The GPGCD Algorithm with the Bézout Matrix for Multiple Univariate Polynomials

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Abstract

We propose a modification of the GPGCD algorithm, which has been presented in our previous research, for calculating approximate greatest common divisor (GCD) of more than 2 univariate polynomials with real coefficients and a given degree. In transferring the approximate GCD problem to a constrained minimization problem, different from the original GPGCD algorithm for multiple polynomials which uses the Sylvester subresultant matrix, the proposed algorithm uses the Bézout matrix. Experiments show that the proposed algorithm is more efficient than the original GPGCD algorithm for multiple polynomials with maintaining almost the same accuracy for most of the cases.

1 Introduction

With the progress of algebraic computation with polynomials and matrices, we are paying more attention to approximate algebraic algorithms. Algorithms for calculating approximate GCD, which are approximate algebraic algorithms, consider a pair of given polynomials $F$ and $G$ that are relatively prime in general, and find $\tilde{F}$ and $\tilde{G}$ which are close to $F$ and $G$, respectively, in the sense of polynomial norm, and have the greatest common divisor of a certain degree. These algorithms can be classified into two categories: 1) for a given tolerance (magnitude) of $\|F - \tilde{F}\|$ and $\|G - \tilde{G}\|$, maximize the degree of approximate GCD, and 2) for a given degree $d$, minimize the magnitude of $\|F - \tilde{F}\|$ and $\|G - \tilde{G}\|$.

In both categories, algorithms based on various methods have been proposed including the Euclidean algorithm ([1], [19], [20]), low-rank approximation of the Sylvester matrix or subresultant matrices ([5], [6], [10], [11], [12], [21], [24], [26], [27], [29]), Padé approximation ([17]), and optimizations ([3], [13], [15], [22], [28]). Among them, the second author of the present paper has proposed the GPGCD algorithm based on low-rank approximation of subresultant matrices by optimization ([24], [25], [26]), which belongs to the second category above. Based on the researches mentioned above, the authors of the present paper have proposed the GPGCD algorithm using the Bézout matrix ([2]), which is the previous research of this paper.

In this paper, we propose the GPGCD algorithm with the Bézout matrix for multiple polynomials, while subresultant matrices have been used in the original GPGCD algorithm for multiple polynomials.
We show that the proposed algorithm is more efficient than the original one with maintaining almost the same accuracy for most of the cases.

The paper is organized as follows. In Section 2, we give a definition of the approximate GCD problem. In Section 3, we give a formulation of the transformation of the approximate GCD problem to the optimization problem using the Bézout matrix. In Section 4, we review the modified Newton method used for optimization. In Section 5, we illustrate the proposed algorithm and give a running time analysis. In Section 6, the results of experiments are shown.

## 2 Approximate GCD Problem

Let $F_1(x), \ldots, F_n(x)$ be univariate polynomials with real coefficients of degree at most $m$:

$$F_i(x) = f_{im}x^m + \cdots + f_{0i}x^0, \quad i = 1, \ldots, n, \quad f_{im} \neq 0. \quad (1)$$

In this paper, for a polynomial $F(x) = f_mx^m + \cdots + f_0x^0$, the norm $\|F\|$ denotes the 2-norm defined as $\|F\|_2 := (f_m^2 + f_{m-1}^2 + \cdots + f_0^2)^{1/2}$. For a vector $(a_1, \ldots, a_n) \in \mathbb{R}^n$, the norm $\|(a_1, \ldots, a_n)\|$ denotes the 2-norm defined as $\|(a_1, \ldots, a_n)\|_2 := (a_1^2 + \cdots + a_n^2)^{1/2}$.

Here, we give a definition of an approximate GCD.

### Definition 1 (Approximate GCD)

For polynomials $F_1(x), \ldots, F_n(x)$ which are relatively prime in general, a positive integer $d$ satisfying that $d < m$, and a positive real number $\varepsilon$, if there exist polynomials $\tilde{F}_1(x), \ldots, \tilde{F}_n(x)$ such that they have a certain GCD as

$$\tilde{F}_i(x) = \tilde{F}_i(x) \times \tilde{H}(x), \quad i = 1, \ldots, n, \quad (2)$$

where $\tilde{H}(x)$ is a polynomial of degree $d$, satisfying that $\|\|\tilde{F}_1 - F_1\|, \ldots, \|\tilde{F}_n - F_n\|\| < \varepsilon$, we call $\tilde{H}(x)$ an approximate GCD of polynomials $F_1(x), \ldots, F_n(x)$ with tolerance $\varepsilon$.

Algorithms for calculating approximate GCD can be classified into two categories: 1) for a given tolerance $\varepsilon$, make the degree of approximate GCD as large as possible, and 2) for a given degree $d$, minimize the magnitude $\|\|\tilde{F}_1 - F_1\|, \ldots, \|\tilde{F}_n - F_n\|\|$. In this paper, we focus on the second category of the approximate GCD algorithms, solving the following problem.

### Problem 2 (Approximate GCD problem)

For given univariate polynomials $F_1(x), \ldots, F_n(x)$ as shown in (1) and a positive integer $d < m$, find polynomials $\tilde{F}_1(x), \ldots, \tilde{F}_n(x)$ and $\tilde{H}(x)$ satisfying (2), with making the perturbation

$$\Delta := \sqrt{\sum_{i=1}^{n} \|F_i(x) - \tilde{F}_i(x)\|^2} \quad (3)$$

as small as possible.

## 3 Transformation of the approximate GCD problem

For solving the approximate GCD problem, we transfer the approximate GCD problem to a constrained minimization problem with the Bézout matrix.
Definition 3 (Bézout Matrix [8])
Let \( F(x) \) and \( G(x) \) be two real polynomials with the degree at most \( m \). Then, the matrix \( \text{Bez}(F, G) = (b_{ij})_{i,j=1,...,m} \), where
\[
\frac{F(x)G(y) - F(y)G(x)}{x - y} = \sum_{i,j=1}^{m} b_{ij} x^{i-1} y^{j-1},
\]
is called the Bézout matrix associated to \( F(x) \) and \( G(x) \). For polynomials \( F_1(x), \ldots, F_n(x) \), the matrix
\[
\text{Bez}(F_1, \ldots, F_n) = \begin{pmatrix}
\text{Bez}(F_1, F_2) \\
\text{Bez}(F_1, F_3) \\
\vdots \\
\text{Bez}(F_1, F_n)
\end{pmatrix}
\]
is called the Bézout matrix associated to \( F_1(x), \ldots, F_n(x) \).

For the Bézout matrix, we have a following theorem.

Theorem 4 (Barnett’s theorem [8])
Let \( F_1(x), \ldots, F_n(x) \) be real polynomials with the degree at most \( m \). Let \( d = \deg(\gcd(F_1(x), \ldots, F_n(x))) \) and \( (b_1, \ldots, b_m) = \text{Bez}(F_1(x), \ldots, F_n(x)) \). Then, the vectors \( b_1, \ldots, b_{m-d} \) are linearly independent, and there exists coefficients \( c_{i,j} \) such that
\[
b_i = \sum_{j=1}^{m-d} c_{i,j} b_j, \quad 1 \leq i \leq d,
\]
(4)

Furthermore, the monic form of the GCD of \( F_1(x), \ldots, F_n(x) \) is represented as
\[
\gcd(F_1(x), \ldots, F_n(x)) = x^d + c_{d,1} x^{d-1} + \cdots + c_{1,1} x^0.
\]
(5)

For polynomials \( \tilde{F}_1(x), \ldots, \tilde{F}_n(x) \) in Problem 2 let \( \tilde{B} = \text{Bez}(\tilde{F}_1(x), \ldots, \tilde{F}_n(x)) = (b_1, \ldots, b_m) \). In the case that polynomials \( \tilde{F}_1(x), \ldots, \tilde{F}_n(x) \) have an exact GCD of degree \( d \), Theorem 4 shows that \( b_1, \ldots, b_{m-d} \) are linearly independent, and \( b_{m-d+1} \) can be represented as a linear combination of \( b_1, \ldots, b_{m-d} \).

So there exists a vector \( y \in \mathbb{R}^{m-d} \), such that \( \langle b_1, \ldots, b_{m-d} \rangle y = b_{m-d+1} \).

Let \( s \) be the vector of the coefficients of the input polynomials: \( s := (f_1, \ldots, f_m, \ldots, f_{n_0}, \ldots, f_{n_m}) \).

Let the Bézout matrix of the input polynomials represented by \( s \) be \( B(s) = (b(s)_1, \ldots, b(s)_m) \). In the same way, let the Bézout matrix of the polynomials we find (\( \tilde{F}_1(x), \ldots, \tilde{F}_n(x) \)) be \( B(s + \Delta s) \), where \( \Delta s = (\tilde{f}_1 - f_1, \ldots, \tilde{f}_m - f_m, \ldots, \tilde{f}_{n_0} - f_{n_0}, \ldots, \tilde{f}_{n_m} - f_{n_m}) \).

From the above, we can transfer Problem 2 to a constrained minimization problem with the objective function:
\[
||\Delta s|| = \sqrt{\sum_{i=1}^{n} \sum_{j=0}^{m} (\tilde{f}_{ij} - f_{ij})^2},
\]
(6)

and the constraints:
\[
(b(s + \Delta s)_1, \ldots, b(s + \Delta s)_{m-d}) y = b(s + \Delta s)_{m-d+1} \quad \text{for some } y,
\]
(7)
with variables:
\[
(\Delta s, y) = (\tilde{f}_1, \ldots, \tilde{f}_m, \ldots, \tilde{f}_{n_0}, \ldots, \tilde{f}_{n_m}, y_1, \ldots, y_{m-d}).
\]
(8)
4 The Modified Newton Method

We consider a constrained minimization problem of minimizing an objective function $f(x) : \mathbb{R}^s \rightarrow \mathbb{R}$ which is twice continuously differentiable, subject to the constraints $g(x) = (g_1(x), \ldots, g_t(x))^T = 0$, where $g_i(x)$ is a function of $\mathbb{R}^s \rightarrow \mathbb{R}$ and is also twice continuously differentiable. For solving the constrained minimization problem, we use the modified Newton method by Tanabe [23], which is a generalization of the Gradient Projection method [18], as in the original GPGCD algorithm [26]. For $x_k$ which satisfies $g(x_k) = 0$, we calculate the search direction $d_k$ and the Lagrange multipliers $\lambda_k$ by solving the following linear system

$$
\begin{pmatrix}
I & -(J_g(x_k))^T \\
J_g(x_k) & 0
\end{pmatrix}
\begin{pmatrix}
d_k \\
\lambda_{k+1}
\end{pmatrix}
= -
\begin{pmatrix}
\nabla f(x_k) \\
g(x_k)
\end{pmatrix},
$$

(9)

where $J_g(x)$ is the Jacobian matrix represented as

$$
J_g(x) = \frac{\partial g_i}{\partial x_j}.
$$

(10)

5 The Algorithm for Calculating Approximate GCD

We give an algorithm for calculating approximate GCD of multiple polynomials using the Bézout matrix in this section.

5.1 Representation of the Jacobian matrix

In the modified Newton method, the Jacobian matrix is represented as follows. From the constraints (7) and the objective function (6), let $d(i) = \text{deg}(F_i(x))$, $i = 1, \ldots, n$, then the Jacobian matrix is represented as:

$$
\begin{pmatrix}
JL(2) & JM(2) & O & \cdots & O \\
\vdots & O & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & O \\
JL(n) & O & \cdots & O & JM(n)
\end{pmatrix}
\begin{pmatrix}
JR
\end{pmatrix},
$$

(11)
where

\[ JL(k) = JL_1(k) + JL_2(k), \]

\[ JL_1(k) = (p_{1k})_{i=1,m,j=1,m+1}, \]

\[ p_{1kij} = \begin{cases} \sum_{l=0}^{\min(m-d-j+i)} f_{kij}y_{j-i+l} & 1 \leq i < j \leq m + 1, \\ -\sum_{l=0}^{\min(m-i-j-d)} f_{kji}y_{j+i-l} & 1 \leq j \leq i \leq m. \end{cases} \]

\[ JL_2(k) = (p_{2k})_{i=1,m,j=1,m+1}, \]

\[ p_{2kij} = \begin{cases} f_{k(i+1-d-i-j)} & 1 \leq j \leq i \leq j + d - 1 \leq m, \\ -f_{k(i+1-d-i-j)} & 1 \leq j - (m + 1 - d) \leq i \leq j - 1 \leq m. \end{cases} \]

\[ JM(k) = JM_1(k) + JM_2(k), \]

\[ JM_1(k) = (p_{3k})_{i=1,m,j=1,d(k)+1}, \]

\[ p_{3kij} = \begin{cases} -\sum_{l=0}^{\min(m-d,k)-j+i} f_{1(i)}y_{j-i+l} & 1 \leq i < j \leq d(k) + 1, \\ \sum_{l=0}^{\min(m-i-j-d,k)} f_{1(i)}y_{j+i-l} & 1 \leq j \leq i \leq m. \end{cases} \]

\[ JM_2(k) = (p_{4k})_{i=1,m,j=1,d(k)+1}, \]

\[ p_{4kij} = \begin{cases} f_{1(i+1-d-i-j)} & 1 \leq j \leq i \leq j + d - 1 \leq m, \\ -f_{1(i+1-d-i-j)} & 1 \leq j - (m + 1 - d) \leq i \leq j - 1 \leq m. \end{cases} \]

\[ JR = (b_1, \ldots, b_{m-d}). \]

### 5.2 Setting the Initial Values

We give the initial values for variables in \( f \) as follows. For the given polynomials \( f \), we set the initial value for variables \( s_0 \) as:

\[ s_0 = (f_1, \ldots, f_m, \ldots, f_{n_1}, \ldots, f_{n_m}). \]  \hspace{1cm} (12)

For the Bézout matrix \( B = \text{Bez}(F_1(x), \ldots, F_n(x)) = (b_1, \ldots, b_m) \), we calculate \( y = (y_0, \ldots, y_{0(m-d)}) \) by solving the following least squares problem:

\[ \min ||(b_1, \ldots, b_{m-d})y - b_{m-d+1}||. \]  \hspace{1cm} (13)

Then we set the initial value for variables \( y_0 \) as:

\[ y_0 = (y_0, \ldots, y_{0(m-d)}). \]  \hspace{1cm} (14)

From above, we give the initial value for variables in \( f \) as

\[ (s_0, y_0) = (f_1, \ldots, f_m, \ldots, f_{n_1}, \ldots, f_{n_m}, y_0, \ldots, y_{0(m-d)}). \]  \hspace{1cm} (15)

### 5.3 Calculating the Approximate GCD and Finding the Approximate polynomials

Let \( (s^*, y^*) = (f_1, \ldots, f_m, \ldots, f_{n_1}, \ldots, f_{n_m}, y_1, \ldots, y_{0(m-d)}) \) be the minimizer of the objective function in \( f \) calculated by the modified Newton method, corresponding to the coefficients of \( F_1(x), \ldots, F_n(x) \). Then, we calculate the approximate GCD from the Bézout matrix \( B(s^*) \) with Theorem 4.

Finally, we find the approximate polynomials \( \tilde{F}_i(x) = \tilde{F}_i(x)\tilde{H}(x), i = 1, \ldots, n \) in \( C \) by solving least squares problems:

\[ \min ||F_i(x) - \tilde{F}_i(x)\tilde{H}(x)|| \quad i = 1, \ldots, n. \]  \hspace{1cm} (16)
5.4 The algorithm and running time analysis

Summarizing above, we give the algorithm for calculating approximate GCD for multiple polynomials as Algorithm [1]. We give an analysis of the arithmetic running time of Algorithm [1] as follows.

In Step 1, we set the initial values by the construction of the Bézout matrix, calculation of the initial values using the least squares solution, and the construction of the Jacobian matrix. Since the dimension of the Bézout matrix and the Jacobian matrix is $mn \times m$ and $mn \times (m - d + \sum_{i=1}^{n} \deg(F_i) + 1)$, respectively, we can estimate the running time of the construction of the Bézout matrix, calculation of the initial values using the least squares solution, and the construction of the Jacobian matrix as $O(m^2n)$ ([3]), $O(m^{3})$ ([15]), $O(m(n(m-d+n)+\sum_{i=1}^{n} \deg(F_i))) = O(mn(m+n-d))$, respectively, where the Jacobian matrix is computed by ([1]).

In Step 3, since the dimension of the Jacobian matrix is $mn(m-d+\sum_{i=1}^{n} \deg(F_i) + 1)$, the running time for solving the linear system is $O((mn + m - d + \sum_{i=1}^{n} \deg(F_i) + 1)^3) = O((mn - d)^3)$ ([7]).

In Step 4, the running time of the construction of the Bézout matrix and the Jacobian matrix is $O(m^2n)$ and $O(mn(m - d + n) + \sum_{i=1}^{n} \deg(F_i))) = O(mn(m+n-d))$, respectively.

In Step 5, the running time for calculating the approximate GCD $\tilde{H}(x)$ and calculation of the approximate polynomials $\tilde{F}_i(x)$, $i = 1, \ldots, n$ is $O(m^2n)$ and $O(mn - d)^3$.

As a consequence, the running time of Algorithm [1] is the number of iteration times $O((mn - d)^3)$.

Algorithm 1 The GPGCD algorithm for multiple polynomials with the Bézout matrix

Inputs:
- $F_1(x), \ldots, F_n(x) \in \mathbb{R}(x)$: the given polynomials with $\max(\deg(F_i(x)), i = 1 \ldots n) = \deg(F_1(x)) > 0$,
- $d \in \mathbb{N}$: the given degree of approximate GCD with $d \leq \min(\deg(F_i(x)), i = 1 \ldots n)$,
- $\epsilon > 0$: the stop criterion with the modified Newton method,
- $0 < \alpha \leq 1$: the step width with the modified Newton method.

Outputs:
- $\tilde{H}(x)$: the approximate GCD, with $\deg(\tilde{H}) = d$,
- $\tilde{F}_1(x), \ldots, \tilde{F}_n(x)$: the polynomials which are close to $F_1(x), \ldots, F_n(x)$, respectively, with the GCD $\tilde{H}$.

Step 1 Generate the Bézout matrix $B = \text{Bez}(F_1(x) \ldots F_n(x))$, the initial values $y_0$ ([14]), and the Jacobian matrix ([11]).

Step 2 Set the initial values $(s_0, y_0)$ as in ([15]).

Step 3 Solve the linear system ([9]) to find the search direction $d_k$.

Step 4 If $\|d_k\| < \epsilon$, obtain the $(s^*, y^*)$ as $(s_k, y_k)$, generate the Bézout matrix $\tilde{B} = \text{Bez}(\tilde{F}_1(x), \ldots, \tilde{F}_n(x))$ from polynomials $\tilde{F}_i(x)$, $i = 1, \ldots, n$, then go to Step 5. Otherwise, let $(s_{k+1}, y_{k+1}) = (s_k, y_k) + \alpha d_k$ and calculate the Bézout matrix and the Jacobian matrix with $(s_{k+1}, y_{k+1})$, then go to Step 3.

Step 5 Calculate the approximate GCD $\tilde{H}(x)$ with Theorem 4. For $i = 1, \ldots, n$, calculate the approximate polynomials $\tilde{F}_i(x)$ by solving least squares problems in ([16]). Return $\tilde{F}_1(x), \ldots, \tilde{F}_n(x)$ and $\tilde{H}(x)$.
6 Experiments

We have implemented our GPGCD algorithm on a computer algebra system Maple 2021. For \(i = 1, \ldots, n\), the test polynomials \(F_i(x)\) are generated as

\[
F_i(x) = \hat{F}_i(x)H(x) + \frac{e}{\|F(x)\|}\hat{F}_i(x),
\]

(17)

Here, \(\hat{F}_i(x)\) and \(H(x)\) are polynomials of degrees \(\deg(F_i) - d\) and \(d\), respectively. Note that \(F_i(x)\) are relatively prime polynomials. They are generated as polynomials that their coefficients are floating-point numbers and their absolute values are not greater than 10. The noise polynomials \(\hat{F}_i(x)\) are polynomials of degree \(\deg(F_i) - 1\), which are randomly generated with coefficients given as the same as for \(F_i(x)\) and \(H(x)\).

We have generated 5 test groups of test polynomials, each group comprising 100 tests. We set the degree of the input polynomials for all groups as 10, and the degree of the approximate GCD as 3, 4, 5, 6, 7, respectively (shown as in Table 1). We set the number of polynomials for each test as \(n = 10\), and set the norm of the noise \(e\) as \(e = 0.01\) in our tests. The stop criterion \(\epsilon\) and the stop width \(\alpha\) in Algorithm 1 are set as 0.1 and 1, respectively.

We have carried out the tests on CPU Intel(R) Xeon(R) Silver 4 210 at 2.20GHz with RAM 256GB under Linux 5.4.0 with Maple 2021.

6.1 The experimental results

We have carried out the tests with the GPGCD algorithm with the Bézout matrix for multiple polynomials (abbreviated as GPGCD-Bézout-multiple algorithm). For comparison, we have also carried out the tests with the original GPGCD algorithm with subresultant matrices for multiple polynomials (abbreviated as GPGCD-Sylvester-multiple algorithm) (125).

For the results, we have compared the norm of the perturbation of the GPGCD-Bézout-multiple algorithm with that of the GPGCD-Sylvester-multiple algorithm. For \(i = 1, \ldots, n\), let \(F_{Bi} = f_{Bi,m}x^m + \cdots + f_{Bi,0}x^0\) be the approximate polynomials from the outputs of the GPGCD-Bézout-multiple algorithm, and \(F_{Si} = f_{Si,m}x^m + \cdots + f_{Si,0}x^0\) be the approximate polynomials from the outputs of the GPGCD-Sylvester-multiple algorithm. Then, let \(\Delta B\) be the norm of the perturbation of the GPGCD-Bézout-multiple algorithm:

\[
\Delta B = \sqrt{\sum_{i=1}^{n} \sum_{j=0}^{m} (f_{Bi,j} - f_{i,j})^2},
\]

and let \(\Delta S\) be the norm of the perturbation of the GPGCD-Sylvester-multiple algorithm:

\[
\Delta S = \sqrt{\sum_{i=1}^{n} \sum_{j=0}^{m} (f_{Si,j} - f_{i,j})^2}.
\]

If we have \(|\Delta B - \Delta S| \leq 0.5\), then we say that the minimizers of both algorithms are closed to each other.

The number of tests which the minimizers are closed to each other is shown in the left part of Table 2. In ‘All’, we show the number for all tests which the minimizers are closed to each other. In ‘Bézout’, we show the number for tests which the GPGCD-Bézout-multiple algorithm has a smaller perturbation than that of GPGCD-Sylvester-multiple algorithm. In ‘Sylvester’, we show the number for tests

1) Coefficients are generated with the Mersenne Twister algorithm by built-in function Generate with RandomTools:-MersenneTwister in Maple, which approximates a uniform distribution on \([-10, 10]\).
Table 1: Degrees of test polynomials

| Group | \( m = \deg(F_i) \) | \( d = \deg(H) \) |
|-------|-------------------|-----------------|
| 1     | 10                | 3               |
| 2     | 10                | 4               |
| 3     | 10                | 5               |
| 4     | 10                | 6               |
| 5     | 10                | 7               |

which GPGCD-Sylvester-multiple algorithm has a smaller perturbation than that of the GPGCD-Bézout-
multiple algorithm. In the same way, the number of tests which the minimizers are not closed to each
other is shown in the right part of Table 2.

For the tests in which the minimizers are closed to each other, the average time and the average
number of iterations are shown in Table 3. In the left part, we show the average time for two algorithms.
In the middle part, we show the average number of iterations. In the right part, we show the average time
per iteration.

For the tests in which the minimizers are closed to each other, we have also calculated the norm of
the remainders of the GPGCD-Bézout-multiple algorithm \( \|F_{BR1}(x)\|, \ldots, \|F_{BRn}(x)\| \) and the norm of
the remainders of the GPGCD-Sylvester-multiple algorithm \( \|F_{SR1}(x)\|, \ldots, \|F_{SRn}(x)\| \), where

\[
F_{Br}(x) = \bar{F}_i(x)\tilde{H}(x) + F_{BRi}(x), \quad \deg(F_{BR}) < d, \quad i = 1, \ldots, n,
\]

\[
F_{Sr}(x) = \bar{F}_i(x)\tilde{H}(x) + F_{SRi}(x), \quad \deg(F_{SR}) < d, \quad i = 1, \ldots, n.
\]

We show the average norm of the remainders in Table 4.

6.2 Comparison with the GPGCD-Sylvester-multiple algorithm

From Table 2, we see that the number of tests in which the minimizers are closed to each other of each
group is larger when the degree of the approximate GCD is higher, where the least of them is over half
of the total number of tests. For all results, we see that the number of tests in which the perturbation of
the GPGCD-Sylvester-multiple algorithm is smaller than that of the GPGCD-Bézout-multiple algorithm
is larger than that of the opposite. That means the GPGCD-Sylvester-multiple algorithm shows higher
stability than the GPGCD-Bézout-multiple algorithm.

From Table 3, we see that total computing time, number of iterations, average computing time per iter-
ation of the GPGCD-Bézout-multiple algorithm are smaller than those of the GPGCD-Sylvester-multiple
algorithm, respectively. By comparing average computing time per iteration in each group, we see that,
except for Group 1 in the GPGCD-Bézout-multiple algorithm, when the degree of approximate GCD is
higher, the average time per iteration of both algorithms is smaller.

In the previous research, the GPGCD algorithm with the Bézout matrix with 2 polynomials (abbrevi-
ated as GPGCD-Bézout algorithm) (2), we have shown that the average of the norm of the remainders
of the tests in the GPGCD-Bézout algorithm is larger than that of the original GPGCD algorithm (ab-
 abbreviated as GPGCD-Sylvester algorithm). From Table 4 we see that there are little difference between
the GPGCD-Bézout-multiple algorithm and the GPGCD-Sylvester-multiple algorithm in the sense of the
average of the norm of the remainders.

7 Conclusions

We have proposed an algorithm using the Bézout matrix based on the GPGCD algorithm for multiple
polynomials.
Table 2: The number of tests which the minimizers are closed to / not closed to each other

| Group | Closed to each other | Not closed to each other |
|-------|----------------------|--------------------------|
|       | All Bézout Sylvester | All Bézout Sylvester    |
| 1     | 55 28 27             | 45 4 41                 |
| 2     | 57 21 36             | 43 9 34                 |
| 3     | 72 29 43             | 28 2 26                 |
| 4     | 72 24 48             | 28 0 28                 |
| 5     | 81 19 62             | 19 2 17                 |

Table 3: Comparison of computing time and the number of iterations (See Remark 1 for detail)

| Group | Average time (sec.) | Average number of iterations | Average time per iteration (sec.) |
|-------|---------------------|------------------------------|----------------------------------|
|       | Bézout   Sylvester | Bézout          Sylvester      | Bézout   Sylvester              |
| 1     | 9.804     166.266  | 14.47            23.47         | 0.677     7.083                   |
| 2     | 11.035    141.235  | 16.14            21.28         | 0.684     6.637                   |
| 3     | 9.587     112.952  | 14.375           18.14         | 0.667     6.227                   |
| 4     | 7.296     99.230   | 11.33            17             | 0.644     5.837                   |
| 5     | 6.272     89.058   | 9.83             16.09         | 0.638     5.536                   |

Table 4: Comparison of norm of the remainders

| Group | Bézout    | Sylvester |
|-------|-----------|-----------|
| 1     | $2.320 \times 10^{-9}$ | $1.875 \times 10^{-9}$ |
| 2     | $1.390 \times 10^{-9}$ | $9.170 \times 10^{-9}$ |
| 3     | $1.150 \times 10^{-9}$ | $1.059 \times 10^{-9}$ |
| 4     | $9.684 \times 10^{-9}$ | $8.427 \times 10^{-10}$ |
| 5     | $7.987 \times 10^{-11}$ | $9.012 \times 10^{-11}$ |
The experimental results show that, in most of the cases, the minimizer of the proposed algorithm is closed to the minimizer of the GPGCD-Sylvester-multiple algorithm. For the cases in which the minimizers of both algorithms are closed to each other, total computing time, the number of iterations, average computing time per iteration of the GPGCD-Bézout-multiple algorithm are smaller than those of the GPGCD-Sylvester-multiple algorithm, respectively. Thus, we see that the proposed algorithm is more efficient than the GPGCD-Sylvester-multiple algorithm. By comparing the computing time per iteration, the experimental results show that when the degree of the approximate GCD is higher, the computing time per iteration of the proposed algorithm is smaller. This is consistent with the arithmetic running time analysis shown in Section 5.4.

The previous research has shown that the GPGCD-Bézout algorithm was not so accurate in the sense of the norm of perturbations compared with the GPGCD-Sylvester algorithm. However, new experimental results show that, with the improvement for calculating the approximate polynomials as shown in [14], the GPGCD-Bézout-multiple algorithm calculate approximate GCDs with almost the same accuracy as the GPGCD-Sylvester-multiple algorithm in the sense of the norm of perturbations, for the tests in which the minimizers of both algorithms are closed to each other.

On the other hand, the experimental results show that the number of tests in which the perturbation of the GPGCD-Sylvester-multiple algorithm is smaller than that of the proposed algorithm is larger than that of the opposite. Thus, the GPGCD-Sylvester-multiple algorithm shows higher stability than the proposed algorithm. Improving stability of the proposed algorithm will be one of the topics of further research.

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