The $SU(2) \times SU(2)$ sector in the string dual of $\mathcal{N} = 6$ superconformal Chern-Simons theory

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Abstract

We examine the string dual of the recently constructed $\mathcal{N} = 6$ superconformal Chern-Simons theory of Aharony, Bergman, Jafferis and Maldacena (ABJM theory). We focus in particular on the $SU(2) \times SU(2)$ sector. We find a sigma-model limit in which the resulting sigma-model is two Landau-Lifshitz models added together. We consider a Penrose limit for which we can approach the $SU(2) \times SU(2)$ sector. Finally, we find a new Giant Magnon solution in the $SU(2) \times SU(2)$ sector corresponding to one magnon in each $SU(2)$. We put these results together to find the full magnon dispersion relation and we compare this to recently found results for ABJM theory at weak coupling.
1 Introduction and summary

For the last decade, the duality between $\mathcal{N} = 4$ superconformal Yang-Mills (SYM) theory and type IIB string theory on $\text{AdS}_5 \times S^5$ have been celebrated as the one example of an exact duality between gauge theory and string theory. Recently, developments, initiated by Bagger, Lambert and Gustavsson [1], in finding the superconformal world-volume theory for multiple M2-branes led Aharony, Bergman, Jafferis and Maldacena to construct a new $\mathcal{N} = 6$ superconformal Chern-Simons theory (ABJM theory) [2] which should be the world-volume theory of multiple M2-brane on $\mathbb{C}^4/\mathbb{Z}_k$. Based on this they conjectured a new duality between ABJM theory and type IIA string theory on $\text{AdS}_4 \times \mathbb{C}P^3$. This is a new exact duality between gauge theory and string theory [3].

The ABJM theory consists of two Chern-Simons theories of level $k$ and $-k$ and each with gauge group $SU(N)$, which means that the total gauge symmetry is $SU(N) \times SU(N)$. In addition it has two pairs of chiral superfields transforming in the bifundamental representations of $SU(N) \times SU(N)$. The R-symmetry is $SU(4)$ in accordance with the $\mathcal{N} = 6$ supersymmetry of the theory. It was observed in [2] that one can define a ’t Hooft coupling $\lambda = N/k$ and that in the ’t Hooft limit $N \rightarrow \infty$ with $\lambda$ fixed one has a continuous coupling $\lambda$ and that the ABJM theory is weakly coupled for $\lambda \ll 1$. The ABJM theory is conjectured to be dual to M-theory on $\text{AdS}_4 \times S^7/\mathbb{Z}_k$ with $N$ units of four-form flux. In the limit of large $k$ one has roughly speaking that $S^7/\mathbb{Z}_k \cong \mathbb{C}P^3 \times S^1$ which thus means that ABJM theory in the ’t Hooft limit is dual to type IIA string theory on $\text{AdS}_4 \times \mathbb{C}P^3$. This duality is valid for $\lambda \gg 1$ and the type IIA string description holds when $k \gg N^{1/5}$.

Having this new $\text{AdS}_4/\text{CFT}_3$ duality naturally brings up the question of how similar it is with the $\text{AdS}_5/\text{CFT}_4$ duality. We see that despite the fact that $k$ is integer valued we can still define a continuous ’t Hooft coupling and we have a weak/strong duality between the ABJM theory and type IIA string theory. Furthermore, Minahan and Zarembo [8] have recently provided evidence that ABJM theory is integrable to second order in $\lambda$ by finding an integrable $SU(4)$ spin chain. This thus brings the hope that ABJM theory is integrable, just as has been seen in the case of $\mathcal{N} = 4$ SYM theory [9]. However, there is one notable difference between the $\text{AdS}_4/\text{CFT}_3$ and $\text{AdS}_5/\text{CFT}_4$ dualities, namely that while one has the maximal number of 32 supercharges in the $\text{AdS}_5/\text{CFT}_4$ case, the number of supercharges in the $\text{AdS}_4/\text{CFT}_3$ duality is 24. This means that it can be more challenging to interpolate from weak to strong coupling in the $\text{AdS}_4/\text{CFT}_3$ duality.

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1 The construction of the $\mathcal{N} = 6$ superconformal Chern-Simons theory is based on [3]. For papers considering the Bagger-Lambert-Gustavsson theory see [4]. For papers considering the ABJM theory see [5, 6, 7, 8].
In this paper we study further the question of integrability in the new AdS$_4$/CFT$_3$ duality. We do this by investigating the $SU(2) \times SU(2)$ sector of the ABJM theory on the string side. For $\lambda \ll 1$ Minahan and Zarembo found that there is a decoupled $SU(2) \times SU(2)$ sector in the $SU(4)$ spin chain [8]. In this sector the spin chain Hamiltonian is that of two $XXX_{1/2}$ Heisenberg spin chains.

We find on the string side a limit of type IIA string theory on AdS$_4 \times \mathbb{C}P^3$ that corresponds to the $SU(2) \times SU(2)$ sector. In this limit the string sigma-model becomes that of two Landau-Lifshitz models, thus in accordance with the results of [8]. As one might expect, this means that the S-matrices matches up to second-order corrections for small momenta. We also find a dispersion relation of the form

$$\Delta = \frac{1}{2} + \frac{\lambda}{2} p^2$$

(1)

This dispersion relation holds in the limit of $p \to 0$ with large but fixed $\lambda$. However, it does not match the one found by Minahan and Zarembo in [8].

To examine further the dispersion relation on the string theory side we consider a Penrose limit corresponding to the $SU(2) \times SU(2)$ sector (see [7] for another Penrose limit dual to ABJM theory). We find in particular the dispersion relation for an $SU(2) \times SU(2)$ magnon

$$\Delta = \sqrt{\frac{1}{4} + \frac{\lambda}{2} p^2}$$

(2)

This holds for $p \to 0$ with $\lambda p^2$ fixed. This result is consistent with our sigma-model limit and is furthermore consistent with the Penrose limit of [7].

We find moreover a new Giant Magnon solution in the $SU(2) \times SU(2)$ sector of type IIA string theory on AdS$_4 \times \mathbb{C}P^3$, following the Giant Magnon solutions in AdS$_5 \times S^5$ [10, 11]. The Giant Magnon solution in the $SU(2) \times SU(2)$ sector that we find has the interesting feature that it consists of two Giant Magnons, one for each $SU(2)$. As for the Hofman-Maldacena Giant Magnon solution on AdS$_5 \times S^5$, this is a closed string solution with open boundary conditions in two azimuthal directions[2].

From our new Giant Magnon solution we get the following result for the dispersion relation (for a single magnon)

$$\Delta = \sqrt{2\lambda} \left| \sin \frac{p}{2} \right|$$

(3)

which holds for $\lambda \to \infty$ and fixed $p$. This result is consistent with the Penrose limit result.

Combining our results from the sigma-model limit, the Penrose limit and the Giant Magnon analysis, we find the dispersion relation

$$\Delta = \sqrt{\frac{1}{4} + 2\lambda \sin^2 \left( \frac{p}{2} \right)}$$

(4)

for $\lambda \gg 1$. For $\lambda \ll 1$ the following dispersion relation has instead been found [8]

$$\Delta = \frac{1}{2} + 4\lambda^2 \sin^2 \left( \frac{p}{2} \right)$$

(5)

It is evident that (4) and (5) cannot match, as one clearly can see in the limit of small momenta.

For the analogous question in the AdS$_5$/CFT$_4$ duality it was found by Beisert that the form of the magnon dispersion relation is fixed up to a function depending only of the ’t Hooft coupling [14]. Assuming that this symmetry argument can be generalized to the AdS$_4$/CFT$_3$ duality, this leads to the proposal that the magnon dispersion relation in the $SU(2) \times SU(2)$ sector for any value of $\lambda$ is of the form

$$\Delta = \sqrt{\frac{1}{4} + h(\lambda) \sin^2 \left( \frac{p}{2} \right)}$$

(6)

2It would be interesting to see if by considering an orbifold of $\mathbb{C}P^3$ it would be possible to identify the string endpoints to make of this a legitimate closed string solution, as was done in [12, 13] for the AdS$_5 \times S^5$ Giant Magnon.
where $h(\lambda)$ is a function of $\lambda$. Then our computations, together with (5), shows that
\begin{equation}
\begin{cases}
4\lambda^2 + \mathcal{O}(\lambda^4) & \text{for } \lambda \ll 1 \\
2\lambda + \mathcal{O}(\sqrt{\lambda}) & \text{for } \lambda \gg 1
\end{cases}
\end{equation}

Thus, $h(\lambda)$ is a non-trivial function of the coupling. This is in contrast with the AdS$_5$/CFT$_4$ duality where the same dispersion relation holds for weak and strong coupling. We believe that this difference is due to the lower amount of supersymmetry of the AdS$_4$/CFT$_3$ duality which indeed makes it more challenging to connect the two sides of the duality.

Note added: After completing this paper, Ref. [16] appeared on the arXive. This paper has substantial overlap with our sections 5 and 6.

2 ABJM theory, its spin chain description and its string dual

The ABJM theory, which is an $\mathcal{N} = 6$ $SU(N) \times SU(N)$ superconformal Chern-Simons theory at level $k$, has two pairs of chiral superfields, each transforming in a bifundamental representation of $SU(N) \times SU(N)$. The theory has an explicit $SU(2) \times SU(2)$ R-symmetry with one pair of superfields being in the spin 1/2 representation of the first $SU(2)$ and the other pair in the second $SU(2)$. Furthermore, the R-symmetry of the theory has been shown to be enhanced to $SU(4)$ (further enhanced to $SO(8)$ for $k = 1, 2$).

ABJM introduced a ’t Hooft coupling $\lambda = N/k$. In the ’t Hooft limit $N \to \infty$ with $\lambda$ fixed, $\lambda$ is a continuous parameter. For $\lambda \ll 1$ the ABJM theory is weakly coupled.

We consider the ABJM theory on $\mathbb{R} \times S^2$, thus the global bosonic symmetry group is $SO(2,3) \times SU(4)$. By the state/operator correspondence a state for the theory on $\mathbb{R} \times S^2$ is mapped to an operator for the theory on $\mathbb{R}^3$ with the scaling dimension $\Delta$ given by the energy in units of the two-sphere radius.

Focusing on the scalars in the theory we have a pair of complex scalars $A_1, A_2$ which transform in the $N \times \bar{N}$ representation of $SU(N) \times SU(N)$ and a pair of complex scalars $B_1, B_2$ which transform in the $\bar{N} \times N$ representation. One can group these scalars into multiplets of the R-symmetry group $SU(4)$
\begin{equation}
Z^a = (A_1, A_2, B_1^\dagger, B_2^\dagger), \quad Z_\alpha = (A_1^\dagger, A_2^\dagger, B_1, B_2)
\end{equation}
with $Z^a$ transforming in the fundamental representation and $Z_\alpha$ in the anti-fundamental representation of $SU(4)$. All scalars have conformal dimension $\Delta = 1/2$ and transform in the trivial representation of the $SO(3)$ symmetry.

We have in addition a covariant derivative $D_\mu$ transforming in the spin 1 representation of $SO(3)$ and in the trivial representation of $SU(4)$. The scaling dimension is $\Delta = 1$. We write the three components as $D_-, D_0$ and $D_+$ according to the Cartan generator $S$ of $SO(3)$ (i.e. with eigenvalues $-1$, $0$ and $1$).

The fermions of the ABJM theory are the superpartners of the scalars, thus they transform in the fundamental and anti-fundamental representations of $SU(4)$, and they transform in the spin 1/2 representation of the $SO(3)$ symmetry.

3See [15] for another case where the dispersion relation depends non-trivially on the coupling.
Scalar operators and the SU(4) spin chain

If we wish to construct gauge-invariant single-trace operators only from scalars we see that this should be done by alternatingly combining the scalars $Z^a$ with the scalars $Z^a_\dagger$ since then we can contract the indices with respect to the $SU(N) \times SU(N)$ gauge group. Thus, we can consider single-trace operators of the form \[\mathcal{O} = W_{a_1a_2\cdots a_n} \text{Tr}(Z^{a_1}Z^{a_2}_\dagger \cdots Z^{a_n}Z^{a_n}_\dagger) \] (9)

In [8] the two-loop dilatation operator was considered for this class of operators interpreting the operator (9) as a spin chain of length $2n$ with the spins in the odd sites transforming in the fundamental and the spins in the even sites in the anti-fundamental representations of $SU(4)$. This is in analogy with the analysis of the scalar operators of $\mathcal{N} = 4$ SYM [9]. The result is the anomalous dimension \[\Delta = \Delta_0 + \frac{\lambda^2}{2} \sum_{i=1}^{2n} (2 - 2P_{i,i+2} + P_{i,i+2}K_{i,i+1} + K_{i,i+1}P_{i,i+2}) \] (10)

with $P$ being the permutation operator and $K$ the trace operator.

Amazingly, it was shown in [8] that (10) is integrable, thus suggesting that ABJM theory in the 't Hooft limit has an integrable structure in analogy with that of $\mathcal{N} = 4$ SYM. This indeed makes it a very interesting theory to study. The explicit Bethe equations and dispersion relation for the integrable $SU(4)$ spin chain are written down in [8].

The AdS$_4$/CFT$_3$ duality

The ABJM theory is conjectured to be the world-volume theory on $N' = Nk$ coincident M2-branes on the orbifold $\mathbb{C}^4/\mathbb{Z}_k$. Taking the near-horizon limit of the geometry of $N'$ M2-branes on $\mathbb{C}^4/\mathbb{Z}_k$ gives the AdS$_4 \times S^7/\mathbb{Z}_k$ geometry

\[ds^2_{11} = \frac{\hat{R}^2}{4} \left( - \cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\hat{\Omega}^2_7 \right) + \hat{R}^2 ds^2_{S^7/\mathbb{Z}_k} \] (11)

with $\hat{R}^2 = (2\pi^2 N')^{1/3} l_p^2$ and with the four form field strength

\[F_{(4)} = \frac{3\hat{R}^3}{8} \epsilon_{\text{AdS}_4} \] (12)

where $\epsilon_{\text{AdS}_4}$ is the unit volume form on AdS$_4$. We can parameterize the $S^7/\mathbb{Z}_k$ geometry using the four complex scalars $z_1, z_2, z_3, z_4$ such that

\[ds^2_{S^7/\mathbb{Z}_k} = \sum_{a=1}^4 dz_a d\bar{z}_a , \quad \sum_{a=1}^4 z_a \bar{z}_a = 1 \] (13)

The orbifolding is implemented as follows. We write

\[z_a = \mu_a e^{i\phi_a} \] (14)

Then we span an $S^7$ if $\sum_{a=1}^4 \mu^2_a = 1$. To each angle $\phi_a$ we associate the angular momentum

\[J_a = -i \partial_{\phi_a} \] (15)

Footnote: These operators resemble scalar operators in the $\mathcal{N} = 2$ superconformal Quiver Gauge Theories [17, 18].
Write now the angles as
\[ \phi_1 = \gamma + \frac{1}{2}(-\eta_1 - \eta_2 - \eta_3), \quad \phi_2 = \gamma + \frac{1}{2}(\eta_1 + \eta_2 - \eta_3), \]
\[ \phi_3 = \gamma + \frac{1}{2}(\eta_1 - \eta_2 + \eta_3), \quad \phi_4 = \gamma + \frac{1}{2}(-\eta_1 + \eta_2 + \eta_3) \] (16)

The orbifold \( S^7/\mathbb{Z}_k \) is now implemented as the identification
\[ \gamma \equiv \gamma + \frac{2\pi}{k} \] (17)

We have that
\[ J_1 + J_2 + J_3 + J_4 = -i \partial_\gamma \] (18)

Thus, we see that the orbifolding is equivalent to the quantization condition
\[ J_1 + J_2 + J_3 + J_4 \in k\mathbb{Z} \] (19)

Introducing the three charges
\[ R_j = -i \partial_{\eta_j} \] (20)

we see that \( R_1, R_2, R_3 \) are the three Cartan generators for the \( SU(4) \) subgroup of \( SO(8) \) which is dual to the \( SU(4) \) R-symmetry of the ABJM theory. In detail,
\[ R_1 = \frac{1}{2}(J_1 - J_2 - J_3 + J_4), \quad R_2 = \frac{1}{2}(-J_1 + J_2 - J_3 + J_4), \quad R_3 = \frac{1}{2}(-J_1 - J_2 + J_3 + J_4) \] (21)

We can identify the four complex scalars \( z_a \) with the four scalar fields \( Z^a \) of the ABJM theory given in (8). In particular we see that \( Z^a \) transforms in the fundamental representation with highest weight \( (1/2, 1/2, 1/2) \) in terms of \((R_1, R_2, R_3)\) while \( Z^a \) transforms in the \((1/2, 1/2, -1/2)\) anti-fundamental representation.

Write now
\[ ds^2_{S^7/\mathbb{Z}_k} = ds^2_{\mathbb{C}P^3} + (d\gamma + A)^2 \] (22)

Thus the eleven-dimensional metric is
\[ ds^2_{11} = \frac{\hat{R}^2}{4} \left( - \cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\hat{\Omega}^2_2 \right) + \hat{R}^2 ds^2_{\mathbb{C}P^3} + \hat{R}^2 (d\gamma + A)^2 \] (23)

Using the standard relation between the M-theory metric and the type IIA metric, along with the relation \( l_s^2 = g_s l_s^2 \) and that the eleven-dimensional radius is \( R_{11} = g_s l_s \), we get the following background of type IIA supergravity given by the metric
\[ ds^2 = \frac{R^2}{4} \left( - \cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\hat{\Omega}^2_2 \right) + R^2 ds^2_{\mathbb{C}P^3} \] (24)

with
\[ \frac{R^2}{l_s^2} = \frac{\sqrt{2^5 \pi^2 N'}}{k} = \sqrt{\frac{2^5 \pi^2 N}{k}} = \sqrt{2^7 \pi^2 \lambda} \] (25)

and moreover given by the string coupling constant
\[ g_s = \frac{(2^5 \pi^2 N')^{1/4}}{k^{3/2}} = \left( \frac{2^5 \pi^2 N}{k^5} \right)^{\frac{1}{4}} \] (26)

the Ramond-Ramond (RR) four-form field strength
\[ F_{(4)} = \frac{3 R^3}{8} e^{AdS_4} \] (27)
and with $A$ being a one-form RR potential corresponding to the two-form RR field strength $F_{(2)} = dA$.

From demanding a small curvature and a small string coupling one finds that this background is a valid background for type IIA string theory when $\lambda \gg 1$ and $N \ll k^5 [2]$.

When considering the type IIA description we should clearly require that the dependence on $\gamma$ is absent, we get therefore that we should only consider operators obeying

$$J_1 + J_2 + J_3 + J_4 = 0$$

(28)

This is in accordance with the construction of single-trace scalars operators (9) in the ABJM theory since we see that these operators indeed obey (28).

Note that for fixed $\rho \gg 1$ the AdS$_4$ part of the metric (24) approaches $\mathbb{R} \times S^2$ as $e^{2\rho} R^2 / 4 (dt^2 + d\hat{\Omega}_2^2)$. Since the conformal dimension $\Delta$ in ABJM theory is the energy in units of the two-sphere radius, we see that we should identify $\Delta$ with

$$\Delta = i \partial_t$$

(29)

3 Subsectors of the ABJM theory

In this section we consider decoupled subsectors in the ABJM theory. A straightforward method to analyze this was provided in [19] for $N = 4$ SYM (for a method based on group theory see [20]). For ABJM theory we should consider the possible inequalities of the form

$$\Delta_0 \geq m_1 R_1 + m_2 R_2 + m_3 R_3 + m_4 S$$

(30)

where $\Delta_0$ is the bare scaling operator, $R_j$ are the three Cartan generators of the $SU(4)$ R-symmetry, $S$ is the Cartan generator of the $SO(3)$ symmetry and $m_i$ are rational numbers. Alternatively using (21) we can express this as

$$\Delta_0 \geq n_1 J_1 + n_2 J_2 + n_3 J_3 + n_4 J_4 + n_5 S$$

(31)

assuming the extra restriction (28) and where $n_i$ are rational numbers. The upshot is that if the inequality is saturated for certain operators then those operators comprise a decoupled sector for the leading contribution to the anomalous dimension operator $\Delta - \Delta_0$.

The $SU(2) \times SU(2)$ sector

Consider the inequality

$$\Delta_0 \geq J_1 + J_2$$

(32)

The operators in the ABJM theory that saturate this inequality, i.e. for which $\Delta_0 = J_1 + J_2$, are the ones made out of the scalars $A_{1,2}$ and $B_{1,2}$. The single-trace operators are thus of the form

$$O = W_{i_1 i_2 \cdots i_J}^{j_1 j_2 \cdots j_J} \text{Tr}(A_{i_1} B_{j_1} \cdots A_{i_J} B_{j_J})$$

(33)

This constitutes an $SU(2) \times SU(2)$ sector of the ABJM theory, as found in [8], since the $A_{1,2}$ and $B_{1,2}$ scalars transform in two separate $SU(2)$ subgroups of the $SU(4)$. From the result (11) of [8] we see furthermore that

$$\Delta - J = \lambda^2 \sum_{l=1}^{2J} (1 - P_{l,l+2}) = \lambda^2 \sum_{l=1}^{J} (1 - P_{2l-1,2l+1} + 1 - P_{2l,2l+2})$$

(34)
We defined here $J = J_1 + J_2 = -J_3 - J_4$. We see that \([34]\) corresponds to two decoupled ferromagnetic $XXX_{1/2}$ Heisenberg spin chains, one living at the odd sites and the other at the even sites [3]. The spectrum is determined by the following dispersion relation, Bethe equations and momentum constraint

$$
\Delta - J = 4\lambda^2 \left[ \sum_{i=1}^{M_1} \sin^2 \left( \frac{p_i^{(1)}}{2} \right) + \sum_{i=1}^{M_2} \sin^2 \left( \frac{p_i^{(2)}}{2} \right) \right] \quad (35)
$$

$$
e^{ip_k^{(a)}J} = \prod_{j=1,j\neq k}^{M_0} S(p_k^{(a)}, p_j^{(a)}) \hspace{1cm} \sum_{i=1}^{M_1} p_i^{(1)} + \sum_{i=1}^{M_2} p_i^{(2)} = 0 \quad (36)
$$

for $a = 1, 2$, with the S-matrix given by

$$
S(p_k, p_j) = -\frac{1 + e^{i(p_k + p_j)} - 2e^{ip_k}}{1 + e^{i(p_k + p_j)} - 2e^{ip_j}} \quad (37)
$$

We see that the two chains affect each other through the momentum constraint which means that the spectrum is not just given by adding together two independent Heisenberg spin chains. We also note that we can infer from \([35]\) that the magnon dispersion relation in the $SU(2) \times SU(2)$ sector is given by \([34]\) which in turn reveals that $h(\lambda) = 4\lambda^2$ for small $\lambda$ in the general dispersion relation \([4]\).

**Other sectors**

Consider the inequality

$$
\Delta_0 \geq J_1 + J_2 + J_3 \quad (38)
$$

We see that the only operators that can saturate this inequality are those that have $B_2$ on the even sites and $(Z^1, Z^2, Z^3) = (A_1, A_2, B_1^1)$ on the odd sites. Thus, we can consider single-trace operators of the form

$$
\mathcal{O} = W_{a_1a_2\cdots a_n} \text{Tr}(Z^{a_1}B_2\cdots Z^{a_n}B_2) \quad (39)
$$

with $a_j = 1, 2, 3$. This is the $SU(3)$ sector found in \([3]\).

It is furthermore interesting to consider sectors with derivatives. We can only get derivatives in the inequality \([31]\) if $n_5 \in \{-1, 1\}$. Consider the inequality

$$
\Delta_0 \geq S + J_1 + J_2 \quad (40)
$$

For this case we see that at odd sites we can either have $D^n_+ A_{1,2}$ or $D^n_+ \chi_{A_{1,2}}$ where $\chi_{A_{1,2}}$ is the component of the superpartner of $A_{1,2}$ with $S = 1/2$. For even sites we can either have $D^n_+ B_{1,2}$ and $D^n_+ \chi_{B_{1,2}}$ where $\chi_{B_{1,2}}$ is the component of the superpartner of $B_{1,2}$ with $S = 1/2$. This sector generalizes the $SU(2) \times SU(2)$ sector to include both the derivative $D_+$ and a superpartner. This sector could be relevant for studying the cusp anomaly in the ABJM theory.

We can also generalize the $SU(3)$ sector inequality to

$$
\Delta_0 \geq S + J_1 + J_2 + J_3 \quad (41)
$$

At odd sites we have $D^n_+ Z^{1,2,3}$ and $D^n_+ \chi_{Z^{1,2,3}}$ with $\chi_{Z^{1,2,3}}$ being the superpartner of $Z^{1,2,3}$ with $S = 1/2$, while at even sites we have $D^n_+ B_2$ and $D^n_+ \chi_{B_2}$ with $\chi_{B_2}$ being the superpartner of $B_2$ with $S = 1/2$. 


4 The $SU(2) \times SU(2)$ sigma-model limit

In this section we take a limit of the type IIA string theory sigma-model on $AdS_4 \times CP^3$ corresponding to zooming in to the $SU(2) \times SU(2)$ sector. The idea is that by taking a limit where $\Delta - J_1 - J_2$ goes to zero, then only the string states of the $SU(2) \times SU(2)$ sector can survive. This corresponds to a limit of small momenta, and the leading contribution to $\Delta - J_1 - J_2$ gives a sigma-model describing the small momentum regime of the strings in the $SU(2) \times SU(2)$ sector. This type of limit was first found in [21] (see also [22, 23]).

In order to understand how to zoom in to the relevant part of the geometry of $AdS_4 \times CP^3$ we first take a step back and consider the M-theory background $AdS_4 \times S^7$ corresponding to M2-branes on $C^4$. As is clear from Section 2, the two $SU(2)$'s are gotten from splitting up $C^4 = \mathbb{C}^2 \times \mathbb{C}^2$. In detail the first $SU(2)$ corresponding to $A_{1,2}$ is then associated to $z_{1,2}$ while the second $SU(2)$ corresponding to $B_{1,2}$ is associated to $\bar{z}_{3,4}$. We therefore split up the $S^7$ into two $S^3$'s, one for each $\mathbb{C}^2$, as follows

$$ds^2_{S^7} = d\theta^2 + \cos^2 \theta d\Omega_3^2 + \sin^2 \theta d\Omega_3'^2$$

We parameterize the two three-spheres as

$$d\Omega_3^2 = d\psi_1^2 + \sin^2 \psi_1 d\phi_1^2 + \cos^2 \psi_1 d\phi_2^2, \quad d\Omega_3'^2 = d\psi_2^2 + \sin^2 \psi_2 d\phi_3^2 + \cos^2 \psi_2 d\phi_4^2$$

with $\phi_i$ being the angles introduced in Section 2. Introduce now the angles

$$\theta_1 = 2\psi_1 - \frac{\pi}{2}, \quad \theta_2 = 2\psi_2 - \frac{\pi}{2}, \quad \phi_1 = \phi_1 - \phi_2, \quad \phi_2 = \phi_4 - \phi_3$$

$$\gamma = \frac{1}{4}(\phi_1 + \phi_2 + \phi_3 + \phi_4), \quad \delta = \frac{1}{4}(\phi_1 + \phi_2 - \phi_3 - \phi_4)$$

With this, we can write

$$d\Omega_3^2 = \frac{1}{4}d\Omega_2^2 + \left(\frac{d\gamma + d\delta + \frac{1}{4}\sin \theta_1 d\phi_1}{2}\right)^2, \quad d\Omega_3'^2 = d\theta_1^2 + \cos^2 \theta_1 d\phi_1^2$$

$$d\Omega_3'^2 = \frac{1}{4}d\Omega_2^2 + \left(\frac{d\gamma - d\delta - \frac{1}{4}\sin \theta_2 d\phi_2}{2}\right)^2, \quad d\Omega_3'^2 = d\theta_2^2 + \cos^2 \theta_2 d\phi_2^2$$

We have

$$S_1^{(1)} = \frac{J_1 - J_2}{2} = -i\partial_{\phi_1}, \quad S_2^{(2)} = \frac{J_3 - J_4}{2} = -i\partial_{\phi_2}$$

$$J_1 + J_2 + J_3 + J_4 = -i\partial_{\gamma}, \quad J_1 + J_2 - J_3 - J_4 = -i\partial_{\delta}$$

We see that the coordinates $(\theta_i, \phi_i)$, $i = 1, 2$, parameterize a pair of two-spheres. These two two-spheres correspond to the two $SU(2)$'s. Moreover, we note that we chose the opposite orientation for $\phi_1$ and $\phi_2$ in the two $\mathbb{C}^2$'s since one $SU(2)$ corresponds to $(z_1, z_2)$ ($A_{1,2}$ in the ABJM theory) while the other $SU(2)$ to $(\bar{z}_3, \bar{z}_4)$ ($B_{1,2}$ in the ABJM theory). This gives the two Cartan generators $S_2^{(i)}$ corresponding to the total spins for the two $SU(2)$'s.

We can now implement the orbifolding of the $S^7$ by the identification (17). In order to zoom in to the $SU(2) \times SU(2)$ sector we set

$$\rho = 0, \quad \theta = \frac{\pi}{4}$$

This can be justified further since in the limit we take below one can check that the transverse excitations in the $\rho$ and $\theta$ directions become infinitely heavy in the limit, just as it happens in the $SU(2)$ sigma-model limit of $AdS_5 \times S^5$ [23]. We should thus consider the eleven-dimensional metric

$$ds^2_{11} = -\frac{\hat{R}^2}{4}dt^2 + \frac{\hat{R}^2}{2}(d\Omega_3^2 + d\Omega_3'^2)$$
with the identification \( \hat{R} \) is given in Section 2. We find that
\[
d s_{11}^2 = -\frac{R^2}{4} dt^2 + \hat{R}^2 (d\gamma + A)^2 + \hat{R}^2 \left[ \frac{1}{8} d\Omega_2^2 + \frac{1}{8} d\Omega_2^2 + (d\delta + \omega)^2 \right]
\] (49)
with the one-forms \( A \) and \( \omega \) given by
\[
A = \frac{1}{4} (\sin \theta_1 d\varphi_1 - \sin \theta_2 d\varphi_2), \quad \omega = \frac{1}{4} (\sin \theta_1 d\varphi_1 + \sin \theta_2 d\varphi_2)
\] (50)
The type IIA background then has the ten-dimensional metric
\[
d s^2 = -\frac{R^2}{4} dt^2 + R^2 \left[ \frac{1}{8} d\Omega_2^2 + \frac{1}{8} d\Omega_2^2 + (d\delta + \omega)^2 \right]
\] (51)
with \( R \) given in (25).
As explained in Section 3 the \( SU(2) \times SU(2) \) sector is obtained by considering states for which \( \Delta - J_1 - J_2 \) is small. To implement this as a sigma-model limit we make the coordinate transformation
\[
\tilde{t} = \frac{1}{J^2} t, \quad \chi = \delta - \frac{1}{2} t
\] (52)
so that
\[
\tilde{H} \equiv -i\partial_{\tilde{t}} = J^2 \left( \Delta - \frac{1}{2} (J_1 + J_2 - J_3 - J_4) \right), \quad J_1 + J_2 - J_3 - J_4 = -i\partial_{\chi}
\] (53)
where we have defined
\[
J \equiv J_1 + J_2
\] (54)
Using the condition \( \ref{J} \) we see that \( \ref{H} \) can be written as
\[
\tilde{H} \equiv i\partial_{\tilde{t}} = J^2 (\Delta - J), \quad 2J = -i\partial_{\chi}
\] (55)
We see here that taking \( J \to \infty \) corresponds to zooming in to the regime where \( \Delta - J \) is of order \( 1/J^2 \). This corresponds to the energy scale in which we see the individual magnon states in the spin chain description. We see from \( \ref{chi} \) that we are zooming in close to \( \delta = t/2 \). This is a null-geodesic in the metric \( \ref{metric} \). This null-geodesic corresponds to a chiral primary of the ABJM theory with \( \Delta = J \).

Employing the coordinate transformation \( \ref{chi} \) we get the type IIA metric
\[
d s^2 = R^2 \left[ (J^2 d\tilde{t} + d\chi + \omega)(d\chi + \omega) + \frac{1}{8} d\Omega_2^2 + \frac{1}{8} d\Omega_2^2 \right]
\] (56)
Consider now the bosonic sigma-model Lagrangian
\[
\mathcal{L} = -\frac{1}{2} G_{\mu\nu} h^{\alpha\beta} \partial_{\alpha} x^\mu \partial_{\beta} x^\nu
\] (57)
We pick the gauge
\[
\tilde{t} = \kappa \tau, \quad p_\chi = \text{const.}, \quad h_{\alpha\beta} = \eta_{\alpha\beta}
\] (58)
with \( 2\pi l_s^2 p_\chi = \partial \mathcal{L}/\partial \partial_{\tau} \chi \). The Lagrangian \( \ref{lagrangian} \) is then found to be
\[
\frac{2}{R^2} \mathcal{L} = (\kappa J^2 + \partial_{\tau} \chi + \omega_\tau)(\partial_{\tau} \chi + \omega_\tau) - (\chi' + \omega_\tau)^2 + \frac{1}{8} \sum_{i=1}^2 \left[ (\partial_{\tau} \theta_i)^2 - \theta_i' \cos^2 \theta_i (\partial_{\tau} \varphi_i - \varphi_i') \right]
\] (59)
with \( \omega = \omega_\tau d\tau + \omega_\sigma d\sigma \) and with prime denoting the derivative with respect to \( \sigma \). The Virasoro constraints are
\[
(\kappa J^2 + \partial_{\tau} \chi + \omega_\tau)(\chi' + \omega_\sigma) + \frac{1}{8} \sum_{i=1}^2 \left[ \partial_{\tau} \theta_i \theta_i' + \cos^2 \theta_i \partial_{\tau} \varphi_i \varphi_i' \right] = 0
\] (60)
We have
\[ p_\chi = \frac{R^2}{2\pi l_s^2} \left( \frac{\kappa J_s^2}{2} + \partial_\tau \chi + \omega_\tau \right) \]  
(61)

Since \( \tilde{t} \) measures the time corresponding to the energy scale \( \hat{H} \) we should consider the velocities with respect to \( \tilde{t} \) to be finite in the \( J \to \infty \) limit. Hence for example \( \partial_\tau \chi = \kappa \partial_\tau \chi \). Inserting this in (61), we see that \( \partial_\tau \chi \to 0 \) and using that \( 2J = \int_0^{2\pi} p_\chi \) we get from (61) that
\[ \kappa = \frac{4l_s^2}{J R^2} \]  
(62)

which is seen to go to zero in the \( J \to \infty \) limit. Taking now the \( J \to \infty \) limit of the Lagrangian (59) and the constraints (60) we get
\[ \frac{2}{R^2} \mathcal{L} = \frac{16l_s^4}{R^4}(\dot{\chi} + \omega_\tau) - (\dot{\chi}' + \omega_\sigma)^2 - \frac{1}{8} \sum_{i=1}^{2} \left[ \theta_i'^2 + \cos^2 \theta_i \varphi_i'^2 \right] \]
(63)
\[ \chi' + \omega_\sigma = 0, \quad \frac{16l_s^4}{R^4}(\dot{\chi} + \omega_\tau) + \frac{1}{8} \sum_{i=1}^{2} \left[ \theta_i'^2 + \cos^2 \theta_i \varphi_i'^2 \right] = 0 \]

Here the dot denote the derivative with respect to \( \tilde{t} \). We see that the constraints fix \( \chi \) in terms of \( \theta_i \) and \( \varphi_i \). Thus we can eliminate \( \chi \) to get the gauge fixed Lagrangian
\[ \frac{2}{R^2} \mathcal{L} = \frac{16l_s^4}{R^4} \omega_\tau - \frac{1}{8} \sum_{i=1}^{2} \left[ \theta_i'^2 + \cos^2 \theta_i \varphi_i'^2 \right] \]  
(64)

From this we finally get the action for the sigma-model model in the \( J \to \infty \) limit as
\[ I = \frac{J}{4\pi} \sum_{i=1}^{2} \int d\tilde{t} \int_0^{2\pi} d\sigma \left[ \sin \theta_i \dot{\varphi}_i - \pi^2 \lambda \left( \theta_i'^2 + \cos^2 \theta_i \varphi_i'^2 \right) \right] \]  
(65)

This is supplemented with the momentum constraint
\[ \sum_{i=1}^{2} \int_0^{2\pi} d\sigma \sin \theta_i \varphi_i' = 0 \]  
(66)

Thus, in conclusion the result of taking the \( SU(2) \times SU(2) \) sigma-model limit is that we obtain two Landau-Lifshitz models added together (65), one for each \( SU(2) \), which only affect each other through the momentum constraint (66). Since the Landau-Lifshitz model corresponds to the long wave-length \( J \to \infty \) limit of the \( \times \times \times \frac{1}{2} \) Heisenberg spin chain our result is consistent with finding two Heisenberg spin chains in the \( SU(2) \times SU(2) \) sector of ABJM theory at \( \lambda \ll 1 \) [8].

It is interesting to compare further the integrable structure that we found here on the string side with the integrable structure (35)-(37) found on the weakly coupled ABJM theory. Using the analysis of [24] we can write the Bethe equations and dispersion relation corresponding to (65)-(66) as
\[ \Delta - J = \frac{\lambda}{2} \left[ \sum_{i=1}^{M_1} (p_i^{(1)})^2 + \sum_{i=1}^{M_2} (p_i^{(2)})^2 \right] \]  
(67)
\[ e^{i p_k^{(a)} J} = \prod_{j=1,j \neq k}^{M_a} S(p_k^{(a)}, p_j^{(a)}), \quad \sum_{i=1}^{M_1} p_i^{(1)} + \sum_{i=1}^{M_2} p_i^{(2)} = 0 \]  
(68)
for \( a = 1, 2 \), with the S-matrix given by
\[ S(p_k, p_j) = \frac{1}{p_k - p_j} \pm \frac{i}{p_k - p_j} \]  
(69)
Comparing this with (35)-(37) found in the weakly coupled ABJM theory we see that the S-matrices coincide for small momenta (up to order $p^2$) which again is as expected since the Landau-Lifshitz model describes the long wave-length expansion of the Heisenberg spin chain. However, the dispersion relations are clearly different, if one compares them in the $p \to 0$ limit. See the introduction in Section 1 for a discussion of this point.

5 The $SU(2) \times SU(2)$ Penrose limit

In this section we consider a Penrose limit of the AdS$_4 \times \mathbb{C}P^3$ background which corresponds to the $SU(2) \times SU(2)$ sigma-model limit of Section 4, following [25, 17]. Another Penrose limit of AdS$_4 \times \mathbb{C}P^3$ have been considered in [7]. We comment on the relation between the two Penrose limits below.

We choose in the following to consider only the bosonic string modes for simplicity. To take the Penrose limit we consider first the metric for AdS$_4 \times \mathbb{C}P^3$

$$ds^2 = \frac{R^2}{4} \left( - \cosh^2 \rho dt^2 + \rho^2 + \sinh^2 \rho d\Omega_2^2 \right) + R^2 ds^2_{\mathbb{C}P^3}$$

where the $\mathbb{C}P^3$ metric is

$$ds^2_{\mathbb{C}P^3} = d\theta^2 + \frac{\cos^2 \theta}{4} d\Omega_2^2 + \frac{\sin^2 \theta}{4} d\Omega_2^2 + 4 \cos^2 \theta \sin^2 \theta (d\delta + \omega)^2$$

with

$$\omega = \frac{1}{4} \sin \theta_1 d\varphi_1 + \frac{1}{4} \sin \theta_2 d\varphi_2$$

where we used the angles introduced in Section 4. Define

$$t' = t , \chi = \delta - \frac{1}{2} t$$

In these coordinates the metric (70) takes the form

$$ds^2 = - \frac{R^2}{4} dt'^2 (1 - 4 \cos^2 \theta \sin^2 \theta + \sinh^2 \rho) + \frac{R^2}{4} (d\rho^2 + \sinh^2 \rho d\Omega_2^2)$$

$$+ R^2 \left[ d\theta^2 + \frac{\cos^2 \theta}{4} d\Omega_2^2 + \frac{\sin^2 \theta}{4} d\Omega_2^2 + 4 \cos^2 \theta \sin^2 \theta (dt' + d\chi + \omega)(d\chi + \omega) \right]$$

We have that

$$\Delta - J = i\partial_{t'} , \quad 2J = -i\partial_{\chi}$$

Here $\Delta - J$ is the energy we are interested in measuring for the $SU(2) \times SU(2)$ sector.

Define now the rescaled coordinates

$$v = R^2 \chi , \quad u_a = R \left( \theta - \frac{\pi}{4} \right) , \quad r = \frac{R}{2} \rho , \quad x_a = R \varphi_a , \quad y_a = R \theta_a$$

with $a = 1, 2$. Then the Penrose limit $R \to \infty$ gives the following metric for a type IIA pp-wave background

$$ds^2 = dv dt' + \sum_{i=1}^4 (du_i^2 - u_i^2 dt'^2) + \frac{1}{8} \sum_{a=1}^2 (dx_a^2 + dy_a^2 + 2dt' y_a dx_a)$$

where $r^2 = \sum_{i=1}^3 u_i^2$ and $dr^2 + r^2 d\Omega_2^2 = \sum_{i=1}^3 du_i^2$. The RR field strengths for this pp-wave background are given by

$$F_{(2)} = dt' du_4 , \quad F_{(4)} = 3 dt' du_1 du_2 du_3$$
The type IIA pp-wave background \([77]-[78]\) has 24 supercharges and it is the same background found from another Penrose limit of \(\text{AdS}_4 \times \mathbb{C}^3\) in [7] though in another coordinate system. The background \([77]-[78]\) was originally found in [26, 27].

Similarly, we shall see below that the Penrose limit given by (76) is particularly well suited for the \(SU(2)\) sector of \(\text{AdS}_5 \times S^5\), as explained in [28, 29]. Similarly, we shall see below that the Penrose limit given by (76) is particularly well suited for the \(SU(2) \times SU(2)\) sector of \(\text{AdS}_4 \times \mathbb{C}^3\).

We choose the gauge
\[
t' = c\tau, \quad h_{\alpha,\beta} = \eta_{\alpha,\beta}
\]
and we get, for the bosonic fields, the following gauge fixed Lagrangian
\[
\mathcal{L} = \frac{1}{2} \sum_{i=1}^{4} \left[ (\partial_\tau u_i)^2 - u_i^2 - c^2 u_i^2 \right] + \frac{c}{8} \sum_{a=1}^{2} y_a \partial_\tau x_a + \frac{1}{16} \sum_{a=1}^{2} \left[ (\partial_\tau x_a)^2 + (\partial_\tau y_a)^2 - x_a^2 - y_a^2 \right]
\]
(80)

The bosonic light-cone Hamiltonian is then given by
\[
ch_{lc} = \frac{1}{2\pi l_s^2} \int_0^{2\pi} d\sigma \left\{ \frac{1}{2} \sum_{i=1}^{4} \left[ (\partial_\tau u_i)^2 + u_i^2 + c^2 u_i^2 \right] + \frac{1}{16} \sum_{a=1}^{2} \left[ (\partial_\tau x_a)^2 + (\partial_\tau y_a)^2 + x_a^2 + y_a^2 \right] \right\}
\]
(81)

The mode expansion for the bosonic fields can be written as
\[
u_i(\tau, \sigma) = \frac{i}{\sqrt{2}} \sum_{n \in \mathbb{Z}} \frac{1}{\sqrt{\Omega_n}} \left[ \hat{a}^i_n e^{-i(\Omega_n \tau - n\sigma)} - (\hat{a}^i_n)^\dagger e^{i(\Omega_n \tau - n\sigma)} \right]
\]
(82)

\[
z_a(\tau, \sigma) = 2\sqrt{2} e^{i\frac{\tau}{2}} \sum_{n \in \mathbb{Z}} \frac{1}{\omega_n} \left[ a^a_n e^{-i(\omega_n \tau - n\sigma)} - (a^a_n)^\dagger e^{i(\omega_n \tau - n\sigma)} \right]
\]
(83)

where \(\Omega_n = \sqrt{c^2 + n^2}, \quad \omega_n = \sqrt{\frac{c^2}{4} + n^2}\) and we defined \(z_a(\tau, \sigma) = x_a(\tau, \sigma) + iy_a(\tau, \sigma)\). The canonical commutation relations \([x_a(\tau, \sigma), p_{x_a}(\tau, \sigma')] = i\delta_{ab}\delta(\sigma - \sigma'), \quad [y_a(\tau, \sigma), p_{y_a}(\tau, \sigma')] = i\delta_{ab}\delta(\sigma - \sigma')\) and \([u_i(\tau, \sigma), p_j(\tau, \sigma')] = i\delta_{ij}\delta(\sigma - \sigma')\) follows from
\[
[a^a_n], (\hat{a}^a_n)^\dagger = \delta_{mn} \delta_{ab}, \quad [\hat{a}^a_n], (\hat{a}^b_n)^\dagger = \delta_{mn} \delta_{ab}, \quad [\hat{a}^a_n], (\hat{a}^b_n)^\dagger = \delta_{mn} \delta_{ij}
\]
(84)

Employing (84) we obtain the bosonic spectrum
\[
c h_{lc} = \sum_{i=1}^{4} \sum_{n \in \mathbb{Z}} \sqrt{n^2 + c^2} \hat{N}_n^i + \sum_{a=1}^{2} \sum_{n \in \mathbb{Z}} \left[ \sqrt{\frac{c^2}{4} + n^2 - \frac{c}{2}} \right] M_n^a + \sum_{a=1}^{2} \sum_{n \in \mathbb{Z}} \left[ \sqrt{\frac{c^2}{4} + n^2 + \frac{c}{2}} \right] N_n^a
\]
(85)

with the number operators \(\hat{N}_n^i = (\hat{a}^i_n)^\dagger \hat{a}^i_n\), \(M_n^a = (a^a_n)^\dagger a^a_n\) and \(N_n^a = (\hat{a}^a_n)^\dagger \hat{a}^a_n\), and with the level-matching condition
\[
\sum_{n \in \mathbb{Z}} n \left[ \sum_{i=1}^{4} \hat{N}_n^i + \sum_{a=1}^{2} (M_n^a + N_n^a) \right] = 0
\]
(86)

The constant \(c\) can be fixed from the term \(\frac{1}{2} \partial_\tau v\) in the full Lagrangian. In fact we have that
\[
2\pi l_s^2 p_v = \partial \mathcal{L} / \partial \partial_\tau v \quad \text{which gives}
\]
\[
c = \frac{4l_s^2 J}{R^2} = \frac{J}{\pi \sqrt{2\lambda}}
\]
(87)

where we again used that \(\int_0^{2\pi} d\sigma p_v = 2J\). Using (87) the spectrum (85) reads
\[
H_{lc} = \sum_{i=1}^{4} \sum_{n \in \mathbb{Z}} \left[ \sqrt{\frac{1}{4} + \frac{2\pi^2 \lambda}{J^2} n^2} \hat{N}_n^i + \sum_{a=1}^{2} \sum_{n \in \mathbb{Z}} \left[ \left( \frac{1}{4} + \frac{2\pi^2 \lambda}{J^2} n^2 - \frac{1}{2} \right) M_n^a + \left( \frac{1}{4} + \frac{2\pi^2 \lambda}{J^2} n^2 + \frac{1}{2} \right) N_n^a \right] \right]
\]
(88)
We see that this spectrum is consistent with the spectrum found in [7]. Here we used that from (85) we have that
\[ H_{lc} = \Delta - J \]  
(89)

From the spectrum (88) we can infer the following dispersion relation for an \( SU(2) \times SU(2) \) magnon
\[ \Delta = \sqrt{\frac{1}{4} + \frac{\lambda l}{2} n^2} \]  
(90)

where \( p = 2\pi n / J \) is the momentum of the magnon. This dispersion relation is clearly consistent with the dispersion relation (87) found from the \( SU(2) \times SU(2) \) sigma-model limit as one can see by taking a \( p \to 0 \) limit. As explained in the introduction in Section 3 this dispersion relation does not match with weakly coupled ABJM theory.

We can now connect the \( SU(2) \times SU(2) \) sigma-model limit of Section 4 to the above Penrose limit. Consider the limit \( J \to \infty \). In this limit \( c \to \infty \). We see therefore from the spectrum (88) that the modes \( N^i_n \) and \( N^a_n \) decouple, i.e. that the \( a^i_n \) and the \( \tilde{a}^a_n \) decouple. Indeed only the \( M^a_n \) modes corresponding to \( a^i_n \) are left, giving the spectrum
\[ H_{lc} = \sum_{a=1}^{2} \sum_{n \in \mathbb{Z}} \frac{2\pi^2 \lambda}{J^2} n^2 M^a_n, \quad \sum_{a=1}^{2} \sum_{n \in \mathbb{Z}} nM^a_n = 0 \]  
(91)

This precisely corresponds to the spectrum of the \( SU(2) \times SU(2) \) sigma-model limit for small \( p \), as can be seen from (87). This resembles the \( SU(2) \) decoupling limit taken of the analogous pp-wave solution for the \( SU(2) \) sector of type IIB string theory on \( AdS_5 \times S^5 \) [28].

We can also connect the above Penrose limit to the \( SU(2) \times SU(2) \) sigma-model limit on the level of the action. Consider the limit
\[ J \to \infty, \quad \sqrt{\frac{J}{R}} x_i = \sqrt{\frac{J}{R}} \varphi_i \text{ fixed}, \quad \sqrt{\frac{J}{R}} y_i = \sqrt{\frac{J}{R}} \theta_i \text{ fixed} \]  
(92)

In this limit we zoom in near a point on each of the two-spheres that are the target spaces of the double Landau-Lifshitz model (65). This gives the action
\[ I = \frac{J}{16\pi^2 \sqrt{2 \Lambda^2}} \sum_{a=1}^{2} \int dt' ds' \left[ y_i \partial_{t'} x_i - \frac{\pi^2 \lambda}{J^2} x_i^2 \right] \]  
(93)

This is the same action as one obtain by taking a \( c \to \infty \) limit of the action corresponding to the Lagrangian (80).

In conclusion we can connect the \( SU(2) \times SU(2) \) sigma-model limit of Section 4 and the Penrose limit (76) in the same way as was done in [23] for the \( SU(2) \) sector of \( AdS_5 \times S^5 \). In particular, the above \( c \to \infty \) limit involves a non-relativistic limit of type IIA string theory on the pp-wave (77)-(78).

6 New Giant Magnon solution in the \( SU(2) \times SU(2) \) sector

In this section we find a new Giant Magnon solution in the \( SU(2) \times SU(2) \) sector of type IIA string theory on \( AdS_4 \times \mathbb{C}P^3 \).

To find the Giant Magnon solution on \( AdS_4 \times \mathbb{C}P^3 \) we consider the string sigma model on this metric background. The coordinates can be taken as a 5-vector \( Y \) and an 8-vector \( X \) where \( X \in S^7 \), \( Y \in AdS_4 \) constrained by
\[ X^2 = \sum_{i=1}^{8} X_i^2 = 1, \quad Y^2 = \sum_{i=1}^{3} Y_i^2 - Y_4^2 - Y_5^2 = -1 \]  
(94)
and we furthermore demand
\[ C_1 \equiv \sum_{i=1,3,5,7} (X_i \partial_\tau X_{i+1} - X_{i+1} \partial_\tau X_i) = 0, \quad C_2 \equiv \sum_{i=1,3,5,7} (X_i \partial_\sigma X_{i+1} - X_{i+1} \partial_\sigma X_i) = 0 \] (95)

defining the background to be \( \mathbb{C}P^3 \).

The bosonic part of the sigma model action in the conformal gauge is
\[ S = -\sqrt{2} \int d\tau d\sigma \left[ \frac{1}{4} \partial_\alpha Y \cdot \partial^\alpha Y + \partial_\alpha X \cdot \partial^\alpha X + \hat{\Lambda}(Y^2 + 1) + \Lambda(X^2 - 1) + \Lambda_1 C_1^2 + \Lambda_2 C_2^2 \right] \] (96)

Here, \( \Lambda, \hat{\Lambda} \) and \( \Lambda_i, i = 1,2 \) are Lagrange multipliers which enforce the coordinate constraints (94) and the constraints (95). Keeping into account the constraints (95), the equations of motion following from the action (96) take the form
\[ (\partial^2 - \Lambda) X_i = 0, \quad i = 1, \ldots, 8 \]
\[ \left( \frac{1}{4} \partial^2 - \hat{\Lambda} \right) Y_i = 0, \quad i = 1, \ldots, 5 \] (97)

and should be supplemented by Virasoro constraints
\[ \partial_\tau X \cdot \partial_\tau X + \partial_\sigma X \cdot \partial_\sigma X + \frac{1}{4} (\partial_\tau Y \cdot \partial_\tau Y + \partial_\sigma Y \cdot \partial_\sigma Y) = 0 \] (98)
\[ 2\partial_\tau X \cdot \partial_\sigma X + \frac{1}{2} \partial_\tau Y \cdot \partial_\sigma Y = 0 \] (99)

From (94) and (97) it follows that the classical values of the Lagrange multipliers \( \Lambda \) and \( \hat{\Lambda} \) are
\[ \Lambda = X \cdot \partial^2 X, \quad \hat{\Lambda} = -\frac{1}{4} Y \cdot \partial^2 Y \] (100)

The Giant Magnon solution will be found as a solution of the classical equations of motion where only coordinates on two \( S^2 \subset S^7 \) and \( R^1 \subset AdS_4 \) are excited. The solution on \( AdS_5 \times S^5 \) was originally found by Hofman and Maldacena [10]. This is a closed string solution with open boundary conditions in one azimuthal direction.

In the case we are studying the solution is point-like in \( AdS_4 \) and extended along the two \( S^2 \) which are subsets of \( S^7 \). The solution lives on an \( R^1 \times S^2 \times S^2 \) subspace of \( AdS_4 \times S^7 \), the \( R^1 \subset AdS_4 \) and \( S^2 \times S^2 \subset S^7 \). We shall choose the solution in such a way that it has opposite azimuthal angles in the two \( S^2 \) and the same polar angle. The boundary conditions are those of closed string theory. All variables are periodic, except for the azimuthal angles of the two \( S^2 \)'s which will be chosen to obey the magnon boundary condition which on one \( S^2 \) is
\[ \Delta \phi_1 \equiv p \] (101)
and on the other one will be\(^6\)
\[ \Delta \phi_3 = -p \] (102)

These identifications correspond to opposite orientations of the string on the two \( S^2 \). The Giant Magnon is then characterized by the momentum \( p \) and by the choice of the point in the transverse directions to the two \( S^2 \), i.e. by 2 two-component polarization vectors. \( p \) has to be interpreted as the momentum of the magnon in the spin chain, these two magnons have equal magnon momentum. They give the same contribution to the total momentum constraint.

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\(^5\) Here \( \partial^2 = \partial_\alpha \partial^\alpha = -\partial_\tau^2 + \partial_\sigma^2 \).

\(^6\) We denote it \( \phi_3 \) since the generator of rotations along the azimuthal direction of the second \( S^2 \) is called \( J_3 \).
We have found a new solution for the equations (97) satisfying the Virasoro constraints (98), (99) and the constraints (94), (95). With our coordinate choice it reads

\[ Y_4 + i Y_5 = e^{i 2 \tau}, \quad Y_1 = Y_2 = Y_3 = 0 \]  

(103)

\[ (X_3, X_4) = \hat{n}_1 \sin \frac{p}{2} \text{sech} u, \quad (X_7, X_8) = \hat{n}_2 \sin \frac{p}{2} \text{sech} u \]

where \( \hat{n}_i \) are two constant unit vectors and

\[ u = (\sigma - \tau \cos \frac{p}{2}) \csc \frac{p}{2} \]  

(105)

\[ X_1 + i X_2 = e^{i \tau} \frac{1}{\sqrt{2}} \left[ \cos \frac{p}{2} + i \sin \frac{p}{2} \tanh u \right] \]

\[ X_5 + i X_6 = e^{-i \tau} \frac{1}{\sqrt{2}} \left[ \cos \frac{p}{2} - i \sin \frac{p}{2} \tanh u \right] \]  

(104)

The solution describes right moving solitons traveling along the worldsheet with velocity \( \cos \frac{p}{2} \).

The solution on AdS\(_4\) in (103) is then chosen so that the energy density, associated with global time translations, is constant. Rather than in this energy we are more interested in the conserved quantity

\[ \Delta - J_1 - J_3 = -2 \sqrt{2} \lambda \int_{-\infty}^{\infty} d\sigma \left[ \frac{1}{4} \left( Y_4 \dot{Y}_5 - Y_5 \dot{Y}_4 \right) + \frac{X_1 \dot{X}_2 - X_2 \dot{X}_1}{2} + \frac{X_6 \dot{X}_5 - X_5 \dot{X}_6}{2} \right] \]  

(107)

where \( J_1 \) is the charge associated with azimuthal translations on one of the two \( S^2 \) and \( J_3 \) is the generator of the azimuthal translations on the other \( S^2 \). The classical value for \( \Delta - J = \Delta - J_1 - J_3 \) on the solution (103)-(104) then is

\[ \Delta - J = 2 \sqrt{2} \lambda \left| \sin \frac{p}{2} \right| \]  

(108)

Note that the above Giant Magnon solution describes two magnons, one for each two-sphere (or \( SU(2) \)). Using this fact we can infer from (108) that the dispersion relation for a single magnon in the \( SU(2) \times SU(2) \) sector of type IIA string theory on AdS\(_4 \times \mathbb{C}P^3\) is

\[ \Delta - J = \sqrt{2} \lambda \left| \sin \frac{p}{2} \right| \]  

(109)

This dispersion relation is seen to be consistent with the dispersion relation (90) found from the \( SU(2) \times SU(2) \) Penrose limit in Section 5.

7 Conclusions

We studied in this paper the \( SU(2) \times SU(2) \) sector in the type IIA string theory on AdS\(_4 \times \mathbb{C}P^3\), the proposed string dual of the recently constructed ABJM theory [2]. We found a sigma-model limit and a Penrose limit corresponding to the \( SU(2) \times SU(2) \) sector and furthermore a new Giant Magnon solution. Comparing this to the weak coupling results of [8] we found (6)-(7), showing that the dispersion relation for ABJM theory has a non-trivial dependence on \( \lambda \).

We note here that beside the dispersion relation (6)-(7) there are other dispersion relations in the theory, corresponding to the AdS\(_4\) directions and one of the \( \mathbb{C}P^3 \) directions. Thus, there might be another independent interpolation function for these modes.
It would obviously be interesting to study both the dispersion relations and the S-matrix in the spin chain description for $\lambda \ll 1$ and $\lambda \gg 1$.

It is also interesting to consider finite-size corrections to the new Giant Magnon solution found in this paper. This will be considered in [30].

Finally, we would like to compare with the results of [23]. In [23] it was argued that one can take a $\lambda \rightarrow 0$ limit of type IIB string theory on $\text{AdS}_5 \times S^5$. This limit corresponds to the $SU(2)$ decoupling limit of [31, 28, 32, 19]. It was argued in [23] that in this limit one can quantitatively match $\mathcal{N} = 4$ SYM with type IIB string theory on $\text{AdS}_5 \times S^5$, and in particular we argued that the one-loop matching was a result of this. Obviously, this cannot be the case for the duality between ABJM theory and type IIA string theory on $\text{AdS}_4 \times \mathbb{C}P^3$. We believe that the difference between the $\text{AdS}_5$/CFT4 case and the $\text{AdS}_4$/CFT3 is that the latter duality only possesses 24 supersymmetries. From this we expect that $\text{AdS}_4 \times \mathbb{C}P^3$ is not an exact type IIA string theory background. Indeed, to show that $\text{AdS}_5 \times S^5$ is exact the full 32 supercharges were used [31]. Therefore, the $\text{AdS}_4 \times \mathbb{C}P^3$ is indeed more challenging than the $\text{AdS}_5$/CFT4 duality.

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