REPRESENTING REGULAR PSEUDOCOMPLEMENTED KLEENE ALGEBRAS IN TERMS OF ROUGH SETS

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Abstract. We introduce Varlet spaces as partially ordered sets equipped with a polarity satisfying certain additional conditions. By applying Varlet spaces, we prove that each pseudocomplemented Kleene algebra is isomorphic to a subalgebra of the rough set pseudocomplemented Kleene algebra defined by a tolerance. We also characterize the Varlet spaces corresponding to the regular pseudocomplemented Kleene algebras satisfying the Stone identity.

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1. Introduction to rough set algebras

Rough set theory was introduced by Z. Pawlak in [Paw82]. His idea was to develop a formalism for dealing with vague concepts and sets. In rough set theory it is assumed that our knowledge is restricted by an indistinguishability relation. Originally Pawlak defined an indistinguishability relation as an equivalence relation $E$ such that two elements of a universe of discourse $U$ are $E$-equivalent if we cannot distinguish these two elements by their properties known by us. For instance, we may know some properties of human beings $U$ such as age, gender, height, and weight. Then $x E y$ means that the persons $x$ and $y$ are indistinguishable by these properties.

Each subset $X$ of $U$ can now be “approximated” by using the indistinguishability relation $E$. Let us denote the equivalence class of $x$ by $[x]_E$, that is, $[x]_E = \{y \in U \mid x E y\}$. The lower approximation

$$X^\wedge = \{x \in U \mid [x]_E \subseteq X\}$$

of $X$ may be viewed as the set of elements belonging certainly to $X$ in view of the knowledge $E$, because if $x \in X^\wedge$, then all elements indistinguishable from $x$ are in $X$. The upper approximation

$$X^\vee = \{x \in U \mid [x]_E \cap X \neq \emptyset\}$$

of $X$ can be seen as the set of elements belonging possible to $X$ by the means of the knowledge $E$. Indeed, if $x \in X^\vee$, then $X$ contains at least one element indistinguishable from $x$.

Let us denote by $\varphi(U)$ the powerset of $U$, that is, $\varphi(U) = \{X \mid X \subseteq U\}$. We define a relation $\equiv$ on $\varphi(U)$ by

$$X \equiv Y \iff X^\wedge = Y^\wedge \text{ and } X^\vee = Y^\vee.$$  

The relation $\equiv$ is an equivalence called rough equality. If $X \equiv Y$, then the same elements belong certainly and possibly to $X$ and $Y$ in view of the knowledge $E$. The equivalence classes of $E$ are called rough sets.

The order-theoretical study of rough sets was initiated by T. B. Iwiński in [Iwi87]. In his approach rough sets on $U$ are the pairs $(X^\wedge, X^\vee)$, where $X \subseteq U$. This is justified because if $C \subseteq \varphi(U)$ is a rough set as defined before, that is, $C$ is an equivalence class of rough sets induced by an irredundant covering.

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\( \equiv \), then \( C \) is uniquely determined by the pair \((X^\triangledown, X^\triangledown)\), where \( X \) is any member of \( C \): a set \( Y \subseteq U \) belongs to \( C \) if and only if \((Y^\triangledown,Y^\triangledown) = (X^\triangledown,X^\triangledown)\). Therefore, we call
\[
RS = \{(X^\triangledown, X^\triangledown) \mid X \subseteq U\}.
\]
the set of rough sets. The set \( RS \) is ordered by the componentwise inclusion:
\[
(X^\triangledown, X^\triangledown) \leq (Y^\triangledown, Y^\triangledown) \iff X^\triangledown \subseteq Y^\triangledown \text{ and } X^\triangledown \subseteq Y^\triangledown.
\]
Let \( X^c = U \setminus X \) denote the set-theoretic complement of \( X \subseteq U \). Iwiński noted that the map \( \sim \colon RS \to RS \) defined by \( \sim(X^\triangledown, X^\triangledown) = (X^c, X^c) = (X^\triangledown, X^\triangledown) \) is a polarity. A polarity \( \uparrow \colon P \to P \) on an ordered set \( P \) is defined so that \( x \uparrow = x \) and \( x \leq y \) implies \( x \uparrow \geq y \uparrow \) for \( x, y \in P \). Such a polarity is an order-isomorphism from \((P, \leq)\) to \((P, \geq)\). Hence \( P \) is isomorphic to its dual.

It was proved in [PPSS] that \( RS \) is a complete sublattice of \( \wp(U) \times \wp(U) \) ordered by the coordinatewise set-inclusion relation, which means that \( RS \) is an algebraic completely distributive lattice such that
\[
\bigwedge \{(X^\triangledown, X^\triangledown) \mid X \in \mathcal{H}\} = \left( \bigcap_{X \in \mathcal{H}} X^\triangledown, \bigcap_{X \in \mathcal{H}} X^\triangledown \right)
\]
and
\[
\bigvee \{(X^\triangledown, X^\triangledown) \mid X \in \mathcal{H}\} = \left( \bigcup_{X \in \mathcal{H}} X^\triangledown, \bigcup_{X \in \mathcal{H}} X^\triangledown \right)
\]
for any \( \mathcal{H} \subseteq \wp(U) \). Note that an algebraic lattice \( L \) is a complete lattice in which each element is a join of compact elements of \( L \), and a completely distributive lattice is a complete lattice in which arbitrary joins distribute over arbitrary meets. They also proved that \( RS \) is a Stone lattice such that \((X^\triangledown, X^\triangledown)^\ast = (X^c, X^c)\) for \( X \subseteq U \). This result was improved by S. D. Comer [Com93] by showing that
\[
RS = (RS, \lor, \land, ^\ast, ^\dagger, (\emptyset, \emptyset), (U, U))
\]
is a regular double Stone algebra, where \((X^\triangledown, X^\triangledown)^\dagger = (X^c, X^c)\) for \( X \subseteq U \). More importantly, he proved that every regular double Stone algebra is isomorphic to a subalgebra of \( RS \) defined by some equivalence relation.

In the literature can be found numerous studies on rough sets that are determined by so-called information relations reflecting distinguishability or indistinguishability of the elements of the universe of discourse; see [Jar07] for further references. The idea is that \( R \) may be an arbitrary information relation, and rough lower and upper approximations are then defined in terms of \( R \). Let us denote for any \( x \in U \), \( R(x) = \{y \mid x \mathcal{R} y\} \). For all \( X \subseteq U \), the lower and upper approximations of \( X \) are defined by
\[
X^\triangledown = \{x \in U \mid R(x) \subseteq X\} \quad \text{and} \quad X^\triangledown = \{x \in U \mid R(x) \cap X \neq \emptyset\},
\]
respectively. The rough equality relation, the set of rough sets \( RS \), and its partial order are defined as in case of equivalences.

As shown in [JRV09], if \( R \) is a quasiorder (a reflexive and transitive binary relation) on \( U \), then \( RS \) is a complete sublattice of \( \wp(U) \times \wp(U) \) ordered by the componentwise inclusion, meaning that \( RS \) is an algebraic completely distributive lattice such that the lattice-operations are defined as in [1] and [2] for all \( \mathcal{H} \subseteq RS \). As in the case of equivalences, the map \( \sim \colon (X^\triangledown, X^\triangledown) \mapsto (X^\triangledown, X^\triangledown) \) is a polarity on \( RS \). In fact (see [MRL13]), if \( R \) is a quasiorder, then the algebra
\[
RS = (RS, \lor, \land, \rightarrow, \sim, 0, 1)
\]
forms a Nelson algebra with the operations:

\[(X^\triangledown, X^\triangledown) \lor (Y^\triangledown, Y^\triangledown) = (X^\triangledown \cup Y^\triangledown, X^\triangledown \cup Y^\triangledown),\]
\[(X^\triangledown, X^\triangledown) \land (Y^\triangledown, Y^\triangledown) = (X^\triangledown \cap Y^\triangledown, X^\triangledown \cap Y^\triangledown),\]
\[(X^\triangledown, X^\triangledown) \Rightarrow (Y^\triangledown, Y^\triangledown) = ((X^\triangledown \cup Y^\triangledown)^\triangledown, X^\triangledown \cup Y^\triangledown),\]
\[\sim(X^\triangledown, X^\triangledown) = (X^\triangledown^*, X^\triangledown^*),\]
\[0 = (\emptyset, \emptyset),\]
\[1 = (U, U).\]

We proved in [JR16] that if \( A \) is a Nelson algebra defined on an algebraic lattice, there exists a set \( U \) and a quasiorder \( R \) on \( U \) such that \( A \) is isomorphic to the Nelson algebra \( RS \) determined by \( R \). In [JR14a], we generalized this representation theorem by stating that for any Nelson algebra \( A \), there exists a set \( U \) and a quasiorder \( R \) on \( U \) such that \( A \) is isomorphic to a subalgebra of the Nelson algebra \( RS \) determined by \( R \).

If \( R \) is a tolerance (a reflexive and symmetric binary relation), then \( RS \) is not necessarily a lattice [Jar99]. A collection \( H \) of nonempty subsets of \( U \) is called a covering of \( U \) if \( \bigcup H = U \). A covering \( H \) is irredundant if \( H \setminus \{X\} \) is not a covering of \( U \) for any \( X \in H \). Each covering \( H \) defines a tolerance \( \bigcup\{X \times X \mid X \in H\} \), called the tolerance induced by \( H \). In [JR14b], we proved that if \( R \) is a tolerance induced by an irredundant covering of \( U \), then \( RS \) is an algebraic and completely distributive lattice such that for any \( H \subseteq \wp(U) \),

\[
\bigwedge \{(X^\triangledown, X^\triangledown) \mid X \in H\} = \left( \bigcap_{X \in H} X^\triangledown, \bigcap_{X \in H} X^\triangledown \right)^\triangledown
\]

and

\[
\bigvee \{(X^\triangledown, X^\triangledown) \mid X \in H\} = \left( \bigcup_{X \in H} X^\triangledown \right)^\triangledown, \bigcup_{X \in H} X^\triangledown.
\]

In addition, we have showed [JR17] that if \( R \) is a tolerance induced by an irredundant covering, then

\[(RS, \lor, \land, \sim^*, 0, 1)\]

is a pseudocomplemented Kleene algebra with the operations:

\[(X^\triangledown, X^\triangledown) \lor (Y^\triangledown, Y^\triangledown) = ((X^\triangledown \cup Y^\triangledown)^\triangledown, X^\triangledown \cup Y^\triangledown),\]
\[(X^\triangledown, X^\triangledown) \land (Y^\triangledown, Y^\triangledown) = (X^\triangledown \cap Y^\triangledown, (X^\triangledown \cap Y^\triangledown)^\triangledown),\]
\[\sim(X^\triangledown, X^\triangledown) = (X^\triangledown^*, X^\triangledown^*),\]
\[(X^\triangledown, X^\triangledown)^* = (X^\triangledown^*, X^\triangledown^*),\]
\[0 = (\emptyset, \emptyset),\]
\[1 = (U, U).\]

We also proved in [JR17] that if \( L = (L, \lor, \land, \sim, 0, 1) \) is a regular pseudocomplemented Kleene algebra defined on an algebraic lattice, then there exists a set \( U \) and a tolerance \( R \) induced by an irredundant covering of \( U \) such that \( L \) is isomorphic to the rough set pseudocomplemented Kleene algebra \( RS = (RS, \lor, \land, \sim^*, 0, 1) \) determined by \( R \). In this work we generalize this result by showing that if \( L \) is any regular pseudocomplemented Kleene algebra, then there exists a set \( U \) and a tolerance \( R \) induced by an irredundant covering of \( U \) such that \( L \) is isomorphic to a subalgebra of the rough set pseudocomplemented Kleene algebra \( RS \) determined by \( R \).

This work is structured as follows: In the next section we recall some notions and facts related to De Morgan and Kleene algebras. In particular, we are interested in pseudocomplemented Kleene algebras and their regularity. In Section 3 we introduce so called Varlet spaces. These structures are essential tools for our main result of the section stating that each regular pseudocomplemented Kleene algebra is isomorphic to a subalgebra of the rough set pseudocomplemented Kleene algebra defined by a tolerance. Section 4...
is devoted to regular pseudocomplemented Kleene algebras satisfying the Stone identity \( x^* \lor x^{**} = 1 \). We show that there is a one-to-one correspondence between these algebras and regular double Stone algebras. The section ends by showing that each regular pseudocomplemented Kleene algebra satisfying the Stone identity is isomorphic to a subalgebra of a rough set Kleene algebra defined by an equivalence relation. The last section considers Varlet spaces of regular pseudocomplemented Kleene algebras satisfying the Stone identity.

2. Regular pseudocomplemented Kleene algebras

An algebra \((L, \lor, \land, *, 0, 1)\) is a \(p\)-algebra if \((L, \lor, \land, 0, 1)\) is a bounded lattice and \(*\) is a unary operation on \(L\) such that \(x \land z = 0\) iff \(z \leq x^*\). The element \(x^*\) is the pseudocomplement of \(x\). A lattice \(L\) in which each element has a pseudocomplement is called a pseudocomplemented lattice. It is well known that \(x \leq y\) implies \(x^* \geq y^*\). We also have for \(x, y \in L\),

\[
\begin{align*}
x^* &= x^{**}, \\
(x \lor y)^* &= x^* \land y^*, \\
(x \land y)^{**} &= x^{**} \land y^{**}.
\end{align*}
\]

An algebra \((L, \lor, \land, *, +, 0, 1)\) is a double \(p\)-algebra if \((L, \lor, \land, *, 0, 1)\) is a \(p\)-algebra and \((L, \lor, \land, +, 0, 1)\) is a dual \(p\)-algebra (i.e. \(z \geq x^+\) iff \(x \lor z = 1\) for all \(x, y \in L\)). The element \(x^+\) is the dual pseudocomplement of \(a\). If \(x \leq y\), then \(x^+ \geq y^+\). In addition,

\[
\begin{align*}
x^+ &= x^{+++}, \\
(x \land y)^+ &= x^+ \lor y^+, \\
(x \lor y)^{+++} &= x^{++} \lor y^{++}.
\end{align*}
\]

Note that by definition \(x \leq x^{**}\) and \(x^{++} \leq x\). Therefore, in a double \(p\)-algebra \(x^{++} \leq x^{**}\).

An algebra is called congruence-regular if every congruence is determined by any class of it: two congruences are necessarily equal when they have a class in common. J. Varlet has proved in [Var72] that double \(p\)-algebras satisfying the condition

\((M)\)

\(x^* = y^*\) and \(x^+ = y^+\) imply \(x = y\).

are exactly the congruence-regular ones. It is also proved by him that for the distributive double \(p\)-algebras, condition \((M)\) is equivalent to condition

\((D)\)

\(x \land x^+ \leq y \lor y^*\).

On the other hand, T. Katriňák [Kat73] has shown that any congruence-regular double pseudocomplemented lattice is distributive. In what follows, we use the shorter term “regular” instead of “congruence-regular”.

A filter \(F\) of a lattice \(L\) is called proper, if \(F \neq L\). A proper filter \(F\) is a prime filter if \(a \lor b \in F\) implies \(a \in F\) or \(b \in F\). The set of prime filters of \(L\) is denoted by \(\mathcal{F}_p(L)\), or by \(\mathcal{F}_p\) if there is no danger of confusion. A filter \(F\) is maximal if \(F\) is proper and there is no proper filter that is strictly greater than \(F\). It can be shown by using Zorn’s Lemma that every proper filter can be extended to a maximal filter. It is also known that in distributive lattices, each maximal filter is a prime filter, but the converse statement is not true in general.

Varlet has proved in [Var72] for distributive double \(p\)-algebras the following characterization in terms of prime filters.

**Proposition 1.** Let \((L, \lor, \land, *, +, 0, 1)\) be a distributive double \(p\)-algebra. The following are equivalent:

(a) \(L\) is regular;
(b) Any chain of prime filters of \(L\) has at most two elements.
A De Morgan algebra is an algebra \((L, \vee, \wedge, \sim, 0, 1)\) such that \((L, \vee, \wedge, 0, 1)\) is a bounded distributive lattice and \(\sim\) is a polarity on \(L\), that is, it satisfies \(\sim x = x\) and \(x \leq y\) implies \(\sim x \geq \sim y\). Note that equationally the operation \(\sim\) can be defined by

\[
x = \sim x \quad \text{and} \quad \sim x \vee \sim y = \sim (x \wedge y).
\]

A Kleene algebra is a De Morgan algebra \((L, \vee, \wedge, \sim, 0, 1)\) satisfying

\[
x \wedge \sim x \leq y \vee \sim y
\]

for all \(x, y \in L\). In [CdG81] Lemma 1.1 it is proved that in a Kleene algebra \((L, \vee, \wedge, \sim, 0, 1)\),

\[
x \wedge y = 0 \text{ implies } y \leq \sim x.
\]

A pseudocomplemented De Morgan algebra \((L, \vee, \wedge, \sim, *, 0, 1)\) is such that \((L, \vee, \wedge, \sim, 0, 1)\) is a De Morgan algebra and \((L, \vee, \wedge, *, 0, 1)\) is a \(p\)-algebra. In fact, any pseudocomplemented De Morgan algebra forms a double \(p\)-algebras, where the pseudocomplement operations determine each other by

\[
\sim x^* = (\sim x)^+ \quad \text{and} \quad \sim x^+ = (\sim x)^*.
\]

By (5) we have that in a pseudocomplemented Kleene algebra

\[
x^* \leq \sim x \leq x^+.
\]

H. P. Sankappanavar [San86] has proved that a pseudocomplemented De Morgan algebra satisfying (M) truly is a congruence-regular pseudocomplemented De Morgan algebra. Therefore, in the sequel we may call pseudocomplemented De Morgan and Kleene algebras regular when they satisfy (M) or (D). Note that in a pseudocomplemented De Morgan algebra, condition (M) is actually in the form

\[
x^* = y^* \text{ and } (\sim x)^* = (\sim y)^* \text{ imply } x = y.
\]

An element \(j\) of a complete lattice \(L\) is called completely join-irreducible if \(j = \bigvee S\) implies \(j \in S\) for every subset \(S\) of \(L\). Note that the least element 0 of \(L\) is not completely join-irreducible. The set of completely join-irreducible elements of \(L\) is denoted by \(J(L)\), or simply by \(J\) if there is no danger of confusion. Note that in a distributive lattice \(L\), the principal filter \(\uparrow j = \{x \in L \mid x \geq j\}\) of each \(j \in J\) is prime.

A complete lattice \(L\) is spatial if for each \(x \in L\),

\[
x = \bigvee \{j \in J \mid j \leq x\}.
\]

An element \(x\) of a complete lattice \(L\) is said to be compact if, for every \(S \subseteq L\),

\[
x \leq \bigvee S \implies x \leq \bigvee F \text{ for some finite subset } F \text{ of } S.
\]

Let us denote by \(K(L)\) the set of compact elements of \(L\). A complete lattice \(L\) is said to be algebraic if for each \(a \in L\),

\[
a = \bigvee \{x \in K(L) \mid x \leq a\}.
\]

It is known (see e.g. [JR17]) that every De Morgan algebra defined on an algebraic lattice is spatial. In case of regular pseudocomplemented Kleene algebras defined on algebraic lattices we presented in [JR17] the following variant of Proposition [1]

**Proposition 2.** Let \((L, \vee, \wedge, \sim, *, 0, 1)\) be a pseudocomplemented De Morgan algebra defined on an algebraic lattice. The following are equivalent:

(a) \(L\) is regular.
(b) Any chain in \(J\) has at most two elements.
3. Alexandrov topologies and Varlet spaces

First recall some facts of Alexandrov topologies from the literature [Ale37, Bir37]. An Alexandrov topology is a topology that contains all arbitrary intersections of its members. Let $\mathcal{T}$ be an Alexandrov topology on $X$. Then, for each $A \subseteq X$, there exists the smallest neighbourhood (i.e. the smallest open set containing $A$):

$$N(A) = \bigcap \{ Y \in \mathcal{T} \mid A \subseteq Y \}.$$ 

In particular, the smallest neighbourhood of a point $x \in X$ is denoted by $N(x)$. The family

$$\mathcal{J}(\mathcal{T}) = \{ N(x) \mid x \in X \}$$

is the smallest base of the Alexandrov topology $\mathcal{T}$. Each member $B$ of $\mathcal{T}$ can be expressed as $B = \bigcup \{ N(x) \mid x \in B \}$.

For an Alexandrov topology $\mathcal{T}$ on $X$, the ordered set $(\mathcal{T}, \subseteq)$ is a complete lattice is which

$$\bigvee \mathcal{H} = \bigcup \mathcal{H} \quad \text{and} \quad \bigwedge \mathcal{H} = \bigcap \mathcal{H}$$

for any $\mathcal{H} \subseteq \mathcal{T}$. This lattice $\mathcal{T}$ is spatial and $\mathcal{J}(\mathcal{T})$ is the set of completely join-irreducible elements. The complete Boolean lattice $\wp(X)$ is known to be algebraic and completely distributive. Because $\mathcal{T}$ is a complete sublattice of $\wp(X)$, it is algebraic and completely distributive.

Let $(X, \preceq)$ be a quasiordered set. We may define an Alexandrov topology $\mathcal{U}(X)$ on $X$ consisting of all upward-closed subsets of $X$ with respect to the relation $\preceq$. Formally,

$$\mathcal{U}(X) = \{ A \subseteq X \mid (\forall x, y \in X) \ x \in A \ \& \ x \preceq y \implies y \in A \}$$

We write $\uparrow x = \{ y \in X \mid x \preceq y \}$ also for the quasiorder filter of $x$. The set $\uparrow x$ is the smallest neighbourhood of the point $x$ in the Alexandrov topology $\mathcal{U}(X)$.

We may now define the rough approximation operators in terms of the quasiorder

$$A^\uparrow = \{ x \in X \mid \uparrow x \subseteq A \} \quad \text{and} \quad A^\downarrow = \{ x \in X \mid \uparrow x \cap A \neq \emptyset \}$$

for any $A \subseteq X$. It is easy to see that

$$\mathcal{U}(X) = \{ A^\uparrow \mid A \subseteq X \},$$

which means that $A \mapsto A^\uparrow$ is the interior operator of the topology $\mathcal{U}(X)$. The lattice $\mathcal{U}(X)$ is pseudocomplemented, in which

$$A^* = A^\uparrow \downarrow = A^\downarrow^\uparrow = \{ x \in X \mid \uparrow x \cap A = \emptyset \}.$$ 

This means that for any quasiordered set $(X, \preceq)$, the algebra

$$\langle \mathcal{U}(X), \cup, \cap, ^*, \emptyset, X \rangle$$

is a distributive $p$-algebra.

We studied in [JR14a] so-called Monteiro spaces in the setting of rough sets defined by quasiorders. Monteiro spaces were introduced by D. Vakarelov in [Vak77], where they were used for giving a representation theorem for Nelson algebras. In this work, we introduce Varlet spaces which resemble Monteiro spaces. First three conditions (J1)–(J3) are the same, but for Monteiro spaces, condition (J4) defines so-called interpolation property. In the case of Varlet spaces, the condition is replaced by a requirement that any chain has at most two elements.

**Definition 3.** Let $(X, \leq, g)$ be a structure such that $(X, \leq)$ is a partially ordered set and $g$ is a map on $X$ satisfying the following conditions for all $x, y$:

- (J1) if $x \leq y$, then $g(x) \geq g(y)$,
- (J2) $g(g(x)) = x$,
- (J3) $x \leq g(x)$ or $g(x) \leq x$,
- (J4) any chain in $(X, \leq)$ has at most two elements.
Then, \((X, \leq, g)\) is called a Varlet space.

**Remark 4.** Let \((X, \leq, g)\) be a Varlet space. Here we present some observations related to the map \(g\).

(a) Conditions (J1) and (J2) mean that \(g : X \to X\) is a polarity on \(X\), that is, \(X\) is isomorphic to its dual.

(b) Condition (J3) means that any \(x \in X\) is comparable with \(g(x)\). Therefore \(X\) can be divided into two disjoint parts in terms of \(g\):

\[
\{x \in X \mid x \leq g(x)\} \quad \text{and} \quad \{x \in X \mid x > g(x)\}.
\]

(c) Condition (J4) says that \(X\) has at most two levels: \(\{x \in X \mid x \leq g(x)\}\) is the “lower level” and \(\{x \in X \mid x > g(x)\}\) is the “upper level”.

(d) If \(g(x) = x\), then \(x\) is not comparable with any \(y \neq x\). Indeed, if \(x < y\), then \(g(y) < g(x) = x < y\) is a chain with more than two elements, which contradicts (J4). Similarly, \(y < x\) implies that \(y \leq x = g(x) < g(y)\) is a chain of three elements, a contradiction again.

For a Varlet space \((X, \leq, g)\), we define a map \(\sim : \mathcal{U}(X) \to \mathcal{U}(X)\) by:

\[
\sim A = \{x \in X \mid g(x) \notin A\}.
\]

The operation \(\sim\) is well defined. Indeed, let \(A \in \mathcal{U}(X), x \in A\) and \(x \leq y\). Then \(x \in \sim A\) means \(g(x) \notin A\), and \(g(y) \leq g(x)\) gives \(g(y) \notin A\), that is, \(y \in \sim A\). Thus, \(\sim A \in \mathcal{U}(X)\).

We can now write the following proposition:

**Proposition 5.** Let \((X, \leq, g)\) be a Varlet space. Then, the algebra

\[
(\mathcal{U}(X), \cup, \cap, \sim, *, \emptyset, X)
\]

is a regular pseudocomplemented Kleene algebra defined on an algebraic lattice.

**Proof.** We already know that \((\mathcal{U}(X), \cup, \cap, \sim, *, \emptyset, X)\) is a pseudocomplemented algebraic lattice. Next we verify that \(\sim\) is a Kleene operation. This proof is modified from the one appearing in [Vak77]. Let \(A, B \in \mathcal{U}(X)\).

- Now, \(x \in A \iff g(x) \in A \iff g(x) \notin \sim A \iff x \in \sim \sim A\). Thus \(A = \sim \sim A\).
- Assume \(A \subseteq B\). If \(x \in \sim B\), then \(g(x) \notin B\). This gives \(g(x) \notin A\) and \(x \in \sim A\). So, \(\sim B \subseteq \sim A\).
- Suppose that \(A \cap \sim A \nsubseteq B \cup \sim B\). Then there exists \(x \in X\) such that \(x \in A \cap \sim A\) and \(x \notin B \cup \sim B\). Therefore,

\[
x \in A, \quad g(x) \notin A, \quad x \notin B, \quad g(x) \in B.
\]

But from these we have

\[
x \notin g(x) \quad \text{and} \quad g(x) \notin x,
\]

a contradiction. We have now proved that \((\mathcal{U}(X), \cup, \cap, \sim, *, \emptyset, X)\) is a pseudocomplemented Kleene algebra.

We end the proof by showing the regularity of this pseudocomplemented Kleene algebra. As we have noted, the family \(\mathcal{J}(\mathcal{U}(X)) = \{\uparrow x \mid x \in X\}\) is the set of completely join-irreducible elements of the complete lattice \(\mathcal{U}(X)\). It is easy to observe that for all \(x, y \in X, x \leq y\) if and only if \(\uparrow y \subseteq \uparrow x\). This implies that any chain in \(\mathcal{J}(\mathcal{U}(X))\) has at most two elements, because \(X\) satisfies this property. Therefore, by Proposition 2, \((\mathcal{U}(X), \cup, \cap, \sim, *, \emptyset, X)\) is a regular pseudocomplemented Kleene algebra.

Let \((L, \lor, \land, \sim, *, 0, 1)\) be a regular pseudocomplemented Kleene algebra with \(\mathcal{F}_p\) as the set of its prime filters. The algebra \((L, \lor, \land, *, ^+, 0, 1)\) is a distributive double \(p\)-algebra, where \(^+\) is defined as in [13]. By Proposition 1, any chain in \(\mathcal{F}_p, \subseteq\) has at most two elements. We define for any \(P \in \mathcal{F}_p\) the set

\[
g(P) = \{x \in L \mid \sim x \notin P\}.
\]

For any \(P \in \mathcal{F}_p, g(P)\) is a prime filter. Indeed:
• Because \( P \) is a proper filter, \( \sim 1 = 0 \notin P \). This means that \( 1 \in g(P) \) and therefore \( g(P) \) is nonempty.

• Assume that \( x \in g(P) \) and \( x \leq y \). Now \( \sim x \notin P \) and \( \sim x \geq \sim y \) imply \( \sim y \notin P \), because \( P \) is a filter. Then \( y \in g(P) \) and \( g(P) \) is upward-closed.

• Suppose \( a, b \in g(P) \). Then, \( \sim a \notin P \) and \( \sim b \notin P \). Assume for contradiction that \( a \land b \notin g(P) \). Then \( \sim (a \land b) = \sim a \lor \sim b \) belongs to \( P \). But because \( P \) is a prime filter, we have that \( \sim a \in P \) or \( \sim b \in P \), a contradiction. Thus \( a \land b \notin g(P) \).

• The filter \( g(P) \) is proper, because \( 0 \in g(P) \) would imply \( \sim 0 = 1 \notin P \), which is impossible.

• Finally, suppose \( g(P) \) is not prime. Then, there are \( a \) and \( b \) in \( L \) such that \( a \lor b \in g(P) \), but \( a \notin g(P) \) and \( b \notin g(P) \). Therefore, \( \sim a \in P \) and \( \sim b \in P \). Because \( P \) is a filter then \( \sim a \land \sim b = \sim (a \lor b) \) is in \( P \). But this gives that \( a \lor b \notin g(P) \), a contradiction. Hence \( g(P) \) is prime.

**Lemma 6.** If \((L, \lor, \land, \sim, *, 0, 1)\) is a regular pseudocomplemented Kleene algebra, then the triple \((\mathcal{F}_p, \subseteq, g)\) forms a Varlet space.

**Proof.** We will show that \( g \) satisfies the conditions (J1)–(J4). Let \( P, P_1 \), and \( P_2 \) be prime filters of \( L \).

\begin{itemize}
  \item \((J1)\) Assume that \( P_1 \subseteq P_2 \). If \( x \in g(P_2) \), then \( \sim x \notin P_2 \). This gives \( \sim x \notin P_1 \) and \( x \in g(P_1) \). Hence, \( g(P_2) \subseteq g(P_1) \).
  \item \((J2)\) For any \( x \in L \), \( x \in P \iff \sim \sim x \in P \iff \sim x \notin g(P) \iff x \in g(g(P)) \).
  \item \((J3)\) Suppose that \( P \nsubseteq g(P) \) and \( g(P) \nsubseteq P \). There are elements \( x, y \in L \) such that \( x \in P \), \( x \notin g(P) \), \( y \in g(P) \), \( y \notin P \).
  \item \((J4)\) This condition is clear by Proposition \ref{prop:primefilter}.$\Box$
\end{itemize}

By combining Proposition \ref{prop:primefilter} and Lemma \ref{lem:regular_pseudocomplemented} we have that any regular pseudocomplemented Kleene algebra \((L, \lor, \land, \sim, *, 0, 1)\) determines a regular pseudocomplemented Kleene algebra

\[(U(\mathcal{F}_p), \cup, \cap, \sim, *, \emptyset, \mathcal{F}_p),\]

defined on an algebraic lattice. Recall that for all \( A \in U(\mathcal{F}_p) \):

\[\sim A = \{ P \in \mathcal{F}_p \mid g(P) \notin A \}\]

and \( A^* = \{ P \in \mathcal{F}_p \mid \uparrow P \cap A = \emptyset \} \),

where \( \uparrow P = \{ Q \in \mathcal{F}_p \mid P \subseteq Q \} \).

For any element \( x \in L \), we denote

\[h(x) = \{ P \in \mathcal{F}_p \mid x \in P \}.
\]

It is easy to see that \( h(x) \in U(\mathcal{F}_p) \). Namely, if \( P \in h(x) \) and \( P \subseteq Q \) for some \( P, Q \in \mathcal{F}_p \), then \( x \in P \subseteq Q \), that is, \( Q \in h(x) \). Therefore, the mapping \( h : L \to U(\mathcal{F}_p) \) is well defined.

**Proposition 7.** The mapping \( h \) is an embedding between pseudocomplemented Kleene algebras.

**Proof.** We first note that \( h \) is an injection. Because \( L \) is a distributive, for any \( x \neq y \) there exists a prime filter \( P \) such that \( x \in P \) and \( y \notin P \), or \( x \notin P \) and \( y \in P \). This means that \( h(x) \neq h(y) \).

\begin{itemize}
  \item \( h(0) = \emptyset \), because prime filters must be proper filters. Therefore, \( 0 \) does not belong to any prime filter.
  \item \( h(1) = \mathcal{F}_p \), because \( 1 \) must belong to all prime filters.
\end{itemize}
• $P \in h(x \lor y) \iff x \lor y \in P$ \iff $x \in P$ or $y \in P$ \iff $P \in h(x)$ or $P \in h(y)$. Thus, $h(x \lor y) = h(x) \cup h(y)$.
• $P \in h(x \land y) \iff x \land y \in P$ \iff $x \in P$ and $y \in P$ \iff $P \in h(x)$ and $P \in h(y)$
• $P \in h(\neg x) \iff \neg x \in P$ \iff $x \notin g(P)$ \iff $g(P) \notin h(x)$ \iff $P \in \neg h(x)$
This means that $h(\neg x) = \neg h(x)$.

Finally, we prove that $h$ preserves the pseudocomplement $\ast$. The structure of the proof is taken from the proof of Lemma 9.10.4 in [ORM13].

$(\Rightarrow)$ Suppose $P \in h(x^\ast)$, that is, $x^\ast \in P$. Let $Q \in \uparrow P$. Then $x^\ast \in Q$, which gives $x \notin Q$, because otherwise $0 = x \land x^\ast$ in $Q$. This is not possible, since $Q$ is a prime filter and thus proper. Then $Q \notin h(x)$ gives $\uparrow P \cap h(x) = \emptyset$ and $P \neq h(x)^\ast$.

$(\Leftarrow)$ Assume $P \notin h(x^\ast)$, that is, $x^\ast \notin P$. Let $Q$ be a filter generated by $P \cup \{x\}$. First we show that $Q$ is proper. Indeed, if $Q$ is not proper, then $0 \in Q$. Because $0 \notin P$ and $x \neq 0$, we have that $0 = x \land y$ for some $y \in P$, because $Q$ is the filter generated by $P \cup \{x\}$. This implies $y \leq x^\ast$. Now $x^\ast \notin P$ gives $y \notin P$, a contradiction. Because $Q$ is a proper filter, there exists a prime filter $W$ such that $P \subseteq Q \subseteq W$. Now $x \in P \cup \{x\} \subseteq Q \subseteq W$ gives $W \subseteq h(x)$. Thus, $\uparrow P \cap h(x) \neq \emptyset$ and $P \neq h(x)^\ast$, as required. $\square$

In [JR17] Theorem 5.3 we proved that any regular pseudocomplemented Kleene algebra defined on an algebraic lattice is isomorphic to a rough set Kleene algebra determined by an irredundant covering. If $(A, \lor, \land, \neg, ^\ast)$ is a regular pseudocomplemented Kleene algebra, then by Proposition 7 it is isomorphic to a subalgebra of $\mathcal{U} = (\mathcal{U}(X), \cup, \cap, \neg, ^\ast, \emptyset, X)$. Because $\mathcal{U}$ is a regular pseudocomplemented Kleene algebra defined on an algebraic lattice, there exists a tolerance induced by an irredundant covering such that its rough set regular pseudocomplemented Kleene algebra $\mathcal{RS}$ is isomorphic to $\mathcal{U}$. Therefore, we can write the following representation theorem.

Theorem 8. Let $L$ be a regular pseudocomplemented Kleene algebra. Then, there exists a set $U$ and a tolerance $T$ induced by an irredundant covering of $U$ such that $L$ is isomorphic to a subalgebra of $\mathcal{RS}$.

4. Regular pseudocomplemented Kleene algebras satisfying the Stone identity

A Stone algebra is a $p$-algebra $(L, \lor, \land, ^\ast, 0, 1)$ satisfying the Stone identity:

$x^\ast \lor x^{**} = 1$. 

In a Stone algebra the identities

$$(x \land y)^\ast = x^\ast \lor y^\ast$$

and

$$(x \lor y)^{**} = x^{**} \lor y^{**}$$

also hold. A double Stone algebra is a $p$-algebra $(L, \lor, \land, ^\ast, ^+, 0, 1)$ satisfying (5) and

$$x^+ \land x^{++} = 0.$$ 

A double Stone algebra satisfies the identity $x^{++} = x^{**}$, because

$$x^{**} = x^{**} \land 1 = x^{**} \land (x^+ \lor x^\ast) = x^{**} \land x^+,$$

and hence $x^{**} \leq x^+$. The inequality $x^{++} \leq x^{**}$ follows from $x^\ast \lor x^+ = 1$. Similarly, we can show $x^{++} = x^{**}$. Because $x^{++} \leq x^{**}$, we have

$$x^\ast = x^{**} \leq x^{++} = x^{**} = x^+.$$ 

A double Stone algebra is called regular if it is regular as a double $p$-algebra, that is, it satisfies (M) or (D).

Varlet has proved in [Var68] that three-valued Lukasiewicz algebras coincide with regular double Stone algebras. Here we use similar technique to prove that regular double Stone
algebras coincide with regular pseudocomplemented Kleene algebras satisfying \[.\] The proof of the following proposition is modified from the proof of [BFGR91, Theorem 4.4].

**Proposition 9.** Let \((L, \lor, \land, *, +, 0, 1)\) be a regular double Stone algebra. If we define an operation \(\sim\) by

\[
\sim x = (x \land x^+) \lor x^*,
\]

then the algebra \((L, \lor, \land, \sim, *, 0, 1)\) is a regular pseudocomplemented Kleene algebra satisfying \([8]\).

**Proof.** By straightforward computation:

\[
(\sim x)^* = (x \land x^+)^* \land x^{**} = (x^* \lor x^{**}) \land x^{**} = (x^* \lor x^+) \land x^{**} = x^{**} = \sim x;
\]

\[
(\sim x)^+ = (x \land x^+)^+ \land x^{**} = (x^+ \lor x^+) \land x^{**} = x^+;
\]

\[
\sim x = (x \land (\sim x)^+) \lor (\sim x)^* = (((x \land x^+) \lor x^*) \land x^{**}) \lor x^{**} = (x \land x^+) \lor x^+;
\]

\[
(\sim x)^* = (x \land x^+)^* \land x^{**} = (x^* \lor x^{**}) \land x^+ = (x^* \lor x^+) \land x^+ = (x^* \lor x^+) \lor x^+ = x^+;
\]

\[
(\sim x)^+ = (x \land x^+)^+ \lor x^{**} = (x^+ \lor x^+) \lor x^{**} = (x^+ \lor x^+) \lor x^+ = x^+.
\]

Because \((\sim x)^* = x^*\) and \((\sim x)^+ = x^+\), we obtain \(x = \sim x\) by (M). Now

\[
(\sim x \lor \sim y)^* = (\sim x)^* \land (\sim y)^* = x^{**} \land y^{**} = (x \land y)^+ = (\sim (x \land y))^*
\]

and

\[
(\sim x \lor \sim y)^+ = (\sim x)^+ \land (\sim y)^+ = x^* \land y^* = (x \land y)^{**} = (\sim (x \land y))^+
\]

From this we have \(\sim x \lor \sim y = \sim (x \land y)\). Therefore, \((L, \lor, \land, \sim, 0, 1)\) is a De Morgan algebra. Furthermore,

\[
x \land \sim x = x \land ((x \land x^+) \lor x^+) = (x \land x^+) \lor (x \land x^*) = x \land x^+
\]

and

\[
y \lor \sim y = y \lor (y \land y^*) \lor y^* = y \lor y^*.
\]

We have

\[
x \land \sim x = x \land x^+ \leq y \lor y^* = y \lor \sim y
\]

by (D). So \((L, \lor, \land, \sim, *, 0, 1)\) is a pseudocomplemented Kleene algebra, which is regular and satisfies \([8]\) by assumption.

As we have noted, \(x^* \leq x^+\) holds for any element \(x\) of a double Stone algebra. This implies that

\[
(x \land x^+) \lor x^* = (x \lor x^*) \land (x^+ \lor x^*) = (x \lor x^*) \land x^+.
\]

Therefore, the operation \(\sim\) may be defined also as \(\sim x = (x \lor x^*) \land x^+\) in a regular double Stone algebra \((L, \lor, \land, *, +, 0, 1)\).

Let \((L, \lor, \land, \sim, *, 0, 1)\) be a pseudocomplemented Kleene algebra. Then \(L\) is a double pseudocomplemented lattice in which the pseudocomplements * and + determine each other. In particular, \(\sim x^+ = (\sim x)^*\). Therefore, if \([8]\) is satisfied in \(L\), then

\[
1 = (\sim x)^* \lor (\sim x)^{**} = (x^+ \lor x^*) \land (x^+ \lor x^+) = (\sim x^+) \land (\sim x^+)^*
\]

\[
\sim x^+ \lor (\sim x^+) = (x^+ \lor x^+) \land (\sim x^+)^+.
\]

This means that \(x^+ \land (x^+)^+ = 0\) and \([8]\) is valid in \(L\). Therefore, we can write the following proposition.

**Proposition 10.** If \((L, \lor, \land, \sim, *, 0, 1)\) is a pseudocomplemented Kleene algebra satisfying \([8]\), then \((L, \lor, \land, \sim, \lor, \land, +, 0, 1)\) is a double Stone algebra which is regular exactly when the pseudocomplemented Kleene algebra \((L, \lor, \land, \sim, *, 0, 1)\) is regular.
We have now shown that each regular double Stone algebra
\[ L = (\mathcal{L}, \vee, \wedge, *, \plane, 0, 1) \]
defines a regular pseudocomplemented Kleene algebra \( \mathcal{L}^{\text{rpK}} \) satisfying the identity \( x^* \vee x^{**} = 1 \), and each regular pseudocomplemented Kleene algebra
\[ \mathcal{K} = (\mathcal{K}, \vee, \wedge, \sim, *, 0, 1) \]
satisfying \( x^* \vee x^{**} = 1 \) defines a regular double Stone algebra \( \mathcal{K}^{\text{rdS}} \). Our next proposition shows that the correspondences \( L \leftrightarrow \mathcal{L}^{\text{rpK}} \) and \( L \leftrightarrow \mathcal{K}^{\text{rdS}} \) are one-to-one and mutually inverse.

**Proposition 11.** Let \( L \) be a regular double Stone algebra and \( \mathcal{K} \) be a regular pseudocomplemented Kleene algebra satisfying \( x^* \vee x^{**} = 1 \). Then the following equalities hold:

(a) \( L = (\mathcal{L}^{\text{rpK}})^{\text{rdS}} \),

(b) \( \mathcal{K} = (\mathcal{K}^{\text{rdS}})^{\text{rpK}} \).

**Proof.** Let us first note that operations \( \vee, \wedge, *, 0, 1 \) are immutable in these transformations, because they are in the signature of the both algebras.

(a) Assume \( L = (\mathcal{L}, \vee, \wedge, *, \plane, 0, 1) \) is a regular double Stone algebra. It defines a regular pseudocomplemented Kleene algebra \( \mathcal{L}^{\text{rpK}} \) in which the operation \( \sim \) is defined by \( \sim x = (x \wedge x^+) \vee x^* \).

In \( \mathcal{L}^{\text{rpK}} \), a dual pseudocomplement is defined in terms of this \( \sim \) and \( * \) by \( x^\oplus = \sim (\sim x)^*. \) Now
\[ x^\oplus = \sim (\sim x)^* = \sim x^{++} = (x^+++ \wedge x^{+++}) \vee x^{+++} = x^{++} = x^+. \]

This means that the algebras \( L \) and \( (\mathcal{L}^{\text{rpK}})^{\text{rdS}} \) coincide.

(b) Let \( \mathcal{K} \) be a regular pseudocomplemented Kleene algebra satisfying \( x^* \vee x^{**} = 1 \). The corresponding regular double Stone algebra is \( \mathcal{K}^{\text{rdS}} \) where the dual pseudocomplement is defined by \( x^+ = \sim (\sim x)^* \). In \( \mathcal{K}^{\text{rdS}} \), a Kleene negation is defined by
\[ \sim x = (x \wedge x^+) \vee x^* \].

According to the proof of Proposition 9 we have \( (\sim x)^* = x^{++} \). Because \( \mathcal{K}^{\text{rdS}} \) is a double Stone algebra, \( x^{++} = x^{**} \). On the other hand, as a pseudocomplemented Kleene algebra \( L \) satisfies (7), and therefore
\[ x^{**} \leq \sim x^+ \leq x^{++}. \]

We have \( \sim x^+ = x^{++} = x^{**} \). By definition, \( \sim x^+ = (\sim x)^* \). Hence,
\[ (\sim x)^* = x^{++} = \sim x^+ = (\sim x)^*. \]

Similarly, by the proof of Proposition 9 \( (\sim x)^+ = x^{**} \). Since \( \mathcal{K}^{\text{rdS}} \) is a double Stone algebra, we have \( x^{++} = x^{**} \). By (7), \( x^{**} \leq \sim x^* \leq x^{**} \), and therefore \( \sim x^* = x^{++} = x^{**} \). Because \( \sim x^* = (\sim x)^+ \), we can write
\[ (\sim x)^+ = x^{**} = (\sim x)^+. \]

We have now proved \( (\sim x)^* = (\sim x)^* \) and \( (\sim x)^+ = (\sim x)^+ \). Because \( \mathcal{K}^{\text{rdS}} \) is a regular double Stone algebra, we have \( \sim x = \sim x \). \( \square \)

Since there is one-to-one correspondence between regular double Stone algebras and regular pseudocomplemented Kleene algebras satisfying (8), and the pseudocomplements and dual pseudocomplements in double Stone algebras are unique, we can write the following corollary.

**Corollary 12.** In any regular pseudocomplemented Kleene algebra satisfying identity (8), the operation \( \sim \) is unique.
Example 13. In a regular pseudocomplemented Kleene algebra, the operation \( \sim \) is not necessarily unique. Let us consider the regular pseudocomplemented Kleene algebra \((L, \vee, \wedge, \sim, \ast, 0, 1)\) depicted in Figure 1(a). There are two ways to define the Kleene operation. The first way is
\[
\sim 0 = 1, \sim a = g, \sim b = f, \sim d = d
\]
and the second is
\[
\sim 0 = 1, \sim a = f, \sim b = g, \sim d = d.
\]
This is possible since \(L\) does not satisfy (8):
\[
a^* \vee a^{**} = b \vee b^* = b \vee a = d \neq 1.
\]

The distributive bounded lattice in Figure 1(b) is a well-known double Stone algebra. The only way to define a Kleene operation in \(L\) is by
\[
\sim 0 = 1, \sim a = g, \sim b = f, \sim c = e, \sim d = d.
\]

We end this section by presenting a representation theorem for regular pseudocomplemented Kleene algebra satisfying the Stone identity. Let \(L = (L, \vee, \wedge, \sim, \ast, 0, 1)\) be a regular pseudocomplemented Kleene algebra satisfying \(x^* \vee x^{**} = 1\). The unique regular double Stone algebra corresponding \(L\) is \(L^{rdS}\). We may now apply the result by Comer mentioned in Section 1, which states that there exists a set \(U\) and an equivalence \(E\) on \(U\) such that \(L^{rdS}\) can be embedded to \(RS = (RS, \vee, \wedge, \sim, \ast, +, 0, 1)\), the pseudocomplemented Kleene algebra defined by \(E\). By the above, \(RS\) uniquely determines a regular pseudocomplemented Kleene algebra \(RS^{pK}\) satisfying \(x^* \vee x^{**} = 1\). Obviously, the original regular pseudocomplemented Kleene algebra \(L\) can be embedded to \(RS^{pK}\). Therefore, we can write the following theorem.

Theorem 14. Let \(L\) be a regular pseudocomplemented Kleene algebra satisfying \(x^* \vee x^{**} = 1\). Then, there exists a set \(U\) and an equivalence \(E\) on \(U\) such that \(L\) is isomorphic to a subalgebra of
\[
(RS, \vee, \wedge, \sim, \ast, (0, \emptyset), (U, U)),
\]
the pseudocomplemented Kleene algebra defined by \(E\).

5. Varlet spaces for regular pseudocomplemented Kleene algebras satisfying the Stone identity

It is proved in [Var66, Theorem 1] that a distributive pseudocomplemented lattice is a Stone lattice if and only if every prime filter is contained in only one proper maximal filter. It is known than in a distributive lattice each maximal proper filter is a (maximal) prime filter. This means that a distributive double \(p\)-algebra is a double Stone algebra if
and only if each prime filter is included in a unique maximal prime filter and includes a unique minimal prime filter.

If we combine this with the claim of Proposition 1 stating that a distributive double \( p \)-algebra is regular if and only if any chain of prime filters of \( L \) has at most two elements, we have that a distributive double \( p \)-algebra is a regular double Stone algebra if and only if the family its prime filters is a disjoint union of chains of at most two elements.

Let \(( X, \leq, g )\) be a Varlet space. We introduce the condition:

\((*)\) For all \( x, y \in X, \uparrow x \cap \uparrow y \neq \emptyset \) implies \( x = y \) or \( x = g(y) \).

Note that for all \( x, y \in X, x = g(y) \) is equivalent to \( y = g(x) \).

**Lemma 15.** Let \(( X, \leq, g )\) be a Varlet space. Condition \((*)\) holds if and only if \(( X, \leq )\) is a disjoint union of chains of at most two elements.

**Proof.** Because \(( X, \leq, g )\) is a Varlet space, any chain in \(( X, \leq )\) has at most two elements. Suppose that \((*)\) holds. It is enough to show that \(( X, \leq )\) has no intersecting maximal chains. If \( C = \{ x \} \) is maximal chain of one element, then \( x = g(x) \). By Remark 4(d), \( x \) is not comparable with any other element in \( X \). Therefore, \( C \) cannot have common elements which other maximal chains.

Suppose that there are maximal two-element chains \( C_1 \) and \( C_2 \) that have a common element \( y \). Hence, \( C_1 = \{ x_1, y \} \) and \( C_2 = \{ x_2, y \} \) for some \( x_1, x_2 \in X \) such that either (a) or (b) in Figure 2 holds. In both these cases, \( \uparrow x_1 \cap \uparrow y \neq \emptyset \) and \( \uparrow x_2 \cap \uparrow y \neq \emptyset \). By \((*)\), we have \( g(x_1) = y = g(x_2) \). This implies \( x_1 = x_2 \) and \( C_1 = \{ x_1, y \} = \{ x_2, y \} = C_2 \).

**Figure 2.**

Conversely, let \(( X, \leq )\) be a disjoint union of chains with at most two elements. By Remark 2 it is clear that \( \uparrow x = \{ x, g(x) \} \) and \( \uparrow y = \{ y, g(y) \} \) Suppose that \( \uparrow x \cap \uparrow y \neq \emptyset \) for some \( x, y \in X \). This is possible only if \( \{ x, g(x) \} = \{ y, g(y) \} \). This gives directly \( x = y \) or \( x = g(y) \).

Let \(( X, \leq, g )\) be a Varlet space. For any \( A \subseteq X \), let us denote

\[ g[A] = \{ g(a) \mid a \in A \} \]

It is clear that for any \( A, B \subseteq X \),

\[ g[A^c] = g[A]^c, \quad g[A \cup B] = g[A] \cup g[B], \quad g[A \cap B] = g[A] \cap g[B]. \]

In addition, \( g[g[A]] = A \) because \( g(g(x)) = x \) for any \( x \in X \).

**Proposition 16.** Let \(( X, \leq, g )\) be a Varlet space. Then the regular pseudocomplemented Kleene algebra \(( \mathcal{U}(X), \cup, \cap, \sim^*, \emptyset, X )\) defined by \(( X, \leq, g )\) satisfies the Stone identity \((\mathcal{S})\) if and only if \(( X, \leq )\) is a union of disjoint chains of at most two elements.

**Proof.** Suppose \(( \mathcal{U}(X), \cup, \cap, \sim^*, \emptyset, X )\) satisfies the Stone identity \((\mathcal{S})\). Because \(( X, \leq, g )\) is a Varlet space, each chain in \(( X, \leq )\) has at most two elements. We will show that if \( C_1 \) and \( C_2 \) are maximal intersecting chains, then \( C_1 = C_2 \). As in the proof of Lemma 15 it is enough to consider chains \( C_1 = \{ x_1, y \} \) and \( C_2 = \{ x_2, y \} \) such that the situation is as in Figure 2 that is, either (a) \( x_1, x_2 \leq y \) or (b) \( y \leq x_1, x_2 \). Note that in (b), \( g(x_1), g(x_2) < g(y) \). Hence, it suffices to consider only (a).
For (a) we show that \(g(x_1) = y = g(x_2)\). Suppose for contradiction that \(g(x_1) \neq y\). As \(x_1 < y\), \(x_1\) belongs to the “lower level” \(\{x \in X \mid x < g(x)\}\) of \(X\) and \(\uparrow g(x_1) = \{g(x_1)\}\) by Remark 2. By the definition of the pseudocomplement in \(\mathcal{U}(X)\),
\[
(\uparrow g(x_1))^* = \{x \in X \mid \uparrow x \cap \{g(x_1)\} = \emptyset\} = \{x \in X \mid x \not\in g(x_1)\}.
\]
Because \(x_1 < g(x_1)\), we get \(x_1 \not\in (\uparrow g(x_1))^*\). Observe that \(y < g(x_1)\) implies that \(x_1 < y < g(x_1)\) is a chain of more than two elements. This is not possible, so \(y \not\in g(x_1)\). Since \(y \neq g(x_1)\) holds by assumption, we obtain \(y \not\in g(x_1)\) and \(y \in (\uparrow g(x_1))^*\). As \(x_1 < y\), we have \(\uparrow x_1 \cap (\uparrow g(x_1))^* \neq \emptyset\). By definition,
\[
(\uparrow g(x_1))^** = \{x \in X \mid \uparrow x \cap (\uparrow g(x_1))^* = \emptyset\}.
\]
Therefore, \(x_1 \not\in (\uparrow g(x_1))^**\). We already showed \(x_1 \not\in (\uparrow g(x_1))^*\). Thus,
\[
x_1 \not\in (\uparrow g(x_1))^* \cup (\uparrow g(x_1))^**.
\]
This contradicts the Stone identity \((\uparrow g(x_1))^* \cup (\uparrow g(x_1))^** = X\), which should hold by assumption. This contradiction means that \(g(x_1) \neq y\) is false and hence \(g(x_1) = y\). In an analogous way we can prove that \(g(x_2) = y\). Now \(g(x_1) = y = g(x_2)\) implies \(x_1 = x_2\) and \(C_1 = C_2\). Thus, \((X, \leq)\) is a union of disjoint chains of at most two elements.

Conversely, suppose that \((X, \leq)\) is a union of disjoint chains of at most two elements. Let \(A \in \mathcal{U}(X)\). First, we show that \(A^* = (A \cup g[A])^c\).

We have that \(A^* = \{x \in X \mid \uparrow x \cap A = \emptyset\}\). Clearly, \(\uparrow x \cap A = \emptyset\) means that \(x \not\in A\). We claim that this yields also \(x \not\in g[A]\), which, by definition, is equivalent to \(g(x) \not\in A\). Indeed, if \(x \leq g(x)\), then \(\uparrow x \cap A = \emptyset\) implies \(g(x) \not\in A\). If \(g(x) < x\), then we get \(g(x) \not\in A\), since \(g(x) \in A\) would imply \(x \in A\), because \(A\) is \(\leq\)-closed. Thus, \(x \not\in g[A]\) and \(A^* \subseteq A^c \cap g[A]^c = (A \cup g[A])^c\).

Now take any \(x \in (A \cup g[A])^c\), and assume that \(\uparrow x \cap A \neq \emptyset\). Then there exists an element \(y \in \uparrow x \cap A\). Now \(\uparrow x \cap \uparrow y \neq \emptyset\). Since \(x \not\in A\), we have \(x \neq y\). By Lemma 15 \((X, \leq, g)\) satisfies condition (1). Hence, \(x = g(y) \in g[A]\), a contradiction. We have \(\uparrow x \cap A = \emptyset\), which gives \(x \in A^*\). We have how proved \(A^* = (A \cup g[A])^c\).

Now \(g[A^*] = g[(A \cup g[A])^c] = g[(A \cup g[A])^c] = (g[A] \cup A)^c = A^c\). We have \(A^* = (A^* \cup g[A^*])^c = (A^* \cup A^c)^c = (A^c)^c = A\). This means that the algebra \((\mathcal{U}(X), \cup, \cap, \sim, *, \emptyset, X)\) satisfies the Stone identity [5].

We end this work showing the connections between regular pseudocomplemented Kleene algebras, Varlet spaces of their prime filters, and Kleene algebras of the Alexandrov topologies of upward-closed of prime filters.

**Corollary 17.** Let \(L = (L, \lor, \land, \sim, *, 0, 1)\) be a regular pseudocomplemented Kleene algebra. The following are equivalent:

(a) The algebra \(L\) satisfies \(x^* \lor x^{**} = 1\).

(b) The Varlet space \((\mathcal{F}_p, \subseteq, g)\) determined by \(L\) is a union of disjoint chains of at most two elements.

(c) The regular pseudocomplemented Kleene algebra \((\mathcal{U}(\mathcal{F}_p), \cup, \cap, \sim, *, \emptyset, \mathcal{F}_p)\) defined by \((\mathcal{F}_p, \subseteq, g)\) satisfies the Stone identity [5].

**Proof.** (a)\(\Rightarrow\) (b): By Lemma 6 \((\mathcal{F}_p, \subseteq, g)\) is a Varlet space, which means that each chain has at most two elements. If \(L\) satisfies \(x^* \lor x^{**} = 1\), then \(L^{\text{dS}}\) is a double Stone algebra. As we noted in the beginning of this section, a distributive double \(p\)-algebra is a double Stone algebra if and only if each prime filter is included in a unique maximal prime filter and includes a unique minimal prime filter. By combining these two observations, we have that \((\mathcal{F}_p, \subseteq)\) is a union of disjoint chains of at most two elements.

(b)\(\Rightarrow\) (c): This follows directly from Proposition 16

(c)\(\Rightarrow\) (a): The mapping \(h(x) = \{P \in \mathcal{F}_p \mid x \in P\}\) is a homomorphism by Proposition 7. This means that
\[
h(x^* \lor x^{**}) = h(x^*) \lor h(x^{**}) = h(x)^* \lor h(x)^{**} = \mathcal{F}_p = h(1).
\]
Because \( h \) is also an embedding, it is an injection. This gives \( x^* \lor x^{**} = 1 \).

References

[Ale37] Paul Alexandroff, *Diskrete räume*, Matematicheskij Sbornik 2 (1937), 501–518.

[BFGR91] V. Boicescu, A. Filipoiu, G. Georgescu, and S. Rudeanu, *Lukasiewicz–Moisil algebras*, Elsevier, Amsterdam, 1991.

[Bir37] Garrett Birkhoff, *Rings of sets*, Duke Mathematical Journal 3 (1937), 443–454.

[CdG81] Roberto Cignoli and Marta S. de Gallego, *The lattice structure of some Lukasiewicz algebras*, Algebra Universalis 13 (1981), 315–328.

[Com93] Stephen D. Comer, *On connections between information systems, rough sets, and algebraic logic*, Algebraic Methods in Logic and Computer Science, Banach Center Publications, no. 28, 1993, pp. 117–124.

[Iwi87] Tadeusz B. Iwiński, *Algebraic approach to rough sets*, Bulletin of Polish Academy of Sciences. Mathematics 35 (1987), 673–683.

[Jär99] Jouni Järvinen, *Knowledge representation and rough sets*, Ph.D. dissertation, Department of Mathematics, University of Turku, Finland, 1999, TUCS Dissertations 14.

[Jär07] Jouni Järvinen, *Lattice theory for rough sets*, Transactions on Rough Sets VI (2007), 400–498.

[JPR13] Jouni Järvinen, Piero Pagliani, and Sándor Radeleczki, *Information completeness in Nelson algebras of rough sets induced by quasiorders*, Studia Logica 101 (2013), 1073–1092.

[JR11] Jouni Järvinen and Sándor Radeleczki, *Representation of Nelson algebras by rough sets determined by quasiorders*, Algebra Universalis 66 (2011), 163–179.

[JR14a] Jouni Järvinen, *Monteiro spaces and rough sets determined by quasiorder relations: Models for Nelson algebras*, Fundamenta Informaticae 131 (2014), 205–215.

[JR14b] Jouni Järvinen, *Rough sets determined by tolerances*, International Journal of Approximate Reasoning 55 (2014), 1419–1438.

[JR17] Jouni Järvinen, *Representing regular pseudocomplemented Kleene algebras by tolerance-based rough sets*, Journal of the Australian Mathematical Society (2017), In press.

[JRV09] Jouni Järvinen, Sándor Radeleczki, and Laura Veres, *Rough sets determined by quasiorders*, Order 26 (2009), 337–355.

[Kat73] T. Katriňák, *The structure of distributive double p-algebras. Regularity and congruences*, Algebra Universalis 3 (1973), 238–246.

[ORM15] E. Orłowska, A. M. Radzikowska, and Rewitzky I. M., *Dualities for structures of applied logics*, Studies in logic, no. 56, College Publications, London, 2015.

[Paw82] Zdzisław Pawlak, *Rough sets*, International Journal of Computer and Information Sciences 11 (1982), 341–356.

[PP88] Jacek Pomykała and Janusz A. Pomykała, *The Stone algebra of rough sets*, Bulletin of Polish Academy of Sciences. Mathematics 36 (1988), 495–512.

[San86] H. P. Sankappanavar, *Pseudocomplemented Ockham and De Morgan algebras*, Mathematical Logic Quarterly 32 (1986), 385–394.

[Vak77] Dimiter Vakarelov, *Notes on N-lattices and constructive logic with strong negation*, Studia Logica 36 (1977), 109–125.

[Var66] Jules Varlet, *On the characterization of Stone lattices*, Acta Scientiarum Mathematicarum 27 (1966), 81–84.

[Var68] Jules Varlet, *Algèbres de Lukasiewicz trivalentes*, Bulletin de la Société Royale des Sciences de Liège 37 (1968), 399–408.

[Var72] Jules Varlet, *A regular variety of type \( \langle 2, 2, 1, 1, 0, 0 \rangle \)*, Algebra Universalis 2 (1972), 218–223.

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