Double-sided Taylor’s approximations and their applications in Theory of analytic inequalities

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Abstract In this paper the double-sided TAYLOR’s approximations are studied. A short proof of a well-known theorem on the double-sided TAYLOR’s approximations is introduced. Also, two new theorems are proved regarding the monotonicity of such approximations. Then we present some new applications of the double-sided TAYLOR’s approximations in the theory of analytic inequalities.

1 Introduction

Consider a real function \( f : (a, b) \longrightarrow \mathbb{R} \) such that there exist finite limits \( f^{(k)}(a^+) = \lim_{x \to a^+} f^{(k)}(x) \), for \( k = 0, 1, \ldots, n \). Let us denote by \( T_n^{f,a^+}(x) \) TAYLOR’s polynomial of degree \( n \), \( n \in \mathbb{N}_0 \), for the function \( f(x) \) in the right neighborhood of \( a \):

\[
T_n^{f,a^+}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a^+)}{k!} (x - a)^k.
\]

We will call \( T_n^{f,a^+}(x) \) the first TAYLOR’s approximation in the right neighborhood of \( a \).

Similarly, the first TAYLOR’s approximation in the left neighborhood of \( b \) is defined by:

\[
T_n^{f,b^-}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(b^-)}{k!} (x - b)^k,
\]

where \( f^{(k)}(b^-) = \lim_{x \to b^-} f^{(k)}(x) \), for \( k = 0, 1, \ldots, n \).
Also, for $n \in \mathbb{N}$, the following functions:

$$R^f_{n,a^+}(x) = f(x) - T^f_{n,a^+}(x)$$

and

$$R^f_{n,b^-}(x) = f(x) - T^f_{n,b^-}(x)$$

are called the remainder of the first TAYLOR's approximation in the right neighborhood of $a$, and the remainder of the first TAYLOR's approximation in the left neighborhood of $b$, respectively.

Polynomials:

$$\mathbb{T}^f_{n,a^+,b^-}(x) = \begin{cases} 
T^f_{n-1,a^+}(x) + \frac{1}{(b-a)^n}R^f_{n,a^+}(b^-)(x-a)^n & : n \geq 1 \\
T^f_{n-1,b^-}(x) - \frac{1}{(a-b)^n}R^f_{n,b^-}(a^+)(x-b)^n & : n \geq 1 \\
f(b^-) & : n = 0,
\end{cases}$$

and

$$\mathbb{T}^f_{n,b^-,a^+}(x) = \begin{cases} 
T^f_{n-1,b^-}(x) + \frac{1}{(a-b)^n}R^f_{n,b^-}(a^+)(x-b)^n & : n \geq 1 \\
T^f_{n-1,a^+}(x) - \frac{1}{(b-a)^n}R^f_{n,a^+}(b^-)(x-a)^n & : n \geq 1 \\
f(a^+) & : n = 0,
\end{cases}$$

are called the second TAYLOR's approximation in the right neighborhood of $a$, and the second TAYLOR's approximation in the left neighborhood of $b$, respectively, $n \in \mathbb{N}_0$.

Theorem 2 in [7] provides an important result regarding TAYLOR’s approximations. We cite it below:

**Theorem 1.** Suppose that $f(x)$ is a real function on $(a, b)$, and that $n$ is a positive integer such that $f^{(k)}(a^+), f^{(k)}(b^-)$, for $k \in \{0, 1, 2, \ldots, n\}$, exist.

(i) Supposing that $(-1)^n f^{(n)}(x)$ is increasing on $(a, b)$, then for all $x \in (a, b)$ the following inequality holds:

$$\mathbb{T}^f_{n,b^-,a^+}(x) < f(x) < T^f_{n,b^-}(x). \quad (1)$$

Furthermore, if $(-1)^n f^{(n)}(x)$ is decreasing on $(a, b)$, then the reversed inequality of (1) holds.

(ii) Supposing that $f^{(n)}(x)$ is increasing on $(a, b)$, then for all $x \in (a, b)$ the following inequality also holds:

$$\mathbb{T}^f_{n,a^+,b^-}(x) > f(x) > T^f_{n,a^+}(x). \quad (2)$$

Furthermore, if $f^{(n)}(x)$ is decreasing on $(a, b)$, then the reversed inequality of (2) holds.
Let us name this theorem the Theorem on double-sided Taylor’s approximations. In papers [23], [24], [25], [26] and [27] Theorem 1 was denoted by Theorem WD. Let us note that the proof of Theorem 1 (Theorem 2 in [7]) was based on the L’Hospital’s rule for monotonicity. The same method was used in proofs of some theorems in [4], [5] and [6], that had been published earlier.

Here, we cite a theorem (Theorem 1.1. from [25]) that represents a natural extension of Theorem 1 over the set of real analytic functions.

**Theorem 2.** For the function $f : (a, b) \rightarrow \mathbb{R}$ let there exist the power series expansion:

$$f(x) = \sum_{k=0}^{\infty} c_k (x-a)^k,$$

for every $x \in (a, b)$, where $\{c_k\}_{k \in \mathbb{N}_0}$ is the sequence of coefficients such that there is only a finite number of negative coefficients, and their indices are all in the set $J = \{j_0, \ldots, j_\ell\}$.

Then, for the function

$$F(x) = f(x) - \sum_{i=0}^{\ell} c_{j_i} (x-a)^{j_i} = \sum_{k \in \mathbb{N}_0 \setminus J} c_k (x-a)^k,$$

and the sequence $\{C_k\}_{k \in \mathbb{N}_0}$ of the non-negative coefficients defined by

$$C_k = \begin{cases} c_k & c_k > 0, \\ 0 & c_k \leq 0; \end{cases}$$

holds that:

$$F(x) = \sum_{k=0}^{\infty} C_k (x-a)^k,$$

for every $x \in (a, b)$.

Also, $F^{(k)}(a^+) = k! C_k$, for $k \in \{0, 1, 2, \ldots, n\}$, and the following inequalities hold:

$$\sum_{k=0}^{n} C_k (x-a)^k < F(x) < \sum_{k=0}^{n-1} C_k (x-a)^k + \frac{1}{(b-a)^n} \left( F(b-) - \sum_{k=0}^{n-1} (b-a)^k C_k \right) (x-a)^n,$$

i.e.
\[ \sum_{k=0}^{n} C_k (x-a)^k + \sum_{i=0}^{\ell} c_j (x-a)^j \ell \sum_{i=0}^{n-1} C_k (b-a)^k - \sum_{i=0}^{\ell} c_j (b-a)^j, \]

for every \( x \in (a, b) \).

**Corollary 1.** Let there hold the conditions from the previous theorem. If

\[ n > \max\{j_0, \ldots, j_\ell\}, \]

then the following holds:

\[ \sum_{k=0}^{n} c_k (x-a)^k < f(x) < \sum_{k=0}^{n-1} c_k (x-a)^k + \frac{1}{(b-a)^n} \left( f(b) - \sum_{k=0}^{n-1} c_k (b-a)^k \right) (x-a)^n, \]

for every \( x \in (a, b) \).

### 2 Some new results on double-sided TAYLOR’s approximations

Consider a real function \( f : (a, b) \to \mathbb{R} \) such that there exist its first and second TAYLOR’s approximations on both sides, for some \( n \in \mathbb{N} \). Let us recall the remainders in LAGRANGE and the integral form, respectively, [17]:

\[ R_{f,a+}^n(x) = \frac{f^{(n)}(\xi_{a,x})}{n!} (x-a)^n, \]

for some \( \xi_{a,x} \in (a, x) \), and

\[ R_{f,a-}^n(x) = \frac{(x-a)^n}{(n-1)!} \int_0^1 f^{(n)}(a+(x-a)t)(1-t)^{n-1} dt. \]

#### 2.1 A new proof of Theorem 1

We consider the case when \( f^{(n)}(x) \) is a monotonically increasing function on \( (a, b) \) for some \( n \in \mathbb{N} \). Other cases from Theorem [1] are proved similarly.

From the LAGRANGE form of the remainder and monotonicity of \( f^{(n)}(x) \) on \( (a, b) \) we get:
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\[ \frac{f^{(n)}(a_+)}{n!} < \frac{f^{(n)}(\xi_{a,x})}{n!} = \frac{f(x) - T_{n-1}^{f,a+}(x)}{(x-a)^n} \implies T_{n}^{f,a+}(x) < f(x). \]

since \( \xi_{a,x} \in (a,x) \) for all \( x \in (a,b) \).

Using the integral form of the remainder we obtain the following inequality for all \( x \in (a,b) \):

\[ R_n^{f,a+}(x) = \frac{\frac{(x-a)^n}{(n-1)!}}{\frac{n!}{(b-a)^n}} \int_0^1 f^{(n)}(a + (x-a)t)(1-t)^{n-1} dt < \frac{\frac{(x-a)^n}{(n-1)!}}{\frac{n!}{(b-a)^n}} \int_0^1 f^{(n)}(a + (b-a)t)(1-t)^{n-1} dt \]

\[ = \frac{(x-a)^n}{(b-a)^n} R_n^{f,a+}(b-) \implies f(x) < T_{n+1}^{f,a+,b-}(x). \]

This completes the proof. \( \square \)

### 2.2 Monotonicity of double-sided Taylor’s approximations

**Proposition 1.** Consider a real function \( f : (a,b) \to \mathbb{R} \) such that there exist its first and second Taylor’s approximations on both sides, for some \( n \in \mathbb{N}_0 \). Then,

\[ \text{sgn} \left( T_n^{f,a+,b-}(x) - T_{n+1}^{f,a+,b-}(x) \right) = \text{sgn} \left( f(b-) - T_{n}^{f,a+}(b) \right), \quad (11) \]

for all \( x \in (a,b) \).

**Proof.** From the definitions of the first and second Taylor’s approximations we have:

\[ T_{n+1}^{f,a+,b-}(x) = T_n^{f,a+}(x) + \frac{f^{(n)}(a_+)b}{n!} \cdot \left( f(b-) - T_n^{f,a+}(b) \right) \]

\[ = T_{n-1}^{f,a+}(x) + \frac{f^{(n)}(a)(x-a)^n}{n!} + \frac{x-a}{b-a} \cdot \left( f(b-) - T_n^{f,a+}(b) \right) \]

\[ = T_{n-1}^{f,a+}(x) + \frac{f^{(n)}(a)(x-a)^n}{n!} + \frac{x-a}{b-a} \cdot \left( f(b-) - T_n^{f,a+}(b) \right) \]

\[ = T_n^{f,a+}(x) + \frac{f^{(n)}(a)(x-a)^n}{n!} + \frac{x-a}{b-a} \cdot \left( f(b-) - T_n^{f,a+}(b) \right) \]

Thus we have:

\[ T_n^{f,a+}(x) - T_{n+1}^{f,a+,b-}(x) = \frac{b-x}{b-a} \left( f(b-) - T_n^{f,a+}(b) \right), \quad (12) \]

and the equality (11) immediately follows. \( \square \)
Now, let us notice that if the real analytic function $f : (a, b) \rightarrow \mathbb{R}$ satisfies the condition $(\forall n \in \mathbb{N}_0) f^{(n)}(a^+) \geq 0$, then, from Proposition 1 directly follows:

$$(\forall n \in \mathbb{N}_0) (\forall x \in (a, b)) \quad T_n^{f, a^+ - b^-}(x) > T_{n+1}^{f, a^+ - b^-}(x).$$

**Theorem 3.** Consider a real function $f : (a, b) \rightarrow \mathbb{R}$ such that the derivatives $f^{(k)}(a^+)$, $k \in \{0, 1, 2, \ldots, n + 1\}$ exist, for some $n \in \mathbb{N}$.

Suppose that $f^{(n)}(x)$ and $f^{(n+1)}(x)$ are increasing on $(a, b)$, then for all $x \in (a, b)$ the following inequalities hold:

$$T_n^{f, a^+}(x) < T_{n+1}^{f, a^+}(x) < f(x) < T_{n+1}^{f, a^+ - b^-}(x) < T_n^{f, a^+ - b^-}(x), \quad (13)$$

for all $x \in (a, b)$. If $f^{(n)}(x)$ and $f^{(n+1)}(x)$ are decreasing on $(a, b)$, then for all $x \in (a, b)$ the reversed inequalities hold.

**Case of the real analytic functions**

In applications, of special interest are the real analytic functions.

**Theorem 4.** Consider the real analytic functions $f : (a, b) \rightarrow \mathbb{R}$:

$$f(x) = \sum_{k=0}^{\infty} c_k (x - a)^k, \quad (14)$$

where $c_k \in \mathbb{R}$ and $c_k \geq 0$ for all $k \in \mathbb{N}_0$. Then,

$$T_0^{f, a^+}(x) \leq \ldots \leq T_n^{f, a^+}(x) \leq T_{n+1}^{f, a^+}(x) \leq \ldots$$

$$\ldots \leq f(x) \leq \ldots \leq T_{n+1}^{f, a^+ - b^-}(x) \leq T_n^{f, a^+ - b^-}(x) \leq \ldots \leq T_0^{f, a^+ - b^-}(x), \quad (15)$$

for all $x \in (a, b)$. If $c_k \in \mathbb{R}$ and $c_k \leq 0$ for all $k \in \mathbb{N}_0$, then the reversed inequalities hold.

**3 An application of the Theorem on double-sided TAYLOR’s approximations**

In this section we discuss an implementation of the Theorem on double-sided TAYLOR’s approximations applied to the sequence of functions:

$$h_n(x) = \frac{\tan x - T_{2n-1}^{\tan, 0}(x)}{x^{2n} \tan x} : (0, \frac{\pi}{2}) \rightarrow \mathbb{R}, \quad (16)$$
for \( n \in \mathbb{N} \). This sequence of functions was considered in papers [3], [10]. In order to obtain estimates of functions \( h_n(x) \), we use the well-known series expansions:

\[
\tan x = \sum_{i=1}^{\infty} \frac{2^i(2^{2i} - 1)|B_{2i}|}{(2i)!} x^{2i-1},
\]

where \( |x| < \frac{\pi}{2} \) and \( B_k \) is the \( k \)-th BERNOULLI number. Then:

\[
T_{2n-1}^{\tan,0}(x) = \sum_{i=1}^{n} \frac{2^i(2^{2i} - 1)B_{2i}}{(2i)!} x^{2i-1},
\]

for \( x \in \left(0, \frac{\pi}{2}\right) \). The main results on the functions \( h_n(x) \), presented in the paper [10] (see also [3]), are cited below in the following two theorems.

**Theorem 5.** For \( x \in \left(0, \frac{\pi}{2}\right) \) and \( n \in \mathbb{N} \), we have:

\[
h_n(x) = \sum_{j=1}^{n} \frac{2^{2(n-j)+1}(2^{2(n-j)+1} - 1)|B_{2(n-j+1)}|}{(2(n-j+1))!} \sum_{k=j}^{\infty} \frac{2^{2k}|B_{2k}|}{(2k)!} x^{2(k-j)}. \tag{19}
\]

**Theorem 6.** For \( x \in \left(0, \frac{\pi}{2}\right) \) and \( n \in \mathbb{N} \), we have:

\[
\frac{2^{2(n+1)}(2^{2(n+1)} - 1)|B_{2(n+1)}|}{(2n+2)!} < h_n(x) < \left(\frac{2}{\pi}\right)^{2n} \tag{20}
\]

where the scalars \( \frac{2^{2(n+1)}(2^{2(n+1)} - 1)|B_{2(n+1)}|}{(2n+2)!} \) and \( \left(\frac{2}{\pi}\right)^{2n} \) in [20] are the best possible.

From Theorem 5 using the change of variables and some algebraic transformations, immediately follows the next theorem.

**Theorem 7.** For \( x \in \left(0, \frac{\pi}{2}\right) \) and \( n \in \mathbb{N} \), functions \( h_n(x) \) are real analytic functions and have the following TAYLOR series expansions:

\[
h_n(x) = \sum_{i=0}^{\infty} \sum_{j=1}^{n} \frac{2^{2(n+j+1)}(2^{2(n+j+1)} - 1)|B_{2(n-j+1)}||B_{2(i+j)}|}{(2(n-j+1))!(2(i+j))!} x^{2i}. \tag{21}
\]

Let us notice that the TAYLOR series expansions of the functions \( h_n(x) \) satisfy the conditions of Theorem 4.

Thus, we get the improvement of the results of Theorem 6.

**Theorem 8.** For \( x \in \left(0, \frac{\pi}{2}\right) \) and \( n \in \mathbb{N} \), we have
\[ T_0^{h_n(x),0+}(x) = \frac{2^{2(n+1)}(2^{2(n+1)} - 1)|B_{2(n+1)}|}{(2n+2)!} < T_0^{h_n(x),0+}(x) < \ldots < T_{2m}^{h_n(x),0+}(x) < T_{2m+2}^{h_n(x),0+}(x) < \ldots \]
\[ \ldots < T_{2m+2}^{h_n(x),0+}(x) < T_{2m+1}^{h_n(x),0-}(x) < \ldots < T_0^{h_n(x),0+}(x) < \ldots \]
\[ \ldots < T_{2m+2}^{h_n(x),0+}(x) < \frac{\pi}{2} - x < \frac{\pi}{2} - x \]
\[ < T_0^{h_n(x),0+}(x) = \left( \frac{2}{\pi} \right)^{2n}. \]  

4 More examples of double-sided Taylor’s approximations

In this section we give two examples of some analytic inequalities recently proved using the results of Theorem 1. Also, we illustrate the application of double-sided Taylor approximations and Theorem 4 in the generalizations and improvements of some analytic inequalities.

Example 1. In [13] the following improvement of Stečkin’s inequality, in the left neighborhood of \( b = \frac{\pi}{2} \), was proposed and proved:
\[ Q_1(x) = \frac{2}{\pi} - \frac{1}{2} \left( \frac{\pi}{2} - x \right) < \tan x - \frac{4x}{\pi(2\pi - x)} < \frac{2}{\pi} - \frac{1}{3} \left( \frac{\pi}{2} - x \right) = R_1(x), \]  
for \( x \in \left( 0, \frac{\pi}{2} \right) \). In [24] the inequality (23) was further generalized. The starting point was the following real function:
\[ g(t) = \cot t - \frac{1}{t} + \frac{2}{\pi} : \left( 0, \frac{\pi}{2} \right) \rightarrow \mathbb{R}, \]  
for which it is fulfilled
\[ g \left( \frac{\pi}{2} - x \right) = \tan x - \frac{4x}{\pi(2\pi - x)}, \]  
for \( x \in \left( 0, \frac{\pi}{2} \right) \). It has been shown that the function \( g(t) \) satisfies the conditions of Theorem 1. Namely, it has the following power series expansion
\[ g(t) = \frac{2}{\pi} - \sum_{k=1}^{\infty} \frac{2^{2k} |B_{2k}|}{(2k)!} 2^{k-1} \]  
which converges for \( t \in \left( 0, \frac{\pi}{2} \right) \), and it is true.
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\[ g(0+) = \lim_{t \to 0^+} g(t) = \frac{2}{\pi} \quad \text{and} \quad g\left(\frac{\pi}{2}\right) = \lim_{t \to \pi/2^-} g(t) = 0. \]

The function \( g(t) \) also satisfies the conditions of Theorem [4]. Based on this, the following result was proposed in [24] (Theorem 3) for the function \( f(x) = g\left(\frac{\pi}{2} - x\right) \):

**Theorem 9.** For every \( x \in \left(0, \frac{\pi}{2}\right) \) and \( m \in \mathbb{N}, m \geq 2 \), the following inequalities hold:

\[
T_{2m-1}^{g, 0+, \pi/2-}\left(\frac{\pi}{2} - x\right) < f(x) < T_{2m-1}^{g, 0, 0}\left(\frac{\pi}{2} - x\right),
\]

where

\[
T_{2m-1}^{g, 0+, \pi/2-}\left(\frac{\pi}{2} - x\right) = \frac{2}{\pi} - \sum_{k=1}^{m-1} \frac{2^{2k} |B_{2k}|}{(2k)!} \left(\frac{\pi}{2} - x\right)^{2k-1} + \sum_{k=1}^{m-1} \frac{2^{2k} |B_{2k}|}{(2k)!} \left(\frac{2}{\pi}\right)^{2(m-k-1)} \left(\frac{\pi}{2} - x\right)^{2m-2k-1}
\]

and

\[
T_{2m-1}^{g, 0, 0}\left(\frac{\pi}{2} - x\right) = \frac{2}{\pi} - \sum_{k=1}^{m} \frac{2^{2k} |B_{2k}|}{(2k)!} \left(\frac{2}{\pi}\right)^{2k-1} \left(\frac{\pi}{2} - x\right)^{2k-1}.
\]

It is easy to check that the function \( g(t) \) also satisfies the conditions of Theorem [4]. Therefore, for the function \( f(x) = g\left(\frac{\pi}{2} - x\right) \) the following assertion directly follows:

**Theorem 10.** For every \( x \in \left(0, \frac{\pi}{2}\right) \) and \( m \in \mathbb{N}, m \geq 2 \), the following inequalities hold:

\[
T_{1}^{g, 0+}\left(\frac{\pi}{2} - x\right) \leq \ldots \leq T_{2m-1}^{g, 0+}\left(\frac{\pi}{2} - x\right) \leq T_{2m+1}^{g, 0+}\left(\frac{\pi}{2} - x\right) \leq \ldots \leq f(x) \leq \ldots
\]

\[
\ldots \leq T_{2m+1}^{g, 0+, \pi/2-}\left(\frac{\pi}{2} - x\right) \leq T_{2m-1}^{g, 0+, \pi/2-}\left(\frac{\pi}{2} - x\right) \leq \ldots \leq T_{1}^{g, 0+, \pi/2-}\left(\frac{\pi}{2} - x\right).
\]

Finally, from the previous two theorems an improvement of the inequality [23] directly follows. For example, if \( m = 1 \), we have:

\[
T_{1}^{g, 0+, \pi/2-}\left(\frac{\pi}{2} - x\right) = \frac{2}{\pi} - \frac{4}{\pi^2} \left(\frac{\pi}{2} - x\right) \leq \tan x - \frac{4x}{\pi(2\pi - x)} \leq \frac{2}{\pi} - \frac{1}{3} \left(\frac{\pi}{2} - x\right) = T_{1}^{g, 0}\left(\frac{\pi}{2} - x\right),
\]

which further implies the following:
\[ Q_1(x) < T_{x}^{r,0+,\pi/2-}\left(\frac{\pi}{2} - x\right) \leq \tan x - \frac{4x}{\pi(2\pi - x)} \leq T_{x}^{r,0+}\left(\frac{\pi}{2} - x\right) = R_1(x), \]
for \( x \in \left(0, \frac{\pi}{2}\right) \).

Note that the same approach (based on Theorem 1 and Theorem 4) enables generalizations of the inequalities from [2] connected with the function
\[ f(x) = \left(\frac{\pi}{2} - 4x^2\right)\tan x: \left(0, \frac{\pi}{2}\right) \rightarrow R. \]

**Example 2.** In [9] (Theorem 5) the following inequality was proved:
\[ 2 + \frac{2}{45}x^4 < \left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x} \quad \text{for} \quad 0 < x < \frac{\pi}{2}. \]  
(31)

In order to refine the previous inequality, the following real function was considered in [23]:
\[ f(x) = \left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x} \quad \text{for} \quad 0 < x < \frac{\pi}{2}. \]

It has been shown that the above function satisfies the conditions of Theorem 1.

Namely, it has the following power series expansion
\[ f(x) = 2 + \sum_{k=2}^{\infty} \frac{|B_{2k}|(2k - 2)4^k}{(2k)!} x^{2k}, \]  
(32)
which converges for \( x \in \left(0, \frac{\pi}{2}\right) \), and it is true
\[ f(0+) = \lim_{x \to 0^+} f(x) = 2 \quad \text{and} \quad f\left(\frac{\pi}{2}^{-}\right) = \lim_{x \to \pi/2^-} f(x) = \frac{\pi^2}{4}. \]

Based on this, the following result was proposed and proved in [23] (Theorem 5):

**Theorem 11.** For every \( x \in \left(0, \frac{\pi}{2}\right) \) and \( m \in \mathbb{N}, m \geq 2 \), the following inequalities hold:
\[ T_{2m}^{f,0}(x) < f(x) < T_{2m}^{r,0+,\pi/2-}(x), \]  
(33)
where
\[ T_{2m}^{f,0}(x) = 2 + \sum_{k=2}^{m} \frac{|B_{2k}|(2k - 2)4^k}{(2k)!} x^{2k} \]  
(34)
and
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\[ T^{f,0+,:\pi/2-}_{2m}(x) = 2 + \sum_{k=2}^{m-1} \frac{|B_{2k}|(2k-2)4^k}{(2k)!} x^{2k} + \left( \frac{\pi}{4} \right)^{2m} - 2 - \sum_{k=2}^{m-1} \frac{|B_{2k}|(2k-2)4^k}{(2k)!} \left( \frac{\pi}{2} \right)^{2k} \right) x^{2m}. \]  

In [23] the polynomials \( T^{f,0+}_{m}(x) \) and \( T^{f,0+,:\pi/2-}_{m}(x) \) are calculated and the concrete inequalities

\[ T^{f,0+}_{m}(x) < f(x) < T^{f,0+,:\pi/2-}_{m}(x) \]

are given for \( x \in \left(0, \frac{\pi}{2}\right) \) and for \( m = 2, 3, 4, 5 \).

It is easy to check that the function \( f(x) \) also satisfies the conditions of Theorem 4 and hence the following generalizations of the inequality (33) i.e. of the inequality (31) are true:

**Theorem 12.** For every \( x \in \left(0, \frac{\pi}{2}\right) \) and \( m \in \mathbb{N}, m \geq 2 \), the following inequalities hold:

\[ T^{f,0+}_{0}(x) \leq \ldots \leq T^{f,0+}_{2m}(x) \leq T^{f,0+}_{2m+2}(x) \leq \ldots \]

\[ \ldots \leq f(x) \leq \ldots \]

\[ \ldots \leq T^{f,0+,:\pi/2-}_{2m+2}(x) \leq T^{f,0+,:\pi/2-}_{2m}(x) \leq \ldots \leq T^{f,0+,:\pi/2-}_{0}(x) \]  

The same approach, based on Theorem 1 and Theorem 4, provides generalizations of the inequalities from [23] related to the function

\[ f(x) = 3 \frac{x}{\sin x} + \cos x : \left(0, \frac{\pi}{2}\right) \rightarrow \mathbb{R}. \]

5 Conclusion

Even though Taylor’s approximations represent a few centuries old topic, they are still present in research nowadays in many areas of science and engineering. Let us note that many results regarding Taylor’s approximations are presented in well-known monographs [2] and [11]. Historically speaking the second Taylor’s approximation was mentioned in 1851 in the proof of the Taylor’s formula with the Lagrange remainder in the paper [1] by H. Cox, see also [17].

Let us mention that in papers [8], [18], [22], [25], [26] and [27] double-sided Taylor’s approximations are used to obtain corresponding inequalities. Results of these papers can be further organized and made more precise using Theorem 4 so we get the order among the functions occurring within these inequalities. Similar to double-sided Taylor’s approximations, in papers [12], [14], [15], [16], [19], [20], [21], [28], [29] i [30] the finite expansions are used in the proofs of some mixed-trigonometric polynomial inequalities, as well as in some inequalities which can be reduced to mixed-trigonometric polynomial inequalities.
Currently, we are working on developing a computer system for automatic proving of some classes of analytic inequalities based on the results in the mentioned papers.

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