Reducibility for a fast driven linear Klein-Gordon equation

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Abstract

We prove a reducibility result for a linear Klein-Gordon equation with a quasi-periodic driving on a compact interval with Dirichlet boundary conditions. No assumptions are made on the size of the driving, however we require it to be fast oscillating. In particular, provided that the external frequency is sufficiently large and chosen from a Cantor set of large measure, the original equation is conjugated to a time independent, diagonal one. We achieve this result in two steps. First, we perform a preliminary transformation, adapted to fast oscillating systems, which puts the original equation in a perturbative setting. Then we show that this new equation can be put to constant coefficients by applying a KAM reducibility scheme, whose convergence requires a new type of Melnikov conditions.

1 Introduction

We consider a linear Klein-Gordon equation with quasi-periodic driving

\[ \ddot{u} - \ddot{u} + \pi^2 u + V(\omega t, x)u = 0, \quad x \in [0, \pi], \quad t \in \mathbb{R}, \]

with spatial Dirichlet boundary conditions \( u(t, 0) = u(t, \pi) = 0 \).

The potential \( V : T^\nu \times [0, \pi] \to \mathbb{R} \), is quasi-periodic in time with a frequency vector \( \omega \in \mathbb{R}^\nu \setminus \{0\} \). The main feature of this driving is that it is not perturbative in size, but we require it to be fast oscillating, namely \( |\omega| \gg 1 \).

The goal of our paper is to provide, for any frequency \( \omega \) belonging to a Cantor set of large measure, a reducibility result for the system (1.1). That is, we construct a change of coordinates which conjugates equation (1.1) into a diagonal, time independent one.

As long as we know, this is the first result of reducibility in an infinite dimensional setting in which the perturbation is not assumed to be small in size, but only fast oscillating.

The proof is carried out in two steps, combining a preliminary transformation, adapted to fast oscillating systems, with a KAM reducibility scheme which completely removes the time dependence from the equation. In particular we first perform a change of coordinates, following [ADRHH17], that conjugates (1.1) to an equation with driving of size \( |\omega|^{-1} \), and thus perturbative in size. The price to pay is that the new equation might not fit in the standard KAM scheme developed by Kuksin in [Kuk87]. The problem is overcome in our model by exploiting the pseudodifferential properties of the operators involved, showing that the new perturbation features regularizing properties.

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The second key ingredient of the proof concerns appropriate balanced Melnikov conditions (see \cite{16}), which allow us to perform a convergent KAM reducibility iteration.

From a mathematical point of view, our result is part of the attempts to extend classical Floquet theory and its quasi-periodic generalization to infinite dimensional systems. While many progresses have been made in the last 20 years to prove non-perturbative reducibility for finite (and actually low) dimensional systems \cite{Eli01, Kri99, Kri01, Cha11, AFK11}, in the infinite dimensional case the only available results nowadays deal with systems which are small perturbations of a diagonal operator, i.e. of the form $D + \epsilon V(\omega t)$, where $D$ is diagonal, $\epsilon$ small and $\omega$ in some Cantor set. In this case the literature splits essentially in two parts: the first one dealing with the case of perturbations which are bounded operators $\epsilon \lesssim 1$, while the second one (of more recent interest) with unbounded ones $\epsilon \gg 1$. In particular, for the wave and Klein-Gordon equations, the papers \cite{EK09, GT11, GP16b, GP16a, WL17} are in the first group, while \cite{BG01, LY10, BBM14, FP15, Bam17, Bam18, BGMR18} are in the second group. In any case, all the previous results require a smallness assumption on the size of the perturbation.

In order to deal with perturbations periodic in time and fast oscillating, Abanin, De Roeck, Ho and Huveneers developed an adapted normal form \cite{ADRH16, ADRHH17a, ADRHH17b}, which generalizes the classical Magnus expansion \cite{Mag54}. Such a normal form, which from now on we call Magnus normal form, allows to extract a time independent Hamiltonian (usually called the effective Hamiltonian), which approximates well the dynamics up to some finite but very long times. In \cite{ADRHH17b}, the authors apply the Magnus normal form to the study of some quantum many-body systems (spin chains) with a fast periodic driving. Although the Magnus normal form was developed for periodic systems, we extend it here for quasi-periodic ones, and we use it as a preliminary transformation which moves the problem in a more favourable setting for starting a KAM reducibility scheme. However, we point out that an important difference between \cite{ADRHH17b} and our work lies in the fact that, while in \cite{ADRHH17b} all the involved operators are bounded, on the contrary our principal operator is an unbounded one.

In case of systems of the form $H_0 + V(t)$, where the perturbation $V(t)$ is neither small in size nor fast oscillating, a general reducibility is not known. However, in some cases it is possible to find some results of "almost reducibility": that is, the original Hamiltonian is conjugated to one of the form $H_0 + Z(t) + R(t)$, where $Z(t)$ commutes with $H_0$, while $R(t)$ is an arbitrary smoothing operator, see e.g. \cite{BGMR17}. This normal form ensures upper bounds on the speed of transfer of energy from low to high frequencies; e.g. it implies that the Sobolev norms of each solution grows at most as $t^\epsilon$ when $t \to \infty$, for any arbitrary small $\epsilon > 0$. This procedure (or a close variant of it), has been applied also in \cite{Del10, MR17, Mon17a}.

There are also examples in \cite{Bou99, Del14, Mas18}, where the authors engineer periodic drivings aimed to transfer energy from low to high frequencies and leading to unbounded growth of Sobolev norms (see also Remark \ref{rem:unbounded} below).

Finally, we want to mention also the papers \cite{BB08, CG17}, where KAM techniques are applied to construct quasi-periodic solutions with $|\omega| \gg 1$. In \cite{BB08} this is shown for a nonlinear wave equation with Dirichlet boundary conditions, however reducibility is not obtained. In \cite{CG17}, KAM techniques are applied to a many-body system with fast driving; the authors construct a periodic orbit with large frequency and prove its asymptotic stability.

Before closing this introduction, we mention that periodically driven systems have also a great interest in physics, both theoretically and experimentally. Indeed such systems often exhibit a rich and surprising behaviour. A first example, from classical mechanics, is the Kapitza pendulum \cite{Kap51}, where the fast periodic driving stabilizes the otherwise unstable equilibrium point in
which the pendulum is upside-down. Another example is the quasi-periodically kicked quantum rotor, where localization or spreading of the initial data depend on the nonresonant properties of the forcing frequency [FGP82, Com90]. More recently, a lot of attention has been dedicated to fast periodically driven many-body systems [JMC15, GD14, KBRD10, JMD`14, SSS14]. Here the interest is the possibility of engineering periodic drivings for realizing novel quantum states of matter; this procedure, commonly called “Floquet engineering” [BDP14], has been implemented in several physical systems, including cold atoms, graphenes and crystals.

1.1 Main result

The potential driving \( V(\omega t, x) \) is treated as a smooth function \( V : \mathbb{T}^\nu \times [0, \pi] \ni (\theta, x) \mapsto V(\theta, x) \in \mathbb{R}, \nu \geq 1 \), which satisfies two conditions:

(V1) The even extension in \( x \) of \( V(\theta, x) \) on the torus \( \mathbb{T}^\nu \approx [-\pi, \pi] \), which we still denote by \( V \), is smooth in both variables and it extends analytically in \( \theta \) in a proper complex neighbourhood of \( \mathbb{T}^\nu \) of width \( \rho \geq 0 \). In particular, for any \( \ell \in \mathbb{N} \), there is a constant \( C_{\ell, \rho} > 0 \) such that

\[
|\partial_x^{\ell} V(\theta, x)| \leq C_{\ell, \rho} \quad \forall x \in \mathbb{T}, \ |\Im \theta| \leq \rho ;
\]

(V2) \( \int_{\mathbb{T}^\nu} V(\theta, x) d\theta = 0 \) for any \( x \in [0, \pi] \).

To state precisely our main result, equation (1.1) has to be rewritten as an Hamiltonian system. We introduce the new variables

\[
\varphi := \frac{1}{2} B^{1/2} u + i B^{-1/2} \partial_t u , \quad \overline{\varphi} := \frac{1}{2} B^{1/2} u - i B^{-1/2} \partial_t u ,
\]

where

\[
B := \sqrt{-\Delta + m^2} ,
\]

so that equation (1.1) is equivalent to

\[
i \partial_t \varphi(t) = B \varphi(t) + \frac{1}{2} B^{-1/2} V(\omega t) B^{-1/2} (\varphi(t) + \overline{\varphi}(t)) .
\]

Taking (1.4) coupled with its complex conjugate, we obtain the following system

\[
i \partial_t \varphi(t) = H(t) \varphi(t) , \quad H(t) := \begin{pmatrix} B & 0 \\ 0 & -B \end{pmatrix} + \frac{1}{2} B^{-1/2} V(\omega t, x) B^{-1/2} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} ,
\]

where, abusing notation, we denoted \( \varphi(t) = \begin{pmatrix} \varphi(t) \\ \overline{\varphi}(t) \end{pmatrix} \) the vector with the components \( \varphi, \overline{\varphi} \). The phase space for (1.5) is \( \mathcal{H}^r \times \mathcal{H}^r \), where, for \( r \geq 0 \),

\[
\mathcal{H}^r := \left\{ \varphi(x) = \sum_{m \in \mathbb{N}} \varphi_m \sin(mx), \ x \in [0, \pi] \ \middle| \ ||\varphi||_{\mathcal{H}^r}^2 := \sum_{m \in \mathbb{N}} \langle m \rangle^{2r} |\varphi_m|^2 < \infty \right\} .
\]

Here we have used the notation \( \langle m \rangle := (1 + |m|^2)^{\frac{r}{2}} \), which will be kept throughout all the article. We define the \( \nu \)-dimensional annulus of size \( M \geq 0 \) by

\[
R_M := \mathbb{B}_M(0) \setminus \mathbb{B}_0(0) \subset \mathbb{R}^\nu .
\]
Theorem 1.1. Consider the system \((1.5)\) and assume (V1) and (V2). Fix arbitrary \(r, m \geq 0\) and \(\alpha \in (0, 1)\). Fix also an arbitrary \(\gamma_\alpha > 0\) sufficiently small.
Then there exist \(M_0 > 1\) and, for any \(M \geq M_0\), a subset \(\Omega^\alpha_x = \Omega^\alpha_x(M, \gamma_\alpha)\) in \(R^m\), fulfilling
\[
\frac{\text{meas}(R^m \setminus \Omega^\alpha_x)}{\text{meas}(R^m)} = O(\gamma_\alpha),
\]
such that the following holds true. For any frequency vector \(\omega \in \Omega^\alpha_x\), there exists an operator \(T(\omega t; \omega)\), bounded in \(L(H^r \times H^r)\), quasi-periodic in time and analytic in a shrunken neighbourhood of \(\mathbb{R}^r\) of width \(\rho/8\), such that the change of coordinates \(\varphi = T(\omega t; \omega)\psi\) conjugates \((1.5)\) to the diagonal time-independent system
\[
\dot{\psi}(t) = H^{x, \alpha} \psi(t), \quad H^{x, \alpha} := \begin{pmatrix} D^{x, \alpha} & 0 \\ 0 & -D^{x, \alpha} \end{pmatrix}, \quad D^{x, \alpha} = \text{diag} \left\{ \lambda_j^{x, \alpha} (\omega) \mid j \in \mathbb{N} \right\}.
\]

The transformation \(T(\omega t; \omega)\) is close to the identity, in the sense that there exists \(C_r > 0\) independent of \(M\) such that
\[
\| T(\omega t; \omega) - 1 \|_{H^r \times H^r} \leq C_r M^{\alpha - 1}.
\]
The new eigenvalues \((\lambda_j^{\alpha}(\omega))_{j \in \mathbb{N}}\) are real, Lipschitz in \(\omega\), and admit the following asymptotics for \(j \in \mathbb{N}\):
\[
\lambda_j^{\alpha}(\omega) = \lambda_j^{x, \alpha}(\omega, \alpha) + \varepsilon_j^{\alpha}(\omega, \alpha) + O \left( \frac{1}{H_j^\alpha} \right),
\]
where \(\lambda_j = \sqrt{j^2 + \Omega^\alpha}\) are the eigenvalues of the operator \(B\).

Remark 1.2. In particular, back to the original coordinates, equation \((1.1)\) is reduced to
\[
\partial_t^r u + (D^{x, \alpha})^2 u = 0.
\]

Remark 1.3. The parameter \(\alpha\), which one chooses and fixes in the real interval \((0, 1)\), influences the asymptotic expansion of the final eigenvalues, as one can read from \((1.10)\). Also the construction of the set of the admissible frequency vectors heavily depends on this parameter.

Let us denote by \(U_r(t, \tau)\) the propagator generated by \((1.5)\) such that \(U_r(\tau, \tau) = 1\), \(\forall \tau \in \mathbb{R}\).
An immediate consequence of Theorem 1.1 is that we have a Floquet decomposition:
\[
U_r(t, \tau) = T(\omega t; \omega)^* \circ e^{-i(t - \tau)H^{x, \alpha}} \circ T(\omega t; \omega).
\]

Another consequence of \((1.12)\) is that, for any \(r \geq 0\), the norm \(\|U_r(t, 0)\varphi_0\|_{H^r \times H^r}\) is bounded uniformly in time:

Corollary 1.4. Let \(M \geq M_0\) and \(\omega \in \Omega^\alpha_x\). For any \(r \geq 0\) one has
\[
\|\varphi_0\|_{H^r \times H^r} \leq \|U_r(t, 0)\varphi_0\|_{H^r \times H^r} \leq C_r \|\varphi_0\|_{H^r \times H^r}, \quad \forall t \in \mathbb{R}, \forall \varphi_0 \in H^r \times H^r,
\]
for some \(C_r > 0, C_r > 0\).
Moreover there exists a constant \(C_r^*\), s.t. if the initial data \(\varphi_0 \in H^r \times H^r\) then
\[
(1 - C_r^* M^{-\frac{\alpha}{2}}) \|\varphi_0\|_{H^r \times H^r} \leq \|U_r(t, 0)\varphi_0\|_{H^r \times H^r} \leq (1 + C_r^* M^{-\frac{\alpha}{2}}) \|\varphi_0\|_{H^r \times H^r}, \quad \forall t \in \mathbb{R}.
\]

Remark 1.5. Corollary 1.4 shows that, if the frequency \(\omega\) is chosen in the Cantor set \(\Omega^\alpha_x\), no phenomenon of growth of Sobolev norms can happen. On the contrary, if \(\omega\) is chosen resonant, one can construct drivings which provoke norm explosion with exponential rate, see [Bou99] (see also [Mas18] for other examples).
1.2 Scheme of the proof

Our proof splits into three different parts, which we now summarize.

The Magnus normal form. In Section 3 we perform a preliminary transformation, adapted to fast oscillating systems, which moves the non-perturbative equation \((1.5)\) into a perturbative one where the size of the transformed quasi-periodic potential is as small as large is the module of the frequency vector. Sketchily, we perform a change of coordinates which conjugates

\[
\begin{cases}
    H(t) = H_0 + W(\omega t) \\
    \text{"size}(W) \sim 1^n
\end{cases}
\]

\[
\Rightarrow
\begin{cases}
    \dot{H}(t) = H_0 + V(\omega t; \omega) \\
    \text{"size}(V) \sim |\omega|^{-1}n
\end{cases}
\]

This change of coordinates, called below Magnus normal form, is an extension to quasi-periodic systems of the one performed in [ADRHH17b].

As we already mentioned, the price to pay is that, in principle, it is not clear that the new perturbation is sufficiently regularizing to fit in a standard KAM scheme (see Remark 3.3 for a more detailed discussion).

Here it is essential to employ pseudodifferential calculus, thanks to which we control the order (as a pseudodifferential operator) of the new perturbation, and prove that it is actually enough regular for the KAM iteration. This is true because the principal term of the new perturbation is a commutator with \(H_0\) (see equation \((3.20)\)), and one can exploit the smoothing properties of the commutator of pseudodifferential operators (see \((2.8)\) in Remark 2.6).

Balanced Melnikov conditions. After the Magnus normal form, we perform a KAM reducibility scheme in order to remove the time dependence on the coefficients of the equation. As usual one needs second order Melnikov conditions on the unperturbed eigenvalues \(\lambda_j = \sqrt{j^2 + \pi^2}\).

The “standard” ones are

\[
|\omega \cdot k + \lambda_j \pm \lambda_l| \geq \frac{\gamma}{\langle k \rangle} \frac{\langle j \pm l \rangle}{|\omega|},
\]

for some \(\gamma, \tau > 0\), where we emphasized the dependence on the size of \(\omega\) in the r.h.s. of \((1.15)\). Such Melnikov conditions are useless in our context; indeed recall that, after the Magnus normal form, the new perturbation has size \(\sim |\omega|^{-1}\) while the small denominators in \((1.15)\) have size \(\sim |\omega|\); so the two of them compensate each others, and the KAM step cannot reduce in size.

To overcome the problem, rather than \((1.15)\), we impose new balanced Melnikov conditions, in which we balance the loss in size (in the denominator) and gain in regularity (in the numerator) in \((1.15)\). More precisely, we show that for any \(\alpha \in [0, 1]\) one can impose

\[
|\omega \cdot k + \lambda_j \pm \lambda_l| \geq \frac{\gamma}{\langle k \rangle} \frac{\langle j \pm l \rangle^\alpha}{|\omega|^{\alpha}},
\]

for a set of \(\omega\)’s in \(R_M\) of large relative measure. This is proved in Section 4. By choosing \(0 < \alpha < 1\), the l.h.s. of \((1.16)\) is larger than the corresponding one in \((1.15)\), and the KAM transformation reduces in size. However note that the choice of \(\alpha\) will influence the regularizing effect given by \(\langle j \pm l \rangle^\alpha\) in the r.h.s. of \((1.16)\); ultimately, this modifies the asymptotic expansion of the final eigenvalues, as one can see in \((1.10)\).

The KAM reducibility. At this point we perform a KAM reducibility scheme. The functional setting that we have decided to employ for this part is introduced in Section 2.2; it consists on a family of norms for \(2 \times 2\) matrix of operators with different smoothing orders on the diagonal.
and anti-diagonal elements controlled by the $s$-decay norm.

Section 5 is devoted to prove Theorem 5.13 for the KAM reducibility.

A final note. In order to focus on the main problem, which is to deal with perturbations large in size and fast oscillating, we decided to eliminate some technical complications which can be addressed with the modern techniques. For example we decided to use Dirichlet boundary conditions (rather than periodic) and perturbations analytic in time and smooth in space (rather than Sobolev in both variables). These restrictions can be eliminated using e.g. the techniques in [Mon17b].

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2 Functional settings

Given a set $\Omega \subset \mathbb{R}^\nu$ and a Fréchet space $\mathcal{F}$, the latter endowed with a system of seminorms $\{\|\cdot\|_n \mid n \in \mathbb{N}\}$, we define for a function $f : \Omega \ni \omega \mapsto f(\omega) \in \mathcal{F}$ the quantities

$$
|f|_{n, \Omega}^\infty := \sup_{\omega \in \Omega} \|f(\omega)\|_n, \quad |f|_{n, \Omega}^{\text{Lip}} := \sup_{\omega_1, \omega_2 \in \Omega, \omega_1 \neq \omega_2} \frac{\|f(\omega_1) - f(\omega_2)\|_n}{|\omega_1 - \omega_2|}.
$$

Given $w \in \mathbb{R}_+$, we denote by $\text{Lip}_w(\Omega, \mathcal{F})$ the space of functions from $\Omega$ into $\mathcal{F}$ such that

$$
|f|_{n, \Omega}^{\text{Lip}(w)} := |f|_{n, \Omega}^\infty + w |f|_{n, \Omega}^{\text{Lip}} < \infty.
$$

**Remark 2.1.** If $\mathcal{F}$ is a Fréchet algebra$^1$ so is $\text{Lip}_w(\Omega, \mathcal{F})$. Moreover, for any $j \in \mathbb{N}$, there exist $N \geq j$, $C > 0$ such that

$$
\|fg\|_{n, \Omega}^{\text{Lip}(a)} \leq C \|f\|_{N, \Omega}^{\text{Lip}(a)} \|g\|_{N, \Omega}^{\text{Lip}(a)}.
$$

2.1 Pseudodifferential operators

The main tool for the construction of the Magnus transform in Section 3 is the calculus with pseudodifferential operators acting on the scale of the standard Sobolev spaces on the torus $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$, which is defined for any $r \in \mathbb{R}$ as

$$
H^r(\mathbb{T}) := \left\{ \varphi(x) = \sum_{j \in \mathbb{Z}} \varphi_j e^{ijx}, x \in \mathbb{T} \left| \|\varphi\|_{H^r(\mathbb{T})}^2 := \sum_{j \in \mathbb{Z}} (j)^{2r} |\varphi_j|^2 < \infty \right. \right\}.
$$

For a function $f : \mathbb{T} \times \mathbb{Z} \to \mathbb{R}$, define the difference operator $\Delta f(x, j) := f(x, j + 1) - f(x, j)$ and let $\Delta^\beta = \Delta \circ \cdots \circ \Delta$ be the composition $\beta$ times of $\Delta$. Then, we have the following:

$^1$we recall that $\mathcal{F}$ is a Fréchet algebra if the multiplication operator $\mathcal{F} \times \mathcal{F} \ni (a, b) \mapsto a \cdot b \in \mathcal{F}$ is continuous in the topology induced by the seminorms. Equivalently, for any $j \in \mathbb{N}$, there exist $N \geq n, C > 0$ s.t.

$$
\|a \cdot b\|_n \leq C \|a\|_N \|b\|_N, \quad \forall a, b \in \mathcal{F}.
$$
Definition 2.2. We say that a function \( f : \mathbb{T} \times \mathbb{Z} \to \mathbb{R} \) is a symbol of order \( m \in \mathbb{R} \) if for any \( j \in \mathbb{Z} \) the map \( x \mapsto f(x,j) \) is smooth and, furthermore, for any \( \alpha, \beta \in \mathbb{N} \), there exists \( C_{\alpha,\beta} > 0 \) such that
\[
|\partial_x^{\alpha} \partial_x^{\beta} f(x,j)| \leq C_{\alpha,\beta} (j)^{m-\beta}, \quad \forall x \in \mathbb{T}.
\]
If this is the case, we write \( f \in \mathcal{S}^m \).

We endow \( \mathcal{S}^m \) with the family of seminorms
\[
\varphi_{\ell}^m(f) := \sum_{\alpha+\beta \leq \ell} \sup_{(x,j) \in \mathbb{T} \times \mathbb{Z}} |\langle j \rangle^{-m+\beta} |\partial_x^{\alpha} \partial_x^{\beta} f(x,j)|, \quad \ell \in \mathbb{N}_0.
\]

Analytic families of pseudodifferential operators. We will consider in our discussion also symbols depending real analytically on the variable \( \theta \in \mathbb{T}^\nu \). To define them, we need to introduce the complex neighbourhood of the torus
\[
\mathbb{T}_\rho := \{ a + ib \in \mathbb{C}^\nu \mid a \in \mathbb{T}^\nu, |b| \leq \rho \}.
\]

Definition 2.3. Given \( m \in \mathbb{R} \) and \( \rho > 0 \), a function \( f : \mathbb{T}^\nu \times \mathbb{T} \times \mathbb{Z} \to \mathbb{R}, (\theta, x, j) \mapsto f(\theta, x, j) \), is called a symbol of class \( S^m_\rho \) if for any \( j \in \mathbb{N} \) it is smooth in \( x \), it extends analytically in \( \theta \) in \( \mathbb{T}^\nu \) and, furthermore, for every \( \alpha, \beta \in \mathbb{N} \) there exists \( C_{\alpha,\beta} > 0 \) such that
\[
|\partial_x^{\alpha} \partial_x^{\beta} f(\theta, x, j)| \leq C_{\alpha,\beta} (j)^{m-\beta} \quad \forall x \in \mathbb{T}, \forall \theta \in \mathbb{C}^\nu, |\text{Im} \theta| \leq \rho.
\]
For such a function we write \( f \in \mathcal{S}^m_\rho \).

We endow \( \mathcal{S}^m_\rho \) with the family of seminorms
\[
\varphi_{\ell}^{m,\rho}(f) := \sup_{|\text{Im} \theta| \leq \rho} \sum_{\alpha+\beta \leq \ell} \sup_{(x,j) \in \mathbb{T} \times \mathbb{Z}} |\langle j \rangle^{-m+\beta} |\partial_x^{\alpha} \partial_x^{\beta} f(\theta, x, j)|, \quad \ell \in \mathbb{N}_0.
\]
We associate to a symbol \( f \in \mathcal{S}^m_\rho \) the operator \( f(\theta, x, D_x) \) by standard quantization
\[
(2.5) \quad \psi(x) = \sum_{j \in \mathbb{Z}} \psi_j e^{ijx} \mapsto (f(\theta, x, D_x)\psi)(x) := \sum_{j \in \mathbb{Z}} f(\theta, x, j) \psi_j e^{ijx};
\]
here \( D_x = D := i^{-1} \partial_x \) is the Hörmander derivative.

Definition 2.4. We say that \( F \in \mathcal{A}^m \) if it is a pseudodifferential operator with symbol of class \( \mathcal{S}^m_\rho \), i.e. if there exists a symbol \( f \in \mathcal{S}^m_\rho \) such that \( F = f(\theta, x, D_x) \).

If \( F \) does not depend on \( \theta \), we simply write \( F \in \mathcal{A}^m \).

Remark 2.5. The operator \( \langle D \rangle := (1 - \partial_{xx})^2 \) is a pseudodifferential operator in \( \mathcal{A}^1 \), being the quantization of the symbol \( \langle j \rangle \in \mathcal{S}^1 \). Similarly, for any \( \sigma \in \mathbb{R} \), the operator \( \langle D \rangle^\sigma = (1 - \partial_{xx})^\frac{\sigma}{2} \) is a pseudodifferential operator in \( \mathcal{A}^\sigma \).

As usual we give to \( \mathcal{A}^m_\rho \) a Fréchet structure by endowing it with the seminorms of the symbols.

Remark 2.6. It is standard that \( F \in \mathcal{A}^m_\rho \), \( G \in \mathcal{A}^n_\rho \) implies that \( F(\theta) \in \mathcal{L}(H^r(\mathbb{T}), H^{r-n}(\mathbb{T})) \) for any \( \theta \in \mathbb{T}^\nu \) and \( r \in \mathbb{R} \), \( FG \in \mathcal{A}^{m+n}_\rho + \mathbb{N} \), \( [F, G] \in \mathcal{A}^{m+n-1}_\rho \). Moreover
\[
(2.6) \quad \forall r \exists N = N(m, r) \text{ s.t. } \sup_{\theta \in \mathbb{T}^\nu} \|F(\theta)\|_{\mathcal{L}(H^r(\mathbb{T}), H^{r-n}(\mathbb{T}))} \leq C_1 \varphi_{N,\rho}(F),
\]
\[
(2.7) \quad \forall j \exists N = N(m, n, j) \text{ s.t. } \varphi_{j,\rho}^{m+n}(FG) \leq C_2 \varphi_{j,\rho}^{m}(F) \varphi_{j,\rho}^{n}(G),
\]
\[
(2.8) \quad \forall j \exists N = N(m, n, j) \text{ s.t. } \varphi_{j,\rho}^{m+n-1}(FG) \leq C_3 \varphi_{j,\rho}^{m}(F) \varphi_{j,\rho}^{n}(G),
\]
for some positive constants \( C_1(m, r), C_2(m, n, j), C_3(m, n, j) \).
Finally we define the class of pseudodifferential operators depending on a Lipschitz way on an external parameter.

**Definition 2.7.** We denote by $\text{Lip}_p(\Omega, A^m_\rho)$ the space of pseudodifferential operators whose symbols belong to $\text{Lip}_p(\Omega, S^m_\rho)$ and by $\left(\varphi^{n,\rho}_j(\cdot)_{\Omega}\right)_{j \in \mathbb{N}}$ the corresponding seminorms.

**Remark 2.8.** Let $F \in \text{Lip}_p(\Omega, A^m_\rho)$ and $G \in \text{Lip}_p(\Omega, A^m_\rho)$. Then (2.7) and (2.8) imply that $FG \in \text{Lip}_p(\Omega, A^{m+n}_\rho)$ and $[F, G] \in \text{Lip}_p(\Omega, A^{m+n-1}_\rho)$, with the quantitative bounds
\[
\forall j \exists N \text{ s.t. } \varphi_j^{m+n,\rho}(FG)_{\Omega}^{\text{Lip}(\varphi)} \leq C_4 \varphi_j^{m,\rho}(F)_{\Omega}^{\text{Lip}(\varphi)} \varphi_j^{n,\rho}(G)_{\Omega}^{\text{Lip}(\varphi)},
\]
\[
\forall j \exists N \text{ s.t. } \varphi_j^{m+n-1,\rho}(F, G)_{\Omega}^{\text{Lip}(\varphi)} \leq C_5 \varphi_j^{m,\rho}(F)_{\Omega}^{\text{Lip}(\varphi)} \varphi_j^{n,\rho}(G)_{\Omega}^{\text{Lip}(\varphi)}.
\]

**Parity preserving operators.** The space $\mathcal{H}^0$ of (1.6) is naturally identified with the subspace of $\mathcal{H}^\nu(\mathbb{T}) \equiv L^2(\mathbb{T})$ of odd functions. Therefore it makes sense to work with pseudodifferential operators preserving the parity. Before describing them, we recall the orthogonal decomposition of the periodic $L^2$-functions on $\mathbb{T}$:
\[
L^2(\mathbb{T}) = L^2_{\text{even}}(\mathbb{T}) \oplus L^2_{\text{odd}}(\mathbb{T})
\]
where, for $u(x) = \sum_{j \in \mathbb{Z}} u_j e^{ijx} \in L^2(\mathbb{T})$, we have for any $j \in \mathbb{Z}$,
\[
(2.9) \quad u \in L^2_{\text{even}}(\mathbb{T}) \iff u_{-j} = u_j \quad \text{and} \quad u \in L^2_{\text{odd}}(\mathbb{T}) \iff u_{-j} = -u_j.
\]

**Definition 2.9.** We denote by $\mathcal{P}S^m_\rho$ the class of symbols $f \in S^m_\rho$ satisfying the property
\[
(2.10) \quad f(\theta, x, j) = f(\theta, -x, -j) \quad \forall \theta \in \mathbb{T}^\nu, \quad x \in \mathbb{T}, \quad j \in \mathbb{Z}.
\]
We denote by $\mathcal{P}A^m_\rho$ the subset of $A^m_\rho$ of parity preserving operators, that is, those operators $A \in A^m_\rho$ such that $A(L^2_{\text{even}}) \subseteq L^2_{\text{even}}$ and $A(L^2_{\text{odd}}) \subseteq L^2_{\text{odd}}$.

**Lemma 2.10.** Let $F \in A^m_\rho$ with symbol $f \in S^m_\rho$. Then $F \in \mathcal{P}A^m_\rho$ if and only if $f \in \mathcal{P}S^m_\rho$.

**Proof.** It is easy to check that $F(L^2_{\text{odd}}(\mathbb{T})) \subseteq L^2_{\text{odd}}(\mathbb{T})$ if and only if the symbol $f(x, j)$ of $F$ fulfills $\text{Im}[(f(x, j) - f(-x, -j)) e^{ijx}] = 0$. Similarly $F(L^2_{\text{even}}(\mathbb{T})) \subseteq L^2_{\text{even}}(\mathbb{T})$ if and only if $\text{Re}[(f(x, j) - f(-x, -j)) e^{ijx}] = 0$.

**Remark 2.11.** The operator $(D)_{\nu}^\sigma \in \mathcal{P}A^\nu$, while, by the assumption (V1), $V(\theta, x) \in \mathcal{P}S^0_\rho$ and so the corresponding operator in $\mathcal{P}A^0_\rho$.

**Remark 2.12.** Parity preserving operators are closed under composition and commutators.

### 2.2 Matrix representation and operator matrices

For the KAM reducibility, a second and wider class of operators without a pseudodifferential structure is needed on the scale of Hilbert spaces $(\mathcal{H}^\nu)_{\nu \in \mathbb{R}}$, as defined as in (1.6). Moreover, let $\mathcal{H}^{\nu+} := \bigcap_{\nu \in \mathbb{R}} \mathcal{H}^\nu$ and $\mathcal{H}^{\nu-} := \bigcup_{\nu \in \mathbb{R}} \mathcal{H}^\nu$.

If $A$ is a linear operator with domain $D(A) \supseteq \mathcal{H}^\nu$, we denote by $A^*$ the adjoint of $A$ with respect to the scalar product of $\mathcal{H}^\nu$, while we denote by $\overline{A}$ the conjugate operator:
\[
\overline{A} \psi := \overline{A^* \psi}, \quad \forall \psi \in D(A).
\]
Matrix representation of operators. To any linear operator \( A : \mathcal{H}^\infty \rightarrow \mathcal{H}^{-\infty} \) we associate its matrix of coefficients \( (A_m^n)_{m,n \in \mathbb{N}} \) on the basis \( (\hat{e}_n := \sin(nx))_{n \in \mathbb{N}} \), defined for \( m,n \in \mathbb{N} \) as

\[
A_m^n = \langle A\hat{e}_m, \hat{e}_n \rangle_{\mathcal{H}}.
\]

**Remark 2.13.** If \( A \) is a bounded operator, the following implications hold:

\[
A = A^* \iff A_m^n = A_{m}^n \forall m,n \in \mathbb{N} ;
\]

\[
A = A^* \iff A_m^n = A_n^m \forall m,n \in \mathbb{N} .
\]

A useful norm we can put on the space of such operators is in the following:

**Definition 2.14.** Given a linear operator \( A : \mathcal{H}^\infty \rightarrow \mathcal{H}^{-\infty} \) and \( s \in \mathbb{R} \), we say that \( A \) has finite \( s \)-decay norm provided

\[
|A|_s := \left( \sum_{h \in \mathbb{N}_0} (\langle h \rangle)^{2s} \sup_{|m-n|=h} |A_{m}^n|^2 \right)^{1/2} < \infty .
\]

One has the following:

**Lemma 2.15 (Algebra of the \( s \)-decay).** For any \( s > \frac{1}{2} \) there is a constant \( C_s > 0 \) such that

\[
|AB|_s \leq C_s |A|_s |B|_s .
\]

The proof of the Lemma is an easy variant of the one in [BBP14] we sketch it in Appendix A.3.

**Remark 2.16.** If \( A : \mathcal{H}^\infty \rightarrow \mathcal{H}^{-\infty} \) has finite \( s \)-decay norm with \( s > \frac{1}{2} \), then for any \( r \in [0,s] \), \( A \) extends to a bounded operator \( \mathcal{H}^r \rightarrow \mathcal{H}^r \). Moreover, by tame estimates, one has the quantitative bound \( \|A\|_{L(\mathcal{H}^r)} \leq C_{r,s}|A|_s \).

Next, we consider operators depending analytically on angles \( \theta \in \mathbb{T}^r \).

**Definition 2.17.** Let \( A \) be a \( \theta \)-depending operator, \( A : \mathbb{T}^r \rightarrow \mathcal{L}(\mathcal{H}^\infty, \mathcal{H}^{-\infty}) \). Given \( s \geq 0 \) and \( \rho > 0 \), we say that \( A \in \mathcal{M}_{\rho,s} \) if one has

\[
|A|_{\rho,s} := \sum_{k \in \mathbb{Z}^r} e^{\rho|k|} \left| \hat{A}(k) \right|_s < \infty ;
\]

here \( \hat{A}(k) \) is the operator obtained as the \( k^{th} \) Fourier coefficient of \( A(\theta) \):

\[
\hat{A}(k) := \frac{1}{(2\pi)^\nu} \int_{\mathbb{T}^r} A(\theta) e^{-i k \cdot \theta} d\theta .
\]

**Remark 2.18.** If \( A \) is a \( \theta \)-depending bounded operator, the following implications hold:

\[
A = A^* \iff \hat{A}(k)^* = \bar{\hat{A}}(-k) \forall k \in \mathbb{Z}^r \iff \hat{A}_m^n(k) = \overline{\hat{A}_n^m(-k)} \forall k \in \mathbb{Z}^r , \forall m,n \in \mathbb{N}
\]

\[
A = A^* \iff \hat{A}(k)^* = \bar{\hat{A}}(-k) \forall k \in \mathbb{Z}^r \iff \hat{A}_m^n(k) = \overline{\hat{A}_n^m(-k)} \forall k \in \mathbb{Z}^r , \forall m,n \in \mathbb{N}
\]

If \( \Omega \ni \omega \mapsto A(\omega) \in \mathcal{M}_{\rho,s} \) is a Lipschitz map, we write \( A \in \text{Lip}_p(\Omega, \mathcal{M}_{\rho,s}) \), provided

\[
|A|^{\text{Lip}(\rho_s)} := \sup_{\omega \in \Omega} |A(\omega)|_{\rho,s} + \sup_{\omega_1 \neq \omega_2 \in \Omega} \frac{|A(\omega_1) - A(\omega_2)|_{\rho,s}}{|\omega_1 - \omega_2|} < \infty .
\]

**Remark 2.19.** For any \( s > \frac{1}{2} \) and \( \rho > 0 \), the spaces \( \mathcal{M}_{\rho,s} \) and \( \text{Lip}_p(\Omega, \mathcal{M}_{\rho,s}) \) are closed with respect to composition, with

\[
|AB|_{\rho,s} \leq C_s |A|_{\rho,s} |B|_{\rho,s} ;
\]

\[
|AB|^{\text{Lip}(\rho_s)} \leq C_s |A|^{\text{Lip}(\rho_s)} |B|^{\text{Lip}(\rho_s)} .
\]

This follows from Lemma 2.15 Remark 2.1 and the algebra properties for analytic functions.
Operator matrices. We are going to meet matrices of operators of the form

\[ A = \begin{pmatrix} A^d & A^o \\ -A^o & -A^d \end{pmatrix}, \]

where \( A^d \) and \( A^o \) are linear operators belonging to the class \( M_{\rho,s} \). Actually, the operator \( A^d \) on the diagonal will have different decay properties than the element on the anti-diagonal \( A^o \). Therefore, we introduce classes of operator matrices in which we keep track of these differences.

**Definition 2.20.** Given an operator matrix \( A \) of the form (2.16), \( \alpha, \beta \in \mathbb{R}, \rho > 0, s \geq 0 \), we say that \( A \) belongs to \( M_{\rho,s}(\alpha, \beta) \) if

\[ [A^d]^* = A^d, \quad [A^o]^* = A^o \]

and one also has

\[ (2.17) \quad \langle D \rangle^\alpha A^d, A^d \langle D \rangle^\alpha \in M_{\rho,s} \],

\[ (2.18) \quad \langle D \rangle^\beta A^o, A^o \langle D \rangle^\beta \in M_{\rho,s} \],

\[ (2.19) \quad \langle D \rangle^\sigma A^d \langle D \rangle^{-\sigma} \in M_{\rho,s}, \quad \forall \sigma \in \{\pm \alpha, \pm \beta, 0\}, \quad \forall \delta \in \{d, o\}. \]

We endow \( M_{\rho,s}(\alpha, \beta) \) with the norm

\[ |A|_{\rho,s}^{\alpha,\beta} := \left| \langle D \rangle^\alpha A^d \right|_{\rho,s} + \left| \langle D \rangle^\beta A^o \right|_{\rho,s} + \left| \langle D \rangle^\delta A^d \langle D \rangle^{-\delta} \right|_{\rho,s} \]

\[ + \sum_{\sigma \in \{\pm \alpha, \pm \beta, 0\}} \left| \langle D \rangle^\sigma A^d \langle D \rangle^{-\sigma} \right|_{\rho,s}, \tag{2.21} \]

with the convention that, in case of repetition (when \( \alpha = \beta, \alpha = 0 \) or \( \beta = 0 \)), the same terms are not summed twice. When \( A \) is independent of \( \theta \in \mathbb{T}^n \), we use the norm \( |A|_{\rho,s}^{\alpha,\beta} \), defined as (2.21), but replacing \(|\cdot|_{\rho,s}\) with the \( s \)-decay norm \(|\cdot|_s\) defined in (2.11).

Let us motivate the properties describing the class \( M_{\rho,s}(\alpha, \beta) \):

- **Condition (2.17)** is equivalent to ask that \( A \) is the Hamiltonian vector field of a real valued quadratic Hamiltonian, see e.g. [Mon17b] for a discussion;

- **Conditions (2.18) and (2.19)** control the decay properties for the coefficient of the coefficients of the matrices associated to \( A^d \) and \( A^o \): indeed the matrix coefficients of \( \langle D \rangle^\alpha A \langle D \rangle^\beta \) are given by

\[ \left[ \langle D \rangle^\alpha A \langle D \rangle^\beta \right]_m^n (k) = \langle m |^\alpha \hat{A}^o_m (k) \langle n |^\beta, \]

therefore decay (or growth) properties for the matrix coefficients of the operator \( A \) are implied by the boundedness of the norms \(|\cdot|_{\rho,s}\);

- **Condition (2.20)** is added for a technical reason, simplifying greatly the computations below.

**Remark 2.21.** Let \( 0 < \rho' \leq \rho, 0 \leq \rho \leq s \leq s' \equiv \alpha', \beta \geq \beta' \). Then \( M_{\rho,s}(\alpha, \beta) \subseteq M_{\rho',s'}(\alpha', \beta') \) with the quantitative bound \(|A|_{\rho',s'}^{\alpha',\beta'} \leq |A|_{\rho,s}^{\alpha,\beta} \).
Remark 2.22. By Definition \([2.17]\) the norms in \([2.21]\) admit the following more operative reformulation:

\[
\tag{2.22} \left| A_{p,s}^{\alpha,\beta} \right| = \sum_{k \in \mathbb{Z}^n} e^{i\beta k} \left| \widehat{A}(k) \right|^\alpha_s \theta , \quad \widehat{A}(k) := \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} A(\theta) e^{-i\theta k} \, d\theta .
\]

Finally, if \(A^\alpha(\omega)\) and \(A^\beta(\omega)\) depend in a Lipschitz way on a parameter \(\omega\), we introduce the Lipschitz norm

\[
\tag{2.23} \left| A_{\rho,s}^{\alpha,\beta}(\omega) \right| := \sup_{\omega \in \Omega} \left| A(\omega) \right|^{\alpha,\beta}_{\rho,s} + \sup_{\omega_1 \neq \omega_2 \in \Omega} \frac{|A(\omega_1) - A(\omega_2)|^{\alpha,\beta}_{\rho,s}}{|\omega_1 - \omega_2|} .
\]

If such a norm is finite, we write \(A \in \text{Lip}_{\rho,s}^{\alpha}(\Omega, \mathcal{M}_{\rho,s}(\alpha, \beta))\).

Embedding of parity preserving pseudodifferential operators. The introduction of the classes \(\mathcal{M}_{\rho,s}(\alpha, \beta)\) is due to the fact that they are closed with respect the KAM reducibility scheme, for a proper choice of \(\alpha\) and \(\beta\). In the next lemma we show how parity preserving pseudodifferential operators embed in such classes.

Lemma 2.23 (Embedding). Given \(\alpha, \beta, \rho > 0\), consider \(F \in \mathcal{PA}_{\rho}^{\alpha}\) and \(G \in \mathcal{PA}_{\rho}^{\beta}\). Assume that

\[
F^* = F , \quad G^* = \bar{G} ,
\]

(where the adjoint is with respect to the scalar product of \(\mathcal{H}^0\)). Define the operator matrix

\[
\tag{2.24} A := \begin{pmatrix} F & G \\ -\bar{G} & -\bar{F} \end{pmatrix} .
\]

Then, for any \(s \geq 0\) and \(0 < \rho' < \rho\), one has \(A \in \mathcal{M}_{\rho',s}(\alpha, \beta)\). Moreover, there exist \(C, c > 0\) such that

\[
\tag{2.25} |A|_{\rho',s}^{\alpha,\beta} \leq \frac{C}{(\rho - \rho')^c} \left( \rho^{\alpha,\beta}_{s+\rho}(F) + \rho^{-\beta,\rho}_{s+c}(G) \right) .
\]

Finally, if \(F \in \text{Lip}_{\rho}^{s}(\Omega, \mathcal{PA}_{\rho}^{\alpha})\), \(G \in \text{Lip}_{\rho}^{s}(\Omega, \mathcal{PA}_{\rho}^{\beta})\), one has \(A \in \text{Lip}_{\rho}^{s}(\Omega, \mathcal{M}_{\rho,s}(\alpha, \beta))\) and \(2.25\) holds with the corresponding weighted Lipschitz norms.

The proof is available in Appendix \([A]\).

Commutators and fluxes. These classes of matrices enjoy also closure properties under commutators and flow generation. We define the adjoint operator

\[
\tag{2.26} \text{ad}_X(V) := i[X, V] ;
\]

note the multiplication by the imaginary unit in the definition of the adjoint map.

Lemma 2.24 (Commutator). Let \(\alpha, \rho > 0\) and \(s > \frac{1}{2}\). Assume \(V \in \mathcal{M}_{\rho,s}(\alpha, 0)\) and \(X \in \mathcal{M}_{\rho,s}(\alpha, \alpha)\). Then \(\text{ad}_X(V)\) belongs to \(\mathcal{M}_{\rho,s}(\alpha, \alpha)\) with the quantitative bound

\[
\tag{2.27} \left| \text{ad}_X(V) \right|_{\rho,s}^{\alpha,\alpha} \leq 2C_s |X|_{\rho,s}^{\alpha,\alpha} |V|_{\rho,s}^{\alpha,0} ;
\]

here \(C_s\) is the algebra constant of \([2.11]\). Moreover, if \(V \in \text{Lip}_{\rho}^{s}(\Omega, \mathcal{M}_{\rho,s}(\alpha, 0))\) and \(X \in \text{Lip}_{\rho}^{s}(\Omega, \mathcal{M}_{\rho,s}(\alpha, \alpha))\), then \(\text{ad}_X(V) \in \text{Lip}_{\rho}^{s}(\Omega, \mathcal{M}_{\rho,s}(\alpha, \alpha))\), with

\[
\tag{2.28} |\text{ad}_X(V)|_{\rho,s}^{\text{Lip}^{(s)}} \leq 2C_s |X|_{\rho,s}^{\text{Lip}^{(s)}} |V|_{\rho,s}^{\text{Lip}^{(s)}} .
\]
Also the proof of this lemma is postponed to Appendix A.

 Lemma 2.25 (Flow). Let $\alpha, \rho > 0$, $s > \frac{1}{2}$. Assume $V \in \mathcal{M}_{p,s}(\alpha, 0)$, $X \in \mathcal{M}_{p,s}(\alpha, \alpha)$. Then the followings hold true:

(i) For any $r \in [0, s]$ and any $\theta \in \mathbb{T}^\nu$, the operator $e^{X(\theta)} \in \mathcal{L}(H^r)$, with the standard operator norm uniformly bounded in $\theta$;

(ii) The operator $e^{XV} e^{-iX}$ belongs to $\mathcal{M}_{p,s}(\alpha, 0)$, while $e^{XV} e^{-iX} e^{X}$ belongs to $\mathcal{M}_{p,s}(\alpha, \alpha)$ with the quantitative bounds:

\[
\left| e^{XV} e^{-iX} e^{X(\theta)} \right|_{\mathcal{M}_{p,s}(\alpha, 0)} \leq e^{2C_s |X|_{p,s}^{\alpha, \alpha}} |V|_{p,s}^{\alpha, 0},
\]

\[
\left| e^{XV} e^{-iX} - V e^{X(\theta)} \right|_{\mathcal{M}_{p,s}(\alpha, \alpha)} \leq 2C_s e^{2C_s |X|_{p,s}^{\alpha, \alpha}} |X|_{p,s}^{\alpha, \alpha} |V|_{p,s}^{\alpha, 0}.
\]

Analogous assertions hold for $V \in \text{Lip}_p(\Omega, \mathcal{M}_{p,s}(\alpha, 0))$ and $X \in \text{Lip}_p(\Omega, \mathcal{M}_{p,s}(\alpha, \alpha))$.

The proof of this lemma is a standard application of (2.27) and the remark that the operator norm is controlled by the $|\cdot|_{p,s}^{\alpha, \alpha}$-norm (see also Remark 2.16).

Remark 2.26. As we are going to fix in our proofs a control level of the decay at some $s_0 > 1/2$, in the next we will omit the constant $C_s$ related to the algebra property that appears in all the previous estimates.

3 The Magnus normal form

To begin with, we recall the Pauli matrices notation. Let us introduce

\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]

and, moreover, define

\[
\sigma_4 := \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, \quad 1 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad 0 := \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
\]

Using Pauli matrix notation, equation (1.5) reads as

\[
i \dot{\varphi}(t) = \mathcal{H}(t) \varphi(t) := (H_0 + W(\omega t)) \varphi(t),
\]

\[
H_0 := B\sigma_3, \quad W(\omega t) := \frac{1}{2} B^{-1/2} V(\omega t) B^{-1/2} \sigma_4.
\]

Note that, by assumption (V1), one has $V \in \mathcal{P} \mathcal{A}_p^0$ (see Remark 2.11); therefore the properties of the pseudodifferential calculus and of the associated symbols (see Remarks 2.6 and 2.12) imply that

\[
B \in \mathcal{P} \mathcal{A}_p^1 \quad \text{and} \quad B^{-1/2} V B^{-1/2} \in \mathcal{P} \mathcal{A}_p^{-1}.
\]

The difficulty in treating equation (3.2) is that it is not perturbative in the size of the potential, so standard KAM techniques do not apply directly.

To deal with this problem, we perform a change of coordinates, adapted to fast oscillating systems, which puts (3.2) in a perturbative setting. We refer to this procedure as Magnus normal...
form. The Magnus normal form is achieved in the following way: the change of coordinates \( \varphi(t) = e^{-iX(\omega t)\omega} \) conjugates (3.2) to \( i\dot{\varphi}(t) = \mathcal{H}(t)\varphi(t) \), where the Hamiltonian \( \mathcal{H}(t) \) is given by (see [Bam18, Lemma 3.2]):

\[
\begin{align*}
(3.4) & \quad \mathcal{H}(t) = e^{iX(\omega t)\omega} \mathcal{H}(t)e^{-iX(\omega t)\omega} - \int_0^1 e^{iX(\omega t)\omega} X(\omega t;\omega)e^{-iX(\omega t)\omega} \, ds \\
(3.5) & \quad = H_0 + i[X, H_0] + W - X + i[X, \ldots].
\end{align*}
\]

In (3.5) we wrote, informally, \([X, \ldots] \) to remark that all the non written terms are commutators with \( X \). Following [ADRHH17b], one chooses \( X \) to solve \( W - X = 0 \). If the frequency \( \omega \) is large and nonresonant, then \( X \) has size \( |\omega|^{-1} \), and the new equation (3.5) is now perturbative in size. The price to pay is the appearance of \( i[X, H_0] \), which is small in size but possibly unbounded as operator. We control this term by employing pseudodifferential calculus and the properties of the commutators.

With this informal introduction, the main result of the section is the following:

**Theorem 3.1 (Magnus normal form).** For any \( 0 < \gamma_0 < 1 \), there exist a set \( \Omega_0 \subset \mathbb{R}^d \subset \mathbb{R}^\nu \) and a constant \( c_0 > 0 \) (independent of \( \Theta \)), with

\[
(3.6) \quad \frac{\text{meas}(R^d(\Omega_0))}{\text{meas}(R^d)} \leq c_0 \gamma_0,
\]

such that the following holds true. For any \( \omega \in \Omega_0 \) and any weight \( \omega > 0 \), there exists a time dependent change of coordinates \( \varphi(t) = e^{-iX(\omega t)\omega}\varphi(t) \), where

\[
X(\omega t;\omega) = X(\omega t;\omega)\sigma_4, \quad X \in \text{Lip}_\omega(\Omega_0, \mathcal{P}A^{-1}_{\rho/2}),
\]

that conjugates equation (3.2) to

\[
(3.7) \quad i\dot{\varphi}(t) = \tilde{\mathcal{H}}(t)\varphi(t), \quad \tilde{\mathcal{H}}(t) := H_0 + V(\omega t;\omega),
\]

where

\[
(3.8) \quad V(\theta;\omega) = \begin{pmatrix} V^d(\theta;\omega) & V^\omega(\theta;\omega) \\ -V^\omega(\theta;\omega) & -V^d(\theta;\omega) \end{pmatrix}, \quad \text{with} \quad [V^d]^* = V^d, \quad [V^\omega]^* = V^\omega
\]

and

\[
(3.9) \quad V^d \in \text{Lip}_\omega(\Omega_0, \mathcal{P}A^{-1}_{\rho/2}), \quad V^\omega \in \text{Lip}_\omega(\Omega_0, \mathcal{P}A^0_{\rho/2}).
\]

Furthermore, for any \( \ell \in \mathbb{N}_0 \), there exists \( C_\ell > 0 \) such that

\[
(3.10) \quad \varphi_\ell^{-1,\rho/2}(V^d)_{|\Omega_0} + \varphi_\ell^{0,\rho/2}(V^\omega)_{|\Omega_0} \leq C_\ell \frac{M}{H}.
\]

**Proof.** The proof is splitted into two parts, one for the formal algebraic construction, the other for checking that the operators that we have found possess the right pseudodifferential properties we are looking for.

**Step 1.** Expanding (3.10) in commutators we have

\[
(3.11) \quad \tilde{\mathcal{H}}(t) = H_0 + i[X, H_0] - \frac{1}{2}[X, [X, H_0]] + W - X + R,
\]
where the remainder $R$ of the expansion is given in integral form by

\[ R := \int_0^1 \frac{(1-s)^2}{2} e^{isX} \text{ad}_X^2(H_0)e^{-isX} ds \]

(3.12)

\[ + i \int_0^1 e^{isX}[X, W]e^{-isX} ds - i \int_0^1 (1-s)e^{isX}[X, \dot{X}]e^{-isX} ds. \]

(3.14)

From the properties of the Pauli matrices, we note that $\sigma^2 = 0$. This means that the terms in (3.12) involving $W$ and $\dot{X}$ are null, and the remainder is given only by

\[ R = \int_0^1 \frac{(1-s)^2}{2} e^{isX} \text{ad}_X^2(H_0)e^{-isX} ds. \]

(3.13)

We ask $X$ to solve the homological equation

\[ 0 = W - \dot{X} = \left( \frac{1}{2} B^{-1/2} V(\omega t) B^{-1/2} - \dot{X}(\omega t; \omega) \right) \sigma_1. \]

(3.14)

Expanding in Fourier coefficients with respect to the angles, its solution is actually given by

\[ \dot{X}(k; \omega) = \frac{1}{2i \omega \cdot k} B^{-1/2} \hat{V}(k) B^{-1/2}, \quad \text{for } k \in \mathbb{Z}^n \setminus \{0\}, \]

(3.15)

\[ \dot{X}(0; \omega) = 0 \]

where the second of (3.15) is a consequence of (V2). It remains to compute the terms in (3.4) and (3.13) involving $H_0$. Using again the structure of the Pauli matrices, we get:

\[ \text{ad}_X(H_0) := [i[X\sigma_4, B\sigma_3] = iXB(1 - \sigma_1) - iB(1 + \sigma_1) = iX, B]_s \sigma_1, \]

(3.16)

where we have denoted by $[X, B]_s := XB + BX$ the anticommutator. Similarly one has

\[ \text{ad}_X(H_0) := -[X\sigma_4, [X\sigma_4, B\sigma_3]] \]

(3.17)

\[ = -(\{X\sigma_4, [X, B]_s\sigma_1\}) \]

\[ = -([X, [X, B]] - [X, [X, B]_s])\sigma_4 \]

\[ = 4BX\sigma_4; \]

thus

\[ \text{ad}_X(H_0) = 4i[X\sigma_4, XBX\sigma_4] = 0. \]

(3.18)

This shows that $R = 0$ and, imposing (3.15) in (3.4), we obtain

\[ \dot{H}(t) = H_0 + V(\omega t; \omega), \]

(3.19)

with

\[ V^d(\theta; \omega) := i[X(\theta; \omega), B] + 2X(\theta; \omega)BX(\theta; \omega), \]

(3.20)

\[ V^c(\theta; \omega) := -i[X(\theta; \omega), B]_s + 2X(\theta; \omega)BX(\theta; \omega). \]

Step II). We show now that $X, V^d$ and $V^c$, defined in (3.15) and (3.20) respectively, are pseudodifferential operators in the proper classes, provided $\omega$ is sufficiently nonresonant. First consider $X$. For $\gamma_0 > 0$ and $\tau_0 > \nu - 1$, define the set of Diophantine frequency vectors

\[ \Omega_0 := \Omega_0(\gamma_0, \tau_0) := \left\{ \omega \in \mathbb{R}^n \mid |\omega \cdot k| \geq \frac{\gamma_0}{\langle k \rangle^{\tau_0}} \quad \forall k \in \mathbb{Z}^n \setminus \{0\} \right\}. \]

(3.21)
We will prove in Proposition \[\text{(3.1)}\] that there exists a constant \(c_0 > 0\), independent of \(M\) and \(\gamma_0\), such that
\[
\text{meas}(R_0 \setminus \Omega_0) \leq c_0 \gamma_0.
\]
This fixes the set \(\Omega_0\) and proves \[\text{(3.10)}\].

We show now that \(X \in \text{Lip}_\omega(\Omega_0, \mathcal{PA}^{-1}_0)\). First note that, by Lemma \[\text{(A.1)}\] (in Appendix \[\text{A}\]) and Remark \[\text{(2.12)}\] one has \(B^{-1/2} \hat{V}(k)B^{-1/2} \in \mathcal{PA}^{-1}\) (both \(B\) and \(V\) are independent from \(\omega\)) with
\[
\hat{\varphi}_t^{-1}(B^{-1/2} \hat{V}(k)B^{-1/2}) \leq 4e^{-\rho|k|} \hat{\varphi}_t^{-1,\rho}(B^{-1/2}VB^{-1/2}) \leq 4e^{-\rho|k|} C_\ell.
\]
Provided \(\omega \in \Omega_0\), it follows that
\[
\hat{\varphi}_t^{-1}(\hat{X}(k; \cdot))_{\Omega_0} \leq \frac{1}{2} \left[ \sup_{x \in \Omega_0} \frac{1}{|\omega \cdot k|} \right] \hat{\varphi}_t^{-1}(B^{-1/2} \hat{V}(k)B^{-1/2}) \leq \frac{4}{\gamma_0 M} C_\ell e^{-\rho|k|} C_\ell.
\]
To compute the Lipschitz norm, it is convenient to use the notation
\[
(3.22) \quad \Delta \omega f(\omega) = f(\omega + \Delta \omega) - f(\omega),
\]
with \(\omega, \omega + \Delta \omega \in \Omega_0\), \(\Delta \omega \neq 0\). In this way one gets
\[
|\Delta \omega \hat{X}(k; \omega)| \leq \frac{|\Delta \omega|}{|\omega \cdot k|} \left| B^{-1/2} \hat{V}(k)B^{-1/2} \right| \Rightarrow \hat{\varphi}_t^{-1}(\hat{X}(k; \cdot))_{\Omega_0} \leq \frac{4}{\gamma_0 M} C_\ell e^{-\rho|k|} C_\ell.
\]
As a consequence \(X(\theta; \omega) = \sum_{k} \hat{X}(k; \omega)e^{ik \theta}\) is a pseudodifferential operator in the class \(\text{Lip}_\omega(\Omega_0, \mathcal{PA}^{-1}_0)\) (see Lemma \[\text{(A.1)}\] (ii) in Appendix \[\text{A}\] for details) fulfilling
\[
(3.23) \quad \hat{\varphi}_t^{-1,\rho/2}(X)_{\Omega_0}^{\text{Lip}(\omega)} \leq \left( \frac{1}{\gamma_0 M} + \frac{\omega}{\gamma_0 M^2} \right) \frac{C_\ell}{\rho^{\gamma_0 M + \nu}} \leq \max(1, \omega) \frac{C_\ell}{\rho^{2\gamma_0 M + \nu}}.
\]
It follows by Remark \[\text{(2.12)}\] that \(V^d \in \text{Lip}_\omega(\Omega_0, \mathcal{PA}^{-1}_0)\) while \(V^\omega \in \text{Lip}_\omega(\Omega_0, \mathcal{PA}_0)\) with the claimed estimates \[\text{(3.10)}\].

Finally, \(V\) is a real selfadjoint operator, simply because it is a real bounded potential, and therefore \(V^* = V = \hat{V}\). It follows by Remark \[\text{(2.13)}\] and the explicit expression \[\text{(3.13)}\] that \(X^* = X = \hat{X}\). Using these properties one verifies by a direct computation that \([V^d]^* = V^d\) and \([V^\omega]^* = V^\omega\). Estimate \[\text{(3.23)}\] and the symbolic calculus of Remark \[\text{(2.12)}\] give \[\text{(3.10)}\].

Remark 3.2. Everything works with the more general assumptions \(V \in \mathcal{PA}_0\).

Remark 3.3. Pseudodifferential calculus is used to guarantee that \(V^d\) has order -1 while \(V^\omega\) has order 0 (see \[\text{(3.10)}\]). Without this information it would be problematic to apply the standard KAM iteration of Kuksin \[\text{Kuk87}\], which requires the eigenvalues to have an asymptotic of the form \(j + O(j^\delta)\) with \(\delta < 0\).

In principle one might circumvent this problem by using the ideas of Berti, Baldi and Montalto \[\text{BBM14}\] (which in turn are a development of those of Plotnikov and Toland \[\text{PT01}\]) to regularize the order of the perturbation. However in our context this smoothing procedure is tricky, since it produces terms of size \(|\omega|\), which are very large and therefore unacceptable for our purposes.
4 Balanced unperturbed Melnikov conditions

As we shall see, in order to perform a converging KAM scheme, we must be able to impose second order Melnikov conditions, namely bounds from below of quantities like \( \omega \cdot k + \lambda_i \pm \lambda_j \), where the \( \lambda_j \)'s are the eigenvalues of the operator \( B \) defined in (13). Explicitly,

\[
\lambda_j := \sqrt{j^2 + m^2} = j + c_j(m),
\]

with \( c_j(m) := j(\sqrt{j^2 + m^2} - j) \). One can check that

\[
0 \leq c_j(m) \leq m^2 \quad \forall j \in \mathbb{N}.
\]

We introduce the notation of the indexes sets:

\[
\mathcal{I}^+ := \mathbb{Z}^r \times \mathbb{N} \times \mathbb{N},
\]

\[
\mathcal{I}^- := \{ (k, j, l) \in \mathcal{I}^+ \mid (k, j, l) \neq (0, a, a), a \in \mathbb{N} \}.
\]

Furthermore, we define the relative measure of a measurable set \( \Omega \) as

\[
m_r(\Omega) := \frac{|\Omega|}{|R_{\mathbb{N}}|} = \frac{|\Omega|}{\nu(2^r - 1)c_\nu},
\]

where \( |C| \) is the Lebesgue measure of the set \( C \) and \( c_\nu \) is the volume of the unitary ball in \( \mathbb{R}^r \).

The main result of this section is the following theorem.

**Theorem 4.1** (Balanced Melnikov conditions). Fix \( 0 \leq \alpha \leq 1 \) and assume that

\[
M \geq M_0 := \min\{m^2, \langle m \rangle^{1/\alpha} \} \quad \text{for } \alpha \in (0, 1],
\]

\[
M \geq M_0 := m^2 \wedge \bar{\gamma}^{2/3} \leq 2 \langle m \rangle^{-1} \quad \text{for } \alpha = 0.
\]

Then, for \( 0 < \bar{\gamma} \leq \min\{\gamma_0^{3/2}, 1/8\} \) and \( \bar{\gamma} \geq 2\nu + 3 \), the set

\[
\mathcal{U}_\alpha := \left\{ \omega \in \Omega_0 \mid |\omega \cdot k + \lambda_j \pm \lambda_i| \geq \bar{\gamma} \frac{\langle j \pm b \rangle^\alpha}{\langle k \rangle^\alpha} \quad \forall (k, j, l) \in \mathcal{I}^\pm \right\}
\]

is of large relative measure, that is

\[
m_r(\Omega_0 \setminus \mathcal{U}_\alpha) \leq C \bar{\gamma}^{1/3},
\]

where \( C > 0 \) is independent of \( M \) and \( \bar{\gamma} \).

**Remark 4.2.** The family of Cantor sets \( (\mathcal{U}_\alpha)_{\alpha \in (0, 1]} \) is completely unordered. Therefore, the choice of the frequency vector \( \omega \) will strongly depend on the choice of the smoothing order \( \alpha \) and, in particular, it will influence the expansion of the final eigenvalues \( \lambda_j^{\alpha} \) as perturbative versions of the initial ones, see Corollary 5.10.

We will use several times the following result, which is an easy variant of Lemma 5 of [Pös96b].

**Lemma 4.3.** Fix \( k \in \mathbb{Z}^r \setminus \{0\} \) and let \( R_\mathbb{N} \ni \omega \mapsto \zeta(\omega) \in \mathbb{R} \) be a Lipschitz function fulfilling

\[
|\zeta|_{R_\mathbb{N}}^{\text{Lip}} \leq c_0 < |k|.
\]

Define \( f(\omega) = \omega \cdot k + \zeta(\omega) \). Then, for any \( \delta > 0 \), the measure of the set \( A := \{ \omega \in R_\mathbb{N} \mid |f(\omega)| \leq \delta \} \) satisfies the upper bound

\[
|A| \leq \frac{2\delta}{|k| - c_0 (4M)^{\nu - 1}}.
\]
Proof. Take \( \omega_1 = \omega + \epsilon k \), with \( \epsilon \) sufficiently small so that \( \omega_1 \in R_\delta \). Then
\[
\frac{|f(\omega_1) - f(\omega)|}{|\omega_1 - \omega|} \geq |k| - |\gamma|_{R_\delta} \geq |\gamma|_{R_\delta} - c_0
\]
and the estimate follows by Fubini theorem.

In the rest of the section we write \( a \lesssim b \), meaning that \( a \leq C b \) for some numerical constant \( C > 0 \) independent of the relevant parameters.

The result of Theorem 4.1 is carried out in two steps. The first one is the following lemma.

Lemma 4.4. Fix \( 0 \leq \alpha < 1 \). There exist \( \tilde{\gamma}_1 > 0 \) and \( \tau_1 > \nu + \alpha \) such that the set
\[
\mathcal{T}_1 := \left\{ \omega \in \Omega_0 \left| |\omega \cdot k + l| \geq \frac{\tilde{\gamma}_1}{\langle k \rangle^{\tau_1}} \frac{|l|^\alpha}{M^\alpha} \right. \forall (k, l) \in \mathbb{Z}^{r+1}\setminus\{0\} \right\}
\]
has relative measure \( m_r(\Omega_0 \setminus \mathcal{T}_1) \leq C_1 \tilde{\gamma}_1 \), where \( C_1 > 0 \) is independent of \( M \) and \( \tilde{\gamma}_1 \).

Proof. If \( k = 0 \) and \( l \neq 0 \), the estimate in (4.8) holds. The same is true if \( k \neq 0 \) and \( l = 0 \). Therefore, let both \( k \) and \( l \) be different from zero. For \( |l| > 4M |k| \), the inequality in (4.8) holds true taking \( \tilde{\gamma}_1 \leq \frac{1}{2} \). Indeed:
\[
|\omega \cdot k + l| \geq |l| - |\omega| |k| \geq |l| - 2M |k| \geq \frac{|l|}{2} \geq \frac{1}{2} |l|^\alpha \geq \frac{\tilde{\gamma}_1}{\langle k \rangle^{\tau_1}} \frac{|l|^\alpha}{M^\alpha}.
\]
Then, consider the case \( 1 \leq |l| \leq 4M |k| \) (so, only a finite number of \( l \in \mathbb{Z} \setminus \{0\} \)). For fixed \( k \) and \( l \), define the set
\[
\mathcal{G}_1^k := \left\{ \omega \in R_\delta \left| |\omega \cdot k + l| \leq \frac{\tilde{\gamma}_1}{\langle k \rangle^{\tau_1}} \frac{|l|^\alpha}{M^\alpha} \right. \right\}.
\]
By Lemma 4.3, the measure of each set can be estimated by
\[
|\mathcal{G}_1^k| \lesssim M^{r-1} \frac{\tilde{\gamma}_1^r}{\langle k \rangle^{\tau_1}} \frac{|l|^\alpha}{M^\alpha} \lesssim \tilde{\gamma}_1 M^{r-1-\alpha} \frac{|l|^\alpha}{\langle k \rangle^{\tau_1+1}}.
\]
Let
\[
\mathcal{G}_1 := \Omega_0 \cap \bigcup \{ \mathcal{G}_1^k \mid (k, l) \in \mathbb{Z}^{r+1}\setminus\{0\}, |l| \leq 4M |k| \}.
\]
Then
\[
|\mathcal{G}_1| \leq \sum_{k \in \mathbb{Z} \setminus \{0\}} \sum_{|l| \leq 4M |k|} |\mathcal{G}_1^k| \lesssim \tilde{\gamma}_1 M^{r-1-\alpha} \sum_{k \neq 0} \sum_{|l| \leq 4M |k|} \frac{|l|^\alpha}{\langle k \rangle^{\tau_1+1}} \lesssim \tilde{\gamma}_1 M^{r-1-\alpha} \sum_{k \neq 0} \frac{1}{\langle k \rangle^{\tau_1+1}} (4M |k|)^{\alpha + 1} \leq \tilde{\gamma}_1 M^{r-1} \sum_{k \neq 0} \frac{1}{\langle k \rangle^{\tau_1-\alpha}} \leq \tilde{\gamma}_1 M^r
\]
provided \( \tau_1 > \nu + \alpha \). It follows that the relative measure of \( \mathcal{G}_1 \) is given by
\[
m_r(\mathcal{G}_1) \leq C_1 \tilde{\gamma}_1,
\]
where \( C_1 > 0 \) is independent of \( M \) and \( \tilde{\gamma}_1 \). The thesis follows, since \( \mathcal{T}_1 = \Omega_0 \setminus \mathcal{G}_1 \).

Remark 4.5. In case \( m = 0 \), Lemma 4.4 implies Theorem 4.1.
From now on assume that \( m > 0 \). The second step is the next lemma.

**Lemma 4.6.** There exist \( 0 < \tilde{\gamma}_2 \leq \min\{\gamma_0, \gamma_1/2\} \) and \( q_2 \geq q_1 + \nu + 1 \) such that the set

\[
\mathcal{T}_2 := \left\{ \omega \in \mathcal{T}_1 \mid |\omega \cdot k + \lambda_j + \lambda_l| \geq \frac{\tilde{\gamma}_2}{\langle k \rangle^2} \frac{(j + l)^\alpha}{M^\alpha} \quad \forall (k, j, l) \in \mathcal{I}^\pm \right\}
\]

fulfills \( m_r(\mathcal{T}_1 \setminus \mathcal{T}_2) \leq C_2 \frac{\tilde{\gamma}_2}{\gamma_1} \), where \( C_2 > 0 \) is independent of \( m, \tilde{\gamma}_1, \tilde{\gamma}_2 \).

**Proof.** Let \((k, j, l) \in \mathcal{I}^\pm\). We can rule out some cases for which the inequality in (4.13) is already satisfied when \( \omega \in \mathcal{T}_1 \subset \Omega_0 \):

- For \( \pm = + \) and \( k = 0 \), we have
  \[ \lambda_j + \lambda_l = j + l + \frac{c_j(m)}{j} \frac{c_l(m)}{l} \geq j + l \geq \frac{\tilde{\gamma}_2}{M^\alpha} (j + l)^\alpha ; \]

- For \( \pm = - \) and \( k \neq 0 \), \( j = l \),
  \[ |\omega \cdot k| \geq \frac{\gamma_0}{\langle k \rangle M} \geq \frac{\tilde{\gamma}_2}{M^\alpha} \frac{1}{M^\alpha} ; \]

- For \( \pm = - \) and \( k = 0 \), \( j \neq l \), it holds that
  \[ |\lambda_j - \lambda_l| \geq \frac{1}{M^\alpha} |j - l| \geq \frac{\tilde{\gamma}_2}{M^\alpha} (j - l)^\alpha . \]

Indeed, for \( 0 < \alpha \leq 1 \),
\[
|\lambda_j - \lambda_l| = \left| \int_l^j \frac{x}{\sqrt{x^2 + m^2}} dx \right| \geq \min_{x \in (1, \infty)} \left| \frac{x}{\sqrt{x^2 + m^2}} \right| \int_l^j dx = \frac{1}{\langle m \rangle} |l - j| \geq \frac{1}{M^\alpha} |l - j| ,
\]
onlyi otherwise, for \( \alpha = 0 \) and provided \( \tilde{\gamma}_2 \leq 2 \langle m \rangle^{-1} \),
\[
|\lambda_j - \lambda_l| \geq \frac{1}{\langle m \rangle} |j - l| \geq \frac{2}{\langle m \rangle} \geq \tilde{\gamma}_2 .
\]

Therefore, for the rest of this argument, let \( k \neq 0 \) and \( j \neq l \). Assume first that \( |j \pm l| \geq 8M |k| \).

In this case, one has:
\[
|\omega \cdot k + \lambda_j + \lambda_l| \geq |j \pm l| - \left| \frac{c_j(m)}{j} \pm \frac{c_l(m)}{l} \right| - |\omega \cdot k|
\]
\[
\geq |j \pm l| - 2M^2 - 2M |k| \geq |j \pm l| - 4M |k| \geq \frac{1}{2} |j \pm l| .
\]

Let now \( |j \pm l| < 8M |k| \). In the region \( j < l \) assume
\[
|j \langle j \pm l \rangle^\alpha \geq R(k) := \frac{4M^2 |k|^{\alpha} \langle k \rangle^{\gamma_1}}{\tilde{\gamma}_1} ,
\]
where $\tilde{\gamma}_1$ and $\tau_1$ are the ones of Lemma 4.4. So, for $\omega \in \mathcal{T}_1$, we get
\begin{equation}
|\omega \cdot k + \lambda_j + \lambda_l| \geq |\omega \cdot k + l| - \frac{c_j(n)}{j} \geq \frac{\tilde{\gamma}_1}{\langle k \rangle^{\alpha/3}} - \frac{2m^2 \gamma_0}{j} \geq \frac{\tilde{\gamma}_1}{\langle k \rangle^{\alpha/3}},
\end{equation}
(4.16)
Thus, we consider just those $j$ and $l$ with $j < l$ for which $l(j + l)^{\alpha} < \mathcal{R}(k)$. The symmetric argument shows that we can take those $l < j$ for which $l(j + l)^{\alpha} < \mathcal{R}(k)$.

Like in the previous proof, consider the set
\begin{equation}
\mathcal{G}_{j,l} := \left\{ \omega \in \mathcal{R} \left| |\omega \cdot k + \lambda_j + \lambda_l| < \frac{\tilde{\gamma}_2}{\langle k \rangle^{\alpha/3}} \right. \right\}
\end{equation}
defined for those $k \neq 0$ and $j \neq l$ in the regions
\begin{equation}
\mathcal{P} := \left\{ |j + l| < 8M|k| \right\} \cap \left\{ \{ j(j + l)^{\alpha} < \mathcal{R}(k), j < l \} \cup \{ l(j + l)^{\alpha} < \mathcal{R}(k), l < j \} \right\}.
\end{equation}
Using Lemma 4.8, the estimate for its Lebesgue measure is
\begin{equation}
|\mathcal{G}_{j,l}| \lesssim \tilde{\gamma}_2 m^{\nu - 1 - \alpha} \left(\frac{j + l}{|k|^{\alpha/3}}\right)^{\nu + 1}.
\end{equation}
Define $\mathcal{G}_j := \mathcal{T}_1 \cup \bigcup \left\{ \mathcal{G}_{j,l} \mid (k, j, l) \in \mathcal{P} \right\}$. By symmetry of the summand, we estimate
\begin{equation}
|\mathcal{G}_j| \lesssim \sum_{(k, j, l) \in \mathcal{P}} \left| \mathcal{G}^e_{j,l} \right| \lesssim \tilde{\gamma}_2 m^{\nu - 1 - \alpha} \sum_{(k, j, l) \in \mathcal{P}} \frac{(j - l)^{\alpha}}{|k|^{\alpha/3}} \sum_{j < l < \mathcal{R}(k)} \frac{\langle j \rangle^{\alpha}}{|k|^{\alpha/3}}
\end{equation}
(4.20)
\begin{equation}
\lesssim \tilde{\gamma}_2 m^{\nu - 1 - \alpha} \sum_{k \neq 0} \sum_{j < \mathcal{R}(k)} \frac{\langle j \rangle^{\alpha}}{|k|^{\alpha/3}} \lesssim \tilde{\gamma}_2 m^{\nu - 1} \sum_{k \neq 0} \frac{1}{|k|^{\alpha/3}} \lesssim \tilde{\gamma}_2 m^{\nu - 1} \sum_{k \neq 0} \frac{1}{|k|^{\alpha/3}} \lesssim \frac{\tilde{\gamma}_2}{\gamma_1}
\end{equation}
provided $\tau_2 > \tau_1 + \nu$. The same computation holds for $\mathcal{G}_j$. We conclude that
\begin{equation}
m_\nu(\mathcal{T}_1 \mathcal{T}_2) \leq m_\nu(\mathcal{G}_j - \mathcal{G}_j) \leq C_1 \frac{\tilde{\gamma}_2}{\gamma_1},
\end{equation}
where $C_1 > 0$ is independent of $M$, $\tilde{\gamma}_1$, $\tilde{\gamma}_2$. \hfill \Box

Proof of Theorem 4.4. Take $\tilde{\gamma}_1 = \tilde{\gamma}^{1/3}$, $\tilde{\gamma}_2 = \tilde{\gamma}^{2/3}$ with some $\tilde{\gamma} > 0$ sufficiently small so that $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ fulfill the assumptions of the previous lemmas. Similarly, choose $\tau_1 = \nu + 2$, $\tau_2 = 2\nu + 3$.

By definition, $\mathcal{U}_0 = \mathcal{T}_2 \subset \Omega_0$. Since $\Omega_0 \mathcal{U}_0 = (\Omega \mathcal{T}_1) \cup (\mathcal{T}_1 \mathcal{T}_2)$, we get by Lemma 4.3 and Lemma 4.6 that
\begin{equation}
m_\nu(\Omega_0 \mathcal{U}_0) \leq C_1 \tilde{\gamma}_1 + C_2 \frac{\tilde{\gamma}_2}{\gamma_1} \leq C_2 \frac{\tilde{\gamma}_1^{1/3}}{\gamma_1},
\end{equation}
with $C = 2(C_1 + C_2)$. \hfill \Box
5 The KAM reducibility transformation

The new potential $V(\omega t; \omega)$ that we have found in Theorem 3.1 is perturbative, in the sense that the smallness of its norm is controlled by the size $M$ of the frequency vector $\omega$. Thus, we are now ready to attack with a KAM reduction scheme, presenting first the algebraic construction of the single iteration, then quantifying it via the norms and seminorms that we have introduced in Section 2. The complete result for this reduction transformation, together with its iterative lemma, is proved at the end of this section. The discussion about the measure estimates for the Melnikov conditions we need to impose on the perturbed eigenvalues is postponed to Section B.2.

5.1 Preparation for the KAM iteration

Actually, for the KAM scheme it is more convenient to work with operators of type $M_{\rho,s}$. Of course, as we have seen in Section 2, pseudodifferential operators analytic in $\theta$ belong to such a class. Define the norm

$$\|V\|_{\text{Lip}(\gamma)} := \sup_{\omega \in \Omega_0} |V(\omega)|_{\rho_0,s_0} + \gamma \sup_{\omega \neq \omega_0} \frac{|\Delta_\omega V(\omega)|_{\rho_0,s_0}}{M |\Delta_\omega|}.$$  

We have following result:

Lemma 5.1. Fix an arbitrary $s_0 > 1/2$ and put $\rho_0 := \rho/4$. Then the operator $V(\omega) = V(\theta; \omega)$ defined in (3.8) belongs to $\text{Lip}_w(\Omega_0, M_{\rho_0,s_0}(1,0))$, with $w = \gamma/M^\alpha$, and there exist constants $C, \gamma > 0$, independent of $M$, such that

$$\|V\|_{\text{Lip}(\gamma)} \leq \frac{C}{M}.$$  

Proof. It is sufficient to apply the embedding Lemma 2.23 and (3.10). \qed

Remark 5.2. Note that neither the smoothing order $\alpha \in (0,1)$ nor the weight $w \in (0,1)$ have played any role neither in the Magnus normal form nor in the construction of the set $\Omega_0$ in Theorem 3.1.

5.2 General step of the reduction

Looking forward to the iterative scheme, which we shall discuss in Section 5.4, we start with a general system

$$i\dot{\psi}(t) = H(t)\psi(t), \quad H(t) := A(\omega) + P(\omega t; \omega),$$

where:

- the frequency vector $\omega$ varies in some set $\Omega \subset \mathbb{R}^\nu$, $M \leq |\omega| \leq 2M$;
- the time-independent operator $A(\omega)$ is diagonal, with

$$A(\omega) = \begin{pmatrix} A(\omega) & 0 \\ 0 & -A(\omega) \end{pmatrix}, \quad A(\omega) := \text{diag}(\lambda_j(\omega) | j \in \mathbb{N}) \subset (0, \infty)^\mathbb{N};$$
- the quasi-periodic perturbation $P(\omega t; \omega)$ has the form

$$P(\omega t; \omega) = \begin{pmatrix} P^d(\omega t; \omega) & P^p(\omega t; \omega) \\ -P^d(\omega t; \omega) & -P^p(\omega t; \omega) \end{pmatrix}, \quad P^d = [P^d]^*, \quad P^p = [P^p]^*.$$


As we shall see, the time independent diagonal operator matrix $A$ keeps track of all the averages we cannot delete from the perturbations of the previous steps. Therefore, except for the very first step of the iteration, the perturbed eigenvalues of $A$ depend (in a Lipschitz way) on the parameter $\omega \in \Omega$.

The goal is to square the size of the perturbation (see Lemma 5.3) and we do it by conjugating the Hamiltonian $H(t)$ through a transformation $\psi := e^{-iX^+(\omega t_0)}\varphi$ of the form

$$
X^+(\omega t; \omega) = \begin{pmatrix} X^d(\omega t; \omega) & X^o(\omega t; \omega) \\ -X^o(\omega t; \omega) & -X^d(\omega t; \omega) \end{pmatrix}, \quad X^d = [X^d]^\ast, \quad X^o = [X^o]^\ast,
$$

so that the transformed Hamiltonian, as in (5.4), is

$$
H^+(t) := e^{iX^+(\omega t; \omega)}H(t)e^{-iX^+(\omega t; \omega)} = \int_0^1 e^{i\omega X^+(\omega t; \omega)}X^+(\omega t; \omega)e^{-i\omega X^+(\omega t; \omega)}ds.
$$

Its expansion in commutators is given by

$$
H^+(t) = A + P + i[X^+, A] - \hat{X}^+ + R,
$$

where $\hat{X}^+$ is the projector on the frequencies smaller than $\omega$.

We then ask now $X^+$ to solve the "quantum" homological equation:

$$
i[X^+(\theta), A] - \omega \cdot \partial_\theta X^+(\theta) + \Pi_N P(\theta) = Z
$$

where $\Pi_N$ is the projector on the frequencies smaller than $N$, i.e.

$$
\Pi_N P(\theta; \omega) := \sum_{|k| \leq N} \hat{P}(k; \omega)e^{ik \theta},
$$

while is the diagonal, time independent part of $P^d$:

$$
Z = Z(\omega) := \begin{pmatrix} Z(\omega) & 0 \\ 0 & -Z(\omega) \end{pmatrix}, \quad Z = \text{diag\{$(P^d)^j(0; \omega)$ | $j \in \mathbb{N}$\}}.
$$

With this choice, the new Hamiltonian becomes

$$
H(t)^+ = A^+ + P(\omega t)^+, \quad A^+ = A + Z.
$$

In order to solve equation (5.3), note that component-wise it reads as

$$
\begin{cases}
i[X^d, A] - \omega \cdot \partial_\theta X^d + P^d = Z \\
-i[X^o, A]^\ast - \omega \cdot \partial_\theta X^o + P^o = 0
\end{cases}.
$$

Expanding both with respect to the exponential basis of $B$ (for the space) and in Fourier in angles (for the time), we get the solutions

$$
(X^d)^j_l(k; \omega) := \begin{cases}
\frac{1}{i(\omega \cdot k + \lambda_j^l(\omega) - \lambda_l^j(\omega))} \frac{(P^d)^j_l(0; \omega)}{ \lambda_j^l(\omega) - \lambda_l^j(\omega)} & (k, j, l) \in \mathcal{I}_N \\
0 & \text{otherwise}
\end{cases}.
$$
Furthermore, we choose as Lipschitz weight $w$ from the following set a direct consequence of Lemma 5.11 and Lemma 5.12 of Section 5.4.

(5.16) \[ I_N^+ := \{ (k, j, l) \in I_N^+ \mid |k| \leq N \} . \]

Furthermore, we track down the behaviour of the new perturbed eigenvalues. By (5.11) and (5.10), we deduce that

(5.17) \[ \sigma(A^+) = \sigma(A + Z) = \{ \lambda_j^+ (\omega) \mid j \in \mathbb{N} \} \]

where the expression of $\lambda_j^+ (\omega)$ is given by

(5.18) \[ \lambda_j^+ (\omega) := \lambda_j^- (\omega) + (P^0)_{j} (0; \omega) = \lambda_j + \varepsilon_j^+ (\omega) . \]

Here, $\lambda_j = \sqrt{j^2 + m^2}$ is the $j^{th}$ unperturbed eigenvalue, while $\varepsilon_j^+ (\omega)$ is the sum of all the corrections coming from all the homological equations up to the current step, included.

### 5.3 Estimates for the general step

Both for well-posing the solutions (5.14) and (5.15) and ensuring convergence of the norms, second order Melnikov conditions are required to be imposed. In particular, we choose the frequency vector from the following set

(5.19) \[ \Omega^+ := \left\{ \omega \in \Omega \mid |\omega \cdot k + \lambda_j^- (\omega) \pm \lambda_j^- (\omega)| \geq \frac{\gamma}{2} \lambda_j^0, \forall \omega, \forall (k, j, l) \in I_N^+ \right\} \]

with $\gamma, \tau > 0$ to be fixed later on. Here $I_N^+$ has been defined in (5.16).

The fact that $\Omega^+$ is actually a set of large measure, that is $m_\nu (\Omega^+) = O(\gamma)$, will be clear as a direct consequence of Lemma 5.11 and Lemma 5.12 of Section 5.4.

From now on, we choose as Lipschitz weight $w := \gamma / M^\alpha$ and, abusing notation, we denote

(5.20) \[ \text{Lip}_s (\Omega, F) := \text{Lip}_{\gamma / M^\alpha} (\Omega, F) . \]

Furthermore, we fix once for all $s_0 > 1 / 2$ and $\alpha \in (0, 1)$.

For $V \in \text{Lip}_s (\Omega, \mathcal{M}_{\rho, s_0} (\alpha, 0))$, we write

(5.21) \[ |V| := |V|_{s_0}^{\alpha, 0} , \]

\[ |V|_{\rho} := |V|_{\rho, s_0}^{\alpha, 0} = \sum_{k \in \mathbb{Z}^s} e^{\rho |k|} \left| \hat{V} (k) \right| , \]

\[ |V|_{\rho, \Omega}^{\text{Lip} (\gamma)} := |V|_{\rho, s_0, \alpha, 0, \Omega}^{\text{Lip} (\gamma / M^\alpha)} = |V|_{\rho, \Omega}^{\infty} + \frac{\gamma}{M^\alpha} |V|_{\rho, \Omega}^{\text{Lip} (\gamma)} , \]

while, for $V \in \text{Lip}_s (\Omega, \mathcal{M}_{\rho, s_0} (\alpha, \alpha))$, we denote

(5.22) \[ \|V\| := |V|_{s_0}^{\alpha, \alpha} , \]

\[ \|V\|_{\rho} := |V|_{\rho, s_0}^{\alpha, \alpha} = \sum_{k \in \mathbb{Z}^s} e^{\rho |k|} \left\| \hat{V} (k) \right\| , \]

\[ \|V\|_{\rho, \Omega}^{\text{Lip} (\gamma)} := |V|_{\rho, s_0, \alpha, \alpha, \Omega}^{\text{Lip} (\gamma / M^\alpha)} = \|V\|_{\rho, \Omega}^{\infty} + \frac{\gamma}{M^\alpha} \|V\|_{\rho, \Omega}^{\text{Lip} (\gamma)} . \]
Remark 5.3. With respect to the norm introduced in (5.1), note that \( |V|^{\text{Lip}(\gamma)}_{\rho_0,0} \leq |V|^{\text{Lip}(\gamma)}_{\rho_0,0} \).

Now, we provide the estimate on the generator \( X^+ \) of the previous transformation. For sake of simplicity during the forthcoming proof, as short notation we define

\[
C^{k,\pm}(\omega) := \omega \cdot k + \lambda^-_j(\omega) \pm \lambda^+_j(\omega)
\]

for \((k,j,l) \in \mathcal{T}^+_N\).

Lemma 5.4. Let \( X^+ = X^+(\omega t, \omega) \) be defined by (5.14) and (5.15). We assume that:

(a) \( P \in \text{Lip}_\gamma(\Omega, \mathcal{M}_{\rho,s_0}(\alpha,0)) \), with an arbitrary \( \rho > 0 \);

(b) There exists \( 0 < C \leq 1 \) such that for any \( j \in \mathbb{N}, \omega, \Delta \omega \in \Omega^+ \) one has

\[
|\Delta\omega \lambda^-_j(\omega)| \leq C.
\]

Then \( X^+ \in \text{Lip}_\gamma(\Omega^+, \mathcal{M}_{\rho,s_0}(\alpha,\alpha)) \) with the quantitative bound

\[
\|X^+\|_{\rho,\Omega^+}^{\text{Lip}(\gamma)} \leq 16 \langle N \rangle^{2^{\tau+1}} \frac{M_0^{\alpha}}{\gamma} \|P\|_{\rho,\Omega}^{\text{Lip}(\gamma)}.
\]

Proof. We start with the seminorm \( \|X^+\|_{\rho,\Omega^+}^{\infty} \). Fix \( \omega \in \Omega^+ \) and \(|k| \leq N\). Then, when \( j \neq l \), we have

\[
\left| \langle X^{d} \rangle^j_l(k;\omega) \right| \leq \frac{1}{|C^{k,\pm}(\omega)|} \left| \langle P^d \rangle^j_l(k;\omega) \right| \leq \frac{2 \langle N \rangle^{2^{\tau+1}} M_0^{\alpha}}{\gamma} \left| \langle P^d \rangle^j_l(k;\omega) \right| \left| \langle D \rangle^j_l(k;\omega) \right|
\]

and similarly, for any \( j, l \in \mathbb{N} \)

\[
\left| \langle X^{\sigma} \rangle^j_l(k;\omega) \right| \leq \frac{2 \langle N \rangle^{2^{\tau+1}} M_0^{\alpha}}{\gamma} \left| \langle P^\sigma \rangle^j_l(k;\omega) \right| \left| \langle D \rangle^j_l(k;\omega) \right|
\]

From the assumption (a), we have that all the terms

\[
\left| \langle D \rangle^\sigma \hat{P^d}(k;\omega) \right|_{s_0}, \left| \hat{P^d}(k;\omega) \langle D \rangle^\sigma \right|_{s_0}, \left| \langle D \rangle^\sigma \hat{P^d}(k;\omega) \langle D \rangle^{-\sigma} \right|_{s_0},
\]

(with \( \sigma = \pm, 0, \delta = d, o \)) are bounded. In order to bound \( \|X^+(k;\omega)\| \), what we have to prove is that we can control also the terms

\[
\left| \langle D \rangle^\sigma \hat{X^d}(k;\omega) \right|_{s_0}, \left| \hat{X^d}(k;\omega) \langle D \rangle^\sigma \right|_{s_0}, \left| \langle D \rangle^\sigma \hat{X^d}(k;\omega) \langle D \rangle^{-\sigma} \right|_{s_0}.
\]

The seminorms involving the diagonal term \( X^d \) can be easily handled, since, by (5.26), they are essentially bounded by the same seminorms for \( P^d \). The similar bound in (5.24) is enough also.
when we consider the terms \(|\langle D \rangle^\alpha \widehat{X}^\alpha(k; \omega)\rangle \). Consider now the term \(|\langle D \rangle^\alpha \widehat{X}^\alpha(k; \omega)\rangle\).

Applying again (5.24), we get

\[
\left| \left( \langle D \rangle^\alpha \widehat{X}^\alpha(k; \omega) \right) \right|^2 = \frac{2 \langle N \rangle^\gamma \omega}{\gamma} \frac{\langle l \rangle^\alpha}{(j + l)^\alpha} \left| \left( \langle P \rangle \right)^\alpha \right| (k; \omega) \leq \frac{2 \langle N \rangle^\gamma \omega}{\gamma} \left| \left( \langle P \rangle \right)^\alpha \right| (k; \omega) .
\]

Therefore, it follows that

\[
\left| \langle D \rangle^\alpha \widehat{X}^\alpha(k; \omega) \right| \leq \frac{2 \langle N \rangle^\gamma \omega}{\gamma} \left| \left( \langle P \rangle \right)^\alpha \right| (k; \omega) ,
\]

The same bound holds for \(|\langle D \rangle^\alpha \widehat{X}^\alpha(k; \omega)\rangle\). We obtain that

\[
\left\| \widehat{X}^\alpha(k; \omega) \right\| \leq \frac{2 \langle N \rangle^\gamma \omega}{\gamma} \left| \left( \langle P \rangle \right)^\alpha \right| (k; \omega) ,
\]

and, consequently,

\[
\left\| \widehat{X}^\alpha \right\|_{p, \Omega^\alpha} \leq \frac{2 \langle N \rangle^\gamma \omega}{\gamma} \left| \left( \langle P \rangle \right)^\alpha \right|_{p, \Omega^\alpha} .
\]

We deal now with the estimates on the Lipschitz seminorm \(|\langle X \rangle^\alpha \|_{L^\infty} \). Using the notation (3.22) that we have introduced during the proof of Theorem 3.11 we have, for \( \delta = \delta, \alpha \):

\[
\Delta_\omega(X^\delta) j\{k; \omega \} = \Delta_\omega \left( \frac{i}{g_{j,l}^\pm (\omega)} \right) \left( \langle P \rangle \right)^\delta j\{k; \omega \} + \frac{i}{g_{j,l}^\pm (\omega + \Delta_\omega)} \Delta_\omega \left( \langle P \rangle \right)^\delta j\{k; \omega \}
\]

\[
\leq -i \Delta_\omega \left( \frac{g_{j,l}^\pm (\omega)}{g_{j,l}^\pm (\omega + \Delta_\omega)} \right) \left( \langle P \rangle \right)^\delta j\{k; \omega \} + \frac{i}{g_{j,l}^\pm (\omega + \Delta_\omega)} \Delta_\omega \left( \langle P \rangle \right)^\delta j\{k; \omega \} .
\]

By the assumption in (5.24), we have that

\[
\Delta_\omega (g_{j,l}^\pm (\omega)) = | \Delta_\omega (k + \Delta_\omega) \gamma_{j,l}^\pm | \leq |k| \gamma |\Delta_\omega | + 2 \gamma |\Delta_\omega | \leq \langle N \rangle |\Delta_\omega |
\]

uniformly for every \( j, \ell \in \mathbb{N} \) and \( k \in \mathbb{Z}^\nu \), \(|k| \leq N \). We can estimate (5.33) by

\[
\Delta_\omega (X^\delta) j\{k; \omega \} \leq \frac{2 \langle N \rangle |\Delta_\omega |}{\gamma^2} \left( \langle P \rangle \right)^\delta j\{k; \omega \} + \frac{1}{\gamma \langle j + \ell \rangle^\alpha} \Delta_\omega \left( \langle P \rangle \right)^\delta j\{k; \omega \} .
\]

Now, arguing as we did for getting the first control on the s-decay (5.31), one obtains that

\[
\left\| \Delta_\omega \widehat{X}^\alpha(k; \omega) \right\| \leq \frac{8 \langle N \rangle^2 |\Delta_\omega |}{\gamma^2} \left| \left( \langle P \rangle \right)^\alpha \right| (k; \omega) + \frac{2 \langle N \rangle^\gamma \omega}{\gamma} \left| \left( \langle P \rangle \right)^\alpha \right| (k; \omega) .
\]
and therefore
\[
\|X^+\|_{\text{Lip}, \Omega^+} \leq \frac{8 \langle N \rangle^{2r+1} M^{2\alpha}}{\gamma^2} \|P\|_{\text{Lip}, \Omega^+}^2 + \frac{2 \langle N \rangle^{2r+1} M^{\alpha}}{\gamma} \|P\|_{\text{Lip}, \Omega^+}^2.
\]

Finally, we plug (5.32) and (5.37) into (5.32) and obtain
\[
\|X^+\|_{\text{Lip}(\gamma)} := \|X^+\|_{\text{Lip}, \Omega^+} + \frac{\gamma}{N} \|X^+\|_{\text{Lip}, \Omega^+} \leq C^+ \|P\|_{\text{Lip}(\gamma)}
\]
where the constant $C^+$ is given by
\[
C^+ := \max\left\{2 \langle N \rangle^{2r+1} \frac{M^\alpha}{\gamma}, 8 \langle N \rangle^{2r+1} \frac{M^\alpha}{\gamma}, 2 \langle N \rangle^{2r+1} \frac{M^\alpha}{\gamma} \right\} \leq 16 \langle N \rangle^{2r+1} \frac{M^\alpha}{\gamma}.
\]

\[\square\]

**Lemma 5.5.** Let $P \in \text{Lip}_\gamma(\Omega, \mathcal{M}_{\rho, \eta}(\alpha, 0))$. Assume (5.24) and
\[
C_{\rho} 16 \langle N \rangle^{2r+1} \frac{M^\alpha}{\gamma} \|P\|_{\text{Lip}(\gamma)} < 1.
\]
Then $P^+ = \Pi_{\rho} P + R$, defined as in (5.32), belongs to $\text{Lip}_\gamma(\Omega^+, \mathcal{M}_{\rho^+, \eta}(\alpha, 0))$ for any $0 < \rho^+ < \rho$, with bounds
\[
\|\Pi_{\rho} P\|_{\text{Lip}(\gamma)} \leq e^{-(\rho^+ - \rho)N} \|P\|_{\text{Lip}(\gamma)}, \quad \|R^+\|_{\text{Lip}(\gamma)} \leq C_{\rho} 2^\alpha \langle N \rangle^{2r+1} \left(\|P\|_{\text{Lip}(\gamma)}\right)^2.
\]

**Remark 5.6.** We will show in Proposition 5.8 that the smallness assumption in (5.40) implies the one in (5.24). This allows us to apply Lemma 5.4 in the following proof.

**Proof.** The operator $\Pi_{\rho}^+ P$ belongs to $\text{Lip}_\gamma(\Omega, \mathcal{M}_{\rho^+, \eta}(\alpha, 0))$, since it holds that:
\[
\|\Pi_{\rho}^+ P\|_{\text{Lip}(\gamma)} = \sum_{|k| > N} e^{\rho^+ |k|} \|\hat{P}(k)\|_{\text{Lip}(\gamma)} \leq e^{-(\rho^+ - \rho)N} \|P\|_{\text{Lip}(\gamma)}.
\]
Consider now the integral remainder
\[
R^+ := \int_0^1 (1 - s)e^{ix^+} \text{ad}_{X^+} (Z - P)e^{-isx^+} ds + \int_0^1 e^{ix^+} \text{ad}_{X^+} (P)e^{-isx^+} ds.
\]

By Lemma 2.25, we have $R^+ \in \text{Lip}_\gamma(\Omega^+, \mathcal{M}_{\rho, \eta}(\alpha, \alpha))$ with
\[
\|R^+\|_{\text{Lip}(\gamma)} \leq 4 C_{\rho} e^{2C_{\rho}} \|X^+\|_{\text{Lip}(\gamma)} \|P\|_{\text{Lip}, \Omega^+} \leq C_{\rho} 2^\alpha \|X^+\|_{\text{Lip}(\gamma)} \|P\|_{\text{Lip}(\gamma)}.
\]
Therefore the estimate follows by Lemma 5.4. By monotonicity of the involved norms, we conclude that $R^+ \in \text{Lip}_\gamma(\Omega^+, \mathcal{M}_{\rho^+, \eta}(\alpha, 0))$. \[\square\]

**Remark 5.7.** Defining the quantities
\[
\eta := \frac{M^\alpha}{\gamma} \|P\|_{\text{Lip}(\gamma)}, \quad \eta^+ := \frac{M^\alpha}{\gamma} \|P^+\|_{\text{Lip}(\gamma)},
\]
with some positive constant $c > 0$, Lemma 5.3 implies that
\[
\eta^+ \leq \left(e^{-\delta N} + \langle N \rangle^{2r+1} \right) \eta, \quad \delta := \rho - \rho^+.
\]
In particular, if $N = -\delta^{-1} \ln \eta$, we find an (almost-)quadratic estimate:
\[
\eta^+ \leq \left(1 + \langle N \rangle^{2r+1}\right) \eta^2 \approx \left(1 + \frac{1}{\delta^{2r+1}} \left(\frac{1}{\eta} \right)^{2r+1}\right) \eta^2.
\]
5.4 Iterative Lemma and KAM reduction

Once that the general step has been illustrated, we are ready for setting our iterative scheme. The Hamiltonian the iteration starts with is the one that we have found after the Magnus normal form in Section 3.

\[
H^{(0)}(t) = H_{0}^{(0)} + V^{(0)}(\omega t; \omega), \quad |V^{(0)}|^{\Lip(\gamma)}_{p_{0}, \Omega_{0}} \lesssim |V^{(0)}|^{\Lip(\gamma)}_{p_{0}, \Omega_{0}} \lesssim C/H,
\]

where \(H_{0}^{(0)} := H_{0}\) and \(V^{(0)} := V\) as in Theorem 3.1. All the iterated items are constructed from the general transformation in Sections 5.2, 5.3 by setting for \(n \geq 0\)

\[
H^{(n)}(t) := A(\omega) + P(\omega t; \omega), \quad A := H_{0}^{(n)}, \quad P := V^{(n)},
\]

\[
Z := Z^{(n)}, \quad X := X^{(n)}, \quad R := R^{(n)},
\]

\[
\rho := \rho_{n}, \quad \Omega := \Omega_{n}, \quad \eta := \eta_{n} := \frac{M^{\alpha}}{\gamma} |V^{(n)}|^{\Lip(\gamma)}_{p_{n}, \Omega_{n}},
\]

\[
\delta := \delta_{n} := \rho_{n} - \rho_{n+1}, \quad \lambda_{j}^{(n)}(\omega) := \lambda_{j} + \sum_{h=0}^{n-1} (V^{(h)}, A)_{j}(0; \omega)
\]

and using the notation \(\ast : n \rightarrow n + 1\) (only exception: the indexes set \(I_{N}^{+}\)). Moreover, we choose \(\delta_{n} := (1 + n^{2})^{-1}\delta_{0}\), with \(\delta_{0} = \frac{1}{\pi \rho_{0}}\) in such a way that

\[
\sum_{n \in \mathbb{N}} \delta_{n} = \frac{1}{\delta_{0}} \sum_{n \in \mathbb{N}} \frac{1}{1 + n^{2}} \lesssim \frac{\rho_{0}}{2} = \frac{\rho}{8}
\]

and we also set

\[
N := N_{n} = \frac{1}{\delta_{0}} \ln \eta_{n}, \quad \chi := \frac{3}{2}.
\]

**Proposition 5.8 (Iterative Lemma).** Fix \(\tau > 0\). There exists \(k_{0} = k_{0}(\tau, \rho_{0}, s_{0}) > 0\) such that for any \(0 < \gamma < \tilde{\gamma}\), any \(H > 0\) for which

\[
\eta_{0} := \frac{M^{\alpha}}{\gamma} |V^{(0)}|^{\Lip(\gamma)}_{p_{0}, \Omega_{0}} \lesssim k_{0} e^{-1},
\]

the following items hold true for any \(n \in \mathbb{N}:

(i) The sets \(\Omega_{n}\) are defined recursively by

\[
\Omega_{n+1} := \left\{ \omega \in \Omega_{n} : \left| \omega \cdot k + \lambda_{j}^{(n)}(\omega) + \lambda_{l}^{(n)}(\omega) \right| > \frac{\gamma}{2 N^{\frac{1}{n}}} \frac{j + l^{\alpha}}{\delta_{0}}, \quad \forall (k, j, l) \in I_{N}^{+} \right\};
\]

(ii) For every \(\omega \in \Omega_{n}\), the operator \(X^{(n)}(\omega, \cdot) \in \Lip_{s}(\Omega_{n}, M_{\rho_{n-1}, s_{n}}(\alpha, \alpha))\) with the quantitative bound

\[
\|X^{(n)}\|^{\Lip(\gamma)}_{\rho_{n-1}, \Omega_{n}} \lesssim \sqrt{k_{0} e^{\frac{1}{2}(1 - \chi^{-1})}}.
\]
We start with item (i). We prove now that (5.40) is fulfilled. Using (5.62) with as long as \( \eta \), and the functions \( \lambda_j^{(n)}(\omega) = \lambda_j^{(n)}(\omega; H, \alpha) \) are defined over all \( \Omega_0 \), fulfilling

\[
(5.58) \quad |\lambda_j^{(n)} - \lambda_j^{(n-1)}|_{\text{Lip}} |_{\Omega_0} \leq \eta_0 e^{1-\chi_{n-1}};
\]

(iv) The new perturbation \( \mathbf{V}^{(n)} \in \text{Lip}_\gamma(\Omega_n, \mathcal{M}_{\rho_n, \eta_0}(\alpha, 0)) \) and

\[
(5.59) \quad \eta_n := \frac{M_n^\alpha}{\gamma} \left| \mathbf{V}^{(n)} \right|_{\text{Lip}(\gamma)} |_{\rho_n, \eta_0} \leq \eta_0 e^{1-\chi_n}.
\]

Proof. We argue by induction. For \( n = 0 \) one requires (5.55). Now, assume that the statements hold true up to a fixed \( n \in \mathbb{N} \). Define \( \Omega_{n+1} \) as in item (i). In order to apply Lemma 5.4 and Lemma 5.5 we need to check that the assumptions in (5.24) and (5.40) are verified, respectively. First, note that, by item (iii),

\[
(5.60) \quad \left| \lambda_j^{(m)} \right|_{\Omega_0} \leq \sum_{m=1}^{n} \left| \lambda_j^{(m)} - \lambda_j^{(m-1)} \right|_{\Omega_0} + \left| \lambda_j \right|_{\Omega_0} \leq \eta_0 e^{\sum_{m=1}^{n} e^{-\chi_{m-1}}} \leq \eta_0 e,
\]

so that (5.24) is satisfied, provided simply

\[
(5.61) \quad \eta_0 e \leq 1.
\]

Now, standard calculus arguments show that, for an arbitrary \( \beta > 0 \),

\[
(5.62) \quad e^{-x} < \frac{1}{x^\beta} \quad \forall x > 1 \quad \Rightarrow \quad \ln \frac{1}{y} < \frac{1}{y^{1/\beta}} \quad \forall y \in (0, 1).
\]

We prove now that (5.40) is fulfilled. Using (5.62) with \( \beta = 2(2\tau + 1) \) and (5.59), we have

\[
\langle N_n \rangle^{2\tau + 1} \frac{M_n^\alpha}{\gamma} \left| \mathbf{V}^{(n)} \right|_{\rho_n, \eta_0} \leq \left\langle -\frac{1}{\delta_n} \ln \eta_n \right\rangle^{2\tau + 1} \eta_n \leq \left( \frac{1 + n^2}{\delta_0} \right)^{2\tau + 1} \frac{\eta_n^\beta}{\eta_0^\beta} \leq (\eta_0 e)^{\beta} \frac{\eta_0^\beta}{\eta_0^\beta} \left( \frac{1 + n^2}{\delta_0} \right)^{2\tau + 1} \leq \frac{1}{2 \cdot 16 \cdot \gamma}.
\]

as long as \( \eta_0 \) satisfies the bound

\[
(5.63) \quad \eta_0 e \leq \frac{1}{2^{10} \cdot \gamma^2} \min_{n \geq 0} \left( e^x \frac{\eta_0^\beta}{\eta_0^\beta} \left( \frac{\delta_0}{1 + n^2} \right)^{2\tau + 1} \right)^2.
\]

Therefore we can apply Lemma 5.4 and Lemma 5.5 with \( P = \mathbf{V}^{(n)} \) and define \( X^{(n+1)} \in \text{Lip}_\gamma(\Omega_{n+1}, \mathcal{M}_{\rho_{n+1}, \eta_0}(\alpha, \alpha)) \), the new eigenvalues

\[
(5.64) \quad \lambda^{(n+1)}(\omega) := \lambda_j^{(n)}(\omega) + (\mathbf{V}_d(\omega)) j(\theta; \omega) \quad \forall j \in \mathbb{N}
\]

and the new perturbation \( \mathbf{V}^{(n+1)} \). We are left only with the quantitative estimates. We start with item (iv). Recall the (almost-)quadratic scheme in Remark 5.7.

\[
(5.65) \quad \frac{M_n^\alpha}{\gamma} \left| \mathbf{V}^{(n+1)} \right|_{\rho_{n+1}, \Omega_{n+1}} \leq \eta_{n+1} \leq (1 + \langle N_n \rangle^{2\tau + 1} \eta_n^\beta) \eta_n^2 \simeq \left( 1 + \frac{1}{\delta_0} \left( \frac{\ln \eta_n}{\eta_n} \right)^{2\tau + 1} \right) \eta_n^2.
\]
We want to show that this implies (5.59) at the level \( n + 1 \). By (5.62) with \( \beta = 4(2n + 1) \), we get
\[
\eta_{n+1} \leq 2 \left( \frac{1 + n^2}{\delta_0} \right)^{2^r + 1} \eta_n^\beta \leq 2 \left( \frac{1 + n^2}{\delta_0} \right)^{2^r + 1} (\eta_0 e)^{2^r - \frac{1}{2} n^2}.
\]
Thus, (5.60) is satisfied at the iteration \( n + 1 \) if
\[
\eta_0 e \leq \frac{1}{2^r} \min_{n \geq 0} \left( e^{\frac{1}{2} n^2} \left( \frac{\delta_0}{1 + n^2} \right)^{2^r + 1} \right)^{\frac{1}{2^r}}.
\]

For item (iii), it is sufficient to note that
\[
\left| \lambda_j^{(n+1)} - \lambda_j^{(n)} \right|^{\text{Lip}}_{\Omega_n} \leq \left| (V^{(d)}(n))_j(0, \cdot) \right|^{\text{Lip}}_{\Omega_n} \leq \frac{M_\gamma}{\gamma} \left| V^{(n)} \right|^{\text{Lip}(\gamma)}_{\rho_n, \Omega_n} \leq \eta_0 e^{1 - \frac{1}{2} n^2}.
\]

Now, by Kirszbraun theorem, we can extend the functions \( \lambda_{j}^{(n)}(\omega, \mathcal{M}) \) to all \( \Omega_0 \) preserving their Lipschitz constant. This proves (iii). Finally we prove item (ii). By (5.25), the inductive assumption, and (5.62) with \( \beta = 4(2n + 1) \), we get
\[
\left\| X^{(n+1)} \right\|_{\text{Lip}(\gamma)}^{\rho_n, \Omega_{n+1}} \leq C N^{2r + 1} \delta_n \leq C \left( \frac{\delta_0^{-1}(n^2 + 1)}{\eta_0 e} \right)^{2^r + 1} (\eta_0 e)^{\frac{1}{2} n^2} e^{-\frac{1}{2} n^2},
\]
provided that
\[
\eta_0 e \leq \min_{n \geq 0} \left( e^{\frac{1}{2} n^2} \left( \frac{\delta_0}{n^2 + 1} \right)^{2^r + 1} \right)^{\frac{1}{2^r}}.
\]

This proves (ii) at step \( n + 1 \). By selecting \( \eta_0 \) in order to fulfill (5.61), (5.68), (5.69) and (5.69), the induction is closed.

A consequence of the iterative lemma is the following result.

**Corollary 5.9** (Final eigenvalues). Fix \( \tau > \bar{\tau} \). Assume (5.55). Then for every \( \omega \in \Omega_0 \) and for every \( j \in \mathbb{N} \), the sequence \( \{\lambda_j^{(n)}(\cdot; \mathcal{M}, \alpha)\}_{n \geq 1} \) is a Cauchy sequence. We denote by \( \lambda_j^{(\infty)}(\omega; \mathcal{M}, \alpha) \) its limit, which is given by
\[
\lambda_j^{(\infty)}(\omega) = \lambda_j + \varepsilon_j^{(\infty)}(\omega), \quad \varepsilon_j^{(\infty)}(\omega) := \sum_{n=0}^{\infty} (V^{(n)}_j(0, \omega)),
\]
and one has the estimate
\[
\sup_{j \in \mathbb{N}} \|j^{\alpha} \varepsilon_j^{(\infty)} \|^{\text{Lip}(\gamma)}_{\Omega_0} \leq \frac{\gamma}{M^{\alpha}} \eta_0 e.
\]

**Proof.** We have
\[
\left| j^{\alpha} (V^{(n)} d)_j(0; \omega) \right| \leq \left| (D^ \alpha V^{(n)} d)(0; \omega) \right|_{\rho_0, \Omega_n} \leq \left| V^{(n)} \right|^{\text{Lip}(\gamma)}_{\rho_n, \Omega_n} \leq \frac{\gamma}{M^{\alpha}} \eta_0 e^{1 - \frac{1}{2} n^2},
\]
and the thesis follows easily.
In the next result we study the convergence of the sequence of transformations.

**Corollary 5.10** (Iterated flow). Under the same assumptions of Corollary 5.9, for any \( \omega \in \cap_n \Omega_n \) and \( \theta \in \mathbb{T}^n \), the sequence of transformations

\[
W^n(\theta; \omega) := e^{-iX^{(1)}(\theta \omega)} \circ \cdots \circ e^{-iX^{(n)}(\theta \omega)}
\]

is a Cauchy sequence in \( \mathcal{L}(\mathcal{H}^r \times \mathcal{H}^r) \) with \( r \in [0, s_0] \) with the following estimates for each \( n \in \mathbb{N} \):

\[
\|W^n(\theta; \omega)\|_{\mathcal{L}(\mathcal{H}^r \times \mathcal{H}^r)} \leq e^{\sqrt{\gamma} \|\omega\|_{\Sigma}}, \quad \Sigma := \sum_{q=0}^{n} e^{-\frac{1}{q+1}} \approx 1.69409\ldots,
\]

\[
\|W^n(\theta; \omega) - I\|_{\mathcal{L}(\mathcal{H}^r \times \mathcal{H}^r)} \leq \sqrt{\gamma} e^{\sqrt{\gamma} \|\omega\|_{\Sigma}}.
\]

We denote by \( W^\omega(\theta; \omega) \) its limit in \( \mathcal{L}(\mathcal{H}^r \times \mathcal{H}^r) \).

The proof of the convergence of the transformations is standard, while the control of the operator norm \( \mathcal{L}(\mathcal{H}^r \times \mathcal{H}^r) \) follows from Remark 5.10; we skip the details.

Since for any \( j \in \mathbb{N} \) the sequence \( \{\lambda_j^{(n)}\}_{n=1}^\infty \) converges to a well defined Lipschitz function \( \lambda_j^x \) defined on \( \Omega_0 \), we can now impose second order Melnikov conditions only on the final frequencies.

**Lemma 5.11.** Consider the set

\[
\Omega_{x, \alpha} := \left\{ \omega \in \mathcal{U}_n \mid |\omega \cdot k + \lambda_j^x(\omega) \pm \lambda_j^\alpha(\omega)| \geq \frac{N}{M^\alpha}, \quad \forall (k, j, l) \in \mathcal{I}_+^+ \right\}.
\]

We have that \( \Omega_{x, \alpha} \subseteq \cap_n \Omega_n \).

**Proof.** By definition \( \Omega_{x, \alpha} \subseteq \Omega_0 \). By induction, assume that it is contained in \( \cap_{m \leq n} \Omega_m \equiv \Omega_n \). We want to show that it is also contained in

\[
\Omega_{n+1} := \left\{ \omega \in \Omega_n \mid |\omega \cdot k + \lambda_j^{(n)}(\omega) \pm \lambda_j^{(n)}(\omega)| \geq \frac{N}{2M^\alpha}, \quad \forall (k, j, l) \in \mathcal{I}_+^+ \right\}.
\]

First recall that

\[
\lambda_j^x(\omega) - \lambda_j^{(n)}(\omega) \leq \sum_{m \geq n} \left| \lambda_j^{(m+1)}(\omega) - \lambda_j^{(m)}(\omega) \right| \leq \sum_{m \geq n} \left| V^{(m)} \right|_{\|\cdot\|_{\Omega_m \wedge \Omega_n}} \leq \frac{1}{4 M^\alpha} \frac{1}{\langle N_m \rangle^{2+1}} \leq \frac{1}{16 M^\alpha} \frac{1}{\langle N_m \rangle^{1+1}} \leq \frac{1}{4 M^\alpha} \frac{1}{\langle N_m \rangle^{1+1}}.
\]

Thus for \( |k| \leq N_n \),

\[
|\omega \cdot k + \lambda_j^{(n)}(\omega) \pm \lambda_j^{(n)}(\omega)| \geq |\omega \cdot k + \lambda_j^{x}(\omega) \pm \lambda_j^{\alpha}(\omega)| - 2 \sup_m \left| \lambda_j^{x}(\omega) - \lambda_j^{(n)}(\omega) \right| \geq \frac{\gamma}{\langle k \rangle^{\alpha}} \frac{|j \pm l|^\alpha}{M^\alpha} - \frac{\gamma}{2 M^\alpha} N^{-\tau} \geq \frac{\gamma}{2 M^\alpha} N^{-\tau} \geq \frac{\gamma}{2 M^\alpha} N^{-\tau}.
\]

The proof of the next lemma is postponed in Appendix 5.2

**Lemma 5.12.** Under the conditions \( \tau > \nu + \frac{\alpha}{M} \), \( 0 < \gamma \leq \gamma_2/2 \) and \( \underline{4.13} \) of Theorem 4.1, there exists a constant \( C_x > 0 \), independent of \( M \) and \( \gamma \), such that

\[
m_\tau(\mathcal{U}_n \setminus \Omega_{x, \alpha}) \leq C_x \gamma.
\]
Now we can prove the KAM reducibility

**Theorem 5.13 (KAM reducibility).** Fix $\alpha \in (0, 1)$, $s_0 > 1/2$, and $\tau > \nu + 1 + \alpha + \frac{\gamma}{2}$. For any $\gamma < \gamma_0$, there exists $M_s = M_s(\alpha, \gamma, \rho_0) > 0$ such that for any $M \geq M_s$ the following holds true. There exist functions $\{\lambda_j^\infty(\alpha, \beta, \gamma, \rho_0)\}_{j \in \mathbb{N}}$, defined and Lipschitz in $\omega$ in the set $R_\omega$ such that:

1. The set $\Omega_{E, \omega} = \Omega_{E, \omega}(\gamma, \tau, M) \subset R_\omega$ defined in (5.7.10) fulfills $m_r(R_\omega \setminus \Omega_{E}) \leq C(\gamma + \gamma_1^{1/3} + \gamma_0)$, where $\gamma_0$ is defined in Theorem 5.7.4 and $\gamma_1$ in Theorem 4.1.
2. For each $\omega \in \Omega_{E, \omega}$ there exists a change of coordinates $\psi = \mathcal{W}_\infty(\omega t, \omega)\phi$ which conjugates the equation

\[ i\dot{\psi} = \mathbf{H}(t)\psi, \quad \mathbf{H}(t) = \mathbf{H}(0)(\omega t; \omega) \]

to a constant-coefficient diagonal one:

\[ i\dot{\phi} = \mathbf{H}_\infty \phi, \quad \mathbf{H}_\infty = \mathbf{H}_\infty(\omega, \alpha) = \text{diag}\{\lambda_j^\infty(\omega, \alpha) \mid j \in \mathbb{N}\} \sigma_3. \]

Furthermore for any $r \in [0, s_0]$ one has

\[ \|\mathcal{W}_\infty - \mathbf{1}\|_{\mathcal{L}(H^r \times H^r)} \leq \sqrt{\gamma_0} \epsilon \Sigma e^{\sqrt{\gamma_0} r \Sigma}. \]

**Proof.** Having fixed $\alpha, s_0$ and $\tau$, we can produce the constant $k_0(\delta_0, \tau)$ of the iterative Lemma 5.8. Having fixed also $0 < \gamma < \gamma_1$, we produce $M_s(\alpha, \gamma, \rho_0, s_0)$ in such a way that for every $M \geq M_s$, the estimate (5.5.7) is fulfilled. More explicitly,

\[ \frac{M^\alpha}{\gamma} \left| \mathcal{V}(0) \right|_{\rho_0, \Omega_0}^{Lip(\gamma)} \leq \frac{C}{\gamma H^{1-\alpha}} \leq k_0 \epsilon^{-1} \Rightarrow M \geq \left( \frac{C \epsilon}{\gamma k_0} \right)^{\frac{1}{1-\alpha}}. \]

We can now apply the iterative Lemma 5.8, Corollary 5.9, and Lemma 5.11 to get the result. □

**Remark 5.14.** Note that, the more $\alpha$ approaches 1, the bigger the threshold $M_s$ is and so $M$.

### 5.5 A final remark

The KAM reducibility scheme that we have presented has transformed Equation (5.7.8), which is the result of the Magnus normal form of Section 5 into

\[ i\dot{\phi}(t) = \text{diag}\{\lambda_j^\infty(\omega, \alpha)\}\phi(t) \]

where the asymptotic for the final eigenvalues are given, using Equation (5.7.11) of Corollary 5.9 by

\[ \lambda_j^\infty(\omega, \alpha) \sim O\left( \frac{\gamma_0}{M^\alpha j^\alpha} \right). \]

Moreover, as we have stated in Corollary 5.10 and Theorem 5.13, the transformations performed in the KAM steps, as well as their limit, are close to the identity: this fact implies an almost conservation of the Sobolev norms, see Corollary 5.7.4.

One can argue that the asymptotic $\lambda_j^\infty(\alpha) \sim O(M^{-1} j^{-\alpha})$ is not that satisfying, since the perturbation $\mathbf{V}^{(0)}$ at the beginning of the KAM scheme belongs to the class $M_{\rho_0, s_0}(1, 0)$ and so its diagonal elements have a smoothing effect of order 1 which could be expected to be preserved in the effective Hamiltonian.
Actually, it is possible to modify our reducibility scheme for achieving this result: we explain now briefly how to do it.

After the Magnus normal form, we conjugate system (5.78) through \( e^{-iY(\omega t)} \), where

\[
(5.83) \quad Y(\omega t) := \begin{pmatrix} 0 & Y^\omega(\omega t) \\ -\overline{Y^\omega(\omega t)} & 0 \end{pmatrix}
\]

so that \( Y^\omega \) solves the homological equation

\[
(5.84) \quad -i[Y^\omega(\theta), B]_+ + V^\omega(\theta) - \omega \cdot \mathcal{E}_0 Y^\omega(\theta) = 0 \quad \Rightarrow \quad \overline{(V^\omega)^\gamma_j(k)} := \frac{(V^\omega)^{\gamma_j}(k)}{i(\omega \cdot k + \lambda_j + \lambda_l)} \quad \forall \ k, j, l .
\]

We ask now the frequency vector \( \omega \) to belong to \( U_1 \cap U_0 \) (see (4.5)). In this way one gets (in the same lines of the proof of Lemma 5.4) that \( Y \in \text{Lip}_{\gamma}^{\gamma}(U_1, M_{\beta_0, s_0}(1, 1)) \), since we have chosen \( \omega \in U_1 \), with the bound

\[
(5.85) \quad |Y\rangle_{\text{Lip}(\gamma/\gamma)} \leq C_{\gamma} \frac{M_{\gamma}}{\gamma} \left| V(0) \right|_{\rho_0, s_0, 1, 1} \leq C_{\gamma} \frac{M_{\gamma}}{\gamma} \left| V(0) \right|_{\rho_0, s_0, 1, 0} \leq \tilde{C} .
\]

and the new perturbation

\[
(5.86) \quad \mathcal{V}(0)(\omega t) := \begin{pmatrix} V^\omega(\omega t) & 0 \\ -\overline{V^\omega(\omega t)} & \end{pmatrix} + \int_0^1 (1 - s)e^{isY(\omega t)} \text{ad}_Y(\omega t)\left| V(0) \right| e^{-isY(\omega t)} ds
\]

belongs to the class \( \text{Lip}_{\gamma}^{\gamma}(U_1, M_{\beta_0, s_0}(1, 1)) \) fulfilling estimate (5.2). Thus, one can perform a KAM reducibility scheme as in Section 5.3–5.4, in which one takes \( \alpha = 0 \) in (5.19), the perturbations appearing in the iterations stay in the class \( \text{Lip}_{\gamma}^{\gamma}(U_1, M_{\beta_0, s_0}(1, 1)) \) and the new final eigenvalues \( \lambda^\gamma \) satisfy the nonresonance condition

\[
(5.87) \quad |\omega \cdot k + \lambda^\gamma_j \pm \lambda^\gamma_l| \geq \frac{\gamma}{\langle k \rangle} , \quad \forall \ (k, j, l) \in \mathbb{I}^\pm .
\]

In particular, we gain a more regularizing asymptotics on the final eigenvalues, that is \( \lambda^\gamma_j \sim O(1/j) \). The price that we pay for this result is that the preliminary change of coordinate \( e^{-iY(\omega t)} \) is not a transformation close to identity, as the generator \( Y(\omega t) \) is just a bounded operator and not small in size, see (5.85). The main consequence is on the effective dynamics of the original system, as Corollary 5.4 is no more valid. In this case, it is possible to conclude just that the Sobolev norms stay uniformly bounded in time and do not grow, but in general their (almost-)conservation is lost.

\section{A \ Technical results}

\subsection{A.1 \ Properties of pseudodifferential operators}

Recall that if \( F \) is an operator, we denote by \( \widehat{F}(k) \) its \( k \)-th Fourier coefficient defined as in (2.14). If \( F \) is a pseudodifferential operator with symbol \( f \), so \( \widehat{F}(k) \) is, with symbol given by

\[
\hat{f}(k, x, j) := \frac{1}{(2\pi)^{2}} \int_{\mathbb{R}^{2}} f(\theta, x, j) e^{-i\theta \cdot k} d\theta .
\]

\textbf{Lemma A.1.} Let \( \rho > 0 \) and \( \mu \in \mathbb{R} \). The following holds true:
(i) If $F \in \mathcal{A}_\rho^\mu$, then the operator $\hat{F}(k)$ belongs to $\mathcal{A}^\mu$ for any $k \in \mathbb{Z}^\nu$ and

$$\varphi_\ell^\mu(\hat{F}(k)) \leq e^{-\rho|k|} \varphi_\ell^\mu(F) \quad \forall \ell \in \mathbb{N}_0.$$ 

(ii) Assume to have $\forall k \in \mathbb{Z}^\nu$ an operator $\hat{F}(k) \in \mathcal{A}^\nu$ fulfilling

$$(A.1) \quad \varphi_\ell^\nu(\hat{F}(k)) \leq \langle k \rangle^\tau e^{-\rho|k|} C_\ell \quad \forall k \in \mathbb{Z}^\nu, \quad \forall \ell \in \mathbb{N}_0,$$

for some $\tau \geq 0$, $\rho > 0$ and $C_\ell > 0$ independent of $k$. Define the operator $F(\theta) := \sum_{k \in \mathbb{Z}^\nu_0} \hat{F}(k)e^{i\theta \cdot k}$. Then, $F$ belongs to $\mathcal{A}_\rho^\nu$ for any $0 < \rho' < \rho$ and one has

$$\varphi_\ell^\nu,\rho(\hat{F}(k)) \leq \frac{C_\ell}{(\rho - \rho')^{\tau + \nu}} \quad \forall \ell \in \mathbb{N}_0.$$ 

On the classes $\text{Lip}_\rho(\Omega, \mathcal{P}\mathcal{A}_\rho^\mu)$, these assertions extend naturally without any further loss of analyticity.

Proof. (i) By Cauchy estimates, it is well-known the analytic decay for the Fourier coefficients of the symbol $f(\theta; x, j)$:

$$\hat{f}(k, x, j) \leq e^{-\rho|k|} \sup_{|\Im \theta| \leq \rho} |f(\theta, x, j)|.$$ 

Plugging it into Definition 2.2 of $\varphi_\ell^\mu(F(\hat{k}))$, we get the claim;

(ii) It is possible to control the seminorm $\varphi_\ell^\nu,\rho(F)$ in terms of the ones for the Fourier coefficients:

$$\varphi_\ell^\nu,\rho(F) \leq \sum_{k \in \mathbb{Z}^\nu} e^{\rho|k|} \varphi_\ell^\nu(\hat{F}(k)) \leq \sum_{k \in \mathbb{Z}^\nu} e^{(\rho'-\rho)|k|} \langle k \rangle^\tau C_\ell \leq \frac{C_\ell}{(\rho - \rho')^{\tau + \nu}}.$$ 

In the next Proposition we essentially prove that pseudodifferential operators as in Definition 2.1 have matrices which belong to the classes $\text{Lip}_\rho(\Omega, \mathcal{M}_{\rho,s})$ extended from Definition 2.17.

**Proposition A.2.** Let $F \in \text{Lip}_\rho(\Omega, \mathcal{P}\mathcal{A}_\rho^\mu)$, with $\rho > 0$. For any $0 < \rho' < \rho$ and $s > \frac{1}{2}$, the matrix of the operator

$$\langle D \rangle^\alpha \langle D \rangle^\beta \langle D \rangle^\mu \quad \alpha + \beta + \mu \leq 0,$$

belongs to $\text{Lip}_\rho(\Omega, \mathcal{M}_{\rho',s})$. Moreover for any $s > \frac{1}{2}$, $\forall \alpha + \beta \leq -\mu$, there exists $\sigma > 0$ such that

$$(A.4) \quad \left| \langle D \rangle^\alpha \langle D \rangle^\beta \langle D \rangle^\mu \right|_{\rho',s,\Omega} \leq \frac{C}{(\rho - \rho')^\sigma} \varphi_\sigma^{\mu,\rho}(F)_{\rho,\Omega}.$$ 

Proof. By Remark 2.1 and since $\langle D \rangle \in \mathcal{P}\mathcal{A}_1$ is clearly independent of parameters, without loss of generality let $F$ belong to $\mathcal{P}\mathcal{A}_\rho^\mu$. We start by proving the result in the case $\mu = \alpha = \beta = 0$. Let an arbitrary $s > \frac{1}{2}$ be fixed.

$$\hat{F}_m^\mu(k) := \frac{1}{(2\pi)^\nu} \int_{\mathbb{T}^\nu \times [0, \pi]} F(\theta, x, D_x)[\sin(mx)]\sin(nx)e^{-ik\theta} d\theta dx$$

$$= \frac{1}{2(2\pi)^\nu} \int_{\mathbb{T}^\nu \times [-\pi, \pi]} F(\theta, x, D_x)[\sin(mx)]\sin(nx)e^{-ik\theta} d\theta dx$$

$$= \frac{1}{2(2\pi)^\nu} \int_{\mathbb{T}^\nu_{\nu+1}} F(\theta, x, D_x) \left[ \frac{e^{imx} - e^{-imx}}{2i} \right] \frac{e^{imx} - e^{-imx}}{2i} e^{-ik\theta} d\theta dx$$

$$= \frac{1}{4(2\pi)^\nu} \int_{\mathbb{T}^\nu_{\nu+1}} f(\theta, x, m)(e^{i(m-n)x} - e^{i(m+n)x})e^{-ik\theta} d\theta dx,$$
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where \( f \in \mathcal{PS}_\rho^m \) is the symbol of \( F \). Consider first the case \( m \neq n \). Then, integrating by parts \( \tilde{s} \)-times in \( x \), with \( \tilde{s} : = \lfloor s + 2 \rfloor + 1 \), and shifting the contour of integration in \( \theta \) to \( T^\nu - i \rho \text{sgn}(k) \) (here \( \text{sgn}(k) := (\text{sgn}(k_1), \ldots, \text{sgn}(k_\nu)) \in \{-1, 1\}^\nu \)), one gets that for any \( m, n \in \mathbb{N}, m \neq n, k \in \mathbb{Z}^\nu \)

\[
\left| \hat{F}_m^n(k) \right| \leq e^{-|\rho|k} \left( \frac{1}{|m + n|} + \frac{1}{|m - n|^s} \right) \sup_{\|\text{meas}\| \leq \rho} \left| \hat{s}_n^\alpha f(\theta; x, m) \right|
\]

\[
\leq 2 e^{-|\rho|k} \left( |x_{m - n}|^{\tilde{s}} \hat{s}_{k_{\rho}}^0(\rho) (f) \right).
\]

If \( m = n \), in a similar way one proves the bound \( \sup_{m \in \mathbb{N}} \left| \hat{F}_m^n(k) \right| \leq e^{-|\rho|k} \left( |x_{m - n}|^{\tilde{s}} \hat{s}_{k_{\rho}}^0(\rho) (f) \right) \). It follows that for any \( 0 < \rho' < \rho \), one has \( |F|_{\rho', s} \leq C(\rho - \rho')^{-\nu} \hat{s}_{k_{\rho}}^0(\rho) < \infty \), which proves (A.3) in the case \( \alpha = \beta = \mu = 0 \). To treat the general case, it is sufficient to note that, by Remarks 2.6, 2.11 and 2.12, the operator \( \langle D \rangle^\alpha F \langle D \rangle^\beta \in \mathcal{PA}^0_\rho \), we so have

\[
(A.6) \quad \left| \langle D \rangle^\alpha F \langle D \rangle^\beta \right|_{\rho', s} \leq \left( \frac{C}{(\rho - \rho')^\nu} \right) \hat{s}_{k_{\rho}}^0(\rho) (F).
\]

\( \square \)

A.2 Proof of Lemma 2.23 (Embedding)

The result now follows immediately by applying Proposition A.2 to \( F \in \text{Lip}_\beta(\Omega, \mathcal{PA}^0_\rho) \) and \( G \in \text{Lip}_\beta(\Omega, \mathcal{PA}^{0,\alpha}_\rho) \). Indeed, we obtain

\[
\left| \langle D \rangle^\alpha F \langle D \rangle^\beta \right|_{\rho', s, \Omega} \leq \left( \frac{C}{(\rho - \rho')^\nu} \right) \hat{s}_{k_{\rho}}^0(\rho) (F) \text{Lip}_\beta(\Omega),
\]

\[
\left| \langle D \rangle^\alpha G \langle D \rangle^\beta \right|_{\rho', s, \Omega} \leq \left( \frac{C}{(\rho - \rho')^\nu} \right) \hat{s}_{k_{\rho}}^0(\rho) (G) \text{Lip}_\beta(\Omega).
\]

A.3 Proof of Lemma 2.15

Denote by \( A_e \) the extension of the operator \( A \) on \( L^2(T) \) which coincides with \( A \) on \( L^2_{\text{even}}(T) = \mathcal{H}_0 \) and is identically zero on \( L^2_{\text{odd}}(T) \). Since \( A_e \) is parity preserving, one verifies for any \( m, n \in \mathbb{Z} \) that \( \langle A_e e^{imx}, e^{in\pi} \rangle_{L^2(T)} = 2 \langle A \sin(mx), \sin(nx) \rangle_{\mathcal{H}_0} \). Therefore, (A.12) is equivalent to the classical algebra property developed on the exponential basis (for instance, see [BB13]). We skip the details.

A.4 Proof of Lemma 2.24 (Commutator)

We start with operators independent of \( \theta \) in \( T^\nu \). Let

\[
X = \begin{pmatrix} X^d & X^o \\ -X^o & -X^d \end{pmatrix}, \quad V = \begin{pmatrix} V^d & V^o \\ -V^o & -V^d \end{pmatrix}.
\]

One has

\[
i[X, V] = i(XV - VX) = \begin{pmatrix} iZ^d & iZ^o \\ -(iZ^o) & -(iZ^d) \end{pmatrix},
\]

\( \mbox{(A.7)} \)
where
\[(A.8)\] \[Z^d := X^dV^d - X^d \overline{V^d} - V^d X^d + \overline{V^d} X^d,\]
\[(A.9)\] \[Z^0 := X^dV^0 - X^d \overline{V^0} - V^d X^0 + \overline{V^0} X^0.\]

Omitting sake of simplicity conjugate operators and labels for diagonal and anti-diagonal elements, by Remark 2.19 the following inequalities hold (here \(\sigma = \pm \alpha, 0\)):
\[\left| \langle D \rangle^\sigma XV \langle D \rangle^{-\sigma} \right|_s \leq C_s \left| \langle D \rangle^\sigma X \langle D \rangle^{-\sigma} \right|_s \left| \langle D \rangle^\sigma V \langle D \rangle^{-\sigma} \right|_s ;\]
\[(A.10)\] \[
\left| \langle D \rangle^\alpha XV \right|_s \leq C_s \left| \langle D \rangle^\alpha X \right|_s \left| V \right|_s ;
\]
\[
\left| XV \langle D \rangle^\alpha \right|_s \leq C_s \left| X \langle D \rangle^\alpha \right|_s \left| \langle D \rangle^\alpha V \right|_s ;
\]
the same for those terms involving \(VX\). All these norms extend easily to the analytic case. Therefore, by the assumption and from the definition in (2.21), properties 2.18, 2.19 and 2.20 are satisfied. It remains to show the symmetries conditions in (2.20). Note that
\[(A.11)\] \[(iZ^d)^* = iZ^d, (iZ^\alpha)^* = \overline{iZ^\alpha} \iff (Z^d)^* = -Z^d, (Z^\alpha)^* = \overline{Z^\alpha}.\]

We check the condition for \(Z^d\). We have
\[(A.12)\] \[\langle Z^d \rangle^* = \langle V^d \rangle^* (X^d)^* - \langle V^\overline{d} \rangle^* (X^\overline{d})^* - \langle X^d \rangle^* (V^d)^* + \langle X^\overline{d} \rangle^* (V^\overline{d})^* \]
\[= V^d X^d - V^\overline{d} \overline{X^d} - X^d V^d + \overline{X^d} \overline{V^d} = -Z^d.\]

In the same way one checks that \((Z^\alpha)^* = \overline{Z^\alpha} \)

The assertion for the Lipschitz dependence on parameters essentially follows from Remark 2.1

\section{Other measure estimates}

\subsection{Diophantine condition for the Magnus normal form}

\textbf{Proposition B.1.} There exist \(\gamma_0 > 0\) and \(\tau_0 > 0\) such that the set
\[(B.1)\] \[\Omega_0 := \left\{ \omega \in R_\theta \left| \omega \cdot k \geq \frac{\gamma_0}{|k|^\alpha M} \right. \forall k \in \mathbb{Z}^\nu \setminus \{0\} \right\} \]
is of large measure, in the sense that \(m_\nu(R_\theta \setminus \Omega_0) = O(\gamma_0)\).

\textbf{Proof.} Consider the sets
\[(B.2)\] \[G^k := \left\{ \omega \in R_\theta \left| \omega \cdot k \leq \frac{\gamma_0}{|k|^\alpha M} \right. \right\}, \quad G := \bigcup_{k \neq 0} G^k = R_\theta \setminus \Omega_0.\]

By Lemma 4.3 it holds that
\[(B.3)\] \[|G^k| \leq \frac{\gamma_0}{|k|^{\gamma_0 + 1} M^\nu}.\]
Therefore
\[ |\mathcal{G}| \leq \sum_{k \neq 0} |\mathcal{G}^k| \leq \gamma_0 H^\nu \sum_{k \neq 0} \frac{1}{|k|^\gamma} \leq C \gamma_0 H^\nu, \quad C = C(\nu) < \infty \]
choosing \( \tau_0 + 1 > \nu \). We conclude that \( m_r(\mathcal{G}) \lesssim \gamma_0 \).

**B.2 Proof of Lemma 5.12**

By Corollary 5.9, we are led to prove the Melnikov conditions

\[ \left| |x \cdot k + \lambda_j^\tau(\omega) \pm \lambda_j^\alpha(\omega)|\right| \geq \frac{\gamma j (j + l)^\alpha}{|k|^{\gamma} M^\alpha}, \]

as a small perturbation of the ones we have shown in Theorem 4.1. Write

\[ \lambda_j^\tau(\omega) := \lambda_j + \frac{\bar{\varepsilon}_j(\omega)}{j^\alpha}, \quad \bar{\varepsilon}_j(\omega) := j^{\alpha} \varepsilon_j(\omega), \]

where, by (B.6), one has

\[ \sup_j \left| \varepsilon_j^{\text{Lip}}(\gamma) \right| \leq \frac{\gamma}{M^\alpha} \eta_0 \varepsilon. \]

Let \( \omega \in \mathcal{U}_0 \) and \( (k, j, l) \in \mathcal{I}^\pm \). Arguing as in the proof of Lemma 4.6, we can rule out some cases:

- For \( \pm = + \) and \( k = 0 \), we have
  \[ \lambda_j^\tau(\omega) + \lambda_j^\alpha(\omega) \geq \frac{1}{2} (\lambda_j + \lambda_l) \geq \frac{1}{2} (j + l) \geq \frac{\gamma}{|k|^{\gamma} M^\alpha} \, (j + l)^\alpha, \]
  which holds true by (B.3), (5.55) and the choice of \( k_0 \);

- For \( \pm = - \) and \( k \neq 0, j = l \), recalling that \( \Omega_\infty \subset \mathcal{U}_0 \subset \Omega_0 \),
  \[ |x \cdot k| \geq \frac{\gamma_0}{|k|^{\gamma} M^\alpha} \geq \frac{\gamma}{|k|^{\gamma} M^\alpha}; \]

- For \( \pm = - \) and \( k = 0, j \neq l \), it holds, by (5.71), (5.55) and (4.4), that
  \[ |\lambda_j^\tau(\omega) - \lambda_l^\tau(\omega)| \geq |\lambda_j - \lambda_l| - 2 \sup_{j \in \mathbb{N}} \left| \varepsilon_j^\alpha(\omega) \right| \geq \frac{1}{2M^\alpha} |j - l|. \]

Thus, we restrict to consider all \( (k, j, l) \in \mathcal{I}^\pm \) for which \( k \neq 0 \) and \( j \neq l \). If \( |j + l| \geq 16 M |k| \), we get (recall \( M > m^2 \) as in (1.21)):

\[ |x \cdot k + \lambda_j^\tau(\omega) \pm \lambda_l^\tau(\omega)| \geq |\lambda_j \pm \lambda_l| - |x \cdot k| - \left| \frac{\bar{\varepsilon}_j(\omega)}{j^\alpha} \pm \frac{\bar{\varepsilon}_l(\omega)}{l^\alpha} \right| \]
\[ \geq |j \pm l| - \left| \frac{c_j(m)}{j} \pm \frac{c_l(m)}{l} \right| - |x \cdot k| - \left| \frac{\bar{\varepsilon}_j(\omega)}{j^\alpha} \pm \frac{\bar{\varepsilon}_l(\omega)}{l^\alpha} \right| \]
\[ \geq |j \pm l| - 2m^2 - 2M |k| - 2 \frac{\gamma}{M^\alpha} \eta_0 \varepsilon \]
\[ \geq \frac{1}{2} |j \pm l|. \]
So, we can work in the regions \(|j \pm l| < 16M |k|\). Now, for \(j < l\) satisfying
\[
(B.8) \quad j \langle j \pm l \rangle \geq \left( \frac{2\eta_0 e \langle k \rangle^\gamma}{c(\gamma, \tilde{\gamma})} \right)^{\frac{1}{\gamma}} =: \tilde{R}(k),
\]
where \(c(\gamma, \tilde{\gamma}) := \frac{\tilde{\gamma}}{\gamma} - 1 > 1\) (recall that \(\tilde{\gamma}/\gamma > \gamma\), we have
\[
|\omega \cdot k + \lambda_j^{\infty}(\omega) \pm \lambda_j^{\infty}(\omega)| \geq |\omega \cdot k + \lambda_j^\alpha(\omega)| - \frac{|\tilde{\gamma}l(\omega)|}{j^\alpha} = \frac{\tilde{\gamma} \langle j \pm l \rangle^\alpha}{\langle k \rangle^\gamma} \geq 2 \gamma \frac{\eta_0 e}{j^\alpha} \langle k \rangle^\gamma - \frac{2 \gamma}{M^\alpha} - \frac{2 \gamma \eta_0 e}{j^\alpha} \langle k \rangle^\gamma.
\]
Therefore, we can further restrict ourselves to consider just those \(j < l\) satisfying \(j \langle j \pm l \rangle < \tilde{R}(k)\).

The symmetric argument leads to work in the sector \(j < l\) under the condition \(l \langle j \pm l \rangle < \tilde{R}(k)\).

Now, define the set
\[
(B.10) \quad G_{j,l}^{k,\pm} := \left\{ \omega \in R_{k} \left| \langle j \pm l \rangle^\alpha \langle k \rangle^{\gamma - \frac{\alpha}{\gamma}} \right. \right\}
\]
for those \(k \neq 0\) and \(j \neq l\) in the region
\[
(R^{\pm} := \{j \pm l < 16M |k|\} \cap \{(j \langle j \pm l \rangle < \tilde{R}(k), j < l\} \cup \{l \langle j \pm l \rangle < \tilde{R}(k), l < j\})
\]
Recall that \(f_k^\infty(\omega) := \omega \cdot k + \lambda_j^{\infty}(\omega) \pm \lambda_j^{\infty}(\omega)\) are Lipschitz functions on \(R_k\). For \(k \neq 0\), since \(\left|\lambda_j^{\infty}\right|^\text{Lip}_{R_k} = \left|\tilde{\gamma}l\right|^\text{Lip}_{R_k} \leq \eta_0 e < |k|/4\), by Lemma 4.3, we get
\[
(B.12) \quad \left|G_{j,l}^{k,\pm}\right| \lesssim M^{\nu - \gamma} \frac{\tilde{\gamma} \langle j \pm l \rangle^\alpha}{|k|^{\gamma + 1}}
\]
Define \(G_{j,l}^{k,\pm} := \left\{ G_{j,l}^{k,\pm} \left| (k, j, l) \in R^{\pm} \right. \right\} \cap U_{\alpha}\). We have
\[
\left|G_{k}^{\pm}\right| \leq 2 \gamma M^{\nu - \alpha} \sum_{k \neq 0} \sum_{j, l \neq 0} \sum_{|j - l| < 16M |k|} \sum_{k \neq 0} \sum_{j, l \neq 0} \sum_{|j - l| < 16M |k|} \frac{\langle j - l \rangle^\alpha}{|k|^{\gamma + 1}} \lesssim \gamma M^{\nu - \alpha} \sum_{k \neq 0} \sum_{j, l \neq 0} \sum_{|j - l| < 16M |k|} \frac{\langle l \rangle^\alpha}{|k|^{\gamma + 1}} \lesssim \gamma M^{\nu - \alpha} \sum_{k \neq 0} \frac{1}{|k|^{\gamma + 1 + \frac{\alpha}{\nu - \alpha}}}
\]
taking \(\tau + 1 - \alpha - \frac{\tau}{\alpha} > \nu\). The same computation holds for \(G_{k}^{\pm}\). We conclude that
\[
m_r(U_{\alpha}, \Omega_{\alpha, \gamma}) \leq m_r(G^{\pm}_{\infty} \cap G^{\pm}_{+}) \leq C_{B} \gamma.
\]
with \(C_{B}\) independent of \(M, \gamma, \tilde{\gamma}\).
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