AHARONOV-BOHM SCATTERING, CONTACT INTERACTIONS AND SCALE INVARIANCE*

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ABSTRACT

We perform a perturbative analysis of the Aharonov-Bohm problem to one loop in a field-theoretic formulation, and show that contact interactions are necessary for renormalizability. In general, the classical scale invariance of this problem is broken quantum mechanically. There exists however a critical point for which this anomaly disappears.

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I. INTRODUCTION

The Aharonov-Bohm (AB) effect, which is essentially the scattering of charged particles by a flux tube [1], has been one of the most studied problems of planar physics in the last thirty years [2]. Much of the recent interest in the subject is a consequence of the discovery that flux-charge composites acquire fractional statistics (becoming anyons) through the AB effect [3]. Such composites can be obtained by coupling ordinary particles (bosons or fermions) to a gauge field, whose dynamics are governed not by the Maxwell action but by the Chern-Simons (CS) action [4]. Thus, scattering of a charged particle from a flux tube, scattering of two anyons, and scattering of two particles coupled to a CS gauge field are all the same problem.

In their original work, Aharonov and Bohm found an exact expression for the scattering amplitude, which has been since rederived in different ways [2,5,6]. Nevertheless, several attempts to obtain the result perturbatively have failed [7]. As recognized by Corinaldesi and Rafeli, the Born approximation misses the s-wave contribution to the scattering amplitude in first order, while the second term in the Born series is infinite.

Not surprisingly, similar problems with the perturbative calculation have appeared more recently in the context of anyon physics in the so-called bosonic end [8,9,10]. In this case the quantities of interest are the eigenenergies and thermodynamic properties of a system of anyons. Sense is made of the perturbation theory by either redefining the unperturbed wave-function [8], or by solving a different but equivalent problem with a transformed wavefunction and a transformed Hamiltonian [9,10]. In the latter method the transformed Hamiltonian is actually not Hermitian, and it is interesting to note that if this is corrected by adding to it its conjugate, a $\delta$-function potential arises [10]. Though these methods reproduce to lowest order the correct result for two anyons, the manipulations necessary for this appear arbitrary.

The aim of our paper is to clarify the perturbative analysis of the AB problem, and to show that a correct treatment must include contact interactions. We shall do this in the framework of quantum field theory by using the second quantized formulation of the AB effect for bosons, developed in [11]. We shall see that the contact interaction, first considered in [12], is necessary to ensure the renormalizability of the theory. In part I we shall write down the Lagrangian density for this theory, review how it reduces to the AB problem, and show that it is formally scale invariant. In part II we shall perform a standard perturbative analysis to one loop, and show that in general renormalization is necessary, and results in the breaking of scale invariance, as occurs in a theory with only contact interaction [13]. However, for a certain strength of the contact interaction, whether attractive or repulsive, the theory has a critical point [14], and scale invariance is regained. In the repulsive case, this value of the strength also reproduces the AB result. In the concluding remarks we comment on how our analysis changes if we solve the constraints for the gauge field before doing perturbation theory, and how it changes for fermions.

II. FIELD-THEORETICAL FORMULATION OF THE AB EFFECT

We begin by considering a system of nonrelativistic bosons in $2+1$ dimensions minimally coupled to a CS gauge field. The Lagrangian density then reads:

$$
\mathcal{L} = \frac{\kappa}{2} \partial_\tau \mathbf{A} \times \mathbf{A} - \kappa A_0 B + \phi^* \left( iD_t + \frac{D^2}{2} \right) \phi - V[\phi, \phi^*],
$$

\text{(II.1)}
where the covariant derivatives are

\[ D_t = \partial_t + ieA_0 \]  
\[ D = \nabla - ieA \]  

and the mass is set to unity. This model, with the potential term \( V[\phi, \phi^*] \) set to zero, was first considered by Hagen [11] as an example of a Galilean invariant gauge theory. The Lagrangian density (II.1) also coincides with the one proposed by other authors [15] as an effective theory for the Fractional Quantum Hall Effect. We shall consider the potential corresponding to the \( \delta \)-function, or contact, interaction:

\[ V[\phi, \phi^*] = \frac{\nu_0}{4} \phi^* \phi \phi \]  

(To study the thermodynamic properties of the system one also includes a quadratic term, whose coefficient is the chemical potential. Since we are only interested in scattering processes in the vacuum, we drop this term.)

It was shown in reference [11] that this model, with \( \nu_0 = 0 \), is a field-theoretical formulation of the AB problem. To see this, first notice that using the Gauss Law

\[ \nabla \times A = -\frac{e}{\kappa} \phi^* \phi \]  

the gauge fields can be expressed in terms of the matter fields (in the Coulomb gauge) as:

\[ A(r, t) = -\frac{e}{\kappa} \nabla \times \int d^2r' G(r - r') \phi^*(r', t) \phi(r', t) \]  

where \( G(r) \) is the Green’s function of the two dimensional Laplacian

\[ G(r) = \frac{1}{2\pi} \ln |r| \]  

After imposing canonical commutation relations

\[ [\phi(r, t), \phi^*(r', t)] = \delta(r - r') \]  

and identifying the Hamiltonian as

\[ H = \int d^2r \left[ \frac{1}{2} (D\phi)^* \cdot (D\phi) + V[\phi, \phi^*] \right] \]  

one can show that the wave function for the two particle sector satisfies the following Schrödinger equation

\[ E\psi(r_1, r_2) = \left[ -\frac{1}{2} \sum_{i,j}^2 \left( \nabla_i - \frac{ie^2}{\kappa} \nabla_i \times G(r_i - r_j) \right)^2 + \frac{\nu_0}{2} \delta(r_1 - r_2) \right] \psi(r_1, r_2) \]
For \( v_0 = 0 \) this becomes in the center of mass (c.m.) frame
\[
\left[ \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} - \frac{(m + \alpha)^2}{r^2} \right] \Psi(r) = -E \Psi(r),
\]
where
\[
\psi(r) = e^{im\theta} \Psi(r)
\]
and
\[
\alpha = \frac{e^{2\pi\kappa}}{2},
\]
which is exactly the equation studied by Aharonov and Bohm. The scattering amplitude can be calculated exactly, and for \(|\alpha| < 1\) is given by
\[
f(k, \theta) = (2\pi i k)^{-1/2} \sin \pi \alpha \left[ \cot \frac{\theta}{2} - i \text{sgn} (\alpha) \right], \quad \theta \neq 0.
\]

For identical particles one must add to this the amplitude with the final states exchanged, which is gotten by letting \( \theta \to \theta - \pi \) in (II.13), resulting in the amplitude
\[
f(k, \theta) = \left( \frac{2}{\pi i k} \right)^{1/2} \sin \pi \alpha \left[ \cot \theta - i \text{sgn} (\alpha) \right], \quad \theta \neq 0, \pi.
\]

The dependence of the amplitude on \( k \) is only through the kinematical factor \((2\pi k)^{-1/2}\), which is a sign of scale invariance. This can be seen from the fact that the parameter \( \alpha \) is dimensionless. Note also that the inclusion of the \( \delta\)-function potential would not modify this property because \( v_0 \) is also dimensionless.

Scale invariance of the system can also be seen in the field theory. As shown by Jackiw and Pi [12], under a dilation
\[
\delta t = 2t \quad \delta r = r,
\]
the following infinitesimal transformation of the fields
\[
\delta \phi = -[1 + r \cdot D + 2t D_t] \phi \quad \text{(II.16a)}
\]
\[
\delta A = [r \times \hat{z} B + 2t E] \quad \text{(II.16b)}
\]
\[
\delta A_0 = r \cdot E \quad \text{(II.16c)}
\]
where \( E \equiv -\nabla A_0 - \partial_t A \) and \( B \equiv \nabla \times A \), leaves the action invariant.

Similarly, under a special conformal transformation
\[
\delta t = -t^2 \quad \delta r = -tr,
\]
the following infinitesimal transformation
\[
\delta \phi = \left[ t - \frac{i}{2} r^2 + tr \cdot D + t^2 D_t \right] \phi \quad \text{(II.18a)}
\]
\[
\delta A = [-tr \times \hat{z} B + t^2 E] \quad \text{(II.18b)}
\]
\[
\delta A_0 = -tr \cdot E \quad \text{(II.18c)}
\]
also leaves the action invariant.

One can prove that the generators of the above two transformations together with the Hamiltonian form a group SO(2,1). Of course invariance of the action does not always guarantee an invariant result, because symmetries can be broken quantum mechanically by anomalies. This is indeed what happens if we set $e = 0$ in (II.1), and consider the bosons interacting only among themselves via the $\delta$-function potential. The classical scale invariance of the $\delta$-function interaction is broken by quantum effects [13,16]. Moreover, we shall show in the next section that this is also the case when one considers (II.1) without the contact interaction, and to obtain a scale invariant result requires a fine tuning between $e$ and $v_0$.

**III. PERTURBATION THEORY**

Let us begin by reviewing the problems with the perturbative treatment of the AB effect. If one applies the Born approximation to the Hamiltonian considered by Aharonov and Bohm, (II.10), instead of obtaining

\[ f(k, \theta) = \alpha \left( \frac{2\pi}{ik} \right)^{1/2} \left[ \cot \theta - i \text{sgn}(\alpha) \right] + \mathcal{O}(\alpha^3), \theta \neq 0, \pi, \quad (III.1) \]

one obtains the incorrect result [7]

\[ f(k, \theta) = \alpha \left( \frac{2\pi}{ik} \right)^{1/2} \cot \theta + \mathcal{O}(\alpha^2), \theta \neq 0, \pi, \quad (III.2) \]

in which the non-analytic part is missing. In addition, the next order approximation, $\mathcal{O}(\alpha^2)$, diverges.

Corinaldesi and Rafeli [7] noticed that the problem can be traced to the s-wave contribution to the scattering amplitude, which is missing in (III.2). As can be seen in (II.10), for the s-wave ($m = 0$) the perturbation is of order $\alpha^2$, and when one looks at the integral equation satisfied by the solution one encounters logarithmic divergences. This is due to the singular nature of the potential and the nonvanishing behavior of the unperturbed s-waves at the origin.

As we shall see, these inconsistencies are resolved by the introduction of the contact interaction, which has been ignored in all the perturbative treatments up until now. In Aharonov and Bohm’s exact treatment such a contact term could be ignored since they imposed the boundary condition

\[ \Psi(0) = 0 \quad (III.3) \]

for the exact wavefunction. Such a boundary condition can not however be imposed on the unperturbed wavefunction in a perturbative treatment.

We analyze the perturbative problem from a field-theoretical point of view, since it is a more familiar setting for dealing with divergences, renormalization and scale symmetry breaking. The quantity we wish to study is the four point function associated with the scattering of two identical particles in the c.m. frame.

The nonrelativistic bosonic propagator is given in momentum space by

\[ D(k) = \frac{1}{k_0 - \frac{1}{2}k^2 + i\epsilon}, \quad (III.4) \]
where we are using the relativistic notation \( k = (k_0, \mathbf{k}) \). Note that the denominator is linear in the energy, resulting in propagation which is only forward in time.

As the gauge field is completely constrained by the equations of motion (II.4), there are no real gauge particles. Nevertheless, it can still be treated dynamically in internal lines. In order to get the gauge propagator we add to the Lagrangian density the following Galilean invariant gauge fixing term

\[
\mathcal{L}_{GF} = \frac{1}{\xi} (\nabla \cdot \mathbf{A})^2 ,
\]

and then take the limit \( \xi \to 0 \) in the propagator. The only nonvanishing components of the propagator are given in momentum space by

\[
D_{i0}(k) = -D_{0i}(k) = \frac{i\epsilon^{ij}k_j}{\kappa|\mathbf{k}|^2} .
\]

Note that since there is no \( k_0 \) dependence, the coordinate space propagator is instantaneous in time, which is another sign that the associated particle is only virtual.

As shown in fig.1, there are three vertices coming from the covariant derivatives

\[
\Gamma_0 = -ie \\
\Gamma_i = \frac{ie}{2}(p_i + p'_i) \\
\Gamma_{ij} = -i\epsilon^2 \delta_{ij} ,
\]

and one from the contact interaction

\[
\Gamma_{v0} = -iv_0 .
\]

The relevant graphs for the tree level scattering amplitude are depicted in fig.2. The amplitude of the one-gauge-particle-exchange graph is given by

\[
A^{(0)}_{exc} = \frac{e^2}{\kappa} \sin \theta \left[ \frac{1}{1 - \cos \theta} - \frac{1}{1 + \cos \theta} \right] \]

\[
= 2\frac{e^2}{\kappa} \cot \theta .
\]

The second term in (III.8) corresponds to the crossed graph, where the final particle states are exchanged. The remaining graph is just the contribution of the contact interaction. The full amplitude to tree level is then

\[
A^{(0)} = 2\frac{e^2}{\kappa} \cot \theta - iv_0 .
\]

The graphs contributing to the one-loop scattering amplitude are shown in fig.3. All other possible one-loop graphs vanish. As happens in many field theories, we expect an infinite contribution that should be regularized. Taking into account that the scale dimensions (under the scale transformation (II.14)) of \( k_0 \) and \( \mathbf{k}^2 \) are equal, naive power counting shows that all
the graphs are potentially divergent. The box diagram, fig.3(a), although naively logarithmically divergent, is finite. Its contribution in the c.m. frame reads, after performing the $k_0$ integration,

$$A^{(1)}_{\text{box}} = \frac{4ie^4}{\kappa^2} \int \frac{d^2k}{(2\pi)^2} \left[ \frac{(k \times p)(k \times p')}{(k + p)^2(k + p')^2(2\kappa^2 - p^2 + i\epsilon)} + p' \to -p' \right]$$

(III.11)

$$= -\frac{ie^4}{2\pi \kappa^2} \left[ \ln(2\sin \theta) + i\pi \right],$$

where $p$ is the relative incident momentum in the c.m., $p'$ is the relative scattered momentum, and $\theta$ is the angle between them, so

$$|p| = |p'|,$$

$$p \cdot p' = p^2 \cos \theta.$$  \hspace{1cm} \text{(III.12)}

The triangle diagram, fig.3(b), gives the following integral

$$A^{(1)}_{\text{tri}} = -\frac{ie^4}{\kappa^2} \int \frac{d^2k}{(2\pi)^2} \left[ \frac{k \cdot (k - p + p')}{k^2(k - p + p')^2} + p' \to -p' \right].$$

(III.13)

This integral is logarithmically divergent. To regularize it, we impose an ultra-violet cutoff $\Lambda$ to obtain

$$A^{(1)}_{\text{tri}}(p, p') = -\frac{ie^4}{2\pi \kappa^2} \ln \frac{\Lambda^2}{2p^2 |\sin \theta|}.$$  \hspace{1cm} \text{(III.14)}

This divergence is nothing more than the divergent contribution of the s-wave that we mentioned before. In fact, this diagram corresponds to the $\frac{\alpha^2 r^2}{\pi}$ term in (II.10), whereas the $\frac{2\pi \alpha r^2}{\kappa}$ term corresponds to the single gauge particle exchange diagram, fig.2(a). The presence of this divergence in the four-point function shows that without the contact term the theory is not renormalizable.

The contribution of the bubble diagram, fig.3(c), is known to be logarithmically divergent [13], and is given by

$$A^{(1)}_{v_0}(p, p') = -\frac{i v_0^2}{2} \int \frac{d^2k}{(2\pi)^2} \frac{1}{p^2 - k^2 + i\epsilon}$$

$$= \frac{i v_0^2}{8\pi} \left[ \ln \frac{\Lambda^2}{p^2} + i\pi \right],$$

and the total one-loop scattering amplitude is given by

$$A^{(1)}(p, p') = \frac{i}{8\pi} \left[ v_0^2 - 4e^4 \kappa^2 \right] \left[ \ln \frac{\Lambda^2}{p^2} + i\pi \right].$$  \hspace{1cm} \text{(III.15)}

Renormalization of this amplitude is carried out by redefining the coupling constant $v_0$:

$$v_0 = v + \delta v$$

$$\delta v = \frac{1}{4\pi} \left( v^2 - \frac{4e^4}{\kappa^2} \right) \ln \frac{\Lambda}{\mu} + \mathcal{O}(v^3, e^6),$$  \hspace{1cm} \text{(III.16)}
and the total renormalized amplitude is given by

\[ A(p, p', \mu) = \frac{2e^2}{\kappa} \cot \theta - iv + \frac{i}{8\pi} \left( v^2 - \frac{4e^4}{\kappa^2} \right) \left( \ln \frac{\mu^2}{p^2} + i\pi \right). \] (III.18)

This amplitude is not scale invariant, as can be seen by the presence of the arbitrary mass scale \( \mu \). A related result was obtained previously by Manuel and Tarrach using a first quantized approach to studying contact interactions of anyons [17].

We see however that at the critical point

\[ v = \pm \frac{2e^2}{|\kappa|}, \] (III.19)
the scale dependent term vanishes, restoring scale invariance to the solution. Upon multiplication by the kinematic factor \((2\pi p)^{-1/2}\), the scattering amplitude then becomes

\[ f(p, \theta) = \alpha \left( \frac{2\pi}{jp} \right)^{1/2} [\cot \theta \mp i \text{sgn}(\alpha)], \theta \neq 0, \pi, \] (III.20)

where \(\alpha\) was defined in (II.12). Choosing the upper sign, corresponding to a repulsive contact interaction, reproduces the AB result with the proper modification for identical particles (II.14). A contact interaction of such strength has also been considered by Ezawa and Iwazaki [18].

**IV. CONCLUDING REMARKS**

In this final section we would like to address two additional points. Namely, how the analysis of the previous section changes if one treats the gauge field not as a dynamical variable, but as a function of the scalar field given by (II.5), and why in the fermion case scale invariance is automatic.

Substituting equation (II.5) into our Lagrangian (II.1) results in a field theory of scalars with local and non-local interactions. It is not difficult to see that the term

\[ \mathcal{L}_1 = \frac{-ie}{2} A \cdot \phi^* \tilde{\nabla} \phi \]

\[ = \frac{ie^2}{2\kappa} \int d^2r' [\nabla \times G(r - r')] \phi^* (r', t) \phi (r', t) \cdot \phi^* (r, t) \tilde{\nabla} \phi (r, t), \] (IV.1)

plays the role of the single gauge particle exchange diagram. The divergent contribution of the triangle diagram appears in this scheme from the term

\[ \mathcal{L}_2 = -\frac{e^2}{2} A^2 \phi^* \phi, \] (IV.2)

when one contracts a scalar field from one of the gauge fields with a conjugate scalar field from the other.
The fermionic case can be analyzed along similar lines as the bosonic case, except that the result is automatically scale invariant. The fermionic Lagrangian does not contain the contact term (II.3), but instead contains the Pauli interaction term

\[ \mathcal{L}_P = \frac{e}{2} s B \psi^* \psi, \quad (IV.3) \]

where \( s \) is the spin projection. The presence of this new vertex gives rise to a new single-gauge-particle-exchange diagram, which plays the role of the contact interaction diagram. Unlike in the bosonic theory, the contribution of this diagram is fixed by the strength of the Pauli interaction, which corresponds to the critical point (III.19). Scale invariance is then automatic.

Note also that, in the attractive case, this value of \( v \) renders the equations of motion self-dual [12]. It is also the value for which the system admits an \( N = 2 \) supersymmetric extension [19].

To summarize, we have shown that the Lagrangian (II.1) corresponds to a theory that in general breaks scale invariance quantum mechanically, and corresponds to a field theoretical formulation of the AB effect only when a special relation between the coupling constants is satisfied, for which scale invariance is preserved.

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