Axiomatic characterization of committee scoring rules

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Abstract

Committee scoring rules form a rich class of aggregators of voters’ preferences for the purpose of selecting subsets of candidates of a given size. We provide an axiomatic characterization of committee scoring rules in the spirit of celebrated Young’s characterization of single-winner scoring rules. We show that committee scoring rules are characterized by the set of four standard axioms: symmetry, consistency, continuity and Pareto dominance. In the course of our proof, we introduce and axiomatically characterize multiwinner decision scoring rules, a class of rules that generalizes the well-known majority relation.

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1. Introduction

We extend Young’s celebrated characterization of single-winner scoring rules (Young, 1974) to the multiwinner setting. Specifically, we show that committee scoring rules, introduced by Elkind et al. (2017), are exactly those multiwinner rules that are symmetric, consistent, continuous, and satisfy Pareto dominance property.

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In our model of multiwinner elections, we are given a set of $m$ candidates, a collection of voters with preferences over those candidates, and an integer $k$. A multiwinner voting rule is based on an algorithm that allows us to compare any two subsets of candidates of size $k$—referred to as committees—on the basis of preferences of the voters; we call such an algorithm a decision rule. ¹ A transitive decision rule is called a multiwinner voting rule. In other words, given a profile of preferences, a multiwinner voting rule produces a weak linear order over the committees (and, in particular, identifies the best committees).

Multiwinner elections have recently been attracting attention due to a wide range of their applications. For example, they can be used for choosing a country’s parliament and, indeed, the multiwinner voting rules of Chamberlin and Courant (1983) and Monroe (1995) were introduced exactly for this purpose, whereas Brill et al. (2017) have shown that various apportionment methods can be expressed in the language of multiwinner voting. However, multiwinner elections also have other applications, ranging from selecting finalists of competitions (Elkind et al., 2017), through solving certain resource allocation problems (Lu and Boutilier, 2011, 2015; Skowron et al., 2016a), to making internal decisions within combinatorial optimization algorithms (Faliszewski et al., 2016a; Pourghanbar et al., 2015). Committee scoring rules form a large and versatile class of multiwinner voting rules that can be fine-tuned to suit many different applications.

This paper is organized as follows. First, in Section 2, we provide formal background regarding multiwinner elections and committee scoring rules. In Section 3 we formally describe the axioms that we use and present our characterization results. Section 4 contains further discussions of our model. The proofs of our results are delegated to the appendix.

2. The model

For each positive integer $t$, we set $[t] = \{1, \ldots, t\}$, and by $[t]_k$ we mean the set of all $k$-element subsets of $[t]$. For a set $X$ and $k \in \mathbb{N}$, by $S_k(X)$ we denote the set of all $k$-element subsets of $X$ (so, in particular, we have that $S_k([t]) = [t]_k$). For a given set $X$, by $\Pi_\succ(X)$ we denote the set of all linear orders over $X$, and by $\Pi_{\succeq}(X)$ we denote the set of all weak orders over $X$.

2.1. Multiwinner elections

Let $A = \{a_1, \ldots, a_m\}$ be the set of candidates, and let $N = \{1, 2, \ldots\}$ be the set of all possible voters. We refer to the members of $S_k(A)$ as size-$k$ committees, or, simply, as committees, when the size $k$ is clear from the context. For each finite subset $V \subseteq N$, by $\mathcal{P}(V)$ we denote the set of all $|V|$-tuples of elements from $\Pi_\succ(A)$, indexed by elements of $V$. We refer to elements of $\mathcal{P}(V)$ as preference profiles for the set of voters $V$. We set $\mathcal{P} = \{P \in \mathcal{P}(V) : V$ is a finite subset of $N\}$ to be the set of all possible preference profiles. For each preference profile $P \in \mathcal{P}$, by $\text{Vot}(P)$ we denote the set of all the voters in $P$ (in particular, we have that $\text{Vot}(P) = V$ for each $P \in \mathcal{P}(V)$). For each profile $P$ and each voter $v \in \text{Vot}(P)$, by $P(v)$ we denote the preference order of $v$ in $P$.

Our proof relies on using what we call $k$-decision rules. A $k$-decision rule $f_k$,

$$f_k : \mathcal{P} \rightarrow \left(S_k(A) \times S_k(A) \rightarrow \{-1, 0, 1\}\right),$$

¹ An example of a decision rule in a single-winner case would be to calculate which of the two candidates is preferred by a majority of voters.
is a function that for each preference profile \( P \in \mathcal{P} \) provides a mapping, \( f_k(P) : S_k(A) \times S_k(A) \to \{-1, 0, 1\} \), such that for each two size-\( k \) committees \( C_1 \) and \( C_2 \) it holds that \( f_k(P)(C_1, C_2) = -f_k(P)(C_2, C_1) \). We interpret \( f_k(P)(C_1, C_2) = 1 \) as saying that at profile \( P \) the society prefers committee \( C_1 \) over committee \( C_2 \) and we denote this as \( C_1 \succ P C_2 \) (we omit \( f_k \) from this notation because it will always be clear from the context). Similarly, we interpret \( f_k(P)(C_1, C_2) = 0 \) as saying that at profile \( P \) the society views the committees as equally good (denoted \( C_1 = P C_2 \)), and \( f_k(P)(C_1, C_2) = -1 \) as saying that at profile \( P \) the society prefers \( C_2 \) to \( C_1 \) (denoted \( C_2 \succ P C_1 \)). In other words, a decision rule maps voters’ preferences over individual candidates to weak tournaments over committees, and, thus, broadly generalize the notion of a majority relation.

A \( k \)-winner election rule \( f_k \) is a \( k \)-decision rule that additionally satisfies the transitivity requirement, i.e., it is a \( k \)-decision rule such that for each profile \( P \) and each three committees \( C_1, C_2, \) and \( C_3 \) of size \( k \) we have that if \( C_1 \succ P C_2 \) and \( C_2 \succ P C_3 \) then \( C_1 \succ P C_3 \). A multiwinner election rule \( f \) is a family \( (f_k)_{k \in \mathbb{N}} \) of \( k \)-winner election rules, with one \( k \)-winner rule for each committee size \( k \). We remark that often multiwinner rules are defined to simply return the set of winning committees, whereas in our case they implicitly define weak orders over all possible committees of a given size. Since the number of such committees is large, we believe that giving a concise algorithm for comparing committees—this is what a transitive decision rule is—is the right way to define a multiwinner analog of a social welfare function.

### 2.2. Committee scoring rules

For a preference order \( \pi \in \Pi_\succ(A) \), by \( \text{pos}_\pi(a) \) we denote the position of candidate \( a \) in \( \pi \) (the top-ranked candidate has position 1 and the bottom-ranked candidate has position \( m \)). A single-winner scoring function \( \gamma : [m] \to \mathbb{R} \) assigns a number of points to each position in a preference order so that \( \gamma(i) \geq \gamma(i+1) \) for all \( i \in [m-1] \). For example, the Borda scoring function, \( \beta : [m] \to \mathbb{N} \), is defined as \( \beta(i) = m - i \).

We extend the notion of a position of a candidate to the case of committees as follows. For a preference order \( \pi \in \Pi_\succ(A) \) and a committee \( C \in S_k(A) \), by \( \text{pos}_\pi(C) \) we mean the set \( \text{pos}_\pi(C) = \{ \text{pos}_\pi(a) : a \in C \} \). By a committee scoring function for committees of size \( k \), we mean a function \( \lambda : [m]_k \to \mathbb{R} \), that for each element of \( [m]_k \), interpreted as a position of a committee in some vote, assigns a score. A committee scoring function must also satisfy the following dominance requirement. Let \( I \) and \( J \) be two sets from \( [m]_k \) (i.e., two possible committee positions) such that \( I = \{i_1, \ldots, i_k\}, J = \{j_1, \ldots, j_k\} \) with \( i_1 < \cdots < i_k \) and \( j_1 < \cdots < j_k \). We say that \( I \) dominates \( J \) if for each \( t \in [k] \) we have \( i_t \leq j_t \) (note that this notion might be referred to as “weak dominance” as well, since a set dominates itself). If \( I \) dominates \( J \), then we require that \( \lambda(I) \geq \lambda(J) \). For each set of voters \( V \subseteq N \) and each preference profile \( P \in \mathcal{P}(V) \), by \( \text{score}_\lambda(C, P) \) we denote the total score that the voters from \( V \) assign to committee \( C \). Formally, we have that \( \text{score}_\lambda(C, P) = \sum_{v \in \text{Vol}(P)} \lambda(\text{pos}_{P(v)}(C)) \). By a committee scoring function we mean a family \( \lambda = (\lambda_k)_{k \in \mathbb{N}}, \) where for each \( k, \lambda_k \) is a committee scoring function for committees of size \( k \).

**Definition 1** (Committee scoring rules). A multiwinner election rule is a committee scoring rule if there exists a committee scoring function \( \lambda \) such that for each two committees \( C_1 \) and \( C_2 \) of equal size, we have that \( C_1 \succ P C_2 \) if and only if \( \text{score}_\lambda(C_1, P) > \text{score}_\lambda(C_2, P) \), and \( C_1 = P C_2 \) if and only if \( \text{score}_\lambda(C_1, P) = \text{score}_\lambda(C_2, P) \).
Committee scoring rules were introduced by Elkkind et al. (2017) and were later studied by Faliszewski et al. (2016c, 2016b) (closely related notions were considered by Thiele (1895), Skowron et al. (2016a), and Aziz et al. (2017, 2015)). In Example 1 below we illustrate the flexibility and diversity of the committee scoring rules.

Example 1. Consider the following preference profile with three groups of voters:

\[ G_1 \ (60 \ \text{voters}): \quad a_1 \succ a_2 \succ a_3 \succ c_1 \succ c_2 \succ c_3 \succ b_1 \succ b_2 \succ b_3, \]

\[ G_2 \ (30 \ \text{voters}): \quad b_1 \succ b_2 \succ b_3 \succ a_1 \succ a_2 \succ a_3 \succ c_1 \succ c_2 \succ c_3, \]

\[ G_3 \ (1 \ \text{voter}): \quad c_1 \succ c_2 \succ c_3 \succ b_1 \succ b_2 \succ b_3 \succ a_1 \succ a_2 \succ a_3. \]

Assume our goal is to select \( k = 3 \) candidates. A committee scoring rule based on function:

\[ \lambda_{k\text{-Borda}}(I) = \sum_{i \in I} \beta(i), \]

would select a committee of three candidates with the highest Borda scores, i.e., \( \{a_1, a_2, a_3\} \). Intuitively, this committee consists of three individually strongest candidates. This rule, known as \( k\)-Borda, would be suitable, e.g., for selecting finalists of contests. The next scoring function defines the Chamberlin–Courant committee scoring rule (Chamberlin and Courant, 1983):

\[ \lambda_{\text{CC}}(I) = \max_{i \in I} \beta(i). \]

According to this rule, the best committee is \( \{a_1, b_1, c_1\} \). This committee represents well the diversity of opinions of the voters (indeed, in our case each voter has her most preferred candidate in the committee). Such rules were advertised in the context of deliberative democracy (Chamberlin and Courant, 1983) and for targeted recommendations (Lu and Boutilier, 2011). Finally, consider the rule based on scoring function:

\[ \lambda_{\text{PAV}}(I) = \sum_{i=1}^{\lfloor \text{top}_k(I) \rfloor} \frac{1}{i}, \quad \text{where} \quad \text{top}_k(I) = \{i \in I \mid i \leq k\}. \]

Such a rule would return a “proportional” committee \( \{a_1, a_2, b_1\} \), where for each group of voters the number of committee members liked most by this group is proportional to the group size. Such rules are suitable for electing representative assemblies, and in fact the committee scoring rule based on \( \lambda_{\text{PAV}} \) extends the d’Hondt method of apportionment (Brill et al., 2017).

2.3. Decision scoring rules

Decision scoring rules are our main example of \( k \)-decision rules. These rules are similar to committee scoring rules, but with the difference that the scores of two committees usually cannot be computed independently. Specifically, for each pair of committee positions \( (I_1, I_2) \) we define a numerical value, the score that a voter assigns to the pair of committees \( (C_1, C_2) \) under the condition that \( C_1 \) and \( C_2 \) stand in this voter’s preference order on positions \( I_1 \) and \( I_2 \), respectively. If the total score of a pair of committees \( (C_1, C_2) \) is positive, then \( C_1 \) is socially preferred over \( C_2 \); if it is negative, then \( C_2 \) is socially preferred over \( C_1 \); if it is equal to zero, then \( C_1 \) and \( C_2 \) are seen under this decision rule as equally good.

Definition 2 (Decision scoring rules). Let \( d : [m]_k \times [m]_k \to \mathbb{R} \) be a decision scoring function, that is, a function that for each pair of committee positions \( (I_1, I_2) \), where \( I_1, I_2 \in [m]_k \), returns a score value (possibly negative), such that for each \( I_1 \) and \( I_2 \), it holds that \( d(I_1, I_2) = -d(I_2, I_1) \). For each preference profile \( P \in \mathcal{P} \) and for each pair of committees \( (C_1, C_2) \), we define the score:
A $k$-decision rule is a decision scoring rule if there exists a decision scoring function $d$ such that for each preference profile $P$ and each two committees $C_1$ and $C_2$ it holds that: (i) $C_1 \succeq_P C_2$ if and only if $\text{score}_d(C_1, C_2, P) \geq 0$, and (ii) $C_1 =_P C_2$ if and only if $\text{score}_d(C_1, C_2, P) = 0$.

One of the arguments in favor of decision scoring rules is that they generalize the notion of a majority relation: For a committee size $k = 1$ we define $d_{\text{maj}}(\{i_1\}, \{i_2\}) = 1$ if $i_1 < i_2$ and $d_{\text{maj}}(\{i_1\}, \{i_2\}) = -1$ if $i_1 > i_2$, so a candidate $x$ is preferred to a candidate $y$ if and only if more voters place $x$ ahead of $y$ than the other way around. Naturally, each committee scoring rule is an example of a (transitive) decision scoring rule as well.

3. Axioms and our characterizations

In this section we provide our characterizations of committee scoring rules and decision rules. The axioms that we use are natural, straightforward generalizations of the respective properties of single-winner elections. We formulate them for the case of $k$-decision rules (for a given value of $k$), but since $k$-winner rules are a type of $k$-decision rules, the properties apply to $k$-winner rules as well. For each of our properties $\mathcal{P}$, we say that a multiwinner election rule $f = \{f_k\}_{k \in \mathbb{N}}$ satisfies $\mathcal{P}$ if $f_k$ satisfies $\mathcal{P}$ for each $k \in \mathbb{N}$.

We start by recalling the definitions of anonymity and neutrality, the two properties that ensure that the election is fair to all voters and all candidates. Anonymity means that no voters are privileged nor discriminated against, whereas neutrality says the same for the candidates.

Definition 3 (Anonymity). We say that a $k$-decision rule $f_k$ is anonymous if for each two (not necessarily different) sets of voters $V, V' \subseteq N$ such that $|V| = |V'|$, for each bijection $\rho: V \to V'$ and for each two preference profiles $P \in \mathcal{P}(V)$ and $Q \in \mathcal{P}(V')$ such that $P(v) = Q(\rho(v))$ for each $v \in V$, it holds that $f_k(P) = f_k(Q)$.

Let $\sigma$ be a permutation of the set of candidates $A$. For a committee $C$, by $\sigma(C)$ we mean the committee $\{\sigma(c) : c \in C\}$. For a linear order $\pi \in \Pi_\succeq(A)$, by $\sigma(\pi)$ we denote the linear order such that for each two candidates $a$ and $b$ we have $a \pi b \iff \sigma(a) \sigma(\pi) \sigma(b)$. For a given $k$-decision rule $f_k$ and profile $P$, by $\sigma(f_k(P))$ we mean the function such that for each two size-$k$ committees $C_1$ and $C_2$ it holds that $\sigma(f_k(P))(\sigma(C_1), \sigma(C_2)) = f_k(P)(C_1, C_2)$.

Definition 4 (Neutrality). A $k$-decision rule $f_k$ is neutral if for each permutation $\sigma$ of $A$ and each two preference profiles $P, Q$ over the same set of voters $V$, such that $P(v) = \sigma(Q(v))$ for each $v \in V$, it holds that $f_k(P) = \sigma(f_k(Q))$.

A rule that is anonymous and neutral is called symmetric (or impartial (Moulin, 1988)). The next axiom describes a situation where two elections with disjoint sets of voters are merged. Given profiles $P$ and $Q$ over the same set of alternatives and with disjoint sets of voters, by $P + Q$ we denote the profile that consists of all the voters from $P$ and $Q$ with their respective preferences.

Definition 5 (Consistency). A $k$-decision rule $f_k$ is consistent if for each two profiles $P$ and $Q$ over disjoint sets of voters, $V \subset N$ and $V' \subset N$, and each two committees $C_1, C_2 \in S_k(A)$, (i) if

$$\text{score}_d(C_1, C_2, P) = \sum_{v \in \text{Vot}(P)} d(\text{pos}_{P(v)}(C_1), \text{pos}_{P(v)}(C_2)).$$
Theorem A (Axiomatic characterization of decision scoring rules). A decision rule is a decision scoring rule if and only if it is symmetric, consistent and continuous.

Theorem B (Axiomatic characterization of committee scoring rules). A multiwinner voting rule is a committee scoring rule if and only if it is symmetric, consistent, continuous, and satisfies Pareto dominance.
setting selects as a winner a candidate with the lowest Borda score), and is required by the definition of the class of committee scoring rules.

4. Discussion

One of the main issues that researchers face in relation to multiwinner voting—and a crucial one—is to choose the way in which the voters should express their preferences over committees. Asking the voters to rank all the committees that can be formed is often unrealistic. Indeed, experiments in psychology showed that humans have difficulties even when asked to rank more than seven alternatives, and with seven candidates and committees of size three they would have to rank 35 committees (while it may not appear completely unthinkable for voters to rank 35 committees, this number grows very quickly; e.g., choosing a committee of size 5 from a set of 20 candidates would require each voter to rank over fifteen thousand committees).

An alternative approach—and this is the way taken in our paper—is to assume that voters’ preferences over individual alternatives can be extended to those over committees in a systematic way. Since there are numerous ways to extend preferences of voters over candidates to preferences over committees, it is very important that this extension is done as a part of a multiwinner voting rule, on the basis of principles that underlie this rule. This method of comparing committees is quite expressive (see Example 1). By choosing the committee scoring function wisely, one can design very different rules, geared for selecting committees with very different properties (Faliszewski et al., 2017).

We note that one could also consider a middle ground approach, where the voters are allowed to include in their preferences information about synergies between candidates (e.g., to say that they would like $c_1$ to be included in a committee only if $c_2$ is not included; or that $c_1$ and $c_2$ can work particularly well together). In other words, the voters can be given some concise language for the sake of expressing their preferences. Indeed, even Fishburn (1981b, 1981a), who pioneered the approach of ranking committees, used such a language, based on ranking individual candidates. Another example of such a language was proposed by Boutilier et al. (2004) (their model of expressing preferences is known under the name of CP-nets); this research direction was then pursued in the area of voting in combinatorial domains (Land and Xia, 2015). However, to the best of our knowledge, all such languages considered in the literature always come with certain restrictions (their applicability depends on the specific context) and assume some logic embedded in a voting rule or the language itself.

5. Conclusion

We have provided an axiomatic characterization of committee scoring rules, a new class of multiwinner voting rules recently introduced by Elkind et al. (2017). Committee scoring rules form a remarkably general class of multiwinner systems that consists of many nontrivial rules with a variety of applications. Thus, our characterization provides a framework for further axiomatic studies of this class and makes a step towards their understanding.

Our Theorem B required to develop the notion of a decision rule, which generalizes the concept of pairwise comparisons to the multiwinner framework. We believe that decision rules are an interesting notion that deserves further study.
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Appendix A. Proofs of main results

We now start proving our main results. Here is the roadmap of the proof.

Since anonymity allows us to ignore the order of linear orders in profiles, in Section A.1 we change the domain of our rules from the set of preference profiles to the set of voting situations. A voting situation is an \( m! \)-dimensional vector with non-negative integers specifying how many times each linear order repeats in the voters’ preferences. We use this new representation of the domain of decision rules in Section A.1 and we conclude this section by proving Theorem A.

In Section A.2 we further extend the domain of our rules to generalized voting situations, allowing fractional and negative multiplicities of linear orders; the voting situations in such extended domain can then be identified with the elements of \( \mathbb{Q}^{m!} \). Next, we use the characterization from Section A.1 to further simplify the reasoning for proving Theorem B. We show that one does not have to consider the rules as functions that map preference profiles to weak orders over committees; it is sufficient to view the rules as tools that determine which committees are ranked on the same position in the output weak order—such functions in the simpler domain extend uniquely to the general one (this observation is summarized and formalized as Lemma 7).

In Sections A.3 and A.4 we then focus solely on proving Theorem B. In Section A.3, we prove Theorem B for the case where \( f \) is used to recognize in which profiles a certain committee \( C_1 \) is preferred over some other committee \( C_2 \), when \( |C_1 \cap C_2| = k - 1 \). If \( |C_1 \cap C_2| = k - 1 \) then there are only two candidates, let us refer to them as \( c_1 \) and \( c_2 \), such that \( C_1 = (C_1 \cap C_2) \cup \{c_1\} \), and \( C_2 = (C_1 \cap C_2) \cup \{c_2\} \). Thus, this case closely resembles the single-winner setting, studied by Young (1975) and Merlin (2003). In the single-winner case one can use geometric arguments to note that the set of profiles in which \( c_1 \) is preferred over \( c_2 \) can be separated from the set of profiles in which \( c_2 \) is preferred over \( c_1 \) by a hyperplane. The coefficients of the linear equation that specifies this hyperplane define a single-winner scoring rule, and this scoring rule is exactly the voting rule that one started with. In Section A.3 we use the same geometric arguments, but the technical details are much more sophisticated.

In Section A.4 we extend the result from Section A.3 to the case of any two committees (irrespective of the size of their intersection). Here, we use a different technique. To deal with committees \( C_1 \) and \( C_2 \) that have fewer than \( k - 1 \) elements in common, we form a third committee, \( C_3 \), whose intersections with \( C_1 \) and \( C_2 \) have more elements than the intersection of \( C_1 \) and \( C_2 \). Then, using an inductive argument, we conclude that the space of profiles \( P \) where \( C_1 =_P C_3 \) is \((m! - 1)\)-dimensional, and that the same holds for the space of profiles \( P \) such that \( C_2 =_P C_3 \). An intersection of two vector spaces with this dimension has dimension at least \( m! - 2 \) and, so, we have a subspace of profiles \( P \) such that \( C_1 =_P C_2 \) whose dimension is at least \( m! - 2 \). Using combinatorial arguments, we find a profile \( P' \) which does not belong to the space but for which \( C_1 =_P C_2 \) still holds. This gives us our \((m! - 1)\)-dimensional space. By applying results from the first part of the proof, we explain that this suffices to conclude that the committee scoring
function that we found in Section A.3 for committees that differ in at most one element works for all other committees as well.

A.1. Characterization of decision rules

This section is devoted to proving Theorem A, i.e., here we will consider $k$-decision rules. Recall that the outcomes of $k$-decision rules do not need to be transitive: for a $k$-decision rule $f_k$ it is possible to have a profile $P$ and three committees such that $C_1 \succ_p C_2$, $C_2 \succ_p C_3$, and $C_3 \succ_p C_1$.

Additional notation For a preference profile $P$, we write $C_1 \succeq_p C_2$ if $C_1 \succ_p C_2$ or $C_1 =_p C_2$, which is equivalent to $f_k(P)(C_1, C_2) \geq 0$. Sometimes, when $P$ is a more involved expression, we write $C_1 \succeq_2 [P] C_2$ instead of $C_1 \succeq_p C_2$ and $C_1 =_2 [P] C_2$ instead of $C_1 =_p C_2$.

Setting up the framework Let us fix, for the rest of the proof, a positive integer $k$, the size of the committee to be elected, and a symmetric, consistent, continuous $k$-decision rule $f_k$. Our goal is to find a function $d : [m]_k \times [m]_k \to \mathbb{R}$ such that for each profile $P$ and each two committees $C_1$, $C_2$ it holds that $C_1 \succeq_p C_2$ if and only if $\text{score}_d(C_1, C_2, P) \geq 0$.

Our function $d$ will be piecewise-defined. For each $s \in [k]$ we will define a function $d_s$ which applies only to pairs $(I_1, I_2) \in [m]_k \times [m]_k$ satisfying $|I_1 \cap I_2| = s$, outputs real values and such that the score:

$$\text{score}_{d_s}(C_1, C_2, P) = \sum_{v \in \text{Vote}(P)} d_s(\text{pos}_{P(v)}(C_1), \text{pos}_{P(v)}(C_2))$$

(2)

calculated with the use of this function satisfies the following condition: if $|C_1 \cap C_2| = s$, then $C_1 \succeq_p C_2$ if and only if $\text{score}_{d_s}(C_1, C_2, P) \geq 0$. Pursuing this idea, for the rest of the proof we will fix $s$ and restrict ourselves to pairs of committees satisfying $|C_1 \cap C_2| = s$. The restriction of $f_k$ to such pairs of committees will be denoted $f_{k,s}$.

The first domain change Since $f_k$ is anonymous, the order of votes in any profile is no longer meaningful. The outcome of the rule is fully determined by the voting situation that specifies how many times each linear order repeats in a given profile (a voting situation can be viewed as an $m!$-dimensional vector with non-negative integer coefficients). For any $\pi \in \Pi_+ (A)$ and voting situation $P$, by $P(\pi)$ we mean the number of voters in $P$ with preference order $\pi$.

Correspondingly, we can view $f_k$ as a function:

$$f_k : \mathbb{N}^{m!} \rightarrow \left( S_k(A) \times S_k(A) \rightarrow \{-1, 0, 1\} \right)$$

with the domain $\mathbb{N}^{m!}$ instead of $\mathcal{P}$. This representation is helpful, since algebraic operations on vectors from $\mathbb{N}^{m!}$ become meaningful: e.g., for a voting situation $P$ and a constant $c \in \mathbb{N}$, $cP$ is the voting situation that corresponds to $P$ in which each vote was replicated $c$ times.

We sometimes treat each vote $v$ (i.e., each preference order) as a standalone voting situation that contains this vote only. When we say that we modify a vote within a voting situation $P$, we mean modifying one copy of this vote, and not all the votes that have the same preference order.

Let $d' : [m]_k \times [m]_k \to \mathbb{R}$ be a decision scoring function. Clearly, we can speak of applying the corresponding decision scoring rule to voting situations instead of applying them to preference profiles as in (1). For a voting situation $P \in \mathbb{N}^{m!}$, the score of a committee pair $(C_1, C_2)$ is:
score\(_d'(C_1, C_2, P) = \sum_{v \in \Pi_>(A)} P(v) \cdot d'(\text{pos}_v(C_1), \text{pos}_v(C_2)). (3)

**Independence of committee comparisons from irrelevant swaps** We will now show that for each two committees \(C_1\) and \(C_2\), the result of their comparison according to \(f_k\) depends only on the positions on which \(C_1\) and \(C_2\) are ranked by the voters (and do not depend on the positions of candidates not belonging to \(C_1 \cup C_2\)).

For \(v \in \Pi_>(A)\), we write \(v[a \leftrightarrow b]\) to denote the vote obtained from \(v\) by swapping candidates \(a\) and \(b\). Further, if \(v\) is a vote in \(P\), by \(P[v, a \leftrightarrow b]\) we denote the voting situation obtained from \(P\) by swapping \(a\) and \(b\) in one copy of vote \(v\), and by \(P[a \leftrightarrow b]\) we denote the voting situation obtained from \(P\) by swapping \(a\) and \(b\) in every vote.

**Lemma 1.** Let \(C_1\) and \(C_2\) be two size-\(k\) committees, \(P\) be a voting situation, \(a, b\) be candidates such that one of the following conditions holds: (i) \(a, b \notin C_1 \cup C_2\), (ii) \(a, b \in C_1 \cap C_2\), (iii) \(a, b \in C_1 \setminus C_2\), or (iv) \(a, b \in C_2 \setminus C_1\). Then for each vote \(v\) in \(P\), \(C_1 \succeq_P C_2\) if and only if \(C_1 \succeq_{P[v,a\leftrightarrow b]} C_2\).

**Proof.** Let \(C_1 \succeq_P C_2\), and towards a contradiction assume that \(C_2 \succ_{P[v, a \leftrightarrow b]} C_1\).

We rename the candidates so that \(C_1 \setminus C_2 = \{a_1, \ldots, a_\ell\}\) and \(C_2 \setminus C_1 = \{b_1, \ldots, b_\ell\}\), and we define \(\sigma\) to be a permutation (over the set of candidates) that for each \(x \in [\ell]\) swaps \(a_x\) with \(b_x\), but leaves all the other candidates intact.

Since \(C_1 \succeq_P C_2\), by neutrality we have that \(C_2 \succeq_{\sigma(P)} C_1\). Due to our assumptions, it holds that \(C_2 \succeq_{P[v, a \leftrightarrow b]} C_1\) and, by consistency,

\[C_2 \succ [\sigma(P) + P[v, a \leftrightarrow b]] C_1. \quad (4)\]

Let \(Q\) be a voting situation consisting of two votes, \(v[a \leftrightarrow b]\) and \(\sigma(v)\). Observe that \(\sigma(P) - \sigma(v) = \sigma(P[v, a \leftrightarrow b]) - v[a \leftrightarrow b]\). This is because \(P[v, a \leftrightarrow b] - v[a \leftrightarrow b]\) is the same as \(P - v\). Since:

\[\sigma(P) + P[v, a \leftrightarrow b] - Q = \underbrace{(\sigma(P) - \sigma(v))}_{R'} + \underbrace{(P[v, a \leftrightarrow b] - v[a \leftrightarrow b])}_{R''},\]

and both summands on the right-hand-side are symmetric with respect to \(\sigma\) (i.e., \(\sigma(R') = R''\), \(\sigma(R'') = R'\), and \(\sigma^2\) is an identity permutation), by symmetry of \(f_k\) we have:

\[C_2 \equiv [\sigma(P) + P[v, a \leftrightarrow b] - Q] C_1. \quad (5)\]

Thus, by consistency—as applied to equations (4) and (5)—we get that \(C_2 \succ Q C_1\). By neutrality, we also infer that \(C_2 \succ_{Q[a \leftrightarrow b]} C_1\). This follows because for each of the four conditions for \(a, b\) from the statement of the lemma it holds that permutation \(a \leftrightarrow b\) maps committee \(C_1\) to committee \(C_1\) and committee \(C_2\) to committee \(C_2\). Next, by consistency we get that \(C_2 \equiv [Q + Q[a \leftrightarrow b]] C_1\).

However, we observe that:

\[Q + Q[a \leftrightarrow b] = \left(\underbrace{v[a \leftrightarrow b] + \sigma(v)}_{Q'} + \underbrace{v + \sigma(v)[a \leftrightarrow b]}_{Q''}\right) = \left(\underbrace{v[a \leftrightarrow b] + \sigma(v)[a \leftrightarrow b]}_{Q'} + \underbrace{v + \sigma(v)}_{Q''}\right).\]
Furthermore, if \(a, b \notin C_1 \cup C_2\), or \(a, b \in C_1 \cap C_2\), then \(\sigma(v[a \leftrightarrow b]) = \sigma(v[a \leftrightarrow b])\). On the other hand, if \(a, b \in C_1 \setminus C_2\) or \(a, b \in C_2 \setminus C_1\), then \(\sigma(v[a \leftrightarrow b]) = (\sigma \circ [a \leftrightarrow b])(v[a \leftrightarrow b])\). In other words, there always exists a permutation \(\tau\) such that \(Q' = \tau(Q')\), \(C_1 = \tau(C_2)\), and \(C_2 = \tau(C_1)\) (\(\tau\) is either \(\sigma\) or \(\sigma \circ [a \leftrightarrow b]\)), and, similarly, we have \(Q'' = \sigma(Q'')\), \(C_2 = \sigma(C_1)\), \(C_1 = \sigma(C_2)\). Thus, by neutrality, we get that:

\[
C_2 = [v[a \leftrightarrow b] + \sigma(v[a \leftrightarrow b])] C_1 \quad \text{and} \quad C_2 = [v + \sigma(v)] C_1.
\]

By consistency, we infer that \(C_2 = [Q + Q[a \leftrightarrow b]] C_1\), which contradicts our previous conclusion. \(\square\)

**Putting the focus on two fixed committees** We fix a pair of size-\(k\) committees, \(C_1\) and \(C_2\) with \(|C_1 \cap C_2| = s\), and define \(f_{C_1, C_2}\) to be the rule that acts on voting situations in the same way as \(f_{k,s}\) does, but that only distinguishes, at any voting situation \(P\), whether (i) \(C_1\) is preferred over \(C_2\), or (ii) \(C_1\) and \(C_2\) are equally good, or (iii) \(C_2\) is preferred over \(C_1\).

**Defining distinguished profiles** For each two committee positions \(I_1\) and \(I_2\) such that \(|I_1 \cap I_2| = |C_1 \cap C_2| = s\), we consider a single-vote voting situation \(v(C_1 \rightarrow I_1, C_2 \rightarrow I_2)\) where \(C_1\) and \(C_2\) are ranked on positions \(I_1\) and \(I_2\), respectively, and all the other candidates are ranked arbitrarily, but in some fixed, predetermined order (see Skowron et al., 2016b, for the construction of such a vote).

Let us consider two cases. First, let us assume that for each two committee positions \(I_1\) and \(I_2\) such that \(|I_1 \cap I_2| = s\), it holds that

\[
C_1 = [v(C_1 \rightarrow I_1, C_2 \rightarrow I_2)] C_2.
\]

By Lemma 1, we infer that for each single-vote voting situation \(v\) we have \(C_1 = v C_2\).

Further, by consistency, we conclude that \(f_{C_1, C_2}\) is trivial, i.e., for every voting situation \(P\) it holds that \(C_1 = P C_2\). By neutrality, we get that \(f_{k,s}\) is also trivial (i.e., it declares equally good each two committees whose intersection has \(s\) candidates). Of course, in this case \(f_{k,s}\) is a decision scoring rule.

If the above case does not hold, then there are some two committee positions, \(I_i^+\) and \(I_i^-\), such that \(|I_i^+ \cap I_i^-| = s\) and \(C_1\) is not equivalent to \(C_2\) relative to \(v(C_1 \rightarrow I_i^+, C_2 \rightarrow I_i^-)\). W.l.o.g.:

\[
C_1 > [v(C_1 \rightarrow I_i^+, C_2 \rightarrow I_i^-)] C_2.
\]

We note that, by neutrality, this implies:

\[
C_2 > [v(C_2 \rightarrow I_i^+, C_1 \rightarrow I_i^-)] C_1.
\]

Let us fix any two such \(I_i^+\) and \(I_i^-\).

For each two committee positions \(I_1\) and \(I_2\) with \(|I_1 \cap I_2| = |I_i^+ \cap I_i^-| = s\), and for each two nonnegative integers \(x\) and \(y\), we define voting situation:

\[
P_{x,y}(v(C_1 \rightarrow I_i^+, C_2 \rightarrow I_i^-)) = y \cdot (v(C_1 \rightarrow I_i^+, C_2 \rightarrow I_i^-)) + x \cdot (v(C_1 \rightarrow I_1, C_2 \rightarrow I_2)).
\]

**Deriving the components for the decision scoring function for \(f_{C_1, C_2}\)** We now proceed toward defining a decision scoring function for \(f_{k,s}\). To this end, we define the value \(\Delta_{I_1, I_2}\) as:
Lemma 2. For each two committee positions \( I_1 \) and \( I_2 \) with \( |I_1 \cap I_2| = s \), we have \( \Delta_{I_1,I_2} = -\Delta_{I_1,I_2} \).

Proof. We assume, w.l.o.g., that \( C_1 > [v(C_1 \rightarrow I_1, C_2 \rightarrow I_2)] C_2 \). Let us consider two sets:

\[
U = \left\{ \frac{y}{x} : C_2 > \left[ P^{y(C_1 \rightarrow I^*_1, C_2 \rightarrow I^*_2)}_{x(C_1 \rightarrow I_2, C_2 \rightarrow I_1)} \right] C_1, \ x, y \in \mathbb{N} \right\}
\]

(\( U \) is the set that we take supremum of in Equation (8)), and:

\[
L = \left\{ \frac{y}{x} : C_2 > \left[ P^{y(C_1 \rightarrow I^*_1, C_2 \rightarrow I^*_2)}_{x(C_1 \rightarrow I_2, C_2 \rightarrow I_1)} \right] C_1, \ x, y \in \mathbb{N} \right\}
\]

(\( L \) is the set that we take infimum of in Equation (8), for \( \Delta_{I_1,I_1} \)). We will show that sup \( U = \inf L \). First, we show that sup \( U \leq \inf L \). Towards a contradiction assume that this is not the case, i.e., that there exists \( \frac{y}{x} \in U \) and \( \frac{y'}{x'} \in L \) such that \( \frac{y}{x} > \frac{y'}{x'} \). Since \( \frac{y}{x} \in U \) and \( \frac{y'}{x'} \in L \), we get:

\[
C_2 > \left[ P^{y(C_1 \rightarrow I^*_1, C_2 \rightarrow I^*_2)}_{x(C_1 \rightarrow I_2, C_2 \rightarrow I_1)} \right] C_1 \quad \text{and} \quad C_2 > \left[ P^{y'(C_1 \rightarrow I^*_1, C_2 \rightarrow I^*_2)}_{x'(C_1 \rightarrow I_2, C_2 \rightarrow I_1)} \right] C_1.
\]

Let us consider the voting situation:

\[
S = y' \cdot P^{y(C_1 \rightarrow I^*_1, C_2 \rightarrow I^*_2)}_{x(C_1 \rightarrow I_2, C_2 \rightarrow I_1)} + y \cdot P^{y'(C_1 \rightarrow I^*_1, C_2 \rightarrow I^*_2)}_{x'(C_1 \rightarrow I_2, C_2 \rightarrow I_1)}.
\]

By consistency, we have that \( C_2 > S \).

On the other hand, there are \( yy' \) voters that rank \( C_1 \) and \( C_2 \) on positions \( I^*_1 \) and \( I^*_2 \), respectively, and the same number \( yy' \) of voters that rank \( C_1 \) and \( C_2 \) on positions \( I^*_2 \) and \( I^*_1 \). Due to neutrality and consistency, these voters cancel each other out. (Formally, if \( S' \) were a voting situation limited to these voters only, we would have \( C_1 = S' \). C_2 \). This is so due to the symmetry
of $f_k$ and the fact that for any permutation $\sigma$ that swaps all the members of $C_1 \setminus C_2$ with all the members of $C_2 \setminus C_1$, we have $S' = \sigma(S')$. Next, there are $x'y$ voters that rank $C_1$ and $C_2$ on positions $I_1$ and $I_2$, and $xy'$ of voters that rank $C_1$ and $C_2$ on positions $I_2$ and $I_1$, respectively. Since we assumed that $\frac{y}{x} > \frac{y'}{x'}$, we have that $x'y > xy'$. So, $xy'$ voters from each of the two aforementioned groups cancel each other out (in the same sense as above), and we are left with considering $x'y - xy' > 0$ voters that rank $C_1$ and $C_2$ on positions $I_1$ and $I_2$. Thus, $C_1 >_S C_2$. Yet, this contradicts $C_2 >_S C_1$ and we conclude that sup $U \leq$ inf $L$.

Next, we prove sup $U \geq$ inf $L$. Assume on the contrary that there exist values $\frac{y}{x}$ and $\frac{y'}{x'}$ such that sup $U < \frac{y}{x} < \frac{y'}{x'}$ in $L$. It must be the case that $\frac{y}{x}$ is not in $U$ and, so, we have:

$$C_1 \geq \left[ P^y_{x,(C_1 \rightarrow I_1^*, C_2 \rightarrow I_2^*)} \right] C_2. \tag{9}$$

Since $\frac{y}{x}$ also cannot be in $L$, we have:

$$C_1 \geq \left[ P^y_{x,(C_1 \rightarrow I_2^*, C_2 \rightarrow I_1^*)} \right] C_2. \tag{10}$$

By applying neutrality to (10) (swapping candidates from $C_1 \setminus C_2$ with those from $C_2 \setminus C_1$):

$$C_2 \geq \left[ P^x_{y,(C_1 \rightarrow I_1^*, C_2 \rightarrow I_2^*)} \right] C_1. \tag{11}$$

By putting together (9) and (11), and repeating the same reasoning for $\frac{y'}{x'}$ instead of $\frac{y}{x}$ we get:

$$C_1 = \left[ P^y_{x,(C_1 \rightarrow I_1^*, C_2 \rightarrow I_2^*)} \right] C_2 \quad \text{and} \quad C_1 = \left[ P^y_{x,(C_1 \rightarrow I_1^*, C_2 \rightarrow I_2^*)} \right] C_2. \tag{12}$$

By applying neutrality to the first voting situation in (12):

$$C_1 = \left[ P^y_{x,(C_1 \rightarrow I_1^*, C_2 \rightarrow I_2^*)} \right] C_2 \quad \text{and} \quad C_1 = \left[ P^y_{x,(C_1 \rightarrow I_1^*, C_2 \rightarrow I_2^*)} \right] C_2. \tag{13}$$

We now define voting situation:

$$Q = x' \cdot P^y_{x,(C_1 \rightarrow I_1^*, C_2 \rightarrow I_2^*)} + x \cdot P^x_{y,(C_1 \rightarrow I_1^*, C_2 \rightarrow I_2^*)}. \quad \text{From Equation (13) (and consistency), we get that } C_1 =_Q C_2. \quad \text{In } Q \text{ there is the same number of voters who rank } C_1 \text{ and } C_2 \text{ on positions } I_1 \text{ and } I_2 \text{ as those that rank them on positions } I_2 \text{ and } I_1, \text{ respectively (so these voters cancel each other out). On the other hand, there are } yx' \text{ voters who rank } C_1 \text{ and } C_2 \text{ on positions } I_1^* \text{ and } I_2^*, \text{ and } y'x \text{ voters who rank these committees on positions } I_1^* \text{ and } I_2^*, \text{ respectively. Since } yx' < y'x, \text{ we get that } C_1 >_Q C_2, \text{ a contradiction. Thus, sup } U \geq \text{inf } L. \quad \square$$

Since sup $U \leq$ inf $L$ and sup $U \geq$ inf $L$, we have sup $U =$ inf $L$. Thus, $\Delta_{I_1,I_2} = -\Delta_{I_2,I_1}$. □

The next lemma shows that $\Delta_{I_1,I_2}$ provides a threshold value for proportions of voters.

**Lemma 3.** Let $I_1$ and $I_2$ be two committee positions such that $|I_1 \cap I_2| = s$, and let $x$, $y$ be two positive integers. The following two implications hold:

1. if $C_1 > \left[ v(C_1 \rightarrow I_1, C_2 \rightarrow I_2) \right] C_2$ and $\frac{y}{x} < \Delta_{I_1,I_2}$, then $C_2 > \left[ P^x_{y,(C_1 \rightarrow I_2, C_2 \rightarrow I_1)} \right] C_1$.
2. if $C_2 > \left[ v(C_1 \rightarrow I_1, C_2 \rightarrow I_2) \right] C_1$ and $\frac{y}{x} > -\Delta_{I_1,I_2}$, then $C_2 > \left[ P^x_{y,(C_1 \rightarrow I_2, C_2 \rightarrow I_1)} \right] C_1$. 
Proof. For the first implication assume \( C_1 >\left\lbrack v(C_1 \rightarrow I_1, C_2 \rightarrow I_2) \right\rbrack C_2 \), and on the contrary that:

\[
C_1 \geq \left\lbrack P_{x(C_1 \rightarrow I_2, C_2 \rightarrow I_1)}^{y(C_1 \rightarrow I_1, I_2 \rightarrow I_2)} \right\rbrack C_2.
\]  

(14)

From the definition of \( \Delta_{I_1, I_2} \) there exist two numbers \( x', y' \in \mathbb{N} \), such that \( \frac{y'}{x'} < \frac{y}{x} \leq \Delta_{I_1, I_2} \) and:

\[
C_2 > \left\lbrack P_{x(C_1 \rightarrow I_2, C_2 \rightarrow I_1)}^{y(C_1 \rightarrow I_1, I_2 \rightarrow I_2)} \right\rbrack C_1.
\]  

(15)

Let us consider voting situation \( P_{x(C_1 \rightarrow I_1, C_2 \rightarrow I_2)}^{y(C_1 \rightarrow I_1, I_2 \rightarrow I_2)} \) that is obtained from the one appearing in (14) by swapping positions of \( C_1 \) and \( C_2 \). In this voting situation \( C_2 \) is weakly preferred over \( C_1 \):

\[
C_2 \succeq \left\lbrack P_{x(C_1 \rightarrow I_2, C_2 \rightarrow I_1)}^{y(C_1 \rightarrow I_1, I_2 \rightarrow I_2)} \right\rbrack C_1.
\]  

(16)

By Equations (16), (15), and consistency of \( f_k \), we observe that in the voting situation:

\[
P = x' \cdot P_{x(C_1 \rightarrow I_2, C_2 \rightarrow I_1)}^{y(C_1 \rightarrow I_1, I_2 \rightarrow I_2)} + x \cdot P_{x(C_1 \rightarrow I_1, C_2 \rightarrow I_2)}^{y(C_1 \rightarrow I_1, I_2 \rightarrow I_2)}
\]

committee \( C_2 \) is strictly preferred over \( C_1 \) (i.e., \( C_2 > P C_1 \)). Let us now count the voters in \( P \).

There are \( xx' \) of them who put \( C_1 \) and \( C_2 \) on positions \( I_1 \) and \( I_2 \), respectively, and there are \( xx' \) voters who put \( C_1 \) and \( C_2 \) on positions \( I_2 \) and \( I_1 \), respectively. By the same arguments as used in the proof of Lemma 2, these voters cancel each other out. Next, there are \( yy' \) voters who put \( C_1 \) and \( C_2 \) on positions \( I_1^* \) and \( I_2^* \), respectively, and \( x'y' \) voters who put \( C_1 \) and \( C_2 \) on positions \( I_2^* \) and \( I_1^* \), respectively. Since \( y'y > xy' \) we conclude that \( C_1 > P C_2 \), a contradiction.

The proof of the second implication is similar. \( \square \)

Putting together the decision scoring function for \( f_{C_1, C_2} \) We are ready to define a decision scoring function \( d_s \) for \( f_{C_1, C_2} \). For all committee positions \( I_1 \) and \( I_2 \), with \( |I_1 \cap I_2| = 1 \), we set:

\[
d_s(I_1, I_2) = \Delta_{I_1, I_2}.
\]  

(17)

We note that our \( d_s \) formally depends on the choice of \( I_1^* \) and \( I_2^* \), however this is not a problem. Each choice of \( I_1^* \) and \( I_2^* \) would give us a valid decision scoring function. Intuitively, we can view \( d_s(I_1, I_2) \) as an (oriented) distance between \( I_1 \) and \( I_2 \). The next lemma shows that we treat the distance between \( I_1^* \) and \( I_2^* \) as a gauge to measure distances between other positions.

Lemma 4. \( \Delta_{I_1^*, I_2^*} = 1 \).

Proof. We note that for each positive integer \( z \), we have \( C_1 = \left\lbrack P_{x(C_1 \rightarrow I_2, C_2 \rightarrow I_1)}^{y(C_1 \rightarrow I_1, I_2 \rightarrow I_2)} \right\rbrack C_2 \). Further, due to consistency of \( f_k \) and by the choice of \( I_1^* \) and \( I_2^* \) (recall Equations (6) and (7)), we observe that:

\[
C_1 > \left\lbrack P_{x(C_1 \rightarrow I_2, C_2 \rightarrow I_1)}^{y(C_1 \rightarrow I_1, I_2 \rightarrow I_2)} \right\rbrack C_2 \quad \text{whenever} \quad y > x \quad \text{and} \quad C_2 > \left\lbrack P_{x(C_1 \rightarrow I_2, C_2 \rightarrow I_1)}^{y(C_1 \rightarrow I_1, I_2 \rightarrow I_2)} \right\rbrack C_1 \quad \text{whenever} \quad y < x.
\]

We conclude that \( \Delta_{I_1^*, I_2^*} = \sup \left\{ \frac{y}{x} : y < x, \text{ for } x, y \in \mathbb{N}_+ \right\} = 1 \). \( \square \)

Lemma 5 below shows that \( d_s \) implements \( f_{C_1, C_2} \). We will later argue that it works for all pairs of committees, not only for \( (C_1, C_2) \), and hence that it also implements \( f_{k,s} \).
Lemma 5. Recall that $C_1$ and $C_2$ are the two fixed committees (see the paragraph below Lemma 1) and $d_i$ is defined in (17). For each voting situation $P$ the following three implications hold: (i) if $\text{score}_{d_i}(C_1, C_2, P) > 0$, then $C_1 \succ_p C_2$; (ii) if $\text{score}_{d_i}(C_1, C_2, P) = 0$, then $C_1 = p C_2$; (iii) if $\text{score}_{d_i}(C_1, C_2, P) < 0$, then $C_2 \succ_p C_1$.

Proof. We start by proving (i). Let $P$ be a voting situation such that $\text{score}_{d_i}(C_1, C_2, P) > 0$. For the sake of contradiction we assume that $C_2 \succeq_p C_1$.

The idea of the proof is to perform a sequence of transformations of $P$ so that the result according to $f_k$ does not change, but, eventually, in the resulting profile each voter puts committees $C_1$ and $C_2$ either on positions $I^*_1$, $I^*_2$ or the other way round. Let $t$ be the total number of transformations that we perform to achieve this and let $P_t$ be the voting situation that we obtain after the $i$-th transformation. We will ensure that for each voting situation $P_i$ it holds that

$$\text{score}_{d_i}(C_1, C_2, P_i) > 0$$

and $C_2 \succeq_p C_1$. In particular, for the final voting situation $P_t$ we will have $C_2 \succeq_p C_1$, $\text{score}_{d_i}(C_1, C_2, P_t) > 0$, and each voter will have committees $C_1$ and $C_2$ on positions $I^*_1$ and $I^*_2$ or the other way round. Therefore, we will have:

$$P_t = x \cdot v(C_1 \rightarrow I^*_1, C_2 \rightarrow I^*_2) + y \cdot v(C_1 \rightarrow I^*_2, C_2 \rightarrow I^*_1)$$

for some nonnegative integers $x$ and $y$, and by Lemmas 2 and 4 we will have:

$$\text{score}_{d_i}(C_1, C_2, P_t) = xd_i(I^*_1, I^*_2) + yd_i(I^*_2, I^*_1) = x - y$$

However, from $\text{score}_{d_i}(C_1, C_2, P_t) > 0$ we get $x > y$, i.e., there must be more voters who put $C_1$ and $C_2$ on positions $I^*_1$ and $I^*_2$ than on positions $I^*_2$ and $I^*_1$. By our choice of $I^*_1$ and $I^*_2$ (recall Equation (6) as in the proof of Lemma 4) we will conclude that $C_1 \succ_p C_2$, a contradiction.

We now describe the transformations. We construct the profiles $P_i$ inductively. We set $P_0 = P$. Assume that we have already constructed profile $P_{i-1}$. If for each voter in $P_{i-1}$, committees $C_1$ and $C_2$ stand on positions $I^*_1$ and $I^*_2$ (or the other way round), we stop and set $t = i - 1$. Otherwise, we perform the $i$-th transformation in the following way. We take a preference order of an arbitrary voter from $P_{i-1}$, for whom the set of committee positions of $C_1$ and $C_2$ is not $\{I^*_1, I^*_2\}$. Let us denote this voter by $v_i$. Let $z$ denote the number of voters in $P_{i-1}$ who rank $C_1$ and $C_2$ on the same positions as $v_i$, including $v_i$ (so $z \geq 1$). Let $I_1$ and $I_2$ denote the positions of the committees $C_1$ and $C_2$ in the preference order of $v_i$, respectively. Let $\epsilon = \text{score}_{d_i}(C_1, C_2, P_{i-1})/2z > 0$.

Case 1: If $C_1 = [v(C_1 \rightarrow I_1, C_2 \rightarrow I_2)]C_2$, then we obtain $P_i$ by removing from $P_{i-1}$ all $z$ voters with the same preference order as $v_i$.

Case 2: If $C_1 \succ [v(C_1 \rightarrow I_1, C_2 \rightarrow I_2)]C_2$, then let $x$ and $y$ be such integers that $\Delta_{I_1, I_2} - \epsilon < \frac{x}{y} \leq \Delta_{I_1, I_2}$

We define two new voting situations:

$$R_{i-1} = z \cdot P_{x}(C_1 \rightarrow I_1, C_2 \rightarrow I_2)$$

and

$$Q_{i-1} = x \cdot P_{i-1} + R_{i-1}.$$

In the voting situation $Q_{i-1}$ at least $xz$ voters have $C_1$ and $C_2$ on positions $I_2$ and $I_1$, respectively (these voters are introduced in $R_{i-1}$). Further, there are exactly $xz$ voters who rank $C_1$ and $C_2$ on positions $I_1$ and $I_2$, respectively (these are the cloned-$x$ times voters that were originally in $P_{i-1}$). We define $P_i$ as $Q_{i-1}$ with these $2xz$ voters removed.

Case 3: If $C_2 \succ [v(C_1 \rightarrow I_1, C_2 \rightarrow I_2)]C_1$, then our construction is very similar to that from Case 2. The details are provided in the technical report (Skowron et al., 2016b).
It is straightforward, yet technical to show that after each such a transformation it holds that 
\( \text{score}_d(C_1, C_2, P) > 0 \) and \( C_2 \preceq_P C_1 \). One can also show that such a procedure finally stops 
and that we will reach a voting situation, \( P_f \), where each voter ranks \( C_1 \) and \( C_2 \) on positions 
\( I^*_1 \) and \( I^*_2 \) (or the other way round). The full proofs of these properties are provided in the 
technical report (Skowron et al., 2016b). As we discussed at the beginning of the proof, this gives a contradiction. 
The proof of part (ii) of the lemma is analogous (for full details, we refer to the technical report (Skowron et al., 2016b)). □

Completing the Proof of Theorem A  
Lemma 5 justifies that \( d_e \) is a decision scoring function for 
\( f_{C_1,C_2} \). From neutrality it follows that \( d_e \) gives a decision scoring function for \( f_{k,s} \). Since, \( f_k \) is 
in fact a collection of independent functions \( f_{k,s} \) for \( s \in \{0 \ldots k - 1\} \), we get Theorem A.

A.2. The tools to deal with committee scoring rules

This section shows how to apply the results obtained so far to committee scoring rules.

Second Domain Change. We distinguish a specific voting situation where each possible vote 
is cast exactly once, \( e = (1, 1, \ldots, 1) \). It is apparent that under a symmetric \( k \)-decision rule \( f_k \), 
each two committees \( C_1, C_2 \) are ranked equally in \( e \), i.e., \( C_1 =_e C_2 \). We use the following result, 
originally stated for single-winner rules but adapted to the multiwinner setting.

Lemma 6 (Young, 1975; Merlin, 2003). Suppose a \( k \)-decision rule \( f_k : \mathbb{N}^{m!} \rightarrow (S_k(A) \times 
S_k(A)) \rightarrow \{-1, 0, 1\} \) is symmetric, and consistent. There exists a unique extension of \( f_k \) to the 
domain \( \mathbb{Q}^{m!} \) (which we also denote by \( f_k \)), satisfying for each positive \( \ell \in \mathbb{N} \), and \( P \in \mathbb{N}^{m!} \) the 
following two conditions: (i) \( f_k(P - \ell e) = f_k(P) \), (ii) \( f_k \left( \frac{P}{\ell} \right) = f_k(P) \).

Lemma 6 allows us to consider voting situations with fractional numbers of linear orders.
From now on, when we speak of voting situations, we mean the elements of \( \mathbb{Q}^{m!} \). We note that 
within our new domain, the score of a pair \( (C_1, C_2) \) of committees relative to a voting situation 
\( P \) under decision scoring function \( d \) can still be expressed as in Equation (3).

\[
\text{score}_d(C_1, C_2, P) = \sum_{\pi \in \Omega, (A)} P(\pi) \cdot d(\text{pos}_\pi(C_1), \text{pos}_\pi(C_2)).
\]

Indeed, for decision scoring rules, this definition gives the unique extension that Lemma 6 speaks 
of. Thus Theorem A extends to decision rules with domain \( \mathbb{Q}^{m!} \).

Constructing a tool for committee scoring rules  
Since \( \mathbb{Q}^{m!} \) is a vector space over the field of rational numbers, from Theorem A (extended to \( \mathbb{Q}^{m!} \) ) we infer that for each two committees \( C_1 \) and 
\( C_2 \), the space of voting situations \( P \) such that \( C_1 =_P C_2 \) is a hyperplane in the \( m! \)-dimensional 
vector space of all voting situations. This is so, because if we treat a voting situation \( P \) as a 
vector of \( m! \) variables, then condition \( \text{score}_d(C_1, C_2, P) = 0 \) is a single linear equation. If this 
is a trivial equation, then in each voting situation \( P \) committees \( C_1 \) and \( C_2 \) are equally good. 
Otherwise, the space of voting situations \( P \) such that \( C_1 =_P C_2 \) is a hyperplane in \( \mathbb{Q}^{m!} \) and has 
dimension \( m! - 1 \). This can be summarized as the following corollary.

Corollary 1. The set \( \{P \in \mathbb{Q}^{m!} : C_1 =_P C_2 \} \) is either the vector space of all voting situations \( \mathbb{Q}^{m!} \), 
or a hyperplane in \( \mathbb{Q}^{m!} \).
From now on, we assume that our \( k \)-decision rule \( f_k \) is a \( k \)-winner election rule, i.e., \( f_k \) is transitive and, so, for each voting situation \( P \) and each three committees \( C_1, C_2, \) and \( C_3 \) it holds that \((C_1 \succeq P C_2) \text{and} (C_2 \succeq P C_3) \implies (C_1 \succeq P C_3)\).

**Lemma 7.** Let \( f_k \) be a symmetric, consistent, Pareto-dominant, continuous \( k \)-winner election rule, and let \( \lambda : [m]^k \to \mathbb{R} \) be a committee scoring function. If for each two committees \( C_1 \) and \( C_2 \) and each voting situation \( P \) we have that: the committee scores of \( C_1 \) and \( C_2 \) are equal (according to \( \lambda \)) if and only if \( C_1 = P C_2 \) are equally good according to \( f_k \), then the following holds: For each two committees \( C_1 \) and \( C_2 \) and each voting situation \( P \), if the committee score of \( C_1 \) is greater than that of \( C_2 \) (according to \( \lambda \)) then \( C_1 \) is preferred over \( C_2 \) according to \( f_k \).

**Proof.** Based on \( \lambda \), we build a decision scoring function \( g \) as follows. For each two committee positions \( I_1 \) and \( I_2 \), we have \( g(I_1, I_2) = \lambda(I_1) - \lambda(I_2) \). The score of a committee pair \((C_1, C_2)\) in voting situation \( P \) under \( g \) is given by:

\[
\text{score}_g(C_1, C_2, P) = \sum_{\pi \in \Pi_n(A)} P(\pi) \cdot g(\text{pos}_\pi(C_1), \text{pos}_\pi(C_2)).
\]

Let us fix \( x \in [k - 1] \) and two arbitrary committees \( C_1^* \) and \( C_2^* \) such that \( |C_1^* \cap C_2^*| = x \). We note that, by the assumptions of the theorem, if it holds that:

\[
\text{score}_g(C_1^*, C_2^*, P) = 0 \iff C_1^* = P C_2^*,
\]

then, by Corollary 1, either \( \text{score}_g(C_1^*, C_2^*, P) = 0 \) in each voting situation (and the statement of the lemma holds), or \( H = \{ P \in \mathbb{Q}^m : C_1^* = P C_2^* \} \) is an \((m! - 1)\)-dimensional hyperplane. More so, this is the same hyperplane as the following two (where \( d = d_x \) is the decision scoring function from the thesis of Lemma 5, built for \( f_k \)):

\[
\{ P \in \mathbb{Q}^m : \text{score}_g(C_1^*, C_2^*, P) = 0 \} \text{ and } \{ P \in \mathbb{Q}^m : \text{score}_d(C_1^*, C_2^*, P) = 0 \}.
\]

We claim that for \( C_1^* \) and \( C_2^* \) one of the following conditions must hold:

1. For each voting situation \( P \), if \( \text{score}_g(C_1^*, C_2^*, P) > 0 \) then \( C_1^* \succeq P C_2^* \).
2. For each voting situation \( P \), if \( \text{score}_g(C_1^*, C_2^*, P) > 0 \) then \( C_2^* \succeq P C_1^* \).

For the sake of contradiction, let us assume that there exist two voting situations, \( P \) and \( Q \), such that \( \text{score}_g(C_1^*, C_2^*, P) > 0 \) and \( \text{score}_g(C_1^*, C_2^*, Q) > 0 \), but \( C_1^* \succeq P C_2^* \) and \( C_2^* \succeq Q C_1^* \). From the fact that \( \text{score}_g(C_1^*, C_2^*, P) > 0 \) and \( \text{score}_g(C_1^*, C_2^*, Q) > 0 \), we see that the points \( P \) and \( Q \) lie on the same side of hyperplane \( H \) and neither of them lies on \( H \). From \( C_1^* \succeq P C_2^* \) and \( C_2^* \succeq Q C_1^* \), and from Lemma 5, we see that \( \text{score}_d(C_1^*, C_2^*, P) \geq 0 \) and \( \text{score}_d(C_1^*, C_2^*, Q) \leq 0 \). That is, at least one of the voting situations \( P \) and \( Q \) lies on the hyperplane, or they both lie on different sides of the hyperplane. This gives a contradiction and proves our claim.

Now, using the Pareto dominance axiom, we exclude the second possibility. For each \( i \in [m - k + 1] \) we set \( I_i = \{i, i + 1, \ldots, i + k - 1\} \). Let \( I \) and \( J \) denote, respectively, the best possible and the worst possible position of a committee, i.e., \( I = I_1 \) and \( J = I_{m-k+1} \). For the sake of contradiction, let us assume that there exists a profile \( P' \), where \( \text{score}_g(C_1^*, C_2^*, P') > 0 \) and \( C_1^* \succeq P' C_2^* \). Since there exists a profile with \( \text{score}_g(C_1^*, C_2^*, P') > 0 \), it must be the case that \( \lambda(I) > \lambda(J) \) (otherwise \( \lambda \) would be a constant function). Thus there must exist \( p \) such that \( \lambda(I_p) > \lambda(I_{p+k-2}) \). Let us consider a profile \( S \) consisting of a single vote where \( C_1^* \) stands
on position \( I_p \) and \( C_2^* \) stands on position \( I_{p+k-x} \) (as \( |C_1^* \cap C_2^*| = x \), this is possible). Since \( \lambda(I_p) > \lambda(I_{p+k-x}) \), we have that \( \text{score}_g(C_1^*, C_2^*, S) > 0 \). By Pareto dominance of \( f_k \), it follows that \( C_1^* \succeq C_2^* \). However, from the reasoning in the preceding paragraph (applied to profile \( S \)), we know that either \( C_1^* \succ C_2^* \) or \( C_2^* \succ C_1^* \). Putting these two facts together, we conclude that \( C_1^* \succeq C_2^* \). Since we have shown a single profile \( S \) such that \( \text{score}_g(C_1^*, C_2^*, S) > 0 \) and \( C_1^* \succeq C_2^* \), by the argument from the previous paragraph, we know that for every profile \( P \) it holds that:

If \( \text{score}_g(C_1^*, C_2^*, P) > 0 \) then \( C_1^* \succ P C_2^* \).

Our choice of committees \( C_1^* \) and \( C_2^* \) was arbitrary and, thus, the above implication holds for all pairs of committees. This completes the proof. \( \square \)

Due to Lemma 7, in our further discussion, given a symmetric, consistent, committee-dominant, continuous \( k \)-winner election rule \( f_k \) we can focus solely on the subspace \( \{ P : C_1 = P C_2 \} \). If we show that committees \( C_1 \) and \( C_2 \) are equivalent if and only if the score of \( C_1 \) is equal to the score of \( C_2 \) according to some committee scoring function \( \lambda \), then we can conclude that \( f_k \) is a committee scoring rule defined by this committee scoring function \( \lambda \).

A.3. Second part of the proof: committees with all but one candidate in common

We start the second part of the proof. The current section is independent from the results of the previous one, but we do use all the notation that was introduced and, in particular, we consider voting situations over \( \mathbb{Q}^{m!} \). We will use results from Sections A.1 and A.2 only in Section A.4.

The setting and our goal As before, the size of committees is denoted as \( k \). Our goal is to show that as long as we consider committees that contain some \( k - 1 \) fixed members and can differ only in the final one, \( f_k \) acts on such committee pairs as a committee scoring rule. The discussion in this section is inspired by that of Young (1975) and Merlin (2003), but the main part of our analysis is original (in particular Lemma 11).

Position-difference function Let \( P \) be a voting situation in \( \mathbb{Q}^{m!} \), \( C \) be a size-\( k \) committee, and \( I \) be a committee position. We define the weight of position \( I \) with respect to \( C \) within \( P \) as:

\[
\text{pos-weight}_I(C, P) = \sum_{\pi \in \Pi_{\geq 1}(A) : \text{pos}_\pi(C) = I} P(\pi).
\]

That is, \( \text{pos-weight}_I(C, P) \) is the (rational) number of votes in which \( C \) is ranked on position \( I \).

For each committees \( C_1, C_2 \) with \( |C_1 \cap C_2| = k - 1 \), we define a committee position-difference function \( \alpha_{C_1, C_2} : \mathbb{Q}^{m!} \to \mathbb{Q}_{(m)}^{(n)} \) that for each voting situation \( P \in \mathbb{Q}^{m!} \) returns a vector of \( \binom{m}{k} \) elements, indexed by committee positions (i.e., elements of \( [m]_k \)), such that for each \( I \in [m]_k \):

\[
\alpha_{C_1, C_2}(P)[I] = \text{pos-weight}_I(C_1, P) - \text{pos-weight}_I(C_2, P).
\]

Naturally, \( \alpha_{C_1, C_2}(P) \) is a linear function of \( P \). We claim that for each voting situation \( P \), we have:

\[
\sum_{I \in [m]_k} \alpha_{C_1, C_2}(P)[I] = 0.
\] (18)
To see why this is the case, we note that 
\[ \sum_{I \in [m]} \text{pos-weight}_I(C_1, P) = \sum_{\pi \in \Pi_2(A)} P(\pi) \]
beca
cause every vote is accounted exactly once.

Before we can use position-difference functions, we need to provide some further tools.

**Johnson graphs and Hamiltonian paths.** We will need the following graph-theoretic results to build certain votes and preference profiles in our following analysis. We mention that the graphs that Lemmas 8 and 9 speak of are called Johnson graphs. Lemma 8 was known before (we found the result in the work of Asplach (2013) and could not trace an earlier reference).

**Lemma 8.** Let \( p \) and \( j \) be integers such that \( 1 \leq j \leq p \). Let \( G(j, p) \) be a graph constructed in the following way. We associate \( j \)-element subsets of \( \{1, \ldots, p\} \) with vertices and we say that two vertices are connected if the corresponding subsets differ by exactly one element (they have \( j - 1 \) elements in common). Such a graph contains a Hamiltonian path, i.e., a path that visits each vertex exactly once, that starts from the set \( \{1, \ldots, j\} \) and ends in the set \( \{p - j + 1, \ldots, p\} \).

**Proof.** We partition the set of vertices of \( \bar{G}(j, p, r) \) into \( r - 1 \) groups \( V(j, p, 1), \ldots, V(j, p, r - 1) \), where for each \( x \in \{1, \ldots, r - 1\} \), group \( V(j, p, x) \) consists of all the sets (i.e., all the vertices) such that \( x \) is their smallest member.

We build our Hamiltonian path for \( \bar{G}(j, p, r) \) as follows. We start with vertex \( \{1, \ldots, j\} \). By Lemma 8, we can continue the path from \( \{1, \ldots, j\} \), traverse all vertices in \( V(j, p, 1) \), and end in \( \{1, p - j + 2, \ldots, p\} \). From \( \{1, p - j + 2, \ldots, p\} \) we can go, over a single edge, to \( \{2, p - j + 2, \ldots, p\} \), and we can traverse all vertices in \( V(j, p, 2) \). Then we can go, over a single edge, to some vertex from \( V(j, p, 3) \), and we can continue analogously. \( \square \)

The range of \( \alpha_{C_1, C_2} \) Let us consider two distinct committees \( C_1 \) and \( C_2 \). Using Lemma 8, we establish the dimension of the range of function \( \alpha_{C_1, C_2} \).

**Lemma 10.** For each \( C_1, C_2 \in S_k(A) \), the range of the function \( \alpha_{C_1, C_2} \) has dimension \((m^2)/(k) - 1\).

**Proof.** From (18), we get that the dimension of the range of function \( \alpha_{C_1, C_2} \) is at most \((m^2)/(k) - 1\). Consider graph \( G = G(k, m) \) from Lemma 8 and the Hamiltonian path specified in this lemma. We can view each vertex in \( G \) as a committee position. For each edge (\( I, I' \)) on our Hamiltonian path, consider a single vote where \( C_1 \) and \( C_2 \) stand on positions \( I \) and \( I' \), respectively (it exists, since \(|I \cap I'| = k - 1 \) whenever \( I, I' \) is an edge in the Hamiltonian path). For such a vote, \( \alpha_{C_1, C_2} \) returns a vector with all zeros except a single 1 on position \( I \) and a single −1 on \( I' \). It is apparent that there are \((m^2)/(k) - 1\) such votes and that so constructed vectors are linearly independent. \( \square \)

\((C_1, C_2)\)-symmetric profiles The final tool that we need to provide before we prove Lemma 11 is the definition of \((C_1, C_2)\)-symmetric profiles. Suppose \( \sigma \) is a permutation of \( A \). Then we can extend its action to linear orders and voting situations in the natural way.
Definition 8. Let $C_1$ and $C_2$ be two size-$k$ committees. We say that a voting situation $P$ is $(C_1, C_2)$-symmetric if there exists a permutation of the set of candidates $\sigma$ and a sequence of committees $F_1, F_2, \ldots, F_k$ such that $P = \sigma(P)$ and:

1. $C_1 = F_1 = F_3$ and $C_2 = F_2$.
2. for each $i \in [x-1]$ it holds that $\sigma(F_i) = F_{i+1}$.

If a voting situation $P$ is $(C_1, C_2)$-symmetric then we know that $C_1 \equiv_P C_2$. Indeed, towards a contradiction assume that $C_1 \neq_P C_2$ and, w.l.o.g., that $C_1 \succ_P C_2$. From $C_1 \succ_P C_2$ (which translates to $F_1 \succ_P F_2$) by neutrality of $f_k$ we infer that $F_2 \succ_{\sigma(P)} F_3$, thus $F_2 \succ_P F_3$. Similarly, we get that $F_1 \succ_P F_2 \succ_P F_3 \succ_P \cdots \succ_P F_k$. Consequently, we get that $C_1 \succ_P C_2$, a contradiction.

Further, we observe that for each $(C_1, C_2)$-symmetric voting situation $P$ it holds that $\alpha_{C_1, C_2}(P) = (0, \ldots, 0)$. Indeed, if $\sigma$ is as in Definition 8, we note that since $\sigma(C_1) = C_2$ and since $\sigma(P) = P$, for each (fractional) vote in $P$ where committee $C_1$ stands on some position $I$ we can uniquely assign a (fractional) vote in $P$ where committee $C_2$ stands on the same position $I$. This shows that $\alpha_{C_1, C_2}(P)[I]$ is a vector of non-positive numbers. By an analogous argument (using the fact that $\sigma^{-1}(C_2) = C_1$ and $\sigma^{-1}(P) = P$) we infer that $\alpha_{C_1, C_2}(P)[I]$ is a vector of nonnegative numbers, and, so, we conclude that $\alpha_{C_1, C_2}(P) = (0, \ldots, 0)$.

Inferring committee equivalence using $\alpha_{C_1, C_2}$ We are ready to present Lemma 11, our main technical tool required in this part of the proof. On the intuitive level, it says that for $|C_1 \cap C_2| = k - 1$ the information provided by the function $\alpha_{C_1, C_2}$ in relation to a given profile $P$ is sufficient to distinguish whether $C_1$ is equivalent to $C_2$ with respect to $P$.

Lemma 11. For each two committees $C_1, C_2 \in S_k(A)$ such that $|C_1 \cap C_2| = k - 1$ and for each voting situation $P \in \mathbb{Q}^m$, if $\alpha_{C_1, C_2}(P) = (0, \ldots, 0)$ then $C_1 \equiv_P C_2$.

Proof. The kernel of a linear function is the space of all vectors for which this function returns the zero vector. In particular, the kernel of $\alpha_{C_1, C_2}$, denoted $\ker(\alpha_{C_1, C_2})$, is the space of all voting situations $P$ such that $\alpha_{C_1, C_2}(P) = (0, \ldots, 0)$. Since the domain of function $\alpha_{C_1, C_2}$ has dimension $m!$ and, by Lemma 10, its range has dimension $\binom{m}{k} - 1$, the kernel of $\alpha_{C_1, C_2}$ has dimension $m! - \binom{m}{k} + 1$. We will construct a basis of this kernel consisting of $(C_1, C_2)$-symmetric voting situations only. Since for each $(C_1, C_2)$-symmetric voting situation $P$ it holds that $C_1 \equiv_P C_2$, $\alpha_{C_1, C_2}(P) = (0, \ldots, 0)$, by consistency of $f_k$ and linearity of $\alpha_{C_1, C_2}$ we will prove the lemma.

We prove the statement by a two-dimensional induction on $k$ (the committee size) and $m$ (the number of candidates). As a base for the induction we will show that the property holds for $k = 1$ and all values of $m$. For the inductive step we will show that from the fact that the property holds for committee size $j - 1$ and for $p - 1$ candidates it follows that the property also holds for committee size $j$ and for $p$ candidates.

For $k = 1$ and for an arbitrary value of $m$, the problem collapses to the single-winner setting. It has been shown by Young (1975) and by Merlin, 2003 that for each two candidates $c_1$ and $c'_1$, there exists a basis of $\ker(\alpha_{\{c_1\}, \{c'_1\}})$ that consists of $m! - (m - 1)$ voting situations which are $((c_1), \{c'_1\})$-symmetric. This gives us the base for the induction.

Let us now prove the inductive step. We want to show that the statement is satisfied for $A_p = \{a_1, a_2, \ldots, a_p\}$, $C_{1,j} = \{a_1, a_2, \ldots, a_j\}$ and $C_{2,j} = \{a'_1, a_2, \ldots, a_j\}$, where we set $a'_1 = a_{j+1}$. (We note that since $f_k$ is symmetric, the exact names of the candidates we use here are irrelevant, and we picked these for notational convenience.) From the sets $A_p$, $C_{1,j}$ and $C_{2,j}$ we take
out element $a_j$ and get $A_{p-1} = \{a_1, a_2, \ldots, a_{j-1}, a_{j+1}, \ldots, a_p\}$, $C_{1,(j-1)} = \{a_1, a_2, \ldots, a_{j-1}\}$ and $C_{2,(j-1)} = \{a'_1, a_2, \ldots, a_{j-1}\}$. Let $V_{j-1}$ be a basis of $\ker(\alpha_{C_{1,(j-1)},C_{2,(j-1)}})$ that consists of $(C_{1,(j-1)}, C_{2,(j-1)})$-symmetric voting situations. We know that it exists from the induction hypothesis. We also know that it consists of $(p - 1)! - \binom{p-1}{j} + 1$ voting situations. We now build the desired basis for $\ker(\alpha_{C_{1,j},C_{2,j}})$ using $V_{j-1}$ as the starting point. Our basis has to consist of $p! - \binom{p-1}{j} + 1$ linearly independent, $(C_{1,j}, C_{2,j})$-symmetric voting situations.

First, for each voting situation $P \in V_{j-1}$ and for each $r \in \{1, \ldots, p\}$ we create a voting situation $P_r$ as follows. We take each vote $v$ in $P$ and we put $a_j$ in the $r$-th position of $v$, pushing the candidates on positions $r, r+1, r+2, \ldots$ back by one position, but keeping their relative order unchanged. There are $p! - p\binom{p-1}{j} + p$ such vectors and it is easy to see that they are linearly independent. Let us refer to the set of these vectors as $B_1$. Naturally, the vectors from $B_1$ do not span the whole space $\ker(\alpha_{\{a_1, \ldots, a_j\},\{a'_1, \ldots, a_j\}})$ as there is too few of them. However, there is also a certain structural reason for this and understanding this reason will help us further in the proof. Let $\text{lin}(B_1)$ denote the set of linear combinations of voting situations from $B_1$. For each $r \in \{1, \ldots, p\}$ and each $T \in \text{lin}(B_1)$, let $T(a_j \rightarrow r)$ denote the voting situation that consists of all votes from $T$ which have $a_j$ on the $r$-th position. We can see that for each $r \in \{1, \ldots, p\}$ and each $T \in \text{lin}(B_1)$, it holds that $\alpha_{C_{1,j},C_{2,j}}(T(a_j \rightarrow r)) = \langle 0, \ldots, 0 \rangle$ (the reason for this is that $T(a_j \rightarrow r)$ is, in essence, a linear combination of voting situations from $V_{j-1}$, with $a_j$ inserted at position $r$). This property certainly does not hold for all the voting situations in $\ker(\alpha_{\{a_1, \ldots, a_j\},\{a'_1, \ldots, a_j\}})$.

We now form the second part of our basis, denoted $B_2$ and consisting of $p\binom{p-1}{j-1} - (p - 1)$ voting situations.

We start constructing each voting situation in $B_2$ by constructing its distinctive vote. To construct a distinctive vote, we first select the position for candidate $a_j$; we consider each position from $\{1, \ldots, p\}$. Let us fix $r \in \{1, \ldots, p\}$ as the position that we picked. Next, we select a set of $j$ positions for the candidates from $\{a_1, \ldots, a_{j-1}, a'_1\}$. To do that, we first construct the following graph. We associate all sets of $j - 1$ positions such that $r$ is greater\(^3\) than at least one of them with vertices (for a fixed $r$ there are $\binom{p-1}{j-1} - \binom{p-r}{j-1}$ such vertices; we choose $j - 1$ positions out of $p - 1$ still available, but we omit the situations where all these $j - 1$ positions are greater than $r$). We say that two vertices are connected if the corresponding sets differ by exactly one element. From Lemma 9 it follows that such a graph contains a Hamiltonian path. Now, for each edge $(X, X')$ on the considered Hamiltonian path we do the following. Let $B = X \cap X'$, and let $b$ and $b'$ be the two elements such that $b < b'$ and $\{b, b'\} = (X \setminus B) \cup (X' \setminus B)$.

Note that $|B| = j - 2$. We form a distinctive vote by putting candidate $a_j$ on position $r$, candidates $a_2, \ldots, a_{j-1}$ on the positions from $B$ (in some arbitrary order), $a_1$ on position $b$, $a'_1$ on position $b'$, and all the other candidates on the remaining positions (in some arbitrary order).

How many distinctive votes have we constructed? There are $p$ possible values for the position of $a_j$, and for each such position we consider a graph. If the position of $a_j$ is $r$, then the graph has $\binom{p-1}{j-1} - \binom{p-r}{j-1}$ vertices. Thus, altogether, the number of vertices is:

$$
\sum_{r=1}^{p} \left( \binom{p-1}{j-1} - \binom{p-r}{j-1} \right) = p \binom{p-1}{j-1} - \sum_{r=1}^{p} \binom{p-r}{j-1} = p \binom{p-1}{j-1} - \binom{p}{j}
$$

\(^3\) By “greater” we mean greater as a number. So, for example, position 7 is greater than position 5 (even though we would say that a candidate ranked on position 5 is ranked higher than candidate ranked on position 7).
\[ = p \left( \frac{p-1}{j-1} \right)^{j-1} \frac{1}{j}. \]

One of the graphs is empty (it is the one that is constructed for \( r = 1 \), because there is no element in \( \{1, \ldots, p\} \) lower than \( r = 1 \)). Thus we have \( p-1 \) non-empty graphs. As a result, the total number of edges in the considered Hamiltonian paths is \( p \left( \frac{p-1}{j-1} \right)^{j-1} - (p-1) \). Every edge corresponds to a distinctive vote, so this is also the number of distinctive votes constructed.

For each distinctive vote \( v \) constructed, we build the following voting situation:

**Case 1.** If \( a_1 \) and \( a'_1 \) are both ranked ahead of \( a_j \), then we let \( \tau \) be permutation \( \tau := (a_1, a_j, a'_1) \) (i.e., we let \( \tau \) be the identity permutation except that \( \tau(a_1) = a_j \), \( \tau(a_j) = a'_1 \), \( \tau(a'_1) = a_1 \)) and we let the voting situation consist of three votes, \( v \), \( \tau(v) \), and \( \tau^{(2)}(v) \):

\[
\begin{align*}
v: & \cdot \cdot \cdot > a_1 > \cdot \cdot \cdot > a'_1 > \cdot \cdot \cdot > a_j > \cdot \cdot \cdot \\
\tau(v): & \cdot \cdot \cdot > a_j > \cdot \cdot \cdot > a_1 > \cdot \cdot \cdot > a'_1 > \cdot \cdot \cdot \\
\tau^{(2)}(v): & \cdot \cdot \cdot > a'_1 > \cdot \cdot \cdot > a_j > \cdot \cdot \cdot > a_1 > \cdot \cdot \cdot 
\end{align*}
\]

The sequence \( F_1 = \{a_1, \ldots, a_j\} \), \( F_2 = \{a'_1, a_2, \ldots, a_j\} \), \( F_3 = \{a_1, \ldots, a_{j-1}, a'_1\} \), \( F_4 = \{a_1, \ldots, a_j\} \) and permutation \( \tau \) witness that this voting situation is \((C_{1,j}, C_{2,j})\)-symmetric.

**Case 2.** If it is not the case that \( a_1 \) and \( a'_1 \) are both ranked ahead of \( a_j \) in distinctive vote \( v \), then we know that there is some other candidate \( a \in \{a_2, \ldots, a_{j-1}\} \) ranked ahead of \( a_j \). This is due to our construction of distinctive votes—we always put \( a_j \) on position \( r \) and make sure that there is some candidate ranked on a position ahead of \( r \). If all the candidates \( a_2, \ldots, a_{j-1} \) were ranked behind \( a_j \), then it would have to be the case that both \( a_1 \) and \( a'_1 \) are ranked ahead of \( a_j \).\(^4\) Since it is not the case that both \( a_1 \) and \( a'_1 \) are ranked ahead of \( a_j \), there must be some other candidate from \( \{a_2, \ldots, a_{j-1}\} \) that is. We call this candidate \( a \). We let \( \rho \) be permutation \( \rho := (a_1, a'_1)(a, a_j) \) (i.e., \( \rho \) is the identity permutation, except that it swaps \( a_1 \) with \( a'_1 \) and \( a \) with \( a_j \)). We form a voting situation that consists of \( v \) and \( \rho(v) \):

\[
\begin{align*}
v: & \cdot \cdot \cdot > a > \cdot \cdot \cdot > a_j > \cdot \cdot \cdot > a_1 > \cdot \cdot \cdot > a'_1 > \cdot \cdot \cdot \\
\rho(v): & \cdot \cdot \cdot > a_j > \cdot \cdot \cdot > a > \cdot \cdot \cdot > a'_1 > \cdot \cdot \cdot > a_1 > \cdot \cdot \cdot 
\end{align*}
\]

Permutation \( \rho \) and the sequence \( F_1 = \{a_1, \ldots, a_j\} \), \( F_2 = \{a'_1, a_2, \ldots, a_j\} \), \( F_3 = \{a_1, \ldots, a_j\} \) witness that this is a \((C_{1,j}, C_{2,j})\)-symmetric voting situation.

Let \( B_2 \) consist of all the voting situations constructed from the distinctive votes.

For each \( r \in \{1, \ldots, p\} \), each set of \( j - 1 \) positions \( R \) from \( \{1, \ldots, p\} \setminus \{r\} \), and each voting situation \( P \), we define \( \gamma_{r,R}(P) \) to be the total (possibly fractional) number of votes from \( P \) that have \( a_j \) on the \( r \)-th position and that have candidates from \( \{a_1, a_2, \ldots, a_{j-1}\} \) on positions

\(^4\) To see why this is the case, recall how the distinctive votes are produced. We have an edge \((X, X')\) on a Hamiltonian path in our graph. We set \( B = X \cap X' \) and \( \{b, b'\} = (X \setminus B) \cup (X' \setminus B) \). \( B \) contains positions of the candidates \( a_2, \ldots, a_{j-1} \), whereas \( b \) and \( b' \) are positions of \( a_1 \) and \( a'_1 \). Without loss of generality, we can take \( X = B \cup \{b\} \) and \( X' = B \cup \{b'\} \). Since—by our assumption here—the positions of \( a_2, \ldots, a_{j-1} \) (i.e., the positions in \( B \)) are greater than the position of \( a_j \) (denoted \( r \) in the description of distinctive votes construction), for \( X \) and \( X' \) to be vertices in the graph, we need both \( b \) and \( b' \) to be smaller than \( r \) (and, in effect, both \( a_1 \) and \( a'_1 \) precede \( a_j \)).
from $R$. We define $\gamma'_{r,R}(P)$ analogously, for the votes where $a_j$ is on position $r$ and candidates $a'_1, a_2, \ldots, a'_{j-1}$ take positions from $R$. We define $\beta_{r,R}(P)$ to be $\gamma_{r,R}(P) - \gamma'_{r,R}(P)$. E.g., $\beta_{r,R}(P) = 0$ for each $P \in B_1$.

Let us consider voting situations from $B_2$ which were created from a single Hamiltonian path in one of the graphs. The distinctive votes for all these voting situations have $a_j$ on the same position; we denote this position by $r$. For each such voting situation $P$, each non-distinctive vote belonging to $P$ has $a_j$ on a position ahead of position $r$. Further, we see that there exist exactly two sets $R_1$ and $R_2$ such that $\beta_{r,R_1}(P) \neq 0$ and $\beta_{r,R_2}(P) \neq 0$. These are the sets that correspond to the vertices connected by the edge from which the distinctive vote for $P$ was created (for one of them, let us say $R_1$, we have $\beta_{r,R_1}(P) = 1$, and for the other we have $\beta_{r,R_2}(P) = -1$; to see that this holds, recall that $a_j$ is ranked on positions ahead of $r$ in non-distinctive votes and, thus, it suffices to consider the distinctive vote only).

Now we are ready to explain why the vectors from $B_1 \cup B_2$ are linearly independent. For each nontrivial linear combination $L$ of the vectors from $B_1 \cup B_2$ we will show that $L$ cannot be equal to the zero vector. For the sake of contradiction let us assume that $L = (0, \ldots, 0)$. We start by showing that all coefficients of vectors from $B_2$ in $L$ are equal to zero. Again, for the sake of contradiction let us assume that this is not the case. Let $B'_2$ consist of those vectors from $B_2$ that appear in $L$ with non-zero coefficients. Let $r$ be the largest position of $a_j$ in some vote in $B'_2$. Let $B'_{2,r}$ be the set of all voting situations from $B'_2$ that have some votes which have $a_j$ on position $r$. Each voting situation in $B'_{2,r}$ consists of either two or three votes. However, the votes belonging to those voting situations which have $a_j$ on position $r$ must be distinctive votes (all non-distinctive votes for voting situations in $B_2$ have $a_j$ on positions ahead of $r$). Each such distinctive vote is built from an edge of a single Hamiltonian path (they come from the same Hamiltonian path because otherwise they would not have $a_j$ on the same position). Let $S$ be a voting situation in $B'_{2,r}$ that has a distinctive vote built from the latest edge on the path, among the edges that contributed voting situations to $B'_{2,r}$ (to make this notion meaningful, we orient the path in one of the two possible ways). Let $R_1$ and $R_2$ be the sets of $j - 1$ positions that form this edge. By the reasoning from the previous paragraph we have that $\beta_{r,R_1}(S) \neq 0$, $\beta_{r,R_2}(S) \neq 0$, and one of the following two conditions must hold (depending on the orientation of the Hamiltonian path that we chose):

1. For each voting situation $Q'$ in $B'_2$ other than $S$ we have $\beta_{r,R_1}(Q') = 0$.
2. For each voting situation $Q'$ in $B'_2$ other than $S$ we have $\beta_{r,R_2}(Q') = 0$.

Further, for each $Q \in B_1$ we have $\beta_{r,R_1}(Q) = \beta_{r,R_2}(Q) = 0$. Thus, since $\beta_{r,R_1}$ and $\beta_{r,R_2}$ are linear functions, we have that either $\beta_{r,R_1}(L) \neq 0$ or $\beta_{r,R_2}(L) \neq 0$. Thus, $L$ cannot be a zero-vector, which gives a contradiction.

We have shown that all coefficients of vectors from $B_2$ used to form $L$ are equal to zero. Thus $L$ must be a linear combination of vectors from $B_1$. However, the vectors from $B_1$ are linearly independent, which means that if $L$ is $(0, \ldots, 0)$, then the coefficients of all the vectors from $B_1$ are zeros. Thus we conclude that the vectors from $B_1 \cup B_2$ are linearly independent.

It remains to show that $B_1 \cup B_2$ indeed forms a basis of the kernel of $\alpha_{C_{1,j}, C_{2,j}}$. Since vectors in $B_1$ and $B_2$ are linearly independent, it suffices to check that the cardinality of $B_1 \cup B_2$ is equal to the dimension of $\ker(\alpha_{C_{1,j}, C_{2,j}})$. The number of vectors in $B_1 \cup B_2$ is equal to:
\[
\begin{align*}
\left( p! - p \frac{p - 1}{j - 1} \right) + \frac{p - 1}{j - 1} - p + 1 &= \frac{p! - p \left( p - 1 \right) + 1}{j - 1} \\
&= p! - \left( \frac{p}{j} \right) + 1.
\end{align*}
\]

This completes our induction. The proof works for arbitrary committees \( C_1 \) and \( C_2 \) with \( |C_1 \cap C_2| = k - 1 \) due to symmetry of \( f_k \). □

We are almost ready to show that for committees that differ by one candidate only, \( f_k \) is a committee scoring rule, and to derive its committee scoring function. However, before we do that we need to change the domain once again. We will also need some notions from topology.

**Topological definitions** For every set \( S \) in some Euclidean space \( \mathbb{R}^n \), by \( \text{int}(S) \) we mean the interior of \( S \), i.e., the largest (in terms of inclusion) open set contained in \( S \). By \( \text{conv}(S) \) we mean the convex hull of \( S \), i.e., the smallest (in terms of inclusion) convex set that contains \( S \). Finally, by \( \overline{S} \) we define the closure of \( S \), i.e., the smallest closed set that contains \( S \). We use the concept of \( \mathbb{Q} \)-convex sets of Young (1975) and we recall his two observations.

**Definition 9 (\( \mathbb{Q} \)-convex sets).** A set \( S \subseteq \mathbb{R}^n \) is \( \mathbb{Q} \)-convex if \( S \subseteq \mathbb{Q}^n \) and for each \( s_1, s_2 \in S \) and each \( q \in \mathbb{Q}, 0 \leq q \leq 1 \), it holds that \( q \cdot s_1 + (1 - q) \cdot s_2 \in S \).

**Lemma 12 (Young, 1975).** Set \( S \subseteq \mathbb{R}^n \) is \( \mathbb{Q} \)-convex if and only if \( S = \mathbb{Q}^n \cap \text{conv}(S) \).

**Lemma 13 (Young, 1975).** If a set \( S \) is \( \mathbb{Q} \)-convex, then \( \overline{S} = \text{conv}(S) \); moreover, \( \overline{S} \) is convex.

**Third domain change** In the following arguments, we fix two arbitrary committees \( C_1 \) and \( C_2 \) such that \( |C_1 \cap C_2| = k - 1 \) and focus on them. (In other words, we consider function \( f_{C_1,C_2} \) instead of \( f_k \).) In this case, Lemma 11 allows us to change the domain of the function.

Let us consider two voting situations \( P \) and \( Q \) such that \( \alpha_{C_1,C_2}(P) = \alpha_{C_1,C_2}(Q) \). Since \( \alpha_{C_1,C_2} \) is a linear function, we have \( \alpha_{C_1,C_2}(P - Q) = \langle 0, \ldots, 0 \rangle \). Thus, by Lemma 11, we know that \( C_1 \succ_p C_2 \) if and only if \( C_1 \succ_Q C_2 \). We can express \( Q \) as \( Q = P + (Q - P) \) and thus, by consistency of \( f_{C_1,C_2} \), we have:

\[
C_1 \succ_p C_2 \iff C_1 \succ_Q C_2.
\]

Consequently, to answer the question “what is the relation between committees \( C_1 \) and \( C_2 \) according to \( f_{C_1,C_2} \) in voting situation \( P \)?” it suffices to know the value \( \alpha_{C_1,C_2}(P) \). This is exactly because for any two profiles, \( P \) and \( Q \), with the same values of function \( \alpha_{C_1,C_2} \) the result of comparison of committees \( C_1 \) and \( C_2 \) according to \( f_{C_1,C_2} \) is the same in \( P \) and \( Q \).

In effect, we can restrict the domain of \( f_{C_1,C_2} \) to an \((\binom{n}{k} - 1)\)-dimensional space \( D \):

\[
D = \left\{ P \in \mathbb{Q}^{\binom{n}{k}} : \sum_{I \in [m]_k} P[I] = 0 \right\}.
\]

We interpret elements of \( D \) as the values of the committee position-difference function \( \alpha_{C_1,C_2} \) and, so, the condition \( \sum_{I \in [m]_k} P[I] = 0 \) corresponds to the property (18). By the previous argument, we know that from the point of view of comparing committees \( C_1 \) and \( C_2 \) using function
$f_{C_1, C_2}$, the vector of values $\alpha_{C_1, C_2}$ provides the same information as a voting situation from which it is obtained. Thus, we can think of elements of $D$ as corresponding to voting situations.

**Separating two committees** We proceed by defining two sets, $D_1, D_2 \subseteq D$, such that:

$$D_1 = \{ P \in D : C_1 \succ_P C_2 \} \quad \text{and} \quad D_2 = \{ P \in D : C_2 \succ_P C_1 \}. $$

From consistency of $f_{C_1, C_2}$, it follows that $D_1$ and $D_2$ are $\mathbb{Q}$-convex.

Let us consider the case where $f_{C_1, C_2}$ is trivial. By neutrality, $f_k$ ranks equally each two committees $C'_1$ and $C'_2$, such that $|C'_1 \cap C'_2| = k - 1$. This means that $f_k$ (for committees with intersection $k - 1$) is implemented by the trivial committee scoring function $\lambda \equiv 0$. So let us assume that $f_{C_1, C_2}$ is nontrivial and there is a voting situation where it does not rank $C_1$ and $C_2$ equally. Then, $D_1$ or $D_2$ is nonempty. From neutrality it follows that so is the other one. Now, we move our analysis from $\mathbb{Q}^m$ to $\mathbb{R}^m$, by analyzing the closures of the sets $D_1$ and $D_2$.

**Lemma 14.** The sets int$(\overline{D_1})$ and $\overline{D_2}$ are disjoint, convex, and nonempty relative to $D$ (i.e., int$(\overline{D_1}) \cap D \neq \emptyset$ and $\overline{D_2} \cap D \neq \emptyset$).

**Recovering the scoring function** We are ready to derive our committee scoring function. From the hyperplane separation theorem, it follows that there exists a vector $\eta \in \mathbb{R}^m_k$ such that (for $P \in D$, by $\eta \cdot P$ we mean the dot product of $P$ and $\eta$, both treated as $(m_k)$ dimensional vectors):

1. For each voting situation $P \in \overline{D_2}$ it holds that $\eta \cdot P \leq 0$.
2. For each voting situation $P \in \text{int}(\overline{D_1})$ it holds that $\eta \cdot P > 0$.

We note that Lemma 14 allows us to directly apply the hyperplane separation theorem as the sets int$(\overline{D_1})$ and $\overline{D_2}$ are disjoint.\(^5\)

We now show that if $P \in D$ and $\eta \cdot P > 0$, then $P \in D_1$. Since $\eta \cdot P > 0$, $P$ cannot belong to $D_2$, but it might be the case that $C_1 \equiv P \cdot C_2$. Towards a contradiction assume that this is the case. We observe that there exists an $(m_k - 1)$-dimensional open ball $B$ in $D$ with $P \in B$, such that for each $S \in B$ we have $C_1 \succeq_S C_2$ (since $D$ does not belong to $\overline{D_2}$). We consider two cases.

**Case 1.** If for each $S \in B$ we have $C_1 \equiv_S C_2$, then we proceed as follows. Consider $Q$ such that $C_1 \succ_Q C_2$. There must exist some (possibly very small) $x$ such that $S = x \cdot Q + (1 - x) \cdot P \in B$. Yet, from consistency we would get that $C_1 \succ_S C_2$, a contradiction.

**Case 2.** If there is $Q \in B$ such that $C_1 \succ_Q C_2$, then there exists $0 < \epsilon < 1$ such that $S = \frac{P - \epsilon Q}{1 - \epsilon} \in B$. Since $S \in B$, we have that $C_1 \succeq_S C_2$. Further, we have that $P = \epsilon Q + (1 - \epsilon) S$. By consistency of $f_{C_1, C_2}$ we get that $C_1 \succ_P C_2$, a contradiction.

Next, we show that if $\eta \cdot P < 0$, then $P \in D_2$. Towards a contradiction assume that there is $P$ such that $\eta \cdot P < 0$ but $C_1 \succeq_P C_2$. Then there exists such $\epsilon$ that if $|Q - P| < \epsilon$ then $\eta \cdot Q < 0$ (and so $Q \notin \text{int}(\overline{D_1})$). Thus there exists an open ball $B$ in $D$ with $P \in B$, such that $B \cap \text{int}(\overline{D_1}) = \emptyset$. Thus, $B \cap D_1 = \emptyset$. We infer that some point $S$ in $B$ could be represented as a linear combination of $P$ and some point from $D_1$. From consistency we would get that $C_1 \succ_S C_2$, a contradiction.

---

\(^5\) This is different from Young’s (1975) and Merlin’s (2003) approach, who operate on sets with disjoint interiors, but which do not have to be disjoint on their own.
Remark 1. We have shown that for each \( P \in D \), (a) \( \eta \cdot P > 0 \) implies that \( P \in D_1 \) (and, so, \( C_1 >_p C_2 \)), and (b) \( \eta \cdot P < 0 \) implies that \( P \in D_2 \) (and, so, \( C_2 >_p C_1 \)). From symmetry, the same vector \( \eta \) works for each pair of committees \( C_1 \) and \( C_2 \) such that \(|C_1 \cap C_2| = k - 1\).

Now we will use continuity to prove that if \( \eta \cdot P = 0 \) then \( C_1 =_p C_2 \). For the sake of contradiction assume that there exists a voting situation \( P \in D \) such that \( \eta \cdot P = 0 \) but \( C_1 \neq_p C_2 \). Without loss of generality, let us assume that \( C_1 >_p C_2 \). Let \( Q \) be a voting situation such that \( \eta \cdot Q < 0 \) and so \( C_2 >_Q C_1 \). For each \( x \) it holds that \( \eta \cdot (xP + Q) < 0 \) and so \( C_2 >_x P + Q C_1 \). However, this contradicts continuity of \( f_k \). Thus, for every \( P \in D \), if \( \eta \cdot P = 0 \) then \( C_1 =_p C_2 \).

From vector \( \eta \), we retrieve a committee scoring function \( \lambda \). For each committee position \( I \in [m_k] \) we set \( \lambda (I) = \eta [I] \). Now, we can see that for each two committees \( C_1, C_2 \), and for each voting situation \( P \in \mathbb{Q}^m \) it holds that (see the comment below for an explanation of what \( Q \) is):

\[
\text{score}_\lambda(C_1, P) = \text{score}_\lambda(C_2, P) = \sum_{I \in [m_k]} (\lambda(I) \cdot \text{pos-weight}_I(C_1, P) - \lambda(I) \cdot \text{pos-weight}_I(C_2, P))
\]

\[
= \sum_{I \in [m_k]} \lambda(I) \cdot \alpha_{C_1, C_2}(P)[I] = \sum_{I \in [m_k]} \eta[I] \cdot \alpha_{C_1, C_2}(P)[I] = \eta \cdot Q,
\]

where \( Q \in D \) is the representation of \( P \) in the space \( D \) (i.e., \( Q \) is the vector of values of the committee position-difference function \( \alpha_{C_1, C_2} \) for profile \( P \)). From the above inequality we see that \( \text{score}_\lambda(C_1, P) > \text{score}_\lambda(C_2, P) \) implies that \( C_1 >_p C_2 \) and that \( \text{score}_\lambda(C_1, P) = \text{score}_\lambda(C_2, P) \) implies that \( C_1 =_p C_2 \). From neutrality we get that the same committee scoring function \( \lambda \) works for every two committees \( C'_1 \) and \( C'_2 \) with \(|C'_1 \cap C'_2| = k - 1\).

Summarizing our discussion from this section, we get our main result, Theorem B, for the committees \( C_1 \) and \( C_2 \), with \(|C_1 \cap C_2| = k - 1\).

A.4. Putting everything together: comparing arbitrary committees

This section concludes the proof of Theorem B by extending the reasoning from the previous section to apply to every two committees \( C_1 \) and \( C_2 \) irrespective of the size of their intersection.

Setting up the proof Let \( f_k \) be a \( k \)-winner election rule that is symmetric, consistent, continuous, and has the Pareto dominance property. Let \( \lambda \) be the scoring function derived for this \( f_k \) as described at the end of the previous section. We know that for each two committees \( C_1 \) and \( C_2 \) such that \(|C_1 \cap C_2| = k - 1\) and each voting situation \( P \in \mathbb{Q}^m \) it holds that \( \text{score}_\lambda(C_1, P) > \text{score}_\lambda(C_2, P) \) if and only if \( C_1 >_p C_2 \), and \( \text{score}_\lambda(C_1, P) = \text{score}_\lambda(C_2, P) \) if and only if \( C_1 =_p C_2 \). We will show that the same holds for all committees \( C_1 \) and \( C_2 \), irrespective of the size of their intersection. We will show this by induction over \( k - |C_1 \cap C_2| \).

Let us fix some value \( k' < k - 1 \) and assume that \( \lambda \) can be used to distinguish whether a committee \( C_1 \) is preferred over a committee \( C_2 \) whenever \(|C_1 \cap C_2| > k' \). We will show that the same \( \lambda \) can be used to distinguish whether \( C_1 \) is preferred over \( C_2 \) when \(|C_1 \cap C_2| = k' \).

Let \( C_1 \) and \( C_2 \) be two arbitrary committees such that \(|C_1 \cap C_2| = k' \). Let us rename the candidates so that \( C_1 \setminus C_2 = \{c_1, \ldots, c_{k-k'}\} \), \( C_1 \cap C_2 = \{c_{k-k'+1}, \ldots, c_k\} \) and \( C_2 \setminus C_1 = \{c_{k+1}, \ldots, c_{2k-k'}\} \).
The case where $k - k'$ is even

If $k - k'$ is even, we consider the following two cases:

**Case 1:** There exists a vector of $2k - k'$ positions $\langle p_1, \ldots, p_{2k-k'} \rangle$ such that:

$$
\lambda([p_1, \ldots, p_k]) + \lambda([p_{k-k'+1}, \ldots, p_{2k-k'}]) \neq 2\lambda([p_{k-k'+1}, \ldots, p_{k-k'+k}]). 
$$

(19)

Let us consider the committee $C_3 = \{c_{\frac{k-k'}{2}+1}, \ldots, c_{\frac{k-k'+k}{2}}\}$. We consider the vector space of voting situations $P \in \mathbb{Q}^{m!}$ such that $C_1 \supseteq C_3$ and $C_2 \supseteq C_2$ (the fact that this is a vector space follows from the inductive assumption; $|C_1 \cap C_3| = |C_2 \cap C_3| > 2$).

The conditions $C_1 = p C_3$ and $C_2 = p C_2$ are not contradictory (consider the profile in which each vote is cast exactly once—in such profile all size-$k$ committees are equivalent with respect to $f_k$). This space has dimension either $m! - 2$ or $m! - 1$. This is so, because each of the conditions $C_1 = p C_3$ and $C_2 = p C_3$ boils down to a single linear equation. If these equations are independent then the dimension is $m! - 2$. Otherwise, it is $m! - 1$. By transitivity of $f_k$ we get that in each voting situation $P$ from this space it holds that $C_1 = p C_2$ and that the committee score of $C_1$ (according to $\lambda$) is equal to the committee score of $C_2$. Let $B$ be a basis of this space. Further, let $v$ be a vote where each candidate $c_i, i \in \{1, \ldots, 2k-k'\}$, stands on position $p_i$ (recall Equation (19) above), and let $v'$ be an identical vote except that candidates from $C_1 \cup C_2$ are listed in the reverse order (i.e., $c_1$ is on position $p_{2k-k'}, c_2$ is on position $p_{2k-k'-1}$ and so on). Let $S_b$ be a voting situation that consists of $v$ and $v'$. The positions of $C_1$ and $C_3$ in $v$ and $v'$ are:

$$
\begin{align*}
\text{pos}_v(C_1) &= \{p_1, \ldots, p_k\} \\
\text{pos}_v(C_3) &= \{p_{\frac{k-k'}{2}+1}, \ldots, p_k\} \\
\text{pos}_v(C_1) &= \{p_{k-k'+1}, \ldots, p_{2k-k'}\} \\
\text{pos}_v(C_3) &= \{p_{k-k'+1}, \ldots, p_{k-k'+k}\}
\end{align*}
$$

Consequently, according to Equation (19), in voting situation $S_b$ the committee score of $C_1$ is not equal to that of $C_3$. By the inductive assumption, it must be the case that $C_1 \supseteq C_3$. This means that the voting situations in $B \cup \{S_b\}$ are linearly independent.

We now show that $C_1 = S_b C_2$. Consider a permutation $\sigma$ (over the candidate set) that swaps $c_1$ with $c_{2k-k'}, c_2$ with $c_{2k-k'-1}$, and so on. We note that $\sigma(C_1) = C_2, \sigma(C_2) = C_1, \text{ and } S_b = \sigma(S_b)$. Thus, by symmetry of $f_k$, it must be the case that $C_1 = S_b C_2$.

Further, the committee scores of $C_1$ and $C_2$ are equal in $S_b$.

Altogether, the basis $B \cup \{S_b\}$ defines an $(m! - 1)$-dimensional space of voting situations $P$ such that $C_1 = p C_2$ and the committee scores of $C_1$ and $C_2$ are equal. From Corollary 1 and nontriviality of $f_{C_1,C_2}$ we know that this space includes all $P$ such that $C_1 = p C_2$.

From Lemma 7 we get that $C_1 \succ S C_2$ whenever the score of $C_1$ is greater than that of $C_2$.

**Case 2:** For each vector of $2k - k'$ positions $\langle p_1, \ldots, p_{2k-k'} \rangle$ it holds that (note that the condition below is a negation of the condition from Case 1):

$$
\lambda([p_1, \ldots, p_k]) - \lambda([p_{\frac{k-k'}{2}+1}, \ldots, p_{\frac{k-k'+k}{2}}]) = \lambda([p_{\frac{k-k'}{2}+1}, \ldots, p_{\frac{k-k'+k}{2}}]) - \lambda([p_{k-k'+1}, \ldots, p_{2k-k'}]).
$$

As before, let $C_3 = \{c_{\frac{k-k'}{2}+1}, \ldots, c_{\frac{k-k'+k}{2}}\}$. Since the above equality must hold for each vector of $2k - k'$ positions, we see that if the committee score of $C_1$ is equal to the committee score of $C_3$, then the committee score of $C_3$ is equal to the committee score of $C_2$. Consequently, by the inductive assumption, we get that $C_1 = p C_3$ implies that

$$
\lambda([p_1, \ldots, p_k]) - \lambda([p_{\frac{k-k'}{2}+1}, \ldots, p_{\frac{k-k'+k}{2}}]) = \lambda([p_{\frac{k-k'}{2}+1}, \ldots, p_{\frac{k-k'+k}{2}}]) - \lambda([p_{k-k'+1}, \ldots, p_{2k-k'}]).
$$
$C_3 = p \cdot C_2$. Thus, by $f_k$’s transitivity, we get that for each voting situation $P$, the condition $C_1 = p \cdot C_3$ implies that $C_1 = p \cdot C_2$. Consequently, there is an $(m! - 1)$-dimensional space of voting situations $P$ such that $C_1 = p \cdot C_2$ and such that $C_1$ has the same committee score as $C_2$. We conclude the reasoning similarly as in Case 1.

The case where $k - k' \geq 3$ and $k - k'$ is odd. Similarly as before we consider two cases:

Case 1: There exists a vector $(p_1, \ldots, p_{2k-k'})$ and a number $x \in \{1, \ldots, k - k'\}$ such that:

$$\lambda((p_1, \ldots, p_k)) + \lambda((p_{k-k'}+1, \ldots, p_{2k-k'})) \neq$$

$$\lambda((p_x, \ldots, p_{k+x-1})) + \lambda((p_{k-k'+2-x}, \ldots, p_{2k-k'+1-x})).$$

We repeat the reasoning from Case 1 in the previous subsection with $C_3 = \{c_x, \ldots, c_{k-x-1}\}$.

Case 2: For each vector $(p_1, \ldots, p_{2k-k'})$ and each number $x \in \{1, \ldots, k - k'\}$ it holds that:

$$\lambda((p_1, \ldots, p_k)) + \lambda((p_{k-k'}+1, \ldots, p_{2k-k'})) =$$

$$\lambda((p_x, \ldots, p_{k+x-1})) + \lambda((p_{k-k'+2-x}, \ldots, p_{2k-k'+1-x})).$$

The above equality for $x = \lfloor \frac{k-k'}{2} \rfloor$ and for $x = \lfloor \frac{k-k'}{2} \rfloor + 1$ gives (since $k - k' - \lfloor \frac{k-k'}{2} \rfloor = \lceil \frac{k-k'}{2} \rceil$):

$$\lambda((p_1, \ldots, p_k)) + \lambda((p_{k-k'}+1, \ldots, p_{2k-k'})) =$$

$$\lambda((p_{\lfloor \frac{k-k'}{2} \rfloor + 1}, \ldots, p_{k+\lfloor \frac{k-k'}{2} \rfloor - 1})) + \lambda((p_{\lfloor \frac{k-k'}{2} \rfloor + 2}, \ldots, p_{k+([\frac{k-k'}{2}]+1)})) =$$

$$\lambda((p_{\lfloor \frac{k-k'}{2} \rfloor + 1}, \ldots, p_{k+\lfloor \frac{k-k'}{2} \rfloor - 1})) + \lambda((p_{\lfloor \frac{k-k'}{2} \rfloor + 2}, \ldots, p_{k+([\frac{k-k'}{2}]+1)})�

By renaming indices in the last above equality, we get that for each set of $k + 3$ positions $(q_1, \ldots, q_{k+3})$ (note that by assumption there exist at least $k + 3$ alternatives) it holds that:

$$\lambda((q_1, \ldots, q_k)) + \lambda((q_{4}, \ldots, q_{k+3})) = \lambda((q_2, \ldots, q_{k+1})) + \lambda((q_3, \ldots, q_{k+2})).$$

After reformulation we get:

$$\lambda((q_1, \ldots, q_k)) - \lambda((q_2, \ldots, q_{k+1})) = \lambda((q_3, \ldots, q_{k+2})) - \lambda((q_4, \ldots, q_{k+3})).$$

(20)

If $k$ is odd, we obtain the following series of equalities (the consecutive equalities, except for the last one, are consequences of applying Equation (20) to the cyclic shifts of the list $(q_1, q_2, \ldots, q_{k+3})$; the last equality breaks the pattern and is a consequence of applying Equation (20) to the list $(q_{k+2}, q_{k+3}, q_1, q_2, \ldots, q_{k-1}, q_{k+1}, q_k)$):

$$\lambda((q_1, \ldots, q_k)) - \lambda((q_2, \ldots, q_{k+1})) = \lambda((q_3, \ldots, q_{k+2})) - \lambda((q_4, \ldots, q_{k+3}))$$

$$= \lambda((q_5, \ldots, q_{k+3}, q_1)) - \lambda((q_6, \ldots, q_{k+3}, q_1, q_2))$$

$$= \lambda((q_7, \ldots, q_{k+3}, q_1, q_2, q_3)) - \lambda((q_8, \ldots, q_{k+3}, q_1, q_2, q_3, q_4))$$

$$\vdots$$

$$= \lambda((q_{k+2}, q_{k+3}, q_1, \ldots, q_{k-2})) - \lambda((q_{k+3}, \ldots, q_1, q_{k-1}))$$

$$= \lambda((q_1, \ldots, q_{k-1}, q_{k+1})) - \lambda((q_2, \ldots, q_{k+1})).$$
In consequence, it must be the case that $\lambda([q_1, \ldots, q_k]) = \lambda([q_1, \ldots, q_{k+1}])$. Thus, by transitivity, we get that $\lambda$ is a constant function (in essence, what we have shown is that we can replace positions in the set of $k$ positions, one by one, without changing the value of the committee scoring function). Let $C_3 = \{c_2, \ldots, c_{k+1}\}$. Since $\lambda$ is a constant function, then by the inductive assumption we have that for every voting situation $P$ it holds that $C_1 = p C_3$ and $C_3 = p C_2$. By transitivity we get that for each voting situation $P$ it holds that $C_1 = p C_2$. Thus our trivial scoring function works correctly on $C_1$ and $C_2$.

Let us now assume that $k$ is even. Now we obtain the following series of equalities (in this case all the consecutive equalities are consequences of applying Equation (20) to the cyclic shifts of the sequence $\langle q_1, q_2, \ldots, q_{k+3} \rangle$):

$$
\begin{align*}
\lambda([q_1, \ldots, q_k]) &- \lambda([q_2, \ldots, q_{k+1}]) = \lambda([q_3, \ldots, q_{k+2}]) - \lambda([q_4, \ldots, q_{k+3}]) \\
&= \lambda([q_5, \ldots, q_{k+3}, q_1]) - \lambda([q_6, \ldots, q_{k+3}, q_1, q_2]) \\
&= \lambda([q_7, \ldots, q_{k+3}, q_1, q_2, q_3]) - \lambda([q_8, \ldots, q_{k+3}, q_1, q_2, q_3, q_4]) \\
&\vdots \\
&= \lambda([q_{k+3}, q_1, \ldots, q_{k-1}]) - \lambda([q_1, \ldots, q_k]) \\
&= \lambda([q_2, \ldots, q_{k+1}]) - \lambda([q_3, \ldots, q_{k+2}]).
\end{align*}
$$

In consequence, it is the case that:

$$
\lambda([q_1, \ldots, q_k]) - \lambda([q_2, \ldots, q_{k+1}]) = \lambda([q_2, \ldots, q_{k+1}]) - \lambda([q_3, \ldots, q_{k+2}]),
$$

and this holds for every sequence $\langle q_1, \ldots, q_{k+2} \rangle$ of positions. Thus, we get that for each voting situation in which $\{c_1, \ldots, c_k\}$ is equivalent to $\{c_2, \ldots, c_{k+1}\}$, it also holds that $\{c_2, \ldots, c_{k+1}\}$ is equivalent to $\{c_3, \ldots, c_{k+2}\}$, it also holds that $\{c_3, \ldots, c_{k+2}\}$ is equivalent to $\{c_4, \ldots, c_{k+3}\}$, etc. Let $C_3 = \{c_2, \ldots, c_{k+1}\}$. From the preceding reasoning we have that for each voting situation $P$ the fact that it holds that $C_1 = p C_3$ implies that $C_1 = p C_2$. We conclude the proof in the same way as in the case of even $k - k'$ (Case 2). Specifically, we conclude that there exists an $(m! - 1)$-dimensional space of voting situations $P$ such that $C_1 = p C_2$ and such that $C_1$ has the same committee score as $C_2$. This means that for each voting situation $P$ the condition $C_1 = p C_2$ is equivalent to the condition that $C_1$ has the same committee score as $C_2$ according to $\lambda$, and that it holds that $C_1 > p C_2$ whenever the committee score of $C_1$ is greater than that of $C_2$ (by Lemma 7).

**Conclusion of the proof** We have shown that if a $k$-winner rule is symmetric, consistent, continuous, and has the Pareto dominance property, then it is a committee scoring rule. On the other hand, committee scoring rules satisfy all these conditions. This completes our proof of Theorem B.

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