INDECOMPOSABILITY OF DERIVED CATEGORIES IN FAMILIES

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Abstract. Using the moduli space of semiorthogonal decompositions in a smooth projective family introduced by the second, the third and the fourth author, we discuss indecomposability results for derived categories in families. In particular, we prove that given a smooth projective family of varieties, if the derived category of the general fibre does not admit a semiorthogonal decomposition, the same happens for every fibre of the family. As a consequence, we deduce that in a smooth family of complex projective varieties, if there exists a fibre such that the base locus of its canonical linear series is either empty or finite, then any fibre of the family has indecomposable derived category.

Then we apply our results to achieve indecomposability of the derived categories of various explicit classes of varieties, as e.g. $n$-fold symmetric products (with $0 < n < \lfloor (g+3)/2 \rfloor$), Horikawa surfaces, an interesting class of double covers of the projective plane introduced by Ciliberto, and Hilbert schemes of points on these two classes of surfaces.

1. Introduction

One of the important tools in the study of derived categories of coherent sheaves on smooth projective varieties is the notion of a semiorthogonal decomposition, which allows one to decompose the derived category into smaller pieces. In this paper we study which smooth projective varieties are “atomic”, in the sense that their derived categories cannot be decomposed.

Let $X$ be a smooth projective variety and let $D^b(X)$ be the bounded derived category of coherent sheaves on $X$. We say that $D^b(X)$ admits a semiorthogonal decomposition if it is generated by full non-zero triangulated subcategories $A,B$ such that $\text{Hom}_{D^b(X)}(B,A) = 0$ for any $A \in A$ and $B \in B$, otherwise we say that $D^b(X)$ is indecomposable. For an overview of the state-of-the-art of semiorthogonal decompositions (from a few years ago), we refer the reader to Kuznetsov’s ICM address [28]. To understand the importance of indecomposability, the relevant points are that

1. semiorthogonal decompositions often reflect properties of the geometry of the variety, e.g. by relating the variety at hand to other, easier varieties, or suggesting it is rational, see e.g. [2, 27];
2. operations in the minimal model program are conjectured to give semiorthogonal decompositions, where the guiding principle is the DK-hypothesis,
which predicts that $K$-equivalent varieties have equivalent derived categories, and that a $K$-inequality between varieties exhibits one derived category as an admissible subcategory of the other, see e.g. [22, 23].

From the general picture in point (2) the derived category being indecomposable is conjectured to imply minimality in the sense of the minimal model program. But there exist several minimal varieties whose derived categories have a non-trivial semiorthogonal decomposition, such as classical Enriques surfaces. Hence it is an important question to understand when derived categories of minimal varieties are indecomposable. In this paper we provide new tools and examples to study this question.

The first two instances where indecomposability was known are:

- varieties with trivial canonical bundle, by using the arguments in [10];
- curves of genus $\geq 2$, by [32, Theorem 1.1].

In [24] these results were generalised, and it was shown how the geometry of the base locus $\text{Bs}|\omega_X|$ of the canonical linear series — we call it the canonical base locus — governs indecomposability, because of its link to the Serre functor for $\mathcal{D}^b(X)$. In particular, it is shown that for a semiorthogonal decomposition $\mathcal{D}^b(X) = \langle A, B \rangle$, either all objects of $A$ or all objects of $B$ are necessarily supported on $\text{Bs}|\omega_X|$. One of the main consequences is the following.

**Theorem 1.1** (Kawatani–Okawa). **Let $X$ be a smooth projective variety. If for all connected components $Z$ of $\text{Bs}|\omega_X|$ there exists a Zariski open subset $U \subseteq X$ containing $Z$ such that $\omega_X|_U \cong \mathcal{O}_U$, then $\mathcal{D}^b(X)$ is indecomposable.**

In particular, if $\dim \text{Bs}|\omega_X| \leq 0$ then $\mathcal{D}^b(X)$ is indecomposable.

In this paper we are concerned with indecomposability of derived categories of varieties deforming in a smooth family. In particular, we prove that if the general fibre of a suitable family has indecomposable derived category, the same happens for special fibres as well. More precisely, we shall prove the following result.

**Theorem A.** **Let $f: \mathcal{X} \to T$ be a smooth projective morphism, where $T$ is an excellent scheme. If the subset**

$$U := \{t \in T \mid \mathcal{D}^b(f^{-1}(t)) \text{ is indecomposable}\} \subseteq T$$

**is dense, then for all $t \in T$ we have that $\mathcal{D}^b(f^{-1}(t))$ is indecomposable.**

This result is an application of the abstract result [8, Theorem A] regarding the geometry of the moduli space of semiorthogonal decompositions.

By Theorem A indecomposability of derived categories can be extended from general to special fibres of a family $f: \mathcal{X} \to T$ as above. On the other hand, in characteristic 0, the dimension of the canonical base locus $\dim \text{Bs}|\omega_{f^{-1}(t)}|$ is an upper-semicontinuous function $T \to \mathbb{N} \cup \{-\infty\}$ (cf. Proposition 2.3). Therefore, if $\dim \text{Bs}|\omega_{f^{-1}(t_0)}| \leq 0$ for some $t_0 \in T$, then $\dim \text{Bs}|\omega_{f^{-1}(t)}| \leq 0$ for general $t \in T$, and hence $\mathcal{D}^b(f^{-1}(t))$ is indecomposable by Theorem 1.1. Combining this fact with Theorem A we then deduce the following.

**Corollary B.** **Let $f: \mathcal{X} \to T$ be a smooth projective geometrically connected morphism, where $T$ is an irreducible excellent scheme in characteristic 0. If there exists a point $t_0 \in T$ such that $\dim \text{Bs}|\omega_{f^{-1}(t_0)}| \leq 0$, then $\mathcal{D}^b(f^{-1}(t))$ is indecomposable for all $t \in T$.**
We note that the assumption on the characteristic of the ground field is essential. In Remark 2.6, we show that Corollary 3 does not hold in general in positive or mixed characteristic.

In the light of the latter result, it seems natural to raise the following question on the invariance under deformation of indecomposability of derived categories.

**Question C.** Let \( f : X \to T \) be a smooth projective morphism, where \( T \) is irreducible, and suppose that \( D^b(f^{-1}(t_0)) \) is indecomposable for some \( t_0 \in T \). Under which additional conditions we can ensure that \( D^b(f^{-1}(t)) \) is indecomposable for all \( t \in T \)?

We prove Theorem A and Corollary B in Section 2. Then, we turn to consider specific classes of complex projective varieties, and we achieve the indecomposability of their derived categories by applying Theorem A and Corollary B.

In Section 3, we consider \( n \)-fold symmetric products \( \text{Sym}^n C \) of smooth projective curves \( C \) of genus \( g \geq 3 \), and we describe the canonical base locus \( B_s|\omega_{\text{Sym}^n C}| \) depending on the gonality \( \text{gon}(C) \) of the curve \( C \) (cf. Proposition 3.1). It is worth noting that the same result has been proved independently in [9], and it is used to show that \( D^b(\text{Sym}^n C) \) is indecomposable for any \( n < \text{gon}(C) \) (see [9, Theorem 1.2]). Using Theorem A and the fact that the gonality of the general curve is \( (g + 3)/2 - 1 \), we achieve the following strengthening of this indecomposability result (cf. Propositions 3.5 and 3.7).

**Corollary D.** Let \( C \) be a smooth complex projective curve of genus \( g \geq 2 \) and let \( \text{Sym}^n C \) be its \( n \)-fold symmetric product. Then

1. the derived category \( D^b(\text{Sym}^n C) \) is indecomposable for any \( 1 \leq n \leq \left\lfloor \frac{g+3}{2} \right\rfloor - 1 \);
2. if \( g = 4 \), the derived category \( D^b(\text{Sym}^3 C) \) is indecomposable.

In particular, Corollary D extends the bound \( n \leq \left\lfloor \frac{g+3}{2} \right\rfloor - 1 \) from general curves to any curve of genus \( g \geq 3 \). We note further that, according to Remark 3.8 and [9, Conjecture 1.4]), we expect \( D^b(\text{Sym}^n C) \) to be indecomposable up to \( n = g - 1 \). However, our techniques do not apply to the range \( \left\lfloor \frac{g+3}{2} \right\rfloor \leq n \leq g - 1 \) as the canonical base locus \( B_s|\omega_{\text{Sym}^n C}| \) is always positive-dimensional.

In Section 4 we examine two classes of surfaces of general type. On one hand we consider Horikawa surfaces, i.e. those surfaces having \( c_1^2 = 2p_g - k \) with \( k = 2, 3, 4 \), which have been classified in [17, 18, 19, 20, 16]. In particular, we complete the analysis in [24, Example 4.6], and we show that their derived categories do not admit a semiorthogonal decomposition (see Theorem 4.1).

We will also study a series of examples introduced by Ciliberto [11] for discussing properties of the canonical ring of surfaces of general type. These surfaces are obtained as the minimal model of certain double planes, and they all have positive-dimensional canonical base locus. However, we prove by degeneration that they have indecomposable derived categories (see Theorem 4.3).

The descriptions of the canonical base loci of both these classes of surfaces are collected in Appendix A (for Horikawa surfaces) and in Appendix B (for Ciliberto double planes).

Finally, in Section 5 we study indecomposability of the derived category of Hilbert schemes \( S^{[n]} \) of points on a surface \( S \). In particular, after showing that the emptiness of \( B_s|\omega_S| \) implies the emptiness of \( B_s|\omega_{S^{[n]}}| \) (see Proposition 5.1),
we discuss the indecomposability of $D^b(S[n])$, for some of the surfaces considered in Section 4.

To conclude, let us point out the following question (stated as a conjecture in [9]), where a positive answer would imply the indecomposability for all the examples studied in this paper. We point out that given a smooth projective variety $X$, the nefness of $\omega_X$ is nothing but the minimality in the sense of the minimal model program, and the existence of a global section of the canonical bundle is reminiscent of, but weaker than, the conditions in [24].

**Question E.** Let $X$ be a smooth projective variety. If $\omega_X$ is nef and $H^0(X, \omega_X) \neq 0$, then is $D^b(X)$ indecomposable?

This question seems completely out of reach at the moment. To see that the converse implication does not hold, observe that bielliptic surfaces are examples of varieties where $\omega_X$ is nef, but $H^0(X, \omega_X) = 0$, and their derived categories are indecomposable by [24, Proposition 4.1].

### 2. Extending indecomposability

In this section we will discuss the proof of Theorem A and Corollary B. To do so, we will first briefly recall the notion of a semiorthogonal decomposition, and how this behaves in families.

**Definition 2.1.** Let $\mathcal{T}$ be a triangulated category. A semiorthogonal decomposition of length $n$ for $\mathcal{T}$ is a sequence $A_1, \ldots, A_n$ of full triangulated subcategories, such that

- **semiorthogonality:** for all $1 \leq j < i \leq n$ and for all $A_i \in A_i$ and $A_j \in A_j$ we have
  \[
  \text{Hom}_{\mathcal{T}}(A_i, A_j) = 0;
  \]

- **generation:** the categories $A_1, \ldots, A_n$ generate $\mathcal{T}$, i.e. the smallest triangulated subcategory of $\mathcal{T}$ containing $A_1, \ldots, A_n$ is $\mathcal{T}$.

In this case we will use the notation

\[
\mathcal{T} = \langle A_1, \ldots, A_n \rangle.
\]

We will only be concerned with semiorthogonal decompositions of the bounded derived category $D^b(X)$ of coherent sheaves on a smooth projective variety $X$ (for which [28] provides an excellent starting point), and $T$-linear semiorthogonal decompositions of the category $\text{Perf}_X$ of perfect complexes on the total space $X$ of a smooth projective morphism $f: X \to T$. The notion of $T$-linearity is introduced in [26, §2.6] and ensures that the semiorthogonal decompositions behave well in families.

**Definition 2.2.** Let $X$ be a smooth projective variety. We say that $D^b(X)$ is **indecomposable** if there exist no non-trivial semiorthogonal decompositions, i.e. in Definition 2.1 all but one of the subcategories are zero.

In [8] a moduli space of semiorthogonal decompositions was constructed, relative to a smooth projective family $f: X \to T$. Because we are only interested in indecomposability results, it suffices to consider the moduli space $\text{ntSOD}_{T^2}$ of non-trivial
semiorthogonal decompositions of length 2, as introduced in [8 Definition 8.26]. By 
[8 Proposition 8.31], the $T'$-valued points for $T' \to T$ are given by 
\[ \text{ntSOD}^2_T(T') \simeq \left\{ \text{Perf} \mathcal{X}_{T'} = \langle A_1, A_2 \rangle \mid T' \text{-linear} \mid A_i|_x \neq 0 \text{ for any } i = 1, 2 \text{ and } x \in T' \right\}, \]
where $A_i|_x$ denotes the base change of the subcategory to a point $x \in T'$.

By [8 Theorem 8.30] we have the following geometric properties of this moduli space.

**Theorem 2.3** (Belmans–Okawa–Ricolfi). Let $f: \mathcal{X} \to T$ be a smooth projective morphism, where $T$ is an excellent scheme. Then $\text{ntSOD}^2_T$ is an algebraic space, which is étale over $T$.

An important consequence of this theorem is [8 Theorem 8.33], which is as follows:

**Theorem 2.4.** Under the assumptions of Theorem 2.3, the set of points $t \in T$ for which $D^b(f^{-1}(t))$ admits a non-trivial semiorthogonal decomposition is a (possibly empty) Zariski open subset of $T$.

**Proof.** The subset of interest is nothing but the image of the étale morphism $\text{ntSOD}^2_T \to T$, hence is Zariski open. □

Note that, in particular, we can take $T$ to be a scheme locally of finite type over $\mathbb{C}$.

We can now obtain the following short proof of the main theorem.

**Proof of Theorem A.** By Theorem 2.4, the subset $T \setminus U \subseteq T$ is open. Since it does not intersect the dense subset $U \subseteq T$, it has to be empty. Hence $U = T$, which is nothing but the assertion. □

In the remainder of this paper we give instances in which we can apply this result, for which we need to exhibit a dense subset $U \subseteq T$ parameterising fibres of $f: \mathcal{X} \to T$ with indecomposable derived category. In this direction, we use Theorem 1.1 and the following result. It is likely standard, but we have not found a reference, so we include a proof.

**Proposition 2.5.** Let $f: \mathcal{X} \to T$ be a smooth and projective morphism of noetherian schemes defined in characteristic 0. Then the function 
\[ \varphi: T \to \mathbb{N} \cup \{-\infty\}, \quad t \mapsto \varphi(t) := \dim B_s|\omega_{\mathcal{X}_t}| \]
is upper semi-continuous.

**Proof.** Define the closed subscheme $\mathcal{B} \subset \mathcal{X}$, which could be called the *relative canonical base locus*, by the ideal sheaf \( \text{Im}(f^*f_*\omega_f \otimes \omega_f^{-1} \to O_{\mathcal{X}}) \), where the morphism is induced by tensoring the counit morphism $f^*f_*\omega_f \to \omega_f$ by $\omega_f^{-1}$. Here $\omega_f = \det \Omega_{\mathcal{X}/T}^1$ is the relative canonical bundle.

We obtain the associated right exact sequence 
\[ (2.3) \quad f^*f_*\omega_f \otimes \omega_f^{-1} \to O_{\mathcal{X}} \to O_{\mathcal{B}} \to 0. \]

In the rest of the proof, we will show that for each $t \in T$ the fibre $\mathcal{B}_t$ coincides with the canonical base locus of the fibre $\mathcal{X}_t = f^{-1}(t)$. As the fibre dimension of the proper morphism $\mathcal{B} \to T$ is upper semi-continuous, we obtain the assertion.
Taking the fibre of \( \phi \) over \( t \in T \) we obtain the right exact sequence
\[
(f \ast \omega_f \otimes \mathcal{O}_T \mathbf{k}(t)) \otimes_{\mathbf{k}(t)} \omega_{X_t}^{-1} \to \mathcal{O}_{X_t} \to \mathcal{O}_{B_t} \to 0.
\]
Therefore all we have to show is that the canonical map
\[
f \ast \omega_f \otimes \mathcal{O}_T \mathbf{k}(t) \to H^0(X_t, \omega_{X_t})
\]
is an isomorphism. But this is a famous consequence of the degeneration of the Hodge-to-de Rham spectral sequence (see, say, [13, Theorem 10.21]) and this is where we need the assumption on the characteristic. \( \square \)

**Remark 2.6.** It is well known that the invariance of (pluri-)genera fails in positive characteristic and mixed characteristic (and for non-Kähler manifolds). In fact, Proposition 2.3 and Corollary 13 completely fail without the assumption on the characteristic. To see this, let \( k \) be an algebraically closed field of characteristic 2, and \( X \) be a supersingular Enriques surface [30]. Then one can find a smooth projective morphism \( X \to T \) to a smooth pointed curve \((T, 0)\) such that
- \( X_0 \cong X \), and
- \( X_t \) is a classical Enriques surface for \( t \in T \setminus \{0\} \).

For a classical Enriques surface \( Y \) we have that \( H^i(Y, \mathcal{O}_Y) = 0 \) for \( i = 1, 2 \), \( \omega_Y \not\cong \mathcal{O}_Y \), and \( \omega_Y^2 \cong \mathcal{O}_Y \). On the other hand, for a singular or a supersingular Enriques surface \( X \) we have that \( \omega_X \cong \mathcal{O}_X \). Therefore the family \( X \to T \) considered above shows that the invariance of genera fails in characteristic 2.

Since \( \omega_X \cong \mathcal{O}_X \), there is no non-trivial semiorthogonal decomposition of \( D^b(X) \). On the other hand, if we let \( \pi \) be the restriction of the morphism \( X \to T \) over \( T \setminus \{0\} \), then the assumption implies that \( \mathbf{R}\pi_* \mathcal{O}_{X \setminus X} \cong \mathcal{O}_{T \setminus \{0\}} \). Hence there is a non-trivial \((T \setminus \{0\})\)-linear semiorthogonal decomposition of \( D^b(X \setminus X) \) which does not extend to the central fibre. If we take \( t_0 = 0 \) in the statement of Corollary 13, this shows that the assertion does not hold in characteristic 2.

It is shown in [30, Theorem 4.10] that any Enriques surface defined in positive characteristic admits a lifting to characteristic 0. Let \( X \) be a supersingular Enriques surface defined in characteristic 2, and let \( X \to T \) be its lifting to characteristic 0. Since any Enriques surface in characteristic 0 is classical, this gives a similar example as above in mixed characteristic (where the characteristic of the residue field is 2).

We can now prove Corollary 13 which implements the explicit check for indecomposability in families.

**Proof of Corollary 13.** Let \( X \to T \) be a smooth projective morphism as in Theorem A. Then the set \( U := \varphi^{-1}(\{-\infty, 0\}) \) is an open subset of \( T \), by Proposition 2.3, which is non-empty by assumption. Applying Theorem 4.1 we can conclude by Theorem A. \( \square \)

### 3. Symmetric products of curves

Let \( C \) be a smooth complex projective curve of genus \( g \geq 3 \), and let \( n \geq 1 \). We denote by \( \text{Sym}^n C \) the \( n \)-fold symmetric product, which is the smooth \( n \)-dimensional variety parameterising unordered \( n \)-tuples \( D = p_1 + \cdots + p_n \) of points \( p_1, \ldots, p_n \in C \). In this section we prove Corollary 13 by applying Theorem A. To this aim we need a good understanding of the canonical base locus of \( \text{Sym}^n C \). For this we recall some standard facts on the Abel–Jacobi map and Brill–Noether theory [1, §IV].
Let us consider the Abel–Jacobi map

\[
AJ_n : \text{Sym}^n C \to \text{Pic}^n C, \quad D \mapsto \mathcal{O}_C(D),
\]

such that \(AJ_n^{-1}(AJ_n(D)) = \mathbb{P}^h(C, \mathcal{O}_C(D))^\vee\). For \(r = 0, \ldots, n\), we have the following standard subschemes of \(\text{Sym}^n C\)

\[
C^n_r := \{ D = q_1 + \cdots + q_n \mid \text{rk}(dAJ_n, D) \leq n - r \} \subseteq \text{Sym}^n C,
\]

and the subschemes of the Picard scheme \(W^r := AJ_n(C^n_r) \subseteq \text{Pic}^n C\), endowed with the morphisms

\[
AJ^n_r = AJ_n|_{C^n_r} : C^n_r \to W^r.
\]

We note that \(\text{Sym}^n C = C^n_0\), and we define \(W_n := W^n_0\). Moreover, ignoring the scheme-theoretic structure, we have set-theoretic identities

\[
\text{Supp} C^n_r = \{ D \in \text{Sym}^n C \mid h^0(C, \mathcal{O}_C(D)) \geq r + 1 \},
\]

\[
\text{Supp} W^n_r = \{ \mathcal{L} \in \text{Pic}^n C \mid h^0(C, \mathcal{L}) \geq r + 1 \}.
\]

Let \(\text{gon}(C)\) denote the gonality of the curve \(C\), that is the least degree of a non-constant morphism \(C \to \mathbb{P}^1\). The following result describes the canonical base locus of \(\text{Sym}^n C\) depending on the gonality of \(C\), and it also appears as [3 Proposition 3.4]. We present an alternative proof, relying on Macdonalds description of canonical divisors on \(\text{Sym}^3 C\) [11] and the geometric version of the Riemann–Roch theorem [1 §1.2].

**Proposition 3.1.** Let \(C\) be a non-hyperelliptic curve of genus \(g \geq 3\) and let \(n = 1, \ldots, g - 1\). Then

(1) \(\text{Bs} | \omega_{\text{Sym}^n C} | = \emptyset\) if and only if \(n < \text{gon}(C)\);

(2) if \(n \geq \text{gon}(C)\), then \(\text{Bs} | \omega_{\text{Sym}^n C} |\) is infinite, and set-theoretically we have an equality

\[
\text{Bs} | \omega_{\text{Sym}^n C} | = C^n_1.
\]

**Proof.** It is well-known that \(\text{Bs} | \omega_C | = \emptyset\), so we can assume that \(n = 2, \ldots, g - 1\). The following two lemmas show that the set-theoretic equality \(\text{Bs} | \omega_{\text{Sym}^n C} | = C^n_1\) holds, so the proof ends by observing that \(C^n_1 = \emptyset\) if and only if \(n < \text{gon}(C)\). \(\square\)

**Lemma 3.2.** If \(q_1 + \cdots + q_n \in \text{Sym}^n C\) is in \(\text{Bs} | \omega_{\text{Sym}^n C} |\) then \(q_1 + \cdots + q_n \in C^n_1\).

**Proof.** Let \(\phi : C \to \mathbb{P}^{g-1}\) be the canonical embedding and let \(P = p_1 + \cdots + p_n \in \text{Sym}^n C\). As in [1 page 12], we denote by \(\phi(P)\) the intersection of the hyperplanes \(H \subseteq \mathbb{P}^{g-1}\) such that \(\phi^*(H) \ni P\). In particular, we note that \(\dim \phi(P) \leq n - 1\), and if the points \(p_1, \ldots, p_n\) are distinct, then \(\phi(P)\) is simply the linear span of \(\phi(p_1), \ldots, \phi(p_n)\).

Given a \((g - 1 - n)\)-plane \(L \subseteq \mathbb{P}^{g-1}\), let \(\mathcal{D}_L\) be the linear series of divisors \(\phi^*(H)\) on \(C\) cut out by the hyperplanes \(H \subseteq \mathbb{P}^{g-1}\) containing \(L\). So \(\mathcal{D}_L\) is a linear system of degree \(2g - 2\) and dimension \(n - 1\), possibly, with base points at \(L \cap \phi(C)\).

Consider the subordinate variety \(\Gamma(\mathcal{D}_L)\), which is a determinantal variety supported on the set

\[
\text{Supp} \Gamma(\mathcal{D}_L) := \{ P = p_1 + \cdots + p_n \in \text{Sym}^n C \mid D - P \geq 0 \text{ for some } D \in |\mathcal{D}_L| \}
\]

(cf. [1 pages 341–342]). By [3 Lemma 2.1], \(\Gamma(\mathcal{D}_L)\) is a canonical divisor of \(\text{Sym}^n C\).
We claim that if \( Q = q_1 + \cdots + q_n \in \text{Sym}^n C \) is a base point of \( |\omega_{\text{Sym}^n C}| \), then \( \dim \phi(Q) \leq n - 2 \). Indeed, for any \((g - 1 - n)\)-plane \( L \subset \mathbb{P}^{g-1} \), \( Q \) lies on the canonical divisor \( \Gamma(\mathcal{O}_L) \). Hence there exists a hyperplane \( H \subset \mathbb{P}^{g-1} \) containing \( L \) such that \( \phi^*(H) \geq Q \), so that \( \phi(Q) \subset H \) as well.

However, if we assumed \( \dim \phi(Q) = n - 1 \) and we considered a \((g - 1 - n)\)-plane \( L \subset \mathbb{P}^{g-1} \) not meeting \( \phi(Q) \), the linear span of \( L \) and \( \phi(Q) \) would be the whole \( \mathbb{P}^{g-1} \), a contradiction.

Thus \( \dim \phi(Q) \leq n - 2 \), and the geometric version of the Riemann–Roch theorem yields that \( \dim |Q| = \deg Q - 1 - \dim \phi(Q) \geq 1 \), that is \( Q \in C_n^1 \).

**Lemma 3.3.** If \( q_1 + \cdots + q_n \in C_n^1 \), then \( q_1 + \cdots + q_n \in \text{Bs} |\omega_{\text{Sym}^n C}| \).

**Proof.** Consider the \( n \)-fold ordinary product \( C^n \) endowed with the natural projections \( \pi_i: C^n \to C \), with \( i = 1, \ldots, n \). We follow [31, §3-8], and we consider a basis \( \{\omega_1, \ldots, \omega_g\} \) of the space \( H^{0}(C, \omega_C) \cong H^{1,0}(C) \). Then for any \( j = 1, \ldots, g \), we can define a holomorphic 1-form on \( C^n \)

\[
\xi_j := \sum_{i=1}^{n} \pi_i^* \omega_j \in H^{1,0}(C^n),
\]

which is invariant under the action of the symmetric group \( \mathfrak{S}_n \), so that it can be considered as a holomorphic 1-form on the \( n \)-fold symmetric product \( \text{Sym}^n C \).

In [31 §8], Macdonald proved that the set of holomorphic \( n \)-forms

\[
\{\xi_{j_1} \wedge \xi_{j_2} \wedge \cdots \wedge \xi_{j_n} \mid 1 \leq j_1 < j_2 < \cdots < j_n \leq g\}
\]

is a basis for the space \( H^{n,0}(\text{Sym}^n C) \cong H^{0}(\text{Sym}^n C, \omega_{\text{Sym}^n C}) \) of canonical forms on \( \text{Sym}^n C \).

Now we consider a point \( Q := q_1 + \cdots + q_n \in C_n^1 \) and we assume furthermore that the points \( q_1, \ldots, q_n \in C \) are distinct, that is \( Q \in C_n^1 \setminus \Delta \), where

\[
\Delta := \{2p + p_3 + \cdots + p_n \in \text{Sym}^n C \mid p, p_3, \ldots, p_n \in C\}
\]

is the diagonal divisor of \( \text{Sym}^n C \). In this case, we may write (with a slight abuse of notation)

\[
\xi_j(Q) = \sum_{i=1}^{n} \omega_j(q_i).
\]

As in the proof of Lemma 3.2, we consider the canonical embedding \( \phi: C \hookrightarrow \mathbb{P}^{g-1} \) and, by the geometric version of Riemann–Roch theorem, we deduce that the linear span \( \phi(Q) \) of \( \phi(q_1), \ldots, \phi(q_n) \) has dimension \( \dim \phi(Q) = \deg Q - 1 - \dim |Q| \leq n - 1 - 1 = n - 2 \).

This is equivalent to saying that the vectors

\[
(\omega_1(q_1), \ldots, \omega_g(q_1)) \quad i = 1, \ldots, n
\]

are linearly dependent, i.e. the rank of the \( n \times g \) matrix \( A := (\omega_j(q_i)) \) is smaller than \( n \). Thus for any multi-index \((j_1, j_2, \ldots, j_n)\) such that \( 1 \leq j_1 < j_2 < \cdots < j_n \leq g \), the columns \( (\omega_{j_1}(q_1), \ldots, \omega_{j_n}(q_n))^\top \) of \( A \) are linearly dependent, and the equality \( \text{rank}(A) \leq n - 1 \) gives that the canonical form \( \xi_{j_1} \wedge \xi_{j_2} \wedge \cdots \wedge \xi_{j_n} \) vanishes at \( Q \). Since these holomorphic \( n \)-forms provide a basis of \( H^{0}(\text{Sym}^n C, \omega_{\text{Sym}^n C}) \), we deduce that if \( Q \in C_n^1 \) and the points \( q_1, \ldots, q_n \in C \) are distinct, then \( Q \in \text{Bs} |\omega_{\text{Sym}^n C}| \).
To conclude the proof, we must assume $Q \in C_n^1 \cap \Delta$ and prove that $Q \in \text{Bs} \left| \omega_{\text{Sym}^n C} \right|$. We recall that $\text{Bs} \left| \omega_{\text{Sym}^n C} \right|$ is a closed subset of $\text{Sym}^n C$ and since $C_n^1 \cap \Delta \subset \text{Bs} \left| \omega_{\text{Sym}^n C} \right|$ by the first part of the proof, it is enough to show that if $Q \in C_n^1 \cap \Delta$, then $Q$ belongs to the Zariski closure of $C_n^1 \setminus \Delta$. To this aim, let us assume that $Q = 2q + q_3 + \cdots + q_n$, where $q, q_3, \ldots, q_n$ are distinct points (the case with more points being equal is analogous) and we consider the complete linear series $|Q|$ on $C$. Since the divisor $Q \in \text{Sym}^n C$ is singular at the point $q \in C$, Bertini’s theorem ensures that either the general divisor $P \in |Q|$ is smooth, or $2q$ is a base point of the complete linear system $|Q|$.

In the former case $Q$ belongs to the Zariski closure of the locus $|Q| \setminus \Delta \subset C_n^1$, which is contained in $\text{Bs} \left| \omega_{\text{Sym}^n C} \right|$ by the first part of the proof.

In the latter case, any divisor in $|Q|$ has the form $2q + p_3 + \cdots + p_n$, with $P = p_3 + \cdots + p_n$ varying in a linear series $|P|$ of degree $n - 2$ and dimension equal to $\dim |Q|$. Therefore, taking a pair of general points $p_1, p_2 \in C$ and a divisor $D \in |P|$, we have that $p_1 + p_2 + D$ lies in $C_n^1 \setminus \Delta$, which is contained in $\text{Bs} \left| \omega_{\text{Sym}^n C} \right|$ by the first part of the proof. Since $Q$ lies in the Zariski closure of the locus described by the points $p_1 + p_2 + D \in C_n^1$ as above, we conclude that $Q \in \text{Bs} \left| \omega_{\text{Sym}^n C} \right|$ as well.

Finally, the case where $Q = \sum_{x \in C} n_x x$ has more than two equal points can be handled recursively in the same way, by taking a general divisor $P = \sum_{x \in C} m_x x \in |Q|$, and distinguishing the following cases: either $m_x = n_x$ for any $x \in C$ such that $n_x > 1$, or there exists some $x \in C$ such that $n_x > 1$ and $0 < m_x < n_x$ (i.e. $P$ is less singular than $Q$ at $x$).

This gives an alternative proof of the main result of [9]. The case $g = 2$ here is covered by [32].

**Proposition 3.4** (Biswas–Gomez–Lee). Let $C$ be a smooth projective curve of genus $g \geq 2$. Let $n = 1, \ldots, \text{gon}(C) - 1$. Then $D^b(\text{Sym}^n C)$ is indecomposable.

**Proof.** By Proposition 3.1(1) we have that in this case the canonical base locus $\text{Bs} \left| \omega_{\text{Sym}^n C} \right|$ is empty, so that by Theorem 1.1 we have that $D^b(\text{Sym}^n C)$ is indecomposable. \hfill \Box

With this result in mind, the following proposition is an application of Corollary 11. Hence our main contribution here is the amplification using Theorem A from a general curve to every curve, removing the dependence of $n$ on the curve.

**Proposition 3.5.** Let $C$ be a smooth projective curve of genus $g \geq 2$. Then $D^b(\text{Sym}^n C)$ is indecomposable for $n = 1, \ldots, \lfloor \frac{g+3}{2} \rfloor - 1$.

**Proof.** The gonality of $C$ is bounded above by $\lfloor \frac{g+3}{2} \rfloor$, and a general curve realises this bound, by [11] page 213. Therefore, we can construct a smooth family $f : C \to T$ of curves parameterised over a curve $T$ mapping finitely to a complete curve in the moduli space $\mathcal{M}_g$, where $f^{-1}(\overline{T}) = C$ for some $\overline{T} \in T$ and the general member of the family is a curve of gonality $\lfloor \frac{g+3}{2} \rfloor$ (see e.g. [15] Exercise 2.10).

Thus the relative symmetric product $f_n : \text{Sym}_n^T C \to T$ is a smooth projective family, whose general fibre $f_n^{-1}(t)$ is the $n$-fold symmetric product of a smooth curve $f^{-1}(t)$ having gonality $\lfloor \frac{g+3}{2} \rfloor$, so that $D^b(f_n^{-1}(t))$ is indecomposable by Proposition 3.4. Hence the morphism $f_n$ satisfies the assumptions of Theorem A and the assertion follows. \hfill \Box
It follows from [9, Theorem 1.3] and Proposition 3.5 that $D^b(\text{Sym}^2 C)$ is indecomposable for any curve $C$ of genus $g \geq 3$. So the first unknown case is $\text{Sym}^3 C$ where $C$ is a curve of genus 4. Let us now explain how one can settle this using a somewhat ad-hoc method.

If $C$ is a general curve of genus $g$, then the loci $C^*_k$ and $W^*_k$ are better understood. In particular, letting $\rho(g, r, n) := g - (r + 1)(g - n + r)$ be the Brill–Noether number, we have the following result (cf. [1] Lemma IV.1.6 and [1] Theorems V.1.6 and V.1.7).

**Theorem 3.6.** Let $C$ be a general curve. For $1 \leq n \leq g$ and $r \geq 0$ we have that

1. $W^*_n \setminus W^*_n$ is smooth, and of dimension $\rho(g, r, n)$;
2. $C^*_n \setminus C^*_n$ is smooth, and of dimension $\rho(g, r, n) + r$.

This allows us to prove the following proposition.

**Proposition 3.7.** Let $C$ be a smooth projective curve of genus $g = 4$. Then $D^b(\text{Sym}^3 C)$ is indecomposable.

**Proof.** Let $C$ be a general curve of genus 4. Then $\rho(4, 1, 3) = 0$, hence $W^*_3$ is a finite set of closed points, and its preimage $C^*_3$ is a disjoint union of finitely many projective lines.

Consider the morphism $AJ_3: \text{Sym}^3 C \to W_3$ from (3.3). We have $W_3 = \Theta$ is the Theta-divisor on $\text{Pic}^3 C$, and the morphism is a flopping contraction. To see this, recall that $\Theta$ has Gorenstein singularities, as it is a divisor on a smooth variety. We know that $AJ_3$ is a small (i.e., isomorphic in codimension one) birational map and $\text{Sym}^3 C$ is smooth, so as to obtain the isomorphism of line bundles $AJ_3^* \omega_{W_3} \cong \omega_{\text{Sym}^3 C}$. In particular, any curve contracted by $AJ_3$ intersects by 0 with $\omega_{\text{Sym}^3 C}$. Thus we see that $AJ_3$ is a flopping contraction.

Since $W_3$ is Gorenstein and its singular locus $W^*_3$ is isolated, for every point $p \in W^*_3$ there exists an open neighbourhood $U \subseteq W_3$ such that $\omega_{W_3}$ is trivial. As $AJ_3^* (\omega_{W_3}) \cong \omega_{\text{Sym}^3 C}$, we see that $\omega_{\text{Sym}^3 C}$ is trivial on the open set $(AJ_3)^{-1} (U)$, and we can apply Theorem 1.1.

If $C$ is an arbitrary curve of genus 4, we can proceed as in the proof of Proposition 3.5 by using the fact that the general curve in $M_4$ satisfies $\rho(4, 1, 3) = 0$ and applying Theorem A.

Hence the first open case is $\text{Sym}^4 C$ where $C$ is a curve of genus 5.

**Remark 3.8.** The derived categories of symmetric powers feature in an interesting way in the (conjectural) semiorthogonal decomposition for other moduli spaces of sheaves on the curve $C$. In [29, Conjecture 1.1] a conjecture due to Narasimhan is discussed, which was obtained independently as [9, Conjecture 7] by Belmans–Galkin–Mukhopadhyay. It states that there exists a semiorthogonal decomposition of the form

$$(3.12) \quad D^b(M_C(2, \mathcal{L})) = \langle D^b(k), D^b(k), D^b(C), D^b(C), D^b(\text{Sym}^2 C), D^b(\text{Sym}^2 C), \ldots, D^b(\text{Sym}^{g-2} C), D^b(\text{Sym}^{g-2} C), D^b(\text{Sym}^{g-1} C) \rangle$$

where $\mathcal{L}$ is a line bundle of degree 1, so that $M_C(2, \mathcal{L})$ is a smooth projective Fano variety of dimension $3g - 3$. Hence the indecomposability suggested by Question 5 for $D^b(\text{Sym}^n C)$ for $n \leq g - 1$ would ensure that this is a decomposition into indecomposable pieces.
Remark 3.9. The derived category $D^b(\text{Sym}^n C)$ is not indecomposable for $n \geq g$.
In [34, Corollary 5.11] a semiorthogonal decomposition was obtained using wall-crossing methods. Remark that $\text{Sym}^n C$ is not minimal in this case.

Alternatively, the geometry of the Abel–Jacobi morphism exhibits the symmetric power as a projective bundle (of a not necessarily locally free sheaf). This description was used in [7, Theorem D] and [21, Corollary 3.8] to give the semiorthogonal decomposition
\[(3.13) \quad D^b(\text{Sym}^n C) = \langle D^b(\text{Jac} C), \ldots, D^b(\text{Jac} C), D^b(\text{Sym}^{2g-2-n} C) \rangle \]
where there are $n - g + 1$ copies of the derived category of the Jacobian.

4. Surfaces of general type

In this section we consider two classes of smooth complex projective surfaces of general type, and we apply Theorem [A] and Corollary [13] in order to achieve the indecomposability of their derived categories. In particular, Section 4.1 concerns the so-called Horikawa surfaces, of which we summarise what is known about their canonical base loci and their behaviour under degeneration in Appendix [A].

Section 4.2 is instead concerned with a series of examples of double coverings of the projective plane introduced by Ciliberto [11], whose construction is retraced in Appendix [B].

In both cases we study surfaces $S$ having positive-dimensional canonical base locus, and our argument for proving the indecomposability of $D^b(S)$ relies on the behaviour of their canonical base loci in families. This description of the canonical base locus will be in turn used in Section 5 to bootstrap indecomposability results for the Hilbert schemes of points $S^{[n]}$.

4.1. Horikawa surfaces. In a series of papers [16, 17, 18, 19, 20] Horikawa classified minimal surfaces of general type which are on (or close to) the Noether line, i.e. for which the inequality
\[(4.1) \quad c_1^2 \geq 2p_g - 4 \]
is (almost) an equality. In particular, he considered the cases where $c_1^2$ equals $2p_g - 4$ [17], $2p_g - 3$ [18, 16] or $2p_g - 2$ [19, 20]. In Appendix [A] we summarise the behaviour of the canonical base locus in the classification of Horikawa surfaces. It follows from this description that it is possible to realise each Horikawa surface in a family where the general member has empty canonical base locus. By applying Theorem [A] we thus obtain the following theorem.

**Theorem 4.1.** Let $S$ be a Horikawa surface, i.e. a minimal smooth projective surface of general type with $c_1^2 = 2p_g - 4$, $2p_g - 3$ or $2p_g - 2$. Then $D^b(S)$ is indecomposable.

**Remark 4.2.** We note that for Horikawa surfaces with empty or finite canonical base locus, the indecomposability of $D^b(S)$ follows immediately from Theorem [1.1].

For all the others (i.e. Horikawa surfaces whose canonical base locus consists of a rational curve $F$ with $F^2 = -2$) one can deduce Theorem [1.1] alternatively by using [24, Theorem 1.10], which asserts that if $S$ is a minimal surface such that $p_g \geq 2$ and every one-dimensional connected component of $Bs[\omega_S]$ has negative definite intersection matrix, then $D^b(S)$ is indecomposable. This result is indeed applied in [24, Example 4.6] to Horikawa surfaces of type III from [20]. However, we will see
in Section 4 how our degeneration argument is crucial to discuss indecomposability of Hilbert schemes of points of Horikawa surfaces.

4.2. Ciliberto double planes. In [11] Ciliberto considered various classes of surfaces of general type, which exhibit interesting properties in their canonical coordinate ring. For our purposes we are interested in [11] Esempio 4.3, where for any integer $h \geq 1$, we obtain a family of minimal smooth projective surfaces $S_h$ of general type of geometric genus $p_g(S_h) = 2h$, whose canonical base locus $Bs |\omega_{S_h}|$ consists of an irreducible curve $F$ with vanishing self-intersection. We refer to Appendix B for a summary of Ciliberto’s construction and for some properties of the surfaces $S_h$.

In this section we prove that any such a surface has indecomposable derived category.

**Theorem 4.3.** Let $h \geq 1$ be an integer and let $C_h \subset \mathbb{P}^2$ be an irreducible curve of degree $8h + 4$, having a $(4h + 2)$-tuple ordinary point at $p_1$, a $4h$-tuple ordinary point at $p_2$, four points of type $[2h + 1, 2h + 1]$ at $p_3, \ldots, p_6$, and no other singularities, where $p_1, \ldots, p_6 \in \mathbb{P}^2$ lie on an irreducible conic transverse to $C_h$ at any $p_i$. Let $S_h$ be the minimal desingularisation of the double plane $S'_h \to \mathbb{P}^2$ branched along $C_h$. Then $D^b(S_h)$ is indecomposable.

**Proof.** We want to construct a smooth family $f : \mathcal{X} \to T$ of projective surfaces such that $\dim Bs |\omega_{f^{-1}(t)}| \leq 0$ and $f^{-1}(t') = S_h$ for some $t, t' \in T$, so that the assertion will follow from Corollary B.

Let $Y \subset \mathbb{P}^2 \times G(1, 2)$ be the universal family of the Grassmannian of lines in $\mathbb{P}^2$, and let $U \subset \mathbb{P}^2 \times \mathbb{P}^2 \times \text{Sym}^4 Y$ be the irreducible open subset parametrising tuples $u = (x_1, x_2, (x_3, [\ell_3]) + \cdots + (x_6, [\ell_6]))$ such that the points $x_1, \ldots, x_6$ are distinct, no three of them lie on a line, and there exists an irreducible quartic with nodes at $x_1, x_2$ and tangent to each $\ell_i$ at $x_i$. For any $u \in U$, let $\mathcal{L}_u \subset \mathbb{P}(H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(8h + 4)))$ be the linear system of plane curves of degree $8h + 4$ having a $(4h + 2)$-tuple point at $x_1$, a $4h$-tuple point at $x_2$, four points of type $[2h + 1, 2h + 1]$ at $x_3, \ldots, x_6$, each with tangent line $\ell_i$. Consider the incidence variety

\begin{equation}
W := \left\{ ([F], u) \in \mathbb{P}(H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(8h + 4))) \times U \mid C = V(F) \in \mathcal{L}_u \right\},
\end{equation}

For any $u \in U$ the fibre via the second projection $\pi_2 : W \to U$ is $\pi_2^{-1}(u) \cong \mathcal{L}_u$.

As in [B.3], $\dim \mathcal{L}_u \geq 8h + 3$ and Lemma B.3 ensures that this is an equality when $x_1, \ldots, x_6$ lie on an irreducible conic. Then we deduce by semi-continuity that $\dim \mathcal{L}_u = 8h + 3$ also for points $x_1, \ldots, x_6$ in general position. Being $\pi_2$ a surjective map with fibres $\pi_2^{-1}(u) \cong \mathbb{P}^{8h+3}$ over the irreducible variety $U$, we conclude that $W$ is itself irreducible. Therefore, for general $w = ([F], u) \in W$, the curve $C_w := V(F) \in \mathcal{L}_u$ is irreducible and its only singular points are a $(4h + 2)$-tuple ordinary point at $x_1$, a $4h$-tuple ordinary point at $x_2$, and four points of type $[2h + 1, 2h + 1]$ at $x_3, \ldots, x_6$, because this happens when the points $x_1, \ldots, x_6$ are special, as we check in Proposition B.1.

Let $T \subset W$ be the locus parametrising irreducible curves $C_w$ as above. Then we can define a family of surfaces $f' : \mathcal{X}' \to T$ such that for any $t \in T$, the fibre

\[1\text{.The latter is an open condition on } \mathbb{P}^2 \times \mathbb{P}^2 \times \text{Sym}^4 Y \text{ which ensures that, although the points } x_1, \ldots, x_6 \text{ may lie on an irreducible conic, the tangent directions are sufficiently general, as in Proposition B.1.]}}
$X'_f = (f')^{-1}(t)$ is the double covering of $\mathbb{P}^2$ branched over the curve $C_f$. Up to base-changing and shrinking $T$, we then obtain a family $f : X \to T$ of smooth surfaces of general type, where $X_t = f^{-1}(t)$ is a minimal desingularisation of $X'_f$. In particular, when the singularities of the curve $C_f$ lie on an irreducible conic, we obtain a surface $S_h$ as in the assertion. Thus it remains to show that there exists $t \in T$ such that $\text{Bs}|\omega_{X_t}|$ does not contain a curve.

To this aim, we consider a general point $t \in T$ and we set $u := \pi_2(t)$. Let $\phi : X_t \to \mathbb{P}^2$ be the generically finite morphism of degree 2 induced by the double covering $X'_f \to \mathbb{P}^2$ As in Remark B.2, the image under $\phi$ of the curves in the canonical linear system $|\omega_{X_t}|$ is the linear system $\mathcal{D}_h$ of curves $D \subset \mathbb{P}^2$ of degree $4h - 1$ having a $2h$-tuple point at $x_1$, a $(2h - 1)$-tuple point at $x_2$ and four points of type $[h, h - 1]$ at $x_3, \ldots, x_6$. Hence $|\omega_{X_t}|$ possesses a fixed curve if and only if $\mathcal{D}_h$ does.

However, if $h = 1$, then $\mathcal{D}_1$ is the pencil of cubics having a node at $x_1$ and passing through $x_2, \ldots, x_6$, whose general element is indeed an irreducible curve.

When $h \geq 2$, we argue as in B.2, and we deduce

$$\dim \mathcal{D}_h \geq \frac{(4h - 1)(4h + 2)}{2} - 2h(2h + 1) - \frac{(2h - 1)2h}{2} - 4 \left(\frac{h(h + 1)}{2} + \frac{(h - 1)h}{2}\right) = 2h - 1.$$ (4.3)

On the other hand, by Remark B.2, the latter is an equality when the points $x_1, \ldots, x_6$ lie on an irreducible conic, hence $\dim \mathcal{D}_h = 2h - 1$ by semicontinuity. Moreover, we recall that for such a general $u = (x_1, x_2, (x_3, [\ell_3]) + \cdots + (x_6, [\ell_6]))$, there exists an irreducible quartic $\Delta \subset \mathbb{P}^2$ having nodes at $x_1, x_2$ and passing through $x_3, \ldots, x_6$ with tangent lines $\ell_3, \ldots, \ell_6$. Therefore, for any cubic $D \in \mathcal{D}_1$ as above, the curve $D' := (h - 1)\Delta + D$ belongs to $\mathcal{D}_h$. Hence $\mathcal{D}_h$ has a fixed curve if and only if $\Delta \subset D_h$ for any $D_h \in \mathcal{D}_h$. If this were the case, residual curves would be such that $\mathcal{D}_h \setminus \Delta \in \mathcal{D}_{h-1}$, and hence $2h - 1 = \dim \mathcal{D}_h = \dim \mathcal{D}_{h-1} = 2(h-1)-1$, a contradiction. Then $\mathcal{D}_h$ has no fixed curves for any $h \geq 2$.

Thus we conclude for generic $t$ that $\text{Bs}|\omega_{X_t}|$ does not contain curves, and the assertion follows from Corollary B.4. 

**Remark 4.4.** Let $S_h$ be a surface as in Theorem 4.3. It follows from the analysis of [11], Esempio 4.3 that the fixed part of $|\omega_{S_h}|$ is a genus 2 curve $F$ such that $F^2 = 0$. Therefore, unlike the case of Horikawa surfaces, it is not possible to apply [24], Theorem 1.10 to achieve the indecomposability of $\mathbf{D}^b(S_h)$, and we do not know alternative arguments for proving Theorem 4.3.

**Remark 4.5.** It follows from the proof that Theorem 4.3 holds even for surfaces $S_h$ obtained when the points $p_1, \ldots, p_6 \in \mathbb{P}^2$ are in general position. Indeed, they are the surfaces $X_t$ with $\dim \text{Bs}|\omega_{X_t}| \leq 0$ appearing in the proof, so that $\mathbf{D}^b(X_t)$ is indecomposable by Theorem 1.1.

Furthermore, one can easily check that for general $t \in T$, the canonical base locus $\text{Bs}|\omega_{X_t}|$ is empty; we will use this fact in the next section to deduce that the Hilbert scheme of points $S_h^{[n]}$ of the surface $S_h$ has indecomposable derived category.
5. Hilbert schemes of points on surfaces

The Hilbert schemes of points, in particular on surfaces, have been studied since [14]. They provide interesting smooth projective varieties (of dimension $2n$), as crepant resolutions of the symmetric power of the surface $S$.

Given a smooth projective surface $S$, let $S^{[n]}$ denote its Hilbert scheme of points, which parametrises 0-dimensional schemes of length $n$ on $S$. The following proposition is our starting point for the study of indecomposability of the Hilbert schemes of points on surfaces.

**Proposition 5.1.** Let $S$ be a smooth projective surface. Let $n \geq 1$. If $Bs|\omega_S| = \emptyset$, then $Bs|\omega_{S^{[n]}}| = \emptyset$.

**Proof.** The Hilbert–Chow morphism

\begin{equation}
\pi: S^{[n]} \to \text{Sym}^n S
\end{equation}

is a crepant resolution of singularities, and $\text{Sym}^n S$ is Gorenstein. Hence there is an isomorphism of line bundles $\omega_{S^{[n]}} \cong \pi^*\omega_{\text{Sym}^n S}$, and combined with the projection formula and the fact that $\pi_*\mathcal{O}_{S^{[n]}} \cong \mathcal{O}_{\text{Sym}^n S}$, it induces the natural isomorphism

\begin{equation}
\pi^*: H^0(\text{Sym}^n S, \omega_{\text{Sym}^n S}) \to H^0(S^{[n]}, \omega_{S^{[n]}}).
\end{equation}

This in particular implies the following identification of closed subschemes of $S^{[n]}$

\begin{equation}
\pi^{-1}(Bs|\omega_{\text{Sym}^n S}|) = Bs|\omega_{S^{[n]}}|.
\end{equation}

To compute the canonical base locus of $\text{Sym}^n S$ we can use the identification

\begin{equation}
H^0(\text{Sym}^n S, \omega_{\text{Sym}^n S}) \cong H^0(S^n, \omega_{S^n})^G_n
\end{equation}

induced by the isomorphism of line bundles $q^*\omega_{\text{Sym}^n S} \cong \omega_{S^n}$, where $(-)^G_n$ are the $G_n$-invariant sections, and $q: S^n \to \text{Sym}^n S$ is the quotient map. For each section $t \in H^0(\text{Sym}^n S, \omega_{\text{Sym}^n S})$, the equality $q^{-1}(V(t)) = V(q^*(t))$ of closed subschemes of $S^n$ holds.

By assumption $Bs|\omega_S| = \emptyset$. Let $z \in \text{Sym}^n S$, and let $(p_1, \ldots, p_n) \in S^n$ be a point such that $q(p_1, \ldots, p_n) = z$. A general section $s \in H^0(S, \omega_S)$ avoids $p_1, \ldots, p_n$ hence, if $\pi_i: S^n \to S$ denotes the projection onto the $i$th factor, the $G_n$-invariant section $s \otimes \cdots \otimes s = \pi_1^*(s) \otimes \cdots \otimes \pi_n^*(s)$ of $\omega_{S^n}$ is non-zero at $(p_1, \ldots, p_n)$. This implies that the corresponding global section of $\omega_{\text{Sym}^n S}$ does not vanish at $z$. Hence $Bs|\omega_{\text{Sym}^n S}| = \emptyset$, and we are done by (5.3). $\square$

**Remark 5.2.** This result is to be contrasted with the behaviour of the canonical base locus for symmetric powers of curves, where the power $n$ plays an important role. The difference between the two cases is that the action of the symmetric group on $C^n$ has as its non-free locus the big diagonal, which has codimension 1. Hence the quotient morphism $C^n \to \text{Sym}^n C$ has a ramification divisor. On the other hand, the action on $S^n$ gives rise to a quotient morphism $S^n \to \text{Sym}^n S$ which is étale in codimension 1, as the non-free locus has codimension 2.

The description of the canonical base locus in Proposition 5.1 shows indecomposability of the derived category of the Hilbert scheme $S^{[n]}$ for any surface $S$ with $Bs|\omega_S| = \emptyset$. Of course, there are several classes of surfaces of general type for which this is the case.
When instead $B_S|\omega_S| \neq \emptyset$, we can sometimes deduce the indecomposability of $D^b(S^{[n]})$ by combining Proposition 5.1 and Corollary B and arguing by degeneration. Namely, suppose to have a smooth family $f: X \rightarrow T$ of projective surfaces such that $f^{-1}(t_0) = S$ and $B_S|\omega_{f^{-1}(t_1)}| = \emptyset$ for some $t_0, t_1 \in T$, and let $f_n: X^{[n]} \rightarrow T$ be the relative Hilbert scheme. Thus Proposition 5.1 ensures that the canonical base locus of the fibre $f_n^{-1}(t_1)$ is empty, and we conclude by Corollary B that $f_n^{-1}(t)$ has indecomposable derived category for any $t \in T$. One instance of this is given by the following.

**Proposition 5.3.** Let $S$ be a Horikawa surface in one of the following cases:
- for $c_1^2 = 2p_g - 4$ and $p_g \geq 3$;
- for $c_1^2 = 2p_g - 3$ and $p_g = 3$;
- for $c_1^2 = 2p_g - 2$ and either $p_g = 3$, or $p_g = 4$ and types I, II, III, IV, or V.

Then for all $n \geq 2$ we have that $D^b(S^{[n]})$ is indecomposable.

**Proof.** For $c_1^2 = 2p_g - 4$ the assertion follows from Proposition 5.1 and Theorem 1.1.

For $c_1^2 = 2p_g - 3$ we apply Corollary B to the family the Hilbert schemes associated to a family of Horikawa surfaces whose generic member is of type I, whose canonical base locus is empty.

For $c_1^2 = 2p_g - 2$ and $p_g = 3$ we use that surfaces of type I have empty canonical base locus, so that Corollary B can be applied to a family whose generic member is of this type. The same argument works for $p_g = 4$, by inspecting the degeneration diagrams (A.1), (A.2). \[\square\]

The ‘types’ mentioned in the previous statement, as well as their relationships in terms of degeneration, are recalled in Appendix A.

**Remark 5.4.** In the light of Remark 4.5, we may argue analogously in order to achieve indecomposability of $D^b(S^{[n]})$, when $S_h$ is a double covering of $\mathbb{P}^2$ as in Section 4.2.

**Remark 5.5.** In higher dimensions the Hilbert scheme of points $X^{[n]}$ is usually singular, except for $n \leq 3$. But a similar indecomposability result is not possible: a semiorthogonal decomposition of $X^{[2]}$ for $\dim X \geq 3$ is exhibited in [25, Theorem B]. In this case the Hilbert–Chow morphism is no longer crepant, hence the canonical line bundle is no longer nef. For $n = 3$ a similar picture is expected (and it is straightforward to write down the expected form), but no semiorthogonal decomposition has been constructed yet.

**APPENDIX A. ON HORIKAWA SURFACES**

In this appendix we summarise the classification of Horikawa surfaces, the description of their canonical base loci, and their behaviour under degeneration. This is used to show that all Horikawa surfaces have indecomposable derived category (Theorem 4.1), and to deduce indecomposability for Hilbert schemes of points on some classes of Horikawa surfaces (Proposition 5.3).

**The case** $c_1^2 = 2p_g - 4$. For surfaces on the Noether line, we have by [17, Lemma 1.1] that the canonical base locus $B_S|\omega_S|$ is empty.

**The case** $c_1^2 = 2p_g - 3$. The canonical base locus now depends on the value of $p_g \geq 2$, and its description is summarised as follows:
• if $p_g = 2$, then as in the proof of [18] Lemma 2.1 the canonical base locus consists of a single point;
• if $p_g = 3$, then by [18] Theorem 2.3 the canonical base locus is either empty (type I) or consists of a single point (type II), and by [18] Lemma 5.3 we know that every surface of type II can be obtained as deformation of surfaces of type I;
• if $p_g = 4$, then by [16] Theorem 1 the canonical base locus is either empty or consists of a single point;
• if $p_g \geq 5$, then by [18] Theorem 1.3 the canonical base locus consists of a single point.

For use in Section 5 we highlight the behaviour for $p_g = 3$, where we have a deformation from empty canonical base locus to non-empty canonical base locus.

The case $c_1^2 = 2p_g - 2$. This is the most interesting case for our purposes. We do not give an exhaustive description, but rather focus on the cases $p_g = 2$ and $p_g = 4$ because the former appears in [24] Example 4.6 and the latter features all interesting behavior for the canonical base locus under deformation. If $A$ and $B$ denote classes of surfaces, then $A \rightarrow B$ means that any surface of type $B$ is the special member of some smooth family of surfaces of type $A$.

- If $p_g = 2$, there are three types, I, II and III. By the discussion in [20] §1 the canonical base locus is either at most two points (type I or II) or a rational curve of self-intersection $-2$ (type III).
  
  By [20] Theorem 1.5(ii) we know that every surface of type III can be obtained as deformation of surfaces of type II, which in turn are obtained from surfaces of type I.

- If $p_g = 4$, there exist several types depending on the behavior of the canonical morphism and its indeterminacy locus, which we have summarised in the following two diagrams. These combine [19] Theorems 4.1, 6.1 and 6.2 and [5] Theorem 0.1. The canonical base locus can be empty, consist of two distinct points, consist of a double point or have fixed part $F$ which is a rational curve of self-intersection $-2$.

\begin{align}
&\text{I}_a : \text{Bs} = \emptyset & \text{IV}_{a-1} : \#\text{Bs} = 2 \\
&\quad \downarrow & \quad \downarrow \\
&\text{IV}_{b-1} : \#\text{Bs} = 2 & \text{IV}_{a-2} : \text{Bs} = F, F^2 = -2 \\
&\quad \downarrow & \downarrow \\
&\text{I}_b : \text{Bs} = \emptyset & \text{IV}_{b-2} : \text{Bs} = F, F^2 = -2 \\
(A.1) & & \\

&\text{III}_a : \#\text{Bs} = 2 \\
&\quad \downarrow \\
&\text{II} : \text{Bs} = \emptyset \rightarrow \text{III}_b : \text{Bs} = k[\epsilon]/(\epsilon^2) \rightarrow V_1 : \#\text{Bs} = 2 \\
(A.2) & & \rightarrow V_2 : \text{Bs} = F, F^2 = -2
\end{align}
For \( p_g = 3 \) the analysis started in [20] §2 is completed in [33] §2, giving rise to 4 types of canonical base locus. Using the notation from [33] §4, type I has empty canonical base locus, type II a single point, type III two distinct points and type \( \Pi_a \) a rational curve \( F \) with \( F^2 = -2 \). There exist degenerations \( I \rightsquigarrow II \) and \( I \rightsquigarrow III \rightsquigarrow \Pi_a \). This suffices to perform an analysis similar to the one for \( p_g = 4 \). For \( p_g \geq 5 \) one is referred to [20] §5, §6 for the proofs that the canonical base locus is either finite, or has fixed part a rational curve of self-intersection \(-2\), and the degeneration hierarchies.

In Section 5 the deformation (for \( p_g = 4 \)) of type II into type \( \Pi_b \) (obtained in [3] Theorem 0.1) and of type \( \Pi_a \) into type \( IV_{b-1} \), deforming from empty base locus to non-empty base locus, will be used.

**Appendix B. On Ciliberto double planes**

In this appendix we retrace and discuss a construction due to Ciliberto (see [11] Esempio 4.3) of families of surfaces of general type, with interesting properties both from the viewpoint of the indecomposability criteria used in the main body of this paper, and from the perspective of classification of surfaces of general type.

The surfaces are obtained as minimal desingularisations of double covers of \( \mathbb{P}^2 \) branched along certain plane curves with prescribed singularities. The existence of the branch curves is ensured by the following proposition, of which we include a proof for the sake of completeness.

As it is customary, given two integers \( a \geq b \geq 0 \), we say that a plane curve \( C \subset \mathbb{P}^2 \) has a point of type \([a,b]\) at \( p \in C \) if the curve has a singularity of multiplicity \( a \) at \( p \) and a singularity of multiplicity \( b \) at a point infinitely near to \( p \). In particular, the general curve having a point of type \([a,b]\) at \( p \) has exactly \( a \) smooth branches passing through \( p \), where \( b \) of them are simply tangent to the same line and the remaining \( a - b \) branches have distinct tangent directions.

**Proposition B.1.** For every integer \( h \geq 1 \) there exists an irreducible curve \( C_h \subset \mathbb{P}^2 \) of degree \( 8h + 4 \) having

- an ordinary singularity of multiplicity \( 4h + 2 \) at \( p_1 \),
- an ordinary singularity of multiplicity \( 4h \) at \( p_2 \),
- four singularities of type \([2h+1,2h+1]\) at \( p_3, \ldots, p_6 \),

and no other singularities, where the points \( p_1, \ldots, p_6 \) lie on an irreducible conic \( \Gamma \subset \mathbb{P}^2 \) transverse to \( C_h \) at any \( p_i \).

**Proof.** Let \( \Delta \subset \mathbb{P}^2 \) be an irreducible quartic having two nodes at \( p_1, p_2 \in \mathbb{P}^2 \). Let \( \Gamma \subset \mathbb{P}^2 \) be an irreducible conic passing through \( p_1, p_2 \) and intersecting \( \Delta \) transversely at four other points \( p_3, \ldots, p_6 \). Let \( \mathcal{L}_h \) be the linear system of curves of degree \( 8h + 4 \) having a \((4h+2)\)-tuple point at \( p_1 \), a \(4h\)-tuple point at \( p_2 \), and points of type \([2h+1,2h+1]\) at \( p_3, \ldots, p_6 \) with tangent directions given by the tangent lines of \( \Delta \) at \( p_3, \ldots, p_6 \). We need to prove that \( \mathcal{L}_h \) contains irreducible curves. To this aim we define two curves \( A := (2h+1)\Delta \) and \( B := 4h\Gamma + \Sigma + \ell \), where \( \ell \) is a general line through \( p_1 \), and \( \Sigma \) is a cubic passing through \( p_1 \) and tangent to \( \Delta \) at \( p_3, \ldots, p_6 \), which exists by a dimension count. We note that \( A, B \in \mathcal{L}_h \), and we consider the pencil \( \Phi := \alpha A + \beta B \).

Since \( \Phi \) has no fixed curves, Bertini’s second theorem implies that the general curve of the pencil is reducible if and only if \( \Phi \) is composed with a pencil, i.e. if there exists a pencil \( \Psi = \lambda C + \mu C' \) and \( n \geq 2 \) such that \( A = C_1 \cup \cdots \cup C_n \).
and $B = C'_1 \cup \cdots \cup C'_n$ for some $C_j, C'_j \in \Psi$. Suppose by contradiction that this is the case. Hence we have that $C_1 = k\Delta$ for some integer $k$, and we can set $C'_1$ to be the component of $B$ containing $\ell$. Then $\Psi = \lambda C_1 + \mu C'_1$, and any curve of $\Psi$ must pass through $C_1 \cap C'_1$. However, $C'_1$ is the unique component of $B$ passing through all the points of $\Delta \cap \ell$, a contradiction.

Furthermore, the general curve of $\Phi$ is smooth at the base points outside $p_1, \ldots, p_6$, because $B$ is. Thus we conclude by Bertini’s theorem that $L_h$ contains irreducible curves having singularities only at $p_1, \ldots, p_6$, and the assertion follows. $\square$

Fixing an integer $h \geq 1$, we consider a curve $C_h \subset \mathbb{P}^2$ as in Proposition B.1 and the double plane $\pi' : S'_h \to \mathbb{P}^2$ branched along $C_h$. Then we define the surface $S_h$ to be the minimal desingularisation of $S'_h$, which is endowed with a generically finite morphism of degree 2 induced by $\pi'$,

$$\pi : S_h \to \mathbb{P}^2.$$  

Remark B.2. The image under $\pi$ of the canonical linear series $|\omega_{S_h}|$ is the linear system $D_h$ of curves $D_h \subset \mathbb{P}^2$ of degree $4h - 1$ having a 2$h$-tuple point at $p_1$, a $(2h - 1)$-tuple point at $p_2$, and points of type $[h, h - 1]$ at $p_3, \ldots, p_6$ (cfr. e. g. [3, Theorem III.7.2]). Therefore, the conic $\Gamma$ is a fixed component of $D_h$ by Bézout’s theorem. Thus the curve $F := \pi^{-1}(\Gamma)$ is a fixed curve of $|\omega_{S_h}|$, which can be proved to be a smooth irreducible curve of genus 2 with $F^2 = 0$.

So the curves in $D_h$ are of the form $D_h = \Gamma + E_h$, where $E_h \subset \mathbb{P}^2$ varies in the linear system $E_h$ of curves of degree $4h - 3$ with a $(2h - 1)$-tuple point at $p_1$, a $(2h - 2)$-tuple point at $p_2$, and points of type $[h - 1, h - 1]$ at $p_3, \ldots, p_6$. Therefore, counting conditions imposed by the prescribed singularities of $E_h$, we deduce

$$\dim E_h \geq \frac{(4h - 3)4h}{2} - \frac{(2h - 1)2h}{2} - \frac{(2h - 2)(2h - 1)}{2} - 4 \cdot \frac{2(h - 1)h}{2} = 2h - 1,$$

and noting that the curves of $E_h$ passing through two extra points $p_7, p_8 \in \Gamma$ are of the form $2\Gamma + E_{h-1}$, one concludes by induction on $h$ that (B.2) is actually an equality. Thus the geometric genus of $S_h$ is $p_g(S_h) = \dim D_h + 1 = \dim E_h + 1 = 2h$, whereas the irregularity is $q(S_h) = 0$ (see e.g. [3, Section V.22]).

Finally, we consider the linear system $L_h$ appearing in the proof of Proposition B.1 consisting of plane curves of degree $8h + 4$ having a $(4h + 2)$-tuple point at $p_1$, a $4h$-tuple point at $p_2$, and points of type $[2h + 1, 2h + 1]$ at $p_3, \ldots, p_6$ (with fixed tangent directions), where $p_1, \ldots, p_6$ lie on the irreducible conic $\Gamma$. In particular, we compute the dimension of $L_h$, which is involved in the degeneration argument of Section 4.2.

Lemma B.3. Given an integer $h \geq 1$ and using the notation above, the linear system $L_h$ has dimension $8h + 3$.

Proof. Arguing as in (B.2), we note that

$$\dim L_h \geq \frac{(8h + 4)(8h + 7)}{2} - \frac{(4h + 2)(4h + 3)}{2} - \frac{4h(4h + 1)}{2} - 4 \cdot \frac{2(2h + 1)(2h + 2)}{2} = 8h + 3.$$

In order to prove that (B.3) is actually an equality, we consider a general element $C = C_h$ of $L_h$ as in Proposition B.1 and the blowup $\widetilde{X} := \text{Bl}_{p_1, \ldots, p_6} \mathbb{P}^2$ of $\mathbb{P}^2$ at
the points \( p_1, \ldots, p_6 \), with exceptional divisors by \( \tilde{E}_1, \ldots, \tilde{E}_6 \). Let \( q_3, \ldots, q_6 \) be the points on \( \tilde{E}_3, \ldots, \tilde{E}_6 \) corresponding to the common tangent directions of the \( 2h+1 \) branches of \( C \) at \( p_3, \ldots, p_6 \), and let \( X := \text{Bl}_{q_3,\ldots,q_6} \bar{X} \) be the blowup at these points, with exceptional divisors by \( F_3, \ldots, F_6 \). We denote by \( E_1, \ldots, E_6 \subset X \) the strict transforms of \( \tilde{E}_1, \ldots, \tilde{E}_6 \), so that \( E_1^2 = E_2^2 = -1, E_3^2 = \cdots = E_6^2 = -2 \) and \( E_i \cdot F_j = \delta_{ij} \).

Let \( \mathcal{L}_h^* \) be the linear system obtained on \( X \) by taking strict transforms of the curves in \( \mathcal{L}_h \), and let \( D \in \mathcal{L}_h^* \) be the strict transform of \( C \), which is a smooth irreducible curve, as the singularities of \( C \) have been resolved by the sequence of blowups. Denoting by \( H \) the class of the strict transform of a line in \( \mathbb{P}^2 \), the numerical equivalence classes of a canonical divisor \( K_X \) and \( D \) can be easily computed to be

\[
\begin{align*}
K_X & \equiv -3H + \sum_{i=1}^{6} E_i + 2 \sum_{j=3}^{6} F_j \\
D & \equiv (8h+4)H - (4h+2)E_1 - 4hE_2 - (2h+1) \sum_{j=3}^{6} E_j - 2(2h+1) \sum_{j=3}^{6} F_j.
\end{align*}
\]

Hence \( D^2 = 16h+4 \) and \( D \cdot K_X = -2 \), so that \( g(D) = 1 + \frac{1}{2}(D^2 + D \cdot K_X) = 8h+2 \) by the adjunction formula.

Finally, we consider the characteristic linear series \( \mathcal{L}_h^*|_D \) cut out on \( D \) by the other curves in \( \mathcal{L}_h^* \), so that

\[
\begin{align*}
\deg \mathcal{L}_h^*|_D &= D^2 = 16h+4 = 2g(D) \quad \text{and} \quad \dim \mathcal{L}_h^*|_D = \dim \mathcal{L}_h^* - 1 \geq 8h+2 \\
\text{by (B.4).}
\end{align*}
\]

On the other hand, Riemann–Roch theorem yields that

\[
\dim \mathcal{L}_h^*|_D \leq \deg \mathcal{L}_h^*|_D - g(D) = 8h+2.
\]

Thus we conclude that \( \dim \mathcal{L}_h = \dim \mathcal{L}_h^* = \dim \mathcal{L}_h^*|_D + 1 = 8h+3 \), as claimed. \( \square \)

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