INFINITELY MANY SOLUTIONS FOR QUASILINEAR ELLIPTIC EQUATIONS INVOLVING DOUBLE CRITICAL TERMS AND BOUNDRY GEOMETRY

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Abstract. Let $1 < p < N$, $p^* = Np/(N - p)$, $0 < s < p$, $p^*(s) = (N - s)p/(N - p)$, and $\Omega \in C^1$ be a bounded domain in $\mathbb{R}^N$ with $0 \in \bar{\Omega}$. In this paper, we study the following problem

$$
\begin{cases}
-\Delta_p u = \mu|u|^{p^* - 2}u + \frac{|u|^{p^*(s) - 2}u}{|x|^s} + a(x)|u|^{p - 2}u, & \text{in } \Omega, \\
u = 0, & \text{on } \partial \Omega,
\end{cases}
$$

where $\mu \geq 0$ is a constant, $\Delta_p$ is the $p$-Laplacian operator and $a \in C^1(\bar{\Omega})$. By an approximation argument, we prove that if $N > p^2 + p$, $a(0) > 0$ and $\Omega$ satisfies some geometry conditions if $0 \in \partial \Omega$, say, all the principle curvatures of $\partial \Omega$ at 0 are negative, then the above problem has infinitely many solutions.

Keywords: Quasilinear elliptic equations; Double critical terms; Boundary geometry condition; Infinitely many solutions; Approximation argument.

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1. INTRODUCTION AND MAIN RESULTS

Let $1 < p < N$, $p^* = Np/(N - p)$, $0 < s < p$, $p^*(s) = (N - s)p/(N - p)$, and $\Omega \in C^1$ be an open bounded domain in $\mathbb{R}^N$ with $0 \in \bar{\Omega}$. In this paper, we study the following quasilinear elliptic equations

$$
\begin{cases}
-\Delta_p u = \mu|u|^{p^* - 2}u + \frac{|u|^{p^*(s) - 2}u}{|x|^s} + a(x)|u|^{p - 2}u, & \text{in } \Omega, \\
u = 0, & \text{on } \partial \Omega,
\end{cases}
$$

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where $\mu \geq 0$ is a constant,
\[
\Delta_p u = \sum_{i=1}^{N} \partial_{x_i} (|\nabla u|^{p-2} \partial_{x_i} u), \quad \nabla u = (\partial_{x_1} u, \ldots, \partial_{x_N} u)
\]
is the $p$-Laplacian operator and $a \in C^4(\Omega)$.

The functional corresponding to equation (1.1) is
\[
I(u) = \frac{1}{p} \int_{\Omega} (|\nabla u|^p - a(x)|u|^p) \, dx - \mu \int_{\Omega} |u|^p \, dx - \frac{1}{p^*} \int_{\Omega} |u|^{p^*(s)} \, dx,
\]
for $u \in W^{1,p}_0(\Omega)$. All of the integrals in energy functional $I$ are well defined, due to the Sobolev inequality
\[
C \left( \int_{\mathbb{R}^N} |\varphi|^{p^*(s)} \, dx \right)^{\frac{1}{p^*(s)}} \leq \int_{\mathbb{R}^N} |\nabla \varphi|^p \, dx, \quad \forall \varphi \in W^{1,p}_0(\Omega),
\]
for $C = C(N, p) > 0$, and due to the Caffarelli-Kohn-Nirenberg inequality (see [5])
\[
C \left( \int_{\Omega} |\varphi|^{p^*(s)} \, dx \right)^{\frac{1}{p^*(s)}} \leq \int_{\mathbb{R}^N} |\nabla \varphi|^p \, dx, \quad \forall \varphi \in W^{1,p}_0(\Omega),
\]
for $C = C(N, p, s) > 0$.

Since the pioneer work of Brézis and Nirenberg [4], there are enormous results on semilinear problems e.g. [2, 3, 9, 10, 14, 16, 28] and on quasilinear problems e.g. [6, 12, 13, 15, 17, 20, 21, 25, 31] with Sobolev exponents.

Without the presence of the Hardy term $|x|^{-s}|u|^{p^*(s)-2}u$ in equation (1.1), Devillanova and Solimini [16] considered equation (1.1) in the semilinear case ($p = 2$). With the assumptions that $\mu > 0$ and $a \equiv \lambda$ in $\Omega$ for some constant $\lambda > 0$, they proved the existence of infinitely many solutions to equation (1.1) if $N > 6$. Then Cao, Peng and Yan [6] generalized their result to the quasilinear case, that is, $1 < p < N$. Under the same assumptions on $\mu$ and $a$ as that of Devillanova and Solimini [16], they proved the existence of infinitely many solutions to equation (1.1) if $N > p^2 + p$.

In the presence of the Hardy term $|x|^{-s}|u|^{p^*(s)-2}u$ in equation (1.1), Yan and Yang [30] considered equation (1.1) in the semilinear case. Under the assumption that $a(0) > 0$ and the following geometry assumption imposed on $\Omega$: $\Omega \in C^3$ and
\[
\text{all the principle curvatures of } \partial \Omega \text{ at 0 are negative if } 0 \in \partial \Omega,
\]
they proved the existence of infinitely many solutions for equation (1.1) if $N > 6$.

So a natural problem is whether in the quasilinear case equation (1.1) has infinitely many solutions. The functional $I$ defined by (1.2) does not satisfy the Palais-Smale condition at large energy level. So it is impossible to apply the mountain pass lemma [1] directly to obtain the existence of infinitely many solutions for equation (1.1). In this paper, we follow the idea of Devillanova and Solimini [16] to study the following perturbed problem:
\[
\left\{ \begin{array}{ll}
-\Delta_p u = \mu|u|^{p^*-2}\epsilon u + \frac{|u|^{p^*(s)-2}u}{|x|^s} + a(x)|u|^{p-2}u, & \text{in } \Omega, \\
u = 0, & \text{on } \partial \Omega,
\end{array} \right.
\]
where $\epsilon > 0$ is a small constant. See also [6, 8, 30] for applications of the same idea. The functional corresponding to equation (1.4) is
\[
I_\epsilon(u) = \frac{1}{p} \int_{\Omega} (|\nabla u|^p - a(x)|u|^p) \, dx - \mu \int_{\Omega} |u|^{p^*-\epsilon} \, dx - \frac{1}{p^*} \int_{\Omega} |u|^{p^*(s)-\epsilon} \, dx,
\]
for $u \in W^{1,p}_0(\Omega)$. 


Now $I_\epsilon$ is an even functional and satisfies the Palais-Smale condition in all energy levels. It follows from the symmetric mountain pass lemma [1, 26] that equation (1.4) has infinitely many solutions. See also [20, 27]. Precisely, for $\epsilon > 0$ fixed, there are positive numbers $c_{\epsilon,l}$ and critical points $u_{\epsilon,l}$, $l = 1, 2, \cdots$, such that

$$I(u_{\epsilon,l}) = c_{\epsilon,l} \to \infty, \quad \text{as } l \to \infty.$$ 

Moreover, for each $l \geq 1$ fixed, the sequence $\{c_{\epsilon,l}\}_{\epsilon > 0}$ is bounded with respect to $\epsilon$ and thus can be assumed to converge to a limit $c_l$ as $\epsilon \to 0$.

To obtain the existence of infinitely many solutions for equation (1.1), the first step is to investigate whether $u_{\epsilon,l}$ converges strongly in $W^{1,p}_0(\Omega)$ as $\epsilon \to 0$. That is, we need to study the compactness of the set of solutions for equation (1.4) for all $\epsilon > 0$ small. If $u_{\epsilon,l}$ is proved to converge to some $u_l \in W^{1,p}_0(\Omega)$ strongly in $W^{1,p}_0(\Omega)$, then the next step is to investigate whether $c_l \to \infty$ as $l \to \infty$. If so, then we obtain infinitely many solutions for equation (1.1) with arbitrarily large energy level.

Throughout the paper, we use $\| \cdot \|$ to denote the norm of $W^{1,p}_0(\Omega)$. We assume that $\Omega \in C^1$ satisfies the following condition:

$$x \cdot \nu \leq 0 \text{ in a neighborhood of } 0 \text{ in } \partial \Omega \text{ if } 0 \in \partial \Omega,$$  

(1.6)

where $\nu$ is the outward unit normal of $\partial \Omega$. Our main result in this paper is the following theorem.

**Theorem 1.1.** Suppose that $a(0) > 0$ and $\Omega \in C^1$ satisfies the condition (1.6). If $N > p^2 + p$, then for any $u_n$ ($n = 1, 2, \cdots$), which is a solution to equation (1.4) with $\epsilon = \epsilon_n \to 0$, satisfying $\|u_n\| \leq C$ for some constant $C$ independent of $n$, $u_n$ converges strongly in $W^{1,p}_0(\Omega)$ up to a subsequence as $n \to \infty$.

As an application of Theorem 1.1, we have the following existence result for equation (1.1).

**Theorem 1.2.** Suppose that $a(0) > 0$ and $\Omega \in C^1$ satisfies the condition (1.6). If $N > p^2 + p$, then equation (1.1) has infinitely many solutions.
which implies that
\[ x \cdot \nu(x) \leq 0 \quad \text{for } x \in \partial \Omega \cap B_{\delta'}(0), \]
for some \( 0 < \delta' < \delta \). That is, (1.6) is satisfied. So we find that condition (1.3) is a special case of condition (1.6).

On the other hand, condition (1.6) does allow more possibilities than that of condition (1.3). As an example, suppose that \( \partial \Omega \) has a piece of concave boundary close to 0 if \( 0 \in \partial \Omega \). Precisely, let \( \varphi \in C^1 \) be given such that (1.7) holds, and
\[ 0 = \varphi(0) \leq \varphi(x') + \sum_{j=1}^{N-1} \partial_{x_j} \varphi(x')(0 - x'_j) \]
for \( x' \) close to 0. Then we have
\[ x \cdot \nu(x) = \frac{\varphi(x') + \sum_{j=1}^{N-1} \partial_{x_j} \varphi(x')(0 - x'_j)}{\sqrt{1 + \sum_{j=1}^{N} |\partial_{x_j} \varphi(x')|^2}} \leq 0 \]
for \( x' \) close to 0. That is, (1.6) is satisfied. In particular, if \( \Omega \) has a piece of flat boundary in a neighborhood of 0 when \( 0 \in \partial \Omega \), then all the principle curvatures of \( \partial \Omega \) vanish at the point 0. So in this case (1.6) is satisfied while (1.3) is not satisfied.

Finally, we point out that in the case when \( 0 \in \partial \Omega \), the mean curvature of \( \partial \Omega \) at 0 plays an important role in the existence of the mountain pass solutions to equation (1.1). See for example \([11, 18, 19, 22, 23]\).

Our paper is organized as follows. In Section 2 we obtain some integral estimates. In Section 3 we obtain estimates for solutions of equation (1.4) in the region which is close to but is suitably away from the blow up point. We prove Theorems 1.1 and 1.2 in Section 4. In order to give a clear line of our framework, we will list some necessary estimates on solutions of quasilinear equation with Hardy potential in Appendix A, a decay estimate for critical Sobolev growth equation in Appendix B, some estimates on solutions of \( p \)-Laplacian equation by Wolff potential in Appendix C, and a global compactness result for the solution \( u_n \) of equation (1.4) in Appendix D, respectively.

Our notations are standard. \( B_R(x) \) is the open ball in \( \mathbb{R}^N \) centered at \( x \) with radius \( R \). We write
\[ \int_E u \, dx = \frac{1}{|E|} \int_E u \, dx, \]
whenever \( E \) is a measurable set with \( 0 < |E| < \infty \), the \( n \)-dimensional Lebesgue measure of \( E \). Let \( D \) be an arbitrary domain in \( \mathbb{R}^N \). We denote by \( C_0^\infty(D) \) the space of smooth functions with compact support in \( D \). For any \( 1 \leq r \leq \infty \), \( L^r(D) \) is the Banach space of Lebesgue measurable functions \( u \) such that the norm
\[ ||u||_{r,D} = \begin{cases} \left( \int_D |u|^r \right)^{\frac{1}{r}}, & \text{if } 1 \leq r < \infty \\ \text{esssup}_D |u|, & \text{if } r = \infty \end{cases} \]
is finite. The local space \( L_{loc}^r(D) \) consists of functions belonging to \( L^r(D') \) for all \( D' \subset \subset D \). We also denote \( d\mu_s = |x|^{-s} \, dx \) and \( ||v||_{q,\mu_s} = \left( \int |v|^q \, d\mu_s \right)^{1/q} \) when there is no confusion on the domain of the integral. A function \( u \) belongs to the Sobolev space \( W^{1,r}(D) \) if \( u \in L^r(D) \) and its first order weak partial derivatives also belong to \( L^r(D) \). We endow \( W^{1,r}(D) \) with the norm
\[ ||u||_{1,r,D} = ||u||_{r,D} + ||\nabla u||_{1,D}. \]
The local space \( W_{loc}^{1,r}(D) \) consists of functions belonging to \( W^{1,r}(D') \) for all open \( D' \subset \subset D \). We recall that \( W_{0}^{1,r}(D) \) is the completion of \( C_0^\infty(D) \) in the norm \( || \cdot ||_{1,r,D} \). For the properties of the Sobolev functions, we refer to the monograph \([32]\).
2. Integral estimates

Let \( u_n, n = 1, 2, \ldots, \) be a solution of equation (1.4) with \( \epsilon = \epsilon_n \to 0, \) satisfying \( \|u_n\| \leq C \) for some constant \( C \) independent of \( n. \) In this section we deduce some integral estimates for \( u_n. \) For any function \( u, \) we define

\[
\rho_{x,\lambda}(u) = \lambda^{\frac{N-p}{p}} u(\lambda \cdot - x))
\]

for any \( \lambda > 0 \) and \( x \in \mathbb{R}^N. \) By Proposition D.1, \( u_n \) can be decomposed as

\[
u_n = u_0 + \sum_{j=1}^{m} \rho_{x_{n,j},\lambda_{n,j}}(U_j) + \omega_n.
\]

Here \( x_{n,j} = 0 \) for \( j = k + 1, \ldots, m. \)

To prove that \( u_n \) strongly converges in \( W_0^{1,p}(\Omega), \) we only need to show that the bubbles \( \rho_{x_{n,j},\lambda_{n,j}}(U_j) \) will not appear in the decomposition of \( u_n. \) Among all the bubbles, we can choose one bubble such that this bubble has the slowest concentration rate. That is, the corresponding \( \lambda \) is the lowest order infinity among all the \( \lambda \) appearing in the bubbles. For simplicity, we denote by \( \lambda_n \) the slowest concentration rate and by \( x_n \) the corresponding concentration point.

For any \( q > 1, \) denote

\[
\|u\|_{*,q} = \left( \int_{\Omega} |u|^q dx \right)^{\frac{1}{q}} + \left( \int_{\Omega} \left| u \right|^{\frac{q(N-q)}{N-q}} d\mu_s \right)^{\frac{N}{N-q}}
\]

and \( q' = \frac{q}{q-1}. \) Recall that \( d\mu_s = |x|^{-s} dx. \)

For any \( p^*/p' < p_2 < p^* < p_1, \) \( \alpha > 0 \) and \( \lambda \geq 1, \) we consider the following relation:

\[
\begin{cases}
\|u_1\|_{*,p_1} \leq \alpha, \\
\|u_2\|_{*,p_2} \leq \alpha \lambda^{\frac{N}{p_2} - \frac{N}{p_1}}.
\end{cases}
\]

(2.2)

and define

\[
\|u\|_{*,p_1,p_2,\lambda} = \inf \alpha,
\]

(2.3)

where the infimum is taken over all \( \alpha > 0 \) for which there exist \( u_1, u_2 \) such that \( |u| \leq u_1 + u_2 \) and (2.2) holds. Our main result in this section is the following estimate.

**Proposition 2.1.** Let \( u_n, n = 1, 2, \ldots, \) be a solution of equation (1.4) with \( \epsilon = \epsilon_n \to 0, \) satisfying \( \|u_n\| \leq C \) for some positive constant \( C \) independent of \( n. \) Then for any \( p_1, p_2 \in (p^*/p', \infty), \)

\( p_2 < p^* < p_1, \) there exists a constant \( C = C(p_1, p_2) > 0, \) independent of \( n, \) such that

\[
\|u_n\|_{*,p_1,p_2,\lambda} \leq C
\]

for all \( n. \) Here \( \lambda_n \) is the slowest concentration rate of \( u_n. \)

Several lemmas are needed to prove Proposition 2.1. In the rest of this section, let us fix a bounded domain \( D \) with \( \Omega \subset \subset D \) and define \( r = \frac{1}{3} \text{dist}(\Omega, \partial D). \)

**Lemma 2.2.** Let \( w \in W_0^{1,p}(D), w \geq 0, \) be the solution of

\[
\begin{cases}
-\Delta_p w = (a_1(x) + \frac{a_2(x)}{|x|^p}) v^{p-1}, & \text{in } D, \\
w = 0, & \text{on } \partial D,
\end{cases}
\]

(2.4)

where \( a_1, a_2, v \geq 0 \) are bounded functions in \( D. \) Then for any \( \frac{p}{p'} < p_2 < p^* < p_1, \) there is a constant \( C = C(p_1, p_2) > 0, \) such that for any \( \lambda \geq 1, \)

\[
\|w\|_{*,p_1,p_2,\lambda} \leq C \left( \|a_1\|_{L^\infty} + \|a_2\|_{L^\infty} \right)^{\frac{1}{p'}} \|v\|_{*,p_1,p_2,\lambda}.
\]

(2.5)
Proof. Let \( \alpha > \|v\|_{s,p_1,p_2,\lambda} \) be an arbitrary constant. Then by the definition of \( \|v\|_{s,p_1,p_2,\lambda} \), there exist \( v_1, v_2 \) such that \( \|v\| = v_1 + v_2 \) and (2.2) holds with \( u_i = v_i, \ i = 1, 2 \).

Let \( w_i \in W_0^{1,p}(D) \), \( w_i \geq 0, \ i = 1, 2 \), be the solution of equation (2.4) with \( v = 2v_1 \). Then Corollary A.2 implies that

\[
\|w_i\|_{s,p_i} \leq C \left( \|a_1\| \frac{\|v\|}{p} + \|a_2\| \frac{\|v\|}{p_2 - \mu_s} \right)^{\frac{1}{p - 1}} \|v_1\|_{s,p_1}. \tag{2.6}
\]

Let \( \tilde{w} \in W_0^{1,p}(D) \), \( \tilde{w} \geq 0 \), be the solution of equation

\[
\begin{cases}
-\Delta_p w = (a_1(x) + \frac{a_2(x)}{|x|^2}) \left((2v_1)^{p-1} + (2v_2)^{p-1}\right), & \text{in } D, \\
w = 0, & \text{on } \partial D.
\end{cases}
\]

Applying Corollary A.2 gives us

\[
\|\tilde{w}\|_{s,p_2} \leq C \left( \|a_1\| \frac{\|v\|}{p} + \|a_2\| \frac{\|v\|}{p_2 - \mu_s} \right)^{\frac{1}{p - 1}} \left( \|v_1\|_{s,p_2} + \|v_2\|_{s,p_2} \right).
\]

Thus for any \( x \in \Omega \), we have

\[
\inf_{B_r(x)} \tilde{w} \leq \left( \int_{B_r(x)} \tilde{w}^{p_2} dy \right)^{\frac{1}{p_2}} \leq C \left( \|a_1\| \frac{\|v\|}{p} + \|a_2\| \frac{\|v\|}{p_2 - \mu_s} \right)^{\frac{1}{p - 1}} \alpha. \tag{2.7}
\]

Note that \( v^{p-1} \leq ((2v_1)^{p-1} + (2v_2)^{p-1}) \). Thus \( w \leq \tilde{w} \) by comparison principle. Applying Proposition C.1 gives us

\[
w(x) \leq \tilde{w}(x) \leq C \inf_{B_r(x)} \tilde{w} + Cw_1(x) + Cw_2(x), \quad \forall x \in \Omega.
\]

Let \( \tilde{w}_1(x) = C \inf_{B_r(x)} \tilde{w} + Cw_1(x) \) and \( \tilde{w}_2(x) = Cw_2(x) \) for \( x \in \Omega \). Then \( w \leq \tilde{w}_1 + \tilde{w}_2 \) in \( \Omega \). By (2.6) and (2.7), we have that

\[
\|\tilde{w}_1\|_{s,p_1} \leq C \left( \|a_1\| \frac{\|v\|}{p} + \|a_2\| \frac{\|v\|}{p_2 - \mu_s} \right)^{\frac{1}{p - 1}} \alpha,
\]

and that

\[
\|\tilde{w}_2\|_{s,p_2} \leq C \left( \|a_1\| \frac{\|v\|}{p} + \|a_2\| \frac{\|v\|}{p_2 - \mu_s} \right)^{\frac{1}{p - 1}} \lambda^{\frac{N}{p} - \frac{N}{p_2}} \alpha.
\]

Hence by definition (2.3), we obtain that

\[
\|w\|_{s,p_1,p_2,\lambda} \leq C \left( \|a_1\| \frac{\|v\|}{p} + \|a_2\| \frac{\|v\|}{p_2 - \mu_s} \right)^{\frac{1}{p - 1}} \alpha.
\]

Since \( \alpha > \|v\|_{s,p_1,p_2,\lambda} \) is arbitrary, we get (2.5). This finishes the proof. \( \Box \)

We also have the following result which will be used in the proof of Proposition 2.1.

Lemma 2.3. Let \( w \in W_0^{1,p}(D) \), \( w \geq 0 \), be the solution of

\[
\begin{cases}
-\Delta_p w = 2\mu v^{p^*-1} + 2\frac{v^{p^*(\gamma)^{-1}}}{|x|^\gamma} + \frac{A}{|x|^\gamma}, & \text{in } D, \\
w = 0, & \text{on } \partial D, \tag{2.8}
\end{cases}
\]

where \( v \geq 0 \) is a bounded function and \( A \geq 0 \) is a constant. Then for any \( p_1, p_2 \in (p^* - 1, \frac{N}{p}(p^* - 1)) \), \( p_2 < p^* < p_1 \), and for any \( \lambda \geq 1 \), there exists a constant \( C = C(p_1, p_2) > 0 \), such that

\[
\|w\|_{s,q_1,q_2,\lambda} \leq C \|v\|_{s,q_1,q_2,\lambda}^{\frac{p^*-1}{p^*}} + C, \tag{2.9}
\]
where \( q_1, q_2 \) are given by
\[
q_1 = \frac{(p-1)N\hat{p}_1}{N - pp_1} \quad \text{with} \quad \hat{p}_1 = \frac{Np_1}{(p^*(s)-1)N + sp_1},
\]
and
\[
q_2 = \frac{(p-1)N\hat{p}_2}{N - pp_2} \quad \text{with} \quad \hat{p}_2 = \frac{p_2}{p^*-1}.
\]

Proof. Let \( \alpha > \|v\|_{s,p_1,p_2,\lambda} \) be an arbitrary constant. Then by the definition of \( \|v\|_{s,p_1,p_2,\lambda} \), there exist \( v_1, v_2 \) such that \( |v| \leq v_1 + v_2 \) and (2.2) holds with \( u_i = v_i, \ i = 1, 2 \).

Let \( w_1 \in W^{1,p}_0(D), w_1 \geq 0 \), be the solution of equation (2.8) with \( v = 2v_1 \). Let
\[
\hat{p}_1 = \min \left\{ \frac{p_1}{p^*-1}, \frac{Np_1}{(p^*(s)-1)N + sp_1} \right\}.
\]
By our assumptions on the parameters \( N, p, s \) and \( p_1, \) we get
\[
\hat{p}_1 = \frac{Np_1}{(p^*(s)-1)N + sp_1} \in \left( 1, \frac{N}{p} \right)
\]
and \( (p^*-1)\hat{p}_1 \leq p_1 \). Applying Proposition A.1 gives us
\[
\|w_1\|_{s,q_1} \leq C \left( \|v_1^p - 1\|_{\hat{p}_1} + \|v_1^p(s)^{-1} - 1\|_{\hat{p}_1} + A\right)
\]
where \( q_1 = (p-1)N\hat{p}_1/(N - pp_1) \).

Similarly, let \( w_2 \in W^{1,p}_0(D), w_2 \geq 0 \), be the solution of equation
\[
\left\{ \begin{array}{ll}
-\Delta_p w = 2\mu w^{p^*-1} + \frac{2\mu w^{p(s)^{-1}}}{|x|}, & \text{in } D, \\
w = 0, & \text{on } \partial D.
\end{array} \right.
\]
Let
\[
\hat{p}_2 = \min \left\{ \frac{p_2}{p^*-1}, \frac{Np_2}{(p^*(s)-1)N + sp_2} \right\}.
\]
Then
\[
\hat{p}_2 = \frac{p_2}{p^*-1} \in \left( 1, \frac{N}{p} \right)
\]
and \( (N-s)\hat{p}_2 \leq \frac{(N-s)p_2}{N} \). Applying Proposition A.1 as above, we obtain that
\[
\|w_2\|_{s,q_2} \leq C \alpha^{\frac{q_2}{p^*-1}} + C,
\]
where \( q_2 = (p-1)N\hat{p}_2/(N - pp_2) \). To obtain the above estimate, we used the equality
\[
\left( \frac{N}{p^*} \right)^\frac{N}{p^* - 1} \frac{N}{p^* - 1} \frac{N}{p^* - q_2}.
\]
Let \( \tilde{w} \in W^{1,p}_0(D), \tilde{w} \geq 0 \), be the solution of equation
\[
\left\{ \begin{array}{ll}
-\Delta_p \tilde{w} = 2\mu \left( (2v_1)^{p^*-1} + (2v_2)^{p^*-1} \right) + \frac{A(2v_1)^{p(s)^{-1}} + (2v_2)^{p(s)^{-1}}}{|x|^s}, & \text{in } D, \\
\tilde{w} = 0, & \text{on } \partial D.
\end{array} \right.
\]
Estimating as above gives that
\[
\|\tilde{w}\|_{s,q_2} \leq C \alpha^{\frac{q_2}{p^*-1}} + C,
\]
which implies that
\[
\inf_{B_r(x)} \bar{w} \leq \left( \frac{\int_{B_r(x)} \bar{w} \rho^2 \, dy}{\rho^2} \right)^{\frac{1}{2}} \leq C \alpha^{\frac{p^*-1}{p^*-2}} + C, \quad \forall x \in \Omega.
\]

Note that \( w \leq \bar{w} \) in \( \Omega \). Applying Proposition C.1 and arguing as that of Lemma 2.2, we prove Lemma 2.3. This completes the proof. \( \square \)

Now define \( u_n = 0 \) in \( D \setminus \Omega \). It is easy to see that
\[
|\mu| |u|^{p^*-2} u + \frac{|u|^{p^*(s)-2} - 2|u|^{p^*-2} u}{|x|^s} + a(x)|u|^{p^*-2} u \leq 2|u|^{p^*-1} + 2|u|^{p^*(s)-1} + A
\]
for sufficiently large constant \( A > 0 \). Let \( w_n \in W^{1,p}_0(D) \), \( w \geq 0 \), be the solution of equation
\[
\begin{cases}
-\Delta_p w = 2\mu|u_n|^{p^*-1} + \frac{2|u_n|^{p^*(s)-1}}{|x|^s} + \frac{A}{|x|^s}, & \text{in } D, \\
w = 0, & \text{on } \partial D.
\end{cases}
\tag{2.10}
\]

Then by comparison principle,
\[
|u_n| \leq w_n \quad \text{in } \Omega.
\tag{2.11}
\]

Moreover, since \( ||u_n|| \leq C \), it is easy to obtain from equation (2.10) that
\[
\|w_n\|_{p^*} + \|w_n\|_{p^*(s),p} \leq C
\tag{2.12}
\]
for some \( C > 0 \) independent of \( n \).

To prove Proposition 2.1, it is enough to prove the estimate of Proposition 2.1 for \( w_n \). We have the following result which shows that Proposition 2.1 holds for \( w_n \) for some \( p_1, p_2 \in (p^*/p', \infty) \), \( p_2 < p^* < p_1 \).

**Lemma 2.4.** There exist \( p_1, p_2 \in (p^*/p', \infty) \), \( p_2 < p^* < p_1 \), and constant \( C = C(p_1, p_2) > 0 \), independent of \( n \), such that
\[
\|w_n\|_{\ast, p_1, p_2, \lambda_n} \leq C.
\tag{2.13}
\]

**Proof.** By Proposition D.1, \( u_n \) can be decomposed as
\[
u_n = u_0 + \sum_{j=1}^k \rho_{x_{n,j}, \lambda_n,j} (U_j) + \sum_{j=k+1}^m \rho_{0, \lambda_n,j} (U_j) + \omega_n.
\]

Write \( x_{n,j} = 0 \) for \( j = k+1, \ldots, m \). In the following proof, we denote
\[
u_{n,0} = u_0, \quad u_{n,1} = \sum_{j=1}^m \rho_{x_{n,j}, \lambda_n,j} (U_j), \quad \text{and } u_{n,2} = \omega_n.
\]

By (2.11), we have
\[
2\mu|u_n|^{p^*-1} + \frac{2|u_n|^{p^*(s)-1}}{|x|^s} + \frac{A}{|x|^s} \leq C \sum_{i=0}^2 \left( |u_{n,i}|^{p^*-p} + \frac{|u_{n,i}|^{p^*(s)-p}}{|x|^s} \right) w_n^{p-1} + \frac{A}{|x|^s}.
\]

Let \( \bar{w}_n \in W^{1,p}_0(D) \), \( \bar{w}_n \geq 0 \), be the solution of equation
\[
\begin{cases}
-\Delta_p w = C \sum_{i=0}^2 \left( |u_{n,i}|^{p^*-p} + \frac{|u_{n,i}|^{p^*(s)-p}}{|x|^s} \right) w_n^{p-1} + \frac{A}{|x|^s}, & \text{in } D, \\
w = 0, & \text{on } \partial D.
\end{cases}
\tag{2.14}
\]

Comparison principle implies that
\[
w_n \leq \bar{w}_n, \quad \text{in } D.
\]
By (2.12), it is easy to derive that
\[
\|\tilde{w}_n\|_{p^*} + \|\tilde{w}_n\|_{p^*(s),\mu_s} \leq C. \tag{2.15}
\]
Thus we have
\[
\inf_{\tilde{B}_i(x)} \tilde{w}_n \leq C, \quad \forall x \in \Omega. \tag{2.16}
\]

Now let \(w_i \in W_0^1, p(D), w_i \geq 0, i = 0, 1, 2,\) be the solution of equation
\[
\begin{cases}
-\Delta_p w = C \left( |u_{n,i}|^{p^*} - p + \frac{|u_{n,i}|^{p^*(s)} - p}{|x|^s} \right) w_n^{p-1} + A\delta_0, & \text{in } D, \\
\quad w = 0, & \text{on } \partial D,
\end{cases}
\]
where \(\delta_0 = 1\) and \(\delta_{10} = \delta_{20} = 0.\)

Then by Proposition C.1 and (2.16), we obtain that
\[
\tilde{w}_n(x) \leq C + C w_0(x) + C w_1(x) + C w_2(x), \quad \forall x \in \Omega. \tag{2.17}
\]
In the following we estimate \(w_i, i = 0, 1, 2,\) term by term.

First we estimate \(w_0.\) We will use Proposition A.1 to estimate \(w_0.\) Since \(0 < s < p,\) we can choose \(q \geq 1\) such that
\[
\frac{s}{N} + \frac{p - 1}{p^*} = \frac{1}{q} = \frac{p - 1}{p^*} = \frac{p^* - 1}{p^*}
\]
and that
\[
q \leq \frac{N}{p}.
\]
Then
\[
\frac{(p - 1)Nq}{N - pq} < p^* \quad \text{and} \quad \frac{(p - 1)(N - s)q}{N - sq} < p^*(s).
\]
Let \(p_1 = \frac{(p - 1)Nq}{N - pq}.\) Applying Proposition A.1 to \(w_0\) gives us
\[
\|w_0\|_{s,p_1} \leq C \left( \left\| |u_{n,0}|^{p^*} - p w_n^{p-1} \right\|_q + \left\| |u_{n,0}|^{p^*(s)} - p w_n^{p-1} + A \right\|_{(N-s)q,\mu_s} \right)^{\frac{1}{p-1}}
\]
\[
\leq C \left( \left\| w_n^{p-1} \right\|_q + \left\| w_n^{p-1} + A \right\|_{(N-s)q,\mu_s} + 1 \right)^{\frac{1}{p-1}}
\]
\[
\leq C \left( \left\| w_n \right\|_{(p-1)q} + \left\| w_n \right\|_{(p-1)(N-s)q,\mu_s} + 1 \right)
\]\n\[
\leq C \left( \left\| w_n \right\|_{p^*} + \left\| w_n \right\|_{p^*(s),\mu_s} + 1 \right)
\]
\[
\leq C.
\]
Here in the second inequality we used the boundedness of \(u_{n,0} = u_0\) and in the last inequality we used (2.12). So this gives estimate for \(w_0.\)

Next we use Corollary A.3 to estimate \(w_1.\) We will choose \(p_2 < p^*,\) \(p_2\) close to \(p^*\) enough such that
\[
\|w_1\|_{s,p_2} \leq C \lambda_n^{\frac{p}{p_2}} - \frac{N}{r_2}.
\]
Indeed, applying Corollary A.3 to \(w_1\) gives us that
\[
\|w_1\|_{s,p_2} \leq C \left( \left\| |u_{n,1}|^{p^*} - p \right\|\right) + \left\| |u_{n,1}|^{p^*(s)} - p \right\|_{r_2,\mu_s} \cdot \|w_n\|_{s,p^*},
\]
where \(r_1, r_2\) are defined by
\[
\frac{1}{r_1} = (p - 1) \left( \frac{1}{p_2} - \frac{1}{p} \right) + \frac{p}{N} \quad \text{and} \quad \frac{1}{r_2} = (p - 1) \left( \frac{N}{(N-s)p_2} - \frac{1}{p^*(s)} \right) + \frac{p - s}{N - s}.
\]
Thus for all $W_{r1}$, we have
\[
\|u_{1,n,1}\|^{p_r_1}_{p_r_1} \leq C \left( \|u_{n,1}|^{p_r_1-p_r_1}_{p_r_1} + \|u_{n,1}|^{p_r_1(s)-p_r_1}_{p_r_1} \right)^{\frac{1}{m}}.
\]
(2.20)

We only need to estimate $\|u_{n,1}|^{p_r_1-p_r_1}_{p_r_1}$ and $\|u_{n,1}|^{p_r_1(s)-p_r_1}_{p_r_1}$. For all $1 \leq j \leq m$, it is easy to see that
\[
\int_{\mathbb{R}^N} |\rho_{x_{n,j},\lambda_{n,j}}(U_j)|^{(p_r_1-p_r_1)}dy = \lambda_{n,j}^{p_1-N} \int_{\mathbb{R}^N} |U_j|^{(p_r_1-p_r_1)}dy.
\]

By Proposition B.1, for all $1 \leq j \leq m$,
\[
|U_j(y)| \leq \frac{C}{1 + |y|^{\frac{N-p_1}{p_1}}}, \quad \forall y \in \mathbb{R}^N.
\]

Since $\frac{N-p_1}{p_1}(p^* - p)_{r1} \rightarrow \frac{pN}{p-1}$ as $p_2 \rightarrow p^*$, we can choose $p_2$ close to $p^*$ enough such that $\frac{N-p_1}{p_1}(p^* - p)_{r1} > N$. Then
\[
\int_{\mathbb{R}^N} |U_j|^{(p_r_1-p_r_1)}dy < \infty.
\]

Thus for all $1 \leq j \leq m$,
\[
\int_{\mathbb{R}^N} |\rho_{x_{n,j},\lambda_{n,j}}(U_j)|^{(p_r_1-p_r_1)}dy \leq C\lambda_{n,j}^{p_1-N}.
\]
Therefore
\[
\|u_{n,1}|^{p_r_1-p_r_1}_{p_r_1} = \|u_{n,1}|^{p_r_1-p_r_1}_{p_r_1} \leq C \sum_{j=1}^{m} \|\rho_{x_{n,j},\lambda_{n,j}}(U_j)|^{p_r_1-p_r_1}_{p_r_1}
\]
\[
\leq C \sum_{j=1}^{m} \lambda_{n,j}^{p_1-N} \lambda_{n,j}^{p_r_1-p_r_1} \leq C\lambda_{n,j}^{p_1-N} \lambda_{n,j}^{p_r_1-p_r_1}.
\]
(2.21)

We used the equality
\[
\frac{pr_1-N}{p_r_1} = \frac{p^* - p}{p-1} = \frac{N}{p^* - p} = \frac{N}{p_2}
\]
in the last inequality of (2.21). This gives estimate for $\|u_{n,1}|^{p_r_1-p_r_1}_{p_r_1}$.

We can also choose $p_2$ close to $p^*$ enough such that for all $1 \leq j \leq m$,
\[
\int_{\mathbb{R}^N} |\rho_{x_{n,j},\lambda_{n,j}}(U_j)|^{(p_r_1(s)-p_r_1)}d\mu_s \leq C\lambda_{n,j}^{(p_r_1(s)-p_r_1)2-N+s}.
\]

Indeed, we have
\[
\int_{\mathbb{R}^N} |\rho_{x_{n,j},\lambda_{n,j}}(U_j)|^{(p_r_1(s)-p_r_1)}d\mu_s = \lambda_{n,j}^{(p_r_1(s)-p_r_1)2-N+s} \int_{\mathbb{R}^N} \frac{|U_j(y)|^{(p_r_1(s)-p_r_1)}}{|y + \lambda_{n,j}x_{n,j}|^s}dy.
\]

Write $y_{n,j} = -\lambda_{n,j}x_{n,j}$. Let
\[
I_1 = \int_{B_1(y_{n,j})} \frac{|U_j(y)|^{(p_r_1(s)-p_r_1)}_{r_2}}{|y - y_{n,j}|^s}dy, \quad \text{and} \quad I_2 = \int_{\mathbb{R}^N \setminus B_1(y_{n,j})} \frac{|U_j(y)|^{(p_r_1(s)-p_r_1)}_{r_2}}{|y - y_{n,j}|^s}dy.
\]

Since $U_j$ is bounded and $0 < s < N$, we have
\[
I_1 \leq C.
\]

Let $\delta > 0$ be a number to be determined. By Hölder’s inequality, we have
\[
I_2 \leq \left( \int_{\mathbb{R}^N \setminus B_1(y_{n,j})} \frac{1}{|y - y_{n,j}|^{N+s}}dy \right)^{\frac{N+s}{N}} \left( \int_{\mathbb{R}^N \setminus B_1(y_{n,j})} |U_j(y)|^{(p_r_1(s)-p_r_1)2(N+s)}dy \right)^{\frac{N+s}{N}}
\]
\[
\leq C\delta \left( \int_{\mathbb{R}^N} |U_j(y)|^{(p_r_1(s)-p_r_1)2(N+s)}dy \right)^{\frac{N+s}{N}}.
\]
Since
\[
\frac{N - p (p^*(s) - p)}{p - 1} \frac{r_2(N + \delta)}{N + \delta - s} \rightarrow \frac{p(N - s)(N + \delta)}{(p - 1)(N + \delta - s)} \quad \text{as } p_2 \to p^*,
\]
and
\[
\frac{p(N - s)(N + \delta)}{(p - 1)(N + \delta - s)} > N \quad \text{for } \delta > 0 \text{ small enough},
\]
we can choose enough and \( \hat{\lambda}_{n,j} \) such that \( \frac{N - p (p^*(s) - p)}{p - 1} \frac{r_2(N + \delta)}{N + \delta - s} > N(p^*-p)_r_2-N+s \). Then
\[
\int_{\mathbb{R}^N} |U_j(y)|^{\frac{(p^*(s)-p)r_2(N+\delta)}{N+s}} dy < \infty.
\]
Then we obtain that
\[
I_2 \leq C.
\]
Combining the estimates of \( I_1 \) and \( I_2 \) we obtain that
\[
\int_{\mathbb{R}^N} |\rho_{x_{n,j},\lambda_{n,j}}(U_j)|^{(p^*(s)-p)r_2} d\mu_s \leq C \lambda_{n,j}^{(p-s)r_2-N+s}.
\]
Hence we have
\[
\left\|u_{n,1}\right\|^{p^*(s)-p}_{r_2,\mu_s} = \left\|u_{n,1}\right\|^{(p^*(s)-p)r_2}_{r_2,\mu_s} \leq C \sum_{j=1}^{m} \lambda_{n,j}^{\frac{(p-s)r_2-N+s}{p^*(s)-p}} \left\|u_{n,1}\right\|^{(p-s)r_2}_{r_2,\mu_s}
\]
\[
\leq C \lambda_n^{\frac{N}{p^*-p}-\frac{N}{p^*}}.
\]
In the above inequality we used the equality
\[
\frac{(p-s)r_2-N+s}{(p^*(s)-p)r_2} = \frac{N}{p^*} - \frac{N}{p^*-p}.
\]
Combining (2.20)-(2.22) gives (2.19).

Finally we use Lemma 2.2 to estimate \( w_2 \). By Lemma 2.2, we have
\[
\left\|w_2\right\|_{*,p,1,p_2,\lambda_n} \leq C \left( \left\|u_{n,1}\right\|^{p^*(s)-p}_{\mu_s} + \left\|u_{n,2}\right\|^{p^*(s)-p}_{\mu_s} \right)^{\frac{1}{p^*}} \left\|w_1\right\|_{*,p,1,p_2,\lambda_n}
\]
\[
\leq \frac{1}{2C'} \left\|w_n\right\|_{*,p,1,p_2,\lambda_n},
\]
since \( \lambda_n \to 0 \) in \( W_0^{1,p}(\Omega) \), where the constant \( C' \) is given by (2.17).

Now combining (2.12), (2.17)-(2.19) and (2.23), we obtain that
\[
\left\|\tilde{w}_n\right\|_{*,p,1,p_2,\lambda_n} \leq C + C \left\|w_0\right\|_{*,p,1,p_2,\lambda_n} + C \left\|w_1\right\|_{*,p,1,p_2,\lambda_n} + C \left\|w_2\right\|_{*,p,1,p_2,\lambda_n}
\]
\[
\leq C + C \left\|w_0\right\|_{*,p_1} + C \left\|w_1\right\|_{*,p_1} + \frac{1}{2} \left\|w_n\right\|_{*,p,1,p_2,\lambda_n}
\]
\[
\leq C + \frac{1}{2} \left\|\tilde{w}_n\right\|_{*,p,1,p_2,\lambda_n},
\]
which completes the proof. \( \square \)

Now we can prove Proposition 2.1.

**Proof of Proposition 2.1.** Since \( w_n \) is a solution to equation (2.10), we can use Lemma 2.2 and Lemma 2.3 to prove Proposition 2.1. See details in e.g. [6]. This finishes the proof of Proposition 2.1. \( \square \)
3. Estimates on safe regions

Since the number of the bubbles of $u_n$ is finite, by Proposition D.1 we can always find a constant $\bar{C} > 0$, independent of $n$, such that the region

$$A_n^1 = \left( B_{(\bar{C}+5)\lambda_n^{-\frac{1}{p}}} (x_n) \right) \cap \Omega$$

does not contain any concentration point of $u_n$ for any $n$. We call this region a safe region for $u_n$.

Let

$$A_n^2 = \left( B_{(\bar{C}+4)\lambda_n^{-\frac{1}{p}}} (x_n) \right) \cap \Omega.$$ 

In this section, we prove the following result.

**Proposition 3.1.** Let $u_n$ be a solution of equation (1.4) with $\epsilon = \epsilon_n \to 0$, satisfying $\| u_n \| \leq C$ for some positive constant $C$ independent of $n$. Then for any constant $q \geq p$, there is a constant $C > 0$ independent of $n$, such that

$$\int_{A_n^2} |u_n|^q dx \leq C \lambda_n^{-\frac{q}{p}}.$$ 

In order to prove Proposition 3.1, we need the following lemma.

**Lemma 3.2.** Let $D$ be a bounded domain with $\Omega \subset \subset D$ and $w_n$ the solution of equation (2.10). Then there exist a number $\gamma > p - 1$ and a constant $C > 0$ independent of $n$, such that

$$\left( \frac{1}{r^N} \int_{B_r(y) \cap \Omega} w_n^\gamma dx \right)^\frac{1}{\gamma} \leq C, \quad \forall y \in \Omega,$$

for all $r \geq \tilde{C} \lambda_n^{-\frac{1}{p}}$.

**Proof.** We will combine Proposition 2.1 and Proposition C.2 to prove Lemma 3.2. Since $w_n$ is the solution of equation (2.10), applying proposition C.2 gives us a number $\gamma \in (p - 1, (p - 1)N/(N - p + 1))$ and a constant $C = C(N, p, \gamma)$ such that

$$\left( \frac{1}{r^N} \int_{B_r(y) \cap \Omega} w_n^\gamma dx \right)^\frac{1}{\gamma} \leq C \int_r^R \left( \frac{1}{t^{N-p}} \int_{B_t(y)} \left( 2\mu |u_n|^{p^*-1} + \frac{2u_n |p^*(s)-1|}{|x|^s} + \frac{A}{|x|^s} \right) dx \right)^\frac{1}{p^*-1} dt,$$

for all $0 < r < R$, where $R = \text{dist}(\Omega, \partial D)$. Let

$$I_1 = \int_r^R \left( \frac{1}{t^{N-p}} \int_{B_t(y)} |u_n|^{p^*-1} dx \right)^\frac{1}{p^*-1} dt,$$

and

$$I_2 = \int_r^R \left( \frac{1}{t^{N-p}} \int_{B_t(y)} \frac{|u_n | |p^*(s)-1|}{|x|^s} dx \right)^\frac{1}{p^*-1} dt,$$

such that

$$\left( \frac{1}{r^N} \int_{B_r(y) \cap \Omega} w_n^\gamma dx \right)^\frac{1}{\gamma} \leq C + CI_1 + CI_2.$$ 

We now estimate $I_1$ and $I_2$ for $r \geq \bar{C} \lambda_n^{-1/p}$.

By Proposition 2.1, $\| u_n \|_{s,p_1, p_2, \lambda} \leq C$ for any $p_1, p_2 \in (p^*/p', \infty)$, $p_2 < p^* < p_1$. 

Let $p_1 > p^*$ be a number to be determined and $p_2 = p^* - 1$. There exist $u_{n,1}$, $u_{n,2}$ with $|u_n| \leq u_{n,1} + u_{n,2}$ such that $\|u_{n,1}\|_{*,p_1} \leq C$ and $\|u_{n,2}\|_{*,p_2} \leq C \lambda_n^{\frac{N}{p^*} - \frac{N}{p_2}}$. Then
\[
\int_{B_t(y)} |u_{n,1}|^{p_1-1} dx \leq C \left( \int_{B_t(y)} |u_{n,1}|^{p_1} dx \right)^{\frac{p_1-1}{p_1}} |B_t(y)|^{1-\frac{p_1-1}{p_1}} \leq C t^{(1-\frac{p_1-1}{p_1})N},
\]
and
\[
\int_{B_t(y)} |u_{n,2}|^{p_2-1} dx = \int_{B_t(y)} |u_{n,2}|^{p_2} dx \leq C \lambda_n^{\frac{N}{p^*} - \frac{N}{p_2}} = C \lambda_n^{\frac{p-N}{p}}.
\]
Thus
\[
\int_{B_t(y)} |u_n|^{p_1-1} dx = \int_{B_t(y)} |u_{n,2}|^{p_2-1} dx \leq C \int_{B_t(y)} |u_{n,1}|^{p_1-1} dx + C \int_{B_t(y)} |u_{n,2}|^{p_2-1} dx \leq C t^{(1-\frac{p_1-1}{p_1})N} + C \lambda_n^{\frac{p-N}{p}}.
\]
Since $\frac{N}{p_1-1} \left( 1 - \frac{p_1-1}{p_1} \right) + \frac{p-N}{p_1-1} < 0$, we can choose $p_1 > p^*$ large enough such that
\[
\int_0^R t^{\left( 1 - \frac{p_1-1}{p_1} \right) \frac{N}{p_1} + \frac{p-N}{p_1}} dt < C.
\]
Note also that for $r \geq C \lambda_n^{-1/p}$, we have
\[
\int_r^\infty \frac{t^{\frac{p-N}{p}}}{t} dt \leq C \lambda_n^{\frac{N-p}{p_1}}.
\]
Therefore
\[
I_1 \leq \int_0^R \left( C t^{(1-\frac{p_1-1}{p_1})N} + C \lambda_n^{\frac{p-N}{p}} \right) t^{\frac{N}{p_1} - \frac{p-N}{p_1}} dt \leq C \int_0^R \frac{t^{\frac{N}{p_1} - (\frac{p_1-1}{p_1} + \frac{p-N}{p_1})}}{t} + C \lambda_n^{\frac{p-N}{p}} \int_r^\infty \frac{t^{p-N}}{t} dt \leq C.
\]
This gives estimate for $I_1$.

Next we estimate $I_2$. Let $p_1 > p^*$ to be determined and $p_2 = N \left( p^*(s-1) \right) / (N-s)$. There exist $\bar{u}_{n,1}$, $\bar{u}_{n,2}$ with $|u_n| \leq \bar{u}_{n,1} + \bar{u}_{n,2}$ such that $\|\bar{u}_{n,1}\|_{*,p_1} \leq C$ and $\|\bar{u}_{n,2}\|_{*,p_2} \leq C \lambda_n^{\frac{N}{p^*} - \frac{N}{p_2}}$. Then
\[
\int_{B_t(y)} |\bar{u}_{n,1}|^{p_1(s)-1} d\mu_s \leq \left( \int_{B_t(y)} |\bar{u}_{n,1}|^{\frac{(p*(s)-1)N}{(N-s)p_1}} d\mu_s \right)^\frac{1}{p_1^{s-1}} \left( \int_{B_t(y)} d\mu_s \right) \leq C t^{N-s-\frac{(p*(s)-1)N}{p_1}},
\]
and
\[
\int_{B_t(y)} |\bar{u}_{n,2}|^{p_2(s)-1} d\mu_s = \int_{B_t(y)} |\bar{u}_{n,2}|^{\frac{Np_2}{p_1}} d\mu_s \leq C \lambda_n^{\frac{p-N}{p}}.
\]
Arguing as above yields that
\[
I_2 \leq C,
\]
if we choose $p_1$ large enough. This gives estimate for $I_3$.

Combining (3.1)-(3.3), we complete the proof of Lemma 3.2. \qed

Now we can prove Proposition 3.1.
Proof of Proposition 3.1. Let $\gamma > p - 1$ be as in Lemma 3.2. Since $|u_n| \leq w_n$, we have
\[
\int_{B_{\frac{1}{2} \lambda_n^{-1/p}(y)}} |u_n|^\gamma \, dx \leq C \lambda_n^{-\frac{\gamma}{p}}, \quad \forall y \in A^2_n. \tag{3.4}
\]

Let $v_n(x) = u_n(\lambda_n^{-\frac{1}{p}} x)$, $x \in \Omega_n = \{ x; \lambda_n^{-\frac{1}{p}} x \in \Omega \}$. Then $v_n$ is a solution to equation
\[
\begin{cases}
-\Delta_p v_n = \lambda_n^{-1} \left( \mu |v_n|^{p^*-p-\epsilon_n} + \frac{\lambda_n^{-\frac{1}{p}} |v_n|^{p^*(s)-p-\epsilon_n}}{|x|^s} + a(\lambda_n^{-\frac{1}{p}} x) \right) |v_n|^{p^*-2} v_n, & x \in \Omega_n, \\
v_n = 0, & \text{on } \partial \Omega_n.
\end{cases}
\]

Let $z = \lambda_n^{-\frac{1}{p}} y$, $y \in A^2_n$. Since $B_{\lambda_n^{-1/p}}(y)$ does not contain any concentration point of $u_n$, we can deduce that
\[
\int_{B_1(z)} |\lambda_n^{-1} \left( \mu |v_n|^{p^*-p-\epsilon_n} + a(\lambda_n^{-\frac{1}{p}} x) \right)|^{\frac{\gamma}{\mu}} \, dx \leq C \int_{B_1(z)} |\lambda_n^{-1} \left( |v_n|^{p^*-p+1} \right)|^{\frac{\gamma}{\mu}} \, dx 
\]
\[
\leq C \int_{B_{\lambda_n^{-\frac{1}{p}}}(y)} |u_n|^{\gamma} \, dx + C \lambda_n^{-\frac{\gamma}{p}} \to 0,
\]
and that
\[
\int_{B_1(z)} \frac{|\lambda_n^{-1} \left( \mu |v_n|^{p^*(s)-p-\epsilon_n} + a(\lambda_n^{-\frac{1}{p}} x) \right)|^{\frac{N-s}{p}}}{|x|^s} \, dx \leq C \int_{B_1(z)} \frac{|\lambda_n^{-\frac{1}{p}} |v_n|^{p*(s)-p+1}|^{\frac{N-s}{p}}}{|x|^s} \, dx 
\]
\[
\leq C \int_{B_{\lambda_n^{-\frac{1}{p}}}(y)} \frac{|u_n|^{p*(s)}}{|x|^s} \, dx + C \lambda_n^{-\frac{N-s}{p}} \to 0,
\]
as $n \to \infty$.

Thus for any $q > p^*$, we obtain by Lemma A.4 and (3.4) that,
\[
\|v_n\|_{q, B_{1/2}(z)} \leq C \left( \int_{B_1(z)} |v_n|^\gamma \, dx \right)^{\frac{1}{\gamma}} = C \left( \int_{B_{\lambda_n^{-1/p}}(y)} |u_n|^\gamma \, dx \right)^{\frac{1}{\gamma}} \leq C.
\]
Equivalently, we arrive at
\[
\int_{B_{\frac{1}{2} \lambda_n^{-1/p}(y)}} |u_n|^q \, dx \leq C \lambda_n^{-\frac{N}{p}}, \quad \forall y \in A^2_n.
\]

Now a simple covering argument proves Proposition 3.1 in the case when $q > p^*$.

If $p \leq q \leq 2p^*$, we apply Hölder’s inequality to obtain that
\[
\left( \int_{A^2_n} |u_n|^q \, dx \right)^{\frac{1}{q}} \leq \left( \int_{A^2_n} |u_n|^{2p^*} \, dx \right)^{\frac{1}{2p^*}} \leq C.
\]

We complete the proof of Proposition 3.1. \(\square\)

Let
\[
A^3_n = \left( B_{(C+3)\Lambda_n^{-\frac{1}{p}}}(x_n) \setminus B_{(C+2)\Lambda_n^{-\frac{1}{p}}}(x_n) \right) \cap \Omega.
\]
In the end of this section, we prove the following estimate for $u_n$.

**Proposition 3.3.** We have
\[
\int_{A^3_n} |\nabla u_n|^p \, dx \leq C \int_{A^3_n} \left( |u_n|^{p^*} + \frac{|u_n|^{p^*(s)}}{|x|^s} + 1 \right) \, dx + C \lambda_n \int_{A^3_n} |u_n|^p \, dx. \tag{3.5}
\]
In particular, we have
\[ \int_{A_n^3} |\nabla u_n|^p dx \leq C \lambda_n^{-\frac{pN}{p-\alpha}}. \tag{3.6} \]

Proof. Let \( \phi \in C_0^\infty(A_n^3) \) be a cut-off function with \( \phi = 1 \) in \( A_n^3 \), \( 0 \leq \phi \leq 1 \) and \( |\nabla \phi| \leq C \lambda_n^\frac{1}{p} \).

Multiplying the equation of \( u_n \) by \( \phi^p u_n \) yields that
\[ \int_{A_n^3} |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla (\phi^p u_n) dx = \int_{A_n^3} \left( \mu |u_n|^{p-2} u_n + \frac{|u_n|^{p^*(s)} - 2\epsilon_n u_n}{|x|^s} + a |u_n|^{p-2} u_n \right) \phi^p u_n dx. \]

It is easy to derive (3.5) from the equality above.

Let \( q > p^*(s) \). By Proposition 3.1, we have
\[ \int_{A_n^3} \frac{\phi^p |u_n|^{p^*(s)}}{|x|^s} dx \leq \left( \int_{A_n^3} \phi^p |u_n|^q dx \right)^{\frac{q}{q-\frac{p^*(s)}{q}}} \left( \int_{A_n^3} \phi^p \left| \frac{x}{|x|^s} \right|^{q-\frac{p^*(s)}{q}} dx \right)^{\frac{q-\frac{p^*(s)}{q}}{q}} \leq C \lambda_n^{-\frac{q}{q-\frac{p^*(s)}{q}}} N \left( \frac{q}{q-\frac{p^*(s)}{q}} \right) \leq C \lambda_n^{-\frac{q}{q-\frac{p^*(s)}{q}}} \lambda_n^\frac{p^*(s)}{q} \leq C \lambda_n^{-\frac{q}{q-\frac{p^*(s)}{q}}} \lambda_n \leq C \lambda_n^{-\frac{q}{q-\frac{p^*(s)}{q}}} \lambda_n^\frac{p^*(s)}{q}. \tag{3.7} \]

Now from (3.7), (3.5) and Proposition 3.1, we obtain that
\[ \int_{A_n^3} |\nabla u_n|^p dx \leq C \lambda_n^{-\frac{p}{p-\alpha}} + C \lambda_n^{-\frac{q}{q-\frac{p^*(s)}{q}}} + C \lambda_n^{-\frac{q}{q-\frac{p^*(s)}{q}}} \leq C \lambda_n^{-\frac{p}{p-\alpha}}. \]

This proves (3.6). We finish the proof. \( \square \)

4. Proof of main results

In this section we prove Theorem 1.1 and Theorem 1.2.

For simplicity, write \( p_n = p^* - \epsilon_n \) and \( p_n(s) = p^*(s) - \epsilon_n \). Choose \( t_n \in [\tilde{C} + 2, \tilde{C} + 3] \) such that
\[ \int_{\partial B_{t_n}^{\frac{1}{p}}(x_n)} \left( \mu |u_n|^p + |u_n|^p + \lambda_n^{-1} |\nabla u_n|^p + \lambda_n^{-\frac{s}{p}} \frac{|u_n|^{p_n(s)}}{|x|^s} \right) d\sigma \geq C \lambda_n^N \int_{A_n^3} \left( \mu |u_n|^p + |u_n|^p + \lambda_n^{-1} |\nabla u_n|^p + \lambda_n^{-\frac{s}{p}} \frac{|u_n|^{p_n(s)}}{|x|^s} \right) dx. \tag{4.1} \]

By Proposition 3.1, (3.6) and (3.7), we obtain that
\[ \int_{\partial B_{t_n}^{\frac{1}{p}}(x_n)} \left( \mu |u_n|^p + |u_n|^p + \lambda_n^{-1} |\nabla u_n|^p + \lambda_n^{-\frac{s}{p}} \frac{|u_n|^{p_n(s)}}{|x|^s} \right) d\sigma \leq C \lambda_n^{1-N}. \tag{4.2} \]
We also have the following Pohozaev identity for $u_n$ on $B_n = B_{\lambda_n} \frac{1}{p} (x_n) \cap \Omega$

\[
\left( \frac{N}{p_n} - \frac{N - p}{p} \right) \mu \int_{B_n} |u_n|^{p_n} \, dx + \int_{B_n} \left[ a(x) - \frac{1}{p} \nabla a(x) \cdot (x - x_0) \right] |u_n|^{p} \, dx \\
+ \left( \frac{N - s}{p_n(s)} - \frac{N - p}{p} \right) \int_{B_n} \frac{|u_n|^{p_n(s)}}{|x|^s} \, dx + \frac{s}{p_n(s)} \int_{B_n} \frac{|u_n|^{p_n(s)}}{|x|^{2+s}} (x_0 \cdot x) \, dx
\]

\[
= \frac{N - p}{p} \int_{\partial B_n} |\nabla u_n|^{p-2} \frac{\partial u_n}{\partial \nu} u_n \, d\sigma + \int_{\partial B_n} |\nabla u_n|^{p-2} \nabla u_n \cdot (x - x_0) \frac{\partial u_n}{\partial \nu} \, d\sigma \\
- \frac{1}{p} \int_{\partial B_n} |\nabla u_n|^p (x - x_0) \cdot \nu \, d\sigma \\
+ \int_{\partial B_n} (x - x_0) \cdot \nu \left[ \frac{1}{p_n} |u_n|^{p_n} + \frac{1}{p_n(s)} \frac{|u_n|^{p_n(s)}}{|x|^s} + \frac{1}{p} a(x) |u_n|^p \right] \, d\sigma,
\]

where $\nu$ is the outward unit normal to $\partial B_n$ and $x_0 \in \mathbb{R}^N$. Since $p_n < p^*$ and $p_n(s) < p^*(s)$, we have the following inequality from above

\[
\int_{B_n} \left[ a(x) - \frac{1}{p} \nabla a(x) \cdot (x - x_0) \right] |u_n|^p \, dx \\
\leq \frac{N - p}{p} \int_{\partial B_n} |\nabla u_n|^{p-2} \frac{\partial u_n}{\partial \nu} u_n \, d\sigma + \int_{\partial B_n} |\nabla u_n|^{p-2} \nabla u_n \cdot (x - x_0) \frac{\partial u_n}{\partial \nu} \, d\sigma \\
- \frac{1}{p} \int_{\partial B_n} |\nabla u_n|^p (x - x_0) \cdot \nu \, d\sigma \\
+ \int_{\partial B_n} (x - x_0) \cdot \nu \left[ \frac{1}{p_n} |u_n|^{p_n} + \frac{1}{p_n(s)} \frac{|u_n|^{p_n(s)}}{|x|^s} + \frac{1}{p} a(x) |u_n|^p \right] \, d\sigma.
\]

Now we can prove Theorem 1.1.

**Proof of Theorem 1.1.** Since $\{x_n\} \subset \Omega$ is a bounded sequence, we may assume that $x_n \to x^* \in \bar{\Omega}$ as $n \to \infty$. We have two cases:

Case 1. $x^* = 0$;

Case 2. $x^* \neq 0$.

In Case 1, choose $x_0 = 0$ in (4.3). Then we obtain that

\[
\int_{B_n} \left[ a(x) - \frac{1}{p} \nabla a(x) \cdot x \right] |u_n|^p \, dx \\
\leq \frac{N - p}{p} \int_{\partial B_n} |\nabla u_n|^{p-2} \frac{\partial u_n}{\partial \nu} u_n \, d\sigma + \int_{\partial B_n} |\nabla u_n|^{p-2} \nabla u_n \cdot x \frac{\partial u_n}{\partial \nu} \, d\sigma \\
- \frac{1}{p} \int_{\partial B_n} |\nabla u_n|^p x \cdot \nu \, d\sigma \\
+ \int_{\partial B_n} x \cdot \nu \left[ \frac{1}{p_n} |u_n|^{p_n} + \frac{1}{p_n(s)} \frac{|u_n|^{p_n(s)}}{|x|^s} + \frac{1}{p} a(x) |u_n|^p \right] \, d\sigma.
\]

Decompose $\partial B_n$ by $\partial B_n = \partial_1 B_n \cup \partial_2 B_n$, where $\partial_1 B_n = \partial B_n \cap \Omega$ and $\partial_2 B_n = \partial B_n \cap \partial \Omega$. 


Consider the case $0 \in \partial \Omega$ first. Observe that $u_n = 0$ on $\partial \Omega$. Thus (4.4) implies that
\[
L_1 := \int_{B_n} \left[ a(x) - \frac{1}{p} \nabla a(x) \cdot x \right] |u_n|^p dx - \left( 1 - \frac{1}{p} \right) \int_{\partial B_n} |\nabla u_n|^p x \cdot \nu d\sigma \\
\leq \frac{N - p}{p} \int_{\partial B_n} |\nabla u_n|^{p-2} \frac{\partial u_n}{\partial \nu} u_n d\sigma + \int_{\partial B_n} |\nabla u_n|^{p-2} \nabla u_n \cdot \frac{\partial u_n}{\partial \nu} d\sigma \\
- \frac{1}{p} \int_{\partial B_n} |\nabla u_n|^p x \cdot \nu d\sigma \\
+ \int_{\partial B_n} x \cdot \nu \left[ \frac{1}{p_n} |u_n|^p + \frac{1}{p_n(s)} \frac{|u_n|^p(s)}{|x|^s} + \frac{1}{p} a(x)|u_n|^p \right] d\sigma.
\]  
(4.5)

By assumption (1.6), we have
\[
\int_{\partial B_n} |\nabla u_n|^p x \cdot \nu d\sigma \leq 0.
\]
Also note that $a(0) > 0$. Thus (4.5) gives us
\[
L_1 \geq \frac{1}{2} a(0) \int_{B_n} |u_n|^p dx.
\]  
(4.6)

On the other hand, since $|x| \leq C\lambda_n^{-1/p}$ for $x \in \partial B_n$, by (4.2), we have
\[
R_1 \leq C\lambda_n^{-p} \int_{\partial B_n} \left[ |u_n|^p + |u_n|^{p_n} + \frac{|u_n|^p(s)}{|x|^s} + \frac{|u_n|^p(s)}{|x|^s} \right] d\sigma \\
+ C \int_{\partial B_n} |\nabla u_n|^{p-1} |u_n| d\sigma \\
\leq C\lambda_n^{-p}.
\]  
(4.7)

Thus combining (4.5), (4.6) and (4.7) implies that
\[
\int_{B_n} |u_n|^p dx \leq C\lambda_n^{\frac{p-N}{p}}.
\]  
(4.8)

Now arguing as that of [6], we have
\[
\int_{B_n} |u_n|^p dx \geq C' \lambda_n^{-p}.
\]

Therefore we arrive at
\[
\lambda_n^{-p} \leq C\lambda_n^{\frac{p-N}{p}}.
\]  
(4.9)

Since $\lambda_n \to \infty$, (4.9) can not happen under the assumption that
\[
N > p^2 + p.
\]

The case $0 \in \Omega$ turns out to be easier than the previous case since $\partial_e B_n = \emptyset$ now. So (4.5) holds as well with $\partial_e B_n = \partial B_n$. Arguing as above, we get a contradiction. So we complete the proof of Theorem 1.1 in Case 1.

Now we consider Case 2. That is, $x^* \neq 0$. We have two possibilities: either $B_{\frac{1}{t_n\lambda_n^p}}(x_n) \subset \subset \Omega$ or $B_{\frac{1}{t_n\lambda_n^p}}(x_n) \cap (\mathbb{R}^N \setminus \Omega) \neq \emptyset$. 
Suppose that \( B_{\frac{1}{t_n \lambda_n^p}}(x_n) \subset \Omega \). Then \( B_n = B_{\frac{1}{t_n \lambda_n^p}}(x_n) \). We take \( x_0 = x_n \) in (4.3) and obtain that

\[
L_2 := \frac{s}{p_n(s)} \int_{B_n} \frac{|u_n|^{p_n(s)}}{|x|^{2+s}}(x_n \cdot x) dx \leq -\int_{B_n} [a(x) - \frac{1}{p} \nabla a(x) \cdot (x - x_n)] |u_n|^p dx + \frac{N - p}{p} \int_{\partial B_n} |\nabla u_n|^{p-2} \frac{\partial u_n}{\partial \nu} u_n d\sigma + \int_{\partial B_n} |\nabla u_n|^{p-2} \nabla u_n \cdot (x - x_n) \frac{\partial u_n}{\partial \nu} d\sigma - \frac{1}{p} \int_{\partial B_n} |\nabla u_n|^p (x - x_n) \cdot \nu d\sigma + \int_{\partial B_n} (x - x_n) \cdot \nu \left[ \frac{1}{p_n} |u_n|^{p_n} + \frac{1}{p_n(s)} \frac{|u_n|^{p_n(s)}}{|x|^s} + \frac{1}{p} a(x) |u_n|^p \right] d\sigma.
\]

(4.10)

Thus

\[
L_2 \geq C \int_{B_n} |u_n|^{p_n(s)} dx.
\]

Again, applying the same argument as that of [6] gives us that

\[
L_2 \geq C' \lambda_n^{-N + p_n(s) \frac{N - p}{p}}.
\]

(4.11)

On the other hand, by arguing as before, we easily get that

\[
R_2 \leq C \lambda_n^{\frac{p-n}{p}} + C \int_{B_n} |u_n|^p dx,
\]

(4.12)

in which the assumption \( a \in C^1(\overline{\Omega}) \) was used. We claim that

\[
\int_{B_n} |u_n|^p dx \leq C \lambda_n^{-p}.
\]

(4.13)

Indeed, let \( p_1 > p^* \) such that \( \frac{N}{p} (1 - \frac{p}{p_1}) > p \). This is possible since \( N > p^2 + p \). Also, let \( p_2 = p \). Then we have \( p^*/p' < p_2 < p^* \). By proposition 2.1, there exist \( v_i \geq 0, i = 1, 2 \), such that \( |u_n| \leq v_1 + v_2 \) and

\[
\|v_1\|_{s, p_1} \leq C, \quad \|v_2\|_{s, p} \leq C \lambda_n^{\frac{N - p}{p}} = C \lambda_n^{-1}.
\]

Hence

\[
\int_{B_n} |u_n|^p dx \leq 2^{p-1} \int_{B_n} |v_1|^p dx + 2^{p-1} \int_{B_n} |v_2|^p dx \leq C \lambda_n^{-\frac{N}{p} (1 - \frac{p}{p_1})} + C \lambda_n^{-p} \leq C \lambda_n^{-p}.
\]

This gives (4.13). Now combining (4.11)-(4.13) gives us

\[
\lambda_n^{-N + \frac{N - p}{p} p_n(s)} \leq C \lambda_n^{-p} + C \lambda_n^{\frac{p-n}{p}} \leq C \lambda_n^{-p},
\]

(4.14)

which is impossible since \( N > p^2 + p \) and \( s < p \).
It remains to consider \( B \frac{1}{t_n \lambda_n^p} (x_n) \cap (\mathbb{R}^N \setminus \Omega) \neq \emptyset \). In (4.3), we take \( x_0 \in \mathbb{R}^N \setminus \Omega \) with \(|x_0 - x_n| \leq 2t_n \lambda_n^p\) and \( \nu \cdot (x - x_0) \leq 0 \) in \( \partial \Omega \cap B_n \). With this choice of \( x_0 \), we get from (4.3),

\[
\frac{8}{p_n(s)} \int_{B_n} |u_n|^{p_n(s)} (x_n \cdot x) dx \\
\leq - \int_{B_n} \left[ a(x) - \frac{1}{p} \nabla a(x) \cdot (x - x_n) \right] |u_n|^p dx \\
+ \frac{N - p}{p} \int_{\partial B_n} |\nabla u_n|^{p-2} \frac{\partial u_n}{\partial \nu} u_n d\sigma + \int_{\partial B_n} |\nabla u_n|^{p-2} \nabla u_n \cdot (x - x_n) \frac{\partial u_n}{\partial \nu} d\sigma \\
- \frac{1}{p} \int_{\partial B_n} |\nabla u_n|^p (x - x_n) \cdot \nu d\sigma \\
+ \int_{\partial B_n} (x - x_n) \cdot \nu \left[ \frac{1}{p_n} |u_n|_{p_n} + \frac{1}{p_n(s)} \frac{|u_n|^{p_n(s)}}{|x|^s} + \frac{1}{p} a(x) |u_n|^p \right] d\sigma.
\]

Arguing as before, we find that (4.14) still holds. Thus we get a contradiction. We complete the proof of Theorem 1.1. \( \square \)

Now we can prove Theorem 1.2.

**Proof of Theorem 1.2.** With Theorem 1.1 at hand, we can prove Theorem 1.2 by the same method as that of [6]. So we omit the details. \( \square \)

**APPENDIX A. ESTIMATES FOR QUASILINEAR PROBLEMS WITH HARDY POTENTIAL**

In this section, we deduce some elementary estimates for solutions of a quasilinear elliptic problem involving a Hardy potential. Let \( D \) be a bounded domain in \( \mathbb{R}^N \) and \( 0 \in D \). For any \( 0 \leq t < p \), write \( d\mu_t = |x|^{-t} dx \) and \( \|w\|_{q, \mu_t} = \int_D |w|^q d\mu_t \). We also use the notation \( \|w\|_q = \|w\|_{q, \mu_0} \).

Let us recall that

\[
\|w\|_{*, q} = \|w\|_q + \|w\|_{\frac{N(q - 1)}{N - pq}, \mu_q}.
\]

**Proposition A.1.** For any \( f_i \geq 0 \) and \( f_i \in L^\infty(D), i = 1, 2 \), let \( w \in W_0^{1,p}(D) \) be the solution of

\[
\begin{align*}
-\Delta_p w &= f_1(x) + \frac{f_2(x)}{|x|^s}, & x \in D, \\
\quad w &= 0, & \text{on } \partial D.
\end{align*}
\]

Then, for any \( 1 < q < N/p \), there is \( C = C(N, p, s, q) > 0 \) such that

\[
\|w\|_{*, (q-1)\frac{N}{N-pq}} \leq C \left( \|f_1\|_q + \|f_2\|_{\frac{N(q-1)}{N - pq}, \mu_q} \right)^{\frac{1}{q}}.
\]

**Proof.** By the maximum principle, we find that \( w \geq 0 \). We claim that if \( r > 1/p' \), then

\[
\|w\|_{*, p^*, r}^{p^*} \leq C \int_D \left( f_1 + \frac{f_2}{|x|^s} \right) w^{1 + p(r-1)} dx,
\]

for some \( C = C(r) > 0 \).

First we suppose that \( r \geq 1 \). Since \( f_1, f_2 \) are bounded functions, it is standard to prove that \( w \in L^\infty(D) \) by Moser’s iteration method [24]. Then we can take a test function \( \xi = w^{1 + p(r-1)} \) so that

\[
\frac{1}{r} \int_D |\nabla w|^{r} dx = \int_D \left( f_1 + \frac{f_2}{|x|^s} \right) w^{1 + p(r-1)} dx.
\]

Applying Sobolev inequality and Caffarelli-Kohn-Nirenberg inequality give us

\[
\|w\|_{p^*, r}^{p^*} + \|w\| \leq C \int_D |\nabla w|^\tau dx.
\]
for some $C = C(N, p, s) > 0$. Thus

$$\|w\|_{s,r}^{p} \leq C \int_{D} |\nabla w|^p dx. \quad (A.3)$$

Therefore, combining (A.2) and (A.3) yields (A.1).

Now let $r \in (1/p', 1)$ and $\epsilon > 0$. Define $\xi = w(w+\epsilon)^{p(r-1)}$. It is easy to verify that $\xi \in W_{0}^{1,p}(D)$ and

$$\nabla \xi = (w+\epsilon)^{p(r-1)} \nabla w + p(r-1)w(w+\epsilon)^{p(r-1)-1} \nabla w.$$ 

Take $\xi$ as a test function. We have

$$\int_{D} |\nabla w|^{p-2} \nabla w \cdot \nabla \xi dx = \int_{D} \left( f_{1} + \frac{f_{2}}{|x|^s} \right) \xi dx.$$ 

A simple calculation gives that

$$\int_{D} |\nabla w|^{p-2} \nabla w \cdot \nabla \xi dx \geq \frac{1 + p(r-1)}{p} \int_{D} (w+\epsilon)^{p(r-1)} |\nabla w|^p dx,$$

$$= \frac{1 + p(r-1)}{p} \int_{D} |\nabla ((w+\epsilon)^r - \epsilon^r)| dx \geq C(r) \left( \|(w+\epsilon)^r - \epsilon^r\|_{p^*}^{p} + \|(w+\epsilon)^r - \epsilon^r\|_{p^*(s),\mu_s}^{p} \right),$$

for $C = C(N, p, s, r) > 0$.

Let $w_{\epsilon} = ((w+\epsilon)^r - \epsilon^r)^{1/r}$. Then there exists $C > 0$ such that

$$\int_{D} |\nabla w|^{p-2} \nabla w \cdot \nabla \xi dx \geq C(r) \|w_{\epsilon}\|_{s,r}^{p}.$$ 

Thus

$$\|w_{\epsilon}\|_{s,r}^{p} \leq C \int_{D} \left( f_{1} + \frac{f_{2}}{|x|^s} \right) w(w+\epsilon)^{p(r-1)} dx.$$ 

Letting $\epsilon \to 0$, we obtain (A.1) in the case when $r \in (1/p', 1)$.

To prove Proposition A.1, we apply Hölder’s inequality to (A.1) and obtain that

$$\|w\|_{s,r}^{p} \leq C \left( \|f_{1}\|_{p^*(s)r - (1+p(r-1)),\mu_r}^{p^*(s)r} + \|f_{2}\|_{p^*(s)r - (1+p(r-1)),\mu_r}^{p^*(s)r} \right) \left( \|w\|_{p^*}^{1+p(r-1)} + \|w\|_{p^*(s),\mu_s}^{1+p(r-1)} \right),$$

which implies that

$$\|w\|_{s,r}^{p} \leq C \left( \|f_{1}\|_{p^*(s)r - (1+p(r-1)),\mu_r}^{p^*(s)r} + \|f_{2}\|_{p^*(s)r - (1+p(r-1)),\mu_r}^{p^*(s)r} \right) \frac{1}{1+p(r-1)}.$$ 

Give $q \in (1, N/p)$. Let $r \in (1/p', \infty)$ be such that $q = \frac{p^*r}{p^*r - (1+p(r-1))}$. Then a simple calculation gives us

$$\frac{p^*(s)r}{p^*(s)r - (1+p(r-1))} = \frac{(N-s)q}{N-sq} \quad \text{and} \quad p^* = \frac{(p-1)Nq}{N-pq}.$$ 

We finish the proof. \qed

As a consequence of Proposition A.1 we have the following corollary.

**Corollary A.2.** Let $w \in W_{0}^{1,p}(D)$ be the solution of

$$\begin{cases}
-\Delta_{p}w = \left( a_{1}(x) + \frac{a_{2}(x)}{|x|^s} \right) v^{p-1}, & x \in D, \\
w = 0, & \text{on } \partial D,
\end{cases}$$

for some $C = C(N, p, s) > 0$. Thus

$$\|w\|_{s,r}^{p} \leq C \int_{D} |\nabla w|^p dx. \quad (A.3)$$

Therefore, combining (A.2) and (A.3) yields (A.1).
where \(a_1, a_2, v \in L^\infty(D)\) are nonnegative functions. Then for any \(\infty > q > p^*/p'\), there holds
\[
\|w\|_{s,q} \leq C \left( \|a_1\|_{\frac{q}{p'}} + \|a_2\|_{\frac{q - s}{p - s}, \mu_s} \right)^{\frac{1}{q - r}},
\]
for \(C = C(N, p, s, q) > 0\).

Proof. Let \(\infty > q > p^*/p'\) and define \(r = Nq/(N(p - 1) + pq)\). Then \(1 < r < N/p\) and \(q = (p - 1)Nr/(N - pr)\).

By applying Proposition A.1 with \(f_i = a_iv^{p - 1}, i = 1, 2\), we obtain that
\[
\|w\|_{s,q} \leq C \left( \|f_1\|_{r} + \|f_2\|_{\frac{(N - s)r}{N - pr}, \mu_s} \right)^{\frac{1}{r - 1}},
\]
for \(C = C(N, p, s, q) > 0\). By Hölder’s inequality and the definition of \(\| \cdot \|_{s,q}\), we have that
\[
\|f_1\|_r \leq \|a_1\|_{\frac{q}{p'}} \|v\|_{q}^{p - 1} \leq \|a_1\|_{\frac{q}{p'}} \|v\|_{s,q}^{p - 1}
\]
and that
\[
\|f_2\|_{\frac{(N - s)r}{N - pr}, \mu_s} \leq \|a_2\|_{\frac{q - s}{p - s}, \mu_s} \|v\|_{s,q}^{p - 1}.
\]
Combining the above inequalities gives Corollary A.2. \(\square\)

We also have the following corollary.

Corollary A.3. Let \(w \in W_0^{1, p}(D)\) be the solution of
\[
\begin{align*}
-\Delta_p w &= \left( a_1(x) + \frac{a_2(x)}{|x|^s} \right) v^{p - 1}, & x \in D, \\
w &= 0, & \text{on } \partial D,
\end{align*}
\]
where \(a_1, a_2, v \in L^\infty(D)\) are nonnegative functions. Then for any \(p_2 \in (p^*/p', p^*)\), there is a constant \(C = C(N, p, s, p_2) > 0\) such that
\[
\|w\|_{s,p_2} \leq C \left( \|a_1\|_{r_1} + \|a_2\|_{r_2, \mu_s} \right)^{\frac{1}{p' - 1}} \|v\|_{s,p^*},
\]
where \(r_1, r_2\) are defined by
\[
\frac{1}{r_1} = (p - 1) \left( \frac{1}{p_2} - \frac{1}{p'} \right) + \frac{p}{N}, \quad \text{and} \quad \frac{1}{r_2} = (p - 1) \left( \frac{N}{(N - s)p_2} - \frac{1}{p'(s)} \right) + \frac{p - s}{N - s}. \tag{A.4}
\]

Proof. The proof is similar to that of Corollary A.2. By applying Proposition A.1 with \(f_i = a_iv^{p - 1}, i = 1, 2\), we obtain
\[
\|w\|_{s,p_2} \leq C \left( \|f_1\|_{\frac{Np_2}{(p - 1)N + p_2p}} + \|f_2\|_{\frac{(N - s)p_2}{(p - 1)N + (p - s)p_2}, \mu_s} \right)^{\frac{1}{p' - 1}}.
\]
Define \(r_1, r_2\) by (A.4). Applying Hölder’s inequality gives us that
\[
\|f_1\|_{\frac{Np_2}{(p - 1)N + p_2p}} \leq \|a_1\|_{r_1} \|v\|_{p^*}^{p - 1},
\]
and that
\[
\|f_2\|_{\frac{(N - s)p_2}{(p - 1)N + (p - s)p_2}, \mu_s} \leq \|a_2\|_{r_2, \mu_s} \|v\|_{p'(s), \mu_s}^{p - 1}.
\]
Combining the above inequalities gives Corollary A.3. \(\square\)

We will need the following lemma in Section 3.
Lemma A.4. Let $w \in W^{1,p}_{\text{loc}}(\mathbb{R}^N)$, $w \geq 0$ be a weak solution of the equation

$$-\Delta_p w \leq \left( a_1(x) + \frac{a_2(x)}{|x|^s} \right) w^{p-1}$$

in $\mathbb{R}^N$, where $a_1, a_2 \in L^\infty_{\text{loc}}(\mathbb{R}^N)$ are nonnegative functions. Then for any unit ball $B_1(y) \subset \mathbb{R}^N$ and for any $q > p^*$, there is a small constant $\delta = \delta(q) > 0$ such that if

$$\left( \int_{B_1(y)} a_1 ^ \frac{p}{p-1} dx \right)^{\frac{p}{p}} + \left( \int_{B_1(y)} \frac{a_2}{a_2 - \frac{s}{p}} d\mu_s \right)^{\frac{p}{p-1}} < \delta,$$

then for any $\gamma \in (0, p^*)$, there has

$$||w||_{p,B_1/2(y)} \leq C ||w||_{\gamma,B_1(y)}$$

for some $C = C(N, p, s, q, \gamma) > 0$.

Proof. It is standard to show that $w \in L^\infty_{\text{loc}}(\mathbb{R}^N)$ by Moser’s iteration method [24]. Thus for any $\eta \in C_c^\infty(B_1(y))$, we can take a test function by $\varphi = \eta^p w^{1+p(\tau-1)}$ for any $\tau \geq 1$. Write $B_r = B_r(y)$ for $r > 0$ in the following proof. Then we have

$$\int_{B_1} |\nabla w|^{p-2} \nabla w \cdot \nabla \varphi dx \leq \int_{B_1} \left( a_1(x) + \frac{a_2(x)}{|x|^s} \right) \eta^p w^{p\tau} dx. \quad (A.5)$$

Firstly, we have

$$\int_{B_1} |\nabla w|^{p-2} \nabla w \cdot \nabla \varphi dx \geq \frac{C}{\tau^{p-1}} \int_{B_1} |\nabla (\eta w)|^p dx - C \int_{B_1} |\nabla \eta|^p w^{p\tau} dx.$$

Secondly, we have

$$\int_{B_1} a_1(x) \eta^p w^{p\tau} dx \leq \left( \int_{B_1} \frac{a_1}{x^s} dx \right) \left( \int_{B_1} (\eta w)^{p\tau} dx \right)^{\frac{p}{p}},$$

and

$$\int_{B_1} a_2(x) \eta^p w^{p\tau} d\mu_s \leq \left( \int_{B_1} \frac{a_2}{x^s} d\mu_s \right) \left( \int_{B_1} (\eta w)^{p\tau(s)} d\mu_s \right)^{\frac{p}{p(s)}}.$$

Thus (A.5) implies that

$$\int_{B_1} |\nabla (\eta w)|^p dx \leq C \int_{B_1} |\nabla \eta|^p w^{p\tau} dx + CA \left( \left( \int_{B_1} (\eta w)^{p\tau} dx \right)^{\frac{p}{p}} + \left( \int_{B_1} (\eta w)^{p\tau(s)} d\mu_s \right)^{\frac{p}{p(s)}} \right), \quad (A.6)$$

where $C = C(\tau) > 0$ and $A$ is given by

$$A = \left( \int_{B_1} \frac{a_1}{x^s} dx \right) \left( \int_{B_1} (\eta w)^{p\tau} dx \right)^{\frac{p}{p}} + \left( \int_{B_1} \frac{a_2}{x^s} d\mu_s \right) \left( \int_{B_1} (\eta w)^{p\tau(s)} d\mu_s \right)^{\frac{p}{p(s)}}.$$

By Sobolev inequality and Caffarelli-Kohn-Nirenberg inequality, we obtain that

$$\left( \int_{B_1} (\eta w)^{p\tau} dx \right)^{\frac{p}{p\tau}} + \left( \int_{B_1} (\eta w)^{p\tau(s)} d\mu_s \right)^{\frac{p}{p(s)}} \leq C(N, p, s) \int_{B_1} |\nabla (\eta w)|^p dx. \quad (A.7)$$

Combining (A.6) and (A.7) yields that

$$\int_{B_1} |\nabla \eta|^p w^{p\tau} dx + CA \left( \left( \int_{B_1} (\eta w)^{p\tau} dx \right)^{\frac{p}{p}} + \left( \int_{B_1} (\eta w)^{p\tau(s)} d\mu_s \right)^{\frac{p}{p(s)}} \right).$$
Thus we can choose
\[ \delta = \delta(\tau) > 0 \] (A.8)
small enough such that if \( A < \delta \), then \( CA < 1/2 \) and
\[ \left( \int_{B_1} (\eta w^*)^{p_r} \, dx \right)^{\frac{1}{p_r}} + \left( \int_{B_1} (\eta w^*)^{p_r(s)} \, dx \right)^{\frac{1}{p_r(s)}} \leq C \int_{B_1} \left| \nabla \eta \right|^p w^{p_r} \, dx. \]
In particular, if \( A < \delta \), we have
\[ \left( \int_{B_1} (\eta w^*)^{p_r} \, dx \right)^{\frac{1}{p_r}} \leq C(\tau) \int_{B_1} \left| \nabla \eta \right|^p w^{p_r} \, dx. \] (A.9)

Let \( 0 < r < R \leq 1 \) and \( \eta \in C_0^\infty(R_R) \) be a cut-off function such that \( 0 \leq \eta \leq 1 \), \( \eta \equiv 1 \) in \( B_r \) and \( |\nabla \eta| \leq 2/(R-r) \). Substituting \( \eta \) into (A.9) gives us that
\[ \left( \int_{B_r} w^{p_x} \, dx \right)^{\frac{1}{p_x}} \leq \frac{C(\tau)}{(R-r)^p} \int_{B_R} w^{p_r} \, dx, \] (A.10)
where \( \chi = p^*/p > 1 \).

Now for any fixed \( q > p^* \), there exists \( k \in \mathbb{N} \) such that \( p\chi^k \leq q < p\chi^{k+1} \). Let \( \tau_i = \chi^i, i = 1, \ldots, k \) and let
\[ \delta = \min\{\delta(\tau_i)\}_{i=1}^k, \]
where \( \delta(\tau_i) \) is defined by (A.8) with \( \tau = \tau_i \). Then if \( A < \delta \), we obtain from (A.10) that, for all \( \tau = \tau_i, i = 1, \ldots, k \),
\[ \left( \int_{B_r} w^{p_{x_i+1}} \, dx \right)^{\frac{1}{p_{x_i+1}}} \leq \frac{C(\tau_i)}{(R-r)^{1/q}} \left( \int_{B_R} w^{p_x} \, dx \right)^{\frac{1}{p_x}}. \]
Let \( r_i = r + (R-r)/2^{i-1}, i \geq 1 \). Take \( r = r_i, R = r_{i-1} \) in the above formula and iterate for finitely many times. We obtain, for any \( 0 < r < R \leq 1 \),
\[ \left( \int_{B_r} w^{p_{x_i+1}} \, dx \right)^{\frac{1}{p_{x_i+1}}} \leq \frac{C}{(R-r)^{\sigma}} \left( \int_{B_R} w^{p_r} \, dx \right)^{\frac{1}{p_r}} \]
for some constants \( C > 0 \) and \( \sigma > 0 \). In particular, we have
\[ \left( \int_{B_r} w^q \, dx \right)^{\frac{1}{q}} \leq \frac{C}{(R-r)^{\sigma}} \left( \int_{B_R} w^{p_r} \, dx \right)^{\frac{1}{p_r}}. \] (A.11)

Fix \( \gamma \in (0, p^*) \). There exists \( \theta \in (0, 1) \) such that
\[ \frac{1}{p^*} = \frac{\theta}{\gamma} + \frac{1-\theta}{q}. \]
Thus by Hölder’s inequality and Young’s inequality, (A.11) implies that
\[ \left( \int_{B_r} w^q \, dx \right)^{\frac{1}{q}} \leq \frac{1}{2} \left( \int_{B_R} w^q \, dx \right)^{\frac{1}{q}} + \frac{C}{(R-r)^{\sigma/\theta}} \left( \int_{B_R} w^\gamma \, dx \right)^{\frac{1}{\gamma}}. \]
Now an iteration argument gives us that
\[ \left( \int_{B_r} w^q \, dx \right)^{\frac{1}{q}} \leq \frac{C}{(R-r)^{\sigma'}} \left( \int_{B_R} w^\gamma \, dx \right)^{\frac{1}{\gamma}} \]
for some constants \( C, \sigma' > 0 \). Choose \( r = 1/2 \) and \( R = 1 \). We complete the proof. \( \square \)
APPENDIX B. A DECAY ESTIMATE

We use $\mathbb{R}_*^N$ to denote either $\mathbb{R}^N$ or $\mathbb{R}_+^N$. Consider the following equation

$$
\begin{align*}
-\Delta_p u &= \mu |u|^{p-2}u + \frac{|u|^{p^*(r)-2}u}{|x|^p}, \quad \text{in } \mathbb{R}_*^N, \\
u \in \mathcal{D}^{1,p}_0(\mathbb{R}_*^N),
\end{align*}
$$

(B.1)

where $\mathcal{D}^{1,p}_0(\mathbb{R}_*^N)$ is the completion of $C_c^{\infty}(\mathbb{R}_*^N)$ in the norm $\|u\|_{\mathcal{D}^{1,p}_0(\mathbb{R}_*^N)} = \|\nabla u\|_{p,\mathbb{R}_*^N}$. In this section, we give an estimate for the decay of solutions to equation (B.1) at the infinity. We have the following result.

**Proposition B.1.** Let $u$ be a solution of (B.1). Then there exists a constant $C > 0$ such that

$$
|u(x)| \leq \frac{C}{1 + |x|^{\frac{2}{p-1}}}, \quad \forall x \in \mathbb{R}_*^N.
$$

To prove Proposition B.1, the following preliminary estimate is needed.

**Lemma B.2.** Let $u$ be a solution of (B.1). Then there is a constant $C > 0$ such that

$$
|u(x)| \leq \frac{C}{1 + |x|^{\frac{N-p}{2} + \sigma}}, \quad \forall |x| \geq 1,
$$

for some $\sigma > 0$.

Lemma B.2 can be proved as that of [6, Lemma B.3] or [7, Proposition 2.1]. So we omit the details. We also need the following comparison principle which is a special case of [7, Theorem 1.5].

**Theorem B.3.** Let $\Omega$ be an exterior domain such that $\Omega^c = \mathbb{R}^N \setminus \Omega$ is bounded and $f \in L^{\frac{N}{p}}(\Omega)$. Let $u \in \mathcal{D}^{1,p}(\Omega)$ be a subsolution of equation

$$
-\Delta_p u = f|u|^{p-2}u \quad \text{in } \Omega,
$$

(B.2)

and $v \in \mathcal{D}^{1,p}(\Omega)$ a positive supersolution of

$$
-\Delta_p v = g|v|^{p-2}v \quad \text{in } \Omega,
$$

(B.3)

such that $\inf_{\partial \Omega} v > 0$, where functions $g$ belongs to $L^{\frac{N}{p}}(\Omega)$ and $f \leq g$ in $\Omega$. Moreover, assume that

$$
\limsup_{R \to \infty} \frac{1}{R} \int_{B_{2R} \setminus B_R} u^p |\nabla \log v|^{p-1} = 0.
$$

(B.4)

If $u \leq v$ on $\partial \Omega$, then

$$
u \leq v \quad \text{in } \Omega.
$$

Now we can prove Proposition B.1.

**Proof of Proposition B.1.** Let $u$ be a weak solution to equation (B.1). In case $\mathbb{R}_*^N = \mathbb{R}_+^N$, we define an odd extension of $u$ by

$$
\tilde{u}(x) = \begin{cases}
  u(x', x_N) & \text{if } x_N \geq 0; \\
  -u(x', -x_N) & \text{if } x_N < 0
\end{cases}
$$

for $x = (x', x_N) \in \mathbb{R}^N$. Then it is direct to verify that $\tilde{u} \in \mathcal{D}^{1,p}(\mathbb{R}^N)$ and $\tilde{u}$ is a solution of equation (B.1) in the whole space $\mathbb{R}^N$. Thus in the rest of the proof we assume that $\mathbb{R}_*^N = \mathbb{R}^N$.

We use Theorem B.3 to prove Proposition B.1. Let $\epsilon > 0$ and denote $\gamma = (N - p)/(p - 1)$. Let $v(x) = |x|^{-\gamma}(1 + |x|^{-\epsilon})$ for $x \neq 0$. A simple calculation gives that

$$
-\Delta_p v = g(x)v^{p-1}, \quad \text{for } x \neq 0,
$$

where $g(x)$ is the right-hand side of equation (B.1). If we can verify that $v(x) \leq \tilde{u}(x)$ for $x = (x', 0) \in \mathbb{R}^N$, then by Theorem B.3 we have

$$
-\Delta_p \tilde{u} \leq g(x)\tilde{u}^{p-1} \quad \text{in } \mathbb{R}^N.
$$

This will give the conclusion we want.

**Proof of the Conclusion.** Let $x = (x', 0) \in \mathbb{R}^N$. We use the comparison principle to show that $v(x) \leq \tilde{u}(x)$. By the definition of $\tilde{u}$, we only need to show that $v(x) \leq u(x')$ for $x_N = 0$. If $x_N = 0$, then $u(x') = 0$; if $x_N > 0$, then $u(x') = u(x')$. Otherwise $x_N < 0$, then $u(x') = -u(x')$. A simple calculation gives that

$$
-\Delta_p v = g(x)v^{p-1}, \quad \text{for } x \neq 0,
$$

where $g(x)$ is the right-hand side of equation (B.1). If we can verify that $v(x) \leq \tilde{u}(x)$ for $x = (x', 0) \in \mathbb{R}^N$, then by Theorem B.3 we have

$$
-\Delta_p \tilde{u} \leq g(x)\tilde{u}^{p-1} \quad \text{in } \mathbb{R}^N.
$$

This will give the conclusion we want.

**Proof of the Conclusion.** Let $x = (x', 0) \in \mathbb{R}^N$. We use the comparison principle to show that $v(x) \leq \tilde{u}(x)$. By the definition of $\tilde{u}$, we only need to show that $v(x) \leq u(x')$ for $x_N = 0$. If $x_N = 0$, then $u(x') = 0$; if $x_N > 0$, then $u(x') = u(x')$. Otherwise $x_N < 0$, then $u(x') = -u(x')$. A simple calculation gives that

$$
-\Delta_p v = g(x)v^{p-1}, \quad \text{for } x \neq 0,
$$

where $g(x)$ is the right-hand side of equation (B.1). If we can verify that $v(x) \leq \tilde{u}(x)$ for $x = (x', 0) \in \mathbb{R}^N$, then by Theorem B.3 we have

$$
-\Delta_p \tilde{u} \leq g(x)\tilde{u}^{p-1} \quad \text{in } \mathbb{R}^N.
$$

This will give the conclusion we want.
where
\[ g(x) = \frac{(p-1)(\gamma + \epsilon)^p - (N-p)(\gamma + \epsilon)^{p-1}}{(1 + |x|^{-\epsilon})^{p-1}|x|^{p+(p-1)\epsilon}}. \]

Since \((p-1)(\gamma + \epsilon)^p - (N-p)(\gamma + \epsilon)^{p-1} > 0 \) for any \( \epsilon > 0 \), it is easy to obtain that
\[ g(x) \geq C|x|^{-p-(p-1)\epsilon}, \quad \text{for } |x| \geq 1, \]
for some constant \( C > 0 \). Thus \( g \in L_+^\infty(\mathbb{R}^N \setminus B_1(0)) \) since \( \epsilon > 0 \).

On the other hand, let \( u \) be a solution to equation (B.1) and denote
\[ f(x) = \mu |u|^{p^* - p} + \frac{|u|^{p^*(s)-p}}{|x|^s}. \]

Lemma B.2 implies that
\[ f(x) \leq C|x|^{-\alpha} \quad \text{for } |x| \geq 1, \]
where
\[ \alpha = \min \left\{ (s-p) \left( \frac{N-p}{p} + \sigma \right), s + (p^*(s)-p) \left( \frac{N-p}{p} + \sigma \right) \right\} = p + (p^*(s)-p)\sigma \]
since \( \sigma > 0 \), we have \( \alpha > p \), and thus \( f \in L_+^\infty(\mathbb{R}^N \setminus B_1(0)) \).

Choose \( \epsilon > 0 \) small such that \( p + (p-1)\epsilon < \alpha \). Then we can find a large number \( R > 1 \) such that
\[ g(x) \geq f(x), \quad \text{for } |x| \geq R. \]

It is easy to verify that the condition (B.4) is satisfied. Therefore, applying Theorem B.3 with \( \Omega = \mathbb{R}^N \setminus B_R(0) \) gives us
\[ \pm u(x) \leq Cv(x), \quad \text{for } |x| \geq R. \]

That is,
\[ |u(x)| \leq C|x|^{-\frac{N-p}{s}}, \quad \text{for } |x| \geq R. \]

So we obtain the decay rate for the solution \( u \) at the infinity. To prove Proposition B.1, one only needs to note that \( u \in L_+^\infty(\mathbb{R}^N) \), which can be done by Moser’s iteration method [24]. This finishes the proof. \( \square \)

**Appendix C. Estimates for \( p \)-Laplacian equation**

In this section, we copy two results on \( p \)-Laplacian equation from [6] without proof. We assume that \( D \) is a bounded domain with \( \Omega \subset \subset D \).

**Proposition C.1.** ([6, Lemma 2.2]) For any functions \( f_1(x) \geq 0 \) and \( f_2(x) \geq 0 \), let \( w \geq 0 \) be the solution of
\[
\begin{align*}
-\Delta_p w &= f_1 + f_2 & \text{in } D, \\
            w &= 0 & \text{on } \partial D.
\end{align*}
\]
Also, let \( w_i, i = 1, 2, \) be the solution of
\[
\begin{align*}
-\Delta_p w &= f_i & \text{in } D, \\
            w &= 0 & \text{on } \partial D,
\end{align*}
\]
respectively. Then, there is a constant \( C > 0 \), depending only on \( r = \frac{1}{2} \text{dist}(\Omega, \partial D) \), such that
\[ w(x) \leq C \inf_{y \in B_r(x)} w(y) + C w_1(x) + C w_2(x), \quad \forall x \in \Omega. \]

Next result gives an estimate for solutions of \( p \)-Laplacian equation by Wolff potential.
Proposition C.2. ([6, Proposition C.1]) There is a constant $\gamma \in (p - 1, (p - 1)N/(N - p + 1))$, such that for any solution $u \in W^{1,p}(D) \cap L^\infty(D)$ to equation

$$-\Delta_p u = f, \quad \text{in } D,$$

where $f \in L^1(D)$, $f \geq 0$, there exists a constant $C = C(N, p, \gamma) > 0$, such that for any $x \in D$ and $r \in (0, \text{dist}(x, \partial D))$,

$$\left(\int_{B_r(x)} u^\gamma dy \right) \overset{\hat{\gamma}}{\leq} C + C \int_r^{\text{dist}(x, \partial D)} \left( \frac{1}{t^{N-p}} \int_{B_t(x)} |f|dy \right) \frac{p-\gamma}{p-\gamma-2} \frac{dt}{t}.$$

APPENDIX D. GLOBAL COMPACTNESS RESULT

Recall that by (2.1) we define, for any function $u$,

$$\rho_{x,\lambda}(u) = \lambda^{\frac{\gamma}{p}} u(\lambda \cdot - x)$$

for any $\lambda > 0$ and $x \in \mathbb{R}^N$. In this section, we give a global compactness result in the following proposition.

Proposition D.1. Let $u_n, n = 1, 2, \ldots$, be a solution of equation (1.4) with $\epsilon = \epsilon_n \to 0$, satisfying $\|u_n\| \leq C$ for some constant $C$ independent of $n$. Then $u_n$ can be decomposed as

$$u_n = u_0 + \sum_{j=1}^{k} \rho_{x_{n,j},\lambda_{n,j}}(U_j) + \sum_{j=k+1}^{m} \rho_{0,\lambda_{n,j}}(U_j) + \omega_n,$$

where $u_0$ is a solution for (1.1), $\omega_n \to 0$ strongly in $W^{1,p}_0(\Omega)$, $x_{n,j} \in \Omega$. And as $n \to \infty$, $\lambda_{n,j} \to \infty$ for all $1 \leq j \leq m, \lambda_{n,j} d(x_{n,j}, \partial \Omega) \to \infty$ for $j = 1, \cdots, k$.

For $j = 1, 2, \cdots, k$, $U_j$ is a solution of

$$\begin{cases}
-\Delta_p u = b_j |u|^{p^*-2}u, & \text{in } \mathbb{R}^N, \\
u \in D^{1,p}(\mathbb{R}^N),
\end{cases}$$

for some $b_j \in (0, 1]$. For $j = k+1, k+2, \cdots, m$, $U_j$ is a solution of

$$\begin{cases}
-\Delta_p u = b_j |u|^{p^*-2}u + b_j \frac{|u|^{p^*(\cdot)-2}u}{|x|}, & \text{in } \mathbb{R}^N_+ \\
u \in D^{1,p}_0(\mathbb{R}^N_+),
\end{cases}$$

for some $b_j \in (0, 1]$, where $\mathbb{R}^N_+ = \mathbb{R}^N$ if $0 \in \Omega$, while $\mathbb{R}^N_+ = \mathbb{R}^N_\ast$ if $0 \in \partial \Omega$.

Moreover, set $x_{n,i} = 0$ for $i = k + 1, \cdots, m$. For $i, j = 1, 2, \cdots, m$, if $i \neq j$, then

$$\frac{\lambda_{n,j}}{\lambda_{n,i}} + \frac{\lambda_{n,i}}{\lambda_{n,j}} + \lambda_{n,i} \lambda_{n,j} |x_{n,i} - x_{n,j}|^2 \to \infty$$

as $n \to \infty$.

Proof. The proof is similar to [6, 8, 29] and we omit the details. $\square$

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REFERENCES

[1] A. Ambrosetti, P. H. Rabinowitz, Dual variational methods in critical point theory and applications. J. Funct. Anal. 14 (1973), 349-381.

[2] A. Bahri, J. Coron, On a nonlinear elliptic equation involving the critical Sobolev exponent: the effect of the topology of the domain. Comm. Pure Appl. Math. 41 (1988), 253-294.

[3] H. Brézis, Nonlinear elliptic equations involving the critical Sobolev exponent-survey and perspectives. Directions in partial differential equations (Madison, WI, 1985), 17-36, Publ. Math. Res. Center Univ. Wisconsin, 54, Academic Press, Boston, MA, 1987.

[4] H. Brézis, L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents. Comm. Pure Appl. Math. 36 (1983), no. 4, 437-477.

[5] L.A. Caffarelli, R. Kohn, L. Nirenberg, First order interpolation inequalities with weights. Compos. Math. 53, (1984), 259-275.

[6] D. Cao, S. Peng, S. Yan, Infinitely many solutions for p-Laplacian equation involving critical Sobolev growth. J. Funct. Anal. 262 (2012), no. 6, 2861-2902.

[7] C.L. Xiang, Asymptotic behaviors of solutions to quasilinear elliptic equations with critical Sobolev growth and Hardy potential. Submitted.

[8] D. Cao, S. Yan, Infinitely many solutions for an elliptic problem involving critical Sobolev growth and Hardy potential. Calc. Var. Partial Differential Equations 38 (2010), no. 3-4, 471-501.

[9] A. Capozzi, D. Fortunato, G. Palmieri, An existence result for nonlinear elliptic problems involving critical exponents. Ann. Inst. H. Poincaré Anal. Non Linéaire 2 (1985), 463-470.

[10] N. Ghoussoub, X. Kang, Hardy-Sobolev critical elliptic equations with boundary singularities. Arch. Ration. Mech. Anal. 203 (2012), no. 3, 943-968.

[11] J.M. Coron, Topologie et cas limite des injections de Sobolev (Topology and limit case of Sobolev embeddings). C. R. Acad. Sci. Paris Sér. I Math. 299 (1984), 209-212 (in French).

[12] M. Degiovanni, S. Lancelotti, Linking solutions for p-Laplace equations with nonlinearity at critical growth. J. Funct. Anal. 256 (2009), 3643-3659.

[13] G. Devillanova, S. Solimini, Concentration estimates and multiple solutions to elliptic problems at critical growth. Adv. Differential Equations 7 (2002), 1257-1280.

[14] H. Egnell, Existence and nonexistence results for m-Laplace equations involving critical Sobolev exponents. Arch. Ration. Mech. Anal. 104 (1988), 57-77.

[15] N. Ghoussoub, X.S. Kang, Hardy-Sobolev critical elliptic equations with boundary singularities. Ann. Inst. H. Poincaré Anal. Non Linéaire 21 (2004), 767-793.

[16] N. Ghoussoub, F. Robert, The effect of curvature on the best constant in the Hardy-Sobolev inequalities. Geom. Funct. Anal. 16 (2006), 1201-1245.

[17] N. Ghoussoub, C. Yuan, Multiple solutions for quasilinear PDEs involving critical Sobolev and Hardy exponents. Trans. Amer. Math. Soc. 352 (2000), 5703-5743.

[18] M. Guedda, L. Veron, Quasilinear elliptic equations involving critical Sobolev exponents. Nonlinear Anal. 13 (1989), 879-902.

[19] C.H. Hsia, C.-S. Lin, H. Wadade, Revisiting an idea of Brézis and Nirenberg. J. Funct. Anal. 259 (2010), 1816-1849.

[20] Y.Y. Li, C.-S. Lin, A nonlinear elliptic PDE and two Sobolev-Hardy critical exponents. Arch. Ration. Mech. Anal. 203 (2012), no. 3, 943-968.

[21] J. Moser, A new proof of De Giorgi’s theorem concerning the regularity problem for elliptic differential equations. Comm. Pure Appl. Math. 13 (1960) 457-468.
[25] E.H. Papageorgiou, N.S. Papageorgiou, A multiplicity theorem for problems with the p-Laplacian. J. Funct. Anal. 244 (2007), 63-77.

[26] P.H. Rabinowitz, Minimax methods in critical points theory with applications to differential equations. CBMS Series, No. 65, Providence, RI, (1986).

[27] Y. Shen, X. Guo, On the existence of infinitely many critical points of the even functional \( \int_{\Omega} F(x, u, Du) dx \) in \( W_0^{1,p} \). Acta Math. Sci. (English Ed.) 7 (1987), 187-195.

[28] M. Struwe, A global compactness result for elliptic boundary value problems involving nonlinearities. Math. Z. 187 (1984), 511-517.

[29] S. Yan, A global compactness result for quasilinear elliptic equations with critical Sobolev exponents. Chinese J. Contemp. Math. 16 (1995), 227-234.

[30] S. Yan, J. Yang, Infinitely many solutions for an elliptic problem involving critical Sobolev and Hardy-Sobolev exponents. Calc. Var. Partial Differential Equations 48 (2013), 587-610.

[31] X. Zhu, Nontrivial solution of quasilinear elliptic equations involving critical Sobolev exponent. Sci. Sin. Ser. A 31 (1988), 1166-1181.

[32] W.P. Ziemer, Weakly differentiable functions. Graduate Texts in Mathematics, 120. Springer-Verlag, New York, 1989.