ON THE QUOTIENT CLASS OF NON-ARCHIMEDEAN FIELDS

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ABSTRACT. The quotient class of a non-archimedean field is the set of cosets with respect to all of its additive convex subgroups. The algebraic operations on the quotient class are the Minkowski sum and product. We study the algebraic laws of these operations. Addition and multiplication have a common structure in terms of regular ordered semigroups. The two algebraic operations are related by an adapted distributivity law.

Keywords: non-archimedean fields, cosets, regular semigroups, convexity.

AMS classification: 26E30, 12J15, 20M17, 06F05.

1. Introduction

We study the algebraic properties of the set of cosets with respect to all possible convex additive subgroups of a non-archimedean field \( F \), typically a field of formal series or a Hardy-field. We will call this set of cosets the quotient class of \( F \). Because the (Minkowski) sum of a nontrivial convex additive subgroup and an arbitrary element can never be zero, the quotient class cannot be a group for addition, and for similar reasons neither for multiplication. Still a quotient class satisfies rather strong algebraic properties, for, as we will see, addition and multiplication are commutative, satisfy the properties of regular semigroups and are related by an adapted distributive law.

The common structure of addition and multiplication is stronger than a regular semigroup and was called assembly in [7]. We will call magnitude a convex additive subgroup \( M \) of \( F \), this is in line with a common interpretation of Hardy-fields as models of orders of magnitude of functions [9][3][1]. There exists a definite relationship between non-archimedean structures and asymptotics [2][4][5][13]; in a sense, a magnitude may be seen as the size of an imprecision. Given a coset with respect to \( M \), the magnitude \( M \) acts as an individualized neutral element for addition. If \( \alpha \) is a coset which is not reduced to a magnitude \( M \) it has an individualized neutral element for multiplication \( M/\alpha \), which with some abuse of language is called unity.

It is easy to see that distributivity does not hold in general. However we show that distributivity holds up to a correction term which has the form of a magnitude. We will identify other properties which relate addition and multiplication and call the resulting structure association.

The order relation in the ordered field \( F \) induces a natural order in the quotient class \( Q \). We show that this is a total dense order relation compatible with the operations. If \( F \) is archimedean, the quotient class reduces to an ordered field. If \( F \)
is non-archimedean, the quotient class contains magnitudes different from \( \{0\} \) and its domain. Clearly \( \{0\} \) is the minimal magnitude of \( Q \), but \( Q \) has also a maximal magnitude which is its domain \( F \) itself; the minimal unity is \( \{1\} \). In general, an association with these properties is called a **solid**. So we will prove that a quotient class of a non-archimedean field \( F \) is a solid. For the sake of clarity we give a full list of the axioms of a solid in the appendix.

As remarked above, in solids distributivity does not hold in general. However, it turns out that in many cases full distributivity does hold, for example for elements of the same sign. Also it is possible to give necessary and sufficient conditions for the distributive law to hold for triples of elements of solids. The proofs are rather involved and are presented in a second paper.

In Section 2 we define the quotient class of an ordered field. We extend the order relation to the quotient class, prove that the property of trichotomy is maintained and show compatibility properties of the order with the algebraic operations. We recall also some basic notions of semigroups. In Section 3 we recall the notion of assembly which amounts to a regular semigroup with an idempotent condition on the magnitude operator. As a consequence the magnitude operator will be linear. We show that the quotient class is an assembly for addition and, leaving out the magnitudes, for multiplication. In Section 4 we define a structure called association which is, roughly speaking, a ring with individualized neutral elements for both addition and multiplication, and an adapted distributive law. Ordered associations are associations equipped with a total order relation respecting the algebraic operations. In Section 5 we define solids which are in a sense weakly distributive ordered fields with generalized neutral elements given by magnitudes and unities. We show that the quotient class of a non-archimedean field is a solid.

By the above, solids arise with non-archimedean fields. Archimedean solids may exist, but only in a set theory with a different axiomatics than conventional set theory. This question is briefly addressed at the end of the last section.

### 2. Quotient Classes

Let \( (F, +, \cdot, \leq, 0, 1) \) be a non-archimedean ordered field. Let \( C \) be the set of all convex subgroups for addition of \( F \) and \( Q \) be the set of all cosets with respect to the elements of \( C \). We will call the elements of \( C \) **magnitudes** and \( Q \) the **quotient class** of \( F \) with respect to \( C \). Observe that \( C \) is not reduced to \( \{0\} \) and \( F \) itself. Indeed, a non-archimedean ordered field necessarily has infinitesimals other than 0. Let \( \Theta \) denote the set of all infinitesimals in \( F \). It is clearly convex and satisfies the group property, so \( \Theta \in C \). An element of \( Q \setminus C \) is called **zeroless**.

For \( \alpha \in Q \), in the remainder of this section we use the notation \( \alpha = a + A \), with \( a \in F \) and \( A \in C \). Clearly \( A \) is unique but \( a \) is not. If \( \alpha \) is zeroless, one proves by induction that \( A/a \subseteq [-1/n,1/n] \) for all \( n \in \mathbb{N} \), hence \( A/a \subseteq \Theta \).

We define (with abuse of notation) addition in \( Q \) pointwise, i.e. by the Minkowski sum

\[
\alpha + \beta := a + b + A + B.
\]

We define (with abuse of notation) multiplication in \( Q \) also pointwise, by

\[
\alpha\beta := ab + aB + bA + AB.
\]
Let also \( \alpha = a' + A \) and \( \beta = b' + B \). Then \( a' + b' - (a + b) \in A + B \) and
\[
\begin{align*}
a' b' - ab &= a(b' - b) + b(a' - a) + (a' - a)(b' - b) \\
&\in aB + bA + AB,
\end{align*}
\]
hence addition and multiplication do not depend on the choice of representatives.

By convexity the sum of two magnitudes \( A \) and \( B \) is equal to one of them, i.e.
\[
(1) \quad A + B = A \lor A + B = B,
\]
according to whether \( B \subseteq A \) or \( A \subseteq B \).

We now use the order relation \( \leq \) on \( F \) to define an order relation, also noted \( \leq \),
on \( Q \); we will see below that it is a total order relation respecting the operations,
extending the order relation on \( F \).

**Definition 2.2.** Given \( \alpha, \beta \in Q \), we say that \( \alpha \) is less than or equal to \( \beta \) and we write (with abuse of notation) \( \alpha \leq \beta \), if and only if
\[
(\forall x \in \alpha)(\exists y \in \beta)(x \leq y).
\]
We say that \( \alpha \) is (strictly) less than \( \beta \) and write \( \alpha < \beta \), if \( \alpha \leq \beta \) and \( \alpha \cap \beta = \emptyset \).

Note that if \( \alpha \) and \( \beta \) are separated, formula (2) is equivalent to \((\forall x \in \alpha)(\forall y \in \beta)(x < y)\).

If the magnitudes of \( \alpha \) and \( \beta \) are \( \{0\} \), these elements may be identified with elements of \( F \), and the order relation (2) corresponds to the original order relation \( \leq \) on \( F \). Let \( A \) and \( B \) be magnitudes. By convexity, \( A \leq B \) if and only if \( A \subseteq B \), i.e. magnitudes are ordered by inclusion. Clearly \( 0 \in A \) for every magnitude \( A \) and as a consequence all magnitudes are positive. Since \( A \subseteq B \) if and only if \( A + B = B \), the order relation on the magnitudes corresponds to the natural partial order restricted to idempotents, see for example [10] p. 14 [15] p. 18.

**Theorem 2.3.** The relation \( \leq \) is a total order relation. It is compatible with addition and multiplication in the following way:

1. \( \forall \alpha \forall \beta \forall \gamma (\alpha \leq \beta \Rightarrow \alpha + \gamma \leq \beta + \gamma) \).
2. \( \forall \alpha (A < \alpha \Rightarrow \forall \beta \forall \gamma (\beta \leq \gamma \Rightarrow \alpha \beta \leq \alpha \gamma)) \).
3. \( \forall A \forall \beta \forall \gamma (B \leq \beta \leq \gamma \Rightarrow A \beta \leq A \gamma) \).
Proof. Let \( \alpha, \beta \) and \( \gamma \) be arbitrary elements of \( Q \). We prove firstly that the relation \( \leq \) is a total order relation on \( Q \). Let \( x \in \alpha \). Because \( x \leq x \) one has \( \alpha \leq \alpha \), so the relation is reflexive. Suppose that \( \alpha \leq \beta \) and \( \beta \leq \gamma \). Then for all \( x \in \alpha \) there exist \( y \in \beta \) and \( z \in \gamma \) such that \( x \leq y \) and \( y \leq z \). Hence \( x \leq z \) and the relation is transitive. Suppose now that \( \alpha \leq \beta \) and \( \beta \leq \alpha \). Let \( x \in \alpha \). Because \( \alpha \leq \beta \) there exists \( y \in \beta \) such that \( x \leq y \). There exists \( y' \in \beta \) such that \( y' \leq x \), if not, all \( x \in \alpha \) and \( y' \in \beta \) satisfy \( x < y' \), in contradiction with the fact that \( \beta \leq \alpha \). By convexity \( x \in [y', y] \subseteq \beta \), hence \( \alpha \subseteq \beta \). In an analogous way one shows that \( \beta \subseteq \alpha \). Hence \( \alpha = \beta \) and the relation is antisymmetric. To prove the totality property suppose that \( \alpha \nleq \beta \). Then there is \( x \in \alpha \) such that \( y < x \) for all \( y \in \beta \). Hence \( \beta \leq \alpha \). We conclude that the relation \( \leq \) is a total order relation.

We finish by proving the three compatibility properties.

1. Suppose that \( \alpha \leq \beta \). Let \( w \in \alpha + \gamma \). Then there are \( x \in \alpha \) and \( z \in \gamma \) such that \( w = x + z \). Now there exists \( y \in \beta \) such that \( x \leq y \). Hence \( x + z \leq y + z \leq \beta + \gamma \) and one concludes that \( \alpha + \gamma \leq \beta + \gamma \).

2. Suppose that \( A < \alpha \) and \( \beta \leq \gamma \). Let \( w \in \alpha \beta \). Then there exist \( x \in \alpha \), \( 0 < \alpha \) and \( y \in \beta \) such that \( w = xy \). Because \( \beta \leq \gamma \) there is \( z \in \gamma \) such that \( y \leq z \). Then \( xy \leq yz \), hence \( \alpha \gamma \leq \beta \gamma \).

3. Suppose that \( B \leq \beta \leq \gamma \). Let \( w \in A \beta \). Because \( B \leq \beta \) the element \( w \) may be supposed positive. Then there exist \( x \in A \), \( 0 \leq x \) and \( 0 \leq y \in \beta \) such that \( w = xy \). Because \( \beta \leq \gamma \) there is \( z \in \gamma \) such that \( y \leq z \). Then \( xy \leq yz \), hence \( A \gamma \leq A \gamma \).

\[ \square \]

The above proposition states that usual compatibility holds for multiplication by strictly positive (zeroless) elements \( \alpha \).

If \( \alpha = A \) is a magnitude, the rule must be restricted to nonnegative \( \beta \) and \( \gamma \), for instance, if \( \omega > 0 \) is infinitely large, one has \( -\omega < -1 \), while \( \ominus \cdot (-1) \leq \ominus \cdot (-\omega) \), for \( \ominus \cdot (-1) = \ominus \) and \( \ominus \cdot (-\omega) = \ominus \cdot \omega \geq 1/\omega \cdot \omega = 1 \).

3. The Magnitude Operator. Assemblies.

Let \( \alpha = a + A \in Q \). Then the magnitude \( A \) is a sort of individualized neutral element, since \( \alpha + A = a + A + A = a + A = \alpha \). As regards to other magnitudes \( B \) which leave \( \alpha \) invariant it distinguishes itself by the property \( A + B = A \) and being uniquely determined by \( \alpha \). Hence we may define a function \( e : Q \to \mathcal{C} \) by putting \( e(\alpha) = A \). The function is linear, for if \( \beta = b + B \in Q \)

\[ e(\alpha + \beta) = A + B = e(\alpha) + e(\beta). \]

Also, by (1)

\[ e(\alpha + \beta) = e(\alpha) \lor e(\alpha + \beta) = e(\beta). \]

With respect to \( e(\alpha) \) we may also identify a distinguished symmetrical element \( s(\alpha) \), having the same magnitude as \( \alpha \), simply by putting \( s(\alpha) = -\alpha = -a + A \). Semigroup structures with the above properties for individualized neutral and symmetrical elements have been called assemblies in [7].

Below we list the axioms of an assembly. It is easy to verify that the element \( e \) of Definition 3 [8, 13] is unique (see Remark A.1) and with some abuse of language we use the same notation as above.
Definition 3.1. A non-empty structure $(\mathcal{A}, +)$ is called an assembly if
(1) $\forall x \forall y \forall z (x + (y + z) = (x + y) + z)$.
(2) $\forall x \forall y (x + y = y + x)$.
(3) $\forall x \exists e (x + e = x \wedge \forall f (x + f = x \rightarrow e + f = e))$.
(4) $\forall x \exists s (x + s = e (x) \wedge e (s) = e (x))$.
(5) $\forall x \forall y (e (x + y) = e (x) \vee e (x + y) = e (y))$.

We will prove that $(Q, +)$ and $(Q \setminus \mathcal{C}, \cdot)$ are assemblies. First we prove that they are completely regular commutative semigroups. Let $A$ be a magnitude. We let $F_A = \{x + A | x \in F\}$ and $R_A = \{x (1 + A/\alpha) | x \in F \setminus \{0\}\}$.

Proposition 3.2. $(Q, +)$ is a completely regular commutative semigroup.

Proof. Clearly addition is associative and commutative. Let $\alpha = a + A \in Q$. Then $\alpha \in F_A$. Observe that $e (\alpha)$ is the neutral element of $F_A$ and $s (\alpha)$ is the symmetric element of $\alpha$ in $F_A$, hence $F_A$ is a subgroup of $Q$. Hence the commutative semigroup $(Q, +)$ is completely regular. □

Proposition 3.3. $(Q \setminus \mathcal{C}, \cdot)$ is a completely regular commutative semigroup.

Proof. Clearly multiplication is associative and commutative. Let $\alpha = a + A \in Q \setminus \mathcal{C}$. We may write $\alpha = a (1 + A/\alpha)$. Recall that $1 + A/\alpha$ is zeroless. Then $\alpha \in R_A$, which is a subgroup of $Q$; this follows from the fact that $(1 + A/\alpha) (1 + A/\alpha) = 1 + A/\alpha + A/a + A/a = (1 + A/a)$, noting that $A/a \cdot A/\alpha \subseteq A/\alpha \cdot 1 = A/\alpha$. Hence the commutative semigroup $(Q \setminus \mathcal{C}, \cdot)$ is completely regular. □

The above structures are not proper groups. Indeed, if $X \in \mathcal{C}, X > 0$, the equation $1 + \odot + (-1 + X) = 0$ does not have a solution, for $\odot + X > X > 0$. Also $(1 + X)/(1 + \odot) \neq 1$ for any convex group $X$, for $(1 + X)/(1 + \odot) = (1 + X)(1 + \odot) = 1 + X + \odot + X \odot > 1$.

An assembly is a completely regular commutative semigroup, for every $a \in A$ is element of the group $A_{e (a)} = \{x \in A | e (x) = e (a)\}$. Conversely, a completely regular commutative semigroup is an assembly with $s (\alpha) = -a + A$ if the operator $e$ satisfies condition (5) of Definition 3.1. By the remarks above and formula (4), we have the following proposition.

Proposition 3.4. The structure $(Q, +)$ is an assembly.

The proposition is also true for the multiplicative structure $(Q \setminus \mathcal{C}, \cdot)$, Let $\mu : Q \setminus \mathcal{C} \rightarrow Q \setminus \mathcal{C}$ be defined by $\mu (\alpha) = 1 + A/\alpha$ and let $d : Q \setminus \mathcal{C} \rightarrow Q \setminus \mathcal{C}$ be defined by $d (\alpha) = \frac{1}{\alpha}$. Note that for all $a$ such that $\alpha = a + A$

\[(5) \quad \frac{1}{\alpha} = \frac{1}{a + A} = \frac{1}{a (1 + A/\alpha)} = \frac{1}{a} \left(1 + \frac{A}{a}\right) = \frac{1}{a} \frac{A}{a^2} \in Q \setminus \mathcal{C}.\]

Then also for all $a$ such that $\alpha = a + A$

\[(6) \quad \frac{A}{\alpha} = \frac{A}{a} + \frac{A^2}{a^2} = \frac{A}{a}\]

and $\mu (\alpha) = 1 + A/\alpha \in Q \setminus \mathcal{C}$. This means that the functions $u$ and $d$ are well-defined. Clearly $u (\alpha)$ is the multiplicative neutral element (unity) of $\alpha$ and $d (\alpha)$ is the inverse of $\alpha$ in the group $R_A$. In particular $a u (\alpha) = \alpha$ and $a d (\alpha) = u (\alpha)$. Note that

\[(7) \quad e (u (\alpha)) = e (\alpha) / \alpha\]
and
\[(8)\quad e(d(\alpha)) = e(\alpha)/\alpha^2.\]

**Proposition 3.5.** The structure \((Q \setminus \mathcal{C}, \cdot)\) is an assembly.

**Proof.** By Proposition 3.3 we need only to verify condition (5) of Definition 3.1. Let \(\alpha, \beta \in Q \setminus \mathcal{C}\). Then \(aB + bA + AB = aB + bA\). Hence
\[
u(\alpha\beta) = \nu(ab + aB + bA + AB) = 1 + \frac{aB + bA}{ab} = 1 + \frac{B}{b} + \frac{A}{a}\]
which is equal to \(\nu(\alpha)\) or \(\nu(\beta)\).

Classical models of orders of magnitude are based on the \(O\)’s and \(o\)'s. They can be seen as sets of real functions, for which addition can be defined pointwise [4]. We give an example where \(O\)’s and \(o\)'s give rise to additive and multiplicative assemblies. This is done in the context of a non-archimedean field, in which all the magnitudes except \(\{0\}\) and the field itself may be determined in terms of \(O\)’s and \(o\)'s; in fact, the \(o\)'s are reduced to \(O\)'s. Let \(\mathcal{R}\) be the set of all rational fractions with coefficients in \(\mathbb{R}\) with the usual addition and multiplication. Let \(n \in \mathbb{Z}\). Clearly \(O(x^n)\) is a magnitude, for \(x \to \infty\) and then \(o(x^n) = O(x^{n-1})\). Conversely, let \(\{0\} \subset M \subset \mathcal{R}\) be a magnitude. Let \(n \in \mathbb{N}\) be minimal such that \(x^n \notin M\). If there exists \(r \in \mathcal{R}\) such that \(\limsup r(x)/x^{n-1} = \lim r(x)/x^{n-1} = \infty\) then the degree of \(r\) is equal to \(n\). Hence \(x^n \in M\), a contradiction. Hence \(M = O(x^{n-1})\). Let \(Q\) be the quotient field of \(\mathcal{R}\). So within \(Q\) the \(O\)'s define additive and multiplicative assemblies.

The following example shows that \(O\)'s and \(o\)'s do not generate assemblies in general.

**Example 3.6.** We will show that condition (5) of Definition 3.1 does not hold for \(O\)'s and \(o\)'s of real functions. Let \(f, g : \mathbb{R} \to \mathbb{R}^+\) be defined by \(f(x) = x + x^2(\sin x, 0)^\top\) and \(g(x) = x + x^2(\cos x, 0)^\top\). For \(x \to +\infty\) we have \(O(f + g) = O(x^2)\), but since \(x^2 \notin O(f)\) and \(x^2 \notin O(g)\), neither \(O(f) = O(f + g)\), nor \(O(g) = O(f + g)\). For the same reason neither \(o(f) = o(f + g)\), nor \(o(g) = o(f + g)\).

We end with examples of assemblies in a different context.

**Example 3.7.**
1. Commutative groups are assemblies on which the function \(e\) is constant.
2. Let \(C\) be a chain for inclusion with the union operation \(\cup\). The structure \((C, \cup)\) is an assembly, with \(e(U) = s(U) = U\) for all \(U \subseteq C\). Note that \(e(U \cup V) = U \cup V\). Hence \(e(U \cup V) = e(U)\) or \(e(U \cup V) = e(V)\).

4. Mixed properties of addition and multiplication

We will see by a simple example that distributivity does not hold in \(Q\). Still an adapted version of distributivity does hold, which requires the introduction of a correcting term in the form of a magnitude. Then we calculate the magnitude and the symmetrical of the product. We introduce the notion of association which roughly speaking stays in relation to rings in the way assemblies are to groups. Associations with a total order relation compatible with the operations are called ordered associations. Finally we show that \(Q\) is indeed an ordered association.

We start by showing that distributivity does not hold in \(Q\).
Example 4.1. 0 = ⊙(1 − 1) ̸= ⊙1 − ⊙1 = ⊙.

In the example the error made has the form of a magnitude. This is generally true and follows from the next two propositions. The first proposition gives the form of the error term and the second one shows that this error is a magnitude.

Proposition 4.2. Let α = a + A, β = b + B, γ = c + C ∈ Q. Then αβ + αγ = α(β + γ) + Aβ + Aγ.

Proof. Because F is a field a (b + c) = ab + ac. Furthermore A (b + c) ⊆ bA + cA, because |b + c| ≤ 2 max(|b|, |c|), 2A = A and bA + cA = max(|b|, |c|)A. Also we have the identity of groups A (B + C) = AB + AC. Hence

α(β + γ) + Aβ + Aγ
= (a + A)(b + c + B + C) + A(b + B) + A(c + C)
= ab + ac + aB + aC + bA + cA + AB + AC
= (a + A)(b + (a + A) B) + (a + A) c + (a + A) C
= (a + A)(b + B) + (a + A)(c + C)
= αβ + αγ.

□

Next proposition shows that the correction term in the adapted version of distributivity is a magnitude.

Proposition 4.3. Let α = a + A, β = b + B ∈ Q. Then there exists δ ∈ Q such that e(α) β = e(δ).

Proof. Put δ = bA + AB. One has

e(α) β = A(b + B) = bA + AB = δ = e(δ).

□

It is not difficult to determine the magnitudes of a product, the unity element and the inverse of a zeroless element in Q. In fact we have the following proposition.

Proposition 4.4. Let α, β ∈ Q. Then

1. e(αβ) = αe(β) + βe(α).
2. −(αβ) = (−α) β = α(−β).

Proof. Let α = a + A, β = b + B ∈ Q.

1. One has βe(α) + αe(β) = A(b + B) + (a + A) B = bA + AB + aB + AB = e(ab + bA + aB + AB) = e(αβ).

2. This is evident, because −(ab + bA + aB + AB) = (−a + A)(b + B) = (a + A)(−b + B).

□

Structures with the properties given by Propositions 4.2, 4.3 and 4.4 and formulas (7) and (3) will be called associations. Let A be an assembly. We denote by N the set of all elements of A which are not zeroless.

Definition 4.5. A structure (A, +, ·) is called an association if the structures (A, +) and (A \ N, ·) are both assemblies and if the following hold:

1. ∀x∀y∀z (xy + xz = x(y + z) + e(x)y + e(x)z).
\(\forall x, y, z \in A (e(xy) = e(x)y + e(y)x)\).

(4) \(\forall x \neq e(x) (e(u(x)) = e(x)d(x))\).

(5) \(\forall x, y \in A (s(x) y = s(y)x)\).

**Theorem 4.6.** The structure \((Q, +, e, s, \cdot, u, d, \leq)\) is an association.

**Proof.** Directly from Proposition 4.2, Proposition 4.3 and Proposition 4.4. □

Theorem 2.3 shows that it is possible to introduce in Q a total order compatible with the operations. We also have to specify the order relation between the magnitudes and general elements. We already saw that if \(A, B\) are magnitudes such that \(A + B = A\), then \(B \leq A\). Condition 2 of the next definition generalizes this to arbitrary elements. Structures satisfying the above properties are called ordered associations.

**Definition 4.7.** We say that a structure \((A, +, \cdot, \leq)\) is an ordered association if \((A, +, \cdot)\) is an association, \(\leq\) is a total order relation and the following hold:

(1) \(\forall x, y, z \in A (x \leq y \implies x + z \leq y + z)\).

(2) \(\forall x, y \in A (y + e(x) = e(x) \implies y \leq e(x) \land s(y) \leq e(x))\).

(3) \(\forall x, y, z \in A ((e(x) < x \land y \leq z) \implies xy \leq xz)\).

(4) \(\forall x, y, z \in A ((e(y) \leq y \leq z) \implies e(x)y \leq e(x)z)\).

**Theorem 4.8.** The structure \((Q, +, e, s, \cdot, u, d, \leq)\) is an ordered association.

**Proof.** By Theorem 2.3 and Theorem 4.6, we only need to show that condition 2 is satisfied. Let \(\alpha = a + A, \beta = b + B \in Q\). Assume that \(\alpha + B = B\), i.e. \(a + A + B = B\). Then \(A \subseteq B\), so \(a + A \subseteq B\). Hence \(\alpha \leq B\). Also \(\alpha + \alpha \subseteq B + B = B\). Then \(-\alpha = \alpha - (\alpha + \alpha) \subseteq B - B = B\). Hence \(-\alpha \leq B\). □

5. **Solids**

Clearly rings with unity are associations and the same is true for fields. As will be shown below associations with a unique magnitude are fields.

In order to distinguish fields from associations we will postulate the existence of particular elements. The resulting structure will be called a solid. We finish by proving that the quotient class of a non-archimedean field is a solid.

**Proposition 5.1.** Let \((A, +, \cdot)\) be an association with a unique magnitude \(e\) then \((A, +, \cdot)\) is a ring. Furthermore if it has a unique unity \(u\) then \((A, +, \cdot, e, u)\) is a field.

**Proof.** With respect to the first part we only need to show that the magnitude is the neutral element and that distributivity holds. Let \(x \in A\) and let \(e\) be the unique magnitude in \(A\). Then \(x = x + e(x) = x + e\). Hence \(e\) is the neutral element for addition. Observe that \(ex = e\), by condition 2 of Definition 4.4. To prove distributivity let \(x, y, z \in A\). Then \(xy + xz = x(y + z) + e(x)y + e(x)z = x(y + z) + e + e = x(y + z)\). Hence \((A, +, \cdot)\) is a ring.

To prove the second part note that \(u \neq e\) because \(u\) is not a magnitude. As above, \((A \setminus \langle \cdot, u \rangle)\) is a group. Hence \((A, +, \cdot, e, u)\) is a field. □

**Definition 5.2.** A structure \((S, +, \cdot, \leq)\) is called a solid if \((S, +, \cdot, \leq)\) is an ordered association such that the following hold:
(1) \( \exists m \forall x \left( m + x = x \right) \).
(2) \( \exists M \forall x \left( e(x) + M = M \right) \).
(3) \( \exists u \forall x \left( ux = x \right) \).
(4) \( \forall x \exists a \left( x = a + e(x) \land e(a) = 0 \right) \).
(5) \( \exists x \left( e(x) \neq m \land e(x) \neq M \right) \).
(6) \( \forall x \forall y \left( x = e(x) \land y = e(y) \land x < y \land x < z \land y < z \Rightarrow \exists z \left( z \neq e(z) \land x < z \land y < z \right) \right) \).

Conditions (1) and (4) are completion properties in the sense that they postulate the existence of (minimal) neutral elements for addition and multiplication (corresponding to 0 and 1 in groups and fields). Condition (2) postulates the existence of a maximal individualized neutral element (denoted \( M \)). The existence of such an absorber is a common procedure in semigroups where it is called "zero element" (see for example [10, p. 2]). In the case of the structure \( (Q, +, e, s, - u, d, \leq) \) it is the field \( F \) which is the largest magnitude. Condition (4) allows to decompose each element in terms of an element with minimal neutral element ("precise element") and an individualized neutral element, like the representation \( a = a + A \) in \( Q \). We may identify \( a \) with an element of \( F \). Condition (5) postulates the existence of nontrivial neutral elements, i.e. neutral elements besides \( m \) and \( M \) and as a consequence that effectively solids have a richer structure than fields. Condition (6) avoids "gaps" in the sense that two magnitudes are separated by an element which is not a magnitude.

**Theorem 5.3.** The structure \( (Q, +, \cdot, \leq) \) is a solid.

**Proof.** By Theorem 1.8 we only need to verify that conditions (1)-(5) of Definition 5.2 are satisfied.

Condition (1) is satisfied by construction. Conditions (1)-(3) are satisfied taking \( m = \{0\} \), \( M = F \) and \( u = \{1\} \). A non-archimedean ordered field necessarily has infinitesimals other than 0. Let \( \otimes \) denote the set of all infinitesimals in \( F \). It is clearly convex and satisfies the group property. Also \( \otimes \neq \{0\} \) and \( \otimes \neq F \), so condition (5) also holds. To show that condition (6) holds let \( A, B \) be magnitudes in \( F \) such that \( A < B \). Let \( b \in B \setminus A \). Then \( A < b < B \).

**Remark 5.4.** Due to the existence of non-trivial magnitudes, within ordinary set theory ZFC any solid must be non-archimedean. Indeed, let \( x \) be such that \( 0 < e(x) < M \). By Definition 5.2.6 there exist \( y \) such that \( e(x) < y < M \). Then \( e(x) = e(x)/y < 1 \). Now \( e(x) + e(x) = e'(x) \), and because the induction scheme holds, one obtains that \( ne'(x) = e'(x) \) for all \( n \in \mathbb{N} \). As a consequence \( ne'(x) < 1 \) for all \( n \in \mathbb{N} \). However, there exists also an Archimedean field such that the quotient class with respect to its magnitudes is a solid. Such a solid exists within the axiomatic approach to Nonstandard Analysis IST (Internal Set Theory) of Nelson [14]. In this approach the set of all real numbers \( \mathbb{R} \) is Archimedean, the axiomatics distinguishes "standard" natural numbers and "nonstandard" natural numbers within \( \mathbb{N} \), the latter numbers being always larger than the first. Then there exist (many) convex ordered groups within \( \mathbb{R} \) which are not reduced to \( \{0\} \) and \( \mathbb{R} \) itself, like the set of all infinitesimals. It has to be noted that they are "external sets" in the sense of the extended axiomatics HST presented in [11]. They were called "(scalar) neutrices" in [12], after the functional neutrices of Van der Corput [5]. In [12] a (mostly external) coset with respect to a convex ordered subgroup within \( \mathbb{R} \) was called "external number". The external set of all possible external numbers was shown to be an assembly for addition in [7] and a solid in [6].

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Remark A.1. The functional notation for magnitudes is justified by the fact that the element $e$ of Axiom 3 is unique. Indeed, if $e'$ satisfies Axiom 3 one has $e' = e + e = e + e' = e$. Also $s$ is unique and may be considered functional. Indeed, if $s'$ satisfies Axiom 4 one has $s' = s' + e(s') = s' + e(x) = s' + x + s = x + s + s = e(x) + s = c(s) + s = s$. In fact we will use the notation $-x$ for $s(x)$. The functional notation for unities is justified in an analogous way where we will use $/x$ instead of $d(x)$.

(1) Axioms for addition
Axiom 1. $\forall x\forall y\forall z(x + (y + z) = (x + y) + z)$.
Axiom 2. $\forall x\forall y(x + y = y + x)$.
Axiom 3. $\forall x\exists e (x + e = x \land \forall f (x + f = x \rightarrow e + f = e))$.
Axiom 4. $\forall x\exists s (x + s = e(x) \land e(s) = e(x))$.
Axiom 5. $\forall x\forall y (e(x + y) = e(x) \lor e(x + y) = e(y))$.

(2) Axioms for multiplication
Axiom 6. $\forall x\forall y\forall z(x(yz) = (xy)z)$.
Axiom 7. $\forall x\forall y(xy = yx)$.
Axiom 8. $\forall x \neq e(x) \exists u (xu = x \land \forall v(xv = x \rightarrow uv = u))$.
Axiom 9. $\forall x \neq e(x) \exists d (xd = u(x) \land u(d) = u(x))$.
Axiom 10. $\forall x \neq e(x) \forall y \neq e(y) (u(xy) = u(x) \lor u(xy) = u(y))$.

(3) Order axioms
Axiom 11. $\forall x(x \leq x)$.
Axiom 12. $\forall x\forall y(x \leq y \land y \leq x \rightarrow x = y)$.
Axiom 13. $\forall x\forall y\forall z(x \leq y \land y \leq z \rightarrow x \leq z)$. 

Appendix A. List of axioms

The first and second group of axioms are the algebraic laws of an additive, respectively multiplicative, assembly. The third group of axioms states that there is a total order relation compatible with addition and multiplication, with some particular rules for the magnitudes. The fourth group of axioms connects addition and multiplication, together with the first three groups they give the algebraic laws of an ordered association. The fifth group permits to distinguish solids from associations, by postulating the existence of particular elements: minimal neutral elements for addition and multiplication, a maximal neutral element for addition, a decomposition, nontrivial magnitudes and finally elements separating two magnitudes. The axioms are written in the first-order language $L = \{+ , \cdot, \leq\}$.

The existence of solids in different settings suggests that it is worthwhile to investigate the algebraic properties of solids. In particular we are able to give necessary and sufficient conditions for distributivity to hold. The proof is rather involved and requires a thorough investigation in the algebra of magnitudes. These results are presented in [5].
Axiom 14. \( \forall x \forall y (x \leq y \lor y \leq x) \).

Axiom 15. \( \forall x \forall y \exists z (x \leq y \rightarrow x + z \leq y + z) \).

Axiom 16. \( \forall x \forall y (y + e(x) = e(x) \rightarrow (y \leq e(x) \land -y \leq e(x))) \).

Axiom 17. \( \forall x \forall y \forall z ((e(x) < x \land y \leq z) \rightarrow xy \leq xz) \).

Axiom 18. \( \forall x \forall y \forall z ((e(y) \leq y \leq z) \rightarrow e(x) y \leq e(x) z) \).

(4) Axioms relating addition and multiplication

Axiom 19. \( \forall x \forall y \exists z (e(x)y = e(z)) \).

Axiom 20. \( \forall x \forall y (e(xy) = e(x)y + e(y)x) \).

Axiom 21. \( \forall x \neq e(x) (e(u(x)) = e(x)/x) \).

Axiom 22. \( \forall x \forall y \forall z (xy + xz = x(y + z) + e(x)y + e(x)z) \).

Axiom 23. \( \forall x \forall y (-xy) = (-x)y \).

(5) Axioms of existence

Axiom 24. \( \exists m \forall x (m + x = x) \).

Axiom 25. \( \exists u \forall x (ux = x) \).

Axiom 26. \( \exists M \forall x (e(x) + M = M) \).

Axiom 27. \( \exists x (e(x) \neq 0 \land e(x) \neq M) \).

Axiom 28. \( \forall x \exists a (x = a + e(x) \rightarrow e(a) = 0) \).

Axiom 29. \( \forall x \exists y (x = e(x) \land y = e(y) \land x < y \rightarrow \exists z (z = e(z) \land x < z < y)) \).

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