VIRASORO CONSTRAINTS IN QUANTUM SINGULARITY THEORIES

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Abstract. We introduce Virasoro operators for any Landau-Ginzburg pair \((W, G)\) where \(W\) is a non-degenerate quasi-homogeneous polynomial and \(G\) is a certain group of diagonal symmetries. We propose a conjecture that the total ancestor potential of the FJRW theory of the pair \((W, G)\) is annihilated by these Virasoro operators. We prove the conjecture in various cases, including: (1) invertible polynomials with the maximal group, (2) some two-variable polynomials with the minimal group, (3) certain Calabi-Yau polynomials with groups. We also discuss the connections among Virasoro constraints, mirror symmetry of Landau-Ginzburg models, and Landau-Ginzburg/Calabi-Yau correspondence.

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0. Introduction

Virasoro constraints have been proposed in Gromov-Witten theory by Eguchi, Hori, and Xiong [16] and S. Katz [32]. It becomes one of the most fundamental and fascinating conjectures in Gromov-Witten theory. Despite significant developments in the literature [16, 17, 47, 23, 14,
24, 52], it remains as one of the most difficult conjectures in Gromov-Witten theory. Virasoro constraints have also been proposed in various topics in enumerative geometry [14, 30, 11, 50, 51].

In this paper, we propose Virasoro constraints for certain Landau-Ginzburg pairs \((W,G)\), with \(W\) a certain quasi-homogeneous polynomial and \(G\) a certain group of symmetries of \(W\). For such a LG pair \((W,G)\), Fan, Jarvis, and Ruan constructed a Cohomological Field Theory (in the sense of Kontsevich and Manin [35]) in [19, 20], based on a proposal of Witten [60]. Originally it was called quantum singularity theory in [20]. Nowadays the theory constructed in [19, 20] are widely called Fan-Jarvis-Ruan-Witten theory (or FJRW for short). The construction used both algebraic and analytic tools. There are purely algebraic constructions of Cohomology LG A-model theories in the study of mirror symmetry [29, 6, 5, 38, 9, 41, 27, 28].

0.1. Admissible Landau-Ginzburg pairs and the state spaces. Following the work of Fan, Jarvis, and Ruan [20], we let \(W : \mathbb{C}^{n} \rightarrow \mathbb{C}\) be a non-degenerate quasi-homogeneous polynomial of \(n\) variables. Here the non-degeneracy means the \(W\) has isolated critical points only at the origin and the weights (or degrees) of the variables are uniquely given by rational numbers. We write

\[
\text{wt}(x_i) := q_i \in (0, \frac{1}{2}] \cap \mathbb{Q}.
\]

Let \(G_W \leq \mathbb{G}^n_m\) be the group of diagonal symmetries of \(W\) defined by

\[
G_W := \{ (\lambda_1, \cdots, \lambda_n) \in \mathbb{G}^n_m \mid W(\lambda_1 x_1, \cdots, \lambda_n x_n) = W(x_1, \cdots, x_n) \}.
\]

There exists the exponential grading element

\[
J_W := (\exp(2\pi \sqrt{-1} q_1), \cdots, \exp(2\pi \sqrt{-1} q_n)) \in G_W.
\]

**Definition 0.1** (Admissible LG pairs). A subgroup \(G \leq G_W\) is called admissible if \(J_W \in G\).

- We call the pair \((W,G)\) an admissible LG pair if \(G\) is admissible.
- We call \(\langle J_W \rangle\) the minimal group of \(W\).
- We call \(G_W\) the maximal group of \(W\) and write \(G_W = G_{\text{max}}\) or \(G_W = \text{Aut}(W)\) sometimes.

For an admissible LG pair \((W,G)\), there exists a super vector space with a non-degenerate bilinear pairing and a bigrading [29, 6, 5, 38]. We call it the state space and denote it by \(\mathcal{H}_{W,G}\). Let us briefly review the construction here.

0.1.1. A state space \(\mathcal{H}_{W,G}\). For each \(\gamma \in G \leq \mathbb{G}^n_m\), we write

\[
\gamma := (\exp(2\pi \sqrt{-1} \theta_1), \cdots, \exp(2\pi \sqrt{-1} \theta_n)),
\]

where for all \(i\) such that \(1 \leq i \leq n, \theta_i \in [0,1) \cap \mathbb{Q}\) are given uniquely. Define the degree shift number of \(\gamma\) to be

\[
\iota_\gamma := \text{age}(\gamma) - \sum_{i=1}^{n} q_i := \sum_{i=1}^{n} (\theta_i - q_i).
\]

The fixed locus of \(\gamma\) in \(\mathbb{C}^n\) is denoted by \(\text{Fix}(\gamma)\). It is a subspace of \(\mathbb{C}^n\) of dimension

\[
N_\gamma := \dim_{\mathbb{C}} \text{Fix}(\gamma).
\]

Let \(W_\gamma := W|_{\text{Fix}(\gamma)}\) be the restriction. We denote \(d\mathbf{x}_\gamma\) the standard top form on \(\text{Fix}(\gamma)\). Recall the Jacobian algebra of \(W\) is given by

\[
\text{Jac}(W) := \mathbb{C}[x_1, \cdots, x_n]/\left( \frac{\partial W}{\partial x_1} = \cdots = \frac{\partial W}{\partial x_n} = 0 \right).
\]
**Definition 0.2 (A state space).** For an LG pair \((W, G)\), the state space \(\mathcal{H}_{W,G}\) is defined to be the direct sum of \(G\)-invariant spaces

\[
\mathcal{H}_{W,G} := \bigoplus_{\gamma \in G} \mathcal{H}_\gamma, \quad \mathcal{H}_\gamma := (\text{Jac}(W_\gamma) \cdot dx_\gamma)^G.
\]

Here \(\gamma = (\lambda_1, \cdots, \lambda_n) \in G\) acts on \(x_i\) and \(dx_i\) both by multiplying \(\lambda_i\). We call an element in the subspace \(\mathcal{H}_\gamma\) narrow if \(N_\gamma = 0\), otherwise it is broad.

Following the notation of \([38]\), the state space \(\mathcal{H}_{W,G}\) is spanned by elements of the form \(\alpha|\gamma\rangle\), where \(\gamma \in G\) and \(\alpha \in \text{Jac}(W_\gamma) \cdot dx_\gamma\). If \(x_i\) is fixed by \(\gamma\), i.e. \(\theta_i = 0\), then we write \(i \in W_\gamma\).

Consider a monomial representative

\[
\alpha = [x_\gamma^m] \cdot dx_\gamma = \left[ \prod_{i \in W_\gamma} x_i^{m_i} \right] \cdot dx_\gamma \in \text{Jac}(W_\gamma) \cdot dx_\gamma.
\]

Let \(\text{wt}(\alpha)\) be the weight of the form \(x_\gamma^m dx_\gamma\), defined by

\[
\text{wt}(\alpha) := \sum_{i \in W_\gamma} (m_i + 1)q_i.
\]

0.1.2. A pairing on \(\mathcal{H}_{W,G}\). We fix a homogeneous basis of \(\mathcal{H}_{W,G}\), denoted by

\[
\{ \phi_a = [f_a]dx_\gamma|\gamma_a\rangle \mid \gamma_a \in G, [f_a] \in \text{Jac}(W_{\gamma_a}) \}.
\]

Let \(\text{Res}_{W_{\gamma_a}}\) be the Grothendieck residue pairing on \(\text{Jac}(W_{\gamma_a})\). Then there is a natural nondegenerate bilinear pairing \((\cdot, \cdot)\) on \(\mathcal{H}_{W,G}\), defined by

\[
\eta_{ab} := (\phi_a, \phi_b) := \begin{cases} 0, & \text{if } \gamma_a \neq \gamma_b^{-1}; \\ \text{Res}_{W_{\gamma_a}}(f_af_b \cdot dx_{\gamma_{\gamma_a}}), & \text{if } \gamma_a = \gamma_b^{-1}. \end{cases}
\]

0.1.3. Bigrading and parity. We denote the central charge of the polynomial \(W\) by

\[
\hat{c}_W := \sum_{i=1}^n (1 - 2q_i).
\]

Following [29, 38], we introduce a bigrading and a parity on \(\mathcal{H}_{W,G}\). For an homogeneous element

\[
\phi_a := \alpha|\gamma\rangle = [x_\gamma^m] \cdot dx_\gamma|\gamma\rangle \in \mathcal{H}_{W,G},
\]

we assign a bigrading \((\mu_+^a, \mu_-^a)\),

\[
\begin{cases} 
\mu_+^a := \text{wt}(\alpha) + \iota_\gamma - \frac{\hat{c}_W}{2}; \\
\mu_-^a := N_\gamma - \text{wt}(\alpha) + \iota_\gamma - \frac{\hat{c}_W}{2}.
\end{cases}
\]

and a parity

\[
|\phi_a| = (-1)^{N_\gamma}.
\]

We also define a complex degree

\[
\text{deg}_C \phi_a := \frac{1}{2} (\mu_+^a + \mu_-^a + \hat{c}_W).
\]
Example 0.3. For the Fermat cubic pair \((W = x_1^3 + x_2^3 + x_3^3, G = \langle J_W \rangle)\), we have the following list for a basis of \(\mathcal{H}_{W,G}\). This example will be discussed in Section 4.2 with more details.

Table 1. The state space of a Fermat cubic

| \(\phi_a\) | \(1|J\rangle\) | \(1|J^2\rangle\) | \(dx|J^0\rangle\) | \(x_1x_2x_3dx|J^0\rangle\) |
|---|---|---|---|---|
| \((\mu_1^+, \mu_2^-)\) | \((-\frac{1}{2}, -\frac{1}{2})\) | \((\frac{1}{2}, \frac{1}{2})\) | \((-\frac{1}{2}, \frac{1}{2})\) | \((\frac{1}{2}, -\frac{1}{2})\) |
| \(|\phi_a|\) | 1 | 1 | -1 | -1 |
| \(\text{deg}_{\mathbb{C}} \phi_a\) | 0 | 1 | \(\frac{1}{2}\) | \(\frac{1}{2}\) |

0.2. Virasoro operators for admissible LG pairs. For \(\ell \in \mathbb{R}\) and \(n \in \mathbb{Z}_{\geq 0}\), we denote the Pochhammer symbol by

\[
(\ell)_n := \begin{cases} 
\ell(\ell+1) \cdots (\ell+n-1), & \text{if } n \geq 1, \\
1, & \text{if } n = 0.
\end{cases}
\]

We assign a variable \(t^a_m\) to each \(\phi_a z^m \in \mathcal{H}_{W,G}[z]\).

Definition 0.4 (Virasoro operators). For each integer \(k \in \mathbb{Z}_{\geq -1}\), we introduce a differential operator

\[
L_k := -\left(\frac{3 - \tilde{c}_W}{2}\right)_{k+1} \frac{\partial}{\partial \eta_{k+1}^a} \\
+ \sum_{m=0}^{\infty} \left(\mu_a^+ + m + \frac{1}{2}\right)_{k+1} t^a_m \frac{\partial}{\partial \eta_{m+k}^a} \\
+ \frac{h^2}{2} \sum_{m=-k}^{-1} (-1)^m \left(\mu_a^+ + m + \frac{1}{2}\right)_{k+1} \eta_{ab} \frac{\partial}{\partial \eta_{-m-1}^a} \frac{\partial}{\partial \eta_{m+k}^b} \\
+ \frac{1}{2h^2} \delta_{-1,k} \eta_{ab} \eta_{00} \eta_{00} \\
- \frac{\delta_{0,k}}{4} \sum_a (-1)^{|\phi_a|} (\mu_a^+ - \frac{1}{2})(\mu_a^+ + \frac{1}{2}).
\]

Here \(\eta_{ab}\) is the \((a,b)\)-th entry of the inverse matrix of the pairing matrix \((\eta_{ab})\).

Following the work of Givental [24], these differential operators are related to quantization of quadratic Hamiltonians. This allows us to verify

Proposition 0.5. The differential operators \(\{L_k\}_{k \geq -1}\) satisfy the relations:

\[
[L_m, L_n] = (m - n)L_{m+n}.
\]

In general, a set of operators \(\{L_k\}_{k \in \mathbb{Z}}\) are called Virasoro operators if

\[
[L_m, L_n] = (m - n)L_{m+n} + \frac{m^3 - m}{12} \cdot \delta_{m+n,0} \cdot c,
\]

with some constant \(c\) (which is called the central charge of the Virasoro algebra). The relations become (0.10) if we restrict to \(k \geq -1\). By abuse of the notations, we will call \(\{L_k\}_{k \geq -1}\) in (0.9) the Virasoro operators for admissible LG (A-model) pairs.
0.3. **Virasoro constraints.** The essential ingredients in each construction [19, 20, 53, 33] are moduli spaces of $W$-spin structures in [19, 20], or $G$-spin structures in [53] and [33], over genus-$g$ orbifold curves decorated by the element

$$\tilde{\gamma} = (\gamma_1, \ldots, \gamma_k) \in G^k.$$ 

We denote the moduli space by $W^G_{g,\tilde{\gamma}}$. The moduli space carries a virtual fundamental classes, which will be denoted by $[W^G_{g,\tilde{\gamma}}]^\text{vir}$.

Despite the different nature of the techniques, each construction gives a CohFT \{\Lambda^{\mathfrak{s}, W,G}_{g,k}\} on the isomorphic underlying state space $H_{W,G}$. Here $g$ and $k$ satisfies the stability condition $2g - 2 + k > 0$. We denote the CohFTs by $\Lambda^{\mathfrak{s}}$, where

$$\mathfrak{s} = \text{FJR}, \text{PV}, \text{KL}$$

stands for the abbreviation of each theory.

We will mainly consider FJR theory in our paper, but the setting also applies to the other two. Choosing a set of elements $\{\phi_i \in H_{\gamma_i}\}$, the linear maps

$$\Lambda^{\mathfrak{s}, W,G}_{g,k} : H_{W,G} \otimes_k \to H^*(\overline{M}_{g,k}, \mathbb{C})$$

produces intersection numbers, called **ancestor invariants**

\begin{equation}
\left\langle \prod_{i=1}^k \tau_{\ell_i}(\phi_i) \right\rangle^{\mathfrak{s}, W,G}_{g,k} = \int_{\overline{M}_{g,k}} \Lambda^{\mathfrak{s}, W,G}_{g,k}(\phi_1 \otimes \cdots \otimes \phi_k) \cdot \prod_{i=1}^k \psi_{\ell_i}^i,
\end{equation}

where $\phi_i \in H_{W,G}$, $\psi_i$’s are the first chern classes of cotangent line bundles on coarse moduli of spaces $\overline{M}_{g,n}$.

0.3.1. **The total ancestor potential.** We write

$$t(z) := \sum_{m \geq 0} \sum_a t^a_m \phi_a z^m.$$ 

As in [22, Section 1.3], we assign parity for $t^a_m$, which coincides with that of $\phi_a$, and moving $t^a_m$ across $\phi_b$ will give a sign, i.e.

$$t^a_m \phi_b = (-1)^{|t^a_m|} |\phi_b| t^a_m.$$ 

The **total ancestor potential** is given by

\begin{equation}
\mathcal{A}^{\mathfrak{s}, W,G} := \exp \left( \sum_g \hbar^{2g-2} \sum_k \frac{1}{k!} \left\langle \prod_{i=1}^k t(\psi_i) \right\rangle^{\mathfrak{s}, W,G}_{g,k} \right).
\end{equation}

The three CohFTs constructed in the work of Fan-Jarvis-Ruan [19, 20], Polishchuk-Vaintrob [53], and Kiem-Li [33] are conjectured to be equivalent. In particular, the pairing in (0.3) matches the intersection pairing of Lefschetz thimbles in FJRW theory [20], the perfect pairing on intersection homology in KL theory [33], and the Mukai pairing on Hochchild homology of $\Gamma$-equivariant matrix factorizations of $W$ [53]. In general, the equivalence is known when restricting to a subspace of narrow elements in $H_{W,G}$ [8]. Numerically, if we consider the total ancestor potentials, the equivalence are verified for more cases [26, 27, 28].
0.3.2. Virasoro conjecture for admissible LG pairs. Now we propose a Virasoro conjecture for the admissible LG A-model pairs.

**Conjecture 0.6** (Virasoro conjecture for admissible LG A-model pairs). For any admissible LG pair \((W,G)\), the total ancestor potentials \(A_{W,G}^{\bullet}\) in FJRW/PV/KL theory satisfy the Virasoro constraints

\[
L_k A_{W,G}^{\bullet} = 0, \quad \text{for all } k \geq -1.
\]

**Remark 0.7.** When \(k = -1\), the constraint \(L_{-1} A_{W,G}^{\bullet} = 0\) always holds true in each theory \([20, 53, 33]\) as it is equivalent to the string equations. When \(k = 0\), the last term in (0.9) can be rewritten in terms of the central charge and Euler characteristic of the LG pair by a supertrace formula (1.6). If \(c_W = 3\), the supertrace term \(\text{Str}(\theta^2 - \frac{1}{4})\) still vanishes and the constraints by \(L_0\) is equivalent to a grading equation (1.12), which is determined by an Euler vector that arises in the quantum singularity theory. By the dilaton equation (1.10), such an equivalence still holds when \(c_W \neq 3\).

0.4. Main results. In this paper, we study Virasoro Conjecture 0.6 in various situations. We will mostly focus on the FJRW theory. Part (2) of Theorem 0.8 is stated for PV theory. The result is slightly stronger there than in the FJRW theory, due to some technique advantages.

0.4.1. Virasoro constraints for semi-simple Frobenius manifolds. The genus zero invariants in (0.11) gives a Frobenius manifold in the sense of Dubrovin [13]. If the Frobenius manifold is generically semi-simple, i.e., the Frobenius algebra is semi-simple at a generic point of the Frobenius manifold, the total ancestor potential is uniquely constructed from the Frobenius manifold by the famous Givental-Teleman formula in (3.5) \([25, 58]\). By writing the Virasoro operators using quantization operators of certain quadratic Hamiltonians, Givental proved the formula (3.5) after a modification is annihilated by Virasoro operators [24, Theorem 7.7].

By applying Givental’s result, we show the LG A-models of two types of admissible LG pairs have generically semisimple Frobenius manifolds and thus Virasoro Conjecture 0.6 hold for them. Both types are from invertible polynomials, which are of the form \(\sum_{i=1}^{n} \prod_{j=1}^{n} x_{ij}^{a_{ij}}\), with a matrix \((a_{ij})_{n \times n} \in \text{GL}(n, \mathbb{Q})\).

**Theorem 0.8.** Consider an admissible pair \((W,G_W)\) with an invertible polynomial \(W\).

1. If \(W\) has no weight \(\frac{1}{2}\) chain variable, then \(L_k A_{W,G_W}^{\bullet=\text{FJRW}} = 0\) for all \(k \geq -1\).
2. The equation \(L_k A_{W,G_W}^{\bullet=\text{PV}} = 0\) holds for all \(k \geq -1\).

According to the mirror theorems in [27, 28], the LG A-models in Theorem 0.8 are equivalent to Saito-Givental B-models of their mirror polynomials. The mirror LG B models are generically semi-simple as the deformed polynomials are of Morse type in general.

Semisimple Frobenius manifolds also exists for admissible LG A-model pairs even if \(G \neq G_W\). For example, for any invertible polynomial of two variables, if we take \(G\) to be the minimal group \(\langle J \rangle\). Then the FJRW theory of \((W, \langle J \rangle)\) may not be isomorphic to any admissible LG pair of invertible polynomial with a maximal group \(G_W\). We consider some examples of two-variable invertible polynomials in [21, Section 4]. We compute the quantum multiplication of the quantum Euler vector field for these examples. Then by applying the criteria of semisimplicity described in [1], we obtain the underlying Frobenius manifolds are generically semisimple. As a consequence, we obtain

**Proposition 0.9.** Virasoro Conjecture 0.6 holds for all LG pairs \((W, \langle J \rangle)\) if

\[
W = x^4 + y^4, x^3 + y^6, x^3 + y^9, x^4 + y^6, x^3 + xy^8.
\]
0.4.2. LG pairs of Calabi-Yau type. If a polynomial $W(x_1, \cdots, x_n)$ is of Calabi-Yau type, i.e., $\hat{c}_W = n-2$, the hypersurface ($W = 0$) and the quotient space induced by the $G$-action are Calabi-Yau varieties. In Witten’s work of Gauge linear sigma models (GLSM) [61], the Gromov-Witten theory of the quotient Calabi-Yau variety has a deep connection to the LG A-model of the pair $(W,G)$. This is called the Landau-Ginzburg/Calabi-Yau correspondence. By comparing the FJRW theory with the Gromov-Witten theory of the Calabi-Yau counterpart, we obtain

**Theorem 0.10.** Let $W$ be an invertible polynomial of Calabi-Yau type. For all $k \geq -1$, the Virasoro constraints $L_k \Lambda_{W,G}^{FJRW} = 0$ hold for the following admissible LG pairs $(W,G)$:

1. $\hat{c}_W \geq 3$ and $G \leq \text{SL}(\mathbb{C})$, under Assumption 1.4;
2. or $W$ is a Fermat CY polynomial of three variables, and $G = \langle J_W \rangle$.

The Frobenius manifolds in these LG models are no longer generically semi-simple. Part (1) of Theorem 0.10 follows from a simple degree calculation, which is an analog of [22, Theorem 7.1]. For part (2), we first recall the LG/CY correspondence proved in [42, 43]. The authors there relates the LG theory of the pair and the GW theory of the elliptic curve by a holomorphic Cayley transformation, which is induced from the theory of quasi-modular forms. The Virasoro operators in LG/CY theories commutes with the holomorphic Cayley transformation. Thus Part (2) follows from the Virasoro constraints for the Gromov-Witten theory of the elliptic curve [52].

**Plan of the paper.** In Section 1, we briefly review some properties of the CohFTs for the admissible LG pairs and discuss their influence on Virasoro constraints of $L_{-1}$ and $L_0$. In Section 2, we follow Givental’s work to express the Virasoro operators as quantization operators of certain quadratic Hamiltonians and prove the Virasoro relations (0.10) in Proposition 0.5. In Section 3, we provide some examples of semisimple Frobenius manifolds. Theorem 0.8 will follow from the mirror symmetry statements and Givental’s proof of Virasoro constraints for semi-simple Frobenius manifolds. We prove Proposition 0.9 by calculating the quantum multiplication of the quantum Euler vector. In Section 4, we discuss the connection between Virasoro constraints and LG/CY correspondence, and prove Theorem 0.10.

**Acknowledgement.** We would like to thank Huijun Fan, Todor Milanov, Alexander Polishchuk, Yongbin Ruan, and Arkady Vaintrob for helpful discussions. W. He would like to thank Jianxun Hu, Huazhong Ke, Xiaowen Hu, Yifan Li, Xiaobo Li and Xin Wang. Y. Shen would like to thank Amanda Francis, Takashi Kimura, Y-P Lee, Hsian-Hua Tseng, and Jie Zhou. We thank the hospitality of Institute for Advanced Study in Mathematics, Zhejiang University. Part of the work was done there during the authors’ visit in September 2019. Y. Shen is partially supported by Simons Collaboration Grant 587119.

1. **Virasoro constraints of $L_{-1}$ and $L_0$**

In this section, we discuss some properties of the cohomology field theories and the connections to Virasoro operators $L_{-1}$ and $L_0$.

1.1. **String equation and the operator $L_{-1}$.** According to [20, Theorem 4.2.2], [53, Theorem 5.1.2], and [33, Theorem 4.6], each of the three CohFTs $\{\Lambda_{g,k}^{W,G}\}$ has an element

$$1 := \phi_0 := 1|J\rangle,$$

called a flat identity, which satisfies the following two conditions:

- The CohFT is compatible with the pairing

$$\int M_{0,3} \Lambda_{0,3}^{W,G}(\phi_a, \phi_b, 1) = (\phi_a, \phi_b).$$
• If \( 2g - 2 + k > 0 \), let \( p : \overline{\mathcal{M}}_{g,k+1} \to \overline{\mathcal{M}}_{g,k} \) be the morphism forgetting the last marking and contracting all the unstable components, then

\[
\Lambda_{g,k+1}^{W,G}(\phi_1, \cdots, \phi_k, 1) = p^* \Lambda_{g,k}^{W,G}(\phi_1, \cdots, \phi_k).
\]

Using the geometry of psi-classes, these two conditions imply:

\[
\begin{align*}
\left\langle \tau_0(\phi_a) \tau_0(\phi_b) \tau_0(1) \right\rangle_{0,3}^{W,G} &= \eta_{ab}, \\
\prod_{i=1}^{k} \tau_{\ell_i}(\phi_i) \tau_0(1)_{g,k+1}^{W,G} &= \sum_{j=1}^{k} \prod_{i=1}^{k} \tau_{\ell_i - \ell_j}(\phi_i)_{g,k}^{W,G}.
\end{align*}
\]

These equations are called string equations. The Virasoro operator \( L_{-1} \) in (0.9) is given by

\[
L_{-1} = -\frac{\partial}{\partial t_0^a} + \sum_{a,m} t_{m+1}^a \frac{\partial^2}{\partial t_m^a} + \frac{1}{2\hbar^2} \sum_{a,b} \eta_{ab} t_0^a t_0^b.
\]

The peculiar ordering of the variables here reflects the potential presence of odd classes. Immediately, we see the string equations (1.2) and (1.1) is equivalent to the Virasoro constraint

\[
L_{-1} \Lambda_{W,G} = 0.
\]

1.2. A grading equation and the operator \( L_0 \). Now we consider the Virasoro constraint of \( L_0 \). We need some preparations. Recall the parity \(|\phi_a|\) defined in Definition 0.6, we define the Euler characteristic of the admissible LG pair \((W,G)\) to be

\[
\chi_{W,G} := \sum_{\phi_a \in H_{W,G}} |\phi_a|.
\]

**Remark 1.1.** In FJRW theory, the parity is induced by the cohomological degree of Lefschetz thimbles; in PV theory, the parity is induced by the Hochschild degree of Hochschild homology class.

Next we define a Hodge grading operator \( \theta \) by

\[
\theta(\phi_a z^m) := \mu_a^+ \cdot \phi_a z^m.
\]

Now we rewrite the last term in the Virasoro operator \( L_k \) in (0.9) using a supertrace

\[
\text{Str}(\theta^2 - \frac{1}{4}) := \sum_a (-1)^{|\phi_a|}(\mu_a^+ - \frac{1}{2})(\mu_a^+ + \frac{1}{2}).
\]

Similar to [22, Proposition 2.6], the super trace term \( \text{Str}(\theta^2 - \frac{1}{4}) \) is related to the Euler characteristic \( \chi_{W,G} \).

**Proposition 1.2** (A supertrace formula). Let \((W,G)\) be an admissible LG pair, then

\[
\text{Str}(\theta^2 - \frac{1}{4}) = \frac{\hat{c}_W - 3}{12} \chi_{W,G}.
\]

This formula will be discussed in Section A. As a consequence, the Virasoro operator \( L_0 \) has the following expression:

\[
L_0 := -\frac{3 - \hat{c}_W}{2} \frac{\partial}{\partial t_1^a} + \sum_{m=0}^{\infty} \left( \mu_a^+ + m + \frac{1}{2} \right) t_m^a \frac{\partial}{\partial t_m^a} - \frac{\hat{c}_W - 3}{48} \chi_{W,G}.
\]
1.2.1. Dilaton equations. The dilaton equations are:

\begin{equation}
\left\langle \tau_1(1) \right\rangle_{1,1}^{W,G} = \frac{\chi_{W,G}}{24}.
\end{equation}

\begin{equation}
\left\langle \prod_{i=1}^{k} \tau_\ell_i(\phi_i) \tau_1(1) \right\rangle_{g,k+1}^{W,G} = (2g - 2 + k)\left\langle \prod_{i=1}^{k} \tau_\ell_i(\phi_i) \right\rangle_{g,k}^{W,G}.
\end{equation}

These equations are obtained by using the geometry of psi-classes and the virtual fundamental cycles. For example, the first equation can be deduced from the tautological relation

\[ \psi_1 = \frac{1}{24} \cdot \delta_{\text{irr}} \in H^2(M_{1,1}), \]

where \( \delta_{\text{irr}} \) is the boundary divisor of \( M_{1,1} \), parametrizing all stable genus one curves with one node and one marking.

Similar to the string equations, the dilaton equations (1.8) and (1.9) can be rewritten as differential equations

\begin{equation}
\left( -\frac{\partial}{\partial t_1^0} + \sum_{a,m} t_m^a \frac{\partial}{\partial t_m^a} + \hbar \frac{\partial}{\partial \hbar} + \frac{\chi_{W,G}}{24} \right) A_{W,G}^{\bullet} = 0.
\end{equation}

Again, the super trace formula (1.6) allows us to replace the scalar multiplication by the Euler characteristic \( \chi_{W,G} \) in (1.10) by a term related to the super trace \( \text{Str}(\theta^2 - \frac{1}{4}) \).

1.2.2. A grading equation. We define a grading operator

\begin{equation}
\tilde{E} := \sum_{m,a} \left( m - 1 + \mu_a^+ + \frac{c_W}{2} \right) t_m^a \frac{\partial}{\partial t_m^a}.
\end{equation}

**Proposition 1.3.** The Virasoro constraint \( L_0 A_{W,G}^{\bullet} = 0 \) is equivalent to the grading equation

\begin{equation}
\tilde{E} A_{W,G}^{\bullet} = \left( \frac{3 - \hat{c}_W}{2} \right) \hbar \frac{\partial}{\partial \hbar} A_{W,G}^{\bullet}.
\end{equation}

**Proof.** There are two situations. If \( \hat{c}_W = 3 \), we compare (1.7) and (1.11). The grading equation \( \tilde{E}(A_{W,G}^{\bullet}) = 0 \) is exactly the Virasoro constraint \( L_0 A_{W,G}^{\bullet} = 0 \).

If \( \hat{c}_W \neq 3 \), then we can cancel the differential operator \( \hbar \frac{\partial}{\partial \hbar} \) by taking a linear combination of the dilaton equation (1.10) and the grading equation (1.12), and get the Virasoro constraint

\begin{equation}
\left( \frac{\hat{c}_W - 3}{2} \frac{\partial}{\partial t_1^0} + \sum_{m,a} (m + \mu_a^+ + \frac{1}{2}) t_m^a \frac{\partial}{\partial t_m^a} + \frac{3 - \hat{c}_W}{48} \chi \right) A_{W,G}^{\bullet} = 0.
\end{equation}

\[ \square \]

1.2.3. A sufficient condition. We provide a sufficient condition for the grading equation (1.12).

**Assumption 1.4.** The quantum invariant

\[ \left\langle \prod_{i=1}^{k} \tau_\ell_i(\phi_i) \right\rangle_{g,k}^{W,G} \neq 0 \]

only if

\[ \sum_{i=1}^{k} \left( \mu_i^+ + \frac{\hat{c}_W}{2} + \ell_i \right) = (3 - \hat{c}_W)(g - 1) + k. \]
One can check directly that for all the examples in Theorem 0.8, Theorem 0.9, and Theorem 0.10, Assumption 1.4 holds true. In fact, using the bigrading in (0.5), we obtain

**Proposition 1.5.** If Assumption 1.4 is satisfied for \( \{ A_{g,k}^{W,G} \} \) of an admissible LG pair \((W,G)\), then \( L_0 A_{W,G} = 0 \).

**Remark 1.6.** The FJRW invariants satisfies a degree constraint \([20, \text{Theorem 4.1.8 (1)}]\)

\[
\deg_C W_{g,\gamma}^{G,\text{vir},\text{FJRW}} = (3 - \hat{c}_W)(g - 1) + n - \sum_{i=1}^n t_i.
\]

The KL theory satisfies the same constraint, which is induced by \([7, (3.20)]\). Such a constraint is conjectural in PV theory, called the Homogeneity Conjecture \([53, \text{Section 5.6}]\). We see Assumption 1.4 is stronger than (1.14).

**Remark 1.7.** A similar condition as Assumption 1.4 exists in Gromov-Witten theory of smooth algebraic varieties. It is closely related to the motive axiom of cohomological field theory \([35]\). The motive axiom states that Gromov-Witten theory is induce by some algebraic cycle via Fourier-Mukai transform, which is proved by Li-Tian \([44]\) and Behrend-Fantechi \([3, 4]\). More explicitly, for \( V \) is a smooth algebraic variety, the GW CoFT

\[
\Lambda_{g,n,\beta}^{GW} : H^*(V)^{\otimes n} \to H^*(\overline{M}_{g,n})
\]

is induced by some cocycle \( C_{g,n,\beta} \) in the Chow ring \( A^*(V^n \times \overline{M}_{g,n}) \) via the following Fourier-Mukai transform:

\[
\Lambda_{g,n,\beta}^{GW} (\phi_1 \otimes \cdots \otimes \phi_n) = (pr_2)_* [pr_1^* (\phi_1 \wedge \cdots \wedge \phi_n) \wedge C_{g,n,\beta}],
\]

where \( pr_1 : V^n \times \overline{M}_{g,n} \to V^n \), \( pr_2 : V^n \times \overline{M}_{g,n} \to \overline{M}_{g,n} \) are the projections. Now consider the Hodge decomposition \( H^*(V^n \times \overline{M}_{g,n}) = \bigoplus_{p,q} H^{p,q} \). Since \( C_{g,n,\beta} \) is an algebraic cycle, we know \( C_{g,n,\beta} \in H^{d,d} \), where \( 2d \) is the cohomological degree of \( C_{g,n,\beta} \). Furthermore, GW invariant is obtained via the following integration:

\[
\langle \phi_1, \cdots, \phi_n \rangle_{g,n,\beta}^{GW} = \int_{\overline{M}_{g,n}} \Lambda_{g,n,\beta}^{GW} (\phi_1 \otimes \cdots \otimes \phi_n).
\]

The invariant does not vanish means that

\[
\Lambda_{g,n,\beta}^{GW} (\phi_1 \otimes \cdots \otimes \phi_n) \in H^{3g-3+n,3g-3+n} (\overline{M}_{g,n})
\]

Suppose \( \phi_i \in H^{p_i,q_i}(V) \), then (1.15) and the above argument will induce the following constraints:

\[
\sum_{i=1}^n p_i = \sum_{i=1}^n q_i.
\]

Assumption 1.4 in GW theory is a consequence of the above equality and dimension axiom of GW theory.

2. **Virasoro relations and Givental formalism**

In this section, we recall Givental’s work on quantization of quadratic Hamiltonians [24]. It implies the Virasoro relation (0.10) in Proposition 0.5.
2.1. Quantization of quadratic Hamiltonians. This section is mainly based on [24]. See also [12] for an exposition.

Fix a $\mathbb{Z}_2$-graded state space $\mathcal{H}$, set
$$\mathbb{H} := \mathcal{H}((z^{-1})).$$

A choice of basis $\{\phi_a\}$ for $\mathcal{H}$ yields a symplectic basis for $\mathbb{H}$, in which the expression for an arbitrary element in Darboux coordinates is
\begin{equation}
    f(z) = \sum_a \sum_{\ell \geq 0} p_{\ell,a} \phi^a(-z)^{-\ell-1} + \sum_b \sum_{m \geq 0} q_b^m \phi_b(z)^m \in \mathbb{H}.
\end{equation}

Here the parity of $p_{\ell,a}$, $q_b^m$ coincides that of $\phi^a$ and $\phi_b$ respectively. The pairing $(,)$ on $\mathcal{H}$ induces a symplectic form on $\mathbb{H}$, denoted by
$$\Omega(f(z), g(z)) := \text{Res}_{z=0}(f(-z), g(z)).$$

**Definition 2.1.** An operator $A : \mathbb{H} \rightarrow \mathbb{H}$ is called an infinitesimal symplectic transformation if
$$\Omega(Af, g) + \Omega(f, Ag) = 0.$$

An operator $T : \mathbb{H} \rightarrow \mathbb{H}$ is called a symplectic transformation if
$$\Omega(Tf, Tg) = \Omega(f, g).$$

Now define the quantization of quadratic terms by
\begin{equation}
\begin{aligned}
\hat{q}_i q_j &= q_i q_j, \\
\hat{p}_i p_j &= \hbar \partial_{q_j} \partial_{q_i}, \\
\hat{q}_i p_j &= q_i \partial_{q_j}.
\end{aligned}
\end{equation}

The quantization of quadratic Hamiltonians forms a projective representation of the Poisson (Lie) algebra.

For an infinitesimal symplectic transformation $A$, define the quadratic Hamiltonian by
\begin{equation}
    h_A := \frac{1}{2} \Omega(A\Phi, \Phi),
\end{equation}
and the quantization of $A$ is defined to be the quantization of $h_A$ via (2.2):
\begin{equation}
    \hat{A} := \hat{\hat{h}_A}.
\end{equation}

For a symplectic transformation $T = \exp A$, define its quantization as
\begin{equation}
    \hat{T} := \exp \hat{A}.
\end{equation}

**Lemma 2.2.** [24] Let $A_1, A_2$ be two infinitesimal symplectic transformations, then
$$[\hat{A}_1, \hat{A}_2] = [A_1, A_2]^\wedge + C(h_{A_1}, h_{A_2}),$$
where $C(,)$ is a cocycle defined by
\begin{equation}
    C(p_a p_b, q_a q_b) = (-1)^{|p_a||q_b|} + \delta_{a,b},
\end{equation}
and $C = 0$ on any other pair of quadratic Darboux monomials. Here cocycle means $C$ is a closed 2-form in the Hochschild cochain of the Lie algebra of quadratic Hamiltonian.

For simplicity, we denote
$$C(A_1, A_2) := C(h_{A_1}, h_{A_2}).$$
Corollary 2.3. Let $A$ be an infinitesimal symplectic transformation and $T$ be a symplectic transformation, we have
\[\hat{T} \circ \hat{A} \circ \hat{T}^{-1} = \hat{TAT}^{-1} + C_T(A)\]
Here $C_T(A)$ is the constant defined to be
\[C_T(A) = C \left( \log T, \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \text{Ad}_{\log T}^n(A) \right).
\]
Proof. Let $T = \exp B$, then (2.7) is just
\[\hat{T} \circ \hat{A} \circ \hat{T}^{-1} = \exp(\text{Ad}_{\hat{B}})(\hat{A}) = \exp(\text{Ad}_{B})(A) + C \left( B, \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \text{Ad}_B^n(A) \right),\]
where $\text{Ad}_B(\hat{A}) := [B, A]$. \qed

In this paper, we mainly consider operators on $\mathbb{H}$ of following two types, lower triangular operators of the form
\[S(z) = S_0 + S_1 z^{-1} + S_2 z^{-2} + \cdots, \quad S_i \in \text{End}(\mathcal{H}),\]
and upper triangular operators of the form
\[R(z) = R_0 + R_1 z + R_2 z^2 + \cdots, \quad R_i \in \text{End}(\mathcal{H}).\]
When we consider pair or Lie bracket between positive sum and negative sum, we always assume one of them is finite, such that the pair or Lie bracket is well-defined.

2.2. Virasoro relations. Now we return to an admissible LG pair $(W, G)$. Recall that $(\mu_+^a, \mu_-^a)$ is the bigrading of the element $\phi_a = [f_a]dx^\gamma_a \in \mathcal{H}_{W,G}$ defined in (0.5). Using (0.5) and (0.3), it is easy to check that

Lemma 2.4. We fix a basis of $\mathcal{H}_{W,G}$ as follows \{\phi_a = [f_a]dx^\gamma_a \} \in \text{Jac}(W_{\gamma_a})\}. If the pairing $\eta_{ab} := (\phi_a, \phi_b) \neq 0$, then $\mu_+^a + \mu_-^b = 0$.

The Hodge grading operator $\theta$ defined in (1.4) is an operator on $\mathbb{H}$. We introduce an auxiliary operator
\[D := z(\partial_z + z^{-1}\theta)z = z^2 \partial_z + z(\theta + 1).\]
It satisfies
\[D, z^{-1} = -1.\]

We will consider a sequence of differential operators
\[L_k := z^{-1/2}D^{k+1}z^{-1/2}, \quad k \geq -1.\]
Using Lemma 2.4 and (2.10), we obtain

Lemma 2.5. For all integers $k \geq -1$, $L_k$ is infinitesimal symplectic, and
\[L_m, L_n = (m - n)L_{m+n}.\]
Recall $\eta^{ab}$ is the $(a, b)$-th entry of the inverse matrix of $(\eta_{ab})$ and $f(z)$ is the Darbourx coordinate defined in (2.1).
Lemma 2.6. The differential operator $L_k$ induces a quadratic Hamiltonian

$$\Omega(L_k f, f) = -\delta_{k,-1} \sum_{a, b} q_0^a q_0^b \eta_{ab}$$

(2.13)

$$-2 \sum_{a} \sum_{m \geq 0} \prod_{j=1}^{k+1} (\mu_a^+ + m - \frac{1}{2} + j)p_{m+k,a} q_m^a$$

(2.14)

$$+ \sum_{a, b, \ell = 0}^{k-1} (-1)\ell \sum_{i=1}^{k+1} (\mu_a^+ - \ell - \frac{3}{2} + i)p_{k-\ell-1,b} p_{\ell,a} \eta^{b,a}.$$  

Proof. The proof follows from direction calculations. Using (2.11), we have

$$L_k f(z) = \sum_{a} \sum_{\ell \geq 0} \prod_{i=1}^{k+1} (\mu_a^+ - \ell - \frac{3}{2} + i)p_{\ell,a} \phi_a(z)^{-\ell-1} z^k$$

$$+ \sum_{b} \sum_{m \geq 0} \prod_{j=1}^{k+1} (\mu_b^+ + m - \frac{1}{2} + j)q_m^b \phi_b m^k.$$  

Here if $k = -1$, we have the convention

$$\prod_{i=1}^{k+1} (\mu_a^+ - \ell - \frac{3}{2} + i) = 1.$$  

The quadratic Hamiltonian $\Omega(L_k f, f)$ contains at most three types of terms: $q_i q_j$-term, $q_i p_j$-term, and $p_i p_j$-term. We discuss each type in details using the formula

$$\Omega(L_k f, f) = -\Omega(f, L_k f).$$

(1) The $q_i q_j$-terms appear in $-\Omega(f, L_k f)$ only if $k = -1$. They are

$$-\operatorname{Res}_{z=0} \left( q_0^a \phi_a, q_0^b \phi_b z^{-1} \right) = -\sum_{a, b} q_0^a q_0^b \eta_{ab}.$$

(2) The $q_i p_j$-terms appear in two situations. The term appears in (2.13) is a combination of the contributions from the two situations.

- In the first situation, we have

$$- \left( \sum_{a} \sum_{\ell \geq 0} p_{\ell,a} \phi_a(z)^{-\ell-1}, \sum_{m \geq 0} \prod_{j=1}^{k+1} (\mu_b^+ + m - \frac{1}{2} + j)q_m^b \phi_b z^m z^k \right)$$

$$= - \sum_{a, b} \prod_{j=1}^{k+1} (\mu_b^+ + m - \frac{1}{2} + j)p_{m+k,a} q_m^b (\phi^a, \phi_b).$$

- In the second situation, we have

$$- \left( \sum_{b} \sum_{m \geq 0} q_m^b \phi_b(-z)^m, \sum_{\ell \geq 0} \prod_{a}^{k+1} (\mu_a^+ - \ell - \frac{3}{2} + i)p_{\ell,a} \phi_a(-z)^{-\ell-1} z^k \right)$$

$$= \sum_{a, b} \prod_{j=1}^{k+1} (\mu_a^+ - \ell - \frac{3}{2} + i)q_m^b p_{m+k,a} (\phi_b, \phi^a).$$
Recall in Lemma 2.4, we have \((\phi_a, \phi^b) = \delta^b_a\). Now taking \(\ell = m + k\), we obtain
\[
\prod_{j=1}^{k+1} (\mu^+_b + m - \frac{1}{2} + j) = \prod_{j=1}^{k+1} (-\mu^+_a + m - \frac{1}{2} + j)
\]
\[- (-1)^{k+1} \prod_{j=1}^{k+1} (\mu^+_a - \ell + k + \frac{1}{2} - j).
\]

So the terms from two situations are equal. So we obtain the total contribution is the term in (2.13).

(3) Finally, the \(p_ip_j\)-terms are (by taking \(m = k - \ell - 1\))
\[
- \left( \sum_b \sum_{m \geq 0} p_{m,b} \phi^b(z) \sum_a \sum_{\ell \geq 0} \prod_{i=1}^{k+1} (\mu^+_a - \ell - \frac{3}{2} + i)p_{\ell,a} \phi^a(-z)^{-\ell - 1}z^k \right)
\]
\[
= \sum_{a,b} \sum_{\ell=0}^{k-1} (-1)^\ell \prod_{i=1}^{k+1} (\mu^+_a - \ell - \frac{3}{2} + i)p_{k-\ell-1}p_{\ell,a}^b q_{ba}.
\]

This is the contribution in (2.14).

\[\square\]

**Definition 2.7.** Let \(\hat{L}_k\) be the quantization operator of the differential operator \(L_k\) in (2.11), defined by the formulas (2.4) and (2.2).

Using Lemma 2.6, we can calculate these quantization operators explicitly.

**Lemma 2.8.** For \(k \geq -1\), the quantization operator \(\hat{L}_k\) has the form
\[
\hat{L}_k(q) = -\frac{\delta_{k-1}}{2\hbar} \sum_{a,b} q^a q^b q_{a,b} + \sum_a \sum_{m \geq 0} \sum_{m \geq -k} \prod_{j=1}^{k+1} (\mu^+_a + m - \frac{1}{2} + j)q^a_m \frac{\partial}{\partial q^{a}_{m+k}}
\]
\[
+ \frac{\hbar}{2} \sum_{a,b} \sum_{\ell=0}^{k-1} (-1)^\ell \prod_{i=1}^{k+1} (\mu^+_a - \ell - \frac{3}{2} + i) \frac{\partial}{\partial q_{k-\ell-1}^b} \frac{\partial}{\partial q_{k+1}^a} q_{ba}.
\]

Now we can relate the quantization operators \(\{\hat{L}_k\}\) to the differential operators \(\{L_k\}\) defined in (0.9) by considering the *dilaton shift*
\[q^a_m := t^a_m - \delta_{m,1} \delta_{a,0}.
\]

**Proposition 2.9.** We have
\[
L_k = \hat{L}_k - \delta_{k,0} \cdot \frac{1}{4} \sum_a (\mu^+_a - \frac{1}{2})(\mu^+_a + \frac{1}{2}), \quad k \geq -1.
\]

**Proof.** Since \(p^+_{a=0} = -\hat{c}_W/2\), using the dilaton shift \(q^0 = t^0 - 1\), we have
\[
- (-1)^{k+1} \prod_{j=1}^{k+1} (\mu^+_a = 0 + 1 - \frac{1}{2} + j)p_{1+k,a=0} = \prod_{j=1}^{k+1} (-\hat{c}_W - \frac{1}{2} + j) \frac{\partial}{\partial q_{k+1}^a}.
\]
Thus

\[ \hat{L}_k(t) = \prod_{j=1}^{k+1} \left( -\frac{\hat{c}_W}{2} - j \right) \frac{\partial}{\partial t_{k+1}^m} - \frac{\delta_{k-1}}{2\hbar} \sum_{a,b} t_{0}^{ab} \eta_{a,b} \]

\[ - \sum_{a} \sum_{m \geq 0} \prod_{j=1}^{k+1} \left( \mu_a^+ + m - \frac{1}{2} + j \right) t_m^a \frac{\partial}{\partial t_{m+k}^a} \]

\[ + \frac{\hbar}{2} \sum_{a,b} \sum_{\ell = 0}^{k-1} (-1)^{\ell} \prod_{i=1}^{k+1} \left( \mu_a^+ - \ell - \frac{3}{2} + i \right) \frac{\partial}{\partial t_{k+1-\ell}^a} \frac{\partial}{\partial t_{\ell}^a} \eta_{ba} \]

\[ = L_k(t) + \delta_{k,0} \cdot \frac{1}{4} \sum_{a} (\mu_a^+ - \frac{1}{2})(\mu_a^+ + \frac{1}{2}) \]

\[ \square \]

A proof of Proposition 0.10. If \( \{m,n\} \neq \{1,-1\} \), by Lemma 2.2 and (2.12), we have

\[ [L_m, L_n] = [\hat{L}_m, \hat{L}_n] = [\hat{L}_m, \hat{L}_n] = (m-n)\hat{L}_{m+n} = (m-n)L_{m+n}. \]

Otherwise, assume \( m = 1 \) and \( n = -1 \), we have

\[ [L_1, L_{-1}] = [\hat{L}_1, \hat{L}_{-1}] = \{\hat{L}_1, \hat{L}_{-1}\} + C(h_1, h_{-1}) \]

\[ = 2\hat{L}_0 - \frac{1}{2} \sum_{a} (\mu_a^+ - \frac{1}{2})(\mu_a^+ + \frac{1}{2}) \]

\[ = 2L_0. \]

Notice that here we use (2.6). \( \square \)

3. Semi-simple Frobenius manifolds and Virasoro constraints

In this section, we first review Givental Theorem on the Virasoro constraints for semi-simple Frobenius manifolds, Theorem 3.2. Then we search some semisimple Frobenius manifolds and apply the Givental Theorem to prove Theorem 0.8 and Proposition 3.8.

3.1. Frobenius manifolds and Givental Theorem.

3.1.1. Conformal Frobenius manifold, quantum connection, and calibration. Dubrovin \[13\] introduced the concepts of Frobenius manifold to study the geometry of 2D topological field theories. Briefly speaking, a Frobenius manifold \( M \) consists of a quadruple \( (M, \langle \cdot, \cdot \rangle, F, 1) \) with

- a flat metric \( \langle \cdot, \cdot \rangle \) on the tangent bundle \( T_tM \) for \( t \in M \);
- a prepotential \( F(t) \), whose 3-rd covariant derivative gives an associative multiplication \( \ast_t \)

\[ F_{abc} = \langle a \ast_t b, c \rangle = \langle a, b \ast_t c \rangle; \]

- a flat vector field \( 1 \), which is an identity of the multiplication \( \ast_t \).
A Frobenius manifold is called conformal if there exists an Euler vector field. See [13] for the definitions and details of these concepts. Examples of conformal Frobenius manifolds exist in Gromov-Witten theory, Fan-Jarvis-Ruan-Witten theory, and Saito’s construction of primitive forms for miniversal deformations of isolated critical points of holomorphic functions [54].

**Remark 3.1.** Dubrovin’s notion of Frobenius manifolds can be generalized to Frobenius super-manifolds, where the quantum multiplication $\star_t$ becomes super-commutative. The examples appear in this section are all Frobenius manifolds, while some examples in Section 4 are Frobenius super-manifolds, see Example 0.3.

Let \( \{\phi_a\} \) be a homogeneous flat basis of the Frobenius manifold and \( t_0^a \) be the coordinate of \( \phi_a \). There exists a quantum connection
\[
\nabla_z := d - z^{-1} \sum_a (\phi_a \star_t) dt_0^a \wedge .
\]

Following [13, 24], one can construct a fundamental solution \( S_t(z) \), called calibration, which is upper-triangular as in (2.8). Such a calibration is unique up to right multiplication by a constant lower upper-triangular operator \( C(z) \) which does not depend on the choice of \( t \).

3.1.2. **Givental Theorem.** A Frobenius manifold is called semi-simple at \( t \in M \) if the Frobenius algebra \( \star_t \) is semi-simple. Let \( M \) be a Frobenius manifold of (complex) dimension \( r \). If \( M \) is semi-simple at \( t \in M \), then there exist a canonical coordinate system \( u = (u_1, \ldots, u_r) \) near \( t \in M \), such that
\[
\frac{\partial}{\partial u_i} \star_t \frac{\partial}{\partial u_j} = \delta_{i,j} \frac{\partial}{\partial u_i}.
\]

Let \( U := \text{diag}(u_1, \ldots, u_r) \) and \( \Psi_t \) be the base change from \( \{\frac{\partial}{\partial u_i}\} \) to \( \{\sqrt{\Delta_i} \frac{\partial}{\partial u_i}\} \), where
\[
\langle \frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j} \rangle = \frac{\delta_{i,j}}{\Delta_i}.
\]

According to [13, 25, 58], we consider a Frobenius manifold of conformal dimension \( K \) with an Euler vector field
\[
E = \sum_a (1 - d_a) t_0^a \frac{\partial}{\partial t_0^a} + \sum_{a,d_a=1} \rho_a \frac{\partial}{\partial t_0^a},
\]
there exists a unique lower triangular symplectic transformation \( R_t(z) \) such that
\[
[R_t(z), z^{-1} E \star] = (z \partial_z + \tilde{\theta}) R_t(z),
\]
and
\[
S_t(z) \sim \Psi_t R_t(z) e^{U/z}.
\]

Here \( \tilde{\theta} := \text{diag}(d_1 - K/2, \ldots, d_r - K/2) \).

Let \( \mathcal{D}_{pt}(h; q) \) be the Witten-Kontsevich \( \tau \)-function. For semisimple Frobenius manifolds, Givental [25] constructed the (abstract) total descendent potential
\[
\mathcal{D}(h; q) := e^{F^1(t)} \hat{S}_t^{-1} \hat{\Psi}_t R_t e^{\hat{U}/z} \prod_{i=1}^N \mathcal{D}_{pt}(h \Delta_i; \sqrt{\Delta_i} q^i).
\]

and the (abstract) total ancestor potential
\[
\mathcal{A}_t(h; q) = \hat{\Psi}_t R_t e^{\hat{U}/z} \prod_{i=1}^N \mathcal{D}_{pt}(h \Delta_i; \sqrt{\Delta_i} q^i).
\]

Here \( q_m^a = t_m^a - \delta_{m,1} \delta_{a,0} \) is the dilaton shift, and \( F^1(t) \) is the genus one potential.
According to Kontsevich Theorem [34] on the Witten conjecture [59], the Witten-Kontsevich \( \tau \)-function \( D_{\mathfrak{M}}(h; q) \) is annihilated by differential operators
\[
(z^{-1/2}(z\partial_z z)^k z^{-1/2}) + \frac{\delta_{k,k}}{16}, \quad k \geq -1.
\]
For a conformal semisimple Frobenius manifold, let \( \rho = \sum \rho_a \phi_a \) in the Euler vector field \( E \). Replacing the auxiliary operator in (2.9) by \( z(\partial_z + z^{-1}\tilde{\theta}) z + \rho \), and defining differential operators by (2.15), which we denote by \( L_{k}^{\text{Giv}} \) here, Givental proved an equality between two operators in [24, Theorem 8.1] and deduced the following result from Kontsevich Theorem.

**Theorem 3.2.** [24, Proposition 7.7] The Virasoro operators \( \{ L_{k}^{\text{Giv}} \}_{k \geq -1} \) annihilate the total descendant potential \( D(h; q) \) of a semi-simple Frobenius manifold. That is,
\[
L_{k}^{\text{Giv}} D(h; q) = 0, \quad k \geq -1.
\]
We call it Givental Theorem on the Virasoro constraints for semi-simple Frobenius manifolds, or simply Givental Theorem in this paper.

3.1.3. **Givental-Teleman Theorem.** A CohFT is called semisimple if the underlying Frobenius manifold is semisimple. The following result is a consequence of the famous Givental-Teleman Theorem [58], which says the semisimple CohFT is uniquely reconstructed from the underlying Frobenius manifold.

**Theorem 3.3.** [58] For a semi-simple CohFT, the abstract total ancestor potential is the same as the formal total ancestor potential defined from the CohFT.

See (3.8) and (4.5) for examples of formal total ancestor potentials.

3.1.4. **Frobenius manifolds in LG A-models.** In LG A-models, there is a conformal formal Frobenius (super)-manifolds (see [20, Corollary 4.2.8] for example)
\[
\left( \mathcal{H}_{W,G}, (\cdot, \cdot), F_{0}^{(W,G)}, 1, J, E \right).
\]
Here
- the metric is given by the pairing \( (\cdot, \cdot) \) in (0.3);
- the homogeneous basis \( \{ \phi_a = [f_a] d\mathbf{x}_a \} \) of \( \mathcal{H}_{W,G} \) is a flat basis;
- the prepotential is given by the genus zero potential
\[
F_{0}^{(W,G)}(t) = \sum_{n \geq 3} \frac{1}{n!} \left\langle \underbrace{t, \cdots, t}_{0,n} \right\rangle^{(W,G)}, \quad t = \sum a t_{a}^{0} \phi_{a}.
\]
- the Euler vector field \( E \) is of the form
\[
E = \sum_{a} \left( 1 - \deg_{C} \phi_{a} \right) t_{a}^{0} \frac{\partial}{\partial t_{a}^{0}}.
\]
- the conformal dimension \( K \) is the central charge \( \tilde{c}_{W} \).

The quantum product \( \ast_{t} \) is given by genus-zero three-point correlators
\[
(\phi_{a} \ast_{t} \phi_{b}, \phi_{c}) = \frac{\partial^{3}}{\partial t_{0}^{a} \partial t_{0}^{b} \partial t_{0}^{c}} F_{0}^{(W,G)}(t) := \langle \langle \phi_{a}, \phi_{b}, \phi_{c} \rangle \rangle^{(W,G)}(t).
\]
For arbitrary \( t \), one can construct a formal total ancestor potential from the CohFT
\[
\mathcal{A}_{W,G}(t) := \exp \left( \sum_{g} h^{2g-2} \sum_{k} \frac{1}{k!} \left\langle \prod_{i=1}^{k} t(\psi_{i}) \right\rangle_{g,k}^{(W,G)}(t) \right).
\]
The quantum invariants in the definition of the formula (0.12) are replaced by the double brackets, which are formal power series of $t$,
\[
\left\langle \prod_{i=1}^{k} t(\psi_i) \right\rangle_{g,k}^{(W,G)}(t) := \sum_{m \geq 0} \frac{1}{m!} \left\langle \prod_{i=1}^{k} t(\psi_i), t, \cdots, t \right\rangle_{g,k+m}^{(W,G)}.
\]

**Remark 3.4.** We make a few remarks when applying the results in Section 3.1 to the LG A-models in Section 3.2 and Section 3.3.

1. According to Givental-Teleman Theorem 3.3, if the Frobenius manifold is generically semisimple at $t$, then the abstract total ancestor potential $A_t(h;\mathbf{q})$ in (3.5) is the same as the geometric total ancestor potential $A_{t,W,G}(t)$ in (3.8).

2. According to [48], the potential $A_{t,W,G}$ in (0.12) is the limit of the geometric total ancestor potential $A_{t,W,G}(t)$ in (3.8) at $t = 0$.

3. We can choose the calibration operator $S_t(z)$ by
\[
(S_t(z)(\phi_a), \phi_b) := (\phi_a, \phi_b) + \left\langle \frac{\phi_a}{z - \psi_1}, \phi_b \right\rangle_{0,2}(t).
\]
Taking the limit $t \to 0$, we see the total descendent potential $\mathcal{D}(h;\mathbf{q})$ in (3.4) in LG A-model theories is the same as the potential $A_{W,G}$, up to a constant.

4. In Section 3.2 and Section 3.3, the LG A-model theories we consider satisfies the following condition,
\[
\mu^+_a = \mu^-_a, \quad \forall \phi_a \in \mathcal{H}_{W,G}.
\]

5. If (3.9) holds, then $\mu^+_a = \deg_C \phi_a - \tilde{c}_W/2$. Since the term $\rho$ in the Euler vector field vanishes, the Virasoro operator $L_k^{\text{Giv}}$ in Theorem 3.2 matches the Virasoro operator $L_k$ defined in (0.9).

Thus if the Frobenius manifold $(\mathcal{H}_{W,G}, (\cdot, \cdot), F_0^{(W,G)}, 1, \mathbb{J}, E)$ is generically semisimple, Conjecture 0.6 is a consequence of Theorem 3.2. Next we will give some examples.

### 3.2. Semisimple Frobenius manifolds via LG mirror symmetry

According to the mirror symmetry statements proved in [28] and [27], we obtain a large class of semisimple Frobenius manifolds in LG A-model theories from admissible LG pairs with an invertible polynomial and the maximal group.

**Definition 3.5** (Invertible polynomials). A polynomial $W$ is called *invertible* if up to a rescaling of the variables, $W$ can be written in the form
\[
W = \sum_{i=1}^{n} \prod_{j=1}^{n} a_{ij} x_j^{a_{ij}}
\]
and its *exponent matrix* $E_W := (a_{ij})_{n \times n}$ is an invertible matrix in $\text{GL}(n, \mathbb{Q})$.

According to the classification by Kreuzer and Skarke [39], an invertible polynomial must be a direct sum of atomic types ($a \geq 2, a_i \geq 2$):

1. **Fermat atomic type**: $x^a$.
2. **Chain atomic type**: $x_1^{a_1} x_2 + x_2^{a_2} x_3 + \cdots + x_k^{a_k-1} x_k + x_k^{a_k}$.
3. **Loop atomic type**: $x_1^{a_1} x_2 + x_2^{a_2} x_3 + \cdots + x_k^{a_k} x_1$.

We slightly abuse the notation and still call $W$ is of *Fermat type* if
\[
W = x_1^{a_1} + \cdots + x_n^{a_n}.
\]
Lemma 3.6. Let \((W,G_W)\) be an admissible LG pair and \(W\) be an invertible polynomial. For any \(\phi_a \in \mathcal{H}_{W,G_W}\), we have \(\mu_+ = \mu_-\).

Proof. If \(W\) is a direct sum of \(W_1, \ldots, W_N\), then \(G_W = G_{W_1} \times \cdots \times G_{W_N}\). Now it suffices to prove this lemma in the case \(W\) is of atomic type. By (0.5), for \(\phi_a = \alpha|\gamma\rangle\), we only need to consider the case \(\phi_a\) is broad, and prove that \(N_\gamma = 2\text{wt}(\alpha)\). For \(\phi_a\) is broad, by [38, Lemma 1.7], we have

(1) If \(W\) is Fermat type, then there is no broad element.
(2) If \(W\) is loop type with \(k\) variables, then \(\gamma = \text{Id}\), \(k\) is even, and

\[
\alpha = \prod_{i=1}^{k} x^{\delta_{\text{even}}(a_i-1)}_i \, dx_i \quad \text{or} \quad \prod_{i=1}^{k} x^{\delta_{\text{odd}}(a_i-1)}_i \, dx_i.
\]

(3) If \(W\) is chain type with \(k\) variables, then \(N_\gamma\) is even, and

\[
\alpha = \prod_{k-N_\gamma+1}^{k} x^{\delta_{\text{odd}}(a_i-1)}_i \, dx_i.
\]

Keeping in mind \(W\) is of weight 1, one can check \(\text{wt}(\alpha) = N_\gamma/2\) easily. □

If \(W\) is an invertible polynomial of the form (3.10), Berglund-Hübsch [6] constructed a mirror polynomial

\[
W^T = \sum_{i=1}^{n} \prod_{j=1}^{n} x^{a_{ji}}_j.
\]

So the exponent matrix \(E_{W^T}\) of \(W^T\) is the transpose matrix of \(E_W\).

Now we recall two mirror symmetry statements proved in [27] and [28].

Theorem 3.7 (All genus mirror symmetry for invertible polynomials). [27, 28] Let \((W,G_W)\) be an admissible LG pair and \(W\) be an invertible polynomial. There exists a primitive form \(\zeta\) and a bigrading preserving mirror map \(\mathcal{H}_{W,G_W} \to \text{Jac}(W^T)\). Under the mirror map, there are:

(1) (Polishchuk-Vaintrob to Saito-Givental mirror theorem [28, Theorem 1.3])

\[
\mathcal{A}_{W,G_W}^{PV} = \mathcal{A}_{W^T,\zeta}^{SG}.
\]

(2) (Fan-Jarvis-Ruan-Witten to Saito-Givental mirror theorem [27, Theorem 1.2]) If \(W\) has no weight-\(1/2\) chain variable,

\[
\mathcal{A}_{W,G_W}^{FJRW} = \mathcal{A}_{W^T,\zeta}^{SG}.
\]

Here \(\mathcal{A}_{W^T,\zeta}^{SG}\) is the Saito-Givental limit potential for \(W^T\), see [27] for the details. The main ingredients in the proof in [27] and [28] are the identification between the Frobenius manifold of the admissible LG A-model \((W,G_W)\) and the Saito’s Frobenius manifold of the isolated singularity \(W^T\) [54], with an appropriate choice of a primitive form \(\zeta\) [40, 41].

The condition in Part (2) is not essential. It is caused by the lack of enough computation tools for the analytic part of FJRW theory. The condition can be removed in some cases of exceptional singularities, when \(W = x^2 + xy^2 + yz^4\) and \(W = x^2 + xy^3 + yz^3\) [41, Section 2.3].

3.2.1. A proof of Theorem 0.8. It is well known that Saito’s Frobenius manifold of \(W^T\) is generically semisimple. Now Theorem 0.8 follows from Theorem 3.7 and Givental Theorem 3.2. □
3.3. Semisimplicity via quantum Euler vector field. In this section, we consider some A-model LG pairs \((W, \langle J \rangle)\) studied in [21], where the group \(\langle J \rangle\) is not the maximal group \(G_W\). We consider the following list

\[
x^4 + y^4, x^3 + y^6, x^3 + y^9, x^4 + y^6, x^3 + xy^8.
\]

The main result of this section is

**Proposition 3.8.** Let \(W\) be an invertible polynomial in (3.12). Then the FJRW theory of the admissible LG pair \((W, \langle J_W \rangle)\) has the following properties:

1. the Frobenius manifold is generically semisimple near the origin of \(H_{W,\langle J \rangle}\);
2. the Virasoro Conjecture 0.9 holds true.

We see part (2) is Proposition 0.9. It follows from part (1) and Givental Theorem 3.2. In the remaining part of this section, we prove part (1) using the criteria described in [1].

**Remark 3.9.** Our method works in much more generality for invertible polynomials with two-variables. In general, the formulas involved will be more complicated, even for the other two examples \(x^3 + xy^6\) and \(x^3 y + y^7\) in [21].

3.3.1. Quantum product \(\star_t\). Let \(\mu = \dim_{\mathbb{C}} H_{W,\langle J \rangle}\). We fix a basis of \(H_{W,\langle J \rangle}\), denoted by \(\{\phi_i \mid i = 1, \ldots, \mu\}\). We write the dual basis by \(\{\phi_i^j \mid i = 1, \ldots, \mu\}\). If \(j_k \in \langle J \rangle\) is a narrow element, the generator \(1|J_k^k\) \(J_k \in H_{W,\langle J \rangle}\) is also denoted by the symbol \(e_{k,j}\) in [21], where \(e_0 = dx dy\). We fix a basis of \(H_{W,\langle J \rangle}\) as follows. The notations here is from [21], except we don’t use \(W\) to avoid confusion. We always have \(\phi_1 = 1 = e_J, \phi_2 = e_{(d-1)J}, \phi_3 = e_{2J}\).

| \(W\) | \(\mu\) | \(\phi_1, \ldots, \phi_\mu\) |
|-------|-------|----------------------------|
| \(x^4 + y^4\) | 6 | 1, \(e_{3J}, e_{2J}, 4y^2 e_0 = 4X, 4x y e_0 = 4Y, 4x^2 e_0 = 4Z\) |
| \(x^3 + y^6\) | 6 | 1, \(e_{5J}, e_{2J}, Z, 3X, 6Y\) |
| \(x^3 + y^9\) | 8 | 1, \(X^2 Y, Y, X, XY, X^2, 3Z, 9xy^2 e_0\) |
| \(x^4 + y^6\) | 9 | 1, \(XY^2, Y, Y^2, X^2, Z, X, XY, \sqrt{24x} y^2 e_0\) |
| \(x^3 + xy^8\) | 10 | 1, \(X^3 Y, Y, X, XY, X^2, Y^2, X^2 Y, X^3, \sqrt{24x} y^3 e_0, \sqrt{-8Z}\) |

Let us denote the dimension of the narrow subspace of \(H_{W,\langle J \rangle}\) by \(\mu_{n\text{ar}}\), the weights of the variables by \((\frac{w_1}{d}, \frac{w_2}{d})\), where \(d, w_1, w_2\) are positive integers and \(\text{gcd}(w_1, w_2) = 1\).

We parametrize \(e_{2J}\) by \(t\) and compute the quantum product \(\star_t\). We give an example to compute \(e_{(d-1)J} \star_t\). It is enough to find all nontrivial invariants of the form

\[
\langle e_{(d-1)J}, \phi_1, \phi_j, e_{2J}, \ldots, e_{2J} \rangle_{0,n+3}.
\]

Since

\[
\deg_{\mathbb{C}} e_{(d-1)J} = 2 - 2 \frac{w_1 + w_2}{d} = \tilde{c}_W, \quad \deg_{\mathbb{C}} e_{2J} = \frac{w_1 + w_2}{d}.
\]

The degree constraint (1.14) implies if the invariant in (3.13) does not vanish, then

\[
\deg_{\mathbb{C}} \phi_i + \deg_{\mathbb{C}} \phi_j = \frac{n}{2} \tilde{c}_W.
\]

Since \(0 \leq \deg_{\mathbb{C}} \phi_i, \deg_{\mathbb{C}} \phi_j \leq \tilde{c}_W\), we must have \(n = 0, 1, 2, 3, 4\). In fact, if \(n = 1\) or \(3\), there is no nontrivial invariant of the form (3.13). Now we list all possible nontrivial contributions as follows. Some of the FJRW invariants (including the 7-point invariant) are obtained by WDVV equations.
We compute \( E(3.14) \)

**Lemma 3.10.** Let \( W \) be an invertible polynomial in (3.12). The quantum Euler vector field of the FJRW theory of \((W, \langle J \rangle)\) along \( e_{2J} \) has the form

\[
E(t) = \mu \cdot e_{(d-1)J} + (2\mu_\text{nar} - 2 - \mu) \frac{w_1 w_2 t^2}{2d^2} \mathbf{1}.
\]
3.3.3. A proof of Proposition 3.8, Part (1): According to [1, Theorem 3.4], the Frobenius manifold is semi-simple at \( v \) if and only if

\[
\det \mathbf{E}(v)_v \neq 0.
\]

In fact, we just need to restrict the multiplication to \( v = t \cdot e_{2J} \). We see the quantum multiplication of second term in (3.16) on the fixed basis gives a multiple of identity matrix. Using (3.14) and (3.16), the multiplication \( \mathbf{E}(t)_t \) on the basis \( \{ \phi_1 = 1, \phi_2 = e_{(d-1)J}, \phi_3, \cdots, \phi_\mu \} \) is given by the matrix

\[
M \bigoplus \left( \bigoplus_{i=3}^{\mu_{\text{nar}}} \frac{(\mu_{\text{nar}} - 1)w_1w_2}{d^2} t^2 \text{Id}_i \right) \bigoplus \left( \bigoplus_{i=\mu_{\text{nar}}+1}^{\mu} \frac{(-1-\mu)w_1w_2}{d^2} t^2 \text{Id}_i \right),
\]

where

\[
M = \left( \begin{array}{cc}
\frac{(2\mu_{\text{nar}} - 2 - \mu)w_1w_2}{2d^2} & \frac{w_1^2w_2^2\mu t^4}{2d^2} \\
\mu & \frac{(2\mu_{\text{nar}} - 2 - \mu)w_1w_2}{2d^2}
\end{array} \right).
\]

Thus we have \( \det \mathbf{E}(t)_t \neq 0 \) if \( t \neq 0 \). Thus \( \det \mathbf{E}(v)_v \neq 0 \) in a small neighborhood of \( v = (0, 0, t, 0, \cdots, 0) \) and the result follows from [1, Theorem 3.4]. \( \square \)

4. Virasoro constraints and LG/CY correspondence

In this section, we consider the Virasoro constraints for admissible LG pairs \((W, G)\) when \( W \) is of Calabi-Yau type.

**Definition 4.1** (Calabi-Yau polynomials). We say a quasihomogeneous polynomial \( W(x_1, \cdots, x_n) \) is of Calabi-Yau type if

\[
\widehat{c}_W := \sum_{i=1}^{n} (1 - 2q_i) = n - 2 \in \mathbb{Z}_+.
\]

Let \( TX_W \) be the tangent bundle of the hypersurface

\[
X_W := (W = 0) \subset \mathbb{P}^{n-1}(w_1, \cdots, w_n).
\]

The condition (4.1) implies that \( c_1(TX_W) = 0 \). On the other hand, the finite group \( \widetilde{G} := G/\langle J_W \rangle \) acts on the hypersurface \( X_W \) and the global quotient

\[
X_{W,G} := X_W/\widetilde{G}
\]

also satisfies \( c_1(TX_{W,G}) = 0 \). We will abuse the notations to call these varieties \( X_{W,G} \) of Calabi-Yau type.

Let \((W, G)\) be an admissible LG pair of CY type. The LG A-model theories of \((W, G)\) are closely related to the Gromov-Witten theory for the Calabi-Yau variety \( X_{W,G} \). Such connections are called Landau-Ginzburg/Calabi-Yau correspondence in the literature [10], inspired by the work of physicists (see [61] for the references therein).

In general, LG pairs of CY type are not generically semi-simple so Givental’s Theorem is no longer applicable. However, the idea of LG/CY correspondence is very useful here.

If the Calabi-Yau \( X_{W,G} \) has complex dimension at least three, (or \( \widehat{c}_W \geq 3 \) equivalently), then Virasoro Conjecture 0.6 can be obtained by a simple degree calculation. We show it in Section 4.1. When the Calabi-Yau side \( X_{W,G} \) is an elliptic curve, we can prove Virasoro Conjecture 0.6 using the Virasoro constraints for the elliptic curve in [52] and the Landau-Ginzburg/Calabi-Yau correspondence studied in [42, 43]. In Section 4.3, we discuss the cases for all the pairs \((W, G)\) when \( X_{W,G} \) is one-dimensional Calabi-Yau.
4.1. Calabi-Yau polynomials with large central charges. For admissible LG pairs of CY type, if the central charge is at least three, as an analog of the Virasoro constraints for Calabi-Yau manifolds that has been proved in [22, Theorem 7.1], we have

**Lemma 4.2.** Let \((W, G)\) be an admissible LG pair of Calabi-Yau type with \(\hat{c}_W \geq 3\). If for any nonzero \(\phi_a = \alpha|\gamma\rangle \in \mathcal{H}_{W,G}\) with \(\gamma \neq J\), we have \(\deg_C \phi_a \geq 1\), then the Virasoro constraints

\[
L_k \mathcal{A}_{W,G}^\bullet = 0, \ k \geq -1
\]

is equivalent to Assumption 1.4.

Now we deduce the second part of Theorem 0.10 from this Lemma.

**Corollary 4.3.** Let \(W\) be an invertible polynomial of Calabi-Yau type. If \(\hat{c}_W \geq 3\) and \(G \leq \text{SL}_n(\mathbb{C})\), then Virasoro Conjecture 0.6 holds for \(\mathcal{A}_{W,G}^{\text{FJRW}}\).

**Proof.** For a fixed homogeneous element \(\phi_a = \alpha|\gamma\rangle \in \mathcal{H}_{W,G}\) with

\[
\gamma = (\exp(2\pi \sqrt{-1} \theta_1), \ldots, \exp(2\pi \sqrt{-1} \theta_n))
\]

the group element given in (0.1). Recall that \(\text{age}(\gamma) = \sum_{i=1}^n \theta_i\) is defined (0.2). Then \(G \leq \text{SL}_n(\mathbb{C})\) implies that \(\text{age}(\gamma) \in \mathbb{Z}_{\geq 0}\). Using the bigrading formula (0.5) and the CY condition (4.1), we have

\[
\deg_C \phi_a = \frac{N_\gamma}{2} + \text{age}(\gamma) - \sum_{i=1}^n q_i = \frac{N_\gamma}{2} + \text{age}(\gamma) - 1.
\]

If \(\gamma = J^0\), then \(\text{age}(\gamma) = 0\) and \(N_\gamma = n = 2 + \hat{c}_W \geq 5\). So \(\deg_C \phi_a \geq 1\) holds.

If \(\gamma \neq J^0\), then \(\text{age}(\gamma) \geq 1\). The condition \(\deg_C \phi_a \geq 1\) could fail if

1. \(\text{age}(\gamma) = 1\) and \(N_\gamma = 0\);
2. \(\text{age}(\gamma) = 1\) and \(N_\gamma = 1\).

For case (1), we claim that \(\gamma = J\). In fact, \(N_\gamma = 0\) implies for each \(i\), we have \(0 < \theta_i < 1\). Recall \(E_W\) is the exponent matrix of \(W\), then \(\gamma \in G_W\) implies

\[
E_W \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_n \end{pmatrix} \in \mathbb{Z}_+^n.
\]

Let \(q_i^T\) be the weight of \(x_i\) in the mirror polynomial \(W^T\), then we have

\[
\begin{pmatrix} q_1^T & \cdots & q_n^T \end{pmatrix} E_W \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_n \end{pmatrix} = \sum_{i=1}^n \theta_i = 1.
\]

Since \(q_i^T\) are positive numbers and

\[
\sum_{i=1}^n q_i^T = \sum_{i=1}^n q_i = 1,
\]

we must have

\[
E_W \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_n \end{pmatrix} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}.
\]
This implies $\theta_i = q_i$ for each $i$ and $\gamma = J$.

On the other hand, case (2) is impossible. In fact, $W_\gamma$ is still an invertible polynomial so we must have $W_\gamma = x_i^{a_i}$ for some $i$. Then it is easy check that there is no $\langle J \rangle$-invariant element in $\text{Jac}(W_\gamma)dx_\gamma$.

In conclusion, if $\gamma \neq J$, condition $\deg_C \phi_a \geq 1$ always holds and the result follows from Lemma 4.2. $\square$

4.2. Fermat CY polynomials of three variables. Fermat CY polynomials with three variables can be written in the form of

\begin{equation}
W_d := x_1^{a_1} + x_2^{a_2} + x_3^{a_3}, \quad \text{with} \quad \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} = 1.
\end{equation}

Here we assume $d = a_1 \geq a_2 \geq a_3$, then we have $\langle a_1, a_2, a_3 \rangle = (3, 3, 3), (4, 4, 2), \text{or } (6, 3, 2)$.

The FJRW theory of the admissible LG pairs \quad $W_d := x_1^{a_1} + x_2^{a_2} + x_3^{a_3}, \langle J \rangle$ has been studied in [43]. When $d = 3, 4, 6$, we choose $h(W_d) := x_1x_2x_3/27, x_1^2x_2^2/32, x_4x_2/36$.

We form a homogeneous basis of $\mathcal{H}_{W_d, \langle J \rangle}$ by

$\left\{1|J, 1|J^2, dx|J^0, h(W_d)dx|J^0\right\}$.

We parametrize them by $t_0^0, t_1^0, s_0^0$, and $s_0^0$ respectively. The choice of $h(W_d)$ is made to match the Poincaré pairing on the GW side.

According to (0.9), the Virasoro operators for the pair $(W_d, \langle J \rangle)$ are

\begin{equation}
L_k := - (k + 1)! \frac{\partial}{\partial t_0^0}
\end{equation}

\begin{equation}
+ \sum_{m \geq 0} \left( (m)_{k+1} t_0^0 \frac{\partial}{\partial t_{k+m}^0} + (m + 1)_{k+1} t_1^0 \frac{\partial}{\partial t_{k+m}^1} \right)
\end{equation}

\begin{equation}
+ \sum_{m \geq 0} \left( (m + 1)_{k+1} s_0^0 \frac{\partial}{\partial s_{k+m}^0} + (m)_{k+1} s_1^0 \frac{\partial}{\partial s_{k+m}^1} \right).
\end{equation}

Recall $\mathcal{A}_{W,G}^{\text{FJRW}}$ is the ancestor FJRW potential for the LG pair $(W, G)$ defined in (0.12). In this section, we will prove Part (2) of Theorem 0.10, which is

**Theorem 4.4.** Let $W_d$ be a Fermat CY polynomial in (4.2) and $L_k$ be the Virasoro operators in (4.3). For all $k \geq -1$, we have Virasoro constraints

$L_k \mathcal{A}_{W_d, \langle J \rangle}^{\text{FJRW}} = 0$.

We will prove Theorem 4.4 in three steps:

(1) We recall the Virasoro constraints for descendent Gromov-Witten potential in [52].

(2) We use the Ancestor/Descendant correspondence (Theorem 4.5) to write the Virasoro constraints for the ancestor Gromov-Witten potential of the elliptic curves.

(3) We use the LG/CY correspondence proved in [43] to complete a proof for Theorem 4.4.
4.2.1. **Ancestor and descendent.** For a compact Kähler manifold \(X\), there are two types of Gromov-Witten invariants, called **ancestor invariants** and **descendent invariants**, depending on the choice of psi-classes.

Let \(\{\alpha_i \in H^*(X, \mathbb{C})\}\) be a set of cohomology classes of \(X\). Let \(\overline{M}_{g,n}(X, d)\) be the moduli stack of degree-\(d\) stable maps from a connected genus \(g\) curve with \(n\) markings to the target \(X\). The moduli stack has a virtual fundamental cycle, denoted by \([\overline{M}_{g,n}(X, d)]^{\text{vir}}\). Let \(\pi : \overline{M}_{g,k}(X, d) \to \overline{M}_{g,k}\) be the forgetful morphism and \(\ev_i : \overline{M}_{g,k}(X, d) \to X\) be the evaluation morphism given by the \(i\)-th marking.

The descendent GW invariants are defined by intersecting the virtual fundamental cycles in GW theory with psi classes \(\{\bar{\psi}_i\}\) on moduli space of stable maps. Let \(\{\alpha_i\}\) be a homogeneous basis of \(H^*(X, \mathbb{C})\). Define the **total descendant GW potential** of \(X\) by

\[
\mathcal{D}^X = \exp \left( \sum_{g \geq 0} \frac{\hbar^{2g-2} \sum \frac{Q^d}{k!} \int_{[\overline{M}_{g,k}(X, d)]^{\text{vir}}} \prod_{i=1}^{k} \sum_{m=0}^{\infty} \left( t^m_{\alpha} \ev_i^* (\alpha_{\alpha}) \bar{\psi}_i^m \right) }{m!} \right).
\]

On the other hand, similar to the ancestor invariants (0.11) in LG A-model theories, the ancestor GW invariants are defined by intersecting the Gromov-Witten CohFT classes with psi classes on \(\overline{M}_{g,k}\). We denote by the **ancestor GW invariants**

\[
\{\alpha_1 \psi_{11}, \ldots, \alpha_n \psi_{n1}\}_{g,n,d}^X = \int_{[\overline{M}_{g,n}(X, d)]^{\text{vir}}} \prod_{k=1}^{n} \ev_k^* (\alpha_k) \pi^* \psi_{k}. \]

Let

\[
t := \sum_a \ell^a \alpha_a \in H^*(X, \mathbb{C}), \quad \tilde{t}(z) := \sum_{m \geq 0} \sum_a \ell^a \alpha_a z^m.
\]

The **ancestor GW correlation function**

\[
\langle \alpha_1 \psi_{11}, \ldots, \alpha_k \psi_{k1} \rangle_{g,k}^X (t) = \sum_d \sum_{m \geq 0} \frac{Q^d}{m!} \langle \alpha_1 \psi_{11}, \ldots, \alpha_k \psi_{k1}, t, \ldots, t \rangle^X_{g,k+m,d}.
\]

Via the dilaton shift

\[
\bar{q}(z) = \tilde{t}(z) - 1 : z,
\]

we denote the **total GW ancestor potential** of the target \(X\) by

\[
\mathcal{A}^X_t(\bar{q}) := \exp \left( \sum_g \frac{\hbar^{2g-2} \sum_k \frac{1}{k!} \langle \tilde{t}(\psi), \ldots, \tilde{t}(\psi) \rangle^X_{g,k}(t) }{m!} \right).
\]

The total ancestor potential depends on a choice of \(t\) while \(\mathcal{D}^X\) does not.

Using **topological recursion relations**, the operators \(S^X_t\) defined by

\[
(S^X_t(z)(\alpha_1), \alpha_2) := (\alpha_1, \alpha_2) + \langle \frac{\alpha_1}{z - \psi_1}, \alpha_2 \rangle_{0,2}^X (t)
\]

is a calibration with respect to the quantum connection in (3.1). The operator \(S^X_t(z)\) is a symplectic transformation.

**Theorem 4.5.** [36, 24] Let \(\mathcal{F}^X_t(t) := \langle \rangle^X_{1,0}(t)\) be the genus-one GW generating function, then

\[
\mathcal{D}^X(\bar{q}) = e^{\mathcal{F}^X_t(t)} S^{-1}_t \mathcal{A}^X_t(\bar{q}).
\]
4.2.2. Virasoro constraints for the descendents of elliptic curves. Let \( W_d \) be the Fermat CY polynomial in (4.2). The hypersurface determined by the \( W_d = x_1^{a_1} + x_2^{a_2} + x_3^{a_3} \) in the weighted projective space \( \mathbb{P}^2 \left( \frac{d}{a_1}, \frac{d}{a_2}, \frac{d}{a_3} \right) \) is an elliptic curve

\[
\mathcal{E}_d := X_{W_d} = (W_d = 0) \subset \mathbb{P}^2 \left( \frac{d}{a_1}, \frac{d}{a_2}, \frac{d}{a_3} \right).
\]

The Gromov-Witten theory does not depend on the choice of the elliptic curves. So we drop the subscript in \( \mathcal{E}_d \) and consider a basis of cohomology \( H^*(\mathcal{E}, \mathbb{C}) \), given by

- the identity class \( 1 \in H^0(\mathcal{E}, \mathbb{C}) \),
- the Poincaré dual of the point \( \omega \in H^2(\mathcal{E}, \mathbb{C}) \),
- the classes \( \alpha, \beta \in H^1(\mathcal{E}, \mathbb{C}) \), which is a symplectic basis of \( H^1(\mathcal{E}, \mathbb{C}) \).

We assign a bigrading and a parity for the basis \( \{1, \alpha, \beta, \omega\} \) as below. The bigrading is the Hodge grading shifted by \(-1/2\).

| \( \phi \) | \( \omega \) | \( \alpha \) | \( \beta \) |
|-----------|-----------|-----------|-----------|
| \( (\mu_{+}^\omega, \mu_{-}^\omega) \) | \( (-\frac{1}{2}, -\frac{1}{2}) \) | \( (\frac{1}{2}, -\frac{1}{2}) \) | \( (\frac{1}{2}, \frac{1}{2}) \) |
| \( |\phi| \) | \( 1 \) | \( 1 \) | \( -1 \) |

Let \( \Psi : H^*(\mathcal{E}, \mathbb{C}) \to \mathcal{H}_{W_d(J)} \) be a linear map defined by

\[
(4.7) \quad \Psi(1) = 1|J), \quad \Psi(\omega) = 1|J^{-1}), \quad \Psi(\alpha) = h(W_d)dx, \quad \Psi(\beta) = dx.
\]

Comparing Table 1 with Table 3, we see the isomorphism \( \Psi \) preserves bigrading and parity.

We use the ordered set of variables \( \{l_0, s_k, \tilde{s}_k, \bar{t}_k\} \) to parametrize both the descendent insertions

\[
\{1, \psi^k, \alpha \psi^k, \beta \psi^k, \omega \psi^k\},
\]

and the ancestor insertions

\[
\{1, \psi^k, \alpha \psi^k, \beta \psi^k, \omega \psi^k\}.
\]

Using our terminology, the Virasoro operators \( \{L_k^\mathcal{E}\}_{k \geq -1} \) for the target elliptic curve \( \mathcal{E} \) are given by

\[
(4.8) \quad L_k^\mathcal{E} = - (k + 1)! \frac{\partial}{\partial t_k^{k+1}} \ + \sum_{\ell \geq 0} \left( (\ell + 1)_{k+1} \frac{\partial}{\partial s_k^\ell} \ + \ (\ell + 1)_{k+1} \frac{\partial}{\partial \bar{t}_k^{k+1}} \right) \ + \sum_{\ell \geq 0} \left( (\ell + 1)_{k+1} \frac{\partial}{\partial \bar{t}_k^\ell} \ + \ (\ell + 1)_{k+1} \frac{\partial}{\partial s_k^{k+1}} \right).
\]

Similar as Proposition 0.10, the operators satisfy

\[
[\tilde{L}_n, \tilde{L}_m] = (n - m) \tilde{L}_{n+m}.
\]

In fact, under the linear isomorphism \( \Psi : H^*(\mathcal{E}, \mathbb{C}) \to \mathcal{H}_{W_d(J)} \) defined by (4.7), we can identify the two Virasoro operators

\[
L_k(t) = L_k^\mathcal{E}(\tilde{t})|_{\tilde{t} = t}.
\]
The Virasoro constraints for the (descendent) Gromov-Witten theory of the elliptic curve are solved by Okounkov and Pandharipande [52]. The following result is a special case of [52, Theorem 3], which is deduced from [52, Theorem 3] by summing over all degree $d$ in the absolute theory.

**Proposition 4.6.** [52] The Virasoro constraints for the absolute descendent Gromov-Witten theory of the elliptic curve $E$ hold:

\[(4.9) \quad L_k^E D^E = 0, \quad \text{for all } k \geq -1.\]

### 4.2.3. Virasoro constraints for ancestor GW theory of elliptic curves.

According to Theorem 4.5, the descendent potential $D^E$ does not depend on the choice of $t$. We restrict to the slice $t = t_0 \omega$, where the genus one potential $F_1^E(t_0)$ and the operator $S^E(t_0)$ can be computed explicitly for the elliptic curve $E$. We make a coordinate change

\[q := e^{t_0}.\]

From now on, we write

\[
\begin{align*}
F_1^E(q) &:= F_1^E(t_0)|_{t_0 = \log q}, \\
S^E(q) &:= S^E(t_0)|_{t_0 = \log q}.
\end{align*}
\]

Now we use Theorem 4.5 and Proposition 4.6 to prove

**Proposition 4.7.** The Virasoro constraints

\[(4.10) \quad L_k^E A_q^E = 0, \quad \forall k \geq -1,\]

holds for the ancestor GW theory of the elliptic curves.

**Proof.** We have

\[F_1^E(q) = -\log(q)_{\infty} = -\log \prod_{n=1}^{\infty} (1 - q^n).\]

Thus $\exp(F_1^E(q))$ commutes with the Virasoro operators

\[(4.11) \quad L_k^E = e^{-F_1^E(q)} L_k^E e^{F_1^E(q)}.\]

Next we consider the operator $S^E(q)$. Using the formula of $S_t^X$ in (4.6) and fixing a basis $(1, \omega, \alpha, \beta) \in H^*(E, \mathbb{C})$, we obtain that $\log S^E(q) \in \text{End}(H^*(E, \mathbb{C}))$ is given by the matrix

\[
\begin{pmatrix}
0 & 0 & 0 & 0 \\
\frac{q}{\bar{z}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

Thus the quadratic Hamiltonian for $\log S_\tau$ is given by

\[(4.13) \quad \Omega(\log S(q)f, f) = -q \left( \frac{q_0}{\bar{z}} \right)^2 - q \sum_{k \geq 0} q_{k+1} \bar{p}_k^0.\]

Now we calculate

\[S(q) \circ \mathcal{L}_k \circ S(q)^{-1}\]

and the cocycle that appears in the conjugation.

By (4.12), we have

\[S(q) \circ (\partial_z + z^{-1} \theta) \circ S(q)^{-1} = \partial_z + z^{-1} \theta.\]

Then (2.11) implies that,

\[S(q) \circ \mathcal{L}_k \circ S(q)^{-1} = \mathcal{L}_k.\]
Since the term (4.13) only contains \((q_0^0)^2\)-term but \(\tilde{L}_k\) does not have any \(p_ap_b\)-terms, all cocycles vanish by Lemma 2.2. We obtain that the quantization operator \(\tilde{S}(q)\) commutes with the Virasoro operators

\[
\tilde{S}(q)L_k^\xi \tilde{S}(q)^{-1} = L_k^\xi.
\]

Combining (4.11), we obtain

\[
L_k^\xi \mathcal{A}_\tau^\xi = \tilde{S}(q)L_k^\xi \tilde{S}(q)^{-1} \mathcal{A}_\tau^\xi
\]

\[
= \tilde{S}(q)e^{-\mathcal{F}_k^\xi(q)} L_k^\xi e^{\mathcal{F}_k^\xi(q)} \tilde{S}(q)^{-1} \mathcal{A}_\tau^\xi
\]

\[
= \tilde{S}(q)e^{-\mathcal{F}_k^\xi(q)} L_k^\xi \mathcal{D}_\tau^\xi
\]

\[
= 0.
\]

The last two equations follow from Theorem 4.5 and Proposition 4.6. □

4.2.4. Virasoro constraints for Fermat LG pairs \((W_d, \langle J \rangle)\). Now we consider the last step in the proof of Theorem 4.4 using LG/CY correspondence in [43]. We remark that the LG/CY correspondence does depend on the choice of the defining polynomial. The approach in [43] relates the GW theory and LG theory by holomorphic Cayley transformations of quasi-modular forms. Let us recall the construction and quasi-modular forms briefly [31, 62, 56].

Let \(\tau\) be the coordinate of the upper half plane. Let \(\Gamma := \text{PSL}(2, \mathbb{Z})\) be the modular group. Let \(E_{2k}(\tau)\) be the weight-2\(k\) Eisenstein series, and

\[
\tilde{M}(\Gamma) := \mathbb{C}[E_2(\tau), E_4(\tau), E_6(\tau)]
\]

be the ring of quasi-modular forms [31]. There is a natural ring isomorphism, called modular completion, from \(\tilde{M}(\Gamma)\) to

\[
\hat{M}(\Gamma) := \mathbb{C}[\hat{E}_2(\tau, \bar{\tau}), E_4(\tau), E_6(\tau)],
\]

the ring of almost holomorphic modular forms, by sending the ring generators \(E_4(\tau)\) and \(E_6(\tau)\) to themselves, and \(E_{2k}(\tau)\) to its modular completion

\[
\hat{E}_{2k}(\tau, \bar{\tau}) := E_{2k}(\tau) - \frac{3}{\pi \Im(\tau)} = E_{2k}(\tau) - \frac{6\sqrt{-1}}{\pi (\tau - \bar{\tau})}.
\]

Fixing a complex multiplication point \(\tau_s\) in the upper half plane, and a constant \(c \in \mathbb{C}\), there is a coordinate change, called Cayley transformation,

\[
s = c \cdot \frac{\tau - \tau_s}{\tau - \bar{\tau}_s}.
\]

(4.14)

This coordinate change induces a linear operator on \(\hat{M}(\Gamma)\), denoted by \(C_{\tau_s}\), which sends a weight 2\(k\) almost holomorphic modular form \(\hat{f}(\tau, \bar{\tau}) \in \hat{M}_{2k}(\Gamma)\), to analytic function of \(s\) and its complex conjugate \(\bar{s}\), denoted by

\[
C_{\tau_s}(\hat{f})(s, \bar{s}) = (2\pi \sqrt{-1}c)^{-k} \left( \frac{\tau(s) - \bar{\tau}_s}{\tau_s - \bar{\tau}_s} \right)^{2k} \hat{f}(\tau(s), \bar{\tau}(s)),
\]

Here \(\tau(s)\) is the inverse of (4.14), and the operator \(C_{\tau_s}\) is called the Cayley transformation on \(\hat{M}(\Gamma)\). The modular completion \(\hat{M}(\Gamma) \to \hat{M}(\Gamma)\) has an inverse, called a holomorphic limit, by taking \(\lim_{q \to \infty}\), or equivalently \(q \to 0\). A similar notion of holomorphic limit can be defined for \(C_{\tau_s}(f)(s, \bar{s})\), and one obtain a holomorphic Cayley transformation on \(\hat{M}(\Gamma)\) [62, 56], denoted by \(C_{\tau_s}^{\text{hol}}\), via the commutative diagram
For each $d = 3, 4, 6$, we make a coordinate change
\[ q = \exp\left(\frac{2\pi \sqrt{-1} \tau}{d}\right). \]

Recall that the ancestor GW functions are Fourier series of quasi-modular forms \[ [52, 43] \]
\[ \langle\langle \alpha_1 \psi_1^{f_1} \cdots \alpha_n \psi_n^{f_n} \rangle\rangle_{E_3}^{g,n}(\tilde{q}) \in \tilde{M}(\Gamma). \]

The following LG/CY correspondence is proved in \[ [43] \] for the Fermat cubic LG pair
\( W_3 = x_1^3 + x_2^3 + x_3^3, \langle J \rangle \).

**Theorem 4.8.** \[ [43, Theorem 1] \]
Let
\[ \Psi : H^*(E_3, \mathbb{C}) \to \mathcal{H}_{W_3,\langle J \rangle}. \]

be the linear isomorphism defined in (4.7). Let \( \tau_\ast = -\frac{\sqrt{-1}}{3} \exp\left(\frac{2\pi \sqrt{-1}}{3} \right) \). There exists a holomorphic Cayley transformation \( C_{\tau_\ast}^{\text{hol}} \), such that
\[ C_{\tau_\ast}^{\text{hol}}\left(\langle\langle \alpha_1 \psi_1^{f_1} \cdots \alpha_n \psi_n^{f_n} \rangle\rangle_{E_3}^{g,n}(q)\right) = \langle\langle \Psi(\alpha_1) \psi_1^{f_1} \cdots \Psi(\alpha_n) \psi_n^{f_n} \rangle\rangle_{F\text{JRW}}^{W_3,\langle J \rangle}(s). \]

Here \( s \) is the parameter of \( 1 | J^2 \rangle \in \mathcal{H}_{W_3,\langle J \rangle}. \) The holomorphic Cayley transformation sends the ancestor GW functions to Taylor series of \( s \), near \( s = 0 \), which is \( \tau = \tau_\ast \). The resulting Taylor series are exactly the corresponding ancestor FJRW functions. We can rewrite Theorem 4.8 in terms of ancestor potentials by
\[ (4.15) \]
\[ C_{\tau_\ast}^{\text{hol}}(A_q^{E_3}(\tilde{q})) = A_{W_3,\langle J \rangle}^{E_3}(q). \]

**A proof of Theorem 4.4:** We deal with the Fermat cubic pair \( (W_3, \langle J \rangle) \) first. By the construction of the holomorphic Cayley transformation, we see it commutes with the Virasoro operators
\[ L_k C_{\tau_\ast}^{\text{hol}} = C_{\tau_\ast}^{\text{hol}} L_k^{E_3}. \]

Now applying (4.15) and (4.10), we have
\[ L_k \left( A_{W_3,\langle J \rangle}^{E_3}(q) \right) = L_k C_{\tau_\ast}^{\text{hol}}(A_q^{E_3}(\tilde{q})) = C_{\tau_\ast}^{\text{hol}} L_k^{E_3}(A_q^{E_3}(\tilde{q})) = 0. \]

Next we discuss for the Fermat quartic and the Fermat sextic. In fact, for all three cases, let \( s \) be the parameter of \( 1 | J^{d-1} \rangle \in \mathcal{H}_{W_d,\langle J \rangle}. \) Then according to [43, Proposition 1], all the ancestor FJRW functions satisfy
\[ \langle\langle \prod_{i=1}^{k} \tau_{\ell_i}(\phi_i) \rangle\rangle_{W_d,\langle J \rangle}^{g,k}(s) \in \mathbb{C}[f, f', f'']. \]
where \( f'(s) = \frac{df}{ds}(s) \), and \( f(s) \) is the genus one FJRW function
\[
f(s) := \langle\langle 1| J^{d-1} \rangle\rangle_{1,1}^{W_d,J}(s).
\]
It satisfies a Chazy equation
\[
2f''' - 2f \cdot f'' + 3(f')^2 = 0
\]
and therefore is completely determined by the first three coefficients, two of which vanish by the construction of the moduli space.

The remaining coefficient of the cubic case \( (W_3, \langle J \rangle) \) is calculated in [42], which completely determines the holomorphic Cayley transformation \( \hat{c}_{\text{hol}}^{W_3} \). Now for the Fermat quartic \( W_4 \) and the Fermat sextic \( W_6 \), the value of the remaining coefficient is not known due to the complexity of the computation. There are two situations:

1. if the coefficient is nonzero, then it determines a holomorphic Cayley transformation as in Theorem 4.8.
2. if the coefficient vanishes, then we have a trivial holomorphic Cayley transformation.

In either case, Virasoro constraints hold.

4.2.5. Additional Virasoro constraints. Following the work of [52], we introduce additional differential operators for \( k \geq -1 \):
\[
D_k = -(k + 1)! \frac{\partial}{\partial s_{k+1}} + \sum_{\ell \geq 0} \left( \ell)_{k+1} t_0^0 \frac{\partial}{\partial s_{k+1}} + (\ell + 1)_{k+1} s_{k+1}^1 \frac{\partial}{\partial t_{k+1}} \right),
\]
\[
\bar{D}_k = -(k + 1)! \frac{\partial}{\partial s_{k+1}} + \sum_{\ell \geq 0} \left( \ell)_{k+1} t_0^0 \frac{\partial}{\partial s_{k+1}} - (\ell + 1)_{k+1} s_{k+1}^0 \frac{\partial}{\partial t_{k+1}} \right).
\]

It is proved that the descendent GW potential of the elliptic curves are annihilated by these differential operators [52, Theorem 4]. As an application, we show that the ancestor potential \( \mathcal{A}_{W_d,J}^{\text{FJRW}} \) are also annihilated by these operators.

**Theorem 4.9.** For LG pairs \( (W_d, \langle J \rangle) \), we have additional constraints
\[
D_k \mathcal{A}_{W_d,J}^{\text{FJRW}} = \bar{D}_k \mathcal{A}_{W_d,J}^{\text{FJRW}} = 0, \quad k \geq -1.
\]

**Proof.** By (4.13), we have
\[
(\log S(q)) = -q \frac{(q_0^0)^2}{2} - q \cdot q_{k+1} \frac{\partial}{\partial q_k}.
\]
Via dilaton shift \( q_0^1 = t_k^1 - \delta^{i,0} \delta_{k,1} \), it is easy to check
\[
[(\log S(q))^\wedge, D_k] = [(\log S(q))^\wedge, \bar{D}_k] = 0.
\]
Then
\[
\hat{S}(q)e^{-F(q)} D_k e^{F(q)} \hat{S}(q)^{-1} = \hat{S}(q) D_k \hat{S}(q)^{-1} = \exp(\text{Ad}_{(\log S(q))^\wedge}) (D_k) = D_k.
\]
The argument for \( \bar{D}_k \) is similar. The remaining proof is the same as that of Theorem 4.4. \( \square \)
4.3. Invertible CY polynomials of three variables. In this section, we discuss invertible Calabi-Yau polynomials of three variables. According to [49], up to permutations of variables, such a Calabi-Yau polynomial \( W(x_1, x_2, x_3) \) must be one of the forms listed below.

**Table 4. Invertible CY polynomials of three variables**

| \((q_1, q_2, q_3)\) | \(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\) | \(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\) | \(\frac{1}{6}, \frac{3}{4}, \frac{1}{2}\) |
|---|---|---|
| \(W\) | \(E_6^{(1,1)}\) | \(E_7^{(1,1)}\) | \(E_8^{(1,1)}\) |
| Fermat | \(x_1^3 + x_2^3 + x_3^3\) | \(x_1^3 + x_2^3 + x_3^3\) | \(x_1^3 + x_2^3 + x_3^3\) |
| Chain | \(x_1^2 x_2 + x_2^2 x_3 + x_3^2\) | \(x_1^3 x_2 + x_2^2 x_3 + x_3^2\) | \(x_1^3 x_2 + x_2^2 x_3 + x_3^2\) |
| Loop | \(x_1^3 x_2 + x_2^2 x_3 + x_3^2 x_1\) | \(x_1^3 x_2 + x_2^2 x_3 + x_3^2 x_1\) | \(x_1^3 x_2 + x_2^2 x_3 + x_3^2 x_1\) |
| Mixed | \(x_1^3 + x_2^3 x_3 + x_3^3\) | \(x_1^3 x_2 + x_2^2 x_3 + x_3^2\) | \(x_1^3 x_2 + x_2^2 x_3 + x_3^2\) |
| \(x_1^2 x_2 + x_1 x_2^2 + x_3^3\) | \(x_1^3 x_2 + x_2^2 x_3 + x_3^2\) | \(x_1^3 x_2 + x_2^2 x_3 + x_3^2\) |
| \(x_1^2 x_2 + x_1 x_2^2 + x_3^3\) | \(x_1^4 + x_2^2 x_3 + x_3^2\) | \(x_1^3 + x_2^3 x_3 + x_3\) |
| \(x_1^3 x_2 + x_2^2 x_3 + x_3^2\) | \(x_1^3 x_2 + x_2^2 x_3 + x_3^2\) | \(x_1^3 x_2 + x_2^2 x_3 + x_3^2\) |
| \(x_1^3 x_2 + x_2^2 x_3 + x_3^2\) | \(x_1^3 x_2 + x_2^2 x_3 + x_3^2\) | \(x_1^3 x_2 + x_2^2 x_3 + x_3^2\) |

Here \((q_1, q_2, q_3)\) are the weights of the variables. The notation \(E_{u-2}^{(1,1)}\) is introduced by Saito for the simple elliptic singularities [55], where \(\mu\) is the Milnor number of the singularity. In [2], they are denoted by \(P_8, X_9\), and \(J_{10}\).

For such a polynomial \(W(x_1, x_2, x_3)\), and an arbitrary group \(G\) such that \(\langle J \rangle < G < GW\), using Theorem 3.2 and Theorem 4.4, the Virasoro Conjecture 0.6 for the LG pairs \((W(x_1, x_2, x_3), G)\) of CY type will follow from the conjecture below.

**Conjecture 4.10.** Let \(W\) be an invertible Calabi-Yau polynomial of three variables. The LG A-model theory of an admissible LG pair \((W, G)\) is either generically semisimple, or equivalent to the LG A-model theory of a Fermat Calabi-Yau pair

\[
(W = x_1^{a_1} + x_2^{a_2} + x_3^{a_3}, \langle J_W \rangle).
\]

This conjecture can be checked case-by-case in two steps.

1. The bigraded superspace \(H_{W,G}\) is isomorphic to one of the five cases:
   \[
   \begin{cases}
   H_{W,G}, & W = x_1^{a_1} + x_2^{a_2} + x_3^{a_3}, \quad (a_1, a_2, a_3) = (3, 3, 3), (4, 4, 2), (6, 3, 2); \\
   H_{W,J_W}, & W = x_1^3 + x_2^3 + x_3^3; \\
   H_{W,J_{10}}, & W = x_1^4 + x_2^4.
   \end{cases}
   \]

2. In these cases, the ranks of the state spaces are 4, 6, 8, 9, 10 respectively. The first three cases are special cases in Theorem 0.8. The fourth case is discussed in Theorem 4.4. The last case is a special case in Proposition 0.9.

**Example 4.11.** For the quartic polynomial \(W = x_1^4 + x_2^4 + x_3^4\), we consider the following choices of admissible groups

\[
\begin{align*}
\langle J_{x_1^4 + x_2^4} \rangle & \times \langle J_{x_3^4} \rangle \\
\langle J_W \rangle & \longrightarrow GW \\
\text{SL}(3, \mathbb{C})
\end{align*}
\]

Here are some observations:
The state spaces $\mathcal{H}_{W,G}$ are isomorphic if $G = \langle J_W \rangle$ or $G = \text{SL}(3, \mathbb{C})$.

By [20, Theorem 4.1.8 (8)], the FJRW theory of the pair 

$$(W, G = \langle J_{x_1^4 + x_2^2} \rangle \times \langle J_{x_1^4 + x_2^2} \rangle)$$

is isomorphic to the FJRW theory of the pair 

$$(x_1^4 + x_2^4, \langle J_{x_1^4 + x_2^2} \rangle).$$

We call $(W, G)$ a pillowcase LG pair if the orbifold curve $(W = 0)/\tilde{G}$ on the Calabi-Yau side is the pillowcase $\mathbb{P}^1_{2,2,2,2}$ [18]. For example, the pair 

$$(W = x_1^4 + x_2^4 + x_3^2, G = \langle J_{x_1^4 + x_2^2} \rangle)$$

is a pillowcase LG pair. For pillowcase LG pairs, a stronger version of Conjecture 4.10 would be the LG A-model theories of all pillowcase LG pairs have isomorphic generically semisimple CohFTs.

**Appendix A. A super trace formula**

In this section, we prove Proposition 1.2. Our proof is a minor modification of the elegant argument in [15], where the cases of $G \subseteq \text{SL}(n, \mathbb{C})$ is proved. The condition $G \subseteq \text{SL}(n, \mathbb{C})$ is only used to define the “variance” $\text{Var}_{(f,G)}$. The variance is equivalent to the super trace $\text{Str}(\theta^2)$ if $G \subseteq \text{SL}(n, \mathbb{C})$. If we use $\text{Str}(\theta^2)$, the condition $G \subseteq \text{SL}(n, \mathbb{C})$ can be removed. We will explore their idea here.

Firstly, fix $W$, we define the Poincaré series of $\text{Jac}(W) \cdot dx$ as following:

$$(A.1) \quad P_W(y) := \sum_{\alpha \in \text{Jac}(W) \cdot \Omega} (-1)^{\text{wt}(\alpha)} \cdot y^\text{wt}(\alpha)$$

Recall $q_i$ is the weight of $x_i$ in $W$. The following is a standard formula due to A. G. Kouchnirenko [37] and J. Steenbrink [57],

**Lemma A.1.** The Poincaré series is

$$(A.2) \quad P_W(y) = \prod_{i=1}^{n} \frac{y - y^{q_i}}{1 - y^{q_i}}.$$

**Proof.** Let $A := \mathbb{C}[x_1, \ldots, x_n]$, and

$$K_j := \bigoplus_{|I| = j} A \cdot dx_I$$

be the $A$-module consist of $j$-differential form in $\mathbb{C}^n$. We consider the following Koszul complex:

$$(A.3) \quad 0 \rightarrow K_0 \xrightarrow{d_1} K_1 \xrightarrow{d_2} \cdots \xrightarrow{d_n} K_n \xrightarrow{\pi} \text{Jac}(W) \cdot dx \rightarrow 0$$

where $\pi$ is the projection, $d_i$ is the differential defined by

$$d_i := \sum_{i=1}^{n} \frac{\partial W}{\partial x_i} \cdot dx_i \wedge .$$

Recall that $W$ is a non-degenerate polynomial, with isolated critical points only at the origin, so the complex (A.3) is exact. Furthermore, assign $q_i$ as the degree of $x_i$ and $dx_i$, we know $d_j$ is an operator of degree 1 for all $j$. Similar as (A.1), we define the Poincaré series of $A$ and $K_j$:

$$P_A(y) = \prod_{i=1}^{n} \frac{1}{y^{q_i} - 1}.$$
\[ P_{K_j}(y) = P_A(y) \cdot \prod_{|I|=j \ i \in I} y^{q_i}. \]

Via the exactness of (A.3) and the fact that \( d_i \) is degree 1, we have
\[
P_W(y) = P_{K_n}(y) - y \cdot P_{K_{n-1}}(y) + \cdots (-1)^n y^n P_{K_0}(y)
= \prod_{i=1}^n \frac{y - y^{q_i}}{1 - y^{q_i}}.
\]

Now let \((W,G)\) be an admissible LG pair. For \( g \in G \), we write its diagonal action on \( \mathbb{C}^n \) by
\[ g : (x_1, \ldots, x_n) \mapsto (\lambda_1(g)x_1, \ldots, \lambda_n(g)x_n). \]
It induces a \( g \)-action on each \( \alpha \in \text{Jac}(W) \cdot dx \), denoted by
\[ g \cdot \alpha = \rho_g(\alpha) \cdot \alpha. \]
We define the Poincaré series of \( \text{Jac}(W) \cdot dx \) coupled with \( g \) as
\[ P_{W,g}(y) := \sum_{\alpha \in \text{Jac}(W) \cdot dx} (-1)^n \rho_g(\alpha) \cdot y^{\text{wt}(\alpha)}. \]

Notice that \( g \) also acts on \( A \) and \( K_j \) naturally, and \( d_j \) is \( g \)-invariants as \( W \) is \( g \)-invariant. Via a similar argument as above, we have
\[
P_{A,g}(y) = \prod_{i=1}^n \frac{1}{\lambda_i(g)y^{q_i} - 1},
\]
\[
P_{W,g}(y) = \prod_{i=1}^n \frac{y - \lambda_i(g)y^{q_i}}{1 - \lambda_i(g)y^{q_i}}.
\]
Then summing over all \( g \in G \), the following formula for the Poincaré series of the \( G \)-invariant part of \( \text{Jac}(W) \cdot dx \) holds [15, Theorem 6]:
\[ P_{W,G}(y) := \sum_{\alpha \in (\text{Jac}(W) \cdot dx)^G} (-1)^n \cdot y^{\text{wt}(\alpha)} = \frac{1}{|G|} \sum_{g \in G} \prod_{i=1}^n \frac{y - \lambda_i(g)y^{q_i}}{1 - \lambda_i(g)y^{q_i}}. \]

We define its Poincaré series by
\[ P_{W,G}(y) := \sum_{\phi_\alpha \in \mathcal{H}_{W,G}} (-1)^{\phi_\alpha} \cdot y^{\mu_\alpha}. \]

By definition of \( \mathcal{H}_{W,G} \), the bigrading and parity, we have
\[ P_{W,G}(y) = \sum_{\gamma \in G} y^{\gamma - \frac{2q}{2}} \cdot P_{W,G}(y) \]
\[ = \sum_{\gamma \in G} y^{\text{age}(\gamma) - \frac{2}{2}} \cdot \frac{1}{|G|} \sum_{g \in G} \prod_{1 \leq i \leq n, \lambda_i(\gamma) = 1} \frac{y - \lambda_i(g)y^{q_i}}{1 - \lambda_i(g)y^{q_i}} \]
\[ = \sum_{\gamma \in G} y^{\text{age}(\gamma) - \frac{a - N_0}{2}} \cdot \frac{1}{|G|} \sum_{g \in G} \prod_{1 \leq i \leq n, \lambda_i(\gamma) = 1} \frac{y^{\frac{1}{2}} - \lambda_i(g)y^{q_i} - \frac{1}{2}}{1 - \lambda_i(g)y^{q_i}}. \]
Notice that in the above equation, we use (A.5) and the fact that for all $\gamma \in G$, $W_\gamma$ is still a non-degenerate quasi-homogeneous polynomial \cite{15, Lemma 2.1.10} \cite{15, Proposition 5}. It is easily to see

**Lemma A.2.** Let $(W, G)$ be an admissible LG pair, we have

$$
\chi_{W,G} = \lim_{y \to 1} P_{(W,G)}(y),
$$

$$
\text{Str}(\theta^2) = \lim_{y \to 1} \frac{d}{dy} \left( y \frac{d}{dy} P_{(W,G)}(y) \right).
$$

Now it suffices to prove:

**Proposition A.3.** Let $(W, G)$ be an admissible LG pair, we have

$$
\lim_{y \to 1} \frac{d}{dy} \left( y \frac{d}{dy} P_{(W,G)}(y) \right) = \hat{c}_W \frac{12}{12} \cdot \lim_{y \to 1} P_{(W,G)}(y).
$$

In fact, from (A.6), we know $P_{(W,G)}(y)$ here coincides with $(-1)^n \chi(W, G)(y)$ defined in \cite{15, Theorem 19}. We omit the detail here, only mention that the proof there works for all $G \subseteq G_W$.

**Appendix B. Genus 0 modified Virasoro constraints**

As we see in Section 1.2, an obstruction in the proof of $L_0$-constraints is the grading assumption 1.4, of which the proof is lacking. However, in FJRW theory or KL theory, the following degree constrain holds:

**Lemma B.1.** For $\clubsuit=FJRW$ or $KL$, the quantum invariant

$$
\langle \prod_{i=1}^{k} \tau_{\ell_i}(\phi_i) \rangle_{\clubsuit,W,G}^{g,k} \neq 0
$$

only if

$$
\sum_{i=1}^{k} \left( \frac{\mu_+^a + \mu_-^a}{2} + \frac{\hat{c}_W}{2} + \ell_i \right) = (3 - \hat{c}_W)(g - 1) + k.
$$

**Proof.** $2\deg_{\mathbb{C}} \phi_i$ is the cohomological degree(after degree shift) of Lefschetz thimbles in FJRW theory \cite{20}, and that of intersection homology in KL theory \cite{33}. This lemma is a corollary of (0.7) and (1.14). \hfill \Box

Now we modify the genus 0 constraints as following(just replace $\mu_i^+$ with $\frac{\mu_i^+ + \mu_i^-}{2}$ in Definition B.2, and restrict it to genus 0 case)

**Definition B.2.** For each integer $k \in \mathbb{Z}_{\geq -1}$, we introduce a differential operator

$$
L_k^M := - \left( \frac{3 - \hat{c}_W}{2} \right)_{k+1} \frac{\partial}{\partial \theta^0_{k+1}}
$$

$$
+ \sum_{m=0}^{\infty} \left( \frac{\mu_+^a + \mu_-^a}{2} + m + \frac{1}{2} \right)_{k+1} \lambda_m \frac{\partial}{\partial \lambda^a_{m+k}}
$$

$$
+ \frac{\hbar^2}{2} \sum_{m=-k}^{-1} (-1)^m \left( \frac{\mu_+^a + \mu_-^a}{2} + m + \frac{1}{2} \right)_{k+1} \eta_{ab} \frac{\partial}{\partial \eta^{a}_{m-1}} \frac{\partial}{\partial \eta^{b}_{m+k}}
$$

$$
+ \frac{1}{2\hbar^2} \delta_{-1,k} \eta_{ab} \frac{\partial}{\partial \lambda^a_0} \frac{\partial}{\partial \lambda^b_0}
$$
\[-\frac{\delta_{0,k}}{4} \sum_a (-1)^{1/2} a \left( \frac{\mu_a^+ + \mu_a^-}{2} - \frac{1}{2} \right) (\mu_a^+ + \mu_a^- + 1/2).\]

Define the genus 0 potential as

\[
F_0^{\bullet}(t) := \sum_k \frac{1}{k!} \left( \prod_{i=1}^k t(\psi_i) \right)_{0,k}^{\bullet_{(W,G)}}.
\]

One can show that

\[
L_0^M \exp \left( \hbar^{-2} F_{0,JRW,KL}^{F} \right) \in \mathbb{C}[[\hbar]] \exp \left( \hbar^{-2} F_{0,JRW,KL}^{F} \right) \text{ via the proof in Section 1.2.}
\]

Furthermore, via the same argument in [47] and [22], we have

**Theorem B.3.**

\[
L_k^M \exp \left( \hbar^{-2} F_{0,JRW,KL}^{F} \right) \in \mathbb{C}[[\hbar]] \cdot \exp \left( \hbar^{-2} F_{0,JRW,KL}^{F} \right), \quad k \geq -1.
\]

Since the coefficients of \( \hbar^{-2} \) in \( L_k^M \exp \left( \hbar^{-2} F_{0,JRW,KL}^{F} \right) / \exp \left( \hbar^{-2} F_{0,JRW,KL}^{F} \right) \) vanishes, one can obtain a series of recursive relations for genus 0 invariants.

**Remark B.4.** In general, one can check \( L_k^M \) can not be a constraints of \( A^\bullet_{W,G} \) for higher genus. In fact, there does not exist a constant \( c \) such that \( L_0^M + c \) behave well, which means that, \( (L_0^M + c) A^\bullet_{W,G} = 0 \) and Virasoro relation (0.10) holds simultaneously. This is because of the absence of supertrace formula (1.6) in this case. See [22, Section 2.10] for this issue for Gromov-Witten theory.

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