QUANTUM DOUBLE INCLUSIONS ASSOCIATED TO A FAMILY OF KAC ALGEBRA SUBFACTORS

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ABSTRACT. In [4] we defined the notion of quantum double inclusion associated to a finite-index and finite-depth subfactor and studied the quantum double inclusion associated to the Kac algebra subfactor $R^H \subset R$ where $H$ is a finite-dimensional Kac algebra acting outerly on the hyperfinite $II_1$ factor $R$ and $R^H$ denotes the fixed-point subalgebra. In this article we analyse quantum double inclusions associated to the family of Kac algebra subfactors given by $\{R^H \subset R \times H \rtimes \cdots : m \geq 1\}$. For each $m > 2$, we construct a model $N^m \subset M$ for the quantum double inclusion of $\{R^H \subset R \times H \rtimes \cdots \}$ with $N^m = (\cdots \times H^{-2} \rtimes H^{-1}) \otimes (H^m \rtimes H^{m+1} \cdots)$, $M = (\cdots \times H^{-1} \rtimes H^0 \rtimes H^1 \rtimes \cdots)^*$ and where for any integer $i$, $H^i$ denotes $H$ or $H^*$ according as $i$ is odd or even. In this article, we give an explicit description of $\P(N^m) \subset M$ ($m > 2$), the subfactor planar algebra associated to $N^m \subset M$, which turns out to be a planar subalgebra of $*^{(m)}\P(H^m)$ (the adjoint of the $m$-cabling of the planar algebra of $H^m$). We then show that for $m > 2$, depth of $N^m \subset M$ is always two. Observing that $N^m \subset M$ is reducible for all $m > 2$, we explicitly describe the weak Hopf $C^*$-algebra structure on $(N^m)'' \cap M_2$, thus obtaining a family of weak Hopf $C^*$-algebras starting with a single Kac algebra $H$.

INTRODUCTION

The motivation for this article primarily stems from the work of the author in [4]. Given a finite-index and finite-depth subfactor $N \subset M$ with $N(= M_0) \subset M(= M_1) \subset M_2 \subset M_3 \subset \cdots$ being the Jones’ basic construction tower associated to $N \subset M$, we defined in [4] the inclusion

$$N \vee (M' \cap M_\infty) \subset M_\infty$$

to be the quantum double inclusion associated to $N \subset M$ where $M_\infty$ denotes the $II_1$ factor obtained as the von Neumann closure $(\cup_{n=0}^\infty M_n)''$ in the GNS representation with respect to the trace on $\cup_{n=0}^\infty M_n$ and $N \vee (M' \cap M_\infty)$ denotes the von Neumann algebra generated by $N$ and $M' \cap M_\infty$. In [4] we studied the quantum double inclusion associated to the Kac algebra subfactor $R^H \subset R$ where $H$ is a finite-dimensional Kac algebra acting outerly on the hyperfinite $II_1$ factor $R$ and $R^H$ denotes the fixed-point subalgebra. The main result of [4] states that the quantum double inclusion of $R^H \subset R$ is isomorphic to $R \subset R \rtimes D(H)^{\text{cop}}$ for some outer action of $D(H)^{\text{cop}}$ on $R$ where $D(H)$ denotes the Drinfeld double of $H$. This result seemed to be quite interesting and motivated us to analyse quantum double inclusions associated to a general class of Kac algebra subfactors given by $\{R^H \subset R \times H \rtimes \cdots : m \geq 1\}$.

One of the main steps towards understanding the quantum double inclusions associated to the family of subfactors $\{R^H \subset R \times H \rtimes \cdots : m \geq 1\}$ is to construct their models. Given any finite-dimensional Kac algebra $H$, let $H^i$, where $i$ is any integer, denote $H$ or $H^*$ according as $i$ is odd or even. For each positive integer $m > 2$, we construct in §2 a hyperfinite, finite-index

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subfactor \( N^m \subset \mathcal{M} \) where \( N^m = ((\cdots \times H^{-3} \times H^{-2} \times H^{-1}) \otimes (H^m \times H^{m+1} \times \cdots))'' \), \( \mathcal{M} = ((\cdots \times H^{-1} \times H^0 \times H^1 \times \cdots)'' \) and show that \( N^m \subset \mathcal{M} \) is a model for the quantum double inclusion of \( R^H \subset R \rtimes H \rtimes H^* \rtimes \cdots \) \( m - 2 \) times.

The heart of the paper is §3 where we compute the basic construction tower associated to \( N^m \subset \mathcal{M} \) and also compute the relative commutants. The proofs all rely on explicit pictorial computations in the planar algebra of \( H \) or \( H^* \).

In §4, we explicitly describe the planar algebra associated to the subfactor \( N^m \subset \mathcal{M} \) \( (m > 2) \) which turns out to be an interesting planar subalgebra of \( S(m)P(H^m) \) (the adjoint of the \( m \)-cabling of the planar algebra of \( H^m \)).

It is evident from the main result of [4] that the quantum double inclusion of \( R^H \subset R \) is of depth two. It is thus a natural question to ask whether the quantum double inclusions associated to the family of subfactors \( \{ R^H \subset R \rtimes H \rtimes H^* \rtimes \cdots : m \geq 1 \} \) have finite depth. In this article we answer to this question in affirmative by proving that for \( m > 2 \), depth of \( N^m \subset \mathcal{M} \) is always 2 (Theorem 11). This is the main result of §5. One primary ingredient of the proof is Lemma 32 where we identify the commutant of the middle \( H \) in \( H^* \rtimes H \rtimes H^* \).

In [4] we constructed a model \( N \subset \mathcal{M} \) for the quantum double inclusion of \( R^H \subset R \). As an immediate consequence of the main result [4] Theorem 40 one obtains that the relative commutant \( N' \cap \mathcal{M}_2 \) is isomorphic to \( D(H)^{\text{cop}} = D(H)^{\text{cop}} \) as Kac algebras. In §6 we explicitly describe the structure maps of \( N' \cap \mathcal{M}_2 \) which will be useful to achieve a simple and nice description of the weak Hopf \( C^* \)-algebra structures on \( (N^m)' \cap \mathcal{M}_2 \) \( (m > 2) \) in §7.

It is well-known (see [16], [2]) that if \( N \subset M \) is a finite-index irreducible depth 2 inclusion of \( II_1 \) factors and if \( N(= M_0) \subset M(= M_1) \subset M_2 \subset M_3 \subset \cdots \) is the Jones’ basic construction tower associated to \( N \subset M \), then the relative commutants \( N' \cap \mathcal{M}_2 \) and \( M' \cap \mathcal{M}_3 \) admit mutually dual weak Hopf \( C^* \)-algebra structures. Now, for each \( m > 2 \), the subfactor \( N^m \subset \mathcal{M} \) being reducible and of depth 2, \( (N^m)' \cap \mathcal{M}_2 \) admits a weak Hopf \( C^* \)-algebra structure. The final §7 is devoted to recovering the weak Hopf \( C^* \)-algebra structure on \( (N^m)' \cap \mathcal{M}_2 \) for all \( m > 2 \). Here, in Theorem 54 we construct a family \( \{ K_m : m \geq 2 \} \) of weak Hopf \( C^* \)-algebras, with underlying vector spaces \( A(H)^{\text{op}}_{m-2} \otimes D(H)^{\text{op}} \otimes A(H)^{\text{op}}_{m-2} \) or \( A(H)^{\text{op}}_{m-2} \otimes D(H)^{\text{op}} \otimes A(H)^{\text{op}}_{m-2} \) according as \( m \) is odd or even, such that \( K_m \cong (N^m)' \cap \mathcal{M}_2 \) as weak Hopf \( C^* \)-algebras where, for any positive integer \( l \) and any finite-dimensional Kac algebra \( K \), \( A(K)_l \) denotes the finite crossed product algebra \( K \rtimes K^* \rtimes \cdots \) \( l \) times.

1. Preliminaries

The prerequisites for this article can be found in §1 and §2 of [4]. For a convenient reading, below we briefly explain the notations and recall some necessary facts that will be frequently used in the sequel.

1.1. Crossed product by Kac algebras. Throughout this article \( H(= H(\mu, \eta, \Delta, \varepsilon, S, *)) \) will denote a finite-dimensional Kac algebra and \( \delta \), the positive square root of \( \dim H \). We set \( H^\delta = H \) or \( H^* \) according as \( i \) is odd or even. The unique non-zero idempotent integrals of \( H^* \) and \( H \) will be denoted by \( \phi \) and \( h \) respectively and moreover, for any non-negative integer \( i \), the symbols \( \phi^i \) and \( h^i \) will always denote a copy of \( \phi \) and \( h \) respectively. It is a fact that \( \phi(h) = \frac{1}{\dim H} \). The letters \( x, y, z, t \) will always denote an element of \( H \) and for any integer \( i \), the symbols \( x^i, y^i, z^i, t^i \) will always represent an element of \( H \). The letters \( f, g, k \) will always denote an element of \( H^* \) and for any integer \( i \), the symbols \( f^i, g^i \) and \( k^i \) will always represent an element of \( H^* \).
Given $x \in H$, $\Delta(x)$ is denoted by $x_1 \otimes x_2$ (a simplified version of the Sweedler coproduct notation). We draw the reader’s attention to a notational abuse of which we will often be guilty. We denote elements of a tensor product as decomposable tensors with the understanding that there is an implied omitted summation (just as in our simplified Sweedler notation). Thus, when we write ‘suppose $f \otimes x \in H^* \otimes H$’, we mean ‘suppose $\sum_i f_i \otimes x^i \in H^* \otimes H$’ (for some $f_i \in H^*$ and $x^i \in H$, the sum over a finite index set).

We refer to [4] §1 for the notion of action of $H$ on a finite-dimensional complex $*$-algebra say, $A$ and the construction of the corresponding crossed product algebra, denoted $A \rtimes H$. Though the vector space underlying $A \rtimes H$ is $A \otimes H$, we denote a general element of $A \rtimes H$ by $a \times x$ instead of $a \otimes x$. There is a natural action of $H^*$ on $H$ given by $f.x = f(x_2)x_1$ for $f \in H^*$, $x \in H$. Similarly we have action of $H$ on $H^*$. If $H$ acts on $A$, then $H^*$ also acts on $A \rtimes H$ just by acting on $H$-part and ignoring the $A$-part, meaning that $f.(a \times x) = a \times f.x = f(x_2)a \times x_1$ for $f \in H^*$ and $a \times x \in A \rtimes H$ and consequently, we can construct $A \rtimes H \rtimes H^*$. Continuing this way, we may construct $A \rtimes H \rtimes H^* \rtimes \cdots$.

For integers $i \leq j$, we define $H_{[i,j]}$ to be the crossed product algebra $H^i \rtimes H^{i+1} \rtimes \cdots \rtimes H^j$. If $i = j$, we will simply write $H_i$ to denote $H_{[i,i]}$ and if $i > j$, we take $H_{[i,j]}$ to be $\mathbb{C}$. A typical element of $H_{[i,j]}$ will be denoted by $x^i/f^i \rtimes f^{i+1}/x^{i+1} \rtimes \cdots \rtimes f^j$ (resp., $x^i/f^i \rtimes f^{i+1}/x^{i+1} \rtimes \cdots \rtimes f^j$). We use the symbol $A(H)_\ell$, where $\ell$ is any positive integer, to denote the crossed product algebra $H \rtimes H^* \rtimes \cdots \rtimes \text{I terms}$.

Following [4] §1, we denote by $H_{(-\infty,\infty)}$ the algebra which, by definition, is the ‘union’ of all the $H_{[i,j]}$. Note that a typical element of $H_{(-\infty,\infty)}$ is a finite sum of terms of the form $\cdots \rtimes x^{-1} \rtimes f^0 \rtimes x^i \rtimes \cdots$ where in any such term all but finitely many of the $f^i$ are $\epsilon$ and all but finitely many of the $x^i$ are $1$. For any integer $m$, $H_{[m,\infty)}$ denotes the subalgebra of $H_{(-\infty,\infty)}$ which consists of all (finite sums of) elements $\cdots \rtimes x^{-1} \rtimes f^0 \rtimes x^1 \rtimes \cdots$ of $H_{(-\infty,\infty)}$ where for $i < m$, $f^i = \epsilon$ if $i$ is even and $x^i = 1$ if $i$ is odd. Similarly, we define the subalgebra $H_{(-\infty,m)}$ of $H_{(-\infty,\infty)}$. It is worth mentioning that the family $\{H_{[-\infty,-1]} \otimes H_{[m,\infty]} \subset H_{(-\infty,\infty)} ; m > 1\}$ of inclusions of infinite iterated crossed product algebras will be used in §2 to construct models for quantum double inclusions associated to the family of Kac algebra subfactors given by $\{R^H \subset R \rtimes H \rtimes H^* \rtimes \cdots ; m \geq 1\}$. The following results will be very useful. We refer to [1] Theorem 2.1, Corollary 2.3(iii)] for the proof of Lemma [8 Lemma 4.5,3] or [5 Proposition 3] for the proof of Lemma [2 and [8 Lemma 4.2.3] for the proof of Lemma 3.

**Lemma 1.** $H \rtimes H^* \rtimes H \rtimes \cdots$ (2k-term) is isomorphic to the matrix algebra $M_{n_k}(\mathbb{C})$ where $n = \dim H$.

**Lemma 2.** For any $p \in \mathbb{Z}$, the subalgebras $H_{(-\infty,p]}$ and $H_{[p+2,\infty)}$ are mutual commutants in $H_{(-\infty,\infty)}$.

Given integers $i \leq j$ and $p \leq q$ such that $j - i = q - p$ and assume that $j$ and $p$ (resp., $i$ and $q$) have the same parity. Given $X \in H_{[i,j]}$, let $X'$ denote the element obtained by ‘flipping $X$ about $i$ (equivalently, $j$)’ and then applying $S^{\otimes (j-i-1)}$ on this flipped element. For instance, if we assume $i$ to be even and $j$ to be even and if $X = x^i \rtimes f^{i+1} \rtimes \cdots \rtimes f^j \in H_{[i,j]}$, then $X'$ is given by $Sf^j \rtimes Sx^{j-1} \rtimes \cdots \rtimes Sf^{i+1} \rtimes Sx^i$. It is evident that $X' \in H_{[p,q]}$.

**Lemma 3.** The map $X \mapsto X'$ is a $*$-anti-isomorphism of $H_{[i,j]}$ onto $H_{[p,q]}$.

The Fourier transform map $F_H : H \to H^*$ is defined by $F_H(a) = \delta \phi_1(a) \phi_2$ and satisfies $F_H F_H = S$. We will usually omit the subscript of $F_H$ and $F_{H^*}$ and write both as $F$ with the argument making it clear which is meant.
1.2. Planar algebras. For the basics of (subfactor) planar algebras, we refer to [9, 11] and [12]. We will use the older notion of planar algebras where \( \text{Col} \), the set of colours, is given by \( \{(0, \pm), 1, 2, \cdots \} \) (note that only 0 has two variants, namely, \((0, +) \) and \((0, -) \)). This is equivalent to the newer notion of planar algebras (see [3, §2.2]) where \( \text{Col} = \{(k, \pm) : k \geq 0 \text{ integer}\} \) and we refer to [3] Proposition 1 for the proof of this equivalence. We will use the notation \( T_{k_1, k_2, \cdots, k_s} \) to denote a tangle \( T \) of colour \( k_0 \) (i.e., the colour of the external box of \( T \) is \( k_0 \)) with \( b \) internal boxes (\( b \) may be zero also) such that the colour of the \( i \)-th internal box is \( k_i \). Given a tangle \( T = T_{k_1, k_2, \cdots, k_s} \) and a planar algebra \( P \), \( \mathbb{Z}_T^P \) will always denote the associated linear map from \( P_{k_1} \otimes P_{k_2} \otimes \cdots \otimes P_{k_s} \) to \( P_{k_0} \) induced by the tangle \( T \).

We will also find it useful to recall the notions of cabling and adjoints for tangles and for planar algebras. Given any positive integer \( m \), and a tangle \( T \), say \( T = T_{k_1, k_2, \cdots, k_s} \), the \( m \)-cabling of \( T \), denoted by \( T^{(m)} \), is the tangle obtained from \( T \) by replacing each string of \( T \) by a parallel cable of \( m \)-strings. It is worth noting that the number of internal boxes of \( T^{(m)} \) and \( T \) are the same and that if \( k_i(T^{(m)}) \) denotes the colour of the \( i \)-th internal disc of \( T^{(m)} \), then

\[
k_i(T^{(m)}) = \begin{cases} mk_i, & \text{if } k_i > 0 \\ (0, +), & \text{if } k_i = (0, +) \\ (0, -), & \text{if } k_i = (0, -) \text{ and } m \text{ is odd} \\ (0, +), & \text{if } k_i = (0, -) \text{ and } m \text{ is even}. \end{cases}
\]

Now given any planar algebra \( P \), construct a new planar algebra \( \langle m \rangle P \), called \( m \)-cabling of \( P \), by setting

\[
\langle m \rangle P_k = \begin{cases} P_{mk}, & \text{if } k > 0 \\ P_{(0,+)}, & \text{if } k = (0, +) \\ P_{(0,-)}, & \text{if } k = (0, -) \text{ and } m \text{ is odd} \\ P_{(0,+)}, & \text{if } k = (0, -) \text{ and } m \text{ is even}. \end{cases}
\]

and defining \( \mathbb{Z}_T^{\langle m \rangle P} = \mathbb{Z}_T^P \) for any tangle \( T \). Similarly, given a planar algebra \( P \), we construct a new planar algebra \( *P \), called the adjoint of \( P \), where for any \( k \in \text{Col} \), \( (*P)_k = P_k \) as vector spaces and given any tangle \( T \), the action \( \mathbb{Z}_T^P \) of \( T \) on \( *P \) is specified by \( \mathbb{Z}_T^P \), where \( T^* \) is the tangle obtained by reflecting the tangle \( T \) across any line in the plane.

\[\begin{array}{c}
\text{Figure 1. trace tangle } : t_k^{(0,+)}(\text{left}) \text{ and rotation tangle } : R_k^k(\text{right})
\end{array}\]

\[\begin{array}{c}
\text{Figure 2. The tangles: } T^3 \text{ (left) and } T^4 \text{ (right)}
\end{array}\]

Observe that Figures 2 and 3 show some elements of two families of tangles. In Figure 2 we have the tangles \( T^n \) of colour \( n \) for \( n \geq 2 \), with exactly \( n - 1 \) internal 2-boxes and no internal regions.
illustrated for $n = 3$ and $n = 4$. In Figure 3 we have tangles $A(2m, 2n)$ defined for $m, n \geq 0$ of colour $2m + 2n + 4$ with exactly $2m + 2n + 3$ internal 2-boxes and no internal regions.

If $P$ is a subfactor planar algebra of modulus $d$, then for each $k \geq 1$, we refer to the (faithful, positive, normalised) trace $\tau : P_k \to \mathbb{C}$ defined for $x \in P_k$ by $\tau(x) = d^{-k}Ztr_{k}(0, +)(x)$ as the normalised pictorial trace on $P_k$ where $tr_{k}(0, +)$ denotes the $(0, +)$ tangle with a single internal $k$-box as shown in Figure 1.

1.3. Planar algebra of a Kac algebra. Suppose that $H$ acts outerly on the hyperfinite $II_1$ factor $M$. Let $P(H, \delta)$ (or, simply, $P(H)$) denote the subfactor planar algebra associated to $M^H \subset M$ where $M^H$ is the fixed-point subalgebra of $M$. We recall from [4, Theorem 8] (see also [13]) the construction of $P(H)$. The planar algebra $P(H)$ is defined to be the quotient of the universal planar algebra on the label set $L = L_2 = H$ by the set of relations in Figures 4 - 7 (where (i) we write the relations as identities - so the statement $a = b$ is interpreted as $a - b \in \text{rel}$; (ii) $\zeta \in \mathbb{K}$ and $a, b \in H$; and (iii) the external boxes of all tangles appearing in the relations are left undrawn and it is assumed that all external $*$-arcs are the leftmost arcs.

\[
\begin{align*}
\begin{array}{c}
\zeta \begin{array}{c}
a + b
\end{array} = \\
\begin{array}{c}
a \\
\begin{array}{c}
\zeta
\end{array}
\end{array} + \\
\begin{array}{c}
b
\end{array} \\
\begin{array}{c}
1
\end{array}
\end{array}
\end{align*}
\]

Figure 4. The L(inearity) and M(odulus) relations

\[
\begin{align*}
\begin{array}{c}
\delta^{-1}
\end{array} = \\
\begin{array}{c}
h
\end{array} \\
\begin{array}{c}
\delta
\end{array}
\end{align*}
\]

Figure 5. The U(nit) and I(ntegral) relations

\[
\begin{align*}
\begin{array}{c}
\epsilon(a)
\end{array} = \\
\begin{array}{c}
a
\end{array} \\
\begin{array}{c}
\delta \phi(a)
\end{array}
\end{align*}
\]

Figure 6. The C(ounit) and T(race) relations
Note that the modulus relation is a pair of relations - one for each choice of shading the circle. Finally, note that the interchange of $\delta$ and $\delta^{-1}$ between the (I) and (T) relations here and those of \cite{13} is due to the different normalisations of $h$ and $\phi$.

A reformulation of Lemma 16 from \cite{13} will be useful. Let $\mathcal{T}(k,p)(p \leq k - 1)$ denote the set of $k$ tangles (interpreted as 0 for $k = 0$) with $p$ internal boxes of colour 2 and no ‘internal regions’. If $p = k - 1$, we will simply write $\mathcal{T}(k)$ instead of $\mathcal{T}(k,k - 1)$. The result then asserts:

**Lemma 4.** For each tangle $X \in \mathcal{T}(k,p)$, the map $Z_{X}^{(P(H))} : (P(H)_{2})^{\otimes p} \to P(H)_{k}$ is an injective linear map and if $p = k - 1$, then $Z_{X}^{(P(H))} : (P(H)_{2})^{\otimes k-1} \to P(H)_{k}$ is a linear isomorphism.

The following lemma (a reformulation of \cite{8} Proposition 4.3.1) establishes algebra isomorphisms between $P(H)_{k}$ and finite iterated crossed product algebras.

**Lemma 5.** For each $k \geq 2$, the map from $H \otimes H^{*} \otimes \cdots$ to $P(H)_{k}$ given by

$$x^{1} \otimes f^{2} \otimes \cdots \mapsto Z_{T}^{(P(H))}(x^{1} \otimes Ff^{2} \otimes \cdots)$$

is a $*$-algebra isomorphism.

We will use this identification of $H \otimes H^{*} \otimes \cdots$ with $P(H)_{k}$ very frequently without mention. Finally, for $i \leq j$, $tr_{H_{[i,j]}}$ denotes the faithful, positive, tracial state on $H_{[i,j]}$ given by

$$\begin{cases}
h^{i} \otimes \phi^{i+1} \otimes h^{i+2} \otimes \cdots (j-i+1\text{-terms}), & \text{if } i \text{ is even} \\
\phi^{i} \otimes h^{i+1} \otimes \phi^{i+2} \otimes \cdots (j-i+1\text{-terms}), & \text{if } i \text{ is odd}
\end{cases}$$

Thus, for instance, if we assume $i$ to be odd, $j$ to be even and if $X \in H_{[i,j]}$, say, $X = x^{i} \otimes f^{i+1} \otimes \cdots \otimes x^{j-1} \otimes f^{j}$, then $tr_{H_{[i,j]}}(X) = \phi^{i}(x^{i})f^{i+1}(h^{i+1})\cdots \phi^{j-1}(x^{j-1})f^{j}(h^{j})$.

1.4. Drinfeld double construction. The Drinfeld double or quantum double construction is a construction that builds a quasitriangular Hopf algebra out of any finite-dimensional Hopf algebra. The Drinfeld double of $H$ is denoted by $D(H)$. The definition of $D(H)$ is not uniform in the literature. As in \cite{6} what we actually is an isomorphic variant of the version of $D(H)$ in \cite{15} which has underlying vector space $H^{*} \otimes H$ and the structure maps are given by the following formulæ:

$$
(f \otimes x)(g \otimes y) = g_{1}(x_{1})g_{3}(Sx_{3})(fg_{2} \otimes yx_{2}),
$$
$$
\Delta(f \otimes x) = (f_{2} \otimes x_{2}) \otimes (f_{1} \otimes x_{1}), \text{ and}
$$
$$
S(f \otimes x) = f_{1}(Sx_{1})f_{3}(x_{3})(S^{-1}f_{2} \otimes Sx_{2}).
$$

![Figure 7. The E(xchange) and A(ntipode) relations](image-url)
One can easily verify that the structure maps of $D(H)^*$ are given by the following formulae:

\[
\begin{align*}
(f \otimes x)(g \otimes y) &= gf \otimes yx, \\
\Delta(f \otimes x) &= \delta^2\phi_2(x_2)\phi_4(S_h)\phi_3(x_1) \otimes (f_1 \otimes h_1), \\
S(f \otimes x) &= \delta^2\phi_4(x_2)\phi_2(h_2)\phi_3 \otimes h_1, \\
\varepsilon(f \otimes x) &= f(1)\varepsilon(x).
\end{align*}
\]

Consider the linear isomorphism $Id_H \otimes F^{-1}_H : H^* \otimes H^* \to D(H)^*$. We can make $H^* \otimes H^*$ into a Kac algebra where the structure maps are obtained by transporting the structure maps on $D(H)^*$ using this linear isomorphism. Thus, by construction, $H^* \otimes H^*$ is isomorphic to $D(H)^*$ as a Kac algebra. The following lemma explicitly describes the structure maps on $H^* \otimes H^*$.

**Lemma 6.** The structure maps on $H^* \otimes H^*$ are given by the following formulae:

\[
\begin{align*}
(g \otimes f)(k \otimes p) &= \delta(Sf_2p)(h)kg \otimes f_1, \\
\Delta(g \otimes f) &= \delta(\phi_1 g_2 S\phi_3 \otimes f\phi_2) \otimes (g_1 \otimes \phi_4), \\
S(g \otimes f) &= f_1 Sg Sf_1 \otimes Sf_2, \\
\varepsilon(g \otimes f) &= \delta f(h)g(1).
\end{align*}
\]

**Proof.** Easy to verify and is left to the reader. \qed

## 2. Construction of models for the quantum double inclusion

In [13], we defined the notion of quantum double inclusion associated to a finite-index and finite-depth subfactor and constructed a model for the quantum double inclusion of $R^H \subset R$. In a similar way we construct in this section models for the quantum double inclusions of the family of subfactors $\{R^H \subset R \times H \times H^\times \times \cdots : m \geq 1\}$.

We begin with recalling from [13] the notion of quantum double inclusion. Given a finite-index and finite-depth subfactor $N \subset M$, let $N(= M_0) \subset M(= M_1) \subset M_2 \subset M_3 \subset \cdots$ denote the basic construction tower of $N \subset M$. Let $M_{\infty}$ denote the $II_1$ factor obtained as the von Neumann closure $(\cup_{n=0}^{\infty} M_n)$ in the GNS representation with respect to the trace on $\cup_{n=0}^{\infty} M_n$. Then the inclusion

\[N \vee (M' \cap M_{\infty}) \subset M_{\infty}\]

is defined to be the quantum double inclusion associated to $N \subset M$.

It is well-known that for any positive integer $k$, $\subset H_k \subset H_{[k-1,k]} \subset H_{[k-2,k]} \subset H_{[k-3,k]} \subset \cdots$ is the basic construction tower associated to the initial (connected) inclusion $\subset H_k$ so that $H_{(-\infty,k)}(= \cup_{i=0}^{\infty} H_{[i-k,i]})$ comes equipped with a tracial state and consequently,

\[H''_{(-\infty,k)} \cong (\cup_{i=0}^{\infty} H_{[i-k,i]})'' = (H_{(-\infty,k)})''\]

turns out to be a hyperfinite $II_1$ factor. It is also well-known (see [11] Theorem 4.11) that the basic construction tower associated to $R^H \subset R$ is given by:

\[R^H \subset R \subset R \times H \subset R \times H \times H^* \subset R \times H \times H^* \times H \subset \cdots .\]

The following lemma ([13] Lemma 17) describes models for $R^H \subset R$ as well as for the basic construction tower of $R^H \subset R$.

**Lemma 7.** $H''_{(-\infty, -1)} \subset H''_{(-\infty, 0)}$ is a model for $R^H \subset R$ for some outer action of $H$ on the hyperfinite $II_1$ factor $R$ and $H''_{(-\infty, -1)} \subset H''_{(-\infty, 0)} \subset H''_{(-\infty, 1)} \subset H''_{(-\infty, 2)} \subset \cdots$ is a model for the basic construction tower of $R^H \subset R$. 
As an immediate consequence of Lemma 7 we obtain that \((\bigcup_{i=-1}^\infty H''_{(\infty,i)})''\) is a hyperfinite \(H_1\) factor. It is not hard to see that \((\bigcup_{i=-1}^\infty H''_{(\infty,i)})'' = (H_{(-\infty,\infty)})''\). We set
\[
H''_{(-\infty,\infty)} := (H_{(-\infty,\infty)})''.
\]
It follows easily from Lemma 7 and [10, Proposition 4.3.6] that:

**Lemma 8.** Given any positive integer \(m\), \(H''_{(-\infty,-1]} \subset H''_{(-\infty,m]}\) is a model for \(R^H \subset R \times H \times H^* \times \cdots\)
\[\text{m times}\]
and \(H''_{(-\infty,-1]} \subset H''_{(-\infty,m]} \subset H''_{(-\infty,2m+1]} \subset H''_{(-\infty,3m+2]} \subset \cdots\) is a model for the basic construction tower of \(R^H \subset R \times H \times H^* \times \cdots\)
\[\text{m times}\]

Thus for any positive integer \(m\), a model for the quantum double inclusion of \(R^H \subset R \times H \times H^* \times \cdots\) is given by
\[
H''_{(-\infty,-1]} \vee ((H''_{(-\infty,m])}' \cap H''_{(-\infty,\infty)}) \subseteq H''_{(-\infty,\infty)}.
\]
By an appeal to [10] Lemma 14(2)], one can easily see that
\[
(H''_{(-\infty,m])}' \cap H''_{(-\infty,\infty)} = H''_{m+2,\infty})
\]
and consequently,
\[
H''_{(-\infty,-1]} \vee ((H''_{(-\infty,m])}' \cap H''_{(-\infty,\infty)}) = H''_{(-\infty,-1]} \vee H''_{m+2,\infty} = (H_{(-\infty,-1]} \otimes H_{m+2,\infty})''
\]

**Definition 9.** For each integer \(m > 2\), set \(N^m = (H_{(-\infty,-1]} \otimes H_{(m,\infty)})''\) and \(M = H''_{(-\infty,\infty)}\).

We have thus shown that:

**Proposition 10.** For each integer \(m > 2\), the subfactor \(N^m \subset M\) is a model for the quantum double inclusion of \(R^H \subset R \times H \times H^* \times \cdots\)
\[\text{m times}\]

3. **Basic construction tower of \(N^m \subset M\), \(m > 2\) and relative commutants**

The purpose of this section is to construct the basic construction tower associated to \(N^m \subset M\) \((m > 2)\) and also to compute the relative commutants.

3.1. **Some finite-dimensional basic constructions.** This subsection is devoted to analysing the basic constructions associated to certain unital inclusions of finite-dimensional \(C^*\)-algebras. We begin with recalling the following lemma (a reformulation of Lemma 5.3.1 of [10]) which provides an abstract characterisation of the basic construction associated to a unital inclusion of finite-dimensional \(C^*\)-algebras.

**Lemma 11.** [10, Lemma 5.3.1] Let \(A \subseteq B \subseteq C\) be a unital inclusion of finite-dimensional \(C^*\)-algebras. Let \(\operatorname{tr}_B\) denote a faithful tracial state on \(B\) and let \(E_A\) denote the \(\operatorname{tr}_B\)-preserving conditional expectation of \(B\) onto \(A\). Let \(f \in C\) be a projection. Then \(C\) is isomorphic to the basic construction for \(A \subseteq B\) with \(f\) as the Jones projection if the following conditions are satisfied:

(i) \(f\) commutes with every element of \(A\) and \(a \mapsto af\) is an injective map of \(A\) into \(C\),

(ii) \(f\) implements the trace-preserving conditional expectation of \(B\) onto \(A\) i.e., \(fbf = E_A(b)f\) for all \(b \in B\), and

(iii) \(BFB = C\).

In the next lemma, we explicitly compute certain conditional expectation map.
Lemma 12. Given integers \( l, p \geq 1 \) and \( s \geq 0 \), let \( \psi_{l,s,p} \) denote the embedding of \( H_{[-l,p+s]} \) inside \( H_{[-l,-1]} \otimes H_{[p,3p+s]} \) specified as follows:

Let \( X = x^{-l}/f^{-l} \times \cdots \times x^{p+s}/f^{p+s} \in H_{[-l,p+s]} \), then \( \psi_{l,s,p}(X) \in H_{[-l,-1]} \otimes H_{[p,3p+s]} \) is given by

\[
(x^{-l}/f^{-l} \times \cdots \times f^{-2} \times x_1^{-1}) \otimes (\underbrace{1 \times \epsilon \times 1 \times \cdots \times \epsilon \times x_2^{-1} \times f^0 \times \cdots \times x^{p+s}/f^{p+s}}_{p-1 \text{ terms}})
\]

or

\[
(x^{-l}/f^{-l} \times \cdots \times f^{-2} \times x_1^{-1}) \otimes (\underbrace{\epsilon \times 1 \times \epsilon \times \cdots \times \epsilon \times x_2^{-1} \times f^0 \times \cdots \times x^{p+s}/f^{p+s}}_{p-1 \text{ terms}})
\]

according as \( p \) is odd or even. Then the trace-preserving conditional expectation \( E \) of \( H_{[-l,-1]} \otimes H_{[p,3p+s]} \) onto \( H_{[-l,p+s]} \) is given by

\[
E((x^{-l}/f^{-l} \times \cdots \times x^{-1}) \otimes (x^p/f^p \times f^{p+1}/x^{p+1} \times \cdots \times x^{3p+s}/f^{3p+s}))
\]

\[
= \phi(Sx_2^{-1}x^{2p-1})tr_{H_{[3p,3p+p]}}((x^p/f^p \times \cdots \times f^{2p-2})x^{-l}/f^{-l} \times \cdots \times f^{-2} \times x_1^{-1} \times f^{2p} \times \cdots \times x^{3p+s}/f^{3p+s})
\]

Proof. In [4, Lemma 21(ii)] we proved the result for \( p = 2 \). The proof for the general case will follow in a similar fashion and hence, we omit the proof. \( \square \)

Next, we apply Lemma 12 to explicitly describe certain basic constructions and their associated Jones projections.

Proposition 13. The following are instances of basic constructions with the Jones projections being specified pictorially in appropriate planar algebras.

1. If \( l \geq 1, s \geq 0 \) are integers, then given any positive integer \( p \), \( H_{[-l,-1]} \otimes H_{[p,p+s]} \subset H_{[-l,p+s]} \subset H_{[-l,-1]} \otimes H_{[p,3p+s]} \subset H_{[-l,3p+s]} \) is an instance of the basic construction with the Jones projection given by the following figure

   \[
   \begin{array}{ccc}
   \delta^{-p} & & p+1 \\
   & \downarrow & \\
   p & \cup & s+2 \\
   \end{array}
   \]

   where the first inclusion is natural and the second inclusion is given by the map \( \psi_{l,s,p} \) as defined in the statement of Lemma 12. Furthermore, \( tr_{H_{[-l,p+s]}} \) is a Markov trace of modulus \( \delta^{2p} \) for the inclusion \( H_{[-l,-1]} \otimes H_{[p,p+s]} \subset H_{[-l,p+s]} \).

2. If \( l \geq 1, s \geq 0 \) are integers, then given any positive integer \( p \), \( H_{[-l,p+s]} \subset H_{[-l,-1]} \otimes H_{[p,3p+s]} \subset H_{[-l,3p+s]} \) is an instance of the basic construction with the Jones projection given by

   \[
   \begin{array}{ccc}
   \delta^{-p} & & l \\
   & \cup & \\
   p & \downarrow & p+s+2 \\
   \end{array}
   \]

   where the first inclusion is given by the map \( \psi_{l,s,p} \) as described in the statement of Lemma 12 and the second inclusion is the natural inclusion. Also, \( tr_{H_{[-l,-1]} \otimes H_{[p,3p+s]}} \) is a Markov trace of modulus \( \delta^{2p} \) for the inclusion \( H_{[-l,p+s]} \subset H_{[-l,-1]} \otimes H_{[p,3p+s]} \).

Proof. In [4, Proposition 22(2), 22(3)] we proved the result for \( p = 2 \). The proof for the general case will follow in a similar fashion. For the sake of completeness, we provide the proof of only one part namely, part 2, of the proposition which is also the harder part.

2. We only present the proof when \( l = 1 \) and \( s = 0 \), omitting the proof for the general case which is analogous. Let \( e \) denote the projection defined in the statement of Proposition 13(2). We identify as usual \( H_{[-1,6m-3]} \) with \( P(H)_{6m} \).
Given $X = x^{-1} \otimes f^0 \otimes \cdots \otimes x^{2m-1} \in H_{[-1,2m-1]}$, its image in $H_{[-1,6m-3]}$ is given by

$$x^{-1} \otimes \epsilon \otimes 1 \cdots \otimes \epsilon \otimes x^{-1} \otimes f^0 \otimes \cdots \otimes x^{2m-1}.$$  

The element $eX$ is shown on the left in Figure 8. An application of the relation (E) shows that $eX$ equals the element on the right in Figure 8. Similarly, by an appeal to the relations (A) and (E), one can easily see that the element $Xe$ equals the element on the right in Figure 8 so that $eX = Xe$. Thus, we conclude that $e$ commutes with $X$. Further, it is evident from the pictorial representation of the element $Xe$ as shown on the right in Figure 8 that the map $X \mapsto Xe$ of $H_{[-1,2m-1]}$ into $H_{[-1,6m-3]}$ is injective, verifying condition (i) of Lemma 11.

**Figure 8.** $eX = Xe$

Given $X = x^{-1} \otimes (x^{2m-1} \otimes f^{2m} \otimes \cdots \otimes x^{6m-3}) \in H_{-1} \otimes H_{[2m-1,6m-3]}$, the element $eXe$ is shown in Figure 9. Repeated application of the relations (T), (C), and (A) reduces the element in Figure 9 to that on the left in Figure 10 where $\alpha = \delta^{-2m}tr_{H_{[2m-1,4m-4]}}(x^{2m-1} \otimes f^{2m} \otimes \cdots \otimes f^{4m-4})$. Again repeated application of the relations (E) and (A), and finally, an application of the relation (T) reduces the element on the left in Figure 10 to that on the right in Figure 10. It follows from Lemma 12 that if $E$ denotes the trace-preserving conditional expectation of $H_{[-1,1]} \otimes H_{[2m-1,6m-3]}$ onto $H_{[-1,2m-1]}$, then

$$E(X) = \phi(Sx_2^{-1}x^{4m-3})tr_{H_{[2m-1,4m-4]}}(x^{2m-1} \otimes \cdots \otimes f^{4m-4}) x_1^{-1} \otimes f^{4m-2} \otimes \cdots \otimes x^{6m-3}.$$  

Now observe that $E(X)e$ equals the element as given by Figure 11 which, after a straight-
The inclusion \( \delta \mathcal{Z} \) of \( \mathcal{Z} \) where \( \mathcal{Z} \) is the tangle \( T \). Then by comparing dimensions of spaces we have that \( \mathcal{Z} \) is the linear isomorphism induced by the tangle \( T \) as shown in Figure 12. Thus, we see that \( (H_{[-1,-1]} \otimes H_{[2m-1,6m-3]}) \text{e}^e \mathcal{Z} = (H_{[-1,-1]} \otimes H_{[2m-1,6m-3]}) \) contains the image of \( \mathcal{Z} \). Then by comparing dimensions of spaces we have that \( H_{[-1,6m-3]} = (H_{[-1,-1]} \otimes H_{[2m-1,6m-3]}) \text{e}^e \mathcal{Z} \).

Finally, a routine computation shows that for any \( X \in H_{[-1,-1]} \otimes H_{[2m-1,6m-3]} \), \( tr(X e) = \delta^{-2(2m-1)} tr(X) \), so that \( tr(H_{[-1,-1]} \otimes H_{[2m-1,6m-3]}) \) is a Markov trace of modulus \( \delta^{-2(2m-1)} \) for the inclusion \( H_{[-1,2m-1]} \subset H_{[-1,-1]} \otimes H_{[2m-1,6m-3]} \), completing the proof.

\[ \square \]
3.2. **Jones’ basic construction tower of** $N^m \subset \mathcal{M}$ **and relative commutants.** Throughout this subsection, $m > 2$ denotes a fixed positive integer. The goal of this subsection is to explicitly determine the basic construction tower of $N^m \subset \mathcal{M}$.

We set $A_{0,0} = C, A_{0,1} = H_{[0,m-1]}, A_{1,0} = H_{-1} \otimes H_m$ or $H_{[-2,-1]} \otimes H_m$ according as $m$ is even or odd and $A_{1,1} = H_{[-1,m]}$ or $H_{[-2,m]}$ according as $m$ is even or odd. It follows from [4, Lemma 23] that the square in Figure 13 is a symmetric commuting square with respect to $tr_{A_{1,1}}$, which is a Markov trace for the inclusion $A_{0,1} \subset A_{1,1}$. Further, here all the inclusions are connected since the lower left corner is $C$ while the upper right corner is a matrix algebra by Lemma 1. For $k \geq 2$, we have

$$A_{0,1} \subset A_{1,1}$$

$$\cup$$

$$A_{0,0} \subset A_{1,0}$$

**Figure 13.** Commuting square

set

$$A_{k,1} = H_{[-k,m+k-1]} \text{ or } H_{[-2k,m+k-1]}$$

according as $m$ is even or odd.

It is then a consequence of [4, Proposition 22(1)(i)] that $A_{0,1} \subset A_{1,1} \subset A_{2,1} \subset A_{3,1} \subset \cdots$ is the basic construction tower associated to the initial inclusion $A_{0,1} \subset A_{1,1}$ and for any $k \geq 0$, if $e'_{k+2}$ denotes the Jones projection lying in $A_{k+2,1}$ for the basic construction of $A_{k,1} \subset A_{k+1,1}$, then $e'_{k+2}$ is given by Figure 14.

**Figure 14.** $e'_{k+2}$: with $m$ even (left) and with $m$ odd (right)

Further, we define inductively

$$A_{k+2,0} = \langle A_{k+1,0}, e'_{k+2} \rangle$$
for each \( k \geq 0 \). It is well-known that \( A_{0,0} \subset A_{1,0} \subset A_{2,0} \subset A_{3,0} \subset \cdots \) is the basic construction tower of \( A_{0,0} \subset A_{1,0} \). Proceeding along the same line as in the proof of [4, Lemma 24], one can show that:

**Lemma 14.** For any \( k > 0 \),

\[
A_{k,0} = \begin{cases} 
H_{[-2k,-1]} \otimes H_{[m, m+k-1]}, & \text{if } m \text{ is odd,} \\
H_{[-k,-1]} \otimes H_{[m, m+k-1]}, & \text{if } m \text{ is even.}
\end{cases}
\]

At this point we need to recall from [14] the notion of finite pre-von Neumann algebras. By a finite pre-von Neumann algebra, we will mean a pair \((\lambda, \tau)\) consisting of a complex \(*\)-algebra \( \lambda \) that is equipped with a normalised trace \( \tau \) such that (i) the sesquilinear form defined by \( \langle a, b \rangle = \tau(b^*a) \) defines an inner-product on \( \lambda \) and such that (ii) for each \( a \in \lambda \), the left-multiplication map \( \lambda(a) : \lambda \to \lambda \) is bounded for the trace induced norm of \( \lambda \). By a compatible pair of finite pre-von Neumann algebras, we will mean a pair \((\lambda, \tau_A)\) and \((B, \tau_B)\) of finite pre-von Neumann algebras such that \( A \subseteq B \) and \( \tau_B|_A = \tau_A \).

If \( A \) is a finite pre-von Neumann algebra with trace \( \tau_A \), the symbol \( L^2(A) \) will always denote the Hilbert space completion of \( A \) for the associated norm. Obviously, the left regular representation \( \lambda_A : A \to \mathcal{L}(L^2(A)) \) is well-defined, i.e., for each \( a \in A \), \( \lambda_A(a) : A \to A \) extends to a bounded operator on \( L^2(A) \). The notation \( A'' \) will always denote the von Neumann algebra \((\lambda_A(A))'' \subset \mathcal{L}(L^2(A))\). The following lemma (a reformulation of [14, Proposition 4.6(1)]) will be useful.

**Lemma 15.** [14, Proposition 4.6(1)] Let \((A, \tau_A)\) and \((B, \tau_B)\) be a compatible pair of finite pre-von Neumann algebras. The inclusion \( A \subseteq B \) extends uniquely to a normal inclusion of \( A'' \) into \( B'' \) with image \((\lambda_B(A))''\).

Note that \( \bigcup_{k=0}^\infty A_{k,0}(= H_{(-\infty,-1]} \otimes H_{[m,\infty)}) \) and \( \bigcup_{k=0}^\infty A_{k,1}(= H_{(-\infty,\infty)}) \) are finite pre-von Neumann algebras and \( \bigcup_{k=0}^\infty A_{k,0} \subset \bigcup_{k=0}^\infty A_{k,1} \) is a compatible pair so that by Lemma 15 the inclusion \( \bigcup_{k=0}^\infty A_{k,0} \subset \bigcup_{k=0}^\infty A_{k,1} \) extends uniquely to a normal inclusion \((\bigcup_{k=0}^\infty A_{k,0})'' \subset (\bigcup_{k=0}^\infty A_{k,1})''\). It follows from Definition 9 that

\[
(\bigcup_{k=0}^\infty A_{k,0})'' = (H_{(-\infty,-1]} \otimes H_{[m,\infty)})'' = \mathcal{N}^m \quad \text{and} \quad (\bigcup_{k=0}^\infty A_{k,1})'' = (H_{(-\infty,\infty)})'' = \mathcal{M}.
\]

Thus we have proved that:

**Lemma 16.** \( \mathcal{N}^m \) and \( \mathcal{M} \) are hyperfinite II\(_1\) factors.

The following lemma shows that \( \mathcal{N}^m \subset \mathcal{M} \) is of finite index equal to \( \delta^2m \).

**Lemma 17.** \([\mathcal{M} : \mathcal{N}^m] = \delta^2m\).

**Proof.** It is well-known that (see [10, Corollary 5.7.4]) \([\mathcal{M} : \mathcal{N}^m]\) equals the square of the norm of the inclusion matrix for \( A_{0,0} \subset A_{0,1} \) which further equals the modulus of the Markov trace \( tr_{A_{0,1}} \) for the inclusion \( A_{0,0}(= \mathbb{C}) \subset A_{0,1}(= H_{[0,m-1]}) \) which, again, by an application of [4, Proposition 22(1)(ii)], equals \( \delta^2m \). \(\square\)

For each \( k \geq 0 \) and \( n \geq 2 \), we now define a finite-dimensional \( C^*\)-algebra, denoted \( A_{k,n} \), as follows.

- **Case (i):** \( m \) is odd

\[
A_{k,n} = \begin{cases} 
H_{[-2k,-1]} \otimes H_{[m,(n+1)m+k-1]}, & \text{if } n \text{ is even and } k > 0, \\
H_{[-2k,nm+k-1]}, & \text{if } n \text{ is odd and } k > 0, \\
H_{[-(n-1)m,m-1]}, & \text{if } k = 0.
\end{cases}
\]
Thus, we have a grid \( \{ A_{k,n} : k, n \geq 0 \} \) of finite-dimensional \( C^* \)-algebras. The following remark contains several useful facts concerning the grid \( \{ A_{k,n} : k, n \geq 0 \} \).

Remark 18. 
(i) We have already seen that the square of finite-dimensional \( C^* \)-algebras as shown in Figure 13 is a symmetric commuting square with respect to \( \text{tr} A_{1,1} \) which is a Markov trace for the inclusion \( A_{1,0} \subset A_{1,1} \) and all the inclusions are connected. Further, by Lemmas 12 and 14 \( (\cup_{k=0}^\infty A_{k,n})'' (= N^m) \) as well as \( (\cup_{k=0}^\infty A_{k,n})' (= M) \) are hyperfinite II\(_1\) factors with \( [M : N^m] = \delta^{2m} \).

(ii) It follows from the embedding prescriptions that the following diagram (see Figure 15) commutes for all \( k, n \geq 0 \).

\[
\begin{array}{ccc}
A_{k,n+1} & \subset & A_{k+1,n+1} \\
\cup & & \cup \\
A_{k,n} & \subset & A_{k+1,n}
\end{array}
\]

**Figure 15.** Commutative diagram

(iii) It is a direct consequence of [4, Proposition 22] that for any \( k, n \geq 0 \), \( A_{k,n} \subset A_{k,n+1} \subset A_{k,n+2} \) is an instance of the basic construction and further, \( \text{tr}_{A_{k,n+1}} \) is a Markov trace of modulus \( \delta^{2m} \) for the inclusion \( A_{k,n} \subset A_{k,n+1} \). Let \( e_{k,n+2}^m \) (\( k \geq 0, n \geq 0 \)) denote the Jones projection lying in \( A_{k,n+2} \) applied to the basic construction \( A_{k,n} \subset A_{k,n+1} \).

(iv) For any \( k \geq 0, n \geq 2 \), the embedding of \( A_{k,n} \) inside \( A_{k+1,n} \) carries \( e_{k,n+2}^m \) to \( e_{k+1,n+2}^m \).

Obviously for any \( n \geq 0 \), \( \cup_{k=0}^\infty A_{k,n} \) is a finite pre-von Neumann algebra. Consider the tower of finite pre-von Neumann algebras

\[
\cup_{k=0}^\infty A_{k,0} \subset \cup_{k=0}^\infty A_{k,1} \subset \cup_{k=0}^\infty A_{k,2} \subset \cdots .
\]

Observe that for any \( n \geq 0 \), \( \cup_{k=0}^\infty A_{k,n} \subset \cup_{k=0}^\infty A_{k,n+1} \) is a compatible pair so that by Lemma 15 the inclusion \( \cup_{k=0}^\infty A_{k,n} \subset \cup_{k=0}^\infty A_{k,n+1} \) extends uniquely to a normal extension \( (\cup_{k=0}^\infty A_{k,n})'' \subset (\cup_{k=0}^\infty A_{k,n+1})'' \). Note also that \( \cup_{k=0}^\infty A_{k,n} = H(-\infty,-1] \otimes H(m,\infty) \) or \( H(-\infty,-\infty) \) according as \( n \) is even or odd. For each \( n \geq 0 \), we define \( M_n := (\cup_{k=0}^\infty A_{k,n})'' \). Then \( M_n = (H(-\infty,-1] \otimes H(m,\infty))'' \) or \( H''(-\infty,-\infty) \) according as \( n \) is even or odd. In view of the facts concerning the grid \( \{ A_{k,n} : k, n \geq 0 \} \) as mentioned in Remark 18 one can conclude that:
Proposition 19. $M_0(=N^m) \subset M_1(=M) \subset M_2 \subset M_3 \subset \cdots$ is the basic construction tower of $N^m \subset M$.

3.3. Computation of the relative commutants. We now proceed to compute the relative commutants. By virtue of Ocneanu’s compactness theorem (see [10, Theorem 5.7.6]), the relative commutant $(N^m)' \cap M_k$ ($k > 0$) is given by

$$(N^m)' \cap M_k = A_{0,k} \cap (A_{1,0})', \ k \geq 1.$$  

The following proposition describes the spaces $A_{0,k} \cap (A_{1,0})'$, $k \geq 1$. The proof of the proposition is similar to that of [3, Proposition 29] and we omit its proof.

Proposition 20. Let $k \geq 1$ be an integer and set

$$\tilde{Q}_{2k}^m = \{X \in H_{[m,(2k+1)m-2]} : X \text{ commutes with } \Delta_{k-1}(x) \in \otimes_{i=1}^{k} H_{2im-1}, \forall x \in H \},$$

$$\tilde{Q}_{2k-1}^m = \{X \in H_{[0,(2k-1)m-2]} : X \text{ commutes with } \Delta_{k-1}(x) \in \otimes_{i=0}^{k-1} H_{2im-1}, \forall x \in H \}.$$  

Then, $A_{0,2k} \cap (A_{1,0})' \cong \tilde{Q}_{2k}^m$ and $A_{0,2k-1} \cap (A_{1,0})' \cong \tilde{Q}_{2k-1}^m$.

It follows from Remark [18](iii) that the Jones projection lying in $N' \cap M_{n+2} = A_{0,n+2} \cap (A_{1,0})'$ ($n \geq 0$) is given by $\tilde{e}^m_{0,n+2}$ (see Figure 16), which, under the identification of $A_{0,n+2} \cap (A_{1,0})'$ with $\tilde{Q}_{n+2}^m$ as given by Proposition 20, is easily seen to be identified with the projection $\tilde{e}^m_{n+2}$ in $\tilde{Q}_{n+2}^m$ as shown on the right in Figure 16.

![Figure 16. $e_{0,n+2}^m$ (left) and $\tilde{e}_{n+2}^m$ (right), $n \geq 0$](image)

Remark 21. It is worth knowing the embedding of $\tilde{Q}_{k}^m$ inside $\tilde{Q}_{k+1}^m$ ($k \geq 1$). It follows easily from the embedding formulae of $A_{1,k}$ inside $A_{1,k+1}$ and $A_{0,k}$ (resp., $A_{0,k+1}$) inside $A_{1,k}$ (resp., $A_{1,k+1}$) and Proposition 20 that given $X \in \tilde{Q}_{k}^m$, it sits inside $\tilde{Q}_{k+1}^m$ as

$$\epsilon \times 1 \times \cdots \times 1 \times X, \text{ if } m \text{ is even}$$

and if $m$ is odd, then the image of $X \in \tilde{Q}_{k}^m$ inside $\tilde{Q}_{k+1}^m$ is given by

$$\epsilon \times 1 \times \cdots \times \epsilon \times X \text{ or } 1 \times \epsilon \times \cdots \times 1 \times X$$

according as $k$ is even or odd. Also, the diagram in Figure 17 commutes where each horizontal arrow indicates the $\ast$-isomorphism.

![Figure 17. A commutative diagram](image)

For each integer $n \geq 1$, we define a subspace $Q_n^m$ of $H_{[1,mn-1]}$ or $H_{[0,mn-2]}$ according as $m$ is odd or even as follows:
Case (i): \( m \) is odd

\[
Q_n^m := \left\{ X \in H_{[1, mn-1]} : X \leftrightarrow \Delta_{k-1}(x) \in \otimes_{i=1}^k H_{m(2i-1)-1}, \forall x \in H \right\}
\]

\[ k = \frac{n}{2} \text{ if } n \text{ is even or } \frac{n + 1}{2} \text{ if } n \text{ is odd} \]

Case (ii): \( m \) is even

\[
Q_n^m := \left\{ X \in H_{[0, mn-2]} : X \leftrightarrow \Delta_{k-1}(x) \in \otimes_{i=1}^k H_{m(2i-1)-1} - 1, \forall x \in H \right\}
\]

\[ k = \frac{n}{2} \text{ if } n \text{ is even or } \frac{n + 1}{2} \text{ if } n \text{ is odd} \]

This is an immediate consequence of Lemma 3 that for any \( n \geq 1, \tilde{Q}_n^m \) is \(*\)-anti-isomorphic to \( Q_n^m \) and let \( \gamma_m : \tilde{Q}_n^m \to Q_n^m \) denote this anti-isomorphism. We then have the following commutative diagram.

Further, if \( e_n^m \in Q_n^m (n \geq 2) \) denotes the projection which is the image of \( \tilde{e}_n^m \in \tilde{Q}_n^m \) under \( \gamma_n^m \), it is then not hard to see that \( e_n^m \) is given by Figure 19. Obviously, the identity map of \( Q_n^m \) onto its opposite algebra, denoted \( Q_n^{m^\text{op}} \), is a \(*\)-isomorphism for each \( n \geq 1 \) and for \( n \geq 2 \), it carries \( e_0^m \) to \( e_n^m \). The commutative diagrams in Figures 17 and 18 together imply commutativity of the diagram in Figure 20.

It will be useful to identify the spaces \( \mathcal{M} \cap \mathcal{M}_n (n \geq 2) \) also. Once again applying Ocneanu’s compactness theorem we obtain that \( \mathcal{M} \cap \mathcal{M}_n (n \geq 2) = A_{0,n} \cap (A_{1,1})'(n \geq 2) \). Proceeding along the same line of argument as in the proof of Proposition 20 one can show that:
Lemma 22. If \( n \) is even, then \( A_{0,n} \cap (A_{1,1})' \) can be identified with
\[
\{ X \in H_{[m, mn-2]} : X \text{ commutes with } \Delta_{k-1}(x) \in \otimes_{i=1}^{k} H_{2m_i-1}, k = \frac{n}{2} \},
\]
and if \( n \) is odd, then \( A_{0,n} \cap (A_{1,1})' \) can be identified with
\[
\{ X \in H_{[0, m(n-1)-2]} : X \text{ commutes with } \Delta_{k}(x) \in \otimes_{i=0}^{k} H_{2m_i-1}, k = \frac{n-1}{2} \}.
\]

As an immediate consequence of this lemma, we obtain that:

Lemma 23. The *-isomorphism \( \Psi_n^m \) of \( (\mathcal{N}^m)'/\mathcal{M}_n \) onto \( Q_n^{op} \) carries \( \mathcal{M}' \cap \mathcal{M}_n \) onto the subspace of \( Q_n^{op} \) given by
\[
\begin{align*}
(i) & \text{ } m \text{ is odd: } \\
& \{ X \in H_{[m+1, mn-1]}^{op} : X \text{ commutes with } \Delta_{k}(x) \in \otimes_{i=1}^{k} H_{2(2i-1)m}, \forall x \in H \}, \\
(ii) & \text{ } m \text{ is even: } \\
& \{ X \in H_{[m, mn-2]}^{op} : X \text{ commutes with } \Delta_{k}(x) \in \otimes_{i=1}^{k} H_{2(2i-1)m-1}, \forall x \in H \},
\end{align*}
\]
where, in either case, \( k = \frac{m}{2} \) or \( \frac{m-1}{2} \) according as \( n \) is even or odd.

In the next lemma we consider the question of irreducibility of \( \mathcal{N}^m \subseteq \mathcal{M} \) for \( m \geq 2 \).

Lemma 24. \( \mathcal{N}^m \subset \mathcal{M} \) is reducible for all \( m > 2 \).

Proof. Applying Lemma 2 one can easily observe that for \( m > 2 \), \( Q_1^n = H_{[1, m-2]} \) or \( H_{[0, m-3]} \) according as \( m \) is odd or even and consequently, \( \mathcal{N}^m \subseteq \mathcal{M} \) is not irreducible. \( \square \)

4. Planar Algebra of \( \mathcal{N}^m \subset \mathcal{M}(m > 2) \)

Let \( m > 2 \) be an integer. In this section we explicitly describe the subfactor planar algebra associated to the subfactor \( \mathcal{N}^m \subset \mathcal{M} \) which turns out to be a planar subalgebra of \( (m)^*P(H^m) \).

For each \( n \geq 1 \), consider the linear map \( \alpha^{m,n} : H \rightarrow \text{End}(P(H^m)_{mn}) \) defined for \( x \in H \) and \( X \in P(H^m)_{mn} \) by Figure 21 where the notation \( \alpha^{m,n}_x \) stands for \( \alpha^{m,n}(x) \).

**Figure 21.** \( \alpha^{m,k}_x(X) \), \( m \) odd (Left) and \( \alpha^{m,k}_x(X) \), \( m \) even (Right)

With the help of the maps \( \alpha^{m,n} \) defined above we give an equivalent description of the spaces \( Q_n^m \).

Proposition 25. For any \( k \geq 1 \),
\[
Q_k^m = \{ X \in P(H^m)_{mk} : \alpha_{h}^{m,k}(X) = X \}.
\]
Lemma 26. Let \( k \geq 1 \) be an integer.

(a) If \( m \) is odd, then for \( X \in H_{[1,m(2k-1)-1]} \), the following are equivalent:
   (i) \( X \times 1 \) commutes with \( \Delta_k^{-1}(x) \in \otimes_{i=1}^{k} H_{m(2i-1)}, \forall x \in H \),
   (ii) \( \Delta_k^{-1}(h_1)(X \times 1)\Delta_k^{-1}(Sh_2) = X \times 1 \), where \( \Delta_k^{-1}(h_1) \otimes \Delta_k^{-1}(Sh_2) \in (\otimes_{i=1}^{k} H_{m(2i-1)})^\otimes \).

(b) If \( m \) is even, then for \( X \in H_{[1,2mk-1]} \), the following are equivalent:
   (i) \( X \times 1 \) commutes with \( \Delta_k^{-1}(x) \in \otimes_{i=1}^{k} H_{m(2i-1)}, \forall x \in H \),
   (ii) \( \Delta_k^{-1}(h_1)X\Delta_k^{-1}(Sh_2) = X \), where \( \Delta_k^{-1}(h_1) \otimes \Delta_k^{-1}(Sh_2) \in (\otimes_{i=1}^{k} H_{m(2i-1)})^\otimes \).

(c) If \( m \) is even, then for \( X \in H_{[0,m(2k-1)-2]} \), the following are equivalent:
   (i) \( X \times 1 \) commutes with \( \Delta_k^{-1}(x) \in \otimes_{i=1}^{k} H_{m(2i-1)-1}, \forall x \in H \),
   (ii) \( \Delta_k^{-1}(h_1)X\Delta_k^{-1}(Sh_2) = X \), where \( \Delta_k^{-1}(h_1) \otimes \Delta_k^{-1}(Sh_2) \in (\otimes_{i=1}^{k} H_{m(2i-1)-1})^\otimes \).

(d) If \( m \) is even, then for \( X \in H_{[0,2mk-2]} \), the following are equivalent:
   (i) \( X \times 1 \) commutes with \( \Delta_k^{-1}(x) \in \otimes_{i=1}^{k} H_{m(2i-1)-1}, \forall x \in H \),
   (ii) \( \Delta_k^{-1}(h_1)X\Delta_k^{-1}(Sh_2) = X \), where \( \Delta_k^{-1}(h_1) \otimes \Delta_k^{-1}(Sh_2) \in (\otimes_{i=1}^{k} H_{m(2i-1)-1})^\otimes \).

We are now ready to prove Proposition 25.

Proof of Proposition 25. When \( m \geq 2 \) is even, the proof of the proposition is similar to that of [1] Proposition 33. Thus, we prove the proposition only when \( m > 2 \) is odd, leaving the other case for the reader. It is an immediate consequence of Lemma 26(a) that the space \( Q_{2k-1}^m \) can equivalently be described as

\[
Q_{2k-1}^m = \{ X \in H_{[1,m(2k-1)-1]} : \Delta_k^{-1}(h_1)(X \times 1)\Delta_k^{-1}(Sh_2) = X \times 1 \}
\]

where \( \Delta_k^{-1}(h_1) \otimes \Delta_k^{-1}(h_2) \in (\otimes_{i=1}^{k} H_{m(2i-1)})^\otimes \). Interpreting this equivalent description of \( Q_{2k-1}^m \) in the language of the planar algebra of \( H \), we note that \( Q_{2k-1}^m \) consists of precisely those elements \( X \in P(H)_{m(2k-1)} \) such that the equation of Figure 22 holds. Now applying the conditional expectation tangle \( E_{m(2k-1)}^{(2k-1)+} \), we reduce the element on the left in Figure 22 to that on the left in Figure 23. On the other hand, an application of the conditional expectation tangle \( E_{m(2k-1)}^{(2k-1)+} \) to the element on the right in Figure 22 and then an appeal to the modulus relation reduces the element on the right in Figure 22 to \( \delta X \) as shown on the right in Figure 23. Now applying the exchange relation first and then the modulus relation, one can easily see that the element on the left in Figure 23 indeed equals \( \delta \alpha_{m,2k-1}^m(X) \) and the desired description of \( Q_{2k-1}^m \) follows.

Similarly, it follows immediately from Lemma 26(b) that the space \( Q_{2k}^m \) can equivalently be described as

\[
Q_{2k}^m = \{ X \in H_{[1,2km-1]} : \Delta_k^{-1}(h_1)X\Delta_k^{-1}(Sh_2) = X \}
\]

where \( \Delta_k^{-1}(h_1) \otimes \Delta_k^{-1}(h_2) \in (\otimes_{i=1}^{k} H_{m(2i-1)})^\otimes \). Now the desired description of \( Q_{2k}^m \) follows at once from the definition of \( \alpha_{h,2k}^m(X) \) and by interpreting this equivalent description of \( Q_{2k}^m \) in the language of \( P(H) \), completing the proof.

Thus for each \( m > 2 \), we have a family \( \{ Q_n^m : n \geq 1 \} \) of vector spaces where for \( n \geq 1, Q_n^m \) is a subspace of \( P(H^n)_{mn} = (\otimes_n P(H^n))_n \). Setting \( Q_{0,\pm}^m = \mathbb{C} \), we note that \( Q^m := \{ Q_n^m : n \in \text{Cot} \} \) is a subspace of \( (\otimes_n P(H^n)) \). The following proposition, whose proof is similar to that of , shows that \( Q^m \) is indeed a planar subalgebra of \( (\otimes_n P(H^n)) \) and we omit its proof.

Proposition 27. For \( m > 2 \), \( Q^m \) is a planar subalgebra of \( (\otimes_n P(H^n)) \).
Proof. By an appeal to Theorem [12, Theorem 3.5], it suffices to prove that $Q^m$ is closed under the action of the following set of tangles

$$\{1^{0,+}, 1^{0,-}\} \cup \{R^k_k : k \geq 2\} \cup \{M^k_{k,k}, E^{k+1}_k : k \in \text{Col}\}$$

where we refer to [4, Figures 2, 3 and 5] for the definition of tangles $M^k_{k,k}, E^{k+1}_k, I^{k+1}_k, 1^{0,+}$ and $1^{0,-}$.

When $m$ is even, the proof of the proposition is similar to that of [4, Proposition 35]. Thus we prove the result only when $m$ is odd.

It is obvious to see that $Q^m$ is closed under the action of the tangles $1^{0,\pm}$ and $M^k_{k,k}, I^{k+1}_k (k \in \text{Col})$.

To see that $Q^m$ is closed under the action of the rotation tangle $R^k_k (k \geq 2)$, we note that for any $X \in Q^m_k (k \geq 2)$, we have

$$Z^{(m)p(H)}_{R^k_k}(X) = Z^{(m)p(H)}_{R^k_k}(\alpha^{m,k}_h(X)) = \alpha^{m,k}_h(Z^{(m)p(H)}_{R^k_k}(X))$$

where the first equality follows from the fact that $\alpha^{m,k}_h(X) = X$ and to see the second equality we need to use the Hopf algebra identity $h_1 \otimes h_2 \otimes \cdots \otimes h_l = h_2 \otimes h_3 \otimes \cdots \otimes h_l \otimes h_1$ ($l \geq 2$) which basically follows from $h_1 \otimes h_2 = h_2 \otimes h_1$ (which essentially expresses traciality of $h$).

Verifying that $Q^m$ is closed under the action of $E^{k+1}_{k+1} (k \geq 1)$ amounts to verification of the following identity

$$Z^{(m)p(H)}_{E^{k+1}_{k+1}}(X) = \alpha^{m,k}_h(Z^{(m)p(H)}_{E^{k+1}_{k+1}}(X))$$
for any $X \in Q^m_{k+1}$. Note that since $\alpha^{m,k+1}_h(X) = X$, we have that

$$Z^{(m)p(H)}_{E_{k+1}^k}(X) = Z^{(m)p(H)}_{E_{k+1}^k}(\alpha^{m,k+1}_h(X)) = Z^{P(H)}_{E_{k+1}^k}(\alpha^{m,k+1}_h(X)).$$

When $k$ is odd (resp., even), representing the element $Z^{P(H)}_{E_{k+1}^k}(\alpha^{m,k+1}_h(X))$ pictorially in $P(H)$ and then applying relation (E) (resp., (C)) one can easily see that

$$Z^{(m)p(H)}_{E_{k+1}^k}(X) = \alpha^{m,k}_h(Z^{P(H)}_{E_{k+1}^k}(X)),$$

finishing the proof.

As an immediate corollary of Proposition 27 we obtain that

**Corollary 28.** $^*Q^m$, the adjoint of $Q^m$, is a planar subalgebra of $^*(m)p(H^m)$.

Finally, similar argument as in the proof of [1] Proposition 36] shows that $P^{N^m} \subset M$, the planar algebra associated to $N^m \subset M$, is given by the adjoint of the planar algebra $Q^m$. Thus we have:

**Proposition 29.** $P^{N^m} \subset M = ^*Q^m$, $m > 2$.

We collect the results of the previous statements into a single main theorem.

**Theorem 30.** For any integer $m > 2$, $^*Q^m$ is a planar subalgebra of $^*(m)p(H^m)$ and $^*Q^m = P^{N^m} \subset M$. If $m$ is odd, $^*Q^m_k (k \geq 1)$ consists of all $X \in P(H)_{mk}$ such that the element on the left in Figure 24 equals $X$ and if $m$ is even, $^*Q^m_k (k \geq 1)$ consists of all $X \in P(H^*)_mk$ such that the element on the right in Figure 24 equals $X$.

**Proof.** It follows immediately from Proposition 29 after observing that $\alpha^{m,k}_h(X)$ for $m$ odd (resp., even) in Figure 21 is equivalent to the element on the left (resp., right) in Figure 24.

---

5. **Depth of $N^m \subset M, m > 2$**

In this section we investigate the depth of the subfactors $N^m \subset M$ for $m > 2$. The main result of this section is contained in the following theorem.

**Theorem 31.** For $m > 2$, the subfactor $N^m \subset M$ is of depth 2.
By virtue of the commutative diagram in Figure 20 one can easily see that $N^m \subset M$ has depth $k$, $k \geq 1$ an integer, is equivalent to $k$ being the smallest positive integer such that $Q^{m \circ p}_k \subset Q^{m \circ p}_k \subset Q^{m \circ p}_{k+1}$ is an instance of the basic construction with the Jones projection $e_k^{m \circ p}$ which obviously is equivalent to saying that $Q^{m \circ p}_{k-1} \subset Q^m_k \subset Q^m_{k+1}$ is an instance of the basic construction with the same Jones projection. Thus, in order to prove Theorem 31 it suffices to show that 2 is the smallest positive integer such that $Q^m_1 \subset Q^m_2 \subset Q^m_3$ is an instance of the basic construction with the Jones projection $e_3^m$.

We find it necessary to explicitly know the elements of the space $Q^m_2$. The following lemma is the main step to this end.

**Lemma 32.** Let $S$ be the space defined by

$$S = \{X \in H^* \times H \times H^* (\cong P(H^*)_4) : X \text{ commutes with } \epsilon \times x \times \epsilon, \forall x \in H\}.$$ 

Then $S$ precisely consists of elements of the form $Z_A^{P(H^*)}(f_2 \otimes f_1 \otimes g)$ where $A \in T(4)$ is the tangle as shown on the left in Figure 3 and $f \otimes g \in H^* \otimes H^*$. Consequently, $S$ has dimension $(\dim H)^2$.

**Proof.** Let $X = Z_A^{P(H^*)}(f_2 \otimes f_1 \otimes g) \in P(H^*)_4$ where $f \otimes g \in H^* \otimes H^*$. For any $t \in H$, let $Y_t$ denote the image of $\epsilon \times t \times \epsilon \in H^* \times H \times H^*$ in $P(H^*)_4$ under the algebra isomorphism between $H^* \times H \times H^*$ and $P(H^*)_4$ as given by Lemma 3, i.e., $Y_t = Z_T(4)(\epsilon \otimes F_t \otimes \epsilon)$ (see Figure 25) where $T^4$ is the tangle of colour 4 as shown on the right in Figure 2. Thus given $t \in H$, we need to show that $X$ commutes with $Y_t$. The element $Y_tX$ (resp., $XY_t$) is shown on the left (resp., right) in Figure 25. Set $k = F_t$. An application of the relations (E) and (T) shows that $XY_t$ equals the element

$$\delta(S_{f_2}k)(h)Z_A^{P(H^*)}(f_3 \otimes f_1 \otimes g)$$

wheras $Y_tX$ equals the element $\delta(Sk_1f_1)(h)Z_A^{P(H^*)}(f_2 \otimes k_2 \otimes g)$. Since, by Lemma 4, $Z_A^{P(H^*)}$ is a linear isomorphism of $H^* \otimes 3$ onto $P(H^*)_4$, in order to see that $XY_t = Y_tX$, it suffices to verify that

$$\delta(S_{f_2}k)(h)f_3 \otimes f_1 \otimes g = \delta(Sk_1f_1)(h)f_2 \otimes k_2 \otimes g.$$ 

Evaluating the expression on the left-hand side on $a \otimes b \otimes c \in H \otimes H \otimes H$ we obtain $f(bSh_1a)k(h_2)g(c)$ whereas evaluating the expression on the right-hand side on the same element we obtain the value $k(Sh_1b)f(h_2a)g(c)$ which, using the Hopf-algebraic formula $Sh_1a \otimes h_2 = Sh_1 \otimes xh_2$ and the fact that $Sh = h$, equals $f(bSh_1a)k(h_2)g(c)$. Thus we see that

$$\{Z_A^{P(H^*)}(f_2 \otimes f_1 \otimes g) : f \otimes g \in H^* \otimes 2\} \subseteq S$$

and consequently the dimension of the space $S$ is $\geq (\dim H)^2$. To finish the proof we just need to see that the dimension of $S$ is $\leq (\dim H)^2$. First observe that for any $X \in S$,

$$X = X(\epsilon \times h_1 \times \epsilon)(\epsilon \times Sh_2 \times \epsilon) = (\epsilon \times h_1 \times \epsilon)X(\epsilon \times Sh_2 \times \epsilon)$$

and thus $S$ is a subspace of

$$W = \{X \in P(H^*)_4 : X = Y_{h_1}XY_{Sh_2}\}.$$
Corollary 35.

Proof. (5.1) $Z = S$

Corollary 34.

Finally observe that $\dim U$ and hence, Lemma 33.

Thus it suffices to see that $\dim W = \dim U$ where

$$U = \{x \otimes y \otimes z \in H^{\otimes 3} : x \otimes y \otimes z = x \otimes h_1 y \otimes zSh_2\}.$$ 

Thus it suffices to see that $\dim U \leq (\dim H)^2$. Note that the space $U$, using the Hopf-algebraic formula $h_1 a \otimes bSh_2 = h_1 \otimes baSh_2$, can alternatively described as

$$U = \{x \otimes y \otimes z \in H^{\otimes 3} : x \otimes y \otimes z = x \otimes h_1 \otimes zySh_2\}.$$ 

Finally observe that $U$ is contained in the image of the injective linear map $\theta$ form $H \otimes H$ to $H \otimes H$ given by $x \otimes y \to x \otimes h_1 \otimes ySh_2$ for if $x \otimes y \otimes z \in U$, then clearly $x \otimes y \otimes z = \theta(x \otimes zy)$ and hence, $\dim U \leq (\dim H)^2$, finishing the proof. \qed

We now present a technical lemma that will be useful in order to precisely express the elements of $Q^m_2$, $m > 2$.

Lemma 33. The following equation holds in $P(H^*)_4$ for $f, g \in H^*$:

$$Z^{P(H^*)}_A(f_2 \otimes f_1 \otimes g) = Z^{P(H^*)}_T(f_1 Sg_3 Sf_3 \otimes f_2 g_2 \otimes Sg_1).$$

Proof. Left as an exercise. \qed

Consequently, the space $S$ can equivalently be described as:

Corollary 34. $S = \{f_1 Sg_3 Sf_3 \times F^{-1}(f_2 g_2) \times Sg_1 \in H^* \times H^*: f \otimes g \in H^{* \otimes 2}\}.$

Proof. Follows immediately from Lemmas 32 and 33. \qed

Corollary 35. Let $m > 2$.

(i) If $m$ is odd, then $Q^m_2$ consists of elements of the form

$$(5.1) \quad x^1 \times f^2 \times \cdots \times x^{m-2} \times g_1 Sg_3 Sg_3 \times F^{-1}(g_2 k_2) \times Sg_1 \times x^{m+2} \times f^{m+3} \times \cdots \times x^{2m-1} \in A(H)_{2m-1}$$

with $(x^1 \otimes \cdots \otimes x^{m-2} \otimes x^{m+2} \otimes \cdots \otimes x^{2m-1}) \otimes (f^2 \otimes \cdots \otimes f^{m+3} \otimes g \otimes k \otimes f^{m+3} \otimes \cdots \otimes f^{2m-2}) \in H^{\otimes (m-1)} \otimes H^{\otimes (m-1)}$.

(ii) If $m$ is even, $Q^m_2$ consists of elements of the form

$$(5.2) \quad f^1 \times x^2 \times \cdots \times x^{m-2} \times g_1 Sg_3 Sg_3 \times F^{-1}(g_2 k_2) \times Sg_1 \times x^{m+2} \otimes \cdots \otimes f^{2m-1} \in A(H)_{2m-1}$$

with $(f^1 \otimes f^{m+3} \otimes g \otimes k \otimes f^{m+3} \otimes \cdots \otimes f^{2m-1}) \otimes (x^2 \otimes x^{m-2} \otimes x^{m+2} \otimes \cdots \otimes x^{2m-2}) \in H^{\otimes (m-2)} \otimes H^{\otimes (m-2)}$. In this case, the elements of $Q^m_2$ can equivalently be expressed as

$$(5.3) \quad Z^{P(H^*)}_A(f_1 \otimes f^2 \otimes \cdots \otimes f^{m-2} \otimes f_2 f^{-1} \otimes f_1 \otimes f^{m+1} \otimes \cdots \otimes f^{2m-2})$$

with $\otimes_{i=1}^{2m-2} f^i \in H^{\otimes (2m-2)}$ (see Figure 3 for the definition of tangles $A(m-2, m-2)$).
observe that, in view of Proposition 25, the space just need to show that dim Q = linear map given by (5.2) and consequently, XeY = A simple computation shows that θ = For notational convenience we use the symbol without any mention in the proofs of both the propositions. the form as given by (5.2) when m is odd, or in the form as given by (5.3) or (5.3) when m is even.

Note that N^m ⊂ M can not be of depth 1 for otherwise it must happen that C ⊂ Q^m_1 ⊂ Q^m_2 is an instance of the basic construction and therefore, dim Q^m_2 must be equal to dim (Q^m_1)^2 which is not possible since dim Q^m_1 = (dim H)^m−2 whereas dim Q^m_2, by an appeal to Corollary 35 equals (dim H)^2(m−1). Consequently, depth of N^m ⊂ M is greater than 1.

In the following two propositions, namely, Proposition 37 and Proposition 38, we prove that Q^m_1 ⊂ Q^m_2 ⊂ Q^m_3 is an instance of the basic construction where Proposition 37 treats the case when m is even while Proposition 38 treats the case when m is odd. We will use Lemma 4 frequently without any mention in the proofs of both the propositions.

Proposition 37. If m > 1, then Q^m_1 ⊂ Q^m_2 ⊂ Q^m_3 is an instance of the basic construction with the Jones projection e_3^m.

Proof. For notational convenience we use the symbol e to denote e_3^m. Since the conditions (i) and (ii) of Lemma 11 are automatically satisfied, we only need to verify the condition (iii). Let us consider the elements of Q^m_2 given by 

\[ X = Z^{P(H^*)}_{A(2m−2,2m−2)}(f^0 \otimes f^1 \otimes \cdots \otimes f^{2m−3} \otimes f_2^{2m−2} \otimes f_1^{2m−2} \otimes f_2^{2m−1} \otimes \cdots \otimes f^{4m−3}), \]

\[ Y = Z^{P(H^*)}_{A(2m−2,2m−2)}((e \otimes F(1)) \otimes g_2^{2m−2} \otimes g_4^{2m−2} \otimes g_2^{2m−1} \otimes g_6^{2m} \otimes \cdots \otimes g^{4m−3}). \]

A simple computation shows that \( XeY \) equals the element 

\[ δ^{−2m}Z^P S_{P(H^*)}(f^0 \otimes \cdots \otimes f^{2m−3} \otimes f^{2m} \otimes \cdots \otimes f^{4m−3} \otimes g^{2m} \otimes \cdots \otimes g^{4m−3} \otimes g^{2m−1} \otimes f^{2m−1} \]

\[ \otimes f_1^{2m−2} \otimes Sg_1^{2m−2} \otimes f_2^{2m−2} \otimes g_2^{2m−2}) \]

where S ∈ T(6m) is as shown in Figure 27. Consider the linear map θ : H^⊕2 → H^⊕3 given by 

\[ f \otimes g \mapsto f_1 \otimes Sg_1 f_2 \otimes g_2. \]

We assert that θ is injective. To see this one can easily verify that the map from H^⊕3 → H^⊕2 given by 

\[ f \otimes g \otimes k \mapsto f(1)k_1 g \otimes k_2 \]

is a left inverse of θ, proving the assertion. Thus clearly Q^m_2 e Q^m_2 contains the image of the injective linear map 

\[ Z_S \circ (1d_{H^* \otimes (6m−4)} \otimes θ) : H^* \otimes (6m−2) \rightarrow P(H^*)_{6m} \]

and consequently, dim (Q^m_2 e Q^m_2) ≥ (dim H)^{6m−2}. Thus in order to see that Q^m_2 e Q^m_2 = Q^m_2 we just need to show that dim Q^m_2 ≤ (dim H)^{6m−2}. To this end we consider the tangle P ∈ T(6m) as shown in Figure 28. Since Z^P S induces a linear isomorphism of H^⊕(6m−4) onto P(H^*)_{6m}, we observe that, in view of Proposition 29 the space Q^m_3 is linearly isomorphic to 

\[ \{f^1 \otimes \cdots \otimes f^{6m−4} \otimes x \otimes y \otimes z \in H^* \otimes (6m−4) \otimes H^⊗3 : α_{h,3}^{2m−3}(Z^P(f^1 \otimes \cdots \otimes f^{6m−4} \otimes Fx \otimes Fy \otimes Fz)) = Z^P(f^1 \otimes \cdots \otimes f^{6m−4} \otimes Fx \otimes Fy \otimes Fz)\}. \]

\[ \square \]
A trivial computation in $P(H^*)$ shows that
\[ \Delta_h^{2m,3}(Z_P^{P(H^*)}(\otimes_{i=1}^{6m-4} f^i \otimes Fx \otimes Fy \otimes Fz)) = Z_P^{P(H^*)}(\otimes_{i=1}^{6m-4} f^i \otimes F(h_1x) \otimes F(h_2y) \otimes F(h_3z)). \]

Now using injectivity of $Z_P$ and invertibility of $F$, we conclude that $Q_3^{2m}$ is linearly isomorphic to the space $W$ defined by
\[ W = \{ x \otimes y \otimes z \otimes f^1 \otimes \cdots \otimes f^{6m-4} \in H^{\otimes 3} \otimes H^* \otimes (6m-4) : h_1x \otimes h_2y \otimes h_3z \otimes f^1 \otimes \cdots \otimes f^{6m-4} = x \otimes y \otimes z \otimes f^1 \otimes \cdots \otimes f^{6m-4} \}. \]

Thus it suffices to see that $\dim W \leq (\dim H)^{6m-2}$. Note that the space $W$, using the Hopf-algebraic formula $h_1a \otimes h_2 = h_1 \otimes h_2 Sa$, can equivalently be described as
\[ W = \{ x \otimes y \otimes z \otimes f^1 \otimes \cdots \otimes f^{6m-4} \in H^{\otimes 3} \otimes H^* \otimes (6m-4) : h_1 \otimes h_2 Sx_2y \otimes h_3 Sx_1z \otimes f^1 \otimes \cdots \otimes f^{6m-4} = x \otimes y \otimes z \otimes f^1 \otimes \cdots \otimes f^{6m-4} \}. \]

Further we observe that $W$ is contained in the range of the linear map $\rho : H^{\otimes 2} \otimes H^* \otimes (6m-4) \to H^{\otimes 3} \otimes H^* \otimes (6m-4)$ given by
\[ a \otimes b \otimes f^i \mapsto h_1 \otimes h_2 a \otimes h_3 b \otimes f^i, \]
for, if $X = x \otimes y \otimes z \otimes f^1 \otimes \cdots \otimes f^{6m-4} \in W$, then \( \rho(Sx_2y \otimes Sx_1z \otimes f^1 \otimes \cdots \otimes f^{6m-4}) = h_1 \otimes h_2 \otimes h_3 \otimes f^1 \otimes \cdots \otimes f^{6m-4} = X \) and consequently, \( \dim W \leq \text{rank of } \rho \leq (\dim H)^{6m-2} \), completing the proof.

**Proposition 38.** Given \( m > 1 \), \( Q_1^{2m-1} \subset Q_2^{2m-1} \subset Q_3^{2m-1} \) is an instance of the basic construction with the Jones projection \( e_3^{2m-1} \).

**Proof.** Since the conditions (i) and (ii) of Lemma 11 are automatically satisfied, we just need to verify the condition (iii). Note that \( Q_1^{2m-1} = H_{[1,2m-3]} \), \( Q_2^{2m-1} = \{ X \in H_{[1,4m-3]} : X \leftrightarrow H_{2m-1} \} \), \( Q_3^{2m-1} = \{ X \in H_{1,6m-4} : X \leftrightarrow \Delta(x) \in H_{2m-1} \otimes H_{6m-3} \} \). Now an application of Lemma 3 shows that the tower \( Q_1^{2m-1} \subset Q_2^{2m-1} \subset Q_3^{2m-1} \) is \( \ast \)-anti-isomorphic to the tower \( A \subset B \subset C \) with \( e = e_3^{2m-1} \) where \( A = H_{[-(2m-3),-1]} \), \( B = \{ X \in H_{[-(4m-3),-1]} : X \leftrightarrow H_{-(2m-1)} \} \), and \( C = \{ X \in H_{[-(6m-3),-1]} : X \leftrightarrow \Delta(x) \in H_{-(6m-3)} \otimes H_{-(2m-1)} \} \). Thus it suffices to prove that \( BeB = C \), or equivalently, \( \dim BeB = \dim C \).

Identify \( H_{[-(6m-3),-1]} \) with \( P(H^*)_{6m-3} \) and regard \( A, B, C \) as subalgebras of \( P(H^*)_{6m-3} \). In view of Corollary 35 we see that a general element of \( B \) is of the form

\[
Z_U^{P(H^*)}(f^1 \otimes f^2 \otimes \cdots \otimes f^{2m-3} \otimes f_2^{2m-2} \otimes f_1^{2m-2} \otimes f_2^{2m-1} \otimes f_1^{2m} \otimes \cdots \otimes f^{4m-4})
\]

where \( \otimes_{i=1}^{4m-4} f_i \in H^* \otimes (4m-4) \) and \( U \) is the tangle with exactly \( 4m-3 \) internal 2-boxes as shown in Figure 29.

![Figure 29. Tangle U](image)

Now, given

\[
X = Z_U(f^1 \otimes f^2 \otimes \cdots \otimes f^{2m-3} \otimes f_2^{2m-2} \otimes f_1^{2m-2} \otimes f_2^{2m-1} \otimes \cdots \otimes f^{4m-4}),
\]
\[
Y = Z_U(g^1 \otimes g^2 \otimes \cdots \otimes g^{2m-3} \otimes g_2^{2m-2} \otimes g_1^{2m-2} \otimes g_2^{2m-1} \otimes g_1^{2m} \otimes \cdots \otimes g \otimes F(1)),
\]

a little manipulation with the relations (E) and (A) shows that the element \( XeY \) equals

\[
\delta^{-(2m-1)} Z_Q^{P(H^*)}(\otimes_{i=1}^{2m-3} f_i \otimes \otimes_{i=2m+1}^{4m-4} f_i \otimes f^{2m-3} f_1^{2m-2} g_1^{2m-2} g_2^{2m-1} f_2^{2m} g_3^{2m-2} f^{2m-1} g_5^{2m-2} f_1^{2m-2} f_2^{2m-2} f_3^{2m-2} f_4^{2m-2} f_5^{2m-2} f_6^{2m-2} f_7^{2m-2} f_8^{2m-2} f_9^{2m-2} f_1^{2m-2})
\]

where \( Q \) is the tangle as shown in Figure 29. Observe that \( Q \in T(6m - 3) \). Let \( \theta : H^* \otimes 5 \to H^* \otimes 6 \)
be the linear map defined by
\[ f \otimes g \otimes k \otimes u \otimes v \mapsto f_1 S g \otimes f_2 k \otimes f_3 u \otimes f_4 S v \otimes f_5 v. \]
We assert that \( \theta \) is injective. To see this one can easily verify that the map \( \theta' : H^* \otimes 6 \rightarrow H^* \otimes 5 \) given by
\[ f \otimes g \otimes k \otimes p \otimes u \otimes v \mapsto p(1) u_4 \otimes S f u_3 \otimes S u_2 g \otimes S u_1 k \otimes v \]
is a left inverse of \( \theta \), proving the assertion. Now clearly \( \mathcal{B} \mathcal{B} \) contains the image of the injective linear map \( Z_Q \circ (Id_{H^* \otimes (6m-10)} \otimes \theta) : H^* \otimes (6m-5) \rightarrow P(H^*)_{6m-3} \) and consequently \( \dim \mathcal{B} \mathcal{B} \geq (\dim H)^{6m-5} \). To finish the proof we just need to show that \( \dim C \leq (\dim H)^{5m-5} \), or equivalently, \( \dim Q_{3}^{2m-1} \leq (\dim H)^{6m-5} \). Let us consider the tangle \( R \in \mathcal{T}(6m-3) \) as shown in Figure 31. Since
\[ Z_{R}^{P(H)} \] is a linear isomorphism of \( H^{\otimes (6m-4)} \) onto \( P(H)_{6m-3} \), we observe, in view of Proposition 25, that the space \( Q_{3}^{2m-1} \) is linearly isomorphic to the space
\[ \{ \otimes_{i=1}^{6m-4} x^i \in H^{\otimes (6m-4)} : \alpha_{h}^{2m-1,3} (Z_{R}^{P(H)} (\otimes_{i=1}^{6m-4} x^i)) = Z_{R}^{P(H)} (\otimes_{i=1}^{6m-4} x^i) \}. \]
A simple computation shows that
\[
\alpha_{h}^{2m-1,3}(Z_{R}^{P}(\otimes_{i=1}^{6m-4}x^{i})) = Z_{R}^{P}(H)(\otimes_{i=1}^{6m-7}x^{i} \otimes h_{1}x^{6m-6} \otimes h_{2}x^{6m-5} \otimes h_{3}x^{6m-4})
\]
and hence, \(Q_{3}^{2m-1}\) is linearly isomorphic to the space \(V\) defined by
\[
V = \{\otimes_{i=1}^{6m-4}x^{i} \in H^{\otimes(6m-4)} : h_{1}x^{i} \otimes h_{2}x^{i} \otimes h_{3}x^{i} \otimes \otimes_{i=1}^{6m-4}x^{i} = \otimes_{i=1}^{6m-4}x^{i}\}.
\]
Proceeding in a similar fashion as in the last part of the proof of Proposition 37 one can easily see that \(\dim V \leq (\dim H)^{6m-5}\), completing the proof. \(\square\)

We are now ready to conclude Theorem 31.

Proof of Theorem 31. Follows immediately from Propositions 37 and 38. \(\square\)

6. Structure maps on \(N' \cap M_{2}\)

The main result of [4] asserts that the quantum double inclusion of \(R_{H} \subset R\) is isomorphic to \(R \subset R \rtimes D(H)^{\text{cop}}\) for some outer action of \(D(H)^{\text{cop}}\) on \(R\). We proved this result by constructing a model \(N \subset M\) (see [4] Definition 18) for the definition of \(N\) for the quantum double inclusion of \(R_{H} \subset R\) and then showing that the planar algebras associated to \(N \subset M\) and \(R \subset R \rtimes D(H)^{\text{cop}}\) are isomorphic. As an immediate consequence of this result, we obtain that the relative commutant \(N' \cap M_{2}\) is isomorphic to \(D(H)^{\text{cop}} (= D(H)^{\text{cop}})\) as Kac algebras. From the proof of the main result of [4], namely [4] Theorem 40, the structure maps on \(N' \cap M_{2}\) can not directly be derived. In this section we explicitly describe the structure maps of \(N' \cap M_{2}\) which will be useful in §7 to achieve a simple and nice description of the weak Hopf \(C^{*}\)-algebra structures on \((N' \cap M_{2})(m > 2)\).

Let \(N \subset M\) be a finite-index, depth two, irreducible subfactor and let \(N(= M_{0}) \subset M(= M_{1}) \subset M_{2} \subset \cdots\) be the associated tower of basic construction. Then the relative commutants \(N' \cap M_{2}\) and \(M' \cap M_{3}\) admit mutually dual Kac algebra structures. Let \(P\) denote the subfactor planar algebra associated to \(N \subset M\) so that \(P = N' \cap M_{2}\). The next Theorem 39 summarises the content of [4] §3 where the authors gave pictorial description of the structure maps on \(P\). Before we state the theorem, we need to specify certain useful tangles. Let \(E, F, G\) denote tangles as shown in Figure 32.

![Figure 32](image)

**Figure 32.** Tangles \(E\)(left), \(F\)(middle) and \(G\)(right)

**Theorem 39.** [3] The counit \(\varepsilon : P_{2} \rightarrow \mathbb{C}\) and the antipode \(S : P_{2} \rightarrow P_{2}\) are defined for \(a \in P_{2}\) by
\[
\varepsilon(a) = [M : N]^{-\frac{1}{2}}Z_{P}^{P}(a), \quad S(a) = Z_{P}^{P}(a),
\]
and the comultiplication \(\Delta : P_{2} \rightarrow P_{2} \otimes P_{2}\) is the unique linear map such that the equation
\[
Z_{P}^{P}(a \otimes x \otimes y) = [M : N]^{-\frac{1}{2}}\text{tr}_{2}^{0}+(a_{1}x, \text{tr}_{2}^{0}+(a_{2}y))
\]
holds for all \(a, x, y \in P_{2}\).
Recall from [4] Theorem 38 that the planar algebra associated to \( N \subset \mathcal{M} \), denoted \( \mathcal{Q} \), is a planar subalgebra of \( *P(H^*) \) and for each integer \( k \geq 1 \), the space \( \mathcal{Q}_k \) is the opposite algebra of \( \{ X \in H_{[0,2k-2]} : X \text{ commutes with } \Delta_{l-1}(x) \in \otimes_{i=1}^{l} H_{4i-3}, \forall x \in H \text{ where } l = \frac{k}{2} \text{ or } \frac{k+1}{2} \} \).

Thus, in particular, \( \mathcal{Q}_2 \) is the opposite algebra of \[ \{ X \in H^* \times H \times H^* : X \text{ commutes with the middle } H \}. \]

That is, \( \mathcal{Q}_2 \) is the opposite algebra of \( \mathcal{S} \) (see Lemma [32] for the definition of \( \mathcal{S} \)) and consequently, by Lemma 32, \( \mathcal{Q}_2 \) precisely consists of elements of the form \( Z_{A}(f_2 \otimes f_1 \otimes g) \) where \( f \otimes g \in H^* \otimes 2 \).

We apply Theorem 39 above to derive the structure maps for \( \mathcal{Q}_2 \).

**Proposition 40.** Let \( X = Z_{A}(f_2 \otimes f_1 \otimes k) \in \mathcal{Q}_2 \), then

- **Comultiplication:** \( \Delta(X) = \delta Z_{A}(fS\phi_2 \otimes fS\phi_1) \otimes Z_{A}(S(f \phi_1 k) \otimes (\phi_4) \otimes k) \).
- **Antipode:** \( S(X) = Z_{A}(Sf_2 \otimes Sf_3 \otimes f_1SkSf_4) \).
- **Counit:** \( \varepsilon(X) = \delta f(h)k(1) \).
- **Involution:** \( X^* = Z_{A}(f_2^* \otimes Sf_3^* \otimes g^*) \).

**Proof.** Applying Theorem 39, the formula for \( S(X) \) follows directly by using the relations (E) and (A) whereas to verify the formula for \( \varepsilon(X) \) one needs to use the relations (T) and (M). To verify the involution formula we just need to observe that \( X^* = Z_{A}(f_2^* \otimes Sf_3^* \otimes g^*) = Z_{A}(f_2^* \otimes Sf_3^* \otimes g^*) \).

We now verify the formula for \( \Delta(X) \). It follows from Theorem 39 that given any \( W \in \mathcal{Q}_2 \), then \( \Delta_{\mathcal{Q}_2}(W) = W_1 \otimes W_2 \) that is equal to the element of \( \mathcal{Q}_2 \) such that

\[
\begin{align*}
Z_{E}^Q(W \otimes Y \otimes Z) &= \delta^{-2}(= |M:N|^{-\frac{1}{2}})tr_2(0,+)tr_1(0)(W_1 \otimes Y \otimes W_2) Z,
\end{align*}
\]

holds for all \( Y, Z \in \mathcal{Q}_2 \). Let \( Y, Z \) be arbitrary elements in \( \mathcal{Q}_2 \), say, \( Y = Z_{A}(g_2 \otimes g_1 \otimes p) \), \( Z = Z_{A}(u_2 \otimes u_1 \otimes v) \). Then the element \( Z_{E}^Q(X \otimes Y \otimes Z) = Z_{E}^P(H^*) \otimes (X \otimes Y \otimes Z) \) is as shown in Figure 38. A very lengthy but completely routine computation in \( P(H^*) \) along with repeated application of the well-known Hopf-algebraic formulae such as \( h_1a \otimes b_2 = h_1 \otimes h_2b_2 \) shows that

\[
\begin{align*}
Z_{E}^Q(X \otimes Y \otimes Z) &= \delta^3 f(Sh_1^2Sh_1^2)g(h_2^2h_1^4k(h_3^2h_4^2u(h_6^2h_4^2S_3^2)\delta(v^2))\delta(v^2))
\end{align*}
\]

verifying the formula for \( \Delta(X) \).

Using Lemma 33, it follows immediately from Proposition 40 that the structure maps of \( \mathcal{Q}_2 \) can also be expressed as:

**Lemma 41.** Given \( X = f_1Sk_3Sf_3 \times f_2k_2 \times Sk_1 \in \mathcal{Q}_2 \) where \( f \otimes k \in H^* \otimes 2 \), then

- **Comultiplication:** \( \Delta(X) = \delta ((fS\phi_2) \otimes ((fS\phi_2) \otimes S(fS\phi_3)_3 \times f^{-1}((fS\phi_2) \otimes (S(f_1k_2Sf_3) \otimes S(f_1k_2Sf_3)_1) \otimes (S(f_1k_2Sf_3) \otimes S(f_1k_2Sf_3)_1) \otimes (\phi_4)_1 \otimes k_3(S(f_1k_2Sf_3)_3 \times F^{-1}((\phi_4)_2 \otimes k_1(k_2) \times S(k_1)_1),

- **Antipode:** \( S(X) = k_1 \times F^{-1}(Sf_2k_2) \times S(f_1Sk_3Sf_3) \).

**Remark 42.** In §1.4, we considered a version of \( D(H)^* \) whose underlying vector space is \( H^* \otimes H^* \) and the structure maps are given by Lemma 4. Consider the linear isomorphism \( \nu : D(H)^* \otimes H^* \otimes \mathcal{Q}_2 \) that takes \( g \otimes f \mapsto Z_{A}(f_2^* \otimes f_1 \otimes g) \). It follows immediately from Proposition 40 and the structure maps on \( D(H)^* \) as given by Lemma 4 that \( \nu \) is an isomorphism of Kac algebras.
7. Weak Hopf $C^*$-algebra structure on $(N^m)' \cap M_2, m > 2$

It is well-known (see [2], [16]) that if $N \subset M$ is a depth two, reducible, finite-index inclusion of $II_1$-factors and if $N (= M_0) \subset M (= M_1) \subset M_2 \subset M_3 \subset \cdots$ is the Jones’ basic construction tower associated to $N \subset M$, then the relative commutants $N' \cap M_2$ and $M' \cap M_3$ admit mutually dual weak Hopf $C^*$-algebra structures. The following Theorem 43 (reformulation of Proposition 4.7 of [2]) explicitly describes the weak Hopf $C^*$-algebra structures on $N' \cap M_2$. Before we state the theorem, we need to fix some notations. Let $P$ be the planar algebra associated to $N \subset M$ so that $P_2 = N' \cap M_2$, $P_{1,2} = M' \cap M_2$. Set $[M : N] = d^2$. Further, let $z_R$ denote the unique element of $P_{1,2}$ for which $tr_L(x) = tr_2(z_R x)$ for all $x \in P_{1,2}$ where $tr_L$ is the trace of left regular representation of $P_{1,2}$ and $tr_2$ denotes the normalised pictorial trace on $P_2$. One can easily see that $z_R$ is a well-defined, central, positive, invertible element of $P_{1,2}$. By $\omega_R$ we denote the unique positive square root of $z_R$ and let $\omega_L$ be $Z_{R2}(\omega_R)$. We will use $\omega$ to denote $\omega_L\omega_R^{-1}$. The following theorem describes the structure maps of $P_2$.

**Theorem 43.** [2] The comultiplication $\Delta : P_2 \to P_2 \otimes P_2$ is the unique linear map such that the equation

$$Z^P_E(\omega_R \omega_L \otimes x \otimes y) = d^{-1} Z^P_{tr_2}(\omega_R a_1 \omega_L x) \times Z^P_{tr_2}(\omega_R a_2 \omega_L y)$$

holds in $P$ for all $a, x, y \in P_2$. The counit $\varepsilon : P_2 \to \mathbb{C}$ and the antipode $S : P_2 \to P_2$ are defined by

$$\varepsilon(a) = d^{-1} Z^P_E(\omega_R a \omega_L), \text{ and } S(a) = Z^P_E(w \otimes a \otimes w^{-1}),$$

for all $a$ in $P_2$ where $E, F, G$ are the tangles as shown in the Figure 33.

We use Theorem 43 to recover the weak Hopf $C^*$-algebra structure on $^*Q^m_2$ for all $m > 2$. Let us use the symbols $\omega^m_R, \omega^m_L$ and $\omega^m$ to denote the elements $\omega_R, \omega_L$ and $\omega$ respectively of $^*Q^m_2$.

Note that in order to find the structure maps of $^*Q^m_2$ using Theorem 43, we must know the elements $\omega^m_R, \omega^m_L$ and $\omega^m$. It follows from Lemma 23 that

$$^*Q^m_{1,2} = \begin{cases} \{X \in H_{[m+1,2m-1]}^p : X \text{ commutes with } H_m \}, & \text{if } m \text{ is odd} \\ \{X \in H_{[m,2m-2]}^p : X \text{ commutes with } H_{m-1} \}, & \text{if } m \text{ is even} \end{cases}$$

![Figure 33. $Z^Q_E(X \otimes Y \otimes Z)$](image-url)
Then by an appeal to Lemma 2 it follows immediately that $\alpha Q_{1,2}^m$ is identified with the subalgebra $H_{[m+2,2m-1]}^{op}$ or $H_{[m+1,2m-2]}^{op}$ of $\alpha Q_2^m$ according as $m$ is odd or even. Thus, for any $m > 2$, $\alpha Q_{1,2}^m \cong A(H)_{m-2}^{op}$ as algebras. One can easily see that if $A$ is any multi-matrix algebra over the complex field, then for any $a \in A$, $tr^L(a) = tr_R^L(a)$ and consequently, $tr^L(a) = tr_R^L(a) = tr^R(a)$ (resp., $tr_R^L(a)$) denotes the trace of the linear endomorphism of $A$ given by left (resp., right) multiplication by $a$. Thus, given any $X$ in $\alpha Q_{1,2}^m = A(H)_{m-2}$, we have $tr^L_{X}^{\alpha Q_{1,2}^2}(X) = tr^L_{A(H)_{m-2}}(X) = tr^L_{A(H)_{m-2}}(X)$. The following lemma computes $tr^L_{A(H)_k}(X)$ for any $X \in A(H)_k$ where $k \geq 1$ is an integer.

**Lemma 44.** $tr^L_{A(H)_k}(X) = (\dim H)^k$ (normalised pictorial trace of $X$).

**Proof.** If $k$ is even, then $A(H)_k$ is a matrix algebra by Lemma 1 and the result follows immediately. Now suppose that $H$ is a finite-dimensional Hopf algebra acting on a finite-dimensional algebra $A$. A simple exercise in linear algebra shows that for any $a \in A$, $tr^L_{A \times H}(a \times 1) = \dim H \cdot tr^L_{A}(a)$. Thus if $k$ is odd, then given $X \in A(H)_k$, we note that $tr^L_{A(H)_k}(X) = \frac{1}{\dim H} \cdot tr^L_{A(H)_{k+1}}(X \times 1) = \frac{1}{\dim H} \cdot \dim H^{k+1}$ (normalised pictorial trace of $X \times 1 = (\dim H)^k$ (normalised pictorial trace of $X$) where the second equality follows since $k+1$ is even, completing the proof.

As an immediate corollary we have

**Corollary 45.** $\omega_L^m = \omega_R^m = (\dim H)^{m-2} \cdot 1_{Q_{1,2}^m}$ and hence, $\omega^m = 1_{Q_{1,2}^m}$ for $m > 2$.

**Proof.** It follows from Lemma 44 and the discussion preceding Lemma 44 that for any $X$ in $\alpha Q_{1,2}^m$, $tr^L_{X}^{\alpha Q_{1,2}^2}(X) = tr^L_{A,m-2}(X) = (\dim H)^{m-2}$ normalised pictorial trace of $X$. Hence, $\omega_R^m = (\dim H)^{m-2} \cdot 1_{Q_{1,2}^m}$ and consequently, $\omega_L^m = (\dim H)^{m-2} \cdot 1_{Q_{1,2}^m}$, $\omega^m = 1_{Q_{1,2}^m}$.  

---

*Figure 34. Definition of $\Delta$*

*Figure 35. Definitions of $S$ and $\epsilon$*
We now proceed towards recovering the structure maps of $\Omega_2^m$. The entire procedure solely relies on Hopf-algebraic as well as pictorial computations in $P(H)$ or $P(H^*)$. At this point, it is worth recalling from Corollary 35 and Remark 36 that a general element of $\Omega_2^m$ is of the form

\begin{equation}
 x^1 \times f^2 \times \cdots \times x^{m-2} \times k_1 S p_3 S k_3 \times F^{-1}(k_2 p_2) \times S p_1 \times x^{m+2} \times f^{m+3} \times \cdots \times x^{2m-1}
\end{equation}

or

\begin{equation}
 f^1 \times x^2 \times \cdots \times x^{m-2} \times k_1 S p_3 S k_3 \times F^{-1}(k_2 p_2) \times S p_1 \times x^{m+2} \times \cdots \times f^{2m-1}
\end{equation}

according as $m$ is odd or even. Moreover, if $m$ is even, the elements of $\Omega_2^m$ can also be expressed as

\begin{equation}
 Z_{A(2m-2,2m-2)}^P(H^*)(f^1 \otimes f^2 \otimes \cdots \otimes f^{m-2} \otimes f^{m-1} \otimes f^m \otimes f^{m+1} \otimes \cdots \otimes f^{2m-2}).
\end{equation}

First we find the formula for antipode in $\Omega_2^m$. It follows from Theorem 43 and Lemma 15 that for any $X \in \Omega_2^m$,

\[
 S(X) = Z_{R_2^2}^Q(X) = Z_{(R_2^2)^m}^P(H^*) = Z_{(R_2^2)^m}^P(H^*) (X) \quad (\text{since } (R_2^2)^* = R_2^2).
\]

Let us consider a general element, say $X$, of $\Omega_2^m$ as given by (7.5) or (7.6) according as $m$ is odd or even. Assume that $m$ is even. Then note that $X$ is identified with

\[
 Z_{T_2^m}^P(H^*) (f^1 \otimes F x^2 \otimes \cdots \otimes F x^{m-2} \otimes k_1 S p_3 S k_3 \otimes k_2 p_2 \otimes S p_1 \otimes F x^{m+2} \otimes f^{m+3} \otimes \cdots \otimes f^{2m-1}).
\]

Consequently,

\[
 S(X) = Z_{(R_2^2)^m}^P(H^*) (Z_{T_2^m}^P(H^*) (f^1 \otimes F x^2 \otimes \cdots \otimes F x^{m-2} \otimes k_1 S p_3 S k_3 \otimes k_2 p_2 \otimes S p_1 \otimes \cdots \otimes f^{2m-1})),
\]

which, by repeated application of the relation (A) in $P(H^*)$ is easily seen to be equal to

\[
 Z_{T_2^m}^P(H^*) (S f^{2m-1} \otimes \cdots \otimes S F x^{m+2} \otimes \cdots \otimes S F x^1, 1)
\]

which, by virtue of the fact that $F S = S F$, is identified with

\[
 S f^{2m-1} \times \cdots \times S x^{m+2} \times \cdots \times S x^2 \times S f^1.
\]

Thus, we obtain the formula for $S(X)$ when $m$ is even. Proceeding exactly the same way, one can show that the formula for $S(X)$, when $m$ is odd, is given by

\[
 S f^{2m-1} \times \cdots \times S x^{m+2} \times \cdots \times S f^2 \times S x^1.
\]

Thus we have proved that:

**Lemma 46.** Let $X \in \Omega_2^m$ be as given by (7.5) or (7.6) according as $m$ is odd or even. Then $S(X)$ is given by

\[
 S f^{2m-1} \times \cdots \times S x^{m+2} \times \cdots \times S f^2 \times S x^1.
\]

Among all the structure maps the hardest is to recover the comultiplication formula and we do it in steps. By an appeal to Corollary 35 and Lemma 17 it follows immediately from Theorem 13 that given $X \in \Omega_2^m$, $\Delta_{Q_2^m}(X) = X_1 \otimes X_2$ is that element of $(\Omega_2^m)^{\otimes 2}$ such that

\begin{equation}
 Z_{E}^{Q^m}(X \otimes Y \otimes Z) = \delta^{-m} \delta^{2m-4} Z_{K_{Q_2^m}}^{Q^m}(X_1 Y) . Z_{K_{Q_2^m}}^{Q^m}(X_2 Z),
\end{equation}

holds for all $Y, Z \in \Omega_2^m$. 

We begin with finding the comultiplication formula for a special class of elements of $\mathcal{Q}^m_2$. Recall from the discussion preceding Proposition 10 in §6 that the space $\mathcal{Q}_2$, where $\mathcal{Q}$ is the planar algebra associated to $\mathcal{N} \subset \mathcal{M}$, is same as $\mathcal{S}$ as vector spaces but as an algebra it is the opposite algebra of $\mathcal{S}$. Given $f \times x \times g \in \mathcal{Q}_2$, we define $X^m_{f \times x \times g}$ to be the element of $\mathcal{Q}^m_2$ given by $1 \times \epsilon \times \cdots \times 1 \times f \times x \times g \times 1 \times \epsilon \times \cdots \times 1$ or $\epsilon \times 1 \times \cdots \times 1 \times f \times x \times g \times 1 \times \epsilon \times \cdots \times \epsilon$ according as $m$ is odd or even. Let $\Delta Q_2(f \times x \times g) = (f \times x \times g)_1 \otimes (f \times x \times g)_2$. The following lemma computes $\Delta Q^m_2(X^m_{f \times x \times g})$.

**Lemma 47.** $\Delta Q^m_2(X^m_{f \times x \times g}) = 1_1 X^m_{(f \times x \times g)_1} \otimes 1_2 X^m_{(f \times x \times g)_2} \in (\mathcal{Q}^m_2) \otimes^2$ where $1_1 \otimes 1_2 = \Delta Q^m_2(1)$.

**Proof.** To avoid notational clumsiness and to elucidate the computational procedure, instead of treating the general case, we explicitly work out the particular case when $m = 4$. The general case when $m > 2$ is even follows in a similar fashion and the case when $m > 2$ is odd follows almost in a similar way with slight modifications.

Let $f \times x \times g \in \mathcal{Q}_2$. Let us consider the element $X^4_{f \times x \times g} \in \mathcal{Q}_2$. Let $Y, Z$ be arbitrary elements of $\mathcal{Q}_2$, say, $Y = Z^{P(H^*)}_{A[2,2]}(k_1 \otimes k_2 \otimes u_2 \otimes u_1 \otimes v \otimes k^3 \otimes k^4)$ and $Z = Z^{P(H^*)}_{A[2,2]}(p^4 \otimes p^3 \otimes \tilde{u}_2 \otimes \tilde{u}_1 \otimes \tilde{v} \otimes p^3 \otimes p^4)$. It follows from (7,8) that in order to verify the comultiplication formula for $X^4_{f \times x \times g}$, we just need to verify that

$$Z^{Q^4}_{tr_2(0,+)}(1_1 X^4_{(f \times x \times g)_1}, Y) Z^{Q^4}_{tr_2(0,+)}(1_2 X^4_{(f \times x \times g)_2}, Z) = Z^{Q^4}_{E^*(4)}(X^4_{f \times x \times g} \otimes Y \otimes Z).$$

Another appeal to (7,8) shows that

$$Z^{Q^4}_{tr_2(0,+)}(1_1 X^4_{(f \times x \times g)_1}, Y) Z^{Q^4}_{tr_2(0,+)}(1_2 X^4_{(f \times x \times g)_2}, Z) = Z^{Q^4}_{E^*(4)}(1 \otimes X^4_{(f \times x \times g)_1} \otimes X^4_{(f \times x \times g)_2}) = Z^{P(H^*)}_{E^*(4)}(1 \otimes Y X^4_{(f \times x \times g)_1} \otimes Z X^4_{(f \times x \times g)_2}).$$

where the last equality follows from the definition of adjoint and cabling of a planar algebra. Now a pleasant but lengthy computation in $P(H^*)$ involving sphericality of $P(H^*)$ and repeated application of the relations (E), (T), (C) and (A), shows that

$$Z^{P(H^*)}_{E^*(4)}(1 \otimes Y X^4_{(f \times x \times g)_1} \otimes Z X^4_{(f \times x \times g)_2}) \quad \begin{align*} &\quad = \delta^4 p^4(h^1) p^3(1) k^3(h^2) k^2(1) (S p^1 k^4)(h^3) (v S(k^3 p^2 p^2)(2)(h^4) \\
&\quad = Z^{P(H^*)}_{E^*(2)}(1 \otimes Z^{P(H^*)}_{A}(u_2 \otimes u_1 \otimes (k^3 p^2 p^2))(f \times x \times g)_1 \otimes Z^{P(H^*)}_{A}(\tilde{u}_2 \otimes \tilde{u}_1 \otimes \tilde{v})(f \times x \times g)_2) \\
&\quad = \delta^4 p^4(h^1) p^3(1) k^3(h^2) k^2(1) (S p^1 k^4)(h^3) (v S(k^3 p^2 p^2)(2)(h^4) \\
&\quad = Z^{Q^4}_{E^*(4)}((f \times x \times g)_1 Z^{P(H^*)}_{A}(u_2 \otimes u_1 \otimes (k^3 p^2 p^2))(f \times x \times g)_1) Z^{Q^4}_{E^*(4)}((f \times x \times g)_2 Z^{P(H^*)}_{A}(u_2 \otimes u_1 \otimes \tilde{v})) \end{align*}$$

Now repeated application of Equation (6,4) shows that

$$\begin{align*}
\delta^4 p^4(h^1) p^3(1) k^3(h^2) k^2(1) (S p^1 k^4)(h^3) (v S(k^3 p^2 p^2)(2)(h^4) \\
Z^{Q^4}_{E^*(4)}((f \times x \times g)_1 Z^{P(H^*)}_{A}(u_2 \otimes u_1 \otimes (k^3 p^2 p^2))(f \times x \times g)_1) Z^{Q^4}_{E^*(4)}((f \times x \times g)_2 Z^{P(H^*)}_{A}(u_2 \otimes u_1 \otimes \tilde{v}))
\end{align*}$$
Finally, a routine computation in $P(H^*)$ shows that
\[
Z_E^Q(X_{f \times x \times g} \otimes Y \otimes Z) = \delta^i p^i(h^i) p^i(1) k^1(h^2) k^2(1) (S p^i_1 k^4)(h^3) (v S(k^3 p^i_2 p^2_2 h^4)) Z_E^P((f \times x \times g) \otimes Z_A^{(H^*)}(u_2 \otimes u_1 \otimes (k^3 p^i_2 p^2_2 1)) \otimes Z_A^{(H^*)}(u_2 \otimes u_1 \otimes \delta)).
\]
Hence, the formula for $\Delta_{Q^2}(X_{f \times x \times g})$ is verified. □

We now proceed towards establishing the comultiplication formula for a general element of $^*Q^m_2$. Let us take a general element $X$ of $^*Q^m_2$ as given by \([m,0]\) or \([m,1]\) according as $m$ is odd or even. The multiplication in $^*Q^m_2$ shows that $X$ can be expressed as
\[
X = X^m_{1} \otimes X^m_{k_1 S p_3 S k_3 \times F^{-1}(k_2 p_2) \times S k_1} X^m_{1}
\]
with $X^m_{1} \otimes X^m_{k_1 S p_3 S k_3 \times F^{-1}(k_2 p_2) \times S k_1} \otimes X^m_{(\in (^*Q^m_2) \otimes 3)}$ being given by
\[
(f^1 \times \cdots \times x^{m-2} \times \epsilon \times 1 \times \cdots \times \epsilon) \otimes X^m_{k_1 S p_3 S k_3 \times F^{-1}(k_2 p_2) \times S k_1} \otimes (\epsilon \times \cdots \times \epsilon \times x^{m+2} \times \cdots \times f^{2m-1})
\]
according as $m$ is even or odd. It is then not hard to show using Equation \([7.8]\) and Lemma \([17]\) that:

**Proposition 48.** $\Delta_{Q^2}(X) = \Delta_{Q^2}(1) \left( X^m_{(k_1 S p_3 S k_3 \times F^{-1}(k_2 p_2) \times S k_1)} X^m_{1} \otimes X^m_{3} X^m_{(k_1 S p_3 S k_3 \times F^{-1}(k_2 p_2) \times S k_1)_{2}} \right)$.

Observe that the comultiplication formula involves $\Delta(1)$. Certain useful facts regarding $\Delta(1)$ are contained in the following lemma.

**Lemma 49.** \([2]\), Proposition 4.12, Corollary 4.13 | If $P = P^{N \subset M}$ denotes the subfactor planar algebra associated to the finite-index, reducible, depth two inclusion $N \subset M$ of $II_1$-factors, then $\Delta_{P_1}(1) = f^1 \otimes S f^2$ where $f^1 \otimes f^2$ is the unique symmetric separability element of $P_{1,2}$ and $\Delta_{P_2}$ denotes the comultiplication in the weak Hopf $C^*$-algebra $P_2$.

Let $U \otimes V$ denote the unique symmetric separability element of $^*Q^m_{1,2}$ and let $U \otimes V(\in ^*Q^m_{1,2} \otimes 2)$ be given by
\[
(\epsilon \times 1 \times \epsilon \times \cdots \times \epsilon \times y^{1} \times g^{2} \times \cdots \times g^{m-2}) \otimes (\epsilon \times 1 \times \epsilon \times \cdots \times \epsilon \times y^{1} \times g^{2} \times \cdots \times g^{m-2})
\]
\[
(1 \times \epsilon \times 1 \times \cdots \times \epsilon \times y^{1} \times g^{2} \times \cdots \times y^{m-2}) \otimes (1 \times \epsilon \times 1 \times \cdots \times \epsilon \times y^{1} \times g^{2} \times \cdots \times y^{m-2})
\]
according as $m$ is even or odd. It then follows from Lemma \([49]\) and Lemma \([56]\) that $\Delta_{Q^m_2}(1)(= U \otimes SV)$ equals
\[
(\epsilon \times 1 \times \epsilon \times \cdots \times \epsilon \times y^{1} \times g^{2} \times \cdots \times g^{m-2}) \otimes (S y^{m-2} \times \cdots \times S y^{1} \times \epsilon \times 1 \times \epsilon \times \cdots \times \epsilon)
\]
\[
(1 \times \epsilon \times 1 \times \cdots \times \epsilon \times y^{1} \times g^{2} \times \cdots \times y^{m-2}) \otimes (S y^{m-2} \times \cdots \times S y^{1} \times \epsilon \times 1 \times \epsilon \times \cdots \times \epsilon)
\]
according as $m$ is even or odd. Hence, it follows from Proposition \([48]\) and Lemma \([49]\) that
\[
\Delta_{Q^m_2}(X) = U \ X^m_{(k_1 S p_3 S k_3 \times F^{-1}(k_2 p_2) \times S p_1)_{1}} X^m_{1} \otimes S(V) \ X^m_{3} X^m_{(k_1 S p_3 S k_3 \times F^{-1}(k_2 p_2) \times S p_1)_{2}}.
\]
Using the comultiplication formula in $\mathcal{Q}_m^2$ as given by Lemma [1], we see that
\[
\Delta_{\mathcal{Q}_m^2}(X) = \delta U X^m_{(kS\phi_2)1} S(\phi_1 p_3 S\phi_3) S(kS\phi_2)_3 \times F^{-1}((kS\phi_2)2 (\phi_1 p_3 S\phi_3)2) \times S(\phi_1 p_2 S\phi_3)_1 \times X^m_1 \otimes S(V) X^m_3 (p_1)3 S(\phi_4)3 \times F^{-1}((\phi_4)2(p_1)2) \times S(p_1)1.
\]
Assume now that $m$ is even. A tedious computation using the multiplication rule in $\mathcal{Q}_m^2$ shows that the formula for $\Delta_{\mathcal{Q}_m^2}(X)$ is given by:
\[
\Delta_{\mathcal{Q}_m^2}(X) = \delta (\phi_4 S\phi_2 S\phi_3)(S\phi_1^1) (f^1 \times \cdots \times x^{m-2} \times (kS\phi_2)_1 S(\phi_1 p_3 S\phi_3)_3 \times F^{-1}((kS\phi_2)2 (\phi_1 p_3 S\phi_3)_2) \times S(\phi_1 p_3 S\phi_3)_1 \times y^1 \times y^{m-2} \otimes S\phi_2^1 \times \cdots \times S\phi_2^3 \times (\phi_3)_1 S(p_1)_3 S(\phi_3)_3 \times F^{-1}((\phi_3)2(p_1)2) \times S(p_1)_1 \times x^{m+2} \times f^{m+3} \times \cdots \times f^{2m-1}).
\]
Similarly, when $m$ is odd, one can show that the formula for $\Delta_{\mathcal{Q}_m^2}(X)$ is given by:
\[
\Delta_{\mathcal{Q}_m^2}(X) = \delta (\phi_4 S\phi_2 S\phi_3)(S\phi_1^1) (x^1 \times \cdots \times x^{m-2} \times (kS\phi_2)_1 S(\phi_1 p_3 S\phi_3)_3 \times F^{-1}((kS\phi_2)2 (\phi_1 p_3 S\phi_3)_2) \times S(\phi_1 p_3 S\phi_3)_1 \times y^1 \times \cdots \times y^{m-2} \otimes S\phi_2^1 \times \cdots \times S\phi_2^3 \times (\phi_3)_1 S(p_1)_3 S(\phi_3)_3 \times F^{-1}((\phi_3)2(p_1)2) \times S(p_1)_1 \times x^{m+2} \times f^{m+3} \times \cdots \times x^{2m-1}).
\]
For each integer $m > 2$, let $K_m$ denote the vector space $A(H)^{op}_{m-2} \otimes D(H)^{op} \otimes A(H)^{op}_{m-2}$ or $A(H^*)^m_{m-2} \otimes D(H^*)^{op} \otimes A(H^*)^m_{m-2}$ according as $m$ is odd or even. Consider the linear isomorphism $\psi$ of $K_m$ onto $\mathcal{Q}_m^2$ given by
\[
a \otimes (g \otimes f) \otimes b \mapsto a \otimes f_1 Sg_3 Sf_3 \times F^{-1}(f_2 g_2) \times Sg_1 \otimes b.
\]
We make $K_m$ into a weak Hopf C*-algebra by transporting the structure maps on $\mathcal{Q}_m^2$ to $K_m$ using this linear isomorphism. Thus, by construction, $K_m$ is isomorphic to $\mathcal{Q}_m^2$ as weak Hopf C*-algebras. The next theorem, which is the main result of this section, explicitly describes the structure maps of $K_m$.

**Theorem 50.** For each $m > 2$, $K_m$ is a weak Hopf C*-algebra with the structure maps given by the following formulae.

**Multiplication:** $(a \otimes (g \otimes f) \otimes b)(\tilde{a} \otimes (\tilde{g} \otimes \tilde{f}) \otimes \tilde{b}) = (f_1 Sg_3 Sf_3 a) \tilde{a} \otimes (g_2 \otimes f) (\tilde{g}_1 \otimes \tilde{f}_2) \otimes b p g_1(\tilde{b}),$

**Comultiplication:** $\Delta(a \otimes (g \otimes f) \otimes b) = \delta(a \otimes (\phi_1 g_3 S\phi_3 \otimes f S\phi_2) \otimes u) \otimes ((\phi_2 g_2 S\phi_2)_1 \otimes (g_1 + \phi_2) \otimes b),$

**Counit:** $\varepsilon(a \otimes (g \otimes f) \otimes b) = \delta m^{-2} f(h) Z_{F_{m-1}}^H((a \times Sg) \otimes b),$

**Antipode:** $S(a \otimes (g \otimes f) \otimes b) = b' \otimes S_{D(H)^{op}}(g_3 \otimes f) \otimes a',$

with $a \otimes (g \otimes f) \otimes b, \tilde{a} \otimes (\tilde{g} \otimes \tilde{f}) \otimes \tilde{b}$ being elements of $K_m$, where

- for any $k \in H^*$ and $(\cdots \times f \times x)$ in $A(H)^{op}_{m-2}$ or $A(H^*)^{op}_{m-2}$ according as $m$ is odd or even, $m-2$ factors
  $k.(\cdots \times f \times x) := k(x_2) \cdots (\cdots \times f \times x_1),$
- $\rho$ denotes the algebra action of $H^*$ on $A(H)^{op}_{m-2}$ defined for $k \in H^*$ and $x \times p \times \cdots \in A(H)^{op}_{m-2}$ by $\rho_k(x \times p \times \cdots) = k(x) x_2 \times p \times \cdots,$
- $u \otimes v$ denotes the unique symmetric separability element of $A(H)^{op}_{m-2},$
- for any positive integer $k$, $F^{(k)}$ denotes the $k$-cabling of the tangle $F$ (see Figure 32) and
- for any $X$ in $H_{[1,1]}$, the symbol $X'$ denotes the element as defined in §1.1 preceding Lemma [3].

**Proof.** We first verify the formula for antipode. Assume without loss of generality that $m$ is even. Let $X \in K_m$ be given by
\[
(f^1 \times x^2 \times \cdots \times x^{m-2}) \otimes (p \otimes k) \otimes (x^{m+2} \times \cdots \times f^{2m-1})
\]
so that
\[ \psi(X) = f^1 \times x^2 \times \cdots \times x^{m-2} \times k_1 \text{Sp}_3 \text{Sp}_3 \times F^{-1}(k_2 p_2) \times \text{Sp}_1 \times x^{m+2} \times \cdots \times f^{2m-1}. \]

By Lemma 46
\[ S(\psi(X)) = \left( f^{2m-1} \times \cdots \times S x^{m+2} \times p_1 \times F^{-1}(k_2 p_2) \times S(k_1 \text{Sp}_3 \text{Sp}_3) \times S x^{m-2} \times \cdots \times S x^2 \times S f^1 \right) \]
\[ = (x^{m+2} \times \cdots \times f^{2m-1})^* \times p_1 \times F^{-1}(k_2 p_2) \times S(k_1 \text{Sp}_3 \text{Sp}_3) \times (f^1 \times x^2 \times \cdots \times x^{m-2})^*. \]

Finally, it follows from the formula for antipode as given in Theorem 50 that each \( K_m \) is a weak Kac algebra.

\[ \psi^{-1}(S(\psi(X))) = (x^{m+2} \times \cdots \times f^{2m-1})^* \otimes S_{D(H)^{op}}(p \otimes k) \otimes (f^1 \times x^2 \times \cdots \times x^{m-2})^*. \]

Thus the formula for antipode is verified. In a similar way, using the comultiplication formula in \( "Q_2^n \) as given by (7.9) or (7.10) according as \( m \) is even or odd, the comultiplication formula in \( K_m \) can easily be verified. The verification of the multiplication and counit formula in \( K_m \) involves tedious computation and we leave these verifications for the reader.

\[ \square \]

**Remark 51.** It follows immediately from the formula for antipode as given in Theorem 50 that each \( K_m \) (\( m > 2 \)) has involutive antipode i.e., square of the antipode is the identity and consequently, each \( K_m \) is a weak Kac algebra.

**Remark 52.** The sole importance of Theorem 50 lies in the fact that it constructs a family of weak Kac algebras out of a given finite-dimensional Kac algebra.

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