hat{N}_{21}) = (\hat{T}_{21}, \hat{M}_{21}, \hat{N}_{21})$. If this identity is hold, we get
\[
\hat{T}W = WT_{21}, \quad \hat{M}W = M_{21}, \quad \hat{N} = WN_{21},
\]
where $W = P_{(31)}W_{21}, W_* = P^*_{(31)}W_{21}$. Thus, $\Sigma_{21} \sim \hat{\Sigma}$ and the Proposition is proved. Note also that $W_*^*W = I$ and $WW_*^* = P_{(31)}$.

We check the desired identities by computations within the functional model. The identities
\[
\begin{align*}
\hat{T}_{21} &= \left( \begin{array}{cc} \hat{T}_1 & \hat{N}_1M_2 \\ 0 & \hat{T}_2 \end{array} \right), \\
\hat{M}_{21} &= \left( \begin{array}{c} \hat{M}_1 \\ \frac{-1}{2\pi i} \int_C \Theta_1(z)[\hat{M}_2(\hat{T}_2 - \zeta) - f_{\hat{\Sigma}}(z)dz] \end{array} \right)
\end{align*}
\]
can be obtained by a straightforward calculation. Indeed, we have

$$
\hat{T}_{21} = W_{21}T_{21}W_{*21}^* = (W_1, \ W_2) \begin{pmatrix} T_1 & N_1M_2 \\ 0 & T_2 \end{pmatrix} \begin{pmatrix} W_{*21}^* \\ W_{*22}^* \end{pmatrix}
$$

$$
= \begin{pmatrix} W_1T_1W_{*1}^* & W_1N_1M_2W_{*2}^* \\ 0 & W_2T_2W_{*2}^* \end{pmatrix} = \begin{pmatrix} \tilde{T}_1 & \tilde{N}_1\tilde{M}_2 \\ 0 & \tilde{T}_2 \end{pmatrix}
$$

and

$$
\tilde{M}_{21}f = M_{21}W_{*21}^*f = (M_1, \ M_{21}^*) \begin{pmatrix} W_{*1}^* \\ W_{*2}^* \end{pmatrix} f
$$

$$
= M_1W_{*1}^*f_{21} + M_{21}^*W_{*2}^*f_{32}
$$

$$
= \tilde{M}_1f_{21} - \frac{1}{2\pi i} \int_C \Theta_1^{-1}(z) [M_2(T_2 - \cdot)^{-1} W_{*2}^*f_{32}]^* \frac{d}{dz}
$$

$$
= \tilde{M}_1f_{21} - \frac{1}{2\pi i} \int_C \Theta_1^{-1}(z) [\tilde{M}_2(\tilde{T}_2 - \cdot)^{-1} f_{32}]^* \frac{d}{dz},
$$

where \( f = f_{21} + f_{32} \in \mathcal{K}_{(21)}\hat{+}\mathcal{K}_{(32)} \).

On the other hand, using Lemma 1.3, Prop. 1.4, and the inclusions \( \mathcal{UD}_{1+} \subset \mathcal{D}_{1+} \), \( \mathcal{UH}_{31+} \subset \mathcal{H}_{31+} \), we get

$$
\tilde{T}_{21}f = (P_{(21)} + P_{(32)})\tilde{TP}_{(31)}f = (P_{(21)} + P_{(32)})P_{(31)}\mathcal{U}P_{(31)}f =
$$

$$
= (P_{(21)} + P_{(32)})\mathcal{U}f = (P_{(21)} + P_{(32)})\mathcal{U}f =
$$

$$
P_{(21)}\mathcal{U}f_{21} + P_{(32)}\mathcal{U}f_{21} + P_{(21)}\mathcal{U}f_{32} + P_{(32)}\mathcal{U}f_{32} =
$$

$$
= \tilde{T}_1f_{21} + 0 + P_{(21)}\mathcal{U}f_{32} + \tilde{T}_2f_{32} =
$$

$$
= \tilde{T}_1f_{21} + P_{(21)}(I - P_{(32)})\mathcal{U}f_{32} + \tilde{T}_2f_{32} =
$$

$$
= \tilde{T}_1f_{21} + P_{(21)}(\mathcal{U}f_{32} - \mathcal{U}f_{32} + \pi_2M_2f_{32}) + \tilde{T}_2f_{32} =
$$

$$
= \tilde{T}_1f_{21} + \tilde{N}_1\tilde{M}_2f_{32} + \tilde{T}_2f_{32},
$$
where \( f = f_{21} + f_{32} \in \mathcal{K}_{(21)} + \mathcal{K}_{(32)} \). Thus we have \( \hat{T}_{21}^* = \hat{T}_{21} \). Further, if we recall Lemma 2.2, we obtain
\[
\begin{align*}
\hat{M}_{21} f &= \hat{M}(P_{31}) f = \hat{M}(I - \pi_1 P_+ \pi_1^1) f_{32} = \\
&= \frac{1}{2\pi i} \int (\pi_1^1 P_{21})(z) dz + \frac{1}{2\pi i} \int (\pi_1^1 (I - \pi_1 P_+ \pi_1^1) f_{32})(z) dz = \\
&= \hat{M}_1 f_{21} + \frac{1}{2\pi i} \int (\pi_{12}^1 f_{32})(z) dz = \hat{M}_1 f_{21} + \frac{1}{2\pi i} \int (\pi_1^1 f_{32})(z) dz = \\
&= \hat{M}_1 f_{21} + \frac{1}{2\pi i} \int \Theta_1(z)(\pi_2^1 f_{32})(z) dz + 0 = \\
&= \hat{M}_1 f_{21} - \frac{1}{2\pi i} \int \Theta_1(z) [\hat{M}_2 (\hat{T}_2 - \zeta)^{-1} f_{32} - f_{32}](z) dz.
\end{align*}
\]

Therefore, \( \hat{M}'_{21} = \hat{M}_{21} \). Similarly, \( \hat{M}'_{21} = \hat{M}_{21} \). We can obtain the residuvial identity \( \hat{N}'_{21} = \hat{N}_{21} \) if we make use of the duality relations
\[
\begin{align*}
(\hat{M}'_{21} f', n) &= (f', \hat{N}_{21} n), & (\hat{N}'_{21} m, g') &= (m, \hat{M}_{21} g')
\end{align*}
\]
and
\[
\begin{align*}
(\hat{M}_{21} f, n) &= (f, \hat{N}_{21} n), & (\hat{N}_{21} m, g) &= (m, \hat{M}_{21} g),
\end{align*}
\]
where \( f' \in \mathcal{K}_{[31]} = \mathcal{K}_{(32)} + \mathcal{K}_{(21)}, g' \in \mathcal{K}_{*[31]} = \mathcal{K}_{*[32]} + \mathcal{K}_{*[21]}, f \in \mathcal{K}_{(31)}, g \in \mathcal{K}_{*[31]}, n \in \mathcal{N}_{1}, \text{ and } m \in \mathcal{N}_{3}. \) Therefore, using relations of duality, we have
\[
(\hat{N}'_{21} m, g) = (m, \hat{M}_{21} g) = (\hat{N}_{21} m, g) \quad \square
\]

**Remark.** Note that we do not claim that \( \mathcal{F}_{sm}(\Pi_2) \mathcal{F}_{sm}(\Pi_1) = \mathcal{F}_{sm}(\Pi_2 \Pi_1) \). The statement and proof of Prop. 2.3 is a good illustration to our remark that the linear similarity (but not unitary equivalence) is the natural kind of equivalence for conservative curved systems.

The following theorem is a direct consequence of Prop. 2.3. We shall use the notation \( \mathcal{F}_{sc} := \mathcal{F}_{sm} \circ \mathcal{F}_{mc} \).

**Theorem A.** Let \( \hat{\Sigma}_1 = \mathcal{F}_{sc}(\Theta_1), \hat{\Sigma}_2 = \mathcal{F}_{sc}(\Theta_2) \) and \( \hat{\Sigma}_{21} = \mathcal{F}_{sc}(\Theta_{21}) \), where \( \Theta_1, \Theta_2; \Theta_{21} = \Theta_2 \Theta_1 \in \text{Cf}n \). Suppose that \( \Sigma_1 \sim (\hat{\Sigma}_1 + \Sigma_{1u}) \) and \( \Sigma_2 \sim (\hat{\Sigma}_2 + \Sigma_{2u}) \), where the systems \( \Sigma_{1u} \) and \( \Sigma_{2u} \) are “purely normal” systems. Then there exists a “purely normal” system \( \Sigma_u = (T_u, 0, 0, 0) \) such that \( \Sigma_2 \cdot \Sigma_1 \sim (\hat{\Sigma}_2 \Sigma_1 + \Sigma_u) \).

**Proof.** By Prop. 2.3 \( \hat{\Sigma}_2 \cdot \hat{\Sigma}_1 \sim \mathcal{F}_{sc}(\Theta_{21} \cdot \Theta_1) = \hat{\Sigma}_{21} + \hat{\Sigma}_u \). Then, using Prop. 2.1 we have
\[
\Sigma_2 \cdot \Sigma_1 \sim (\hat{\Sigma}_2 + \Sigma_{2u}) \cdot (\hat{\Sigma}_1 + \Sigma_{1u}) \sim (\hat{\Sigma}_2 \cdot \hat{\Sigma}_1) + \Sigma_{1u} + \Sigma_{2u}
\]
and therefore \( \Sigma_2 \cdot \Sigma_1 \sim (\hat{\Sigma}_{21} + \Sigma_u) \), where \( \Sigma_u = \hat{\Sigma}_u + \Sigma_{1u} + \Sigma_{2u} \). \( \square \)

Thus we see that the definition of product of conservative curved systems (Prod) is tightly linked to functional model though we do not refer to it explicitly. On
the other hand, its formal independence from functional model characterizes the comparative autonomy of conservative curved system well enough. Moreover, we have explicit formulas for $\Sigma_2 \cdot \Sigma_1$ of systems and the product depends only on the factors $\Sigma_2, \Sigma_1$ and their characteristic functions (theoretically, the dependence on characteristic functions is undesirable, but, in author’s opinion, we cannot count on having more than we have).

Now we turn to the associativity of multiplication of systems.

**Proposition 2.4.** One has \[
\Sigma_3 \cdot (\Sigma_2 \cdot \Sigma_1) \sim (\Sigma_3 \cdot \Sigma_2) \cdot \Sigma_1, \quad \text{where } \Sigma_k \in \text{Sys}, \ k = 1, 3.
\]

**Proof.** Let $\Sigma_k = \mathcal{F}_{sc}(\Theta_k), \ \Pi_k = \mathcal{F}_{ms}(\Theta_k), \ \Pi = \Pi_3 \cdot \Pi_2 \cdot \Pi_1$ and $\Sigma = \mathcal{F}_{sm}(\Pi)$. For the functional model $\Pi$, we consider the following subspaces $\mathcal{K}_1 = \mathcal{K}_{(21)}$, $\mathcal{K}_2 = \mathcal{K}_{(32)}$, $\mathcal{K}_3 = \mathcal{K}_{(43)}$, $\mathcal{K}_{21} = \mathcal{K}_{(31)}$, $\mathcal{K}_{32} = \mathcal{K}_{(42)}$, $\mathcal{K}_{321} = \mathcal{K}_{(41)}$, and dual to them. We will denote by $W_1: H_1 \to \mathcal{K}_1$, $W_2: H_2 \to \mathcal{K}_2$, $W_3: H_3 \to \mathcal{K}_3$ the operators that realize similarities of the systems $\Sigma_k, \ k = 1, 2, 3$ with the corresponding systems $\Sigma_k$ in the model $\Pi$ (see the proof of Prop. 2.3). Denote by $W_{k}\ , \ k = 1, 2, 3$ the dual operators. As in the proof of Prop. 2.3 we get that the operator $W_{21} = P_{(31)}(W_1, W_2)$ realizes similarity $\Sigma_2 \cdot \Sigma_1 \sim \mathcal{K}_{(21)}$. Similarly, the operator $W_{32} = P_{(42)}(W_2, W_3)$ realizes similarity $\Sigma_3 \cdot \Sigma_2 \sim \mathcal{K}_{(32)}$. By the same argument, we get that the operator $W_{3(21)} = P_{(41)}(W_1, W_3)$ realizes similarity $\Sigma_3 \cdot (\Sigma_2 \cdot \Sigma_1) \sim \mathcal{K}_{(321)}$. Thus, the operators

\[
W_{3(21)} = P_{(41)}(P_{(31)}(W_1, W_2), W_3), \quad W_{(32)1} = P_{(41)}(W_1, P_{(42)}(W_2, W_3))
\]

realize the similarities $\Sigma_3 \cdot (\Sigma_2 \cdot \Sigma_1) \sim \mathcal{K}_{(321)}$ and $\Sigma_3 \cdot \Sigma_2 \sim \mathcal{K}_{(321)}$, respectively. Therefore, $\Sigma_3 \cdot (\Sigma_2 \cdot \Sigma_1) \sim (\Sigma_3 \cdot \Sigma_2) \cdot \Sigma_1$. \hfill $\Box$

Recall that the operator $P_{(41)} = P_{(21)}(I - P_{(32)})(I - P_{(43)}) + P_{(23)}(I - P_{(32)}) + P_{(43)} = P_{(21)}(I - P_{(43)}) + P_{(32)} + P_{(43)}$ is a projection in $\mathcal{H}$ onto the subspace $\mathcal{K}_{(21)} + \mathcal{K}_{(32)} + \mathcal{K}_{(43)}$ and its components $P_{(21)}(I - P_{(43)}), P_{(32)}, P_{(43)}$ are commuting projections onto the subspaces $\mathcal{K}_{(21)}, \mathcal{K}_{(32)}, \mathcal{K}_{(43)}$, respectively. Then we have

\[
W_{3(21)} = P_{(41)}(P_{(31)}(W_1, W_2), W_3) = P_{(41)}(P_{(31)}(P_{(21)}(I - P_{(43)}) + P_{(32)}) + P_{(43)})(W_1, W_2, W_3)
\]

and

\[
W_{(32)1} = P_{(41)}(W_1, P_{(42)}(W_2, W_3)) = P_{(41)}(P_{(21)}(I - P_{(43)}) + P_{(32)} + P_{(43)})(W_1, W_2, W_3).
\]

Thus, $W_{3(21)} = Y(W_1, W_2, W_3)$ and $W_{(32)1} = Z(W_1, W_2, W_3)$, where $Y, Z: \mathcal{K}_{(21)} + \mathcal{K}_{(32)} + \mathcal{K}_{(43)} \to \mathcal{K}_{(41)}$ and

\[
Y = P_{(41)}(P_{(31)}(P_{(21)}(I - P_{(43)}) + P_{(32)}) + P_{(43)}),
\]

\[
Z = P_{(41)}(P_{(21)}(I - P_{(43)}) + P_{(42)}(P_{(32)} + P_{(43)})).\]
Let us show that $Z^{-1} = [P_{(21)} + (P_{(32)} + P_{(43)})P_{(42)}][K_{(41)}]$. Indeed, using Lemma (iii) and Corollary of Lemma (iii) we have

$$(P_{(21)} + (P_{(32)} + P_{(43)})P_{(42)})Z$$

$$= (P_{(21)} + (P_{(32)} + P_{(43)})P_{(42)})P_{(41)}(P_{(21)}(I - P_{(43)}) + P_{(42)}(P_{(32)} + P_{(43)}))$$

$$= (P_{(21)}P_{(41)} + (P_{(32)} + P_{(43)})P_{(42)})P_{(21)}(I - P_{(43)}) + P_{(42)}(P_{(32)} + P_{(43)}))$$

$$= P_{(21)}P_{(41)}P_{(21)}(I - P_{(43)}) + P_{(41)}P_{(21)}P_{(42)}(P_{(32)} + P_{(43)})$$

$$+ (P_{(32)} + P_{(43)})P_{(42)}P_{(21)}(I - P_{(43)}) + (P_{(32)} + P_{(43)})P_{(42)}(P_{(32)} + P_{(43)}))$$

$$= P_{(21)}(I - P_{(43)}) + P_{(21)}P_{(41)}P_{(42)}(P_{(32)} + P_{(43)}) + (P_{(32)} + P_{(43)})$$

$$= I + P_{(21)}P_{(41)}P_{(42)}(P_{(32)} + P_{(43)})$$

Since

$$P_{(21)}P_{(41)}P_{(42)} = (P_{(21)} + P_{(42)})P_{(41)}P_{(42)} - P_{(42)}P_{(41)}P_{(42)}$$

$$= (P_{(21)} + P_{(42)})P_{(41)}P_{(42)} - P_{(42)}$$

$$= (P_{(21)} + P_{(42)})P_{(41)}(P_{(21)} + P_{(42)}) - (P_{(21)} + P_{(42)})P_{(41)}P_{(21)} - P_{(42)}$$

$$= (P_{(21)} + P_{(42)}) - (P_{(21)} + P_{(42)})P_{(41)}P_{(21)} - P_{(42)}$$

$$= P_{(21)} + P_{(42)} - P_{(21)} - P_{(42)} = 0,$$

we obtain $(P_{(21)} + (P_{(32)} + P_{(43)})P_{(42)})Z = I[K_{(21)} + K_{(32)} + K_{(43)}]$. On the other hand, we have

$$Z(P_{(21)} + (P_{(32)} + P_{(43)})P_{(42)})$$

$$= P_{(41)}(P_{(21)}(I - P_{(43)}) + P_{(42)}(P_{(32)} + P_{(43)}))(P_{(21)} + (P_{(32)} + P_{(43)})P_{(42)})$$

$$= P_{(41)}(P_{(21)} + P_{(42)}(P_{(32)} + P_{(43)})P_{(42)}) = P_{(41)}(P_{(21)} + P_{(42)})[K_{(41)}]$$

$$= P_{(41)}(P_{(21)} + P_{(42)})P_{(41)}[K_{(41)}] = P_{(41)}[K_{(41)}] = I[K_{(41)}].$$

We need to compute the operator $Z^{-1}Y$:

$$Z^{-1}Y = (P_{(21)} + (P_{(32)} + P_{(43)})P_{(42)})P_{(41)}(P_{(31)}(P_{(21)}(I - P_{(43)}) + P_{(32)}) + P_{(43)})$$

$$= (P_{(21)} + (P_{(32)} + P_{(43)})P_{(42)})P_{(31)}(P_{(21)}(I - P_{(43)}) + P_{(32)}) + P_{(41)}P_{(43)})$$

$$= P_{(21)}P_{(31)}(P_{(21)}(I - P_{(43)}) + P_{(32)}) + P_{(21)}P_{(41)}P_{(43)}$$

$$+(P_{(32)} + P_{(43)})P_{(42)}P_{(31)}(P_{(21)}(I - P_{(43)}) + P_{(32)}) + (P_{(32)} + P_{(43)})P_{(42)}P_{(41)}P_{(43)}$$

$$= P_{(21)}(I - P_{(43)}) + P_{(21)}P_{(31)}P_{(32)} + P_{(21)}P_{(41)}P_{(43)}$$

$$+ P_{(32)}P_{(42)}P_{(31)}(P_{(21)}(I - P_{(43)}) + P_{(32)}) + (P_{(32)} + P_{(43)})P_{(42)}P_{(43)}$$

$$= P_{(21)}(I - P_{(43)}) + P_{(21)}P_{(31)}P_{(32)} + P_{(21)}P_{(41)}P_{(43)}$$

$$+ P_{(32)}P_{(42)}P_{(21)}(I - P_{(43)}) + P_{(32)}P_{(42)}P_{(31)}P_{(32)} + P_{(32)}P_{(42)}P_{(43)} + P_{(43)}.$$
Since
\[ P_{(21)}P_{(31)}P_{(32)} = \frac{(P_{(21)} + P_{(32)})P_{(31)}(P_{(21)} + P_{(32)}) - P_{(21)}P_{(31)}P_{(21)}}{-P_{(32)}P_{(31)}P_{(21)} - P_{(32)}P_{(31)}P_{(32)}} \]

\[ = P_{(21)} + P_{(32)} - P_{(21)} - P_{(32)}P_{(21)} = 0 \]

and
\[ P_{(32)}P_{(42)}P_{(43)} = \frac{(P_{(32)} + P_{(43)})P_{(42)}(P_{(32)} + P_{(43)}) - P_{(32)}P_{(42)}P_{(32)}}{-P_{(43)}P_{(42)}P_{(32)} - P_{(43)}P_{(42)}P_{(43)}} \]

\[ = P_{(32)} + P_{(43)} - P_{(32)} - P_{(43)}P_{(32)} - P_{(43)} = 0 , \]

we obtain
\[ Z^{-1}Y = P_{(21)}(I - P_{(43)}) + P_{(21)}P_{(41)}P_{(43)} + P_{(32)}P_{(42)}P_{(31)}P_{(32)} + P_{(43)} . \]

Taking into account that \( \text{Ran} P_{(43)} \subset H_{41^+} \) and using Lemma \([iii]\) we have
\[ P_{(21)}P_{(41)}P_{(43)} = (P_{(21)} + P_{(42)})P_{(41)}P_{(43)} - P_{(42)}P_{(41)}P_{(43)} \]

\[ = (P_{(21)} + P_{(42)})P_{(43)} - P_{(42)}P_{(43)} = P_{(21)}P_{(43)} . \]

Likewise, taking into account that \( \text{Ran} P_{(32)} \subset H_{41^+} \) and using Lemma \([iii]\) we have
\[ P_{(32)}P_{(42)}P_{(31)}P_{(32)} = P_{(32)}(P_{(21)} + P_{(42)})P_{(31)}P_{(32)} \]

\[ = P_{(32)}P_{(21)}P_{(32)} + P_{(32)}P_{(42)}P_{(32)} = P_{(32)}P_{(32)} = P_{(32)} . \]

Therefore,
\[ Z^{-1}Y = P_{(21)}(I - P_{(43)}) + P_{(21)}P_{(43)} + P_{(32)} + P_{(43)} \]

\[ = (P_{(21)} + P_{(43)}) + P_{(32)} + P_{(43)} \]

\[ = (I + P_{(21)}P_{(43)})|\mathcal{K}_{(21)} + \mathcal{K}_{(32)} + \mathcal{K}_{(43)} . \]

Note also that \( Y^{-1}Z = (I - P_{(21)}P_{(43)})|\mathcal{K}_{(21)} + \mathcal{K}_{(32)} + \mathcal{K}_{(43)} . \) Indeed, we have
\[ P_{(21)}P_{(43)}P_{(21)}P_{(43)} = 0 . \]

Further, for the operator \((W_{1}, W_{2}, W_{3}) : H_{1} \oplus H_{2} \oplus H_{3} \to \mathcal{K}_{(21)} + \mathcal{K}_{(32)} + \mathcal{K}_{(43)} , \)

it can easily be checked that \((W_{1}, W_{2}, W_{3})^{-1} = \begin{pmatrix} W_{01}P_{(21)}(I - P_{(43)}) & W_{02}P_{(32)} & W_{03}P_{(43)} \end{pmatrix} . \) Thus we obtain
\[ W_{(32,1)}^{-1}W_{(32,1)} = (W_{1}, W_{2}, W_{3})^{-1}Z^{-1}Y(W_{1}, W_{2}, W_{3}) \]

\[ = \begin{pmatrix} I & 0 & W_{*1}P_{(21)}P_{(43)}W_{3} \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \]

and therefore \( W_{(32,1)}^{-1}W_{(32,1)} \neq I \) because, in general, we have no the property \( P_{(21)}P_{(43)} = 0 . \) Thus the identity \( \Sigma_{3} \cdot (\Sigma_{2} \cdot \Sigma_{1}) = (\Sigma_{3} \cdot \Sigma_{2}) \cdot \Sigma_{1} \) does not hold.
Example. We continue the example from Section 1. Consider the systems
\[ \Sigma_1 = \Sigma_2 = \mathcal{F}_{sc}(\theta) = ((0), (1), (1)), \quad \Sigma_3 = \mathcal{F}_{sc}(\theta^2) = \left( \begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right), (1, 0), \left( \begin{array}{c} 2\varepsilon \\ 1 \end{array} \right) \].

Then we can easily calculate that
\[ \Sigma_3 \cdot (\Sigma_2 \cdot \Sigma_1) = \left( \begin{array}{ccc} 1 & 0 & \varepsilon \\ 0 & 1 & -\varepsilon^2 \\ 0 & 0 & 0 \end{array} \right), (1, \varepsilon, -\varepsilon^2, 2\varepsilon^3), \left( \begin{array}{c} \varepsilon^3 \\ -\varepsilon^2 \\ 2\varepsilon \\ 1 \end{array} \right) \]
and
\[ (\Sigma_3 \cdot \Sigma_2) \cdot \Sigma_1 = \left( \begin{array}{ccc} 0 & 1 & \varepsilon \\ 0 & 1 & -\varepsilon^2 \\ 0 & 0 & 0 \end{array} \right), (1, \varepsilon, 0, 0), \left( \begin{array}{c} \varepsilon^3 \\ -\varepsilon^2 \\ 2\varepsilon \\ 1 \end{array} \right) \].

Thus, \( \Sigma_3 \cdot (\Sigma_2 \cdot \Sigma_1) \neq (\Sigma_3 \cdot \Sigma_2) \cdot \Sigma_1 \). The matrix
\[ X = \left( \begin{array}{ccc} 1 & 0 & -\varepsilon^2 & 2\varepsilon^3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right) \]
realizes the similarity \( \Sigma_3 \cdot (\Sigma_2 \cdot \Sigma_1) \sim (\Sigma_3 \cdot \Sigma_2) \cdot \Sigma_1 \).

3. Regular factorizations

We start with extension of the notion of regularity to the class of n-characteristic functions.

Definition. We shall say that a n-characteristic function \( \Theta \) is regular (and write \( \Theta \in \text{Cfn}_n^{reg} \)) if \( \forall i \geq j \geq k \)
\[ \text{Ran}(I - \Theta_{ij}(z)\Theta_{ij}(z))^{1/2} \cap \text{Ran}(I - \Theta_{jk}(z)\Theta_{jk}(z))^{1/2} = \{0\}, \ a.e. \ z \in C. \]

Note that it suffices to check these conditions for \( k = 1, \ i = n, \ j = 1, m \) (it follows from [1], Lemma VII.4.1).

Taking into account the fact that we identify n-characteristic function with the factorization of Schur class function, we obtain the definition of regularity for factorization of Schur class function in the case when \( n = 3 \). If additionally \( \Xi_k \equiv 1, \ k = 1, 3 \), we arrive at the standard definition [1][2] (see the Introduction).

In the context of functional models the corresponding notion is the following.

Definition. Let \( \text{Mod}_n^{reg} := \{ \Pi \in \text{Mod}_n : \text{Ran} \pi_1 \lor \text{Ran} \pi_n = \mathcal{H} \} \). We shall say that an n-model \( \Pi \in \text{Mod}_n \) is regular if \( \Pi \in \text{Mod}_n^{reg} \).

We are going to show that these two notions of regularity (for n-characteristic functions and for n-models) agree. With that end in mind we employ the construction of Prop. [1][1]. It is easy to show that for any two contractive operators \( A_{21} : \mathcal{N}_1 \to \mathcal{N}_2 \) and \( A_{32} : \mathcal{N}_2 \to \mathcal{N}_3 \) there exist three isometries \( V_1 : \mathcal{N}_1 \to \mathcal{H}, \ V_2 : \mathcal{N}_2 \to \mathcal{H}, \) and \( V_3 : \mathcal{N}_3 \to \mathcal{H} \) such that
\[ A_{21} = V_2^* V_1, \quad A_{32} = V_3^* V_2, \quad A_{32} A_{21} = V_3^* V_1. \]

Note that we need no to assume as in Prop. [1][1] that the operators \( A_{21} \) and \( A_{32} \) are operator valued functions of weighted Schur class: it suffices to assume that
they are merely contractive operators. Evolving this approach, we obtain the following Lemmas and Let $V_1$, $V_2$, $V_3$ be isometries: $A_{21} = V_2^* V_1$, $A_{32} = V_3^* V_2$, $A_{31} = V_3^* V_1$; $\mathcal{E}_1 = \text{Ran} V_1$, $\mathcal{E}_2 = \text{Ran} V_2$, and $\mathcal{E}_3 = \text{Ran} V_3$.

**Lemma 3.1.** The following conditions are equivalent:

1) $A_{31} = A_{32} A_{21}$; 2) $V_3^* V_1 - V_3^* V_2 V_2^* V_1 = 0$; 3) $((\mathcal{E}_1 \vee \mathcal{E}_2) \ominus \mathcal{E}_2) \perp ((\mathcal{E}_3 \vee \mathcal{E}_2) \ominus \mathcal{E}_2)$.

**Proof.** 1) $\iff$ 2) is obvious. To prove 2) $\iff$ 3) we need the following

**Lemma (iv).** One has $\text{clos Ran}(I - V_2 V_2^*) V_1 = (\text{Ran} V_1 \lor \text{Ran} V_2) \ominus \text{Ran} V_2$.

**Proof.** Let $f \in \text{Ran}(I - V_2 V_2^*) V_1$. Then $f = (I - V_2 V_2^*) V_1 u \in \text{Ran} V_1 \lor \text{Ran} V_2$. On the other hand, $V_2^* f = V_2^* (I - V_2 V_2^*) V_1 u = (I - V_2^* V_2) V_2^* V_1 u = 0$, that is, $f \perp \text{Ran} V_2$ and therefore $\text{Ran}(I - V_2 V_2^*) V_1 \subset (\text{Ran} V_1 \lor \text{Ran} V_2) \ominus \text{Ran} V_2$.

Conversely, let $f \in (\text{Ran} V_1 \lor \text{Ran} V_2) \ominus \text{Ran} V_2$. Then we have $V_2^* f = 0$ and $f = \lim_{n \to \infty} (V_1 u_{1n} + V_2 u_{2n})$. Hence,

$$f = (I - V_2 V_2^*) f = \lim_{n \to \infty} (I - V_2 V_2^*) V_1 u_{1n} + (I - V_2 V_2^*) V_2 u_{2n})$$

$$= (I - V_2 V_2^*) f = \lim_{n \to \infty} (I - V_2 V_2^*) V_1 u_{1n} \in \text{clos Ran}(I - V_2 V_2^*) V_1.$$  \[ \square \]

To complete the proof of Lemma 3.1 we need only to make use of the following observation

$$\text{Ran}(I - V_2 V_2^*) V_1 \perp \text{Ran}(I - V_2 V_2^*) V_3 \iff V_3^* (I - V_2 V_2^*) V_1 = 0.$$  \[ \square \]

**Lemma (v).** Assume that $V_3^* V_1 - V_3^* V_2 V_2^* V_1 = 0$. Then

$$(\mathcal{E}_1 \lor \mathcal{E}_3) \ominus \mathcal{E}_3 \subset ((\mathcal{E}_1 \vee \mathcal{E}_2) \ominus \mathcal{E}_2) \ominus ((\mathcal{E}_3 \vee \mathcal{E}_2) \ominus \mathcal{E}_3).$$

**Proof.** Using Lemma (iv) and the obvious identity

$$(I - V_2 V_2^*) V_1 = (I - V_2 V_2^*) V_1 + (I - V_3 V_3^*) V_2 V_2^* V_1$$

we get

$$(\mathcal{E}_1 \lor \mathcal{E}_3) \ominus \mathcal{E}_3 \subset ((\mathcal{E}_1 \vee \mathcal{E}_2) \ominus \mathcal{E}_2) \lor ((\mathcal{E}_3 \vee \mathcal{E}_2) \ominus \mathcal{E}_3).$$

By Lemma 3.1 $((\mathcal{E}_1 \vee \mathcal{E}_2) \ominus \mathcal{E}_2) \perp ((\mathcal{E}_3 \vee \mathcal{E}_2) \ominus \mathcal{E}_2)$.

Then

$$((\mathcal{E}_1 \lor \mathcal{E}_3) \ominus \mathcal{E}_3) \perp ((\mathcal{E}_3 \lor \mathcal{E}_2) \ominus \mathcal{E}_2) \ominus \mathcal{E}_3 = \mathcal{E}_3 \lor \mathcal{E}_2$$

and therefore $((\mathcal{E}_1 \lor \mathcal{E}_2) \ominus \mathcal{E}_2) \perp ((\mathcal{E}_3 \lor \mathcal{E}_2) \ominus \mathcal{E}_3)$.

**Remark.** If we define the isometries $\tau_{ij}$: $\text{clos Ran}(I - V_j^* V_i V_i^* V_j)^{1/2} \to \mathcal{H}$ by the formula $\tau_{ij}(I - V_j^* V_i V_i^* V_j)^{1/2} = (I - V_i V_i^*) V_j$, we can rewrite the identity

$$(I - V_3 V_3^*) V_1 = (I - V_2 V_2^*) V_1 + (I - V_3 V_3^*) V_2 V_2^* V_1$$

in the form

$$\tau_{131}(I - A_{31}^* A_{31})^{1/2} = \tau_{121}(I - A_{21}^* A_{21})^{1/2} + \tau_{232}(I - A_{32}^* A_{32})^{1/2} A_{21}.$$  

Note that $\text{clos Ran} \tau_{ij} = \text{clos Ran}(I - V_i V_i^*) V_j$. Then, by Lemma 3.1, the condition $V_j^* (I - V_2 V_2^*) V_1 = 0$ means $\tau_{232} \tau_{121} = 0$ and $\tau_{121} \tau_{232} = 0$. Therefore we have

$$Z(I - A_{31}^* A_{31})^{1/2} = \left( \begin{array}{l} (I - A_{21}^* A_{21})^{1/2} \\ (I - A_{32}^* A_{32})^{1/2} A_{21} \end{array} \right)$$

and the operator $Z = (\tau_{121}^* + \tau_{232}^* \tau_{131})$ is an isometry. We need the following Lemma established in [2].
**Lemma (v).** The following conditions are equivalent:
1) \( \text{Ran} \left( I - A_{12}^* A_{32} \right)^{1/2} \cap \text{Ran} \left( I - A_{21}^* A_{21} \right)^{1/2} = \{0\} \);
2) \( A_{21}^* (I - A_{12}^* A_{32})^{1/2} m + (I - A_{21}^* A_{21})^{1/2} n = 0 \), \( m \in \text{clos Ran} \left( I - V_2^* V_3 V_2^* V_1 \right)^{1/2} \) and \( n \in \text{clos Ran} \left( I - V_1^* V_2^* V_1 \right)^{1/2} \) \( \implies m = 0, n = 0 \);
3) The operator \( Z \) defined by the condition (Z) is an unitary operator.

Now we have prepared to prove the Lemma 3.2, which allow us to translate factorization problems into geometrical language and now we can point out the purely geometrical nature of the notion of regularity. Note that this fact is the underlying basis of the generalization of Sz.-Nagy-Foiaş’s regularity criterion in [18], where the authors drop the condition of analyticity.

**Lemma 3.2.** Assume that \( V_1^* V_1 - V_3^* V_2 V_1^* V_1 = 0 \).
Then the following conditions are equivalent:
1) \( \text{Ran} \left( I - A_{12}^* A_{32} \right)^{1/2} \cap \text{Ran} \left( I - A_{21}^* A_{21} \right)^{1/2} = \{0\} \);
2) \( \text{clos Ran} \left( I - V_1^* V_2^* \right) V_1 = \text{clos Ran} \left( I - V_2^* V_3^* \right) V_2 \);
3) \( E_2 \subset E_1 \vee E_3 \).

**Proof.**
1) \( \iff \) 2) By Lemma [12], condition 1) is equivalent to the condition that the operator \( Z \) is unitary. Since under our assumptions \( Z \) is always isometrical, we can check only that \( Z^* = \tau_{131}(\tau_{121} + \tau_{232}) \) is an isometrical operator. The latter is equivalent to the condition \( \text{Ran} \tau_{121} \oplus \text{Ran} \tau_{232} \subset \text{Ran} \tau_{131} \). The inverse inclusion is Lemma [14]

2) \( \implies \) 3) Since we have \( (E_1 \vee E_3) \oplus E_3 = ((E_1 \vee E_2) \oplus E_2) \oplus ((E_3 \vee E_2) \oplus E_3) \), we obtain
\[
E_1 \vee E_3 = E_3 \oplus (E_1 \vee E_3) \oplus E_3 = E_3 \oplus ((E_3 \vee E_2) \oplus E_3) \oplus ((E_1 \vee E_2) \oplus E_2)
\]
\[
= (E_3 \vee E_2) \oplus ((E_1 \vee E_2) \oplus E_2).
\]
Hence, \( E_2 \subset E_1 \vee E_3 \). Then
\[
E_3 \oplus ((E_2 \vee E_3) \oplus E_3) = E_2 \vee E_3 \subset E_1 \vee E_3 = E_3 \oplus ((E_1 \vee E_3) \oplus E_3)
\]
and therefore \( (E_2 \vee E_3) \oplus E_3 \subset (E_1 \vee E_3) \oplus E_3 \).

On the other hand, we have \( ((E_1 \vee E_2) \oplus E_2) \perp E_2 \) and \( ((E_1 \vee E_2) \oplus E_2) \perp ((E_3 \vee E_2) \oplus E_2) \). Hence, \( ((E_1 \vee E_2) \oplus E_2) \perp E_2 \). Then, we get
\[
E_3 \oplus ((E_1 \vee E_2) \oplus E_2) \subset E_1 \vee E_2 \subset E_1 \vee E_3 \subset E_1 \vee E_3 = E_3 \oplus ((E_1 \vee E_3) \oplus E_3)
\]
and therefore \( (E_1 \vee E_2) \oplus E_2 \subset (E_1 \vee E_3) \oplus E_3 \). Thus, we obtain
\[
((E_1 \vee E_2) \oplus E_2) \oplus ((E_2 \vee E_3) \oplus E_3) \subset (E_1 \vee E_3) \oplus E_3.
\]
The inverse inclusion is Lemma [17].

The following assertion is a straightforward consequence of the lemmas.

**Proposition 3.3.** One has \( \Pi = F_{mc}(\Theta) \in \text{Mod}_{n}^{reg} \iff \Theta \in \text{Cfn}_{n}^{reg} \).

We have defined the notions of regularity for \( \text{Cfn}_{n} \) and \( \text{Mod}_{n} \). Now we pass over to curved conservative systems looking for a counterpart of the regularity in this new context. In the Introduction we have defined the notion of simple curved conservative systems. For them, we have
Proposition 3.4. 1) Let $\hat{\Sigma} = \mathcal{F}_{sm}(\Pi)$, $\Pi \in \text{Mod}$ and $\rho(\hat{T}) \cap G_+ \neq \emptyset$. Then the system $\hat{\Sigma}$ is simple; 2) If $\Sigma \sim \Sigma'$, then $\Sigma$ is simple $\iff \Sigma'$ is simple; 3) If a system $\Sigma$ is simple, then the system $\Sigma^*$ is also simple; 4) If $\Sigma \not\sim \Sigma'$, $\Sigma \not\sim \Sigma'$ and the system $\Sigma$ is simple, then $X = X'$; 5) If the system $\Sigma = \Sigma_2 \cdot \Sigma_1$ is simple and $\rho(T) \cap \rho(T_1) \cap G_+ \neq \emptyset$, then the systems $\Sigma_1$ and $\Sigma_2$ are simple.

Proof. 1) By [17], we have

$$-\hat{M}(\hat{T} - z)^{-1}f = \begin{cases} \Theta^+(z)^{-1}(\pi_+^f)(z), & z \in \rho(\hat{T}) \cap G_+ \\ (\pi_+^f)(z), & z \in G_-. \end{cases}$$

and therefore

$$\bigcap_{z \in \rho(T)} \text{Ker } \hat{M}(\hat{T} - z)^{-1} = \{f \in \mathcal{K} : \pi_+^f = 0, \pi_-^f = 0\} = \{f \in \mathcal{K} : f \perp \text{Ran } \pi_+, f \perp \text{Ran } \pi_-\} = \{0\}.$$

2) Let $\Sigma \not\sim \Sigma'$. Then $M(T - z)^{-1} = M'(T' - z)^{-1}X$ and the property follows straightforwardly from this identity.

3) is a direct consequence of properties 1) and 2).

4) It is sufficient to check that $\Sigma \not\sim \Sigma \Rightarrow X = I$. We have

$$M(T - z)^{-1}Xf = MX(T - z)^{-1}f = M(T - z)^{-1}f \implies$$

$$Xf - f \in \bigcap_{z \in \rho(T)} \text{Ker } M(T - z)^{-1} = \{0\} \implies Xf = f.$$

5) It can easily be checked that $\rho(T) \cap \rho(T_1) \subset \rho(T_2)$. Then we have

$$(T_21 - z)^{-1} = \left( \begin{array}{cc} (T_1 - z)^{-1} & -(T_1 - z)^{-1}N_1M_2(T_2 - z)^{-1} \\ 0 & (T_2 - z)^{-1} \end{array} \right)$$

and therefore $\forall f_1 \in H_1, M_1(T_1 - z)^{-1}f_1 = M_1(T_1 - z)^{-1}f_1$. Hence,

$$\bigcap_{z \in \rho(T)} \text{Ker } M_1(T_1 - z)^{-1} \subset \bigcap_{z \in \rho(T)} \text{Ker } M_2(T_2 - z)^{-1} = \{0\},$$

that is, the system $\Sigma_1$ is simple.

Further, by property 3), the system $\Sigma^*$ is simple. Then, using the same arguments as above, it follows that the system $\Sigma_2$ is simple. Hence the system $\Sigma_2$ is simple too. $\square$

Definition. The product of systems $\Sigma_21 = \Sigma_2 \cdot \Sigma_1$ is called regular if the system $\Sigma_{21}$ is simple.

Proposition 3.5. Let $\hat{\Sigma}_1 = \mathcal{F}_{sm}(\Pi_1)$, $\hat{\Sigma}_2 = \mathcal{F}_{sm}(\Pi_2)$. Suppose $\Sigma_1 \sim \hat{\Sigma}_1$, $\Sigma_2 \sim \hat{\Sigma}_2$, $\Sigma_{21} = \Sigma_2 \cdot \Sigma_1$, and $\rho(T_{21}) \cap G_+ \neq \emptyset$. Then the product $\Sigma_2 \cdot \Sigma_1$ is regular $\iff$ the product $\Pi_2 \cdot \Pi_1$ is regular.

Proof. Without loss of generality (see Prop. [28] and Prop. [31]) it can be assumed that $\Sigma_{21} = \hat{\Sigma} = \mathcal{F}_{sm}(\Pi_2 \cdot \Pi_1)$ and $T_{21} = \hat{T}$. As above (see the proof of Prop. [34]), we get

$$\mathcal{K}_u = \bigcap_{z \in \rho(T)} \text{Ker } \hat{M}(\hat{T} - z)^{-1} = \{f \in \mathcal{K}(31) : \pi_+^f = 0, \pi_-^f = 0\}.$$
and therefore \( \mathcal{K}_u \subset (\operatorname{Ran} \pi_+ \vee \operatorname{Ran} \pi_-)^\perp \). On the hand, if \( f \in (\operatorname{Ran} \pi_+ \vee \operatorname{Ran} \pi_-)^\perp \), then \( P_{3(1)} f = f \) and \( f \in \mathcal{K}_{3(1)} \). Thus, \( \mathcal{K}_u = (\operatorname{Ran} \pi_+ \vee \operatorname{Ran} \pi_-)^\perp \). It remains to note that the product \( \Sigma_2 \cdot \Sigma_1 \) is regular iff \( \mathcal{K}_u = \{0\} \) and the product \( \Pi_2 \cdot \Pi_1 \) is regular iff \( \operatorname{Ran} \pi_+ \vee \operatorname{Ran} \pi_- = \mathcal{H} \) (recall that \( \pi_+ = \pi_3 \) and \( \pi_- = \pi_1 \)). \( \square \)

Combining Prop. 3.3 and Prop. 3.5 we arrive at

**Proposition 3.6** (Criterion of regularity). Let \( \tilde{\Sigma}_1 = \mathcal{F}_{sc}(\Theta_1) \), \( \tilde{\Sigma}_2 = \mathcal{F}_{sc}(\Theta_2) \).

Suppose \( \Sigma_1 \sim \tilde{\Sigma}_1 \), \( \Sigma_2 \sim \tilde{\Sigma}_2 \), \( \Sigma_{21} = \Sigma_2 \cdot \Sigma_1 \), and \( \rho(T_{21}) \cap G_+ \neq \emptyset \). Then the product \( \Sigma_2 \cdot \Sigma_1 \) is regular \( \iff \) the factorization \( \Theta_2 \cdot \Theta_1 \) is regular.

Thus we obtain the correspondence between regular factorizations of characteristic functions and regular products of systems.

**Remark.** It can easily be shown that the inner-outer factorization \( \Pi \) of Schur class functions is regular (see \( \square \)). Hence, using the criterion of regularity, one can prove that the product of colligations with \( C_{11} \) and \( C_{00} \) contractions is regular. It is possible to extend this result to the case of weighted Schur functions employing the generalization of regularity criterion (Prop. 3.6). Note that, for \( J \)-contractive analytic operator functions, \( J \)-inner-outer factorization is regular too \( [19] \). However, since in this situation we lose such a geometrical functional model as is the Sz.-Nagy-Foiaş model for contractions (and such a geometrical description of regularity), we have to establish at first the regularity of the product of “absolutely continuous” and “singular” colligations (analogous of \( C_{11} \) and \( C_{00} \) contractions) and then to obtain the uniqueness of \( J \)-inner-outer factorization \( [19] \).

4. FACTORIZATIONS AND INVARIANT SUBSPACES

The most remarkable feature of the product of systems is its connection with invariant subspaces. We see that the subspace \( H_1 \) in the definition (Prod) is invariant under the operator \( T_{21} \) (and under its resolvent \( (T_{21} - z)^{-1} \), \( z \in G_- \)). In the context of functional model this implies that the subspace \( \mathcal{K}(21) \) is invariant under the operator \( \hat{T} \) (see Prop. 2.3). Following B. Sz.-Nagy and C. Foiaş, we shall work within the functional model and use the model as a tool for studying the correspondence “factorizations \( \leftrightarrow \) invariant subspaces”. Let \( \Theta \in \text{Mod}_3 \), \( \Pi = \mathcal{F}_{mc}(\Theta) = (\pi_1, \pi_2, \pi_3) \in \text{Mod}_3 \). We define the transformation \( L = \mathcal{F}_{ic}(\Theta) \) as a mapping that takes each 3-characteristic function \( \Theta \) (which we identify with factorization of Schur class function) to the invariant subspace \( L := \mathcal{K}(21) = \operatorname{Ran} P_{(21)} \).

To study the transformation \( \mathcal{F}_{ic} \) (and its ingenious extension to \( n \)-characteristic functions), we need to make some preliminary work.

Let \( \Pi \in \text{Mod}_n \). Consider the chain of subspaces \( \mathcal{H}_{11} \subset \ldots \subset \mathcal{H}_{n1} \) (see the definition of \( \mathcal{H}_{ij} \) after Lemma 1.3). These subspaces are invariant under the resolvent \( (\mathcal{U} - z)^{-1} \), \( z \in G_- \). The inverse is also true accurate up to the “normal” part of the chain.

**Proposition 4.1.** Suppose \( \mathcal{U} \in \mathcal{L}(\mathcal{H}) \) is a normal operator, \( \sigma(\mathcal{U}) \subset C \), and \( \mathcal{H}_{1+} \subset \ldots \subset \mathcal{H}_{n+} \) is a chain of invariant under \( (\mathcal{U} - z)^{-1} \), \( z \in G_- \) subspaces. Then there exists an \( n \)-model \( \Pi \in \text{Mod}_n \) such that \( \mathcal{H}_{k1} \subset \mathcal{H}_{k+} \), \( k = 1, n \) and the subspaces \( \mathcal{H}_{uk} := \mathcal{H}_{k+} \ominus \mathcal{H}_{k1} \) reduce the operator \( \mathcal{U} \). If an \( n \)-model \( \Pi' \in \text{Mod}_n \) satisfies the same conditions, then \( \mathcal{H}''_{k+} = \mathcal{H}_{k+} \) and \( \exists \psi_k \) such that \( \psi_k, \psi_k^{-1} \in H^\infty(G_+, \mathcal{L}(\mathcal{H}_k)) \) and \( \pi_k = \pi_k \psi_k \). Besides, we have \( \mathcal{H}_{11} \subset \ldots \subset \mathcal{H}_{nn} \).
Proof. Consider the Wold type decomposition $\mathcal{H}_{k^+} = \mathcal{H}^{par}_{k^+} \oplus \mathcal{H}^{nor}_{k^+}$ with respect to the normal operator $\mathcal{U}$, $\sigma(\mathcal{U}) \subset C$ (see [10]). The operator $\mathcal{U}|\mathcal{H}^{par}_{k^+}$ is the pure subnormal part of $\mathcal{U}|\mathcal{H}_{k^+}$ and $\mathcal{U}|\mathcal{H}^{nor}_{k^+}$ is a normal operator. This decomposition is unique. We set

$$\mathcal{E}_{k^+} = \mathcal{H}^{par}_{k^+}, \quad \mathcal{E}_k = \vee_{z \in C}(\mathcal{U} - z)^{-1}\mathcal{E}_{k^+}, \quad \mathcal{E}_{k^-} = \mathcal{E}_k \ominus \mathcal{E}_{k^+}$$

Obviously, $\mathcal{E}_{k^-} \subset \mathcal{H}_{k^-}$ and $\mathcal{U}^*|\mathcal{E}_{k^-}$ is the pure subnormal part of $\mathcal{U}^*|\mathcal{H}_{k^+}$. For $i \geq j \geq k$, we have $\mathcal{E}_{i^-} \perp \mathcal{E}_{j^+}, \mathcal{E}_{k^+} \subset \mathcal{H}_{j^+}$, and $\mathcal{E}_{i^-} \subset \mathcal{H}_{j^+}$. Hence,

$$\mathcal{E}_k \subset \vee_{z \in C}(\mathcal{U} - z)^{-1}\mathcal{H}_{j^+} \quad \text{and} \quad \mathcal{E}_i \subset \vee_{z \in C}(\mathcal{U}^* - \bar{z})^{-1}\mathcal{H}_{j^+}^\perp.$$  

This implies that

$$\mathcal{E}_j \ominus ((\mathcal{E}_k \cap \mathcal{E}_j) \ominus \mathcal{E}_j) = \mathcal{E}_k \cap \mathcal{E}_j \subset \mathcal{E}_j \cap \mathcal{H}_{j^+} = \mathcal{E}_j \ominus \mathcal{H}_{j^+}.$$  

Therefore we get

$$(\mathcal{E}_k \cap \mathcal{E}_j) \ominus \mathcal{E}_j \subset \mathcal{H}^{nor}_{j^+} \quad \text{and} \quad \mathcal{E}_{j^+} \ominus ((\mathcal{E}_k \cap \mathcal{E}_j) \ominus \mathcal{E}_j) \subset \mathcal{H}_{j^+}.$$  

In the same way, $\mathcal{E}_{j^-} \ominus ((\mathcal{E}_k \cap \mathcal{E}_j) \ominus \mathcal{E}_j) \subset \mathcal{H}_{j^+}$. And finally,

$$((\mathcal{E}_i \cap \mathcal{E}_j) \ominus \mathcal{E}_j) \perp ((\mathcal{E}_k \cap \mathcal{E}_j) \ominus \mathcal{E}_j).$$

We need to make use of the following lemma.

**Lemma.** Suppose $\mathcal{U} \in \mathcal{L}(\mathcal{H})$ is a normal operator, $\sigma(\mathcal{U}) \subset C$, $\mathcal{E}_+ \subset \mathcal{H}$ and $\mathcal{U}|\mathcal{E}_+$ is a pure subnormal operator. Then there exists an operator $\pi \in \mathcal{L}(L^2(C, \mathfrak{M}), \mathcal{H})$ such that $\text{Ran} \pi = \vee_{\lambda \in C}(\mathcal{U} - \lambda)^{-1}\mathcal{E}_+$, $\ker \pi = \{0\}$, $\pi E^2(G_+, \mathfrak{M}) = \mathcal{E}_+$ and $\mathcal{U}\pi = \pi \mathcal{U}$.

**Proof.** Without loss of generality we can assume that $\mathcal{H} = \vee_{\lambda \in C}(\mathcal{U} - \lambda)^{-1}\mathcal{E}_+$. By [10], there exists an unitary operator $Y_0 \in \mathcal{L}(E^2_0(G_+, \mathfrak{M}), \mathcal{E}_+)$ such that $\mathcal{U}Y_0 = Y_0\mathcal{U}$, where $E^2_0(G_+, \mathfrak{M})$ is the Smirnov space of character-automorphic functions (see the comments between Prop. [4] and Prop. [14]). By Mlak's lifting theorem [20], the operator $Y_0$ can be extended to the space $L^2(C, \mathfrak{M})$ lifting the intertwining condition. This extension will be denoted by $\pi_0 \in \mathcal{L}(L^2(C, \mathfrak{M}), \mathcal{H})$. So, we have $\mathcal{U}\pi_0 = \pi_0\mathcal{U}$. Similarly, there exists an extension $X_0 \in \mathcal{L}(\mathcal{H}, L^2(C, \mathfrak{M}))$ of the operator $Y_0^{-1}$ such that $X_0\mathcal{U} = \mathcal{U}X_0$. Thus, $X_0\pi_0|E^2_0(G_+, \mathfrak{M}) = \mathcal{U}E^2_0(G_+, \mathfrak{M})$. Since $L^2(C, \mathfrak{M}) = \vee_{\lambda \in C}(\mathcal{U} - \lambda)^{-1}E^2_0(G_+, \mathfrak{M})$, we get $X_0\pi_0 = \mathcal{I}$. Likewise, since $\pi_0X_0|\mathcal{E}_+ = \mathcal{I}|\mathcal{E}_+$ and $\mathcal{H} = \vee_{\lambda \in C}(\mathcal{U} - \lambda)^{-1}\mathcal{E}_+$, we get $\pi_0X_0 = \mathcal{I}$ and therefore $\pi_0^{-1} = X_0 \in \mathcal{L}(\mathcal{H}, L^2(C, \mathfrak{M}))$.

According to [10], the “bundle” shift $\mathcal{Z}|E^2_0(G_+, \mathfrak{M})$ is similar to the trivial shift $\mathcal{Z}|E^2(G_+, \mathfrak{M})$. The similarity is realized by operator valued function $\chi \in L^\infty(C, \mathcal{L}(\mathfrak{M}))$ such that $\chi^{-1} \in L^\infty(C, \mathcal{L}(\mathfrak{M}))$ and $\chi E^2(G_+, \mathfrak{M}) = E^2_\alpha(G_+, \mathfrak{M})$. Then we put $\pi := \pi_0\chi$. □

Since $\mathcal{U}|\mathcal{E}_{j^+}$ is the pure subnormal part of $\mathcal{U}|\mathcal{H}_j$, there exists operators $\pi_j \in \mathcal{L}(L^2(C, \mathfrak{M}), \mathcal{H})$ such that $\text{Ran} \pi_j = \mathcal{E}_j$, $\pi_j E^2(G_+, \mathfrak{M}) = \mathcal{E}_{j^+}$, and $\mathcal{U}\pi_j = \pi_j\mathcal{U}$. In terms of operators $\pi_j$ we rewrite the relations obtained earlier. The relation $\mathcal{E}_{i^-} \perp \mathcal{E}_{j^+}$ implies $P_-(\pi_i^{-1}\pi_j)\mathcal{P}_+ = 0$ and the orthogonality $(\mathcal{E}_i \cap \mathcal{E}_j) \perp ((\mathcal{E}_k \ominus \mathcal{E}_j) \ominus \mathcal{E}_j)$ means that $\text{Ran}(I - \pi_j^{1/2})\pi_i \perp \text{Ran}(I - \pi_j^{1/2})\pi_k$. Hence, $\pi_i^{1/2}\pi_k = \pi_k^{1/2}\pi_i^{1/2}\pi_k$. Thus, the n-tuple $\Pi = (\pi_1, \ldots, \pi_n)$ is an n-model.
We put $\mathcal{H}_{j1+} = \mathcal{H}_{\pi_j,\cdots,\pi_1} \cap \ker \pi_j P_{-\pi_j^\dagger}$. Then,
\begin{align*}
\mathcal{H}_{j1+} &= \operatorname{Ran} P_{\pi_j,\cdots,\pi_1} \cap \operatorname{Ran} (I - \pi_j P_{-\pi_j^\dagger}) = \operatorname{Ran} P_{\pi_j,\cdots,\pi_1} (I - \pi_j P_{-\pi_j^\dagger}) \\
&= \operatorname{Ran} P_{\pi_j,\cdots,\pi_1} ((I - \pi_j^\dagger) + \pi_j P_{\pi_j^\dagger}) = \mathcal{E}_{j+} \oplus \operatorname{Ran} (I - \pi_j^\dagger) P_{\pi_j,\cdots,\pi_1} \\
&= \mathcal{E}_{j+} \oplus (\bigvee_{k=1}^{j-1} \operatorname{clo} \operatorname{Ran} (I - \pi_j^\dagger) \pi_k = \mathcal{E}_{j+} \oplus (\bigvee_{k=1}^{j-1} ((\mathcal{E}_k \vee \mathcal{E}_j) \oplus \mathcal{E}_j))).
\end{align*}

Hence we get $\mathcal{H}_{j1+} \subset \mathcal{H}_{j+} = \mathcal{E}_{j+} \oplus \mathcal{H}_{j+}^{\text{nor}}$ and

$$
\mathcal{H}_{uj} = \mathcal{H}_{j+} \oplus \mathcal{H}_{j1+} = \mathcal{H}_{j+}^{\text{nor}} \oplus (\bigvee_{k=1}^{j-1} ((\mathcal{E}_k \vee \mathcal{E}_j) \oplus \mathcal{E}_j)).
$$

It is obvious that the subspace $\mathcal{H}_{uj}$ reduces the operator $\mathcal{U}$.

Assume that $\mathcal{H}_{j1+} = \mathcal{H}_{\pi_j,\cdots,\pi_1} \cap \ker \pi_j^\dagger P_{\pi_j^\dagger}$, $\mathcal{H}_{j1+} \subset \mathcal{H}_{j+}$ and the subspace $\mathcal{H}_{j+} \oplus \mathcal{H}_{j1+}$ reduces the operator $\mathcal{U}$, where $\Pi' = (\pi_1, \ldots, \pi_n) \in \text{Mod}_n$. Then we have the generalized Wold decomposition \[ \mathcal{H}_{j+} = \mathcal{E}_{j+} \oplus (\bigvee_{k=1}^{j-1} ((\mathcal{E}_k \vee \mathcal{E}_j) \oplus \mathcal{E}_j)) \oplus (\mathcal{H}_{j+} \oplus \mathcal{H}_{j1+}). \] Since this decomposition is unique, we obtain $\mathcal{E}_{j+} = \mathcal{E}_j$ and, by induction, $\mathcal{H}_{j1+} = \mathcal{H}_{j1+}$. Then, $\pi_j = \pi_j \psi_j$, where $\psi_j = \pi_j^\dagger \pi_j$, $\psi_j^{-1} = \pi_j^\dagger \pi_j \in H^\infty(G_+, \mathcal{L}(\mathcal{H}_j))$.

Since $\mathcal{H}_{uj} \perp ((\mathcal{E}_j \oplus \mathcal{E}_j) \oplus \mathcal{E}_j))$ and $\mathcal{H}_{uj} \subset \mathcal{H}_{j+}$, we get $\mathcal{H}_{uj} \perp (\mathcal{E}_k \vee \mathcal{E}_j)$.

For $i > j$, we have $\mathcal{E}_{i-} \subset \mathcal{H}_{j+} \subset \mathcal{H}_{uj}$. Hence, $\mathcal{H}_{uj} \perp \mathcal{E}_i$ and $\mathcal{H}_{uj} \perp \mathcal{H}_{n,\pi_\cdots,\pi_1}$. Since $\mathcal{H}_{j1+} \subset \mathcal{H}_{\pi_\cdots,\pi_1}$, $\mathcal{H}_{uj} \subset \mathcal{H}_{\pi_\cdots,\pi_1}$ and $\mathcal{H}_{j+} = \mathcal{H}_{uj} \oplus \mathcal{H}_{j1+}$, we have $\mathcal{H}_{uj} = \mathcal{H}_{\pi_\cdots,\pi_1} \oplus \mathcal{H}_{j+}$ and therefore $\mathcal{H}_{uj} \subset \ldots \subset \mathcal{H}_{uj}$. $\square$

Let $\theta \in \text{Cfn}$. We fix $\theta$ and define $\text{Mod}_n^\theta := \{ \Pi \in \text{Mod}_n : \pi_1^\dagger \pi_1 = \theta \}$. Then we can consider the chain of subspaces $\mathcal{F}_{\text{im}}^\theta(\Pi) := \{ \mathcal{K}_{(11)} \subset \mathcal{K}_{(21)} \subset \ldots \subset \mathcal{K}_{(n1)} \}$, where $\mathcal{K}_{(kj)} = \operatorname{Ran} P_{(kj)}$. The subspaces $\mathcal{K}_{(kj)}$ are invariant under the operator $\mathcal{F}$ and this observation motivates the following definition.

Let $\theta = \pi_1^\dagger \pi_1$, where the operators $\pi_{\pm} \in \mathcal{L}(L^2(\Xi_\pm), \mathcal{H})$ are isometries. Let $\mathcal{U} \in \mathcal{L}(\mathcal{H})$ be a normal operator such that $\mathcal{U} \pi_{\pm} = \pi_{\pm} \pi_{\pm}$ and $\sigma(\mathcal{U}) \subset C$. Let also $\mathcal{K} = \operatorname{Ran} \mathcal{P}$, $\mathcal{P} = (I - \pi_+^\dagger P_{-\pi_+^\dagger})(I - \pi_-^\dagger P_{-\pi_-^\dagger})$, and $\mathcal{T} = \mathcal{P} \mathcal{U} \mathcal{K}$.

Definition. A chain of subspaces $\mathcal{L} = (L_1 \subset L_2 \subset \ldots \subset L_n)$ is called $n$-invariant if $L_n \subset \mathcal{K}$, $(T - z)^{-1} L_k \subset L_k$, $z \in G_-$, $k = 1, n$, and the subspaces $L_1, \mathcal{K} \oplus L_n$ reduce the operator $\mathcal{U}$. We will denote the class of all $n$-invariant chains by $\text{Inv}_n^\theta$.

In fact, we have already defined the transformation $\mathcal{F}_{\text{im}}^\theta : \text{Mod}_n^\theta \to \text{Inv}_n^\theta$, which takes each $\Pi \in \text{Mod}_n^\theta$ to the $n$-invariant chain of subspaces $(\mathcal{K}_{(11)} \subset \mathcal{K}_{(21)} \subset \ldots \subset \mathcal{K}_{(n1)}) \in \text{Inv}_n^\theta$. This transformation is surjective accurate up to the “normal” part of the chain.

**Proposition 4.2.** Suppose a chain $L$ is $n$-invariant. Then there exists an $n$-model $\Pi \in \text{Mod}_n^\theta$ such that $\mathcal{K}_{(kj)} \subset L_k$, $k = 1, n$ and the subspaces $L_k := L_k \oplus \mathcal{K}_{(kj)}$ reduce the operator $\mathcal{U}$. If an $n$-model $\Pi' \in \text{Mod}_n$ satisfies the same conditions, then $\mathcal{K}_{(kj)}' = \mathcal{K}_{(kj)}$ and $\exists \psi_k$ such that $\psi_k \psi_k^{-1} \in H^\infty(G_+, \mathcal{L}(\mathcal{R}_k))$ and $\pi_k = \pi_k \psi_k$. Besides, we have $L_{uj} \subset \ldots \subset L_{uj}$. $\square$

**Proof.** We put $\mathcal{H}_{k+} = L_k \oplus \mathcal{D}_+$, where $\mathcal{D}_+ = \operatorname{Ran} q_+$, $q_+ = \pi_+ P_+ \pi_+^\dagger$. Then, for $z \in G_-$, we get $(U - z)^{-1} \mathcal{D}_+ \subset \mathcal{D}_+ \subset \mathcal{H}_k$ and

$$(U - z)^{-1} L_k \subset P(U - z)^{-1} L_k \oplus q_+(U - z)^{-1} L_k \subset (T - z)^{-1} L_k \oplus \mathcal{D}_+ \subset \mathcal{H}_k.$$
Therefore the chain $\mathcal{H}_1+ \ldots + \mathcal{H}_{n+}$ is invariant under $(\mathcal{U} - z)^{-1}$, $z \in G_-$. By Prop. 4.1 there exists an $n$-model $\Pi \in \text{Mod}_n$ such that $\mathcal{H}_k{\mathcal{H}}_{n+k}$, $k = 1, n$ and the subspaces $\mathcal{H}_{uk} = \mathcal{H}_{k+} \oplus \mathcal{H}_{k+1}$ reduce the operator $\mathcal{U}$. Since $L_1$ reduces $\mathcal{U}$, we have that $\mathcal{H}_1+ = \mathcal{D}_+ + L_1$ is the generalized Wold decomposition of $\mathcal{H}_1+$. Taking into account the uniqueness of Wold decomposition, we obtain $\pi_+ = \pi_1\psi_1$. Comparing the Wold decompositions of the equal subspaces $\mathcal{K} \oplus \mathcal{D}_+$ and $(\mathcal{K} \oplus L_n) \oplus \mathcal{H}_{n+n}$, we obtain $\pi_+ = \pi_1\psi_n$. Thus we can assume without loss of generality (see the proof of Prop. 4.1) that $\pi_1 = \pi_+$ and $\pi_n = \pi_-$, i.e., $\Pi \in \text{Mod}_n^n$.

Since $L_k \subset \mathcal{K}$, we have $L_k = (I - \pi_+ P_+ \pi_1') \mathcal{H}_{k+} + \mathcal{H}_{k+1}$, taking into account that $K_{(k)} = (I - \pi_+ P_+ \pi_1') \mathcal{H}_{k+}$ and $\mathcal{H}_{k+1} \subset \mathcal{H}_{k+}$, we get $K_{(k)} \subset L_k$. Since $\mathcal{H}_{uk} = H_{\pi_1 \vee \ldots \vee \pi_n} \cap \mathcal{H}_{k+}$, we have $\mathcal{H}_{uk} = \mathcal{H}_{uk} \subset \mathcal{H}_{k+}$ and therefore $\mathcal{H}_{uk} \oplus K'_{(k)} \subset L_k$. In fact, these two spaces are equal. Consider the operator $q_+ = \pi_1 P_+ + \pi_1'$, which is the orthogonal counterpart to $q_- = \pi_+ P_+ \pi_1'$ (see the comments between Prop. 4.1 and Prop. 4.5). Put $L_k' = (I - q_+) L_k$ and $K_{(k)'} = (I - q_+) K_{(k)}$. By Corollary of Lemma (iii) $L_k = (I - q_+) L_k'$ and $K_{(k)} = (I - q_+) K_{(k)'}$. Further, we have

$$L_k \oplus K_{(k)'} = (L_k' \oplus D_+ \oplus (K_{(k)}' \oplus D_+)) = \mathcal{H}_{k+} \oplus \mathcal{H}_{k+1} = \mathcal{H}_{uk}.$$ 

Then

$$L_k = (I - q_+) L_k' = (I - q_+) (K_{(k)'} \oplus \mathcal{H}_{uk}) = K_{(k)} \oplus \mathcal{H}_{uk} = K_{(k)} \oplus \mathcal{H}_{uk}$$

and therefore $L_k = K_{(k)} \oplus \mathcal{H}_{uk}$. Hence, $L_{uk} = L_k \oplus K_{(k)} = \mathcal{H}_{uk}$. Then, by Prop. 4.1 we have $L_{u1} \subset \ldots \subset L_{un}$.

Let $\Pi \in \text{Mod}_n^n$ be a $n$-model such that $K_{(k)} \subset L_k$, $k = 1, n$ and the subspaces $L_{uk} = \mathcal{L} \oplus K_{(k)}$ reduce the operator $\mathcal{U}$. Then $\mathcal{H}_{k+} = K_{(k)} \oplus D_+ \subset L_k \oplus D_+ = \mathcal{H}_{k+}$ and the subspaces $\mathcal{H}_{k+} \oplus \mathcal{H}_{k+1} = L_{uk}$ reduce the operator $\mathcal{U}$. By Prop. 4.1 we get $\mathcal{H}_{k+1} = \mathcal{H}_{k+1}$. Hence, $K_{(k)} = (I - \pi_+ P_+ \pi_1') \mathcal{H}_{k+1} = (I - \pi_+ P_+ \pi_1') \mathcal{H}_{k+} = K_{(k)}$. 

Remark. In the case $n = 2$ this proposition is an analogue of the well-known decomposition of a contraction $T$ into the orthogonal sum $T = T_{cnu} + T_u$ of the completely non-unitary part $T_{cnu}$ and the unitary part $T_u$ (see [1]). In this connection, we will use the notation

$$\text{Inv}_n^{\theta, cnu} := \{(L_1 \subset L_2 \subset \ldots \subset L_n) \in \text{Inv}_n^{\theta} : L_{un} = \{0\}\}.$$ 

In this notation Prop. 4.1 means merely that $\text{Ran} \mathcal{F}_{im}^{\theta} = \text{Inv}_n^{\theta, cnu}$. Note also that the condition $L_{un} = \{0\}$ is equivalent to the condition

$$\forall n \in \mathbb{N} \left[\forall z \in \mathbb{C} (\mathcal{U} - z)^{-1} (\mathcal{U} - z)^{-1}(L_k \oplus D_+) \oplus (L_k \oplus D_+)) = \mathcal{H} \right].$$

Let us now return to the transformation $\mathcal{F}_{ic}$. Fix $\theta \in \text{Cfn}$ and define $\text{Mod}_n^\theta := \{\Theta \in \text{Mod} : \Theta_n = \theta\}$. Then we can consider the restriction $\mathcal{F}_{ic} | \text{Mod}_n^\theta$, which takes each 3-characteristic function $\Theta \in \text{Mod}_n^\theta$ to the invariant subspace $L := K_{(21)} \subset \mathcal{H}$. The main difficulty to handle effectively factoriavations of the function $\theta$ is the fact that the space $\mathcal{H}$ is variable and we cannot compare invariant subspaces when we run over factorizations of $\theta$. To avoid this effect we shall restrict ourselves to models for which $\mathcal{H} = \mathcal{H}_{\pi_+ \vee \pi_-} = \text{Ran} \pi_+ \vee \text{Ran} \pi_-$ and $\Pi = (\pi_+, \pi_2, \pi_-)$, where
\[ \pi_{\pm} \in \mathcal{L}(L^2(\mathbb{C}_\pm), \mathcal{H}) \] are some fixed isometries such that \( \theta = \pi_+^\dagger \pi_- \). Then we obviously have \( \text{Ran} \, \pi_2 \subset \text{Ran} \, \pi_1 \lor \text{Ran} \, \pi_3 \) and therefore \( \Pi = (\pi_1, \pi_2, \pi_3) \in \text{Mod}_{3}^{\text{reg}} \).

In this connection we define the subclasses
\[ \text{Cfn}^n_{\theta \text{reg}} := \text{Cfn}^\theta_n \cap \text{Cfn}^{\text{reg}}_n, \quad \text{Mod}^\theta_n := \text{Mod}^\theta_n \cap \text{Mod}^{\text{reg}}_n \]
and
\[ \text{Inv}^\theta_n := \{ L \in \text{Inv}^\theta_n : \text{Ran} \, \pi_+ \lor \text{Ran} \, \pi_- = \mathcal{H} \} \].

By Prop. 3.3 it can easily be shown that
\[ \mathcal{F}_{\text{ic}}^\theta(\Pi) \in \text{Inv}^\theta_n \iff \Pi \in \text{Mod}^\theta_n \]
Besides, it is clear that \( \text{Inv}^\theta_n \subset \text{Inv}^\theta_{\text{cnu}} \).

Finally, we define the transformation \( \mathcal{F}_{\text{ic}}^\theta : \text{Cfn}^\theta_n \to \text{Inv}^\theta_n \) by the following procedure. Let \( \Theta \in \text{Cfn}^\theta_n \) and \( \Pi = \mathcal{F}_{\text{ic}}(\Theta) \in \text{Mod}_n \) (in fact, by Prop. 3.3 \( \Pi \in \text{Mod}^\theta_n \)). Then, \( \mathcal{H} = \mathcal{H}_{\pi_1 \lor \pi_2} = \text{Ran} \, \pi_1 \lor \text{Ran} \, \pi_2 \) and \( \theta = \pi_1^\dagger \pi_2 \). By Prop. 3.4 there exists an unique unitary operator \( \mathcal{H} : \mathcal{H}_{\pi_1 \lor \pi_2} \to \mathcal{H}_{\pi_+ \lor \pi_-} \) such that \( \pi_1 = X \pi_1 \) and \( \pi_- = X \pi_- \). Then we put \( \mathcal{F}_{\text{ic}}(\Theta) := \mathcal{F}_{\text{ic}}(\Pi) \in \text{Inv}^\theta_n \), where \( \Pi = (X \pi_1, X \pi_2, \ldots, X \pi_n) \in \text{Mod}^\theta_n \). This definition of the fundamental transformation \( \mathcal{F}_{\text{ic}}^\theta \) is rather indirect. As justification of it we note that in the case of the unit disk the known approaches \[ \text{[11,12,13]} \] are not simpler than our procedure.

The following Proposition is a straightforward consequence of Prop. 3.4.

**Proposition 4.3.** One has 1) \( \text{Ran} \, \mathcal{F}_{\text{ic}}^\theta = \text{Inv}^\theta_n \); 2) If \( \mathcal{F}_{\text{ic}}^\theta(\Theta') = \mathcal{F}_{\text{ic}}^\theta(\Theta) \), \( \Theta, \Theta' \in \text{Cfn}^\theta_n \), then \( \Theta' \sim \Theta \), where \( \sim \) is equivalence relation: \( \Theta' \sim \Theta \) if \( \exists \psi_k, k = 2, n - 1 \) such that \( \psi_k, \psi_k^{-1} \in H^\infty(G_+, \mathcal{L}(\mathcal{K}_k)) \) and \( \Theta_{i_2, i_1} = \psi_i^{-1} \Theta_{1, i_1} \psi_i \), therefore \( \Xi_k = \psi_k^* \Xi_k \psi_k, \psi_1 = I, \psi_n = I \).

Thus, one can consider the quotient space \( \text{Cfn}^n_{\theta \text{reg}} := \text{Cfn}^\theta_n / \sim \) and the corresponding one-to-one transformation \( \mathcal{F}_{\text{ic}}^\theta : \text{Cfn}^\theta_n \to \text{Inv}^\theta_n \). Note that the functions \( \psi_k \) can be regarded as \( \Xi \)-unitary constants, i.e., \( \psi_k^* = \psi_k^{-1} \in H^\infty(G_+, \mathcal{L}(\mathcal{K}_k)) \), where \( \psi_k \) are adjoint to \( \psi_k : L^2(\Xi_k) \to L^2(\Xi_k) \).

Let us consider particular cases. In the case of \( n = 3 \) we obtain that the transformation \( \mathcal{F}_{\text{ic}}^\theta : \text{Cfn}^3_{\theta \text{reg}} \to \text{Inv}^\theta_n \) is an one-to-one correspondence between regular factorizations of a characteristic function and invariant subspaces of the corresponding model operator.

Consider the case \( n = 4 \). Let \( L = (L_1, L_2, L_3, L_4) \in \text{Inv}^\theta_n \). By Prop. 3.5 there exists \( \Theta \in \text{Cfn}^4_{\theta \text{reg}} \) such that \( L = \mathcal{F}_{\text{ic}}^\theta(\Theta) \). If we rename \( L' = L_2, L'' = L_3 \) (recall that \( L_1 = \{0\}, L_4 = \mathcal{K}_0 \)) and \( \theta = \Theta_{31}, \theta_{1} = \Theta_{21}, \theta_{2} = \Theta_{42}, \theta_{3} = \Theta_{31}, \theta_{3} = \Theta_{42}, \Xi_+ = \Xi_1, \Xi_+ = \Xi_2, \Xi'' = \Xi_3, \Xi_+ = \Xi_4, \) then we have
\[ \theta = \theta_{1}^* \theta_{1}^* = \theta_{2}^* \theta_{2}^* \]
and \( \exists \vartheta \in S_\Xi \) such that \( \theta_{1}^* = \theta_{1}^* \) and \( \theta_{2}^* = \theta_{1}^* \). (\( \sim \))

Certainly, \( \vartheta = \Theta_{32} \) and \( \Xi = (\Xi, \Xi'') \). We shall say that the factorization \( \theta = \theta_{1}^* \theta_{1}^* \) precedes the factorization \( \theta = \theta_{2}^* \theta_{2}^* \) (and write \( \theta_{2}^* \theta_{1}^* \sim \theta_{2}^* \theta_{1}^* \)) if the condition (\( \sim \)) is satisfied. Thus, \( L' \subset L'' \iff \theta_{2}^* \theta_{1}^* \sim \theta_{2}^* \theta_{1}^* \).

Conversely, suppose that factorizations \( \theta_{2}^* \theta_{1}^* = \theta_{2}^* \theta_{1}^* \) are regular and \( \theta_{2}^* \theta_{1}^* \sim \theta_{2}^* \theta_{1}^* \). After backward renaming we have \( \Theta \in \text{Cfn}^4_{\theta \text{reg}} \). Let \( L = \mathcal{F}_{\text{ic}}^\theta(\Theta) \). Then we have \( L = L_2, L'' = L_3 \). Therefore, \( \theta_{2}^* \theta_{1}^* \sim \theta_{2}^* \theta_{1}^* \iff L' \subset L'' \). Finally, we have
\[ \mathcal{F}_{\text{ic}}^\theta(\Theta_{32} \Theta_{21}) \subset \mathcal{F}_{\text{ic}}^\theta(\Theta_{32} \Theta_{31}) \iff \Theta_{42} \Theta_{21} \sim \Theta_{42} \Theta_{32} \Theta_{31} . \]
It is easy to check that \( \theta'_2 \theta'_1 \prec \theta''_2 \theta''_1 \), \( \theta'_2 \theta'_1 \sim \theta''_2 \theta''_1 \), \( \theta''_2 \theta''_1 \sim \theta''_2 \theta''_1 \Rightarrow \theta'_2 \theta'_1 \sim \theta''_2 \theta''_1 \), i.e., the order relation \( \prec \) is well defined on the quotient space \( \mathcal{C} \mathcal{H}^2_{\ast} \). Taking all this into account, we arrive at the main result of the Section.

**Theorem B.** There is an order preserving one-to-one correspondence between regular factorizations of a characteristic function (up to the equivalence relation) and invariant subspaces of the resolvent \((\hat{T} - z)^{-1}, z \in G_-\) of the corresponding model operator.

This Theorem is an extension of the fundamental result from \cite{1} (Theorems VII.1.1 and VII.4.3; see also \cite{21} for some refinement).

**Corollary.** Suppose that factorizations \( \theta'_2 \theta'_1, \theta''_2 \theta''_1 \) are regular, \( \theta'_2 \theta'_1 \prec \theta''_2 \theta''_1 \) and \( \theta'_2 \theta'_1 \sim \theta''_2 \theta''_1 \). Then \( \theta'_2 \theta'_1 \sim \theta''_2 \theta''_1 \).

**Proof.** Let \( L' = \mathcal{F}_{ic}^{reg}(\theta'_2 \theta'_1) \) and \( L'' = \mathcal{F}_{ic}^{reg}(\theta''_2 \theta''_1) \). By Theorem B, we get \( L' \subseteq L'' \subseteq L' \) and therefore \( L' = L'' \). Then, by Prop. \( \ref{prop} \) we have \( \theta'_2 \theta'_1 \sim \theta''_2 \theta''_1 \). \( \Box \)

Note that the Corollary can be proved independently from Theorem B. The corresponding argumentation make use of Lemmas \( \ref{lem} \) and \( \ref{lem2} \) and therefore we can drop the assumptions that \( \theta'_2, \theta'_1, \theta''_2, \theta''_1 \) are operator valued functions.

**Proposition.** Let \( A_{21}, A_{42}, A_{31}, A_{43} \) be contractions. Suppose that factorizations \( A_{42} \cdot A_{21}, A_{43} \cdot A_{31} \) are regular, \( A_{42} \cdot A_{21} \prec A_{43} \cdot A_{31} \) and \( A_{43} \cdot A_{31} \prec A_{42} \cdot A_{21} \). Then there exists an unitary operator \( U \) such that \( A_{31} = U A_{21} \) and \( A_{43} = A_{42} U^{-1} \).

**Proof.** We shall make use of the following two lemmas.

**Lemma (vii).** Suppose that \( ||A|| \leq 1 \) and \( A|H_1 = I|H_1 \). Then \( A^*|H_1 = I|H_1 \).

**Proof.** We have \( A = \begin{pmatrix} I & a_{12} \\ 0 & a_{22} \end{pmatrix} \). Then

\[
0 \leq ((I - A^* A) \begin{pmatrix} f_1 \\ 0 \end{pmatrix}, \begin{pmatrix} f_1 \\ 0 \end{pmatrix}) = -(a_{12}^* f_1, a_{12}^* f_1) \leq 0.
\]

Therefore, \( a_{12} = 0 \) and \( A = \begin{pmatrix} I & 0 \\ 0 & a_{22} \end{pmatrix} \). \( \Box \)

**Lemma (viii).** Let \( A_{21}, A_{32} \) be contractions. Suppose that factorization \( A_{32} \cdot A_{21} \) is regular. Then \( (\text{Ran } A_{32} \vee \text{Ran } A_{21})^+ = \{0\} \).

**Proof.** Let \( f \perp (\text{Ran } A_{32} \vee \text{Ran } A_{21}) \). Then \( f \in \text{Ker } A_{32} \) and \( f \in \text{Ker } A_{21}^* \). Hence, \( (I - A_{21}^* A_{21}) f = f \) and therefore \( (I - A_{21} A_{21}^*)^{1/2} f = f \). Similarly, we have \( (I - A_{32} A_{32}^*)^{1/2} f = f \). Then \( f \in \text{Ran } (I - A_{32}^* A_{32})^{1/2} \cap \text{Ran } (I - A_{21} A_{21}^*)^{1/2} = \{0\} \). \( \Box \)

From the definition of the order relation \( \prec \) we get that there exists contractions \( A_{32}, A_{23} \) such that \( A_{42} = A_{43} A_{32}, A_{31} = A_{32} A_{21}, A_{43} = A_{42} A_{23}, \) and \( A_{32} = A_{23} A_{31} \). Let \( A = A_{23} A_{32} \). Then we have \( A_{31} = A A_{21} = A_{42} A \) and therefore \( A| \text{Ran } A_{21} = I| \text{Ran } A_{21} \) and \( A^*| \text{Ran } A_{42} = I| \text{Ran } A_{42} \). By Lemma \( \ref{lem} \) \( A| \text{Ran } A_{42} = I| \text{Ran } A_{42} \). Finally, by Lemma \( \ref{lem2} \) we get \( A = A((\text{Ran } A_{21} \vee \text{Ran } A_{42}) = I((\text{Ran } A_{21} \vee \text{Ran } A_{42}) = I \), that is, \( A_{23} A_{32} = I \).

Likewise, we get \( A_{32} A_{23} = I \). Since \( A_{23} \) and \( A_{32} \) are contraction, they are unitary operators. It remains to put \( U = A_{32} \). \( \Box \)
In conclusion we again consider curved conservative systems. The following assertion is just a translation of Prop. 4.4 into the language of systems.

**Proposition 4.4.** Suppose $\Sigma = (T, M, N) \in \text{Sys}$ and a subspace $L$ is invariant under the resolvent $(T - z)^{-1}$, $z \in G_\infty$. Then there exist systems $\Sigma_1, \Sigma_2 \in \text{Sys}$ and an operator $X : H_1 \oplus H_2 \rightarrow H$ such that $\Sigma \sim \Sigma_2 \cdot \Sigma_1$ and $L = XH_1$.

*Proof.* Let $\Sigma \sim \hat{\Sigma} = \mathcal{F}_{sc}(\theta)$ and $\Pi = (\pi_1, \pi_3) = \mathcal{F}_{mc}(\theta)$. Then $\hat{L} = YL$ is an invariant subspace for the model operator. By Theorem B, there exists an regular factorization $\theta = \theta_2 \cdot \theta_1$ such that $\hat{L} = \mathcal{F}_{sc}^{reg}(\theta_2 \cdot \theta_1) = \text{Ran} P_{(21)}$. Besides, $\theta_1 = \pi_1^1 \pi_1$ and $\theta_2 = \pi_3^1 \pi_2$. We put $\Pi_1 = (\pi_1, \pi_2)$, $\Pi_2 = (\pi_2, \pi_3)$, $\hat{\Sigma}_1 = \mathcal{F}_{sc}(\Pi_1)$, and $\hat{\Sigma}_2 = \mathcal{F}_{sc}(\Pi_2)$. Let $\Sigma_1 \sim \hat{\Sigma}_1$ and $\Sigma_2 \sim \hat{\Sigma}_2$. By Prop. 4.5 we get $\Sigma_1 \cdot \Sigma_2 \sim \hat{\Sigma}$ realizing the similarity. It can easily be checked that $\hat{L} = P_{(31)}(Y_1, Y_2)H_1$. Then, for $X = Y^{-1}P_{(31)}(Y_1, Y_2)$, we get $\Sigma \sim \Sigma_2 \cdot \Sigma_1$ and $L = XH_1$. □

Besides, we have the following assertion.

**Proposition 4.5.** Suppose the system $\Sigma_2 \cdot \Sigma_1 = \Sigma_2' \cdot \Sigma_1'$ is simple, $H_1 = H_1'$, and $\Theta_2 \Theta_1 = \Theta_2' \Theta_1'$. Then there exists $\psi$ such that $\psi, \psi^{-1} \in H^\infty(G_+, \mathcal{L}(\Pi))$ and $\Sigma_1' \sim \Sigma_1'' = (T_1, M_1, N_1'')$, where $N_1'' f_1 = -\frac{1}{2\pi i} \int_C \psi(z)^* \left[ N_1'^* (T_1^* - \cdot)^{-1} f_1 \right] \cdot (z) d\nu$, $f_1 \in H_1$.

*Proof.* Let $\Sigma = \Sigma_2 \cdot \Sigma_1 \sim \hat{\Sigma} = \mathcal{F}_{sc}(\theta)$ and $\Pi = (\pi_1, \pi_3) = \mathcal{F}_{mc}(\theta)$. Using the same notation as in the proof of Prop. 4.4 we obtain that the operators $P_{(31)}(Y_1, Y_2)$ and $P_{(31)}(Y_1', Y_2')$ realize the similarities $\Sigma_2 \cdot \Sigma_1 \sim \hat{\Sigma}$ and $\Sigma_2' \cdot \Sigma_1' \sim \hat{\Sigma}$, respectively. Since the system $\hat{\Sigma} \sim \Sigma$ is simple, by Prop. 3.4, we get $P_{(31)}(Y_1, Y_2) = P_{(31)}(Y_1', Y_2')$ and therefore $P_{(31)}(Y_1, Y_2)H_1 = P_{(31)}(Y_1', Y_2')H_1'$. Then, by Prop. 4.5 there exists an operator valued function $\psi$ such that $\psi, \psi^{-1} \in H^\infty(G_+, \mathcal{L}(\Pi_2))$, $\theta_1' = \psi^{-1} \theta_1$, and $\theta_2' = \theta_2 \psi$. According to 4.5, $\Sigma_1' \sim \Sigma_1'' = \mathcal{F}_{sc}(\psi^{-1} \theta_1)$. Since $\Sigma_1' \sim \hat{\Sigma} = \mathcal{F}_{sc}(\theta_2')$, we get $\Sigma_1' \sim \Sigma_1''$. □

Further, we shall say that a system $\Sigma \in \text{Sys}$ possesses the property of uniqueness of characteristic function if there exists an unique characteristic function $\Theta \in \text{Cfn}$ such that $\Sigma = \mathcal{F}_{cs}(\Theta)$. Recall (see the Introduction) the sufficient condition for this property: the transfer function $\Upsilon(z)$ of the system $\Sigma$ is an operator valued function of Nevanlinna class. For products of systems we have the following (non-trivial) fact: *suppose that a system $\Sigma = \Sigma_2 \Sigma_1$ is simple, possesses the property of uniqueness, and $\rho(T_1) \cap G_+ \neq \emptyset$; then the system $\Sigma_1$ possesses the same property too.*

**Proposition 4.6.** Suppose the system $\Sigma_2 \cdot \Sigma_1 = \Sigma_2' \cdot \Sigma_1$ is simple and possesses the property of uniqueness. Suppose also $\rho(T_1) \cap G_+ \neq \emptyset$. Then $\Sigma_2 = \Sigma_2'$.

*Proof.* Let $\Sigma = \Sigma_2 \cdot \Sigma_1 = \Sigma_2' \cdot \Sigma_1$ and $\theta = \theta_2 \theta_1 = \theta_2' \theta_1'$ be the corresponding factorizations. Then $\theta_1 = \theta_1'$ (see the comments before the Proposition). Since $\forall \lambda \in \rho(T_1) \cap G_+ \neq \emptyset \exists \theta_1(\lambda)^{-1}$, we get $\theta_2 = \theta_2'$. Then $\Sigma_2 \sim \mathcal{F}_{sc}(\theta_2)$,
$\Sigma_2' \sim F_{\theta_2}$ and therefore $\Sigma_2 \overset{X_2}{\sim} \Sigma_2'$. Taking this into account, we have $\Sigma_1 \overset{X_1}{\sim} \Sigma_1'$ and $\Sigma \overset{X_2}{\sim} \Sigma$. By Prop. 3.4(4), we get $X_2 = I$. □

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