Abstract

For some absolute constants $c$, $n_0$ and any $n \geq n_0$, we show that with probability close to one the convex hull of the $n$-dimensional Brownian motion $\text{conv}\{\text{BM}_n(t) : t \in [1, 2^{cn}]\}$ does not contain the origin. The result can be interpreted as an estimate of the minimax of the Gaussian process $\{\langle \bar{u}, \text{BM}_n(t) \rangle, \bar{u} \in S^{n-1}, t \in [1, 2^n]\}$.

1 Introduction

Our paper is motivated by the following question raised by I. Benjamini and considered by R. Eldan in [2]:

Let $t_1, t_2, \ldots, t_N$ be points in $[0, 1]$ generated by a homogeneous Poisson point process with intensity $\alpha$. Estimate the value $\alpha = \alpha_0$ such that the convex hull of $\text{BM}_n(t_i), i = 1, 2, \ldots, N,$ contains the origin with probability $1/2$.

Here, $\text{BM}_n$ is the standard Brownian motion in $\mathbb{R}^n$. Eldan [2] showed that $\alpha_0$ satisfies

$$e^{c_1n/\log n} \leq \alpha_0 \leq e^{c_2n\log n},$$

for some universal constants $c_1$ and $c_2$. Related results were obtained in [2] for the standard random walk on $\mathbb{Z}^n$ and the spherical Brownian motion. The right-hand side estimate in [1] was recently improved to $e^{c_2n}$ by the authors [9]. In fact, [9] provides a rather general method for estimating from below the probability of the event $0 \in \{W(t)\}$ for various types of random walks $W$ in $\mathbb{R}^n$. At the same time, the question of optimizing the lower bound for $\alpha_0$ in [1] remained open.

The main result of this paper is the following theorem.

**Theorem 1.** There exist universal constants $c > 0$ and $n_0 \in \mathbb{N}$ with the following property: let $n \geq n_0$ and $\text{BM}_n(t) (0 \leq t < \infty)$ be the Brownian motion in $\mathbb{R}^n$. Then

$$\mathbb{P}\{0 \in \text{conv}\{\text{BM}_n(t) : t \in [1, 2^{cn}]\}\} \leq \frac{1}{n}.$$ 

**Remark 1.** The bound $\frac{1}{n}$ in the above theorem can be replaced with $\frac{1}{n^L}$ for any constant $L > 0$ at expense of decreasing $c$ and increasing $n_0$. 

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As an immediate corollary of Theorem 1 we get

**Corollary 2.** There exist universal constants \( \tilde{c} > 0 \) and \( n_0 \in \mathbb{N} \) with the following property: Let \( n \geq n_0 \) and let \( BM_n(t) (t \in [0, \infty)) \) be the standard Brownian motion in \( \mathbb{R}^n \). Further, let \( t_1, t_2, \ldots, t_N \) be points generated by the homogeneous Poisson process on \([0, 1]\) of intensity \( \alpha > 0 \), which is independent from the process \( BM_n \). If \( \alpha \leq \exp(\tilde{c} n) \) then

\[
\mathbb{P}\{0 \in \text{conv}\{BM(t_i) : i \leq N\} \} \leq \frac{1}{n}.
\]

In particular, we improve the left-hand side estimate in (1) to \( e^{-c_1 n} \leq \alpha_0 \) and, together with the aforementioned result of [9], provide the optimal bounds for \( \alpha_0 \), up to the choice of \( c_1 \) and \( c_2 \).

The main result of this paper is equivalent to the estimate

\[
\mathbb{P}\{\min_{u \in S^{n-1}} \max_{t \in [1,2^n]} \langle u, BM_n(t) \rangle < 0 \} \geq 1 - \frac{1}{n}.
\]

We note that the minimax of certain Gaussian processes was studied in [3], [4] (see also [6, Theorem 3.16]). Those results found applications in Asymptotic Geometric Analysis (Dvoretzky’s Theorem) and the theory of compressed sensing (see [1]).

We think that it may be of interest to consider the following generalization of the question studied in this paper:

**Let** \( X(t) \) **be a centered Gaussian process in** \( \mathbb{R}^n \). **Estimate the distribution of**

\[
\min_{u \in S^{n-1}} \max_{t \in [1,2^n]} \langle u, X(t) \rangle
\]

**in terms of the covariance structure of the process** \( X \). **The corresponding question of estimating (up to a constant multiple)** \( \mathbb{E}\sup_t Y(t) \) **for a 1-dimensional Gaussian process** \( Y \) **was solved by Fernique and Talagrand (see [8] and references therein).**

Let us give an informal description of the proof of the main result. We construct a random unit vector \( \bar{n} \) in \( \mathbb{R}^n \) such that with probability close to one

\[
\langle \bar{n}, BM_n(t) \rangle > 0 \quad \text{for any } t \in [1,2^n].
\]

The construction procedure shall be divided into a series of steps. At the initial step, we produce a random vector \( \bar{n}_0 \) such that

\[
\langle \bar{n}_0, BM_n(2^i) \rangle > 0 \quad \text{for any } i = 0, 1, \ldots, cn.
\]

(In fact, \( \bar{n}_0 \) will satisfy a stronger condition). At a step \( k, k \geq 1 \), we “update” the vector \( \bar{n}_{k-1} \) by adding a small “perturbation” in such a way that

\[
\langle \bar{n}_k, BM_n(2^{j2^{-k}}) \rangle > 0 \quad \text{for any } j = 0, 1, \ldots, 2^k cn.
\]

(Again \( \bar{n}_k \) will in fact satisfy a stronger condition). Finally, using some standard properties of the Brownian bridge, we verify that \( \bar{n} := \bar{n}_{\ln \ln n} \) satisfies (2) with a large probability.
2 Preliminaries

In this section we introduce some notation and state several auxiliary results that will be used within the proof.

By \( \{e_i\}_{i=1}^n \) we denote the standard unit basis in \( \mathbb{R}^n \), by \( \| \cdot \| \) — the canonical Euclidean norm and by \( \langle \cdot, \cdot \rangle \) — the corresponding inner product. For \( N \geq n \) and an \( N \times n \) matrix \( A \), let \( s_{\max}(A) \) and \( s_{\min}(A) \) be its largest and smallest singular values, respectively, i.e. \( s_{\max}(A) = \|A\| \) (the operator norm of \( A \)) and \( s_{\min}(A) = \inf_{y \in S^{n-1}} \|Ay\| \). For a finite set \( I \), let \( |I| \) be its cardinality. By \( c, c_1, \tilde{c}, \) etc. we denote universal constants. To avoid difficult to read formulas, we do not use any notation for truncation of a real number to the nearest integer. For example, the product \( cn \) in the next section is always treated as an integer, as well as several other quantities depending on \( n \).

Let \( (\Omega, \Sigma, \mathbb{P}) \) be the probability space. Throughout the text, \( \gamma \) denotes the standard Gaussian variable. The following estimate is well known (see, for example, \[3\] Lemma VII.1.2):

\[
\mathbb{P}\{\gamma \geq \tau\} = \frac{1}{\sqrt{2\pi}} \int_{\tau}^{\infty} \exp(-t^2/2) \, dt < \frac{1}{\sqrt{2\pi\tau}} \exp(-\tau^2/2), \quad \tau > 0.
\] (3)

Let \( n \geq m \) and let \( G \) be the standard \( n \times m \) Gaussian matrix. Then for any \( t \geq 0 \)

\[
\mathbb{P}\{\sqrt{n} - \sqrt{m} - t \leq s_{\min}(G) \leq s_{\max}(G) \leq \sqrt{n} + \sqrt{m} + t\} \geq 1 - 2\exp(-t^2/2)
\] (4)

(see, for example, \[10\] Corollary 5.35).

The proof of the next Lemma is straightforward, so we omit it.

**Lemma 3.** Let \( \text{BM}_n(t) \ (0 \leq t < \infty) \) be the standard Brownian motion in \( \mathbb{R}^n \) and let \( 0 < a < b \). Fix any \( s \in (a, b) \) and set

\[
w(s) := \frac{b-s}{b-a} \text{BM}_n(a) + \frac{s-a}{b-a} \text{BM}_n(b); \quad u(s) := \text{BM}_n(s) - w(s).
\]

Then

1. \( u(s) \sim N(0, \frac{(b-s)(s-a)}{b-a}I_n) \).
2. The random vector \( u(s) \) is independent from \( (\text{BM}_n(t)), t \in (0, a] \cup [b, \infty) \).

**Lemma 4.** Let \( d, m \in \mathbb{N} \) be such that \( m \leq d/2 \). Let \( X_1, X_2, \ldots, X_m \) be independent standard Gaussian vectors in \( \mathbb{R}^d \). Then for any \( b \in S^{m-1} \), there exists a random unit vector \( \hat{u} \in \mathbb{R}^d \) such that

\[
\mathbb{P}\left\{ \langle \hat{u}, X_i \rangle \geq \sqrt{d} |b_i|, \text{ for all } i = 1, 2, \ldots, m \right\} \geq 1 - \exp(-\varepsilon d),
\]

where \( \varepsilon \) is a universal constant and \( b_i \)'s are the coordinates of \( b \). Moreover, \( \hat{u} \) can be defined as a Borel function of \( X_i \)'s and \( b \).

**Proof.** Without loss of generality, we can assume that \( b_i \neq 0 \) for any \( i \leq m \) and that \( X_i \)'s are linearly independent on the entire probability space. Denote by \( E \) the
affine subspace spanned by \( \{ |b_i|^{-1} X_i \}_{i \leq m} \). Define \( \bar{u} \) as the unique unit vector in span\( \{ X_1, \ldots, X_m \} \) such that \( \bar{u} \) is orthogonal to \( E \) and for any \( i \leq m \) we have
\[
\langle \bar{u}, |b_i|^{-1} X_i \rangle = \operatorname{dist}(0, E),
\]
where \( \operatorname{dist}(0, E) \) stands for the distance from the origin to \( E \). Then we have
\[
\sum_{i \leq m} \langle \bar{u}, X_i \rangle^2 = \sum_{i \leq m} \langle \bar{u}, X_i / |b_i| \rangle^2 = \sum_{i \leq m} \operatorname{dist}(0, E)^2 \cdot |b_i|^2 = \operatorname{dist}(0, E)^2. \tag{5}
\]
Let \( G \) be the \( m \times d \) standard Gaussian matrix with rows \( X_i, i = 1, 2, \ldots, m \). Using the definition of \( \bar{u} \) together with (5), we obtain for any \( \tau > 0 \):
\[
\operatorname{Pr}\left\{ \langle \bar{u}, X_i \rangle \geq \tau \sqrt{d}|b_i| \text{ for all } i = 1, 2, \ldots, m \right\} = \operatorname{Pr}\left\{ d(0, E) \geq \tau \sqrt{d} \right\}
\]
\[
\quad = \operatorname{Pr}\left\{ \sqrt{\sum_{i \leq m} \langle \bar{u}, X_i \rangle^2} \geq \tau \sqrt{d} \right\}
\]
\[
\quad = \operatorname{Pr}\left\{ \|G\bar{u}\| \geq \tau \sqrt{d} \right\}
\]
\[
\quad \geq \operatorname{Pr}\left\{ s_{\text{min}}(G) \geq \tau \sqrt{d} \right\}.
\]
The proof is finished by choosing a sufficiently small \( \lambda := \tau \) and applying (4). \( \square \)

**Lemma 5.** Let \( q \in \mathbb{N} \) and \( r \in \mathbb{R} \) with \( e \leq r \leq \sqrt{\ln q} \), and let \( \gamma_1, \gamma_2, \ldots, \gamma_q \) be independent standard Gaussian variables. Define a random vector \( b = (b_1, b_2, \ldots, b_q) \in \mathbb{R}^q \) by \( b_i := \max(0, \gamma_i - r), i \leq q \). Then
\[
\operatorname{Pr}\left\{ \|b\| \leq 4 \sqrt{q} \exp(-r^2/8) \right\} \geq 1 - \exp(-2 \sqrt{q}).
\]

**Proof.** Let \( \lambda \in (0, 1/2) \). We have
\[
\mathbb{E} e^{\lambda \|b\|^2} = \prod_{i=1}^q \mathbb{E} e^{\lambda b_i^2} = \left( 1 + \int_1^{\infty} \mathbb{P}\{ e^{\lambda b_i^2} \geq \tau \} d\tau \right)^q.
\]
Next, using (3), we get
\[
\int_1^{\infty} \mathbb{P}\{ e^{\lambda b_i^2} \geq \tau \} d\tau \leq (r-1)\mathbb{P}\{ \gamma_1 > r \} + \int_r^{\infty} \mathbb{P}\{ e^{\lambda b_i^2} \geq \tau \} d\tau
\]
\[
\leq e^{-r^2/2} + \int_r^{\infty} \mathbb{P}\{ \gamma_1 \geq \sqrt{\ln \tau} / \lambda \} d\tau
\]
\[
\leq e^{-r^2/2} + \int_r^{\infty} \tau^{-1/2 \lambda} d\tau
\]
\[
= e^{-r^2/2} + r^{1-1/2 \lambda} \frac{1}{2 \lambda - 1}.
\]
Now, take \( \lambda = \left( 2 + \frac{r^2}{\ln r} \right)^{-1} \) so that \( \frac{1}{2 \lambda} - 1 = \frac{r^2}{2 \ln r} \). After replacing \( \lambda \) with its value, we deduce that
\[
\mathbb{E} e^{\lambda \|b\|^2} \leq \left( 1 + 2e^{-r^2/2} \right)^q \leq \exp(2qe^{-r^2/2}). \tag{6}
\]
Using Markov's inequality together with (6), we obtain
\[
P(\lambda \|b\|^2 \geq 4e^{-r^2/2}) \leq \exp(-2e^{-r^2/2}) \leq \exp(-2\sqrt{q}),
\]
where the last inequality holds since \( r \leq \sqrt{\ln q} \). To finish the proof, it remains to note that
\[
\frac{4e^{-r^2/2}}{\lambda} \leq 8q\lambda e^{-r^2/2} \leq 16q e^{-r^2/4}.
\]

3 The proof

Throughout the section, we assume that \( c > 0 \) and \( n_0 \in \mathbb{N} \) are appropriately chosen constants (with \( c \) sufficiently small and \( n_0 \) sufficiently large) and \( n \geq n_0 \) is fixed. The precise conditions on \( c \) and \( n_0 \) can be recovered from the proof, however, we prefer to avoid these technical details. To prove our main result, we shall construct a random unit vector \( \tilde{n} \in \mathbb{R}^n \) such that
\[
\langle \tilde{n}, BM_n(t) \rangle > 0 \quad \text{for any } t \in [1, 2^{c_n}]
\]
with probability close to one.

Our construction shall be iterative; in fact, we shall produce a sequence of random vectors \( \tilde{n}_k, k = 0, 1, \ldots, M \) (with \( M = \log_2 \ln n \)), where each \( \tilde{n}_k \) satisfies \( \langle \tilde{n}_k, BM_n(t) \rangle > 0 \) for certain discrete subset of \([1, 2^{c_n}]\) with a high probability (the precise condition shall be given later). The size of those discrete subsets shall grow with \( k \) in such a way that the vector \( \tilde{n} := \tilde{n}_M \) shall possess the required property (7) with probability close to one.

Given any \( 0 < k \leq M \), the vector \( \tilde{n}_k \) shall be a “small perturbation” of the vector \( \tilde{n}_{k-1} \). The operation of constructing \( \tilde{n}_k \) will be referred to as the \( k\)-th step of the construction. We must admit that the construction is rather technical. In fact, each step itself shall be divided into a sequence of substeps. To make the exposition of the proof as clear as possible, we won’t provide all the details at once but instead introduce them sequentially.

Let \( M' = \frac{1}{4} \log_2 \ln n \). We split \( \mathbb{R}^n \) into \((M + 1) \times M'\) coordinate subspaces. Precisely, we write
\[
\mathbb{R}^n := \prod_{k=0}^{M} \prod_{\ell=1}^{M'} \mathbb{R}^{J^k_\ell},
\]
where \( J^k_\ell \) are pairwise disjoint subsets of \( \{1, 2, \ldots, n\} \) with \( |J^k_\ell| = c_J n 2^{-(k+\ell)/8} \) for an appropriate constant \( c_J \) and \( \mathbb{R}^{J^k_\ell} = \text{span}\{e_i\}_{i \in J^k_\ell} \). For every \( k \leq M, \ell \leq M' \), define \( P^k_\ell : \mathbb{R}^n \to \mathbb{R}^n \) as the orthogonal projection onto \( \mathbb{R}^{J^k_\ell} \).

Let \( N = cn \) and define
\[
a_0 = 0 \quad \text{and} \quad a_i := 2^{i-1}, \quad i = 1, 2, \ldots, N + 1.
\]

We shall split the interval \([0, a_{N+1}]\) into “blocks”. The zero block is the interval \([0, 1]\); for each admissible \( i \geq 0 \), the \( i\)-th block is the interval \([a_i, a_{i+1}]\). With the
The vector $\bar{n}_0$ shall be constructed directly using Lemma 7. For $k \geq 1$, the vectors $\bar{n}_k$ are obtained via an embedded iteration procedure realized as a sequence of substeps. First, let us give a “partial” description of the procedure, omitting some details.
Fix \( k \geq 1 \) and set \( \bar{n}_{k,0} := \bar{n}_{k-1} \). We shall inductively construct random vectors \( \bar{n}_{k,\ell}, 1 \leq \ell \leq M' \) using the following notion. For each \( \ell = 1, 2, \ldots, M' + 1 \) and every block \( i = 0, 1, 2, \ldots, N \) the \( i \)-th block statistic is

\[
B_i(k, \ell) := \max\left(0, \max_{t \in I_i^k} \left\langle \bar{n}_{k,\ell-1}, \frac{BM_n(a_i) - BM_n(t)}{\sqrt{a_i}} \right\rangle - h(k, \ell), \right.
\]

\[
\left. \frac{BM_n(a_i - BM_n(a_{i+1}))}{\sqrt{a_{i+1}}} + f(k, \ell) \right\rangle.
\]

Note that for the zero block the corresponding statistic is simply

\[
\max\left(0, -\langle \bar{n}_{k,\ell-1}, BM_n(a_1) \rangle + f(k, \ell) \right).
\]

The \((N + 1)\)-dimensional vector \((B_0(k, \ell), \ldots, B_N(k, \ell))\) shall be denoted by \( B(k, \ell) \). Let us also denote

\[
I(k, \ell) := \{i : B_i(k, \ell) \neq 0\}.
\]

Given \( \bar{n}_{k,\ell-1} \), the goal of the \( \ell \)-th substep is to construct a random unit vector \( \bar{n}_{k,\ell} \) such that

1. \( \bar{n}_{k,\ell} \in \prod_{(p, q) \prec(k, \ell)} \mathbb{R}^p_q \), with \((p, q) \prec(k, \ell)\) meaning “\( p < k \) or \( p = k, q \leq \ell \)”;
   \( (11) \)

2. \( \bar{n}_{k,\ell} \) is measurable with respect to the \( \sigma \)-algebra generated by \( P_{p_q}BM_n(t) \),
   for all \((p, q) \prec(k, \ell)\) and \( t \in \{a_1, \ldots, a_{N+1}\} \cup I_1^k \cup I_2^k \cup \cdots \cup I_N^k \);
   \( (12) \)

3. \( B(k, \ell + 1) \) “typically” has a smaller Euclidean norm than \( B(k, \ell) \).

The third property shall be made more precise later. For now, we note that the “typical” value of \( B(k, \ell) \) shall decrease with \( \ell \) in such a way that, after the \( M' \)-th substep, the vector \( \bar{B}(k, M' + 1) \) shall be zero with probability close to one. Juxtaposing the definition of the block statistics with that of \( E_k \), it is easy to see that, by setting \( \bar{n}_k := \bar{n}_{k,M'} \), we get

\[
\mathbb{P}(E_k) = \mathbb{P}\{I(k, M' + 1) = \emptyset\} = \mathbb{P}\{B(k, M' + 1) = 0\} \approx 1.
\]

The vector \( \bar{n}_{k,\ell} \) shall be defined as

\[
\bar{n}_{k,\ell} = \frac{\bar{n}_{k,\ell-1} + \alpha_{k,\ell} \Delta_{k,\ell}}{\sqrt{1 + \alpha_{k,\ell}^2}},
\]

where \( \Delta_{k,\ell} \) is a random unit vector (“perturbation”) and \( \alpha_{k,\ell} = 16^{-k-\ell} \).
The vector $\bar{\Delta}_{k,\ell}$ shall satisfy the following properties:

1. $\bar{\Delta}_{k,\ell} \in \mathbb{R}^k$;  
2. $\bar{\Delta}_{k,\ell}$ is measurable with respect to the $\sigma$-algebra generated by $P^p_k\mathcal{B}_n(t)$ for all admissible $(p,q) \preceq (k,\ell)$ and $t \in \{a_1, \ldots, a_{N+1}\} \cup I_k^1 \cup I_k^2 \cup \cdots \cup I_k^N$;  
3. For any fixed subset $I \subset \{0,1,\ldots,N\}$ such that $P\{I(k,\ell) = I\} > 0$, $\bar{\Delta}_{k,\ell}$ is conditionally independent from the collection of random vectors $\{P^p_k(\mathcal{B}_n(t) - \mathcal{B}_n(a_i)) \ (t \in I_k^1 \cup \{a_i+1\}, \ i \notin I\}$ given the event $\{I(k,\ell) = I\}$.
4. The event $E_{k,\ell} := \{B_i(k,\ell+1) = 0 \text{ for all } i \in \mathcal{I}(k,\ell)\}$ has probability close to one.

Again, we shall make the last statement more precise later. Before that, we need to verify certain quantitative properties of the block statistics. The next Lemma deals with the statistics for the initial substep; it is followed by a corresponding statement for $\mathcal{B}(k,\ell), \ell > 1$.

**Lemma 6** (Initial substep for block statistics). Fix any $1 \leq k \leq M$ and assume that a random unit vector $\bar{n}_{k,0} := \bar{n}_{k-1}$ satisfying properties (9) and (10) has been constructed. Then

$$
\mathbb{P}\left\{|\mathcal{I}(k,1)| \leq N \exp(-C_h 2^{k/2}/16) \text{ and } \|\mathcal{B}(k,1)\| \leq \frac{8\sqrt{N}}{\exp(C_h 2^{k/2}/32)}\right\} \\
\geq \mathbb{P}(E_{k-1}) - 2 \exp(-2\sqrt{N}).
$$

**Proof.** Let $i > 0$ so that $I_{k-1}^i \neq \emptyset$. For each $t \in I_k^i \setminus I_{k-1}^i$, let $t_L$ be the maximal number in $\{a_i\} \cup I_{k-1}^i$ strictly less than $t$ ("left neighbour") and, similarly, $t_R$ be the minimal number in $I_{k-1}^i \cup \{a_i+1\}$ strictly greater than $t$ ("right neighbour"). For every such $t$, let

$$w_t := \frac{t_R - t}{t_R - t_L}\mathcal{B}_n(t_L) + \frac{t - t_L}{t_R - t_L}\mathcal{B}_n(t_R); \ u_t := \mathcal{B}_n(t) - w_t.
$$

It is not difficult to see that

$$
\langle \bar{n}_{k,0}, \frac{\mathcal{B}_n(a_i) - w_t}{\sqrt{a_i}} \rangle \\
\leq \max\left(\langle \bar{n}_{k,0}, \frac{\mathcal{B}_n(a_i) - \mathcal{B}_n(t_L)}{\sqrt{a_i}} \rangle, \langle \bar{n}_{k,0}, \frac{\mathcal{B}_n(a_i) - \mathcal{B}_n(t_R)}{\sqrt{a_i}} \rangle \right) \\
\leq \max\left(0, \max_{\tau \in I_{k-1}^i} \langle \bar{n}_{k,0}, \frac{\mathcal{B}_n(a_i) - \mathcal{B}_n(\tau)}{\sqrt{a_i}} \rangle, \langle 2\bar{n}_{k,0}, \frac{\mathcal{B}_n(a_i) - \mathcal{B}_n(a_{i+1})}{\sqrt{a_{i+1}}} \rangle \right).
$$

Hence, the $i$-th block statistic (for $i = 0, 1, \ldots, N$) can be (deterministically) bounded...
as
\[
\mathcal{B}_i(k, 1) \leq \max_{t \in I_k^i \setminus I_{k-1}^i} \left( 0, \max_{t \in I_k^i \setminus I_{k-1}^i} \left\langle \tilde{\eta}_{k, 0}, \frac{\text{BM}_n(a_i) - \text{BM}_n(t)}{\sqrt{a_i}} \right\rangle - h(k, 1), \right.
\]
\[
\left. \max_{t \in I_k^i \setminus I_{k-1}^i} \left\langle \tilde{\eta}_{k, 0}, \frac{\text{BM}_n(a_i) - w_i}{\sqrt{a_i}} \right\rangle - h(k, 1) + \max_{t \in I_k^i \setminus I_{k-1}^i} \left\langle \tilde{\eta}_{k, 0}, \frac{-u_t}{\sqrt{a_i}} \right\rangle, \right.
\]
\[
\left. \langle \tilde{\eta}_{k, 0}, \frac{\text{BM}_n(a_i) - \text{BM}_n(a_{i+1})}{\sqrt{a_{i+1}}} \rangle + f(k, 1) \right) \right)
\]
\[
\leq \max_{t \in I_k^i \setminus I_{k-1}^i} \left( 0, \max_{t \in I_k^i \setminus I_{k-1}^i} \left\langle \tilde{\eta}_{k, 0}, \frac{\text{BM}_n(a_i) - \text{BM}_n(t)}{\sqrt{a_i}} \right\rangle - h(k, 0), \right.
\]
\[
\left. \langle 2\tilde{\eta}_{k, 0}, \frac{\text{BM}_n(a_i) - \text{BM}_n(a_{i+1})}{\sqrt{a_{i+1}}} \rangle + 2f(k, 0) \right) + \max_{t \in I_k^i \setminus I_{k-1}^i} \left( 0, \max_{t \in I_k^i \setminus I_{k-1}^i} \left\langle \tilde{\eta}_{k, 0}, \frac{-u_t}{\sqrt{a_i}} \right\rangle + h(k, 0) - h(k, 1) \right). \]

Let us denote the first summand in the last estimate by \( \xi_i \), so that
\[
\mathcal{B}_i(k, 1) \leq \xi_i + \max_{t \in I_k^i \setminus I_{k-1}^i} \left( 0, \max_{t \in I_k^i \setminus I_{k-1}^i} \left\langle \tilde{\eta}_{k, 0}, \frac{-u_t}{\sqrt{a_i}} \right\rangle + h(k, 0) - h(k, 1) \right). \]

Note that
\[
\mathcal{E}_{k-1} = \{ \xi_i = 0 \text{ for all } i = 0, 1, \ldots, N \}. \tag{17}
\]
Further, the property (10) of the vector \( \tilde{\eta}_{k, 0} = \tilde{\eta}_{k-1} \), together with Lemma 3 and properties of the Brownian motion, imply that the Gaussian variables \( \langle \tilde{\eta}_{k, 0}, \frac{-u_t}{\sqrt{a_i}} \rangle \) are jointly independent for \( t \in I_k^i \setminus I_{k-1}^i, i = 1, 2, \ldots, N \), and the variance of each one can be estimated from above by \( 2^{1-k} \). Thus, the vector \( \mathcal{B}(k, 1) \) can be majorized coordinate-wise by the vector
\[
(\xi_i + \max_{t \in I_k^i \setminus I_{k-1}^i} (0, 2^{1-k}/2 \gamma_t + h(k, 0) - h(k, 1)))^{N} \]
where \( \gamma_t \) (\( t \in I_k^i \setminus I_{k-1}^i, i = 0, 1, \ldots, N \)) are i.i.d. standard Gaussians (in fact, appropriate scalar multiples of \( \langle \tilde{\eta}_{k, 0}, \frac{-u_t}{\sqrt{a_i}} \rangle \)). Denoting by \( \gamma \) the standard Gaussian variable, we get from the definition of \( h \):
\[
P\left\{ \max_{t \in I_k^i \setminus I_{k-1}^i} (0, 2^{1-k}/2 \gamma_t + h(k, 0) - h(k, 1)) > 0 \right\} \leq 2^k P\{ \gamma > C_h 2^{k/4}/2 \}
\]
\[
\leq 2^k \exp(-C_h 2^{k/2}/8)
\]
\[
\leq \frac{1}{2} \exp(-C_h 2^{k/2}/16). \]
(In the last two inequalities, we assumed that \( C_h \) is sufficiently large). Applying Hoeffding’s inequality to corresponding indicators, we infer
\[
|\mathcal{I}(k, 1)| \leq |\{i : \xi_i \neq 0\}| + N \exp(-C_h 2^{k/2}/16)
\]
with probability at least \( 1 - \exp(-2\sqrt{N}) \) (we note that, in view of the inequality \( k \leq M_i \), we have \( \frac{1}{2} \exp(-C_h 2^{k/2}/16) \geq N^{-1/4} \)). Next, it is not hard to see that the Euclidean norm of \( \mathcal{B}(k, 1) \) is majorized (deterministically) by the sum
\[
\| (\xi_i)_{i=0}^{N} \| + 2^{(1-k)/2} \left( \max(0, \gamma_t - C_h 2^{k/4}/2) \right) \|,
\]
with the second vector having $\sum_{i=0}^{N} |I_k^i \setminus I_{k-1}^i| \leq 2^k N$ coordinates. Applying Lemma 5 to the second vector (note that for sufficiently large $n$ we have $C_h 2^{k/4}/2 \leq \sqrt{\ln N}$), we get
\[
\| \mathcal{B}(k, 1) \| \leq \| (\xi_i)_{i=0}^{N} \| + \frac{8\sqrt{N}}{\exp(C_h 2^{k/2}/32)}
\]
with probability at least $1 - \exp(-2\sqrt{N})$. Combining the estimates with (17), we obtain the result.

**Lemma 7** (Subsequent substeps for block statistics). Fix any $1 \leq k \leq M$ and $1 < \ell \leq M'+1$ and assume that the random unit vectors $\hat{n}_{k, \ell-2}$ and $\Delta_{k, \ell-1}$ satisfying properties (11) — (12) and (14) — (16), respectively, have been constructed, and $\hat{n}_{k, \ell-1}$ is defined according to formula (13). Then
\[
\mathbb{P}\{ |\mathcal{I}(k, \ell)| \leq N \exp(-C_h 2^{(k+\ell)/2}) \text{ and } \| \mathcal{B}(k, \ell) \| \leq \frac{\sqrt{N}}{\exp(C_h 2^{(k+\ell)/2})} \} \\
\geq \mathbb{P}(\xi_{k, \ell-1}) - 2 \exp(-2\sqrt{N}).
\]
Moreover,
\[
\mathbb{P}\{ |\mathcal{I}(k, \ell) \| \neq 0 \} \leq N \exp(-C_h 2^{2/\alpha_{k, \ell-1}}) + 1 - \mathbb{P}(\xi_{k, \ell-1}).
\]

**Proof.** To shorten the notation, we shall use $\alpha$ in place of $\alpha_{k, \ell-1}$ within the proof. Using the definition of $\hat{n}_{k, \ell-1}$ in terms of $\hat{n}_{k, \ell-2}$ and $\Delta_{k, \ell-1}$, we get for every $i = 0, 1, \ldots, N$
\[
\mathcal{B}_i(k, \ell) = \max_{t \in I_k^i} \left( \frac{\hat{n}_{k, \ell-2} + \alpha \Delta_{k, \ell-1}}{\sqrt{1 + \alpha^2}}, \frac{BM_n(a_i) - BM_n(t)}{\sqrt{a_i}} \right) - h(k, \ell),
\]
\[
= \left( \frac{\hat{n}_{k, \ell-2} + \alpha \Delta_{k, \ell-1}}{\sqrt{1 + \alpha^2}}, \frac{BM_n(a_i) - BM_n(a_{i+1})}{\sqrt{a_{i+1}}} \right) + f(k, \ell)
\]
\[
\leq \frac{\mathcal{B}_i(k, \ell-1)}{\sqrt{1 + \alpha^2}}
\]
\[
+ \max_{t \in I_k^i} \left( \frac{\alpha \Delta_{k, \ell-1}}{\sqrt{a_i}}, \frac{BM_n(a_i) - BM_n(t)}{\sqrt{a_i}} \right) + h(k, \ell-1) - h(k, \ell),
\]
\[
\leq \left( \frac{\alpha \Delta_{k, \ell-1}}{\sqrt{a_{i+1}}}, \frac{BM_n(a_i) - BM_n(a_{i+1})}{\sqrt{a_{i+1}}} \right) + \sqrt{1 + \alpha^2} f(k, \ell) - f(k, \ell-1).
\]
Let us denote the second summand by $\eta_i$ so that
\[
\mathcal{B}_i(k, \ell) \leq \frac{\mathcal{B}_i(k, \ell-1)}{\sqrt{1 + \alpha^2}} + \eta_i.
\]

Fix for a moment any subset $I$ of $\{0, 1, \ldots, N\}$ such that $\mathbb{P}\{ |\mathcal{I}(k, \ell-1) = I \} > 0$. A crucial observation is that, conditioned on the event $|\mathcal{I}(k, \ell-1) = I \}$, the variables $\eta_i$, $i \notin I$, are jointly independent. This follows from properties (14), (16) of $\Delta_{k, \ell-1}$ and properties of the Brownian motion. Next, the same properties tell us that, conditioned on $|\mathcal{I}(k, \ell-1) = I \}$, each variable $\langle \Delta_{k, \ell-1}, \frac{BM_n(a_i) - BM_n(t)}{\sqrt{a_i}} \rangle$, $t \in I_k^i$, and $\langle \Delta_{k, \ell-1}, \frac{BM_n(a_i) - BM_n(a_{i+1})}{\sqrt{a_{i+1}}} \rangle$ have Gaussian distributions with variances at most 1. Further, note that, by the choice of $\alpha$ and the functions $f$ and $h$, we have
\[
\sqrt{1 + \alpha^2} f(k, \ell) - f(k, \ell-1) \leq h(k, \ell-1) - h(k, \ell) = -C_h 2^{(-k-\ell)/4}.
\]
Thus, denoting by $\gamma$ the standard Gaussian variable, we get
\[
\mathbb{P}\{\eta_i > 0 \mid \mathcal{I}(k, \ell - 1) = I\} \leq 2^k \mathbb{P}\{\gamma > \alpha^{-1} C_h 2^{(-k-\ell)/4}\} \\
\leq \frac{1}{2} \exp(-C_h^2 \alpha^{-1}), \ i \in \{0, 1, \ldots, N\} \setminus I.
\] (18)
Hence, by Hoeffding’s inequality (note that $\exp(-C_h^2 2^{(k+\ell)/2}) > 2N^{-1/4}$):
\[
\mathbb{P}\{\{i \notin I : \eta_i > 0\} \geq N \exp(-C_h^2 2^{(k+\ell)/2}) \mid \mathcal{I}(k, \ell - 1) = I\} \leq \exp(-2\sqrt{N}).
\]
Next, it is not difficult to see that for any $\tau > 0$ and $i \notin I$
\[
\mathbb{P}\{\eta_i^2 \geq \tau \mid \mathcal{I}(k, \ell - 1) = I\} \leq 2^k \mathbb{P}\{\max(0, \alpha \gamma - C_h 2^{(-k-\ell)/4})^2 \geq \tau\} \\
\leq 1 - \exp(-2^{k+1} \mathbb{P}\{\max(0, \alpha \gamma - C_h 2^{(-k-\ell)/4})^2 \geq \tau\}) \\
\leq 1 - \mathbb{P}\{\max(0, \alpha \gamma - C_h 2^{(-k-\ell)/4})^2 < \tau\}^{2^{k+1}} \\
\leq \mathbb{P}\left\{\sum_{j=1}^{2^{k+1}} \max(0, \alpha \gamma_j - C_h 2^{(-k-\ell)/4})^2 \geq \tau\right\} \\
\leq \mathbb{P}\left\{\sum_{j=1}^{2^{k+1}} \max(0, \alpha \gamma_j - 4C_h 2^{(k+\ell)/4})^2 \geq \tau\right\},
\]
where $\gamma_j$ ($j = 1, 2, \ldots, 2^{k+1}$) are i.i.d. copies of $\gamma$. Hence, the conditional cdf of $\|\eta_i\|_{i \notin I}$ given $\mathcal{I}(k, \ell - 1) = I$ majorizes the cdf of
\[
\alpha \sqrt{\left(\max(0, \gamma_j - 4C_h 2^{(k+\ell)/4})\right)_{j=1}^{2^{k+1}N}}
\]
for i.i.d. standard Gaussians $\gamma_j$, $j = 1, 2, \ldots, 2^k N$. Applying Lemma 5 (note that $4C_h 2^{(k+\ell)/4} \leq \sqrt{\ln N}$), we obtain
\[
\mathbb{P}\left\{\|\eta_i\|_{i \notin I} \geq \frac{\sqrt{N}}{\exp(C_h^2 2^{(k+\ell)/2})} \mid \mathcal{I}(k, \ell - 1) = I\right\} \\
\leq \mathbb{P}\{\left(\max(0, \gamma_j - 4C_h 2^{(k+\ell)/4})\right)_{j=1}^{2^{k+1}N} \geq \frac{\alpha^{-1} \sqrt{N}}{\exp(C_h^2 2^{(k+\ell)/2})} \mid \mathcal{I}(k, \ell - 1) = I\} \\
\leq \mathbb{P}\{\left(\max(0, \gamma_j - 4C_h 2^{(k+\ell)/4})\right)_{j=1}^{2^{k+1}N} \geq \frac{4\sqrt{2^{2k+1}N}}{\exp(2C_h^2 2^{(k+\ell)/2})} \mid \mathcal{I}(k, \ell - 1) = I\} \\
\leq \exp(-2\sqrt{N}).
\]
Now, clearly $B_i(k, \ell - 1) = 0$ for all $i \notin I$ given $\mathcal{I}(k, \ell - 1) = I$. Hence, the above estimates give
\[
\mathbb{P}\left\{\mathcal{I}(k, \ell) \geq N \exp(-C_h^2 2^{(k+\ell)/2}) \right\} \\
\text{or } \|B(k, \ell)\| \geq \frac{\sqrt{N}}{\exp(C_h^2 2^{(k+\ell)/2})} \mid \mathcal{I}(k, \ell - 1) = I\} \\
\leq \mathbb{P}\{B_i(k, \ell) > 0 \text{ for some } i \in I \mid \mathcal{I}(k, \ell - 1) = I\} + 2\exp(-2\sqrt{N}).
\]
Now, summing over all admissible subsets $I$, we get
\[
\mathbb{P}\left\{ |I(k, \ell)| \geq N \exp(-C_h 2^{(k+\ell)/2}) \text{ or } \|B(k, \ell)\| > \frac{\sqrt{N}}{\exp(C_h 2^{(k+\ell)/2})} \right\} \\
\leq 2 \exp(-2\sqrt{N}) + \sum_I \mathbb{P}\{B_i(k, \ell) > 0 \text{ for some } i \in I | I(k, \ell - 1) = I\} \mathbb{P}\{I(k, \ell - 1) = I\} \\
= 2 \exp(-2\sqrt{N}) + \mathbb{P}\{B_i(k, \ell) > 0 \text{ for some } i \in I(k, \ell - 1)\} \\
= 2 \exp(-2\sqrt{N}) + 1 - \mathbb{P}(\mathcal{E}_{k,\ell-1}).
\]

By analogous argument, as a corollary of (18),
\[
\mathbb{P}\{I(k, \ell) \neq 0\} \leq N \exp(-C_h 2^{\alpha-1}) + 1 - \mathbb{P}(\mathcal{E}_{k,\ell-1}).
\]

\[\square\]

**Lemma 8** (Construction of $\Delta_{k,\ell}$). Let $1 \leq k \leq M$ and $1 \leq \ell \leq M'$ and assume that the random unit vector $\vec{u}_{k,\ell-1}$ satisfying properties (11) and (12) has been constructed. Then one can construct a random unit vector $\Delta_{k,\ell}$ satisfying properties (13)–(16) and such that
\[
\mathbb{P}(\mathcal{E}_{k,\ell}) \geq \mathbb{P}(\mathcal{E}_{k,\ell-1}) - 3 \exp(-\sqrt{N})
\]
if $\ell > 1$, or
\[
\mathbb{P}(\mathcal{E}_{k,\ell}) \geq \mathbb{P}(\mathcal{E}_{k-1}) - 3 \exp(-\sqrt{N})
\]
if $\ell = 1$.

**Proof.** Fix for a moment any subset $I \subset \{0, 1, \ldots, N\}$ such that the event
\[
\mathcal{E}_I = \{I(k, \ell) = I\}
\]
has a non-zero probability. If $|I| > N \exp(-C_h 2^{(k+\ell)/2}/32)$ then define a “random” vector $\Delta^I_{k,\ell}$ on $\mathcal{E}_I$ by setting $\Delta^I_{k,\ell} := u$ for a fixed unit vector $u \in \mathbb{R}^k_{\ell}$. Otherwise, if $|I| \leq N \exp(-C_h 2^{(k+\ell)/2}/32)$, we proceed as follows:

For each $i \in I \setminus \{0\}$, define $2^k$ “increments” on $\mathcal{E}_I$:
\[
X_{i,p} := \frac{\text{P}^k_{\ell}(BM_n(t_{i,p+1}) - BM_n(t_{i,p}))}{\sqrt{t_{i,p+1} - t_{i,p}}}, \quad p = 0, 1, \ldots, 2^k - 1,
\]
where $t_{i,p} = 2^{i-1+p/2^k}$ for $p = 0, 1, \ldots, 2^k$. Additionally, if $0 \in I$, then define
\[
X_{0,0} := \text{P}^k_{\ell}BM_n(1).
\]

Let us denote by $T_I$ the set of all pairs of indices $(i, p)$ corresponding to the “increments” $X_{i,p}$. Note that the process $\text{P}^k_{\ell}BM_n(t)$ is independent from $\mathcal{E}_I$; in particular, \{$(X_{i,p}, (i, p) \in T_I)$\} is a collection of standard Gaussian vectors on $\mathcal{E}_I$ with values in $\mathbb{R}^k_{\ell}$, such that all $X_{i,p}$ and the vector $B(k, \ell)$ are jointly independent given $\mathcal{E}_I$. Let us define a random vector $\vec{b}^I \in \mathbb{R}^{T_I}$ on $\mathcal{E}_I$ by
\[
\vec{b}^I_{i,p} = \begin{cases} 2^{-k/2}B_i(k, \ell)/\|B(k, \ell)\|, & \text{if } B(k, \ell) \neq 0; \\ 0, & \text{otherwise.} \end{cases}
\]

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It is easy to see that $\|\bar{b}^I\| \leq 1$ (deterministically) and that

$$|T_I| \leq 2^k |I| \leq 2^k N \exp\left(-C_h 2^{(k+h)/2}/32\right) \leq \frac{1}{2} |J^k|.$$  

(In the last estimate, we used the assumption that $C_h$ is a large constant). Hence, in view of Lemma 5 there exists a random unit vector $\Delta_{k,\ell}^I \in \mathbb{R}^{J^k}$ on $\mathcal{E}_I$ (which is a Borel function of $X_{i,p}$ and $\bar{b}^I$) such that

$$\mathbb{P}\left\{ \langle \Delta_{k,\ell}^I, X_{i,p} \rangle \geq \sqrt{|J^k|} \|\bar{b}^I\| \text{ for all } (i, p) \in T_I | \mathcal{E}_I \right\} \geq 1 - \exp\left(-\frac{1}{2} |J^k|\right) \geq 1 - \exp\left(-\sqrt{N}\right).$$

It will be convenient for us to denote by $\tilde{\mathcal{E}}_I$ the event

$$\{ \langle \Delta_{k,\ell}^I, X_{i,p} \rangle \geq \sqrt{|J^k|} \|\bar{b}^I\| \text{ for all } (i, p) \in T_I \} \subset \mathcal{E}_I.$$ 

By “glueing together” $\bar{\Delta}_{k,\ell}^I$ for all $I$, we obtain a random vector $\bar{\Delta}_{k,\ell}$ on the entire probability space.

Clearly, $\bar{\Delta}_{k,\ell}$ satisfies properties (14) and (15). Next, on each $\mathcal{E}_I$ with $\mathbb{P}(\mathcal{E}_I) > 0$ the vector $\bar{\Delta}_{k,\ell}$ was defined as a Borel function of $B(k, \ell)$ and $\mathbb{P}^k(\mathbb{B}(\ell) - \mathbb{B}(\tau))$, $t, \tau \in I_k \cup \{a_i, a_{i+1}\}$, $i \in I$, so, in view of the properties of the Brownian motion, $\bar{\Delta}_{k,\ell}$ satisfies (16).

Finally, we shall estimate the probability of $\mathcal{E}_{k,\ell}$. Define

$$\mathcal{E} = \left\{ |I(k, \ell)| \leq N \exp\left(-C_h 2^{(k+h)/2}/32\right) \text{ and } \|B(k, \ell)\| \leq \frac{\sqrt{N}}{\exp\left(C_h 2^{(k+h)/2}/64\right)} \right\}.$$  

Note that, according to Lemmas 5 and 6 the probability of $\mathcal{E}$ can be estimated from below by $\mathbb{P}(\mathcal{E}_{k,\ell-1}) - 2 \exp(-2\sqrt{N})$ for $\ell > 1$ and $\mathbb{P}(\mathcal{E}_{k-1}) - 2 \exp(-2\sqrt{N})$ for $\ell = 1$.

Take any subset $I \subset \{0, 1, \ldots, N\}$ with $|I| \leq N \exp(-C_h 2^{(k+h)/2}/32)$ and such that $\tilde{\mathcal{E}}_I \cap \mathcal{E} \neq \emptyset$, and let $\omega \in \tilde{\mathcal{E}}_I \cap \mathcal{E}$. If $I(k, \ell) = \emptyset$ at point $\omega$ then, obviously, $\omega \in \mathcal{E}_{k,\ell}$. Otherwise, we have

$$\langle \bar{\Delta}_{k,\ell}(\omega), \frac{\mathbb{B}_n(t_{i,p+1})(\omega) - \mathbb{B}_n(t_{i,p})(\omega)}{\sqrt{t_{i,p+1} - t_{i,p}}} \rangle \geq \frac{2^{k-1/2} \sqrt{|J^k|} \|B(k, \ell)(\omega)\|}{\|B(k, \ell)(\omega)\|} \text{ for all } (i, p) \in T_I,$$

whence, using the estimate $t_{i,p+1} - t_{i,p} \geq \frac{2^{i+h}}{4}$ ($(i, p) \in T_I$), we obtain for any $i \in I$ and $t \in I_k \cup \{a_{i+1}\}$:

$$\langle \bar{\Delta}_{k,\ell}(\omega), \mathbb{B}_n(t)(\omega) - \mathbb{B}_n(a_{i})(\omega) \rangle = \sum_{p : t_{i,p} < t} \langle \bar{\Delta}_{k,\ell}(\omega), \mathbb{B}_n(t_{i,p+1})(\omega) - \mathbb{B}_n(t_{i,p})(\omega) \rangle \geq \frac{2^{k-1} \sqrt{a_{i+1} |J^k|} \|B(k, \ell)(\omega)\|}{\|B(k, \ell)(\omega)\|}.$$
Further,
\[
\frac{N^{2-k-1} \sqrt{|J_k|}}{\|B(k, \ell)(\omega)\|} \geq \frac{N^{2-k-1} \sqrt{cJ_n2(-k-\ell)/8} \exp(CH^{2/2(k+\ell)/2}/64)}{\sqrt{N}} \geq \frac{1}{\alpha_{k,\ell}}.
\]

Using the definition of \( \bar{n}_{k,\ell} \) in terms of \( \bar{n}_{k,\ell-1} \) and \( \bar{\Delta}_{k,\ell} \) and the above estimates, we get
\[
\langle \bar{n}_{k,\ell}(\omega), BM_n(t)(\omega) - BM_n(a_i)(\omega) \rangle \geq \frac{-h(k, \ell) - B_i(k, \ell)(\omega)}{\sqrt{1 + \alpha_{k,\ell}^2}} + \frac{\alpha_{k,\ell}}{\sqrt{1 + \alpha_{k,\ell}^2}} \langle \bar{\Delta}_{k,\ell}(\omega), BM_n(t)(\omega) - BM_n(a_i)(\omega) \rangle \geq \frac{-h(k, \ell)}{\sqrt{1 + \alpha_{k,\ell}^2}} \geq -h(k, \ell + 1), \quad t \in \tilde{I}_k, \quad i \in I,
\]
and similarly,
\[
\langle \bar{n}_{k,\ell}(\omega), BM_n(a_i+1)(\omega) - BM_n(a_i)(\omega) \rangle \geq \frac{f(k, \ell)}{\sqrt{1 + \alpha_{k,\ell}^2}} \geq f(k, \ell + 1), \quad i \in I.
\]
Thus, by the definition of the event \( \mathcal{E}_{k,\ell} \), we get \( \omega \in \mathcal{E}_{k,\ell} \).

The above argument shows that
\[
\mathbb{P}(\mathcal{E}_{k,\ell}) \geq \sum_I \mathbb{P}(\tilde{E}_I \cap \mathcal{E}),
\]
where the sum is taken over all \( I \) with \( |I| \leq N \exp\left(-C'h^{2/2(k+\ell)/2}/32\right) \). Finally,
\[
\sum_I \mathbb{P}(\tilde{E}_I \cap \mathcal{E}) \geq \sum_I \mathbb{P}(E_I \cap \mathcal{E}) - \sum_I \mathbb{P}(E_I \setminus \tilde{E}_I) \geq \mathbb{P}(\mathcal{E}) - \mathbb{P}(\mathcal{E} - \exp(-\sqrt{N})],
\]
and we get the result. \( \square \)

**Lemma 9 (k-th Step).** Let \( 1 \leq k \leq M \) and assume that a random unit vector \( \bar{n}_{k-1} \) satisfying properties \( (9) \) \( (10) \) has been constructed. Then there exists a random unit vector \( \bar{n}_k \) satisfying \( (9) \) \( (11) \) and such that
\[
\mathbb{P}(\mathcal{E}_k) \geq \mathbb{P}(\mathcal{E}_{k-1}) - \frac{1}{n^2}.
\]

**Proof.** As before, we set \( \bar{n}_{k,0} := \bar{n}_{k-1} \). Consecutively applying Lemma \( 8 \) and formula \( (13) \) \( M' \) times, we obtain a random unit vector \( \bar{n}_{k,M'} \) satisfying \( (11) \) \( (12) \). Moreover, the same lemma provides the estimate
\[
\mathbb{P}(\mathcal{E}_{k,M'}) \geq \mathbb{P}(\mathcal{E}_{k-1}) - M' \exp(-\sqrt{N}).
\]
Then, in view of Lemma \( 7 \) and the definition of \( M' \), we have
\[
\mathbb{P}\{I(k, M' + 1) \neq \emptyset\} \leq N \exp(-C'h^{2/\alpha_{k,M'}}) + 1 - \mathbb{P}(\mathcal{E}_{k,M'}) \leq \frac{1}{n^2} + 1 - \mathbb{P}(\mathcal{E}_{k-1}).
\]
Combining the above estimate with the definition of \( \mathcal{E}_k \), we get for \( \bar{n}_k := \bar{n}_{k,M'}; \)
\[
\mathbb{P}(\mathcal{E}_k) \geq \mathbb{P}(\mathcal{E}_{k-1}) - \frac{1}{n^2}. \quad \square
Proof of Theorem 1. Define a vector \( b = (b_0, b_1, \ldots, b_N) \) by
\[
b_i := 2^{-1} f(1, 0), \quad i = 0, 1, \ldots, N.
\]
In view of the definition of \( f \) and the relation (8), we have \( \|b\| \leq \sqrt{|J_1^0|} \), and, as we have chosen \( c \) to be small, \( N + 1 \leq |J_1^0|/2 \). Hence, in view of Lemma 4, there exists a random unit vector \( \tilde{n}_0 \in \mathbb{R}^{d_1} \) measurable with respect to the \( \sigma \)-algebra generated by vectors \( P^1_1(BM_n(a_{i+1}) - BM_n(a_i)), i = 0, 1, \ldots, N \), and such that
\[
\mathbb{P}(\mathcal{E}_0) = \mathbb{P}\{\langle \tilde{n}_0, BM_n(a_{i+1}) - BM_n(a_i) \rangle \geq f(1, 0)\sqrt{a_{i+1}} \text{ for all } i = 0, 1, \ldots, N\} \\
\geq \mathbb{P}\{\langle \tilde{n}_0, BM_n(a_{i+1}) - BM_n(a_i) \rangle \geq \frac{1}{2} f_1 \text{ for all } i = 0, 1, \ldots, N\} \\
\geq 1 - \exp(-\frac{1}{4} |J_1^0|) \\
\geq 1 - \frac{1}{n^2}.
\]
Applying Lemma 4 \( M \) times, we obtain a random unit vector \( \tilde{n}_M \) satisfying (9)-(10) such that
\[
\mathbb{P}(\mathcal{E}_M) \geq 1 - \frac{M + 1}{n^2}.
\]
Note that for any \( \omega \in \mathcal{E}_M \), we have
\[
\langle \tilde{n}_M, BM_n(a_{i+1})(\omega) \rangle \geq \langle \tilde{n}_M, BM_n(a_{i+1})(\omega) - BM_n(a_i)(\omega) \rangle \geq C_f \sqrt{a_{i+1}}
\]
and
\[
\langle \tilde{n}_M, BM_n(t)(\omega) \rangle \geq \langle \tilde{n}_M, BM_n(a_i)(\omega) \rangle - \frac{C_f}{2} \sqrt{a_i} \geq \frac{C_f}{2} \sqrt{a_i}, \quad t \in I_i
\]
for all \( i = 0, 1, \ldots, N \). Hence, denoting \( Q := \{a_1, a_2, \ldots, a_{N+1}\} \cup \bigcup_{i=1}^{N} I_i \), we get
\[
\mathcal{E}_M \subset \left\{ \begin{array}{l}
\langle \tilde{n}_M, \frac{BM_n(t)}{\sqrt{t}} \rangle \geq \frac{C_f}{4}, \quad t \in Q
\end{array} \right\}.
\]
(19)

Now, take any two adjacent (i.e. neighbour) points \( t_1 < t_2 \) from \( Q \). Note that, conditioned on a realization of vectors \( BM_n(t), t \in Q \), the random process
\[
X(s) = \langle \tilde{n}_M, \frac{BM_n(t_2) + (1-s)BM_n(t_1)}{\sqrt{t_2 - t_1}} \rangle - \langle \tilde{n}_M, \frac{BM_n(t_1 + s(t_2 - t_1))}{\sqrt{t_2 - t_1}} \rangle, \quad s \in [0, 1],
\]
is the standard Brownian bridge. Hence (see, for example, [7] p. 34), we have for any \( \tau > 0 \)
\[
\mathbb{P}\{X(s) \geq \tau \text{ for some } s \in [0, 1]\} = \exp(-2\tau^2).
\]
Taking \( \tau := 2\sqrt{\ln n} \), we obtain
\[
\mathbb{P}\left\{ \langle \tilde{n}_M, BM_n(t) \rangle \leq \max(\langle \tilde{n}_M, BM_n(t_1) \rangle, \langle \tilde{n}_M, BM_n(t_2) \rangle) \right. \\
\left. - 2\sqrt{t_2 - t_1} \sqrt{\ln n} \text{ for some } t \in [t_1, t_2] \right\} \\
\leq \frac{1}{n^3}.
\]
Finally, note that, in view of (19), everywhere on $E_M$ we have

\[
(t_2 - t_1)^{-1/2} \max\{\langle \bar{n}_M, B_{M_n}(t_1) \rangle, \langle \bar{n}_M, B_{M_n}(t_2) \rangle\} - 2\sqrt{\ln n} \\
\geq \frac{C_f}{4} \sqrt{\frac{t_2}{t_2 - t_1}} - 2\sqrt{\ln n} \\
\geq 2^{M/2 - 2} C_f - 2\sqrt{\ln n} \\
> 0.
\]

Taking the union bound over all adjacent pairs in $Q$ (clearly, $|Q| \leq n^2$), we come to the relation

\[
P\{\langle \bar{n}_M, B_{M_n}(t) \rangle > 0 \text{ for all } t \in [1, 2^{cn}]\} \geq P(E_M) - \frac{|Q|}{n^8} \geq 1 - \frac{1}{n}.
\]

\[\Box\]

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