Formulation of Cell Petri Nets

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Summary. Based on the Petri net definitions and theorems already formalized in the Mizar article [13], in this article we were able to formalize the definition of cell Petri nets. It is based on [12]. Colored Petri net has already been defined in [11]. In addition, the conditions of the firing rule and the colored set to this definition, that defines the cell Petri nets are further extended to CPNT:i further. The synthesis of two Petri nets was introduced in [11] and in this work the definition is extended to produce the synthesis of a family of colored Petri nets. Specifically, the extension to a CPNT family is performed by specifying how to link the outbound transitions of each colored Petri net to the place elements of other nets to form a neighborhood relationship. Finally, the activation of colored Petri nets was formalized.

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The notation and terminology used in this paper have been introduced in the following articles: [1], [15], [10], [5], [6], [7], [17], [2], [3], [4], [8], [16], [13], [11], [19], [14], [18], and [9].

1. Preliminaries

Let \( I \) be a non empty set and \( C_1 \) be a many sorted set indexed by \( I \). We say that \( C_1 \) is colored Petri net family-like if and only if

(Def. 1) Let us consider an element \( i \) of \( I \). Then \( C_1(i) \) is a colored place/transition net.

Note that there exists a many sorted set indexed by \( I \) which is colored Petri net family-like.

A colored Petri net family of \( I \) is a colored Petri net family-like many sorted set indexed by \( I \). Let \( C_1 \) be a colored Petri net family of \( I \) and \( i \) be an element
of \( I \). One can check that the functor \( C_1(i) \) yields a colored place/transition net. Let \( C_2 \) be a colored Petri net family of \( I \). We say that \( C_2 \) is disjoint valued if and only if

(Def. 2) Let us consider elements \( i, j \) of \( I \). Suppose \( i \neq j \). Then

(i) the carrier of \( C_2(i) \) misses the carrier of \( C_2(j) \),

(ii) the carrier’ of \( C_2(i) \) misses the carrier’ of \( C_2(j) \).

Now we state the propositions:

(1) Let us consider a set \( I \) and many sorted sets \( F, D, R \) indexed by \( I \). Suppose

(i) for every element \( i \) such that \( i \in I \) there exists a function \( f \) such that

\( f = F(i) \) and \( \text{dom } f = D(i) \) and \( \text{rng } f = R(i) \), and

(ii) for every elements \( i, j \) and for every functions \( f, g \) such that \( i, j \in I \)

and \( i \neq j \) and \( f = F(i) \) and \( g = F(j) \) holds \( \text{dom } f \) misses \( \text{dom } g \).

Then there exists a function \( G \) such that

(iii) \( G = \bigcup \text{rng } F \), and

(iv) \( \text{dom } G = \bigcup \text{rng } D \), and

(v) \( \text{rng } G = \bigcup \text{rng } R \), and

(vi) for every elements \( i, x \) and for every function \( f \) such that \( i \in I \) and

\( f = F(i) \) and \( x \in \text{dom } f \) holds \( G(x) = f(x) \).

**Proof:** For every element \( z \) such that \( z \in \bigcup \text{rng } F \) there exist elements \( x, y, i \) such that \( z = \langle x, y \rangle \) and \( z \in F(i) \) and \( i \in I \). For every element \( z \) such that \( z \in \bigcup \text{rng } F \) there exist elements \( x, y \) such that \( z = \langle x, y \rangle \). Reconsider \( G = \bigcup \text{rng } F \) as a binary relation. \( G \) is a function. For every element \( x, x \in \text{dom } G \) iff \( x \in \bigcup \text{rng } D \) by [5, (3)]. For every element \( x, x \in \text{rng } G \) iff \( x \in \bigcup \text{rng } R \) by [5, (3)]. For every elements \( i, x \) and for every function \( f \) such that \( i \in I \) and \( f = F(i) \) and \( x \in \text{dom } f \) holds \( G(x) = f(x) \) by [5, (1), (3)]. □

(2) Let us consider a set \( I \) and many sorted sets \( Y, Z \) indexed by \( I \). Suppose elements \( i, j \). If \( i, j \in I \) and \( i \neq j \), then \( Y(i) \cap Z(j) = \emptyset \). Then \( \bigcup (Y \setminus Z) = \bigcup Y \setminus \bigcup Z \). **Proof:** Set \( X = Y \setminus Z \). For every element \( x, x \in \bigcup \text{rng } X \) iff \( x \in \bigcup \text{rng } Y \setminus \bigcup \text{rng } Z \) by [5, (3)]. □

(3) Let us consider a set \( I \) and many sorted sets \( X, Y, Z \) indexed by \( I \). Suppose

(i) \( X \subseteq Y \setminus Z \), and

(ii) for every elements \( i, j \) such that \( i, j \in I \) and \( i \neq j \) holds \( Y(i) \cap Z(j) = \emptyset \).

Then \( \bigcup X \subseteq \bigcup Y \setminus \bigcup Z \). The theorem is a consequence of (2).
2. Synthesis of CPNT and I

Let $I$ be a non trivial set. The functor $\text{XorDelta } I$ yielding a non empty set is defined by the term

(Def. 3) $\{(i, j), \text{ where } i, j \text{ are elements of } I : i \neq j\}.$

Now we state the proposition:

(4) Let us consider a non trivial finite set $I$ and a colored Petri net family $C_2$ of $I$. Then $\bigcup\{(\text{the carrier of } C_2(j))^{\text{Outbds}(C_2(i))}, \text{ where } i, j \text{ are elements of } I : i \neq j\}$ is not empty.

Let $I$ be a non trivial finite set and $C_2$ be a colored Petri net family of $I$. A connecting mapping of $C_2$ is a many sorted set indexed by $\text{XorDelta } I$ and is defined by

(Def. 4) (i) $\text{rng } it \subseteq \bigcup\{(\text{the carrier of } C_2(j))^{\text{Outbds}(C_2(i))}, \text{ where } i, j \text{ are elements of } I : i \neq j\}$, and

(ii) for every elements $i, j$ of $I$ such that $i \neq j$ holds $it((i, j))$ is a function from $\text{Outbds}(C_2(i))$ into the carrier of $C_2(j)$.

Now we state the proposition:

(5) Let us consider colored place/transition nets $C_4, C_5$, a function $O_1$ from $\text{Outbds } C_4$ into the carrier of $C_5$, and a function $q_1$. Suppose

(i) $\text{dom } q_1 = \text{Outbds } C_4$, and

(ii) for every transition $t_1$ of $C_4$ such that $t_1$ is outbound holds $q_1(t_1)$ is a function from the thin cylinders of the colored set of $C_4$ and $\ast\{t_1\}$ into the thin cylinders of the colored set of $C_4$ and $O_1^\circ t_1$.

Then $q_1 \in (\bigcup\{(\text{the thin cylinders of the colored set of } C_4 \text{ and } O_1^\circ t_1)^\alpha, \text{ where } t_1 \text{ is a transition of } C_4 : t_1 \text{ is outbound}\})^{\text{Outbds } C_4}$, where $\alpha$ is the thin cylinders of the colored set of $C_4$ and $\ast\{t_1\}$.

Let $I$ be a non trivial finite set, $C_2$ be a colored Petri net family of $I$, and $O$ be a connecting mapping of $C_2$. A connecting firing rule of $O$ is a many sorted set indexed by $\text{XorDelta } I$ and is defined by

(Def. 5) Let us consider elements $i, j$ of $I$. Suppose $i \neq j$. Then there exists a function $O_2$ from $\text{Outbds}(C_2(i))$ into the carrier of $C_2(j)$ and there exists a function $q_2$ such that $q_2 = it((i, j))$ and $O_2 = O((i, j))$ and $\text{dom } q_2 = \text{Outbds}(C_2(i))$ and for every transition $t_1$ of $C_2(i)$ such that $t_1$ is outbound holds $q_2(t_1)$ is a function from the thin cylinders of the colored set of $C_2(i)$ and $\ast\{t_1\}$ into the thin cylinders of the colored set of $C_2(i)$ and $O_2^\circ t_1$. 
3. Extension to a Family of Colored Petri Nets

Let $I$ be a non trivial finite set, $C_2$ be a colored Petri net family of $I$, $O$ be a connecting mapping of $C_2$, and $q$ be a connecting firing rule of $O$. Assume $C_2$ is disjoint valued and for every elements $i, j_1, j_2$ of $I$ such that $i \neq j_1$ and $i \neq j_2$ and there exist elements $x, y_1, y_2$ such that $\langle x, y_1 \rangle \in q(\langle i, j_1 \rangle)$ and $\langle x, y_2 \rangle \in q(\langle i, j_2 \rangle)$ holds $j_1 = j_2$. The functor synthesis $q$ yielding a strict colored place/transition net is defined by

(Def. 6) There exist many sorted sets $P, T, S_1, T_1, C_3, F$ indexed by $I$ and there exist functions $U, U_1$ such that for every element $i$ of $I$, $P(i) = \text{the carrier of } C_2(i)$ and $T(i) = \text{the carrier' of } C_2(i)$ and $S_1(i) = \text{the S-T arcs of } C_2(i)$ and $T_1(i) = \text{the T-S arcs of } C_2(i)$ and $C_3(i) = \text{the colored set of } C_2(i)$ and $F(i) = \text{the firing rule of } C_2(i)$ and $U = \bigcup \text{rng } F$ and $U_1 = \bigcup \text{rng } q$ and the carrier of $it = \bigcup \text{rng } P$ and the carrier' of $it = \bigcup \text{rng } T$ and the S-T arcs of $it = \bigcup \text{rng } S_1$ and the T-S arcs of $it = \bigcup \text{rng } T_1 \cup \bigcup \text{rng } O$ and the colored set of $it = \bigcup \text{rng } C_3$ and the firing rule of $it = U + U_1$.

4. Definition of Cell Petri Nets

Let $I$ be a non empty finite set and $C_2$ be a colored Petri net family of $I$. We say that $C_2$ is cell Petri nets if and only if

(Def. 7) There exists a function $N$ from $I$ into $2^{\text{rng } C_2}$ such that for every element $i$ of $I$, $N(i) = \{C_2(j) \mid j \text{ is an element of } I : j \neq i\}$.

Let $N$ be a function from $I$ into $2^{\text{rng } C_2}$ and $O$ be a connecting mapping of $C_2$. We say that $(N, O)$ is cell Petri nets if and only if

(Def. 8) Let us consider an element $i$ of $I$. Then $N(i) = \{C_2(j) \mid j \text{ is an element of } I : j \neq i \text{ and there exists a transition } t \text{ of } C_2(i) \text{ and there exists an element } s \text{ such that } \langle t, s \rangle \in O(\langle i, j \rangle)\}$.

Now we state the proposition:

(6) Let us consider a non trivial finite set $I$, a colored Petri net family $C_2$ of $I$, a function $N$ from $I$ into $2^{\text{rng } C_2}$, and a connecting mapping $O$ of $C_2$. Suppose

(i) $C_2$ is one-to-one, and

(ii) $(N, O)$ is cell Petri nets.

Let us consider an element $i$ of $I$. Then $C_2(i) \notin N(i)$. 

5. Activation of Petri Nets

Let $C_6$ be a colored place/transition net structure. We say that $C_6$ has nontrivial colored set if and only if

(Def. 9) The colored set of $C_6$ is not trivial.

One can verify that there exists a strict colored-PT-net-like colored Petri net which has nontrivial colored set.

Let $C_2$ be a colored place/transition net with nontrivial colored set. One can verify that the colored set of $C_2$ is non trivial.

Let $C_6$ be a colored place/transition net with nontrivial colored set, $S$ be a subset of the carrier of $C_6$, and $D$ be a thin cylinder of the colored set of $C_6$ and $S$. A color threshold of $D$ is a function from $\text{loc} D$ into the colored set of $C_6$.

Let $C_6$ be a colored place/transition net. A color count of $C_6$ is a function from the colored set of $C_6$ into $\mathbb{N}$. The colored states of $C_6$ yielding a non empty set is defined by the term

(Def. 10) the set of all $e$ where $e$ is a color count of $C_6$.

A colored state of $C_6$ is a function from $C_6$ into the colored states of $C_6$.

From now on $C_6$ denotes a colored place/transition net with nontrivial colored set, $m$ denotes a colored state of $C_6$, and $t$ denotes an element of the carrier of $C_6$.

Let $C_6$ be a colored place/transition net with nontrivial colored set, $m$ be a colored state of $C_6$, and $p$ be a place of $C_6$. Observe that the functor $m(p)$ yields a color count of $C_6$. Let $m_1$ be a color count of $C_6$ and $x$ be an element. Let us observe that the functor $m_1(x)$ yields an element of $\mathbb{N}$. Let us consider $C_6$, $m$, and $t$. Let $D$ be a thin cylinder of the colored set of $C_6$ and $\{t\}$ and $C_7$ be a color threshold of $D$. We say that $t$ is firable on $m$ and $C_7$ if and only if

(Def. 11) (i) (the firing rule of $C_6$)\((t, D)\) $\neq \emptyset$, and

(ii) for every place $p$ of $C_6$ such that $p \in \text{loc} D$ holds $1 \leq m(p)(C_7(p))$.

The firable set on $m$ and $t$ yielding a set is defined by the term

(Def. 12) $\{D, \text{ where } D \text{ is a thin cylinder of the colored set of } C_6 \text{ and } \{t\} : \text{there exists a color threshold } C_7 \text{ of } D \text{ such that } t \text{ is firable on } m \text{ and } C_7\}$.

Now we state the proposition:

(7) Let us consider a thin cylinder $D$ of the colored set of $C_6$ and $\{t\}$. Then there exists a color threshold $C_7$ of $D$ such that $t$ is firable on $m$ and $C_7$ if and only if $D \in \text{ the firable set on } m \text{ and } t$.

Let us consider $C_6$, $m$, and $t$. Let $D$ be a thin cylinder of the colored set of $C_6$ and $\{t\}$, $C_7$ be a color threshold of $D$, and $p$ be an element of $C_6$. Assume $t$ is firable on $m$ and $C_7$. The Petri subtraction($C_7, m, p)\) yielding a function from the colored set of $C_6$ into $\mathbb{N}$ is defined by

(Def. 13) Let us consider an element $x$ of the colored set of $C_6$. Then
(i) if \( p \in \text{loc} \, D \) and \( x = C_7(p) \), then \( it(x) = m(p)(x) - 1 \), and

(ii) if it is not true that \( p \in \text{loc} \, D \) and \( x = C_7(p) \), then \( it(x) = m(p)(x) \).

Let \( D \) be a thin cylinder of the colored set of \( C_6 \) and \( \{t\} \). The Petri addition\((C_7, m, p)\) yielding a function from the colored set of \( C_6 \) into \( \mathbb{N} \) is defined by

**Def. 14** Let us consider an element \( x \) of the colored set of \( C_6 \). Then

(i) if \( p \in \text{loc} \, D \) and \( x = C_7(p) \), then \( it(x) = m(p)(x) + 1 \), and

(ii) if it is not true that \( p \in \text{loc} \, D \) and \( x = C_7(p) \), then \( it(x) = m(p)(x) \).

Let \( D \) be a thin cylinder of the colored set of \( C_6 \) and \( *\{t\} \) and \( E \) be a thin cylinder of the colored set of \( C_6 \) and \( \{t\} \). Let \( C_{10} \) be a color threshold of \( E \). The firing result\((C_7, C_{10}, m, p)\) yielding a function from the colored set of \( C_6 \) into \( \mathbb{N} \) is defined by the term

\[
\begin{align*}
\text{the Petri subtraction}(C_7, m, p), & \quad \text{if } t \text{ is firable on } m \text{ and } C_7, \text{ and } p \in \text{loc} \, D \setminus \text{loc} \, E, \\
\text{the Petri addition}(C_{10}, m, p), & \quad \text{if } t \text{ is firable on } m \text{ and } C_7, \text{ and } p \in \text{loc} \, E \setminus \text{loc} \, D, \\
m(p), & \quad \text{otherwise}.
\end{align*}
\]

**Def. 15** Let us consider a thin cylinder \( D_1 \) of the colored set of \( C_6 \) and \( *\{t\} \), a thin cylinder \( D_2 \) of the colored set of \( C_6 \) and \( \{t\} \), a color threshold \( C_8 \) of \( D_1 \), a color threshold \( C_9 \) of \( D_2 \), an element \( x \) of the colored set of \( C_6 \), and an element \( p \) of \( C_6 \). Now we state the propositions:

(8) \( m(p)(x) - 1 \leq (\text{the firing result}(C_8, C_9, m, p))(x) \leq m(p)(x) + 1 \).

(9) If \( t \) is outbound, then \( m(p)(x) - 1 \leq (\text{the firing result}(C_8, C_9, m, p))(x) \leq m(p)(x) \).

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