Indices of inseparability in towers of field extensions

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Abstract

Let $K$ be a local field whose residue field has characteristic $p$ and let $L/K$ be a finite separable totally ramified extension of degree $n = ap^\nu$. The indices of inseparability $i_0, i_1, \ldots, i_\nu$ of $L/K$ were defined by Fried in the case $\text{char}(K) = p$ and by Heiermann in the case $\text{char}(K) = 0$; they give a refinement of the usual ramification data for $L/K$. The indices of inseparability can be used to construct “generalized Hasse-Herbrand functions” $\phi^j_{L/K}$ for $0 \leq j \leq \nu$. In this paper we give an interpretation of the values $\phi^j_{L/K}(c)$ for natural numbers $c$. We use this interpretation to study the behavior of generalized Hasse-Herbrand functions in towers of field extensions.

1 Introduction

Let $K$ be a local field whose residue field $\overline{K}$ is a perfect field of characteristic $p$, and let $K^{\text{sep}}$ be a separable closure of $K$. Let $L/K$ be a finite totally ramified subextension of $K^{\text{sep}}/K$. The indices of inseparability of $L/K$ were defined by Fried $[2]$ in the case $\text{char}(K) = p$, and by Heiermann $[5]$ in the case $\text{char}(K) = 0$. The indices of inseparability of $L/K$ determine the ramification data of $L/K$ (as defined for instance in Chapter IV of $[7]$), but the ramification data does not always determine the indices of inseparability. Therefore the indices of inseparability of $L/K$ may be viewed as a refinement of the usual ramification data of $L/K$.

Let $\pi_K$, $\pi_L$ be uniformizers for $K$, $L$. The most natural definition of the ramification data of $L/K$ is based on the valuations of $\sigma(\pi_L) - \pi_L$ for $K$-embeddings $\sigma : L \to K^{\text{sep}}$; this is the approach used in Serre’s book $[7]$. The ramification data can also be defined in terms of the relation between the norm map $N_{L/K}$ and the filtrations of the unit groups of $L$ and $K$, as in Fesenko-Vostokov $[1]$. This approach can be used to derive the
well-known relation between higher ramification theory and class field theory. Finally, the ramification data can be computed by expressing $\pi_K$ as a power series in $\pi_L$ with coefficients in the set $R$ of Teichmüller representatives for $\overline{K}$. This third approach, which is used by Fried and Heiermann, makes clear the connection between ramification data and the indices of inseparability.

Heiermann [5] defined “generalized Hasse-Herbrand functions” $\phi^j_{L/K}$ for $0 \leq j \leq \nu$. In Section 2 we give an interpretation of the values $\phi^j_{L/K}(c)$ of these functions at non-negative integers $c$. This leads to an alternative definition of the indices of inseparability which is closely related to the third method for defining the ramification data. In Section 3 we consider a tower of finite totally ramified separable extensions $M/L/K$. We use our interpretation of the values $\phi^j_{L/K}(c)$ to study the relations between the generalized Hasse-Herbrand functions of $L/K$, $M/L$, and $M/K$.

**Notation**

$\mathbb{N}_0 = \mathbb{N} \cup \{0\} = \{0, 1, 2, \ldots \}$

$v_p = p$-adic valuation on $\mathbb{Z}$

$K$ = local field with perfect residue field $\overline{K}$ of characteristic $p > 0$

$K^{sep} =$ separable closure of $K$

$v_K =$ valuation on $K^{sep}$ normalized so that $v_K(K^\times) = \mathbb{Z}$

$\mathcal{O}_K = \{\alpha \in K : v_K(\alpha) \geq 0\} =$ ring of integers of $K$

$\pi_K =$ uniformizer for $K$

$R =$ set of Teichmüller representatives for $\overline{K}$

$L/K =$ finite totally ramified subextension of $K^{sep}/K$ of degree $n > 1$, with $v_p(n) = \nu$

$M/L =$ finite totally ramified subextension of $K^{sep}/L$ of degree $m > 1$, with $v_p(m) = \mu$

$v_K$, $\mathcal{O}_K$, and $\pi_K$ have natural analogs for $L$ and $M$

2 Generalized Hasse-Herbrand functions

We begin by recalling the definition of the indices of inseparability $i_j$ ($0 \leq j \leq \nu$) for a nontrivial totally ramified separable extension $L/K$ of degree $n = ap^\ell$, as formulated by Heiermann [5]. Let $R \subset \mathcal{O}_K$ be the set of Teichmüller representatives for $\overline{K}$. Then there is a unique series $\hat{F}(X) = \sum_{h=0}^{\infty} a_h X^{h+n}$ with coefficients in $R$ such that $\pi_K = \hat{F}(\pi_L)$. For $0 \leq j \leq \nu$ set

$$i_j = \min\{h \geq 0 : v_p(h + n) \leq j, a_h \neq 0\}. \quad (2.1)$$

If $\text{char}(K) = 0$ it may happen that $a_h = 0$ for all $h \geq 0$ such that $v_p(h + n) \leq j$, in which case we set $i_j = \infty$. The indices of inseparability are defined recursively in terms of $i_j$ by $i_\nu = i_\nu = 0$ and $i_j = \min\{i_{j_1} + v_L(p) : j_1 \leq j \leq 1\}$ for $j = \nu - 1, \ldots, 1, 0$. Thus

$$i_j = \min\{i_{j_1} + (j_1 - j)v_L(p) : j_1 \leq j \leq 1\}.$$

It follows from the definitions that $0 = i_\nu < i_{\nu-1} \leq i_{\nu-1} \leq \cdots \leq i_0$. If $\text{char}(K) = p$ then $(j_1 - j)v_L(p) = 0$ for $j_1 = j$ and $(j_1 - j)v_L(p) = \infty$ for $j_1 > j$, so $i_j = \tilde{i_j}$ in this
case. If char$(K) = 0$ then $i_j$ can depend on the choice of $\pi_L$, and it is not obvious that $i_j$ is a well-defined invariant of the extension $L/K$. We will have more to say about this issue in Remark 2.5.

Following [3, (4.4)], for $0 \leq j \leq \nu$ we define functions $\tilde{\phi}^j_{L/K} : [0, \infty) \to [0, \infty)$ by $\tilde{\phi}^j_{L/K}(x) = i_j + p^j x$. The generalized Hasse-Herbrand functions $\phi^j_{L/K} : [0, \infty) \to [0, \infty)$ are then defined by

$$\phi^j_{L/K}(x) = \min \{ \tilde{\phi}^j_{L/K}(x) : 0 \leq j_0 \leq j \}. \quad (2.2)$$

Hence we have $\phi^j_{L/K}(x) \leq \phi^{j'}_{L/K}(x)$ for $0 \leq j' \leq j$. Let $\phi_{L/K} : [0, \infty) \to [0, \infty)$ be the usual Hasse-Herbrand function, as defined for instance in Chapter IV of [7]. Then by [5, Cor. 6.11] we have $\phi^j_{L/K}(x) = n\phi_{L/K}(x)$.

We wish to reformulate the definition of $\phi^j_{L/K}(x)$. We will use the following elementary fact about binomial coefficients, which is proved in [3, Lemma 5.6].

**Lemma 2.1** Let $b \geq c \geq 1$. Then $v_p \left( \binom{b}{c} \right) \geq v_p(b) - v_p(c)$, with equality if $v_p(b) \geq v_p(c)$ and $c$ is a power of $p$.

**Proposition 2.2** For $0 \leq j \leq \nu$ and $x \geq 0$ we have

$$\phi^j_{L/K}(x) = \min \left\{ h + v_L \left( \binom{h + n}{p^{j_0}} \right) + p^{j_0} x : 0 \leq j_0 \leq j, a_h \neq 0 \right\}. \quad \text{(2.1)}$$

**Proof:** Using (2.1)–(2.2) we get

$$\phi^j_{L/K}(x) = \min \{ h + (j_1 - j_0) v_L(p) + p^{j_0} x : 0 \leq j_0 \leq j, j_0 \leq j_1 \leq \nu, v_p(h + n) \leq j_1, a_h \neq 0 \}. \quad \text{(2.2)}$$

If $j_0 > v_p(h + n)$ then we can replace $j_0$ with $j_0 - 1$ and $j_1$ with $j_1 - 1$ without increasing the value of $h + (j_1 - j_0) v_L(p) + p^{j_0} x$. Hence we may assume $j_0 \leq v_p(h + n)$ and $j_1 = v_p(h + n)$. It follows that

$$\phi^j_{L/K}(x) = \min \left\{ h + (v_p(h + n) - j_0) v_L(p) + p^{j_0} x : 0 \leq j_0 \leq j, j_0 \leq v_p(h + n), a_h \neq 0 \right\}$$

$$= \min \left\{ h + v_L \left( \binom{h + n}{p^{j_0}} \right) + p^{j_0} x : 0 \leq j_0 \leq j, j_0 \leq v_p(h + n), a_h \neq 0 \right\}$$

$$= \min \left\{ h + v_L \left( \binom{h + n}{p^{j_0}} \right) + p^{j_0} x : 0 \leq j_0 \leq j, a_h \neq 0 \right\},$$

where the second and third equalities follow from Lemma 2.1. \hfill \Box

For $d \geq 0$ set $B_d = \mathcal{O}_L / \mathcal{M}_L^{n+d}$ and let $A_d = (\mathcal{O}_K + \mathcal{M}_L^{n+d}) / \mathcal{M}_L^{n+d}$ be the image of $\mathcal{O}_K$ in $B_d$. For $0 \leq j \leq \nu$ set $B_d[\epsilon_j] = B_d[\epsilon] / (\epsilon^{p^j+1})$, so that $\epsilon_j = \epsilon + (\epsilon^{p^j+1})$ satisfies $\epsilon_j^{p^j+1} = 0$. 

3
Proposition 2.3  Let \( 0 \leq j \leq \nu \), let \( d \geq c \geq 0 \), and let \( u \in \mathcal{O}_L[\epsilon_j]^{\times} \). Also let \( F(X) \in \mathcal{O}_K[[X]] \) be a power series with Weierstrass degree \( n \) such that \( F(\pi_L) = \pi_K \). Then the following are equivalent:

1. \( F(\pi_L + u\pi_L^{c+1}\epsilon_j) \equiv \pi_K \) (mod \( \pi_L^{n+d} \)).
2. There exists an \( A_d \)-algebra homomorphism \( s_d : B_d \to B_d[\epsilon_j] \) such that \( s_d(\pi_L) = \pi_L + u\pi_L^{c+1}\epsilon_j \).
3. There exists an \( A_d \)-algebra homomorphism \( s_d : B_d \to B_d[\epsilon_j] \) such that
   \[
   s_d \equiv \text{id}_{B_d} \pmod{\pi_L^{c+1}\epsilon_j} \\
   s_d \not\equiv \text{id}_{B_d} \pmod{\pi_L^{c+1}\epsilon_j \cdot (\pi_L, \epsilon_j)}.
   \]

Proof: Suppose Condition 1 holds. Let \( \tilde{u}(X, \epsilon_j) \) be an element of \( \mathcal{O}_K[[X]][\epsilon_j] \) such that \( \tilde{u}(\pi_L, \epsilon_j) \equiv u \) (mod \( \pi_L^{n+d} \)). Since the Weierstrass polynomial of \( F(X) - \pi_K \) is the minimum polynomial of \( \pi_L \) over \( K \) we have \( \mathcal{O}_L \cong \mathcal{O}_K[[X]]/(F(X) - \pi_K) \). Therefore the \( \mathcal{O}_K \)-algebra homomorphism \( \tilde{s} : \mathcal{O}_K[[X]] \to \mathcal{O}_K[[X]][\epsilon_j] \) defined by \( \tilde{s}(X) = X + \tilde{u}X^{c+1}\epsilon_j \) induces an \( A_d \)-algebra homomorphism \( s_d : B_d \to B_d[\epsilon_j] \) such that \( s_d(\pi_L) = \pi_L + u\pi_L^{c+1}\epsilon_j \).

Therefore Condition 2 holds. On the other hand, if Condition 2 holds then applying the homomorphism \( s_d \) to the congruence \( F(\pi_L) \equiv \pi_K \) (mod \( \pi_L^{n+d} \)) gives Condition 1. Hence the first two conditions are equivalent. Suppose Condition 2 holds. Since \( d \geq c \) and \( n \geq 2 \) we see that \( s_d \) satisfies the requirements of Condition 3. Suppose Condition 3 holds. Then \( s_d(\pi_L) = \pi_L + v\pi_L^{c+1}\epsilon_j \) for some \( v \in B_d[\epsilon_j]^{\times} \). Let \( \gamma : B_d[\epsilon_j] \to B_d[\epsilon_j] \) be the \( B_d \)-algebra homomorphism such that \( \gamma(\epsilon_j) = uv^{-1}\epsilon_j \), and define \( s'_d : B_d \to B_d[\epsilon_j] \) by \( s'_d = \gamma \circ s_d \). Then \( s'_d \) satisfies the requirements of Condition 2. \( \square \)

The assumptions on \( F(X) \) imply that \( F(\pi_L + \pi_L^{c+1}\epsilon_j) \equiv \pi_K \) (mod \( \pi_L^{n+c} \)). Therefore the conditions of the proposition are satisfied when \( d = c \). On the other hand, since \( L/K \) is separable we have \( F(\pi_L + u\pi_L^{c+1}\epsilon_j) \neq \pi_K \). Hence for \( d \) sufficiently large the conditions in the proposition are not satisfied. We define a function \( \Phi_{L/K}^j : \mathbb{N}_0 \to \mathbb{N}_0 \) by setting \( \Phi_{L/K}^j(c) \) equal to the largest integer \( d \) satisfying the equivalent conditions of Proposition 2.3. By Condition 3 we see that this definition does not depend on the choice of \( \pi_L, u, \) or \( F \).

We now show that \( \Phi_{L/K}^j \) and \( \phi_{L/K}^j \) agree on nonnegative integers. This gives an alternative description of the restriction of \( \phi_{L/K}^j \) to \( \mathbb{N}_0 \) which does not depend on the indices of inseparability.

Proposition 2.4  For \( c \in \mathbb{N}_0 \) we have \( \Phi_{L/K}^j(c) = \phi_{L/K}^j(c) \).

Proof: Let \( c \in \mathbb{N}_0 \). Since \( \tilde{F}(X) \) satisfies the hypotheses for \( F(X) \) in Proposition 2.3 \( \Phi_{L/K}^j(c) \) is equal to the largest \( d \in \mathbb{N}_0 \) such that

\[
\tilde{F}(\pi_L + \pi_L^{c+1}\epsilon_j) \equiv \tilde{F}(\pi_L) \pmod{\pi_L^{n+d}}.
\]
For \( m \geq 0 \) define
\[
(D^m \hat{F})(X) = \sum_{h=0}^{\infty} \left( \begin{array}{c} h + n \\ m \end{array} \right) a_h X^{h+n-m}.
\]

Then
\[
\hat{F}(X + \epsilon_j X^{c+1}) = \sum_{m=0}^{p^{j+1}-1} (D^m \hat{F})(X) \cdot (\epsilon_j X^{c+1})^m.
\]

Since \( \epsilon_j, \epsilon_j^2, \ldots, \epsilon_j^{p^{j+1}-1} \) are linearly independent over \( \mathcal{O}_L \), (2.3) holds if and only if
\[
(D^m \hat{F})(\pi_L) \cdot \pi_L^{(c+1)m} \in \mathcal{M}_L^{n+d} \text{ for } 1 \leq m < p^{j+1}.
\] (2.4)

Hence by Proposition 2.2 it is sufficient to prove that (2.4) is equivalent to the following:
\[
h + v_L \left( \left( \begin{array}{c} h + n \\ p^{j_0} \end{array} \right) \right) + cp^{j_0} \geq d \text{ for all } j_0, h \text{ such that } 0 \leq j_0 \leq j \text{ and } a_h \neq 0. \] (2.5)

Assume first that (2.5) holds. Choose \( m \) such that \( 1 \leq m < p^{j+1} \) and write \( m = rp^{j_0} \) with \( p \nmid r \) and \( j_0 \leq j \). Choose \( h \geq 0 \) such that \( a_h \neq 0 \) and set \( l = v_p(h+n) \). If \( m > h+n \) then \( \left( \begin{array}{c} h + n \\ m \end{array} \right) = 0 \), so we have
\[
\left( \begin{array}{c} h + n \\ m \end{array} \right) a_h \pi_L^{h+n-m} \cdot \pi_L^{(c+1)m} \in \mathcal{M}_L^{n+d}.
\] (2.6)

Suppose \( m \leq h+n \) and \( l \geq j_0 \). Using Lemma 2.1 we get
\[
v_p \left( \left( \begin{array}{c} h + n \\ m \end{array} \right) \right) \geq l - j_0 = v_p \left( \left( \begin{array}{c} h + n \\ p^{j_0} \end{array} \right) \right).
\]

Combining this with (2.5) we get
\[
h + v_L \left( \left( \begin{array}{c} h + n \\ m \end{array} \right) \right) + cm + n \geq h + v_L \left( \left( \begin{array}{c} h + n \\ p^{j_0} \end{array} \right) \right) + cp^{j_0} + n \geq n + d.
\]

Hence (2.6) holds in this case. Finally, suppose \( m \leq h+n \) and \( l < j_0 \leq j \). It follows from Lemma 2.1 that \( v_L \left( \left( \begin{array}{c} h + n \\ p^l \end{array} \right) \right) = 0 \), so by (2.5) we have \( h + c p^l \geq d \). Since \( m \geq p^{j_0} > p^l \) we get
\[
h + v_L \left( \left( \begin{array}{c} h + n \\ m \end{array} \right) \right) + cm + n \geq h + c p^l + n \geq n + d.
\]

Therefore (2.6) holds in this case as well. It follows that every term in \( (D^m \hat{F})(\pi_L) \) lies in \( \mathcal{M}_L^{n+d} \), so (2.4) holds.
Assume conversely that (2.4) holds. Among all the nonzero terms that occur in any of the series
\[(D^m \hat{F})(\pi_L) \cdot \pi_L^{(c+1)p^i} = \sum_{h=0}^{\infty} a_h \left(\frac{h+n}{p^i}\right) \pi_L^{h+n+cp^i}\]
for \(0 \leq i \leq j\) let \(a_h \left(\frac{h+n}{p^i}\right) \pi_L^{n+h+cp^i}\) be a term whose \(L\)-valuation \(w\) is minimum. If \(\text{char}(K) = p\) then for each \(m \geq 1\) the nonzero terms of \((D^m \hat{F})(\pi_L)\) have distinct \(L\)-valuations, so it follows from (2.4) that \(w \geq n + d\). Suppose \(\text{char}(K) = 0\) and set \(l = v_p(h + n)\). If \(i > l\) then since \(v_L \left(\left(\frac{h+n}{p^i}\right)\right) = 0\) we have
\[v_L \left(\left(\frac{h+n}{p^i}\right) \pi_L^{n+h+cp^i}\right) \leq v_L \left(\left(\frac{h+n}{p^i}\right) \pi_L^{n+h+cp^i}\right) = w.\]
Therefore we may assume \(i \leq l\). Since \(v_p \left(\left(\frac{n}{p^i}\right)\right) = \nu - i\) and \(a_0 \neq 0\) we have \(l \leq \nu\).
Suppose \(w < n + d\). Then it follows from (2.4) that there is \(h' \neq h\) such that \(a_{h'} \neq 0\) and
\[v_L \left(\left(\frac{h'+n}{p^i}\right) \pi_L^{n+h'+cp^i}\right) = v_L \left(\left(\frac{h+n}{p^i}\right) \pi_L^{n+h+cp^i}\right). \tag{2.7}\]
Since \(n \mid v_L(p)\) this implies \(h' \equiv h \pmod{n}\). Since \(v_p(h + n) \leq \nu\) and \(v_p(h' + n) \leq \nu\) we get \(v_p(h' + n) = v_p(h + n) = l\). Therefore by Lemma 2.1 we have
\[v_p \left(\left(\frac{h'+n}{p^i}\right)\right) = v_p \left(\left(\frac{h+n}{p^i}\right)\right) = l - i.\]
Combining this with (2.7) gives \(h' = h\), a contradiction. Therefore \(w \geq n + d\) holds in general. Hence by the minimality of \(w\) we get (2.5). \(\Box\)

**Remark 2.5** If \(\text{char}(K) = 0\) then the value of \(i_j\) may depend on the choice of uniformizer \(\pi_L\) for \(L\). It was proved in [3, Th. 7.1] that \(i_j\) is a well-defined invariant of the extension \(L/K\). This can also be deduced from Proposition 2.4 by setting \(c = 0\).

**Remark 2.6** Let \(0 \leq j \leq \nu\). Even though the function \(\phi_{L/K}^j : [0, \infty) \to [0, \infty)\) may not be determined by its restriction to \(N_0\), it is determined by the sequence \((i_0, i_1, \ldots, i_j)\). Since \(i_j = \phi_{L/K}^j(0)\) this implies that the collection consisting of the restrictions of \(\phi_{L/K}^j\) to \(N_0\) for \(0 \leq j_0 \leq j\) determines \(\phi_{L/K}^j\).

For \(0 \leq j \leq \nu\) let \(B_d[\epsilon] = B_d[\epsilon]/(e^{p^i+1})\), so that \(\epsilon_j = \epsilon + (e^{p^i+1})\) satisfies \(\epsilon_j^{p^i+1} = 0\). Define \(\Phi_{L/K}\) : \(N_0 \to N_0\) analogously to \(\Phi_{L/K}^j\), using \(\epsilon_j\) in place of \(\epsilon_j\). Then the arguments in this section remain valid with \(\epsilon_j, \Phi_{L/K}^j\) replaced by \(\epsilon_j, \Phi_{L/K}\). (In particular, note that the proof that (2.4) implies (2.5) only uses the fact that (2.4) holds with \(m = p^i\) for \(0 \leq i \leq j\).) Hence by Propositions 2.3 and 2.4 and their analogs for \(\Phi_{L/K}\) we get the following:
Corollary 2.7 Let \( c, d \in \mathbb{N}_0 \), let \( u \in \mathcal{O}_L[\epsilon_j]^{\times} \), and let \( \pi \in \mathcal{O}_L[\epsilon_j]^{\times} \). In addition, let \( F(X) \in \mathcal{O}_K[[X]] \) be a power series with Weierstrass degree \( n \) such that \( F(\pi_L) = \pi_K \). Then the following are equivalent:

1. \( \phi^j_{L/K}(c) \geq d \).
2. \( F(\pi_L + u\pi_L^{c+1}\epsilon_j) \equiv F(\pi_L) \pmod{\pi_L^{n+d}} \).
3. \( F(\pi_L + \pi_L^{c+1}\epsilon_j) \equiv F(\pi_L) \pmod{\pi_L^{n+d}} \).

Some of the proofs in Section 3 depend on “tame shifts”: Let \( e \geq 1 \) be relatively prime to \( p[L : K] = pn \), let \( K_e/K \) be a totally ramified subextension of \( K^{sep}/K \) of degree \( e \), and set \( L_e = LK_e \). Then \( L_e/K_e \) is a totally ramified extension of degree \( n \) which is closely related to \( L/K \):

Lemma 2.8 Let \( x \geq 0 \). Then for \( 0 \leq j \leq \nu \) we have

\[
\tilde{\phi}^j_{L_e/K_e}(x) = e \tilde{\phi}^j_{L/K}(x/e) \\
\phi^j_{L_e/K_e}(x) = e \phi^j_{L/K}(x/e).
\]

Proof: It suffices to show that \( e^{i_0}, e^{i_1}, \ldots, e^{i_\nu} \) are the indices of inseparability of \( L_e/K_e \).

By Proposition 2.4 this is equivalent to showing that \( \Phi^j_{L_e/K_e}(0) = e \Phi^j_{L/K}(0) \). There are uniformizers \( \pi_K, \pi_{K_e}, \pi_L, \pi_{L_e} \) for \( K, K_e, L, L_e \) such that \( \pi_{L_e} = \pi_K \) and \( \pi_{L_e} = \pi_L \). As above we let \( \tilde{\mathcal{F}}(X) \) be a series with coefficients in \( R \) such that \( \pi_K = \tilde{\mathcal{F}}(\pi_L) \). Then \( \pi_{K_e} = \tilde{\mathcal{F}}(\pi_{L_e})^{1/e} \). Hence the series \( \tilde{\mathcal{F}}_e(X) = \tilde{\mathcal{F}}(X^{1/e}) \in \mathcal{O}_K[[X]] \) satisfies \( \tilde{\mathcal{F}}_e(\pi_{L_e}) = \pi_{K_e} \).

If \( \Phi^j_{L/K}(0) \geq d \) then

\[
\tilde{\mathcal{F}}_e(\pi_{L_e} + \pi_{L_e}\epsilon_j) = \tilde{\mathcal{F}}(\pi_L(1 + \epsilon_j)^e)^{1/e} \\
\equiv \tilde{\mathcal{F}}(\pi_L)^{1/e} \pmod{\pi_K \cdot \pi_L^d} \\
\equiv \tilde{\mathcal{F}}_e(\pi_{L_e}) \pmod{\pi_{L_e}^{n+de}},
\]

and hence \( \Phi^j_{L_e/K_e}(0) \geq de \). Conversely, if \( \Phi^j_{L_e/K_e}(0) \geq d \) then

\[
\tilde{\mathcal{F}}(\pi_L + \pi_L\epsilon_j) = \tilde{\mathcal{F}}_e(\pi_{L_e}(1 + \epsilon_j)^e)^e \\
\equiv \tilde{\mathcal{F}}_e(\pi_{L_e})^e \pmod{\pi_K \cdot \pi_L^{de}} \\
\equiv \tilde{\mathcal{F}}(\pi_L) \pmod{\pi_L^{n+[d/e]}},
\]

and hence \( \Phi^j_{L/K}(0) \geq [d/e] \). By combining these results we get \( \Phi^j_{L_e/K_e}(0) = e \Phi^j_{L/K}(0) \). □
3 Towers of extensions

In this section we consider a tower $M/L/K$ of finite totally ramified subextensions of $K^{sep}/K$. Our goal is to determine relations between the generalized Hasse-Herbrand functions $\phi^j_{M/K}$ of the extension $M/K$ and the corresponding functions for $L/K$ and $M/L$. It is well-known that the indices of inseparability of $L/K$ and $M/L$ do not always determine the indices of inseparability of $M/K$ (see for instance Example 5.8 in [3] or Remark 7.8 in [5]). Therefore we cannot expect to obtain a general formula which expresses $\phi^j_{M/K}$ in terms of $\phi^j_{L/K}$ and $\phi^k_{M/L}$. However, we do get a lower bound for $\phi^j_{M/K}(x)$, and we are able to show that this lower bound is equal to $\phi^j_{M/K}(x)$ in certain cases.

Set $[L : K] = n$, $[M : L] = m$, $\nu = v_p(n)$, and $\mu = v_p(m)$. Let $\pi_K$, $\pi_L$, $\pi_M$ be uniformizers for $K$, $L$, $M$. Let $F(X) \in \mathcal{O}_K[[X]]$ be a power series with Weierstrass degree $n$ such that $F(\pi_L) = \pi_K$ and define

$$F^*(\epsilon) = \pi_K^{-1}(F(\pi_L + \pi_L\epsilon) - \pi_K).$$

Then $F^*(\epsilon) \in \mathcal{O}_L[[\epsilon]]$ is uniquely determined by $\pi_L$ up to multiplication by an element of $\mathcal{O}_L[[\epsilon]]^\times$.

Write $F^*(\epsilon) = c_1\epsilon + c_2\epsilon^2 + \cdots$ and define the “valuation function” of $F^*$ with respect to $\nu_K$ by

$$\Psi_{F^*(\epsilon)}^K(x) = \min \{v_K(c_i) + ix : i \geq 1\} \quad (3.1)$$

for $x \in [0, \infty)$. The graph of $\Psi_{F^*(\epsilon)}^K$ is the Newton copolygon of $F^*(\epsilon)$ with respect to $\nu_K$. Gross [4, Lemma 1.5] attributes the following observation to Tate:

**Proposition 3.1** For $x \geq 0$ we have $\phi_{L/K}(x) = \Psi_{F^*(\epsilon)}^K(x)$.

Suppose we also have $G(X) \in \mathcal{O}_K[[X]]$ with Weierstrass degree $m = [M : L]$ such that $G(\pi_M) = \pi_L$. Set $H(X) = F(G(X))$. Then $H(X) \in \mathcal{O}_K[[X]]$ has Weierstrass degree $nm = [M : K]$ and $H(\pi_M) = \pi_K$. It follows that we can use the series

$$G^*(\epsilon) = \pi_L^{-1}(G(\pi_M + \pi_M\epsilon) - \pi_L)$$

$$H^*(\epsilon) = \pi_K^{-1}(H(\pi_M + \pi_M\epsilon) - \pi_K)$$

to compute the Hasse-Herbrand functions for the extensions $M/L$ and $M/K$. As Lubin points out in [6, Th. 1.6], by applying Proposition 3.1 to the relation $H^*(\epsilon) = F^*(G^*(\epsilon))$, we obtain the well-known composition formula $\phi_{M/K} = \phi_{L/K} \circ \phi_{M/L}$.

We wish to extend the results above to apply to the generalized Hasse-Herbrand functions $\phi^j_{L/K}$. For $0 \leq j \leq \nu$ let $F^*(\epsilon_j)$ denote the image of $F^*(\epsilon)$ in $\mathcal{O}_L[[\epsilon]]/(\epsilon^{p^{j+1}}) \cong \mathcal{O}_L[\epsilon_j]$. Alternatively, we may view $F^*(\epsilon_j)$ as the polynomial obtained by discarding all the terms of $F^*(\epsilon)$ of degree $\geq p^{j+1}$. Therefore it makes sense to consider the valuation function $\Psi_{F^*(\epsilon_j)}^L(x)$ of $F^*(\epsilon_j)$.

**Proposition 3.2** $\phi^j_{L/K}(x) = \Psi_{F^*(\epsilon_j)}^L(x)$ for all $x \in [0, \infty)$. 
Proof: We first prove that $\phi_{L/K}^j$ and $\Psi_{F^*(\epsilon)}^L$ agree on $\mathbb{N}_0$. Let $d \geq b \geq 0$. Then $\Phi_{L/K}^j(b) \geq d$ if and only if $F^*(\pi_{L}^j \epsilon) \equiv 0 \pmod{\pi_{L}^d}$. By (3.1) this is equivalent to $\Psi_{F^*(\epsilon)}^L(b) \geq d$. Since $\Phi_{L/K}^j$ and $\Psi_{F^*(\epsilon)}^L$ map $\mathbb{N}_0$ to $\mathbb{N}_0$, this implies $\Phi_{L/K}^j(c) = \Psi_{F^*(\epsilon)}^L(c)$ for all $c \in \mathbb{N}_0$. Using Proposition 2.7 we deduce that $\Phi_{L/K}^j(c) = \Psi_{F^*(\epsilon)}^L(c)$ for $c \in \mathbb{N}_0$.

Now choose $e \geq 1$ relatively prime to $p | [L : K] = pn$ and let $K_e/K$ be a totally ramified subextension of $K^sep/K$ of degree $e$. Then $L_e = L K_e$ is a totally ramified extension of $L$ degree $e$. Let $\pi_K, \pi_K, \pi_L, \pi_{L_e}$ be uniformizers for $K, K_e, L, L_e$ such that $\pi_{K_e}^x = \pi_K$ and $\pi_{L_e}^x = \pi_L$. Then $\pi_K = F(\pi_{L_e}^e)^{1/e}$, so the series $F_e(X) = F(X^e)^{1/e}$ satisfies $F_e(\pi_{L_e}) = \pi_{K_e}$. Let

$$F_e^*(\epsilon) = \pi_{K_e}^{-1}(F_e(\pi_{L_e} + \pi_{L_e} \epsilon) - \pi_{K_e}) = (1 + F^*(1 + \epsilon)^e - 1)^{1/e} - 1.$$ 

Then $F_e^*(\epsilon) = \eta^{-1}(F_e^* (\eta(\epsilon)))$, where $\eta(\epsilon) = (1 + \epsilon)^e - 1$ and $\eta^{-1}(\epsilon) = (1 + \epsilon)^{1/e} - 1$ have coefficients in $\mathcal{O}_K$. It follows that for $0 \leq j \leq \nu$ we have $F_e^*(\epsilon_j) = \eta^{-1}(F_e^*(\eta(\epsilon_j)))$, so for $c \in \mathbb{N}_0$ we get

$$\Psi_{F^*(\epsilon)}^L(c) = \Psi_{F^*(\epsilon)}^L(c) = e \Psi_{F^*(\epsilon)}^L(c/e).$$

By Lemma 2.7 we have $\phi_{L/K}^j(c/e) = e^{-1}\phi_{L/K}^j(c_e)$. Since the proposition holds for the extension $L_e/K_e$ with $x = c$ this implies

$$\phi_{L/K}^j(c/e) = e^{-1}\phi_{L/K}^j(c/e) = \Psi_{F^*(\epsilon)}^L(c/e).$$

Since the set $\{c/e : c, e \in \mathbb{N}, \gcd(e, pn) = 1\}$ is dense in $[0, \infty)$, and $\phi_{L/K}^j, \Psi_{F^*(\epsilon)}^L$ are continuous on $[0, \infty)$, we conclude that $\phi_{L/K}^j(x) = \Psi_{F^*(\epsilon)}^L(x)$ for all $x \in [0, \infty)$. $\square$

Following [5] (4.4), for $0 \leq j \leq \nu$ and $m \in \mathbb{N}$ we define functions on $[0, \infty)$ by

$$\tilde{\phi}_{L/K}^j(m) = m \phi_{L/K}^j(x/m) = mi_j + p^j x$$

$$\phi_{L/K}^j(m) = m \phi_{L/K}^j(x/m) = \min \{ \phi_{L/K}^j(x/m) : 0 \leq j_0 \leq j \}.$$ 

For $0 \leq l \leq \nu + \mu$ let

$$\Omega_l = \{(j, k) : 0 \leq j \leq \nu, 0 \leq k \leq \mu, j + k = l \},$$

and for $x \geq 0$ define

$$\lambda_{M/K}^l(x) = \min \{ \phi_{L/K}^j(m) : (j, k) \in \Omega_l \} = \min \{ \phi_{L/K}^j(m) : (j, k) \in \Omega_{l_0} \text{ for some } 0 \leq l_0 \leq l \}.$$

For $0 \leq a \leq l$ set

$$S^m_l(x) = \{(j, k) \in \Omega_a : \phi_{L/K}^j(m) = \lambda_{M/K}^l(x) \}.$$
Theorem 3.3 Let \( 0 \leq l \leq \nu + \mu \) and \( x \in [0, \infty) \). Then

(a) \( \phi_{M/K}^l(x) \geq \lambda_{M/K}^l(x) \).

(b) Suppose there exists \( l_0 \leq l \) such that \( |S_l(x)| = 1 \). Then \( \phi_{M/K}^l(x) = \lambda_{M/K}^l(x) \).

The rest of the paper is devoted to proving this theorem. We first consider the cases where \( x = c \in \mathbb{N}_0 \). The proof in these cases is based on Proposition 2.4. To get information about \( \Phi_{M/K}^l(c) \) we compute the most significant terms of \( \hat{F}(\hat{G}(\pi_M + \pi_M^c + 1)) \).

It follows from Proposition 2.4 that for \( 0 \leq j \leq \nu \) we have

\[ \hat{F}(\pi_L(1 + \epsilon)) \equiv \pi_K \pmod{\pi^n_L \cdot (\pi_{\nu}^{i_j}, \epsilon^{p^j} + 1)}. \]  

(3.2)

In addition, since \( X^n \) divides \( \hat{F}(X) \) we have

\[ \hat{F}(\pi_L(1 + \epsilon)) \equiv \pi_K \pmod{\pi^n_L \cdot (\pi_{\nu}^{i_j}, \epsilon^{p^j + 1})}. \]  

(3.3)

Define an ideal in \( \mathcal{O}_L[[\epsilon]] \) by

\[ I_F = (\pi_{0}^{i_0}, \epsilon^{p_0}) \cap (\pi_{1}^{i_1}, \epsilon^{p_1}) \cap \cdots \cap (\pi_{\nu}^{i_\nu}, \epsilon^{p_{\nu} + 1}) \cap (\epsilon) \]

\[ = (\pi_{0}^{i_0} \cdot \epsilon^{p_0}, \pi_{1}^{i_1} \cdot \epsilon^{p_1}, \ldots, \pi_{\nu}^{i_\nu} \cdot \epsilon^{p_{\nu}}). \]

It follows from (3.2) and (3.3) that

\[ \hat{F}(\pi_L(1 + \epsilon)) \equiv \pi_K \pmod{\pi^n_L \cdot I_F}. \]  

(3.4)

Let \( i'_0, i'_1, \ldots, i'_\mu \) be the indices of inseparability of \( M/L \). As above we find that

\[ \hat{G}(\pi_M(1 + \epsilon)) \equiv \pi_L \pmod{\pi^m_M \cdot I_G}, \]

where \( I_G \) is the ideal in \( \mathcal{O}_M[[\epsilon]] \) defined by

\[ I_G = (\pi_M^{i'_0} \cdot \epsilon^{p_0}, \pi_M^{i'_1} \cdot \epsilon^{p_1}, \ldots, \pi_M^{i'_\mu} \cdot \epsilon^{p_{\mu}}). \]

By replacing \( \epsilon \) with \( \pi_M^{i'_c} \epsilon \) we get

\[ \hat{G}(\pi_M(1 + \pi_M^c \epsilon)) \equiv \pi_L \pmod{\pi^m_M \cdot I'_G}, \]  

(3.5)

where \( I'_G \) is the ideal in \( \mathcal{O}_M[[\epsilon]] \) defined by

\[ I'_G = (\pi_M^{i'_0} \cdot \epsilon^{p_0}, \pi_M^{i'_1} \cdot \epsilon^{p_1}, \ldots, \pi_M^{i'_\mu} \cdot \epsilon^{p_{\mu}}). \]
It follows from (3.4) and (3.5) that there are \( r_j, s_k \in R, \delta \in (\pi, \epsilon) \cdot I \), and \( \delta \in (\pi, \epsilon) \cdot I_g \) such that
\[
\hat{F}(\pi_L(1 + \epsilon)) = \pi_K \cdot \left( 1 + \sum_{j=0}^{\nu} r_j \pi_L^i p^j + \delta \right) \tag{3.6}
\]
\[
\hat{G}(\pi_M(1 + \pi^c \epsilon)) = \pi_L \cdot \left( 1 + \sum_{k=0}^{\mu} s_k \pi_M^k \pi_M(\epsilon) \pi^k + \delta \right). \tag{3.7}
\]

Define an ideal in \( \mathcal{O}_M[[\epsilon]] \) by
\[
I_{FG} = \left( \frac{\hat{G}}{\pi^c \cdot \pi_M^{\nu} \cdot \pi_M(\epsilon) \cdot \pi^k + \delta} : 0 \leq j \leq \nu, 0 \leq k \leq \mu \right) = \left( \frac{\lambda^g_{\pi, M \cdot K}(\epsilon)}{\pi^c \cdot \pi_M^{\nu} \cdot \pi_M(\epsilon) \cdot \pi^k + \delta} : 0 \leq g \leq \nu + \mu \right).
\]

Hence for \( d \geq 0 \) and \( 0 \leq g \leq \nu + \mu \) we have \( \pi_M^d \pi^g \in I_{FG} \) if and only if \( d \geq \lambda^g_{\pi, M \cdot K}(\epsilon) \). We also define \( u = \pi_L / \pi_M^{\nu} \in \mathcal{O}_M^\times \).

**Lemma 3.4** Let \( 0 \leq j \leq \nu \). Then
\[
\pi_L^{i_j} \left( \sum_{k=0}^{\mu} s_k \pi_M^k \pi_M(\epsilon) \pi^k + \delta \right) \equiv u^{i_j} \sum_{k=0}^{\mu} s_k \pi_M^k \pi_M(\epsilon) \pi^k \pmod{(\pi, \epsilon) \cdot I_{FG}}.
\]

**Proof:** For \( 0 \leq j \leq \nu \) define ideals in \( \mathbb{Z}[X_0, X_1, \ldots, X_\mu] \) by
\[
H_j = (p^h X_k^{j-h} : 1 \leq h \leq j, 0 \leq k \leq \mu).
\]

By induction on \( j \) we get
\[
(X_0 + X_1 + \cdots + X_\mu)^{p^j} \equiv X_0^{p^j} + X_1^{p^j} + \cdots + X_\mu^{p^j} \pmod{H_j}.
\]

Since both sides of this congruence are homogeneous polynomials of degree \( p^j \), it follows that
\[
(X_0 + X_1 + \cdots + X_\mu)^{p^j} \equiv X_0^{p^j} + X_1^{p^j} + \cdots + X_\mu^{p^j} \pmod{H_j}, \tag{3.8}
\]

where
\[
H_j = (p^h X_k^{j-h} X_w : 1 \leq h \leq j, 0 \leq k \leq \mu, 0 \leq w \leq \mu).
\]

Since \( \delta \in (\pi, \epsilon) \cdot I_g \) there are \( \tilde{s}_k \in \mathcal{O}_M[[\epsilon]] \) such that \( \tilde{s}_k \equiv s_k \pmod{(\pi, \epsilon)} \) and
\[
\sum_{k=0}^{\mu} \tilde{s}_k \pi_M^k \pi_M(\epsilon) \pi^k + \delta \equiv \sum_{k=0}^{\mu} s_k \pi_M^k \pi_M(\epsilon) \pi^k.
\]
Hence by replacing $X_k$ with $\tilde{s}_k\tilde{\phi}_M^{\Delta} e^{p^k}$ for $0 \leq k \leq \mu$ in (3.8) we get
\[
\left( \sum_{k=0}^{\mu} s_k\tilde{\phi}_M^{\Delta} e^{p^k} + \delta_G \right)^{p^j} \equiv \sum_{k=0}^{\mu} s_k^{p^j} \tilde{\phi}_M^{\Delta} e^{p^{j+k}} \pmod{\epsilon \cdot A},
\]
where $A$ is the ideal in $\mathcal{O}_M[[\epsilon]]$ defined by
\[
A = (p^h(\tilde{\phi}_M^{\Delta} e^{p^k})^{p^j-j} : 1 \leq h \leq j, 0 \leq k \leq \mu).
\]

Let $1 \leq h \leq j$ and $0 \leq k \leq \mu$. Since $i_j + hv_L(p) \geq i_{j-h}$ we have
\[
v_M(\pi_L^{i_j} p^h \pi_M^{i_{j-h}} \tilde{\phi}_M^{\Delta}) \geq mi_{j-h} + p^{j-h} \tilde{\phi}_M^{\Delta}(c)
\]
\[= \tilde{\phi}_L^{j-h,m}(\tilde{\phi}_M^{\Delta}(c))
\]
\[\geq \lambda_{j-M/K}(c).
\]

It follows that $\pi_L^{i_j} \epsilon \cdot p^h(\tilde{\phi}_M^{\Delta} e^{p^k})^{p^j-j} \in \epsilon \cdot I_{FG}$, and hence that $\pi_L^{i_j} \epsilon \cdot A \subset \epsilon \cdot I_{FG}$. Therefore
\[
\pi_L^{i_j} \left( \sum_{k=0}^{\mu} s_k\tilde{\phi}_M^{\Delta} e^{p^k} + \delta_G \right)^{p^j} \equiv \pi_L^{i_j} \sum_{k=0}^{\mu} s_k^{p^j} \tilde{\phi}_M^{\Delta} e^{p^{j+k}} \pmod{\epsilon \cdot I_{FG}}
\]
\[\equiv u^{i_j} \sum_{k=0}^{\mu} s_k^{p^j} \tilde{\phi}_L^{i_j,m}(\tilde{\phi}_M^{\Delta}) e^{p^{j+k}} \pmod{\epsilon \cdot I_{FG}}.
\]

Since $\tilde{s}_k \equiv s_k \pmod{\pi_M, \epsilon}$ the lemma follows. $\Box$

We now replace $\epsilon$ with $\sum_{k=0}^{\mu} s_k\tilde{\phi}_M^{\Delta} e^{p^k} + \delta_G$ in (3.6). With the help of Lemma 3.4 we get
\[
\hat{F}(\tilde{G}(\pi_M(1 + \pi_M^c))) = \pi_K \cdot \left( 1 + \sum_{j=0}^{\nu} r_j u^{i_j} \sum_{k=0}^{\mu} s_k^{p^j} \tilde{\phi}_L^{i_j,m}(\tilde{\phi}_M^{\Delta}) e^{p^{j+k}} + \delta_{FG} \right)
\]
\[= \pi_K \cdot \left( 1 + \sum_{g=0}^{\nu+\mu} \left( \sum_{(j,k) \in \Omega} u^{i_j} r_j s_k^{p^j} \tilde{\phi}_L^{i_j,m}(\tilde{\phi}_M^{\Delta}) e^{p^g} + \delta_{FG} \right) \right) \quad (3.9)
\]
for some $\delta_{FG} \in (\pi_M, \epsilon) \cdot I_{FG}$.

To prove (a) in the case $x = c \in \mathbb{N}_0$ we define an ideal $J_l = (\pi_M^{nm+\lambda_M^{l-M/K}(c)}, e^{p^{l+1}}$ in $\mathcal{O}_M[[\epsilon]]$. Since $\pi_K \cdot I_{FG} \subset J_l$, by (3.9) we get
\[
\hat{F}(\tilde{G}(\pi_M(1 + \pi_M^c))) \equiv \pi_K \pmod{J_l}.
\]
It follows from Corollary 2.7 that $\phi_{M/K}^l(c) \geq \lambda_{M/K}^l(c)$.

Now let $e \geq 1$ be relatively prime to $p[M : K] = pnm$. Let $K_e/K$ be a totally ramified extension of degree $e$ and set $L_e = L K_e$, $M_e = M K_e$. Let $0 \leq h \leq \nu$, $0 \leq i \leq \mu$, and $0 \leq l \leq \nu + \mu$. Using Lemma 2.8 we get

$$\tilde{\phi}_{M/L}^l(x) = e^{-1} \phi_{M_e/L_e}^i(ex) \quad (3.10)$$

$$\tilde{\phi}_{L/K}^{h,m}(x) = e^{-1} \phi_{L_e/K_e}^{h,m}(ex) \quad (3.11)$$

$$\phi_{M/K}^l(x) = e^{-1} \phi_{M_e/K_e}^l(ex) \quad (3.12)$$

$$\lambda_{M/K}^l(x) = e^{-1} \lambda_{M_e/K_e}^l(ex). \quad (3.13)$$

We know from the preceding paragraph that $\phi_{M_e/K_e}^l(c) \geq \lambda_{M_e/K_e}^l(c)$ for every $c \in \mathbb{N}_0$. By applying (3.12) and (3.13) with $x = c/e$ we get $\phi_{M/K}^l(c/e) \geq \lambda_{M/K}^l(c/e)$. It follows that (a) holds whenever $x = c/e$ with $c \geq 0$, $e \geq 1$, and $\gcd(e, pnm) = 1$. Since numbers of this form are dense in $[0, \infty)$, by continuity we get $\phi_{M/K}^l(x) \geq \lambda_{M/K}^l(x)$ for all $x \geq 0$. This proves (a).

To facilitate the proof of (b) we define a subset of the nonnegative reals by

$$T_l(M/K) = \{ t \geq 0 : \exists l_0 \leq l \text{ with } |S_{l_0}^l(t)| = 1 \text{ and } |S_a^l(t)| = 0 \text{ for } 0 \leq a < l_0 \}. \quad (3.14)$$

Suppose $t > 0$ and $(t, \lambda_{M/K}^l(t))$ is not a vertex of the graph of $\lambda_{M/K}^l$. Then there is a unique $0 \leq l_0 \leq l$ such that $|S_{l_0}^l(t)| \geq 1$; in fact, $l_0$ is determined by the condition $\lambda_{M/K}^l(t) = p^{l_0}$. Hence if the hypotheses of (b) are satisfied with $x = t$ then $t \in T_l(M/K)$.

**Lemma 3.5** Suppose the hypotheses of (b) are satisfied with $x = 0$. Then $0 \in T_l(M/K)$.

**Proof:** Suppose $0 \notin T_l(M/K)$, and let $l_0$ be the minimum integer satisfying the hypotheses of (b) with $x = 0$. Also let $l_1 < l_0$ be maximum such that $|S_{l_1}^l(0)| \neq 0$. Then $|S_{l_0}^l(0)| = 2$. Hence there is $(j, k) \in S_{l_1}^l(0)$ such that $k < \mu$. Since

$$\tilde{\phi}_{M/L}^{k+1}(0) = i_{k+1}' \leq i_k' = \tilde{\phi}_{M/L}^k(0)$$

we get

$$\lambda_{M/K}^l(0) \leq \tilde{\phi}_{L/K}^{j,m}(\tilde{\phi}_{M/L}^{k+1}(0)) \leq \tilde{\phi}_{L/K}^{j,m}(\tilde{\phi}_{M/L}^k(0)) = \lambda_{M/K}^l(0).$$

It follows that $\tilde{\phi}_{L/K}^{j,m}(\tilde{\phi}_{M/L}^{k+1}(0)) = \tilde{\phi}_{L/K}^{j,m}(\tilde{\phi}_{M/L}(0))$, so we have $i_k' = i_{k+1}'$ and $(j, k + 1) \in S_{l_1}^{l+1}(0)$. Hence by the maximality of $l_1$ we get $l_1 = l_0 - 1$. Since $|S_{l_1}^{l_0}(0)| = 1$ we must have $|S_{l_0}^{l_1}(0)| = 2$ and $(l_0 - \mu - 1, \mu) \in S_0^{l_0-1}(0)$. Since

$$mi_{l_0 - \mu} \leq mi_{l_0 - \mu - 1} = \tilde{\phi}_{L/K}^{l_0 - \mu, m}(\tilde{\phi}_{M/L}^{\mu}(0)) = \lambda_{M/K}^l(0) \leq \tilde{\phi}_{L/K}^{l_0 - \mu, m}(\tilde{\phi}_{M/L}^{\mu}(0)) = mi_{l_0 - \mu}$$

we have $\lambda_{M/K}^l(0) = \tilde{\phi}_{L/K}^{l_0 - \mu, m}(\tilde{\phi}_{M/L}^{\mu}(0))$. Hence $(l_0 - \mu, \mu) \in S_{l_0}^{l_0}(0)$. Since $(j, k+1) \in S_{l_0}^{l_0}(0)$, and $|S_{l_0}^{l_0}(0)| = 1$, we get $k + 1 = \mu$, and hence $i_{k-1}' = i_k' = i_{k+1}' = i_{\mu}' = 0$. Since $i_{\mu-1}' > 0$, this is a contradiction. Therefore $0 \in T_l(M/K)$. \qed
Lemma 3.6 Let $c \in \mathbb{N}_0 \cap T_l(M/K)$, let $l_0$ be the integer specified by \((3.4)\) for \(t = c\), and let \((j, k)\) be the unique element of $\Omega_{t_0}$ such that $\lambda_{M/K}^j(c) = \hat{\phi}_{L/K}^j(\hat{\phi}_{M/L}^k(c))$. Then $r_j$ and $s_k$ are nonzero.

Proof: Since $c \in T_l(M/K)$, for $0 \leq j' < j$ we have $\hat{\phi}_{M/L}^{j'}(\hat{\phi}_{M/L}^k(c)) > \hat{\phi}_{L/K}^{j'}(\hat{\phi}_{M/L}^k(c))$. It follows that $i_{j'} > i_j$, and hence that $\pi_K \cdot (\pi_L, \epsilon) \cdot I_F \subset (\pi_L^{n+i_j+1}, e^{i_j+1})$. Therefore by \((3.6)\) we get

$$\hat{\mathcal{F}}(\pi_L(1 + \epsilon)) \equiv \pi_K \cdot (1 + r_j \pi_L e^{i_j}) \pmod{\pi_L^{n+i_j+1}, e^{i_j+1}}.$$ 

If $r_j = 0$ then by Corollary 2.7 we have $i_j = \hat{\phi}_L^j(0) \geq i_j + 1$, a contradiction. It follows that $r_j \neq 0$.

Suppose there is $0 \leq k' < k$ such that $\hat{\phi}_{M/L}^{k'}(c) < \hat{\phi}_{M/L}^k(c)$. Since $c \in T_l(M/K)$ we have $(j, k') \not\in S_i^{c+k'}(c)$, and hence

$$\lambda_{M/K}^j(c) < \hat{\phi}_L^j(\hat{\phi}_{M/L}^{k'}(c)) \leq \hat{\phi}_L^j(\hat{\phi}_{M/L}^k(c)) = \lambda_{M/K}^j(c).$$

This is a contradiction, so we must have $\hat{\phi}_{M/L}^{j'}(c) > \hat{\phi}_{M/L}^k(c)$ for $0 \leq k' < k$. Hence $\hat{\phi}_{M/L}^{k'}(c) = \hat{\phi}_{M/L}^k(c)$. Set $d = \phi_{M/L}^k(c)$. Then $\pi_L \cdot (\pi_M, \epsilon) \cdot I_\mathfrak{g} \subset (\pi_M^{m+d+1}, e^{p^{k+1}})$. Using \((3.7)\) we get

$$\hat{\mathcal{G}}(\pi_M(1 + \pi_M^c)) \equiv \hat{\mathcal{G}}(\pi_M)(1 + s_k \pi_M^d e^{p^k}) \pmod{\pi_M^{m+d+1}, e^{p^{k+1}}}.$$ 

If $s_k = 0$ then by Corollary 2.7 we have $\hat{\phi}_{M/L}^k(c) \geq d + 1$, a contradiction. It follows that $s_k \neq 0$. \(\square\)

We now prove \((b)\) for $x = c \in \mathbb{N}_0 \cap T_l(M/K)$. Let $l_0$ be the minimum integer satisfying the hypotheses of \((b)\) for $x = c$. Then there is a unique pair $(j, k) \in \Omega_{t_0}$ such that $\lambda_{M/K}^j(c) = \hat{\phi}_{L/K}^{j_0}(\hat{\phi}_{M/L}^j(c))$. Furthermore, we have $\lambda_{M/K}^{j_0}(c) = \lambda_{M/K}^j(c)$ and $\lambda_{M/K}^j(c) > \lambda_{M/K}^{j_0}(c)$ for $l_1 < l_0$. Define $J'_0 = (\pi_M^{n+m+\lambda_{M/K}^{j_0}(c)}, e^{p^{j_0}+1})$. Then $\pi_L \cdot (\pi_M, \epsilon) \cdot I_{\mathfrak{g}} \subset J'_0$, so by \((3.9)\) we get

$$\hat{\mathcal{F}}(\hat{\mathcal{G}}(\pi_M(1 + \pi_M^c))) \equiv \pi_K \cdot (1 + u^{j_0} \pi_L s_k \pi_M^{j_0} \lambda_{M/K}^{j_0}(c) e^{p_0}) \pmod{J'_0}.$$ 

It follows from Lemma 3.6 that $r_j, s_k \in R \setminus \{0\}$ are units. Therefore we have

$$\hat{\mathcal{F}}(\hat{\mathcal{G}}(\pi_M(1 + \pi_M^c))) \equiv \pi_K \pmod{J'_0}.$$ 

Hence by \((a)\) and Corollary 2.7 we get

$$\lambda_{M/K}^{j_0}(c) \leq \phi_{M/K}^{j_0}(c) < \lambda_{M/K}^j(c) + 1.$$ 

Since $\lambda_{M/K}^{j_0}(c)$ and $\phi_{M/K}^{j_0}(c)$ are integers this implies that $\lambda_{M/K}^{j_0}(c) = \phi_{M/K}^{j_0}(c)$. Using \((a)\) we get

$$\lambda_{M/K}^j(c) \leq \phi_{M/K}^j(c) \leq \phi_{M/K}^{j_0}(c) = \lambda_{M/K}^{j_0}(c) = \lambda_{M/K}^j(c),$$
and hence $\lambda^l_{M/K}(c) = \phi^l_{M/K}(c)$. Thus (b) holds for $x \in \mathbb{N}_0 \cap T_l(M/K)$. In particular, it follows from Lemma 3.5 that (b) holds for $x = 0$.

As in the proof of (a) let $e \geq 1$ be relatively prime to $pnm$ and let $K_e/K$ be a totally ramified extension of degree $e$. Also set $L_e = LK_e$ and $M_e = MK_e$. Let $c \in \mathbb{N}_0$ be such that $c/e \in T_l(M/K)$ and the hypotheses of (b) are satisfied for the extensions $M/L/K$ with $x = c/e$. Then it follows from (3.10)–(3.13) that $c \in T_l(M_e/K_e)$ and the hypotheses of (b) are satisfied for the extensions $M_e/L_e/K_e$ with $x = c$. Hence by the preceding paragraph we get $\phi^l_{M_e/K_e}(c) = \lambda^l_{M_e/K_e}(c)$. Using (3.12) and (3.13) we deduce that $\phi^l_{M/K}(c/e) = \lambda^l_{M/K}(c/e)$.

Now let $r$ be any positive real number such that the hypotheses of (b) are satisfied with $x = r$, and let $l_0$ be the minimum integer which satisfies the hypotheses. Then there is a unique element $(j, k) \in \Omega_{l_0}$ such that $\tilde{\phi}^l_{L/K} \circ \tilde{\phi}^k_{M/L}(r) = \lambda^l_{M/K}(r)$. Let $0 \leq a \leq l_0$ and let $(u, v) \in \Omega_a$. Then the graph of $\tilde{\phi}^u_{L/K} \circ \tilde{\phi}^v_{M/L}$ is a line of slope $p^{u+v} = p^a \leq p^0$. Hence if $(u, v) \neq (j, k)$ and $0 \leq t < r$ then $\tilde{\phi}^u_{L/K} \circ \tilde{\phi}^v_{M/L}(t) > \tilde{\phi}^l_{L/K} \circ \tilde{\phi}^l_{M/L}(t)$. It follows that $S^l_{l_0}(t) = \{ (j, k) \}$ and $S^l_{l_0}(t) = \emptyset$ for $0 \leq a < l_0$. Hence $t \in T_{l_0}(M/K)$ and the hypotheses of (b) are satisfied with $x = t$ and $l$ replaced by $l_0$.

Suppose $\phi^l_{M/K}(r) > \lambda^l_{M/K}(r)$. Then there are $c, e \geq 1$ such that $\gcd(e, pnm) = 1$ and

$$0 < r - \frac{c}{e} < \frac{\phi^l_{M/K}(r) - \lambda^l_{M/K}(r)}{p^{u+v}}. \tag{3.15}$$

Since $\lambda^l_{M/K}(r) = \lambda^l_{M/K}(r)$ we get

$$\phi^l_{M/K}(r) - \lambda^l_{M/K}(r) \geq 0. \tag{3.16}$$

Since $\phi^l_{M/K}$ and $\lambda^l_{M/K}$ are continuous increasing piecewise linear functions with derivatives at most $p^{u+v}$ it follows from (3.15) that $\phi^l_{M/K}(c/e) - \lambda^l_{M/K}(c/e) > 0$. On the other hand, by the preceding paragraph we know that $c/e \in T_{l_0}(M/K)$ and the hypotheses of (b) are satisfied with $x = c/e$ and $l$ replaced by $l_0$. Hence $\phi^l_{M/K}(c/e) = \lambda^l_{M/K}(c/e)$. This contradicts (3.16), so we must have $\phi^l_{M/K}(r) \leq \lambda^l_{M/K}(r)$. By combining this inequality with (a) we get $\phi^l_{M/K}(r) = \lambda^l_{M/K}(r)$. This completes the proof of (b).

By setting $x = 0$ in Theorem 3.3 we get the following. A special case of this result is given in [3] Prop. 5.10.

**Corollary 3.7** For $0 \leq l \leq \nu + \mu$ let $i''_l$ denote the $l$th index of inseparability of $M/K$. Then

$$i''_l \leq \min\{mi_j + p^l i''_k : (j, k) \in \Omega_{l_0} \text{ for some } 0 \leq l_0 \leq l\},$$

with equality if there exists $0 \leq l_0 \leq l$ such that there is a unique pair $(j, k) \in \Omega_{l_0}$ which realizes the minimum.
References

[1] I. B. Fesenko and S. V. Vostokov, *Local fields and their extensions. A constructive approach*, Amer. Math. Soc., Providence, RI, 1993.

[2] M. Fried, Arithmetical properties of function fields II, The generalized Schur problem, Acta Arith. 25 (1973/74), 225–258.

[3] M. Fried and A. Mézard, Configuration spaces for wildly ramified covers, appearing in *Arithmetic Fundamental Groups and Noncommutative Algebra*, Proc. Sympos. Pure Math. 70 (2002), 353–376.

[4] B. Gross, Ramification in $p$-adic Lie extensions, Astérisque 65 (1979), 81–102.

[5] V. Heiermann, De nouveaux invariants numériques pour les extensions totalement ramifiées de corps locaux, J. Number Theory 59 (1996), 159–202.

[6] J. Lubin, Elementary analytic methods in higher ramification theory, J. Number Theory 133 (2013), 983–999.

[7] J.-P. Serre, *Corps Locaux*, Hermann, Paris (1962).