Euler top and freedom in supersymmetrization of one-dimensional mechanics

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Recently A. Galajinsky has suggested the $N = 1$ supersymmetric extension of Euler top and made a few interesting observations on its properties \[1\]. In this paper we use the formulation of the Euler top as a system on complex projective plane, playing the role of phase space, i.e. as a one-dimensional mechanical system. Then we suggest the supersymmetrization scheme of the generic one-dimensional systems with positive Hamiltonian which yields à priori integrable family of $N = 2k$ supersymmetric Hamiltonians parameterized by $N/2$ arbitrary real functions.

I. INTRODUCTION

The Euler top is a textbook integrable system describing the rotation of free rigid body, see, e.g. \[2\]. Conventionally it is described by the Hamiltonian system with degenerated Poisson brackets parameterized by the components of angular momentum $\ell = (x_1, x_2, x_3)$,

$$\{x_i, x_j\} = \varepsilon_{ijk} x_k, \quad H = \sum_{i=1}^{3} \frac{x_i^2}{2I_i},$$

where $H$ is the Hamiltonian, and $I_i > 0$ are the principal momenta of inertia. Since $x_i$ form $so(3)$ algebra, the system has a Casimir function

$$C = \sum_{i=1}^{3} x_i^2 : \{C(x), x_i\} = 0$$

Its fixation leads to the Hamiltonian system with two-dimensional non-degenerated phase space, i.e. one-dimensional system. Hence, Euler top is a priori integrable. Being introduced centuries ago, Euler top has been studied as completely as the one-dimensional oscillator both at classical and quantum-mechanical levels. So, none of the open questions is to be studied there, except various aspects of its perturbations and generalizations.

Very recently A. Galajinsky noticed the absence of the relevant supersymmetric extensions of Euler top \[1\]. He suggested its $N = 1$ supersymmetrization via extension of the degenerated Poisson brackets (1) by three real Grassmann coordinates, stating that in general the resulting system lacks integrability. It seems to us that many questions asked in that paper come from the improper supersymmetrization procedure formulated in terms of degenerated phase space. As a consequence, it yields overcompleted number of fermionic variables which do not have impact on the actual properties of the system. Moreover, the invention of three Grassmann variables could yield the problems with the physical interpretation of the quantized version of system (though quantum aspects were not touched in that paper). Furthermore, the $N \geq 2$ supersymmetric extensions of the Euler top, which at the quantum level create qualitative corrections to the initial spectrum (e.g. degeneracy of the energy levels etc), were not considered there at all.

In this paper we propose to supersymmetrize the Euler top formulated in terms of nondegenerated phase space. First, fixing the value of the Casimir (2), we formulate the Hamiltonian system (1) in terms of two-dimensional nondegenerated phase space given by the complex projective plane $\mathbb{CP}^1$, i.e. as a one-dimensional mechanical system. Since any two-dimensional manifold can be provided with the Kähler structure, the initial system can be quantized by the well-developed technique of geometric quantization on Kähler manifold (see, e.g. \[3\]).

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Then we present the procedure of $N = 2k$ supersymmetry of the systems with generic two-dimensional nondegenerated phase space which results in á priori integrable supersymmetric extension of the initial system. Suggested procedure provides us with the family of the supersymmetric systems parameterized by the $N/2$ angle-like arbitrary functions. Similar functional freedom was noticed earlier in the one-dimensional $N = 2$ supersymmetric mechanics [3] and in the two-dimensional $N = 4$ supersymmetric ones [5]. In proposed supersymmetrization scheme the fermionic variables are splitted from the bosonic ones, in contrast with $N = 2$ supersymmetrization procedure of the systems with generic Kähler phase space suggested in [6]. They form, with respect to the Poisson brackets, the Clifford algebra. Thus, quantization of the supersymmetric system is straightforward: we should perform geometric quantization of the initial bosonic system and then replace the fermionic variables by the respective gamma-matrices.

The paper is organized as follows.
In Section 2 we review the description of Euler top on the phase space given by the complex projective plane $\mathbb{C}P^1$.
In Section 3 we present the $N = 2k$ supersymmetry scheme for the system with generic one-(complex)dimensional Kähler spaces.
In Section 4 we summarize obtained results and discuss the possible extensions of the proposed scheme to the Lagrange and Kowalewski tops.

II. EULER TOP

For the description of the Euler top (1) in terms of the nondegerated phase space, let us introduce instead of $x_i$, the coordinates $j, z, \bar{z}$

$$j := \sqrt{\sum_{i=1}^{3} x_i^2}, \quad z := \frac{x_1 + ix_2}{j - x_3}. \quad (3)$$

Clearly, $j$ is the complete angular momentum.

In these coordinates the Poisson brackets read

$$\{\bar{z}, z\} = -\frac{i}{2j} (1 + z\bar{z})^2, \quad \{z, j\} = \{z, z\} = 0, \quad (4)$$

while the momentum generators look as follows

$$j^2 := \sum_{i=1}^{3} x_i^2, \quad x_1 := h_1 = j \frac{z + \bar{z}}{1 + z\bar{z}}, \quad x_2 := h_2 = j \frac{(\bar{z} - z)}{1 + z\bar{z}}, \quad x_3 := h_3 = j \frac{z\bar{z} - 1}{1 + z\bar{z}}. \quad (5)$$

However, the point $x_3 = j$ cannot be described in these terms. To improve this lack we should introduce, instead of $z$, another coordinates $\tilde{z}, \bar{\tilde{z}}$

$$\tilde{z} := \frac{x_1 - ix_2}{j + x_3}; \quad \{\bar{\tilde{z}}, \tilde{z}\} = -\frac{i}{2j} (1 + \tilde{z}\bar{\tilde{z}})^2. \quad (6)$$

Out of the points $x_3 = \pm j$ these coordinates are related with each other as follows

$$\tilde{z} = \frac{1}{z}. \quad (7)$$

The Poisson brackets (4) and (6) transform to each other under this transformation. Thus, fixing $j$ to be constant we arrive at the two-dimensional phase space covered by two charts (parameterized by the single complex coordinate $z$ or $\tilde{z}$) and equipped with the one-(complex)dimensional Kähler structure - the complex projective plane $\mathbb{C}P^1$ with the Fubini-Study metrics

$$g(\tilde{z}, \bar{\tilde{z}})d\tilde{z}d\bar{\tilde{z}} := 2j \frac{d\tilde{z}d\bar{\tilde{z}}}{(1 + z\bar{z})^2}, \quad (8)$$

which corresponds to the Kähler potential

$$K(z, \tilde{z}) = 2j \log(1 + z\bar{z}). \quad (9)$$
The generators (5) become Killing potentials of \( \mathbb{C}P^1 \), and define the Hamiltonian holomorphic vector fields,
\[
\{ h_1, \} = i(1 - z^2)\partial_z + \text{c.c.}, \quad \{ h_2, \} = -(1 + z^2)\partial_z + \text{c.c.}, \quad \{ h_3, \} = 2iz\partial_z + \text{c.c.},
\]
In these terms the Hamiltonian of Euler top reads
\[
H = \sum_{i=1}^{3} \frac{x_i^2}{2I_i} = -j\frac{b(z^2 + \bar{z}^2) + 2az\bar{z}}{2(1 + z\bar{z})^2} + \frac{j^2}{2I_3},
\]
where
\[
a := \frac{2}{I_3} - \frac{1}{I_1} - \frac{1}{I_2}, \quad b := \frac{1}{I_2} - \frac{1}{I_1}.
\]

Now, let us rewrite the Euler top in canonical coordinates. For this purpose we notice that \( \mathbb{C}P^1 \) is just the two-dimensional sphere \( S^2 \) formulated in the projective coordinates
\[
z = \cot \frac{\theta}{2}e^{i\varphi},
\]
where \( \theta, \varphi \) are the spherical coordinates. In these terms the Poisson bracket (4) reads
\[
\{ \varphi, j \cos \theta \} = 1
\]
Hence, the function \( p := j \cos \theta \) defines the canonical momentum conjugated to \( \varphi \). In terms of these canonical coordinates the angular momentum generators (and Killing potentials) (5) read
\[
x_1 + ix_2 = j \sin \theta e^{i\varphi} = \sqrt{b^2 - p^2}e^{i\varphi}, \quad x_3 = j \cos \theta = p,
\]
while the Hamiltonian of the Euler top takes the form
\[
H = \frac{1}{4}(a + b \cos 2\varphi)p^2 + j^2 \frac{2}{4} \left( \frac{2}{I_3} - (a + b \cos 2\varphi) \right).
\]

Without loss of generality we assume
\[
I_3 \leq I_2 \leq I_1,
\]
and perform canonical transformation \((p, \varphi) \rightarrow (P, Q)\), where
\[
P = \sqrt{\frac{a + b \cos 2\varphi}{2}} p, \quad Q = \sqrt{\frac{2}{a + b}} \int \frac{d\varphi}{\sqrt{1 - \frac{2b}{a+b} \sin^2 \varphi}} = \sqrt{\frac{2}{a + b}} F(\varphi, k): \quad \{Q, P\} = 1,
\]
where \( F(\varphi, k) \) is an elliptic integral of the first kind, with \( k = \sqrt{2b/(a+b)} \) being its modulus, and \( \varphi \) is the so-called Jacobi amplitude (which defines the Jacobi elliptic functions) \[2\],
\[
\varphi = F^{-1}(F, k) = \text{amp}(F, k).
\]
Respectively
\[
\sin \varphi = \sin(\text{amp}(F, k)) = \text{sn}(F, k)
\]
is the Jacobi sine amplitude of the elliptic functions.

In this terms the Hamiltonian of Euler top reads
\[
H = \frac{1}{2}p^2 + \frac{j^2b}{2} \text{sn}^2 \left( \sqrt{\frac{a + b}{2}} Q, \sqrt{\frac{2b}{a+b}} \right) + \frac{j^2}{2I_3}.
\]
In the particular case of the symmetric top \((I_1 = I_2 := I)\) it reduces to the one-dimensional free particle Hamiltonian
\[
H = \frac{1}{2} \left( \frac{1}{I_3} - \frac{1}{I} \right) p^2 + \frac{j^2}{2I}.
\]

So, the Euler top is the one-dimensional Hamiltonian system with \( \mathbb{C}P^1 \) phase space and with the Hamiltonian given by the quadratic functions of its Killing potentials. In the canonical coordinates it results in the one-dimensional nonlinear oscillator.
III. SUPERSYMMETRY

In the previous section we formulated the Euler top in terms of one-(complex)dimensional phase space given by complex projective plane. Being one-dimensional system, the Euler top allows many ways of supersymmetrization, including supersymmetrization in canonical coordinates. However, we are interested in the supersymmetrization compatible with the Kähler geometry describing the phase space of the Euler top.

One of the ways to supersymmetrize the Euler top is to use the approach suggested in [8], which is based on the extension of the Kähler phase space to the super-Kähler one defined by the potential $K(z, \bar{z}, \theta_a, \bar{\theta}^a) = K(z, \bar{z}) + F(ig(z, \bar{z})\theta_a\bar{\theta}^a)$, where $F(x)$ is the real function with $F'(0) \neq 0$, with $K(z, \bar{z})$, $g(z, \bar{z})$ given by (9) and (8), while the fermionic variables $\theta_a$ are associated with $dz$, in complete similarity with the superfield approach.

Another particular way of supersymmetrization is to extend the complex projective plane to the complex projective $\mathbb{C}P^2$ for the construction of $\mathcal{N}$-extended one-dimensional superconformal mechanics. Later on, this approach was generalized to the higher-dimensional systems in [10]. Below we suggest different, less geometric approach, which is applicable not only for the Euler top, but for any one-dimensional system. We will consider the systems with generic two-(real)dimensional phase space. Such phase spaces can be always equipped with the one-dimensional Kähler structure, so that the Poisson brackets will be given by the relation

$$\{z, \bar{z}\} = \frac{i}{g(z, \bar{z})}. \tag{25}$$

For the construction of $\mathcal{N}$-supersymmetric extensions of these systems (with even $\mathcal{N}$) we extend this phase space by the canonical complex Grassmann variables $\psi_a$, $a = 1, \ldots, \mathcal{N}$

$$\{\psi_a, \psi^b\} = i\delta^b_a, \tag{26}$$

where $\bar{\psi}^a := \psi_a$ and $\mathcal{F}_1\mathcal{F}_2 = \mathcal{F}_2\mathcal{F}_1$.

With these Poisson brackets at hands we can construct the $\mathcal{N}$ supersymmetric extensions of two-dimensional systems defined by the Poisson brackets (26) and by any positive Hamiltonian $H(z, \bar{z}) > 0$,

$$\left\{Q_a, Q^b\right\} = i\delta^b_a H, \quad H := H(z, \bar{z}) + \text{fermions}. \tag{27}$$

In accordance with the generalization of Liouville theorem to the supermanifolds [10] (see also [4]) these supersymmetric extensions will be apriori integrable, since we will get the system with $(2|\mathcal{N})_{\mathbb{R}}$-dimensional phase space with one bosonic constant of motion $H$ and $\mathcal{N}$ fermionic constants of motion $Q_a, \bar{Q}^b$ commuting with the bosonic integral $H$.

$\mathcal{N} = 2$ supersymmetry

For the construction of $\mathcal{N} = 2$ supersymmetric extension of the system with Hamiltonian $H(z, \bar{z}) > 0$ we choose, following [4], the appropriate Ansatz for supercharges and arrive the family of $\mathcal{N} = 2$ supersymmetric extensions of the Hamiltonian $H$, parameterized by the arbitrary real function $\Phi(z, \bar{z})$

$$Q = \sqrt{H}e^{i\Phi}\psi, \quad \bar{Q} = \sqrt{H}e^{-i\Phi}\bar{\psi} \implies H = H + \{\Phi, H\}\bar{\psi}\psi. \tag{28}$$

Specifying the Poisson brackets and Hamiltonian we will get the respective supersymmetric extension of the Euler top.

Direct extension of construction to the $\mathcal{N} \geq 4$ supersymmetric mechanics fixes the function $\Phi$ and leads to the trivial family of the supersymmetric Hamiltonians. Namely, choosing $Q_a = \sqrt{H}e^{i\Phi}\psi_a$, we get that the superalgebra (27) is fulfilled when $\{H, \Phi\} = 0$. Hence, the resulting supersymmetric Hamiltonian is trivial: it coincides with the initial bosonic Hamiltonian.
\( \mathcal{N} = 4 \) supersymmetry

For the construction of nontrivial \( \mathcal{N} = 4 \) supersymmetric system we choose the following Ansatz for supercharges

\[
Q_a = f_1(z, \bar{z}) \psi_a + f_2(z, \bar{z}) \psi_a \sum_{b=1}^{N/2} \psi_b \bar{\psi}_b, \quad \overline{Q}^a = \bar{f}_1(z, \bar{z}) \bar{\psi}^a - \bar{f}_2(z, \bar{z}) \bar{\psi}^a \sum_{b=1}^{N/2} \psi^b \psi_b , \tag{29}
\]

with

\[
f_1(z, \bar{z}) := \sqrt{H} e^{\phi_1(z, \bar{z})}, \quad f_2 = R(z, \bar{z}) e^{(\phi_1 - \phi_2)} \quad \tilde{R} = R, \quad \Phi_a(z, \bar{z}) = \Phi_a(z, \bar{z}). \tag{30}
\]

Then, we require that the supercharges (29) form the \( \mathcal{N} = 4 \) Poincaré superalgebra (27), which results in the following conditions on the functions involved

\[
\{ f_1, \bar{f}_1 \} = f_1 \bar{f}_2 + \bar{f}_1 f_2 \quad \Leftrightarrow \quad \{ \sqrt{H}, \Phi_1 \} = R \cos \Phi_2, \tag{31}
\]

with the Hamiltonian \( \mathcal{H} \) acquiring the form

\[
\mathcal{H} = f_1 \bar{f}_1 + i \left\{ f_1, \bar{f}_1 \right\} \sum_{a=1}^{N/2} \psi_a \bar{\psi}_a + \frac{i}{2} \left( \left\{ f_1, \bar{f}_2 \right\} + \left\{ f_2, \bar{f}_1 \right\} \right) \left( \sum_{a=1}^{N/2} \psi_a \bar{\psi}_a \right)^2 \tag{32}
\]

\[
= H + \left\{ H, \Phi_1 \right\} \psi_a \bar{\psi}_a + A(\sqrt{H}, \Phi_{1,2}) \left( \sum_{a=1}^{N/2} \psi_a \bar{\psi}_a \right)^2 , \tag{33}
\]

with

\[
A(\sqrt{H}, \Phi_{1,2}) := \frac{i}{2} \left( \left\{ f_1, \bar{f}_2 \right\} + \left\{ f_2, \bar{f}_1 \right\} \right) = \left( \left\{ \sqrt{H}, \Phi_1 \right\} \right)^2 - \frac{\left\{ \sqrt{H}, \Phi_1 \right\} \left\{ \sqrt{H}, \Phi_2 \right\} }{\cos^2 \Phi_2} + \left\{ \sqrt{H}, \Phi_1 \right\} \sqrt{H} + \left\{ \sqrt{H}, \Phi_1 \right\} \sqrt{H} \tan \Phi_2. \tag{34}
\]

Thus, we get the \( \mathcal{N} = 4 \) supersymmetric mechanics parametrized by two arbitrary functions \( \Phi_{1,2} \).

We can use the Ansatz (29) for the construction of \( \mathcal{N} > 4 \) supersymmetric systems as well. However, in that case we get the additional constraints on the functions \( f_1, f_2 \),

\[
\mathcal{N} = 6 : \{ f_1, \bar{f}_2 \} + \{ f_2, \bar{f}_1 \} = 2i f_2 \bar{f}_2 \tag{36}
\]

\[
\mathcal{N} = 8 : \{ f_1, \bar{f}_2 \} + \{ f_2, \bar{f}_1 \} = 2i f_2 \bar{f}_2 , \quad \{ f_2, \bar{f}_2 \} = 0. \tag{37}
\]

These constraints, obviously, restrict the functional freedom existing in \( \mathcal{N} = 4 \) systems.

In the \( \mathcal{N} = 6 \) case this constraint leads to the following restriction

\[
A(\sqrt{H}, \Phi_1) + \left( \frac{\left\{ \sqrt{H}, \Phi_1 \right\}}{\cos \Phi_2} \right)^2 = 0, \tag{38}
\]

with \( A \) given by (33). Hence, the system has single functional degree of freedom parameterized by \( \Phi_1 \), as in the \( \mathcal{N} = 2 \) case.

The requirement of \( \mathcal{N} = 8 \) supersymmetry further fixes the value of \( \Phi_1 \)

\[
\{ \sqrt{H}, \Phi_1, - \Phi_2 \} = \left\{ \Phi_1, \Phi_2 \right\} \left\{ \sqrt{H}, \Phi_1 \right\} \tan \Phi_2. \tag{39}
\]

As a result, we get the \( \mathcal{N} = 8 \) Hamiltonian with no functional freedom.

The evident way to construct the \( \mathcal{N} > 4 \) systems with wide functional freedom is to extend the supercharges Ansatz by higher 5- and 7- fermionic terms.
IV. $\mathcal{N} = 6, 8$ SUPERSYMMETRIC MECHANICS

In the previous section, we have shown that the supercharges with cubic fermionic terms allow to construct $\mathcal{N} = 4$ supersymmetric mechanics with two functional degrees of freedom, $\mathcal{N} = 6$ supersymmetric mechanics with single functional degree of freedom, and $\mathcal{N} = 8$ supersymmetric mechanics without any functional freedom.

One can guess that the supercharges with fifth-order fermionic term could lead to the $\mathcal{N} = 6$ supersymmetric mechanics with three functional degrees of freedom and to $\mathcal{N} = 8$ supersymmetric mechanics with two functional degrees of freedom. Furthermore, one can expect that the supercharges with seventh-order fermionic terms could lead to the $\mathcal{N} = 8$ supersymmetric mechanics with four functional degrees of freedom and so on. Let us show that it is indeed the case.

In order to construct the $\mathcal{N} = 6$ supersymmetric systems with three functional degrees of freedom, we consider the following Ansatz for the supercharges

$$Q_a = f_1(z, \bar{z})\psi_a + f_2(z, \bar{z})\psi_a \left( \sum_{b=1}^{N/2} \psi_b \tilde{\psi}^b \right) + f_3(z, \bar{z})\psi_a \left( \sum_{b=1}^{N/2} \psi_b \bar{\psi}^b \right)^2,$$

$$\bar{Q}^a = \bar{f}_1(z, \bar{z})\bar{\psi}^a + \bar{f}_2(z, \bar{z})\bar{\psi}^a \left( \sum_{b=1}^{N/2} \psi_b \bar{\psi}^b \right) + \bar{f}_3(z, \bar{z})\bar{\psi}^a \left( \sum_{b=1}^{N/2} \psi_b \bar{\psi}^b \right)^2,$$

with $a, b = 1, 2, 3$.

Then, requiring that these functions form $\mathcal{N} = 6$ Poincaré superalgebra (27), we get the following restrictions to the functions $f_a$

$$f_1 \bar{f}_2 + \bar{f}_1 f_2 - i \left\{ f_1, \bar{f}_1 \right\} = 0, \quad 2f_3 \bar{f}_1 + 2f_2 \bar{f}_3 + 2f_1 \bar{f}_3 - i \left\{ f_1, \bar{f}_2 \right\} + i \left\{ f_1, \bar{f}_3 \right\} = 0,$$

(41)

The respective Hamiltonian then reads

$$\mathcal{H} = \frac{1}{2} f_1 \bar{f}_1 + \frac{1}{2} (f_1 \bar{f}_2 + \bar{f}_1 f_2) \sum_{a=1}^{N/2} \psi_a \bar{\psi}^a + \frac{1}{2} (f_3 \bar{f}_1 + f_1 \bar{f}_3 + \bar{f}_1 f_3) \left( \sum_{a=1}^{N/2} \psi_a \bar{\psi}^a \right)^2 + \frac{1}{2} (f_2 \bar{f}_3 + f_3 \bar{f}_2) \left( \sum_{a=1}^{N/2} \psi_a \bar{\psi}^a \right)^3,$$

(42)

Representing $f_a$ in the form

$$f_1 = \sqrt{\mathcal{H}} e^{\Phi_1}, \quad f_2 = R_2 e^{i(\Phi_1 - \Phi_2)}, \quad f_3 = R_3 e^{i(\Phi_1 - \Phi_2 - \Phi_3)},$$

(43)

and re-writing in these terms the conditions (41), we conclude that the functions $\Phi_1, \Phi_2, \Phi_3$ remains unfixed. Therefore, we arrive at the family of $\mathcal{N} = 6$ supersymmetric mechanics parameterized by three arbitrary real functions.

In order to construct the $\mathcal{N} = 8$ supersymmetric systems with four functional degrees of freedom, we introduce the following generalization of the Ansatz (29).

$$Q_a = f_1(z, \bar{z})\psi_a + f_2(z, \bar{z})\psi_a \left( \sum_{b=1}^{N/2} \psi_b \bar{\psi}^b \right) + f_3(z, \bar{z})\psi_a \left( \sum_{b=1}^{N/2} \psi_b \bar{\psi}^b \right)^2 + f_4(z, \bar{z})\psi_a \left( \sum_{b=1}^{N/2} \psi_b \bar{\psi}^b \right)^3,$$

$$\bar{Q}^a = \bar{f}_1(z, \bar{z})\bar{\psi}^a + \bar{f}_2(z, \bar{z})\bar{\psi}^a \left( \sum_{b=1}^{N/2} \psi_b \bar{\psi}^b \right) + \bar{f}_3(z, \bar{z})\bar{\psi}^a \left( \sum_{b=1}^{N/2} \psi_b \bar{\psi}^b \right)^2 + \bar{f}_4(z, \bar{z})\bar{\psi}^a \left( \sum_{b=1}^{N/2} \psi_b \bar{\psi}^b \right)^3,$$

(44)

with $a, b = 1, 2, 3, 4$.

Then, requiring that these functions form $\mathcal{N} = 8$ Poincaré superalgebra (27), we get the following restrictions on the functions $f_a$

$$f_1 \bar{f}_2 + \bar{f}_1 f_2 - i \left\{ f_1, \bar{f}_1 \right\} = 0, \quad 2f_3 \bar{f}_1 + 2f_2 \bar{f}_3 + 2f_1 \bar{f}_3 - i \left\{ f_1, \bar{f}_2 \right\} + i \left\{ f_1, \bar{f}_3 \right\} = 0,$$

$$3f_4 \bar{f}_1 + 3f_3 \bar{f}_2 + 3f_2 \bar{f}_3 + 3f_1 \bar{f}_4 - i \left\{ f_1, \bar{f}_3 \right\} + i \left\{ f_1, \bar{f}_4 \right\} - i \left\{ f_2, \bar{f}_2 \right\} = 0.$$

(45)

(46)
The respective Hamiltonian then reads
\[
\mathcal{H} = \frac{1}{2} f_1 \dot{f}_1 + \frac{1}{2} (f_1 \dot{f}_2 + \dot{f}_1 f_2) \left( \sum_{a=1}^{N/2} \psi^a \bar{\psi}^a \right) + \frac{1}{2} (f_3 \dot{f}_1 + f_1 \dot{f}_3 + f_2 \dot{f}_2) \left( \sum_{a=1}^{N/2} \psi^a \bar{\psi}^a \right)^2
\]
\[
+ \frac{1}{2} (f_4 \dot{f}_1 + f_1 \dot{f}_4 + f_2 \dot{f}_3 + f_3 \dot{f}_2) \left( \sum_{a=1}^{N/2} \psi^a \bar{\psi}^a \right)^3 + \frac{1}{8} \left( \{ f_1, \dot{f}_4 \} + \{ f_2, \dot{f}_3 \} + \{ f_3, \dot{f}_2 \} + \{ f_4, \dot{f}_1 \} \right) \left( \sum_{a=1}^{N/2} \psi^a \bar{\psi}^a \right)^4
\]

Let us notice that the restriction for \( N = 8 \) system coincides with the restrictions for \( N = 6 \) case. While the additional constraint contains extra complex function \( f_4(z, \bar{z}) \). Hence, representing \( f_a \) in the form
\[
f_1 = \sqrt{H} e^{i \Phi_1}, \quad f_2 = R_2 e^{i(\Phi_1 - \Phi_2)}, \quad f_3 = R_3 e^{i(\Phi_1 - \Phi_2 - \Phi_3)}, \quad f_4 = R_4 e^{i(\Phi_1 - \Phi_2 - \Phi_3 - \Phi_4)},
\]
we conclude that the functions \( \Phi_1, \ldots, \Phi_4 \) remain unfixed. Therefore, the \( N = 8 \) supersymmetric Hamiltonian depends on four arbitrary real functions.

So, specifying the formulae given in the Third and Fours Sections to the particular case of Euler top given in the Section 2 by \( \mathbb{S} \) we will get its integrable \( N = 2, 4, 6, 8 \) supersymmetric extensions.

From the consideration above it is easy to deduce that for the construction of \( N = 10, 12, \ldots, 2k \) superextensions of initial Hamiltonian we should choose the following Ansätze for the supercharges
\[
Q_a = f_1(z, \bar{z}) \psi_a + \sum_{l=1}^{N/2} f_{l+1}(z, \bar{z}) \bar{\psi}_b \left( \sum_{b=1}^{N/2} \psi^b \right)^l
\]
\[
\tilde{Q} = \bar{f}_1(z, \bar{z}) \bar{\psi}_a + \sum_{l=1}^{N/2} \bar{f}_{l+1}(z, \bar{z}) \psi^b \left( \sum_{b=1}^{N/2} \bar{\psi}_b \right)^l
\]
with \( a, b = 1, \ldots, N/2k \). Then, requiring that they form Poincaré superalgebra \( [\mathbb{S}] \) we will get the family of \( N = 2k \) supersymmetric Hamiltonians parameterized by \( k \) arbitrary real functions.

V. CONCLUDING REMARKS

In this paper we formulated the Euler top as a system with phase space \( \mathbb{CP}^1 \), i.e. as an one-dimensional system. Then we proposed the procedure of \( N = 2k \) \( \check{\text{a priori} } \) integrable supersymmetrization of the generic one-dimensional systems which provides the family of \( N \)-supersymmetric extensions depending on \( N/2 \) arbitrary real functions. Thus, we gave the \( N = 2k \) supersymmetric extensions of the Euler top as well.

One may ask whether it is possible to construct the family of supersymmetric extensions of the Lagrange and Kowalewski tops (see, e.g., \( [11] \)) which are parameterized by \( k \) arbitrary functions?

Here we present some preliminary remarks on this issue. The phase spaces of Lagrange and Kowalewski tops could be identified with cotangent bundle of complex projective plane. This supermanifold can be equipped with three symplectic (and complex) structures, parameterized by the coordinates \( u_A = (z, \pi) \),
\[
\omega_1 = 0\pi \wedge dz + 0\bar{\pi} \wedge d\bar{z}, \quad \omega_2 = id\pi \wedge dz - id\bar{\pi} \wedge d\bar{z},
\]
\[
\omega_3 = i^{n_{\pi} + n_{\bar{\pi}}} du^A \wedge d\bar{\mu}^B, \quad K = K(z, \bar{z}) + F(g^{-1}\pi, \bar{\pi}),
\]
with \( K(z, \bar{z}) \) and \( g(z, \bar{z}) \) are given by \( [9] \) and \( [8] \) respectively, while \( F(x) \) is the real function obeing condition \( F'(0) \neq 0 \). Within appropriate choice of the function \( F(x) \) these symplectic structures provide the manifold \( T^*\mathbb{CP}^1 \) with hyper-Kähler structure \( [12] \). Formulating Lagrange and Kowalewski tops in terms of symplectic structures \( [50] \), we can try to construct their conventional \( N = 2, 4 \) supersymmetric extensions, extending these simplectic structure by fermionic variables associated with \( dz \). However, we are expecting that using the symplectic structure \( [11] \) will be more useful for the construction of the supersymmetry extensions of Lagrange and Kowalewski tops.

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