Supersymmetric Index
Of Three-Dimensional Gauge Theory

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In $N = 1$ super Yang-Mills theory in three spacetime dimensions, with a simple gauge group $G$ and a Chern-Simons interaction of level $k$, the supersymmetric index $\text{Tr} (-1)^F$ can be computed by making a relation to a pure Chern-Simons theory or microscopically by an explicit Born-Oppenheimer calculation on a two-torus. The result shows that supersymmetry is unbroken if $|k| \geq h/2$ (with $h$ the dual Coxeter number of $G$) and suggests that dynamical supersymmetry breaking occurs for $|k| < h/2$. The theories with large $|k|$ are massive gauge theories whose universality class is not fully described by the standard criteria.

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1. Introduction

If a $d+1$-dimensional supersymmetric quantum field theory is quantized on $T^d \times \mathbb{R}$ (with $T^d$ understood as space and $\mathbb{R}$ parametrizing the time), the spectrum is often discrete. If so, one can define a supersymmetric index $\text{Tr} (-1)^F$, the number of zero energy states that are bosonic minus the number that are fermionic. The index is invariant under smooth variations of parameters (such as masses, couplings, and the flat metric on $T^d$) that can be varied while preserving supersymmetry. For this reason, it often can be computed even in strongly coupled theories [1].

When $\text{Tr} (-1)^F$ is nonzero, there are supersymmetric states for any volume of $T^d$, and hence the ground state energy is zero regardless of the volume. When one has a reasonable control on the behavior of the theory for large field strengths (to avoid for example the possibility that a supersymmetric state goes off to infinity as the volume goes to infinity), it follows that the ground state energy is zero and supersymmetry is unbroken in the infinite volume limit. Conversely, if $\text{Tr} (-1)^F = 0$, this gives a hint that supersymmetry might be spontaneously broken in the quantum theory, even if it appears to be unbroken classically.

There are interesting examples of theories (e.g., nonlinear sigma models in two dimensions, and pure $N=1$ supersymmetric Yang-Mills theory in four dimensions) in which a nonzero value of $\text{Tr} (-1)^F$ has been used to show that supersymmetry remains unbroken even for strong coupling. But there are in practice very few instances in which vanishing of this index has served as a clue to spontaneous supersymmetry breaking. One reason for this is that many interesting supersymmetric theories have a continuous spectrum if compactified on a torus, making $\text{Tr} (-1)^F$ difficult to define, or have a nonzero value of $\text{Tr} (-1)^F$, so that supersymmetry cannot be broken. In other examples, $\text{Tr} (-1)^F$ is defined and equals zero, but does not give a useful hint of supersymmetry breaking because this phenomenon is either obvious classically or is obstructed by the existence, classically, of a mass gap, or for other reasons.

The present paper is devoted to a case in which the index does seem to give a clue about when supersymmetry is dynamically broken. This example is the pure $N=1$ supersymmetric gauge theory in three spacetime dimensions, with simple compact gauge group $G$. The theory can be described in terms of a gauge field $A$ and a gluino field $\lambda$ (a Majorana fermion in the adjoint representation). We include a Chern-Simons interaction, so the Lagrangian with Euclidean signature reads

$$L = \frac{1}{4e^2} \int d^3 x \text{Tr} (F_{IJ} F^{IJ} + \overline{\lambda} \Gamma \cdot D \lambda) - \frac{ik}{4\pi} \int \text{Tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A + \overline{\lambda} \lambda \right). \quad (1.1)$$
The parameter \( k \) is quantized topologically \([2]\). If \( h \) denotes the dual Coxeter number of \( G \), then the quantization condition is actually that \( k - h/2 \) should be an integer, as was pointed out in \([3,4]\), using a mechanism of \([5,6]\). The situation will be reviewed in section 2.

Let \( I(k) \) denote the supersymmetric index as a function of \( k \). We will show that \( I(k) \neq 0 \) for \( |k| \geq h/2 \), but \( I(k) = 0 \) for \( |k| < h/2 \). For example, for \( G = SU(n) \), we have \( h = n \), and

\[
I(k) = \frac{1}{(n-1)!} \prod_{j=-n/2+1}^{n/2-1} (k - j).
\] (1.2)

So \( I(k) \) vanishes precisely if \( |k| < n/2 = h/2 \).

From this it follows that supersymmetry is unbroken quantum mechanically for \( |k| \geq h/2 \). But we conjecture that in the “gap,” \( |k| < h/2 \), supersymmetry is spontaneously broken. For this we offer two bits of evidence beyond the vanishing of the index. One is that an attempt to disprove the hypothesis of spontaneous supersymmetry breaking for \( |k| < h/2 \) by considering an \( SU(n)/\mathbb{Z}_n \) theory (instead of \( SU(n) \)) fails in a subtle and interesting way. The second is that, as we will see, if the theory is formulated on a two-torus of finite volume, spontaneous breaking of supersymmetry occurs. Of course, these considerations do not add up to a proof, but they are rather suggestive.

The paper is organized as follows. In section 2, we compute the index for sufficiently large \( k \) by using low energy effective field theory and the relation \([4]\) of Chern-Simons gauge theory to two-dimensional conformal field theory. In the process, we also review the anomaly that sometimes shifts \( k \) to half-integer values, and we explain the failure of a plausible attempt to disprove the hypothesis of symmetry breaking in the gap via \( SU(n)/\mathbb{Z}_n \) gauge theory. In section 3, we make a more precise microscopic computation of the index, and show that for finite volume symmetry breaking does occur in the gap. Finally, in section 4, we consider three-dimensional Chern-Simons theories in the light of the familiar classification \([8]\) of massive phases of gauge theories, and show that such massive phases are not fully classified by the usual criteria. This is true even in four dimensions, but the full classification of massive phases is particularly rich in three dimensions.

For other recent results on dynamics of supersymmetric Chern-Simons theories in three dimensions, see \([9]\).
2. Computation Via Low Energy Effective Field Theory

The index can be computed very quickly if $k$ is sufficiently large. At the classical level, the theory has a mass gap for $k \neq 0$. The mass is of order $e^2 |k|$, which if $|k| \gg 1$ is much greater than the scale $e^2$ set by the gauge couplings. So for $|k| \gg 1$, the classical computation is reliable, the theory has a mass gap, and in particular (as there is no Goldstone fermion) supersymmetry is unbroken.

Moreover, we can compute the index for sufficiently large $|k|$ using low energy effective field theory. For large enough $|k|$, the mass gap implies that the fermions can be integrated out to give a low energy effective action that is still local. Integrating out the fermions gives a shift in the effective value of $k$. The shift can be computed exactly at the one-loop level. In fact, integrating out the fermions shifts the effective value of $k$ in the low energy effective field theory to

$$k' = k - \frac{h}{2} \text{sgn}(k),$$

(2.1)

where $\text{sgn}(k)$ is the sign of $k$. (The shift in $k$ is proportional to the sign of $k$, because this sign determines the sign of the fermion mass term.) So for example if $k$ is positive, as we assume until further notice, then $k' = k - h/2$. For the low energy theory to make sense, $k'$ must be an integer, and hence $k$ must be congruent to $h/2$ modulo $\mathbb{Z}$. So if $h$ is odd, then $k$ is half-integral, rather than integral. For example, for $SU(n)$, $h = n$ and $k$ is half-integral if $n$ is odd.

Since the factor of $h/2$ will be important in this paper, we pause to comment on how it emerges from Feynman diagrams. The basic parity anomaly is the assertion that for an $SU(2)$ gauge theory with Majorana fermions consisting of two copies of the two-dimensional representation, the one-loop shift in $k$ is $1/2$. (We must take two copies of the $2$ of $SU(2)$, not one, because the $2$ is a pseudoreal representation, but Majorana fermions are real.) For any other representation, the one-loop shift is scaled up in proportion to the trace of the quadratic Casimir for that representation. For three Majorana fermions in the adjoint representation of $SU(2)$, the trace of the quadratic Casimir is twice as big as for two $2$'s, so the shift in $k$ is 1, which we write as $h/2$, with $h = 2$ for $SU(2)$. The result $h/2$ is universal, since $h$ is the group theory factor in the one-loop diagram for any

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1 There are many ways to prove this. For example, the $s$-loop effective action for $s > 1$ is the integral of a gauge-invariant local density, which the Chern-Simons functional is not, so a renormalization of the effective value of $k$ can only come at one loop. Alternatively, an $s$-loop diagram is proportional to $e^{2s-2}$, and so can only renormalize an integer $k$ if $s = 1$. 
group. This argument does not explain the minus sign in the formula \( k' = k - h/2 \), which depends on some care with orientations. This sign can be seen in Feynman diagrams \(^3\), and also has a topological meaning that we will see in section 3.

Now, for very large \( k \), though the theory has a mass gap, it is not completely trivial at low energies. Rather, there is a nontrivial dynamics of zero energy states governed by the Chern-Simons theory at level \( k' \). At low energies, we can ignore the Maxwell-Yang-Mills term in the action, and approximate the theory by a “pure Chern-Simons” theory, with the Chern-Simons action only. This is a topological field theory and in fact is a particularly interesting one. In general, if the pure Chern-Simons theory at level \( k' \) is formulated on a Riemann surface \( \Sigma \) of genus \( g \), then \(^7\) the number of zero energy states equals the number of conformal blocks of the WZW model of \( G \) at level \( k' \). Moreover, these states are all bosonic. \(^2\) For our present application, \( \Sigma = T^2 \) and the genus is 1. In this case, the number \( J(k') \) of conformal blocks is equal to the number of representations of the affine Lie algebra \( \hat{G} \) at level \( k' \). This number is positive for all \( k' \geq 0 \), and for large \( k' \) is of order \((k')^r\), with \( r \) the rank of \( G \). For more detail on the canonical quantization of the Chern-Simons theory, see \(^{10-12}\).

The following paragraph is aimed to avoid a possible confusion. In Chern-Simons theory at level \( k' \), many physical results, like expectation values of products of Wilson loops, are conveniently written as functions of \( k' + h \). From the point of view of Feynman diagram calculations, this arises because a one-loop diagram with internal gauge bosons shifts \( k' \) to \( k' + h \) \(^{13,14}\). In a Hamiltonian approach to Chern-Simons gauge theory without fermions, one sees in another way that if the parameter in the Lagrangian is \( k' \), many physical answers are functions of \( k' + h \) \(^{10,11}\). (This Hamiltonian approach is much closer to what we will do in section 3 for the theory with fermions.) In asserting that the effective coefficient of the Chern-Simons interaction is \( k' = k - h/2 \), we are referring to an effective Lagrangian in which the fermions have been integrated out, but one has not yet tried to solve for the quantum dynamics of the gauge bosons.

Since the pure Chern-Simons theory is a good low energy description for sufficiently large \( k \), the index of the supersymmetric theory at level \( k \) can be identified for sufficiently large \( k \) with the number of supersymmetric states of the pure Chern-Simons theory at level \( k' \):

\[
I(k) = J(k').
\]  

\(^2\) Or they are all fermionic. In finite volume, there is a potentially arbitrary sign choice in the definition of the operator \((-1)^F\), as we will see in more detail in section 3.
For example, suppose $G = SU(2)$. The representations of the $SU(2)$ affine algebra at level $k'$ have highest weights of spin $0, 1/2, 1, \ldots, k'/2$; there are $k'+1$ such representations in all. As $k'+1 = k$ for $SU(2)$, we get
\[ I(k) = k, \] (2.3)

at least for sufficiently big $k$ where the effective description by $SU(2)$ Chern-Simons theory at level $k'$ is valid.

The formula, however, has a natural analytic continuation for all $k$, and we may wonder if (2.3) holds for all $k$. We will show this in the next section by a microscopic computation, but in the meantime, a hint that this is so is as follows. The sign reversal $k \to -k$ is equivalent in the Chern-Simons theory to a reversal of spacetime orientation, so one might expect $I(-k) = I(k)$. Actually, in general, the sign of the operator $(-1)^F$ in finite volume can depend on an arbitrary choice, as in some examples in [1]. If a parity-invariant choice of this sign cannot be made in general, then we should expect only $I(-k) = \pm I(k)$. This is consistent with (2.3), which gives $I(-k) = -I(k)$. We will see in section 3 that the general formula, for a gauge group $G$ of rank $r$, is
\[ I(-k) = (-1)^r I(k). \] (2.4)

In (2.3), we can also see the claim made in the introduction: $I(k) = 0$ for $|k| < h/2$, and $I(k) \neq 0$ for $|k| \geq h/2$. For $SU(2)$, as $h/2 = 1$, this is equivalent to the statement that $I(k)$ vanishes precisely if $k = 0$. We thus learn that for $G = SU(2)$, supersymmetry is unbroken for all $k \neq 0$, and we conjecture that it is spontaneously broken for $k = 0$. (If this is so, then in particular there is a Goldstone fermion for $k = 0$, and the pure Chern-Simons theory on which we have based our initial derivation of (2.3) is not a good low energy description for $k = 0$.)

A similar structure holds for other groups. For example, for $G = SU(n)$ one has
\[ J(k') = \frac{1}{(n-1)!} \prod_{j=1}^{n-1} (k' + j). \] (2.5)

(One way to compute this formula – and its generalization to other groups – will be reviewed in section 3.) When expressed in terms of $k$, this gives the formula for $I(k)$ already presented in the introduction:
\[ I(k) = \frac{1}{(n-1)!} \prod_{j=-n/2+1}^{n/2-1} (k - j). \] (2.6)

We see the characteristic properties $I(-k) = (-1)^{n-1} I(k)$ and $I(k) = 0$ for $|k| < n/2$.  

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2.1. Microscopic Derivation Of Parity Anomaly

The shift in the effective value of $k$ – namely $k' = k - \frac{1}{2} h \text{sgn}(k)$ – has played an important role in this discussion. As we have already noted, the existence of this shift implies – since the effective Chern-Simons coupling must be an integer – that $k$ is congruent modulo $\mathbb{Z}$ to $h/2$. When $h$ is odd – for example, for $SU(n)$ with odd $n$ – it follows that $k$ is not an integer and in particular cannot be zero. Such a phenomenon in three-dimensional gauge theories is known as a parity anomaly [5,6], the idea being that the theory conserves parity if and only if $k$ vanishes, so the non-integrality of $k$ means that parity cannot be conserved.

The derivation of the parity anomaly from the shift in the effective value of $k$ is valid for sufficiently large $k$ – where there is an effective low energy description as a Chern-Simons theory – but is not valid for small $k$. One would like to complement this low energy explanation by an explanation at short distances, in terms of the elementary degrees of freedom, that does not depend on knowledge about the dynamics at long distances.

We will now review how this is done [5,6]. In this discussion, we assume to begin with that the gauge group $G$ is simply-connected (and connected), so that the gauge bundle over the three-dimensional spacetime manifold $X$ is automatically trivial. For most of the discussion below, the topology of $X$ does not matter, but for eventual computation of $\text{Tr} \left( (-1)^F \right)$, one is most interested in $T^3$ or $T^2 \times \mathbb{R}$.

The path integral in a three-dimensional gauge theory with fermions has two factors: the definition of whose phases requires care. One is the exponential of the Chern-Simons functional. The other is the fermion path integral. As the fermions are real, the fermion path integral equals the square root of the determinant of the Dirac operator $\mathcal{D} = i \Gamma \cdot \mathcal{D}$. (When we want to make explicit the dependence of the Dirac operator on a gauge field $A$, we write it as $\mathcal{D}_A$. Note that we consider the massless Dirac operator. The topological considerations of interest for the moment are independent of the mass.) Thus, the factors that we must look at are

$$\sqrt{\det \mathcal{D}} \exp \left( \frac{ik}{4\pi} \int \text{Tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) \right). \quad (2.7)$$

First let us recall the issues in defining $\sqrt{\det \mathcal{D}}$. The operator $\mathcal{D}$ is hermitian, so its eigenvalues are real. Moreover, in three dimensions, for fermions taking values in a real bundle such as the adjoint bundle, the eigenvalues are all of even multiplicity. This follows
from the existence of an antiunitary symmetry analogous to CPT in four dimensions. The determinant of the Dirac operator is defined roughly as

$$\det D = \prod_i \lambda_i,$$  \hspace{1cm} (2.8)

where the infinite product is regularized with (for example) zeta function or Pauli-Villars regularization. Note in particular that the determinant is formally positive – there are infinitely many negative $\lambda$’s, but they come in pairs – and this positivity is preserved in the regularization. Now consider the square root of the determinant, which is defined roughly as

$$\sqrt{\det D} = \prod_i '\lambda_i,$$  \hspace{1cm} (2.9)

where the product runs over all pairs of eigenvalues and the symbol $\prod_i'$ means that (to get the square root of the determinant) we take one eigenvalue from each degenerate pair. This infinite product of course needs regularization. Since $\det D$ has already been defined, to make sense of $\sqrt{\det D}$ we must only define the sign. For this we must determine, formally, whether the number of negative eigenvalue pairs is even or odd; it is here that an anomaly will come in.

It suffices to determine the sign of $\sqrt{\det D}_A$ up to an overall $A$-independent sign (which cancels out when we compute correlation functions). For this, we fix an arbitrary connection $A_0$ (chosen generically so that the Dirac operator $D_{A_0}$ has no zero eigenvalues), and declare that $\sqrt{\det D_{A_0}}$ is, say, positive. Then to determine the sign of $\sqrt{\det D}_A$ for any other connection $A$ on the same bundle, we interpolate from $A_0$ to $A$ via a one-parameter family of connections $A_t$, with $A_{t=0} = A_0$, and $A_{t=1} = A$. We follow the spectrum of $D_{A_t}$ as $t$ evolves from 0 to 1, and denote the net number of eigenvalue pairs that change sign from positive to negative as the spectral flow $q$. (If $A_t$ is a generic one-parameter family, then there are no level crossings for $0 \leq t \leq 1$, and the spectral flow for $0 \leq t \leq 1$ is as follows: every eigenvalue pair flows upwards or downwards by $|q|$ units.) Then we define

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3 Use standard gamma matrix conventions such that, in a local Lorentz frame, the gamma matrices are the $2 \times 2$ Pauli spin matrices, which are real and symmetric or imaginary and antisymmetric. The Dirac operator then commutes with the antiunitary transformation $T : \lambda^\alpha \rightarrow \epsilon^{\alpha\beta} \overline{\lambda}_\beta$. Since $T$ is antiunitary and $T^2 = -1$, $\lambda$ and $T\lambda$ are always linearly independent, so the eigenstates of the Dirac operator come in pairs.

4 For example, we can take the family $A_t = tA_0 + (1 - t)A$.  

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the sign of \( \sqrt{\det D_A} \) to be \((-1)^q\), the intuitive idea being that the sign of the product in (2.3) should change whenever an eigenvalue pair crosses zero. The only potential problem with this definition is that it might depend on the path from \( A_0 \) to \( A \).

A problem arises precisely if there is a path dependence in the value of \( q \) modulo 2. There is such path dependence if and only if there is a closed path, in the space of connections modulo gauge transformations, for which the spectral flow is odd. To determine whether this occurs, we proceed as follows. Let \( A_t \), for \( 0 \leq t \leq 1 \), be a family of gauge fields such that \( A_1 \) is gauge-equivalent to \( A_0 \) by a gauge transformation \( \Omega \). Such an \( \Omega \) is classified by its “winding number” \( \nu \) which takes values in \( \pi_3(G) = \mathbb{Z}^\mathbb{F} \). In this situation, there is a nice formula for the spectral flow. Each \( A_t \) is a connection on a trivial bundle over \( X \). The family \( A_t \), \( 0 \leq t \leq 1 \), can be fit together to make a connection on a trivial bundle over \( I \times X \), where \( I = [0, 1] \) is the closed unit interval. Gluing together the endpoints of \( I \) to make a circle \( S^1 \) – and identifying the gauge bundles over the boundaries of \( I \times X \) using the gauge transformation \( \Omega \) – one can reinterpret the family \( A_t \) as a connection on a possibly nontrivial bundle \( E \) over \( S^1 \times X \). This bundle has instanton number \( \nu \), determined by the topological twist of \( \Omega \). The spectral flow is then

\[
q = h\nu. \tag{2.10}
\]

This relation between spectral flow and the topology of the bundle \( \mathbb{F} \), which is important in instanton physics \( \mathbb{F} \), is proved roughly as follows using the index theorem for the four-dimensional Dirac operator on \( S^1 \times X \). We call that operator \( D_4 \) and let \( D_{4,+} \) and \( D_{4,-} \) be the restrictions of \( D_4 \) to spinors of positive or negative chirality, namely

\[
D_{4,\pm} = \pm \frac{\partial}{\partial t} + \epsilon D_{A_t}, \tag{2.11}
\]

where \( \epsilon \) is a positive real number that one can introduce by scaling the metric on \( X \). For small \( \epsilon \), the Dirac equation \( D_{4,\pm} \psi = 0 \) can be studied in terms of the \( t \)-dependent spectrum of \( D_{A_t} \). If \( \lambda_i(t) \) are the eigenstates of \( D_{A_t} \) with eigenvalues \( s_i(t) \), then one can approximately solve the four-dimensional Dirac equation with the formula

\[
\Psi_i(t) = \sum_{k \in \mathbb{Z}} \exp \left( \mp \epsilon \int_0^{t+k} s_i(t')dt' \right) \lambda_i(t + k) \tag{2.12}
\]

5 The winding number completely specifies the topology of \( \Omega \) because we are taking \( G \) to be connected and simply-connected. Otherwise, depending on the topology of spacetime, \( \Omega \) may have additional topological invariants.
where the sign in the exponent is $\mp$ to give zero modes of $D_{4,\pm}$. The sum over $k$ has been included to ensure $\Psi_i(t+1) = \Psi_i(t)$. Different $\lambda_i$ that are related by spectral flow (that is by $t \to t + 1$) give the same $\Psi_i$, so for generic spectral flow there are $|2q|$ linearly independent four-dimensional solutions of this kind. For $\Psi_i$ to be square integrable, the exponential factor in (2.12) must vanish for $t' \to \pm \infty$, so $\mp s_i(t')$ must be negative for $t' \to \infty$ and also for $t' \to -\infty$. This determines that the chirality of the solutions is the same as the sign of the spectral flow $q$. The upshot is that the index $I(D_4)$ of $D_4$ equals $2q$. (The factor of 2 arises because we defined $q$ by counting pairs of eigenvalues; each pair contributes two four-dimensional zero modes.) On the other hand, the index theorem for the Dirac operator gives $I(D_4) = 2h\nu$. Combining the formulas for $I(D_4)$ gives (2.10).

Now we put our results together. Under the gauge transformation $\Omega$, or in other words in interpolating from $t = 0$ to $t = 1$, the sign of $\sqrt{\det D}$ changes by $(-1)^q$. In view of (2.10), this factor is $(-1)^{h\nu}$. On the other hand, the change in the Chern-Simons functional under a gauge transformation of winding number $\nu$ is $2\pi \nu$. So under the gauge transformation $\Omega$, the dangerous factors (2.7) in the path integral pick up a factor

$$(-1)^{h\nu} \exp(2\pi ik\nu).$$  \hfill (2.13)

Gauge invariance of the theory amounts to the statement that this factor must be an integer for arbitrary integer $\nu$, and this gives us the restriction on $k$:

$$k \approx \frac{h}{2} \text{ modulo } \mathbb{Z}. \quad \text{ (2.14)}$$

### 2.2. The $SU(n)/\mathbb{Z}_n$ Theory

Now we have assembled the ingredients to put the hypothesis of dynamical supersymmetry breaking for $|k| < h/2$ to an apparently rather severe test. The discussion is most interesting for the case $G = SU(n)$, so we focus on that case.

The idea is to consider $\text{Tr}(-1)^F$ for an $SU(n)/\mathbb{Z}_n$ theory on $T^2$. The key difference between $SU(n)$ and $SU(n)/\mathbb{Z}_n$ is that any $SU(n)$ bundle on $T^2$ is trivial, but an $SU(n)/\mathbb{Z}_n$ bundle on $T^2$ is characterized by a “discrete magnetic flux” $w$ that takes values in $\mathbb{Z}_n$. (For $n = 2$, $SU(2)/\mathbb{Z}_2 = SO(3)$, and the discrete flux is the second Stieffel-Whitney class of the bundle.) An example of a bundle with any required value of $w$ is as follows. Consider a flat $SU(n)/\mathbb{Z}_n$ bundle whose holonomies $U$ and $V$ around the two directions in $T^2$, if lifted to $SU(n)$, obey

$$UV = VU \exp(2\pi ir/n).$$  \hfill (2.15)
Such a flat bundle has \( w = r \).

The computation of \( \text{Tr} (-1)^F \) for this theory can be made very easily in case \( r \) and \( n \) are relatively prime, for instance \( r = 1 \). (The computation can be done for any \( r \) by using the relation to the WZW model of \( SU(n)/\mathbb{Z}_n \), along the lines of section 2.1 above, or more explicitly using the techniques of section 3.) The idea is simply \( 1 \) that zero energy quantum states are obtained, for weak coupling, by quantizing the space of zero energy classical states (including possible bosonic or fermionic zero modes). A zero energy classical configuration of the gauge fields is a flat connection. For \( r \) and \( n \) relatively prime, a flat connection – that is a pair of matrices \( U \) and \( V \) obeying (2.15) – is unique up to gauge transformation. Moreover, in expanding around such a flat connection, there are no bosonic or fermionic zero modes. Hence, the quantization is straightforward: quantizing a unique, isolated classical state of zero energy, with no zero modes, gives a unique quantum state.\(^6\) The index is therefore \( \pm 1 \) (with the sign possibly depending on a choice of sign in the definition of the operator \( (-1)^F \)).

Note that \( k \) plays no role in this argument. Hence, for any \( k \) for which the \( SU(n)/\mathbb{Z}_n \) theory exists, this theory, if formulated on a bundle with \( r \) prime to \( n \), has a supersymmetric vacuum state for any volume of \( T^2 \). Taking the limit of infinite volume, it follows that the \( SU(n)/\mathbb{Z}_n \) theory, for any such \( k \), has zero vacuum energy and hence unbroken supersymmetry.

But in infinite volume, the \( SU(n) \) and \( SU(n)/\mathbb{Z}_n \) theories are equivalent.\(^7\) Hence for any \( k \) for which the \( SU(n)/\mathbb{Z}_n \) theory is defined, the \( SU(n) \) theory has unbroken supersymmetry.

Does this not disprove the hypothesis that the \( SU(n) \) theory has spontaneously broken supersymmetry in the “gap,” that is for \( |k| < n/2 \)? In fact, there is an elegant escape which we will now describe.

The allowed values of \( k \) were determined for \( SU(n) \) by requiring that

\[
(-1)^{n\nu} \exp(2\pi ik\nu) \tag{2.16}
\]

\(^6\) There is actually a potential subtlety in this statement, though it is inessential in the examples under discussion. One must verify that the one state in question obeys Gauss’s law, in other words that it is invariant under the gauge symmetries left unbroken by the classical solution that is being quantized.

\(^7\) Except for questions of which operators one chooses to probe them by; such questions are irrelevant for the present purposes.
should equal 1 for all integer values of the instanton number $\nu$. (We have rewritten (2.13) using the fact that $h = n$ for $SU(n)$.) For $SU(n)/\mathbb{Z}_n$, there is a crucial difference: the instanton number $\nu$ is not necessarily an integer, but takes values in $\mathbb{Z}/n$. (For example, setting $X = T^3$, an $SU(n)/\mathbb{Z}_n$ bundle on $S^1 \times T^3 = T^4$ that has unit magnetic flux in the 1-2 and 3-4 directions and other components vanishing has instanton number $1/n$ modulo $\mathbb{Z}$. In fact, on a four-manifold that is not spin, the instanton number takes values in $\mathbb{Z}/2n$, but for our present purpose – as the supersymmetric theory has fermions – only spin manifolds are relevant.) Hence gauge invariance of the theory requires that (2.16) should equal 1 not just for all $\nu \in \mathbb{Z}$, but for all $\nu \in \mathbb{Z}/n$. This gives the relation

$$k \cong \frac{n}{2} \text{ modulo } n. \quad (2.17)$$

Thus, for $SU(n)/\mathbb{Z}_n$, $k$ cannot be in the “gap” $|k| < n/2$, and the behavior of the $SU(n)/\mathbb{Z}_n$ theory in finite volume cannot be used to exclude the hypothesis that in the gap supersymmetry is dynamically broken. Though this does not prove that supersymmetry is broken in the gap for $SU(n)$, the elegant escape does suggest that that is the right interpretation.

3. Microscopic Computation Of The Index

In this section, we will make a microscopic computation of $\text{Tr} \ (-1)^F$ in the $N = 1$ supersymmetric pure gauge theory in three spacetime dimensions. We consider first the case that the gauge group $G$ is simply connected.

We thus formulate the theory on a spatial torus $T^2$ (times time) and look for zero energy states. As in [1], the computation will be done by a “Born-Oppenheimer approximation,” quantizing the space of classical zero energy states, and is valid for weak coupling or small volume of $T^2$. To be more exact, we work on a torus of radius $r$, and let $e$ and $k$ denote, as before, the gauge and Chern-Simons couplings. Particles with momentum on $T^2$ have energies of order $1/r$, while the fermion and gauge boson bare mass is $e^2 k$. We work in the region

$$e^2 k << \frac{1}{r}. \quad (3.1)$$

We will write an effective Hamiltonian that describes states with energies of order $e^2 k$ (or less) but omits states with energies of order $1/r$.

A zero energy classical gauge field configuration is a flat connection and is determined up to gauge transformation by its holonomies $U, V$ around the two directions in $T^2$. These
holonomies, since they commute, can simultaneously be conjugated to the maximal torus $U$ of $G$ in a way that is unique up to a Weyl transformation. The moduli space $\mathcal{M}$ of flat $G$-connections on $T^2$ is thus a copy of $(U \times U)/W$, where $W$ is the Weyl group. Concretely, a flat connection on $T^2$ can be represented by a constant gauge field

$$A_i = \sum_{a=1}^{r} c_i^a T^a,$$

where the $T^a$, $a = 1, \ldots, r$ are a basis of the Lie algebra of $U$ and the $c_i^a$ can be regarded as constant abelian gauge fields on $T^2$. The flat metric on $T^2$ determines a complex structure on $T^2$; it also determines a complex structure on $\mathcal{M}$ in which the complex coordinates are the components $c^a_i$ of the one-forms $c^i$. The $c^i$ are defined modulo $2\pi$ shifts.

To construct the right quantum mechanics on $\mathcal{M}$, we must also look at the fermion zero modes. Actually, by “zero modes” we mean modes whose energy is at most of order $e^2 k$, rather than $1/r$. In finding these modes, we can ignore the fermion bare mass and look for zero modes of the massless two-dimensional Dirac operator $\mathcal{D}$. We then will write an effective Lagrangian that incorporates the effects of the fermion bare mass. Let $\lambda_+$ and $\lambda_-$ be the gluino fields of positive and negative chirality on $T^2$. (They are hermitian conjugates of each other.) For a diagonal flat connection such as we have assumed, the equation $\mathcal{D}\lambda = 0$ has a very simple structure. For generic $U$ and $V$ (or equivalently for generic $c^i$), the “off-diagonal” fermions have no zero modes, while the “diagonal” fermions have “constant” zero modes. In other words, the zero modes of $\lambda_\pm$ are given by the ansatz

$$\lambda_\pm = \sum_{a=1}^{r} \eta^a_\pm T^a,$$

with $\eta^a_\pm$ being anticommuting constants.

Now let us discuss quantization of the fermions. Quantization of the nonzero modes gives a Fock space. Quantization of the zero modes $\eta^a_\pm$ is, as usual, more subtle. The canonical anticommutation relations of the $\eta$’s are, with an appropriate normalization,

$$\{\eta^a_+, \eta^b_-\} = \delta^{ab}, \quad \{\eta_+, \eta_+\} = \{\eta_-, \eta_-\} = 0.$$

Thus, we can regard the $\eta_+$ and $\eta_-$ as creation and annihilation operators. For example, we can introduce a state $|\Omega_-\rangle$ annihilated by the $\eta^a_-$, and build other states by acting with

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8 The analogous statement can fail – see the appendix of [18] – for the case of three commuting elements of $G$. 

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η⁺_a’s; or we can introduce a state |Ω₊⟩ annihilated by the η⁺_a’s, and build the rest of the Hilbert space by acting with η₋_a’s. The relation between the two descriptions is of course

|Ω₊⟩ = \prod_{a=1}^{r} η₊_a |Ω₋⟩.

(3.5)

Now let us try to define the operator (-1)^F. It is clear how we want (-1)^F to act on the Fock space built by quantizing the nonzero modes of the fermions: it leaves the ground state invariant and anticommutes with all nonzero modes of λ. The only subtlety is in the action on the zero mode Hilbert space. There is in general no natural choice for the sign of (-1)^F. If we pick, say (-1)^F |Ω₋⟩ = + |Ω₋⟩, then (3.5) implies that (-1)^F |Ω₊⟩ = (-1)^r |Ω₊⟩. Thus, if r is even, we can fairly naturally pick both of the states |Ω₊⟩ and |Ω₋⟩ to be bosonic. But if r is odd, then inevitably one is bosonic and one is fermionic; which is which depends on a completely arbitrary choice. Now we can explain an assertion in section 2, namely that

I(−k) = (-1)^r I(k).

(3.6)

Changing the sign of the Chern-Simons level is equivalent to a transformation that reverses the orientation of T^2. Such a transformation exchanges λ₊ with λ₋, and so exchanges |Ω₊⟩ with |Ω₋⟩. This exchange reverses the sign of the (-1)^F operator if r is odd, and that leads to (3.6).

The Hilbert space made by quantizing the fermion zero modes has a very natural interpretation. The quadratic form (3.4) has the same structure as the metric ds^2 = Σ_{a,b} δ_{ab} c^a_z ⊗ d^-_z of M, so the η’s can be interpreted as gamma matrices on M. Hence the Hilbert space obtained by quantizing the zero modes is the space of spinor fields on M, with values in a line bundle W that we have not yet identified. (Such a line bundle may appear because, for example, for a given point on M, the state |Ω₊⟩ is unique up to a complex multiple, but as one moves on M, it varies as the fiber of a not-yet-determined complex line bundle.)

Because M is a complex manifold, spinor fields on M have a particularly simple description. Let K be the canonical line bundle of M, and assume for the time being that there exists on M a line bundle K^{1/2}. Then the space of spinors on M is the same as the space Ω^{0,q}(M) ⊗ K^{1/2} of (0,q)-forms on M (for 0 ≤ q ≤ s) with values in K^{1/2}. In this identification, we regard |Ω₊⟩ as a (0,0)-form on M (with values in a line bundle), and we identify a general state in the fermionic space

η₋_a^1 ... η₋_a^q |Ω₊⟩

(3.7)
as a \((0, q)\) form on \(\mathcal{M}\). From this point of view, we identify \(\eta^-\) with the \((0, 1)\) form \(dc^a_{\bar{z}}\), and \(\eta^+\) with the “contraction” operator that removes the one-form \(dc^a_{\bar{z}}\) from a differential form, if it is present. (Of course, by exchanging the role of \(\eta^-\) and \(\eta^+\), we could instead regard the spinors on \(\mathcal{M}\) as \((q, 0)\)-forms, with values in a line bundle.)

The relation of spinors on \(\mathcal{M}\) to \((0, q)\)-forms has the following consequence. The Dirac operator \(\mathcal{D}\) acting on sections of a holomorphic line bundle \(\mathcal{W}\) over a complex manifold \(\mathcal{M}\) has a decomposition

\[
\mathcal{D} = \overline{\partial} + \overline{\partial}^\dagger,
\]

where \(\overline{\partial}\) is the \(\overline{\partial}\) operator acting on \((0, q)\)-forms with values in \(\mathcal{W} \otimes K^{1/2}\), and \(\overline{\partial}^\dagger\) is its adjoint. These operators obey

\[
\{\overline{\partial}, \overline{\partial}^\dagger\} = H, \quad \overline{\partial}^2 = (\overline{\partial}^\dagger)^2 = 0,
\]

where \(H = \mathcal{D}^2\). If we identify \(\overline{\partial}\) and \(\overline{\partial}^\dagger\) with the two supercharges and \(H\) with the Hamiltonian, then (3.3) coincides with the supersymmetry algebra of a \(2 + 1\)-dimensional system with \(N = 1\) supersymmetry, in a sector in which the momentum vanishes. In the present discussion the momentum vanishes because the classical zero energy states that we are quantizing all have zero momentum. This strongly suggests that, in the approximation of quantizing the space of classical zero energy states, the supercharges reduce to (a multiple of) \(\overline{\partial}\) and \(\overline{\partial}^\dagger\).

It is not difficult to show this and at the same time identify the line bundle \(\mathcal{W}\). In canonical quantization of the Yang-Mills theory with Chern-Simons coupling, the momentum conjugate to \(A_i^a\) is

\[
\Pi_i^a = \frac{F_{0i}^a}{e^2} - \frac{k}{4\pi} \epsilon_{ij} A_j^a.
\]

Writing formally \(\Pi_i^a = -i \delta / \delta A_i^a\), we have

\[
\frac{F_{0i}^a(x)}{e^2} = -i \frac{D}{DA_i^a(x)},
\]

where \(D / DA_i^a(x)\) is a “covariant derivative in field space,”

\[
\frac{D}{DA_i^a(x)} = \frac{\delta}{\delta A_i^a(x)} + i \frac{k}{4\pi} \epsilon_{ij} A_j^a.
\]

The object \(D / DA_i^a\) is a connection on a line bundle \(\mathcal{W}\) over the space of connections. The connection form of \(\mathcal{W}\) is \((k/4\pi) \epsilon_{ij} A^j\). Requiring that the curvature form of \(\mathcal{W}\) should...
have periods that are integer multiples of $2\pi$ gives a condition that is equivalent to the quantization \[2\] of the Chern-Simons coupling (see \[10\]-\[12\] for more on such matters), so if we set $k = 1$ the line bundle that we get is the most basic line bundle $L$ over the phase space, in the sense that it has positive curvature and all other line bundles over the phase space are of the form $L^n$ for some integer $n$. The factor of $k$ in \(3.12\) means that the line bundle $W$ over the phase space $M$ is $W = L^k$. So the states are spinors with values in $L^k$, or equivalently $(0,q)$-forms with values in $L^k \otimes K^{1/2}$.

As for the supercharges, they are

$$Q_\alpha = \frac{1}{e^2} \int_{T^2} \Gamma^{IJ}_{\alpha\beta} \text{Tr} F_{IJ} \lambda^\beta. \quad (3.13)$$

To write an effective formula in the space of zero energy states, we set the spatial part of $F$ to zero. The supercharges $Q_\pm$ of definite two-dimensional chirality then become

$$Q_- = \frac{1}{e^2} \int_{T^2} \text{Tr} F_{0\bar{z}} \lambda_+ = \int_{T^2} \text{Tr} \lambda_+ \frac{D}{DA_{\bar{z}}}$$
$$Q_+ = \frac{1}{e^2} \int_{T^2} \text{Tr} F_{0\bar{z}} \lambda_- = \int_{T^2} \text{Tr} \lambda_- \frac{D}{DA_{\bar{z}}} \quad (3.14)$$

Evaluating this expression in the space of zero modes, the $\lambda$’s become gamma matrices (or raising and lowering operators) on spinors over the moduli space $M$; and $D/DA_{\bar{z}}$ and $D/DA_z$ are holomorphic and antiholomorphic covariant derivatives on $M$. Altogether, the supercharges $Q_-$ and $Q_+$ reduce to $e$ times the $\overline{\partial}$ and $\overline{\partial}^\dagger$ operators on spinors valued in $W = L^k$.

In this discussion, we have not incorporated explicitly the fermion bare mass $e^2 k$. But that bare mass is related by supersymmetry to the Chern-Simons coupling, which we have incorporated, so the supersymmetric effective Hamiltonian $H = \{\overline{\partial}, \overline{\partial}^\dagger\}$ that we have written inevitably includes the effects of the fermion bare mass. This arises as follows: because there is a “magnetic field” on $M$ proportional to $k$ (with connection form on the right hand side of \(3.12\)), the operator $H = e^2 \{\overline{\partial}, \overline{\partial}^\dagger\}$, if written out more explicitly, contains a term $e^2 k \eta^a \eta^b \delta_{ab}$. This coupling is the bare mass term, written in the space of $\eta$’s.

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9 The factor of $e$ arises because the $\lambda$ kinetic energy in the original Lagrangian was $\overline{\lambda} D \lambda / e^2$, so $\lambda/e$ is a canonically normalized fermion. As the supercharges are properly normalized as $e \overline{\partial}$ and $e \overline{\partial}^\dagger$, the Hamiltonian is $H = e^2 \{\overline{\partial}, \overline{\partial}^\dagger\}$.
3.1. Calculations

Now we will perform calculations of $\text{Tr} \left( (-1)^F \right)$. First we consider the case that $G = SU(n)$.

For $SU(n)$, the moduli space $\mathcal{M}$ is a copy of $\mathbb{C}P^{n-1}$. The basic line bundle over $\mathcal{M}$ is $\mathcal{L} = \mathcal{O}(1)$, the bundle whose sections are functions of degree one in the homogeneous coordinates of $\mathbb{C}P^{n-1}$. The canonical bundle of $\mathbb{C}P^{n-1}$ is $K = \mathcal{L}^{-n}$.

The quantum Hilbert space found in the above Born-Oppenheimer approximation is the space of spinors with values in $\mathcal{W} = \mathcal{L}^k$, or equivalently $(0,q)$-forms with values in $\mathcal{W} \otimes K^{1/2} = \mathcal{L}^{k-n/2}$. Since only integral powers of $\mathcal{L}$ are well-defined as line bundles over $\mathcal{M}$, we get the restriction

$$k \cong \frac{n}{2} \mod \mathbb{Z}. \quad (3.15)$$

This is the restriction found in [3] and reviewed in section 2; we have now given a Hamiltonian explanation of it.

Since the supersymmetry generators are the $\bar{\partial}$ and $\bar{\partial}^\dagger$ operators, the space of supersymmetric states, in this approximation, is

$$\bigoplus_{i=0}^n H^i(\mathbb{C}P^{n-1}, \mathcal{L}^{k-n/2}). \quad (3.16)$$

The supersymmetric index is

$$I(k) = \sum_{i=0}^n (-1)^i \dim H^i(\mathbb{C}P^{n-1}, \mathcal{L}^{k-n/2}). \quad (3.17)$$

This can be computed by a Riemann-Roch formula, which implies in particular that $I(k)$ is a polynomial in $k$ of order $n$.

However, for a more precise description – and in particular to see supersymmetry breaking in the “gap,” $|k| < n/2$ – we wish to compute the individual cohomology groups, and not just the index. For this computation, see for example [20]. For $-n < t < 0$, one has $H^i(\mathbb{C}P^{n-1}, \mathcal{T}^t) = 0$ for all $i$. Hence, for

$$-\frac{n}{2} < k < \frac{n}{2}, \quad (3.18)$$

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10 For example, for $n = 2$, the maximal torus $U$ is a circle and the Weyl group is $W = \mathbb{Z}_2$, so $\mathcal{M} = (U \times U)/W = T^2/\mathbb{Z}_2$, which is an orbifold version of $S^2 = \mathbb{C}P^1$. For general $n$, the standard proof that $\mathcal{M} = \mathbb{C}P^{n-1}$ can be found, for example, in section 2.1 of [19].
there are no zero energy states at all in the present approximation. Thus, for this range of \( k \), supersymmetry is spontaneously broken if the theory is formulated on a two-torus with sufficiently weak coupling that our analysis is a good approximation. (Because the ground state energy in finite volume is a real analytic function of the volume, it also follows that supersymmetry is unbroken for any generic volume on \( \mathbb{T}^2 \).) This hints but certainly does not prove that also for infinite volume, supersymmetry is spontaneously broken if \(|k| < n/2\).

For \( t \geq 0 \), \( H^i(\mathbb{CP}^{n-1}, \mathcal{L}^t) = 0 \) for \( i > 0 \), and \( H^0(\mathbb{CP}^{n-1}, \mathcal{L}^t) \) is the space of homogeneous polynomials of degree \( t \) in the \( n \) homogeneous coordinates of \( \mathbb{CP}^{n-1} \). The dimension of this space is \( n(n+1) \ldots (n+t-1)/t! = (n+t-1)!/t!(n-1)! \). Setting \( t = k - n/2 \) and interpreting this dimension as the supersymmetric index \( I(k) \), we get the formula for \( I(k) \) that was stated in the introduction:

\[
I(k) = \frac{1}{(n-1)!} \prod_{j=-n/2+1}^{n/2-1} (k-j).
\tag{3.19}
\]

Finally, Serre duality determines what happens for \( t \leq -n \) in terms of the results for \( t \geq 0 \). In particular, for \( t \leq -n \), the cohomology \( H^i(\mathbb{CP}^{n-1}, \mathcal{L}^t) \) vanishes except for \( i = n-1 \), and is dual to \( H^0(\mathbb{CP}^{n-1}, \mathcal{L}^{-n-t}) \). From this, we get a formula for \( I(k) \) with \( k \leq -n/2 \) which coincides with (3.19). Note that for \( k \geq n/2 \), all supersymmetric states are bosonic, and for \( k \leq -n/2 \), all supersymmetric states have statistics \((-1)^{n-1}\). Serre duality gives directly \( I(-k) = (-1)^{n-1}I(k) \).

**Generalization To Other Groups**

We will now more briefly summarize the generalization for an arbitrary simple, connected and simply-connected gauge group \( G \) of rank \( r \).

First of all, the moduli space \( \mathcal{M} \) is a weighted projective space \( \text{WCP}^{s_0,s_1,\ldots,s_r}_{s_i} \), where the weights \( s_i \) are 1 and the coefficients of the highest coroot of \( G \). This is a theorem of Looijenga; for an alternative proof see \[19\]. In particular, the weights obey \( \sum_{i=0}^{r} s_i = h \). The basic line bundle over \( \mathcal{M} \) is \( \mathcal{L} = O(1) \), characterized by the fact that sections of \( \mathcal{L}^t \) for any \( t \) are functions of weighted degree \( t \) in the homogeneous coordinates of \( \mathcal{M} \). The canonical bundle of \( \mathcal{M} \) is \( K = \mathcal{L}^{-h} \). So, in the Born-Oppenheimer approximation, the low-lying states are spinors valued in \( \mathcal{L}^{k-h/2} \). Integrality of the exponent gives again the result that \( k - h/2 \) must be integral.
The space of supersymmetric states is, again, 

\[ \bigoplus_{i=0}^{r} H^i(\mathcal{M}, \mathcal{L}^{k-h/2}). \]  

(3.20)

A weighted projective space has certain properties in common with an ordinary projective space. One of these is that \( H^i(\mathcal{M}, \mathcal{L}^t) = 0 \) for all \( i \) if \( -h < t < 0 \). This implies (in finite volume) supersymmetry breaking in the “gap,” \( |k| < h/2 \). For \( t \geq 0 \), the cohomology groups vanish except in dimension 0, and \( H^0(\mathcal{M}, \mathcal{L}^t) \) is the space of polynomials homogeneous and of weighted degree \( t \) in the homogeneous coordinates of \( \mathcal{M} \). In particular, \( I(k) > 0 \) for \( k \geq h/2 \), and supersymmetry is unbroken. Serre duality asserts that \( H^i(\mathcal{M}, \mathcal{L}^t) \) is dual to \( H^{r-i}(\mathcal{M}, \mathcal{L}^{-h-t}) \), and relates the region \( k \leq -h/2 \) to \( k \geq h/2 \). In particular, for \( k \leq h/2 \), the only nonzero cohomology group is in dimension \( r \), the states have statistics \((-1)^r\), and the index is determined by \( I(-k) = (-1)^r I(k) \) and so is in particular nonzero.

3.2. Orbifolds And Anyons

Here, we make a few miscellaneous comments on the problem.

The moduli space \( \mathcal{M} \) is an orbifold \( \mathcal{M} = (\mathbf{U} \times \mathbf{U})/W \), a quotient of a flat manifold by a finite group. However, we have not used this fact in computing the index. The reason is that although the moduli space \( \mathcal{M} \) is an orbifold, the quantum mechanics on \( \mathcal{M} \) is not orbifold quantum mechanics, that is, it is not obtained from supersymmetric free particle motion on \( \mathbf{U} \times \mathbf{U} \) by imposing \( W \)-invariance. Rather, the quantum mechanics on \( \mathcal{M} \) depends on the line bundle \( \mathcal{L}^k \).

One can ask whether there are values of \( k \) at which the quantum mechanics on \( \mathcal{M} \) reduces to orbifold quantum mechanics. We will approach this as follows. We begin with a system consisting of \((0,q)\)-forms on \( \mathbf{U} \times \mathbf{U} \) with the Hamiltonian being simply the Laplacian (relative to the flat metric on \( \mathbf{U} \times \mathbf{U} \)). In orbifold quantum mechanics, we want \( W \)-invariant states of zero energy.

A zero energy state must have a wave function that is invariant under translations on \( \mathbf{U} \times \mathbf{U} \). This means that the bosonic part of the wave-function, being a constant function, is \( W \)-invariant. Hence, \( W \)-invariance must be imposed on the fermionic part of the wave function, which as we recall takes values in a fermionic Fock space with basis obtained by acting with creation operators on a vacuum \( |\Omega_-\rangle \) or \( |\Omega_+\rangle \).

The states \( |\Omega_-\rangle \) and \( |\Omega_+\rangle \) transform as one-dimensional representations of \( W \), since the condition that a state be annihilated by all \( \eta^a_+ \)'s (or by all \( \eta^a_- \)'s) is Weyl-invariant.
The group $W$ has two one-dimensional representations: the trivial representation; and a representation $R$ in which each elementary reflection acts by $-1$. Since

$$|\Omega_\pm\rangle = \prod_{a=1}^{r} \eta_a^{q_a}|\Omega_-\rangle,$$  \hspace{1cm} (3.21)

and the product $\prod_{a=1}^{r} \eta_a^{q_a}$ is odd under every elementary reflection, the two states $|\Omega_-\rangle$ and $|\Omega_+\rangle$ transform oppositely: one transforms in the trivial representation of $W$, and the other transforms as $R$.

Suppose that we take the $W$ action such that $|\Omega_-\rangle$ transforms trivially. Then $|\Omega_-\rangle$ itself (times a constant function on $U \times U$) is a $W$-invariant state of zero energy, and is in fact the only one. To prove the uniqueness, one can use the fact that the $W$ action on the fermion Fock space is the same as that on the $(0, q)$-forms on $U \times U$. The $W$-invariant and translation-invariant states on $U \times U$ can therefore be identified with the cohomology group $H^i((U \times U)/W, O)$, where $O$ is a trivial holomorphic line bundle. This cohomology is one-dimensional for $i = 0$ and vanishes for $i > 0$, since $(U \times U)/W = \mathbb{C}P^{n-1}$.

Now let us compare this orbifold quantum mechanics to the Born-Oppenheimer quantization of the gauge theory. In the latter, at general level $k$, we identified the supersymmetric states with elements of $H^i(M, \mathcal{L}^{k-\hbar/2})$, where $M = (U \times U)/W$. This agrees with the orbifold answer $H^i(M, O)$ if and only if $k = \hbar/2$, so that must be the correct value of $k$ corresponding to orbifold quantum mechanics with $|\Omega_-\rangle$ assumed to be Weyl-invariant.

The other possibility, that $|\Omega_+\rangle$ is Weyl-invariant, is obtained by reversal of orientation, which is equivalent to $k \to -k$; so this other orbifold quantum mechanics should correspond to $k = -\hbar/2$.

These arguments strongly suggest that the low energy quantum mechanics is just orbifold quantum mechanics for these special values of $k$. As we discuss below and in section 4, these are apparently the values for which the theory is confining.

**Anyons**

It is perhaps surprising that the “simple” orbifold cases correspond not to the obvious case $k = 0$ but to $k = \pm \hbar/2$. Let us see instead consider what happens for $k = 0$. For simplicity, we take $G = SU(2)$, so that $U$ is a circle and $W = \mathbb{Z}_2$.

Even though $M = (U \times U)/\mathbb{Z}_2$ is an orbifold, the quantum mechanics is, as we have seen, not orbifold quantum mechanics for $k = 0$. To measure the failure, let us see what happens near the orbifold singularities of $(U \times U)/\mathbb{Z}_2$. For example, we can consider the
singularity associated with the trivial flat connection, where the $c_i$ introduced in (3.2) all vanish. (For $SU(2)$, the index $a$ takes only one value, so we write the $c_i^a$ just as $c_i$.) The Weyl group acts as $c_i \to -c_i$, and there is a singularity at $c_i = 0$. How can we best understand this singularity? Near $c_i = 0$, it is more illuminating to consider not compactification from $2 + 1$ dimensions to $0 + 1$ – as we have done so far – but dimensional reduction to $0 + 1$ dimensions, in which one starts with $2 + 1$-dimensional super Yang-Mills thoery and by fiat one requires the fields to be invariant under spatial translations. Dimensional reduction and compactification differ in that the compactified theory also has modes of non-zero momentum along $T^2$, and has periodic identifications of the $c_i$ under $c_i \to c_i + 2\pi$. These are irrelevant for studying the singularity near $c_i = 0$.

In the dimensionally reduced theory, there is a $U(1)$ symmetry under rotations of the $c_1 - c_2$ plane. (The compactified theory only has a discrete subgroup of this symmetry; that is one of the main reasons to consider the dimensionally reduced theory in the present discussion.) We will call the generator of this $U(1)$ the angular momentum. The fermion Fock space has, for $SU(2)$, only the two states $|\Omega_-\rangle$ and $|\Omega_+\rangle$ (as there is only one creation operator and one annihilation operator). Since a fermion creation operator, of spin $1/2$, maps one to the other, their angular momenta are $j$ and $j - 1/2$ for some $j$. But the dimensionally reduced theory has also a parity symmetry exchanging these two states and reversing the sign of the angular momentum. Hence $j = 1/4$, and the two states have angular momenta $1/4, -1/4$. This contrasts with orbifold quantum mechanics on $R^2/Z_2$, where the spins are half-integral. Thus a precise measure of the difference of the $k = 0$ system from an orbifold is that the $k = 0$ system generates “anyons,” states whose angular momentum does not take values in $Z/2$.

Such anyons may arise in the compactification of Type IIB superstring theory to three dimensions on a seven-manifold $X$ of $G_2$ holonomy. Consider a system of $m$ parallel sevenbranes wrapped on $X$. This system is governed by $2 + 1$-dimensional $U(m)$ super Yang-Mills theory with two supercharges, dimensionally reduced to $0 + 1$ dimensions. This suggests that the wrapped sevenbranes may be anyons of spins $\pm 1/4$, but to be certain, one would need to look closely at the definition of angular momentum for these string theory excitations.
3.3. Discrete Electric And Magnetic Flux

We will briefly discuss the generalization of the computation to incorporate discrete electric and magnetic flux. (We will consider only the simplest versions of electric or magnetic flux. It is of course possible to mix the two constructions.)

Including magnetic flux simply means taking the gauge group $G$ not to be simply connected and working on a non-trivial gauge bundle $E$ over $T^2$. The moduli space $\mathcal{M}$ of zero energy gauge configurations is now the moduli space of flat connections on $E$. In certain cases, mentioned in section 2.2 above, $\mathcal{M}$ is a single point, and the quantization is then completely straightforward. In general, $\mathcal{M}$ is always a weighted projective space (which can be constructed using the technique in [19]), and the quantum ground states are always, as above, $H^i(\mathcal{M}, L^{k-h/2})$. In particular, the index is always nonvanishing for $|k| \geq h/2$.

Including electric flux means that one goes back to the case that $G$ is simply connected. One considers a gauge transformation $U$ that, in going around, say, the first circle in $T^2 = S^1 \times S^1$, transforms as

$$U \rightarrow U\omega,$$

where $\omega$ is an element of the center of $G$. This transformation is a symmetry $T_\omega$ of the theory. If, for example, $\omega$ is of order $s$, then $U^s$ generates a gauge transformation that is homotopic to the identity, and $T_\omega^s = 1$ on all physical states. The eigenvalues of $T_\omega$ are thus of the form $\exp(2\pi ir/s)$ where $r$ is an integer called the discrete electric flux.

A simple way to determine the action of $T_\omega$ on the space $\mathcal{H}$ of supersymmetric ground states of our supersymmetric gauge theory is to use the relation of pure Chern-Simons theory at level $k'$ to the WZW model, also at that level. The Hilbert space of the pure Chern-Simons theory in quantization on $T^2$ has a basis that can be described as follows. Regard $T^2 = S^1 \times S^1$ as the boundary of $S^1 \times D$, where $D$ is a two-dimensional disc. Consider the Chern-Simons path integral on $S^1 \times S^1$, with an insertion of a Wilson line operator

$$W_R(C) = \text{Tr}_R P \exp \int_C A.$$  \hspace{1cm} (3.23)

Here $R$ is a representation of $G$, and $C \subset S^1 \times D$ is a circle of the form $C = S^1 \times P$, with $P$ being a point in the disc $D$. For any given $R$, the path integral on $S^1 \times D$ with insertion of $W_R(C)$ gives a state $\Psi_R$ in the Chern-Simons Hilbert space on $S^1 \times S^1$ at level $k'$, and this space, as we have argued, is the same as $\mathcal{H}$. As $R$ ranges over the highest weights of
integrable representations of the \(\hat{G}\) affine algebra at level \(k'\), the \(\Psi_R\) furnish a basis of \(\mathcal{H}\). (The \(\Psi_R\)'s for other representations are zero or a multiple of one of the \(\Psi_R\)'s for an integrable representation.)

Going back to the electric flux operator \(T_\omega\), its action on the state \(\Psi_R\) is now clear. It maps \(W_R(C)\) to \(\omega(R)W_R(C)\), where the central element \(\omega\) of \(G\) acts in the irreducible representation \(R\) as multiplication by \(\omega(R)\). So it likewise maps \(\Psi_R\) to \(\omega(R)\Psi_R\).

If there is a zero energy state carrying electric flux for any value of the spatial volume, this means that the theory is not confining. Confinement of electric flux can therefore occur only if the center of \(G\) acts trivially on all \(\Psi_R\), or equivalently on all integrable representations of the WZW model at level \(k'\). This, however, is so only at \(k' = 0\) (where only the trivial representation is integrable). For example, for \(SU(2)\), at level \(k'\), the integrable representations have highest weights \(0, 1/2, 1, \ldots, k'/2\), so whenever \(k' > 0\), there is an integrable representation of half-integer spin, on which the center acts nontrivially. Among the theories with unbroken supersymmetry, only the theory with \(k' = 0\) – and thus \(k = \pm h/2\) – might be interpreted as confining.

The theories with \(|k| < h/2\) may very well also be confining, but as they conjecturally have spontaneously broken supersymmetry, we cannot probe their dynamics by looking for supersymmetric states.

4. Classification Of Massive Phases

This concluding section will be devoted to some remarks about the classification of massive phases of gauge theories.\(^\text{11}\)

Consider a gauge theory with a mass gap. Let us look at the behavior of Wilson loop operators \(W_R(C) = \text{Tr}_R P \exp \int_C A\), with \(R\) some representation of \(G\) and \(C\) a loop in spacetime. Let \(L(C)\) be the circumference of \(C\), and \(A(C)\) the minimal area of a surface that it spans. The renormalization of \(W_R(C)\) that we allow is local along the loop:

\[
W_R(C) \rightarrow e^{\alpha_R L(C)} W_R(C). \tag{4.1}
\]

Here \(\alpha_R\) is a renormalization parameter. We want to study the behavior of \(\langle W_R(C) \rangle\) as the loop \(C\) is scaled up in size. In some theories (and for some representations), \(\langle W_R(C) \rangle \sim \)\

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\(^{11}\) These remarks were suggested in part by discussions ca. 1990 with M. Marcu – see \[21\] – and M. F. Atiyah.
\[ \exp(-\beta_R A(C)), \] with \( \beta_R \) a positive constant, modulo subleading terms which vary with the circumference rather than the area. Such a statement is invariant under a renormalization of the form (4.1). If \( \langle W_R(C) \rangle \) behaves this way, then it vanishes in the limit that \( C \) is scaled up no matter what renormalization is used. On the other hand, it may happen that the area coefficient \( \beta_R \) vanishes. In this case, \( \langle W_R(C) \rangle \) has with generic renormalization an exponential dependence on the circumference. We can pick the renormalization constant in (4.1) to cancel this term. What happens then?

In a theory with a mass gap, one would expect that for large loops,

\[ \ln\langle W_R(C) \rangle = \beta_R A(C) + \gamma_R L(C) + \ldots, \] (4.2)

where the \( \ldots \) terms are constant in the limit of large loops. (In the absence of a mass gap, there can be much more complicated behavior; for example, Feynman diagrams in a theory with massless fields can give terms \( L(C) \ln L(C)^m \) for all integers \( m \).) If \( \beta_R \) vanishes and we choose the renormalization to cancel \( \gamma_R \), then \( \ln\langle W_R(C) \rangle \) should have a limit as \( C \) becomes large. We conclude then that with our renormalization

\[ N_R(C) = \lim_{C \to \infty} \langle W_R(C) \rangle \] (4.3)

should exist in a massive gauge theory. The confining case \( -\beta_R > 0 \) – is the case that \( N_R(C) = 0 \).

Above three dimensions, this construction exhibits one constant for every representation. In three dimensions, the construction is much richer, because the loop \( C \) in spacetime may be knotted. Thus, in the three-dimensional case, in a massive gauge theory, we get an invariant for every representation and every knot class. In the examples we have been examining in the present paper, there is a mass gap for sufficiently large \( |k| \) (conjecturally that this is so precisely if \( |k| \geq h/2 \), and the “low energy” theory is a Chern-Simons theory at level \( k' = k - h/2 \cdot \text{sgn} \, k \). The large loop limits of the expectation values of the \( \langle W_R(C) \rangle \) can \[ \] be expressed in terms of the celebrated Jones polynomial of knots and its generalizations.

At least in the three-dimensional examples, it seems fairly clear that the \( N_R(C) \) must depend only on the universality class of the theory. The effective theory at very long distances is characterized just by the integer \( k' \) – which controls the knot invariants – and this integer cannot change under continuous variation of parameters.
In four-dimensional massive gauge theories, the meaning and significance of the $N_R(C)$ are less apparent. It seems probable, however, that they are invariants of the universality class of a theory.

Analysis Of Phases In Three Dimensions

The dependence on $k'$ makes it clear that the usual Higgs/confinement dichotomy is not the whole story for classification of massive phases of gauge theories in three dimensions. We have, on the contrary, infinitely many inequivalent universality classes, parameterized by $k'$; none of the theories with $k' > 0$ is confining as they all have some nonzero $N_R(C)$ with $R$ a representation in which the center of the gauge group acts nontrivially. This follows from the formulas in [7] for expectation values of Wilson loops in pure Chern-Simons theory. Alternatively, the theories with $k' > 0$ are not confining, since we showed in section 3.3 that in these theories, electric flux winding on a torus has no cost in energy.

The case that $k' = 0$, when the low energy theory is a “pure gauge theory” without Chern-Simons interaction, might be confining. Some evidence for this appeared in section 3.3, where we saw that in this case, there is no zero energy state on $T^2$ with electric flux. It follows from this fact that all $N_R(C)$, with the center of $G$ acting nontrivially on $R$, vanish if $k' = 0$. For one could factorize the evaluation of $N_R(C)$ by “cutting” the three-dimensional spacetime on a two-torus $S$ that consists of all points a distance $\epsilon$ from $C$, for some small $\epsilon$. Upon scaling $C \to \infty$, one can also take $\epsilon \to \infty$, and the path integral on the solid torus bounded by $S$ and containing $C$ gives a state carrying electric flux in the Hilbert space obtained by quantizing the pure Chern-Simons theory on $C$. As the pure Chern-Simons theory has no physical states on $S$ that have electric flux, the path integral for $\langle W_R(C) \rangle$ will vanish for $C \to \infty$.

The above reasoning used a possibly risky analytic continuation of the Chern-Simons results (which are usually considered for $k' > 0$) to $k' = 0$. I will now describe somewhat more explicitly how this analytic continuation works, taking $G = SU(2)$ as an illustration. In $SU(2)$ Chern-Simons theory at level $k'$, the loop expectation value $N_R(C)$, for a nontrivial representation $R$ with highest weight of spin $j$, vanishes if $j$ is congruent to $-1/2$ mod $(k' + 2)/2$, but not otherwise for a generic $C$. For $k' = 0$, this means that $N_R(C)$

\[ S_{jj'} = \frac{\sqrt{2}}{(k + 2)} \sin \left( \pi (2j + 1)(2j' + 1)/(k' + 2) \right). \]

This shows the vanishing if $j$ or $j'$ is congruent to $-1/2$ mod $(k' + 2)/2$.

\[ ^{12} \text{All Chern-Simons observables can be expressed in terms of quantities in the WZW model such as the matrix $S$ that generates the modular transformation $\tau \to -1/\tau$ on the characters. In a basis of representations of highest weight $j$, the matrix elements of $S$ for $SU(2)$ at level $k'$ are } S_{jj'} = \frac{\sqrt{2}}{(k + 2)} \sin \left( \pi (2j + 1)(2j' + 1)/(k' + 2) \right). \text{ This shows the vanishing if } j \text{ or } j' \text{ is congruent to } -1/2 \text{ mod } (k' + 2)/2. \]
vanishes if $j$ is a half-integer, and not otherwise. This is the usual statement of confinement: there is an area law precisely if the representation transforms nontrivially under the center of the gauge group.

Thus it seems likely that precisely at $k = \pm h/2$, the theories studied in the present paper have a mass gap, unbroken supersymmetry and confinement. For $|k| > h/2$, they are in inequivalent Higgs-like phases with a mass gap and unbroken supersymmetry, and for $|k| < h/2$, they conjecturally have a massless Goldstone fermion (and perhaps confinement).

Going back to the three-dimensional examples, let us examine the other standard criterion for confinement, which is whether external magnetic flux is screened. In three dimensions, one considers a local ’t Hooft operator $\mathcal{O}(P; w)$ defined by removing a point $P$ from spacetime and inserting a nontrivial magnetic flux $w$ on a small sphere surrounding $P$. (In four dimensions, one has instead an ’t Hooft loop operator defined by removing a loop $C$ from spacetime and inserting magnetic flux on a sphere that links $C$.) In the three-dimensional case, a restriction on $k$ is needed in introducing the operators $\mathcal{O}(P; w)$. For instance, as we saw in section 2, if $G = SU(n)$ and $w$ is prime to $n$, then the restriction is that $k$ should be congruent to $n/2$ modulo $n$.

’t Hooft’s criterion for a Higgs phase of a massive gauge theory in four dimensions is that the ’t Hooft loop should show area law in four dimensions; in three dimensions the criterion is that the expectation value $\langle \mathcal{O}(P; w) \rangle$ should vanish. In our three-dimensional examples, one might expect this criterion to be obeyed for $k > h/2$, as these theories are not confining. This is so. On $S^2$, the modulo space of flat connections on a bundle with nonzero magnetic flux is empty, and hence the Chern-Simons theory if quantized with such a bundle on $S^2 \times \mathbb{R}$ (with $\mathbb{R}$ understood as the “time” direction) has no physical states. Because of the topological invariance of the Chern-Simons theory at long distances, the expectation value $\langle \mathcal{O}(P; w) \rangle$ can be computed in radial quantization – where the radius measures the distance from $P$. In other words, we consider the operator $\mathcal{O}(P; w)$ to prepare an initial state at $r = 0$ ($r$ being the distance from $P$), and propagate outward to $r = \infty$. This propagation should project onto zero energy states. But there are no zero energy states to project onto, so the expectation value vanishes. Or more prosaically, the expectation value $\langle \mathcal{O}(P; w) \rangle$ vanishes because – with a gauge bundle that is nontrivial when restricted to any arbitrarily large sphere surrounding $P$ – the classical equation of motion $F = 0$ of the long distance effective Chern-Simons theory cannot be obeyed even near spatial infinity.
Now let us consider a different but also standard criterion for “screening of magnetic flux.” In this alternative formulation, we quantize the theory on $T^2 \times \mathbb{R}$ and interpret magnetic screening to mean that in the limit of large volume of $T^2$, the ground state energy is independent of the magnetic flux on $T^2$. With this criterion, magnetic screening does occur in the three-dimensional $N = 1$ gauge theories for all allowed $k \geq h/2$ since, as we have seen in sections 2 and 3, $\text{Tr} (-1)^F$ is nonzero and hence the ground state energy vanishes whether there is magnetic flux or not.

Thus, the two standard criteria for magnetic confinement give different answers in these theories. The key difference between the two criteria is that there are flat connections on a bundle over $T^2$ with magnetic flux, but not on such a bundle over $S^2$. As far as I know, the distinction between the two notions of magnetic screening has not been important in massive phases of gauge theories that have been studied previously.

**Generalization**

Part of the above story is special to three dimensions, but part is not.

One basic question is how to describe the long distance limit of a theory. It is conventionally claimed that the long distance limit of a massive theory is “trivial,” but the very idea of ’t Hooft and Wilson loops as criteria for confinement shows that there is more to say about the long distance limit of a massive theory than just this.

The lesson from the above discussion is that a massive theory may give at long distances a nontrivial topological field theory, which governs the possible vacuum states in different conditions, even though there are no “physical excitations” at very long wavelengths. Topological field theory is particularly interesting in three-dimensions because of the existence of the Chern-Simons theories. In four dimensions, the known unitary examples are less interesting. The most obvious example of a topological field theory in four dimensions (or indeed in any dimension) is a gauge theory with a finite gauge group $\Gamma$. If such a theory is quantized on a three-manifold $X$, the number of physical states is the number of conjugacy classes of representations of the fundamental group of $X$ into $\Gamma$. (These theories have been discussed in [22], including also a generalization involving a group cohomology class. The generalization leads to a somewhat more elaborate formula for the number of physical states.) In particular, these theories do depend on $\Gamma$.

Consider a weakly coupled four-dimensional theory with mass gap in which, perturbatively, a connected simple gauge group $G$ is spontaneously broken to a finite subgroup

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13 Donaldson theory is based on a non-unitary topological field theory in four dimensions.
Γ. From the point of view of the usual criteria involving 't Hooft and Wilson loops, these theories are all Higgs theories. For example, by virtue of the perturbative Higgs mechanism, the Wilson loops can be computed reliably in perturbation theory and show no area law. 't Hooft loops do show an area law, since a bundle on $S^2$ with nontrivial magnetic flux cannot have a $\Gamma$-valued connection, so a path integral with such a bundle (on an $S^2$ that links an 't Hooft loop) cannot receive a contribution in the low energy theory.

Nevertheless, order parameters distinguishing such theories have been constructed [23]. We can reformulate this discussion to some extent and say that a basic order parameter is the topological field theory that prevails at long distances. It is simply a gauge theory of the finite group $\Gamma$.

Here is a simple yet interesting example. Take $G = SO(3)$ and let $\Gamma$ be the subgroup – isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ – consisting of diagonal matrices with entries $\pm 1$ (and determinant 1). As we have already explained, magnetic flux wrapped on $S^2$ is unscreened, and the 't Hooft loops show area law. However, in such a theory, magnetic flux on $T^2$ is screened. This is so simply because a bundle on $T^2$ with nonzero magnetic flux admits a flat connection with holonomies in $\Gamma$. The holonomies around the two directions in $T^2$ can be the matrices

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad V = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (4.4)$$

A flat bundle with these holonomies has nonzero magnetic flux since the matrices $U$ and $V$, if lifted to $SU(2)$, do not commute. This gives an elementary four-dimensional example in which standard criteria for magnetic screening give different results, somewhat as we found in three dimensions.

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