Periodic-parabolic eigenvalue problems with a large parameter and degeneration

Daniel Daners and Christopher Thornett*

School of Mathematics and Statistics University of Sydney, NSW 2006, Australia
daniel.daners@sydney.edu.au thornett@cims.nyu.edu

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Abstract

We consider a periodic-parabolic eigenvalue problem with a non-negative potential \(\lambda m\) vanishing on a non-cylindrical domain \(D_m\) satisfying conditions similar to those for the parabolic maximum principle. We show that the limit as \(\lambda \to \infty\) leads a periodic-parabolic problem on \(D_m\) having a unique periodic-parabolic principal eigenvalue and eigenfunction. We substantially improve a result from [Du & Peng, Trans. Amer. Math. Soc. 364 (2012), p. 6039–6070]. At the same time we offer a different approach based on a periodic-parabolic initial boundary value problem. The results are motivated by an analysis of the asymptotic behaviour of positive solutions to semilinear logistic periodic-parabolic problems with temporal and spacial degeneracies.

1 Introduction

We consider a periodic-parabolic eigenvalue problem arising in the study of the asymptotic behaviour of positive solutions to a \(T\)-periodic logistic type population problem such as first studied in [25, 7] and later in [2, 3, 20, 21, 22, 31]. The limiting behaviour of the eigenvalue problem allows to deduce information about the corresponding logistic-type semilinear problem. Our focus is in on the case of temporal and spacial degeneracies motivated in particular in [21].

More precisely, we are interested in the behaviour of the principal eigenvalue for the periodic-parabolic eigenvalue problem

\[
\begin{align*}
\frac{\partial u}{\partial t} + A(t)u + \lambda m(x, t)u &= \mu(\lambda)u \quad \text{in } \Omega \times (0, T), \\
B(t)u &= 0 \quad \text{in } \partial\Omega \times (0, T), \\
u(x, 0) &= u(x, T) \quad \text{in } \Omega,
\end{align*}
\]

*Current address: Courant Institute of Mathematical Sciences, New York University, 251 Mercer Street, New York, NY 10012, USA

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as \( \lambda \to \infty \), where \( m \in L^\infty(\Omega \times (0, T)) \) is a non-negative weight function that has a non-trivial zero set satisfying suitable assumptions. Moreover, \( \Omega \subseteq \mathbb{R}^N \) is a bounded domain, and

\[
A(t)u := -\text{div}(D(x, t)\nabla u + a(x, t)u) + (b(x, t) \cdot \nabla u + c_0(x, t)u) \tag{1.2}
\]

is a uniformly strongly elliptic operator with bounded and measurable coefficients and \( B(t) \) a boundary operator of Dirichlet, Neumann or Robin type (for precise assumptions see Section 2).

As in [25], a principal eigenvalue of (1.1) is an eigenvalue having a positive eigenvector. If \( m(x, t) > 0 \) on \( \Omega \times (0, T) \) nothing interesting happens, so we focus on the case where \( m(x, t) = 0 \) in some region \( D_m \subseteq \Omega \times [0, T] \) of non-zero measure. Such problems have been looked at in particular for the corresponding elliptic problem in [2, 9, 31]. The most general weights \( m \) are considered in [1, 21, 32], where spacial and temporal degeneration is allowed. Our aim is to simplify and generalise some of these results using an alternative method and allowing fully non-autonomous operators \( (A(t), B(t)) \) including the principal part.

The approach we take is quite different from previous work and related to the one used in [16] for elliptic systems. Rather than studying the eigenvalue problem (1.1) directly we study what happens to the solution to

\[
\frac{\partial u}{\partial t} + A(t)u + \lambda m(x, t)u = 0 \quad \text{in } \Omega \times (s, T),
\]

\[
B(t)u = 0 \quad \text{in } \partial \Omega \times (s, T),
\]

\[
u(x, s) = u_s(x) \quad \text{in } \Omega,
\]

as \( \lambda \to \infty \), where \( s \in [0, T) \). We consider the behaviour of weak solutions of (1.3) with a non-zero right hand side as \( \lambda \to \infty \) in Section 2. In Section 3 we show that for every initial value \( u_0 \in L^p(\Omega) \) the problem (1.3) has a unique solution \( u \in C([s, T], L^p(\Omega)) \). This in particular allows us to define the evolution operator \( U_\lambda(t, s) \) by

\[
U_\lambda(t, s)u_s := u(t). \tag{1.4}
\]

Letting \( \lambda \to \infty \) we get an evolution operator \( U_\infty(t, s) \) (not necessarily strongly continuous at \( t = s \)). We show that \( U_\lambda(t, s) \) has uniform Gaussian kernel estimates, which lead to uniform \( L^p-L^q \) estimates for solutions of (1.3) and the limit problem as \( \lambda \to \infty \).

Our main results on the convergence and existence of principal periodic-parabolic eigenvalues and eigenfunctions are then given in Section 4, in particular Theorem 4.2. The idea is to use continuity properties of eigenvalues similarly as in [11, 13]. The result generalises [21, Theorem 3.3 and 3.4] significantly by allowing much more general conditions on \( m \). In the final section we show that our conditions on \( m \) are in some sense optimal.

## 2 Convergence of weak solutions

Before stating our main result we make our assumptions precise. We consider a boundary operator of the form (1.2) with \( D \in L^\infty(\Omega \times (0, T), \mathbb{R}^{N \times N}) \), \( a, b \in L^\infty(\Omega \times (0, T)) \), and \( c_0 \in L^1(\Omega \times (0, T)) \).
and \( c_0 \in L^\infty(\Omega \times (0, T), \mathbb{R}) \). We assume that \( A(t) \) is uniformly strongly elliptic, that is, the matrix \( D(x, t) \) is positive definite uniformly with respect to \((x, t)\). Hence there exists \( \alpha > 0 \) such that
\[
y^T D(x, t) y \geq \alpha |y|^2
\]
for all \( y \in \mathbb{R}^N \) and almost all \((x, t) \in \Omega \times (0, T)\). We admit boundary operators of the form
\[
B(t) u := \begin{cases} u_{|\partial \Omega} & \text{Dirichlet boundary operator} \\ (D \nabla u + au) \cdot \nu + b_0 u & \text{Neumann or Robin boundary operator} \end{cases}
\]
where \( \nu \) is the outward pointing unit normal to \( \Omega \) and \( b_0 \in L^\infty(\partial \Omega) \). If \( b_0 = 0 \) we have (natural) Neumann boundary conditions, and if \( b_0 \neq 0 \) we have Robin boundary conditions. In case of Dirichlet boundary conditions we admit any bounded domain \( \Omega \subseteq \mathbb{R}^N \). In case of Neumann or Robin boundary we assume that \( \Omega \) is a Lipschitz domain. In we can then assume without loss of generality that \( b_0 > 0 \) on \( \partial \Omega \) as shown in [15]. We could also have Dirichlet and Robin boundary conditions on disjoint parts of \( \partial \Omega \). We finally assume that \( m \in L^\infty(\Omega \times (0, T)) \) is non-negative and that \( \lambda \in \mathbb{R} \).

We use the theory of variational evolution equations as presented in [18, 30] to study the initial value problem
\[
\frac{\partial u}{\partial t} + A(t)u + \lambda mu = f(x, t) \quad \text{in } \Omega \times (s, T),
\]
\[
B(t)u = 0 \quad \text{in } \partial \Omega \times (s, T),
\]
\[
u(x, s) = u_s \quad \text{in } \Omega,
\]
as \( \lambda \to \infty \). We first look at the \( L^2 \)-theory, and then by means of heat kernel estimates generalise to an \( L^p \)-theory. For \( t \in [0, T] \) we introduce the bilinear forms
\[
a(t, u, v) := \int_{\Omega} (D \nabla u + au) \nabla v + (b \cdot \nabla u + c_0 u) v \, dx + \int_{\partial \Omega} b_0 uv \, d\sigma,
\]
where \( d\sigma \) denotes integration with respect to surface measure on \( \partial \Omega \). That bilinear form is defined on the space
\[
V := \begin{cases} H^1_0(\Omega) & \text{in case of Dirichlet boundary conditions,} \\ H^1(\Omega) & \text{in case of Robin or Neumann boundary conditions.} \end{cases}
\]
For Dirichlet (or Neumann) boundary conditions the boundary integral is not present. From the assumptions on the coefficients of \( A(t) \) there exists a constant \( M \geq 0 \) such that
\[
|a(t, u, v)| \leq M \|u\|_V \|v\|_V
\]
for all \( u, v \in V \) and \( t \in [0, T] \). This is also true for Robin boundary conditions since in that case we assume that \( \Omega \) is a Lipschitz domain and therefore we can use a trace inequality to estimate the boundary integral. Further, one can show that
\[
\frac{\alpha}{2} \|u\|_V^2 \leq a(t, u, u) + \gamma \|u\|^2_V
\]
for all \( u \in V \) and \( t \in [0, T] \), where
\[
\gamma \geq \gamma_0 := \frac{\|a\|_\infty + \|b\|_\infty}{2\alpha} + \|c_0\|_\infty \tag{2.2}
\]
(see for instance [15, Prop 2.4]). Naturally, \( V \hookrightarrow L^2(\Omega) \) is compactly embedded. Identifying \( L^2(\Omega) \) with its dual, \( V \hookrightarrow L_2(\Omega) \hookrightarrow V' \) are dense and compact embeddings, where \( V' \) is the dual of \( V \). In that case, duality is given by
\[
\langle u, v \rangle := \int_\Omega u(x)v(x) \, dx
\]
for \( u \in L^2(\Omega), v \in V \). Given \( f \in L^2((s, T), V') \) we call \( u \in L^2((s, T), V) \) a weak solution of (2.1) if
\[
- \int_s^T \langle \dot{v}(t), u(t) \rangle \, dt - \langle u_s, v(s) \rangle + \int_s^T a(t, u(t), v(t)) \, dt \\
+ \lambda \int_s^T \langle mu(t), v(t) \rangle \, dt = \int_s^T \langle f(t), v(t) \rangle \, dt \tag{2.3}
\]
for all \( v \in C_c^\infty((s, T)) \otimes V \). We then introduce the spaces
\[
W(s, T, V, V') := \{ u \in L^2((s, T), V) : \dot{u} \in L^2((s, T), V') \},
\]
for \( s \in [0, T] \), where \( V' \) is the dual space of \( V \) and \( \dot{u} \) is the derivative with respect to \( t \) in the sense of distributions with values in \( V' \). The space \( W(s, T, V, V') \) is a Hilbert space with the norm
\[
\|u\|_W := \left( \int_s^T \|u(t)\|^2_V \, dt + \int_s^T \|\dot{u}(t)\|_{V'} \, dt \right)^{1/2}.
\]
The space \( W(s, T, V, V') \) has some useful properties. First of all we have the embedding
\[
W(s, T, V, V') \hookrightarrow C([s, T], L^2(\Omega)). \tag{2.4}
\]
For this reason it makes sense to write \( u(t) \) for \( t \in [0, T] \). Moreover the embedding
\[
W(s, T, V, V') \hookrightarrow L^2((s, T), L^2(\Omega)) \tag{2.5}
\]
is compact. We also have the formula of integration by parts
\[
\int_s^t \langle \dot{u}(\tau), v(\tau) \rangle \, d\tau + \int_s^t \langle u(\tau), \dot{v}(\tau) \rangle \, d\tau = \langle u(t), \dot{v}(t) \rangle - \langle u(s), \dot{v}(s) \rangle \tag{2.6}
\]
for all \( u, v \in W(0, T, V, V') \) and \( 0 \leq s \leq t \leq T \). Finally, one may show that \( C_c^\infty((s, T)) \otimes V \) is dense in \( \{ v \in W(s, T, V, V') : v(T) = 0 \} \). This implies that we may test all \( v \in W(s, T, V, V') \) with \( v(T) = 0 \) in the definition (2.3) of a weak solution. By defining the operators \( A(t) \in \mathcal{L}(V, V') \) by \( \langle A(t)u, v \rangle = a(t, u, v) \), we note that \( u \) is a weak solution if and only if \( u \in W(s, T, V, V') \) and
\[
\dot{u}(t) + A(t)u(t) + \lambda m(t)u(t) = f(t) \quad \text{for } t \in (s, T],
\]
\[
u(s) = u_s \tag{2.7}
\]
where equality in the first line is in the sense of \( L^2((s, T), V') \). For all these facts see [18, Section XVIII.§1.2] and [29, Theorem 1.5.1] for the compact embedding (2.5). We first prove some a priori estimates.
Proposition 2.1. Suppose that \( f \in L^2((0,T),V') \) and \( \gamma \) as in (2.2). If \( u \) is a weak solution of (2.1), then
\[
\frac{1}{2} \|u(t)\|_2^2 + \frac{\alpha}{4} \int_s^t \|e^{\gamma(t-\tau)}u(\tau)\|_{V'}^2 d\tau + \lambda \int_s^t \langle m(\tau)u(\tau), u(\tau) \rangle e^{2\gamma(t-\tau)} d\tau \leq \frac{1}{2} \|u(s)e^{\gamma(s)}\|_2^2 + \frac{1}{\alpha} \int_s^t e^{2\gamma(t-\tau)} \|f(\tau)\|_{V'}^2 d\tau
\]
for all \( 0 \leq s \leq t \leq T \) and all \( \lambda \geq 0 \).

Proof. As \( e^{-\gamma t}u(t) \) solves (2.1) with \( \mathcal{A} \) replaced by \( \mathcal{A} + \gamma \) and \( f(t) \) replaced by \( e^{-\gamma t}f(t) \) we conclude from the definition of a weak solution and (2.6) that
\[
\frac{\alpha}{2} \int_s^t \|e^{-\gamma \tau}u(\tau)\|_{V'}^2 d\tau + \lambda \int_s^t \langle m(\tau)u(\tau), u(\tau) \rangle e^{-2\gamma \tau} d\tau \\
\leq \int_s^t a(\tau, e^{-\gamma \tau}u(\tau), e^{-\gamma \tau}u(\tau)) d\tau + \lambda \int_s^t \langle e^{-\gamma \tau}m(\tau)u(\tau), e^{-\gamma \tau}u(\tau) \rangle d\tau \\
= -\frac{1}{2} \|u(t)e^{-\gamma t}\|_2^2 + \frac{1}{2} \|u(s)e^{-\gamma s}\|_2^2 + \int_s^t \langle e^{-\gamma \tau}f(\tau), e^{-\gamma \tau}u(\tau) \rangle d\tau.
\]
By an elementary inequality
\[
\int_s^t \langle e^{-\gamma \tau}f(\tau), e^{-\gamma \tau}u(\tau) \rangle d\tau \leq \frac{\alpha}{4} \int_s^t \|e^{-\gamma \tau}u(\tau)\|_{V'}^2 d\tau + \frac{1}{\alpha} \int_s^t e^{-2\gamma \tau} \|f(\tau)\|_{V'}^2 d\tau,
\]
Putting everything together and multiplying the inequality by \( e^{2\gamma t} \) we get the required estimate. \( \square \)

Using the above estimate we can get a compactness result.

Theorem 2.2. Suppose that \( (f_n) \) is a bounded sequence in \( L_2((0,T),V') \) and that \( \lambda_n \to \infty \). Also assume \( u_{0n} \) is bounded in \( L^2(\Omega) \). Let \( u_n \) be the solution of (2.1) with \( \lambda \) replaced by \( \lambda_n \), \( f \) replaced by \( f_n \) and \( u_0 \) replaced by \( u_{0n} \). For \( \varepsilon > 0 \) let
\[
S_\varepsilon := \{(x,t) \in \Omega \times (0,T) : m(x,t) \geq \varepsilon \} \subseteq \Omega \times (0,T)
\]
Then the following assertions are true.
(i) \( (u_n) \) is bounded in \( L^\infty((0,T),L^2(\Omega)) \) and in \( L^2((0,T),V) \).
(ii) \( u_n \to 0 \) in \( L^2(S_\varepsilon) \) for all \( \varepsilon > 0 \).
(iii) There exists a subsequence \( (n_k) \) such that \( u_{n_k} \rightharpoonup u \) weakly in \( L^2((0,T),V) \), \( u_{0n_k} \rightharpoonup u_0 \) weakly in \( L^2(\Omega) \) and \( f_{n_k} \rightharpoonup f \) weakly in \( L^2((0,T),V') \). Moreover,
\[
-\int_0^T \langle \dot{v}(t), u(t) \rangle dt - \langle u_0, v(0) \rangle + \int_0^T a(t, u(t), v(t)) dt = \int_0^T \langle f(t), v(t) \rangle dt \quad (2.8)
\]
for all \( v \in W(0,T,V,V') \) with \( v = 0 \) on
\[
S_0 := \{(x,t) \in \Omega \times (0,T) : m(x,t) > 0 \} = \bigcap_{\varepsilon > 0} S_\varepsilon. \quad (2.9)
\]
Proof. (i) By Proposition 2.1,
\[
\|u_n(t)\|_2^2 + \frac{\alpha}{4} \int_0^t e^{\gamma(t-\tau)} \|u_n(\tau)\|_V^2 \, d\tau \leq e^{2\gamma T} \|u_0n\|_2^2 + \frac{2e^{2\gamma T}}{\alpha} \|f_n\|_{L^2(0,T,V')} \tag{2.10}
\]
for all \(0 \leq t \leq T\), which remains uniformly bounded as with respect to \(t \in [0,T]\) as \(n \to \infty\). Hence, \((u_n)\) is bounded in \(L^\infty([0,T], L^2(\Omega))\). Since \(\gamma \geq 0\) we have \(e^{\gamma(T-\tau)} \geq 1\) for all \(\tau \in [0,T]\). Setting \(t = T\) it therefore follows from (2.10) that \((u_n)\) is bounded in \(L^2((0,T), V)\).

(ii) By Proposition 2.1 the sequence
\[
\lambda_n \int_0^T \langle m(\tau)u_n(\tau), u_n(\tau) \rangle \, d\tau
\]
remains bounded as \(n \to \infty\). As \(\lambda_n \to \infty\) we conclude that
\[
\|u_{nk}\|_{L^2(U_s)} \leq \frac{1}{\varepsilon} \int_0^T \langle m(\tau)u_{nk}(\tau), u_{nk}(\tau) \rangle \, d\tau \to 0
\]
as \(k \to \infty\).

(iii) By (i) the sequence \((u_n)\) is bounded in \(L^2((0,T), V)\). By assumption, \(u_{0n}\) is bounded in \(L^2(\Omega)\) and \((f_n)\) is bounded in \(L^2((0,T), V')\). Since all spaces are Hilbert spaces, we can select a subsequence such that \(u_{nk} \rightharpoonup u\) weakly in \(L^2((0,T), V)\), \(u_{0nk} \rightharpoonup u_0\) weakly in \(L^2(\Omega)\) and \(f_{nk} \rightharpoonup f\) weakly in \(L^2((0,T), V')\). If \(v \in W(0,T,V,V')\) with \(v = 0\) on \(S_0\), then
\[
\int_0^T \langle mu_{nk}(\tau), v(\tau) \rangle \, d\tau = 0
\]
for all \(k \in \mathbb{N}\) and thus (2.3) reduces to
\[
- \int_0^T \langle \dot{v}(t), u_{nk}(t) \rangle \, dt - \langle u_{0nk}, v(s) \rangle + \int_s^T a(t, u_{nk}(t), v(t)) \, dt = \int_s^T \langle f_{nk}(t), v(t) \rangle \, dt
\]
for all \(k \in \mathbb{N}\). Now (2.8) follows by letting \(k \to \infty\). \qed

In the above theorem we make only minimal assumptions on the weight functions \(m\). In particular, we do not require that the set \(S_0\) given by (2.9) is an open set, nor that it has any regularity. We can say something more about the limit problem if we make some stronger assumptions. We assume that \(\text{supp}(m)\) is topologically regular in the sense that
\[
\text{supp}(m) = \overline{\text{int}(\text{supp}(m))}. \tag{2.11}
\]
We furthermore define the possibly non-cylindrical set
\[
D_m := (\Omega \times [0,T]) \setminus \text{supp}(m) \tag{2.12}
\]
and for \(0 \leq t \leq T\)
\[
\Omega_t := \{x \in \Omega; (x,t) \in D_m\}. \tag{2.13}
\]
Intuitively, the limit problem satisfies Dirichlet boundary conditions on $\partial D_m \cap (\Omega \times (0, T])$ because the solution of the limit problem is forced to be zero outside $D_m$. For this to be really true we need to assume that $\Omega_t$ is stable in the sense of Keldysh [26] for all $t \in [0, T]$. This means that

$$H^1_0(\Omega_t) = \{ w \in H^1_0(\Omega) : w = 0 \text{ a.e. on } \Omega \setminus \Omega_t \};$$

(2.14)

see the discussions in [5, 8, 24, 31, 34]. We have the following corollary, where as usual all boundary conditions are satisfied in the weak sense.

**Corollary 2.3.** Suppose that the assumptions of Theorem 2.2 are satisfied, that supp$(m)$ satisfies (2.11) and that $D_m \neq \emptyset$. Let $u$ be the limit of $(u_m)$ as in Theorem 2.2(iii).

(i) Then $u$ is a (local) weak solution of the parabolic limit problem

$$\begin{align*}
\frac{\partial u}{\partial t} + A(t)u &= f(x, t) \quad \text{in } D_m, \\
B(t)u &= 0 \quad \text{on } \partial D_m \cap (\partial \Omega \times (0, T)) \\
u(x, 0) &= u_0(x) \quad \text{on } \Omega_0.
\end{align*}$$

(2.15)

(ii) Suppose that $\Omega_t$ is stable in the sense of (2.14) for all $t \in [0, T]$. Then the solution $u$ of (2.15) satisfies Dirichlet boundary conditions $u = 0$ on $\partial \Omega_t \cap \Omega$ for almost all $t \in [0, T]$.

**Proof.** (i) Note that (2.8) is equivalent to the statement that $u$ is a weak solution of (2.15).

(iii) It follows from Theorem 2.2(ii) and the regularity assumption (2.11) that $u = 0$ on supp$(m)$. As $u \in L^2((0, T), V)$ we have that $u(t) \in V \subseteq H^1(\Omega)$ for almost all $t \in (0, T)$ with $u(t) = 0$ on $\Omega \setminus \Omega_t$. Since stability is a local property of the boundary of $\Omega_t$ it follows that $u(t) \in H^1_0(\Omega_t \cap U)$ for every open set $U$ with $U \cap \partial \Omega = \emptyset$. Hence $u$ satisfies Dirichlet boundary conditions in the weak sense on $\partial \Omega_t \cap \Omega$. □

## 3 The evolution operator and $L^p$-theory

It is shown in [12] that the evolution operator $U_\lambda(t, s)$, $0 \leq s \leq t \leq T$ defined in (1.4) is acting on $L^p(\Omega)$ for $1 \leq p \leq \infty$. A key fact we establish is that $U_\lambda(t, s)$ has a kernel satisfying Gaussian estimates uniformly with respect to $\lambda \geq 0$.

**Theorem 3.1.** Let $m \in L^\infty(\Omega \times (0, \infty))$ be non-negative. For all $1 \leq p \leq q \leq \infty$ the evolution operator $U_\lambda(t, s) \in L(L^p(\Omega), L^q(\Omega))$ is a compact positive irreducible operator having a kernel $k_\lambda(x, y, t, s)$ satisfying a Gaussian estimate. More precisely, there exist constants $M \geq 1$, $\omega \in \mathbb{R}$ and $c > 0$ such that

$$0 < k_\lambda(x, y, t, s) \leq k_{\lambda_1}(x, y, t, s) \leq M e^{\omega(t-s)}(t-s)^{-N/2} e^{-c(x-y)^2 / (t-s)}$$

(3.1)

for all $0 \leq \lambda_1 \leq \lambda < \infty$, $0 \leq s < t \leq T$ and $x, y \in \Omega$. Moreover,

$$\|U_\lambda(t, s)\|_{\mathcal{L}(L^p, L^q)} \leq M t^{-\frac{N}{2} \left(\frac{1}{p} - \frac{1}{q} \right)} e^{\omega(t-s)}$$

(3.2)

If $u_0 \geq 0$, then $U_\lambda(t, s)u_0$ is decreasing as $\lambda \to \infty$. 

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Proof. From [12, Section 6 and 8] the evolution operator $U_\lambda(t,s)$ is a positive operator on $L^p(\Omega)$ with kernel $k_\lambda(x,y,t,s)$ satisfying a Gaussian estimate. Assume now that $\lambda_1 \leq \lambda_2$ and that $u_0 \in L^p(\Omega)$ is non-negative. For $i=1,2$ set $u_i := U_{\lambda_i}(\cdot,s)u_0$. We want to show that $u_2 \leq u_1$. Clearly $u_2$ is the solution of
\[
\frac{\partial u_2}{\partial t} + \mathcal{A}(t)u_2 + \lambda_1 m u_2 = -(\lambda_2 - \lambda_1) m u_2 \quad \text{in } \Omega \times (s,T),
\]
\[
\mathcal{B}(t)u = 0 \quad \text{in } \partial \Omega \times (s,T),
\]
\[
u(x,s) = u_0 \quad \text{in } \Omega,
\]
and therefore by the variation of constants formula for variational evolution equations (see for instance [10, Section 4])
\[0 \leq u_2(t) = u_1(t) - (\lambda_2 - \lambda_1) \int_s^t U_{\lambda_1}(t,\tau)m(\tau)u_2(\tau)\,d\tau.
\]
for all $s \in [0,T)$ and all $t \in [s,T]$. Here we used that $u_1(t) = U_{\lambda_1}(t,s)u_0$. We already know that $u_1, u_2 \geq 0$. As $m \geq 0$ and $\lambda_2 - \lambda_1 \geq 0$ and $U(t,\tau)$ is a positive operator it follows that $u_2 \leq u_1$. In particular, $U_\lambda(t,s)u_0$ is decreasing in $\lambda$ if $u_0 \geq 0$. In terms of the heat kernels the above writes
\[\int_\Omega k_\lambda(x,y,t,s)u_0(y)\,dy = u_2(x,t) \leq u_1(x,t) = \int_\Omega k_\lambda(x,y,t,s)u_0(y)\,dy
\]
for all non-negative $u_0 \in L^p(\Omega)$. Hence (3.1) follows from [12, Theorem 7.1] by taking the estimate of the kernel for $\lambda=0$. As $|U_\lambda(t,s)u_0| \leq U_\lambda(t,s)|u_0|$ we get
\[\|U_\lambda(t,s)\|_{L(L^p,L^q)} \leq \|U_0(t,s)\|_{L(L^p,L^q)}
\]
for all $\lambda \geq 0$. Now (3.2) follows from [12, Corollary 7.2].

We next look at convergence and compactness properties of the evolution operator.

**Theorem 3.2.** Under the assumptions of Theorem 3.1, for every $0 \leq s < t \leq T$ and $1 < p \leq q < \infty$
\[U_\infty(t,s) := \lim_{\lambda \to \infty} U_\lambda(t,s)
\]
exist in $L(L^p(\Omega),L^q(\Omega))$. Moreover, $U_\infty(t,s)$ is a positive compact operator on $L^p(\Omega)$ with kernel $k_\infty(x,y,t,s)$ satisfying a Gaussian estimate, and
\[U_\infty(t,s) = U_\infty(t,\tau)U_\infty(\tau,s)
\]
whenever $0 \leq s < \tau < t \leq T$. Finally, the linear operator defined by
\[u_0 \mapsto U_\lambda(\cdot,s)u_0
\]
converges to the corresponding operator in $L(L^p(\Omega),L^r((s,T),L^q(\Omega)))$ as $\lambda \to \infty$ whenever $1 < r < \infty$ and $1 < p \leq q < \infty$ are such that
\[\frac{N}{2} \left( \frac{1}{p} - \frac{1}{q} \right) < \frac{1}{r}.
\]
Proof. First we look at strong convergence of $U_\lambda(t, s)$. For $u_0 \in L^p(\Omega)$ non-negative we know from Theorem 3.1 that $U_\lambda(\cdot, s)u_0$ decreases as $\lambda \to \infty$ on the cylinder $\Omega \times (s, T]$ and therefore

$$u(x, t) := \lim\limits_{\lambda \to \infty} [U_\lambda(t, s)u_0](x) = \lim\limits_{\lambda \to \infty} \int_\Omega k_\lambda(t, s, x, y) u_0(y) \, dy$$

exists for all $(x, t) \in \Omega \times (s, T]$. By (3.1) also

$$0 \leq k_\infty(t, s, x, y) := \lim\limits_{\lambda \to \infty} k_\lambda(t, s, x, y) \leq M e^{\omega(t-s)}(t-s)^{-N/2} e^{-\frac{k^2}{1-k^2}}$$

exists for all $0 \leq s < t \leq T$ and $x, y \in \Omega$. By the dominated convergence theorem

$$u(x, t) = \lim\limits_{\lambda \to \infty} \int_\Omega k_\lambda(t, s, x, y) u_0(y) \, dy = \int_\Omega k_\infty(t, s, x, y) u_0(y) \, dy,$$

so $U_\infty(t, s)$ has a kernel with a Gaussian estimate. By splitting an arbitrary initial condition into its positive and negative part the above limit exists for every $u_0 \in L^p(\Omega)$.

Let now $1 < p \leq q < \infty$. By Hölder’s inequality

$$\| (U_\lambda(t, s) - U_\infty(t, s)) u_0 \|_q$$

$$= \left( \int_\Omega \left( \int_\Omega (k_\lambda(x, y, t, s) - k_\infty(x, y, t, s)) u_0(y) \, dy \right)^q \, dx \right)^{1/q}$$

$$\leq \left( \int_\Omega \left( \int_\Omega (k_\lambda(x, y, t, s) - k_\infty(x, y, t, s))^\frac{q}{p'} \, dy \right)^{\frac{q}{p'}} \, dx \right)^{1/q} \| u_0 \|_p.$$

By (3.1), (3.6) and the dominated convergence theorem

$$\| U_\lambda(t, s) - U_\infty(t, s) \|_{L^r(\Omega)}$$

$$\leq \left( \int_\Omega \left( \int_\Omega (k_\lambda(x, y, t, s) - k_\infty(x, y, t, s))^\frac{q}{p'} \, dy \right)^{\frac{q}{p'}} \, dx \right)^{1/q} \to 0$$

as $\lambda \to \infty$. Hence $U_\lambda(t, s) \to U_\infty(t, s)$ in $L^r(\Omega)$ whenever $1 < p \leq q < \infty$. As the limit of compact operators is a compact operator, we conclude that $U_\infty(t, s) \in L^r(\Omega)$ is compact.

We next prove convergence of $u_0 \to U_\lambda(\cdot, s)u_0$ as a linear operator with respect to the norm in $L^r(\Omega)$ for suitable $r, p, q$. We already know from what we proved above that

$$\| U_\lambda(t, s) - U_\infty(t, s) \|_{L^r(\Omega)} \to 0$$

for every $t \in (s, T]$ if $1 < p \leq q < \infty$. We need to show that

$$\int_s^T \| U_\lambda(t, s) - U_\infty(t, s) \|_{L^r(\Omega)} \, dt \to 0$$

as $\lambda \to \infty$. We deduce from (3.2) that

$$\| U_\lambda(t, s) - U_\infty(t, s) \|_{L^r(\Omega)}$$

$$\leq \| U_\lambda(t, s) \|_{L^r(\Omega)} + \| U_\infty(t, s) \|_{L^r(\Omega)} \leq 2 M (t-s) e^{-\frac{\omega}{2} (t-s)}.$$
for all $0 < s < t \leq T$ with constants $M$ and $\omega$ independent of $\lambda > 0$. We note that
\[
\int_s^T (t - s)^{-\frac{\lambda}{2}(\frac{1}{p} - \frac{1}{q})} e^{\omega(t-s)^r} dt \leq e^{\omega T} \int_s^T (t - s)^{-\frac{\lambda}{2}(\frac{1}{p} - \frac{1}{q})} dt < \infty
\]
if and only if (3.5) is satisfied. Hence, (3.7) follows from the dominated convergence theorem.

\[\square\]

Remark 3.3. The family $U_\infty(t, s)$, $0 \leq s \leq t \leq T$, is not in general an evolution operator since in general $U_\infty(s, s)$ is not the identity, but only a projection. In the extreme case where $m(x, t) > 0$ in $\Omega \times [0, T]$, then $U_\infty(t, s) = 0$ is the zero operator. Hence we need conditions that guarantee that $U_\infty(\cdot, \cdot)$ is non-trivial.

Proposition 3.4. Suppose that $m \in L^\infty(\Omega \times (0, T))$ and that there exists a non-empty open set $\Omega_0 \subset \Omega$, $s_0 \in [0, T)$ and $\varepsilon > 0$ so that $m = 0$ almost everywhere on $\Omega_0 \times (s_0, s_0 + \varepsilon)$. Then $U_\infty(t, s) \neq 0$ for $s_0 < s \leq t < s_0 + \varepsilon$. More precisely if $K(x, y, t, s)$ is the kernel of the evolution operator of the problem
\[
\frac{\partial u}{\partial t} + A(t)u = 0 \quad \text{in } \Omega_0 \times (s_0, s_0 + \varepsilon),
\]
\[
u = 0 \quad \text{on } \partial \Omega_0 \times (s_0, s_0 + \varepsilon),
\]
\[
u(x, s_0) = u_{s_0}(x) \quad \text{in } \Omega_0,
\]
then $k_\infty(x, y, t, s) \geq K(x, y, t, s) > 0$ for all $x, y \in \Omega_0$ and all $s_0 \leq s \leq t \leq s_0 + \varepsilon$.

Proof. Clearly the operators $A(t) + \lambda m$ and $A(t)$ coincide on $\Omega_0 \times (s_0, s_0 + \varepsilon)$. Hence from [12, Theorem 8.3] we deduce that
\[
k_\lambda(x, y, t, s) \geq K(x, y, t, s) > 0
\]
for all $x, y \in \Omega_0$ and all $s_0 \leq s \leq t \leq s_0 + \varepsilon$. Here we also use that the kernel of the problem with Neumann or Robin boundary conditions dominates that of the problem with Dirichlet boundary conditions. Now the assertion of the theorem follows from (3.6). \[\square\]

Using the evolution operator we can generalise the notion of solution of (1.1) for right hand sides not necessarily in $L^2((0, T), V')$.

Definition 3.5. Let $1 \leq r, p \leq \infty$, $u_0 \in L^p(\Omega)$ and $f \in L^r(0, T, L^p(\Omega))$. We call
\[
u(t) = U_\lambda(t, 0)u_0 + \int_0^t U_\lambda(t, \tau) f(\tau) d\tau,
\]
(3.9)
t $\in [0, T]$ a mild solution of (2.1). Likewise we call a $u$ a mild solution of the limit problem as $\lambda \to \infty$ if
\[
u(t) = U_\infty(t, 0)u_0 + \int_0^t U_\infty(t, \tau) f(\tau) d\tau,
\]
(3.10)
for all $t \in [0, T]$.\[10\]
Remark 3.6. By the Sobolev embedding theorem $V \hookrightarrow L^q(\Omega)$ for $q \leq 2N/(N - 2)$ if $N \geq 3$ and $q < \infty$ if $N = 2$. Hence $L^p(\Omega) \hookrightarrow V'$ for $p \geq 2N/(N + 2)$ if $N \geq 3$ and $p > 1$ if $N = 2$. Thus, if $r \geq 2$, then
\[ L^r((0, T), L^p(\Omega)) \hookrightarrow L^2((0, T), V') \]

for $p \geq 2N/(N + 2)$ if $N \geq 3$ and $p > 1$ if $N = 2$. The above embedding always holds if $N = 1$ and $1 \leq p \leq \infty$. In these cases every mild solution of (1.1) is a weak solution of (1.1).

Now that we know that the limit problem is non-trivial in general we strengthen some results from Theorem 2.2.

**Theorem 3.7.** Assume that $m$ satisfies (2.11). Suppose that $1 < p \leq q \leq \infty$ and $1 < r < \infty$ such that
\[ \frac{N}{2} \left( \frac{1}{p} - \frac{1}{q} \right) < \min \left\{ \frac{1}{r}, 1 - \frac{1}{r} \right\}, \tag{3.11} \]

Assume that $u_{0n} \rightharpoonup u_0$ weakly in $L^p(\Omega)$ and that $f_n \rightharpoonup f$ in $L^r((0, T), L^p(\Omega))$. Let $u_n$ be the mild solution of (2.1) with $\lambda_n$ replaced by $\lambda_n$, $f$ replaced by $f_n$ and $u_0$ replaced by $u_{0n}$. Finally suppose that $\lambda_n \to \infty$. Then $u_n(t) \to u(t)$ in $L^q(\Omega)$ for all $t \in (0, T]$ and $u$ satisfies (3.10). Moreover, $u_n \to u$ in $L^r((0, T), L^q(\Omega))$.

**Proof.** We know that
\[ u_n(t) = U_{\lambda_n}(t, 0)u_{0n} + \int_0^t U_{\lambda_n}(t, \tau)f_n(\tau) \, d\tau \tag{3.12} \]

for all $t \in (0, T]$. As $(u_{0n})$ is weakly convergent in $L^p(\Omega)$ there exists $c_1 > 0$ such that $\|u_{0n}\|_p \leq c_1$ for all $n \in \mathbb{N}$. Moreover, since $U_\infty(t, 0) \in L(L^p(\Omega), L^q(\Omega))$ is compact and $U_{\lambda_n}(t, 0) \to U_\infty(t, 0)$ in $L(L^p(\Omega), L^q(\Omega))$ by Theorem 3.2 we see that
\[
\|U_{\lambda_n}(t, 0)u_{0n} - U_\infty(t, 0)u_0\|_q \\
\leq \|U_{\lambda_n}(t, 0) - U_\infty(t, 0)\|_{L(L^p, L^q)}\|u_{0n}\|_p + \|U_\infty(t, 0)(u_{0n} - u_0)\|_q \to 0
\]

for every $t \in (0, T]$ as $n \to \infty$. Using the uniform kernel estimates from Theorem 3.1 we see that
\[
\|U_{\lambda_n}(t, 0)u_{0n} - U_\infty(t, 0)u_0\|_q \leq 2e^{|\omega|T}M c_1 t^{-\frac{N}{2}(\frac{1}{p} - \frac{1}{q})}
\]

for all $t \in (0, T]$. As $t^{-\frac{N}{2}(\frac{1}{p} - \frac{1}{q})}$ is integrable on $(0, T)$ by (3.11), the dominated convergence theorem implies that $U_{\lambda_n}(\cdot, 0)u_{0n} \to U_\infty(\cdot, 0)u_0$ in $L^r((0, T), L^q(\Omega))$.

We next deal with the integral term in (3.12). Using Hölder’s inequality,
\[
\int_0^t \|U_{\lambda_n}(t, \tau)f_n(\tau) - U_\infty(t, \tau)f(\tau)\|_q \, d\tau
\leq \int_0^t \|U_{\lambda_n}(t, \tau) - U_\infty(t, \tau)\|_{L(L^p, L^q)}\|f_n(\tau)\|_p \, d\tau
\]
\[ + \int_0^t \|U_\infty(t, \tau)\|_{L(L^p, L^q)}\|f_n(\tau) - f(\tau)\|_q \, d\tau \tag{3.13} \]
\[
\leq \left( \int_0^t \|U_{\lambda_n}(t, \tau) - U_\infty(t, \tau)\|_{L(L^p, L^q)}\right)^{\frac{1}{2}} \|f_n\|_{L^r((0, T), L^p)}
\]
\[ + \left( \int_0^t \|U_{\lambda_n}(t, \tau)\|_{L(L^p, L^q)}\right)^{\frac{1}{2}} \|f_n - f\|_{L^r((0, T), L^p)}
\]
for all $t \in [0,T]$. By again using the uniform kernel estimates

$$\|U_{\lambda_n}(t, \tau) - U_{\infty}(t, \tau)\|_{L(L^p, L^q)} \leq 2e^{\omega|T|}M(t-\tau)^{-\frac{N}{2p} - \frac{1}{r}}$$

for all $0 \leq \tau < t \leq T$. It follows from (3.11) that

$$\eta := \frac{N}{2} \left( \frac{1}{p} - \frac{1}{q} \right) \frac{r}{r-1} < 1.$$ 

and hence $(t-\tau)^{-\eta}$ is integrable on $(0,t)$. As $f_n \to f$ in $L^r((0,T), L_p(\Omega))$ we conclude from Theorem 3.2 and (3.13) that

$$\int_0^t U_{\lambda_n}(t, \tau)f_n(\tau) \, d\tau \to \int_0^t U_{\infty}(t, \tau)f(\tau) \, d\tau$$

in $L^q(\Omega)$ for all $t \in (0,T]$. It also follows that

$$\left\| \int_0^t U_{\lambda_n}(t, \tau)f_n(\tau) \, d\tau \right\| q \leq e^{\omega|T|CT^{1-\eta}}$$

for all $t \in [0,T]$, where $C$ is a constant independent of $n$ and $1-\eta > 0$. In particular, the integral part in (3.12) converges in $L^r((0,T), L^q(\Omega))$ as well.

In the above theorem we have excluded the case $r, q = \infty$, that is, uniform convergence. The next theorem shows local convergence in a space of (locally) Hölder continuous functions on $D_m \cap (\Omega \times (\varepsilon, T])$ for every $\varepsilon \in (0, T)$.

**Theorem 3.8.** Assume that $m$ satisfies (2.11). Suppose that $N/2 < p \leq \infty$ and $2 \leq r < \infty$ such that

$$\frac{N}{2p} + \frac{1}{r} < 1. \tag{3.14}$$

Assume that $u_{n_0} \to u_0$ weakly in $L^p(\Omega)$ and that $f_n \to f$ in $L^r((0,T), L_p(\Omega))$. Let $u_n$ be the mild solution of (2.1) with $\lambda$ replaced by $\lambda_n$, $f$ replaced by $f_n$ and $u_0$ replaced by $u_{n_0}$. Finally suppose that $\lambda_n \to \infty$. Then for every $\varepsilon \in (0,T)$ and every compact subset $K \subseteq D_m \cap (\Omega \times [\varepsilon, T])$ there exists $\beta \in (0,1)$ such that $u_n \to u$ in $C^\beta(K)$.

**Proof.** First note that (3.14) implies that $f_n \in L^2((0,T), V')$, and that the sequence $(f_n)$ is bounded in that space; see Remark 3.6. Hence, by Theorem 2.2 $u_n \to u$ weakly in $L^2((0,T), V)$. By Corollary 2.3 $u$ is a weak solution of (2.15). The weak solution of (2.1) is given by

$$u_n(t) = U_{\lambda_n}(t,0)u_{n_0} + \int_0^t U_{\lambda_n}(t, \tau)f_n(\tau) \, d\tau.$$ 

Using the uniform kernel estimates from Theorem 3.1 we see that

$$\|U_{\lambda_n}(t,0)u_{n_0}\|_\infty \leq M t^{-\frac{N}{2p}} e^{\omega t} \|u_{n_0}\|_p$$

$$\|U_{\lambda_n}(t, \tau)f_n(\tau)\|_\infty \leq M(t-\tau)^{-\frac{N}{2p}} e^{\omega(t-s)} \|f_n(\tau)\|_p \tag{3.15}$$
for all $n \in \mathbb{N}$ and all $0 < \tau < t \leq T$. Hence by Hölder’s inequality

$$
\left\| \int_0^t U_{n_m}^n(t, \tau) f_n(\tau) \, d\tau \right\|_\infty \leq Me^{\omega_1 T} \left( \int_0^t \left( t - \tau \right)^{-N\frac{N}{2p} - \frac{N}{2} - \frac{1}{2}} \, d\tau \right)^{1 - \frac{1}{2}} \left\| f_n \right\|_{L^p((0,T),L^p)} \quad (3.16)
$$

The second integral in (3.16) is finite if and only if (3.14) holds. Putting everything together we see that the sequence $(u_n)$ is bounded in $L^\infty(\Omega \times (\varepsilon, T))$ for every $\varepsilon > 0$. Since $u_n$ is a solution of (2.15) with $f$ replaced by $f_n$ we conclude from [6, Theorem 4] that for $\varepsilon \in (0, T)$ and every compact subset $K \subseteq D_m \cap ([\varepsilon, T] \times \Omega)$ there exists $\gamma \in (0, 1)$ such that $u_n$ is bounded in $C^\gamma(K)$. As we know that $u_n \to u$ weakly in $L^2(\Omega \times (0, T))$, we conclude that $u_n \to u$ in $C^\beta(K)$ for $\beta \in (0, \gamma)$. Here we use that Hölder spaces with different exponents embed compactly. □

**Remark 3.9.** If we strengthen the regularity assumptions on the coefficients of $(A(t), B(t))$, $m$ and $f_n$ we obtain (local) convergence in $D_m$ in stronger norms. In particular, assume that the the coefficients of the diffusion matrix $D$ and the vector field $u$ in (1.2) are in $C^{1+\beta,\beta/2}$ and $b, c, b_0, m, f_n$ are in $C^{\beta,\beta/2}$ for some $\beta \in (0, 1)$. Then the Schauder theory in [27, Theorem VI.10.1] or [28, Theorem 3.4.9] shows that on every compact subset $K \subseteq D_m \cap (\Omega \times [\varepsilon, T])$ there exists $\gamma \in (0, 1)$ such that $u_n$ is bounded in $C^{2+\gamma,1+\gamma/2}(K)$. Hence $u_n \to u$ in $C^{2+\beta,1+\beta/2}(K)$ for $\beta \in (0, \gamma)$. Convergence in $C^{\beta,\beta/2}$ may also be true up to the boundary depending on the regularity of $D_m$, in particular if $D_m$ contains parabolic cylinders with sufficiently smooth boundary such as the situation considered in [21] corresponding to the example on the right in Figure 3.1; see also Example 3.13.

Based on Proposition 3.4 we show that $U_{\infty}(t, s)$ is has some nice properties for all $0 \leq s < t \leq T$ if $m$ satisfies certain conditions.

**Assumption 3.10.** Let $m \in L^\infty(\Omega \times [0, T])$ and assume that the support of $m$ is topologically regular, that is, (2.11) is satisfied. We define the sets $D_m$ and $\Omega_t$ as in (2.12) and (2.13) respectively. Assume that $\Omega_t \neq \emptyset$ for every $t \in [0, T]$. Suppose that for every pair of points $y \in \Omega_t$ and $x \in \Omega_t$ with $t \in (0, T]$ there exist a continuous function $\varphi: [0, t] \to \Omega$ with $\varphi(0) = y$, $\varphi(1) = x$ and such that $(\varphi(\tau), \tau) \in D_m$ for all $\tau \in [0, t]$; see Figure 3.1.

**Remark 3.11.** (a) The condition about the existence of the curve $\varphi$ in Assumption 3.10 is related to the condition on non-cylindrical regions in the parabolic maximum principle. The condition for the validity of the maximum principle is that the point is to be reached by a continuous path that only goes “horizontal” or “upwards”, that is, “forward” in time; see [33, p169] or [23], where also a counter example is shown if the condition is violated. As a consequence the limit problem is well behaved in the sense that the parabolic maximum principle is valid for the non-cylindrical domain $D_m$ and hence there are uniqueness theorems.

(b) Our condition also guarantees that $D_m$ is connected. If $D_m$ is not connected, we apply our arguments to every connected component. Examples are shown in Figure 3.1.

(c) The diagram on the right in Figure 3.1 is the special situation considered in [1, 21], where $T^r$ is as in these references.
Theorem 3.12. Suppose that $m$ satisfies Assumption 3.10 and let $k_\infty$ be the kernel of the limit evolution system $U_\infty$ as in Theorem 3.2. Then $k_\infty(x, y, t, 0) > 0$ for all $t \in (0, T]$ and all $(x, y) \in \Omega_t \times \Omega_0$.

Proof. Fix $(y, 0), (x, t) \in D_m$ with $0 < t \leq T$ and let $\varphi : [0, t] \to \Omega$ be as in Assumption 3.10. We consider the set

$$I := \{ s \in (0, t] : k_\infty(\varphi(\tau), y, \tau, 0) > 0 \text{ for all } \tau \in [0, s] \}.$$  

We need to show that $I = (0, t]$. Because $(0, t]$ is connected it is sufficient to show that $I$ is non-empty, open and closed in $(0, t]$. To do so we use the fact that the function

$$(0, t] \to [0, \infty), \tau \mapsto k_\infty(\varphi(\tau), y, \tau, 0)$$

is continuous. (3.17)

Further note that if $s \in I$, then $(0, s] \subseteq I$ by definition of $I$.

We first show that $I$ is non-empty. As $(y, 0) \in D_m$ and $D_m$ is open there exists an open neighbourhood $V_0 \subseteq \Omega$ of $y$ and an interval $[0, s_0]$ such that $V_0 \times [0, s_0] \subseteq D_m$. Proposition 3.4 implies that $k_\infty(z, y, \tau, 0) > 0$ for all $(z, \tau) \in V_0 \times J_0$. By (3.17) there exists $\tau_0 \in (0, s_0]$ such that $\varphi(\tau) \in V$ for all $\tau \in (0, \tau_0]$ and hence $k_\infty(\varphi(\tau), y, \tau, 0) > 0$ for all $\tau \in (0, \tau_0]$. Hence, $\tau_0 \in I$ and so $I \neq \emptyset$.

We next show that $I$ is open. If $s \in I$, then $k(\varphi(\tau), y, \tau, 0) > 0$ for all $\tau \in (0, s)$. In particular, $k(\varphi(s), y, s, 0) > 0$. By (3.17) there exists $s_1 > s$ so that $k(\varphi(\tau), y, \tau, 0) > 0$ for all $\tau \in (0, s_1]$. Hence $I$ is open.

We finally show that $I$ is closed. If $s > 0$ is in the closure of $I$, then $k_\infty(\varphi(\tau), y, \tau, 0) > 0$ for all $\tau \in (0, s)$. Because $D_m$ is open there exists a non-empty open set $V \subseteq \Omega$ and an open interval $J \subseteq \mathbb{R}$ such that $(\varphi(s), s) \in V \times J \subseteq D_m$. Now Proposition 3.4 implies that $k_\infty(z, w, s, \tau) > 0$ for all $z, w \in V$ and all $\tau \in J$ with $\tau < s$. Due to (3.17) we can choose $\tau_0 \in J$ with $\tau_0 < s$ such that $\varphi(\tau_0) \in V$. Then, by (3.4)

$$k_\infty(x, y, s, 0) = \int_{\Omega_s} k_\infty(x, z, s, \tau_0) k_\infty(z, y, \tau_0, 0) dz. \quad (3.18)$$

We have chosen $V$ and $J$ such that $k_\infty(x, z, s, \tau_0) > 0$ for all $z \in V$. In particular $k_\infty(x, z, s, \tau_0) > 0$ for $z$ in a neighbourhood of $\varphi(\tau_0)$. By the continuity of $k_\infty(x, z, s, \tau_0)$ as a function of $z$ we also deduce that $k_\infty(x, z, s, \tau_0) > 0$ for all $z$ in a neighbourhood of $\varphi(\tau_0)$. As $k_\infty \geq 0$ we conclude from (3.18) that $k_\infty(x, y, \tau_0, 0) > 0$. Hence, $s \in I$ and thus $I$ is closed. \qed

Figure 3.1: Parabolic cylinder with $\text{supp } m$ (shaded) and $D_m$ with $0 \leq t \leq T$. 

![Diagram](image-url)
We complete this section by reviewing the special case of \( m \) treated in [21].

**Example 3.13.** The special case considered in [21] is \( m \) of the form \( m(x,t) = p(x)q(t) \) with \( \text{supp}(p) = \Omega \setminus U_0 \) for some non-empty open set \( U_0 \) and \( \text{supp} q = [T^*, T] \) for some \( T^* \in (0, T) \), or the slightly more general situation given in [21, condition (3.2)]. The situation is depicted in Figure 3.1 on the right. We only assume that \( m \in L^\infty(\Omega \times (0, T)) \). The set \( D_m \) consists of two cylindrical regions: \( \Omega \times (0, T^*) \) and \( U_0 \times (T^*, T) \). Let now \( r \geq 2 \) and \( f \in L^r((0, T), L^p(\Omega)) \) with \( p \) satisfying (3.14). If \( \Omega \) and \( U_0 \) are regular enough, such as in [21], then standard regularity theory for parabolic equations imply that the convergence is actually uniform in \( \Omega \times [\epsilon, T^* - \epsilon] \) and (or) in \( U_0 \times [T^* + \epsilon, T] \). According to [27, Theorem III.10.1] the H"older estimates in Theorem 3.8 are not just local, but global in the above cylinders for every \( \epsilon > 0 \) sufficiently small. A sufficient condition is that \( \Omega \) and (or) \( U_0 \) satisfies a uniform exterior cone condition. Such a condition is satisfied in [21].

### 4 The periodic-parabolic eigenvalue problem

In this section we study the principal eigenvalue of the periodic-parabolic eigenvalue problem (1.1) as a function of \( \lambda \). In particular we assume throughout that the coefficients of \((A(t), B(t))\) as well as the weight function \( m(x,t) \) are \( T \)-periodic as a function of time \( t \in \mathbb{R} \).

It is well known that there is a one-to-one correspondence between the real eigenvalues and corresponding eigenfunctions of (1.1) and the positive eigenvalues of \( U_\lambda(T,0) \) and their eigenfunctions; see [25, Prop 14.4]. Indeed, the following lemma is easily checked, see also [17, 11, 25].

**Lemma 4.1.** Let the assumptions of Section 2 be satisfied. Then \( \beta(\lambda) \in \mathbb{R} \) is an eigenvalue of \( U_\lambda(T,0) \) with eigenfunction \( w_\lambda \in L^2(\Omega) \) if and only if

\[
\mu(\lambda) := -\frac{1}{T} \log(\beta(\lambda))
\]

is a periodic-parabolic eigenvalue of (1.1) with \( T \)-periodic eigenfunction \( u_\lambda \in C(\mathbb{R}, L^2(\Omega)) \) given by

\[
u_\lambda(t) := e^{\mu(\lambda)t} U_\lambda(t,0) w_\lambda
\]

for all \( t \in \mathbb{R} \).

We next want show that under Assumption 3.10 the limit problem as \( \lambda \to \infty \) has a periodic-parabolic principal eigenvalue \( \mu_\infty \) that can be obtained as the limit of \( \mu(\lambda) \). We also show that the corresponding eigenfunctions can be chosen so that they converge in \( L^p(\Omega \times (0,T)) \) for \( 1 \leq p < \infty \) and in \( C^\beta(D_m) \), that is, locally in a Hölder norm. As mentioned already the theorem generalises and simplifies a results in [21, Theorem 3.3 and 3.4], where a very special case of Assumption 3.10 is covered.

**Theorem 4.2.** Suppose \( m \in L^\infty(\Omega \times \mathbb{R}) \) is \( T \)-periodic and satisfies Assumption 3.10. Let \( \mu(\lambda) \) be the principal eigenvalue of the periodic-parabolic problem (1.1). Then \( \mu(\lambda) \) is an increasing function of \( \lambda > 0 \) and

\[
\mu_\infty := \lim_{\lambda \to \infty} \mu(\lambda) \in \mathbb{R}
\]
exists. Furthermore, we can choose eigenfunctions \( u_\lambda \in L^\infty(\Omega \times (0, T)) \) of (1.1) such that
\[
u_\infty(t) = \lim_{\lambda \to \infty} u_\lambda(t) = e^{\mu_\infty t} U_\lambda(t, 0) u_\infty(0)
\]
in \( L^q(\Omega) \) for all \( t \in \mathbb{R} \) whenever \( 1 \leq q < \infty \). Moreover, for every compact subset \( K \subseteq D_m \cap (\Omega \times [0, T]) \) there exists \( \beta \in (0, 1) \) such that \( u_\lambda \to u_\infty \) in \( C^\beta(K) \) as \( \lambda \to \infty \). Finally, \( \mu_\infty \) is the unique principal eigenvalue of
\[
\frac{\partial u}{\partial t} + A(t) u = \mu_\infty u \quad \text{in } D_m,
\]
\[
B(t) u = 0 \quad \text{on } \partial D_m \cap (\partial \Omega \times (0, T)),
\]
\[
u(x, 0) = u(x, T) \quad \text{on } \Omega_0,
\]
and \( u_\infty \) is the unique positive eigenfunction up to scalar multiples. If \( \Omega \) is regular for all \( t \in [0, T] \) as in Corollary 2.3, then \( u_\infty \) satisfies Dirichlet boundary conditions on \( \partial \Omega_0 \cap \Omega \) for almost all \( t \in \mathbb{R} \).

**Proof.** If \( \mu(\lambda) \) is the principal eigenvalue of (1.1), then by Lemma 4.1 we have
\[
r(\lambda) := \text{spr}(U_\lambda(T, 0)) = e^{-\mu(\lambda) T}.
\]

We have proved that \( U_\lambda(t, s) \) is decreasing as a function of \( \lambda \) and therefore standard theory of positive operators on the Banach lattice \( L^2(\Omega) \) implies that \( r(\lambda) \) is decreasing in \( \lambda \). Hence \( \mu(\lambda) \) is an increasing function of \( \lambda \).

We know from Theorem 3.2 that \( U_\lambda(T, 0) \to U_\infty(T, 0) \) in \( L(L^2(\Omega)) \), and that \( U_\infty(T, 0) \) is compact. We have proved in Theorem 3.12 that \( U_\infty(T, 0) \) has a kernel \( k_\infty(x, y, T, 0) \geq 0 \) for all \( (x, y) \in \Omega_T \times \Omega_0 \). Hence, if \( w \in L^2(\Omega_0) \) is non-negative with \( w \geq 0 \) on a set of positive measure, then
\[
(U_\infty(T, 0) w)(x) = \int_{\Omega_0} k_\infty(x, y, T, 0) w(y) dy > 0
\]
for all \( x \in \Omega_T \). By the \( T \)-periodicity we have \( \Omega_T = \Omega_0 \) and therefore \( (U_\infty(T, 0) w)(x) > 0 \) for all \( x \in \Omega_0 \) if \( w > 0 \) on \( \Omega_0 \). Hence, the restriction of \( U_\infty(T, 0) \) to \( L^2(\Omega_0) \) is a compact positive and irreducible operator on \( L^2(\Omega_0) \). Therefore, the spectral radius \( r_\infty := \text{spr}(U_\infty(T, 0)) \) is an algebraically simple eigenvalue of \( U_\infty(T, 0) \) and \( r_\infty > 0 \) by a generalisation of the Krein-Rutman theorem due to [19].

By using perturbation results involving extensions and restrictions to sub-domains such as [14, Section 4.3], we have
\[
r_\infty = \lim_{\lambda \to \infty} r(\lambda) = \text{spr}(U_\infty(T, 0)).
\]
Moreover, we can choose eigenfunctions \( w_\lambda > 0 \) of \( U_\lambda(T, 0) \) such that \( w_\lambda \to w_\infty \) in \( L^2(\Omega) \) as \( \lambda \to \infty \). In particular, \( w_\infty \) is an eigenfunction of \( U_\infty(T, 0) \) corresponding to the eigenvalue \( r_\infty \), and
\[
\mu_\infty = \frac{1}{T} \log r_\infty < \infty.
\]

We know from Lemma 4.1 that
\[
u_\lambda(t) = e^{\mu(\lambda) t} U_\lambda(t, 0) w_\lambda, \quad t \in \mathbb{R}
\]
is a positive periodic-parabolic eigenfunction of (1.1). It follows from Theorem 3.7 that
\[ u_\lambda(t) \to u_\infty(t) := e^{\mu_\infty t}U(t, 0)w_\infty \]
in \( L^2(\Omega) \) for all \( t > 0 \), and hence by the \( T \)-periodicity for all \( t \in \mathbb{R} \). The above argument also implies the uniqueness of the periodic-parabolic eigenvalue and eigenfunction up to scalar multiples.

Applying an estimate similar to (3.15) as well as the fact that \( \mu_\infty \geq \mu(\lambda) \) we see that
\[ \|u_\lambda(t)\|_\infty \leq e^{2|\omega|T}e^{\mu_\infty t}T^{-N/4}\|w_\lambda\|_2 \]
for all \( t \in [T, 2T] \). As \( w_\lambda \) is bounded in \( L^2(\Omega) \) and \( u_\lambda \) is \( T \)-periodic it follows that the family of periodic-parabolic eigenfunctions \( (u_\lambda) \) is bounded in \( L^\infty(\mathbb{R} \times \Omega) \). Let \( K \subseteq D_m \cap (\Omega \times [0, T]) \) and consider the compact set
\[ \tilde{K} := \{(x, t + T) : (x, t) \in K\} \subseteq D_m \cap (\Omega \times [T, 2T]). \]
By Theorem 3.8 there exists \( \beta \in (0, 1) \) such that \( u_\lambda \to u_\infty \) in \( C^\beta(\tilde{K}) \) as \( \lambda \to \infty \). By the \( T \)-periodicity we also have \( u_\lambda \to u_\infty \) in \( C^\beta(K) \) as \( \lambda \to \infty \). Finally, Theorem 3.12 and periodicity it is strictly positive on \( D_m \).

**Remark 4.3.** Under stronger assumptions on the regularity of the coefficients such as those in Remark 3.9, for every compact subset \( K \subseteq D_m \) we have \( u_\lambda \to u_\infty \) in \( C^{2+\beta, 1+\beta/2}(K) \) for some \( \beta \in (0, 1) \). Due to Corollary 2.3 we can also deduce H"older regularity of \( u_\infty \) up to some parts of \( \partial D_m \), but depending on geometry and smoothness assumptions on \( D_m \). This recovers the regularity result in [21, Theorem 3.3].

We next make some comparisons to the earlier work in [21].

**Example 4.4.** If we combine the comments in Example 3.13 with Theorem 4.2 we can strengthen the convergence result in [21]. Rather than having local uniform convergence of the periodic-parabolic eigenfunction \( u_\lambda \) in \( D_m \), we have global uniform convergence of \( u_\lambda \) in the cylinders \( [\varepsilon, T^*-\varepsilon] \times \Omega \) and (or) in \( [T^*+\varepsilon, T] \times U_0 \), depending on the regularity assumptions on \( \Omega \) and (or) \( U_0 \). These conditions are satisfied in [21].

5 Optimality of the conditions on the weight function

We note that Assumption 3.10 is not necessary to guarantee a solution to the limit problem. Indeed, suppose there exists \( T \)-periodic \( \tilde{m} \in L^\infty(\Omega \times \mathbb{R}) \) satisfying Assumption 3.10 such that \( m \leq \tilde{m} \). Then Theorem 4.2 applied to the problem with \( \tilde{m} \), along with a similar argument to the proof of (3.1) in Theorem 3.1, implies that \( \mu_\infty \) is an eigenvalue of the limit problem. In particular, the vanishing set \( D_m \) need only satisfy the conditions of Assumption 3.10 on a nonempty open subset. However, in this case we cannot guarantee that the eigenfunction of the limit problem is strictly positive, nor that it is unique. Non-uniqueness can occur if the set \( D_m \) has for instance two connected components, both satisfying Assumption 3.10.
Nevertheless, we show now that the condition just described cannot be omitted. In an extreme case we could consider a situation where \( m > 0 \) on a set \( \Omega \times I \), where \( I \subseteq (0, T) \) is a non-trivial interval. Then clearly \( U_\infty(T, 0) = 0 \) and there is no periodic-parabolic eigenvalue and eigenfunction associated with the limit problem.

Even if \( D_m \) is path-connected, such a situation can arise. In the following example, the set \( D_m \) is path-connected but any path connecting \((x, 0)\) to \((y, T)\) must go “back in time”, violating Assumption 3.10. Certainly this implies there is no dominating function \( \tilde{m} \) for \( m \) satisfying Assumption 3.10.

**Example 5.1.** Let \( 0 < t_0 < t_1 < \cdots < t_5 < T \), \( x_0 < x_1 < \cdots < x_5 \) and let \( \Omega = (x_0, x_5) \). Consider the problem with \( A(t)u = -\partial u/\partial x^2 \) in \( \Omega \times (0, T) \) and \( B(t)u = u \) in \( \partial \Omega \times (0, T) \). Let \( m = 1_X \) where \( X \) is given by Figure 5.1. If \( \Delta_\Omega \) is the operator associated with \((A, B)\) on \( \Omega \), then clearly

\[
U_\lambda(t, s) = \begin{cases}
\exp((t-s)(\Delta_\Omega - \lambda 1_{[x_1,x_5]})) & \text{if } t_0 < s < t < t_1, \\
\exp((t-s)(\Delta_\Omega - \lambda 1_{[x_1,x_4]})) & \text{if } t_1 < s < t < t_2, \\
\exp((t-s)(\Delta_\Omega - \lambda 1_{[x_1,x_3]})) & \text{if } t_2 < s < t < t_3, \\
\exp((t-s)(\Delta_\Omega - \lambda 1_{[x_2,x_4]})) & \text{if } t_3 < s < t < t_4, \\
\exp((t-s)(\Delta_\Omega - \lambda 1_{[x_3,x_4]})) & \text{if } t_4 < s < t < t_5.
\end{cases}
\]

Using strong continuity we can extend extends these to \( t_j \leq s \leq t \leq t_{j+1} \). Standard absorption semigroup techniques (see [4]) show that, if \( \Omega_0 \subset \Omega \) is open,

\[
\exp(t(\Delta_\Omega - \lambda 1_{\Omega_0^c})) \to e^{t\Delta_{\Omega_0}} 1_{\Omega_0}
\]

in \( L(L^p(\Omega)) \) for \( 1 < p < \infty \), where \( \Delta_{\Omega_0} \) is the Dirichlet Laplacian on \( \Omega_0 \). Moreover, if \( A, B \subseteq \Omega \) are open and \( A \cap B = \emptyset \), then

\[
e^{t\Delta_{A \cup B}} = e^{t\Delta_A} 1_A + e^{t\Delta_B} 1_B.
\]

Taking \( u_0 \in L^p(\Omega) \) and defining \( u_j := U_\infty(t_j, t_{j-1})u_{j-1} \), we then find that \( u_1, u_2 \) and \( u_3 \) vanish outside \( (x_0, x_1) \), implying \( u_4 \) vanishes outside \( (x_0, x_3) \). But then

\[
u_5 = U_\infty(t_5, t_4)u_4 = e^{(t_5-t_4)\Delta_{(x_3,x_5)}} 1_{(x_3,x_5)} u_4 = 0
\]

and hence \( U_\infty(t_5, t_0) = 0 \). In particular, this means \( U_\infty(T, 0) = 0 \) and so the limit problem is trivial.

![Figure 5.1: Graph of set X (shaded) and set D_m (white).](image-url)
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