THE NORM RESIDUE SYMBOL FOR FORMAL DRINFELD MODULES

MARWA ALA EDDINE

Abstract. In this paper, we study the Kummer pairing associated with formal Drinfeld modules having stable reduction of height one. We give an explicit description of the pairing à la Kolyvagin, in terms of the logarithm of the formal Drinfeld module, a certain derivation, torsion points and the trace. The results obtained give a generalization of the results of Anglès [1] proved for Carlitz modules, and of Bars and Longhi [2] proved for sign-normalized rank one Drinfeld modules. It also presents an extension of our previous formulas proved in [3] to arbitrary finite extensions of local fields containing enough torsion points.

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1. Introduction

Let $K$ be a local field, $p$ be its characteristic, and let $\mu_K$ be its normalized discrete valuation. We denote $\mathcal{O}_K$ the valuation ring of $K$ and $p_K$ its maximal ideal. Let $q$ be the order of the residue field $\mathcal{O}_K/p_K$. Then $q$ is a power of $p$. Fix an algebraic closure $\Omega$ of $K$, and let $\mu$ be the unique extension of $\mu_K$ to $\Omega$. All the extensions $F$ of $K$ considered in this paper are supposed to be such that $F \subset \Omega$. We also denote $\pi_F$ a uniformizer of $F$, $\mathcal{O}_F$ the valuation ring of $F$ and $p_F$ its maximal ideal. Let $K_{ur} \subset \Omega$ be the maximal unramified extension of $K$ in $\Omega$, and $H \subset K_{ur}$ be a finite unramified extension of $K$.

Let

$$\rho : \mathcal{O}_K \longrightarrow \mathcal{O}_H\{\{\tau\}\}$$

$$a \mapsto \rho(a)$$

be a formal Drinfeld module having stable reduction of height one, as defined by Rosen in [4, §1]. Here, $\tau$ is the $q$-Frobenius element satisfying

$$\tau x = x^q \tau, \quad \forall x \in \Omega.$$  

(1.1)

The completion $\tilde{\Omega}$ of $\Omega$ is an $\mathcal{O}_K$-module for the following action of $\rho$

$$a \cdot_\rho x = \rho(a)(x) \quad \forall x \in \tilde{\Omega}.$$  

(1.2)

For an integer $n \geq 0$, let

$$V^n_\rho = \{\alpha \in \Omega; \rho(\alpha)(0) = 0 \quad \forall a \in p_K^n\}$$

be the $p_K^n$ torsion submodule of $\tilde{\Omega}$ for the action (1.2). It is isomorphic as an $\mathcal{O}_K$-module to $\mathcal{O}_K/p_K^n$. Any element $v_0 \in V^n_\rho \setminus V^{n-1}_\rho$ is therefore a generator of $V^n_\rho$ and the extension $H^n_\rho = H(V^n_\rho)$ is equal to $H(v_0)$. For more details see [5, 6].

E-mail: marwa.alaeddine@univ-fcomte.fr
Now let $m_0 \geq 1$ be an integer dividing $[H : K]$, and $\eta \in K$ of valuation $\mu(\eta) = m_0$. Let
\[ W^n_\rho = V^{n m_0}_\rho = \{ \alpha \in \mathfrak{p}_\Omega; \rho^{nm}(\alpha) = 0 \}, \quad \text{and} \quad W^n_\rho = \bigcup_n V^n_\rho = \bigcup_n W^n_\rho. \]

Fix once and for all a generator $v_n$ of $W^n_\rho$. Let
\[ E^n_\rho = H(W^n_\rho) = H^{nm_0}. \]

Let $\mathcal{O}_n$ be the valuation ring of $E^n_\rho$ and $p_n$ be its maximal ideal. If $L$ is a finite extension of $E^n_\rho$, then we denote by
\[ \Phi_L : L^\times \to \text{Gal}(L^{ab}|L) \]
the norm residue map. By [3, Lemma 2.1], for each $\alpha \in \mathfrak{p}_L$, there exists $\xi \in L^{ab}$ such that $\rho^{nm}(\xi) = \alpha$. Therefore we can define the map $(\cdot, \cdot)_{\rho, L, n} : \mathfrak{p}_L \times L^\times \to W^n_\rho$ such that
\[ (\alpha, \beta)_{\rho, L, n} = \Phi_L(\beta)(\xi) - \xi; \quad \rho^{nm}(\xi) = \alpha, \]
for $\alpha \in \mathfrak{p}_L$ and $\beta \in L^\times$. We omit $\rho$ in the index when there is no risk of confusion.

The main result in this paper is the following (cf. Theorem 3.8 for the precise formulation).

**Theorem 1.1.** Suppose that $L|K$ is a separable extension, then there exists an $\mathcal{O}_K$-derivation $\overline{D}_{L,v_n}$ from $\mathcal{O}_L$ into a certain $\mathcal{O}_L$-submodule $W$ of $L$ such that
\[ (\alpha, \beta)_{L,n} = T_{L|K}(\lambda_\rho(\alpha) \text{dlog} \overline{D}_{L,v_n}(\beta)) \cdot \rho \cdot v_n \]
for all $\beta \in L^\times$ and $\alpha \in L$ of valuation $\mu(\alpha) > \frac{nm_0}{q} + \frac{1}{q-1} + \frac{1}{e(L|K)}$. Here, $\lambda_\rho$ is the logarithm of $\rho$ and $e(L|K)$ is the ramification index of $L|K$. For $\beta = u\pi_L^k \in L^\times$, the logarithmic derivative $\text{dlog} \overline{D}_{L,v_n}$ associated with the derivation $\overline{D}_{L,v_n}$ is defined as follows
\[ \text{dlog} \overline{D}_{L,v_n}(\beta) = \frac{\overline{D}_{L,v_n}(u)}{u} + k \frac{\overline{D}_{L,v_n}(\pi_L)}{\pi_L}. \]

An advantage of having a derivation is that it is determined and explicitly constructible in terms of its value at a uniformizer $\pi_L$ of $L$ as follows. For $x \in \mathcal{O}_L$, we can write
\[ \overline{D}_{L,v_n}(x) = f'(\pi_L) \overline{D}_{L,v_n}(\pi_L), \]
where $f$ is the unique power series in $\mathbb{F}_q[x]$ such that $x = f(\pi_L)$. Here, $\mathbb{F}_q$ denotes the residue field of $L$. In the particular case where $L = E^n_\rho$ and $\pi_L = v_m$, our previous work in [3] implies the subsequent proposition.

**Proposition 1.2.** (Proposition 3.13) Suppose $\rho$ is such that $\rho_\eta \equiv \tau^{m_0} \mod \mathfrak{p}_H$. For all $m \geq n$, we have
\[ \overline{D}_{E^n_\rho, v_n}(v_m) = \frac{1}{\eta^m}. \]

Finally, using invariants attached to the representation $r : \text{Gal}(\Omega|H) \to \text{GL}_1(\mathcal{O}_K) = U_K$, which is induced by the action of $\text{Gal}(\Omega|H)$ on the module $\lim \mathcal{O}_L$, we get the following congruence, of which we do not have a direct proof.

**Proposition 1.3.** (Corollary 3.14) Suppose $\rho$ is such that $\rho_\eta \equiv \tau^{m_0} \mod \mathfrak{p}_H$ and let $L = E^n_\rho$ for $m \geq n$. Let $u$ be a unit of $L$ such that $\mu(1-u) > \max\{\frac{nm_0}{q}, \frac{1}{q-1}\} + \frac{1}{q-1}$. Then
\[ N_{L|K}(u^{-1}) - 1 \equiv T_{L|K}((\frac{1-u}{u})(1 - \frac{g'(v_m)}{g(v_m)})) \mod \mathfrak{p}_K^{(n+m)m_0}, \]
where $g(X) \in \mathbb{F}_q[x]$ is such that $g(v_m) = u$. 


The method used to obtain (1.4) was inspired by the work of Kolyvagin [7], in which he proved explicit formulas for the Kummer pairing in the case of formal groups of finite height in zero characteristic. The results of Kolyvagin extended those of Iwasawa [8] and Wiles [9], who proved explicit laws for the Kummer pairing associated to the multiplicative group and to general Lubin-Tate formal groups respectively. The results obtained here extend those of Anglès [1] proved for Carlitz module, and of Bars and Longhi [2] proved for sign-normalized rank one Drinfeld modules. In his turn, Florez [10, 11] followed Kolyvagin’s method to generalize the latter’s work and prove explicit laws in the case of formal groups and Lubin-Tate formal groups defined over arbitrary higher local field of mixed characteristic. Whence, one may ask if we can generalize our results as well to local fields of higher dimension.

2. Properties of the Kummer pairing

In this section, we state some of the main properties of the pairing $\langle , \rangle_{L,n}$. Throughout this section, fix a positive integer $n$ and a finite extension $L$ of $E_p^n$.

**Proposition 2.1.** The map $\langle , \rangle_{L,n}$ satisfies the following properties

(i) The map $\langle , \rangle_{L,n}$ is bilinear and $O$-linear in the first coordinate for the action (1.2).

(ii) We have $\langle \lambda, \beta \rangle_{L,n} = 0 \iff \beta$ is a norm from $L(\xi)$, where $\rho_{q^n}(\xi) = \alpha$.

(iii) Let $M$ be a finite separable extension of $L$, let $\alpha \in p_L$ and $\beta \in M^x$. Then $\langle \alpha, \beta \rangle_{M,n} = (\alpha, N_{M/L}(\beta))_{L,n}$.

(iv) Let $M$ be a finite separable extension of $L$ of degree $d$, let $\alpha \in p_M$ and $\beta \in L^x$. Then $\langle \alpha, \beta \rangle_{M,n} = (\tau_{M/L}(\alpha), \beta)_{L,n}$.

(v) Suppose $L \supset E^n_p$ for $m \geq n$. Then $\langle \alpha, \beta \rangle_{L,n} = \rho_{q^{m-n}}((\alpha, \beta)_{L,m}) = \langle \rho_{q^{m-n}}(\alpha), \beta \rangle_{L,m}$.

(vi) Let $\rho'$ be a formal Drinfeld $O$-module isomorphic to $\rho$, i.e. there exists a power series $t$ invertible in $O_H\{\{\tau\}\}$ such that $\rho'_a = t^{-1} \circ \rho_a \circ t$ for all $a \in O$. Then we have $\langle \alpha, \beta \rangle_{\rho', L,n} = t^{-1}(\langle t(\alpha), \beta \rangle_{\rho, L,n})$.

We omit the proof of these properties. The interested reader may find a detailed proof in [7, §3.3]. As mentioned in the introduction, there exists explicit formulas for the pairing $\langle , \rangle_{L,n}$ which include the logarithm of the Drinfeld module $\rho$. This so-called logarithm was defined by Rosen in [4, §2] as follows.

**Lemma 2.2.** There exists a unique power series $\lambda_\rho \in H\{\{\tau\}\}$, called the logarithm of $\rho$, such that $\lambda_\rho(X) \equiv X \mod \deg 2$ and $\lambda_\rho \rho_a = a \lambda_\rho$ for all $a \in O$. Moreover, we have

(i) If $\lambda_\rho = \sum_{i \geq 0} c_i \tau^i$, then $\mu(c_i) \geq -i$ for all $i \geq 0$. Thus the element $\lambda_\rho(x) = \sum_{i \geq 0} c_i x^i$ is well defined in $L$ for any $x \in p_L$.

(ii) If $x \in p_{\Omega}$, then $\lambda_\rho(X) = 0$ if and only if $x \in V_{\rho}$. Put $W_L = L \cap W_\rho \subset p_L$. Then the map $\lambda_\rho : p_L/W_L \to \lambda_\rho(p_L)$ is an isomorphism of $O$-modules.

(iii) Let $p_{\Omega,1}$ denote the set of all the elements $x$ of $p_\Omega$ such that $\mu(x) > 1/(q-1)$. The logarithm $\lambda_{\rho}$ gives an isomorphism of $O$-modules from $p_{\Omega,1}$, viewed as an $O$-module under the action (1.2), to itself, viewed as an $O$-module under the multiplication in $\Omega$. In particular, if we denote $p_{L,1} = p_L \cap p_{\Omega,1}$, the logarithm $\lambda_{\rho}$ also induces an isomorphism from the ideal $p_{L,1}$ to itself. This follows from the fact that $\mu(\lambda_{\rho}(x)) = \mu(x)$ for all $x \in p_{\Omega,1}$.
Inspired by [7], we proved in [3, §3] the following explicit formula for $(\cdot, \cdot)_{L,n}$. We denote by $X_{L,1} \subset L$ the fractional ideal of all elements $y$ such that $T_{L|K}(xy) \in \mathcal{O}_K$ for all $x \in \lambda_{\rho}(p_{L,1})$. We have

$$X_{L,1} = \{ y \in L; \ T_{L|K}(\lambda_{\rho}(\alpha) y) \in \mathcal{O}_K \ \forall \alpha \in p_{L,1} \}$$

(2.1)

$$= \{ y \in L; \ T_{L|K}(\alpha' y) \in \mathcal{O}_K \ \forall \alpha' \in p_{L,1} \}$$

$$= \{ y \in L; \ \mu(y) \geq -\frac{1}{q-1} - \frac{1}{e(L|K)} - \mu(D_{L|K}) \}. $$

Proposition 2.3. Suppose that the extension $L|K$ is separable. Then, there exists a unique map $\Psi_{L,v_n} : \mathcal{X} \to \mathcal{X}_{L,1}/\eta^n\mathcal{X}_{L,1}$ such that

$$\tag{2.2} (\alpha, \beta)_{L,n} = T_{L|K}(\lambda_{\rho}(\alpha)\Psi_{L,v_n}(\beta)) \cdot \rho v_n$$

for all $\alpha \in p_{L,1}$ and $\beta \in L^\times$. Furthermore, $\Psi_{L,v_n}$ is a continuous group homomorphism.

Remark 2.4. (i) In (2.2), we view $\Psi_{L,v_n}(\beta)$ as an element of $X_{L,1}$. It is easy to see that for any $\alpha \in p_{L,1}$, the value of $T_{L|K}(\lambda_{\rho}(\alpha)\Psi_{L,v_n}(\beta)) \cdot \rho v_n$ does not depend on the choice of the representative of the class of $\Psi_{L,v_n}(\beta)$ in $X_{L,1}/\eta^nX_{L,1}$.

(ii) Let $v_n'$ be another generator of $\mathcal{W}^n\rho$, then $v_n' = \rho u(v_n)$ for a unit $u$ of $K$. We have

$$\tag{2.3} \Psi_{L,v_n} = u\Psi_{L,v_n'}.$$

Exactly as in [7, §3.5], our $\Psi_{L,v_n}$ satisfies the properties $\varphi_1$, $\varphi_2$, $\varphi_3$, $\varphi_4$, $\varphi_5$ and $\varphi_6$ of loc. cit. The equality (2.2) gives an expression of the pairing $(\cdot, \cdot)_{L,n}$ in terms of the trace of $L|K$, the logarithm of $\rho$, and the map $\Psi_{L,v_n}$. However, we do not have an explicit expression of $\Psi_{L,v_n}$. Therefore, we will use $\Psi_{L,v_n}$ to construct a derivation $\bar{D}_{L,v_n}$ (see §3.2 below), which will help us prove an explicit formula for $(\cdot, \cdot)_{L,n}$. In fact, we will see that $\bar{D}_{L,v_n}$ is determined by its value at a prime $\pi_L$, which is, by its turn, determined by invariants from representation theory.

Proposition 2.5. There exists a unique power series $r = r_n \in \mathcal{O}_H\{\{\tau\}\}$ such that

$$\prod_{\omega \in \mathcal{W}^n\rho} (X - \omega) = r \circ \rho_\eta^\rho(X).$$

Furthermore, the power series $r$ is invertible in $\mathcal{O}_H\{\{\tau\}\}$ and satisfies

$$(x, r(x))_{L,n} = 0, \ \forall x \in \mathcal{P}_L.$$

Proof. See [3, Proposition 4.2].

Lemma 2.6. Let $r = r_n$ be the power series defined in Proposition 2.5. Let $\rho'$ be defined by

$$\tag{2.4} \rho'_a = r \circ \rho_a \circ r^{-1}$$

for all $a \in \mathcal{O}$. Then $\rho'$ is a formal Drinfeld module having a stable reduction of height 1, and we have

$$\tag{2.5} (x, x)_{\rho', L,n} = 0 \ \text{for all } x \in \mathcal{P}_L.$$

Proof. See [3, Lemma 4.3].

As we will see in the sequel, it will be easier to deal with formal Drinfeld modules satisfying the property (2.5). Lemma 2.6 will ensure that, starting any formal Drinfeld module having a stable reduction of height 1, we will be able to reach, by isomorphism, a formal Drinfeld module having a stable reduction of height 1, satisfying (2.5).
3. Derivations

3.1. Recall on Derivations. In this paragraph, we give a brief recall on derivations and their main properties that will be useful for us in the sequel. Let \( R \) be a commutative ring with unit, and \( O \) be a subring of \( R \). If \( W \) is an \( R \)-module, a map \( D : R \to W \) is said to be an \( O \)-derivation of \( R \) into \( W \) if it is \( O \)-linear and satisfies the Leibniz rule

\[
D(xy) = xD(y) + yD(x) \quad \forall x, y \in R.
\]

In particular, a derivation \( D : R \to W \) also fulfills the following:

(i) \( D(x + y) = D(x) + D(y) \quad \forall x, y \in R, \)

(ii) \( D(a) = 0 \quad \forall a \in O. \)

The set of all such derivations \( D_O(R, W) \) is an \( R \)-module, where \( aD \) is defined by \( (aD)(x) = aD(x) \) for all \( a, x \in R \). We will show that there exists a universal derivation, in other words, an \( R \)-module \( \Omega_O(R) \), and a derivation

\[
d : R \to \Omega_O(R)
\]

such that for every derivation \( D : R \to W \), there exists a unique homomorphism of \( R \)-modules \( f : \Omega_O(R) \to W \) such that the diagram

\[
\begin{array}{ccc}
R & \xrightarrow{d} & \Omega_O(R) \\
\downarrow D & & \downarrow f \\
W & \xrightarrow{\exists ! f} & W
\end{array}
\]

commutes. Let \( \mathcal{R} \) be the direct sum of the modules \( (R)_{x \in R}. \) Then \( \mathcal{R} \) is the submodule of \( \prod_{x \in R} R \) which consists of families \( (a_x)_{x \in R} \) having finite support. For each element \( x \in R \), we associate a symbol \( d x \), so that an element \( (a_x)_{x \in R} \) in \( \mathcal{R} \) can be written as a finite sum \( \sum_{x \in R} a_x d x \). Here, the symbols \( d x \) are supposed to be distinct for distinct elements of \( R \). Consider the submodule of \( \mathcal{R} \) generated by the set

\[
\{ d(xy) - y d x - x d y, \ d(x + y) - d x - d y, \ d a; \ x, y \in R, a \in O \}.
\]

The quotient of \( \mathcal{R} \) by this submodule, which we denote by \( \Omega_O(R) \), together with the derivation \( d : \mathcal{R} \to \Omega_O(R) \) that sends \( x \) to the class of \( d x \) in \( \Omega_O(R) \), form the universal derivation we are looking for. Indeed, let \( W \) be an \( R \)-module and \( D : R \to W \) be a derivation, and consider the unique homomorphism of \( R \)-modules from \( \mathcal{R} \) to \( W \) that maps \( d a \) to \( D(a) \). This homomorphism is trivial on the submodule of \( \mathcal{R} \) generated by the set (3.3), thus it factors through \( \Omega_O(R) \), whence the universal property. We call \( \Omega_O(R) \) the module of differentials of \( R \) over \( O \).

The universal derivation yields an isomorphism of \( R \)-modules

\[
D_O(R, W) \simeq \text{Hom}_R(\Omega_O(R), W).
\]

Let \( M \) be a local field and \( N \) be a finite separable extension of \( M \). We denote by \( \mathcal{D}(N|M) \) the different of \( N|M \). In the special case where \( R = O_N \) and \( O = O_M \), we have the following results.

**Proposition 3.1.** There exists an isomorphism of \( O_N \)-modules

\[
\Omega_{O_M}(O_N) \simeq O_N/\mathcal{D}(N|M).
\]

Furthermore, if \( \pi_N \) is a prime of \( N \), then \( d\pi_N \) is a generator of \( \Omega_{O_M}(O_N) \).

**Proof.** Kolyvagin proved this proposition in the case of zero characteristic [7, Proposition 5.1]. His proof is suitable for our case. \( \square \)
**Corollary 3.2.** Let $W$ be an $\mathcal{O}_N$-module and $\pi_N$ be a prime of $N$. Let
\[(3.6) \quad S := \{x \in W, \ ax = 0 \ \forall a \in D(N|M)\}\]
be the $D(N|M)$-torsion submodule of $W$. Then, the map
\[(3.7) \quad D_{\mathcal{O}_M}(\mathcal{O}_N, W) \to S \quad D \mapsto D(\pi_N)\]
is an isomorphism of $\mathcal{O}_N$-modules.

**Proof.** The proof of [7, Corollary 5.2] is convenient for our case as well. \hfill \square

**Remark 3.3.** With the notations of Corollary 3.2, the inverse homomorphism of (3.7) associates to an element $x \in S$ a derivation $D_x$ satisfying
\[(3.8) \quad D_x(t(\pi_N)) = t'(\pi_N)x\]
for all $t \in \mathcal{O}_N[[X]]$, where $\tilde{N}$ is the inertia field of $N|M$. This follows from the fact that a derivation in $D_{\mathcal{O}_M}(\mathcal{O}_N, W)$ is continuous for the discrete topology on $W$.

### 3.2. The derivation $\overline{D}_{L,v_n}$

In this section, we assume that $L|K$ is a separable extension. We define the map $D_{L,v_n} : \mathcal{O}_L \to \mathfrak{X}_{L,1}/\eta^P\mathfrak{X}_{L,1}$ by $D_{L,v_n}(0) = 0$ and $D_{L,v_n}(\alpha) = \alpha\Psi_{L,v_n}(\alpha)$ for $\alpha \in \mathcal{O}_L \setminus \{0\}$, where $\Psi_{L,v_n}$ is the homomorphism defined in (2.3). In this section we will prove that $D_{L,v_n}$, reduced modulo a convenient submodule of $\mathfrak{X}_{L,1}$, is a derivation, and it satisfies (1.4).

It is clear that the map $D_{L,v_n}$ satisfies the Leibniz rule
\[(3.9) \quad D_{L,v_n}(xy) = xD_{L,v_n}(y) + yD_{L,v_n}(x) \quad \forall x, y \in \mathcal{O}_L.\]
This follows from the fact that $\Psi_{L,v_n}$ is a group homomorphism. Using this rule, we can prove by induction that
\[(3.10) \quad D_{L,v_n}(x^m) = mx^{m-1}D_{L,v_n}(x) \quad \forall x \in \mathcal{O}_L \text{ and } \forall m \geq 1.\]
We will now prove that $D_{L,v_n}$ is additive.

**Lemma 3.4.** Suppose $\rho$ is such that $(x, x)_{\rho, L,n} = 0$ for all $x \in p_L$. Let $\alpha \in p_L \setminus \{0\}$ and let $u$ be a unit of $L$ such that $\mu(\alpha(1-u)) > \frac{nm}{q} + \frac{1}{q-1}$. We have
\[(3.11) \quad (\alpha u, u)_{L,n} = T_{L|K}((1-u)D_{L,v_n}(\alpha)) \cdot \rho v_n.\]

**Proof.** We have
\[(\alpha u, u)_{L,n} = (\alpha u, \frac{\alpha u}{\alpha})_{L,n} = (\alpha u, \alpha u)_{L,n} - (\alpha u, \alpha)_{L,n} = (\alpha, \alpha)_{L,n} - (\alpha u, \alpha)_{L,n} = (\alpha - \alpha u, \alpha)_{L,n} = T_{L|K}(\lambda_{\rho}(\alpha - \alpha u)\Psi_{L,v_n}(\alpha)) \cdot \rho v_n\]
by Proposition 2.3. Let $\gamma = \alpha(1-u)$, we will show that
\[(3.12) \quad T_{L|K}(\lambda_{\rho}(\gamma)\Psi_{L,v_n}(\alpha)) \cdot \rho v_n = T_{L|K}(\gamma\Psi_{L,v_n}(\alpha)) \cdot \rho v_n.\]
By the hypothesis on the valuations, we have \( \mu(\gamma) > \frac{nm_0}{q} + \frac{1}{q-1} \). Hence
\[
\mu(\theta(\gamma) - \gamma) = \mu\left(\sum_{i \geq 1} c_i \gamma^i\right) \\
\geq \min_{i \geq 1}\left\{\mu(c_i) + q^i \mu(\gamma)\right\} \\
> \min_{i \geq 1}\left\{\frac{-i + q^i (nm_0)}{q} + \frac{1}{q-1}\right\} \\
\geq nm_0 + \frac{1}{q-1}
\]
Therefore, we can write \( \theta(\gamma) - \gamma = \eta^n \delta \), where \( \delta \) is an element of \( p_{L,1} \). Thus, by (2.1),
\[
T_{L|K}\left( (\theta(\gamma) - \gamma) \Psi_{L,v_n}(\alpha) \right) \cdot \rho v_n = 0
\]
because \( \Psi_{L,v_n}(\alpha) \in \mathfrak{X}_{L,1} \). This concludes the proof. \( \square \)

**Proposition 3.5.** Suppose \( \rho \) is such that \((x, x)_\rho, L, n = 0\) for all \( x \in p_L \). Let \( \gamma \) be an element of \( \mathcal{O}_L \setminus \{0\} \) of valuation \( \mu(\gamma) = \max\left\{\frac{nm_0}{q} - \frac{1}{q-1}\right\} \), that is \( \mu(\gamma) = \frac{nm_0}{q} \) if \( nm_0 \geq 2 \), and \( \mu(\gamma) = \frac{1}{q-1} \) if \( nm_0 = 1 \). Then
\[
(D_{L,v_n}(x + y) \equiv D_{L,v_n}(x) + D_{L,v_n}(y) \quad \text{mod} \quad \frac{\eta^n}{\gamma} \mathfrak{X}_{L,1})
\]
for all \( x, y \in \mathcal{O}_L \).

**Proof.** Let us prove first why such a \( \gamma \) exists. Since \( E^n_\rho \subset L \), the ramification index of \( L|K \) is a multiple of the ramification index of \( E^n_\rho|K \), which is equal to \( q^{nm_0-1}(q-1) \). Hence, there exists elements in \( L \) of valuation \( \frac{qnm_0-1}{q-1} \), whence the existence of \( \gamma \). Now let us prove (3.13). Let \( x, y \in \mathcal{O}_L \), then, by Lemma 3.4, we have
\[
(\gamma(x + y), u)_{L,n} = T_{L|K}\left( (1-u) D_{L,v_n}(\gamma(x + y)) \right) \cdot \rho v_n \\
\]
(3.14) for all \( u \in 1 + p_{L,1} \). However, again by Lemma 3.4, we have
\[
(\gamma(x + y), u)_{L,n} = (\gamma xu, u)_{L,n} + (\gamma yu, u)_{L,n} \\
\]
(3.15) for all \( u \in 1 + p_{L,1} \). Therefore, (3.14) and (3.15) being equal, we conclude that
\[
(\gamma D_{L,v_n}(x + y) \equiv \gamma(D_{L,v_n}(x) + D_{L,v_n}(y)) \quad \text{mod} \quad \frac{\eta^n}{\gamma} \mathfrak{X}_{L,1})
\]
by the very definition (2.1) of \( \mathfrak{X}_{L,1} \). Hence, we have
\[
(D_{L,v_n}(x + y) \equiv D_{L,v_n}(x) + D_{L,v_n}(y) \quad \text{mod} \quad \frac{\eta^n}{\gamma} \mathfrak{X}_{L,1})
\]
\( \square \)

**Corollary 3.6.** Let \( \gamma \) be as in Proposition 3.5. Then
\[
(D_{L,v_n}(x + y) \equiv D_{L,v_n}(x) + D_{L,v_n}(y) \quad \text{mod} \quad \frac{\eta^n}{\gamma} \mathfrak{X}_{L,1})
\]
for all \( x, y \in \mathcal{O}_L \).
Proof. Let \( r \) be the series defined in Proposition 2.5 and let \( \rho' \) the Drinfeld module defined by

\[
\rho'_a = r \circ \rho_a \circ r^{-1}.
\]

Then \( r \) defines an isomorphism of \( \mathcal{O}_K \)-modules \( r: W^n_{\rho} \to W^n_{\rho'} \). Furthermore, if we denote by \( D_{\rho,L,v_n} \) (respectively \( \rho' \)) the map defined in the beginning of §3.2 associated to \( \rho \) (respectively \( \rho' \)), we have

\[
D_{\rho,L,v_n}(x) = r'(0) D_{\rho',L,v_n}(x).
\]

(3.19)

Here, \( r'(0) \) is a unit in \( H \) because \( \bar{D}_{\rho,L,v_n}(x) \in \mathcal{O}_H[[X]] \) is invertible. Since \( (x,x)_{\rho',L,v}=0 \) for all \( x \in p_L \), by Lemma 2.6, we can apply Proposition 3.5 for \( \rho' \) so that

\[
D_{\rho',L,v_n}(x+y) \equiv D_{\rho',L,v_n}(x) + D_{\rho',L,v_n}(y) \mod \frac{\eta^n}{\gamma} \bar{x}_{L,1}
\]

(3.20)

for all \( x, y \in \mathcal{O}_L \). Thus, using (3.19), we conclude that

\[
D_{\rho,L,v_n}(x+y) \equiv D_{\rho,L,v_n}(x) + D_{\rho,L,v_n}(y) \mod \frac{\eta^n}{\gamma} \bar{x}_{L,1}
\]

(3.21)

for all \( x, y \in \mathcal{O}_L \).

\[\square\]

Proposition 3.7. Let

\[
\bar{x}_{L,1}^{(n)} = \{ y \in L; \mu(y) \geq nm_0 - \max\left\{ \frac{nm_0}{q}, \frac{1}{q-1} \right\} - \frac{1}{q-1} - \frac{1}{e(L|K)} - \mu(D(L|K)) \} \subset \bar{x}_{L,1}.
\]

The reduction of \( D_{L,v_n} \) modulo \( \bar{x}_{L,1}^{(n)} \), denoted by \( \bar{D}_{L,v_n}: \mathcal{O}_L \to \bar{x}_{L,1}/\bar{x}_{L,1}^{(n)} \), is an \( \mathcal{O}_K \)-derivation.

Proof. Let \( \gamma \in \mathcal{O}_L \setminus \{0\} \) be as in Proposition 3.5, then

\[
\bar{x}_{L,1}^{(n)} = \frac{\eta^n}{\gamma} \bar{x}_{L,1}.
\]

(3.23)

Let \( \pi_L \) be a prime of \( L \) and let \( w = \bar{D}_{L,v_n}(\pi_L) \in \bar{x}_{L,1}/\bar{x}_{L,1}^{(n)} \). Since \( \mu(\bar{x}_{L,1}^{(n)}) = nm_0 - \mu(\gamma) \leq nm_0 - \frac{1}{q-1} \leq \mu(D(L|K)) \), we have \( D(L|K)w = 0 \). Hence, by Corollary 3.2, there exists a derivation \( D: \mathcal{O}_L \to \bar{x}_{L,1}/\bar{x}_{L,1}^{(n)} \) such that \( D(\pi_L) = w \) and

\[
D(g(\pi_L)) = g'(\pi_L)w
\]

(3.24)

for every power series \( g \in \mathcal{O}_L[[X]] \), where \( L \) is the maximal subextension of \( L \) unramified over \( K \). In particular, (3.24) is true for all the power series defined over the residue field of \( L \), which is equal to the residue field of \( L \). We will prove that \( D \) and \( \bar{D}_{L,v_n} \) are equal. Indeed, let \( x \in \mathcal{O}_L \), and let \( g(\pi_L) = \sum_{i \geq 0} a_i X^i \) be the unique power series defined over the residue field \( \mathbb{F}_L \) of \( L \) such that \( g(\pi_L) = x \). We have

\[
\bar{D}_{L,v_n}(x) = \bar{D}_{L,v_n}(g(\pi_L)) = \sum_{i \geq 0} \bar{D}_{L,v_n}(a_i \pi_L^i)
\]

(3.25)

because \( \bar{D}_{L,v_n} \) is additive by Proposition 3.6, and continuous by Proposition 2.3. Let \( q_L \) be the cardinal of \( \mathbb{F}_{q_L} \), then \( q_L \) is a power of \( p \). Hence, for all \( i \geq 0 \), we have

\[
\bar{D}_{L,v_n}(a_i) = \bar{D}_{L,v_n}(a_i^{q_L}) = 0
\]

by (3.10). Therefore, applying the Leibniz rule (3.9) to (3.25), we get

\[
\bar{D}_{L,v_n}(x) = \sum_{i \geq 0} a_i \bar{D}_{L,v_n}(\pi_L^i) = \sum_{i \geq 0} a_i \times i \times \pi_L^{i-1} \times \bar{D}_{L,v_n}(\pi_L)
\]

(3.26)

again by (3.10). However, this is equal to \( g'(\pi_L)\bar{D}_{L,v_n}(\pi_L) \), which is, by (3.24), equal to \( D(x) \) □
Now, we will define the logarithmic derivative $d\log \bar{D}_{L,v_n}$ associated to the derivation $\bar{D}_{L,v_n}$ as follows. Let

$$f : \mathcal{X}_{L,1}/\mathcal{X}_{L,1}^{(n)} \to \pi_L^{-1}\mathcal{X}_{L,1}/\pi_L^{-1}\mathcal{X}_{L,1}^{(n)}$$

be the natural map induced by the inclusion $\mathcal{X}_{L,1} \hookrightarrow \pi_L^{-1}\mathcal{X}_{L,1}$, and

$$g_{\pi_L} : \mathcal{X}_{L,1}/\mathcal{X}_{L,1}^{(n)} \to \pi_L^{-1}\mathcal{X}_{L,1}/\pi_L^{-1}\mathcal{X}_{L,1}^{(n)}$$

be the multiplication by $\pi_L^{-1}$ map. For $x = u\pi_L^{k} \in L^\times$, where $u$ is a unit in $L$, we define

$$d\log \bar{D}_{L,v_n}(x) = f(u^{-1}\bar{D}_{L,v_n}(u)) + kg_{\pi_L}(\bar{D}_{L,v_n}(\pi_L)).$$

The map $d\log \bar{D}_{L,v_n} : L^\times \to \pi_L^{-1}\mathcal{X}_{L,1}/\pi_L^{-1}\mathcal{X}_{L,1}^{(n)}$ is a group homomorphism. Furthermore, its definition does not depend on the choice of the uniformizer $\pi_L$. Indeed, let $\pi'_L$ be another uniformizer of $L$ and let $x = u\pi_L^{k} = u'\pi'_L^{k} \in L^\times$, where $u$ and $u'$ are units of $L$. Let $u_0$ be the unit of $L$ such that $\pi'_L = u_0\pi_L$. Then,

$$f(u^{-1}\bar{D}_{L,v_n}(u)) + kg_{\pi_L}(\bar{D}_{L,v_n}(\pi_L)) = f(u'^{-1}u_0^{-k}\bar{D}_{L,v_n}(u'u_0^k)) + kg_{\pi_L}(\bar{D}_{L,v_n}(\pi_L))$$

$$= f(u'^{-1}\bar{D}_{L,v_n}(u') + u_0^{-k}\bar{D}_{L,v_n}(u_0^k)) + kg_{\pi_L}(\bar{D}_{L,v_n}(\pi_L))$$

$$= f(u'^{-1}\bar{D}_{L,v_n}(u')) + f(u_0^{-k}\bar{D}_{L,v_n}(u_0^k)) + kg_{\pi_L}(\bar{D}_{L,v_n}(\pi_L))$$

$$= f(u'^{-1}\bar{D}_{L,v_n}(u')) + k\bar{D}_{L,v_n}(u_0^{-1}) + kg_{\pi_L}(\bar{D}_{L,v_n}(\pi_L)).$$

(3.30)

On the other hand, we have

$$g_{\pi'_L}(\bar{D}_{L,v_n}(\pi_L)) = g_{\pi'_L}(\bar{D}_{L,v_n}(u_0\pi_L))$$

$$= g_{\pi'_L}(u_0\bar{D}_{L,v_n}(\pi_L) + \pi_L\bar{D}_{L,v_n}(u_0))$$

$$\equiv \pi'_L^{-1}(u_0\bar{D}_{L,v_n}(\pi_L) + \pi_L\bar{D}_{L,v_n}(u_0)) \mod \pi_L^{-1}\mathcal{X}_{L,1}^{(n)}$$

$$\equiv u_0^{-1}\pi_L^{-1}(u_0\bar{D}_{L,v_n}(\pi_L) + \pi_L\bar{D}_{L,v_n}(u_0)) \mod \pi_L^{-1}\mathcal{X}_{L,1}^{(n)}$$

$$\equiv \pi^{-1}_L\bar{D}_{L,v_n}(\pi_L) + u_0^{-1}\bar{D}_{L,v_n}(u_0) \mod \pi_L^{-1}\mathcal{X}_{L,1}^{(n)}$$

(3.31)

Therefore, (3.30) and (3.31) yield that $d\log \bar{D}_{L,v_n}$ does not depend on the choice of $\pi_L$.

**Theorem 3.8.** The derivation $\bar{D}_{L,v_n} : \mathcal{O}_L \to \mathcal{X}_{L,1}/\mathcal{X}_{L,1}^{(n)}$ satisfies

$$\langle \alpha, \beta \rangle_{L,n} = T_{L|K}(\lambda_\rho(\alpha) \ d\log \bar{D}_{L,v_n}(\beta)) \cdot v_n$$

for all $\alpha$ such that $\mu(\alpha) > \max\left\{ \frac{2m}{q-1}, \frac{1}{q-1} + \frac{1}{\pi L | \bar{K}} \right\}$ and for all $\beta \in L^\times$.

**Proof.** To prove (3.32) is equivalent to prove that

$$d\log \bar{D}_{L,v_n}(\beta) - \Psi_{L,v_n}(\beta) \in \pi_L^{-1}\mathcal{X}_{L,1}^{(n)}$$

for all $\beta \in L^\times$, where $d\log \bar{D}_{L,v_n}(\beta)$ and $\Psi_{L,v_n}(\beta)$ are regarded as elements of $\pi_L^{-1}\mathcal{X}_{L,1}$. Indeed, let $\beta \in L^\times$. Since Proposition 2.3 shows that

$$\langle \alpha, \beta \rangle_{L,n} = T_{L|K}(\lambda_\rho(\alpha) \Psi_{L,v_n}(\beta)) \cdot v_n$$

for all $\alpha \in p_{L,1}$, then (3.32) is equivalent to say that

$$T_{L|K}(\lambda_\rho(\alpha) d\log \bar{D}_{L,v_n}(\beta)) \cdot v_n = T_{L|K}(\lambda_\rho(\alpha) \Psi_{L,v_n}(\beta)) \cdot v_n$$

(3.35)
for all $\alpha$ in $L$ such that $\mu(\alpha) > \max\{\frac{nm}{q}, \frac{1}{q-1}\}$. Obviously, (3.35) is equivalent to
\begin{equation}
T_{L|K}(\lambda_\rho(\alpha)(d\log \bar{D}_{L,v_n}(\beta) - \Psi_{L,v_n}(\beta)) \in \eta^n \mathcal{O}_K
\end{equation}
for all $\alpha \in \gamma L \mathfrak{p}_{L,1}$, where $\gamma \in L$ is of valuation $\mu(\gamma) = \max\{\frac{nm}{q}, \frac{1}{q-1}\}$. However, since $\mathfrak{p}_{L,1} = \lambda_\rho(\mathfrak{p}_{L,1})$ and $\mu(\lambda_\rho(\alpha)) = \mu(\alpha)$ whenever $\alpha \in \mathfrak{p}_{L,1}$ (see Lemma 2.2), then (3.36) is in turn equivalent to
\begin{equation}
T_{L|K}(\gamma \pi L \alpha(\lambda L,v_n(\beta) - \Psi_{L,v_n}(\beta)) \in \eta^n \mathcal{O}_K
\end{equation}
for all $\alpha \in \mathfrak{p}_{L,1}$. Finally, by the very definition of $\chi_{L,1}$, (3.37) is equivalent to
\begin{equation}
d\log \bar{D}_{L,v_n}(\beta) - \Psi_{L,v_n}(\beta) \in \pi^{-1} \eta^n \gamma L \chi_{L,1} = \pi^{-1} \chi_{L,1}^{(n)}
\end{equation}
Let us now prove (3.38). Let $\beta = u \pi L \in L^\times$, then $d\log \bar{D}_{L,v_n}(\beta) - \Psi_{L,v_n}(\beta)$ is equal to $u^{-1} \bar{D}_{L,v_n}(u) + k \pi^{-1} \bar{D}_{L,v_n}(\pi L) - \Psi_{L,v_n}(u) - k \Psi_{L,v_n}(\pi L)$ modulo $\pi^{-1} \chi_{L,1}^{(n)}$. However, by the very definition of $\bar{D}_{L,v_n}$, we have
\begin{equation}
\bar{D}_{L,v_n}(u) \equiv u \Psi_{L,v_n}(u) \mod \chi_{L,1}^{(n)}
\end{equation}
But as $\chi_{L,1}^{(n)} \subset \pi^{-1} \chi_{L,1}^{(n)}$, the congruence (3.39) implies that
\begin{equation}
\bar{D}_{L,v_n}(u) \equiv u \Psi_{L,v_n}(u) \mod \pi^{-1} \chi_{L,1}^{(n)}
\end{equation}
Thus, we have
\begin{equation}
u^{-1} \bar{D}_{L,v_n}(u) \equiv \Psi_{L,v_n}(u) \mod \pi^{-1} \chi_{L,1}^{(n)}
\end{equation}
Moreover, we have
\begin{equation}
\bar{D}_{L,v_n}(\pi L) \equiv \pi L \Psi_{L,v_n}(\pi L) \mod \chi_{L,1}^{(n)}
\end{equation}
and thus,
\begin{equation}
\pi^{-1} \bar{D}_{L,v_n}(\pi L) \equiv \Psi_{L,v_n}(\pi L) \mod \pi^{-1} \chi_{L,1}^{(n)}
\end{equation}
This concludes the proof. \hfill \Box

3.3. Values of $\bar{D}_{L,v_n}$ in terms of representation theory. Let $\mathcal{U}_K$ be the group of units of $K$. In this section, we will consider the continuous representation $\mathfrak{r} : \text{Gal}(\Omega|H) \to \text{GL}_1(\mathcal{O}_K) = \mathcal{U}_K$ defined in [6, Proposition 2.5]. The image $\mathfrak{r}(\sigma)$ of an element $\sigma \in \text{Gal}(\Omega|H)$ is the unique unit $u$ of $K$ such that $\sigma(\alpha) = \rho_\alpha(\alpha)$ for all $\alpha \in W_\rho$. This representation is induced by the action of $\text{Gal}(\Omega|H)$ on the module $\lim_{\rho} W_\rho$. We will show that we can obtain explicit formulas in terms of invariants of this representation. It is obvious that the kernel of $\mathfrak{r}$ is $\text{Gal}(\Omega|H_\rho)$. Thus, $\mathfrak{r}$ induces an imbedding $\text{Gal}(H_\rho|H) \to \mathcal{U}_K$. Reducing modulo $\mathcal{U}_{K,n} = 1 + \mathfrak{p}_K^n$, we get the map $\mathfrak{r}_n$, which, restricted to $\text{Gal}(H_\rho^n|H)$, defines an isomorphism
\begin{equation}
\mathfrak{r}_n : \text{Gal}(H_\rho^n|H) \to \mathcal{U}_K/\mathcal{U}_{K,n}.
\end{equation}
For an algebraic extension $F$ of $H$, we also denote by $\mathfrak{r} : \text{Gal}(\Omega|F) \to \mathcal{U}_K$ the restriction of $\mathfrak{r}$ to $\text{Gal}(\Omega|F)$, and by $\mathfrak{r}_n : \text{Gal}(F(V_\rho^n)|F) \to \mathcal{U}_K/\mathcal{U}_{K,n}$ the restriction to $\text{Gal}(F(V_\rho^n)|F)$.

Proposition 3.9. Let $m \geq n$ and suppose $L \supset E_\rho^n$. There exists a character $\chi_{L,m,n} : L^\times \to \mathcal{O}_K/p_{K,m}^{nm}$ such that
\begin{equation}
\mathfrak{r}_n(m)(\Phi_L(\beta)) = 1 + \eta^m \chi_{L,m,n}(\beta) \in \mathcal{U}_K/\mathcal{U}_{K,(m+n)ma}
\end{equation}
for all $\beta \in L^\times$. Furthermore, $\chi_{L,m,n}$ satisfies the following.
(i) $\chi_{L,m,n}(\beta) = \chi_{E_\rho^n,m,n}(N_L|E_\rho^n(\beta))$. 

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(ii) Let \( v = \rho_a(v_m) \in W^m_\rho \), where \( v_m \) is a generator of \( W^m_\rho \) such that \( \rho_{\eta^{m-n}}(v_m) = v_n \). Then
\[
(v, \beta)_{L,n} = (a \chi_{L,m,n}(\beta)) \cdot \rho v_n
\]
for all \( \beta \in L^\times \). In particular if \( v = v_m \), then for every \( \beta \in L^\times \), we have
\[
(3.46) \quad (v_m, \beta)_{L,n} = \chi_{L,m,n}(\beta) \cdot \rho v_n.
\]
Proof. Let \( \beta \in L^\times \). As \( \Phi_L(\beta) \) fixes \( L \), thus in particular fixes \( E^m_\rho \), we have
\[
(3.47) \quad r_{m_0(m+n)}(\Phi_L(\beta)) \equiv 1 \mod \eta^m.
\]
Thus, there exists an element \( \chi_{L,m,n}(\beta) \in \mathcal{O}_K/p^n_K \) such that (3.45) holds. It is easy to check that \( \chi_{L,m,n} : L^\times \to \mathcal{O}_K/p^n_K \) is a group homomorphism. Moreover, the properties of the reciprocity map \( \Phi_L \) imply (i). To prove (ii), let \( \xi \in \mathfrak{p}_\Omega \) be such that \( \rho_{\eta^n}(\xi) = v \). Such a \( \xi \) exists by [3, Lemma 2.1]. Since \( v \in W^m_\rho \), then \( \xi \in W^{m+n} \) and
\[
(3.46) \quad (v, \beta)_{L,n} = \Phi_L(\beta)(\xi) - \xi
= r_{m_0(m+n)}(\Phi_L(\beta)) \cdot \rho \xi - \xi
= (r_{m_0(m+n)}(\Phi_L(\beta)) - 1) \cdot \rho \xi
= (\eta^m \chi_{L,m,n}(\beta)) \cdot \rho \xi
= (\eta^{m-n} \chi_{L,m,n}(\beta)) \cdot \rho v
= (\eta^{m-n} \chi_{L,m,n}(\beta)) \cdot (\rho(a \cdot v_m))
= (a \chi_{L,m,n}(\beta)) \cdot \rho v_n.
\]

Lemma 3.10. The character \( \chi_{L,m,n} : L^\times \to \mathcal{O}_K/p^n_K \) is stable by isomorphism class of \( \rho \). In other words, if \( t \) is an invertible power series in \( \mathcal{O}_H \{\{\tau\}\} \) such that \( \rho_a' = t^{-1} \circ \rho_a \circ t \) for all \( a \in \mathcal{O}_K \), then the characters defined in Proposition 3.9 associated to \( \rho \) and \( \rho' \) are equal.

Proof. Let \( v_m \) be such that \( \rho_{\eta^{m-n}}(v_m) = v_n \) and let \( v'_i = t^{-1}(v_i) \) for \( i = m, n \). Denote by \( \chi_{L,m,n} \) (respectively \( \chi'_{L,m,n} \)) the character defined in Proposition 3.9 associated to \( \rho \) (respectively \( \rho' \)). Then by (3.46), we have \( \chi'_{L,m,n}(\beta) \cdot \rho' v'_i = (v'_m, \beta)_{\rho',L,n} \) which is equal to \( t^{-1}((t(v'_m), \beta)_{\rho,L,n}) = t^{-1}((v_m, \beta)_{\rho,L,n}) \) by Proposition 2.1 (vi). Again from (3.46), we conclude that
\[
\chi'_{L,m,n}(\beta) \cdot \rho' v'_n = t^{-1}((v_m, \beta)_{\rho,L,n})
= t^{-1}(\chi_{L,m,n}(\beta) \cdot \rho v_n)
= \chi_{L,m,n}(\beta) \cdot \rho t^{-1}(v_n)
= \chi_{L,m,n}(\beta) \cdot \rho v_n.
\]

Proposition 3.11. Let \( m \geq n \) and suppose \( L \supset E^m_\rho \) is such that \( p \) does not divide the ramification index of the extension \( L|E^m_\rho \). Let \( u \) be a unit in \( L \) such that \( \mu(1-u) > \max\{\frac{q}{q-1}, \frac{1}{q-1}\} + \frac{1}{q-1} \). Let \( f(X) \) and \( g(X) \) be power series in \( \mathbb{F}_q[[X]] \) such that \( f(\pi_L) = v_m \) and \( g(\pi_L) = u \). Then,
\[
(3.48) \quad \frac{\tilde{g}(\pi_L)}{f(\pi_L)} \frac{v_m}{v_n} \in \mathfrak{p}_L,
\]
and
\[
(3.49) \quad \chi_{L,m,n}(u) \equiv T_{L|K}(\frac{1-u}{u})(1-\frac{\tilde{g}(\pi_L)}{f(\pi_L)} \frac{v_m}{v_n})\overline{D}_{L,v_n}(v_m) \mod p^n_K.
\]
Proof. Since $p$ does not divide the ramification index of $L|E^m_\rho$, we have $\mu(f'(\pi_L)) = \mu(f(\pi_L)) - \mu(\pi_L)$. Furthermore, since $\mu(1-u) > \max\{\frac{mn}{q-1}, \frac{1}{q-1}\}$, we can write $g(X) = 1 + \sum a_iX^i$, where $i \geq 2$ and $a_i \in \mathbb{F}_q$. Hence, $\mu(g'(\pi_L)) > \mu(\pi_L)$ and therefore, we have (3.48). Now, let us prove (3.49). By Lemma 2.6 and Lemma 3.10, we can suppose that $\rho$ is such that $(x, x)_{\rho, L, n} = 0$ for all $x \in p_L$. For such a $\rho$ and for $u \in L^x$ such that $\mu(1-u) > \max\{\frac{mn}{q}, \frac{1}{q-1}\}$, we have

$$F(\pi_L) = 1 + \sum \alpha \pi m \times \chi_{\rho, L, n}(u) \cdot \nu_n$$

for all $\alpha \in p_L \setminus \{0\}$ by Lemma 3.4. We note that the hypothesis on the valuation of $1-u$ allows us to replace $D_{\rho, L, n}(\alpha)$ by $D_{\pi, L, n}(\alpha)$ in (3.50). Let $\alpha$ be such that $\alpha u = v_m$, where $v_m$ is a generator of $W^m_\rho$ such that $\rho_{\eta m - n}(v_m) = \nu_n$. Hence, (3.50) together with (3.46) give us

$$D_{\rho, L, n}(\alpha u, u, L, n = T_L\chi(L)((1-u) \nu_n(\pi_L) \cdot \nu_n).$$

However, $D_{\rho, L, n}(\alpha u, u, L, n) = \frac{1}{v_m}(u \nu_n(v_m) - v_n \nu_n(v_n))$. Moreover, we have

$$D_{\rho, L, n}(u) = g'(\pi_L) \bar{D}_{\rho, L, n}(\pi_L) \text{ and } D_{\rho, L, n}(v_m) = f'(\pi_L) \bar{D}_{\rho, L, n}(\pi_L).$$

This implies that

$$f'(\pi_L) \bar{D}_{\rho, L, n}(u) - f'(\pi_L)g'(\pi_L) \bar{D}_{\rho, L, n}(\pi_L) \in \chi_{L, 1}^{(n)}$$

and

$$g'(\pi_L) \bar{D}_{\rho, L, n}(v_m) - f'(\pi_L)g'(\pi_L) \bar{D}_{\rho, L, n}(\pi_L) \in \chi_{L, 1}^{(n)},$$

so that

$$f'(\pi_L) \bar{D}_{\rho, L, n}(u) - g'(\pi_L) \bar{D}_{\rho, L, n}(v_m) \in \chi_{L, 1}^{(n)}.$$ 

Now, since the calculation in the beginning of this proof shows that $\frac{v_m}{f(\pi_L)} \in p_L$, we can multiply (3.52) by $\frac{v_m}{f(\pi_L)}$ in the fractional ideal $\chi_{L, 1}^{(n)}$. Therefore, we get

$$v_m \bar{D}_{\rho, L, n}(u) = v_m \frac{g'(\pi_L)}{f'(\pi_L)} \bar{D}_{\rho, L, n}(v_m) \in \chi_{L, 1}^{(n)}.$$ 

Finally, we can write

$$\chi_{L, m, n}(u) \equiv T_L\chi((1-u) \frac{1}{u^2} \nu_n(v_m) - v_m \frac{g'(\pi_L)}{f'(\pi_L)} \bar{D}_{\rho, L, n}(v_m)) \mod p_{K, n}^{(n)}$$

$$\equiv T_L\chi((1-u)(1 - \frac{g'(\pi_L)}{f'(\pi_L)} \bar{D}_{\rho, L, n}(v_m)) \mod p_{K, n}^{(n)}.$$

Lemma 3.12. Let $m \geq n$ and suppose $L \supset E^m_\rho$ is such that $p$ does not divide the ramification index of the extension $L|E^m_\rho$. Then, $D_{\rho, L, n}(v_m) \in \chi_{L, 1}^{(n)} \chi_{L, 1}^{(n)}$ is uniquely determined by (3.49).

Proof. Let $x$ and $x'$ be two elements in $\chi_{L, 1}^{(n)}$ such that

$$T_L\chi((1-u)(1 - \frac{g'(\pi_L)}{f'(\pi_L)} \bar{D}_{\rho, L, n}(v_m)) \mod p_{K, n}^{(n)},$$

for all $u \in U_L$ such that $\mu(1-u) > \max\{\frac{mn}{q}, \frac{1}{q-1}\}$. This means that

$$T_L\chi((1-u)(1 - \frac{g'(\pi_L)}{f'(\pi_L)} \bar{D}_{\rho, L, n}(v_m)) \equiv \chi_{L, 1}^{(n)}(x - x') \in p_{K, n}^{(n)}.$$
for all units $u \in L$ such that $\mu(1 - u) > \max\{\frac{nmq}{q - 1}, \frac{1}{q - 1}\}$. We need to prove that $x - x' \in \mathfrak{x}_{L,1}^{(n)}$. Since we are considering any $u$ such that $\mu(1 - u) > \max\{\frac{nmq}{q - 1}, \frac{1}{q - 1}\}$, then we can write $1 - u = \gamma\alpha$, where $\gamma \in L$ is of valuation $\max\{\frac{nmq}{q - 1}, \frac{1}{q - 1}\}$ and $\alpha$ varies in $p_{L,1} = \lambda_\rho(p_{L,1})$. Furthermore, the element $1 - \frac{g'(\sigma_\rho)}{v_m}$ is a unit in $L$. Therefore, $x$ and $x'$ are such that

\[(3.55)\quad T_{L/K}(\gamma\alpha(x - x')) \in p_K^{nm_0}\]

for all $\alpha \in p_{L,1}$. This yields that $x - x' \in \mathfrak{x}_{L,1}^{(n)}$. \qed

### 3.4. Explicit formulas in a particular case.

In this section, we place ourselves in the case where $\rho_\eta \equiv \tau^{m_0} \mod p_H$ and $L = E^m_\rho \supset E^n_\rho$ for an integer $m \geq n$. As previously shown in Theorem 3.8, we have

\[(3.56)\quad (\alpha, v_m)_{L,n} = T_{L/K}(\lambda_\rho(\alpha)\frac{1}{v_m}) \cdot (L,v_n) \cdot \rho v_n\]

for all $\alpha \in p_L$ such that $\mu(\alpha) \geq \frac{\nu_{m_0}q}{q - 1} + \frac{1}{q - 1} + \frac{1}{q^{nm_0}(q - 1)}$. On the other hand, we can prove using [3], that for the same condition on $\alpha$, we have

\[(3.57)\quad (\alpha, v_m)_{L,n} = \frac{1}{\eta^m} T_{L/K}(\lambda_\rho(\alpha)\frac{1}{v_m}) \cdot \rho v_n.\]

Indeed,

\[(\alpha, v_m)_{L,n} = (\rho_{\eta^{m-n}}(\alpha), v_m)_{L,m}\]

(by [3, Proposition 2.2])

\[= \frac{1}{\eta^m} T_{L/K}(\lambda_\rho(\rho_{\eta^{m-n}}(\alpha))\frac{1}{v_m}) \cdot \rho v_m\]

(by [3, Theorem 5.7])

\[= \frac{1}{\eta^m} T_{L/K}(\eta^{m-n}\lambda_\rho(\alpha)\frac{1}{v_m}) \cdot \rho v_m\]

\[= \frac{1}{\eta^m} T_{L/K}(\lambda_\rho(\alpha)\frac{1}{v_m}) \cdot \rho v_n.\]

Here, we can apply [3, Theorem 5.7] for $(\rho_{\eta^{m-n}}(\alpha), v_m)_{L,m}$ because $\mu(\rho_{\eta^{m-n}}(\alpha)) \geq \frac{\nu_{m_0}q}{q - 1} + \frac{1}{q - 1} + \frac{1}{q^{nm_0}(q - 1)}$ for all $m \geq n$ (see proof of [3, Lemma 4.1]).

**Proposition 3.13.** We have

\[(3.58)\quad \overline{D}_{E^m_\rho,v_n}(v_m) = \frac{1}{\eta^m}.\]

**Proof.** This is a direct consequence of the explicit formulas (3.56) and (3.57). \qed

**Corollary 3.14.** Let $u$ be a unit of $L$ such that $\mu(1 - u) > \max\{\frac{nmq}{q - 1}, \frac{1}{q - 1}\}$. Then

\[(3.59)\quad N_{L/K}(u^{-1}) - 1 \equiv T_{L/K}(\frac{1 - u}{u})(1 - \frac{g'(v_m)}{u}v_m) \mod p_K^{(n+m)m_0},\]

where $g(X) \in F_{q_L}[[X]]$ is such that $g(v_m) = u$.

**Proof.** Since we proved in Lemma 3.1 that $\overline{D}_{L,v_n}(v_m) = \frac{1}{\eta^m} \in \mathfrak{x}_{L,1}^{(n)}$ is uniquely determined by (3.49), we have

\[(3.60)\quad \chi_{L,m,n}(u) \equiv \frac{1}{\eta^m} T_{L/K}(\frac{1 - u}{u})(1 - \frac{g'(v_m)}{u}v_m) \mod p_K^{nm_0}\]

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for all units $u$ of $L$ such that $\mu(1-u) > \max\{\frac{mn}{q}, \frac{1}{q-1}\} + \frac{1}{q-1}$. Moreover, we know by (3.46) that
\begin{equation}
(v_m, u)_{L,n} = \chi_{L,m,n}(u) \cdot \rho v_n = \eta^m \chi_{L,m,n}(u) \cdot \rho v_{m+n},
\end{equation}
where $v_{m+n} \in W^m_{m+n}$ is such that $\rho^m (v_{m+n}) = v_m$. On the other hand, by the definition of $(\cdot, \cdot)_{L,n}$, we have
\begin{equation}
(v_m, u)_{L,n} = \Phi_L(u)(v_{m+n}) - v_{m+n} = \Phi_K(N_{L/K}(u))(v_{m+n}) - v_{m+n}.
\end{equation}
But $\Phi_K(N_{L/K}(u))(v_{m+n}) = \rho N_{L/K}(u^{-1}) (v_{m+n})$ by [3, Proposition 5.1]. Therefore, $(v_m, u)_{L,n} = (N_{L/K}(u^{-1}) - 1) \cdot \rho v_{m+n}$ and hence,
\begin{equation}
N_{L/K}(u^{-1}) - 1 \equiv \eta^m \chi_{L,m,n}(u) \mod p_K^{(n+m)m_0}.
\end{equation}
Finally, (3.59) follows from (3.60) and (3.63). □

References
[1] B. Anglès. On explicit reciprocity laws for the local Carlitz-Kummer symbols. J. Number Theory, 78(2):228–252, 1999.
[2] F. Bars and I. Longhi. Coleman’s power series and Wiles’ reciprocity for rank 1 Drinfeld modules. J. Number Theory, 129(2):789–805, 2009.
[3] M. Ala Eddine. Explicit reciprocity laws for formal drinfeld modules. arXiv:2202.02348.
[4] M. Rosen. Formal Drinfeld modules. J. Number Theory, 103(2):234–256, 2003.
[5] K. Iwasawa. Local class field theory. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1986. Oxford Mathematical Monographs.
[6] H. Oukhaba. On local fields generated by division values of formal Drinfeld modules. Glasg. Math. J., 62(2):459–472, 2020.
[7] V. A. Kolyvagin. Formal groups and the norm residue symbol. Izv. Akad. Nauk SSSR Ser. Mat., 43(5):1054–1120, 1198, 1979.
[8] K. Iwasawa. On some modules in the theory of cyclotomic fields. J. Math. Soc. Japan, 16:42–82, 1964.
[9] A. Wiles. Higher explicit reciprocity laws. Ann. of Math. (2), 107(2):235–254, 1978.
[10] J. Flórez. Explicit reciprocity laws for higher local fields. J. Number Theory, 213:400–444, 2020.
[11] J. Flórez. The norm residue symbol for higher local fields. Journal of Number Theory, 2021, in press.