BRST Quantization of Noncommutative Gauge Theories

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Abstract

In this paper, the BRST symmetry transformation is presented for the noncommutative $U(N)$ gauge theory. The nilpotency of the charge associated to this symmetry is then proved. As a consequence for the space-like non-commutativity parameter, the Hilbert space of physical states is determined by the cohomology space of the BRST operator as in the commutative case. Further, the unitarity of the S-matrix elements projected onto the subspace of physical states is deduced.

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1 Introduction

In the past few years, a lot of work has been devoted to study of noncommutative gauge theories. The main motivation for these studies is related to the realization of such theories in the framework of string theory. It turns out that noncommutative supersymmetric gauge theories appear as the low energy effective theory on a D-brane in the presence of a nonzero NS-NS two form background field[1, 2, 3]. Apart from the string theory realization, the analysis of quantum mechanical features of noncommutative gauge theories is important from field theoretical point of view.

As it is clear, the BRST symmetry which makes explicit the fact that the quantization is independent of a choice of a particular gauge is a method to quantization of gauge theories. This procedure includes two stages. The first stage is the introduction of ghosts, as in the standard Faddeev-Popov method. The theory included ghosts is then quantized in the usual way by ignoring the gauge symmetry. Since ghosts violate spin-statistics, the space of physical states of the quantum theory has pseudo-Hermitian rather than Hermitian inner product. The second stage, however, restores positivity of the inner product defined in the space of states and also brings back the gauge symmetry. In this process, a continuous global symmetry transformation (the BRST transformation) is defined on the algebra of local operators. The nilpotency of the charge associated to this global symmetry is then proved. Ultimately, the Hilbert space structure on the quantum states is determined by the cohomology space of the BRST operator (the charge associated to the BRST symmetry). Consequently, the induced inner product on the cohomology space will be positive.

In this paper, the BRST quantization procedure is followed for the noncommutative $U(N)$ gauge theory. In Sect. 2, after gauge fixing and introducing ghost fields, the full action of the theory is considered. The BRST symmetry transformation is then found in Sect. 3 in a way that the full action of the theory remains invariant under it. In Sect. 4, however, it is shown that the charge associated to the BRST symmetry is nilpotent.

Sect. 5 includes several parts. First in Sect. 5.1, we discuss the BRST symmetry leads to a conserved charge which commutes with the Hamiltonian of the theory only for the space-like non-commutativity although it is found for a general non-commutativity parameter. In the next part, Sect. 5.2, we prove that the BRST symmetry is preserved at quantum level. More precisely, we have to show that the path integral measure of the partition function of the theory
remains invariant under the BRST transformation. Arguments presented in the last part of this section, Sect. 5.3, are restricted to the case of the space-like non-commutativity. We will show the space of physical states is determined by the cohomology space of the BRST operator just as the commutative case. Finally, the unitarity of the S-matrix elements projected onto the subspace of physical states is proved. Sect. 6 is devoted to a brief discussion.

2 Full Action of the Noncommutative Gauge Theory

In this section, we fix our notation. Meanwhile, we introduce the action of the noncommutative $U(N)$ gauge theory. The classical action of the noncommutative pure Yang-Mills theory is given by:

$$S_g[A] = -\frac{1}{4} \int \text{Tr}(F_{\mu\nu} \star F^{\mu\nu}) ,$$

where $\star$ denotes the Moyal star product and curvature $F_{\mu\nu}^a$ is defined as follows:

$$F_{\mu\nu}^a = \partial_{[\mu} A_{\nu]}^a - ig [A_{\mu}^b t^b, A_{\nu}^c t^c]^a_\star .$$

In order to separate the effect of non-commutativity, we introduce some notations. Defining $h_{abc} t^a \equiv t^b t^c$, we can express $F_{\mu\nu}^a$ as:

$$F_{\mu\nu}^a = \partial_{[\mu} A_{\nu]}^a - ig h_{abc} A_{\mu}^b \star A_{\nu}^c + ig h_{acb} A_{\nu}^c \star A_{\mu}^b .$$

The structure constants $f_{abc}$ and totally symmetric $d_{abc}$ factors of $U(N)$ are introduced by:

$$[t^a, t^b] = i f_{abc} t^c , \quad \{t^a, t^b\} = d_{abc} t^c .$$

Expressing $h_{abc}$ with respect to $f_{abc}$ and $d_{abc}$ factors:

$$h_{abc} = \frac{i}{2} f_{abc} + \frac{1}{2} d_{abc} ,$$

we immediately conclude:

$$F_{\mu\nu}^a = \partial_{[\mu} A_{\nu]}^a + \frac{1}{2} g f_{abc} [A_{\mu}^b, A_{\nu}^c]_\star - \frac{i}{2} g d_{abc} [A_{\mu}^b, A_{\nu}^c]_\star .$$

$^b$ It should be noticed that constructing the noncommutative gauge theory with other famous matrix Lie groups $SU(N)$, $SO(N)$ or $Sp(N)$ is not possible, since Lie algebras of these groups are not closed under the Moyal bracket$[4, 5]$.

$^c$ We suppose that the Killing form has been defined, whether the indices are up or down is irrelevant in the orthonormal basis.
Now it is clear if we put $\theta = 0$, the last term will vanish and the commutative expression will be obtained for $F^\mu_{\nu\rho}$.

The matter field part includes the connection, defined by:

$$D = \partial - igA^\star ,$$  \hspace{1cm} (2.7)

where we considered the fundamental representation of the matter field with respect to the star product. The matter part of the action is then given by:

$$S_{\text{mat}}[A, \psi, \bar{\psi}] = \int \bar{\psi} (iD - m) \psi .$$  \hspace{1cm} (2.8)

The two parts $S_g$ and $S_{\text{mat}}$ of the action are separately invariant under the following gauge transformations of matter and gauge fields:

$$\psi \to U \star \psi ,$$  \hspace{1cm} (2.9)

$$A \to U \star A \star U^{-1} + \frac{i}{g} U \star \partial U^{-1} ,$$  \hspace{1cm} (2.10)

where $U \in U_\ast(N)$. However, the infinitesimal gauge transformations of the matter and gauge fields take the forms:

$$\delta \psi = ig\omega^a t^a \star \psi ,$$  \hspace{1cm} (2.11)

$$\delta A^a = (D^{\text{adj}}\omega)^a - i gh^{abd}[A^d, \omega^b] \star ,$$  \hspace{1cm} (2.12)

where $\omega^a$'s are infinitesimal local parameters of the gauge transformation and $D^{\text{adj}}$ is the connection associated to the commutative $U(N)$ gauge group in the adjoint representation and is introduced by:

$$(D^{\text{adj}})^{ac} = \delta^{ac} \partial + g f^{abc} A^b \star .$$  \hspace{1cm} (2.13)

Obviously for the case $\theta = 0$, the second term in the r.h.s. of Eq. (2.12) vanishes and $\delta A^a$ becomes the expression of the commutative case.

In order to quantize the theory with the gauge symmetry, we have to follow the standard Faddeev-Popov procedure. More precisely, to compute the partition function of the theory, we should integrate over the quotient space $\mathcal{F}/\mathcal{G}$, where $\mathcal{F}$ and $\mathcal{G}$ are the space of field configurations and the group associated to the gauge symmetry respectively. Imposing the covariant
gauge condition on the decomposition of unity, we find the gauge fixing and ghost parts of the action as:

\[
S_{gf}[A] = -\frac{1}{2}\xi^{-1}\int (\partial \cdot A^a)^2 ,
\]

\[
S_{gh}[c, \bar{c}, A] = -\int \{ \bar{c}^a \ast (\partial \cdot D_{adj} c)^a - igh^{ab} \bar{c}^a \ast \partial \mu [A^d_{\mu}, c^b] \ast \},
\]

where \( c \) and \( \bar{c} \) denote the ghost and the antighost fields. Ultimately, the full action of the noncommutative gauge theory is found as follows:

\[
S[A, \psi, \bar{\psi}, c, \bar{c}] = S_g[A] + S_{mat}[A, \psi, \bar{\psi}] + S_{gh}[A, c, \bar{c}] + S_{gf}[A] .
\]

\(3\) \textbf{BRST Symmetry}

As in the commutative case, any gauge fixing procedure destroys gauge invariance of the noncommutative gauge theory. Therefore, in order to obtain a sensible quantization, we must make sure that in the final result the gauge symmetry is restored. The BRST quantization method provides a powerful tool to achieve this aim. The essential element for applying this method is first introducing a global symmetry transformation whose associated charge is nilpotent. In this section, we present the BRST symmetry transformation for the noncommutative \( U(N) \) gauge theory.

The BRST symmetry is a global transformation (the parameter of the transformation is global.) which leaves the full action of the theory invariant. Since the pure gauge and matter parts of the action are invariant under gauge transformations Eq. (2.11) and Eq. (2.12), the most natural BRST transformations which can be considered for the gauge and matter fields are the ordinary gauge transformations. Therefore, we introduce the BRST transformations for the gauge field \( A^a \) and the matter field \( \psi \) as:

\[
\delta A^a = (D_{adj} c)^a \delta \lambda - i gh^{acb} [A^b, c^c], \delta \lambda ,
\]

\[
\delta \psi = -ig (c^a \ast \psi) \delta \lambda ,
\]

where \( \delta \lambda \) (the parameter of the transformation) is a global Grassmann variable. It should be noticed that \( \delta \lambda \) is not necessarily infinitesimal. But since \( (\delta \lambda)^2 = 0 \), Eq. (3.1) and Eq. (3.2) introduce infinitesimal gauge transformations. Hence, we have:

\[
\delta (S_g[A] + S_{mat}[A, \psi, \bar{\psi}]) = 0 .
\]
Now we are going to find the transformations of the ghost and antighost fields in a way that the action is invariant under this transformation. Therefore, we conclude:

\[ \delta S = \delta \left( S_{gh} + S_{gf} \right) = 0 . \]  

(3.4)

Using Eq. (3.1), the variation of the action is then given by:

\[ \delta S = \int \left\{ -\left( \delta \bar{c}^a - \xi^{-1}(\partial \cdot A^a)\delta \lambda \right) \star \left( \partial \cdot (D^{adj}c)^a - i gh^{abd}[A^d, c^b]_\star \right) \right. \]
\[ \left. - \bar{c}^a \star \partial \cdot \left[ \delta \left( (D^{adj}c)^a - i gh^{abd}[A^d, c^b]_\star \right) \right] \right\} , \]  

(3.5)

which will vanish, if we require that:

\[ \delta \bar{c}^a = \xi^{-1}(\partial \cdot A^a)\delta \lambda , \]  

(3.6)

\[ \delta \left( (D^{adj}c)^a - i gh^{abd}[A^d, c^b]_\star \right) = 0 . \]  

(3.7)

The first one introduces the BRST transformation of the antighost field and the other one will then introduce the BRST transformation of the ghost field. Considering the first term of Eq. (3.7), we obtain:

\[ \delta (D^{adj}c)^a = \partial (\delta c^a) + g f^{abc} \delta A^b \star c^c + g f^{abc} A^b \star \delta c^c \]
\[ = \partial (\delta c^a) - g f^{abc} \left( \partial c^b + g f^{bde} A^d \star c^e \right) \star c^c \delta \lambda \]
\[ + i g^2 f^{abc} h^{bed}[A^d, c^e]_\star \star c^e \delta \lambda + g f^{abc} A^b \star \delta c^c , \]  

(3.8)

where in the last line, we used Eq. (3.1). For the second term of Eq. (3.7), however, we have:

\[ \delta [A^d, c^b]_\star = [\delta A^d, c^b]_\star + [A^d, \delta c^b]_\star , \]  

(3.9)

again using Eq. (3.1), the first term in the r.h.s. of the above equality itself reads:

\[ [\delta A^d, c^b]_\star = -\left\{ (D^{adj}c)^d, c^b \right\}_\star \delta \lambda + i gh^{def} \left\{ [A^e, c^f]_\star, c^b \right\}_\star \delta \lambda . \]  

(3.10)

Now substitute Eq. (3.8), Eq. (3.9), and Eq. (3.10) for Eq. (3.7). Then as the first step in Eq. (3.7), considering all terms including space-time derivatives, we find:

\[ \text{terms including derivative} = \partial (\delta c^a) - g f^{abc} \partial c^b \star c^c \delta \lambda + i gh^{abd} \{ \partial c^d, c^b \}_\star \delta \lambda \]
\[ = \partial \left( \delta c^a - \frac{1}{2} g f^{abc} c^b \star c^d \delta \lambda + \frac{i}{4} g d^{abd} \{ c^d, c^b \}_\star \delta \lambda \right) . \]  

(3.11)
To obtain the last expression, we used Eq. (2.5). Notice that to derive the above expression, there is no necessity to exchange the position of fields. We only have exchanged the position of the gauge group indices of fields. The remained terms not including space-time derivatives of Eq. (3.7) are:

\[-g^2 f^{abc} f^{bde} \varepsilon^d \varepsilon^a \varepsilon^e \delta \lambda + ig^2 f^{abc} h^{bed} [A^d, \varepsilon^e] \varepsilon^d \varepsilon^c \delta \lambda + g f^{abc} A^b \varepsilon^d \varepsilon^c \delta \lambda \]

\[= \frac{i}{2} f^{abc} h^{bed} [A^d, \varepsilon^e] \varepsilon^d \varepsilon^c \delta \lambda . \quad (3.12)\]

Now the claim is that Eq. (3.7) can ultimately be rewritten in the form of Eq. (3.15). To arrive to this equation, we present the way that leads only to one of its term. One can find more details in appendix A. For instance, consider the second term of the first line of the above expression. Using Eq. (2.5), we can write it as follows:

\[ig^2 f^{abc} h^{bed} [A^d, \varepsilon^e] \varepsilon^d \varepsilon^c \delta \lambda = \frac{-1}{2} f^{abc} f^{bed} \varepsilon^d \varepsilon^a \varepsilon^c \delta \lambda + \frac{1}{2} g^2 f^{abc} f^{bed} \varepsilon^d \varepsilon^c A^d \varepsilon^c \delta \lambda \]

\[+ \frac{i}{2} f^{abc} d^{bed} [A^d, \varepsilon^e] \varepsilon^d \varepsilon^c \delta \lambda . \quad (3.13)\]

Adding the first term of the r.h.s. of the above equality to the first term of Eq. (3.12), it immediately results:

\[\frac{-1}{2} g^2 f^{abc} f^{bed} \varepsilon^d \varepsilon^a \varepsilon^c \delta \lambda = \frac{-1}{2} g^2 (f^{abd} f^{bec} + f^{abc} f^{bed}) A^d \varepsilon^c \varepsilon^c \delta \lambda , \quad (3.14)\]

where we used the Jacobi identity in the r.h.s. of the above expression. The first term in the r.h.s. of the above equality is exactly the second term appeared in the second line of Eq. (3.15). Therefore, Eq. (3.7) yields:

\[\partial (\delta c^a - \frac{1}{2} g f^{abcd} c^b \varepsilon^c \varepsilon^d \delta \lambda + \frac{i}{4} g d^{abc} [c^b, c^d] \varepsilon^c \delta \lambda ) \]

\[+ g f^{abc} A^b \varepsilon^d \varepsilon^d \varepsilon^c \delta \lambda + \frac{i}{4} g d^{cde} [c^d, c^e] \varepsilon^c \delta \lambda ) \]

\[- i g h^{abd} [A^d, \delta c^b - \frac{1}{2} g f^{bea} A^b \varepsilon^c \varepsilon^d \varepsilon^c \delta \lambda + \frac{i}{4} g d^{bde} [c^e, c^f] \varepsilon^c \delta \lambda ] \varepsilon = 0 . \quad (3.15)\]

In order to obtain vanishing result for the l.h.s. of the above equality, it is sufficient to require that:

\[\delta c^a = \frac{1}{2} g f^{abcd} c^b \varepsilon^c \varepsilon^d \delta \lambda - \frac{i}{4} g d^{abc} [c^d, c^b] \varepsilon \delta \lambda , \quad (3.16)\]

which introduces the BRST transformation of the ghost field. Therefore, we have proved that the full action of the noncommutative $U(N)$ gauge theory remains invariant under the following
transformation:
\[
\begin{align*}
\delta A^a &= (D^\text{adj} c)^a \delta \lambda - i g h^{acb}[A^b, c^c], \\
\delta \psi &= -i g (c^a \eta^a \psi) \delta \lambda, \\
\delta c^a &= \xi^{-1} (\partial \cdot A^a) \delta \lambda, \\
\delta \bar{c}^a &= \frac{1}{2} g f^{a b c} c^b c^d \delta \lambda - \frac{i}{4} g d^{a b c} c^b c^d, \delta \lambda.
\end{align*}
\]

To compare the above result with the commutative case, the above symmetry transformation takes its commutative form if we put \(\theta = 0\). Further considering the \(U(1)\) gauge group, we can study the situation of the noncommutative QED. It is easy to see that the above transformation leads to the BRST transformation which was first found in [6].

## 4 Nilpotent Charge of the BRST Symmetry

In this section, we will show that the charge of the BRST symmetry is nilpotent. The charge \(Q\) associated to the BRST symmetry satisfies the following relation:
\[
\delta \phi = i [\delta \lambda Q, \phi],
\]
where \(\delta \phi\) is introduced by the transformations Eq. (3.17). Notice that the commutator appeared in Eq. (4.1) is not the Moyal bracket. For simplicity, we define the action of charge \(Q\) on each field \(\phi\) as:
\[
\delta \phi = \delta \lambda (Q \phi) = i [\delta \lambda Q, \phi],
\]
where \(Q = i [Q, \ ]\) satisfies the super Leibnitz rule. Considering the transformations Eq. (3.17), we find the action of charge \(Q\) on each field of the theory as follows:
\[
\begin{align*}
Q A^a &= -(D^\text{adj} c)^a + i g h^{a b c}[A^b, c^c], \\
Q \psi &= -i g (c^a \eta^a \psi), \\
Q c^a &= \xi^{-1} (\partial \cdot A^a), \\
Q \bar{c}^a &= \frac{1}{2} g f^{a b c} c^b c^d - \frac{i}{4} g d^{a b c} c^b c^d.
\end{align*}
\]

Now we want to find the action of \(Q^2\) on each field of the theory. The reason is that one can easily show:
\[
Q^2 \phi = [Q^2, \phi],
\]
which indicates the relation between $Q^2$ and $Q^2$. First, consider the antighost field:

$$Q^2 c^a = Q(Qc^a) = \xi^{-1} \partial \cdot (QA^a) = -\xi^{-1} \partial \cdot \left( (D^{\text{adj}}_c)^a - ig h^{ab}[A^d, c^b] \right). \quad (4.8)$$

Using the equation of motion governed on the antighost field for the above expression, we immediately conclude:

$$Q^2 c^a = 0 . \quad (4.9)$$

Considering Eq. (4.4) and Eq. (4.6), the action of $Q^2$ on the matter field yields:

$$Q^2 \psi = Q(Q\psi) = -ig(Qc^a)t^a \star \psi + igc^a t^a \star (Q\psi) = -\frac{1}{2} g^2 f^{ab} b^c \star c^d t^a \star \psi - \frac{1}{4} g^2 d^{ab} \{c^b, c^d\} \star t^a \psi + g^2 c^a \star c^b (t^a t^b) \star \psi , \quad (4.10)$$

where $c^a \star c^b (t^a t^b)$ can be written in the form:

$$c^a \star c^b (t^a t^b) = h^{dab} c^a \star c^b t^d = \left( \frac{i}{2} f^{abcd} c^a \star c^d + \frac{1}{4} d^{abcd} \{c^b, c^d\} \right) t^a , \quad (4.11)$$

which leads to the vanishing result for $Q^2 \psi = 0$. For the gauge field, however; we have:

$$Q^2 A^a = Q(QA^a) = -Q \left( (D^{\text{adj}}_c)^a - ig t^{abc} [A^b, c^c] \right) \quad (4.12)$$

substituting the following equation:

$$Q[A^b, c^c] = \{Q[A^b, c^c], + [A^b, Qc^c] \} , \quad (4.13)$$

for Eq. (4.12), we arrive at:

$$Q^2 A^a = -\left[ \partial(Qc^a) + \frac{1}{2} g f^{abc} [QA^b, c^c] + \frac{1}{2} g f^{abc} [A^b, Qc^c] \right] - \frac{i}{2} g d^{abc} \{QA^b, Qc^c\} - \frac{i}{2} g d^{abc} [A^b, Qc^c] . \quad (4.14)$$
Applying Eq. (4.3) and Eq. (4.6) for the above relation, we have:

\[
Q^2 A^a = -\partial \left( \frac{1}{2} g f^{abc} c^b \ast c^c - \frac{i}{4} g d^{abc} \{ c^b, c^c \} \ast \right) \\
+ \frac{1}{2} g f^{abc} [ (D^{adj})^b - i g h^{bed} [ A^d, c^e ] \ast, c^c ] \\
- \frac{1}{2} g f^{abc} \{ A^b, \frac{1}{2} g f^{cde} c^d \ast c^e - \frac{i}{4} g d^{cde} \{ c^d, c^e \} \ast \} \\
+ \frac{i}{2} g d^{abc} [ A^b, \frac{1}{2} g f^{cde} c^d \ast c^e - \frac{i}{4} g d^{cde} \{ c^d, c^e \} \ast ] \\
- \frac{i}{2} g d^{abc} \{ (D^{adj})^b - i g h^{bed} [ A^d, c^e ] \ast, c^c \} \ast. 
\]

(4.15)

Now consider the terms include space-time derivative:

\[
\text{terms including derivative} = \frac{1}{2} g f^{abc} \partial (c^b \ast c^c) - \frac{i}{4} g d^{abc} \partial \{ c^b, c^d \} \ast, \\
- \frac{1}{2} g f^{abc} \{ \partial c^b, c^c \} \ast + \frac{i}{2} g d^{abc} \{ \partial c^b, c^c \} \ast = 0. 
\]

(4.16)

For the rest terms, use the following identity:

\[
f^{bde} A^d \ast c^e - i h^{bed} A^d \ast c^e = -i h^{bde} A^d \ast c^e + i h^{bed} c^e \ast A^d, 
\]

we will obtain:

\[
Q^2 A^a = \frac{1}{2} g^2 (h^{abc} h^{cde} - h^{ace} h^{cbd}) A^b \ast c^d \ast c^e \\
+ \frac{1}{2} g^2 (h^{ade} h^{ceb} - h^{acb} h^{cde}) c^d \ast c^e \ast A^b \\
+ \frac{1}{2} g^2 (h^{abe} h^{bcd} - h^{acb} h^{bde}) c^c \ast A^d \ast c^e. 
\]

(4.18)

Using the following Jacobi identity (see appendix B.) between \( h^{abc} \)’s constants:

\[
h^{abc} h^{bcd} - h^{acb} h^{bde} = 0, 
\]

(4.19)

we easily find \( Q^2 A^a = 0 \). As the last step, consider the action of \( Q^2 \) on the ghost field. To accomplish calculations in an easier way, notice that we can rewrite Eq. (4.6) as follows:

\[
Q c^a = -i g h^{abd} c^b \ast c^d. 
\]

(4.20)

Now using Eq. (4.6), we gain the action of \( Q^2 \) on the ghost field as:

\[
Q^2 c^a = Q (Q c^a) \\
= -i g h^{abd} \left( (Q c^b) \ast c^d - c^b \ast (Q c^d) \right) \\
= -i g h^{abd} \left( -i g h^{bef} c^e \ast c^f \ast c^d + i g h^{def} c^b \ast c^e \ast c^f \right) \\
= -g^2 (h^{abe} h^{bde} - h^{acb} h^{bde}) c^e \ast c^f \ast c^d, 
\]

(4.21)
which will vanish if we use the Jacobi identity Eq. (4.19). In this manner, we found that:

\[ Q^2 \phi = 0 , \quad \phi \in \{ A^a, \psi, \epsilon^a, \epsilon^a \} . \quad (4.22) \]

Remembering Eq. (4.7), the above result implies that:

\[ [Q^2, \phi] = 0 . \quad (4.23) \]

For this to be satisfied for all operators \( \phi \), it is necessary for \( Q^2 \) either to vanish or be proportional to the unit operator. But \( Q^2 \) cannot be proportional to the unit operator because it has a non-vanishing ghost quantum number, so it must vanish:

\[ Q^2 = 0 . \quad (4.24) \]

## 5 The Space of Physical States and Unitarity

In keeping with the BRST quantization method, we found a global symmetry transformation (the BRST transformation) for noncommutative gauge theory such that its associated charge is nilpotent. In this section, however, we will establish the Hilbert space of physical states, following the rest of our discussion in the Hamiltonian formalism. First, in Sect. 5.1, we discuss that the BRST symmetry leads to a conserved charge only for the space-like non-commutativity parameter. In the next part, Sect. 5.2, we make sure that the BRST symmetry preserves at quantum level. Finally in Sect. 5.3, it turns out the subspace of physical states is the cohomology space of the BRST operator.

### 5.1 Conserved Charge and Space-like Non-commutativity

Although the BRST symmetry was proved for a general non-commutativity parameter, we discuss this symmetry leads to a conserved charge only for the space-like non-commutativity parameter \( (\theta^{0i} = 0) \). The rest of our arguments is therefore restricted to this case.

The common way to obtain the charge of a continuous global symmetry of the theory is replacing the local function for the global parameter of the symmetry transformation. In this manner, there is no necessity for the new transformation to be the symmetry of the theory, too. In fact, the variation of the action, under this new transformation, introduces the current associated to the global symmetry of the theory. Although this variation vanishes for all local
parameters of transformation, we cannot conclude that the divergence of the current has to vanish, in contrast to the commutative case. Since the Moyal star product can be removed in any quadratic term of the action, the most general result which can be deduced is that the divergence of the current is equal to the Moyal bracket of some functions\cite{8}. The mentioned Moyal bracket disappears only for the space-like non-commutativity when we integrate on the continuity equation over all spatial coordinates in order to obtain the time variation of the charge. Consequently, the charge associated to a symmetry transformation commutes with the Hamiltonian of the theory in this case. Hence, the rest of our arguments are restricted to the space-like non-commutativity.

5.2 BRST Symmetry at Quantum Level

So far, we have shown that the full action of the noncommutative gauge theory, Eq. (2.16), is invariant under the BRST transformation, Eq. (3.17). But in this stage, we cannot still claim that this symmetry is preserved at quantum level. On other words, we must show that the path integral measure of the partition function of the theory remains invariant under the BRST transformation. The sourceless partition function of the noncommutative gauge theory is given by:

$$Z = \int \mathcal{D}A \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}c \mathcal{D}\bar{c} e^{iS[A,\psi,\bar{\psi},c,\bar{c}]} .$$

(5.1)

One can immediately convince himself that the above path integral measure is invariant under the BRST transformation. For instance, the Jacobian describes the change in $\mathcal{D}c$ is the determinant of:

$$\frac{\delta}{\delta c^b(y)}(c^a(x) + \delta c^a(x)) = \left(\delta^{ab}\delta_y - ig(\delta \lambda)h^{abd}\delta_y \star \epsilon^d - ig(\delta \lambda)h^{adb}\epsilon^d \star \delta_y \right)(x) ,$$

(5.2)

where we defined $\delta_y(x) \equiv \delta(x-y)$. Obviously, this determinant is equal to one since $(\delta \lambda)^2 = 0$. Therefore, the partition function of the noncommutative gauge theory is invariant under the BRST transformation.

5.3 Hilbert Space of Physical States

We are already able to establish the subspace of physical states. Since ghosts violate spin-statistics (being scalar fermions), the space of states of the theory included ghosts cannot be an
actual space. Further, the state space has a pseudo-Hermitian rather Hermitian inner product. But it is possible to construct a certain subspace $\mathcal{H}$ which does not include ghosts and has a Hermitian norm, analogous to the Hilbert space in actual physical theories.

In order to obtain BRST invariant S-matrix elements, it is necessary for physical states to belong in the kernel of $Q$ (i.e. $Q$–closed):

$$Q|\psi\rangle = 0,$$  \hspace{1cm} (5.3)

just as the commutative case. Since charge $Q$ is nilpotent, each two physical states that differ only by a state in the image of $Q$ (i.e. $Q$–exact), have the same matrix elements with all other physical states, and are therefore physically equivalent. Hence, we define an equivalence relation to put equivalent states in the same class. Two states $|\psi\rangle$ and $|\psi'\rangle$ are equivalent: $|\psi'\rangle \sim |\psi\rangle$ if there is some state $|\chi\rangle$ such that:

$$|\psi'\rangle = |\psi\rangle + Q|\chi\rangle .$$  \hspace{1cm} (5.4)

The set of equivalent classes is nothing but the cohomology of $Q$. Thus, the space of physical states is isomorphic to the quotient space:

$$\mathcal{H} \sim \text{Ker}Q / \text{Im}Q .$$  \hspace{1cm} (5.5)

As the final discussion, we will prove the unitarity of the S-matrix elements in the subspace of physical states. Consider a typical S-matrix element:

$$\text{out} \langle \alpha | \beta \rangle_{\text{in}} = \langle \alpha | S^\dagger S | \beta \rangle ,$$  \hspace{1cm} (5.6)

where $\text{out} \langle \alpha |$ and $| \beta \rangle_{\text{in}}$ are asymptotic states, whereas $| \alpha \rangle, | \beta \rangle$ are two physical states of the theory. Since $Q$ commutes with the Hamiltonian of the theory, the time evolution of any such states must also be annihilated by $Q$:

$$Q S |\beta\rangle = 0 .$$  \hspace{1cm} (5.7)

The above expression states that $S |\beta\rangle$ is a linear combination of states in $\text{Ker}Q$. On other words, we have:

$$S |\beta\rangle = \sum_{|\gamma\rangle \in \mathcal{P}_1} |\gamma\rangle \langle \gamma | S |\beta\rangle ,$$  \hspace{1cm} (5.8)

where $\mathcal{P}_1 = \{ \gamma : \text{Ker}Q = \text{span}\{ |\gamma\rangle\} \}$.
where \( P_1 \) is a basis for subspace \( \ker Q \). Again for the states, in the kernel of \( Q \), can be written in the form \(|\gamma\rangle + Q|\chi\rangle\), we find:
\[
(\langle \gamma \rangle + \langle \chi \rangle Q) S|\beta\rangle = \langle \gamma \rangle S|\beta\rangle + \langle \chi \rangle QS|\beta\rangle,
\]
but the second term of the r.h.s. of the above equality vanishes since \([Q, H] = 0\). This result implies that:
\[
S|\beta\rangle = \sum_{|\gamma\rangle \in P_2} |\gamma\rangle \langle \gamma | S|\beta\rangle, \quad P_2 = \{|\gamma\rangle : \mathcal{H} = \text{span}\{|\gamma\rangle\}\},
\]
where \( P_2 \) is a basis for the Hilbert space of physical states. This relation guarantees the unitarity of S-matrix elements for the subspace of physical states.

### 6 Conclusion

In this paper, the BRST quantization method is followed for noncommutative gauge theories. Of course, our discussions are presented in the framework of the Moyal noncommutative gauge theory. Nevertheless, all obtained results are valid for any noncommutative gauge theory whose \( C^* \) algebra of functions on \( \mathcal{R}^d \) is associative with respect to its star product and also the trace defined on this algebra satisfies the cyclicity property.

After introducing the full action of the Moyal noncommutative \( U(N) \) gauge theory in Sect. 2, however, the BRST symmetry transformation is presented for this theory in Sect. 3. In analogy to the commutative case, the BRST transformations for the gauge and matter fields are nothing but infinitesimal gauge transformations. Moreover, as expected, this transformation will take its commutative form if we put \( \theta = 0 \).

In Sect. 4, the nilpotency of the charge associated to the BRST symmetry is then proved. Each nilpotent operator which commutes with the Hamiltonian of the theory has many useful advantages. But in the first part of Sect. 5, it is proved the charge associated to a continuous global symmetry of the theory is not conserved in general, in contrast to the commutative case. Due to the cyclicity property of the trace with respect to the star product, the associated charge is conserved only for the space-like non-commutativity parameter. In this case, the expected commutation relation between the charge and the Hamiltonian of the theory is restored.

In the next part of this section, Sect. 5.2, the BRST symmetry transformation is considered at quantum level. To preserve this transformation as a symmetry at quantum level, it
is necessary for the path integral measure of the partition function to remain invariant under the BRST transformation. This is easily shown since the global parameter of the BRST transformation is a Grassmann variable.

Since the ghost and antighost fields violate the spin-statistics theorem, the space of states included ghosts is not a Hilbert space. Hence, in the last part of Sect. 5, the BRST cohomology is taken to produce the physical Hilbert space just as the commutative case. Ultimately, it is proved that the S-matrix elements projected onto the physical space of states are unitary.

Since the BRST symmetry is a continuous transformation, it generates a set of Ward identities for noncommutative non-Abelian gauge theories. Therefore, the BRST symmetry will provide a powerful tool to study the renormalization properties of noncommutative gauge theories in all orders of the perturbative expansion.

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A More Details to Obtain Eq. (3.15)

In this section, we present more details for calculations that lead to Eq. (3.15). First, let us extract term \( \frac{i}{2} g^2 f^{abc} d^{cde} \{ c^d, c^e \}_* \delta \lambda \) which is appeared in the second line of Eq. (3.15). To do this, consider the last term in the r.h.s. of Eq. (3.13) which is:

\[
\frac{i}{2} g^2 f^{abc} d^{bed} [A^d, c^e]_* c^c \delta \lambda = \frac{i}{2} g^2 f^{abc} d^{bed} (A^d \star c^e \star c^c - c^e \star A^d \star c^c) \delta \lambda . \tag{A.1}
\]

We use the following Jacobi identity (see appendix B.) for the r.h.s. of the above relation:

\[
f^{abc} d^{bed} = f^{aeb} d^{bcd} - f^{abd} d^{ceb}, \tag{A.2}
\]

the fist term in the r.h.s. of Eq. (A.1) itself becomes two terms that one of them is just we need. Up to now, the remained terms only for the first line of Eq. (3.12) are:

\[
\frac{1}{2} g^2 f^{abc} f^{bed} c^e \star A^d \star c^c \delta \lambda - \frac{1}{2} g^2 f^{abc} f^{bed} c^c \star c^e \delta \lambda \\
- \frac{i}{2} g^2 f^{abc} d^{bed} c^e \star A^d \star c^c \delta \lambda + \frac{i}{2} g^2 f^{abc} d^{bed} A^d \star c^e \star c^c \delta \lambda , \tag{A.3}
\]

which can be written in the form:

\[
-ig^2 f^{abc} h^{bed} \{ A^d \star c^e, c^e \}_* \delta \lambda . \tag{A.4}
\]

We release this remained term from the first line of Eq. (3.12) for a moment, and pay attention to the second line of Eq. (3.12). We want to form terms are appeared in the last line of Eq. (3.15). Considering the second term of the second line of Eq. (3.12):

\[
g^2 h^{abl} h^{def} \{ A^e, c^f \}_* c^b \delta \lambda = g^2 h^{abl} h^{def} \left( \left\{ A^e \star c^f, c^b \right\}_* - \left\{ c^f \star A^e, c^b \right\}_* \right) \delta \lambda , \tag{A.5}
\]

where we rewrite its first term in the r.h.s. as follows:

\[
g^2 h^{abl} h^{def} \left\{ A^e \star c^f, c^b \right\}_* = g^2 h^{abl} \left( \frac{i}{2} h^{df e} + \frac{1}{2} d^{df e} \right) \left\{ A^e \star c^f, c^b \right\}_*. \tag{A.6}
\]

Adding the first term of the r.h.s. of the above equality to the first term of the second line of Eq. (3.12), we obtain:

\[
\frac{i}{2} g^2 h^{abl} f^{def} \left\{ A^e \star c^f, c^b \right\}_* \delta \lambda . \tag{A.7}
\]

Rewrite the second term of Eq. (A.5) as Eq. (A.6) and then add its \( h^{abl} f^{def} \) term to the above result (Eq. (A.7)), we gain:

\[
\frac{i}{2} g^2 h^{abl} f^{def} \left\{ A^e, c^f \right\}_*, c^b \right\}_* \delta \lambda . \tag{A.8}
\]
Using the following Jacobi identity between $h^{abc}$ and $f^{abc}$ constants (see appendix B.):

$$h^{abd}f^{dce} + h^{dbe}f^{cad} - h^{dea}f^{bcd} = 0, \quad (A.9)$$

for Eq. (A.8), we obtain:

$$\frac{i}{2} g^2 h^{abc} f^{def} \{ \{ A^e, c^f \}^*, c^b \}^* \delta \lambda = \frac{i}{2} g^2 (-h^{ade} f^{dbf} + f^{adf} h^{dfe}) \{ \{ A^e, c^f \}^*, c^b \}^* \delta \lambda, \quad (A.10)$$

where its first term, $-i gh^{ade} [A^e, -\frac{1}{2} g f^{dfe} c^f \star c^b \delta \lambda]_*$, is exactly one of the terms has been appeared in the third line of Eq. (3.15). To find the last term in the third line of Eq. (3.15), we add the second term in the r.h.s. of Eq. (A.6) to the similar term ($f^{abd}d^{dfe}$ term) derived from the second term of the r.h.s. of Eq. (A.5):

$$\frac{1}{2} g^2 h^{abd} d^{dfe} \left( \{ A^e \star c^f, c^b \}_* - \{ c^f \star A^e, c^b \}_* \right) \delta \lambda. \quad (A.11)$$

and use the following Jacobi identity (see appendix B.):

$$h^{abc} d^{dfe} - h^{ade} d^{dbf} - i f^{ad} h^{dfe} = 0, \quad (A.12)$$

we immediately find Eq. (A.11) as follows:

$$\frac{1}{2} g^2 (h^{ade} d^{dbf} - i f^{ad} h^{dfe}) \{ [A^e, c^f]_*, c^b \}^* \delta \lambda. \quad (A.13)$$

The first term of the above expression:

$$-i gh^{ade} [A^e, \frac{i}{4} g d^{dbf} \{ c^b, c^f \}_*], \quad (A.14)$$

is just the needed term. As the final task, we must show that the all rest terms cancel themselves. All remained terms from the second line of Eq. (3.12) are the second term in the r.h.s. of Eq. (A.10) and the second term of Eq. (A.13). The sum of these two terms is:

$$i g^2 f^{adf} h^{dfe} \{ A^e \star c^f, c^b \}_* \delta \lambda = i g^2 f^{abc} h^{bed} \{ A^d \star c^e, c^b \}_* \delta \lambda. \quad (A.15)$$

This result is just minus of what we obtained in Eq. (A.4). Therefore, the rest term of the first line of Eq. (3.12) cancels the rest term of the second line. Hence, the BRST transformation which introduced by Eq. (3.17) indeed remains the action of the theory (Eq. (2.11)) invariant and therefore is the symmetry of the theory.
B Identities Between the Structure Constants of \( U(N) \)

- Jacobi identities between \( f^{abc} \) and \( d^{abc} \) are:

\[
\begin{align*}
    f^{abd}f^{bce} + f^{cba}f^{bde} + f^{dcb}f^{bae} & = 0 , \\
d^{abd}d^{bce} - d^{cba}d^{bde} - f^{dcb}f^{bae} & = 0 , \\
f^{abd}d^{bce} + f^{cba}d^{bde} + f^{dcb}d^{bae} & = 0 , \\
d^{abd}d^{bce} + f^{cba}d^{bde} - f^{dcb}f^{bae} & = 0 ,
\end{align*}
\]

- Jacobi identities between \( h^{abc} \) and \( f^{abc} \) can easily be obtained by combining above relations:

\[
\begin{align*}
    h^{abd}f^{bce} + h^{cba}f^{bde} + h^{dcb}f^{bae} & = 0 , \\
h^{abd}h^{bce} - h^{cba}h^{bde} - h^{dcb}f^{bae} & = 0 , \\
f^{abd}h^{bce} + h^{cba}f^{bde} + h^{dcb}f^{bae} & = 0 , \\
h^{abd}h^{bce} - h^{cba}h^{bde} + f^{dcb}f^{bae} & = 0 .
\end{align*}
\]

- Jacobi identities between \( h^{abc} \) and \( d^{abc} \) can also be obtained in a similar way:

\[
\begin{align*}
    h^{abd}d^{bce} + i h^{cba}f^{bde} - h^{dcb}d^{bae} & = 0 , \\
h^{abd}d^{bce} - h^{cba}d^{bde} + i f^{dcb}h^{bae} & = 0 , \\
h^{abd}d^{bce} - h^{cba}d^{bde} - h^{dcb}d^{bae} & = 0 , \\
h^{abd}d^{bce} + i f^{cba}h^{bde} - h^{dcb}d^{bae} & = 0 .
\end{align*}
\]

- Applying the above various Jacobi identities, we can prove an identity only for \( h^{abc} \)'s:

\[
h^{abe}h^{bcd} - h^{acb}h^{bde} = 0 .
\]

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