KAPPA-DEFORMED SPACE-TIME UNCERTAINTY RELATIONS

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ABSTRACT: We discuss the $\kappa$-deformed phase space obtained as a cross product algebra of the deformed translations algebra and its dual configuration space. We consider two kinds of the $\kappa$-deformed uncertainty relations.

1. Introduction

Deformations of space-time symmetries are extensively investigated in last years. In this approach the notion of symmetries is generalized to quantum groups i.e. Hopf algebras and in consequence we deal with noncommuting space-time. Having a deformed configuration space it is interesting to build and investigate a deformed phase space. For a given configuration space one can define a momentum space using the concept of duality. It appears that such a momentum space has also a Hopf algebra structure.

The problem arises when we need to define a deformed phase space as a generalization of the deformed configuration and momentum spaces. Roughly speaking, one can define the commutators between position and momentum in different ways. Therefore, the generalization procedure is ambiguous.

It is known that from a pair of dual Hopf algebras describing the quantum symmetry group and its dual quantum Lie algebra one can construct three different double algebras: Drinfeld double and Drinfeld codouble, both with Hopf algebra structure, and Heisenberg double (or more generally, cross product algebra), which is not a Hopf algebra.

It is easy to see that deformed phase space with a Drinfeld double structure does not give us in the nondeformed limit the standard quantum mechanical phase space, because the canonical commutation relations between momentum and position operators are broken. Therefore, deformed phase space cannot be a Hopf algebra.

In the present note we consider a deformed phase space which is an extension of the $\kappa$-deformed Minkowski space [1], dual to the momentum sector of $\kappa$-Poincaré algebra [2].

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Choosing different realizations of $\kappa$-Poincaré algebra we can obtain different $\kappa$-deformed phase spaces by the cross product algebra construction. In particular, we discuss two realizations of $\kappa$-Poincaré basis: bicrossproduct basis \cite{4} and standard basis \cite{2}.

2. $\kappa$-deformed phase space as cross product algebra

Let us begin our considerations assuming the bicrossproduct basis for $\kappa$-Poincaré algebra \cite{4}, then the $\kappa$-deformed Hopf algebra of translations $\mathcal{P}_\kappa$ is given by ($k = 1, 2, 3$)

i) bicrossproduct basis

\[
[P_0, P_k] = 0 \quad (1.a)
\]
\[
\Delta(P_0) = P_0 \otimes 1 + 1 \otimes P_0
\]
\[
\Delta(P_k) = P_k \otimes 1 + e^{-\frac{\kappa}{\hbar}} \otimes P_k \quad (1.b)
\]
and we define the antipode and counit as follows

\[
S(P_\mu) = -P_\mu \quad \epsilon(P_\mu) = 0 \quad (2)
\]

where two constants are introduced: $\hbar$ - Planck’s constant and $\kappa$ - massive deformation parameter.

Using the duality relations ($\mu, \nu = 0, 1, 2, 3$)

\[
< x_\mu, P_\nu > = i\hbar g_{\mu\nu} \quad g_{\mu\nu} = (-1, 1, 1, 1) \quad (3)
\]

we obtain the noncommutative $\kappa$-deformed configuration space $\mathcal{X}_\kappa$ as a Hopf algebra with the following algebra and coalgebra structure \cite{1}

\[
[x_0, x_k] = \frac{\kappa}{\hbar} x_k
\]
\[
[x_k, x_l] = 0 \quad (4a)
\]
\[
\Delta(x_\mu) = x_\mu \otimes 1 + 1 \otimes x_\mu
\]
\[
S(x_\mu) = -x_\mu \quad \epsilon(x_\mu) = 0 \quad (4b)
\]

It is obvious that the algebra $\mathcal{P}_\kappa$ is a left (right) $\mathcal{X}_\kappa$-module and vice versa the algebra $\mathcal{X}_\kappa$ is $\mathcal{P}_\kappa$-module if we introduce the following actions (we use the Sweedler’s notation)

-**left actions**

\[
\triangleright : \mathcal{X}_\kappa \otimes \mathcal{P}_\kappa \to \mathcal{P}_\kappa : x \otimes p \to x \triangleright p = < x, p(2) > p(1) \quad (5.a)
\]
\[
\triangleright : \mathcal{P}_\kappa \otimes \mathcal{X}_\kappa \to \mathcal{X}_\kappa : p \otimes x \to p \triangleright x = < p, x(2) > x(1) \quad (5.b)
\]

-**right actions**

\[
\langle : \mathcal{P}_\kappa \otimes \mathcal{X}_\kappa \to \mathcal{P}_\kappa : p \otimes x \to p \langle x = < p, x(1) > p(2) \quad (5.c)
\]
\[
\langle : \mathcal{X}_\kappa \otimes \mathcal{P}_\kappa \to \mathcal{X}_\kappa : x \otimes p \to x \langle p = < p, x(1) > x(2) \quad (5.d)
\]
in chosen basis the actions are the following

\[ \triangleright : \mathcal{X}_\kappa \otimes \mathcal{P}_\kappa \rightarrow \mathcal{P}_\kappa : \]

\[
\begin{align*}
x_0 \triangleright P_0 &= -i\hbar \\
x_k \triangleright P_0 &= 0 \\
x_0 \triangleright P_k &= 0 \\
x_k \triangleright P_l &= i\hbar \delta_{kl}e^{-\frac{\theta}{\hbar}}
\end{align*}
\] (6.a)

\[ \triangleright : \mathcal{P}_\kappa \otimes \mathcal{X}_\kappa \rightarrow \mathcal{X}_\kappa : \]

\[
\begin{align*}
P_0 \triangleright x_0 &= -i\hbar \\
P_k \triangleright x_0 &= 0 \\
P_0 \triangleright x_k &= 0 \\
P_k \triangleright x_l &= i\hbar \delta_{kl}
\end{align*}
\] (6.b)

\[ \triangleleft : \mathcal{P}_\kappa \otimes \mathcal{X}_\kappa \rightarrow \mathcal{P}_\kappa : \]

\[
\begin{align*}
P_0 \triangleleft x_0 &= -i\hbar \\
P_k \triangleleft x_0 &= \frac{i}{\kappa}P_k \\
P_0 \triangleleft x_k &= 0 \\
P_k \triangleleft x_l &= i\hbar \delta_{kl}
\end{align*}
\] (6.c)

\[ \triangleleft : \mathcal{X}_\kappa \otimes \mathcal{P}_\kappa \rightarrow \mathcal{X}_\kappa : \]

\[
\begin{align*}
x_0 \triangleleft P_0 &= -i\hbar \\
x_k \triangleleft P_0 &= 0 \\
x_0 \triangleleft P_k &= 0 \\
x_k \triangleleft P_l &= i\hbar \delta_{kl}
\end{align*}
\] (6.d)

In order to construct a deformed phase space \( \Pi_\kappa \) one has to extend the commutation relations (1a) and (4b) by adding a cross commutators between \( \mathcal{X}_\kappa \) and \( \mathcal{P}_\kappa \). In this way \( \Pi_\kappa \sim \mathcal{X}_\kappa \otimes \mathcal{P}_\kappa \) as a vector space and becomes an associative algebra.

We find the cross commutators using the notion of a left (right) cross product (smash product) algebra.

We recall the following definition

**Def.** (cross product algebra) [5]

Let \( \mathcal{P} \) be a Hopf algebra and \( \mathcal{X} \) a left (right) \( \mathcal{P} \)-module algebra. Left (right) cross product algebra \( \mathcal{X} \times \mathcal{P} \) (\( \mathcal{P} \times \mathcal{X} \)) is a vectors space \( \mathcal{X} \otimes \mathcal{P} \) with product

\[ (x \otimes p)(\tilde{x} \otimes \tilde{p}) = x(p_{(1)} \triangleright \tilde{x}) \otimes p_{(2)}\tilde{p} \quad \text{(left cross product)} \] (7.a)

and

\[ (p \otimes x)(\tilde{p} \otimes \tilde{x}) = p\tilde{p}_{(1)} \otimes (x \triangleleft \tilde{p}_{(2)})\tilde{x} \quad \text{(right cross product)} \] (7.b)

with unit element \( 1 \otimes 1 \), where \( x, \tilde{x} \in \mathcal{X} \) and \( p, \tilde{p} \in \mathcal{P} \).

In our case the notion of cross product algebra is equivalent to the Heisenberg double algebra [3].

The obvious isomorphism \( \mathcal{X} \sim \mathcal{X} \otimes 1 \), \( \mathcal{P} \sim \mathcal{P} \otimes 1 \) gives us the following cross relations between the configuration and momentum space

\[ p \circ x = (p_{(1)} \triangleright x) \circ p_{(2)} \quad \text{(left cross product)} \] (8.a)
\[ x \circ p = p_{(1)} \circ (x \triangleleft p_{(2)}) \quad \text{(right cross product)} \] (8.b)

which one can rewrite as a cross commutation relations \([x, p] = x \circ p - p \circ x\).
Let us notice that the following isomorphisms hold between left and right cross product algebras
\[ \mathcal{X}_\kappa \bowtie \mathcal{P}_\kappa \sim \mathcal{X}_\kappa \bowtie \mathcal{P}_\kappa \] (9.a)
\[ \mathcal{P}_\kappa \bowtie \mathcal{X}_\kappa \sim \mathcal{P}_\kappa \bowtie \mathcal{X}_\kappa \] (9.b)
therefore, it is enough to consider left cross product algebras \( \mathcal{X}_\kappa \bowtie \mathcal{P}_\kappa \) and \( \mathcal{P}_\kappa \bowtie \mathcal{X}_\kappa \) as the models of our \( \kappa \)-deformed phase space \( \Pi_\kappa \).

From the relations (1.a), (4.a) and the cross relations (7.a) it follows that we have the commutation relations in the form
- the case \( \Pi_\kappa \sim \mathcal{X}_\kappa \bowtie \mathcal{P}_\kappa \)

\[ [P_0, P_k] = 0 \] (10.a)
\[ [x_\mu, x_\nu] = \frac{i}{\kappa} (\delta_{\mu0} x_\nu - \delta_{\nu0} x_\mu) \] (10.b)
\[ [x_\mu, P_0] = i\hbar \delta_{\mu0} \] (10.c)
\[ [x_\mu, P_k] = -i\hbar \delta_{\mu k} - \frac{i}{\kappa} \delta_{\mu0} P_k \] (10.d)

In analogous way, from (1.a), (4.a) and (7.b) we obtain
- the case \( \Pi_\kappa \sim \mathcal{P}_\kappa \bowtie \mathcal{X}_\kappa \)

\[ [P_0, P_k] = 0 \] (11.a)
\[ [x_\mu, x_\nu] = \frac{i}{\kappa} (\delta_{\mu0} x_\nu - \delta_{\nu0} x_\mu) \] (11.b)
\[ [x_0, P_\mu] = -i\hbar \delta_{\mu0} \] (11.c)
\[ [x_k, P_\mu] = i\hbar \delta_{\mu k} e^{-\frac{P_0}{2\kappa}} \] (11.d)

If we choose \( g_{\mu\nu} \) with opposite signs in (3) then the relations (10.b – d) and (11.b – d) change signs.

On the other hand, if we choose instead (1.b) transposed coproduct

\[ \tilde{\Delta}(P_k) = P_k \otimes e^{-\frac{P_0}{2\kappa}} + 1 \otimes P_k \]

the commutators (10) change signs and we obtain \( \kappa \)-deformed phase space discussed in [3].

For our choice we obtain in the limit \( \kappa \to \infty \) the quantum-mechanical phase space in the case \( \Pi_\kappa \sim \mathcal{P}_\kappa \bowtie \mathcal{X}_\kappa \) (relations (11)).

If we consider the \( \kappa \)-Poincaré algebra in the standard basis [2], the translations subalgebra \( \Pi_\kappa \) is given by

ii) standard basis
\[ [P_0, P_k] = 0 \] (12.a)
\[ \Delta(P_0) = P_0 \otimes 1 + 1 \otimes P_0 \]
\[ \Delta(P_k) = P_k \otimes e^{\frac{P_0}{2\kappa}} + e^{-\frac{P_0}{2\kappa}} \otimes P_k \] (12.b)
In analogous way we obtain two cross product algebras

- the case $\Pi_\kappa \sim X_\kappa \gg P_\kappa$

\[
[P_0, P_k] = 0 \quad \text{(13.a)}
\]
\[
[x_\mu, x_\nu] = \frac{i}{\kappa} (\delta_{\mu0} x_\nu - \delta_{\nu0} x_\mu) \quad \text{(13.b)}
\]
\[
[x_\mu, P_0] = i\hbar \delta_{\mu0} \quad \text{(13.c)}
\]
\[
[x_\mu, P_k] = -i\hbar \delta_{\kappa \mu} e^{-\frac{P_0}{2\kappa \hbar}} - \frac{i}{2\kappa} \delta_{0\mu} P_k \quad \text{(13.d)}
\]

- the case $\Pi_\kappa \sim P_\kappa \gg X_\kappa$

\[
[P_0, P_k] = 0 \quad \text{(14.a)}
\]
\[
[x_\mu, x_\nu] = \frac{i}{\kappa} (\delta_{\mu0} x_\nu - \delta_{\nu0} x_\mu) \quad \text{(14.b)}
\]
\[
[x_\mu, P_0] = -i\hbar \delta_{\mu0} \quad \text{(14.c)}
\]
\[
[x_\mu, P_k] = i\hbar \delta_{\kappa \mu} e^{-\frac{P_0}{2\kappa \hbar}} - \frac{i}{2\kappa} \delta_{0\mu} P_k \quad \text{(14.d)}
\]

Let us notice that only for the algebra (14) we get the correct limit for $\kappa \to \infty$, so one can assume that the relations (14) define $\kappa$-deformed phase space.

3. $\kappa$-deformed uncertainty relations

Introducing the dispersion of the observable $a$ in quantum mechanical sense by

\[
\Delta(a) = \sqrt{<a^2> - <a>^2} \quad \text{(15.a)}
\]

we have

\[
\Delta(a) \Delta(b) \geq \frac{1}{2} | <c> | \quad \text{where} \quad c = [a, b] \quad \text{(15.b)}
\]

In the first case (11), we obtain $\kappa$-deformed uncertainty relations in $\Pi_\kappa$ in the form

\[
\Delta(x_0) \Delta(x_k) \geq \frac{1}{2\kappa} | <x_k> | \quad \text{(16.a)}
\]
\[
\Delta(P_k) \Delta(x_l) \geq \frac{1}{2} \hbar \delta_{kl} | <e^{-\frac{P_0}{2\kappa \hbar}}> | \quad \text{(16.b)}
\]
\[
\Delta(P_0) \Delta(x_0) \geq \frac{1}{2} \hbar \quad \text{(16.c)}
\]
\[
\Delta(P_k) \Delta(x_0) \geq 0 \quad \text{(16.d)}
\]

and in the second case (14) we get

\[
\Delta(x_0) \Delta(x_k) \geq \frac{1}{2\kappa} | <x_k> | \quad \text{(17.a)}
\]
\[
\Delta(P_k)\Delta(x_l) \geq \frac{1}{2}\hbar \delta_{kl} \left< e^{-\frac{P_0}{2\hbar}} \right> | \tag{17.b}
\]
\[
\Delta(P_0)\Delta(x_0) \geq \frac{1}{2}\hbar \tag{17.c}
\]
\[
\Delta(P_k)\Delta(x_0) \geq \frac{1}{2\kappa} | < P_k > | \tag{17.d}
\]

In both cases we obtain the standard uncertainty relations in the limit \( \kappa \to \infty \). The physical implications of these deformed uncertainty relations will be considered elsewhere.

4. Final remarks

In this note we presented two realizations of \( \kappa \)-deformed phase space. The cross relations between position and momentum operators form two kinds of generalized Heisenberg commutation relations depending on two parameters: \( \hbar \) - Planck’s constant and \( \kappa \) - deformation parameter with mass dimension.

We would like to stress that the realizations of the cross product algebras of \( \kappa \)-deformed translations \( P_\kappa \) and positions \( X_\kappa \) strongly depend on the choice of the basis of \( \kappa \)-Poincaré algebra as well on the metric tensor \( g_{\mu\nu} \) (see also [3]).

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