Asymptotic for critical value of the
large-dimensional SIR epidemic on clusters

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Abstract: In this paper we are concerned with the SIR (Susceptible-Infective-Removed) epidemic on open clusters of bond percolation on the squared lattice. For the SIR model, a susceptible vertex is infected at rate proportional to the number of infective neighbors while an infective vertex becomes removed at a constant rate. A removed vertex will never be infected again. We assume that there is only one infective vertex at $t = 0$ and define the critical value of the model as the maximum of the infection rates with which infective vertices die out with probability one, then we show that the critical value is $(1 + o(1))/(2dp)$ as $d \to +\infty$, where $d$ is the dimension of the lattice and $p$ is the probability that a given edge is open. Our result is a counterpart of the main theorem in [4] for the contact process.

Keywords: SIR model, critical value, percolation.

1 Introduction

In this paper, we are concerned with the SIR (susceptible-infective-removed) epidemic model on open clusters of bond percolation in squared lattices $\{\mathbb{Z}^d\}_{d \geq 1}$ (see a survey of percolation in [3]). For later use, we identify $\mathbb{Z}^d$ with the vertices set of it and denote by $E_d$ the edges set of $\mathbb{Z}^d$. We denote by $O$ the origin of the lattice. We assume that $\{X(e)\}_{e \in E_d}$ are i. i. d. random Bernoulli variables such that

$$P(X(e) = 1) = p = 1 - P(X(e) = 0)$$

for some $p \in (0, 1]$. For later use, we write $X(e)$ as $X(x,y)$ when $e$ connects vertices $x$ and $y$. For vertices $x$ and $y$, we write $x \sim y$ when and only when there is an edge $e$ connecting $x, y$ and $X(e) = 1$. Intuitively, we delete each edge in state 0 while remain those in state 1, then $x \sim y$ when and only when they are neighbors on the consequent graph.

We denote by $\mathcal{P}(\mathbb{Z}^d)$ the set of all the subsets of $\mathbb{Z}^d$, then the SIR model is a Markov process with state space

$$\Omega = \{ (A, B) : A, B \in \mathcal{P}(\mathbb{Z}^d), A \cap B = \emptyset \}.$$
We denote by \((S_t, I_t)\) the state of the process at moment \(t\) for any \(t \geq 0\), then the SIR epidemic evolves as follows.

\[
(S_t, I_t) \rightarrow \begin{cases} 
(S_t, I_t \setminus \{x\}) \quad \text{at rate 1 if } x \in I_t, \\
(S_t \setminus \{x\}, I_t \cup \{x\}) \quad \text{at rate } \lambda \sum_{y : y \sim x} 1_{\{y \in I_t\}} \quad \text{if } x \in S_t,
\end{cases}
\]

where \(\lambda\) is a positive parameter called the infection rate and we denote by \(1_A\) the indicator function of the random event \(A\).

Intuitively, the process \(\{(S_t, I_t)\}_{t \geq 0}\) describes the spread of an epidemic. Vertices in \(S_t\) are susceptible which can be infected while vertices in \(I_t\) are infective which can infect neighbors. Vertices in \(\mathbb{Z}^d \setminus (S_t \cup I_t)\) are removed which will never be infected again. A susceptible vertex is infected at rate proportional to the number of infective neighbors while an infective vertex becomes removed at rate one. Note that here we say \(x\) and \(y\) are neighbors when the edge \(e\) connecting them satisfies \(X(e) = 1\) as we introduced.

The main topic we are concerned with in this paper is the estimation of the critical value of our model, which is the maximum of the infection rates with which infective vertices die out with probability one when at \(t = 0\) there are finite infective vertices. The critical value of infection is first studied for another type of epidemic which is the SIS model, where an infective vertex will become healthy and then may be infected again. The SIS model is also named as the contact process. See a survey of the contact process in Chapter 6 of [6] and Part one of [7]. A direct corollary of our main result given in the next section can be seen as a counterpart of the asymptotic behavior of critical value of the large-dimensional contact process obtained in [4] by Holley and Liggett. For mathematical details, see the next section.

Let \(p_c\) be the maximum of \(p\) with which the open cluster containing \(O\) is finite with probability one, then when \(p < p_c\) infective vertices die out almost surely since the infection spreads on finite graphs and the critical value of infection rate is infinity as a result. However, Kesten proves \(\lim_{d \to +\infty} 2dp_c(d) = 1\) in [5] and hence \(p > p_c(d)\) for given \(p > 0\) and sufficiently large \(d\), which makes the critical infection rate of the epidemic nontrivial in large dimension. We are inspired by the technique introduced in [5] a lot when proving the main result of this paper.

We are inspired a lot by recent references about the SIR epidemic on percolation models. The percolation on complete graph is also known as the ER (Erdős-Rényi) graph. In [8], Neal studies a discrete-time version of SIR on the ER graph and gives limit distribution of the process. In [10], Xue considers a law of large numbers of the SIR on ER graph inspired by the theory of density dependent population model introduced by Ethier and Kurtz in [2]. In [9] and [11], Xue considers the SIR epidemics on open clusters of oriented site and oriented bond percolation models on lattices as auxiliary tools to study corresponding contact processes.

2 Main result

In this section we give our main result. First we introduce some notations, definitions and basic assumptions. For each \(d \geq 1\), we assume that \(\{X(e)\}_{e \in E_d}\) are defined under the probability space \((Y_d, \mathcal{F}_d, \mu_d)\). We denote by \(E_{\mu_d}\) the expectation operator with respect to \(\mu_d\). For any \(\omega \in Y_d\), we denote by \(P_{\lambda, \omega}\) the probability measure of the process \(\{(S_t, I_t)\}_{t \geq 0}\) with infection rate \(\lambda\) in the random environment on \(\mathbb{Z}^d\) with respect to \(\{X(\omega, e)\}_{e \in E_d}\). \(P_{\lambda, \omega}\)
is called the quenched measure. We denote by $E^\omega_\lambda$ the expectation operator with respect to $P^\omega_\lambda$. We define

$$P_{\lambda,d}(\cdot) = E_{\mu_d} \left( P^\omega_\lambda(\cdot) \right) = \int P^\omega_\lambda(\cdot) \mu_d(d\omega),$$

which is called the annealed measure. We denote by $E_{\lambda,d}$ the expectation operator with respect to $P_{\lambda,d}$. When there is no misunderstanding, we write $Y_d, F_d, \mu_d, E\mu_d, P_\lambda, E\lambda$.

Throughout this paper we assume that

$$(S_0, I_0) = (\mathbb{Z}_d \setminus \{O\}, \{O\})$$

for the process on $\mathbb{Z}_d$ and each $d \geq 1$. Note that $O$ is the origin of the lattice as we introduced. According to the basic coupling of Markov processes (see Section 3.1 of [6]), for any $\lambda_1 < \lambda_2$,

$$P_{\lambda_1}(I_t \neq \emptyset, \forall t > 0) \leq P_{\lambda_2}(I_t \neq \emptyset, \forall t > 0).$$

As a result, it is reasonable to define

$$\lambda_c(d) = \sup \{ \lambda : P_{\lambda,d}(I_t \neq \emptyset, \forall t > 0) = 0 \}$$

for each $d \geq 1$. That is to say, $\lambda_c$ is the maximum of the infection rates with which the infective vertices die out almost surely when there are finite infective vertices at $t = 0$.

The following theorem is our main result, which gives the asymptotic behavior of $\lambda_c(d)$ as $d$ grows to infinity.

**Theorem 2.1.** If $\lambda_c(d)$ is defined as in Equation (2.2), then

$$\lim_{d \to +\infty} d\lambda_c(d) = \frac{1}{2p}.$$ 

Note that $p$ is the probability that a given edge is in state 1 as we defined at the beginning of this paper. When $p = 1$, our model reduces to the classic SIR epidemic on lattices and Theorem 2.1 shows that

$$\lim_{d \to +\infty} 2d\lambda_c(d) = 1.$$ 

Let $\hat{\lambda}_c(d)$ be the counterpart of $\lambda_c(d)$ with respect to the contact process, then it is proved in [4] that

$$\lim_{d \to +\infty} 2d\hat{\lambda}_c(d) = 1$$

for the classic case where $p = 1$. For general case, it is easy to see that $\hat{\lambda}_c(d) \leq \lambda_c(d)$ according to basic coupling of Markov processes. Hence as a direct corollary of Theorem 2.1,

$$\limsup_{d \to +\infty} d\hat{\lambda}_c(d) \leq \frac{1}{2p}.$$ 

This result has been proved in [12] in a general case where each infective vertex recovers at i. i. d. random rates. We believe that $\liminf_{d \to +\infty} d\hat{\lambda}_c(d) \geq \frac{1}{2p}$ and hence $\lim_{d \to +\infty} d\hat{\lambda}_c(d) = \frac{1}{2p}$ but have not found a proof yet.

We give an intuitive explanation of Theorem 2.1 according to a mean-field analysis. When $d$ is large, an vertex has about $2dp$ neighbors according to the law of large numbers.
Each infective vertex becomes removed at rate one while infects a given neighbor at rate $\lambda$, hence the number $I$ of infective vertices approximately follows the ODE

$$\frac{dI}{dt} = (2dp\lambda - 1)I.$$ 

Then $I$ converges to 0 when and only when $\lambda < \frac{1}{2dp}$.

The proof of Theorem 2.1 is divided into two sections. In Section 3, we will show that

$$\limsup_{d \to +\infty} d\lambda_c(d) \leq \frac{1}{2p}.$$  \hspace{1cm} (2.3)$$

Our proof of Equation (2.3) is inspired by the technique introduced in [5]. We consider the self-avoiding paths on which the infection spreads from the beginning to the end. We call such paths the infection paths. If $\lambda$ satisfies that the probability that there exist infection paths with arbitrary lengths is positive, then $\lambda$ is a upper bound of $\lambda_c$. In Section 4, we will show that

$$\liminf_{d \to +\infty} d\lambda_c(d) \geq \frac{1}{2p}.$$  \hspace{1cm} (2.4)$$

The proof of Equation (2.4) is relative easy. We consider the number $N$ of the vertices which have ever been infective. If $\lambda$ satisfies that the mean of $N$ is finite, then $\lambda$ is a lower bound of $\lambda_c$. For mathematical details, see Sections 3 and 4.

3 Proof of Equation (2.3)

In this section we will give the proof of Equation (2.3). Since Equation (2.3) is about the asymptotic behavior of $\lambda_c(d)$ as $d \to +\infty$, we assume that the dimension $d$ of the lattice satisfies $d \geq 20$ throughout this section. We will explain the reason of this assumption later. First we introduce some definitions and notations. We denote by $\| \cdot \|_1$ the $l_1$ norm on $\mathbb{Z}^d$ such that

$$\|x - y\|_1 = \sum_{i=1}^{d} |x_i - y_i|$$

for any $x = (x_1, \ldots, x_d)$, $y = (y_1, \ldots, y_d) \in \mathbb{Z}^d$. For $1 \leq i \leq d$, we use $e_i$ to denote

$$(0, \ldots, 0, 1_i, 0, \ldots, 0),$$

which is the $i$th elementary vector on $\mathbb{Z}^d$. For any $x \in \mathbb{Z}^d$, we assume that $T(x)$ is an exponential time with rate 1. For any $x, y \in \mathbb{Z}^d$ such that $\|x - y\|_1 = 1$, we assume that $U(x, y)$ is an exponential time with rate $\lambda$. Note that here we care about the order of $x$ and $y$, hence $U(x, y) \neq U(y, x)$. We assume that all these exponential times are independent and are independent with the random environment $\{X(e)\}_{e \in \mathbb{Z}^d}$. Intuitively, $T(x)$ is the time interval $x$ waits for to become removed after being infected while $U(x, y)$ is the time interval $x$ waits for to infect $y$ after $x$ being infected if $X(x, y) = 1$.

For each integer $K \geq 1$, we define

$$L_K = \left\{ \vec{l} = (l_0, l_1, \ldots, l_K) \in (\mathbb{Z}^d)^K : l_0 = O; \|l_{i+1} - l_i\|_1 = 1, \right.$$  

$$\forall 0 \leq i \leq K - 1; l_i \neq l_j, \forall i \neq j \right\}$$

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as the set of self-avoiding paths on $\mathbb{Z}^d$ starting at the origin $O$ with length $K$. For each $K \geq 1$ and any $\vec{I} = (l_0, \ldots, l_K) \in L_K$, we define

$$A_\vec{I} = \{ U(l_i, l_{i+1}) \leq T(l_i), \forall 0 \leq i \leq K-1; \ X(l_i, l_{i+1}) = 1, \forall 0 \leq i \leq K-1 \}.$$  

Note that $A_\vec{I}$ is a random event. According to the definition of the SIR model, a vertex can not be infected repeatedly and hence any vertex that has ever been infective must be infected through a self-avoiding path from $O$ to it since $O$ is the only infective vertex at $t = 0$. Therefore, in the sense of coupling,

$$A_\vec{I} \subseteq \{ l_K \in I_t \text{ for some } t > 0 \} \quad (3.1)$$

for $\vec{I} = (l_0, \ldots, l_K) \in L_K$ and

$$\{ x \in I_t \text{ for some } t > 0 \} = \bigcup_{K=1}^{\infty} \bigcup_{\vec{I} \in L_K} A_\vec{I} \quad (3.2)$$

for any $x \neq O$. By direct calculation, for any $\vec{I} \in L_K$,

$$P_{\lambda}(A_\vec{I}) = \left( P(U \leq T)P(X(e) = 1) \right)^K = \frac{\lambda^K p^K}{(\lambda + 1)^K}, \quad (3.3)$$

where $U$ and $T$ are independent exponential times with rates $\lambda$ and 1 respectively, since all the edges on the self-avoiding path are different with each other and $\{ X(e) \}_{e \in E_d}$ are i.i.d.. For later use, we need to give an upper bound of $P(A_\vec{I} \cap A_{\vec{s}})$ for $\vec{I}, \vec{s} \in L_K$. For this purpose, for $\vec{I} = (l_0, \ldots, l_K), \vec{s} = (s_0, \ldots, s_K) \in L_K$, we define

$$D(\vec{I}, \vec{s}) = \{ 0 \leq i \leq K : s_i = l_j \text{ for some } j \in \{0, \ldots, K\} \}$$

and

$$F(\vec{I}, \vec{s}) = \{ 0 \leq i \leq K-1 : s_i = l_j \text{ and } s_{i+1} = l_{j+1} \text{ for some } j \in \{0, \ldots, K\} \}.$$ 

We use $\text{card}(A)$ or $|A|$ to denote the cardinality of the set $A$, then $|D(\vec{I}, \vec{s})|$ is the number of vertices that both $\vec{I}$ and $\vec{s}$ visit while $|F(\vec{I}, \vec{s})|$ is the number of edges that $\vec{I}$ and $\vec{s}$ visit through the same direction. We have the following lemma which gives an upper bound of $P(A_\vec{I} \cap A_{\vec{s}})$.

**Lemma 3.1.** For $\vec{I}, \vec{s} \in L_K$,

$$P(A_\vec{I} \cap A_{\vec{s}}) \leq \left( \frac{\lambda p}{\lambda + 1} \right)^{2K - |D(\vec{I}, \vec{s})|} \left( \frac{2}{p} \right)^{|D(\vec{I}, \vec{s}) \setminus F(\vec{I}, \vec{s})|}.$$ 

**Proof.** By Equation (3.3),

$$P(A_\vec{I} \cap A_{\vec{s}}) = P(A_{\vec{s}} | A_\vec{I}) P(A_\vec{I}) = \left( \frac{\lambda p}{\lambda + 1} \right)^K P(A_{\vec{s}} | A_\vec{I}).$$

For any $i \notin D(\vec{I}, \vec{s})$, $X(s_i, s_{i+1})$, $T(s_i)$ and $U(s_i, s_{i+1})$ are independent with $A_\vec{I}$, therefore $P(A_{\vec{s}} | A_\vec{I})$ has the factor

$$\left( P(U \leq T)P(X(e) = 1) \right)^{K - |D(\vec{I}, \vec{s})|} = \left( \frac{\lambda p}{\lambda + 1} \right)^{K - |D(\vec{I}, \vec{s})|}.$$
For each $i \in D(\vec{i}, \vec{s}) \setminus F(\vec{i}, \vec{s})$, there exist $0 \leq j \leq K - 1$ and $x, y, z$ such that $s_i = l_j = x$, $s_{i+1} = y$, $l_{j+1} = z$ and $y \neq z$. Hence there is a factor at most

$$P(U(x, y) < T(x) | U(x, z) < T(x)) \leq \frac{2\lambda}{\lambda + 1}$$

in the expression of $P(A_x | A_f)$ for each $i \in D(\vec{i}, \vec{s}) \setminus F(\vec{i}, \vec{s})$. Therefore, $P(A_x | A_f)$ has a factor at most $(\frac{2\lambda}{\lambda + 1})^{|D(\vec{i}, \vec{s}) \setminus F(\vec{i}, \vec{s})|}$. For each $i \in F(\vec{i}, \vec{s})$, $X(s_i, s_{i+1}) = 1$ and $U(s_i, s_{i+1}) \leq T(s_i)$ occurs with probability one conditioned on $A_f$ since there exists $j$ such that $s_i = l_j$ and $s_{i+1} = l_{j+1}$. In conclusion,

$$P(A_f | A_x) \leq \left( \frac{\lambda p}{\lambda + 1} \right)^{\lambda - |D(\vec{i}, \vec{s})|} \left( \frac{2\lambda}{\lambda + 1} \right)^{|D(\vec{i}, \vec{s}) \setminus F(\vec{i}, \vec{s})|}$$

and hence

$$P(A_f \cap A_x) = \left( \frac{\lambda p}{\lambda + 1} \right)^{K} P(A_x | A_f) \leq \left( \frac{\lambda p}{\lambda + 1} \right)^{2K - |F(\vec{i}, \vec{s})|} \left( \frac{2}{p} \right)^{|D(\vec{i}, \vec{s}) \setminus F(\vec{i}, \vec{s})|}.$$

Inspired by the approach introduced in \[5\] by Kesten, we consider a special type of self-avoiding paths on $\mathbb{Z}^d$. For each $k \geq 1$, we define

$$\Gamma_k = \left\{ \vec{r} = (l_0, \ldots, l_k) \in L_k : l_{i+1} - l_i \in \{ \pm e_j : 1 \leq j \leq d - \lfloor \frac{d}{\log d} \rfloor \} \right\}$$

for any $i$ such that $|\log d| \nmid (i + 1)$; $l_{i+1} - l_i \in \{ e_j : d - \lfloor \frac{d}{\log d} \rfloor + 1 \leq j \leq d \}$

for any $i$ such that $|\log d| \nmid (i + 1)$,

where we use $a \mid b$ to denote that $b$ is divisible by $a$ and $\{ e_j \}_{1 \leq j \leq d}$ are the elementary vectors on $\mathbb{Z}^d$ as we defined in Section \[4\]. We introduce a random walk $\{ S_n \}_{n=0}^{+\infty}$ on $\mathbb{Z}^d$ with paths in $\bigcup_{k \geq 1} \Gamma_k$. For $n = 0$, we assume that $S_0 = O$. For $n \geq 1$, $\{ S_n \}_{n=1}^{+\infty}$ evolves as follows. For each $j \geq 1$ such that $|\log d| \nmid j$, assuming that we have already obtained the first $j$ steps $S_0, S_1, S_2, \ldots, S_{j-1}$, then

$$P(S_j = z | S_i, 0 \leq i \leq j - 1) = \frac{1}{|H(j)|}$$

for any $z \in H(j)$, where

$$H(j) = \left\{ y : y - S_{j-1} \in \{ \pm e_l : 1 \leq l \leq d - \lfloor \frac{d}{\log d} \rfloor \}, S_i \neq y \text{ for all } 0 \leq i \leq j - 1 \right\}$$

which is a random set depending on $\{ S_0, S_1, \ldots, S_{j-1} \}$. For each $j \geq 1$ such that $|\log d| \mid j$,

$$P(S_j = S_{j-1} + e_l | S_i, 0 \leq i \leq j - 1) = \frac{1}{|H(j)|}.$$
for each \(d - \lfloor \frac{d}{\log d} \rfloor + 1 \leq l \leq d\). For any \(x = (x_1, \ldots, x_d) \in \mathbb{Z}^d\), we define

\[
\beta(x) = \sum_{j=d-\lfloor \frac{d}{\log d} \rfloor + 1}^{d} x_j,
\]

then it is easy to check that, for each \(k \geq 0\) and \(k[\log d] \leq j \leq (k + 1)[\log d] - 1\),

\[
\beta(S_j) = k.
\]

For each \(j\) such that \([\log d] \nmid j\), we claim that

\[
|H(j)| \geq 2(d - \lfloor \frac{d}{\log d} \rfloor) - \lfloor \log d \rfloor.
\]  

(3.4)

This is because \(\beta(y) = \beta(S_{j-1})\) for each \(y\) such that \(y - S_{j-1} \in \{\pm e_l : 1 \leq l \leq d - \lfloor \frac{d}{\log d} \rfloor\}\) while \(\text{card}\{u : \beta(S_u) = \beta(S_j)\} = \lfloor \log d \rfloor\).

For each \(k \geq 1\), we use \(S_k\) to denote the path \((S_0, S_1, \ldots, S_k)\) on \(\mathbb{Z}^d\), then it is easy to check that

\[
\bar{S}_k[\log d] \in \Gamma_k
\]

for each \(k \geq 1\). We denote by \(\{V_n\}_{n=0}^{+\infty}\) an independent copy of \(\{S_n\}_{n=0}^{+\infty}\) with \(V_0 = 0\), and use \(\bar{V}_k\) to denote the path \((V_0, V_1, \ldots, V_k)\) for each \(k \geq 1\), then we define

\[
D(\bar{S}, \bar{V}) = \bigcup_{k \geq 1} D(\bar{S}_k[\log d], \bar{V}_k[\log d]) = \left\{i : V_i = S_j \text{ for some } j \geq 0 \right\}
\]

and

\[
F(\bar{S}, \bar{V}) = \bigcup_{k \geq 1} F(\bar{S}_k[\log d], \bar{V}_k[\log d]) = \left\{i : V_i = S_j \text{ and } V_{i+1} = S_{j+1} \text{ for some } j \geq 0 \right\},
\]

where we use \(\bar{S}\) and \(\bar{V}\) to denote the entire paths of \(\{S_n\}_{n \geq 0}\) and \(\{V_n\}_{n \geq 0}\) respectively. Here we claim that \(|D(\bar{S}, \bar{V})| < +\infty\) almost surely under our assumption that \(d \geq 20\). The reason is as follows. When \(d \geq 20\), \(\lfloor \frac{d}{\log d} \rfloor \geq 4\). The path of the latter \(\lfloor \frac{d}{\log d} \rfloor\) coordinates of \(\bar{S}\) is a \(\lfloor \log d \rfloor\) times slower oriented random walk on \(\mathbb{Z}^{\lfloor \frac{d}{\log d} \rfloor}\) while former reference shows that two independent oriented simple random walks on \(\mathbb{Z}^u\) with \(u \geq 4\) collides with each other finitely many times almost surely.

We use \(\bar{P}\) to denote the probability measure of \(\{S_n\}_{n \geq 0}^{+\infty}\) and \(\{V_n\}_{n \geq 0}^{+\infty}\) while denote by \(\bar{E}\) the expectation operator with respect to \(\bar{P}\), then the following lemma is crucial for us to prove Equation (2.3).

**Lemma 3.2.** If \(\lambda\) satisfies that

\[
\bar{E}\left(\frac{\lambda + 1}{\lambda p} \frac{|F(\bar{S}, \bar{V})|}{D(\bar{S}, \bar{V})} \frac{2}{\bar{P}(D(\bar{S}, \bar{V}) \setminus F(\bar{S}, \bar{V}))}\right) < +\infty,
\]

then \(\lambda \geq \lambda_\bullet(d)\).

The following lemma is utilized in the proof of Lemma 3.2.
Lemma 3.3. If \( C_1, C_2, \ldots, C_n \) are \( n \) arbitrary random events defined under the same probability space such that \( P(C_i) > 0 \) for \( 1 \leq i \leq n \) and \( q_1, q_2, \ldots, q_n \) are \( n \) positive constants such that \( \sum_{j=1}^{n} q_j = 1 \), then

\[
P \left( \bigcup_{j=1}^{+\infty} C_j \right) \geq \frac{1}{\sum_{i=1}^{n} \sum_{j=1}^{n} q_i q_j \frac{P(C_i \cap C_j)}{P(C_i) P(C_j)}}.
\]

Proof of Lemma \[\text{3.3}\]. For each \( 1 \leq i \leq n \), we define

\[
Y_i = \begin{cases} \frac{q_i}{P(C_i)} & \text{on } C_i, \\ 0 & \text{on } C_i^c, \end{cases}
\]

where \( C_i^c \) is the complement set of \( C_i \), then

\[
\left( E \left( \sum_{i=1}^{n} Y_i \right) \right)^2 = \left( \sum_{i=1}^{n} \frac{q_i}{P(C_i)} P(C_i) \right)^2 = 1^2 = 1
\]

and

\[
E \left( \left( \sum_{i=1}^{n} Y_i \right)^2 \right) = \sum_{i=1}^{n} \sum_{j=1}^{n} E(Y_i Y_j) = \sum_{i=1}^{n} \sum_{j=1}^{n} q_i q_j \frac{P(C_i \cap C_j)}{P(C_i) P(C_j)}.
\]

According to H"older’s inequality,

\[
P \left( \bigcup_{i=1}^{n} C_i \right) = P \left( \sum_{i=1}^{n} Y_i > 0 \right) \geq \frac{\left( E \left( \sum_{i=1}^{n} Y_i \right) \right)^2}{E \left( \left( \sum_{i=1}^{n} Y_i \right)^2 \right)} = \frac{1}{\sum_{i=1}^{n} \sum_{j=1}^{n} q_i q_j \frac{P(C_i \cap C_j)}{P(C_i) P(C_j)}}
\]

and the proof is complete. \( \square \)

Now we give the proof of Lemma \[\text{3.2}\].

Proof of Lemma \[\text{3.2}\]. On the event \( +\infty \bigcup_{k=1}^\infty \bigcup_{i \in \Gamma_k} A_i \) there exist vertices which have ever been infected with arbitrary large norms and hence \( I_t \neq \emptyset \) for any \( t > 0 \). Therefore, according to the Dominated Convergence Theorem,

\[
P_{\lambda} (I_t \neq \emptyset, \forall t > 0) \geq P_{\lambda} \left( \bigcup_{k=1}^{+\infty} \bigcup_{i \in \Gamma_k} A_i \right) = \lim_{k \to +\infty} P_{\lambda} \left( \bigcup_{i \in \Gamma_k} A_i \right). \tag{3.5}
\]

For each \( \vec{t} \in \Gamma_k \), we define \( q_{\vec{t}} = P_{\lambda} (\vec{S}_{k[\log d]} = \vec{t}) \), then \( \sum_{\vec{t} \in \Gamma_k} q_{\vec{t}} = 1 \) since \( \vec{S}_{k[\log d]} \in \Gamma_k \) almost surely. Then by Lemma \[\text{3.3}\],

\[
P_{\lambda} \left( \bigcup_{i \in \Gamma_k} A_i \right) \geq \frac{1}{\sum_{\vec{t} \in \Gamma_k} \sum_{\vec{t} \in \Gamma_k} q_{\vec{t}} \frac{P(A_{\vec{t}} \cap A_{\vec{t}'})}{P(A_{\vec{t}}) P(A_{\vec{t}'})}}.
\]

By Equation \[\text{3.3}\] and Lemma \[\text{3.1}\],

\[
P \left( A_{\vec{t}} \cap A_{\vec{t}'} \right) \leq \frac{1}{\lambda p} |F(\vec{t}, \vec{s})| \left( \frac{2}{p} \right)^{|D(\vec{t}, \vec{s}) \setminus F(\vec{t}, \vec{s})|}.
\]
As a result,

\[
\begin{align*}
P_\lambda \left( \bigcup_{l \in \Gamma_k} A_l \right) & \geq \frac{1}{\sum_{l \in \Gamma_k} \sum_{x \in \Gamma_k} \sum_{y \in \Gamma_k} q_l q_x \left( \frac{2}{p} \right)^{D(l, x) \wedge F(l, x)} \left( \frac{2}{p} \right)^{D(l, y) \wedge F(l, y)}} \\
& = \frac{1}{E \left( \left( \frac{\lambda_p}{\lambda} \right)^{|F(S_k \log d), V_k \log d|} \left( \frac{2}{p} \right)^{|D(S_k \log d) \wedge F(S_k \log d), V_k \log d|} \right)},
\end{align*}
\]

since \( P_\lambda (S_k \log d) = \tilde{r}, \tilde{V}_k \log d = \tilde{s} \) = \( q_l q_x \). According to the Dominated Convergence Theorem,

\[
\lim_{k \to +\infty} E \left( \left( \frac{\lambda_p}{\lambda} \right)^{|F(S_k \log d), V_k \log d|} \left( \frac{2}{p} \right)^{|D(S_k \log d) \wedge F(S_k \log d), V_k \log d|} \right) = E \left( \left( \frac{\lambda_p}{\lambda} \right)^{|F(S, V)|} \left( \frac{2}{p} \right)^{|D(S, V) \wedge F(S, V)|} \right).
\]

Then by Equations (3.5) and (3.6),

\[
P_\lambda (I_t \neq \emptyset, \forall \ t > 0) \geq \frac{1}{E \left( \left( \frac{\lambda_p}{\lambda} \right)^{|F(S, V)|} \left( \frac{2}{p} \right)^{|D(S, V) \wedge F(S, V)|} \right)} > 0
\]

and \( \lambda \geq \lambda_\psi(d) \) consequently when

\[
E \left( \left( \frac{\lambda_p}{\lambda} \right)^{|F(S, V)|} \left( \frac{2}{p} \right)^{|D(S, V) \wedge F(S, V)|} \right) < +\infty.
\]

We do not check whether \( S_k \log d \) is uniformly distributed on \( \Gamma_k \) since our proof of Lemma 3.2 does not require this property to hold. We leave this to readers good at calculation.

According to Lemma 3.2, we need to give upper bound of \( E \left( \left( \frac{\lambda_p}{\lambda} \right)^{|F(S, V)|} \left( \frac{2}{p} \right)^{|D(S, V) \wedge F(S, V)|} \right) \).

We have the following related lemma.

**Lemma 3.4.** There exists a constant \( M_1 < +\infty \) which does not depend on \( d \) such that for any \( \theta, \psi > 0 \),

\[
E \left( \theta |F(S, V)| \cdot \psi |D(S, V) \wedge F(S, V)| \right) \leq \sum_{k=0}^{+\infty} \sum_{j=1}^{3} \Phi_{\theta, \psi}^k (1, j),
\]

where \( \Phi_{\theta, \psi} \) is a \( 3 \times 3 \) matrix such that

\[
\Phi_{\theta, \psi} = \begin{pmatrix}
\left( \frac{\log d}{\log d} \right)^{\frac{3}{d}} & \theta & M_1 (\log d)^{\frac{5}{d}} \\
\left( \frac{\log d}{\log d} \right)^{\frac{3}{d}} & \theta & M_1 (\log d)^{\frac{5}{d}} \\
\left( \frac{\log d}{\log d} \right)^{\frac{3}{d}} & \theta & M_1 (\log d)^{\frac{5}{d}} \\
\end{pmatrix}.
\]

We give the proof of Lemma 3.4 at the end of this section. Now we show how to utilize Lemma 3.4 to prove Equation (2.3).
Proof of Equation (2.3). For given $r > 1$, let $\lambda = \frac{r}{2dp}$, $\theta = \frac{\lambda + 1}{\lambda p}$ and $\psi = \frac{2}{p}$, then

$$\max \left\{ \sum_{j=1}^{3} \Phi_{\theta,\psi}(i,j) : 1 \leq i \leq 3 \right\} < 1$$

for sufficiently large $d$ according to the definition of $\Phi_{\theta,\psi}$. As a result, by Lemma 3.4

$$\tilde{E} \left( \theta_{\{F(\bar{S},\bar{V})\}}, \psi_{\{D(\bar{S},\bar{V})\} \cap \{F(\bar{S},\bar{V})\}} \right) < +\infty$$

for sufficiently large $d$, where $\theta = \frac{\lambda + 1}{\lambda p}$, $\psi = \frac{2}{p}$ and $\lambda = \frac{r}{2dp}$ for $r > 1$. Then by Lemma 3.2

$$\lambda_c(d) \leq \lambda = \frac{r}{2dp}$$

for sufficiently large $d$ and $r > 1$. Therefore,

$$\limsup_{d \to +\infty} d\lambda_c(d) \leq \frac{r}{2p}$$

for any $r > 1$. Let $r \to 1$, then the proof is complete.

At last, we only need to prove Lemma 3.4. For this purpose, we introduce some notations and definitions. We use $\tau$ to denote $|D(\bar{S},\bar{V})|$, which is the number of vertices both $\bar{S}$ and $\bar{V}$ visit. For $1 \leq i \leq \tau$, we define $t(1) = 0$ and

$$t(i) = \inf \left\{ j : j > t(i-1) \text{ and } j \in D(\bar{S},\bar{V}) \right\}.$$

Note that $t(1) = 0$ because $S_0 = V_0 = O$. We divide $\{t(i)\}_{1 \leq i \leq \tau}$ into three different types. If $t(i) \in F(\bar{S},\bar{V})$ and $|\log d| \nmid (t(i) + 1)$, then we say that $t(i)$ is with type 1. If $t(i) \in F(\bar{S},\bar{V})$ and $|\log d| \mid (t(i) + 1)$, then we say that $t(i)$ is with type 2. If $t(i) \in D(\bar{S},\bar{V}) \setminus F(\bar{S},\bar{V})$, then we say that $t(i)$ is with type 3. From now on, we assume that $\theta$ and $\psi$ are given. For $i = 1, 2, 3$, we define

$$\alpha(i) = \begin{cases} \theta & \text{if } i = 1, 2, \\ \psi & \text{if } i = 3. \end{cases}$$

For $k \geq 2$ and $1 \leq i, j \leq 3$, we use $\Upsilon(k,i,j)$ to denote

$$\alpha(j) \tilde{P} \left( \tau \geq k + 1, t(k) \text{ is with type } j \bigg| \bar{S} ; t(u), u \leq k ; V_u, u \leq t(k) \right)$$

on the event $\{\tau \geq k ; t(k-1) \text{ is with type } i\}$. For $1 \leq j \leq 3$, we define

$$\nu(j) = \alpha(j) \tilde{P} \left( \tau \geq 2, t(1) \text{ is with type } j \bigg| \bar{S} \right).$$

For $k \geq 2$ and $1 \leq i \leq 3$, we use $b(k,i)$ to denote

$$\psi \tilde{P} \left( \tau = k \bigg| \bar{S} ; t(u), u \leq k ; V_u, u \leq t(k) \right)$$
on the event \( \{ \tau \geq k; t(k-1) \text{ is with type } i \} \). Note that \( t(k) \) is with type 3 when \( \tau = k \).

For each \( m \geq 1 \), we define

\[
W_m = \left\{ \vec{i} = (i_1, i_2, \ldots, i_m) \in \mathbb{Z}^m : i_l \in \{1, 2, 3\} \text{ for } 1 \leq l \leq m \right\}.
\]

According to the Total Probability Theorem, for \( m \geq 2 \) and \( \vec{i} = (i_1, \ldots, i_m) \in W_m \),

\[
\tilde{E} \left( \theta^{\mathcal{F}(\vec{S}, \vec{V})}, \psi^{\mathcal{D}(\vec{S}, \vec{V}) \setminus \mathcal{F}(\vec{S}, \vec{V})} : \tau = m + 1, t(l) \text{ is with type } i_l \text{ for } 1 \leq l \leq m \right)
= \tilde{E} \left( \nu(i_1) \left[ \prod_{k=2}^m \Upsilon(k, i_{k-1}, i_k) \right] b(m + 1, i_m) \right)
\]

and hence

\[
\tilde{E} \left( \theta^{\mathcal{F}(\vec{S}, \vec{V})}, \psi^{\mathcal{D}(\vec{S}, \vec{V}) \setminus \mathcal{F}(\vec{S}, \vec{V})} \right)
= \psi \tilde{P}(\tau = 1) + \tilde{E} \left( \sum_{i=1}^3 \nu(i) b(2, i) + \sum_{m=2}^{+\infty} \sum_{\vec{i} \in W_m} \nu(i_1) \left[ \prod_{k=2}^m \Upsilon(k, i_{k-1}, i_k) \right] b(m + 1, i_m) \right)
\leq \psi + \psi \tilde{E} \left( \sum_{i=1}^3 \nu(i) + \sum_{m=2}^{+\infty} \sum_{\vec{i} \in W_m} \nu(i_1) \left[ \prod_{k=2}^m \Upsilon(k, i_{k-1}, i_k) \right] \right)
\]

(3.7) since \( b(k, i) \leq \psi \). Note that here we utilize the assumption that \( d \geq 20 \), which ensures that \( \tau < +\infty \) almost surely.

To prove Lemma \ref{lem:3.3}, we need the following lemma.

**Lemma 3.5.** For each \( k \geq 2 \) and each \( 1 \leq i \leq 3 \),

\[
\Upsilon(k, i, 1) \leq \frac{\theta}{2(d - \lfloor \frac{d}{\log d} \rfloor) - \lfloor \log d \rfloor} \quad \text{and} \quad \nu(1) \leq \frac{\theta}{2(d - \lfloor \frac{d}{\log d} \rfloor) - \lfloor \log d \rfloor}.
\]

(3.8)

For each \( k \geq 2 \) and each \( 1 \leq i \leq 3 \),

\[
\Upsilon(k, i, 2) \leq \frac{\theta}{\lfloor \log d \rfloor} \quad \text{and} \quad \nu(2) \leq \frac{\theta}{\lfloor \log d \rfloor}.
\]

(3.9)

For each \( k \geq 2 \) and each \( 1 \leq i \leq 3 \), there exists a constant \( M_1 \) which does not depend on \( d, \theta, \psi \) such that

\[
\Upsilon(k, i, 3) \leq \frac{\psi M_1 (\log d)^2}{d} \quad \text{and} \quad \nu(3) \leq \frac{\psi M_1 (\log d)^2}{d^2}.
\]

(3.10)

**Proof.** For Equation \ref{eq:3.3}, conditioned on \( \tau \geq k \), there exists a unique \( j \) such that \( S_j = V_{t(k)} \). Note that \( j \) is unique according to the fact that \( \vec{S} \) is self-avoiding. Then, \( t(k) \) is with type 1 when and only when \( V_{t(k)+1} = S_{j+1} \) and \( \lfloor \log d \rfloor \nmid (t(k) + 1) \). As a result,

\[
\Upsilon(k, i, 1) \leq \theta \max \left\{ \tilde{P}(V_l = y) : y \in \mathbb{Z}^d, \lfloor \log d \rfloor \nmid l \right\} \leq \frac{\theta}{2(d - \lfloor \frac{d}{\log d} \rfloor) - \lfloor \log d \rfloor}
\]

by Equation \ref{eq:3.3}. \( \nu(1) \leq \frac{\theta}{2(d - \lfloor \frac{d}{\log d} \rfloor) - \lfloor \log d \rfloor} \) follows from a similar analysis.
For Equation (3.9), according to a similar analysis with that in the proof of Equation (3.8),

\[ \gamma(k, i, 2) \leq \theta \max \left\{ \tilde{P}(V_i = y) : y \in \mathbb{Z}^d, \lfloor \log d \rfloor \right\} \]

\[ = \frac{\theta}{\lfloor \log d \rfloor}. \]

\[ \nu(2) \leq \frac{\theta}{\lfloor \log d \rfloor} \] follows from a similar analysis.

For Equation (3.10),

\[ \gamma(k, i, 3) \leq \psi \tilde{P} \left( \exists j > t(k), j \in D(\vec{S}, \vec{V}) \mid \vec{S}; t(u), u \leq k; V_u, u \leq t(k) \right) \]

while

\[ \tilde{P} \left( \exists j > t(k), j \in D(\vec{S}, \vec{V}) \mid \vec{S}; t(u), u \leq k; V_u, u \leq t(k) \right) = A + B, \]

where

\[ A = \tilde{P} \left( \exists j > t(k), \beta(V_j) = \beta(V_{t(k)}) \text{ and } j \in D(\vec{S}, \vec{V}) \mid \vec{S}; t(u), u \leq k; V_u, u \leq t(k) \right) \]

and

\[ B = \tilde{P} \left( \exists j > t(k), \beta(V_j) > \beta(V_{t(k)}) \text{ and } j \in D(\vec{S}, \vec{V}) \mid \vec{S}; t(u), u \leq k; V_u, u \leq t(k) \right). \]

If \( j \in D(\vec{S}, \vec{V}) \) for some \( j \) such that \( j > t(k) \) and \( \beta(V_j) = \beta(V_{t(k)}) \), then there exists \( i \) such that \( \beta(S_i) = \beta(V_{t(k)}) \) and \( V_j = S_i \). The value of the function \( \beta \) increases by one every \( \lfloor \log d \rfloor \) steps of the random walk, hence the number of possible choices of such \((j, i)\) is at most \( \lfloor \log d \rfloor^2 \). As a result,

\[ A \leq \lfloor \log d \rfloor^2 \max \left\{ \tilde{P}(V_i = y) : y \in \mathbb{Z}^d, \lfloor \log d \rfloor \right\} \leq \frac{\lfloor \log d \rfloor^2}{2(d - \lfloor \log d \rfloor) - \lfloor \log d \rfloor}. \]

Note that in the above equation we utilize the fact that \( \lfloor \log d \rfloor \) when \( j > t(k) \) and \( \beta(V_j) = \beta(V_{t(k)}) \).

Now we deal with \( B \). For each \( x = (x_1, \ldots, x_d) \in \mathbb{Z}^d \), we define

\[ \xi(x) = (x_{d-\lfloor \log d \rfloor+1}, \ldots, x_d) \in \mathbb{Z}^{\lfloor \log d \rfloor}. \]

Then \( \{\xi(S_k|\lfloor \log d \rfloor)\}_{k \geq 0} \) and \( \{\xi(V_k|\lfloor \log d \rfloor)\}_{k \geq 0} \) are two independent oriented simple random walks on \( \mathbb{Z}^{\lfloor \log d \rfloor} \) starting at the origin according to our definition of \( \vec{S} \) and \( \vec{V} \). It is shown in [1] that there exists \( M_2 > 0 \) such that two independent oriented simple random walk on \( \mathbb{Z}^d \), both starting at \( O \), collide with each other at least once after leaving \( O \) with probability at most \( 1/d + M_2/d^2 \), where \( M_2 \) does not depend on \( d \). Let \( c = \left\lfloor \frac{t(k)}{\lfloor \log d \rfloor} \right\rfloor \), then

\[ B \leq \tilde{P} \left( \exists f > c, \xi(V_f|\lfloor \log d \rfloor) = \xi(S_{t(k)}|\lfloor \log d \rfloor) \right) \left( \xi(V_c|\lfloor \log d \rfloor) = \xi(S_c|\lfloor \log d \rfloor) \right) \]

\[ \leq \frac{1}{\lfloor \log d \rfloor} + \frac{M_2}{\lfloor \log d \rfloor^2}. \]
As a result,
\[
\Upsilon(k,i,3) = \psi(A + B)
\]
\[
\leq \psi\left(\frac{\lfloor \log d \rfloor^2}{2(d - \lfloor \frac{d}{\log d} \rfloor)} - \lfloor \log d \rfloor + \frac{1}{\lfloor \frac{d}{\log d} \rfloor^2} + \frac{M_2}{\lfloor \frac{d}{\log d} \rfloor^2} \leq \frac{\psi M_1 (\log d)^2}{d},
\]
where we can choose \( M_1 \) which does not depend on \( d \). \( \nu(3) \leq \frac{\psi M_1 (\log d)^2}{d} \) follows from a similar analysis.

\[ \square \]

At last we give the proof of Lemma 3.4.

**Proof of Lemma 3.4.** According to Equation (3.7) and Lemma 3.5,
\[
\widetilde{E}\left(\varphi^{F(S,V)} \cdot \varphi^{D(S,V) \setminus F(S,V)}; \tau = m + 1, t(l) \text{ is with type } i_l \text{ for } 1 \leq l \leq m \right)
\]
\[
= \widetilde{E}\left(\nu(i_1) \prod_{l=2}^{m} \Upsilon(l, i_{l-1}, i_l) b(m + 1, i_m) \right) \leq \psi \Lambda(1, i_1) \prod_{l=2}^{m} \Lambda(i_{l-1}, i_l)
\]
for any \( \vec{\tau} = (i_1, i_2, \ldots, i_m) \in W_m \) with \( m \geq 2 \) and hence
\[
\widetilde{E}\left(\varphi^{F(S,V)} \cdot \varphi^{D(S,V) \setminus F(S,V)}; \tau = m + 1, t(l) \text{ is with type } i_l \text{ for } 1 \leq l \leq m \right)
\]
\[
= \psi \sum_{k=0}^{+\infty} \sum_{j=1}^{3} \Lambda^k(1, j), \tag{3.11}
\]
where \( \Lambda \) is a \( 3 \times 3 \) matrix such that
\[
\Lambda = \begin{pmatrix}
\frac{\theta}{2(d - \lfloor \frac{d}{\log d} \rfloor) - \lfloor \log d \rfloor} & \frac{\theta}{\lfloor \frac{d}{\log d} \rfloor} & \frac{\theta}{\lfloor \frac{d}{\log d} \rfloor} \\
\frac{\theta}{2(d - \lfloor \frac{d}{\log d} \rfloor) - \lfloor \log d \rfloor} & \frac{\theta}{\lfloor \frac{d}{\log d} \rfloor} & \frac{\theta}{\lfloor \frac{d}{\log d} \rfloor} \\
\frac{\theta}{2(d - \lfloor \frac{d}{\log d} \rfloor) - \lfloor \log d \rfloor} & \frac{\theta}{\lfloor \frac{d}{\log d} \rfloor} & \frac{\theta}{\lfloor \frac{d}{\log d} \rfloor}
\end{pmatrix}.
\]
Note that the expression of \( \Lambda \) is different with that of \( \Phi_{\theta, \psi} \). We can replace \( \Lambda \) by \( \Phi_{\theta, \psi} \) in Equation (3.11) according to the following analysis. \( \Phi_{\theta, \psi} \) is generated from \( \Lambda \) through multiplying the first column of \( \Lambda \) by \( \lfloor \log d \rfloor \) \( \left\{ \frac{1}{\lfloor \frac{d}{\log d} \rfloor} \right\} \), multiplying the second column of \( \Lambda \) by \( \frac{\psi M_1 (\log d)^2}{d} \) \( \left\{ \frac{1}{\lfloor \frac{d}{\log d} \rfloor} \right\} \), and multiplying the third column of \( \Lambda \) by \( \frac{\psi M_1 (\log d)^2}{d} \) \( \left\{ \frac{1}{\lfloor \frac{d}{\log d} \rfloor} \right\} \). Between two adjacent type 2 moments, there is either at least one type 3 moment or \( \lfloor \log d \rfloor - 1 \) consecutive type 1 moments. Therefore,
\[
\Lambda(1, i_1) \prod_{l=2}^{m} \Lambda(i_{l-1}, i_l) \leq \Phi_{\theta, \psi}(1, i_1) \prod_{l=2}^{m} \Phi_{\theta, \psi}(i_{l-1}, i_l) \tag{3.12}
\]
for any \( \vec{\tau} = (i_1, i_2, \ldots, i_m) \in W_m \) with \( m \geq 2 \) such that
\[
\widetilde{F}(t(l)) \text{ is with type } i_l \text{ for } 1 \leq l \leq m > 0.
\]
In other words, any \( \vec{\tau} \in W_m \) not satisfying Equation (3.12) cannot be the vector indicating the types of \( (t(1), \ldots, t(m)) \). As a result, we can replace \( \Lambda \) by \( \Phi_{\theta, \psi} \) in Equation (3.11) and the proof is complete.

\[ \square \]
4 Proof of Equation (2.4)

In this section we give the proof of Equation (2.4).

Proof of Equation (2.4). We define

\[ N = \text{card}\{ x \in \mathbb{Z}^d : x \in I_t \text{ for some } t > 0 \}, \]

then

\[ E_\lambda N = 1 + \sum_{x \neq O} P_\lambda(x \in I_t \text{ for some } t > 0) \]

since \( O \in I_0 \), where \( E_\lambda \) is the expectation operator with respect to \( P_\lambda \). By Equations (3.2) and (3.3),

\[ P_\lambda(x \in I_t \text{ for some } t > 0) \leq \sum_{K=1}^{+\infty} \frac{\lambda^K p^K \text{card}\{ \bar{l} = (l_0, \ldots, l_K) \in L_K : l_K = x \}}{(\lambda + 1)^K}, \]

for \( x \neq O \) and hence

\[ E_\lambda N \leq 1 + \sum_{K=1}^{+\infty} \frac{\lambda^K p^K \text{card}(L_K)}{(\lambda + 1)^K} \leq 1 + \sum_{K=1}^{+\infty} \frac{\lambda^K p^K 2d(2d-1)^{K-1}}{(\lambda + 1)^K}, \]

(4.1)

since \( \text{card}(L_K) \leq 2d(2d-1)^{K-1} \). By Equation (4.1), for sufficiently large \( d \) such that \( p(2d-1) > 1 \), \( E_\lambda N < +\infty \) when \( \lambda < \frac{1}{(2d-1)p-1} \). Therefore, \( P_\lambda(N < +\infty) = 1 \) when \( \lambda < \frac{1}{(2d-1)p-1} \). On the event \( \{ N < +\infty \} \), infective vertices die out when all these \( N \) vertices become removed. As a result, \( P_\lambda(I_t = \emptyset \text{ for some } t > 0) = 1 \) when \( \lambda < \frac{1}{(2d-1)p-1} \) and

\[ \frac{1}{(2d-1)p-1} \leq \lambda_c(d). \]

Therefore

\[ \liminf_{d \to +\infty} d\lambda_c(d) \geq \lim_{d \to +\infty} \frac{d}{(2d-1)p-1} = \frac{1}{2p} \]

and the proof is complete.

Acknowledgments. The author is grateful to the financial support from the National Natural Science Foundation of China with grant number 11501542 and the financial support from Beijing Jiaotong University with grant number KSRC16006536.

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