Finite VEVs from a Large Distance Vacuum Wave Functional

Alfonso Jaramillo 
Departamento de Física Teorica and I.F.I.C.
Centro Mixto Universidad de Valencia – C.S.I.C.
E-46100 Burjassot (Valencia), Spain.

Paul Mansfield 
Department of Mathematical Sciences
University of Durham
South Road
Durham, DH1 3LE, England.

Abstract

We show how to compute vacuum expectation values from derivative expansions of the vacuum wave functional. Such expansions appear to be valid only for slowly varying fields, but by exploiting analyticity in a complex scale parameter we can reconstruct the contribution from rapidly varying fields.

\[a\] Alfonso.Jaramillo@uv.es
\[b\] P.R.W.Mansfield@durham.ac.uk
1 Introduction

Although canonical quantisation provides the basic formalism of quantum field theory, the corresponding Schrödinger Representation, in which the field operators are diagonal, has not received commensurate attention. This is partly due to the popularity of the functional integral which displays space-time symmetries manifestly, and partly because the existence of wave functionals was only shown by Symanzik as late as 1981, [1]. Nonetheless there has been growing interest in the subject as a result of the search for new tools in field theory; and also because the Schrödinger Representation is implicit in much recent work on field theories defined on space-times with boundaries, see for example [2].

The vacuum wave-functional (VWF), $\Psi_0$, may be constructed as a functional integral over the Euclidean space-time $x^0 < 0$ which has the quantisation surface $x^0 = 0$ as boundary. $W \equiv \log(\Psi)$ is then a functional of the values the field takes on the boundary. As we will see, these boundary values act as source terms in the functional integral. Symanzik showed that, at least in perturbation theory, this functional has a finite limit as the cut-off is removed, subject to the inclusion of the usual counter-terms together with additional ones localised to $x^0 = 0$ which result in the boundary values of the field undergoing an additional field renormalisation, [1]. These boundary counter-terms are absent in fewer than three dimensions, as they are for Yang-Mills theory in four dimensions due to gauge invariance. He also proved the existence of the Schrödinger equation for $\phi^4$ in four dimensions.

In [3] it was shown that the vacuum functional of the scaled Yang-Mills field $A^s(x) \equiv A(x/\sqrt{s})/\sqrt{s}$ extends to an analytic function of $s$ in the complex $s$-plane with the negative real axis removed. This also applies to scalar field theory. This allows the vacuum functional to be reconstructed for arbitrary $A(x)$ in terms of the scaled field $A^s(x)$ for large $s$ using Cauchy’s theorem. The scaled field is slowly varying for large $s$, and for such a field we would expect to be able to expand $W$ in powers of derivatives divided by the lightest glueball mass (in our appendix C we give an argument to justify the possibility of performing this local expansion). Thus $W$ can be obtained for arbitrary $A(x)$ from a knowledge of this derivative expansion. The existence of this expansion was originally considered by Greensite [4] who found the leading terms from a Monte-Carlo simulation of lattice gauge theory. We emphasize that this procedure is valid for arbitrary $A(x)$, it does not amount to restricting attention to slowly varying field configurations, it is simply a way of parametrising the function space of sources in the functional integral in terms of a derivative expansion.

$\Psi$ satisfies the Schrödinger equation, but this takes a special form when using the derivative expansion for $W$ due to the employment of Cauchy’s theorem. This was described in [3] where a non-perturbative approximation scheme was outlined. The semiclassical expansion of this equation was shown to agree with the direct semiclassical evaluation of $W$ via Feynman diagrams in [3], and extended to Yang-Mills theory in [3] where the beta-function was correctly reproduced from the derivative.
expansion. This is not as obvious as it might appear because a naive insertion of a local expansion into the usual Schrödinger equation will not converge for momenta greater than the mass of the lightest particle and so will not lead to the correct behavior as the ultra-violet cut-off in that equation is removed. What was missing from this work was a method for constructing vacuum expectation values (VEVs) directly from the derivative expansion, which is the subject of this paper. We will show that when these are written as functional integrals over the boundary values of fields they are analytic in an ultra-violet momentum cut-off in the plane cut as above. Again, Cauchy’s theorem may be used to compute VEVs for large cut-off from a knowledge of the corresponding functional integral for small cut-off, which in turn can be obtained from the derivative expansion, or some other systematic approach. Notice that if we try to compute the VEV in the most obvious way, by expanding the logarithm of the vacuum functional in a local expansion and doing the usual perturbative approach then we would get a sum of contractions which would, in general, lead to ultra-violet divergences of all orders. These divergences cannot be absorbed into renormalisation of the wave-functional to obtain finite VEVs because the wave-functional is already finite according to Symanzik’s work. (The only possibility for such renormalisation is if the inner product involves a non-trivial weight functional with coefficients that can be chosen to cancel divergences. The form of these weights is determined by the hermiticity of the Hamiltonian operator, and is very restricted. In 1+1 dimensional scalar field theory, and Yang-Mills theory such weights are absent.) The origin of these ultra-violet divergences is that we would be attempting to compute the integral for field configurations beyond the convergence radius of the derivative expansion, and this is inconsistent. In this paper we propose a method for computing VEVs in which the same expansion is employed but with a cut-off that lies inside the convergence radius of the series. Typically this means that the cut-off is smaller than the mass of the lightest particle. It therefore does not appear to be an ultraviolet cut-off. However we will be able to send the cut-off to infinity in this expansion because, as we will show, that VEVs are analytic in the cut-off when we continue to complex values. Thus we can use Cauchy’s theorem to relate the large cut-off behaviour which we need to compute, to the small cut-off behaviour which we can calculate using the local expansion of the functional integral.

We concentrate throughout on the toy model of $\phi^4$-theory in 1+1 dimensions as this is particularly straightforward given the absence of boundary counter-terms resulting in there being no wave-function renormalization. The absence of boundary counter-terms is shared by Yang-Mills theory which also has a correspondingly simple Schrödinger equation. Super-renormalizable theories are, in any case, of interest in their own right by virtue of their connection with integrable theories and with String Theory. We will only discuss the VEV of operators which will be diagonal in field configuration space (we do not expect that our conclusions will change if we consider more general operators, $A(\pi, \phi)$, as we can see in [3] where the analyticity of $H\Psi_0$
is shown). Finally, with an analytical continuation it is often difficult to estimate truncation errors, but we will see that they can be controlled.

Several authors [1, 7, 8, 9, 10, 11] have devised perturbative and non-perturbative approches to compute the vacuum wave-functional. In section 2 we describe its representation as a functional integral. In the next section we give a general discussion of the construction of VEVs in the Schrödinger Representation in terms of Feynman diagrams. We display the mechanism whereby the Feynman diagram expansion of $W$, which makes use of a propagator on the space-time $x^0 < 0$ with Dirichlet boundary conditions, leads to the usual Feynman diagrams for VEVs on the full space-time with the standard propagator. We end the section by giving an operator approach which uses the results of appendix A. In section 4 we translate the calculations of the previous section into the language of first quantisation in which the vacuum functional can be expressed in terms of random paths that are reflected at the quantisation surface. In section 5 we describe the analyticity of VWF and VEVs and describe the recombination of the series in the cut-off. Section 6 illustrates our method in the simpler context of non-relativistic quantum mechanics and we have left to the appendix B mathematical details of our method of analytic continuation. In the section 7 we will discuss the computation of the equal-time two point function through diagrams in a dimensionally reduced effective theory. The last section is devoted to our conclusions.

2 Representations for the vacuum wave functional

The VWF is the inner product $\langle \varphi | 0 \rangle$ of the vacuum $| 0 \rangle$ and an eigenbra of the field operator $\hat{\phi}$ restricted to the quantization surface (which we take to be $t = 0$) belonging to the eigenvalue $\phi(x)$:

$$\langle \varphi | \phi(x, 0) = \phi(x) \langle \varphi |$$

(1)

The $\varphi$-dependence of the eigenbra may be made explicit by writing

$$\langle \varphi | = \langle D | \exp(i \int d x \: \varphi(x) \hat{\pi}(x, 0))$$

(2)

where $\langle D |$ is annihilated by $\hat{\phi}(x, 0)$, i.e. it is the state $\langle \varphi = 0 |$, $D$ stands for Dirichlet, and $\hat{\pi}$ is canonically conjugate to $\hat{\phi}$. The canonical commutation relations then yield

$$\frac{\delta}{\delta \varphi(x)} \langle \varphi | = i \langle \varphi | \hat{\pi}(x)$$

(3)

if we apply the Euclidean time evolution operator $\exp(-\hat{H}T)$ to any state, $|v\rangle$, not orthogonal to the vacuum, then for large times

$$\exp(-\hat{H}T) | v \rangle \sim | 0 \rangle e^{-E_0 T} \langle 0 | v \rangle \quad (T \to \infty)$$

(4)

where $E_0$ is the energy of the vacuum. Thus

$$\Psi[\varphi] = \lim_{T \to \infty} N \langle D | e^{i \int d x \: \varphi(x) \hat{\pi}(x, 0)} e^{-\hat{H}T} | v \rangle$$

(5)
Where $N$ is a normalization constant depending on $|\psi\rangle$. Using $\hat{\pi} = \frac{\partial}{\partial \varphi}$ this, as we will explicitly show later, may be written as the functional integral

$$\int \mathcal{D}\varphi \ e^{-S_E[\varphi]} \int dx \ (\varphi(x)\dot{\varphi}(x,0)+\Lambda \varphi^2(x))$$

(6)

where $S_E$ is the Euclidean action for the space $t \leq 0$. $\Lambda$ is a regularization of $\delta(0)$ that arises from the differentiation of the time ordered product that is represented by the functional integral, i.e.

$$T(\hat{\pi}(x,t)\hat{\pi}(x',t')) = \frac{\partial^2}{\partial t \partial t'} T(\hat{\varphi}(x,t)\hat{\varphi}(x',t')) - i\delta(x-x')\delta(t-t')$$

(7)

On the boundary $t = 0$ the integration variable $\varphi$ should vanish, reflecting the fact that $\langle \mathcal{D}\rangle \hat{\varphi}(x,0) = 0$.

Alternatively, we can obtain this path integral representation for the VWF, by beginning with

$$\Psi_0[\varphi] = N \lim_{\tau \to \infty} e^{\tau E_0} \int \mathcal{D}\varphi(x,t) \ e^{-S_E[\varphi,\dot{\varphi}]} \bigg|_{\varphi(x,0)=0}$$

(8)

where $\varphi(x, -\infty)$ can be anything, and performing a functional change of variables in the path integral in such a way that we do not have the $\phi(x)$-field dependence in the integration limit. Our change of variables is formally

$$\tilde{\varphi}(x,t) = \varphi(x,t) - 2\theta(t)\phi(x)$$

$$\mathcal{D}\tilde{\varphi} = \mathcal{D}\varphi$$

(9)

$$S_E[\varphi,\tilde{\varphi}] = \int_{-T}^{0} dt \int dx \mathcal{L} (\tilde{\varphi} + 2\theta(t)\phi, \dot{\tilde{\varphi}} + 2\delta(t)\phi)$$

where $\theta$ is the step function and we take $\theta(0) = \frac{1}{2}$. Naively the $\theta$ terms do not contribute to the potential. Therefore our path integral on removing the tilde can be written as

$$\int \mathcal{D}\varphi(x,t) \ e^{-S_E[\varphi,\dot{\varphi}+2\delta(t)\phi]} \bigg|_{\varphi(x,0)=0}$$

(10)

In the scalar case we will have

$$S_E = \int_{-\infty}^{0} dt \int dx \left\{ \frac{1}{2} (\dot{\varphi})^2 + V(\varphi) \right\} + \int dx \ \dot{\varphi}(x,0)\phi + \int dx \ \delta(0)\phi^2$$

(11)

Therefore, the logarithm of the VWF $(W[\phi])$ is given by the sum of connected diagrams constructed from the new action (11) and with a boundary at $t = 0$ where the field vanishes. The new action contains a source term on the boundary and a $\delta(0)$ term (to be regulated) coupled to the boundary fields. This argument has been rather too formal. To be more careful we should smooth the $\theta$ functions in (9), taking them to
be non-constant in a region of size $1/\Lambda$, and this will regulate $\delta(0)$. So we replace $\theta$ by $\theta_\Lambda$. With a cutoff this function will be given by

$$\theta_\Lambda(t) = \frac{i}{2\pi} \int_{-\Lambda}^{\Lambda} d\omega \frac{1}{\omega + i\epsilon} e^{-i\omega t}, \quad (12)$$

and we have $\theta_\Lambda(t) = \theta(t) - \frac{1}{\pi} \frac{\cos(\Delta t)}{\Delta t} (1 + O(\frac{1}{\Delta t})).$ If we had already a cutoff in space-time then the $\theta$ functions will be regulated by this cutoff. Therefore, because of the appearance of the $\delta(0)$ terms, it is necessary to regulate both the space-like and the time-like dimensions. If we regulate the space-like dimensions keeping unregulated the time-like direction (for instance using dimensional regularization for the space-like dimensions, as is done by Symanzik [1]) then we need to introduce another regulator for the time direction. Assuming that we have a cutoff in space-time, then the terms proportional to some power of $\theta_\Lambda$, in $V(\varphi + 2\theta_\Lambda(t)\dot{\phi})$, will vanish when $\Lambda \to \infty$ because the time integral (which occurs in the definition of the action) will only be non-zero in a region of size $1/\Lambda$ around the endpoint $t = 0$. When we compute the perturbative quantum corrections we may get some divergent loop diagram that may compensate for the vanishing contribution of the $\theta_\Lambda$-terms insertions. But for that to happen we need a linear divergence (because $\theta_\Lambda(t) \sim 1/\Lambda$) or a time derivative acting on the $\theta_\Lambda$.

In $1 + 1$ dimensions we do not get linear divergences and in our case (where we do not have derivative interactions) we do not have vertices with both $\theta_\Lambda$ and $\dot{\phi}$, so we can ignore such terms.

We can also see how to get \textup{(3)} from \textup{(2)} together with the $\theta_\Lambda$-terms within the canonical operator formalism. We will use the following identity

$$e^A e^B = T e^{\int_{-1}^{0} dt e^{Bt}(B+A)e^{-Bt}} \quad (13)$$

where $A$ and $B$ are arbitrary matrices (or operators) and $T$ is the time-ordering operator. As usual, the $T$-ordering implies that the first term in the exponential is to be thought of as $B(t)$ and then, at the end, set to a constant. This relation can be derived by considering the operator $U(t) \equiv e^{At} e^{Bt}$, then we calculate $\frac{d}{dt} U(t)$ and then we integrate it back to get the integral equation

$$U(t) = 1 + \int_{0}^{t} dt' U(t') (B + A(-t'))$$

with $A(t) \equiv e^{Bt} A e^{-Bt}$. Once we solve the integral equation in terms of a $T$-ordered exponential, we set $t = 1$ and we get \textup{(13)}.

Now we will consider

$$\langle D | e^{i\int d\varphi(x)\hat{\pi}(x,0)} e^{-\int \varphi(x)\hat{\pi}(x,0)} e^{-\epsilon\hat{H} e^{-(T-\epsilon)\hat{H}}} | \nu \rangle = \langle D | e^{i\int d\varphi(x)\hat{\pi}(x,0)} e^{-\epsilon\hat{H} e^{-(T-\epsilon)\hat{H}}} | \nu \rangle \quad (14)$$

and we will use our relation \textup{(13)} to combine the $e^{i\int d\varphi(x)\hat{\pi}(x,0)} e^{-\epsilon\hat{H} e^{-(T-\epsilon)\hat{H}}}$ term into a single exponential, but before that we will follow some intermediate steps. Firstly we take (and to shorten the notation $\int d\varphi(x)\hat{\pi}(x,0) \equiv \phi\hat{\pi}$)

$$e^{-\epsilon\hat{H}} e^{i\phi\hat{\pi}} = T e^{-\int_{-1}^{0} \epsilon dt \left( \frac{1}{\epsilon} \phi\hat{\pi} + e^{i\phi\hat{\pi}} \hat{H} e^{-i\phi\hat{\pi}} \right)} = T e^{-\int_{-1}^{0} \epsilon dt \left( \frac{1}{\epsilon} \phi\hat{\pi} + \hat{H}(\hat{\pi},\dot{\phi} + \phi\hat{L}) \right)} \quad (15)$$
which (after hermitian conjugate, $\phi \to -\phi$, $t \to -t$ and later $t \to t - \epsilon$) will become

$$e^{i\phi\hat{\pi}} e^{-\epsilon \hat{H}} = Te^{-\int_{-\epsilon}^{0} dt \left( \frac{i}{\epsilon} \phi \hat{\pi} + \hat{H}(\hat{\pi}, \hat{\phi} + (\frac{1}{t} + 1)\phi) \right)}$$

Therefore

$$\langle \phi | e^{-T \hat{H}} | \chi \rangle = \langle D | e^{i\phi\hat{\pi}} e^{-T \hat{H}} | \chi \rangle = \langle D | Te^{-\int_{-T}^{0} dt' \hat{H}'(t')} | \chi \rangle$$

with $\hat{H}'(t) \equiv \hat{H}(\hat{\pi}, \hat{\phi} + 2\theta_\epsilon(t)\phi) - 2i\delta_\epsilon(t)\phi\hat{\pi}$, where the $\theta_\epsilon(t)$ is defined by

$$\theta_\epsilon(t) = \begin{cases} 0 & \text{if } t \leq -\epsilon \\ \frac{t}{\epsilon} + \frac{1}{2} & \text{if } -\epsilon < t < \epsilon \\ 1 & \text{if } t \geq \epsilon \end{cases}$$

and the $\delta_\epsilon(t)$ is obtained by taking its derivative. Notice that we needed to put a factor 2 in front because $\int_{-\epsilon}^{0} dt \frac{1}{t} = 1 = 2 \int_{-\epsilon}^{0} dt \delta_\epsilon(t)$. With such a definition of the Hamiltonian (we have $\hat{H}(t) = \hat{H}$ for $t < -\epsilon$) the time evolution until $t < -\epsilon$ is reproduced by the third exponential in the r.h.s. of eq. (14). As we see, we have the same result that we obtained by shifting the integration variable in the path-integral.

The path-integral representation now follows from the standard construction. Just the last step ($t < -\epsilon$) is different from usual. Let us illustrate it for the case of quantum mechanics

$$\langle q | e^{-\epsilon(\frac{1}{2}\hat{p}^2 + V(q))} | q_1 \rangle = \langle q = 0 | e^{-\epsilon(\frac{1}{2}\hat{p}^2 - \frac{\epsilon}{2} \hat{\phi} + V(q + 2\theta_\epsilon(t)q))} | q_1 \rangle = N e^{-\frac{\epsilon}{2}(\frac{2\theta_\epsilon(t)q}{2})^2 - V(q + 2\theta_\epsilon(t)q)} = N e^{-S(q, \dot{q})}$$

where the $\frac{2\theta_\epsilon(t)}{2}$ (in the last equality) is interpreted as $\dot{q}_1$ and $2$ as $2\delta_\epsilon(t)\dot{q}$. As we see we get the same action as before. Finally we can write the VWF as

$$\Psi[\phi] = \lim_{T \to \infty} N \langle D | Te^{-\int_{-T}^{0} dt' \hat{H}'(t')} | \phi \rangle = \lim_{T \to \infty} \int_{\phi, t=-T}^{0, t=0} D\chi(x, t) e^{-S_E(\chi(x, t) + 2\theta_\epsilon(t)\phi(x), \dot{\chi}(x, t) + 2\delta_\epsilon(t)\dot{\phi}(x))}$$

Notice that we could have got the following relation

$$e^{i\phi\hat{\pi}} e^{-T \hat{H}} = Te^{-\int_{-\epsilon}^{0} dt \left( \frac{i}{\epsilon} \phi \hat{\pi} + \hat{H}(\hat{\pi}, \hat{\phi} + (\frac{1}{t} + 1)\phi) \right)}$$

which could be derived with (11) using $2\theta(t) = 1 + t/T$. Lastly, we may also construct a relation

$$e^{i\phi\hat{\pi}} e^{-T \hat{H}} = Te^{-\int_{-T}^{0} dt \left( \hat{H} - \frac{i}{\epsilon} \phi \hat{\pi} + \frac{1}{t} \phi \hat{\pi} \right)}$$

with $\hat{\pi}(t) = e^{-i\hat{H} t} \hat{\pi} e^{i\hat{H} t}$ but this time we cannot use (11) to obtain the corresponding path integral version. We have seen from a range of methods that the (time-independent) VWF will be given by

$$\Psi[\phi(x)] = N \int D\phi(x, t) e^{-S_E - \int dx \dot{\phi}(x) \phi(x) - \int dx \delta_\epsilon(t) \phi^2} \bigg|_{\phi(x, 0) = 0}$$
We can also obtain a path integral representation for the VWF (with \( \phi \)-independent boundary conditions) without any delta function by using a different shift in the field in (9). Consider shifting \( \phi \) by a solution to the free field equations denoted by \( \Omega(t) \phi \), where \( \Omega(t) \) will be defined by

\[
\Omega(t) = \frac{\sinh(t\sqrt{-\partial^2 + m^2})}{\sinh(\Lambda\sqrt{-\partial^2 + m^2})}.
\]

Thus \( S_0(\phi + \Omega(t) \phi) = S_0(\phi) + S_0(\Omega(t) \phi) \) which yields

\[
\int D\phi \, e^{-S_0(\phi)} \int V(\phi) \bigg|_{\phi(x,0)=0,\phi(x,\Lambda)=\phi(x)} = \int D\phi \, e^{-S_0(\phi)} \int V(\phi + \Omega(t) \phi) - S_0(\Omega(t) \phi) \bigg|_{\phi(x,0)=\phi(x,\Lambda)=0}.
\]

This provides an alternative derivation of the starting point used in [7] to set up a non-perturbative algorithm to compute the VWF by resumming the perturbative expansion of (24). Another possibility is to shift the field by a solution to the Euler-Lagrange equations of the full action with boundary conditions \( \phi'(x,0) = \phi \) and \( \phi'(x,-T) = 0 \) (where \( T \to \infty \)). In the background field technique a \( \phi'(x,t) = \langle \hat{\phi}(x,t) \rangle \phi \) (the expectation value is taken with the boundary condition \( \phi(x,0) = \phi \)) is taken (which at first order coincides with the one which minimises the action) and then the \( \Psi_0 \) gives the exponential of the effective action evaluated at the configuration \( \phi' \). This is the method used in [14] where the gauge theory case is analyzed and \( \phi' \) plays the role of their induced background field, although they do not expand in the background field, and in [9] where they study general scalar theories with curved boundaries using dimensional regularization for the space-time dimensions. In this way they give a method to use dimensional (space-time) regularization with the Schrödinger representation where the time is given by the coordinate perpendicular to the boundary of the \( d \)-dimensional space-time. Note that usually [1] the space-like dimensions are regulated differently (for example with dimensional regularization) from the time (for example by time-splitting).

We end this section by deriving the important path integral representation introduced by Symanzik [1], where the Dirichlet boundary conditions are only introduced through a boundary term. Again we can derive this representation in an easy way, following steps similar to those of eq. (3). Consider (as before, we understand the delta functions to be regularized)

\[
\int D\varphi(x,t) \, e^{-\int_0^t dt \int d\chi \{ \frac{1}{2}(\varphi')^2 + V(\varphi) \}} \int DC(x) \, e^{\int dx \delta(0)(C-\varphi)^2} = \int DC(x) \, e^{\int dx \delta(0)(C-\varphi)^2}
\]

where \( \varphi(x,0) = \phi(x) \) and \( \varphi(x,-\infty) = 0 \). Now absorb the denominator into a proportionality constant and commute the \( D\varphi \) and \( DC \) integrals. Shifting the variables
\[ \varphi(x, t) = \varphi(x, t) - 2\theta(t)\dot{\phi}(x) + 2\theta(t)C(x) \] gives

\[ N \int DC(x) D\varphi(x, t) e^{-\int_0^0 dt \int dx \left\{ \frac{1}{2} \varphi^2 + V(\varphi) + 2\dot{\varphi}(t)(\phi-C) + 2(\theta(t))^2 (\phi-C)^2 \right\}} e^{\int dx \delta(0)(\phi-C)^2} \tag{26} \]

where now \( \varphi(x, 0) = C(x) \) and \( \varphi(x, -\infty) = 0 \). After the cancellation of the exponential our VWF will be given by

\[ \Psi_0[\phi] = N' \int DC(x) D\varphi(x, t) e^{-\int_0^0 dt \int dx \varphi(x, 0)(\phi(x)-\varphi(x, 0))} , \tag{27} \]

where the \( \int dx \) integral has to be understood as \( \int dt \int dx 2\delta(t) \). We can now interpret \( \int DC(x) \int D\varphi(x, t) \) with the conditions \( \varphi(x, 0) = C(x) \) and \( \varphi(x, -\infty) = 0 \) as an ordinary \( \int D\varphi(x, t) \) with \( t \leq 0 \) and free boundary at \( t = 0 \). This formula agrees with Symanzik’s one (eq. (2.10) of [1]) but for an arbitrary potential. We expect that quantum corrections will renormalize it.

As Symanzik has discussed, placing source terms on the boundary leads to divergences, [1]. These appear in perturbation theory because the field is placed at the same point as the image charges that enforce the boundary conditions on propagators. In order to regulate these divergences we should split in time the fields (thus the fields in (28) will be defined at different ordered times, but with their \( |t| < \epsilon \) ). These divergences appear as the coefficient of local operators forming a boundary operator expansion [8, 12] (analogous to the usual operator product expansion). These coefficients will scale with some non-trivial dimension (in perturbation theory they are calculated as a series in the coupling), which (order by order in the coupling) will lead to logarithms of \( t \). In a perturbatively renormalizable theory (where the anomalous dimensions cannot make relevant an irrelevant operator) we need to only consider the operators of dimension smaller or equal than the product of operators at the boundary because \( \epsilon \to 0 \). Therefore, in order to use the Schrödinger representation (where \( \epsilon = 0 \) is implied), we should subtract the previous divergences (Wilson-boundary coefficients) in the original lagrangian. Extension of the validity of the boundary operator expansion to the non-perturbative domain suggests that we may have to consider additional relevant fields, we assume that this is not the case. Symanzik [1] showed that in \( \phi^4 \) theory in \( 3 + 1 \) dimensions only two counterterms where needed (\( \phi^2 \) and \( \phi\delta_0\phi \)) and conjectured that in a general perturbatively renormalizable field theory in four dimensions we only need operators of dimension less or equal than three. Because in lower dimensions we find fewer UV infinities, we will limit our discussion to scalar \( \phi^4 \) theory in \( 1 + 1 \) dimensions, where there is no wave-function renormalization and (28) is valid.

3 Feynman diagram expansion of VEVs

The purpose of this section is to describe how the Feynman diagram expansion of the VWF, \( \Psi_0[\phi] \), generates the usual diagrams of equal time Green’s functions via the
following relation
\[ \langle \phi(x_1, 0) \cdots \phi(x_n, 0) \rangle = \int \mathcal{D}\phi(x) \prod_n \phi(x_n) |\Psi_0[\phi]|^2 \]  

When we use representation (8) for \( \langle \phi | \Psi_0 \rangle \) in (28) we formally obtain the usual path integral for the time ordered product of field operators, as we should. If instead we first perturbatively compute the vacuum functionals using (22) then we get some unusual diagrams that generate an effective action which will be used to compute (28) with new propagators and (non-local) vertices. It is of interest to see how these combine to produce the usual result for the VEV. This will also allow us to use ordinary Feynman diagrams to compute (28) (which will be used in the section 7 to compute an equal-time propagator).

Given the comments at the end of the last section we take the Euclidean action in 1+1 dimensions as
\[ S_E = \int dxdt \left( \frac{1}{2}(\dot{\phi}^2 + \phi'^2 + M^2(\Lambda)\phi^2) + \frac{g}{4!}\phi^4 \right) \]  

In perturbation theory the only divergent diagrams with external legs are tadpoles which can be removed by normal ordering. This enables the dependence of \( M \) on the cut-off \( \Lambda \) to be calculated. We will regulate with a cut-off on the spatial component of the momentum (in 1+1 dimensions this is almost sufficient, as we will see), so that
\[ M^2(\Lambda) = M^2 + \hbar \delta M^2 - \frac{g\hbar}{4} \int_{p^2 < \Lambda} \frac{dp}{2\pi} \frac{1}{p^2 + M^2} = \]
\[ = M^2 + \hbar \delta M^2 - \frac{g\hbar}{4} \int_{p^2 < \Lambda} \frac{dp}{2\pi} \frac{1}{\sqrt{p^2 + M^2}} = \]
\[ = M^2 + \hbar M_c^2 \]  

The Feynman diagram expansion of the VWF can now be constructed so that its logarithm, \( W[\varphi] \), is a sum of connected diagrams in which \( \varphi \) is the source for \( \dot{\phi} \) restricted to the boundary \( t = 0 \). The major difference from usual Feynman diagrams encountered in free space is that the propagator satisfies Dirichlet boundary conditions, which means that it should vanish when either end lies on the boundary. Such a propagator is given by the method of images as
\[ G(x, t, y, t') = G_0(x, t, y, t') - G_I(x, t, y, t') \]  

where \( G_0 \) is the free-space propagator and
\[ G_I(x, t, y, t') = G_0(x, -t, y, t') = G_0(x, t, y, -t') \]

The tree-level diagrams that contribute up to the \( \varphi^6 \) term in \( W[\varphi] \) are given in figure (1). The heavy line denotes the boundary, \( t = 0 \), and the dots denote the differentiation.
Figure 1: Tree level contribution to $\Psi_0[\phi]$.  

with respect to $t$ that results from $\varphi$ being coupled to $\dot{\phi}$. When the propagator ends on the boundary this differentiation leads to $G_0$ and the image propagator $G_I$ contributing equally:

$$
\frac{\partial}{\partial t} G(x, t, y, t')|_{t=0} = 2 \frac{\partial}{\partial t} G_0(x, t, y, t')|_{t=0}
$$

(32)

The $\Lambda \varphi^2$ term in (3) cancels a divergence in the first diagram of figure (1) since this is

$$
\int dxdy \varphi(x)\varphi(y) \frac{\partial^2}{\partial t \partial t'} 2G_0(x, t, y, t')|_{t=t'=0} =
$$

(33)

$$
= \int \frac{dp}{2\pi^2} \tilde{\varphi}(p)\tilde{\varphi}(-p) \int dp_0 \frac{p_0^2}{p_0^2 + p^2 + m^2}
$$

and the $\Lambda \varphi^2$ term leads to a subtraction so that the $p_0$ integral is replaced by

$$
\int dp_0 \left( \frac{p_0^2}{p_0^2 + p^2 + m^2} - 1 \right) = -\pi \sqrt{p^2 + m^2}
$$

(34)

All the diagrams that occur in figure (1) involve integrals over the time-like components of Euclidean momenta as this integration forces the source terms to be on the boundary, but this is the only divergent integration since the coupling, $g$, has dimensions of $M^2$.

Another way to deal with this divergence is take the two times $t$ and $t'$ to be distinct, say $t = \epsilon < 0$ and $t' = 0$. Then the last term of (7) is $-i\hbar \delta(x - x')\delta(\epsilon) = 0$ so with this prescription $\Lambda = 0$ and the integral in (34) is replaced by

$$
\int dp_0 \frac{p_0^2}{p_0^2 + p^2 + m^2} e^{i \hbar \epsilon}
$$

(35)

The exponent allows the $p_0$ contour to be closed in the lower half-plane giving (34), as before. The $p_i^0$ integrals can be done in a straightforward way by contour integration (with semi-circle in the upper plane) using

$$
\delta(\sum_i p_i^0) = \frac{-1}{\pi} \text{Im} \frac{1}{\sum_i p_i^0 + i\epsilon}
$$

(36)
Now, we expand \( W \) as
\[
W[\phi] = \frac{1}{\hbar} \sum_{n=1}^{\infty} \int dp_1 \cdots dp_{2n} \ \tilde{\phi}(p_1) \cdots \tilde{\phi}(p_{2n}) \Gamma_{2n}(p_1, \cdots, p_{2n}) \delta(p_1 + \cdots + p_{2n}) \tag{37}
\]
and the tree-level contributions to the kernels, \( \Gamma'_{2n} \), are given by
\[
\Gamma'_2(p, -p) = -\frac{1}{4\pi} \sqrt{p^2 + M^2} = -\frac{\omega(p)}{4\pi} \\
\Gamma'_4(p_1, \cdots, p_4) = -\frac{g}{(2\pi)^3 4!(\omega(p_1) + \cdots + \omega(p_4))} 
\tag{38}
\]
and, for \( r > 2 \), the recursion relation
\[
\Gamma'_{2r}(p_1, \cdots, p_{2r}) = \frac{4\pi}{\sum_{i=1}^{2r} \omega(p_i)} \sum_{n=2}^{r-1} n(r + 1 - n) \times \left\{ \Gamma'_{2n}(k, p_1, \cdots, p_{2n-1}) \Gamma'_{2(r+1-n)}(q, p_{2n}, \cdots, p_{2r}) \right\} 
\tag{39}
\]
where \( S \) symmetrises the momenta, \( k = -(p_1 + \cdots + p_{2n-1}) \), and \( q = -(p_{2n} + \cdots + p_{2r}) \).

The one-loop diagrams up to fourth order in \( \phi \) are shown in figure (2), where the cross

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{One-loop diagrams for \( \Psi_0[\phi] \), up to fourth order in \( \phi \).}
\end{figure}

denotes the mass counter-term given by the 1-loop term \( \delta M^2 \) in (30), which we chose to be \( \delta M^2 = \frac{g}{4\pi} \) (so \( \Gamma^h_{2}(0, 0) = 0 \)). These yield the \( O(h) \) contribution
\[
\Gamma^h_2(p, -p) = \frac{hg}{32\pi^2 p} \sinh^{-1}\left(\frac{p}{M}\right) - \frac{\hbar \delta M^2}{8\pi \omega(p)} 
\tag{40}
\]
and
\[ \Gamma_4^h(p_1, \ldots, p_4) = -\frac{g^2 h}{(2\pi)^4!(\omega(p_1) + \cdots + \omega(p_4))} \]

\[ S\{ \int_0^\infty \frac{dq}{2\omega(q) + \sum_{i=1}^4 \omega(p_i)} \cdot \left( -\frac{1}{\omega(q)(\omega(q) + \omega(p_1))} \right)^3 \frac{1}{(\omega(q) + \omega(p_1) + \omega(p_2) + \omega(q + p_1 + p_2))} \}
\]

We will now study how the perturbative calculation of \( W[\phi] \) yields the Feynman diagram expansion of equal time Green’s functions when substituted in (28). Keeping only \( \Gamma'_2 \) in the exponent and expanding the other contributions to \( W[\phi] \), which we call \( \tilde{W}[\phi] \), yields the Fourier transform of the equal time Green’s functions as

\[ \int d\mathbf{x}_1 \cdots d\mathbf{x}_n \langle 0 | \hat{\phi}(\mathbf{x}_1, 0) \cdots \hat{\phi}(\mathbf{x}_n, 0) | 0 \rangle e^{-i\sum_{i=1}^n \mathbf{p}_i \cdot \mathbf{x}_i} = \]

\[ \int \mathcal{D}\hat{\phi} e^{i\mathbf{p} \cdot \hat{\phi}(\mathbf{p})} \Gamma_2(\mathbf{p}, -\mathbf{p}) \hat{\phi}(\mathbf{p}) \sum_{n} \frac{(2\tilde{W}[\phi])^n}{n!} \hat{\phi}(\mathbf{p}_1) \cdots \hat{\phi}(\mathbf{p}_n) \]

So we have to contract \( \hat{\phi}(\mathbf{p}_1) \cdots \hat{\phi}(\mathbf{p}_n) \) with the \( \hat{\phi} \) in the diagrams contributing to \( \tilde{W} \) using the inverse of \( -2\Gamma_2 \) which is the Fourier transform of the equal time propagator in free-space (a cut-off in spatial momenta will be implied, but not for the time-like momenta which will be integrated out)

\[ \int \frac{d\mathbf{p}}{2\pi} e^{i\mathbf{p} \cdot \mathbf{x}} \frac{1}{2\sqrt{\mathbf{p}^2 + M^2}} = \int \frac{d\mathbf{p}_0}{2\pi} \frac{d\mathbf{p}}{2\pi} \frac{1}{\mathbf{p}_0^2 + \mathbf{p}^2 + M^2} e^{i\mathbf{p} \cdot \mathbf{x}} = G_0(\mathbf{x}, 0, 0, 0) \]

If we denote this by a line above the boundary then the diagrams contributing to the equal time two-point function are, to one-loop order, those shown in figure (3), which is to be compared with the usual Feynman diagram calculation of the VEV of two fields in terms of the free space propagator which we denote by a double line in figure (4). We can understand the equivalence of these two sets of diagrams, and those of

Figure 3: Diagrams contributing to the equal time two-point function to one-loop.
other VEVs, by studying the *gluing together* of two free-space propagators in $D + 1$ dimensions on a $D$-dimensional plane. Consider

\[
\int dy \, G_0(x_1, t_1, y, t) \frac{\partial}{\partial t} G(y, t, x_2, t_2) = \int \frac{dp \, dp_0 \, dq \, dq_0 \, dy}{(2\pi)^{2(D+1)}} \frac{e^{i[p(x_1-y)+p_0(t_1-t)+q(y-x_2)+q_0(t-t_2)]}}{(p_0^2 + p^2 + M^2)(q_0^2 + q^2 + M^2)} = (44)
\]

The $q_0$ contour may be closed in the upper or lower half-plane, depending on the sign of $t - t_2$, $\epsilon(t - t_2)$, giving

\[
\frac{1}{2} \epsilon(t - t_2) \int \frac{dp \, dp_0}{(2\pi)^{D+1}} \frac{e^{i[p_0(t_1-t)+\omega(y)(t-t_2)]}}{p_0^2 + \omega^2(y)} = (45)
\]

and a similar treatment of the $p_0$ integration leads to

\[
\frac{1}{2} \epsilon(t - t_2) \int \frac{dp}{(2\pi)^D} \frac{e^{i[\omega(p)(t_1-t)+\omega(y)(t-t_2)+p(x-y)]}}{2\omega(p)} = (46)
\]

Given that

\[
G(x, t_1, y, t_2) = \int \frac{dp}{(2\pi)^D} \frac{1}{2\omega(p)} e^{i[\omega(p)|t_1-t|+\omega(y)]},
\]

it follows that

\[
2 \int dy \, G_0(x_1, t_1, y, t) \frac{\partial}{\partial t} G(y, t, x_2, t_2) = \begin{cases} 
G_0(x_1, t_1, x_2, t_2) & t_1 > t, t > t_2 \\
-G_0(x_1, t_1, x_2, t_2) & t_1 < t, t < t_2 \\
G_I(x_1, t_1, x_2, t_2) & t_1 < t, t > t_2 \\
-G_I(x_1, t_1, x_2, t_2) & t_1 > t, t < t_2
\end{cases} (48)
\]

where $G_I$ is the “image propagator” equal to the free space propagator for the points $(x_1, t_1)$ and the reflection of $(x_2, t_2)$ in the plane at time $t$. In short, if the two points are on opposite sides of the plane at time $t$, the two propagators are “glued” to form the usual propagator, up to a sign, if they are on the same side the gluing produces the image propagator. In the next section we will interpret this relation in terms of the geometry of random paths. It should not be confused with the self-reproducing
property of heat-kernels, but plays a nonetheless fundamental role in field theory. For example, applying it twice leads to

\[
\int d\mathbf{x}_2 d\mathbf{x}_3 \; G_0(\mathbf{x}_1, t_1, \mathbf{x}_2, t_2) \left( \frac{-\partial^2}{\partial t_2 \partial t_3} 4G_0(\mathbf{x}_2, t_2, \mathbf{x}_3, t_3) \right) G_0(\mathbf{x}_3, t_3, \mathbf{x}_4, t_4) = \]

\[
= G_0(\mathbf{x}_1, t_1, \mathbf{x}_4, t_4) \quad \text{for } t_1 > t_2 > t_3 > t_4
\]  

(49)

Taking all the \( t_i \) to zero gives a relation which may be expressed graphically as in

Figure 5: The propagator at equal times is the inverse of \( 2\Gamma_2 \).

\[
\int dy \; G_0(\mathbf{x}, t_1, y, t) \frac{\partial}{\partial t} 2G_0(y, t, \mathbf{x}, t_2) = G_0(\mathbf{x}, 0, \mathbf{x}_2, t_2)
\]

(50)

i.e. gluing \( (-2\Gamma_2^1)^{-1} \) onto the Dirichlet propagator on the boundary turns it into the free-space propagator restricted to the boundary. The second component, figure (6), is also simplified using the gluing relation with \( t_1 < t < t', t' > t_2 \) and both \( t, t' \to 0 \)

\[
\int dy \; dz \; \left( \frac{\partial}{\partial t} 2G_0(\mathbf{x}, t_1, y, t) \right) G_0(y, t, z, t') \frac{\partial}{\partial t'} 2G_0(z, t', \mathbf{x}_2, t_2) = G_I(\mathbf{x}, t_1, \mathbf{x}_2, t_2)
\]

(51)
Figure 8: The equal time two-point function.

Figure 9: Equal time two-point function once we have taken into account the “gluings”.

Figure 10: Diagram to be cancelled.
So the effect of \((-2\Gamma_1^2)\)^{-1} as an internal line in a diagram is to produce an image propagator. This cancels against the image propagator part of the Dirichlet propagator contributing from another diagram. So if we denote the image propagator by a dotted line, (and the free-space propagator by a double line, as before) then the equal time two-point function is shown in figure (8), and with the above “gluings” we get the figure (9), which is just the figure (4) with the free end-points restricted to \(t = 0\). This cancellation may be made explicit at the level of integrals where the diagram in figure (10) will be written as

\[
\int \frac{d\mathbf{q} \, dq_0 \, dj_0 \, dp_0}{(2\pi)^4} \frac{1}{\omega(p)} \frac{i p_0}{p_0^2 + \omega^2(p)} \frac{1}{q_0^2 + \omega^2(q)} \frac{1}{\omega(q)} \frac{i j_0}{j_0^2 + \omega^2(q)} (-i(p_0 + q_0 + j_0)) \frac{1}{(p_0 + q_0 + j_0)^2 + \omega^2(p)} \frac{1}{\omega(p)} .
\]  

(52)

Spatial momentum is conserved at each gluing, momentum and energy are conserved at the four-point vertex. Whereas in the diagram in figure (11), which is given by

\[
- \int \frac{d\mathbf{q} \, dq_0 \, dp_0}{(2\pi)^3} \frac{1}{\omega(p)} \frac{i p_0}{p_0^2 + \omega^2(p)} \frac{1}{q_0^2 + \omega^2(q)} \frac{1}{\omega(q)} \frac{(-i)(p_0 + 2q_0)}{(p_0 + q_0 + j_0)^2 + \omega^2(p)} \frac{1}{\omega(p)} ,
\]  

(53)

the image propagator causes energy not to be conserved at the vertex, or rather to be “conserved” with a change of sign.

The two diagrams may be written as

\[
D_1 = \int dq \, dq_0 \, dj_0 \frac{q_0 j_0 f(q_0 - j_0)}{(q_0^2 + \omega^2(q))(j_0^2 + \omega^2(q))\omega(q)} \]  

(54)

and

\[
D_2 = - \int dq \, dq_0 \frac{f(2q_0)}{q_0^2 + \omega^2(q)} \]  

(55)

with the same function \(f\). If we change variables in \(D_1\) from \(q_0, j_0\) to \(p_\pm \equiv \frac{1}{2}(q_0 \pm j_0)\) then

\[
D_1 = \int dp_+ \, dp_- \frac{(p_+ + p_-)(p_+ - p_-) f(2p_-)}{(p_+ + p_-)^2 + \omega^2(q)((p_+ - p_-)^2 + \omega^2(q))\omega(q)} = \int dp_- \, dq \frac{f(2p_-)}{p_-^2 + \omega^2(q)} = -D_2 ,
\]  

(56)
so that $D_1$ and $D_2$ cancel, as our general argument implied. Another way to obtain the same result is by using (36).

To end this section we will relate our diagrammatic method to the canonical operator formalism. For that, we will use the formula (derived in appendix A)

$$\Psi_0[\phi] = \lim_{t \to \infty} e^{tE_0^{(i)}} \langle \phi(x) | T e^{-\int_0^t dt' \hat{H}_i(t')} | \Psi_0^{(0)} \rangle \quad (57)$$

where $\hat{H}_i(t)$ is the interaction Hamiltonian with the free evolution and $\Psi_0^{(0)}$ the free VWF. This equation has the advantage that it can be easily related to the Rayleigh-Schrödinger perturbation expansion (see Appendix A) and that it gives the interaction-picture VEVs. Previously, we have shown that we could calculate the VWF if we introduced Dirichlet-propagators and a new interaction term in the action. We will outline how (57) will give the same diagrammatic procedure.

We can derive the same boundary diagrammatics as before if we move the $\phi$-dependence in (57) to the interaction by using the eq. (13). Now we only need to consider the dynamics of free fields with a Dirichlet boundary. Again we can implement this boundary condition by using the method of image methods, with image charges on the other side of the boundary:

$$\hat{\phi}(x, t) = \frac{1}{\sqrt{2}} (\hat{\phi}_0(x, t) - \hat{\phi}_0(x, -t)) \quad (58)$$

where $\hat{\phi}_0(x, t)$ is the field operator with free evolution and no boundary. With this definition, the field operator vanishes at the boundary and the propagator is just the previous Dirichlet propagator (31). Expanding the exponential and using Wick’s theorem reduces (57) to a combination of Dirichlet propagators and boundary interactions term with the same diagrammatic interpretation as before.

Now we can construct the VEVs using the equations (124) and its conjugate (116)

$$\langle \Psi_0 | \phi(x_1, 0) \cdots \phi(x_n, 0) | \Psi_0 \rangle = \lim_{t \to \infty} e^{2tE_0^{(i)}}$$

$$\langle \Psi_0^{(0)} | T e^{-\int_0^t dt' \hat{H}_i(t')} \phi(x_1, 0) \cdots \phi(x_n, 0) T e^{-\int_0^t dt' \hat{H}_i(t')} | \Psi_0^{(0)} \rangle = \lim_{t \to \infty} e^{2tE_0^{(i)}} \langle \Psi_0^{(0)} | T \phi(x_1, 0) \cdots \phi(x_n, 0) e^{-\int_0^t dt' \hat{H}_i(t')} | \Psi_0 \rangle \quad (59)$$

We recognize in the last equality the usual interaction-picture formula which, by expanding the exponential, gives the usual Feynman diagrams. The previous interpretation of (57) in terms of boundary diagrams implies that, when we compute the VEVs, all these boundary diagrams combine to reproduce the usual Feynman diagrams, as they should.
4 Interpretation of the Vacuum functional in terms of random paths

The Feynman diagram expansion of the VWF has a simple interpretation in terms of the random paths of first quantisation allowing a geometric understanding of the ‘gluing relation’\(^{(18)}\). It is well known that the Euclidean free-space propagator from \(A\) to \(B\) may be written as a sum over all paths from \(A\) to \(B\) of a Boltzman weight given by the exponential of the length of the path. Explicitly this sum may be expressed in the path integral form

\[
\int D\vec{x} \, e^{-M \int_0^1 d\xi \sqrt{\dot{x} \dot{x}}} \tag{60}
\]

where \(\xi\) parametrises the path and \(\vec{x}(0)\) is the point \(A\), \(\vec{x}(1)\) the point \(B\). Alternatively this may be written in the form

\[
\int D\vec{x} D\epsilon \, e^{-\int_0^1 d\xi \left( \frac{\dot{x} \dot{x}}{\epsilon} + Me \right)} \tag{61}
\]

where the square-root has been eliminated at the cost of introducing an additional degree of freedom, \(\epsilon\), which plays the role of an intrinsic metric. The Feynman diagram expansion of \(\Psi\) needs the Dirichlet propagator rather than the free space one. We will now show that this is given as a sum over paths on the space-time in which points are identified with their reflection in the quantisation surface, \(t = 0\), and paths are weighted with a minus sign every time they cross this surface. Even though the path integrals (60) and (61) are one-dimensional (i.e. quantum mechanical) they still require regularization. This can be done by expanding \(\vec{x}\) about a classical solution to the Euler-Lagrange equations (satisfying the boundary conditions that paths run from \(A\) to \(B\)) as a Fourier sine-series truncated at some short wave-length

\[
\vec{x}(\xi) = \vec{x}^{\text{class}} + \sum_{n=1}^{N} x_n \sin(n \pi) \tag{62}
\]

This restricts us to differentiable paths. When we sum over paths from \(A\) to \(B\) we have to include paths from \(A\) to \(B'\), the reflection of \(B\), since \(B\) and \(B'\) are considered
equivalent (see figure [13]). Such paths cross the quantisation surface an odd number of times and so acquire an overall minus sign, whereas paths directly from \(A\) to \(B\) cross an even number of times and so are weighted with an overall plus sign. The contribution from the latter paths gives \(G_0\) and from the former \(-G_I\) in expression (31) for the Dirichlet propagator.

\[
\int d\mathbf{y} \, G_0(\mathbf{x}_1, t_1, \mathbf{y}, t) 2 \frac{\partial}{\partial t} G_0(\mathbf{y}, t, \mathbf{x}_2, t_2) = G_0(\mathbf{x}_1, t_1, \mathbf{x}_2, t_2) \quad \text{for} \quad t_1 > t > t_2
\]

simply reflects the fact that paths from \((\mathbf{x}_1, t_1)\) to \((\mathbf{x}_2, t_2)\) must cross the plane at time \(t\) at least once (if \(t_1 > t > t_2\)) allowing the sum over such paths to be factorized.

Figure 13: Previous diagram where we identify each path that crosses \(t = 0\) with another path that is reflected at \(t = 0\).

Alternatively we can identify each path that crosses \(t = 0\) with another path that is reflected at \(t = 0\) and so confined to \(t < 0\). For example the paths in figure [12] are identified with those in figure [13]. Now we attach a minus for each reflection. The paths have the same lengths as previously and so again lead to the Dirichlet propagator. There is no double counting because the reflected paths in figure [13] do not appear in the previous sum as they are not differentiable at \(t=0\). When this representation of the Dirichlet propagator is inserted into the Feynman diagram expansion we arrive at an expression for \(W\) as a sum over networks of paths in \(t < 0\) that are reflected at the boundary.

Finally, the gluing property of the free-space propagator \(G_0\):

\[
\int d\mathbf{y} \, G_0(\mathbf{x}_1, t_1, \mathbf{y}, t) 2 \frac{\partial}{\partial t} G_0(\mathbf{y}, t, \mathbf{x}_2, t_2) = G_0(\mathbf{x}_1, t_1, \mathbf{x}_2, t_2) \quad \text{for} \quad t_1 > t > t_2
\]

Figure 14: Paths from \((\mathbf{x}_1, t_1)\) to \((\mathbf{x}_2, t_2)\) must cross the plane at time \(t\) at least once.
(see figure (14)) so that formally

\[
\sum_{\text{paths } AB} e^{-\text{length}(AB)} = \\
\sum_y \left( \sum_{\text{paths } AC} e^{-\text{length}(AC)} \right) \left( \sum_{\text{paths } CB} e^{-\text{length}(CB)} \right)
\]

(63)

We note in passing that the first quantized path integrals (60) and (61) have well-known generalizations to string theory suggesting that our considerations in this section have relevance to String Field Theory, and extend immediately to the Anti de Sitter space-time considered in [2].

5 Analyticity of VEVs in the cut-off

The purpose of this paper is to show how VEVs can be reconstructed from a derivative expansion of the VWF. This relies on a knowledge of the domain of analyticity of VEVs in a momentum cut-off which we will now discuss. We begin with the representation of the VWF (6) and use the time splitting regularization for convenience. Our basic assumption is that \( \Psi_0[\phi] \) can be expanded in positive powers of \( \phi \) (we will give an argument to justify this), so that

\[
\Psi_0[\phi] = \int D\phi \ e^{-S_3[\phi]} \left( \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{i=1}^{n} \int dt \ dx \ \phi(x) \hat{\phi}(x, t) \delta(t) \right)
\]

(64)

We can now reinterpret this Euclidean field theory on the space-time \( t < 0 \) by interchanging the roles of space and Euclidean time, so that we obtain a theory on the space-time \( x > 0 \). \( \hat{\phi}(x, t) \) is then replaced by \( \phi'(x, t) \) and the functional integral is interpreted in the canonical formalism as

\[
\Psi_0[\phi] = \sum_{n} \frac{1}{n! (2\pi)^n} \int \prod dt_i \ dx_i \ \varphi(t_i) \langle 0_r | T \prod_{i=1}^{n} \phi'(x_i, t_i) | 0_r \rangle
\]

(65)

where \( | 0_r \rangle \) is the vacuum of the rotated theory and the integration over \( x \) is understood to be through the interval \([-\epsilon, 0]\). Therefore we can interpret \( \Psi_0[\phi] \) as the VEV of the evolution operator between \( t = -\infty \) and \( t = \infty \) of a time dependent Hamiltonian \( H(\varphi(t)) \) corresponding to the rotated theory at \( x > 0 \). The field \( \varphi(t) \) will now play the role of a time dependent coupling. Consider slowly varying configurations for which the interaction term \( \int dt \ \varphi(t) \phi'(0, t) \) is adiabatic, then \( | 0_r \rangle \) (originally an eigenstate of \( H(\varphi(-\infty)) \)) will remain at the instantaneous ground state of \( H(\varphi(t)) \). As a result of this, \( \Psi_0[\phi] \) is given by the exponential of \(- \int_{-\infty}^{\infty} dt \ E_0(\varphi(t)) \). The \( E_0(\varphi(t)) \) energy of the instantaneous Hamiltonian (\( t \) is fixed) is then approximately given by the effective potential at its minimum. If we change the sign of the coupling \( \varphi(t) \), then we do not
expect that the interaction term can change the position of the vacuum \( \int dt \varphi \phi' < H \) for large \( \phi \). Therefore \( E_0(\varphi) \) will be analytic for small \( \varphi \) and expandable, as we wanted to show. The adiabatic approximation will be good for slowly varying \( \varphi(t) \), roughly constant on the time scale \( \sim 1/(E_1 - E_0) \), with \( E_1 \) being the energy of the first excitation.

The time integrals in (65) can be done if we Fourier analyze the sources \( \varphi \), and use

\[
\hat{\varphi}'(\epsilon, t) = e^{iH_r t} \hat{\varphi}'(\epsilon, 0) e^{-iH_r t}
\]

so that (we have normal ordered the Hamiltonian and assumed an \( i\epsilon \) prescription)

\[
\Psi_0[\varphi] = \sum_{n=0}^{\infty} \int \prod_{i=1}^{n} dk_i \hat{\varphi}(k_1) \cdots \hat{\varphi}(k_n) \delta(k_1 + \cdots + k_n) \\
\times \langle 0_r | \hat{\varphi}' \frac{1}{H_r + i \sum_{i=1}^{n-1} k_i} \cdots \frac{1}{H_r + i(k_1 + k_2)} \hat{\varphi}' \frac{1}{H_r + ik_1} | 0_r \rangle
\]

\[
\equiv \sum_{n=0}^{\infty} \int \prod_{i=1}^{n} dk_i \hat{\varphi}(k_1) \cdots \hat{\varphi}(k_n) \delta(k_1 + \cdots + k_n) G(k_1, \ldots, k_n)
\]

We now focus on the analyticity properties of each Green’s function \( G(k_1, \ldots, k_n) \) when we scale all the momenta by \( s^{-1/2} \)

\[
G_n \left( \frac{k_1}{\sqrt{s}}, \ldots, \frac{k_n}{\sqrt{s}} \right) = \\
\langle 0_r | \hat{\varphi}' \frac{1}{H_r + i \sum_{i=1}^{n-1} k_i/\sqrt{s}} \cdots \frac{1}{H_r + i(k_1 + k_2)/\sqrt{s}} \hat{\varphi}' \frac{1}{H_r + ik_1/\sqrt{s}} | 0_r \rangle
\]

In order to decompose this matrix element into simpler elements we will insert the resolution of the identity in terms of a basis of energy eigenstates. The sums over the energy eigenvalues converge, since the vacuum functional itself is finite. Even in the higher dimensional case when an infinite wave-function renormalisation is needed along with boundary counter-terms (which depend on \( \varphi \) and are polynomial in the momenta) to ensure this finiteness these sums will continue to converge as the cut-off is removed. If they did not, then in the presence of a regulator these sums would be finite, but dominated by large energy contributions for which the \( k_i \) would be negligible, so that when the cut-off is removed the vacuum functional would have trivial dependence on the momenta. Because these sums converge their singularities occur where the denominators vanish. Since the eigenvalues of \( \hat{H}_r \) are real, as are the \( k_i \), the singularities of our expression lie on the negative real axis of the complex \( s \)-plane.

We can illustrate this analyticity in perturbation theory. The expansion of eq. (119) (where the eq. (120) is its leading term) gives the VWF to an arbitrary order. We will scale the field momenta by a factor of \( s^{1/2} \) and then we perform a change of
variables in the momenta integrations (scaling the momenta by \( s^{-1/2} \)). To a given order of the perturbation expansion \( \Psi_0[\phi] \) is given by a finite sum of terms containing energy denominators \((E_0^{(0)} - E_n^{(0)}) = \sum \omega_k, \text{with} \ \omega_k = \sqrt{k^2/s + m^2} \), expectation values (like \( \langle \Psi_n^{(0)} | H_{\text{int}} | \Psi_0^{(0)} \rangle \equiv H_I^{(0)} \)) and wave functionals \( \Psi_n^{(0)}[\phi; k_1, \cdots, k_n] \). In general we have

\[
\Psi_0[\phi] = \Psi_0^{(0)}[\phi] + \sum_{k_1 \cdots k_n} \sum_{n_1 \cdots n_k} \Psi_n^{(0)}[\phi] \frac{H_{n_1 n_2} H_{n_2 n_3} \cdots H_{n_k 0}}{(E_0^{(0)} - E_{n_1}^{(0)} - i\epsilon) \cdots (E_0^{(0)} - E_{n_k}^{(0)} - i\epsilon)}
\]

(69)

Where the \( n_i \) sums also involve the momenta integrations. The energy denominators have the expected analytic behavior: analytic in the whole complex \( s \)-plane with a cut from \( s = -\Lambda^2/m^2 \) (the \( \omega_k \)'s are integrated up to the cut-off \( \Lambda \)) to \( s = 0 \). The matrix elements will also be a combination of \( \omega_k \) because they are evaluated with Wick’s theorem (remember that the creation operator is given by \( a^\dagger(k) = \int dx \ e^{-ikx} (\omega_k \phi(x) - \delta/\delta \phi(x)) \) and therefore they have the same analytic behavior. Finally the wave functionals, which can be constructed operating \( a^\dagger(k) \) several times into the free VWF (which can be written as \( N e^{-\frac{1}{2} \int \frac{dk}{2} \omega_k \phi(k) \delta(\phi(-k))) \), are also analytic. Therefore we obtain that the scaled VWF is analytic in the whole complex \( s \)-plane with a cut from \( s = -\Lambda^2/m^2 \) to \( s = 0 \).

Relying also on the assumption that we can treat \( \Psi_0 \) as a power series in \( \phi \) allows us to take the logarithm of \( \Psi_0 \) to obtain the \( \Gamma_{2n} \) of (57) as sums of products of \( G_m \) with \( m \leq 2n \) so that \( \Gamma_{2n}(p_1/\sqrt{s}, \ldots, p_{2n}/\sqrt{s}) \) is also analytic in this domain.

Now consider evaluating the Fourier transform of the equal time VEV (28) by expanding all but the terms quadratic in \( \phi \) in \( |\Psi_0|^2 = \exp(2\bar{W}) \):

\[
\int \mathcal{D}\phi \ |\Psi_0[\phi]|^2 \hat{\phi}(p_1) \cdots \hat{\phi}(p_n) =
\]

\[
\int \mathcal{D}\phi \ e^{\frac{1}{2}\int dp \ \bar{\phi}(p) \Gamma_2(p, -p) \phi(p)} \sum_n \frac{(2\bar{W}[\phi])^n}{n!} \hat{\phi}(p_1) \cdots \hat{\phi}(p_n)
\]

(70)

The expansion is similar to that of (54), but we keep contributions to \( \Gamma_2 \) of all orders in \( \hbar \) in the exponent, and \( \bar{W} \) consists of the remaining terms in \( W \). Integrating over \( \hat{\phi} \) with a momentum cut-off \( p^2 < \Lambda \) leads to a sum of terms in which we make contractions with \( (\frac{2}{\hbar} \Gamma_2)^{-1} \), so if we write a typical term in the expansion of \( \bar{W}^n \) as

\[
\int \prod_{i} dp_i \ \hat{\phi}(p_i) H(p_1, \ldots, p_n)
\]

(71)

then these contractions lead to sums of terms like

\[
\int_{q^2 < \Lambda} \prod_{i=1}^m dq_i \ H(q_1, -q_1, q_2, -q_2, \ldots, q_m, -q_m, \ldots, p_1, \ldots, p_n).
\]

(72)
In the $1 + 1$ dimensions we have been working in this is finite as $\Lambda \to \infty$. We want to show that it is computable from a knowledge of an expansion of $W$ in positive powers of momentum. Such an expansion would appear to be convergent only for small momenta, if at all, so it would not appear to be useful for the limit $\Lambda \to \infty$. However analyticity in $\Lambda$ allows us to calculate large $\Lambda$ behavior from small $\Lambda$ behavior using Cauchy’s theorem. Consider the effect of scaling $\Lambda$ by $1/s$ and the momenta by $1/\sqrt{s}$

$$K\left(\frac{p_1}{\sqrt{s}}, \ldots, \frac{p_n}{\sqrt{s}}, \frac{\Lambda}{s}\right) = \int_{q^2 < \Lambda} \prod_{i=1}^{m} \frac{dq_i}{\sqrt{s}} H\left(\frac{q_1}{\sqrt{s}}, \ldots, \frac{q_m}{\sqrt{s}}, -\frac{q_m}{\sqrt{s}}, \ldots, \frac{p_1}{\sqrt{s}}, \ldots, \frac{p_n}{\sqrt{s}}\right).$$

(73)

For $s$ large all the arguments of $H$ and $\Gamma_2$ are small so we can use a small momentum expansion for these quantities. We have already shown that $H$ and $\Gamma_2$ are analytic in the complex $s$-plane cut along the negative real axis so it follows that $K$ is analytic in this cut-plane provided that $\Gamma_{-1/2}^2$ is also analytic. This last result follows if the only zeroes of $\Gamma_2^2(p, -p)$ lie on the negative real axis. Since $\Gamma_2(p, -p)$ is even in $p$, we obtain from (67)

$$\Gamma_2\left(\frac{p}{\sqrt{s}}, -\frac{p}{\sqrt{s}}\right) = \langle 0_r | \hat{\phi}' | 0_r \rangle$$

(74)

Inserting a basis of eigenstates of $\hat{H}_r$ gives

$$\Gamma_2\left(\frac{p}{\sqrt{s}}, -\frac{p}{\sqrt{s}}\right) = \sum_\epsilon |\langle 0_r | \hat{\phi}' | \epsilon \rangle|^2 \frac{\epsilon}{\epsilon + \frac{p^2}{s}}$$

(75)

The imaginary part of this is

$$\text{Im}\Gamma_2\left(\frac{p}{\sqrt{s}}, -\frac{p}{\sqrt{s}}\right) = \left( \sum_\epsilon \frac{|\langle 0_r | \hat{\phi}' | \epsilon \rangle|^2 \epsilon p^2}{\epsilon^2 + \frac{p^4}{s^2}} \right) \text{Im}\left(\frac{1}{s}\right)$$

(76)

Each term in the sum is greater than or equal to zero, and some terms must be non-zero, else $\hat{\phi}' | 0_r \rangle = 0$. So the imaginary part of $\Gamma_2$ can only be zero for finite $s$ when $s$ is real. The real part of $\Gamma_2$ is, for real $s$

$$\text{Re}\Gamma_2\left(\frac{p}{\sqrt{s}}, -\frac{p}{\sqrt{s}}\right) = \sum_\epsilon \frac{|\langle 0_r | \hat{\phi}' | \epsilon \rangle|^2 \epsilon p^2}{\epsilon^2 + \frac{p^4}{s^2}} \left( \epsilon + \frac{p^2}{s} \right)$$

(77)

which is a sum of positive terms for $s > 0$ and hence can only vanish when $s < 0$. Thus the zeroes of $\Gamma_2\left(\frac{p}{\sqrt{s}}, -\frac{p}{\sqrt{s}}\right)$ lie on the negative real axis and so (73) is analytic in the cut $s$-plane. (A boundary counter term such as $\Lambda \varphi^2$ cannot spoil this, because,
if present, it affects only the momentum independent part of $\Gamma_2$ cancelling any divergence.) Furthermore, for large $s$ we expect that $\Gamma_2\left(\frac{q}{\sqrt{s}}, \frac{q}{\sqrt{s}}\right)$ and $H\left(\frac{q}{\sqrt{s}}, \ldots, \frac{q}{\sqrt{s}}\right)$ have a finite limit, so that (74) behaves like $s^{-m/2}$. We conclude that the VEV with scaled momenta and cut-off is a sum of terms each of which is analytic in the cut $s$-plane (plus a neighborhood of $s = \infty$) and drops off like a negative power of $s$ for large $s$.

Let $\tilde{K}$ be the sum of such contributions and define the contour integral

$$I(\lambda) = \frac{1}{2\pi i} \int_C ds \frac{e^{\lambda(s-1)}}{s-1} \tilde{K}\left(\frac{p_1}{\sqrt{s}}, \ldots, \frac{p_n}{\sqrt{s}}, \frac{\Lambda}{\sqrt{s}}\right)$$

where $C$ is the circle at infinity starting below the cut on the negative real axis and ending above it. On $C$ all the momenta in (73) are small and we can use the small momentum expansions for $H$ and $\Gamma$, which leads to a power series in $1/\sqrt{s}$. Since $s$ is large we can rewrite this as a power series in $1/(s-1)$

$$\tilde{K}\left(\frac{p_1}{\sqrt{s}}, \ldots, \frac{p_n}{\sqrt{s}}, \frac{\Lambda}{\sqrt{s}}\right) \sim \sum A_n(p_1, \ldots, p_n, \Lambda) \frac{1}{(s-1)^n} \quad (s \to \infty)$$

and the analyticity around $s = \infty$ implies that the coefficients $A_n$ will grow as $C^n$ ($C$ is a constant). Because the convergence radius around $s = \infty$ is proportional to the cut-off we expect that these coefficients will grow with the cut-off (and if the cut-off is eliminated then the $C^n$ behavior at large $n$ is spoiled). Now

$$I(\lambda) = \sum A_n(p_1, \ldots, p_n, \Lambda) \frac{\lambda^n}{n!}$$

is the Borel transform of (79). This series, due to the $n!$, will be convergent for all $\lambda$ (thanks to the previous $C^n$ growth which was the consequence of the analyticity around $s = \infty$). If we now collapse the contour we obtain a contribution from the pole at $s = 1$ which is the VEV we seek, $\tilde{K}(p_1, \ldots, p_n, \Lambda)$, together with a contribution from the cut which is suppressed by the exponential factor $e^{\lambda(s-1)}$ as $\lambda \to \infty$ (given that the discontinuity in $\tilde{K}$ across the cut goes to zero as $s \to -\infty$ as a power of $s$). Thus we recover the VEV as the limit of the Borel transform

$$\tilde{K}(p_1, \ldots, p_n, \Lambda) = \lim_{\lambda \to \infty} I(\lambda) = \lim_{\lambda \to \infty} \sum A_n(p_1, \ldots, p_n, \Lambda) \frac{\lambda^n}{n!}$$

And the coefficients $A_n$ are obtainable from the local expansion of the VWF. We can also see from the integral (78) why $I(\lambda)$ is finite: the analyticity around $s = \infty$ allows us to change the integration contour in such a way that it will have a finite length and it will not cross any singularity, and thanks to the exponential suppression the limit $\lim_{\lambda \to \infty} I(\lambda)$ exists and is finite. This implies that this series is alternating and, as we have seen previously, also convergent for all $\lambda$. Although we need to evaluate the power series for large $\lambda$, we can obtain the value of the limit $\lambda \to \infty$ in the same way that we can estimate the value of the limit $\lim_{x \to \infty} e^{-x}$ from the $x$ behavior of $\sum_{n=0}^{N} (-1)^n x^n/n!$ for $N$ as large as $x^N/N! \ll 1$ (if we take $N = 10$ then for $x < 4$ this
series gives $e^{-x}$ with an error of $O(0.1)$ and therefore our estimate for $\lim_{x \to \infty} e^{-x}$ will be $
olimits \sum_{n=0}^{10} (-1)^n 4^n / n! = 0.097$ which is zero up to $O(0.1))$.

This leads to a controllable approximation in which we truncate the series at some order in $\lambda$. The error is then bounded by the highest order term retained which tells us how large we can take $\lambda$. How close this truncated series comes to displaying the limiting behavior for large $\lambda$ can then be judged from how flat the truncated series is as a function of $\lambda$ in the region of the largest value for which the truncation is trustworthy.

Before ending this section, it is interesting to see the connection of our resummation technique and the usual dispersion relation method. Consider the equation (78), rescale $s$ to $s/\lambda$ and consider the contour to go from $s = -\infty - i\epsilon$ to $s = -i\epsilon$, then to $s = i\epsilon$ and finally to $s = -\infty + i\epsilon$ (we have separated the pole term). We get (to simplify, we just write the $s$-dependence)

$$I(\lambda) = \tilde{K}(1) + \frac{e^{-\lambda}}{2\pi i} \int_0^\infty ds \frac{e^{-s}}{s + \lambda} \Delta \tilde{K}(s)$$

(82)

where $\Delta \tilde{K}(s) = \tilde{K}(-s/\lambda + i\epsilon) - \tilde{K}(-s/\lambda - i\epsilon)$ is the discontinuity across the cut and we have assumed an infinitesimal $\epsilon$. Therefore, for large $\lambda$, $\tilde{K}(1)$ is determined by the low momentum expansion which will give us $I(\lambda)$, and by an “exponentially subtracted” dispersive integral in contrast to the usual subtractions by polynomials. We also see that in the limit $\lambda \to \infty$ the integral in (83) only goes to zero as a power of $1/\lambda$, because we have assumed that $\Delta \tilde{K}(s)$ was polynomially bounded, and therefore we get the expected result (81).

### 6 Matrix elements in Quantum Mechanics by cut-off resummation

In this section we will apply our method to a one-dimensional non-relativistic quantum mechanical bound state problem. Consider a Hamiltonian such that its ground state wave-function is short ranged, vanishing at least exponentially for large distances, and non-singular. This condition implies that its Fourier transform $\tilde{\Psi}_0(k)$

$$\tilde{\Psi}_0(k) = \int_\infty dx e^{-ikx}\Psi_0(x)$$

(83)

is analytic for $|k| < 1/R_0$ if $\Psi_0(x) \sim e^{-|x|/R_0}$ for large $|x|$ because the integral, and all its derivatives in $x$, are convergent. We can say more about the analyticity of $\tilde{\Psi}_0(k)$, it is analytic in the whole complex plane with poles at $k = i/R_0$ due to the upper integration limit and at $k = -i/R_0$ due to the $x \to -\infty$ limit. If $\Psi_0(x)$ decays faster than that at large distances then $R_0 \to \infty$. It is enough for our purposes to consider systems with a short range potentials and then $R_0$ will be finite. Now we will introduce
the projector
\[
\hat{P} \equiv \int_{|k|<1/\sqrt{s}} |k\rangle\langle k|
\] (84)
to eliminate the degrees of freedom \( k > 1/\sqrt{s} \) in the computation of matrix elements.
We refer the reader to the appendix B for some examples and mathematical details.

Let us study the analyticity of a simple expectation value
\[
E(s) = \langle \Psi_0 | \hat{P} \frac{k^2}{\sqrt{s}} \hat{P} \frac{k^2}{\sqrt{s}} | \Psi_0 \rangle = \int_{|k|<1/\sqrt{s}} dk \, |k\rangle\langle k| \frac{1}{s^{3/2}} \int_{|k|<1} dk' \, |k'\rangle\langle k'| \) (85)
\]
the \(|\tilde{\Psi}_0(\frac{k}{\sqrt{s}})|^2\) has poles at \( \sqrt{s} = \pm ikR_0 \). Then the integral is analytic in the complex \( s \)-plane with a cut in the negative real axis. We see that \( s^{3/2}E(s) \) has the cut from \( s = -R_0 \) to \( s = 0 \). Notice that in field theory, this projector would correspond to one in the Fourier field amplitudes which is not the one we have used before (where we have restricted the Fourier components), but the main purpose of a regulator is to discard some kind of degree of freedom and we do not consider that our method depends on the form of the regulator. In fact, in field theory will be sometimes more useful to use some soft cut-off regulators which will preserve some wanted symmetries of the theory. More generally, we can consider
\[
\langle \Psi_n | \hat{P} \frac{k^2}{\sqrt{s}} \hat{A} \frac{k^2}{\sqrt{s}} | \Psi_0 \rangle = \frac{1}{s} \int_{|k|<1} dk \, \frac{k}{\sqrt{s}} \tilde{\Psi}_n^* (\frac{k}{\sqrt{s}}) \tilde{\Psi}_m (\frac{k}{\sqrt{s}}) \langle \frac{k}{\sqrt{s}} | \hat{A} | \frac{k}{\sqrt{s}} \rangle (86)
\]
and assume that \( \langle x | \hat{A} | x' \rangle \) vanishes, when \( |x - x'| \to \infty \), at least as \( e^{-|x-x'|/R} \) (in particular, that \( \langle k | \hat{A} | k' \rangle \) is proportional to \( \delta(k - k') \) and has poles for \( k = \pm i/R \) with non-zero \( R \)). For such local operators we can perform both integrals and we get analyticity in the \( s \)-plane with a cut in the negative real axis (again, sometimes we may multiply the matrix element by some power of \( s \) to get an analytic function in \( s \) with a cut from \( s = -\max(R_0, R) \) to \( s = 0 \). Alternatively we could have said that because the wave functions and \( \langle x | \hat{A} | x' \rangle \) decrease exponentially we obtain convergent integrals even if we take an arbitrary number of derivatives of them. In field theory we were only able to study the analyticity by expanding the wave functions (and only considering ultralocal functional \( \Lambda \) operators) because we had still unbounded integrals (\( \int d\phi_k \) integrals) which can change the analyticity domain if these integrals diverge.

Now, suppose that we got the ground state wave function for small \( k \), then we expand it around \( k = 0 \) (this is the analogous to the derivative expansion of the VWF) and use the resummation method to compute a matrix element like \( \frac{1}{\sqrt{s}} \) (we will use, as cut-off, \( \Lambda \) for simplicity, in the appendix B it is explained the difference with \( 1/\sqrt{s} \))
\[
\int_{|k|<\Lambda} dk \, k^2 \langle \Phi_0(k) |^2 = \int_{|k|<\Lambda} dk \, k^2 (a_0 - k^2 a_2 + \cdots) = (\Lambda^3 \frac{a_0}{3} - \Lambda^5 \frac{a_2}{5} + \cdots) (87)
\]
if we only consider these two terms, the resummed value will be estimated by the value of $\Lambda^3a_3^0 - \Lambda^5a_5^0$ at its local maximum, which gives $4a_0^{5/2}/a_2^{3/2}$. As an example, we will consider a $\delta(x)$, attractive, potential. Then

$$\Psi_0(x) = e^{-\alpha|x|}$$

$$\tilde{\Psi}_0(k) = \frac{2\alpha}{\alpha^2 + k^2}$$

and

$$|\tilde{\Psi}_0(k)|^2 = \frac{4}{\alpha^2} - \frac{8}{\alpha^4}k^2 + \frac{12}{\alpha^6}k^4 + \cdots$$

In the appendix B we discuss the details of the resummation of (87) and of $\langle \Psi_0 | \Psi_0 \rangle$. Although the quantum mechanical case is trivial it is useful as a test ground. Notice that we do not attempt to construct the local expansion coefficients (i.e. $\tilde{\Psi}_0(k)$ at small $k$) from a given Hamiltonian. We only give a method to compute amplitudes once a wave-function at large distances is given.

### 7 Effective expansion

In this section we will describe how the computation of VEVs with a given local expansion of the VWF automatically leads to a new “infrared” diagramatic approach. The VEVs will arise a the result of the resummation (i.e. we will find the analytic continuation of the series for the region of large cut-off in a systematic way). Suppose we know the leading terms of the local expansion of $W[\tilde{\varphi}]$

$$2W[\tilde{\varphi}] = -\frac{1}{2} \int \frac{dp}{2\pi} \tilde{\varphi}(p)\tilde{\varphi}(-p) \left(\alpha_0 + \alpha_2p^2 + \cdots\right) -$$

$$\frac{1}{4!} \int \frac{dp_1}{2\pi} \cdots \int \frac{dp_4}{2\pi} 2\pi\delta\left(\sum p_i\right)\tilde{\varphi}(p_1)\cdots\tilde{\varphi}(p_4)$$

$$\times(\beta_0 + \beta_2(p_1^2 + \cdots + p_4^2) + \cdots) + O(\tilde{\varphi}^6)$$

where we have discarded the $O(p^4)$ terms. Having this information at hand, we want to compute an equal-time connected Green’s function. For simplicity, we choose the two-point correlation function, which we compute using eq. (70). We have to perform a perturbation expansion with a $\varphi^4$ model in $1+1$ dimensions, with derivative interactions and an explicit small cut-off $\Lambda$. We will obtain our correlator as a series in $\Lambda, p$. Because we have truncated $W[\tilde{\varphi}]$ we will discard the terms $\Lambda O(\Lambda^4, p^4, \Lambda^2p^2)$. In figure (15) we show the diagrams corresponding to the contribution to the correlator
at zero momenta. Due to the fact that the cut-off is small, we consider the $\alpha_2$ term as a $(\partial \varphi)^2$ insertion (which we represent by a dot) and our (momentum-independent) propagator will be $1/\alpha_0$. The vertex with a dot is a $\beta_2$ vertex, with the arrow indicating where the momenta are sitting. Then we respectively get for each diagram:

$$
\frac{1}{\alpha_0} - \frac{1}{2} \frac{\beta_0}{\alpha_0^2} \int dq \frac{1}{(2\pi)^2} - \frac{1}{2} \frac{\beta_0}{\alpha_0} \int dq \frac{1}{(2\pi)^2} \frac{(-\alpha_2 q^2)}{\alpha_0} - \frac{\beta_2}{\alpha_0^2} \int dq \frac{q^2}{(2\pi)^2} \tag{91}
$$

which gives

$$
\frac{1}{\alpha_0} - \frac{1}{2} \frac{\beta_0}{\alpha_0} \frac{\Lambda}{\alpha_0} + \frac{\beta_0}{6\pi \alpha_0} \frac{\Lambda^3}{\alpha_0} - \frac{\beta_2}{3\pi \alpha_0} \frac{\Lambda^3}{\alpha_0} \tag{92}
$$

What are the dimensionless expansion parameters? To answer that, we scale the field $\varphi$ to absorb the $\alpha_2$ coefficient and, at the same time, we note that (because we have a super-renormalizable model) the perturbative expansion is performed on the coupling divided by the mass term. Therefore, we will get as dimensionless couplings

$$
\tilde{\beta}_0 = \frac{\beta_0}{\sqrt{\alpha_0/\alpha_2}} \quad \text{and} \quad \tilde{\beta}_2 = \frac{\beta_2}{\sqrt{\alpha_0/\alpha_2}},
$$

with $\tilde{\Lambda} = \frac{\Lambda}{\sqrt{\alpha_0/\alpha_2}}$ as a cut-off. Then we may rewrite (92) as

$$
\frac{1}{\alpha_0}(1 - \frac{1}{2\pi} \tilde{\beta}_0 \tilde{\Lambda} + \frac{1}{6\pi} \tilde{\beta}_0 \tilde{\Lambda}^3 - \frac{1}{3\pi} \tilde{\beta}_2 \tilde{\Lambda}^3) \tag{93}
$$

We assume that $\tilde{\beta}_0, \tilde{\beta}_2 \ll 1$. Our approximation to the $\tilde{\Lambda} \to \infty$ limit is obtained by applying (81) to the resummation of the $\tilde{\Lambda}$-series (see appendix B). Although we have very few terms, we can still do the resumation. Because the series has a finite limit for $\tilde{\Lambda} \to \infty$ our best estimate will be a stationary value (see also the remarks given in the appendix B).

Applying this to the local expansion of the perturbative $W[\varphi]$ to $O(g)$ as given by eq. (38) and (40), we have

$$
\Gamma_2(p,-p) = \frac{-1}{8\pi}(\alpha_0 + \alpha_2 p^2 + \cdots) = \frac{-\omega(p)}{4\pi} + \frac{g}{32\pi} \int dq \frac{1}{2\pi \omega(q)(\omega(q) + \omega(p))} - \frac{\delta M^2}{8\pi \omega(p)} \tag{94}
$$

$$
\Gamma_4(p_1,\ldots,p_4) = \frac{-1}{2(2\pi)^3} \frac{1}{4!} (\beta_0 + \beta_2 (p_1^2 + \cdots + p_4^2) + \cdots) = \frac{-g}{(2\pi)^3} 4! (\omega(p_1) + \cdots + \omega(p_4)) \tag{95}
$$

using $\delta M^2 = g/(4\pi)$, we obtain the local expansion coefficients by expanding the right-hand sides in powers of $p$, getting

$$
\alpha_0 = 2m \quad \alpha_2 = (m - \frac{m}{12\pi m^2}) \frac{1}{m^2} \tag{96}
$$

and

$$
\beta_0 = \frac{m g}{2 m^2} \quad \beta_2 = -\frac{m g}{16 m^2 m^2} \tag{97}
$$
when we substitute these values into the zero momentum correlator (eq. (92)) we get

\[ \frac{1}{2m} - \frac{1}{32\pi m m^2} \frac{g \Lambda}{m} + \frac{1}{128\pi m m^2} \frac{g \Lambda^3}{m^3} + \frac{1}{m} O(\frac{g^2}{m^4}, \frac{\Lambda^5}{m^5}) \]  

(98)

We can also compute the \( p^2 \) term of the two-point correlator. This is given by the

\[ \text{Figure 16: } p^2 \text{ contribution to the two-point Green’s function at } O(\beta \Lambda p^2). \]

diagrams of the figure (16), where we have neglected the \( O(\Lambda^2 p^2/m^4) \) terms, from which we get

\[ -\frac{\alpha_2}{\alpha_0} p^2 - \frac{p^2}{\alpha_0} \beta_2 \int_{|q|<\Lambda} dq \frac{1}{2\pi \alpha_0} + p^2 \frac{\alpha_2 \beta_0}{\alpha_0^2} \int_{|q|<\Lambda} dq \frac{1}{2\pi \alpha_0} \]  

(99)

which, after the substitutions (96) and (97), gives

\[ \left( -\frac{1}{4m} + \frac{1}{48\pi m m^2} + \frac{5}{128\pi m m^2} \right) \frac{p^2}{m^2} + O(\frac{g^2}{m^3}) \]  

(100)

if we want to get right the \( g\Lambda^3 p^2/m^7 \) term, we have to include (in addition to the vertices already present in the action, but omitted in figure (16)) \( \beta_4 \) vertices which will also give \( q^2 p^2 \) terms. Notice that we only got a \( O(\Lambda/m) \) contribution in (100), in contrast with the \( O(\Lambda^3/m^3) \) of (98).

The resummation of (98) gives

\[ \frac{1}{2m} - \frac{1.88}{32\pi m m^2} \frac{g}{m} \]  

(101)

we can check these results by computing our connected correlation function without locally expanding the \( W[\tilde{\varphi}] \) term

\[ \int \mathcal{D}\varphi \, \tilde{\varphi}(p)\tilde{\varphi}(-p) e^{2W[\varphi]} = \frac{1}{2\tilde{\Gamma}_2(p,-p)} - \frac{1}{2} \left( \frac{1}{2\sqrt{p^2 + m^2}} \right)^2 \int_{|q|<\Lambda} dq \frac{1}{2\tilde{\Gamma}_2(q,-q)} \frac{g}{\omega(q) + \omega(p)} + O(g^2) \]  

(102)

where \( \tilde{\Gamma}_2 = -4\pi \Gamma_2 \), then \( 2\tilde{\Gamma}_2 = \alpha_0 + \alpha_2 p^2 + \cdots \). The second term of the right-hand side is the tadpole diagram with dressed propagator and vertex (we have a non-local \( \varphi^4 \) vertex). The propagator is

\[ \frac{1}{2\tilde{\Gamma}_2(p,-p)} = \frac{1}{2\sqrt{p^2 + m^2}} + \frac{g}{16} \left( \frac{1}{\sqrt{p^2 + m^2}} \right)^2 \int_{|q|<\Lambda} dq \frac{1}{2\pi} \frac{1}{\omega(q)(\omega(q) + \omega(p))} - \frac{g}{16\pi} \left( \frac{1}{\sqrt{p^2 + m^2}} \right)^2 \frac{1}{\omega(p)} + O(g^2) \]  

(103)
then we get for (102)

\[
\frac{1}{2\sqrt{p^2 + m^2}} + \frac{g}{16} \left( \frac{1}{\sqrt{p^2 + m^2}} \right)^2 \int \frac{dq}{2\pi} \frac{1}{\omega(q)(\omega(q) + \omega(p))} -
\]

\[-\frac{g}{16\pi} \left( \frac{1}{\sqrt{p^2 + m^2}} \right)^2 \frac{1}{\omega(p)} - \frac{g}{16} \left( \frac{1}{\sqrt{p^2 + m^2}} \right)^2 \int \frac{dq}{2\pi} \frac{1}{\omega(q)(\omega(q) + \omega(p))} + O(g^2)\]

(104)

We realize that when \(\Lambda \to \infty\), the second and fourth term cancel. This is precisely the cancellation between the diagram of figure (11) with the one of figure (10) respectively. Now, we expand (104) in powers of \(\Lambda/m, p/m\)

\[
\left( \frac{1}{2m} - \frac{1}{32\pi m^2} \frac{g}{m^2} + \frac{1}{128\pi m^3} \frac{g}{m^3} \right) +
\]

\[+ \left( \frac{-1}{4m} + \frac{1}{48\pi m^2} \frac{g}{m^2} + \frac{5}{128\pi m^3} \frac{g}{m^3} \right) \frac{p^2}{m^2} + O\left( \frac{p^4}{m^4} \right)\]

(105)

we have dropped the terms \(\Lambda^3/m^3, p^2/m^2\). If instead of expanding in the cut-off, we take the \(\Lambda \to \infty\) limit of (104) and expand it in powers of \(p^2/m^2\) we get

\[
\left( \frac{1}{2m} - \frac{1}{16\pi m^2} \frac{g}{m^2} \right) + \left( \frac{-1}{4m} - \frac{3}{32\pi m^2} \frac{g}{m^2} \right) \frac{p^2}{m^2} + \cdots
\]

(106)

we find that the resummed value (101) is a good estimate of the first term. We need to go to the next order to perform a resummation of the \(p^2/m^2\) coefficient in eq. (100), so we cannot compare it with the \(\Lambda \to \infty\) value.

The approach we have just described fails when \(\alpha_0 = 0\), which happens for example in the perturbative treatment of massless theories. However the Schrödinger functional

\[
\Psi_t[\varphi, \varphi'] = \langle \varphi | e^{-\hat{H} t} | \varphi' \rangle
\]

(107)

does have a local expansion in which \(1/t\) plays the role of the mass-gap, and from which the functional can be constructed for arbitrary \(\varphi, \varphi'\) using analyticity as before. (This was used in [6] to obtain the standard result for the one-loop beta function for Yang-Mills). Since for large \(t\)

\[
\Psi_t[\varphi, \varphi'] \sim \Psi_0[\varphi] \Psi_0[\varphi']^*\]

(108)

we could compute VEVs from the large \(t_1, t_2\) behaviour of

\[
\int \mathcal{D}\phi(x) \phi(x_1) \cdots \phi(x_n) \Psi_{t_1}[\nu, \phi] \Psi_{t_2}[\phi, \nu]
\]

(109)

using our previous method.
8 Conclusions

In this paper we have shown how to compute vacuum expectation values (VEVs) from a knowledge of the vacuum wave-functional expressed as a local expansion as would arise, for example, by solving the field theoretic Schrödinger equation along the lines of [3]-[6]. Such expansions are valid for slowly varying fields and so would conventionally be thought of as making only a small contribution to VEVs. However we have shown that VEVs are analytic in a momentum cut-off, so that the large cut-off behaviour which we want to compute can be obtained from a knowledge of the VEVs at small momenta where the local expansion is valid.

We gave several path-integral representations of VEVs. We also derived a diagrammatic approach to compute VEVs from a perturbative vacuum functional as input, this not only showed how we recover the usual perturbative expansion, but also how we could get the same VEVs with a dimensionally reduced, and non-local, Euclidean effective action \(2W[\phi]\). We have justified the local expansion of the vacuum functional in appendix C. We have also interpreted the vacuum functional in terms of random paths, which suggests a generalisation to String Field Theory.

We discussed in detail the non-perturbative analyticity properties of VEVs in the cut-off, for the scalar 1 + 1 theory, illustrating this using perturbation theory. We believe that these results generalise to higher dimensional theories and also to higher spin fields, having shown that analyticity is unaffected by the inclusion of the boundary counter-terms that are the new features of this generalisation.

Acknowledgments

A. Jaramillo has been partially supported by DGES under contract PB95-1096, by a doctoral fellowship from IVEI, by a travel grant from the British Council and by some financial aid from the Mathematical Sciences Department of Durham University (where part of this work has been performed) whose hospitality is also thanked. He wishes to acknowledge fruitful discussions with E. Fradkin, E. Gallego, V.P. Nair, R. Perry, R. Schiappa, A. Vainshtein, V. Vento and U.J. Wiese. P. Mansfield is grateful to the British Council for a travel grant and to the University of Valencia for hospitality.

Appendix A

In this appendix we will construct an alternative representation of the VWF which we will discuss in the context of perturbation theory. In particular, we will derive the relation (57). Consider

\[
|\Psi_0 \rangle = \lim_{t \rightarrow \infty} e^{-(\hat{H}-E_0)t} |\Psi_0^{(0)} \rangle
\]

(110)

where \(\Psi_0^{(0)}\) is the free VWF, \(\hat{H}\) is the full Hamiltonian and \(E_0\) is the vacuum energy. This equation will show us how to compute the VWF in perturbation theory (provided...
that \( \langle \Psi_0 | \Psi_0^{(0)} \rangle \neq 0 \). We can modify this equation by introducing a new term which will not alter the formula

\[
| \Psi_0 \rangle = \lim_{t \to \infty} e^{-(\hat{H} - E_0)t} e^{(\hat{H}_0 - E_0^{(i)})t} | \Psi_0^{(0)} \rangle \equiv \lim_{t \to \infty} \hat{U}(t) | \Psi_0^{(0)} \rangle \quad (111)
\]

We will separate the interaction part from the free one, \( \hat{H} = \hat{H}_0 + \hat{H}_i \) and \( E_0 = E_0^{(0)} + E_0^{(i)} \) where \( \hat{H}_0 \) and \( E_0^{(0)} \) are the free Hamiltonian and free vacuum energy respectively. If we take a time derivative of \( \hat{U}(t) \) we get

\[
- \frac{d}{dt} \hat{U}(t) = \hat{U}(t) \hat{V}(-t) \quad (112)
\]

where \( \hat{V}(t) \equiv e^{(\hat{H}_0 - E_0^{(0)})t} \hat{V}(0)e^{-(\hat{H}_0 - E_0^{(0)})t} = e^{H_0t} \hat{V}(0)e^{-\hat{H}_0t} \) and \( \hat{V}(0) \equiv \hat{H}_i - E_0^{(i)} \). By integrating this equation between 0 and \( t \) we get

\[
\hat{U}(t) = 1 - \int_0^t dt' \hat{U}(t') \hat{V}(-t') \quad (113)
\]

As usual, solving this equation by iteration gives

\[
\hat{U}(t) = T e^{- \int_0^t dt' \hat{V}(-t')} = T e^{- \int_0^t dt' \hat{H}_i(t')}
\]

Therefore

\[
| \Psi_0 \rangle = T e^{- \int_{-\infty}^t dt (\hat{H}_i(t) - E_0^{(i)})} | \Psi_0^{(0)} \rangle = \lim_{t \to \infty} e^{tE_0^{(i)}} T e^{- \int_{-\infty}^t dt' \hat{H}_i(t')} | \Psi_0^{(0)} \rangle =
\]

\[
= Ne^{-\epsilon \hat{H}_i(0)} e^{-\epsilon \hat{H}_i(-\epsilon)} \ldots e^{-\epsilon \hat{H}_i(-\infty)}
\]

where \( \hat{H}_i(t) \) has the same time-evolution than \( \hat{V}(t) \). Now we can expand the exponential to get the perturbative series. The conjugate relation will be

\[
\langle \Psi_0 | = \langle \Psi_0^{(0)} | T e^{- \int_{-\infty}^t dt (\hat{H}_i(t) - E_0^{(i)})} = \lim_{t \to \infty} e^{tE_0^{(i)}} \langle \Psi_0^{(0)} | T e^{- \int_0^t dt' \hat{H}_i(t')} \quad (116)
\]

The previous relation \( (113) \) is easily related to the usual Rayleigh-Schrödinger one if we substitute in equation \( (113) \) the definition of \( \hat{U}(t) \) given in \( (111) \)

\[
\hat{U}(t) = 1 - \int_0^t dt' e^{-\hat{H}t} (\hat{H}_i - E_0^{(i)}) e^{-(\hat{H}_0 - E_0^{(i)})t'}
\]

(\text{notice that the only } t\text{-dependence is on the exponentials}) which gives

\[
| \Psi_0 \rangle = \hat{U}(\infty) | \Psi_0^{(0)} \rangle = | \Psi_0^{(0)} \rangle - \int_0^\infty dt e^{-\hat{H}t} (\hat{H}_i - E_0^{(i)}) | \Psi_0^{(0)} \rangle \quad (117)
\]

We realize that the exponential becomes 1 when it is applied to \( | \Psi_0 \rangle \) and therefore we will substitute \( \hat{H} - E_0 \) by \( \hat{H} - E_0 + i\epsilon \) and we will demand that the limit \( \epsilon \to 0 \) will give a finite value. Now, we perform the \( t \)-integration

\[
| \Psi_0 \rangle = | \Psi_0^{(0)} \rangle - \frac{1}{\hat{H} - E_0 + i\epsilon} (\hat{H}_i - E_0^{(i)}) | \Psi_0^{(0)} \rangle \quad (119)
\]
\( E^{(i)}_0 \) will be given by the condition that the limit \( \epsilon \to 0 \) should be finite (in other words, \( \langle \Psi_0 | \hat{H}_i - E^{(i)}_0 | \Psi_0^{(0)} \rangle = 0 \)). This is essentially the Lippmann-Schwinger equation (which here also applies to the discrete part of the spectrum). Now, if we insert \( \sum_n | \Psi_n^{(0)} \rangle \langle \Psi_n^{(0)} | \) after the \( \frac{1}{H - E_0 + i\epsilon} \) term and we perform an expansion in the coupling constant \( \lambda (\hat{H}_i \equiv \lambda \hat{H}_{\text{int}}) \) we get the usual Rayleigh-Schrödinger perturbation expansion \[13\]. For example, to first order we will obtain

\[
\Psi_0[\phi] = \Psi_0^{(0)}[\phi] + \lambda \sum_{n \neq 0} \int d\mathbf{k}_1 \cdots d\mathbf{k}_n
\]

\[
\langle \Psi_n^{(0)}(\mathbf{k}_1, \cdots, \mathbf{k}_n) | \hat{H}_{\text{int}} | \Psi_0^{(0)} \rangle \frac{\langle \Psi_n^{(0)}[\phi] \rangle}{E_0^{(0)} - E_n^{(0)}}
\]

where we have taken the \( \epsilon \to 0 \) limit. This relation hides the diagrammatic method of section 3, but at the end of that section this equivalence is shown using the previous formula \[13\].

Of course, in order to derive the perturbative series \[120\] we could directly expand

\[
\Psi_0[\phi] = \lim_{t \to \infty} e^{tE^{(i)}_0} \langle \phi(\mathbf{x}) | T e^{-\int_{-t}^{0} dt' \hat{H}_i(t')} | \Psi_0^{(0)} \rangle
\]

getting to first order

\[
\Psi_0^{(0)}[\phi] - \lambda \int_{-t}^{0} dt' \langle \phi(\mathbf{x}) | e^{t'\hat{H}_0} \hat{H}_{\text{int}} e^{-t'\hat{H}_0} | \Psi_0^{(0)} \rangle = \Psi_0^{(0)}[\phi] -
\]

\[
- \lambda \int_{-t}^{0} dt' \sum_{n,m=0}^{\infty} e^{t' (E_0^{(0)} - E_n^{(0)})} \langle \Psi_n^{(0)} | \hat{H}_{\text{int}} | \Psi_m^{(0)} \rangle \langle \Psi_m^{(0)} | \Psi_0^{(0)} \rangle \Psi_n^{(0)}[\phi]
\]

When both, the \( t \)-integral and the limit, are done we get

\[
\Psi_0^{(0)}[\phi] - \lambda t \langle \Psi_0^{(0)} | \hat{H}_{\text{int}} | \Psi_0^{(0)} \rangle \Psi_0^{(0)}[\phi] -
\]

\[
- \lambda \sum_{n \neq 0} \frac{1}{E_0^{(0)} - E_n^{(0)}} \langle \Psi_n^{(0)} | \hat{H}_{\text{int}} | \Psi_0^{(0)} \rangle \Psi_n^{(0)}[\phi]
\]

Finally

\[
\Psi_0[\phi] = \lim_{t \to \infty} e^{tE^{(i)}_0} \left\{ (1 - \lambda t \langle \Psi_0^{(0)} | \hat{H}_{\text{int}} | \Psi_0^{(0)} \rangle) \Psi_0^{(0)}[\phi] +
\]

\[
+ \lambda \sum_{n \neq 0} \frac{1}{E_0^{(0)} - E_n^{(0)}} \langle \Psi_n^{(0)} | \hat{H}_{\text{int}} | \Psi_0^{(0)} \rangle \Psi_n^{(0)}[\phi] \right\}
\]

As before, we choose \( E^{(i)}_0 \) in such a way that the limit exists (although that this time is another limit) and therefore if we take \( E^{(i)}_0 = \lambda \langle \Psi_0^{(0)} | \hat{H}_{\text{int}} | \Psi_0^{(0)} \rangle \) the \( t \)-limit will be finite until \( O(\lambda^2) \). As we can see, we got the same result than in equation \[120\].
Appendix B

In this appendix we will discuss the mathematical details of the resummation program through the use of several examples. Let us begin with the integral

\[ K(\Lambda) = \int_0^\Lambda dx \frac{1}{1 + x^2} = \arctan(\Lambda) \]  

(125)

\( K(\Lambda) \) is analytic in the cut \( \Lambda \)-plane (with the cut chosen from \( \pm i \) to \( \infty \)). We want to compute \( K(\infty) \) from a series expansion for small \( \Lambda \). For \( \Lambda < 1 \) we get

\[ K(\Lambda) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n + 1} \Lambda^{2n+1} \]

We can compute its Borel transform by constructing a new series with a \((2n + 1)!\) in the denominator.

\[ I(\lambda) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n + 1} \frac{\lambda^{2n+1}}{(2n + 1)!} = \int_0^\lambda dx \frac{\sin(x)}{x} \]  

(126)

This is the analogous step to that from eq. (79) to (80) but with a different variable. Although an integral representation similar to (78) is known (see [14] for details on the Borel summation of series), it is not used. Instead, \( K(\Lambda) \) is usually recovered from \( I(\lambda) \) through the “inverse Borel transform”

\[ K(\Lambda) = \int_0^\infty d\lambda \frac{1}{\Lambda} I(\Lambda \lambda) e^{-\lambda} = \frac{1}{\Lambda} \int_0^\infty d\lambda I(\lambda) e^{-\lambda/\Lambda} \]  

(127)

If \( I(\lambda) \) is given in terms of a series (as in our discussion of VEVs), then the integral would trivially remove the \( n! \) term and we would get back the previous series. But usually with Borel-summable asymptotic series one only knows \( I(\lambda) \) in a series with finite convergence radius which is insufficient to compute the infinite integral (127). One then has to find \( I(\lambda) \) for larger \( \lambda \) by analytic continuation and this is in general does not yield a polynomial. Conformal mapping and Padé approximants are used for that purpose. In our case, we work with series \( K(\Lambda) \) with finite convergence radius which implies that \( I(\lambda) \) series will be valid for all the integration domain, so the analytic continuation cannot provide us with a non-polynomial form and we have to change the method to recover \( K(\Lambda) \). We solve the problem by using the integral representation (78) which allows us to compute \( K(\Lambda) \) from \( I(\infty) \) as we will show in an example below. We prefer this method over the conformal mapping or the Padé approximants because it is more systematic to implement in a field theoretical large distance expansion, where a series in the cut-off naturally arises.

If \( I(\lambda) \) has a finite \( \Lambda \to \infty \) limit, eq. (127) gives \( K(\infty) = I(\infty) \), which we apply to our example (126)

\[ K(\infty) = \int_0^\infty dx \frac{\sin(x)}{x} = \frac{\pi}{2} \]  

(128)
agreeing with the $K(\infty)$ computed directly from (125). Usually we will have to compute $I(\infty)$ approximately, by truncating the series. The truncation error in this alternating series is bounded by the first neglected term. But we have treated the exact expression to be able to study the convergence.

We can adapt our method to the case of $K(\Lambda)$ for finite $\Lambda$, greater than 1. Consider

$$K(\Lambda, s) = \sqrt{\pi} \int_0^{\Lambda/\sqrt{\pi}} dx \frac{1}{1 + x^2}$$

(129)

which, as a function of $s$, is analytic in the cut $s$-plane (with the cut from $-\Lambda^2$ to 0). Then we expand $K(\Lambda, s)$ into powers of $(s - 1)^{-n}$ (we have analyticity for large $s$) as in eq. (79).

$$K(\Lambda, s) = \Lambda + \sum_{n=1}^{\infty} (-1)^n \frac{P_{2n+1}(\Lambda)}{(s - 1)^n}$$

(130)

with $P_{2n+1}(\Lambda) = \int_0^\Lambda dx x^2 (1 + x^2)^{n-1}$. Although the point $s = 1$ (where we recover $K(\Lambda)$) is beyond the convergence radius, we will use the relation (80), written as

$$I(\lambda) = \Lambda + \sum_{n=1}^{\infty} (-1)^n \frac{P_{2n+1}(\Lambda) \lambda^n}{n!}.$$  

(131)

Again, we can obtain the whole sum in a closed form

$$I(\lambda) = \Lambda + \int_0^\lambda dx \frac{e^{-x}}{4x^{3/2}} (2\sqrt{x}e^{-x\Lambda} - \sqrt{\pi}\text{erf}(\sqrt{x}\Lambda))$$

(132)

where erf($x$) is the error function. The limit $\lambda \to \infty$ can also be obtained exactly

$$\lim_{\lambda \to \infty} I(\lambda) = \arctan(\Lambda)$$

(133)

using this result into the relation (81) gives $K(\Lambda) = I(\infty) = \arctan(\Lambda)$, which agrees with (125).

As another example we will compute the integral

$$K(\Lambda) = \int_{|k|<\Lambda} \frac{dk}{2\pi} \left(\frac{2}{1 + k^2}\right)^2$$

(134)

for $\Lambda \to \infty$, which corresponds to the normalization of the wave-function (88). Its Borel transform is given by

$$I(\lambda) = 4 \pi \sum_{n=0}^{\infty} (-1)^n \frac{n + 1}{2n + 1} \frac{\lambda^{2n+1}}{(2n + 1)!} = \frac{2}{\pi} (\sin(\lambda) + \text{Si}(\lambda))$$

(135)

Where Si denotes the sine integral function. We realize that there is no $\lambda \to \infty$ limit of $I(\lambda)$ because of the oscillatory behavior. The same thing happens with the $\langle \Psi_0 | k^2 | \Psi_0 \rangle$ integral

$$K(\Lambda) = \int_{|k|<\Lambda} \frac{dk}{2\pi} \left(\frac{k^2}{1 + k^2}\right)^2$$

(136)
which has the Borel transform

\[ I(\lambda) = \frac{2}{\pi} (\text{Si}(\lambda) - \sin(\lambda)) \]  \hspace{1cm} (137)

If we take a truncated series for \( K(\Lambda) \) then we will estimate \( I(\infty) \) as the value of the approximant at its first stationary point (if we take the highest \( \lambda^n \) term to have \( n \) even then the stationary point will be a local maximum). For the case of \( |\tilde{\Psi}_0|^2 \) to \( O(k^2) \) we get that (134) gives 1.47 (to be compared with 1, the \( \Lambda \to \infty \) value). For (136) with the wave-function squared expanded to \( O(k^2) \) we get 0.89 (to be also compared with \( \langle \Psi_0 | k^2 | \Psi_0 \rangle = 1 \)). Higher order terms will give worse estimates due to the fact that they are estimating the first maximum (the \( O(k^6) \) truncated \( |\tilde{\Psi}_0|^2 \) gives 1.56, the \( O(k^8) \) gives 1.7 like all higher approximants).

We can ask when does this oscillation occur? From eq. (127) we can see that if we substitute \( I(\lambda) \), for large \( \lambda \), by \( e^{i\alpha \lambda} \) then we find that \( K(\Lambda) \) has to have a pole at \( \Lambda = -i/\alpha \). To solve this no-convergence problem we will do a mapping, to move the pole. In fact, in eq. (129) we may think that we have performed a mapping \( \Lambda \to \mu/(1 + 1/|z|^{1/2}) \) with \( \mu = \Lambda \), but now the pole has moved to the negative real axis. But we should keep \( \Lambda \) finite. Then the idea to compute (134) would be to choose first a high value of \( \Lambda \) (for instance, \( \Lambda = 4 \) where \( K(4) = 0.994 \)), then we use eq. (78) to eq. (81) to compute \( K(4) \) (as we did for (129)). We get that \( K(\Lambda, s) = \sum A_n(\Lambda)s^{-n} = \sum B_n(\Lambda)(s - 1)^{-n} \) and then \( I(\lambda) = \sum B_n(\Lambda)\lambda^n/n! \). But because \( B_n(4) \sim 18^n(1)^n \), we will have to truncate the series at least at order \( n = 46 \). Therefore we have to improve the method. In fact an easy solution is the one used in section 5. Consider

\[ K(s) = \int_{|k|<1/\sqrt{s}} \frac{dk}{2\pi} \left( \frac{2}{1 + k^2} \right)^2 \]  \hspace{1cm} (138)

to compute the \( s \to 0 \) limit we will perform its Borel transform, which we define

\[ I(\lambda) = \frac{1}{2\pi i} \int_C \frac{ds}{s} K(s) e^{\lambda s} \]  \hspace{1cm} (139)

where \( C \) is a large, almost closed, circular (counterclockwise) path which does not cross the negative real axis. With the help of

\[ \frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \int_C dx \ x^{-z} e^x \]  \hspace{1cm} (140)

(where the contour \( C \) surrounds the negative real axis clockwise), we see that the Borel transform of

\[ K(s) = \sum_{n=0}^{\infty} (-1)^n a_n \frac{1}{s^{n+1/2}} \]  \hspace{1cm} (141)

is given by

\[ I(\lambda) = \sum_{n=0}^{\infty} (-1)^n a_n \frac{\lambda^{n+1/2}}{\Gamma(n + 3/2)} \]  \hspace{1cm} (142)
We see that we have divided by $(n + 1/2)! \equiv \Gamma(n + 3/2)$. This series has a good convergence (by only taking two terms we get $K(0) = I(\infty) = 0.83$ as estimate), but the function $I(\lambda)$ has one oscillation before decaying. It has a maximum at $\lambda = 2.24$ and $I(2.24) = 1.042$. If we estimate $I(\infty)$ by its value at the first local maximum, then 1.042 will be our result when computing the series truncated at any $n > 7$. Therefore we have to truncate $I(\lambda)$ for $n = 3$ getting $I(\infty) = 0.978$ which is close enough to $K(0) = 1$. 

We have used our resummation method to compute finite integrals. If we consider a divergent integral, then the integral will remain divergent after resummation as we can see with the following example

$$K(\Lambda) = \int_0^\Lambda dx \frac{1}{1 + x}$$

we can compute $I(\Lambda)$ in a closed form

$$I(\lambda) = \int_0^\lambda dx \frac{1 - e^{-x}}{x}$$

and we see that $I(\Lambda) \sim \log(\Lambda)$ for large $\Lambda$, reproducing the previous divergence.

Appendix C

In this appendix we will give argument for the validity of the local expansion of the VWF for a scalar theory with a mass gap. Again, it would be very convenient to use a path integral representation for the VWF.

$$|\Psi_0[\phi]|^2 = \int D J(x) \exp(-G_c[J(x)\delta(t)]) - i \int dx \ J(x)\phi(x))$$

where $G_c[J(x,t)]$ is the generator of connected Green’s functions in Euclidean space. This formula can be derived by interpreting $e^{-G_c[J(x)\delta(t)]}$ (in terms of $\Psi_0[\phi]$) as the (functional) Fourier transform of $|\Psi_0[\phi]|^2$. When we Fourier transform back we get (143). The relation (145) has the advantage that we can think of $|\Psi_0[\phi]|^2$ as a partition function with a non-local action $G_c[J(x)\delta(t)]$. Because we expect that the connected Green’s functions will be analytic at zero momenta (with the nearest pole given by the mass), they will have an exponential decay in configuration space (even if we put $t = 0$) which implies that the equal-time connected Green’s functions in momentum space will be analytic in a neighborhood of zero. This applies even if we had a massless Lagrangian (but non-zero mass gap). Because the logarithm of the square of the VWF is the generator of the connected Green’s functions (of the field $J$), if we have slowly varying $\phi$-configurations (low external momenta) we can substitute the non-local action by a local action without changing the infrared amplitudes (i.e. giving the same slowly varying $|\Psi_0[\phi]|^2$). Thanks to this assumed universality, we can now work with an effective field theory (for the quantum field $J(x)$) in the usual way.
We see that the connected two-point function at zero momentum will give a mass term for the field $J$. Thanks to this mass term, we will have analyticity at low momenta of the $J$-Green’s functions which means that we can perform the local expansion of the VWF. The fact that the effective theory is non-renormalizable does not cause trouble as it is common with effective field theories. But we still have to assume that the quantum corrections to the mass are small. Finally, we should realize that although the VWF is finite when we remove the cut-off (and then also the coefficients of its local expansion), the coefficients of the effective action are (generally) not finite (because they take into account the change in the theory at short distances due to the cut-off).

References

[1] K. Symanzik, *Nucl. Phys.* B **190** (1983) 1; M. Lüscher, *Nucl. Phys.* B **254** (1985) 52.

[2] Witten, E., *Anti de Sitter Space and Holography*, hep-th/9802150.

[3] P. Mansfield, *Phys. Lett.* B **358** (1995) 287; *Phys. Lett.* B **365** (1996) 207.

[4] J. Greensite, *Nucl. Phys.* B **158** (1979) 469; *Nucl. Phys.* B **166** (1980) 113; *Phys. Lett.* B **191** (1987) 431; J. Greensite and J. Iwasaki, *Phys. Lett.* B **223** (1989) 207; H. Arisue, *Phys. Lett.* B **280** (1992) 85; Q.Z. Chen, X.Q. Luo, and S.H. Guo, *Phys. Lett.* B **341** (1995) 349; M. Halpern, *Phys. Rev.* D **19** (1979) 517; J. Ambjorn, P. Olesen, C. Petersen, *Nucl. Phys.* B **240** (1984) 189.

[5] P. Mansfield, M. Sampaio and J. Pachos, hep-th/9702072.

[6] P. Mansfield, and M. Sampaio, hep-th/9807163.

[7] J. Cornwall, *Phys. Rev.* D **38** (1988) 656.

[8] B. Basista and P. Suranyi, *Phys. Rev.* D **48** (1993) 3826.

[9] D.M. McAvity and H. Osborn, *Nucl. Phys.* B **394** (1993) 728.

[10] M. Lüscher, R. Narayanan, P. Weisz and U. Wolff, *Nucl. Phys.* B **384** (1992) 168.

[11] E. Fradkin, *Nucl. Phys.* B **389** (1993) 587.

[12] H.W. Diehl, 'The theory of boundary critical phenomena’, cond-mat/9610143.

[13] B. Hatfield, ‘Quantum field theory of point particles and strings’, Addison-Wesley, Redwood City (USA), 1992.
[14] J. C. Le Guillou and J. Zinn-Justin, (Eds.) “Large order behavior of perturbation theory”. Amsterdam, Netherlands: North-Holland (1990).