The Propus Construction for Symmetric Hadamard Matrices

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Abstract

Propus (which means twins) is a construction method for orthogonal ±1 matrices based on a variation of the Williamson array called the propus array

\[
\begin{pmatrix}
A & B & B & D \\
B & D & -A & -B \\
B & -A & -D & B \\
D & -B & B & -A \\
\end{pmatrix}
\]

This construction designed to find symmetric Hadamard matrices was originally based on circulant symmetric ±1 matrices, called propus matrices. We also give another construction based on symmetric Williamson-type matrices.

We give constructions to find symmetric propus-Hadamard matrices for 57 orders 4n, n < 200 odd.

We give variations of the above array to allow for more general matrices than symmetric Williamson propus matrices. One such is the Generalized Propus Array (GP).

Keywords: Hadamard Matrices, D-optimal designs, conference matrices, propus construction, Williamson matrices; Cretan matrices; 05B20.

1 Introduction

Hadamard matrices arise in statistics, signal processing, masking, compression, combinatorics, weaving, spectroscopy and other areas. They been studied extensively. Hadamard showed [13] the order of an Hadamard matrix must be 1, 2 or a multiple of 4. Many constructions for ±1 matrices and

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similar matrices such as Hadamard matrices, weighing matrices, conference matrices and $D$-optimal designs use skew and symmetric Hadamard matrices in their construction. For more details see Seberry and Yamada [29].

An Hadamard matrix of order $n$ is an $n \times n$ matrix with elements $\pm 1$ such that $HH^\top = H^\top H = nI_n$, where $I_n$ is the $n \times n$ identity matrix and $\top$ stands for transposition. A skew Hadamard matrix $H = I + S$ has $S^\top = -S$.

For more details see the books and surveys of Jennifer Seberry (Wallis) and others [29, 33] cited in the bibliography.

Theorems of the type for every odd integer $n$ there exists a $t$ dependent on $n$ so that Hadamard, regular Hadamard, co-cyclic Hadamard and some full orthogonal designs exist for all orders $2^tn$, $t$ integer are known [36, 37, 38, 39].

A similar result for symmetric Hadamard and skew-Hadamard matrices has not yet been published but is conjectured.

Propus is a construction method for orthogonal $\pm 1$ matrices, $A$, $B = C$, and $D$, where

$$AA^\top + 2BB^\top + DD^\top = \text{constant } I,$$

$I$ the identity matrix, based on the array

$$
\begin{array}{cccc}
A & B & B & D \\
B & D & -A & -B \\
B & -A & -D & B \\
D & -B & B & -A.
\end{array}
$$

This construction, based on circulant symmetric $\pm 1$ matrices, called propus matrices, gives symmetric Hadamard matrices. It also gives aesthetically pleasing visual images (pictures) when converted using MATLAB (we show some below).

We give methods to find propus-Hadamard matrices: using Williamson matrices and $D$-optimal designs. These are then generalized to allow non-circulant and/or non-symmetric matrices with the same aim to give symmetric Hadamard matrices.

We show that for

- $q \equiv 1 \pmod{4}$, a prime power, such matrices exist for order $t = \frac{1}{2}(q+1)$, and thus propus-Hadamard matrices of order $2(q+1)$;
- $t \equiv 3 \pmod{4}$, a prime, such that $D$-optimal designs, constructed using two circulant matrices, one of which must be circulant and symmetric, exist of order $2t$, then such propus-Hadamard matrices exist for order $4t$.
- $4 - \{t; s_1, s_2, s_3, s_4; \sum_{j=1}^{4} s_j(s_j-1)\} \equiv \{a, b, c, d\} \equiv t \pmod{4}$, $a = 2s_1 - t, b = 2s_2 - t, c = 2s_3 - t, d = 2s_4 - t$, where one of the supplementary difference sets is symmetric then such propus-Hadamard matrices exist for order $4t$.  

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symmetric variant propus matrices \([29]\) may be used to find symmetric propus-Hadamard matrices for orders described below (see Corollary 1).

We note that appropriate Williamson type matrices may also be used to give propus-Hadamard matrices but do not pursue this avenue in this paper. There is also the possibility that this propus construction may lead to some insight into the existence or non-existence of symmetric conference matrices for some orders. We refer the interested reader to mathscinet.ru/catalogue/propus/.

1.1 Definitions and Basics

Two matrices \(X\) and \(Y\) of order \(n\) are said to be amicable if \(XY^\top = YX^\top\).

We define the following classes of propus like matrices. We note that there are slight variations in the matrices which allow variant arrays and non-circulant matrices to be used to give symmetric Hadamard matrices. All propus like matrices \(A, B, C, D\) of order \(n\) satisfy the additive property

\[
AA^\top + 2BB^\top + DD^\top = 4nI_n,
\]

where \(I\) the identity matrix, \(J\) the matrix of all ones.

We make the following definitions:

- **propus matrices**: four circulant symmetric \(\pm 1\) matrices, \(A, B, B, D\) of order \(n\), satisfying the additive property (use \(P\));
- **propus-type matrices**: four pairwise amicable \(\pm 1\) matrices, \(A, B, B, D\) of order \(n\), \(A^\top = A\), satisfying the additive property (use \(P\));
- **generalized-propus matrices**: four pairwise commutative \(\pm 1\) matrices, \(A, B, B, D\) of order \(n\), \(A^\top = A\), which satisfy the additive property (use \(GP\)).

We use two types of arrays into which to plug the propus like matrices: the Propus array, \(P\), or the generalized-propus array, \(GP\). These can also be used with generalized matrices ([32]).

\[
P = \begin{bmatrix}
A & B & B & D \\
B & D & -A & -B \\
B & -A & -D & B \\
D & -B & B & -A
\end{bmatrix}
\quad \text{and} \quad
GP = \begin{bmatrix}
A & BR & BR & DR \\
BR & D^\top R & -A & -B^\top R \\
BR & -A & -D^\top R & B^\top R \\
DR & -B^\top R & B^\top R & -A
\end{bmatrix}.
\]

Symmetric Hadamard matrices made using propus like matrices will be called symmetric propus-Hadamard matrices.
2 Symmetric Propus-Hadamard Matrices

We first give the explicit statements of two well known theorem, Paley’s Theorem [27], for the Legendre core $Q$, and Turyn’s Theorem [30], in the form in which we will use them.

**Theorem 1. [Paley’s Legendre Core [27]]** Let $p$ be a prime power, either $\equiv 1 \pmod{4}$ or $\equiv 3 \pmod{4}$ then there exists a matrix, $Q$, of order $p$ with zero diagonal and other elements $\pm 1$ satisfying $QQ^T = (q + 1)I - J$, $Q$ is symmetric or skew-symmetric according as $p \equiv 1 \pmod{4}$ or $p \equiv 3 \pmod{4}$.

**Theorem 2. [Turyn’s Theorem [30]]** Let $q \equiv 1 \pmod{4}$ be a prime power then there are two symmetric matrices, $P$ and $S$ of order $\frac{1}{2}(q + 1)$, satisfying $PP^T + SS^T = qI$: $P$ has zero diagonal and other elements $\pm 1$ and $S$ elements $\pm 1$.

2.1 Propus-Hadamard Matrices from Williamson Matrices

**Lemma 1.** Let $q \equiv 1 \pmod{4}$, be a prime power, then propus matrices exist for orders $n = \frac{1}{2}(q + 1)$ which give symmetric propus-Hadamard matrices of order $2(q + 1)$.

**Proof.** We note that for $q \equiv 1 \pmod{4}$, a prime power, Turyn (Theorem 2 [30]) gave Williamson matrices, $X + I$, $X - I$, $Y$, $Y$, which are circulant and symmetric for orders $n = \frac{1}{2}(q + 1)$. Then choosing

$$A = X + I, \quad B = C = Y, \quad D = X - I$$

gives the required propus-Hadamard matrices. \qed

We now have propus-Hadamard matrices for orders $4n$ where $n$ is in

$$\{1, 3, [5], 7, 9, [13], 15, 19, 21, [25], 27, 31, 37, [41], 45, 49, 51, 55, 57, 59, [61], [63], 67, 69, 75, 79, 81, [85], 87, 89, 91, 97, 99, 105, 111, 115, 117, 119, 121, 127, 129, 135, 139, 141, [145], 147, 157, 159, 169, 175, 177, [181], 187, 195, 199.\}$$

The cases written in square brackets [5],[13],[25],[41],[61],[63],[85],[113],[145],[181] arise when $q$ is a prime power, however the Delsarte-Goethals-Seidel-Turyn construction means the required circulant matrices also exist for these prime powers (see Figure 2).
2.1.1 Propus matrices of small order and from $q$ prime power

This family is considered to contain the two trivial propus-Hadamard matrices of orders 12 and 20 based on symmetric Paley cores $A = J$, $B = C = J - 2I = QR$, $D = J = 2I = QR$ for $n = 3$, and $A = Q + I$, $B = C = J - 2I$, $D = Q - I$ (constructed using Legendre symbols) for $n=5$. This special set can be continued with back-circulant matrices $C = B$ which allows the symmetry property of $A$ to be conserved.

2.2 Propus-Hadamard matrices from $D$-optimal designs

**Lemma 2.** Let $n \equiv 3 \pmod{4}$, be a prime, such that $D$-optimal designs, constructed using two circulant matrices, one of which is symmetric, exist for order $2n$. Then propus-Hadamard matrices exist for order $4n$.

Djoković and Kotsireas in [22, 8] give $D$-optimal designs, constructed using two circulant matrices, for $n \in \{3, 5, 7, 9, 13, 15, 19, 21, 23, 25, 27, 31, 33, 37, 41, 43, 45, 49, 51, 55, 57, 59, 61, 63, 69, 73, 75, 77, 79, 85, 87, 91, 93, 97, 103, 113, 121, 131, 133, 145, 157, 181, 183\}, n < 200. We are interested in those cases where the $D$-optimal design is constructed from two circulant matrices one of which must be symmetric.
Suppose $D$-optimal designs for orders $n \equiv 3 \pmod{4}$, a prime, are constructed using two circulant matrices, $X$ and $Y$. Suppose $X$ is symmetric. Let $Q + I$ be the Paley matrix of order $n$. Then choosing

$$A = X, \quad B = C = Q + I, \quad D = Y,$$

to put in the array $GP$ gives the required propus-Hadamard matrices.

Hence we have propus-Hadamard matrices, constructed using $D$-optimal designs, for orders $4n$ where $n$ is in

$$\{3, 7, 19, 31\}.$$

The results for $n = 19$ and 31 were given to us by Dragomir Djoković.

Figure 3: D-optimal designs for orders $2n$ propus-Hadamard matrices for orders $4n$
2.3 A Variation of a Theorem of Miyamoto

In Seberry and Yamada [29] one of Miyamoto’s results [25] was reformulated so that symmetric Williamson-type matrices can be obtained. The results given here are due to Miyamoto, Seberry and Yamada.

Lemma 3 (Propus Variation). Let $U_i, V_j, i, j = 1, 2, 3, 4$ be $(0, +1, -1)$ matrices of order $n$ which satisfy

(i) $U_i, U_j, i \neq j$ are pairwise amicable,

(ii) $V_i, V_j, i \neq j$ are pairwise amicable,

(iii) $U_i \pm V_i, (+1, -1)$ matrices, $i = 1, 2, 3, 4$,

(iv) the row sum of $U_1$ is 1, and the row sum of $U_i, i = 2, 3, 4$ is zero,

(v) $\sum_{i=1}^{4} U_i U_i^T = (2n + 1) I - 2J, \sum_{i=1}^{4} V_i V_i^T = (2n + 1) I$.

Let $S_1, S_2, S_3, S_4$ be four $(+1, -1)$-matrices of order $2n$ defined by

$$S_j = U_j \times \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ \end{bmatrix} + V_j \times \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ \end{bmatrix},$$

where $S_2 = S_3$.

Then there are 4 propus-Williamson type matrices of order $2n + 1$. If $U_i$ and $V_i$ are symmetric, $i = 1, 2, 3, 4$ then the Williamson-type matrices are symmetric. Hence there is a symmetric propus-type Hadamard matrix of order $4(2n + 1)$.

Proof. With $S_1, S_2, S_3, S_4$, as in the theorem enunciation the row sum of $S_1 = 2$ and of $S_i = 0, i = 2, 3, 4$. Now define

$$X_1 = \begin{bmatrix} 1 \\ -e_2n \\ \end{bmatrix} - e_2n S_1$$

and

$$X_i = \begin{bmatrix} 1 \\ e_2n \\ \end{bmatrix} e_i S_i,$$

where $i = 2, 3, 4$.

First note that since $U_i, U_j, i \neq j$ and $V_i, V_j, i \neq j$ are pairwise amicable,

$$S_i S_j^T = \left( U_i \times \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ \end{bmatrix} + V_i \times \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ \end{bmatrix} \right) \left( U_j^T \times \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ \end{bmatrix} + V_j^T \times \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ \end{bmatrix} \right)$$

$$= U_i U_j^T \times \begin{bmatrix} 2 & 2 \\ 2 & 2 \\ \end{bmatrix} + V_i V_j^T \times \begin{bmatrix} 2 & -2 \\ -2 & 2 \\ \end{bmatrix}$$

$$= S_j S_i^T.$$

(Note this relationship is valid if and only if conditions (i) and (ii) of the theorem are valid.)
\[
\sum_{i=1}^{4} S_iS_i^T = \sum_{i=1}^{4} U_iU_i^T \times \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} + \sum_{i=1}^{4} V_iV_i^T \times \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} 2(2n+1)I - 2J & -2J \\ -2J & 2(2n+1)I - 2J \end{bmatrix} = 4(2n+1)I_{2n} - 4J_{2n}
\]

Next we observe
\[
X_1X_i^T = \begin{bmatrix} 1 - 2n & e_{2n} \\ e_{2n}^T & -J + S_iS_i^T \end{bmatrix} = X_iX_1^T \quad i = 2, 3, 4,
\]
and
\[
X_iX_j^T = \begin{bmatrix} 1 + 2n & e_{2n} \\ e_{2n}^T & J + S_iS_j^T \end{bmatrix} = X_jX_i^T \quad i \neq j, \quad i, j = 2, 3, 4.
\]

Further
\[
\sum_{i=1}^{4} X_iX_i^T = \begin{bmatrix} 1 + 2n & -3e_{2n} \\ -3e_{2n}^T & J + S_iS_i^T \end{bmatrix} + \sum_{i=2}^{4} \begin{bmatrix} 1 + 2n & e_{2n} \\ e_{2n}^T & J + S_iS_i^T \end{bmatrix} = \begin{bmatrix} 4(2n+1) & 0 \\ 0 & 4J + 4(2n+1)I - 4J \end{bmatrix}.
\]

Thus we have shown that \(X_1, X_2, X_3, X_4\) are pairwise amicable, symmetric Williamson type matrices of order \(2n + 1\), where \(X_2 = X_3\). These can be used as in (ii) of Theorem using the additive property to obtain the required symmetric propus Hadamard matrix of order \((4n+1)\). \(\square\)

Many powerful corollaries arose and new results were obtained by making suitable choices in the theorem. We choose \(X_1, X_2, X_3, X_4\) to ensure that the propus construction can be used to form symmetric Hadamard matrices of order \((4n+1)\).

From Paley’s theorem (Corollary 1) for \(p \equiv 3 \pmod{4}\) we use the backcirculant or type 1, symmetric matrices \(QR\) and \(R\) instead of \(Q\) and \(I\); whereas for \(p \equiv 1 \pmod{4}\) we use the symmetric Paley core \(Q\). If \(p\) is a prime power \(\equiv 3 \pmod{4}\) we set \(U_1 = I, U_2 = U_3 = QR, U_4 = 0\) of order \(p\), and if \(p\) is a prime power \(\equiv 1 \pmod{4}\), we set \(U_1 = I, U_2 = U_3 = Q, U_4 = 0\) of order \(p\).

Hence \(\sum_{i=1}^{4} V_iV_i^T = (q+2)I\).

From Turyn’s result (Corollary 2) we set, for \(p \equiv 1 \pmod{4}\) \(U_1 = P, U_2 = U_3 = I\) and \(U_4 = S\), and for \(p \equiv 3 \pmod{4}\) \(V_1 = P, V_2 = V_3 = R\) and \(V_4 = S\), so \(\sum_{k=1}^{4} U_kU_k^T = (q+2)I\).

Hence we have:
Corollary 1. Let \( q \equiv 1 \pmod{4} \) be a prime power and \( \frac{1}{2}(q+1) \) be a prime power or the order of the core of a symmetric conference matrix (this happens for \( q = 89 \)). Then there exist symmetric Williamson type matrices of order \( 2q+1 \) and a symmetric propus-type Hadamard matrix of order \( 4(2q+1) \).

This gives the previously unresolved cases for 11 and 83.

2.3.1 Three Equal

The family given above includes two starting Hadamard matrices of orders 12 and 28 based on the skew Paley core \( B = C = D = Q + I \) (constructed using Legendre symbols). This special set is finite because 12 = 3^2 + 1^2 + 1^2 + 1^2 and 28 = 5^2 + 1^2 + 1^2 + 1^2 and these are the only orders for which a symmetric circulant \( A \) can exist with \( B = C = D \).

![Figure 4: Propus-Hadamard matrices using \( D \)-optimal designs](image)

3 Propus-Hadamard matrices from conference matrices: even order matrices

A powerful method to construct propus-Hadamard matrices for \( n \) even is using conference matrices.

Lemma 4. Suppose \( M \) is a conference matrix of order \( n \equiv 2 \pmod{4} \). Then \( MM^\top = M^\top M = (n-1)I \), where \( I \) is the identity matrix and \( M^\top = M \). Then using \( A = M + I \), \( B = C = M - I \), \( D = M + I \) gives a propus-Hadamard matrix of order \( 4n \).

We use the conference matrix orders from [1] and so have propus-Hadamard matrices of orders \( 4n \) where \( n \in \{6,10,14,18,26,30,38,42,46,50,54,62,74,82,90,98\} \).

The conference matrices in Figure 5 are made two circulant matrices \( A \) and \( B \) of order \( n \) where both \( A \) and \( B \) are symmetric.
Then using the matrices $A + I$, $B = C$ and $D = A - I$ in $P$ gives the required construction.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure5}
\caption{Conference matrices for orders $2n$ using two circulants: propus-Hadamard matrices for orders $4n$}
\end{figure}

The conference matrices in Figure 6 are made from two circulant matrices $A$ and $B$ of order $n$ where both $A$ and $B$ are symmetric. However here we use $A + I$, $BR = CR$ and $D = A - I$ in $P$ to obtain the required construction.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure6}
\caption{Conference matrices for orders $2n$ using two circulant and back-circulants: propus-Hadamard matrices for orders $4n$}
\end{figure}

There is another variant of this family which uses the symmetric Paley cores $A = Q + I$, $D = Q - I$ (constructed using Legendre symbols) and one circulant matrix of maximal determinant $B = C = Y$. 

3.1 Propus-Hadamard matrices for $n$ even

Figure 7 gives visualizations (images/pictures) of propus-Hadamard matrices orders 16, 32. These have even $n$.

![Figure 7: Matrices P16 and P32](image)

4 Conclusion and Future Work

Using the results of Lemma 1 and Corollary 1 and the symmetric propus-Hadamard matrices of Di Matteo, Djoković, and Kotsireas given in [4], we see that the unresolved cases for symmetric propus-Hadamard matrices for orders $4n$, $n < 200$ odd, are where $n \in$

$$\{17, 23, 29, 33, 35, 39, 47, 53, 65, 71, 73, 77, 93, 95, 97, 99,$$
$$\quad 101, 103, 107, 109, 113, 125, 131, 133, 137, 143, 149, 151, 153, 155,$$
$$\quad 161, 163, 165, 167, 171, 173, 179, 183, 185, 189, 191, 193, 197.\}$$

There are many constructions and variations of the propus theme to be explored in future research. Visualizing the propus construction gives aesthetically pleasing examples of propus-Hadamard matrices. The visualization also makes the construction method clearer. There is the possibility that these visualizations may be used for quilting.

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