ON EQUIVARIANT MAPS RELATED TO THE SPACE OF PAIRS OF EXCEPTIONAL JORDAN ALGEBRAS

RYO KATO AND AKIHIKO YUKIE

ABSTRACT. Let $\mathcal{J}$ be the exceptional Jordan algebra and $V = \mathcal{J} \oplus \mathcal{J}$. We construct an equivariant map from $V$ to $\text{Hom}_k(\mathcal{J} \oplus \mathcal{J}, \mathcal{J})$ defined by homogeneous polynomials of degree 8 such that if $x \in V$ is a generic point, then the image of $x$ is the structure constant of the isotope of $\mathcal{J}$ corresponding to $x$. We also give an alternative way to define the isotope corresponding to a generic point of $\mathcal{J}$ by an equivariant map from $\mathcal{J}$ to the space of trilinear forms.

1. INTRODUCTION

Let $k$ be a field of characteristic not equal to 2, 3, $k_{\text{sep}}$ the separable closure of $k$ and $\overline{k}$ the algebraic closure of $k$. Let $\overline{O}$ be the split octonion over $k$. It is the normed algebra over $k$ obtained by the Cayley–Dickson process (see [2] pp.101–110]). If $A$ is the algebra of $2 \times 2$ matrices, $\overline{O} = A(+) \oplus c$ with the notation of [2]. An octonion is, by definition, a normed algebra which is a $k$-form of $\overline{O}$. Let $O$ be an octonion. We use the notation $\|x\|$ for the norm of $x \in O$. If $a \in k$, $\|ax\| = a^2\|x\|$. Also if $x, y \in O$, then $\|xy\| = \|x\|\|y\|$. For $x, y \in O$, let

$$Q(x, y) = \frac{1}{2}(\|x + y\| - \|x\| - \|y\|).$$

This is a non-degenerate symmetric bilinear form such that $Q(x, x) = \|x\|$. Let $W \subset O$ be the orthogonal complement of $k \cdot 1$ with respect to $Q$. If $x = x_1 + x_2$ where $x_1 \in k \cdot 1, x_2 \in W$, then we define $\overline{x} = x_1 - x_2$ and call it the conjugate of $x$. Note that $\|x\| = x\overline{x}$. For $x \in O$, we define the trace $\text{tr}(x)$ by $\text{tr}(x) = x + \overline{x}$. It is easy to verify that

$$\text{tr}(xy) = \text{tr}(yx), \quad 2Q(x, y) = \text{tr}(xy).$$

Let $\text{GL}(n)$ be the group of $n \times n$ invertible matrices over $k$. If $V$ is a finite dimensional vector space, then we denote the group of invertible linear maps from $V$ to $V$ by $\text{GL}(V)$. Let $\mathcal{J}$ be the exceptional Jordan algebra over $k$. Elements of $\mathcal{J}$ are of the form:

$$X = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}, \quad x_i \in k, x_i \in O \quad (i = 1, 2, 3),$$

The multiplication of $\mathcal{J}$ is defined as follows:

$$X \circ Y = \frac{1}{2}(XY + YX),$$

where the multiplication used on the right-hand side is the multiplication of matrices.

The algebraic groups $E_6$ and $GE_6$ are given by:

$$E_6 = \{ L \in \text{GL}(\mathcal{J}) \mid \forall X \in \mathcal{J}, \det(LX) = \det(X) \},$$

$$GE_6 = \{ L \in \text{GL}(\mathcal{J}) \mid \forall X \in \mathcal{J}, \det(LX) = c(L) \det(X) \text{ for some } c(L) \in \text{GL}(1) \}$$

respectively. Then $c : GE_6 \to \text{GL}(1)$ is a character and there exists an exact sequence

$$0 \to E_6 \to GE_6 \to \text{GL}(1) \to 0.$$

2000 Mathematics Subject Classification. 11S90, 11R45.

Key words and phrases. prehomogeneous vector spaces, Jordan algebra, cubic fields.

The first author was partially supported by Grant-in-Aid (B) (24340001)
It is known that $E_6$ is a smooth connected quasi-simple simply-connected algebraic group of type $E_6$ (see [3 p.181, Theorem 7.3.2]). The terminology “quasi-simple” means that its inner automorphism group is simple (see [4 p.136]). The smoothness of the group follows from the fact that the dimension of $E_6$ as a variety and the dimension of the Lie algebra of $E_6$ coincide (see the proof of [5 p.181, Theorem 7.3.2]).

Let $H_1 = E_6$, $G_1 = G E_6$, $H = H_1 \times \text{GL}(2)$ and $G = G_1 \times \text{GL}(2)$. Let $V = J \otimes \text{Aff}^2$. We regard elements of $V$ as the set of $x = x_1 v_1 + x_2 v_2$ where $x_1, x_2 \in J$ and two variables $v_1, v_2$. The action of $g = (g_1, g_2) \in G$ where $g_2 = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right)$ on $V$ is given by

\begin{equation}
g(x_1 v_1 + x_2 v_2) = g_1(x_1)(av_1 + cv_2) + g_1(x_2)(bv_1 + dv_2).
\end{equation}

For $x = x_1 v_1 + x_2 v_2 \in V$, we put $F_3(v) = F_3(v_1, v_2) = \det(x)$. Then $F_3(v)$ is a binary cubic form. Let $\Delta(x)$ be the discriminant of $F_3(v)$ as a polynomial of $v$. Let

\begin{equation}
w = v_1 v_1 + v_2 v_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} v_1 + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} v_2 \in J \otimes \text{Aff}^2.
\end{equation}

Then it is easy to see that $F_w(v) = v_1 v_2 (v_1 - v_2)$ and $\Delta(w) = 1$. The pair $(G, V)$ is an irreducible regular prehomogeneous vector space (see [3, Proposition 3.2] and [6]) and the polynomial $\Delta(x)$ is what we call a relative invariant polynomial. We define $V^{ss} = \{ x \in V \mid \Delta(x) \neq 0 \}$. Points in $V^{ss}$ are called semi-stable points. As we pointed out as above, $w \in V^{ss}_k$. The polynomial $\Delta(x)$ is of degree 12. If we put $\chi(g) = c(g_1)^4 \det(g_2)^3$ for $g = (g_1, g_2) \in G$, then $\Delta(gx) = \chi(g) \Delta(x)$.

A Jordan algebra $M$ is called an isotope of $J$ if $M \otimes k^{sep} \cong J \otimes k^{sep}$ and the “determinant” of $M$ is a constant multiple of that of $J$. For details, the reader should see [5 pp.154–158]. If $M_1, M_2$ are isotope of $J$ and $n_1 \subset M_1, n_2 \subset M_2$ are cubic étale subalgebras, then the pairs $(M_1, n_1), (M_2, n_2)$ are defined to be equivalent if there exists a $k$-isomorphism $M_1 \rightarrow M_2$ which induces an isomorphism from $n_1$ to $n_2$. Let $JIC(k)$ be the set of equivalence classes of pairs $(M, n)$ as above.

In [3], to each point in $V^{ss}_k$, a pair $(M, n) \in JIC(k)$ was associated. It is proved in [3 Theorem 5.8] that there is a bijection correspondence between the set $G_k \setminus V^{ss}_k$ of rational orbits and $JIC(k)$. Moreover, an equivariant map $m : V \rightarrow J$ was defined and the Jordan algebra corresponding to $x \in V^{ss}_k$ was explicitly described using the point $m(x)$ (see [3 Section 5]).

Let $t \subset J$ be the subalgebra of diagonal matrices, which is isomorphic to $k^3$. Let $\text{Aut}(J)$ be the algebraic group of automorphisms of the Jordan algebra $J$. We define $\text{Aut}(J, t)$ to be the subgroup of $\text{Aut}(J)$ consisting of automorphisms $L$ such that $L(t) = t$. It is proved in ([3 Theorem 3.1, Lemma 4.2]) that $G_w \cong \text{GL}(1) \times \text{Aut}(J, t)$. The first Galois cohomology set $H^1(k, \text{GL}(1) \times \text{Aut}(J, t))$ can be identified with $H^1(k, \text{Aut}(J, t))$ and there is a natural map $H^1(k, \text{Aut}(J, t)) \rightarrow H^1(k, \text{Aut}(J))$. One can show by standard argument that elements of $H^1(k, \text{Aut}(J))$ correspond bijectively with $k$-forms of $J$.

Suppose that $x = g_x w \in V^{ss}_k$ where $g_x \in G_w^{sep}$. Then $h_x : \text{Gal}(k^{sep}/k) \ni \sigma \mapsto g_x^{-1} g_x^{\sigma} \in G_w^{sep}$ is a 1-cocycle, which defines an element, say $c_x \in H^1(k, G_w)$. This element $c_x$ does not depend on the choice of $g_x$. Since there is a natural map $H^1(k, G_w) \cong H^1(k, \text{Aut}(J)) \rightarrow H^1(k, \text{Aut}(J)) \cong \text{Hom}_k(J \otimes J, J)$ (we call this element the “structure constant”).

If we choose bases of $V, J$ as $k$-vector spaces, the map $m : V \rightarrow J$ is given by homogeneous polynomials of degree 4 on $V$. The structure constant which is associated to elements of $J$ is, with the denominator multiplied, given by homogeneous polynomials of degree 11 on $J$. So the structure constant of $M$ is given by homogeneous polynomials of degree 44 on $V$.

The first purpose of this paper is to prove the following theorem (see Section 2).

**Theorem 1.4.** There is an equivariant map $S : V \ni x \mapsto s_x \in \text{Hom}_k(J \otimes J, J)$ defined by homogeneous polynomials of degree 8 on $V$ such that if $x \in V^{ss}_k$ corresponds to the pair $(M, n) \in JIC(k)$, then $\Delta(x)^{-1} s_x$ is the structure constant of the Jordan algebra $M$. 
Let \( a \in J \) and \( \det(a) \neq 0 \). In \([5, p. 155]\), for \( X, Y \in J \), the product structure \( X \circ_a Y \) defining the isotope \( J_a \) and the corresponding symmetric bilinear form \( Q_a(X, Y) \) are given (see \( [3,5] \)). Then \( J^3 \ni (X, Y, Z) \mapsto T_a(X, Y, Z) = Q_a(X \circ_a Y, Z) \) is a trilinear form on \( J \). To construct \( T_a \) first equivariantly to provide an alternative way to define the product structure on \( J_a \) is another purpose of this paper.

We prove the following theorem in Section \([3]\).

**Theorem 1.5.** There is an equivariant map \( T : J \ni a \mapsto T_a \in \text{Hom}_k(J \otimes J \otimes J, k) \) defined by homogeneous polynomials of degree 6 such that if for \( X, Y \in J \), \( X \circ_a Y \in J \) is the element such \( Q_a(X \circ_a Y, Z) = \det(a)^{-1} T_a(X, Y, Z) \) for all \( Z \in J \), then this product structure coincides with that of the isotope \( J_a \).

2. **Equivariant map I**

We prove Theorem \([1,4]\) in this section. We first define basic notions and then define the desired equivariant map.

We denote the \( 3 \times 3 \) diagonal matrix with diagonal entries \( a_1, a_2, a_3 \in k^* \) by \( \text{diag}(a_1, a_2, a_3) \). \( J^{n \otimes} \) is the tensor product of \( n \) copies of \( J \). We define a symmetric trilinear form \( D \) on \( J \) by

\[
6D(X, Y, Z) = \det(X + Y + Z) - \det(X + Y) - \det(Y + Z) - \det(Z + X) + \det(X) + \det(Y) + \det(Z)
\]

for \( X, Y, Z \in J \). Let \( \text{Tr}(X) \) be the sum of diagonal entries of \( X \in J \). For \( X, Y \in J \), we define a symmetric bilinear form \( \langle \cdot, \cdot \rangle \) on \( J \) by

\[
\langle X, Y \rangle = \text{Tr}(X \circ Y).
\]

One can verify by direct computation that the symmetric bilinear form \( \langle \cdot, \cdot \rangle \) satisfies the following equation:

\[
(2.1) \quad \langle X \circ Y, Z \rangle = \langle X, Y \circ Z \rangle, \quad \forall X, Y, Z \in J.
\]

For \( X, Y \in J \), the cross product \( X \times Y \) is, by definition, the element satisfying the following equation:

\[
\langle X \times Y, Z \rangle = 3D(X, Y, Z), \quad \forall Z \in J.
\]

Let \( e = I_3 \). Then the following equations are satisfied (see \([5, p.122, \text{Lemma 5.2.1}]\)).

\[
(2.2) \quad \forall X \in J, \quad X \circ (X \times X) = \det(X)e, \quad e \times e = e.
\]

For any \( g \in \text{GL}(J) \), we define \( \tilde{g} \in \text{GL}(J) \) by

\[
\langle \tilde{g}(X), \tilde{g}(Y) \rangle = \langle X, Y \rangle, \quad \forall X, Y \in J.
\]

The following lemma is proved in \([5, p.180, \text{Proposition 7.3.1}]\).

**Lemma 2.3.** The map \( g \mapsto \tilde{g} \) is an automorphism of \( H_1 \) with order 2 and for all \( X, Y \in J \),

\[
g(X \times Y) = \tilde{g}(X) \times \tilde{g}(Y), \quad \tilde{g}(X \times Y) = g(X) \times g(Y).
\]

**Corollary 2.4.** If \( g \in G_1 \) and \( X, Y, Z, W \in J \), then

\[
g((X \times Y) \times (Z \times W)) = c(g)^{-1} (g(X) \times g(Y)) \times (g(Z) \times g(W)).
\]

**Proof.** There exists \( t \in k \) such that \( t^3 = c(g) \). Then \( t^{-1}g \in H_1 \). So

\[
g((X \times Y) \times (Z \times W)) = t(t^{-1}g) ((X \times Y) \times (Z \times W)) = t \left( t^{-1}g(X) \times t^{-1}g(Y) \right) \times \left( t^{-1}g(Z) \times t^{-1}g(W) \right) = c(g)^{-1} (g(X) \times g(Y)) \times (g(Z) \times g(W)).
\]

\[\square\]
We now define an equivariant map
\[ S : V \ni x \mapsto S_x \in \text{Hom}_k(\mathcal{J} \otimes \mathcal{J}, \mathcal{J}) \]
such that \( S_w(X, Y) = X \circ Y \) for all \( X, Y \in \mathcal{J} \).

Let
\[
X = \begin{pmatrix} s_1 & x_3 & x_2 \\ x_3 & s_2 & x_1 \\ x_2 & x_1 & s_3 \end{pmatrix}, \quad Y = \begin{pmatrix} t_1 & y_3 & y_2 \\ y_3 & t_2 & y_1 \\ y_2 & y_1 & t_3 \end{pmatrix}
\]
where \( s_i, t_i \in k, x_i, y_i \in O \) for \( i = 1, 2, 3 \). By computation, \( X \circ Y \) is the following matrix:
\[
\frac{1}{2} \begin{pmatrix}
2s_1t_1 + \text{tr}(x_3y_3 + x_2y_2) & s_1y_3 + t_2x_3 + \overline{x_2y_1} + t_1x_3 + s_2y_3 + \overline{y_2x_1} & s_1y_2 + x_3y_1 + t_3x_2 \\
\text{tr}(x_3y_3 + x_2y_2) + t_1x_3 + s_2y_3 + x_1y_2 & 2s_2t_2 + \text{tr}(x_3y_3 + x_1y_1) + \overline{y_3x_2} + t_2x_1 + s_3y_1 & \overline{x_3y_2} + s_2y_1 + t_3x_1 \\
\text{tr}(x_3y_3 + x_2y_2) + t_1x_2 + s_1y_3 + s_3y_2 & \text{tr}(x_3y_3 + x_1y_1) + t_2x_3 + s_1y_2 + s_4y_1 & 2s_3t_3 + \text{tr}(x_2y_2 + x_1y_1)
\end{pmatrix}.
\]

Note that \( \text{tr}(x_2y_2) = \text{tr}(y_2x_2) = \text{tr}(x_2y_2) \), etc. In particular, if \( Y \) is diagonal, then
\[
X \circ Y = \frac{1}{2} \begin{pmatrix}
2s_1t_1 & (t_1 + t_2)x_3 & (t_1 + t_3)x_3 \\
(t_1 + t_2)x_3 & 2s_2t_2 & (t_2 + t_3)x_1 \\
(t_1 + t_3)x_3 & (t_2 + t_3)x_1 & 2s_3t_3
\end{pmatrix}.
\]

It is known (see \{5\} p.122) that
\[
X \times Y = X \circ Y - \frac{1}{2} (X, Y)X - \frac{1}{2} (Y, Y)X - \frac{1}{2} (X, Y)X + \frac{1}{2} (X, Y)X + \frac{1}{2} \text{Tr}(X) \text{Tr}(Y)X.
\]

For \( a, b \in \mathcal{J} \) and \( i_1, \ldots, i_8 = 1, 2 \), we define
\[
[[i_1, i_2], [i_3, i_4], [i_5, i_6], [i_7, i_8]]_{(a,b)} = v_1 \otimes \cdots \otimes v_8
\]
where \( v_j = a \) (resp. \( v_j = b \)) if \( i_j = 1 \) (resp. \( i_j = 2 \)). For example,
\[
[[1, 2, 2, 1, 2, 1, 2]]_{(a,b)} = a \otimes b \otimes b \otimes a \otimes b \otimes a \otimes a \otimes b.
\]

We only consider \([i_1, i_2], [i_3, i_4], [i_5, i_6], [i_7, i_8]]_{(a,b)} \) such that \( \{i_{2j-1}, i_{2j}\} = \{1, 2\} \) (\( j = 1, \ldots, 4 \)). Let
\[
t_4(a, b) = \underbrace{(a \otimes b \otimes b \otimes a \otimes b \otimes a \otimes a \otimes b)}_{4} \in \mathcal{J}^{\otimes 8}.
\]

Note that \( t_4(a, b) \) depends only on \( a \wedge b \in \wedge^2 \mathcal{J} \). By expanding all the terms,
\[
t_4(a, b) = \left[ [1, 2, 1, 2, 1, 2, 1]_{(a,b)} - [1, 2, 1, 2, 1, 2, 1]_{(a,b)} \right]_4
- \left[ [1, 2, 1, 2, 1, 2, 1]_{(a,b)} + [1, 2, 1, 2, 1, 2, 1]_{(a,b)} \right]_4
+ \left[ [1, 2, 1, 2, 1, 2, 1]_{(a,b)} - [1, 2, 1, 2, 1, 2, 1]_{(a,b)} \right]_4
- \left[ [2, 1, 2, 1, 2, 1, 2]_{(a,b)} + [2, 1, 2, 1, 2, 1, 2]_{(a,b)} \right]_4
+ \left[ [2, 1, 2, 1, 2, 1, 2]_{(a,b)} - [2, 1, 2, 1, 2, 1, 2]_{(a,b)} \right]_4
- \left[ [2, 1, 2, 1, 2, 1, 2]_{(a,b)} + [2, 1, 2, 1, 2, 1, 2]_{(a,b)} \right]_4.
\]

For \( v_1, \ldots, v_8, X, Y \in \mathcal{J} \), we put
\[
\Psi_1(v_1 \otimes \cdots \otimes v_8)(X, Y) = D(v_2, v_5, v_7)D(v_4, v_6, v_8)(v_1 \times v_3) \times (X \times Y),
\]
\[
\Psi_2(v_1 \otimes \cdots \otimes v_8)(X, Y) = D(v_2, v_5, v_7)D(v_4, v_6, v_8)(D(v_1, v_3, X)Y + D(v_1, v_3, Y)X).
\]
Then $\Psi_i$ induces a $k$-linear map $\mathcal{J}^8 \to \text{Hom}_k(\mathcal{J} \otimes \mathcal{J}, \mathcal{J})$ for $i = 1, 2$. We define

$$\Phi_{i,(a,b)}(X,Y) = \Psi_i(t_4(a,b))(X,Y)$$

for $i = 1, 2$. Then

$$\Phi_i : V \to (\wedge \mathcal{J}^2)^{4\otimes} \to \text{Hom}_k(\mathcal{J} \otimes \mathcal{J}, \mathcal{J})$$

is a $k$-linear map defined by degree 8 polynomials of $(a,b)$.

**Lemma 2.11.** $\Phi_{i,gx}(g_1X, g_1Y) = c(g_1)^3 \det(g_2)^4 g_1(\Phi_{i,x}(X,Y))$ for $i = 1, 2$ and all $g = (g_1, g_2) \in G$, $X, Y \in \mathcal{J}$.

**Proof.** If $x = (x_1, x_2)$, then $t_4(x_1, x_2)$ depends only on $x_1 \wedge x_2$. If $g_2 \in \text{GL}(2)$ and $y = (y_1, y_2) = g_2 x$, then $y_1 \wedge y_2 = (\det(g_2)) x_1 \wedge x_2$. Since $x_1 \wedge x_2 - x_2 \wedge x_1$ can be identified with $x_1 \wedge x_2$ and it appears in the definition of $t_4(x_1, x_2)$ four times, $t_4(y_1, y_2) = (\det(g_2))^4 t_4(x_1, x_2)$. So we may assume that $g_2 = 1$ and only consider the action of $G_1$.

Suppose that $v_1 \otimes \cdots \otimes v_8$ is a term which appears in the expansion of $t_4(x_1, x_2)$. If $g_1 \in G_1$ and $x$ is replaced by $g_1 x = (g_1 x_1, g_1 x_2)$, then terms which appear in the expansion of $t_4(g_1 x_1, g_1 x_2)$ are in the form $g_1 v_1 \otimes \cdots \otimes g_1 v_8$. By Corollary 2.4,

$$D(g_1 v_2, g_1 v_5, g_1 v_7) D(g_1 v_4, g_1 v_6, g_1 v_8)(g_1 v_1 \times g_1 v_3) \times (g_1 X \times g_1 Y),$$

$$= c(g_1)^3 D(v_2, v_5, v_7) D(v_4, v_6, v_8) g_1 ((v_1 \times v_3) \times (X \times Y)).$$

Also

$$D(g_1 v_2, g_1 v_5, g_1 v_7) D(g_1 v_4, g_1 v_6, g_1 v_8)(D(g_1 v_1, g_1 v_3, g_1 X) g_1 Y + D(g_1 v_1, g_1 v_3, g_1 Y) g_1 X)$$

$$= c(g_1)^3 D(v_2, v_5, v_7) D(v_4, v_6, v_8) g_1 (D(v_1, v_3, X) Y + D(v_1, v_3, Y) X).$$

Therefore, $\Phi_{i,gx}(g_1 X, g_1 Y) = c(g_1)^3 g_1(\Phi_{i,x}(X,Y))$ for $i = 1, 2$. \qed

We first evaluate $\Phi_{1,w}$ (see (1.3)).

**Proposition 2.12.** For all $X, Y \in \mathcal{J}$, $-18 \Phi_{1,w}(X,Y) = X \circ Y - \frac{1}{2} \text{Tr}(X)Y - \frac{1}{2} \text{Tr}(Y)X$.

**Proof.** Suppose that $(a,b) = w = (w_1, w_2)$ in the following. If we expand $t_4(a,b)$, then the coefficient of $[\begin{array}{cccc} i_1 & i_2 & i_3 & i_4 \\ i_5 & i_6 & i_7 & i_8 \end{array}]_{(a,b)}$ is 1 (resp. −1) if the number of $j = 1, \ldots, 4$ such that $(i_2j - 1, i_2j) = (2, 1)$ is even (resp. odd). We list the sign in $t_4(a,b)$, $D(v_2, v_5, v_7) D(v_4, v_6, v_8)$ and $v_1 \times v_3$ in the following table assuming that $[\begin{array}{cccc} i_1 & i_2 & i_3 & i_4 \\ i_5 & i_6 & i_7 & i_8 \end{array}]_{(a,b)}$ is in the form (2.9).

| $(1)$ | $(D(v_2, v_5, v_7) D(v_4, v_6, v_8))_1$ | $v_1 \times v_3$ |
|-------|-----------------------------------------|-----------------|
| $+\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$ | $D(b,a,a)D(b,b,b)$ | $a \times a$ |
| $-\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$ | $D(b,a,b)D(b,b,a)$ | $a \times a$ |
| $-\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \end{bmatrix}$ | $D(b,b,a)D(b,a,a)$ | $a \times a$ |
| $+\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \end{bmatrix}$ | $D(b,b,b)D(b,a,a)$ | $a \times a$ |
| $-\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$ | $D(b,a,b)D(b,a,a)$ | $a \times b$ |
| $+\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$ | $D(b,b,b)D(b,a,a)$ | $a \times b$ |
| $-\begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$ | $D(b,b,a)D(b,a,a)$ | $b \times a$ |
| $+\begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$ | $D(b,a,a)D(b,a,a)$ | $b \times a$ |
| $-\begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$ | $D(a,b,a)D(a,b,a)$ | $b \times b$ |
| $+\begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$ | $D(a,a,a)D(a,a,a)$ | $b \times b$ |
Proof. (2.15) For all $X$, if we put 

$$D(a,a,a) = D(b,b,b) = 0, D(a,a,b) = \frac{1}{3}, D(a,b,b) = -\frac{1}{3}.$$ 

So we can ignore terms with the second column including either $D(a,a,a)$ or $D(b,b,b)$. Removing these terms, we obtain the following table.

| $(1)$ | $9D(v_2, v_5, v_7)D(v_4, v_6, v_8)$ | $e_1 \times e_3$ |
|-------|------------------------------------|------------------|
| $[1,2,1,2,1,2]$ | $1$ | $a \times a$ |
| $[1,2,1,2,1,1,1]$ | $1$ | $a \times a$ |
| $[1,2,2,1,1,1,2]$ | $-1$ | $a \times b$ |
| $[1,2,2,1,1,2,1]$ | $-1$ | $a \times b$ |
| $[1,2,2,1,2,1,2]$ | $-1$ | $b \times a$ |
| $[2,1,2,2,1,1,2]$ | $-1$ | $b \times a$ |
| $[2,1,2,2,1,2,1]$ | $-1$ | $b \times b$ |
| $[2,1,2,1,1,2,1]$ | $1$ | $b \times b$ |
| $[2,1,2,1,2,1,2]$ | $1$ | $b \times b$ |

By the above table,

$$9\Phi_{1,x}(X,Y) = -(2a \times a + a \times b + a \times b + 2b \times b) \times (X \times Y).$$

If we put $c = a + b = \text{diag}(1,0,-1)$, then

$$9\Phi_{1,x}(X,Y) = -(a \times a + b \times c \times c) \times (X \times Y).$$

By (2.7) and (2.8)

$$a \times a = \text{diag}(0,0,-1), b \times b = \text{diag}(-1,0,0), c \times c = \text{diag}(0,-1,0).$$

Therefore, $9\Phi_{1,x}(X,Y) = e \times (X \times Y)$. Since $e$ is the unit element of $J$, replacing $X,Y$ in (2.8) by $e, X \times Y$, we obtain

$$9\Phi_{1,x}(X,Y) = X \times Y - \frac{1}{2}\text{Tr}(e)X \times Y - \frac{1}{2}\text{Tr}(X \times Y)e - \frac{1}{2}\text{Tr}(X \times Y)e + \frac{1}{2}\text{Tr}(e)\text{Tr}(X \times Y)e.$$ 

$$= -\frac{1}{2}X \times Y + \frac{1}{2}\text{Tr}(X \times Y)e.$$ 

By (2.8),

$$\text{Tr}(X \times Y) = \text{Tr}(X \circ Y) - \text{Tr}(X)\text{Tr}(Y) - \frac{3}{2}\text{Tr}(X \circ Y) + \frac{3}{2}\text{Tr}(X)\text{Tr}(Y)$$ 

$$= -\frac{1}{2}\text{Tr}(X \circ Y) + \frac{1}{2}\text{Tr}(X)\text{Tr}(Y).$$

Therefore, again by (2.8),

$$9\Phi_{1,x}(X,Y) = -\frac{1}{2}X \circ Y + \frac{1}{4}\text{Tr}(Y)X + \frac{1}{4}\text{Tr}(X)Y.$$ 

Multiplying $-2$, we obtain the proposition. \hfill \Box

**Proposition 2.14.** For all $X,Y \in J$, $3\Phi_{2,x}(X,Y) = \text{Tr}(Y)X + \text{Tr}(X)Y$.

**Proof.** By very similar computations as in the case of $\Phi_{1,x}(X,Y)$, we obtain

$$3\Phi_{2,x}(X,Y) = -3(D(a,a,X) + D(b,b,X) + D(c,c,X))Y$$ 

$$-3(D(a,a,Y) + D(b,b,Y) + D(c,c,Y))X$$ 

$$= (s_3 + s_1 + s_2)Y + (t_3 + t_1 + t_2)X$$ 

$$= \text{Tr}(Y)X + \text{Tr}(X)Y.$$ 

\hfill \Box
For \((a, b) \in V_k\), we define

\[
S_{(a,b)}(X, Y) = -18\Phi_{1,(a,b)}(X, Y) + \frac{3}{2}\Phi_{2,(a,b)}(X, Y).
\]

The following proposition follows from Lemma 2.11 and (2.13), (2.15).

**Proposition 2.17.** (1) If \(x \in V_k\), \(g = (g_1, g_2) \in G_k\) and \(X, Y \in J\), then

\[
S_{gX}(g_1X, g_1Y) = c(g_1)^3 \det(g_2)^4g_1(S_x(X, Y)).
\]

(2) \(S_w(X, Y) = X \circ Y\).

Let \(\Delta(x)\) be the relaive invariant polynomial of degree 12 and \(\Delta(w) = 1\) which we defined in Introduction. For \(x \in V_k^s\), we define a \(k\)-algebra structure \(\circ_x\) on \(J\) by

\[
X \circ_x Y \overset{\text{def}}{=} \Delta(x)^{-1}S_x(X, Y).
\]

We denote this \(k\)-algebra on the underlying vector space \(J\) by \(M_x\). Proposition 2.17 (2) implies that \(M_w\) is isomorphic to \(J\) (the original Jordan algebra structure).

For \(g = (g_1, g_2) \in G\), we define an element of \(G_1\) by

\[
\mu_g = c(g_1) \det(g_2)^2g_1 \in G_1.
\]

Note that the map \(g \to \mu_g\) is a homomorphism.

**Theorem 2.20.** If \(x, y \in V_k^s\), \(g \in G_{w^{\text{sep}}}\) and \(y = gx\), then \(\mu_g : M_{x^{\text{sep}}} \to M_{y^{\text{sep}}}\) is an isomorphism of \(k\)-algebras

**Proof.** Let \(X, Y \in J_{w^{\text{sep}}} = M_{x^{\text{sep}}}\). Then

\[
\mu_g(X) \circ_y \mu_g(Y) = \Delta(gx)^{-1}S_{gx}(c(g_1) \det(g_2)^2g_1(X), c(g_1) \det(g_2)^2g_1(Y))
\]

\[
= \Delta(gx)^{-1}c(g_1)^2 \det(g_2)^4S_{gx}(g_1(X), g_1(Y))
\]

\[
= \Delta(gx)^{-1}c(g_1)^2 \det(g_2)^4c(g_1)^3 \det(g_2)^4g_1(S_x(X, Y)) \text{ by Proposition 2.17}
\]

\[
= \Delta(gx)^{-1}c(g_1)^4 \det(g_2)^6c(g_1) \det(g_2)^2g_1(\Delta(x)^{-1}S_x(X, Y))
\]

\[
= c(g_1) \det(g_2)^2g_1(X \circ_x Y) = \mu_g(X \circ_x Y).
\]

Therefore, \(\mu_g\) is a homomorphism. Since \(\mu_g\) is obviously bijective, it is an isomorphism.

The above theorem implies that if \(g_x = (g_1, g_2) \in G_{w^{\text{sep}}}\) and \(x = gx, w \in V_k^s\), then \(\mu_g^{-1}(M_x) \subset J_{w^{\text{sep}}} = M_{w^{\text{sep}}}\) is a \(k\)-form of \(J\). If \((M, n) \in \text{JIC}(k)\) is the pair corresponding to \(x\) (see [3, Section 4.5]), \(M\) was characterized in the same manner (see [3, (4.9)]). Therefore, Theorem 1.4 follows.

3. **Equivariant Map II**

In this section, we prove Theorem 1.5

Let

\[
T_a(X, Y, Z) = 27D(a, a, X)D(a, a, Y)D(a, a, Z) - 24D(a, a, a)D(a \times X, a \times Y, a \times Z)
\]

for \(a, X, Y, Z \in J\). Then the map

\[
T : J \ni a \mapsto T_a \in \text{Hom}_k(J \otimes J \otimes J, k)
\]

is \(k\)-linear.

**Lemma 3.2.** If \(a \in J\), \(g \in G_1\) and \(X, Y, Z \in J\), then

\[
T_{ga}(gX, gY, gZ) = c(g)^3g(T_a(X, Y, Z)).
\]
Proof. There exists $t \in \mathbb{k}$ such that $t^3 = c(g)$. We put $g_1 = t^{-1}g$. Then $g = tg_1$ and $g_1 \in H_\mathbb{F}$. So,
\[
D(ga \times gX, ga \times gY, ga \times gZ) = t^6D(g_1a \times g_1X, g_1a \times g_1Y, g_1a \times g_1Z)
\]
\[
= t^6D(g_1(a \times X), g_1(a \times Y), g_1(a \times Z))
\]
\[
= t^6D(a \times X, a \times Y, a \times Z)
\]
\[
= c(g)^2D(a \times X, a \times Y, a \times Z).
\]
Since $D(ga, ga, gX) = c(g)D(a, a, X)$, etc., the lemma follows. □

The following proposition plays a crucial role in proving Theorem 1.5.

**Proposition 3.3.** $T_e(X, Y, Z) = \text{Tr}((X \circ Y) \circ Z) = \text{Tr}(X \circ (Y \circ Z))$ for all $X, Y, Z \in \mathcal{J}$.

**Proof.** Since $D(e, e, e) = 1$,
\[
T_e(X, Y, Z) = 27D(e, e, X)D(e, e, Y)D(e, e, Z) - 24D(e \times X, e \times Y, e \times Z).
\]
Let
\[
X = \left(\begin{array}{ccc}
    s_1 & x_3 & \overline{x_2} \\
    x_2 & s_2 & x_1 \\
    \overline{x_1} & x_1 & s_3
\end{array}\right), \quad Y = \left(\begin{array}{ccc}
    t_1 & y_3 & \overline{y_2} \\
    y_2 & t_2 & y_1 \\
    \overline{y_1} & y_1 & t_3
\end{array}\right), \quad Z = \left(\begin{array}{ccc}
    u_1 & z_3 & \overline{z_2} \\
    z_2 & u_2 & z_1 \\
    \overline{z_1} & z_1 & u_3
\end{array}\right) \in \mathcal{J}.
\]
Note that
\[
6D(X, Y, Z) = \sum_{\{i, j, k\} = \{1, 2, 3\}} s_it_ju_k + \sum_{\{i, j, k\} = \{1, 2, 3\}} \text{tr}(x_iy_jz_k)
\]
\[
- \sum_i s_i \text{tr}(y_i\overline{z_i}) - \sum_i t_i \text{tr}(x_i\overline{z_i}) - \sum_i u_i \text{tr}(x_iy_i\overline{t_i}).
\]
By (2.7) and (2.8),
\[
e \times X = \frac{1}{2} \left(\begin{array}{ccc}
    -(s_2 + s_3) & x_3 & \overline{x_2} \\
    x_2 & -(s_1 + s_3) & x_1 \\
    \overline{x_1} & x_1 & -(s_1 + s_2)
\end{array}\right)
\]
Therefore,
\[
48D(e \times X, e \times Y, e \times Z) = [(s_2 + s_3)(t_1 + t_3)(u_1 + u_2) + (s_2 + s_3)(t_1 + t_2)(u_1 + u_3)
\]
\[
+ (s_1 + s_3)(t_2 + t_3)(u_1 + u_2) + (s_1 + s_3)(t_1 + t_2)(u_2 + u_3)
\]
\[
+ (s_1 + s_2)(t_2 + t_3)(u_1 + u_3) + (s_1 + s_2)(t_1 + t_3)(u_2 + u_3)]
\]
\[
- \sum_{\{i, j, k\} = \{1, 2, 3\}} \text{tr}(x_iy_jz_k)
\]
\[
- (s_2 + s_3) \text{tr}(y_1\overline{z_1}) - (s_1 + s_3) \text{tr}(y_2\overline{z_2}) - (s_1 + s_2) \text{tr}(y_3\overline{z_3})
\]
\[
- (t_2 + t_3) \text{tr}(x_1\overline{z_1}) - (t_1 + t_3) \text{tr}(x_2\overline{z_2}) - (t_1 + t_2) \text{tr}(x_3\overline{z_3})
\]
\[
- (u_2 + u_3) \text{tr}(x_1y_1\overline{t_1}) + (u_1 + u_3) \text{tr}(x_2y_2\overline{t_2}) + (u_1 + u_2) \text{tr}(x_3y_3\overline{t_3})
\]
\[
= 2 \sum_{\{i, j, k\} = \{1, 2, 3\}} s_it_ju_k + 2(s_1t_1u_2 + \cdots) - \sum_{\{i, j, k\} = \{1, 2, 3\}} \text{tr}(x_iy_jz_k)
\]
\[
- \sum_{i \neq j} s_i \text{tr}(y_i\overline{z_j}) - \sum_{i \neq j} t_i \text{tr}(x_i\overline{z_j}) - \sum_{i \neq j} u_i \text{tr}(x_iy_i\overline{t_j}).
\]
Since $6D(e, e, X) = 2(s_1 + s_2 + s_3)$, we have
\[
27D(e, e, X)D(e, e, Y)D(e, e, Z) = (s_1 + s_2 + s_3)(t_1 + t_2 + t_3)(u_1 + u_2 + u_3)
\]
\[
= \sum_i s_it_1u_i + (s_1t_1u_2 + \cdots) + \sum_{\{i, j, k\} = \{1, 2, 3\}} s_it_ju_k.
Therefore,
\[
T_e(X, Y, Z) = \sum_i s_i t_i u_i + \frac{1}{2} \sum_{i,j,k = (1,2,3)} \tr(x_i y_j z_k) \\
+ \frac{1}{2} \sum_{i,j} s_i \tr(y_j z_j) + \frac{1}{2} \sum_{i \neq j} t_i \tr(x_i z_j) + \frac{1}{2} \sum_{i \neq j} u_i \tr(x_i y_j).
\]

(3.4)

Replacing \(X, Y\) in (2.6) by \(X \circ Y, Z\) respectively and taking the sum of diagonal entries, we can express \(4 \tr((X \circ Y) \circ Z)\) in the following manner:
\[
2(2s_1 t_1 + \tr(x_3 y_3 + x_2 y_2)) u_1 \\
+ \tr((s_1 y_3 + x_3 t_2 + \frac{1}{2} x_1 x_3 + y_3 s_2 + \frac{1}{2} y_2 x_1) z_3 + (s_1 y_2 + x_3 y_1 + x_2 t_3 + t_1 x_3 + y_3 x_1 + y_2 s_3) z_2) \\
+ 2(2s_2 t_2 + \tr(x_1 y_3 + y_3 y_3)) u_2 \\
+ \tr((x_3 y_2 + s_2 y_1 + x_1 t_3 + \frac{1}{2} y_3 x_2 + t_2 x_1 + y_1 s_3) z_3 + (s_3 t_1 + s_2 y_2 + x_1 y_3 + y_3 s_1 + t_2 x_3 + y_1 x_2) z_3) \\
+ 2(2s_3 t_3 + \tr(x_2 y_2 + x_1 y_3)) u_3 \\
+ \tr((x_2 t_1 + \frac{1}{2} x_1 y_3 + s_3 y_2 + y_2 s_1 + \frac{1}{2} y_3 x_2 + t_3 x_2) z_3 + (x_2 y_3 + x_1 t_2 + s_2 y_1 + y_2 x_3 + \frac{1}{2} y_1 t_2 + \frac{1}{2} x_3 y_1) z_1).
\]

This coincides with 4 times (3.4).

□

The construction of the isotope defined for \(a \in \mathcal{J}\) (det(\(a\)) \neq 0) is given in [5, p.155]. Let \(Q_a(X, Y) = -6 \det(a) D(X, Y, a) + 9D(X, a, a)D(Y, a, a),\)
\[
\Phi_a(X, Y) = 4 \det(a)^3 (X \times a) \times (Y \times a) + \frac{1}{2} (\det(a)^2 Q_a(X, Y) - Q_a(X, a)Q_a(Y, a)) a.
\]

Then the product structure and the associated bilinear form of the isotope corresponding to \(a \in \mathcal{J}\) is given by
\[
\mathcal{J}^2 \ni (X, Y) \mapsto X \circ_a Y \overset{\text{def}}{=} \det(a)^{-4} \Phi_a(X, Y) \in \mathcal{J},
\]
\[
\mathcal{J}^2 \ni (X, Y) \mapsto \langle X, Y \rangle_a \overset{\text{def}}{=} \det(a)^{-2} Q_a(X, Y) \in k.
\]

(see [5, p.147, Proposition 5.6.2], [5, p.153, Proposition 5.8.2], [5, p.155, Proposition 5.9.2]).

Here we provide an alternative way to define the product structure on \(\mathcal{J}_a\).

**Definition 3.6.** Suppose that \(a \in \mathcal{J}\), det(\(a\)) \neq 0. For \(X, Y \in \mathcal{J}\), we define \(X \circ_a Y \in \mathcal{J}\) to be the element such that \(Q_a(X \circ_a Y, Z) = \det(a)^{-1} T_a(X, Y, Z)\) for all \(Z \in \mathcal{J}\).

Since \(Q_a\) is a non-degenerate bilinear form, the definition of \(X \circ_a Y\) is well-defined.

**Theorem 3.7.** If \(a \in \mathcal{J}\), det(\(a\)) \neq 0, \(g_a \in GE_{k \text{sep}}\), then \(\mathcal{J} \otimes k \text{sep} \ni X \mapsto g(X \circ_a Y) \in \mathcal{J}_a \otimes k \text{sep}\) induces an isomorphism of \(k\)-algebras \(\mathcal{J} \otimes k \text{sep} \cong \mathcal{J}_a \otimes k \text{sep}\).

**Proof.** Suppose that \(X, Y, Z \in \mathcal{J} \otimes k \text{sep}\). Since det(\(a\)) = det(\(e\)) = det(\(g\)),
\[
Q_a(g(X \circ Y), gZ) = c(g)^2 Q_a(X \circ Y, Z) = c(g)^2 T_a(X, Y, Z) = c(g)^{-1} T_a(gX, gY, gZ) = \det(a)^{-1} T_a(gX, gY, gZ) = Q_a(gX \circ_a gY, gZ).
\]

Since this holds for all \(Z \in \mathcal{J} \otimes k \text{sep}\), \(g(X \circ Y) = g(X \circ_a Y)\). Therefore, \(g\) induces an isomorphism of \(k\)-algebras \(\mathcal{J} \otimes k \text{sep} \cong \mathcal{J}_a \otimes k \text{sep}.\)

□

In the situation of Theorem 3.7, \(\mathcal{J}_a\) can be identified with the \(k\)-form \(g^{-1}(\mathcal{J}_a) \subset \mathcal{J}_{k \text{sep}}.\) Therefore, this \(\mathcal{J}_a\) coincides with the isotope \(\mathcal{J}_a\) constructed in [5].

Let \(m : V \to \mathcal{J}\) be the equivariant map in [3, Section 5]. Since \(m\) is defined by homogeneous polynomials of degree 4, \(Q_{m(x)}, T_{m(x)}\) are defined by homogeneous polynomials of degrees 16, 24 respectively. Can we construct lower degree equivariant maps from \(V\) to the space of bilinear forms and trilinear forms on \(\mathcal{J}\) to give an alternative way to define the product structure on \(\mathcal{M}_2\), (see (2.18))? It does not seem so. For example, if there were such a quadratic form \(Q_4\) depending on \(x\), the degree must be \(16 - 12 = 4.\) However, there seems to be only one equivariant map of degree
from $\wedge^2 J$ to the space of bilinear forms on $J$ by the calculation of the software “LiE” ([1]) as follows.

\begin{verbatim}
> alt_tensor(2, [1,0,0,0,0,0])
1X[0,0,1,0,0,0]
> sym_tensor(2, [0,0,1,0,0,0])
1X[0,0,0,0,0,2] +1X[0,0,2,0,0,0] +1X[1,0,0,0,1,0] +1X[1,0,0,0,0,0] +1X[2,0,0,0,0,1]
\end{verbatim}

We can construct an equivariant map from $\wedge^2 J$ to the space of bilinear forms on $J$ such that $w_1 \wedge w_2$ corresponds to the bilinear form

$$J^2 \ni (X, X) \mapsto \sum_{i=1}^{3} s_i^2 - \sum_{i=1}^{3} \|x_i\| \in k$$

where as

$$\text{Tr}(X \circ X) = \sum_{i=1}^{3} s_i^2 + \sum_{i=1}^{3} \|x_i\|.$$ 

Therefore, we cannot obtain $\text{Tr}(X \circ X)$.

**References**

[1] A.M. Cohen, M. van Leeuwen, and B. Lisser. *Lie Version 2.2.2*. 2012.

[2] F.R. Harvey. *Spinors and calibrations*. Academic Press, New York, San Francisco, London, 1990.

[3] R. Kato and A. Yukie. Rational orbits of the space of pairs of exceptional jordan algebras. preprint.

[4] T.A. Springer. *Linear algebraic groups*, volume 9 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, second edition, 1998.

[5] T.A. Springer and F.D. Veldkamp. *Octonions, Jordan algebras and exceptional groups*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2000.

[6] A. Yukie. A remark on the regularity of prehomogeneous vector spaces. arXiv:math/9708209.