Equivalence Principle, Higher Dimensional Möbius Group and the Hidden Antisymmetric Tensor of Quantum Mechanics

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Abstract

We show that the recently formulated Equivalence Principle (EP) implies a basic cocycle condition both in Euclidean and Minkowski spaces, which holds in any dimension. This condition, that in one–dimension is sufficient to fix the Schwarzian equation [6], implies a fundamental higher dimensional Möbius invariance which in turn univocally fixes the quantum version of the Hamilton–Jacobi equation. This holds also in the relativistic case, so that we obtain both the time–dependent Schrödinger equation and the Klein–Gordon equation in any dimension. We then show that the EP implies that masses are related by maps induced by the coordinate transformations connecting different physical systems. Furthermore, we show that the minimal coupling prescription, and therefore gauge invariance, arises quite naturally in implementing the EP. Finally, we show that there is an antisymmetric two–tensor which underlies Quantum Mechanics and sheds new light on the nature of the Quantum Hamilton–Jacobi equation.
1 Introduction

The consistent synthesis of the 20th century most important philosophical advances, Quantum Mechanics (QM) and General Relativity (GR), remains elusive. These two theories have changed the human experience of reality and allowed it to probe into the smallest and largest possible scales. Yet these two pillars of modern science remain incompatible at a fundamental level, despite enormous efforts devoted to formulating the proper mathematical theory that will embrace both QM and GR. It also seems that none of the current approaches to quantum gravity provides a satisfactory resolution. So, for example, the issues of the vacuum energy and generation of mass remain unsolved. Thus, it is fair to say that at present there does not exist a proper framework for the consistent formulation of quantum gravity, and what may be needed is a new paradigm. For example, one usually considers GR as the natural framework to describe gravitation seen as one of the four fundamental forces. On the other hand, QM is seen as the natural framework to describe interactions. So, the current view considers QM and GR as playing qualitatively rather different roles.

Our view is going in another direction. Namely, suppose that QM and GR are in fact two facets of the same medal. If so, then we should need a reformulation of QM and a better understanding about the nature of GR and of the other interactions. Recently, in [1]–[6], it has been proposed that QM can follow from an Equivalence Principle (EP) which is reminiscent of the Einstein EP. This principle requires that it is possible to connect all physical systems by coordinate transformations. In particular, there should always exist a coordinate transformation connecting a physical system with a non–trivial potential \( V \) and energy \( E \), to the one with \( V - E = 0 \). Conversely, any allowed physical state should arise by a coordinate transformation from the state with \( V - E = 0 \). That is, under coordinate transformations, the trivial state should transform with an inhomogeneous term into a non–trivial one. In this context we stress that the EP has been formulated for states composed by one particle. However, its formulation can be suitably generalized.

The above aspects are intimately related with the concept of space–time. Actually, the removal of the peculiar degeneration arising in the classical concepts of rest frame and time parameterization is at the heart of the EP [6]. In [2, 6] it was shown that this univocally leads to the Quantum Stationary HJ Equation (QSHJE). This is a third–order non–linear differential equation which provides a trajectory representation of QM. After publishing [1], the authors became aware that this equation was assumed in [7] as a starting point to formulate a trajectory interpretation of QM (see also [8]). In [1, 6] it was shown that the trajectories depend on the Planck length through hidden variables which arise as initial conditions. So we see that QM may in fact need gravity.
A property of the formulation is the manifest $p$–$q$ duality, which in turn is a consequence of the involutive nature of the Legendre transformation and of its recently observed relation with second–order linear differential equations \[9\]. The role of the Legendre transformation in QM is related to the prepotential which appears in expressing the space coordinate in terms of the wave–function \[10\]|\[11\]|\[12\].

The $p$–$q$ duality is deeply related to the Möbius symmetry underlying the EP, which in turn fixes the QSHJE. This is also at the basis of energy quantization \[4\]|\[5\]. In particular, the QSHJE is defined only if the ratio $w = \psi^D/\psi$ of a pair of real linearly independent solutions of the Schrödinger equation is a local homeomorphism of the extended real line $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ into itself. This is an important feature as the $L^2(\mathbb{R})$ condition, which in the Copenhagen formulation is a consequence of the axiomatic interpretation of the wave–function, directly follows as a basic theorem which only uses the geometrical glueing conditions of $w$ at $q = \pm\infty$ as implied by the EP. In particular, denoting by $q_-$ ($q_+$) the lowest (highest) $q$ for which $V(q) - E$ changes sign, we have that \[4\]|\[5\]

If

\[
V(q) - E \geq \begin{cases} 
  P^2 > 0, & q < q_-, \\
  P^2_+ > 0, & q > q_+, 
\end{cases}
\]

(1.1)

then $w = \psi^D/\psi$ is a local self–homeomorphism of $\hat{\mathbb{R}}$ if and only if the corresponding Schrödinger equation has an $L^2(\mathbb{R})$ solution.

Thus, since the QSHJE is defined if and only if $w$ is a local self–homeomorphism of $\hat{\mathbb{R}}$, this theorem implies that energy quantization directly follows from the QSHJE itself. Thus, we have that basic characteristics of QM are predicted by the EP as they arise by self–consistency from the EP without further assumptions. This is a fundamental aspect as in the standard formulation of QM the $L^2(\mathbb{R})$ condition is a consequence of the probabilistic interpretation of the wave–function.

An important observation is that the Equivalence Postulate cannot be formulated consistently in Classical Mechanics (CM). To see this observe that if $S_0^{cl}(q)$ and $S_0^{cl}(q^\nu)$ denote the classical Hamiltonian characteristic function, also called reduced actions, of two classical systems, then the coordinate transformation connecting the two systems can be defined by setting

\[
S_0^{cl}(q^\nu) = S_0^{cl}(q),
\]

(1.2)

which implies $\partial_{q^\nu}S_0^{cl}(q^\nu) = (\partial_{q^\nu}q)\partial_qS_0^{cl}(q)$. On the other hand, comparing the Classical Stationary Hamilton–Jacobi Equation (CSHJE) for $S_0^{cl}(q)$ ($W(q) \equiv V(q) - E$)

\[
\frac{1}{2m} \left( \frac{\partial S_0^{cl}(q)}{\partial q} \right)^2 + W(q) = 0,
\]

(1.3)
with the CSHJE satisfied by \( S_0^{cl}\nu(q^\nu) \)
\[
\frac{1}{2m} \left( \frac{\partial S_0^{cl}\nu(q^\nu)}{\partial q^\nu} \right)^2 + W^\nu(q^\nu) = 0,
\]
we see that \( W^\nu(q^\nu) = (\partial_{q^\nu} q) W(q) \). This implies that the state corresponding to \( W = W^0 \equiv 0 \) is a fixed point, that is any coordinate transformation leaves \( W^0 \) invariant as \( W^0 \rightarrow (\partial_{q^\nu} q) W^0 \equiv 0 \). This aspect can be also understood by observing that in CM the state corresponding to \( W^0 \) has a trivial reduced action, thus the transformation is highly singular in this case. This can be seen as the impossibility of implementing covariance of CM under the transformations defined by (1.2). Thus, in CM it is not possible to generate all non–trivial states by a coordinate transformation from the trivial one. Consistent implementation of the EP requires a modification of CM. This univocally leads to QM. The starting point is to observe that the obstacle to the implementation of the EP is the transformation property \( W^\nu(q^\nu) = (\partial_{q^\nu} q)^2 W(q) \) which in turn is a consequence of the CSHJE. It follows that implementation of the EP has a highly dynamical content as it requires modifying the classical HJ equation. Therefore we should add to the CSHJE a still unknown term \( Q \)
\[
\frac{1}{2m} \left( \frac{\partial S_0(q)}{\partial q} \right)^2 + W(q) + Q(q) = 0,
\]
where, in the \( Q \rightarrow 0 \) limit, \( S_0 \) corresponds to the classical reduced action.

According to the EP, all physical systems composed by one particle under an external potential, labeled by the function \( W(q) \equiv V(q) - E \), can be connected by a coordinate transformation \( q^a \rightarrow q^b = q^b(q^a) \), defined by
\[
S_0^b(q^b) = S_0^a(q^a).
\]
Observe that at this stage we have not any dynamical information. It is just the implementation of the EP which will univocally fix the term \( Q \) in (1.5). Furthermore, it is worth stressing that the equivalence concerns all physical systems. In particular, note that we are not restricting the equivalence to different energy levels of a system with a fixed potential. The only restriction we are considering here concerns the number of particles composing the systems. As we said, we are considering the simplest case of systems composed by a single particle under an external potential. Nevertheless the EP can be suitably generalized to higher degrees of freedom.

It is immediate to see that the implementation of the EP has dramatic consequences. In fact, since the state with \( W = W^0 \equiv 0 \) corresponds to a fixed point, we see that the only way to implement the EP is to admit an inhomogeneous term in the transformation properties of \( W \)
\[
W^\nu(q^a) \rightarrow W^b(q^b) = \left( \partial_{q^a} q^a \right)^2 W^a(q^a) + (q^a; q^b).
\]
On the other hand, by (1.5) and (1.6) we have
\[ W^b(q^b) + Q^b(q^b) = (\partial_{q^b} q^a)^2 [W^a(q^a) + Q^a(q^a)], \]
so that
\[ Q^a(q^a) \rightarrow Q^b(q^b) = \left( \partial_{q^b} q^a \right)^2 Q^a(q^a) - (q^a; q^b). \] (1.8)

We used the notation \((q^a; q^b)\) to stress that the unknown term depends on the functional relation between \(q^a\) and \(q^b\). The fundamental fact is that this term is fixed by the basic cocycle condition
\[ (q^a; q^c) = \left( \partial_{q^c} q^b \right)^2 [(q^a; q^b) - (q^c; q^b)], \] (1.9)
which follows as consistency condition for (1.7) or, equivalently, (1.8). Actually, in one dimension a key point was the following result [6]
The cocycle condition (1.9) uniquely defines the Schwarzianderivative up to a multiplicative constant and a coboundary term.

In particular, one obtains \((q^a; q^b) = -\beta^2 \{q^a, q^b\}/4m\), where \(\{f, q\} = f'''/f' - 3(f''/f')^2/2\) denotes the Schwarzianderivative and \(\beta\) is a constant with the dimension of an action. Since in the classical case the term \((q^a; q^b)\) must disappear from (1.7), we have that the classical limit is reached for \(\beta \rightarrow 0\). Thus \(\beta\) is naturally identified with \(\hbar\). Furthermore, one sees that the inhomogeneous term \((q^a; q^b)\) has a purely quantum origin.

An important issue of the present formulation concerns the similarity between the postulate equivalence of states and the Einstein EP. According to the Einstein EP it is always possible to choose a locally inertial coordinate system such that the physical laws have the same form as in unaccelerated coordinate systems in the absence of gravitation. The EP we formulated states that it is always possible to choose a coordinate system in such a way that the reduced action corresponds to the one of the free particle with vanishing energy. While in the case of the Einstein EP, it is the gravitational field which is “locally balanced” by a coordinate transformation, here there is an arbitrary external potential which is “globally balanced” by a coordinate transformation. Another fundamental difference concerns the framework in which this is formulated. While the Einstein EP is formulated at the level of the equation of motions, here the formulation is implemented in the framework of HJ theory. This is a quite crucial difference. This becomes particularly transparent if we consider the case of a time–independent potential. In this case we can use the reduced action so that time never appears directly in the relevant equations. Only after the QSHJE is solved, one introduces time parameterization according to Jacobi theorem [7], that is \(t - t_0 = \partial_q S_0\). The fact that the QSHJE differs from the classical version implies that the conjugate momentum \(\partial_q S_0\) does not coincide with the mechanical one \(\dot{q}\). Thus, a feature of the EP is that time arises as parameter for trajectories

\[1\]We refer to [3] for several explicit examples of the formulation.
and it is not introduced a priori. We believe this is a distinctive feature of HJ theory whose power 
fully manifests in the present formulation of QM. The different role of time in the formulation 
of the two equivalence principles can be also seen in considering the equation of motion of a 
particle in an external gravitational field $m\ddot{q} = mg$. Performing the time–dependent coordinate 
transformation $q' = q - gt^2/2$, we have $m\ddot{q}' = 0$, for any value of the energy $E$ of the particle, 
including the free particle at rest for which $E = 0$. So that, depending on the initial conditions 
of $m\ddot{q}' = 0$, we may have $q'$ to be constant, say $q' = 0$. Therefore, there are no selected frames 
if one uses time–dependent coordinate transformations. In other words, while the classical 
reduced action, which is not a function of time, is trivial, the equation of motions contain the 
time parameter which continues to flow. Hence, while with the CSHJE description it is not 
always possible to connect two systems by a coordinate transformation, this is not the case if 
one describes the dynamics using Newton’s equation. In particular, in finding the coordinate 
transformation reducing the CSHJE description of $m\ddot{q} = mg$, one has 

$$\frac{1}{2m} \left( \frac{\partial S_{cl}^d(q)}{\partial q} \right)^2 - mgq + E = 0,$$

(1.10) 

for which there is no coordinate transformation $q \rightarrow q^v(q)$ such that $S_{cl}^d(q^v) = S_{cl}^d(q)$ with 
$S_{cl}^d(q^v)$ the reduced action of the free particle with $E = 0$. Time parameterization can be 
seen as a way to express a constant, say 0, by means of the solution of the equation of motions, 
$q = f(t)$. For example, for a particle with constant velocity, we have $0 = q - vt$, so that particle’s 
position can be denoted by either $q$ itself or $vt$. In this way one can always reduce to the particle 
at rest by simply setting $q' = q - f(t)$. While in the case of the CSHJE description there is 
the degenerate case $cnst = mvq$, corresponding to $S_{cl}^d(q^v) = S_{cl}^d(q)$, time parameterization 
provides a well–defined and invertible transformation i.e. $q' = q - f(t) \rightarrow q = q' + f(t)$. The 
reason underlying the differences in considering the role of space and time is that fixed values 
of $q$ and $t$ correspond to quite different situations. Even if the particle is at rest, say at $q = 0$, 
time continues to flow. It is just the use of time that allows to connect different systems by a 
coordinate transformation.

There is a common feature underlying both the Einstein EP and the one we formulated. 
Namely, note that the existence of the classical systems with $S_{cl}^d = 0$, is essentially the reason 
of the impossibility of implementing covariance of CM under the transformations defined by 
(1.2). To be more precise, note that Eqs.(1.3) and (1.4) can be seen as a first step in checking 
covariance. However, these equations have not any particular content. The problem of covari-
ance arises when one tries to connect them by some transformation. We have seen that there 
is an inconsistency if we consider the coordinate transformation $S_{cl}^d(q^v) = S_{cl}^d(q)$. It is just 
the removal of this inconsistency which allows the implementation of the EP and therefore to 
have covariance. This univocally leads to QM. Thus, similarly to GR which can be derived by
implementing the principle of general covariance under diffeomorphisms, also QM arises from a covariance principle. In our case covariance essentially relies on the request that the transformation $S'_0(q^+)=S_0(q)$ be always defined. This immediately discards CM and modifies the classical concept of particle at rest. Thus we see that, similarly to the case of GR, the EP implies a principle of covariance.

One may wonder whether the properties of the Schwarzian derivative, an intrinsically one-dimensional (possibly complex) object, extend to higher dimension. Experience with string theory and CFT would indicate that similar properties are in fact strictly related to low-dimensional spaces. Nevertheless, these are related to the appearance of the QSHJE, and so, since the essence of QM manifests itself already in one-dimension, one may in fact believe that the higher dimensional generalization exists. We will in fact show that the basic fact underlying the construction is that the EP implies a Möbius symmetry in any dimension. More precisely, the EP implies the higher dimensional analogue of the cocycle condition.

One of the main results of the present paper is the proof that the above condition leads, in the case of the Euclidean metric, to an invariance under D-dimensional Möbius transformations. In the case of the Minkowski metric the relevant invariance is with respect to the (D+1)-dimensional conformal group. This result is also non-trivial from the mathematical point of view, and may have implications for the higher dimensional diffeomorphisms. This Möbius symmetry will then univocally lead to the time-dependent Schrödinger equation in higher dimension.

Remarkably, we will see that the EP in fact implies also the higher dimensional Relativistic Quantum HJ Equation (RQHJE) with external potentials. Furthermore, while considering an external potential leads to a mixing between the kinetic and potential part in deriving the RQHJE, this equation is obtained quite naturally once one considers the minimal coupling prescription. This aspect is a relevant feature of the EP which in fact corresponds to a sort of naturalness. Namely, the right framework to formulate it is the exact one, that is special relativity. So, for example, the time-dependent Schrödinger equation is simply derived as the non-relativistic approximation from the RQHJE. Furthermore, as we will see in the case of the Klein–Gordon equation in the presence of the electromagnetic field, the minimal coupling prescription is in fact the natural one. This indicates that gauge theories are deeply related to the EP. In this context, it is useful to stress that the standard Schrödinger problems one usually considers correspond to ideal situations. So for example, a potential well cannot be seen as a fundamental interaction. Actually, the Schrödinger problems one may consider at the level of fundamental interactions essentially concern the electromagnetic one. It is then interesting that the Schrödinger equation for minimal coupled potentials simply follows from the EP as a non-relativistic limit.
Another key ingredient in the one-dimensional derivation of QM from the EP was the following identity involving the Schwarzian derivatives

\[(\partial_q S_0)^2 = \hbar^2 2 \left( \{ e^{\frac{i}{\hbar} S_0}, q \} - \{ S_0, q \} \right). \tag{1.11} \]

Again, in the present paper we will find the generalization of this identity to higher dimension and in the relativistic case.

We started the introduction by arguing for the need for a radical new paradigm for QM. The fact that QM arises from the EP may suggest that masses have a quantum origin. We will show that indeed this may be the case. The point is that in the relativistic case one has

\[ W = \frac{1}{2} mc^2, \tag{1.12} \]

then we have that mass of a particle is obtained from the state corresponding to \( W^0 \equiv 0 \) and is due to the inhomogeneous term which arises from coordinate transformations.

Another basic feature of the present approach concerns the appearance of a new field which underlies QM. This is one of the new points one meets in considering the higher dimensional generalization of our formulation. As we will see, this field arises by solving the continuity equation associated to the QHJE. In particular, this equation defines a \((D - 2)\)-form which in turn defines an antisymmetric 2-tensor.

Our paper is organized as follows. In section 2 we set the notation and derive the higher dimensional cocycle condition. In section 3 we will prove the invariance of the cocycle condition under the \(D\)-dimensional Möbius transformations. In section 4 we derive the higher-dimensional Schrödinger equation and discuss a possible connection with the holographic principle. In section 5 we then discuss the generalization to the relativistic case. We show that in the case of the Minkowski metric, the cocycle condition is invariant under the \((D + 1)\)-dimensional conformal group. We derive the generalization of our approach for the Klein–Gordon equation and show how the time-dependent non-relativistic limit correctly reproduces the time-dependent Schrödinger equation. In section 6 we discuss the generalization to the case with a four-vector including covariant derivatives. We also investigate the generation of mass in our approach and the appearance of the hidden antisymmetric tensor field which underlies QM. Finally, Appendices A and B are devoted to some more technical aspects of sections 3 and 5.

## 2 EP and cocycle condition

Let us consider the case of two physical systems with Hamilton’s characteristic functions \( S_0 \) and \( S_0^v \) and denote the coordinates of the two systems by \( q \) and \( q^v \) respectively. Let us set

\[ S_0^v(q^v) = S_0(q). \tag{2.1} \]
Observe that there are no particular assumptions in making the above identification. The point is that the physical content is in the functional dependence of $S_v^0$ and $S_0$ on their arguments $q^v$ and $q$ respectively, and (2.4) simply defines a functional relation between $q^v$ and $q$. One may also choose another rule connecting $S_v^0(q^v)$ and $S_0(q)$. However, as the one–dimensional case shows [1]–[6], the formulation would result much more cumbersome. Thus, in a certain sense, we can say that $S_0$ transforms as a scalar. In particular, the true assumption which underlies (2.1) is that there is a functional relation between the coordinates of two arbitrary physical systems characterized by the systems themselves. This is essentially the content of the EP we will formulate. Note that the existence of a non–singular functional relation between the coordinate of two physical systems cannot hold for all states of CM. In fact, (2.1) does not make sense once one considers the classical state with $W = 0$, corresponding to $S_0(q) = \text{cnst}$. Thus, requiring that (2.1) is defined for all systems implies that $S_0(q) = \text{cnst}$ cannot corresponds to a physical state. This corresponds to a criticism of the concept of rest frame in CM. It is just the removal of the peculiar degeneration arising in the classical concepts of rest frame and time parameterization, discussed in great detail in sect.2 of Ref.[6], which provides the physical motivation for formulating the EP.

As in the one–dimensional case, we will see that the implementation of the EP, not only excludes CM, but also uniquely leads to the quantum version of the HJ equation.

Note that Eq.(2.1) induces, in the one–dimensional case, the map

$$q \longrightarrow q^v = v(q), \tag{2.2}$$

where

$$v = S_v^0^{-1} \circ S_0, \tag{2.3}$$

with $S_v^0^{-1}$ denoting the inverse of $S_v^0$. This construction is equivalent to say that the map (2.2) induces the transformation $S_0 \rightarrow S_v^0 = S_0 \circ v^{-1}$, that is $S_0(q) \rightarrow S_v^0(q^v) = S_0(q(q^v))$.

In the higher dimensional case, the relation $S_v^0(q^v) = S_0(q)$ defines infinitely many maps $q \longrightarrow q^v = v(q)$. Since, as we will see, the EP requires that two arbitrary physical systems can be always connected by a coordinate transformation, the only condition we need is that there exists the inverse of the map $v$. This is not sufficient to fix the particular form of the $v(q)$. However, as we will see, we only need that for any pair of states there exists an invertible map (2.2) satisfying (2.1). We will call such maps $v$–transformations.

2Tensorial properties are characterized by giving specific rules under given transformations of coordinates. Here we are not giving the transformation rules of $S_0$ under a set of coordinate transformations. Rather, we are defining a set of coordinate transformations starting from the knowledge of the functional structure of $S_v^0(q^v)$ and $S_0(q)$. For this reason, strictly speaking, even if $S_v^0(q^v) = S_0(q)$, the reduced action cannot be considered a scalar function.
One of the main results in [1]–[6] was that the reduced action $S^0_\theta(q^0)$ corresponding to the free system with vanishing energy is not a constant but the “self–dual state”

$$e^{\frac{i}{\hbar} S^0_\theta} = \frac{q_0^0 + i\ell_0}{q_0^0 - i\ell_0},$$

with $\ell_0$, $\text{Re} \ell_0 \neq 0$, a complex integration constant. This corresponds to the overlooked zero mode of the conformal factor in the quantum analogue of the Hamilton–Jacobi equation [1][2][6]. Furthermore, in Ref.[1] the function $T_\theta(p)$, defined as the Legendre transform of the reduced action, was introduced

$$T_\theta(p) = \sum_{k=1}^{D} q_k p_k - S_\theta(q), \quad S_\theta(q) = \sum_{k=1}^{D} p_k q_k - T_\theta(p).$$

While $S_\theta(q)$ is the momentum generating function, its Legendre dual $T_\theta(p)$ is the coordinate generating function

$$p_k = \frac{\partial S_\theta}{\partial q_k}, \quad q_k = \frac{\partial T_\theta}{\partial p_k}.$$ (2.6)

Let us now consider the Classical Stationary Hamilton–Jacobi Equation (CSHJE) in $D$–dimensions

$$\frac{1}{2m} \sum_{k=1}^{D} \left( \frac{\partial S^0_{cl}}{\partial q_k} \right)^2 + \mathcal{W}(q) = 0,$$ (2.7)

where

$$\mathcal{W}(q) \equiv V(q) - E,$$ (2.8)

with $V(q)$ the potential and $E$ the energy. We denote by $\mathcal{H}$ the space of all possible $\mathcal{W}$’s corresponding to physical systems composed by one particle (the extension to more general cases will be investigated elsewhere).

In [1] the following Equivalence Principle has been formulated

For each pair $\mathcal{W}^a, \mathcal{W}^b \in \mathcal{H}$, there is a $v$–transformation such that

$$\mathcal{W}^a(q) \longrightarrow \mathcal{W}^{av}(q^v) = \mathcal{W}^b(q^v).$$ (2.9)

We will see that the implementation of the EP will univocally lead to the QSHJE. This implies that there always exists the trivializing coordinate $q^0$ for which $\mathcal{W}(q) \longrightarrow \mathcal{W}^0(\theta^0)$, where

$$\mathcal{W}^0(q^0) = 0.$$ (2.10)

\footnote{We note that a common shift of $V$ and $E$ by a constant does not change $\mathcal{W} \equiv V - E$. Since this is the combination in which the data $V$ and $E$ enter in the equation of motions, we see that the case $V - E = 0$ is indistinguishable from $V = E = 0$.}
In particular, since the inverse transformation should exist as well, it is clear that the trivializing transformation should be locally invertible. We will also see that since classically \( W^0 \) is a fixed point, implementation of (2.9) requires that \( W^b(q^v) \) is given in terms of \( W^a(q) \) (times a suitable Jacobian) together with an additive term. In other words, the EP immediately implies that the \( W \) states transform inhomogeneously.

The fact that the EP cannot be consistently implemented in Classical Mechanics (CM) is true in any dimension. To show this let us consider the coordinate transformation induced by the identification

\[
S_{0}^{clv}(q^v) = S_{0}^0(q).
\]

(2.11)

Then note that the CSHJE

\[
\frac{1}{2m} \sum_{k=1}^{D} (\partial_{q_k} S_{0}^{cl}(q))^2 + W(q) = 0,
\]

(2.12)

provides a correspondence between \( W \) and \( S_{0}^{cl} \) that we can use to fix, by consistency, the transformation properties of \( W \) induced by that of \( S_{0} \). In particular, since \( S_{0}^{clv}(q^v) \) must satisfy

\[
\frac{1}{2m} \sum_{k=1}^{D} (\partial_{q_k} S_{0}^{clv}(q^v))^2 + W^v(q^v) = 0,
\]

(2.13)

by (2.11) we have

\[
p_k \rightarrow p'_k = \frac{\partial S_{0}^{clv}(q^v)}{\partial q_k} = \sum_{i=1}^{D} \frac{\partial q_i}{\partial q'_k} \frac{\partial S_{0}^{cl}(q)}{\partial q_i} = \sum_{i=1}^{D} J_{ki} p_i,
\]

(2.14)

where \( J \) is the Jacobian matrix

\[
J_{ki} = \frac{\partial q_i}{\partial q'_k}.
\]

(2.15)

Let us introduce the notation

\[
(p^v|p) = \frac{\sum_k p_k'^2}{\sum_k p_k^2} = \frac{p^f J^t J p}{p^t p}.
\]

(2.16)

Note that in the 1–dimensional case

\[
(p^v|p) = \left( \frac{p^v}{p} \right)^2 = \left( \frac{\partial S_0}{\partial q} \frac{\partial q}{\partial q^v} \frac{\partial S_0}{\partial q} \right)^2 = \left( \frac{\partial q^v}{\partial q} \right)^{-2},
\]

(2.17)

so that the ratio of momenta corresponds to the Jacobian of a coordinate transformation. By (2.12), we have

\[
W(q) \rightarrow W^v(q^v) = (p^v|p)W(q),
\]

(2.18)

that for the \( W^0 \) state gives

\[
W^0(q^0) \rightarrow W^v(q^v) = (p^v|p^0)W^0(q^0) = 0.
\]

(2.19)
Thus we have [1]

\[ W \text{ states transform as quadratic differentials under classical } v \text{-maps. It follows that } W^0 \text{ is a fixed point in } \mathcal{H}. \text{ Equivalently, in CM the space } \mathcal{H} \text{ cannot be reduced to a point upon factorization by the classical } v \text{-transformations. Hence, the EP (2.3) cannot be consistently implemented in CM. This can be seen as the impossibility of implementing covariance of CM under the coordinate transformation defined by (2.11)}. \]

It is therefore clear that in order to implement the EP we have to deform the CSHJE. As we will see, this requirement will determine the equation for \( S_0 \) in any dimension.

Let us discuss its general form. First of all observe that adding a constant to \( S_0 \) does not change the dynamics. Actually, Eqs.(2.5)(2.6) are unchanged upon adding a constant to either \( S_0 \) or \( T_0 \). Then, the most general differential equation \( S_0 \) should satisfy has the structure

\[ \mathcal{F}(\nabla S_0, \Delta S_0, \ldots) = 0. \quad (2.20) \]

Let us write down Eq.(2.20) in the general form

\[ \frac{1}{2m} \sum_{k=1}^{D} (\partial_k S_0(q))^2 + \mathcal{W}(q) + Q(q) = 0. \quad (2.21) \]

The transformation properties of \( \mathcal{W} + Q \) under the \( v \)-maps (2.2) are determined by the transformed equation

\[ \frac{1}{2m} \sum_{k=1}^{D} (\partial_k S_0^v(q^v))^2/2m + \mathcal{W}^v(q^v) + Q^v(q^v) = 0, \quad (2.22) \]

which by (2.1) and (2.21) yields

\[ \mathcal{W}^v(q^v) + Q^v(q^v) = (p^v|p) [\mathcal{W}(q) + Q(q)]. \quad (2.23) \]

A basic guidance in deriving the differential equation for \( S_0 \) is that in some limit it should reduce to the CSHJE. In [1][2] it was shown that the parameter which selects the classical phase is the Planck constant. Therefore, in determining the structure of the \( Q \) term we have to take into account that in the classical limit

\[ \lim_{\hbar \to 0} Q = 0. \quad (2.24) \]

The only possibility to reach any other state \( \mathcal{W}^v \neq 0 \) starting from \( \mathcal{W}^0 \) is that it transforms with an inhomogeneous term. Namely as \( \mathcal{W}^0 \to \mathcal{W}^v(q^v) \neq 0 \), it follows that for an arbitrary \( \mathcal{W}^a \) state

\[ \mathcal{W}^v(q^v) = (p^v|p^a) \mathcal{W}^a(q^a) + (q^a; q^v), \quad (2.25) \]
and by (2.23)

\[ Q^\nu(q^\nu) = (p^\nu|p^\nu) Q^\alpha(q^\alpha) - (q^\alpha; q^\nu). \]  

Let us stress that the purely quantum origin of the inhomogeneous term \((q^\alpha; q^\nu)\) is particularly transparent once one consider the compatibility between the classical limit (2.24) and the transformation properties of \(Q\) in Eq. (2.26).

The \(W^0\) state plays a special role. Actually, setting \(W^\nu = W^0\) in Eq. (2.25) yields

\[ W^\nu(q^\nu) = (q^0; q^\nu), \]  

so that, according to the EP (2.9), all the states correspond to the inhomogeneous part in the transformation of the \(W^0\) state induced by some \(v\)-map.

Let us denote by \(a, b, c, \ldots\) different \(v\)-transformations. Comparing

\[ W^b(q^b) = (p^b|p^b) W^a(q^a) + (q^a; q^b) = (q^0; q^b), \]  

with the same formula with \(q^a\) and \(q^b\) interchanged we have

\[ (q^b; q^a) = - (p^a|p^b)(q^a; q^b), \]  

in particular

\[ (q; q) = 0. \]  

More generally, comparing

\[ W^b(q^b) = (p^b|p^c) W^c(q^c) + (q^c; q^b) = (p^b|p^a) W^a(q^a) + (p^b|p^c)(q^a; q^c) + (q^c; q^b), \]  

with (2.28) we obtain the basic cocycle condition

\[ (q^a; q^c) = (p^c|p^b) \left[ (q^a; q^b) - (q^c; q^b) \right], \]  

which expresses the essence of the EP.

### 3 \((q^a; q^b)\) and the higher dimensional Möbius group

In this section, we will show that \((q^a; q^b)\) vanishes identically if \(q^a\) and \(q^b\) are related by a Möbius transformation. This is a consequence of Eqs. (2.20) and (2.32) and generalizes the one-dimensional result obtained in [1][2][3], which states that \((q^b; q^a) = 0\) if and only if \(q^b\) is a linear fractional transformation of \(q^a\)

\[ q^b = \frac{Aq^a + B}{Cq^a + D}, \quad AD - BC \neq 0. \]  

As in the one dimensional case, the Möbius symmetry will fix the \(Q\)-term in Eq. (2.21). Before going into the details of the proof, we will give a brief overview of the Möbius group (see, for example, [13]).
3.1 Higher dimensional Möbius group

Let us denote by \( q = (q_1, \ldots, q_D) \) an arbitrary point in \( \mathbb{R}^D \). A similarity is the affine mapping

\[
q \rightarrow Mq + b,
\]

where \( b \in \mathbb{R}^D \) and the matrix \( M = A \Lambda \) is the composition of a dilatation

\[
q \rightarrow Aq, \quad A \in \mathbb{R},
\]

and a rotation

\[
q \rightarrow \Lambda q,
\]

where \( \Lambda \in O(D) \). Similarities are naturally extended to the compactified space \( \hat{\mathbb{R}}^D = \mathbb{R}^D \cup \{\infty\} \). A similarity maps \( \infty \) to itself.

Let us consider the hyperplane

\[
P(a, t) = \{ q \in \mathbb{R}^D | q \cdot a = t, a \in \mathbb{R}^D, t \in \mathbb{R} \}.
\]

The reflection with respect to \( P(a, t) \) is given by

\[
f(q) = q - 2 \left( \frac{q \cdot a - t}{a \cdot a} \right) a.
\]

Let us set

\[
r^2 = q_1^2 + \cdots + q_D^2.
\]

The last generator of the Möbius group is the inversion or reflection in the unit sphere \( S^{D-1} \). If \( q \neq 0 \)

\[
q \rightarrow q^* = \frac{q}{r^2},
\]

otherwise

\[
0 \rightarrow \infty, \quad \infty \rightarrow 0.
\]

The Möbius group \( M(\hat{\mathbb{R}}^D) \) is defined as the set of transformations generated by all similarities together with the inversion. Actually, an arbitrary Möbius transformation is the composition of a number of reflections and inversions. Furthermore, a Möbius transformation is conformal with respect to the euclidean metric and a theorem due to Liouville states that the conformal group and \( M(\hat{\mathbb{R}}^D) \) actually coincide for \( D > 2 \).
3.2 Translations and dilatations

We now begin to study the properties of \((q^b; q^a)\) when \(q^b\) and \(q^a\) are related by dilatations and translations. Let us start by noticing that if \(B\) and \(C\) are arbitrary constant vectors, then from (2.32) we have

\[
(q + B + C; q) = (q + B + C; q + B) + (q + B; q) = (q + B + C; q + C) + (q + C; q),
\]

(3.10)

so that

\[
(q + B + C; q + B) - (q + B + C; q + C) = (q + C; q) - (q + B; q),
\]

(3.11)

where \((q + B)_k = q_k + B_k, k = 1, \ldots, D\). We will show that the unique solution of (3.11) is

\[
(q + B; q) = F(q + B) - F(q),
\]

(3.12)

where \(F\) is an arbitrary function of \(q\). Pick \(j \in [1, D]\) and let \(B_j\) and \(C_j\) be the only non–vanishing components of \(B\) and \(C\)

\[
B = (0, \ldots, B_j, 0, \ldots), \quad C = (0, \ldots, C_j, 0, \ldots).
\]

(3.13)

Let us set

\[
f(B, q) = (q + B; q),
\]

(3.14)

with \(B\) given by (3.13). Eq.(3.11) reads

\[
f(C, q + B) - f(B, q + C) = f(C, q) - f(B, q).
\]

(3.15)

Taking the derivative of both sides of Eq.(3.13) with respect to \(B_j\), we get

\[
\partial_{q_j} f(C, q + B) - \partial_{B_j} f(B, q + C) = -\partial_{B_j} f(B, q).
\]

(3.16)

By (2.30)

\[
f(B, q) = \sum_{n=1}^{\infty} c_n(q) B_j^n.
\]

(3.17)

Plugging this expression into Eq.(3.16), we find

\[
\sum_{n=1}^{\infty} \partial_{q_j} c_n(q + B) C_j^n - \sum_{n=1}^{\infty} n c_n(q + C) B_j^{n-1} = - \sum_{n=1}^{\infty} n c_n(q) B_j^{n-1}.
\]

(3.18)

Furthermore, expanding \(c_n(q + B)\) and \(c_n(q + C)\), it follows by Eq.(3.18) that

\[
\frac{1}{(m - 1)!} \partial_{q_j}^m c_n(q) = \frac{m}{n!} \partial_{q_j}^m c_m(q),
\]

(3.19)

that for \(n = 1\) reads

\[
\frac{1}{m!} \partial_{q_j}^m c_1(q) = \partial_{q_j} c_m(q).
\]

(3.20)
It follows that
\[
\partial_q f(B, q) = \sum_{n=1}^{\infty} \partial_q c_n(q) B_j^n = \sum_{n=1}^{\infty} \frac{1}{n!} \partial_q^n c_1(q) B_j^n = c_1(q + B) - c_1(q),
\] (3.21)

which upon integration on \(q_j\) yields
\[
f(B, q) = c(q + B) - c(q) + g(B, q),
\] (3.22)

with \(\hat{q}\) denoting all the variables other than \(q_j\). Moreover, by (2.30) \(g(0, \hat{q}) = 0\). Let us show that \(g(B, \hat{q})\) is identically vanishing. By (2.32)
\[
f(B + C, q) = (q + B + C; q) = (q + B + C; q + B) + (q + B; q) = f(C, q + B) + f(B, q),\]
(3.23)

that by (3.22) implies
\[
g(B + C, \hat{q}) = g(B, \hat{q}) + g(C, \hat{q}),
\] (3.24)

that is \(g(B, \hat{q}) = g'(0, \hat{q}) B_j\). However, by (3.17) (3.21) and (3.22)
\[
\partial_B f(B, q) = \partial_q c(q + B) + g'(0, \hat{q}) = c_1(q + B) + g'(0, \hat{q}) = \sum_{n=1}^{\infty} n c_n(q) B_j^{n-1}.
\] (3.25)

Then, setting \(B_j = 0\), we find from the last equality that \(g'(0, \hat{q}) = 0\). Finally, we are left with
\[
f(B, q) = c(q + B) - c(q).
\] (3.26)

From this equation it is then possible to derive Eq.(3.12). The technical details are reported in Appendix A. Related reasonings, reported in Appendix A, show that
\[
(Aq; q) = A^2 F(Aq) - F(q),
\] (3.27)

where now
\[
F(0) = 0,
\] (3.28)

with \(F\) the same function appearing in Eq.(3.12)

### 3.3 Rotations

Let us consider \((\Lambda q; q)\), where \(\Lambda \in O(D)\). First of all, if \(q^b = \Lambda q^a\), we see that \((p^b|p^a) = 1\), because \(\Lambda^t \Lambda = \Lambda \Lambda^t = 1_D\). Hence, by (2.32)
\[
(\Lambda(q + B); q + B) = (\Lambda q + \Lambda B; q + B) = (\Lambda q + \Lambda B; \Lambda q) + (\Lambda q; q + B) =
\]
\[
(\Lambda q + \Lambda B; \Lambda q) + (\Lambda q; q) + (q; q + B),
\] (3.29)
which implies
\[
(\Lambda(q + B); q + B) - (\Lambda q; q) = F(\Lambda(q + B)) - F(\Lambda q) + F(q) - F(q + B). \tag{3.30}
\]
Therefore, \((\Lambda q; q) = F(\Lambda q) - F(q) + c\). However, since \((\Lambda q; q)\) evaluated at \(q = 0\) cannot depend on \(\Lambda\), we have \((\Lambda q; q)_{q=0} = (q; q)_{q=0} = (q; q) = 0\). Then
\[
(\Lambda q; q) = F(\Lambda q) - F(q). \tag{3.31}
\]

### 3.4 Inversion

Let us consider the inversion \(q^*(q)\) \((3.8)\). The Jacobian matrix of this mapping is given by
\[
J_{kl} = \frac{\partial q^*_l}{\partial q_k} = \frac{\partial}{\partial q_k} \left( \frac{q_l}{r^2} \right) = \frac{\delta_{kl}}{r^2} - 2 \frac{q_k q_l}{r^4}. \tag{3.32}
\]
Then
\[
(J^t J)_{jk} = (J^2)_{jk} = \sum_{i=1}^{D} J_{ji} J_{ik} = \sum_{i=1}^{D} \left( \frac{\delta_{ji}}{r^2} - 2 \frac{q_i q_j}{r^4} \right) \left( \frac{\delta_{lk}}{r^2} - 2 \frac{q_l q_k}{r^4} \right) = \frac{\delta_{jk}}{r^4}, \tag{3.33}
\]
which implies
\[
(p|p_*) = (p_*|p)^{-1} = \frac{\sum_k p_{k*}^2}{\sum_k p^2} = \frac{p_*^t J^t J p_*}{p_*^t p_*} = \frac{1}{r^4}. \tag{3.34}
\]
Note that \(q^*\) is involutive since
\[
r^2_* = \sum_{k=1}^{D} q^*_k q^*_k = \frac{1}{r^4} \sum_{k=1}^{D} q_k q_k = \frac{1}{r^2}, \tag{3.35}
\]
and therefore
\[
(q^*_k) = q^*_k = \frac{q_k}{r^2 r_*} = q_k. \tag{3.36}
\]
Observe that since rotations leave \(r\) invariant, we have
\[
(\Lambda q)^* = \frac{(\Lambda q)_j}{r^2_A} = \frac{\Lambda_{jk} q_k}{r^2} = \Lambda_{jk} q^*_k = (\Lambda q^*)_j. \tag{3.37}
\]
Finally, we recognize the following behaviour under dilatations
\[
(Aq)^* = \frac{(Aq)_j}{r^2_A} = \frac{A q_j}{A^2 r^2} = A^{-1} \frac{q_j}{r^2} = A^{-1} q^*_j, \tag{3.38}
\]
where \(r^2_A = \sum_{k=1}^{D} A q_k A q_k = A^2 r^2\). By \((3.34)\) and \((3.36)\)
\[
(q^*; q) = -(p|p^*)(q; q^*) = -\frac{1}{r^4} ((q^*)^*; q^*), \tag{3.39}
\]
which implies that \((q^*; q)\) vanishes when evaluated at any \(q_0\) solution of \(q^* = q\), that is
\[
(q^*; q)|_{q=q_0} = 0. \tag{3.40}
\]
From this one derives the following result
\[
(q^*; q) = \frac{1}{r^4} F(q^*) - F(q), \tag{3.41}
\]
whose proof is reported in Appendix A.
3.5 Fixing the coboundary

Let us denote by $\gamma(q)$ a Möbius transformation of $q$. By (3.12)(3.27)(3.31) and (3.41) we have

$$(\gamma(q); q) = (p|p^{\gamma})F(\gamma(q)) - F(q).$$  \hfill (3.42)

Given a function $f(q)$, we have that if $(f(q); q)$ satisfies the cocycle condition (2.32), then this is still satisfied under the substitution

$$(f(q); q) \rightarrow (f(q); q) + (p|p^{f})G(f(q)) - G(q),$$  \hfill (3.43)

where $G$ has to satisfy the condition $G(0) = 0$. This condition is a consequence of the fact that $(Aq; q)$ evaluated at $q = 0$ is independent of $A$, so that it vanishes at $q = 0$. Therefore, if $(Aq; q)$ satisfies (2.32), then also $(Aq; q) + A^2G(Aq) - G(q)$ should vanish at $q = 0$, implying that $G(0)(A^2 - 1) = 0$, that is $G(0) = 0$. The term $(p|p^{f})G(f(q)) - G(q)$ can be seen as a coboundary term. We now show how the coboundary ambiguity (3.43) is fixed. First of all observe that (2.1) implies

$$S^0_0(q^0) = S_0(q),$$  \hfill (3.44)

where $S^0_0$ denotes the reduced action associated to the $W^0 \equiv 0$ state. On the other hand, by (2.27) we have that the equation of motion for $S_0(q)$ we are looking for is

$$(q^0; q) = W(q).$$  \hfill (3.45)

Comparing (3.43) with (2.20) and (3.44) we see that a necessary condition to satisfy (2.20) is that $(q^0; q)$ depends only on the first and higher derivatives of $q^0$. In fact, by (3.44) we have $q^0 = S^0_0^{-1} \circ S_0(q)$, that is $q^0$ is a functional of $S_0$. Therefore, a possible dependence of $(q^0; q)$ on $q^0$ itself, would imply that (3.45) has the form $F(S_0, \nabla S_0, \Delta S_0, \ldots) = 0$ rather than (2.20). Therefore, the only possibility is that the function $F$ in (3.42) be vanishing

$$F = 0.$$  \hfill (3.46)

Therefore, we arrived at the following basic result

Eq. (2.20) and the cocycle condition (2.32) imply that $(q^a; q^b)$ vanishes when $q^a$ and $q^b$ are related by a Möbius transformation, that is

$$(q + B; q) = 0,$$  \hfill (3.47)

$$(Aq; q) = 0,$$  \hfill (3.48)

$$(\Lambda q; q) = 0,$$  \hfill (3.49)
\[(q^*; q) = 0. \quad (3.50)\]

The above equations are equivalent to \((\gamma(q); q) = 0\). Furthermore, by \((2.32)\) we have
\[
(\gamma(q^a); q^b) = (q^a; q^b), \quad (q^*; \gamma(q^b)) = (p^{\gamma(b)}|p^b)(q^a; q^b). \quad (3.51)
\]
Let us consider the Jacobian factor \((p^{\gamma(b)}|p^b)\). First of all observe that the Möbius transformation is conformal with respect to the Euclidean metric. Namely, we have
\[
d s^2 = \sum_{j,k,l=1}^D d\gamma(q)_j d\gamma(q)_j = \sum_{j,k,l=1}^D \frac{\partial\gamma(q)_j}{\partial q_k} \frac{\partial\gamma(q)_j}{\partial q_l} dq_k dq_l = \sum_{j=1}^D e^{\phi_\gamma(q)} dq_j dq_j. \quad (3.52)
\]
Therefore
\[
(p^{\gamma(b)}|p^b) = e^{-\phi_\gamma(q^b)}. \quad (3.53)
\]
Note that in the case of translations and rotations the conformal re–scaling is the identity. For dilatations \(\exp \phi^A = A^2\), whereas for the inversion \(\exp \phi^* = r^{-4}\).

Note that the above conformal structure arises by setting \(S^v_0(q^v) = S_0(q)\). Let us make clear that this is not an assumption. Any transformation we choose other than \(S^v_0(q^v) = S_0(q)\) would yield the same results. In particular, the absence of assumptions in setting \(S^v_0(q^v) = S_0(q)\) results from the fact that \(q\) and \(q^v\) represent the spatial coordinates in their own systems. So, \(S^v_0(q^v) = S_0(q)\) can be seen just as the simplest way to set the coordinate transformations from the system with reduced action \(S^v_0\) (since physics is determined by the functional structure of \(S^v_0\), we can denote the coordinate as we like) to the one with reduced action \(S_0\). Nevertheless, there is a hidden apparently “innocuous” assumption: that the position \(S^v_0(q^v) = S_0(q)\) actually makes sense. This is not the case in CM, as for the free particle of vanishing energy we have \(S_0(q) = \text{cnst}\). In this case the above position does not make sense. Requiring that this is well–defined for any system is essentially the same as imposing the EP. However, on the one hand we have seen that the existence of the transformation implies the conformal structure, on the other we will see that the EP, and therefore existence of the transformation, implies QM. Thus, we have that the Möbius group, that for \(D \geq 3\) coincides with the conformal group, is intimately related to QM itself.

### 4 The Schrödinger equation

In this section, we will derive the quantum Hamilton–Jacobi equation in \(D\) dimensions and then show that the latter is equivalent to the stationary Schrödinger equation.

Let us start with the Quantum Stationary HJ Equation (QSHJE) in one dimension
\[
\frac{1}{2m} \left( \frac{\partial S_0}{\partial q} \right)^2 + V(q) - E + \frac{\hbar^2}{4m} \{S_0, q\} = 0. \quad (4.1)
\]
This equation was univocally derived from the EP in \[1\]–[2]. After publishing [1], the authors became aware that this equation was assumed in [7] as a starting point to formulate a trajectory interpretation of QM. In particular, Floyd [7] introduced the concept of trajectories by using Jacobi’s theorem according to which

$$t - t_0 = \frac{\partial S_0}{\partial E},$$

(4.2)

from which one sees that the conjugate momentum \( p = \partial_q S_0 \) does not in general correspond to the mechanical one, that is \( p \neq m \dot{q} \). This is a basic difference with respect to Bohm’s theory [14]–[18]. Furthermore, Floyd noted that Bohm’s assumption \( \psi = \Re e \bar{h} \hat{S}_0 \) does not work in this case [7]. Apparently one may infer that (4.1) is equivalent to the standard version of the quantum stationary HJ equation

$$\frac{1}{2m} \left( \frac{\partial \hat{S}_0}{\partial q} \right)^2 + V(q) - E - \frac{\hbar^2}{2m} \frac{\partial^2 R}{R} = 0,$$

(4.3)

$$\partial_q (R^2 \partial_q \hat{S}_0) = 0.$$  

(4.4)

In fact, solving (4.4) would give

$$R = \frac{c}{\sqrt{\partial_q \hat{S}_0}},$$

(4.5)

which is equivalent to

$$\{ \hat{S}_0, q \} = -2 \frac{\partial^2 R}{R},$$

(4.6)

so that \( \hat{S}_0 \) would satisfy the same equation as \( S_0 \). Nevertheless, there is a problem in the above derivation. Namely, in Bohm’s assumption, like in the usual formulation of quantum HJ theory, the identification is not between a general solution of the Schrödinger equation and \( \Re e \bar{h} \hat{S}_0 \), but between the wave–function and \( \Re e \bar{h} \hat{S}_0 \). Thus, suppose that the wave–function describes a bound state so that it must be proportional to a real function. According to Bohm and the usual approach, this would imply

\[ \hat{S}_0 \] is a constant for bound states.

This in turn implies rather peculiar properties. For example, quantum mechanically the conjugate momentum is vanishing for bound states. This seems to be an unsatisfactory feature of (4.3)–(4.4). To be more precise, Eqs.(4.3)–(4.4) are good equations unless one forces the identification of \( \Re e \bar{h} \hat{S}_0 \) with the wave–function. In general \( \Re e \bar{h} \hat{S}_0 \) should be identified with a linear

4This is a consequence of reality of \( W \) as this implies that if \( \psi \) solves the Schrödinger equation, then this is the case also of \( \tilde{\psi} \). If \( \tilde{\psi} \not= \psi \), then \( \psi \tilde{\psi}' - \psi' \tilde{\psi} = \text{const} \not= 0 \), so that \( \psi \) is never vanishing. In particular, if \( \psi \in L^2(\mathbb{R}) \), then \( \tilde{\psi} \propto \psi \) (see also sections 14 and 17 of Ref.[3]).

5To be precise, bound states would correspond to \( \hat{S}_0 = \text{const} \) outside the nodes of the wave–function.
combination of two linearly independent solutions of the Schrödinger equation. So, the general expression for the wave–function is

$$\psi = R \left( A e^{-\frac{i}{\hbar}S_0} + B e^{\frac{i}{\hbar}S_0} \right),$$

so that since \( \bar{\psi} \propto \psi \) gives

$$|A| = |B|. \quad (4.8)$$

Thus for \( S_0 \) there is no trace of the condition \( \hat{S}_0 = \text{cnst} \) one has for bound states setting \( \psi = R e^{\frac{i}{\hbar}\hat{S}_0} \). Let us note that whereas \( \hat{S}_0 = \text{cnst} \) would be consistent in the case of classically forbidden regions, as there \( S_0^d = \text{cnst} \) problems arise in regions which are not classically forbidden, i.e. where \( S_0^d \neq \text{cnst} \). For example, in the case of the harmonic oscillator, one has \( \hat{S}_0 = \text{cnst} \ \forall q \in \mathbb{R} \), which follows by identifying \( R \exp(i\hat{S}_0/\hbar) \) with the wave–function, while in some region one has \( S_0^d \neq \text{cnst} \). As a consequence, while quantum mechanically the particle would be at rest, after taking the \( \hbar \to 0 \) limit, the particle should start moving. We refer to Holland’s book \cite{15} for an interesting analysis concerning the classical limit of the harmonic oscillator in Bohmian theory.

The above analysis can be summarized by the following basic fact

*If \( \hat{S}_0 \) is the quantum analogue of the reduced action, and therefore reduces to the classical one in the \( \hbar \to 0 \) limit, then the wave–function cannot be generally identified with \( R \exp(i\hat{S}_0/\hbar) \). In particular, this cannot be the case for bound states, such as the harmonic oscillator, in which the wave–function is proportional to a real function also in regions which are not classically forbidden.*

However, we have seen that if \( R \exp(i\hat{S}_0/\hbar) \) is not identified with real solutions of the Schrödinger equation, then we have equivalence between \( (4.1) \) and \( (4.3)-(4.4) \). We also note that with the formulation \( (4.1) \) one directly sees that the situation \( S_0 = \text{cnst} \) can never occur. In fact, one has rather stringent conditions connected with the existence of \( \{ S_0, q \} \), which in turn reflects the basic nature of the cocycle condition and therefore of the EP. In this respect we recall that existence of \( \{ S_0, q \} \) implies that the ratio of two real linearly independent solutions of the Schrödinger equation must be a local self–homeomorphism of the extended real line \( \hat{\mathbb{R}} = \mathbb{R} \cup \{ \infty \} \). This is a basic fact as it implies energy quantization without any assumption \( \mathbb{E} \). \footnote{Note also that having \( \hat{S}_0 = \text{cnst} \) in the classically forbidden regions would imply a trivial trajectory, since \( p = 0 \) there.}
4.1 Hidden variables, Planck length and holographic principle

The above remarks are related to the proposal of changing the Bohmian definition of mechanical momentum \( m\dot{q} \) \[18\]. This proposal is related to the fact that (4.3)-(4.4) allow a rearrangement of \( \hat{S}_0 \) and \( R \). On the other hand, these symmetries are particularly evident working directly with Eq.(4.1), which in turn is equivalent to the Schwarzian equation

\[
\left\{ e^{\frac{2\pi i}{\hbar}S_0}, q \right\} = -\frac{4m}{\hbar^2} \mathcal{W}. \tag{4.9}
\]

In fact these symmetries correspond to the invariance of (4.9) under Möbius transformations of \( e^{\frac{2\pi i}{\hbar}S_0} \).

Eq.(4.1) implies that \( S_0 \) can be expressed in the “canonical form” \[1\]-\[6\]

\[
e^{\frac{2\pi i}{\hbar}S_0(\delta)} = e^{i\alpha w + i\ell \bar{w}} - i\ell, \tag{4.10}
\]

which is equivalent to the one considered by Floyd \[7\]. Here \( \delta = \{\alpha, \ell\} \), where \( \alpha \in \mathbb{R} \) and \( \ell_1 = \text{Re} \ell \neq 0, \ell_2 = \text{Im} \ell \) are integration constants, and \( w = \psi^D/\psi \in \mathbb{R} \), where \( \psi^D \) and \( \psi \) are real linearly independent solutions of the stationary Schrödinger equation

\[
\left[ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial q^2} + V(q) \right] \psi = E\psi. \tag{4.11}
\]

Observe that the condition \( \ell_1 \neq 0 \) is equivalent to having \( S_0 \neq \text{cnst} \) which is a necessary condition to define \( \{S_0, q\} \) in the QSHJE.

A basic feature of (4.10) is that it explicitly shows the existence of Möbius states \[1\]-\[6\], called microstates by Floyd \[7\]. In particular, the constants \( \ell_1 \) and \( \ell_2 \) correspond, together with \( \alpha \), to the initial conditions of Eq.(4.1). These initial conditions do not appear in the Schrödinger equation, so that \( \ell_1 \) and \( \ell_2 \) can be seen as a sort of hidden variables. Their role is quite basic. In particular, it has been shown in \[4\]-\[6\] that in order to have a well-defined classical limit, \( \ell \) should depend on fundamental lengths which in turn should depend on \( \hbar \). This dependence arises in considering the \( E \to 0 \) and \( \hbar \to 0 \) limits. In particular, let us consider the conjugate momentum in the case of the free particle with energy \( E \)

\[
p_E = \pm \frac{\hbar(\ell_E + \bar{\ell}_E)}{2|k| \sin kq - i\ell_E \cos kq|q^2}, \tag{4.12}
\]

where \( k = \sqrt{2mE}/\hbar \). The first condition is that in the \( \hbar \to 0 \) limit the conjugate momentum reduces to the classical one

\[
\lim_{\hbar \to 0} p_E = \pm \sqrt{2mE}. \tag{4.13}
\]

On the other hand, we should also have

\[
\lim_{E \to 0} p_E = p_0 = \pm \frac{\hbar(\ell_0 + \bar{\ell}_0)}{2|q - i\ell_0|}, \tag{4.14}
\]

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Eqs. (4.12)-(4.13) show that, due to the factor $\bar{h}$ in $\cos kq$, the quantity $\ell_E$ should depend on $E$. Let us set

$$\ell_E = k^{-1} f(E, \bar{h}) + \lambda_E,$$

(4.15)

where, since $\lambda_E$ is still arbitrary, we can choose the dimensionless function $f$ to be real. By (4.12) we have

$$p_E = \pm \sqrt{2mE} f(E, \bar{h}) + mE (\lambda_E + \bar{\lambda}_E) / \bar{h},$$

(4.16)

Note that if one ignores $\lambda_E$ and sets $\lambda_E = 0$, then by (4.13)

$$\lim_{{\bar{h} \to 0}} f(E, \bar{h}) = 1.$$  (4.17)

We now consider the properties that $\lambda_E$ and $f$ should have in order that (4.17) be satisfied in the physical case in which $\lambda_E$ is arbitrary but for the condition $\text{Re} \ell_E \neq 0$, as required by the existence of the QSHJE. First of all note that cancellation of the divergent term $E^{-1/2}$ in

$$p_E \sim \pm \frac{2h^2(2mE)^{-1/2} f(E, \bar{h}) + \bar{h}(\lambda_E + \bar{\lambda}_E)}{2|q - ih(2mE)^{-1/2} f(E, \bar{h}) - i\lambda_E|^2},$$

(4.18)

yields

$$\lim_{{E \to 0}} E^{-1/2} f(E, \bar{h}) = 0,$$

(4.19)

so that $k$ must enter in the expression of $f(E, \bar{h})$. Since $f$ is a dimensionless constant, we need at least one more constant with the dimension of length. Two fundamental lengths one can consider are the Compton length $x_c = \hbar/mc$, and the Planck length $x_p = \sqrt{\hbar G/c^3}$. Two dimensionless quantities depending on $E$ are

$$x_c = k \lambda_c = \sqrt{\frac{2E}{mc^2}},$$

(4.20)

and

$$x_p = k \lambda_p = \sqrt{\frac{2mEG}{\hbar c^3}}.$$  (4.21)

On the other hand, since $x_c$ does not depend on $\hbar$ it cannot be used to satisfy (4.17), so that it is natural to consider $f$ as a function of $x_p$. Let us set

$$f(E, \bar{h}) = e^{-\alpha(x_p^{-1})},$$

(4.22)

where $\alpha(x_p^{-1}) = \sum_{k \geq 1} \alpha_k x_p^{-k}$. The conditions (4.17)-(4.19) correspond to conditions on the coefficients $\alpha_k$. In order to consider the structure of $\lambda_E$, we note that although $e^{-\alpha(x_p^{-1})}$ cancelled the $E^{-1/2}$ divergent term, we still have some conditions to be satisfied. To see this note that

$$p_E = \pm \frac{\sqrt{2mE} e^{-\alpha(x_p^{-1})} + mE (\lambda_E + \bar{\lambda}_E) / \bar{h}}{[e^{ikq} + (e^{-\alpha(x_p^{-1})} - 1 + k\lambda_E) \cos kq]^2},$$

(4.23)
so that (4.13) implies
\[ \lim_{\hbar \to 0} \frac{\lambda_E}{\hbar} = 0. \] (4.24)

To discuss this limit, we first note that
\[ p_E = \pm \frac{2\hbar k^{-1} e^{-\alpha(x_p^{-1})} + \hbar (\lambda_E + \bar{\lambda}_E)}{2 |k^{-1} \sin kq - i (k^{-1} e^{-\alpha(x_p^{-1})} + \lambda_E) \cos kq|^2}. \] (4.25)

So that, since \( \lim_{E \to 0} k^{-1} e^{-\alpha(x_p^{-1})} = 0 \), by (4.14) and (4.25) we have
\[ \lambda_0 = \lim_{E \to 0} \lambda_E = \lim_{E \to 0} \ell_E = \ell_0. \] (4.26)

Let us now consider the limit
\[ \lim_{\hbar \to 0} p_0 = 0. \] (4.27)

First of all note that, since
\[ p_0 = \pm \frac{\hbar (\ell_0 + \bar{\ell}_0)}{2 |q^0 - i \ell_0|^2}, \] (4.28)

we have that the effect on \( p_0 \) of a shift of \( \text{Im} \ell_0 \) is equivalent to a shift of the coordinate. Therefore, in considering (4.27) we can set \( \text{Im} \ell_0 = 0 \) and distinguish the cases \( q^0 \neq 0 \) and \( q^0 = 0 \). Note that as we always have \( \text{Re} \ell_0 \neq 0 \), it follows that the denominator in the right hand side of (4.28) is never vanishing. Let us define \( \gamma \) by
\[ \text{Re} \ell_0 \xrightarrow{h \to 0} h^\gamma. \] (4.29)

We have
\[ p_0 \xrightarrow{h \to 0} \begin{cases} h^{\gamma+1}, & q_0 \neq 0, \\ h^{1-\gamma}, & q_0 = 0. \end{cases} \] (4.30)

and by (4.27)
\[ -1 < \gamma < 1. \] (4.31)

A constant length having powers of \( \hbar \) can be constructed by means of \( \lambda_c \) and \( \lambda_p \). We also note that a constant length which is independent of \( \hbar \) is provided by \( \lambda_e = e^2/mc^2 \) where \( e \) is the electric charge. Thus \( \ell_0 \) can be considered as a suitable function of \( \lambda_c, \lambda_p \) and \( \lambda_e \) satisfying the constraint (4.31).

The above investigation indicates that a natural way to express \( \lambda_E \) is given by
\[ \lambda_E = e^{-\beta(x_p)} \lambda_0, \] (4.32)

where \( \beta(x_p) = \sum_{k \geq 1} \beta_k x_p^k \). Any possible choice of \( \beta(x_p) \) should satisfy the conditions (4.24) and (4.26). For example, for the modulus \( \ell_E \) built with \( \beta(x_p) = \beta_1 x_p \), one should have \( \beta_1 > 0 \).
Summarizing, by (4.15) (4.22) (4.26) and (4.32) we have
\[ \ell_E = k^{-1}e^{-\alpha(x_p^{-1})} + e^{-\beta(x_p)}\ell_0, \] (4.33)
where \( \ell_0 = \ell_0(\lambda_c, \lambda_p, \lambda_e) \), and for the conjugate momentum of the state \( \mathcal{W} = -E \) we have
\[ p_E = \pm \frac{2k^{-1}\hbar e^{-\alpha(x_p^{-1})} + \hbar e^{-\beta(x_p)}(\ell_0 + \tilde{\ell}_0)}{2k^{-1}\sin kq - i\left(k^{-1}e^{-\alpha(x_p^{-1})} + e^{-\beta(x_p)}\ell_0\right)} \cdot \cos kq \] (4.34)

We stress that the appearance of the Planck length is strictly related to \( p-q \) duality and to the existence of the Legendre transformation of \( S_0 \) for any state. This \( p-q \) duality has a counterpart in the \( \psi^D-\psi \) duality [1]–[6] which sets a length scale that already appears in considering linear combinations of \( \psi^D_0 = q^0 \) and \( \psi^0 = 1 \). This aspect is related to the fact that we always have \( S_0 \neq \text{cnst} \) and \( S_0 \not\propto q + \text{cnst} \), so that also for the states \( \mathcal{W}_0 \) and \( \mathcal{W} = -E \) one has a non–constant conjugate momentum. In particular, the Planck length naturally emerges in considering \( \lim_{E \to 0} p_E = p_0 \), together with the analysis of the \( \hbar \to 0 \) limit of both \( p_E \) and \( p_0 \). As a result the Compton length and \( \lambda_e \) appear as well.

We also note that in [2][6] it has been shown that the wave–function remains invariant under suitable transformations of \( \alpha \) and \( \ell \). These transformations constitute the basic symmetry group of the wave–function. To see this we consider the case of the wave–function \( \psi_E \) corresponding to a state of energy \( E \). Since \( \psi_E \) solves the Schrödinger equation, for any fixed set of integration constants \( \alpha \) and \( \ell \), there are coefficients \( A \) and \( B \) such that
\[ \psi_E\{\delta\} = \frac{1}{\sqrt{\mathcal{S}_0\{\delta\}}} \left(Ae^{-\hat{S}_0\{\delta\}} + Be^{\hat{S}_0\{\delta\}}\right). \] (4.35)
Performing a transformation of the moduli \( \delta \to \delta' = \{\alpha', \ell'\} \), we have (we refer to [2][3] for notation)
\[ \psi_E\{\delta'\} = \left(\frac{2i}{\mu_{\hbar\partial_q\gamma_{S_0}}}\right)^{1/2} \left[Ad + B\tilde{b} + (A\tilde{c} + B\tilde{a})\gamma_{S_0}\right]. \] (4.36)
Requiring that \( \psi_E\{\delta\} \) remains unchanged up to some multiplicative constant \( c \), that is
\[ \psi_E\{\delta\} \longrightarrow \psi_E\{\delta'\} = c\psi_E\{\delta\}, \] (4.37)
we have by (4.36)
\[ A^2\tilde{b} + AB\tilde{a} = AB\tilde{a} + B^2\tilde{b}. \] (4.38)
This defines the symmetry group of the wave–function. Thus, we have seen that there are hidden variables depending on the Planck length and that these can be suitably changed without any effect on the wave–function. Therefore, we can say that there is a sort of information loss in
considering the wave–function. So, the probabilistic interpretation of the wave–function seems due to our ignorance about Planck scale physics.

The above analysis can be summarized as follows

1. QM follows from the EP\([1]-[6]\). The formulation is strictly related to \(p-q\) duality, which in turn is a consequence of the involutive nature of the Legendre transformation. In this context QM is described in terms of trajectories where, according to Floyd \([7]\), time parameterization is defined by Jacobi’s theorem.

2. The theory shows the existence of Möbius states \([1]-[6]\), called microstates by Floyd \([7]\). These states cannot be seen in the framework of ordinary QM. In particular, these states appear in the context of the quantum HJ equation whose initial conditions depend on the Planck length \([4][6]\). Furthermore, from the symmetries of the wave–function under change of hidden variables, we explicitly see that there are equivalence classes of the moduli \(\delta\) which correspond to the same wave–function \([2][6]\).

3. The role of \(p-q\) duality is a fundamental one. In fact, this reflects in the appearance in the formulation of a pair of real linearly independent solutions of the Schrödinger equation. So there is a \(\psi^D-\psi\) duality \([1]-[3]\) which reflects the basic Möbius symmetry and therefore the existence of Möbius states. This directly shows that in considering solutions of the basic Schrödinger equation \(\partial_q^2 \psi^0 = 0\), one has to introduce a length to consider linear combinations of \(\psi^D = q^0\) and \(\psi^0 = 1\). Since in this case \(\mathcal{W} \equiv V - E = 0\), the Schrödinger problem does not provide any scale, so that we are forced to introduce a universal length.

4. Implementation of the EP implies that the trivializing map, expressed as the Möbius transform of \(\psi^D/\psi\), must be a local self–homeomorphism of \(\hat{\mathbb{R}}\) \([1][6]\). This in turn implies that for suitable \(\mathcal{W}\)’s the corresponding Schrödinger equation must admit an \(L^2(\mathbb{R})\) solution \([1][6]\). This implies that the EP itself implies energy quantization. So basic facts of QM, such as tunnelling and energy quantization, are derived without axiomatic assumptions concerning the interpretation of the wave–function. Furthermore, the appearance of the \(L^2(\mathbb{R})\) condition shows that the Hilbert space structure starts emerging.

The above shortly summarizes some of the main aspects of the theory. In this context we note that the appearance of Planck length in hidden variables has been recently advocated by ’t Hooft \([19]\). ’t Hooft argues that such hidden variables must play a role in the implementation of the holographic principle \([20]\). In ’t Hooft’s paper it is also argued that due to information loss, Planck scale degrees of freedom must be combined into equivalence classes. The presence
of equivalence classes moduli $\delta$, corresponding to symmetries of the wave–function, seems to be a possible framework for ’t Hooft’s proposal.

4.2 The higher dimensional case

Let us now consider the problem of finding the equation for $S_0$ in the higher dimensional case. To this end, let us first consider a potential of the form

$$V(q) = \sum_{k=1}^{D} V_k(q_k),$$

(4.39)

so that

$$W(q) = \sum_{k=1}^{D} W_k(q_k),$$

(4.40)

where

$$W_k(q_k) = V_k(q_k) - E_k.$$  

(4.41)

In this case, since

$$\frac{1}{2m} (\partial_k S_{0,k}(q_k))^2 + W_k(q_k) + Q_k(q_k) = 0, \quad k = 1, \ldots, D,$$

(4.42)

we have

$$\frac{1}{2m} \sum_{k=1}^{D} (\partial_k S_0(q))^2 + W(q) + Q(q) = 0,$$

(4.43)

where

$$S_0(q) = \sum_{k=1}^{D} S_{0,k}(q_k), \quad Q(q) = \sum_{k=1}^{D} Q_k(q_k),$$

(4.44)

and

$$Q_k(q_k) = \frac{\hbar^2}{4m} \{ S_{0,k}(q_k), q_k \}.$$  

(4.45)

Note that by (2.23)(2.26)(3.47) and (3.49), both $W(q)$ and $Q(q)$ are invariant under rotations and translations.

$$\tilde{W}(\tilde{q}) = W(q(\tilde{q})), \quad \tilde{Q}(\tilde{q}) = Q(q(\tilde{q})), \quad \tilde{q} = \Lambda q + b.$$  

(4.46)

However, observe that

$$\tilde{Q}(\tilde{q}) = Q(q(\tilde{q})) = \frac{\hbar^2}{4m} \sum_{k=1}^{D} \{ S_0(q), q_k \} = \frac{\hbar^2}{4m} \sum_{k=1}^{D} \{ \tilde{S}_0(\tilde{q}), q_k(\tilde{q}) \} \neq \frac{\hbar^2}{4m} \sum_{k=1}^{D} \{ \tilde{S}_0(\tilde{q}), \tilde{q}_k \}. $$

(4.47)

We observe that in an interesting paper, Floyd has recently considered related issues. It is worth stressing that these transformations are not a symmetry of the physical system as in general the functional structures change, that is $\tilde{W}(x) \neq W(x)$, $\tilde{Q}(x) \neq Q(x)$. Thus, Eq. (4.46) should not be confused with true symmetries, e.g. invariance of the potential under rotations, expressed as $W(\Lambda q) = W(q)$. 

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This means that expressing $Q(q)$ in terms of sums of Schwarzian derivatives does not provide a convenient, i.e. covariant, formulation. In the following, we will express the quantum potential $Q$, and consequently the QSHJE in a way that makes this invariance manifest. First of all, note that any $Q_k$ can be written as

$$Q_k(q_k) = -\hbar^2 \frac{\Delta_k R_k}{2m R_k}, \quad \partial_k(R_k^2 \partial_k S_{0,k}(q_k)) = 0. \quad (4.48)$$

In fact, as we have seen above, since the implementation of the EP implies that $S_0$ is never a constant, we have (4.6). Therefore, we have

$$Q(q) = \sum_{k=1}^{D} Q_k(q_k) = -\hbar^2 \frac{\sum_{k=1}^{D} \Delta_k R_k}{2m} = -\hbar^2 \frac{\Delta R}{2m R}, \quad (4.49)$$

where $R(q) = \prod_{k=1}^{D} R_k(q_k)$ satisfies the continuity equation

$$\sum_{k=1}^{D} \partial_k(R^2 \partial_k S_0) = 0, \quad (4.50)$$

where $S_0(q) = \sum_{k=1}^{D} S_{0,k}(q_k)$. Now consider the following basic identity, which generalizes the one–dimensional version [1]–[6]

$$\alpha^2 (\nabla S_0)^2 = \frac{\Delta (Re^{\alpha S_0})}{Re^{\alpha S_0}} - \frac{\Delta R}{R} - \frac{\alpha}{R^2} \nabla \cdot (R^2 \nabla S_0), \quad (4.51)$$

which holds for any constant $\alpha$ and any functions $R$ and $S_0$. Then, if $R$ satisfies the continuity equation

$$\nabla \cdot (R^2 \nabla S_0) = 0, \quad (4.52)$$

and setting $\alpha = i/\hbar$, we have

$$\frac{1}{2m} (\nabla S_0)^2 = -\hbar^2 \frac{\Delta (Re^{\alpha S_0})}{2m Re^{\alpha S_0}} + \frac{\hbar^2}{2m} \frac{\Delta R}{R}. \quad (4.53)$$

Comparing Eq.(4.43) and (4.49) with Eq.(4.53) we can make the following identification

$$\mathcal{W}(q) = V(q) - E = \frac{\hbar^2}{2m} \frac{\Delta (Re^{\alpha S_0})}{Re^{\alpha S_0}},$$

$$Q(q) = -\frac{\hbar^2}{2m} \frac{\Delta R}{R}. \quad (4.54)$$

Let us now consider an arbitrary state, not necessarily corresponding to a $\mathcal{W}$ of the kind (4.40), with some reduced action $S_0$. We consider $R$ solution of (4.52). Note that, as (4.51) is independent of the form of $\mathcal{W}$, we have that (4.53) holds for arbitrary $S_0$ and $R$ satisfying (4.52). We now start showing that (4.55) holds in general, not only in the case (4.40).
Let us set
\[ W(q) = \frac{\hbar^2}{2m} \Delta \left( Re \hat{\Phi}_S^0 \right) + g(q). \] (4.56)
Eqs. (2.21) (4.53) and (4.56) imply
\[ Q(q) = -\frac{\hbar^2}{2m} \frac{\Delta R}{R} - g(q). \] (4.57)

We have seen in (4.46) that the system described by \( \tilde{S}_0(\tilde{q}) \), where \( \tilde{q} = \Lambda q + b \), has the important property that \( \tilde{W}(\tilde{q}) = W(q(\tilde{q})) \) and \( \tilde{Q}(\tilde{q}) = Q(q(\tilde{q})) \). Furthermore, using \( \tilde{S}_0(\tilde{q}) = S_0(q) \), we find
\[ \tilde{\nabla} \cdot (\tilde{R}^2(\tilde{q}) \tilde{\nabla} \tilde{S}_0(\tilde{q})) = \nabla \cdot (\tilde{R}^2(\tilde{q}(q)) \nabla S_0(q)) = 0. \] (4.58)

Now observe that the continuity equation implies that \( \tilde{R}^2(\tilde{q}(q)) \nabla S_0(q) \) is the curl of some vector. In general we have \( R^2 \partial_i S_0 = \epsilon_{i_2 \ldots i_D} \partial_{i_2} F_{i_3 \ldots i_D} \), where \( F \) is a \((D-2)\)-form. Later we will exploit the field \( F \). Therefore \( \tilde{R}^2(\tilde{q}(q)) \nabla S_0(q) \) must be a vector. On the other hand, since also \( \nabla S_0 \) is a vector, we have that \( \tilde{R}(\tilde{q}(q)) \) must be a scalar under rotations and translations
\[ \tilde{R}(\tilde{q}) = R(q(\tilde{q})). \] (4.59)
in agreement with the fact that \( \nabla \cdot (\tilde{R}^2(q) \nabla S_0(q)) = 0 \). Therefore, we have
\[ \tilde{g}(\tilde{q}) = -\frac{\hbar^2}{2m} \frac{\Delta \tilde{R}}{\tilde{R}} - \tilde{Q}(\tilde{q}) = -\frac{\hbar^2}{2m} \frac{\Delta R}{R} - Q(q) = g(q), \] (4.60)
that is \( g \) is scalar under rotations and translations. This implies that \( g \) may depend only on \((\nabla S_0)^2, \Delta S_0, R, \Delta R, (\nabla R)^2\) and higher derivatives which are invariant under rotations and translations, that is
\[ g = H((\nabla S_0)^2, \Delta S_0, R, \ldots). \] (4.61)

Let us now consider the case in which \( W = \sum_{k=1}^D W_k(q_k) \), so that the problem reduces to a one dimensional one. In this case we have \( g = 0, S_0 = \sum_{k=1}^D S_{0,k}(q_k) \) and \( R = \prod_{k=1}^D R_k(q_k) \), where \( R_k \propto (\partial_q S_{0,k})^{-1/2} \). This provides the following constraints
\[ H \left( \sum_k (\partial_k S_{0,k}), \sum_k \partial^2_k S_{0,k}, \prod_k R_k(q_k), \ldots \right) = 0. \] (4.62)
This implies that \( H \equiv 0 \). Let us show this in two related way. First note that by summing the one dimensional QSHJE and then performing a rotation, we arrive to a \( D \)-dimensional QSHJE which does not decompose in 1D QSHJE in the transformed coordinates. Thus also implies \( H((\nabla S_0)^2, \Delta S_0, R, \ldots) = 0. \)

\(^9\)A possible dependence of \( g \) on \( S_0 \) would imply, against Eq. (2.20), that \( S_0 \) satisfies a differential equation involving \( S_0 \) itself.
A similar reasoning to prove that $H$ vanishes identically is to note that by (4.61) and (4.62) the only possibility to have a non–trivial $H$ is that it depends in a suitable way on terms that cancel when $S_0 = \sum_{k=1}^{D} S_{0,k}(q_k)$. The building blocks to construct such terms have the form $\partial_j \partial_k S_0$, $j \neq k$, which vanish when $S_0 = \sum_{k=1}^{D} S_{0,k}(q_k)$. On the other hand, such terms are not scalar under rotations, so that they may enter in $H$ only if suitably saturated with other indices. Since the only vectorial indices at our disposal are provided by derivatives, we see that there are not terms which are scalar under rotations and vanish identically when $W = \sum_{k=1}^{D} W_k(q_k)$. Hence

$$g = 0.$$  \hfill (4.63)

Therefore, we have the basic result that the EP actually implies that in any dimension the reduced action satisfies the QSHJE

$$\frac{1}{2m}(\nabla S_0)^2 + W - \frac{\hbar^2}{2m} \frac{\Delta R}{R} = 0,$$ \hfill (4.64)

and the continuity equation

$$\nabla \cdot (R^2 \nabla S_0) = 0.$$ \hfill (4.65)

This equation implies the $D$–dimensional Schrödinger equation

$$\left[ -\frac{\hbar^2}{2m} \Delta + V(q) \right] \psi = E \psi.$$ \hfill (4.66)

We stress that also in the higher dimensional case there is a fundamental difference between the correspondence (4.64)(4.65) and (4.66) and the one usually considered in the literature. Namely, we have seen in the one–dimensional case that in general $Re^{\pm S_0}$ cannot be identified with the wave–function. In particular, this would cause trouble in the case of bound states, as $S_0$ would be a constant and inconsistencies arise in the classical limit. This was also evident from the fact that $\{S_0, q\}$ is not defined for $S_0 = cns$. In the higher dimensional case, this would lead to the degeneration of the continuity equation (4.63), with (4.64) resulting in

$$W = \frac{\hbar^2}{2m} \frac{\Delta R}{R},$$ \hfill (4.67)

that in the classical limit, that by definition corresponds to $Q = 0$, would lead to the contradiction

$$W = 0.$$ \hfill (4.68)

Therefore, we have

The general relationship between the wave–function, $R$ and $S_0$ has the form

$$\psi = R \left( Ae^{-\frac{i}{\hbar} S_0} + Be^{\frac{i}{\hbar} S_0} \right).$$ \hfill (4.69)
In particular, for bound states we have

$$|A| = |B|. \quad (4.70)$$

Furthermore, $S_0$ is never a constant.

Finally, we note that by Eqs. (2.26) and (4.55)

$$\langle q^a; q^b \rangle = (p^b|p^a)Q^a(q^a) - Q^b(q^b) = -\frac{\hbar^2}{2m} \left[ (p^b|p^a) \frac{\Delta^a R^a}{R^a} - \frac{\Delta^b R^b}{R^b} \right]. \quad (4.71)$$

### 4.3 Inversion II

$R^b(q^b)$ and $R^a(q^a)$ are related in a simple fashion in the case of rotations, reflections, dilatations and translations. Namely, $R^a(q^a)$ solves the continuity equation in the $q^a$–system if and only if $R^a(q^a(q^b))$ solves the continuity equation in the $q^b$–system. Thus, as we already proved in section 3, we explicitly verify by (4.71) that $\langle q^a; q^b \rangle$ vanishes identically in such cases. The analogous relation for the inversion is

$$R^*(q^*) = r^{D-2}R(q) = \frac{1}{r^{D-2}}R(q(q^*)). \quad (4.72)$$

Basically, we will show that, given a solution $R(q)$ of the continuity equation in the $q$–system, $R^*(q^*)$ in Eq. (4.72) solves the continuity equation in the $q^*$–system. Then, we will verify that $\langle q; q^* \rangle = 0$, as we proved in section 3 as a direct consequence of the EP.

The continuity equation for $R^*(q^*)$ reads

$$\nabla^* \cdot (R^{*2}(q^*)\nabla^* S_0^*(q^*)) =$$

$$r^4 \nabla \cdot (R^{*2}(q^*)\nabla S_0(q)) + (4 - 2D)r^2 R^{*2} \sum_{k=1}^{D} q_k \partial_k S_0(q) = 0, \quad (4.73)$$

that, after setting $R^{*2}(q^*) = h(q)R^2(q)$, becomes

$$hr^4 \nabla \cdot \left( R^2 \nabla S_0 \right) + r^4 R^2 \nabla h \cdot \nabla S_0 + (4 - 2D)r^2 R^2 h \sum_{k=1}^{D} q_k \partial_k S_0 = 0, \quad (4.74)$$

implying by Eq. (4.50)

$$\left( r^2 \nabla h + (4 - 2D)\vec{h} \vec{q} \right) \cdot \nabla S_0 = r^{2D-2} \nabla (r^{4-2D} h) \cdot \nabla S_0 = 0, \quad (4.75)$$

which is solved by

$$h(q) = r^{2D-4}. \quad (4.76)$$
We now show that \((q = (q^*); q^*)\) vanishes identically. By (4.71) we have
\[
(q; q^*) = (p_*, p) Q(q) - Q^*(q^*) = -\frac{\hbar^2}{2m} \left( r^4 \frac{\Delta R}{R} - \frac{\Delta^* R^*}{R^*} \right).
\] (4.77)
On the other hand, by (4.72)
\[
\frac{\Delta^* R^*}{R^*} = \frac{\Delta (r^{D-2} R)}{r^{D-6} R} + (4 - 2D) \frac{\sum_{k=1}^{D} q_k \partial_k (r^{D-2} R)}{r^{D-4} R} = r^4 \frac{\Delta R}{R},
\] (4.78)
so that
\[
(q; q^*) = 0.
\] (4.79)

5 Relativistic extension and Klein–Gordon equation

A basic property of the EP is that it has a universal character. In general, the implementation of the EP leads to a deformation of the corresponding classical HJ equation. In this respect, we note that existence of a fixed point in the non–relativistic stationary case demands the principle to be implemented in all the other circumstances. If we did not modify the time–dependent case as well, then taking the stationary limit would lead to inconsistencies. In other words, since modifying the stationary classical equation comes from a modification of the classical transformation properties of \(W\), which in general gets an inhomogeneous contribution, such as \((q; q^*)\), consistency implies that also in the time–dependent case the potential cannot transform as in the classical case.

We start by deriving the Relativistic Quantum HJ Equation (RQHJE). Here we will consider the case in which the external potential is described by an arbitrary potential \(V(q, t)\). This form will be particularly useful in deriving the time–dependent Quantum HJ Equation (QHJE), which in turn implies the time–dependent Schrödinger equation, as the non–relativistic limit of the RQHJE. Later on, we will consider the case in which the interaction is given in terms of the electromagnetic four–vector \(A_\mu\).

The Relativistic Classical Hamilton–Jacobi Equation (RCHJE) reads
\[
\frac{1}{2m} \sum_{k=1}^{D} (\partial_k S^{cl}(q, t))^2 + W_{rel}(q, t) = 0,
\] (5.1)
where
\[
W_{rel}(q, t) = \frac{1}{2mc^2} \left[ m^2 c^4 - (V(q, t) + \partial_t S^{cl}(q, t))^2 \right].
\] (5.2)
In the time–independent case one has \(S^{cl}(q, t) = S^{cl}_{0}(q) = Et\), and (5.1) (5.2) become
\[
\frac{1}{2m} \sum_{k=1}^{D} (\partial_k S^{cl}_{0})^2 + W_{rel} = 0,
\] (5.3)
and
\[ W_{\text{rel}}(q) = \frac{1}{2mc^2}[m^2c^4 - (V(q) - E)^2]. \] (5.4)

In the latter case, we can go through the same steps as in the non-relativistic case and the stationary RQHJE reads
\[ \frac{1}{2m}(\nabla S_0)^2 + W_{\text{rel}} - \frac{\hbar^2}{2m} \Delta R = 0, \] (5.5)

where \( R \) satisfies the continuity equation
\[ \nabla \cdot (R^2 \nabla S_0) = 0. \] (5.6)

Furthermore, (5.5)(5.6) imply the stationary Klein–Gordon equation
\[ -\hbar^2 c^2 \Delta \psi + (m^2 c^4 - V^2 + 2EV - E^2)\psi = 0, \] (5.7)

where \( \psi = R \exp(iS_0/\hbar) \).

5.1 Time–dependent case

Let us start by noticing that in the time–dependent case, the \((D+1)\)–dimensional RCHJE can be cast in the form (later on summation on repeated indices is understood)
\[ \frac{1}{2m}\eta^{\mu\nu} \partial_\mu S^{ct} \partial_\nu S^{ct} + W'_{\text{rel}} = 0, \] (5.8)

where \( \eta^{\mu\nu} \) is the Minkowski metric diag \((-1, 1, \ldots, 1)\), and
\[ W'_{\text{rel}}(q) = \frac{1}{2mc^2}[m^2c^4 - V^2(q) - 2cV(q)\partial_\alpha S^{ct}(q)], \] (5.9)

where \( q \equiv (ct, q_1, \ldots, q_D) \). We thus recognize that Eq.(5.8) has the same structure as Eq.(5.3), the Euclidean metric being replaced by the Minkowskian one. Also in this case, in order to implement the EP, we have to modify the classical equation by adding a function to be determined, namely
\[ \frac{1}{2m}(\partial S)^2 + W_{\text{rel}} + Q = 0. \] (5.10)

Observe that since now \( W'_{\text{rel}} \) depends on \( S^{ct} \), we have to make the identification
\[ W_{\text{rel}}(q) = \frac{1}{2mc^2}[m^2c^4 - V^2(q) - 2cV(q)\partial_\alpha S(q)], \] (5.11)

which differs from \( W'_{\text{rel}} \) for the Hamiltonian principal function as now \( S \) appears rather than \( S^{ct} \).

Implementation of the EP requires that for an arbitrary \( W^a \) state
\[ W^b_{\text{rel}}(q^b) = (p^{b}|p^a)W^a_{\text{rel}}(q^a) + (q^a; q^b), \] (5.12)
and

\[ Q^b(q^b) = (p^b|p^a)Q^a(q^a) - (q^a; q^b), \]  

(5.13)

where this time

\[ (p^b|p) = \frac{\eta^{\mu\nu}p^b_\mu p^b_\nu}{\eta^{\mu\nu}p_\mu p_\nu} = \frac{p^\dagger \eta Jp}{p^\dagger \eta \eta}, \]  

(5.14)

and \( J \) is the Jacobian matrix

\[ J^\mu_\nu = \frac{\partial q^\mu}{\partial q^{b\nu}}. \]  

(5.15)

Furthermore, we obtain the cocycle condition

\[ (q^a; q^c) = (p^c|p^b) \left[ (q^a; q^b) - (q^c; q^b) \right]. \]  

(5.16)

As reported in Appendix B, this cocycle condition and the condition

\[ F_{rel}(\partial \mu S, \Box S, \ldots) = 0, \]  

(5.17)

which is the analogue of the condition \((2.20)\), imply that

\[ (\gamma(q); q) = 0. \]  

(5.18)

### 5.2 The RQHJE

Let us now consider the following identity

\[ \alpha^2(\partial S)^2 = \frac{\Box(Re^{aS})}{Re^{aS}} - \frac{\Box R}{R} - \frac{\alpha}{R^2} \partial \cdot (R^2 \partial S), \]  

(5.19)

which holds for any constant \( \alpha \) and any functions \( R, S \). Then, if \( R \) satisfies the continuity equation \( \partial(R^2 \cdot \partial S) = 0 \), and setting \( \alpha = i/\hbar \) we have

\[ \frac{1}{2m}(\partial S)^2 = -\frac{\hbar^2}{2m} \frac{\Box(Re^{aS})}{Re^{aS}} + \frac{\hbar^2}{2m} \frac{\Box R}{R}. \]  

(5.20)

Our aim is to prove that

\[ W_{rel} = \frac{\hbar^2}{2m} \frac{\Box(Re^{aS})}{Re^{aS}}, \]  

(5.21)

that by \((5.20)\) implies

\[ Q_{rel} = -\frac{\hbar^2}{2m} \frac{\Box R}{R}. \]  

(5.22)

Suppose that

\[ W_{rel} = \frac{\hbar^2}{2m} \frac{\Box(Re^{aS})}{Re^{aS}} + g, \]  

(5.23)
and correspondingly
\[ Q_{\text{rel}} = -\frac{\hbar^2}{2m} \square R - g. \] (5.24)

First of all, we have seen that the system described by \( \tilde{S}(\tilde{q}) = S(q) \), where \( \tilde{q} = \Lambda q + b \), has the important property that \( \tilde{W}_{\text{rel}}(\tilde{q}) = W_{\text{rel}}(q) \). Furthermore, we find
\[ \tilde{\partial} \cdot (\tilde{R}^2 \tilde{\partial} \tilde{S}) = \partial \cdot (\tilde{R}^2 \partial S) = 0. \] (5.25)

Comparing this with the continuity equation \( \partial \cdot (R^2 \partial S) = 0 \), we have that under Poincaré transformations, the \( R \)-function of the transformed system defined by \( \tilde{S}(\tilde{q}) = S(q) \) has the transformation property
\[ \tilde{R}(\tilde{q}) = R(q). \] (5.26)

Therefore, by (5.18) and (5.26) we have
\[ \tilde{g} = -\frac{\hbar^2}{2m} \square \tilde{R} - \tilde{Q} = -\frac{\hbar^2}{2m} \square R - Q = g. \] (5.27)

that is \( g \) is a scalar under the Poincaré transformations. This implies that \( g(q) \) may depend only on \( \square S_0, (\partial S_0)^2, R, \square R \) and \( (\partial R)^2 \) and higher derivatives which are invariant under Poincaré transformations. However, in the time–independent limit the RQHJE must reduce to (5.5), in particular \( Q_{\text{rel}} \rightarrow Q \). Therefore, \( g \) must vanish in this limit implying
\[ g = 0. \] (5.28)

Then, the RQHJE reads
\[ \frac{1}{2m} (\partial S)^2 + W_{\text{rel}} - \frac{\hbar^2}{2m} \frac{\square R}{R} = 0, \] (5.29)

where \( R \) and \( S \) satisfy the continuity equation
\[ \partial \cdot (R^2 \partial S) = 0. \] (5.30)

### 5.3 Non–relativistic limit

In this section we will consider the non–relativistic limit of the RQHJE. This will yield the time–dependent non–relativistic QHJE together with the time–dependent Schrödinger equation.

To perform the classical limit we first need to make the usual substitution \( S = S' - mc^2t \) and then taking the limit \( c \rightarrow \infty \). We have
\[ W_{\text{rel}} \rightarrow \frac{1}{2} mc^2 + V, \] (5.31)
\[ -\frac{1}{2m} (\partial_0 S)^2 \rightarrow \frac{\partial}{\partial t} S' - \frac{1}{2} mc^2, \] (5.32)
\[ \partial \cdot (R^2 \partial S) = 0 \rightarrow m \frac{\partial}{\partial t} R^2 + \nabla \cdot (R^2 \nabla S') = 0. \] (5.33)

Therefore, in the non–relativistic limit Eq.(5.29) becomes (we remove the ' from R and S)

\[ \frac{1}{2m} (\nabla S)^2 + V + \frac{\partial}{\partial t} S - \frac{\hbar^2}{2m} \Delta R \frac{\partial}{\partial t} R = 0, \] (5.34)

with the time–dependent non–relativistic continuity equation being

\[ m \frac{\partial}{\partial t} R^2 + \nabla \cdot (R^2 \nabla S) = 0. \] (5.35)

It is then easy to see that (5.34) and (5.35) imply

\[ i\hbar \frac{\partial}{\partial t} \psi = \left( -\frac{\hbar^2}{2m} \Delta + V \right) \psi, \] (5.36)

where \( \psi = R \exp(iS/\hbar) \). Note that if we used \( \psi = R \exp(-iS/\hbar) \), then we would get the complex conjugate of (5.36).

6 Gauge invariance and EP

In section 5, we derived the RQHJE with an arbitrary potential. As a byproduct, we obtained the time–dependent Schrödinger equation in the non–relativistic limit. This was a nice step as it indicates that, to be correctly implemented, the EP must be formulated in the exact framework, that is the relativistic one. In other words, even if the time–dependent Schrödinger equation can be derived directly in the non–relativistic framework [3], the natural realm to implement the EP is not in the approximate theory. In fact, while the Klein–Gordon equation follows naturally from the EP, the derivation of the time–dependent Schrödinger equation is less straightforward if one derives it directly in the non–relativistic theory.

However, even if the derivation of the RQHJE is perfectly consistent, the formulation becomes particularly transparent if one works with gauge theories.

6.1 Minimal coupling from the EP

The point is that in general we considered \( W \) as an external fixed quantity, then the corrections concerned \( S^{cl} \), as \( S \) solves an equation which is modified by the quantum potential. Nevertheless, we saw that if one considers the relativistic extension, then \( W \) contains \( S \) itself. So special relativity leads to consider \( W \) as composed by an external potential and \( S \) (see (5.3)). On the other hand, standard QM problems generally correspond to effective potentials. So, for example, the potential well, does not exist as a fundamental interaction. Thus, the nature of
the EP indicates that it should be formulated in the framework of fundamental interactions. On the other hand, since we are in the relativistic framework, interactions cannot be strictly separated in kinetic and potential part. So the only possibility is that both are included in a generalized kinetic term, with $W$ being space–time independent. It is clear that this fixes the interaction to be described in terms of the minimal coupling. On the other hand, the minimal coupling prescription is at the heart of gauge theories.

We now show how the EP is simply implemented once one considers the minimal coupling prescription. Let us consider the interaction to be described in terms of the electromagnetic four–vector $A_\mu$. Let us set $P^d_\mu = p^d_\mu + eA_\mu$ where $p^d_\mu$ is particle’s momentum and $P^d_\mu = \partial_\mu S^d$ is the generalized one. In this case the RCHJE reads

$$\frac{1}{2m} (\partial S^d - eA)^2 + \frac{1}{2}mc^2 = 0,$$

where $A_0 = -\frac{V}{c^2}$. Note that now

$$W = \frac{1}{2}mc^2,$$

and the critical case corresponds to the limit situation in which $m = 0$. As usual, in order to implement the EP, we are forced to add a correction to (6.1)

$$\frac{1}{2m} (\partial S - eA)^2 + \frac{1}{2}mc^2 + Q = 0.$$

Furthermore, we have the transformation properties

$$W^b(q^b) = (p^b|p^a)W^a(q^a) + (q^a;q^b),$$

and

$$Q^b(q^b) = (p^b|p^a)Q^a(q^a) - (q^a;q^b),$$

where

$$(p^b|p) = \frac{(p^b - eA)^2}{(p - eA)^2} = \frac{(p - eA)^tJ^t(p - eA)}{(p - eA)^t\eta(p - eA)}.$$  

and $J$ is the Jacobian matrix

$$J^\mu_\nu = \frac{\partial q^\mu}{\partial q^{b\nu}}.$$  

These transformations imply the cocycle condition

$$(q^a;q^b) = (p^c|p^b) [(q^a;q^b) - (q^a;q^b)].$$  

As we proved in subsection 5.1, $(q^a;q^b)$ vanishes if $q^a$ and $q^b$ are related by a conformal transformation.
As usual we now have to consider the relevant identity for the (generalized) kinetic term. We have
\[ \alpha^2 (\partial S - eA)^2 = \frac{D^2 \Re \alpha S}{\Re \alpha S} - \frac{\Box R}{R} - \frac{\alpha}{R^2} \partial \cdot (R^2 (\partial S - eA)). \] (6.9)
where
\[ D_\mu = \partial_\mu - \alpha eA_\mu, \] (6.10)
and
\[ D^2 \equiv D^\mu D_\mu = \Box - 2\alpha eA \partial + \alpha^2 e^2 A^2 - \alpha e(\partial A). \] (6.11)
Since the identity (6.9) holds for any \( R, S \) and \( \alpha \), we can require \( \partial \cdot (R^2 (\partial S - eA)) = 0 \), and then set \( \alpha = i/\hbar \) to have
\[ (\partial S - eA)^2 = \hbar^2 \left( \frac{\Box R}{R} - \frac{D^2 (\Re \alpha S)}{\Re \alpha S} \right). \] (6.12)
We stress that there is no loss of generality in considering (6.12) since, by \( \partial \cdot (R^2 (\partial S - eA)) = 0 \), this is an identity.

We now show that
\[ W = \frac{\hbar^2}{2m} \frac{D^2 (\Re \alpha S)}{\Re \alpha S}, \] (6.13)
which, by (6.3) and (6.12) implies
\[ Q = - \frac{\hbar^2}{2m} \frac{\Box R}{R}. \] (6.14)
To prove (6.13) we first define the function \( g(q) \) by
\[ W = \frac{\hbar^2}{2m} \frac{D^2 (\Re \alpha S)}{\Re \alpha S} + g. \] (6.15)
Similarly to the case of the function \( g \) in Eq.(5.27), also here \( g \) is a scalar function under Poincaré transformations. Furthermore, since in the \( A \to 0 \) limit we should reproduce Eq.(5.29) with \( W_{\text{rel}} = mc^2/2 \), we see that
\[ g = 0, \] (6.16)
and the RQHJE reads
\[ (\partial S - eA)^2 + m^2 c^2 - \hbar^2 \frac{\Box R}{R} = 0, \] (6.17)
where \( R \) and \( S \) satisfy the continuity equation
\[ \partial \cdot (R^2 (\partial S - eA)) = 0. \] (6.18)
Let us stress that the same result can be directly obtained from (5.1)(5.2) by observing that (5.8) coincides with (6.1) after setting \( W_{\text{rel}} = mc^2/2 \) and replacing \( \partial_\mu S^{\text{cl}} \) by \( \partial_\mu S^{\text{cl}} - eA_\mu \).
One can check that Eqs. (6.17) (6.18) imply the Klein–Gordon equation

\[(i\hbar \partial + eA)^2 \psi + m^2 c^2 \psi = 0, \tag{6.19}\]

where

\[\psi = Re^{i\hbar S}. \tag{6.20}\]

If we considered \(\psi = Re^{-i\hbar S}\), then we would have the complex conjugate of (6.19)

\[(i\hbar \partial - eA)^2 \psi + m^2 c^2 \psi = 0. \tag{6.21}\]

In the time–independent limit \(A_\mu = (-\frac{V}{c}, 0, \ldots, 0), \partial_t V = 0\), both (6.19) and (6.21) reduce to the stationary Klein–Gordon equation (5.7). Correspondingly, Eq. (6.18) reduces to the stationary continuity equation (5.6)

\[\partial \cdot (R^2 (\partial S - eA)) = \nabla \cdot (R^2 \nabla S_0) - \frac{1}{c^2} (\partial_t R^2 (V - E) + R^2 \partial_t V) = \nabla \cdot (R^2 \nabla S_0) = 0. \tag{6.22}\]

### 6.2 EP and mass generation

A special property of the EP is that it cannot be implemented in CM because of the fixed point corresponding to \(W^0 \equiv 0\). Implementing the EP then forces us to introduce a univocally determined piece to the classical HJ equation. A remarkable fact is that in the case of the RCHJE (6.1), the fixed point \(W^0(q^0) \equiv 0\) corresponds to \(m = 0\). The EP then implies that from this all the other masses can be generated by a coordinate transformation. Thus, we have

*Masses correspond to the inhomogeneous term in the transformation properties of the \(W^0\) state*

\[\frac{1}{2} mc^2 = (q^0; q). \tag{6.23}\]

*Furthermore, by (6.4), (6.3) masses are expressed in terms of the quantum potential*

\[\frac{1}{2} mc^2 = (p|p^0)Q^0(q^0) - Q(q). \tag{6.24}\]

A basic feature of the formulation is that the EP implies that \(S\) is never trivial. So, for example also in the case of the non–relativistic particle with \(V - E = 0\), we have a non–trivial quantum potential [1]–[6]. In particular, in [3] the role of the quantum potential was seen as a sort of intrinsic self–energy which is reminiscent of the relativistic self–energy. Eq. (6.24) provides a more explicit evidence of such an interpretation.

Furthermore, in [3, 6] it has been shown that tunnelling is a direct consequence of the quantum potential. In particular, \(Q\) provides the energy to make \(p\) real, and can be seen as
the response of the particle self-energy to external potentials. This example also shows that
external potential and particle energy are strictly related, and so they should be considered as
components of a single object. In part, this is what the minimal coupling prescription provides.
However, the EP, which naturally leads to such a prescription, also implies the additional
quantum potential. In [1]–[6] it has been shown that this contribution is not fixed, rather it
may change once the “hidden variables” are changed. In particular, a change of $\ell$ corresponds
to a mixing between the $p^2/2m$ term and the quantum potential. We now show that in higher
dimension there is a new degree of freedom, represented by an antisymmetric tensor, which is
related to the hidden variables.

6.3 EP and the hidden antisymmetric tensor of QM

A basic property of the formulation immediately appears in one dimension once we consider the
QSHJE. To understand this point it is useful to recall that the difference between the QSHJE
and the one considered by Bohm, is that the QSHJE is written in terms of one function only:
the reduced action $S_0$. While we always have $S_0 \neq \text{cnst}$, in Bohm theory one has $S_0 = \text{cnst}$ for
bound states. However, if one excludes, as implied by the EP, the trivial solution, then one can
obtain the QSHJE from the standard version. In doing this one has to express $R$ in terms of
$S_0$ by solving the continuity equation. We now show that if one tries to write down the QHJE
in higher dimension by solving the continuity equation, then a new field appears. We already
encountered this situation in subsection 4.2. Namely, we saw that the continuity equation of
the QSHJE implies that $R^2 \nabla S_0$ is given by the generalized curl of a $(D-2)$–form $F$. More
precisely, we saw that

$$ R^2 \partial_i S_0 = \epsilon^{i_2 \ldots i_D}_i \partial_{i_2} F_{i_3 \ldots i_D}. $$

This equation is equivalent to

$$ R^2 \partial_i S_0 = \partial^j B_{ij}, $$

where $B_{ij}$ is the antisymmetric two–tensor

$$ B_{ij} = \epsilon^{i_3 \ldots i_D}_{ij} F_{i_3 \ldots i_D}. $$

In other words, the 2–form $B$ is the Hodge dual of $F$

$$ B = * F, $$

and the continuity equation is

$$ d^+ d^+ * F = * ddF = 0. $$
In the time-dependent relativistic case \( F \) is a \((D - 1)\)-form. We have
\[
R^2(\partial_\mu S - eA_\mu) = \epsilon_\mu^{\sigma_1 \ldots \sigma_D} \partial_{\sigma_1} F_{\sigma_2 \ldots \sigma_D} = \partial^\nu B_{\mu\nu},
\]
that is
\[
R^2 = \frac{(\partial^\mu S - eA^\mu)}{(\partial S - eA)^2} \partial^\nu B_{\mu\nu},
\]
or, equivalently,
\[
R^4 = \frac{\partial^\nu B_{\mu\nu} \partial_\sigma B^{\mu\sigma}}{(\partial S - eA)^2}.
\]
In terms of \( B \) and \( R \) the RQHJE (6.17) reads
\[
\partial^\nu B_{\mu\nu} \partial_\sigma B^{\mu\sigma} + R^4 m^2 c^2 - \bar{\hbar}^2 R^3 \Box R = 0.
\]

The EP itself and the appearance of a new field indicates that now the RQHJE should be considered in a different context with respect to the usual one. So, for example, one may wonder if the \( B \)–field may help in considering a possible quantum origin of fundamental interactions. In the Introduction we suggested that QM and GR are facets of the same medal. More generally one should understand if there is a possible role of QM underlying the fundamental interactions or, more precisely, whether the EP underlies the structure of fundamental interactions through QM. We already saw evidence of the dynamical role of QM through the quantum potential [6], e.g. in considering the tunnel effect. In this context one should understand whether Eq.(6.33) may provide a different understanding of the usual problems one meets in considering the Klein–Gordon equation.

Let us find how \( R \) transforms under general \( v \)–maps, \( q \longrightarrow \tilde{q}(q) = v(q) \). In the \( \tilde{q} \) system we have
\[
\tilde{F}_{\sigma_2 \ldots \sigma_D} = \frac{\partial q^{\nu_2}}{\partial \tilde{q}^{\sigma_2}} \cdots \frac{\partial q^{\nu_D}}{\partial \tilde{q}^{\sigma_D}} F_{\nu_2 \ldots \nu_D},
\]
therefore
\[
\epsilon^{\nu_0 \ldots \nu_D}(\partial_\nu S - eA_\nu) \tilde{F}_{\sigma_2 \ldots \sigma_D} = \epsilon^{\nu_0 \ldots \nu_D} \frac{\partial q^{\nu_0}}{\partial \tilde{q}^{\sigma_0}} \partial_{\sigma_0} S - eA_{\nu_0}) \frac{\partial q^{\nu_1}}{\partial \tilde{q}^{\sigma_1}} \frac{\partial q^{\nu_2}}{\partial \tilde{q}^{\sigma_2}} \cdots \frac{\partial q^{\nu_D}}{\partial \tilde{q}^{\sigma_D}} \partial_{\nu_1} F_{\nu_2 \ldots \nu_D} = \det \left( \frac{\partial q}{\partial \tilde{q}} \right) \epsilon^{\nu_0 \ldots \nu_D} (\partial_\nu S - eA_{\nu_0}) \tilde{F}_{\nu_2 \ldots \nu_D},
\]
where we used the fact that the second derivatives of the Jacobian matrix get cancelled due to the antisymmetry of the Levi–Civita tensor. Finally, by (6.31) and (6.31), we have
\[
\tilde{R}^2 = \det \left( \frac{\partial q}{\partial \tilde{q}} \right) (\rho|\tilde{p}) R^2,
\]
that holds also in the stationary non–relativistic case with \((p|\tilde{p})\) given by (2.10). Given (6.36), we easily re–derive the transformation property of \(R\) under the inversion \(q^*(q)\) we derived in subsection 4.3. In fact by (3.32) and (3.33) we have \(\det^2(J^{-1}) = \det(r^4 1_D)\), that is
\[
\det^2 \left( \frac{\partial q}{\partial q^*} \right) = \det(r^4 1_D) = r^{4D},
\] (6.37)
then, as by (3.34) \((p|p^*) = r^{-4}\), we obtain
\[
R^{*2}(q^*) = r^{2D-4}R^2(q).
\] (6.38)

It is then convenient to use (6.36) to find the transformation property of \(R\) under the inversion (B.4). By (B.9) and (B.10), we have
\[
R^{*2}(q^*) = (q^2)^{D-1}R^2(q).
\] (6.39)

A

In this appendix we report the proof of Eqs.(3.12)(3.27) and (3.41) concerning the structure of the function \((q^a; q^b)\) in the cases in which \(q^a\) and \(q^b\) are related by a translations, dilatations and inversion respectively.

Let us start by considering the function
\[
G(D, q) = (q + D; q),
\] (A.1)
where \(D\) is an arbitrary constant vector. In terms of \(G(D, q)\), Eq.(3.11) yields
\[
G(D, q + B) - c(q + B + D) + c(q + D) = G(D, q) - c(q + B) + c(q).
\] (A.2)
Taking the derivative of both sides of Eq.(A.2) with respect to \(B_j\), we get
\[
\partial_{q_j} G(D, q + B) - \partial_{q_j} c(q + B + D) = -\partial_{q_j} c(q + B).
\] (A.3)
After setting \(B_j = 0\) and integrating
\[
G(D, q) = c(q + D) - c(q) + \hat{G}(D, \hat{q}),
\] (A.4)
where, as before, by \(\hat{q}\) we denote all the components of \(q\) other than \(q_j\), and, by Eq.(3.20), \(\hat{G}(D, \hat{q})\) vanishes if all the components of \(D\) other than \(D_j\) are zero. Furthermore, plugging Eq.(A.4) into Eq.(3.10), we realize that \(\hat{G}(D, \hat{q})\) shares the same properties as \(G(D, q)\). Therefore, the analogue of Eq.(A.4) holds for \(\hat{G}(D, \hat{q})\) as well. Hence, applying this reasoning recursively we end up with
\[
(q + D; q) = F(q + D) - F(q) + H(D),
\] (A.5)
where $H(D)$ vanishes whenever only one component of $D$ is not zero. However, by (3.10), we find that $H$ is linear

$$H(D+E) = H(D) + H(E),$$  \hspace{1cm} (A.6)

which implies

$$H(D) = \sum_{k=1}^{D} a_k D_k,$$  \hspace{1cm} (A.7)

so that

$$H = 0,$$  \hspace{1cm} (A.8)

and we arrive to Eq.(3.12), that is

$$(q + D; q) = F(q + D) - F(q),$$  \hspace{1cm} (A.9)

Note that the right hand side remains invariant under the constant shift

$$F \longrightarrow F + c.$$  \hspace{1cm} (A.10)

Let us now analyze the consequences of (A.9) on $h(A, q) = (Aq, q)$. By (2.32) and noting that $(p|p_A) = A^2$, we have

$$(A(q + B); q) = (A(q + B); q + B) + (q + B; q) = A^2(Aq + AB; Aq) + (Aq; q),$$  \hspace{1cm} (A.11)

which is equivalent to

$$h(A, q + B) - h(A, q) = A^2[F(Aq + AB) - F(Aq)] - F(q + B) + F(q),$$  \hspace{1cm} (A.12)

where now $B$ is an arbitrary vector. Taking the derivative with respect to $B_j$

$$\partial_q h(A, q + B) = A^2\partial_q F(Aq + AB) - \partial_q F(q + B),$$  \hspace{1cm} (A.13)

and setting $B = 0$ we find

$$\partial_q h(A, q) = A^2\partial_q F(Aq) - \partial_q F(q),$$  \hspace{1cm} (A.14)

that upon integration yields

$$h(A, q) = A^2 F(Aq) - F(q) + g(A).$$  \hspace{1cm} (A.15)

A useful observation is that $h(A, q) = (Aq; q)$ evaluated at $q = 0$ cannot depend on $A$, so that $h(A, 0) = h(1, 0) = 0$. Therefore

$$g(A) = -(A^2 - 1)F(0),$$  \hspace{1cm} (A.16)
and $h(A,q) = A^2(F(Aq) - F(0)) - (F(q) - F(0))$, that, upon the re-labeling $F(q) \rightarrow F(q) - F(0)$, coincides with (3.27), that is

$$(Aq; q) = A^2F(Aq) - F(q), \quad \text{(A.17)}$$

where now

$$F(0) = 0. \quad \text{(A.18)}$$

Note that this fixes the ambiguity (A.11).

We now complete the proof of Eq.(3.41). First of all note that by (2.32)

$$((Aq)^*; q) = (p|p_A)((Aq)^*; Aq) + (Aq; q). \quad \text{(A.19)}$$

On the other hand, by (2.32) and (3.38)

$$((Aq)^*; q) = (A^{-1}q^*; q) = (p|p_*)(A^{-1}q^*; q^*) + (q^*; q), \quad \text{(A.20)}$$

so that

$$A^2((Aq)^*; Aq) + (Aq; q) = r^{-4}(A^{-1}q^*; q^*) + (q^*; q), \quad \text{(A.21)}$$

and by (3.27)

$$A^2((Aq)^*; Aq) + A^2F(Aq) - F(q) = \frac{1}{r^4}[A^{-2}F(A^{-1}q^*) - F(q^*)] + (q^*; q). \quad \text{(A.22)}$$

Picking a $q_0$ such that $q_0^* = q_0$ and noticing that $r_0 = 1$, we have by (3.40) and (A.22) that

$$((Aq_0)^*; Aq_0) = A^{-4}F(A^{-1}q_0) - F(Aq_0). \quad \text{(A.23)}$$

Now observe that any $q$ can be expressed as $Aq_0$, where $A = r$ and with $q_0$ a suitable solution of $q^* = q$. Furthermore, by (3.38) we have $A^{-1}q_0 = (Aq_0)^*$, so that (A.23) is equivalent to Eq.(B.41), that is

$$(q^*; q) = \frac{1}{r^4}F(q^*) - F(q). \quad \text{(A.24)}$$

**B**

In section B we showed that, as a consequence of Eq.(2.32), $(q^a; q^b)$ vanishes identically if $q^a$ and $q^b$ are related by a Möbius transformation. In the present case, we can prove the analogous result that $(q^a; q^b)$ vanishes if $q^a$ and $q^b$ are related by a conformal transformation, where the conformal group, with respect to the Minkowski metric, is generated by translations

$$q \rightarrow q + b, \quad b \in \mathbb{R}^{D+1}, \quad \text{(B.1)}$$
dilatations
\[ q \rightarrow Aq, \quad A \in \mathbb{R}, \quad (B.2) \]

Lorentz transformations
\[ q \rightarrow \Lambda q, \quad \Lambda \in O(D, 1), \quad (B.3) \]

and the inversion
\[ q^* = \frac{q}{q^2}, \quad (B.4) \]

where \( q^2 = \eta^{\mu\nu} q_\mu q_\nu \). Note that for the inversion to be well-defined, \( \mathbb{R}^{D+1} \) must be completed by a cone at infinity \(^{[22]}\). This space is the analogue of \( \hat{\mathbb{R}}^{D+1} \). The proof is the same as the one provided in section \(^{[3]}\) and in Appendix A. In particular, we have
\[ (q + B; q) = F(q + B) - F(q), \quad (B.5) \]
\[ (Aq; q) = A^2 F(Aq) - F(q), \quad (B.6) \]
\[ (\Lambda q; q) = F(\Lambda q) - F(q), \quad (B.7) \]

where \( F \) is an arbitrary function satisfying \( F(0) = 0 \). As far as the inversion is concerned, the proof needs to be slightly modified. The Jacobian matrix of this mapping is given by
\[ J_{\mu\nu} = \partial_\nu q^{*\mu} = \partial_\nu \frac{q^{\mu}}{q^2} = \frac{\delta_{\mu\nu}}{q^2} - 2 \frac{q^{\mu} q_\nu}{q^4}, \quad (B.8) \]

where \( q^4 \equiv (q^2)^2 \). Then
\[ (J \eta J^t)^{\mu\nu} = J^\mu_\rho \eta^{\rho\sigma} J^{\nu \sigma} = \left( \frac{\delta_{\mu\nu}}{q^2} - 2 \frac{q^{\mu} q_\nu}{q^4} \right) \eta^{\rho\sigma} \left( \frac{\delta_\rho_\sigma}{q^2} - 2 \frac{q^{\rho} q_\sigma}{q^4} \right) = \frac{1}{q^4} \eta^{\mu\nu}, \quad (B.9) \]

which implies
\[ (p|p_*) = (p_*|p)^{-1} = \frac{p_\mu \eta^{\mu\nu} p_\nu}{p_*\mu \eta^{\mu\nu} p_*\nu} = \frac{p_*^\nu J \eta J^\nu p_*}{p_*^\nu \eta p_*} = \frac{1}{q^4}, \quad (B.10) \]

Note that \( q^* \) is involutive since
\[ q^{*2} = q^{*\mu} \eta_{\mu\nu} q^{*\nu} = \frac{1}{q^4} q^{\mu} \eta_{\mu\nu} q^{\nu} = \frac{1}{q^2}, \quad (B.11) \]

and therefore
\[ (q^*)^{*\mu} = \frac{q^{*\mu}}{q^2} = \frac{q^{\mu}}{q^2 q^2} = q^{\mu}. \quad (B.12) \]

Since \( q^2 \) is invariant under Lorentz transformations, these commute with the inversion
\[ (\Lambda q)^* = \Lambda q^*. \quad (B.13) \]

Finally, under dilatations
\[ (Aq)^* = \frac{(Aq)^\mu}{q_A^2} = \frac{Aq^\mu}{A^2 q^2} = A^{-1} \frac{q^\mu}{q^2} = A^{-1} q^{*\mu}, \quad (B.14) \]
where $q_A^2 = Aq^\mu\eta_{\mu\nu}Aq^\nu = A^2 q^2$. By (B.10) and (B.12)

\[(q^*; q) = -(p|p^*)(q; q^*) = -\frac{1}{q^4}((q^*)^*; q^*),\]  

(B.15)

which implies that $(q^*; q)$ vanishes when evaluated at any $q_0$ solution of $q^* = q$

\[(q^*; q)|_{q=q_0} = 0,\]  

(B.16)

and that

\[(q^*; q)|_{q=q_1} = -(q; q^*)|_{q=q_1},\]  

(B.17)

where $q_1$ satisfies $q^* = -q$. By (5.16)

\[((Aq)^*; q) = (p|p_A)((Aq)^*; Aq) + (Aq; q).\]  

(B.18)

On the other hand, by (B.10) and (B.14)

\[((Aq)^*; q) = (A^{-1}q^*; q) = (p|p_*)(A^{-1}q^*; q^*) + (q^*; q),\]  

(B.19)

therefore, by (B.6)

\[A^2((Aq)^*; Aq) + A^2 F(Aq) - F(q) = \frac{1}{q^4}[A^{-2}F(A^{-1}q^*) - F(q^*)] + (q^*; q).\]  

(B.20)

Picking a $q_0$ such that $q_0^* = q_0$, Eq. (B.20) yields

\[((Aq_0)^*; Aq_0) = A^{-4}F(A^{-1}q_0) - F(Aq_0).\]  

(B.21)

Now observe that any $q$, such that $q^2 > 0$, can be expressed as $Aq_0$, where $A^2 = q^2$, and with $q_0$ a suitable solution of $q^* = q$. Furthermore, by (B.14) we have $A^{-1}q_0 = (Aq_0)^*$, so that (B.21) is equivalent to

\[(q^*; q) = \frac{1}{q^4}F(q^*) - F(q), \quad q^2 > 0.\]  

(B.22)

On the other hand, picking a $q_1$ such that $q_1^* = -q_1$, Eq. (B.20) yields

\[((Aq_1)^*; Aq_1) = A^{-4}F(A^{-1}q_1) - F(Aq_1) + A^{-2}(F(q_1) - F(q_1^*)) + A^{-2}(q^*; q)|_{q=q_1}.\]  

(B.23)

Taking $A = -1$, (B.23) becomes

\[(q_1; q_1^*) = F(q_1) - F(-q_1) + F(q_1) - F(-q_1) + (q_1^*; q_1),\]  

(B.24)

which, by virtue of (B.14), is equivalent to

\[(q_1^*; q_1) = F(q_1^*) - F(q_1).\]  

(B.25)
Hence, (B.23) becomes

\[(Aq_1)^*; Aq_1) = A^{-4} F(A^{-1}q_1^*) - F(Aq_1). \] \hspace{1cm} (B.26)

Now observe that any \(q\), such that \(q^2 < 0\), can be expressed as \(Aq_1\), where \(A^2 = -q^2\), and with \(q_1\) a suitable solution of \(q^* = -q\). Furthermore, by (B.14) we have \(A^{-1}q_1^* = (Aq_1)^*\), so that (B.26) is equivalent to

\[(q^*; q) = \frac{1}{q^4} F(q^*) - F(q), \quad q^2 < 0. \] \hspace{1cm} (B.27)

Thus, in general

\[(q^*; q) = \frac{1}{q^4} F(q^*) - F(q). \] \hspace{1cm} (B.28)

However, also in the relativistic case we have the analogue of condition (2.20)

\[\mathcal{F}_{\text{rel}}(\partial_\mu S, \square S, \ldots) = 0. \] \hspace{1cm} (B.29)

Then, the same reasoning as in subsection 3.5 leads to

\[F = 0. \] \hspace{1cm} (B.30)

Therefore, we can state the following result

Eq. (B.23) and the cocycle condition (5.10) imply that \((q^a; q^b)\) vanishes when \(q^a\) and \(q^b\) are related by a conformal transformation

\[(\gamma(q); q) = 0. \] \hspace{1cm} (B.31)

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