ON THE MANY-TO-ONE STRONGLY STABLE FRACTIONAL MATCHING SET

Pablo A. Neme * Jorge Oviedo *

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Abstract

For a many-to-one matching market where firms have strict and $q$-responsive preferences, we give a characterization of the strongly stable fractional matching set as the union of the convex hull of connected sets of stable matchings.

keywords: Matching Markets; Many-to-one Matching Market; Strongly Stable Fractional Matchings; Linear Programming.

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1 Introduction:

A large part of the matching literature studies many-to-one matching markets. The agents in these markets are divided into two disjoint sets: The many-side of the market, namely resident doctors, students, workers, etc, and the one-side, namely hospitals, college, firms, etc. The main property studied in the matching literature is the stability. A matching is called stable if all agents have acceptable partners and there is no unmatched pair (hospital-doctor, college-student, firm-worker, etc.), where both agents would prefer to be matched to each other rather than staying with their current partners under the proposed matching. Each agent has a preference list that determines an order over the agents or sets of agents on the other side of the market, with the possibility of staying unmatched. In this paper the firms have $q$-responsive and strict preferences.

Linear programming is a widely used mathematical tool in matching theory. Each matching can be represented by an assignment matrix called the incidence vector of the matching.

Vande Vate [19] and Rothblum [19] present a system of linear inequalities that characterizes the stable matching of the marriage market for two different restrictions of the market. Both papers show that the stable matchings for the marriage market correspond to the set of incidence vectors (integer solutions for linear inequalities). In other words, the stable matchings are exactly the extreme points of the polytope generated by the system of linear inequalities. Roth et al. [15], for the marriage market, introduce a linear program that characterizes all stable matchings as the integer solutions.

*paneme@unsl.edu.ar (P. Neme); joviedo@unsl.edu.ar (J. Oviedo).
Linear programming approach has been developed to the theory of stable matching markets also by Abeledo and Rothblum [4, 3], Abeledo and Blum [1], Abeledo et al. [2], Fleiner [9, 8], Sethuraman and Teo [18] and by others.

Ba˘ıou and Balinski [5] present two characterizations of the convex polytope for the many-to-one matching market. We focus on one of these characterizations.

Lotteries over stable matchings has been study in many instance in the literature. Roth et al. [15] for the marriage market, studied lotteries over stable matching via linear programming. When the extreme points of the convex polytope generated by the constraints of a linear program are exactly the stable matchings of the market, for instance the marriage market, a random matching coincides with the concept of stable fractional matching. Roth et al. [15] defines stable fractional matching as a not necessarily integer solution of the linear program. When the extreme points are not all integer, these two concept are not the same, for instance a many-to-one matching market with q-responsive and strict preferences. That is to say, a random matching is always a stable fractional matching, but some stable fractional matching can not be written as a lottery over stable matchings. Example 1 expose a many-to-one matching market with an extreme point that is a stable fractional matching.

Each entry of an incidence vector of a stable fractional matching can be interpreted as the time that each agent spends with one agent on the other side of the market. For a stable fractional matching, it can happen that two agents, one of each side of the market, have an incentive to increase the time that they spend together at the expense of those matched agents that they like less than each other at a stable fractional matching. To study a ”good” fractional solution, the idea is to avoid this and prevent that agents have incentive ”block” the stable fractional matching in a fractional way. For a Marriage Market, Roth et al. [15] define a strongly stable fractional matching as a stable fractional matching that fulfill a non-linear equalities that represent this non-blocking condition mentioned above. In other words, a stable fractional matching that fulfill the non-linear equalities from Roth et al. [15], is a strongly stable fractional matching.

Neme and Oviedo [13] give a characterization of the strongly stable fractional matching for the marriage market. Our work extends their result and provides a characterization for the many-to-one strongly stable fractional matching set. We analyze the linear programming structure for the many-to-one matching market when the firms have q-responsive preferences. We extend the strong stability condition from Roth et al. [15] to a many-to-one matching market. We focus on one of the characterizations of Ba˘ıou and Balinski [5]. As we mentioned before, the convex polytope generated by the linear inequalities of this characterization may have fractional extreme points. We prove that these fractional extreme points violate our strong stability condition.

In the school choice set-up, strong stability for lotteries has been introduced by Kesten and Ünver [11], they called ex-ante stability for lotteries. For this market, they deal with indifferences in the preferences of the schools. Kesten and Ünver [11] also present a fractional deferred-acceptance algorithm that computes a unique strongly ex-ante random matching. Their paper analyses strategy proofness and efficiency of this mechanism. Our characterization goes in another direction, we study the relationship between the stable matchings that are involve in the lotteries.

Bansal et al. [6] and Cheng et al. [7] study the concept of cycles in preferences and cyclic matchings for many-to-many and many-to-one matching markets, respectively. These papers are an extension of Irving and Leather [10]. To seek for cycles in preferences, these authors first
reduce the preference lists of all agents. We present the reduction procedure for our market in the Appendix. This reduction procedure allows us to find cycles in preferences. Since the cycles of a reduced list are disjoint, we extend the definition of cyclic matching to a set of cycles in the reduced preference profile.

Following the extension of cyclic matching used by Bansal et al. [6] and Cheng et al. [7], we define a connected set generated by a stable matching $\mu$ as the set of all cyclic matching of $\mu$ (including $\mu$). Then, we characterize a strongly stable fractional matching as a lottery over stable matchings that belong to the same connected set. Moreover, we prove that these stable matching of the lottery, have a decreasing order on the eyes of all firms. In this way, we characterize the set of all strongly stable fractional matchings as the union of the convex hull of these connected sets.

Roth et al. [15], (in Corollary 21) proves that in a strongly stable fractional matching, each agent is matched with at most two agents of the other side of the market. Schlegel [17] generalizes this result for the school choice set-up with strict priorities. They show that an ex-ante stable lottery fulfills that each worker has a positive probability with at most two distinct firms, and for each firm, all but possibly one position are assigned deterministically. For the one position that is assigned by a lottery, two workers have a positive probability of been employed. Our characterization gives an alternative prove for this two result, for the school choice set-up due to Schlegel [17] is straigtforogard, and for the marriage marker due to Roth et al. [15], its necessary only to set all quotas of all firms equal to one.

This paper is organized as follows. Section 2 formally introduces the market, preliminary results, and one of Bâıou and Balinski’s characterizations of stable matchings [5]. Section 3 discusses the definition of strongly stable fractional matching and some properties of these fractional matchings. We also discuss cycles and cyclic matching properties. In section 4, we present our characterization of a strongly stable fractional matching. The Appendix contains the reduction procedure, lemmas and proofs of the lemmas needed for our characterization.

2 Preliminary Results.

In the many-to-one matching markets that we study, there are two sets of agents, the set of firms $F = \{f_1, \ldots, f_n\}$ and the set of workers $W = \{w_1, \ldots, w_m\}$. Each worker $w$ has an antisymmetric, transitive, and complete preference relation $P_w$ over $F \cup \{w\}$, and each firm $f$ an antisymmetric, transitive, and complete preferences relation $P_f$ over set of workers, $2^W$. Also, each firm $f$ has a maximum number of positions to fill: their quota, denoted by $q_f$. Given $W_0, W_1 \subseteq W$, we write $W_0 \geq_f W_1$ to indicate that the firm $f$ likes $W_0$ as much as $W_1$. Given the preference relation $P_f$, we say that $W_0 >_f W_1$ when $W_0 \geq_f W_1$ and $W_0 \neq W_1$. Analogously, for each worker $w$, and any two firms, $f_0, f_1 \in F$, we write $f_0 \geq_w f_1$ and $f_0 >_w f_1$.

The preference profiles are $(n+m)$ - tuples of preference relations and they are denoted by $P = (P_{f_1}, \ldots, P_{f_n}, P_{w_1}, \ldots, P_{w_m})$. The matching market for the sets $W$ and $F$ with the preference profiles $P$ will be denoted by $(F, W, P)$.

We say that a pair $(f, w) \in F \times W$ is an acceptable pair if $w$ is acceptable for the firm $f$, and $f$ is acceptable for the worker $w$, that is $\{w\} \geq_f \emptyset$ and $f \geq_w w$. Let us denote by $A(P)$ the set of all acceptable pairs of the matching market $(F, W, P)$, (simply $A$, when no confusion arises).

The assignment problem consist in matching workers with firms keeping the bilateral nature
of their relationship and allowing for the possibility that firm and workers remain unmatched. Formally,

**Definition 1.** A matching \( \mu \) is a mapping from the set \( F \cup W \) into the set of all subsets of \( F \cup W \) such that, for all \( w \in W \) and \( f \in F \):

1. \( |\mu(w)| = 1 \) and \( \mu(w) \subseteq F \) or \( \mu(w) = \{w\} \) if \( \mu(w) \not\subseteq F \).
2. \( \mu(f) \in 2^W \) and \( |\mu(f)| \leq q_f \).
3. \( \mu(w) = \{f\} \) if and only if \( w \in \mu(f) \).

Usually we will omit the curly brackets, for instance, instead of condition 1. and 3., we will write: 1. \( |\mu(w)| = 1 \) and \( \mu(w) \subseteq F \) or \( \mu(w) = w \) if \( \mu(w) \not\subseteq F \) and 3. \( \mu(w) = f \) if and only if \( w \in \mu(f) \).

One of the most important results of the matching theory is called the **Rural Hospital Theorem.** It states that any firm that does not fill its quota in some stable matching will be matched to the same workers in every stable matching. Formally:

Let \( \mu >_F \mu' \) denote that all firms like \( \mu \) at least as well as \( \mu' \) with at least one firm preferring \( \mu \) to \( \mu' \) outright, that is, that \( \mu(f) \geq_f \mu'(f) \) for all \( f \), and \( \mu(f') >_f \mu'(f') \) for at least one firm \( f' \). Analogously \( \mu >_W \mu' \) for the set of workers. We say that \( \mu >_F \mu' \) means that either \( \mu >_F \mu' \) or \( \mu = \mu' \). Analogously \( \mu \geq_W \mu' \) for the set of workers.

We say that a matching \( \mu \) is **individually rational**, if \( \mu(w) = f \) for some worker \( w \) and firm \( f \), such that the pair \( (f, w) \) is an acceptable pair. Similarly, a pair \( (f, w) \) is a **blocking pair** for the matching \( \mu \), if the worker \( w \) is not employed by the firm \( f \), but they both prefer to be matched to one another. That is, a matching \( \mu \) is blocked by a **firm-worker pair** \( (f, w) \):

a) If \( |\mu(f)| = q_f, \mu(w) \neq f, f >_w \mu(w) \) and \( w >_f w' \) for some \( w' \in \mu(f) \).

b) If \( |\mu(f)| < q_f, \mu(w) \neq f \) and \( f >_w \mu(w) \) and \( w >_f f \).

In that way, a matching \( \mu \) is **stable** if it is individually rational and has no blocking pairs.

One of the most important results of the matching theory is called the **Rural Hospital Theorem.** It states that any firm that does not fill its quota in some stable matching will be matched to the same workers in every stable matching. Formally:
Theorem 1. (Roth [14]) When preferences over individuals are strict, any hospital that does not fill its quota at some stable matching is assigned precisely the same set of students at every stable matching.

2.1 Linear Programming Approach.

For the marriage market, Rothblum [16] characterizes stable matchings as extreme points of a convex polytope generated by a linear inequality system. Baïou and Balinski [5] present two generalizations of the convex polytope for the many-to-one matching market \((F, W, P)\) with \(q_f\)-responsive preferences.

Given a matching \(\mu\), a vector \(x^\mu \in \{0, 1\}^{F \times |W|}\) is an incidence vector when \(x^\mu_{f,w} = 1\) if and only if \(\mu(w) = f\) and \(x^\mu_{f,w} = 0\) otherwise. When no confusion arises, we identify each matching with its incidence vector.

Let \(PC\) be the convex polytope generated by the following linear inequalities:

\[
\begin{align*}
\sum_{j \in W} x_{f,j} &\leq q_f \quad f \in F \\
\sum_{i \in F} x_{i,w} &\leq 1 \quad w \in W \\
\sum_{j > f, w} x_{f,j} + q_f \sum_{i > w, f} x_{i,w} + q_f x_{f,w} &\geq q_f \quad (f, w) \in A \\
x_{f,w} &\geq 0 \quad (f, w) \in F \times W \\
x_{f,w} &= 0 \quad (f, w) \in F \times W \setminus A
\end{align*}
\]

Lemma 1. Let \((F, W, P)\) be a many-to-one matching market. \(\mu\) is a stable matching for \((F, W, P)\) if and only if its incidence vector satisfies the linear inequalities (1)-(5).

Remark 1. Notice that, an integer vector that satisfies the linear inequalities (1), (2) and (4), represent the incidence vector of a matching for the many-to-one matching market. A not integer solution of linear inequalities (1), (2) and (4) we will called a fractional matching.

We will define a stable fractional matching as a not necessarily integer point of the convex polytope \(PC\). That is, a stable fractional matching is a not necessarily integer solution for the system of linear inequalities (1)-(5). For the marriage market, i.e. \(q_f = 1\) for all \(f \in F\), Rothblum [16] prove that the extreme points of the associated convex polytope, are the stable matchings. It is naturally expected that this result carries over to the more general case, a many-to-one matching market. But this is not true for the convex polytope \(PC\). Here, we present an example taken from Baïou and Balinski [5] that shows a many-to-one market, where the convex polytope has fractional extreme points. This also shows that a lottery over stable matchings is also a stable fractional matching. However, the opposite case does not always hold.

Example 1. Let \(F = \{f_1, f_2\}\) and \(W = \{w_1, w_2, w_3, w_4\}\) with the following lists of preferences:

\[
\begin{align*}
P(f_1) &= \{w_1, w_2, w_3, w_4\} & P(w_1) &= \{f_2, f_1\} \\
P(f_2) &= \{w_4, w_3, w_2, w_1\} & P(w_2) &= \{f_2, f_1\} \\
P(w_3) &= \{f_2, f_1\} & P(w_3) &= \{f_1, f_2\}
\end{align*}
\]
The quotas for the firms are \( q_1 = q_2 = 2 \). The only two stable matching for this market are:

\[
x_{\mu F} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}; \quad x_{\mu W} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}.
\]

Baïou and Balinski observe that the stable fractional matching

\[
x^1 = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix},
\]

is a vertex of the convex polytope \( PC \).

Observing that the convex polytope has fractional extreme points, Baïou and Balinski present a second generalization for the many-to-one matching market. In this second generalization, the extreme points of the convex polytope \( PC_{B-B} \), are exactly the stable matchings for a many-to-one market. This assures that this last convex polytope, \( PC_{B-B} \), is a subset of the convex polytope \( PC \). For that reason, our study is based on the convex polytope \( PC \).

### 3 The Strongly Stable Fractional Matchings.

Each entry of the matrix that represent a stable fractional matching, \( x_{f,w} \), can be interpreted as the time that firm \( f \) and worker \( w \) spends with each other. For a stable fractional matching \( x \), it can happen that two agents, one of each side or the market, have an incentive to increase the time that they spend together at the expense of those they like less at a stable fractional matching \( x \). The importance of a strongly stable fractional matching, is to avoid this and prevent that agents have incentive "block" the stable fractional matching in a fractional way. We formally present the definition of a strongly stable fractional matching for our market.

**Definition 3.** Let \((F, W, P)\) be a many-to-one matching market. Let us consider a stable fractional matching \( \bar{x} \). We say that \( \bar{x} \) is **strongly stable** if for all \((f, w) \in A\), \( \bar{x} \) satisfies the strong stability condition

\[
q_f - \sum_{j \geq f w} \bar{x}_{f,j} = 0 \quad \text{and} \quad 1 - \sum_{i \geq w f} \bar{x}_{i,w} = 0.
\]

**Remark 2.** The incidence vector of a stable matching, also fulfill condition (7).

**Lemma 2.** Let \((F, W, P)\) be a many-to-one matching market. Let \( \bar{x} \) be a strongly stable fractional matching, such that \( \bar{x} = \alpha y + (1 - \alpha)z \), with \( 0 < \alpha < 1 \), where \( y \) and \( z \) satisfy linear inequalities (1), (2), (4) and (5)(i.e. \( y \) and \( z \) are fractional matchings). Then, for all \((f, w) \in A\) we have that either

\[
q_f - \sum_{j \geq f w} y_{f,j} = 0 \quad \text{and} \quad q_f - \sum_{j \geq f w} z_{f,j} = 0
\]

or

\[
1 - \sum_{i \geq w f} y_{i,w} = 0 \quad \text{and} \quad 1 - \sum_{i \geq w f} z_{i,w} = 0.
\]

**Proof.** Let \((F, W, P)\) be a many-to-one matching market. Let \( \bar{x} \) be a strongly stable fractional matching, such that \( \bar{x} = \alpha y + (1 - \alpha)z \), with \( 0 < \alpha < 1 \) where \( y \) and \( z \) satisfy linear inequalities
(1), (2), (4) and (5). Then, for all \((f, w) \in A(P)\),

\[
q_f - \sum_{j \geq f w} x_{f,j} \cdot \left[ 1 - \sum_{i \geq u} x_{i,w} \right] = 0
\]

\[
q_f - \sum_{j \geq f w} \alpha y_{f,j} + (1 - \alpha)z_{f,j} \cdot \left[ 1 - \sum_{i \geq u} \alpha y_{i,w} + (1 - \alpha)z_{i,w} \right] = 0
\]

\[
q_f - \alpha \sum_{j \geq f w} y_{f,j} - (1 - \alpha) \sum_{j \geq f w} z_{f,j} \cdot \left[ 1 - \alpha \sum_{i \geq w} y_{i,w} - (1 - \alpha) \sum_{i \geq w} z_{i,w} \right] = 0
\]

From inequality (1) and (2) for \(y\) and \(z\), we have that

\[
q_f - \sum_{j \geq f w} y_{f,j} \geq 0, \quad q_f - \sum_{j \geq f w} z_{f,j} \geq 0,
\]

\[
1 - \sum_{i \geq w} y_{f,j} \geq 0 \text{ and } 1 - \sum_{j \geq f w} z_{f,j} \geq 0.
\]

Then, we can assure that either

\[
\left( q_f - \sum_{j \geq f w} y_{f,j} = 0 \right) \text{ and } \left( q_f - \sum_{j \geq f w} z_{f,j} = 0 \right)
\]

or

\[
\left( 1 - \sum_{i \geq w} y_{i,w} = 0 \right) \text{ and } \left( 1 - \sum_{i \geq w} z_{i,w} = 0 \right).
\]

Notice that the extreme points of the convex polytope generated by linear inequalities (1),(2), (4) and (5) are all integer points. This convex polytope is known as the polytope of the transportation problem. For more detail, see Luenberger and Ye [12].

The following proposition states that a strongly stable fractional matching can be written as a lottery over stable matchings. Formally:

**Proposition 1.** Let \((F, W, P)\) be a many-to-one matching market. Let \(\bar{x}\) be a strongly stable fractional matching. Then, there is a collection of stable matchings \(\mu^p\), such that \(\bar{x} = \sum_{p=1}^{k} \alpha_p \mu^p\) with \(0 < \alpha_p \leq 1\) and \(\sum_{p=1}^{k} \alpha_p = 1\).

**Proof.** Let \((F, W, P)\) be a many-to-one matching market. Let \(\bar{x}\) be a strongly stable fractional matching. Then, \(\bar{x}\) fulfill linear inequalities (1),(2), (4) and (5). Hence, \(\bar{x} = \sum_{p=1}^{k} \alpha_p \mu^p\), with \(0 < \alpha_p \leq 1\) and \(\sum_{p=1}^{k} \alpha_p = 1\), where \(\mu^p\) are individually rational matchings. That is, for
fulfill linear inequalities (1), (2), (4) and (5). We need to prove \( \mu^p \) is stable for all \( p = 1, \ldots, k \). Then,

\[
\bar{x} = \sum_{p=1}^{k} \alpha_p \mu^p = \alpha_k \mu^k + \sum_{p=1}^{k-1} \alpha_p \mu^p = \alpha_k \mu^k + (1 - \alpha_k)y,
\]

where \( y = \sum_{p=1}^{k-1} \alpha_p \mu^p \). By Lemma 2, \( \mu^k \) and \( y \) fulfill Condition (6). That is, \( \mu^k \) is stable and \( y \) is strongly stable. Since \( y \) fulfill linear inequalities (1), (2), (4) and (5), then \( y = \sum_{p=1}^{k-1} \beta_p \mu^p \), with \( 0 < \beta_p < 1 \) and \( \sum_{p=1}^{k} \beta_p = 1 \), where \( \mu^p \) are individually rational matchings. Then,

\[
y = \sum_{p=1}^{k-1} \beta_p \mu^p = \beta_k \mu^{k-1} + \sum_{p=1}^{k-2} \beta_p \mu^p = \beta_k \mu^{k-1} + (1 - \beta_{k-1})z,
\]

where \( z = \sum_{p=1}^{k-2} \beta_p \mu^p \). Notice that \( \beta_p = \frac{\alpha_p}{1 - \alpha_k} \). Hence, by Lemma 2, \( \mu^{k-1} \) and \( z \) fulfill Condition (6). That is, \( \mu^{k-1} \) is stable and \( z \) is strongly stable. Continuing with the same argument, we have that \( \mu^p \) is stable for all \( p = 1, \ldots, k \).

We denote \( \text{supp}(x) \) to be the support of the fractional matching \( x \), that is, \( \text{supp}(x) = \{(f, w) : x_{f,w} > 0\} \). The following theorem states that no non-integer extreme points of the convex polytope \( PC \) is a strongly stable fractional matching.

**Theorem 2.** Let \((F, W, P)\) be a many-to-one matching market. Let \( x \) be a non-integer extreme point of the convex polytope \( PC \). Then, \( x \) cannot be a strongly stable fractional matching.

**Proof.** Let \((F, W, P)\) be a many-to-one matching market. Let \( x \) be a non-integer extreme point of the convex polytope \( PC \), then \( x \) cannot be written as convex combination of points of the convex polytope \( PC \). Since \( x \) satisfies linear inequalities (1)-(5), \( x \) can be written as a convex combination of the extreme points of the convex polytope generated only by the linear inequalities (1), (2), (4) and (5). That is, \( x \) is a convex combination of matchings. Then, there exits a (not stable) matching whose support is included in the support of the stable fractional matching \( x \). Then, by Proposition 1, \( x \) is not a strongly stable fractional matching.

**Remark 3.** The previous theorem assures that if a non-integer matching is strongly stable, then it is an stable fractional matching. Hence, the set of the strongly stable fractional matchings is included in the set of the lottery over stable matchings, and this one in turn is included in the set of stable fractional matchings.

### 3.1 Cycles in Preferences.

For a marriage market, Irving and Leather [10] define a cycle in preference and cyclic matching in order to present an algorithm that finds all stable matchings for this market. Bansal et al. [6] and Cheng et al. [7] extend the concept of cycles and cyclic matchings for a many-to-many and many-to-one matching markets, respectively. We will state some properties of cycles that are taken from these authors. They refer to the cycles as rotations.

Given a stable matching \( \mu \) for a many-to-one matching market \((F, W, P)\), we define a **reduced preference profile** \( P^\mu \), as the preference profile obtained after the the reduction procedure. This reduction procedure is presented in the Appendix. The order in the reduced preference list
of firm $f$, $P^\mu(f)$ is denoted by $>_f^\mu$ ($>_F^\mu$). In the same way, $>_w^\mu$ ($>_W^\mu$) is the order in worker $w$ reduced list of preference.

**Lemma 3** (Bansal et al. [8]). Let $(F,W,P)$ be a many-to-one matching market, and let $\mu'$ be a matching. Then, $\mu'$ is stable under $P^\mu$ if and only if it is stable under the original preference profile and $\mu>_F \mu'$.

**Definition 4.** Let $(F,W,P)$ be a many-to-one matching market. Given a stable matching $\mu$, and the reduced preference profile $P^\mu$, a set of firms $\sigma = \{e_1, ..., e_r\} \subseteq F$ defines a cycle if:

- a) $w_{e_d} \not\in \mu(e_d), w_{e_d} \supseteq w'$ for all $w' \not\in \mu(e_d)$, and $w_{e_d} \in \mu(e_{d+1})$ for all $d = 1, ..., r - 1$.
- b) Moreover, $w_{e_r} \not\in \mu(e_r), w_{e_r} \supseteq w'$ for all $w' \not\in \mu(e_r)$, and $w_{e_r} \in \mu(e_1)$.

Given a cycle $\sigma$, we can define a cyclic matching as follows:

**Definition 5.** Let $(F,W,P)$ be a many-to-one matching market. Given a stable matching $\mu$, and the reduced preference profile $P^\mu$, let $\sigma = \{e_1, ..., e_r\}$ be a cycle in $P^\mu$, and $\{w_{e_1}, ..., w_{e_r}\}$ defined by the cycle $\sigma$. We define the matching $\mu[\sigma]$ as follows:

$$
\begin{align*}
\mu[\sigma](e_1) &= \mu(e_1) - \{w_{e_r}\} \cup \{w_{e_1}\}, \\
\mu[\sigma](e_d) &= \mu(e_d) - \{w_{e_{d-1}}\} \cup \{w_{e_d}\} \quad \text{for } d = 2, ..., r - 1, \\
\mu[\sigma](e_r) &= \mu(e_r) - \{w_{e_{r-1}}\} \cup \{w_{e_r}\}, \\
\mu[\sigma](f) &= \mu(f) \quad \text{for all } f \not\in \sigma.
\end{align*}
$$

Notice that if a firm $f$ belongs to a cycle $\sigma$, this means that it has different sets of workers assigned in $\mu$ as well as in $\mu[\sigma]$. Then, by the Rural Hospital Theorem, we have that $|\mu(f)| = q_f$.

**Lemma 4** (Bansal et al. [8]). Let $(F,W,P)$ be a many-to-one matching market. For a stable matching $\mu$, the reduced preference profile $P^\mu$, and a cycle $\sigma$ in $P^\mu$, the cyclic matching $\mu[\sigma]$ is a stable matching in the original preference profile.

Let $\Phi(\mu)$ denote the set of cycles of the reduced preference profile $P^\mu$. Now, we can extend the definition of a cyclic matching as follows.

**Definition 6.** Let $(F,W,P)$ be a many-to-one matching market. For a stable matching $\mu$, and the reduced preference profile $P^\mu$, let $K = \{\sigma_1, ..., \sigma_n\} \subseteq \Phi(\mu)$, define the cyclic matching $\mu[K]$ 

$$
\mu[K](f) = \begin{cases} 
\mu[\sigma_h](f) & f \in \sigma_h, \ h = 1, ..., n \\
\mu(f) & \text{otherwise.}
\end{cases}
$$

**Remark 4.** Let $K \subseteq \Phi(\mu)$ be a subset of cycles of $P^\mu$. By Lemma [8], we have that $\mu[K](f) = \mu[\sigma](f)$ for each $f \in \sigma$ with $\sigma \in K$.

**Lemma 5** (Cheng et al. [7]). Let $(F,W,P)$ be a many-to-one matching market, for the reduced preference profile $P^\mu$, and two different cycles $\sigma_s$ and $\sigma_t$ in $\Phi(\mu)$. Then, $\sigma_s \cap \sigma_t = \emptyset$.

Notice that Lemma [4] and Lemma [5] assure that the cyclic matching $\mu[K]$ from Definition [6] is stable under the original preference profile.

**Lemma 6** (Cheng et al. [7]). Let $(F,W,P)$ be a many-to-one matching market. Let $P^\mu$ be the reduced preference profile, and let $\mu'$ be a stable matching in $P^\mu$. If $\mu \neq \mu'$, then a cycle $\sigma \in P^\mu$ exists such that $\mu[\sigma] >_F \mu'$.
Lemma 7. Let $(F, W, P)$ be a many-to-one matching market. For the reduced preference profile $P^\mu$, and two different cycles $\sigma_s$ and $\sigma_l$ in $\Phi(\mu)$. Then, $\sigma_s$ is a cycle of $P^\mu = P^{\mu[\sigma_l]}$.

Proof. See the Appendix.

The matching obtained by applying different cycles is independent from the order in which they are applied.

Lemma 8. Let $(F, W, P)$ be a many-to-one matching market. Let $P^\mu$ be the reduced preference profile, and let $\sigma_s$ and $\sigma_l$ be two different cycles in $\Phi(\mu)$. Then,

$$\mu[\sigma_1, \sigma_s] = \mu[\sigma_s, \sigma_l].$$

Proof. See the Appendix.

4 A characterization of the strongly stable fractional matching set.

The characterization of a strongly stable fractional matching that we present is based on the idea of cyclic matchings. So, we need that a stable fractional matching that is strongly stable in a reduced preference profile is also strongly stable in the original preference profile. This statement is proved on Lemma 10 in the Appendix.

In order to present our characterization, we define the connected set generated by a stable matching.

Definition 7. Let $(F, W, P)$ be a many-to-one matching market. A set of stable matchings $\mathcal{M}$ is connected if there is a stable matching $\mu$ such that:

$$\mathcal{M} = \{ \mu[K] : K \subseteq \Phi(\mu) \}.$$

In this case, we say that $\mathcal{M}$ is generated by $\mu$, and we write $\mathcal{M} = \mathcal{M}_\mu$.

Notice that from definition of $\mu[K]$, we can see that $\mu$ is also a cyclic matching of itself. That is, if $K = \emptyset$, then $\mu[K] = \mu$, hence $\mu \in \mathcal{M}_\mu$. The following theorem presents a property that fulfills any convex combination of stable matchings from a connected set.

Theorem 3. Let $(F, W, P)$ be a many-to-one matching market. Then, any convex combination of stable matchings from a connected set is a strongly stable fractional matching.

Proof. Let $(F, W, P)$ be a many-to-one matching market. Let $\tilde{x}$ be a convex combination of stable matchings from a connected set. That is, there exists a stable matching $\mu$ and its reduced preference profile $P^\mu$. In $P^\mu$, there exist $K_1, ..., K_t \subseteq \Phi(\mu)$ and the corresponding cyclic matching $\mu^1, ..., \mu^t$, such that

$$\tilde{x} = \sum_{l=1}^{t} \alpha_l \mu^l$$

with $0 \leq \alpha_l \leq 1$ and $\sum_{l=1}^{t} \alpha_l = 1$.

Since $\tilde{x}$ is a convex combination of stable matchings from $\mathcal{M}_\mu$, we have that $\mu \geq_F \mu^l$ for all $l = 1, ..., t$. Then, we have that $\tilde{x}$ is a stable fractional matching for the matching market $(F, W, P^\mu)$. Moreover, since $S(P^\mu) \subseteq S(P)$, we have that $\tilde{x}$ is also a stable fractional matching for the matching market $(F, W, P)$. By Lemma 10, we only need to prove that $\tilde{x}$ is strongly stable in the reduced preference profile $P^\mu$. 

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If $\alpha_l = 1$ for some $l = 1, \ldots, t$, we have that $\bar{x} = x^\mu$. Since $\mu^l$ is also a stable matching in the original preferences, then we have that $\bar{x}$ is strongly stable. Hence, we assume $0 < \alpha_l < 1$. We will prove that $\bar{x}$ fulfill Condition (3) for all pair $(f, w) \in A(P^\mu)$.

Fix $f \in F$. Assume that firm $f$ does not fill its quota, the Rural Hospital Theorem assures that this firm will always be assigned to the same set of workers in every stable matching. Then $\bar{x}_{f,j} = x^\mu_{f,j}$ for $l = 1, \ldots, t$ and for all $j$ such that $(f, j) \in A(P^\mu)$. Since $\mu^l$ is a stable matching for $l = 1, \ldots, t$, it fulfills Condition (4) for each $(f, j) \in A(P^\mu)$. Then, by Remark 2 we have that $\bar{x}$ also fulfill Condition (3) for each $(f, j) \in A(P^\mu)$.

Assume now that $f$ does fill its quota. Let $\mathcal{K} = \bigcup_{l=1}^t K_l$. Let $\mu(f) = \{w_1, \ldots, w_{q_f}\}$ and $w_i \succ f w_{i+1}$. We analyze two cases separately.

1. There is no $\sigma \in \mathcal{K}$ such that $f \in \sigma$. By Definition 3 we have that $\mu[\mathcal{K}](f) = \mu(f)$. That is, for all $j \in W$, $\bar{x}_{f,j} = x^\mu_{f,j}$.

   Thus, if $w \leq_f w_{q_f}$ we have that
   \[
   \sum_{j \geq_{f,w}^\mu} \bar{x}_{f,j} = \sum_{j \geq_{f,w}^\mu} x^\mu_{f,j} = q_f.
   \]

   If $w >_f w_{q_f}$, we have that
   \[
   \sum_{j \geq_{f,w}^\mu} \bar{x}_{f,j} = \sum_{j \geq_{f,w}^\mu} x^\mu_{f,j} < q_f.
   \]

   Then, we also have that $\sum_{j >_w^\mu \sigma} x^\mu_{f,j} < q_f$. Moreover, by linear inequality (3) for the stable matching $\mu$, we have that $\sum_{i \geq_w^\mu \sigma} x^\mu_{i,w} > 0$. Therefore, $\sum_{i \geq_w^\mu \sigma} x^\mu_{i,w} = 1$. Lemma 11 states that $\bar{x} \geq_W x^\mu$, i.e.,
   \[
   \sum_{i \geq_w^\mu \sigma} \bar{x}_{i,w} \geq \sum_{i \geq_w^\mu \sigma} x^\mu_{i,w} = 1.
   \]

   By the linear inequality (2), we have that $\sum_{i \geq_w^\mu \sigma} \bar{x}_{i,w} = 1$. Thus, for $(f, w) \in A(P^\mu)$, we have that
   \[
   \left[ q_f - \sum_{j \geq_{f,w}^\mu} \bar{x}_{f,j} \right] \cdot \left[ 1 - \sum_{i \geq_{f,w}^\mu} \bar{x}_{i,w} \right] = 0.
   \]

2. There is $\sigma_f \in \mathcal{K}$ such that $f \in \sigma_f$. By Lemma 5 we have that there is a unique cycle $\sigma_f \in K_l$ such that $f \in \sigma_f$. But $\sigma_f$ may be in more than one set $K_l$. We denote $L_f = \{l : \sigma_f \in K_l\}$.

   Therefore,
   \[
   \bar{x}_{f,j} = \sum_{l=1}^t \alpha_l x^\mu_{f,j} = \sum_{l \in L_f} \alpha_l x^\mu_{f,j} + \sum_{l \notin L_f} \alpha_l x^\mu_{f,j}.
   \]

   Since $\sigma_f$ is unique, by Lemma 5 we have that $\mu[K_l](f) = \mu[\sigma_f](f)$ and $x^\mu_{f,j} = x^\mu_{f,j}^{[\sigma_f]}$ for those $l \in L_f$.

   Also, $\mu[K_l](f) = \mu(f)$ and $x^\mu_{f,j} = x^\mu_{f,j}$ for those $l \notin L_f$. Hence,
   \[
   \sum_{l \in L_f} \alpha_l x^\mu_{f,j} + \sum_{l \notin L_f} \alpha_l x^\mu_{f,j} = \sum_{l \in L_f} \alpha_l x^\mu_{f,j}^{[\sigma_f]} + \sum_{l \notin L_f} \alpha_l x^\mu_{f,j} =
   \]
Now we will prove that \( \bar{x}_{f, j} = \sum_{l \in L_f} \alpha_l + \bar{x}_{f, j}^\mu \sum_{q \in L_j} \alpha_l \).  

Since \( \sum_{l \in L_f} \alpha_l + \sum_{q \notin L_j} \alpha_l = 1 \), then we define \( \bar{\alpha} = \sum_{l \in L_f} \alpha_l \).

Then (7) is equal to \( \bar{x}_{f, j} = (1 - \bar{\alpha}) \bar{x}_{f, j}^\mu + \bar{x}_{f, j}^\mu \). That is, \( \bar{x}_{f, j} \) is the convex combination of \( \bar{x}_{f, j}^\mu \) and \( \bar{x}_{f, j}^\mu \).

Since \( f \in \sigma_f \), we have that

\[
|\text{supp}(x_{f, \cdot}^\mu)| = |\text{supp}(x_{f, \cdot}^\mu)| = q_f \text{ and } |\text{supp}(x_{f, \cdot}^\mu) \cap \text{supp}(x_{f, \cdot}^\mu)| = q_f - 1.
\]

Then, there is a unique \( w_a \) such that \((f, w_a) \in \text{supp}(x^\mu) \setminus \text{supp}(x_\cdot^\sigma)\) and a unique \( w_b \) such that \((f, w_b) \in \text{supp}(x_\cdot^\sigma) \setminus \text{supp}(x^\mu)\), and we also denote by \( T = \{w_a : (f, w_a) \in \text{supp}(x^\mu) \cap \text{supp}(x_\cdot^\sigma)\}\). Then,

\[
\bar{x}_{f, j} = \begin{cases} 
1 & \text{if } j \in T \\
1 - \bar{\alpha} & \text{if } j = w_a \\
\bar{\alpha} & \text{if } j = w_b.
\end{cases}
\]

Now we will prove that \( \bar{x} \) fulfills Condition (6) in \( P^\mu \) for all \( w_a \):

i) If \( w \preceq_f w_b \), then

\[
\sum_{j \geq w} \bar{x}_{f, j} = \sum_{s \in T} \bar{x}_{f, w_a} + \bar{x}_{f, w_a} + \bar{x}_{f, w_b} = (q_f - 1) + (1 - \bar{\alpha}) + \bar{\alpha} = q_f.
\]

Then,

\[
\begin{bmatrix} q_f - \sum_{j \geq w} \bar{x}_{f, j} \end{bmatrix} = 0,
\]

hence,

\[
\begin{bmatrix} q_f - \sum_{j \geq w} \bar{x}_{f, j} \end{bmatrix} \cdot \begin{bmatrix} 1 - \sum_{i \geq w} \bar{x}_{i, w} \end{bmatrix} = 0.
\]

ii) If \( w \succ_f w_qf \), then

\[
\sum_{j \geq w_qf} \bar{x}_{f, j} < q_f.
\]

Since

\[
q_f = \sum_{j \geq \bar{\sigma} w_qf} \bar{x}_{f, j} = (1 - \bar{\alpha}) \sum_{j \geq \bar{\sigma} w_qf} x_{f, j}^\mu + \bar{\alpha} \sum_{j \geq \bar{\sigma} w_qf} x_{f, j}^\mu \sigma_f,
\]

then we have that

\[
\sum_{j \geq \bar{\sigma} w_qf} x_{f, j}^\mu < q_f \text{ and } \sum_{j \geq \bar{\sigma} w_qf} x_{f, j}^\mu \sigma_f < q_f.
\]

But \( \mu \) and \( \mu[\sigma] \) are stable matchings, and these stable matchings fulfill Condition (6).
So we have that
\[ \sum_{i \geq \mu_f} x_{i,w}^\mu = 1 \] and \[ \sum_{i \geq \mu_f} x_{i,w}^{\mu[f]} = 1, \]
in which case we can assure that
\[ \sum_{i \geq \mu_f} \tilde{x}_{i,w} = (1 - \bar{\alpha}) \sum_{i \geq \mu_f} x_{i,w}^\mu + \bar{\alpha} \sum_{i \geq \mu_f} x_{i,w}^{\mu[f]} = 1. \]
That is,
\[ \begin{pmatrix} q_f - \sum_{j \geq \mu_f} \bar{x}_{f,j} \\ \bar{\alpha} \end{pmatrix} \cdot \begin{pmatrix} 1 - \sum_{i \geq \mu_f} \tilde{x}_{i,w} \\ \bar{\alpha} \end{pmatrix} = 0. \]

iii) If \( w = w_{q_f} \). From definition of \( \mu \), we have that \( \mu(w_{q_f}) = f \). Also, we have that \( \mu^l(w_{q_f}) > \mu, \mu \geq w \) for all \( l = 1, ..., t \). In particular, \( \mu^l(w_{q_f}) \geq w_{q_f} \mu(w_{q_f}) = f \) for all \( l = 1, ..., t \). This implies that
\[ \sum_{i \geq w_{q_f}} x_{i,w_{q_f}}^{\mu^l} = 1 \]
for all \( l = 1, ..., t \). Hence
\[ \sum_{i \geq w_{q_f}} \tilde{x}_{i,w_{q_f}} = \sum_{i \geq w_{q_f}} \sum_{l=1}^t \alpha_l x_{i,w_{q_f}}^{\mu^l} = \sum_{l=1}^t \alpha_l \sum_{i \geq w_{q_f}} x_{i,w_{q_f}}^{\mu^l} = \sum_{l=1}^t \alpha_l = 1. \]
Then,
\[ \begin{pmatrix} q_f - \sum_{j \geq w_{q_f}} \bar{x}_{f,j} \\ \bar{\alpha} \end{pmatrix} \cdot \begin{pmatrix} 1 - \sum_{i \geq w_{q_f}} \tilde{x}_{i,w_{q_f}} \\ \bar{\alpha} \end{pmatrix} = 0. \]
From Cases 1 and 2, we have that for the pair \((f, w)\),
\[ \begin{pmatrix} q_f - \sum_{j \geq w} \bar{x}_{f,j} \\ \bar{\alpha} \end{pmatrix} \cdot \begin{pmatrix} 1 - \sum_{i \geq w} \tilde{x}_{i,w} \\ \bar{\alpha} \end{pmatrix} = 0. \]

Therefore, \( \bar{x} \) is a strongly stable fractional matching in the reduced preference profile \( P^\mu \), and by Lemma 10, \( \bar{x} \) is a strongly stable fractional matching in the original preference profile \( P \).

The following corollary extends Corollary 21 from Roth et al. [15], that states that for the marriage market set-up, under a strongly stable fractional matching, each agent is fractionally matched with at most two agents of the other side of the market (set \( q_f = 1 \) for all \( f \in F \)). Another extension is due to Schlegel [17] for a school choice matching market with strict preferences (Similar set-up than ours).

**Corollary 1.** Let \((F, W, P)\) be a many-to-one matching market. Each strongly stable fractional matching fulfills the following 2 conditions:

1. Each worker has a positive probability with at most two distinct firms.
2. Each firm, all but possibly one position are assigned deterministically. For the one position that is assigned by a lottery, two workers have a positive probability of been employed.

For the particular case with all quotas equal to one, the marriage market \((M, W, P)\), and for a stable fractional matching \(x\), Rothblum \[16\] defines a stable matching such that it fulfills the following condition: assign to each man \(m\) the most preferred woman among those that \(x_{m,j} > 0\), for all \(j \in w\). Here we generalize this definition for the many-to-one matching market \((F, W, P)\).

For a many-to-one matching market \((F, W, P)\), and for a given stable fractional matching \(x\), we define the set of workers employed in the best \(q_f\) positions.

Let \(C_f^0(x) = \{w : (f, w) \in \text{supp}(x)\}\). Let \(C_f^1(x) = \{w \in C_f^0(x) : \text{there does not exists } w' \in C_f^0(x), w' >_f w\}\). Notice that the set \(C_f^1(x)\), only has one element, the best worker for firm \(f\) in the \(\text{supp}(x)\). Let \(C_f^2(x) = \{w \in C_f^0(x) : \text{there does not exists } w' \in C_f^0(x) \setminus C_f^1(x), w' >_f w\}\). The set \(C_f^2(x)\), has the two bests workers for firm \(f\) in the \(\text{supp}(x)\). In the same way, let \(C_f^k(x) = \{w \in C_f^0(x) : \text{there does not exists } w' \in C_f^0(x) \setminus C_f^{k-1}(x), w' >_f w\}\). In this way, \(C_f^k(x)\) is the set of the \(k\)-best workers in the \(\text{supp}(x)\). Now, we define the matching where each firm employs the best \(q_f\) workers in the \(\text{supp}(x)\). Formally:

**Definition 8.** Let \((F, W, P)\) be a many-to-one matching market. Let \(x\) be a stable fractional matching. For each firm \(f\), we define \(\mu_x\) as:

\[
\mu_x(f) = \{w \in C_f^0(x) : w \in C_f^{q_f}(x)\}.
\]

**Remark 5.** If for some firm \(f\) we have that \(|C_f^0(x)| \leq q_f\), then \(x_{f,i}^a = 1\) for all \(i \in C_f^0(x)\).

The following lemma generalizes Lemma 12 of Roth et al. \[15\].

**Lemma 9.** Let \((F, W, P)\) be a many-to-one matching market. Let \(\bar{x}\) be a strongly stable fractional matching. Then, \(\mu_x\) is a stable matching.

**Proof.** Let \((F, W, P)\) be a many-to-one matching market. Let \(\bar{x}\) be a strongly stable fractional matching. First, we will prove that \(\mu_x\) is a matching. Assume that all positive entries of \(\bar{x}\) are equal to 1, then we have by Definition \[8\] that \(x^{\mu_x} = \bar{x}\). Since \(\bar{x}\) is a strongly stable fractional matching, by Lemma \[1\] we have that \(\mu_x\) is a stable matching.

Assume now that not all positive entries of \(\bar{x}\) are equal to 1. Also, assume that \(\mu_x\) is not a matching. That is, there exists a worker \(w\) and two different firms \(f\) and \(f'\), such that \(w \in \mu_x(f)\) and \(w \in \mu_x(f')\). Since the preferences of the worker \(w\) are strict, without loss of generality, we can assume that \(f >_w f'\). We will show that \(\sum_{i \geq w} x_{i,w} = 1\), and we analize two cases:

i) \(|C_f^0(\bar{x})| \leq q_f\). We have that if \(\sum_{j \geq f} x_{f,j} < q_f\), since \(\bar{x}\) is a stable fractional matching.

Then Condition \[14\] implies that

\[
\sum_{i \geq w} \bar{x}_{i,w} = 1.
\]

Hence, \(\sum_{i < w} \bar{x}_{i,w} = 0\), and \(\bar{x}_{f',w} = 0\), which contradicts the assumption of \(\bar{x}_{f',w} > 0\).

If \(\sum_{j \geq f} x_{f,j} = q_f\), then \(w \in C_f^{q_f}\) and \(\bar{x}_{f,w} = 1\). This implies that

\[
\sum_{i \geq w} \bar{x}_{i,w} = 1.
\]
ii) If $|C_f^0(x)| > q_f$, that is $C_f^0(x) \subset C_f^0(\bar{x})$. Since $w \in \mu_x(f)$, then $w \in C_f^0(\bar{x})$, and we have that

$$\sum_{j \geq f} \bar{x}_{f,j} \leq \sum_{j \in C_f^0(x)} \bar{x}_{f,j} < \sum_{j \in C_f^0(x)} \bar{x}_{f,j} \leq q_f$$

Hence, $\sum_{j \geq f} \bar{x}_{f,j} < q_f$. Since $\bar{x}$ is a strongly stable fractional matching, Condition $\text{ii)}$ implies that

$$\sum_{i \geq w} \bar{x}_{i,w} = 1.$$

From cases i) and ii), we have that $\sum_{i \geq w} \bar{x}_{i,w} = 1$, then $\sum_{i < f} \bar{x}_{i,w} = 0$, and also $\bar{x}_{f',w} = 0$ since $f > w f'$, which contradicts the assumption of $\bar{x}_{f',w} > 0$. Therefore $\mu_x$ is a matching. Let $w \in \mu_x(f)$, then $w \in C_f^0(\bar{x}) \subseteq C_f^0(x)$. Hence $w >_f f$ and $f >_w w$. Then $\mu_x$ in an individually rational matching.

Now we will prove that $\mu_x$ has no blocking pairs. Assume that there exists a blocking pair $(\bar{f}, \bar{w})$ of $\mu_x$. Then, we have that:

a) $\bar{w} \not\in \mu_x(\bar{f})$.

b) There exists $w' \in \mu_x(\bar{f})$ such that $\bar{w} > f w'$ if $|\mu_x(\bar{f})| = q_f$. Or $\bar{w} > f \bar{f}$ if $|\mu_x(\bar{f})| < q_f$.

c) $\bar{f} > w \mu_x(\bar{w})$.

Since $\bar{w} \not\in \mu_x(\bar{f})$, we have that $\bar{w} \not\in C_f^0(\bar{x})$, then we will show that $\bar{x}_{f,w} = 0$:

- If $|C_f^0(\bar{x})| < q_f$, then $C_f^0(\bar{x}) = C_f^0(\bar{x})$. Hence, since $\mu_x(\bar{f}) = C_f^0(\bar{x})$ and $\bar{w} \not\in \mu_x(\bar{f})$, we have that $\bar{x}_{f,w} = 0$.

- If $|C_f^0(\bar{x})| = q_f$, since $\bar{w} \not\in \mu_x(\bar{f})$, we have that $\bar{w} > f w'$ for all $w' \in C_f^0(\bar{x})$, then $\bar{x}_{f,w} = 0$.

Since $\bar{x}_{f,w} = 0$, by item b) we have that $\sum_{j \geq f} \bar{x}_{f,j} < \sum_{j \geq f} \bar{x}_{f,j} \leq q_f$, and since $\bar{x}$ is a strongly stable fractional matching, we have that

$$1 = \sum_{i < f} \bar{x}_{i,w} = \bar{x}_{f,w} + \sum_{i \geq w} \bar{x}_{i,w}.$$ 

Then, $\sum_{i < w} \bar{x}_{i,w} = 0$, but this is a contradiction since from Definition 8 we have that $\bar{x}_{\mu_x(w),w} > 0$ and item c) $\bar{f} > w \mu_x(\bar{w})$. Therefore, $\mu_x$ is a stable matching.

The main result of this paper is the characterization that states that $\bar{x}$ is a strongly stable fractional matching if and only if it belongs to the convex hull of a connected set. Formally:

**Theorem 4.** Let $(F, W, P)$ be a many-to-one matching market and let $\bar{x}$ be a stable fractional matching in $(F, W, P)$. $\bar{x}$ is strongly stable if and only if there is a collection of connected stable matchings such that $\bar{x} = \sum_{l=1}^{k} \alpha_l x^{l}$, $0 < \alpha_l \leq 1$, $\sum_{l=1}^{k} \alpha_l = 1$.

**Proof.** Let $(F, W, P)$ be a many-to-one matching market. Theorem 3 proves $\Leftarrow$.

For $\Rightarrow$, let $\bar{x}$ be a strongly stable fractional matching. Proposition 11 and Lemma 13 assures that

$$\bar{x} = \sum_{l=1}^{k} \alpha_l x^{l}, \quad 0 < \alpha_l \leq 1, \quad \sum_{l=1}^{k} \alpha_l = 1,$$

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with \( \mu' \in S(P) \) for \( l = 1, \ldots, k \) and \( \mu^1 >_F \mu^2 >_F \ldots >_F \mu^k \).

Assume that \( \bar{x} \notin \text{Conv} \{ \mathcal{M}_{\mu_2} \} \). Then, we have that \( \mu^t \notin \mathcal{M}_{\mu_2} \). Let \( t = \min \{ l : \mu^t \notin \mathcal{M}_{\mu_2} \} \). Since \( \mu^1 = \mu_2 \in \mathcal{M}_{\mu_2} \), we have that \( t \geq 2 \).

Given that \( \mu^t \notin \mathcal{M}_{\mu_2} \), and \( \mu^{t-1} \in \mathcal{M}_{\mu_2} \), and \( \mu^{t-1} \neq \mu^t \), we have \( \mu^{t-1} >_F \mu^t \).

Then by Lemma 6 there exists a cycle \( \sigma^* \in \Phi(\mu^{t-1}) \) such that \( \mu^{t-1}[\sigma^*] \notin \mathcal{M}_{\mu_1} \) and \( \mu^{t-1}[\sigma^*] >_F \mu^t \). Notice that this implies that \( \sigma^* \notin \Phi(\mu^1) \).

Here we analyze two cases:

a) If there exists \( \sigma \in \Phi(\mu^1) \) such that \( \sigma^* \cap \sigma \neq \emptyset \). Then for any \( \tilde{f} \in \sigma^* \cap \sigma \), we have that, \( \mu^1 > \tilde{f} \mu^{t-1} > \tilde{f} \mu^{t-1}[\sigma^*] \geq \tilde{f} \mu^t \). If \( t = 2 \), we have that \( \sigma^* \in \Phi(\mu^1) \), and since \( \sigma \cap \sigma^* \neq \emptyset \), then \( \sigma = \sigma^* \) which results in a contradiction, and \( t \geq 3 \).

By the Rural Hospital Theorem, there exist \( w^* \), \( w_1 \), \( w_2 \) such that:

\[
\begin{align*}
w_1 & \in \mu^1(\tilde{f}) - (\mu^{t-1}(\tilde{f}) \cup \mu^{t-1}[\sigma^*](\tilde{f})) , \\
w^* & \in \mu^{t-1}(\tilde{f}) - \mu^t(\tilde{f}) , \\
w_2 & \in \mu^{t-1}[\sigma^*](\tilde{f}) - (\mu^{t-1}(\tilde{f}) \cup \mu^t(\tilde{f})) ,
\end{align*}
\]

and \( w_1 > \tilde{f} w^* > \tilde{f} w_2 \). Now, we will prove that for the pair \( (\tilde{f}, w^*) \), Condition 4 fails. That is,

\[
q_{\tilde{f}} - \sum_{j \geq f w^*} \bar{x}_{\tilde{f},j} - \left[ 1 - \sum_{i \geq w^* \tilde{f}} \bar{x}_{i,w^*} \right] \neq 0 .
\]

We will analyze the two factors separately:

i) \[
q_{\tilde{f}} - \sum_{j \geq f w^*} \bar{x}_{\tilde{f},j} = q_{\tilde{f}} - \sum_{j \geq f w^*} \left( \sum_{l=1}^{k} \alpha_l x_{\mu^l,j}^l \right) = \\
q_{\tilde{f}} - \sum_{l=1}^{k} \alpha_l \left( \sum_{j \geq f w^*} x_{\mu^l,j} \right).
\]

Since \( \mu^l \leq \tilde{f} \mu^{t-1}[\sigma^*] \), we have that

\[
\sum_{j \geq f w} x_{\mu^l,j} \leq \sum_{j \geq f w} x_{\tilde{f},j} \mu^{t-1}[\sigma^*]
\]

for all \( w \in W \).

In particular for \( w = w^* \), and the fact that \( w > \tilde{f} w^2 \) with \( w^2 \in \mu^{t-1}[\sigma^*](\tilde{f}) \), we have that

\[
\sum_{j \geq f w^*} x_{\mu^l,j} \leq \sum_{j \geq f w^*} x_{\tilde{f},j} \mu^{t-1}[\sigma^*] < q_{\tilde{f}} .
\]

By 6 we also have that for all \( l = 1, \ldots, k \)

\[
\sum_{j \geq f w^*} x_{\mu^l,j} \leq q_{\tilde{f}} .
\]
Using the decreasing sequence of stable matchings of Lemma 13, we have that \( \alpha_l > 0 \) for \( l = 1, \ldots, k \). Then, we have that

\[
\left[ q_f - \sum_{j \geq f} \bar{x}_{f,j} \right] = \left[ q_f - \sum_{l=1}^k \alpha_l \left( \sum_{j \geq f} x_{f,j}^l \right) \right] > \left[ q_f - \left( \sum_{l=1}^k \alpha_l q_f^l \right) \right] = 0
\]

ii)

\[
1 - \sum_{i \geq w^*} \bar{x}_{i,w^*} = 1 - \sum_{i \geq w^*} \left( \alpha_1 x_{i,w^*}^1 + \sum_{l=2}^k \alpha_l \sum_{i \geq w^*} x_{i,w^*}^l \right) = 1 - \left( \alpha_1 \sum_{i \geq w^*} x_{i,w^*}^1 + \sum_{l=2}^k \alpha_l \sum_{i \geq w^*} x_{i,w^*}^l \right)
\]

By (8), we have that \( w^* \notin \mu^1(f) \). Since \( \mu^1 >_f \mu^{t-1} \), we have that \( \mu^{t-1}(w^*) = \tilde{f} > w^* \), \( \mu^1(w^*) \). Therefore,

\[
\sum_{i \geq w^*} x_{i,w^*}^l = 0.
\]

Since \( \alpha_l > 0 \) for \( l = 1, \ldots, k \), then

\[
1 - \left( \alpha_1 \sum_{i \geq w^*} x_{i,w^*}^1 + \sum_{l=2}^k \alpha_l \sum_{i \geq w^*} x_{i,w^*}^l \right) = 1 - \sum_{l=2}^k \alpha_l > 0.
\]

Then from cases i) and ii), we have that for the pair \((\tilde{f}, w^*)\),

\[
q_{\tilde{f}} - \sum_{j \geq \tilde{f}} \bar{x}_{\tilde{f},j} \cdot \left[ 1 - \sum_{i \geq w^*} \bar{x}_{i,w^*} \right] \neq 0.
\]

That is, for the pair \((\tilde{f}, w^*)\) Condition (6) fails.

b) If \( \sigma \cap \sigma^* = \emptyset \) for all \( \sigma \in \Phi(\mu^1) \). Notice that \( \mu^{t-1}(f) \neq \mu^{t-1}[\sigma^*](f) \) for all \( f \in \sigma^* \).

In this case, we have that there exist \( \tilde{f} \in \sigma^* \) and \( \bar{w} \in W \), such that for \( \mu^1(f) = \{w_1^\mu, \ldots, w_{\tilde{f}}^\mu\} \), and \( \mu^{t-1}(f) = \{w_1^{t-1}, \ldots, w_{\tilde{f}}^{t-1}\} \). Notice that if \( q_f = 1 \), we have that \( w_i^\mu = w_i^{t-1} \) and \( w_i^{t-1} = w_i^{t-1} \), but \( w_i^\mu > f w_i^{t-1} \). Then, we have that \( w_i^\mu > f w_i^{t-1} \). Otherwise, for all \( f \in \sigma^* \), we have that \( \mu^1(f) > f \mu^{t-1}[\sigma^*](f) > f w \), for all \( w \notin \mu^1(f) \cup \mu^{t-1}[\sigma^*](f) \). Here \( \mu^{t-1}[\sigma^*](f) > f w \) denotes that worker \( w \) is less preferred for the firm \( f \) than all workers matched to firm \( f \) under the stable matching \( \mu^{t-1}[\sigma^*] \). Since \( \sigma \cap \sigma^* = \emptyset \), we have that \( \mu^1(f) = \mu^{t-1}(f) \) for all \( f \in \sigma^* \). Then, let \( \{w^f\} = \mu^{t-1}[\sigma^*](f) \setminus \mu^1(f) \) for each \( f \in \sigma^* \). That is, \( w^f \) is the most preferred worker in the reduced preference list \( P^{t-1}(f) \) such that it does not belong to \( \mu^1(f) \). This implies that \( \sigma^* \in \Phi(\mu^1) \), and it is a contradiction since \( \sigma^* \notin \Phi(\mu^1) \).
Therefore, we can assure that $\bar{f} \in \sigma^*$ and $\bar{w} \in W$ exist, such that

$$\mu^1(\bar{f}) = \mu^{t-1}([\bar{f}] >_w \bar{w} >_f \mu^{t-1}[\sigma^*](\bar{f}) \tag{9}$$

Since $\sigma^* \in \Phi(\mu^{t-1})$, in order to obtain the reduced preference lists $P^{t-1}$, $\bar{f}$ should have eliminated $\bar{w}$ by means of the third step of the reduction procedure. Then, we have that

$$\mu^t(\bar{w}) \geq \mu^{t-1}(\sigma^*)(\bar{w}) >_w \bar{w} >_f \mu^{t-1}(\bar{w}) >_w \mu^1(\bar{w}) \tag{10}$$

Since $\bar{x}$ can be written as in Lemma 13, $(\mu^1, \mu^{t-1} \in \mathcal{M}_{\mu_Z}$ and $\mu^{t-1} > F \mu^t)$, and using inequalities (9) and (10), we have that

$$\sum_{j \geq \bar{f} \bar{w}} \bar{x}_{f,j} < q_f \quad \text{and} \quad \sum_{i \geq \bar{w} f} \bar{x}_{i,w} < 1.$$ 

Then,

$$\left[ q_f - \sum_{j \geq \bar{f} \bar{w}} \bar{x}_{f,j} \right] \cdot \left[ 1 - \sum_{i \geq \bar{w} f} \bar{x}_{i,w} \right] > 0.$$

That is, Condition (14) fails for the pair $(\bar{f}, \bar{w})$.

Therefore, from cases a) and b), we have that $\bar{x} \in \text{Conv}(\mathcal{M}_{\mu_Z})$. \hfill $\square$

Once we have characterized a strongly stable fractional matching for the many-to-one matching market $(F, W, P)$, we can characterize the set of all strongly stable fractional matchings as follows:

**Corollary 2.** Let $(F, W, P)$ be a many-to-one matching market. Let $SS^q(P)$ be the set of all strongly stable fractional matchings. Then,

$$SS^q(P) = \bigcup_{\mu \in S(P)} \text{Conv}(\mathcal{M}_\mu)$$

5 **APPENDIX**

The reduction procedure:

Let $(F, W, P)$ be a many-to-one matching market. Let $f \in F$, $w \in W$, $\mu_F$ the optimal stable matching for all firms, and $\mu_W$ the optimal stable matching for all workers.

**Step 1:** Remove all $w$ who are more preferred than the most preferred worker matched under $\mu_F(f)$ from $f$’s list of acceptable workers. Remove all $f$ who are more preferred than $\mu_W(w)$ from $w$’s list of acceptable firms.

Therefore, the most preferred worker matched in $\mu_F(f)$ will be the first entry in $f$’s reduced list, and $\mu_W(w)$ will be the first entry in $w$’s reduced list.

**Step 2:** Remove all $f$ who are less preferred than $\mu_F(w)$ from $w$’s list of acceptable firms. Remove all $w$ who are less preferred than the least worker matched under $\mu_W(f)$ from $f$’s list of acceptable workers.

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Thus, $\mu_F(w)$ will be the last entry in $w$’s reduced list, and the least preferred worker in $\mu_W(f)$ will be the last entry in $f$’s reduced list.

**Step 3:** After steps 1 and 2, if $f$ is not acceptable for $w$ (i.e., if $f$ is not on $w$’s preference list as now modified), then remove $w$ from $f$’s list of acceptable workers, and similarly, remove from $w$’s list of acceptable firms, any firm $f$ to whom $w$ is no longer acceptable.

Hence, $f$ will be acceptable for $w$ if and only if $w$ is acceptable for $f$ after Step 3.

In general, if $\mu$ is any stable matching, and we replace $\mu_F$ by $\mu$ in the reduction process, the resulting profile will be called a **profile of reduced lists $P^\mu$**.

For the matching market $(F, W, P^\mu)$, the stable matching $\mu$ is the $F$–optimal stable matching, that is the stable matching that all firm prefers in the matching market $(F, W, P^\mu)$.

**Lemmas and Proofs.**

**Proof.** Proof of Lemma 7

Let $P^\mu$ be a reduced preference profile. Let $\sigma_s$ and $\sigma_l$ be two different cycles in $P^\mu$. By Lemma 5 we have that $\sigma_s \cap \sigma_l = \emptyset$. Then, we can assume that $\sigma_l = \{e_1, \ldots, e_r\}$ and $\sigma_s = \{e_{r+1}, \ldots, e_{r+r'}\}$. By Definition 6,

$$
\mu[\sigma_l](f) = \begin{cases} 
\mu(e_1) & \text{if } f = e_r \\
\mu(e_{k+1}) & \text{if } f = e_k, \ k = 1, \ldots, r - 1 \\
\mu(f) & \text{if } f \notin \{e_1, \ldots, e_r\}.
\end{cases}
$$

That is, the firms that do not belong to the cycle $\sigma_l$ do not change the set of workers in both stable matchings ($\mu$ and $\mu[\sigma_l]$). Then, by Lemma 5 the cycle $\sigma_s$ is a subset of those firms that do not change. That is, $\sigma_s$ is a cycle of $P^\mu[\sigma_l]$. □

**Proof.** Proof of Lemma 8

Let $P^\mu$ be a profile of reduced lists for $(F, W, P)$, and let $\sigma_s$ and $\sigma_l$ be two different cycles in $P^\mu$. Let $K = \{\sigma_s, \sigma_l\}$. Since $K$ is a set of cycles (not an ordered set), by Definition 6 and Lemma 5 we have that

$$
\mu[K](f) = \begin{cases} 
\mu[\sigma_s](f) & \text{if } f \in \sigma_s \\
\mu[\sigma_l](f) & \text{if } f \in \sigma_l \\
\mu(f) & \text{otherwise}.
\end{cases}
$$

Then, $\mu[\sigma_l, \sigma_s] = \mu[\sigma_s, \sigma_l]$. □

We define a way to compare fractional matchings as follows:

**Definition 9.** A fractional matching $x$ weakly dominates a fractional matching $y$ with respect to the preference of the firm $f$, if for all worker $w$,

$$
\sum_{j \geq f}^w x_{f,j} \geq \sum_{j \geq f}^w y_{f,j},
$$

and it will be denoted by $x \succeq_f y$.

We say that $x$ strongly dominates $y$, denoted by $x \succ_f y$, if the previous inequality holds strictly for at least one worker $w$. Weak and strong domination under a worker’s preferences are defined analogously. We say that $x \succeq_F y$ when $x \succeq_f y$ for all $f \in F$. Analogously, for $x \succeq_W y$. 19
Lemma 10. Let \((F, W, P)\) be a many-to-one matching market. Let \(\mu \in S(P)\), and \(P^\mu\) the reduced preference profile. Let \(\bar{x}\) be a stable fractional matching for a many-to-one matching market \((F, W, P)\). If \(\bar{x}\) is a strongly stable fractional matching for a many-to-one matching market \((F, W, P^\mu)\), then \(\bar{x}\) is a strongly stable fractional matching for a many-to-one matching market \((F, W, P)\).

Proof. Let \((F, W, P)\) be a many-to-one matching market. Let \(\mu \in S(P)\), and \(P^\mu\) be the reduced preference profile. Let \(\bar{x}\) be a stable fractional matching for a many-to-one matching market \((F, W, P)\). Let \(\bar{x}\) be a strongly stable fractional matching for a many-to-one matching market \((F, W, P^\mu)\), that is:

\[
\left[ q_f - \sum_{j \geq f} \bar{x}_{f,j} \right] \cdot \left[ 1 - \sum_{i \geq u_f} \bar{x}_{i,w} \right] = 0,
\]

for all \((f, w) \in A(P^\mu)\).

We need to prove that, for all \((f, w) \in A(P)\), \(\bar{x}\) fulfills

\[
\left[ q_f - \sum_{j \geq f} \bar{x}_{f,j} \right] \cdot \left[ 1 - \sum_{i \geq u_f} \bar{x}_{i,w} \right] = 0.
\]

We will consider the following two cases:

1. Let \((f, w) \in A(P^\mu)\). That is, \((f, w)\) was not eliminated in \(P^\mu\). So,

\[
\sum_{j \geq f} \bar{x}_{f,j} \geq \sum_{j \geq f} \bar{x}_{f,j}
\]

holds, since for each firm \(f\), there are more workers in the original preference list than in the reduced preference list.

Hence,

\[
q_f - \sum_{j \geq f} \bar{x}_{f,j} \leq q_f - \sum_{j \geq f} \bar{x}_{f,j}.
\]

With a similar argument we have that

\[
1 - \sum_{i \geq u_f} \bar{x}_{i,w} \leq 1 - \sum_{i \geq u_f} \bar{x}_{i,w}.
\]

By hypothesis, and linear inequalities (1) and (2) of \(PC\),

\[
0 = \left[ q_f - \sum_{j \geq f} \bar{x}_{f,j} \right] \cdot \left[ 1 - \sum_{i \geq u_f} \bar{x}_{i,w} \right] \geq \left[ q_f - \sum_{j \geq f} \bar{x}_{f,j} \right] \cdot \left[ 1 - \sum_{i \geq u_f} \bar{x}_{i,w} \right] \geq 0.
\]

Then, for \((f, w) \in A(P^\mu)\), we have that

\[
\left[ q_f - \sum_{j \geq f} \bar{x}_{f,j} \right] \cdot \left[ 1 - \sum_{i \geq u_f} \bar{x}_{i,w} \right] = 0.
\]

2. Let \((f, w) \in A(P) - A(P^\mu)\). We will analyze two sub-cases:

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We have that \( x^\mu \succeq_F \bar{x} \), then
\[
\sum_{j > f \, w} \bar{x}_{f,j} \leq \sum_{j > f \, w} x^\mu = 0.
\] (11)

Since, \( \bar{x} \) is a stable fractional matching, \( \bar{x} \) satisfies inequalities (2) and (3) of \( PC \), i.e.
\[
\sum_{i \in F} \bar{x}_{i,w} \leq 1 \quad \text{and} \quad \sum_{j > f \, w} \bar{x}_{j,w} + q_f \sum_{i \geq w} \bar{x}_{i,w} + q_{f,j,f,w} \geq q_f.
\]

Then, by condition (11)
\[
q_f \sum_{i \geq w} \bar{x}_{i,w} \geq q_f,
\]
and for all \( i \geq w \, f \), \( \bar{x}_{i,w} = 1 \). Hence,
\[
\sum_{i \geq w \, f} \bar{x}_{i,w} \geq 1,
\]
and by linear inequality (2), we have that
\[
\sum_{i \geq w \, f} \bar{x}_{i,w} = 1,
\]
then we have that
\[
\left[q_f - \sum_{j \geq f \, w} \bar{x}_{f,j}\right] \cdot \left[1 - \sum_{i \geq w \, f} \bar{x}_{i,w}\right] = 0.
\]

ii) \( w_1 > f \, w \). We will analyze two sub-cases, if the firm \( f \) does or does not fill its quota.

a) If the firm \( f \) does not fill its quota, by the Rural Hospital Theorem, the firm \( f \) will be assigned to the same set of workers in every stable matching. Assume that \( \mu(f) = \{w_1, ..., w_p\} \) with \( p < q_f \). If \( w < f \, w_1 \), we have that
\[
0 < \sum_{j > f \, w} \bar{x}_{f,j} < q_f.
\]

Then, we can have two cases, if \( \sum_{i \geq w \, f} \bar{x}_{i,w} = 1 \) or \( 0 < \sum_{i \geq w \, f} \bar{x}_{i,w} < 1 \).

If \( 0 < \sum_{i \geq w \, f} \bar{x}_{i,w} < 1 \), then \( \sum_{i < w \, f} \bar{x}_{i,w} > 0 \). Since \( \bar{x} \in SS(P^\mu) \), then by Corollary \[ there exist \( \mu^p \) stable matchings in \( (F, W, P^\mu) \) and \( \alpha_p \) real numbers, such that
\[
\bar{x} = \sum_{p=1}^k \alpha_p \mu^p, \quad \text{with} \quad 0 < \alpha_p < 1 \quad \text{and} \quad \sum_{p=1}^k \alpha_p = 1.
\]

Since \( \sum_{i \geq w \, f} \bar{x}_{i,w} > 0 \), then there exists a stable matching \( \mu^p \) for some \( p = 1, ..., k \) such that \( \sum_{i < w \, f} \bar{x}_{i,w}^\mu = 1 \). Then, since \( (f, w) \in A(P) \), the firm \( f \) does not fill its quota, and \( \sum_{i < w \, f} \bar{x}_{i,w}^\mu = 1 \), we have that \( \mu^p(w) < w \, f \). Hence, \( (f, w) \) is a blocking pair for \( \mu^p \) for some \( p = 1, ..., k \). Thus, this case does not occur.

Then \( \sum_{i \geq w \, f} \bar{x}_{i,w} = 1 \), therefore \( \bar{x} \) fulfills Condition (11).

b) If the firm \( f \) fill its quota, then we will consider the following 3 sub-cases: Without loss of generality, we assume that \( \mu(f) = \{w_1, ..., w_q\} \) and \( \mu_W(f) = \{w'_1, ..., w'_q\} \) and \( w_l > f \, w_{l+1} \) and \( w'_l > f \, w'_{l+1} \) for \( l = 1, ..., q-1 \). Notice that,
\( \mu(f) \cap \mu_W(f) \) is not necessarily empty.

b1) \( w_1 >_f w >_f w_q \).

Then, we have that
\[
\sum_{j \geq f} x_{f,j}^\mu \leq q_f,
\]

since all stable matchings fulfill Condition (6), then
\[
\sum_{i \geq w} x_{i,w}^\mu = 1.
\]

But \( x^\mu \succeq_P \bar{x} \), and by Lemma (11), we have that \( \bar{x} \succeq_W x^\mu \), then
\[
1 = \sum_{i \geq w} x_{i,w}^\mu \leq \sum_{i \geq w} \bar{x}_{i,w} \leq 1,
\]

then
\[
\sum_{i \geq w} \bar{x}_{i,w} = 1,
\]
and
\[
\left[ q_f - \sum_{j \geq f} \bar{x}_{f,j} \right] \cdot \left[ 1 - \sum_{i \geq w} \bar{x}_{i,w} \right] = 0.
\]

b2) \( w_q >_f w >_f w'_q \).

Since \( (f, w) \not\in A(P^\mu) \), then \( f \) was eliminated from the worker \( w \)'s preference list \( P^\mu \). Then, for the worker \( w \) we have that \( f >_w \mu_W(w) \) or \( \mu(w) >_w f \).

If \( f >_w \mu_W \), then the pair \( (f, w) \) blocks the matching \( \mu_W \), then \( \mu(w) >_w f \).

Therefore, by Lemma (11) we have that \( \bar{x} \succeq_W x^\mu \). Hence,
\[
\sum_{i \geq w} \bar{x}_{i,w} \geq \sum_{i \geq w} x_{i,w}^\mu = x_{\mu(w),w}^\mu = 1.
\]

Since \( \bar{x} \) satisfies linear inequality (2), we have that \( \sum_{i \geq w} \bar{x}_{i,w} = 1 \), then
\[
\left[ q_f - \sum_{j \geq f} \bar{x}_{f,j} \right] \cdot \left[ 1 - \sum_{i \geq w} \bar{x}_{i,w} \right] = 0.
\]

b3) \( w'_q >_f w \).

By Lemma (11) we have \( \bar{x} \succeq_P x^\mu_W \). Hence,
\[
\sum_{j \geq f} \bar{x}_{f,j} \geq \sum_{j \geq f} x_{f,j}^{\mu_W} = q_f.
\]

Since \( \bar{x} \) satisfies linear inequality (1), we have that \( \sum_{j \geq f} \bar{x}_{f,j} = q_f \), then
\[
\left[ q_f - \sum_{j \geq f} \bar{x}_{f,j} \right] \cdot \left[ 1 - \sum_{i \geq w} \bar{x}_{i,w} \right] = 0.
\]
From cases 1 and 2, we conclude that \( \bar{x} \) is a strongly stable fractional matching for the many-to-one matching market \((F, W, P)\).

**Lemma 11.** Let \((F, W, P)\) be a many-to-one matching market. Let \( \bar{x} \) be a strongly stable fractional matching. Then, \( x^{\mu_F} \succeq_f \bar{x} \succeq_f x^{\mu_W} \) for all \( f \in F \) and \( x^{\mu_W} \succeq_w \bar{x} \succeq_w x^{\mu_F} \) for all \( w \in W \).

**Proof.** Let \((F, W, P)\) be a many-to-one matching market. Let \( \bar{x} \) be a strongly stable fractional matching. Since \( \bar{x} \) is a convex combination of stable matchings, we have that exists \( \mu^k \in S(P) \) and real numbers \( \alpha_k \) such that

\[
\bar{x} = \sum_{k=1}^{t} \alpha_k x^{\mu^k},
\]

with \( 0 \leq \alpha_k \leq 1 \), and \( \sum_{k=1}^{t} \alpha_k = 1 \). Since \( \mu^k >_F \mu^j >_F \mu_W \) for all \( \mu^k \in S(P) \), then for \( f \in F \) we have that

\[
\sum_{j \geq f} x^{\mu^k}_{f,j} = \sum_{j \geq f} \left( \sum_{k=1}^{t} \alpha_k x^{\mu^k}_{f,j} \right) = \sum_{k=1}^{t} \alpha_k \left( \sum_{j \geq f} x^{\mu^k}_{f,j} \right) \geq \sum_{k=1}^{t} \alpha_k \left( \sum_{j \geq f} x^{\mu^j}_{f,j} \right) = \sum_{j \geq f} \bar{x}_{f,j},
\]

for all \( w \in W \). Then \( x^{\mu_F} \succeq_f \bar{x} \).

To prove that \( \bar{x} \succeq_f x^{\mu_W} \),

\[
\sum_{j \geq f} \bar{x}_{f,j} = \sum_{j \geq f} \left( \sum_{k=1}^{t} \alpha_k x^{\mu^k}_{f,j} \right) = \sum_{k=1}^{t} \alpha_k \left( \sum_{j \geq f} x^{\mu^k}_{f,j} \right) \geq \sum_{k=1}^{t} \alpha_k \left( \sum_{j \geq f} x^{\mu^w}_{f,j} \right) = \sum_{j \geq f} x^{\mu_W}_{f,j},
\]

for all \( w \in W \). Then \( \bar{x} \succeq_f x^{\mu_W} \).

A similar argument proves that \( x^{\mu_W} \succeq_w \bar{x} \succeq_w x^{\mu_F} \). \(\)

**Lemma 12.** Let \((F, W, P)\) be a many-to-one matching market. Let \( \bar{x} \) be a strongly stable fractional matching and \( \bar{x} \neq x^{\mu_F} \). Let \( \alpha = \min\{\bar{x}_{f,w} : (f, w) \in \text{supp}(x^{\mu_F})\} \). Then, \( y \) defined as:

\[
y = \frac{\bar{x} - \alpha x^{\mu_F}}{1 - \alpha}
\]

is a strongly stable fractional matching, such that \( \text{supp}(y) \subset \text{supp}(\bar{x}) \).

**Proof.** Let \( \bar{x} \) be a strongly stable fractional matching. Then, by Lemma \(\) we have that \( \mu_F \) is a stable matching. By Definition \(\) we have that \( \text{supp}(x^{\mu_F}) \subset \text{supp}(\bar{x}) \), and by the fact that \( \bar{x} \neq x^{\mu_F} \) we have that

\[
\bar{x} = \alpha x^{\mu_F} + (1 - \alpha)y,
\]

We need to prove that

\[
y = \frac{\bar{x} - \alpha x^{\mu_F}}{1 - \alpha}
\]

is a strongly stable fractional matching. That is, \( y \) satisfies linear inequalities (1),(2),(4),(5) and Condition \(\).
From \( \bar{x} \neq x^{\mu_x} \), we have that \( \alpha > 0 \). From definition of \( \alpha \), we have that \( \alpha < 1 \).

Assume that \( \alpha = \bar{x}_{f,w} \), with \( \bar{w} \in C_f^\mu(\bar{x}) \).

- **Inequality (1)**

Following from the definition of \( y \) and the definition of \( \alpha \), we have that:

If \( |C_f^\mu(\bar{x})| \geq q_f \), then

\[
\sum_{j \in W} \bar{x}_{f,j} - \alpha \sum_{j \in W} x_{f,j}^{\mu_x} = \sum_{i \in W} x_{f,j} - \alpha q_f \leq q_f - \alpha q_f.
\]

Therefore,

\[
\sum_{j \in W} y_{f,j} = \frac{1}{1 - \alpha} \left[ \sum_{j \in W} \bar{x}_{f,j} - \alpha \sum_{j \in W} x_{f,j}^{\mu_x} \right] \leq q_f.
\]

If \( |C_f^\mu(\bar{x})| = r < q_f \), then

\[
\sum_{j \in W} \bar{x}_{f,j} - \alpha \sum_{j \in W} x_{f,j}^{\mu_x} = \sum_{i \in W} \bar{x}_{f,j} - \alpha r \leq r - \alpha r = r(1 - \alpha) < q_f(1 - \alpha).
\]

Then,

\[
\sum_{j \in W} y_{f,j} = \frac{1}{1 - \alpha} \left[ \sum_{j \in W} \bar{x}_{f,j} - \alpha \sum_{j \in W} x_{f,j}^{\mu_x} \right] \leq q_f.
\]

That is, \( y \) satisfy linear inequality (1).

- **Inequality (2)**

Similar argument that is used for Inequality (1), proves that \( y \) satisfy linear inequality (2).

- **Inequality (4)**

If \( (f, w) \in supp(\bar{x}) \), by definition of \( \mu_x \), we shall consider two cases:

- If \( (f, w) \not\in supp(x^{\mu_x}) \), that is, \( x_{f,w}^{\mu_x} = 0 \). Then, \( y_{f,w} = \frac{\bar{x}_{f,w}}{1 - \alpha} = 0 \).
- If \( (f, w) \in supp(x^{\mu_x}) \), that is, \( x_{f,w}^{\mu_x} = 1 \) and also \( x_{f,w}^{\mu_x} = 1 \).

Then, \( y_{f,w} = \frac{\bar{x}_{f,w} - \alpha x_{f,w}^{\mu_x}}{1 - \alpha} = \frac{\bar{x}_{f,w} - \alpha}{1 - \alpha} \geq 0 \). Then, for \( (f, w) \in A \), \( y_{f,w} \geq 0 \), that is \( y \) satisfies linear inequality (4).

- **Inequality (5)**

Inequality (5) it is straight forward satisfy form definition of \( y \).

- **Condition (6)**

Since, \( x^{\mu_x} \) and \( y \) satisfies linear inequalities (1), (2), (4) and (5), and \( \bar{x} = \alpha x^{\mu_x} + (1 - \alpha)y \), by Lemma 2 the fractional matching \( y \) is a strongly stable fractional matching.

Since \( supp(x^{\mu_x}) \subseteq supp(\bar{x}) \), we have that \( supp(y) \subseteq supp(\bar{x}) \). Moreover, since \( y_{f,w} = 0 \), and \( x_{f,w}^{\mu_x} = 1 \), then \( supp(y) \subset supp(\bar{x}) \).

Note that here we use "\( \subset \)" to denote the strict inclusion; that is, \( A \subset B \) means that \( A \) is a proper subset of \( B \).
Lemma 13. Let \((F, W, P)\) be a many-to-one matching market. Let \(\bar{x}\) be a strongly stable fractional matching. Then, \(\bar{x}\) can be written as:

\[
x = \sum_{i=1}^{k} \alpha_i x^{\mu_i}, \quad 0 < \alpha_i \leq 1, \quad \sum_{i=1}^{k} \alpha_i = 1,
\]

with \(\mu^l \in S(P)\) for \(l = 1, \ldots, k\) and \(\mu^1 >_F \mu^2 >_F \ldots >_F \mu^k\).

Proof. Let \((F, W, P)\) be a many-to-one matching market and let \(\bar{x}\) be a strongly stable fractional matching.

Denote by \(\mu^1 = \mu_x\), where \(\mu_x\) is defined by Definition 8 and by Lemma 9 is stable.

If \(\bar{x} = x^{\mu^1}\) (i.e., \(\bar{x}\) is a stable matching), then \(\bar{x}\) can be written according to (12) for \(k = 1\), and \(\alpha_1 = 1\).

If \(\bar{x} \neq x^{\mu^1}\) (i.e., \(\bar{x}\) is not a stable matching). Then, according to Lemma 12, there is a strongly stable fractional matching, \(x^2\), defined by

\[
x^2 = \frac{\bar{x} - \alpha_1^1 x^{\mu_1}}{1 - \alpha_1^1},
\]

for some \(0 < \alpha_1^1 < 1\) and \(\text{supp} (x^2) \subset \text{supp} (\bar{x})\). Then,

\[
\bar{x} = (1 - \alpha_1^1) x^3 + \alpha_1^1 x^{\mu_1}.
\]

for \(0 < \alpha_1^1 < 1\) and \(\text{supp} (x^{\mu_1}) \subset \text{supp} (\bar{x})\).

For the strongly stable fractional matching \(x^2\), denote by \(\mu^2 = \mu_{x^2}\) to the stable matching according to Definition 8.

Notice that, from the fact that \(\text{supp} (x^{\mu_1}) \subset \text{supp} (\bar{x})\), \(\text{supp} (x^2) \subset \text{supp} (\bar{x})\) and definitions of \(\mu^1\) and \(x^2\), we can conclude that \(\mu^1 >_F \mu^2\).

If \(x^2 = x^{\mu^2}\) (i.e., \(x^2\) is a stable matching), then \(\bar{x}\) can be written according to (12).

If \(x^2 \neq x^{\mu^2}\) (i.e., \(x^2\) is not a stable matching), again by Lemma 12 there is a strongly stable fractional matching \(x^3\), defined by

\[
x^3 = \frac{x^2 - \alpha_2^1 x^{\mu_2}}{1 - \alpha_2^1},
\]

for some \(0 < \alpha_2^1 < 1\) and \(\text{supp} (x^3) \subset \text{supp} (x^2)\). That is,

\[
x^2 = (1 - \alpha_2^1) x^3 + \alpha_2^1 x^{\mu_2}.
\]

Since \(0 < \alpha_2^1 < 1\), we have that \(\text{supp} (x^{\mu_2}) \subset \text{supp} (x^2)\). Denote by \(\mu^3 = \mu_{x^3}\) to the stable matching according to Definition 8. Since \(\text{supp} (x^3) \subset \text{supp} (x^2)\), we have that \(\mu^2 >_F \mu_3\). Then, \(\mu^1 >_F \mu^2 >_F \mu^3\).

If \(x^3 = x^{\mu^3}\) (i.e., \(x^3\) is a stable matching), from equalities (13) and (14) we have that

\[
\bar{x} = (1 - \alpha_1^1) x^2 + \alpha_1^1 x^{\mu_1} = (1 - \alpha_1^1) \left( (1 - \alpha_2^1) x^3 + \alpha_2^1 x^{\mu_2} \right) + \alpha_1^1 x^{\mu_1}
\]

\[
= (1 - \alpha_1^1) (1 - \alpha_2^1) x^3 + (1 - \alpha_1^1) \alpha_2^1 x^{\mu_2} + \alpha_1^1 x^{\mu_1}.
\]
Then $\bar{x}$ can be written according to (12) for $k = 3$, with $\alpha_1 = \alpha_1', \alpha_2 = (1 - \alpha_1')\alpha_2'$, and $\alpha_3 = (1 - \alpha_1')(1 - \alpha_2')$. Notice that $\alpha_1 + \alpha_2 + \alpha_3 = 1$.

If $x^3 \neq x^{\mu_3}$ (i.e., $x^3$ is not a stable matching), then we continue this procedure. The finiteness of the $\text{supp}(\bar{x})$ guarantees that this procedure ends by constructing a stable matching. This proves that $\bar{x}$ can be written according to (12) for some $k \geq 1$. \hfill \Box

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