Evolution of thin-wall configurations of texture matter

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Abstract

We consider the free matter of global textures within the framework of the perfect fluid approximation in general relativity. We examine thermodynamical properties of texture matter in comparison with radiation fluid and bubble matter. Then we study dynamics of thin-wall selfgravitating texture objects, and show that classical motion can be elliptical (finite), parabolical or hyperbolical. It is shown that total gravitational mass of neutral textures in equilibrium equals to zero as was expected. Finally, we perform the Wheeler-DeWitt’s minisuperspace quantization of the theory, obtain exact wave functions and discrete spectra of bound states with provision for spatial topology.

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1 Introduction

The scalar field theories, in which the global symmetry $G$ is spontaneously broken to $H$ in such a way that vacuum manifold $G/H$ has nontrivial homotopy group $\pi_3(G/H)$, predict the existence of the matter with an equation of state $\varepsilon + 3p = 0$ called as the texture matter or k-matter.\(^1\)

Of course, the terminology ”texture matter” does not seem to be perfectly apposite because the real $\sigma$-model textures are dynamical defects and the equation of state above is not valid in general case. However, in numerous papers such a terminology was fixedly settled (see \(^\text{1}\) and \(^\text{2}\)).

\(^1\)The late LANL e-version is slightly extended with respect to that published in GRG (some complementary speculations, footnotes and suggested references were added).

\(^2\)Also, the matter with such an equation of state naturally appears in string cosmology theories. One can obtain it from the “string-driven” effective equation of state $\varepsilon + dp - nq = 0$, where $d$ and $n$ are respectively the numbers of expanding and internal contracting dimensions, $p$ and $q$ are respectively the pressure in the expanding space and shrinking dimensions \(^\text{17}\). Indeed, in the “after-string” era the internal dimensions were compactified (therefore, $n \to 0$), the number of expanding dimensions became the usual one, $d = 3$. This texture-dominated era continued till epochs of standard-model particles, quarks and leptons, when the decreasing temperature and density led textures to couple up. Nowadays we probably could observe some tracks of it not only at cosmological scales but also among the fundamental properties of recently observed particles \(^\text{13}\).
references therein) thus we will follow it in present paper as well. Some known properties of textures say that it is probably another kind of vacuum similar to the de Sitter vacuum $\varepsilon + p = 0$ (the latter is known also as the bubble matter). Let us consider, for instance, the $O(4) \rightarrow O(3)$ textures arising in the scalar fourplet theory described by the action

$$S(\vec{\phi}) = \int \left[ \partial^\mu \vec{\phi} \partial_\mu \vec{\phi} + \lambda (\vec{\phi} \cdot \vec{\phi} - \eta^2)^2 \right] \sqrt{-g} d^4 x$$

in a closed FRW universe ($0 \leq \xi \leq \pi$)

$$ds^2 = dt^2 - a^2(t)[d\xi^2 + \sin^2 \xi (d\theta^2 + \sin^2 \theta d\varphi^2)].$$

Then the texture solution of winding number one,

$$\vec{\phi} = \eta \begin{bmatrix} \cos \varphi \sin \theta \sin \xi \\ \sin \varphi \sin \theta \sin \xi \\ \cos \theta \sin \xi \\ \cos \xi \end{bmatrix},$$

has the following stress-energy tensor,

$$T^\mu_\nu = \frac{\eta^2}{2a^2} \text{diag}(3, 1, 1, 1),$$

which evidently satisfies with the above-mentioned equation of state. The zero-zero component of this tensor will be compared in Sec. 3 with a surface case.

The gravitational effects caused by 3D texture matter were intensively studied in many works. The main aim of present paper is to study the 2D fluid of global textures which forms spherically symmetric singular hypersurfaces (surfaces of discontinuities of second kind). These hypersurfaces can be interpreted both as the thin-wall approximation of the layer of bulk matter and as the brane-like objects embedded in spacetime of higher dimensionality. As such, the singular model turns to be simple enough to obtain important and instructive exact results not only when studying classical dynamics but also when considering quantum aspects. With respect to the 3D case this model appears to be the thin-wall approximation, which can elicit main features common for 2D and 3D cases.

The paper is organized as follows. In section 2 we give a comparative description of thermodynamics of 2D and 3D texture matter at finite temperature with respect to each other and with respect to bubble matter and ordinary matter represented by radiation fluid. Section 3 is devoted to classical dynamics of the isentropic singular shells “made” from 2D texture fluid. In section 4 we perform minisuperspace quantization of the singular model with provision for both the through (wormhole-like) and ordinary topology. Conclusions are made in section 5.

## 2 Comparative thermodynamics

Let us consider the thermodynamical properties of texture matter as such and in comparison with those for radiation fluid (quasi-counterpart of texture) and bubble matter $\varepsilon + p = 0$. First

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3. The dualities found in some string cosmology models suggest that the matter with this equation of state can be regarded as the matter of low-energy string origin which in some sense is dual to radiation fluid (incoherent radiation). Therefore, the term “dual-radiation matter” seems to be appropriate for it as well.
of all, we try to answer the question, what is thermodynamical information we can obtain from an equation of state.

The first thermodynamical law says:

\[ dE = T dS - pdV. \] (1)

On the other hand, following the definition of the entropy as a function of volume and temperature, one can write

\[ dS = \frac{\partial S}{\partial T} dT + \frac{\partial S}{\partial V} dV. \] (2)

Comparing these equations, we obtain

\[ \frac{\partial S}{\partial T} = \frac{1}{T} \frac{\partial E}{\partial T}, \] (3)

\[ \frac{\partial S}{\partial V} = \frac{1}{T} \left( p + \frac{\partial E}{\partial V} \right). \] (4)

Then the equality of mixed derivatives yields the expression

\[ p + \frac{\partial E}{\partial V} = T \frac{\partial p}{\partial T}, \] (5)

which gives opportunities to obtain internal energy as a function of volume and temperature from an equation of state. Let us introduce the densities of energy and entropy such that

\[ E = \varepsilon(T)V, \quad S = s(T)V, \] (6)

and consider barotropic matter with linear equation of state (LEOS)

\[ p = \eta \varepsilon. \] (7)

Then (5) reads

\[ \eta T \frac{d\varepsilon}{dT} = (\eta + 1) \varepsilon, \] (8)

and we obtain the energy density

\[ \varepsilon = \varepsilon_0 T^{1+1/\eta}. \] (9)

For instance, for 3D radiation fluid this expression yields the expected Stefan-Boltzmann law describing energy of incoherent radiation with respect to temperature:

\[ \varepsilon = \alpha_{SB} T^4. \]

The internal energy and pressure are, respectively,

\[ E = \varepsilon_0 T^{1+1/\eta} V, \] (10)

\[ p = \eta \varepsilon_0 T^{1+1/\eta}. \] (11)

Further, from (4), (6), (7) and (9) one can see that entropy has to be

\[ S = (\eta + 1) \varepsilon_0 T^{1/\eta} V + S_0. \] (12)

The above-mentioned special cases of LEOS matter are illustrated in table 1.

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4The features presented there (first of all, the unusual inverse dependence of energy and entropy on temperature which means that increasing of temperature is energetically favorable) can serve also for revealing of free texture matter in present era, e.g., inside superhot objects with appropriate energy density (of order of GUT scale, 10^{13} TeV).
Table 1: Comparative thermodynamical properties of ordinary and vacuum-like matter.

| Matter                  | EOS        | Energy density | Entropy                  | Comments                        |
|------------------------|------------|----------------|--------------------------|---------------------------------|
| 3D radiation fluid     | $\varepsilon - 3p = 0$ | $\varepsilon_0 T^4$ | $\frac{1}{3} \varepsilon_0 T^3 V$ | Stefan-Boltzmann law            |
| 2D radiation fluid     | $\varepsilon - 2p = 0$ | $\varepsilon_0 T^3$ | $\frac{2}{3} \varepsilon_0 T^2 V$ |                                  |
| de Sitter bubble       | $\varepsilon + p = 0$ | $\varepsilon_0$ | $S_0$                   | no dependence on $T$            |
| 2D texture             | $\varepsilon + 2p = 0$ | $\varepsilon_0 T^{-1}$ | $\frac{1}{2} \varepsilon_0 T^{-2} V + S_0$ | $T \neq 0$                     |
| 3D texture             | $\varepsilon + 3p = 0$ | $\varepsilon_0 T^{-2}$ | $\frac{3}{2} \varepsilon_0 T^{-3} V + S_0$ | $T \neq 0$                     |

law of thermodynamics) because its energy diverges; bubble and texture matter have nonzero minimal energy unlike ordinary matter $\eta > 0$ including ultrarelativistic radiation fluid. It seems to be another argument to the advantage of interpretation of the texture matter as a specific vacuum state similar to the de Sitter one.

3 Thin-wall model

Beginning from the classical works [3, 4, 5] formalism of surface layers has been widely described in the literature (see Refs. [6, 7] for details). The three-dimensional singular embeddings appear to be both interesting extended objects as such, and simple (but realistic) models of four-dimensional phenomena. From the viewpoint of the general physics of extended objects the concept “singular hypersurface” has to be the next-order approximation, after the “point particle” one, which takes into account both external, kinetic and dynamical, properties and internal structure (surface pressure, mass density, temperature etc.).

So, one considers the infinitely thin isentropic layer of matter with the surface stress-energy tensor of a perfect fluid in general case (we use the units $\gamma = c = 1$, where $\gamma$ is the gravitational constant)

$$ S_{ab} = \sigma u_{a} u_{b} + p (u_{a} u_{b} + (3)g_{ab}), $$

where $\sigma$ and $p$ are the surface mass-energy density and pressure respectively, $u$ is the unit tangent vector, $(3)g_{ab}$ is the three-metric of the shell’s hypersurface. We suppose that this shell is spherically symmetric, closed, and hence divides the whole manifold into the two regions $\Sigma^\pm$. Also we suppose the metrics of the space-times outside $\Sigma^+$ and inside $\Sigma^-$ of a spherically symmetric shell to be of the form

$$ ds^2 = -(1 + \Phi^\pm(r)) dt^2 + \left[1 + \Phi^\pm(r)\right]^{-1} dr^2 + r^2 d\Omega^2, $$

(13)

where $d\Omega^2$ is the metric of the unit two-sphere. Of course, we have some loss of generality but it is enough for further. It is possible to show that if one introduces the proper time $\tau$, then the 3-metric of a shell can be written in the form

$$ (3) ds^2 = -d\tau^2 + R^2 d\Omega^2, $$

(14)

where $R(\tau)$ is a proper radius of a shell. Define a simple jump of the second fundamental forms across a shell as $[K^a_b] = K^a_+ - K^a_-$, where

$$ K^a_\pm = \lim_{n \rightarrow \pm 0} \frac{1}{2} (3)g^{ac} \frac{\partial}{\partial n} (3)g_{cb}, $$

(15)
where \( n \) is a proper distance in normal direction. The Einstein equations on a shell then yield equations which are the well-known Lichnerowicz-Darmois-Israel junction conditions

\[
(K^a_b)^+ - (K^a_b)^- = 4\pi\sigma(2u^a u_b + \delta^a_b),
\]

Besides, an integrability condition of the Einstein equations is the energy conservation law for shell matter. In terms of the proper time it can be written as

\[
d\left(\sigma (3)g\right) + p d\left( (3)g \right) + (3)g [T] d\tau = 0,
\]

where \([T] = (T^{\tau n})^+ - (T^{\tau n})^-\), \(T^{\tau n} = T^\beta_\alpha u^\alpha n_\beta\) is the projection of stress-energy tensors in the \(\Sigma^\pm\) space-times on the tangent and normal vectors, \((3)g = \sqrt{-\det(3)g_{ab}} = R^2 \sin \theta\).

We assume that our shell carries no charges on a surface and contains no matter inside itself. If we define \(M\) to be the total mass-energy of the shell then one can suppose the external and internal spacetimes to be Schwarzschild and Minkowskian respectively:

\[
\Phi^+ = -\frac{2M}{R}, \quad \Phi^- = 0.
\]

After straightforward computation of extrinsic curvatures the \(\theta\theta\) component of (16) yields the equation of motion of the perfect fluid neutral hollow shell

\[
\epsilon_+ \sqrt{1 + \dot{R}^2} - \frac{2M}{R} - \epsilon_- \sqrt{1 + \dot{R}^2} = \frac{m}{R},
\]

where

\[
m = 4\pi\sigma R^2
\]

is interpreted as the (effective) rest mass, \(\dot{R} = \frac{dR}{d\tau}\) is a proper velocity of the shell, \(\epsilon_+ = \text{sign}\left(\sqrt{1 + \dot{R}^2} - \frac{2M}{R}\right), \epsilon_- = \text{sign}\left(\sqrt{1 + \dot{R}^2}\right)\). It is well-known that \(\epsilon = +1\) if \(R\) increases in the outward normal direction to the shell, and \(\epsilon = -1\) if \(R\) decreases. Thus, under the choice \(\epsilon_+ = \epsilon_-\) we have an ordinary (black hole type) shell, whereas at \(\epsilon_+ = -\epsilon_-\) we have the thin-shell traversable wormhole [8].

Let us consider the conservation law (17). One can obtain that \([T]\) is identically zero for the spacetimes (18). Further, if we assume the 2D texture equation of state of the shell’s matter,

\[
\sigma + 2p = 0,
\]

then, solving the differential equation (17) with respect to \(\sigma\), we obtain

\[
\sigma = \frac{\alpha}{2\pi R},
\]

hence

\[
m = 2\alpha R,
\]

where \(\alpha\) is the dimensionless integration constant which can be determined via surface mass density (or pressure) at fixed \(R\). The surface energy density determined by (22) appears to be the 2D analogue of the cosmological \(T^0_0\) component from the Sec. [8] if one takes into account the reduction of dimensionality. This is an expected result: from the viewpoint of the 2D observer “living” on the shell it seems for him to be the whole universe with the scale factor \(R\). Thus, our 2D fluid model indeed not only considers the established trace properties of the texture
stress-energy tensor but also restores its components for the surface case. In this connection the integration constant \( \alpha \) obtains the sense of the (squared) topological charge \( \eta \). The topological nature of the textures will brightly show itself at the end of this section when we will study the texture fluid singular layers with the vanishing total gravitational mass-energy.

Equations (19) and (23) together with the choice of the signs \( \epsilon_{\pm} \) completely determine the motion of the thin-wall texture. In conventional general relativity it is usually supposed that masses are nonnegative. However, keeping in mind possible wormhole and quantum extensions of the theory [9], we will not restrict ourselves by positive values and consider general case of arbitrary (real) masses. Then forbidden and permitted signs of this values can be determined from table 2. Let us find now the trajectories of 2D textures. Integrating (19) we obtain the transcendental equation for \( R(\tau) \)

\[
\frac{\tau}{M} = J(R/M) - J(R_0/M),
\]

where

\[
J(y) = \begin{cases} \frac{1}{\alpha^2 - 1} \left\{ \frac{1}{\alpha} \sqrt{Z_1} + \frac{1}{2\sqrt{1 - \alpha^2}} \arcsin Z_2 \right\}, & \alpha^2 < 1, \\ \pm \frac{1}{6} \sqrt{4y + 1(2y - 1)}, & \alpha = \pm 1, \\ \frac{1}{\alpha^2 - 1} \left\{ \frac{1}{\alpha} \sqrt{Z_1} - \frac{1}{2\sqrt{1 - \alpha^2}} \arccosh Z_2 \right\}, & \alpha^2 > 1, \end{cases}
\]

\[
Z_1 = \alpha^2(\alpha^2 - 1)y^2 + \alpha^2 y + 1/4,
\]

\[
Z_2 = 2 \alpha(\alpha^2 - 1)y + \alpha.
\]

Thus, in dependence on the parameter \( \alpha^2 \) one can distinguish elliptical, parabolical and hyperbolical trajectories. Let us consider below the consistency conditions which yield permitted domains of \( \alpha \) and \( y = R/M \) for each from three cases \( \alpha^2 \).

(a) Hyperbolic trajectories (\( \alpha^2 > 1 \)).

Following (24) the next two conditions should be satisfied jointly:

\[
Z_1 \geq 0, \quad Z_2 \geq 0.
\]

Define

\[
y_{\pm} = -\frac{1}{2\alpha} \frac{1}{\alpha \pm 1}, \quad \bar{y} = \frac{1}{2(1 - \alpha^2)},
\]

and consider the two subcases:

(a.1) \( \alpha < -1 \). Then \( y_+ < \bar{y} < y_- < 0 \) and inequalities (25) can be reduced respectively to

\[
\{ y \leq y_+ \} \cup \{ y \geq y_- \}, \quad y \leq \bar{y},
\]
that yields
\[ y \leq y_+. \]  
(28)

(a.2) \( \alpha > 1 \). Then \( y_- < \bar{y} < y_+ < 0 \) and inequalities (25) can be reduced respectively to
\[ \{y \leq y_-\} \cup \{y \geq y_+\}, \ y \geq \bar{y}, \]
that yields
\[ y \geq y_+. \]  
(29)
Thus, inequalities (28) and (29) determine permitted regions \( \{\alpha, R/M\} \) for hyperbolical trajectories.

(b) Elliptic trajectories \( (\alpha^2 < 1) \).
In the same way as above we can obtain the next restrictions:
\[ Z_1 \geq 0, \quad -1 \leq Z_2 \leq 1, \]  
(30)
and consider the two subcases:

(b.1) \(-1 < \alpha < 0\). Then \( y_+ > 0 \) and \( y_- < 0 \), and inequalities (30) read
\[ y_- \leq y \leq y_+. \]  
(31)

(b.2) \( 0 < \alpha < 1 \). Then \( y_+ < 0 \) and \( y_- > 0 \), and
\[ y_+ \leq y \leq y_- . \]  
(32)

(c) Parabolic trajectories \( (\alpha^2 = 1) \).
We obtain that \( y \) should obey
\[ y \geq -1/4. \]  
(33)
The cases (a)-(c) are illustrated in figure 1 which represents dependence \( y = R/M \) on \( \alpha \). Note, we did not restrict signs of mass, therefore, table 2 should be kept in mind.

Let us study now equilibrium states of thin-wall textures. Differentiating (19) with respect
\[ \dot{\tau}, \quad \ddot{\tau}, \quad \frac{\ddot{R} + M/R^2}{\epsilon_+ \sqrt{1 + R^2 - 2M/R}} - \frac{\ddot{R}}{\epsilon_- \sqrt{1 + R^2}} = 0, \]  
(34)
which independently could be obtained from junction conditions (16). Then in equilibrium state \( \dot{R} = \ddot{R} = 0 \) we obtain
\[ M = 0, \]
i. e., the texture fluid in equilibrium has zero total gravitational mass that is already well-known [1]. Another way to show this feature is to generalize (19), (34) by inserting the mass \( M_\pm \) inside the shell, then the external and internal spacetimes turn to be the Schwarzschild ones with masses \( M^+ \) and \( M^- \) respectively. Performing the analogical calculations we would obtain that in equilibrium: at \( \epsilon_+ = \epsilon_- \) the static masses \( M^+ = M^- \) and \( \alpha = 0 \) (that evidently corresponds to the already decayed shell because \( \alpha \) is the genuine criterion of existence and distinguishability of the shell) whereas at \( \epsilon_+ = -\epsilon_- \) the static masses \( M_\pm \) should vanish but \( \alpha \) should not, giving the nonzero value for static radius, i.e. again the static texture shell makes no contribution to the total gravitational mass of the system.

In other words, if in the (generalized) equations (19), (34) we even suppose \( M^+ = M^- = 0 \) identically then we do not obtain \( \alpha \equiv 0 \) with necessity. It illustrates the fact that at some choice
of signs $\epsilon_\pm$ we come to a non-trivial case despite the total masses are zero. Indeed, at $\dot{R} = \ddot{R} = 0$ we have $\alpha \neq 0$ if $\epsilon_+ = -\epsilon_-$, i.e., for wormhole shells (as for the ordinary hollow texture-shells, then always $\alpha_{st} = 0$, and thus they cannot have equilibrium states). Thus there exists the so-called zeroth traversable texture wormhole (ZTTW): one can see that junction remains to be possible at $\epsilon_+ = -\epsilon_-$ and $\alpha \neq 0$ (among the rest linear equations of state the texture’s one (21) appears to be unique in this sense). Therefore, ZTTW represents itself the specific vacuum-like topological barrier (characterized only by $\alpha$, see eq. (22) and comments after it) between two flat spacetimes which has no observable mass but possesses nontrivial internal structure and inertial external dynamics

$$\sqrt{1 + \dot{R}^2} = |\alpha|, \quad \ddot{R} \equiv 0,$$

thereby the restriction $\epsilon_- \alpha > 0$ should be satisfied as it can be readily seen from eqs. (19), (23), (34).

4 Minisuperspace quantization

Following the Wheeler-DeWitt’s approach, in quantum cosmology the whole Universe is considered quantum mechanically and is described by a wave function. The minisuperspace approach appears to be the direct application of Wheeler-DeWitt’s quantization procedure for (2+1)-dimensional singular hypersurfaces having own internal three-metric (see [10, 11, 12, 13] and references therein). So, let us consider the minisuperspace model described by the Lagrangian:

$$L = \frac{m\dot{R}^2}{2} - \alpha(1 - \alpha^2)R + \alpha M + \frac{M^2}{4\alpha R},$$

(35)

where $m$ was defined by (23). If we define

$$U = \alpha(1 - \alpha^2)R - \alpha M - \frac{M^2}{4\alpha R},$$

then the equation of motion following from this Lagrangian is

$$\frac{d}{d\tau}(m\dot{R}) = \frac{mR\dot{R}^2}{2} - U_R,$$

(36)

where subscript “$R$” means derivative with respect to $R$. Using time symmetry we can easy decrease order of this differential equation and obtain

$$\dot{R}^2 = \frac{2}{\alpha R}(H - U),$$

(37)

where $H$ is the integration constant. This equation coincides with double squared (19) at (23) when we suppose $H = 0$ as a constraint. Thus, our Lagrangian indeed describes dynamics of the thin-wall texture up to the topological wormhole/blackhole division which was described by the signs $\epsilon_\pm$. However, we always can restore the topology $\epsilon_\pm$ both at classical (rejecting redundant roots) and quantum (considering appropriate boundary conditions for the corresponding Wheeler-DeWitt equation, see below) levels.

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6It does mean also that in very general case the flat spacetime cannot be regarded as absolutely matter-free one even on the classical level: despite textures are the defects of quantum-field nature and (after)string origin, their condensate, texture matter, is macroscopic. The locally flat spacetime is globally determined at least up to the foliation by nested ZTTW’s walls.
Further, at $\Pi = m \dot{R}$ the (super)Hamiltonian is
\[ H = \Pi \dot{R} - L = H = 0. \tag{38} \]

The prefix “super” means that in general case $H$ has to be a functional defined on the superspace which is the space of all admissible metrics and accompanying fields. In spherically symmetric case the world sheet of a singular hypersurface is determined by a single function, proper radius $R(\tau)$.

To perform quantization we replace momentum by the operator $\hat{\Pi} = -i \partial / \partial R$ (we assume Planckian units), and (38) yields the Wheeler-DeWitt equation for the wave function $\Psi(R)$:
\[ \Psi_{RR} + \left[ M^2 + 4M\alpha^2 R - 4\alpha^2 (1 - \alpha^2) R^2 \right] \Psi = 0. \tag{39} \]

One can see the main advantage of the minisuperspace approach, namely, it does not require any time slicing on the basic manifold.

Further, the important remark should be made now. Last equation can be reduced to that for quantum harmonical oscillator, but not in all cases: the oscillator’s equation is defined on the line $(-\infty, +\infty)$ whereas in our case the extension of an application domain on the whole axis $R \in (-\infty, +\infty)$ seems to be physically ill-grounded in the major cases, therefore we should study the quantum theory on the half-line $[0, \infty)$. Strictly speaking, such a situation happens also in quantum field theory then the mathematical procedure known as the Langer modification had been used there [14]. The similar transformation we perform below to obtain the required solution.

In the case $R \in [0, +\infty)$ equation (39) in general cannot be resolved in terms of the parabolical cylinder functions and Hermite polynomials. Fortunately, a solution can be expressed in terms of the functions well-defined on the half-line $[0, +\infty)$. To show up this feature let us perform, at first, the following substitution
\[ x = R - b/2a \Rightarrow x \in [-b/2a, +\infty), \tag{40} \]
where $a = 4\alpha^2 (1 - \alpha^2)$, $b = 4\alpha^2 M$. Then (39) can be rewritten as
\[ \Psi_{xx} + \frac{\Delta - 4a^2 x^2}{4a} \Psi = 0, \tag{41} \]
where $\Delta = (4\alpha M)^2$. Further, considering (40) it can be seen that at $a > 0$ ($\alpha^2 < 1$) the substitution
\[ z = \sqrt{a} x^2 \tag{42} \]
has not to be the injective mapping and acts like the baker’s transformation [16] around $x = 0$ that provides the important property
\[ z \in [0, +\infty). \tag{43} \]
Then the transformation
\[ \Psi(x) = e^{-z/2} \sqrt{2\omega(z)} \tag{44} \]
rewrites (41) in a form of the confluent hypergeometric equation
\[ z\omega_{zz} + (c - z)\omega_z - a^2 \omega = 0, \tag{45} \]
whose solutions are the regular Kummer functions $M(a', c' ; z)$, where

$$a' = \frac{3}{4} - \frac{\Delta}{16a^{3/2}}, \quad c' = \frac{3}{2}.$$ 

Therefore, the true solutions of (39) at $R \in [0, +\infty)$ are the functions (up to multiplicative constant):

$$\Psi = e^{-z/2}\sqrt{z}M\left(\frac{3}{4} - \frac{\Delta}{16a^{3/2}} ; \frac{3}{2} ; z\right),$$

where

$$z = 2|\alpha|\sqrt{1-\alpha^2}\left[R - \frac{M}{2(1-\alpha^2)}\right]^2.$$ 

Further, if we wish to determine bound states we should require $\Psi(R = +\infty) = 0$; the also required condition $\Psi(R = 0) = 0$ (that corresponds, according to aforesaid, to $\Psi(z = 0) = 0$) has been already satisfied by the choice of the solution (46) when a one integration constant was used (the second constant always remains to normalize a solution) [12, 13]. In this case

$$a' = -n,$$ 

$n$ is a nonnegative integer, and the Kummer function moves to the Laguerre polynomials. From last expression we obtain the mass spectrum of the thin-wall ordinary texture in a bound state:

$$M_n = \pm \sqrt{2|\alpha|(4n + 3)(1-\alpha^2)^{3/4}},$$

which evidently has to be a subset of the oscillator’s spectrum, the Laguerre polynomials are connected with the Hermite ones through the transformation (42). Thus, our procedure has cut out from the oscillator’s eigenfunctions and eigenvalues those which satisfy with the boundary conditions on a half-line.

## 5 Conclusion

In present paper we considered texture matter and singular hypersurfaces made from it. First of all, we studied thermodynamical properties of 2D and 3D texture matter in comparison with radiation fluid and bubble matter. These properties say that textures can be imagined as the specific vacuum state having the congeniality with the already known de Sitter vacuum and duality with radiation fluid [17]. We obtained equations of motion of selfgravitating texture objects, showed that classical motion can be elliptical (finite), parabolical or hyperbolical, thereby permitted and forbidden regions of motion was determined. We showed up that neutral textures in equilibrium have zero total gravitational mass as was expected. Moreover, it was established that there can exist the nontrivial wormhole-textures having vanishing total mass and matching two flat spacetimes. Finally, we considered quantum aspects of the theory by means of Wheeler-DeWitt’s minisuperspace quantization procedure, obtained the exact wave function and spectrum of bound states.

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Figure 1: Forbidden and permitted (dashed) regions of thin-wall texture motion. Permitted parabolical trajectories are the vertical half-lines $y \geq -/4$ at $\alpha = \pm 1$; curve $a$ is $y = y_+$, $b$ is $y = y_-$. 