Abstract. We give an analytic version of the injectivity theorem by using multiplier ideal sheaves, and prove some extension theorems for the adjoint bundle of dlt pairs. Moreover, by combining techniques of the minimal model program, we obtain some results for semi-ampleness related to the abundance conjecture in birational geometry and the Strominger-Yau-Zaslow conjecture for hyperKähler manifolds.

1. Introduction

Throughout this paper, we work over \( \mathbb{C} \), the complex number field, and freely use the standard notation in [BCHM], [Dem], [KaMM], and [KoM]. Further we interchangeably use the words “Cartier divisors”, “line bundles”, and “invertible sheaves”.

The following conjecture is one of the most important conjectures in the classification theory of algebraic varieties:

Conjecture 1.1 (Generalized abundance conjecture). Let \( X \) be a normal projective variety and \( \Delta \) be an effective \( \mathbb{Q} \)-divisor such that \( (X, \Delta) \) is a klt pair. Then \( \kappa(X, K_X + \Delta) = \kappa_\sigma(X, K_X + \Delta) \). In particular, if \( K_X + \Delta \) is nef, then it is semi-ample. (For the definition of \( \kappa(\cdot) \) and \( \kappa_\sigma(\cdot) \), see [N].)

Toward the abundance conjecture, we need to solve the non-vanishing conjecture and the extension conjecture, see [DHP], [F1, Introduction], [FG, Section5]. One of the purposes of this paper is to study the following extension conjecture for the adjoint bundle of dlt pairs formulated in [DHP, Conjecture 1.3]:

Conjecture 1.2 (Extension conjecture for dlt pairs). Let \( X \) be a normal projective variety and \( S + B \) be an effective \( \mathbb{Q} \)-divisor with the following assumptions:

- \( (X, S + B) \) is a dlt pair.
- \( [S + B] = S \).
- \( K_X + S + B \) is nef.
- \( K_X + S + B \) is \( \mathbb{Q} \)-linearly equivalent to an effective divisor \( D \) with \( S \subseteq \text{Supp} D \subseteq \text{Supp} (S + B) \).

Then the restriction map

\[
H^0(X, \mathcal{O}_X(m(K_X + S + B))) \rightarrow H^0(S, \mathcal{O}_S(m(K_X + S + B)))
\]

is surjective for all sufficiently divisible integers \( m \geq 2 \).

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When $S$ is a normal irreducible variety (namely $(X, S+B)$ is a plt pair), Demailly–Hacon–Păun proved the above conjecture in [DHP] by using technical methods based on a version of the Ohsawa-Takegoshi extension theorem. This result can be seen as an extension theorem for plt pairs.

In this paper, we study the extension conjecture for dlt pairs by giving an analytic version of the injectivity theorem instead of the Ohsawa-Takegoshi extension theorem. Thanks to the injectivity theorem, we can obtain some extension theorems for not only plt pairs but also dlt pairs. This is one of our advantages. Our injectivity theorem is as follows:

**Theorem 1.3** (A version of the injectivity theorem = Theorem 3.1). Let $(F, h_F)$ and $(L, h_L)$ be (singular) hermitian line bundles with semi-positive curvature on a compact Kähler manifold $X$. Assume that there exists an effective $\mathbb{R}$-divisor $\Delta$ with 

$$h_F = h_L^a \cdot h_\Delta,$$

where $a$ is a positive real number and $h_\Delta$ is the singular metric defined by $\Delta$.

Then for a (non-zero) section $s$ of $L$ satisfying $\sup_X |s|_{h_L} < \infty$, the multiplication map 

$$\Phi_s : H^q(X, K_X \otimes F \otimes I(h_F)) \to H^q(X, K_X \otimes F \otimes L \otimes I(h_F h_L))$$

is (well-defined and) injective for any $q$. Here $I(h)$ denotes the multiplier ideal sheaf associated to a singular metric $h$.

The injectivity theorem has been studied by several authors, for example, Tankeev [T], Kollár [Ko1], Enoki [E], Ohsawa [O], Enault–Viehweg [EV-book], Fujino [F2], [F5], [F4], [F-book], and Ambro [A1], [A2]. Recently the second author gave an injectivity theorem with multiplier ideal sheaves, which corresponds to the case of $\Delta = 0$ (see [Mat2]). Our proof is a generalization of the methods of [E], [O], [F5], and [Mat2].

By applying this injectivity theorem to an extension problem, we obtain the following extension theorem. Even if $K_X + \Delta$ is semi-positive (namely admits a smooth metric with semi-positive curvature), it seems to be not able to prove Conjecture 1.2 for dlt pairs by the Ohsawa-Takegoshi extension theorem in at least current situation, and thus we need our injectivity theorem (Theorem 1.3).

**Theorem 1.4** (= Theorem 4.1). Let $X$ be a compact Kähler manifold and $S + B$ be an effective $\mathbb{Q}$-divisor with the following assumptions:

- $S + B$ is a simple normal crossing divisor with $0 \leq S + B \leq 1$ and $\lfloor S + B \rfloor = S$.
- $K_X + S + B$ is $\mathbb{Q}$-linearly equivalent to an effective divisor $D$ with $S \subseteq \text{Supp} D$.
- $K_X + S + B$ admits a (singular) metric $h$ with semi-positive curvature.

Then for an integer $m \geq 2$ with Cartier divisor $m(K_X+S+B)$ and any section $u \in H^0(S, O_S(m(K_X+S+B)))$ that comes from $H^0(S, O_S(m(K_X+S+B)) \otimes I(h^{m-1}h_B))$, the section $u$ can be extended to a section in $H^0(X, O_X(m(K_X+S+B)))$.

In particular, we can prove Conjecture 1.2 under the assumption that $K_X + \Delta$ admits a (singular) metric whose curvature is semi-positive and Lelong number is identically zero (see Corollary 4.5). This assumption is stronger than the assumption that $K_X + \Delta$
is nef, but weaker than the assumption that $K_X + \Delta$ is semi-positive. Remark that Verbitsky proved the non-vanishing conjecture on hyperKähler manifolds under the same assumption (see [V]).

As compared with Conjecture 1.2, one of our advantages is removing the condition $\text{Supp} \, D \subseteq \text{Supp}(S + B)$ (this version of the extension conjecture is [FG, Conjecture 5.8]). Thanks to removing the condition $\text{Supp} \, D \subseteq \text{Supp}(S + B)$ in Conjecture 1.2, we can run the MMP preserving the good condition for metrics (cf. [DHP, Section 8] and [FG, Theorem 5.9]). By applying the above theorem and techniques of the MMP, we obtain the following theorem related to the abundance conjecture:

**Theorem 1.5** (=Theorem 5.1). Assume that Conjecture 1.1 holds in dimension $n - 1$. Let $X$ be an $n$-dimensional normal projective variety and $\Delta$ be an $\mathbb{Q}$-divisor with the following assumptions:

- $(X, \Delta)$ is a klt pair.
- There exists a projective birational morphism $\varphi : Y \to X$ such that $Y$ is smooth and $\varphi^*(\mathcal{O}_X(m(K_X + \Delta)))$ admits a (singular) metric whose curvature is semi-positive and Lelong number is identically zero. Here $m$ is a positive integer such that $m(K_X + \Delta)$ is Cartier.

If $\kappa(K_X + \Delta) \geq 0$, then $K_X + \Delta$ is semi-ample.

Finally, by combining Verbitsky’s non-vanishing theorem ([V, Theorem 4.1]) on hyperKähler manifolds (holomorphic symplectic manifolds), we obtain a result for semi-ampleness on four dimensional projective hyperKähler manifolds, which is closely related to the Strominger-Yau-Zaslow conjecture for hyperKähler manifolds. See [AC] for recent related topics and [COP] for non-algebraic cases.

**Theorem 1.6** (=Corollary 5.5). Let $X$ be a 4-dimensional projective hyperKähler manifold and $L$ be a line bundle admitting a (singular) metric whose curvature is semi-positive and Lelong number is identically zero. Then $L$ is semi-ample.

We summarize the contents of this paper. In Section 2, we collect the basic definitions and facts needed later. In Section 3, we prove the injectivity theorem (Theorem 1.3). In Section 4, we give direct applications of the injectivity theorem to the extension theorem (Theorem 1.4 and Corollary 4.5). Section 3 is devoted to results for semi-amplicity related to the abundance conjecture (Theorem 1.5 and Theorem 1.6).

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2. Preliminaries

In this section, we first recall the definitions and basic properties of singular metrics and multiplier ideal sheaves. For more details, refer to [Dem], [Dem-book].

We denote by $X$ a compact complex manifold and by $F$ a line bundle on $X$ unless otherwise mentioned. Further we fix a smooth (hermitian) metric $g$ on $F$.

**Definition 2.1** (Singular metrics and their curvature currents).

1. For an $L^1$-function $\varphi$ on $X$, the metric $h$ defined by
   
   $h := ge^{-\varphi}$

   is called a singular metric on $F$. Further $\varphi$ is called the weight of $h$ with respect to the fixed smooth metric $g$.

2. A (singular) metric $h$ on $F$ is said to have algebraic (resp. analytic) singularities, if there exists an ideal sheaf $I \subseteq O_X$ such that a weight $\varphi$ of $h$ can be locally written as
   
   $\varphi = \frac{c}{2} \log \left( |f_1|^2 + |f_2|^2 + \cdots + |f_k|^2 \right) + v$,

   where $c$ is a positive rational number (resp. real number), $f_1, \ldots, f_k$ are local generators of $I$, and $v$ is a smooth function.

3. The curvature current $\sqrt{-1}\Theta_h(F)$ associated to $h$ is defined by
   
   $\sqrt{-1}\Theta_h(F) = \sqrt{-1}\Theta_g(F) + dd^c \varphi$,

   where $\sqrt{-1}\Theta_g(F)$ is the Chern curvature of $g$.

In this paper, we simply abbreviate the singular metric (resp. the curvature current) to the metric (resp. the curvature). The Levi form $dd^c \varphi$ is taken in the sense of distributions, and thus the curvature is a $(1,1)$-current but not always a smooth $(1,1)$-form. The curvature $\sqrt{-1}\Theta_h(F)$ of $h$ is said to be semi-positive if $\sqrt{-1}\Theta_h(F) \geq 0$ in the sense of currents.

**Definition 2.2** (Multiplier ideal sheaves). Let $h$ be a metric on $F$ such that $\sqrt{-1}\Theta_h(F) \geq \gamma$ for some smooth $(1,1)$-form $\gamma$ on $X$. The ideal sheaf $I(h)$ defined to be

$I(h)(B) := I(\varphi)(B) := \{ f \in O_X(B) \mid |f| e^{-\varphi} \in L^2_{\text{loc}}(B) \}$

for every open set $B \subseteq X$, is called the multiplier ideal sheaf associated to $h$.

It is known that multiplier ideal sheaves are coherent sheaves. The following is a typical example of singular metrics that often appear in algebraic geometry.

**Example 2.3.** For given holomorphic sections $\{s_i\}_{i=1}^N$ of the $m$-th tensor powers $F^m$ of $F$, the (singular) metric $ge^{-\varphi}$ can be defined by

$\varphi := \frac{1}{2m} \sum_{i=1}^N \log |s_i|_{s^m}^2.$
Then the curvature of this metric is semi-positive, and further the multiplier ideal sheaf can be algebraically computed (see [Dem]). In particular, for an effective \( \mathbb{R} \)-divisor \( D \) on \( X \), the metric \( h_D \) on \( \mathcal{O}_X(D) \) can be constructed by the natural section of \( D \). We can easily check \( I(h_D) = \mathcal{O}_X([D]) \) if \( D \) is a simple normal crossing divisor.

We recall the definition of the Lelong number of singular metrics and Skoda’s lemma which gives a relation between the multiplier ideal sheaf and the Lelong number of singular metrics.

**Definition 2.4** (Lelong numbers). Let \( \varphi \) be a (quasi-)psh function on an open set \( B \) in \( \mathbb{C}^n \). The Lelong number \( \nu(\varphi, x) \) of \( \varphi \) at \( x \in B \) is defined by

\[
\nu(\varphi, x) = \liminf_{z \to x} \frac{\varphi(z)}{\log |z-x|}.
\]

For a singular metric \( h \) on \( F \) such that \( \sqrt{-1} \Theta_h(F) \geq \gamma \) for some smooth \( (1,1) \)-form, we define the Lelong number \( \nu(h, x) \) of \( h \) at \( x \in X \) by \( \nu(h, x) := \nu(\varphi, x) \), where \( \varphi \) is a weight of \( h \).

**Theorem 2.5** (Skoda’s lemma). Let \( \varphi \) be a (quasi-)psh function on an open set \( B \) in \( \mathbb{C}^n \).

- If \( \nu(\varphi, x) < 1 \), then we have \( I(\varphi)_x = \mathcal{O}_{B,x} \).
- If \( \nu(\varphi, x) \geq n + s \) for some integer \( s \geq 0 \), then we have \( I(\varphi)_x \subseteq \mathcal{M}_{B,x}^{s+1} \), where \( \mathcal{M}_{B,x} \) is the maximal ideal of \( \mathcal{O}_{B,x} \).

Next we give the definition of singularities of pairs.

**Definition 2.6** (Singularities of pairs). Let \( X \) be a normal variety and \( \Delta \) be an effective \( \mathbb{Q} \)-divisor on \( X \) such that \( K_X + \Delta \) is \( \mathbb{Q} \)-Cartier. Let \( \varphi : Y \to X \) be a log resolution of \( (X, \Delta) \). We set

\[
K_Y = \varphi^*(K_X + \Delta) + \sum a_i E_i,
\]

where \( E_i \) is a prime divisor on \( Y \) for every \( i \). The pair \((X, \Delta)\) is called

- **kawamata log terminal** (klt, for short) if \( a_i > -1 \) for all \( i \),
- **log canonical** (lc, for short) if \( a_i \geq -1 \) for all \( i \).

**Definition 2.7** (Semi-log canonical, [F1 Definition 1.1]). Let \( X \) be a reduced \( S_2 \)-scheme. We assume that it is pure \( n \)-dimensional and is normal crossing in codimension 1. Let \( \Delta \) be an effective \( \mathbb{Q} \)-Weil divisor on \( X \) such that \( K_X + \Delta \) is \( \mathbb{Q} \)-Cartier.

Let \( X = \bigcup X_i \) be the decomposition into irreducible components, and \( \nu : X' := \bigsqcup X'_i \to X = \bigsqcup X_i \) the normalization, where the normalization \( \nu : X' := \bigsqcup X'_i \to X = \bigsqcup X_i \) means that \( \nu|_{X_i} : X'_i \to X_i \) is the usual normalization for any \( i \). We call \( X \) a normal scheme if \( \nu \) is isomorphic. Define the \( \mathbb{Q} \)-divisor \( \Theta \) on \( X' \) by \( K_X + \Theta = \nu^*(K_X + \Delta) \) and set \( \Theta_i = \Theta|_{X'_i} \).

We say that \((X, \Delta)\) is semi-log canonical (for short, slc) if \((X'_i, \Theta_i)\) is an lc pair for every \( i \).
3. A Version of the Injectivity Theorem

The purpose of this section is to give an analytic version of the injectivity theorem by using multiplier ideal sheaves. This theorem is a generalization of [Mat2, Theorem 1.3] and it is applied in order to obtain the extension theorem (Theorem 1.4). See [F5] and [Mat2] for relations of various injectivity theorems and vanishing theorems.

**Theorem 3.1** (A version of the injectivity theorem). Let \((F, h_F)\) and \((L, h_L)\) be (singular) hermitian line bundles with semi-positive curvature on a compact Kähler manifold \(X\). Assume that there exists an effective \(\mathbb{R}\)-divisor \(\Delta\) with

\[
h_F = h_L^a \cdot h_\Delta,
\]

where \(a\) is a positive real number and \(h_\Delta\) is the singular metric defined by \(\Delta\).

Then for a (non-zero) section \(s\) of \(L\) satisfying \(\sup_X |s|_{h_L} < \infty\), the multiplication map

\[
\Phi_s : H^q(X, K_X \otimes F \otimes I(h_F)) \overset{\cong}{\to} H^q(X, K_X \otimes F \otimes L \otimes I(h_F h_L))
\]

is (well-defined and) injective for any \(q\). Here \(I(h)\) denotes the multiplier ideal sheaf associated to a singular metric \(h\).

**Remark 3.2.**

1. The multiplication map is well-defined thanks to the assumption of \(\sup_X |s|_{h_L} < \infty\). When \(h_L\) is a metric with minimal singularities on \(L\), this assumption is always satisfied for any section \(s\) of \(L\) (see [Dem] for the definition of metrics with minimal singularities).

2. The case of \(\Delta = 0\) corresponds to the main result in [Mat2]. To obtain the extension theorem (Theorem 1.4), it is important to consider the case of \(\Delta \neq 0\).

3. If \(h_L\) and \(h_F\) are smooth on a Zariski open set, the same conclusion holds under the weaker assumption of \(\sqrt{-1} \Theta_{h_L}(F) \geq a \sqrt{-1} \Theta_{h_L}(L)\) (see [F5], [Mat1]).

**Proof.** The proof is a generalization of the proof of the main result in [Mat2] which corresponds to the case of \(\Delta = 0\). First of all, we recall Enoki’s techniques to generalize Kollar’s injectivity theorem, which give a proof of the special case where \(h_L\) is smooth and \(\Delta = 0\). In this case, the cohomology group \(H^q(X, K_X \otimes F)\) is isomorphic to the space of the harmonic forms with respect to \(h_F\)

\[
\mathcal{H}^{n,q}(F)_{h_F} := \{u \mid u \text{ is a smooth } F\text{-valued } (n,q)\text{-form on } X \text{ such that } \overline{\partial} u = D_{h_F}^{\ast} u = 0\},
\]

where \(D_{h_F}^{\ast}\) is the adjoint operator of the \(\overline{\partial}\)-operator. For an arbitrary harmonic form \(u \in \mathcal{H}^{n,q}(F)_{h_F}\), we can conclude that \(D_{h_L}^{\ast} s u = 0\) thanks to semi-positivity of the curvature and \(h_F = h_L^a\). This step strongly depends on semi-positivity of the curvature. Then the multiplication map \(\Phi_s\) induces the map from \(\mathcal{H}^{n,q}(F)_{h_F}\) to \(\mathcal{H}^{n,q}(F \otimes L)_{h_F h_L}\), and thus the injectivity is obvious.

In our situation, we must consider singular metrics with transcendental (non-algebraic) singularities. It is quite difficult to directly handle transcendental singularities, and thus in Step 1, we approximate a given metric \(h_F\) by metrics \(\{h_F\}_{\varepsilon > 0}\) that are smooth on a Zariski open set. Then we represent a given cohomology class in \(H^q(X, K_X \otimes F \otimes I(h_F))\) by the associated harmonic form \(u_\varepsilon\) with respect to \(h_\varepsilon\) on the Zariski open set. We want to show
that $su_\varepsilon$ is also harmonic by using the same method as Enoki’s proof. However, the same argument as Enoki’s proof fails since the curvature of $h_\varepsilon$ is not semi-positive. For this reason, in Step 2, we investigate the asymptotic behavior of the harmonic forms $u_\varepsilon$ with respect to a family of the regularized metrics $\{h_\varepsilon\}_{\varepsilon>0}$. Then we show that the $L^2$-norm $\|D_{h_\varepsilon h_{\varepsilon L}}^\ast su_\varepsilon\|$ converges to zero as letting $\varepsilon$ go to zero, where $h_{\varepsilon L}$ is a suitable approximation of $h_L$. Moreover, in Step 3, we construct solutions $\gamma_\varepsilon$ of the $\bar{\partial}$-equation $\bar{\partial}\gamma_\varepsilon = su_\varepsilon$ such that the $L^2$-norm $\|\gamma_\varepsilon\|$ is uniformly bounded, by applying the Čech complex with the topology induced by the local $L^2$-norms. In Step 4, we see

$$\|su_\varepsilon\|^2 = \langle su_\varepsilon, \bar{\partial}\gamma_\varepsilon \rangle \leq \|D_{h_\varepsilon h_{\varepsilon L}}^\ast su_\varepsilon\|\|\gamma_\varepsilon\| \to 0 \quad \text{as } \varepsilon \to 0.$$  

From these observations, we can conclude that $u_\varepsilon$ converges to zero in a suitable sense, which completes the proof.

**Step 1 (The equisingular approximation of $h_F$)**

Throughout the proof, we fix a Kähler form $\omega$ on $X$. For the proof, we want to apply the theory of harmonic integrals, but the metric $h_F$ may not be smooth. For this reason, we approximate $h_F$ by metrics $\{h_\varepsilon\}_{\varepsilon>0}$ that are smooth on a Zariski open set. By [DPS, Theorem 2.3], we can obtain metrics $\{h_\varepsilon\}_{\varepsilon>0}$ on $F$ with the following properties:

- (a) $h_\varepsilon$ is smooth on $X \setminus Z_\varepsilon$, where $Z_\varepsilon$ is a subvariety on $X$.
- (b) $h_\varepsilon_2 \leq h_\varepsilon_1 \leq h_F$ holds for any $0 < \varepsilon_1 < \varepsilon_2$.
- (c) $I(h_F) = I(h_\varepsilon)$.
- (d) $\sqrt{-1}\Theta(h_\varepsilon(F) \geq -\varepsilon\omega$.

Since the point-wise norm $|s|_{h_\varepsilon}$ is bounded on $X$ and $h_F = h_\varepsilon h_\Lambda$, the set $\{x \in X | \nu(h_F, x) > 0\}$ is contained in the subvariety $Z$ defined by $Z := s^{-1}(0) \cup \text{Supp } \Delta$. Therefore we may assume a stronger property than property (a) (for example see [Mat2, Theorem 2.3]), namely

- (e) $h_\varepsilon$ is smooth on $Y := X \setminus Z$, where $Z = s^{-1}(0) \cup \text{Supp } \Delta$.

Now we construct a “complete” Kähler form on $Y$ with suitable potential function. Take a quasi-psh function $\psi$ on $X$ such that $\psi$ has a logarithmic pole along $Z$ and $\psi$ is smooth on $Y$. Since quasi-psh functions are upper semi-continuous, the function $\psi$ is bounded above, and thus we may assume $\psi \leq -\varepsilon$. Then we define the $(1, 1)$-form $\tilde{\omega}$ on $Y$ by

$$\tilde{\omega} := \ell\omega + dd^c\Psi,$$

where $\ell$ is a positive number and $\Psi := \frac{1}{\log(-\psi)}$. We can show that the $(1, 1)$-form $\tilde{\omega}$ satisfies the following properties for a sufficiently large $\ell > 0$:

- (A) $\tilde{\omega}$ is a complete Kähler form on $Y$.
- (B) $\Psi$ is bounded on $X$.
- (C) $\tilde{\omega} \geq \omega$.

Indeed, properties (B), (C) are obvious by the definition of $\Psi$ and $\tilde{\omega}$, and property (A) follows from straightforward computations (see [F5], Lemma 3.1) for the precise proof of property (A)).
In the proof, we mainly consider harmonic forms on \( Y \) with respect to \( h_\varepsilon \) and \( \bar{\omega} \). Let \( L^n(\alpha; Y, F)_{h_\varepsilon, \bar{\omega}} \) be the space of the \( L^2 \)-integrable \( F \)-valued \((n, q)\)-forms \( \alpha \) with respect to the inner product \( \| \cdot \|_{h_\varepsilon, \bar{\omega}} \) defined by

\[
\| \alpha \|^2_{h_\varepsilon, \bar{\omega}} := \int_Y |\alpha|^2_{h_\varepsilon, \bar{\omega}} \bar{\omega}^n.
\]

Then we can obtain the following orthogonal decomposition:

\[
L^n(\alpha; Y, F)_{h_\varepsilon, \bar{\omega}} = \text{Im} \bar{\partial} \oplus \mathcal{H}^n(\alpha; F)_{h_\varepsilon, \bar{\omega}} \oplus \text{Im} D''_{h_\varepsilon}.
\]

Here the operator \( D'_{h_\varepsilon} \) (resp. \( D''_{h_\varepsilon} \)) denotes the closed extension of the formal adjoint of the \((1, 0)\)-part \( D'_{h_\varepsilon} \) (resp. \((0, 1)\)-part \( D''_{h_\varepsilon} = \bar{\partial} \)) of the Chern connection \( D_{h_\varepsilon} = D'_{h_\varepsilon} + D''_{h_\varepsilon} \). Note that they coincide with the Hilbert space adjoints since \( \bar{\omega} \) is complete. Further \( \mathcal{H}^n(\alpha; F)_{h_\varepsilon, \bar{\omega}} \) denotes the space of the harmonic forms with respect to \( h_\varepsilon \) and \( \bar{\omega} \), namely

\[
\mathcal{H}^n(\alpha; F)_{h_\varepsilon, \bar{\omega}} := \{ \alpha \mid \alpha \text{ is an } F \text{-valued } (n, q)\text{-form with } \bar{\partial} \alpha = D''_{h_\varepsilon} \alpha = 0 \}.
\]

Harmonic forms in \( \mathcal{H}^n(\alpha; F)_{h_\varepsilon, \bar{\omega}} \) are smooth by the regularization theorem for elliptic operators. These results are known to specialists. The precise proof of them can be found in [FS] Claim 1.

For every \((n, q)\)-form \( \beta \), we have \( |\beta|^2_{\bar{\omega}} \bar{\omega}^n \leq |\beta|^2_{\omega} \omega^n \) since the inequality \( \bar{\omega} \geq \omega \) holds by property (C). From this inequality and property (b) of \( h_\varepsilon \), we obtain

\[
(1) \quad \| \alpha \|_{h_\varepsilon, \bar{\omega}} \leq \| \alpha \|_{h_\varepsilon, \omega} \leq \| \alpha \|_{h_\varepsilon, \bar{\omega}}
\]

for an \( F \)-valued \((n, q)\)-form \( \alpha \), which plays a crucial role in the proof.

Take an arbitrary cohomology class \([u]\) \( \in H^n(X, K_X \otimes F \otimes I(h_\varepsilon)) \) represented by an \( F \)-valued \((n, q)\)-form \( u \) with \( \| u \|_{h_\varepsilon, \bar{\omega}} < \infty \). In order to prove that the multiplication map \( \Phi \) is injective, we assume that the cohomology class of \( su \) is zero in \( H^n(X, K_X \otimes F \otimes L \otimes I(h_\varepsilon h_\bar{\varepsilon})) \). Our final goal is to show that the cohomology class of \( u \) is actually zero under this assumption.

By inequality (1), we have \( \| u \|_{h_\varepsilon, \bar{\omega}} < \infty \) for any \( \varepsilon > 0 \). Therefore by the above orthogonal decomposition, there exist \( u_\varepsilon \in \mathcal{H}^n(\alpha; F)_{h_\varepsilon, \bar{\omega}} \) and \( v_\varepsilon \in L^n(\alpha; Y, F)_{h_\varepsilon, \bar{\omega}} \) such that

\[
u = u_\varepsilon + \bar{\partial} v_\varepsilon.
\]

Note that the component of \( \text{Im} D''_{h_\varepsilon} \) is zero since \( u \) is \( \bar{\partial} \)-closed.

At the end of this step, we explain the strategy of the proof. In Step 2, we show that \( \| D''_{h_\varepsilon} u_\varepsilon \|_{h_\varepsilon, h_{\varepsilon, \bar{\omega}}} \) converges to zero as letting \( \varepsilon \) go to zero. Here \( h_{L, \varepsilon} \) is the singular metric on \( L \) defined by

\[
h_{L, \varepsilon} := h_\varepsilon^{1/a} h_{\Delta}^{-1/a}.
\]

Since the cohomology class of \( su \) is zero, there are solutions \( \gamma_\varepsilon \) of the \( \bar{\partial} \)-equation \( \bar{\partial} \gamma_\varepsilon = su_\varepsilon \). For the proof, we need to obtain \( L^2 \)-estimates of them. In Step 3, we construct solutions
\(\gamma_\varepsilon\) of the \(\overline{\partial}\)-equation \(\overline{\partial} \gamma_\varepsilon = su_\varepsilon\) such that the norm \(\|\gamma_\varepsilon\|_{h_\varepsilon h_{L,\varepsilon}, \tilde{\omega}}\) is uniformly bounded. Then we have

\[
\|su_\varepsilon\|_{h_\varepsilon h_{L,\varepsilon}, \tilde{\omega}}^2 \leq \|D_{h_\varepsilon h_{L,\varepsilon}}^{\gamma_\varepsilon} su_\varepsilon\|_{h_\varepsilon h_{L,\varepsilon}, \tilde{\omega}} \|\gamma_\varepsilon\|_{h_\varepsilon h_{L,\varepsilon}, \tilde{\omega}}.
\]

By Step 2 and Step 3, we can conclude that the right hand side goes to zero as letting \(\varepsilon\) go to zero. In Step 4, from this convergence, we prove that \(u_\varepsilon\) converges to zero in a suitable sense, which implies that the cohomology class of \(u\) is zero.

**Step 2 (A generalization of Enoki’s proof of the injectivity theorem)**

The aim of this step is to prove the following proposition, whose proof can be seen as a generalization of Enoki’s injectivity theorem.

**Proposition 3.3.** As letting \(\varepsilon\) go to zero, the norm \(\|D_{h_\varepsilon h_{L,\varepsilon}}^{\gamma_\varepsilon} su_\varepsilon\|_{h_\varepsilon h_{L,\varepsilon}, \tilde{\omega}}\) converges to zero.

**Proof of Proposition 3.3** We have the following inequality:

\[
\|u_\varepsilon\|_{h_\varepsilon, \tilde{\omega}} \leq \|u\|_{h_\varepsilon, \tilde{\omega}} \leq \|u\|_{h, \omega}.
\]

This inequality is often used in the proof. The first inequality follows from the definition of \(u_\varepsilon\) and the second inequality follows from inequality (1). Remark the right hand side does not depend on \(\varepsilon\). By applying Nakano’s identity and the density lemma to \(u_\varepsilon\) (for example see [Mat2, Proposition 2.4]), we obtain

\[
0 = \langle \sqrt{-1}\Theta_{h_\varepsilon}(F)\Lambda_{\tilde{\omega}} u_\varepsilon, u_\varepsilon \rangle_{h_\varepsilon, \tilde{\omega}} + \|D_{h_\varepsilon}^{\gamma_\varepsilon} u_\varepsilon\|_{h_\varepsilon, \tilde{\omega}}^2.
\]

Note that the left hand side is zero since \(u_\varepsilon\) is harmonic. Let \(A_\varepsilon\) be the first term and \(B_\varepsilon\) be the second term of the right hand side of equality (3). First, we show that the first term \(A_\varepsilon\) and the second term \(B_\varepsilon\) converge to zero. For simplicity, we denote the integrand of \(A_\varepsilon\) by \(g_\varepsilon\), namely

\[
g_\varepsilon := \langle \sqrt{-1}\Theta_{h_\varepsilon}(F)\Lambda_{\tilde{\omega}} u_\varepsilon, u_\varepsilon \rangle_{h_\varepsilon, \tilde{\omega}}.
\]

Then there exists a positive constant \(C > 0\) (independent of \(\varepsilon\)) such that

\[
g_\varepsilon \geq -\varepsilon C\|u_\varepsilon\|_{h_\varepsilon, \tilde{\omega}}^2.
\]

It is easy to check this inequality. Indeed, let \(\lambda_1^{\varepsilon} \leq \lambda_2^{\varepsilon} \leq \cdots \leq \lambda_n^{\varepsilon}\) be the eigenvalues of \(\sqrt{-1}\Theta_{h_\varepsilon}(F)\) with respect to \(\tilde{\omega}\). Then for any point \(y \in Y\) there exists a local coordinate \((z_1, z_2, \ldots, z_n)\) centered at \(y\) such that

\[
\sqrt{-1}\Theta_{h_\varepsilon}(F) = \sum_{j=1}^{n} \lambda_j^{\varepsilon} dz_j \wedge d\bar{z}_j \quad \text{and} \quad \tilde{\omega} = \sum_{j=1}^{n} dz_j \wedge d\bar{z}_j \text{ at } y.
\]

When we locally write \(u_\varepsilon\) as \(u_\varepsilon = \sum_{|K|=q} f_K^{\varepsilon} dz_1 \wedge \cdots \wedge dz_n \wedge d\bar{K}\), we have

\[
g_\varepsilon = \sum_{|K|=q} \left( \sum_{j \in K} \lambda_j^{\varepsilon} \right) f_K^{\varepsilon} h_\varepsilon
\]
by a straightforward computation. On the other hand, from property (C) of \( \tilde{\omega} \) and property (d) of \( h_L \), we have \( \sqrt{-1}\Theta_{h_L}(F) \geq -\varepsilon \omega \geq -\varepsilon \tilde{\omega} \). This implies \( \lambda_j^\varepsilon \geq -\varepsilon \), and thus we obtain inequality (4).

From equality (3) and inequality (4), we obtain

\[
0 \geq A_\varepsilon = \int_Y g_\varepsilon \tilde{\omega}^n
\geq -\varepsilon C \int_Y |u_{L,h_L,\tilde{\omega}}|^2 \tilde{\omega}^n
\geq -\varepsilon C\|u\|_{h_L,\tilde{\omega}}^2.
\]

The last inequality follows from inequality (2). Therefore \( A_\varepsilon \) converges to zero, and further we can conclude that \( B_\varepsilon \) also converges to zero by equality (3).

To apply Nakano’s identity to \( su_\varepsilon \), again, we first check \( su_\varepsilon \in L^2(U_n^h(Y,F \otimes L)_{h_L,h_L,\tilde{\omega}}) \). By the assumption, the point-wise norm \( |s|_{h_L} \) with respect to \( h_L \) is bounded, and further we have \( |s|_{h_L} \leq |s|_{h_L} \) from property (b) of \( h_L \). They imply

\[
|su_\varepsilon|_{h_L,h_L,\tilde{\omega}} \leq \sup_x |s|_{h_L} |u_\varepsilon|_{h_L,\tilde{\omega}} \leq \sup_x |s|_{h_L} \|u\|_{h_L,\tilde{\omega}} < \infty.
\]

Remark that the right hand side does not depend on \( \varepsilon \). By applying Nakano’s identity to \( su_\varepsilon \), we obtain

\[
\|D'_{h_L,h_L} su_\varepsilon\|_{h_L,h_L,\tilde{\omega}}^2
= \langle \sqrt{-1}\Theta_{h_L}(F \otimes L)\Lambda_\tilde{\omega} su_\varepsilon, su_\varepsilon \rangle_{h_L,h_L,\tilde{\omega}} + \|D'_{h_L,h_L} su_\varepsilon\|_{h_L,h_L,\tilde{\omega}}^2
\]

(5)

Here we used \( \partial \bar{\partial} s_\varepsilon = 0 \). We first see that the second term of the right hand side converges to zero. Since \( s \) is a holomorphic \((0,0)\)-form, we have \( D'_{h_L,h_L} su_\varepsilon = s D'_{h_L} u_\varepsilon \). Thus we have

\[
\|D'_{h_L,h_L} su_\varepsilon\|_{h_L,h_L,\tilde{\omega}}^2 \leq \sup_x |s|_{h_L}^2 \int_Y |D'_{h_L} u_\varepsilon|^2_{h_L,\tilde{\omega}} < \infty.
\]

Since \( |s|_{h_L}^2 \) is bounded and \( B_\varepsilon \) converges to zero, the second term \( \|D'_{h_L,h_L} su_\varepsilon\|_{h_L,h_L,\tilde{\omega}}^2 \) also converges to zero.

For the proof of the proposition, it remains to show that the first term of the right hand side of equality (5) converges to zero. We can easily see \( \sqrt{-1}\Theta_{h_L}(F \otimes L) = (1 + 1/a) \sqrt{-1}\Theta_{h_L}(F) \) from \( \sqrt{-1}\Theta_{h_L} = 0 \) on \( Y \) and the definition of \( h_L \). Therefore we obtain

\[
\langle \sqrt{-1}\Theta_{h_L,h_L}(F \otimes L)\Lambda_\tilde{\omega} su_\varepsilon, su_\varepsilon \rangle_{h_L,h_L,\tilde{\omega}} = (1 + 1/a) \int_Y |s|_{h_L}^2 g_\varepsilon \tilde{\omega}^n
\]

Now we investigate \( A_\varepsilon \) in details. By the definition of \( A_\varepsilon \), we have

\[
A_\varepsilon = \int_{\{g_\varepsilon \geq 0\}} g_\varepsilon \tilde{\omega}^n + \int_{\{g_\varepsilon \leq 0\}} g_\varepsilon \tilde{\omega}^n.
\]
It is easy to see that the second term converges to zero. Indeed, simple computations and inequality (4) imply
\[
0 \geq \int_{\{g \leq 0\}} g \omega^n - \varepsilon C \int_{\{g \leq 0\}} |u|_{h, \omega}^2 \omega^n
\geq -\varepsilon C \int_Y |u|_{h, \omega}^2 \omega^n
\geq -\varepsilon C \|u\|_{h, \omega}^2.
\]
Therefore the first term also converges to zero. Now we have
\[
\int_Y |s|_{h_L}^2 g \omega^n = \left\{ \int_{\{g \geq 0\}} |s|_{h_L}^2 g \omega^n + \int_{\{g \leq 0\}} |s|_{h_L}^2 g \omega^n \right\}.
\]
On the other hand, we have the inequalities
\[
\bullet \quad 0 \leq \int_{\{g \geq 0\}} |s|_{h_L}^2 g \omega^n \leq \sup_X |s|_{h_L}^2 \int_{\{g \geq 0\}} g \omega^n
\leq \sup_X |s|_{h_L}^2 \int_{\{g \geq 0\}} g \omega^n,
\]
\[
\bullet \quad 0 \geq \int_{\{g \leq 0\}} |s|_{h_L}^2 g \omega^n \geq \sup_X |s|_{h_L}^2 \int_{\{g \leq 0\}} g \omega^n
\geq \sup_X |s|_{h_L}^2 \int_{\{g \leq 0\}} g \omega^n.
\]
Therefore the right hand side of equality (5) converges to zero. We obtain the conclusion of Proposition 3.3.

Step 3 (A construction of solutions of the $\overline{\partial}$-equation via the Čeck complex)
By the absolutely same method as [Mat2, Step 3], we can prove the following proposition. See [Mat2, Step 3] for the proof.

Proposition 3.4. There exist $F$-valued $(n, q - 1)$-forms $\alpha_\varepsilon$ on $Y$ with the following properties:

1. $\overline{\partial} \alpha_\varepsilon = u - u_\varepsilon$.
2. The norm $\|\alpha_\varepsilon\|_{h, \omega}$ is uniformly bounded.

Step 4 (The limit of the harmonic forms)
In this step, we investigate the limit of $u_\varepsilon$ and complete the proof of Theorem 3.1. First we prove the following proposition.

Proposition 3.5. There exist $F \otimes L$-valued $(n, q - 1)$-forms $\gamma_\varepsilon$ on $Y$ with the following properties:

1. $\overline{\partial} \gamma_\varepsilon = su_\varepsilon$.
2. The norm $\|\gamma_\varepsilon\|_{h, \omega}$ is uniformly bounded.
Proposition 3.7. Then we prove the following proposition.

For every positive number $\varepsilon$, there exists a subsequence of $L$-valued $(n, q - 1)$-form $\gamma$ such that $\partial \gamma = su$ and $\|\gamma\|_{h_t L, \omega} < \infty$. (Recall that we are assuming that the cohomology class of $su$ is zero in $H^q(X, K_X \otimes F \otimes L \otimes I(h_t L))$.) If we take $\alpha_\varepsilon$ with the properties in Proposition 3.4 and put $\gamma_\varepsilon := -s\alpha_\varepsilon + \gamma$, then we have $\partial \gamma_\varepsilon = su_\varepsilon$. Further an easy computation yields

$$\|\gamma_\varepsilon\|_{h_t L, \omega} \leq \|s\alpha_\varepsilon\|_{h_t L, \omega} + \|\gamma\|_{h_t L, \omega}.$$  

Since $\|\gamma\|_{h_t L, \omega} < \infty$ and the norm $\|\alpha_\varepsilon\|_{h_t L, \omega}$ is uniformly bounded, the right hand side can be estimated by a constant independent of $\varepsilon$. $\square$

We consider the limit of the norm $\|su_\varepsilon\|_{h_t L, \omega}$.

Proposition 3.6. The norm $\|su_\varepsilon\|_{h_t L, \omega}$ converges to zero as letting $\varepsilon$ go to zero.

Proof. By taking $\gamma_\varepsilon \in L^{n,q-1}(Y,F \otimes L)_{h_t L, \omega}$ satisfying the properties in Proposition 3.5, we obtain

$$\|su_\varepsilon\|^2_{h_t L, \omega} = \langle su_\varepsilon, \overline{\partial} \gamma_\varepsilon \rangle_{h_t L, \omega} = \langle D^{n,*}_{h_t L} su_\varepsilon, \gamma_\varepsilon \rangle_{h_t L, \omega} \leq ||D^{n,*}_{h_t L} su_\varepsilon||_{h_t L, \omega} \langle \gamma_\varepsilon \rangle_{h_t L, \omega}.$$  

By Proposition 3.5, the norm $\|\gamma_\varepsilon\|_{h_t L, \omega}$ is uniformly bounded. On the other hand, the norm $\|D^{n,*}_{h_t L} su_\varepsilon\|_{h_t L, \omega}$ converges to zero by Proposition 3.3. Therefore the norm $\|su_\varepsilon\|_{h_t L, \omega}$ also converges to zero. $\square$

Fix a small number $\varepsilon_0 > 0$. Then for any positive number $\varepsilon$ with $0 < \varepsilon < \varepsilon_0$, by property (b) of $h_t$, we obtain

$$\|u_\varepsilon\|_{h_t L, \omega} \leq \|u_\varepsilon\|_{h_t L, \omega} \leq \|u\|_{h_t L, \omega}.$$  

It says that the norm of $u_\varepsilon$ with respect to $h_\varepsilon$ is uniformly bounded. Therefore there exists a subsequence of $\{u_\varepsilon\}_{\varepsilon > 0}$ that converges to $\alpha \in L^{n,q}_{(2)}(Y,F)_{h_\varepsilon, \omega}$ with respect to the weak $L^2$-topology. For simplicity, we denote this subsequence by the same notation $\{u_\varepsilon\}_{\varepsilon > 0}$. Then we prove the following proposition.

Proposition 3.7. The weak limit $\alpha$ of $\{u_\varepsilon\}_{\varepsilon > 0}$ in $L^{n,q}_{(2)}(Y,F)_{h_\varepsilon, \omega}$ is zero.

Proof. For every positive number $\delta > 0$, we define the open subset $A_\delta$ of $Y$ by $A_\delta := \{x \in Y \mid |s|^2_{h_t L, 0} > \delta\}$. By an easy computation, we have

$$\|su_\varepsilon\|^2_{h_t L, \omega} \geq \|su_\varepsilon\|^2_{h_t L, \omega} \geq \int_{A_\delta} |s|^2_{h_t L, 0} |u_\varepsilon|^2_{h_\varepsilon, \omega} \overline{\omega}^n \geq \delta \int_{A_\delta} |u_\varepsilon|^2_{h_\varepsilon, \omega} \overline{\omega}^n \geq 0.$$
for any $\delta > 0$. Since the left hand side converges to zero, the norm $\|u_\varepsilon\|_{h_0,\omega, A_\delta}$ on $A_\delta$ also converges to zero. Notice that $u_\varepsilon|_{A_\delta}$ converges to $\alpha|_{A_\delta}$ with respect to the weak $L^2$-topology in $L^{n,q}_{(2)}(A_\delta, F)_{h_0,\omega}$. Here $u_\varepsilon|_{A_\delta}$ (resp. $\alpha|_{A_\delta}$) denotes the restriction of $u_\varepsilon$ (resp. $\alpha$) to $A_\delta$. Indeed for every $\gamma \in L^{n,q}_{(2)}(A_\delta, F)_{h_0,\omega}$, the inner product $\langle u_\varepsilon|_{A_\delta}, \gamma \rangle_{A_\delta} = \langle u_\varepsilon, \tilde{\gamma} \rangle_Y$ converges to $\langle \alpha, \tilde{\gamma} \rangle_Y = \langle \alpha|_{A_\delta}, \tilde{\gamma} \rangle_{A_\delta}$, where $\gamma$ denotes the zero extension of $\gamma$ to $Y$. Since $u_\varepsilon|_{A_\delta}$ converges to $\alpha|_{A_\delta}$, we obtain

$$\|\alpha|_{A_\delta}\|_{h_0,\omega, A_\delta} \leq \liminf_{\varepsilon \to 0} \|u_\varepsilon|_{A_\delta}\|_{h_0,\omega, A_\delta} = 0.$$ (Recall the norm of the weak limit can be estimated by the limit inferior of the norms of sequences.) Therefore we have $\alpha|_{A_\delta} = 0$ for any $\delta > 0$. By the definition of $A_\delta$, the union of $\{A_\delta\}_{\delta>0}$ is equal to $Y = X \setminus Z$, which asserts that the weak limit $\alpha$ is zero on $Y$.

By using Proposition 3.7, we complete the proof of Theorem 3.1. By the definition of $u_\varepsilon$, we have

$$u = u_\varepsilon + \overline{\partial}v_\varepsilon.$$ Proposition 3.7 says that $\overline{\partial}v_\varepsilon$ converges to $u$ with respect to the weak $L^2$-topology. Then it is easy to see that $u$ is a $\overline{\partial}$-exact form (that is, $u \in \text{Im} \overline{\partial}$). This is because the subspace $\text{Im} \overline{\partial}$ is closed in $L^{n,q}_{(2)}(Y,F)_{h_0,\omega}$ with respect to the weak $L^2$-topology. Indeed, for every $\gamma = \gamma_1 + D^{n,*}_{h_0} \gamma_2 \in \mathcal{H}^{n,q}_{(2)}(F)_{h_0,\omega} \oplus \text{Im} D^{n,*}_{h_0}$, we have $\langle u, \gamma \rangle = \lim_{\varepsilon \to 0} \langle \overline{\partial}v_\varepsilon, \gamma_1 + D^{n,*}_{h_0} \gamma_2 \rangle = 0$. Therefore we can conclude $u \in \text{Im} \overline{\partial}$.

In summary, we proved that $u$ is a $\overline{\partial}$-exact form in $L^{n,q}_{(2)}(Y,F)_{h_0,\omega}$, which says that the cohomology class $[u]$ of $u$ is zero in $H^q(X, K_X \otimes F \otimes I(h_{e_0}))$. By property (c), we obtain the conclusion of Theorem 3.1.

\[\square\]

4. Proof of Corollaries related to the Extension conjecture

The purpose of this section is to obtain some extension theorems as applications of our injectivity theorem. For this purpose, by making use of the injectivity theorem (Theorem 3.1), we first prove the following extension theorem, which can be seen as a special case of the extension conjecture for dlt pairs.

**Theorem 4.1.** Let $X$ be a compact Kähler manifold and $\Delta = S + B$ be an effective $\mathbb{Q}$-divisor with the following assumptions:

1. $\Delta$ is a simple normal crossing divisor with $0 \leq \Delta \leq 1$ and $|\Delta| = S$.
2. $K_X + \Delta$ is $\mathbb{Q}$-linearly equivalent to an effective divisor $D$ with $S \subset \text{Supp} D$.
3. $K_X + \Delta$ admits a (singular) metric $h$ with semi-positive curvature.

Then for an integer $m \geq 2$ with Cartier divisor $m(K_X + S + B)$ and any section $u \in H^0(S, O_S(m(K_X + S + B)))$ that belongs to the image of $H^0(S, O_S(m(K_X + S + B)) \otimes I(h^{n-1})h_B) \rightarrow H^0(S, O_S(m(K_X + S + B)))$, the section $u$ can be extended to a section in $H^0(X, O_X(m(K_X + S + B)))$. 

Moreover if we assume that $h \leq Ch_D$ for some $C > 0$, where $h_D$ is the singular metric induced by $D$, then any cohomology class $u \in H^q(S, O_S(m(K_X + \Delta)) \otimes I(h^{m-1}h_B))$ can be extended to a class in $H^q(X, O_X(m(K_X + \Delta)) \otimes I(h^{m-1}h_B))$ for any $q \geq 0$.

**Remark 4.2.** (1) If $X$ is projective and $S$ is smooth (namely $(X, \Delta)$ is plt) and further if the divisor induced by a given $u \in H^0(S, O_S(m(K_X + S + B)))$ is larger than the extension obstruction divisor (which is zero if $K_X + \Delta$ is nef), then $K_X + \Delta$ admits a (singular) metric $h$ with $\sup_S |u|_h < \infty$. (see [DHP] Corollary 1.8). 

(2) Even if $S$ has singularities, we can construct a (singular) metric $h$ with $\sup_S |u|_h < \infty$ in the special case where $u$ is zero along the singular locus of $S$, by the same method as in [DHP].

**Proof.** We may add the additional assumption of $h \leq h_D$, where $h_D$ is the singular metric on $O_X(K_X + \Delta)$ defined by the effective divisor $D$. Indeed, for a smooth metric $g$ on $O_X(K_X + \Delta)$ and an $L^1$-function $\varphi$ (resp. $\varphi_D$) with $h = g e^{-\varphi}$ (resp. $h_D = g e^{-\varphi_D}$), the metric defined by $g e^{-\max(\varphi, \varphi_D)}$ satisfies assumption (3) again.

We prove only the first conclusion since the second conclusion follows from the same argument as the first conclusion. We put $G := m(K_X + \Delta)$, and consider the following exact sequence:

$$0 \to O_X(G - S) \otimes I(h^{m-1}h_B) \to O_X(G) \otimes I(h^{m-1}h_B) \to O_S(G) \otimes I(h^{m-1}h_B) \to 0.$$ 

We prove the induced homomorphism

$$H^q(X, O_X(G - S) \otimes I(h^{m-1}h_B)) \to H^q(X, O_X(G) \otimes I(h^{m-1}h_B))$$

is injective by the injectivity theorem. Then the conclusion follows from the induced long exact sequence.

By the assumption on the support of $D$, we can take an integer $a > 0$ such that $aD$ is a Cartier divisor and $S \leq aD$. Then we have the following commutative diagram:

$$H^q(X, O_X(G) \otimes I(h^{m-1}h_B)) \supseteq \text{Im}(+S)$$

$$\xrightarrow{\text{+S}} \quad H^q(X, O_X(G - S) \otimes I(h^{m-1}h_B)) \quad \xrightarrow{+aD} \quad H^q(X, O_X(G - S + aD) \otimes I(h^{a+m-1}h_B)),$$

where the map $+S : H^q(X, O_X(G - S) \otimes I(h^{m-1}h_B)) \to H^q(X, O_X(G) \otimes I(h^{m-1}h_B))$. Our goal is to show that the map to the upper right is injective. For this goal, we show that the horizontal map is injective as an application of Theorem 3.1.

By the definition of $G$, we have

$$G - S = m(K_X + \Delta) - S = K_X + (m - 1)(K_X + \Delta) + B.$$ 

Then the line bundle $F := O_X((m - 1)(K_X + \Delta) + B)$ equipped with the metric $h_F := h^{m-1}h_B$ and the line bundle $L := O_X(aD)$ equipped with the metric $h_L := h^a$ satisfy the assumptions in Theorem 3.1. Indeed, we have $h_F = h_L^{(m-1)/a}h_B$ by the construction, and further the point-wise norm $|s_{aD}|_{h_L}$ is bounded on $X$ by the inequality $h \leq h_D$, where $s_{aD}$ is the natural section of $aD$. Therefore the horizontal map is injective by Theorem 3.1. $\square$
To obtain some results related to the abundance conjecture (Theorem 5.1 and Corollary 5.3), we need the following corollary, which is a slight generalization of Theorem 4.1.

**Corollary 4.3.** Under the same situation as in Theorem 4.1, instead of assumption (3), we assume the following assumption:

\( (3') \) There exist effective \( \mathbb{Q} \)-divisors \( E \) and \( F \) and a (singular) metric \( h \) on \( \mathcal{O}_X(F) \) with semi-positive curvature such that

- \( K_X + \Delta \sim_{\mathbb{Q}} E + F \),
- \( E + B \) is simple normal crossing and \( E \) has no common component with \( S \),
- \( \nu(h, x) = 0 \) at every point \( x \in S \).

Let \( \widetilde{s} \) be a section on \( X \) with \( \text{div} \; \widetilde{s} = mE \). Then for a section \( u \in H^0(S, \mathcal{O}_S(mE)) \), the section \( u \cdot \widetilde{s} \in H^0(S, \mathcal{O}_S(m(K_X + \Delta))) \) can be extended to a section in \( H^0(X, \mathcal{O}_X(m(K_X + \Delta))) \).

**Proof.** Let \( h_E \) be the singular metric on \( \mathcal{O}_X(E) \) induced by the section \( \widetilde{s} \in H^0(X, \mathcal{O}_X(mE)) \). By the definition, the metric \( h_E \) satisfies \( \sqrt{-1} \Theta_{h_E} \geq 0 \) and \( \sup |\widetilde{s}|_{h_E} < \infty \). The product \( h \cdot h_E \) determines the singular metric on \( K_X + \Delta \) with semi-positive curvature. Therefore it is sufficient to show that \( u \cdot \widetilde{s} \) belongs to \( H^0(S, \mathcal{O}_S(m(K_X + \Delta))) \) \( \otimes \mathcal{F} \), where we put \( \mathcal{F} := I(h^{m-1}_E h_B^{-1}) \) for simplicity.

In the first step, we see that

\[ \mathcal{F}_x = I(h^{m-1}_E h_B)_x \]

for every \( x \in S \), where \( \mathcal{F}_x \) denotes the stalk of a sheaf \( \mathcal{F} \) at \( x \). Let \( f \) be a holomorphic function on an open neighborhood \( U_x \) of \( x \in S \) with \( f \in I(h^{m-1}_E h_B)_x \), and let \( \varphi \) (resp. \( \varphi_E, \varphi_B \)) be a local weight of \( h \) (resp. \( h_E, h_B \)). By taking a real number \( p > 1 \) with \( I(h^{p(m-1)}_E h_B^p) = I(h^{m-1}_E h_B) \), we may assume that \( |f| e^{-p(m-1)\varphi_E - p\varphi_B} \) is \( L^2 \)-integrable on \( U_s \). Then, by taking the positive number \( q \) with \( 1/p + 1/q = 1 \), we obtain

\[ \int_{U_x} |f|^2 e^{-2(m-1)\varphi - 2(m-1)\varphi_E - 2\varphi_B} \leq \left( \int_{U_x} |f|^2 e^{-2p(m-1)\varphi_E - 2\varphi_B} \right)^{1/p} \cdot \left( \int_{U_x} e^{-2q(m-1)\varphi} \right)^{1/q} \]

by Hölder’s inequality. The function \( e^{-2q(m-1)\varphi} \) is locally \( L^2 \)-integrable for any \( q > 0 \) by Skoda’s lemma and the assumption of the Lelong number. On the other hand, as mentioned above, \( |f|^p e^{-p(m-1)\varphi_E - p\varphi_B} \) is also locally \( L^2 \)-integrable. Therefore we have \( \mathcal{F}_x = I(h^{m-1}_E h_B)_x \) for every \( x \in S \).

In the second step, we prove

\[ u \cdot \widetilde{s} \in H^0(S, \mathcal{O}_S(m(K_X + \Delta))) \otimes \mathcal{F}|_S, \]

where \( \mathcal{F}|_S \) is the restriction of \( \mathcal{F} \) defined by

\[ \mathcal{F}|_S := \mathcal{F} \cdot \mathcal{O}_S = \mathcal{F} / (\mathcal{F} \cap I_S). \]

Let \( \widetilde{u} \) be a local extension of \( u \) on an open neighborhood \( U_x \) of \( x \in S \). By the klt condition of \( B \), we can take a real number \( p > 1 \) with \( I(h^p_B) = \mathcal{O}_X \). Then for the holomorphic
function $g := \tilde{u} \cdot \tilde{s}$, by taking the positive number $q$ with $1/p + 1/q = 1$, we obtain

$$\int_{U_x} |g|^2 e^{-2(m-1)\nu_E} \lesssim (\int_{U_x} |g|^{2p} e^{-2p(m-1)\nu_E})^{1/p} \cdot (\int_{U_x} 1)^{1/q} \leq \sup_{U_x} |g|^{2p} e^{-2p(m-1)\nu_E} \left( \int_{U_x} e^{-2p\nu_E} \right)^{1/p} \cdot (\int_{U_x} 1)^{1/q}$$

by Hölder’s inequality again. The point-wise norm $|g|^2 p e^{-2p(m-1)\nu_E}$ is bounded by the choice of $h_E$, and $e^{-2p(m-1)\nu_E}$ is locally $L^2$-integrable for any $q > 0$ by the assumption on the Lelong number. It implies that $u \cdot \tilde{s}$ (locally) belongs to $\mathcal{I}(h_E^m h_B)|_S = \mathcal{I}|_S$.

Finally we show

$$u \cdot \tilde{s} \in H^0(S, \mathcal{O}_S(m(K_X + \Delta)) \otimes \mathcal{J}).$$

By simple computations we have $\mathcal{O}_S \otimes \mathcal{J} = \mathcal{O}_X \otimes \mathcal{J}/(\mathcal{J} \cdot I_S)$, and thus, by the second step, it is sufficient to see

$$\mathcal{J} \cap I_S = \mathcal{J} \cdot I_S.$$

Here $I_S$ denotes the ideal sheaf defined by $S$. By the first step and the assumption on the support of $E + B$, we have

$$\mathcal{J}_x = \mathcal{I}(h_E^{m-1} h_B)_x = \mathcal{O}_X([- (m-1) E + B])$$

for every $x \in S$. Since the divisors $S$ and $E + B$ has no common component by the assumption on $E$, we can easily see $\mathcal{J} \cap I_S = \mathcal{J} \cdot I_S$. Therefore $u \cdot \tilde{s}$ actually belongs to $H^0(S, \mathcal{O}_S(m(K_X + \Delta)) \otimes \mathcal{J})$. The conclusion follows from Theorem 4.1. $\square$

**Remark 4.4.** When we apply the injectivity theorem in order to extend sections, we need to handle $\mathcal{O}_S \otimes \mathcal{I}(\varphi)$ (not $\mathcal{I}(\varphi)|_S$). When we apply the Ohsawa-Takegoshi extension theorem, we usually use the restriction of multiplier ideal sheaves $\mathcal{I}(\varphi)|_S$. It is relatively difficult to handle $\mathcal{O}_S \otimes \mathcal{I}(\varphi)$. However the support condition of $E$ (the second assumption of the above corollary) fortunately appears in the proof of the applications related to the abundance conjecture, which asserts $\mathcal{O}_S \otimes \mathcal{I}(\varphi) = \mathcal{I}(\varphi)|_S$.

In a special case of the above corollary when $E = \mathcal{O}_X$ and $\tilde{s} = 1 \in H^0(X, \mathcal{O}_X)$, we obtain the following:

**Corollary 4.5.** Under the same situation as in Theorem 4.1, instead of assumption (3), we assume the following assumption:

(3’’) $K_X + \Delta$ admits a (singular) metric $h$ such that $\sqrt{-1} \Theta_h \geq 0$ and $v(h, x) = 0$ at every point $x \in S$.

Then for an integer $m \geq 2$ with Cartier divisor $m(K_X + \Delta)$, a section $u \in H^0(S, \mathcal{O}_S(m(K_X + \Delta)))$ can be extended to a section in $H^0(X, \mathcal{O}_S(m(K_X + \Delta)))$.

For further applications of the above theorems, we prepare the following lemma.

**Lemma 4.6.** Let $\varphi$ be a (quasi)-psh function on a complex manifold $X$. If the Lelong number $\nu(\varphi, x_0)$ is zero at $x_0 \in X$, then for any modification $\pi : Y \to X$, the Lelong number $\nu(\pi^* \varphi, y)$ is zero at every point $y \in \pi^{-1}(x_0)$. 
Proof. For a contradiction, we assume that $\nu(\pi^*\varphi, y_0) > 0$ for some point $y_0 \in \pi^{-1}(x_0)$. By Skoda’s lemma (see Theorem 2.5), we can take a sufficiently large number $m > 0$ such that $\pi^*dV_X e^{-2m\pi^*\varphi}$ is not integrable on a neighborhood of $y_0$, where $dV_X$ is a standard volume form on a neighborhood $U$ of $x_0$. By the change of variable formula, we have

$$\int_U e^{-2m\varphi}dV_X = \int_{\pi^{-1}(U)} e^{-2m\varphi}\pi^*dV_X.$$

By the assumption of $\nu(\varphi, x_0) = 0$, the left hand side is finite for a sufficiently small $U$. It is a contradiction to the choice of $m$. Therefore $\nu(\pi^*\varphi, y) = 0$ at every point $y \in \pi^{-1}(x_0)$. □

5. Proof of Corollaries related to the abundance conjecture

In this section, we prove some applications related to the abundance conjecture. The proof of the following theorem is based on [DHP, Section 8] and [FG, Theorem 5.9]. We use the different MMP from them to preserve metric conditions.

**Theorem 5.1.** Assume that Conjecture 1.1 holds in dimension $n - 1$. Let $X$ be an $n$-dimensional normal projective variety and $\Delta$ be a Q-divisor with the following assumptions:

- $(X, \Delta)$ is a klt pair such that there exists an effective Q-divisor $D$ such that $K_X + \Delta \sim_Q D$.
- There exists a projective birational morphism $\varphi : Y \to X$ such that $\varphi^*(O_X(m(K_X + \Delta)))$ admits a (singular) metric whose curvature is semi-positive and Lelong number is identically zero on $\text{Supp} \varphi^*D$. Here $m$ is a positive integer such that $m(K_X + \Delta)$ is Cartier.

Then $K_X + \Delta$ is semi-ample.

**Proof.** Note that Conjecture 1.1 in dimension $n - 1$ implies that the existence of good minimal model for $(n - 1)$-dimensional klt pairs by [GL, Theorem 4.3] or [DHP, Remark 2.6]. First we may assume that $\kappa(K_X + \Delta) = 0$ by Kawamata’s theorem [Ka, Theorem 7.3] (see also [KeMMc, 5.6 Lemma]). There exists the effective Q-divisor $D$ such that

$D \sim_Q K_X + \Delta$

by the assumption. For a contradiction, assume $D \neq 0$. We may assume that $\varphi$ is a log resolution of $(X, \Delta + D)$. We write

$$K_Y + B = \varphi^*(K_X + \Delta) \sim_Q \varphi^*D.$$

Let $l \geq \text{lct}(\varphi^*D; Y, B)$. Set effective Q-divisors $C$, $S$, and $G$ such that $C + S = (B + l\varphi^*D)^{\geq 0}$, $[C + S] = S$, and $G = (B + l\varphi^*D)^{\leq 0}$. Then there are no common components of any two divisors of $C$, $S$, and $G$, and we have

$$K_Y + C + S \sim_Q (1 + l)\varphi^*D + G,$$

$$S + C - G = B + l\varphi^*D.$$

Note that $S \neq 0$ and $G$ is $\varphi$-exceptional. Then we see that $S \subseteq \text{Supp} \varphi^*D$. Let us consider the MMP for $(Y, C + S)$ over $X$ by [BCHM] and [F3, Theorem 2.3]. Then this program...
contracts only the divisor $G$. Let $f : Y \to Y'$ be a minimal model of this program and $\varphi' : Y' \to X$ the induced morphism. Then $S$ is not contracted by $f$ since $S$ and $G$ have no common component. Denote $C'$ and $S'(\neq 0)$ by the strict transforms on $Y$ of $C$ and $S$. Then it follows that

$$K_{Y'} + C' + S' \sim_Q (1 + l)\varphi'^*D.$$ 

Thus $K_{Y'} + C' + S'$ is nef.

**Claim 5.2.** For a sufficiently large and divisible $m' \in \mathbb{Z}$, the restriction map

$$H^0(Y', m'(K_{Y'} + C' + S')) \to H^0(S', m'(K_{Y'} + C' + S'))$$

is surjective.

**Proof of Claim 5.2.** Let $u$ be a non-zero section in $H^0(S', m'(K_{Y'} + C' + S'))$ and let $\alpha$ and $\beta$ be common log resolutions such that $\beta$ is isomorphic over the generic point of every lc center of $(Y', C' + S')$. Then we write

$$K_W + S_W + F \sim_Q (1 + l)\beta^*\varphi'^*D + E,$$

where $S_W$ is the strict transform of $S'$, and $E$ and $F$ are effective $\mathbb{Q}$-divisor such that $[F] = 0$ and any two divisors of $S_W, F$ and $E$ have no common component. Note that $m'F$ and $m'E$ are Cartier since $m'$ is sufficiently large and divisible. Let $u_{m'E} = h_{m'E}$ be the global section and the metric associated to $m'E$ respectively. Then by the assumption we have a semi-positive singular metric $h$ on $m'(1 + l)\beta^*\varphi'^*D$ induced by the pullback (we use the same symbol since it contains no confusion).

By applying Corollary 4.3 for $F = (1 + l)\beta^*\varphi'^*D$ (cf. Lemma 4.6) and $\tilde{s} = u_{m'E}$, we have a section $U \in H^0(m'(K_W + S_W + F))$ such that $U_{|S_W} = \beta^*u \otimes (u_{m'E})_{|S_W}$. Thus we see that

$$H^0(Y', m'(K_{Y'} + C' + S')) \to H^0(S', m'(K_{Y'} + C' + S'))$$

is surjective. Indeed we see that

$$H^0(W, m'(K_W + S_W + F)) \to H^0(Y', m'(K_{Y'} + C' + S'))$$

is isomorphic by mapping $s \mapsto \beta^*s \otimes u_{m'E}$ for a section $s \in H^0(W, m'(K_W + S_W + F))$, and

$$H^0(S', m'(K_{Y'} + C' + S')) \to H^0(S_W, m'(K_{S_W} + F_{|S_W})).$$

is injection by mapping $t \mapsto \beta^*t \otimes (u_{m'E})_{|S_W}$ for a section $t \in H^0(S', m'(K_{Y'} + C' + S'))$ by $\beta_*O_{S_W} = O_{S'}$ from Kollár–Shokurov’s connectedness Theorem [Ko2, 17.4 Theorem]. We finish the proof of Claim 5.2. 

□
On the other hand, this restriction map is zero map since
\[ \kappa(K_X + \Delta) = \kappa(K_Y + C' + S') = 0 \]
and \( S' \subseteq \text{Supp} \varphi^*D \). The pair \((S', C'_S)\) is a projective semi-log canonical pair such that \( K_{S'} + C'_S \) is nef, where \( (K_Y + C' + S')|_{S'} = (K_{S'} + C'_S) \). This implies \( K_{S'} + C'_S \) is semiample, because of the abundance conjecture (Conjecture 1.1) holds in dimension \( n - 1 \) and [FG] Theorem 1.5] or [HX] (here we need the assumption of projectivity). This is a contradiction to Claim 5.2. Thus \( D = 0 \). We finish the proof. □

By combining the abundance theorem in dimension 3 ([Ka2, Theorem 1.1], [KeMMC, 1.1 Theorem], [F1, Theorem 0.1]), we obtain the following results:

**Corollary 5.3.** Let \((X, \Delta)\) be a 4-dimensional projective klt pair. Assume that there exists a projective birational morphism \( \varphi : Y \to X \) on to each components such that \( Y \) is smooth and \( \varphi^*(\mathcal{O}_X(m(K_X + \Delta))) \) admits a singular metric whose curvature is semi-positive and Lelong number is identically zero (in particular it is satisfied if \( h \) is smooth). Here \( m \) is an integer with Cartier divisor \( m(K_X + \Delta) \). If \( \kappa(K_X + \Delta) \geq 0 \), then \( K_X + \Delta \) is semi-ample.

**Remark 5.4.** When \( h \) is smooth, we can show Corollary 5.3 without using our injectivity theorem. By replacing Theorem 1.3 to the generalized Enoki’s injectivity theorem after Fujino [F5, Theorem 1.2 and Corollary 1.3] in the proof of Theorems 4.1, 5.1, and Corollary 4.3, we can obtain it.

Finally we give a result for semi-ampleness by combining with Verbitsky’s non-vanishing theorem ([V, Theorem 4.1]).

**Corollary 5.5.** Let \( X \) be a 4-dimensional projective hyperKähler manifold and \( L \) be a line bundle admitting a (singular) metric whose curvature is semi-positive and Lelong number is identically zero (in particular it is satisfied if \( h \) is smooth). Then \( L \) is semi-ample.

**Proof.** It is enough to show \( \kappa(L) \geq 0 \) by Corollary 5.3 since if there exists an effective \( \mathbb{Q} \)-divisor such that \( D \sim_{\mathbb{Q}} L \), the pair \((X, \varepsilon D)\) is klt and \( K_X + \varepsilon D \sim_{\mathbb{Q}} \varepsilon L \) for sufficiently small \( \varepsilon > 0 \). If \( q(L, L) > 0 \), then \( L \) is big, where \( q(\cdot, \cdot) \) is the Bogomolov–Beauville–Fujiki form. In the case of \( q(L, L) = L^{\dim X} = 0 \), then \( \kappa(L) \geq 0 \) follows from [V, Theorem 4.1]. □

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