STABILIZATION AND COSTABILIZATION WITH RESPECT TO AN ACTION OF A MONOIDAL CATEGORY

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Abstract. Given a monoidal category $\mathcal{V}$ that acts on a 0-cell $A$ in a 2-category $\mathcal{M}$, we give constructions of stabilization and costabilization of $A$ with respect to the $\mathcal{V}$-action. This provides a general unified treatment for the stabilization of homotopy theories. The constructions of stabilization and costabilization are defined via universal properties and they define two endofunctors on the 2-category of 0-cells with $\mathcal{V}$-actions in $\mathcal{M}$. We show that several examples that exist in the literature fit into our setting after fixing the 0-cell $A$ and the $\mathcal{V}$-action on it. In particular, our constructions establish a duality between stable homotopy categories and Spanier-Whitehead categories.

1. Introduction

Given a relative category $\mathcal{A}$ equipped with a family of suspension functors $\Sigma^a : \mathcal{A} \to \mathcal{A}$ its stabilization is obtained by formally inverting all these suspensions. The standard construction is given as a category generated by spectrum objects in $\mathcal{A}$. If there is only one such suspension functor to invert, spectrum objects consist of sequence of objects $X_n$ in $\mathcal{A}$ together with structure maps $\Sigma X_n \to X_{n+1}$. The stabilization construction that assigns a homotopy theory to a stable homotopy theory, however, is not functorial, see e.g., [14, Sec. 7]. In order to get over the non-functoriality of the stabilization one uses Goodwillie calculus to get functorial approximations via Goodwillie-Taylor tower, [20, 13].

The non-functoriality of the stabilization is due to non-equivariance of the functors between homotopy theories. This can be understood best in set level constructions. Let $A$ be a finite set on which $\mathbb{N}$ acts. Let $\mu_1$ denote the map $\mu_1 : A \to A$ given by $\mu_1(a) = 1 \cdot a$, where $\cdot$ is the $\mathbb{N}$-action. If we universally lift such a $\mathbb{N}$ action on $A$ to a $\mathbb{Z}$ action (i.e., a set on which $\mathbb{N}$ acts by bijections) then we obtain the largest subset of $A$ on which the restriction of $\mu_1$ is a bijection, see [6]. Now, if $f : A \to B$ is a map between finite $\mathbb{N}$-sets, then $f$ does not have to induce a map on such maximal subsets unless $f$ is $\mathbb{N}$-equivariant. In other words, the association that sends a finite $\mathbb{N}$-set to its largest subset on which $\mathbb{N}$ acts by bijections does not define an endofunctor from the category of $\mathbb{N}$-sets and functions; however, it defines an endofunctor on the category of $\mathbb{N}$-sets and $\mathbb{N}$-equivariant functions.

In this paper, by introducing a notion of equivariant functors between categories with monoidal category actions, we give another way of tackling the non-functoriality of stabilization. In fact, one can consider a collection suspension functors (or loop space functors) on a relative category $\mathcal{A}$ as an action of a monoidal category on

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and construct stabilization as a functor from a category of relative categories equipped with actions of a fixed monoidal category and certain equivariant functors between them. The essential ingredient of our viewpoint is this notion of $\mathcal{V}$-equivariance. We state our main result in an arbitrary 2-category $\mathcal{M}$. Given a monoidal category $\mathcal{V}$, we consider a 2-category $\mathcal{VM}$ of $\mathcal{V}$-0-cells, $\mathcal{V}$-equivariant 1-cells and 2-cells, and take the full sub-2-category $\text{st} \mathcal{VM}$ of stable 0-cells. Then the stabilization is the universal way of turning a $\mathcal{V}$-0-cell into a stable $\mathcal{V}$-0-cell, that has the 2-categorical terminal property. Thus, the stabilization defines a 2-functor from $\mathcal{VM}$ to $\text{st} \mathcal{VM}$. We also obtain a dual version of the stabilization, which we call costabilization.

**Statements of the main results.** Let $\mathcal{V}$ be a monoidal category and let $\mathcal{M}$ be any (possibly large) strict 2-category. In section 2, we discuss $\mathcal{V}$-actions and we introduce the notions of centralizer and cocentralizer of a $\mathcal{V}$-action and using this we introduce the notion of $\mathcal{V}$-equivariant 1-cells between 0-cells. We show that the data defined by 0-cells of $\mathcal{M}$ with $\mathcal{V}$-bactions, $\mathcal{V}$-equivariant 1-cells between 0-cells and 2-cells between equivariant 1-cells form a 2-category, which we denote by $\mathcal{VM}$. A 0-cell in this 2-category will be called $\mathcal{V}$-object. There is a distinguished sub-2-category of $\mathcal{VM}$ that consists of stable $\mathcal{V}$-objects, where a $\mathcal{V}$-object is stable if the $\mathcal{V}$-action on $A$ has an inverse up to 2-cells, see Definition 2.6. This sub-2-category is denoted by $\text{st} \mathcal{VM}$. One of the main results of the present paper is the following:

**Theorem 1.1.** If $\mathcal{M}$ is a complete (resp. cocomplete) 2-category, then $\text{st} \mathcal{VM}$ is coreflective (resp. reflective) in $\mathcal{VM}$.

Here by (co)completeness we mean 2-categorical completeness; i.e., $\text{Cat}$-(co)complete. Thus, $\mathcal{M}$ is powered and copowered over $\text{Cat}$. The coreflector in the statement of the theorem is called stabilization and the reflector is called costabilization. The theorem is still valid in the lax sense, and in this case the coreflector is called lax stabilization and the reflector is called colax costabilization. The above theorem has a particular interest in homotopy theory, when we choose $\mathcal{M}$ to be the 2-category $\mathcal{RelCat}$ of relative categories, relative functors and relative natural transformations, see [1]. Under certain conditions, an action on a relative category induces an action on the category of functors that satisfy generalized versions of (co)homology axioms of Eilenberg-Steenrod. In particular, we show that (co)homology theories are objects in the stabilization of (co)homology functors with respect to actions induced by suspensions.

We later establish that stable homotopy categories indexed by a symmetric monoidal category $\mathcal{V}$ are homotopy categories of lax stabilizations of pointed relative categories with respect to $\mathcal{V}$-actions and Spanier-Whitehead categories indexed by $\mathcal{V}$ are homotopy categories of colax costabilizations with respect to $\mathcal{V}$-actions.

**Organization of the paper.** In Section 2 we give the definition of $\mathcal{V}$-actions and the central notion of equivariance. We introduce the 2-category $\mathcal{VM}$ of $\mathcal{V}$-0-cells, $\mathcal{V}$-equivariant 1-cells and 2-cells between them. In section 3 we give definitions and constructions of stabilizations and cotabilizations with respect to $\mathcal{V}$-actions. We also give the proof of our main result. In section 4 we show that homology and cohomology theories are stabilizations of $\mathcal{V}$-actions on categories of homology and cohomology functors. In section 5 we consider the special case of actions on relative categories and obtain the various categories of spectra as stabilizations, and
Spanier-Whitehead categories as homotopy categories of costabilizations, of relative categories with respect to actions of symmetric monoidal categories.

2. Actions of monoidal categories and equivariance

In this section, we present a categorification of the definitions of monoid actions on sets and equivariant functions given in [6]. In particular, we define the action of a monoidal category on an object in a strict 2-category, generalizing the definition in [10], and explore the properties of this notion.

2.1. $\mathcal{V}$-actions. Let $(\mathcal{V}, \otimes, \mathbb{1})$ be a strict monoidal category. There is an associated strict 2-category $B\mathcal{V}$ with a single 0-cell $\ast$, called the delooping bicategory of $\mathcal{V}$. The 1-cells (i.e., 1-morphisms) from $\ast$ to $\ast$ are objects of $\mathcal{V}$ and the composition of two 1-cells in $B\mathcal{V}$ is given by the monoidal product on $\mathcal{V}$. We write $B\mathcal{V}^{op}$ for the 1-cell dual of the 2-category $B\mathcal{V}$; that is, the 2-category obtained by reversing its 1-cells but not the 2-cells. For the rest of the paper we use $B\mathcal{W}$ for the 2-category $B\mathcal{V} \times B\mathcal{V}^{op}$. Let $\mathfrak{M}$ be a strict 2-category and $A$ be a 0-cell in $\mathfrak{M}$. An action of $\mathcal{V}$ on $A$ (or a $\mathcal{V}$-action on $A$) is a strict 2-functor

$$\alpha : B\mathcal{W} \to \mathfrak{M}$$

where $\alpha(\ast, \ast) = A$.

This definition is, in fact, the same as the definition of ordinary biaction. However, we construct a 2-category endowed with an exotic notion of equivariance so that the category equivalent to one sided actions, see Section 2.2.

Clearly, ordinary actions of monoids, considered as 2-categories in the trivial way, are trivial examples.

Remark 2.1. Note that we can define actions of a nonstrict monoidal category after strictification. In other words, if $\mathcal{V}$ is a monoidal category (not necessarily strict) we can define a $\mathcal{V}$-action as a $\text{str}\mathcal{V}$-action where $\text{str}\mathcal{V}$ denotes the strictification of the monoidal category $\mathcal{V}$. Equivalently, for an arbitrary monoidal category $\mathcal{V}$ we can construct $B\mathcal{W}$ as above and any pseudofunctor from $B\mathcal{W}$ to a strict 2-category can be considered as a $\mathcal{V}$-action by the strictification adjunction given in [3].

Some examples where higher morphisms are more interesting are given in [10] which can be considered as examples to our actions by composing with the projection from $B\mathcal{V} \times B\mathcal{V}^{op}$ onto its left component $B\mathcal{V}$ and considering Remark 2.1.

Example 2.2. If $\mathcal{V}$ is a strict monoidal category, then $\mathcal{V}$ (or any of its monoidal subcategory) acts on $\mathcal{V}$ by its tensor product. One can define an action by tensoring from the left, or from the right or from both sides. More generally, if $\mathcal{A}$ is (co)powered over $\mathcal{V}$, then the (co)power defines a $\mathcal{V}$-action on $\mathcal{A}$.

Example 2.3. Any 1-endomorphism on a 0-cell $A$ in a strict 2-category $\mathfrak{M}$ generates a $\mathbb{N}$ action by considering $\mathbb{N}$ as a monoidal category with only identity morphisms. More generally, one can choose a collection of 1-endomorphisms of $A$ and take the monoidal category generated by these endomorphisms. The resulting monoidal category acts on $A$.

Some examples that have interest in homotopy theory are discussed in Section 4.2.

Notation 2.4. We use the notation $A_\alpha$ for a 0-cell $A$ with $\mathcal{V}$-action $\alpha$. 
2.2. $\mathcal{V}$-equivariant 1-cells. Since $\mathcal{M}$ is a strict 2-category, we have a functor from $\mathcal{M}^{\text{op}} \times \mathcal{M}$ to $\text{Cat}$ which sends $(\mathcal{A}, \mathcal{B})$ to the category $\mathcal{M}(\mathcal{A}, \mathcal{B})$ which we simply denote by $[\mathcal{A}, \mathcal{B}]$. Given a 2-functor $F : \mathcal{C}^{\text{op}} \times \mathcal{C} \to \mathcal{M}$ between strict 2-categories, its lax end $\int_{x \in \mathcal{C}} F(x, x)$ is defined by the categorical equivalence

$$[Z, \int_{x \in \mathcal{C}} F(x, x)] \simeq \text{Lax}(\mathcal{C}^{\text{op}} \times \mathcal{C}, \text{Cat}) (\text{hom}_{\mathcal{C}}(-, -), [Z, F(-, -)])$$

natural in $Z$. Here $\text{Lax}(\mathcal{C}^{\text{op}} \times \mathcal{C}, \text{Cat})$ denotes the 2-category of 2-functors, lax-natural transformations and modifications. Similarly, the end $\int_{x \in \mathcal{C}} F(x, x)$ is defined by replacing Lax with Psd, the category of 2-functors, pseudo-natural transformations and modifications; that is,

$$[Z, \int_{x \in \mathcal{C}} F(x, x)] \simeq \text{Psd}(\mathcal{C}^{\text{op}} \times \mathcal{C}, \text{Cat}) (\text{hom}_{\mathcal{C}}(-, -), [Z, F(-, -)])$$

In other words, the (lax) end of $F$ is a homc weighted (lax) bilimit of $F$.

Let $\mathcal{A}$ be a 0-cell in the strict 2-category $\mathcal{M}$, and $\alpha : B\mathcal{V} \to \mathcal{M}$ be a $\mathcal{V}$-action on $\mathcal{A}$. Since $B\mathcal{V}$ has a single 0-cell, The category

$$\text{Psd}(B\mathcal{V}, \text{Cat}) (\text{hom}_{B\mathcal{V}^{\text{op}}}(\mathcal{A}, \mathcal{A}), [Z, \alpha(-, -)])$$

on the right hand side of the categorical equivalence that defines the end

$$\int_{x \in B\mathcal{V}^{\text{op}}} \alpha(x, x)$$

is equivalent to the category whose objects are 1-cells

$$\omega : Z \to \mathcal{A}$$

and invertible 2-cells (i.e., a 2-isomorphisms)

$$\sigma(u) : \alpha(\mathcal{1}, u^{\text{op}}) \circ \omega \Rightarrow \alpha(u, \mathcal{1}^{\text{op}}) \circ \omega$$

assigned to all objects $u$ in $\mathcal{V}$ so that we have

$$\sigma(v) \bullet (\alpha(\mathcal{1}, u^{\text{op}}) \circ \omega) = (\alpha(f, \mathcal{1}^{\text{op}}) \circ \omega) \bullet \sigma(u)$$

(2.1)

for every morphism $f : u \to v$ in $\mathcal{V}$ and the following diagram of 2-cells

$$\begin{array}{ccc}
\alpha(\mathcal{1}, v^{\text{op}}) \circ \alpha(\mathcal{1}, u^{\text{op}}) \circ \omega = \alpha(\mathcal{1}, v^{\text{op}}) \circ \alpha(u, \mathcal{1}^{\text{op}}) \circ \omega \\
\alpha(\mathcal{1}, v^{\text{op}})(\sigma(u)) & \Rightarrow & \alpha(u, \mathcal{1}^{\text{op}})(\sigma(v)) \\
\alpha(\mathcal{1}, v^{\text{op}}) \circ \alpha(\mathcal{1}, u^{\text{op}}) \circ \omega & \Rightarrow & \alpha(u, \mathcal{1}^{\text{op}}) \circ \alpha(v, \mathcal{1}^{\text{op}}) \circ \omega \\
\alpha(\mathcal{1}, (u \otimes v)^{\text{op}}) \circ \omega & \Rightarrow & \alpha(u \otimes v, \mathcal{1}^{\text{op}}) \circ \omega \\
\sigma(u \otimes v) & \Rightarrow & \sigma(u \otimes v)
\end{array}$$

(2.2)

commutes for every $u, v$ in $\mathcal{V}$ and

$$1_{\alpha(\mathcal{1}, \mathcal{1}^{\text{op}}) \circ \omega} = \sigma(\mathcal{1})$$

(2.3)

Morphisms from $(\omega, \sigma)$ to $(\overline{\omega}, \overline{\sigma})$ are 2-cells $\theta$ from the 1-cell $\omega$ to the 1-cell $\overline{\omega}$ so that

$$\overline{\sigma}(u) \bullet (\alpha(\mathcal{1}, u^{\text{op}}) \circ \theta) = (\alpha(u, \mathcal{1}^{\text{op}}) \circ \theta) \bullet \sigma(u)$$

for all objects $u$ in $\mathcal{V}$.
Remark 2.5. In the case when $\mathcal{M}$ has an object $*$ such that $*, \mathcal{A} \cong \mathcal{A}$ for every 0-cell $\mathcal{A}$, than one obtains that

$$\int_{x: \mathcal{B}^{op}} \alpha(x, x)$$

has objects as pairs $(a, \sigma)$ where $a$ is a 1-cell $* \to \mathcal{A}$ and $\sigma$ is the natural transformation as above that satisfies 2.1, 2.2 and 2.3. A morphism from $(a, \sigma)$ to $((\tau, \sigma))$ is a morphism $\theta: a \to \tau$ in $\mathcal{A}$ that satisfies

$$\tau(u) \bullet (\alpha(1, u^{op}) < \theta) = (\alpha(u, 1^{op}) < \theta) \bullet \sigma(u).$$

$\mathcal{M} = \mathcal{C}at$, that is $\mathcal{A}$ is a category, by choosing $Z$ as the terminal category. Now let

$$\psi: \mathcal{B} \mathcal{W} \to \mathcal{B}W^{op} \times \mathcal{B}W$$

be the 2-functor given by $\psi(u, v^{op}) = ((u, v^{op})^{op}, (u, v^{op}))$. Given two $\mathcal{V}$-actions $\alpha$ and $\beta$ on 0-cells $\mathcal{A}_\alpha$ and $\mathcal{B}_\beta$, we define a $\mathcal{V}$-action on $[\mathcal{A}_\alpha, \mathcal{B}_\beta]$ by the composition

$$\mathcal{B}W \xrightarrow{\psi} \mathcal{B}W^{op} \times \mathcal{B}W \xrightarrow{(\alpha^{op}, \beta)} \mathcal{M}^{op} \times \mathcal{M} \xrightarrow{\mathcal{M}, \mathcal{M}, \mathcal{M}} \mathcal{C}at.$$

We denote this $\mathcal{V}$-action by $[\alpha, \beta]$. We define the category $\mathcal{F}un_\mathcal{V}(\mathcal{A}_\alpha, \mathcal{B}_\beta)$ of $\mathcal{V}$-equivariant 1-cells from $\mathcal{A}_\alpha$ to $\mathcal{B}_\beta$ as the end

$$\mathcal{F}un_\mathcal{V}(\mathcal{A}_\alpha, \mathcal{B}_\beta) = \int_{x: \mathcal{B}^{op}} [\alpha, \beta](x, x)$$

provided that it exists. Any object in $\mathcal{F}un_\mathcal{V}(\mathcal{A}_\alpha, \mathcal{B}_\beta)$ is called $\mathcal{V}$-equivariant 1-cell from $\mathcal{A}_\alpha$ to $\mathcal{B}_\beta$. The category $\mathcal{F}un_\mathcal{V}(\mathcal{A}_\alpha, \mathcal{B}_\beta)$ will in fact be considered as the hom-category of a 2-category. Therefore, we give an explicit description of $\mathcal{F}un_\mathcal{V}(\mathcal{A}_\alpha, \mathcal{B}_\beta)$ that is functorial on $(\mathcal{A}_\alpha, \mathcal{B}_\beta)$. Objects of $\mathcal{F}un_\mathcal{V}(\mathcal{A}_\alpha, \mathcal{B}_\beta)$ are pairs $(f, \tau)$ such that $f$ is an object in $[\mathcal{A}_\alpha, \mathcal{B}_\beta]$ and $\tau$ is a natural isomorphism from $[\alpha, \beta](\mathcal{1}, -^{op})(f): \mathcal{V} \to [\mathcal{A}_\alpha, \mathcal{B}_\beta]$ to $[\alpha, \beta](-, 1^{op})(f): \mathcal{V} \to [\mathcal{A}_\alpha, \mathcal{B}_\beta]$. We define $\omega(f, \tau) = f$ and for every object $u$ in $\mathcal{V}$, we define the $(f, \tau)$ component of the 2-cell $\sigma(u)$ as $\tau(u)$. So that the above compatibility conditions are satisfied. In particular, given a $\mathcal{V}$-equivariant 1-cell $(f, \tau)$ from $\mathcal{A}_\alpha$ to $\mathcal{B}_\beta$ we have an morphism

$$\sigma(u)(f, \tau) = \tau(u) : (\beta(1, u^{op}) \circ f \circ \alpha(u, 1^{op}) \to \beta(u, 1^{op}) \circ f \circ \alpha(1, u^{op})$$

in $[\mathcal{A}_\alpha, \mathcal{B}_\beta]$ for every $u$ in $\mathcal{V}$.

2.3. The 2-category $\mathcal{V}\mathcal{M}$. We define two functors $\pi_l, \pi_r: \mathcal{B}W \to \mathcal{B}W$ as $\pi_l: (u, v^{op}) \mapsto (u, 1^{op})$ and $\pi_r: (u, v^{op}) \mapsto (1, v^{op})$ on 1-cells and corresponding projections on 2-cells. We say a $\mathcal{V}$-action $\alpha$ is a right (respectively left) action if $\alpha \circ \pi_r = \alpha$ (respectively $\alpha \circ \pi_l = \alpha$). An action will be called an action mute on one side if it is a right action or a left action. We will say a left action is mute on right and a right action is mute on left. Now we start defining the 2-category $\mathcal{V}\mathcal{M}$. The 0-cells of $\mathcal{V}\mathcal{M}$ are the 0-cells of $\mathcal{M}$ equipped with a $\mathcal{V}$-action mute on at least one side. The morphisms of $\mathcal{V}\mathcal{M}$ are $\mathcal{V}$-equivariant 1-cells as defined in the previous section. Now we define the composition of two equivariant 1-cells.

Let $\mathcal{A}_\alpha$, $\mathcal{B}_\beta$, and $\mathcal{C}_\gamma$ be 0-cells in $\mathcal{V}\mathcal{M}$. Assume $(f, \tau_f)$ is a $\mathcal{V}$-equivariant 1-cell from $\mathcal{A}_\alpha$ to $\mathcal{B}_\beta$ and $(g, \tau_g)$ is a $\mathcal{V}$-equivariant 1-cell from $\mathcal{B}_\beta$ to $\mathcal{C}_\gamma$. We define the composition of $(f, \tau_f)$ and $(g, \tau_g)$ as the pair $(g \circ f, \tau_{g \circ f})$ where $\tau_{g \circ f}(u)$ for $u$ in $\mathcal{V}$ is defined as follows: In case $\beta$ is mute on left and hence a right action, for $u$ in $\mathcal{V}$,
we have $\beta(u, 1_{\mathbb{V}^{op}}) = 1_B$, so we can define $\tau_{gof}(u)$ as the isomorphism given by the vertical composition of the isomorphisms:

In case $\beta$ is mute on the right and hence a left action, for $u$ in $\mathbb{V}$ we have $\beta(1, u_{\mathbb{V}^{op}}) = 1_B$ and so we can define $\tau_{gof}(u)$ as the isomorphism given by the vertical composition of the following isomorphisms:

Notice in case $\beta$ is mute on both sides, these definitions coincide by the interchange law.

Let $A_\alpha, B_\beta, C_\gamma$ and $D_\delta$ be 0-cells in $\mathbb{V}\mathbb{M}$ and $(f, \tau_f) : A_\alpha \to B_\beta, (g, \tau_g) : B_\beta \to C_\gamma$ and $(h, \tau_h) : C_\gamma \to D_\delta$ be $\mathbb{V}$-equivariant 1-cells. In the case when $\beta$ and $\gamma$ are both...
mute on the left, the associativity can be obtained from the following diagram:

\[
\tau_{h\circ(g\circ f)}(u) \text{ is obtained from the following diagram}
\]
and $\tau_{(h \circ g) \circ f}(u)$ is obtained from the following diagram

![Diagram](image)

Using the exchange law, since every unlabeled square is identity, we obtain that

$$\tau_{(h \circ g) \circ f}(u) = \tau_{h \circ (g \circ f)}(u).$$

The remaining cases are similar to one of the cases above. It is also straightforward to check that the diagram in 2.2 commutes. Therefore, we obtain that the composition in $\mathcal{VM}$ is strictly associative. It is now easy to see that the unitors in $\mathcal{VM}$ are also identity, which makes $\mathcal{VM}$ a strict 2-category.

2.4. Stable 0-cells with respect to a $\mathcal{V}$-action. We define the notion of stability for $\mathcal{V}$-objects with respect to a $\mathcal{V}$-action.

**Definition 2.6.** We say a 0-cell $A_\alpha$ in $\mathcal{VM}$ is stable if $\alpha$ is mute on the right (resp. left) and there exists a $\mathcal{V}$-action $\beta$ on a 0-cell $B$ that is mute on the left (resp. right) such that the 0-cells of $A_\alpha$ and $B_\beta$ are 1-equivalent in $\mathcal{VM}$. The full sub-2-category of stable $\mathcal{V}$-objects is denoted by $\text{st}\mathcal{VM}$.

3. Stabilizations and costabilizations with respect to $\mathcal{V}$-actions

Suppose that $\mathcal{M}$ is a strict 2-category that is powered and copowered over $\mathcal{Cat}$. We denote by

$$\ni: \text{Cat}^{\text{op}} \times \mathcal{M} \to \mathcal{M} : (\mathcal{C}, \mathcal{A}) \mapsto \ni (\mathcal{C}, \mathcal{A})$$

the powering of a 0-cell $\mathcal{A}$ in $\mathcal{M}$ by a category $\mathcal{C}$, and we denote by

$$\boxtimes : \text{Cat} \times \mathcal{M} \to \mathcal{M} : (\mathcal{C}, \mathcal{A}) \mapsto \mathcal{C} \boxtimes \mathcal{A}$$

the copowering of $\mathcal{A}$ by $\mathcal{C}$.

We define two functors $\mu_r, \mu_l : BW \times BW^{\text{op}} \to \text{Cat}^{\text{op}}$ as follows: For $s$ in $\{l, r\}$ define $\mu_s((*, *), (*, *)) = \mathcal{V}$ considered as a category after forgetting the monoidal structure and $\mu_s((x, y^{\text{op}}), (z, w^{\text{op}})^{\text{op}}) : \mathcal{V} \to \mathcal{V}$ is the functor given by

$$\mu_s((x, y^{\text{op}}), (z, w^{\text{op}})^{\text{op}})(u) = \begin{cases} (y \otimes u) \otimes w, & \text{if } s = r \\ z \otimes (u \otimes x), & \text{if } s = l \end{cases} \quad (3.6)$$
Let \( \alpha : BW \to M \) be a \( \mathcal{V} \)-action on a 0-cell \( A \) in \( M \). Define \( \Theta_{s,\alpha} \) as the composition
\[
BW \times BW \xrightarrow{id \times \psi} BW \times BW^{op} \times BW^{\mu_{s,\alpha}} \xrightarrow{id} \text{Cat}^{op} \times M \xrightarrow{\Theta} M.
\]
(3.7)
Define a \( \mathcal{V} \)-action \( \text{Inv}_s(\alpha) : BW \to M \)
\[
\text{Inv}_s(\alpha)(-) = \int_{x:BV^{op}} \Theta_{s,\alpha}(-(x,x)).
\]
(3.8)
We use the notation \( \text{Inv}_s(A_\alpha) \) for \( \text{Inv}_s(\alpha)(\ast,\ast) \).

Let \( \tilde{\mu}_l = \mu((\mathbb{1},1^{op}),-) \) and \( \tilde{\mu}_r = \mu((-,(1,1^{op})) \). Let \( s, s' \in \{l, r\} \) with \( s \neq s' \). Note that \( \Theta_{s,\alpha}(\ast,\ast, \ast) \) is equal to the composition given by
\[
BW \xrightarrow{\Theta} BW^{op} \times BW^{\mu_{s',\alpha}} \xrightarrow{id \times \Theta} \text{Cat}^{op} \times M \xrightarrow{\Theta} M.
\]
which we denote by \( \Theta(\tilde{\mu}_{s'}, \alpha) \). Thus, \( \text{Inv}_s(A_\alpha) \) is isomorphic to
\[
\int_{x:BV^{op}} \Theta(\tilde{\mu}_{s'}, \alpha)(x,x)
\]
(3.9)
The inclusion \( \{1\} \to \mathcal{V} \) induces a \( \mathcal{V} \)-equivariant 1-cell
\[
\epsilon_s : \text{Inv}_s(A_\alpha) \to A_\alpha
\]
called the **evaluation at** \( 1 \), given by the composition of \( \omega_s : \text{Inv}_s(A_\alpha) \to \Theta(\mathcal{V}, A) \), the universal wedge of the end, and \( \epsilon : \Theta(\mathcal{V}, A) \to \Theta(1, A) \cong A \), the usual evaluation. We can define
\[
\sigma : [\text{Inv}_s(A_\alpha), \alpha](1, u^{op})(\epsilon_s) \to [\text{Inv}_s(A_\alpha), \alpha](u, 1^{op})(\epsilon_s)
\]
as follows: Let \( \tilde{\mu}_l = \mu((-,(1,1^{op})) \) and \( \tilde{\mu}_r = \mu((\mathbb{1},1^{op}),-) \). For every \( u \in \mathcal{V} \), we have a map
\[
\epsilon \circ \Theta(\tilde{\mu}_{s'}, \alpha)(1, u^{op}) \circ \omega_s \to \epsilon \circ \Theta(\tilde{\mu}_{s'}, \alpha)(u, 1^{op}) \circ \omega_s
\]
By naturality of \( \epsilon \), we obtain
\[
\alpha(1, u^{op}) \circ \epsilon \circ \Theta(\tilde{\mu}_{s'}, id_A)(1, u^{op}) \circ \omega_s \to \alpha(u, 1^{op}) \circ \epsilon \circ \Theta(\tilde{\mu}_{s'}, id_A)(u, 1^{op}) \circ \omega_s
\]
Notice that \( \epsilon \circ \Theta(\tilde{\mu}_{s'}, id_A)(u, 1^{op}) = \epsilon \circ \Theta(\tilde{\mu}_{s'}, id_A)(1, u^{op}) \) and \( \epsilon \circ \Theta(\tilde{\mu}_{s'}, id_A)(1, u^{op}) = \epsilon \circ \Theta(\tilde{\mu}_{s'}, id_A)(u, 1^{op}) \). Then we obtain
\[
\alpha(1, u^{op}) \circ \epsilon \circ \Theta(\tilde{\mu}_{s'}, id_A)(u, 1^{op}) \circ \omega_s \to \alpha(u, 1^{op}) \circ \epsilon \circ \Theta(\tilde{\mu}_{s'}, id_A)(1, u^{op}) \circ \omega_s
\]
By Fubini theorem, we get
\[
\alpha(1, u^{op}) \circ \epsilon \circ \omega_s \circ \text{Inv}_s(\alpha)(u, 1^{op}) \to \alpha(u, 1^{op}) \circ \epsilon \circ \omega_s \circ \text{Inv}_s(\alpha)(1, u^{op})
\]
We define \( \sigma \) as
\[
\sigma(u) : \alpha(1, u^{op}) \circ \epsilon \circ \text{Inv}_s(\alpha)(u, 1^{op}) \to \alpha(u, 1^{op}) \circ \epsilon \circ \text{Inv}_s(\alpha)(1, u^{op}).
\]

**Theorem 3.1.** Let \( \alpha \) be an left (respectively right) \( \mathcal{V} \)-action on a 0-cell \( A \) in a strict 2-category \( M \) that is powered and copowered over \( \text{Cat} \). Then \( A_\alpha \) is stable if and only if \( \epsilon_r \) (respectively \( \epsilon_l \)) is a 1-equivalence in \( \mathcal{V}M \).

**Proof.** The \"if\" part follows from the definition of being stable. Now for the \"only if\" part. Assume that \( A_\alpha \) is stable. Without loss of generality assume \( \alpha \) is a right action. Then \( A_\alpha \) is 1-equivalent to \( B_\beta \) for some left \( \mathcal{V} \)-action \( \beta \). Now it is straightforward to check that \( \epsilon_l \) from \( \text{Inv}_l(\beta) \) to \( B_\beta \) is a 1-equivalence. Also notice that the 1-equivalence from \( A_\alpha \) to \( B_\beta \) induces a 1-equivalence from \( \text{Inv}_l(A_\alpha) \) and...
Inv\textsubscript{\(V\)}(\(B_{\beta}\)). Hence by naturality of the evaluation map \(\epsilon_l\) from Inv\textsubscript{\(V\)}(\(A_\alpha\)) to \(A_\alpha\) is a 1-equivalence.

Let \(A_\alpha\) be a 0-cell in \(\mathcal{VM}\) where \(\mathcal{V}\) and \(\mathcal{M}\) are as above.

**Definition 3.2.** The stabilization of \(A_\alpha\) is a stable 0-cell Stab\(\mathcal{V}\)(\(A_\alpha\)), together with a \(\mathcal{V}\)-equivariant 1-cell \(\epsilon : \text{Stab}\mathcal{V}(A_\alpha) \rightarrow A_\alpha\) such that for every stable 0-cell \(B_{\beta}\), the induced functor \(\epsilon_* : \text{Fun}_{\mathcal{V}}(B_{\beta}, \text{Stab}\mathcal{V}(A_\alpha)) \rightarrow \text{Fun}_{\mathcal{V}}(B_{\beta}, A_\alpha)\) is a categorical equivalence.

**Remark 3.3.** Notice that the definition above implies for every \(\mathcal{V}\)-equivariant 1-cell \(f : B_{\beta} \rightarrow A_\alpha\) there exists a \(\mathcal{V}\)-equivariant 1-cell \(\tilde{f} : \text{Stab}\mathcal{V}(A_\alpha) \rightarrow B_{\beta}\) together with a 2-isomorphism

\[
\begin{align*}
B_{\beta} & \xrightarrow{f} A_\alpha \\
\text{Stab}\mathcal{V}(A_\alpha) & \xrightarrow{\epsilon} B_{\beta}
\end{align*}
\]

where \(\tilde{f}\) is unique up to unique 1-isomorphism.

Given any 0-cell \(A_\alpha\) in \(\mathcal{VM}\) let

\[
\text{Inv}_{\mathcal{V}}(A_\alpha) = \text{Inv}_{1}(\text{Inv}_{\mathcal{V}}(A_\alpha))
\]

and \(\epsilon = \epsilon_* \circ \epsilon_l\). Let Inv\textsubscript{\(V\)}\textsuperscript{\(n\)}(\(A_\alpha\)) = Inv\textsubscript{\(V\)}(\(A_\alpha\)) and Inv\textsubscript{\(V\)}\textsuperscript{\(n\)-1}(\(A_\alpha\)) = Inv\textsubscript{\(V\)}(Inv\textsubscript{\(V\)}\textsuperscript{n-1}(\(A_\alpha\))) for every \(n > 1\). Define Inv\textsubscript{\(V\)}\textsuperscript{\(\infty\)}(\(A_\alpha\)) as the limit of the diagram

\[
\cdots \rightarrow \text{Inv}_{\mathcal{V}}(A_\alpha) \xrightarrow{\epsilon} \text{Inv}_{\mathcal{V}}^2(A_\alpha) \xrightarrow{\epsilon} \text{Inv}_{\mathcal{V}}^3(A_\alpha) \xrightarrow{\epsilon} \cdots
\]

in the 2-category \([BV, \mathcal{M}]\). When limits in \(\mathcal{M}\) exists, so does this limit. Note that every 0-cell in the above diagram is mute on the right, so that the object is isomorphic to an object in \([BV, \mathcal{M}]\) (which is a subcategory of \(\mathcal{VM}\)) under the inclusion induced by the functor \(BV \rightarrow BV : (u, v^{op}) \mapsto u\).

Let \(\epsilon_{\infty} : \text{Inv}_{\mathcal{V}}^\infty(A_\alpha) \rightarrow A_\alpha\) be the \(\epsilon_l\) composed with the transfinite compositions of the maps in the diagram.

**Theorem 3.4.** Assume that limits in \(\mathcal{M}\) exist. Then Inv\textsubscript{\(V\)}\textsuperscript{\(\infty\)}(\(A_\alpha\)) together with \(\epsilon_{\infty} : \text{Inv}_{\mathcal{V}}^\infty(A_\alpha) \rightarrow A_\alpha\) is a stabilization of \(A_\alpha\).

**Proof.** Since Inv\textsubscript{\(V\)} is given by an end of a powering, it commutes with limits. Therefore, Inv\textsubscript{\(V\)}(Inv\textsubscript{\(V\)}\textsuperscript{\(\infty\)}(\(A_\alpha\))) is isomorphic to the limit \(\lim_{n \geq 1} \text{Inv}_{\mathcal{V}}(\text{Inv}_{\mathcal{V}}(A_\alpha))\). For each \(n \geq 1\), we have 1-cells \(\epsilon_* : \text{Inv}_{\mathcal{V}}(\text{Inv}_{\mathcal{V}}^n(A_\alpha)) \rightarrow \text{Inv}_{\mathcal{V}}(A_\alpha)\), so that there is a universal 1-cell

\[
F : \text{Inv}_{\mathcal{V}}(\text{Inv}_{\mathcal{V}}^n(A_\alpha)) \rightarrow \text{Inv}_{\mathcal{V}}^n(A_\alpha)
\]

and we have 1-cells \(\epsilon_l : \text{Inv}_{\mathcal{V}}^{n+1}(A_\alpha) \rightarrow \text{Inv}_{\mathcal{V}}(\text{Inv}_{\mathcal{V}}^n(A_\alpha))\), so that there is a universal 1-cell

\[
G : \text{Inv}_{\mathcal{V}}^\infty(A_\alpha) \rightarrow \text{Inv}_{\mathcal{V}}(\text{Inv}_{\mathcal{V}}^\infty(A_\alpha))
\]

Hence, Inv\textsubscript{\(V\)}\textsuperscript{\(\infty\)}(\(A_\alpha\)) is stable.

Given any stable \(B_{\beta}\) in \(\mathcal{VM}\) we have a \(\mathcal{V}\)-equivariant 1-isomorphism \(B_{\beta} \rightarrow \text{Inv}_{\mathcal{V}}(B_{\beta})\), so that Inv\textsubscript{\(V\)}(\(B_{\beta}\)) is a also stable. Inductively, we obtain \(\mathcal{V}\)-equivariant 1-isomorphisms \(B_{\beta} \rightarrow \text{Inv}_{\mathcal{V}}^n(B_{\beta})\) for all \(n \geq 1\). Thus, there is a 1-isomorphisms
$\mathcal{B}_\beta \to \text{Inv}_{\mathcal{V}}^\infty(\mathcal{A}_\alpha)$. Then, for any given $\mathcal{V}$-equivariant 1-cell $f : B \to A$, there exist a unique $\tilde{f} : B \to \text{Inv}_{\mathcal{V}}^\infty(\mathcal{A}_\alpha)$ given by the composition

$$\tilde{f} : B \xrightarrow{f} \text{Inv}_{\mathcal{V}}^\infty(\mathcal{B}_\beta) \xrightarrow{\text{Inv}_{\mathcal{V}}^\infty(f)} \text{Inv}_{\mathcal{V}}^\infty(\mathcal{A}_\alpha)$$

such that $\varepsilon^\infty \circ \tilde{f}$ is naturally isomorphic to $f$. This implies that

$$\varepsilon^\infty : \text{Fun}_{\mathcal{V}}(\mathcal{B}_\beta, \text{Inv}_{\mathcal{V}}^\infty(\mathcal{A}_\alpha)) \to \text{Fun}_{\mathcal{V}}(\mathcal{B}_\beta, \mathcal{A}_\alpha)$$

is an equivalence of categories. \qed

In the case when $\mathcal{V}$ is a symmetric monoidal category, we can consider the bi-reverse as a stabilization functor due to the following theorem.

**Theorem 3.5.** Assume that $\mathcal{V}$ is a symmetric monoidal category. Then $\text{Inv}_\mathcal{V}(\mathcal{A}_\alpha)$ is equivalent to $\text{Inv}_\mathcal{V}^\infty(\mathcal{A}_\alpha)$.

**Proof.** Let $E : \text{Inv}_\mathcal{V}(\mathcal{A}_\alpha) \to \text{Inv}(\text{Inv}_\mathcal{V}(\mathcal{A}_\alpha))$ be the functor induced by $\pi_2 \times \mu \circ (\pi_1 \times \pi_3) : \mathcal{V} \times \mathcal{V} \times \mathcal{V} \to \mathcal{V} \times \mathcal{V} : (u, v, w) \mapsto (v, u \otimes w)$. It is straightforward to check that $E$ with $\varepsilon_r : \text{Inv}_r(\text{Inv}_\mathcal{V}(\mathcal{A}_\alpha)) \to \text{Inv}_\mathcal{V}(\mathcal{A}_\alpha)$ is a 1-equivalence. \qed

If $\mathcal{M} = \text{Cat}$, then the objects in this limit can be viewed as follows: The objects of $\text{Inv}_{\mathcal{V}}^\infty(\mathcal{A}_\alpha)$ are sequences of pairs $\{(f_n, \phi_n)\}_{n \in \mathbb{N}}$ where $f_n \in \text{Inv}_{\mathcal{V}}(\mathcal{A}_\alpha)$ and $\phi_n : f_{n+1}(\mathbb{1}) \to f_n$ is an isomorphism in $\text{Inv}_{\mathcal{V}}(\mathcal{A}_\alpha)$. A morphism $\zeta$ between two objects $\{(f_n, \phi_n)\}$ and $\{(g_n, \varphi_n)\}$ is a set of arrows $\zeta_n : f_n \to g_n$ satisfying the evident compatibility conditions. The action on $\text{Inv}_{\mathcal{V}}^\infty(\mathcal{A}_\alpha)$ is defined pointwise; that is, $\alpha^\infty(u, v_{\text{op}})\{(f_n, \phi_n)\} = \{(\tilde{\alpha}^\infty_n(u, v_{\text{op}})(f_n), \tilde{\alpha}^\infty_n(u, v_{\text{op}})(\phi_n))\}$ where $\tilde{\alpha}^\infty_n$ is the action on $\text{Inv}_{\mathcal{V}}^\infty(\mathcal{A}_\alpha)$.

We can also see that $\text{Inv}_{\mathcal{V}}^\infty(\mathcal{A}_\alpha)$ is stable under the action by using the object-wise description given above. Define

$$\psi : \text{Inv}_{\mathcal{V}}^\infty(\mathcal{A}_\alpha) \to \text{Inv}_r(\text{Inv}_{\mathcal{V}}^\infty(\mathcal{A}_\alpha))$$

by

$$\psi\{(f_n, \phi_n)\}(u) = \{(f_{n+1}(\mathbb{1}), \phi_{n+1}(\mathbb{1}, u))\}$$

for any $\{(f_n, \phi_n)\}$ in $\text{Inv}_{\mathcal{V}}^\infty(\mathcal{A}_\alpha)$. Then

$$(\varepsilon_l \circ \psi)\{(f_n, \phi_n)\} = \psi\{(f_n, \phi_n)\}(\mathbb{1}) = \{(f_{n+1}(\mathbb{1}), \phi_{n+1}(\mathbb{1}, \mathbb{1}))\} \cong \{(f_n, \phi_n)\}$$

and for any $g : \mathcal{V} \to \text{Inv}_{\mathcal{V}}^\infty(\mathcal{A}_\alpha)$ in $\text{Inv}(\text{Inv}_{\mathcal{V}}^\infty(\mathcal{A}_\alpha))$ such that $g(\mathbb{1}) = \{(f_n^{(1)}, \phi_n^{(1)})\}$. Note that

$$g(u) \cong \tilde{\alpha}^\infty_n(u, v_{\text{op}})(f_{n+1}(\mathbb{1})) \cong \tilde{\alpha}^\infty_n(u, v_{\text{op}})(f_n^{(1)}, \phi_n^{(1)})$$

Therefore

$$(\psi \circ \varepsilon_l)(g)(u) = \psi(g(\mathbb{1}))(u) = \psi\{(f_{n+1}(\mathbb{1}), \phi_{n+1}(\mathbb{1}, u))\}$$

$$= (f_{n+1}(\mathbb{1}, u), \phi_{n+1}(\mathbb{1}, u))$$

$$\cong (f_n^{(1)}(u), \phi_n^{(1)}(u))$$

$$= (f_n^{(1)}(u), \phi_n^{(1)}(u))$$

$$= (f_n^{(1)}(u), \phi_n^{(1)}(u)) = g(u)$$
Hence by induction \( \phi(e_i(\tilde{f})) = \tilde{f} \).

In the case when \( \mathcal{V} \) is a symmetric monoidal category, the inverse equivalence \( E : \text{Inv}_\mathcal{V}(\mathcal{A}_\alpha) \to \text{Inv}_r(\text{Inv}_\mathcal{V}(\mathcal{A}_\alpha)) \) of \( e_r : \text{Inv}_r(\text{Inv}_\mathcal{V}(\mathcal{A}_\alpha)) \to \text{Inv}_\mathcal{V}(\mathcal{A}_\alpha) \) in the proof of Theorem 3.5 is given by \( E(f)(v)(w) = f(v)(u \otimes w) \). Observe that for every \( g \) in \( \text{Inv}_r(\text{Inv}_\mathcal{V}(\mathcal{A}_\alpha)) \),

\[
E(\epsilon(g))(v)(w) = \epsilon(g)(v)(u \otimes w) \\
= g(1)(v)(u \otimes w) \\
\cong g(u)(v \otimes u)(u \otimes w) \\
\cong g(u)(v)(w).
\]

Dualizing the definition of the stabilization and obtain the costabilization.

**Definition 3.6.** The costabilization of \( \mathcal{A}_\alpha \) is a stable 0-cell \( \text{coStab}_\mathcal{V}(\mathcal{A}_\alpha) \), together with a \( \mathcal{V} \)-equivariant 1-cell \( \eta : \mathcal{A}_\alpha \to \text{coStab}_\mathcal{V}(\mathcal{A}_\alpha) \) in \( \mathcal{V} \mathcal{M} \) such that for every stable 0-cell \( \mathcal{B}_\beta \), the induced functor \( \eta^* : \text{Fun}_\mathcal{V}(\mathcal{A}_\alpha, \mathcal{B}_\beta) \to \text{Fun}_\mathcal{V}(\text{coStab}_\mathcal{V}(\mathcal{A}_\alpha), \mathcal{B}_\beta) \) is an equivalence of categories.

**Remark 3.7.** Similar to the definition of stabilization, the definition above implies for every \( \mathcal{V} \)-equivariant 1-cell \( f : \mathcal{B}_\beta \to \mathcal{A}_\alpha \) in \( \mathcal{V} \mathcal{M} \) such that \( \beta \) is stable, there exists a 1-cell \( \tilde{f} : \mathcal{B}_\beta \to \text{Stab}_\mathcal{V}(\mathcal{A}_\alpha) \) together with a 2-isomorphism

\[
\begin{array}{ccc}
\mathcal{A}_\alpha & \xrightarrow{\eta} & \text{coStab}_\mathcal{V}(\mathcal{A}_\alpha) \\
\downarrow & \searrow & \downarrow \\
\mathcal{B}_\beta & \xrightarrow{\tilde{f}} & \text{Stab}_\mathcal{V}(\mathcal{A}_\alpha)
\end{array}
\]

(3.11)

where \( \tilde{f} \) is unique up to unique 1-isomorphism.

Let \( \Delta : \mathcal{B} \mathcal{W} \to \mathcal{B} \mathcal{W} \times \mathcal{B} \mathcal{W} \) be the diagonal functor. Let \( \nu : \mathcal{B} \mathcal{W} \times \mathcal{B} \mathcal{W} \to \mathbf{Cat} \) be the functor given by

\[
\nu_s((x, y^\text{op}),(z, w^\text{op}))(u) = \left\{ \begin{array}{ll}
(x \otimes u) \otimes w & \text{if } s = r \\
 z \otimes (u \otimes y) & \text{if } s = l
\end{array} \right. \quad (3.12)
\]

Given a \( \mathcal{V} \)-action \( \alpha : \mathcal{B} \mathcal{W} \to \mathcal{M} \) on a 0-cell \( \mathcal{A} \), define \( \Upsilon^{s,<}_\alpha \) as the composition

\[
\mathcal{B} \mathcal{W} \times \mathcal{B} \mathcal{W} \xrightarrow{\text{id} \times \Delta} \mathcal{B} \mathcal{W} \times \mathcal{B} \mathcal{W} \xrightarrow{(\nu_s, \alpha)} \mathbf{Cat} \times \mathcal{M} \xrightarrow{\otimes} \mathcal{M}.
\]

Define a \( \mathcal{V} \)-action \( \text{coInv}_\alpha : \mathcal{B} \mathcal{W} \to \mathcal{M} \) as

\[
\text{coInv}_\alpha(\cdot) = \int x : \mathcal{B} \mathcal{W}^{\text{op}} \Upsilon^{s,<}_\alpha(-, (x, x)) \quad (3.14)
\]

We write \( \text{coInv}_\alpha(\mathcal{A}_\alpha) \) for \( \text{coInv}_\alpha(\mathcal{A}_\alpha)(s, s) \).

Let \( \tilde{\nu}_l = \nu_l((1, 1^\text{op}), -) \) and \( \tilde{\nu}_r = \nu_r((-1, 1^\text{op})) \). Then, for \( s \neq s' \) in \( \{l, r\} \), \( \Upsilon^{s,<}_\alpha(s, s, -) \) is naturally isomorphic to the composition

\[
\mathcal{B} \mathcal{W} \xrightarrow{\Delta} \mathcal{B} \mathcal{W} \times \mathcal{B} \mathcal{W} \xrightarrow{(\tilde{\nu}_{s'}, \alpha)} \mathbf{Cat} \times \mathcal{M} \xrightarrow{\otimes} \mathcal{M}
\]

which we denote by \( \tilde{\nu}_{s'} \otimes \alpha \). Thus, \( \text{coInv}_\alpha(\mathcal{A}_\alpha) \) is isomorphic to

\[
\int x : \mathcal{B} \mathcal{W}^{\text{op}} \tilde{\nu}_{s'} \otimes \alpha(x, x)
\]
In this case, the inclusion \( \{1\} \to \mathcal{V} \) induces a \( \mathcal{V} \)-equivariant 1-cell
\[
\eta_\alpha : \mathcal{A}_\alpha \to \text{coInv}_\alpha(\mathcal{A}_\alpha)
\]
which we call the coevaluation at \( 1 \). The \( \mathcal{V} \)-equivariance of \( \eta_\alpha \) is similar to the case of \( \epsilon_\alpha \) above.

**Theorem 3.8.** Let \( \alpha \) be an left (respectively right) \( \mathcal{V} \)-action on a 0-cell \( \mathcal{A} \) in a strict 2-category \( \mathcal{M} \) that is powered and copowered over \( \text{Cat} \). Then \( \mathcal{A}_\alpha \) is stable if and only if \( \eta_{\alpha} \) (respectively \( \eta_{\alpha} \)) is a 1-equivalence in \( \mathcal{V}\mathcal{M} \).

**Proof.** This is dual to Theorem 3.1. \( \square \)

Given any 0-cell \( \mathcal{A}_\alpha \) in \( \mathcal{V}\mathcal{M} \) let \( \eta = \eta_\alpha \circ \eta_{\alpha} \). Let \( \text{coInv}_1^V(\mathcal{A}_\alpha) = \text{coInv}_1(\mathcal{A}_\alpha) \) and \( \text{coInv}_2^V(\mathcal{A}_\alpha) = \text{coInv}_V(\text{coInv}_{V^{-1}}^V(\mathcal{A}_\alpha)) \) for every \( n > 1 \). Define \( \text{coInv}_V^\infty(\mathcal{A}_\alpha) \) as the colimit of sequence
\[
\text{coInv}_1^V(\mathcal{A}_\alpha) \xrightarrow{\eta} \text{coInv}_2^V(\mathcal{A}_\alpha) \xrightarrow{\eta} \cdots,
\]
provided that it exists. Let \( \eta^\infty : \mathcal{A}_\alpha \to \text{coInv}_V^\infty(\mathcal{A}_\alpha) \) be \( \eta_\alpha \) composed with the transfinite compositions of the maps in the diagram.

**Theorem 3.9.** Assume that colimits in \( \mathcal{M} \) exist. Then \( \text{coInv}_V^\infty(\mathcal{A}_\alpha) \) together with \( \eta^\infty : \mathcal{A}_\alpha \to \text{coInv}_V^\infty(\mathcal{A}_\alpha) \) is a costabilization of \( \mathcal{A}_\alpha \).

**Proof.** The proof is dual to the proof of Theorem 3.4. \( \square \)

In the case when \( V \) is a symmetric monoidal category, we can consider the bi-reverse as a stabilization functor due to the following theorem.

**Theorem 3.10.** If \( V \) is a symmetric monoidal category, then \( \text{coInv}_V(\mathcal{A}_\alpha) \) is equivalent to \( \text{coInv}_V^\infty(\mathcal{A}_\alpha) \).

**Proof.** The proof is dual to the proof of Theorem 3.5. \( \square \)

Let \( V \) be a symmetric monoidal category and \( \mathcal{M} = \text{Cat} \). Note that if \( \mathcal{A}_\alpha \) is mute on the right, then \( \text{coInv}_V^\infty(\mathcal{A}_\alpha) \) is equivalent to \( \text{coInv}_V(\mathcal{A}_\alpha) \). The category \( \text{coInv}_V(\mathcal{A}_\alpha) \) can be constructed (up to equivalence of categories) as follows. Let \( \text{coInv}_V(\mathcal{A}_\alpha) \) be the category whose objects are the collections of all pairs \( (u,a) \) and triples \( (u,v,a) \) where \( u,v \) are objects in \( V \) and \( a \) an object in \( \mathcal{A}_\alpha \). A morphism between pairs is a morphism in \( V \times \mathcal{A} \) and a morphism triples is a morphism in \( V \times V \times \mathcal{A} \). A morphism from the triple \( (u,v,a_1) \) to a pair \( (w,a_2) \) is either a morphism from \( (u,a_1) \) to \( (w,a_2) \) or a morphism from \( (u \otimes v, \alpha(v, \mathbb{1}^{\text{op}})(a_1)) \) to \( (w,a_2) \). There is no morphism from a pair to a triple in \( \text{coInv}_V(\mathcal{A}_\alpha) \). Let \( S \) be the collection of morphisms from triples to pairs for which are identity in both coordinates; that is, a morphism from \( (u,v,a_1) \) to \( (w,a_2) \) is in \( S \) if either \( a_1 = a_2 \) and \( u = w \) and the corresponding map \( (u,a_1) \) to \( (w,a_2) \) is identity or \( u \otimes v = w \) and \( \alpha(v, \mathbb{1}^{\text{op}})(a_1) = a_2 \) and the corresponding map from \( (u \otimes v, \alpha(v, \mathbb{1}^{\text{op}})(a_1)) \) to \( (w,a_2) \) is identity. Then the category \( \text{coInv}_V(\mathcal{A}_\alpha) \) is given by the localization of \( \text{coInv}_V(\mathcal{A}_\alpha) \) at \( S \). Observe that every triple in \( \text{coInv}_V(\mathcal{A}_\alpha) \) become isomorphic to a pair after passing to localization, \( \text{coInv}_V(\mathcal{A}_\alpha) \). Besides, every pair of the form \( (u \otimes v, \alpha(v, \mathbb{1}^{\text{op}})(a_1)) \) becomes isomorphic to the pair \( (u,a_1) \). The \( V \)-action on \( \text{coInv}_V(\mathcal{A}_\alpha) \) is given by tensoring of first coordinate from the left; that is, if \( x \) is in \( V \), then

If \( \mathcal{A} \) is stable, then \( \eta_\alpha : \mathcal{A}_\alpha \to \text{coInv}_V(\mathcal{A}_\alpha) \) is an equivalence of categories. This can be realized via the isomorphism given by \( (u,a) \mapsto \alpha(u, \mathbb{1}^{\text{op}})(a). \)
Combining the results of Theorems 3.4 and 3.9, we obtain our main result.

**Corollary 3.11 (Theorem 1.1).** If \( \mathcal{M} \) is a complete (resp. cocomplete) 2-category, then \( st\mathcal{M} \) is coreflective (resp. reflective) in \( \mathcal{V}\mathcal{M} \).

### 3.1. Lax stabilization and spectrification

Let \( \mathcal{V} \) be symmetric monoidal and \( \alpha : B\mathcal{V} \to \mathcal{M} \) be a \( \mathcal{V} \)-action. Assume \( \{s, s'\} = \{l, r\} \). Define a \( \mathcal{V} \)-action \( \ell \text{Inv}_s(\alpha) : B\mathcal{V} \to \mathcal{M} \) as

\[
\ell \text{Inv}_s(\alpha)(-) = \int_{x:B\mathcal{V}^{op}} \Theta_{s,\alpha}(-, (x, x))
\]

so that \( \ell \text{Inv}_s(\alpha)(\ast, \ast) = \ell \text{Inv}_s(\mathcal{A}_\alpha) \) is isomorphic to

\[
\int_{x:B\mathcal{V}^{op}} \Theta_{\mu,\alpha}(x, x)
\]

Assume that \( \mu \) is mute on the right (left) and \( s = r \) (\( s = l \)). Then we can call \( \ell \text{Inv}_s(\mathcal{A}_\alpha) \) the lax stabilization of \( \mathcal{A}_\alpha \). There is a canonical 1-cell

\[
\iota_\alpha : \text{Inv}_s(\mathcal{A}_\alpha) \to \ell \text{Inv}_s(\mathcal{A}_\alpha).
\]

We call the left adjoint of \( \iota_\alpha \) in \( \mathcal{V}\mathcal{M} \) the *spectrification* whenever it exists.

Assume that \( \alpha \) factors through the category of adjunctions \( \text{Adj}(\mathcal{M}) \) whose 1-cells are adjunctions in \( \mathcal{M} \). Then, there is an adjoint action \( \tilde{\alpha} \) on \( \mathcal{A} \) so that for each \( u, v \) in \( \mathcal{V} \), \( \alpha(u, u^{op}) \) is right adjoint to \( \tilde{\alpha}(v, u^{op}) \). Then there exist a equivariant 1-cell

\[
\hat{\alpha} : \mathcal{A}_{\tilde{\alpha}} \to \ell \text{Inv}_s(\mathcal{A}_\alpha)
\]

induced by the isomorphism \( [\mathcal{A}, \mathcal{V} \mathcal{C}, \mathcal{A}] \cong \mathcal{V} \mathcal{C}(\mathcal{V}, [\mathcal{A}, \mathcal{A}]) \). Therefore, if there is a spectrification, then there exist a \( \mathcal{V} \)-equivariant 1-cell \( \mathcal{A}_{\tilde{\alpha}} \to \text{Inv}_s(\mathcal{A}_\alpha) \). Composing with \( \epsilon_s \) we obtain a \( \mathcal{V} \)-equivariant 1-cell \( \mathcal{A}_{\tilde{\alpha}} \to \mathcal{A}_\alpha \).

In the case \( \mathcal{M} = \mathcal{C} \mathcal{A} \) and \( s = r \), then \( \hat{\alpha} \) is a pair \( (\hat{\alpha}, \sigma_{\hat{\alpha}}) \) where \( \hat{\alpha}(a)(u) = \alpha(1, u^{op})(a) \) and

\[
\sigma_{\hat{\alpha}} : a \to \alpha(1, u^{op})(\hat{\alpha}(1, u^{op})(a)) = \alpha(1, u^{op})(\hat{\alpha}(a))
\]

is the adjunct of the identity map \( \text{id}_{\hat{\alpha}(1, u^{op})(a)} : \hat{\alpha}(1, u^{op})(a) \to \hat{\alpha}(1, u^{op})(a) \). If \( L \) denotes the spectrification, then the \( \mathcal{V} \)-equivariant functor \( \mathcal{A}_{\tilde{\alpha}} \to \mathcal{A}_\alpha \) is given by the composition

\[
\mathcal{A}_{\tilde{\alpha}} \xrightarrow{\hat{\alpha}} \ell \text{Inv}_r(\mathcal{A}_\alpha) \xrightarrow{\iota_r \text{Inv}_r(\mathcal{A}_\alpha)} \iota_{\epsilon_r \text{Inv}_r(\mathcal{A}_\alpha)} \implies \mathcal{A}_\alpha.
\]

### 4. Examples of stable actions and stabilizations

Our primary examples of stable actions and stabilizations come from reduced (co)homology theories. Traditionally, (co)homology theories are measurement tools that compare homotopy properties of objects via comparing corresponding algebraic data. However, one does not have to use algebraic data to make such comparisons. It can be done by means of any functor between any two category. If \( \mathcal{A} \) and \( \mathcal{B} \) are two categories, and \( F : \mathcal{A} \to \mathcal{B} \) is a functor between them than if \( Fx \not\cong Fy \) in \( \mathcal{B} \) then \( x \not\cong y \) in \( \mathcal{A} \). More generally, if \( F \) is a functor that preserves certain structure (such as homotopy) than \( Fx \) and \( Fy \) not sharing this structure implies neither do \( x \) and \( y \). In this paper we propose a more general and unorthodox definitions of a homology and cohomology functors, whose source and target are homotopy categories, where the target has the notion of “exact sequence”. For the sake of computational tools, as in the classical (co)homology theories, we require these functors to satisfy certain axioms similar to the classical Eilenberg-Steenrod axioms. Notions of suspension...
and loop functors, which are in fact actions of certain monoidal categories, are also
generalized accordingly.

**Some conventions about categories.** We assume the axiom of Grothendieck
universes, cite [4]. For a universe \( \mathcal{U} \), when a monoidal category \( \mathcal{V} \) acts on a category
\( \mathcal{A} \), we assume \( \mathcal{V} \) is \( \mathcal{U} \)-small if \( \mathcal{A} \) is a \( \mathcal{U} \)-category, and the category \( \mathcal{A} \) will belong to
a \( \mathcal{U}^+ \)-category of \( \mathcal{U} \)-categories.

4.1. \( \mathcal{V} \)-graded (co)homology theories. For this section we let \( \mathcal{V} \) be symmetric
monoidal and consider \( \mathcal{V} \)-actions on categories that are \( \pi_0 \) of pointed \((\infty,1)\)-categories; in particular, homotopy categories of some pointed homotopy theories.
Therefore, in the categories on which \( \mathcal{V} \) acts, homotopy fiber and cofiber sequences
exists. Let \( f : A \to S \) be a functor between such categories where \( S \) admits a notion
of exact sequences (e.g., an Abelian category). We say \( f \) is left exact if it sends
homotopy fiber sequences to exact sequences, and right exact if it sends homotopy
cofiber sequences to exact sequences.

We first consider a general definition of cohomology and homology functors in
view of Eilenberg-Steenrod axioms. Let \( A \) and \( S \) be two pointed categories such
\( S \) has exact sequences.

**Cohomology Functors:** A product preserving right exact functor \( h : A^{op} \to S \)
is called a cohomology functor. The full subcategory of cohomology
functors in \( [A^{op}, S] \) is denoted by \( \operatorname{cohml}(A, S) \).

**Homology Functors:** A coproduct preserving left exact functor \( h : A \to S \)
is called a homology functor. The full subcategory of homology functors in
\( [A, S] \) is denoted by \( \operatorname{hml}(A, S) \).

Let \( \alpha \) be a \( \mathcal{V} \)-action on \( A^{op} \) that is mute on the left. Consider \( S \) with the
trivial action. Then the functor category \( [A^{op}, S] \) admits a \( \mathcal{V} \)-action as described
in 2.5, which is mute on the right. We denote this action by \( [\alpha, 1] \). Under certain
conditions \( [\alpha, 1] \) restricts to the subcategory \( \operatorname{cohml}(A, S) \) of cohomology functors.
For example, it is enough to assume \( \alpha \) is cocontinuous (preserves \((\infty,1)\)-colimits
that exist in \( A \)).

We define category of cohomology theories graded over \( \mathcal{V} \) as follows:

**Definition 4.1.** The category of cohomology theories graded over \( \mathcal{V} \)
is defined as
\[
\operatorname{COHML}_\mathcal{V}(A, S) = \operatorname{Stab}_\mathcal{V}(\operatorname{cohml}(A, S)_{[\alpha, 1]});
\]
i.e., the stabilization of cohomology functors with respect to \([\alpha, 1] \). We say that
a cohomology theory graded over \( \mathcal{V} \) is an object in the category of cohomology theories.

We can similarly define homology theories. Suppose now that \( \alpha \) is a \( \mathcal{V} \)-action on
\( A \) that is mute on the left, so that the induced action \([\alpha, 1] \) on \([A, S] \) restricts to
\( \operatorname{hml}(A, S) \) (e.g., \( \alpha \) is cocontinuous).

**Definition 4.2.** The category of homology theories graded over \( \mathcal{V} \) is defined as
\[
\operatorname{HML}_\mathcal{V}(A, S) = \operatorname{Stab}_\mathcal{V}(\operatorname{hml}(A, S)_{[\alpha, 1]});
\]
i.e., the stabilization of homology functors with respect to action induced by \([\mathcal{T}, 1] \).
We say that a homology theory graded over \( \mathcal{V} \) is an object in the category of homology theories.
4.1.1. Axiomatic interpretation of the definitions above. Unfolding the definitions above we can see that a cohomology theory graded over $\mathcal{V}$ consists of a functor $h : \mathcal{V} \times A^{\text{op}} \to S$ such that each $h^u = h(1)(u)$ is a cohomology functor, together with natural isomorphisms

$$\sigma_{u,v} : h^u \to h^{u \otimes v} \circ \alpha(1, v^{\text{op}})$$

such that the following diagram commutes

$$h^u(a) \xrightarrow{\sigma_{u,v}} h^{u \otimes v}(\alpha(1, v^{\text{op}})(a)) \xrightarrow{\sigma_{u \otimes v, w}} h^{u \otimes v \otimes w}(\alpha(1, v \otimes w^{\text{op}})(a))$$

(4.17)

for every $u, v, w$ in $\mathcal{V}$. This diagram is induced by the commuting triangle 2.2. Moreover, naturality of $\sigma$ implies that the following diagram commutes

$$h^u(a) \xrightarrow{\sigma_{u,v}} h^{u \otimes v}(\alpha(1, v^{\text{op}})(a)) \xrightarrow{\sigma_{u \otimes v, w}} h^{u \otimes v \otimes w}(\alpha(1, v \otimes w^{\text{op}})(a)) \xrightarrow{\sigma_{u, v \otimes w}} h^{u \otimes v \otimes w}(\alpha(1, v \otimes w^{\text{op}})(a))$$

(4.18)

for any morphism $m : v \to w$ in $\mathcal{V}$.

Similarly, a homology theory graded over $\mathcal{V}$ consists of a functor $h : \mathcal{V} \times A \to S$ such that for every $u$ in $\mathcal{V}$, the functor $h_u$ given by $h_u(a) = h(1, u)(a)$ is a homology functor, together with natural isomorphisms

$$\sigma_{u,v} : \alpha(1, v^{\text{op}}) \circ h_{u \otimes v} \to h_u$$

satisfying similar conditions dual to the ones above.

4.2. Some known examples of (co)homology theories. Definitions of several existing cohomology and homology theories fit into the setting that described above, after choosing actions appropriately. Some examples of actions below are non-strict; however, they are particular cases of the setting above in view of Remark 2.1. In the examples below, a space means a compactly generated and weakly Hausdorff topological space.

4.2.1. Generalized ordinary (co)homology theories. Let $\mathbb{N}$ be the natural numbers considered as a monoidal category with identity morphisms as the only morphisms. Let $\text{Ab}$ denote the category of abelian groups. Consider $\text{Ab}$ with the trivial $\mathbb{N}$-action. Let $\text{ho}\mathcal{S}$ be the homotopy category of pointed spaces with respect to Quillen model structure. Define the $\mathbb{N}$-action $\Sigma$ by

$$\Sigma(n, m^{\text{op}})(X) = X \wedge S^n = \Sigma^n X$$

for $n, m$ in $\mathbb{N}$ and $X$ in $\text{ho}\mathcal{S}$. Then a generalized (co)homology theory $h$ in our setting gives a generalized (co)homology theory satisfying the Eilenberg-Steenrod (co)homology axioms.
4.2.2. Equivariant cohomology theories graded over representations. Let $G$ be a compact Lie group and $U$ be a complete $G$-universe; that is, a countably infinite dimensional orthogonal $G$-representation having non-zero $G$-fixed points and contains the direct sum $V^\otimes \lambda$ for every finite dimensional representation $V$ and every cardinal $\lambda \leq \aleph_0$, see [22, Defn. 1.1.12] or [15, Ch. II, Defn. 1.1]. Let $\mathcal{V}$ be the monoidal category $RO(G; U)$ whose objects are orthogonal $G$-representations embeddable in $U$ and whose morphisms are $G$-linear isometric isomorphisms and monoidal product is the direct sum. For $V$ an object in $\mathcal{V}$, denote by $S^V$ the one point compactification of $V$. Let $hRO(G; U)$ be the quotient of $RO(G; U)$ with the relation given by $f \sim g : V \to W$ if the induced maps $f_* \simeq g_* : S^V \to S^W$ are stably homotopic [17, see pp.130]. Let $G\mathcal{F}$ denote the category of pointed $G$-spaces and pointed $G$-maps with the standard model structure and $ho\ G\mathcal{F}$ be its homotopy category. Let $Ab$ be the category of abelian groups and 1 denote the trivial $\mathcal{V}$-action on $Ab$. We define an action $\Sigma$ on $ho\ G\mathcal{F}^{op}$ as follows:

$$\Sigma(V,W^{\text{op}})(X) = \Sigma^W X = X \wedge S^W;$$

that is, the usual suspension action. For each $V, W$ in $\mathcal{V}$ the suspension functor $\Sigma^W$ preserves homotopy colimits in $G\mathcal{F}$, see [11]. Hence $\Sigma$ induces an action on $\text{cohom}(ho\ G\mathcal{F}, Ab)$. An RO$(G)$-graded cohomology theory is an object in the stabilization of $\text{cohom}(ho\ G\mathcal{F}, Ab)$ with respect to the action $[\Sigma, 1]$. Therefore, a cohomology theory is a pair $(h, \sigma)$ where $h : hRO(G; U) \times ho\ G\mathcal{F}^{op} \to Ab$ is a functor with $h^V = h(V, -)$ a cohomology functor for every $V$ in $hRO(G; U)$, and

$$\sigma_{V,W} : h^V \to h^{V \oplus V \circ \Sigma^W}$$

are natural isomorphisms. For each pair of representations $W, Z$, as above, we have the following diagram commutes

If $m : W \to W'$ is a morphism in $hRO(G; U)$, then by naturality of $\sigma$ we have the following diagram commutes

This definition of cohomology theory is the same as the one given in [17, VII, Defn. 1.1]. Dualizing the definition, one obtains $RO(G)$-graded equivariant homology theories.

4.2.3. Equivariant cohomology theories graded over actions on spheres. Let $\mathcal{V}$ be the category whose objects are pointed spheres (with $S^{-1} = *$) with continuous base-point preserving $G$-actions, morphisms are pointed $G$-isomorphisms between them. It is a monoidal category with $G$-smash product with diagonal action and $S^{-1} = *$ as the unit. We denote an object in $\mathcal{V}$ by $S^m_\mu$, where $n$ is the dimension
of the sphere and $\mu$ is the $G$-action. Note that $RO(G; U)$ faithfully embeds in $\mathcal{V}$. Define $\Sigma$ on $\text{ho}\,G\mathcal{F}^{\text{op}}$ similar to the above; i.e.,

$$\Sigma(S^n_\xi, (S^n_\mu)^{\text{op}})(X) = \Sigma^{S^n_\mu}X = X \wedge S^n;$$

with the diagonal action. Clearly, $\Sigma$ induces an action on $\text{cohml}(\text{ho}\,G\mathcal{F}, A\mathbb{B})$. An $\mathcal{V}$-graded cohomology theory is an object in the stabilization of $\text{cohml}(\text{ho}\,G\mathcal{F}, A\mathbb{B})$ with respect to the action $[\Sigma, 1]$. Therefore, a $\mathcal{V}$-graded cohomology theory is a pair $(h, \sigma)$ where $h : \mathcal{V} \times \text{ho}\,G\mathcal{F}^{\text{op}} \to A\mathbb{B}$ is a functor with $h^{S^n_\mu} = h(S^n_\mu, -)$ a cohomology functor for every $S^n_\mu$ in $\mathcal{V}$, and

$$\sigma^{S^n_\mu} : h^{S^n_\mu} \to h^{S^n_\mu \wedge S^n_\xi} \circ \Sigma^{S^n_\mu}$$

are natural isomorphisms. The compatibility diagrams commute as above.

4.2.4. Parameterized cohomology theories graded over vector bundles. For the original definitions of parameterized reduced cohomology theories, see [18]. These theories are cohomology theories for the categories of ex-spaces. Let $B$ be a space and $\mathcal{F}/B$ be the category of spaces over $B$. The terminal object in this category is the identity of $B$. The category of based spaces over $B$, $\mathcal{F}_B$, is the under category $\mathcal{F}_B = \text{id}_B/(\mathcal{F}/B)$. Objects of this category are often called ex-spaces of $B$, see e.g. [9]. Let $\text{ho}\,\mathcal{F}_B$ be the homotopy category of $\mathcal{F}_B$ with respect to the model structure of [18, Ch. 6]. Ordinary parameterized cohomology theories takes values from $\text{ho}\,\mathcal{F}_B$ and graded over integers in [18]. On the other hand, this category admits other obvious suspensions than the ordinary one.

Let $\mathcal{V}$ be the monoidal category whose objects are real vector bundles over $B$ with fiberwise inner products (i.e., taking values in $B \times R$ the trivial $R$-bundle over $B$), fiberwise linear isometric embeddings as morphisms, Whitney sum $\oplus_B$ as the monoidal product and $0$-bundle as the monoidal unit. For $\xi$ in $\mathcal{V}$, denote by $S^n_\mathcal{V}$ the associated fiber-wise one point compactification, which is a sphere bundle over $B$ with point as the section induced by the zero section of $\xi$. Given bundles $\xi, \eta$ in $\mathcal{V}$, define $\Sigma_B$ by

$$\Sigma_B(\xi, \eta^{\text{op}})(t : X \to B) = t \wedge_B S^n,$$

see [18, Definition 1.3.3] for $\wedge$. The construction $\Sigma_B$ defines an action on $\text{ho}\,\mathcal{F}_B$. Moreover, for every $\xi, \eta$ in $\mathcal{V}$ $\Sigma_B(\xi, \eta^{\text{op}})$ preserves homotopy colimits in $\mathcal{F}_B$. Consider $A\mathbb{B}$ with the trivial $\mathcal{V}$-action 1. Then the $\mathcal{V}$-action $[\Sigma_B, 1]$ on $[\text{ho}\,\mathcal{F}_B^{\text{op}}, A\mathbb{B}]$ induces a $\mathcal{V}$-action on $\text{cohml}(\text{ho}\,\mathcal{F}_B, A\mathbb{B})$. A parameterized cohomology theory graded over $\mathcal{V}$ is an object $\text{Stab}_\mathcal{V}(\text{cohml}(\text{ho}\,\mathcal{F}_B, A\mathbb{B}))$. Since $\mathcal{V}$ is symmetric monoidal, the category $\text{Stab}_\mathcal{V}(\text{cohml}(\text{ho}\,\mathcal{F}_B, A\mathbb{B}))$ is equivalent to $\text{Inv}_\mathcal{V}(\text{cohml}(\text{ho}\,\mathcal{F}_B, S))$. Then a parameterized cohomology theory graded over $\mathcal{V}$ is a pair $(h, \sigma)$ where $h : \mathcal{V} \to \text{cohml}(\text{ho}\,\mathcal{F}_B, A\mathbb{B})$ is a functor and $\sigma$ is the associated desuspension. More precisely, a parameterized cohomology theory graded over $\mathcal{V}$ consists of a functor $h : \mathcal{V} \times \text{ho}\,\mathcal{F}_B^{\text{op}} \to A\mathbb{B}$ such that for every $\xi$ in $\mathcal{V}$ $h^\xi = h(1)(\xi)$ is a cohomology functor in $\text{ho}\,\mathcal{F}_B$, together with natural isomorphisms

$$\sigma^{\xi, \eta} : h^\xi \to h^{\xi \oplus_B \eta} \circ \Sigma_B$$
for every $\xi, \eta$ in $V$. Moreover, for every object $\tau : X \to B$ in $\mathcal{F}_B$ the following diagram commutes

$$
\begin{array}{ccc}
\sigma_{\xi, \eta} & \xrightarrow{h\xi(\tau)} & h\xi \oplus_B \eta (\Sigma_B^\eta \tau) \\
\downarrow{\sigma_{\xi, \eta \oplus_B \zeta}} & & \downarrow{h\xi \oplus_B \eta \oplus_B \zeta (\Sigma_B^{\eta \oplus_B \zeta} \tau)} \\
\sigma_{\xi, \eta \oplus_B \zeta} & \xrightarrow{h\xi(\tau)} & h\xi \oplus_B \eta \oplus_B \zeta (\Sigma_B^{\eta \oplus_B \zeta} \tau) \\
\end{array}
$$

If $f : \eta \to \eta'$ is a morphism in $V$, then we have a commutative diagram as follows:

$$
\begin{array}{ccc}
h\xi(\tau) & \xrightarrow{\sigma_{\xi, \eta}} & h\xi \oplus_B \eta (\Sigma_B^\eta \tau) \\
\downarrow{\sigma_{\xi, \eta \oplus_B \zeta}} & & \downarrow{h\xi \oplus_B \eta \oplus_B \zeta (\Sigma_B^{\eta \oplus_B \zeta} \tau)} \\
h\xi \oplus_B \eta' (\Sigma_B^{\eta'} \tau) & \xrightarrow{h\xi \oplus_B \eta' (id \oplus_B f)} & h\xi \oplus_B \eta' (\Sigma_B^{\eta'} \tau) \\
\end{array}
$$

since $\sigma$ is natural transformation. Passing to isomorphism classes in $V$, one obtains $KO_0(B)$-graded parameterized cohomology theories (for a slightly different definition see [21, Sec.1]).

One can also pass to bundles fibrewise embeddable in a universe bundle for convenience. Let $u$ be a vector bundle in $V$, such that for every finite dimensional vector bundle $\xi$ in $V$ and for every $\lambda \leq \aleph_0$, there is a monomorphism $\xi \otimes_B \lambda \to u$ of vector bundles. Let $V_u$ be the subcategory of $V$ whose objects are such $\xi$’s and whose morphisms are isomorphisms in $V$ between them. Denote by $\Sigma_B^u$ the restriction of the action $\Sigma_B$. Then, the associated stabilization of cohomology functors with respect to the action $V_u$-action $[\Sigma_B^u, 1]$ gives the parametrized cohomology theories indexed by a universe.

One can go further and grade parameterized cohomology theories over the symmetric monoidal category $\mathcal{V}$ of sphere bundles over $B$ admitting sections, with fibrewise isomorphisms of pointed bundle maps as morphisms. More precisely, objects of $\mathcal{V}$ are pairs $(\xi, s)$ where $\xi : E \to B$ is a sphere bundle and $s : B \to E$ is a section. The monoidal product is given by the fiber-wise smash product on the first coordinate and the unique section of the pushout in [18, Definition 1.3.4] on the second coordinate. The action is defined similarly; that is, given objects $(\xi, s), (\eta, t)$ in $\mathcal{V}$, $\Sigma_B^u$ is given by

$$
\Sigma_B^u((\xi, s), (\eta, t)^{op})(\tau : X \to B) = \tau \land_B \eta.
$$

Then $\Sigma_B^u$ induces an action on $\text{cohml}(ho \mathcal{F}_B, Ab)$ and an object in the stabilization of $\text{cohml}(ho \mathcal{F}_B, Ab)$ with respect to $\Sigma_B^u$ defines a parameterized cohomology theory graded over sphere bundles admitting sections. The properties enjoyed by such a theory can be obtained similar to the one described above.

5. (Co)stabilizations of relative categories with respect to $V$-actions

The category $\text{RelCat}$ of relative categories and relative functors between them admits a model structure due to Barwick-Kan [1, 6.1]. Besides, $\text{RelCat}$ is a cartesian closed symmetric monoidal category [1, 7.1]. It is in particular, a 2-category ($\text{Cat}$-enriched) and powered and copowered over any small category. If $I$ is a small category and $\mathcal{A} = (A, W_A)$ is a relative category, then the powering $\triangleleft (I, \mathcal{A})$ is
given by \((A^I, W_A^I)\), where \(W_A^I\) denotes the pointwise weak equivalences. Similarly, the copowering \(I \boxtimes A\) is given by \((I \times A, I \times W_A)\).

5.1 Stabilization of relative categories with respect to \(V\)-actions. Let \(\alpha\) be a \(V\)-action on \(A = (A, W_A)\) that is mute on the right (i.e., a left action). Then, by Theorem 3.5, \(\text{Stab}_V(A_\alpha)\) is equivalent to \(\text{Inv}(A_\alpha)\). If \(\alpha\) is mute on the right, then we have also an equivalence \(\text{Inv}(A_\alpha) \cong \text{Inv}_l(A_\alpha)\). The objects of \(\text{Inv}_l(A_\alpha)\) are pairs \((E, \sigma)\) where \(E : V \to A : u \mapsto E_u\) is a relative functor (with trivial relative structure on \(V\)) and for every \(v\) in \(V\)

\[
\sigma(v) : E \Rightarrow \natural (\mu_r, \alpha)(v, 1^\text{op})(E)
\]

is a natural isomorphism. Here, we note that \(\natural (\mu_r, \alpha)(1, u^\text{op})(E) = E\) as \(\alpha\) is mute on the right. Write \(\alpha^v = \alpha(v, 1^\text{op})\) and \(\sigma(v)_u = \sigma_{u,v}\) (while noting that \(\alpha(v, 1^\text{op}) = \alpha(v, w^\text{op})\) for every \(w\) in \(V\)). Then, for every \(u, v, w, z\) in \(V\), \(\sigma\) defines an isomorphism \(\sigma_{u,v,w} : E_u \to \alpha^v E_{u \otimes v}\). The diagram 2.2 implies that for every \(w\) in \(V\) the following triangle commutes

\[
\begin{array}{ccc}
E_u & \xrightarrow{\alpha^v E_{u \otimes v}} & \alpha^v \otimes w E_{u \otimes v \otimes w} \\
\alpha^v E_{u \otimes v} & \downarrow {\sigma_{u,v,w}} & \alpha^v \otimes w E_{u \otimes v \otimes w} \\
\end{array}
\]

where the bottom horizontal map is the composition

\[
\alpha^v E_{u \otimes v} \xrightarrow{\alpha^v (\sigma_{u \otimes v, w})} \alpha^v \otimes w E_{u \otimes v \otimes w} \cong \alpha^v \otimes w E_{u \otimes v \otimes w}
\]

Morphisms are natural transformations that are coherent with the \(V\)-action, and weak equivalences are defined levelwise. This category admits a right action \(\tilde{\alpha}\) given by \(\tilde{\alpha}(u, u^\text{op})(f, \sigma) = (f \circ \mu_r(u), \tilde{\sigma}(u))\) where \(\tilde{\sigma}(u)_{e,w} = \sigma(u \otimes w)\).

On the other hand, the lax stabilization of \(A\) with respect to \(\alpha\) is the same except that \(\sigma\) in the above construction is not required to be an isomorphism but just a natural transformation. In the case when the underlying relative category \(A\) is a model category and \(V\) acts by left Quillen functors, the lax stabilization coincides with the lax homotopy limit in \([2]\); and therefore, admits a model structure where weak equivalences and cofibrations defined levelwise.

One equivalently uses \(\text{Inv}(A_\alpha)\) for the stabilization, which has a natural left action (without using \(V\) is symmetric monoidal). In this case, objects can be viewed as pairs \((E, \zeta)\) where \(E : V \times V \to A\) and is a functor and \(\zeta : E \Rightarrow \natural (\hat{\mu}_r \times \hat{\mu}_l, \alpha)(E)\) is a natural isomorphism; so that, for every \(u, v, w, z \in V\), \(\zeta_{u,v,w,z} : E(u, v) \to \alpha(w, z^\text{op})E(u \otimes w, z \otimes v)\) is a natural isomorphism. Writing \(E^w \otimes v = E(u, v)\) and \(\alpha^w \otimes z = \alpha(w, z^\text{op})\), we can see that \(\text{Inv}(A_\alpha)\) just extends the indexing in \(\text{Inv}_l(A_\alpha)\) to formal differences in \(V\). Again, in the lax version, the assumption on \(\zeta\) being invertible is dropped.

In some classical cases, the lax stabilization given in the present paper coincides with the notion of prespectra in the literature while stabilization coincides with spectra.

5.1.1 A special case: \(V\) is a monoidal groupoid acting on a model category. Consider the special case when the symmetric monoidal category \(V\) is a groupoid; that is, the 2-category \(BV\) is a \((2, 1)\)-category and \(\alpha\) is an action on a relative category \(A\) given by a 2-functor from \(BW\) to the 2-truncation of the \((\infty, 1)\)-localization of the
model category \textbf{RelCat} with respect to the Barwick-Kan model structure. Then \(\text{Inv}_\ast(\mathcal{A}_n)\), as given by a 2-end, coincides with the homotopy end (i.e., \((2,1)\)-limit).

In the case when the relative category \(\mathcal{A}\) is a model category and \(\mathcal{V}\) acts on \(\mathcal{A}\) by left Quillen functors, then following \cite{2} one obtains a construction of \(\text{Inv}_\ast(\mathcal{A}_n)\) in this case same as above. Objects of \(\text{Inv}_\ast(\mathcal{A}_n)\) are pairs \((E,\sigma)\) where \(E : \mathcal{V} \to \mathcal{A} : u \to E_u\) is a functor and

\[\sigma(v) : E \Rightarrow \ominus (\mu_{\mathcal{V}},\alpha)(v, 1^{\mathcal{V}})(E)\]

is a natural weak equivalence for every \(v\) in \(\mathcal{V}\); i.e., \(\sigma_{u,v}\) is a weak equivalence. The diagram 5.19 commutes, which corresponds to the compatibility condition mentioned in \cite[Defn. 3.1]{2}.

5.1.2. A construction of spectrification. For the case of the actions on relative categories, we can construct spectrification as in \cite{5} or \cite{12}. Let \(\mathcal{V}\) be symmetric monoidal and \(\alpha\) be a \(\mathcal{V}\)-action of a relative category \(\mathcal{A}\). Assume that \(\alpha\) is mute on the right. Recall that spectrification is left adjoint to the natural inclusion \(\iota_\alpha : \text{Inv}_\ast(\mathcal{A}_n) \to \ell \text{Inv}_\ast(\mathcal{A}_n)\). Let \(T(\mathcal{V})\) denote the transport category of \(\mathcal{V}\); i.e., category whose objects are objects of \(\mathcal{V}\) and a morphism \(u \to v\) in \(T(\mathcal{V})\) is a pair \((w, f)\) where \(w\) is an object and \(f : u \otimes w \to v\) is an isomorphism in \(\mathcal{V}\). The composition of \((w, f)\) and \((z, g)\) is \((z \otimes w, g \circ f \otimes id_w)\). Suppose that \(\alpha\) is an action that commutes with \(T(\mathcal{V})\) shaped colimits. Then the inclusion \(\iota_\alpha\) admits a left adjoint \(L\) given as follows: For \((E, \sigma)\) an object in \(\ell \text{Inv}_\ast(\mathcal{A}_n)\), let \(F^{\alpha,E,z} : T(\mathcal{V}) \to \mathcal{A}\) be the functor given by \(F^{\alpha,E,z}(u) = \alpha(u, 1)E(z \otimes u)\) and \(F^{\alpha,E,z}((w, f))\) is the composition

\[\alpha(u, 1)E(z \otimes u) \xrightarrow{w \sigma} \alpha(u \otimes w, 1)E(z \otimes u \otimes w) \xrightarrow{f \sigma} \alpha(v, 1)E(z \otimes v)\]

Then \(LE : \mathcal{V} \to \mathcal{A}\) is the functor given by

\[LE(z) = \colim_{T(\mathcal{V})} F^{\alpha,E,z}\]

We call the left adjoint of this inclusion \textit{spectrification}. The \(x\) component of \(\sigma\) is given by the following composition

\[\alpha(x, 1)(LE(z \otimes x)) = \alpha(x, 1)(\colim_{u \in T(\mathcal{V})} \alpha(u, 1)E(z \otimes x \otimes u))\]

\[= \colim_{u \in T(\mathcal{V})} \alpha(u \otimes u, 1)E(z \otimes x \otimes u) \cong \colim_{v \in T(\mathcal{V})} \alpha(u, 1)E(z \otimes u) = LE(z)\]

so that \(LE\) is an 0-cell in \(\mathcal{V}\text{MR}\). The \(\ast\) follows from the naturality of colimit. The 1-cell \(L\) is \(\mathcal{V}\)-equivariant via \(LE(x \otimes -) \cong L(E(x \otimes -))\).

5.2. Some known examples of stabilizations of relative categories.

5.2.1. Sequential and Symmetric Spectra. Let \(\mathcal{V} = \mathbb{N}\) with addition and identity maps as morphisms, and let \(\mathcal{A} = \mathcal{T}\) be the category of pointed spaces with the standard model structure. Let \(\Omega : BW \to \textbf{RelCat}\) be the action given by usual loop space functors; that is, \(\Omega(n,m)(X) = \Omega^nX\), the \(n\)-fold loop space of \(X\). This action is in fact the action generated by the functor \(\Omega : \mathcal{T} \to \mathcal{T}\). Then \(\text{Stab}_\ast(\mathcal{T})\) has objects as pairs \(E : \mathbb{N} \to \mathcal{T} : n \to E_n\) functors together with (natural) isomorphisms \(\sigma_m : E_n \to \Omega^mE_{n+m}\). The triangular diagram 5.19 above already commutes in this case. Such objects are known as spectra in the literature. If we consider the lax stabilization \(\ell \text{Stab}_\ast(\mathcal{T})\), then \(\sigma_m\) is just a map. Objects of \(\ell \text{Stab}_\ast(\mathcal{T})\) are then called prespectra. The model structure on \(\ell \text{Stab}_\ast(\mathcal{T})\) coincides with the level model structure and the spectrification functor in 5.1.2 is
the usual spectrification of prespectra. In this case, since \( \mathcal{V} \) is a monoidal groupoid, the construction in 5.1.1 gives the category of \( \Omega \)-spectra.

If instead we choose \( \mathcal{V} \) as the permutation category; whose objects are finite sets \([n] = \{1, \ldots, n\}\) and whose morphisms are permutations (i.e., \( \mathcal{V} \) is the skeleton of the groupoid of finite sets and permutations), and cardinal sum as the monoidal product and \([0] = \emptyset\) as the monoidal unit. Defining \( \Omega([n], [m]) = \Omega^nX \) on \( \mathcal{F} \) one obtains that \( \ell \text{Stab}_V(\mathcal{F}) \) has objects of pairs \( E : \mathbb{N} \to \mathcal{F} : [n] \to E_n \) functors together with maps \( \sigma_{n,m} : E_n \to \Omega^mE_{n+m} \). Since \( \mathcal{V}([n], [n]) \cong S_n \); the symmetric group, each \( E_n \) admits an action of \( S_n \). By the naturality of \( \sigma \), the map is \( \sigma_{n,m} \) \( S_n \times S_m \)-equivariant where \( S_n \times S_m \) action on \( E_{n+m} \) the action is the restricted action of \( S_{n+m} \) and on \( \Omega^mE_{n+m} \) the \( S_m \)-action is the conjugation action and on \( E_n \), the second coordinate of \( S_n \times S_m \) acts trivially. This implies the \( S_n \times S_m \)-equivariance of the dual maps \( \sigma_{n,m}^* : \Sigma^mE_n \to E_{n+m} \). The objects of \( \ell \text{Stab}_V(\mathcal{F}) \) are symmetric spectra [8].

5.2.2. Coordinate Free and orthogonal Spectra. Let \( \text{Inn} \) be the monoidal category real inner product spaces with direct sum and linear isometric embeddings. For an inner product space \( V \), let \( S^V \) denote its one point compactification. Define a \( \text{Inn} \)-action \( \Omega \) on the relative category \( \mathcal{F} \) by \( \Omega(V, 1^\text{op}) = S^V \). Then \( \text{Stab}_{\text{Inn}}(\mathcal{F}) \) is equivalent to the category with objects as pairs \((E, \sigma)\) where \( E : \text{Inn} \to \mathcal{F} : V \to Ev \) is a functor and \( \sigma_{V,W} : Ev \to \Omega^WEv \oplus W \) is a natural isomorphism. Moreover, for any \( Z \in \mathcal{V} \) the diagram

\[
\begin{array}{ccc}
E_V & \xrightarrow{\sigma_{V,W \oplus Z}} & \Omega^W \oplus Z Ev \oplus W \oplus Z \\
\sigma_{V,W} & & \downarrow \cong \\
\Omega^W Ev \oplus W & \xrightarrow{\Omega^W(\sigma_{V,W \oplus Z})} & \Omega^W \Omega^Z Ev \oplus W \oplus Z
\end{array}
\]

commutes. Note that this diagram is the same as the diagram 5.19 after composing bottom arrow with inverse of the right vertical isomorphism.

Such objects are closely related to the objects known as coordinate free spectra in the literature. Let \( U = \mathbb{R}^{\infty} \) be a countably infinite dimensional real inner product space. Let \( \mathcal{V} \) be the category whose objects are finite dimensional subspaces of \( U \) and morphisms are linear isometric isomorphisms. Again, define \( \Omega \) on \( \mathcal{F} \) by \( \Omega(1, V)(X) = S^V \). Then the stabilization \( \text{Stab}_V(\mathcal{F}) \) is equivalent to the category of coordinate free spectra. In fact, if \( V \subset W \) by writing \( W - V \) for the orthogonal complement of \( V \) in \( W \), we obtain structure maps of the form \( \sigma_{V,W - V} : Ev \to \Omega^{W - V}Ev \). Similarly, the lax stabilization \( \ell \text{Stab}_V(\mathcal{F}) \) gives the coordinate free prespectra, which admits the level model structure in which weak equivalences and cofibrations defined levelwise. Since \( \mathcal{V} \) is a monoidal groupoid, the construction in 5.1.1 gives the category of coordinate free \( \Omega \)-spectra.

Note that for each \( V \) in \( \mathcal{V} \), the hom-set \( \mathcal{V}(V, V) \cong O(V) \), the orthogonal group on \( V \). Let \( \mathcal{O} \) be the full-subcategory of \( \mathcal{V} \) consisting of objects \( \mathbb{R}^n \) for each \( n \in \mathbb{N} \). Then \( \mathcal{O} \) is monoidal with unit \( \mathbb{R}^0 = 0 \) and the symmetric monoidal product given as \( \mathbb{R}^k \oplus \mathbb{R}^l = \mathbb{R}^{k+l} \), for every \( k, l \in \mathbb{N} \). On morphisms, the product is defined via the diagonal embedding \( O(k) \times O(l) \to O(k+l) \). Consider the \( \mathcal{O} \)-action \( \Omega \) on \( \mathcal{F} \) by \( \Omega(\mathbb{R}^1, 0^\text{op})(X) = \Omega X = \mathcal{F}(S^1, X) \) and \( \Omega(\mathbb{R}^n, 0^\text{op})(X) = \Omega^n X \cong \mathcal{F}(S^n, X) \), where \( S^n \) denotes the one point compactification of \( \mathbb{R}^n \). A
A morphism \( m : \mathbb{R}^\oplus n \to \mathbb{R}^\oplus n \) is sent to the induced self map on loop spaces. In this case, an object \( \text{Stab}_G(\mathcal{T}) \) is a pair \((E, \sigma)\) where \( E : \mathcal{O} \to \mathcal{T} : \mathbb{R}^n \to E_n \) is a functor and a natural isomorphism \( \sigma_{n,m} : E_n \to \Omega^m E_{n+m} \). Since \( \mathcal{O}(\mathbb{R}^\oplus n, \mathbb{R}^\oplus n) \cong O(n) \), \( E_n \) admits an \( O(n) \)-action. By naturality of \( \sigma_{n,m} \) in \( n \), for every element \( g : \mathbb{R}^\oplus n \to \mathbb{R}^\oplus n \) in \( O(n) \), we have the following diagram commutes

\[
\begin{array}{ccc}
E_n & \xrightarrow{\sigma_{n,m}} & \Omega^m E_{n+m} \\
g \downarrow & & \downarrow \sigma_{n,m} \\
E_n & \xrightarrow{\sigma_{n,m}} & \Omega^m E_{n+m}
\end{array}
\]

Therefore, \( \sigma_{n,m} \) is \( O(n) \)-equivariant with respect to the restricted action on codomains via the diagonal embedding. For any \( h : \mathbb{R}^\oplus m \to \mathbb{R}^\oplus m \), there is an induced map of one point compactifications \( h_S : S^m \to S^m \). Thus, there is an induced conjugation action \( O(m) \)-action on \( \Omega^n X = \mathcal{T}(S^m, X) \) for every \( X \). The space \( E_{n+m} \) admits a \( O(m+n) \) action, and thus, an \( O(m) \)-action after restriction, so that there is an \( O(m) \)-action on \( \Omega^n E_{n+m} \) by conjugation. By 2.1, the following diagram commutes

\[
\begin{array}{ccc}
E_n & \xrightarrow{\sigma_{n,m}} & \Omega^m E_{n+m} \\
h_* = \text{id} \downarrow & & \downarrow h_S^* \\
E_n & \xrightarrow{\sigma_{n,m}} & \Omega^m E_{n+m}
\end{array}
\]

Thus, \( \sigma_{n,m} \) is \( O(n) \times O(m) \)-equivariant. This gives the notion of the orthogonal spectra. In fact, the action \( \Omega \) induces an adjoint action \( \Sigma \) on \( \mathcal{T} \), for which \( \Sigma(0, n) = \Sigma^n \) is the left adjoint to \( \Omega^n \); namely, \( n \)-fold suspension functor. The adjunct of \( \sigma_{n,m} \) gives a map \( \sigma'_{n,m} : \Sigma^n E_n \to E_{n+m} \). This map is \( O(k) \times O(l) \) equivariant since \( \sigma'_{n,m} \) is.

Generalizing the idea we can find another model for the highly structured spectra by replacing \( O \) with \( G \); the groupoid whose objects are \( S^n \), \( n \)-sphere with a basepoint for all \( n \in \mathbb{N} \), whose morphisms are \( G_n \) the group of self homeomorphisms of \( S^n \). This is topological version of the orthogonal spectra.

5.2.3. Genuine G-Spectra. Now, let \( G \) be a group and \( G\text{Inn} \) be the monoidal category of \( G \)-representations with direct sum and \( G \)-equivariant linear maps. Consider the category \( G\mathcal{T} \) of pointed \( G \)-spaces and pointed \( G \)-maps with the standard model structure. Define a \( G\text{Inn} \)-action \( \Omega : G\text{Inn} \times G\text{Inn}^{\text{op}} \to \text{RelCat} \) on \( G\mathcal{T} \) by \( \Omega(V, 1^{\text{op}})(X) = \Omega^V X = G\mathcal{T}_*(S^V, X) \), the set of pointed maps from the one point compactification of \( V \) to \( X \) considered with the conjugation action. Then \( \text{Stab}_V(G\mathcal{T}) \) is equivalent to the category with objects as pairs \((E, \sigma)\) where \( E : G\text{Inn} \to G\mathcal{T} : V \mapsto E_V \) is a functor and \( \sigma_{V,W} : E_V \to \Omega^W E_{V \oplus W} \) is a natural isomorphism. The diagram analogues to 5.19 commutes.

For convenience one considers the more classical notion of equivariant spectra indexed by a \( G \)-universe. Let \( \mathcal{V} \) be as in 4.2.2. Define a \( \mathcal{V} \)-action \( \Omega \) on \( G\mathcal{T} \) as above. Then \( \text{Stab}_V(G\mathcal{T}) \) is the category of genuine \( G \)-spectra and \( \ell \text{Stab}_V(G\mathcal{T}) \) gives the category of genuine orthogonal \( G \)-prespectra, see [17, Ch. XII, 2]. Here, we note that \( \mathcal{V}(V, V) = O(V) \), so that for each object \((E, \sigma)\) in \( \ell \text{Stab}_V(G\mathcal{T}) \), \( E(V) \) admits a \( O(V) \)-action and \( \sigma_{V,W} \) is \( O(V) \times O(W) \)-equivariant analogous to the coordinate
free case above. The construction analogous to the one in 5.1.1 gives the category of \( G-\Omega \)-spectra.

5.2.4. **Parameterized Spectra.** Let \( B \) be a space and \( \mathcal{T}_B \) be the model category given in [18, Ch. 6]. Let \( \mathcal{V} \) be the monoidal category as given in 4.2.4. Consider the model structure on \( \mathcal{T}_B \) [18, Ch. 6]. For \( \xi \in \mathcal{V} \), denote by \( \mathcal{S}^\xi \) the associated fiber-wise one point compactification, which is a sphere bundle over \( B \) with point as the section induced by the zero section of \( \xi \). Define the \( \mathcal{V} \)-action \( \Omega_B : BW \rightarrow \mathbf{RelCat} \) on \( \mathcal{T}_B \) as follows: Given \( \xi, \eta \) in \( \mathcal{V} \), define \( \Omega_B \) by

\[
\Omega_B(\xi, \eta_\otimes \phi)(t : X \rightarrow B) = \mathcal{T}_B(\mathcal{S}^\xi, t).
\]

Denote by \( \Omega_B^\xi \) the functor \( \mathcal{T}_B(\mathcal{S}^\xi, -) \). Then \( \text{Stab}_\mathcal{V}(\mathcal{T}_B) \) has objects as pairs \((F, \sigma)\) where \( F : \mathcal{V} \rightarrow \mathcal{T}_B : \xi \mapsto F\xi \) is a functor and \( \sigma_{\xi, \eta} : F\xi \rightarrow \Omega_B^\xi F \xi \oplus \eta \) such that for any \( \zeta \in \mathcal{V} \) the following diagram commutes

\[
\begin{array}{ccc}
F\xi & \xrightarrow{\phantom{\otimes}} & \Omega_B^\xi \eta F \xi \oplus \eta \\
\sigma & \downarrow \cong & \downarrow \\
\Omega_B^\eta F \xi \oplus \eta & \xrightarrow{\phantom{\otimes}} & \Omega_B^\xi \Omega_B^\eta F \xi \oplus \eta \oplus \zeta
\end{array}
\]

Such objects are called as genuine spectra parameterized by \( B \). The lax stabilization \( \ell \text{Stab}_\mathcal{V}(\mathcal{T}_B) \) then gives the category of genuine prespectra parameterized by \( B \). The category \( \ell \text{Stab}_\mathcal{V}(\mathcal{T}_B) \) admits the level model structure as in the previous examples.

5.2.5. **Parameterized \( G \)-Spectra.** Parameterized \( G \)-spectra and prespectra can be obtained just like the transition from coordinate free spectra to genuine \( G \)-spectra.

5.2.6. **Diagram Spectra.** Let \( \mathbb{T} \) be a closed symmetric monoidal model category with internal hom \( \text{hom}_\mathbb{T} \) and \( D \) be a \( \mathbb{T} \)-enriched small symmetric monoidal category. Assume both \( \mathbb{T} \) and \( D \) are pointed as categories. Let \( \mathcal{D} \mathbb{T} \) be the category of \( \mathbb{T} \)-enriched functors from \( D \) to \( \mathbb{T} \). The category \( \mathcal{D} \mathbb{T} \) admits a symmetric monoidal product with Day convolution. Let \( R \) be any monoid in the symmetric monoidal category \( \mathcal{D} \mathbb{T} \). Define a \( D \) action on \( \Omega \) on \( \mathbb{T} \) as

\[
\Omega(u, v_\otimes \phi)(X) = \Omega^u X = \text{hom}_\mathbb{T}(R(u), X).
\]

Since \( R \) is a monoid, the action is well-defined. The lax stabilization with respect to \( \Omega \), \( \ell \text{Stab}_\mathcal{D}(\mathbb{T}) \), has objects as pairs \((E, \zeta)\) where \( E : D \rightarrow \mathbb{T} : u \mapsto E_u \) and \( \zeta : E_v \rightarrow \Omega^u E_{v_\otimes u} \) is a natural transformation. If \( \Omega \) factors through the category of adjunctions; i.e., there exist an adjoint action \( \Sigma \) on \( \mathbb{T} \), then an object in the lax stabilization coincides with the notion of \( \mathcal{D} \)-spectra of [16].

5.3. **Costabilization of relative categories with respect to \( \mathcal{V} \)-actions.** Let \( \alpha : BW \rightarrow \mathbf{RelCat} \) be a \( \mathcal{V} \)-action on \( \mathcal{A} = (A, W_A) \) that is mute on the left. The colax costabilization of \( \mathcal{A} \) with respect to \( \alpha \), denoted by \( \ell \text{coStab}_\mathcal{V}(\mathcal{A}) \), can be defined as the colax coend of \( \alpha \), which can be seen as a colax colimit over \( \mathcal{V} \). This colax colimit can be given in terms of Gröthendieck construction. Since \( \alpha \) is mute on the left, this 2-colimit can be given as \( \ell \text{coStab}_\mathcal{V}(\mathcal{A}) = \int_\mathcal{V}(\mu_v : \alpha(v) \otimes \cdot) \circ \iota_v \). Then objects of \( \ell \text{coStab}_\mathcal{V}(\mathcal{A}) \) are pairs \((u, a)\) with \((u \in \mathcal{V} \) and \( a \in \mathcal{M} \), and a morphism between \((u, a) \rightarrow (u', a')\) is a triple \((v, \varphi, f)\) such that \( v \in \mathcal{V} \) and \((\varphi, f) : (u \otimes v, \alpha(v, 1_\mathbb{V})(a)) \rightarrow (u', a') \). A triple \((v, \varphi, f)\) is a weak equivalence if \((\varphi, f) :

(u \otimes v, \alpha(v, 1^\op)(a)) \to (u', a') is a weak equivalence; that is, $u \otimes v \cong u'$ in $\mathcal{V}$ and $f : \alpha(v, 1^\op)(a) \to a'$ is a weak equivalence in $\mathcal{M}$. In particular, for every $v$ in $\mathcal{V}$, $(u, a)$ is weakly equivalent to $(u \otimes v, \alpha(v, 1^\op)(a))$. In sufficiently nice cases, if $\mathcal{A}$ is a model category $\ell \coStab_{\mathcal{V}}(\mathcal{A})$ admits a model structure; e.g., when $\mathcal{V}$ is bicomplete (so a model category with trivial model structure), and $\alpha$ is relative and proper functor, see [7]. The construction coincides with the homotopy colimit; and thus, homotopically correct with respect to the Barwick-Kan model structure (see [7] and also [19]).

The costabilization is obtained by localizing $\ell \coStab_{\mathcal{V}}(\mathcal{A})$ at the op lax cartesian morphisms in the Gröthendieck construction.

5.4. **Some examples of (colax) costabilizations of relative categories.**

5.4.1. **Spanier Whitehead Category.** Let $\mathcal{V} = \mathbb{N}$ with only identity maps as morphisms and $\mathcal{A} = \mathcal{F}$ pointed topological spaces with with standard model structure and $\Sigma : BW \to \RelCat$ be the action given by usual suspensions; that is, $\Sigma(n, m)(X) = \Sigma^n X$ the $n$-fold suspension of $X$. Then $\ell \coStab_{\mathcal{V}}(\mathcal{F})$ has objects as pairs $(n, X)$ and for $n \leq m$ a morphism from $(n, X)$ to $(m, Y)$ is a map $f : \Sigma^m X \to Y$. Its homotopy category with respect to the weak equivalences described above gives the usual Spanier-Whitehead category (see also [19]).

5.4.2. **Coordinate Free Spanier-Whitehead Category.** Let $\mathcal{V}$ be as in the case of the coordinate free spectra and $\Sigma : BW \to \RelCat$ be the action given by $\Sigma(V, W)(X) = \Sigma W X := S^W \wedge X$ where $\wedge$ denotes the smash product of spaces. Then $\ell \coStab_{\mathcal{V}}(\mathcal{F})$ has objects as pairs $(W, X)$ and for $\dim(W) \leq \dim(Z)$ a morphism from $(W, X)$ to $(Z, Y)$ is a triple $(V, \phi, f)$ where $\phi : V \oplus W \to Z$ is an isometric isomorphism and $f : \Sigma W X \to Y$ is a map. Such a triple $(V, \phi, f)$ is a weak equivalence if $f$ is a weak equivalence. Then, following the homotopy category $\ell \coStab_{\mathcal{V}}(\mathcal{F})$ with these weak equivalences gives the coordinate free version of the Spanier-Whitehead category.

5.4.3. **G-Equivariant Spanier-Whitehead Category.** This is the $G$-equivariant version of coordinate free Spanier-Whitehead category where $\mathcal{V}$ is chosen as in 5.2.3 and $\Sigma : BW \to \RelCat$ is the action given by $\Sigma(V, W)(X) = \Sigma W X := \mathcal{F}(S^W, X)$ where $X$ is a $G$ space and $S^W$ is the one point compactification of $W$ with the induced $G$-action. Then $\ell \coStab_{\mathcal{V}}(\mathcal{F})$ has objects as pairs $(W, X)$ and for $\dim(W) \leq \dim(Z)$ a morphism from $(W, X)$ to $(Z, Y)$ is a triple $(V, \phi, f)$ where $\phi : V \oplus W \to Z$ is an equivalence class of an isometric $G$-isomorphisms and $f : \Sigma W X \to Y$ is a $G$-map.

5.4.4. **Parameterized Spanier-Whitehead Category.** This is the parameterized version of coordinate free Spanier-Whitehead category where $\mathcal{V}$ is chosen as in 5.2.4. Let $\Sigma_B : BW \to \RelCat$ be the $\mathcal{V}$-action on $\mathcal{F}_B$ given by $\Sigma_B(\xi, t)(X) = \Sigma_t := S^t \wedge_B t$ where $t : X \to B$ be a pointed map over $B$ (i.e., a pointed object in the over category $\mathcal{F}/B$) and $S^t$ is the sphere bundle over $B$ given by the fiber-wise one point compactification of $\xi$, and $\wedge_B$ is the fiber-wise smash product. In this case objects are pairs $(\xi, t)$ where $\xi$ is an object in $\mathcal{V}$ and $t : X \to B$ a pointed map. A morphism from $(\xi, t)$ to $(\zeta, z)$ is a triple $(\chi, m, f)$ where $\chi$ is an object in $\mathcal{V}$ and $m : \xi \oplus X \to \zeta$ a morphism in $\mathcal{V}$ and $f : S^x \wedge_B t \to z$ is a pointed map over $B$. 

5.4.5. **Equivariant Parameterized Spanier-Whitehead Category.** The description of Equivariant Parameterized Spanier-Whitehead Category just the equivariant generalization of the Parameterized Spanier-Whitehead Category above, which is the costabilization of $G\mathcal{T}_B = \text{id}_B/(G\mathcal{T}/B)$ with respect to the action given by the fibrewise suspensions as above, but equipped with $G$-actions.

5.4.6. **Diagram Spanier-Whitehead Category.** Let $T$, $D$ and $R$ be as in 5.2.6. Define a $D$ action on $\Sigma$ on $T$ as

$$\Sigma(u, v^{op})(X) := \Sigma v X := X \wedge_T R(u).$$

Then $\mathcal{L}\text{coStab}_D(T)$ has objects as pairs $(v, X)$ where $v$ in $D$ and $X$ in $T$. A morphism from $(w, X)$ to $(v, Y)$ is a triple $(u, \phi, f)$ where $\phi : u \wedge_D w \to v$ is a morphisms in $D$ and $f : X \wedge_T R(u) \to Y$ is a morphism in $T$. Weak equivalences are the triples in which $\phi$ is an isomorphism and $f$ is a weak equivalence.

**Remark 5.1.** Observe that the duality between lax stabilization and oplax costabilization establishes a duality between stable homotopy categories and Spanier-Whitehead categories.

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