Relating $c < 0$ and $c > 0$ Conformal Field Theories

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ABSTRACT

A ‘canonical mapping’ is established between the $c = -1$ system of bosonic ghosts and the $c = 2$ complex scalar theory and, a similar mapping between the $c = -2$ system of fermionic ghosts and the $c = 1$ Dirac theory. The existence of this mapping is suggested by the identity of the characters of the respective theories. The respective $c < 0$ and $c > 0$ theories share the same space of states, whereas the spaces of conformal fields are different. Upon this mapping from their $c < 0$ counterparts, the $(c > 0)$ complex scalar and the Dirac theories inherit hidden nonlocal $sl(2)$ symmetries.
1. Introduction

Two dimensional conformal field theories with central charge \( c < 0 \) have recently attracted much attention, especially in the context of condensed matter applications such as unusual (paired) Quantum Hall states \[1,2,3\], disordered systems \[4,5,6\], including the plateau transition in the Integer Quantum Hall effect \[9\] and polymer theories \[10\]. All of these systems involve the so called \((\beta, \gamma)\) system of conformal weight \((1/2, 1/2)\) “bosonic ghost”, of central charge \( c = -1 \), or a version of the fermionic ghost \((\xi, \eta)\) system of central charge \( c = -2 \). (Both these systems were first introduced in the context of string theories \[11\]).

Both are non-unitary conformal field theories, the space of states created by the \( c < 0 \) Virasoro algebra containing negative norm states. This is a complication for various reasons. For example, in the context of the mentioned Quantum Hall systems, the spectrum of the physical Hamiltonian of the edge states of the system is given in terms of the Virasoro zero mode \( L_0 \). Clearly, the Hilbert space of states on which it acts must be positive definite \[12\]. In the context of disordered systems, the bosonic ghost system interacts with a Dirac fermion. Certain exactly marginal interactions can be treated exactly \[4,7\]. However, in general the system flows off to strong coupling. Very little intuition is available to help access the strong coupling behavior, especially because of the non-unitary nature of the bosonic ghost part of the theory. Therefore, a better understanding, and a better intuition about the properties of the \( c < 0 \) ghost systems is highly desirable.

In this paper we show that there is a simple relationship (i) between the fermionic ghost theory at \( c = -2 \) and the free Dirac theory at \( c = 1 \), as well as (ii) between the \( c = -1 \) bosonic ghost theory and a \( c = 2 \) theory of a complex scalar field. First, we observe that the characters of the respective theories, i.e. (i) of \( c = -1 \) and \( c = +2 \) as well of (ii) of \( c = -2 \) and of \( c = +1 \), are identical, in the various sectors [periodic and antiperiodic]. This suggests a deeper connection between the respective theories. Indeed there is. We establish a ‘canonical’ mapping from one theory to the other. This mapping implies that the space of states of the \( c < 0 \) theories is identical to that of the respective \( c > 0 \) theories, and so are the \( L_0 \)-generators (upto a shift in ground state energy due to different normal ordering prescriptions). Therefore, we have two different conformal field theories which share the same spectrum of \((L_0 - c/24)\) and the same space of states (implying that the characters are the same). The spaces of conformal fields of the respective \( c < 0 \) and \( c > 0 \) theories are different, and the fields are nonlocally related to each other. The Virasoro generators \( L_n \) with \( n \neq 0 \) are not simply related upon this mapping.

This mapping allows us to establish the existence of hidden non-abelian symmetries in the theories with \( c > 0 \). These are not visible in the usual way from the Lagrangian, since there is no corresponding Noether current. The bosonic ghost theory is known to possess an \( sl(2) \) Kac-Moody current algebra symmetry at level \( k = -1/2 \). Our mapping establishes a \( sl(2) \) current algebra symmetry also for the \( c = 2 \) theory of a complex scalar field, for half-integer moding. A complex scalar theory has an obvious \( U(1) \) symmetry, and an associated global charge. However, its Noether current does not give rise to a Kac-Moody current algebra symmetry, because of logarithms in its correlation functions. But our mapping shows that the global \( U(1) \) charge is nevertheless the zero mode of a hidden
\( U(1) \) KM current algebra. Moreover, we show that there are two additional sets of current algebra generators, that extend this \( U(1) \) current algebra symmetry to an \( sl(2) \) current algebra. Already the global \( sl(2) \) symmetry, present for integer and half-integer moding, is interesting. No such symmetry can be identified from the Lagrangian of the \( c = 2 \) theory of a complex scalar field. The two additional charges, besides the \( U(1) \) charge, are non-local when expressed in terms of the complex scalar field variables. Physically, these two generators effect continuous rotations of the complex scalar field \( \phi(x) \) into its charge conjugate \( \phi^\dagger(x) \). In the Hilbert space of states, the symmetry corresponds to global rotations between positive and negative frequency modes. In addition, we observe that the space of states of the the bosonic ghost and complex scalar field theories has a structure isomorphic to the multiplets of the \( SU(2) \)-level-1 Yangian \(^{13,14,15}\).

The fermionic theory at \( c = -2 \), on the other hand, is known \(^{16,17}\), to possess a global \( sl(2) \) symmetry. No non-abelian symmetry can be identified from the Lagrangian for the Dirac fermion. Using our mapping, the theory of a Dirac fermion (at \( c = 1 \)) inherits a hidden \( sl(2) \) symmetry from the \( c = -2 \) theory. The origin of this symmetry is the charge conjugation (“particle-hole”) symmetry of the Dirac theory. One of the non-abelian global charges is the ordinary Dirac charge. The remaining two non-abelian charges are non-local in the Dirac field variables. Physically, those two generators effect continuous rotations of the Dirac field \( \psi(x) \) into its charge conjugate \( \psi^\dagger(x) \). (This may be viewed as a one-species version of the hidden \( SU(2) \) symmetry, discovered in the Hubbard model \(^{17}\). In that model this symmetry is local.)

The organization of this paper is as follows: In section 2, we define the theories in a language that will be useful for our discussion and exhibit their respective characters. We see that the characters of the \( c < 0 \) bosonic and fermionic theories are identical to those of the corresponding \( c > 0 \) bosonic and fermionic theories for periodic and antiperiodic boundary conditions on the torus. In section 3, we give the exact relationship between these theories using a ‘canonical mapping’, showing that the spaces of states of these theories are identical. In section 4, we discuss the hidden non-abelian \( sl(2) \) global and current algebra symmetries that the \( c > 0 \) theories, Dirac and the complex scalar, inherit from this mapping and we derive the corresponding non-local generators. Some concluding remarks are made in section 5. (In Appendix A we summarize the results for the characters in the Ramond and Neveu-Schwarz sectors for the \( c < 0 \) and \( c > 0 \) bosonic and fermionic theories in question, and in Appendix B we discuss the relationship of the bosonic ghost characters with those derived by Kac and Wakimoto for the \( sl(2)_{-1/2} \) current algebra.)

2. Conformal field theories at \( c = \pm 1 \) and \( c = \pm 2 \)

**Dirac Fermion (\( c = +1 \)):**

Let us start considering a free Dirac fermion described by the action

\[
S_\psi = \int d\tau \int dx \{ \psi_L^\dagger [\partial_\tau + i\partial_x] \psi_L + \psi_R^\dagger [\partial_\tau - i\partial_x] \psi_R \} 
\]
on a torus of size $\beta$ in the “euclidean time” direction $\tau$ and of size $l$ in the “space” direction $x$. We will only consider the, say, L-moving part, and drop the subscript $L$, i.e. we write $\psi_L \to \psi$. We will do this for all theories considered in this paper. For the chiral field, we consider a general twisted boundary condition in the functional integral:

$$
\psi(\tau, x + l) = (-1)e^{2\pi i \lambda'}\psi(\tau, x), \quad \psi(\tau + \beta, x) = (-1)e^{2\pi i \mu}\psi(\tau, x).
$$

Note that when $\mu = \lambda' = 0$ we have antiperiodic b.c.’s along the spatial and temporal cycles, “natural” for a fermionic theory. The Hamiltonian can be expressed in terms of momentum modes of the fermion operators

$$
\psi(\tau, x) = \sum_{s \in \mathbb{Z} + 1/2 - \lambda'} e^{-(\tau + ix)2\pi s/l} \psi_s, \quad \psi^\dagger(\tau, x) = \sum_{s \in \mathbb{Z} + 1/2 - \lambda'} e^{(\tau + ix)2\pi s/l} \psi^\dagger_s,
$$

where $p = \frac{2\pi}{l}s$ are the momenta of the system on a space of size $l$:

$$
H_\psi = \left(\frac{2\pi}{l}\right) \sum_s s\psi^\dagger_s \psi_s.
$$

We write the general Dirac character as

$$
\chi_{c=+1}(q; \mu; \lambda') = \text{Tr} e^{-\beta H_\psi} = q^{-1/24} q^{(\lambda'^2)/2} \prod_{n=0}^{\infty} (1 + e^{2\pi i \mu} q^{n+1/2 - \lambda'})(1 + e^{-2\pi i \mu} q^{n+1/2 + \lambda'}).
$$

where the trace is taken over the Hilbert space of the fermion modes (the L-moving part of the theory), acting on the Fermi sea vacuum [giving rise to the prefactor $q^{-1/24}$]. We have set $q = e^{-2\pi \beta/l}$ and $e^{-2\pi i \mu}$ is a “fugacity” which keeps track of the $U(1)$ charge of the fermion state (with respect to filled Fermi sea). The parameter $\lambda'$ plays the role of a simple “phase shift” of the Dirac fermion, and shifts the single particle energy levels occupied by the fermions by an amount $(2\pi/l)\lambda'$. In particular, it allows us to continuously interpolate between antiperiodic and periodic boundary conditions in the spatial direction. The prefactor $q^{(\lambda'^2)/2}$ arises from the shift of the ground state energy of the filled Fermi sea. For $\lambda' = 1/2$, one has periodic spatial boundary conditions corresponding to the Ramond sector. From the character in (2.1), we see that, in this case, the vacuum is two fold degenerate, the two ground states having $U(1)$ charges $Q = 0, +1$, as seen from the powers of the fugacities. The zero modes appear since for $\lambda' = 1/2$ there is a single particle state sitting right at the Fermi level, and therefore, occupying it does not cost any energy. The vacuum with periodic boundary conditions (the Ramond vacuum) has conformal weight which is an amount of $\Delta = 1/8$ above the conformal weight $= 0$ of the ground state with antiperiodic boundary conditions (Neveu-Schwarz vacuum).

**Bosonic ghost ($c = -1$):**

The graded supersymmetry approach to random systems of recent interest is based on the observation that the Dirac partition function of (2.1) (corresponding to a first
derivative action and hamiltonian) is exactly cancelled by a bosonic counterpart. What is needed for this cancellation to occur is a theory of bosons with first order action having the same boundary conditions along both cycles of the torus as the fermionic counterpart. For $\mu = 0$ this implies anti-periodic b.c.’s along the time direction, ‘unnatural’ for bosons. A suitable bosonic action is that of the bosonic ghost system (known as the “beta-gamma” system in string theory[11]). Since this cancellation occurs for arbitrary inhomogeneous quadratic potentials (“random potentials”), it can be used to represent averages over those random potentials in terms of a theory of interacting Dirac fermions and bosonic ghosts[13,4].

The first order action (for the L-moving part) is given [19] in terms of two real variables $\beta_1(\tau, x), \beta_2(\tau, x)$:

$$S_\beta = \frac{-i}{2} \int d\tau \int dx \epsilon^{ab} \beta_a [\partial_\tau + i \partial_x] \beta_b$$

$\epsilon^{12} = -\epsilon^{21} = 1$, is the antisymmetric tensor. (Often the notation $\beta_1 \to \gamma$ and $i\beta_2 \to \beta$ is used [11]). The action is invariant under the symplectic group $Sp(2)$, under which the fields transform as a doublet. We define the linear combinations

$$\beta = \frac{1}{\sqrt{2}} [\beta_1 + i \beta_2], \quad \bar{\beta} = \frac{1}{\sqrt{2}} [\beta_1 - i \beta_2].$$

The hamiltonian, as obtained from the canonical stress tensor, can be written in terms of momentum modes,

$$\beta(\tau, x) = \sum_{s \in Z + \frac{1}{2} - \lambda'} e^{-(\tau + ix)2\pi s/l} \beta_s, \quad \bar{\beta}(\tau, x) = \sum_{s \in Z + \frac{1}{2} - \lambda'} e^{(\tau + ix)2\pi s/l} \bar{\beta}_s$$

satisfying canonical commutations relations $[\beta_s, \bar{\beta}_t] = \delta_{s,t}$, as:

$$H_\beta = (2\pi/l) \sum_s s \bar{\beta}_s \beta_s.$$

Since the momentum mode index $s$ can take on negative values, the hamiltonian is normal ordered on a vacuum satisfying $\beta_s|0 >= \bar{\beta}_{-s}|0 >= 0$ for $s \geq 0$ (operators annihilating this vacuum are moved to the right). The normal ordered hamiltonian is bounded from below. [In the context of disordered systems, this choice of ghost vacuum is dictated by the convergence of the bosonic part of the functional integral that cancels the corresponding fermionic one.] Since the partition function factorizes it is sufficient to consider a single momentum $p = \frac{2\pi}{l} s$. Consider the bosonic partition function for a single mode (excluding the ground state energy)

$$Z_{\beta_s} \equiv \text{Tr} e^{-[\beta (2\pi/l)s - 2\pi i(\frac{1}{2} + \mu)] : \bar{\beta}_s \beta_s :}$$

where the trace is taken over the bosonic Fock space of the canonical operators. Note that for $\mu = 0$ we have inserted $(-1)^N$ where $N$ is the boson number. This insertion corresponds
to antiperiodic b.c.’s in euclidean time $\tau$ in the functional integral formulation, in which the bosonic fields satisfy the general twisted b.c.’s:

$$\beta(\tau, x + l) = (-1) e^{2\pi i \lambda'} \beta(\tau, x), \quad \beta(\tau + \frac{1}{T}, x) = (-1) e^{2\pi i \mu} \beta(\tau, x)$$

where we denote the inverse temperature by $\beta = 1/T$ [ No confusion should arise between the inverse temperature and bosonic ghost field.]

This yields the following chiral partition function

$$\chi_{c=-1}(q, \mu, \lambda') = q^{-(-1/24)} q^{-\frac{1}{8} + \frac{1}{2}(\lambda')^2/2} \prod_{n=0}^{\infty} \frac{1}{(1 + e^{2\pi i \mu q^{n+1/2}})} \frac{1}{(1 + e^{-2\pi i \mu q^{n+1/2}})}$$

(2.2)

[The prefactor of $q^{-(-1/24)} q^{-\frac{1}{8} + \frac{1}{2}(\lambda')^2/2}$ arises from normal ordering on the ghost vacuum.] For half-integer moding ($\lambda' = 0$), the combined theory of a chiral Dirac and a bosonic ghost field, relevant in the context of disordered systems, possesses graded supersymmetry, reflected in the fact that the product of the two partition functions is identically unity on the torus (Appendix A). The case of integer moding, represented e.g. by a phase shift of $\lambda' = 1/2$ requires extra discussion. In this case, the cancellation of the fermionic partition function by the bosonic one is achieved by an extra single bosonic zero mode $\beta_0$ (annihilating the vacuum). The part of the partition function coming from the bosonic zero mode needs to be regularized:

$$Z_{\beta_0} = \frac{1}{(1 + e^{2\pi i \mu})} = \lim_{\eta \to 0^+} \sum_{n=0}^{\infty} (-1)^n e^{n(2\pi i \mu - \beta \eta)}.$$

This cancels precisely the contribution from the Dirac zero mode, $\psi_0$, which is

$$Z_{\psi_0} = (1 + e^{2\pi i \mu}).$$

Note that the expansion in powers of the fugacity shows that the bosonic ghost ground state has an infinite degeneracy, arising from the zero mode. [The zero-mode Hilbert space is to be thought of as a sum of the two irreducible lowest weight representations of the non-compact group $SU(1, 1)$ of a single boson (see Appendix B for a brief discussion)].

**Complex scalar field ($c = +2$):**

Consider a theory of two real scalar fields, $\varphi_j$ ($j = 1, 2$), or equivalently, a complex scalar field, $\phi^\dagger, \phi$, with action

$$S = \frac{1}{8\pi} \sum_{j=1,2} \int d\tau \int dx (\partial \varphi_j)^2 = \frac{1}{8\pi} \int d\tau \int dx (\partial \phi^\dagger) (\partial \phi)$$

(2.3)

where

$$\phi(\tau, x) = \frac{1}{\sqrt{2}} [\varphi_1(\tau, x) + i \varphi_2(\tau, x)], \quad \phi^\dagger(\tau, x) = \frac{1}{\sqrt{2}} [\varphi_1(\tau, x) - i \varphi_2(\tau, x)].$$
As usual, from the equations of motion, $\phi = \phi_L(z) + \phi_R(z^*)$ and we focus on $\phi_L \to \phi$ (dropping subscript $L$). First, consider antiperiodic boundary conditions in space. There is no zero mode and the usual momentum space mode expansion reads

$$\varphi_j(\tau, x) = \sum_{s \in \mathbb{Z} + \frac{1}{2}, s > 0} \frac{1}{\sqrt{s}} \{ b_{j,s} e^{-(\tau + ix)(2\pi/l)s} + b_{j,s}^\dagger e^{+(\tau + ix)(2\pi/l)s} \}, \quad (j = 1, 2)$$

where

$$[b_{j_1,s_1}, b_{j_2,s_2}^\dagger] = \delta_{j_1,j_2} \delta_{s_1,s_2}, \quad (s_1, s_2 > 0)$$

are oscillator modes, quantized on the usual vacuum $b_{j,s}|0 > = 0$, and occupying single particle levels with positive momenta $p = \frac{2\pi}{l}(n + 1/2), n = 0, 1, 2, 3, ...$

So far we have used antiperiodic b.c.’s. However, for the complex scalar we may define more general twisted boundary conditions.

$$\phi(\tau, x + l) = (-1)e^{2\pi i \lambda'}\phi(\tau, x).$$

When $-1/2 < \lambda' < 1/2$, this implies the following mode expansion of the complex scalar field. In the formula below, the mode indices of the canonical operators take on values $s \in \mathbb{Z} + 1/2 - \lambda'$ for $B_{+,s}$ and values $t \in \mathbb{Z} + 1/2 + \lambda'$ for $B_{-,t}$:

$$\phi(\tau, x) = \sum_{s > 0} \frac{B_{+,s}}{\sqrt{s}} e^{-(\tau + ix)(2\pi/l)s} + \sum_{t > 0} \frac{B_{-,t}^\dagger}{\sqrt{t}} e^{+(\tau + ix)(2\pi/l)t}$$

and

$$\phi^\dagger(\tau, x) = \sum_{t > 0} \frac{B_{-,t}}{\sqrt{t}} e^{-(\tau + ix)(2\pi/l)t} + \sum_{s > 0} \frac{B_{+,s}^\dagger}{\sqrt{s}} e^{+(\tau + ix)(2\pi/l)s}$$

($B_{\pm,s} = [b_{1,s} \pm ib_{2,s}]/\sqrt{2}$, when $\lambda' = 0$.) The Hamiltonian is

$$H = \frac{2\pi}{l} \left\{ \sum_{s > 0} s \left[ B_{+,s}^\dagger B_{+,s} + 1/2 \right] + \sum_{t > 0} t \left[ B_{-,t}^\dagger B_{-,t} + 1/2 \right] \right\}.$$  (2.6)

The partition function factorizes so that for each mode, say $B_{+,s}$ we define (excluding the ground state energy)

$$Z_{B_{+,s}} \equiv \text{Tr} \exp\{ -[\beta \frac{2\pi}{l}s - 2\pi i(\frac{1}{2} + \mu)]B_{+,s}^\dagger B_{+,s} \}$$

which gives the character for the complex scalar

$$\chi_{c=+2}(q, \mu, \lambda') = q^{-(-2/24)} q^\lambda \prod_{n=0}^{\infty} \frac{1}{1 + e^{2\pi i \mu q^{n+1/2 - \lambda'}}} \frac{1}{1 + e^{-2\pi i \mu q^{n+1/2 + \lambda'}}},$$

(2.7)
[The prefactor of $q^{-(2/24)} q^{1/2 - (\lambda')^2/2}$ arises from the ground state energy of the oscillators.] We see that this is the character of the bosonic ghost, up to an overall power of $q$. As we move $\lambda'$ from 0 to $\lambda' = 1/2$ a single bosonic zero mode $B_{+0}$ occurs. Thus, for integer mode ($\lambda' = 1/2$), the character is a product of that of a complex scalar field, at zero boson radius, and a decoupled single oscillator zero mode.

\textit{Fermionic ghost and Symplectic fermion ($c = -2$):}

Finally we consider the fermionic analog of the complex scalar theory, known also as the “symplectic fermion” for reasons that will be obvious shortly. This theory is related to the conformal weight $(0,1)$ fermionic ghost system $(\xi, \eta)$ with $c = -2$ system in string theory. We follow here Ref. [3], where details relating the $(\xi, \eta)$ system to the symplectic fermions can also be found. [A similar analysis of this theory can be found in [3].] The symplectic fermions are described by a Grassman functional integral over two fermion fields, $\Phi^a$, $a = \pm$, based on the second derivative action

$$S = \frac{1}{4\pi} \int d\tau \int dx \epsilon_{ab} \partial_\mu \Phi^a \partial^\mu \Phi^b$$

where $\epsilon^{+-} = -\epsilon^{-+}$, $\epsilon_{ab} = -\epsilon^{ab}$. The action is invariant under symplectic transformations, the field transforming as a doublet. Just as for the bosonic scalar field, the general solution of the equation of motion shows that the field is a sum of analytic and anti-analytic parts ($z = \tau + ix$):

$$\Phi^a(\tau, x) = \Phi^a_L(z) + \Phi^a_R(z^*)$$

Invariance of the action under shift by a constant Grassman number, $\delta \Phi^a = \theta^a$, results in (anti-) analytic conserved fermionic currents

$$\mathcal{J}^a_L(z) \equiv i \partial_z \Phi^a, \quad \mathcal{J}^a_R(z^*) \equiv i \partial_{z^*} \Phi^a.$$  

From the action one obtains the canonical stress tensor,

$$T_L(z) = -\frac{1}{2} \epsilon_{ab} (\partial_z \Phi^a)(\partial_z \Phi^b), \quad T_R(z^*) = -\frac{1}{2} \epsilon_{ab} (\partial_{z^*} \Phi^a)(\partial_{z^*} \Phi^b)$$

which yields the hamiltonian

$$H = H_L + H_R$$

where

$$H_L = -\frac{1}{2} \int dx [(\partial_x \Phi^+_L)(\partial_x \Phi^-_L) - (\partial_x \Phi^-_L)(\partial_x \Phi^+_L)]$$  \hspace{1cm} (2.8)

(and a similar expression for the R-moving part).

First consider a field $\Phi^a_L$ which is antiperiodic on finite space (size $l$). Using an expansion in Fourier modes, and dropping the subscript $L$

$$\Phi^\pm(\tau, x) = \sum_{s \in \mathbb{Z} + 1/2} \frac{\mathcal{J}^\pm}{s} e^{2\pi i s(\tau + ix)}$$  \hspace{1cm} (2.9)
(due to antiperiodicity the zero mode is absent) where the "fermionic currents" in (2.9) satisfy the following anticommutation relations:

$$\{J^a_{s_1}, J^b_{s_2}\} = s_1 \epsilon^{ab} \delta_{s_1 + s_2, 0}, \quad (a, b = \pm). \quad (2.10)$$

Equation (2.10) remains valid for general moding, using modes $s \in \mathbb{Z} + 1/2 \pm \lambda'$ for $\Phi^\pm$, and so does (2.9) as long as $-1/2 < \lambda' < 1/2$. This corresponds to twisted b.c.'s

$$\Phi^\pm(\tau, x + l) = (-1)^{\mp 2\pi i \lambda'} \Phi^\pm(\tau, x)$$

The theory is quantized on a vacuum $|0\rangle$ satisfying

$$J^a_s |0\rangle = 0, \quad \text{for} \ s \geq 0. \quad (2.11)$$

The character for the general twisted case reads [16]

$$\chi_{c=-2}(q, \mu, \lambda') = \text{Tr} \ e^{-\beta H} = q^{-(2/24)} q^{-(1/2-\lambda')^2/2} \prod_{n=0}^\infty (1 + e^{2\pi i \mu q^{n+1/2}-\lambda'})(1 + e^{-2\pi i \mu q^{n+1/2}+\lambda'}). \quad (2.12)$$

As we move $\lambda'$ from 0 to $\lambda' = 1/2$, an extra fermionic zero mode occurs, whose contribution to the partition function is analogous to the one for the Dirac fermion.

We see from the expressions (2.2), (2.7) and (2.12) that the bosonic and fermionic characters for the $c < 0$ and $c > 0$ theories are identical, respectively, up to multiplicative power of $q$. This is no accident. In fact, in the following section, we show that the identity of the characters is related to an identity of the space of states of the respective theories. There is a 'canonical transformation' relating the respective fermionic and bosonic operators, of the $c > 0$ and $c < 0$ theories, in momentum (mode) space.

In Appendix A we summarize the respective characters, described above, for the special cases of half-integer ($\lambda' = 0$) and integer ($\lambda' = 1/2$) moding. For these cases the $c < 0$ and $c > 0$ fermionic and bosonic characters are completely identical, respectively [20], including the multiplicative power of $q$.

3. Mapping between theories with $c > 0$ and $c < 0$:

The fact that the characters of the $(\bar{\beta}, \beta)$ system and the symplectic fermion system are identical to those associated with the complex scalar and the Dirac fermions, respectively, in both the half-integer and the integer moded sectors suggests the existence of some 'canonical mapping' between the two theories at the level of the operators that create the space of states, used when performing the trace giving rise to the respective characters, $\text{Tr} \ q^{L_0-c/24}$ ($H = 2\pi (L_0 - c/24)$ is the hamiltonian).

That is, we have $L_0 - c/24$ and $L_0' - c'/24$, for the $c > 0$ theory and the $c' < 0$ theory, respectively, in a sector with the same b.c. And there is a set of eigenstates for both of these. The characters are the same, when we form the trace of $q^{L_0-c/24}$ over one set of eigenvectors and the trace of $q^{L_0'-c'/24}$ over the other set of eigenvectors. Since the trace is
invariant under basis change, one may expect a similarity transformation between the two sets of eigenstates. Thus one would expect a ‘canonical mapping’ between the operators of the $c > 0$ theory and those of the $c' < 0$ theory.

We now construct explicitly this canonical mapping relating the $c > 0$ and $c < 0$ theories. It maps the Hamiltonians $H = (2\pi/l)[L_0 - c/24]$ into each other. The other Virasoro generators $L_n$ for the two theories are not simply related, and the space of fields of the two theories is different.

**3.1. Bosonic theories: Canonical mapping between a complex scalar field and bosonic ghost**

We use the notations of section (2). The bosonic ghost system may be viewed as arising from an attempt to construct a theory of conformal fields in position space, whose Fourier modes are precisely the oscillator modes of the complex scalar field themselves (without the square-roots in (2.4), (2.5)). This is not possible. It becomes however possible, when the following canonical transformation is performed on the oscillator modes of the complex scalar $s \in \mathbb{Z} + 1/2 - \lambda'$,

\[
B^\dagger_{+,s} \equiv \bar{\beta}_s \quad (= \sqrt{s} \phi^\dagger_s) \\
B_{+,s} \equiv \beta_s \quad (= \sqrt{s} \phi_s) \\
B^\dagger_{-,t} \equiv \beta_{-t} \quad (= \sqrt{t} \phi_{-t}) \\
B_{-,t} \equiv -\bar{\beta}_{-t} \quad (= \sqrt{t} \phi^\dagger_{-t})
\]  

(3.1)

where $\phi_s (s \in \mathbb{Z} + 1/2 - \lambda')$ are the Fourier modes of the complex scalar. The so-defined $\beta, \bar{\beta}$ system is nothing but the bosonic ghost theory. Note a crucial minus sign in the last equation of (3.1), which ensures canonical commutation relations for the $\beta$-modes:

\[ [\beta_{s_2}, \bar{\beta}_{s_1}] = \delta_{s_1,s_2}, \quad \text{for all } s_1, s_2 \in \mathbb{Z} + 1/2 - \lambda' \quad \text{(positive and negative)} \]

The “ghost vacuum”, derived this way, satisfies,

\[ \beta_s |0> = 0, \quad \bar{\beta}_{-s} |0> = 0, \quad s > 0. \]

( This is analogous to $\phi_s |0> = \phi^\dagger_{-t} |0> = 0, \quad \text{for } s > 0, t > 0. )

Using the canonical transformation (3.1) we see that the hamiltonians, $H = (2\pi/l)[L_0 - c/24]$ of the two theories are identical (giving the results displayed in Section (2)). This is easily seen by re-writing the complex scalar hamiltonian, of section (2), in terms of the bosonic ghost modes, using (3.1)

\[
H^{(\phi,\phi^\dagger)}/(2\pi/l) = \sum_{s \in \mathbb{Z} + 1/2 - \lambda', s > 0} s \bar{\beta}_s \beta_s - \sum_{t \in \mathbb{Z} + 1/2 + \lambda', t > 0} t \beta_{-t} \bar{\beta}_{-t}
\]

= \sum_{s \in \mathbb{Z} + 1/2 - \lambda'} s : \bar{\beta}_s \beta_s : + \delta(\lambda')

where $\delta(\lambda')$ is the shift in the ground state energy. For integer moding it is the $\bar{\beta}_s, \beta_s$ ghost modes with $s \neq 0$ that map to the complex scalar modes by this transformation, leaving behind a zero mode oscillator, decoupled from the complex scalar field [see also the characters in section (2)].
3.2. Fermionic theories: relating symplectic fermions and Dirac fermions

Let us recall the anticommutation relations of the modes of the fermionic currents, $\mathcal{J}^a(z) \equiv i\partial_z \Phi^a$

\[
\{\mathcal{J}^a_{s_1}, \mathcal{J}^b_{s_2}\} = s_1 \epsilon^{ab}\delta_{s_1+s_2,0} = |s_1| \text{ sign}(s_1) \delta_{s_1+s_2,0}, \quad (a, b = \pm).
\]

where $s \in \mathbb{Z} + 1/2 \pm \lambda$ are mode indices for $\mathcal{J}^\pm_s$. Comparison with the canonical anticommutation relations of the modes of the Dirac fermion,

\[
\{\psi^\dagger_{s_1}, \psi_{s_2}\} = \delta_{s_1-s_2,0}
\]

[where $s_1, s_2 \in \mathbb{Z} + 1/2 - \lambda'$] suggests the following mapping:

\[
\mathcal{J}^\pm_s \text{sign}(-s) = \sqrt{|s|} \psi^\dagger_s
\]

\[
\mathcal{J}^-_s = \sqrt{|s|} \psi_s
\]

or equivalently, since $\mathcal{J}^\pm_s = \Phi^\pm_s$,

\[
\sqrt{|s|} \Phi^+_s = -\psi^\dagger_s
\]

and

\[
\sqrt{|s|} \Phi^-_s = -\text{sign}(s) \psi_s.
\]

The vacuum of the symplectic fermion now satisfies (Eq. (2.11))

\[
\psi^\dagger_{-s}|0> = 0, \quad \psi_s|0> = 0, \quad \text{for } s > 0
\]

which is nothing but the Dirac vacuum.

In analogy to the the bosonic case, we can use the transformation (3.3) to express the hamiltonian of the symplectic fermion theory, given in section (2), in terms of the Dirac fermion modes. Using (2.8), the former reads when expressed in terms of the modes $\mathcal{J}^\pm_s$:

\[
L^{\text{sym}}_0 = -\frac{1}{2} \sum_{s} : [\mathcal{J}^+_s \mathcal{J}^-_s - \mathcal{J}^-_s \mathcal{J}^+_s] : + \text{fermionic normal ordering: :}
\]

where : ... : denotes (fermionic) normal ordering on the vacuum defined in (2.11). Splitting the sum into $s > 0, s < 0$ this is equal, upto a constant shift in ground state energy, to:

\[
= \sum_s s : \psi^\dagger_s \psi_s : + \text{fermionic normal ordering: :}
\]

This establishes the identity of the hamiltonians of the symplectic fermion and the Dirac theories [up to constant shifts, displayed in section (2)]. At the same time, this establishes, as mentioned in section (2), that the character of the symplectic fermion theory is identical to that of the Dirac theory.
4. Hidden non-local symmetries

The existence of a simple mapping between the modes of the $c < 0$ theories and the $c > 0$ theories, has very interesting consequences. It can be used to exhibit hidden non-abelian symmetries in the simple Dirac theory, as well as in the theory of the complex scalar field. Clearly, these two theories have no obvious non-abelian symmetries. The origin of these symmetries is however easily understood from our mapping.

Consider first the bosonic theories. The bosonic ghost is known to have an $sl(2)_{-1/2}$ Kac-Moody[KM] current algebra symmetry. The currents can be written as bilinears in the ghost modes. For half-integer moding, we use the canonical map of the previous section, to express these generators in terms of modes of the complex scalar. This establishes by construction, that the free complex scalar field possesses $sl(2)$ KM current algebra symmetry, for half-integer moding. Particularly simple expressions are obtained for the global $sl(2)$ generators $I^a_0$ $(a = \pm, 3)$, the KM zero mode. We also establish the existence of a global $sl(2)$ symmetry for the complex scalar with integer moding. Under this $sl(2)$, the complex scalar field $(\phi, \phi^\dagger)$ transforms as a doublet. Interestingly, the $I^\pm_0$ generators are non-local, whereas $I^3_0$ is local, in the complex scalar field variables. In the Hilbert space of the complex scalar, the $sl(2)$ symmetry rotates the two sets of oscillator modes of the complex scalar into each other. In addition we observe, interestingly, that the Hilbert space has the same structure as that encountered in the Yangian description of the level-1 $sl(2)$ current algebra [13,14,15]. This suggests the existence of a Yangian symmetry algebra for both, the complex scalar and the bosonic ghost theory.

As discussed above, the fermionic $c = -2$ theory possess a global symplectic $Sp(2)$ symmetry [3,16]. This does not extend to a current algebra symmetry (at least not in an obvious way, since the corresponding Noether current has logarithmic factors in its correlation functions). Mapping the corresponding symmetry generators into the Dirac fermion language, we establish that the latter also has a nonabelian (global) $sl(2)$ symmetry. The pair $(\psi^\dagger, \psi)$ transforms as a doublet under this symmetry. Again, the corresponding $Q^\pm$ generators are non-local, whereas $Q^3$ is the ordinary local Dirac charge [22].

We now discuss bosonic and fermionic non-abelian symmetries in turn.

4.1. Complex scalar theory

The action of the complex scalar field (2.3) has an obvious $U(1)$ symmetry. However, the associated Noether current (we consider again only the L-chiral part) $J(\tau, x) = \frac{1}{2}[\phi^* \partial_x \phi - \phi \partial_x \phi^*]$ does not generate a current algebra, due to logarithmic factors in its correlation functions. Nevertheless, of course, its spatial integral is the conserved $U(1)$ charge

$$J_0 = \frac{1}{2t} \int dx [\phi^\dagger \partial_x \phi - \phi \partial_x \phi^\dagger] \quad (4.1)$$

We now show that, for half-integer moding, the global $U(1)$ generator (4.1) is the zero mode of a $U(1)$ KM current algebra, using the mapping from the $c = -1$ ghost theory. Moreover, this procedure will give us a set of $sl(2)$ current algebra generators of the complex scalar theory, for half-integer moding. For integer moding, we will construct global $sl(2)$ generators for the complex scalar.
The \( c = -1 \) bosonic ghost theory possesses three conformal weight = 1 currents:

\[
I^+(z) \equiv \frac{1}{2} \bar{\beta}^2(z), \quad I^-(z) \equiv -\frac{1}{2} \beta^2(z), \quad I^3(z) \equiv \frac{1}{2} : \bar{\beta} \beta : (z).
\]

Using the correlator of the ghost fields, \(< \bar{\beta}(z_1) \beta(z_2) > = - < \beta(z_1) \bar{\beta}(z_2) > = 1/(z_1 - z_2)\) one finds for the singular part of the short distance expansions

\[
I^\pm(z) I^\mp(0) \sim \frac{k}{z^2} + \frac{\pm 2I^3(0)}{z}
\]

\[
I^\pm(z) I^3(0) \sim \frac{\pm I^\pm}{z}
\]

\[
I^3(z) I^3(0) \sim \frac{k/2}{z^2}
\]

with \( k = -1/2 \). Hence, the modes

\[
I^+_n = \frac{1}{2} \sum_s \bar{\beta}_s \beta_{-s-n}
\]

\[
I^-_n = -\frac{1}{2} \sum_s \beta_{s+n} \beta_{-s}
\]

\[
I^3_n = \frac{1}{2} \sum_s : \bar{\beta}_s \beta_{s+n} :\] (4.2)

of the currents satisfy an \( sl(2) \) Kac-Moody current algebra at level \( k = -1/2 \).

We focus first on half-integer moding. Expressing these modes in terms of those of the complex scalar, using (3.1), gives us a set of \( sl(2) \) current algebra generators that commute with the hamiltonian of the complex scalar field in (2.6). The presence of a current algebra symmetry of the complex scalar field seems rather unexpected.

Let us now focus in more detail on the global \( sl(2) \) generators, induced by this mapping on the scalar field theory. Using this mapping, we express the global generators \( I^a_0 \) \( (a = \pm, 3) \) of (4.2) in terms of the creation/annihilation operators, of the complex scalar. First consider the case of half-integer moding, for which we find

\[
I^+_0 = (-1) \sum_{s>0} B^\dagger_{+,s} B_{-,s}
\]

\[
I^-_0 = (-1) \sum_{s>0} B^\dagger_{-,s} B_{+,s}
\]

\[
I^3_0 = \frac{1}{2} \sum_{s>0} [B^\dagger_{+,s} B_{+,s} - B^\dagger_{-,s} B_{-,s}]
\]

(4.3)

[where \( s \in Z + 1/2 \)]. We see that the global \( sl(2) \) symmetry rotates the two sets of oscillator modes of the complex scalar field into each other. It is this symmetry that extends to the
$sl(2)$ current algebra symmetry. The expressions for the modes $I^n_a$ with $n \neq 0$ are not quite so simply expressed in terms of the oscillator modes; nevertheless they form an $sl(2)$ current algebra, by construction.

We observe that the Hilbert space possesses a structure isomorphic to the multiplets of the (nonlocal) Yangian symmetry present in the $sl(2)$ current algebra representations at level $k = 1$ \[13,14,15\]: Consider a general state in the complex boson Hilbert space

$$B^+_{\alpha, n_1} B^+_{\alpha_2, n_2} B^+_{\alpha_1, n_1}|0>, \quad n_N \geq ... n_2 \geq ... n_1 \geq 0 \quad (4.4)$$

where $\alpha_i = \pm (i = 1, ..., N)$. This state has $L_0$-eigenvalue $N/2 + \sum_{i=1}^N n_i$. Consider a fixed set of mode indices $n_i$. When all $n_i$ are different this is a $2^N$ dimensional space. On the other hand, if some mode indices are equal, the number of states is reduced, the corresponding product of $sl(2)$ doublets with equal mode index being projected on the totally symmetric combination only (due to bose statistics). For example, for $N = 2$, there are four states when $n_1 \neq n_2$, but only three states when $n_1 = n_2$. This is the same structure of Hilbert space \[23\] encountered in the Yangian description of the $sl(2)_1$ current algebra modules \[13,14,15\], suggesting the existence of Yangian symmetry generators in the complex scalar and bosonic ghost theory. Interestingly, this structure is realized in the ‘permanent’ Quantum Hall state \[13,14\].

Finally, consider the expressions for the global $sl(2)$ generators in terms of the complex scalar field itself. When expressed in terms of the momentum modes of the scalar field $\phi_s, \phi^+_s$, we get

$$I^+_0 = -\frac{1}{2} \sum_s |s| \phi^+_s \phi^+_s$$

$$I^-_0 = -\frac{1}{2} \sum_s |s| \phi_s \phi^-_s$$

$$I^3_0 = \frac{1}{2} \sum_s s : \phi^+_s \phi^-_s : \quad (4.5)$$

We see that whereas $I^3_0$, which is just $(1/2)$ times the charge generator \[4.1\], can be locally expressed in terms of the the complex scalar field, the other generators $I^\pm_0$ give rise to non-local expressions in the complex scalar field variables in position space.

For integer moding the generators $I^a_n$ of the bosonic ghost theory [Eq.(4.2)] still generate a current algebra (see Appendix B for a brief discussion). Let us focus on the *global* generators $I^a_0$. These can be split into a sum of two commuting sets of generators, a ghost zero mode contribution $j^a$, and a piece $\tilde{I}^a$ that comes from all $s \neq 0$ modes, $J^a_0 = j^a + \tilde{I}^a$. The generators $\tilde{I}^a$ can be mapped into the complex scalar theory, using \[3.1\]. They give rise to a global $sl(2)$ symmetry, acting on the momentum modes with non-zero mode index (which still has the structure discussed below \[4.4\]). These can be expressed in terms of the complex scalar field itself as in \[4.5\]. Again, the last generator in that equation is the ordinary $U(1)$ charge of the complex scalar. In summary, we established that the complex scalar field (at zero boson radius) has a global $sl(2)$ symmetry, for integer moding.
4.2. Dirac fermions

The symplectic fermion action is invariant under \( sl(2) \), under which the \( \Phi^\pm \) transform as a doublet. The Noether currents associated with this symmetry are

\[
J_\alpha^L(z,z^*) = \frac{i}{2} t_\alpha^{ab} : \Phi^a_L \partial_z \Phi^b_L
\]

and likewise for \( J_R \). We treat only the \( L \)-moving (holomorphic) sector again, dropping the subscript.

Following the notation of [16], we chose the following basis for the \( sl(2) \) generators, \( t_\alpha^{ab} \):

\[
t_0^{ab} = \frac{1}{2} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad t_1^{ab} = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad t_2^{ab} = \frac{1}{2} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.
\]

(4.7)

The \( sl(2) \) symmetry is present for antiperiodic (\( \lambda' = 0 \)) and periodic (\( \lambda' = 1/2 \)) b.c.’s. In the following expressions, the sums run either over \( s \in \mathbb{Z} + 1/2 \), or over \( s \in \mathbb{Z}, s \neq 0 \). The \( sl(2) \) charges \( Q^\alpha \) can be computed explicitly [16] and we write them in terms of \( \Phi^\pm \), or the fermionic currents \( J^\pm \).

\[
Q^\alpha = t_\alpha^{ab} \sum_s \frac{J^a_{-s} J^b_s}{s}
\]

(4.8)

where \( \alpha = 0, 1, 2 \) and \( a, b = \pm \). The \( sl(2) \) charges of the \( \Phi^\pm \) theory can now be translated into corresponding charges in the Dirac theory using the canonical mapping between the fermions \( \Phi^\pm \) and Dirac fermions. Explicitly, we have:

\[
Q^0 = t_0^{0+} \sum_s \frac{J^+_s J^+_{-s}}{-s} + t_0^{0-} \sum_s \frac{J^{-}_s J^{-}_{-s}}{s}
\]

with \( t_0^{0+} = t_0^{0-} = -1/2 \). The charge \( Q^0 \) in terms of the Dirac fields is *non-local*:

\[
Q^0 = -1/2 \sum_s \text{sign}(s) \left[ \psi^+_s \psi^+_s - \psi^-_s \psi^-_s \right].
\]

The charge \( Q^1 \) is given by

\[
Q^1 = t_1^{1+} \sum_s \frac{J^+_s J^+_{-s}}{-s} - t_1^{1-} \sum_s \frac{J^{-}_s J^{-}_{-s}}{s}
\]

with \( t_1^{1+} = -t_1^{1-} = 1/2 \) and in the Dirac fermion language, this becomes

\[
Q^1 = 1/2 \sum_s \text{sign}(s) \left[ \psi^+_s \psi^+_s - \psi^-_s \psi^-_s \right]
\]

which is again *non-local*. Finally,

\[
Q^2 = t_2^{2-} \sum_s \frac{J^+_s J^+_{-s}}{-s} + t_2^{2+} \sum_s \frac{J^{-}_s J^{-}_{-s}}{s}
\]
with $t_{2_+}^2 = t_{2_-}^2 = -1/2$, giving, in terms of Dirac fermions, a local charge:

$$Q^2 = -1/2 \sum_s \left[ -\psi^{\dagger}_{-s} \psi_{-s} + \psi_{-s} \psi^{\dagger}_{-s} \right] = \sum_s [\psi^{\dagger}_s \psi_s - \frac{1}{2}],$$

In summary, using the basis $Q^\pm, Q^3$, the global $sl(2)$ generators of the symplectic fermionic system have the following form in the theory of free Dirac fermions:

$$Q^+ = \sum_{s>0} \psi^{\dagger}_{-s} \psi^+_s,$$

$$Q^- = \sum_{s>0} \psi^+_s \psi_{-s},$$

$$Q^3 = \frac{1}{2} \sum_s [\psi^{\dagger}_s \psi_s - \frac{1}{2}]. \quad (4.9)$$

$$([Q^+, Q^-] = 2Q^3, \quad [Q^3, Q^\pm] = \pm Q^\pm)$$

[As mentioned above, the last sum excludes the zero mode, $s = 0$, for integer moding.]

We end this section with a simple application of the non-local symmetries: Consider a Dirac fermion on a half infinite space, initially with a boundary condition $\psi_L(x = 0) = \psi_R(x = 0)$. We wish to add a term to the action that would give a boundary condition $\psi^{\dagger}_L(x = 0) = C\psi_R(x = 0) + S\psi^{\dagger}_R(x = 0), \quad |C|^2 + |S|^2 = 1$. This is achieved by addition of a non-local term

$$\delta A = (\lambda/2i) \int dy_1 \int dy_2 \frac{1}{(y_1 - y_2)} [\psi^{\dagger}(x = 0, y_1)\psi^{\dagger}(x = 0, y_2) - \psi(x = 0, y_1)\psi(x = 0, y_2)]$$

(where $y$ is the coordinate along the boundary), corresponding to a linear combination of the two non-local generators.

5. Conclusions and comments

In this paper we have found a canonical mapping between the $c = -1$ bosonic ghost theory and the $c = +2$ complex scalar theory and also between the $c = -2$ theory of "symplectic" fermions and the $c = +1$ Dirac fermion theory, mapping the Hamiltonians $H = (2\pi/l)[L_0 - c/24]$ into each other. As a consequence, the characters of the respective theories that are related by this canonical mapping are the same in a general "twisted" sector up to a power of $q = e^{-2\pi \beta/l}$ and are identical in the sectors with twist $\lambda' = 0, 1/2$ (antiperiodic, and periodic). Moreover, this canonical mapping reveals a hidden $sl(2)$ current algebra symmetry of the complex scalar for half-integer moding, as well as a hidden global $sl(2)$ symmetry for integer moding and for the Dirac fermion theories, that these theories inherit from their $c < 0$ counterparts. One of the global generators in these two
theories is the $U(1)$ charge. The remaining generators are non-local when expressed in the language of the $c > 0$ fields and are therefore not visible from the Lagrangian (i.e. without the knowledge of the mapping with the $c < 0$ theories that we use).

The fact that the non-unitary $c < 0$ theories have the same space of states and the same hamiltonian as their unitary counterparts, permits the $c < 0$ ghost theories to represent physical paired Quantum Hall states [3,24], which require a positive definite Hilbert space. It is remarkable that, for these systems, the bulk Quantum Hall wave functions are represented by conformal blocks of correlation functions of the $c < 0$ theories [1], whereas the hamiltonian of the quantum hall edge states associated with these bulk wavefunctions possesses the Hilbert space of the unitary $c = 1$ and $c = 2$ theories. This is clearly necessary for those theories to represent physical Quantum Hall systems. The correlation functions of the edge mode fields are also those of a $c < 0$ theory. It would be interesting to study further examples of conformal field theories, that are non-unitary, but, at the same time, share their space of states, and Hamiltonian with a unitary theory.

The presence of non-abelian symmetries, in the $c = 1,2$ theories is unexpected. The non-locality of the symmetry generators has the ‘flavor’ of Yangian symmetries, whose generators are generally non-local. Yangian symmetries are thought to be responsible for the integrability of Bethe Ansatz solvable models. [In a sense, they are the “symmetries” of Bethe Ansatz integrability]. At a more general level, there has been much discussion recently, about different bases for a given conformal field theory. Some physical applications have come from viewing a certain conformal field theory as a Yang-Baxter (Bethe Ansatz) integrable system. Integrability, then, generates a basis of the Hilbert space of the conformal field theory. Such bases are in general not the Verma module bases, but of a very different kind, in that they possess a Fock-space like structure. The Verma module basis is isomorphic to the space of fields. The Bethe Ansatz basis is a basis for the space of states. In general, these bases are completely different in structure. The non-unitary ghost theories provide thus examples of a similar phenomenon. The possibility to view a given conformal field theory in different bases, has been at the root of much progress, in solving strong coupling problems in interacting theories. The reason is simply that the interaction may look simple in a suitably chosen basis, and become analytically tractable. One set of such examples are various Kondo models [25]. Another example is recent progress on Quantum Hall point contacts devices [26], involving the Bethe Ansatz basis mentioned.

We hope that our analysis of the ghost theories will lend some intuition that would help us get a better understanding of random systems in the strong coupling regime. Some aspects of this will be addressed in a separate piece of work [27].

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Appendix A. Characters for half-integer and integer moding

In this appendix we summarize the respective characters for the cases of half-integer ($\lambda' = 0$) and integer ($\lambda' = 1/2$) moding. As we mentioned in section 2, for these cases, the
c = −1 character of the (β, β) system and the c = 2 character of the complex scalar theory are identical [20]. Similarly, so are the c = −2 character of the “symplectic fermions” and the c = 1 character of the Dirac fermions.

We denote the characters by the boundary conditions in time and space, (Bτ, Bx), when µ = 0. The variable µ then defines a non-specialized character, permitting us to extract more detailed information about the space of states.

A.1. Bosonic characters

A.1.1. Half-integer moding (λ′ = 0)

**Bosonic ghost**

\[ \chi^{(\bar{\beta}, \beta), AA}_{c=-1}(q, \mu) = q^{-\frac{1}{24}} \prod_{n=0}^{\infty} \frac{1}{1 + e^{2\pi i \mu q^n+1/2}} \frac{1}{1 + e^{-2\pi i \mu q^n+1/2}}. \]  

(A.1)

Lowest c = −1 conformal weights: \( \Delta^{(\bar{\beta}, \beta), AA}_{c=-1} = 0, 1/2, 1, \ldots \).

**Complex scalar**

\[ \chi^{(\phi^*, \phi), AA}_{c=2}(q, \mu) = q^{-2/24} q^{1/8} \prod_{n=1}^{\infty} \frac{1}{1 + e^{2\pi i \mu q^n+1/2}} \frac{1}{1 + e^{-2\pi i \mu q^n+1/2}}. \]  

(A.2)

Lowest c = +2 conformal weights: \( \Delta^{(\phi^*, \phi), AA}_{c=2} = 1/8, 1/8 + 1/2, 1/8 + 1, \ldots \). Note that the prefactor can be written as

\[ q^{-2/24} q^{1/8} = q^{-(1/24)} \]  

(A.3)

exhibiting the identity of the c = −1 and the c = +2 bosonic characters:

\[ \chi^{(\phi^*, \phi), AA}_{c=2}(q, \mu) = \chi^{(\bar{\beta}, \beta), AA}_{c=-1}(q, \mu). \]  

(A.4)

We could also use the partition function of the complex scalar theory to cancel the partition function of the Dirac theory. In the first case, the combined (fermion plus boson) theory has c = 3, and N = 2 superconformal symmetry. In the latter case the combined theory has c = 0, and graded SUSY. (The partition function is unity, in both cases.)

A.1.2. Integer moding (λ′ = 1/2)

**Bosonic ghost**

\[ \chi^{(\bar{\beta}, \beta), AP}_{c=-1}(q, \mu) = q^{-\frac{1}{24}} q^{-1/8} \prod_{n=1}^{\infty} \frac{1}{1 + e^{2\pi i \mu q^n}} \frac{1}{1 + e^{-2\pi i \mu q^n}}. \]  

(A.5)
the first factor being the contribution from the bosonic zero mode. Lowest \( c = -1 \) conformal weights: \( \Delta_{c=-1}^{(\beta,\beta)} = -1/8, -1/8 + 1, \ldots \)

**Complex Scalar**

\[
\chi_{c=2}^{(\phi^*,\phi),AP}(q,\mu) = q^{-2/24} \frac{1}{(1+e^{2\pi i\mu})} \prod_{n=1}^{\infty} \frac{1}{(1+e^{2\pi i\mu}q^n)} \frac{1}{(1+e^{-2\pi i\mu}q^n)}, \tag{A.6}
\]

Lowest \( c = +2 \) conformal weights: \( \Delta_{c=2}^{(\phi^*,\phi),AP} = 0, 1, 2, \ldots \)

Again, we find that the two bosonic characters are identical:

\[
\chi_{c=-1}^{(\beta,\beta),AP}(q,\mu) = \chi_{c=2}^{(\phi^*,\phi),AP}(q,\mu)
\]

**A.2. Fermionic characters**

**A.2.1. Half-integer moding**

**Dirac fermion**

\[
\chi_{c=1}^{Dirac,AA}(q,\mu) = q^{-1/24} \prod_{n=0}^{\infty} (1 + e^{2\pi i\mu}q^{n+1/2})(1 + e^{-2\pi i\mu}q^{n+1/2}), \tag{A.7}
\]

Lowest \( c = 1 \) conformal weights: \( \Delta_{c=1}^{Dirac,AA} = 0, 1/2, \ldots \)

**Symplectic Fermion**

\[
\chi_{c=-2}^{sympl,AA}(q,\mu) = q^{(-2/24)} q^{-1/8} \prod_{n=0}^{\infty} (1 + e^{2\pi i\mu}q^{n+1/2})(1 + e^{-2\pi i\mu}q^{n+1/2}), \tag{A.8}
\]

Lowest \( c = -2 \) conformal weights: \( \Delta_{c=-2}^{sympl,AA} = -1/8, -1/8 + 1/2, \ldots \) Eq.(A.3) then shows the identity of this and the Dirac character,

\[
\chi_{c=1}^{Dirac,AA}(q,\mu) = \chi_{c=-2}^{sympl,AA}(q,\mu) \tag{A.9}
\]

**A.2.2. Integer moding**

**Dirac fermion**

\[
\chi_{c=1}^{Dirac,AP}(q,\mu) = q^{-1/24} q^{1/8} (1 + e^{2\pi i\mu}) \prod_{n=1}^{\infty} (1 + e^{2\pi i\mu}q^n)(1 + e^{-2\pi i\mu}q^n), \tag{A.10}
\]
where the extra factor comes from the zero mode. Lowest \( c = 1 \) conformal weights: \( \Delta_{c=1}^{\text{Dirac,AP}} = 1/8, 1/8 + 1, \ldots \).

**Symplectic Fermion**

\[
\chi^{\text{sympl,AP}}_{c=-2}(q, \mu) = q^{(-2/24)}(1 + e^{2\pi i \mu}) \prod_{n=1}^{\infty} (1 + e^{2\pi i \mu} q^n)(1 + e^{-2\pi i \mu} q^n), \tag{A.11}
\]

Lowest \( c = -2 \) conformal weights: \( \Delta_{c=-2}^{\text{sympl,AP}} = 0, 1, 2, \ldots \). Again Eq.\( (A.3) \) show the exact identity of this and the Dirac character:

\[
\chi^{\text{Dirac,AP}}_{c=1}(q, \mu) = \chi^{\text{sympl,AP}}_{c=-2}(q, \mu) \tag{A.12}
\]

**Appendix B. Characters of the \( sl(2)_{-1/2} \) current algebra**

In this Appendix we briefly discuss the relationship of the characters of the bosonic ghost theory of Appendix A, for both integer and half-integer moding, with the expressions obtained by Kac and Wakimoto for the fractional level \( sl(2)_{-1/2} \) current algebra obeyed by the generators \( I^n_a \) (Eq.\( (4.2) \)). Projecting the characters \( (A.1) \) of Appendix A, onto even and odd ghost boson number, we obtain two characters at \( c = -1 \) with scaling dimensions \( \Delta = 0, 1/2 \) (modulo integers):

\[
\chi^{(\bar{\beta}, \beta),0}_{c=-1} \equiv \frac{1}{2} [\chi^{(\bar{\beta}, \beta),AA}_{c=-1}(q, \mu) + \chi^{(\bar{\beta}, \beta),AA}_{c=-1}(q, \mu + 1/2)]
\]

\[
\chi^{(\bar{\beta}, \beta),1/2}_{c=-1} \equiv \frac{1}{2} [\chi^{(\bar{\beta}, \beta),AA}_{c=-1}(q, \mu) - \chi^{(\bar{\beta}, \beta),AA}_{c=-1}(q, \mu + 1/2)]
\]

Similarly those with periodic spatial b.c.’s, give two characters at \( c = -1 \) with scaling dimensions \( \Delta = -1/8 \) (modulo integers):

\[
\chi^{(\bar{\beta}, \beta),-1/8,+}_{c=-1} \equiv \frac{1}{2} [\chi^{(\bar{\beta}, \beta),AP}_{c=-1}(q, \mu) + \chi^{(\bar{\beta}, \beta),AP}_{c=-1}(q, \mu + 1/2)]
\]

\[
\chi^{(\bar{\beta}, \beta),-1/8,-}_{c=-1} \equiv \frac{1}{2} [\chi^{(\bar{\beta}, \beta),AP}_{c=-1}(q, \mu) - \chi^{(\bar{\beta}, \beta),AP}_{c=-1}(q, \mu + 1/2)] \tag{B.1}
\]

On the other hand, the characters of the fractional level \( sl(2)_{-1/2} \) current algebra have been derived by Kac and Wakimoto \[28\], and expressed in terms of theta functions. There are four characters, \( \chi^{KW}_{n,k}(q, \mu) \) with \( n, k = 0, 1 \), transforming into each other under modular transformations. Comparison of those with the expressions in \( (B.1) \) shows that \( \chi^{KW}_{0,0}, \chi^{KW}_{1,0}, \chi^{KW}_{0,1}, \chi^{KW}_{1,1} \) correspond precisely to \( \chi^{(\bar{\beta}, \beta),0}_{c=-1}, \chi^{(\bar{\beta}, \beta),1/2}_{c=-1}, \chi^{(\bar{\beta}, \beta),1/8,+}_{c=-1}, \chi^{(\bar{\beta}, \beta),1/8,-}_{c=-1} \), respectively.
From the comparison with the characters in (B.1), the interpretation of the Kac-Wakimoto characters is as follows: For half-integer moding, the generators (4.2) act on the Hilbert space without zero mode. The even- and odd- boson number characters correspond thus to $\chi_{0,0}^{KW}, \chi_{1,0}^{KW}$. On the eigenspace of lowest $L_0$ eigenvalue the global generators $I_0^a$ transform in a unitary representation of $SU(2)$ of spin $= 0$ and $= 1/2$ respectively, for even and odd boson numbers. These characters correspond therefore to $SU(2)_{-1/2}$ primaries.

For integer moding, on the other hand, the generators (4.2) act on the Hilbert space with the extra zero mode boson $\beta_0 |0> = 0$. The eigenspace of lowest $L_0$ eigenvalue is now infinite dimensional, corresponding to the states $\bar{\beta}_0^N |0 >, (N=0,1,2,...)$. When splitting into even and odd $N$, these are two unitary irreducible lowest weight representations of $SU(1,1)$. The global generators $I_0^a$ do not act unitarily on this space. However, the generators redefined by $I_n^+ \rightarrow \hat{I}_n^+, I_n^- \rightarrow (-1)^n \hat{I}_n^-, I_n^0 \rightarrow \hat{I}_n^3$, satisfy an $SU(1,1)$ current algebra [29]. In particular, the global generators $\hat{I}_n^3$ act unitarily on the space of even and odd boson modes, and correspond to the (infinite dimensional) irreducible lowest weight representations of “angular momentum” $j = -3/4$ and $j = -1/4$ [30]. The (integer moded) characters $\chi_{0,1}^{KW}, \chi_{1,1}^{KW}$ correspond to these representations of the $SU(1,1)_{1/2}$ current algebra.
References

[1] G. Moore and N. Read, Nucl. Phys. B360 (1991) 362.
[2] X. G. Wen, Y. S. Wu, T. Hatsugai, Nucl. Phys. B422 (1994) 476; X. G. Wen, Y. S. Wu, Nucl. Phys. B419 (1994) 455.
[3] M. Milovanovic and N. Read, Phys. Rev. B53 (1996) 13559.
[4] C. Mudry, C. Chamon and X. G. Wen, Phys. Rev. B53 (1996) R7638; Nucl. Phys. B466 (1996) 383.
[5] D. Bernard, Cargese Lectures; hep-th/9509137.
[6] Z. Maassarani and D. Serban, hep-th/9605062.
[7] A. W.W. Ludwig, M. P.A. Fisher, R. Shankar and G. Grinstein, Phys. Rev. B50 (1994) 7526.
[8] D.H.Lee, Phys Rev B50 (1994) 10788.
[9] The description of the Integer Quantum Hall transition using $c = -1$ bosonic ghosts (and graded supersymmetry) is based on [7] and [8].
[10] H. Saleur, Nucl. Phys. B382 (1992) 486.
[11] A comprehensive discussion is given in: D. Friedan, E. Martinec and S. Shenker, Nucl. Phys. B271 (1986) 93.
[12] For a discussion as to how this is achieved within the framework of the respective $c < 0$ theories, see [3].
[13] F. D. M. Haldane et al, Phys Rev Lett 69 (1992) 2021.
[14] P. Bouwknegt, A. W.W. Ludwig, K. Schoutens, Phys Lett B338(1994) 448.
[15] D. Bernard, V. Pasquier, D. Serban, Nucl. Phys. B428 (1994) 612.
[16] H. Kausch, hep-th/9510149.
[17] C. N. Yang, Phys Rev Lett (1989) 2144.
[18] K. B. Efetov, Adv. in Physics 32 (1983) 53.
[19] P. Goddard, D. Olive and G. Watterson, Commun. Math. Phys. 112 (1987) 591.
[20] The $c = 2$ character arising this way should be viewed here, as mentioned above, as the product of the character of a complex scalar at zero boson radius, and a single decoupled oscillator zero mode.
[21] A. Leclair, Nucl. Phys. B415 (1994) 734.
[22] The $\hat{s}l(2)$ symmetry discussed in [21] appears to be local, and thus different from ours.
[23] We thank K. J. Schoutens for clarifying discussions on this point.
[24] N. Read and I. Rezayi, cond-mat/9609079.
[25] I. Affleck and A. W.W. Ludwig, Nucl. Phys. B360 (1991) 641.
[26] P. Fendley, A. W.W. Ludwig and H. Saleur, Phys. Rev. Lett. 74 (1995) 3005.
[27] S. Guruswamy and A. W.W. Ludwig, in preparation.
[28] V. G. Kac and M. Wakimoto, Proc. Nat. Acad. Sci. 85 (1988) 4956.
[29] See also [24].
[30] See for example: N. Ja. Vilenkin, A. U. Klimyk, “Representation of Lie Groups and Special Functions”, (Kluwer Academic Publ., Boston, 1993).