PEIERLS BRACKETS: FROM FIELD THEORY
TO DISSIPATIVE SYSTEMS

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Abstract. Peierls brackets are part of the space-time approach to quantum field theory,
and provide a Poisson bracket which, being defined for pairs of observables which are group
invariant, is group invariant by construction. It is therefore well suited for combining the
use of Poisson brackets and the full diffeomorphism group in general relativity. The present
paper provides at first an introduction to the topic, with applications to gauge field theory.
In the second part, a set of brackets for classical dissipative systems, subject to external
random forces, are derived. The method is inspired by the old procedure of Peierls, for
deriving the canonical brackets of conservative systems, starting from an action principle.
It is found that an adaptation of Peierls’ method is applicable also to dissipative systems,
when the friction term can be described by a linear functional of the coordinates, as is
the case in the classical Langevin equation, with an arbitrary memory function. The
general expression for the brackets satisfied by the coordinates, as well as by the external
random forces, at different times, is determined, and it turns out that they all satisfy the
Jacobi identity. Upon quantization, these classical brackets are found to coincide with the
commutation rules for the quantum Langevin equation, that have been obtained in the
past, by appealing to microscopic conservative quantum models for the friction mechanism.
1. Introduction

Although the Hamiltonian formalism provides a powerful tool for studying general relativity [1], its initial-value problem and the approach to canonical quantization [2], it suffers from severe drawbacks: the space + time split of \((M, g)\) disagrees with the aims of general relativity, and the space-time topology is taken to be \(\Sigma \times \mathbb{R}\), so that the full diffeomorphism group of \(M\) is lost [3,4].

However, as was shown by DeWitt in the sixties [5], it remains possible to use a Poisson-bracket formalism which preserves the full invariance properties of the original theory, by relying upon the work of Peierls [6]. In our paper, whose aims are pedagogical, we begin by describing the general framework, assuming that the reader has been introduced to the DeWitt covariant approach to quantum field theory [5]. Let us therefore consider a gauge field theory with classical action functional \(S\) and generators of infinitesimal gauge transformations denoted by \(R^i_\alpha\). The small disturbances \(\delta \varphi^i\) are ruled by the invertible differential operator

\[
F_{ij} \equiv S_{,ij} + \gamma_{ik} R^k_\alpha \tilde{\gamma}^{\alpha\beta} \gamma_{ji} R^l_\beta,
\]

where \(\gamma_{ij}\) is a local and symmetric matrix which is taken to transform like \(S_{,ij}\) under group transformations, and \(\tilde{\gamma}^{\alpha\beta}\) is a local, non-singular, symmetric matrix which transforms according to the adjoint representation of the infinite-dimensional invariance group (hence one gets \(R^i_\alpha \equiv \gamma_{ij} R^j_i\alpha\) and \(R^i_\alpha \equiv \tilde{\gamma}^{\alpha\beta} R^j_i\beta\), respectively). We are interested in advanced and retarded Green functions \(G^\pm\) which are left inverses of \(-F\), i.e.

\[
G^\pm_{ij} F_{jk} = -\delta^i_k.
\]

Furthermore, the form of \(F_{ij}\) and arbitrariness of Cauchy data imply that \(G^\pm\) are right inverses as well, i.e.

\[
F_{ij} G^\pm_{jk} = -\delta^i_k.
\]

If symmetry of \(F\) is required, one also finds

\[
G^{+ij} = G^{-ji}, \quad G^{-ij} = G^{+ji},
\]

(4)
because in general
\[ G^{\pm ij} - G^{\mp ji} = G^{\pm ik}(F_{kl} - F_{lk})G^{\mp jl}. \] (5)

Thus, the supercommutator function defined as
\[ \tilde{G}^{ij} \equiv G^{+ij} - G^{-ij} \] (6)
is antisymmetric in that \( \tilde{G}^{ij} = -\tilde{G}^{ji} \). These properties show that, on defining \( \delta^\pm_A B \equiv \varepsilon B_i G^{\pm ij} A_j \), one has, on relabelling dummy indices,
\[ \delta^\pm_A B = \varepsilon B_j G^{\pm ji} A_i = \varepsilon A_i G^{\mp ji} B_j = \delta^\mp_B A. \] (7)

These are the reciprocity relations, which express the idea that the retarded (resp. advanced) effect of \( A \) on \( B \) equals the advanced (resp. retarded) effect of \( B \) on \( A \). Another cornerstone of the formalism is a relation involving the Green function \( \hat{G} \) of the operator \(-\hat{F}\), having set \( R_{k\beta} R^k_\alpha \equiv \hat{F}_\beta\alpha \); this is
\[ R^i_\alpha \hat{G}^{\pm \alpha \beta}_i \tilde{\gamma}_{\beta \delta} = R^i_\alpha \hat{G}^{\pm \alpha \delta}_i = G^{\pm ij} \gamma_{jk} R^k_\delta = G^{\pm ij} R_{j\delta}. \] (8)

This holds because, for background fields satisfying the field equations, one finds that
\[ F_{ik} R^k_\alpha = R^\beta_\alpha R_{k\beta} R^k_\alpha = R^\beta_\alpha \hat{F}_\beta\alpha. \] (9)

On multiplying this equation on the left by \( G^{\pm ji} \) and on the right by \( \hat{G}^{\pm \alpha \beta} \) one gets
\[ R^j_\alpha \hat{G}^{\pm \alpha \beta}_j = G^{\pm ji} R^i_\beta, \] (10)
i.e. the desired formula (8) is proved. Moreover, by virtue of (4), the transposed equations
\[ \hat{G}^{\pm \alpha \beta} R^j_\beta = R^\alpha_i G^{\pm ij} \] (11)
also hold. We are now in a position to define the Peierls bracket of any two observables \( A \) and \( B \). First, we consider the operation
\[ D_A B \equiv \lim_{\varepsilon \to 0} \varepsilon^{-1} \delta^\pm_A B, \] (12)
with $D_B A$ obtained by interchanging $A$ with $B$ in (12). The *Peierls bracket* of $A$ and $B$ is then defined by

$$
(A, B) \equiv D_A B - D_B A = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[ \varepsilon A_1 G^+ B_1 - \varepsilon A_1 G^- B_1 \right] = A_1 \tilde{G} B_1 = A_i \tilde{G}^{ij} B_{,j}, \quad (13)
$$

where we have used (7) and (12) to obtain the last expression. Following DeWitt [7], it should be stressed that the Peierls bracket depends only on the behaviour of infinitesimal disturbances.

In classical mechanics, following Peierls [6], we may arrive at the derivatives in (12) and (13) starting from the action functional $S \equiv \int L \, d\tau$ and considering the extremals of $S$ and those of $S + \lambda A$, where $\lambda$ is an infinitesimal parameter and $A$ any function of the path $\gamma$. Next we consider solutions of the modified equations as expansions in powers of $\lambda$, and hence the new set of solutions to first order reads

$$
\gamma'(\tau) = \gamma(\tau) + \lambda D_A \gamma(\tau). \quad (14)
$$

This modified solution is required to obey the condition that, in the distant past, it should be identical with the original one, i.e.

$$
D_A \gamma(\tau) \to 0 \text{ as } \tau \to -\infty. \quad (15)
$$

Similarly to the construction of the above “retarded” solution, we may define an “advanced” solution

$$
\gamma''(\tau) = \gamma(\tau) + \lambda D_A \gamma(\tau), \quad (16)
$$

such that

$$
D_A \gamma(\tau) \to 0 \text{ as } \tau \to +\infty. \quad (17)
$$

From these modified solutions one can now find $D_A \gamma(\tau)$ along the solutions of the unmodified action and therefore, to first order, the changes in any other function $B$ of the field variables, and these are denoted by $D_A B$ and $D_B A$, respectively.
2. Mathematical properties of Peierls brackets

We are now aiming to prove that \((A, B)\) satisfies all properties of a Poisson bracket. The first two, anti-symmetry and bilinearity, are indeed obvious:

\[
(A, B) = -(B, A),
\]

\[
(A, B + C) = (A, B) + (A, C),
\]

whereas the proof of the Jacobi identity is not obvious and is therefore presented in detail. First, by repeated application of (13) one finds

\[
P(A, B, C) \equiv (A, (B, C)) + (B, (C, A)) + (C, (A, B))
\]

\[
= A_{,il} \bar{G}^{il} (B_{,j} \bar{G}^{jk} C_{,k})_{,l} + B_{,jl} \bar{G}^{jl} (C_{,k} \bar{G}^{ki} A_{,i})_{,l} + C_{,kl} \bar{G}^{kl} (A_{,i} \bar{G}^{ij} B_{,j})_{,l}
\]

\[
= A_{,il} B_{,j} C_{,k} \left( \bar{G}^{ij} \bar{G}^{kl} + \bar{G}^{jl} \bar{G}^{ki} \right) + A_{,jl} B_{,j} C_{,k} \left( \bar{G}^{jk} \bar{G}^{il} + \bar{G}^{kl} \bar{G}^{ij} \right)
\]

\[
+ A_{,kl} B_{,j} C_{,k} \left( \bar{G}^{ki} \bar{G}^{jl} + \bar{G}^{ij} \bar{G}^{kl} \right) + A_{,il} B_{,j} C_{,k} \left( \bar{G}^{il} \bar{G}^{jk} + \bar{G}^{jl} \bar{G}^{ki} \right)
\]

\[
+ A_{,jl} B_{,j} C_{,k} \left( \bar{G}^{jl} \bar{G}^{ki} + \bar{G}^{ij} \bar{G}^{kl} \right) + A_{,kl} B_{,j} C_{,k} \left( \bar{G}^{kl} \bar{G}^{ij} \right).
\]

Now the antisymmetry property of \(\bar{G}\), jointly with commutation of functional derivatives: \(T_{,il} = T_{,li}\) for all \(T = A, B, C\), implies that the first three terms on the last equality in (20) vanish. For example one finds

\[
A_{,il} B_{,j} C_{,k} \left( \bar{G}^{ij} \bar{G}^{kl} + \bar{G}^{jl} \bar{G}^{ki} \right) = A_{,il} B_{,j} C_{,k} \left( \bar{G}^{lj} \bar{G}^{ki} + \bar{G}^{ij} \bar{G}^{kl} \right)
\]

\[
= -A_{,il} B_{,j} C_{,k} \left( \bar{G}^{jl} \bar{G}^{ki} + \bar{G}^{ij} \bar{G}^{kl} \right) = 0,
\]

and an entirely analogous procedure can be applied to the terms containing the second functional derivatives \(B_{,jl}\) and \(C_{,kl}\). The last term in (20) requires new calculations because it contains functional derivatives of \(\bar{G}^{ij}\). These can be dealt with after taking infinitesimal variations of Eq. (3), so that

\[
F \delta G^\pm = -(\delta F)G^\pm,
\]
and hence
\[ G^\pm F \delta G^\pm = FG^\pm \delta G^\pm = -\delta G^\pm = -G^\pm (\delta F)G^\pm, \]  
(23)
i.e.
\[ \delta G^\pm = G^\pm (\delta F)G^\pm. \]  
(24)

Thus, the desired functional derivatives of advanced and retarded Green functions read
\[ G^{\pm ij}_{,c} = G^{\pm ia} F_{ab,c} G^{\pm bj} = G^{\pm ia} \left( S_{ab} + R_{a\alpha} R_{b}^{\alpha} \right)_{,c} G^{\pm bj} \\
= G^{\pm ia} S_{abc} G^{\pm bj} + G^{\pm ia} R_{a\alpha,c} R_{b}^{\alpha} G^{\pm bj} + G^{\pm ia} R_{a\alpha} R_{b}^{\alpha,c} G^{\pm bj}. \]  
(25)

In this formula the contractions \( R_{b}^{\alpha} G^{\pm bj} \) and \( G^{\pm ia} R_{a\alpha} \) can be re-expressed with the help of Eqs. (10) and (11), and eventually one gets
\[ G^{\pm ij}_{,c} = G^{\pm ia} S_{abc} G^{\pm bj} + G^{\pm ia} R_{a\alpha,c} \widehat{G}^{\alpha\beta} R_{j}^{\beta} + R_{i}^{i} \widehat{G}^{\pm \beta} R_{b}^{\alpha,c} G^{\pm bj}. \]  
(26)

By virtue of the group invariance property satisfied by all physical observables, the second and third term on the right-hand side of Eq. (26) give vanishing contribution to (20). One is therefore left with the contributions involving third functional derivatives of the action. Bearing in mind that \( S_{abc} = S_{acb} = S_{bca} = \ldots \), one can relabel indices summed over, finding eventually (upon using (4))
\[ P(A, B, C) = A_{,i} B_{,j} C_{,k} \left[ (G^{+ic} - G^{-ic})(G^{+ja} G^{+bk} - G^{-ja} G^{-bk}) \right. \]
\[ + (G^{+jc} - G^{-jc})(G^{+ka} G^{+bi} - G^{-ka} G^{-bi}) \]
\[ + (G^{+kc} - G^{-kc})(G^{+ia} G^{+bj} - G^{-ia} G^{-bj}) \left. \right] S_{abc} \\
= A_{,i} B_{,j} C_{,k} \left[ (G^{+ia} - G^{-ia})(G^{+jb} G^{-kc} - G^{-jb} G^{+kc}) \right. \]
\[ + (G^{+jb} - G^{-jb})(G^{+kc} G^{-ia} - G^{-kc} G^{+ia}) \]
\[ + (G^{+kc} - G^{-kc})(G^{+ia} G^{-jb} - G^{-ia} G^{+jb}) \left. \right] S_{abc} = 0. \]  
(27)
This sum vanishes because it involves six pairs of triple products of Green functions with opposite signs. The Jacobi identity is therefore fulfilled. Moreover, the fourth fundamental property of Poisson brackets, i.e.

\[(A, BC) = (A, B)C + B(A, C)\]  \hspace{1cm} (28)

is also satisfied, because

\[(A, BC) = A, i\tilde{G}^{ik}(BC), k = A, i\tilde{G}^{ik}B, kC + BA, i\tilde{G}^{ik}C, k = (A, B)C + B(A, C).\]  \hspace{1cm} (29)

Thus, the Peierls bracket defined in (13) is indeed a Poisson bracket of physical observables. Equation (28) can be regarded as a compatibility condition of the Peierls bracket with the product of physical observables.

It should be stressed that the idea of Peierls [6] was to introduce a bracket related directly to the action principle without making any reference to the Hamiltonian. This implies that even classical mechanics should be considered as a “field theory” in a zero-dimensional space, having only the time dimension. This means that one deals with an infinite-dimensional space of paths \(\gamma : \mathbb{R} \to Q\), therefore we are dealing with functional derivatives and distributions even in this situation where modern standard treatments rely upon \(C^\infty\) manifolds and smooth structures. Thus, the present treatment is hiding most technicalities involving infinite-dimensional manifolds. In finite dimensions on a smooth manifold, any bracket satisfying (19) and (28) is associated with first-order bidifferential operators [8,9]; in this proof it is important that the commutative and associative product \(BC\) is a local product. In any case these brackets at the classical level could be a starting point to define a \(\ast\)-product in the spirit of non-commutative geometry [10] or deformation quantization [11].

3. The most general Peierls bracket

The Peierls bracket is a group invariant by construction, being defined for pairs of observables which are group invariant, and is invariant under both infinitesimal and finite
changes in the matrices $\gamma_{ij}$ and $\tilde{\gamma}_{\alpha\beta}$. DeWitt [5] went on to prove that, even if independent
differential operators $P_i^\alpha$ and $Q_{i\alpha}$ are introduced such that

$$F_{ij} \equiv S_{ij} + P_i^\alpha Q_{j\alpha}, \quad \tilde{F}_{\alpha\beta} \equiv Q_{i\alpha} R^i_{\beta}, \quad F_{\alpha}^\beta \equiv R^i_{\alpha} P_i^\beta,$$

are all non-singular, with unique advanced and retarded Green functions, the reciprocity
theorem expressed by (7) still holds, and the resulting Peierls bracket is invariant under
changes in the $P_i^\alpha$ and $Q_{i\alpha}$, by virtue of the identities

$$Q_{i\alpha} G^{ij}_{} = G_{\alpha}^\beta R_j^\beta, \quad (31)$$

$$G^{ij} P_j^\beta = R_i^\alpha \tilde{G}_{\alpha\beta}. \quad (32)$$

This is proved as follows. The composition of $F_{ik}$ with the infinitesimal generators of gauge transformations yields

$$F_{ik} R_k^\alpha = P_i^\beta F_{\beta\alpha}, \quad (33)$$

and hence

$$G^{ji} F_{ik} R_k^\alpha = -R_j^\alpha = G^{ji} P_i^\gamma F_{\gamma\alpha}, \quad (34)$$

which implies

$$R_i^\alpha G^{\alpha\beta} = -G^{ji} P_i^\gamma F_{\gamma\alpha} G^{\alpha\beta} = G^{ji} P_i^\beta, \quad (35)$$

i.e. Eq. (32) is obtained. Similarly,

$$R_i^\alpha F_{ij} = F_{\alpha}^\beta Q_{j\beta}, \quad (36)$$

and hence

$$G_{\alpha}^\gamma R_i^\alpha F_{ij} = -Q_{j\alpha}, \quad (37)$$

which implies

$$Q_{i\alpha} G^{ij} = -G_{\alpha}^\gamma R_{\gamma}^k F_{ki} G^{ij} = G_{\alpha}^\beta R_j^\beta, \quad (38)$$

i.e. Eq. (31) is obtained. Now we use the first line of Eq. (7) for $\delta^\pm_A B$, jointly with Eq. (5), so that

$$\delta^\pm_A B - \varepsilon B_{iG} G^{ij} A_{j} - \varepsilon B_{iR^k} G^{\gamma\alpha Q_{i\alpha}} G^{\gamma j} A_{j} - \varepsilon B_{iP^\alpha} G_{\pm ik} Q_{k\alpha} G^{\pm j} A_{j}. \quad (39)$$
Since $B$ is an observable by hypothesis, the first term on the right-hand side of (39) vanishes. Moreover one finds, from (32)

$$G^{\pm ik}P^{\alpha}_i Q^{\mp j}_\alpha = G^{\pm il} R^{j}_\beta G^{\mp \beta \alpha} Q_{l\alpha}. \quad (40)$$

and hence also the second term on the right-hand side of (39) vanishes ($A$ being an observable, for which $R^j_{\beta A, j} = 0$), yielding eventually the reciprocity relation (7). Moreover, the invariance of the Peierls bracket under variations of $P^\alpha_i$ and $Q^\alpha_i$ holds because

$$\delta(\delta^+ A) = \varepsilon B_i \delta G^{\pm ij} A_{, j} = \varepsilon B_i G^{\pm ik}(\delta F_{kl})G^{\pm lj} A_{, j}$$

$$= \varepsilon B_i G^{\pm ik}[(\delta P^\alpha_k)Q_{l\alpha} + P^\alpha_k(\delta Q_{l\alpha})]G^{\pm lj} A_{, j}$$

$$= \varepsilon B_i G^{\pm ik}(\delta P^\alpha_k)Q_{l\alpha} G^{\pm lj} A_{, j} + \varepsilon B_i G^{\pm ik} P^\alpha_k(\delta Q_{l\alpha}) G^{\pm lj} A_{, j}$$

$$= \varepsilon B_i G^{\pm ik}(\delta P^\alpha_k)G^{\pm \beta}_\alpha R^{j}_{\beta} A_{, j} + \varepsilon B_i R^i_{, \gamma} G^{\pm \gamma \alpha}(\delta Q_{l\alpha}) G^{\pm lj} A_{, j} = 0, \quad (41)$$

where Eqs. (31) and (32) have been exploited once more.

4. Quantum dissipative systems

The study of quantum dissipative systems has attracted, over the last decades, a lot of interest, in view of its broad spectrum of applications, ranging from quantum optics through statistical mechanics. The standard approach to deal with quantum dissipation, is based on the idea that the physical origin of dissipation is the interaction of the system with a heat bath, consisting of a large number of degrees of freedom. One considers then some microscopic, conservative model for the heat bath (and its interaction with the system), and tries to recover the macroscopic quantum behavior of the dissipative system alone, by eliminating from the description the degrees of freedom describing the bath. In Ref. [12], it is shown, indeed, that the most general quantum Langevin equation, which is one of the most popular models for dissipation, can be obtained from a simple microscopic model, where the heat bath is described by a set of independent oscillators, linearly coupled to the system of interest.

One may wonder whether it is possible to find a quantization method for dissipative systems, which is based on the macroscopic description of dissipation only, and makes
therefore no use of microscopic models. As is well known, quantization of dissipative systems is by no means straightforward, because in general they admit neither a Lagrangian nor a Hamiltonian formulation. Moreover, even in those special instances where such a formulation can be given, the application of the conventional canonical quantization methods leads to physically incorrect results [13]. In this paper, we show that new classical brackets can be consistently built for dissipative systems, by generalizing the covariant definition of Poisson brackets for Lagrangian systems, discovered long ago by Peierls [6] (see also Refs. [5,14–16]). Our bracket is defined on the infinite-dimensional functional space consisting of all possible classical trajectories, that are accessible to the system under the influence of the random force. It turns out that, when dissipation is present, the random external force has a non-vanishing bracket with the system coordinates, which implies that it cannot be consistently taken to be zero. This seems to be in agreement with the fluctuation-dissipation theorem, which requires fluctuating forces, in the presence of dissipation.

By the correspondence principle, our classical brackets can be eventually quantized, upon substituting them by $\frac{1}{i\hbar}$ times commutators. In this way, we recover the same expressions for the commutators between the system coordinates and the random forces, which were derived from the independent oscillator microscopic model of Ref. [12].

In what follows, we make no attempt at mathematical rigor, and the presentation is totally heuristic. We hope to clarify elsewhere the delicate issues involved in the consideration of Poisson structures in infinite-dimensional functional spaces.

5. The classical brackets

We consider a mechanical system, with coordinates $(q^1, ..., q^n)$, described by an action functional $S = \int dt \mathcal{L}(q^i, \dot{q}^i, t)$, where dot denotes a time derivative. We assume that the Lagrangian is a polynomial of second degree in the velocities $\dot{q}^i$, and that its Hessian $\frac{\partial^2 \mathcal{L}}{\partial \dot{q}^i \partial \dot{q}^j}$ is a constant, non-degenerate matrix $M_{ij}$. We imagine that the system is in contact with a heat bath, and we assume that the influence of the heat bath can be described, effectively, by a mean force, characterized by a bounded memory function
\( \mu_{ij}(t-t') \), and a random force \( F_i(t) \). The time evolution of the system is then described by the following equation of Langevin type:

\[
-\frac{\delta S}{\delta q^i(t)} + \int_{-\infty}^{t} dt' \mu_{ij}(t-t') \dot{q}^j(t') = F_i(t).
\] (42)

Here, \( \delta S/\delta q^i(t) \) denotes the functional derivative of the action \( S \):

\[
\frac{\delta S}{\delta q^i(t)} \equiv -\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^i} \right) + \frac{\partial L}{\partial q^i}.
\] (43)

The original form of the Langevin equation results from the singular limit, where \( \mu_{ij}(t-t') \) approaches \( \gamma_{ij}\delta(t-t') \).

Mimicking the procedure found by Peierls, to compute the Poisson brackets of a conservative Lagrangian system [6], one can consider the effect, on the system evolution, of a small disturbance, produced by an infinitesimal change \( \delta S \) in the form of the action. We consider changes of the form \( \delta S = \epsilon A \), where \( \epsilon \) is an infinitesimal constant and \( A \) is a local functional of the trajectory \( q^j(t) \), taken from a finite time interval. The small disturbance causes an infinitesimal shift \( \delta q^j(t) \) in the trajectory \( q^j(t) \), and it is easy to see that, to first order in \( \epsilon \), \( \delta q^j \) satisfies the following linear integro-differential equation:

\[
(L q^j)_{ij}(t) \equiv -\int dt' \frac{\delta^2 S}{\delta q^i(t) \delta q^j(t')} \delta q^j(t')
+ \int_{-\infty}^{t} dt' \mu_{ij}(t-t') \delta \dot{q}^j(t') = \epsilon \frac{\delta A}{\delta q^i(t)};
\] (44)

where it is understood that all functional derivatives are evaluated along the undisturbed trajectory. When writing the above equation, we have also assumed that the random force does not undergo any variation, to first order in \( \epsilon \). We point out that, by virtue of our assumptions on the Lagrangian, the coefficients of Eq. (44) depend only on the coordinates \( q^i(t) \) and the velocities \( \dot{q}^i(t) \) of the undisturbed trajectory, while they are independent of the accelerations. This is reassuring, because, by virtue of the random external force, the classical trajectories possess, in general, smooth velocities, while the acceleration does not exist, in the ordinary sense of time-derivatives of the velocity [17].
Since the disturbance $A$ is localized in a finite time interval, it makes sense to consider the solution $\delta^+_A q^j(t)$ of Eq. (44) satisfying \textit{retarded} boundary conditions, i.e.

$$\delta^+_A q^j(t) = 0 \quad \text{at early times.}$$

(45)

The non-degeneracy condition for the Hessian $M_{ij}$ of the Lagrangian, ensures that $\delta^+_A q^j(t)$ exists and is unique. We consider also the \textit{advanced} solution $\delta^-_A q^j(t)$:

$$\delta^-_A q^j(t) = 0 \quad \text{at late times}$$

(46)

of the \textit{adjoint} equation of Eq. (44):

$$(L^T \delta^+_A q)_i(t) = - \int dt' \frac{\delta^2 S}{\delta q^j(t') \delta q^i(t)} \delta^+_A q^j(t')$$

$$- \int_t^\infty dt' \mu_{ji}(t' - t) \delta^+_A \dot{q}^j(t') = \epsilon \frac{\delta A}{\delta q^i(t)},$$

(47)

where the superscript $T$ stands for transpose (the transpose coincides with the adjoint, because we are in the real field). If $B$ is another functional of the trajectory, with support in a finite time interval, we define the bracket $\{A, B\}$ as the following expression, involving the quantities $\delta^\pm_A q^j(t)$:

$$\{A, B\} \equiv \frac{1}{\epsilon} \int dt \frac{\delta B}{\delta q^i(t)} (\delta^+_A q^j(t) - \delta^-_A q^j(t)).$$

(48)

It is immediate to verify that the bracket is bilinear and satisfies the Leibniz rule:

$$\{AB, C\} = \{A, C\} B + A\{B, C\},$$

(49)

$$\{A, BC\} = \{A, B\} C + B\{A, C\}.$$

(50)

To verify that the bracket (48) is also antisymmetric and that it satisfies the Jacobi identity, it is useful to re-express it in terms of the Green functions $G^{\pm ij}(t, t'; q)$, defined so that

$$\delta^+_A q^i(t) = \epsilon \int dt' G^{\pm ij}(t, t'; q) \frac{\delta A}{\delta q^j(t')}.$$
The Green functions $G^{\pm ij}(t, t')$ satisfy the following boundary conditions:

\begin{align}
G^{-ij}(t, t'; q) &= 0, \quad \text{for } t \leq t', \\
G^{+ij}(t, t'; q) &= 0, \quad \text{for } t \geq t', \\
\lim_{t \to t' \pm} \frac{\partial G^{\pm ij}}{\partial t}(t, t'; q) &= \mp (M^{-1})^{ij}.
\end{align}

We define now the commutator function $\tilde{G}^{ij}(t, t'; q)$:

\[\tilde{G}^{ij}(t, t'; q) \equiv G^{+ij}(t, t'; q) - G^{-ij}(t, t'; q).\]

Note that, by virtue of the boundary conditions satisfied by the retarded and the advanced Green functions, $\tilde{G}^{ij}(t, t')$ and $\partial_t \tilde{G}^{ij}(t, t')$ are continuous, in the coincidence time limit, $t \to t'$. By using $\tilde{G}^{ij}(t, t')$, we can rewrite the bracket (48) as (cf. (13))

\[\{A, B\} = \int dt \int dt' \frac{\delta B}{\delta q^i(t)} \tilde{G}^{ij}(t, t'; q) \frac{\delta A}{\delta q^j(t')}.
\]

The antisymmetry of the bracket (48) follows from the fact that the commutator function $\tilde{G}^{ij}$ is antisymmetric, as a consequence of the following reciprocity relation, satisfied by the advanced and retarded Green functions:

\[G^{+ij}(t, t'; q) = G^{-ji}(t', t; q).
\]

Before turning to the proof of Eq. (57), it is useful to recall that, with the condensed index notation devised by DeWitt [5], the trajectory $q^i(t)$ is just denoted as $q^i$, with the single Latin index $i$ playing the rôle of both the discrete index, and the time variable. Thus, repeated condensed indices mean a summation on the discrete indices as well as a time integration. For example, Eq. (44), with the condensed notation, is written as

\[L_{ij} \delta_A q^j \equiv (-S_{ij} + \kappa_{ij}) \delta_A q^j = \epsilon_{A,i},
\]

where commas denote functional differentiation, and $\kappa_{ij} \delta_A q^j$ is a symbolic notation for the integral linear operator, depending on the memory function, in Eq. (44). To prove the
reciprocity relation (57), we point out that the Green functions satisfy by definition the equations

\[ L_{ij} G^{-jk} = \delta_i^k, \quad (L^T)_{ij} G^{+jk} = \delta_i^k. \]  

(59)

Upon multiplying the second of the above equations by \( G^{-il} \), we obtain

\[ G^{-il} (L^T)_{ij} G^{+jk} = G^{-il} \delta_i^k = G^{-kl}. \]  

(60)

However, upon using the first of Eq. (59), we can rewrite the l.h.s. of the above equation as

\[ G^{-il} (L^T)_{ij} G^{+jk} = G^{-il} (L)_{ji} G^{+jk} = \delta_j^i G^{+jk} = G^{+lk}. \]  

(61)

Upon comparing the r.h.s. of Eq. (60) and Eq. (61), the reciprocity relation (57) follows.

We can now verify the Jacobi identity. Direct evaluation of the quantity \( \{ \{ A, B \}, C \} + \{ \{ C, A \}, B \} + \{ \{ B, C \}, A \} \), using Eq. (56) shows that:

\[ \{ \{ A, B \}, C \} + \text{c.p.} = A_{si} B_{ij} C_{kl} T^{ijk}, \]  

(62)

where c.p. stands for cyclic permutations of the functionals \( A, B, C \). The terms involving second-order functional derivatives of \( A, B \) and \( C \) cancel exactly, by virtue of the anti-symmetry of \( \tilde{G}^{ij} \). In the above expression, \( T^{ijk} \) denotes the following quantity, built out of functional derivatives of the commutator function:

\[ T^{ijk} = \tilde{G}^{il} \tilde{G}^{jk}, l + \tilde{G}^{jl} \tilde{G}^{ki}, l + \tilde{G}^{kl} \tilde{G}^{ij}, l. \]  

(63)

By using the reciprocity relation, the quantity \( T^{ijk} \) can be written solely in terms of the retarded Green function \( G^{-ij} \), and its functional derivatives. On the other hand, the functional derivatives \( G^{-jk}, l \) can be computed by functionally differentiating the first of Eqs. (59):

\[ L_{ij,l} G^{-jk} + L_{ij} G^{-jk}, l = 0. \]  

(64)

Multiplication by \( G^{+mi} \) then gives

\[ G^{-mk}, l = -G^{+mi} L_{ij,l} G^{-jk} = -G^{-im} L_{ij,l} G^{-jk}, \]  

(65)
where in the last passage use has been made again of the reciprocity relation. By using this expression into Eq. (63), it is possible to verify that:

\[ T^{ijk} = (G^{-li}G^{-mj}G^{-nk} + \text{c.p.})(L_{mn,l} - L_{nm,l}), \]

(66)

where c.p. stands for cyclic permutations of the indices \( ijk \). It is easy to check now that \( T^{ijk} \) vanishes. Indeed, in view of Eq. (58), we see that the quantity between the brackets of the r.h.s. is equal to:

\[ S_{mn,l} - S_{nml} + \kappa_{mn,l} - \kappa_{nm,l}. \]

The terms involving third-order functional derivatives of the action functional cancel each other, because functional derivatives commute with each other. On the other hand, the quantities \( \kappa_{ij} \) are independent of the trajectories \( q^j \), and hence their functional derivatives vanish identically. It follows then that \( T^{ijk} \) vanishes, and hence the Jacobi identity holds.

We have therefore shown that it is possible to define a bracket on the space of all trajectories. We can now evaluate the brackets satisfied by the random force \( F_i(t) \). To do this, we can use the expression for \( F_i(t) \), provided by the Langevin equation (42). In this way, we find

\[ \{F_i, q^k\} = \{-S_{i,i} + \kappa_{ij} q^j, q^k\} = L_{ij}\{q^j, q^k\} = L_{ij}(G^{+jk} - G^{-jk}) \]

\[ = (L - L^T)_{ij} G^{+jk} + (L^T)_{ij} G^{+jk} - L_{ij} G^{-jk} \]

\[ = (S_{ij} - S_{jii} + \kappa_{ij} - \kappa_{ji}) G^{-kj} = (\kappa - \kappa_T)_{ij} G^{-kj}, \]

(67)

and

\[ \{F_i, F_j\} = \{-S_{ii} + \kappa_{ik} q^k, -S_{ij} + \kappa_{jl} q^l\} = L_{ik} L_{jl}\{q^k, q^l\} \]

\[ = L_{ik} L_{jl}(G^{+kl} - G^{-kl}) = L_{ik} L_{jl}(G^{-lk} - G^{-kl}) = L_{ij} - L_{ji} \]

\[ = S_{ij} - S_{jii} + \kappa_{ij} - \kappa_{ji} = (\kappa - \kappa_T)_{ij}. \]

(68)

It is useful to write the above bracket in plain form:

\[ \{F_i(t), F_j(t')\} = \frac{d\mu_{ij}}{dt} (t - t') + \frac{d\mu_{ji}}{dt} (t' - t). \]

(69)
We see from these equations that, when friction is present, the external forces have non-vanishing brackets, which implies that they cannot be set to zero.

Using Eq. (56), it is possible to verify that the equal-time brackets for the coordinates \( q^i(t) \) and the momenta \( p_i(t) = \partial L / \partial \dot{q}^i(t) \) have the familiar canonical form:

\[
\{ q^i(t), q^j(t) \} = 0, \quad (70) \\
\{ q^i(t), p_j(t) \} = \delta_j^i, \quad (71) \\
\{ p_i(t), p_j(t) \} = 0. \quad (72)
\]

The verification is similar to the conservative case [5], because the memory function contributes to \( \tilde{G}^{ij}(t, t') \) only to order \((t - t')^3\). This can be seen by inserting the expansions of \( G^{\pm ij}(t, t') \) in powers of \((t - t')\) into Eqs. (59), and exploiting the boundedness of the memory function.

6. Concluding remarks

The Peierls-bracket formalism is equivalent to the conventional canonical formalism when the latter exists. The proof can be given starting from point Lagrangians, as is shown in Ref. [5]. Current applications of Peierls brackets deal with string theory [18,19], path integration and decoherence [20], supersymmetric proof of the index theorem [21], classical dynamical systems involving parafermionic and parabosonic dynamical variables [22], while for recent literature on covariant approaches to a canonical formulation of field theories we refer the reader to the work in Refs. [23-30].

In the infinite-dimensional setting which, strictly, applies also to classical mechanics, as we stressed at the end of Sec. 2, we hope to elucidate the relation between a covariant description of dynamics as obtained from the kernel of the symplectic form, and a parametrized description of dynamics as obtained from any Poisson bracket, including the Peierls bracket.

In the second part of our paper, we have constructed a set of brackets for a classical dissipative system, described by a Langevin equation, with an arbitrary memory function.
The brackets satisfy the usual properties enjoyed by Poisson brackets of Hamiltonian systems. It is worth pointing out the essential rôle played, in our treatment, by external random forces. When dissipation occurs, they have non-vanishing brackets with the system coordinates, and thus cannot be consistently set to zero. As a result, our brackets are a priori defined on the infinite-dimensional functional space of all possible trajectories, accessible to the system under the action of arbitrary external forces. However, in the absence of friction, when the dynamics is conservative, the brackets can be restricted onto the finite-dimensional classical phase-space, spanned by the solutions of the classical equations of motion, with no external forces. In this case, our construction reproduces Peierls’ covariant definition of the Poisson brackets, for dynamical systems admitting an action principle. Within the framework of conservative systems, the possibility of extending the brackets from the phase space to the space of all trajectories, was considered some-time ago [14], and our brackets coincide with those of Ref. [14], in the absence of friction.

Quantization can be carried out according to the traditional procedure, by replacing the classical brackets with commutators [31,32]. The resulting commutation rules coincide with those that are obtained in standard treatments of quantum dissipation, by making recourse to a microscopic model for the heat bath, after elimination of the bath degrees of freedom (see, for example, Ref. [12] and references therein).

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