Galois covers of degree $p$ : semi-stable reduction and Galois action

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0. Abstract. In this paper we study the semi-stable reduction of Galois covers of degree $p$ above semi-stable curves over a complete discrete valuation ring of inequal characteristics $(0,p)$. We are also able to describe the Galois action on these covers in terms of some geometric and combinatorial datas in characteristic $p$ endowed with the action of the Galois group of the residue field.

0. Introduction. In this paper we study the semi-stable reduction of Galois covers of degree $p$ above semi-stable curves over a complete discrete valuation ring of inequal characteristics $(0,p)$, as well as the Galois action on these datas.

More precisely, let $R$ be a complete discrete valuation ring of inequal characteristics $(0,p)$, with fraction field $K$, and residue field $k$. In the first part of the paper we consider the following local situation. Let $f : Y \to X$ be a Galois cover of degree $p$ between formal germs of $R$-curves at the closed points $y$ and $x$ respectively, with $Y$ normal, and $X$ semi-stable i.e. $X$ is the formal germ of an $R$-curve at a closed point $x$ which is either a smooth point or an ordinary double point. It follows from the theorem of semi-stable reduction for curves that the germ $Y$ admits potentially a semi-stable reduction i.e. there exists, after eventually a finite extension of $R$, a semi-stable model $\tilde{Y} \to Y$ which is “essentially” unique. In particular, the Galois group of the cover $f$ acts on $\tilde{Y}$ and the quotient $\tilde{X}$ of $\tilde{Y}$ by this action is a semi-stable blow-up of $X$. We have a canonical morphism $\tilde{f} : \tilde{Y} \to \tilde{X}$ which is Galois of degree $p$. To the above cover $f : Y \to X$ we associate canonically some degeneration datas which determine completely the special fibre $\tilde{Y}_k := \tilde{Y} \times_R k$ of $\tilde{Y}$. These degeneration datas consist of the (canonically marked) tree associated to the special fibre $\tilde{X}_k := \tilde{X} \times_R k$ of $\tilde{X}$, plus some geometric datas which consists of a torsor $f_i : V_i \to U_i$ under a finite and flat group scheme of rank $p$ above each irreducible marked component $U_i$ of $\tilde{X}_k$ such that the “conductors” of these torsors at the marked points satisfy certain “compatibility” conditions (these are what we called in [Sa] Kummerian mixed torsors) (cf. 1.2, 1.3 for a more precise definition). For an illustration of this situation we refer to the
example 1.2.1. Let’s denote by $\text{Deg}_p$ the set of isomorphism classes of such degeneration datas which are defined over an algebraic closure $\overline{k}$ of $k$ (cf. 1.2.2, 1.3.1 for a precise definition). We show that the set $\text{Deg}_p$ is a $G_k$-set, where $G_k$ is the Galois group of the separable closure of $k$ contained in $\overline{k}$ (cf. 1.2.3). Let $\overline{K}$ be an algebraic closure of $K$, and let $\text{Spec} \, L$ be the geometric generic point of $X$. The étale cohomology group $H^1_{\text{ét}}(\text{Spec} \, L, \mu_p)$ classifies (pointed)-geometric Galois covers of $X$ of degree $p$. Moreover our result which exhibit the above degeneration datas can be translated as the existence of a canonical specialisation map $\text{Sp} : H^1_{\text{ét}}(\text{Spec} \, L, \mu_p) \to \text{Deg}_p$. Note that both $H^1_{\text{ét}}(\text{Spec} \, L, \mu_p)$ and $\text{Deg}_p$ are $G_K$-sets, where $G_K$ is the Galois group of $K$ over $K$, and $G_K$ acts on $\text{Deg}_p$ via its canonical quotient $G_k$. Our main result in this part of the paper is the following:

**Theorem (1.2.7, 1.3.2).** The above specialisation map $\text{Sp} : H^1_{\text{ét}}(\text{Spec} \, L, \mu_p) \to \text{Deg}_p$ is surjective and $G_K$-equivariant.

In other words the association of degeneration datas to a cover $f : \mathcal{Y} \to X$ as above is compatible with the action of the Galois group on covers and on degeneration datas, Moreover one has a realisation result for such degeneration datas (cf. example 1.2.5 which illustrates the realisation of degeneration datas). The proof of the above result relies heavily on the results in [Sa] and [Sa-1]. We also use formal patching techniques à la Harbater. As an application of the above result we construct in 2.1.8 an example of a covers $f : \mathcal{Y} \to X$ as above where $X$ is smooth at the closed point $x$, and where the special fibre $\mathcal{Y}_k := \mathcal{Y} \times_R k$ of $\mathcal{Y}$ is singular and unibranche at the closed point $y$, and such that the configuration of the special fibre of the semi-stable model $\tilde{\mathcal{Y}}$ of $\mathcal{Y}$ is not a tree-like. This indeed answers a question raised by Lorenzini whether such a situation can occur in the unequal characteristic case.

In the second part of the paper we study the global situation of a Galois cover $f : Y \to X$ of degree $p$ above a proper and semi-stable $R$-curve $X$ with $Y$ normal. As above, and using the theorem of semi-stable reduction for curves (cf. [De-Mu]), one obtains after eventually a finite extension of $R$ a semi-stable model $\tilde{Y} \to Y$ of $Y$ and a Galois cover $f : \tilde{Y} \to \tilde{X}$ of degree $p$, where $\tilde{X}$ is a semi-stable blow-up of $X$. We associate then canonically to the cover $f : Y \to X$ some “degeneration datas” which determine completely the special fibre $\tilde{Y}_k := \tilde{Y} \times_R k$ of the semi-stable model $\tilde{Y}$ of $Y$ (cf. 2.2). This consists of the graph associated to the semi-stable $k$-curve $X_k$, and a given “mixed torsor” above $X_k$, plus given local degeneration datas at the critical points of this mixed torsor (cf. 2.2.1 for a more precise definition). For an illustration of this we refer to the example 2.2.3. Let’s denote by $\text{Deg}_p(X_k)$ the set of isomorphism classes of such degeneration datas which are defined over $k$. We show that this set is a $G_k$-set in a canonical way. Let $\overline{\eta}$ be the geometric generic point of $X$. The étale cohomology group $H^1_{\text{ét}}(\overline{\eta}, \mu_p)$ classifies
(pointed)-geometric Galois covers of \(X\) of degree \(p\). Moreover our result which exhibit the above degeneration datas can be translated as the existence of a canonical specialisation map \(\text{Sp} : H^1_{\text{et}}(\eta, \mu_p) \to \text{Deg}_p(X_k)\). Note that both \(H^1_{\text{et}}(\eta, \mu_p)\) and \(\text{Deg}_p(X_k)\) are \(G_K\)-sets, where \(G_K\) is the Galois group of \(\overline{K}\) over \(K\), and \(G_K\) acts on \(\text{Deg}_p\) via its canonical quotient \(G_k\). Our main result in the second part of the paper is the following:

**Theorem (2.3.2).** The above specialisation map \(\text{Sp} : H^1_{\text{et}}(\eta, \mu_p) \to \text{Deg}_p(X_k)\) is \(G_K\)-equivariant.

The above specialisation map \(\text{Sp}\) can not be surjective in this global case. However we prove the following result of realisation of degeneration datas:

**Theorem (2.3.1).** Let \(X_k\) be a proper and semi-stable \(k\)-curve. Let \(R\) be a complete discrete valuation ring of unequal characteristics with residue field \(k\), and fractions field \(K\). Suppose given a \(\eta\)-degeneration data \(\text{deg}(X_k)\) of rank \(p\) associated to \(X_k\) (in other words suppose given an element of \(\text{Deg}_p(X_k)\)). Then there exists, after eventually a finite extension of \(R\), a proper and semi-stable \(R\)-curve \(\tilde{X}\) with smooth generic fibre and a special fibre \(\tilde{X}_k := \tilde{X} \times_R k\) isomorphic to \(X_k\), and a Galois cover \(\tilde{f} : \tilde{Y} \to \tilde{X}\), such that the degeneration data associated to \(f\) is isomorphic to the given data \(\text{deg}(X_k)\).

We refer to the example 2.3.4 which illustrate the realisation of global degeneration datas. Finally, I believe the ideas presented in this paper provide the framework to construct Hurwitz spaces over \(\mathbb{Z}\) whose fibre in characteristic zero classifies Galois covers of degree \(p\) plus extra datas. Also in the case of Galois covers of degree \(p\) of the projective line over a \(p\)-adic field these results should lead to an algorithm which compute the semi-stable reduction of these covers at least in the case where the number of branched points is smaller than \(p\).

**I. Semi-stable reduction of Galois covers of degree \(p\) above formal germs of curves with Galois action.**

**1.0.** In what follows we use the following notations: \(R\) is a complete discrete valuation ring of unequal characteristics, with residue characteristic \(p > 0\), and which contains a primitive \(p\)-th root of unity \(\zeta\). We denote by \(K\) the fraction field of \(R\), \(\pi\) a uniformising parameter, and \(k\) the residue field of \(R\). Let \(\overline{K}\) be a fixed algebraic closure of \(K\), and let \(G_K\) be the Galois group of \(\overline{K}\) of \(K\). Let \(\overline{R}\) be the integral closure of \(R\) in \(\overline{K}\) which is a valuation ring, and let \(\overline{k}\) be the residue field of \(\overline{R}\) which is an algebraic closure of \(k\). Let \(G_k\) be the Galois group of a separable closure of \(k\) contained in \(\overline{k}\). We have the following
canonical exact sequence, where $I_K$ denotes the inertia subgroup:

$$0 \to I_K \to G_K \to G_k \to 0$$

In this section we will consider a formal germ $X$ of a semi-stable $R$-curve at a closed point, a Galois cover $f : \mathcal{Y} \to \mathcal{X}$ with group $G = \mathbb{Z}/p\mathbb{Z}$, and we will study the semi-stable reduction of $\mathcal{Y}$, as well as the action of $G_K$ on these data.

1.1. Let $X := \text{Spec} \hat{O}_{X,x}$ be the formal germ of an $R$-curve $X$ at a closed point $x$, and let $f : \mathcal{Y} \to \mathcal{X}$ be a Galois cover with group $G \simeq \mathbb{Z}/p\mathbb{Z}$, such that $\mathcal{Y}$ is normal and local. It follows then easily from the theorem of semi-stable reduction for curves (cf. [De-Mu], as well as the compactification process in [Sa-1] 2.3), that after eventually a finite extension $R'$ of $R$ with residue field $k'$, and fractions field $K'$, the formal germ $\mathcal{Y}$ has a semi-stable reduction. More precisely there exists a birational and proper morphism $\tilde{f} : \tilde{\mathcal{Y}} \to \mathcal{Y}'$, where $\mathcal{Y}'$ is the normalisation of $\mathcal{Y} \times_R R'$, such that $\tilde{\mathcal{Y}}_{K'} \simeq \mathcal{Y}'_{K'}$, and the following conditions hold:

(i) The special fibre $\tilde{\mathcal{Y}}_k := \tilde{\mathcal{Y}} \times_{\text{Spec} R'} \text{Spec} k'$ of $\tilde{\mathcal{Y}}$ is reduced.

(ii) $\tilde{\mathcal{Y}}_k$ has only ordinary double points as singularities.

Moreover there exists such a semi-stable model $\tilde{f} : \tilde{\mathcal{Y}} \to \mathcal{Y}'$ which is minimal for the above properties. In particular the action of $G$ on $\mathcal{Y}'$ extends to an action on $\tilde{\mathcal{Y}}$. Let $\tilde{\mathcal{X}}$ be the quotient of $\tilde{\mathcal{Y}}$ by $G$, which is a semi-stable model of $\mathcal{X}$. One has the following commutative diagram:

$$
\begin{array}{ccc}
\tilde{\mathcal{Y}} & \xrightarrow{\tilde{f}} & \mathcal{Y}' \\
\downarrow g & & \downarrow f' \\
\tilde{\mathcal{X}} & \xrightarrow{\tilde{g}} & \mathcal{X}'
\end{array}
$$

With the same notations as above, one can moreover choose the semi-stable models $\tilde{\mathcal{Y}}$ and $\tilde{\mathcal{X}}$ such that the set of points $B_{K'} := \{x_{i,K'}\}_{1 \leq i \leq r}$, consisting of the branch locus in the morphism $\mathcal{Y}'_{K'} \to \mathcal{X}'_{K'}$, which we may assume to be rational, specialise in smooth (resp. smooth distincts) points of $\mathcal{X}'_{K'}$. We may also, after eventually a finite extension of $K$, suppose that the double points of $\tilde{\mathcal{X}}_k$ are rational. Moreover one can choose such $\tilde{\mathcal{X}}$ and $\tilde{\mathcal{Y}}$ which are minimal for these properties. We will denote by $\tilde{f}^{\text{ns}} : \tilde{\mathcal{Y}}^{\text{ns}} \to \mathcal{Y}'$ (resp. $\tilde{f}^{\text{sp}} : \tilde{\mathcal{Y}}^{\text{sp}} \to \mathcal{Y}'$) the minimal semi-stable model of $\tilde{f}$ such that the points of $B_K$ specialise in smooth (resp. smooth distincts) points of $\tilde{\mathcal{X}}^{\text{ns}} := \tilde{\mathcal{Y}}^{\text{ns}}/G$ (resp. of $\tilde{\mathcal{X}}^{\text{sp}} := \tilde{\mathcal{Y}}^{\text{sp}}/G$). We call $\tilde{\mathcal{Y}}^{\text{ns}}$ (resp. $\tilde{\mathcal{Y}}^{\text{sp}}$) the minimal non split (resp. split) semi-stable model of $\mathcal{Y}$. We have the following commutative diagrams:
Moreover the morphism $\tilde{f}^{sp} : \tilde{Y}^{sp} \to Y'$ factors through $f^{nsp}$ as follows: $\tilde{Y}^{sp} \to \tilde{Y}^{nsp} \to Y'$, and the later commutative diagram factors through the first one. In the case where $X$ is the formal germ of a semi-stable $R$-curve at a closed point $x$, the fibre $(\tilde{g}^{nsp})^{-1}(x)$ (resp. $(\tilde{g}^{sp})^{-1}(x)$) of the closed point $x$ in $\tilde{X}^{nsp}$ (resp. in $\tilde{X}^{sp}$) is a tree $\Gamma^{nsp}$ (resp. $\Gamma^{sp}$) of projective lines. This tree is canonically endowed with some “degeneration datas” that we will exhibit below, and which follow mainly from the results in [Sa] and [Sa-1].

1.2. We will use the same notations as in 1.1. We first consider the case where $X \simeq \text{Spf } A$ is the formal germ of a semi-stable $R$-curve at a smooth point $x$. Let $R'$ be a finite extension of $R$ as in 1.1, and let $\pi'$ be a uniformiser of $R'$. We assume that $X$ is geometrically connected, in which case $Y$ is also geometrically connected. Below we exhibit the degeneration datas associated to the non split (resp. split) semi-stable reduction of $Y$, and which are consequences of the results in [Sa] and [Sa-1].

Deg.1. Let $\varphi := (\pi')$ be the ideal of $A' := A \otimes_R R'$ generated by $\pi'$, and let $\hat{A}_\varphi'$ be the completion of the localisation of $A'$ at $\varphi$. Let $X'_\varphi := \text{Spf } \hat{A}_\varphi'$ be the boundary of $X'$, and let $X'_\eta \to X'$ be the canonical morphism. Consider the following cartesian diagram:

$$
\begin{array}{c}
\tilde{Y}^{nsp} \xrightarrow{f^{nsp}} Y' \\
\downarrow g^{nsp} \downarrow f' \\
\tilde{X}^{nsp} \xrightarrow{g^{nsp}} X'
\end{array}
$$

and

$$
\begin{array}{c}
\tilde{Y}^{sp} \xrightarrow{f^{sp}} Y' \\
\downarrow g^{sp} \downarrow f' \\
\tilde{X}^{sp} \xrightarrow{g^{sp}} X'
\end{array}
$$

Then $f_\eta : Y_\eta \to X'_\eta$ is a torsor under a commutative finite and flat $R'$-group scheme $G_{R'}$ of rank $p$. Let $\delta$ be the degree of the different associated to the torsor $f_\eta$. One has the degeneration type $(G_{k'}, m, h)$ of the torsor $f_\eta$ (cf. [Sa] 3.2) which is canonically associated to $f$. The geometric genus $g_y$ of the point $y$ equals $(r - m - 1)(p - 1)/2$ (cf. [Sa-1] 3.1.1), where $r$ is the cardinality of $B_{K'}$. 

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**Deg.2.** The fibre $(\tilde{g}^{\text{sp}})^{-1}(x)$ (resp. $(\tilde{g}^{\text{nspp}})^{-1}(x)$) of the closed point $x$ of $X'$ in $\tilde{X}^{\text{nspp}}$ (resp. in $X^{\text{sp}}$) is a tree $\Gamma^{\text{nspp}}$ (resp. $\Gamma^{\text{sp}}$) of projective lines, $\Gamma^{\text{nspp}}$ is a sub-tree of $\Gamma^{\text{sp}}$. Let $\text{Vert}(\Gamma^{\text{nspp}}) := \{X_i\}_{i \in I^{\text{nspp}}}$ (resp. $\text{Vert}(\Gamma^{\text{sp}}) := \{X_i\}_{i \in I^{\text{sp}}}$) be the set of irreducible components of $(\tilde{g}^{\text{nspp}})^{-1}(x)$ (resp. of $(\tilde{g}^{\text{sp}})^{-1}(x)$), which are also the vertices of the tree $\Gamma^{\text{nspp}}$ (resp. $\Gamma^{\text{sp}}$). The tree $\Gamma^{\text{sp}}$ (hence also $\Gamma^{\text{nspp}}$) is canonically endowed with an origin vertex $X_{i_0}$, which is the unique irreducible component of $(\tilde{g}^{\text{sp}})^{-1}(x)$ which meets the point $x$. We fix an orientation of the tree $\Gamma^{\text{sp}}$ starting from $X_{i_0}$ in the direction of the ends. Such an orientation induces of course an orientation of the subtree $\Gamma^{\text{nspp}}$.

**Deg.3.** For each $i \in I^{\text{nspp}}$ (resp. $I^{\text{sp}}$), let $\{x_{i,j}\}_{j \in S_i}$ be the set of points of $X_i$ in which specialise some points of $B_{K'}$ ($S_i$ may be empty), say in each point $x_{i,j}$ specialise $r_{i,j}$ points of $B_K$. If $i \in I^{\text{sp}}$, and $S_i$ is non empty, we have $r_{i,j} = 1$. Also let $\{z_{i,j}\}_{j \in D_i}$ be the set of points of the irreducible component $X_i$ where $X_i$ meets the rest of the components of $\tilde{X}_k^{\text{nspp}}$ (resp. of $\tilde{X}_k^{\text{sp}}$). These are the double points of $\tilde{X}_k^{\text{nspp}}$ (resp. $\tilde{X}_k^{\text{sp}}$) supported by $X_i$. We denote by $B_k^{\text{sp}}$ (resp. $B_k^{\text{sp}}$) the set of all points $\bigcup_{i \in I^{\text{sp}} \{x_{i,j}\}_{j \in S_i}}$ (resp. $\bigcup_{i \in I^{\text{sp}} \{x_{i,j}\}_{j \in S_i}}$), which is the set of specialisation of the branch locus $B_{K'}$, and by $D_k^{\text{sp}}$ (resp. $D_k^{\text{sp}}$) the set of double points of $\tilde{X}_k^{\text{nspp}}$ (resp. $\tilde{X}_k^{\text{sp}}$). In particular $x_{i_0,j_0} := x$ is a double point of $\tilde{X}_k^{\text{nspp}}$ (resp. of $\tilde{X}_k^{\text{sp}}$).

**Deg.4.** Let $\mathcal{U}^{\text{sp}} := \mathcal{X}^{\text{sp}} - \{B_k^{\text{sp}} \cup D_k^{\text{sp}}\}$ (resp. $\mathcal{U}^{\text{nspp}} := \mathcal{X}^{\text{nspp}} - \{B_k^{\text{sp}} \cup D_k^{\text{sp}}\}$). The cover $f$ induces a mixed torsor $f^{\text{sp}} : \mathcal{V} \to \mathcal{U}^{\text{sp}}$ above $\mathcal{U}^{\text{sp}}$ (resp. $f^{\text{nspp}} : \mathcal{V} \to \mathcal{U}^{\text{nspp}}$ above $\mathcal{U}^{\text{sp}}$), whose special fibre $f_k^{\text{sp}} : \mathcal{V}_k \to \mathcal{U}_k^{\text{sp}}$ (resp. $f_k^{\text{sp}} : \mathcal{V}_k \to \mathcal{U}_k^{\text{sp}}$) is an element of $H^1_{\text{fppf}}(\mathcal{U}_k^{\text{sp}})_p$ (resp. an element of $H^1_{\text{fppf}}(\mathcal{U}_k^{\text{sp}})_p$) (cf. [Sa] 1.6 for the definition of $H^1_{\text{fppf}}(\mathcal{U}_k^{\text{sp}})_p$). In particular the restriction of $f^{\text{sp}}$ (resp. $f^{\text{nspp}}$) to each connected component $\mathcal{U}_i$ of $\mathcal{U}^{\text{sp}}$ (resp. $\mathcal{U}^{\text{nspp}}$) is a torsor $f_i : \mathcal{V}_i \to \mathcal{U}_i$ under a commutative finite and flat $R'$-group scheme $G_{R',i}$ of rank $p$, and $f_i : \mathcal{V}_i \to \mathcal{U}_i$ is a torsor under the $k'$-group scheme $G_{k',i} := G_{R',i} \times_{R'} k'$. Moreover if $G_{k',i}$ is radical, and if $\omega_i$ is the associated differential form, then the set of zeros and poles of $w_i$ is necessarily contained in $\{x_{i,j}\}_{j \in S_i} \cup \{z_{i,j}\}_{j \in D_i}$, as $\mathcal{V}_i$ is smooth (cf. [Sa], I).

**Deg.5.** Each smooth point $x_{i,j} \in B_k^{\text{sp}}$ (resp. $B_k^{\text{sp}}$) is endowed via $f$ with degeneration datas on the boundary of the formal fibre at $x_{i,j}$, as in Deg.1 above, and which satisfy certain compatibility conditions. More precisely for each smooth point $x_{i,j}$ we have the reduction type $(G_{k',i}, m_{i,j}, h_{i,j})$ on the boundary of the formal fibre at this point, induced by $g^{\text{sp}}$ (resp. $g^{\text{nspp}}$), and such that $r_{i,j} = m_{i,j} + 1$, as a consequence of [Sa-1] 3.11. Also if $x_{i,j} \in B_k^{\text{sp}}$ (in other words if $i \in I^{\text{sp}}$, and $S_i$ is non empty), then necessarily $G_{k',i} = \mu_p$, $m_{i,j} = -1$ and $h_{i,j} = 0$ (cf. [Sa-1, 3.1.2]).

**Deg.6.** Each double point $z_{i,j} = z_{i',j'} \in X_i \cap X_{i'}$ of $\tilde{X}^{\text{nspp}}$ (resp. $\tilde{X}^{\text{sp}}$) is endowed with degeneration datas $(G_{i,k'}, m_{i,j}, h_{i,j})$ and $(G_{i',k'}, m_{i',j'}, h_{i',j'})$, induced by $g^{\text{sp}}$ (resp. $g^{\text{sp}}$),
on the two boundaries of the formal fibre at this point as in Deg.1, such that \( m_{i,j} + m_{i',j'} = 0 \), and \( h_{i,j} + h_{i',j'} = 0 \), as a consequence of [Sa-1] 3.2.3. In particular \( m_{i_0,j_0} + m = 0 \), and \( h_{i_0,j_0} + h = 0 \). In other words the above induced by \( f \) element of \( H^1_{fppf}(U_{k^p}^{\text{sp}}) \) (resp. of \( H^1_{fppf}(U_{k'}^{\text{sp}}) \)) is indeed an element of \( H^1_{fppf}(U_{k'}^{\text{sp}})^{\text{kum}} \) (resp. an element of \( H^1_{fppf}(U_{k'}^{\text{sp}})^{\text{kum}} \)) (cf. [Sa] 1.7 for the definition of \( H \)).

**Deg.7.** For each double point \( z_{i,j} = z_{i',j'} \in X_i \cap X_{i'} \) of \( \tilde{X}^{\text{sp}} \) (resp. \( \tilde{X}^{\text{sp}} \)), let \( e_{i,j} \) be the thickness of \( z_{i,j} \). Then \( e_{i,j} = pt_{i,j} \) is necessarily divisible by \( p \). For each irreducible component \( X_i \), \( i \in I^{\text{sp}} \) (resp. \( i \in I^{\text{sp}} \)), let \( \eta_i \) be the generic point of \( X_i \). Let \( \delta_i \) be the degree of different above \( \eta_i \) in the cover \( g^{\text{sp}} \) (resp. \( g^{\text{sp}} \)). Then we have \( \delta_i - \delta_i = t_{i,j} m_{i,j}(p-1) \) as follows from [Sa-1] 3.2.5. In particular \( \delta - \delta_{i_0} = t_{i_0,j_0} m(p-1) \), and \( |D_i| \delta_i = \sum_{j \in D_i} (\delta_j + t_{i,j} m_{i,j}(p-1)) \), where \( |D_i| \) denotes the cardinality of \( D_i \).

**Deg.8.** The contribution to the arithmetic genus \( g_y \) of the point \( y \) was computed in [Sa-1] 3.1.1, in terms of the degeneration data \( (G_{k'}, m, h) \) on the boundary of the formal fibre at \( x \), and the cardinality \( r \) of \( B_K \). It follows also from the above considerations and after easy calculation that: \( g_y = \sum_{i \in I^{\text{sp}}} (-2 + \sum_{j \in S_i} (m_{i,j} + 1) + \sum_{j \in D_i} (m_{i,j} + 1))(p-1)/2 \), where \( I^{\text{sp}} \) denote the subset of \( I^{\text{sp}} \), or \( I^{\text{sp}} \), which among to the same, consisting of those \( i \) for which \( G_{k',i} \) is étale.

### 1.2.1. Example
In the following we give an example where one can exhibit the degeneration datas associated to a Galois cover \( f : \mathcal{Y} \to \mathcal{X} \) of degree \( p \) where \( \mathcal{X} \simeq \text{Spf } R[[T]] \) is the formal germ of a smooth point. More precisely, for \( m > 0 \) an integer prime to \( p \) consider the cover given generically by the equation \( X^p = 1 + \lambda^p(T^{-m} + \pi T^{-m+1}) \) (this is the example 1 in [Sa-1], 3.2.4, with \( m' = m + 1 \)). Here \( r = m + 2 \) and this cover has a reduction of type \((\mathbb{Z}/p\mathbb{Z}, m, 0)\) on the boundary. In particular the geometric genus \( g_y \) of the closed point \( y \) of \( \mathcal{Y} \) equals \((p-1)/2\). The non split degeneration data associated to the above cover consists necessarily of a tree with only one vertex and no edges, i.e. a unique projective line \( X_1 \) with a marked \( \overline{k} \)-point \( x_1 \), and an étale torsor \( f_1 : V_1 \to U_1 := X_1 - \{x_1\} \) above \( U_1 \) with conductor 2 at the point \( x_1 \).

\[
\begin{array}{ccc}
\tilde{y} & \xrightarrow{V_1} & \tilde{x} \\
\downarrow & & \downarrow \\
y & \xrightarrow{U_1} & x
\end{array}
\]

The above considerations leads naturally to the following abstract geometric and com-
binatorial definition of degeneration datas.

1.2.2. Definition. Let \( k' \) be an algebraic extension of \( k \). A \( k' \)-simple non split (resp. split) degeneration data \( \text{Deg}(x) \) of type \((r, (G_{k'}, m, h))\) and rank \( p \) consists of the following datas:

**Deg.1.** \( G_{k'} \) is a commutative finite and flat \( k' \)-group scheme of rank \( p \), \( r \geq 0 \) is an integer, \( m \) is an integer prime to \( p \) such that \( r - m - 1 \geq 0 \), and \( h \in \mathbb{F}_p \) equals 0 unless \( G_{k'} = \mu_p \) and \( m = 0 \).

**Deg.2.** \( \Gamma := X_{k'} \) is an oriented tree of projective lines with vertices \( \text{Vert}(\Gamma) := \{X_i\}_{i \in I} \) (resp. \( \text{Vert}(\Gamma) := \{X_i\}_{i \in I_{\text{sp}}} \)), endowed with an origin vertex \( X_{i_0} \), and a marked smooth point \( x := x_{i_0,j_0} \) on \( X_{i_0} \) which is a \( k' \)-rational point. We denote by \( \{z_{i,j}\}_{j \in D_i} \) the set of double points, or (non oriented) edges, of \( \Gamma \) which are supported by \( X_i \), and which we assume to be \( k' \)-rational points.

**Deg.3.** For each vertex \( X_i \) of \( \Gamma \) is given a set, may be empty, of smooth marked \( k' \)-rational points \( \{x_{i,j}\}_{j \in S_i} \).

We will summarise the datas Deg.2 and Deg.3 by saying that we are given a \( k' \)-rational marked tree of projective lines, or a marked semi-stable \( k' \)-curve with arithmetic genus 0.

**Deg.4.** For each \( i \in I \) (resp. \( i \in I_{\text{sp}} \)), is given a torsor \( f_{i,k'} : V_{i,k'} \rightarrow U_{i,k'} := X_i,k' - \{\{x_{i,j}\}_{j \in S_i} \cup \{z_{i,j}\}_{j \in D_i}\} \) under a commutative finite and flat \( k' \)-group scheme \( G_{k',i} \) of rank \( p \), where \( X_{i,k'} := X_i \times_{\text{Spec} k} \text{Spec} k' \), with \( V_{i,k'} \) smooth. Let \( U_{k'} \) be the open subscheme of \( X_{k'} \) obtained by deleting the double and marked points of \( X_{k'} \). Then the above data determines an element of \( H^1_{\text{fppf}}(U_{k'})_p \) (cf. [Sa], I). Moreover if \( G_{k',i} \) is radicial, and if \( \omega_i \) is the associated differential form, then the set of zeros and poles of \( u_i \) is necessarily contained in \( \{x_{i,j}\}_{j \in S_i} \cup \{z_{i,j}\}_{j \in D_i} \), as \( V_{i,k'} \) is smooth. Also if \( i \in I_{\text{sp}} \), and \( S_i \) is non empty, we assume that \( G_{i,k'} = \mu_p \).

**Deg.5.** For each \( i \in I \) (resp. \( i \in I_{\text{sp}} \)), are given for each \( j \in S_i \) the integers \( (m_{i,j}, h_{i,j}) \), where \( m_{i,j} \) is the conductor of the torsor \( f_i \) at the point \( x_{i,j} \), and \( h_{i,j} \) its residue at this point. Let \( r_{i,j} := m_{i,j} + 1 \). If \( i \in I_{\text{sp}} \) then we assume that \( r_{i,j} = 1 \). Also for the marked point \( x_{i_0,j_0} \) is given the data \( (m_{i_0,j_0}, h_{i_0,j_0}) \), such that \( m_{i_0,j_0} + m = 0 \), and \( h_{i_0,j_0} + h = 0 \).

**Deg.6.** For each double point \( z_{i,j} = z_{i',j'} \in X_i \cap X_{i'} \) is given the pair \( (m_{i,j}, h_{i,j}) \) (resp. \( (m_{i',j'}, h_{i',j'}) \)), where \( m_{i,j} \) (resp. \( m_{i',j'} \)) is the conductor of the torsor \( f_i \) (resp. \( f_{i'}, k' \)) at the point \( z_{i,j} \) (resp. \( z_{i',j'} \)), and \( h_{i,j} \) (resp. \( h_{i',j'} \)) its residue at this point. These datas must satisfy \( m_{i,j} + m_{i',j'} = 0 \), and \( h_{i,j} + h_{i',j'} = 0 \). In other words the element of \( H^1_{\text{fppf}}(U_{k'})_p \) determined by the data Deg.4 must belong to the subgroup \( H^1_{\text{fppf}}(U_k)^{\text{num}} \) (cf. [Sa], I).
Deg.7. For each irreducible component $X_i$ of $\Gamma$ is given an integer $\delta_i \leq v_K(p)$ which is divisible by $p - 1$. For each double point $z_{i,j} = z_{i',j'} \in X_i \cap X_{i'}$ of $\Gamma$ is given an integer $e_{i,j} = pt_{i,j}$ divisible by $p$, such that with the same notations as above we have $\delta_i - \delta_{i'} = m_{i,j}t_{i,j}(p-1)$.

Deg.8. Let $I_{et}$ denote the subset of $I$ (resp. of $I^p$) consisting of those $i$ for which $G_{k',i}$ is étale. Then the following equality should hold: $(r - m - 1)(p - 1)/2 = \sum_{i \in I_{et}}(-2 + \sum_{j \in S_i}(m_{i,j} + 1) + \sum_{j \in D_i}(m_{i,j} + 1))(p - 1)/2$. The integer $g_x := (r - m - 1)(p - 1)/2$ is called the genus of the degeneration data $Deg(x)$.

There is a natural notion of isomorphism of simple split (resp. non split) degeneration data of a given type over a given algebraic extension of $k$. We will denote by $SS-Deg_p$ (resp. $SNS-Deg_p$) the set of isomorphism classes of $\bar{k}$-simple split (resp. $k'$-simple non split) degeneration data of rank $p$.

1.2.3. Galois action on degeneration data.

The Galois group $G_k$ of the separable closure of $k$ contained in $\bar{k}$ acts in a canonical way on the set $SS-Deg_p$ (resp. $SNS-Deg_p$) of isomorphism classes of $\bar{k}$-simple split (resp. non split) degeneration data of rank $p$, in a way that is compatible with the group law on $G_k$. In other words $SS-Deg_p$ and $SNS-Deg_p$ are $G_k$-sets. More precisely, to a $\bar{k}$-simple split (resp. non split) degeneration data $Deg(x)$ of type $(r, (G_{\bar{k}}, m, h))$, and an element $\sigma \in G_k$, we associate the simple degeneration data $Deg(x)^{\sigma}$ of the same type $(r, G_{\bar{k}}, m, h))$. We explain briefly the defining data for $Deg(x)^{\sigma}$:

Deg$^{\sigma}$.1. The data $Deg^{\sigma}$.1 are the same as in Deg.1.

Deg$^{\sigma}$.2. and Deg$^{\sigma}$.3. The $\bar{k}$-rational marked tree $\Gamma^{\sigma}$ is the marked tree $\Gamma$.

Deg$^{\sigma}$.4. For each $i \in I$ (resp. $i \in I^p$), is given the torsor $f_i^{\sigma} : \nabla_i^{\sigma} \to \bar{U}_i := \bar{X}_i - \{(x_{i,j}) : j \in S_i \cup \{z_{i,j} : j \in D_i\} \}$ under the commutative finite and flat $\bar{k}$-group scheme $G_{\bar{k},i}$ of rank $p$, where $\bar{X}_i := X_i \times_k \bar{k}$, and $f_i^{\sigma}$ is the image of the torsor $f_i$ above $\bar{U}_i$ given by the data of $Deg$4, via the action of $\sigma$ on torsors (cf. [Sa], I). In other words the element $(f_i^{\sigma})_t$ of $H^1_{fppf}(U_{\bar{k},i})_p$ determined by the data of $Deg^{\sigma}$ of rank $p$, is the transform $(f_i)_{\sigma}^{\tau}$ of the element $(f_i)_{\tau}$ of $H^1_{fppf}(U_{\bar{k}})_p$ determined by the data of $Deg.4$, under the canonical action of $\sigma$ on $H^1_{fppf}(U_{\bar{k}})_p$ (cf. [Sa], I).

Deg$^{\sigma}$.5. For each $i \in I$ (resp. $i \in I^p$), are given for each $j \in S_i$ the integers $(m_{i,j}^{\sigma}, h_{i,j}^{\sigma})$, where $m_{i,j}^{\sigma}$ is the conductor of the torsor $f_i^{\sigma}$ at the point $x_{i,j}^{\sigma}$ of $S_i^{\sigma}$, and $h_{i,j}^{\sigma}$ its residue at this point. We have $m_{i,j}^{\sigma} = m_{i,j}$ and $h_{i,j}^{\sigma} = h_{i,j}$.

Deg$^{\sigma}$.6. For each double point $z_{i,j}^{\sigma} = z_{i',j'}^{\sigma} \in X_i \cap X_{i'}$ is given the pair $(m_{i,j}^{\sigma}, h_{i,j}^{\sigma})$ (resp. $(m_{i',j'}^{\sigma}, h_{i',j'}^{\sigma})$), where $m_{i,j}^{\sigma}$ (resp. $m_{i',j'}^{\sigma}$) is the conductor of the torsor $f_i^{\sigma}$ (resp. $f_{i'}^{\sigma}$) at the
point \( z_{i,j}^\sigma \) (resp. \( z_{i,j}'^\sigma \)), and \( h_{i,j}^\sigma \) (resp. \( h_{i,j}'^\sigma \)) its residue at this point. We have \( m_{i,j}^\sigma = m_{i,j} \) and \( h_{i,j}^\sigma = h_{i,j} \).

**Deg**.7. For each double point \( z_{i,j}^\sigma = z_{i,j}'^\sigma \in X_\sigma \cap X_\sigma' \) of \( \Gamma^\sigma \) is given the integer \( e_{i,j}^\sigma = pt_{i,j}^\sigma \) divisible by \( p \). For each irreducible component \( X_i \) of \( \Gamma \) is given the integer \( \delta_i^\sigma \leq v_K(p) \) which is divisible by \( p - 1 \), such that with the same notations as above \( \delta_i^\sigma - \delta_i'^\sigma = m_{i,j}^\sigma t_{i,j}^\sigma (p - 1) \). We have \( e_{i,j}^\sigma = e_{i,j} \), and \( \delta_i^\sigma = \delta_i \).

**Deg**.8. The integer \( g_x := (r - m - 1)(p - 1)/2 \) is the genus of the degeneration data \( D_x^\sigma \).

In conclusion, the action of the group \( G_k \) on the set of isomorphism classes of simple degeneration datas is given, and completely determined, through the canonical action of \( G_k \) on the data Deg.4, which is essentially given by the canonical action of \( G_k \) on the group \( H^1_{\text{fppf}}(U_k) \) and which was studied in [Sa].

Let \( \mathcal{X} := \text{Spf} \ A \) be the formal germ of an \( R \)-curve at a smooth closed point \( x \), and let \( \mathfrak{X} := \text{Spf} \ A \) where \( A := A \otimes_R \mathfrak{R} \). Let \( L := \text{Fr} \ A \) be the fractions field of \( A \). In 1.2 above we associated to a Galois cover \( f : \mathcal{Y} \to \mathcal{X} \) of degree \( p \), above a formal germ of an \( R \)-curve at a smooth point \( x \), a simple non split (resp. split) degeneration data \( D_x \). This indeed can be interpreted as the existence of a canonical specialisation map:

\[
\text{Sp} : H^1_{\text{et}}(\text{Spec} \ L, \mu_p) \to \text{SS-Deg}_p
\]

resp.

\[
\text{Sp} : H^1_{\text{et}}(\text{Spec} \ L, \mu_p) \to \text{SNS-Deg}_p
\]

Reciprocally, we have the following result of realisation for degeneration datas for such covers:

**1.2.4. Theorem.** Let \( k' \) be a finite extension of \( k \). Let \( \text{Deg}(x) \) be a \( k' \)-simple non split (resp. split) degeneration data of type \((r, (G_k, m, h)) \) and rank \( p \). Then there exists, after eventually a finite ramified extension of \( R \), a non-uniquely determined Galois cover \( f : \mathcal{Y} \to \mathcal{X} \) of degree \( p \), where \( \mathcal{X} \) is a formal germ of an \( R \)-curve at a smooth point \( x \), and such that the degeneration data associated to \( f \) via 1.2. is isomorphic to \( \text{Deg}(x) \). In other words the above specialisation maps \( \text{Sp} : H^1_{\text{et}}(\text{Spec} \ L, \mu_p) \to \text{SS-Deg}_p \) and \( \text{Sp} : H^1_{\text{et}}(\text{Spec} \ L, \mu_p) \to \text{SNS-Deg}_p \) are surjective.

**Proof.** One uses the formal patching results as explained in [Sa-1], plus the examples given in [Sa-1] 3.1.3, 3.1.4 and 3.2.4. We may also assume that \( k = k' \). First, for each \( i \in I^{\text{nsp}} \) (resp. \( i \in I^{\text{sp}} \)), let \( \mathcal{U}_i \) be a formal affine scheme with special fibre \( U_i := U_{k,i} \). The given torsor \( f_i : V_i \to U_i \) is admissible (cf. [Sa], IV), hence can be lifted, after eventually a
ramified extension of $R$, to a torsor $f'_i : V_i \to U_i$ under a finite and flat $R$-group scheme of rank $p$, which is either $\mu_p$ or $\mathcal{H}_{R,n}$ for $0 < n \leq v_K(\lambda)$ where $n := (v_K(p) - \delta_i)/(p-1)$. Such a lifting is non-unique in the radial case (cf. loc. cit). Moreover, for each marked point $x_{i,j}$ (resp. double point $z_{i,j}$) one can find a Galois cover $f_{i,j} : Y_{i,j} \to X_{i,j}$ of degree $p$ where $X_{i,j}$ is a formal germ of a smooth point $x_{i,j}$, and $Y_{i,j}$ is smooth (resp. $f_{i,j} : Y_{i,j} \to Z_{i,j}$ of degree $p$, where $Z_{i,j}$ is a formal germ of a double point $z_{i,j} = z_{i',j'}$ of thickeness $e_{i,j}$, and $Y_{i,j}$ is semi-stable) and with reduction type $(G_{i,k}, m_{i,j}, h_{i,j})$ (resp. $(G_{k,i}, m_{i,j}, h_{i,j}), (G_{k,i'}, m_{i',j}, h_{i',j})$) on the boundaries) (cf. [Sa-1] 3.1.3, 3.1.4, 3.2.6). Also one can find a Galois cover $f_x : Y_x \to X_x$ of degree $p$, above a formal germ of a double point $x$, and with reduction type $(G_k, m, h)$ and $(G_{i_0}, -m, -h)$ on the boundaries of the formal fibre at $x$, as well as a Galois cover $f'_x : Y'_x \to X'_x$ above a formal closed disc $X'_x := Spf R < T >$, with degeneration data $(G_k, m, h)$ on the boundary of the formal fibre at the closed point $T = 0$ (cf. [Sa-1] 2.3.1). Now using the formal patching result as in [Sa-1] I, as well as the result 3.1 in [Sa], and after carefully adjusting the Galois action on the $f_{i,j}$ and the $f'_i$, in order to obtain Galois patching data, one can patch the covers $f'_i, f_{i,j}, f_x$ and $f'_x$ along the points $x_{i,j}, z_{i,j}$ and $x$, in order to obtain a Galois cover $f' : \tilde{Y} \to \tilde{X}$ of degree $p$, where $\tilde{X}$ is a formal proper semi-stable $R$-curve, whose special fibre consists of the tree $\Gamma$ plus a projective line linked to $\Gamma$ via the double point $x$. A $G$-equivariant contraction of the vertices $(X_i)_i$ of the tree $\Gamma$ in $\tilde{X}_k$, will yield to a Galois cover $f : Y' \to X'$ of degree $p$, where $X'$ is a formal proper and smooth projective line, with a marked point $x$ on $X'_k$. A formal localisation now at the point $x$ will give the desired cover $f : Y_x \to X_x := Spf \hat{O}_{X',x}$, and by construction the simple degeneration data associated to $f$ via 1.2 is isomorphic to $D_x$.

1.2.5. Example. This in particular is the realisation given by the above theorem 1.2.4 of the degeneration data which arises in the example 1.2.1. Consider the non split simple degeneration data of rank $p$ which consists of a tree with one vertex and no edges. Hence a projective line $X_1$ with one $k$-marked point $x = x_1$, and a given étale torsor $f_1 : V_1 \to U_1 := X_1 - \{x_1\}$ with conductor $m' - m + 1$ at the point $x_1$ where $m' > m$ are positif integers. Then by 1.2.4 one can construct (after eventually a finite extension of $R$) a Galois cover $f : Y \to X$ of degree $p$ above a formal germ $X$ at a smooth point $x$, such that the singularity of $Y$ at the closed point $y$ is unibranche and the geometric genus of the point $y$ equals $(m' - m)(p - 1)/2$, moreover the semi-stable reduction of $Y$ consists of one irreducible component $Y_1$ of genus $(m' - m)(p - 1)/2$, which is the projective completion of the above affine curve $V_1$, and which is linked to the point $y$ by a double point.

1.2.6. Now we will study the action of $G_K$ on Galois covers of degree $p$ above formal germs of smooth curve at closed points. This action as we will see extends in a functorial way to an action of $G_K$ on the corresponding semi-stable models. More precisely, let $X := Spf A$
be the formal germ of an $R$-curve at a smooth closed point $x$, and let $\overline{X} := \text{Spf} \overline{A}$ where $\overline{A} := A \otimes_R \overline{R}$. Let $L := \text{Fr} \overline{A}$ be the fractions field of $\overline{A}$. Then the Galois group $G_K$ acts in a natural way on cyclic extentions of degree $p$ of $L$, and we have a canonical homomorphism:

$$G_K \to \text{Aut} H^1_{et}(\text{Spec } L, \mu_p)$$

The above action (1) is equivalent to the action of $G_K$ on normal Galois covers of degree $p$ above $\overline{X}$. More precisely to each such a cover $\tilde{f} : Y \to \overline{X}$ with $Y$ local and normal, which corresponds to the extension of functions fields: $\text{Spec } L' \to \text{Spec } L$, and an element $\sigma \in G_K$, one associates the Galois cover $\tilde{f}^\sigma : Y^\sigma \to \overline{X}$ of degree $p$, with $Y^\sigma$ local and normal, which corresponds to the extension of functions fields: $\text{Spec } L'^\sigma \to \text{Spec } L$. This action of $G_K$ on covers extends to an action of $G_K$ on corresponding minimal semi-stable models (because of the minimality condition). Here by a minimal semi-stable model we mean either a split or a non split one. More precisely, let $\tilde{f} : \tilde{Y} \to Y$ (resp. $\tilde{f}' : \tilde{Y}' \to Y'^\sigma$) be a minimal semi-stable model of $Y$ (resp. of $Y'^\sigma$), then the element $\sigma \in G_K$ maps the semi-stable model $\tilde{f}$ to the semi-stable model $\tilde{f}'$, in the sens that we have a commutative diagram:

$$\begin{array}{ccc}
\tilde{Y} & \xrightarrow{\tilde{f}} & Y \\
\sigma \downarrow & & \sigma \downarrow \\
\tilde{Y}' & \xrightarrow{\tilde{f}'^\sigma} & Y'^\sigma \\
\end{array}$$

where the horizontal arrows are $\overline{X}$-automorphisms. We first show how the above action of $G_K$ induces an action of $G_K$ on simple degeneration datas via its canonical quotient $G_k$. Our main result then is that this action coincides with the canonical action of $G_k$ on degeneration datas as defined in 1.2.2, and that the specialisation maps $\text{Sp} : H^1_{et}(\text{Spec } L, \mu_p) \to \text{SS-Deg}_p$ and $\text{Sp} : H^1_{et}(\text{Spec } L, \mu_p) \to \text{SNS-Deg}_p$ are $G_K$-equivariant.

First we may assume that the Galois cover $f : Y \to \mathcal{X}' := \mathcal{X} \times_R R'$, as well as its Galois conjugate via $\sigma$, are defined over a finite extension $R'$ of $R$. Let $\pi'$ be a uniformiser of $R'$, and let $A' := A \otimes_R R'$. Let $\wp := (\pi')$ be the ideal of $A'$ generated by $\pi'$, and let $\hat{A}'_{\wp}$ be the completion of the localisation of $A'$ at $\wp$. Let $\mathcal{X}' : = \text{Spf } \hat{A}'_{\wp}$ be the boundary of $\mathcal{X}'$, and let $\mathcal{X}'_{\eta} \to \mathcal{X}'$ be the canonical morphism. Consider the following cartesian diagram:

$$\begin{array}{ccc}
Y & \xrightarrow{f} & \mathcal{X}' \\
\downarrow & & \downarrow \\
Y_{\eta} & \xrightarrow{f_{\eta}} & \mathcal{X}'_{\eta} \\
\end{array}$$
Then \( f_\eta : Y_\eta \to X'_\eta \) is a torsor under a commutative finite and flat \( R' \)-group scheme \( G_{R'} \) of rank \( p \). Let \((G_{k'}, m, h)\) be the degeneration type of \( f_\eta \) where \( k' \) is the residue field of \( R' \). We have also the following cartesian diagram:

\[
\begin{array}{ccc}
Y^\sigma & \xrightarrow{f^\sigma} & X' \\
\downarrow & & \downarrow \\
Y^\sigma_\eta & \xrightarrow{f^\sigma_\eta} & X'_\eta 
\end{array}
\]

Moreover the torsor \( f^\sigma_\eta \) is the Galois conjugate of the torsor \( f_\eta \) by \( \sigma \) as one sees easily. In particular both of these torsors have the same reduction type.

The Galois group \( G \simeq \mathbb{Z}/p\mathbb{Z} \) of the cover \( f \) (resp. \( f^\sigma \)) acts in a canonical way on \( \tilde{Y} \) (resp. \( \tilde{Y}' := \tilde{Y}^\sigma \)). Let \( X := \tilde{Y}/G \) which also equals \( \tilde{Y}'/G \). We have a commutative diagram:

\[
\begin{array}{ccc}
\tilde{Y} & \xrightarrow{\sigma} & \tilde{X} \\
\downarrow & & \downarrow \\
\tilde{Y}^\sigma & \xrightarrow{\sigma^\sigma} & \tilde{X}^\sigma 
\end{array}
\]

Where the horizontal arrows are Galois covers with group \( G \), and \( \sigma \) is an \( \tilde{X} \)-automorphism. Let \( \text{Deg}(x_1) \) (resp. \( \text{Deg}(x_2) \)) be the simple degeneration data associated to the cover \( f \) (resp. \( f^\sigma \)). Let \( \sigma \) be the image of \( \sigma \) in \( G_k \) via the canonical surjective homomorphism \( G_K \to G_k \). We will prove that \( \text{Deg}(x_1) \) is the transform \( \text{Deg}(x_1)^\sigma \) of \( \text{Deg}(x,1) \) by \( \sigma \), via the canonical action of \( G_k \) on degeneration datas as explained in [Sa], I. The only thing we have to check to prove this is that this is true when considering the degeneration data \( \text{Deg}4 \). Let \( X_i \) be an irreducible component of \( \tilde{X}_{k'} := \tilde{X} \times_k k' \). Let \( D_i \) be the set of double points of \( \tilde{X}_{k'} \) supported by \( X_i \), and let \( S_i \) be the set of smooth points of \( X_i \) in which specialise some branched points of \( f \). Let \( U_i := X_i - \{S_i \cup D_i\} \), and let \( U_i \) be a formal open subset of \( \tilde{X} \) with special fibre \( U_i \). Let \( f_i : V_i \to U_i \) (resp. \( f'_i : V'_i \to U_i \)) be the torsor above \( U_i \) induced by \( f \) (resp. by \( f^\sigma \)). Then \( f'_i \) is the transform \( f^\sigma_i \) of \( f_i \) by \( \sigma \) of the torsor \( f_i \), via the canonical action of \( G_K \) on these torsors. Let \( f_{i,k'} : V_{i,k'} \to U_{i,k'} \) (resp. \( f'_{i,k'} : V'_{i,k'} \to U_{i,k'} \)) be the special fibre of the torsor \( f_i \) (resp. \( f'_i := f^\sigma_i \)). Let \( \sigma \) be the image of \( \sigma \) in \( G_k \) via the canonical homomorphism \( G_K \to G_k \). Then \( f'_{i,k'} \) is nothing else but the transform \( f_{i,k'}^\sigma \) of \( f_{i,k'} \) via \( \sigma \) (cf. [Sa], I). So indeed we have proven the following:

**1.2.7. Theorem.** Let \( G_K \) acts on the sets \( \text{SS-Deg}_p \) (resp. \( \text{SNS-Deg}_p \)) of isomorphism classes of simple split (resp. simple non split) degeneration data of rank \( p \), via its canonical quotient \( G_k \), and through the natural action of \( G_k \) on these sets as defined in 1.2.3. Then the surjective specialisation maps \( \text{Sp} : H^1_{\text{et}}(\text{Spec } L, \mu_p) \to \text{SS-Deg}_p \) and \( \text{Sp} : H^1_{\text{et}}(\text{Spec } L, \mu_p) \to \text{SNS-Deg}_p \) are \( G_K \)-equivariant.
1.2.8. Remark. It is easy to construct examples of covers $f : \mathcal{Y} \to \mathcal{X}$ as in [Sa-1] 3.1.4, where the special fibre $\mathcal{Y}_k$ is singular and unibranche at the closed point $y$ of $\mathcal{Y}$, and such that the configuration of the special fibre of a semi-stable model $\tilde{\mathcal{Y}}$ of $\mathcal{Y}$ is not a tree-like. This indeed answers a question raised by Lorenzini whether such a situation can occur in the unequal characteristic case. More precisely, consider the non split simple degeneration data $\text{Deg}(x)$ of type $(2, (\mu_p, -1, 0))$ which consists of a graph $\Gamma$ with two vertices $X_1$ and $X_2$ linked by a unique edge $y$, with given $\bar{k}$-marked points $x = x_1$ on $X_1$ and $\bar{k}$-marked point $x_2$ on $X_2$, and given étale torsors of rank $p : f_1 : V_1 \to U_1 := X_1 - \{x_1\}$ with conductor $m_1 = 1$ at $x = x_1$ and $f_2 : V_2 \to U_2 := X_2 - \{x_2\}$ with conductor $m_2 = 1$ at $x_2$. Then it follows from 1.2.4 that there exists, after eventually a finite extension of $R$, a Galois cover $f : \mathcal{Y} \to \mathcal{X}$ of degree $p$ above the formal germ $\mathcal{X} \simeq \text{Spf} R[[T]]$ at the smooth $R$-point $x$, such that the $\bar{k}$-simple non split degeneration data associated to the above cover $f$ is the above given one. Moreover by construction the singularity of the closed point $y$ of $\mathcal{Y}$ is unibranche, and the configuration of the (non-split) semi-stable reduction of $\mathcal{Y}$ consists of two projective lines which meet at $p$-double points (the above cover will be étale in reduction above the double point $y$), in particular one has $p - 1$ cycles in this configuration.

1.3. We use the same notations as in 1.1. We consider now the case where $x$ is an ordinary double point. Let $f : \mathcal{Y} \to \mathcal{X}$ be a Galois cover of degree $p$, where $\mathcal{X}$ is the formal germ of an $R$-curve at an ordinary double point $x$. In a similar way as in 1.2, and using the results of [Sa] and [Sa-1], one can associate a double degeneration data to $f$ whose definition we
summarise in the following:

1.3.1. **Definition.** Let \( k' \) be an algebraic extension of \( k \). A \( k' \)-**double non split (resp. split) degeneration data** \( \text{Deg}(x) \) of type \((r, (G_{k',1}, m_1, h_1), (G_{k',2}, m_2, h_2)) \) and rank \( p \) consists of the following:

**Deg.1.** For \( i \in \{1, 2\} \), \( G_{k',i} \) is a finite and flat \( k' \)-group scheme of rank \( p \), \( r \geq 0 \) is an integer, \( m_i \) is an integer prime to \( p \) such that \( r - m_1 - m_2 \geq 0 \), and \( h_i \in \mathbb{F}_p \) equals 0 unless \( G_{k',i} = \mu_p \) and \( m_i = 0 \).

**Deg.2.** \( \Gamma := X_{k'} \) is an oriented tree of projective lines with vertices \( \text{Vert}(\Gamma) := (X_i)_{i \in I} \) (resp. \( \text{Vert}(\Gamma) := (X_i)_{i \in I_p} \)), and \( \tilde{\Gamma} = \bigcup_{i \in \tilde{I}} X_i \), where \( \tilde{I} \subset I \) is a geodesic in \( \Gamma \) linking two given vertices \( X_{i_1} \) and \( X_{i_2} \), with a given marked smooth \( k' \)-rational point \( x_1 \in X_{i_1} \) (resp. \( x_2 \in X_{i_2} \)). We denote by \( \{z_{i,j}\}_{j \in D_i} \) the set of double points, or edges, of \( \Gamma \) which are supported by \( X_i \).

**Deg.3.** For each vertex \( X_i \) of \( \Gamma \) is given a set, may be empty, of smooth marked \( k' \)-rational points \( \{x_{i,j}\}_{j \in S_i} \).

**Deg.4.** For each \( i \in I \) (resp. \( i \in I^p \)), is given a torsor \( f_{i,k'} : V_{i,k'} \to U_{i,k'} := X_{i,k'} - \{(x_{i,j})_{j \in S_i} \cup \{z_{i,j}\}_{j \in D_i}\} \) under a commutative finite and flat \( k' \)-group scheme \( G_{k',i} \) of rank \( p \), where \( X_{i,k'} := X_i \times_{\text{Spec} k} \text{Spec} k' \), with \( V_{i,k'} \) **smooth.** Let \( U_{k'} \) be the open subscheme of \( X_{k'} \) obtained by deleting the double and marked points of \( X_{k'} \). Then the above data determines an element of \( H^1_{\text{fppf}}(U_{k'})_p \). Moreover if \( G_{k',i} \) is radicial, and if \( \omega_i \) is the associated differential form, then the set of zeros and poles of \( \omega_i \) is necessarily contained in \( \{x_{i,j}\}_{j \in S_i} \cup \{z_{i,j}\}_{j \in D_i} \), as \( V_{i,k'} \) is smooth. Also if \( i \in I^p \), and \( S_i \) is non empty, we assume that \( G_{i,k'} = \mu_p \).

**Deg.5.** For each \( i \in I \) (resp. \( i \in I^p \)), are given for each \( j \in S_i \) integers \( (m_{i,j}, h_{i,j}) \), where \( m_{i,j} \) is the conductor of the torsor \( f_i \) at the point \( x_{i,j} \), and \( h_{i,j} \) its residue at this point. Let \( r_{i,j} := m_{i,j} + 1 \). If \( i \in I^p \) then \( r_{i,j} = 1 \).

**Deg.6.** For each double point \( z_{i,j} = z_{i',j'} \in X_i \cap X_{i'} \) is given the pair \( (m_{i,j}, h_{i,j}) \) (resp. \( (m_{i',j'}, h_{i',j'}) \)), where \( m_{i,j} \) (resp. \( m_{i',j'} \)) is the conductor of the torsor \( f_{i,k'} \) (resp. \( f_{i',k'} \)) at the point \( z_{i,j} \) (resp. \( z_{i',j'} \)), and \( h_{i,j} \) (resp. \( h_{i',j'} \)) its residue at this point. These data must satisfy \( m_{i,j} + m_{i',j'} = 0 \), and \( h_{i,j} + h_{i',j'} = 0 \). In other words the element of \( H^1_{\text{fppf}}(U_{k'})_p \) determined by the data \( \text{Deg.4} \) must belong to the subgroup \( H^1_{\text{fppf}}(U_{k'})_{p}^{\text{num}} \). In particular for the double point \( x_{i_1,j_1} := x_1 \) (resp. \( x_{i_2,j_2} := x_2 \)) is given the data \( (m_{i_1,j_1}, h_{i_1,j_1}) \) (resp. \( (m_{i_2,j_2}, h_{i_2,j_2}) \)), such that \( m_{i_1,j_1} + m_1 = 0 \) (resp. \( m_{i_2,j_2} + m_2 = 0 \)), and \( h_{i_1,j_1} + h_1 = 0 \) (resp. \( h_{i_2,j_2} + h_2 = 0 \)).

**Deg.7.** For each irreducible component \( X_i \) of \( \Gamma \) is given an integer \( \delta_i \leq v_K(p) \) which
is divisible by $p - 1$. For each double point $z_{i,j} = z'_{i',j'} \in X_i \cap X_{i'}$ of $\Gamma$ is given an integer $e_{i,j} = pt_{i,j}$ divisible by $p$, such that with the same notations as above we have $\delta_i - \delta_{i'} = m_{i,j}t_{i,j}(p - 1)$.

**Deg.8.** Let $I_{et}$ denote the subset of $I$ (resp. of $I^{sp}$) consisting of those $i$ for which $G_{k',i}$ is étale, then: $(r - m - 1)(p - 1)/2 = \sum_{i \in I_{et}} (-2 + \sum_{j \in S_i} (m_{i,j} + 1) + \sum_{j \in D_i} (m_{i,j} + 1))(p - 1)/2$.

The integer $g_x := (r - m_1 - m_2)(p - 1)/2$ is called the **genus** of the degeneration data $\text{Deg}(x)$.

There is a natural notion of isomorphism of double split (resp. non split) degeneration datas of a given type over a given algebraic extension of $k$. We will denote by $\text{DS-Deg}_p$ (resp. $\text{DNS-Deg}_p$) the set of isomorphism classes of $\overline{k}$-double split (resp. $\overline{k}$-double non split) degeneration data of rank $p$. Also as in 1.2.3 one shows that there exists a canonical action of $G_k$ on these sets, via the canonical action of $G_k$ on the degeneration data $\text{Deg}$, and these sets are $G_k$-sets, hence also canonically $G_K$-sets. The following theorem is proven in the same way as in 1.2.4 and 1.2.6.

**1.3.2. Theorem.** Let $X := \text{Spf } A$ be the formal germ of an $R$-curve at an ordinary double point. Let $\overline{A} := A \otimes_R \overline{R}$, and let $L := \text{Fr}(\overline{A})$. Then the canonical specialisation maps: $\text{Sp} : H^1_{et}(\text{Spec } L, \mu_p) \to \text{DS-Deg}_p$ and $\text{Sp} : H^1_{et}(\text{Spec } L, \mu_p) \to \text{DNS-Deg}_p$ are surjective and $G_K$-equivariant.

**1.3.3. Remark.** The surjectivity result in 1.2.4 and 1.2.7 was shown in [He] in the case of degeneration datas of genus 0 in the framework of automorphisms of open discs and annuli. Our method based on the techniques developed in [Sa] and [Sa-1] avoid the use of this language and uses only kummer theory.

II. Semi-stable reduction of Galois covers of degree $p$ above proper curves.

**2.0.** In this section we will use the same notations as in 1.0. We will consider a proper and semi-stable $R$-curve $X$, a Galois cover $f : Y \to X$ with group $\mathbb{Z}/p\mathbb{Z}$, and with $Y$ normal. We will study the semi-stable reduction of $Y$ as well as the Galois action on these datas.

**2.1.** Let $X$ be a proper and semi-stable $R$-curve, with smooth and connected generic fibre $X_K := X \times_R K$. We assume that the double points of $X_K$ are $k$-rationals. Let $f : Y \to X$ be a Galois cover with group $G \simeq \mathbb{Z}/p\mathbb{Z}$, and with $Y$ normal. It follows from the theorem of semi-stable reduction for curves (cf. [De-Mu]) that, after eventually a finite extension $R'$ of $R$, with fractions field $K'$ and residue field $k'$, the curve $Y$ has a semi-stable reduction. More precisely, there exists a birational and proper morphism $\tilde{f} : \tilde{Y} \to Y' := Y \times_R R'$
such that $\tilde{Y}_{K'} \simeq Y_{K'}$, and the following conditions hold:

(i) The special fibre $\tilde{Y}_{k'}$ of $\tilde{Y}$ is reduced.

(ii) $\tilde{Y}_{k'}$ has only ordinary double points as singularities.

Moreover there exists such a semi-stable model $\tilde{Y}$ which is minimal. In particular the action of $G$ on $Y'$ extends to an action on $\tilde{Y}$. Let $\tilde{X}$ be the quotient of $\tilde{Y}$ by $G$ which is a semi-stable model of $X' := X \times_R R'$. In the following we will choose $\tilde{Y}$ and $\tilde{X}$ as above such that the set of points $B_K := \{x_{i,K}\}_{i=1}^r$ consisting of the branch locus of the morphism $f_K : Y_K \to X_K$ are rational. Moreover we choose such $\tilde{X}$ and $\tilde{Y}$ which are minimal for these properties. One has the following commutative diagram:

$$
\begin{array}{ccc}
\tilde{Y} & \xrightarrow{g} & \tilde{X} \\
\downarrow f & & \downarrow g \\
Y' & \xrightarrow{f'} & X'
\end{array}
$$

Let $(X_i)_{i \in I}$ be the irreducible components of the special fibre $X'_{k'}$ of $X'$. For each $i \in I$, let $(x_{i,j})_{j \in S_i}$ be the set (may be empty) of those smooth points of $X_i$ in which specialise some points of $B_K$, say in $x_{i,j}$ specialise $r_{i,j}$ points of $B_K$, and let $(z_t)_{t \in J}$ be the set of double points of $X_k$. For each $t \in J$, let $r_t$ be the number of points of $B_K$ which specialise in $z_t$, we have $r = \sum_{t \in J} r_t + \sum_{i \in I} \sum_{j \in S_i} r_{i,j}$. Let $U := X' - \{D \cup S\}$, where $S := \bigcup_{i \in I} \{x_{i,j}\}_{j \in S_i}$ and $D = \{z_t\}_{t \in J}$. We will denote by $\{U_i\}_{i \in I}$ the set of irreducible components of $U$.

2.2. Below we will use the notations of 2.0 and 2.1 and will exhibit the degeneration datas associated to the semi-stable reduction of $Y$. Let $R'$ be a finite extension of $R$ such that the conditions of 2.1 hold. The following datas are canonically associated to the Galois cover $f : Y \to X$:

Deg.1. For each irreducible component $X_i$ of $X'_{k'}$, let $\eta_i$ be the generic point of $X_i$, and let $X'_{\eta_i}$ be the spectrum of the completion of the localisation of $X'$ at $\eta_i$. We have a canonical morphism : $X'_{\eta_i} \to X'$. Consider the following cartesian diagram:

$$
\begin{array}{ccc}
Y_{\eta_i} & \xrightarrow{f_{\eta_i}} & X'_{\eta_i} \\
\downarrow & & \downarrow \\
Y' & \xrightarrow{f'} & X'
\end{array}
$$

Then either $f_{\eta_i} : Y_{\eta_i} \to X'_{\eta_i}$ is completely split, or is a torsor under a finite and flat commutative $R'$-group scheme $G_i$ of rank $p$.

Deg.2. For each irreducible component $X_i$ of $X'_{k'}$, let $U_i := X_i - \{\{x_{i,j}\}_{j \in S_i} \cup \{z_t\}_{t \in J_i}\}$, where $J_i \subset J$ denotes the index subset indexing those double points of $X'_{k'}$ which are
supported by $X_i$. Then $V_i := f^{-1}(U_i) \to U_i$ is an admissible torsor under the finite and flat group scheme $G_{k',i} := G_i \times_{R'} k'$. Moreover, if $G_{k',i}$ is étale then $V_i$ is smooth, and if $G_{k',i}$ is radicial then the only singularities of $V_i$ lie above the zeros $\{y_l\}_{l \in Z_i}$ of the differential form $\omega_i$ associated to the above torsor (cf. [Sa], I).

**Deg.3.** For each double point $z_j \in X_i \cup X_{i'}$, let $\mathcal{X}_j$ be the completion of the localisation of $X'$ at $z_j$, and let $\mathcal{X}_j \to X'$ be the canonical morphism. Consider the cartesian diagram:

$$
\begin{array}{ccc}
\mathcal{Y}_j & \xrightarrow{f_j} & \mathcal{X}_j \\
\downarrow & & \downarrow \\
Y' & \xrightarrow{f'} & X'
\end{array}
$$

Then either the cover $f_j : \mathcal{Y}_j \to \mathcal{X}_j$ is split, or it is a Galois cover of degree $p$ with $\mathcal{Y}_j$ connected, in which case is associated to $f_j$ a double degeneration data $\text{Deg}(z_j)$. Let $m_{j,1}$ and $h_{j,1}$ (resp. $m_{j,2}$ and $h_{j,2}$) be the conductor and the residue at the double point $z_j$ associated to the torsor $V_i := f^{-1}(U_i) \to U_i$ (resp. $V_{i'} := f^{-1}(U_{i'}) \to U_{i'}$). Then the above degeneration data $\text{Deg}(z_j)$ is of type $(r_j, (G_{k',i}, m_{j,1}, h_{j,1})), (G_{k',i'}, m_{j,2}, h_{j,2}))$, and genus $(r_j - m_{j,1} - m_{j,2})(p - 1)/2$.

**Deg.4.** For each irreducible component $X_i$ of $X'_k$, and a smooth point $x_{i,j}, j \in S_i$, let $\mathcal{X}_{i,j}$ be the completion of the localisation of $X'$ at $x_{i,j}$, and let $\mathcal{X}_{i,j} \to X'$ be the canonical morphism. Consider the cartesian diagram:

$$
\begin{array}{ccc}
\mathcal{Y}_{i,j} & \xrightarrow{f_{i,j}} & \mathcal{X}_{i,j} \\
\downarrow & & \downarrow \\
Y' & \xrightarrow{f'} & X'
\end{array}
$$

The cover $f_{i,j} : \mathcal{Y}_{i,j} \to \mathcal{X}_{i,j}$ is necessarily a Galois cover of degree $p$ with $\mathcal{Y}_{i,j}$ connected, and to $f_{i,j}$ is associated a simple degeneration data $\text{Deg}(x_{i,j})$. Let $m_{i,j}$ and $h_{i,j}$ be the conductor and the residue at the smooth point $x_{i,j}$ associated to the torsor $V_i := f^{-1}(U_i) \to U_i$. Then the above degeneration data $\text{Deg}(x_{i,j})$ is of type $(r_{i,j}, (G_{k',i}, m_{i,j}, h_{i,j}))$, and genus $(r_{i,j} - m_{i,j} - 1)(p - 1)/2$.

**Deg.5.** For each $i \in I$ such that the group scheme $G_{k',\tilde{i}}$ is radicial, let $\{y_l\}_{l \in Z_i}$ be the zeros of the differential form $\omega_i$ associated to the above torsor $V_i := f^{-1}(U_i) \to U_i$, and which we may assume to be rational over $k'$. For each $l \in Z_i$, let $Z_l$ be the completion of the localisation of $X'$ at $y_l$, and let $Z_l \to X'$ be the canonical morphism. Consider the cartesian diagram:
The cover \( Z'_i \to Z_i \) is a Galois cover of degree \( p \) with \( Z'_i \) connected, and to \( f_i \) is associated a simple degeneration data \( \text{Deg}(y_i) \). Let \( m_i \) and \( h_i \) be the conductor and the residue at the smooth point \( y_i \) associated to the torsor \( V_i := f^{-1}(U_i) \to U_i \). Then the above degeneration data \( \text{Deg}(y_i) \) is of type \((0,(G_{k',i},m_i,h_i))\), and genus \((-m_i - 1)(p - 1)/2\).

The above considerations lead to the following definition of degeneration datas associated to a Galois cover of degree \( p \) above a proper and semi-stable \( R \)-curve.

**2.2.1. Definition.** Let \( X_k \) be a proper and semi-stable \( k \)-curve, and let \( \{X_i \}_{i \in I} \) be the irreducible components of \( X_k \). We assume that the double points \( \{z_i \}_{i \in J} \) of \( X_k \) are \( k \)-rational. Let \( k' \) be an algebraic extension \( k \). A \( k' \)-degneration data of rank \( p \), associated to \( X_k \), consists of the following datas:

**Deg.1.** For each irreducible component \( X_i \) of \( X_k \), is given a set of smooth \( k' \)-rational points \( \{x_{i,j} \}_{j \in S_i} \) of \( X_k \) which are supported by \( X_i \).

**Deg.2.** For each irreducible component \( X_i \) of \( X_k \), let \( U_i := X_i - \{(x_{i,j})_{j \in S_i} \cup \{z_i \}_{i \in J_i} \} \), where \( J_i \subset J \) denotes the index subset indexing those double points of \( X_k \) which are supported by \( X_i \). Then we assume given an admissible torsor \( f_i : V_i \to U'_i := U_i \times_k k' \) (cf. [Sa], 4, for the definition of admissibility) under a finite and flat \( k' \)-group scheme \( G_{k',i} \), and we allow the torsor \( f_i \) to be trivial. If \( G_{k',i} \) is radicial then we assume that the zeros \( \{y_i \}_{i \in Z_i} \) of the differential form \( \omega_i \) associated to the above torsor \( f_i \) are \( k' \)-rational.

**Deg.3.** For each double point \( z_j \in X_i \cap X' \) of \( X_k \), let \( m_{j,1} \) and \( h_{j,1} \) (resp. \( m_{j,2} \) and \( h_{j,2} \)) be the conductor and the residue at the double point \( z_j \) associated to the torsor \( V_i := f^{-1}(U_i) \to U_i \) (resp. \( V'_i := f^{-1}(U'_i) \to U'_i \)). Let \( r_j \) be an integer such that \( r_j - m_{j,1} - m_{j,2} \geq 0 \). Then we assume given a \( k' \)-double degeneration data \( \text{Deg}(z_j) \) of type \((r_j,(G_{k',i},m_{j,1},h_{j,1}),(G_{k',i},m_{j,2},h_{j,2}))\), and genus \((r_j - m_{j,1} - m_{j,2})(p - 1)/2\).

**Deg.4.** For each irreducible component \( X_i \) of \( X_{k'} \), and a smooth point \( x_{i,j} \), \( j \in S_i \), let \( m_{i,j} \) and \( h_{i,j} \) be the conductor and the residue at the smooth point \( x_{i,j} \) associated to the torsor \( V_i := f^{-1}(U_i) \to U_i \). Let \( r_{i,j} \) be an integer such that \( r_{i,j} - m_{i,j} - 1 \geq 0 \). Then we assume given a \( k' \)-simple degeneration data \( \text{Deg}(x_{i,j}) \) of type \((r_{i,j},(G_{k',i},m_{i,j},h_{i,j}))\), and genus \((r_{i,j} - m_{i,j} - 1)(p - 1)/2\).

**Deg.5.** For each \( i \in I \) such that the group scheme \( G_{k',i} \) is radicial, let \( \{y_i \}_{i \in Z_i} \) be the zeros of the differential form \( \omega_i \) associated to the above torsor \( V_i := f^{-1}(U_i) \to U_i \), and which
we may assume to be rational over $k'$. For each $l \in \mathbb{Z}_i$, let $m_l$ and $h_l$ be the conductor and the residue at the smooth point $y_l$ associated to the torsor $V_i := f^{-1}(U_i) \to U_i$. Then we assume given a degeneration data $\text{Deg}(y_l)$ of type $(0, (G_{k', i}, m_l, h_l))$, and genus $(-m_l - 1)(p - 1)/2$.

2.2.2. Remark. The admissibility condition on the torsors $\{f_i\}_{i \in I}$ which is part of the above definition of degeneration datas is satisfied when the $U_i$ are opens of the projective line, e.g. if $X_k$ is a totally degenerate Mumford curve, and in the case where $U_i$ is proper.

2.2.3. Example. In this example we consider a semi-stable and proper $R$-curve $X$ with smooth and geometrically connected generic fibre $X_K$, and whose special fibre $X_k$ consists of two irreducible smooth components $X_1$ of genus $g_1 \geq 2$ and $X_2$ of genus $g_2 \geq 2$ which intersect at the double point $x$. We suppose moreover that both $X_1$ and $X_2$ are generic (for this one has to choose $R$ to be rather large).

Consider a Galois cover $f : Y \to X$ of degree $p$ with $Y$ normal and such that the Galois cover $f_K : Y_K \to X_K$ induced by $f$ above the generic fibre is a $\mu_p$-torsor (hence in particular is étale). We assume that the special fibre $Y_k := Y \times_R k$ of $Y$ is reduced. For $i \in \{1, 2\}$, let $f_i : V_i \to U_i := X_i \setminus \{x\}$ be the torsor above $U_i$ induced by the cover $f$, and let $m_i$ be the conductor of $f_i$ at the point $x$. Then only the following two cases can occur:

Case 1) Either $f_i$ is an étale torsor in which case either $f_i$ ramify above $x$ or $f_i$ extends to an étale torsor above $X_i$. In both cases $V_i$ is smooth.

Case 2) Or $f_i$ is a torsor under $\mu_p$ in which case is associated to $f_i$ canonically a logarithmic differential form $\omega_i$, and the singularities of $V_i$ lie above the zeros of $\omega_i$. In this case where $X_i$ is generic it was shown in [Ra] (proposition 4) that such a differential form $\omega_i$ has $g_i - 1$ double zeros if $p = 2$ and $2g_i - 2$ simple zeros if $p \neq 2$. In the later case $m_i = -2$ or $m_i = -1$ depending on whether $x$ is a zero of $\omega_i$ or not.

In what follows we assume for simplicity that $p \neq 2$. Suppose we are in case 2, and let $y_j$ be a point of $V_i$ above a zero $z_j$ of $\omega_i$ which is different from the point $x$. Then $V_i$ is singular at $y_j$, the singularity at $y_j$ is unibranche, and the geometric genus of $y_j$ equals $(p - 1)/2$. Let $\text{Deg}(z_j)$ be the non split $k$-degeneration data associated to $z_j$ via the semi-stable reduction of $Y$. Then $\text{Deg}(z_j)$ is necessarily of type $(0, (\mu_p, -2, 0))$, and consists of
one projective line $P_i$ with a marked $\overline{k}$-point $x_i = x$ and an étale torsor $T_i \to P_i - \{x_i\}$ with conductor 2 at the point $x_i$.

Above the double point $x$ three situations can occur:

Case a) Both $f_1$ and $f_2$ are étale in which case $f$ is necessarily étale above $x$.

Above the double point $x$ three situations can occur:

Case a) Both $f_1$ and $f_2$ are étale in which case $f$ is necessarily étale above $x$.

Apart from case a) $Y$ must be local above, we will denote by $y$ the unique point of $Y$ above $x$.

Case b) Say $f_1$ is étale and $f_2$ is radicial, and assume for simplicity that $Y$ has two branches at the point $y$. In this case the geometric genus of $y$ equals $(m_1 + 2)(p - 1)/2$ if $\omega_2$ has a zero at $x$, or $(m_1 + 1)(p - 1)/2$ otherwise.

Case c) Both $f_1$ and $f_2$ are radicial in which case the geometric genus of $y$ equals $2(p - 1), 3(p - 1)/2$ or $p - 1$ depending on whether both $f_i$ have a zero at the point $x$, or just one of them do, or none of them.

Let $\text{Deg}(x)$ be the double non split $\overline{k}$-degeneration data associated to $x$ via the semi-stable reduction of $Y$. Assume for example that we are in case c) and that the genus of $y$ equals $p - 1$. Then the following two cases can occur:

First case: either $\text{Deg}(x)$ consists of one projective line $P$ with two marked $\overline{k}$-points $x_1 = x$ and $x_2 = x$, and an étale torsor $T \to P - \{x_1, x_2\}$ with conductor 2 at each of $x_1$ and $x_2$. In this case the special fibre of the semi-stable reduction of $Y$ is a tree like which consist of the components $V_1$ and $V_2$ which are linked by the projective completion of $V$ (which has genus $p - 1$) at the points above $x$, and at each of the points of $V_i$ above a zero of $\omega_i$ is linked a curve of genus $(p - 1)/2$.

Second case: $\text{Deg}(x)$ consists of two projective line $P_1$ and $P_2$ with a marked $\overline{k}$-points $x_1 = x$ and $x_2 = x$ on each one, and an étale torsor $T_i \to P_i - \{x_i\}$ with conductor 2 at $x_i$. In this case the special fibre of the semi-stable reduction of $Y$ contains $(p - 1)$ cycles and it consists of the components $V_1$ and $V_2$ which are linked by the projective completion of $T_1$ and $T_2$ (each has genus $(p - 1)/2$) at the points above $x$, moreover both $T_1$ and $T_2$ meet at $p$ double points, and at each of the points of $V_i$ above a zero of $\omega_i$ is linked a curve of genus $(p - 1)/2$ as in the preceeding case.
2.3. There is a natural notion of isomorphism of degeneration data of rank $p$ associated to a proper and semi-stable $k$-curve $X_k$, over a given algebraic extension of $k$. We will denote by $\text{Deg}_p(X_k)$ the set of isomorphism classes of $\bar{k}$-degeneration datas of rank $p$ associated to $X_k$. The set $\text{Deg}_p(X_k)$ is equipped in a natural way with a $G_k$-action, via the natural action of $G_k$ on the above datas which was explained in [Sa] and I, and hence is a $G_k$-set. The following result is analogous to the result of realisation for degeneration datas obtained in paragraph I in the local situation.

2.3.1. Theorem. Let $X_k$ be a proper and semi-stable $k$-curve. Let $R$ be a complete discrete valuation ring of inequal characteristics with residue field $k$, and fractions field $K$. Suppose given a $\bar{k}$-degeneration data $\text{deg}(X_k)$ of rank $p$ associated to $X_k$ (in other words suppose given an element of $\text{Deg}_p(X_k)$). Then there exists, after eventually a finite extension of $R$, a proper and semi-stable $R$-curve $\tilde{X}$ with smooth generic fibre and a special fibre isomorphic to $X_k$, and a Galois cover $f : \tilde{Y} \to \tilde{X}$, such that the degeneration data associated to $f$ as in 2.2 is isomorphic to the given data $\text{deg}(X_k)$. The above cover $f$ is not unique in general, moreover it can be constructed such that the cover $f_K : \tilde{Y}_K \to \tilde{X}_K$ induced by $f$ on the generic fibres is ramified above $r = \sum_{t \in J} r_t + \sum_{i \in I} \sum_{j \in S} r_{i,j}$ geometric points of $X_K$ (hier $r_t$ and $r_{i,j}$ are the one given in the definition of the degeneration data in 2.2.1).
The next result expresses the compatibility of degeneration with the Galois action.

2.3.2. Theorem. Let $R$ be a complete discrete valuation ring of inequal characteristics with residue field $k$, and fractions field $K$, and let $X$ be a proper and semi-stable $R$-curve with smooth and geometrically connected generic fibre $X_K$ and with special fibre $X_k$. Let $\overline{K}$ be an algebraic closure of $K$, and let $L$ be the function field of the geometric generic fibre $X_{\overline{K}} := X \times_R \overline{K}$ of $X$. Then the canonical specialisation map $Sp : H^1_{et}(\text{Spec } L, \mu_p) \to \text{Deg}_p(X_k)$ is $G_K$-equivariant, where here we consider the natural action of $G_K$ on $\text{Deg}_p(X_k)$ via its canonical quotient $G_k$.

Note that the above specialisation map can not be surjective in this case since the set $\text{Deg}_p(X_k)$ with fixed ramification datas on the "generic fibre" is infinite in general. The proof of the above two results is very similar to the one obtained in paragraph I in the local case and are left to the reader. They are based on formal patching techniques and the results in [Sa] and [Sa-1]. However in order to prove 2.3.1 one will also need the following result on lifting of admissible torsors with given degeneration datas at the "critical points". More precisely let $R$ be as in 1.0. Let $X$ be a formal smooth $R$-curve, whose special fibre $X_k$ is irreducible. Let $f_k : Y_k \to X_k$ be an admissible torsor, under a finite and flat $k$-group scheme $G_k$ of rank $p$, which is radicial. Let $\omega$ be the associated differential form, and let $Z := \{x_j\}_{j \in J}$ be the set of zeros of $\omega$ which are contained in $X_k$, which we call the critical points of $f_k$, in particular the singularities of $Y_k$ lie above the points of $Z$. Assume that $f : Y \to X$ is a torsor under a finite and flat $R$-group scheme $G$ which lifts $f_k$. To each point $x_j \in Z$ is associated via $f$, a simple degeneration data $\text{Deg}(x_j)$ of type $(0, (G_k, m_j, h_j))$, where $m_j$ is the conductor of $f_k$ at $x_j$, and $h_j$ its residue at this point. The following result shows that we can choose such a lifting $f$ of $f_k$ which gives rise to a given set of degeneration datas at the critical points.

2.3.3. Proposition / Definition. Let $f_k : Y_k \to X_k$ be an admissible torsor under a finite and flat $k$-group scheme $G_k$ of rank $p$, which is radicial, and let $Z := \{x_j\}_{j \in J}$ be the set of critical point of $f_k$ which we assume to be $k$-rational. For each point $x_j \in Z$ consider the pair $(m_j, h_j)$, where $m_j$ is the conductor of $f_k$ at $x_j$, and $h_j$ its residue at this point. Suppose given for each critical point $x_j$, a simple degeneration data $\text{Deg}(x_j)$ of type $(0, (G_k, m_j, h_j))$. Then there exists, after eventually a finite extension of $R$, a smooth formal $R$-curve $\tilde{X}$ with a special fibre $\tilde{X}_k := \tilde{X} \times_R k$ which is isomorphic to $X_k$, and a torsor $\tilde{f} : \tilde{Y} \to \tilde{X}$ under a finite and flat $R$-group scheme $G$ of rank $p$, such that the restriction of $\tilde{f}$ to $U_k := X_k - Z$ is isomorphic to the restriction of $f_k$ to $U_k$, and such that the simple degeneration data associated to each point $x_j \in Z$, via $\tilde{f}$, is isomorphic to $\text{Deg}(x_j)$. We call such a torsor $\tilde{f}$ a lifting of $f_k$, with the given degeneration datas $\{\text{Deg}(x_j)\}_{j \in Z}$ at the critical points $\{x_j\}_{j \in Z}$.
Proof. The torsor \( f_k : Y_k \to X_k \) is admissible hence by definition can be lifted, after eventually a ramified extension of \( R \), to a torsor \( f : Y' \to X \) under a finite and flat \( R \)-group scheme of rank \( p \), which is either \( \mu_p \) or \( H_n \) for \( 0 < n < v_K(\lambda) \) where \( v_K \) is the normalised valuation of the fractions field of \( R \) (such a lifting is not unique). Also for each point \( x_j \in Z \) one can find a Galois cover \( f_j : Y_j \to X_j \) of degree \( p \), where \( X_j \) is the formal germ of \( X \) at \( x_j \), and such that the degeneration data associated to \( x_j \) via \( f_j \) is isomorphic to \( \text{Deg}(x_j) \) (cf. 1.2.3). Let \( U_k := X_k - Z \), and let \( g_k \) be the restriction of \( f_k \) to \( U_k \). Then \( f \) induces a lifting \( g : V \to U \) of \( g_k \), which is a torsor under a finite and flat \( R \)-group scheme of rank \( p \), and where \( U \) is obtained from \( X \) by deleting the formal fibres at the points \( \{x_j\}_{j \in Z} \). Now a patching of the covers \( g \) and \( f_j \) along the formal fibres at the points \( \{x_j\}_{j} \) will produce the desired torsor \( \tilde{f} : \tilde{Y} \to \tilde{X} \).

2.3.4. Example. Hier we use the same notations as in the example 2.2.3. First we will explain the Galois action on a \( \mu_p \)-torsor \( f_\sigma : Y_\sigma \to X_\sigma \) above the geometric generic fibre \( X_\sigma \) of \( X \). We assume that the above torsor induces in reduction the situation in case c) where \( g_y = p - 1 \) and where the semi-stable reduction of \( Y_\sigma \) is a tree like. Let \( \sigma \in G_K \), and let \( \sigma \) be the image of \( \sigma \) in \( G_k \). Consider the \( \mu_p \)-torsor \( f_\sigma : Y_\sigma \to X_\sigma \) which is the conjugate of \( f_\sigma \) via \( \sigma \). Then using the result in 2.3.2 one can describe the semi-stable reduction of \( Y_\sigma \). More precisely the graph of the semi-stable reduction of \( Y_\sigma \) is isomorphic to the one of \( Y_\sigma \), in particular it is also a tree like, and it consists of the components \( V_i \), for \( i = 1, 2 \), where \( V_i \) is the conjugate of \( V_i \), which is linked by an irreducible curve \( T \) of genus \( p - 1 \) which is the conjugate of \( T \) by \( \sigma \). Moreover at each point of \( V_i \) above a zero of \( \omega_i \) is linked a curve of genus \( (p-1)/2 \) which is the conjugate by \( \sigma \) of the curve that is linked to the corresponding point (via the action of \( \sigma \)) of \( V_i \) in the graph of semi-stable reduction of \( Y_\sigma \).

Also one can establish the converse of the situation in 2.2.3, i.e. reconstruct a \( \mu_p \)-torsor \( f_K \) as above from the degeneration datas, using 2.3.1. More precisely, let \( X_k \) be the semi-stable curve considered in 2.2.3 and consider the non split degeneration data \( \text{Deg}(X_k) \) of rank \( p \) which consists in the following:

For \( i = 1, 2 \) is given a \( \mu_p \)-torsor \( Y_i \) associated to the logarithmic differential form \( \omega_i \), and assume that the zeros \( \{z_{j,i}\}_j \) of \( \omega_i \) lie outside the point \( x \). For each \( i, j \) suppose given a non split simple degeneration data \( \text{Deg}(z_{j,i}) \) of type \((0, (\mu_p, -2, 0))\) and which consists of one projective line \( P_{j,i} \) with one \( \sigma \)-marked point \( z_{j,i} \) and an étale torsor above \( P_{j,i} - \{z_{j,i}\} \) with conductor \( 2 \) at the point \( z_{j,i} \). At the double point \( x \) is given a non split double degeneration data of type \((0, (\mu_p, -2, 0), (\mu_p, -2, 0))\) and which consists of one projective line \( P \) and two \( \sigma \)-marked points \( x_1 = x \) and \( x_2 = x \) and an étale torsor above \( P - \{x_1, x_2\} \) with conductor \( 2 \) at both points \( x_1 \) and \( x_2 \). Then by 2.3.1 one can find a proper and semi-stable curve \( \tilde{X} \) with smooth generic fibre and a special fibre \( \tilde{X}_k := \tilde{X} \times R_k \) isomorphic to \( X_k \), and (eventually after a finite extension of \( R \)) a Galois cover \( \tilde{f} : \tilde{Y} \to \tilde{X} \).
of degree $p$, such that the degeneration data of rank $p$ associated to $X_k$ via the semi-stable reduction of $\tilde{Y}$ is isomorphic to the above given data $\text{Deg}(X_k)$.

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