ORDERS ON FREE METABELIAN GROUPS

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Abstract. A bi-order on a group $G$ is a total, bi-multiplication invariant order. A subset $S$ in an ordered group $(G, \leq)$ is convex if for all $f \leq g$ in $S$, every element $h \in G$ satisfying $f \leq h \leq g$ belongs to $S$. In this paper, we show that the derived subgroup of the free metabelian group of rank 2 is convex with respect to any bi-order. Moreover, we study the convex hull of the derived subgroup of a free metabelian group of higher rank. As an application, we prove that the space of bi-order of non-abelian free metabelian group of finite rank is homeomorphic to the Cantor set. In addition, we show that no bi-order for these groups can be recognised by a regular language.

1. Introduction

A group $G$ is bi-orderable if there exists a total order $\leq$ which is invariant under multiplication from both sides, i.e., if for $g, h \in G$ with $g \leq h$ then $f_1 g f_2 \leq f_1 h f_2$ for all $f_1, f_2 \in G$. Such a total order is called a bi-invariant order or bi-order for short on the group $G$. Similarly, a group $G$ is left-orderable (right-orderable) if there exists a left-invariant (right-invariant) order on $G$, in which the order is invariant under left-multiplication (right-multiplication). It is not hard to see that right-orders and left-orders have a one-to-one correspondence. Thus, in this paper, we will only discuss left-orders and bi-orders on a group. For every order $\leq$ on $G$, the positive cone $P_\leq$ consists of all positive elements in $G$ under $\leq$. It is a semigroup and $G = P \sqcup P^{-1} \sqcup \{1\}$. If $\leq$ is bi-invariant, then $P$ is invariant under conjugation. A positive cone completely determines the corresponding order, and vice versa. Hence, in this paper, we will identify positive cones and their associated orders when it is convenient.

The free metabelian group of rank $n$ is the quotient of the free group of rank $n$ by its second derived subgroup, which processes the following presentation.

$$M_n = \langle a_1, a_2, \ldots, a_n \mid [u, v], [w, z] = 1, \forall u, v, w, z \in \{a_1, a_2, \ldots, a_n\}^* \rangle.$$ 

Recall that for any set $X$, the notation $X^*$ denotes the free monoid (including the empty word) generated by $X$ and $X^{-1}$. $M_n$ is bi-orderable since by Magnus embedding $\text{Mag39}$ it is a subgroup of $\mathbb{Z}^n \wr \mathbb{Z}^n$, which is bi-orderable as bi-orderability is closed under taking wreath products $\text{BMR77}$ Theorem 2.1.1).

In this paper, we study the convex hull of the derived subgroup of a free metabelian group with respect to a bi-order, where the convex hull $\overline{H}$ of a subgroup $H$ is the smallest convex subgroup containing $H$. Let $M_n$ be the free metabelian group of rank $n$. We show that the derived subgroup is always convex when $n = 2$.

**Theorem A** (Theorem 6.2). $M_2'$ is convex with respect to any bi-invariant order on $M_2$.

When $n \geq 3$, we construct a bi-order such that $\overline{M}_n \neq M_n'$ in Theorem 7.8 But we can still obtain some information about the order from the restriction of the order on the derived subgroup.
Theorem B (Theorem 7.5). Let \( \leq \) be a bi-invariant order on \( M_n \) then the rank of \( M_n / M_n' \) is greater or equal to 2, where \( M_n' \) is the convex hull of the derived subgroup with respect to \( \leq \).

Let \( \mathcal{LO}(G) \) be the set of all left-orders on \( G \). It carries a natural topology whose subbasis is the family of sets of the form \( V_g = \{ P \leq | 1 \leq g \} \) for \( g \in G \). The space \( \mathcal{LO}(G) \) is a closed subset of the Cantor set and is metrizable (See, for example, [DNR14], [CR16]). And the space of all bi-orders \( \mathcal{O}(G) \) is a closed subspace of \( \mathcal{LO}(G) \). A lot has been known about the structure of these spaces. The space of bi-orders of a non-cyclic free abelian group is homeomorphic to the Cantor set [Sik04] and the same holds true for a non-abelian free group [McC89], [DM23]. For the Braid group \( B_n, n \geq 3 \), the space \( \mathcal{LO}(B_n) \) is infinite and has isolated points [DD01]. Tararin gave a complete classification of groups which have finite space of left-orders [KM96, Proposition 5.2.1]. Due to Theorem A and Theorem B we have:

Theorem C (Corollary 7.6). The space \( \mathcal{O}(M_n) \) is homeomorphic to the Cantor set for \( n \geq 2 \).

Note that the space of left orders of a free metabelian group of rank 2 or higher is also a Cantor set [RT16].

Let \( X \) be a generating set of \( G \). A language \( \mathcal{L} \) over \( X \) is a subset of the free monoid \( X^* \). A language is regular if it is accepted by a finite state automaton, and is context-free if it is accepted by a pushdown machine. We refer to [HU79] for the definitions of finite state automata and pushdown machines.

An order is computable if there exists an algorithm deciding if \( u \leq v \) in \( G \) for any pair of words \( u, v \in X^* \). An order is regular (context-free) if the positive cone can be recognised by a regular (context-free) language. Computability of left-orders and bi-orders has gained a lot of interest lately. Harrison-Trainor [HT18] have shown that there exists a left-orderable group with solvable word problem but no computable left-orders, while Darbinyan [Dar20] has constructed an example for the case of bi-orders. Šunić [Š13a, Š13b] showed that there exists a one-counter left-orders on the free groups and, later with Hermiller [Hv17], proved that left-orders on free products are never regular which implies that such left-orders constructed by Šunić are the computationally simplest orders in the sense of Chomsky hierarchy. Antolín, Rivas and Su [ARS21] have studied regular and context-free left-orders on groups and have shown that the metabelian Baumslag-Solitar group \( BS(1, q) \), \( |q| > 1 \) does not admit a regular bi-invariant order.

Recall that a group is computably bi-orderable if the group admits a computable bi-order. As a consequence of a more general Theorem 8.3, we have that:

Theorem D (Theorem 8.1, Corollary 8.4). Let \( M_n \) be the free metabelian group of rank \( n \). Then every \( M_n \) is computably bi-orderable. Moreover, \( M_n \) admits a regular bi-order if and only if \( n = 1 \).

When \( n = 2 \), it can be shown that \( M_2 \) admits a context-free bi-order. It remains unknown if the same holds true for \( n \geq 3 \).

The paper is organised in the following way. In Section 2 we introduce the notion of comparison index as in Section 3 we introduce \( Q \)-invariant orders for \( \mathbb{Z}Q \)-modules where \( Q \) is a free abelian group. In Section 4 and 5 we study \( Q \)-invariant orders for varies cases. We analyse the convex hull of the derived subgroups of free metabelian groups and prove the main theorems in Section 6 and 7. In Section 8 we show that a free metabelian group of
finite rank is computably bi-orderable, but the order is never regular unless the group is the infinite cyclic group.

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2. Comparison Index on Orderable Abelian Groups

It is convenient to firstly make our convention throughout the paper. For elements \( f, g \) in a group \( G \), we set that \( f^g = g^{-1}fg \) and \([f, g] = f^{-1}g^{-1}fg\).

An abelian group is bi-orderable if and only if it is torsion-free. Orders on abelian groups of finite \( \mathbb{R} \)-rank have been well-understood (See [CR16, Section 10.2][Teh61]). In particular, consider a free abelian group \( A \) of rank \( k \) and fix a basis of it. As \( A \) is canonically embedded into \( \mathbb{R}^k \) as a lattice, every order on \( A \) corresponds to a hyperplane passing through the origin, where the hyperplane separates elements in \( A \) into positive elements, negative elements and the identity. If the intersection of the hyperplane and \( A \) is non-empty, we also need to choose an order on the hyperplane itself. Such a hyperplane can be determined by its normal vector \( n \) which points to the positive side of the lattice, and an element \( g \) in \( A \) is positive if the dot product \( g \cdot n \) is positive.

To capture this structure of order on a free abelian group, we introduce the notion of comparison index.

**Definition 2.1.** Let \( x, y \) be positive elements in an ordered abelian group \((A, \leq)\). Then we define the comparison index of \( x, y \) with respect to \( \leq \) as the following.

\[
\text{CI}(x, y; \leq) = \lim_{n \to \infty} -\frac{m(n)}{n}, \quad m(n) = \min\{m \mid mx + ny \geq 0_A\}, \quad n > 0.
\]

If the limit or the minimal element do not exist, we will denote \( \text{CI}(x, y; P_\leq) = \infty \). And we make the convention that \( \text{CI}(0, 0) = 1 \). If there is no ambiguity with the order considered, we will use \( \text{CI}(x, y) \) instead of \( \text{CI}(x, y; \leq) \) to relax the notation.

Notice that \( mx + ny \geq 0_A \) if and only if \( -mx - ny \leq 0_A \), and hence \( |m(n) + m(-n)| \leq 1 \). Therefore, an alternative definition of the comparison index is

\[
\text{CI}(x, y; \leq) = \lim_{n \to -\infty} -\frac{m(n)}{n}, \quad m(n) = \min\{m \mid mx + ny \geq 0_A\}, \quad n < 0.
\]

We also define the absolute value of an element in an ordered group \((G, \leq)\) as

\[
|g| = \begin{cases} 
g & \text{if } g > 1_G, \\
g^{-1} & \text{if } g \leq 1_G. 
\end{cases}
\]

Fix an order \( \leq \) on \( A \). Given \( x, y \in A \), we define the vector \( r(x, y) \) as follows.

\[
r(x, y) = \begin{cases} (1, \text{CI}(|x|, |y|)) & \text{if } \text{CI}(|x|, |y|; \leq) \neq \infty, \\
(0, 1) & \text{if } \text{CI}(|x|, |y|; \leq) = \infty. 
\end{cases}
\]

The next proposition gives a geometric interpretation of the comparison index.
Proposition 2.2 (Geometric interpretation of CI). Let \((A, \leq)\) be an ordered abelian group and \(x, y\) be two positive elements. We have

(i) \(\text{CI}(x, y) \in [0, \infty) \cup \{\infty\}\).

(ii) Suppose that \((x, y)\) has rank 2 and \(r = r(x, y)\). Let \((\langle x, y \rangle, \leq) \to (\mathbb{Z}^2, \prec)\) be the order-preserving isomorphism such that \(x \mapsto (1, 0)\) and \(y \mapsto (0, 1)\). The order of \((\mathbb{Z}^2, \prec)\) can be extended to \((\mathbb{R}^2, \prec')\). The vector \(r\) is the normal vector to the hyperplane defining \(\prec\) and \((0, 0) \prec' r\). In particular,
\[
(m, n) \cdot r > 0 \Rightarrow mx + ny > 0_A.
\]

Proof. (i) Since \(x, y\) are both positive, then so is \(mx + ny\) for any \(m \geq 0, n > 0\). Thus, for a fixed \(n > 0\) the minimum \(m(n) = \min\{m \mid mx + ny \geq 0_A\}\) is non-positive. Hence, we have that \(-\frac{m(n)}{n} \geq 0\) for all \(n > 0\).

If \(x > ny\) for any \(n \in \mathbb{Z}\), then
\[
\text{CI}(x, y; \leq) = \lim_{n \to \infty} -\frac{m(n)}{n} = \lim_{n \to \infty} -\frac{0}{n} = 0.
\]

If \(y > mx\) for any \(m\), then the limit
\[
\lim_{n \to \infty} -\frac{m(n)}{n}
\]
does not exist, which implies that \(\text{CI}(x, y) = \infty\).

Therefore, \(\text{CI}(x, y) \in [0, \infty) \cup \{\infty\}\).

(ii) We will abuse the notations \(x\) and \(y\), and denote the axes containing \((1, 0)\) and \((0, 1)\) by \(x\)-axis and \(y\)-axis, respectively, of plane \((\mathbb{R}^2, \prec')\).

First, we consider the case that \(x < ny\) for some \(n\). Let \(l : y = kx\) be the hyperplane defining \(\prec'\) where \(k < 0\), since both \(x\) and \(y\) are positive. We want to show that \(-k \cdot \text{CI}(x, y) = 1\).

Note that since \((\langle x, y \rangle, \leq)\) and \((\mathbb{Z}^2, \prec)\) are isomorphic, where the isomorphism defined in the statement is order-preserving, we have
\[
mx + ny \geq 0_A \iff (m, n) \succeq 0_{\mathbb{Z}^2}.
\]

Since \(l\) defines \(\prec\), then for a fixed \(n > 0\),
\[
([\frac{n}{k}] + 1, n) \succeq 0_{\mathbb{Z}^2}, \text{ and } ([\frac{n}{k}] - 1, n) \prec 0_{\mathbb{Z}^2}.
\]

Hence
\[
[\frac{n}{k}] - 1 < m(n) \leq [\frac{n}{k}] + 1,
\]
where \(m(n) = \min\{m \mid mx + ny \geq 0_A\}\). Thus
\[
\frac{1}{n}([\frac{n}{k}] + 1) \leq \text{CI}(x, y) \leq \frac{1}{n}([\frac{n}{k}] - 1).
\]

Therefore, we have \(-k \cdot \text{CI}(x, y) = 1\). Consequently, \(r \perp l\) and \(r \succ' 0_{\mathbb{Z}^2}\).

If \(x > ny\) for all \(n \in \mathbb{Z}\), then \(l : y = 0\) is the hyperplane defining \(\prec\). Thus, \(r = (1, 0) \perp l\) and \(r \succ 0_{\mathbb{Z}^2}\).

The rest of the statement follows immediately.

\[\square\]

With this geometric interpretation, we have that:
Proof. Let \((x_1, x_2, \ldots, x_n, \leq)\) be the order-preserving isomorphism such that \(x_i \mapsto (0, 1, \ldots, 0)\), in which the \(i\)-th coordinate is 1. Let \(H\) be the hyperplane defining the order \(\prec\) in the space \(\mathbb{R}^n\). Then the intersection, denoted by \(l_i\), of \(H\) and the \(\mathbb{R}\)-subspace spanned by \(\{x_1, x_i\}\) is perpendicular to \(r_i = (1, 0, \ldots, r_i, \ldots, 0)\) in which the \(i\)-th coordinate is \(r_i\). Therefore, \(r = (1, r_2, r_3, \ldots, r_n)\) is perpendicular to all \(l_i\)'s for \(i = 2, \ldots, n\). Since all \(r_i\) are non-negative, \(\sum_i r_i \geq 0\), and all \(x_i\) are positive, \(\sum_i m_i x_i > 0\). It follows that

\[
(m_1, m_2, \ldots, m_n) \cdot (1, r_2, \ldots, r_n) > 0 \Rightarrow \sum_{i=1}^n m_i x_i > 0_A.
\]

Remark. Since \(g > 0\) if and only if \(-g < 0\), then we have that

\[
(m_1, m_2, \ldots, m_n) \cdot (1, r_2, \ldots, r_n) < 0 \Rightarrow \sum_{i=1}^n m_i x_i < 0_A.
\]

The following proposition provides some useful properties of the comparison index.

Proposition 2.4. Let \((A, \leq)\) be an ordered abelian group and \(x, y, z\) be positive elements. We have

(i) \(\text{CI}(x, x) = 1\). More generally, if \(\langle x, y \rangle\) is cyclic and \(mx = ny\), then \(\text{CI}(x, y) = \frac{m}{n}\).

(ii) \(\text{CI}(x, y) = 1/\text{CI}(y, x)\) with the convention \(\frac{1}{0} = \infty\) and \(\frac{1}{\infty} = 0\).

(iii) \(\text{CI}(x, y) \cdot \text{CI}(y, z) = \text{CI}(x, z)\) if \(\{\text{CI}(x, y), \text{CI}(y, z)\} \neq \{0, \infty\}\).

(iv) Let \(Q\) be a group acting on \(A\) by order-preserving isomorphisms, then for all \(q \in Q\), \(\text{CI}(x, y) = \text{CI}(q \cdot x, q \cdot y)\).

Proof. (i) For positive elements \(x, y\) such that \(-mx + ny = 0\) we have \(-kmx + kny = 0\), for all \(k > 0\). Thus

\[
\text{CI}(x, y) = \lim_{n \to \infty} -\frac{km}{kn} = \frac{m}{n}.
\]

(ii) It follows directly from the geometric interpretation of comparison index and (i).

(iii) If \(\langle x, y, z \rangle\) has rank 1, then the result follows from (i).

If \(\langle x, y, z \rangle\) has rank 2, WLOG, we assume that \(z = ax + by\) for \(a, b \in \mathbb{Z}\). For the case that \(\text{CI}(x, y) \in (0, \infty)\). It is enough to show that \(\text{CI}(x, z) = a + b \text{CI}(x, y)\). We let

\[
m_y(n) = \min\{m \mid mx + ny \geq 0_A\}, \quad m_z(n) = \min\{m \mid mx + nz \geq 0_A\}.
\]

Notice that

\[
mx + nz = (m + an)x + bny.
\]

Thus

\[
m_z(n) = m_y(bn) - an.
\]
It follows that
\[
\text{CI}(x, z) = \lim_{n \to \infty} -\frac{m_z(n)}{n} = \lim_{n \to \infty} -\frac{m_y(bn)}{bn} + a = a + b \text{CI}(x, y).
\]
Therefore, if \( z = ax + by \) we have
\[
\text{CI}(x, y) \cdot \text{CI}(y, z) = \text{CI}(x, y) \cdot \text{CI}(y, ax + by) = \text{CI}(x, y) \cdot \left(\frac{a}{\text{CI}(x, y)} + b\right)
\]
\[
= a + b \text{CI}(x, y)
\]
\[
= \text{CI}(x, z).
\]
If \( \text{CI}(x, y) = 0 \), then \( x \geq ny \) for all \( n \). Since \( z = ax + by \) is positive, we have that \( a \geq 0 \). By the assumption \( \{\text{CI}(x, y), \text{CI}(y, z)\} \neq \{0, \infty\} \) we have \( \text{CI}(y, z) \neq \infty \), which implies that there exists \( n \) such that \( ny > ax + by \). It forces \( a = 0 \). Hence, \( \text{CI}(x, z) = 0 \).

By (i) we have that \( \text{CI}(y, z) = b \), then \( \text{CI}(x, y) \cdot \text{CI}(y, z) = \text{CI}(x, z) \). The case where \( \text{CI}(x, y) = \infty \) is similar.

(iv) Since \( Q \) acts by order-preserving isomorphism, we have
\[
m(n) = \min\{m \mid mx + ny \geq 0_A\} = \min\{m \mid m(q \cdot x) + n(q \cdot y) \geq 0_A\}.
\]
Therefore
\[
\text{CI}(x, y) = \lim_{n \to \infty} -\frac{m(n)}{n} = \text{CI}(q \cdot x, q \cdot y).
\]

\[\Box\]

**Definition 2.5.** Let \((A, \leq)\) be an ordered abelian group and \(x, y\) be non-trivial elements. We say that \(x\) and \(y\) are comparable with respect to \(\leq\) if \(\text{CI}(|x|, |y|; \leq) \in (0, \infty)\), and we write \(x \sim y\). We say that \(x\) is lexicographically less than \(y\) with respect to \(\leq\) if \(\text{CI}(|x|, |y|; \leq) = \infty\), and we write \(x \ll y\). Furthermore, we say that \(x\) is lexicographically greater than \(y\) with respect to \(\leq\) if \(\text{CI}(|x|, |y|; \leq) = 0\), and we write \(x \gg y\).

Note that 0 is lexicographically less than any non-trivial element.

**Proposition 2.6.** Let \((A, \leq)\) be an ordered abelian group.

(i) The relation \(\sim\) (being comparable) is an equivalence relation.
(ii) For all non-trivial elements \(x, y \in A\), \(x \sim y\) if and only if there exist \(m, n \in \mathbb{Z}\) such that \(x \leq ny\) and \(y \leq mx\).
(iii) \(x \ll y\) if and only \(y \gg x\).
(iv) \(\ll\) defines a strict partial order on \(A\).
(v) \(\ll\) induces a strict total order on \(A/\sim\).

**Proof.** (i) It follows from (i),(ii) and (iii) of Proposition 2.4

(ii) Let \(\varepsilon, \eta \in \{-1, 1\}\) be the unique elements such that \(\varepsilon x, \eta y\) are positive. If \(x \sim y\), then by Proposition 2.2, we have that
\[
(m, n) \cdot r(\varepsilon x, \eta y) > 0 \Rightarrow m\varepsilon x + n\eta y > 0_A.
\]
Thus, we pick \(m, n\) such that \((m, -\eta) \cdot r > 0\) and \((-\varepsilon, n) \cdot r > 0\). Then \(x \leq ny\) and \(y \leq mx\).

The converse is straightforward.

(iii) It follows from (ii) of Proposition 2.4

(iv) It follows from (iii) of Proposition 2.4


(v) It follows from (i) and (iii).

3. Q-invariant orders on finitely generated free \( \mathbb{Z}Q \)-modules

We begin with a study on orders on the group ring of free abelian group of finite rank, since the derived subgroup \( M'_2 \) is isomorphic to \( \mathbb{Z}(x, y) \) (See [Bac65], [GM86]). Note that since the order on \( M'_2 \) given by the restriction of an order on \( M_2 \) is compatible with not only the group operation but also the action of \( Q := M_2/M'_2 \), we consider the following ordering structure on modules over the group ring of free abelian group of finite rank.

**Definition 3.1.** Let \( Q \) be a free abelian group of finite rank and \( M \) a finitely generated \( \mathbb{Z}Q \)-module. By a \( Q \)-invariant order \( \leq \) on \( M \) we mean an order satisfying:

1. if \( m_1 \leq m_2 \) then \( m_1 + m_3 \leq m_2 + m_3 \) for any \( m_3 \in M \);
2. if \( m_1 \leq m_2 \) then \( q \cdot m_1 \leq q \cdot m_2 \) for any \( q \in Q \).

If a \( \mathbb{Z}Q \)-module \( M \) admits a \( Q \)-invariant order, we say that \( M \) is \( Q \)-orderable.

Let \( Q = \mathbb{Z}^n \) with basis \( \{x_1, x_2, \ldots, x_n\} \) and let \( F_k \) be a free \( \mathbb{Z}Q \)-module of rank \( k \) with basis \( \{e_1, \ldots, e_k\} \). \( F_k \) is \( Q \)-orderable since \( F_k \) embeds as the base group into the wreath product \( \mathbb{Z}^k \wr \mathbb{Z}^n \), which is bi-orderable [BMR77, Theorem 2.1.1], and the inner automorphism action of \( \mathbb{Z}^n \) on the base group is order-preserving.

We fix a \( Q \)-invariant order \( \leq \) on \( F_k \) and WLOG, we can assume that \( e_1 > e_2 > \cdots > e_n > 0_{F_k} \). For each element \( m \in F_k \) and \( q, q' \in Q \) we define that \( q \sim_m q' \) if \( q \cdot m \sim q' \cdot m \) and \( q \ll_m q' \) if \( q \cdot m \ll q' \cdot m \).

**Lemma 3.2.** Let \( Q \) and \( F_k \) as above and \( m, m' \in F_k \).

(i) \( \sim_m \) is an equivalence relation and \( \ll_m \) is a partial order on \( Q \).

(ii) If \( q_1 \sim_m q_2 \), then \( q_1 q_1 \sim_m q_2 q_2 \) for all \( q \in Q \). In particular, the set \( Q_m = \{q \in Q \mid 1_Q \sim_m q\} \) is a subgroup of \( Q \) and \( \ll_m \) induces a bi-order on \( Q/ \sim_m \).

(iii) If \( m \sim m' \), then \( Q_m = Q_{m'} \).

**Proof.** (i) It follows from Proposition 2.6.

(ii) By Proposition 2.6 (ii) we have that there exist \( k, k' \in \mathbb{Z}_{>0} \) such that

\[
q_1 \cdot m \leq k q_2 \cdot m, \quad \text{and} \quad q_2 \cdot m \leq k' q_1 \cdot m.
\]

Since \( \leq \) is \( Q \)-invariant, we have

\[
q_1 \cdot m \leq k q_2 \cdot m, \quad \text{and} \quad q_2 \cdot m \leq k' q_1 \cdot m.
\]

Therefore, if \( q_1 \sim_m q_2 \), then \( q_1 q_1 \sim_m q_2 q_2 \) for all \( q \in Q \). Thus, \( Q_m \) is a subgroup since \( 1_Q \sim_m q \) implies \( q^{-1} \sim_m 1_Q \) and

\[
1_Q \sim_m q_1 \sim_m q_2 \Rightarrow q_1 \sim_m q_2, q_1 \sim_m q_2 \sim_m q_1 q_2.
\]

Consequently, \( \ll_m \) induces a bi-order on \( Q/ \sim_m \).

(iii) If \( m \sim m' \), then for \( q \in Q_m \) and \( q' \in Q_{m'} \), we have

\[
m \sim q \cdot m \sim q \cdot m' \sim q' \cdot m' \sim m'.
\]

The second equivalence comes from the fact that \( \leq \) is \( Q \)-invariant. Therefore, \( q \in Q_{m'} \) and \( q' \in Q_m \).

□
Let \((A, \leq)\) be an ordered abelian group. For a subset \(S\) of \(A\), we define:
\[
\text{Max}(S) = \{ s \in S \mid s \gg t \text{ or } s \sim t, \forall t \in S \}.
\]

**Proposition 3.3.** Let \(Q\) and \(F_k\) as above. The basis \(\{e_1, e_2, \ldots, e_k\}\) of \(F_k\) satisfies that \(e_1 > e_2 > \cdots > e_n > 0_{F_k}\). Suppose that \(Q_{e_1} = Q\). Then the homomorphism \(\varphi : F_k \rightarrow \mathbb{R}\) given on the abelian generators \(\{q \cdot e_i \mid q \in Q, i = 1, 2, \ldots, k\}\) of \(F_k\) by
\[
\varphi(q \cdot e_i) = \text{CI}(e_1, q \cdot e_i)
\]
satisfies that \(\varphi^{-1}((0, \infty))\) lies in the positive cone \(P_{\leq}\)

**Proof.** Let \(m \in F_k\). Let \(B_1 = \text{Max}\{e_1, e_2, \ldots, e_n\}\). We first notice that
\[
\text{CI}(e_1, q \cdot e_i) = 0, \forall q \in Q, e_i \notin B_1.
\]
Since \(e_1 \in B_1\) and \(e_i \notin B_1\), thus \(e_1 \gg e_i\), which implies that
\[
e_1 \sim q \cdot e_1 \gg q \cdot e_i.
\]

We want to show that if \(\varphi(m) > 0\), then \(m > 0_{F_k}\). Denote by \(\text{supp} m\) the support of \(m\), where \(\text{supp} m\) consists of all abelian generators from \(\{q \cdot e_i \mid q \in Q, i = 1, 2, \ldots, k\}\) with nonzero coefficients.

To simplify notation, the element \(m\) can be uniquely written as
\[
m = a_1 m_1 + a_2 m_2 + \cdots + a_l m_l + m', a_i \in \mathbb{Z}, m_i \in \text{supp} m \cap (Q \cdot B_1),
\]
and \(m'\) lies in the abelian subgroup generated in \(F_k\) by \(\{\text{supp} m \setminus (Q \cdot B_1)\}\). Note that \(\varphi(m') = 0\). Thus
\[
\varphi(m) = a_1 \text{CI}(e_1, m_1) + a_2 \text{CI}(e_1, m_2) + \cdots + a_l \text{CI}(e_1, m_l).
\]

Consider the abelian subgroup \((A, \leq)\) of \((F_k, \leq)\) generated by \(\{m_1, m_2, \ldots, m_l\}\). Notice that
\[
\varphi(m) = (a_1, a_2, \ldots, a_l) \cdot \text{CI}(e_1, m_1)(1, r_2, r_3, \ldots, r_l),
\]
where \(r_i = \text{CI}(m_1, m_i)\). Since \(e_1 \sim m_1\), and hence \(\text{CI}(e_1, m_1) > 0\), then by Theorem \[2, 3\] we have
\[
\varphi(m) > 0 \Rightarrow (a_1, a_2, \ldots, a_l) \cdot (1, r_2, r_3, \ldots, r_l) \Rightarrow m > 0_A = 0_{F_k}.
\]

\(\square\)

4. **The case where \(Q\) is cyclic and \(F_k\) has rank 1**

Let \((A, \leq)\) be an orderable abelian group and let \(Q\) be a group acting on \(A\) by order-preserving isomorphisms. Recall that a subgroup \(H\) of \(A\) is **convex** if for any \(g \in A\) satisfying that there exists \(h_1, h_2 \in H\) such that \(h_1 \leq g \leq h_2\), then \(g \in H\). A subgroup \(H\) is **\(Q\)-convex** if \(H\) is a convex subgroup and is invariant under the action of \(Q\). Note that if \(H_1, H_2\) are convex subgroups of \(A\), then either \(H_1 \subseteq H_2\) or \(H_2 \subseteq H_1\).

For the case \(Q\) is cyclic and \(F_k\) has rank 1, we can identify \(F_1\) with the Laurent polynomials \(\mathbb{Z}(x)\), on which the cyclic group \(\langle x \rangle\) acts.

**Lemma 4.1.** Let \(\prec\) be an \(\langle x \rangle\)-invariant order on \(\mathbb{Z}(x)\). Let \(\varepsilon \in \{-1, 1\}\) be the unique element such that \(\varepsilon \succ 0\) and let \(r = \text{CI}(\varepsilon, e; x; \prec)\). Moreover, for \(f(x) \in \mathbb{Z}(x)\) let \(a_-, a_+\) be the coefficients of the terms of the lowest degree and highest degree of \(f(x)\) respectively. We have:

(i) when \(r = 0\), \(f(x) \succ 0\) if and only if \(\varepsilon a_- > 0\),
(ii) when \(r = \infty\), then \(f(x) \succ 0\) if and only if \(\varepsilon a_+ > 0\),
Proof. For the case $r = 0$, then $x \ll 1$. Thus, we can write any element $f(x)$ as
\[ f(x) = a_-x^k + r(x), \]
where $k$ is the lowest degree of $f(x)$ and $r(x)$ is the rest of $f(x)$. We claim that $a_-x^k \succ r(x)$. Since $x \ll 1$, we have
\[ \varepsilon x^i \succ nx^{i+1}, \forall n \in \mathbb{Z}. \]
Thus, inductively we have
\[ \varepsilon x^k \succ nr(x), \forall n \in \mathbb{Z} \]
since the lowest degree of $r(x)$ is greater than $i$. In particular, $x^i \succ r(x)$. Notice that $a_-x^k \succ 0$ if and only if $\varepsilon a_- > 0$. If $\varepsilon a_- > 0$, then $a_-x^k \succ nr(x)$ for all $n \in \mathbb{Z}$. Thus, $a_-x^k \succ -r(x)$, and hence $f(x) \succ 0$. On the other hand, if $f(x) \succ 0$, then $a_-x^k \succ -r(x)$. Because $a_-x^k \succ r(x)$, it forces $a_-x^k \succ nr(x)$ for all $n \in \mathbb{Z}$. In particular, $a_-x^k \succ 0$. Thus, $\varepsilon a_- > 0$. This completes the proof of the case $r = 0$. The case where $r = \infty$ is similar.

Now we assume $r \in (0, \infty)$. WLOG, we also assume that $\varepsilon = 1$. Then $\text{CI}(1, x^n) = r^n$ for all $n \in \mathbb{Z}$. Suppose $f(x) = \sum_{i=s}^t a_ix^i$ where $a_s, a_t \neq 0$. Then
\[ f(r) = (a_s, a_{s+1}, \ldots, a_t) \cdot (r^s, r^{s+1}, \ldots, r^t) = (a_s, a_{s+1}, \ldots, a_t) \cdot \text{CI}(1, x^s)(1, r, \ldots, r^{t-s}). \]
Note that $\text{CI}(x^s, x^i) = r^{i-s}$ for $i = s + 1, s + 2, \ldots, t$. Then by Theorem 2.3 we have
\[ f(r) > 0 \Rightarrow f(x) \succ 0. \]

\[ \square \]

Lemma 4.2. Let $<$ be an $\langle x \rangle$-invariant order on $\mathbb{Z}(x)$. Let $H$ be the maximal proper $\langle x \rangle$-convex subgroup of $\mathbb{Z}(x)$.

(i) $H$ is trivial if and only if $r$ is either a positive transcendental number, 0, or $\infty$.

(ii) If $r$ is algebraic and $p(x)$ is the primitive irreducible polynomial of $r$ in $\mathbb{Z}[x]$, then $H = p(x)\mathbb{Z}(x)$. In particular, $H$ is isomorphic to $\mathbb{Z}(x)$.

Proof. (i) Suppose $r = 0$, or $\infty$ and $H$ is not trivial. We claim that, in both cases, $1 \in H$. Since $H$ is non-trivial, there exists $f(x) \neq 0 \in H$. WLOG we can assume $f(x) \succ 0$ and $1 \succ 0$. There exists $x^k$ such that $x^k f(x) \succ 1$ by Lemma [1.1] (i), (ii). The claim is proved.

Thus, $x^n \in H$ for all $n \in \mathbb{Z}$ because $H$ is $\langle x \rangle$-invariant. Then by Lemma [1.1] again, $H = \mathbb{Z}(x)$ since for any $g(x) \in \mathbb{Z}[x]$ there exists $x^i, x^j$ such that $-x^i \prec g(x) \prec x^j$. This leads to a contradiction. Thus, $H$ is trivial if $r = 0$ or $\infty$.

Suppose $r$ is transcendental and $H$ is not trivial. Let $f(x) \in H$ be a non-trivial element in $H$. WLOG, we assume that $1 \succ 0$. Since $r$ is transcendental, $g(r) \neq 0$ for every element in $\mathbb{Z}(x)$. Then for every $g(x) \in \mathbb{Z}(x)$ there exists $k \in \mathbb{Z}$ such that
\[ -kf(r) < g(r) < kf(r). \]

Therefore, $g(x) \in H$. Hence, we have a contradiction.

Conversely, if $r$ is algebraic, we claim that $I = \{ f(x) \mid f(r) = 0 \}$ is a non-trivial proper $\langle x \rangle$-convex subgroup. WLOG, we assume that $1 \succ 0$. Since $r$ is algebraic, $I$ is non-trivial. Suppose $g(x) \in \mathbb{Z}(x)$ such that $g(r) \notin I$. If $g(r) > 0$, then $g(x) - f(x) \succ 0$ for all $f(x) \in I$ by Lemma [1.1] (iii). Thus, $g(x) \succ I$. Similarly, if $g(r) < 0$, then $g(x) \prec I$. Thus, $I$ is convex. This completes the proof of (i).
(ii) By the discussion above, \( I = p(x)\mathbb{Z}(x) \subset H \). Suppose there exists \( f(x) \in H \setminus I \). WLOG, we assume that \( 1 > 0 \). Then \( f(r) \neq 0 \) and for any \( g(x) \in \mathbb{Z}(x) \) there exists \( k \in \mathbb{Z} \) such that

\[-kf(r) < g(r) < kf(r).\]

Thus, \( g(x) \in H \), which implies \( H = \mathbb{Z}(x) \). A contradiction.

Combining the above lemmas, we have a classification of all \( \langle x \rangle \)-invariant orders on \( \mathbb{Z}(x) \).

**Theorem 4.3.** Let \( \prec \) be an \( \langle x \rangle \)-invariant order on \( \mathbb{Z}(x) \). We inductively define a sequence \( (r_1, r_2, \ldots, ) \) in \( \mathbb{R} \cup \{\infty\} \) and a sequence \( (\varepsilon_1, \varepsilon_2, \ldots, ) \) in \( \{-1, 1\} \) as follows.

Let \( p_1(x) = 1 \) and suppose we have already defined \( (r_1, \ldots, r_{i-1}), (\varepsilon_1, \ldots, \varepsilon_{i-1}) \) and \( p_1, \ldots, p_s \).

Let \( \varepsilon_s \) be the unique element in \( \{-1, 1\} \) such that \( \varepsilon_s \prod_{i=1}^{s} p_i(x) > 0 \). Then let

\[ r_s = \text{CI}(\varepsilon_s \prod_{i=1}^{s} p_i(x), \varepsilon_s x \prod_{i=1}^{s} p_i(x)), \]

where \( p_{s+1}(x) \) is the primitive irreducible polynomial of \( r_s \) in \( \mathbb{Z}[x] \).

Then the \( \langle x \rangle \)-invariant order \( \prec \) is codified by \( (r_1, r_2, \ldots, ) \) and \( (\varepsilon_1, \varepsilon_2, \ldots, ) \) as follows.

(i) Let \( H_0 = \mathbb{Z}(x) \) and \( H_i = p_i(x)H_{i-1} \) for \( i = 1, 2, \ldots \). Then we have a sequence of nested subgroups

\[ \cdots \subset H_n \subset \cdots \subset H_2 \subset H_1 \subset H_0 = \mathbb{Z}(x), \]

where \( H_i \) is the maximal proper \( \langle x \rangle \)-convex subgroup of \( H_{i-1} \) for \( i = 1, 2, \ldots \). The nested sequence is finite if and only if there exists \( r_k \) that is either transcendental, 0 or \( \infty \).

(ii) For \( f(x) \in H_s \setminus H_{s+1} \) let \( a_-\), \( a_+ \) be the coefficient of the term of the lowest degree and highest degree of \( f(x) \) respectively. Then \( f(x) \succ 0 \) if and only if

\[
\begin{align*}
\varepsilon_s \left( \prod_{i=1}^{s} \frac{1}{p_i(r_s)} \right) f(r_s) & > 0 \quad \text{if } r_s \neq 0, \infty, \\
\varepsilon_s a_- & > 0 \quad \text{if } r_s = 0, \\
\varepsilon_s a_+ & > 0 \quad \text{if } r_s = \infty.
\end{align*}
\]

**Proof.** The base case is handled by Lemma \ref{lem:base} and Lemma \ref{lem:inductive}.

We define an isomorphism \( \varphi : H_n \rightarrow \mathbb{Z}(x) \) such that

\[ f(x) \mapsto \left( \prod_{i=1}^{n} \frac{1}{p_i(x)} \right) f(x) \]

We have an induced order \( \prec_\varphi \) on \( \mathbb{Z}(x) \). Let \( r = \text{CI}(\varepsilon, \varepsilon x; \prec_\varphi) \), where \( \varepsilon \in \{-1, 1\} \) is the unique element such that \( \varepsilon \prec_\varphi 0 \). Thus, the maximal proper \( \langle x \rangle \)-convex subgroup \( H \) of \( (\mathbb{Z}(x), \prec_\varphi) \) is either trivial for the case where \( r \) is transcendental, 0 and \( \infty \), or \( H = p(x)\mathbb{Z}(x) \) where \( p(x) \) is the primitive irreducible polynomial of \( r \) in \( \mathbb{Z}[x] \). Note that

\[ r = \text{CI}(\varepsilon, \varepsilon x; \prec_\varphi) = \text{CI}(\varepsilon_n \prod_{i=1}^{n} p_i(x), \varepsilon_n x \prod_{i=1}^{n} p_i(x); \prec) = r_n, \text{ and } \varepsilon = \varepsilon_n. \]

Moreover, \( H_{n+1} = \varphi^{-1}(H) \). Therefore, \( H_{n+1} \) is trivial if and only if \( r_n \) is transcendental, 0 or \( \infty \). And \( H_{n+1} = p(x)H_n = p_n(x)H_n \). The rest of the theorem follows immediately. \( \square \)
Consider a $Q$-orderable $\mathbb{Z}Q$-module $A$. Let $Q\mathcal{O}(A)$ be the space of $Q$-invariant orders on $A$. Note that since $\mathcal{O}(A)$ is a compact Hausdorff space and $Q$ acts on $\mathcal{O}(A)$ by homeomorphism, then $Q\mathcal{O}(A)$ is a closed subspace of $\mathcal{O}(A)$. Hence, $Q\mathcal{O}(A)$ is also a closed subspace of the Cantor set.

**Corollary 4.4.** The space of $\langle x \rangle$-invariant orders on $\mathbb{Z}(x)$ is a Cantor set.

**Proof.** For a fixed order $\prec$ it is enough to show that for any $f_1, \ldots, f_k \succ 0$, there exists a $\langle x \rangle$-invariant order $\prec'$ such that $P_{\prec} \neq P_{\prec'}$ and $f_1, \ldots, f_k \succ' 0$, where $P_{\prec}$ and $P_{\prec'}$ are positive cones for $\prec$ and $\prec'$ respectively.

By Theorem [4.3], we have a sequence of nested subgroups
\[
\cdots \subset H_n \subset \cdots \subset H_2 \subset H_1 \subset H_0 = \mathbb{Z}(x),
\]
where $H_i$ is the maximal proper $\langle x \rangle$-convex subgroup of $H_{i-1}$ for $i = 1, 2, \ldots$. And $\prec$ is codified by $(r_1, r_2, \ldots)$ and $(\varepsilon_1, \varepsilon_2, \ldots)$.

Suppose $f_1, \ldots, f_k \in \mathbb{Z}(x) \setminus H_s$ where $H_s$ is non-trivial. Then let $\prec'$ be the order such that $\prec' = \prec$ for $\mathbb{Z}(x) \setminus H_s$ and $\prec' = -\prec$ for $H_s$. Then $\prec'$ is the order we are looking for. In this case, $\prec'$ is codified as $(r_1, r_2, \ldots)$ and $(\varepsilon_1, \ldots, -\varepsilon_{s+1}, \ldots)$, where $\prec$ and $\prec'$ differ only at $\varepsilon_{s+1}$.

If $H_s$ is trivial, WLOG, we can assume that $s = 1$ and $\varepsilon_1 = 1$. We claim that there exists a transcendental number $r_1'$ such that $r_1' \neq r_1$, and $f_i(r_1') > 0$ for all $i = 1, \ldots, k$.

For the case $r_1 = \infty$, we can assume that each $f_i$ does not consist of any negative power of $x$, since $f(x) > 0 \iff x^nf(x)$ for any $n \in \mathbb{Z}$. Then the leading coefficient is positive for all $f_i(x)$. Thus
\[
\lim_{t \to \infty} f_i(t) = \infty, \forall i = 1, 2, \ldots, k.
\]
Therefore, there exists a transcendental number $r_1'$ large enough such that $f_i(r_1') > 0$ for all $i = 1, 2, \ldots, k$. The case when $r = 0$ is similar.

For the case $r_1$ is transcendental, we again assume that $f_i$ does not consist of any negative power of $x$. Each $f_i(x)$ can be written as
\[
f_i(x) = (x - r_1)g_i(x) + c_i,
\]
where $c_i = f_i(r_1), g_i(x) \in \mathbb{R}(x)$. By the assumption, $c_i > 0$ for all $i$. Notice that
\[
\lim_{s \to r_1} (x - r_1)g_i(x) = 0, \forall i = 1, 2, \ldots, k.
\]
Then there exists a number $\varepsilon$ small enough such that $\varepsilon g_i(r_1 + \varepsilon) > -c_i$ for all $i$. Therefore, such $r_1'$ exists.

Let $\prec'$ be the order codified by $(r_1')$ and (1). Since $r_1 \neq r_1'$, there exists a rational number $m/n, m, n \in \mathbb{N}$, between $r_1$ and $r_1'$. WLOG, we assume that $r_1 < m/n < r_1'$. Thus, $nx - m \in P_{\prec} \setminus P_{\prec'}$. And hence $\prec$ and $\prec'$ are different. Moreover, $f_i(x) \succ' 0$ for all $i = 1, 2, \ldots, k$, since $f_i(r_1') > 0$. Therefore, we finish the proof. \qed

Recall that an order is *Archimedean* if for every pair of positive elements $f, g$ there exists a natural number $n$ such that $g < f^n$. One immediate observation is that:

**Corollary 4.5.** An $\langle x \rangle$-invariant order on $\mathbb{Z}(x)$ is Archimedean if and only if it is codified by $(r_1)$ and $(\varepsilon_1)$ such that $r_1$ is transcendental.
5. **The case when** $Q = \mathbb{Z}^n, n > 1$

Let $Q = \mathbb{Z}^n$ for $n > 1$. We fix a $Q$-invariant order $\leq$ on $\mathbb{Z}Q$. Let $Q_1 = \{ q \in Q_0 \mid 1_q \sim 1_q \}$. Recall that $q \sim q'$ if and only if $q \cdot 1_{QQ} \sim q' \cdot 1_{QQ}$. By Lemma 3.2, $Q_1$ is a subgroup and the order $\leq$ induces the order $\ll$ on $Q/Q_1$. In particular, if $Q_1$ is a proper subgroup of $Q$, then $Q/Q_1$ is orderable hence torsion-free.

We pick a transversal $\tilde{Q}$ of cosets $Q/Q_1$. An element $m \in \mathbb{Z}Q$ can be uniquely written as

$$m = q_1 \cdot m_1 + q_2 \cdot m_2 + \cdots + q_l \cdot m_l,$$

where $q_i \in \tilde{Q}, m_i \in \mathbb{Z}Q_1$ and $q_1 \gg q_2 \gg \cdots \gg q_l$. Each $m_i$ is called the coefficient of $q_i$, and $m_1$ is called the leading coefficient of $m$, denoted by $\text{LC}(m)$.

**Proposition 5.1.** Let $(\mathbb{Z}Q, \leq)$, and $Q_1$ as above. Suppose $1 \succ 0$. For any transversal $\tilde{Q}$ of $Q/Q_1$ we have that for all $m \in \mathbb{Z}Q$

$$\varphi(\text{LC}(m)) > 0 \Rightarrow m > 0,$$

where $\varphi$ is the homomorphism $\varphi : \mathbb{Z}Q_1 \to \mathbb{R}$ given by

$$\varphi(q') = \text{CI}(1, q'), \forall q' \in Q_1.$$

**Proof.** Let $m \in \mathbb{Z}Q$. The $m$ can be written as

$$m = \sum_{i=1}^{l} \sum_{j=1}^{t_i} a_{ij} q_{ij},$$

where $q_{ij} \sim_1 q_{ij'}, a_{ij} \in \mathbb{Z}$.

We pick a transversal $\tilde{Q}$ of cosets $Q/Q_1$. Then

$$m = \sum_{i=1}^{l} q_i \sum_{j=1}^{t_i} a_{ij} q_i^{-1} q_{ij}, q_i \in \tilde{Q}, q_i \sim_1 q_{i1}.$$ 

Since $q_i \sim_1 q_{i1}$, then $1 \sim_1 q_i^{-1} q_{ij}$. Thus, $m_i := \sum_{j=1}^{t_i} a_{ij} q_i^{-1} q_{ij} \in \mathbb{Z}Q_1$.

We claim that the sign of $\varphi(m_i)$ does not depend on the choice of $\tilde{Q}$. Take $q_i' \in q_iQ_1$. Then the coefficient becomes

$$m_i' := \sum_{j=1}^{t_i} a_{ij} q_i'^{-1} q_{ij}.$$

We have that

$$\varphi(m_i) = \sum_{j=1}^{t_i} a_{ij} \text{CI}(1, q_i^{-1} q_{ij}) = \text{CI}(q_i, q_i') \sum_{j=1}^{t_i} a_{ij} \text{CI}(1, q_i'^{-1} q_{ij}) = \text{CI}(q_i, q_i') \varphi(m_i'),$$

since by Proposition 2.4

$$\text{CI}(q_i, q_i') \text{CI}(1, q_i'^{-1} q_{ij}) = \text{CI}(q_i, q_i') \text{CI}(q_i', q_{ij}) = \text{CI}(q_i, q_{ij}) = \text{CI}(1, q_i^{-1} q_{ij}).$$

Because $q_i' \in q_iQ_1$, the comparison index $\text{CI}(q_i, q_i')$ is a positive real number. Therefore, the signs of $\varphi(m_i)$ and $\varphi(m_i')$ are always the same.

Next, we will show that if $\varphi(\text{LC}(m)) > 0$ then $m > 0$. Since we have a decomposition of $m$ as

$$m = \sum_{i=1}^{l} q_i \sum_{j=1}^{t_i} a_{ij} q_i^{-1} q_{ij}, q_i \in \tilde{Q}, q_i \sim_1 q_{i1}.$$
We assume $q_1 \gg q_i$ for $i = 2, 3, \ldots, l$. Consider the abelian subgroup generated by $\{q_{ij} \mid i = 1, 2, \ldots, l, j = 1, 2, \ldots, t_i\}$. Note that $CI(q_1, q_{11}) \in (0, \infty)$ and $CI(q_{11}, q_{ij}) = 0$ for $i \geq 2$. Moreover, we have that

$$\varphi(LC(m)) = CI(q_1, q_{11}) \sum_{j=1}^{t_i} a_{ij} CI(1, q_{11}^{-1} q_{1j}) = CI(q_1, q_{11}) \sum_{i=1}^{l} \sum_{j=1}^{l_i} a_{ij} CI(q_{11}, q_{ij}).$$

By Theorem 2.3

$$\varphi(LC(m)) = CI(q_1, q_{11}) \sum_{i=1}^{l} \sum_{j=1}^{l_i} a_{ij} CI(q_{11}, q_{ij}) > 0 \Rightarrow m > 0.$$

When the rank of the free $\mathbb{Z}Q$-module exceeds 1, the situation is much more complicated, as $Q_{e_1}$ can be different from $Q_{e_2}$ for different basis elements $e_1, e_2$. The following proposition is a variation of Theorem 2.3 and it is heavily used in the later part of the paper.

**Proposition 5.2.** Let $Q = \mathbb{Z}^n$ and $(F_k, \leq)$ be the free $\mathbb{Z}Q$-module of rank $k$, where the basis element $e_1, e_2, \ldots, e_k$ of $F_k$ are all positive. Let $f \in F_k$. Then $f$ can be written as

$$f = a_1 f_1 + a_2 f_2 + \ldots + a_s f_s + f_r, a_i \in \mathbb{Z}, f_i \in \{q \cdot e_j \mid q \in Q\}, f_r \in F_k,$$

where $\{f_1, f_2, \ldots, f_s\} = \text{Max}(\text{supp} f)$ and $f_1 \gg f_r$. Then we have

$$(a_1, a_2, \ldots, a_s) \cdot (1, r_2, \ldots, r_s) > 0 \Rightarrow f > 0_{F_k},$$

where $r_i = CI(f_1, f_i)$ for $i = 2, 3, \ldots, s$.

**Proof.** It directly follows from Theorem 2.3

In the decomposition of $f$, the part $a_1 f_1 + a_2 f_2 + \ldots + a_s f_s$ is called the leading term of $f$.

6. Bi-orders on the Free Metabelian Group of Rank 2

The following commutator formulas are used throughout the paper.

**Lemma 6.1.** Let $G$ be a group. Then for $a, b, c \in G$ we have

$$[a, bc] = [a, c][a, b]^c, \text{ and } [ab, c] = [a, c]^b[a, c].$$

In addition, if $G$ is metabelian, we have

$$[a^m, b^n] = [a, b]^{(1+a)^m-1(1+b)^n-1}, \forall m, n \in \mathbb{N}.$$

**Proof.** The proof is straightforward.

Now we are ready to prove our main theorem for the free metabelian group of rank 2.

**Theorem 6.2.** $M'_2$ is convex with respect to any bi-invariant order on $M_2$.

**Proof.** Let $a, b$ be generators of $M_2$, and $Q = \mathbb{Z}^2 \cong M_2/M'_2$. The quotient map from $M_2$ to $Q$ is denoted by $\pi$. Then $x = \pi(a)$ and $y = \pi(b)$ form a basis of $Q$.

Let $\leq$ be a bi-order on $M_2$. The restriction of $\leq$ on $M'_2$ gives a $Q$-invariant order on the free $\mathbb{Z}Q$-module of rank 1. By replacing $a, b$ by $a^{-1}$ and $b^{-1}$ if necessary, we can always assume that $[a, b] > 1$ and $a > 1$. We then have an isomorphism $\iota : M'_2 \to \mathbb{Z}(x, y)$ such that

$$\iota([a, b]) = 1, \iota([a, b]^a) = x, \iota([a, b]^b) = y.$$
The order $\leq$ on $M'_2$ induces a $Q$-invariant order $\prec$ on $\mathbb{Z}(x, y)$.

To prove the theorem, it suffices us to show that $|ab^i| > M'_2$. We first claim that it is enough to prove the theorem for the case $a > M'_2$. For $a^ib^j$ where $i, j$ are coprime, there exists an automorphism of $M_2$ such that it sends $a^ib^j$ to $ga$ where $g \in M'_2$. Note that the image of a positive cone is again a positive cone (possibly for a different order). We denote the induced order by $\leq'$. We observe that $|ga| > M'_2$ if and only if $|a| > M'_2$ if and only if $|a| > M'_2$. Moreover, if $|a^ib^j| > M'_2$ for coprime $i, j$ then for every positive integer $k$ we have $|a^kib^j| > M'_2$. In summary, if $|a| > M'_2$ under any bi-order, then so does $|a^ib^j|$. Hence, the claim is proved.

Let $S \subset \mathbb{Z}(x, y)$ be a set consisting of all elements which pre-images are less than $a$, i.e.,

$$S = \{ f(x, y) \in \mathbb{Z}(x, y) \mid \iota^{-1}(f) < a \}.$$

Note that $S$ is convex by its definition.

We have the following properties of $S$. If $f \in S$, then

1. $0 \in S$;
2. $x^n f \in S$ for all $n \in \mathbb{Z}$;
3. $y^n f - (1 + y)^{n-1} \in S$ for all $n \in \mathbb{N}$;
4. $y^{-n} f + y^{-n}(1 + y)^{n-1} \in S$ for all $n \in \mathbb{N}$;
5. $f + (x - 1)g \in S$ for all $g \in \mathbb{Z}(x, y)$.

We will provide the proof of these properties after the proof of the theorem. Thus, to prove $a > M'_2$ it is enough to show that $S = \mathbb{Z}(x, y)$.

Let $\text{Cl}(1, x; \prec) = r, \text{Cl}(1, y; \prec) = s$. We have three cases to consider.

The first case is that $r, s \in (0, \infty)$. Thus, by Proposition 5.1, we have a homomorphism $\varphi : \mathbb{Z}(x, y) \to \mathbb{R}$, where $x \mapsto r, y \mapsto s$, such that $\varphi^{-1}((0, \infty)) \subset P_\prec$.

Using the properties we have

$$0 \in S \xrightarrow{\text{by (3)}} y^{-n}(1 + y)^{n-1} \in S \xrightarrow{\text{by (2) for } n = 1} y^{n-1}(1 + y)^{n-1} - 1 \in S \xrightarrow{\text{by (2) for } n = 1} (1 + y)^{n-1} - y^{n-1} - (1 + y)^{n-2}.$$

If $s < 1$, we have

$$\lim_{n \to \infty} \varphi(y^{-n}(1 + y)^{n-1}) = \infty.$$

And if $s \geq 1$, we have

$$\lim_{n \to \infty} \varphi((1 + y)^{n-1} - y^{n-1} - (1 + y)^{n-2}) = \infty.$$

In either cases, for any $t > 0$ there exists $f \in S$ such that $\varphi(f) > t$. Equivalently, for any $g \in \mathbb{Z}(x, y)$ there exists $f \in S$ such that $f \succ g$. Therefore, $S = \mathbb{Z}(x, y)$.

The second case is $r = 0$ or $r = \infty$. By Proposition 5.1, $Q_1 = \{ q \mid 1 \sim_1 q \}$ is proper. Take a transversal $\tilde{Q}$ of $Q/Q_1$. Again by Proposition 5.1, we have a homomorphism $\varphi' : \mathbb{Z}Q_1 \to \mathbb{R}$ such that $\varphi'^{-1}((0, \infty))$ is positive.

Let $g \in \mathbb{Z}(x, y)$ be a positive element. Then it can be expressed in the following form.

$$g = q_1 \cdot g_1 + q_2 \cdot g_2 + \cdots + q_l \cdot g_l, g_i \in \tilde{Q}, g_i \in \mathbb{Z}Q_1,$$
where \( q_1 \gg q_2 \gg \cdots \gg q_l \). Since \( g \) is positive, \( \varphi'(g_1) \geq 0 \). We let
\[
g'(r) = \begin{cases} 
q_1 \cdot (g_1 + 1) + q_2 \cdot g_2 + \cdots + q_l \cdot g_l & \text{if } r = \infty, \\
-x^{-1}(q_1 \cdot (g_1 + 1) + q_2 \cdot g_2 + \cdots + q_l \cdot g_l) & \text{if } r = 0.
\end{cases}
\]
By the construction (4), we have \( (x-1)g' \in S \). Note that if \( r = \infty \), we have \( x \gg 1 \) and hence \( xq_1 \gg xq_i, xq_1 \gg q_i \) for \( i \geq 2 \). If \( r = 0 \), we have \( 1 \gg x \), and hence \( x^{-1}q_1 \gg x^{-1}q_i, x^{-1}q_1 \gg q_i \). In either cases,
\[
\text{LC}((x-1)g' - g) = g_1 + 1.
\]
Then
\[
\varphi'(\text{LC}((x-1)g' - g)) = \varphi'(g_1) + 1 > 0.
\]
Therefore, \( (x-1)g' \in S \) and \( (x-1)g' > g \), which implies \( g \in S \). Since the choice of \( g \) is arbitrary, we have \( S = \mathbb{Z}(x, y) \).

The last case is that \( r \in (0, \infty) \) and \( s \in \{0, \infty\} \). In this case, \( Q_1 = \langle x \rangle \). By the discussion of the first case, we have already shown that
\[
y^{-n}(1+y)^{n-1}, (1+y)^{n-1} - y^{n-1} - (1+y)^{n-2} \in S, \forall n \in \mathbb{N}.
\]
Note that if \( 1 \ll y \), we have
\[
(1+y)^{n+1} - y^{n+1} - (1+y)^n > y^n > y^{-n}, \forall n \in \mathbb{N}.
\]
And if \( 1 \gg y \), we have
\[
y^{-n}(1+y)^{n-1} > y^{-n} > y^n, \forall n \in \mathbb{N}.
\]
Since \( S \) is convex, we have that \( y^{-n}, y^n \in S \) for all \( n \in \mathbb{N} \). It immediately follows that \( S = \mathbb{Z}(x, y) \).

Therefore, \( a > \iota^{-1}(S) = M'_2 \) in all cases. The theorem is proved. \( \square \)

**Lemma 6.3.** Let \( S \) be as above. If \( f \in S \), then

1. \( 0 \in S; \)
2. \( x^n f \in S \) for all \( n \in \mathbb{Z}; \)
3. \( y^n f - (1+y)^{n-1} \in S \) for all \( n \in \mathbb{N}; \)
4. \( y^{-n} f + y^n (1+y)^{n-1} \in S \) for all \( n \in \mathbb{N}; \)
5. \( f + (x-1)g \in S \) for all \( g \in \mathbb{Z}(x, y) \).

**Proof.** (1) is obvious, since \( a \) is positive.

Let \( f \in S \), we have
\[
\iota^{-1}(f) < a \Rightarrow \iota^{-1}(f)^{a^n} < a, \forall n \in \mathbb{Z}.
\]
Thus
\[
\iota(\iota^{-1}(f)^{a^n}) = x^n f \in S.
\]
The construction (2) is proved.

The proof of the rest constructions is similar. Construction (3) and (4) follow from conjugating \( \iota^{-1}(f) < a \) by \( b^n, b^{-n} \) respectively for \( n \in \mathbb{N} \). Construction (5) follows from conjugating \( \iota^{-1}(f) < a \) by \( h \in M'_2 \) where \( \iota(h) = g \). \( \square \)

It is well-known that left-orderability is preserved under group extensions. Let \( G \) be an extension of \( A \) by \( Q \), where \( \pi : G \to Q \) is the quotient map, and suppose \( A, Q \) are left-orderable. In addition, if we assume \( P_A \) and \( P_Q \) are positive cones of \( A \) and \( Q \) respectively, then \( P := P_A \cup \pi^{-1}(P_Q) \) is a positive cone of a left-order on \( G \), and thus \( G \) is also left-orderable. While in general, bi-orderability is not preserved under group extensions. But
if we assume that $P_A$ is invariant under the action of $Q$, then in this case $P_A \cup \pi^{-1}(P_T)$
defines a bi-order on $G$. An order given by such a construction is called a lexicographical order leading by the quotient.

**Proposition 6.4.** Let $G$ be a finitely generated orderable group that is an extension of $A$ by $Q$. If $A$ is convex with respect to order $\leq$, then $\leq$ is a lexicographical order leading by the quotient where the order on the quotient $Q$ is induced by $\leq$.

**Proof.** Let $\pi : G \to Q$ be the canonical quotient map. Then we define an order $\tilde{\leq}$ on $Q$ in the following way: $q_1 \tilde{\leq} q_2$ in $Q$ if $\pi^{-1}(q_1) \leq \pi^{-1}(q_2)$ in $G$. It is well-defined, since $A$ is convex. Let $P_Q$ be the positive cone in $Q$ associated with $\tilde{\leq}$ and $P_A$ the positive cone in $A$ associated with the restriction of $\leq$ on $A$. Then it is not hard to check that $P_A \cup \pi^{-1}(P_Q)$ is the positive cone associated with $\leq$ in $G$. Hence $\leq$ is a lexicographical order leading by the quotient. \qed

Thus, one immediate consequence of Theorem [5.2] is the following.

**Corollary 6.5.** Any bi-invariant order $\leq$ on $M_2$ is a lexicographical order leading by the quotient with respect to the extension of $M'_2$ by $M_2/M'_2 \cong \mathbb{Z}^2$.

7. **Bi-orders on Free Metabelian Groups of Higher Rank**

Since $M'_n$ is no longer a free $\mathbb{Z}Q$-module when $n > 2$, we have to consider $Q$-invariant orders on general $Q$-orderable $\mathbb{Z}Q$-modules. The following lemma allows us to lift the $Q$-invariant order on a finitely generated $\mathbb{Z}Q$-module to a $Q$-invariant order on a free $\mathbb{Z}Q$-module.

**Lemma 7.1.** Let $M$ be a finitely generated $\mathbb{Z}Q$-module and $M \cong F_k/S$ where $F_k$ is a free $\mathbb{Z}Q$-module. If $M$ is $Q$-orderable, then for every $Q$-invariant order $\leq$ on $M$ there exists a $Q$-invariant order $\tilde{\leq}$ such that $\tilde{\leq}$ is the lexicographical order leading by the quotient with respect to $S$ and $(M, \leq)$. In particular, $S$ is convex under $\tilde{\leq}$, and $\tilde{\leq}$ is called a lift of $\leq$ to $F_k$.

**Proof.** Since $F_k$ is $Q$-orderable, so is $S$. We pick a $Q$-invariant order on $S$ and form a lexicographic order $\leq$ with respect to the order on $S$ and $(M, \leq)$. The rest is straightforward. \qed

Let $M_n$ be the free metabelian group of rank $n$ for $n > 2$. Let $\{a_1, a_2, \ldots, a_n\}$ be the canonical free generators of $M_n$ and $Q$ be the free abelian group with basis $\{x_1, x_2, \ldots, x_n\}$. Then it is not hard to check that $M_n$ satisfies Jacobi’s identities

$$[a_i, [a_j, a_k]][a_j, [a_k, a_i]][a_k, [a_i, a_j]] = 1, \forall i, j, k \in \{1, 2, \ldots, n\}.$$  

Let $D_n$ be a $\mathbb{Z}Q$-module with the module presentation

(*)

$$D_n = \langle e_{ij}, 1 \leq i < j \leq n \mid J(i, j, k) = 0, \forall 1 \leq i < j < k \leq n \rangle,$$

where

$$J(i, j, k) = (1 - x_i)e_{jk} - (1 - x_j)e_{ik} + (1 - x_k)e_{ij}.$$  

Let $\iota : M'_n \to D_n$ be a $\mathbb{Z}Q$-module homomorphism defined by

$$\iota([a_i, a_j]) = e_{ij}, \forall i < j, \iota([a_i, a_j]^{ak}) = x_k \cdot e_{ij}, \forall i < j, \forall k.$$  

It has been shown that $\iota$ is an isomorphism (See [Bac65], [GM86]).

Every bi-invariant order $\leq$ on $M_n$ induces a $Q$-invariant order $\prec$ on $D_n$. Let $F$ be the free $\mathbb{Z}Q$-module generated by $\{e_{ij} \mid 1 \leq i < j \leq n\}$. In addition, let $J$ be the submodule
generated by all Jacobi identities \( \{ J(i, j, k) \mid 1 \leq i < j < k \leq n \} \). Thus, \( D_n \cong F/J \) and the quotient map is denoted by \( \rho : F \to D_n \). We lift \( \prec \) to a \( Q \)-invariant order \( \ll \) on \( F \) under which \( J \) is convex.

For each element \( g \in M_n \), let
\[
S_g = \{ f \in D_n \mid \iota^{-1}(f) < |g| \}.
\]

We denote by \( B_1 = \text{Max}(\{ e_{ij} \mid 1 \leq i < j \leq n \}) \) with respect to \( \gg \). Recall that for a subset \( S \) of an ordered abelian group \((A, \leq)\), we define
\[
\text{Max}(S) = \{ s \in S \mid s \gg t \text{ or } s \sim t, \forall t \in S \}.
\]

**Lemma 7.2.** Let \( e_{ij} \in B_1 \). If \( Qe_{ij} = Q \), then
\[
S_{a_1^la_2^m} = D_n, \forall (l, m) \in \mathbb{Z} \times \mathbb{Z} - (0, 0).
\]

**Proof.** We lift everything to the free \( \mathbb{Z}Q \)-module \((F, \tilde{\ll})\). It is enough to show that
\[
\tilde{S}_{a_1^la_2^m} = \{ f \in F \mid \iota^{-1}(\rho(f)) < |a_1^la_2^m| \} = F.
\]

Since \( Qe_{ij} = Q \), then \( e_{ij} \sim x_k e_{ij} \) for all \( k \). WLOG, we assume \( e_{12} \in B_1 \), \( e_{12} \ll 0 \), and \( Qe_{12} = Q \). First, we notice that
\[
e_{12} \gg qe_{ij}, \forall e_{ij} \notin B_1, q \in Q.
\]

Since \( qe_{12} \sim e_{12} \) and \( qe_{12} \gg qe_{ij} \). Let \( \varepsilon_{ij} \in \{ \pm 1 \} \) be the unique element such that \( \varepsilon_{ij} e_{ij} > 0 \). Therefore, by Proposition 3.3 there exists a homomorphism \( \varphi : F \to \mathbb{R} \), where
\[
\varphi(qe_{ij}) = \varepsilon_{ij} \text{ CI}(e_{12}, \varepsilon_{ij} qe_{ij}; \ll), \forall q \in Q,
\]
such that \( \varphi^{-1}((0, \infty)) \subset P_{\mathbb{Z}} \).

To simplify the proof, we will assume that \( a_1^la_2^m > 1_{M_n} \) and \( l > 0 \). Other cases are similar.

Note that for \( g \in M_n' \)
\[g < a_1^la_2^m \iff g^{a_k} < (a_1^la_2^m)^{a_k} = a_1^{la_k}a_2^{-a_k}, \forall k \in \mathbb{N}.
\]

Thus, if \( f \in \tilde{S}_{a_1} \), then
\[
(1) \quad x_2^k f - (1 + x_1)^{l-1}(1 + x_2)^{k-1}e_{12} \in \tilde{S}_{a_1}, \text{ for } k \in \mathbb{N},
\]
\[
(2) \quad x_2^k f + (1 + x_1)^{l-1}x_2^{-k}(1 + x_2)^{k-1}e_{12} \in \tilde{S}_{a_1}, \text{ for } k \in \mathbb{N}.
\]

Similar to the proof of Theorem 6.2 we have
\[
(1 + x_1)^{l-1}x_2^{-k}(1 + x_2)^{k-1}e_{12} \in \tilde{S}_{a_1}, (1 + x_1)^{l-1}((1 + x_2)^{k-1} - x_2^{-k-1} - (1 + x_2)^{k-2})e_{12} \in \tilde{S}_{a_1}, \forall k \in \mathbb{N}.
\]

Thus, for any real number \( t \), there exists \( g \in \tilde{S}_{a_1} \) such that \( \varphi(g) > t \). Hence, \( \tilde{S}_{a_1} = F \) for \( (l, m) \neq (0, 0) \). The lemma follows immediately.

Let \( \pi : M_n \to M_n/M_n' \cong Q \) be the canonical quotient map.

**Lemma 7.3.** If \( e_{ij} \in B_1 \) and \( qe_{ij} \not\sim e_{ij} \), then
\[
S_g = D_n,
\]
where \( \pi(g) = q \).
Proof. Again, we lift everything to \((F, \lesssim)\). We assume \(g > 1_{M_n}\). Then for \(h_1, h_2 \in M'_n\), we have
\[
h_1 < g \iff h_1^{h_2} < g^{h_2}.
\]
Thus, if \(f \in \widetilde{S}_g\) then
\[
f + (1 - q)h \in \widetilde{S}_g, \forall h \in F.
\]
Here we use the property of metabelian groups that \(g^{h_1} = g^{h_2}\) as long as \(\pi(h_1) = \pi(h_2)\) in a metabelian group. Thus, \(\iota(h_2^g) = q \cdot \iota(h_2)\).

Fix an arbitrary \(h \in F\). WLOG, we suppose \(e_{ij} \lesssim 0\). Since \(e_{ij} \in B_1\), there exists \(q_0\) such that \(q_0 e_{ij} \supseteq \text{supp } h\). If \(q e_{ij} \gg e_{ij}\). Then we have
\[
(1 - q)(-q_0 e_{ij}) \in \widetilde{S}_g.
\]
Note that the leading term of \((1 - q)(-q_0 e_{ij}) - h = q q_0 e_{ij}\). Thus, by Proposition 5.2
\[
(1 - q)(-q_0 e_{ij}) - h \gg 0 \iff (1 - q)(-q_0 e_{ij}) \gtrsim h \Rightarrow h \in \widetilde{S}_g.
\]
Similarly, if \(e_{ij} \gg q e_{ij}\) we have
\[
(1 - q)(q e_{ij}) \in \widetilde{S}_g,
\]
where the leading term of \((1 - q)q_0 e_{ij} - h = q_0 e_{ij}\). Thus, in this case \(h \in \widetilde{S}_g\).

Since the choice of \(h\) is arbitrary, we have that \(\widetilde{S}_g = F\). \(\square\)

Lemma 7.4. If \(e_{ij} \in B_1\) and \(x_{ij}^k e_{ij} \gg \text{Max}\{x_{k}^{\pm 1} e_{ij} \mid k = 1, 2, \ldots, n, e_{ij} \in B_1\}\), for some \(k \in \mathbb{Z}\), then
\[
S_{a_{ij} a_{ij}^m} = D_n, \forall (l, m) \in \mathbb{Z} \times \mathbb{Z} - (0, 0).
\]

Proof. We lift everything to \((F, \lesssim)\) as above. WLOG, we assume \(e_{12} \in B_1\) and \(x_{1}^{k} e_{12} \gg \text{Max}\{x_{k}^{\pm 1} e_{ij} \mid k = 1, 2, \ldots, n, e_{ij} \in B_1\}\). Since there exists \(k\) such that \(x_{1}^{k} e_{12} \gg \text{Max}\{x_{k}^{\pm 1} e_{ij} \mid k = 1, 2, \ldots, n, e_{ij} \in B_1\}\), then \(Q_{e_{12}} \neq Q\). Thus, \(S_{a_{ij} a_{ij}^m} = D_n\) for all \(l \in \mathbb{Z} - \{0\}\) by Lemma 7.3

The only remaining part is to prove \(S_{a_{ij} a_{ij}^m} = D_n\) for \(m \neq 0\). WLOG, we assume that \(m > 0\). As the discussion in Lemma 7.2, we have
\[
g \leq a_{1} a_{2}^{m} \iff g^{a_{1}^{s}} \leq a_{1} a_{2}^{ma_{1}^{s}} \iff g^{a_{1}^{s} a_{2}^{m}} \leq a_{1} a_{2}^{ma_{1}^{s}}, \forall s \in \mathbb{N}.
\]
Thus, we have that if \(f \in \widetilde{S}_{a_{ij} a_{ij}^m}\), then
\[
(1) \ x_{1}^{s} f + (1 + x_{1})^{s-1}(1 + x_{2})^{m-1} e_{12} \in \widetilde{S}_{a_{ij}} \text{ for } s \in \mathbb{N},
(2) \ x_{1}^{s} f - x_{1}^{-s}(1 + x_{1})^{s-1}(1 + x_{2})^{m-1} e_{12} \in \widetilde{S}_{a_{ij}} \text{ for } s \in \mathbb{N}.
\]
Using those constructions, we can deduce that
\[
0 \in \widetilde{S}_{a_{ij}} \overset{(1)}{\rightarrow} (1 + x_{1})^{s-1}(1 + x_{2})^{m-1} e_{12} \in \widetilde{S}_{a_{ij}} \overset{(2)}{\rightarrow} (x_{1}^{-1}(1 + x_{1})^{s-1} - x_{1}^{-1})(1 + x_{2})^{m-1} e_{12} \in \widetilde{S}_{a_{ij}} \overset{(2)}{\rightarrow} (x_{1}^{-s}(1 + x_{1})^{s-1} - x_{1}^{-s} - x_{1}^{-(s-1)}(1 + x_{1})^{s-2})(1 + x_{2})^{m-1} e_{12} \in \widetilde{S}_{a_{ij}}.
\]
Since \((1 + x_{1})^{s-1}(1 + x_{2})^{m-1} e_{12} \in \widetilde{S}_{a_{ij}}\), and \((x_{1}^{-s}(1 + x_{1})^{s-1} - x_{1}^{-s} - x_{1}^{-(s-1)}(1 + x_{1})^{s-2})(1 + x_{2})^{m-1} e_{12} \in \widetilde{S}_{a_{ij}}\), then \(x_{1}^{s} e_{12} \in \widetilde{S}_{a_{ij}}\) for all \(s \in \mathbb{Z}\) no matter \(x_{2} e_{12}\) is comparable to \(e_{12}\) or
not. Since $x_k^i e_{12} \gg \text{Max}\{x_k^{l} e_{ij} \mid k = 1, 2, \ldots, n, e_{ij} \in B_1\}$ for some $k$ then for any choice of $q \in Q$ and $e_{ij}$ there exists $s \in \mathbb{Z}$ such that

$$q \cdot e_{ij} \ll x_1^s e_{12} \in \widetilde{S}_0^l a_2^m.$$  

Therefore, $\widetilde{S}_0^l a_2^m = F$.  

Now we are ready to prove the following theorem.

**Theorem 7.5.** For $n \geq 3$, the rank of $M_n/\overline{M_n}$ is greater or equal to 2.

**Proof.** We use the notation as above. The free metabelian group $M_n$ is generated by $\{a_1, a_2, \ldots, a_n\}$, $M_n'$ is the derived subgroup of $M_n$, and $Q \cong \mathbb{Z}^n$ is the abelianization of $M_n$ with basis $\{x_1, x_2, \ldots, x_n\}$. The canonical quotient map is denoted by $\pi : M_n \to Q$ where $\pi(a_i) = x_i$.

Let $D_n$ be the $\mathbb{Z}Q$-module with presentation $\mathbb{Z}Q$. Moreover, $F$ is the free $\mathbb{Z}Q$-module with basis $\{e_{ij} \mid 1 \leq i < j \leq n\}$ and $J$ is the submodule generated by Jacobi identities $\{J(i, j, k) \mid 1 \leq i < j < k \leq n\}$. We denote by $\iota : (M_n', \leq) \to (D_n, \leq)$ the canonical isomorphism and by $\rho : (F, \prec) \to (D_n, \leq)$ the quotient homomorphism. And

$$S_g = \{f \in D_n \mid \iota^{-1}(f) \prec |g|\} \text{ and } \tilde{S}_g = \{f \in F \mid \iota^{-1} \circ \rho(f) \prec |g|\}.$$  

The goal is to prove that there exist distinct $a_i, a_j$ such that

$$S_0^l a_2^m = D_n, \forall (l, m) \in \mathbb{Z} \times \mathbb{Z} - (0, 0),$$

or equivalently $|a_i^l a_j^m| > M_n'$ whenever $|a_i^l a_j^m| \neq 1_{M_n}$. It implies $a_i^l a_j^m > \overline{M}_n$. Hence, $\mathbb{Z}^2 = \langle x_i, x_j \rangle \subset M_n/\overline{M}_n$.

Let $B_1 = \text{Max}\{e_{ij} \mid 1 \leq i < j \leq n\}$. WLOG, we assume $e_{12} \in B_1$ and $e_{12} \prec 0_F$. If $Q_{e_{12}} = Q$, the result follows from Lemma 7.2. If not, then there exists $x_i e_{12} \prec e_{12}$. We have two cases.

The first case is that

$$x_i^k e_{12} \text{ or } x_i^k e_{12} \gg T := \text{Max}\{x_i^{\pm l} e_{ij} \mid l = 1, 2, \ldots, n, e_{ij} \in B_1\} \text{ for some } k \in \mathbb{Z}.$$  

Then the result follows from Lemma 7.3. Note that if $x_i e_{12} \in T$, then $x_i^2 e_{12} \gg T$.

Therefore, we only need to consider the case $x_i^k e_{12} \in T$ for some $i > 2, \varepsilon \in \{\pm 1\}$ and $x_i^k e_{12} \gg x_i^l x_j^m e_{12}$ for all $(l, m) \in \mathbb{Z} \times \mathbb{Z}$.

If $x_i e_{12} \in T$ for $i > 2$, then we consider the Jacobi identity

$$J(1, 2, i) = (1 - x_1)e_{2i} - (1 - x_2)e_{1i} + (1 - x_i)e_{12}.$$  

We notice that if $x_i e_{12}$ is lexicographically greater than every other term in $J(1, 2, i)$, then $-x_i e_{12}$ is the leading term of $J(1, 2, i) + e_{12}$. Thus, by Proposition 5.2

$$J(1, 2, i) + e_{12} \prec 0_F \Rightarrow J(1, 2, i) \prec e_{12} \prec 0_F.$$  

It contradicts the fact that $J$ is convex under $\prec$. Therefore $x_i e_{12}$ must be comparable to one of the other terms in $J(1, 2, i)$. Since $Q_{e_{12}} \neq Q$ and $x_i e_{12} \in T$, then $x_i e_{12} \succeq e_{12}$.

If $x_i e_{12} \sim e_{1i}$ or $x_i e_{12} \sim e_{2i}$, then

$$e_{1i} \sim x_i e_{12} \gg e_{12} \text{ or } e_{2i} \gg e_{12},$$

which contradicts $e_{12} \in B_1$. Thus, the only possible choice is $x_i e_{12} \sim x_1 e_{2i}$ or $x_i e_{12} \sim x_2 e_{1i}$.

Consider the case where $x_i e_{12} \sim x_1 e_{2i}$. The other case is similar. The assumption implies
that $e_{12} \sim e_{2i}$ otherwise $x_1e_{12} \gg x_1e_{2i} \sim x_ie_{12}$ contradicting to the fact that $x_ie_{12} \in T$. Therefore, $x_1e_{12} \sim x_1e_{2i} \sim x_ie_{12}$. It leads to a contradiction, since $x_i^2e_{12} \gg T$.

If $x_i^{-1}e_{12} \in T$, then $e_{12} \gg x_i e_{12}$. As the discussion above, $e_{12}$ must be comparable with at least one of $e_{2i}, x_1e_{2i}, e_{11}$ and $x_2e_{11}$.

If $e_2 \sim e_{2i}$, then $x_i^{-1}e_{12} \sim x_i^{-1}e_{2i}$. Thus, $x_i^{-1}e_{12} \in T$ and hence by Lemma 7.4, we have

$$S_{\circ \delta a^n_m} = F, \forall (l, m) \in \mathbb{Z} \times \mathbb{Z} - (0, 0).$$

If $e_{12} \gg e_{2i}$ and $e_{12} \sim x_1e_{2i}$, then $x_1e_{12} \gg e_{12}$. By our assumption $x_i^{-1}e_{12} \gg x_i^k e_{12}$ for all $k \in \mathbb{Z}$. Therefore we have

$$x_i^k x_i^m e_{12} \not\sim e_{12}, \forall (l, m) \in \mathbb{Z} \times \mathbb{Z} - (0, 0)$$

It follows from Lemma 7.3 that

$$S_{\circ a_i^n_m} = D_n, \forall (l, m) \in \mathbb{Z} \times \mathbb{Z} - (0, 0).$$

Therefore, in this case, $\mathbb{Z}^2 = \langle x_1, x_i \rangle \subset M_n/\overline{M}_n$. This completes the proof. \qed

Combining Theorem 6.2 and Theorem 7.5, we immediately have:

**Corollary 7.6.** The space $\mathcal{O}(M_n)$ is homeomorphic to the Cantor set for $n \geq 2$.

**Proof.** Since the rank of the free abelian group $M_n'/\overline{M}_n$ is greater or equal to 2, then every bi-order is a lexicographical order with respect to the extension of $\overline{M}_n$ by $\mathbb{Z}^k$ for $k > 1$. Since the space of orders of $\mathbb{Z}^k$ has no isolated point, so does $\mathcal{O}(M_n)$. \qed

Next, we will show that $M_n'$ is not always convex when $n > 2$.

**Lemma 7.7.** Let $M$ be a $Q$-orderable $\mathbb{Z}Q$-module. Suppose there exists a homomorphism $\varphi : M \to \mathbb{R}$ such that if $\varphi(m) \neq 0$ then $\varphi(q \cdot m)/\varphi(m) > 0$ for all $q \in Q$. Then there exists a $Q$-invariant order $\leq$ such that $\varphi^{-1}((0, \infty)) \subset P_\leq$.

**Proof.** Since $M$ is $Q$-orderable, so is ker $\varphi$. Note that by the assumption, the positive cone $P = \{ \varphi(m) \mid \varphi(m) > 0 \}$ of $\varphi(M)$ is $Q$-invariant, where the action of $Q$ on $\varphi(M)$ is defined by

$$q \cdot \varphi(m) := \varphi(q \cdot m).$$

Thus, any lexicographic order $\leq$ formed by the extension of ker $\varphi$ by $\varphi(M)$ is what we are looking for. \qed

With the above lemma, we are ready to prove the following.

**Theorem 7.8.** Let $a_1^{i_1} a_2^{i_2} \ldots a_n^{i_n}$ be a non-trivial element in $M_n$ for $n > 2$. Then there exists a bi-invariant order such that $a_1^{i_1} a_2^{i_2} \ldots a_n^{i_n} \in \overline{M}_n$. In particular, for $n > 2$ there exists a bi-order on $M_n$ such that $M_n'$ is not convex.

**Proof.** By utilizing the automorphism of $M_n$, the problem reduces to proving that there exists an order such that $a_1 \in \overline{M}_n$.

We define a map $\varphi : M_n' \to \mathbb{R}$ as follows.

$$\varphi([a_1, a_i]) = 0, \varphi([a_i, a_j]^q) = 1, \text{ for } i < j \in \{2, \ldots, n\}, q \in Q.$$ 

Since $\varphi$ sends all Jacobi identities to 0, $\varphi$ extends to a homomorphism from $M_n'$ to $\mathbb{R}$. Then by Lemma 7.7, there exists a $Q$-invariant order on $M_n'$ such that $\varphi^{-1}((0, \infty)) \subset P_\leq$. 


Since $\varphi([a_1, a_i]) = 0$ for all $i = 2, \ldots, n$, by Lemma 6.1 we have $\varphi([a_1, g]) = 0$ for all $i \in \mathbb{Z}, g \in M_n$. Moreover, $Q/\langle \bar{a}_1 \rangle$ is a torsion-free finitely generated abelian group and hence bi-orderable, where $\bar{a}_1$ is the image of $a_1$ in $Q$. We denote $\pi : M_n \to Q/\langle \bar{a}_1 \rangle$ to be the canonical quotient map, the composition of the quotient maps of $M_n \to Q$ and $Q \to \langle \bar{a}_1 \rangle$.

Let $P_{Q/\langle \bar{a}_1 \rangle}$ be a positive cone on $Q/\langle \bar{a}_1 \rangle$. We then let
\[
P' := P_\leq \sqcup \left( \bigsqcup_{i \neq 0} \varphi^{-1}((0, \infty)) a_1^i \right) \sqcup \left( \bigsqcup_{i > 0} \ker \varphi a_1^i \right) \sqcup M_n^* \pi^{-1}(P_{Q/\langle \bar{a}_1 \rangle}).
\]

Let us check that $P'$ is a positive cone.

Since $P_\leq$ and $P_{Q/\langle \bar{a}_1 \rangle}$ are semigroups, $g_1 g_2 \in P'$ if $g_1, g_2 \in P_\leq \sqcup M_n^* \pi^{-1}(P_{Q/\langle \bar{a}_1 \rangle})$. Note that
\[
g_1 a_1^i g_2 = g_1 g_2 a_1^i g_2 a_1^{-i}, \forall g_1, g_2 \in M_n,
\]
and $\varphi([a_1^i, a_2^j]) = 0$. Then it is not hard to check the product $g_1 g_2 \in P'$ where $g_1 \in \left( \bigsqcup_{i \neq 0} \varphi^{-1}((0, \infty)) a_1^i \right) \sqcup \left( \bigsqcup_{i > 0} \ker \varphi a_1^i \right)$ and $g_2 \in P_\leq \sqcup M_n^* \pi^{-1}(P_{Q/\langle \bar{a}_1 \rangle})$ since $\varphi(P_\leq) \geq 0$ and $\varphi(M_n^*) = 0$. Therefore, $P'$ is a semigroup.

The inverse of $P'$ can be written as follows.
\[
P'^{-1} = P_\leq \sqcup \left( \bigsqcup_{i \neq 0} \varphi^{-1}((-\infty, 0)) a_1^i \right) \sqcup \left( \bigsqcup_{i < 0} \ker \varphi a_1^i \right) \sqcup M_n^* \pi^{-1}(P_{Q/\langle \bar{a}_1 \rangle}).
\]

Here we use the fact that
\[
(g a_1^i)^{-1} = a_1^{-i} g^{-1} = [a_1^i, g] g^{-1} a_1^{-i}.
\]

Thus, we have $M_n = P' \sqcup P'^{-1} \sqcup \{1\}$.

It is not hard to check that $P'$ is invariant under conjugation with elements in $M_n$. Therefore, $P'$ defines a bi-order $<'$ on the free metabelian group $M_n$. And by its definition, we immediately have
\[
\ker \varphi < a_1 < \varphi^{-1}((0, \infty)).
\]

Therefore, $M_n'$ is not convex under $<'$ and any power of $a_1 \in \overline{M_n}$.

\[\square\]

Remark. When $n = 2$, the map we define is a zero map. Thus, $M_2' = \ker \varphi$.

8. ORDERS ON FREE METABELIAN GROUPS ARE NEVER REGULAR

Let $G$ be a finitely generated group and $X$ a finite generating set of $G$. An order $\leq$ on $G$ is said to be regular (context-free) if there exists a regular (context-free) language $\mathcal{L} \subset X^*$ such that $\pi(\mathcal{L}) = P_\leq$. An order $\leq$ is computable if there exists an algorithm to decide if $g \leq h$ for any pair of $g, h \in G$. All those properties are independent of the choice of the finite generating set [ARS21, Lemma 2.11].

By Theorem 6.2 and Theorem 7.5, there are uncountably many bi-orders on $M_n$ for $n \geq 2$. Hence, there always exist uncountably many orders that are not computable on $M_n$.

In this section, we will show that there exist computable orders on $M_n$ but none of them are regular when $n \geq 2$.

Recall that by Magnus embedding (See [Mag30], [Bau73]), a free metabelian group of rank $n$ embeds into the wreath product of two free abelian groups of rank $n$. It naturally inherits a computable left-order from the wreath product [ARS21]. However, the regular lexicographical left-order on the wreath product where the base group leads is not bi-invariant. One
workaround is to replace the lexicographical order by one which leads by the quotient. The order will become computably bi-invariant, but no longer regular.

Let $M_n$ be the free metabelian group of rank $n$ and $A_n, T_n$ free abelian groups of rank $n$. The generating sets of $M_n, A_n, T_n$ are respectively $X = \{x_1, x_2, \ldots, x_n\}$, $A = \{a_1, a_2, \ldots, a_n\}$ and $T = \{t_1, t_2, \ldots, t_n\}$. The Magnus embedding $\varphi : M_n \to A_n \triangleright T_n$ is given by the homomorphism $\varphi(x_i) = a_i t_i$.

Let $P_A$ and $P_T$ be regular positive cones of $A_n$ and $T_n$ respectively. We define a bi-order on the base group $B = \bigoplus_{i \in T_n} A_n$ as follows. Firstly, note that an element $f$ in $B$ can be uniquely written as a product of conjugates of elements in $A_n$ in the following fashion:

$$f = g_1^{h_1} g_2^{h_2} \ldots g_s^{h_s}, g_i \in A_n, h_i \in T_n,$$

such that $h_1 > h_2 > \cdots > h_s$ with respect to the order on $T_n$. Thus we define $f > 1$ if the leading term $g_1 > 1$. It is not hard to check that this order on $B$ is invariant under the action of $T_n$. The lexicographical order on $A_n \triangleright T_n$ is given by the positive cone $P = P_B \cup \pi^{-1}(P_T)$ where $\pi : A_n \triangleright T_n \to T_n$ is the canonical quotient map.

One remark is that while the lexicographical order where the base group leads is regular, the new order we just define is not regular (it can be shown using the same idea as [ARS21, Lemma 3.11]). Next we will show that $P$ can be recognised by a context-free language.

Let $L_A, L_T$ be the regular languages evaluate onto $P_A$ and $P_T$ respectively. We then define

$$L = \{vw_1 w_2 w_3 \ldots w_n z \mid u_1 \in L_A, w_i \in L_T, u_2, \ldots, u_n \in (A \cup A^{-1})^*,$$

$$v, z \in (T \cup T^{-1})^*, z = F_T(vw_1 w_2 \ldots w_n)^{-1}, n \neq 0 \}$$

$$\cup \{vw_1 w_2 w_3 \ldots w_n z z' \mid w_i, z' \in L_T, u_i \in (A \cup A^{-1})^*, z = F_T(vw_1 w_2 \ldots w_n)^{-1} \}.$$ 

The first part of $L$ covers the positive elements in the base group, and the second part covers the rest. The actual pushdown machine is not hard to construct, as it needs a stack to store the information on $vw_1 w_2 \ldots w_n$. Thus, $L$ is context-free.

The membership problem of $M_n$ in $A_n \triangleright T_n$ is not hard to solve. Therefore, $P \cap M_n$ is recursive at the very least. Whether the set $P \cap M_n$ can be recognised as a context-free language remains unknown.

But for $M_2$ we can indeed construct a context-free bi-order on it. Let $X = \{a, b, c\}$, where $a, b$ generate $M_2$ and $c = [a, b]$. The quotient is generated by the set $T := \{\bar{a}, \bar{b}\}$, where $\bar{a}, \bar{b}$ are the images of $a, b$ respectively. Let $P_Q$ be a regular positive cone on $Q = M_2/M'_2$ and $L_Q$ be the corresponding regular language. We then define

$$L = \{vc^t_1 w_1 c^t_2 w_2 \ldots c^t_n w_n z \mid t_1 \in \mathbb{N}, w_i \in \mathbb{L}_Q, t_2, \ldots, t_n \in \mathbb{Z} \setminus \{0\},$$

$$v, z \in (T \cup T^{-1})^*, z = F_T(vw_1 w_2 \ldots w_n)^{-1}, n \neq 0 \}$$

$$\cup \{vc^t_1 w_1 c^t_2 w_2 \ldots c^t_n w_n z z' \mid w_i, z' \in L_Q, t_i \in \mathbb{Z} \setminus \{0\}, z = F_T(vw_1 w_2 \ldots w_n)^{-1} \}.$$ 

It is not hard to check $L$ is context-free and recognises a bi-invariant positive cone of $M_2$.

In summary, we have that:

**Theorem 8.1.** Free metabelian groups of finite rank are computably bi-orderable. Moreover, $M_1 \cong \mathbb{Z}$ admits a regular bi-order and $M_2$ admits a context-free bi-order.

Next we will show that none of the bi-orders are regular for $M_n, n \geq 2$ by proving a more general theorem for Conradian orders. Recall that a left-order is Conradian if for all positive elements $f, g$, there exists $n \in \mathbb{N}$ such that $g^{-1} f g^n$ is positive. A bi-order is always Conradian, since it is invariant under conjugation.
Recall that a subset $S$ of a metric space $(X, d)$ is coarsely connected if there is $R > 0$ such that the $R$-neighbourhood of $S$ is connected. The following lemma gives a description of a regular positive cone from a geometric perspective.

Lemma 8.2 ([AABR22 Proposition 7.2]). Let $G$ be a finitely generated group. If $\leq$ is a regular order on $G$, then $P_{\leq}$ and $P_{\leq}^{-1}$ are coarsely connected subsets of the Cayley graph of $G$.

Next, recall that for a finitely generated group $G$ a non-trivial homomorphism $\varphi : G \to \mathbb{R}$ belongs to $\Sigma^1(G)$, the Bieri-Neumann-Strebel invariant (BNS invariant for short), if and only if $\varphi^{-1}((0, \infty))$ is coarsely connected. For details of BNS invariant, we refer [BNS87].

Following the idea of [ARS21, Lemma 3.11], we have

**Theorem 8.3.** Let $G$ be a finitely generated Conradian orderable group with $\Sigma^1(G) = \emptyset$. Then no Conradian order of $G$ is regular.

**Proof.** Let $G$ be a finitely generated Conradian orderable group with a Conradian order $\leq$. Consider the maximal proper subgroup $H \leq G$. Such $H$ exists and is normal [BMR77]. We have that $G/H$ is Archimedean with respect to the induced order. In particular, by Hölder’s theorem $G/H$ is a free abelian group of finite rank [Hol01].

Let $\pi : G \to G/H$ be the canonical quotient map. Since $H$ is convex, $P_{\leq}$ induces an order $\leq$ on $G/H$.

We claim that there exists a homomorphism $\varphi : G/H \to \mathbb{R}$ such that $\varphi^{-1}((0, \infty))$ consists of positive elements with respect to $\leq$. Since $G/H$ is a free abelian group of finite rank, then $\leq$ corresponds to a hyperplane. Let $n$ be a normal vector to the hyperplane, where $n$ points to the positive side. Then the map $\varphi(g) := g \cdot n$ is the homomorphism we are seeking for.

Now let $f := \varphi \circ \pi : G \to \mathbb{R}$. Note that ker $\varphi$ is finitely generated as it is a subgroup of a free abelian group of finite rank. Let $P_1$ be the positive cone of $H$ and $g_1 H, g_2 H, \ldots, g_t H$ be the generating set of ker $\varphi$. Then we have

$$P_{\leq} = f^{-1}((0, \infty)) \cup P_1 \cup \left( \bigcup_{g_1^{s_1}g_2^{s_2}\ldots g_t^{s_t} H \geq 1} g_1^{s_1}g_2^{s_2}\ldots g_t^{s_t} H \right), s_i \in \mathbb{Z}.$$  

Since there exists a generator that is sent to a positive number, thus in the Cayley graph of $G$ the distance between $f^{-1}((0, \infty))$ and $P_1$ or any coset $g_1^{s_1}g_2^{s_2}\ldots g_t^{s_t} H$ is 1. Thus, $f^{-1}((0, \infty))$ is coarsely connected if and only if $P_{\leq}$ is coarsely connected. By BNS theory, $f \in \Sigma^1(G)$ if and only $P_{\leq}$ is coarsely connected. Since $H$ contains the derived subgroup, we have

$$(g_1^{s_1}g_2^{s_2}\ldots g_t^{s_t} H)^{-1} = g_1^{-s_1}g_2^{-s_2}\ldots g_t^{-s_t} H.$$  

Thus, $-f \in \Sigma^1(G)$ if and only $P_{\leq}^{-1}$ is coarsely connected.

If $P_{\leq}$ is a regular positive cone, both $P_{\leq}$ and $P_{\leq}^{-1}$ are coarsely connected by Lemma 8.2. Thus, $f$ and $-f$ belong to $\Sigma^1(G)$, which is a contradiction since $\Sigma^1(G)$ is empty.

Note that for free metabelian group $M_n$ where $n \geq 2$, the centraliser of $M_n'$ in $\mathbb{Z}Q$ is trivial. Thus, $\Sigma^1(M_n)$ is empty [BS80].

**Corollary 8.4.** The non-abelian free metabelian group of finite rank does not admit a regular Conradian order. In particular, no bi-order on a non-abelian free metabelian of finite rank is regular.
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