MINIMAL MASS NON-SCATTERING SOLUTIONS OF THE FOCUSING \(L^2\)-CRITICAL HARTREE EQUATIONS WITH RADIAL DATA

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Abstract. We prove that for the Cauchy problem of focusing \(L^2\)-critical Hartree equations with spherically symmetric \(H^1\) data in dimensions 3 and 4, the global non-scattering solution with ground state mass must be a solitary wave up to symmetries of the equation. The approach is a linearization analysis around the ground state combined with an in-out spherical wave decomposition technique.

1. Introduction. Consider the Cauchy problem of focusing \(L^2\)-critical nonlinear Schrödinger equation with Hartree type nonlinearity (Hartree equation)

\[
\begin{aligned}
\begin{cases}
 i \partial_t u + \Delta u = -(|\cdot|^2 * |u|^2)u, & (t, x) \in \mathbb{R} \times \mathbb{R}^d, \\
 u(0, x) = u_0(x) \in H^1_x(\mathbb{R}^d),
\end{cases}
\end{aligned}
\]

where \(d \geq 3\), \(u(t, x)\) is a complex-valued function, and \(*\) denotes convolution on \(\mathbb{R}^d\). The nonlinear Schrödinger equation of Hartree type describes the dynamics of mean-field limits of many-body quantum systems, such that coherent states, condensates. In particular, it provides an effective models for quantum systems with long-range interactions, see, e.g., [4]. The equation in (1) is \(L^2\)-critical since it is preserved by the scaling

\[u_\lambda(t, x) = \lambda^{d/2} u(\lambda^2 t, \lambda x)\]

which keeps the \(L^2\)-norm of the initial datum invariant.

The equation also admits the following symmetries:

- Translation: \(u(t, x) \mapsto u(t + t_0, x + x_0), \quad t_0 \in \mathbb{R}, x_0 \in \mathbb{R}^d\),
- Phase rotation: \(u(t, x) \mapsto e^{i\theta} u(t, x), \quad \theta \in \mathbb{R}\),

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\begin{itemize}
  \item Time reversal: \( u(t, x) \mapsto u(-t, x). \)

  The phase rotation and the time translation symmetries respectively lead to mass and energy conservation laws:
  \begin{itemize}
    \item Mass:
      \[ N(u(t)) = \int_{\mathbb{R}^d} |u(t, x)|^2 \, dx \equiv N(u_0), \]
    \item Energy:
      \[ E(u(t)) = \frac{1}{2} \int_{\mathbb{R}^d} |\nabla u(t, x)|^2 \, dx - \frac{1}{4} \int_{\mathbb{R}^d} (|\cdot - 2 \ast |u|^2)|u|^2(t, x) \, dx \equiv E(u_0). \]
  \end{itemize}

Throughout this paper, we always use \( \mathcal{P}(u) \) to denote \( \frac{1}{4} \int_{\mathbb{R}^d} (|\cdot - 2 \ast |u|^2)|u|^2(x) \, dx. \)

Let \( I \) be a time interval containing 0. By \( u(t, x) : I \times \mathbb{R}^d \mapsto \mathbb{C} \) a strong \( L^2 \) solution of \( (1) \), we mean \( u \in C^0_\text{loc} L^2_x \cap L^6_x L^{6/3}_t(K \times \mathbb{R}^d) \) for any compact \( K \subset \mathbb{R}^d \) and \( u \) obeys the Duhamel formula

\[ u(t) = e^{i t \Delta} u_0 + i \int_0^t e^{i(t-\tau) \Delta} (|\cdot - 2 \ast |u|^2) u(\tau) \, d\tau, \quad \forall t \in I. \]

\( I \) is called the maximal life-span if it can not be extended to any strictly larger interval. If \( I = \mathbb{R}, \) we say \( u \) is global.

The local theory for \( (1) \) with \( L^2_x \) data has been established in \([27]\) (see also \([1]\)). We summarize the results in the following.

**Theorem 1.1** (Local well-posedness). Let \( d \geq 3 \). Let \( u_0 \in L^2_x(\mathbb{R}^d) \). There exists a unique solution \( u \) of \( (1) \) with maximal life-span \( I \). Moreover, \( u \) has the following properties:

- (i) \( I \) is an open neighborhood of 0.
- (ii) (Mass conservation) \( N(u(t)) = N(u_0) \) for all \( t \in I \).
- (iii) (Blow-up criterion) If \( \sup I < \infty \), then there exists \( t_1 \in I \) such that
  \[ \|u\|_{L^6_t L^{6/3}_x((t_1, \sup I) \times \mathbb{R}^d)} = \infty; \quad \text{if } \inf I < \infty, \text{ then there exists } t_2 \in I \text{ such that } \|u\|_{L^6_t L^{6/3}_x((\inf I, t_2) \times \mathbb{R}^d)} = \infty. \]
- (iv) (Scattering) If \( \sup I = \infty \) and \( \|u\|_{L^6_t L^{6/3}_x([0, \infty) \times \mathbb{R}^d)} < \infty \), then \( u \) scatters in this direction. Namely, there exists \( u_+ \in L^2_x(\mathbb{R}^d) \) such that
  \[ \lim_{t \to \infty} \|u(t) - e^{i t \Delta} u_+\|_{L^2_x(\mathbb{R}^d)} = 0; \]

Conversely, given \( u_+ \in L^2_x \), there exists a unique solution \( u \) of \( (1) \) such that \( (1) \) holds. The analog statement holds for the negative time direction.

- (v) (Small data global existence and scattering) If \( N(u_0) \) is sufficiently small, then \( u \) is global and \( \|u\|_{L^6_t L^{6/3}_x(\mathbb{R} \times \mathbb{R}^d)} < \infty \).

For the Cauchy problem \( (1) \), there is a solitary wave \( e^{it} Q \). Here \( Q \) is the unique positive radial smooth decreasing solution \([17]\), called ground state, of the following elliptic equation

\[ -\Delta Q + Q = (|\cdot - 2 \ast Q|^2)Q, \quad x \in \mathbb{R}^d. \]

The ground state \( Q \) is a minimizer of the functional

\[ J(f) := \frac{\|f\|_{L^2_x(\mathbb{R}^d)}^2 \|\nabla f\|_{L^2_x(\mathbb{R}^d)}^2}{\int_{\mathbb{R}^d} (|\cdot - 2 \ast |f|^2)|f|^2 \, dx}. \]
In [17], the uniqueness of the ground state was solved when $d = 4$. In this case, the nonlinearity is a Newtonian potential, and by the Newton’s charge theorem [22], the uniqueness can be achieved using the ODE technique in [18]. If $d = 3$, the nonlinearity is non-Newtonian, and the ODE trick is no longer available. In this paper, we always assume the resolution of the uniqueness for $d = 3$.

**Assumption:** The ground state of (5) is unique when $d = 3$.

In [20], Miao-Xu-Zhao established the variational characterization of the ground state when $d = 4$. The argument can be adapted to the case $d = 3$ on the condition that the ground state is unique. In particular, the following holds.

**Proposition 1** ([20]). Let $d = 3, 4$. For $f \in H^1(\mathbb{R}^d)$,

$$
\int_{\mathbb{R}^d} (|\cdot|^{-2} * |f|^2)|f|^2 \, dx \leq \frac{2}{\|Q\|_{L^2(\mathbb{R}^d)}} \|f\|_{L^2(\mathbb{R}^d)}^2 \|\nabla f\|_{L^2(\mathbb{R}^d)}^2.
$$

(6)

If $\mathcal{N}(f) = \mathcal{N}(Q)$ and $E(f) = 0$. Then $f(x) = e^{i\theta} \Lambda^{d/2} Q(\lambda x + b)$ for some $\theta \in \mathbb{R}$, $\lambda > 0$, $b \in \mathbb{R}^d$.

The scattering problem for critical Hartree equations, especially energy-critical and mass-critical equations, has been well studied by Miao C., et al. Specifically, in [30], Miao-Xu-Zhao used the induction on energy method to prove for defocusing energy critical Hartree equation that spherically symmetric solutions are globally well-posed and scatter. The radial assumption was removed by these authors in [30]. The scattering theory for the focusing energy-critical Hartree equation was established in [19], using a concentration-compactness technique. For the mass critical equation, Miao-Xu-Zhao showed in [28] that the spherically symmetric solution is globally wellposed and scatters if its mass is below the ground state. For the scattering result of NLS, see e.g., [2, 3, 10, 14, 13].

Notice that the equation in (1) admits the pseudo-conformal invariance

$$
u(t, x) \to \frac{1}{|t|^{d/2}} e^{i \frac{|x|^2}{|t|}} u(\frac{1}{t}, \frac{1}{x}).$$

Applying this transformation to $e^{itQ}$, one obtains a finite time blow-up solution

$$PS(e^{itQ}) := \frac{1}{|t - t_0|^{d/2}} e^{i \frac{|x|^2}{|t - t_0|}} e^{-i \frac{|x|^2}{|t - t_0|}} Q(\frac{x}{t - t_0})$$

with mass $\mathcal{N}(Q)$. It is believed that $e^{itQ}$ and $PS(e^{itQ})$ are the only obstructions to scattering when the solution has ground state mass. The characterization of minimal mass blow-up solution for the mass critical dispersive equation was first studied by F. Merle who proved in [24] that if an $H_x^1$ solution of mass-critical NLS with minimal mass blows up in finite time, then it equals to $PS(e^{itQ})$ up to symmetries of the equation. In [12], Killip-Li-Visan-Zhang showed for NLS in $d \geq 4$ that the minimal mass global non-scattering solutions with spherically symmetric $H_x^1$ data must be coincide with solitary wave up to the symmetries of the equation.

**Definition 1.2** (Non-scattering). Let $I$ be a time interval. We say a solution $u$ of (1) with maximal life-span $I$ is non-scattering forward in time if $\sup I < \infty$ and $\|\nabla u(t)\|_{L^2(\mathbb{R}^d)} \to \infty(t \searrow \sup I)$, or $I = \mathbb{R}^+, \|u\|_{L^{\infty}(\mathbb{R}^d)} \to \infty$. Similarly, we can define non-scattering backward in time.

In lower dimensions, i.e., $1 \leq d \leq 3$, the difficulty arises from the low dispersion of the solutions. In [21], Li-Zhang employed the linearized analysis around the ground
state beyond the frequency localization technique, and generalized the results to $d = 2, 3$. The problem for $d = 1$ remains open.

For the Hartree equation, Miao-Xu-Zhao [29] used a slightly different approach from [24] and proved that if an $H^1$ solution with minimal mass blows up in finite time, then it equals to $PS(e^{it}Q)$ up to symmetries of the equation. There they used a simplified argument from [8]. In [20], Li-Zhang showed that when $d \geq 5$ and $u_0 \in H^1_\text{rad}(\mathbb{R}^d)$, the only non-scattering solution of (1) with ground state mass is the solitary wave or the pseudo-conformal solitary wave up to the symmetries of the equation.

Inspired by [21], we consider in this paper (1) in dimensions 3 and 4. The main result sounds as follows.

**Theorem 1.3.** Let $d = 3, 4$. Let $u_0 \in H^1_\text{rad}(\mathbb{R}^d)$ be such that $N(u_0) = N(Q)$. Assume that the corresponding solution $u$ of (1) exists globally but does not scatter in at least one direction. Then there exist $\theta_0 \in \mathbb{R}$ and $\lambda_0 > 0$ such that

$$u(t, x) = e^{i\theta_0} \lambda_0^{-\frac{d}{2}} e^{it/\lambda_0^2} Q(\frac{x}{\lambda_0}).$$

This theorem combined with the results in [20] gives a complete characterization of minimal mass non-scattering solutions for $L^2$-critical Hartree equations with radial data in $H^1$.

The main ingredient in proving Theorem 1.3 is the following localization property of kinetic energy.

**Theorem 1.4.** Let $d = 3, 4$. Let $u_0 \in H^1_\text{rad}(\mathbb{R}^d)$ satisfy $N(u_0) = N(Q)$. Assume the corresponding solution $u$ globally exists but does not scatter. Then for any $\eta > 0$, there exists $C(\eta) > 0$ such that

$$\|\phi_{>C(\eta)} \nabla u(t)\|_{L^2(\mathbb{R}^d)} \leq \eta, \quad \forall t \geq 0.$$ 

Here $\phi_{>C(\eta)}$ is a radial bump function (see Section 2.1).

**Outline of proof for Theorem 1.3.** Note that by Proposition 1, $N(u_0) = N(Q)$ implies $E(u_0) \geq 0$. Moreover, if $E(u_0) = 0$, then $u_0 = Q$ up to rotation, scaling and translation. By uniqueness, $u$ equals to the solitary wave up to the symmetries. Thus, we only need to exclude the case where $E(u_0) > 0$. This is done by using a localized virial argument. On one hand, the localized virial quantity is always bounded above. On the other hand, by virtue of Theorem 1.4, the second derivative of the localized virial quantity has positive lower bound. These yield a contradiction.

So the main task of the paper is to achieve Theorem 1.4. Following the idea of [21], we first employ the ideas of modulation stability theory from Merle-Raphaël [25] to show a regular decomposition for minimal mass solutions of Hartree equation. This allows us to capture weak localization of the kinetic energy of the solution. In particular, we shall show the following proposition.

**Proposition 2 (Weak localization of kinetic energy).** Let $d = 3, 4$, and let $u_0 \in H^1_\text{rad}(\mathbb{R}^d)$ satisfy $N(u_0) = N(Q)$. Let $u$ be the corresponding solution with maximal life-span $I$. Then for all $t \in I$, we have

$$\|\phi_{\geq 1} \nabla u(t)\|_{L^2(\mathbb{R}^d)} \lesssim 1,$$

where $\phi$ is a bump function (see Section 2.1).
In this first step, we shall make use of the structure of the linearized operator $L_+$, and $L_-$ around the ground state,

$$
L_+ = -\Delta + 1 - (| \cdot |^{-2} * Q^2) - 2Q(| \cdot |^{-2} * (Q \cdot)) \\
L_- = -\Delta + 1 - (| \cdot |^{-2} * Q^2).
$$

From the discussion in [5], $L_+$ has exactly one negative eigenvalue and the continuous spectra of $L_-$ and $L_+$ are $[1, \infty)$, see also [17]. Moreover, in [17], Krieger-Lenzmann-Raphaël proved for $d = 4$ the non-degeneracy of $L_+$, and obtained

$$\text{null space of } L_+ = \text{span}_\mathbb{R} \{\nabla Q\}. \quad (7)$$

The argument for the non-degeneracy property relies on the uniqueness of the ground state. In this paper, we assume the resolution of this null space property for $d = 3$. With this and by an adaption of [31] (see also [23]), one may infer the following coercivity of $L_+$, and $L_-$. 

Lemma 1.5. Let $\omega$ be the eigenfunction corresponding to the negative eigenvalue of $L_+$. There exists a constant $\sigma > 0$ such that the following holds.

(i) If $(h, \omega) = (h, \partial_j Q) = 0$ for $1 \leq j \leq d$, then $(h, L_+ h) \geq \sigma \|h\|_{L^2}^2$;

(ii) If $(h, Q) = 0$, then $(h, L_- h) \geq \sigma \|h\|_{L^2}^2$.

Here, $(\cdot, \cdot)$ denotes the inner production in $L^2$.

The second step for achieving the main result is to invoke the compactness property of a non-scattering solution with minimal mass to upgrade Proposition [2] to Theorem [14]. Indeed, it has been shown in [15, 13, 28] that the non-scattering solution with minimal mass is an almost periodic solution in the following sense: there exist functions $N : I \rightarrow \mathbb{R}^+$, $\xi : I \rightarrow \mathbb{R}^d$, $x : I \rightarrow \mathbb{R}^d$ and $C : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that for all $t \in I$ and $\eta > 0$,

$$
\int_{|x-x(t)| \geq C(\eta)/N(t)} |u(t, x)|^2 \, dx + \int_{|\xi-\xi(t)| \geq C(\eta)/N(t)} |\hat{u}(t, x)|^2 \, d\xi \leq \eta.
$$

An important consequence of the almost periodic solution is the following no waste Duhamel formula. 

Lemma 1.6. Let $u$ be an almost periodic solution of (1) on $[0, \infty)$. Then for all $t \geq 0$,

$$
u(t) = -i \lim_{T \rightarrow \infty} \int_t^T e^{i(t-\tau)\Delta} (| \cdot |^{-2} * |u|^2) u(\tau) \, d\tau \quad (8)$$

as a weak limit in $L^2_{\text{loc}}$.

We shall use (8) together with the in-out decomposition technique to derive both frequency and spatial decay estimates for the minimal mass solution and then prove Theorem [14]. The nonlocal nonlinearity makes things a bit complicated. To deal with it, we shall do delicate analysis in different integration regions.

This paper is structured as follows: In Section 2, we present some preliminaries. In Section 3, we give a regular decomposition for the solution with minimal mass, and as a consequence, we obtain Proposition [2]. In Section 4, we prove a frequency and spatial decay for the global non-scattering solution. Theorems [13] and [14] shall be proved in Section 5.
2. Preliminaries. The notation $X \lesssim Y$ or $Y \gtrsim X$ denotes $X \leq CY$, where $C > 0$ is a constant that can depend on exponents (such as dimension), as well as the energy and the mass, but not parameters $t$ or functions $u$. We use $O(Y)$ to denote any quantity $X$ such that $|X| \lesssim Y$. We denote by $X \pm \varepsilon$ any quantity of the form $X \pm \varepsilon$ for any $\varepsilon > 0$. We use the Japanese bracket convention $\langle x \rangle := (1 + |x|^2)^{1/2}$, $(\cdot, \cdot)$ means the inner product in $L^2$.

We use $L_t^q L_x^r$ to denote the Banach space with norm
\[
\|f\|_{L_t^q L_x^r} := \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}^d} |f(t,x)|^q \, dx \right)^{r/q} \, dt \right)^{1/q},
\]
with the usual modifications when $q$ or $r$ is equal to infinity, or the region $\mathbb{R} \times \mathbb{R}^d$ is replaced by some $I \times \mathbb{R}^d$. When $q = r$ we abbreviate $L_t^q L_x^r$ as $L_t^q L_x^q$.

For $s \in \mathbb{R}$, we use $H_{rad}^s(\mathbb{R}^d)$ to denote the space of functions in $H^s(\mathbb{R}^d)$ that are spherically symmetric.

2.1. Basic harmonic analysis. Let $\phi \in C^\infty(\mathbb{R}^d)$ be a radial and non-negative bump function supported in the ball $\{ x \in \mathbb{R}^d ; |x| \leq \frac{R}{2} \} = \mathbb{B}(0, \frac{R}{2})$ and equal to 1 on the ball $\{ x \in \mathbb{R}^d ; |x| \leq 1 \}$. For any constant $c > 0$, we denote $\phi_{\leq c}(x) := \phi(c \xi)$ and $\phi_{> c}(x) := 1 - \phi_{\leq c}(x)$. For each number $N > 0$, we define the multipliers
\[
\tilde{P}_{\leq N} := \phi_{\leq N}(\xi) \hat{f}(\xi),
\]
\[
\tilde{P}_{> N} := \phi_{> N}(\xi) \hat{f}(\xi),
\]
and similarly $P_{\leq N}$ and $P_{\geq N}$. We also define
\[
P_{M < \leq N} := P_{\leq N} - P_{\leq M} = \sum_{M < N' \leq N} P_{N'},
\]
whenever $M < N$. We will always use these multipliers when $M$ and $N$ are dyadic numbers; in particular, all summations over $N$ or $M$ are understood to be over dyadic numbers. Note that $P_2^N$ is not truly a projection, $P_2^N \neq P_N$. We shall use modified Littlewood-Paley operators:
\[
P_{2} := P_{N/2} + P_N + P_{2N}.
\]
These obey $P_N \tilde{P}_N = \tilde{P}_N P_N = P_N$.

Lemma 2.1 (Bernstein estimates). For $1 \leq p \leq q \leq \infty$, $s \in \mathbb{R}$,
\[
\|P_{\leq N} f\|_{L_t^q L_x^r(\mathbb{R}^d)} \lesssim N^{\frac{d}{2} - \frac{d}{q}} \|P_{\leq N} f\|_{L_t^r L_x^q(\mathbb{R}^d)}
\]
\[
\|P_N f\|_{L_t^q L_x^r(\mathbb{R}^d)} \lesssim N^{\frac{d}{2} - \frac{d}{q}} \|P_{\leq N} f\|_{L_t^r L_x^q(\mathbb{R}^d)}
\]
\[
\|\nabla |f|\|_{L_t^q L_x^q(\mathbb{R}^d)} \lesssim N^s \|P_N f\|_{L_t^q L_x^q(\mathbb{R}^d)}.
\]

Lemma 2.2 (Mismatch estimates, $\|\cdot\|_{L_t^q L_x^r(\mathbb{R}^d)}$). Let $R > 0$, and $N, M > 0$ be such that $4 \min\{N, M\} \leq \max\{N, M\}$. Then
\[
(i) \quad \|P_N \phi \lesssim R P_{M} f\|_{L_t^q L_x^r(\mathbb{R}^d)} \lesssim_m \max\{N, M\}^{-m} R^{-m} \|f\|_{L_t^r L_x^q(\mathbb{R}^d)},
\]
\[
(ii) \quad \|\phi_R P_{\leq N} \phi_{\leq R/2} f\|_{L_t^q L_x^r(\mathbb{R}^d)} \lesssim_m N^{-m} R^{-m} \|f\|_{L_t^r L_x^q(\mathbb{R}^d)}, \quad 1 \leq p \leq \infty,
\]
\[
(iii) \quad \|\phi_R P_{N} \phi_{\leq R/2} f\|_{L_t^q L_x^r(\mathbb{R}^d)} \lesssim_m N^{1-m} R^{-m} \|f\|_{L_t^r L_x^q(\mathbb{R}^d)}, \quad 1 \leq p \leq \infty,
\]
for any $m \geq 0$. The estimate (i) holds if $\phi_{\leq R}$ is replaced by $\phi_{> R}$, or substitution $P_M$ with $P_{\leq N/4}$. 
2.2. Strichartz estimates.

**Lemma 2.3** (Strichartz, [7, 9]). Let \( d \geq 3 \). Let \( I \) be an interval containing 0, and let \( u_0 \in L^2_x(\mathbb{R}^d) \) and \( F \in L^2_t L^{\frac{2d}{d-2}}_x(\mathbb{R}^d) \). Then the function \( u \) defined by

\[
 u(t) := e^{it\Delta} u_0 - i \int_0^t e^{i(t-\tau)\Delta} F(\tau) \, d\tau
\]

obeys the estimate

\[
 \|u\|_{L^\infty_t L^2_x(I \times \mathbb{R}^d)} + \|u\|_{L^2_t L^{\frac{2d}{d-2}}_x(I \times \mathbb{R}^d)} \lesssim \|u_0\|_{L^2_x(\mathbb{R}^d)} + \|F\|_{L^2_t L^{\frac{2d}{d-2}}_x(I \times \mathbb{R}^d)}.
\]

We will also need the following weighted Strichartz estimate.

**Lemma 2.4** (Weighted Strichartz, [13, 16]). Let \( I \) be an interval containing \( t_0 \). Let \( F : I \times \mathbb{R}^d \to \mathbb{C} \) be spherically symmetric. Then

\[
 \left\| \int_{t_0}^t e^{i(t-\tau)\Delta} F(\tau) \, d\tau \right\|_{L^2_x(\mathbb{R}^d)} \lesssim \|x|^{-\frac{2(d-1)}{q}} F\|_{L^\infty_t L^{\frac{2d}{d-2}}_x(I \times \mathbb{R}^d)}
\]

for all \( 4 \leq q \leq \infty \).

2.3. In-out decomposition. For a spherically symmetric function \( f : \mathbb{R}^d \to \mathbb{C} \), define operators \( P^\pm \) by

\[
 P^\pm f(r) := \frac{1}{2} f(r) \pm \frac{i}{\pi} \int_0^\infty r^{2-d} f(\rho) \rho^{d-1} \, d\rho.
\]

\( P^+ \) and \( P^- \) are regarded as projections onto outgoing and incoming spherical waves, respectively. For \( N > 0 \), let \( P^\pm_N \) denote the product \( P^\pm P_N \) where \( P_N \) is the Littlewood-Paley operator.

**Lemma 2.5** (Properties of \( P^\pm \), [13, 16]). (i) \( P^+ + P^- \) acts as the identity on \( L^2_{\text{rad}} \). (ii) Fix \( N > 0 \). For any \( f \in L^2_{\text{rad}}(\mathbb{R}^d) \),

\[
 \|P^\pm_N f\|_{L^2_x(|x| \geq \frac{1}{100} N^{-1})} \lesssim \|f\|_{L^2_x(\mathbb{R}^d)}
\]

with an \( N \)-independent constant.

**Lemma 2.6** (Kernel estimate, [13, 16]). For \( |x| \gtrsim N^{-1} \) and \( t \gtrsim N^{-2} \), the integral kernel obeys

\[
 |P^\pm_N e^{it\Delta}(x, y)| \lesssim \begin{cases} 
 (|x| |y|)^{-\frac{d-1}{4}} |t|^{-\frac{1}{2}}, & \text{if } |y| - |x| \sim N t \\
 \frac{N^d}{(N|x|)^{\frac{d}{2}} (N|y|)^{\frac{d}{2}}} (N^2 t + N|x| - N|y|)^{-m}, & \text{otherwise}
\end{cases}
\]

for any \( m \geq 0 \). For \( |x| \gtrsim N^{-1} \) and \( t \lesssim N^{-2} \), the integral kernel obeys

\[
 |P^\pm_N e^{it\Delta}(x, y)| \lesssim \frac{N^d}{(N|x|)^{\frac{d}{2}} (N|y|)^{\frac{d}{2}}} (N|x| - N|y|)^{-m}
\]

for any \( m \geq 0 \).

We end this section by giving a bubble decomposition from [11, 8, 13].

**Lemma 2.7.** Let \( d \geq 3 \). Let \( \{\phi_n\}_{n=1}^\infty \) be a bounded sequence in \( H^1_x(\mathbb{R}^d) \). Then there exist a subsequence of \( \{\phi_n\} \) (still denoted \( \{\phi_n\} \)), a family of sequence \( \{x^n_n\}\) in \( \mathbb{R}^d \) and a sequence \( \{\psi^j\} \) of \( H^1_x \) functions such that

(i) for every \( j \neq k \), \( |x^n_k - x^n_k| \to \infty \) as \( n \to \infty \).
(ii) for every $J \geq 1$ and every $x \in \mathbb{R}^d$, we have
\[
\varphi_n(x) = \sum_{j=1}^{J} \psi^j(x - x_n^j) + r_n^J(x)
\]
with
\[
\lim_{J \to \infty} \limsup_{n \to \infty} \|r_n^J\|_{L^p_x(\mathbb{R}^d)} = 0, \quad 2 < p < \frac{2d}{d - 2}.
\]
Moreover, we have for every $J \geq 1$ that
\[
\lim_{n \to \infty} \left[ \|\varphi_n\|_{H_s^x}^2 - \sum_{j=1}^{J} \|\psi^j\|_{H_s^x}^2 - \|r_n^J\|_{H_s^x}^2 \right] = 0, \quad s = 0, 1.
\]
As a consequence of the above decomposition, we have the decoupling of the nonlinear energy.

**Corollary 1.**
\[
\lim_{J \to \infty} \limsup_{n \to \infty} \left| \mathcal{P}(\varphi_n) - \sum_{j=1}^{J} \mathcal{P}(\psi^j) \right| = 0.
\]

**Proof.** See [6, 29].

### 3. Regular decomposition of the solution

In this section, we shall give a regular decomposition of the solution with ground state mass. We start by discussing $H^1$ function $v$ that satisfies
\[
\mathcal{N}(v) = \mathcal{N}(Q), \quad \|\nabla v\|_{L^2_x(\mathbb{R}^d)} = \|
abla Q\|_{L^2_x(\mathbb{R}^d)},
\]
where $Q$ is the ground state of $[5]$. $E(Q) = 0$.

**Lemma 3.1.** There exist $\theta_0 \in \mathbb{R}$ and $x_0 \in \mathbb{R}^d$ such that
\[
||e^{i\theta_0}v(\cdot + x_0) - Q||_{H^1_x(\mathbb{R}^d)} \leq \eta(E(v))
\]
with $\eta(E(v)) \to 0$ as $E(v) \to 0$.

**Proof.** We argue by contradiction. Assume there exist an $\eta_0 > 0$ and a sequence \( \{v_n\} \subset H^1_x(\mathbb{R}^d) \) satisfying
\[
\mathcal{N}(v_n) = \mathcal{N}(Q), \quad \|
\nabla v_n\|_{L^2} = \|
\nabla Q\|_{L^2}, \quad E(v_n) \to 0,
\]
and
\[
\inf_{\theta \in \mathbb{R}, x \in \mathbb{R}^d} \|e^{i\theta}v_n(\cdot + x) - Q\|_{H^1} \geq \eta_0.
\]
Applying the bubble decomposition (Lemma 2.7) to \( \{v_n\} \), we obtain
\[
v_n(x) = \sum_{j=1}^{J} \psi^j(x - x_n^j) + r_n^J(x),
\]
with
\[
\lim_{n \to \infty} \mathcal{N}(v_n) = \lim_{n \to \infty} \left[ \sum_{j=1}^{J} \mathcal{N}(\psi^j) + \mathcal{N}(r_n^J) \right], \quad \forall J \geq 1,
\]
\[
\lim_{n \to \infty} \|
\nabla v_n\|_{L^2}^2 = \lim_{n \to \infty} \left[ \sum_{j=1}^{J} \|
\nabla \psi^j\|_{L^2}^2 + \|
\nabla r_n^J\|_{L^2}^2 \right], \quad \forall J \geq 1.
\]
Since $E(v_n) \to 0 (n \to \infty)$, $E(Q) = 0$ and $\|\nabla v_n\|_{L^2_\xi} = \|\nabla Q\|_{L^2_\xi}$, we have
\[
\lim_{n \to \infty} \mathcal{P}(v_n) = \lim_{n \to \infty} \left( \frac{1}{2} \|\nabla v_n\|_{L^2_\xi}^2 - E(v_n) \right) = \frac{1}{2} \|\nabla Q\|_{L^2_\xi}^2 = \mathcal{P}(Q). \tag{18}
\]
Furthermore, by Corollary 3, we have
\[
\frac{1}{2} \|\nabla Q\|_{L^2_\xi}^2 = \lim_{n \to \infty} \mathcal{P}(v_n) \leq \lim_{n \to \infty} \limsup_{j \to \infty} \sum_{j=1}^J \mathcal{P}(\psi^j) \leq \limsup_{n \to \infty} \sum_{j=1}^\infty \mathcal{P}(\psi^j). \tag{19}
\]
Invoking the sharp Gagliardo-Nirenberg inequality [6] and (17), we get
\[
\text{RHS of (19)} \leq \limsup_{n \to \infty} \sum_{j=1}^\infty \frac{1}{2 \|Q\|_{L^2_\xi}^2} \mathcal{N}(\psi^j) \|\nabla \psi^j\|_{L^2_\xi}^2 \leq \frac{1}{2 \|Q\|_{L^2_\xi}^2} (\sup_j \mathcal{N}(\psi^j)) \limsup_{n \to \infty} \sum_{j=1}^\infty \|\nabla \psi^j\|_{L^2_\xi}^2 \leq \frac{1}{2 \|Q\|_{L^2_\xi}^2} (\sup_j \mathcal{N}(\psi^j)) \|\nabla Q\|_{L^2_\xi}^2,
\]
which together with (19) implies that
\[
\sup_j \mathcal{N}(\psi^j) \geq \mathcal{N}(Q).
\]
Since $\sum_j \mathcal{N}(\psi^j)$ converges, we see there exists $j_0$ such that
\[
\mathcal{N}(\psi^{j_0}) \geq \mathcal{N}(Q).
\]
Note that from (16), $\mathcal{N}(\psi^{j_0}) \leq \mathcal{N}(Q)$. Thus,
\[
\mathcal{N}(\psi^{j_0}) = \mathcal{N}(Q). \tag{20}
\]
It follows that $\psi^j = 0$ if $j \neq j_0$ and $\lim_{n \to \infty} \mathcal{N}(r^{j_0}_n) = 0$. Therefore,
\[
v_n(x) = \psi^{j_0}(x - x^{j_0}_n) + r^{j_0}_n(x) \quad \text{with} \quad \lim_{n \to \infty} \mathcal{N}(r^{j_0}_n) = 0. \tag{21}
\]
Reviewing the argument from (18)-(19) and using (17), we also get
\[
\|\nabla \psi^{j_0}\|_{L^2_\xi} = \|\nabla Q\|_{L^2_\xi}, \quad \lim_{n \to \infty} \|\nabla r^{j_0}_n\|_{L^2_\xi} = 0.
\]
This together with (20) implies that
\[
v_n(x + x^{j_0}_n) \to \psi^{j_0} \quad \text{in} \quad H^1_\xi(\mathbb{R}^d) \quad \text{as} \quad n \to \infty,
\]
which in turn gives by the Hardy-Littlewood-Sobolev inequality that
\[
\lim_{n \to \infty} \mathcal{P}(v_n) = \mathcal{P}(\psi^{j_0}).
\]
Thus,
\[
E(\psi^{j_0}) = \frac{1}{2} \int_{\mathbb{R}^d} |\nabla \psi^{j_0}(x)|^2 \, dx - \mathcal{P}(\psi^{j_0}) \leq \liminf_{n \to \infty} E(v_n) = 0.
\]
On the other hand, since $\mathcal{N}(\psi^{j_0}) = \mathcal{N}(Q)$, we have $E(\psi^{j_0}) \geq 0$. Thus,
\[
E(\psi^{j_0}) = 0.
\]
By Proposition 1 and $\|\nabla \psi^{j_0}\|_{L^2_\xi} = \|\nabla Q\|_{L^2_\xi}$, there exist $\theta_0 \in \mathbb{R}$ and $x_0 \in \mathbb{R}^d$ such that
\[
\psi^{j_0}(x) = e^{i\theta_0} Q(x + x_0).
\]
Therefore,
\[
v_n(x + x^{j_0}_n) \to e^{i\theta_0} Q(x + x_0),
\]
which is in contradiction to our assumption \((15)\), \(\square\)

**Lemma 3.2.** Let \(e_0\) be the negative eigenvalue of \(\mathcal{L}_+\), and \(\omega\) be the corresponding eigenfunction. Then there exist constants \(\delta > 0, C > 1, K > 0\) such that if \(v \in H^1_0(\mathbb{R}^d)\) satisfying \((12)\) and \(E(v) \leq \delta\), then there exist \(\theta_0 \in \mathbb{R}, x_0 \in \mathbb{R}^d, \lambda_0 > 0\) such that the decomposition

\[
\varepsilon(x) = e^{i\theta_0} \lambda_0^{d/2} v(\lambda_0 x + x_0) - Q(x), \quad (\varepsilon = \varepsilon_1 + i\varepsilon_2) \tag{22}
\]
satisfies the following

1. \((\varepsilon_1, \omega) = (\varepsilon_1, \partial_j Q) = (\varepsilon_2, Q) = 0\), for \(1 \leq j \leq d\).
2. \(\frac{1}{\lambda_0} \leq \lambda_0 \leq C, \quad \|\varepsilon\|_{H^1_0(\mathbb{R}^d)} \leq K \sqrt{E(v)}\).
3. If \(v\) is spherically symmetric, then \(x_0 = 0\).

The proof is achieved by applying the implicit function theorem and was essentially given in \(25\). For sake of convenience, we present it.

**Proof.** Define for \(\alpha > 0\) the neighborhood

\[
U_\alpha = \{ v \in H^1_0(\mathbb{R}^d); \|v - Q\|_{H^1_0(\mathbb{R}^d)} < \alpha \}.
\]

For any \(\theta \in \mathbb{R}, x \in \mathbb{R}^d, \lambda > 0\), define

\[
\varepsilon_{\lambda, \theta, x}(y) = e^{i\theta} \lambda^{d/2} v(\lambda y + x) - Q(y). \tag{23}
\]

**Claim:** There exists \(a_0 > 0\) and a unique \(C^1\) map: \(U_{a_0} \rightarrow \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^d\) such that if \(v \in U_{a_0}\), then there exist \(\theta_1 \in \mathbb{R}, \tilde{x} \in \mathbb{R}^d, \lambda_1 > 0\) satisfying

\[
\left((\varepsilon_{\lambda_1, \theta_1, \tilde{x}})_1, \omega \right) = \left((\varepsilon_{\lambda_1, \theta_1, \tilde{x}})_1, \partial_j Q \right) = \left((\varepsilon_{\lambda_1, \theta_1, \tilde{x}})_2, Q \right) = 0, \quad \text{for } 1 \leq j \leq d. \tag{24}
\]

Moreover, a constant \(C_1\) exists such that for \(0 < \alpha < a_0, v \in U_\alpha\),

\[
\|\varepsilon_{\lambda_1, \theta_1, \tilde{x}}\|_{H^1} + |\lambda_1 - 1| + |\theta_1| + |\tilde{x}| \leq C_1 \alpha. \tag{25}
\]

Here, \((\varepsilon_{\lambda_1, \theta_1, \tilde{x}})_1 = \text{Re} \, \varepsilon_{\lambda_1, \theta_1, \tilde{x}}, (\varepsilon_{\lambda_1, \theta_1, \tilde{x}})_2 = \text{Im} \, \varepsilon_{\lambda_1, \theta_1, \tilde{x}}\).

To prove the claim, define the functionals of \((\lambda, \theta, x, v)\) by

\[
\rho^1(v) = \int_{\mathbb{R}^d} (\varepsilon_{\lambda, \theta, x})_1 \omega(y) \, dy, \quad \rho^{j+1}(v) = \int_{\mathbb{R}^d} (\varepsilon_{\lambda, \theta, x})_1 \partial_j Q(y) \, dy, \quad \text{for } 1 \leq j \leq d,
\]

\[
\rho^{d+2}(v) = \int_{\mathbb{R}^d} (\varepsilon_{\lambda, \theta, x})_2 Q(y) \, dy.
\]

A direct computation at \((1, 0, 0, Q)\) yields

\[
\frac{\partial \varepsilon_{\lambda, \theta, x}}{\partial \lambda} = \frac{d}{2} v + y \cdot \nabla v, \quad \frac{\partial \varepsilon_{\lambda, \theta, x}}{\partial x_j} = \partial_j v, \quad \frac{\partial \varepsilon_{\lambda, \theta, x}}{\partial \theta} = i v.
\]

A further calculation at \((1, 0, 0, Q)\) gives

\[
\begin{align*}
\frac{\partial \rho^1}{\partial \lambda} &= -\frac{1}{\varepsilon_0} \int_{\mathbb{R}^d} Q \omega \, dy, \quad \frac{\partial \rho^1}{\partial \theta} = 0, \quad \frac{\partial \rho^1}{\partial x_j} = \int_{\mathbb{R}^d} \partial_j Q \omega \, dy = 0, \\
\frac{\partial \rho^{j+1}}{\partial \lambda} &= 0, \quad \frac{\partial \rho^{j+1}}{\partial \theta} = 0, \quad \frac{\partial \rho^{j+1}}{\partial x_i} = \int_{\mathbb{R}^d} \partial_j Q \partial_i Q(y) \, dy = \delta_{ij} \|\partial_j Q\|^2_2, \\
\frac{\partial \rho^{d+2}}{\partial \lambda} &= 0, \quad \frac{\partial \rho^{d+2}}{\partial \theta} = \int_{\mathbb{R}^d} Q^2 \, dy, \quad \frac{\partial \rho^{d+2}}{\partial x_j} = 0.
\end{align*}
\]
where we used the fact $L_+ \left( \frac{d}{8} Q + x \cdot \nabla Q \right) = -2Q$. This implies that

$$\left| \frac{\partial (\rho^1, \rho^2, \cdots, \rho^{d+2})}{\partial (\lambda, \theta, x_1, \cdots, x_d)} \right| = (-1)^{d+1} \frac{2}{c_0} \| Q \|_{L^2}^2 \prod_{j=1}^d \| \partial_j Q \|_{L^2} \int_{\mathbb{R}^d} Q(y) \omega(y) \, dy \neq 0.$$ 

Thus, by the implicit function theorem, there exists $\alpha_0 > 0$, a neighborhood $V$ of $(1, 0, 0)$ in $\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^d$ and a unique $C^1$ map $U_{\alpha_0} \mapsto V$ such that (24) and (25) hold.

If $v$ is spherically symmetric, then one defines

$$\varepsilon_{\lambda, \phi}(y) = e^{i\theta} \lambda^{d/2} v(\lambda y) - Q(y),$$

and the functionals

$$\rho^1(v) = \int_{\mathbb{R}^d} (\varepsilon_{\lambda, \phi})_1(y) \omega(y) \, dy, \quad \rho^2(v) = \int_{\mathbb{R}^d} (\varepsilon_{\lambda, \phi})_2(y) Q(y) \, dy.$$ 

Also by the implicit function theorem, we can obtain the desired result.

Since $\mathcal{N}(v) = \mathcal{N}(Q), \| \nabla v \|_{L^2} = \| \nabla Q \|_{L^2}$, by Lemma 3.1 there exist $\tilde{\theta}_0 \in \mathbb{R}$, $\tilde{x}_0 \in \mathbb{R}^d$ such that

$$\| e^{i\tilde{\theta}_0} v(\cdot + \tilde{x}_0) - Q \|_{H^1_0(\mathbb{R}^d)} < \eta(E(v)) \quad \text{with} \quad \eta(E(v)) \to 0 \quad \text{as} \quad E(v) \to 0.$$ 

Denote $\tilde{v}(y) = e^{i\tilde{\theta}_0} v(y + \tilde{x}_0)$. Now take $E(v) \leq \delta$ sufficiently small such that $\eta(E(v)) < \alpha_0$. Then by the above claim, there exist $\lambda_0, \tilde{\theta}, \tilde{x}$ such that

$$\varepsilon_{\lambda_0, \tilde{\theta}, \tilde{x}}(y) = e^{i\tilde{\theta}} \lambda_0^{d/2} \tilde{v}(\lambda_0 y + \tilde{x}) - Q(y)$$

obeys (24) and (25).

Setting $\tilde{\theta}_0 = \tilde{\theta} + \tilde{\theta}_0, x_0 = \tilde{x}_0 + \tilde{x}$, (22) and the orthogonality condition (1) follows.

Next, we prove smallness for $\| \varepsilon \|_{H^1}$. Since $\mathcal{N}(v) = \mathcal{N}(Q)$, we have

$$\int_{\mathbb{R}^d} |\varepsilon_1|^2 \, dx + \int_{\mathbb{R}^d} |\varepsilon_2|^2 \, dx + 2 \int_{\mathbb{R}^d} \varepsilon_1 Q \, dx = 0, \quad \varepsilon_1 = \text{Re} \varepsilon, \varepsilon_2 = \text{Im} \varepsilon \quad (26)$$

A direct computation with energy formula gives

$$2\lambda_0^2 E(v) = \int_{\mathbb{R}^d} |\nabla \varepsilon|^2 \, dx + \int_{\mathbb{R}^d} |\nabla Q|^2 \, dx + 2(\nabla \varepsilon, \nabla Q) - \frac{1}{2} \int_{\mathbb{R}^d} (| \cdot |^{-2} \ast Q^2) Q^2 \, dx$$

$$- 2 \int_{\mathbb{R}^d} (| \cdot |^{-2} \ast Q \varepsilon_1) Q \varepsilon_1 \, dx - \int_{\mathbb{R}^d} (| \cdot |^{-2} \ast Q) |\varepsilon|^2 \, dx - 2 \int_{\mathbb{R}^d} (| \cdot |^{-2} \ast Q^2) Q \varepsilon_1 \, dx$$

$$- 2 \int_{\mathbb{R}^d} (| \cdot |^{-2} \ast Q \varepsilon_1) |\varepsilon|^2 \, dx - \frac{1}{2} \int_{\mathbb{R}^d} (| \cdot |^{-2} \ast |\varepsilon|^2) |\varepsilon|^2 \, dx.$$ 

Invoking $E(Q) = 0$, the equation [5] of $Q$ and (26), we have

$$2\lambda_0^2 E(v) = \int_{\mathbb{R}^d} |\nabla \varepsilon|^2 \, dx + \int_{\mathbb{R}^d} |\varepsilon|^2 \, dx - 2 \int_{\mathbb{R}^d} (| \cdot |^{-2} \ast Q \varepsilon_1) Q \varepsilon_1 \, dx$$

$$- \int_{\mathbb{R}^d} (| \cdot |^{-2} \ast Q^2) |\varepsilon|^2 \, dx - 2 \int_{\mathbb{R}^d} (| \cdot |^{-2} \ast Q \varepsilon_1) |\varepsilon|^2 \, dx - \frac{1}{2} \int_{\mathbb{R}^d} (| \cdot |^{-2} \ast |\varepsilon|^2) |\varepsilon|^2 \, dx$$

$$= (\varepsilon_1, L_+ \varepsilon_1) + (\varepsilon_2, L_- \varepsilon_2) - F(\varepsilon),$$

where

$$F(\varepsilon) = 2 \int_{\mathbb{R}^d} (| \cdot |^{-2} \ast Q \varepsilon_1) |\varepsilon|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^d} (| \cdot |^{-2} \ast |\varepsilon|^2) |\varepsilon|^2 \, dx.$$
By Lemma 1.5 and the Hardy-Littlewood-Sobolev inequality, we have
\[ \sigma \| \varepsilon \|_{L^2}^2 \leq (\varepsilon_1, \mathcal{L}_+ \varepsilon_1) + (\varepsilon_2, \mathcal{L}_- \varepsilon_2) \leq 2\lambda_0^2 E(v) + C\| \varepsilon \|_{L^2}^2 \| \varepsilon \|_{H^1_2}. \]

If \( E(v) < \delta \) is sufficiently small such that \( C\eta(E(v)) < \sigma \), then
\[ \| \varepsilon \|_{L^2}^2 \leq C\lambda_0^2 E(v). \]

By expressions of \( \mathcal{L}_+, \mathcal{L}_- \), the Hardy-Littlewood-Sobolev inequality and \( (27) \), we have
\[
\| \varepsilon \|_{H^1_2}^2 = (\varepsilon_1, \mathcal{L}_+ \varepsilon_1) + (\varepsilon_2, \mathcal{L}_- \varepsilon_2) \\
\quad + 2 \int_{\mathbb{R}^d} (|\cdot|^{-2} * Q_\delta^2)|\varepsilon|^2 \, dx + 2 \int_{\mathbb{R}^d} (|\cdot|^{-2} * Q_\delta^2)Q_\varepsilon_1 \, dx \\
\quad \leq 2\lambda_0^2 E(v) + C\lambda_0 E(v)^{1/2}\| \varepsilon \|_{H^1_2}. \tag{28}
\]
This implies that
\[ \| \varepsilon \|_{H^1_2} \leq C\lambda_0 \sqrt{E(v)}. \]
Taking \( K = C\lambda_0 \), we get the desired result. \( \square \)

With the above lemma, we shall get a decomposition for the solution with ground state mass.

**Proposition 3.** Let \( d = 3, 4 \), and let \( u \) be an \( H^1 \) solution of \( (1) \) with \( \mathcal{N}(u) = \mathcal{N}(Q) \). Then there exist constants \( K_1, C_1, C_2 > 0 \), functions \( x(t) : \mathbb{R}^+ \rightarrow \mathbb{R}^d \), \( \theta(t) : \mathbb{R}^+ \rightarrow \mathbb{R} \), \( \lambda(t) : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \), \( \varepsilon(t) \in H^1_2(\mathbb{R}^d) \) such that for all \( t \geq 0 \),
\[
u(t,y) = e^{i\theta(t)} \lambda(t)^{d/2} Q(\lambda(t)(y-x(t))) + \varepsilon(t,y),
\]
where
\[ \| \varepsilon(t) \|_{H^1_2} \leq K_1(\sqrt{E(u)} + 1). \tag{30} \]

The scaling parameter satisfies
\[
\frac{1}{C_2} \frac{\| \nabla u(t) \|_{L^2}}{\| \nabla Q \|_{L^2}} \leq \lambda(t) \leq \frac{\| \nabla u(t) \|_{L^2}}{\| \nabla Q \|_{L^2}}, \quad \text{if} \quad \| \nabla u(t) \|_{L^2}^2 \geq C_1 E(u), \tag{31}
\]
and
\[ \lambda(t) = 1, \quad \text{if} \quad \| \nabla u(t) \|_{L^2}^2 < C_1 E(u). \tag{32} \]

If \( u \) is spherically symmetric, we may choose \( x(t) \equiv 0 \).

**Proof.** Let \( \delta, C, K \) be as in Lemma 3.2. Take \( C_1 = \| \nabla Q \|_{L^2}^2 / \delta, C_2 = C, K_1 = \max\{K(\| \nabla Q \|_{L^2}^2 + C), \sqrt{C_1} + 2\| \mathcal{N} \|_{H^1_2}\} \). Fix \( t \geq 0 \).

If \( \| \nabla u(t) \|_{L^2}^2 < C_1 E(u) \) Then set \( \theta(t) = 0, x(t) = 0, \lambda(t) = 1 \) and \( \varepsilon(t) = u(t) - Q \) so that \( \| \varepsilon \|_{H^1_2} \leq \sqrt{C_1} + 2\| \mathcal{N} \|_{H^1_2} \), the conclusion follows.

We next consider the case \( \| \nabla u(t) \|_{L^2}^2 \geq C_1 E(u) \). Denote
\[ \mu(t) = \| \nabla Q \|_{L^2}^2 / \| \nabla u(t) \|_{L^2}^2. \]
Define the rescaled function \( v(t,x) = \mu(t)^{d/2} u(t,\mu(t)x) \). Then \( v(t) \) satisfies \( (12) \). Moreover, \( E(v(t)) = \mu(t)^2 E(u(t)) \). It follows that \( E(v(t)) \leq \delta \). By Lemma 3.2 there exist \( \theta(t) \in \mathbb{R}, \lambda(t) > 0, \bar{x}(t) \in \mathbb{R}^d \) such that
\[
\bar{\varepsilon}(t,y) = e^{i\theta(t)} \lambda(t)^{d/2} v(t,\lambda(t)y + \bar{x}(t)) - Q(x) \tag{33}
\]
with
\[
\frac{1}{C} \leq \bar{\lambda}(t) \leq C, \quad \|\bar{\varepsilon}(t)\|_{H^1} \leq K\sqrt{E(v(t))}.
\]  
Inserting \( v(t, x) = \mu(t)^{d/2} u(t, \mu(t)x) \) into (33), we get
\[
u(t, x) = e^{-i\bar{\theta}(t) (\bar{\lambda}(t)\mu(t))^{d/2}(\bar{\varepsilon}(t) + Q)} \left( \frac{x - \mu(t)\bar{x}(t)}{\lambda(t)\mu(t)} \right).
\]
Set \( \theta(t) = -\bar{\theta}(t), \lambda(t) = (\bar{\lambda}(t)\mu(t))^{-1}, x(t) = \mu(t)\bar{x}(t) \),
\[
\varepsilon(t, y) = e^{-i\bar{\theta}(t) (\bar{\lambda}(t)\mu(t))^{d/2}\bar{\varepsilon}(t, \frac{y - \mu(t)\bar{x}(t)}{\lambda(t)\mu(t)})},
\]
we obtain (29).
Since \( E(v(t)) = \mu(t)^2 E(u) \) and \( \|\nabla u(t)\|_2^2 \geq C_1 E(u) \), it follows that
\[
\|\varepsilon(t)\|_{H^1} \leq (1 + \lambda(t))K\sqrt{E(v(t))} \leq (\mu(t) + \bar{\lambda}(t))^{d/2}\sqrt{E(v)} \leq K_1(1 + \sqrt{E(u)}).
\]  
By our choice of \( \lambda(t) \), we see
\[
\frac{1}{C_2} \|\nabla u(t)\|_{L^2} \leq \lambda(t) \leq C_2 \|\nabla u(t)\|_{L^2}.
\]
Finally, if \( u \) is spherically symmetric, then by Lemma 3.2, \( \bar{x}(t) = 0 \) in (33). Hence, \( x(t) = 0 \). This completes the proof of the proposition. \( \square \)

**Proof of Proposition 2.** Let \( d = 3, 4, c_0 > 0 \) be a given number. Let \( u(t) \in H^1(\mathbb{R}^d) \) be a spherically symmetric solution of (1) with \( \mathcal{N}(u) = \mathcal{N}(Q) \) and maximal life-span \( I \). Our aim is to show
\[
\|\phi_{c_0} u(t)\|_{H^1(\mathbb{R}^d)} \lesssim_{E(u), c_0, d} 1, \quad \text{for } c \geq c_0 > 0, \forall t \in I.
\]  
**Proof.** Let \( C_1, C_2 > 0 \) be as in Proposition 3. For fixed \( t \in I \), by Proposition 3 we may consider two cases: \( \|\nabla u(t)\|_{L^2}^2 < C_1 E(u) \) and \( \|\nabla u(t)\|_{L^2}^2 \geq C_1 E(u) \). The former case is trivial. We consider the latter. By virtue of (29), (34), the support property of \( \phi_{c_0} \) and the decay of ground state \( Q \), we obtain
\[
\|\phi_{c_0} u(t)\|_{H^1} \leq \|\phi_{c_0} \varepsilon(t)\|_{H^1} + \|\phi_{c_0} \lambda(t)^{d/2} Q(\lambda(t)x)\|_{H^1} \lesssim_{c_0} K_1(1 + E(u)^{1/2}) + \left( \int_{|x| > \frac{c_0}{d}} \lambda(t)^{d/2} |\nabla Q|^2(\lambda(t)x) \, dx \right)^{1/2}
\]
\[
+ \left( \int_{|x| > \frac{c_0}{d}} \lambda(t)^{d/2} |\nabla Q|^2(\lambda(t)x) \, dx \right)^{1/2}
\]
\[
\lesssim_{E(u), c_0, d} 1 + \lambda(t) \left( \int_{|y| > \frac{c_0}{d}} |\nabla Q|^2(y) \, dy \right)^{1/2}
\]
\[
\lesssim_{E(u), c_0, d} 1 + \left( \int_{\mathbb{R}^d} \frac{|y|^2 |\nabla Q(y)|^2 \, dy}{\lambda(t)^d} \right)^{1/2}
\]
which is the desired result. \( \square \)

**4. Localization of the non-scattering solution.** In this section, we prove a frequency decay estimate and a spatial decay estimate for the global non-scattering solution with minimal mass.
4.1. **Frequency decay estimate.** In this subsection, we prove the following proposition.

**Proposition 4.** Assume $u \in H^1_{rad}(\mathbb{R}^d)$ is a global solution of (1) that is non-scattering and satisfies

$$
\|\phi_{\geq 1} \nabla u(t)\|_{L^2(\mathbb{R}^d)} \lesssim 1, \quad \forall t \geq 0.
$$

Then there exists $\beta = \beta(d) > 0$ such that for any dyadic number $N \geq 1$, we have for all $t \geq 0$ that

$$
\|\phi_{> 1} P_N u(t)\|_{L^2(\mathbb{R}^d)} \lesssim \|\tilde{P}_N u_0\|_{L^2(\mathbb{R}^d)} + N^{-1-\beta}.
$$

The proof of this proposition is completed by several lemmas. We first project $u(t)$ onto the incoming and outgoing waves and use the Duhamel formula backward in time for the incoming wave and (8) forward in time for the outgoing wave. Namely,

$$
\phi_{> 1} P_N u(t) = \phi_{> 1} (P_N^- + P_N^+) u(t)
$$

$$
= \phi_{> 1} P_N^- e^{i \Delta t} u_0 + i \phi_{> 1} \int_0^t P_N^- e^{i \Delta (t - \tau)} (|\cdot|^2 * |u|^2) u(t - \tau) \, d\tau
$$

(37)

$$
- i \phi_{> 1} \int_0^\infty P_N^+ e^{-i \Delta \tau} (|\cdot|^2 * |u|^2) u(t + \tau) \, d\tau,
$$

(38)

where the equality holds in the sense of weak $L^2$ limit. By the boundedness of $\phi_{> 1} P_N^- e^{i \Delta t}$, we have

$$
\|\phi_{> 1} P_N^- e^{i \Delta t} u_0\|_{L^2(\mathbb{R}^d)} \lesssim \|\tilde{P}_N u_0\|_{L^2(\mathbb{R}^d)}.
$$

We rewrite the second term of (37) and (38) as follows

$$
- i (37)_2 = \phi_{> 1} \int_0^{N-1} P_N^- e^{i \Delta \tau} (|\cdot|^2 * |u|^2) u(t - \tau) \, d\tau
$$

(40)

$$
+ \phi_{> 1} \int_{N-1}^t P_N^- e^{i \Delta \tau} (|\cdot|^2 * |u|^2) u(t - \tau) \, d\tau,
$$

(41)

$$
\text{i(38)} = \phi_{> 1} \int_0^{N-1} P_N^+ e^{-i \Delta \tau} (|\cdot|^2 * |u|^2) u(t + \tau) \, d\tau
$$

(42)

$$
+ \phi_{> 1} \int_{N-1}^1 P_N^+ e^{-i \Delta \tau} (|\cdot|^2 * |u|^2) u(t + \tau) \, d\tau
$$

(43)

$$
+ \phi_{> 1} \int_1^\infty P_N^+ e^{-i \Delta \tau} (|\cdot|^2 * |u|^2) u(t + \tau) \, d\tau.
$$

(44)

**Lemma 4.1 (Short-time estimate).** Let $u$ be as in Proposition 4. For $N \geq 1$, we have

$$
\| (40) \|_{L^2(\mathbb{R}^d)} = \| \phi_{> 1} \int_0^{N-1} P_N^- e^{i \Delta \tau} (|\cdot|^2 * |u|^2) u(t - \tau) \, d\tau \|_{L^2(\mathbb{R}^d)} \lesssim N^{-3/2}.
$$

The analog holds for (42).
Lemma 4.2. For any \( \phi \) and we need the following lemma, which is a consequence of integrating by parts.

Now it suffices to estimate \( I_1 \). By the Strichartz estimate, Hölder’s inequality in time, Bernstein’s inequality, and the fact \( \phi_{1/2}(\phi_{1/2}^2 u) = \phi_{1/2}(\phi_{1/2}^2 \phi_{1/2}^2 u) \), we obtain

\[
\| I_1 \|_{L^2_t L^2_x} \lesssim \| \mathcal{P}_N \phi_{1/2}^2 \|_{L^{2(d+2)}_t L^2_x} \lesssim N^{-1/2} \| \mathcal{P}_N \phi_{1/2}^2 \|_{L^{2(d+2)}_t L^2_x} \lesssim N^{-3/2} \| \nabla (\phi_{1/4} \phi_{1/2}^2) \|_{L^{2(d+2)}_t L^2_x} \lesssim N^{-3/2} \| \phi_{1/4} \nabla (\phi_{1/2}^2) \|_{L^{2(d+2)}_t L^2_x}.
\]

By Sobolev’s embedding,

\[
\| \phi_{1/4} \|_{L^4_t L^8_x} \lesssim \| \phi_{1/4} \|_{L^{2(d+2)}_t L^2_x} \lesssim 1.
\]

We claim that \( \| \nabla (\phi_{1/4}) \|_{L^{2(d+2)}_t L^2_x} \) and \( \| \phi_{1/4} \nabla (\phi_{1/2}^2) \|_{L^{2(d+2)}_t L^2_x} \) are bounded. Thus,

\[
\| I_1 \|_{L^2_t L^2_x} \lesssim N^{-3/2}.
\]

The claim shall be reformulated as Lemma \ref{lemma} of nonlinear estimates and be proven in the end of this subsection.

To estimate \( I_2 \), we shall use the equation to replace \( \phi_{1/2}^2 u(t) \) with \( -(\phi_+ + \Delta)u \). And we need the following lemma, which is a consequence of integrating by parts.

Lemma 4.2. For any \( a, b \in \mathbb{R} \), \( c \geq 0 \), we have

\[
\int_a^b e^{i \tau \Delta} \phi_{\leq c}(\phi_+ + \Delta)u(t - \tau) \, d\tau = -i e^{i b \Delta} \phi_{\leq c} u(b - \tau) + i e^{i a \Delta} \phi_{\leq c} u(a - \tau) + 2 \int_a^b e^{i \tau \Delta} \nabla \phi_{\leq c} \cdot \nabla u(t - \tau) \, d\tau.
\]

By Lemma 4.2 we can write

\[
I_2 = -i \phi_{1/2} P_N e^{i \Delta} \phi_{1/2} u(t - N^{-1}) + i \phi_{1/2} P_N \phi_{1/2} u(t + N^{-1})
\]

\[
- \phi_{1/2} P_N e^{i \Delta} \phi_{1/2} u(t - \tau) \, d\tau
\]

\[
- \phi_{1/2} P_N e^{i \Delta} \phi_{1/2} \cdot \nabla u(t - \tau) \, d\tau
\]

\[
=: I_{21} + I_{22} + I_{23} + I_{24}.
\]
Now it suffices to estimate $I_{21} \sim I_{24}$. We only estimate $I_{24}$, since the estimates for the other three terms are in the same manner. By Lemma 2.6, the kernel satisfies

$$|\phi_{>1} P_N e^{i\tau \Delta} \chi_{2/3} (x, y)| \lesssim N^d (N|x| - N|y|)^{-2m} \phi_{>1} (x) \chi_{2/3} (y)$$

$$\lesssim N^d N^{-m} (x - y)^{-m}, \quad \tau \in [0, N^{-2}], \quad \forall m \geq 0,$$

$$|\phi_{>1} P_N e^{i\tau \Delta} \chi_{2/3} (x, y)| \lesssim N^d (N^2 \tau + N|x| - N|y|)^{-2m} \phi_{>1} (x) \chi_{2/3} (y)$$

$$\lesssim N^d (N|x| - N|y|)^{-m} \phi_{>1} (x) \chi_{2/3} (y)$$

$$\lesssim N^d N^{-m} (x - y)^{-m}, \quad \tau \in [N^{-2}, N^{-1}], \quad \forall m \geq 0,$$

where $\chi_{2/3}$ is a characteristic function on the ball $\{x \in \mathbb{R}^d; |x| \leq 2/3\}$. These together with Minkowski's inequality, Schur's test lemma and Hölder's inequality give

$$\left\| \phi_{>1} P_N \int_0^t e^{i\tau \Delta} \nabla \phi_{\leq 1/2} \cdot \nabla u (t - \tau) \, d\tau \right\|_{L^2_t(\mathbb{R}^d)}$$

$$= \left\| \int_0^t \int_{\mathbb{R}^d} \phi_{>1} P_N e^{i\tau \Delta} \chi_{2/3} (x, y) \nabla \phi_{\leq 1/2} \cdot \nabla u (t - \tau, y) \, dy \, d\tau \right\|_{L^2_t(\mathbb{R}^d)}$$

$$\leq \int_0^t \left\| \int_{\mathbb{R}^d} \phi_{>1} P_N e^{i\tau \Delta} \chi_{2/3} (x, y) \nabla \phi_{\leq 1/2} \cdot \nabla u (t - \tau, y) \, dy \right\|_{L^2_y(\mathbb{R}^d)} \, d\tau$$

$$\lesssim N^{-1} \|\phi_{>1} P_N e^{i\tau \Delta} \chi_{2/3}\|_{L^\infty_t L^1_x} \|\nabla \phi_{\leq 1/2}\|_{L^\infty_t L^2_y} \|\nabla u\|_{L^\infty_t L^2_x} \lesssim N^{-10} \|\phi_{>1/4} \nabla u\|_{L^\infty_t L^2_x} \lesssim N^{-10}.$$

Thus, we get

$$\|I_2\|_{L^2_t(\mathbb{R}^d)} \lesssim N^{-10}.$$

This ends the proof of Lemma 4.1. ∎

**Lemma 4.3** (Long time contribution). *Let $u$ be as in Proposition 4. For $N \geq 1$, we have

$$\|I_{41}\|_{L^2_t(\mathbb{R}^d)} = \left\| \phi_{>1} \int_{N^{-1}}^t P_N^* e^{i\tau \Delta} (|\cdot|^{-2} * |u|^2) u (t - \tau) \, d\tau \right\|_{L^2_t(\mathbb{R}^d)} \lesssim N^{-11/6}.$$

The analog holds for $I_{43}$.

**Proof.** We split the integral into three pieces:

$$\phi_{>1} \int_{N^{-1}}^t P_N^* e^{i\tau \Delta} (|\cdot|^{-2} * |u|^2) u (t - \tau) \, d\tau$$

$$= \phi_{>1} \int_{N^{-1}}^t P_N^* e^{i\tau \Delta} \phi_{>N^{\tau/2}} P_N \phi_{\leq N^{\tau/2}} \phi_{\leq \frac{N^{\tau/2}}{N}} (|\cdot|^{-2} * |u|^2) \phi_{\leq \frac{N^{\tau/2}}{N}} u (t - \tau) \, d\tau \quad (45)$$

$$+ \phi_{>1} \int_{N^{-1}}^t P_N^* e^{i\tau \Delta} \phi_{>N^{\tau/2}} P_N \phi_{\geq \frac{N^{\tau/2}}{N}} (|\cdot|^{-2} * |u|^2) \phi_{\geq \frac{N^{\tau/2}}{N}} u (t - \tau) \, d\tau \quad (46)$$

$$+ \phi_{>1} \int_{N^{-1}}^t P_N^* e^{i\tau \Delta} \phi_{\leq 2/3} (|\cdot|^{-2} * |u|^2) u (t - \tau) \, d\tau. \quad (47)$$
Estimate of (45): Noticing that \( \phi_{>1} P_N \) is bounded on \( L^2_x(\mathbb{R}^d) \), by Minkowski's inequality, mismatch estimate (Lemma 2.2), we have

\[
\| (45) \|_{L^2_x(\mathbb{R}^d)} \lesssim \int_{N-1}^t \left\| \widehat{P_N e^{irA \phi_{>N/2}} P_N^\perp \phi_{>N/2} (| \cdot |^{-2} * |u|^2) \phi_{>N/2} u(t-\tau) \right\|_{L^2_x(\mathbb{R}^d)} d\tau 
\]

\[
\lesssim \int_{N-1}^t \left\| \widehat{P_N \phi_{>N/2} P_N^\perp (| \cdot |^{-2} * |u|^2) \phi_{>N/2} u(t-\tau) \right\|_{L^2_x(\mathbb{R}^d)} d\tau 
\]

\[
\lesssim \int_{N-1}^t N^{-9} (N\tau)^{-9} \left\| \phi_{>N/2} (| \cdot |^{-2} * |u|^2) \phi_{>N/2} u(t-\tau) \right\|_{L^2_x(\mathbb{R}^d)} d\tau 
\]

\[
\lesssim N^{-18} \int_{N-1}^\infty \tau^{-9} \left\| \phi_{>N/2} (| \cdot |^{-2} * |u|^2) \phi_{>N/2} u \right\|_{L^2_x \times L^4_x \times \mathbb{R}^d} d\tau \lesssim 1. 
\]

By H"older's inequality, the nonlinear estimate (Lemma 4.5(iii)) and Sobolev's embedding theorem, we have

\[
\| \phi_{>N/2} (| \cdot |^{-2} * |u|^2) \phi_{>N/2} u \|_{L^2_x \times L^4_x \times \mathbb{R}^d} \lesssim \| \phi_{>N/2} u \|_{L^\infty_x \times L^4_x \times \mathbb{R}^d} \lesssim 1.
\]

Thus,

\[
\| (45) \|_{L^2_x(\mathbb{R}^d)} \lesssim N^{-18} \int_{N-1}^\infty \tau^{-9} d\tau \lesssim N^{-10}.
\]

Estimate of (46): We use weighted Strichartz estimate, H"older's inequality to get

\[
\| (46) \|_{L^2_x(\mathbb{R}^d)} \lesssim \| x^{-\frac{2(d-1)}{q}} \phi_{>N} P_N^\perp \phi_{>N} \phi_{>N} (| \cdot |^{-2} * |u|^2) \phi_{>N} u \|_{L^2_x \times L^{\frac{2d}{d-1}}_x \times \mathbb{R}^d} \lesssim N^{-\frac{2(d-1)}{q}} \| x^{-\frac{2(d-1)}{q}} \phi_{>N} \|_{L^\infty_x \times L^{\frac{2d}{d-1}}_x \times \mathbb{R}^d} \times \| P_N^\perp \phi_{>N} (| \cdot |^{-2} * |u|^2) \phi_{>N} u \|_{L^\infty_x \times L^{\frac{2d}{d-1}}_x \times \mathbb{R}^d} \lesssim N^{-\frac{2(d-1)}{q}} \| P_N^\perp \phi_{>N} (| \cdot |^{-2} * |u|^2) \phi_{>N} u \|_{L^\infty_x \times L^{\frac{2d}{d-1}}_x \times \mathbb{R}^d},
\]

where \( q = 2d \geq 4 \).

By Bernstein's inequality, the Leibniz rule and Lemma 4.5 it follows that

\[
\| P_N^\perp \phi_{>N} (| \cdot |^{-2} * |u|^2) \phi_{>N} u \|_{L^\infty_x \times L^{\frac{2d}{d-1}}_x} \lesssim N^{-1} \| \nabla (\phi_{>N} (| \cdot |^{-2} * |u|^2)) \|_{L^\infty_x L^2_\tau} \| \phi_{>N} u \|_{L^\infty_x L^2_\tau} + N^{-1} \| \phi_{>N} (| \cdot |^{-2} * |u|^2) \|_{L^\infty_x L^2_\tau} \| \nabla (\phi_{>N} u) \|_{L^\infty_x L^2_\tau} \lesssim N^{-1}.
\]

This together with (48) yields

\[
\| (46) \|_{L^2_x(\mathbb{R}^d)} \lesssim N^{-1 - \frac{2(d-1)}{2d}}.
\]
Putting all these estimates together, we conclude Lemma 4.3.

\[ (47) = i \phi_1 P_N e^{i \Delta} \phi \leq \chi \leq u_0 - i \phi_1 P_N e^{i \Delta} \phi \leq \chi \leq u(t - N^{-1}) \quad (49) \]

\[ + 2 \phi_1 P_N \int_{N-1}^{t} e^{i \Delta} \frac{y}{N \tau^2} \cdot \nabla \psi \leq \chi \leq u(t - \tau) \, d\tau \]

\[ + \phi_1 P_N \int_{N-1}^{t} e^{i \Delta} \phi \leq \chi \leq u(t - \tau) \, d\tau \]

\[ + \phi_1 P_N \int_{N-1}^{t} e^{i \Delta} \nabla \psi \leq \chi \leq u(t - \tau) \, d\tau. \quad (52) \]

Since the estimates for these terms are similar, we only give the estimate of (50).

Note that the integral kernel obeys

\[ |\phi_1 P_N e^{i \Delta} \chi \leq \chi \leq u(x, y)| \lesssim \frac{N^d}{(N|x|)^{\frac{m}{2}} (N|y|)^{\frac{m}{2}}} (N^2 t + N|x| - N|y|)^{-2m} \lesssim N^{d-m} (x - y)^{-m}, \quad \forall \tau \geq N^{-1}, \forall m \geq 0. \]

Using this, Minkowski’s inequality and Schur’s test lemma, we obtain

\[ \left\| (50) \right\|_{L^2(\mathbb{R}^d)} \lesssim \int_{N-1}^{t} \left\| \int_{\mathbb{R}^d} \phi_1 P_N e^{i \Delta} \chi \leq \chi \leq u(t - \tau) \, d\psi \right\|_{L^2} \, d\tau \lesssim N^{-10} \int_{N-1}^{t} \frac{1}{(N \tau^2)} \, d\tau \left\| \nabla \psi \leq \chi \right\|_{L^2} \lesssim N^{-10}. \]

Putting all these estimates together, we conclude Lemma 4.3. \( \square \)

To establish Proposition 4.3, we are now left to estimate (44). This is the following lemma.

**Lemma 4.4 (The tail).**

\[ \left\| \phi_1 \int_{1}^{\infty} P_N^+ e^{-i \Delta}(|\cdot|^{-2} * |u|^2)u(t + \tau) \, d\tau \right\|_{L^2(\mathbb{R}^d)} \lesssim N^{-2}. \]

**Proof.** We decompose the integral as follows

\[ (44) = \sum_{k=0}^{\infty} \phi_1 \int_{2^k}^{2^{k+1}} P_N^+ e^{-i \Delta}(|\cdot|^{-2} * |u|^2)u(t + \tau) \, d\tau \]

\[ = \sum_{k=0}^{\infty} \phi_1 \int_{2^k}^{2^{k+1}} P_N^+ e^{-i \Delta} \phi \leq \chi \leq u_0 \, d\tau \phi \leq \chi \leq u(t + \tau) \, d\tau \]

\[ + \sum_{k=0}^{\infty} \phi_1 \int_{2^k}^{2^{k+1}} P_N^+ e^{-i \Delta} \phi \leq N_{2^k} \phi \leq \chi \leq u(t + \tau) \, d\tau \]

\[ + \sum_{k=0}^{\infty} \phi_1 \int_{2^k}^{2^{k+1}} P_N^+ e^{-i \Delta} \phi \leq N_{2^k} (|\cdot|^{-2} * |u|^2) \phi \leq \chi \leq u(t + \tau) \, d\tau \]

\[ + \sum_{k=0}^{\infty} \phi_1 \int_{2^k}^{2^{k+1}} P_N^+ e^{-i \Delta} \phi \leq N_{2^k} (|\cdot|^{-2} * |u|^2) \phi \leq \chi \leq u(t + \tau) \, d\tau. \quad (55) \]
Thus, to prove the lemma, it suffices to estimate (53)-(55). By Strichartz’s inequality, mismatch estimate, nonlinear estimate (Lemma 4.5), we have for the \( k \)-th piece that
\[
\| \mathcal{P}_k \|_{L^2_x(\mathbb{R}^d)} \lesssim \| P_N \phi_{\leq 2k} P_{\leq 2k} \phi_{\leq 2k} (|\cdot|^{-2} \ast |u|^2) \phi_{\leq 2k} u \|_{L^2_{t,x}(\mathbb{R}^d)} \lesssim N^{-10} (N^2)^{-10} \| \phi_{\leq 2k} (|\cdot|^{-2} \ast |u|^2) \phi_{\leq 2k} u \|_{L^\infty_{t,x}(\mathbb{R}^d)} \lesssim N^{-2} 2^{-9k} \| \phi_{\leq 2k} (|\cdot|^{-2} \ast |u|^2) \phi_{\leq 2k} u \|_{L^\infty_{t,x} L^q_{x}} \lesssim N^{-2} 2^{-9k}.
\]

We use weighted Strichartz estimate to estimate (54), and consider two cases according to the dimension. Indeed, we have by weighted Strichartz estimate, Hölder’s inequality and Bernstein’s inequality that
\[
\| \mathcal{P}_k \|_{L^2_x(\mathbb{R}^d)} \lesssim \| x^{-\frac{2d-2}{4}} \phi_{\leq 2k} P_{\leq 2k} \phi_{\leq 2k} (|\cdot|^{-2} \ast |u|^2) \phi_{\leq 2k} u \|_{L^2_{t,x}(\mathbb{R}^d)} \lesssim N^{-\frac{2d-11}{4} - \frac{2d-9}{4}} \| \phi_{\leq 2k} (|\cdot|^{-2} \ast |u|^2) \phi_{\leq 2k} u \|_{L^\infty_{t,x} L^q_{x}} \lesssim 1,
\]
where \( q = 4 \) if \( d = 3 \), and \( q = 6 \) if \( d = 4 \).

Furthermore, by Hölder’s inequality, Lemma 4.5, it follows that
\[
\| \nabla (\phi_{\leq 2k} (|\cdot|^{-2} \ast |u|^2) \phi_{\leq 2k} u) \|_{L^\infty_{t,x} L^2_{x}} \lesssim 1.
\]

Hence,
\[
\| \mathcal{P}_k \|_{L^2_x(\mathbb{R}^d)} \lesssim N^{-2} (2^k)^{-1/6}.
\]

So,
\[
\| \mathcal{P}_k \|_{L^2_x(\mathbb{R}^d)} \lesssim N^{-2}.
\]

To estimate (55), using Lemma 4.2, we are reduced to estimating
\[
i \phi_{\leq 1} P^+_N e^{-i 2k+1 \Delta} \phi_{\leq 2k} u(t + 2^k) + \phi_{\leq 1} \int_{2^k}^{2^{k+1}} P^+_N e^{-i \tau \Delta} \nabla \phi_{\leq 2k} \cdot \nabla u(t + \tau) \, d\tau + \phi_{\leq 1} \int_{2^k}^{2^{k+1}} P^+_N e^{-i \tau \Delta} \phi_{\leq 2k} u(t + \tau) \, d\tau.
\]

Invoking the kernel estimate
\[
| \phi_{\leq 1} P^+_N e^{-i \tau \Delta} \phi_{\leq 2k} (x, y) | \lesssim \frac{N^d}{(N|x|)^{\frac{d-2}{4}} (N|y|)^{\frac{d+2}{4}}} (N^2 + N|x| - N|y|)^{-2m} \lesssim N^{d(N^2 2^k)^{-m} (N(x - y))^{-m}}, \quad \forall m \geq 0, \tau \in [2^k, 2^{k+1}],
\]
and by Schur’s test lemma, we obtain
\[
\| \mathcal{P}_k \|_{L^2_x(\mathbb{R}^d)} \lesssim N^{-9} 2^{-5k}.
\]

Thus,
\[
\| \mathcal{P}_k \|_{L^2_x(\mathbb{R}^d)} \lesssim N^{-9}.
\]

Putting the estimate of (53) with (55) together, we achieve Lemma 4.4
Combining the equations 37-44, and Lemmas 4.1, 4.3 and 4.4, we get the desired result in Proposition 4. We finish this subsection by proving the following lemma.

Lemma 4.5 (Nonlinear estimates). Let $d = 3, 4$, and let $u$ be as in Proposition 4. Let $\frac{d}{2} < p < d + 1$. We have for $c \geq c_0 > 0$ that

(i) $\| \nabla \phi_{>c/4}(| \cdot |^{-2} * |u|^2) \|_{L_x^\infty L_t^2} \lesssim_{c_0,d} 1$;

(ii) $\| \phi_{>c} \nabla (| \cdot |^{-2} * |u|^2) \|_{L_x^p L_t^2} \lesssim_{c_0,d} 1$;

(iii) $\| \phi_{>c}(| \cdot |^{-2} * |u|^2) \|_{L_x^\infty L_t^p} \lesssim_{c_0,d,p} 1$.

Proof. Using Hölder’s inequality, we get

$$\| \nabla \phi_{>c/4}(| \cdot |^{-2} * |u|^2) \|_{L_x^\infty L_t^2} \lesssim \| \cdot |^{-2} * |u|^2 \|_{L_x^2(\mathcal{F}|c|<\mathcal{F})} \lesssim_{c_0,d} 1.$$ 

This implies that (i) is a consequence of (iii) for $p = d$.

We first prove (iii). Since $p > 2/d$, we have $\| |x|^{-2} \|_{L_x^p(|x|>\mathcal{F})} \lesssim_{c_0,d} 1$. Moreover, for $\frac{d}{2} < p < d + 1$ it follows that $2 < \frac{2dp}{dp+d-2p} < \frac{2d}{d-2}$. So by the triangle inequality, Young’s inequality, Hardy-Littlewood-Sobolev’s inequality and Sobolev’s embedding, we get

$$\| \phi_{>c}(| \cdot |^{-2} * |u|^2) \|_{L_x^p L_t^2} \leq \| \text{phi}_{<c/4}(| \cdot |^{-2} * \phi_{\leq c} |u|^2) \|_{L_x^\infty L_t^2} + \| \text{phi}_{>c/4}(| \cdot |^{-2} * \phi_{>c/4} |u|^2) \|_{L_x^p L_t^2}$$

$$\lesssim \| |x|^{-2} \|_{L_x^p(|x|>\mathcal{F})} u^2_{L_x^2} + \| \phi_{>c/4} |u|^2 \|_{L_x^p L_t^{2p/(dp+d-2p)}}$$

$$\lesssim \| u \|_{L_x^2}^2 + \| \phi_{>c/4} u \|_{L_x^{2p/(dp+d-2p)}}^2$$

$$\lesssim 1 + \| \phi_{>c/4} u \|_{H^1_x}^2 \lesssim_{c_0,d,p} 1.$$ 

To prove (ii), we also divide the proof into two cases: $d = 4$ and $d = 3$. In the case $d = 4$, we estimate by the triangle inequality, Minkowski’s inequality and the Hardy-Littlewood-Sobolev inequality that

$$\| \phi_{>c/4}(| \cdot |^{-3} * |u|^2) \|_{L_x^p L_t^2}$$

$$\leq \| \phi_{>c/4}(\phi_{\leq c/8} | \cdot |^{-3} * |u|^2) \|_{L_x^\infty L_t^2} + \| \phi_{>c/4} \phi_{>c/8} | \cdot |^{-3} * |u|^2 \|_{L_x^\infty L_t^2}$$

$$\lesssim \| |x|^{-2} \|_{L_x^p(|x|>\mathcal{F})} u^2_{L_x^2} + \| u \|_{L_x^2 L_t^2} \| \phi_{>c/8} |x|^{-3} \|_{L_x^\infty L_t^2}$$

$$\lesssim \| \phi_{>c/8} u \|_{L_x^2 L_t^2}^2 + \| u \|_{L_x^2 L_t^2} \lesssim \| \phi_{>c/8} u \|_{H^1_x} + 1 \lesssim 1.$$ 

In the case $d = 3$, by the triangle inequality, we have

$$\| \phi_{>c/4} \nabla(| \cdot |^{-2} * |u|^2) \|_{L_x^p L_t^2}$$

$$\leq \| \phi_{>c/4}(\phi_{\leq c/8} | \cdot |^{-2} \cdot \nabla u \bar{u}) \|_{L_x^p L_t^2} + \| \phi_{>c/4}(\phi_{>c/8} | \cdot |^{-2} * |u|^2) \|_{L_x^\infty L_t^2}$$

$$\lesssim \| |x|^{-2} \|_{L_x^p(|x|>\mathcal{F})} u^2_{L_x^2} + \| \phi_{>c/8} \nabla (\phi_{>c/8} | \cdot |^{-2} * |u|^2) \|_{L_x^\infty L_t^2}$$

$$\lesssim \| \phi_{>c/4}(\phi_{\leq c/8} | \cdot |^{-3} * |u|^2) \|_{L_x^\infty L_t^2}.$$ 

By the support property of $\phi$, the Hardy-Littlewood-Sobolev inequality and Hölder’s inequality, it follows that

$$\| \phi_{>c/8} \nabla(\phi_{>c/8} | \cdot |^{-2} * \nabla u \bar{u}) \|_{L_x^\infty L_t^2} \lesssim \| \phi_{>c/8} \nabla u \phi_{>c/8} \bar{u} \|_{L_x^\infty L_t^2} \lesssim \| \phi_{>c/8} \nabla u \|_{L_x^\infty L_t^2} \| \phi_{>c/8} u \|_{L_x^2 L_t^2} \lesssim 1.$$
Using Young’s inequality, we obtain
\[ \| \cdot |^{-2} \ast |\phi_{>\gamma}\|_{L^\infty L^2_t L^2_x} \lesssim \| \phi_{>\gamma}\|_{L^\infty L^{12/5}_t L^{12}_x} \lesssim \| \phi_{>\gamma}\|_{L^\infty H^1} \lesssim 1. \]

Also,
\[ \| u \|_{L^\infty L^2_t} \| \phi_{\gamma/8} \|_2 \| x \|^{-3} \| u \|_{L^\infty L^2_x} \lesssim 1. \]

Thus, we establish (ii). This completes the proof of the lemma.

4.2. Spatial decay estimate. In this subsection, we show a spatial decay estimate for the ground state mass solution.

**Proposition 5.** Assume \( u \in H^1_{\text{rad}}(\mathbb{R}^d) \) is a global solution of (1) that is non-scattering and satisfies

\[ \| \phi_{\geq 1} \nabla u(t) \|_{L^2_{\infty}(\mathbb{R}^d)} \lesssim 1, \quad \forall t \geq 0. \]

Let \( N_1 \) be a dyadic number. Then there exist \( \gamma = \gamma(d) > 0 \) and \( R_0 = R_0(N_1, u) \) such that for \( N > N_1 \) and \( R > R_0 \), we have

\[ \| \phi_{>R} P_N u(t) \|_{L^2_{\infty}(\mathbb{R}^d)} \lesssim \| \phi_{>R/2} u_0 \|_{L^2_{\infty}(\mathbb{R}^d)} + R^{-\gamma}, \quad t \geq 0. \]

**Proof.** By Duhamel’s formula and the in-out decomposition,

\[
\phi_{>R} P_N u(t) = \phi_{>R} P_N e^{it\Delta} u_0 + i \phi_{>R} \int_0^t P_N^{-} e^{i\tau\Delta} (|\cdot|^{-2} \ast |\cdot|^2) u(t - \tau) \, d\tau \\
- i \phi_{>R} \int_0^\infty P_N^+ e^{-i\tau\Delta} (|\cdot|^{-2} \ast |\cdot|^2) u(t + \tau) \, d\tau \tag{58}
\]

where the last integral should be understood in weak \( L^2 \) sense. By the triangle inequality,

\[
\| \phi_{>R} P_N^{-} e^{it\Delta} u_0 \|_{L^2_{\infty}(\mathbb{R}^d)} \leq \| \phi_{>R} P_N^{-} e^{it\Delta} \phi_{>R/2} u_0 \|_{L^2_{\infty}(\mathbb{R}^d)} + \| \phi_{>R} P_N^{-} e^{it\Delta} \phi_{\leq R/2} u_0 \|_{L^2_{\infty}(\mathbb{R}^d)}. \]

It is easily seen that

\[
\| \phi_{>R} P_N^{-} e^{it\Delta} \phi_{>R/2} u_0 \|_{L^2_{\infty}(\mathbb{R}^d)} \lesssim \| \phi_{>R/2} u_0 \|_{L^2_{\infty}(\mathbb{R}^d)} \text{ for } R > \frac{1}{100} N_1^{-1}. \]

By Lemma 2.6

\[
| \phi_{>R} P_N^{-} e^{it\Delta} \phi_{\leq R/2}(x, y) | \lesssim N^{d-m} R^{-m/2} (x - y)^{-m/2}, \quad \forall m \geq 0. \]

This together with Young’s inequality implies that

\[
\| \phi_{>R} P_N^{-} e^{it\Delta} \phi_{\leq R/2} u_0 \|_{L^2_{\infty}(\mathbb{R}^d)} \lesssim N^{-4} R^{-5} \| u_0 \|_{L^2_{\infty}(\mathbb{R}^d)}. \tag{59}
\]
Next, we estimate the second and the third terms on the RHS of (58). We first split the integrals into

\[ -i[58]_2 = \phi_R \int_0^t P_N^{-} e^{i\tau \Delta} \phi_{\leq R/2}(| \cdot |^{-2} |u|^2) u(t - \tau) d\tau + \phi_R \int_0^t P_N^{-} e^{i\tau \Delta} \phi_{> R/2}(| \cdot |^{-2} |u|^2) u(t - \tau) d\tau =: I1 + I2 \]

\[ -i[58]_3 = \phi_R \int_0^1 P_N^+ e^{-i\tau \Delta} \phi_{\leq R/2}(| \cdot |^{-2} |u|^2) u(t + \tau) d\tau + \phi_R \int_0^1 P_N^+ e^{-i\tau \Delta} \phi_{> R/2}(| \cdot |^{-2} |u|^2) u(t + \tau) d\tau + \sum_{k=0}^{\infty} \phi_R \int_{2^k}^{2^{k+1}} P_N^+ e^{-i\tau \Delta} \phi_{\leq \hat{R}}(| \cdot |^{-2} |u|^2) u(t + \tau) d\tau + \sum_{k=0}^{\infty} \phi_R \int_{2^k}^{2^{k+1}} P_N^+ e^{-i\tau \Delta} \phi_{> \hat{R}}(| \cdot |^{-2} |u|^2) u(t + \tau) d\tau =: I3 + I4 + I5 + I6, \]

where \( \hat{R} = (12R)^\frac{1}{d} \left( \frac{1}{2^k} \right)^{\frac{1}{2}} \).

We remark that the estimate for \( I1, I3, I5, I6 \), namely those terms where the nonlinearity lies in small radii, can be done in the same way by substituting the nonlinearity with the linear part of the equation. \( I2, I4, I6 \) in which the nonlinearity is confined on large radii will be treated by using weighted Strichartz estimate. We shall only give the estimate for \( I5, I6 \).

**Estimate of \( I5 \):** Using Lemma 4.2, we turn to estimate

\[ i\phi_R P_N^+ e^{-i2^{k+1} \Delta} \phi_{\leq \hat{R}} u(t + 2^{k+1}) + i\phi_R P_N^+ e^{i2^k \Delta} \phi_{\leq \hat{R}} u(t + 2^k) + \phi_R \int_{2^k}^{2^{k+1}} P_N^+ e^{-i\tau \Delta} (u \Delta \phi_{\leq \hat{R}} + 2\nabla u \nabla \phi_{\leq \hat{R}})(t + \tau) u d\tau. \]

Note that for \( |y| \leq \hat{R}, |x| \geq R \), and \( \tau \in [2^k, 2^{k+1}] \), by Young’s inequality,

\[ |y| - |x| \leq R + 2^k - R = 2^k. \]

Thus, the integral kernel has the estimate

\[ |\phi_R P_N^+ e^{-i\tau \Delta} \phi_{\leq \hat{R}}(x, y)| \lesssim \frac{N^d}{(N|x|)^{\frac{d+1}{2}}} \frac{(N^2 + N|x| - N|y|)^{-20}}{\tau} \lesssim N^{d-10} R^{-5} 2^{-5} (|x - y|)^{-5}, \quad \tau \in [2^k, 2^{k+1}]. \]

With this, using Young’s inequality and Schur’s test lemma, we get

\[ \| [60] \|_{L^2(R^d)} \lesssim N^{c} R^{-5} 2^{-k}, \]

where \( c \) is an absolute constant.

Summing over \( k \) gives

\[ \| I5 \|_{L^2(R^d)} \lesssim R^{-5}. \]
Estimate of $\mathcal{H}_6$: By weighted Strichartz estimate, Hölder’s inequality, and Lemma 4.5 we obtain

\[
\| (\mathcal{H}_6)_k \|_{L^2} \lesssim \| |x|^{-\frac{2(d-1)}{q}} \phi_{>R} \left( \phi_{>\frac{q}{2}} (|\cdot|^2 + |u|^2) \phi_{>\frac{q}{2}} u \right) \|_{L^\frac{3}{2}, \infty} L^{\frac{2q}{q+1}} (\mathbb{R}^d \times \mathbb{R}^d)
\]

\[
\lesssim \| |x|^{-\frac{2(d-1)}{q}} \phi_{>R} \|_{L^\frac{q}{q-1}} L^{\frac{q}{q-1}} (\mathbb{R}^d \times \mathbb{R}^d) \| \phi_{>\frac{q}{2}} (|\cdot|^2 + |u|^2) \phi_{>\frac{q}{2}} u \|_{L^\infty} \| q \|_{L^2} \| q \|_{L^2}
\]

\[
\lesssim (2^k)^{-\frac{1}{q}} R^{-\frac{1}{q}}
\]

where $q = 4$ if $d = 3$, and $q = 6$ if $d = 4$.

Combining (59), the estimate of $\mathcal{H}_5, \mathcal{H}_6$, and taking $\gamma = \frac{1}{12}$, we conclude the proposition.

5. Proof for Theorems 1.3 and 1.4. We first prove Theorem 1.4.

Proof of Theorem 1.4. That is for any $\eta > 0$, there exists $C(\eta) > 0$ such that

\[
\| \phi_{>C(\eta)} \nabla u(t) \|_{L^2} \leq \eta, \quad \forall t \geq 0.
\]  

(61)

Let $N_1(\eta), N_2(\eta)$ be dyadic numbers and $C(\eta)$ a large constant to be determined momentarily. By the triangle inequality, we have

\[
\| \phi_{>C(\eta)} \nabla u(t) \|_{L^2} \lesssim \| \phi_{>C(\eta)} \nabla P_{\leq N_1(\eta)} u(t) \|_{L^2} + \| \phi_{>C(\eta)} \nabla P_{N_1(\eta)<\leq N_2(\eta)} u(t) \|_{L^2} + \| \phi_{>C(\eta)} \nabla P_{N_2(\eta)<} u(t) \|_{L^2} =: \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3.
\]

By discarding the real space cutoff and Bernstein’s inequality, we have for $\mathcal{J}_1$ that

\[
\mathcal{J}_1 \lesssim N_1(\eta) \| u(t) \|_{L^2} \lesssim N_1(\eta).
\]

For $\mathcal{J}_2$, it follows from the triangle inequality, Bernstein’s estimate, mismatch estimate and Proposition 5 that

\[
\mathcal{J}_2 \lesssim \sum_{N_1(\eta)<\leq M \leq N_2(\eta)} \| \phi_{>C(\eta)} \nabla P_{M \phi_{>C(\eta)/2}} u(t) \|_{L^2} + \sum_{N_1(\eta)<\leq M \leq N_2(\eta)} \| \phi_{>C(\eta)} \nabla P_{M \phi_{>C(\eta)/2}} \tilde{u}(t) \|_{L^2}
\]

\[
\lesssim \sum_{N_1(\eta)<\leq M \leq N_2(\eta)} \left( M^{-9} C(\eta)^{-10} + \| \phi_{>C(\eta)/2} u \|_{L^2} + C(\eta)^{-\gamma(d)} \right)
\]

\[
\lesssim C(N_1(\eta), N_2(\eta)) \| u \|_{L^2} + \| \phi_{>C(\eta)/4} u_0 \|_{L^2} + C(\eta)^{-\gamma(d)}.
\]

To estimate $\mathcal{J}_3$, we first use the Leibniz rule to get

\[
\mathcal{J}_3 \lesssim \| \nabla (\phi_{>C(\eta)/2} P_{N_2(\eta)} u(t)) \|_{L^2} + \frac{1}{C(\eta)} \| u \|_{L^2}.
\]
Thus, it suffices to estimate \( \| \nabla(\phi_{> C_N}u(t)) \|_{L^2_x(R^d)} \). We use Bernstein’s inequality, mismatch estimate and Proposition 4 to estimate
\[
\begin{align*}
\| \nabla(\phi_{> C_N}u(t)) \|_{L^2_x(R^d)}^2 & \lesssim \| \nabla P_{\leq C_N}(\phi_{> C_N}u(t)) \|_{L^2_x(R^d)}^2 \\
& \quad + \sum_{N > C_N} \| \nabla P_N(\phi_{> C_N}u(t)) \|_{L^2_x(R^d)}^2 \\
& \lesssim N_2(\eta)^2 \| P_{\leq N_2}(\phi_{> C_N}u(t)) \|_{L^2_x(R^d)}^2 \\
& \quad + \sum_{N > N_2(\eta)/4} \| \nabla P_N(\phi_{> C_N}(2P_{\leq 4N}u(t))^2) \|_{L^2_x(R^d)}^2 \\
& \quad + \sum_{N > N_2(\eta)/4} \| \nabla P_N(\phi_{> C_N}(2P_{\leq 4N}u(t))^2) \|_{L^2_x(R^d)}^2 \\
& \lesssim 1 + \sum_{N > N_2(\eta)/4} \frac{N^2}{C(\eta)^{10}} + \sum_{N > N_2(\eta)/4} \frac{N^2}{C(\eta)^{10}N^4} + \frac{1}{C(\eta)^{2+2\beta(d)}} \\
& \lesssim C(\eta)^{-10}N_2(\eta)^{-8} + \| \nabla P_{\geq N_2}\|_{L^2_x(R^d)}^2 + N_2(\eta)^{-2\beta(d)}.
\end{align*}
\]
This gives
\[
\mathcal{I}_3 \lesssim C(\eta)^{-5}N_2(\eta)^{-4} + \| \nabla P_{\geq N_2}\|_{L^2_x(R^d)} + N_2(\eta)^{-\beta(d)} + \frac{1}{C(\eta)} \| u \|_{L^2_x(R^d)}.
\]
Combining the estimates of \( \mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3 \), we obtain
\[
\| \phi_{> C_N}(u(t)) \|_{L^2_x(R^d)} \lesssim C(N_1(\eta), N_2(\eta)) \left( \frac{1}{C(\eta)^{10}} + \| \phi_{> C_N}u(t) \|_{L^2_x(R^d)} + \frac{1}{C(\eta)} \| \nabla u \|_{L^2_x(R^d)} \right) \\
+ \frac{1}{C(\eta)^{5}N_2(\eta)^4} + \| P_{\geq N_2}\|_{L^2_x(R^d)} + N_1(\eta) \\
+ \frac{1}{N_2(\eta)^{2\beta(d)}} + \frac{1}{C(\eta)} \| u \|_{L^2_x(R^d)}.
\]
Choosing \( N_1(\eta) \) sufficiently small and \( N_2(\eta) \) sufficiently large depending on \( \eta, u_0 \), and then \( C(\eta) \) sufficiently large depending on \( N_1(\eta), N_2(\eta), u_0 \), we obtain 61. \( \square \)

We are now in position to prove Theorem 1.3.

Proof of Theorem 1.3. Let \( u \) be the spherically symmetric solution of (1) with \( N(u) = N(Q) \) and \( \| u \|_{L^4_tL^\infty_x(R \times R^d)} = \infty \). As mentioned in the introduction, we only need to disprove the case where \( E(u) > 0 \). Let \( a_R(x) = R^2 \varphi(|x|^2/4R^2) \), where \( \varphi \) is a smooth function on \((0, \infty)\) such that \( \varphi(r) = r \) for \( r \in (0, 1) \) and \( \varphi(r) = 0 \) for \( r > \frac{2R}{3} \). To this end, define the localized virial quantity
\[
V_R(t) = \frac{1}{2} \int_{R^d} a_R(x)|u(t, x)|^2 \, dx.
\]
It is easily seen that
\[
V_R(t) \lesssim R^2, \quad \forall t \in R.
\]

(62)
A direct computation yields
\[
\partial_t^2 V_R(t) = 2 \int_{\mathbb{R}^4} \partial_k \partial_j (a_R(x)) \partial_k u \partial_j \bar{u} \, dx - \frac{1}{2} \int_{\mathbb{R}^4} \partial_k^2 \partial_j^2 (a_R(x)) |u(t, x)|^2 \, dx
\]
\[
- \int_{\mathbb{R}^4} (\nabla a_R(x) - \nabla a_R(y))(x - y) \frac{|u(t, x)|^2 |u(t, y)|^2}{|x - y|^4} \, dx \, dy.
\] (63)

Since
\[
\partial_k \partial_j (a_R(x)) \partial_k u \partial_j \bar{u} = 2 \varphi'(\frac{|x|^2}{R^2}) |\nabla u|^2 + \frac{4}{R^2} \varphi''(\frac{|x|^2}{R^2}) |x \cdot \nabla u|^2,
\]
by the definition of \( \varphi \) we see
\[
\int_{\mathbb{R}^4} \partial_k \partial_j (a_R(x)) \partial_k u \partial_j \bar{u} \, dx = 2 \int_{\mathbb{R}^4} |\nabla u(t, x)|^2 \, dx + O\left( \int_{|x| \geq R} |\nabla u(t, x)|^2 \, dx \right). \] (64)

Since \( |\Delta a(x)| \leq \frac{C}{R^2} \) and \( \text{supp} \Delta a(x) \subset \{ x : R \leq |x| \leq 2R \} \), we have
\[
\int_{\mathbb{R}^4} \partial_k^2 \partial_j^2 (a_R(x)) |u(t, x)|^2 \, dx \leq \frac{C}{R^2} \int_{|x| \leq 2R} |u(t, x)|^2 \, dx. \] (65)

By (63), (64), (65) and by the symmetry with \( x \) and \( y \), we obtain
\[
\partial_t^2 V_R(t) = 8E(u) + O\left( \int_{|x| \geq R} |\nabla u(t, x)|^2 \, dx \right) + O\left( \frac{1}{R^2} \int_{R \leq |x| \leq 2R} |u(t, x)|^2 \, dx \right)
\]
\[
+ 2 \int_{|x| \geq R, |y| \geq R} \frac{|u(t, x)|^2 |u(t, y)|^2}{|x - y|^2} \, dx \, dy + 4 \int_{|x| \geq R, |y| \leq R} \frac{|u(t, x)|^2 |u(t, y)|^2}{|x - y|^2} \, dx \, dy
\]
\[
- \int_{|x| \geq R, |y| \geq R} (\nabla (a_R(x)) - \nabla (a_R(y)))(x - y) \frac{|u(t, x)|^2 |u(t, y)|^2}{|x - y|^4} \, dx \, dy
\]
\[
- 2 \int_{|x| \geq R, |y| \leq R} (\nabla (a_R(x)) - \nabla (a_R(y)))(x - y) \frac{|u(t, x)|^2 |u(t, y)|^2}{|x - y|^4} \, dx \, dy. \] (66)
Choosing \( R \) sufficiently large and invoking Lemma A.1 - Lemma A.4 in Appendix, we see that the last four terms on the right hand can be bounded above by \( \frac{1}{4} E \). Thus, we get
\[
\partial_t^2 V_R(t) \gtrsim E,
\]
which yields a contradiction with \( (62) \). So, \( E(u_0) \) must be zero. By the variational characterization of the ground state, there exist \( \theta_0, \lambda_0 \) such that
\[
u_0(x) = e^{i\theta_0} \lambda_0^{-d/2} Q(\frac{x}{\lambda_0}).
\]
This completes the proof of Theorem 1.3. \( \square \)

Appendix A. In the following we shall show the smallness for integrals in (66).

Lemma A.1. Let \( I \) be the life-span of \( u \). Then
\[
\lim_{R \to \infty} \sup_{t \in I} \int_{|x| \geq R, |y| \geq R} \frac{|u(t, x)|^2 |u(t, y)|^2}{|x - y|^2} \, dx \, dy = 0. \] (67)

Proof. Noticing that
\[
\int_{|x| \geq R, |y| \geq R} \frac{|u(t, x)|^2 |u(t, y)|^2}{|x - y|^2} \, dx \, dy \leq \int_{\mathbb{R}^4 \times \mathbb{R}^4} \frac{|\phi_{\geq R/2}(x) u(t, x)|^2 |\phi_{\geq R/2}(y) u(t, y)|^2}{|x - y|^2} \, dx \, dy,
\]
\[\leq \int_{\mathbb{R}^4 \times \mathbb{R}^4} \frac{|\phi_{\geq R/2}(x) u(t, x)|^2 |\phi_{\geq R/2}(y) u(t, y)|^2}{|x - y|^2} \, dx \, dy,
\]
by the Hardy-Littlewood-Sobolev inequality and mass conversion law we obtain
\[
\iint_{|x| \geq R/2, |y| \geq R} \frac{|u(t, x)|^2|u(t, y)|^2}{|x - y|^2} \, dxdy \\
\leq C \|\phi_{\geq R/2} u(t)\|_{L^2}^2 \|\nabla (\phi_{\geq R/2} u(t))\|_{L^2}^2 \\
\leq C \|\phi_{\geq R/2} u(t)\|_{L^2}^2. \tag{68}
\]
Since
\[
\|\nabla (\phi_{\geq R/2} u(t))\|_{L^2} \leq \|\nabla (\phi_{\geq R/2}) u(t)\|_{L^2} + \|\phi_{\geq R/2} \nabla u(t)\|_{L^2}
\]
this lemma follows from (61), mass conversion law and (68). \qed

\textbf{Lemma A.2.}
\[
\lim_{R \to \infty} \left( \sup_{t \in I} \iint_{|x| \geq R, |y| \leq R} \frac{|u(t, x)|^2|u(t, y)|^2}{|x - y|^2} \, dxdy \right) = 0. \tag{69}
\]
\textbf{Proof.} Since
\[
\iint_{|x| \geq R, |y| \leq R/2} \frac{|u(t, x)|^2|u(t, y)|^2}{|x - y|^2} \, dxdy \\
\quad \leq \iint_{|x| \geq R, |y| \leq R/2} \frac{|u(t, x)|^2|u(t, y)|^2}{|x - y|^2} \, dxdy \\
\quad \quad + \iint_{|x| \geq R/2, |y| \geq R/2} \frac{|u(t, x)|^2|u(t, y)|^2}{|x - y|^2} \, dxdy,
\]
by (67) we only need to estimate
\[
\iint_{|x| \geq R, |y| \leq R/2} \frac{|u(t, x)|^2|u(t, y)|^2}{|x - y|^2} \, dxdy.
\]
In this case, \(|x - y| \geq |x| - |y| \geq R/2\), thus
\[
\iint_{|x| \geq R, |y| \leq R/2} \frac{|u(t, x)|^2|u(t, y)|^2}{|x - y|^2} \, dxdy \\
\quad \leq \frac{4}{R^2} \iint_{|x| \geq R, |y| \leq R/2} |u(t, x)|^2 |u(t, y)|^2 \, dxdy \\
\quad \leq \frac{4}{R^2} \iint_{\mathbb{R}^d} |u(t, x)|^2 |u(t, y)|^2 \, dxdy = \frac{4}{R^2} \|u\|_{L^2}^4,
\]
which yields the desired result. \qed

\textbf{Lemma A.3.} Let \(I\) be the life-span of \(u\). Then
\[
\lim_{R \to \infty} \left( \sup_{t \in I} \iint_{|x| \geq R, |y| \geq R} \left( \nabla a_R(x) - \nabla a_R(y) \right) (x - y) \frac{|u(t, x)|^2|u(t, y)|^2}{|x - y|^4} \, dxdy \right) = 0.
\]
Lemma A.4.

Proof. By the symmetry with $x$ and $y$, we see that
\[
\int \int_{|x| \geq R, |y| \geq R} \left| \nabla a_R(x) - \nabla a_R(y) \right| |x - y| \frac{|u(t, x)|^2 |u(t, y)|^2}{|x - y|^4} \, dx \, dy
\]
\[
\leq \int \int_{|x| \geq 2R, |y| \geq 2R} \left| \nabla a_R(x) - \nabla a_R(y) \right| |x - y| \frac{|u(t, x)|^2 |u(t, y)|^2}{|x - y|^4} \, dx \, dy
\]
\[
+ \int \int_{R \leq |x| \leq 2R, |y| \leq 2R} \left| \nabla a_R(x) - \nabla a_R(y) \right| |x - y| \frac{|u(t, x)|^2 |u(t, y)|^2}{|x - y|^4} \, dx \, dy
\]
\[
+ 2 \int \int_{R \leq |x| \leq 2R, |y| \geq 2R} \left| \nabla a_R(x) - \nabla a_R(y) \right| |x - y| \frac{|u(t, x)|^2 |u(t, y)|^2}{|x - y|^4} \, dx \, dy
\]
\[
:= I_1(t) + I_2(t) + 2 I_3(t). \quad (70)
\]

By the definition of $a_R$ we have
\[
I_1(t) = 0. \quad (71)
\]

By the definition of $a_R$ and mass conservation law, we obtain
\[
I_3(t) = \int \int_{R \leq |x| \leq 2R, |y| \geq 2R} \left\| \nabla a_R(x) \right\| \frac{|u(t, x)|^2 |u(t, y)|^2}{|x - y|^3} \, dx \, dy
\]
\[
\leq C \frac{R^2}{R^2} \int \int_{R \leq |x| \leq 2R, |y| \geq 2R} |u(t, x)|^2 |u(t, y)|^2 \, dx \, dy \leq C \frac{R^2}{R^2} |u|_{L^2}^4.
\]

Now we estimate $I_2(t)$. By the definition of $a_R$, we have that
\[
\nabla a(x) = 2 \varphi'(|x|^2/R^2)x,
\]

thus by the triangle inequality,
\[
|\nabla a_R(x) - \nabla a_R(y)| = 2 |\varphi'(|x|^2/R^2)x - \varphi'(|y|^2/R^2)y|
\]
\[
\leq 2 |\varphi'(|x|^2/R^2)x - \varphi'(|x|^2/R^2)y| + 2 |\varphi'(|x|^2/R^2)y - \varphi'(|y|^2/R^2)y|
\]
\[
\leq C |x - y| + C |\varphi'(|x|^2/R^2)| |y|
\]
\[
\leq C |x - y| + C |y| \frac{|x|^2 - |y|^2}{R^2} \leq C |x - y| + C |y| \frac{(|x| + |y|)(|x| - |y|)}{R^2}.
\]

Since $|x|, |y| \sim R$, (73) implies that
\[
|\nabla a_R(x) - \nabla a_R(y)| \leq C |x - y|. \quad (74)
\]

It follows that
\[
I_2(t) \leq C \int \int_{|x| \geq R/2, |y| \geq R/2} \frac{|u(t, x)|^2 |u(t, y)|^2}{|x - y|^2} \, dx \, dy.
\]

Then $\lim_{R \to \infty} \sup_{t \in I} I_2(t) = 0$ follows from Lemma A.1. \qed

Lemma A.4. Let $I$ be the life-span of $u$. Then
\[
\lim_{R \to \infty} \left( \sup_{t \in I} \int \int_{|x| \geq R, |y| \leq R} \left( \nabla a_R(x) - \nabla a_R(y) \right) (x - y) \frac{|u(t, x)|^2 |u(t, y)|^2}{|x - y|^4} \, dx \, dy \right) = 0.
\]
Proof. Since

\[
\int_{|x| \geq R, |y| \leq R} \left| \nabla a_R(x) - \nabla a_R(y) \right| \left| x - y \right| \frac{|u(t,x)|^2|u(t,y)|^2}{|x-y|^4} \, dx \, dy
\]

\[
\leq \int_{|x| \geq R, |y| \leq R} \left| \nabla a_R(x) - \nabla a_R(y) \right| \left| x - y \right| \frac{|u(t,x)|^2|u(t,y)|^2}{|x-y|^4} \, dx \, dy
\]

\[
+ \int_{R \leq |x| \leq 2R, R/2 \leq |y| \leq R} \left| \nabla a_R(x) - \nabla a_R(y) \right| \left| x - y \right| \frac{|u(t,x)|^2|u(t,y)|^2}{|x-y|^4} \, dx \, dy
\]

\[
+ \int_{R \leq |x| \leq 2R, R/2 \leq |y| \leq R} \left| \nabla a_R(x) - \nabla a_R(y) \right| \left| x - y \right| \frac{|u(t,x)|^2|u(t,y)|^2}{|x-y|^4} \, dx \, dy
\]

\[
=: J_1(t) + J_2(t) + J_3(t).
\]

By the estimates of $I_2(t)$ and $I_3(t)$ in Lemma A.3 we see that

\[
J_1(t) \leq \frac{C}{R^2} \|u(t)\|_{L^2}^4
\]

\[
J_2(t) \leq C \int \int_{|x| \geq R/2, |y| \geq R/2} \frac{|u(t,x)|^2|u(t,y)|^2}{|x-y|^2} \, dx \, dy.
\]

In the region $R \leq |x| \leq 2R$ and $|y| \leq R/2$ we have $|\nabla a_R(x)| \leq C|x|$, $|\nabla a_R(y)| = 2|y|$, and $|x-y| \geq \frac{R}{2}$. Thus

\[
J_3(t) \leq C \int \int_{R \leq |x| \leq 2R, |y| \leq R} \frac{|x| + |y|}{|x-y|^4} |u(t,x)|^2|u(t,y)|^2 \, dx \, dy
\]

\[
\leq C \int \int_{R \leq |x| \leq 2R, |y| \leq R} \frac{R}{R^4} |u(t,x)|^2|u(t,y)|^2 \, dx \, dy
\]

\[
\leq \frac{C}{R^2} \int \int_{R^2 \times R^2} |u(t,x)|^2|u(t,y)|^2 \, dx \, dy = \frac{C}{R^2} \|u(t)\|_{L^2}^4.
\]

The lemma follows from (75), (76), (77) and Lemma A.1. \hfill \Box

REFERENCES

[1] T. Cazenave, *Semilinear Schrödinger Equations* vol. 10 of Courant Lecture Notes in Mathematics, New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2003.

[2] B. Dodson, Global well-posedness and scattering for the mass critical nonlinear Schrödinger equation with mass below the mass of the ground state. *Adv. Math.*, 285 (2015), 1580–1618.

[3] T. Duyckaerts, J. Holmer and S. Roudenko, Scattering for the non-radial 3D cubic nonlinear Schrödinger equation. *Math. Res. Lett.*, 15 (2008), 1233–1250.

[4] J. Fröhlich and E. Lenzmann, Mean-field limit of quantum Bose gases and nonlinear Hartree equation, in *Séminaire: Équations aux Dérivées Partielles. 2003–2004*, Sémin. Équ. Dériv. Partielles, École Polytech., Palaiseau, 2004, Exp. No. XIX, 26pp.

[5] J. Fröhlich, T.-P. Tsai and H.-T. Yau, On the point-particle (Newtonian) limit of the nonlinear Hartree equation. *Comm. Math. Phys.*, 225 (2002), 223–274.

[6] Y. Gao and H. Wu, Scattering for the focusing $H^{1/2}$-critical Hartree equation in energy space. *Nonlinear Anal.*, 73 (2010), 1043–1056.

[7] J. Ginibre and G. Velo, *Smoothing properties and retarded estimates for some dispersive evolution equations* *Comm. Math. Phys.*, 144 (1992), 163–188.

[8] T. Hamidi and S. Keraani, Blowup theory for the critical nonlinear Schrödinger equations revisited *Int. Math. Res. Not.*, 2005 (2005), 2815–2828.

[9] M. Keel and T. Tao, Endpoint Strichartz estimates *Amer. J. Math.*, 120 (1998), 955–980.
[10] C. E. Kenig and F. Merle, Global well-posedness, scattering and blow-up for the energy-critical, focusing, non-linear Schrödinger equation in the radial case, *Invent. Math.*, 166 (2006), 645–675.

[11] S. Keraani, On the blow up phenomenon of the critical nonlinear Schrödinger equation, *J. Funct. Anal.*, 235 (2006), 171–192.

[12] R. Killip, D. Li, M. Visan and X. Zhang, Characterization of minimal-mass blowup solutions to the focusing mass-critical NLS, *SIAM J. Math. Anal.*, 41 (2009), 219–236.

[13] R. Killip, T. Tao and M. Visan, The cubic nonlinear Schrödinger equation in two dimensions with radial data, *J. Eur. Math. Soc. (JEMS)*, 11 (2009), 1203–1258.

[14] R. Killip and M. Visan, The focusing energy-critical nonlinear Schrödinger equation in dimensions five and higher, *Amer. J. Math.*, 132 (2010), 361–424.

[15] R. Killip and M. Visan, Nonlinear Schrödinger equations at critical regularity, in *Evolution equations*, vol. 17 of Clay Math. Proc., Amer. Math. Soc., Providence, RI, 2013, 325–437.

[16] R. Killip and M. Visan, The focusing energy-critical nonlinear Schrödinger equation in dimensions $d \geq 2$, *Sci. China Math.*, 55 (2012), 385–424.

[17] E. H. Lieb and M. Loss, *Analysis*, vol. 14 of Graduate Studies in Mathematics, 2nd edition, American Mathematical Society, Providence, RI, 2001.

[18] Y. Martel and F. Merle, Instability of solitons for the critical generalized Korteweg-de Vries equation, *Geom. Funct. Anal.*, 11 (2001), 74–123.

[19] F. Merle, Determination of blow-up solutions with minimal mass for nonlinear Schrödinger equations with critical power, *Duke Math. J.*, 69 (1993), 427–454.

[20] F. Merle and P. Raphael, The blow-up dynamic and upper bound on the blow-up rate for critical nonlinear Schrödinger equation, *Ann. of Math. (2)*, 161 (2005), 157–222.

[21] M. I. Weinstein, Modulational stability of ground states of nonlinear Schrödinger equations, *SIAM J. Math. Anal.*, 16 (1985), 472–491.

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