Point and Interval Estimation on the
Degree and the Angle of Polarization.
– A Bayesian approach –

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ABSTRACT

Linear polarization measurements provide access to two quantities, the degree (DOP) and the angle
of polarization (AOP). The aim of this work is to give a complete and concise overview of how to analyze
polarimetric measurements. We review interval estimations for the DOP with a frequentist and a Bayesian
approach. Point estimations for the DOP and interval estimations for the AOP are further investigated
with a Bayesian approach to match observational needs. Point and interval estimations are calculated
numerically for frequentist and Bayesian statistics. Monte Carlo simulations are performed to clarify the
meaning of the calculations.

Under observational conditions, the true DOP and AOP are unknown, so that classical statistical con-
siderations – based on true values – are not directly usable. In contrast, Bayesian statistics handles un-
known true values very well and produces point and interval estimations for DOP and AOP, directly. Using
a Bayesian approach, we show how to choose DOP point estimations based on the measured signal-to-
noise ratio. Interval estimations for the DOP show great differences in the limit of low signal-to-noise
ratios between the classical and Bayesian approach. AOP interval estimations that are based on observa-
tional data are presented for the first time. All results are directly usable via plots and parametric fits.

Subject headings: polarization – confidence limits – Bayesian statistics – Methods: statistical – Methods: numerical

1. Introduction

Polarization of electromagnetic waves can be de-
scribed with the concept of elliptical polarization,
which implies linear and circular polarization as spe-
cial cases. In this paper, the term ‘polarization’ is
used to refer to linear polarization. Polarization mea-
surements provide access to two quantities, the degree
of polarization, which describes the relative amount
of polarized photons to all observed photons, and the
angle of polarization, which gives the orientation of
the electric field of the electromagnetic wave. Point
and interval estimations can be performed on both ob-
servables with the help of frequentist and Bayesian
statistics.

Chandrasekhar (1950) rediscovered the Stokes pa-
rameters in 1950. Statistical considerations on the in-
terpretation of polarimetric measurements are still an

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Fig. 1.— Point and interval estimations (p.e. and i.e.)
for the degree and angle of polarization result in four
key values that can be addressed by a frequentist and
a Bayesian approach. The existing works by Simmons
& Stewart (1985) (red), Naghizadeh-Khouei & Clarke
(1993) (orange), and Vaillancourt (2006) (green) are
labeled with colored boxes. The contribution of this
work is colored in blue. Point estimations for the an-
gle of polarization are trivial and are not mentioned
explicitly in literature, see §4.2.1 and §5.2.1
ongoing point of discussion. Figure 1 summarizes the main achievements in the field to which this work refers. Serkowski (1958) emphasized that the observed degree of polarization is subject to biasing effects in a way that the observed value is preferentially greater than the true one. Simmons & Stewart (1985) presented several point estimators to correct for this effect. Furthermore, they constructed confidence intervals to determine the reliability of the estimated degrees of polarization. These works are based on frequentist statistics. Naghizadeh-Khouei & Clarke (1993) constructed confidence intervals for the angle of polarization as a function of the true value of polarization. These works rested on the assumption that the true value of polarization is known a priori; this is a fundamental concept in frequentist statistics.

However, in many situations with unknown true values, there is a need to construct point and interval estimations as functions of observational data. Based on a Bayesian approach, Vaillancourt (2006) proposed a method to construct credibility intervals that, in comparison to the classical confidence intervals, show significant differences in the region of low signal-to-noise ratios. Using Bayesian analysis, Quinn (2012) showed a complete summary of probability distributions for the degree and angle of polarization and possible priors suited to the experiment. The aim of this work is to extend the Bayesian approach to point estimations on the degree of polarization and to interval estimations for the angle of polarization.

§2 summarizes the foundations of the frequentist and the Bayesian approach. §3 presents the underlying statistics that governs polarimetric measurements. Methods to construct point and interval estimations on the basis of frequentist statistics are reviewed in §4 and recalculated with a Bayesian approach in §5. This section includes a new method to choose the best estimator for the degree of polarization and a new method to construct interval estimations on the polarimetric angle. Monte Carlo simulated credibility intervals for the degree and angle of polarization are presented in §6. Results and conclusions follow in §7 and §8.

2. Frequentist and Bayesian approach

The basis of frequentist statistics are probabilities of random events. Deductive reasoning leads to systems where unknown consequences can be studied for a known cause. In the example of polarimetric measurements, this means that the true degree and angle of polarization of radiation that enters a polarimeter are known and the different possible manifestations within the observed data are investigated.

In contrast, Bayesian statistics extends the concept of probabilities to statements that become not only true and false, but more or less plausible. This reasoning allows one to deduce the plausibility of the cause on the basis of observed consequences. This is the usual case for all physical measurements: true values are estimated on the basis of observed data. In the case of polarimetric measurements, the true degree and angle of polarization are unknown parameters that shall be determined with the help of observed data.

Thus, frequentist and Bayesian statistics address two different points of view. A direct comparison of both methods is not reasonable because both methods result in different statements. To find the appropriate method for a given problem, one has to answer the question of whether the true values or the observables are known or unknown.

3. Statistics and Polarimetry

The normalized Stokes parameters \( q \) and \( u \) are appropriate variables to describe the linear polarization state of electromagnetic radiation (see Clarke et al. (1983) for more details). The degree and angle of polarization, \( P \) and \( \Psi \), can be expressed as

\[
P = \sqrt{q^2 + u^2} \quad \text{and} \quad (1)
\]
\[
\Psi = 0.5 \arctan(u/q). \quad (2)
\]

Eq. (1) clearly shows that random noise \( \sigma \) in \( q \) and \( u \) results in a positively biased degree of polarization, \( \sigma_q = \sigma_u \) is assumed. As the outcome of all type of polarimeters can be expressed in terms of Stokes parameters (Vaillancourt 2006), the following considerations are independent on the specific polarimeter type.

The applied terminology for a variable \( X \) is the following: \( x_0 \) labels the true value of \( X \), \( \hat{x} \) is the estimated value, and \( x \) is the observed value. The main variables are:

- \( P \) degree of polarization
- \( \Psi \) sky-angle of polarization
- \( p \) signal-to-noise ratio of \( p \) (Eq. 3)
- \( \sigma \) error in \( q \) and \( u \); \( \sigma = \sigma_q = \sigma_u \)
- \( \rho_q(p) \) probability density of \( q \) (or of \( p \))
- \( \rho(X|A) \) probability density of a variable set \( X \) given the parameter set \( A \)
3.1. The signal-to-noise ratio of polarization

The probability densities used in the next sections depend on different parameters that determine their shape. These parameters are the degree of polarization of the source $P$, and the uncertainties of the measurement $\sigma$. They can be combined to derive the signal-to-noise ratio

$$ p = \frac{P}{\sigma}. \quad (3) $$

The value of $\sigma$ depends on the specific polarimeter type and observational conditions. See, for example, Elsner et al. (2012) for a calculation of $\sigma$ for a counting based measurement with known background in the limit of low polarization.

With Eq. (3), the signal-to-noise ratio of any source observed with a specific polarimeter in a defined observation can be calculated. The signal-to-noise ratio $p$ is independent from specific experimental conditions and serves as an appropriate basis for the following statistical treatment.

3.2. The probability density function $\rho$

The detection principle of a polarimeter is to measure a sinusoidal variation of the signal intensity as a function of the azimuthal angle. Based on that idea, Elsner et al. (2012) derived, in the case of low polarizations ($P^2 \ll 1$), the bivariate probability density function $\rho$ which represents the probability of observing a signal-to-noise ratio $p$ within the interval $[p, p + dp]$ and a polarization angle $\Psi$ within $[\Psi, \Psi + d\Psi]$ while the true values are $p_0$ and $\Psi_0$:

$$ \rho(p, \Psi | p_0, \Psi_0) \, dp \, d\Psi = \frac{p}{\pi} \exp \left( -\frac{(p - p_0)^2 + p_0(2\sin(\Psi - \Psi_0))^2}{2} \right) \, dp \, d\Psi. \quad (4) $$

Equivalent results are obtained under the assumption of Gaussian distributed Stokes parameters $q$ and $u$, but without any limitations on $p$, are presented by Quinn (2012). Integrating Eq. (4), with respect to $p$, yields the Rice distribution (Rice 1945) – the univariate probability density function for the degree of polarization $\rho_p$ (Elsner et al. 2012) – while integrating Eq. (4), with respect to $\Psi$, results in the univariate probability density function for the angle of polarization $\rho_{\Psi}$ (Naghizadeh-Khouei & Clarke 1993):

$$ \rho_p(p | p_0) = p \cdot \exp \left( -\frac{p^2 + p_0^2}{2} \right) \cdot I_{1,0} (pp_0) \quad (5) $$

$$ \rho_{\Psi}(\Psi | p_0, \Psi_0) = \frac{1}{\sqrt{\pi}} \left( \frac{1}{\sqrt{\pi}} + \eta \exp(\eta) \right) e^{-\eta^2/2} \quad (6) $$

where $I_{1,0}$ is the modified Bessel function of first kind and zeroth order, erf is the Gauss error function, and $\eta = p_0/\sqrt{2} \cdot \cos(2(\Psi - \Psi_0))$. The probability density functions are plotted in Fig. 2 and 3.

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1. The following calculations are based on the ideal case that observed polarimetric data are distributed continuously. Effects resulting from discontinuities, such as data binning, are not considered.

2. This work is always related to the sky-angle and not the angle in the q-u-plane. Because $-\pi/2 < \Psi \leq \pi/2$ holds for Eq. (5) a factor of 1/2 is missing compared to other works that consider $-\pi < \Psi \leq \pi$.

3. Statistical considerations on the degree of polarization (see §4.1 and §4.1) can be generalized to any fields in which a quantity is estimated from a quadrature sum of other quantities.
4. Point and interval estimation on the basis of frequentist statistics

4.1. Degree of polarization

The \( p \)-distribution \( \rho_p \) in Fig. 2 shows that even unpolarized light \( (p_0 = 0) \) will yield an observed polarization \( p > 0 \). Simmons & Stewart (1985) showed several methods to correct for this biasing effect. They also presented a suggestion on how to construct confidence intervals around an estimated polarization. For completeness, we briefly review their results.

4.1.1. Point estimation

An estimator \( \hat{p}_0 \) can be constructed in different ways. Simmons & Stewart (1985) presented estimators based on the maximum, the median, and the mean of the \( \rho_p(p|p_0) \) distribution. They also proposed an estimator based on the maximum of the corresponding likelihood function \( \rho_{p_0}(p_0|p) \). In addition to this, we present a very simple estimator \( \text{ebal} \) that uses an approximate behavior of the maximum likelihood estimator for high signal-to-noise ratios (Vaillancourt 2006) in combination with a cut-off at \( p = 1 \):

\[
\hat{p}_{0, \text{ebal}}(p) = \begin{cases} 
0 & p \leq 1 \\
\sqrt{p^2 - 1} & p > 1.
\end{cases}
\] (7)

Polynomial fits for the other estimators are listed in Stewart (1991). Fig. 4 shows the graphs of all estimators. The estimated difference between the estimation value and the true value, the so-called bias, can be used to decide which estimator works best:

\[
bias(p_0) = \int_0^\infty \rho_p(p|p_0) \cdot \hat{p}_0(p) \, dp - p_0. \tag{8}
\]

The best estimator is the one with the smallest bias. Fig. 5 shows that all estimators work best in different regions of \( p_0 \). Simmons & Stewart (1985) concluded that the maximum likelihood estimator should be used for low signal-to-noise ratios and the maximum estimator for high signal-to-noise ratios \( p_0 \). This reasoning is not practically applicable because the estimators are used to obtain values for \( p_0 \), so \( p_0 \) is unknown at the moment the best estimator must be chosen. See §5.1.1 for a solution to this problem. Beside this problem, Simmons & Stewart (1985) did not consider the promising ebal-estimator, which has a bias smaller than 2.4% for signal-to-noise ratios \( p_0 > 1.4 \), and also has a simple analytical description.

4.1.2. Confidence intervals

When the best estimator \( \hat{p}_0 \) for an observed polarization \( p \) is known, the confidence interval makes a statement about the reliability of this estimation. The definition of a confidence interval \( \Delta p_0 = [p_{\text{lower}}, p_{\text{upper}}] \), with confidence level \( C \), is related to a set of repeated measurements of an observable \( p \) for a true, but unknown, parameter \( p_0 \). The probability that the constructed \( \Rightarrow \) confidence intervals, corresponding to the observables
p, contain the true value \( p_0 \) is C:

\[
\forall (p \Rightarrow \Delta p_0) : \text{Prob}(p_0 \in \Delta p_0) = C, \text{ fixed } p_0. \tag{9}
\]

In analogy with the Gaussian distribution, confidence intervals are labeled as \( \sigma_1, \sigma_2, \) and \( \sigma_3 \) for \( C = 68.3\% \), \( C = 95.5\% \), and \( C = 99.7\% \). It should be noted that \( \sigma_n \neq n \cdot \sigma_1 \) as in the Gaussian case.

Confidence intervals make a statistical statement about the intervals containing the true value, but not about the true value being within a specific interval. The construction of a confidence interval is not a trivial calculation because it is related to the true polarization \( p_0 \), but the distribution \( \rho_p(p | p_0) \) (Eq. (5)) is a function of \( p \). Simmons & Stewart (1985) constructed intervals \( \Delta p = [p, \bar{p}] \) by integrating \( \rho_p \) numerically, so that

\[
\int_{p}^{\bar{p}} \rho_p(p | p_0) dp = C \quad \text{[} p, \bar{p} \text{] minimal.} \tag{10}
\]

The additional demand that \( \Delta p \) should be as narrow as possible is due to the fact that Eq. (10) is valid for an infinite number of intervals \( \Delta p \). This demand, in combination with the asymmetric distribution \( \rho_p \), results in intervals \( \Delta p \) that are not symmetric with respect to the maximum of \( \rho_p \), as well.

Now, instead of an interval \( \Delta p \) obtained for a given \( p_0 \), an interval \( \Delta p_0 \) for a given \( p \) is needed. This projection is done in two steps:

1. For all \( p_0 \) values corresponding \( p \) and \( \bar{p} \) can be calculated with Eq. (10).

2. For a specific \( p \) value, only two \( p_0 - \bar{p} \) triplets are of interest: the one triplet with its upper limit \( \bar{p} \) equal to \( p \) defines the lower limit of \( p_0 \) \( (p_0) \); and the other triplet with its lower limit \( p \) equal to \( p \) defines the upper limit of \( p_0 \) \( (\bar{p}_0) \), see Fig. 6.

The following example may illustrate the situation for \( \sigma_1 \) confidence intervals (see Fig. 7 in pale colors): for \( p = 3, \Delta p_0 \approx [1.8, 3.8] \) is well defined. For \( p = 1.25 \), the \( p_0 \)-projection (Fig. 6) is no longer applicable because the lower limit is not zero, but undefined. Nevertheless, the confidence limit \( \Delta p_0 \approx [0, 1.9] \) is in accordance with Eq. (9), if it is assumed that \( \Delta p_0 \) does not exist for \( p \lesssim 0.4 \) (the intersection point of the upper limit curve with the \( p \)-axis).

To avoid the case of non-existing confidence limits, Simmons & Stewart (1985) departed from the term of narrowest \( \Delta p \) for small values of \( p_0 \). They proposed a
tangential progression of the upper limit \( \bar{p}_0 \) to zero for \( p \to 0 \) and calculated corresponding new values of \( p_0 \), so that the definition of Eq. (9) still holds. This way, confidence intervals can be set for all \( p > 0 \). Fig. 7 shows \( \sigma_1 \), \( \sigma_2 \), and \( \sigma_3 \) confidence intervals with and without the tangential progression. See Tab. 1 for parametric fitting results.

Another example shows the interpretation of confidence intervals: for \( p = 0.25 \) the \( \sigma_1 \) confidence interval (by tangential construction) is \( \Delta p_0 \approx [0, 0.4] \). This does not mean that the observed radiation is measured very precisely, but that a set of repeated measurements will result in different – most likely higher – values of \( p \) and new \( \Delta p \) that contain the true value \( p_0 \) in about 68% of all cases.

4.2. Angle of polarization

4.2.1. Point estimation

Since there is no biasing effect in the angle of polarization (see Fig. 3), the trivial estimator \( \hat{\Psi}_0(\Psi) = \Psi \) is sufficient.

4.2.2. Confidence intervals

Naghizadeh-Khouei & Clarke (1993) presented plots for \( \Delta \Psi \) equal to \( \sigma_1 \), \( \sigma_2 \), and \( \sigma_3 \) confidence intervals \( [\Psi_0 - \Delta \Psi/2, \Psi_0 + \Delta \Psi/2] \) by numerically integrating \( \rho_\Psi \) (see Fig. 8):

\[
\int_{\Psi_0 - \Delta \Psi/2}^{\Psi_0 + \Delta \Psi/2} \rho_\Psi(\Psi | p_0, \Psi_0) \, d\Psi = 0.683. \tag{11}
\]

The basic idea in Eq. (11) is the same as in Eq. (10), with the only difference being that the symmetry of \( \rho_\Psi \) around \( \Psi_0 \) makes the narrowest confidence intervals symmetric around \( \Psi_0 \), as well. The dependence of \( \rho_\Psi \) on \( p_0 \) is difficult because \( p_0 \) is unknown from an observational point of view.

Even though the plots of Fig. 8 are a function of \( p_0 \) and therefore not directly usable for observational data, it should be noted that the uncertainty of the polarization angle is a strictly monotonically decreasing function of \( p_0 \). As Eq. (11) is independent of \( p \), we see no way to construct confidence intervals for \( \Psi_0 \) as a function of \( p \). See hereto the discussion in § 5.2.2.

5. Point and interval estimation on the basis of Bayesian statistics

So far, the frequentist approach – taking the true polarization \( p_0 \) as a fixed parameter – leads to the strange situation that the unknown true polarization \( p_0 \) must be known, or at least estimated, to select the best estimator \( \hat{p}_0 \). Furthermore, the frequentist approach with it’s \( p \to p_0 \) projection, results in a difficult construction of confidence intervals \( \Delta p_0 \) at low signal-to-noise ratios \( p \) and it seems impossible to construct usable confidence intervals \( \Delta \Psi \) as a function of \( p \).

All of these problems disappear with a Bayesian approach because now, the true polarization \( p_0 \) can be treated as a stochastic variable. In general, the posterior density \( \rho(p_0 | p) \) can be computed with the likelihood \( \rho(p | p_0) \) and the prior density \( \rho(p_0) \) with Bayes theorem:

\[
\rho(p_0 | p) = \frac{\rho(p | p_0) \cdot \rho(p_0)}{\int_0^\infty \rho(p | p_0) \cdot \rho(p_0) \, dp_0}. \tag{12}
\]

In the following, we are considering a non-informative polar prior density: \( \rho(p_0) = \text{const} \) \cite{Quinn2012} studied the impact of non-informative prior densities, in general, and the difference of Jeffrey’s prior, which is uniform in the Stokes parameters \( q_0 \) and \( u_0 \) to the

\[
\text{Technically speaking, this statement is critical because there is no uniform distribution living on the non-negative half-line } R_+, \text{ that can be normalized. This mathematical problem can be overcome by considering a maximal possible true degree of polarization } p_{0,\text{max}}, \text{ so that } \rho = \frac{1}{p_{0,\text{max}}}. \text{ Finally: } p_{0,\text{max}} \to \infty.
\]
Fig. 9.— The posterior density distribution $\rho_{p_0}(p_0 | p)$ represents the probability density that an observed polarization $p$ results from a true polarization $p_0$. These distributions (Eq. (13)) were first calculated by Vaillancourt (2006).

5.1. Degree of polarization

5.1.1. Point estimation

The probability density $\rho_{p_0}$ can be used to transform any quantity that is a function of $p_0$ to an estimation value as a function of $p$. Using the already calculated bias($p_0$) of Eq. (8) leads to

$$bias(p) = \int_0^\infty \text{bias}(p_0) \cdot \rho_{p_0}(p_0 | p) \, dp_0.$$  \hfill (15)

Each value of bias($p_0$) is weighted with the probability that this $p_0$ value results from an observed value $p$. Fig. The estimators in Fig. 10 show a similar behavior to those presented in Fig. 5, but now, the regions of the best performance are in terms of $p$. This allows the best estimator to be chosen directly for the first time on the basis of the observed data, cf. §7.1.

5.1.2. Credibility intervals

Analogous to confidence intervals in frequentist statistics, credibility intervals in Bayesian statistics make a statement on the reliability of an estimated value. Different from confidence intervals, credibility intervals give the probability that the true value is within a specific interval. For all possible values of $p_0$ that can cause ($\Rightarrow$) the observed $p$, the credibility interval includes those $p_0$ that cause $p$ in a fraction $C$ of all cases:

$$\forall p_0 : \text{Prob} \left( p_0 \in [p_0, \overline{p_0}] \Rightarrow p \right) = C \text{, fixed } p.$$  \hfill (16)

Integrating the density distribution $\rho_{p_0}$ over $p_0$ leads directly to credibility intervals (Vaillancourt 2006):

$$\int_{\overline{p_0}}^{\overline{p_0}} \rho_{p_0}(p_0 | p) \, dp_0 = C \mid [\overline{p_0}, \overline{p_0}] \text{ minimal}.$$  \hfill (17)

The resulting credibility limits are shown in Fig. 11 for $\sigma_1$, $\sigma_2$, and $\sigma_3$ credibility intervals. See Table 2 for parametric fitting results.
5.2. Angle of polarization

5.2.1. Point estimation

As in §4.2.1 the trivial estimator $\hat{\Psi}_0(\Psi) = \Psi$ is sufficient for point estimations on the angle of polarization.

5.2.2. Credibility intervals

Before proposing our method, we want to list the difficulties we recognized in constructing interval estimations $\Delta \Psi(p)$ and clarify why it is incorrect to proceed in the following way:

- Using the best estimator $p_0$ as a parameter for Eq. (11) leads to incorrect results because the uncertainty of $p_0$ does not propagate into the uncertainty of $\Psi$.
- Calculating $\Delta \Psi$ with the lower limit $p_0$ that was computed in §4.1 overestimates the uncertainty in $\Psi$ because the interval $\Delta p_0$ is constructed to be minimal in $p_0$ but not in $\Psi$ (the upper limit $\bar{p}_0$ underestimates $\Delta \Psi$, cf. Fig. 8).
- Transforming $p_0$ to $p$ as in the transformation of the bias (Eq. (15)),

$$\Delta \Psi(p) = \int_0^{\infty} \Delta \Psi(p_0) \cdot \rho_{p_0}(p_0|p) \, dp_0$$  \hspace{1cm} (18)

averages $\Delta \Psi$. The result is reasonable, but does not match the definition of credibility intervals in Eq. (16).

Our idea is to recalculate $\Delta p_0$ credibility intervals with the bivariate probability distribution $\rho(p, \Psi | p_0, \Psi_0)$ of Eq. (4). After integrating over $p_0$ and normalizing the bivariate distribution for fixed $p$’s with respect to $\Psi$, the credibility interval $\Delta \Psi$ can be computed ($\sim$) directly (without loss of generality $\Psi_0 = 0$):

$$\rho^*(p, \Psi) = \int_0^{\infty} \rho(p, \Psi | p_0, 0) \, dp_0.$$  \hspace{1cm} (19)

$$\rho(p, \Psi) = \frac{\rho^*(p, \Psi)}{N(p)} , \quad N(p) = \int_0^{\infty} \rho^*(p, \Psi) \, d\Psi,$$  \hspace{1cm} (20)

$$\Delta \Psi(p) \sim \int_{-\Delta \Psi/2}^{\Delta \Psi/2} \rho(p, \Psi) \, d\Psi = C.$$  \hspace{1cm} (21)

The numerically computed results are shown in Fig. 12. See Tab. 3 for parametric fitting results.
The credibility interval \([p_0, \overline{p}_0]\) for each \(p\)-value can finally be calculated as the narrowest interval in \(p_0\) that contains \(C \cdot N_{p_0}\) data points. Fig. 13 explains the described method graphically for \(p \approx 2\).

### 6.2. Angle of polarization

Simulating interval estimations on the polarization angle (\(\Psi_0 = 0\) is assumed) is done in two steps. First, appropriate random data must be simulated:

1. Choose a true signal-to-noise ratio \(p_0 \in [0, 10]\).
2. Guess a value \(p\) following \(\rho_0(p \mid p_0)\), cf. Eq. (5).
3. Guess a value \(\Psi\) following \(\rho(p, \Psi \mid p_0, 0)\), Eq. (4).

Repeating these steps numerous times yields a large number of \((p_0/p/\Psi)\)-triplets. In the second step, analyzing these triplets will result in the desired credibility intervals:

1. Select all triplets with a specific value of \(p \in [p, p + 0.01]\), independent of \(p_0\) and \(\Psi\).
2. Count the selected triplets \(\rightarrow N_{p_0, \Psi}\).
3. Sort the triplet list with respect to \(\Psi\).
4. Starting counting at \(\Psi = 0\), the credibility interval \(\Delta \Psi/2\) can be obtained as the \(\Psi\) value of the data triplet at list number \(C/2 \cdot N_{p_0, \Psi}\).

In this way, the uncertainty in \(\Psi\) can be estimated on the basis of the observed signal to noise ratio \(p\). Repeating these steps for a set of different \(p\)-values results in the data points plotted in Fig. 12. Again, the method used is explained graphically in Fig. 14.

### 7. Results

#### 7.1. Degree of polarization

Based on the results shown in Fig. 10 the best estimator \(\hat{p}_0\) can be chosen by means of lowest expected bias. The excellent results of the approximation \(\hat{p}_{0, \text{bal}} = (p^2 - 1)^{0.5}\) for \(p > 2.8\) in combination with its analytical form makes this estimator a good choice for high signal-to-noise ratios \(p\). The regions of the best performance and least square fits for \(\hat{p}_0\) within those regions are the following:

\[
\hat{p}_{0, \text{ml}} = 0 \quad p \in [0, \sqrt{2}] \quad (22)
\]

\[
\hat{p}_{0, \text{fit ml}} \approx (p - \sqrt{2})^{0.4542} + (p - \sqrt{2})^{0.4537} + (p - \sqrt{2})/4
\Delta = [-0.0078, 0.011], \quad p \in [\sqrt{2}, 1.70] \quad (23)
\]

\[
\hat{p}_{0, \text{fit mean}} \approx 22p^{0.11} - 22.076
\Delta = [-0.0039, 0.0035], \quad p \in [1.70, 2.23] \quad (24)
\]

\[
\hat{p}_{0, \text{fit med}} \approx 1.8p^{0.76} - 1.328
\Delta = [-0.0024, 0.002], \quad p \in [2.23, 2.83] \quad (25)
\]

\[
\hat{p}_{0, \text{bal}} = (p^2 - 1)^{0.5} \quad p \in [2.83, \infty] \quad (26)
\]

The Eq. (22)-(26) are not one combined estimator for all values of \(p\), but a collection of different estimators,

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\(^5\) The \(\Psi\)-symmetry in Eq. (4) allows to restrict all calculations on the half credibility interval.
each working in the region of best expected performance. Therefore, discontinuous jumps at the interval edges are not a lack of accuracy, but a result of different estimators. The motivation for the functional forms in Eq. (23) - (25) and in Eq. (27) and (28) are not physically driven but determined by mathematical intuition and the best fitting results. \( \Delta = \hat{\rho}_{0.5 \text{fit}} - \hat{\rho}_{0.6} \) indicates the range of maximal deviation between the fitted curves and the respective estimator \( e \).

Concerning the construction of interval estimations, confidence and credibility intervals show very similar results for \( p > 6 \). At low signal-to-noise ratios \( p < 3 \) they differ significantly. The choice of which method to use depends on the question which shall be answered: is it the chance \( C \) that the confidence interval includes the true parameter, or the probability \( C \) that the true parameter is within the credibility interval?

Lower and upper limits for \( \sigma_1, \sigma_2, \) and \( \sigma_3 \) confidence intervals are presented in Fig. 7. A functional description for the tangential construction method is obtained by fitting the function

\[
f(p) = A p^B - C p^D + E p
\]  

(27)

to the computed data points with the least square method. The results are listed in Table 1.

Table 2: Fitting results of Eq. (27) for the lower and upper credibility interval limits for the degree of polarization. See also Fig. 11. The deviation between the numerically calculated data points and the fit \( f \) is maximal \( \pm 0.025 \). \( p < 6 \) for all cases and \( f = 0 \) for all remaining undefined regions.

| \( f \) | \( A \) | \( B \) | \( C \) | \( D \) | \( E \) | Validity |
|-------|-------|-------|-------|-------|-------|---------|
| \( p_1(p) \) | 4.241 | 1.021 | 2.286 | 1.134 | -3.535 | \( p \geq 1.72 \) |
| \( p_2(p) \) | 0.468 | 1.177 | 3.974 | 0.874 | 0.145 | \( p \geq 2.54 \) |
| \( p_3(p) \) | 1.327 | 1.121 | 7.599 | 1.131 | -1.00 | \( p \geq 3.45 \) |
| \( \bar{p}_1(p) \) | 0.292 | 2.063 | -1.00 | 0.000 | 0.000 | \( p \leq 1.72 \) |
| \( \bar{p}_2(p) \) | 0.855 | 0.020 | 17.87 | 6.012 | 1.009 | \( p > 1.72 \) |
| \( \bar{p}_3(p) \) | 1.819 | 1.185 | -2.00 | 0.000 | -1.345 | \( p \leq 2.54 \) |
| \( \bar{p}_4(p) \) | 1.910 | -0.028 | 13.794 | 11.21 | 1.018 | \( p > 2.54 \) |
| \( \bar{p}_5(p) \) | 0.564 | 1.632 | -3.00 | 0.000 | 0.000 | \( p \leq 1.40 \) |
| \( \bar{p}_6(p) \) | 1.058 | 1.000 | -2.47 | 0.000 | 0.000 | 1.4 \( < p \leq 3.5 \) |
| \( \bar{p}_7(p) \) | 4.130 | -0.38 | 2.0e5 | 10.65 | 1.140 | \( p > 3.50 \) |

7.2. Angle of polarization

Despite its symmetrical probability distribution function \( \rho_p \), constructing confidence intervals for the angle of polarization is not trivial because of its dependency on \( p_0 \). Calculated lower and upper limits for \( \sigma_1, \sigma_2, \) and \( \sigma_3 \) credibility intervals are presented in Fig. 11. A functional description is obtained by fitting the function

\[
g(p) = A \left( B + \tanh \left( C (D - p) \right) \right) - E p \quad \text{[in deg.]} \]  

(28)

to the computed data points with the least square method.

For \( p > 6 \), a very simple description can be obtained using Gaussian error propagation on Eq. (1) and (2) with \( \sigma_a = \sigma_q = \sigma \) (Serkowski 1962, Eq. (70)):

\[
\sigma_1(p) = \frac{28.65^\circ}{p} \quad p > 6 \]  

(29)
\[
\sigma_2(p) = \frac{57.30^\circ}{p} \quad p > 6 \]  

(30)
\[
\sigma_3(p) = \frac{85.95^\circ}{p} \quad p > 6 \]  

(31)
Table 3: Fitting results of Eq. (28) for the lower and upper credibility interval limits for the angle of polarization. See also Fig. 12. The deviation between the numerically calculated data points and $g$ is maximal $\pm 1.7^\circ$.

| $g$         | A   | B   | C   | D   | E   | validity |
|-------------|-----|-----|-----|-----|-----|----------|
| $\sigma_1(p)$ | 32.50 | 1.350 | 0.739 | 0.801 | 1.154 | 0.0 < $p$ ≤ 6.0 |
| $\sigma_2(p)$ | 65.65 | 0.323 | 0.858 | 2.688 | 0.000 | 0.0 < $p$ ≤ 2.2 |
|             | 517  | 1.044 | 0.806 | 0.015 | 2.186 | 2.2 < $p$ ≤ 6.0 |
| $\sigma_3(p)$ | 62.88 | 0.423 | 1.385 | 3.546 | 0.000 | 0.0 < $p$ ≤ 3.2 |
|             | 102  | 1.380 | 1.327 | 2.506 | 3.958 | 3.2 < $p$ ≤ 6.0 |

8. Conclusions

Polarimetric measurements incorporate a non-Gaussian statistic in terms of the degree and the angle of polarization. The aim of this work was to present a systematic overview of the statistics that are necessary to analyze such measurements. In particular, we calculated point estimations for the degree of polarization and interval estimations for the angle of polarization with a Bayesian approach for the first time.

From an observational point of view, the Bayesian analysis shows substantial advantages compared to frequentist analysis. It allows direct access for the best estimator to be chosen on the basis of observational data and produces interval estimations with a meaningful interpretation.

In conclusion, observational polarimetric data can be recalculated in terms of signal-to-noise ratios $p$ using Eq. (3). The choice of the best estimator, the best estimated value $\hat{p}_0$, confidence and credibility intervals for the degree of polarization, and credibility intervals for the angle of polarization can then be obtained directly with the approximated formulas in §7.1 and §7.2. Using Eq. (3) reversed, all calculated values for the signal-to-noise ratio of the degree of polarization can be expressed as degree of polarization.

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