Electromagnetic field enhancement in a subwavelength rectangular open cavity

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Abstract
Consider the transverse magnetic polarization of the electromagnetic scattering of a plane wave by a perfectly conducting plane surface, which contains a two-dimensional subwavelength rectangular cavity. The enhancement is investigated fully for the electric and magnetic fields arising in such an interaction. The cavity wall is assumed to be a perfect electric conductor, while the cavity bottom is allowed to be either a perfect electric conductor or a perfect magnetic conductor. We show that the significant field enhancement may be achieved in both nonresonant and resonant regimes. The proofs are based on variational approaches, layer potential techniques, boundary integral equations, and asymptotic analysis. Numerical experiments are also presented to confirm the theoretical findings.

Keywords Cavity scattering problem · Electromagnetic field enhancement · Scattering resonances · Helmholtz equation · Variational formulation · Boundary integral equation · Asymptotic analysis

Mathematics Subject Classification 45A05 · 35C20 · 35Q60 · 35C15
1 Introduction

The electromagnetic scattering by open cavities has significant applications in many scientific areas. For instance, the radar cross section (RCS) is a quantity which measures the detectability of a target by a radar system. Clearly, it is of high importance in military and civil use for a deliberate control in the form of enhancement or reduction of the RCS of a target. Since the cavity RCS can dominate the total RCS, it is critical to have a thorough understanding of the electromagnetic scattering by a target, particularly a cavity, for successful implementation of any desired control of its RCS. The underlying scattering problems have received much attention by many researchers in both the engineering and mathematical communities. There has been a rapid development of the mathematical theories and computational methods in this area [3,8,9]. We refer to [27] for a survey of open cavity scattering problems.

Recently, the electromagnetic wave scattering problems for optical device surfaces containing subwavelength cavities have created a lot of interest in electromagnetic field enhancement and extraordinary optical transmission effects, due to the fact that the light can be localized and greatly enhanced near subwavelength cavities or apertures. Such features have important potential applications in diverse scientific areas, such as biological and chemical sensing, microscopy, spectroscopy and communication [6,10,13,18,19,25,33–35,37]. However, there are many mathematical and computational issues for the underlying problems. The complication arises from the multiscale nature of the structures and various enhancement behaviors that they induce. For instance, the enhancement can be attributed to surface plasmonic resonance [18,19,29], non-plasmonic resonances [35,37], or even without the resonant effect [28].

It is very helpful to provide a rigorous mathematical analysis for these phenomena, which turns out to be extremely challenging. The particular light patterns observed in the far-field or in the near-field may be the results of complex multi-interactions of surface waves, cavity resonances, resonant tunneling of plasmon waves, skin depth effects, etc. It requires the resolution of the full three-dimensional Maxwell equations in non-smooth geometries even for a quantitative description of the diffractive properties of electromagnetic waves in subwavelength structures. For most resonance phenomena, the localization or enhancement of light is very sensitive to geometrical parameters and the frequency. We may begin with simple geometries, where the analytical approaches are made possible to obtain an accurate description of the electromagnetic field. In [7,11], the field enhancement was considered for a single rectangle cavity and double rectangle cavities. Using asymptotic expansions of Green’s function, they showed that the limiting Green function is a perfectly conducting plane with a dipole in place of the cavity when the width of the cavity shrinks to zero. In [14,22,23], an asymptotic expansion method was proposed to study the solution of the Helmholtz equation in a domain including a single thin slot. A quantitative analysis was made for the field enhancement of the Helmholtz equation for a single narrow open slit [28,30], or periodic slits [12,31,32]. We refer [4,5] for the study of closely related Helmholtz resonators and resonances in bubbly media.

Motivated by [28,30], we consider the electromagnetic scattering of a plane wave by a perfectly electrically conducting plane surface, which contains a subwavelength rectangular cavity. More specifically, the cavity is assumed to be invariant in the $x_3$-axis and only the transverse magnetic polarization (TM) is studied. Hence the three-dimensional Maxwell equations can be reduced into the two-dimensional Helmholtz equation. The cavity wall is assumed to be a perfect electric conductor (PEC), and the cavity bottom is assumed to be either a perfect magnetic conductor (PMC) or a perfect electric conductor (PEC). A cavity
Table 1  Electromagnetic field enhancement. $\epsilon$: width of the cavity, $d$: depth of the cavity, $\lambda$: wavelength of the incident field

|                | PEC-PMC cavity | PEC-PEC cavity |
|----------------|----------------|----------------|
|                | Nonresonant    | Resonant       | Nonresonant    | Resonant       |
| Electric field | $O(1/\sin\kappa d)$ | $O(1/\epsilon)$ | $O(\lambda\sqrt{\epsilon}/d)$ | $O(1/\epsilon)$ |
| Magnetic field | no             | $O(1/\epsilon)$ | no             | $O(1/\epsilon)$ |

is called the PEC-PMC type when it has a PEC wall and a PMC bottom, while a cavity is called the PEC-PEC type when it has a PEC wall and a PEC bottom. We investigate the electromagnetic field enhancement for both types of cavities. Using a combination of variational approaches, layer potential techniques, boundary integral equations, and asymptotic analysis, we demonstrate the field enhancement mechanisms for the underlying scattering problems different boundary conditions. Denote by $\lambda, \epsilon, d$ the wavelength of the incident field, the width of the cavity, and the depth of the cavity, respectively. For the PEC-PMC cavity, we prove in the nonresonant regime that the electric field enhancement has an order $O(1/\sin(\kappa d))$ and the magnetic field has no enhancement in the cavity provided that the length scales satisfy $\epsilon \ll d \ll \lambda$. Moreover, if $\epsilon \ll \lambda$, we show that the Fabry–Perot type resonance occurs. In the resonant regime, we derive an asymptotic expansion for the resonant frequencies and deduce an order $O(1/\epsilon)$ enhancement for both the electric and magnetic fields. For the PEC-PEC cavity, we carry the same analysis in both the nonresonant and resonant regimes. The electric field is shown to have a weak enhancement of order $O(\lambda\sqrt{\epsilon}/d)$ in the nonresonant regime with length scales satisfying $\epsilon \ll d \ll \lambda$. In the resonant regime, a similar quantity analysis is done for the resonant frequencies and the same order $O(1/\epsilon)$ is achieved for the electric and magnetic field enhancement. These field enhancement results are summarized in Table 1. Numerical experiments are presented and used to confirm our theoretical findings.

This work is closely related to [28,30], where a transmission problem was considered for small gaps. We take the similar approaches and some of the results are parallel. But the model problem is different and thus the Green’s functions take different forms, which lead to the difference of the details for the analysis. In this work, we give a comprehensive account of the analysis for the enhancement of the fields and the resonant frequencies in both the PEC-PMC and PEC-PEC cavities. Specifically, below is a brief summary of the similarity and difference between our results and those presented in [11,28,30]. In the non-resonant region, for the PEC-PMC cavity problem, we obtain the same order of enhancement for the electric field as that of the PEC gap in [28]; for the PEC-PEC cavity problem, we achieve a weak enhancement for the electric field which was not obtained in [11,28]. In the resonant region, for both the PEC-PMC and PEC-PEC cavity problems, due to influence of the bottom boundary condition, we obtain the same enhancement order as that in [30], but the resonance sets are different.

The rest of the paper is organized as follows. The PEC-PMC cavity is examined in Sects. 2–3 and the PEC-PEC cavity is studied in Sects. 4–5. Sections 2 and 4 focus on the field enhancement for the non-resonant case. In each section, the model problems are introduced, the approximate problems are proposed, the error estimates are derived between the solutions of the exact and approximate problems, and the electric and magnetic field enhancement are discussed. Sections 3 and 5 address the field enhancement for the resonant case. In each section, the boundary integral equations are given, the asymptotic analysis is carried for the Fabry–Perot resonances, and the electric and magnetic field enhancement are presented under
the resonant frequencies. The paper is concluded with some general remarks and directions for future work in Sect. 6.

2 PEC-PMC cavity without resonance

In this section, we discuss the electromagnetic field enhancement for the PEC-PMC cavity problem when the wavenumber is sufficiently small, i.e., no resonance occurs.

2.1 Problem formulation

As is shown in Fig. 1, the problem geometry is assumed to be invariant in the $x_3$-axis. Let $x = (x_1, x_2) \in \mathbb{R}^2$. Consider a two-dimensional rectangular open cavity $D_\epsilon = \{x \in \mathbb{R}^2 : 0 < x_1 < \epsilon, -d < x_2 < 0\}$, where $\epsilon > 0$ and $d > 0$ are constants. Denote by $\Gamma^+_\epsilon = \{x \in \mathbb{R}^2 : 0 < x_1 < \epsilon, x_2 = 0\}$ and $\Gamma^-_\epsilon = \{x \in \mathbb{R}^2 : 0 < x_1 < \epsilon, x_2 = -d\}$ the open aperture and the bottom side of the cavity, respectively. Let $\mathbb{R}^2_+ = \{x \in \mathbb{R}^2 : x_2 > 0\}$ and $\Gamma_0 = \{x \in \mathbb{R}^2 : x_2 = 0\}$. Denote $\Omega_1 = \mathbb{R}^2_+ \cup D_\epsilon$.

We assume that the cavity and the upper half-space are filled with the same homogeneous medium. The electromagnetic wave propagation is governed by the time-harmonic Maxwell equations (time dependence $e^{-i\omega t}$):

$$\nabla \times E - i\omega \mu H = 0, \quad \nabla \times H + i\omega \varepsilon E = 0 \quad \text{in } \Omega \times \mathbb{R},$$

(2.1)

where $\omega > 0$ is the angular frequency, $\varepsilon > 0$ is the electric permittivity, $\mu > 0$ is the magnetic permeability, $E$ and $H$ are the electric field and the magnetic field, respectively. Furthermore, we assume that the boundary $\partial \Omega \setminus \Gamma^-_\epsilon$ is a perfect electrical conductor (PEC) while the boundary $\Gamma^-_\epsilon$ is a perfect magnetic conductor (PMC). Hence $(E, H)$ satisfy the following boundary conditions:

$$\nu \times E = 0 \text{ on } (\partial \Omega \setminus \Gamma^-_\epsilon) \times \mathbb{R}, \quad \nu \times H = 0 \text{ on } \Gamma^-_\epsilon \times \mathbb{R},$$

(2.2)

where $\nu = (\nu, 0)$ is the unit outward normal vector on $\partial \Omega \times \mathbb{R}$, i.e., $\nu$ is the unit outward normal vector on $\partial \Omega$.

We consider a polarized time-harmonic electromagnetic wave $(E^{\text{inc}}, H^{\text{inc}})$ that impinges on the cavity from above. In the transverse magnetic polarization (TM), the magnetic field is parallel to the $x_3$-axis, which implies that the incident magnetic wave $H^{\text{inc}} = (0, 0, u^{\text{inc}})$. Here $u^{\text{inc}}(x) = e^{ix \cdot d} = e^{i\kappa d}$ is a plane wave propagating in the direction $d = (\sin \theta, -\cos \theta)$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ is the incident angle and $\kappa = \omega \sqrt{\varepsilon \mu}$ is the wavenumber. The corresponding
incident electric field \( E^{\text{inc}} \) is determined by Ampere’s law \( E^{\text{inc}} = i(\omega \varepsilon)^{-1} \nabla \times H^{\text{inc}} = \sqrt{\mu / \varepsilon}(\cos \theta, \sin \theta, 0)e^{ix \cdot d} \). Denote by \( \lambda = 2\pi / \kappa \) the wavelength of the incident wave. In this section, we assume that the length scale of the underlying geometry satisfies \( \varepsilon \ll d \ll \lambda \).

Let \( H = (0, 0, u_\varepsilon) \) be the total magnetic field. It can be verified from (2.1)–(2.2) that \( u_\varepsilon \) satisfies

\[
\begin{align*}
\Delta u_\varepsilon + \kappa^2 u_\varepsilon &= 0 \quad \text{in } \Omega, \\
\partial_n u_\varepsilon &= 0 \quad \text{on } \partial \Omega \setminus \Gamma_\varepsilon^-, \\
u_\varepsilon &= 0 \quad \text{on } \Gamma_\varepsilon^-.
\end{align*}
\]

(2.3)

Denote by \( u^{\text{ref}} \) the reflected field due to the interaction between the incident field \( u^{\text{inc}} \) and the PEC plane \( \Gamma_0 \). It can be shown that \( u^{\text{ref}}(x) = e^{iux \cdot d'} \), where \( d' = (\sin \theta, \cos \theta) \). In \( \mathbb{R}^+ \), the total field \( u_\varepsilon \) consists of the incident wave \( u^{\text{inc}} \), the reflected wave \( u^{\text{ref}} \), and the scattered field \( u^{\text{sc}}_\varepsilon \). The scattered fields \( u^{\text{sc}}_\varepsilon \) is required to satisfy the Sommerfeld radiation condition:

\[
\lim_{r \to \infty} \sqrt{r} \left( \partial_r u^{\text{sc}}_\varepsilon - i\kappa u^{\text{sc}}_\varepsilon \right) = 0, \quad r = |x|.
\]

(2.4)

### 2.2 An approximate model

We introduce an approximate model for the problem (2.3)–(2.4) in order to estimate the field enhancement inside the cavity.

Let

\[
\phi_0(x_1) = \frac{1}{\sqrt{\varepsilon}}, \quad \phi_n(x_1) = \sqrt{\frac{2}{\varepsilon}} \cos \left( \frac{n\pi x_1}{\varepsilon} \right), \quad n \geq 1
\]

be an orthonormal basis on the interval \((0, \varepsilon)\). It follows from the Neumann boundary condition in (2.3) that \( u_\varepsilon \) can be expanded as the series of waveguide modes:

\[
u_\varepsilon(x) = \sum_{n=0}^{\infty} \left( \alpha_n^+ e^{-i\beta_n x_2} + \alpha_n^- e^{i\beta_n (x_2+d)} \right) \phi_n(x_1), \quad x \in D_\varepsilon,
\]

(2.5)

where the coefficients \( \beta_n \) are defined as

\[
\beta_n = \begin{cases} 
\kappa, & n = 0, \\
\sqrt{i(\pi n \varepsilon)^2 - \kappa^2^2}, & n \geq 1.
\end{cases}
\]

(2.6)

If \( \varepsilon \) is small enough (in fact it only needs \( \varepsilon < \lambda / 2 \) here), it follows that \( \beta_n \) is are pure imaginary numbers for all \( n \geq 1 \). It is easy to note that for each \( n \), if \( n\pi / \varepsilon \leq \kappa \), the series consists of two propagating wave modes traveling upward and downward respectively; if \( n\pi / \varepsilon > \kappa \) the series consists of two evanescent wave modes decaying exponentially away from the bottom side and the open aperture of the cavity, respectively.

Using the series (2.5), we may reformulate (2.3)–(2.4) into the following coupled problem:

\[
\begin{align*}
\Delta u_\varepsilon + \kappa^2 u_\varepsilon &= 0 \quad \text{in } \mathbb{R}^+^+, \\
u_\varepsilon &= \sum_{n=0}^{\infty} \left( \alpha_n^+ e^{-i\beta_n x_2} + \alpha_n^- e^{i\beta_n (x_2+d)} \right) \phi_n(x_1) \quad \text{in } D_\varepsilon, \\
\partial_n u_\varepsilon &= 0 \quad \text{on } \partial \Omega \setminus \Gamma_\varepsilon^-, \\
u_\varepsilon &= 0 \quad \text{on } \Gamma_\varepsilon^-, \\
u_\varepsilon(x_1, 0+) &= u_\varepsilon(x_1, 0-), \partial_{x_2} u_\varepsilon(x_1, 0+) = \partial_{x_2} u_\varepsilon(x_1, 0-) \quad \text{on } \Gamma_\varepsilon^+, \\
\lim_{r \to \infty} \sqrt{r} \left( \partial_r u^{\text{sc}}_\varepsilon - i\kappa u^{\text{sc}}_\varepsilon \right) &= 0 \quad \text{in } \mathbb{R}^+^+,
\end{align*}
\]

(2.7)
where $u_{\epsilon}^{\text{sc}} = u_{\epsilon} - (u^{\text{inc}} + u^{\text{ref}})$ in $\mathbb{R}_2^+$. The continuity conditions are imposed along the open aperture $\Gamma_{\epsilon}^+$, where $0^+$ and $0^-$ indicate the limits taken from above and below $\Gamma_{\epsilon}^+$, respectively.

For simplicity of notation, we define $u_{\epsilon}^+(x_1) = u_{\epsilon}(x_1, 0)$ and $u_{\epsilon}^-(x_1) = u_{\epsilon}(x_1, 0)$. Let $u_{\epsilon,n}^{\pm} = \langle u_{\epsilon}^{\pm}, \phi_n \rangle_{\Gamma_{\epsilon}^\pm}$ be the Fourier coefficients for $u_{\epsilon}^+$ and $u_{\epsilon}^-$, respectively, where the inner product $\langle \cdot, \cdot \rangle_{\Gamma_{\epsilon}^\pm}$ is defined by

$$\langle u_{\epsilon}^{\pm}, \phi_n \rangle_{\Gamma_{\epsilon}^\pm} := \int_0^\epsilon u_{\epsilon}^{\pm}(x_1) \phi_n(x_1) \, dx_1.$$  

For any $s \in \mathbb{R}$, denote by $H^s(\Gamma_{\epsilon}^\pm)$ the trace functional space on $\Gamma_{\epsilon}^\pm$ with the norm given by

$$\|u_{\epsilon}^+\|^2_{H^s(\Gamma_{\epsilon}^\pm)} = \sum_{n=0}^\infty \left(1 + \left(\frac{n\pi}{\epsilon}\right)^2\right)^s |u_{\epsilon,n}^+|^2.$$  

Matching the series (2.5) on $\Gamma_{\epsilon}^+$, $\Gamma_{\epsilon}^-$ and using the continuity conditions in (2.7), we get

$$\begin{cases}
\alpha_n^+ + \alpha_n^- e^{\beta_n d} = u_{\epsilon,n}^+, \\
\alpha_n^+ e^{\beta_n d} + \alpha_n^- = u_{\epsilon,n}^-.
\end{cases}$$

Solving the above equations by Cramer’s rule yields that

$$\alpha_n^+ = \frac{e^{\beta_n d} u_{\epsilon,n}^+ - u_{\epsilon,n}^-}{e^{2\beta_n d} - 1}, \quad \alpha_n^- = \frac{e^{\beta_n d} u_{\epsilon,n}^+ - u_{\epsilon,n}^-}{e^{2\beta_n d} - 1}.$$  

Therefore, in the cavity $D_{\epsilon}$, the total field

$$u_{\epsilon}(x) = \sum_{n=0}^\infty \left(\frac{e^{\beta_n d} u_{\epsilon,n}^+ - u_{\epsilon,n}^-}{e^{2\beta_n d} - 1} e^{i\beta_n x_2} + \frac{e^{\beta_n d} u_{\epsilon,n}^+ - u_{\epsilon,n}^-}{e^{2\beta_n d} - 1} e^{i\beta_n (x_2+d)}\right) \phi_n(x_1).$$

Since $u_{\epsilon}$ satisfies the homogeneous Dirichlet boundary condition on $\Gamma_{\epsilon}^-$, we have

$$\partial_{\Gamma_{\epsilon}^\pm} u_{\epsilon}(x_1, 0^-) = \sum_{n=0}^\infty i\beta_n \left(\frac{e^{\beta_n d} u_{\epsilon,n}^+ - u_{\epsilon,n}^-}{e^{2\beta_n d} - 1} e^{i\beta_n x_2} - \frac{e^{\beta_n d} u_{\epsilon,n}^+ - u_{\epsilon,n}^-}{e^{2\beta_n d} - 1} e^{i\beta_n (x_2+d)}\right) \phi_n(x_1) = 0$$  

on $\Gamma_{\epsilon}^-$,  

It follows from (2.12) that

$$u_{\epsilon,n}^- = 0.$$  

Substituting (2.13) into (2.11) yields

$$\partial_{\Gamma_{\epsilon}^\pm} u_{\epsilon}(x_1, 0^-) = \sum_{n=0}^\infty i\beta_n \frac{e^{\beta_n d}}{e^{2\beta_n d} - 1} u_{\epsilon,n}^+ \phi_n(x_1)$$  

on $\Gamma_{\epsilon}^+$.

Given a function $u \in H^1(\mathbb{R}_2^+)$ and let $u^+ = u(x_1, 0)$, we define a Dirichlet-to-Neumann (DtN) operator:

$$\mathcal{B}^{\text{PMC}}[u] = \sum_{n=0}^\infty i\beta_n \frac{e^{\beta_n d}}{e^{2\beta_n d} - 1} u_{\epsilon,n}^+ \phi_n(x_1)$$  

on $\Gamma_{\epsilon}^+$.
where the Fourier coefficients \( u_n^+ = \langle u^+, \phi_n \rangle_{\Gamma^+_e} \).

**Lemma 2.1** The DtN operator \( \mathcal{B}^{PMC} \) is bounded from \( H^{1/2}(\Gamma^+_e) \to H^{-1/2}(\Gamma^+_e) \), i.e.,

\[
\| \mathcal{B}^{PMC}[u] \|_{H^{-1/2}(\Gamma^+_e)} \leq C(\lambda, \epsilon) \| u \|_{H^{1/2}(\Gamma^+_e)} \quad \forall u \in H^{1/2}(\Gamma^+_e),
\]

where the constant \( C(\lambda, \epsilon) = \max \left\{ 4, \frac{k}{\tan \kappa d} \right\} \).

**Proof** For any \( u, w \in H^{1/2}(\Gamma^+_e) \), they have the Fourier series expansions

\[
u = \sum_{n=0}^{\infty} u_n^+ \phi_n(x_1), \quad w = \sum_{n=0}^{\infty} w_n^+ \phi_n(x_1),
\]

where \( u_n^+ = \langle u, \phi_n \rangle_{\Gamma^+_e} \) and \( w_n^+ = \langle w, \phi_n \rangle_{\Gamma^+_e} \). It follows from (2.15) that

\[
\langle \mathcal{B}^{PMC}[u], w \rangle_{\Gamma^+_e} = \sum_{n=0}^{\infty} i \beta_n e^{i2\beta_n d} + 1 u_n^+ \bar{w}_n^+.
\]

Note that the coefficients \( \beta_n \) are given by (2.6). For \( n = 0 \), a straightforward calculation yields that

\[
|e^{i2\kappa d} + 1 u_0^+ \bar{w}_0^+| = \frac{\kappa |\cos \kappa d|}{|\sin \kappa d|} |u_0^+||\bar{w}_0^+| \leq \frac{\kappa}{|\tan \kappa d|} |u_0^+||\bar{w}_0^+|,
\]

where we use the fact that \( d \ll \lambda \), which implies \( \kappa d = \frac{2\pi d}{\lambda} \) tends to zero and \( \kappa / \tan(\kappa d) \to 1/d \). For \( n \geq 1 \), using the fact that \( \epsilon \ll \lambda \), we get

\[
|e^{i2\beta_n d} + 1 u_n^+ \bar{w}_n^+| \leq \frac{4n\pi}{\epsilon} |u_n^+||\bar{w}_n^+| \leq 4 \left( 1 + \left( \frac{n\pi}{\epsilon} \right)^2 \right)^{1/2} |u_n^+| \left( 1 + \left( \frac{n\pi}{\epsilon} \right)^2 \right)^{1/2} |\bar{w}_n^+|,
\]

where we use the estimates

\[
|\beta_n| = \sqrt{\left( \frac{n\pi}{\epsilon} \right)^2 - \kappa^2} \leq \frac{n\pi}{\epsilon}, \quad \frac{e^{i2\beta_n d} + 1}{e^{i2\beta_n d} - 1} = \frac{e^{-2\sqrt{(n\pi/\epsilon)^2 - \kappa^2} d} + 1}{e^{-2\sqrt{(n\pi/\epsilon)^2 - \kappa^2} d} - 1} \leq 4.
\]

Combining (2.16)–(2.17) and using (2.8), we obtain from the Cauchy–Schwarz inequality that

\[
|\langle \mathcal{B}^{PMC}[u], w \rangle_{\Gamma^+_e}| \leq C(\lambda, \epsilon) \| u \|_{H^{1/2}(\Gamma^+_e)} \| w \|_{H^{1/2}(\Gamma^+_e)},
\]

which completes the proof. \( \square \)

Using (2.14)–(2.15) and the continuity of \( \partial_{\kappa^\epsilon} u_\epsilon \) on \( \Gamma^+_e \), we obtain the transparent boundary condition (TBC):

\[
\partial_{\kappa^\epsilon} u_\epsilon = \mathcal{B}^{PMC}[u_\epsilon] \quad \text{on} \quad \Gamma^+_e.
\]

Hence the problem (2.3)–(2.4) can be reduced to the following boundary value problem:

\[
\begin{align*}
\Delta u_\epsilon + \kappa^2 u_\epsilon &= 0 \quad \text{in} \quad \mathbb{R}^2_+, \\
\partial_\nu u_\epsilon &= 0 \quad \text{on} \quad \Gamma_0 \setminus \Gamma^+_e, \\
\partial_{\kappa^\epsilon} u_\epsilon &= \mathcal{B}^{PMC}[u_\epsilon] \quad \text{on} \quad \Gamma^+_e, \\
\lim_{\tau \to \infty} \sqrt{\tau} (\partial_\tau u_\epsilon^{sc} - i\kappa u_\epsilon^{sc}) &= 0 \quad \text{in} \quad \mathbb{R}^2_+.
\end{align*}
\]
By calculating the Fourier coefficients $u_{e,n}^+$, which give the coefficients $u_{e,n}^-$ by (2.13), we may obtain the solution in the cavity $D_\epsilon$ from the formula (2.10).

To find an approximate model to (2.18), we examine the series (2.5) more closely. Note that if $\epsilon \ll \lambda$, the wave modes $e^{-i\beta_n x} \phi_n(x_1)$ and $e^{i\beta_n (x_2+d)} \phi_n(x_1)$ decay exponentially in the cavity, with a decaying rate of $O(e^{-\eta/\epsilon})$ for all $n \geq 1$. Only the leading wave modes $e^{-ix_2} \phi_0(x_1)$ and $e^{ix(x_2+d)} \phi_0(x_1)$ propagate in the cavity. The observation motivates us to approximate the DtN map (2.15) by dropping the high order modes and define an approximate DtN map by

$$\mathcal{R}_0^{\text{PMC}}[v] = i k \frac{e^{i 2 \kappa d} + 1}{e^{i 2 \kappa d} - 1} v_0^+ \phi_0(x_1).$$  \hfill (2.19)

Now we arrive at an approximate model problem:

$$\begin{cases}
\Delta v_\epsilon + \kappa^2 v_\epsilon = 0 & \text{in } \mathbb{R}^2_+,
\partial_\nu v_\epsilon = 0 & \text{on } \Gamma_0 \setminus \Gamma_\epsilon^+,
\partial_{x_2} v_\epsilon = \mathcal{R}_0^{\text{PMC}}[v_\epsilon] & \text{on } \Gamma_\epsilon^+,
\lim_{r \to \infty} \sqrt{r} (\partial_r v_\epsilon^{\text{sc}} - i \kappa v_\epsilon^{\text{inc}}) = 0 & \text{in } \mathbb{R}^2_+,
\end{cases}$$  \hfill (2.20)

where $v_\epsilon = 0$ on $\Gamma_\epsilon^-$ and $v_\epsilon^{\text{sc}} = v_\epsilon - (u_\epsilon^{\text{inc}} + u_\epsilon^{\text{ref}})$ in $\mathbb{R}^2_+$. Accordingly, we may approximate $u_\epsilon$ by one single mode:

$$v_\epsilon(x) = (\tilde{a}_0^+ e^{-i x_2} + \tilde{a}_0^- e^{i x(x_2 + d)}) \phi_0(x_1), \quad x \in D_\epsilon,$$  \hfill (2.21)

where the coefficients $\tilde{a}_0^+$ and $\tilde{a}_0^-$ are given by

$$\tilde{a}_0^+ = \frac{e^{i x d} v_\epsilon,0 - v_\epsilon^+}{e^{i 2 \kappa d} - 1}, \quad \tilde{a}_0^- = \frac{e^{i x d} v_\epsilon,0 - v_\epsilon^-}{e^{i 2 \kappa d} - 1}.$$  \hfill (2.22)

It follows from $v_\epsilon = 0$ on $\Gamma_\epsilon^-$ that $\tilde{a}_0^- = -\tilde{a}_0^+ e^{i x d}$, which yields

$$v_\epsilon,0 = 0.$$  \hfill (2.23)

### 2.3 Enhancement of the approximated field

This section introduces the estimates for the solution of the approximate model problem.

**Theorem 2.2** Let $v_\epsilon$ be the solution of the approximate model problem (2.20) and be given by (2.21) in the cavity, then the following estimate holds:

$$\|\nabla v_\epsilon\|_{L^2(D_\epsilon)} = \frac{\kappa \sqrt{\epsilon d}}{\sin \kappa d} (1 + O(\epsilon \ln \epsilon)).$$

**Proof** In $\mathbb{R}^2_+$, since the incident field $u_\epsilon^{\text{inc}}$ and the reflected field $u_\epsilon^{\text{ref}}$ satisfy the Helmholtz equation in (2.20), the scattered field $v_\epsilon^{\text{sc}}$ also satisfies

$$\Delta v_\epsilon^{\text{sc}} + \kappa^2 v_\epsilon^{\text{sc}} = 0 \quad \text{in } \mathbb{R}^2_+.$$  

Noting that $\partial_{x_2} u_\epsilon^{\text{inc}} + \partial_{x_2} u_\epsilon^{\text{ref}} = 0$ on $\Gamma_0$, we have

$$\partial_{x_2} v_\epsilon^{\text{sc}} = 0 \quad \text{on } \Gamma_0 \setminus \Gamma_\epsilon^+.$$  

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Let $G$ be the half-space Green function of the Helmholtz equation with Neumann boundary condition, i.e., it satisfies
\[
\begin{aligned}
\Delta G(x, y) + \kappa^2 G(x, y) &= \delta(x, y), \quad x, y \in \mathbb{R}_+^2, \\
\frac{\partial G(x, y)}{\partial y} &= 0, \quad \text{on } \Gamma_0.
\end{aligned}
\]

It is easy to note that
\[
G(x, y) = -\frac{i}{4} \left( H_0^{(1)}(\kappa |x - y|) + H_0^{(1)}(\kappa |x' - y|) \right),
\]
where $y = (y_1, y_2)$, $H_0^{(1)}$ is the Hankel function of the first kind with order 0, and $x'$ is the reflection of the point $x$ with respect to $\Gamma_0$, i.e., $x' = (x_1, -x_2)$.

It follows from Green’s identity that
\[
v^sc_\epsilon(x) = \int_{\Gamma_+^\epsilon} G(x, y) \frac{\partial v^sc_\epsilon(y)}{\partial y} \, ds_y, \quad x \in \mathbb{R}_+^2,
\]
which gives
\[
v_\epsilon(x) = u^{\text{inc}}(x) + u^{\text{ref}}(x) + \frac{i}{2} \int_{\Gamma_+^\epsilon} H_0^{(1)}(\kappa |x-y|) \frac{\partial v^sc_\epsilon(y)}{\partial y} \, ds_y, \quad x \in \mathbb{R}_+^2.
\]

By the fact that $\partial_{x_2} u^{\text{inc}} + \partial_{x_2} u^{\text{ref}} = 0$ on $\Gamma_0$, especially on $\Gamma_+^\epsilon$, we have
\[
\frac{\partial v^sc_\epsilon}{\partial y} = \frac{\partial v_\epsilon}{\partial y} - \frac{\partial (u^{\text{inc}} + u^{\text{ref}})}{\partial y} = \frac{\partial v_\epsilon}{\partial y} \quad \text{on } \Gamma_+^\epsilon.
\]

It follows from the continuity of single layer potentials \([16,17]\) and the above equality that
\[
v_\epsilon(x) = u^{\text{inc}}(x) + u^{\text{ref}}(x) - \frac{i}{2} \int_{\Gamma_+^\epsilon} H_0^{(1)}(\kappa |x-y|) \frac{\partial v_\epsilon(y)}{\partial y} \, ds_y, \quad x \in \Gamma_+^\epsilon. \tag{2.25}
\]

Using (2.21)-(2.23) yields that
\[
\partial_{x_2} v_\epsilon = i\kappa \left(-\tilde{\alpha}_0^+ + \tilde{\alpha}_0^- e^{i\kappa d}\right) \phi_0(x_1) \quad \text{on } \Gamma_+^\epsilon.
\]

Substituting the above equality into (2.25) and using the fact that $\phi_0(x_1) = \frac{1}{\sqrt{\epsilon}}$, we get
\[
v_\epsilon(x_1, 0) = u^{\text{inc}}(x_1, 0) + u^{\text{ref}}(x_1, 0) + \frac{\kappa}{2} \left(-\tilde{\alpha}_0^+ + \tilde{\alpha}_0^- e^{i\kappa d}\right) \frac{1}{\sqrt{\epsilon}} h_1(x_1), \quad x_1 \in (0, \epsilon),
\]
where
\[
h_1(x_1) = \int_0^\epsilon H_0^{(1)}(\kappa |x_1 - y|) \, dy_1. \tag{2.26}
\]

Therefore, the Fourier coefficients $v_{\epsilon,0}^+$ may be expressed as
\[
v_{\epsilon,0}^+ = (u^{\text{inc}}, \phi_0)_{\Gamma_+^\epsilon} + (u^{\text{ref}}, \phi_0)_{\Gamma_+^\epsilon} + \frac{\kappa}{2} \left(-\tilde{\alpha}_0^+ + \tilde{\alpha}_0^- e^{i\kappa d}\right) \frac{1}{\sqrt{\epsilon}} (h_1, \phi_0)_{\Gamma_+^\epsilon}.
\]

It follows from the fact $u^{\text{inc}}(x_1, 0) = u^{\text{ref}}(x_1, 0)$ and (2.21) that
\[
\tilde{\alpha}_0^+ + \tilde{\alpha}_0^- e^{i\kappa d} = 2 (u^{\text{inc}}, \phi_0)_{\Gamma_+^\epsilon} + \frac{\kappa}{2\sqrt{\epsilon}} (h_1, \phi_0)_{\Gamma_+^\epsilon} \left(-\tilde{\alpha}_0^+ + \tilde{\alpha}_0^- e^{i\kappa d}\right).
\]
By $\tilde{\alpha}_0^- = -\tilde{\alpha}_0^+ e^{i\kappa d}$ and the above equation, we obtain

$$\tilde{\alpha}_0^+ = \frac{2\langle u^{inc}, \phi_0 \rangle_{\Gamma}^+}{(1 + c_0) - (1 - c_0)e^{i2\kappa d}}, \quad \tilde{\alpha}_0^- = -\frac{2e^{i\kappa d} \langle u^{inc}, \phi_0 \rangle_{\Gamma}^+}{(1 + c_0) - (1 - c_0)e^{i2\kappa d}}, \quad (2.27)$$

where $c_0 := \frac{\kappa}{2}\sqrt{\epsilon} \langle h_1, \phi_0 \rangle_{\Gamma}^+$. Note that $x_1, y_1 \in \Gamma_\epsilon^+$ for small $|x_1 - y_1|$, asymptotically, it holds that (cf. [36])

$$H_0^{(1)}(\kappa |x_1 - y_1|) = \frac{2i}{\pi} \ln |x_1 - y_1| + \frac{2i}{\pi} \ln \frac{\kappa}{2} + \gamma_0 + O(|x_1 - y_1|^2 \ln |x_1 - y_1|),$$

where $\gamma_0$ is the Euler constant defined by $\gamma_0 = \lim_{\tau \to \infty} \sum_{m=1}^{\tau} \frac{1}{m} - \ln \tau$. A direct calculation yields

$$c_0 = \frac{\kappa}{2\sqrt{\epsilon}} \langle h_1, \phi_0 \rangle_{\Gamma}^+ = \frac{\kappa}{2\epsilon} \int_0^{\epsilon} \int_0^{\epsilon} H_0^{(1)}(\kappa |x_1 - y_1|) dy_1 dx_1$$

$$= \frac{i\kappa}{\pi} \left( \epsilon \ln \epsilon - \frac{3}{2} \epsilon \right) + \frac{\kappa}{2} \left( \frac{2i}{\pi} \ln \frac{\kappa}{2} + \gamma_0 \right) \epsilon + O(\epsilon^3 \ln \epsilon), \quad (2.28)$$

which gives

$$|(1 + c_0) - (1 - c_0)e^{i2\kappa d}|$$

$$= \left| \kappa \gamma_0 \epsilon \cos \kappa d + i2 \left( \frac{\kappa}{\pi} \left( \epsilon \ln \epsilon - \frac{3}{2} \epsilon + \epsilon \ln \frac{\kappa}{2} \right) \cos \kappa d - \sin \kappa d \right) + O(\epsilon^3 \ln \epsilon) \right|. \quad (2.29)$$

It follows from the assumption $\epsilon \ll d \ll \lambda$ that

$$|(1 + c_0) - (1 - c_0)e^{i2\kappa d}| = 2 \sin(\kappa d) + O(\epsilon \ln \kappa d). \quad (2.29)$$

By the definition of the incident wave, we get

$$|\langle u^{inc}, \phi_0 \rangle_{\Gamma}^+| = \left| \int_0^{\epsilon} e^{ik \sin \theta x_1} \frac{1}{\sqrt{\epsilon}} dx_1 \right| = \frac{1}{\sqrt{\epsilon}} \frac{2 \sin(\kappa \sin \theta \epsilon)}{\kappa \sin \theta} = \sqrt{\epsilon}(1 + O(\kappa^2 \epsilon^2)) \quad (2.30)$$

Using (2.27)–(2.30) gives

$$|\tilde{\alpha}_0^+| = |\tilde{\alpha}_0^-| = \frac{\sqrt{\epsilon} + O(\kappa^2 \epsilon^{3/2})}{2 \sin \kappa d + O(\epsilon \ln \kappa d)} = \frac{\sqrt{\epsilon}}{2 \sin \kappa d}(1 + O(\epsilon \ln \epsilon)).$$

It follows from (2.21) that

$$\|\nabla u_\epsilon\|_{L^2(D_\epsilon)}^2 \leq \int_0^{\epsilon} \int_{-d}^{d} \left( \kappa (|\tilde{\alpha}_0^+|^2 + |\tilde{\alpha}_0^-|^2) \phi_0(x_1) \right)^2 dx_2 dx_1 \leq \frac{\epsilon \kappa^2 d}{\sin^2(\kappa d)}(1 + O(\epsilon \ln \epsilon)).$$

Substituting (2.27) into (2.21) yields

$$v_\epsilon(x) = \left( \frac{2\langle u^{inc}, \phi_0 \rangle_{\Gamma}^+ e^{-ikx_2}}{(1 + c_0) - (1 - c_0)e^{i2\kappa d}} - \frac{2e^{i\kappa d} \langle u^{inc}, \phi_0 \rangle_{\Gamma}^+ e^{ik(x_2 + d)}}{(1 + c_0) - (1 - c_0)e^{i2\kappa d}} \right) \phi_0(x_1). \quad (2.31)$$

We have from a simple calculation that

$$\frac{\partial v_\epsilon}{\partial x_2} = -i\kappa e^{i\kappa d} \frac{2\langle u^{inc}, \phi_0 \rangle_{\Gamma}^+}{(1 + c_0) - (1 - c_0)e^{i2\kappa d}} \phi_0(x_1) \left( e^{-ik(x_2 + d)} + e^{ik(x_2 + d)} \right)$$

$$= -i\kappa e^{i\kappa d} \frac{2\langle u^{inc}, \phi_0 \rangle_{\Gamma}^+}{(1 + c_0) - (1 - c_0)e^{i2\kappa d}} 2\cos \kappa (x_2 + d) \phi_0(x_1). \quad (2.32)$$
Noting that $\epsilon$ is small enough and $\kappa(x_2 + d) \ll 1$, we have from (2.29)–(2.30) and (2.32) that there exists a positive constant $C_1$ such that
\[
\left| \frac{\partial v_\epsilon}{\partial x_2} \right| \geq \frac{\kappa}{\sin \kappa d} (1 + O(\epsilon \ln \epsilon)).
\]
Combining the above estimates yields
\[
\| \nabla v_\epsilon \|_{L^2(D_\epsilon)}^2 \geq \int_0^\epsilon \int_{-d}^0 \left| \frac{\partial v_\epsilon}{\partial x_2} \right|^2 dx_2 dx_1 \geq \frac{\epsilon \kappa^2 d}{\sin^2 \kappa d} (1 + O(\epsilon \ln \epsilon)),
\]
which completes the proof.

**Theorem 2.3** Let $v_\epsilon$ be the solution of the approximate model problem (2.20) and be given by (2.21), then the following estimates hold:
\[
\| v_\epsilon \|_{L^2(D_\epsilon)} = \frac{1}{\sin(\kappa d)} \left( \frac{\epsilon(2\kappa d - \sin(2\kappa d))}{\kappa} \right) (1 + O(\epsilon \ln \epsilon)).
\]

**Proof** It follows from (2.31) that
\[
\int_0^\epsilon \int_{-d}^0 |v_\epsilon(x_1, x_2)|^2 dx_1 dx_2 = \frac{\left| 2(\mu^{inc}, \phi_0)_{\Gamma^+_\epsilon} e^{ikd} \right|^2}{\left| (1 + c_0) - (1 - c_0) e^{2ikd} \right|^2} \int_0^0 \left| e^{-ik(x_2+d)} - e^{ik(x_2+d)} \right|^2 dx_2
\]
\[
= \frac{\left| 2(\mu^{inc}, \phi_0)_{\Gamma^+_\epsilon} \right|^2}{\left| (1 + c_0) - (1 - c_0) e^{2ikd} \right|^2} \int_{-d}^0 4 \sin^2 \kappa(x_2 + d) dx_2.
\]
Noting that $d \ll \lambda$, we have
\[
\int_{-d}^0 4 \sin^2 \kappa(x_2 + d) dx_2 = \frac{2\kappa d - \sin(2\kappa d)}{\kappa}.
\]
It follows from (2.29)–(2.30) that
\[
\frac{\left| 2(\mu^{inc}, \phi_0)_{\Gamma^+_\epsilon} e^{ikd} \right|^2}{\left| (1 + c_0) - (1 - c_0) e^{2ikd} \right|^2} = \frac{\epsilon}{\sin^2(\kappa d)} (1 + O(\epsilon \ln \epsilon)).
\]
Combining the above estimates, we obtain
\[
\| v_\epsilon \|_{L^2(D_\epsilon)}^2 = \frac{\epsilon(2\kappa d - \sin(2\kappa d))}{\kappa \sin^2(\kappa d)} (1 + O(\epsilon \ln \epsilon)),
\]
which completes the proof. \qed

### 2.4 Accuracy of the approximate model

To show the accuracy of the approximate model, we introduce a TBC in $\mathbb{R}^2_+$ and reformulate the problems (2.18) and (2.20) in a bounded domain. Let $B^+_R = \{ x \in \mathbb{R}^2 : |x - x_e| < R, \ x_2 > 0 \}$ be the upper half disc with radius $R$ centered at $x_e = (\epsilon/2, 0)$, where $R$ is a sufficiently large positive real number. Denote by $\partial B^+_R = \{ x \in \mathbb{R}^2 : |x - x_e| = R, \ x_2 > 0 \}$ be the upper half circle. Let $\Gamma^+_0 = \{ x \in \mathbb{R}^2 : |x_1 - \epsilon/2| < R, \ x_2 = 0 \}$ be the line segment on $\Gamma_0$ with length $2R$. The problem geometry is shown in Fig. 2.
In the exterior domain $\mathbb{R}^2_+ \setminus \tilde{B}_R^+$, by noting the PEC condition on $\Gamma_0 \setminus \Gamma^+_R$ and the radiation condition (2.4), the scattered field can be expressed as the Fourier series:

$$u^{sc}_\epsilon (r, \eta) = \sum_{n=0}^{\infty} H^{(1)}_n (\kappa r) u^{sc, n}_{\epsilon, \cos(n\eta)}, \quad 0 < \eta < \pi,$$

where $u^{sc, n}_{\epsilon, \cos(n\eta)} = \frac{2}{\pi} \int_0^\pi u^{sc}_\epsilon (R, \eta) \cos(n\eta) d\eta$, $H^{(1)}_n$ is the Hankel function of the first kind with order $n$, and the polar coordinate $r = \sqrt{(x_1 - \epsilon/2)^2 + x_2^2}$. Taking the normal derivative of $u^{sc}_\epsilon$ on $\partial B^+_R$ yields

$$\frac{\partial u^{sc}_\epsilon (R, \eta)}{\partial r} = \sum_{n=0}^{\infty} \frac{\kappa (H^{(1)}_n)'(\kappa R)}{H^{(1)}_n (\kappa R)} u^{sc, n}_{\epsilon, \cos(n\eta)}. \quad (2.34)$$

For any $v \in H^{1/2}(\partial B^+_R)$ with the Fourier expansion

$$v = \sum_{n=0}^{\infty} v_n \cos(n\eta), \quad v_n = \frac{2}{\pi} \int_0^\pi v(\eta) \cos(n\eta) d\eta,$$

we define the DtN operator on $\partial B^+_R$:

$$(\mathcal{T} v)(\eta) = \sum_{n=0}^{\infty} \frac{\kappa (H^{(1)}_n)'(\kappa R)}{H^{(1)}_n (\kappa R)} v_n \cos(n\eta), \quad 0 < \eta < \pi. \quad (2.35)$$

It follows from (2.34) and (2.35) that the normal derivative of total field on $\partial B^+_R$ can be written as

$$\partial_r u_\epsilon = \partial_r u^{sc}_\epsilon + \partial_r (u^{inc} + u^{ref}) = \mathcal{T} (u_\epsilon) + g, \quad (2.36)$$

where $g = \partial_r (u^{inc} + u^{ref}) - \mathcal{T} (u^{inc} + u^{ref})$.

**Lemma 2.4** The DtN operator $\mathcal{T} : H^{1/2}(\partial B^+_R) \to H^{-1/2}(\partial B^+_R)$ is bounded.
**Proof** Let $A_n = \frac{(H_n^{(1)}(\kappa R))'}{H_n^{(1)}(\kappa R)}$. By the relation of the Hankel function and the modified Bessel function (cf. [1]), it yields that

$$|A_n| = \left| \frac{K_n'(i\kappa R)}{K_n(i\kappa R)} \right|,$$

where $K_n$ is the modified Bessel function of the second kind. Using the formula $zK_n'(z) = -nK_n(z) - zK_{n-1}(z)$, we obtain

$$|A_n| = \left| \frac{K_n'(i\kappa R)}{K_n(i\kappa R)} \right| = \left| \frac{n}{i\kappa R} + \frac{K_{n-1}(i\kappa R)}{K_n(i\kappa R)} \right| \leq \frac{n}{\kappa R} + 1 \leq c\sqrt{1 + n^2},$$

where $c$ is a positive constant.

For any $v, w \in H^{1/2}(\partial B^+_R)$, we have from the definition of (2.35) that

$$\left| (\mathcal{T}v, w)_{\partial B^+_R} \right| = \kappa R \int_0^\pi \sum_{n=0}^\infty A_n v_n \cos(n\eta) \bar{w}_n d\eta \leq \frac{c\kappa R}{2} \sum_{n=0}^\infty \sqrt{1 + n^2} |v_n| |\bar{w}_n|$$

$$\leq \frac{c\kappa R}{2} \left( \sum_{n=0}^\infty \sqrt{1 + n^2} |v_n|^2 \right)^{1/2} \left( \sum_{n=0}^\infty \sqrt{1 + n^2} |w_n|^2 \right)^{1/2}$$

$$= C(R)\kappa \|v\|_{H^{1/2}(\partial B^+_R)} \|w\|_{H^{1/2}(\partial B^+_R)}.$$

Thus we have

$$\|\mathcal{T}v\|_{H^{1/2}(\partial B^+_R)} = \sup_{w \in H^{1/2}(\partial B^+_R)} \frac{|(\mathcal{T}v, w)_{\partial B^+_R}|}{\|w\|_{H^{1/2}(\partial B^+_R)}} \leq C(R)\kappa \|v\|_{H^{1/2}(\partial B^+_R)},$$

which completes the proof. \(\Box\)

**Lemma 2.5** We have

$$\text{Re} (\mathcal{T}v, v)_{\partial B^+_R} \leq 0, \quad \text{Im} (\mathcal{T}v, v)_{\partial B^+_R} > 0, \quad \forall v \in H^{1/2}(\partial B^+_R).$$

**Proof** It follows from the definition (2.35) that

$$(\mathcal{T}v, v)_{\partial B^+_R} = \frac{\kappa R}{2} \sum_{n=0}^\infty A_n |v_n|^2.$$ \hfill (2.37)

Note that $H_n^{(1)} = J_n + iY_n$, where $J_n$ and $Y_n$ are the Bessel functions of the first and second kind, respectively, and the modulus of each Hankel function is decreasing function [1], then

$$\text{Re} A_n = \frac{J_n(kR)J_n'(kR) + Y_n(kR)Y_n'(kR)}{J_n^2(kR) + Y_n^2(kR)} = \frac{1}{2} \left( \frac{J_n^2'(kR) + Y_n^2(kR)}{J_n^2(kR) + Y_n^2(kR)} \right) \leq 0. \hfill (2.38)$$

It follows from the Wronskian formula [1],

$$\text{Im} A_n = \frac{J_n(kR)Y_n'(kR) - Y_n(kR)J_n'(kR)}{J_n^2(kR) + Y_n^2(kR)} = \frac{W(J_n(kR), Y_n(kR))}{J_n^2(kR) + Y_n^2(kR)} = \frac{2}{\kappa R} \left( \frac{1}{J_n^2(kR) + Y_n^2(kR)} \right) > 0, \hfill (2.39)$$
where $W(\cdot, \cdot)$ denotes the Wronskian determinant of $J_n$ and $Y_n$. The proof is completed by combining (2.37)–(2.39).

By the TBC operator (2.36), the model problem (2.18) can be reformulated as follows:

$$
\begin{aligned}
\Delta u_\epsilon + \kappa^2 u_\epsilon &= 0 \quad \text{in } B_R^+, \\
\partial_\nu u_\epsilon &= 0 \quad \text{on } \Gamma_R^+ \setminus \Gamma_\epsilon^+, \\
\partial_\nu u_\epsilon &= \mathcal{B}\text{PMC}[u_\epsilon] \quad \text{on } \Gamma_\epsilon^+, \\
\partial_\nu u_\epsilon &= \mathcal{T}[u_\epsilon] + g \quad \text{on } \partial B_R^+.
\end{aligned}
$$

The variational formulation of (2.40) is in the sense of (2.35), we have

$$
\text{where the sesquilinear form } a(\cdot, \cdot) \text{ yields}
$$

$$
a(u_\epsilon, w) = \langle g, w \rangle_{\partial B_R^+} \quad \forall w \in H^1(B_R^+),
$$

where the sesquilinear form

$$
a(u_\epsilon, w) = \int_{B_R^+} (\nabla u_\epsilon \cdot \nabla \bar{w} - \kappa^2 u_\epsilon \bar{w}) \, dx - \int_{\partial B_R^+} \mathcal{T}[u_\epsilon] \bar{w} \, d\gamma + \int_{\Gamma_\epsilon^+} \mathcal{B}\text{PMC}[u_\epsilon] \bar{w} \, d\gamma.
$$

**Theorem 2.6** The variational problem (2.41) has a unique solution $u_\epsilon \in H^1(B_R^+)$. 

**Proof** By the definition (2.15), we have for sufficiently small $\epsilon$ that

$$
\langle \mathcal{B}\text{PMC}[u_\epsilon], u_\epsilon \rangle_{\Gamma_\epsilon^+} = \sum_{n=0}^{\infty} i\beta_n \frac{e^{i2\beta_n \epsilon} + 1}{e^{i2\beta_n \epsilon} - 1} |u_{\epsilon,n}|^2
$$

$$
= \frac{\kappa \cos(\kappa d)}{\sin(\kappa d)} |u_{\epsilon,0}|^2 + \sum_{n=1}^{\infty} \sqrt{(n\pi/\epsilon)^2 - \kappa^2} \frac{e^{-\sqrt{(n\pi/\epsilon)^2 - \kappa^2}} + 1}{1 - e^{-\sqrt{(n\pi/\epsilon)^2 - \kappa^2}}} |u_{\epsilon,n}|^2 > 0.
$$

It follows from Lemma 2.5 and the above inequality that

$$
\text{Re } \{a(u_\epsilon, u_\epsilon)\} = \| \nabla u_\epsilon \|_{L^2(B_R^+)}^2 - \kappa^2 \| u_\epsilon \|_{L^2(B_R^+)}^2 - \text{Re } \langle \mathcal{T}[u_\epsilon], u_\epsilon \rangle_{\partial B_R^+} + \langle \mathcal{B}\text{PMC}[u_\epsilon], u_\epsilon \rangle_{\Gamma_\epsilon^+}
$$

$$
\geq \| \nabla u_\epsilon \|_{L^2(B_R^+)}^2 - \kappa^2 \| u_\epsilon \|_{L^2(B_R^+)}^2,
$$

which shows that the sesquilinear form $a(\cdot, \cdot)$ satisfies a Gårding type inequality.

Next is show the uniqueness. It suffices to show that $u_\epsilon = 0$ if $g = 0$. A simple calculation yields

$$
\text{Im } a(u_\epsilon, u_\epsilon) = -\text{Im } \langle \mathcal{T}[u_\epsilon], u_\epsilon \rangle_{\partial B_R^+} = -\frac{\kappa \pi R}{2} \sum_{n=0}^{\infty} \text{Im } A_n |u_{\epsilon,n}|^2 = 0.
$$

It follows from Lemma 2.5 that $u_{\epsilon,n} = 0$ for all $n$. Therefore $u_\epsilon = 0$ on $\partial B_R^+$. By the definition (2.35), we have $\mathcal{T} u_\epsilon = 0$, which implies $\partial_\nu u_\epsilon = \mathcal{T} u_\epsilon = 0$ on $\partial B_R^+$ by (2.36). Hence we have $u_\epsilon = 0$ in $\mathbb{R}^2_+ \setminus B_R^+$. It follows from the unique continuation [21] that $u_\epsilon = 0$ in $B_R^+$. The proof is completed by applying the Fredholm alternative theorem.

Correspondingly, the approximate model problem (2.20) can be written as follows:

$$
\begin{aligned}
\Delta v_\epsilon + \kappa^2 v_\epsilon &= 0 \quad \text{in } B_R^+, \\
\partial_\nu v_\epsilon &= 0 \quad \text{on } \Gamma_R^+ \setminus \Gamma_\epsilon^+, \\
\partial_\nu v_\epsilon &= \mathcal{B}_0\text{PMC}[v_\epsilon] \quad \text{on } \Gamma_\epsilon^+, \\
\partial_\nu v_\epsilon &= \mathcal{T}[v_\epsilon] + g \quad \text{on } \partial B_R^+.
\end{aligned}
$$
The variational formulation of (2.43) is to find \( v_\epsilon \in H^1(B_R^+) \) such that
\[
a_0(v_\epsilon, w) = \langle g, w \rangle_{\partial B_R^+} \quad \forall w \in H^1(B_R^+),
\]
where the sesquilinear form
\[
a_0(v_\epsilon, w) = \int_{B_R^+} (\nabla v_\epsilon \cdot \nabla \epsilon - \kappa^2 v_\epsilon \tilde{w}) \, dx - \int_{\partial B_R^+} \mathcal{T}[v_\epsilon] \tilde{w} \, d\gamma + \int_{\Gamma_\epsilon^+} \mathcal{B}_0^{PMC}[v_\epsilon] \tilde{w} \, d\gamma.
\]

The well-posedness of the variational problem (2.44) can be shown similarly to Theorem 2.6. The proof is omitted for brevity.

In the cavity \( D_\epsilon \), the wave fields \( u_\epsilon \) and \( v_\epsilon \) are given by (2.10) and (2.21)–(2.22), respectively. Their Fourier coefficients can be calculated from the solutions obtained in (2.40) and (2.43). The following lemma gives the accuracy of approximate model (2.43) and (2.21)–(2.22).

**Lemma 2.7** Let \( u_\epsilon \) be the expansion of (2.10) inside the cavity with the Fourier coefficients given by the model (2.40), and let \( v_\epsilon \) be the expansion of (2.21)–(2.22) with the Fourier coefficients given by the approximate model (2.43), then the following estimates hold:
\[
\|\nabla u_\epsilon - \nabla v_\epsilon\|_{L^2(D_\epsilon)} \leq C(R) \frac{\kappa^2 \sqrt{d}}{\cos(\kappa d)} \epsilon, \quad \|u_\epsilon - v_\epsilon\|_{L^2(D_\epsilon)} \leq C(R) \frac{\kappa \sqrt{d}}{\cos(\kappa d)} \epsilon \sqrt{\epsilon |\ln \epsilon|},
\]
where \( C(R) \) is a positive constant independent of \( \epsilon \).

**Proof** The proof consists of five steps in order to show the estimates between \( u_\epsilon \) and \( v_\epsilon \).

Step 1: we first give the estimate of \( \|u_\epsilon - u_0\|_{H^1(B_R^+)} \) and \( \|v_\epsilon - u_0\|_{H^1(B_R^+)} \), where \( u_0 \) is the solution of (2.40) when there is no cavity, i.e., \( u_0 \) satisfies the variational problem
\[
b(u_0, w) = \langle g, w \rangle_{\partial B_R^+} \quad \forall w \in H^1(B_R^+),
\]
where the sesquilinear form
\[
b(u_0, w) = \int_{B_R^+} (\nabla u_0 \cdot \nabla \tilde{w} - \kappa^2 u_0 \tilde{w}) \, dx - \int_{\partial B_R^+} \mathcal{T}[u_0] \tilde{w} \, d\gamma.
\]

It follows from (2.41) and (2.46) that
\[
a(u_\epsilon - u_0, w) - b(u_0, w) = \langle g, w \rangle_{\partial B_R^+} - a(u_0, w) = b(u_0, w) - a(u_0, w) = - \int_{\Gamma_\epsilon^+} \mathcal{B}^{PMC}[u_0] \tilde{w} \, d\gamma.
\]

Using (2.15) and (2.19), we have from Lemma 2.1 that
\[
|b(u_0, w) - a(u_0, w)| \leq \left| \int_{\Gamma_\epsilon^+} \mathcal{B}_0^{PMC}[u_0] \tilde{w} \, d\gamma \right| + \left| \int_{\Gamma_\epsilon^+} (\mathcal{B}^{PMC} - \mathcal{B}_0^{PMC})[u_0] \tilde{w} \, d\gamma \right| \\
\leq \frac{\kappa}{|\cot(\kappa d)|} |u_{0+}^+| |\tilde{w}_0^+| + C \|u_0\|_{H^1/2(\Gamma_\epsilon^+)} \|w\|_{H^1/2(\Gamma_\epsilon^+)},
\]
where \( u_{0+}^+ = \langle u_0, \phi_0 \rangle_{\Gamma_\epsilon^+} \) and \( w_{0+}^+ = \langle w, \phi_0 \rangle_{\Gamma_\epsilon^+} \).

Using the estimates [28, Lemma A.1, Lemma A.2]
\[
|u_{0+}^+| \leq \bar{C}(R) \sqrt{\epsilon} \|u_0\|_{H^2(B_R^+)} \quad \text{if } u_0 \in H^2(B_R^+),
\]
\[
|w_{0+}^+| \leq \bar{C}(R) \sqrt{\epsilon} |\ln \epsilon| \|w\|_{H^1(B_R^+)} \quad \text{if } w \in H^1(B_R^+),
\]

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\[ \|u_0\|_{H^{1/2}(\Gamma^+_e)} \leq \tilde{C}(R)\epsilon \|u_0\|_{H^3(B^+_R)} \quad \text{if } u_0 \in H^3(B^+_R), \]
\[ \|w\|_{H^{1/2}(\Gamma^+_e)} \leq \tilde{C}(R)\|w\|_{H^1(B^+_R)} \quad \text{if } w \in H^1(B^+_R), \]

we obtain from (2.48) that
\[
|b(u_0, w) - a(u_0, w)| \leq \tilde{C}(R)\frac{\kappa}{\cot(\kappa d)} \left( \epsilon \sqrt{\ln \frac{\epsilon}{\epsilon}} \|u_0\|_{H^2(B^+_R)} \|w\|_{H^1(B^+_R)} + \epsilon \|u_0\|_{H^3(B^+_R)} \|w\|_{H^1(B^+_R)} \right). 
\]

Since \( u_0 = u^{\text{inc}} + u^{\text{ref}} = e^{i\xi x - d} + e^{-i\xi x - d} \), we have from straightforward calculations that
\[
|b(u_0, w) - a(u_0, w)| \leq \tilde{C}(R)\frac{\kappa}{\cot(\kappa d)} \epsilon \sqrt{\ln \frac{\epsilon}{\epsilon}} \|w\|_{H^1(B^+_R)}. \quad (2.49) 
\]

Denote by \( \mathcal{L}_{\epsilon} \) be the induced operator for the sesquilinear form (2.42) such that
\[
a_{\epsilon}(u_{\epsilon}, w) = (\mathcal{L}_{\epsilon}u_{\epsilon}, w),
\]
where \((\cdot, \cdot)\) is the inner product on \( L^2(B^+_R) \). It follows from [28, Lemma A.3] (or [22, Lemma 4.4]) that there exists a positive constant \( \epsilon_0 \) such that
\[
\|\mathcal{L}_{\epsilon}^{-1}\|_{\mathcal{L}(H^1(B^+_R))} \leq \mathcal{C}(R) \quad \forall \ 0 < \epsilon < \epsilon_0. \quad (2.50) 
\]
where \( \mathcal{C}(R) \) is a constant independent of \( \epsilon \). Combing (2.47)–(2.50) yields
\[
\|u_{\epsilon} - u_0\|_{H^1(B^+_R)} \leq C(R)\frac{\kappa}{\cot(\kappa d)} \epsilon \sqrt{\ln \frac{\epsilon}{\epsilon}},
\]
where \( C(R) \) depends on \( \tilde{C}(R) \) and \( \mathcal{C}(R) \).

Since \( v_{\epsilon} \) satisfies the variational problem (2.44), following the same arguments as above, we may show for sufficiently small \( \epsilon \) that
\[
\|v_{\epsilon} - u_0\|_{H^{3/2}(B^+_R)} \leq C(R)\frac{\kappa}{\cot(\kappa d)} \epsilon \sqrt{\ln \frac{\epsilon}{\epsilon}}. \quad (2.51) 
\]

Step 2: we estimate \( \|v_{\epsilon} - u_0\|_{H^{3/2}(B^+_R)} \). Let \( \xi = v_{\epsilon} - u_0 \). It is easy to verify that \( \xi \) satisfies
\[
\begin{cases}
\Delta \xi + \kappa^2 \xi = 0 & \text{in } B^+_R, \\
\partial_\nu \xi = 0 & \text{on } \Gamma^+_R \setminus \Gamma^+_e, \\
\partial_{x_2} \xi = \partial_{x_3} \xi = 0 & \text{on } \partial B^+_R \\
\partial_\nu \xi = \nabla \xi & \text{on } \partial B^+_R.
\end{cases}
\]

It follows from the standard regularity estimate for the Neumann problem that
\[
\|\xi\|_{H^{3/2}(B^+_R)} \leq C(R)\|\mathcal{B}_{\text{PMC}}^0[v_{\epsilon}]\|_{L^2(\Gamma^+_e)} \\
\leq C(R)\frac{\kappa}{\cot(\kappa d)} \left( \|u_{0,0}^+ + |v_{w,0}^+ - u_{0,0}^+| \|\phi_0\|_{L^2(0,\epsilon)} \right) \\
\leq C(R)\frac{\kappa}{\cot(\kappa d)} \left( \sqrt{\epsilon} \|u_0\|_{H^2(B^+_R)} + \sqrt{\epsilon} \ln \frac{\epsilon}{\epsilon} \|v_{\epsilon} - u_0\|_{H^1(B^+_R)} \right) \\
\leq C(R)\frac{\kappa}{\cot(\kappa d)} \sqrt{\epsilon}, \quad (2.52)
\]

where the last inequality is obtained by using (2.51) and \( C(R) \) may be different constant.

Step 3: This step is to estimate \( \|u_{\epsilon} - v_{\epsilon}\|_{H^1(B^+_R)} \). From (2.41) and (2.44), we have
\[
a(u_{\epsilon} - v_{\epsilon}, w) = a_0(v_{\epsilon}, w) - a(v_{\epsilon}, w). \quad (2.53)
\]
Comparing (2.42) and (2.45) yields
\[
|a_0(v_\epsilon, w) - a(v_\epsilon, w)| \leq \left| \int_{\Gamma_\epsilon^+} (\mathcal{B}_{PMC} - \mathcal{B}_{0}^{PMC}) [v_\epsilon] \tilde{w} \, d\gamma \right|
\leq \left| \int_{\Gamma_\epsilon^+} (\mathcal{B}_{PMC} - \mathcal{B}_{0}^{PMC}) [u_0] \tilde{w} \, d\gamma \right|
+ \left| \int_{\Gamma_\epsilon^+} (\mathcal{B}_{PMC} - \mathcal{B}_{0}^{PMC}) [v_\epsilon - u_0] \tilde{w} \, d\gamma \right|.  \tag{2.54}
\]

Using the estimates (2.48)—(2.49) gives
\[
\left| \int_{\Gamma_\epsilon^+} (\mathcal{B}_{PMC} - \mathcal{B}_{0}^{PMC}) [u_0] \tilde{w} \, d\gamma \right| \leq C(R) \frac{\kappa}{\cot(kd)} \epsilon \|w\|_{H^1(B_\Gamma^+)}.
\]

For the second term in (2.54), we have from the estimate of \(\|v_\epsilon - u_0\|_{H^{1/2}(\Omega)}\) that
\[
\left| \int_{\Gamma_\epsilon^+} (\mathcal{B}_{PMC} - \mathcal{B}_{0}^{PMC}) [v_\epsilon - u_0] \tilde{w} \, d\gamma \right| \leq C(R) \frac{\kappa}{\cot(kd)} \|v_\epsilon - u_0\|_{H^{1/2}(\Omega)} \|w\|_{H^1(B_\Gamma^+)}
\leq C(R) \frac{\kappa}{\cot(kd)} \sqrt{\epsilon} \|v_\epsilon - u_0\|_{H^{1/2}(B_\Gamma^+)} \|w\|_{H^1(B_\Gamma^+)}
\leq C(R) \frac{\kappa}{\cot(kd)} \epsilon \|w\|_{H^1(B_\Gamma^+)}.  \tag{2.55}
\]

For the variational problem (2.53), a combination of the above estimates and the boundedness of inverse of the linear operator \(\mathcal{L}_\epsilon\) in (2.50) lead to the following estimate
\[
\|u_\epsilon - v_\epsilon\|_{H^1(B_\Gamma^+)} \leq C(R) \frac{\kappa}{\cot(kd)} \epsilon.  \tag{2.55}
\]

Step 4: we estimate \(\|\nabla u_\epsilon - \nabla v_\epsilon\|_{L^2(D_\lambda)}\). We have from the expansions of \(u_\epsilon\) in (2.7) and \(v_\epsilon\) in (2.21) that
\[
\|\nabla u_\epsilon - \nabla v_\epsilon\|_{L^2(D_\lambda)}^2 = \int_{-d}^0 k^2 \left| - (\alpha_0^+ - \tilde{\alpha}_0^+) e^{-i\kappa x_2} + (\alpha_0^- - \tilde{\alpha}_0^-) e^{i\kappa (x_2 + d)} \right|^2 \, dx_2
+ \int_{-d}^0 \sum_{n=1}^\infty \left( \frac{n\pi}{\epsilon} \right)^2 \left| \alpha_n^+ e^{-i\beta_n x_2} + \alpha_n^- e^{i\beta_n (x_2 + d)} \right|^2
+ (i\beta_n)^2 \left| - \alpha_n^+ e^{-i\beta_n x_2} + \alpha_n^- e^{i\beta_n (x_2 + d)} \right|^2 \, dx_2
\leq 2\kappa^2 d \left( |\alpha_0^+ - \tilde{\alpha}_0^+|^2 + |\alpha_0^- - \tilde{\alpha}_0^-|^2 \right)
+ \sum_{n=1}^\infty \frac{1 + (n\pi e/2)^2}{n\pi/2\epsilon} \left( |\alpha_n^+|^2 + |\alpha_n^-|^2 \right),
\]
where the Parseval identity is used in the last inequality.

For the first term \((n = 0)\) in the above inequality, following from (2.9) and (2.22) and noting that \(d \ll \lambda\), we get
\[
|\alpha_0^+ - \tilde{\alpha}_0^+|^2 = \frac{|u_{\epsilon,0}^+ - v_{\epsilon,0}^+|^2}{|e^{i2\kappa d} - 1|^2} = \frac{1}{4 \sin^2(kd)} |u_{\epsilon,0}^+ - v_{\epsilon,0}^+|^2,
|\alpha_0^- - \tilde{\alpha}_0^-|^2 = \frac{|e^{i\kappa d}(u_{\epsilon,0}^+ - v_{\epsilon,0}^+)|^2}{|e^{i2\kappa d} - 1|^2} = \frac{1}{4 \sin^2(kd)} |u_{\epsilon,0}^+ - v_{\epsilon,0}^+|^2. \tag{2.56}
\]
For the second term \((n \geq 1)\), by (2.9) and \(i\beta_n d = - d \sqrt{(n\pi/\epsilon)^2 - \kappa^2}\), we have

\[
|\alpha_n|^2 = \frac{|-u_{e,n}^+|^2}{|e^{i2\beta_n d} - 1|^2} \leq 2|u_{e,n}^+|^2, \quad |\alpha_n^-|^2 = \frac{|e^{i\beta_n d} u_{e,n}^+|^2}{|2e^{i\beta_n d} - 1|^2} \leq 2|u_{e,n}^+|^2. \tag{2.57}
\]

Consequently, we obtain

\[
\|\nabla u_e - \nabla v_\epsilon\|_{L^2(D_\epsilon)}^2 \leq C \frac{2d}{\sin^2(\kappa d)} \left( |u_{e,0}^+ - v_\epsilon^+|^2 + \sum_{n=1}^{\infty} \left( 1 + \left( \frac{n\pi}{\epsilon} \right)^2 \right) |u_{e,n}^+|^2 \right) \leq C \frac{2d}{\sin^2(\kappa d)} \left( |u_{e,0}^+ - v_\epsilon^+|^2 + \|u_\epsilon\|_{H^{1/2}(\Gamma_\epsilon^+)}^2 \right),
\]

It follows from (2.52) and (2.55) that

\[
\|u_e\|_{H^{1/2}(\Gamma_\epsilon^+)} \leq \|u_0\|_{H^{1/2}(\Gamma_\epsilon^+)} + \|v_\epsilon - u_0\|_{H^{1/2}(\Gamma_\epsilon^+)} + \|u_\epsilon - v_\epsilon\|_{H^{1/2}(\Gamma_\epsilon^+)} \leq C(R) \frac{\kappa}{\cot(\kappa d)} \left( \frac{\epsilon}{\|u_\epsilon\|_{H^1(B_\epsilon^+)}^2} + \sqrt{\epsilon} \|v_\epsilon - u_0\|_{H^{1/2}(\Gamma_\epsilon^+)}^2 + \|u_\epsilon - v_\epsilon\|_{H^1(B_\epsilon^+)}^2 \right) \leq C(R) \frac{\kappa}{\cot(\kappa d)} \epsilon. \tag{2.58}
\]

On the other hand, we have from (2.55) that

\[
|u_{e,0}^- - v_\epsilon^-| \leq C(R) \sqrt{\epsilon} \ln \epsilon \|u_\epsilon - v_\epsilon\|_{H^1(B_\epsilon^+)} \leq C(R) \frac{\kappa}{\cot(\kappa d)} \sqrt{\epsilon} \ln \epsilon. \tag{2.59}
\]

Combining the above estimates yields

\[
\|\nabla u_e - \nabla v_\epsilon\|_{L^2(D_\epsilon)}^2 \leq C^2(R) \frac{\kappa^4d}{\cos^2(\kappa d)} \epsilon^2.
\]

Step 5: the last step is to estimate \(\|u_e - v_\epsilon\|_{L^2(D_\epsilon)}\). Using the similar method as that in Step 4, we can obtain from the Parseval identity and (2.56)–(2.57) that

\[
\|u_e - v_\epsilon\|_{L^2(D_\epsilon)}^2 \leq 2d \left( |\alpha_e^+ - \tilde{\alpha}_0^+|^2 + |\alpha_e^- - \tilde{\alpha}_0^-|^2 \right) + \sum_{n=1}^{\infty} \frac{1}{n\pi/2\epsilon} \left( |\alpha_n^+|^2 + |\alpha_n^-|^2 \right)
\]

\[
\leq C \frac{d}{\sin^2(\kappa d)} \left( |u_{e,0}^+ - v_\epsilon^+|^2 + \sum_{n=1}^{\infty} \left( \frac{\epsilon}{n\pi} \right)^2 |u_{e,n}^+|^2 \right)
\]

\[
\leq C \frac{d}{\sin^2(\kappa d)} \left( |u_{e,0}^+ - v_\epsilon^+|^2 + \epsilon^2 \sum_{n=1}^{\infty} \left( 1 + \left( \frac{n\pi}{\epsilon} \right)^2 \right) |u_{e,n}^+|^2 \right)
\]

\[
\leq C \frac{d}{\sin^2(\kappa d)} \left( |u_{e,0}^+ - v_\epsilon^+|^2 + \epsilon^2 \|u_\epsilon\|_{H^{1/2}(\Gamma_\epsilon^+)}^2 \right)
\]

\[
\leq C^2(R) \frac{\kappa^2d}{\cos^2(\kappa d)} \left( \epsilon^3 \ln \epsilon + \epsilon^4 \right)
\]

\[
\leq 2C^2(R) \frac{\kappa^2d}{\cos^2(\kappa d)} \left( \epsilon \ln \epsilon \right)^2,
\]

where the estimates (2.58)–(2.59) are used.
2.5 Field enhancement without resonance

The electromagnetic field enhancement factor is defined as the ratio of the energy between the total field and the incident field in the cavity. Hence the electric and magnetic field enhancement factors are given by

\[ Q_E = \frac{\|E_e\|_{L^2(D_e)}}{\|E^{\text{inc}}\|_{L^2(D_e)}} \quad \text{and} \quad Q_H = \frac{\|H_e\|_{L^2(D_e)}}{\|H^{\text{inc}}\|_{L^2(D_e)}}. \]

**Theorem 2.8** There exists a constant \( \epsilon_0 \) depending on \( \lambda \) and \( d \) such that for all \( 0 < \epsilon < \epsilon_0 \) the following estimates hold:

\[ (1 - C(R)\kappa \sqrt{\epsilon} \tan(\kappa d) + O(\sin(\kappa d) \kappa d \epsilon \ln \epsilon)) \frac{1}{\sin \kappa d} \leq Q_E \]

\[ \leq \left(1 + C(R)\kappa \sqrt{\epsilon} \tan(\kappa d) + O(\sin(\kappa d) \kappa d \epsilon \ln \epsilon)\right) \frac{1}{\sin \kappa d} \]

\[ \text{and} \]
\[ \frac{4}{3} C(R)\kappa \epsilon \sqrt{|\ln \epsilon|} + O(\epsilon \ln \epsilon) \leq Q_H \leq \frac{4}{3} C(R)\kappa \epsilon \sqrt{|\ln \epsilon|} + O(\epsilon \ln \epsilon) \quad \text{as} \quad d/\lambda \to 0. \]

**Remark 2.9** The electric field enhancement has an order \( O(1/\sin(\kappa d)) \), which is enormous due to the length scale \( d \ll \lambda \), while the magnetic field has no significant enhancement in this situation.

**Proof** Recalling the scale assumption \( \epsilon \ll d \ll \lambda \), we have from Theorem 2.2 and Lemma 2.7 that

\[ \|\nabla u_e\|_{L^2(D_e)} \leq \|\nabla v_e\|_{L^2(D_e)} + \|\nabla u_e - \nabla v_e\|_{L^2(D_e)} \]
\[ \leq \frac{\kappa \sqrt{\epsilon d}}{\sin(\kappa d)} (1 + O(\epsilon \ln \epsilon)) + C(R) \frac{\kappa^2 \sqrt{d}}{\cos(\kappa d)} \epsilon \]
\[ \leq \left(1 + C(R)\kappa \sqrt{\epsilon} \tan(\kappa d) + O(\sin(\kappa d) \kappa d \epsilon \ln \epsilon)\right) \frac{\kappa \sqrt{\epsilon d}}{\sin \kappa d}. \]

On the other hand,

\[ \|\nabla u_e\|_{L^2(D_e)} \geq \|\nabla v_e\|_{L^2(D_e)} - \|\nabla u_e - \nabla v_e\|_{L^2(D_e)} \]
\[ \geq \frac{\kappa \sqrt{\epsilon d}}{\sin(\kappa d)} (1 + O(\epsilon \ln \epsilon)) - C(R) \frac{\kappa^2 \sqrt{d}}{\cos(\kappa d)} \epsilon \]
\[ \geq \left(1 - C(R)\kappa \sqrt{\epsilon} \tan(\kappa d) + O(\sin(\kappa d) \kappa d \epsilon \ln \epsilon)\right) \frac{\kappa \sqrt{\epsilon d}}{\sin \kappa d}. \]

A straightforward calculation yields

\[ \|\nabla u^{\text{inc}}\|_{L^2(D_e)}^2 = \int_{-d}^d \int_0^\epsilon |\nabla u^{\text{inc}}(x)|^2 \, dx = \kappa^2 \epsilon d. \]

Combining the above estimates gives

\[ \left(1 - C(R)\kappa \sqrt{\epsilon} \tan(\kappa d) + O(\sin(\kappa d) \kappa d \epsilon \ln \epsilon)\right) \frac{1}{\sin \kappa d} \leq \frac{\|\nabla u_e\|_{L^2(D_e)}}{\|\nabla u^{\text{inc}}\|_{L^2(D_e)}} \]
\[ \leq \left(1 + C(R)\kappa \sqrt{\epsilon} \tan(\kappa d) + O(\sin(\kappa d) \kappa d \epsilon \ln \epsilon)\right) \frac{1}{\sin \kappa d}. \]
Noting that $|\nabla \times \mathbf{H}_\epsilon| = |\nabla \times (0, 0, u_\epsilon)| = |\nabla u_\epsilon|$ and $\nabla \times \mathbf{H}_\epsilon = -i\omega \mathbf{E}_\epsilon$, we obtain

$$
\left(1 - C(R)\kappa \sqrt{\epsilon} \tan(\kappa d) + O(\sin(\kappa d)\kappa d \ln \epsilon)\right) \frac{1}{\sin \kappa d} \leq \frac{\|\nabla u_\epsilon\|_{L^2(D_\epsilon)}}{\|\nabla u^{\text{inc}}\|_{L^2(D_\epsilon)}} = \frac{\|\mathbf{E}_\epsilon\|_{L^2(D_\epsilon)}}{\|\mathbf{E}^{\text{inc}}\|_{L^2(D_\epsilon)}} = Q_E
$$

$$
\leq \left(1 + C(R)\kappa \sqrt{\epsilon} \tan(\kappa d) + O(\sin(\kappa d)\kappa d \ln \epsilon)\right) \frac{1}{\sin \kappa d}.
$$

For the magnetic field, it follows from Theorem 2.3 and Lemma 2.7 that

$$
\|u_\epsilon\|_{L^2(D_\epsilon)} \leq \|v_\epsilon\|_{L^2(D_\epsilon)} + \|u_\epsilon - v_\epsilon\|_{L^2(D_\epsilon)}
$$

$$
\leq \frac{1}{\sin(\kappa d)} \sqrt{\frac{\epsilon(2 \kappa d - \sin(2 \kappa d))}{\kappa}} (1 + O(\epsilon \ln \epsilon)) + C(R) \frac{\kappa \sqrt{d}}{\cos(\kappa d)} \epsilon \sqrt{\epsilon |\ln \epsilon|}
$$

$$
\leq \sqrt{\epsilon d} \left( \frac{1}{\sin(\kappa d)} \sqrt{2 - \frac{\sin(2 \kappa d)}{\kappa d}} + C(R) \frac{\kappa \epsilon \sqrt{|\ln \epsilon|}}{\cos(\kappa d)} + O(\epsilon \ln \epsilon) \right)
$$

and

$$
\|u_\epsilon\|_{L^2(D_\epsilon)} \geq \|v_\epsilon\|_{L^2(D_\epsilon)} + \|u_\epsilon - v_\epsilon\|_{L^2(D_\epsilon)}
$$

$$
\geq \sqrt{\epsilon d} \left( \frac{1}{\sin(\kappa d)} \sqrt{2 - \frac{\sin(2 \kappa d)}{\kappa d}} - C(R) \frac{\kappa \epsilon \sqrt{|\ln \epsilon|}}{\cos(\kappa d)} + O(\epsilon \ln \epsilon) \right).
$$

Clearly, we have

$$
\|u^{\text{inc}}\|_{L^2(D_\epsilon)}^2 = \int_{-d}^0 \int_0^\epsilon |u^{\text{inc}}(x)|^2 \, dx = \epsilon d.
$$

Therefore, we obtain

$$
\left( \frac{1}{\sin(\kappa d)} \sqrt{2 - \frac{\sin(2 \kappa d)}{\kappa d}} - C(R) \frac{\kappa \epsilon \sqrt{|\ln \epsilon|}}{\cos(\kappa d)} + O(\epsilon \ln \epsilon) \right)
$$

$$
\leq \frac{\|u_\epsilon\|_{L^2(D_\epsilon)}}{\|u^{\text{inc}}\|_{L^2(D_\epsilon)}} = \frac{\|\mathbf{H}_\epsilon\|_{L^2(D_\epsilon)}}{\|\mathbf{H}^{\text{inc}}\|_{L^2(D_\epsilon)}} = Q_H
$$

$$
\leq \left( \frac{1}{\sin(\kappa d)} \sqrt{2 - \frac{\sin(2 \kappa d)}{\kappa d}} + C(R) \frac{\kappa \epsilon \sqrt{|\ln \epsilon|}}{\cos(\kappa d)} + O(\epsilon \ln \epsilon) \right).
$$

Furthermore, the enhancement coefficients satisfy

$$
\lim_{d/\lambda \to 0} \left( \frac{1}{\sin(\kappa d)} \sqrt{2 - \frac{\sin(2 \kappa d)}{\kappa d}} - C(R) \frac{\kappa \epsilon \sqrt{|\ln \epsilon|}}{\cos(\kappa d)} + O(\epsilon \ln \epsilon) \right) = \frac{4}{3} - C(R) \kappa \epsilon \sqrt{|\ln \epsilon|} + O(\epsilon \ln \epsilon),
$$

$$
\lim_{d/\lambda \to 0} \left( \frac{1}{\sin(\kappa d)} \sqrt{2 - \frac{\sin(2 \kappa d)}{\kappa d}} + C(R) \frac{\kappa \epsilon \sqrt{|\ln \epsilon|}}{\cos(\kappa d)} + O(\epsilon \ln \epsilon) \right) = \frac{4}{3} + C(R) \kappa \epsilon \sqrt{|\ln \epsilon|} + O(\epsilon \ln \epsilon),
$$

which completes the proof. \qed
3 PEC-PMC cavity with resonances

This section is to discuss the Fabry–Perot type resonance and derive the asymptotic expansions for those resonance; analyze quantitatively the field enhancement at the resonance frequencies in a narrow cavity, i.e., $\epsilon \ll \lambda$.

Recall the model problem (2.3)–(2.4):

\[
\begin{align*}
\Delta u_\epsilon + \kappa^2 u_\epsilon &= 0 \quad \text{in } \Omega, \\
\partial_\nu u_\epsilon &= 0 \quad \text{on } \partial \Omega \setminus \Gamma^-_\epsilon, \\
\lim_{r \to \infty} \sqrt{r} \left( \partial_r u^{sc}_\epsilon - i \kappa u^{sc}_\epsilon \right) &= 0 \quad \text{in } \mathbb{R}^2_+. 
\end{align*}
\]

(3.1)

It is known that the scattering problem (3.1) has a unique solution for a complex wavenumber $k$ with $\text{Im} k \geq 0$ (cf., [3]). By analytic continuation, the solution has a meromorphic extension to the whole complex plane, except for a countable number of points, which are poles of the resolvent associated with the scattering problem (3.1). These poles are called the resonances of the scattering problem, and the associated nontrivial solutions are called resonance modes. If the frequency of the incident wave is taken to be close to the real part of the resonance, an enhancement of the field is expected if the imaginary part of the resonance is small.

3.1 Boundary integral equation

Define by $g^{\text{PMC}}_\epsilon(x, y)$ the Green function in the PEC-PMC cavity $D_\epsilon$, i.e., it satisfies

\[
\begin{align*}
\Delta g^{\text{PMC}}_\epsilon(x, y) + \kappa^2 g^{\text{PMC}}_\epsilon(x, y) &= -\delta(x - y), \quad x, y \in D_\epsilon, \\
\frac{\partial g^{\text{PMC}}_\epsilon(x, y)}{\partial \nu_y} &= 0 \quad \text{on } \partial D_\epsilon \setminus \Gamma^-_\epsilon, \\
g^{\text{PMC}}_\epsilon(x, y) &= 0 \quad \text{on } \Gamma^-_\epsilon.
\end{align*}
\]

Since the width of the cavity $\epsilon$ tends to zero, we may assume that $\kappa^2$ is not the eigenvalue of the Laplacian with the above boundary conditions in the cavity. Thus the Green function of the Helmholtz operator in $D_\epsilon$ exists and can be expressed as [15]

\[
g^{\text{PMC}}_\epsilon(x, y) = \sum_{m,n=0}^{\infty} c_{m,n} \phi_{m,n}(x) \phi_{m,n}(y),
\]

where

\[
c_{m,n} = \frac{1}{\kappa^2 - (m\pi/\epsilon)^2 - ((n+1/2)\pi/d)^2},
\]

\[
\phi_{m,n}(x) = \sqrt{\frac{\alpha_{m,n}}{d \epsilon}} \cos \left( \frac{m\pi x_1}{\epsilon} \right) \sin \left( \frac{(n+1/2)\pi}{d} (x_2 + d) \right),
\]

\[
\alpha_{m,n} = \begin{cases} 
2, & m = 0, n \geq 0, \\
4, & m \geq 1, n \geq 0.
\end{cases}
\]

(3.2)
Lemma 3.1 The scattering problem (3.1) is equivalent to the following boundary integral equation:

\[
\int_{\Gamma^+_{\epsilon}} \left( \left( -\frac{i}{2} \right) H^1_0(\kappa|x-y|) + \delta^{\text{PMC}}(x, y, \epsilon) \right) \frac{\partial u^\epsilon_{\text{inc}}(y)}{\partial \nu_y} \, ds_y + u^\text{inc}(x) + u^\text{ref}(x) = 0, \quad x \in \Gamma^+_{\epsilon}.
\]  
(3.3)

**Proof** Noting that \( \partial_{\nu}u^\text{inc} + \partial_{\nu}u^\text{ref} = 0 \) on \( \Gamma_0 \), especially on \( \Gamma^+_{\epsilon} \), we have

\[
\partial_{\nu}u^\epsilon_{\text{sc}} = \partial_{\nu}(u^\text{inc} + u^\text{ref}) = \partial_{\nu}u^\epsilon \quad \text{on} \quad \Gamma^+_{\epsilon}.
\]

It follows from Green’s identity in \( \mathbb{R}^2_+ \) that

\[
u^\epsilon (x) = u^\text{inc}(x) + u^\text{ref}(x) + u^\text{sc}(x) = u^\text{inc}(x) + u^\text{ref}(x) + \int_{\Gamma^+_{\epsilon}} G(x, y) \frac{\partial u^\epsilon_{\text{sc}}(y)}{\partial \nu_y} \, ds_y
\]

\[
= u^\text{inc}(x) + u^\text{ref}(x) + \int_{\Gamma^+_{\epsilon}} G(x, y) \frac{\partial u^\epsilon_{\text{sc}}(y)}{\partial \nu_y} \, ds_y, \quad x \in \mathbb{R}^2_+.
\]  
(3.4)

where the half-space Green function \( G \) is given in (2.24). Using Green’s identity inside the cavity yields

\[
u^\epsilon(x) = -\int_{\Gamma^+_{\epsilon}} g^{\text{PMC}}_e(x, y) \frac{\partial u^\epsilon_{\text{sc}}(y)}{\partial \nu_y} \, ds_y, \quad x \in D_e.
\]  
(3.5)

When \( x \) respectively tends to \( \Gamma^+_{\epsilon} \) from above or below, by the continuity of the single layer potential, we have from (3.4) and (3.5) that

\[
u^\epsilon(x) = u^\text{inc}(x) + u^\text{ref}(x) + \int_{\Gamma^+_{\epsilon}} \left( -\frac{i}{2} \right) H^1_0(\kappa|x-y|) \frac{\partial u^\epsilon_{\text{sc}}(y)}{\partial \nu_y} \, ds_y, \quad x \in \Gamma^+_{\epsilon}
\]

and

\[
u^\epsilon(x) = -\int_{\Gamma^+_{\epsilon}} g^{\text{PMC}}_e(x, y) \frac{\partial u^\epsilon_{\text{sc}}(y)}{\partial \nu_y} \, ds_y, \quad x \in \Gamma^+_{\epsilon}.
\]

The proof is finished by imposing the continuity of the solution on the open aperture \( \Gamma^+_{\epsilon} \). \(\square\)

It is clear to note that

\[
\partial_{\nu}u^\epsilon_{\Gamma^+_{\epsilon}} = \partial_{x_2}u^\epsilon(x_1, 0), \quad u^\text{inc} + u^\text{ref} = 2e^{ik \sin \theta x_1}, \quad x_1 \in (0, \epsilon).
\]

Introduce new variables by rescaling with respect to \( \epsilon: X = x_1/\epsilon, Y = y_1/\epsilon \). Define three functions:

\[
\varphi(Y) = -\partial_{y_2}u^\epsilon(\epsilon Y, 0),
\]

\[
G^\epsilon_e(X, Y) = \left( -\frac{i}{2} \right) H^1_0(\epsilon k|X - Y|),
\]

\[
G^i_e(X, Y) = \delta^{\text{PMC}}(\epsilon X, 0; \epsilon Y, 0) = \sum_{m,n=0}^{\infty} \frac{c_{m,n} \epsilon^{m,n}}{\epsilon d} \cos(m\pi X) \cos(m\pi Y),
\]  
(3.6)

and the boundary integral operators:

\[
(T^\epsilon \varphi)(X) = \int_0^1 G^\epsilon_e(X, Y) \varphi(Y) \, dY, \quad X \in (0, 1).
\]
Lemma 3.2

Hence the boundary integral equation (3.3) is equivalent to the following operator equation:

\[(T^e + T^i)\varphi = f / \epsilon,\]  

(3.8)

where \( f(X) := (u^{inc} + u^{ref})(\epsilon X, 0) = 2e^{i\epsilon \sin \theta e^X}. \)

3.2 Asymptotics of the integral operators

In this subsection, we study the asymptotic properties of the integral operators in (3.7). For clarity, we first introduce several notation below.

\[\Gamma_1(\kappa, \epsilon) = \frac{1}{\pi} (\ln \kappa + \gamma_1) + \frac{1}{\pi} \ln \epsilon, \quad \Gamma_2(\kappa, \epsilon) = -\frac{\tan \kappa d}{\epsilon \kappa} + \frac{2 \ln 2}{\pi}, \]

\[R_1^e(X, Y) = -\frac{\kappa^2}{4\pi} |X - Y|^2 \epsilon^2 \ln \epsilon, \]

\[R_2^e(X, Y) = \left( \frac{\gamma_2 - \ln(\kappa|X - Y|)}{4\pi} \right) \kappa^2 |X - Y|^2 \epsilon^2 + o((\kappa \epsilon |X - Y|)^2), \]

\[R_3^e(X, Y) = -\frac{\epsilon^2 \kappa^2}{\pi} \left( \frac{1}{\pi^2} \sum_{m=1}^{\infty} \frac{1}{m^3} + \frac{(X + Y)^2}{4} \ln(\pi(X + Y)) + \frac{(X - Y)^2}{4} \ln(\pi(X - Y)) \right) + o((X + Y)^2 + (X - Y)^2), \]

where \( \gamma_1 = \gamma_0 - \ln 2 - \frac{\pi^2}{4}, \quad \gamma_2 = -\frac{1}{4\pi} + \frac{i}{8} + \frac{1}{4\pi} (\ln 2 - \gamma_0), \) and \( \gamma_0 \) is the Euler constant.

For the fixed depth of the cavity \( d \) and the wave number \( \kappa \), if \( \epsilon \) small enough, we have the following asymptotic expansions for the kernels \( G_e^e \) and \( G_i^i \).

Lemma 3.2 If \( \epsilon \ll \lambda \), then the following estimates hold:

\[G_e^e = \Gamma_1(\kappa, \epsilon) + \frac{1}{\pi} \ln |X - Y| + R_1^e(X, Y) + R_2^e(X, Y), \]

\[G_i^i = \Gamma_2(\kappa, \epsilon) + \frac{1}{\pi} \left( \ln \left| \sin \left( \frac{\pi(X + Y)}{2} \right) \right| + \ln \left| \sin \left( \frac{\pi(X - Y)}{2} \right) \right| \right) + R_3^e(X, Y), \]

where

\[|R_1^e(X, Y)| \leq C_1 \epsilon^2 |\ln \epsilon|, \quad |R_2^e(X, Y)| \leq C_2 \epsilon^2, \quad |R_3^e(X, Y)| \leq C_3 \epsilon^2. \]

Here \( C_j \), \( j = 1, 2, 3 \) are positive constants independent of \( \epsilon \).

Proof It follows from the asymptotic of the Hankel function near zero (cf. [1]) that

\[G_e^e(X, Y) = \left( -\frac{i}{2} \right) H_0^1(\epsilon \kappa |X - Y|) \]

\[= \frac{1}{\pi} (\gamma_1 + \ln(\epsilon \kappa |X - Y|)) - \frac{1}{4\pi} |\epsilon k(X - Y)|^2 \ln |\epsilon \kappa |X - Y|| \]

\[+ \gamma_2(\epsilon \kappa |X - Y|)^2 + o((\kappa \epsilon |X - Y|)^2) \]

\[= \Gamma_1(\kappa, \epsilon) + \frac{1}{\pi} \ln |X - Y| + R_1^e(X, Y) + R_2^e(X, Y). \]
The integral kernel $G^i_\epsilon(X, Y)$ can be expressed as

$$G^i_\epsilon(X, Y) = \frac{1}{\epsilon d} \sum_{m,n=0}^{\infty} c_{m,n} \alpha_{m,n} \cos(m\pi X) \cos(m\pi Y). \quad (3.9)$$

Recalling the definitions of $c_{m,n}$ and $\alpha_{m,n}$ in (3.2), we set

$$C_m(\epsilon, \kappa) = \sum_{n=0}^{\infty} c_{m,n} \alpha_{m,n} = \sum_{n=0}^{\infty} \frac{\alpha_{m,n}}{\kappa^2 - \left(\frac{m\pi}{\epsilon}\right)^2 - \left(\frac{(n+\frac{1}{2})\pi}{d}\right)^2}.$$ 

Using the fact (cf. [20]) that

$$\tan\left(\frac{\pi x}{2}\right) = \frac{4x}{\pi} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2 - x^2}, \quad x \neq n + \frac{1}{2}, n \in \mathbb{Z},$$

we may check for $m = 0$ that

$$C_0(\kappa) = \sum_{n=0}^{\infty} \frac{2}{\kappa^2 - \left(\frac{(n+\frac{1}{2})\pi}{d}\right)^2} = -\frac{d \tan(\kappa d)}{\kappa}.$$ 

Using the identity

$$\tanh\left(\frac{\pi x}{2}\right) = \frac{4x}{\pi} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2 + x^2},$$

we have for $m \geq 1$ that

$$C_m(\epsilon, \kappa) = \sum_{n=0}^{\infty} \frac{4}{\kappa^2 - \left(\frac{m\pi}{\epsilon}\right)^2 - \left(\frac{(n+\frac{1}{2})\pi}{d}\right)^2}$$

$$= -\frac{2d}{\sqrt{\left(\frac{m\pi}{\epsilon}\right)^2 - \kappa^2}} \tanh\left(d \sqrt{\left(\frac{m\pi}{\epsilon}\right)^2 - \kappa^2}\right)$$

$$= -\frac{2d\epsilon}{m\pi} - \frac{\epsilon^3 k^2 d}{m^3 \pi^3} + O\left(\frac{\epsilon^5 k^4 d}{4! m^5 \pi^5}\right).$$

Using the following facts (cf. [24,26]):

$$\sum_{m=1}^{\infty} \frac{\cos m\pi x}{m} = -\left(\ln 2 + \ln\left|\sin\left(\frac{\pi x}{2}\right)\right|\right), \quad 0 < x < 2,$$

$$\sum_{m=1}^{\infty} \frac{\cos m\pi x}{m^3} = \sum_{m=1}^{\infty} \frac{1}{m^3} + \frac{(\pi x)^2}{2} \ln(\pi x) + O(x^2), \quad 0 < x < 2,$$

and substituting $C_0(\kappa), C_m(\epsilon, \kappa)(m \geq 1)$ into (3.9), we obtain

$$G^i_\epsilon(X, Y) = \frac{1}{\epsilon d} \left(\frac{1}{2} C_0(\kappa) - \sum_{m=1}^{\infty} \frac{2d\epsilon}{m\pi} \cos(m\pi X) \cos(m\pi Y)\right)$$. 

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\[-\sum_{m=1}^{\infty} \frac{\varepsilon^3 k^2 d}{m^3 \pi^5} \cos(m\pi X) \cos(m\pi Y) + O\left(\frac{-2\varepsilon^5 k^4 d}{4!m^5 \pi^5}\right)\]

\[-\frac{\tan(kd)}{\varepsilon \kappa} + \frac{1}{\pi} \left(2 \ln 2 + \ln \left|\sin\left(\frac{\pi (X + Y)}{2}\right)\right| + \ln \left|\sin\left(\frac{\pi (X - Y)}{2}\right)\right|\right)\]

\[-\frac{\varepsilon^2 k^2}{\pi} \left(\frac{1}{\pi^2} \sum_{m=1}^{\infty} \frac{1}{m^3} + \frac{(X + Y)^2}{4} \ln(\pi (X + Y)) + \frac{(X - Y)^2}{4} \ln(\pi (X - Y))\right)\]

\[+ O((X + Y)^2 + (X - Y)^2),\]

which complete the proof. \[\square\]

Let

\[\Gamma = \Gamma_1(\kappa, \varepsilon) + \Gamma_2(\kappa, \varepsilon), \quad k_{\infty}^1(X, Y) = R_1^1(X, Y), \quad k_{\infty}^2 = R_2^0(X, Y) + R_2^3(X, Y),\]

\[k(X, Y) = \frac{1}{\pi} \left(\ln |X - Y| + \ln \left|\sin\left(\frac{\pi (X + Y)}{2}\right)\right| + \ln \left|\sin\left(\frac{\pi (X - Y)}{2}\right)\right|\right).\] (3.10)

Denote by \(K, K_{\infty}^1, K_{\infty}^2\) the integral operators corresponding to the Schwarz kernels \(k(X, Y), k_{\infty}^1(X, Y)\) and \(k_{\infty}^2(X, Y)\), respectively.

Let \(I\) be a bounded open interval in \(\mathbb{R}\) and define

\[H^s(I) := \{u = U|_I, \ U \in H^s(\mathbb{R})\},\]

where \(s \in \mathbb{R}\). Then \(H^s(I)\) is a Hilbert space with the norm

\[\|u\|_{H^s(I)} = \inf \left\{ \|U\|_{H^s(\mathbb{R})} \mid U \in H^s(\mathbb{R}) \text{ and } U|_I = u\right\}.\]

Define

\[\tilde{H}^s(I) := \left\{u = U|_I \mid U \in H^s(\mathbb{R}) \text{ and } \text{supp}(U) \subset \bar{I}\right\}.\]

It can be shown that the space \(\tilde{H}^s(I)\) is the dual of \(H^{-s}(I)\) and the norm for \(\tilde{H}^s(I)\) can be defined via the duality (cf. [2]).

We also define the operator \(P : \tilde{H}^{-1/2}(0, 1) \to H^{1/2}(0, 1)\) by

\[P \phi(X) = \langle \phi, 1_{(0,1)} \rangle_{L^2(0,1)} 1_{(0,1)},\]

where the duality between \(\tilde{H}^{-1/2}(0, 1)\) and \(H^{1/2}(0, 1)\) is defined by \(\langle \cdot, \cdot \rangle_{L^2(0,1)}, 1_{(0,1)}\) is the function defined on the interval \((0, 1)\) and is equal to one. It is easy to note that \(1_{(0,1)} \in H^{1/2}(0, 1)\).

**Lemma 3.3** The following conclusions hold:

(i) The operator \(T^e + T^i\) admits the decomposition:

\[T^e + T^i = \Gamma P + K + K_{\infty}^1 + K_{\infty}^2.\]

Moreover, \(K_{\infty}^1\) and \(K_{\infty}^2\) are bounded from \(\tilde{H}^{-1/2}(0, 1)\) to \(H^{1/2}(0, 1)\) with the operator norm satisfying \(\|K_{\infty}^1\| \leq \varepsilon^2 \ln |\varepsilon|, \|K_{\infty}^2\| \leq \varepsilon^2\) uniformly for wavenumber \(\kappa\).

(ii) The operator \(K\) is invertible from \(\tilde{H}^{-1/2}(0, 1)\) to \(H^{1/2}(0, 1)\). Moreover, the constant \(g_0 := \langle K^{-1} 1_{(0,1)}, 1_{(0,1)} \rangle_{L^2(0,1)} \neq 0\).
Proof The proof of (i) follows directly from the definitions of the operators $T^e, T^i$ in (3.7) and the asymptotic expansions of their kernels given by Lemma 3.2. The proof of (ii) follows directly from [11, Theorem 4.1, Lemma 4.2].

3.3 Asymptotics of the resonances

The scattering resonance of (3.1) is defined as a complex wavenumber $\kappa$ with negative imaginary part such that there is a nontrivial solution to (3.1) when the incident field is zero. This is the characteristic value of the operator $\Gamma P + K + K_1^1 + K_2^2$ with respect to the variable $\kappa$. For simplicity, we write

$$\Gamma P + K + K_1^1 + K_2^2 := \mathcal{P} + \mathcal{L},$$

where $\mathcal{P} = \Gamma P$, $\mathcal{L} = K + K_1^1 + K_2^2$. By Lemma 3.3, it is easy to see that $\mathcal{L}$ is invertible for sufficiently small $\epsilon$. Now assume that there exists $\varphi_0$ such that

$$(\mathcal{P} + \mathcal{L})\varphi_0 = 0.$$

Then

$$(\mathcal{L}^{-1}\mathcal{P} + \mathcal{I})\varphi_0 = 0,$$

where $\mathcal{I}$ is the identity operator. It is straightforward to check that the eigenvalues of operator $\mathcal{L}^{-1}\mathcal{P} + \mathcal{I}$ are

$$\lambda(\kappa, \epsilon) = 1 + \Gamma(\kappa, \epsilon)\langle \mathcal{L}^{-1}1_{(0,1)}, 1_{(0,1)} \rangle_{L^2(0,1)}.$$

Therefore, the characteristic values of the operator $\mathcal{P} + \mathcal{L}$ are the roots of the analytic functions $\lambda(\kappa, \epsilon)$, and the associated characteristic function is given by

$$\varphi_0 = \Gamma(\kappa, \epsilon)\mathcal{L}^{-1}1_{(0,1)}.$$

Lemma 3.4 The resonance of the scattering problem (3.1) are the roots of the analytic functions $\lambda(\kappa, \epsilon) = 0$. Moreover,

$$\mathcal{L}^{-1}1_{(0,1)} = K^{-1}1_{(0,1)} + O(\epsilon^2 \ln \epsilon) + O(\epsilon^2), \quad \langle \mathcal{L}^{-1}1_{(0,1)}, 1_{(0,1)} \rangle_{L^2(0,1)} = q_0$$

$$+ O(\epsilon^2 \ln \epsilon) + O(\epsilon^2). \quad (3.11)$$

Proof For given roots of $\lambda(\kappa, \epsilon)$, it is easy to check that they are the characteristic values of the operator $\mathcal{P} + \mathcal{L}$ with corresponding characteristic function defined above.

It follows from the asymptotic expansions in Lemma 3.3 and the Neumann series that

$$\mathcal{L}^{-1} = (K + K_1^1 + K_2^2)^{-1} = K^{-1} \left( \sum_{j=0}^{\infty} (-1)^j (K^{-1}(K_1^1 + K_2^2))^j \right)$$

$$= K^{-1} + O(\epsilon^2 \ln \epsilon) + O(\epsilon^2),$$

which gives (3.11).
Theorem 3.5  The scattering problem (3.1) has a set of resonances \{k_n\}, which satisfy
\[ k_n = \frac{n\pi}{d} + \frac{n\pi}{d^2}\left(\frac{1}{\pi} \ln \epsilon + \left(\frac{1}{q_0} + \frac{1}{\pi}(2\ln 2 + \ln \frac{n\pi}{d} + \gamma_1)\right)\epsilon\right) + O(\epsilon^2 \ln \epsilon), \quad n = 1, 2, \cdots. \]  

(3.12)

Proof  By Lemma 3.4, we consider the root of
\[ \lambda(\kappa, \epsilon) = 1 + (\Gamma_1(\kappa, \epsilon) + \Gamma_2(\kappa, \epsilon))(\mathcal{L}^{-1}1_{(0,1)}, 1_{(0,1)})_{L^2(0,1)} = 0. \]

Recall that \( \Gamma_1(\kappa, \epsilon) = \frac{1}{\pi}(\ln \kappa + \gamma_1) + \frac{1}{\pi} \ln \epsilon \) and \( \Gamma_2(\kappa, \epsilon) = -\frac{\tan \kappa d}{\epsilon \kappa} + \frac{2\ln 2}{\pi}. \) The above equation can be written as
\[ 1 + \left(-\frac{\tan \kappa d}{\epsilon \kappa} + \frac{1}{\pi}(2\ln 2 + \ln \kappa + \gamma_1) + \frac{1}{\pi} \ln \epsilon\right)(\mathcal{L}^{-1}1_{(0,1)}, 1_{(0,1)})_{L^2(0,1)} = 0. \]

Using Lemma 3.4, we obtain
\[ p(\kappa, \epsilon) := \epsilon \lambda(\kappa, \epsilon) = \epsilon + \left(-\frac{\tan \kappa d}{\kappa} + \epsilon \rho(\kappa) + \frac{1}{\pi} \epsilon \ln \epsilon\right)(q_0 + O(\epsilon^2 \ln \epsilon) + O(\epsilon^2)), \]

(3.13)

where \( \rho(\kappa) = \frac{1}{\pi}(2\ln 2 + \ln \kappa + \gamma_1). \)

Note that the axis \( \{z \in \mathbb{C} : \arg z = \pm \frac{\pi}{2d}\} \) is the branch cut for \( p(\kappa, \epsilon), \) hence we choose a small number \( \theta_0 > 0 \) and consider the domain \( \{z \in \mathbb{C} : -\frac{\pi}{2d} + \theta_0 < \arg z < \frac{\pi}{2d} - \theta_0 \text{ or } -\frac{\pi}{2d} + \theta_0 < \arg z < \frac{3\pi}{2d} - \theta_0\}. \) On the other hand, we are only interested in those resonances which are not in the high frequency regime. Therefore, we only need to find all the roots of \( p(\kappa, \epsilon) \) in the domain
\[ \Omega_{\theta_0,M} = \{z \in \mathbb{C} : |z| \leq M, \quad -\frac{\pi}{2d} + \theta_0 < \arg z < \frac{\pi}{2d} - \theta_0 \text{ or } \frac{\pi}{2d} + \theta_0 < \arg z < \frac{3\pi}{2d} - \theta_0\} \]
for some fixed number \( M > 0. \) Since \( p(\kappa, \epsilon) \) blows up as \( \kappa \to 0 \) or \( \kappa \to (j + 1/2)\pi/d, j \in \mathbb{Z}. \) As a result, there exists \( \delta_0 > 0 \) such that all the roots of \( p \) in \( \Omega_{\theta_0,M} \) actually lies in the smaller domain
\[ \Omega_{\delta_0,\theta_0,M} := \{z \in \mathbb{C} : |z| \geq \delta_0 \text{ or } |z - (j + 1/2)\pi/d| \geq \delta_0, j \in \mathbb{Z}\} \cap \Omega_{\theta_0,M}. \]

It is clear that \(-\frac{\tan \kappa d}{\kappa}\) is analytic in \( \Omega_{\delta_0,\theta_0,M} \) and its roots are given by \( k_{n,0} = n\pi/d, n = 1, 2, \cdots. \) Note that each root is simple. Denote by \( k_n \) the roots of \( \lambda(\kappa, \epsilon). \) From Rouche’s theorem, we deduce that \( k_n \) are also simple and are close to \( k_{n,0} \) if \( \epsilon \) is sufficiently small. We now derive the leading order asymptotic terms for these roots. Define
\[ p_1(\kappa, \epsilon) = \epsilon + \left(-\frac{\tan \kappa d}{\kappa} + \epsilon \rho(\kappa) + \frac{1}{\pi} \epsilon \ln \epsilon\right)(q_0. \]

(3.14)

Expanding \( p_1(\kappa, \epsilon) \) at \( k_{n,0} \) yields
\[ p_1(\kappa, \epsilon) = \epsilon + \left(-\frac{\tan \kappa d}{\kappa}\right)' \bigg|_{\kappa=k_{n,0}} (\kappa - k_{n,0}) + \epsilon \rho(k_{n,0}) + \epsilon \rho'(k_{n,0})(\kappa - k_{n,0}) + O(\kappa - k_{n,0})^2 + \frac{1}{\pi} \epsilon \ln \epsilon \right)q_0 \]
\[ = \epsilon + \left(-\frac{1}{n\pi \kappa} (\kappa - k_{n,0}) + \frac{\epsilon}{\pi} \left(2\ln 2 + \ln \frac{n\pi}{d} + \gamma_1\right) + \frac{\epsilon d}{n\pi^2} (\kappa - k_{n,0}) \right)q_0. \]
\[ + O(\kappa - k_{n0})^2 + \frac{1}{\pi} \epsilon \ln \epsilon \)q_0.\]

We can conclude that \( p_1(\kappa, \epsilon) \) has simple roots in \( \Omega_{\theta_0, k_0, \theta_0} \), which are close to \( k_{n0} \)’s. Moreover, these roots are analytic with respect to the variables \( \epsilon \) and \( \epsilon \ln \epsilon \). Denoting the roots of \( p_1 \) by \( k_{n1} \) and expanding them in terms of \( \epsilon \) and \( \epsilon \ln \epsilon \), we obtain

\[ k_{n1} = k_{n0} + \frac{k_{n0}}{d} \left( \frac{1}{\pi} \epsilon \ln \epsilon + \left( \frac{1}{q_0} + \frac{1}{\pi} (2 \ln 2 + \ln \frac{n\pi}{d} + \gamma_1) \right) \epsilon \right) + O(\epsilon^2 \ln \epsilon). \quad (3.15) \]

Using Rouche’s theorem, we claim that \( k_{n1} \) gives the leading order for the asymptotic expansion of the roots \( k_n \). More precisely, we have

\[ k_n = k_{n1} + O(\epsilon^2 \ln \epsilon). \quad (3.16) \]

In the following, we give a proof of the claim. Note that

\[ p(\kappa, \epsilon) - p_1(\kappa, \epsilon) = \left( -\frac{\tan \frac{\kappa d}{\kappa}}{\kappa} + \frac{1}{\pi} \epsilon \ln \epsilon + \rho(\kappa) \epsilon \right) O(\epsilon^2 \ln \epsilon) \]

and

\[ p_1(\kappa, \epsilon) = -\frac{\tan \frac{\kappa d}{\kappa}}{\kappa} q_0 + \frac{q_0}{\pi} \epsilon \ln \epsilon + (1 + \rho(\kappa) q_0) \epsilon. \]

Hence there exists a constant \( C_n \) such that

\[ |p(\kappa, \epsilon) - p_1(\kappa, \epsilon)| < |p_1(\kappa, \epsilon)|, \quad \forall \kappa \in \{ |\kappa - k_{n1}| = C_n \epsilon^2 \ln \epsilon \}. \]

which implies the claim (3.16) by Rouche’s theorem.

\[ 3.4 \text{ The field enhancement with resonance} \]

First we give an asymptotic expansion of \( p(\kappa, \epsilon) \) at the resonant frequencies.

**Lemma 3.6** At the resonant frequencies \( \kappa = \text{Re} k_n \) where \( k_n \) is given in (3.12), we have

\[ p(\kappa, \epsilon) = -\frac{i q_0}{2} \epsilon + O(\epsilon^2 \ln^2 \epsilon). \]

**Proof** Assume that \( |\kappa - \text{Re} k_n| \leq \epsilon \ln \epsilon \). It follows from the definition of \( p_1 \) in (3.14) that

\[ p(\kappa, \epsilon) = p_1(\kappa, \epsilon) + O(\epsilon^2 \ln \epsilon) \]

\[ = p_1'(k_n)(\kappa - k_n) + O(\kappa - k_n)^2 + O(\epsilon^2 \ln \epsilon) \]

\[ = \left( \frac{-d}{k_{n0}} + \frac{\epsilon}{\pi k_{n0}} + O(|k_n - k_{n0}|) \right) q_0(\kappa - k_n) + O(\epsilon^2 \ln^2 \epsilon) \]

\[ = \left( \frac{-d}{k_{n0}} + \frac{\epsilon}{\pi k_{n0}} \right) q_0(\kappa - \text{Re} k_n - i \text{Im} k_n) + O(\epsilon^2 \ln^2 \epsilon). \]

Recalling \( \gamma_1 = \gamma_0 - \ln 2 - \frac{\pi i}{2} \), we have from the expansion of \( k_{n1} \) in (3.15) that

\[ \text{Im} k_n = \text{Im} k_{n1} + O(\epsilon^2 \ln \epsilon) = \frac{k_{n0}}{d \pi} (\text{Im} \gamma_1) \epsilon = -\frac{k_{n0} \epsilon}{2d}. \]

When \( \kappa = \text{Re} k_n \), we have

\[ p(\kappa, \epsilon) = i q_0 \left( -\frac{1}{2} \epsilon + \frac{1}{2d} \epsilon^2 \right) + O(\epsilon^2 \ln^2 \epsilon) = -\frac{i q_0}{2} \epsilon + O(\epsilon^2 \ln^2 \epsilon), \]

\[ \square \]
which completes the proof. □

**Lemma 3.7** The solution \( \varphi \) of equation (3.8) has the following asymptotic expansion in \( H^{-1/2}(0, 1) \):
\[
\varphi = K^{-1}1_{(0,1)} \cdot \frac{(2 + \sin \theta \cdot O(\kappa \epsilon))}{p} + O(\epsilon^2 \ln \epsilon).
\]

Moreover,
\[
\langle \varphi, 1_{(0,1)} \rangle_{L^2(0,1)} = \frac{q_0 (2 + \sin \theta \cdot O(\kappa \epsilon))}{p} + O(\epsilon^2 \ln \epsilon).
\]

**Proof** It follows from Lemma 3.3 that equation (3.8) is equivalent to
\[
(L^{-1} \mathcal{P} + \mathcal{I}) \varphi = L^{-1}(f/\epsilon).
\]
Recall that the operator \( L^{-1} \mathcal{P} + \mathcal{I} \) has the eigenvalue \( \lambda(\kappa, \epsilon) \). Thus
\[
\langle \varphi, 1_{(0,1)} \rangle_{L^2(0,1)} = \frac{1}{\lambda(\kappa, \epsilon)} \langle L^{-1}(f/\epsilon), 1_{(0,1)} \rangle_{L^2(0,1)},
\]
and
\[
\varphi = L^{-1}(f/\epsilon) - L^{-1}1_{(0,1)} = L^{-1}(f/\epsilon) - L^{-1}1_{(0,1)} \frac{\Gamma}{\lambda(\kappa, \epsilon)} \langle L^{-1}(f/\epsilon), 1_{(0,1)} \rangle_{L^2(0,1)}.
\]
Recall that \( \lambda(\kappa, \epsilon) = 1 + \Gamma \langle L^{-1}1_{(0,1)}, 1_{(0,1)} \rangle_{L^2(0,1)} \). A simple calculation yields
\[
\varphi = L^{-1}(f/\epsilon) - L^{-1}1_{(0,1)} \frac{(\lambda(\kappa, \epsilon) - 1)/\langle L^{-1}1_{(0,1)}, 1_{(0,1)} \rangle_{L^2(0,1)}}{\lambda(\kappa, \epsilon)} \langle L^{-1}(f/\epsilon), 1_{(0,1)} \rangle_{L^2(0,1)}.
\]
Using the definition of \( f \) and expanding it with respect to \( \epsilon \), we get
\[
\frac{f}{\epsilon} = \frac{2e^{i\kappa X \sin \theta}}{\epsilon} = \frac{2}{\epsilon} + i\kappa X \sin \theta + O(\kappa^2 \epsilon),
\]
which gives
\[
L^{-1}(f/\epsilon) = \frac{1}{\epsilon} \left( 2 + \sin \theta \cdot O(\kappa \epsilon) \right) \left( K^{-1}1_{(0,1)} + O(\epsilon^2 \ln \epsilon) \right).
\]
Combining the above equation with (3.11), we obtain
\[
\epsilon \varphi = (2 + \sin \theta \cdot O(\kappa \epsilon)) \left( K^{-1}1_{(0,1)} + O(\epsilon^2 \ln \epsilon) \right)
+ \left( 1 - \frac{\lambda}{\lambda(\kappa, \epsilon)} \right) (q_0 + O(\epsilon^2 \ln \epsilon)) \cdot (2q_0 + q_0 \sin \theta \cdot O(\kappa \epsilon)) \cdot \left( K^{-1}1_{(0,1)} + O(\epsilon^2 \ln \epsilon) \right)
= \frac{(2 + \sin \theta \cdot O(\kappa \epsilon))}{\lambda} \left( K^{-1}1_{(0,1)} + O(\epsilon^2 \ln \epsilon) \right).
\]
Thus
\[
\varphi = K^{-1}1_{(0,1)} \cdot \frac{(2 + \sin \theta \cdot O(\kappa \epsilon))}{p} + O(\epsilon^2 \ln \epsilon),
\]
and
\[
\langle \varphi, 1_{(0,1)} \rangle_{L^2(0,1)} = \frac{q_0 (2 + \sin \theta \cdot O(\kappa \epsilon))}{p} + O(\epsilon^2 \ln \epsilon),
\]
which completes the proof. □
Combining Lemmas 3.6 and 3.7, we obtain at the resonant frequencies $\kappa = \text{Re} k_n$ that
\begin{equation}
\varphi = \frac{f_1}{\epsilon} + O(\epsilon^2 \ln \epsilon), \quad \langle \varphi, 1_{(0,1)} \rangle_{L^2(0,1)} = \frac{c_2}{\epsilon} + O(\epsilon^2 \ln \epsilon), \tag{3.17}
\end{equation}
where
\begin{equation}
f_1 = K^{-1} \bar{1}_{(0,1)} \cdot \frac{2 + \sin \theta \cdot O(\kappa \epsilon)}{-\frac{1}{2} q_0 + O(\epsilon \ln \epsilon)} \in \tilde{H}^{-1/2}(0, 1), \quad c_2 = \frac{q_0 (2 + \sin \theta \cdot O(\kappa \epsilon))}{-\frac{1}{2} q_0 + O(\epsilon \ln \epsilon)}.
\end{equation}

### 3.4.1 Enhancement in the far field

We first investigate the scattered field in the domain $\mathbb{R}_+^2 \setminus \tilde{B}_R^+$. Recall that
\begin{equation}
u^{\text{sc}}(x) = \int_{\Gamma_1^+} G(x, y) \frac{\partial \nu^{\text{sc}}(y)}{\partial y} \ ds_y = \int_{\Gamma_1^+} G(x, y) \frac{\partial \nu(y)}{\partial y} \ ds_y, \quad x \in \mathbb{R}_+^2.
\end{equation}
Here we use the fact $\partial_i \nu^{\text{inc}} + \partial_i \nu^{\text{ref}} = 0$ on $\Gamma_0$, especially on $\Gamma_1^+$. Using
\begin{equation}
\left. \frac{\partial \nu(y)}{\partial y} \right|_{\Gamma_1^+} = \frac{\partial \nu(y_1, 0)}{\partial y_2} = -\varphi \left( \frac{y_1}{\epsilon} \right),
\end{equation}
we have
\begin{equation}
u^{\text{sc}}(x) = -\int_{\Gamma_1^+} G(x, (y_1, 0)) \varphi \left( \frac{y_1}{\epsilon} \right) \ dy_1 = -\epsilon \int_0^1 G(x, (\epsilon Y, 0)) \varphi(Y) \ dy.
\end{equation}
Since
\begin{equation}
G(x, (\epsilon Y, 0)) = G(x, (0, 0))(1 + O(\epsilon)), \quad x \in \mathbb{R}_+^2 \setminus \tilde{B}_R^+.
\end{equation}
we get from Lemma 3.7 that
\begin{equation}
\int_0^1 \varphi(Y) \ dy = \langle \varphi, 1_{(0,1)} \rangle_{L^2(0,1)} = \frac{q_0 (2 + \sin \theta \cdot O(\kappa \epsilon))}{p} + O(\epsilon^2 \ln \epsilon).
\end{equation}
Hence
\begin{equation}
u^{\text{sc}}(x) = -\epsilon G(x, (0, 0))(1 + O(\epsilon)) \int_0^1 \varphi(Y) \ dy
\end{equation}
\begin{equation}
= -\epsilon G(x, (0, 0)) \cdot \frac{q_0 (2 + \sin \theta \cdot O(\kappa \epsilon))}{p} + O(\epsilon^2 \ln \epsilon).
\end{equation}
In the resonance case when $\kappa = \text{Re} k_n$, the enhancement comes from the term $\frac{1}{p}$. It follows from Lemma 3.6 that
\begin{equation}
\frac{1}{p} = \frac{2i}{q_0 \epsilon} (1 + O(\epsilon \ln^2 \epsilon)).
\end{equation}
Correspondingly, we have
\begin{equation}
u^{\text{sc}}(x) = -2iG(x, (0, 0)) \cdot (2 + \sin \theta \cdot O(\kappa \epsilon)) + O(\epsilon \ln^2 \epsilon),
\end{equation}
which shows that the enhancement of the scattered magnetic field in the far field region has an order $O(1/\epsilon)$ compared to the non-resonance case when $\kappa \neq \text{Re} k_n$.

It follows from the Ampere law $\nabla \times \mathbf{H}_e = -i \omega \epsilon \mathbf{E}_e$ and $\mathbf{H}_e = (0, 0, u_e)$ that the enhancement of the scattered electric field also has an order $O(1/\epsilon)$ in the far field region when $\kappa = \text{Re} k_n$. 

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3.4.2 Enhancement in the cavity

Now we study the field enhancement in the cavity. It follows from (2.5) that the total field $u_\epsilon$ may be expanded as the sum of waveguide modes:

$$u_\epsilon (x) = \frac{1}{\sqrt{\epsilon}} \left( \alpha_0^+ e^{-i\kappa x_2} + \alpha_0^- e^{i\kappa(x_2+d)} \right) + \sum_{m \geq 1} \sqrt{\frac{2}{\epsilon}} \alpha_m^+ \cos \left( \frac{m\pi x_1}{\epsilon} \right) e^{-i\beta_m x_2}$$

$$+ \sum_{m \geq 1} \sqrt{\frac{2}{\epsilon}} \alpha_m^- \cos \left( \frac{m\pi x_1}{\epsilon} \right) e^{i\beta_m(x_2+d)}, \quad (3.18)$$

where $\beta_m = i\sqrt{(m\pi/\epsilon)^2 - \kappa^2}$.

**Lemma 3.8** The coefficients in (3.18) have the following expansions:

$$\frac{\alpha_0^+}{\sqrt{\epsilon}} = -q_0(2 + \sin \theta \cdot O(\kappa \epsilon)) e^{-i\kappa d} \frac{2i\kappa \cos(\kappa d) \rho}{2i\kappa \cos(\kappa d) \rho} + O(\epsilon^2 \ln \epsilon),$$

$$\frac{\alpha_0^-}{\sqrt{\epsilon}} = q_0(2 + \sin \theta \cdot O(\kappa \epsilon)) \frac{2i\kappa \cos(\kappa d) \rho}{2i\kappa \cos(\kappa d) \rho} + O(\epsilon^2 \ln \epsilon), \quad (3.19)$$

and

$$\sqrt{\frac{2}{\epsilon}} |\alpha_m^+| \leq \frac{C}{\sqrt{m}}, \quad \sqrt{\frac{2}{\epsilon}} |\alpha_m^-| \leq \frac{C}{\sqrt{m}},$$

where the positive constant $C$ independents of $\epsilon, \kappa$ and $m$.

**Proof** Recall that $u_\epsilon = 0$ on $\Gamma_\epsilon^-$ then equation (3.18) becomes

$$u_\epsilon (x) = \frac{\alpha_0^-}{\sqrt{\epsilon}} \left( -e^{-i\kappa(x_2+d)} + e^{i\kappa(x_2+d)} \right)$$

$$+ \sum_{m \geq 1} \sqrt{\frac{2}{\epsilon}} \alpha_m^- \left( -e^{-i\beta_m(x_2+d)} + e^{i\beta_m(x_2+d)} \right) \cos \left( \frac{m\pi x_1}{\epsilon} \right). \quad (3.20)$$

Taking the derivative of (3.20) with respect to $x_2$ yields

$$\frac{\partial u_\epsilon(x)}{\partial x_2} = \frac{\alpha_0^-}{\sqrt{\epsilon}} i\kappa \left( e^{-i\kappa(x_2+d)} + e^{i\kappa(x_2+d)} \right)$$

$$+ \sum_{m \geq 1} \sqrt{\frac{2}{\epsilon}} \alpha_m^- i\beta_m \left( e^{-i\beta_m(x_2+d)} + e^{i\beta_m(x_2+d)} \right) \cos \left( \frac{m\pi x_1}{\epsilon} \right),$$

which gives

$$\frac{\partial u_\epsilon(x_1, 0)}{\partial x_2} = \frac{\alpha_0^-}{\sqrt{\epsilon}} 2i\kappa \cos(\kappa d) + \sum_{m \geq 1} \sqrt{\frac{2}{\epsilon}} \alpha_m^- 2i\beta_m \cos(\beta_m d) \cos \left( \frac{m\pi x_1}{\epsilon} \right). \quad (3.21)$$
Multiplying (3.21) with $\phi_0(x_1)$ and integrating it over $\Gamma_\epsilon^+$, we get
\[
\frac{\alpha_0^-}{\sqrt{\epsilon}} e^{ik_d} = \frac{1}{\epsilon} \int_{\Gamma_\epsilon^+} \frac{\partial u_\epsilon}{\partial x_2}(x_1,0) \, dx_1 = \frac{1}{\epsilon} \int_{\Gamma_\epsilon^+} \varphi \left( \frac{x_1}{\epsilon} \right) \, dx_1
\]
\[
= \int_0^1 \varphi(X) \, dX = (\varphi, 1_{(0,1)})_{L^2(0,1)}
\]
\[
= q_0 \left( 2 + \sin \theta \cdot O(\epsilon) \right) \frac{\pi}{p} + O(\epsilon^2 \ln \epsilon),
\]
which gives the formulas for $\alpha_0^+$ and $\alpha_0^-$ in (3.19).

For $m \geq 1$, it follows from multiplying (3.21) with $\cos \left( \frac{m \pi x_1}{\epsilon} \right)$ and integrating over $\Gamma_\epsilon^+$ that
\[
\sqrt{\frac{2}{\epsilon}} \alpha_m^- \beta_m \cos \left( \beta_m d \right) = \frac{1}{\epsilon} \int_{\Gamma_\epsilon^+} \frac{\partial u_\epsilon}{\partial x_2}(x_1,0) \cos \left( \frac{m \pi x_1}{\epsilon} \right) \, dx_1
\]
\[
= - \int_0^1 \varphi(X) \cos(m \pi X) \, dX.
\]
Note that $i \beta_m = O \left( \frac{m}{\epsilon} \right)$ for $m \geq 1$, and by (3.17) that $\|\varphi\|_{H^{-1/2}(0,1)} \lesssim \frac{1}{\epsilon}$, $\|\cos(m \pi X)\|_{H^{1/2}(0,1)} \lesssim \sqrt{m}$. Thus, we get
\[
\sqrt{\frac{2}{\epsilon}} |\alpha_m^-| \leq \frac{C}{\sqrt{m}},
\]
which completes the proof. \(\square\)

In the following, we give the field enhancement inside the cavity.

**Theorem 3.9** Denote by $\hat{D}_\epsilon = \{x \in D_\epsilon \mid -d + \epsilon \leq x_2 \leq -\epsilon\}$ the interior of the cavity. If $\kappa = \text{Re} k_n$ where $k_n$ is given in (3.12), then we have for $\epsilon \ll d$ that
\[
uu(x) = \left( \frac{2}{\epsilon} + \sin \theta \cdot O(1) + O(\ln^2 \epsilon) \right) \frac{2i \sin \kappa (x_2 + d)}{\kappa \cos(\kappa_d)} + O(\epsilon^2 \ln \epsilon), \quad x \in \hat{D}_\epsilon.
\]
Moreover, the electric and magnetic field enhancements are of an order $O(1/\epsilon)$.

**Proof** By the expansion (3.18) and Lemma 3.8, it is clear that in the region $\hat{D}_\epsilon$
\[
uu(x) = \frac{q_0}{2i \kappa \cos(\kappa_d) \epsilon} \left( -e^{-i\kappa (x_2 + d)} + e^{i\kappa (x_2 + d)} \right) + O(\epsilon^2 \ln \epsilon) + O(e^{-1/\epsilon})
\]
\[
= \frac{q_0}{\kappa \cos(\kappa_d) \epsilon} \sin \kappa (x_2 + d) \frac{1}{p} + O(\epsilon^2 \ln \epsilon) + O(e^{-1/\epsilon}).
\]
Noting when $\kappa = \text{Re} k_n$, we have
\[
\frac{1}{p} = \frac{2i}{q_0 \epsilon} \left( 1 + O(\epsilon \ln^2 \epsilon) \right),
\]
which yields that
\[
uu(x) = \left( \frac{2}{\epsilon} + \sin \theta \cdot O(1) + O(\ln^2 \epsilon) \right) \frac{2i \sin \kappa (x_2 + d)}{\kappa \cos(\kappa_d)} + O(\epsilon^2 \ln \epsilon).
\]
Therefore, the magnetic field enhancement due to the resonance is of an order $O(1/\epsilon)$ in $\hat{D}_\epsilon$. We conclude from Ampere’s law that the electric field enhancement is also of an order $O(1/\epsilon)$. \(\square\)
3.4.3 Field enhancement on the open aperture

Now we present the enhancement of the scattered field on the open aperture of the cavity. Recall (3.4) for the scattered field on $\Gamma^+_e$:

$$u^{sc}_e(x_1, 0) = \int_{\Gamma^+_e} G((x_1, 0), y) \frac{\partial u_e(y)}{\partial v_y} dy.$$ 

Since $x_1 = \epsilon X, y_1 = \epsilon Y, \varphi = -\partial_v u_e |_{\Gamma^+_e}$, we have

$$u^{sc}_e(x_1, 0) = -\int_0^1 G^e_e(X, Y) \epsilon \varphi(Y) dY.$$ 

It follows from the asymptotic expansion of $G^e_e$ in Lemma 3.2 and Lemma 3.7 that

$$u^{sc}_e(x_1, 0) = -\epsilon \int_0^1 \left( \Gamma_1 + \frac{1}{\pi} \ln |X - Y| + O(\epsilon^2 \ln \epsilon) \right) \varphi(Y) dY$$

$$= -\epsilon \Gamma_1 \langle \varphi, 1_{(0,1)} \rangle_{L^2(0,1)} - \epsilon \frac{2 + \sin \theta \cdot O(\kappa \epsilon)}{p} \frac{1}{\pi} h_2(X) + O(\epsilon^2 \ln \epsilon)$$

$$= -\epsilon \frac{1}{\pi} (\ln \kappa + \gamma_1) + \frac{1}{\pi} \ln \epsilon \left( \frac{q_0 (2 + \sin \theta \cdot O(\kappa \epsilon))}{p} + O(\epsilon^2 \ln \epsilon) \right)$$

$$- \epsilon \frac{2 + \sin \theta \cdot O(\kappa \epsilon)}{p} \frac{1}{\pi} h_2(X) + O(\epsilon^2 \ln \epsilon)$$

$$= -\frac{1}{\pi} \frac{2 q_0}{p} \epsilon \ln \epsilon - \frac{1}{\pi} \frac{2}{p} (q_0 (\ln \kappa + \gamma_1) + h_2(X)) \epsilon$$

$$- \frac{1}{\pi} \left( q_0 (\ln \kappa + \gamma_1) + h_2(X) \right) \frac{\sin \theta \cdot O(1)}{p} \epsilon^2$$

$$- \frac{1}{p} q_0 \sin \theta \cdot O(1) \epsilon^2 \ln \epsilon + O(\epsilon^2 \ln \epsilon).$$

where $h_2(X) := \int_0^1 K^{-1} 1_{(0,1)}(Y) \ln |X - Y| dY$. Note that at the resonant frequencies $\kappa = \text{Re} k_n$,

$$\frac{1}{p} = \frac{2i}{q_0 \epsilon} (1 + O(\epsilon \ln^2 \epsilon)).$$

Thus we have

$$u^{sc}_e(x_1, 0) = -\frac{4i}{\pi} \ln \epsilon - \frac{4i}{\pi} \left( \ln \kappa + \gamma_1 \right) + \frac{h_2(x_1/\epsilon)}{q_0} + O(\epsilon \ln^3 \epsilon).$$

It is clear to note that the leading order of the resonant mode is a constant on the open aperture with an order of $O(\ln \epsilon)$ and the magnetic field enhancement is of an order $O(1/\epsilon)$.

On the other hand, since

$$\partial_{x_1} u^{sc}_e(x_1, 0) = -\frac{2 + \sin \theta \cdot O(\kappa \epsilon)}{p} \frac{1}{\pi} h_2'(x_1/\epsilon) + O(\epsilon^2 \ln \epsilon),$$

which implies that at the resonant frequencies $\kappa = \text{Re} k_n$, on the top aperture $\Gamma^+_e$, the scatter field satisfies

$$\partial_{x_1} u^{sc}_e(x_1, 0) = -\frac{4i(1 + O(\epsilon \ln^2 \epsilon))}{\pi q_0 \epsilon} h'(x_1/\epsilon) - \frac{2i \sin \theta \cdot O(1)}{\pi q_0} h_2'(x_1/\epsilon) + O(\epsilon \ln^2 \epsilon).$$
By Ampere’s law, we know the electric field enhancement is also of an order $O(1/\varepsilon)$ on the top aperture $\Gamma_1^+$ when $\kappa = \text{Re}k_n$.

### 3.5 Numerical experiments

In this section, we present some numerical experiments to verify the theoretical studies of the field enhancement for the PEC-PMC cavity. As a representative example, we take the cavity width $\varepsilon = 0.005$, the cavity depth $d = 1$, and the angle of incidence $\theta = \pi/3$ in the numerics. Figure 3 (left) and (right) show the plot of the electric field enhancement factor $Q_E$ and the magnetic field enhancement factor $Q_H$ against the wavenumber $\kappa$, respectively. We observe that both the electric and magnetic enhancement factors do reach peaks at the resonant frequencies (3.12), which are close to $\kappa = n\pi$, $n = 1, 2, \ldots$. Besides the peaks at the resonant frequencies, the electric field enhancement factor $Q_E$ also displays a peak when the wavenumber $\kappa$ is sufficiently small, i.e., when the wavelength of the incident field $\lambda$ is sufficiently large (see Fig. 4). In this situation, the scales satisfy $\varepsilon \ll d \ll \lambda$. The electric field should have an enormous enhancement according to our theoretical result in Theorem 2.8. In contrast, the magnetic field enhancement factor $Q_H$ does not show a peak when the wavenumber approaches zero, which is also confirmed in Theorem 2.8 that the magnetic field has no enhancement when $\varepsilon \ll d \ll \lambda$.

### 4 PEC-PEC cavity without resonance

In this section, assuming $\varepsilon \ll d \ll \lambda$, we show that there is a weak enhancement of the electric field and there is no enhancement of the magnetic field in the PEC-PEC cavity.

#### 4.1 Problem formulation

Consider the following model problem of the electromagnetic scattering by an open cavity:

$$
\begin{align*}
\Delta u_\varepsilon + \kappa^2 u_\varepsilon &= 0 & \text{in } \Omega, \\
\partial_\nu u_\varepsilon &= 0 & \text{on } \partial\Omega, \\
\lim_{r \to \infty} \sqrt{r} \left( \partial_r u_\varepsilon - i\kappa u_\varepsilon^\text{sc} \right) &= 0 & \text{in } \mathbb{R}^2_+.
\end{align*}
$$

(4.1)
where $u_e^{sc} = u_e - (u^{inc} + u^{ref})$ in $\mathbb{R}^+$. Here the incident field $u^{inc} = e^{ix \cdot d}$ is a plane wave propagating in the direction $d = (\sin \theta, -\cos \theta)^\top$ and the reflected field $u^{ref} = e^{ix \cdot d'}$ with the propagating direction $d' = (\cos \theta, \sin \theta)$.

Let $\phi_0(x_1) = \frac{1}{\sqrt{\epsilon}}$ and $\phi_n(x_1) = \sqrt{\frac{2}{\epsilon}} \cos \left( \frac{n\pi x_1}{\epsilon} \right) (n \geq 1)$ be an orthonormal basis on the interval $(0, \epsilon)$. Since the total field satisfies the Neumann boundary condition on the bottom and lateral sides of the cavity, we have from (4.1) that $u_e$ can be expressed as the sum of waveguide modes:

$$u_e(x) = \sum_{n=0}^{\infty} \left( \alpha_n^+ e^{-i\beta_n x_2} + \alpha_n^- e^{i\beta_n (x_2 + d)} \right) \phi_n(x_1), \quad x \in D_\epsilon, \quad (4.2)$$

where the coefficients $\beta_n$ are defined as

$$\beta_n = \begin{cases} 
\kappa, & n = 0, \\
\sqrt{(n\pi/\epsilon)^2 - \kappa^2}, & n \geq 1.
\end{cases}$$

If $\epsilon < \lambda/2$, then $\beta_n$ are pure imaginary numbers for all $n \geq 1$. For each $n$, it can be seen if $n\pi/\epsilon \leq \kappa$, the expansion consists of two propagating wave modes traveling upward and downward respectively; if $n\pi/\epsilon > \kappa$, the expansion consists of two evanescent wave modes decaying exponentially away from the bottom side and open aperture of the cavity, respectively.

Using the expansion (2.5), we may reformulate the model (4.1) into the following coupled problem:
we may show that the DtN map $B$.

A straightforward calculation yields

$$\text{Substituting (4.7) into (4.5) yields}$$

It follows from (4.6) that

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Substituting (4.7) into (4.5) yields

Given a function $u \in H^{1/2}(\Gamma_\epsilon^+)$, we let $u^+ = u(x_1, 0)$ and define a DtN map on $\Gamma_\epsilon^+$:

where the Fourier coefficients $u^+_n = \langle u^+, \phi_n \rangle_{\Gamma_\epsilon^+}$. Following a similar proof to Lemma 2.1, we may show that the DtN map $\mathcal{B}^\text{PEC}$ is also bounded. The proof is omitted for brevity.
Lemma 4.1 The DtN map $B_{\text{PEC}}^\epsilon$ defined by (2.15) is bounded from $H^{1/2}(\Gamma_\epsilon^+)$ to $H^{-1/2}(\Gamma_\epsilon^+)$, i.e.,

$$\|B_{\text{PEC}}^\epsilon[u]\|_{H^{-1/2}(\Gamma_\epsilon^+)} \leq C\|u\|_{H^{1/2}(\Gamma_\epsilon^+)} \quad \forall u \in H^{1/2}(\Gamma_\epsilon^+),$$

where the constant $C = \max \left\{ \frac{\kappa}{\cot(\kappa d)}, \frac{1-e^{-2\sqrt{\pi(\kappa^2/\epsilon)}}}{1+e^{-2\sqrt{\pi(\kappa^2/\epsilon)}}} \right\} < 1$.

By (4.8), we derive the TBC:

$$\partial_{\nu}\varepsilon = B_{\text{PEC}}^\epsilon[u_\epsilon^\pm] \quad \text{on } \Gamma_\epsilon^+,$$

which helps to reduce (4.1) into a boundary value problem in $\mathbb{R}_2^+$:

$$\begin{align*}
\Delta u_\epsilon + \kappa^2 u_\epsilon &= 0 \quad \text{in } \mathbb{R}_2^+, \\
\partial_{\nu} u_\epsilon &= 0 \quad \text{on } \Gamma_0 \setminus \Gamma_\epsilon^+, \\
\partial_{x_2} u_\epsilon &= B_{\text{PEC}}^\epsilon[u_\epsilon] \quad \text{on } \Gamma_\epsilon^+, \\
\lim_{r \to \infty} \sqrt{r} \left( \partial_{\nu} u_\epsilon^\text{sc} - i\kappa u_\epsilon^\text{sc} \right) &= 0 \quad \text{in } \mathbb{R}_2^+.
\end{align*}$$

The solution $u_\epsilon$ in the cavity $D_\epsilon$ may be obtained from (4.4) once the Fourier coefficients $u_{\epsilon,n}^+$ are available (note that $u_{\epsilon,n}^+$ can be computed from from (4.7)).

To find an approximate the model problem to (4.9), we similarly drop all the nonzero wave modes and define

$$B_{0,\text{PEC}}^\epsilon[v] = i\kappa \frac{e^{ikd} - e^{-ikd}}{e^{ikd} + e^{-ikd}} v_0^+ \phi_0(x_1).$$

Hence we may consider the following problem:

$$\begin{align*}
\Delta v_\epsilon + \kappa^2 v_\epsilon &= 0 \quad \text{in } \mathbb{R}_2^+, \\
\partial_{\nu} v_\epsilon &= 0 \quad \text{on } \Gamma_0 \setminus \Gamma_\epsilon^+, \\
\partial_{x_2} v_\epsilon &= B_{0,\text{PEC}}^\epsilon[v_\epsilon^\pm] \quad \text{on } \Gamma_\epsilon^+, \\
\lim_{r \to \infty} \sqrt{r} \left( \partial_{\nu} v_\epsilon^\text{sc} - i\kappa v_\epsilon^\text{sc} \right) &= 0 \quad \text{in } \mathbb{R}_2^+,
\end{align*}$$

where $v_\epsilon^\pm = v_\epsilon - (u^\text{inc} + u^\text{ref})$ in $\mathbb{R}_2^+$. Inside the cavity, we may approximate $u_\epsilon$ with one single mode:

$$v_\epsilon(x) = \left( \tilde{a}_0^+ e^{-ikx_2} + \tilde{\alpha}_0^- e^{ik(x_2+d)} \right) \phi_0(x_1),$$

where the coefficients $\tilde{a}_0^+$ and $\tilde{\alpha}_0^-$ are given by

$$\tilde{a}_0^+ = \frac{e^{ikd} v_\epsilon^- - v_\epsilon^+}{e^{i2kd} - 1}, \quad \tilde{\alpha}_0^- = \frac{e^{ikd} v_\epsilon^+ - v_\epsilon^-}{e^{i2kd} - 1}.$$

It follows from $\partial_{\nu} v_\epsilon = 0$ on $\Gamma_\epsilon^-$ that $\tilde{\alpha}_0^- = \tilde{a}_0^+ e^{ikd}$, which yields

$$v_\epsilon^- = \frac{2v_\epsilon^+}{e^{ikd} + e^{-ikd}}.$$
4.2 Enhancement of the approximated field

This section presents the estimates of the solution for the approximate model problem (4.11).

**Theorem 4.2** Let \( v_\epsilon \) be the solution of (4.11) and be given by (4.12) inside the cavity, then the following holds

\[
\| \nabla v_\epsilon \|_{L^2(D_\epsilon)} = \sqrt{\kappa \epsilon (2\kappa d - \sin(2\kappa d))} + O(\kappa^3 d^{3/2} \epsilon^{3/2}).
\]

**Proof** For the approximate model (4.11), we obtain from Green’s formula that

\[
v_\epsilon(x) = u^{\text{inc}}(x) + u^{\text{ref}}(x) + \int_{\Gamma^+_\epsilon} G(x, y) \frac{\partial v_\epsilon(y)}{\partial n_y} ds_y, \quad x \in \mathbb{R}_2^+,
\]

where \( G \) is the half-space Green function satisfying the Neumann boundary condition and is given in (2.24). It follows from the continuity of the single layer potential that

\[
v_\epsilon(x) = u^{\text{inc}}(x) + u^{\text{ref}}(x) - \frac{i}{2} \int_{\Gamma^+_\epsilon} H_0^{(1)}(k|x - y|) \frac{\partial v_\epsilon(y)}{\partial n_y} ds_y, \quad x \in \Gamma^+_\epsilon.
\]  

(4.15)

In light of (4.12)–(4.14), it yields that

\[
\partial_{x_2} v_\epsilon = i \kappa \left( -\tilde{\alpha}^+_0 + \tilde{\alpha}^-_0 e^{ikd} \right) \phi_0(x_1) \quad \text{on } \Gamma^+_\epsilon.
\]

Substituting the above equality into (4.15) and using the fact that \( \phi_0(x_1) = \frac{1}{\sqrt{\epsilon}} \), we get

\[
v_\epsilon(x_1, 0) = u^{\text{inc}}(x_1, 0) + u^{\text{ref}}(x_1, 0) + \frac{\kappa}{2} \left( -\tilde{\alpha}^+_0 + \tilde{\alpha}^-_0 e^{ikd} \right) \frac{1}{\sqrt{\epsilon}} h_1(x_1), \quad x_1 \in (0, \epsilon),
\]

where \( h_1(x_1) \) is given in (2.26). Therefore, the Fourier coefficients \( v_{\epsilon, 0}^+ \) may be expressed as

\[
v_{\epsilon, 0}^+ = \langle u^{\text{inc}}, \phi_0 \rangle_{\Gamma^+_\epsilon} + \langle u^{\text{ref}}, \phi_0 \rangle_{\Gamma^+_\epsilon} + \frac{\kappa}{2} \left( -\tilde{\alpha}^+_0 + \tilde{\alpha}^-_0 e^{ikd} \right) \frac{1}{\sqrt{\epsilon}} \langle h_1, \phi_0 \rangle_{\Gamma^+_\epsilon}.
\]

It follows from the fact \( u^{\text{inc}}(x_1, 0) = u^{\text{ref}}(x_1, 0) \) and the inner product of (4.12) with \( \phi_0(x_1) \) that

\[
\tilde{\alpha}^+_0 + \tilde{\alpha}^-_0 e^{ikd} = 2\langle u^{\text{inc}}, \phi_0 \rangle_{\Gamma^+_\epsilon} + \frac{\kappa}{2 \sqrt{\epsilon}} \langle h_1, \phi_0 \rangle_{\Gamma^+_\epsilon} \left( -\tilde{\alpha}^+_0 + \tilde{\alpha}^-_0 e^{ikd} \right).
\]

Using \( \tilde{\alpha}^-_0 = \tilde{\alpha}^+_0 e^{ikd} \) and the above equation gives

\[
\tilde{\alpha}^+_0 = \frac{2\langle u^{\text{inc}}, \phi_0 \rangle_{\Gamma^+_\epsilon}}{(1 + c_0) + (1 - c_0) e^{2kd}}, \quad \tilde{\alpha}^-_0 = \frac{2 e^{ikd} \langle u^{\text{inc}}, \phi_0 \rangle_{\Gamma^+_\epsilon}}{(1 + c_0) + (1 - c_0) e^{2kd}},
\]

(4.16)

where \( c_0 = \frac{\kappa}{2 \sqrt{\epsilon}} \langle h_1, \phi_0 \rangle_{\Gamma^+_\epsilon} \).

Using the assumption \( \epsilon \ll d \ll \lambda \) and expression of \( c_0 \) in (2.28), it gives that

\[
|(1 + c_0) + (1 - c_0) e^{2kd}| = |2 \cos(\kappa d) + \frac{2\kappa}{\pi} \left( \ln \frac{\kappa}{2} + \epsilon \ln \epsilon - \frac{3}{2} \epsilon \right) \sin(\kappa d) + i\kappa \gamma_0 \epsilon \sin(\kappa d) + O(\epsilon^3 \ln \epsilon)|.
\]

(4.17)

Recalling the definition of the incident wave, we get

\[
|\langle u^{\text{inc}}, \phi_0 \rangle_{\Gamma^+_\epsilon}| = \left| \frac{1}{\sqrt{\epsilon}} \int_0^{\epsilon} e^{i \kappa \sin(\theta) x_1} dx_1 \right| = \sqrt{\epsilon} + O(\kappa^2 \epsilon^{5/2}).
\]

(4.18)
Combing (4.16)-(4.18), it yields
\[ |\tilde{\alpha}_0^+| = |\tilde{\alpha}_0^-| = \sqrt{\epsilon} + O(\kappa^2 \epsilon^{5/2}). \]

It follows from (4.12) and (2.33) that
\[
\|\nabla v_\epsilon\|^2_{L^2(D_\epsilon)} \leq \int_0^\epsilon \int_{-d}^0 \kappa (|\tilde{\alpha}_0^+| + |\tilde{\alpha}_0^-|) \phi_0 x_1^2 4 \sin^2 \kappa x_2 \, dx_1 dx_2 \\
\leq \kappa \epsilon (2\kappa d - \sin(2\kappa d)) + O(\kappa^6 d^3 \epsilon^3).
\]

Substituting (4.16) into (4.12) yields
\[
v_\epsilon(x) = \left( \frac{2(u^{\text{inc}}, \phi_0)_{\Gamma^+}}{(1 + c_0) + (1 - c_0)e^{i2\kappa d}} e^{-i\kappa x_2} + \frac{2e^{i\kappa d}(u^{\text{inc}}, \phi_0)_{\Gamma^+}}{(1 + c_0) + (1 - c_0)e^{i2\kappa d}} e^{i\kappa x_2 + d} \right) \phi_0(x_1).
\]

A simple calculation yields
\[
\frac{\partial v_\epsilon}{\partial x_2} = \kappa e^{i\kappa d} \frac{2(u^{\text{inc}}, \phi_0)_{\Gamma^+}}{(1 + c_0) + (1 - c_0)e^{i2\kappa d}} \phi_0(x_1) \left( -e^{-i\kappa x_2 + d} + e^{i\kappa x_2 + d} \right)
\]
\[
= -\kappa e^{i\kappa d} \frac{2(u^{\text{inc}}, \phi_0)_{\Gamma^+}}{(1 + c_0) + (1 - c_0)e^{i2\kappa d}} 2 \sin \kappa x_2 \, dx_2.
\]
which gives
\[
\|\nabla v_\epsilon\|^2_{L^2(D_\epsilon)} \geq \kappa^2 \left| \frac{2(u^{\text{inc}}, \phi_0)_{\Gamma^+}}{(1 + c_0) + (1 - c_0)e^{i2\kappa d}} \right|^2 \int_0^\epsilon \int_{-d}^0 4 \sin^2 \kappa x_2 \, dx_1 dx_2.
\]

Noting that \( d \ll \lambda \), we also have (2.33) holds. It follows from (4.17)–(4.18) that
\[
\|\nabla v_\epsilon\|_{L^2(D_\epsilon)} \geq \sqrt{\kappa \epsilon (2\kappa d - \sin(2\kappa d)) + O(\kappa^3 d^3 \epsilon^3/2)},
\]
which completes the proof. \( \square \)

**Theorem 4.3** Let \( v_\epsilon \) be the solution of (4.11) and be given by (4.12) inside the cavity, then we have the estimate
\[
\|v_\epsilon\|_{L^2(D_\epsilon)} = \sqrt{\epsilon d} \frac{\sqrt{2(1 + \sin(2\kappa d)/(2\kappa d))}}{\cos(\kappa d)} + O(\kappa \epsilon).
\]

**Proof** By (4.19), we have
\[
\|v_\epsilon\|^2_{L^2(D_\epsilon)} = \int_0^\epsilon \int_{-d}^0 |v_\epsilon|^2 \, dx_1 dx_2 \\
\leq \left| \frac{2(u^{\text{inc}}, \phi_0)_{\Gamma^+} e^{i\kappa d}}{(1 + c_0) + (1 - c_0)e^{i2\kappa d}} \right|^2 \int_0^\epsilon \int_{-d}^0 |e^{-i\kappa x_2 + d} + e^{i\kappa x_2 + d}|^2 \, dx_1 dx_2 \\
= \left| \frac{2(u^{\text{inc}}, \phi_0)_{\Gamma^+}}{(1 + c_0) + (1 - c_0)e^{i2\kappa d}} \right|^2 \int_{-d}^0 4 \cos^2 \kappa (x_2 + d) \, dx_2.
\]
It follows from \( d \ll \lambda \) that
\[
\int_{-d}^0 4 \cos^2 \kappa (x_2 + d) \, dx_2 = 2d \left( 1 + \frac{\sin(2\kappa d)}{2\kappa d} \right).
\]
In light of (4.17)–(4.18), it yields
\[ \| v_\epsilon \|_{L^2(D_\epsilon)}^2 = \frac{2\epsilon d}{\cos^2(\kappa d)} \left( 1 + \frac{\sin(2\kappa d)}{2\kappa d} \right) + O(\kappa^2 \epsilon^2), \]
which completes the proof.

\[ \Box \]

4.3 Field enhancement without resonance

Following from the same steps as those in Sect. 2.4, we may also show that
\[ C(R) \frac{\kappa \sqrt{d}}{\sin(\kappa d)} \epsilon (1 - O(\sqrt{\epsilon} \ln \epsilon)) \leq \| \nabla u_\epsilon - \nabla v_\epsilon \|_{L^2(D_\epsilon)} \leq C(R) \frac{\kappa \sqrt{d}}{\sin(\kappa d)} \epsilon, \]
\[ \| u_\epsilon - v_\epsilon \|_{L^2(D_\epsilon)} \leq C(R) \frac{\kappa \sqrt{d}}{\sin(\kappa d)} \epsilon \sqrt{\epsilon} \ln \epsilon, \] 
(4.20)
where \( C(R) \) is a positive constant depending on \( R \) but is independent of \( \lambda, \epsilon, d \). The details are omitted for brevity.

Remark 4.4 In (4.20), the constant \( C(R) \) comes from two places: one is the estimate of the DtN operator \( \beta^{\text{PEC}} \) in Lemma 4.1, where the constant \( C \) is independent of \( \lambda, d \), and the other is the Sobolev embedding results in \( \mathbb{R}^2_+ \), which depends only on \( R \).

The following estimates are the main results of the electromagnetic field enhancement for the PEC-PEC cavity when \( \epsilon \ll d \ll \lambda \).

**Theorem 4.5** If \( \epsilon \) is small enough such that \( C(R) \sqrt{\epsilon} < \frac{2}{3} \kappa^2 d \sin(\kappa d) \), then the electric field has no enhancement. If \( d \ll \lambda \) such that \( \frac{d}{\lambda} \leq \frac{1}{4\pi \sqrt{\frac{2}{3} C(R) \epsilon}} \), then the electric field enhancement factor satisfies
\[ \frac{C(R) \lambda}{4\pi \frac{d}{\lambda} \sqrt{\epsilon}} \leq Q_E \leq \frac{3C(R) \lambda}{4\pi \frac{d}{\lambda} \sqrt{\epsilon}} \]
and the magnetic field enhancement factor satisfies
\[ C_3 \leq Q_H \leq C_4, \]
where the constants \( C_3 = 2 - C(R) \sqrt{\ln \epsilon} / d + O(\kappa \epsilon), \ C_4 = 2 + C(R) \sqrt{\ln \epsilon} / d + O(\kappa \epsilon) \) as \( \frac{d}{\lambda} \rightarrow 0 \), and the constant \( C(R) \) is given in (4.20).

**Proof** By Theorem 4.2 and (4.20), we can obtain
\[ \| \nabla v_\epsilon \|_{L^2(D_\epsilon)} - \| \nabla u_\epsilon - \nabla v_\epsilon \|_{L^2(D_\epsilon)} \leq \| \nabla u_\epsilon \|_{L^2(D_\epsilon)} \leq \| \nabla v_\epsilon \|_{L^2(D_\epsilon)} + \| \nabla u_\epsilon - \nabla v_\epsilon \|_{L^2(D_\epsilon)}. \]
We consider two cases:
(i) If \( \epsilon \) is small enough such that \( C(R) \sqrt{\epsilon} < \frac{2}{3} \kappa^2 d \sin(\kappa d) \), then we have
\[ \| \nabla u_\epsilon \|_{L^2(D_\epsilon)} \leq \sqrt{\kappa \epsilon (2\kappa d - \sin(2\kappa d))} + C(R) \frac{\kappa \sqrt{d}}{\sin(\kappa d)} + O(\kappa^3 d^{3/2} \epsilon^{3/2}) \leq 2\kappa^2 d^2 \sqrt{\epsilon/d} \]
and
\[
\|\nabla u_e\|_{L^2(D_\epsilon)} \geq C_1 (\kappa d)^2 \sqrt{\frac{\epsilon}{d}} - C(R) \epsilon \sqrt{\kappa \epsilon (2 \kappa d - \sin(2 \kappa d))} + C(R) \frac{\kappa \sqrt{d}}{\sin(\kappa d)} + O(\kappa^3 d^{3/2} \epsilon^{3/2}) \geq \frac{2}{3} \kappa^2 d^2 \sqrt{\epsilon/d}.
\]

Since \(\|u_{\text{inc}}\|^2_{L^2(D_\epsilon)} = \kappa^2 \epsilon d\), we obtain
\[
\frac{4 \epsilon}{3 \pi} \lambda \leq \frac{\|u_{\text{inc}}\|_{L^2(D_\epsilon)}}{\|\nabla u_{\text{inc}}\|_{L^2(D_\epsilon)}} \leq 4\pi \frac{d}{\lambda}.
\]

Thus there is no enhancement in this case.

(ii) If \(d \ll \lambda\) such that \(\frac{d}{\lambda} \leq \frac{1}{4\pi} \sqrt{\frac{3}{2} \pi} C(R) \epsilon\), then we get
\[
\frac{C(R) \kappa \sqrt{d}}{2 \sin(\kappa d)} \epsilon \leq \|u_{\text{inc}}\|_{L^2(D_\epsilon)} \leq \frac{3C(R) \kappa \sqrt{d}}{2 \sin(\kappa d)} \epsilon,
\]
which gives
\[
\frac{C(R) \lambda}{4\pi} \sqrt{\epsilon} \leq \frac{\|u_{\text{inc}}\|_{L^2(D_\epsilon)}}{\|\nabla u_{\text{inc}}\|_{L^2(D_\epsilon)}} \leq \frac{3C(R) \lambda}{4\pi} \sqrt{\epsilon}.
\]

Noting that \(|\nabla \times H_e| = |\nabla u_e|\) and using Ampere’s law \(\nabla \times H_e = -i\omega \epsilon E_e\), we obtain
\[
Q_E = \frac{\|E\|_{L^2(D_\epsilon)}}{\|E_{\text{inc}}\|_{L^2(D_\epsilon)}} = \frac{\|\nabla u_e\|_{L^2(D_\epsilon)}}{\|\nabla u_{\text{inc}}\|_{L^2(D_\epsilon)}},
\]
which shows that for case (i) the electric field \(E\) has no enhancement; for case (ii) the electric field enhancement factor has an order \(O(\lambda \sqrt{\epsilon/d})\).

For the magnetic field, by Theorem 4.3 and (4.20), we have
\[
\sqrt{\epsilon d} \left( \frac{\sqrt{2(1 + \sin(2 \kappa d)/(2 \kappa d))}}{\cos(\kappa d)} - \frac{C(R) \kappa \epsilon \sqrt{\ln \epsilon}}{\sin(\kappa d)} + O(\kappa \epsilon) \right) \leq \|u_e\|_{L^2(D_\epsilon)}
\]
\[
\leq \sqrt{\epsilon d} \left( \frac{\sqrt{2(1 + \sin(2 \kappa d)/(2 \kappa d))}}{\cos(\kappa d)} + \frac{C(R) \kappa \epsilon \sqrt{\ln \epsilon}}{\sin(\kappa d)} + O(\kappa \epsilon) \right).
\]

Since \(\|u_{\text{inc}}\|^2_{L^2(D_\epsilon)} = \epsilon d\), we obtain
\[
C_3 \leq Q_H = \frac{\|H\|_{L^2(D_\epsilon)}}{\|H_{\text{inc}}\|_{L^2(D_\epsilon)}} = \frac{\|u_e\|_{L^2(D_\epsilon)}}{\|u_{\text{inc}}\|_{L^2(D_\epsilon)}} \leq C_4,
\]
where
\[
C_3 = \sqrt{\epsilon d} \left( \frac{\sqrt{2(1 + \sin(2 \kappa d)/(2 \kappa d))}}{\cos(\kappa d)} - \frac{C(R) \kappa \epsilon \sqrt{\ln \epsilon}}{\sin(\kappa d)} + O(\kappa \epsilon) \right)
\]
\[
\rightarrow 2 - \frac{C(R) \epsilon \sqrt{\ln \epsilon}}{d} + O(\kappa \epsilon),
\]
\[
C_4 = \sqrt{\epsilon d} \left( \frac{\sqrt{2(1 + \sin(2 \kappa d)/(2 \kappa d))}}{\cos(\kappa d)} + \frac{C(R) \kappa \epsilon \sqrt{\ln \epsilon}}{\sin(\kappa d)} + O(\kappa \epsilon) \right)
\]
\[
\rightarrow 2 + \frac{C(R) \epsilon \sqrt{\ln \epsilon}}{d} + O(\kappa \epsilon)
\]
as \(\frac{d}{\lambda} \rightarrow 0\). This completes the proof.
4.4 Numerical experiments

We present a numerical example to illustrate and verify the results stated in Theorem 4.5. The width of the cavity is $\epsilon = 0.005$, the depth of the cavity is $d = 1$, and the incident angle is $\theta = \pi/3$. Figure 5 plots the electric field enhancement factor $Q_E$ against the wavenumber $\kappa \in (0, 10^{-8})$. It shows that when $\kappa$ is sufficiently small, i.e., when the wavelength $\lambda$ is sufficiently large, the electric field does have an enhancement, which corresponds to case (ii) in Theorem 4.5; when the wavenumber $\kappa$ increases (the wavelength $\lambda$ decreases) but is still small, the electric field has no enhancement, which corresponds to case (i) in Theorem 4.5. The plot is not shown for the magnetic field since it has no enhancement in this region.

5 PEC-PEC cavity with resonance

In this section, we show the field enhancement at resonant frequencies for the PEC-PEC cavity. Consider the model problem:

\[
\begin{cases}
\Delta u_\epsilon + \kappa^2 u_\epsilon = 0 & \text{in } \Omega, \\
\partial_n u_\epsilon = 0 & \text{on } \partial \Omega, \\
\lim_{r \to \infty} \sqrt{r} (\partial_r u_\epsilon^{sc} - i\kappa u_\epsilon^{sc}) = 0 & \text{in } \mathbb{R}^+_2.
\end{cases}
\]  

(5.1)

Here we assume that $\epsilon \ll \lambda$. 
5.1 Boundary integral equations

Let \( g^{\text{PEC}}_{\epsilon} \) be the Green function for the Helmholtz equation with Neumann boundary condition in \( D_{\epsilon} \), i.e., it satisfies

\[
\begin{align*}
\Delta g^{\text{PEC}}_{\epsilon}(x, y) + \kappa^2 g^{\text{PEC}}_{\epsilon}(x, y) &= -\delta(x, y), & x, y &\in D_{\epsilon}, \\
\frac{\partial g^{\text{PEC}}_{\epsilon}(x, y)}{\partial \nu_y} &= 0 &\text{on } \partial D_{\epsilon}.
\end{align*}
\]

It can be verified that

\[
g^{\text{PEC}}_{\epsilon}(x, y) = \sum_{m,n=0}^{\infty} c_{m,n} \phi_{m,n}(x) \phi_{m,n}(y),
\]

where

\[
c_{m,n} = \frac{1}{\kappa^2 - (m\pi/\epsilon)^2 - (n\pi/d)^2},
\]

\[
\phi_{m,n}(x) = \sqrt{\frac{\alpha_{m,n}}{\epsilon d}} \cos \left( \frac{m\pi}{\epsilon} x_1 \right) \cos \left( \frac{n\pi}{d} (x_2 + d) \right),
\]

\[
\alpha_{m,n} = \begin{cases} 
1, & m = 0, n = 0, \\
2, & m = 0, n \geq 1 \text{ or } n = 0, m \geq 1, \\
4, & m \geq 1, n \geq 1.
\end{cases} \tag{5.2}
\]

Using Green’s formula, we obtain

\[
u_{\epsilon}(x) = \int_{\Gamma_{\epsilon}^+} G(x, y) \frac{\partial u_{\epsilon}(y)}{\partial \nu_y} \, ds_y + u_{\text{inc}}^\epsilon(x) + u_{\text{ref}}^\epsilon(x), \quad x \in \mathbb{R}_2^+,
\]

\[
u_{\epsilon}(x) = -\int_{\Gamma_{\epsilon}^+} g^{\text{PEC}}_{\epsilon}(x, y) \frac{\partial u_{\epsilon}(y)}{\partial \nu_y} \, ds_y, \quad x \in D_{\epsilon}.
\]

It follows from the continuity of the single layer potential that we have

\[
u_{\epsilon}(x) = \int_{\Gamma_{\epsilon}^+} \left( -\frac{i}{2} \right) H_0^1(\kappa|x - y|) \frac{\partial u_{\epsilon}(y)}{\partial \nu_y} \, ds_y + u_{\text{inc}}^\epsilon(x) + u_{\text{ref}}^\epsilon(x), \quad x \in \Gamma_{\epsilon}^+
\]

and

\[
u_{\epsilon}(x) = -\int_{\Gamma_{\epsilon}^+} g^{\text{PEC}}_{\epsilon}(x, y) \frac{\partial u_{\epsilon}(y)}{\partial \nu_y} \, ds_y, \quad x \in \Gamma_{\epsilon}^+.
\]

By imposing the continuity of the solution on \( \Gamma_{\epsilon}^+ \), we have the following lemma.

**Lemma 5.1** The scattering problem (5.1) is equivalent to the boundary integral equation:

\[
\int_{\Gamma_{\epsilon}^+} \left( -\frac{i}{2} \right) H_0^1(\kappa|x - y|) + g^{\text{PEC}}_{\epsilon}(x, y) \frac{\partial u_{\epsilon}(y)}{\partial \nu_y} \, ds_y + u_{\text{inc}}^\epsilon(x) + u_{\text{ref}}^\epsilon(x) = 0, \quad x \in \Gamma_{\epsilon}^+. \tag{5.3}
\]

Similarly, we introduce the rescaling variables \( X = x_1/\epsilon, Y = y_1/\epsilon \) and define the following boundary integral operators:
\[(T^e \varphi)(X) := \int_0^2 G^e_\epsilon(X, Y) \varphi(Y) dY, \quad X \in (0, 1),\]
\[(T^i \varphi)(X) := \int_0^1 G^i_\epsilon(X, Y) \varphi(Y) dY, \quad X \in (0, 1).\]  

(5.4)

where \(G^e_\epsilon, G^i_\epsilon, \varphi\) are give in (3.6) with \(s_\epsilon^{PMC}\) being replaced by \(g_\epsilon^{PEC}\). Then the boundary integral equation (5.3) is equivalent to the operator equation:

\[(T^e + T^i) \varphi = f / \epsilon,\]  

(5.5)

where \(f = (u^{inc} + u^{ref}) |_{\Gamma^+} = 2 e^{i \kappa \sin \theta} \sin \theta X\).

5.2 Asymptotics of the integral operators

In this subsection, we study the asymptotic properties of the integral operators in (5.4).

**Lemma 5.2**  If \(\epsilon \ll \lambda\), then we have the following asymptotic formulas:

\[G^e_\epsilon = \Gamma_1(\kappa, \epsilon) + \frac{1}{\pi} \ln |X - Y| + R^e_1(X, Y) + R^e_2(X, Y),\]
\[G^i_\epsilon = \Gamma_2(\kappa, \epsilon) + \frac{1}{\pi} \left( \ln \left| \sin \frac{\pi (X + Y)}{2} \right| + \ln \left| \sin \frac{\pi (X - Y)}{2} \right| \right) + R^i_3(X, Y),\]

where \(\Gamma_2(\kappa, \epsilon) = \frac{\cot \frac{x d}{\epsilon \kappa}}{\epsilon \kappa} + \frac{2 \ln 2}{\pi}\), and \(\Gamma_1(\kappa, \epsilon), R^e_1(X, Y), R^e_2(X, Y), R^i_3(X, Y)\) are given in Lemma 3.2.

**Proof**  The proof is similar to that for Lemma 3.2. The integral kernels \(G^e_\epsilon(X, Y)\) and \(G^i_\epsilon(X, Y)\) can be expressed as

\[G^e_\epsilon(X, Y) = \Gamma_1(\kappa, \epsilon) + \frac{1}{\pi} \ln |X - Y| + R^e_1(X, Y) + R^e_2(X, Y),\]
\[G^i_\epsilon(X, Y) = \frac{1}{\epsilon d} \sum_{m, n=0}^{\infty} c_{m,n} \alpha_{m,n} \cos(m \pi X) \cos(m \pi Y).\]

Recalling the definitions of \(c_{m,n}\) and \(\alpha_{m,n}\) in (5.2), we let

\[C_m(\epsilon, \kappa) = \sum_{n=0}^{\infty} c_{m,n} \alpha_{m,n} = \sum_{n=0}^{\infty} \frac{\alpha_{m,n}}{\kappa^2 - (m \pi / \epsilon)^2 - (n \pi / d)^2}.\]

Using the fact (cf. [20]) that

\[\cot(\pi x) = \frac{1}{\pi x} + \frac{2x}{\pi} \sum_{n=1}^{\infty} \frac{1}{x^2 - n^2}, \quad x \neq n,\]

for \(m = 0\), we get

\[C_0(\kappa) = \frac{1}{\kappa^2} + \sum_{n=1}^{\infty} \frac{2}{\kappa^2 - (n \pi / d)^2} = \frac{d}{\kappa} \left( \frac{1}{\pi (d \kappa / \pi)} + \frac{2d \kappa}{\pi} \sum_{n=1}^{\infty} \frac{1}{(d \kappa / \pi)^2 - n^2} \right) = \frac{d \cot(d \kappa)}{\kappa}.\]
Following from the identity
\[
\coth(\pi x) = \frac{1}{\pi x} + \frac{2x}{\pi} \sum_{n=1}^{\infty} \frac{1}{x^2 + n^2},
\]
we have for \(m \geq 1\) that
\[
C_m(\epsilon, \kappa) = \frac{2}{\kappa^2 - (m\pi/\epsilon)^2} + \sum_{n=1}^{\infty} \frac{4}{\kappa^2 - (n\pi/d)^2}
\]
\[
= \frac{-2d}{\sqrt{(m\pi/\epsilon)^2 - \kappa^2}} \left( \frac{\kappa^2}{4\pi} \sqrt{(m\pi/\epsilon)^2 - \kappa^2} - \kappa \right)
\]
\[
+ \sum_{n=1}^{\infty} \frac{1}{n^2 + (d\pi/\epsilon)^2((m\pi/\epsilon)^2 - \kappa^2)}
\]
\[
= \frac{-2d}{\sqrt{(m\pi/\epsilon)^2 - \kappa^2}} \coth(d\sqrt{(m\pi/\epsilon)^2 - \kappa^2})
\]
\[
= -\frac{2d\epsilon}{m\pi} \frac{\epsilon^2\kappa^2 d}{m^2\pi^3} + O\left(\frac{\epsilon^5}{m^5}\right).
\]

Here we use the asymptotic expression of \((1 - x^2)^{-1/2}\) and \(\coth(d\sqrt{(m\pi/\epsilon)^2 - \kappa^2}) \to 1\) as \(\epsilon \to 0\).

Substituting \(C_0(\kappa), C_m(\epsilon, \kappa)(m \geq 1)\) into \(G_{\epsilon}^i\), we obtain
\[
G_{\epsilon}^i(X, Y) = \frac{1}{\epsilon d} \left( \frac{d \cot(\kappa d)}{\kappa} - \sum_{m=1}^{\infty} \frac{2d\epsilon}{m\pi} + \frac{\epsilon^2\kappa^2 d}{m^3\pi^3} \cos(m\pi X) \cos(m\pi Y) + O\left(\sum_{m=1}^{\infty} \frac{\epsilon^5}{m^5}\right) \right)
\]
\[
= \frac{d \cot(\kappa d)}{\kappa \epsilon} + \frac{1}{\pi} \left( 2 \ln 2 + \ln \left| \sin \frac{\pi(X + Y)}{2} \right| + \ln \left| \sin \frac{\pi(X - Y)}{2} \right| \right)
\]
\[
- \frac{\epsilon^2\kappa^2}{\pi} \left( \frac{1}{\pi^2} \sum_{m=1}^{\infty} \frac{1}{m^2} + \frac{(X + Y)^2}{4} \ln(\pi(X + Y)) + \frac{(X - Y)^2}{4} \ln(\pi(X - Y)) \right)
\]
\[
+ O((X + Y)^2 + (X - Y)^2)
\]
\[
= \Gamma_2(\kappa, \epsilon) + \frac{1}{\pi} \left( \ln \left| \sin \frac{\pi(X + Y)}{2} \right| + \ln \left| \sin \frac{\pi(X - Y)}{2} \right| \right) + R_3^\epsilon(X, Y),
\]
which completes the proof.

Let \(\Gamma = \Gamma_1(\kappa, \epsilon) + \Gamma_2(\kappa, \epsilon), K, K_1^\infty, K_2^\infty\) are defined same as in (3.10). Then for the operator \(T^\epsilon + T^i\) in (5.5), we have a similar decomposition
\[
T^\epsilon + T^i = \Gamma P + K + K_1^\infty + K_2^\infty,
\]
where \(K, K_1^\infty, K_2^\infty\) have the same properties as those in Lemma 3.3.
5.3 Asymptotics of the resonances

For convenience, we write

\[ \Gamma P + K + K^1_\infty + K^2_\infty := \mathcal{P} + \mathcal{L}, \]

where \( \mathcal{P} = \Gamma P, \mathcal{L} = K + K^1_\infty + K^2_\infty \). The eigenvalues of operator \( \mathcal{L}^{-1} \mathcal{P} + \mathcal{I} \) are

\[ \lambda(\kappa, \epsilon) = 1 + \Gamma(\kappa, \epsilon) \langle \mathcal{L}^{-1} 1_{(0,1)}, 1_{(0,1)} \rangle_{L^2(0,1)}. \]

Therefore, the characteristic values of the operator \( \mathcal{P} + \mathcal{L} \) are the roots of the analytic functions \( \lambda(\kappa, \epsilon) \), and the associated characteristic function is given by

\[ \phi_0 = \Gamma(\kappa, \epsilon) \mathcal{L}^{-1} 1_{(0,1)}. \]

**Theorem 5.3** The resonance of the scattering problem (5.1) are the roots of the analytic function \( \lambda(\kappa, \epsilon) = 0 \). Moreover, the resonance set \( \{k_n\}, n = 0, 1, 2, \cdots \) satisfies the following asymptotic expansion

\[ k_n = \frac{(n + \frac{1}{2})\pi}{d} + \frac{(n + \frac{1}{2})\pi}{d^2} \left( \frac{1}{\pi} \epsilon \ln \epsilon + \left( \frac{1}{\epsilon q_0} + \frac{1}{\epsilon} (2 \ln 2 + \ln \frac{(n + \frac{1}{2})\pi}{d} + \gamma_1) \right) \epsilon \right) + O(\epsilon^2 \ln \epsilon). \]

(5.6)

**Proof** Given roots of \( \lambda(\kappa, \epsilon) \), it is easy to check that they are the characteristic values of the operator \( \mathcal{P} + \mathcal{L} \) with corresponding characteristic function defined above.

Consider the roots of

\[ \lambda(\kappa, \epsilon) = 1 + (\Gamma_1(\kappa, \epsilon) + \Gamma_2(\kappa, \epsilon)) \langle \mathcal{L}^{-1} 1_{(0,1)}, 1_{(0,1)} \rangle_{L^2(0,1)} = 0. \]

Recall that \( \Gamma_1(\kappa, \epsilon) = \frac{1}{\pi} (\ln \kappa + \gamma_1) + \frac{1}{\pi} \ln \epsilon \) and \( \Gamma_2(\kappa, \epsilon) = \frac{\cot \kappa d}{\epsilon \kappa} + \frac{2 \ln 2}{\epsilon} \). The above equation can be written as

\[ 1 + \left( \frac{\cot \kappa d}{\epsilon \kappa} + \frac{1}{\pi} (2 \ln 2 + \ln \kappa + \gamma_1) + \frac{1}{\pi} \ln \epsilon \right) \langle \mathcal{L}^{-1} 1_{(0,1)}, 1_{(0,1)} \rangle_{L^2(0,1)} = 0. \]

By Lemma 3.4, similar as (3.13), we let

\[ p(\kappa, \epsilon) := \epsilon \lambda(\kappa, \epsilon) = \epsilon + \left( \frac{\cot \kappa d}{\kappa} + \epsilon \rho(\kappa) + \frac{1}{\pi} \epsilon \ln \epsilon \right) (\epsilon q_0 + O(\epsilon^2 \ln \epsilon)). \]

where \( \rho(\kappa) := \frac{1}{\pi} (2 \ln 2 + \ln \kappa + \gamma_1) \). All the roots of \( p(\kappa, \epsilon) \) lie in the domain \( \Omega_{\delta_0, \theta_0, M} := \{ z \in \mathbb{C} : |z - 2 j \pi/d| \geq \delta_0, j \in \mathbb{Z} \} \cap \Omega_{\theta_0, M} \), where

\[ \Omega_{\theta_0, M} = \left\{ z \in \mathbb{C} : |z| \leq M, -(\pi/d) - \theta_0 \leq \arg z \leq \pi/d - \theta_0 \right\}. \]

It is clear to note that \( \frac{\cot \kappa d}{\kappa} \) is analytic in the domain \( \Omega_{\delta_0, \theta_0, M} \) and the roots are given by

\[ k_{n,0} = \left( n + \frac{1}{2} \right) \frac{\pi}{d}, \quad n = 0, 1, 2, \cdots. \]

Denote by \( k_n \) the roots of \( \lambda(\epsilon, \kappa) \). Applying Rouche’s theorem, we deduce that \( k_n \) are simple and close to \( k_{n,0} \). The corresponding \( p_1(\kappa, \epsilon) \) are defined by

\[ p_1(\kappa, \epsilon) = \epsilon + \left( \frac{\cot \kappa d}{\kappa} + \epsilon \rho(\kappa) + \frac{1}{\pi} \epsilon \ln \epsilon \right) q_0. \]
Expanding $p_1(\kappa, \epsilon)$ at $k_n, 0$ yields

$$p_1(\kappa, \epsilon) = \epsilon + \left( \frac{-d^2}{(n + \frac{1}{2})\pi} (\kappa - k_n) + \frac{\epsilon}{\pi} \left( 2\ln 2 + \ln \left( \frac{n + \frac{1}{2}}{d} \right) + \gamma_1 \right) \right) + \frac{\epsilon d}{(n + \frac{1}{2})\pi^2} (\kappa - k_n) + O(\kappa - k_n)^2 + \frac{1}{\pi} \epsilon \ln \epsilon) q_0.$$

We conclude that $p_1(\kappa, \epsilon)$ has simple roots in $\Omega_{k_0, \theta_0, M}$ which are close to $k_n, 0$. Moreover, these roots are analytic with respect to the variable $\epsilon$ and $\epsilon \ln \epsilon$. Denote the roots of $p_1(\kappa, \epsilon)$ by $k_n, 1$ and expand them in terms of $\epsilon$ and $\epsilon \ln \epsilon$, we obtain

$$k_n, 1 = k_n, 0 + \frac{k_n, 0}{d} \left( \frac{1}{\pi} \epsilon \ln \epsilon + \left( \frac{1}{q_0} + \frac{1}{\pi} (2\ln 2 + \ln \left( \frac{n + \frac{1}{2}}{d} \right) + \gamma_1) \right) \right) + O(\epsilon^2 \ln \epsilon).$$

Using Rouche’s theorem again, we obtain

$$k_n = k_n, 1 + O(\epsilon^2 \ln \epsilon),$$

which completes the proof. \hfill \Box

5.4 The field enhancement with resonance

In the far field region, we can follow the same steps as those in Sect. 3.4.1 with the new $p(\kappa, \epsilon)$. Then we have the following theorem with the detailed proof omitted.

Theorem 5.4 At the resonant frequencies $\kappa = \text{Re} k_n$ where $k_n$ is given in (5.6), the scattered electric and magnetic fields have enhancement of an order $O(1/\epsilon)$ in the far field region.

Next we demonstrate the field enhancement inside the cavity. It follows from (4.2) that the total field $u_\epsilon$ can be expanded as the sum of waveguide modes:

$$u_\epsilon(x) = \frac{1}{\sqrt{\epsilon}} \left( \alpha_0^+ e^{-i\kappa x_2} + \alpha_0^- e^{i\kappa (x_2 + d)} \right) + \sum_{n=1}^{\infty} \sqrt{\frac{2}{\epsilon}} \left( \alpha_n^+ e^{-i\beta_n x_2} + \alpha_n^- e^{i\beta_n (x_2 + d)} \right) \cos \left( \frac{m\pi x_1}{\epsilon} \right). \quad (5.7)$$

Lemma 5.5 The coefficients in (5.7) have the following properties:

$$\frac{\alpha_0^+}{\sqrt{\epsilon}} = \frac{q_0}{i\kappa} \left( 2 + \frac{\sin \theta}{-1 + e^{2i\kappa d}} \right) + O(\epsilon^2 \ln \epsilon), \quad \frac{\alpha_0^-}{\sqrt{\epsilon}} = \frac{q_0}{-2\kappa \sin(\kappa d)} + O(\epsilon^2 \ln \epsilon)$$

and

$$\left| \sqrt{\frac{2}{\epsilon}} \alpha_n^+ \right| \leq \frac{C}{\sqrt{n}}, \quad \left| \sqrt{\frac{2}{\epsilon}} \alpha_n^- \right| \leq \frac{C}{\sqrt{n}}, \quad n \geq 1,$$

where the positive constant $C$ is independent of $\lambda, \epsilon$ and $n$.

**Proof** It follows from the boundary condition $\partial \nu u_\epsilon = 0$ on $\Gamma_\epsilon$ that the expansion of (5.7) becomes

$$u_\epsilon(x) = \frac{\alpha_0^+}{\sqrt{\epsilon}} \left( e^{-i\kappa x_2 + d} + e^{i\kappa x_2 + d} \right) \cos \left( \frac{m\pi x_1}{\epsilon} \right).$$
Taking the derivative of the above equality with respect to $x_2$ on $\Gamma_\epsilon^+$ yields

$$\frac{\partial u_\epsilon(x_1, 0)}{\partial x_2} = \frac{\alpha^+_0}{\sqrt{\epsilon}} i \kappa (-1 + \epsilon i^{2k}) + \sum_{n=1}^{\infty} \sqrt{\frac{2}{\epsilon}} \alpha^+_n i \beta_n (-1 + \epsilon i^{2\beta nd}) \cos \left( \frac{n \pi x_1}{\epsilon} \right).$$

Multiplying the above equality by $\phi_0(x_1)$ and integrating it on $\Gamma_\epsilon^+$ yields

$$\frac{\alpha^+_0}{\sqrt{\epsilon}} i \kappa (-1 + \epsilon i^{2k}) = \frac{1}{\epsilon} \int_{\Gamma_\epsilon^+} \frac{\partial u_\epsilon}{\partial x_2} (x_1, 0) \mathrm{d}x_1 = \frac{1}{\epsilon} \int_{\Gamma_\epsilon^+} -\varphi(x_1/\epsilon) \mathrm{d}x_1 = \int_0^1 \varphi(X) \mathrm{d}X = \frac{q_0 (2 + \sin \theta \cdot O(\kappa \epsilon))}{p} + O(\epsilon^2 \ln \epsilon).$$

Here we use $\langle \varphi, 1_{(0,1)} \rangle_{L^2(0,1)} = \frac{q_0 (2 + \sin \theta \cdot O(\kappa \epsilon))}{p} + O(\epsilon^2 \ln \epsilon)$, which is shown in the proof of Lemma 3.7. By $\alpha^+_0 = \alpha^+_0 e^{i k d}$, we get

$$\frac{\alpha^+_0}{\sqrt{\epsilon}} = \frac{q_0 (2 + \sin \theta \cdot O(\kappa \epsilon))}{i \kappa (-1 + \epsilon i^{2k} \epsilon)} + O(\epsilon^2 \ln \epsilon),$$

$$\frac{\alpha^+_0}{\sqrt{\epsilon}} = \frac{q_0 (2 + \sin \theta \cdot O(\kappa \epsilon))}{-2 \kappa \sin (k \epsilon) \epsilon} + O(\epsilon^2 \ln \epsilon).$$

Similarly, for $n \geq 1$, we may obtain

$$\sqrt{\frac{2}{\epsilon}} \alpha^+_n (-1 + \epsilon i^{2\beta nd}) = -\int_0^1 \varphi(X) \cos (n \pi X) \mathrm{d}X.$$  

Note that $i \beta_n = O(\frac{n}{\epsilon})$, $\| \varphi \|_{H^{-1/2}(0,1)} \lesssim \frac{1}{\epsilon}$, $\| \cos (n \pi X) \|_{H^{1/2}(0,1)} \lesssim \sqrt{n}$. Thus we have

$$\left| \sqrt{\frac{2}{\epsilon}} \alpha^+_n \right| \leq \frac{C}{\sqrt{n}}, \quad \left| \sqrt{\frac{2}{\epsilon}} \alpha^-_n \right| \leq \frac{C}{\sqrt{n}}, \quad n \geq 1,$$

where the positive constant $C$ is independent of $\lambda, \epsilon, n$. \qed

The following result gives the field enhancement inside the cavity.

**Theorem 5.6** Let $\hat{D}_\epsilon := \{ x \in D_\epsilon : -d \leq x_2 \leq -\epsilon \}$ be the interior of the cavity $D_\epsilon$. If $\kappa = \text{Re} k_n$ where $k_n$ is given in (5.6), then we have for $\epsilon \ll \lambda$ that

$$u_\epsilon(x) = \left( \frac{2}{\epsilon} + \sin \theta \cdot O(1) + O(\ln^2 \epsilon) \right) \frac{2i \cos \kappa (x_2 + d)}{\kappa \sin (k \epsilon)} + O(\epsilon^2 \ln \epsilon), \quad x \in \hat{D}_\epsilon.$$  

Moreover, the electric and magnetic field enhancements are of an order $O(1/\epsilon)$.

**Proof** In the region $\hat{D}_\epsilon$, we have from the expansion of (5.7) and Lemma 5.5 that...
The field enhancement factor is plotted against the wavenumber $\kappa$ for the PEC-PEC cavity: (left) the electric field enhancement factor $Q_E$; (right) the magnetic field enhancement factor $Q_H$.

\[ u_\epsilon(x) = \frac{q_0(2 + \sin \theta \cdot O(\kappa \epsilon))}{\sqrt{-\kappa \sin \kappa d}} \left( e^{-i\kappa(x_2 + d)} + e^{i\kappa(x_2 + d)} \right) + O(\epsilon^2 \ln \epsilon) + O(e^{-1/\epsilon}) \]

Using the fact that at the resonant frequencies $\kappa = \text{Re} k_n$,

\[ \frac{1}{p} = \frac{2i}{q_0 \epsilon} (1 + O(\epsilon \ln^2 \epsilon)). \]

we obtain

\[ u_\epsilon(x) = \left( \frac{2}{\epsilon} + \sin \theta \cdot O(1) + O(\ln^2 \epsilon) \right) \frac{2i \cos \kappa (x_2 + d)}{\kappa \sin(\kappa d)} + O(\epsilon^2 \ln \epsilon). \]

Therefore, the magnetic field enhancement has an order of $O(1/\epsilon)$. We conclude from Ampere’s law that the electric field enhancement also has an order of $O(1/\epsilon)$.

Finally, on the open aperture $\Gamma_\epsilon^+$, we may follow similar steps as those in Sect. 3.4.3 and show the electromagnetic field enhancement. The proof is omitted again for brevity.

**Theorem 5.7** At the resonant frequencies $\kappa = \text{Re} k_n$ where $k_n$ is given in (5.6), the enhancement of the scattered electric and magnetic fields has an order $O(1/\epsilon)$ on the open aperture $\Gamma_\epsilon^+$.

5.5 Numerical experiments

We show some numerical experiments to verify the theoretical findings of the field enhancement for the PEC-PEC cavity. We take the cavity width $\epsilon = 0.005$, the cavity depth $d = 1$, and the angle of incidence $\theta = \pi/3$. Figure 6 (left) and (right) show the plot of the electric field enhancement $Q_E$ and the magnetic field enhancement factor $Q_H$ against the wavenumber $\kappa$, respectively. We observe that both the electric and magnetic enhancement factors do obtain peaks at the resonant frequencies (5.6), which are close to $\kappa = (n + \frac{1}{2})\pi, n = 0, 1, 2, \ldots$
6 Concluding remarks

We have studied the electromagnetic field enhancement for the scattering of a plane wave by a subwavelength rectangular open cavity. The TM polarized wave model problem is considered for both the PEC-PMC and the PEC-PEC cavities. The electric and magnetic field enhancements are examined in both the nonresonant and resonant regimes. These results provide an understanding on some of the mechanisms for the wave field amplification and localization in a subwavelength structure. This work focuses on the TM polarization of the electromagnetic wave in a single cavity. We will investigate the influence of multiple cavities, whether the enhancement may occur in the transverse electric polarization (TE), the more complicated three-dimensional Maxwell equations, and the elastic wave equation. The results will be reported elsewhere in the future.

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