FACTORIZABLE GRADED RINGS OF COMPLEXITY ONE

JÜRGEN HAUSEN AND ELAINE HERPPICH

Abstract. We consider finitely generated normal algebras over an algebraically closed field of characteristic zero that come with a complexity one grading by a finitely generated abelian group such that the conditions of a UFD are satisfied for homogeneous elements. Our main results describe these algebras in terms of generators and relations. We apply this to write down explicitly the possible Cox rings of normal complete rational varieties with a complexity one torus action.

1. Statement of the results

The subject of this note are finitely generated normal algebras $R = \bigoplus K R_w$ over some algebraically closed field $K$ of characteristic zero graded by a finitely generated abelian group $K$. We are interested in the following homogeneous version of a unique factorization domain: $R$ is called factorially (K-)graded if every homogeneous nonzero nonunit is a product of $K$-primes, where a $K$-prime element is a homogeneous nonzero nonunit $f \in R$ with the property that whenever $f$ divides a product of homogeneous elements, then it divides one of the factors. For free $K$, the properties factorial and factorially graded are equivalent [3], but for a $K$ with torsion the latter is more general. Our motivation to study factorially graded algebras is that the Cox rings of algebraic varieties are of this type, see for example [2].

We focus on effective $K$-gradings of complexity one, i.e., the $w \in K$ such that $R_w \neq 0$ generate $K$ and $K$ is of rank $\dim(R) - 1$. Moreover, we suppose that the grading is pointed in the sense that $R_0 = K$ holds. The case of a free grading group $K$ and hence factorial $R$ was treated in [5, Section 1]. Here we settle the more general case of factorial gradings allowing torsion. Our results enable us to write down explicitly the possible Cox rings of normal complete rational varieties with a complexity one torus action. This complements [6], where the Cox ring of a given variety was computed in terms of the torus action.

In order to state our results, let us fix the notation. For $r \geq 1$, let $A = (a_0, \ldots, a_r)$ be a sequence of vectors $a_i = (b_i, c_i)$ in $K^2$ such that any pair $(a_i, a_k)$ with $k \neq i$ is linearly independent, $n = (n_0, \ldots, n_r)$ a sequence of positive integers and $L = (l_{ij})$ a family of positive integers, where $0 \leq i \leq r$ and $1 \leq j \leq n_i$. For every $0 \leq i \leq r$, define a monomial

$$T_i := T_{i1}^{l_{i1}} \cdots T_{in_i}^{l_{in_i}} \in S := K[T_{ij}; 0 \leq i \leq r, 1 \leq j \leq n_i].$$

Moreover, for any two indices $0 \leq i, j \leq r$, set $\alpha_{ij} := \det(a_i, a_j) = b_i c_j - b_j c_i$ and for any three indices $0 \leq i < j < k \leq r$ define a trinomial

$$g_{i,j,k} := \alpha_{jk} T_i^{l_{i}} + \alpha_{ki} T_j^{l_{j}} + \alpha_{ij} T_k^{l_{k}} \in S.$$

We define a grading of $S$ by an abelian group $K$ such that all the $g_{i,j,k}$ become homogeneous of the same degree. For this, consider the free abelian groups

$$F := \bigoplus_{i=0}^{r} \bigoplus_{j=1}^{n_i} \mathbb{Z} f_{ij} \cong \mathbb{Z}^n, \quad N := \mathbb{Z}^r.$$
where we set \( n := n_0 + \ldots + n_r \). Set \( l_i := (l_{i1}, \ldots, l_{in_i}) \). Then we have a linear map \( P: F \to N \) defined by the \( r \times n \) matrix

\[
P = \begin{pmatrix}
-l_0 & l_1 & \ldots & 0 \\
\vdots & \vdots & & \vdots \\
-0 & 0 & \ldots & l_r
\end{pmatrix}.
\]

Let \( P^*: M \to E \) be the dual map, set \( K := E/P^*(M) \) and let \( Q: E \to K \) be the projection. Let \( (e_{ij}) \) be the dual basis of \((f_{ij})\) and define a \( K \)-grading on \( S \) by \( \deg(T_{ij}) := Q(e_{ij}) \). Then all \( g_{i,j,k} \) are homogeneous of the same degree and we obtain a \( K \)-graded factor algebra

\[
R(A, n, L) := \mathbb{K}[T_{ij}; 0 \leq i \leq r, 1 \leq j \leq n_i] / \langle g_{i,i+1,i+2}; 0 \leq i \leq r - 2 \rangle.
\]

We say that the triple \((A, n, L)\) is sincere, if \( r \geq 2 \) and \( n_i l_{ij} > 1 \) for all \( i, j \) hold; this ensures that there exist in fact relations \( g_{i,j,k} \) and none of these relations contains a linear term. Note that for \( r = 1 \) we obtain the diagonal complexity one gradings of the polynomial ring of the polynomial ring \( S \).

**Theorem 1.1.** Let \((A, n, L)\) be any triple as above.

(i) The algebra \( R(A, n, L) \) is factorially \( K \)-graded; the \( K \)-grading is effective, pointed and of complexity one.

(ii) Suppose that \((A, n, L)\) is sincere. Then \( R(A, n, L) \) is factorial if and only if the group \( K \) is torsion free.

The second part of this theorem provides examples of factorially graded algebras which are not factorial; note that \( K \) is torsion free if and only if the numbers \( l_i := \gcd(l_{i1}, \ldots, l_{in_i}) \) are pairwise coprime.

**Example 1.2.** Let \( A \) consist of the vectors \((1, 0), (1, 1)\) and \((0, 1)\), take \( n = (1, 1, 1) \) and take the family \( L \) given by \( l_{01} = l_{11} = l_{21} = 2 \). Then the matrix

\[
\begin{pmatrix}
-2 & 2 & 0 \\
-2 & 0 & 2
\end{pmatrix}
\]

describes the map \( P: \mathbb{Z}^3 \to \mathbb{Z}^2 \). Thus the grading group is \( K = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \). Concretely this grading can be realized as

\[
\deg(T_{01}) = (1, \overline{0}, \overline{0}), \quad \deg(T_{11}) = (1, \overline{1}, \overline{0}), \quad \deg(T_{21}) = (1, \overline{0}, \overline{1}).
\]

The associated algebra \( R(A, n, L) \) is factorially \( K \)-graded but not factorial. It is explicitly given by

\[
R(A, n, L) = \mathbb{K}[T_{01}, T_{11}, T_{21}]/\langle T_{01}^2 - T_{11}^2 + T_{21}^2 \rangle.
\]

Obvious further examples of algebras with an effective pointed factorial grading are \( R(A, n, L)[S_1, \ldots, S_m] \), graded by \( K \times \mathbb{Z}^m \) via \( \deg(S_i) := e_i \), where \( e_i \in \mathbb{Z}^m \) denotes the \( i \)-th canonical basis vector. The methods of [6, Section 3] apply directly to our situation and show that there are no other examples, i.e., we arrive at the following.

**Theorem 1.3.** Every finitely generated normal \( \mathbb{K} \)-algebra with an effective, pointed, factorial grading of complexity one is isomorphic to some \( R(A, n, L)[S_1, \ldots, S_m] \).

We turn to Cox rings. Roughly speaking, the Cox ring of a complete normal variety \( X \) with finitely generated divisor class group \( \text{Cl}(X) \) is given as

\[
\mathcal{R}(X) := \bigoplus_{\text{Cl}(X)} \Gamma(X, \mathcal{O}_X(D)),
\]

see [2] for the details of the precise definition. As mentioned, our aim is to write down all possible Cox rings of normal rational complete varieties with a complexity one torus action. They are obtained from \( R(A, n, L)[S_1, \ldots, S_m] \) by coarsening the
(K × ℤ^m)-grading as follows. Let 0 < s < n + m - r, consider an integral s × n matrix d, an integral s × m matrix d' and the block matrix

\[ \hat{P} = \begin{pmatrix} P & 0 \\ d & d' \end{pmatrix}, \]

where d and d' are chosen in such a manner that the columns of the matrix \( \hat{P} \) are pairwise different primitive vectors in \( \mathbb{Z}^{r+s} \) which generate \( \mathbb{Q}^{r+s} \) as a cone. Consider the linear map of lattices \( \hat{P} : \hat{F} \to \hat{N} \), where

\[ \hat{F} := F \oplus \mathbb{Z}f_1 \oplus \ldots \oplus \mathbb{Z}f_m, \quad \hat{N} := \mathbb{Z}^{r+s}. \]

Let \( \hat{P}^* : \hat{M} \to \hat{E} \) be the dual map and \( \hat{Q} : \hat{E} \to \hat{K} \) the projection, where \( \hat{K} := \hat{E}/\hat{P}^*(\hat{M}) \). Denoting by \( e_{ij} \), \( e_k \) the dual basis to \( f_{ij} \), \( f_k \), we obtain a \( K \)-grading of \( R(A, n, L)[S_1, \ldots, S_m] \) by setting

\[ \deg(T_{ij}) := \hat{Q}(e_{ij}), \quad \deg(S_k) := \hat{Q}(e_k). \]

**Theorem 1.4.** In the above notation, the following holds.

(i) The \( K \)-grading of \( R(A, n, L)[S_1, \ldots, S_m] \) is effective, pointed and factorial. Moreover, \( T_{ij}, S_k \) define pairwise nonassociated \( K \)-prime generators.

(ii) The \( K \)-graded algebra \( R(A, n, L)[S_1, \ldots, S_m] \) is the Cox ring of a \( Q \)-factorial rational projective variety with a complexity one torus action.

**Theorem 1.5.** Let \( X \) be a normal rational complete variety with a torus action of complexity one. Then the Cox ring of \( X \) is isomorphic as a graded ring to some \( R(A, n, L)[S_1, \ldots, S_m] \) with a \( K \)-grading as constructed above.

### 2. Proof of the Results

A very first observation lists basic properties of the \( K \)-grading of \( R(A, n, L) \). We denote by \( \mathbb{T}^n := (\mathbb{K}^+)^n \) the standard n-torus. Moreover, we work with the diagonal action of the quasitorus \( H := \text{Spec} \mathbb{K}[K] \) on \( \mathbb{K}^n \) given by \( t \cdot z = (\chi_{ij}(t)z_{ij}) \), where \( \chi_{ij} \in \mathbb{K}(H) \) is the character corresponding to \( Q(e_{ij}) = \deg(T_{ij}) \). By definition, this action stabilizes the zero set

\[ \tilde{X} := V(\mathbb{K}^n; g_{i,i+1,i+2}, 0 \leq i \leq r-2) \subseteq \mathbb{K}^n. \]

**Proposition 2.1.** The algebra \( R(A, n, L) \) is normal, the \( K \)-grading is pointed, effective and of complexity one.

**Proof.** Effectivity of the \( K \)-grading is given by construction, because the degrees \( \deg(T_{ij}) = Q(e_{ij}) \) generate \( K \). From [5, Prop. 1.2], we infer that \( R(A, n, L) \) is a normal complete intersection. Thus, we have

\[ \dim(R(A, n, L)) = n - r + 1 = \dim(\ker(P)) + 1 \]

which means that the \( K \)-grading is of complexity one. Now consider the action of the quasitorus \( H := \text{Spec} \mathbb{K}[K] \) on \( \mathbb{K}^n \) given by the \( K \)-grading. Note that \( H \subseteq \mathbb{T}^n \) is the kernel of the homomorphism of tori

\[ \mathbb{T}^n \to \mathbb{T}^r, \quad (t_0, \ldots, t_r) \mapsto \left( \frac{t_1}{t_0}, \ldots, \frac{t_r}{t_0} \right). \]

The set \( \tilde{X} \subseteq \mathbb{K}^n \) of common zeroes of all the \( g_{i,i+1,i+2} \) is \( H \)-invariant and thus it is invariant under the one-parameter subgroup of \( H \) given by

\[ \mathbb{K}^* \to H, \quad t \mapsto (t^{\zeta_{ij}}), \quad \zeta_{ij} := n_i^{-1}t_i^{-1} \prod_k n_k \prod_l t_{km}. \]

Since all \( \zeta_{ij} \) are positive, any orbit of this one-parameter subgroup in \( \mathbb{K}^n \) has the origin in its closure. Consequently, every \( H \)-invariant function on \( \tilde{X} \) is constant. This shows \( R(A, n, L)_0 = \mathbb{K}. \) \( \square \)
We say that a Weil divisor on $\tilde{X}$ is $H$-prime if it is non-zero, has only multiplicities zero or one and the group $H$ permutes transitively the prime components with multiplicity one. Note that the divisor $\div(f)$ of a homogeneous function $f \in R(A, n, L)$ on $\tilde{X}$ is $H$-prime if and only if $f$ is $K$-prime [4 Prop. 3.2]. The following is an essential ingredient of the proof.

**Proposition 2.2.** Regard the variables $T_{ij}$ as regular functions on $\tilde{X}$.

(i) The divisors of the $T_{ij}$ on $\tilde{X}$ are $H$-prime and pairwise different. In particular, the $T_{ij}$ define pairwise nonassociated $K$-prime elements in $R(A, n, L)$.

(ii) If the ring $R(A, n, L)$ is factorial and $n_1 l_{ij} > 1$ holds, then the divisor of $T_{ij}$ on $\tilde{X}$ is even prime.

**Proof.** For (i), we exemplarily show that the divisor of $T_{01}$ is $H$-prime. First note that by [5 Lemma 1.3] the zero set $V(\tilde{X}; T_{01})$ is described in $\mathbb{K}^n$ by the equations

$$
T_{01} = 0, \quad \alpha_{s+1} T_s^l + \alpha_0 T_{s+1}^l = 0, \quad 1 \leq s \leq r - 1.
$$

Let $h \in S$ denote the product of all $T_{ij}$ with $(i, j) \neq (0, 1)$. Then, in $\mathbb{K}^n_h$, the above equations are equivalent to

$$
T_{01} = 0, \quad \frac{\alpha_{s+1} T_s^l}{\alpha_0 T_{s+1}^l} = 1, \quad 1 \leq s \leq r - 1.
$$

Now, choose a point $\tilde{x} \in \mathbb{K}^n_h$ satisfying these equations. Then $\tilde{x}_{01}$ is the only vanishing coordinate of $\tilde{x}$. Any other such point is of the form

$$(0, t_{02} \tilde{x}_{02}, \ldots, t_{rn} \tilde{x}_{rn}, \ldots), \quad t_{ij} \in \mathbb{K}^+, \quad t_s^l = t_{s+1}^l, \quad 1 \leq s \leq r - 1.
$$

Setting $t_{01} := t_{02} \cdots t_{0n_0} T_{11}^l$, we obtain an element $t = (t_{ij}) \in H$ such that the above point equals $t \cdot \tilde{x}$. This consideration shows

$$
V(\tilde{X}; T_{01}) = H \cdot \tilde{x}.
$$

Using [5 Lemma 1.4], we see that $V(\tilde{X}; T_{01}, T_{ij})$ is of codimension at least two in $\tilde{X}$ whenever $(i, j) \neq (0, 1)$. This allows to conclude

$$
V(\tilde{X}; T_{01}) = \overline{H \cdot \tilde{x}}.
$$

Thus, to obtain that $T_{01}$ defines an $H$-prime divisor on $\tilde{X}$, we only need that the equations (2.1) define a radical ideal. This in turn follows from the fact that their Jacobian at the point $\tilde{x} \in V(\tilde{X}; T_{01})$ is of full rank.

To verify (ii), let $R(A, n, L)$ be factorial. Assume that the divisor of $T_{ij}$ is not prime. Then we have $T_{ij} = h_1 \cdots h_s$ with prime elements $h_l \in R(A, n, L)$. Consider their decomposition into homogeneous parts

$$
h_l = \sum_{w \in K} h_{l,w}.
$$

Plugging this into the product $h_1 \cdots h_s$ shows that $\deg(T_{ij})$ is a positive combination of some $\deg(T_{kl})$ with $(k, l) \neq (i, j)$. Thus, there is a vector $(c_{kl}) \in \ker(Q) \subseteq E$ with $c_{ij} = 1$ and $c_{kl} \leq 0$ whenever $(k, l) \neq (i, j)$. Since, $\ker(Q)$ is spanned by the rows of $P$, we must have $n_i = 1$ and $l_{ij} = 1$, a contradiction to our assumptions. \qed

We come to the main step of the proof, the construction of a $T$-variety having $R(A, n, L)[S_1, \ldots, S_m]$ as its Cox ring. We will obtain $X$ as a subvariety of a toric variety $Z$ and the construction of $Z$ is performed in terms of fans. As before, consider the lattice

$$
\tilde{F} := F \oplus \mathbb{Z} f_1 \oplus \cdots \oplus \mathbb{Z} f_m.
$$
Let $\tilde{\Sigma}$ be the fan in $\tilde{F}$ having the rays $\tilde{\varrho}_{ij}$ and $\tilde{\varrho}_k$ through the basis vectors $f_{ij}$ and $f_k$ as its maximal cones. Let $0 < s < n + m - r$, choose an integral $s \times n$ matrix $d$ and an integral $s \times m$ matrix $d'$. Consider the (block) matrices
\[
\begin{pmatrix}
  P \\
  d \\
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
  P \\
  0 \\
\end{pmatrix}
\]
and suppose that the columns of the first one are primitive, pairwise different and generate $\tilde{N}_{\mathbb{Z}}$ as a cone, where $\tilde{N} := \mathbb{Z}^{r+s}$. With $N := \mathbb{Z}^r$, we have the projection $B: \tilde{N} \to N$ onto the first $r$ coordinates and the linear maps $\tilde{P}: \tilde{F} \to \tilde{N}$ and $\tilde{P}: \tilde{F} \to N$ respectively given by the above matrices.

Let $\Sigma$ be the fan in $N$ with the rays $\varrho_{ij} := \tilde{P}(\tilde{\varrho}_{ij})$ and $\varrho_k := \tilde{P}(\tilde{\varrho}_k)$ as its maximal cones. The ray $\varrho_{ij}$ through the $ij$-th column of $P$ is given in terms of the canonical basis vectors $v_1, \ldots, v_r$ in $N = \mathbb{Z}^r$ as
\[
\varrho_{ij} = Q_{\geq 0}v_i, \quad 1 \leq i \leq r, \quad \varrho_{ij} = -Q_{\geq 0}(v_1 + \ldots + v_r).
\]
For fixed $i$ all $\varrho_{ij}$ are equal to each other; we list them nevertheless all separately in a system of fans $\Sigma$ having the zero cone as the common gluing data; see $[\text{P}]$ for the formal definition of this concept. Finally, we have the fan $\Delta$ in $\mathbb{Z}^r$ with the rays $Q_{\geq 0}v_i$ and $-Q_{\geq 0}(v_1 + \ldots + v_r)$ as its maximal cones.

The toric variety $\tilde{Z}$ associated to $\tilde{\Sigma}$ has $\text{Spec} \mathbb{K}[\tilde{E}] \cong \mathbb{T}^{n+m}$ as its acting torus, where $\tilde{E}$ is the dual lattice of $\tilde{F}$. The fan $\Sigma$ in $\tilde{N}$ defines a toric variety $\tilde{Z}$ and the system of fans $\Sigma$ defines a toric variety $Z$. The toric prime divisors corresponding to the rays $\tilde{\varrho}_{ij}, \tilde{\varrho}_k \in \tilde{\Sigma}$, $\tilde{\varrho}_{ij}, \tilde{\varrho}_k \in \Sigma$ and $\varrho_{ij} \in \Sigma$, are denoted as
\[
\tilde{D}_{ij}, \tilde{D}_k \subseteq \tilde{Z}, \quad \tilde{D}_{ij}, \tilde{D}_k \subseteq \tilde{Z}, \quad D_{ij} \subseteq \tilde{Z}.
\]

The toric variety associated to $\Delta$ is the open subset $\mathbb{P}_\mathbb{R}^{(1)} \subseteq \mathbb{P}_\mathbb{R}$ of the projective space obtained by removing all toric orbits of codimension at least two. The maps $\tilde{P}$ and $\tilde{P}$ define toric morphisms $\tilde{\pi}: \tilde{Z} \to \tilde{Z}$ and $\tilde{\pi}: \tilde{Z} \to \tilde{Z}$. Moreover, $B: \tilde{N} \to N$ defines a toric morphism $\beta: \tilde{Z} \to Z$ and the identity $\mathbb{Z}^r \to \mathbb{Z}^r$ defines a toric morphism $\kappa: Z \to \mathbb{P}_\mathbb{R}^{(1)}$. These morphisms fit into the commutative diagram
\[
\begin{array}{ccc}
\tilde{Z} & \xrightarrow{\tilde{\pi}} & \tilde{Z} \\
\xrightarrow{\pi} & & \xrightarrow{\beta} \\
Z & & Z \\
\xrightarrow{\kappa} & & \mathbb{P}_\mathbb{R}^{(1)}
\end{array}
\]
Note that $\kappa: Z \to \mathbb{P}_\mathbb{R}^{(1)}$ is a local isomorphism which, for fixed $i$, identifies all the divisors $D_{ij}$ with $1 \leq j \leq n_i$. Let $\tilde{H} \subseteq \mathbb{T}^{n+m}$ and $\check{H} \subseteq \mathbb{T}^{n+m}$ be the kernels of the toric morphisms $\tilde{\pi}: \tilde{Z} \to \tilde{Z}$ and $\check{\pi}: \tilde{Z} \to \tilde{Z}$ respectively. Here are the basic features of our construction.

**Proposition 2.3.** In the above notation, the following statements hold.

(i) With $\tilde{Z}_0 := \tilde{Z} \setminus (\tilde{D}_1 \cup \ldots \cup \tilde{D}_m)$, the restriction $\check{\pi}: \tilde{Z}_0 \to \tilde{Z}$ is a geometric quotient for the action of $\check{H}$ on $\tilde{Z}_0$. 


(ii) The quasitorus \( \hat{H} \) acts freely on \( \hat{Z} \) and \( \hat{\pi}: \hat{Z} \rightarrow \hat{Z} \) is the geometric quotient for this action.

(iii) The factor group \( G := \hat{H}/\hat{H} \) is isomorphic to \( \mathbb{T}^s \) and it acts canonically on \( \hat{Z} \).

(iv) The \( G \)-action on \( \hat{Z} \) has infinite isotropy groups along \( \hat{D}_1, \ldots, \hat{D}_m \) and isotropy groups of order \( l_{ij} \) along \( \hat{D}_{ij} \).

(v) With \( \hat{Z}_0 := \hat{Z} \setminus (\hat{D}_1 \cup \ldots \cup \hat{D}_m) \), the restriction \( \hat{\beta}: \hat{Z}_0 \rightarrow Z \) is a geometric quotient for the action of \( G \) on \( \hat{Z} \).

Proof. The fact that \( \hat{\pi}: \hat{Z}_0 \rightarrow Z \) and \( \hat{\pi}: \hat{Z} \rightarrow \hat{Z} \) are geometric quotients is due to known characterizations of these notions in terms of (systems of) fans, see e.g. [1]. As a consequence, also \( \hat{\beta}: \hat{Z}_0 \rightarrow Z \) is a geometric quotient for the induced action of \( G = \hat{H}/\hat{H} \).

We verify the remaining part of (i). According to [2, Prop. II.1.4.2], the isotropy group of \( \hat{H} = \ker(\hat{\pi}) \) at a distinguished point \( z_\delta \in \hat{Z} \) has character group isomorphic to

\[
\ker(\hat{P}) \cap \text{lin}_Q(\hat{\delta}) \oplus \left( \hat{P} / (\text{lin}_Q(\hat{\delta}) \cap \hat{N}) / \hat{P} \right).
\]

By the choice of \( d \) and \( d' \), the map \( \hat{P} \) sends the primitive generators of the rays of \( \hat{\Sigma} \) to the primitive generators of the rays of \( \Sigma \). Thus we obtain that the isotropy of \( z_{\delta_{ij}} \) and \( z_{\delta_k} \) are all trivial.

We turn to (iii). With the dual lattices \( \check{M} \) of \( \check{N} \) and \( M \) of \( N \), we obtain the character groups of \( \hat{H} \) and \( \hat{\check{H}} \) and the factor group \( \hat{\check{H}}/\hat{H} \) as

\[
\chi(\hat{H}) = \check{E}/\check{P}^+(\check{M}), \quad \chi(\hat{\check{H}}) = \check{E}/\check{P}^+(\check{M}), \quad \chi(\hat{\check{H}}/\hat{H}) = \check{P}^+(\check{M})/\check{P}^+(\check{M}).
\]

By definition of the matrices of \( \hat{P} \) and \( \check{P} \), we have \( \check{P}^+(\check{M})/\check{P}^+(\check{M}) \cong \mathbb{Z}^s \). This implies \( G \cong \mathbb{T}^s \) as claimed.

To see (iv), note first that the group \( G \) equals \( \ker(\beta) \) and hence corresponds to the sublattice \( \ker(B) \subseteq \mathbb{Z}^{s+1} \). Thus, the isotropy group \( G_{z_\delta} \) for the distinguished point \( z_\delta \in \hat{Z} \) corresponding to \( \delta \in \hat{\Sigma} \) has character group isomorphic to

\[
\ker(B) \cap \text{lin}_Q(\delta) \oplus \left( B / (\text{lin}_Q(\delta) \cap \hat{N}) \right).
\]

Consequently, for \( \delta = \delta_k \) the isotropy group \( G_{z_\delta} \) is infinite and for \( \delta = \delta_{ij} \) it is of order \( l_{ij} \).

□

Now we come to the construction of the embedded variety. Let \( \delta \subseteq \check{E}_Q \) be the orthant generated by the basis vectors \( f_{ij} \) and \( f_k \). The associated affine toric variety \( \check{Z} \cong \mathbb{K}^{n+m} \) is the spectrum of the polynomial ring

\[
\mathbb{K}[\check{E} \cap \delta^\vee] = \mathbb{K}[T_{ij}, S_k; 0 \leq i \leq r, 1 \leq j \leq n_i, 1 \leq k \leq m] = S[S_1, \ldots, S_m].
\]

Moreover, \( \check{Z} \) contains \( Z \) as an open toric subvariety and the complement \( \check{Z} \setminus Z \) is the union of all toric orbits of codimension at least two. Regarding the trinomials \( g_{i,j,k} \in S \) as elements of the larger polynomial ring \( S[S_1, \ldots, S_m] \), we obtain an \( \check{H} \)-invariant subvariety

\[
\check{X} := V(g_{i,i+1,i+2}; 0 \leq i \leq r - 2) \subseteq \check{Z}.
\]

Proposition 2.4. Set \( \check{X} := \check{X} \cap \hat{Z} \). Consider the images \( \check{X} := \check{X}(\check{X}) \subseteq \hat{Z} \) and \( Y := \beta(\check{X}) \subseteq Z \).

(i) \( \check{X} \subseteq \hat{Z} \) is a normal closed \( G \)-invariant \( s+1 \)-dimensional variety, \( Y \subseteq Z \) is a closed non-separated curve and \( \kappa(Y) \subseteq \mathbb{P}_r \) is a line.

(ii) The intersection \( \check{C}_{ij} := \check{X} \cap \hat{D}_{ij} \) with the toric divisor \( \hat{D}_{ij} \subseteq \hat{Z} \) is a single \( G \)-orbit with isotropy group of order \( l_{ij} \).

(iii) The intersection \( \check{C}_k := \check{X} \cap \hat{D}_k \) with the toric divisor \( \hat{D}_k \subseteq \hat{Z} \) is a smooth rational prime divisor consisting of points with infinite \( G \)-isotropy.
(iv) For every point \( x \in \hat{X} \) not belonging to some \( \hat{C}_{ij} \) or to some \( \hat{C}_k \), the isotropy group \( G_x \) is trivial.

(v) The variety \( \hat{X} \) satisfies \( \Gamma(\hat{X}, \mathcal{O}) = \mathbb{K} \), its divisor class group and Cox ring are given by
\[
\text{Cl}(\hat{X}) \cong \hat{K}, \quad \mathcal{R}(\hat{X}) \cong R(A, n, L)[S_1, \ldots, S_m].
\]
The variables \( T_{ij} \) and \( S_k \) define pairwise nonassociated \( \hat{K} \)-prime elements in \( R(A, n, L)[S_1, \ldots, S_m] \).

(vi) There is a \( G \)-equivariant completion \( \hat{X} \subseteq X \) with a \( \mathbb{Q} \)-factorial projective variety \( X \) such that \( \mathcal{R}(X) = \mathcal{R}(\hat{X}) \) holds.

Proof. By the definition of \( \hat{P} \) and \( \hat{H} \), the closed subvariety \( \overline{\hat{X}} \subseteq \overline{Z} \) is invariant under the action of \( \hat{H} \). In particular, \( \hat{X} \) is \( \hat{H} \)-invariant and thus the image \( \hat{X} := \pi(\hat{X}) \) under the quotient map is closed as well. Moreover, the dimension of \( \hat{X} \) equals \( \dim(\hat{X}/H) = s + 1 \). Analogously we obtain closedness of \( Y = \pi(\hat{X}) \). The image \( \kappa(Y) = \kappa(\pi(\hat{X})) \) is given in \( \mathbb{P}_r \) by the equations
\[
\alpha_{ij}U_i + \alpha_{ki}U_j + \alpha_{ij}U_k = 0
\]
with the variables \( U_0, \ldots, U_r \) on \( \mathbb{P}_r \) corresponding to the toric divisors given by the rays \( \mathbb{Q}_{\geq 0}v_i \) and \( -\mathbb{Q}_{\geq 0}(v_0 + \ldots + v_{r-1}) \) of \( \Delta_i \); to see this, use that pulling back the above equations via \( \kappa \circ \pi \) gives the defining equations for \( \hat{X} \). Consequently \( \kappa(Y) \) is a projective line. This shows (i).

We turn to (ii). According to Proposition 2.3, the intersection \( \hat{X} \cap \hat{D}_{ij} \) is a single \( \hat{H} \)-orbit. Since \( \hat{\pi} : \hat{X} \to \hat{X} \) is a geometric quotient for the \( \hat{H} \)-action, we conclude that \( \hat{C}_{ij} = \hat{\pi}(\hat{D}_{ij}) \) is a single \( \hat{G} \)-orbit. Moreover, since \( \hat{H} \) acts freely, the isotropy group of \( G = \hat{H}/\hat{H} \) along \( \hat{C}_{ij} \) equals that of \( \hat{H} \) along \( \hat{D}_{ij} \) which, by Proposition 2.3 (iv), is of order \( l_{ij} \).

For (iii) note first that the restrictions \( \beta : \hat{D}_k \to \hat{Z} \) are isomorphisms onto the acting torus of \( \hat{Z} \). Moreover, the restricting \( \kappa \) gives an isomorphism of the acting tori of \( \hat{Z} \) and \( \mathbb{P}_r \). Consequently, \( \beta \) maps \( \hat{C}_k \) isomorphically onto the intersection of the line \( Y \) with the acting torus of \( \mathbb{P}_r \). Thus, \( \hat{C}_k \) is a smooth rational curve. Proposition 2.3 (iv) ensures that \( \hat{C}_k \) consists of fixed points. Assertion (iv) is clear.

We prove (v). From Proposition 2.1 we infer \( \Gamma(\hat{X}, \mathcal{O}) = \mathbb{K} \) which implies \( \Gamma(\hat{X}, \mathcal{O}) = \mathbb{K} \). The next step is to establish a surjection \( \hat{K} \to \text{Cl}(\hat{X}) \), where \( \hat{K} := \hat{E}/\hat{P}^+(\hat{M}) \) is the character group of \( \hat{H} \). Consider the push forward \( \hat{\pi}_* \) from the \( \hat{H} \)-invariant Weil divisors on \( \hat{X} \) to the Weil divisors on \( \hat{X} \) sending \( \hat{D} \) to \( \hat{\pi}(\hat{D}) \). For every \( \hat{w} \in \hat{K} \), we fix a \( \hat{w} \)-homogeneous rational function \( f_{\hat{w}} \in \mathbb{K}(\hat{X}) \) and define a map
\[
\mu : \hat{K} \to \text{Cl}(\hat{X}), \quad \hat{w} \mapsto [\hat{\pi}_*\text{div}(f_{\hat{w}})].
\]
One directly checks that this does not depend on the choice of the \( f_{\hat{w}} \) and thus is a well defined homomorphism. In order to see that it is surjective, note that due to Proposition 2.1 we obtain \( \hat{C}_{ij} = \hat{\pi}_*\text{div}(T_{ij}) \) and \( \hat{C}_k = \hat{\pi}_*\text{div}(T_k) \). The claim then follows from the observation that removing all \( \hat{C}_{ij} \) and \( \hat{C}_k \) from \( \hat{X} \) leaves the set \( \hat{X} \cap \mathbb{T}^{r+s} \) which is isomorphic to \( V \times \mathbb{T}^r \) with a proper open subset \( V \subseteq \kappa(Y) \) and hence has trivial divisor class group.

Now \( \text{[6] Theorem 1.3} \) shows that the Cox ring of \( \hat{X} \) is \( R(A, n, L)[S_1, \ldots, S_m] \) with the \( \text{Cl}(\hat{X}) \)-grading given by \( \text{deg}(T_{ij}) = [\hat{C}_{ij}] \) and \( \text{deg}(S_k) = [\hat{C}_k] \). Consequently, \( R(A, n, L)[S_1, \ldots, S_m] \) is factorially \( \text{Cl}(\hat{X}) \)-graded and thus also the finer \( \hat{K} \)-grading is factorial. Since \( \hat{H} \) acts freely on \( \hat{X} \), we can conclude \( \text{Cl}(\hat{X}) = \hat{K} \).

Finally, we construct a completion of \( \hat{X} \subseteq X \) as wanted in (vi). Choose any simplicial projective fan \( \Sigma' \) in \( N \) having the same rays as \( \Sigma \), see [7 Corollary 3.8]. The associated toric variety \( \hat{Z}' \) is projective and it is the good quotient of an open toric subset \( \hat{Z}' \subseteq \overline{Z} \) by the action of \( \hat{H} \). The closure \( X \) of \( \hat{X} \) in \( \hat{Z}' \) is projective and,
as the good quotient of the normal variety $\overline{X} \cap \overline{Z}'$, it is normal. By Proposition 2.2, the complement $X \setminus \hat{X}$ is of codimension at least two, which gives $R(X) = R(\hat{X})$. From [4, Cor. 4.13] we infer that $X$ is $\mathbb{Q}$-factorial. □

Remark 2.5. We may realize any given $R(A, n, L)$ as a subring of the Cox ring of a surface: For every $l_i = (l_{i1}, \ldots, l_{in})$ choose a tuple $d_i = (d_{i1}, \ldots, d_{in})$ of positive integers with $\gcd(l_{ij}, d_{ij}) = 1$ and $d_{i1}/l_{i1} < \ldots < d_{in}/l_{in}$. Then take

$$P := \begin{pmatrix} P & 0 & 0 \\ d & 1 & -1 \end{pmatrix}.$$  

We are ready to verify the main results. As mentioned, Theorem 1.3 follows from [6, Sec. 3]. Moreover, the statements of Theorem 1.4 are contained in Proposition 2.4.

Proof of Theorem 1.1. According to Proposition 2.1, the algebra $R(A, n, L)$ is normal and the $K$-grading is effective, pointed and of complexity one. By Proposition 2.4, the algebra $R(A, n, L)[S_1, \ldots, S_m]$ is a Cox ring and hence it is factorially graded, see [4]. Clearly the graded subring $R(A, n, L)$ inherits the latter property. Now suppose that $(A, n, L)$ is a sincere triple. If $K$ is torsion free, then $K$-factoriality of $R(A, n, L)$ implies factoriality, see [3, Theorem 4.2]. Conversely, if $R(A, n, L)$ is factorial, then the generators $T_{ij}$ are prime by Proposition 2.2. From [5, Lemma 1.5], we then infer that the numbers $\gcd(l_{i1}, \ldots, l_{in})$ are pairwise coprime. This implies that $\tilde{P} : F \to N$ is surjective and thus $K$ is torsion free. □

Proof of Theorem 1.5. According to [6, Theorem 1.3], the Cox ring $R(X)$ is isomorphic to a ring $R(A, n, L)[S_1, \ldots, S_m]$ with a grading by $\overline{K} := \text{Cl}(X)$ such that the variables $T_{ij}$ and $S_k$ are homogeneous. In particular, $X$ is the quotient by the action of $\tilde{H} = \text{Spec} \mathbb{K}[\overline{K}]$ on an open subset $\hat{X}$ of $X = V(g_{i,i+1,i+2}; 0 \leq i \leq r - 2) \subseteq \mathbb{Z}$.

For $r < 2$, the variety $X$ is toric, we may assume that $T$ acts as a subtorus of the big torus and the assertion follows by standard toric geometry. So, let $r \geq 2$. By construction, the grading of $R(A, n, L)[S_1, \ldots, S_m]$ by $\overline{K} := K \times \mathbb{Z}^m$ is the finest possible such that all variables $T_{ij}$ and $S_k$ are homogeneous. Consequently, we have exact sequences of abelian groups fitting into a commutative diagram

\begin{equation}
0 \to \tilde{M} \to \tilde{E} \to \overline{K} \to 0
\end{equation}

\begin{equation}
0 \to \tilde{M}/\tilde{M} \to 0 \to 0
\end{equation}
In particular we extract from this the following two commutative triangles, where the second one is obtained by dualizing the first one

\[ \begin{array}{ccc}
\mathcal{E} & \xrightarrow{\rho^*} & \mathcal{M} \\
\downarrow \ & & \downarrow \\
\mathcal{F} & \xrightarrow{\rho} & \mathcal{N}
\end{array} \]

We claim that the kernel \( \tilde{K} \) is free. Consider \( \mathcal{H} := \text{Spec} \mathbb{K}[\mathcal{K}] \) and the isotropy group \( \mathcal{H}_{ij} \subseteq \mathcal{H} \) of a general point \( \tilde{x}(i, j) \in \tilde{X} \cap V(T_{ij}) \). Then we have exact sequences

\[ \begin{array}{cccc}
1 & \longrightarrow & \mathcal{H}/\mathcal{H}_{ij} & \longrightarrow \mathcal{H} & \longrightarrow \mathcal{H}_{ij} & \longrightarrow & 1 \\
& & 0 & \longrightarrow & \mathcal{K}(i, j) & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{K}/\mathcal{K}(i, j) & \longrightarrow & 0
\end{array} \]

where the second one arises from the first one by passing to the character groups. Note that the subgroup \( \mathcal{K}(i, j) \subseteq \mathcal{K} \) is given by

(2.4) \( \mathcal{K}(i, j) = \text{lin}_{\mathbb{Z}}(\deg T_{kl}; (k, l) \neq (i, j)) + \text{lin}_{\mathbb{Z}}(\deg T_p; 1 \leq p \leq m) \).

Now [6, Theorem 1.3] tells us that each variable \( T_{ij} \) defines a \( \mathcal{K} \)-prime element in \( \mathcal{R}(X) \) and thus its divisor is \( \mathcal{H} \)-prime. Consequently, \( \mathcal{H}/\mathcal{H}_{ij} \) is connected and has a free character group

\[ \chi(\mathcal{H}/\mathcal{H}_{ij}) = \tilde{K}(i, j) := \mathcal{K} \cap \mathcal{K}(i, j). \]

Mimicking equation (2.4), we define a subgroup \( \tilde{K}(i, j) \subseteq \mathcal{K} \) fitting into a commutative net of exact sequences

(2.5)

\[ \begin{array}{cccc}
0 & \longrightarrow & \tilde{K}(i, j) & \longrightarrow & \mathcal{K} & \longrightarrow & \tilde{K}/\mathcal{K}(i, j) & \longrightarrow & 0 \\
& & 0 & \longrightarrow & \tilde{K}(i, j) & \longrightarrow & \mathcal{K} & \longrightarrow & \tilde{K}/\mathcal{K}(i, j) & \longrightarrow & 0 \\
& & 0 & \longrightarrow & \tilde{K}(i, j) & \longrightarrow & \mathcal{K} & \longrightarrow & \tilde{K}/\mathcal{K}(i, j) & \longrightarrow & 0 \\
& & 0 & \longrightarrow & \tilde{K}(i, j) & \longrightarrow & \mathcal{K} & \longrightarrow & \tilde{K}/\mathcal{K}(i, j) & \longrightarrow & 0
\end{array} \]

By general properties of Cox rings [4, Prop. 2.2], we must have \( \tilde{K}/\tilde{K}(i, j) = 0 \) and thus we can conclude

(2.6)

\[ \tilde{K}/\mathcal{K}(i, j) \cong \mathbb{Z}/l_{ij}\mathbb{Z}. \]

Consider again \( \tilde{x}(i, j) \in V(T_{ij}) \cap \tilde{X} \), set \( x(i, j) = p_X(\tilde{x}(i, j)) \) and let \( T \) denote the torus acting on \( X \). Then [6, Prop. 2.6] and its proof provide us with a commutative diagram

\[ \begin{array}{ccc}
\mathcal{H} & \supseteq & \mathcal{H}_{ij} \\
\uparrow & & \uparrow \\
\mathcal{T} & \supseteq & \mathcal{T}_{x(i, j)}
\end{array} \]
where $\tilde{H} = \bar{H}/\dot{H}$ and $\tilde{H}_{ij} = \bar{H}/\dot{H}\bar{H}_{ij}$. Using (2.6) and passing to the character groups we arrive at a commutative diagram

$$
\begin{array}{cccccc}
0 & \rightarrow & \tilde{K}(i,j) & \rightarrow & \tilde{K} & \rightarrow & \mathbb{Z}/l_{ij}\mathbb{Z} & \rightarrow & 0 \\
& & \downarrow & & \cong & & \downarrow \\
& & \mathbb{X}(T) & \rightarrow & \mathbb{Z}/l_{ij}\mathbb{Z} & \rightarrow & 0
\end{array}
$$

with exact rows. As seen before, the group $\tilde{K}(i,j)$ is free abelian. Consequently, also $\tilde{K}$ must be free abelian.

Now the snake lemma tells us that $\dot{M}/M$ is free as well. In particular, the first vertical sequence of (2.2) splits. Thus, we obtain the desired matrix presentation of $\dot{P}$ from rewriting the second commutative triangle of (2.3) as

$$
\begin{array}{ccc}
\tilde{F} & \rightarrow & N \oplus \hat{N}/M^\perp \\
\downarrow & & \downarrow \\
\hat{P} & \rightarrow & N
\end{array}
$$

References

[1] A. A’Campo-Neuen, J. Hausen: Toric prevarieties and subtorus actions. Geom. Dedicata 87 (2001), no. 1-3, 35–64.
[2] I. Arzhantsev, U. Derenthal, J. Hausen, A. Laface: Cox rings, arXiv:1003.4229, see also the authors’ webpages.
[3] D.F. Anderson: Graded Krull domains. Comm. Algebra 7 (1979), no. 1, 79–106.
[4] J. Hausen: Cox rings and combinatorics II. Mosc. Math. J. 8 (2008), no. 4, 711–757.
[5] J. Hausen, E. Herppich, H. Süss: Multigraded factorial rings and Fano varieties with torus action. Documenta Math. 16 (2011), 71–109.
[6] J. Hausen, H. Süss: The Cox ring of an algebraic variety with torus action. Advances Math. 225 (2010), 977–1012.
[7] T. Oda, H.S. Park: Linear Gale transforms and Gelfand-Kapranov-Zelevinskij decompositions. Tohoku Math. J. (2) 43 (1991), no. 3, 375–399.