Some maximal isotropic distributions and their relation to field theory

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Abstract
We study the behavior of differential forms in a manifold having at least one of their maximal isotropic local distributions endowed with the special algebraic property of being decomposable. We show that they can be represented as the sum of a form with constant coefficients and one that vanishes whenever contracted with vector fields in the former distribution, provided some simple integrability conditions are ensured. We also classify possible 'canonical coordinates' for a certain class of forms with potential applications in classical field theory.

Introduction
The importance of the study of maximal isotropic (local) distributions with respect to a \((n+1)\)-differential form \(\omega\) on a manifold \(P\) relies, at least but not last, in its connection with a covariant and finite dimensional approach to the classical theory of fields started with de Donder (\([2]\)) and Weyl (\([1]\)). The existence of a special kind of such a distribution ensures that there are canonical coordinates \((x^\mu, q_i, p, p^\mu_i)\) in which \(\omega\) locally emulates a preexisting canonical form \(\Omega_0\) (\([3, 4]\)), described by the formula

\[
\Omega_0 = dq^i \wedge dp^\mu_i \wedge d^n x - dp \wedge d^n x ,
\]

where \(d^n x = dx^1 \wedge \ldots \wedge dx^n\) and \(d^n x^\mu = i_{\partial^\mu} d^n x\). The local description of \(\Omega_0\) given by equation (1) permits the association of this geometric object to the de Donder-Weyl equations in field theory (\([5-7]\)).

As a generalization of \([3, 4]\), this paper is devoted to the study of differential forms on a manifold \(P\) with local isotropic and decomposable distributions associated to them, that is, we shall consider a general \((n+1)\)-form \(\omega\) on a manifold \(P\) that at each point \(p\) of \(P\) admits a subspace \(L_p\) of \(T_pP\) having the following algebraic properties:

(i) It is maximal isotropic with respect to \(\omega_p\):

\[
\forall v, u \in L_p \quad i_v \wedge u \omega_p = 0 ,
\]

and \(L_p\) is maximal in \(T_pP\) with this property.

(ii) It is decomposable with respect to \(\omega_p\): \(L_p\) has a basis \(\{v_1, \ldots, v_m\}\) satisfying

\[
i_{v_i} \omega_p \quad \text{is a decomposable} \ n\text{-form for each } i = 1, \ldots, m.
\]
The main goal of this article is to show that whenever \( L \) is an integrable distribution, we can represent \( \omega \) locally as a sum

\[
\omega = \Omega_F + \omega_F
\]  

(4)

where \( F \) is a local foliation in \( P \) such that \( TP = L \oplus TF, \omega_F \) is the restriction of \( \omega \) to \( F \) and \( \Omega_F \) can be represented by coordinates preserving the decomposition \( TP = L \oplus TF \) and in which its coefficients are constant, provided it is closed. As its main application we determine a necessary and sufficient condition to the existence of 'canonical coordinates' for differential forms. As a special case, we determine when it is possible to find coordinates like the ones in equation (1).

The principal novelties presented here are:

- The entire study is made in the general context of a manifold, different from \([3, 4]\), where fiber bundles are used from the beginning.
- The approach given here to characterize a differential form \( \omega \) which has canonical coordinates \((x^\mu, q_i, p, p^\mu_i)\) just like (1) is extended naturally, and with no extra effort, to include the degenerate case given by the restriction of \( \omega \) to any submanifold characterized by the relations

\[
dq^i = 0, \quad dp^\mu_k = 0, \quad \text{for some indexes } i, k, \mu.
\]  

(5)

Under the existence of these constraints \( \omega \) is described by

\[
\omega = \sum_{i, \mu \in I} dq^i \wedge dp^\mu_i \wedge d^nx^\mu - dp \wedge d^n x,
\]  

(6)

where \( I \) is the subset of indexes that exclude those appearing in the constraints (5).
- The term '\( \omega_F \)' in the r.h.s. of equation (1) is new and appears as a "horizontal" obstruction to describe \( \omega \) with coordinates in which it has constant coefficients. It might be possible that, under certain conditions, it describes the analog of a background electromagnetic field coupled to the classical fields, as in \([8]\), where we have to add a 'horizontal perturbation' \( \pi^* \omega_F \) to the form \( \Omega_0 \), the last one being the canonical form (1),

\[
\omega = \Omega_0 + \pi^* \omega_F,
\]  

(7)

to get the right description of a bosonic string in a background electromagnetic field. In this example \( \pi \) is a surjective map and \( F \) is the electromagnetic field strength.

We proceed as follows: in section 1 we review some basic algebraic concepts. In section 2 we go further in multilinear algebra and derive the key results that will be of fundamental importance later. In section 3 we start the differential geometric description and prove our main theorem before we show how to translate our results to fibered manifolds. In the last section we work with examples, showing that differential forms with maximal isotropic and decomposable distributions are quite general, although they do not seem to be generic.

1 Basic definitions in the Grassmann Algebra

Let \( F \) be a finite dimensional vector space and \( F^* \) its dual. Fixing the notation, if \( S \subset F^* \) is any subset, then the subspace defined by

\[
S^\perp := \{ v \in F | \, v \cdot \alpha = 0 \quad \forall \alpha \in S \}
\]  

(8)

is the annihilator of the set \( S \).
For a $n$-form $\beta$ on $F$, i.e., $\beta \in \bigwedge^n F^*$, we define the contraction map by

$$\beta^\flat : F \to \bigwedge^{n-1} F^* \quad \beta^\flat(v) := i_v \beta,$$

its kernel to be the subspace of $F$ given by

$$\ker \beta := \{ v \in F \mid i_v \beta = 0 \}$$

and its support to be the subspace of $F^*$ given by

$$S_\beta = \bigcap \{ S \mid S \text{ is a subspace of } F^* \text{ and } \beta \in \bigwedge^n S \} ,$$

which is well defined, since the wedge product has the property

$$\bigwedge^n S_1 \cap \bigwedge^n S_2 = \bigwedge^n (S_1 \cap S_2)$$

for any subspaces $S_1, S_2$ of $F^*$. Hence, $S_\beta$ is the smallest subspace of $F^*$ such that $\beta \in \bigwedge^n S_\beta$. One can verify that ($[3,4]$)

$$S^\bot_\beta = \ker \beta \quad \text{and} \quad S_\beta = (\ker \beta)^\bot .$$

Therefore

$$\dim F = \dim S_\beta + \dim \ker \beta .$$

When $\beta$ has a trivial kernel we say that it is non-degenerate, which is equivalent to say that $S_\beta = F^*$.

Pick a basis $B = \{ e^1, \ldots, e^{n+N} \}$ of $F^*$, put

$$\beta = a_{i_1 \ldots i_n} e^{i_1} \wedge \ldots \wedge e^{i_n} ,$$

and define $\ell_B(\beta)$ to be the length of $\beta$ with respect to $B$, that is,

$$\ell_B(\beta) = \# \{ \vec{i} \in \Omega^{n+N}_n \mid a_{i_1 \ldots i_n} \neq 0 \} ,$$

where we use the notation

$$\Omega^{n+N}_n = \{ \vec{i} := (i_1, \ldots, i_n) \mid 1 \leq i_1 \ldots < i_n \leq n + N \} \quad n + N = \dim F .$$

The length of $\beta$ is the minimum of the relative lengths among all possible basis, i.e.,

$$\ell(\beta) = \min_B \ell_B(\beta) .$$

We say that $\beta$ is decomposable if $\ell(\beta) \leq 1$. If $\ell(\beta) = 1$, there is a L.I. set $\{ \alpha^1, \ldots, \alpha^n \}$ such that

$$\beta = \alpha^1 \wedge \ldots \wedge \alpha^n$$

This set forms a basis for $S_\beta$. Therefore, for any $\beta \neq 0$, $\dim S_\beta = n$ if, and only if, $\beta$ is decomposable (For additional information on decomposable elements see [9]).

2 Maximal Isotropic Decomposable Subspaces

Let $W$ be a finite dimensional vector space, $L$ a subspace of $W$, $\omega$ a $(n+1)$-form on $W$ and $k$ an integer satisfying $0 \leq k \leq n$. The $k$-orthogonal complement of $L$ in $W$ with respect to $\omega$ is the subspace of $W$ given by

$$L^{\omega,k} = \{ v \in W \mid i_{v_1} \ldots i_{v_k} \omega = 0 \quad \text{for all } v_1, \ldots, v_k \in L \} .$$

In the case $k = 0$ we have $L^{\omega,0} = \ker \omega$. The subspace $L$ is said to be, with respect to $\omega$,

1Trivially, $\ell(\beta) = 0$ implies $\beta = 0$. 

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(i) **$k$-isotropic** when $L \subset L^{\omega,k}$. For $k = 1$ we say just **isotropic**;

(ii) **strict $k$-isotropic** if it is $k$-isotropic but not $(k-1)$-isotropic;

(iii) **maximal $k$-isotropic** if it is $k$-isotropic and not a proper subspace of another $k$-isotropic subspace;

Note that

$$L \text{ is maximal } k\text{-isotropic } \implies \ker \omega \subset L,$$

for, in this case, we have that $\ker \omega + L$ is $k$-isotropic, and so, $\ker \omega + L \subset L$ by the maximality condition.

A vector $v \in W$ is **decomposable** with respect to $\omega$ if

$$i_v \omega \in \bigwedge^n W^* \text{ is decomposable.}$$

We say that a subspace $L$ is **decomposable** with respect to $\omega$ if there is a basis for $L$ made of decomposable vectors, with respect to $\omega$.

Any subspace spanned by $\{e_1, \ldots, e_k\}$, with $k \leq n$, is $k$-isotropic (and strict $k$-isotropic if $i_{e_1} \cdots i_{e_k} \omega \neq 0$). Therefore, maximal $k$-isotropic subspaces always exist, but are not necessarily decomposable if $n > 1$. When $k = n = 1$ and $\omega$ is non-degenerate, maximal isotropic subspaces are called Lagrangian and they always admit a complementary maximal isotropic subspace, which is lagrangian as well. Extending this result for forms of any degree:

**Theorem 1**

Let $L$ and $V$ be subspaces of $W$ such that $L \subset V$, $L$ is maximal isotropic decomposable and $V$ is $r$-isotropic with respect to a $(n+1)$-form $\omega$ on $W$. There is a $n$-isotropic subspace $F$ such that

$$W = L \oplus F \quad (F \cap V) \oplus L = V$$

and $F \cap V$ is $(r-1)$-isotropic.

**Proof:**

Assume that $\omega$ is non-degenerate, i.e., $\ker \omega = 0$. This proof is by induction on $m + 1 = \dim L$. Pick a decomposable basis $\{v_0, \ldots, v_m\}$ for $L$ and a 1-form $\alpha_0 \in W^*$ such that $\alpha_0(v_i) = \delta^0_i$. There are 1-forms $u^1, \ldots, u^n \in L^\perp$ such that

$$i_{v_0} \omega = u^1 \wedge \cdots \wedge u^n,$$

with at least $(n-r+1)$ of them in $V^\perp$, since $L \subset V$ and $V$ is $r$-isotropic. Define $L_1 = L \cap \ker \alpha_0$, which is generated by $\{v_1, \ldots, v_m\}$, and the $(n+1)$-form $\omega_1$

$$\omega = \omega_1 + \alpha_0 \wedge u^1 \wedge \cdots \wedge u^n.$$  

Take any subspace $\bigoplus F_1 = \langle u_1, \ldots, u_k \rangle \subset \ker \alpha_0$, such that

(i) $\ker \omega_1 = \langle u_1, \ldots, u_s \rangle \oplus \langle v_0 \rangle$, $s \leq k$;

(ii) $i_{u_1 \wedge \cdots \wedge u_k} u^1 \wedge \cdots \wedge u^n \neq 0$;

(iii) If $n \neq s$ then $i_{u_{s+1} \wedge \cdots \wedge u_k} \omega_1 \notin L^\perp$;

\[\text{Notation: } \langle u_1, \ldots, u_k \rangle \text{ is the subspace spanned by the L.I. set } \{u_1, \ldots, u_k\}.\]
Such a nontrivial \((k \geq 1)\) subspace \(F_1\) always exists. For \(k \leq n\) maximal satisfying such properties, we shall prove that \(k = n\). If \(n = s\), it is trivial. If \(s \leq k < n\), there is a vector \(u_{k+1} \in \ker(\alpha_0 \wedge u^1 \wedge \ldots \wedge u^s)\) such that
\[
i_{u_{k+1} \wedge \ldots \wedge u_{k+1}} \omega_1 \notin \bigwedge^k L^\perp
\]
by the property (iii) and the fact that \(L\) is isotropic. Applying the same reasoning recursively, we get \(k = n\). Therefore,
\[
i_u \omega_1 \in \bigwedge^n L^\perp \implies i_u \omega_1 = 0. \tag{25}
\]
for each \(u \in F_1\).

Define \(W_1 = \ker(\alpha_0 \wedge u^1 \wedge \ldots \wedge u^n)\), where we are assuming that \(F_1 = \langle u_1, \ldots, u_n \rangle\) has the first \(s\) vectors, together with \(v_0\), composing the kernel of \(\omega_1\) and \(w^i(u_i) = \delta_i^1\). Note that \(\omega_1\) is non-degenerate in \(W_1\) and \(L_1 \subset W_1\) is isotropic and decomposable w.r.t. \(\omega_1\), since \(L_1 = \langle v_1, \ldots, v_m \rangle\). To see that \(L_1\) is maximal isotropic w.r.t. \(\omega_1\) in \(W_1\), let \(e = e_1 + u \in W_1\) be written according to the decomposition
\[
W_1 = E \oplus U \quad U = \langle u_{s+1}, \ldots, u_n \rangle \quad E = \ker(\alpha_0 \wedge u^1 \wedge \ldots \wedge u^n)
\]
with \(e_1 \notin L_1\). By the maximal 1-isotropy of \(L\) w.r.t. \(\omega\) in \(W\), there are vectors \(e_2, \ldots, e_n \in W_1\) and \(v \in L_1\) such that
\[
\omega(v, e_1, \ldots, e_n) = \omega_1(v, e_1, \ldots, e_n) = 1
\]
Furthermore, noticing that the annihilator \(L^\perp\) of \(L\) is generated by the 1-forms \(i_{e^1 \wedge e^2 \wedge \ldots \wedge e^n_1} \omega_1\), with \(e^2_2, \ldots, e^n_1 \in W/L\) and \(e' \in L\), we can assume that
\[
\omega(v, u, e_2, \ldots, e_n) = \omega_1(v, u, e_2, \ldots, e_n) = 0.
\]
Therefore,
\[
i_{e_1 + u} \omega_1 \notin \bigwedge^n L_1^\perp. \tag{26}
\]
So, if \(e = e_1 + u \in W_1\) is such that
\[
i_{e_1 \wedge u} \omega_1 = 0
\]
for all \(v \in L_1\), then \(e_1 \in L_1\), and by \(25\), \(u = 0\). In other words, \(L_1\) is maximal isotropic w.r.t. \(\omega_1\) in \(W_1\), implying that it is maximal isotropic decomposable w.r.t. \(\omega_1\). By the induction hypothesis, let \(F' \subset W_1\) be \(n\)-isotropic w.r.t. \(\omega_1\) and complementary to \(L_1\) in \(W_1\). Then \(F = \langle u_1, \ldots, u_s \rangle \oplus F'\) is \(n\)-isotropic w.r.t. \(\omega\) and complementary to \(L\) in \(W\). Furthermore, \(F \cap V \cap W_1\) is \((r - 1)\)-isotopic, and so is \(F \cap V\).

□

An useful lemma that also helps understanding the content of a maximal isotropic decomposable subspace is given below.

**Lemma 1** Let \(L\) be a maximal isotropic decomposable subspace with respect to a \((n + 1)\)-form \(\omega\) on \(W\), and \(f_1, \ldots, f_n\) vectors in \(W\) such that
\[
i_{f_1 \wedge \ldots \wedge f_n} \omega \notin L^\perp
\]
Then there are 1-forms \(f_1, \ldots, f^n\) such that \(f^j(f_i) = \delta_i^j\) and \(f^1 \wedge \ldots \wedge f^n \in \omega^j(L)\).

**Proof:** Since \(i_{f_1 \wedge \ldots \wedge f_n} \omega \notin L^\perp\), there is a \(\omega\)-decomposable vector \(v \in L\) such that \(\omega(v, f_1, \ldots, f_n) = 1\). Since \(i_v \omega\) is decomposable, its kernel has codimension \(n\) in \(W\), and therefore we can write
\[
W = \langle f_1, \ldots, f_n \rangle \oplus \ker i_v \omega.
\]
This implies that for \(f_1, \ldots, f^n \in (\ker i_v \omega)^\perp\) such that \(f^j(f_i) = \delta_i^j\) we have \(i_v \omega = f^1 \wedge \ldots \wedge f^n\). □
Let $L$ be a maximal isotropic decomposable subspace and $F$ a complementary $n$-isotropic subspace in $W$, both with respect to $\omega$. Pick any basis $B_F = \{f_1, \ldots, f_{N+n}\}$ of $F$ and define the number of non-vanishing indexes of $B_F$ with respect to $\omega$:

$$N(B_F) = \# B_F$$

where $B_F = \{\vec{i} \in \mathbb{S}^{n+n}_F | i_1 \wedge \ldots \wedge f_n \omega \neq 0\}$ (27)

and $\mathbb{S}^{n+n}_F$ is given by (17). Let $B = \{e_1, \ldots, e_m, f_1, \ldots, f_{N+n}\}$ be a basis of $W$ adapted to the decomposition (47), that is, $\{e_1, \ldots, e_m\}$ is a basis of $L$ and $B_F \subset B$, and let $B^* = \{e^1, \ldots, e^m, f^1, \ldots, f^{N+n}\}$ be its dual. Hence

$$\omega = \sum_{\vec{i} \in \mathbb{S}^{n+n}_F} \alpha_{\vec{i}} \wedge f^{i_1} \wedge \ldots \wedge f^{i_n}, \quad \alpha_{\vec{i}} \in F^\perp \cong L^*,$$

where for each $\vec{i} \in \mathbb{S}^{n+n}_F$, $\alpha_{\vec{i}}$ is a non-trivial linear combination of the 1-forms $e^1, \ldots, e^m$. Therefore,

$$N(B_F) = \ell_{B_F}(\omega)$$

and

$$N(B_F) \geq \ell(\omega).$$

Taking the minimum among all possible basis of $F$, we can define the number

$$N_L = \min_{B_F} N(B_F) - \dim(L/\ker \omega) \geq 0.$$ (30)

It is not difficult to check that $N_L$ does not depend on the choice of the complementary $n$-isotropic subspace.

**Theorem 2**

Let $L$, $F$ and $V$ be subspaces of $W$ satisfying the relations in theorem 1 with respect to $\omega \in \bigwedge^n W^*$, and suppose that $N_L = 0$, which is equivalent to

$$\dim(L/\ker \omega) = \ell(\omega).$$ (31)

There is a basis $B_F$ for $F$ adapted to $V$, with $B^*_F = \{e^1, \ldots, e^{N+n}\}$ its unique dual in $L^\perp$, and a L.I. set $\{\hat{e}_i^1, \ldots, \hat{e}_i^m\}$ in $F^\perp$ such that

$$\omega = \sum_{\vec{i} \in \mathbb{S}^{n+n}_F} \hat{e}_{\vec{i}} \wedge e^{i_1} \wedge \ldots \wedge e^{i_n}$$ (32)

where $\mathbb{S}^{n+n}_F$ is given by equation (27). Each term $e^{i_1} \wedge \ldots \wedge e^{i_n}$ in equation (32) vanishes when contracted with $r$ vectors in $V$.

**Proof:** Take a basis $B_F$ for $F$ with $\# \mathbb{S}^{n+n}_F = \dim L - \dim \ker \omega$, which exists since $N_L = 0$. Define

$$\hat{e}_{\vec{i}} = i_{e_{i_1} \wedge \ldots \wedge e_{i_n}} \omega$$

Using the hypothesis on the dimension of $L/\ker \omega$, the $n$-isotropy of $F$ and the the $1$-isotropy of $L$, it is easy to check that they form a basis for $(L/\ker \omega)^*$ and that equation (32) holds. Relation (31) follows from the formula (32).
3 Maximal Isotropic Decomposable Distributions

3.1 Maximal Isotropic Decomposable Distributions and Flatness

Let $\omega \in \Omega^{n+1}(P)$ be a $(n+1)$-differential form on a manifold $P$. We say that it is flat if, around each point of $P$, there is a coordinate representation in which it has constant coefficients, that is,

$$\omega = \omega_{i_1 \ldots i_{n+1}} dx^{i_1} \wedge \ldots \wedge dx^{i_{n+1}} \quad \text{and} \quad \frac{\partial}{\partial x^i} \omega_{i_1 \ldots i_{n+1}} = 0, \quad \text{for each } i, i_1, \ldots, i_{n+1}. \tag{33}$$

A distribution $L$ on $P$ is maximal isotropic decomposable with respect to $\omega$ if it is pointwise maximal isotropic decomposable with respect to $\omega$.

**Theorem 3 (Principal Part of $\omega$)**

If $L$ is a maximal isotropic decomposable distribution with respect to $\omega$, then for any differential form $\delta \omega \in \Omega^{n+1}(P)$, we have

$$\delta \omega^\flat(L) = 0 \quad \Rightarrow \quad L \text{ is maximal isotropic decomposable with respect to } \omega + \delta \omega. \tag{34}$$

Therefore, $L$ defines a class of forms admitting it as a maximal isotropic decomposable distribution,

$$[\omega]_L = \{ \omega + \delta \omega \mid \delta \omega^\flat(L) = 0 \} \tag{35}$$

which we will call the principal part of $\omega$ with respect to $L$.

**Proof:** To check this, note that $L$ is isotropic and decomposable with respect to $\omega + \delta \omega$, and if $u$ is a vector fields on $P$ such that $i_u \omega(v + \delta \omega) = 0$ for every $v \in L$, we have

$$i_u \omega = i_u \omega(v + \omega^\flat(L) = 0, \tag{36}$$

for every $v \in L$, implying that $u \in L$, since $L$ is maximal isotropic with respect to $\omega$. Then, $L$ is also maximal isotropic with respect to $\omega + \delta \omega$. \qed

**Theorem 4 (Flatness)**

Let $\omega \in \Omega^{n+1}(P)$ satisfies the regular condition of constant dimension of the kernel distribution. If $L$ is a maximal isotropic decomposable distribution, then for each foliation $\mathcal{F}$ such that $TP = TF \oplus L$, we have that

$$\Omega_{\mathcal{F}} = \omega - \omega_{\mathcal{F}} \in [\omega]_L, \tag{37}$$

where $\omega_{\mathcal{F}}$ is the restriction of $\omega$ to $\mathcal{F}$, defines a representation of the principal part of $\omega$ and admits $TF$ as a $n$-isotropic distribution. If $L$ is integrable, then

$$\Omega_{\mathcal{F}} \text{ is closed if, and only if, it is flat.}$$

Moreover, a coordinate system for $P$ on which $\Omega_{\mathcal{F}}$ has constant coefficients can be chosen to be adapted to the decomposition $TP = TF \oplus L$.

**Proof:**

Since the character of this theorem is local, we will avoid to use the label “local”, keeping in mind that there is no need for global constructions here. Hence, we will assume that $P = X \oplus L$, is a vector space where $L$ is a subspace identified with the maximal isotropic decomposable distribution and $X$ a

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4That is, coordinates $(x, y)$ such that $TF$ is given by $dy = 0$ and $L$ by $dx = 0$. 

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subspace such that $X \times \{y\}$ are the leaves of the foliation $\mathcal{F}$, which is always possible since $L$ and $T\mathcal{F}$ are simultaneously integrable (see appendix A). Moreover, since the distribution $L$ is maximal isotropic decomposable with respect to $\Omega_\mathcal{F}$ (see lemma 3), we can prove the theorem for the case $\omega = \Omega_\mathcal{F}$, that is, $\omega_\mathcal{F} = 0$. Therefore the foliation $\mathcal{F}$ will be, just in this case, $n$-isotropic with respect to $\omega$.

Let $\omega_0$ be the constant form obtained by spreading $\omega(p_0)$, the value of $\omega$ at the origin $p_0 = (0,0)$, all over the vector space $X \oplus L$. Clearly, the distribution $L$ is maximal isotropic decomposable with respect to $\omega_0$.

**Lemma 2** In a small neighborhood of the origin, $\omega_0^\flat(L) = \omega^\flat(L)$.

**Proof:** The vector fields $\{f_\mu := \frac{\partial}{\partial x^\mu}\}$ form a basis for the $n$-isotropic distribution $T\mathcal{F}$. Define the 1-forms

$$\alpha_{\mu_1 \ldots \mu_n} := \frac{1}{n!} f_{\mu_1} \wedge \ldots \wedge f_{\mu_n} \omega$$

$$\alpha_{\mu_1 \ldots \mu_n}^0 := \alpha_{\mu_1 \ldots \mu_n}(p_0) = \frac{1}{n!} f_{\mu_1} \wedge \ldots \wedge f_{\mu_n} \omega_0$$

(38) By continuity, there is a small neighborhood of the origin where we have $\alpha_{\mu_1 \ldots \mu_n}^0 \neq 0$ implies $\alpha_{\mu_1 \ldots \mu_n} \neq 0$, that is, for each point $p$ in this neighborhood

$$(i_v \omega_0)(v_1 \wedge \ldots \wedge v_n) \neq 0 \Rightarrow (i_v \omega)(v_1 \wedge \ldots \wedge v_n) \neq 0$$

(39) for any vectors $v_1, \ldots, v_n \in T_p\mathcal{F}$ and $v \in L_p$. This implies that the annihilators satisfy $(\omega^\flat(L_p))^\perp \subset (\omega_0^\flat(L_p))^\perp$, therefore $\omega_0^\flat(L_p) \subset \omega^\flat(L_p)$. Since $\dim \ker \omega$ is constant by hypothesis and $L$ is integrable, $\dim \omega^\flat(L) = \dim L/\ker \omega$ must be constant, and so, there must exist a small neighborhood where $\omega_0^\flat(L) = \omega^\flat(L)$.

□

**Lemma 3** $\exists \, \theta, \theta_0 \in \omega^\flat(L) : \omega = d\theta \quad \omega_0 = d\theta_0$

**Proof:** Analogously to the “canonical” proof of Poincare lemma, define the “$L$-contraction” $\Phi_t(x,y) = (x,ty)$, for each $t \in \mathbb{R}$. Denote its time dependent vector field by $\xi_t$, that is,

$$\xi_t(\Phi_t(x,y)) = \frac{d}{ds} \Phi_s(x,y) \bigg|_{s=t} \quad \xi_t(x,y) = t^{-1}(0, y).$$

(40)

Although the vector field $\xi_t$ have a singularity at $t = 0$, the forms

$$\theta_t = (\Phi_t)^*(i_{\xi_t} \omega)$$

(41) are well defined for every $t \in \mathbb{R}$. Since $\xi_t \in L$, we have $i_{\xi_t} \omega \in \omega^\flat(L)$, that is, $\theta_t \in \omega^\flat(L)$. Defining the $n$-form

$$\theta = \int_0^1 dt \, \theta_t,$$

(42) it is clear that $\theta \in \omega^\flat(L)$. Moreover, since $d\omega = 0$, $\Phi_1 = Id_p$, and $(\Phi_0)^*(\omega) = 0$, the last relation following by the $n$-isotropy of $T\mathcal{F}$, we have

$$d\theta = \int_0^1 dt \, (\Phi_t)^*d(i_{\xi_t} \omega) = \int_0^1 dt \, (\Phi_t)^*(L_{\xi_t} \omega) = \int_0^1 dt \, \frac{d}{dt}((\Phi_t)^* \omega) = \omega.$$

We can apply the theorem for $\omega_0$, and since $\omega_0^\flat(L) = \omega^\flat(L)$, prove the assertion of the lemma. □

Consider the family of $(n+1)$-forms given by $\omega_t = \omega_0 + t(\omega - \omega_0)$, for every $t \in \mathbb{R}$. By lemma 2, we have $\omega_t^\flat(L_p) \subset \omega^\flat(L_p)$, and the continuity of the argument implies that there is an open neighborhood of the origin $p_0 = (0,0)$ where, for all $t$ satisfying $0 \leq t \leq 1$ and all points $p$ in it, $\omega_t(p)$ satisfies

$$\omega_t^\flat(L_p) = \omega^\flat(L_p).$$

(43)
By lemma 3 we have n-forms \( \theta, \theta_0 \in \omega^q(L) \) satisfying \( \omega = d\theta \) and \( \omega_0 = d\theta_0 \). Therefore,

\[
\alpha := \theta_0 - \theta \quad \text{satisfies} \quad \alpha \in \omega^q(L) \quad \text{and} \quad d\alpha = \omega_0 - \omega, \tag{44}
\]

for every \( 0 \leq t \leq 1 \) and every point in this “small” neighborhood of the origin where the relation holds. This implies that we can pick a time dependent vector field \( X_t \in L \) defined on this neighborhood satisfying

\[
i_{X_t} \omega_t = \alpha.
\]

Let \( \Phi^X_t \equiv \Phi^X_t((0,0)) \) be its flux beginning at the origin, which is well defined for \( 0 \leq t \leq 1 \), in some open neighborhood of the origin. Then it follows that

\[
\frac{d}{ds} \bigg|_{s=t} (\Phi^X_s)^* \omega_s = (\Phi^X_t)^* \left( \frac{d}{ds} \bigg|_{s=t} \omega_s \right) + \frac{d}{ds} \bigg|_{s=t} (\Phi^X_s)^* \omega_t
\]

\[
= (\Phi^X_t)^* (\omega - \omega_0 + L_{X_t} \omega_t)
\]

\[
= (\Phi^X_t)^* (\omega - \omega_0 + d(i_{X_t} \omega_t))
\]

\[
= (\Phi^X_t)^* (\omega - \omega_0 + d\alpha)
\]

\[
= 0
\]

Therefore, \( \Phi^X_t \) is the desired coordinate transformation, since \( (\Phi^X_1)^* \omega = (\Phi^X_1)^* \omega_1 = (\Phi^X_0)^* \omega_0 = \omega_0 \).

Since \( X_t \) is in \( L \), its flux acts as the identity in the leaves of \( \mathcal{F} \).

### 3.2 Maximal Isotropic Decomposable Distributions on Fibered Manifolds

Let \( P \) be a fibered manifold over \( M \), that is, a surjective submersion \( P \xrightarrow{\pi} M \). Denote its vertical distribution by \( V := \ker T\pi \).

When \( \mathfrak{N}_L \equiv 0 \), we can apply theorems 2 and 4 to find “canonical coordinates” for the principal part of \( \omega \). Before stating this theorem, let’s first fix the notation: for each \( 0 \leq s \leq n \)

\[
\exists^s_n (\pi, L) = \{ \bar{i}_s \times \bar{\mu}_s := (i_1, \ldots, i_s, \mu_1, \ldots, \mu_n) \} \quad \text{for} \quad 1 \leq i_1 < \ldots < i_s \leq n' \quad \text{and} \quad 1 \leq \mu_1 < \ldots < \mu_n \leq \ell', \tag{45}
\]

where \( n' = \dim(V/L) \) and \( n' = \dim M \). In important applications we have \( n = n' \), but in general it does not obey this equality.

**Corollary 1 (Canonical Coordinates)**

Let \( \omega \in \Omega^{n+1}(P) \) be a nondegenerate \( \mathfrak{N}_L \equiv 0 \) form on \( P \) and \( L \) an integrable maximal isotropic decomposable distribution such that \( \mathfrak{N}_L \equiv 0 \), that is, the length of \( \omega \) satisfies the relation

\[
\ell(\omega) = \dim L. \tag{46}
\]

Let \( P \xrightarrow{\pi} M \) be any fibered manifold, with vertical distribution \( V \), and \( \mathcal{F} \) any foliation such that

\[
TP = L \oplus T\mathcal{F} \quad \text{and} \quad V = L \oplus (T\mathcal{F} \cap V). \tag{47}
\]

If \( \Omega_\mathcal{F} \), the representation of the principal part of \( \omega \) given by \( \mathcal{F} \), is closed and admits the vertical bundle \( V \) as an \( r \)-isotropic distribution, then, in a small neighborhood of each point of \( P \), there are coordinates \( (p_j, q^i, x^{\mu}) \) such that

\[
\Omega_\mathcal{F} = \omega - \omega_\mathcal{F} = \sum_{s=0}^{r-1} \sum_{\bar{i}_s \times \bar{\mu}_s \in \exists^s_n} dp_{\bar{i}_s \times \bar{\mu}_s} \wedge dq^{\bar{i}_s} \wedge dx^{\bar{\mu}_s}, \tag{48}
\]

\( ^5 \)This could be changed by the regular condition of constant dimension of the kernel distribution. In this case, we should have \( \ell(\omega) + \dim \ker \omega = \dim L \).
where the distribution \( T \mathcal{F} \) is given by \( dp = 0 \), \( V \) by \( dx = 0 \) and \( L \) by \( dq = dx = 0 \). Here we are using the notations: \( \omega \) is the restriction of \( \omega \) to \( \mathcal{F} \), for each \( 0 \leq s \leq r - 1 \), \( \mathcal{I}_s \) is a fixed subset of \( \mathcal{I}_n(\pi, L) \),

\[
dq^i := dq^i \wedge \ldots \wedge dq^s, \quad dx^a := dx^{a_1} \wedge \ldots \wedge dx^{a_{r-s}} \quad \text{and} \quad dp_{i_1^r j_r} := dp_{i_1^r(\bar{j}_r)},
\]

with \( I: \mathcal{I}_0 \cup \ldots \cup \mathcal{I}_{r-1} \to \mathbb{N} \) an injection.

Now we shall single out the right parameters to make formula (48) turn into formula (1). First of all, we must have

\[
\omega \in \Omega^{n+1}(P) \quad n = \dim M \quad N = \dim(V/L) \quad r = 2 \quad \dim L = Nn + 1.
\]

Under these conditions, if we just assume that \( L \) is isotropic, we can show that it is decomposable and maximal isotropic, for the map \( \Omega^\flat_\mathcal{F} \) takes \( L \) onto the decomposable subspace \( \bigwedge^n L^\perp \) of \( n \)-forms which vanishes whenever contracted with two vectors in \( V \) or any vector in \( L \), arriving to the conditions presented in [3, 4]. According to them, in this case we have:

for \( n \geq 1 \), \( L \) is the unique maximal isotropic decomposable distribution in \( V \) and,

\[
\text{if } n \geq 2 \text{ and } \Omega \mathcal{F} \text{ is closed, then } L \text{ is integrable.}
\]

Moreover, applying corollary 4 with the conditions (50), we can find coordinates \((p, p^\nu_j, q^i, x^\mu)\) such that

\[
\Omega_{\mathcal{F}} = dp \wedge d^n x + dp^\mu_i \wedge dq^i \wedge d^n x_\mu
\]

just like the equation (1). If instead of \( \Omega_{\mathcal{F}} \), \( \omega \) itself satisfies the hypothesis above, we can choose \( \mathcal{F} \), at least locally, such that \( \omega_{\mathcal{F}} = 0 \), that is, \( \Omega_{\mathcal{F}} = \omega \).

### 4 Examples

**Example 1 (Decomposable Forms)**

If \( \omega \in \Omega^{n+1}(P) \) is a decomposable form on a manifold \( P \), a local distribution \( L \) is maximal isotropic decomposable if, and only if, at each point where \( \omega \neq 0 \)

\[
L = \ker \omega \oplus L_0 \quad \dim L_0 = 1.
\]

Therefore, every decomposable form admits a local maximal isotropic decomposable distribution. This includes all \((n+1)\)-forms if \( \dim P \leq n + 2 \). For instance, densities and volume forms on \( P \).

**Example 2 (Product of Manifolds)**

Given two manifolds \( P_1 \) and \( P_2 \) together with the \((n+1)\)-forms \( \omega_1 \) and \( \omega_2 \) such that they admit (local) maximal isotropic decomposable distributions \( L_1 \) and \( L_2 \), respectively. Then the manifold \( P_1 \times P_2 \) with the \((n+1)\)-form \( \omega_1 \oplus \omega_2 \) admits the (local) maximal isotropic decomposable distribution \( L_1 \oplus L_2 \).

**Example 3 (Isotropic Distribution of Maximal Dimension)**

Every distribution \( L \) on a manifold \( P \) which is isotropic with respect to \( \omega \in \Omega^{n+1}(P) \), that is, \( \omega^\flat(L) \subset \bigwedge^n L^\perp \), clearly satisfies the dimension constraint

\[
\dim L - \dim \ker \omega = \dim \omega^\flat(L) \leq \dim \bigwedge^n L^\perp = \binom{N+n}{n}.
\]
where $N + n = \dim P - \dim L$. Furthermore, there is the equivalence

$$\dim L = \dim \ker \omega + \binom{N + n}{n} \iff \omega^\wedge(L) = \bigwedge^n L^\perp. \quad (54)$$

Therefore, if $L$ has the maximal dimension allowed to an isotropic distribution and $\mathcal{R}_L = 0$. In [4, 10] it is proved that such a distribution is unique, and if it is integrable and $\omega$ is closed, then there are local coordinates such that formula $\omega^\wedge$ becomes

$$\omega = \sum_{\bar{t} \in \mathbb{N}^{n+N}} dp_{\bar{t}} \wedge dq_{\bar{t}} = \sum_{\bar{t} \in \mathbb{N}^{n+N}} dp_{\bar{t}} \wedge dq^{i_1} \wedge \ldots \wedge dq^{i_n} \quad (55)$$

**Example 4 (Canonical Examples)**

Here we summarize part of the work done in [3, 4]. Let $P \xrightarrow{\pi} M$ be a fibred manifold such that its vertical distribution $V$ is $r$-isotropic with respect to $\omega \in \Omega^{n+1}(P)$. Suppose we have a distribution $L \subset V$ satisfying the relation

$$\omega^\wedge(L) = \bigwedge^n \mathcal{R}^\perp. \quad (56)$$

This is equivalent to say that $L$ is isotropic and

$$\dim L = \dim \ker \omega + \sum_{s=0}^{r-1} \binom{N'}{s} \binom{n'}{n-s}. \quad (57)$$

where $\binom{p}{q} = 0$ if $q > p$, $N' = \dim V/L$ and $n' = \dim M$. Such a distribution is maximal isotropic decomposable with $\mathcal{R}_L \equiv 0$. Furthermore,

$L$ is unique in $V$ if $\binom{n'}{n+1-r} \geq 2$ and for $d\omega = 0$,

$L$ is involutive if $\binom{n'}{n+1-r} \geq 3$.

In this case it is possible to find a foliation $\mathcal{F}$ such that $\omega|_\mathcal{F} = 0$ and coordinates such that formula $\omega^\wedge$ becomes

$$\omega = \sum_{s=0}^{r-1} \sum_{\bar{t}_s \times \bar{\mu}_s \in \mathbb{N}^*_{\times}(\pi, L)} dp_{\bar{t}_s \bar{\mu}_s} \wedge dq_{\bar{t}_s} \wedge dx_{\bar{\mu}_s}. \quad (58)$$

Note that the difference between this and formula $\omega^\wedge$ is the set of indexes in which they are summed.

**Example 5 (A Maximal Isotropic Decomposable Distribution with $\mathcal{R}_L = 1$)**

Let $\omega$ be the 4-form in $\mathbb{R}^{11}$ given by

$$\omega = dp_1 \wedge dq^1 \wedge dx_1^1 \wedge dx_2 + dp_2 \wedge dq^2 \wedge dx_1^2 \wedge dx_2 + dp_3 \wedge (dq^1 + dq^2) \wedge dx_1^3 \wedge dx_2^3$$

Any of the three 3-dimensional subspaces generated by the vectors $\frac{\partial}{\partial p^i}$‘s or $\frac{\partial}{\partial x_1^i}$‘s or $\frac{\partial}{\partial x_2^i}$‘s is maximal isotropic decomposable with $\mathcal{R}_L = 1$. To see this, pick the $p$‘s subspace, that is, the one generated by the relation $dx_1 = dx_2 = dq = 0$, and call it $L$. Also, denote by $F$ the 3-isotropic subspace generated by $dp = 0$. $\omega^\wedge(L)$ is spanned by the three 3-forms on $F$

$$\alpha_1 = dq^1 \wedge dx_1^1 \wedge dx_2, \quad \alpha_2 = dq^2 \wedge dx_1^2 \wedge dx_2 \quad \alpha_3 = (dq^1 + dq^2) \wedge dx_1^3 \wedge dx_2^3.$$
There is no basis $\mathfrak{B}_F$ of $F^* \cong L^*$ such that, if $(\mathfrak{B}_F)^n$ is the basis of $\bigwedge^n F^*$ generated by $\mathfrak{B}_F$,\[ \#((\mathfrak{B}_F)^n \cap \omega^3(L)) = 3. \]

Therefore $\min_{\mathfrak{B}_F}((\mathfrak{B}_F)^n \cap \omega^3(L)) = 4$, implying $\mathfrak{R}_L = 4 - \dim L = 1$.

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A Simultaneously integrable Distributions

Two distributions $L_1$ and $L_2$ on a manifold $P$ are simultaneous integrable if around each point $p \in P$ there are integral manifolds $M_1$ of $L_1$ and $M_2$ of $L_2$ passing in $p$ such that $M_1 \cap M_2$ is an integral manifold of $L_1 \cap L_2$.

**Theorem 5** Let $L_1$ and $L_2$ be two integrable distributions such that $\dim(L_1 \cap L_2)$ is constant. Then, they are simultaneously integrable if, and only if, $L_1 + L_2$ is integrable.

**Proof:**

We will prove just the “if” part and assume that $P$ is a vector space with $TP = L_1 \oplus L_2$. Let $\{v_1^{(1)}, \ldots, v_n^{(1)}, u_1, \ldots, u_n\}$ be a commutative moving frame on $P$ such that $v_1^{(1)}, \ldots, v_n^{(1)}$ is a basis for $L_1$, which exists by the integrability hypothesis on $L_1$. Define functions $a_i^j$ on $P$ such that $e_i^{(2)} = u_i + a_i^j v_j^{(1)} \in L_2$.

Using the fact that $[v_i^{(1)}, v_j^{(1)}] = [v_i^{(1)}, u_i] = [u_j, u_i] = 0$ and $L_2$ is involutive, we conclude that $[e_i^{(2)}, e_j^{(2)}] \in L_1 \cap L_2 = 0$, and therefore $[e_i^{(2)}, e_j^{(2)}] = 0$ and $[e_i^{(2)}, v] \in L_1 \quad \forall v \in L_1$.

In the same way, we can find a basis for $L_1$, and therefore obtain the moving frame $\{e_1^{(1)}, \ldots, e_n^{(1)}, e_1^{(2)}, \ldots, e_n^{(2)}\}$ with $[e_i^{(1)}, e_j^{(2)}] = 0$, since it must be in $L_1 \cap L_2 = 0$.

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\[ \text{To see this, one can use the fact that the } x_1 \text{’s and } x_2 \text{’s subspaces are maximal isotropic decomposable and 3-dimensional, while their complement in } F \text{ just 2-dimensional.} \]
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