Rigid $N=2$ superconformal hypermultiplets

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ABSTRACT

We discuss superconformally invariant systems of hypermultiplets coupled to gauge fields associated with target-space isometries.

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1 Introduction

Hypermultiplets played an important role in the work of Victor I. Ogievetsky, to whose memory this meeting is dedicated. We remember him as a devoted scientist, but above all as a dear friend and colleague who is sorely missed.

In this contribution we discuss hypermultiplets coupled to gauge fields whose action is invariant under rigid $N = 2$ superconformal symmetries. This study is both motivated by recent interest in superconformal theories [1] and by our attempts to understand the coupling of hypermultiplets to supergravity in a way that is more in parallel with the special-geometry formulation of vector multiplets [2]. In this respect it is important that we employ an on-shell treatment of hypermultiplets (to avoid an infinite number of fields), while the vector multiplets and the superconformal theory are considered fully off-shell. This implies that the algebra of the superconformal and gauge symmetries is known up to the hypermultiplet field equations.

2 Hypermultiplet Lagrangians

Hyper-Kähler spaces serve as target spaces for nonlinear sigma models based on hypermultiplets [3]. We start here by summarizing some results on the formulation of these theories following [2]. With respect to the results of [3] this formulation differs in that it incorporates both a metric $g_{AB}$ for the hyper-Kähler target space and a metric $G_{\bar{\alpha} \beta}$ for the fermions. Here we assume that the $n$ hypermultiplets are described by $4n$ real scalars $\phi^A$, $2n$ positive-chirality spinors $\zeta^\alpha$ and $2n$ negative-chirality spinors $\bar{\zeta}^{\bar{\alpha}}$. The latter two are related by complex conjugation (so that we have $2n$ Majorana spinors) under which indices are converted according to $\alpha \leftrightarrow \bar{\alpha}$, while SU(2) indices $i,j,\ldots$ are raised and lowered.

The presence of the fermionic metric is important in obtaining the correct transformation rules under symplectic transformations induced by the so-called c-map from the electric-magnetic duality transformations on a corresponding theory of vector multiplets. In formulations based on $N = 1$ superfields (such as in [4]) one naturally has a fermionic metric but of a special form.

The supersymmetry transformations are parametrized in terms of certain $\phi$-dependent quantities $\gamma^A$ and $V_A$ as

$$\delta_Q \phi^A = 2(\gamma^A_{\bar{\alpha}} \bar{\epsilon}^{\bar{\alpha}} \zeta^\alpha + \bar{\gamma}_\alpha^{Ai} \bar{\epsilon}_i \zeta^\alpha), \quad \delta_Q \zeta^\alpha = V_{A +}^\alpha \phi^A \epsilon^i - \delta_Q \phi^A \Gamma_{A,\alpha}^{\alpha} \zeta^\beta,$$

$$\delta_Q \bar{\zeta}^{\bar{\alpha}} = \bar{V}_{\bar{A}}^{\bar{\alpha}} \phi^A \epsilon_i - \delta_Q \phi^A \bar{\Gamma}^{\bar{\alpha}}_{\bar{A}} \bar{\zeta}^{\bar{\beta}}. \quad (1)$$

Observe that these variations are consistent with a $U(1)$ chiral invariance under which the scalars remain invariant, which we will denote by $U(1)_R$ to indicate that it is a subgroup of the automorphism group of the supersymmetry algebra. In section 4 this $U(1)$ will be included as one of the conformal gauge groups. However, for generic $\gamma^A$ and $V_A$, the SU(2)$_R$ part of the automorphism group
cannot be realized consistently on the fields. In the above, we only used that \( \zeta^\alpha \) and \( \zeta^\bar{\alpha} \) are related by complex conjugation.

The Lagrangian takes the following form

\[
\mathcal{L} = -\frac{1}{2} g_{AB} \partial_\mu \phi^A \partial^\mu \phi^B - G_{\alpha\beta} (\bar{\zeta}^\alpha \partial^\beta \zeta^\bar{\alpha} + \bar{\zeta}^\beta \partial^\alpha \zeta^\bar{\alpha}) - \frac{1}{4} W_{\alpha\beta\gamma\delta} \bar{\zeta}^\alpha \zeta^\beta \bar{\zeta}^\gamma \zeta^\delta,
\]

where we use the covariant derivatives

\[
D_\mu \zeta^\alpha = \partial_\mu \zeta^\alpha + \partial_\mu \phi^A \Gamma_{A\beta}^\alpha \zeta^\beta, \quad D_\mu \zeta^\bar{\alpha} = \partial_\mu \zeta^\bar{\alpha} + \partial_\mu \phi^A \bar{\Gamma}_{A\bar{\beta}}^\bar{\alpha} \zeta^\bar{\beta}.
\]

Besides the Riemann curvature \( R_{ABCD} \) we will be dealing with another curvature \( R_{AB}^{\alpha\beta} \) associated with the connections \( \Gamma_{A\beta}^\alpha \), which takes its values in \( sp(n) \cong usp(2n; \mathbb{C}) \). The tensor \( W \) is defined by

\[
W_{\alpha\beta\gamma\delta} = R_{AB} \, \gamma_{\alpha\alpha} \gamma_{\beta\beta} \, G_{\gamma\delta} = \frac{1}{2} R_{ABCD} \, \gamma_{\alpha\alpha} \gamma_{\beta\beta} \, \gamma_{C\gamma} \gamma_{D\delta}.
\]

Most of these quantities are not independent, as we shall specify below, and the models are entirely characterized by the target-space geometry (for instance, encoded in the metric \( g_{AB} \) and the \( Sp(n) \times Sp(1) \) one-forms \( V^\alpha_i = V^\alpha_{Ai} \, d\phi^A \). The \( Sp(1) \) factor is associated with the indices \( i, j, \ldots \), and coincides with the \( SU(2) \) group mentioned above.

The metric \( g_{AB} \), the tensors \( \gamma^A, \gamma_A \) and the fermionic metric \( G_{\alpha\beta} \) are all covariantly constant with respect to the Christoffel connection and the connections \( \Gamma_{A\beta}^\alpha \). Furthermore we note the following relations,

\[
\gamma^A_{\alpha\alpha} V^\beta_B + \gamma^A_{\alpha\beta} V^\alpha_B i = \delta^A_i \delta^A_B,
\]

\[
g_{AB} \gamma^B_{i\alpha} = G_{\alpha\beta} V^\beta_{Ai}, \quad V^\alpha_{Ai} \gamma^A_{j\beta} = \delta^j_i \delta^\beta_\beta.
\]

These conditions define a number of useful relations between bilinears\(^2\) which include three antisymmetric covariantly constant target-space tensors,

\[
J^{ij}_{AB} = \gamma_{A\alpha} \varepsilon^{k(i} V_{B}^{j)} \varepsilon^{\alpha},
\]

that span the complex structures of the hyper-Kähler target space. They satisfy

\[
(J^{ij}_{AB})^* = \varepsilon_{ik} \varepsilon_{jl} J^{kl}_{AB}, \quad J^{ij}_{AB} J^{kl}_{CB} = \frac{1}{2} \varepsilon^{i(k} \varepsilon^{l)j} g_{AB} + \varepsilon^{(i(k} J^{lj)}_{AB}).
\]

In addition we note the following useful identities,

\[
\gamma_{A\alpha} \bar{V}^j_B = \varepsilon_{ik} J^{kj}_{AB} + \frac{1}{2} g_{AB} \delta^j_i, \quad J^{ij}_{AB} \gamma_{A\alpha} = -\delta^j_i \varepsilon^{(i(k} J^{lj)}_{AB} \gamma_{A\alpha}.
\]

We also note the existence of covariantly constant antisymmetric tensors,

\[
\Omega_{\alpha\beta} = \frac{1}{2} \varepsilon^{ij} g_{AB} \gamma^A_{i\alpha} \gamma^B_{j\beta}, \quad \Omega^{\alpha\beta} = \frac{1}{2} \varepsilon^{ij} g^{AB} \bar{V}^i_{A\alpha} \bar{V}^j_{B\beta},
\]

\(^2\)Such as \( \gamma^{ij}_{Aa} V^a_{B_i} = \gamma_{Ba} \bar{V}^j_A = -\bar{\gamma}^{ij}_{Ba} V^a_{iA} + \delta^j_i g_{AB} \).
satisfying $\Omega^{\alpha\gamma} \bar{\Omega}^{\gamma\beta} = -\delta^\beta_\alpha$.

The existence of the covariantly constant tensors implies a variety of integrability conditions for the curvature tensors. For instance, one proves that the Riemann curvature and the $\text{Sp}(n)$ curvature are related, as indicated in (4). The tensor $W$ defined in (4) can also be written as $W_{\alpha\beta\gamma\delta}$ by contracting with the metric $G$ and the antisymmetric tensor $\Omega$. It then follows that $W_{\alpha\beta\gamma\delta}$ is symmetric in symmetric index pairs $(\alpha\beta)$ and $(\gamma\delta)$. Using the Bianchi identity for Riemann curvature, which implies $g_{D[A} R_{BC]}^{\beta} \gamma_{\alpha\delta} = 0$, one shows that it is in fact symmetric in all four indices. For further results and discussion we refer to [2].

In the next section we consider the gauging of invariances of the hypermultiplet action. Such invariances are related to isometries of the hyper-Kähler manifold. These isometries have been studied earlier in the literature [3, 4, 5, 6, 7, 8] but our purpose is to incorporate them into the set-up discussed in this section.

### 3 Hypermultiplets with gauged target-space isometries

The above Lagrangian and transformation rules are subject to two classes of equivalence transformations associated with the target space. One class consists of the target-space diffeomorphisms associated with $\phi \rightarrow \phi'(\phi)$. The other refers to reparametrizations of the fermion ‘frame’ of the form $\zeta^\alpha \rightarrow S^{\alpha}_{\beta}(\phi) \zeta^\beta$, and similar redefinitions of other quantities carrying indices $\alpha$ or $\bar{\alpha}$. For example, the fermionic metric transforms as $G_{\bar{\alpha}\beta} \rightarrow [S^{-1}]^{\alpha}_{\bar{\alpha}} [S^{-1}]^{\beta}_{\beta} G_{\bar{\alpha}\beta}$. Under these rotations the quantities $\Gamma_{A}\alpha\beta$ play the role of connections.

The above transformations do not constitute invariances of the theory. This is only the case when the metric $g_{AB}$ and and the $\text{Sp}(n) \times \text{Sp}(1)$ one-form $V_{i}^\alpha$ (and thus the related geometric quantities) are left invariant under (a subset of) them. To see how this works, let us consider the scalar fields transforming under a certain isometry (sub)group $G$ characterized by a number of Killing vectors $k_{I}^{A}(\phi)$, with parameters $\theta^{I}$. Hence under infinitesimal transformations,

$$\delta_{G} \phi^{A} = g \theta^{I} k_{I}^{A}(\phi) ,$$

where $g$ is the coupling constant and the $k_{I}^{A}(\phi)$ satisfy the Killing equation

$$D_{A} k_{IB} + D_{B} k_{IA} = 0 .$$

The quantities such as $V_{i}^{\alpha}$ that carry $\text{Sp}(n)$ indices are only required to be invariant under isometries up to fermionic equivalence transformations. Thus $-g(k_{I}^{B} \partial_{B} V_{i}^{\alpha} + \partial_{A} k_{I}^{B} V_{i}^{\alpha})$ must be cancelled by a suitable infinitesimal rotation on the index $\alpha$. Here we make the important assumption that the effect of the diffeomorphism is entirely compensated by a rotation that affects the indices $\alpha$. In principle, one can also allow a compensating $\text{Sp}(1)$ transformation acting on
the indices $i, j, \ldots$. However, we will not do this here, as this would imply that the isometry group would neither commute with Sp(1) nor with supersymmetry. Instead we return to this option in the next section.

Let us parametrize the compensating transformation acting on the Sp($n$) indices by $\delta_G \zeta^\alpha = g[t_I - k^B_I \Gamma_A]^{\alpha}_\beta \zeta^\beta$, where the ($\phi$-dependent) matrices $t_I(\phi)$ remain to be determined,

$$- k^B_I \partial_B V^\alpha_{A_i} - \partial_A k^B_I V^\alpha_{B_i} + (t_I - k^B_I \Gamma_B)^{\alpha}_\beta V^\beta_{A_i} = 0. \quad (12)$$

Obviously similar equations apply to the other geometric quantities, but as those are not independent we do not need to consider them.

Subsequently we derive the main consequences of the two equations (11) and (12). First of all the isometries must constitute an algebra with certain structure constants. This is expressed by

$$k^B_I \partial_B k^A_J - k^B_J \partial_B k^A_I = -f^K_{IJ} k^A_K, \quad (13)$$

where our definitions are such that the gauge fields that are needed once the $\theta^I$ become spacetime dependent, transform according to $\delta G W^I_\mu = \partial_\mu \theta^I - gf^K_{IJ} W^J_\mu \theta^K$. The Killing equation implies the following property

$$D_A D_B k^I_{IC} = -R_{BCAD} k^D_I. \quad (14)$$

Then, using the covariant constancy of $V_A$, we find from (12),

$$(t_I)^{\alpha}_\beta = \frac{1}{2} V^\alpha_{A_i} \bar{\gamma}^B_i \partial_B k^A_I. \quad (15)$$

Target-space scalars satisfy algebraic identities, e.g.,

$$(t_I)^{\gamma}_\alpha \bar{G}^{\bar{\gamma}\beta} + (t_I)^{\gamma}_\beta \bar{G}^{\bar{\alpha}\gamma} = (t_I)^{\gamma}_\beta \bar{\Omega}^{\bar{\beta}\gamma} = 0, \quad (16)$$

which shows that the field-dependent matrices $t_I$ take values in $sp(n)$. An explicit calculation, making use of the equations (11) and (14), shows that

$$D_A t^\alpha_B = k^B_I R^\alpha_{AB} \beta, \quad (17)$$

for any infinitesimal isometry. From the group property of the isometries it follows that the matrices $t_I$ satisfy the commutation relation

$$[t_I, t_J]^{\alpha}_\beta = f^K_{IJ} (t_K)^{\alpha}_\beta + k^A_I k^B_J R^\alpha_{AB} \beta. \quad (18)$$

The apparent lack of closure represented by the presence of the infinitesimal Sp$(n)$ holonomy transformation is related to the fact that the coordinates $\phi^A$ on which the matrices depend, transform under the action of the group. One can show that this result is consistent with the Jacobi identity.
Furthermore we derive from (12) that the complex structures \( J_{AB}^{ij} \) are invariant under the isometries,
\[
k_I^C \partial_C J_{AB}^{ij} - 2 \partial_A k_I^C J_{BC}^{ij} = 0.
\] (19)
This means that the isometries are \textit{tri-holomorphic}. From (19) one shows that
\[
\partial_A (J_{BC}^{ij} k_I^C) - \partial_B (J_{AC}^{ij} k_I^C) = 0,
\]
so that, locally, one can associate three Killing potentials (or moment maps) \( P_I^{ij} \) to every Killing vector, according to
\[
\partial_A P_I^{ij} = J_{AB}^{ij} k_B^I.
\] (20)
Observe that this condition determines the moment maps up to a constant. Up to constants one can also derive the equivariance condition,
\[
J_{BC}^{ij} k_B^I = - f_{JK}^{ij} P_K^I,
\] (21)
which implies that the moment maps transform covariantly under the isometries,
\[
\delta_G P_I^{ij} = \theta^I k_J^{ij} \partial_A P_I^{ij} = - f_{JK}^{ij} P_K^I \theta^J.
\] (22)
Summarizing, the invariance group of the isometries acts as follows,
\[
\delta_G \phi = g \theta^I k_A^I, \quad \delta_G \zeta^\alpha = g (\theta^I t_I)^\alpha_\beta \zeta^\beta - \delta_G \phi^A \Gamma_A^\alpha_\beta \zeta^\beta.
\] (23)
When the parameters of these isometries become spacetime dependent we introduce corresponding gauge fields and fully covariant derivatives,
\[
D_\mu \phi^A = \partial_\mu \phi^A - g W_\mu^I k_A^I, \quad D_\mu \zeta^\alpha = \partial_\mu \zeta^\alpha + \partial_\mu \phi^A \Gamma_A^\alpha_\beta \zeta^\beta - g W_\mu^\alpha \zeta^\beta,
\] (24)
where \( W_\mu^\alpha_\beta = W_\mu^I (t_I)^\alpha_\beta \). The covariance of \( D_\mu \zeta^\alpha \) depends crucially on (17) and (18). The gauge fields \( W_\mu^I \) are accompanied by complex scalars \( X^I \), spinors \( \Omega^I_i \) and auxiliary fields \( Y_{ij}^I \), constituting off-shell \( N = 2 \) vector multiplets. For our notation of vector multiplets, the reader may consult [2].

The minimal coupling to the gauge fields requires extra terms in the supersymmetry transformation rules for the hypermultiplet spinors as well as in the Lagrangian, in order to regain \( N = 2 \) supersymmetry. The extra terms in the transformation rules are
\[
\delta'_Q \zeta^\alpha = 2 g X^I k_D^I V_{A_1}^\alpha \epsilon^i_\beta \epsilon_j \zeta^\beta, \quad \delta'_Q \zeta^\alpha = 2 g X^I k_D^I V_{A_1}^\alpha \bar{\epsilon}_j \epsilon^i_\beta \zeta^\beta.
\] (25)
These terms can be conveniently derived by imposing the commutator of two supersymmetry transformations on the scalars, as this commutator should yield the correct field-dependent gauge transformation.

We distinguish three additional couplings to the Lagrangian. The first one is quadratic in the hypermultiplet spinors and reads
\[
\mathcal{L}^{(1)}_g = g X^I \bar{\gamma}_A^i \epsilon_\beta \gamma_B^j D_B k_{AB} \zeta^\alpha \epsilon_i \epsilon_j + \text{h.c.} = 2 g X^I t_I^\alpha_\beta \Omega^\beta_\gamma \zeta^\alpha \zeta^\beta + \text{h.c.}.
\] (26)

The second one is proportional to the vector multiplet spinor $\Omega^I$ and takes the form
$$\mathcal{L}_g^{(2)} = -2g k^A_i V^a_{AI} \bar{\zeta}^\beta \Omega^{Ii} \bar{\zeta}^\alpha \zeta^a \Omega^{Ij} + \text{h.c.}.$$  

Finally there is a potential given by
$$\mathcal{L}_g^{\text{scalar}} = -2g^2 k^A_i k^B_j g_{AB} X^I \bar{X}^J + g P^{ij}_I Y_{ij},$$  

where $P^{ij}_I$ is the triplet of moment maps on the hyper-Kähler space. These terms were determined both from imposing the supersymmetry algebra and from the invariance of the action. To prove (28), one has to make use of the equivariance condition (21). Actually, gauge invariance, which is prerequisite to supersymmetry, already depends on (22).

4 Superconformally invariant hypermultiplets

In this last section we determine the restrictions from superconformal invariance on the hypermultiplets by evaluating some of the couplings to the fields of $N = 2$ conformal supergravity. At this stage we have only a modest goal, namely to determine the restrictions on the hyper-Kähler geometry that arise from requiring invariance under rigid superconformal transformations. This is the situation that arises when freezing all the fields of conformal supergravity to zero in a flat spacetime metric. In that case the superconformal transformations acquire an explicit dependence on the spacetime coordinates.

We start by implementing the $N = 2$ superconformal algebra [9] on the hypermultiplet fields. We assume that the scalars are invariant under special conformal and special supersymmetry transformations, but they transform under $Q$-supersymmetry and under the additional bosonic symmetries of the superconformal algebra, namely chiral $[\text{SU}(2) \times \text{U}(1)]_{R}$ and dilatations denoted by $D$. At this point we do not assume that these transformations are symmetries of the action and we simply parametrize them as follows,
$$\delta \phi^A = \theta_D k^A_d(\phi) + \theta_{U(1)} k^A_{U(1)}(\phi) + (\theta_{\text{SU}(2)})^i_k \varepsilon^{jk} k^A_{ij}(\phi),$$

where the $k^A$ are left arbitrary. Note that $k^A_{ij}(\phi)$ is assigned to the same symmetric pseudoreal representation of SU(2) as the complex structures, while $\theta_{\text{SU}(2)}$ is antihermitean and traceless.

An important difference with the situation described in the previous section, is that in the conformal superalgebra the dilatations and chiral transformations do not appear in the commutator of two $Q$-supersymmetries, but in the commutator of a $Q$- and an $S$-supersymmetry. To evaluate the $S$-supersymmetry variation of the fermions, we use that $\delta_S \phi^A = \delta_S \zeta^a = 0$ and covariantize the derivative in the fermionic transformations with respect to dilatations. Subsequently we impose
the commutator, \([\delta_K(\Lambda_K), \delta_Q(\epsilon)] = -\delta_S(\Lambda_K \epsilon)\) on the spinors. This expresses the S-supersymmetry variations in terms of \(k^A_D\),

\[
\delta_S(\eta) \zeta^\alpha = V^\alpha_I k^A_D \eta^I, \quad \delta_S(\eta) \zeta^\alpha = V^\alpha_I k^A_D \eta^I.
\] (30)

With this result we first evaluate the commutator of an S- and a Q-supersymmetry transformation on the scalars. This yields

\[
[\delta_S(\eta), \delta_Q(\epsilon)] \phi^A = (\bar{\epsilon}^i \eta_i + \bar{\epsilon}_i \eta^i) k^A_D + 2 J^A_{ik} B^B (\bar{\epsilon}^i \eta_j - \bar{\epsilon}_j \eta^i) k^B_D.
\] (31)

This result can be confronted with the universal result from \(N=2\) conformal supergravity, which reads

\[
[\delta_S(\eta), \delta_Q(\epsilon)] = \delta_M (2 \bar{\eta}^i \sigma^{ab} \epsilon_i + \text{h.c.}) + \delta_D (\eta_\bar{\epsilon} + \text{h.c.})
+ \delta_{U(1)} (i \eta \bar{\epsilon} + \text{h.c.}) + \delta_{SU(2)} (-2 \bar{\eta}^i \epsilon_j - \text{h.c.; traceless} ).
\] (32)

Comparison thus shows that \(k^A_D\) vanishes and that the SU(2) vectors satisfy

\[
k^A_{ij} = J^A_{ij} k^B_D.
\] (33)

Now we proceed to impose the same commutator on the fermions, where on the right-hand side we find a Lorentz transformation, a U(1) transformation and a dilatation, iff we assume the following condition on \(k^A_D\),

\[
D_A k^B_D = \delta^B_A.
\] (34)

This condition suffices to show that the kinetic term of the scalars is scale invariant, provided one includes a spacetime metric or, in flat spacetime, include corresponding scale transformations of the spacetime coordinates. Nevertheless, observe that \(k^A_D\) is not a Killing vector of the hyper-Kähler space, but a special example of a conformal homothetic Killing vector (we thank G. Gibbons for an illuminating discussion regarding such vectors). An immediate consequence of (34) is that \(k^A_D\) can (locally) be expressed in terms of a potential \(\chi_D\), according to \(k^A_{ij} = \partial_A \chi_D\). Another consequence is that the SU(2) vectors \(k^A_{ij}\), as expressed by (33), are themselves Killing vectors, because their derivative is proportional to the corresponding antisymmetric complex structure,

\[
D_A k^B_D = -J^B_A.
\] (35)

Therefore, the bosonic action is also invariant under these SU(2) transformations.

From the \([\delta_S, \delta_Q]\) commutator we also find the fermionic transformation rules under the chiral transformations and the dilatations,

\[
\delta_{SU(2)} \zeta^\alpha + \delta_{SU(2)} \phi^A \Gamma_A^{\alpha \beta} \zeta^\beta = 0, \quad \delta_{U(1)} \zeta^\alpha + \delta_{U(1)} \phi^A \Gamma_A^{\alpha \beta} \zeta^\beta = -\frac{1}{2} \theta_{U(1)} \zeta^\alpha, \quad \delta_D \zeta^\alpha + \delta_D \phi^A \Gamma_A^{\alpha \beta} \zeta^\beta = \frac{3}{2} \theta_D \zeta^\alpha.
\] (36)
Note that the U(1) transformation further simplifies because $\delta_{U(1)} \phi^A = 0$.

To establish that the model as a whole is now invariant under the superconformal transformations it remains to be shown that the tensor $V^\alpha_{A_i}$ is invariant under the diffeomorphisms generated by $k^A_{ij}$, $k^A_{U(1)}$ and $k^A_D$ up to compensating transformations that act on the Sp($n \times$Sp(1) indices in accordance with the transformations of the $\zeta^\alpha$ given above and the symmetry assignments of the supersymmetry parameters $\epsilon^i$. To emphasize the systematics we ignore the fact that $k^A_{U(1)}$ actually vanishes and we write

$$- k^B_{kl} \partial_B V^\alpha_{A_i} - \partial_A k^B_{kl} V^\alpha_{B_i} - k^B_{kl} \Gamma^\alpha_{\beta \gamma} V^\beta_{A_i} + [ - \delta^j_{(k \varepsilon l)} ] V^\alpha_{A_j} = 0;$$
$$- k^B_{U(1)} \partial_B V^\alpha_{A_i} - \partial_A k^B_{U(1)} V^\alpha_{B_i} + [ - \frac{1}{2} i \delta^\alpha_\beta - k^B_{U(1)} \Gamma^\alpha_{\beta \gamma} ] V^\beta_{A_i} + [ \frac{1}{2} i \delta^i_j ] V^\alpha_{A_j} = 0;$$
$$- k^B_D \partial_B V^\alpha_{A_i} - \partial_A k^B_D V^\alpha_{B_i} + [ \frac{3}{2} \delta^\alpha_\beta - k^B_D \Gamma^\alpha_{\beta \gamma} ] V^\beta_{A_i} + [ - \frac{1}{2} i \delta^i_j ] V^\alpha_{A_j} = 0. \quad (37)$$

In these equations the first two terms on the left-hand side represent the effect of the isometry, the third terms represent a uniform scale and chiral U(1) transformation on the indices associated with the Sp($n$) tangent space, and the last terms represent an SU(2), a U(1) and a scale transformation, respectively, on the indices associated with the Sp(1) target space. Eq. (37) should be regarded as a direct extension of (12).

We close with a few comments. First of all, the SU(2) isometries induce a rotation on the complex structures,

$$k^C_{kl} \partial_C J^{ij}_{AB} = 2 \partial_{[A} k^C_{kl} J^{ij}_{B]} = -2 J^{\varepsilon_l |i m} J^{jm}_{AB} = 2 \delta \delta^{(i}_j \varepsilon_l |m} J^{jm}_{AB}, \quad (38)$$

as should be expected. Secondly, one can verify that the isometries discussed in section 3 commute with the scale and chiral transformations, provided that $k^A_I = k^B_D D_B k^A_I$. This condition is also required for the scale invariance of the full action. Observe also that $k^A_D$ satisfies (14), in spite of the fact that it is not a Killing vector. Finally, it is straightforward to write down actions for the vector multiplets that are invariant under rigid $N = 2$ superconformal transformations. Those are based on a holomorphic function that is homogeneous of degree two.

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