Quantum Valley Hall Effect in Massive Dirac Systems Coupled to a Scalar Field

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We use Pseudo Quantum Electrodynamics to study massive (2+1)D Dirac systems interacting electromagnetically via a U(1) gauge field in (3+1)D. It was recently found in Ref. [1], that an interaction-induced Quantum Hall Effect (QHE) and Quantum Valley Hall Effect (QVHE) occur in these systems, when considering a two-component fermion representation. Here, we study the corrections to these effects when coupling the fermions to a (2+1)D massive scalar field via a quartic interaction. We find no correction to the QHE and a non-universal correction to the QVHE, which depends on the ratio of the fermion and scalar-field masses.

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I. INTRODUCTION

With the experimental realization of graphene, a honeycomb lattice of carbon atoms, massless 2D Dirac fermions moving with the Fermi velocity \( v_F \) have been observed in condensed matter [2]. This discovery has triggered the application of the tools of relativistic quantum-field theories in condensed-matter systems.

Initially, it was believed that the electrons in graphene were very weakly interacting. However, the measurement of the fractional Quantum Hall Effect [3] and of the renormalization of the Fermi velocity [4, 5] have proven that interactions are indeed important at low temperatures in sufficiently clean samples.

Considering static electron-electron interactions, both ac- [6] and dc-conductivity [7] have been calculated. The possibility of a gap opening due to strong Coulomb or electron-phonon interactions has also been investigated [8], as well as the conductivity in the presence of both interactions and an external magnetic field [9].

More recently, it was shown that dynamical electromagnetic interactions may lead to a QVHE [10]. In addition, the correction to the bare spin-\( g \)-factor due to dynamical interactions has also been calculated [11], and were found to exhibit good agreement with experiments [12, 13] (see Ref. [14] for a comprehensive review of electron-electron interactions in graphene).

Since the synthesis of graphene, many other 2D materials consisting of a honeycomb lattice have been experimentally realized. One of these is silicene, a honeycomb lattice made of silicon atoms [15]. The larger ionic radius of the silicon compared to carbon causes the lattice to buckle and leads to a band-gap that can be tuned by a perpendicular electric field. The low-energy excitations of silicene are thus massive Dirac fermions [16]. The buckled lattice also increases the intrinsic spin-orbit coupling [16].

By describing silicene within a tight-binding Hamiltonian, including spin-orbit coupling and a perpendicular electric field that explicitly breaks the inversion symmetry, a non-universal QVHE was predicted [17]. At the neutrality point, however, the result becomes universal and depends only on the sign of the spin-orbit and electric-field terms. Non-universal corrections to the QVHE were also obtained in Ref. [18] by including a finite chemical potential, and in Ref. [19] by including a Rashba term that breaks the spin \( s_z \)-symmetry.

On the other hand, it was found in Ref. [11] that not only a QVHE, but also a QHE may emerge due to dynamical interactions in massive Dirac systems, as a consequence of a dynamically driven parity anomaly. In this case, the Hall (transverse) conductivity and the Valley Hall conductivity assume universal values, depending only on the Planck constant \( h \) and the electron charge \( e \). It is remarkable that the effect arises in the absence of a magnetic field or any other perturbation that breaks time-reversal symmetry a priori.

Here, we investigate the fate of this universal QHE and QVHE when we couple the system in Ref. [11] to a (2+1)D massive scalar field \( \sigma \), via a quartic interaction. Scalar fields have been used both in the context of electron-phonon interactions and optomechanics to describe mechanical oscillations either of a lattice [20] or a movable mirror [21]. Although the first term of the interaction in each of these systems is linear in the scalar field, higher-order contributions can be considered. For optomechanical systems, the quadratic term in the scalar field, which generates the quartic interaction, would represent a quadratic displacement of the oscillator’s position [21] and it has already been observed in a cold-atom setup [22]. Moreover, this quartic coupling can be also found in a supersymmetric generalization of Chern-Simons Higgs theory [23], which was recently used in the non-relativistic limit to describe the fractional quantum Hall effect [24].

We consider relativistic massive (2+1)D Dirac electrons, propagating with a Fermi velocity \( v_F \) and interacting via a U(1) gauge field that lives in (3+1)D. This dimensional mismatch is accounted for within the framework of Pseudo Quantum Electrodynamics (PQED), the
effective theory that is obtained by integrating out the extra dimension of the gauge field [25]. The name Pseudo QED stems from the fact that the theory involves pseudodifferential operators. This theory is also sometimes called reduced QED in the literature [26,28].

Using the Kubo formalism, we obtain the correction to the transverse conductivity induced by the coupling to the scalar field. We find a non-universal correction to the QVHE, depending on the ratio of the fermion and scalar-field masses, but no correction to the QHE.

The outline of this paper is as follows: in Sec. I we introduce the model. In Sec. II we calculate the current-current correlation function, which we use in Sec. III to obtain the correction to the conductivity. In Sec. IV we consider the massless case and in Sec. V we present our conclusions. In the appendices we provide additional details of our calculation.

II. THE MODEL

In 2D systems such as graphene and silicene, electrons interact via a U(1) gauge field that propagates in (3+1)D. To describe this system, one can start from QED in (3+1)D and confine the matter current $j^\mu$ to a plane by writing

$$j^\mu(x^0, x^1, x^2, x^3) = \begin{cases} j^\mu_{2+1}(x^0, x^1, x^2)\delta(x^3) & \mu = 0, 1, 2 \\ 0 & \mu = 3. \end{cases}$$

The extra dimension of the gauge field can then be integrated out, thus leading to a non-local theory, that is non-universal to the transverse conductivity, we have to identify the terms proportional to $\epsilon^{ij}$. The current-current correlation function is nothing but the polarization tensor $\Pi^{ij}$. Our strategy is to obtain the conductivity by computing the polarization tensor, and then to apply Kubo’s formula.

We focus on the transverse conductivity, since the longitudinal conductivity was shown to be zero for massive Dirac systems in the two-component fermion representation, up to first order [1]. We will calculate the lowest-order correction to the transverse part of the vacuum polarization tensor coming from the scalar field $\sigma$, to verify whether the coupling to the scalar field may destroy the universal features of the transverse current. The lowest-order contribution comes from the 2-loop diagram depicted in Fig. 1. The corresponding expression is

$$i\Pi^{ij}_2(p, \Delta) = 2e^2g_\sigma^2 \int \frac{d^3k}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \left\{ i \frac{\delta}{k^2 - m_\sigma^2} \times \text{Tr} \left[ \gamma^i S_F(q)^2 \gamma^j S_F(q - p) \right] \right\},$$

where

$$S_F(q) = i \frac{\delta}{q_0^2 - v_F^2 q^2 - \Delta^2},$$

is the fermion propagator. Note that there is a minus sign coming from the fermionic loop, and a symmetry factor of two. We compute the $k$-integral using dimensional regularization [31,33], which yields

$$\int \frac{d^3k}{(2\pi)^3} \frac{1}{k^2 - m_\sigma^2} = \frac{m_\sigma}{4\pi}. \quad (4)$$

It is interesting to observe that using dimensional regularization, we find a finite result. Since we have chosen the fermions to be two-component spinors, the gamma matrices will also be two-dimensional. In this representation, we can choose the gamma matrices such that they are equal to the Pauli matrices, and we find

$$\text{Tr} \left[ \gamma^\mu \gamma^\nu \gamma^\rho \right] = 2i\epsilon^{\mu\nu\rho},$$

where $\epsilon^{\mu\nu\rho}$ is the Levi-Civita tensor. To find the transverse conductivity, we have to identify the terms proportional to $\epsilon^{ij}$. From Eq. (5), we see that these terms will
where we have introduced the notation $\Pi_{ij}^{2\text{-loop}}$. From these two valley currents, we can define the transverse conductivity due to the scalar field. Because of the valley degree of freedom, there are two valley currents in our model, which are connected by time-reversal symmetry. From these two valley currents, we can define the total conductivity
\[
\sigma_{ij}^{\text{tot}} = \lim_{\omega \to 0, p \to 0} \left\{ \frac{i \langle j^i j^j \rangle}{\omega} + i \frac{\langle j^i j^j \rangle^T}{\omega} \right\} = \sigma_{xx}^{\text{tot}} \delta_{ij} + \sigma_{xy}^{\text{tot}} \epsilon_{ij},
\]
and the valley conductivity
\[
\sigma_{ij}^{\text{val}} = \lim_{\omega \to 0, p \to 0} \left\{ \frac{i \langle j^i j^j \rangle}{\omega} - i \frac{\langle j^i j^j \rangle^T}{\omega} \right\} = \sigma_{xx}^{\text{val}} \delta_{ij} + \sigma_{xy}^{\text{val}} \epsilon_{ij},
\]
where $\langle j^i j^j \rangle^T$ is the time-reversed current-current correlation function. In Ref. [1], it was found that
\[
\sigma_{xy}^{\text{val}} = 4 \left( n + \frac{1}{2} \right) \frac{e^2}{h},
\]
and
\[
\sigma_{xy}^{\text{tot}} = \frac{2 e^2}{h}.
\]

From the polarization tensor in Eq. (7), we find the correction to the current
\[
\lim_{\omega \to 0, p \to 0} \frac{i \langle j^i j^j \rangle}{\omega} = \lim_{\omega \to 0, p \to 0} \frac{i \Pi_{ij}^{2\text{-loop}}(p, \Delta)}{\omega},
\]
\[
\lim_{\omega \to 0, p \to 0} \frac{i \langle j^i j^j \rangle^T}{\omega} = \lim_{\omega \to 0, p \to 0} \frac{\left[ \Pi_{ij}^{2\text{-loop}}(p, \Delta) \right]^T}{\omega},
\]
where we recall that $p_0 = \omega$. At first glance, it seems that the expressions in Eq. (6) are not well defined in the Kubo limit. However, when taking all the terms together and considering the Taylor expansion of the logarithms for small $p_0$, we find that the divergences cancel (see appendix B for details). We find
\[
\lim_{\omega \to 0, p \to 0} F_1(p, \Delta) = -\frac{2}{3} \frac{i}{|\Delta|},
\]
\[
\lim_{\omega \to 0, p \to 0} F_2(p, \Delta) p^2 = 0,
\]
\[
\lim_{\omega \to 0, p \to 0} F_3(\Delta, p) \Delta^2 = -\frac{1}{30} \frac{i}{|\Delta|}.
\]

Finally, after substituting Eqs. (15)-(17) into Eq. (7), and this into Eqs. (13) and (14), taking into account a factor of 2 for the spin degree of freedom, and reintroducing $h$ to make the result dimensional, we find a non-universal correction to the valley conductivity
\[
\delta \sigma_{xy}^{\text{val}} = - \frac{1}{(2\pi)^2} \frac{7}{10} \frac{m_{\sigma}}{|\Delta|^2} \frac{e^2}{h},
\]
and no correction to the total conductivity
\[
\delta \sigma_{xy}^{\text{tot}} = 0.
\]
Combining this result with the result from Ref. [1] [Eqs. (11) and (12)], we find that
\[
\sigma_{xy}^{\text{val}} = 2 \frac{e^2}{h} \left[ 2 n + 1 - \frac{1}{(2\pi)^2} \frac{7}{20} \frac{m_{\sigma}}{|\Delta|} \right],
\]
\[
\sigma_{xy}^{\text{tot}} = 2 \frac{e^2}{h}.
\]
V. THE MASSLESS CASE

Let us now investigate the correction to the polarization tensor for massless fermions (as found, for example, in graphene). In the massless case $\Delta = 0$, Eq. (6) then reduces to

$$i\Pi_{2l}^{ij} = -ie^2 g^2 \frac{m_{\sigma}}{\pi} \int \frac{d^3q}{(2\pi)^3} \left\{ \frac{\epsilon^{ij0}(q_0 - p_0)}{q^2(q - \bar{p})^2} \right\}. \quad (21)$$

Combining the denominators and calculating the integrals, we find

$$i\Pi_{2l}^{ij} = \frac{m_{\sigma}e^2 g}{16\pi} \frac{p_0}{\sqrt{p_0^2 - v_F^2 p^2}}. \quad (22)$$

In this case, the Kubo formula is not well defined since dividing by $\omega$ and taking the zero limit of the momentum yields a divergence. We could already expect this result on dimensional grounds. The polarization tensor in our theory has mass dimension one. Integrating out the bosonic loop, we find something proportional to the mass $m_{\sigma}$. The final result should thus be $m_{\sigma}$ multiplied by a dimensionless term. We also know that the transverse part will be proportional to $e^2 p_0$, and thus we need to divide by a term with mass dimension one in order to make the Kubo formula well-defined. In the massless case, we can only divide by a term containing $p_0$ and $v_F p$. This means that the Kubo limit will not be well-defined. When we have a fermion mass, we can also divide by this mass to make the limit finite, and indeed this is exactly what happens in Eqs. (13) and (14).

VI. CONCLUSION

It has been known for some time that for QED in (2+1)D, in the two-component spinor representation, radiative corrections generate a topological gauge field mass term [34, 35], giving rise to a non-vanishing gauge field transverse current in the system [36, 37]. Although this induced mass term emerges when one couples the fermions minimally to the vector potential $A_\mu$, interactions between fermions and other fields could lead to additional contributions to the current. Recently, it was shown that dynamical interactions described within the PQED formalism lead to quantized Hall and valley Hall conductivities [1]. At one-loop order, the results for QED and PQED in (2+1)D are the same. At higher order, however, they differ for the longitudinal conductivity, but remain the same for the transverse one [38]. Here, we investigated the fate of these quantized conductivities in the presence of an additional scalar field quartically coupled to the fermions.

We started by calculating the corrections to the interaction induced QVHE and QHE in massive Dirac systems using the PQED formalism, which takes into account the full dynamical electromagnetic interactions of the electrons. The corrections to the transverse conductivity and transverse valley conductivity were obtained by calculating the polarization tensor diagram up to 2-loop orders, and then using the Kubo formula. We found a non-universal correction to the QVHE, which depends on the ratio of the masses of the scalar field and the fermions, but no correction to the QHE. In addition, we investigated the case of massless fermions ($\Delta = 0$), and showed that the Kubo formula is not well defined in this limit. In the case of massless bosons ($m_{\sigma} = 0$), there is no correction to either the QVHE or the QHE.

Here, we considered a not so explored quartic coupling between the scalar and the fermionic fields. A Yukawa-like coupling was used in the context of electron-phonon interaction in graphene [39]. A theory involving an exponential of a scalar field was recently proposed to describe fractionalization in a square lattice [40]. A second-order expansion of an exponential containing a scalar field would inevitably lead to a theory involving a Yukawa term plus the quartic interaction considered here. We hope that our paper will motivate further research on these non-standard couplings.

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APPENDIX A: CALCULATION OF THE DIAGRAM

In this appendix, we show the detailed calculation of the correction to the polarization tensor. Because Lorentz invariance is broken by the Fermi velocity $v_F$ of the fermions, we must treat the $q_0$ and $q$ integrals separately. Let us start from Eq. (3), and first compute the trace of the gamma matrices. As explained in the main text, we are interested in the terms proportional to $\epsilon^{ij0} p_0$, which can only arise from the trace of three or five gamma matrices. The full trace is (we use the notation $\eta = (q_0, v_F q)$)

$$\text{Tr} \left[ \gamma^i (\gamma^\alpha \eta_\alpha + \Delta) \gamma^j (\gamma^\beta (\eta - \bar{p})_\beta + \Delta) \right].$$

The terms with three gamma matrices are

$$2\Delta^2 \text{Tr} \left[ \gamma^i \gamma^0 \gamma^j \right] q_0 + \Delta^2 \text{Tr} \left[ \gamma^i \gamma^j \gamma^0 \right] (q_0 - p_0)$$
$$=2\Delta^2 \text{Tr} \left[ \gamma^i \gamma^0 \gamma^j \right] q_0 - \Delta^2 \text{Tr} \left[ \gamma^i \gamma^0 \gamma^j \right] (q_0 - p_0)$$
$$=\Delta^2 \text{Tr} \left[ \gamma^i \gamma^0 \gamma^j \right] (q_0 + p_0)$$
$$=2\Delta^2 i \epsilon^{ij0} (q_0 + p_0),$$
where we have used Eq. (5). There is one term containing five gamma matrices, which is

\[ \text{Tr} \left[ \gamma^i \gamma^\alpha \gamma^\beta \gamma^\gamma \gamma^\delta \right] \bar{q}_a \gamma_\beta \left( \bar{q} - \bar{p} \right) \delta = \left\{ - \text{Tr} \left[ \gamma^i \gamma^\alpha \gamma^\beta \gamma^\gamma \gamma^\delta \right] + 2 g^{\alpha \beta} \text{Tr} \left[ \gamma^i \gamma^\gamma \right] \right\} \bar{q}_a \gamma_\beta \left( \bar{q} - \bar{p} \right) \delta \]

which it follows

\[ \text{Tr} \left[ \gamma^i \gamma^\alpha \gamma^\beta \gamma^\gamma \gamma^\delta \right] \bar{q}_a \gamma_\beta \left( \bar{q} - \bar{p} \right) \delta = 2 i \epsilon^{ij0} \hat{q} \left( q_0 - p_0 \right). \]

Substituting the result for the trace in Eq. (3), we obtain Eq. (6). We now combine the denominators using the Feynman trick

\[ \frac{1}{A^2 B} = 2 \int_0^1 dx \left( \frac{1-x}{\left( 1-x \right) A + x B } \right)^3. \]

The denominator becomes

\[ (1-x) A + x B = (q_0 - x p_0)^2 - \Sigma_1, \]

with

\[ \Sigma_1 \equiv - x (1-x) p_0^2 + v_F^2 q^2 + \Delta^2 + x \left[ v_F^2 p^2 - 2 v_F q p q \right]. \]

Rewriting Eq. (6) using Eqs. (23) and (24), then making the shift \( q_0 \to q_0 + x p_0 \), and noticing that the terms odd in \( q_0 \) vanish, the polarization tensor becomes

\[ i \Pi^{ij}_{2i} = - i e^2 g v_F^2 \frac{m_\sigma}{8 \pi} \int_0^1 dx \int \frac{d^3 q}{(2 \pi)^3} \left\{ \frac{\epsilon^{ij0} \left( q_0^2 C + D \right) \left( 1-x \right) }{v_F^2 \left( q_0^2 - \Sigma_1 \right)^3} \right\}, \]

with

\[ C \equiv (x-1) p_0 + 2 x p_0, \]

\[ D \equiv x^2 (1-x) p_0^2 - v_F^2 q^2 (x-1) p_0 - \Delta^2 (1+x) p_0. \]

We can now perform the \( q_0 \) integrals

\[ \int \frac{dq_0}{(2 \pi)} \frac{q_0^2}{\left( q_0^2 - \Sigma_1 \right)^3} = - i \frac{3}{16} \Sigma_1^{-5/2}, \]

\[ \int \frac{dq_0}{(2 \pi)} \frac{1}{\left( q_0^2 - \Sigma_1 \right)} = \frac{3}{16} \Sigma_1^{-5/2}. \]

Rewriting

\[ \Sigma_1 = v_F^2 \left( q - x p \right)^2 - \Sigma_2, \]

with

\[ \Sigma_2 \equiv - x (1-x) p^2 - \frac{\Delta^2}{v_F^2} + x (1-x) p_0^2 \frac{1}{v_F^2}, \]

we find

\[ i \Pi^{ij}_{2i} = - i e^2 g v_F^2 \frac{m_\sigma}{8 \pi} \int_0^1 dx \int \frac{d^3 q}{(2 \pi)^3} \left\{ \frac{- i e^{ij0} \left( 1-x \right) }{v_F^3 \left( q - x p \right)^2 - \Sigma_2^{3/2} + 3 i e^{ij0} D \left( 1-x \right) } \right\}. \]

We now shift \( q \to q + x p \) and notice that the terms odd in \( q \) vanish. The \( q \) integrals may be performed using

\[ \int \frac{d^2 q}{(2 \pi)^2} \frac{1}{(q^2 - \Sigma_2)^{3/2}} = \frac{- i}{2 \pi} \frac{1}{\sqrt{\Sigma_2}}, \]

\[ \int \frac{d^2 q}{(2 \pi)^2} \frac{q^2}{(q^2 - \Sigma_2^{5/2})} = \frac{- i}{2 \pi} \frac{1}{3 \sqrt{\Sigma_2}}, \]

\[ \int \frac{d^2 q}{(2 \pi)^2} \frac{1}{(q^2 - \Sigma_2^{3/2})} = \frac{i}{2 \pi} \frac{1}{3 (\Sigma_2)^{3/2}}. \]

The polarization tensor then becomes

\[ i \Pi^{ij}_{2i} = - i e^2 g v_F^2 \frac{m_\sigma}{16 \pi^2} \int_0^1 dx \left\{ \frac{\epsilon^{ij0} \left[ - C - 2 (x-1) p_0 \right] \left( 1-x \right) }{v_F^3 \left( \Sigma_2 \right)^{1/2}} + \frac{\epsilon^{ij0} E \left( 1-x \right) }{v_F^3 \left( \Sigma_2^{3/2} \right)} \right\}, \]

with \( E \equiv x^2 (1-x) p_0^3 - v_F^2 x^2 p^2 (x-1) p_0 - \Delta^2 (1+x) p_0. \)

We are now left with only the parametric integral over \( x \). The first term of the integral is

\[ I_1 = \epsilon^{ij0} p_0 \frac{1}{v_F^3} \int_0^1 dx \left\{ \frac{3 - 5 x \left( 3 - 5 x \right) \left( 1-x \right) }{\left[ x \left( 1-x \right) p^2 - \Delta^2 \right]^{1/2}} \right\} \]

\[ = \epsilon^{ij0} p_0 \left\{ \frac{- i}{8 \pi^3} \left[ 20 \left| \Delta \right| \left| \Delta - | x p \rangle \right| \right] \right\}, \]

and the second term is

\[ I_2 = \epsilon^{ij0} \frac{1}{v_F^3} \int_0^1 dx \left\{ \frac{I_{2A}}{x \left| x \left( 1-x \right) p^2 - \Delta^2 \right|^{3/2}} \right\}, \]

with

\[ I_{2A} = \frac{I_{2A}}{x \left| x \left( 1-x \right) p^2 - \Delta^2 \right|^{3/2}}. \]
Substituting Eqs. (29)-(32) into Eq. (28) leads to Eq. (7).

\[ I_{2A} = \int_0^1 dx \frac{−x^2(1−x)^2p^2p_0}{[x(1−x)p^2 − Δ^2]^{3/2}} \]

\[ = \frac{−1}{4Δ^4p^4 − Δ^2p^6} \left\{ 2i|Δ||p|(2Δ^2 + p^2) \right. \]

\[ + iΔ(4Δ^2 − p^2)\ln \left( \frac{2|Δ| − |p|}{2|Δ| + |p|} \right) p^2p_0 \]

\[ = F_{2A}(Δ, p)p^2p_0. \] (31)

And

\[ I_{2B} = \int_0^1 dx \frac{−(1−x^2)Δ^2p^2p_0}{[x(1−x)p^2 − Δ^2]^{3/2}} \]

\[ = \frac{−1}{8p^2(−4Δ^2 + p^2)} \left\{ 4i|Δ|p(−12Δ^2 + p^2) \right. \]

\[ − i(48Δ^4 − 8Δ^2p^2 − p^4)\ln \left( \frac{2|Δ| − |p|}{2|Δ| + |p|} \right) Δ^2p_0 \]

\[ = F_{2B}(Δ, p)Δ^2p_0. \] (32)

Substituting Eqs. (29)-(32) into Eq. (28) leads to Eq. (7) in the main text.

**APPENDIX B: TAKING THE KUBO LIMIT**

In this section we calculate the Kubo limit of Eq. (7):

\[ \lim_{p_0 \to 0, \mathbf{p} \to 0} \frac{\Pi^{ij}_{ij}}{p_0} = \lim_{p_0 \to 0, \mathbf{p} \to 0} \frac{−m_σc^2g_σ\epsilon^{i0}}{(4π^2)^2} \times \left[ F_1(Δ, p) + F_2(Δ, p)p^2 + F_3(Δ, p)Δ^2 \right]. \] (33)

where the expressions for \( F_1(Δ, p), F_2(Δ, p)p^2 \) and \( F_3(Δ, p)Δ^2 \) are given in Eq. (8). We first note that Eq. (33) is only dependent on \([\mathbf{p}],\) and taking the limit \(\mathbf{p} \to 0,\) thus amounts to replacing \([\mathbf{p}] \to |p_0|.\) To take the limit of \(p_0 \to 0\) we have to consider the Taylor expansion

\[ \ln \left( \frac{2|Δ| − |p_0|}{2|Δ| + |p_0|} \right) = \frac{|p_0|}{|Δ|} − \frac{1}{12} \frac{|p_0|^3}{|Δ|^3} − \frac{1}{80} \frac{|p_0|^5}{|Δ|^5} + O(p_0^6). \]

We calculate the limit for each term separately. For the first term, we find

\[ \lim_{p_0 \to 0} F_1(Δ, p_0) \]

\[ = \lim_{p_0 \to 0} \frac{−i |Δ|}{8p_0} \frac{−i (20Δ^2 − 7p_0^2)}{8 |p_0|^5} \ln \left( \frac{2|Δ| − |p_0|}{2|Δ| + |p_0|} \right) \]

\[ + O(p_0) \]

\[ = \lim_{p_0 \to 0} \frac{−i}{24 |Δ|} \]

\[ = \frac{−2i}{3 |Δ|}. \]

The second term becomes

\[ \lim_{p_0 \to 0} F_2(Δ, p_0)p_0^2 \]

\[ = \lim_{p_0 \to 0} \frac{−i |Δ| |p_0|^3}{4π^2 |p_0|^3 − Δ^2 |p_0|^5} \left\{ 2|p_0| (2Δ^2 + p_0^2) \right. \]

\[ + |Δ| (4Δ^2 − p_0^2)\ln \left( \frac{2|Δ| − |p_0|}{2|Δ| + |p_0|} \right) \]

\[ = \lim_{p_0 \to 0} \frac{−4i |Δ|}{4Δ^2 − p_0^2} − \frac{2i |p_0|^3}{Δ (4Δ^2 − p_0^2)} = \frac{−i |Δ|}{4 |Δ|^2 + |Δ|} \]

\[ = 0, \]

and the third term becomes

\[ \lim_{p_0 \to 0} F_3(Δ, p_0)Δ^2 \]

\[ = \lim_{p_0 \to 0} \frac{−Δ^2}{8p_0^2 (−4Δ^2 + p_0^2)} \left\{ 4i |Δ| p_0 (−12Δ^2 + p_0^2) \right. \]

\[ − i(48Δ^4 − 8Δ^2 p_0^2 − p_0^4)\ln \left( \frac{2|Δ| − |p_0|}{2|Δ| + |p_0|} \right) \]}

\[ = \lim_{p_0 \to 0} \frac{i |Δ|}{12(p_0^2 − 4|Δ|^2)} + \frac{i |Δ|}{8(p_0^2 − 4|Δ|^2)} \]

\[ = \frac{−i |Δ|}{80(p_0^2 − 4|Δ|^2)} + O(p_0) \]

\[ = \frac{−i}{30 |Δ|}. \]

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