CRITICAL EXPONENTS OF THE TWO DIMENSIONAL COULOMB GAS AT THE BEREZINSKII-KOSTERLITZ-THOULESS TRANSITION

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Abstract. The two dimensional Coulomb gas is the prototypical model of statistical mechanics displaying a special kind of phase transition, named after Berezinskii, Kosterlitz and Thouless. Physicists and mathematicians proposed several predictions about this system. Two of them, valid along the phase transition curve and for small activity, are: a) the long-distance decay of the "fractional charge" correlation is power law, with a multiplicative logarithmic correction; b) in such a decay, the exponent of the power law, as well as the exponent of the logarithmic correction, have a certain precise dependence upon the charge value. In this paper we provide a proof of these two long standing conjectures.

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1. Introduction

The Coulomb gas is the infinite system of point particles which carry positive or negative unit electric charges, interact via the electrostatic potential and are subject to thermal disorder. In this paper we consider the neutral case, in which the total charge of the particle system is zero. (This is the case of major importance in physics; a non-neutral Coulomb gas could also be defined, see Section II.B.2 of [Minnhagen, 1987], and has a different phenomenology.) The mathematical difficulty of the model, as well as the reason of physical interest, stem from the fact that the electrostatic potential in dimension two is very long range: for large \(|x|\) it is

\[
V(x) = -\frac{1}{2\pi} \ln |x| + c_E + o(1),
\]

where the constant \(c_E\) depends on the microscopic regularization.

The study of the two-dimensional Coulomb gas began in theoretical physics with the suggestion of Berezinskii [1971] and of Kosterlitz and Thouless [1973] that this model, as well as the related classical XY model, undergo a new kind of phase transition, named after them. Shortly after, the Berezinskii-Kosterlitz-Thouless (BKT) transition became one of the fundamental paradigms of the theory of critical phenomena: on the one hand, BKT transitions were predicted for several other two-dimensional toy models, including solid-on-solid models, vertex models, interacting dimers and other lattice systems that can be described in terms of a “height function” (see [José et al., 1977; Kadanoff, 1978; den Nijs, 1983; Nienhuis, 1984; Alet et al., 2005]); on the other hand, the BKT transition turned out to explain the outcomes of several experiments on real-world systems, such as trapped atomic gases, liquid helium films and arrays of Josephson junctions (see [Nelson and Kosterlitz, 1977; Resnick et al., 1981; Minnhagen, 1987; Hadzibabic et al., 2006; Hung et al., 2011] and references therein).

A precise description of the phase diagram of the Coulomb gas was elaborated by Kosterlitz [1974]; José et al. [1977]; Giamarchi and Schulz [1989]. The properties of the gas are determined by two parameters: the activity \(z\) (large \(z\) corresponds to high density of particles) and the inverse temperature \(\beta\) (large \(\beta\) corresponds
to small thermal disorder). With a non-rigorous renormalization group (RG) argument, Kosterlitz found the picture given in Fig. 1, which has the following interpretation. The thicker line, \( \beta = \beta_{\text{BKT}}(z) \), called BKT transition line, divides the \( \beta-z \) plane into two regions, the dipole phase on the right and the plasma phase on the left, which are characterized by a different behavior of the correlations. Let us call charge-\( \eta \) correlation, \( \rho_{\eta}(x-y) \), the system response to a probe of charge \( \eta \in (0,1] \) in position \( x \) and a probe of charge \( -\eta \) in position \( y \); and let us call charge-\( \eta \) density, \( \rho_{1,\eta} \), the system response to a probe of charge \( \eta \) at a point \( x \). (A more precise definitions of the former quantity will be given below. The latter quantity is by definition non-zero only if \( \eta = 1 \)). The truncated charge correlation is

\[
\rho^T_{\eta}(x-y) = \rho_{\eta}(x-y) - \rho_{1,\eta} \rho_{1,-\eta}.
\]

It is expected that:

1. For \( \beta > \beta_{\text{BKT}}(z) \), the truncated charge correlation display a power law decay

\[
\rho^T_{\eta}(x-y) \sim \frac{C}{|x-y|^{2\kappa}},
\]

where \( C \equiv C(z,\beta) \) is a prefactor and \( \kappa \) is the correlation critical exponent. Each thinner line in Fig. 1 is the locus of \((\beta,z)\) corresponding to a constant value of the critical exponent

\[
\kappa = \frac{\beta_{\text{eff}}}{4\pi \eta^2}
\]

with a \( \beta_{\text{eff}} \equiv \beta_{\text{eff}}(z,\beta) > 8\pi \).

2. Along the BKT line \( \beta = \beta_{\text{BKT}}(z) \) the truncated charge correlations decay as a power law, but with a multiplicative logarithmic correction

\[
\rho^T_{\eta}(x-y) \sim \begin{cases} 
\frac{C}{|x-y|^{2\kappa}} (\ln |x-y|)^{\kappa} & \text{for } \eta = \frac{1}{2}, \\
\frac{C}{|x-y|^{2\kappa}} (\ln |x-y|)^{-\kappa} & \text{otherwise}.
\end{cases}
\]

The critical exponent \( \kappa \) is constant and given by (1.3) for \( \beta_{\text{eff}} = 8\pi \).

3. For \( \beta < \beta_{\text{BKT}}(z) \), truncated charge correlations decay exponentially (but only if specific boundary conditions are imposed).

The curves in Fig. 1 were obtained by Kosterlitz as orbits of the ODE

\[
\dot{s}(\ell) = -8\pi^2 e^{8\pi c z(\ell)} \ell^2
\]

\[
\dot{z}(\ell) = -2s(\ell)z(\ell)
\]

where \( \ell \) is a length parameter and \( s(\ell) = 1 - \frac{8\pi c}{\beta(\ell)} \). \( \beta(\ell) \) and \( z(\ell) \) are effective parameters, obtained by averaging fluctuations over \( \ell \)-size subparts of the systems:
hence $\beta(0) = \beta$ and $z(0) = z$ are the true parameters of the model; while $\beta(\infty)$ and $z(\infty)$ are the parameters that determine the long-distance asymptotic behavior of the correlations. The orbits of (1.5) are hyperbolas in the $s, z$ variables; a sketch of them is in Fig. 2. Only the initial data $(s(0), z(0))$ on the right of a separatrix asymptotically evolve to one of the fixed points of the horizontal axis. The separatrix is then identified as the BKT line. The speed of convergence towards the fixed point turns out to be exponential, except when the initial data are along the separatrix: in this case the convergence is much slower

$$s(\ell) = 2\pi e^{4\pi \varepsilon z(\ell)} = \frac{s(0)}{1 + 2s(0)\ell}$$

and this explains the appearance of a logarithmic correction in the truncated charge correlations along the BKT line.

This description of the phases diagram was a breakthrough discovery in physics for the theoretical and the experimental implications mentioned at the beginning of this Introduction; however, it has eluded a mathematical validation for a long time. Indeed physicists’ results relied on an RG computation at second order in $z$ only; higher orders are difficult to be taken into account for it is not known whether the perturbation theory is ultimately convergent even for small $z$ (see [Gallavotti and Nicolò, 1985]). Besides, in the plasma region, the second order approximation of the RG flow is divergent and so scarcely reliable.

Remarkably, the exponential decay of the charge correlations in the plasma phase was proven to hold by Yang [1987], although only in a region of the $\beta - z$ plane that is far from the BKT line and only for $\eta = 1$. His approach was not based on an RG argument, but rather on an expansion about mean field theory which was used by Brydges and Federbush [1980] to prove the Debye screening in dimension three. That said, from now on we will focus on the dipole phase and the BKT line.

The fundamental step towards the mathematical understanding of the dipole phase was made by Fröhlich and Spencer [1981]: first, by Jensen’s inequality, they obtained a power law lower bound for $\rho_\eta(x - y)$; second, they developed a sophisticated multi-scale decomposition of $\rho_\eta(x - y)$ that provides an upper bound that is

![Figure 2. Diagram of phases. The BKT line is the separatrix of the dynamical system; the asterisks denote the semi-line of fixed points.](image-url)
also power law. Their result, among the most celebrated ones in rigorous statistical mechanics, had however three substantial limitations: 1) the multi-scale method applied only to fractional charge correlation, i.e. for \( \eta \in (0,1) \), and 2) only in a region of the dipole phase that is far from the BKT line; 3) the upper and lower bounds, being power-laws with different exponents, cannot rule out the presence of multiplicative logarithmic corrections. Their multi-scale method was later improved in a series of papers \[ \text{Marchetti et al., 1990; Marchetti, 1990; Braga, 1991; Marchetti and Klein, 1991} \] so to make it applicable in a region of dipole phase that touches the BKT line at \( z = 0 \); but the other limitations remained. Noteworthily, Fröhlich-Spencer’s calculations suggested an important refinement of the conjectures: for \( \beta \geq \beta_{\text{BKT}}(z) \), the correct formula for the critical exponent \( \kappa \) cannot be \[(1.3)\] but one should rather expect that

\[
\kappa = \begin{cases} 
\frac{\beta_{\text{eff}}}{4\pi} \eta^2 & \text{if } \eta \in \left(0, \frac{1}{2}\right] \\
\frac{\beta_{\text{eff}}}{4\pi} (1 - \eta)^2 & \text{if } \eta \in \left[\frac{1}{2}, 1\right) \\
4 & \text{if } \eta = 1.
\end{cases}
\] (1.7)

To our understanding, the second and third of \((1.7)\) were overlooked by physicists, who mostly had in mind applications with \( \eta \in \left(0, \frac{1}{2}\right] \).

Several authors advocated the use of a rigorous RG approach to have a more direct access to the conjectures. This direction was followed by Dimock and Hurd [2000], who used the general RG approach of Brydges and Yau [1990] and some new bounds for the charged clusters of particles to obtain a convergent series representation of the free energy of the Coulomb gas. This was an important work because it provides a method to obtain, in the RG scheme, some of the “power counting estimates” which are implicit in Kosterlitz’s analysis. However, it is based on some technical ideas that appear to be applicable neither to the study of charge correlations anywhere in the dipole phase, nor to the evaluation of the free energy at the BKT transition line. These technical problems have prevented further mathematical progress in the study the two dimensional Coulomb gas for the last ten years.

The aim of this and of a previous paper, [Falco, 2012], is to show that the Brydges-Yau’s technique is truly an effective method to deal with the BKT line of the Coulomb gas. In [Falco, 2012], building on a technical suggestion due to D. Brydges and on the general scheme of [Brydges, 2009] (see also [Dimock, 2009; Brydges and Slade, 2010]), we already showed that some difficulties of [Dimock and Hurd, 2000] can be avoided; and that a convergent series representation for the free energy along the BKT line, for \( z \) small enough, can be provided. In this paper we take up the mathematically more sophisticated and physically more interesting objective of studying the long-distance decay of fractional charge correlations \((1.4)\), again along the BKT curve and for \( z \) small enough.

Sharp upper bounds for correlations had already been obtained in the general Brydges-Yau’s scheme in the case of a different model, the Dipole gas, [Dimock and Hurd, 1992; Brydges and Keller, 1994]; however, those approaches do not appear to be directly applicable to correlations displaying an anomalous decay, such as the power law with logarithmic factors that is expected along the BKT line. Besides, our interest here is the critical exponents, therefore we rather need exact long-distance asymptotic formulas. For these reasons we introduce in this paper a new method to
deal with correlations, which is inspired, partially, on the study of Benfatto et al. [1994] of fermion systems with anomalous critical exponents.

For clarity’s sake in this paper we only consider the most interesting aspect of the dipole phase: the correlation of two fractional charges for \((\beta, z)\) along the BKT line. However, our approach is also applicable to the case of integer charges and everywhere in the dipole phase, at least if \(z\) is small enough. Furthermore, we believe that results on the \(n\)-points correlations and their scaling limits can also be obtained building on a method which was introduced in [Falco, 2006; Benfatto et al., 2007] to deal with fermion systems \(n\)-points correlations.

2. Definition and Results

The electrostatic interaction is usually defined as the inverse Laplacian; in dimension two, however, the subtraction of a divergent term is needed to make sense of it. For \(L\) an odd integer and \(R\) another integer, consider the finite square lattice

\[
\Lambda = \left\{ (x_0, x_1) \in \mathbb{Z}^2 : \max\{|x_0|, |x_1|\} < \frac{L R}{2} \right\}
\]

equipped with periodic boundary condition. Define the Yukawa interaction on \(\Lambda\) with inverse Debye screening length \(m^*\) as

\[
W_{\Lambda}(x;m) := \frac{1}{|\Lambda|} \sum_{k \in \Lambda^*} \frac{e^{ikx}}{m^2 - \hat{\Delta}(k)}, \tag{2.1}
\]

where: \(\Lambda^* = \left\{ \frac{2\pi}{2\pi} (n_0, n_1) : (n_0, n_1) \in \Lambda \right\}\) is the reciprocal lattice of \(\Lambda\); \(|\Lambda| = L^2 R\) is the volume of \(\Lambda\); \(\hat{\Delta}(k) = -2 \sum_{j=0,1} (1 - \cos k_j)\) is the Fourier transform of the discrete Laplacian on \(\Lambda\). The two dimensional electrostatic potential is

\[
W_{\Lambda}(x|0) := \lim_{m \to 0} [W_{\Lambda}(x;m) - W_{\Lambda}(0;m)] = \frac{1}{|\Lambda|} \sum_{k \in \Lambda^* \setminus \{0\}} e^{ikx} - \frac{1}{\hat{\Delta}(k)}, \tag{2.2}
\]

It is a classical result, [Stolz, 1950], that the large \(|x|\) asymptotic formula for the infinite volume limit of \(W_{\Lambda}(x|0)\) is (1.1), for the \(o(1)\) term that is actually \(O\left(\frac{1}{|x|^2}\right)\) and for \(c_E = \frac{2\gamma_E}{\log 4\pi}\), where \(\gamma_E\) is the Euler’s constant.

We can now define the probabilistic model. Consider a system of point particles labeled with numbers \(j = 1, 2, 3, \ldots, n\); a configuration \(\omega\) is the assignment to each particle \(j\) of a charge \(\sigma_j = \pm 1\) and of a position \(x_j \in \Lambda\). Let \(\Omega^0_n\) be the set of the neutral configurations of \(n\) particles, i.e. the configurations of \(n\) particles such that \(\sigma_1 + \cdots + \sigma_n = 0\). The total energy of \(\omega \in \Omega^0_n\) is

\[
H_{\Lambda}(\omega) := \sum_{i<j=1}^n \sigma_i \sigma_j W_{\Lambda}(x_i - x_j). \tag{2.3}
\]

We consider \(\Omega^0_n\) as made of one configuration, the “no particle” one, with zero total energy. For activity \(z \geq 0\) and inverse temperature \(\beta > 0\), the Grand Canonical partition function of the two dimensional Coulomb gas is

\[
Z_{\Lambda}(\beta, z) := \sum_{n \geq 0} \frac{z^n}{n!} \sum_{\omega \in \Omega^0_n} e^{-\beta H_{\Lambda}(\omega)}. \tag{2.4}
\]
In the previous paper, [Falco, 2012], we studied the free energy,

\[ p(\beta, z) := - \lim_{\Lambda \to \infty} \frac{1}{\beta |\Lambda|} \ln Z_\Lambda(\beta, z). \tag{2.5} \]

In this paper we focus on the fractional charge correlation, which is defined as a ratio of partition functions. Consider two probes: \( p_1 \), which is a particle of charge \( \eta \in (0, 1) \) at the lattice site \( x \); and \( p_2 \), which is a particle of charge \(-\eta\) at the lattice site \( y \). Let \( \omega \wedge \{p_1, p_2\} \) be the configuration \( \omega \) augmented of the two probes. Set

\[ Z^{p_1, p_2}_\Lambda(\beta, z) := \sum_{n \geq 0} \frac{z^n}{n!} \sum_{\omega \in \Omega_n^\alpha} e^{-\beta H_\Lambda(\omega \wedge \{p_1, p_2\})} \tag{2.6} \]

(namely the probes contribute to the energy but not to the entropy of the system). The precise definition of \( \rho_\eta(x - y) \) in the Introduction is then

\[ \rho_\eta(x - y) := \lim_{\Lambda \to \infty} \frac{Z^{p_1, p_2}_\Lambda(\beta, z)}{Z_\Lambda(\beta, z)}. \tag{2.7} \]

The invariance of (2.7) under translations of the probes is a consequence of the definition. The existence of the infinite volume limits will be proved in the theorem below. When \( z = 0 \), the BKT point is at \( \beta = 8\pi \); at these values of the parameters and for every \( \eta \), a simple computation gives

\[ \rho_\eta(x) = \frac{e^{8\pi \eta^2 c_E}}{|x|^{4\pi^2}} (1 + o(1)), \tag{2.8} \]

where \( o(1) \) is vanishing in the limit of \(|x| \to \infty\). When \( z \neq 0 \) the situation is more complicated.

**Theorem 2.1.** Fixed \( \eta \in (0, 1) \), there exist an \( L_0 \equiv L_0(\eta) > 1 \), a \( z_0 \equiv z_0(\eta) > 0 \) and an inverse temperature \( \beta_{BKT}(z) \geq 8\pi \) such that if \( L \geq L_0, 0 < z \leq z_0 \) and \( \beta = \beta_{BKT}(z) \), the limit (2.7) exists and:

1. If \( \eta \neq \frac{1}{2} \), then

\[ \rho_\eta(x) = \rho_\eta^{(a)}(x) + \rho_\eta^{(b)}(x), \tag{2.9} \]

where, for \( x \)-independent \( f_a, f_b, f \),

\[ \rho_\eta^{(a)}(x) = \frac{e^{8\pi \eta^2 c_E} + f_a}{|x|^{4\pi^2} (1 + f \ln |x|)^{2\pi^2}} (1 + o(1)), \]

\[ \rho_\eta^{(b)}(x) = \frac{f_b}{|x|^{4(1-\eta)^2} (1 + f \ln |x|)^{2(1-\eta)^2}} (1 + o(1)). \tag{2.10} \]

2. If \( \eta = \frac{1}{2} \), then, for \( x \)-independent \( f_a, f \),

\[ \rho_{\frac{1}{2}}(x) = \frac{e^{2\pi c_E}}{2} f_a \left( 1 + f \ln |x| \right)^{\frac{1}{2}} (1 + o(1)). \tag{2.11} \]

In the above formulas, \( o(1) \) are vanishing terms for \(|x| \to \infty\); \( f = cz \) for \( c > 0 \), \( f_b = c(\eta)^2 z^2 (1 + f_b) \) for \( c(\eta) > 0 \); \( f_a, f_b \) are vanishing in the limit \( z \to 0 \). Besides \( z_0(\eta) \) is such that, for every \([a, b] \subset (0, 1)\), one has \( \inf \{z_0(\eta) : \eta \in [a, b]\} > 0 \).

This is the main result of the paper.

**Remarks.**
1. In the limit $|x| \to \infty$, (2.9) and (2.11) coincide with (1.4) for exponent (1.7) and $\beta_{\text{eff}} = 8\pi$. For this reason, we identify the curve $\beta = \beta_{BKT}(z)$ with the Berezinskii-Kosterlitz-Thouless transition line. Whether $\beta < \beta_{BKT}(z)$ implies an exponential decay of truncated correlations is an open problem; the only available rigorous result, [Yang, 1987], is for $\beta \ll \beta_{BKT}(z)$.

2. A heuristic interpretation of the result, neglecting for a moment the logarithmic corrections, is the following. A probe charge $\eta$ at an inverse temperature $\beta = \beta_{BKT}(z)$ placed inside the interacting system is equivalent to a “virtual” point charge $\eta + m$ at inverse temperature $8\pi$ placed inside a free system. Here $m$ represents a local fluctuation of unit charges and can be any positive or negative integer value. By choosing the two smallest values of the virtual charge critical exponent, $2(\eta + m)^2$, one obtains the leading parts of $\rho^{(a)}$ and $\rho^{(b)}$ in (2.9).

3. A justification of the logarithmic factor in (2.9) is more subtle and will emerge from the multi-scale approach used in the proof. The different formula for the case $\eta = \frac{1}{2}$ is related to the fact that, continuing with the argument in the previous point, only at this value of $\eta$ there are two different values of $m$ that minimize the virtual charge correlation exponent.

4. If $\eta \in (\frac{1}{2}, 1)$, the critical exponent of the free case, $2\eta^2$, differs from the one of the interacting case, $2(1 - \eta)^2$; despite that, there is no discontinuity in the behavior of the correlation at $z = 0$. Indeed, note that $\rho^{(a)}_\eta$ has a prefactor $O(1)$, whereas $\rho^{(b)}_\eta$ has a prefactor $O(z)$. Therefore, the smaller $z$, the larger the threshold distance passed which $\rho^{(b)}_\eta$ dominates over $\rho^{(a)}_\eta$; for $z \to 0$ such threshold distance is infinite, and the free case critical exponent is recovered.

5. Since the logarithmic corrections have $O(z)$ prefactors, by the same argument of the previous point, in the limit $z \to 0$ the purely power law decay of the free case is recovered.

6. The prefactor $\frac{1}{2}$ in (2.9) is absent in formula for the correlation in the case $z = 0$. Again, this is not a sign of discontinuity: as it can be traced in the proof of the Theorem, among the $o(1)$ terms in (2.9) there is one that in the limit $z \to 0$ does not vanish, ceases to be subleading and, with its contribution, restores the prefactor 1 in the leading term.

In the next section we provide the detailed renormalization group construction that directly implies Theorem 2.1. The reader with some familiarity with physicists’ renormalization group jargon will recognize in the right hand side of (3.30) the beta function of the model; and in the right hand side of (3.32) the gamma function. The major technical novelty of [Falco, 2012], with respect to [Dimock and Hurd, 2000], was the derivation, in the setting of the Brydges-Yau’s expansion, of the dynamical system (3.30) and of new bounds to control it. That allowed us to obtain a convergent series representation of the free energy at the BKT transition. The most important contribution of this paper is the introduction, again in the framework of the Brydges-Yau’s technique, of renormalization constants for the observables—namely for the fractional charges—which are described by the dynamical system (3.30). That allows us to obtain (2.9) and (2.11), partly by bounds and partly by an explicit computation of the leading term of the solution of (3.30).

In the forthcoming analysis, we will work with five parameters: given a charge $\eta \in (0, 1)$ and $0 < \tau \leq \tau_0$, we will need $L \geq L_0(\eta, \tau)$, $A \geq A_0(\eta, \tau, L)$ and $0 < z \leq z_0(\eta, \tau, L, A)$ in order for the results to be valid. We will also have other two parameters, $\alpha$ and $h \equiv h(\alpha)$, which however are eventually fixed by
the condition $\alpha^2 = 8\pi$. Finally, in our notation, $C$, $C_0$, $C_1$ or $c_0$ might represent different prefactors when they appear in different bounds.

3. Strategy of the Proof

3.1. Multiscale approach. Since $W_\Lambda(x - y; m)$ has strictly positive Fourier transform, a Gaussian field $\{\varphi_x : x \in \Lambda\}$ is defined by assigning zero mean and covariance

$$E_{m,\beta} [\varphi_x \varphi_y] = \beta W_\Lambda(x - y; m).$$

By means of the sine-Gordon transformation, such a finite-dimensional measure provides a functional integral representation for the partition function

$$Z_\Lambda(\beta, z) = \lim_{m \to 0} E_{m,\beta} \left[ e^{2z \sum_{x \in \Lambda} \cos \varphi_x} \right],$$

as well as for the correlation

$$\rho_\eta(x - y) = \lim_{\Lambda \to \infty} \left( e^{i\eta(\varphi_x - \varphi_y)} \right)_\Lambda,$$

where

$$(\cdot)_\Lambda := \lim_{m \to 0} E_{m,\beta} \frac{e^{2\sum_{x \in \Lambda} \cos \varphi_x}}{E_{m,\beta} \left[ e^{2\sum_{x \in \Lambda} \cos \varphi_x} \right]}.$$

The proof of (3.2) and (3.3) is in Appendix A. In the RG approach it is natural to study (3.2) and (3.3) through the generating functional of the correlations of $e^{i\eta\varphi_x}$: define

$$\Omega(J, \Lambda) := \lim_{m \to 0} E_{m,\beta} \left[ e^{2\sum_{x \in \Lambda} \cos \varphi_x + \sum_{x \in \Lambda} \left(J_x, + e^{i\eta\varphi_x} + J_x, - e^{-i\eta\varphi_x}\right)} \right]$$

where $\{J_x, \sigma : x \in \Lambda, \sigma = \pm 1\}$ are real variables; then

$$p(\beta, z) = - \lim_{R \to \infty} \frac{1}{|\beta\Lambda|} \ln \Omega(J, \Lambda) \Big|_{J = 0},$$

$$\rho_\eta(x - y) = \lim_{R \to \infty} \frac{1}{Q(J, \Lambda)} \frac{\partial^2 \Omega(J, \Lambda)}{\partial J_x, + \partial J_x, -} \Big|_{J = 0}.$$

The point of departure of the RG analysis is a multi-scale representation of $\Omega(J, \Lambda)$. We need some further notations. The two independent unit vector of the lattice are $e_0 = (1, 0)$ and $e_1 = (0, 1)$. Consider the set of unit vectors $\hat{u} = \{\pm e_0, \pm e_1\}$: for any $\mu \in \hat{u}$ define the discrete partial derivative as $\partial^\mu \varphi_x := \varphi_{x+\mu} - \varphi_x$ if $\mu = e_0, e_1$, or as $\partial^\mu \varphi_x := \varphi_x - \varphi_{x+\mu}$ if $\mu = -e_0, -e_1$. Correspondingly define the vector component $x^\mu := x \cdot \mu$ if $\mu = e_0, e_1$, and $x^\mu := -x \cdot \mu$ if $\mu = -e_0, -e_1$. In our notation, every sum $\sum_{\mu \in \hat{u}}$ will also imply a factor $\frac{1}{2}$ that we do not write explicitly. This means, for example, that the Fourier transform of $\sum_{\mu \in \hat{u}} \partial^\mu f_x$ coincides with $\tilde{\Delta}(k)$ defined after (2.1); and that the discrete form of the first order Taylor expansion of a lattice function $f_y - f_x$ is $\sum_{\mu \in \hat{u}} (\partial^\mu f_x)(y^\mu - x^\mu)$. In Appendix A we prove the multiscale functional integral representation

$$\Omega(J, \Lambda) = e^{E[\Lambda]} \lim_{m \to 0} \mathbb{E}_R \cdots \mathbb{E}_0 \left[ e^{\mathcal{Q}(J, \zeta^{(0)} + \zeta^{(1)} + \cdots + \zeta^{(R)})} \right],$$

1In the sense that, for any lattice path $p_{x,x}$ that joins $x = (x_0, x_1)$ with $y = (y_0, y_1)$ and has length $|y_0 - x_0| + |y_1 - x_1|,$

$$\left| f_y - f_x - \sum_{\mu \in \hat{u}} (\partial^\mu f_x)(y^\mu - x^\mu) \right| \leq 4 \max_{j=0,1} |y_j - x_j|^2 \max_{z \in p_{x,y}} \max_{\mu_1, \mu_2 \in \hat{u}} |\partial^{\mu_1} \partial^{\mu_2} f_z|$$
where, fixed any $s \in (0, \frac{1}{2})$ and for $\alpha^2 := \beta(1 - s)$:

1. $E = \frac{1}{2} \ln(1 - s)$ and the interaction $\mathcal{V}(J, \varphi)$ is
\[
\mathcal{V}(J, \varphi) := \frac{s}{2} \sum_{\mu \in \Lambda} (\partial^{\mu} \varphi_x)^2 + z \sum_{\mu \in \Lambda, \sigma = \pm 1} e^{i\sigma \alpha \varphi_x} \sum_{\mu \in \Lambda, \sigma = \pm 1} J_{x,\sigma} e^{i\eta \alpha \varphi_x}. \tag{3.8}
\]

2. $\zeta^{(0)} \ldots, \zeta^{(R)}$ are two-by-two independent Gaussian fields, each of which has zero mean and covariance
\[
\mathbb{E}_J[\zeta_j^{(j)} \zeta_y^{(j)}] = \begin{cases} 
\Gamma_j(x - y) & \text{for } j = 0, 1, \ldots, R - 1 \\
\Gamma_R(x - y) & \text{for } j = R.
\end{cases} \tag{3.9}
\]

Each $\Gamma_j$ is independent of $m$ and $\Lambda$ and, for positive $C_q$ and $c$,
\[
\Gamma_j(x) = 0 \quad \text{for } |x| \geq L^{j+1}/2, \tag{3.10}
\]
\[
|\partial^{\mu_1} \ldots \partial^{\mu_q} \Gamma_j(x)| \leq C_q L^{-jq} \quad \text{for any } \mu_j \in \Lambda \text{ and any } q \geq 1, \tag{3.11}
\]
\[
\Gamma_j(0) = \frac{1}{2\pi} \ln L + c_j(L) \quad \text{for } |c_j(L)| \leq cL^{-\frac{\pi}{2}}, \tag{3.12}
\]

The covariance $\Gamma_R'$, instead, depends upon $m$ and $\Lambda$. One has
\[
\lim_{m \to 0} \Gamma_R'(0) = +\infty, \tag{3.13}
\]
while, if $\Gamma_R'(x|0) := \Gamma_R'(x) - \Gamma'(0)$,
\[
\lim_{R \to \infty} \lim_{m \to 0} \Gamma_R'(x|0) = 0. \tag{3.14}
\]

The limit (3.14) implies
\[
\Gamma_{\infty,0}(x|0) := \sum_{j=0}^{\infty} [\Gamma_j(x) - \Gamma_j(0)] = -\frac{1}{2\pi} \log |x| + c_E + o(1). \tag{3.15}
\]

The meaning of (3.11) and (3.10) is that $\Gamma_j$ carries a typical momentum $O(L^{-j})$ and has a compact support of side length $O(L^{j+1})$. The precise construction of $\Gamma_R'$ and of $\Gamma_0, \ldots, \Gamma_{R-1}$ was given in [Falco, 2012] building on [Brydges et al., 2004]; a review is in Appendix A.

Note that the expectations in (3.9) are independent of $\beta$, while the interaction in (3.8) is dependent on the new parameters $\alpha$ and $s$. The relationship among $\alpha$, $s$ and $\beta$ and their role in the forthcoming analysis is the following. The parameter $s = s(z)$ is introduced so that the curve in Fig. 1 that corresponds to a system with effective inverse temperature $\alpha^2$ has graph $\beta = \beta_\alpha(z)$, where
\[
\beta_\alpha(z) = \frac{\alpha^2}{1 - s(z)}. \tag{3.16}
\]

Although in many sub-results we will leave an explicit dependence on $\alpha$, for Theorem 2.1 we will eventually set $\alpha^2 = 8\pi$, which means that in the statement of that Theorem $\beta_{BKT}(z) \equiv \beta_{\sqrt{8\pi}}(z)$.

The RG approach consists in computing the integrals in (3.7) progressively from the random variable with highest momentum to the one with lowest. First, set
\[
\Omega_1(J, \varphi, \Lambda) := e^{E[\Lambda]} \mathbb{E}_0 e^{\mathcal{V}(J, \varphi + \zeta^{(0)})}; \tag{3.17}
\]
then, inductively for $j = 2, \ldots, R$, set
\[
\Omega_j(J, \varphi, \Lambda) := \mathbb{E}_{j-1} \left[ \Omega_{j-1}(\Lambda; J, \varphi + \zeta^{(j-1)}) \right]; \tag{3.18}
\]
at last, one finds
\[
\Omega(J, \Lambda) = \lim_{m \to 0} E_R \left[ \Omega_R(\Lambda; J, \zeta^{(R)}) \right].
\] (3.19)

In this way the evaluation of the partition function is transformed into the evaluation of a sequence of effective generating functionals \( \Omega_1, \ldots, \Omega_R, \Omega \).

### 3.2. Polymer gas representation

Following [Brydges and Yau, 1990; Brydges, 2009; Brydges and Slade, 2010], each \( \Omega_j \) can be efficiently represented as a polymer gas. Before describing this formulation, we have to introduce a multiscale decomposition of the lattice and, correspondingly, special types of lattice domains.

#### a) Blocks

Set \(|x| := \max\{|x_0|, |x_1|\} \). Recall that each side of the square lattice \( \Lambda \) is made of \( L^R \) sites, where \( L \) is odd; for \( j = 0, 1, \ldots, R \), pave the periodic lattice \( \Lambda \) with \( L^{2(R-j)} \) disjoint squares of \( L^{2j} \) sites, in such a way that there is a central square,

\[ \{ x \in \Lambda : |x| \leq L^j/2 \} \]

and all the other squares are translations of this one by vectors in \( L^j \mathbb{Z} \). An example is in Fig. 3. We call such squares \( j \)-blocks, and we denote the set of all \( j \)-blocks by \( B_j \equiv B_j(\Lambda) \). 0–blocks are made of single points: \( B_0 = \Lambda \).

#### b) Polymers

A union of two-by-two different \( j \)-blocks is called \( j \)-polymer, and the set of all \( j \)-polymers in \( \Lambda \) is denoted \( P_j \equiv P_j(\Lambda) \). Suppose \( X \) is a \( j \)-polymer: \( \partial X \) is the set of sites in \( X \) with a nearest neighbor outside \( X \); \( \partial_{ext} X \) is the set of sites outside \( X \) with a nearest neighbor inside \( X \); \( B_j(X) \) is the set of the \( j \)-blocks in \( X \); \( |X|_j \) is the cardinality of \( B_j(X) \); the closure \( \overline{X} \) is the smallest polymer in \( P_{j+1}(\Lambda) \) that contains \( X \).

#### c) Connectivity

A polymer made of two different blocks, \( B, B' \in B_j \), is connected

![Figure 3. Lattice paving with blocks of different sizes in the case \( L = 3 \) and \( R = 3 \).](image-url)
if there exist \( x \in B \) and \( x' \in B' \) s.t. \(|x - x'| = 1\); the definition extends to connected polymers of more blocks in the usual way. For example, in Fig. 3 there is one connected 2-polymer, which is the closure of three connected 1-polymer, which in turn are the closure of ten connected 0-polymer. \( \mathcal{P}_j^e \equiv \mathcal{P}_j^e(\Lambda) \) is the set of the connected \( j \)-polymer; the collection of the maximal connected parts of a \( j \)-polymer \( X \) (each of which is a \( j \)-polymer by construction) is called \( \mathcal{C}_j(X) \).

d) Small polymers. The polymer \( X \) is small if it is connected and \(|X|_j \leq 4\). The set of the small \( j \)-polymers will be called \( \mathcal{S}_j = \mathcal{S}_j(\Lambda) \); the set of the connected \( j \)-polymers that are not small will be called \( \mathcal{S}_j^e = \mathcal{S}_j^e(\Lambda) \); the number of small \( j \)-polymer that contain a given \( j \)-block is independent of \( j \) and will be called \( S \). The small set neighborhood of a \( j \)-polymer \( X \) is the set \( X^* := \{ Y \in \mathcal{S}_j : Y \cap X \neq \emptyset \} \).

d) Empty set. The empty set is considered as an element of \( \mathcal{P}_j \), but not of \( \mathcal{P}_j^e \).

We will assume that \( L \geq 16 \) so that, if \( X \in \mathcal{P}_j^e \), then the set \( X^* \setminus X \) is a “small margin” around \( X \), in the following sense: if \( X, Y \in \mathcal{P}_j^e \) and \( X, Y \) are separated by at least one \( j + 1 \) block, then

\[
\min\{ |x - y| : x \in X^*, y \in Y^* \} \geq L^{j+1} - 8L^j \geq \frac{1}{2}L^{j+1} \tag{3.20}
\]

which, by (3.10), is larger than the range of \( \Gamma_j \).

Now we pass to the polymer gas representation of the generating functional. Set \( \Phi = (J, \varphi) \). For each scale \( j = 1, \ldots, R \), assume that five real parameters, \( E_j \) and \( t_j := (s_j, z_j, Z_j, \underline{Z}_j) \) are given; and assume that \( \Omega_j(\Phi, \Lambda) \) has the form

\[
\Omega_j(\Phi, \Lambda) = e^{[\Lambda]E_j} \sum_{X \in \mathcal{P}_j} e^{U_j(\Phi, \Lambda \setminus X)} \prod_{Y \in \mathcal{C}_j(X)} K_j(\Phi, Y), \tag{3.21}
\]

where the definitions of the interaction \( U_j \) and of the polymer activity \( K_j \) follow. Given a \( j \)-block \( B \), the interaction is

\[
U_j(\Phi, B) = V_j(\Phi, B) + W_j(\Phi, B). \tag{3.22}
\]

The first term, \( V_j \), is similar to the initial interaction (3.8) and is the sum of \( V_{0,j} \) and \( V_{1,j} \), for

\[
V_{0,j}(\varphi, B) = \frac{s_j}{2} \sum_{x \in B, \mu \in \bar{U}} (\partial^\mu \varphi_x)^2 + z_j L^{-2j} \sum_{x \in B, \sigma = \pm 1} e^{\eta \sigma \varphi_x},
\]

\[
V_{1,j}(\Phi, B) = Z_j L^{-2j} \sum_{x \in B, \sigma = \pm 1} J_{x,\sigma} e^{i\eta \sigma \varphi_x} + Z_j L^{-2j} \sum_{x \in B, \sigma = \pm 1} J_{x,\sigma} e^{t \eta \sigma \varphi_x}. \tag{3.23}
\]

Here, \( \eta := \eta - 1 \); therefore, as \( \eta \in (0, 1) \), also \( -\eta \in (0, 1) \). The factors \( L^{-2j} \) make \( V_j \) explicitly dependent on the scale \( j \); besides, \( V_j \) depends upon the fields \( \{ \varphi_x : x \in B \cup \partial_{ext}(B) \} \) and \( \{ J_{x,\sigma} : x \in B, \sigma = \pm 1 \} \) and upon the parameters \( t_j \). Note that \( z_j \) and \( s_j \) play the role of the the effective parameters discussed in the Introduction; whereas \( Z_j \) and \( \underline{Z}_j \) are the “fractional charge renormalization constants”.

The second term in (3.22), \( W_j(\Phi, B) \), is generated by the multi-scale integration: \( W_0(\Phi, B) = 0 \); while, for \( j \geq 1 \), inductively assume that \( W_j(\Phi, B) \) depends upon the scale \( j \), upon the fields \( \{ \varphi_x, J_{x,\sigma} : x \in B^*, \sigma = \pm 1 \} \), and upon the parameters \( t_j \). We give now a partially explicit formula for \( W_j \); the \( w \)’s functions that appear in (3.24), (3.25) and (3.26) will be defined in Section 6.1. \( W_j \) is the sum of three terms: \( W_{0,j}(\varphi, B) \), \( W_{1,j}(\Phi, B) \) and \( W_{2,j}(\Phi, B) \), where the enumeration corresponds
to the powers of \( J \) as we now explain. \( W_{0,j} \) contains terms that are independent of \( J \) and quadratic in \( s_j, z_j \):

\[
W_{0,j}(\varphi, B) = -s_j^2 \sum_{y \in \mathbb{Z}^2} w_{0,j}^{\mu}(y) \left( \sum_{x \in B} \frac{\partial \varphi_x}{\partial z_{x+y}} - \frac{\partial \varphi_{-x}}{\partial z_{-x+y}} \right)
\]

\[
+ z_j^2 \sum_{y \in \mathbb{Z}^2} w_{0,b,j}(y) \left( \sum_{x \in B} e^{i\sigma x \varphi_x - \varphi_{x+y}} - 1 + |y|^2 \sum_{\mu, \nu} (\partial \varphi_x)^2 \right)
\]

\[
+ z_j^2 \sum_{y \in \mathbb{Z}^2} w_{0,c,j}(y) \sum_{x \in B} e^{i\sigma x \varphi_x + \varphi_{x+y}}
\]

\[
+ z_j s_j \sum_{y \in \mathbb{Z}^2} w_{0,d,j}(y) \left( \sum_{x \in B} i\sigma x \varphi_x - \varphi_{x+y} \right)
\]

\[
- z_j s_j \sum_{y \in \mathbb{Z}^2} w_{0,e,j}(y) \left( e^{i\sigma x \varphi_x} - e^{i\sigma x \varphi_{-x}} \right) .
\] (3.24)

\( W_{1,j} \) contains terms linear in \( J \), and linear in \( s_j \) or \( z_j \):

\[
W_{1,j}(\Phi, B) = z_j Z_j L^{-2j} \sum_{y \in \mathbb{Z}^2} w_{1,b,j}(y) \left( \sum_{x \in B} J_{x,\sigma} e^{i\sigma (\varphi_x + \varphi_{x+y})} \right)
\]

\[
+ z_j Z_j L^{-2j} \sum_{y \in \mathbb{Z}^2} w_{1,b,j}(y) \left( \sum_{x \in B} J_{x,\sigma} e^{i\sigma (\varphi_x - \varphi_{x+y})} \right) \]

\[
+ z_j Z_j L^{-2j} \sum_{y \in \mathbb{Z}^2} w_{1,c,j}(y) \left( \sum_{x \in B} J_{x,\sigma} e^{i\sigma (\varphi_x + \varphi_{x+y})} \right)
\]

\[
+ z_j Z_j L^{-2j} \sum_{y \in \mathbb{Z}^2} w_{1,c,j}(y) \left( \sum_{x \in B} J_{x,\sigma} e^{i\sigma (\varphi_x - \varphi_{x+y})} \right)
\]

\[
+ z_j Z_j L^{-2j} \sum_{y \in \mathbb{Z}^2} w_{1,d,j}(y) \left( \sum_{x \in B} J_{x,\sigma} e^{i\sigma (\varphi_x + \varphi_{x+y})} \right)
\]

\[
+ z_j Z_j L^{-2j} \sum_{y \in \mathbb{Z}^2} w_{1,d,j}(y) \left( \sum_{x \in B} J_{x,\sigma} e^{i\sigma (\varphi_x - \varphi_{x+y})} \right) .
\] (3.25)

Finally, \( W_{2,j} \) contains the terms quadratic in \( J \), and independent of \( s \) or \( z \):

\[
W_{2,j}(\Phi, B) = \sum_{y \in \mathbb{Z}^2} w_{2,a,j}(y) \left( \sum_{x \in B} J_{x,\sigma} J_{x+y,\sigma} e^{i\sigma (\varphi_x + \varphi_{x+y})} \right)
\]

\[
+ \sum_{y \in \mathbb{Z}^2} w_{2,a,j}(y) \left( \sum_{x \in B} J_{x,\sigma} J_{x+y,\sigma} e^{i\sigma (\varphi_x + \varphi_{x+y})} \right)
\]

\[
+ \sum_{y \in \mathbb{Z}^2} w_{2,b,j}(y) \left( \sum_{x \in B} J_{x,\sigma} J_{x+y,\sigma} e^{i\sigma (\varphi_x + \varphi_{x+y})} \right)
\]

\[
+ \sum_{y \in \mathbb{Z}^2} w_{2,c,j}(y) \left( \sum_{x \in B} J_{x,\sigma} J_{x+y,\sigma} e^{i\sigma (\varphi_x + \varphi_{x+y})} \right) .
\] (3.26)
We extend these definitions from \( j \)-blocks to \( j \)-polymers additively: for \( X \in \mathcal{P}_j \):

\[
U_j(\Phi, X) := \sum_{B \in \mathcal{B}_j(X)} U_j(\Phi, B); \tag{3.27}
\]

\( V_j(\Phi, X) \) and \( W_j(\Phi, X) \) are defined in the same way.

Returning to the explanation of \( (3.21) \), the polymer activity, \( K_j(\Phi, X) \), is also generated by the multi-scale integration: \( K_0(\Phi, X) = 0 \); while, for \( j \geq 1 \), \( K_j(\Phi, X) \) depends upon \( \{ \varphi_x, J_x, \sigma : x \in X^*, \sigma = \pm 1 \} \), and is the sum of four terms,

\[
K_j(\Phi, X) = K_{0,j}(\varphi, X) + K_{1,j}(\Phi, X) + K_{2,j}(\Phi, X) + K_{\geq3,j}(\Phi, X) \tag{3.28}
\]

where, again, the enumeration refers to the powers of \( J \). The last term is proportional to the third power or an higher power of \( J \); it will not play any role in the analysis of this paper, since eventually we are only interested in up to two derivatives in \( J \) at \( J = 0 \). The second and third terms can be further decomposed:

\[
K_{1,j}(\Phi, X) = L^{-2j} \sum_{x \in X, \sigma = \pm 1} J_{x,\sigma} \left[ Z_j K_{1,j}(\varphi, X, x, \sigma) + Z_j K_{1,j}^\dagger(\varphi, X, x, \sigma) \right],
\]

\[
K_{2,j}(\Phi, X) = \sum_{x_1 \in X, x_2 \in X^*, \sigma_1, \sigma_2 = \pm 1} J_{\sigma_1, x_1} J_{\sigma_2, x_2} K_{2,j}(\varphi, X, x_1, \sigma_1, x_2, \sigma_2). \tag{3.29}
\]

Note that \( K_{1,j}(\varphi, X, x, \sigma) \) and \( K_{1,j}^\dagger(\varphi, X, x, \sigma) \) are “pinned” at \( x \) in the sense that they are defined by \( (3.29) \) only for \( x \in X \); we extend their definitions by setting \( K_{1,j}(\varphi, X, x, \sigma) = K_{1,j}^\dagger(\varphi, X, x, \sigma) = 0 \) whenever \( x \notin X \). In the same way, \( K_{2,j}(\varphi, X, x_1, \sigma_1, x_2, \sigma_2) \) is pinned at \( x_1 \) and \( x_2 \) and we set \( K_{2,j}(\varphi, X, x_1, \sigma_1, x_2, \sigma_2) = 0 \) if \( x_1 \notin X \) or \( x_2 \notin X^* \). Besides note that at least one power of \( J \) is assumed to be restricted to the set \( X \) (indeed, the same sort of dependence in \( J \) is assumed in \( (3.25) \) and \( (3.26) \)).

This completes the explanation of the inductive assumption \( (3.21) \). As we read from \( (3.7) \) and \( (3.8) \), \( (3.21) \) holds at \( j = 0 \), for

\[
E_0 \equiv E = \frac{1}{2} \ln(1 - s), \quad (s_0, z_0, Z_0, \bar{Z}_0) = (s, z, 1, 0), \quad W_0 \equiv K_0 \equiv 0.
\]

We shall see that it also holds by induction for any \( j = 1, 2, \ldots, R \), with:

1. Effective couplings \( s_j, z_j \) and effective polymer activity \( K_{0,j} \) given by

\[
\begin{align*}
    s_{j+1} &= s_j - a_j z_j^2 + F_j, \\
    z_{j+1} &= L^2 e^{-\frac{L^2}{2} \Gamma_j(0)} \left[ z_j - b_j s_j z_j + M_j \right], \\
    K_{0,j+1} &= L_{0,j} + R_{0,j}, \tag{3.30}
\end{align*}
\]

for coefficients \( a_j, b_j \), and functionals \( F_j \equiv F_j(K_{0,j}), M_j \equiv M_j(K_{0,j}), R_{0,j} \equiv R_{0,j}(z_j, s_j, K_{0,j}) \) and \( L_j \equiv L_j(K_{0,j}) \). The functionals \( F_j, M_j \) and \( R_{0,j} \) will play the role of “remainder parts” with respect to the other terms in the equation. The functional \( L_{0,j} \) will be a contraction with respect to suitable norms.

2. Effective free energy \( E_j \) given by

\[
E_{j+1} = E_j + L^{-2j} \left[ \mathcal{E}_{1,j} + s_j \mathcal{E}_{2,j} + s_j^2 \mathcal{E}_{3,j} + z_j^2 \mathcal{E}_{4,j} \right], \tag{3.31}
\]

for coefficients \( \mathcal{E}_{2,j}, \mathcal{E}_{3,j}, \mathcal{E}_{3,j} \) and for a functional \( \mathcal{E}_{1,j} \equiv \mathcal{E}_{1,j}(K_{0,j}) \).
3. Fractional charge renormalization constants $Z_j$ and $\bar{Z}_j$,

$$Z_{j+1} = L^2 e^{-\pi^2 \bar{Z}_{j+1}^2 / 4} \left[ (1 - s_j m_{1,1,j} + M_{1,1,j}) Z_j + (z_j m_{2,1,j} + M_{2,1,j}) \bar{Z}_j \right];$$

$$\bar{Z}_{j+1} = L^2 e^{-\pi^2 \bar{Z}_{j+1}^2 / 4} \left[ (1 - s_j m_{2,1,j} + M_{2,2,j}) Z_j + (z_j m_{2,1,j} + M_{2,1,j}) \bar{Z}_j \right],$$

$$K_{1,j+1} = L_{1,j} + \mathcal{R}_{1,j},$$

(3.32)

for coefficients $\{n_{p,q,j} : p, q = 1, 2\}$ and functionals $\{M_{p,q,j} = \mathcal{M}_{p,q,j}(K_{1,j}) : p, q = 1, 2\}$, $L_{1,j} \equiv \mathcal{L}_{1,j}(K_{1,j})$ and $\mathcal{R}_{1,j} \equiv \mathcal{R}_{1,j}(s_j, z_j, K_{0,j}, K_{1,j})$. The functional $\mathcal{L}_{1,j}$ will be a contraction with respect to suitable norms.

For every $j = 0, 1, \ldots, R$, all the coefficients and functionals appearing in (3.30), (3.31) and (3.32) are independent of $\Lambda$: this will simplify the discussion of the calculation of the limit $\Lambda \to \infty$. Note that at $\alpha^2 = 8\pi$, because of (3.12), $L^2 e^{-\pi^2 \bar{Z}_{j+1}^2 / 4} \sim 1$ and the map (3.30) is our rigorous counterpart of Kadanoff’s ODE for the effective coupling constants, (1.5). Note also that (3.31) and (3.32) depend on the flow (3.30), but do not affect it; therefore the study of (3.30) done in [Falco, 2012] remains valid for the developments of this paper.

The last step of the RG is

$$\Omega(J, \Lambda) = e^{E_R[\Lambda]} \lim_{m \to 0} \mathbb{E}_R \left[ e^{U_R(J, \zeta^{(r)}, \Lambda)} + K_R(J, \zeta^{(r)}, \Lambda) \right].$$

(3.33)

Suppressing the dependence in the set $\Lambda$ of interactions and polymer activities, and setting $\delta E_R := E_{R+1} - E_R$, $\xi_x = \xi^{(r)}_x$, $\Phi_x = (J, \xi^{(r)}_x)$, we have:

1. For the free energy,

$$\frac{1}{|\Lambda|} \ln \Omega(0, \Lambda) := E_{R+1} = E_R + L^{-2R} \lim_{m \to 0} \mathbb{E}_R \left[ 1 + \left( e^{V_0, R(\xi^{(r)}) + W_0, R(\xi^{(r)})} - 1 \right) + K_{0,R}(\xi^{(r)}) \right].$$

(3.34)

2. For the fractional charge correlation

$$\frac{\partial^2 \Omega}{\partial J_{x,+} \partial J_{0,-}}(0, \Lambda) = e^{-\delta E_R[\Lambda]} \times \lim_{m \to 0} \mathbb{E}_R \left[ e^{V_0, R(\xi^{(r)}) + W_0, R(\xi^{(r)})} \left( \frac{\partial V_{1,R}(\Phi)}{\partial J_{x,+}} + \frac{\partial W_{1,R}(\Phi)}{\partial J_{x,+}} \right) \left( \frac{\partial V_{1,R}(\Phi)}{\partial J_{0,-}} + \frac{\partial W_{1,R}(\Phi)}{\partial J_{0,-}} \right) \right]_{J=0} + e^{-\delta E_R[\Lambda]} \lim_{m \to 0} \mathbb{E}_R \left[ e^{V_0, R(\xi^{(r)}) + W_0, R(\xi^{(r)})} \frac{\partial^2 W_{2,R}(\Phi)}{\partial J_{x,+} \partial J_{0,-}} + \frac{\partial^2 K_{2,R}(\Phi)}{\partial J_{x,+} \partial J_{0,-}} \right]_{J=0}. \quad (3.35)$$

3.3. Bounds on the RG map. To control the limit $R \to \infty$ of (3.34) and (3.35), we need bounds for all the intermediate steps of the RG map. In the previous paper, [Falco, 2012], we dealt with (3.30), (3.31) and the formula for the free energy (3.5). We showed that there exists a unique choice of the initial value $s$ as function of $z$ such that the limit for $j \to \infty$ of $s_j z_j$ and $K_j$ is vanishing. More precisely, we found the following results.

Lemma 3.1 ([Falco, 2012]). Consider the coefficients $a_j$ and $b_j$ in (3.30). For $\alpha^2 = 8\pi$, there exists a $j$-independent $C \equiv C(L)$ and a number $\tilde{c}_E$ such that

$$|a_j - a| \leq CL^{-\frac{1}{2}}, \quad |b_j - b| \leq CL^{-\frac{1}{2}},$$

(3.36)

where $a = 8\pi^2 e^{-8\pi \tilde{c}_E} \ln L$ and $b = 2 \ln L$. 
The constant $\tilde{c}_E$ in this Lemma is not the same as $c_E$ in (1.1) —although it has a similar origin; note, however, that $\tilde{c}_E$ will not explicitly appear in the final results (2.9) and (2.11). For stating the next results, set, for any $j \geq 1$,
\[
q_j := \frac{q_1}{1 + |q_1|(j - 1)}, \quad q_1 := \sqrt{abz_1}.
\] (3.37)
Hence $q_1 = z_1 4\pi e^{4\pi E} \ln L$ and $q_j$ is almost a discrete version of $2s(\ell)$ in (1.6).

For two parameters, $h > 0$ and $A > 1$, in Section 4 we will introduce the norm $\| \cdot \|_{h,T} \equiv \| \cdot \|_{h,T}(A)$, that will measure the size of polymer activities.

**Theorem 3.2 ([Falco, 2012])**, Given a $\tau > 0$ small enough, for $L$ and $A$ large enough, there exists an $\varepsilon \equiv \varepsilon(A,L,\tau)$ such that the following statement holds. If $0 < z \leq \varepsilon$, there exists a unique $s \equiv s(z)$ such that the solution of (3.30) with initial data $(z_0, s_0) = (z, s(z))$ satisfies
\[
|s_j - \frac{|q_j|}{b}| \leq \frac{\tau}{b} \frac{|q_1|}{1 + |q_1|(j - 1)^{\frac{1}{2}}},
\]
\[
|z_j - \frac{q_j}{\sqrt{ab}}| \leq \frac{\tau}{\sqrt{ab}} \frac{|q_1|}{1 + |q_1|(j - 1)^{\frac{1}{2}}},
\]
\[\|K_{0,j}\|_{h,T} \leq \frac{\tau^2|q_1|^2}{1 + |q_1|(j - 1)^{\frac{5}{2}}}.
\] (3.38)
for all $j = 1, \ldots, R$. Besides, the choice of the parameters $L$, $A$, $\varepsilon$ and the function $s(z)$ are independent of $|A|$.

As anticipated, the $s(z)$ found in this Theorem determines the graph of the BKT transition line, $\beta = \beta_{\text{BKT}}(z)$, via (3.16). This result was instrumental to control (3.31) and to prove the convergence of (3.5).

**Theorem 3.3 ([Falco, 2012])**, There exists $C \equiv C(\alpha, L)$ such that, given any $j = 0, 1, \ldots, R$, if $|s_j|, |z_j|, \|K_{0,j}\|_{h,T} \leq c_0|q_j|$, then
\[
|E_{j+1} - E_j| \leq CL^{-2j}|q_j|.
\] (3.39)
Besides, $E_0, \ldots, E_R$ (but not $E_{R+1}$) are independent of $|A|$.

The consequence of this result is a convergent series representation of the free energy
\[
p(\beta, z) = -\frac{1}{2\beta} \log(1 - s(z)) - \frac{1}{\beta} \sum_{j \geq 0} (E_{j+1} - E_j),
\]
which was the main result of [Falco, 2012]. In this paper we study (3.32) and (3.35). For this task, we need to introduce a norm for activities with one pinning point, $\| \cdot \|_{1,h,T}$, and a norm for activities with two pinning points, $\| \cdot \|_{2,h,T}$, see discussion after (3.29); such norms will be defined in Section 4. In the following result, we control the activities $K_{1,j}$ and $K_{1,j}^\dagger$.

**Theorem 3.4.** There exists $C \equiv C(\alpha) > 0$ such that, under the same hypothesis of Theorem 3.2,
\[
\|K_{1,j}\|_{1,h,T} \leq C|q_j|^2, \quad \|K_{1,j}^\dagger\|_{1,h,T} \leq C|q_j|^2.
\] (3.40)
The proof is in Section 7.1. Next, we study the coefficients in the flow (3.32).
Lemma 3.5. There exists a \( j \)-independent \( C = C(\alpha, L) \) such that, for any \( p = 1, 2 \),
\[
|\mathcal{M}_{p,1,j}| \leq CA^{-1}||K_{1,j}||_{1,h,T}, \quad |\mathcal{M}_{p,2,j}| \leq CA^{-1}||K_{1,j}||_{1,h,T}.
\] (3.41)

Lemma 3.6. Consider \( a \) and \( b \) given in Lemma 3.1. There exists a \( j \)-independent \( C = C(\alpha, L) \) such that: if \( \alpha^2 \geq 8\pi \) and \( p, q = 1, 2 \),
\[
|m_{q,p,j}| \leq C;
\] (3.42)
besides, if \( \alpha^2 = 8\pi \),
\[
|m_{1,1,j} - \eta^2 b| \leq CL^{-\frac{q}{4}}, \quad |m_{2,2,j} - \eta^2 b| \leq CL^{-\frac{q}{4}};
\] (3.43)
finally, if \( \alpha^2 = 8\pi \) and \( \eta = -\overline{\eta} = \frac{1}{2} \), then \( \mathcal{M}_{1,1,j} = \mathcal{M}_{2,2,j}, \mathcal{M}_{1,2,j} = \mathcal{M}_{1,2,j} \) and
\[
|m_{2,1,j} - \frac{\sqrt{ab}}{2}| \leq CL^{-\frac{q}{4}}, \quad |m_{1,2,j} - \frac{\sqrt{ab}}{2}| \leq CL^{-\frac{q}{4}}.
\] (3.44)

This Lemma does not provide the exact asymptotic values of \( m_{2,1,j} \) and \( m_{1,2,j} \) if \( \eta \neq \frac{1}{2} \); however, they will not be necessary for studying (3.32). To formulate the next result, set \( Z_j^+ := Z_j + \overline{Z}_j, Z_j^- := Z_j - \overline{Z}_j \) and
\[
g_j := -\pi \sum_{k=1}^{j} [\Gamma_k(0) - \frac{1}{2\pi} \log L],
\]
which is a bounded sequence because of (3.12).

Theorem 3.7. In the same hypothesis of Theorem 3.2, for \( j = 1, \ldots, R \):
1. If \( \eta = -\overline{\eta} = \frac{1}{2} \), there exist two coefficients \( \{c_{\sigma} : \sigma = \pm\} \) that are vanishing for \( z \to 0 \) and are such that
\[
Z_{j+1}^+ = Z_j^+ L^j(1 + |q_j|^2)e^{g_j} + c_+ r_{1,j}, \quad Z_{j+1}^- = Z_j^- L^j(1 + |q_j|^2)e^{g_j} + c_- r_{2,j}.
\] (3.45)
in the above formulas, for a constant \( C \) and for \( m = 1, 2 \),
\[
|r_m,j| \leq C \frac{\tau}{\sqrt{1 + |q_j|^2}}.
\]
2. If \( 0 \leq \eta < \frac{1}{2} \), there exist two coefficients, \( c_1, c_2 \), which are vanishing in the limit \( z \to 0 \) and are such that
\[
Z_{j+1} = L^j(1 - \eta^2) (1 + |q_j|^2)e^{4\eta^2 g_j} + c_1 \left[ e^{r_{1,j}} Z_1 + c_2 e^{s_{1,j} Z_1} \right], \quad Z_{j+1}^- = L^j(1 - \eta^2) (1 + |q_j|^2)e^{4\eta^2 g_j} [r_{2,j} Z_1 + s_{2,j} Z_1],
\] (3.46)
where, for a \( C_0 \equiv C_0(\eta) \) and any \( m = 1, 2 \),
\[
|r_m,j| \leq C_0 \frac{\tau}{\sqrt{1 + |q_j|^2}}, \quad |s_m,j| \leq C_0 \frac{1}{\sqrt{1 + |q_j|^2}} + C_0 L^{-2(\eta^2 - \eta^2 j)}.
\]
A formula for \( c_2 \) is, for a \( c(\eta) > 0 \),
\[
c_2 = ze^{4\pi(\eta^2 - \eta^2) \Gamma_{0,0}} [c(\eta) - m_{1,2,0}] + O(z^\frac{1}{2}).
\]
3. If \( \frac{1}{2} \leq \eta < 1 \), (3.46) holds after interchanging \( Z_j \) with \( \overline{Z}_j \) and \( \eta \) with \( -\overline{\eta} \) (hence the formula for \( c_2 \) becomes \( ze^{4\pi(\eta^2 - \eta^2) \Gamma_{0,0}} [c(-\overline{\eta}) - m_{2,1,0}] + O(z^\frac{1}{2}) \)).
Finally, for every \( \eta \in (0, 1) \),
\[
Z_1 = \frac{d^2}{\hbar} e^{-\eta^2/2} \Gamma_0(0)(1 + O(z)),
\]
\[
Z_1 = \frac{d^2}{\hbar} e^{-\eta^2/2} \Gamma_0(0) \text{m}_{2,1,0}.z.
\]

**Theorem 3.8.** Under the same hypothesis of Theorem 3.2 and if \( A \geq \varepsilon^2 \), there exists a \( C > 0 \) such that, for any \( j = 1, 2, \ldots, R \) (suppressing the dependence in the variables \( \varphi, x, \sigma_1, x_2, \sigma_2 \)),
\[
K_{2,j} = \sum_{k=0}^{j} 2^{-(j-k)} \frac{d}{\hbar} \frac{d^k}{\hbar} \frac{d^{2k}}{\hbar} \frac{d^{2k}}{\hbar} |x_1 - x_2| \frac{d^{4k}}{\hbar} \left[ Z_k^{(a,k)} \right. \frac{d}{\hbar} \frac{d^k}{\hbar} \frac{d^{2k}}{\hbar} \frac{d^{2k}}{\hbar} + Z_k \frac{d^k}{\hbar} K^{(b,k)} \right],
\]
where, for any \( \delta = a, \bar{a}, b \),
\[
\|K_{2,j}^{(\bar{\delta},k)}\|_{2,\hbar,T_j} \leq C|q_k|.
\]

As a consequence of the above Theorems we can finally turn to the calculation of the fractional charge correlation. Consider the \( w \)’s function in (3.26).

**Theorem 3.9.** The limits
\[
w_{2,a}^{-}(x) := \lim_{R \to \infty} w_{2,a,R}^{-}(x) \quad w_{2,\bar{a}}^{-}(x) := \lim_{R \to \infty} w_{2,\bar{a},R}^{-}(x)
\]
\[
w_{2,b}^{-}(x) := \lim_{R \to \infty} w_{2,b,R}^{-}(x) \quad w_{2,c}^{-}(x) := \lim_{R \to \infty} w_{2,c,R}^{-}(x)
\]
exist and, under the same hypothesis of Theorem 3.2,
\[
\lim_{R \to \infty} \frac{\partial^2 \Omega}{\partial J_{x_1} \partial J_{x_2}}(0, \Lambda) = 2w_{2,a}^{-}(x) + 2w_{2,\bar{a}}^{-}(x) + 2w_{2,c}^{-}(x).
\]
(While \( w_{2,b}^{-}(x) \) does not contribute to the correlation.)

The last ingredient for the proof of the main Theorem is then an exact evaluation of the long \( |x| \) asymptotic formulas for the functions in (3.50).

**Theorem 3.10.** For coefficients \( f, f_a, f_{\bar{a}}, \tilde{f}_b \) that are vanishing for \( z \to 0 \), and for a constant \( C \):

1. If \( \eta = -\eta = \frac{1}{2} \), then, for \( \delta = a, \bar{a} \),
\[
\left| \frac{d}{\hbar} \frac{d^k}{\hbar} \frac{d^{2k}}{\hbar} \frac{d^{2k}}{\hbar} \right| \frac{d^{4k}}{\hbar} \left[ Z_k^{(a,k)} \right. \frac{d}{\hbar} \frac{d^k}{\hbar} \frac{d^{2k}}{\hbar} \frac{d^{2k}}{\hbar} + Z_k \frac{d^k}{\hbar} K^{(b,k)} \right]
\]
\[
\left| w_{2,c}^{-}(x) \right| \leq \frac{C}{|x|} (1 + f \ln |x|)^{-\frac{1}{2}}.
\]

2. If \( \eta \neq \frac{1}{2} \), then, for the same \( c(\eta) \) of Theorem 3.7,
\[
\left| \frac{d}{\hbar} \frac{d^k}{\hbar} \frac{d^{2k}}{\hbar} \frac{d^{2k}}{\hbar} \right| \frac{d^{4k}}{\hbar} \left[ Z_k^{(a,k)} \right. \frac{d}{\hbar} \frac{d^k}{\hbar} \frac{d^{2k}}{\hbar} \frac{d^{2k}}{\hbar} + Z_k \frac{d^k}{\hbar} K^{(b,k)} \right]
\]
\[
\left| w_{2,c}^{-}(x) \right| \leq \frac{C}{|x|^{4\eta^2}} (1 + f \ln |x|)^{-2\eta^2 + 1} + \frac{C}{|x|^{4\eta^2}} (1 + f \ln |x|)^{-2\eta^2 - 1}.
\]
(While, for every \( \eta \in (0, 1) \), \( w_{-\eta,\beta}(x) = 0 \).) Besides, \( f = 4\pi e^{4\pi\bar{\varepsilon}E}L^2e^{-4\pi\Gamma_0(0)z} \).

Our main result, Theorem 2.1, is then a direct consequence of Theorem 3.9 and Theorem 3.10.

4. Dimensional bounds

Here we set up scale dependent norms that we will use to control the size of the polymer activities. We will also show how norms encode the dimensional analysis used in physics to adapt renormalization group ideas to this model.

4.1. Norms and regulators: definitions. We mainly follow [Brydges, 2009]. Let \( j \in \mathbb{N} \). For \( n = 0, 1, 2 \) and for \( \partial^n \) the discrete derivative introduced before (3.7), define
\[
\| \nabla_j^n \varphi \|_{L^\infty(X^*)} := \max_{\nu_j \in \mathbb{N}^n, x \in X^*} L^{n_j} | \partial^{\nu_1} \cdots \partial^{\nu_n} \varphi_x |. \tag{4.1}
\]
For \( X \) a connected \( j \)-polymer, let \( \mathcal{C}_j^2(X) \) be the linear space of the functions \( \varphi : X^* \to \mathbb{C} \) with norm
\[
\| \varphi \|_{\mathcal{C}_j^2(X)} := \max_{n=0,1,2} \| \nabla^n_j \varphi \|_{L^\infty(X^*)}.
\]
Observe that \( \nabla_j \) is \( L^j \partial \), which makes the norm explicitly scale dependent; besides, we are using the notation \( \mathcal{C}_j^2(X) \) even though the domain involved in the definition of the norm is the set \( X^* \). Let \( \mathcal{N}_j(X) \) be the space of the smooth complex activities of the polymer \( X^* \), i.e. the set of \( C^\infty \) functions \( F(\varphi, X) : \mathcal{C}_j^2(X) \to \mathbb{C} \). The \( n \)-order derivative of \( F \) along the directions \( f_1, \ldots, f_n \in \mathcal{C}_j^2(X) \) is
\[
D^n_j F(\varphi, X) \cdot (f_1, \ldots, f_n) := \sum_{x_1, \ldots, x_n \in X^*} (f_1)_{x_1} \cdots (f_n)_{x_n} \frac{\partial^n F}{\partial \varphi_{x_1} \cdots \partial \varphi_{x_n}}(\varphi, X). \tag{4.2}
\]
Again, despite the notation \( \mathcal{N}_j(X) \), the relevant set here is the bigger set \( X^* \). The size of the differential of order \( n \) is given by
\[
\| D^n_j F(\varphi, X) \|_{T^*_j(\varphi, X)} := \sup_{\| \varphi \|_{\mathcal{C}_j^2(X)} = 1} \left| D^n_j F(\varphi, X) \cdot (f_1, \ldots, f_n) \right|. \tag{4.3}
\]
Then, given any \( h > 1 \), define the norm
\[
\| F(\varphi, X) \|_{h, T_j(\varphi, X)} := \sum_{n \geq 0} \frac{h^n}{n!} \| D^n_j F(\varphi, X) \|_{T^*_j(\varphi, X)} . \tag{4.4}
\]
In order to control the norm of the activities as function of the field \( \varphi \), for any scale \( j \) and any \( X \in \mathcal{P}_j^c \) introduce the field regulators, \( G_j(\varphi, X) \geq 1 \), that depends upon derivatives of \( \varphi \) only. An explicit choice will be provided below. Then, define
\[
\| F(X) \|_{h, T_j(X)} := \sup_{\varphi \in \mathcal{C}_j^2(X)} \frac{\| F(\varphi, X) \|_{h, T_j(\varphi, X)}}{G_j(\varphi, X)} . \tag{4.5}
\]
Finally, we have to weight the polymer activity w.r.t. the size of the set. Given a parameter \( A > 1 \), define
\[
\| F \|_{h, T_j} \equiv \| F \|_{h, T_j(A)} := \sup_{X \in \mathcal{P}_j^c} A^{|X|} \| F(X) \|_{h, T_j(X)}. \tag{4.6}
\]
Likewise, given two charges $\sigma, \sigma'$ chosen large enough in various points below. Next, we have to choose $A$ reduced to $h$ and functions that were involved in the definition. The parameter $h$ is chosen to be $h := \max\{1, 2h_j(\alpha) : j \geq 0\}$, where

$$h_j(\alpha) := \max\{\|h_j\|_{c_j^2(X)} : 0 \in X \in S_j\}$$

and $h_j(x)$ is the function $\alpha[\Gamma_j(x) - \Gamma_j(0)]$. The usefulness of this choice will become clear in Appendix B.3. It is not difficult to see that, by (3.11), $h_j$ is bounded in $j$ and so the definition of the constant $h$ makes sense. The parameter $A$ will be chosen large enough in various points below. Next, we have to choose $G_j$. Here we follow [Falco, 2012]. Given two positive constants $c_1, c_3$, and a positive function of $L$, $\kappa_L$, if $X \in P_j$, the function $G_j$ is such that

$$\ln G_j(\varphi, X) = c_1\kappa_L\|\nabla_j^* \varphi\|_{L_j^2(X)}^2 + c_3\kappa_L\|\nabla_j^* \varphi\|_{L_j^2(\partial X)}^2 + c_1\kappa_L W_j(\nabla_j^2 \varphi, X)^2,$$

where we have used $L^2$-type norms

$$\|\nabla_j^* \varphi\|_{L_j^2(X)}^2 := L^{-2j} \sum_{x \in X} \sum_{\mu_1, \ldots, \mu_n} L^{2nj} |\partial^{\mu_1} \cdots \partial^{\mu_n} \varphi_x|^2,$$

$$\|\nabla_j^* \varphi\|_{L_j^2(\partial X)}^2 := L^{-j} \sum_{x \in \partial X} \sum_{\mu_1, \ldots, \mu_n} L^{2nj} |\partial^{\mu_1} \cdots \partial^{\mu_n} \varphi_x|^2,$$

$$W_j(\varphi, X)^2 := \sum_{B \in B_j(X)} \|\varphi\|_{L^2(B)}^2.$$

To control the field dependence of $U_j$ we shall occasionally use an auxiliary field regulator, called strong field regulator, $G_j^{\text{str}}$: for $B \in B_j$ and $X \in P_j$,

$$\ln G_j^{\text{str}}(\varphi, B) := \kappa_L \max_{n=1,2} \|\nabla_j^* \varphi\|_{L^2(B)}^2, \quad G_j^{\text{str}}(\varphi, X) := \prod_{B \in B_j(X)} G_j^{\text{str}}(\varphi, B).$$

(4.11)

4.2. Norms and regulators: properties. First, it is important to observe that $N_j(X)$ with the norm $\|\cdot\|_{h,T_j(X)}$ is a Banach space. We now list some useful features of the field regulators. As apparent from the definition, if $X \in P_{j+1}$,

$$G_j^{\text{str}}(\varphi', X) \leq G_{j+1}^{\text{str}}(\varphi', X).$$

(4.12)
Consider a polymer \( X \in \mathcal{P}_j \). From the definitions, we have \( L^2_j(X) = \sum_{Y \in \mathcal{C}_j(X)} L^2(Y) \). Besides, since two \( Y \)'s in \( \mathcal{C}_j(X) \) have disjoint boundaries, we also have \( L^2_j(\partial X) = \sum_{Y \in \mathcal{C}_j(X)} L^2(\partial Y) \). Therefore

\[
\prod_{Y \in \mathcal{C}_j(X)} G_j(\phi, Y) = G_j(\phi, X). \tag{4.13}
\]

For the following results to hold, \( c_3 \) and \( c_1 \) must be large enough, but independently of the scale \( j \) and the size \( L \). Unless otherwise stated, \( j = 0, 1, \ldots, R - 1 \).

**Lemma 4.1.** For any polymer \( X \in \mathcal{P}_j \),

\[
G_{j}^{\text{est}}(\phi, X) \leq G_j(\phi, X). \tag{4.14}
\]

For any polymer \( X \in \mathcal{P}_j \) and any block \( B \in \mathcal{B}_j \), but \( B \) not inside \( X \),

\[
G_{j}^{\text{est}}(\phi, B)G_j(\phi, X) \leq G_j(\phi, B \cup X). \tag{4.15}
\]

This Lemma corresponds to formula (6.52) of [Brydges, 2009]: the proof can be found in that paper after Lemma 6.21. The role of the field regulators in the forthcoming analysis is to have a standard function to integrate with respect to the Gaussian measures.

**Lemma 4.2.** Let \( \kappa_L = c(\log L)^{-1} \) with \( c > 0 \) and small enough.

1. For \( j = 0, 1, \ldots, R - 1 \) and any connected polymer \( X \in \mathcal{P}_j \),

\[
\mathbb{E}_j \left[ G_j(\phi, X) \right] \leq 2^{|X|} G_{j+1}(\phi', \overline{X}); \tag{4.16}
\]

if instead \( j = R \),

\[
\mathbb{E}_R \left[ G_R(\phi, \Lambda) \right] \leq 2. \tag{4.17}
\]

2. For \( j = 0, 1, \ldots, R - 1, m = 1, 2, 3 \) and any small polymer \( X \in \mathcal{S}_j \), there exists a \( C_m > 1 \) such that

\[
\left( 1 + \max_{n=1,2} \|\nabla_{j+1} \phi' \|_{L_\infty(X^*)} \right)^m \mathbb{E}_j \left[ G_j(\phi, X) \right] \leq \frac{C_m}{\kappa_L^{m/2}} 2^{|X|} G_{j+1}(\phi', \overline{X}); \tag{4.18}
\]

besides the last formula holds even if \( G_j(\phi, X) \) on the left hand side member is replaced by \( \sup_{t \in [0,1]} G_j(t \phi' + \zeta, X) \).

The proof is in Section D of [Falco, 2012]. From the definitions set up so far, we can derive some simple bounds that will be needed in the next section. For any \( \phi \in \mathcal{C}_{j+1}^2(X) \), we have \( \|\phi\|_{C_j^2(X)} \leq \|\phi\|_{C_j^{m+1}(X)} \), so that, for any \( F \in \mathcal{N}_j(X) \)

\[
\|F(\phi, X)\|_{h, T_{j+1}(\phi, X)} \leq \|F(\phi, X)\|_{h, T_j(\phi, X)}. \tag{4.19}
\]

If \( Y \subset X \), for any \( \phi \in \mathcal{C}_j^2(X) \) we have \( \|\phi\|_{C_j^2(Y)} \leq \|\phi\|_{C_j^2(X)} \), so that \( \mathcal{C}_j^2(X) \subset \mathcal{C}_j^2(Y) \) and

\[
\|F(\phi, X)\|_{h, T_j(\phi, X)} \leq \|F(\phi, X)\|_{h, T_j(\phi, Y)}. \tag{4.20}
\]

For any two polymers \( Y_1, Y_2 \) not necessarily disjoint and such that \( Y_1 \cup Y_2 \subset X \), and any two polymer activities, \( F_1 \in \mathcal{N}_j(Y_1) \) and \( F_2 \in \mathcal{N}_j(Y_2) \), we have: a generalized triangular inequality

\[
\|F_1(\phi, Y_1) + F_2(\phi, Y_2)\|_{h, T_j(\phi, X)} \leq \|F_1(\phi, Y_1)\|_{h, T_j(\phi, Y_1)} + \|F_2(\phi, Y_2)\|_{h, T_j(\phi, Y_2)}. \tag{4.21}
\]
(which is stronger than the usual triangular inequality because different norms appear in the two members); and the factorization property
\[ \| F_1(\varphi, Y_1)F_2(\varphi, Y_2) \|_{h_{T_j}(\varphi, X)} \leq \| F_1(\varphi, Y_1) \|_{h_{T_j}(\varphi, Y_1)} \| F_2(\varphi, Y_2) \|_{h_{T_j}(\varphi, Y_2)}. \] (4.22)
Details of the proofs of these inequalities are in [Brydges, 2009].

Finally, given \( \| F(X) \|_{h_{T_{j+1}(X)}} \), in order to have an estimate of the size of \( \| F \|_{h_{T_{j+1}}} \) one needs to sum over the position of the polymer. Let us consider separately the case of configurations on small sets and on large sets. For \( \lambda \in (0,1) \) and \( \rho = s, l \), set
\[ k_\rho(A, \lambda) := \sup_{V \in P_{j+1} \rho} A|V|_{j+1} \sum_{Y \in O_\rho(V)} (\lambda A)^{-|Y|_j}, \] (4.23)
where \( O_s = S_j \) and \( O_l = S_j \). Besides, consider also the case of a pinning point in the sum and set
\[ k_\rho^*(A, \lambda) := \sup_{V \in P_{j+1} \rho} A|V|_{j+1} \sup_{x \in V} \sum_{Y \in S_j(V) \supset x} (\lambda A)^{-|Y|_j}, \] (4.24)
Note that \( k_s(A, \lambda) \), \( k_l(A, \lambda) \) and \( k_l^*(A, \lambda) \) are \( j \)-independent, and so the notation is consistent.

**Lemma 4.3.** There exist \( c > 0 \) and \( \vartheta > 0 \) such that, for \( A \) large enough
\[ k_s(A, \lambda) \leq cL^2, \quad k_l^*(A, \lambda) \leq c, \quad k_l(A, \lambda) \leq A^{-\vartheta}. \] (4.25)
For the proof see Lemma 6.19 and Lemma 6.18 in [Brydges, 2009]. In brief, when the sum is over small sets and there is no pinning point, the bound is proportional to a volume factor \( L^2 \); when the sum is over large sets, the bound is finite in \( L \) and vanishing for large \( A \).

### 4.3. Dimensional analysis
We now return to the actual polymer activities of our RG treatment of the Coulomb Gas. To reproduce the physicists’ analysis we first need to decompose the polymer activity into terms which represents clusters of particles with given total charge. To do so, note that \( K_{0,j} \) contains terms that, as functions of the fields, either are periodic of period \( 2\pi/\alpha \) or are derivative terms; therefore \( K_{0,j} \) is invariant under \( \varphi_x \to \varphi_x + \frac{2\pi}{\alpha} t \) for any constant, integer field \( t \). As explained Appendix B.2, such invariance provides via a Fourier analysis the following decomposition into charged components for \( K_{0,j} \) as well as for \( K_{1,j}, K_1^\dagger, K_2^\dagger, K_2, K_{\delta,k}^\dagger \).

**Lemma 4.4.** For \( j = 0, 1, \ldots, R \) and for any \( X \in P_{j}^\rho \),
\[
K_{0,j}(\varphi, X) = \sum_{q \in \mathbb{Z}} \hat{K}_{0,j}(q, \varphi, X),
K_{1,j}(\varphi, X, x, \sigma) = \sum_{q \in \mathbb{Z}} \hat{K}_{1,j}(q, \varphi, X, x, \sigma),
K_1^\dagger(\varphi, X, x, \sigma) = \sum_{q \in \mathbb{Z}} \hat{K}_1^\dagger(q, \varphi, X, x, \sigma),
K_2^\dagger(\varphi, X, x, \sigma) = \sum_{q \in \mathbb{Z}} \hat{K}_2^\dagger(q, \varphi, X, x, \sigma). \] (4.26)
and, for $\delta = a, \pi, b$

$$K^{(\delta, k)}_{2,j}(\varphi, X, x, \sigma, x', \sigma') = \sum_{q \in \mathbb{Z}} \hat{K}^{(\delta, k)}_{2,j}(q, \varphi, X, x, \sigma, x', \sigma').$$ (4.27)

The above series are absolutely convergent and, if $\vartheta$ is a constant field,

$$\hat{K}_{0,j}(q, \varphi, X) = e^{iqx\vartheta} \hat{K}_{0,j}(q, \varphi - \vartheta, X),$$

$$\hat{K}_{1,j}(q, \varphi, X, x, \sigma) = e^{i(q+\sigma\alpha)\vartheta} \hat{K}_{1,j}(q, \varphi - \vartheta, X, x, \sigma),$$

$$\hat{K}_{1,j}^\dagger(q, \varphi, X, x, \sigma) = e^{i(q+\sigma\alpha)\vartheta} \hat{K}_{1,j}^\dagger(q, \varphi - \vartheta, X, x, \sigma),$$

$$\hat{K}_{2,j}^{(a, k)}(q, \varphi, X, x, \sigma, x', \sigma') = e^{i(q+\eta\sigma+\nu\sigma')\vartheta} \hat{K}_{2,j}^{(a, k)}(q, \varphi - \vartheta, X, x, \sigma, x', \sigma'),$$

$$\hat{K}_{2,j}^{(\pi, k)}(q, \varphi, X, x, \sigma, x', \sigma') = e^{i(q+\pi\sigma+\nu\sigma')\vartheta} \hat{K}_{2,j}^{(a, k)}(q, \varphi - \vartheta, X, x, \sigma, x', \sigma'),$$

$$\hat{K}_{2,j}^{(b, k)}(q, \varphi, X, x, \sigma, x', \sigma') = e^{i(q+\eta\sigma+\nu\sigma')\vartheta} \hat{K}_{2,j}^{(a, k)}(q, \varphi - \vartheta, X, x, \sigma, x', \sigma').$$ (4.28)

Besides,

$$\|\hat{K}_{0,j}\|_{h,T} \leq \|K_{0,j}\|_{h,T},$$ (4.29)

$$\|\hat{K}_{1,j}\|_{1,h,T} \leq \|K_{1,j}\|_{1,h,T}, \quad \|\hat{K}_{1,j}^\dagger\|_{1,h,T} \leq \|K_{1,j}^\dagger\|_{1,h,T},$$ (4.30)

and for $\delta = a, \pi, b$

$$\|\hat{K}_{2,j}^{(\delta, k)}\|_{2,h,T} \leq \|K_{2,j}^{(\delta, k)}\|_{2,h,T}.$$ (4.31)

The meaning of (4.28) is: $\hat{K}_{0,j}(q, \varphi, X)$ represents clusters of particles with a total charge $q$; $\hat{K}_{1,j}(q, \varphi, X, x, \sigma)$ and $\hat{K}_{1,j}^\dagger(q, \varphi, X, x, \sigma)$ represent clusters of particle with total charge $q+\sigma\eta$ and $q+\pi\sigma$ respectively; similarly for $\hat{K}_{2,j}^{(a, k)}(q, \varphi, X, x, \sigma, x', \sigma')$.

Now we can discuss the typical bound we need in the rest of the paper. By (4.19) and (4.16), for any connected polymer $X \in \mathcal{P}^j$

$$\|E_j[K_{0,j}(\varphi, X)]\|_{h,T_{j+1}(\varphi', X)} \leq \|K_{0,j}\|_{h,T_j} \left(\frac{A}{2}\right)^{-|X_j|} G_{j+1}(\varphi', \underline{X});$$ (4.32)

and, by (4.29), for the charged component $\hat{K}_{0,j}(q, \varphi, X)$,

$$\|E_j[\hat{K}_{0,j}(\varphi, X)]\|_{h,T_{j+1}(\varphi', X)} \leq \|K_{0,j}\|_{h,T_j} \left(\frac{A}{2}\right)^{-|X_j|} G_{j+1}(\varphi', \underline{X});$$ (4.33)

similar bounds can be derived for $\hat{K}_{1,j}(q, \varphi, X, x, \sigma)$ and $\hat{K}_{1,j}^\dagger(q, \varphi, X, x, \sigma)$; and also for $\hat{K}_{2,j}^{(a, k)}(q, \varphi, X, x, \sigma, x', \sigma')$. Then we could use (4.25) to sum over the polymer $X$. However, following this procedure, the sum over the small polymers $X$ will generate a bound proportional to the volume factor $L^2$, which would exponentially increase the size of the bound for $\|K_{0,j}\|_{h,T_j}$ at each step. To avoid that, we need to improve (4.32) and (4.33) whenever $X$ is a small set to beat such an $L^2$. Observe that we passed from scale $j+1$ to scale $j$ by the bound (4.19) which is of general validity. Under special circumstances, this step can be done in a more efficient way. To formulate the next results in a simplified notation, in general we will say that $F(\varphi, X)$ is a charge $p$ activity if, for any constant complex field $\vartheta$, one has

$$F(\varphi, X) = e^{iap\vartheta} F(\varphi - \vartheta, X).$$
Theorem 4.5. Consider a charge \( p \) activity \( F(\varphi, X) \), with \( X \in \mathcal{S}_j \). There exists a \( C \equiv C(\alpha) \) such that,

\[
\| \mathbb{E}_j [F(\varphi, X)] \|_{h, T_{j+1}(\varphi', X)} \leq \rho(p, \alpha) \| F \|_{h, T_j} \left( \frac{A}{2} \right)^{-|X_j|} C_{j+1}(\varphi', \overline{X})
\]

(4.34)

for a “dimensional factor”

\[
\rho(p, \alpha) = C^{1+|p|}L^{-d(p)\frac{3}{2}},
\]

where \( d(p) = p^2 \) if \( |p| \leq 1 \) and \( d(p) = 2|p| - 1 \) otherwise.

(4.34) differs from (4.32) by the prefactor \( \rho(p, \alpha) \). The proof, mostly borrowed from [Dimock and Hurd, 2000], is in Appendix B.3. As an application consider the charged components of \( K_{0,j} \) and of \( K_{1,j} \) with total charge \( p : |p| > 1 \). Setting \( F(\varphi) := \hat{K}_{0,j}(q, \varphi, X) \), the hypothesis of the theorem is satisfied for \( p = q \); therefore

\[
\| \mathbb{E}_j \left[ \hat{K}_{0,j}(q, \varphi, X) \right] \|_{h, T_{j+1}(\varphi', X)} \leq C^{1+|q|}L^{-(2|q|-1)\frac{3}{2}} \| K_{0,j} \|_{h, T_j} \left( \frac{A}{2} \right)^{-|X_j|} G_{j+1}(\varphi', \overline{X}).
\]

(4.35)

Considering that \( \alpha^2 \geq 8\pi \), if \( |q| \neq 0, 1 \) and \( L \) is large enough, the prefactor \( C^{1+|q|}L^{-(2|q|-1)\frac{3}{2}} \leq (C^2L^{-3})\frac{3}{2} \) beats the volume factor \( L^2 \) that will be generated by (4.25) once we sum the above bound over \( X \in \mathcal{S}_j \). The same conclusion holds for \( \hat{K}_{1,j}(q, \varphi, X, x, \sigma) \). Indeed, the theorem applies with \( p = q + \eta\sigma \) and we have

\[
\| \mathbb{E}_j \left[ \hat{K}_{1,j}(q, \varphi, X, x, \sigma) \right] \|_{h, T_{j+1}(\varphi', X)} \leq C^{2|q+\eta\sigma|}L^{-d(q+\eta\sigma)\frac{3}{2}} \| K_{1,j} \|_{1, h, T_j} \left( \frac{A}{2} \right)^{-|X_j|} G_{j+1}(\varphi', \overline{X}).
\]

(4.36)

Therefore, if \( |q + \eta\sigma| > 1 \), the prefactor is \( C^{2|q+\eta\sigma|}L^{-(2|q+\eta\sigma|-1)\frac{3}{2}} \leq (C^2L^{-2})|q+\eta\sigma| \) and beats the volume factor \( L^2 \). For completeness, we also state that

\[
\| \mathbb{E}_j \left[ \hat{K}_{1,j}^{\dagger}(q, \varphi, X, x, \sigma) \right] \|_{h, T_{j+1}(\varphi', X)} \leq C^{2|q+\eta\sigma|}L^{-d(q+\eta\sigma)\frac{3}{2}} \| K_{1,j}^{\dagger} \|_{1, h, T_j} \left( \frac{A}{2} \right)^{-|X_j|} G_{j+1}(\varphi', \overline{X}).
\]

(4.37)

Finally, for \( \delta = a, \overline{a}, b \) and \( 0 \leq k \leq j \),

\[
\| \mathbb{E}_j \left[ \hat{K}_{2,j}^{(\delta,k)}(q, \varphi, X, x, \sigma, x', \sigma') \right] \|_{h, T_{j+1}(\varphi', X)} \leq C^{1+|p|}L^{-d(p)\frac{3}{2}} \| K_{2,j}^{(\delta,k)} \|_{2, h, T_j} \left( \frac{A}{2} \right)^{-|X_j|} G_{j+1}(\varphi', \overline{X}),
\]

(4.38)

where

\[
p = \begin{cases} 
q + \eta(\sigma + \sigma') & \text{if } \delta = a \\
q + \overline{\eta}(\sigma + \sigma') & \text{if } \delta = \overline{a} \\
q + (\eta\sigma + \overline{\eta}\sigma') & \text{if } \delta = b
\end{cases}
\]

For other terms for which the above power counting improvement is not sufficient we need to extract some finite order of the Taylor expansion, which we now define.
Let \( F(\xi, X) \) be a smooth function of the field \( \{ \xi_x : x \in X^* \} \); the \( n \)-order Taylor expansion of \( F(\xi, X) \) at \( \xi = 0 \) is

\[
(Tay F)(\xi, X) := \sum_{m=0}^{n} \frac{1}{m!} \sum_{x_1, \ldots, x_m \in X^*} \xi_{x_1} \cdots \xi_{x_m} \frac{\partial^m F}{\partial \xi_{x_1} \cdots \partial \xi_{x_m}}(0, X); \quad (4.39)
\]

the \( n \)-order remainder is

\[
(Rem F)(\xi, X) := F(\xi, X) - (Tay F)(\xi, X). \quad (4.40)
\]

The next theorem provides the power counting improvement in such cases.

**Theorem 4.6.** Consider a charge \( p \) activity \( F(\varphi, X) \) with support \( X \in S_j \) and fix any point \( x_0 \in X \). For any \( m \in \mathbb{N} \), there exist \( C \equiv C(\alpha) \) and \( C_m \) such that, if \((\delta \varphi)_x := \varphi_x - \varphi_{x_0}\)

\[
\| \text{Rem}_m [F(\varphi, X)] \|_{h, T_{j+1}(\varphi', X)} \leq \rho_m(p, \alpha) \| F \|_{h, T_j} \left( \frac{A}{2} \right)^{-|X_j|} G_{j+1}(\varphi', \Xi) \quad (4.41)
\]

for a “dimensional factor”

\[
\rho_m(p, \alpha) := C^{1+|p|} C_m L^{-d(p)} \frac{a^2}{\pi} (\sqrt{\kappa L})^{-(m+1)}
\]

where, again, \( d(p) = p^2 \) if \( |p| \leq 1 \) and \( d(p) = 2|p| - 1 \) otherwise.

The proof of this theorem, mostly borrowed from [Falcò, 2012], is in Appendix B.4. \( \kappa_L = c(\log L)^{-1} \) as stated in Lemma 4.2. There are various consequences of this Theorem that interest us. First, it applies to the neutral components of \( K_{0,j} \). Setting \( F(\varphi, X) := \hat{K}_{0,j}(0, \varphi, X) \),

\[
\| \text{Rem}_j \left[ \hat{K}_{0,j}(0, \varphi, X) \right] \|_{h, T_{j+1}(\varphi', X)} \leq \rho_2(0, \alpha) \| K_{0,j} \|_{h, T_j} \left( \frac{A}{2} \right)^{-|X_j|} G_{j+1}(\varphi', \Xi). \quad (4.42)
\]

For \( L \) large enough, the dimensional factor \( \rho_2(0, \alpha) = CC_2(\sqrt{\kappa L})^{-3} \) beats the volume factor \( L^2 \). Second, this theorem applies to the components of \( \hat{K}_{0,j} \) with charges \( q = \pm 1 \). Indeed, for \( F(\varphi, X) := \hat{K}_{0,j}(q, \varphi, X) \) the hypothesis holds for \( p = q \) and then

\[
\| \text{Rem}_j \left[ \hat{K}_{0,j}(q, \varphi, X) \right] \|_{h, T_{j+1}(\varphi', X)} \leq \rho_0(q, \alpha) \| K_{0,j} \|_{h, T_j} \left( \frac{A}{2} \right)^{-|X_j|} G_{j+1}(\varphi', \Xi). \quad (4.43)
\]

For \( q = \pm 1 \) and \( L \) large enough, the dimensional factor \( \rho_0(1, \alpha) = C_0 C^2(\sqrt{\kappa L})^{-1} L^{-\frac{a^2}{\pi}} \) is smaller than the volume factor \( L^2 \). The third application is the charged components of \( \hat{K}_{1,j} \). We find

\[
\| \text{Rem}_j \left[ \hat{K}_{1,j}(q, \varphi, X, \sigma) \right] \|_{h, T_{j+1}(\varphi', X)} \leq \rho_1(q + \eta \sigma, \alpha) \| K_{1,j} \|_{1, h, T_{j}} \left( \frac{A}{2} \right)^{-|X_j|} G_{j+1}(\varphi', \Xi). \quad (4.44)
\]
Finally, for $\delta = a, \pi, b$ and $0 \leq k \leq j$,

$$\| \text{Rem}_{j} \left[ \hat{K}_{2,j}^{(d,k)} (q, \varphi, X, x, \sigma, x', \sigma') \right] \|_{h, T_{j+1} (\varphi', X)}$$

$$\leq C_{1+|p|} C_{0} L^{-2d(p)} \left( \sqrt{k_{L} L} \right)^{-1} \| K_{2,j}^{(d,k)} \|_{2, h, T_{j}} \left( \frac{1}{\pi} \right) -|X|_{j} \quad \text{G}_{j+1} (\varphi', \overline{X}), \quad (4.45)$$

where

$$p = \begin{cases} q + \eta (\sigma + \sigma') & \text{if } \delta = a \\ q + \pi (\sigma + \sigma') & \text{if } \delta = \pi \\ q + (\eta \sigma + \pi \sigma') & \text{if } \delta = \alpha. \end{cases}$$

We can now describe the “power counting” argument that will drive our analysis in the rest of the paper: a) by Theorem 4.5, terms with charge $q$ contract by a factor $L^{-\frac{2d(q)}{2}}$; b) by Theorem 4.6, terms proportional to $(\partial \varphi')^{n}$ contract by a factor $L^{-n}$; c) as a consequence of Lemma 4.3, all terms are increased by a volume factor $L^{2}$. Therefore, at $\alpha^{2} = 8\pi$, the action of the RG to contract the size of: i) the terms of total integer charge $p$, with $|p| \geq 2$; ii) the terms of total charge $p$, $|p| = 1$, after that the 0-th order Taylor expansion in $\partial \varphi'$ has been extracted; iii) the terms of total charge $p$, with $|p| = \eta$ or $\pi$, after that the 1-th order Taylor expansion in $\partial \varphi'$ has been extracted; iv) neutral terms, after that the 2-th order Taylor expansion in $\partial \varphi'$ has been extracted. The terms that are extracted at points ii), iii) and iv) are absorbed into $E_{j}, t_{j}$ (see definitions before (3.21)) to generate $E_{j+1}, t_{j+1}$. These ideas will be made precise in the next sections.

### 5. Renormalization Group Map

In the present and in the following section we adopt an abridged notation for the fields. In general, we remove the labels $j$ because they will be clear from the context, and we label the sum of the fields on higher scales with a prime, so that $\zeta_{x} := \zeta_{x}^{(j)}$ and $\varphi_{x} := \varphi_{x}^{(R)} + \zeta_{x}^{(R-1)} + \cdots + \zeta_{x}^{(j+1)}$; besides, $\varphi_{x} := \varphi_{x}^{(j)} + \zeta_{x}$. We also set $\Phi = (J, \varphi)$ and $\Phi' = (J, \varphi')$. We indicate with $O(F_{1}, \ldots, F_{n})$ a term that is proportional to the first power, at least, of each of $F_{j}$’s. Besides in the context of the inductive hypothesis described in Section 3, we will also assume the following symmetry properties. Define the $\pi/2$ rotation $R(x_{0}, x_{1}) := (-x_{1}, x_{0})$ and the translation $T_{y} x := x + y$; and extend these transformations in a natural way to lattices subsets; besides, let $(R \varphi)_{x} := \varphi_{Rx}$ and $(T_{y} \varphi)_{x} := \varphi_{x+y}$. We inductively assume that, for $S = R, T_{y}$,

$$\widehat{R}_{0,j} (q, S \varphi, SY) = \widehat{R}_{0,j} (q, \varphi, Y), \quad (5.1)$$

$$\widehat{R}_{1,j} (q, S \varphi, SY, Sx, \sigma) = \widehat{R}_{1,j} (q, \varphi, Y, x, \sigma), \quad (5.2)$$

$$\widehat{R}_{1,j}^{\dagger} (q, S \varphi, SY, Sx, \sigma) = \widehat{R}_{1,j}^{\dagger} (q, \varphi, Y, x, \sigma), \quad (5.3)$$

$$\widehat{K}_{2,j}^{(d,k)} (q, S \varphi, SY, Sx, \sigma, Sx', \sigma') = \widehat{K}_{2,j}^{(d,k)} (q, \varphi, Y, x, \sigma, x', \sigma'). \quad (5.4)$$

Besides,

$$\widehat{R}_{0,j} (-q, -\varphi, Y) = \widehat{R}_{0,j} (q, \varphi, Y), \quad (5.5)$$

$$\widehat{R}_{1,j} (-q, -\varphi, Y, x, -\sigma) = \widehat{R}_{1,j} (q, \varphi, Y, x, \sigma), \quad (5.6)$$

$$\widehat{R}_{1,j}^{\dagger} (-q, -\varphi, Y, x, -\sigma) = \widehat{R}_{1,j}^{\dagger} (q, \varphi, Y, x, \sigma), \quad (5.7)$$

$$\widehat{K}_{2,j}^{(d,k)} (-q, -\varphi, Y, x, -\sigma, x', -\sigma') = \widehat{K}_{2,j}^{(d,k)} (q, \varphi, Y, x, \sigma, x', \sigma'). \quad (5.8)$$
We now discuss the RG procedure at a generic scale \( j = 1, \ldots, R - 1 \); subsequently we will discuss the slightly different procedure at scales \( j = 0 \).

5.1. **General RG step.** Assume by induction that at a given scale \( j = 1, 2, \ldots, R - 1 \) the formula (3.21) holds. We want to provide a useful way to recast \( \Omega_{j+1} = E_j [\Omega_j (J, \varphi' + \zeta)] \) into the same form of (3.21):

\[
\Omega_{j+1} (\Phi') = e^{E_{j+1} [\Lambda]} \sum_{X \in \mathcal{P}_{j+1}} e^{U_{j+1} (\Phi', \Lambda \setminus X)} \prod_{Y \in \mathcal{C}_{j+1} (X)} K_{j+1} (\Phi', Y). \tag{5.9}
\]

We have the freedom to decide what to include in \( K_{j+1} \) and what in \( U_{j+1} \). Our aim will be to have a formula for \( K_{j+1} \) of the form \( \mathcal{L}_j + \mathcal{R}_j \) where \( \mathcal{L}_j \) contains the linear order in \( K_j \), and the linear and quadratic orders in \( s_j \) and \( z_j \); besides, we want \( \mathcal{L}_j \) to be a contraction. To obtain that, as explained in the end of the previous section, we need to implement the extraction based on the power counting argument. The next Lemma can be read in this way: there is a natural tentative choice for \( K_{j+1} \), which at lowest orders contains the terms \( E_j [K_j] \) and \( E_j [V_j; V_j] \); from such a choice, a term \( Q_j = O(K_j) \) is extracted from \( E_j [K_j] \) and a term \( Q_j^* = O(V_j^2) \) is extracted from \( E_j [V_j; V_j] \); next, \( Q_j \) and \( Q_j^* \) are stored into \( U_{j+1} \) and generate the new-scale parameters, \( E_{j+1}, t_{j+1} \), from the old ones, \( E_j, t_j \).

Before stating the Lemma, we need some definitions. Introduce the two "extraction activities":

1. The activity \( Q_j (\Phi', B, X) \), which is nonzero only for \( X \in \mathcal{S}_j \) and \( B \in \mathcal{B}_j (X) \).
   It is assumed to depend upon the fields \{\( \varphi'_x, J_{x, \sigma} : x \in X^*, \sigma = \pm 1 \}\}; however, it is also assumed that the dependence in at least one power of \( J \) is restricted to the block \( B \) (as opposed to the larger \( X^* \)).

2. The activity \( Q_j^* (\Phi', D, Y) \), which is nonzero only for \( |Y|_{j+1} \leq 2 \) and \( D \in \mathcal{B}_{j+1} (Y) \). It is assumed to depend upon the fields \{\( \varphi'_x, J_{x, \sigma} : x \in D^*, \sigma = \pm 1 \}\); but, again, one power of \( J \) is in fact restricted to the set \( D \) (as opposed to \( D^* \)).

Then define a new polymer activity \( J_j \), which contains the extraction activities:

\[
J_j (\Phi', D, Y) := Q_j^* (\Phi', D, Y) + \sum_{B \in \mathcal{B}_j (D)} \sum_{X \in \mathcal{S}_j} Q_j (\Phi', B, X) - \delta_{D,Y} \sum_{Y' \in \mathcal{S}_{j+1}} \left[ Q_j^* (\Phi', D, Y') + \sum_{B \in \mathcal{B}_j (D)} \sum_{X \in \mathcal{S}_j} Q_j (\Phi', B, X) \right]. \tag{5.10}
\]

Hence \( J_j (\Phi', D, Y) \) is zero unless \( Y \in \mathcal{S}_{j+1} \) and \( D \in \mathcal{B}_{j+1} (Y) \). As the conditions \( X \in \mathcal{S}_j \) and \( X \supset B \) together imply \( X^* \subset D^* \) for \( D = \overline{B} \), then \( J_j (\Phi', D, Y) \) depends upon \{\( \varphi'_x, J_{x, \sigma} : x \in D^*, \sigma = \pm 1 \}\); however, one power of \( J \) is actually restricted to \( D \). The second line of (5.10) (with \( \delta_{D,Y} = 1 \) if \( Y = D \) and \( \delta_{D,Y} = 0 \) otherwise) has been included so to obtain the crucial property of zero average:

\[
\sum_{Y \in \mathcal{P}_{j+1}} J_j (\Phi', D, Y) = 0. \tag{5.11}
\]
For $Y \in \mathcal{P}_{j+1}^c$, define
\[
\tilde{K}_j(\Phi, Y) := \sum_{X' \subseteq Y} e^{U_j(\Phi, X') \setminus X'} \prod_{Y' \in \mathcal{C}(X')} K_j(\Phi, Y'),
\]
which depends on $\{\varphi_x, J_{x, \sigma} : x \in Y^*, \sigma = \pm 1\}$. Now we are ready for the extractions. For every block $D \in \mathcal{B}_{j+1}$, define
\[
P_j(\Phi', \zeta, D) := e^{U_j(\Phi, D) - e^{U_{j+1}(\Phi', D) + (E_{j+1} - E_j)|D|}},
\]
which depends on $\{\zeta_x : x \in \bigcup_{B \in \mathcal{B}_j(D)} B^*\}$ and on $\{\varphi_x', J_{x, \sigma} : x \in D^*, \sigma = \pm 1\}$. For every connected polymer $Y \in \mathcal{P}_{j+1}^c$, define
\[
R_j(\Phi', \zeta, Y) := \tilde{K}_j(\Phi, Y) - \sum_{D \in \mathcal{B}_{j+1}(Y)} J_j(\Phi', D, Y),
\]
which depends on $\{\zeta_x : x \in Y^*\}$ and on $\{\varphi_x', J_{x, \sigma} : x \in D^*, \sigma = \pm 1\}$. Note that in (5.12) and (5.14) one power of $J$ is restricted to $Y$; likewise, in (5.13) one power of $J$ is restricted to $D$.

**Lemma 5.1.** Given formula (3.21) with certain $t_j$, $E_j$ and $K_j$; given any two extraction activities as defined above and such that
\[
Q_j(\Phi', B, X) = O(K_j), \quad Q'_j(\Phi', D, Y) = O(V_j^2);
\]
and given parameters $E_{j+1}, t_{j+1}$ that satisfy
\[
(E_j + E_{j+1})|D| + V_{j+1}(\Phi', D) - E_j[V_j(\Phi, D)] = O(K_j, V_j^2),
\]
the following holds. A possible choice for $K_{j+1}$ in (5.9) is
\[
K_{j+1}(\Phi', Y') = \sum_{X_0, X_1, Z, (5.15)} e^{-(E_{j+1} - E_j)|W| + U_{j+1}(\Phi', Y'\setminus W)} \times E_j[P_j(\Phi', \zeta)^2 R_j(\Phi', \zeta)^X_j] J_j(\Phi')^{X_0(D)},
\]
where the notation is:
1. The sum with label $\rightarrow Y'$ indicates the sum over three $j+1$–polymers $X_0, X_1, Z$, contained in $Y'$, and over one $j+1$–block, $D_Y \in \mathcal{B}_{j+1}(Y)$, per each polymer $Y \in \mathcal{C}_{j+1}(X_0)$, such that: a) $X_0$ and $X_1$ are separated by at least by one $j+1$–block, namely $\mathcal{C}_{j+1}(X_0 \cup X_1) = \mathcal{C}_{j+1}(X_0) + \mathcal{C}_{j+1}(X_1)$; b) $Z \in \mathcal{P}_{j+1}(Y \setminus (X_0 \cup X_1))$; c) each connected component of $X_0$ is $j+1$–small; d) $Y \cup Y' \cup Z \cup X_1 = Y'$. Besides, $W \equiv X_0 \cup X_1 \cup Z$.
2. For polymers $Z, X \in \mathcal{P}_{j+1}$, we set
\[
P_j(\Phi', \zeta)^Z := \prod_{D \in \mathcal{B}_{j+1}(Z)} P_j(\Phi', \zeta, D), \quad R_j(\Phi', \zeta)^X := \prod_{Y \in \mathcal{C}_{j+1}(X)} R_j(\Phi', \zeta, Y).
\]
3. Given $X_0 \in \mathcal{P}_{j+1}$ and one $D_Y \in \mathcal{B}_{j+1}(Y)$ for each $Y \in \mathcal{C}_{j+1}(X_0)$, we set
\[
J_j(\Phi')^{X_0(D)} := \prod_{Y \in \mathcal{C}_{j+1}(X_0)} J_j(\Phi', D_Y, Y).
\]
Such choice of $K_{j+1}(\Phi',Y')$ can be decomposed in the sum of two parts, the leading one, $L_j(\Phi',Y')$ and the remainder one $R_j(\Phi',Y')$ in a way that: the latter is an higher order correction in the sense that if $V_j$, $V_{j+1}$ are scaled by $t$ and $W_j$, $W_{j+1}$, $K_j$ are scaled by $t^2$, for small parameter $t$, then $R_j(\Phi',Y') = O(t^3)$; while the former has an explicit formula

$$L_j(\Phi',Y') = L_j^{(a)}(\Phi',Y') + L_j^{(b)}(\Phi',Y') + L_j^{(c)}(\Phi',Y'),$$

where, for $\delta E_j := E_{j+1} - E_j$,

$$L_j^{(a)}(\Phi',Y') = \sum_{X \in \mathcal{P}_j(Y')} E_j[K_j(\Phi, X)] - \sum_{B \in B_j(X)} Q_j(\Phi', B, X),$$

$$L_j^{(b)}(\Phi',Y') = \frac{1}{2} \sum_{B_0,B_1 \in B_j(Y')} \mathbb{E}_j^T \left[ V_j(\bar{t}_j, \Phi, B_0); V_j(\bar{t}_j, \Phi, B_1) \right] - \sum_{D \in B_{j+1}(Y')} Q_j(\Phi', D, Y'),$$

$$L_j^{(c)}(\Phi',Y') = - \sum_{D \in B_{j+1}} W_{j+1}(\Phi', D) - \sum_{Y \in \mathcal{S}_{j+1}} Q_j^*(\Phi', D, Y)$$

for any $t_j$ such that $\bar{t}_j - t_j = (O(z^2), O(z^2), O(z), O(z))$.

Besides the scale $j+1$ activity, $K_{j+1}$, can be decomposed into charged terms as stated in (3.28), (3.29), (4.26) and (4.28) for the scale $j$ activity.

Proof. Starting from (3.21) and re-blocking the polymers on scale $j+1$, we obtain an equivalent formulation for $\Omega_j$:

$$\Omega_j(\Phi) = e^{E_{j+1}(X)} \sum_{X \in \mathcal{P}_{j+1}(X)} \prod_{D \in B_{j+1}(X)} e^{U_j(\Phi, D)} \prod_{Y \in \mathcal{S}_{j+1}(X)} K_j(\Phi, Y)$$

for $K_j$ given by (5.12). Plugging (5.13) and (5.14) in (5.21) and expanding, we find (5.9), for $K_{j+1}$ given by (5.17). Observe that, to derive it, we also used the factorization of $\mathbb{E}_j$ over sets that are in two different connected components of a $j+1$-polymer as explained in (3.20). Besides, in some terms we have the parameters $\bar{t}_j$ instead of the more natural $t_j$ because the difference can be left inside $R_j$. Finally, by construction, $W \subset Y'$ so that that $K_{j+1}(\Phi', Y)$ depends on the fields $\{\varphi'_x, J_x, \sigma : x \in Y^*, \sigma = \pm 1\}$; and, in particular, one power of $J$ is restricted to $Y$, as required.

We have to prove that the linear part in $K_j$, $Q_j$, $Q_j^*$ and second order part in $V_j$ of this choice of $K_{j+1}$ is (5.20); expanding formula (5.17), using (5.15) and (5.16), we obtain (5.20) via two simple identities,

$$\sum_{D \in B_{j+1}(Y')} J_j(\Phi', D, Y') = \sum_{D \in B_{j+1}(Y')} Q_j^*(\Phi', D, Y') + \sum_{X \in \mathcal{S}_j} \sum_{B \in B_j(X)} Q_j(\Phi', B, X)$$

$$- \sum_{D \in B_{j+1}} \sum_{Y \in \mathcal{S}_{j+1}} Q_j^*(\Phi', D, Y) - \sum_{B \in B_j, \mathcal{S}_{j+1}} Q_j(\Phi', B, X);$$

(5.22)
and, by (5.11),
\[
\sum_{Y \in S_{t+1}} \sum_{D \in B_{t+1}(Y)} J_j(\Phi', D, Y) = \sum_{D \in B_{t+1}} \sum_{Y \in S_{t+1}} J_j(\Phi', D, Y) = 0. \tag{5.23}
\]
This completes the proof of the Lemma. ■

The usefulness of (5.20) is that, as planned before, in \( L_j^{(a)}(\Phi', Y') \) and \( L_j^{(b)}(\Phi', Y') \) we read the extraction of \( Q_j \) and \( Q_j^* \) from \( E_j[K_j] \) and \( E_j[\bar{V}_j; V_j] \) respectively; in \( L_j^{(c)}(\Phi', Y') \) the same terms are re-absorbed into \( E_j[t_j] \) so generating \( E_{j+1}, t_{j+1} \).

Note that by construction \( L_j \) depends on \( t_j, K_j, Q_j, Q_j^*, \bar{t}_j, \delta E_j \) and \( t_{j+1} \); however, in Section 6, we will determine the last five of them as function of \( t_j \) and \( K_j \); so that also \( L_j \) is ultimately only a function on \( t_j \) and \( K_j \). In fact, as stated in the next Theorem, also the dependence on \( t_j \) disappears from \( L_j \). Decompose
\[
L_j(\Phi', Y) = L_{0,j}(\varphi', Y) + L_{1,j}(\Phi', Y) + L_{2,j}(\Phi', Y) + L_{3,j}(\Phi', Y)
\]
where the enumeration refers to the powers of \( J \). The term that is in linear in \( J \) is
\[
L_{1,j}(\Phi', Y) = L^{-2(j+1)} + \sum_{x \in Y, \sigma \in \Sigma} J_x, \sigma L_{1,j}(\varphi', Y, x, \sigma)
\]
+ \( L^{-2(j+1)} \sum_{x \in Y, \sigma \in \Sigma} J_x, \sigma L_{1,j}^\dagger(\varphi', Y, x, \sigma) \). \tag{5.24}

The term that is quadratic in \( J \) is
\[
L_{2,j}(\Phi', Y) = \sum_{x_1, x_2 \in Y, \sigma_1, \sigma_2 \in \Sigma} J_{\sigma_1, x_1} J_{\sigma_2, x_2} L_{2,j}(\varphi', Y, x_1, \sigma_1, x_2, \sigma_2) \tag{5.25}
\]
and can be further decomposed into (suppressing the dependence in \( \varphi', Y, x_1, \sigma_1, x_2, \sigma_2 \))
\[
L_{2,j} = \sum_{k=0}^j 2^{-(j-k)} L^{-2k} L^{-k} \left[ Z_k^2 L_{2,j}^{(a,k)} + Z_k^2 L_{2,j}^{(b,k)} + Z_k Z_k L_{2,j}^{(c,k)} \right].
\]

**Theorem 5.2.** For a suitable choice of \( Q_j, Q_j^*, \bar{t}_j, \delta E_j \) and \( t_{j+1} \) as functions of \( t_j, K_j \), the leading part \( L_j \) is independent of \( t_j \) and is linear in \( K_j \). Besides, under the inductive assumption that
\[
\left| \frac{Z_j}{Z_{j+1}} \right| \leq 1, \quad \left| \frac{Z_j}{Z_{j+1}} \right| \leq 1,
\]
\( L_j \) satisfies the following bounds:

1. for the term of \( L_j \) that is independent of \( J \),
\[
\| L_{0,j} \|_{h,T_{j+1}} \leq \rho(L, A) \| K_{0,j} \|_{h,T_j}; \tag{5.26}
\]
where \( \rho(L, A) \) is arbitrarily small for \( L \) and \( A \) large enough;

2. for the terms of \( L_j \) that are linear or quadratic in \( J \),
\[
\| L_{1,j} \|_{1,h,T_{j+1}} \leq \rho(L, A, \eta) \| K_{1,j} \|_{1,h,T_j};
\]
\[
\| L_{1,j}^\dagger \|_{1,h,T_{j+1}} \leq \rho(L, A, \eta) \| K_{1,j}^\dagger \|_{1,h,T_j};
\]
\[
\| L_{2,j} \|_{2,h,T_{j+1}} \leq \rho(L, A, \eta) \| K_{2,j} \|_{2,h,T_j} \tag{5.27}
\]
\[
\| L_{2,j}^{(b,k)} \|_{2,h,T_{j+1}} \leq \rho(L, A, \eta) \| K_{2,j}^{(b,k)} \|_{2,h,T_j} \tag{5.28}
\]
for a \( \rho(L, A, \eta) \) that is arbitrarily small for any \( \eta \in (0, 1) \) if \( L \) and \( A \) are large enough.

The first point of the result was already proven in Falco [2012]. The proof of the second point is a direct consequence of Lemma 6.1, Lemma 6.2, Lemma 6.3 and Lemma 6.4 in Section 6. There we will also explain how to obtain (3.30), (3.31), (3.32) and the following formula for \( \overline{t}_j \),

\[
\overline{t}_j = (s_{j+1}, z_{j+1}L^{-2}e^{\frac{\alpha^2}{4} \Gamma_j(0)}, Z_{j+1}L^{-2}e^{\alpha^2 \Gamma_j(0)}, \overline{z}_j L^{-2}e^{\frac{\alpha^2}{4} \Gamma_j(0)}). \tag{5.29}
\]

Consider now the remainder part. Using (5.17) for \( K_{j+1} \) and formula (5.20) for its leading part, we obtain the following formula for \( \mathcal{R}_j \):

\[
\mathcal{R}_j(\Phi', Y') := \sum_{n=1}^{9} \mathcal{R}^{(n)}_j(\Phi', Y') \tag{5.30}
\]

where, suppressing the dependence in the field (again \( \delta E_j := E_{j+1} - E_j \)),

\[
\mathcal{R}_j^{(1)}(Y') = \sum_{D \in B_{j+1}} \left[ E_j [P_j(D)] + V_{j+1}(D) - E_j[V_j(D)] - \delta E_{j+1}|D| \right. \\
\left. - \frac{1}{2}E^T[V_j(D); V_j(D)] + W_{j+1}(D) - E_j[W_j(D)] \right],
\]

\[
\mathcal{R}_j^{(2)}(Y') = \frac{1}{2} \sum_{D_1 \cup D_2 = Y', D_1 \neq D_2} \left[ E_j [P_j(D_1)P_j(D_2)] - 2E_j[V_j(D_1); V_j(D_2)] - E_j[V_j(\overline{t}_j; D_1); V_j(\overline{t}_j; D_2)] \right],
\]

\[
\mathcal{R}_j^{(3)}(Y') = \sum_{D \in B_{j+1}} \left[ E_j[W_j(D)] - E_j[W_j(\overline{t}_j; D)] \right],
\]

\[
\mathcal{R}_j^{(4)}(Y') = \frac{1}{2} \sum_{D_1, D_2 \in B_{j+1}} \left[ E_j^T[V_j(D_1); V_j(D_2)] - E_j^T[V_j(\overline{t}_j; D_1); V_j(\overline{t}_j; D_2)] \right],
\]

\[
\mathcal{R}_j^{(5)}(Y') = \sum_{|C_{j+1}(X_0 \cup X_1)| \geq 1} \sum_{|D_j| \geq 2} E_j \left[ P_j^Z \sum_{X_j = Y'} \left( e^{-|D_j(Y')|} + U_{j+1}(Y' \backslash W) - 1 \right) \mathcal{E}_j \left[ P_j^Z R_j X_j \right] J_j^{Y_0(D)} \right],
\]

\[
\mathcal{R}_j^{(6)}(Y') = \sum_{Z = Y'} E_j \left[ P_j^Z \sum_{X_j \in Y'} \left( e^{-|D_j(Y')| - 1} \mathcal{E}_j \left[ P_j^Z \right] \right) \right],
\]

\[
\mathcal{R}_j^{(7)}(Y') = \sum_{X_j \in P_j, |X_j| \geq 2} E_j \left[ \prod_{Y_j \in C_j(X)} K_j(Y) \right],
\]

\[
\mathcal{R}_j^{(8)}(Y') = \sum_{X_j \in P_j, |X_j| \geq 2} E_j \left[ \prod_{Y_j \in C_j(X)} K_j(Y) \right],
\]

\[
\mathcal{R}_j^{(9)}(Y') = \sum_{X_j \in P_j} E_j \left[ \left( e^{U_j(Y' \backslash X)} - 1 \right) \prod_{Y_j \in C_j(X)} K_j(Y) \right]. \tag{5.31}
\]
$R_j$, as well as each $R_j^{(n)}$, can be decomposed in terms with increasing powers of $J$,

$$R_j(\Phi', Y) = R_{0,j}(\Phi', Y) + R_{1,j}(\Phi', Y) + R_{2,j}(\Phi', Y) + R_{\geq 3,j}(\Phi', Y).$$  \hfill (5.32)

The term that is linear in $J$ is

$$R_{1,j}(\Phi', Y) = L^{-2(j+1)} Z_{j+1} \sum_{x \in Y} J_{x,\sigma} R_{1,j}(\phi', Y, x, \sigma) + L^{-2(j+1)} Z_{j+1} \sum_{x \in Y} J_{x,\sigma} R_{1,j}(\phi', Y, x, \sigma).$$  \hfill (5.33)

The term that is quadratic in $J$ is

$$R_{2,j}(\Phi', Y) = \sum_{x_1 \in Y, x_2 \in Y^{+*}} J_{\sigma_1, x_1} J_{\sigma_2, x_2} R_{2,j}(\phi', Y, x_1, \sigma_1, x_2, \sigma_2)$$  \hfill (5.34)

and can be further decomposed into (suppressing the dependence in $\phi', Y, x_1, \sigma_1, x_2, \sigma_2$)

$$R_{2,j} = \sum_{k=0}^{j} 2^{-(j-k)} L^{-4k} e^{-L^{-k}|x_1 - x_2|} \left[ Z_k^2 R_{2,j}^{(a,k)} + \sum_{\sigma} Z_k^2 R_{2,j}^{(b,k)} \right].$$  \hfill (5.35)

**Theorem 5.3.** If $z > 0$ is small enough and $|s_j|, |z_j| \leq c_0 |q_j|, \|K_{0,j}\|_{h,T_j} \leq c_0 |q_j|^2$, there exists $C = C(A, L, \alpha)$ such that,

1. for the term of $R_j$ that is independent of $J$

   $$\|R_{0,j} - \hat{R}_{0,j}\|_{h,T_{j+1}} \leq C \left[ |q_j|^2 |s_j - \hat{s}_j| + |q_j|^2 |z_j - \hat{z}_j| + |q_j||K_{0,j} - \hat{K}_{0,j}\|_{h,T_j} \right]$$  \hfill (5.36)

   where $\hat{R}_{0,j}$ is obtained from $R_{0,j}$ by replacing $s_j, z_j, K_{0,j}$ with any $\hat{s}_j, \hat{z}_j, \hat{K}_{0,j}$ that satisfy $|s_j|, |z_j| \leq c_0 |q_j|$ and $\|K_{0,j}\|_{h,T_j} \leq c_0 |q_j|^2$;

2. for the terms of $R_j$ that are linear in $J$,

   $$\|R_{1,j}\|_{1,h,T_{j+1}} \leq C \left[ |q_j|^2 + |q_j||K_{1,j}\|_{1,h,T_j} + |q_j||K_{1,j}\|_{1,h,T_j} \right]$$  \hfill (5.37)

   and the same bound is valid for $\|R_{1,j}\|_{1,h,T_{j+1}}$;

3. for the terms of $R_j$ that are quadratic in $J$, with the extra assumption that $\|K_{1,j}\| \leq c_0 |q_j|^2$ and $\|K_{1,j}\| \leq c_0 |q_j|^2$,

   $$\|R_{2,j}^{(a,k)}\|_{2,h,T_{j+1}} \leq \begin{cases} C |q_k| & \text{for } k = j \\ C |q_j||K_{2,j}^{(a,k)}\|_{2,h,T_j} & \text{for } 0 \leq k \leq j - 1 \end{cases}$$  \hfill (5.38)

The first point was already proven in [Falco, 2012]. The second and third points are consequence of Lemma 7.1 and Lemma 7.4.

### 5.2. First RG step

The starting point is formula (3.17) that in our current notations reads

$$\Omega_1(\Phi) = e^{E_0[A]} E_0 \left[ e^{\sum_{\Phi, A}[\Phi, A]} \right].$$  \hfill (5.39)

As already noted, the term in the square brackets of (5.39) has the form (3.21) for $j = 0$ for $W_0(\Phi, B) = 0$, $K_0(\Phi, Y) = 0$, and for parameters $E_0 = E$ and $t_0 = (s, z, 1, 0)$. We want to recast $\Omega_1$ into the form (3.21) for $j = 1$. For doing so,
we apply Lemma 5.1 to the scale \( j = 0 \): since \( K_0 = 0 \), in Section 6 we will see that there exists a choice of \( Q^*_0, l_0, \delta E_0 \) and \( t_1 \) such that

\[
\mathcal{L}_0(\Phi, Y) \equiv 0. \tag{5.40}
\]

However, since the choice for \( \frac{Q^*_0}{j} \) will differ from the general formula for \( Q^*_j \) in the part that does not depend on \( J \) (we do this for merging with the treatment of [Falco, 2012]) the remainder part is slightly different from (5.30) at \( j = 0 \), \( W_0 = 0 \) and \( K_0 = 0 \); indeed we have

\[
\mathcal{R}_0(\Phi', Y') := \sum_{n=1}^{6} \mathcal{R}_0^{(n)}(\Phi', Y') \tag{5.41}
\]

where, suppressing the dependence in the field,

\[
\begin{align*}
\mathcal{R}_0^{(1)}(Y') &= \sum_{D = D'}^{\mathcal{D} = \mathcal{D}'} \left[ \mathbb{E}_0 \left[ P_0(D) \right] + V_1(D) - \mathbb{E}_0 V_0(D) - \delta E_0 D \right] \\
&\quad - \frac{1}{2} \mathbb{E}_0^T \left[ V_0(D); V_0(D) \right] + W_1(D) + \frac{1}{2} \mathbb{E}_0^T \left[ V_0,0(D); V_0,0(D) \right], \\
\mathcal{R}_0^{(2)}(Y') &= \frac{1}{2} \sum_{D_1, D_2 \in \mathcal{D}_1} \left[ \mathbb{E}_0 \left[ P_0(D_1) P_0(D_2) \right] - \mathbb{E}_0^T \left[ V_0(D_1); V_0(D_2) \right] + \mathbb{E}_0^T \left[ V_0,0(D_1); V_0,0(D_2) \right] \right], \\
\mathcal{R}_0^{(3)}(Y') &= \frac{1}{2} \sum_{D_1, D_2 \in \mathcal{D}_1} \left[ \mathbb{E}_0^T \left[ V_0(D_1); V_0(D_2) \right] - \mathbb{E}_0^T \left[ V_0,0(D_1); V_0,0(D_2) \right] \right], \\
\mathcal{R}_0^{(4)}(Y') &= \sum_{Y'} \mathbb{E}_0 \left[ P_0^Z R_0 X_1 \right] J_0^{X_0(D)}, \\
\mathcal{R}_0^{(5)}(Y') &= \sum_{Z=Z'} \left( e^{-\delta E_0 |Y'| + U_1(Y', W)} - 1 \right) \mathbb{E}_0 \left[ P_0^Z R_0 X_1 \right] J_0^{X_0(D)}, \\
\mathcal{R}_0^{(6)}(Y') &= \sum_{Z=Z'} \mathbb{E}_0 \left[ P_0^Z \right] + \left( e^{-\delta E_0 |Y'|} - 1 \right) \mathbb{E}_0 \left[ P_0^{Y'} \right]. \tag{5.42}
\end{align*}
\]

The decompositions (5.32), (5.33),(5.34) and (5.35) are valid also at \( j = 0 \).

**Theorem 5.4.** Under the same hypothesis of Theorem 5.3,

1. for the term of \( \mathcal{R}_0 \) that is independent of \( J \)

\[
\|\mathcal{R}_0,0\|_{h,T_1} \leq C|q_0| \left[ |s_0 - \hat{s}_0| + |z_0 - \hat{z}_0| \right]; \tag{5.43}
\]

2. for the terms of \( \mathcal{R}_0 \) that are linear in \( J \),

\[
\|\mathcal{R}_1,0\|_{1,h,T_1} \leq C|q_0|^2, \tag{5.44}
\]

and the same bound is valid for \( \|\mathcal{R}_1,0\|_{1,h,T_1} \).
3. for the terms of $R_0$ that are quadratic in $J$,

$$\|R_{2,0}^{(6,0)}\|_{2,h,T_1} \leq C|q_0|.$$ \hspace{1cm} (5.45)

As for Theorem 5.3, we only need to prove the second and third points, which are consequence of Lemma 7.1 and Lemma 7.4. Note that (5.44) and (5.45) coincide with (5.37) and (5.38) at $j=0$; while (5.43) differs from (5.36) at $j=0$ and is the same as in [Falco, 2012].

6. Leading part of the RG map

6.1. Running coupling constants. The choice of $Q_j$ requires Taylor expansion in $\nabla \varphi'$. For any point $x_0 \in X$, if $(\delta \varphi')_x := \varphi'_x - \varphi'_{x_0}$ (which is a sum of $\nabla \varphi'$'), using (4.28), we have

$$\tilde{K}_{0,j}(q, \varphi, X) = e^{i\alpha \varphi'_{x_0}} \tilde{K}_{0,j}(q, \delta \varphi' + \zeta, X),$$

$$\tilde{K}_{1,j}(q, \varphi, X, x, \sigma) = e^{i\alpha(q+\eta \sigma)\varphi'_{x_0}} \tilde{K}_{1,j}(q, \delta \varphi' + \zeta, X, x, \sigma),$$

$$\tilde{K}_{2,j}^{(a,k)}(q, \varphi, X, x, \sigma, \varphi', \sigma') = e^{i(q+\sigma+\eta \sigma')\varphi'_{x_0}} \tilde{K}_{2,j}^{(a,k)}(q, \delta \varphi' + \zeta, X, x, \sigma, \varphi', \sigma'),$$

$$\tilde{K}_{3,j}^{(a,k)}(q, \varphi, X, x, \sigma, \varphi', \sigma') = e^{i(q+\sigma+\eta \sigma')\varphi'_{x_0}} \tilde{K}_{3,j}^{(a,k)}(q, \delta \varphi' + \zeta, X, x, \sigma, \varphi', \sigma').$$ \hspace{1cm} (6.1)

We now choose $Q_j$. Set $Q_j(\Phi', B, X) = 0$ if $X \not\in S_j$ or $B \not\in B_j(X)$; otherwise $Q_j(\Phi', B, X)$ is the sum of the following four terms.

1. A term proportional to $K_{0,j}$:

$$Q_{0,j}(\varphi', B, X) = \frac{1}{|X|} \sum_{x_0 \in B \cap \delta \varphi'} \text{Tay} E_j \left[ \tilde{K}_{0,j}(0, \delta \varphi' + \zeta, X) \right]$$

$$+ \frac{1}{|X|} \sum_{x_0 \in B \cap \delta \varphi'} \text{Tay} E_j \left[ \tilde{K}_{0,j}(\sigma, \delta \varphi' + \zeta, X) \right].$$ \hspace{1cm} (6.2)

2. Two terms proportional to $K_{1,j}$:

$$Q_{1,j}(\Phi', B, X) = Z_j L^{-2j} \sum_{x_0 \in B \cap \delta \varphi'} \text{Tay} E_j \left[ \tilde{K}_{1,j}(0, \delta \varphi' + \zeta, X, x, \sigma) \right]$$

$$+ Z_j L^{-2j} \sum_{x_0 \in B \cap \delta \varphi'} \text{Tay} E_j \left[ \tilde{K}_{1,j}(-\sigma, \delta \varphi' + \zeta, X, x, \sigma) \right],$$

$$Q_{1,j}^t(\Phi', B, X) = Z_j L^{-2j} \sum_{x_0 \in B \cap \delta \varphi'} \text{Tay} E_j \left[ \tilde{K}_{1,j}^t(0, \delta \varphi' + \zeta, X, x, \sigma) \right]$$

$$+ Z_j L^{-2j} \sum_{x_0 \in B \cap \delta \varphi'} \text{Tay} E_j \left[ \tilde{K}_{1,j}^t(\sigma, \delta \varphi' + \zeta, X, x, \sigma) \right],$$ \hspace{1cm} (6.3)

where the special point in $\delta \varphi'$ is $x$. Even though the pinning at $x$ prevents the generation of the volume factor $L^2$ when we sum these terms over $X$, note that
here the extraction is guided by the standard power counting. As it will be clear in Section 6.3, the reason for doing so is that we want to preserve the prefactor $L^{-2j}$ at each scale, which costs an $L^2$ factor at each step.

3. A term proportional to $K_{2,j}$:

$$Q_{2,j}(\Phi', B, X) = \sum_{k=0}^{j} 2^{-(j-k)} L^{-4k} \sum_{x_1 \in \mathcal{X}} e^{-L^{-4k}|x_1|} \times \sum_{\sigma, \sigma' = \pm 1} \sum_{\delta \phi'} \delta \phi' \sum_{x_2 \in \mathcal{X}^*} e^{i\alpha(\sigma + \sigma')(\eta - \frac{\delta \phi'}{2})^2}$$

Finally, we have the following result:

$$L_{2,j}^{(a)}(\Phi', Y') = L_{2,j}^{(a)}(Y') + L_{2,j}^{(a)}(\Phi', Y') + L_{2,j}^{(a)}(\Phi', \Phi')$$

where

$$L_{1,j}^{(a)}(\Phi', Y') = L^{-2(j+1)} Z_{j+1} \sum_{\sigma = \pm 1} J_{x,\sigma} L_{1,j}^{(a)}(\Phi', \Phi', x, \sigma)$$

and

$$L_{2,j}^{(a)}(\Phi', Y') = \sum_{x_1 \in \mathcal{X}, x_2 \in \mathcal{X}^*} J_{x_1, \sigma_1} J_{x_2, \sigma_2} L_{2,j}^{(a)}(\Phi', \Phi', x_1, \sigma_1, x_2, \sigma_2)$$

for (neglecting the variables that are $\Phi', Y', x_1, x_2, \sigma_1, \sigma_2$ in each term)

$$L_{2,j}^{(a)} = \sum_{k=0}^{j} 2^{-(j-k)} L^{-4k} e^{-L^{-4k}|x_1 - x_2|} \left[ Z_k^2 L_{2,j}^{(a,k)} + \mathcal{Z}_k L_{2,j}^{(a,k)} + Z_k \mathcal{Z}_k L_{2,j}^{(b,k)} \right].$$

Lemma 6.1. Assume by induction that

$$\left| \frac{Z_j}{Z_{j+1}} \right| \leq 1 \quad \text{and} \quad \left| \frac{\mathcal{Z}_j}{Z_{j+1}} \right| \leq 1;$$

then, for large enough $L$, there exist $\rho(L, A)$ and $\rho(L, A, \eta)$ such that

$$\|L_{2,j}^{(a)}\|_{h, T_{j+1}} \leq \rho(L, A) \|K_{0,j}\|_{h, T_j};$$

(6.7)
If instead $j = 0$, we do not include in (6.10) the first line, i.e. the one proportional to $V_{0,0}$. This was also the choice in [Falco, 2012]; and this explains why right hand side of (5.43) is quadratic (as opposed to cubic) in $s$ and $z$. With this definition of $Q_j^*$, the proof of the following Lemma is a computational verification.

**Lemma 6.2.**

$$L_j^{(b)}(\Phi', Y') = 0$$

Finally, we have to deal with $L_j^{(c)}(\Phi', Y')$: namely we have to show how $E_j, t_j$ and $Q_j, Q_j^*$ generate $E_{j+1}, t_{j+1}$ so that (5.16) holds and $L_j^{(c)}(\Phi', Y')$ is a contraction. We split this term in two pieces. First define intermediate effective parameters:

1. Intermediate effective couplings $s_j^*$ and $z_j^*$

$$s_j^* := s_j + \mathcal{F}_j,$$

$$z_j^* := L^2 e^{-\xi^2/2} e_{(0)} \{z_j + \mathcal{M}_j\}$$

where $\mathcal{F}_j$ and $\mathcal{M}_j$ are functionals of the fields

$$\mathcal{F}_j \equiv \mathcal{F}_j(K_j) = \sum_{X \in S_j} \frac{L - 2j}{|X|} \sum_{x \in X} \mathbb{E}_j \left[ \frac{\partial^2 \hat{K}_{0,j}}{\partial \phi_{x_1} \partial \phi_{x_2}}(0, \zeta, X) \right] \sum_{\mu \in \mathbb{U}} (x_1 - x_0)^\mu (x_2 - x_0)^\mu,$$

$$\mathcal{M}_j \equiv \mathcal{M}_j(K_j) = \frac{e^{-\xi^2/2} e_{(0)}}{2} \sum_{\sigma = \pm 1} \sum_{X \in S_j} \frac{1}{X_j} \mathbb{E}_j \left[ \hat{K}_{0,j}(\sigma, \zeta, X) \right].$$

2. Intermediate effective free energy

$$E_j^* = E_j + L^{-2j} \left[ E_{1,j} + s_j E_{2,j} \right]$$

Besides, fixed any $\eta \in (0, 1)$, the prefactors $\rho(L, A)$ and $\rho(L, A, \eta)$ are arbitrarily small for $L$ and $A$ large enough.

The proof is in Section 6.3. The next step is to choose $Q_j^*$: set $Q_j^*(\Phi', D, Y') := 0$ if $|Y'|_{j+1} \geq 3$ or $D \not\in B_{j+1}(Y')$; otherwise, if $j \geq 1$,

$$Q_j^*(\Phi', D, Y') := \frac{1}{2} \sum_{B_0 \cup B_2 = Y'} \mathbb{E}_j^T \left[ V_{0,j}(\tilde{t}_j, \Phi, B_0); V_{0,j}(\tilde{t}_j, \varphi, B_1) \right] + \sum_{B_0 \cup B_2 = Y'} \mathbb{E}_j^T \left[ V_{1,j}(\tilde{t}_j, \Phi, B_0); V_{0,j}(\tilde{t}_j, \varphi, B_1) \right] + \frac{1}{2} \sum_{B_0 \cup B_2 = Y'} \mathbb{E}_j^T \left[ V_{1,j}(\tilde{t}_j, \Phi, B_0); V_{1,j}(\tilde{t}_j, \Phi, B_1) \right].$$

(6.10)
3. Intermediate renormalization constants

\[ Z^*_j = L^2 e^{-\eta^2 \frac{a^2}{2} \Gamma_j(0)} \left[ (1 + M_{1,1,j}) Z_j + M_{1,2,j}(K_j) \overline{Z}_j \right], \]
\[ \overline{Z}_j = L^2 e^{-\eta^2 \frac{a^2}{2} \Gamma_j(0)} \left[ M_{2,1,j} Z_j + (1 + M_{2,2,j}) \overline{Z}_j \right], \]

where the functionals \( M_{p,q,j} \) are

\[ M_{1,1,j} \equiv M_{1,1,j}(K_j) = \frac{e^{\eta^2 \frac{a^2}{2} \Gamma_j(0)}}{2} \sum_{\sigma=\pm 1} \sum_{X \in S_j} E_j \left[ \hat{K}_{1,j}(0, \zeta, X, 0, \sigma) \right], \]
\[ M_{1,2,j} \equiv M_{1,2,j}(K_j) = \frac{e^{\eta^2 \frac{a^2}{2} \Gamma_j(0)}}{2} \sum_{\sigma=\pm 1} \sum_{X \in S_j} E_j \left[ \hat{K}_{1,j}(\sigma, \zeta, X, 0, \sigma) \right], \]
\[ M_{2,1,j} \equiv M_{2,1,j}(K_j) = \frac{e^{\eta^2 \frac{a^2}{2} \Gamma_j(0)}}{2} \sum_{\sigma=\pm 1} \sum_{X \in S_j} E_j \left[ \hat{K}_{1,j}(-\sigma, \zeta, X, 0, \sigma) \right], \]
\[ M_{2,2,j} \equiv M_{2,2,j}(K_j) = \frac{e^{\eta^2 \frac{a^2}{2} \Gamma_j(0)}}{2} \sum_{\sigma=\pm 1} \sum_{X \in S_j} E_j \left[ \hat{K}_{1,j}(0, \zeta, X, 0, \sigma) \right]. \]

Note that, in the definition of \( M_{m,n,j} \) we only retained the \( \text{Tr}_0 \) part of (6.3); this is because of cancellations due to (5.2) and (5.3). For example, (5.2) for \( S = R^2 \) gives for any \( m = 0, 1 \)

\[ \sum_{X \in S_j} \sum_{y \in X} E_j \left[ \frac{\partial \hat{K}_{1,j}(0, \zeta, X, 0, \sigma)}{\partial \zeta_y} \right] y^\mu = 0. \]

Besides, we used also the symmetry under charge conjugation (5.6) and (5.7), which implies, for example,

\[ E_j \left[ \hat{K}_{1,j}(0, \zeta, X, 0, 1) \right] = \frac{1}{2} \sum_{\sigma=\pm 1} E_j \left[ \hat{K}_{1,j}(0, \zeta, X, 0, \sigma) \right] \]

These points will be detailed in Section 6.4.

Next, split \( L_j^{(c)}(\Phi', Y') \) into two terms

\[ L_j^{(c)}(\Phi', Y') = L_j^{(c1)}(\Phi', Y') + L_j^{(c2)}(\Phi', Y') \]

for

\[ L_j^{(c1)}(\Phi', Y') := \sum_{B \in B_j} \left\{ (E_j^* - E_j)|B| + V_{j+1}(t_j^*, \varphi', B) - E_j |V_j(\varphi, B)| \right\} + \sum_{X \supset B} \left[ Q_j(\varphi', B, X) - Q_{2,j}(\varphi', B, X) \right] \]

(6.19)
By construction, $\mathcal{L}_j^{(c)}(\Phi', Y')$ is made of a part that is $J$-independent, which we call $\mathcal{L}_{0,j}^{(c)}(\Phi', Y')$, and a part that is linear in $J$, which we call $\mathcal{L}_{1,j}^{(c)}(\Phi', Y')$ and which can be further decomposed

$$\mathcal{L}_{1,j}^{(c)}(\Phi', Y') = L^{-2(j+1)}Z_{j+1} \sum_{x,\sigma} J_{x,\sigma} \mathcal{L}_{1,j}^{(c)}(\varphi', Y', x, \sigma) + L^{-2(j+1)}Z_{j+1} \sum_{x,\sigma} J_{x,\sigma} \mathcal{L}_{1,j}^{(c)}(\varphi', Y', x, \sigma).$$

(6.20)

**Lemma 6.3.** For large enough $L$, there exist $\rho(L, A)$ and $\rho(L, A, \eta)$ such that

$$\|\mathcal{L}_{0,j}^{(c)}\|_{h, T_{j+1}} \leq \rho(L, A)\|K_{0,j}\|_{h, T_j},$$

(6.21)

$$\|\mathcal{L}_{1,j}^{(c)}\|_{1, h, T_{j+1}} \leq \rho(L, A, \eta)\|K_{1,j}\|_{1, h, T_j},$$

(6.22)

$$\|\mathcal{L}_{1,j}^{(c)}\| \leq \rho(L, A, \eta)\|K_{1,j}\|_{h, T_j},$$

(6.22)

Besides, $\rho(L, A)$ and $\rho(L, A, \eta)$ are arbitrarily small for $L$ and $A$ large enough.

The proof is in Section 6.4. By subtraction, the other part of $\mathcal{L}_j^{(c)}(\Phi', Y')$ is

$$\mathcal{L}_j^{(c)}(\Phi', Y') = \sum_{D \in B_{j+1}} \left\{ (E_{j+1} - E_j^*)|D| + V_{j+1}(t_{j+1} - t_j^*, \Phi', D) + W_{j+1}(t_{j+1}, \Phi', D) - \Phi_j(D) \right\}$$

(6.23)

We want to choose $E_{j+1}$, $s_{j+1}$ and $z_{j+1}$ so that $\mathcal{L}_j^{(c)}(\Phi', Y')$ vanishes. Because of the identity

$$\sum_{Y \in S_{j+1}} Q_j^* (\Phi', D, Y) = \frac{1}{2} E_j^T \left[ V_0(j, \tilde{t}_j, \varphi, D); V_0(j, \tilde{t}_j, \varphi, D^*) \right]$$

$$+ \frac{1}{2} E_j^T \left[ V_1(j, \tilde{t}_j, \Phi, D); V_0(j, \tilde{t}_j, \varphi, D^*) \right]$$

(6.24)

and because of computations in Section 6.5, we finally set:

1. Effective couplings $s_{j+1}$ and $z_{j+1}$,

$$s_{j+1} = s_j^2 - a_j z_j^2$$

$$z_{j+1} = z_j^2 - L^2 e^{-\alpha^2 \Gamma_j(0)} b_j s_j z_j$$

(6.25)

where, setting $\Gamma_j(x) := \sum_{m=n}^{j} \Gamma_m(x)$ and $\Gamma_j(0|x) := \Gamma_j(0) - \Gamma_j(x)$, the coefficients in (6.25) are $a_0 = 0$, $b_0 = 0$, and, for any $j \geq 1$,

$$a_j := \frac{\alpha^2}{2} \sum_{y \in \mathbb{Z}} |y|^2 \left[ w_{b_j}(y) \left( e^{-\alpha^2 \Gamma_j(0)} - 1 \right) + e^{-\alpha^2 \Gamma_j(0)} \left( e^{\alpha^2 \Gamma_j(0)} - 1 \right) L^{-4j} \right],$$
\[ b_j := \frac{\alpha^2}{2} \sum_{\mu \in \mathbb{Z}} \left( (\partial^\mu \Gamma_j)^2 (y) + 2 \sum_{n=0}^{j-1} (\partial^\mu \Gamma_n)(y)(\partial^\mu \Gamma_j)(y) e^{-\frac{\alpha^2}{2} \Gamma_j_{-1,n}(0)} L^{2(j-n)} \right). \]

(6.26)

2. Effective free energy \( E_{j+1} \)

\[ E_{j+1} = E_j^* + L^{-2j} \left[ s_3^2 \mathcal{E}_{3,j} + z_2^2 \mathcal{E}_{4,j} \right] \]

(6.27)

where the coefficients in (6.29) are \( \mathcal{E}_{3,0} = \mathcal{E}_{4,0} = 0 \) and, for any \( j \geq 1 \),

\[ \mathcal{E}_{3,j} := \frac{L^{2j}}{4} \sum_{y \in \mathbb{Z}^d, \mu, \nu \in \mathbb{Z}} \left[ (\partial^{-\mu} \partial^\nu \Gamma_j)(y) + 2(\partial^{-\mu} \partial^\nu \Gamma_{j-1,1})(y) \right](\partial^{-\mu} \partial^\nu \Gamma_j)(y), \]

\[ \mathcal{E}_{4,j} := 2L^{2j} \sum_y w_{0,b,j}(y) \left[ e^{-\alpha^2 \Gamma_j(0)y} - 1 - \frac{\alpha^2}{2} |y|^2 \sum_{\mu \in \mathbb{Z}} (\partial^{-\mu} \partial^\nu \Gamma_j)(0) \right] + L^{-2j} \sum_y e^{-\alpha^2 \Gamma_j(0)} \left( e^{\alpha^2 \Gamma_j(y)} - 1 \right). \]

(6.28)

3. Fractional charge renormalization constants \( Z_{j+1} \) and \( \overline{Z}_{j+1} \)

\[ Z_{j+1} = Z_j + L^2 e^{-\eta \alpha^2 \Gamma_j(0)} (-m_{1,1,j} s_j Z_j + m_{1,2,j} \overline{Z}_j), \]

\[ \overline{Z}_{j+1} = \overline{Z}_j + L^2 e^{-\eta \alpha^2 \Gamma_j(0)} (-m_{2,1,j} s_j \overline{Z}_j + m_{2,2,j} \overline{Z}_j), \]

(6.29)

where the coefficients are, for any \( j \geq 0 \),

\[ m_{1,1,j} = \frac{\alpha^2 \eta^2}{4} \sum_{y \in \mathbb{Z}^d, \mu, \nu \in \mathbb{Z}} \left[ (\partial^{-\mu} \partial^\nu \Gamma_j)^2(y) + 2 \sum_{n=0}^{j-1} (\partial^{-\mu} \Gamma_n)(y) [\partial^{-\mu} \Gamma_j(0) - (\partial^\nu \Gamma_j)(0)] L^{2(j-n)} e^{-\eta \alpha^2 \Gamma_j_{-1,n}(0)} \right], \]

\[ m_{2,2,j} = \frac{\alpha^2 \eta^2}{4} \sum_{y \in \mathbb{Z}^d, \mu, \nu \in \mathbb{Z}} \left[ (\partial^{-\mu} \partial^\nu \Gamma_j)^2(y) + 2 \sum_{n=0}^{j-1} (\partial^{-\mu} \Gamma_n)(y) [\partial^{-\mu} \Gamma_j(0) - (\partial^\nu \Gamma_j)(0)] L^{2(j-n)} e^{-\eta \alpha^2 \Gamma_j_{-1,n}(0)} \right], \]

\[ m_{1,2,j} = \sum_{y \in \mathbb{Z}^d} w_{2,c,j}(y) \left( e^{-\alpha^2 \pi \Gamma_j(y)} - 1 \right) + L^{-2j} e^{-\eta \alpha^2 \Gamma_j(0)} \left( e^{-\pi \alpha^2 \Gamma_j(0)} - 1 \right), \]

\[ m_{2,1,j} = \sum_{y \in \mathbb{Z}^d} w_{2,c,j}(y) \left( e^{\alpha^2 \eta \Gamma_j(y)} - 1 \right) + L^{-2j} e^{-\eta \alpha^2 \Gamma_j(0)} \left( e^{\eta \alpha^2 \Gamma_j(0)} - 1 \right). \]

(6.30)

4. The functions \( w \)’s in (3.24) are all vanishing for \( j = 0, 1 \); while, for \( j \geq 2 \),

\[ w_{0,a,j}(y) = \frac{1}{2} \sum_{n=1}^{j-1} (\partial^{-\mu} \partial^\nu \Gamma_n)(y), \]

\[ w_{0,b,j}(y) = \frac{1}{2} \sum_{n=1}^{j-1} e^{-\alpha^2 \Gamma_{j-1,n+1}(0)y} e^{-\alpha^2 \Gamma_n(0)} \left( e^{\alpha^2 \Gamma_n(y)} - 1 \right) L^{-4n}, \]

\[ w_{0,c,j}(y) = \frac{1}{2} \sum_{n=1}^{j-1} e^{-\alpha^2 \Gamma_{j-1,n+1}(0)+\Gamma_{j-1,n+1}(y)} e^{-\alpha^2 \Gamma_n(0)} \left( e^{-\alpha^2 \Gamma_n(y)} - 1 \right) L^{-4n}, \]
\[ w_{0,d,j}^{\mu}(y) = \frac{\alpha}{2} \sum_{n=1}^{j-1} e^{-\frac{\alpha^2}{2} \Gamma_{j-1,n}(0)} (\partial^\mu \Gamma_n)(y) L^{-2n}, \]

\[ w_{0,e,j}^{\mu}(y) = \frac{\alpha^2}{4} \sum_{n=1}^{j-1} e^{-\frac{\alpha^2}{2} \Gamma_{j-1,n}(0)} \sum_{\mu \in \mathbb{N}} \left[ (\partial^\mu \Gamma_{j-1,n})^2(y) - (\partial^\mu \Gamma_{j-1,n+1})^2(y) \right] L^{-2n}. \]

(6.31)

5. The functions \( w \)'s in (3.25) are

\[ w_{1,b,j}(y) = \sum_{n=0}^{j-1} L^{-2n} e^{-\frac{\alpha^2}{2} \Gamma_{j-1,n}(0)} e^{-\eta^2 \Gamma_{j-1,n+1}(y)} \left( e^{-\eta^2 \Gamma_n(y)} - 1 \right), \]

\[ \bar{w}_{1,b,j}(y) = \sum_{n=0}^{j-1} L^{-2n} e^{-\frac{\alpha^2}{2} \Gamma_{j-1,n}(0)} e^{\eta^2 \Gamma_{j-1,n+1}(y)} \left( e^{\eta^2 \Gamma_n(y)} - 1 \right), \]

\[ w_{1,c,j}(y) = \sum_{n=0}^{j-1} L^{-2n} e^{-\frac{\alpha^2}{2} \Gamma_{j-1,n}(0)} e^{\eta^2 \Gamma_{j-1,n+1}(y)} \left( e^{\eta^2 \Gamma_n(y)} - 1 \right), \]

\[ \bar{w}_{1,c,n}(y) = \sum_{n=0}^{j-1} L^{-2n} e^{-\frac{\alpha^2}{2} \Gamma_{j-1,n}(0)} e^{-\eta^2 \Gamma_{j-1,n+1}(y)} \left( e^{-\eta^2 \Gamma_n(y)} - 1 \right), \]

\[ w_{1,d,j}^{\nu}(y) = i\alpha \eta \sum_{n=0}^{j-1} (\partial^\nu \Gamma_n)(y), \]

\[ \bar{w}_{1,d,j}^{\nu}(y) = i\alpha \eta \sum_{n=0}^{j-1} (\partial^\nu \Gamma_n)(y). \]

(6.32)

6. The functions \( w \)'s in (3.26) are

\[ w_{2,a,j}^{\xi}(y) = \frac{1}{2} \sum_{n=0}^{j-1} Z^2_n L^{-4n} e^{-\eta^2 (1+\epsilon) \alpha^2 \Gamma_{j-1,n+1}(y) e^{-\eta^2 \alpha^2 \Gamma_{j-1,n+1}(y)}} \]

\[ \times \left( e^{-\eta^2 \alpha^2 \Gamma_n(y)} - 1 \right), \]

\[ \bar{w}_{2,a,j}^{\xi}(y) = \frac{1}{2} \sum_{n=0}^{j-1} Z^2_n L^{-4n} e^{-\eta^2 (1+\epsilon) \alpha^2 \Gamma_{j-1,n+1}(y)} e^{-\eta^2 \alpha^2 \Gamma_{j-1,n+1}(y)}} \]

\[ \times \left( e^{-\eta^2 \alpha^2 \Gamma_n(y)} - 1 \right), \]

\[ w_{2,b,j}^{\xi}(y) = \frac{1}{2} \sum_{n=0}^{j-1} Z^2_n L^{-4n} e^{-\eta^2 (1+\epsilon) \alpha^2 \Gamma_{j-1,n+1}(y)} e^{-\eta^2 \alpha^2 \Gamma_{j-1,n+1}(y)}} \]

\[ \times \left( e^{-\eta^2 \alpha^2 \Gamma_n(y)} - 1 \right), \]

\[ w_{2,c,j}^{\xi}(y) = \sum_{k=0}^{j-1} L^{-4k} e^{-L^{-k} \eta^2} \sum_{n=0}^{j-1} e^{-\frac{\alpha^2}{2} (1+\epsilon)^2 \Gamma_{j-1,n+1}(y)-\eta^2 \alpha^2 \Gamma_{j-1,n+1}(y)} \]

\[ \times \left\{ Z^2_n \sum_{\sigma=\pm 1} \sum_{X \in \mathbb{R}^n} E_j \left[ \bar{w}_{2,n}^{(a,k)} \left( -\sigma \frac{1+\epsilon}{2}, \xi, X, 0, \sigma, y, \sigma \xi \right) \right] \right\} \]
\[ + \frac{Z_k}{2} \sum_{\sigma = \pm 1} \sum_{X \in S_n} E_j \left[ \hat{K}_{2,n}^{(\pi,k)} \left( \sigma \frac{1 + \varepsilon}{2}, \zeta, X, 0, \sigma, y, \sigma \varepsilon \right) \right] \]

\[ + Z_k \frac{1}{2} \sum_{\sigma = \pm 1} \sum_{X \in S_n} E_j \left[ \hat{K}_{2,n}^{(b,k)} \left( -\sigma \frac{1 - \varepsilon}{2}, \zeta, X, 0, \sigma, y, \sigma \varepsilon \right) \right] \right). \quad (6.33) \]

Note that, because of the smallness condition on \( X \), \( w_{2,c,j}^2(y) = 0 \) for \( |y| \geq 8L^{-j} \).

**Lemma 6.4.**

\[ \mathcal{L}^{(c,2)}_j (\Phi', Y') = 0. \]

By (3.10), \( W_j (\varphi, B) \) depends on the fields \( \varphi_x \) and \( J_{x,\sigma} \) for \( x \) in a neighborhood of \( B \) of diameter \( L^j / 2 \), which is a subset of \( B^* \). Finally, joining (6.25) with (6.11) we obtain (3.30) and (5.29); and condition (5.16) is fulfilled.

6.2. **Proof of Lemma 3.5.** The formulas for \( \{ M_{m,n,j} : m, n = 1, 2 \} \) are in (6.16). The bounds (3.41) directly descend from (4.36) and (4.37) at \( \varphi' = 0 \). For example, for \( \lambda = \frac{1}{2} \) and a \( C \equiv C(\alpha, L) \),

\[ |M_{1,1,j}| \leq \frac{e^2 \pi^2 \Gamma_j(0)}{2} \sum_{\sigma = 1} X_{n \geq 0} \sum_{X \in S_j} ||E_j|| \left[ \hat{K}_{1,j}(0, \zeta, X, 0, \sigma) \right] ||L_{1,j+1}(0, X) \]

\[ \leq C \|K_{1,j}\|_{1, h, T_j} \sum_{X \in S_j} \frac{A}{2}^{-|X_j|} \leq CS^2 k^*_s (A, \lambda) A^{-1} \|K_{1,j}\|_{1, h, T_j}. \quad (6.34) \]

The other \( M_{p,q,j} \)'s can be studied in a similar way.

6.3. **Proof of Lemma 6.1.** In [Falco, 2012] we already proved formula (6.7), for \( \rho(L, A) = C(L^{-\vartheta} + A^{-\vartheta'}) \), where \( C > 1 \) and \( \vartheta, \vartheta' > 0 \). We only need to derive (6.8) and (6.9). Consider \( \mathcal{L}^{(a)}_{1,j} (\varphi', V, x, \sigma) \) and decompose

\[ \mathcal{L}^{(a)}_{1,j} (\varphi', V, x, \sigma) = \sum_{n=1}^{3} \mathcal{L}^{(n)}_j (\varphi', V, x, \sigma), \]

where, with Taylor expansions in \( \delta \varphi' \),

\[ \mathcal{L}^{(1)}_j (\varphi', V, x, \sigma) := \frac{Z_j}{Z_{j+1}} L^2 \sum_{\gamma = V} E_j [K_{1,j}(\varphi, Y, x, \sigma)], \quad (6.35) \]

\[ \mathcal{L}^{(2)}_j (\varphi', V, x, \sigma) := \frac{Z_j}{Z_{j+1}} L^2 \sum_{\gamma = V} E_j \left[ \hat{K}_{1,j}(q, \varphi, Y, x, \sigma) \right], \quad (6.36) \]

\[ \mathcal{L}^{(3)}_j (\varphi', V, x, \sigma) := \frac{Z_j}{Z_{j+1}} L^2 \sum_{\gamma = V} E_j \left[ \hat{K}_{1,j}(q, \varphi, Y, x, \sigma) \right]. \quad (6.37) \]

Let us consider each of the terms, assuming \( |Z_j / Z_{j+1}| \leq 1. \)
1. **Norm of $\mathcal{L}^{(1)}$.** Use (4.21), as well as a simple extension of (4.32) to activities with a pinning point, to find

$$\|\mathcal{L}^{(1)}_j(\varphi', V, x, \sigma)\|_{h, T_{j+1}(\varphi', V)} \leq L^2 \sum_{Y \in S_j} \sum_{Y' \supseteq x} \left\| \mathbb{E}_j \left[ K_{1,j}(\varphi, Y, x, \sigma) \right] \right\|_{h, T_{j+1}(\varphi', Y)}$$

$$\leq G_{j+1}(\varphi', V) \| K_{1,j} \|_{1,h,T_j} L^2 \sum_{Y \in S_j} \sum_{Y' \supseteq x} A^{-|V|} 2^{|V|}$$

$$\leq G_{j+1}(\varphi', V) A^{-|V|+1} \| K_{1,j} \|_{1,h,T_j} L^2 k_l(A, 1/2). \quad (6.38)$$

By (4.25), we find

$$\|\mathcal{L}^{(1)}_j\|_{1,h,T_{j+1}} \leq \delta_1(L, A) \| K_{1,j} \|_{1,h,T_j}$$

with $\delta_1(A, L) = L^2 A^{-\eta}$; this quantity can be made as small as needed since $A$ is chosen after $L$.

2. **Norm of $\mathcal{L}^{(2)}$.** Use (4.21) and (4.36) to find, for $C \equiv C(\alpha)$ and if $\alpha^2 \geq 8\pi$,

$$\|\mathcal{L}^{(2)}_j(\varphi', V, x, \sigma)\|_{h, T_{j+1}(\varphi', V)} \leq L^2 \sum_{Y \in S_j(V)} \sum_{Y' \supseteq x} \| \mathbb{E}_j \left[ \mathcal{K}_{1,j}(q, \varphi, Y, x, \sigma) \right] \|_{h, T_{j+1}(\varphi', Y)}$$

$$\leq G_{j+1}(\varphi', V) A^{-|V|+1} \| K_{1,j} \|_{1,h,T_j} k^*_l(A, 1/2) L^2 \sum_{q \in \mathbb{Z}, q \neq 0} \rho_1(q + \eta \sigma, \alpha); \quad (6.39)$$

by (4.25) we obtain

$$\|\mathcal{L}^{(2)}_j\|_{1,h,T_{j+1}} \leq \delta_2(L, A) \| K_{1,j} \|_{1,h,T_j}$$

with $\delta_2(A, L) = CL^{-\min\{\eta\|, \pi\}}$; this quantity can be made small by taking $L$ large enough, given the choice of $\eta$.

3. **Norm of $\mathcal{L}^{(3)}$.** By (4.21) and (4.44)

$$\|\mathcal{L}^{(3)}_j(\varphi', V, x, \sigma)\|_{h, T_{j+1}(\varphi', V)} \leq L^2 \sum_{Y \in S_j(V)} \sum_{Y' \supseteq x} \| \mathbb{E}_j \left[ \mathcal{K}_{1,j}(q, \varphi, Y, x, \sigma) \right] \|_{h, T_{j+1}(\varphi', Y)}$$

$$\leq G_{j+1}(\varphi', V) A^{-|V|+1} \| K_{1,j} \|_{1,h,T_j} k^*_l(A, 1/2) L^2 \sum_{q \in \mathbb{Z}, q \neq 0} \rho_1(q + \eta \sigma, \alpha); \quad (6.40)$$

by (4.25) we obtain

$$\|\mathcal{L}^{(3)}_j\|_{1,h,T_{j+1}} \leq \delta_3(L, A) \| K_{1,j} \|_{1,h,T_j}$$

with $\delta_3(L, A) = \frac{\kappa_l}{\kappa} L^{-\eta \pi}$. This proves the former of (6.8) for $\rho(L, A) = \delta_1(L, A) + \delta_2(L, A) + \delta_3(L, A)$. The latter of (6.8) has a similar proof. Let us now consider (6.9). For $\mathcal{L}^{(a,k)}_2$ we have

$$\mathcal{L}^{(a,k)}_{2,j}(\varphi', V, x_1, \sigma_1, x_2, \sigma_2) = \sum_{p=1}^{3} \mathcal{L}^{(a,p,k)}_{2,j}(\varphi', V, x_1, \sigma_1, x_2, \sigma_2)$$
where
\begin{equation}
\mathcal{L}_{2,j}^{(a,1,k)}(\varphi', V, x_1, x_2, \sigma_2) := \sum_{\mathcal{Y} \in \mathcal{F}(V)} \sum_{Y \in \mathcal{I}(V)} \mathbb{E}_j \left[ K_{2,j}^{(a,k)}(\varphi, Y, x_1, \sigma_1, x_2, \sigma_2) \right],
\end{equation}
(6.41)

\begin{equation}
\mathcal{L}_{2,j}^{(a,2,k)}(\varphi', V, x_1, x_2, \sigma_2) := \sum_{\mathcal{Y} \in \mathcal{F}(V)} \sum_{Y \in \mathcal{I}(V)} \mathbb{E}_j \left[ \tilde{K}_{2,j}^{(a,k)}(\varphi, Y, x_1, \sigma_1, x_2, \sigma_2) \right],
\end{equation}
(6.42)

\begin{equation}
\mathcal{L}_{2,j}^{(a,3,k)}(\varphi', V, x_1, x_2, \sigma_2) := \sum_{\mathcal{Y} \in \mathcal{F}(V)} \sum_{Y \in \mathcal{I}(V)} \mathbb{E}_j \left[ \tilde{K}_{2,j}^{(a,k)}(-\frac{\sigma_1 + \sigma_2}{2}, \varphi, Y, x_1, \sigma_1, x_2, \sigma_2) \right].
\end{equation}
(6.43)

With estimates similar to the ones used in the previous discussions, we have a bound
\begin{equation}
\|L_{2,j}^{(a,p,k)}\|_{2,h,T_{j+1}} \leq \delta_{a,p,k}(L, A, \eta)\|K_{2,j}^{(a,k)}\|_{2,h,T_j}
\end{equation}
(6.44)

where possible choices of the prefactors \(\delta_{a,p,k}(L, A, \eta)\)'s are: using the third of (4.25), \(\delta_{a,1,k}(L, A, \eta) = A^{-\theta} \) for a \(\theta > 0\); using the second of (4.25) and (4.38), \(\delta_{a,2,k}(L, A, \eta) = CL^{-d(2\eta)}\) for an \(\eta\) independent \(C\) and for \(d(2\eta)\) defined in Theorem 4.5; using the second of (4.25) and (4.45), \(\delta_{a,3,k}(L, A, \eta) = C (\sqrt{KL})^{-1}\). This proves (6.9) for \(\delta = a\) and \(\rho(L, A, \eta) = \sum_{p=1}^{3} \delta_{a,p,k}(L, A, \eta)\). The proof of (6.9) for \(\delta = \pi, b\) is similar.

6.4. Proof of Lemma 6.3.

Lemma 6.5. For \(V_j\) given in (3.23),
\begin{equation}
\mathbb{E}_j \left[ V_j (t_j, \Phi, B) \right] = (\tilde{E}_j - E_j) + V_{j+1}(\tilde{t}_j, \Phi', B),
\end{equation}
(6.45)

where
\begin{equation}
\tilde{E}_j := E_j - \frac{S_j}{2} |B| \sum_{\mu \in \tilde{u}} (\partial^{-\mu} \partial^{\mu} \Gamma_j)(0)
\end{equation}
(6.46)

and \(\tilde{t}_j\) is defined in (5.20).

Proof. From standard results on the correlations of Gaussian measures,
\begin{equation}
\mathbb{E}_j \left[ V_{0,j} (t_j, \Phi, B) \right] = \frac{S_j}{2} \sum_{x \in B} \frac{\varphi'}{\varphi}_{x}^2 |B| \sum_{\mu \in \tilde{u}} (\partial^{-\mu} \partial^{\mu} \Gamma_j)(0)
+ z_j L^{-2j} e^{-\frac{d^2}{2} \tilde{\Gamma}_j(0)} \sum_{x \in B} \sum_{\sigma = \pm 1} e^{i\sigma \varphi_{x}'}
\end{equation}
(6.47)

\begin{equation}
\mathbb{E}_j \left[ V_{1,j} (t_j, \Phi, B) \right] = Z_j L^{-2j} e^{-\frac{d^2}{2} \tilde{\eta} \tilde{\eta} \Gamma_j(0)} \sum_{x \in B} \sum_{\sigma = \pm 1} J_{\sigma, x} e^{i\sigma \eta \varphi_{x}'}
+ Z_j L^{-2j} e^{-\frac{d^2}{2} \tilde{\eta} \tilde{\eta} \tilde{\Gamma}_j(0)} \sum_{x \in \tilde{u}} \sum_{\sigma = \pm 1} J_{\sigma, x} e^{i\sigma \tilde{\eta} \varphi_{x}'}
\end{equation}
(6.48)

These identities give (6.45).
Via this Lemma, be obtain the following formulas for \( L_{1,j}^{(e)} \) and \( L_{1,j}^{(e)} \):

\[
L_{1,j}^{(e)} (\varphi', V, x, \sigma) = \frac{Z_j}{Z_{j+1}} L^2 \sum_{B \in S_j(V)} \left[ e^{-\eta^2 \frac{1}{2} \Gamma_j(0) M_{1,1,j} - \sum_{X \supseteq B} \text{Tay} \mathbb{E}_j \left[ \hat{K}_{1,j} (0, \delta \varphi' + \zeta, X, x, \sigma) \right]} \right] e^{i \eta \sigma \varphi'_x} \\
+ \frac{Z_j}{Z_{j+1}} L^2 \sum_{B \in S_j(V)} \left[ e^{-\eta^2 \frac{1}{2} \Gamma_j(0) M_{2,2,j} - \sum_{X \supseteq B} \text{Tay} \mathbb{E}_j \left[ \hat{K}_{1,j} (\sigma, \delta \varphi' + \zeta, X, x, \sigma) \right]} \right] e^{i \eta \sigma \varphi'_x}
\]

(6.49)

and

\[
L_{1,j}^{(e)} (\varphi', V, x, \sigma) = \frac{Z_j}{Z_{j+1}} L^2 \sum_{B \in S_j(V)} \left[ e^{-\eta^2 \frac{1}{2} \Gamma_j(0) M_{2,2,j} - \sum_{X \supseteq B} \text{Tay} \mathbb{E}_j \left[ \hat{K}_{1,j} (0, \delta \varphi' + \zeta, X, x, \sigma) \right]} \right] e^{i \eta \sigma \varphi'_x} \\
+ \frac{Z_j}{Z_{j+1}} L^2 \sum_{B \in S_j(V)} \left[ e^{-\eta^2 \frac{1}{2} \Gamma_j(0) M_{1,1,j} - \sum_{X \supseteq B} \text{Tay} \mathbb{E}_j \left[ \hat{K}_{1,j} (\sigma, \delta \varphi' + \zeta, X, x, \sigma) \right]} \right] e^{i \eta \sigma \varphi'_x}
\]

(6.50)

Let us consider the two terms in the square brackets in the first line of (6.49): by (6.17) and (6.18) they equal to

\[
\sum_{X \supseteq x} \sum_{X \in S_j} \text{Tay} \mathbb{E}_j \left[ \frac{\partial \hat{K}_{1,j}}{\partial y} (0, \zeta, X, x, \sigma) \right] \left( \sum_{\mu \in \mathbb{U}} (y - x)^{\mu} \partial^\mu \varphi'_x - (\varphi'_y - \varphi'_x) \right)
\]

(6.51)

Note that (6.51) depends on \( \varphi' \) only via the factor \( u_x(y, \varphi') := \sum_{\mu \in \mathbb{U}} (y - x)^{\mu} \partial^\mu \varphi'_x - (\varphi'_y - \varphi'_x) \) and that, with the notation of (4.2),

\[
D^n u_x(y, \varphi') \cdot (f_1, \ldots, f_n) = \begin{cases} u_x(y, f_1) & \text{if } n = 1 \\ 0 & \text{if } n \geq 2 \end{cases}
\]

(6.52)

As \( X \in S_j \), we have \( X^* \subset V^* \) and \( |y - x| \leq CL^1 \), so that

\[
\| u_x (\cdot, \varphi) \|_{C^2(X)} \leq CL^{-2} \| \nabla^2_{j+1} \varphi \|_{L^\infty(V^*)}
\]

(6.53)

Finally the \( \| \cdot \|_{h, T_{j+1}(\varphi', X)} \) norm of (6.51) is bounded by

\[
\sum_{X \supseteq x} \| \text{Tay} \mathbb{E}_j \left[ \hat{K}_{1,j} (0, \zeta, X, x, \sigma) \right] \|_{h, T_{j+1}(0, X)} \\
\times \left( \| u_x (\cdot, \varphi) \|_{C^2(X)} + \sup_{\| f \|_{C^2_{j+1}} = 1} \| u_x (\cdot, f) \|_{C^2(X)} \right)
\]

\[
\leq CL^{-2(1 + \eta^2)} \| K_{1,j} \|_{1, h, T_j} (1 + \| \nabla^2_{j+1} \varphi \|_{L^\infty(V^*)}) \sum_{X \supseteq x} (A/2)^{-|X|} \\
\leq C'\kappa L^{-1} L^{-2(1 + \eta^2)} \| K_{1,j} \|_{1, h, T_j} G_{j+1}^a (\varphi', V)(A/2)^{-1}
\]

(6.54)
where, to obtain the second line we used (4.34) at \( \varphi' = 0 \) and (6.53). The other lines in (6.49) and (6.50) can be dealt with exactly the same procedure. Finally, as \(|V_{j+1}| = 1\), \(Z_j/Z_{j+1} \leq 1\) and because of (4.14), we obtain
\[
\|L_{1,j}^{(c)}(\varphi', V, x, \sigma)\|_{h,T_j+1(\varphi', V)} \leq C\kappa_L^{-1} L^{-2\eta'} \|K_{1,j}\|_{1,h,T_j} G_{j+1}(\varphi', V) A^{-1}|V|_{j+1}
\]
which proves the first of (6.22) for \(\rho(L, A, \eta) = C\kappa_L^{-1} L^{-2\min(\eta', \eta)}\).

6.5. **Proof of Lemma 6.4.** This proof is a detailed calculation of the second order part of the RG map.

**Lemma 6.6.** If the choice of the \( w \)'s functions is the one in (6.31), (6.32) and (6.33), and the choice for \( E_{j+1} \) and \( t_{j+1} \), \( \bar{t}_j, \bar{t}_j^* \) is the one in Section 6.1, then, for any \( D \in B_{j+1} \) and \( B \in B_j(D) \),
\[
\frac{1}{2} \mathbb{E}_j^T \left[ V_{1,j}(\bar{t}_j, \Phi, B); V_{1,j}(\bar{t}_j, \Phi, D^*) \right] + \mathbb{E}_j^T \left[ V_{0,j}(\bar{t}_j, \varphi, B); V_{0,j}(\bar{t}_j, \varphi, D^*) \right] + \sum_{X \in \mathcal{B}_j} Q_2,b_j(\Phi', B, X)
\]
\[
= W_{j+1}(\Phi', B) - \mathbb{E}_j \left[ W_j(\bar{t}_j, \Phi, B) \right] + (E_{j+1} - E_j^*)|B| + V_{j+1}(t_{j+1} - \bar{t}_j^*, \Phi', B). \tag{6.55}
\]

**Proof.** An explicit computation of Gaussian correlations (for \( \alpha \) and \( \alpha' \) any real parameter) yields:
\[
\begin{align*}
\mathbb{E}_j^T \left[ (\partial^\mu \zeta_x)^2; (\partial^\nu \zeta_{x+y})^2 \right] &= 2(\partial^{-\mu} \partial^\nu \Gamma_j)(y)^2, \\
\mathbb{E}_j^T \left[ (\partial^\mu \zeta_x); (\partial^\nu \zeta_{x+y}) \right] &= - (\partial^{-\mu} \partial^\nu \Gamma_j)(y), \\
\mathbb{E}_j^T \left[ e^{i\alpha \zeta_{x+y}}; (\partial^\mu \zeta_x)^2 \right] &= - \alpha^2 e^{-\frac{\alpha^2}{2} \Gamma_j}(0) (\partial^{-\mu} \Gamma_j)(y)^2, \\
\mathbb{E}_j^T \left[ e^{i\alpha \zeta_{x+y}}; (\partial^\mu \zeta_x)^2 \right] &= - \alpha^2 e^{-\frac{\alpha^2}{2} \Gamma_j}(0) (\partial^{-\mu} \Gamma_j)(y)^2, \\
\mathbb{E}_j^T \left[ e^{i\alpha \zeta_{x+y}}; (\partial^\mu \zeta_{x+y}) \right] &= i \alpha e^{-\frac{\alpha^2}{2} \Gamma_j}(0) (\partial^{-\mu} \Gamma_j)(y), \\
\mathbb{E}_j^T \left[ e^{i\alpha \zeta_{x+y}}; (\partial^\mu \zeta_x) \right] &= - i \alpha e^{-\frac{\alpha^2}{2} \Gamma_j}(0) (\partial^{-\mu} \Gamma_j)(y), \\
\mathbb{E}_j^T \left[ e^{i\alpha \zeta_{x+y}}; e^{i\alpha' \zeta_{x+y}} \right] &= e^{-\frac{\alpha^2}{2} \Gamma_j}(0) e^{-\frac{\alpha'^2}{2} \Gamma_j}(0) \left( e^{-\alpha' \Gamma_j}(y) - 1 \right). \tag{6.56}
\end{align*}
\]

Let \( Y := D^* \in \mathcal{P}_{j+1} \). Let us separate the discussion of (6.55) into three parts.

1. **First part.** Our goal is to determine the functions \( w_{0,\alpha,j}(y) \)'s and the coefficients \( t_{j+1} \) so to satisfy Lemma (6.4) for the part that doesn’t depend on \( j \):
\[
\begin{align*}
\frac{1}{2} \mathbb{E}_j^T \left[ V_{0,j}(\bar{t}_j, \Phi, B); V_{0,j}(\bar{t}_j, \Phi, Y) \right] &= W_{0,j+1}(\Phi', B) - \mathbb{E}_j \left[ W_{0,j}(\bar{t}_j, \Phi, B) \right] \\
+ (E_{j+1} - E_j^*)|B| + V_{0,j+1}(t_{j+1} - \bar{t}_j^*, \Phi', B). \tag{6.57}
\end{align*}
\]

This identity was already verified in [Falco, 2012]. However, here we want to show how to re-derive it by means of an ansatz that can be generalized to the more sophisticated second an third parts. We look for \( w_{0,\alpha,j}(y) \), where \( \alpha \) collects the various labels that appear in (3.24), into the form of sum of contribution gathered
at each scale \( n \leq j - 1 \):
\[
\mathbf{w}_{0,\alpha,j}(y) = \sum_{n=1}^{j-1} R_{0,\alpha,n}^{(j-1)}(y).
\]

By use of (6.56),
\[
\frac{1}{2} \mathcal{E}_{j+2}^T [\mathcal{V}_{0,j}(\Phi, B); \mathcal{V}_{0,j}(\varphi, Y)] = s^2 \sum_{\mu, \nu \in \hat{u}} \sum_{y \in \hat{u}^2} (\partial^{-\mu} \partial^\nu \Gamma_j)(y)^2 \\
- s^2 \sum_{\mu, \nu \in \hat{u}} \sum_{y \in \hat{u}^2} (\partial^{-\mu} \partial^\nu \Gamma_j)(y) \sum_{x \in \hat{B}} (\partial^\mu \varphi_x')(y) (\partial^\nu \varphi_{x+y}) \\
+ z_j^2 \frac{\mathcal{L}_{-4j}}{2} \sum_{y \in \hat{u}^2} e^{-\alpha^2 \Gamma_j(y)} \left( e^{\alpha^2 \Gamma_j(y)} - 1 \right) \sum_{x \in \hat{B}} e^{i\alpha \sigma (\varphi_x' - \varphi_{x+y})} \\
+ z_j^2 \frac{\mathcal{L}_{-4j}}{2} \sum_{y \in \hat{u}^2} e^{-\alpha^2 \Gamma_j(y)} \left( e^{-\alpha^2 \Gamma_j(y)} - 1 \right) \sum_{x \in \hat{B}} e^{i\alpha \sigma (\varphi_x' + \varphi_{x+y})} \\
+ z_j s^2 \frac{\mathcal{L}_{-4j}^2}{2} \sum_{y \in \hat{u}^2} e^{-\alpha^2 \Gamma_j(y)} (\partial^\nu \Gamma_j)(y) \sum_{x \in \hat{B}} \sigma \left[ e^{i\alpha \sigma \varphi_x'} (\partial^\nu \varphi_{x+y}) - e^{i\alpha \sigma \varphi_{x+y}} (\partial^\nu \varphi_x') \right] \\
- z_j s^2 \frac{\mathcal{L}_{-4j}^2}{4} \sum_{y \in \hat{u}^2} e^{-\alpha^2 \Gamma_j(y)} (\partial^\nu \Gamma_j)^2(y) \sum_{x \in \hat{B}} \left[ e^{i\alpha \sigma \varphi_x' + e^{i\alpha \sigma \varphi_{x+y}}} \right]. 
\]

The above terms have to be re-arranged according to the following rule: each term is to be either power-counting irrelevant (see discussion after (4.45)) or local (namely with all the fields \( \varphi' \) dependent by a same point \( x \)), or constant (namely independent of \( \varphi' \)). Let us discuss each line of the right hand side member. The first line is a constant, which will be absorbed into \( E_{j+1} \). The second line appears to be marginal; in fact it is irrelevant, because one can plug in the identity
\[
(\partial^\mu \varphi_x')(\partial^\nu \varphi_{x+y}) = (\partial^\mu \varphi_x') \left[ (\partial^\nu \varphi_{x+y}) - (\partial^\nu \varphi_x') \right] + (\partial^\mu \varphi_x')(\partial^\nu \varphi_x')
\]

and neglect the last term because of the cancellation
\[
\sum_{\mu, \nu \in \hat{u}} \sum_{y \in \hat{u}^2} (\partial^{-\mu} \partial^\nu \Gamma_j(y)) = \sum_{i,j=0,1} \sin k_i \sin k_j \tilde{\Gamma}_j(k) |_{k=0} = 0. 
\]

The third line is relevant. To write it as the sum of an irrelevant term plus a local one use the identity
\[
e^{i\alpha \sigma (\varphi_x' - \varphi_{x+y})} = e^{i\alpha \sigma (\varphi_x' - \varphi_{x+y})} - 1 + \frac{\alpha^2}{4} |y|^2 \sum_{\mu \in \hat{u}} (\partial^\mu \varphi_x')^2 \\
+ 1 - \frac{\alpha^2}{4} |y|^2 \sum_{\mu \in \hat{u}} (\partial^\mu \varphi_x')^2.
\]

Again, by symmetries, we have neglected a term, the linear order of the Taylor expansion in \( y \): when plugged into (6.58) this term cancels because it is odd in \( \sigma \). Besides, the term proportional to \( |y|^2 \) is chosen with a special form thanks to the
partial cancellation, for $m, n = 0, 1$,
\[
\sum_{y \in \mathbb{Z}^2} e^{-\alpha^2 \Gamma_j(0)} \left( e^{\alpha^2 \Gamma_j(y)} - 1 \right) y_m y_n = \frac{\delta_{m,n}}{2} \sum_{y \in \mathbb{Z}^2} e^{-\alpha^2 \Gamma_j(0)} \left( e^{\alpha^2 \Gamma_j(y)} - 1 \right) |y|^2,
\]
which makes irrelevant the sum of the terms in the square brackets. The fourth and the fifth lines of (6.58) are irrelevant. The only remaining relevant term is the sixth line. To write it as the sum of an irrelevant term plus a local one use the identity
\[
e^{i \alpha \sigma \varphi_{x+y}} = \left[ e^{i \alpha \sigma \varphi_x} - e^{i \alpha \sigma \varphi_y} \right] + 2 e^{i \alpha \sigma \varphi_x}.
\]
In conclusion, after all such operations, we obtain a new equivalent formula for (6.58):
\[
\frac{1}{2} \mathbb{E}^T_\nu \left[ V_{0,j} (\Phi, \bar{B}); V_{0,j} (\varphi, Y) \right]
= s_j^2 \left| B \right| \frac{1}{4} \sum_{\nu \in \mathbb{Z}^2} (\partial^{-\nu} \partial^\nu \Gamma_j(y))^2(y) + s_j^2 \left| B \right| L^{-4j} \sum_{y \in \mathbb{Z}^2} e^{-\alpha^2 \Gamma_j(0)} \left( e^{\alpha^2 \Gamma_j(y)} - 1 \right)
- s_j^2 \frac{1}{2} \sum_{\nu \in \mathbb{Z}^2} (\partial^{-\nu} \partial^\nu \Gamma_j(y)) \sum_{x \in \bar{B}} (\partial^\nu \varphi'_x) \left[ (\partial^\nu \varphi'_{x+y}) - (\partial^\nu \varphi'_x) \right]
+ s_j^2 \frac{1}{2} \sum_{y \in \mathbb{Z}^2} e^{-\alpha^2 \Gamma_j(0)} \left( e^{\alpha^2 \Gamma_j(y)} - 1 \right) \sum_{x \in \bar{B}} \left[ e^{i \alpha \sigma \varphi'_x - i \alpha \sigma \varphi'_{x+y}} - 1 + |y|^2 \sum_{\mu \in \mathbb{N}} (\partial^\mu \varphi_x)^2 \right]
- s_j^2 \frac{1}{2} \sum_{y \in \mathbb{Z}^2} e^{-\alpha^2 \Gamma_j(0)} \left( e^{\alpha^2 \Gamma_j(y)} - 1 \right) \sum_{x \in \bar{B}} \left[ e^{i \alpha \sigma \varphi'_x + i \alpha \sigma \varphi'_{x+y}} \right]
+ \frac{\alpha L^{-2j}}{2} \sum_{y \in \mathbb{Z}^2} \sum_{\nu \in \mathbb{Z}^2} e^{-\alpha^2 \Gamma_j(0)} (\partial^\nu \Gamma_j(y)) \sum_{x \in \bar{B}} \sum_{\sigma = \pm 1} i \sigma \left[ e^{i \alpha \sigma \varphi_x} (\partial^\nu \varphi'_{x+y}) - e^{i \alpha \sigma \varphi_x} (\partial^\nu \varphi'_x) \right]
+ \frac{\alpha L^{-2j}}{4} \sum_{y \in \mathbb{Z}^2} \sum_{\nu \in \mathbb{Z}^2} e^{-\alpha^2 \Gamma_j(0)} (\partial^\nu \Gamma_j(y))^2 \sum_{x \in \bar{B}} \sum_{\sigma = \pm 1} \left[ e^{i \alpha \sigma \varphi'_{x+y}} - e^{i \alpha \sigma \varphi'_x} \right]
- \frac{\alpha L^{-2j}}{2} \sum_{y \in \mathbb{Z}^2} \sum_{\nu \in \mathbb{Z}^2} e^{-\alpha^2 \Gamma_j(0)} (\partial^\nu \Gamma_j(y))^2 \sum_{x \in \bar{B}} \sum_{\sigma = \pm 1} e^{i \alpha \sigma \varphi'_x}.
\tag{6.61}
\]

The first part of our ansatz is that the irrelevant terms which were generated in the above integration provide the $R_{0,0,j}^{(j)}(y)$'s; more precisely, plugging in (3.24) $\bar{t}_j$ instead of $t_j$ (namely replacing $s_j$ and $z_j$ with $s_{j+1}$ and $z_{j+1}$) and then comparing the irrelevant lines with (3.24), we set
\[
R_{0,0,j}^{(j)}(y) = \frac{1}{2} (\partial^{-\nu} \partial^\nu \Gamma_j)(y),
\]
\[
R_{0,b,j}^{(j)}(y) = \frac{1}{2} e^{-\alpha^2 \Gamma_j(0)} \left( e^{\alpha^2 \Gamma_j(y)} - 1 \right) L^{-4j},
\]
Next consider \( E_j [W_{j,0}] \). As we have done in passing from (6.58) to (6.61), we write the result as a sum of terms each of which is either irrelevant or local, or constant. To do that, we need again some partial cancellations such as, for \( m, n = 0, 1 \),

\[
\sum_{y \in \mathbb{Z}^2} w_{0,b,j}(y) \left( e^{\alpha^2 \Gamma_j(y)} - 1 \right) y_m y_n = \delta_{m,n} \sum_{y \in \mathbb{Z}^2} w_{0,b,j}(y) \left( e^{\alpha^2 \Gamma_j(y)} - 1 \right) \frac{|y|^2}{2},
\]

that is a consequence of the invariance of \( w_{0,b,j}(y) \) under the interchange of \( y_0 \) and \( y_1 \): this property will be apparent in the final choice of \( w_{0,b,j}(y) \) given in (6.31).

The outcome the initial integration and subsequent re-arrangement is

\[
E_j [W_{j,0}(t_j, \varphi', B)] = s_j^2 |B| \sum_{y \in \mathbb{Z}^2} w_{0,a,j}(y) (\partial^{-\mu} \partial^{\mu} \Gamma_j)(y) + s_j^2 2 |B| \sum_{y \in \mathbb{Z}^2} w_{0,b,j}(y) \left( e^{\alpha^2 \Gamma_j(y)} - 1 \right)
\]

\[
- z_j^2 |B| \sum_{y \in \mathbb{Z}^2} w_{0,b,j}(y) \left| y \right|^2 (\partial^{-\mu} \partial^{\mu} \Gamma_j)(0)
\]

\[
- s_j^2 \sum_{y \in \mathbb{Z}^2} \left( \partial^\mu \varphi_x \right) \frac{\alpha^2}{2} \sum_{y \in \mathbb{Z}^2} w_{0,b,j}(y) \left( \partial^{-\mu} \partial^{\mu} \Gamma_j \right)(0)
\]

\[
+ z_j^2 \sum_{y \in \mathbb{Z}^2} w_{0,b,j}(y) e^{\alpha^2 \Gamma_j(y)} \sum_{x \in B} \sum_{\sigma = \pm} e^{i \sigma \alpha (\varphi_x' - \varphi_{x+y})} - 1 + \left| y \right|^2 \frac{\alpha^2}{4} \sum_{\mu \in \mathbb{C}} \left( \partial^\mu \varphi_x' \right)^2
\]

\[
- z_j^2 \sum_{y \in \mathbb{Z}^2} w_{0,b,j}(y) \left( e^{\alpha^2 \Gamma_j(y)} - 1 \right) \frac{\left| y \right|^2}{2} \sum_{x \in B} \sum_{\mu \in \mathbb{C}} (\partial^\mu \varphi_x' \right)^2
\]

\[
+ z_j^2 \sum_{y \in \mathbb{Z}^2} w_{0,c,j}(y) e^{-\alpha^2 (\Gamma_j(0) + \Gamma_j(y))} \sum_{x \in B} e^{i \sigma \alpha \varphi_x' \varphi_{x+y}'}
\]

\[
+ z_j s_j \sum_{y \in \mathbb{Z}^2} \sum_{\mu \in \mathbb{C}} \sum_{\sigma = \pm} \frac{\alpha^2}{2} \sum_{x \in B} e^{i \sigma \alpha \varphi_x' \varphi_{x+y}' - e^{i \sigma \alpha \varphi_x' \varphi_{x+y}'}}
\]

\[
- z_j s_j \sum_{y \in \mathbb{Z}^2} \sum_{\mu \in \mathbb{C}} \sum_{\sigma = \pm} \frac{\alpha^2}{2} \sum_{x \in B} e^{i \sigma \alpha \varphi_x' \varphi_{x+y}' - e^{i \sigma \alpha \varphi_x' \varphi_{x+y}'}}
\]

\[
+ z_j s_j 2 \alpha \sum_{y \in \mathbb{Z}^2} \sum_{x \in B} e^{i \sigma \alpha \varphi_x' \varphi_{x+y}' - e^{i \sigma \alpha \varphi_x' \varphi_{x+y}'}}.
\]

(6.63)

The second part of our ansatz is that the factors produced in the above integration transform \( R_{0,c,j}^{(J-1)}(y) \) into \( R_{0,c,j}^{(j)}(y) \); more precisely, plugging in (6.63) \( t_j \) instead of
Finally, it is straightforward to solve (6.64) with boundary data (6.62); the result is

$$R_{0,a,n}^{(j-1)\mu}(y) = \frac{1}{2} (\partial^{-\nu} \partial^\mu \Gamma_n)(y),$$

$$R_{0,0,n}^{(j)}(y) = \frac{1}{2} e^{-\alpha^2 \Gamma_n(0)} e^{\alpha^2 \Gamma_n(0)} \left( e^{\alpha^2 \Gamma_n(0)} - 1 \right) L^{-4n},$$

$$R_{0,0,d,n}^{(j-1)}(y) = \frac{1}{2} e^{-\alpha^2 \Gamma_n(0) + \Gamma_{j-1,n}(y)} e^{-\alpha^2 \Gamma_n(0)} \left( e^{-\alpha^2 \Gamma_n(0)} - 1 \right) L^{-4n},$$

$$R_{0,d,n}^{(j-1)\mu}(y) = \frac{1}{2} e^{-\alpha^2 \Gamma_n(0)} (\partial^\mu \Gamma_n)(y) L^{-2n},$$

$$R_{0,c,n}^{(j-1)}(y) = \frac{1}{2} e^{-\alpha^2 \Gamma_n(0)} \sum_{\mu} \left[ e^{\mu^2 \Gamma_{j-1,\mu}^2}(y) - e^{\mu^2 \Gamma_{j-1,n+1,\mu}^2}(y) \right] L^{-2n}.$$  

(6.65)

Besides, collecting the marginal and relevant terms from (6.61) and (6.63) we obtain

$$a_j := \alpha^2 \sum_{y \in \mathbb{Z}} |y|^2 \left[ \sum_{n=0}^{j} R_{0,b,n}^{(j)}(y) - \sum_{n=0}^{j-1} R_{0,b,n}^{(j-1)}(y) \right],$$

$$b_j := \sum_{\nu \in \mathbb{Z}} \left[ 2 \alpha L^2 w_{0,d,\nu}^{(j)}(y) \partial^\mu \Gamma_j(y) + \frac{\alpha^2}{2} (\partial^\mu \Gamma_j)^2(y) \right].$$  

(6.66)

This proves that (6.31) and (6.26) yield (6.57).

2. Second term. This term contains one factor of external field $J_{x,\sigma}$. We look for $w_{1,a,j}(y)$ into the form

$$w_{1,a,j}(y) = \sum_{n=0}^{j-1} R_{1,a,n}^{(j-1)}(y)$$

where $R_{1,a,n}^{(j-1)}(y)$ will be determined by means of an ansatz to obtain

$$\mathbb{E}_j^T [V_{1,j}(\bar{t}_j, \Phi, B); V_{0,j}(\bar{t}_j, \Phi, Y)] = W_{1,j+1}(\Phi', B) - \mathbb{E}_j [W_{1,j}(\bar{t}_j, \Phi, B)] + V_{1,j+1}(t_{j+1} - t'_j, \Phi', B).$$  

(6.67)

We find

$$\mathbb{E}_j^T [V_{1,j}(\Phi, B); V_{0,j}(\varphi, Y)] = Z_j z_{j} L^{-4j} \sum_{n \in \mathbb{Z}^2} e^{-(1+\eta^2)\xi^2 \Gamma_{j}(0)} \left( e^{-\alpha^2 \eta \Gamma_{j}(y)} - 1 \right) \sum_{\sigma = \pm 1} J_{x,\sigma} e^{i \sigma \pi (\varphi_{\xi} + \varphi'_{\xi+y})}.$$
\begin{align*}
+ Z_j z_j L^{-4j} \sum_{y \in \mathbb{Z}^2} e^{-(1+\eta^2)\frac{2\pi}{Z} \Gamma_j(y)} \left( e^{\alpha^2 \Gamma_j(y)} - 1 \right) \sum_{x \in B \atop \sigma \in \pm 1} J_{x,\sigma} e^{i\alpha \sigma (\varphi_x - \varphi_{x+y})} \\
+ Z_j z_j L^{-4j} \sum_{y \in \mathbb{Z}^2} e^{-(1+\eta^2)\frac{2\pi}{Z} \Gamma_j(y)} \left( e^{\alpha^2 \eta \Gamma_j(y)} - 1 \right) \sum_{x \in B \atop \sigma \in \pm 1} J_{x,\sigma} e^{i\alpha \sigma (\eta \varphi_x - \varphi_{x+y})} \\
+ Z_j z_j L^{-4j} \sum_{y \in \mathbb{Z}^2} e^{-(1+\eta^2)\frac{2\pi}{Z} \Gamma_j(y)} \left( e^{-\alpha^2 \Gamma_j(y)} - 1 \right) \sum_{x \in B \atop \sigma \in \pm 1} J_{x,\sigma} e^{i\alpha \sigma (\varphi_x + \varphi_{x+y})} \\
+ Z_j s_j \alpha \eta L^{-2j} \sum_{y \in \mathbb{Z}^2 \atop \nu \in \mathcal{G}} e^{-\eta^2 \frac{2\pi}{Z} \Gamma_j(y)} (\partial^\nu \Gamma_j)(y) \sum_{x \in B \atop \sigma \in \pm 1} J_{x,\sigma} e^{i\eta \alpha \sigma \varphi_x} (\partial^\nu \varphi_{x+y}) \\
+ Z_j s_j \alpha \eta L^{-2j} \sum_{y \in \mathbb{Z}^2 \atop \nu \in \mathcal{G}} e^{-\eta^2 \frac{2\pi}{Z} \Gamma_j(y)} (\partial^\nu \Gamma_j)(y) \sum_{x \in B \atop \sigma \in \pm 1} J_{x,\sigma} e^{i\eta \alpha \sigma \varphi_x} (\partial^\nu \varphi_{x+y}) \\
&- Z_j s_j \frac{\alpha^2 \eta^2 L^{-2j}}{2} \sum_{y \in \mathbb{Z}^2 \atop \nu \in \mathcal{G}} e^{-\eta^2 \frac{2\pi}{Z} \Gamma_j(y)} (\partial^\nu \Gamma_j)^2(y) \sum_{x \in B \atop \sigma \in \pm 1} J_{x,\sigma} e^{i\eta \alpha \sigma \varphi_x} \\
&- Z_j s_j \frac{\alpha^2 \eta^2 L^{-2j}}{2} \sum_{y \in \mathbb{Z}^2 \atop \nu \in \mathcal{G}} e^{-\eta^2 \frac{2\pi}{Z} \Gamma_j(y)} (\partial^\nu \Gamma_j)^2(y) \sum_{x \in B \atop \sigma \in \pm 1} J_{x,\sigma} e^{i\eta \alpha \sigma \varphi_x}. \tag{6.68}
\end{align*}

We want to reorganize the summation (6.68) so that every term is either irrelevant or local. The first two lines are irrelevant, because the absolute value of their total charge is $|\eta| + 1 > 1$ or $|\eta| - 1 > 1$. The third and fourth lines are relevant; to write it as sum of an irrelevant term and a local one we extract the Taylor expansion in $y^\mu$ up to the first order: for example, for the third line this means that we plug in the identity
\begin{align*}
e^{i\alpha \sigma (\eta \varphi_x - \varphi_{x+y})} &= e^{i\alpha \sigma \eta \varphi_x} \left[ e^{i\alpha \sigma (\varphi_x - \varphi_{x+y})} - 1 + i\alpha \sigma \sum_{\mu \in \mathbb{G}} y^\mu (\partial^\mu \varphi_x) \right] \\
&+ e^{i\alpha \sigma \eta \varphi_x} + i\alpha \sigma e^{i\alpha \sigma \eta \varphi_x} \sum_{\mu \in \mathbb{G}} y^\mu (\partial^\mu \varphi_x). \tag{6.69}
\end{align*}

However, since $\sum_{y \in \mathbb{Z}^2} \left( e^{\alpha^2 \eta \Gamma_j(y)} - 1 \right) y^\mu = 0$, the last term (once replaced into the third line) cancels. The fifth and sixth lines are apparently relevant; in fact, they are irrelevant as one can see by plugging in the identity
\begin{align*}(\partial^\nu \varphi_{x+y}) &= \left[ (\partial^\nu \varphi_{x+y}) - (\partial^\nu \varphi_x) \right] + (\partial^\nu \varphi_x)
\end{align*}
and observing that $\sum_{y \in \mathbb{Z}^2} (\partial^\nu \Gamma_j)(y) = 0$ so that, the last term, which is $y$-independent, give vanishing contribution. Finally, the seventh and eighth lines are relevant; however, they are already local. In conclusion, an equivalent formulation of (6.68) is
\begin{align*}
\mathbb{E}_j^T [V_{1,j}(\Phi,B); V_{0,j}(\varphi,Y)] &= Z_j z_j L^{-4j} \sum_{y \in \mathbb{Z}^2} e^{-(1+\eta^2)\frac{2\pi}{Z} \Gamma_j(y)} \left( e^{-\alpha^2 \eta \Gamma_j(y)} - 1 \right) \sum_{x \in B \atop \sigma \in \pm 1} J_{x,\sigma} e^{i\alpha \sigma (\eta \varphi_x + \varphi_{x+y})}
\end{align*}
\[ + \sum_{y \in \mathbb{Z}} z_j L^{-4j} \sum_{y \in \mathbb{Z}} e^{-(1+\tau^2)\frac{\alpha^2}{2} \Gamma_j(0)} \left( e^{\alpha^2 \eta \Gamma_j(y)} - 1 \right) \sum_{x \in B} J_{x,\sigma} e^{i\alpha \sigma (\varphi'_x - \varphi'_{x+y})} \]
\[ + Z_j z_j L^{-4j} \sum_{y \in \mathbb{Z}} e^{-(1+\eta^2)\frac{\alpha^2}{2} \Gamma_j(0)} \left( e^{\alpha^2 \eta \Gamma_j(y)} - 1 \right) \sum_{x \in B} J_{x,\sigma} e^{i\alpha \sigma \varphi'_x} \]
\[ \times \left[ e^{-i\alpha (\varphi'_x - \varphi'_{x+y})} - 1 - i\alpha \sum_{\mu} y^{\mu} \partial^{\mu} \varphi'_x \right] \]
\[ + Z_j z_j L^{-4j} \sum_{y \in \mathbb{Z}} e^{-(1+\eta^2)\frac{\alpha^2}{2} \Gamma_j(0)} \left( e^{\alpha^2 \eta \Gamma_j(y)} - 1 \right) \sum_{x \in B} J_{x,\sigma} e^{i\alpha \sigma \eta \varphi'_x} \]
\[ + \sum_{y \in \mathbb{Z}} z_j L^{-4j} \sum_{y \in \mathbb{Z}} e^{-(1+\tau^2)\frac{\alpha^2}{2} \Gamma_j(0)} \left( e^{-\alpha^2 \eta \Gamma_j(y)} - 1 \right) \sum_{x \in B} J_{x,\sigma} e^{i\alpha \sigma \eta \varphi'_x} \]
\[ \times \left[ e^{-i\alpha (\varphi'_x - \varphi'_{x+y})} - 1 + i\alpha \sum_{\mu} y^{\mu} \partial^{\mu} \varphi'_x \right] \]
\[ + Z_j z_j L^{-4j} \sum_{y \in \mathbb{Z}} e^{-(1+\tau^2)\frac{\alpha^2}{2} \Gamma_j(0)} \left( e^{-\alpha^2 \eta \Gamma_j(y)} - 1 \right) \sum_{x \in B} J_{x,\sigma} e^{i\alpha \sigma \eta \varphi'_x} \]
\[ + Z_j \eta \alpha L^{-2j} \sum_{y \in \mathbb{Z}} e^{-\eta^2 \frac{\alpha^2}{2} \Gamma_j(0)} (\partial^\nu \Gamma_j)(y) \sum_{x \in B} J_{x,\sigma} e^{i\eta \sigma \varphi'_x} \left[ (\partial^{\nu} \varphi'_{x+y}) - (\partial^{\nu} \varphi'_{x}) \right] \]
\[ + Z_j \eta \alpha L^{-2j} \sum_{y \in \mathbb{Z}} e^{-\eta^2 \frac{\alpha^2}{2} \Gamma_j(0)} (\partial^\nu \Gamma_j)(y) \sum_{x \in B} J_{x,\sigma} e^{i\eta \sigma \varphi'_x} \left[ (\partial^{\nu} \varphi'_{x+y}) - (\partial^{\nu} \varphi'_{x}) \right] \]
\[ - Z_j \frac{\alpha^2 \eta^2 L^{-2j}}{2} \sum_{y \in \mathbb{Z}} e^{-\eta^2 \frac{\alpha^2}{2} \Gamma_j(0)} (\partial^\nu \Gamma_j)^2(y) \sum_{x \in B} J_{x,\sigma} e^{i\eta \sigma \varphi'_x} \]
\[ - Z_j \frac{\alpha^2 \eta^2 L^{-2j}}{2} \sum_{y \in \mathbb{Z}} e^{-\eta^2 \frac{\alpha^2}{2} \Gamma_j(0)} (\partial^\nu \Gamma_j)^2(y) \sum_{x \in B} J_{x,\sigma} e^{i\eta \sigma \varphi'_x}. \quad (6.70) \]

Replacing \( t_j \) with \( \bar{t}_j \) and then comparing (6.70) with (3.25), we formulate the ansatz

\[ R_{1,\bar{t}_j}(y) = L^{-2j} e^{-\frac{\alpha^2}{2} \Gamma_j(0)} \left( e^{-\alpha^2 \eta \Gamma_j(y)} - 1 \right), \]
\[ \bar{R}_{1,\bar{t}_j}(y) = L^{-2j} e^{-\frac{\alpha^2}{2} \Gamma_j(0)} \left( e^{\alpha^2 \eta \Gamma_j(y)} - 1 \right), \]
\[ R_{1,c,t_j}(y) = L^{-2j} e^{-\frac{\alpha^2}{2} \Gamma_j(0)} \left( e^{\alpha^2 \eta \Gamma_j(y)} - 1 \right), \]
\[ \bar{R}_{1,c,t_j}(y) = L^{-2j} e^{-\frac{\alpha^2}{2} \Gamma_j(0)} \left( e^{-\alpha^2 \eta \Gamma_j(y)} - 1 \right), \]
\[ R_{1,d,t_j}(y) = i\alpha \eta (\partial^{\nu} \Gamma_j)(y) \]
\[ \bar{R}_{1,d,t_j}(y) = i\alpha \eta (\partial^{\nu} \Gamma_j)(y). \quad (6.71) \]

Next consider \( E_j[W_{1,j}] \): the outcome of the integration and the of the rearrangement into terms that are either irrelevant or local is

\[ E_j \left[ W_{1,j} \Phi, B \right] \]
\[
\begin{align*}
&= z_j Z_j L^{-2j} \sum_{y \in \mathbb{Z}^2} w_{1,h,j}(y) e^{-\frac{\alpha^2}{2}(1+\eta^2)\Gamma_j(y)} e^{-\alpha^2 \eta \Gamma_j(y)} \sum_{x \in B \atop \sigma = \pm} J_{x,\sigma} e^{i\alpha \sigma \eta \varphi_x + \varphi_{x+y}} \\
&+ z_j Z_j L^{-2j} \sum_{y \in \mathbb{Z}^2} w_{1,b,j}(y) e^{-\frac{\alpha^2}{2}(1+\eta^2)\Gamma_j(y)} e^{\alpha^2 \eta \Gamma_j(y)} \sum_{x \in B \atop \sigma = \pm} J_{x,\sigma} e^{i\alpha \sigma (\eta \varphi_x - \varphi_{x+y})} \\
&+ z_j Z_j L^{-2j} \sum_{y \in \mathbb{Z}^2} w_{1,c,j}(y) e^{-\frac{\alpha^2}{2}(1+\eta^2)\Gamma_j(y)} e^{\alpha^2 \eta \Gamma_j(y)[0]} \sum_{x \in B \atop \sigma = \pm} J_{x,\sigma} e^{i\alpha \sigma \eta \varphi_x} \\
&\quad \times \left[ e^{-i\alpha \sigma (\varphi_x + y - \varphi_y)} - 1 + i\alpha \sigma y^\mu \sum_{\mu \in \mathbb{Z}} (\partial^\mu \varphi_x) \right] \\
&+ z_j Z_j L^{-2j} \sum_{y \in \mathbb{Z}^2} w_{1,c,j}(y) e^{-\frac{\alpha^2}{2}(1+\eta^2)\Gamma_j(y)} e^{\alpha^2 \eta \Gamma_j(y)[0]} \sum_{x \in B \atop \sigma = \pm} J_{x,\sigma} e^{i\alpha \sigma \eta \varphi_x} \\
&\quad \times \left[ e^{i\alpha \sigma (\varphi_x + y - \varphi_y)} - 1 - i\alpha \sigma y^\mu \sum_{\mu \in \mathbb{Z}} (\partial^\mu \varphi_x) \right] \\
&+ s_j Z_j L^{-2j} \sum_{y \in \mathbb{Z}^2} w_{1,d,j}(y) e^{-\frac{\alpha^2}{2} \eta^2 \Gamma_j(y)} \sum_{x \in B \atop \sigma = \pm} J_{x,\sigma} e^{i\eta \sigma \varphi_x} \sigma \left( (\partial^\nu \varphi_x) - (\partial^\nu \varphi_x) \right) \\
&+ s_j Z_j L^{-2j} \eta \alpha \sum_{y \in \mathbb{Z}^2} w_{1,d,j}(y) \partial^\nu \Gamma_j(y) e^{-\frac{\alpha^2}{2} \eta^2 \Gamma_j(y)} \sum_{x \in B \atop \sigma = \pm} J_{x,\sigma} e^{i\eta \sigma \varphi_x} \\
&+ s_j Z_j L^{-2j} \eta \alpha \sum_{y \in \mathbb{Z}^2} w_{1,d,j}(y) \partial^\nu \Gamma_j(y) e^{-\frac{\alpha^2}{2} \eta^2 \Gamma_j(y)} \sum_{x \in B \atop \sigma = \pm} J_{x,\sigma} e^{i\eta \sigma \varphi_x} \\
&+ s_j Z_j L^{-2j} \eta \alpha \sum_{y \in \mathbb{Z}^2} w_{1,d,j}(y) \partial^\nu \Gamma_j(y) e^{-\frac{\alpha^2}{2} \eta^2 \Gamma_j(y)} \sum_{x \in B \atop \sigma = \pm} J_{x,\sigma} e^{i\eta \sigma \varphi_x} . \tag{6.72}
\end{align*}
\]

Note that, as we did to derive (6.70), we used some cancellations, which in this case are consequence of the parity of \( w_{1,c,j}(y) \) and \( \overline{w}_{1,c,j}(y) \) in \( y \) as seen from (6.32):

\[
\sum_{y \in \mathbb{Z}^2} w_{1,c,j}(y) e^{-\frac{\alpha^2}{2} \eta^2 \Gamma_j(y)[0]} e^{\alpha^2 \eta \Gamma_j(y)} y^\mu = \sum_{y \in \mathbb{Z}^2} \overline{w}_{1,c,j}(y) e^{-\frac{\alpha^2}{2} \eta^2 \Gamma_j(y)[0]} e^{\alpha^2 \eta \Gamma_j(y)} y^\mu = 0 .
\]

Therefore, replacing \( t_j \) with \( \tilde{t}_j \), we formulate the second part of the ansatz

\[
\begin{align*}
P_{1,h,j}(y) &= R_{1,h,n}(y) e^{-\frac{\alpha^2}{2} \Gamma_j(y)} e^{-\alpha^2 \eta \Gamma_j(y)}, \\
\overline{P}_{1,h,j}(y) &= \overline{R}_{1,h,n}(y) e^{-\frac{\alpha^2}{2} \Gamma_j(y)} e^{\alpha^2 \eta \Gamma_j(y)}, \\
P_{1,c,n}(y) &= R_{1,c,n}(y) e^{-\frac{\alpha^2}{2} \Gamma_j(y)} e^{\eta \alpha \Gamma_j(y)}, \\
\overline{P}_{1,c,n}(y) &= \overline{R}_{1,c,n}(y) e^{-\frac{\alpha^2}{2} \Gamma_j(y)} e^{-\eta \alpha \Gamma_j(y)} ,
\end{align*}
\]
\[ R_{1,d,n}^{(j)\nu} (y) = R_{1,d,n}^{(j-1)\nu} (y), \]
\[ \overline{R}_{1,d,n}^{(j)\nu} (y) = \overline{R}_{1,d,n}^{(j-1)\nu} (y). \] (6.73)

Finally it is easy to solve (6.73) with initial data (6.71): we obtain
\[ R_{1,b,n}^{(j-1)} (y) = L^{-2n} e^{-\alpha^2 \Gamma_j - 1,n (0)} e^{-\eta^2 \Gamma_{j-1,n+1} (y)} \left( e^{-\eta^2 \Gamma_n (y)} - 1 \right), \]
\[ \overline{R}_{1,b,n}^{(j-1)} (y) = L^{-2n} e^{-\alpha^2 \Gamma_j - 1,n (0)} e^{\eta^2 \Gamma_{j-1,n+1} (y)} \left( e^{\eta^2 \Gamma_n (y)} - 1 \right), \]
\[ R_{1,c,n}^{(j-1)} (y) = L^{-2n} e^{-\alpha^2 \Gamma_j - 1,n (0)} e^{\eta^2 \Gamma_{j-1,n+1} (y)} \left( e^{\eta^2 \Gamma_n (y)} - 1 \right), \]
\[ \overline{R}_{1,c,n}^{(j-1)} (y) = L^{-2n} e^{-\alpha^2 \Gamma_j - 1,n (0)} e^{-\eta^2 \Gamma_{j-1,n+1} (y)} \left( e^{-\eta^2 \Gamma_n (y)} - 1 \right), \]
\[ R_{1,d,n}^{(j-1)\nu} (y) = i\alpha \eta (\partial^\nu \Gamma_n) (y), \]
\[ \overline{R}_{1,d,n}^{(j-1)\nu} (y) = i\alpha \eta (\partial^\nu \Gamma_n) (y). \] (6.74)

Besides, comparing with (6.29), the marginal and relevant terms of (6.70) and (6.72) give
\[ m_{2,1,j} = \sum_{y \in \mathbb{Z}^2} \left[ \sum_{n=0}^{j} R_{1,c,n}^{(j)} (y) e^{-\alpha^2 (2\eta - 1) \Gamma_n (0)} - \sum_{n=0}^{j-1} R_{1,c,n}^{(j-1)} (y) \right], \]
\[ m_{1,2,j} = \sum_{y \in \mathbb{Z}^2} \left[ \sum_{n=0}^{j} \overline{R}_{1,c,n}^{(j)} (y) e^{2 \eta^2 (2\eta + 1) \Gamma_n (0)} - \sum_{n=0}^{j-1} \overline{R}_{1,c,n}^{(j-1)} (y) \right], \]
\[ m_{1,1,j} = \frac{\alpha^2 \eta^2}{2} \sum_{y \in \mathbb{Z}^2} \sum_{\nu \in \mathbb{Z}} \left[ (\partial^\nu \Gamma_j)^2 (y) + 2 (\partial^\nu \Gamma_{j-1,0}) (y) (\partial^\nu \Gamma_j) (y) \right], \]
\[ m_{2,2,j} = \frac{\alpha^2 \eta^2}{2} \sum_{y \in \mathbb{Z}^2} \sum_{\nu \in \mathbb{Z}} \sum_{\nu' \in \mathbb{Z}} \left[ (\partial^\nu \Gamma_j)^2 (y) + 2 (\partial^\nu \Gamma_{j-1,0}) (y) (\partial^\nu \Gamma_j) (y) \right]. \] (6.75)

This proves that (6.32) and (6.30) yield (6.67).

3. Third term. This term is quadratic in \( J \). We look for \( w_{2,\alpha,j} (y) \) (where again \( \alpha \) is the collections of various labels, compare with (3.26)) into the form
\[ w_{2,\alpha,j} (y) = \sum_{n=1}^{j-1} R_{2,\alpha,n}^{(j-1)} (y); \]
then \( R_{2,\alpha,n}^{(j-1)} (y) \) will be determined by an ansatz to obtain
\[ \frac{1}{2} \mathcal{E}_j^T \left[ V_{1,j} (\overline{I}_j, \Phi, B); V_{1,j} (\overline{I}_j, \Phi, Y) \right] + \sum_{X \supset B} \sum_{X \in S_j} Q_{2,j} (\Phi', B, X) \]
\[ = W_{2,j+1} (\Phi', B) - E_j \left[ W_{2,j} (\overline{I}_j \Phi, B) \right]. \] (6.76)

The first term in (6.76) is
\[ \frac{1}{2} \mathcal{E}_j^T \left[ V_{1,j} (\Phi, B); V_{1,j} (\Phi, Y) \right] \]
where the parameters \( t_j \) have to be replaced with \( \tilde{t}_j \); taking into account also the second term in (6.76), we set

\[
\begin{align*}
R_{2,a,j}^{(j)}(y) & := \frac{1}{2} Z_j^2 L^{-4j} e^{-\frac{\eta^2 a^2}{2} \Gamma_j(0)} \left( e^{-\frac{\eta^2 a^2}{2} \Gamma_j(y)} - 1 \right), \\
\overline{R}_{2,a,j}^{(j)}(y) & := \frac{1}{2} Z_j^2 L^{-4j} e^{-\frac{\eta^2 a^2}{2} \Gamma_j(0)} \left( e^{-\frac{\eta^2 a^2}{2} \Gamma_j(y)} - 1 \right), \\
R_{2,b,j}^{(j)}(y) & := \frac{1}{2} Z_j^2 L^{-4j} e^{-\frac{(\eta^2 + \eta^2)}{2} \Gamma_j(0)} \left( e^{-\frac{\eta^2 a^2}{2} \Gamma_j(y)} - 1 \right), \\
R_{2,c,j}^{(j)}(y) & := \sum_{k=0}^{j} 2^{-j-k} \frac{1}{2} L^{-4k} e^{-L^{-4} |y|} \\
& \quad \times \left\{ Z_k^2 \frac{1}{2} \sum_{\sigma = \pm 1}^{X \geq 0} \sum_{X \in S_j} E_j \left[ \hat{K}_{2,j}^{(s,k)} \left( -\frac{\sigma + \varepsilon}{2}, \xi, X, 0, \sigma, y, \sigma \varepsilon \right) \right] \\
& \quad + \overline{Z}_k^2 \frac{1}{2} \sum_{\sigma = \pm 1}^{X \geq 0} \sum_{X \in S_j} E_j \left[ \hat{K}_{2,j}^{(s,k)} \left( \frac{\sigma + \varepsilon}{2}, \xi, X, 0, \sigma, y, \sigma \varepsilon \right) \right] \\
& \quad + Z_k \overline{Z}_k \frac{1}{2} \sum_{\sigma = \pm 1}^{X \geq 0} \sum_{X \in S_j} E_j \left[ \hat{K}_{2,j}^{(b,k)} \left( -\frac{\sigma - \varepsilon}{2}, \xi, X, 0, \sigma, y, \sigma \varepsilon \right) \right] \right\}. \quad (6.78)
\end{align*}
\]

Next, we find

\[
E_j [W_{2,j}(\Phi, B)] = \sum_{y \in S_j} u_{2,a,j}(y) e^{-\eta^2 a^2 (1+\varepsilon) \Gamma_j(0)} e^{-\eta^2 a^2 \varepsilon \Gamma_j(y) |y|} \sum_{\sigma \in B} J_{x,\sigma} J_{\sigma \varepsilon, \xi, x+y} e^{i \eta a \sigma (\varepsilon \gamma' \pm \varepsilon \gamma' \pm \gamma' \pm \gamma') + i \eta \sigma (\varepsilon \gamma' \pm \varepsilon \gamma' \pm \gamma' \pm \gamma')}
\]

\[
+ \sum_{y \in S_j} u_{2,a,j}(y) e^{-\eta^2 a^2 (1+\varepsilon) \Gamma_j(0)} e^{-\eta^2 a^2 \varepsilon \Gamma_j(y) |y|} \sum_{\sigma \in B} J_{x,\sigma} J_{\sigma \varepsilon, \xi, x+y} e^{i \eta a \sigma (\varepsilon \gamma' \pm \varepsilon \gamma' \pm \gamma' \pm \gamma') + i \eta \sigma (\varepsilon \gamma' \pm \varepsilon \gamma' \pm \gamma' \pm \gamma')}
\]

\[
+ \sum_{y \in S_j} u_{2,b,j}(y) e^{-\eta^2 a^2 \varepsilon \Gamma_j(0)} e^{-\eta^2 a^2 \varepsilon \Gamma_j(y) |y|} \sum_{\sigma \in B} J_{x,\sigma} J_{\sigma \varepsilon, \xi, x+y} \frac{e}{i \alpha \sigma (\varepsilon \gamma' \pm \varepsilon \gamma' \pm \gamma' \pm \gamma') + e^{i \eta \sigma (\varepsilon \gamma' \pm \varepsilon \gamma' \pm \gamma' \pm \gamma')}}
\]

\[
\times [e^{i \alpha \sigma (\varepsilon \gamma' \pm \varepsilon \gamma' \pm \gamma' \pm \gamma') + e^{i \eta \sigma (\varepsilon \gamma' \pm \varepsilon \gamma' \pm \gamma' \pm \gamma')}].
\]
Hence the second part of the ansatz is
\[ R_2(y) = R_2(y) e^{-\eta^2 \alpha^2 (1+\epsilon) \Gamma_j(0)} e^{-\eta^2 \alpha^2 \Gamma_j(0)}, \]
\[ R_2(y) = R_2(y) e^{-\eta^2 \alpha^2 (1+\epsilon) \Gamma_j(0)} e^{-\eta^2 \alpha^2 \Gamma_j(0)}, \]
\[ R_2(y) = R_2(y) e^{-\eta^2 \alpha^2 (1+\epsilon) \Gamma_j(0)} e^{-\eta^2 \alpha^2 \Gamma_j(0)}, \]
\[ R_2(y) = R_2(y) e^{-\eta^2 \alpha^2 (1+\epsilon) \Gamma_j(0)} e^{-\eta^2 \alpha^2 \Gamma_j(0)}, \]
\[ (6.79) \]

Solving (6.80) with initial data (6.78) we obtain
\[ R_2^{(j-1)\epsilon}(y) = \frac{1}{2} Z_n^2 L^{-4n} e^{-\eta^2 (1+\epsilon) \Gamma_j(0)} \]
\[ \times e^{-\eta^2 \alpha^2 \Gamma_j(0)} (e^{-\eta^2 \alpha^2 \Gamma_n(0)} - 1) \]
\[ R_2^{(j-1)\epsilon}(y) = \frac{1}{2} Z_n^2 L^{-4n} e^{-\eta^2 (1+\epsilon) \Gamma_j(0)} \]
\[ \times e^{-\eta^2 \alpha^2 \Gamma_j(0)} (e^{-\eta^2 \alpha^2 \Gamma_n(0)} - 1) \]
\[ R_2^{(j-1)\epsilon}(y) = \frac{1}{2} Z_n^2 L^{-4n} e^{-\eta^2 (1+\epsilon) \Gamma_j(0)} \]
\[ \times e^{-\eta^2 \alpha^2 \Gamma_j(0)} (e^{-\eta^2 \alpha^2 \Gamma_n(0)} - 1) \]
\[ R_2^{(j-1)\epsilon}(y) = \frac{1}{2} Z_n^2 L^{-4n} e^{-\eta^2 (1+\epsilon) \Gamma_j(0)} \]
\[ \times e^{-\eta^2 \alpha^2 \Gamma_j(0)} (e^{-\eta^2 \alpha^2 \Gamma_n(0)} - 1) \]
\[ (6.80) \]

In conclusion, (6.33) yield (6.76). This concludes the proof of Lemma 6.6.

7. REMAINDER PART OF THE RG MAP

Lemma 7.1. If \( z > 0 \) is small enough and \( |s_j|, |z_j| \leq c_0|q_j|, \| K_{0,j}\|_{1,h,T_j} \leq c_0|q_j|^2 \),

there exists \( C = C(A, L, \alpha) \) such that,

\[ \| R_1 \|_{1,h,T_{j+1}} \leq C \left[ |q_j|^2 + |q_j| \| K_{1,j}\|_{1,h,T_j} + |q_j| \| K_{1,j}\|_{1,h,T_j} \right]; \]

(7.1)

besides the same bound is valid for \( \| R_1 \|_{1,h,T_{j+1}} \).

Proof. We begin with an exact formula for \( R_1 \). From (5.13) we have

\[ P_j(D) = P_{0,j}(D) + P_{1,j}(D) + P_{2,j}(D) + P_{3,j}(D) \]
where, if \( \tilde{V}_{0,j}(D) := V_{0,j}(D) - E_j[V_{0,j}(D)] \) and \( \tilde{V}_{1,j}(D) := V_{1,j}(D) - E_j[V_{1,j}(D)] \),

\[
P_{0,j}(D) = \tilde{V}_{0,j}(D) - (V_{0,j+1} + \delta E_j|D| - E_j[V_{0,j}]) + (e^{U_{0,j}(D)} - 1 - V_{0,j}(D))
- \left( e^{U_{0,j+1}(D) + \delta E_j|D|} - 1 - (V_{0,j+1}(D) + \delta E_j|D|) \right),
\]

\[
P_{1,j}(D) = \tilde{V}_{1,j}(D) + \left( e^{U_{0,j}(D)} - 1 \right) \tilde{V}_{1,j}(D)
+ \left( e^{U_{0,j}(D)} - e^{U_{0,j+1}(D) + \delta E_j|D|} \right) E_j[V_{1,j}(D)]
- e^{U_{0,j}(D)} \left( V_{1,j+1}(D) - E_j[V_{1,j}(D)] \right)
+ e^{U_{0,j}(D)} W_{1,j}(D) - e^{U_{0,j+1}(D) + \delta E_j|D|} W_{1,j+1}(D),
\]

\[
P_{2,j}(D) = \frac{1}{2} e^{U_{0,j}(D)} (V_{1,j}(D)^2 + W_{1,j}(D)^2 + 2W_{2,j}(D))
- \frac{1}{2} \left( e^{U_{0,j+1}(D) + \delta E_j|D|} (V_{1,j+1}(D)^2 + W_{1,j+1}(D)^2 + 2W_{2,j+1}(D)) \right); \quad (7.2)
\]

while \( P_{\geq 3,j}(D) \) contains the rest of \( P_j(D) \). Therefore we find

\[
\mathcal{R}_{1,j}(Y') = \sum_{D \in B_{j+1}}^D \sum_{D' \in B_{j+1}}^{D'} E_j \left[ \left( e^{U_{0,j}(D)} - 1 - V_{0,j}(D) \right) V_{1,j}(D) \right]
- \sum_{D \in B_{j+1}}^D \sum_{D' \in B_{j+1}}^{D'} \left( e^{U_{0,j+1}(D) + \delta E_j|D|} - 1 - V_{0,j+1}(D) - \delta E_j|D| \right) E_j[V_{1,j}(D)]
- \sum_{D \in B_{j+1}}^D \sum_{D' \in B_{j+1}}^{D'} \left( e^{U_{0,j}(D) + \delta E_j|D|} - 1 \right) \left( V_{1,j+1}(D) - E_j[V_{1,j}(D)] \right)
- \sum_{D \in B_{j+1}}^D \sum_{D' \in B_{j+1}}^{D'} \left( V_{0,j+1}(D) + \delta E_j|D| - E_j[V_{0,j}(D)] \right) V_{1,j+1}(D)
- \sum_{D \in B_{j+1}}^D \sum_{D' \in B_{j+1}}^{D'} \left( e^{U_{0,j+1}(D) + \delta E_j|D|} - 1 \right) W_{1,j+1}(D)
+ \sum_{D \in B_{j+1}}^D \sum_{D' \in B_{j+1}}^{D'} E_j \left[ \left( e^{U_{0,j}(D)} - 1 \right) W_{1,j}(D) \right],
\]

\[
\mathcal{R}_{1,j}(Y') = \sum_{D_1, D_2 \in B_{j+1}}^{D_1 \cup D_2 = Y'} E_j \left[ \left( P_{0,j}(D_1) - \tilde{V}_{0,j}(D_1) \right) \tilde{V}_{1,j}(D_2) \right]
+ \sum_{D_1, D_2 \in B_{j+1}}^{D_1 \cup D_2 = Y'} E_j \left[ P_{0,j}(D_1) \left( P_{1,j}(D_2) - \tilde{V}_{1,j}(D_2) \right) \right],
\]

\[
\mathcal{R}_{1,j}(Y') = \sum_{D \in B_{j+1}}^D \sum_{D' \in B_{j+1}}^{D'} \left[ E_j[W_{1,j}(D)] - E_j[W_{1,j}(\tilde{I}_j, D)] \right],
\]
\[ R_{1,j}^{(4)}(Y') = \sum_{D_1, D_2 \in B_{j+1}} \left[ E_j^T [V_{0,j}(D_1); V_{1,j}(D_2)] - E_j^T [V_{0,j}(\tilde{t}_j, D_1); V_{1,j}(\tilde{t}_j, D_2)] \right], \]

\[ R_{1,j}^{(5)}(Y') = \sum_{Y_0 \in C_{j+1}(X_0) \geq 1} E_j \left[ P_{0,j}^Z R_{0,j}^{X_0} \right] J_{0,j}^{X_0 \setminus Y_0}(D) J_{1,j}(D_{Y_0}, Y_0) \]

\[ + \sum_{Y_1 \in C_{j+1}(X_1) \geq 1} E_j \left[ P_{1,j}^Z R_{1,j}^{Y_1} \right] J_{1,j}^{Y_1}(D) \]

\[ R_{1,j}^{(6)}(Y') = \sum_{c_{j+1}(X_0 \cup X_1) \geq 1} \sum_{B \in B_{j+1}(Z)} e^{-\delta E_j |Y'| + U_{0,j+1} + U_{1,j+1}(Y' \setminus W)} \left( V_{1,j+1}(B) + W_{1,j+1}(B) \right) E_j \left[ P_{0,j}^Z R_{0,j}^{X_0} \right] J_{0,j}^{X_0}(D) \]

\[ R_{1,j}^{(7)}(Y') = \sum_{Z \ni Y'} \sum_{Z \in \mathcal{P}_{j+1}^{B_{j+1}(Z)}} E_j \left[ P_{0,j}^Z \right] \sum_{B \in B_{j+1}(Z)} e^{-\delta E_j |Y'| - 1} \sum_{Z \ni Y'} \sum_{B \in B_{j+1}(Z)} E_j \left[ P_{j}^{Y'} \right] \]

\[ R_{1,j}^{(8)}(Y') = \sum_{X \in \mathcal{P}_j \cap B_{j+1}(Z)} \sum_{Y_0 \in C_j(X)} E_j \left[ K_{1,j}(Y_0) \prod_{Y \in C_j(X \setminus Y_0)} K_{0,j}(Y) \right], \]

\[ R_{1,j}^{(9)}(Y') = \sum_{X \in \mathcal{P}_j \cap B_{j+1}(Z \setminus X)} E_j \left[ V_{1,j}(B) + W_{1,j}(B) \right] e^{U_{0,j}(Y' \setminus X)} \prod_{Y \in C_j(X)} K_{0,j}(Y) \]
exists a $C$ (Lemma 7.3 and a similar one proportional to $K$ factor of either $s$ least two factors of $V$ the bounds in Lemma 14 of [Brydges, 2009] and a simpler version of the bounds in Lemma 14 of [Falco, 2012].

Lemma 7.2 ([Brydges, 2009]). There exists a $\vartheta > 0$ such that, for any $X \in \mathcal{P}_j$,

$$\| A \|_{h,T_j(\varphi,D)} - 1 \|_{h,T_j(\varphi,D)} \leq C|q_j|G_{j+1}^{str}(\varphi,D),$$  

(7.5)

The reason of (7.1) is that each of the above terms falls into one of two classes: a) those terms which, besides containing a factor of $V_{1,j}$ or $W_{1,j}$, also contain at least two factors of $s_j, z_j$, or one factor of $K_{0,j}$; b) those terms which contain one factor of either $K_{1,j}$ or $K_{1,j}^{\dagger}$ and at least one factor of $s_j, z_j$ or $K_{0,j}$.

Lemma 7.3 ([Falco, 2012]). Under the hypothesis of Lemma 7.1, for a $\vartheta > 0$, there exists a $C \equiv C(A, L, \alpha)$ such that

$$\| e^{U_{0,j}(\varphi,D)} - 1 \|_{h,T_j(\varphi,D)} \leq C|q_j|G_{j}^{str}(\varphi,D),$$

(7.6)

The second line we also used that in the sum in the first line there are no more than $|\mathcal{Y}|$ terms. Let us consider, for example, $\mathcal{R}^{(8)}_j(Y)$. Extracting the dependence in $J$ be obtain, as usual, two terms:

$$\mathcal{R}^{(8)}_j(\varphi', Y, x, \sigma) = \sum_{\varphi \in \mathcal{P}_j} \sum_{|C_{j}(X)| \geq 2} \sum_{Y \in \mathcal{C}_j(X) \setminus \mathcal{Y}_0} \mathcal{E}_j \left[ K_{1,j}(\varphi, Y_0, x, \sigma) \prod_{Y \in \mathcal{C}_j(X) \setminus \mathcal{Y}_0} K_{0,j}(Y) \right],$$

(7.10)

and a similar one proportional to $K_{1,j}^{\dagger}(\varphi, Y_0, x, \sigma)$. Using (4.13), (4.16) and the inequality $A^{-|X|} \leq A^{-(1+2\vartheta)|Y_{j+1}+A^{8(1+2\vartheta)|C_{j}(X)|}}$ which is a consequence of (7.4), a bound for $\| \mathcal{R}^{(8)}_j(\varphi', Y, x, \sigma) \|_{h,T_j(\varphi', Y)}$ is, for a $C \equiv C(A, L, \alpha)$,

$$G_{j+1}(\varphi', Y)\| K_{1,j} \|_{1,h,T_j} \sum_{X,|C_{j}(X)| \geq 2} A^{-|X|} \|_{1,h,T_j} A^{2|X|} (C|q_j|)^{C_j(X)} |C_{j}(X)|^{-1}$$

$$\leq G_{j+1}(\varphi', Y)\| K_{1,j} \|_{1,h,T_j} A^{2|Y_{j+1}|} A^{-(1+2\vartheta)|Y_{j+1}|+1} \sum_{p \geq 2} A^{8(1+2\vartheta)p(C|q_j|)}$$

$$\leq G_{j+1}(\varphi', Y)\| K_{1,j} \|_{1,h,T_j} A^{2|Y_{j+1}|} C_1|q_j|,$$

(7.11)

where the last inequality holds if one first chooses $A$ large enough so that $4L^2 A^{-2\vartheta} < 1$, and then chooses $|q_1|$ small enough so that the series in $p$ is convergent. To obtain the second line we also used that in the sum in the first line there are no more than $2|Y| < 2L^2|Y_{j+1}|$ terms.
As a second sample case, consider one of the terms in \( R_{1,j}^{(6)}(Y) \), which, after the extraction of \( Z_j L^{2j} J_{x,\sigma} \), is

\[
\frac{Z_{j+1}}{Z_j} L^{-2} e^{i\alpha p \varphi'} \sum_{c_{j+1}(X_{0} + X_{1}) \geq 1} \sum_{b \in B_{j+1}(Y \setminus W), B \geq x} e^{-\delta E_j |Y| + U_{0,j+1}(Y \setminus W)} E_j \left[ p_{0,j}^{\mathbb{Z}} R_{0,j}^{X_i} \right] J_{0,j}(D). \tag{7.12}
\]

A bound for the norm \(| \cdot |_{h,T}(\varphi', Y)\) of this term is made of three kinds of factors: a product of field regulators, a product of factors of \( A^{-1} \), and a product of factors of \(|q_j|\). Collecting all the factors of field regulators we obtain

\[
G_{j+1}^{\text{str}}(\varphi', Y \setminus W) \prod_{D \in B_{j+1}(Z)} \left[ G_j^{\text{str}}(\varphi, D) + G_{j+1}^{\text{str}}(\varphi', D) \right] \times \prod_{Y \in C_{j+1}(X_1)} G_j(\varphi, Y) + G_{j+1}^{\text{str}}(\varphi', Y) \right] G_{j+1}^{\text{str}}(\varphi', X_0) \\
\leq \sum_{W_1 \in P_{j+1}(Z)} G_j(\varphi, W_1 \cup W_2) G_{j+1}^{\text{str}}(\varphi', Y \setminus (W_1 \cup W_2)), \tag{7.13}
\]

where \((C(X_1))_{j+1}\) is the collection of all the possible unions of connected parts of \(X_1\). Since the number of terms in the sum over \(W_1\) and \(W_2\) is not larger than \(2^{2|Y_{j+1} + C(X_1)|}\), by (4.16) the expectation of such factors is bounded by

\[
2^{2|Y_{j+1} + C(X_1)|} 2L^2 |Y_{j+1} + L^2 |X_{j+1}| G_{j+1}^{\text{str}}(\varphi', Y).
\]

Next, collecting the \(A^{-1}\) factors coming from (7.7), (7.8), (7.9), we obtain a factor not larger than \(A^{-(1+\theta)|Y_{j+1}|}\). In conclusion, a bound for (7.12) is, for a \(C \equiv (A, L, \alpha),\)

\[
G_{j+1}^{\text{str}}(\varphi', Y) 2^{1+L^2} |Y_{j+1}| A^{-(1+\theta)|Y_{j+1}|} \\
\times \sum_{c_{j+1}(X_0 \cup X_1) \geq 1} \left( C|q_j| \right)^{|Y_{j+1}|+2 |c_{j+1}(X_0 \cup X_1)|} \\
\leq G_{j+1}^{\text{str}}(\varphi', Y) 2^{1+L^2} |Y_{j+1}| A^{-(1+\theta)|Y_{j+1}|} \sum_{p \geq 1} (C|q_j|)^{2p} \\
\leq C_1 G_{j+1}^{\text{str}}(\varphi', Y) A^{-|Y_{j+1}|} |q_j|^2, \tag{7.14}
\]

where we used that the sum in the second line has no more than \(4^{|Y_{j+1}| + |c_{j+1}(X_0)|}\) (indeed each connected component of \(X_0\) has to be a small polymer); besides we assumed \(A\) large enough so that \(2^{1+L^2} A^{-\theta} \leq 1\), as well as \(|q_j|\) small enough so that the series in \(p\) is convergent. The other terms of (7.3) can be studied in a similar manner.

**Lemma 7.4.** If \(z > 0\) is small enough and \(|s_j|, |z_j| \leq c_0 |q_j|, \|K_{0,j}\|_{h,T_j} \leq c_0 |q_j|^2\) and \(\|K_{1,j}\|_{1,h,T_j} \leq c_0 |q_j|^2\), there exists \(C \equiv C(A, L, \alpha)\) such that,

\[
\|R_{2,j}^{(6)}\|_{2,h,T_{j+1}} \leq \begin{cases} C|q_j| & \text{for } k = j \\ C|q_j| \|K_{2,j}^{(6)}\|_{2,h,T_j} & \text{for } 0 \leq k \leq j - 1 \end{cases} \tag{7.15}
\]

Proof. When \( k = j \) the term \( R_{2,j}^{k,j,k} \) is generated by a term in \( K_{2,j+1} \) which contain at least two factors of \( V_{1,j} \), \( W_{1,j+1} \) or \( \Phi_{1,j} \) and at least one factor of \( s_j \), \( z_j \) or \( K_{0,j} \). When \( k \leq j + 1 \), term \( R_{2,j}^{k,j,k} \) is generated by terms in \( K_{2,j+1} \) which contain at least one factor of \( K_{2,j}^{k,j} \) and one factor of \( s_j \), \( z_j \) or \( K_{0,j} \). Note that in the case \( k = j \) the norms on the right hand side are \( \| \cdot \|_{1,h,T_j} \) or \( \| \cdot \|_{1,b,T_j} \), in which the size of the sets are weighed with a factor \( A \), whereas on the norm on the left hand side is \( \| \cdot \|_{2,b,T_j} \) in which the size of the sets are weighed with a factor \( \sqrt{A} \): this provides the factor \( e^{-L^{-k}|x_1-x_2|} \) in (3.48).

7.1. **Proof of Theorem 3.4.** Let us consider the first of (3.40). We have

\[
K_{1,j+1}(\Phi', Y, x, \sigma) = L_{1,j}(\Phi', Y, x, \sigma) + R_{1,j}(\Phi', Y, x, \sigma).
\]  

(7.16)

From (3.38) we see that the assumption of Lemma 7.1 is satisfied; therefore, using also (5.27), we obtain

\[
\|K_{1,j+1}\|_{1,h,T_{j+1}} \leq \rho(L, A, \eta)\|K_{1,j}\|_{1,h,T_j} + C\left(|q_j|^2 + |q_j|\|K_{1,j}\|_{1,h,T_j} + |q_j|\|K_{1,j}^1\|_{1,h,T_j}\right)
\]  

(7.17)

Assuming by induction that \( \|K_{1,j}\|_{1,h,T_j} \leq 2C|q_j|^2 \) and that \( \|K_{1,j}^1\|_{1,h,T_j} \leq 2C|q_j|^2 \), we obtain that

\[\|K_{1,j+1}\|_{1,h,T_{j+1}} \leq 2C|q_{j+1}|^2\]  

(7.18)

(We also used that \( \rho(L, A, \eta) \leq 1/4 \) for \( L \) and \( A \) large enough; that \( |q_j|/|q_{j+1}| \leq 1 + \sqrt{ab}|z| \leq 2 \) for \( |z| \) small enough; and that \( |q_j| \leq c_0|z| \leq \frac{1}{2} \) for \( |z| \) small enough.) This proves the first of (3.40). The second can be obtained in a similar way.

7.2. **Proof of Theorem 3.8.** Let us consider the bound for \( K_{2,j+1}^{(a,k)} \). From the previous definitions, suppressing the dependence in \( (\Phi', Y, x_1, x_2, \sigma_2) \), we have

\[
K_{2,j+1}^{(a,k)} = 2L_{2,j}^{(a,k)} + 2R_{2,j}^{(a,k)}
\]  

(7.19)

where the factors 2 stem from the prefactor \( 2^{-(j-k)} \) in (3.48). Because of (3.38) the assumptions in Lemma 7.4 are satisfied; therefore, with the aid of (5.28), we have

\[
\|K_{2,j+1}^{(a,k)}\|_{2,h,T_{j+1}} \leq 2\rho(L, A, \eta)\|K_{2,j}^{(a,k)}\|_{2,h,T_j} + \begin{cases} 2C|q_k| & \text{for } k = j \\ 2C|q_j|\|K_{2,j}^{(a,k)}\|_{2,h,T_j} & \text{for } 0 \leq k \leq j - 1 \end{cases}
\]  

(7.19)

Therefore, for \( L \) and \( A \) large enough and \( |z| \) small enough it is easy to show inductively a bound such as \( \|K_{2,j}^{(a,k)}\|_{2,h,T_j} \leq 4C|q_k| \). \( K_{2,j+1}^{(a,k)} \) and \( K_{2,j+1}^{(b,k)} \) can be dealt with in a similar way.

8. **Flow of the fractional charge renormalization**

Merging (6.29) and (6.15) we obtain (3.32), namely the equation that describe the flow of the renormalization parameters \( Z_j \) and \( Z_j \). To study such flows we need an explicit computation of some of the coefficients. In this section we set \( \alpha^2 = 8\pi \).

The calculation of (3.36), was already done in [Falco, 2012]. Note that (3.44) is only valid for \( \eta = \frac{1}{2} \); for other values of \( \eta \) in \( (0, 1) \) we just need that \( |m_{2,1,j}|, |m_{1,2,j}| \) are
bounded, see below. Using (3.38) and (3.36), (3.43), the equation for the fractional charge renormalization constants (3.32) becomes

\[
\begin{pmatrix}
Z_{j+1} \\
\tilde{Z}_{j+1}
\end{pmatrix} = \begin{pmatrix}
L^2 e^{-4\pi^2 \gamma_j^{(0)}} & 0 \\
0 & L^2 e^{-4\pi^2 \gamma_j^{(0)}}
\end{pmatrix}
\begin{pmatrix}
1 - \eta^2 |q_j| + \tilde{\mathcal{M}}_{1,j} m_{1,2,j} z_j + \mathcal{M}_{1,2,j} \\
m_{2,1,j} z_j + \mathcal{M}_{2,1,j} 1 - \eta^2 |q_j| + \tilde{\mathcal{M}}_{2,2,j}
\end{pmatrix}
\begin{pmatrix}
Z_j \\
\tilde{Z}_j
\end{pmatrix}
\]

(8.1)

where

\[
\tilde{\mathcal{M}}_{1,j} = -(m_{1,1,j} s_j - \eta^2 |q_j| + \mathcal{M}_{1,1,j},
\]

\[
\tilde{\mathcal{M}}_{2,2,j} = -(m_{2,2,j} s_j - \eta^2 |q_j| + \mathcal{M}_{2,2,j}.
\]

Because of (3.38), (3.40) and (3.43), for a \( C \equiv C(L) \) and \( m = 1, 2, \)

\[|\tilde{\mathcal{M}}_{m,m,j}| \leq C \left[ |q_j| L^{-\frac{4}{2}} + \frac{\tau |q_1|}{|1 + |q_1|(j-1)|^{\frac{3}{2}}}.\right].\]

(8.3)

Let us consider two different cases, \( |\eta| = |\tilde{\eta}|, \) or \( |\eta| < |\tilde{\eta}|; \) the case \( |\tilde{\eta}| < |\eta| \) gives the same formulas after interchanging \( Z_j \) and \( \eta \) with \( \tilde{Z}_j \) and \( -\tilde{\eta}. \)

8.1. Case \( |\eta| = |\tilde{\eta}|. \) In this case \( \eta = -\tilde{\eta} = \frac{1}{2} \) and (3.44) holds. Therefore, if we introduce \( Z_j^+ := Z_j + Z_j \) and \( Z_j^- := Z_j - Z_j; \) then

\[
Z_{j+1}^+ = L^2 e^{-\pi \gamma_j^{(0)}} \left( 1 - \frac{1}{4} |q_j| + \frac{1}{2} q_j + \mathcal{M}_{+,j} \right) Z_j^+,
\]

\[
Z_{j+1}^- = L^2 e^{-\pi \gamma_j^{(0)}} \left( 1 - \frac{1}{4} |q_j| - \frac{1}{2} q_j + \mathcal{M}_{-,j} \right) Z_j^-,
\]

(8.4)

where

\[
\mathcal{M}_{+,j} := \left( m_{1,2,j} z_j - \frac{1}{2} q_j \right) + \tilde{\mathcal{M}}_{1,1,j} + \mathcal{M}_{2,1,j},
\]

\[
\mathcal{M}_{-,j} := - \left( m_{1,2,j} z_j - \frac{1}{2} q_j \right) + \tilde{\mathcal{M}}_{1,1,j} - \mathcal{M}_{2,1,j}.
\]

(8.5)

It is easy to see that also \( \mathcal{M}_{+,j} \) and \( \mathcal{M}_{-,j} \) satisfy the bound (8.3). In the physical case \( z > 0, \) one has \( |q_j| = q_j \) and then

\[
Z_{j+1}^+ = Z_1^+ e^{2\sqrt{\ln L - \pi \gamma_j^{(0)}} + \frac{1}{4} \sum_{k=1}^{j} q_k + \sum_{k=1}^{j} m_{+,k},
\]

\[
Z_{j+1}^- = Z_1^+ e^{2\sqrt{\ln L - \pi \gamma_j^{(0)}} - \frac{3}{4} \sum_{k=1}^{j} q_k + \sum_{k=1}^{j} m_{-,k},
\]

(8.6)

where

\[
m_{+,j} := \log \left( 1 + \frac{1}{4} q_j + \mathcal{M}_{+,j} \right) - \frac{1}{4} q_j,
\]

\[
m_{-,j} := \log \left( 1 - \frac{3}{4} q_j + \mathcal{M}_{-,j} \right) + \frac{3}{4} q_j.
\]

(8.7)

Hence \( |m_{+,j}| \) and \( |m_{-,j}| \) satisfy a bound like (8.3). Therefore, for a \( C \equiv C(L) \) and for three constants \( \{C_\sigma : \sigma = 0, \pm \} \) that are vanishing for \( \tau, |q_1| \to 0, \) one has: for
\[\sigma = \pm, \ m_{\sigma,k} \text{ is summable and} \]
\[
\left| \sum_{k=1}^{j} m_{\sigma,k} - \bar{c}_\sigma \right| \leq C \frac{|q_1| + \tau}{\sqrt{1 + |q_1|(j - 1)}};
\]
while \(q_k\) is not summable but
\[
\left| \sum_{k=1}^{j} q_k - \ln(1 + |q_1|j) - \bar{c}_0 \right| \leq C|q_j|.
\]

Setting \(c_+ := \bar{c}_+ + \bar{c}_0\) and \(c_- := \bar{c}_- + \bar{c}_0\), from (8.6) one finds the explicit formula for \(Z_j^+\) and \(Z_j^-\) in (3.45).

8.2. **Case** \(|\eta| < \bar{\eta}\). When \(0 < \eta < \frac{1}{2}\), we expect that the sequence \((Z_j)\) dominates \((\bar{Z}_j)\); therefore we recast (8.1) as
\[
\left( \begin{array}{c} Z_{j+1} \\ \bar{Z}_{j+1} \end{array} \right) = L^2 e^{-4\pi i \eta^2 \Gamma_j(0) - \eta^2 |q_j| + m_j} \left[ \begin{array}{cc} 1 & 0 \\ 0 & \ell_j \end{array} \right] \left( \begin{array}{c} Z_j \\ \bar{Z}_j \end{array} \right)
\] (8.8)
where
\[
m_j := \ln \left( 1 - \eta^2 |q_j| + \bar{M}_{1.1,j} \right) + \eta^2 |q_j|,
\]
\[
\ell_j := e^{-4\pi i (\bar{\eta}^2 - \eta^2) \Gamma_j(0) + \eta^2 |q_j| - m_j} \left( 1 - \eta^2 |q_j| + \bar{M}_{2.2,j} \right),
\]
\[
m_{-j} := e^{\eta^2 |q_j| - m_j} (m_{1.2,j} \bar{z}_j + M_{1.2,j}),
\]
\[
m_{+j} := e^{-4\pi i (\bar{\eta}^2 - \eta^2) \Gamma_j(0) + \eta^2 |q_j| - m_j} (m_{2.1,j} \bar{z}_j + M_{2.1,j}).
\] (8.9)

For a \(C \equiv C(L)\) and \(\sigma = \pm\) one has
\[
|m_j| \leq C \left[ |q_1| L^{-\frac{j}{2}} + \tau \frac{|q_1|}{\sqrt{1 + |q_1|(j - 1)}} \right],
\]
\[
|\ell_j| \leq L^{-2(\bar{\eta}^2 - \eta^2)} \left[ 1 + C|q_j| + CL^{-\frac{j}{2}} \right],
\]
\[
|m_{\sigma,j}| \leq C|q_j|.
\] (8.10)
The difference with the case \(\eta = -\bar{\eta}\) is that in (8.10) the coefficient \(m_j\) is absolutely summable in \(j\), while \(m_{+j}\) and \(m_{-j}\) are not. This will be compensated by the presence of several factors of \(\ell_j < 1\). For \(z > 0\), the solution of (8.8) is
\[
\left( \begin{array}{c} Z_{j+1} \\ \bar{Z}_{j+1} \end{array} \right) = L^{2j} e^{-4\pi i \eta^2 \Gamma_j(0) - \eta^2 \sum_{k=1}^{j} q_k + \sum_{k=1}^{j} m_k Q(j, 1)} \left( \begin{array}{c} Z_1 \\ \bar{Z}_1 \end{array} \right)
\] (8.11)
where \(Q(f, i)\) is a two-by-two matrix parametrized by two integers \(f \geq i\):
\[
Q(f, i) = \prod_{n=i}^{f} \left[ \begin{array}{cc} 1 & 0 \\ 0 & \ell_n \end{array} \right] + \left( \begin{array}{cc} 0 & m_{-n} \\ m_{+n} & 0 \end{array} \right).
\] (8.12)

From the definition (8.12), for any \(1 \leq j_0 \leq j\), we have the factorization \(Q(j, 1) = Q(j, j_0) Q(j_0 - 1, 1)\). We will take advantage of it by choosing a \(j_0\) that is large when the difference \(\eta^2 - \bar{\eta}^2\) is small; and estimating \(Q(j, j_0)\) and \(Q(j_0 - 1, 1)\) in different ways. This will avoid that \(L^{-1}\) (and hence \(z\)) be vanishing in the limit \(\eta \to \frac{1}{2}\).
Lemma 8.1. If \( 0 \leq z \leq \frac{1}{4} \), for every \( 0 \leq \eta < \frac{1}{2} \) there exist a scale \( j_0 \equiv j_0(\eta) \) and a constant \( C_0 \equiv C_0(\eta) \) such that:

1. Estimates for the entries of \( Q(j_0 - 1, 1) \) are

\[
|Q(j_0 - 1, 1)_{1,1} - 1| \leq C_0|q_1|^2,
\]

\[
|Q(j_0 - 1, 1)_{1,2} - \sum_{d=1}^{j_0-1} \ell_1 \cdots \ell_{d-1}m_{-d}| \leq C_0|q_1|^3,
\]

\[
|Q(j_0 - 1, 1)_{2,1}| \leq C_0|q_1|,
\]

\[
|Q(j_0 - 1, 1)_{2,2} - \ell_1 \cdots \ell_{j_0-1}| \leq C_0|q_1|^2.
\]

(8.13)

2. For \( m = 1, 2 \) the limits \( \overline{Q}_{1,m}(j_0) := \lim_{j \to \infty} Q(j, j_0)_{1,m} \) exist and

\[
|\overline{Q}_{1,1}(j_0) - 1| \leq C_0\sqrt{|q_1|},
\]

\[
|\overline{Q}_{1,1}(j_0)_{1,2} - \sum_{d \geq j_0} \ell_{j_0} \cdots \ell_{d-1}m_{-d}| \leq C_0|q_1|^2;
\]

(8.14)

Besides, estimates for the speed of convergence of the above limits are

\[
|Q(j, j_0)_{1,1} - \overline{Q}(j_0)_{1,1}| \leq C_0|q_j|,
\]

\[
|Q(j, j_0)_{1,2} - \overline{Q}(j_0)_{1,2}| \leq C_0|q_j|,
\]

\[
|Q(j, j_0)_{2,1}| \leq C_0|q_j|,
\]

\[
|Q(j, j_0)_{2,2} - \ell_{j_0} \cdots \ell_j| \leq C_0|q_j||q_1|.
\]

(8.15)

In the limit \( \eta \to \frac{1}{2} \) the constant \( C_0(\eta) \) is divergent.

Proof. Consider (8.12) and expand the product of the sum. The interpretation of the result can be given in terms of the process of two "states", \( A_1 \) and \( A_2 \), and one "particle": at each time \( n = i, i + 1, i + 2, \ldots, f \) the particle can either hold in one of the two states or jump to the other. The "cost" of staying in state \( A_1 \) or \( A_2 \) at time \( n \) is 1 and \( \ell_n \), respectively. The cost of jumping form \( A_1 \) to \( A_2 \) or from \( A_2 \) to \( A_1 \) at time \( n \) is \( m_{+n} \) and \( m_{-n} \), respectively. Let us denote

\[
\sum_{u_1, d_1, \ldots, u_n, d_n}^{[i,f]} \sum_{u_1, d_1, \ldots, u_n, d_n}^{[i,f]} \prod_{s=1}^{n} m_{+u_s} \ell_{u_s+1} \cdots \ell_{d_{s-1}m_{-d_s}},
\]

the sum with constraint \( i \leq u_1 < d_1 < \cdots < u_n < d_n \leq f \).

1. The entry \( Q(f, i)_{1,1} \) is the sum of the cost of all the patterns that start and end at \( A_1 \),

\[
Q(f, i)_{1,1} = 1 + \sum_{n \geq 1} \sum_{u_1, d_1, \ldots, u_n, d_n}^{[i,f]} \prod_{s=1}^{n} m_{+u_s} \ell_{u_s+1} \cdots \ell_{d_{s-1}m_{-d_s}},
\]

where: \( n \) is the number of intervals of time that the particle has spent in \( A_2 \); \( u_s \)'s are the times in which the particle jumps from \( A_1 \) to \( A_2 \), and \( d_s \)'s are the times in which the particle jumps from \( A_2 \) to \( A_1 \). As it is easy to see from (8.10), there exists a constant \( C \) such that, if \( 0 \leq z \leq \frac{1}{4} \), then \( \ell_n \leq C \) and \( |m_{+u_s}m_{-d_s}| \leq C|q_1|^2 \). Hence, for a \( C_0 \equiv C_0(j_0) \) (and divergent in the limit \( j_0 \to \infty \))

\[
|Q(j_0 - 1, 1)_{1,1} - 1| \leq C_0|q_1|^2.
\]

(8.16)

However, if \( j_0 \) is larger that a \( j_0' \equiv j_0'(\eta) \), then one has the better bound \( \ell_n \leq L^{-\eta^2 - \rho} \) for every \( n \geq j_0 \). Therefore, if \( d_s - 1 \geq u_s \geq j_0 \), then \( |m_{+u_s}m_{-d_s}| \leq \eta^2 - \rho \).
2. The entry $Q(f,i)_{1,2}$ is the sum of the cost of all the patterns that start at $A_2$ and end at $A_1$.

\[
Q(f,i)_{1,2} = \sum_{d=1}^{f} \ell_i \cdots \ell_{d-1}m_{d-1} = 1 + \sum_{n \geq 1} s_{n+1} \sum_{n \geq 1} s_{n+1} \prod_{s=1}^{n-1} \ell_{m,n} \ell_{m,n+1} \cdots \ell_{m,d}m_{d}.
\]

Therefore, for constants $C$ and $C_0 \equiv C_0(j_0)$, we have

\[
\left| Q(j_0 - 1, 1)_{1,2} - \sum_{d=1}^{j_0-1} \ell_1 \cdots \ell_{d-1}m_{d-1} \right| \\
\leq j_0 C_0 |q_1| \sum_{d \leq j_0-1} |Q(j_0 - 1, d)_{1,1} - 1| \leq C_0 |q_1|^3.
\]

Besides, $\lim_{j \to \infty} Q(j, j_0)_{1,1}$ exists and for a constant $C_1 \equiv C_1(q_1, j_0)$

\[
\left| Q(j, j_0)_{1,2} - \sum_{d=j_0}^{j} \ell_{j_0} \cdots \ell_{d-1}m_{d-1} \right| \\
\leq \frac{|q_1|}{1 - L^{-2}} \sum_{d \geq j_0} |Q(j, d)_{1,1} - 1| \leq C_1 |q_1|^2.
\]
To study the speed of convergence of the limit consider an \( f \geq j + 1 \) and the difference
\[
Q(f, j_0)_{1,2} - Q(j, j_0)_{1,2} = \sum_{d=j_0}^j \ell_i \cdots \ell_{d-1} m_{-d} \left[ Q(f, d + 1)_{1,1} - Q(j, d + 1)_{1,1} \right]
+ \sum_{d=j+1}^f \ell_i \cdots \ell_{d-1} m_{-d} Q(f, d + 1)_{1,1}.
\]

(8.23)

Then
\[
|Q(f, j_0)_{1,2} - Q(j, j_0)_{1,2}| \leq \frac{C|q_1|}{1 - L^{-|\eta^2 - \eta^2|}} \sup_{j_0 \leq d \leq j} |Q(f, d + 1)_{1,1} - Q(j, d + 1)_{1,1}|
+ \frac{C|q_j|}{1 - L^{-|\eta^2 - \eta^2|}} \sup_{j+1 \leq d \leq f} |Q(f, d + 1)_{1,1}| \leq C_0|q_j|.
\]

(8.24)

3. The entry \( Q(f, i)_{2,1} \) is the sum of the cost of all the patterns that start at \( A_1 \) and end at \( A_2 \),
\[
Q(f, i)_{2,1} = \sum_{u=i}^f \left[ 1 + \sum_{n \geq u+1} \sum_{d_1, \ldots, d_n} \prod_{s=1}^{n-1} m_{+} u \ell_{u+1} \cdots \ell_{d_s} m_{-d_s} \right] m_{+} u \ell_{u+1} \cdots \ell_f.
\]

(8.25)

For constants \( C \) and \( C_0 \equiv C_0(j_0) \), we have
\[
|Q(j_0 - 1, i)_{2,1}| \leq j_0 C |q_1| \sup_{u \leq j_0 - 1} |Q(u, i)_{1,1}| \leq C_0 |q_1|.
\]

(8.26)

If \( j - 1 \geq u \geq j_0 \), for a constant \( C_1 \equiv C_1(\eta) \) we have \( |m_{+} u \ell_{u+1} \cdots \ell_j| \leq C_1 |q_j| L^{\frac{1}{2}(|\eta^2 - \eta^2|)(j - u)} \); hence
\[
|Q(j, j_0)_{2,1}| \leq 2C_1 |q_j|.
\]

(8.27)

4. The entry \( Q(f, i)_{2,2} \) is the total cost of all the patterns that start and end at \( A_2 \):
\[
Q(f, i)_{2,2} = \ell_i \cdots \ell_f
+ \sum_{d < u = i}^f \ell_i \cdots \ell_{d-1} m_{-d} Q(u - 1, d + 1)_{1,1} m_{+} u \ell_{u+1} \cdots \ell_f.
\]

(8.28)

For \( C_0 \equiv C_0(j_0) \),
\[
|Q(j_0 - 1, i)_{2,2} - \ell_1 \cdots \ell_{j_0-1}| \leq C_0 |q_1|^2 ;
\]

(8.29)

besides
\[
|Q(j, j_0)_{2,2} - \ell_{j_0} \cdots \ell_j| \leq C_0 |q_1| |q_j|.
\]

(8.30)

From these formulas, one obtains (8.13), (8.14) and (8.15).

Combining the bounds of this Lemma, we obtain:
\[
Q(j, 1)_{1,1} = e^5 + \tilde{r}_{i,j},
\]
9. Exact asymptotic formulas

9.1. Proof of Lemma 3.1 and Lemma 3.6. A key result is the following Lemma, in which we introduce a continuous approximation of the covariances \( \Gamma_j \) which has a simpler scaling transformation.

**Lemma 9.1.** Consider the set of “continuous covariances” \( \tilde{\Gamma}_j(x) \), \( j = 0, 1, \ldots R - 1 \), defined for \( x \in \mathbb{R}^2 \) as

\[
\tilde{\Gamma}_j(x) := \int \frac{d^2p}{(2\pi)^2} e^{ipx} \frac{u(L^j p) - u(L^{j+1} p)}{p^2}
\]  

(9.1)

where \( u(p) \) is a differentiable even function such that \( u(L^j p) - u(L^{j+1} p) \geq 0 \) for every \( j \) and

\[
u(0) = 1, \quad |u(p)| \leq \frac{C}{1 + |p|^4}.
\]  

(9.2)

There exists a special choice of \( u \) and a constant \( C > 0 \) such that, for every \( x \in \mathbb{Z}^2 \),

\[
|\Gamma_j(x) - \tilde{\Gamma}_j(x)| \leq C L^{-\frac{4}{3} j}, \quad |\partial^\mu \Gamma_j(x) - \tilde{\Gamma}_j^\mu(x)| \leq C L^{-\frac{4}{3} j},
\]  

(9.3)

where the upper label \( ^\mu \) indicates the continuous derivative (as opposed to \( \partial^\mu \) that is the lattice one).
The proof is in Appendix A.3. of [Falco, 2012]. (9.1) and (9.1) have important consequences: first, for every $x \in \mathbb{R}^2$
\[
\tilde{\Gamma}_j(x) = \tilde{\Gamma}_0(L^{-j}x);
\] (9.4)
second,
\[
\tilde{\Gamma}_j(0) = \int \frac{d^2p}{(2\pi)^2} \frac{u(L^j p) - u(L^{j+1} p)}{p^2} = \frac{1}{2\pi} \ln L;
\] (9.5)
finally, $\tilde{\Gamma}_{\infty,0}(x|0)$ is a differentiable function and, asymptotically for large $|x|$
\[
\tilde{\Gamma}_{\infty,0}(x|0) := \sum_{j=0}^{\infty} \left[ \tilde{\Gamma}_j(x) - \tilde{\Gamma}_j(0) \right] = -\frac{1}{2\pi} \ln |x| + \tilde{c}_E + o(1)
\] (9.6)
where $o(1)$ is a vanishing term in the limit $|x| \to \infty$ and $\tilde{c}_E$ is a constant.

Consider the coefficient $a_j$ in (6.66). Let $\tilde{R}_{0,b,n}^{(j)}(y)$ and $\tilde{\Gamma}_j(0|y)$ be the same function as $R_{0,b,n}^{(j-1)}$ and $\Gamma_j(0|y)$, respectively, but with $\tilde{\Gamma}_j(x)$ in place of $\Gamma_j(x)$ for any $j$. Using (9.3) we have
\[
\sum_{y \in \mathbb{Z}^2} |y|^2 \left| R_{0,b,n}^{(j)}(y) - R_{0,b,n}^{(j-1)}(y) - \tilde{R}_{0,b,n}^{(j)}(y) + \tilde{\Gamma}_0^{(j-1)}(y) \right| \leq C(L)L^{-\frac{2}{3}}
\] (9.7)
therefore in the definition of $a_j$ we can replace $R_{0,b,n}^{(j-1)}$ with $\tilde{R}_{0,b,n}^{(j-1)}$ up to an error $CL^{-\frac{2}{3}}$. Besides,
\[
\left| \sum_{y \in \mathbb{Z}^2} |y|^2 \tilde{\Gamma}_{0,b,n}^{(j)}(y) - \int d^2 y \left| y \right|^2 \tilde{R}_{0,b,n}^{(j)}(y) \right| \leq C(L)L^{-\frac{2}{3}}
\]
therefore in the formula for $a_j$ replacing the sum with an integral generates and error not larger than $CL^{-\vartheta j}$ for a $\vartheta < 1$. In conclusion, an equivalent formula for $a_j$ is, up to an $O(L^{-\frac{2}{3}})$ error,
\[
\frac{\alpha^2}{2} \int d^2 y \ y^2 \left[ \sum_{n=0}^{j} \tilde{R}_{0,b,n}^{(j)}(y) - \sum_{n=0}^{j-1} \tilde{R}_{0,b,n}^{(j-1)}(y) \right] \leq C(L)L^{-\frac{2}{3}}
\] (9.8)
We now take advantage of the exact scale transformation (9.4). We have $\tilde{R}_n^{(j-1)}(y) = L^4 \tilde{R}_{n+1}^{(j)}(yL)$; hence the two sums in (9.8) cancel each others almost completely, and (9.8) becomes
\[
\frac{\alpha^2}{2} \int d^2 y \ y^2 \tilde{R}_{0,b,0}^{(j)}(y) = \frac{\alpha^2}{2} \int d^2 y \ y^2 e^{-\alpha^2 \tilde{\Gamma}_{\infty,0}^{(j)}(y)} e^{-\alpha^2 \tilde{\Gamma}_{0}^{(j)}(y)} \left( e^{\alpha^2 \tilde{\Gamma}_{0}^{(j)}(y)} - 1 \right) + O(L^{-j})
\]
\[
= \frac{\alpha^2}{2} \int d^2 y \ y^2 \left[ w(y) - w(yL^{-1})L^4 \frac{4\alpha^2}{\alpha^2 + \alpha^2} + O(L^{-j}) \right]
\] (9.9)
for $w(y) = y^4 e^{-\alpha^2 \tilde{\Gamma}_{\infty,0}^{(j)}(y)}$; a new $O(L^{-j})$ error in the first line is due to the replacement of $e^{-\alpha^2 \tilde{\Gamma}_{\infty,1}^{(j)}(0|y)}$ with $e^{-\alpha^2 \tilde{\Gamma}_{\infty,1}^{(j)}(0|y)}$. At $\alpha^2 = 8\pi$, the last integral can be exactly computed only using the differentiability of $w(y)$ and the boundary values $w(0) = 0$ and $\lim_{y \to \infty} w(y) = e^{-8\pi\tilde{c}_E}$, see (9.6). This proves the first of (3.36).
Now consider the coefficient \(m_{2,1,j}\) in (6.75). Arguing as done for \(a_j\), up to \(O(L^{-\delta j})\) corrections, it is given by

\[
\int d^2y \left[ \sum_{n=0}^{j} \bar{R}_{1,c,n}^{(j)}(y) e^{-\frac{a^2}{2}(2\eta^{-1})\bar{\Gamma}_j(0)} - \sum_{n=0}^{j-1} \bar{R}_{1,c,n}^{(j-1)}(y) \right] \quad (9.10)
\]

where the formula for \(\bar{R}_{1,c,n}^{(j-1)}(y)\) is obtained from the formula for \(R_{j,c,n}^{(j-1)}(y)\) by replacing \(\Gamma_j(x)\) with \(\bar{\Gamma}_j(x)\). In the case \(\eta = -\overline{\eta} = \frac{1}{2}\), by means of the exact scaling \(\bar{R}_{1,c,n}^{(j-1)}(y) = L^2 \bar{R}_{1,c,n+1}^{(j)}(yL)\), (9.10) becomes

\[
\int d^2y \bar{R}_{1,c,0}^{(j)}(y) = \int d^2y e^{-\frac{a^2}{2}\overline{\Gamma}_{\infty,1}(0) e^{-\frac{a^2}{2}\overline{\Gamma}_0(0)} (e^{-\frac{a^2}{2}\overline{\Gamma}_0(y)} - 1) + O(L^{-j})
\]

\[
= \frac{d^2y}{y^2} \left[ \sqrt{w(y)} - \sqrt{w(yL^{-1})}L^{2-\frac{a^2}{2}} \right] + O(L^{-j}) \quad (9.11)
\]

where \(w(y)\) is the same function introduced for \(a_j\). At \(\alpha^2 = 8\pi\) the last integral can be exactly computed only using the differentiability of \(\sqrt{w(y)}\) for \(y \neq 0\) and the boundary values \(\sqrt{w(0)} = 0\) and \(\lim_{y \to \infty} \sqrt{w(y)} = e^{-4\eta^2}\). This proves the first of (3.44); the second is also proven because at \(\eta = -\overline{\eta} = \frac{1}{2}\) one has \(m_{1,2,j} = m_{2,1,j}\).

Finally, consider the coefficient \(b_j\) in (6.66) and \(m_{1,1,j}, m_{2,2,j}\) in (6.75). With the same argument used for \(a_j\), an equivalent formula the last two coefficients, up to an \(\eta^2\) or \(\overline{\eta}^2\) prefactor, is

\[
\frac{\alpha^2}{2} \sum_{\mu = \epsilon} \int d^2y \left[ (\bar{\Gamma}_j^\mu(y))^2 + 2\bar{\Gamma}_j^\mu(y)\bar{\Gamma}_j^\mu(y) \right] + O(L^{-\frac{1}{2}})
\]

\[
= \frac{\alpha^2}{2} \sum_{\mu = \epsilon} \int d^2y \left[ (\bar{\Gamma}_j^\mu(y))^2 - (\bar{\Gamma}_j^\mu(y))^2 \right] + O(L^{-\frac{1}{2}}) \quad (9.12)
\]

At \(\alpha^2 = 8\pi\), (9.12) is also an equivalent formula for \(b_j\). As \(\bar{\Gamma}_j^\mu_{j-1,0}(y) = L\bar{\Gamma}_j^\mu_{j,1}(yL)\), the last integral in (9.12) becomes

\[
\frac{\alpha^2}{2} \sum_{\mu = \epsilon} \int d^2y \left[ (\bar{\Gamma}_j^\mu_{j,0}(y))^2 - (\bar{\Gamma}_j^\mu_{j,1}(y))^2 \right]
\]

\[
= \frac{\alpha^2}{2} \int \frac{d^2p}{(2\pi)^2} \left[ |u(p)|^2 - |u(Lp)|^2 \right] + \alpha^2 \int \frac{d^2p}{(2\pi)^2} \left[ |u(p)| - |u(Lp)| \right] u(L^{j+1}p) \quad (9.13)
\]

In the last line, the former integral can be exactly computed while the latter, using the boundedness of the derivatives of \(u\) is \(O(L^{-j})\). This proves (3.43) and the second of (3.36).

9.2. Proof of Theorem 3.9. From (6.33) we have

\[
w_{2,0,R}(y) = \frac{1}{2} \sum_{n=0}^{R-1} Z_n^2 L^{-4n} e^{\eta^2\alpha^2 \Gamma_{R,n+1}(y0)} e^{-\eta^2\alpha^2 \Gamma_n(y0)} (e^{\eta^2\alpha^2 \Gamma_n(y0)} - 1) \quad (9.14)
\]

Use the inequality \(Z_n^2 L^{-4n} \leq C_0 L^{-\delta n}\) for a \(\delta < \min\{4\eta^2, 4\overline{\eta}^2, 1\}\), to replace the function \(\Gamma_{R,n+1}(y0)\) with \(\Gamma_{\infty,n+1}(y0)\) up to an \(O(L^{-R\delta})\) error term. Note that each \(y\) can be uniquely written as \(y = L^n \tau\) for \(|\tau| \in [1, L]\) and an integer \(n_0\); then
\[ e^{q^2 \alpha^2 \Gamma_n(y) - 1} = 0 \text{ every time } n \leq n_0 - 1 \text{ so that, in (9.14), one can actually start} \]

the sum from \( n = n_0 \). Accordingly, a formula for \( \lim_{R \to \infty} w_{2,a,R}(y) \) is

\[
w_{2,a}(y) = \frac{1}{2} \sum_{n \geq n_0} Z_n^2 L^{-4n} e^{-q^2 \alpha^2 \Gamma_{\infty,n+1}(y(0))} e^{-q^2 \alpha^2 \Gamma_n(y(0))} \left( e^{q^2 \alpha^2 \Gamma_n(y) - 1} \right). \tag{9.15}
\]

Using the same argument, a formula for \( w_{2,a}(y) = \lim_{R \to \infty} w_{2,a,R}(y) \) is given by

(9.15) after replacing \( Z_j \) and \( \eta \) with \( \overline{Z}_j \) and \(-\eta\). Let us consider three different cases.

### 9.2.1. Case \( 0 < \eta < \frac{1}{2} \)

It is convenient to write (9.15) as the difference of two convergent series

\[
\frac{1}{2} \sum_{n \geq n_0} Z_n^2 L^{-4n} e^{q^2 \alpha^2 \Gamma_{\infty,n}(y(0))} - \frac{1}{2} \sum_{n \geq n_0} Z_n^2 L^{-4n} e^{-q^2 \alpha^2 \Gamma_n(y(0))} e^{q^2 \alpha^2 \Gamma_{\infty,n+1}(y(0))}. \tag{9.16}
\]

By replacing in the second series the factor \( Z_n^2 L^{-4n} e^{-q^2 \alpha^2 \Gamma_n(y(0))} \) with the almost identical factor \( Z_{n+1}^2 L^{-4(n+1)} \), each term of the latter series cancels a term in the former, so that only the term for \( n = n_0 \) in the first series survives; besides, by definition of \( n_0 \) we have \( \Gamma_{\infty,n_0}(y(0)) = \Gamma_{\infty,0}(y(0)) + \Gamma_{n_0-1,0}(y(0)) \). Hence

\[
w_{2,a}(y) = \frac{1}{2} Z_{n_0}^2 L^{-4n_0} e^{q^2 \alpha^2 \Gamma_{n_0-1,0}(y(0))} e^{-q^2 \alpha^2 \Gamma_{\infty,0}(y(0))}
- \frac{1}{2} \sum_{n = n_0}^{\infty} \left( Z_n^2 L^{-4n} e^{-q^2 \alpha^2 \Gamma_n(y(0))} - Z_{n+1}^2 L^{-4(n+1)} \right) e^{q^2 \alpha^2 \Gamma_{\infty,n+1}(y(0))}. \tag{9.17}
\]

The first term in (9.17) is the leading one. Indeed, from (3.46) we have

\[
Z_n^2 L^{-4n} e^{-q^2 \alpha^2 \Gamma_n(y(0))} [1 + |q_1| (j - 1)]^{-2q^2} e^{\bar{c}_1 + \bar{r}_1,1},
\]

\[
Z_n^2 L^{-4n} e^{-q^2 \alpha^2 \Gamma_n(y(0))} [1 + |q_1| (j - 1)]^{-2q^2} e^{\bar{r}_2,2}. \tag{9.18}
\]

where \( |r_{m,j}| \leq C(1 + |q_1| j)^{-\frac{1}{2}} \) for any \( m = 1, 2 \) and \( \bar{c}_1, \bar{c}_2 \) are vanishing in the limit \( z \to 0 \). Therefore the first term of (9.17) is

\[
e^{\frac{\delta \pi q^2 \alpha_E}{2 |y|^{4q^2}} (1 + |q_1| \log |y|)^{-2q^2} e^z (1 + o(1)), \tag{9.19}
\]

for \( o(1) \) a term bounded by \( C(\log |y|)^{-\frac{3}{2}} \). The second term in (9.17) is subleading by a factor \( (\log |y|)^{-\frac{3}{2}} \) at least. Indeed from (9.27) we also find

\[
|Z_n^2 L^{-4n} e^{-q^2 \alpha^2 \Gamma_n(y(0))} - Z_{n+1}^2 L^{-4(n+1)}|
\leq Z_{n+1}^2 L^{-4(n+1)} \left[ \left( 1 + |q_1| \frac{n}{1 + |q_1| (n - 1)} \right)^{2q^2} \left( 1 + \bar{s}_{1,n} \frac{1}{1 + \bar{s}_{1,n+1}} \right) - 1 \right]
\leq C \frac{L^{-4q^2 n_0}}{(1 + |q_1| n_0)^{2q^2 + \frac{1}{2}}} L^{-4q^2 (n - n_0)}. \tag{9.20}
\]

Summing over \( n \geq n_0 \), one obtains the bound \( C|y|^{-4q^2} (1 + |q_1| n_0)^{-2q^2 - \frac{1}{2}} \), which is subleading with respect to (9.19). Instead, for \( w_{2,a}(y) \) the method used above
does not work; however, we can provide an upper bound that shows that \( w_{2, \pi}(y) \) is subleading with respect to \( w_{2, a}(y) \): as

\[
e^{T^{2} \alpha^{2} \Gamma_{w, n+1}(y(0))} e^{-2 \pi^{2} \alpha^{2} \Gamma_{n}(0)} - 1 \leq C_{\pi}^{2} \alpha^{2} |\Gamma_{n}(y)|,
\]

\( w_{2, a}(y) \) is bounded by

\[
C_{\pi}^{2} \alpha^{2} \sum_{n \geq 1} \Gamma_{n}(y) \leq C \frac{L^{-4n} \eta}{(1 + |q_1| \eta_0)^{2n^{2} + 2}}.
\]

Next consider \( w_{2, b, R}^{-} \) in (6.33). For any \( 0 \leq \vartheta < 1 \) and a corresponding constant \( C_{\vartheta} \equiv C_{\vartheta}(\eta) \), we have the bound

\[
|w_{2, b, R}^{-}(y)| \leq \eta C_{\vartheta} \alpha^{2} \sum_{n \geq 1} Z_{n} L^{-2n} \sum_{\sigma = \pm 1} \sum_{X S_{n}} \left[ \tilde{K}_{2, n}^{(a, k)}(0, \zeta, X, 0, \sigma, y, -\sigma) \right]
\]

\[
+ Z_{k}^{1/2} \sum_{\sigma, X \in S_{n}} \sum_{n \geq k} \left[ \tilde{K}_{2, n}^{(b, k)}(\sigma, \zeta, X, 0, \sigma, y, -\sigma) \right]
\]

\[
+ Z_{k}^{1/2} \sum_{\sigma, X \in S_{n}} \sum_{n \geq k} \left[ \tilde{K}_{2, n}^{(\pi, k)}(0, \zeta, X, 0, \sigma, y, -\sigma) \right]
\]

\[
\leq S A^{-\frac{1}{2}} k_{y}^{2} (1/2) \sum_{k \geq 0} L^{-4k} e^{-L^{-k}} \sum_{n \geq k} 2^{-n- (n-k)}
\]

\[
\times \left\{ Z_{k}^{2} \| K_{2, n}^{(a, k)} \|_{2, h, T_j} + Z_{k}^{2} \| K_{2, n}^{(\pi, k)} \|_{2, h, T_j} + Z_{k} \| K_{2, n}^{(b, k)} \|_{2, h, T_j} \right\}.
\]

Hence, from (3.49), we obtain

\[
|w_{2, c}^{-}(y)| \leq \frac{|q_1| C}{|y|^{4n^{2}}(1 + |q_1| \log_{L} |y|)^{2n^{2} + 1}},
\]

(9.23)

9.2.2. Case \( \frac{1}{3} < \eta < 1 \). The fundamental difference with the previous case is in the formula for the renormalization constants. From (3.46) we have

\[
Z_{j}^{2} L^{-4j} e^{- \pi^{2} \alpha^{2} \Gamma_{j-1, 0}(0)} = [1 + |q_1| (j - 1)]^{-\frac{2 \vartheta^{2}}{4 \vartheta} c(\eta)^2 z^{2} e^{\tilde{\varphi}} e^{\tilde{\varphi}_1},
\]

\[
Z_{j}^{2} L^{-4j} e^{- \pi^{2} \alpha^{2} \Gamma_{j-1, 0}(0)} = [1 + |q_1| (j - 1)]^{-\frac{2 \vartheta^{2}}{4 \vartheta} c(\eta)^2 z^{2} e^{\tilde{\varphi}} e^{\tilde{\varphi}_2},
\]

(9.24)

where \( c(\eta) \) is the positive constant in Theorem 3.7. Now, proceeding with \( w_{2, \pi}(y) \) with the same method that in the previous section was used for \( w_{2, a}(y) \) we obtain the formula

\[
z^{2} c(\eta)^2 \frac{e^{-4 \eta^{2} \pi^{2} e^{c}}}{2|y|^{2n^{2} + 1}} (1 + |q_1| \log_{L} |y|)^{-2n^{2}} e^{\tilde{\varphi}} e^{\tilde{\varphi} + o(1)},
\]

(9.25)
for $o(1)$ a term bounded by $C(\log |y|)^{-\frac{3}{2}}$. Conversely, proceeding with $w_{2,a}(y)$ with the same method that in the previous section was used for $w_{2,a}(y)$ we obtain the bound

$$C\eta^2 \alpha^2 \sum_{n \geq n_0} Z^2_n L^{-4n} |\Gamma_n(y)| \leq C \frac{L^{-4n} |\eta|}{(1 + |q_1| |n_0|)^{3\eta^2 + \frac{1}{2}}}$$

which is subleading with respect to (9.29). Finally, with the same arguments of the previous section, $w_{2,a}(y) = 0$ and

$$|w_{2,c}(y)| \leq \frac{|q_1| C}{|y|^{4\eta^2} (1 + |q_1| \log L |y|)^{3\eta^2 + 1}}.$$  (9.26)

9.2.3. Case $\eta = \frac{1}{2}$. From (3.45), we have

$$Z_j = \frac{Z_j^+ + Z_j^-}{2} = \frac{1}{2} L^{2j} e^{-\pi \Gamma_j - (0)(1 + |q_1|(j - 1))} e^{\bar{c} + \bar{s}_j}$$   \hspace{1cm} (9.27)

where $\bar{c}$ vanishes for $z \to 0$ and $|\bar{s}_j| \leq \frac{C}{\sqrt{1 + |q_1| j}}$. Hence

$$Z^2_{n_0} L^{-4n_0} e^{q\bar{\alpha} + \Gamma_{n_0} - (0)} = \frac{1}{4} (1 + |q_1|(n_0 - 1)) \frac{1}{2} e^{2\bar{c} + 2\bar{s}_n_0}$$  \hspace{1cm} (9.28)

and the formula for $w_{2,a}(y)$ is

$$\frac{e^{2\pi cE}}{8|y|} (1 + |q_1| \log L |y|) e^{\bar{c}} (1 + o(1)).$$  \hspace{1cm} (9.29)

Now consider $w_{2,\pi}(y)$. Since a formula for $\overline{Z}_j = (Z_j^+ - Z_j^-)/2$ is again given by (9.27), but for numerically different $\bar{c}$ and $\bar{s}_j$—also the formula for $w_{2,\pi}(y)$ is (9.29). Finally, consider $w_{2,c}(y)$.

$$|w_{2,c}(y)| \leq \frac{|q_1| C}{|y| (1 + |q_1| \log L |y|)^{\frac{3}{2}}}.$$  \hspace{1cm} (9.30)

This completes the proof of point 1 of Theorem 3.10.

9.3. Proof of Theorem 3.9. For the sake of brevity in this section we denote $\overline{E}_R$ the limiting expectation $\lim_{m \to 0} \overline{E}_R$. Let us consider the last term of (3.35)

$$e^{-\delta E_{\lambda}|N|} \sum_{J=0}^{\delta K_2 R(\Phi)} \frac{\partial^2 K_2 R(\Phi)}{\partial J_{+x} \partial J_{0,-}} = e^{-\delta E_{\lambda}|N|} \sum_{k=0}^{R} 2^{-(R-k)} e^{-L^{-k}|x|} L^{-4k}$$

\times \left\{ Z^2_{k \overline{E}_R} [K_{2,R}^{(a,k)}] + 2 Z^2_{k \overline{E}_R} [K_{2,R}^{(b,k)}] + Z_k \overline{\zeta}_k \overline{E}_R [K_{2,R}^{(b,k)}] \right\}$$  \hspace{1cm} (9.31)

where we suppressed in $K^{(b,k)}_{2,R}$ the dependence in $(\zeta, \lambda, x, +, 0, -)$. Using (3.39),(4.17) and (3.49), an upper bound for the absolute value of (9.31) is

$$C' A^{-1} e^{C |q_R|} |q_R| \sum_{k=0}^{R} 2^{-(R-k)} e^{-L^{-k}|x|} \left( L^{-4k} Z^2_{k} + L^{-4k} Z^2_{k} \right) |q_k|.$$  \hspace{1cm} (9.32)

In the limit $R \to \infty$, this bound is vanishing: indeed $|q_R| \to 0$ while the sum remains bounded by the fact that $Z_k L^{-2k}, \overline{Z}_k L^{-2k} \leq C$. Next, consider the first
In conclusion, the absolute value of (3.35): expanding the product inside the square brackets, one obtains four terms. Since they can all be studied in similar way, let us consider one of them:

\[
e^{-\delta E_R |\Lambda|} \mathcal{E}_R \left[ e^{V_0, R(\zeta) + W_0, R(\zeta)} \frac{\partial V_2, R(\Phi)}{\partial J_{x^+}} \frac{\partial V_2, R(\Phi)}{\partial J_{0^+}} \right]_{J=0} \leq e^{-\delta E_R |\Lambda|} L^{-4R} \mathcal{E}_R \left[ e^{V_0, R(\zeta) + W_0, R(\zeta)} \left( Z_R^2 e^{i\alpha(\zeta - \zeta_0)} + Z_R^2 e^{i\alpha(\zeta - \zeta_0)} \right) \right]
\]

+ \frac{e^{-\delta E_R |\Lambda|} L^{-4R} \mathcal{E}_R}{\mathcal{E}_R} \left[ e^{V_0, R(\zeta) + W_0, R(\zeta)} Z_R Z_R \left( e^{i\alpha(\zeta - \zeta_0)} + e^{i\alpha(\zeta - \zeta_0)} \right) \right] (9.32)

It is easy to see that, for a \( C \equiv C(\alpha) \) and for \( z \) smaller than a \( z(L, \alpha) \),

\[
\| V_0, R(\zeta, \Lambda) \|_{h, T_R(\Lambda)} \leq C|q_R| \left( 1 + \max_{n=1,2} \| \nabla_R \zeta \|_{L^2(\Lambda)}^2 \right) \leq C|q_R| + \frac{1}{2} \ln G^{\text{str}}(\zeta, \Lambda),
\]

\[
\| W_0, R(\zeta, \Lambda) \|_{h, T_R(\Lambda)} \leq C|q_R| \left( 1 + \max_{n=1,2} \| \nabla_R \zeta \|_{L^2(\Lambda)}^2 \right) \leq C|q_R| + \frac{1}{2} \ln G^{\text{str}}(\zeta, \Lambda),
\]

\[
\| e^{\mu_1 \zeta} \|_{h, T_R(\Lambda)} \leq e^{h|\alpha|},
\]

therefore, for any \( \alpha_1, \alpha_2 \in \mathbb{R} \),

\[
\left| e^{V_0, R(\zeta) + W_0, R(\zeta)} e^{i(\alpha_1 \zeta + \alpha_2 \zeta_0)} \right| \leq \| e^{V_0, R(\zeta) + W_0, R(\zeta)} \|_{h, T_R(\Lambda)} \| e^{i\alpha_1 \zeta} \|_{h, T_R(\Lambda)} \| e^{i\alpha_2 \zeta} \|_{h, T_R(\Lambda)} \leq e^{h|\alpha_1| + h|\alpha_2|} e^{2C|q_R|} G^{\text{str}}(\zeta, \Lambda).
\]

In conclusion, the absolute value of (9.32) can be bounded by

\[
C(\alpha)e^{C|q_R| \left( L^{-4R} Z_R^2 + L^{-4R} Z_R^2 \right)}.
\]

In the limit \( R \to \infty \) this bound is vanishing since \( |q_R|, L^{-2R} Z_R, L^{-2R} Z_R \to 0 \). The remaining term of (9.32) is the one that gives the right hand side of (3.51). To prove this fact, we need to study the difference

\[
e^{-\delta E_R |\Lambda|} \mathcal{E}_R \left[ e^{V_0, R(\zeta) + W_0, R(\zeta)} \frac{\partial^2 W_2, R(\Phi)}{\partial J_{x^+} \partial J_{0^+}} \right]_{J=0} - 2 \left( w_{2, a, R}(x) + w_{2, a, R}(x) + w_{2, c, R}(x) \right)
\]

\[
= 2w_{2, a, R}(x) \left\{ e^{-\delta E_R |\Lambda|} \mathcal{E}_R \left[ V_0, R(\zeta) + W_0, R(\zeta) e^{i\alpha(\zeta - \zeta_0)} \right] - 1 \right\}
\]

\[
+ 2w_{2, b, R}(x) e^{-\delta E_R |\Lambda|} \mathcal{E}_R \left[ V_0, R(\zeta) + W_0, R(\zeta) e^{i\alpha(\zeta - \zeta_0)} \right] - 1 \right\}
\]

\[
+ 2w_{2, c, R}(x) e^{-\delta E_R |\Lambda|} \mathcal{E}_R \left[ e^{V_0, R(\zeta) + W_0, R(\zeta)} \right] - 1 \right\}.
\]

(9.34)

Observe that, by (3.13) and (3.14)

\[
\lim_{R \to \infty} \mathcal{E}_R \left[ e^{i\alpha(\zeta - \zeta_0)} \right] = 1, \quad \mathcal{E}_R \left[ e^{i\alpha(\zeta - \zeta_0)} \right] = 0.
\]

From them it is easy to show that (9.34) is vanishing in the limit \( R \to \infty \).
Appendix A. Functional Integral Formulation

A.1. Sine-Gordon transformation. It has been long known that free-energy and correlations of the Coulomb gas can be formulated as expectations with respect to a Gaussian measure [Kac, 1959; Siegert, 1960]. Since the Yukawa potential $W_{\Lambda}(x,m)$ in (2.1) is strictly positive definite, a finite dimensional Gaussian field $\{\varphi_x : x \in \Lambda\}$ is defined by assigning zero mean and covariance (3.1). Therefore, for real $\sigma_1, \ldots, \sigma_n$ and $x_1, \ldots, x_n \in \Lambda$, we have

$$\mathbb{E}_{m,\beta} \left[ \exp \left( \sum_{j=1}^{n} \sigma_j \varphi_{x_j} \right) \right] = e^{-\frac{1}{2}Q^2W_{\Lambda}(0;0)\exp \left\{ -\frac{\beta}{2} \sum_{i,j=1}^{n} \sigma_i \sigma_j \left[ W_{\Lambda}(x_i - x_j;m) - W_{\Lambda}(0;0) \right] \right\}}$$

(A.1)

where $Q := \sum_{j=1}^{n} \sigma_j$. Now note that in the limit $m \to 0$ the coefficient $W_{\Lambda}(0;0)$ is positively divergent; whereas under the same limit $W_{\Lambda}(x;m) - W_{\Lambda}(0;0)$ converges to $W_{\Lambda}(x|0)$ in (2.2); hence

$$\lim_{m \to 0} \mathbb{E}_{m,\beta} \left[ \exp \left( \sum_{j=1}^{n} \sigma_j \varphi_{x_j} \right) \right] = \begin{cases} \exp \left\{ -\frac{\beta}{2} \sum_{i,j=1}^{n} \sigma_i \sigma_j W_{\Lambda}(x_i - x_j|0) \right\} & \text{if} \sum_j \sigma_j = 0 \\ 0 & \text{otherwise.} \end{cases}$$

(A.2)

Using Taylor expansion in $z$ and (A.2) we have

$$\lim_{m \to 0} \mathbb{E}_{m,\beta} \left[ e^{\Delta \sum_{x \in \Lambda} \cos \varphi_x} \right] = Z_{\Lambda}(\beta, z),$$

$$\lim_{m \to 0} \mathbb{E}_{m,\beta} \left[ e^{i\eta(\varphi_x - \varphi_y)} e^{\Delta \sum_{x \in \Lambda} \cos \varphi_x} \right] = Z_{\Lambda}^{p_1,p_2}(\beta, z);$$

(A.3)

from these the functional integral formulation (3.3) follows.

A.2. Multiscale decomposition of the Gaussian measure. We review the construction of the multiscale representation; for details the reader can consult Appendix A of [Falco, 2012]. In that paper we gave an explicit procedure to obtain the decomposition

$$W_{\Lambda}(x;m) = \sum_{j=0}^{R-1} \Gamma_j(x;m) + \Gamma'_R(x;m) := \Gamma_{R-1,0}(x;m) + \Gamma'_R(x;m),$$

(A.4)

where the terms involved are such that $\Gamma_j(x) \equiv \Gamma_j(x;0)$ and $\Gamma'_R(x) \equiv \Gamma'_R(x;m)$ satisfy the properties discussed after (3.16). For any $s \in (0,\frac{1}{2})$, consider the non-negative definite, $s$-dependent potential

$$\tilde{W}_{\Lambda}(x;m) := \frac{1 - s}{|\Lambda|} \sum_{p \in \Lambda^*} \frac{\hat{\Gamma}_{R-1,0}(p;0) + \hat{\Gamma}_R(p;m)}{1 + s\hat{\Delta}(p) \left[ \hat{\Gamma}_{R-1,0}(p;0) + \hat{\Gamma}_R(p;m) \right]} e^{ixp}$$

where $\hat{\Gamma}_{R-1,0}(p;m)$ and $\hat{\Gamma}_R(p;m)$ are the Fourier transforms of $\Gamma_{R-1,0}(x;m)$ and $\Gamma'_R(x;m)$. Call $\tilde{\mathbb{E}}_{m,\beta}$ the associated Gaussian expectation. In the limit $m \to 0$,
regardless of \( s \), \( \tilde{W}_\Lambda(0; m) \) is positively divergent, while \( \tilde{W}_\Lambda(x; m) - \tilde{W}_\Lambda(0; m) \) converges to \( W_\Lambda(x; 0) \) —indeed, since \( W_\Lambda(x; m) \) is the inverse of \( -\Delta + m^2 \), one has 
\[
\tilde{\Delta}(p) \left[ \tilde{\Gamma}_{R-1,0}(p; 0) + \tilde{\Gamma}_R(p; 0) \right] = -1.
\]
Therefore, using (A.1) and (A.2), it is easy to see that (A.3) are still valid if we replace \( \mathbb{E}_{m,\beta} \) with \( \tilde{\mathbb{E}}_{m,\beta} \).

Now let us consider (A.3) with the latter Gaussian expectation. For reason related to the RG procedure, \( \tilde{W}_\Lambda \) has been chosen so to be able to extract from the measure and to add to the interaction a counterterm proportional to \( \frac{1}{2} (\partial^\mu \varphi_x)^2 \). However note that it is not known whether \( \tilde{W}_\Lambda(x; m) \) is strictly positive definite; therefore, to have a Gaussian measure with a density, define \( g(x; m) \) such that 
\[
\beta \tilde{W}_\Lambda(x; m) = \sum_{y \in \Lambda} g(x - y; m)g(y; m); \tag{A.4}
\]
then, for any integrable function \( F(\varphi) \), such as the ones in (A.3), we have
\[
\tilde{\mathbb{E}}_{m,\beta}[F(\varphi)] = \mathbb{E}_I[F(g^\varphi)]
\]
where \( \mathbb{E}_I \) is the Gaussian expectation such that \( \mathbb{E}_I[\varphi_x \varphi_y] = \delta_{x,y} \), and \( g^\varphi_x := \sum_{y \in \Lambda} g(x - y; m)\varphi_y \). If \( \alpha^2 := \beta(1 - s) \) and \( m_\ast := m/\sqrt{1 - s} \), we have
\[
\tilde{\mathbb{E}}_{m_\ast,\beta}[F(\varphi)] = \mathbb{E}_A \left\{ \exp \left[ \frac{s}{2\alpha^2} \sum_{x \in \Lambda} (\partial^\mu g^\varphi_x)^2 \right] \right\} \mathcal{N}_\Lambda(s; m) \]
\[
= \mathbb{E}_B \left\{ \exp \left[ \frac{s}{2} \sum_{x \in \Lambda} (\partial^\mu \varphi_x)^2 \right] \right\} \mathcal{N}_\Lambda(s; m) \tag{A.5}
\]
where \( \mathbb{E}_A \) and \( \mathbb{E}_B \) are the expectations with respect to the Gaussian measure with covariances
\[
\mathbb{E}_A[\varphi_x \varphi_y] = \frac{1}{|\Lambda|} \sum_{p \in \Lambda^*} \left\{ 1 + s\tilde{\Delta}(p) \left[ \tilde{\Gamma}_{R-1,0}(p; 0) + \tilde{\Gamma}_R(p; m) \right] \right\},
\]
\[
\mathbb{E}_B[\varphi_x \varphi_y] = \frac{1}{|\Lambda|} \sum_{p \in \Lambda^*} \left[ \tilde{\Gamma}_{R-1,0}(p; 0) + \tilde{\Gamma}_R(p; m) \right];
\]
and \( \mathcal{N}_\Lambda(s; m) \) takes into account the different normalization of two measures,
\[
\mathcal{N}_\Lambda(s; m) = \prod_{k \in \Lambda^*} \left[ m^2 - (1 - s)\tilde{\Delta}(k) \right]^{\frac{1}{2}}. \tag{A.6}
\]
Finally (3.7) follows from the identity
\[
\mathbb{E}_B[F(\varphi)] = \mathbb{E}_R\mathbb{E}_{R-1} \cdots \mathbb{E}_1 \mathbb{E}_0 [F(\zeta^{(R)} + \zeta^{(R-1)} + \cdots + \zeta^{(0)})] \tag{A.7}
\]
where \( \zeta^{(R)}, \zeta^{(R-1)}, \ldots, \zeta^{(0)} \) are two-by-two independent Gaussian fields with covariances
\[
\mathbb{E}_j[\zeta_x^{(j)} \zeta_y^{(j)}] = \begin{cases} 
\Gamma_j(x - y; 0) \equiv \Gamma_j(x - y) & \text{for } j = 0, 1, \ldots, R - 1, \\
\Gamma_R(x - y; m) \equiv \Gamma_R(x - y) & \text{for } j = R.
\end{cases}
\]
Appendix B. Proof of the Power Counting Theorems

B.1. Some Preliminary Inequalities.

Lemma B.1. Let $F \in \mathcal{N}_j(X)$ with $X \in \mathcal{S}_j$. For any $x_0 \in X$, if $(\delta \varphi)_x := \varphi_x - \varphi_{x_0}$ and $\rho := 5L^{-1}$,
\[
\|F(\delta \varphi)\|_{h,T_j(\varphi,X)} \leq \|F(\xi)\|_{\rho h,T_j(\xi,X)} \bigg|_{\xi = \delta \varphi_x}; \quad (B.1)
\]
Proof. (B.1) follows from the identity
\[
\sum_{x \in X} f_x \frac{\partial F(\delta \varphi)}{\partial \varphi_x} = \sum_{x \in X} (\delta f)_x \frac{\partial F}{\partial \varphi_x} \bigg|_{\xi = \delta \varphi}
\]
and the fact that, for $X$ small, $\|\delta f\|_{C^1_j(X)} \leq 5L^{-1}\|f\|_{C^2_j(X)}$. ■

Lemma B.2. Let $F \in \mathcal{N}_j(X)$ and $X \in \mathcal{P}_j$. Given $\psi \in C^2_j(X)$, if $\Delta := \|\psi\|_{C^2_j(X)}$,
\[
\|F(\varphi + \psi)\|_{h,T_j(\varphi,X)} \leq \|F(\varphi)\|_{h,\Delta,T_j(\varphi,X)}; \quad (B.2)
\]
Proof. From the definition of the norm $T^m_j(\varphi,X)$,
\[
\|D^n F(\varphi + \psi)\|_{T^m_j(\varphi,X)} \leq \sum_{m \geq 0} \frac{\Delta^n}{m!} \|D^{m+n} F(\varphi)\|_{T^{m+n}_j(\varphi,X)}.
\]
From this (B.2) follows. ■

Lemma B.3. Let $F \in \mathcal{N}_j(X)$ with $X \in \mathcal{P}_j$. For $h > 0$ and $m \in \mathbb{N}$
\[
\|\text{Rem}_m F(\xi)\|_{h,T_j(\xi,X)} \leq 2(1 + h^{-1}\|\xi\|_{C^1_j(X)})^{m+1} \sup_{t \in [0,1]} \|F(t\xi)\|_{h,T_j^{m+1}(t\xi,X)}; \quad (B.3)
\]
where $\|F(\varphi)\|_{h,T_j^{m+1}(\varphi,X)} := \sum_{n \geq m} \frac{K^n}{n!} \|D^n F(\varphi)\|_{T^n_j(\varphi,X)}$
Proof. For $n \geq m + 1$, obviously
\[
\|D^n \text{Rem}_m F(\xi)\|_{T^n_j(\xi,X)} = \|D^n F(\xi)\|_{T^n_j(\xi,X)}; \quad (B.4)
\]
For $0 \leq n \leq m$,
\[
\begin{align*}
D^n \text{Rem}_m F(\xi) & \cdot (f_1, \ldots, f_n) \\
& = D^n F(\xi) \cdot (f_1, \ldots, f_n) - \text{Tay}_{m-n,\xi} [D^n F(\xi) \cdot (f_1, \ldots, f_n)] \\
& = \text{Rem}_{m-n,\xi} [D^n F(\xi) \cdot (f_1, \ldots, f_n)] \\
& = \int_0^1 dt \frac{(1-t)^{m-n}}{(m-n)!} D^{m+1}_\xi F(t\xi) \cdot (f_1, \ldots, f_n, \xi, \ldots, \xi); \quad (B.5)
\end{align*}
\]
then,
\[
\|D^n \text{Rem}_m F(\xi)\|_{T^n_j(\varphi,X)} \leq \frac{\|\xi\|_{C^2_j(X)}^{m+1}}{(m+1-n)!} \sup_{t \in [0,1]} \|D^{m+1}_\xi F(t\xi)\|_{T^{m+1}_j(t\xi,X)}; \quad (B.6)
\]
From (B.4) and (B.6) we obtain
\[
\sum_{n=0}^{m+1} \frac{h^n}{n!} \|D^n \text{Rem}_m F(\xi)\|_{T^n_j(\varphi,X)}
\]
\[
\leq (1 + h^{-1}||\xi||c^2_j(\text{X}))^{m+1}\frac{h^{m+1}}{(m+1)!} \sup_{t \in [0,1]} \|D^{m+1}F(t\xi)\|_{T^{m+1}_j(t\xi,\text{X})}
\]
\[
\leq (1 + h^{-1}||\xi||c^2_j(\text{X}))^{m+1} \sup_{t \in [0,1]} \|F(t\xi)\|_{h,T_j^{\geq m+1}(t\xi,\text{X})}
\]

(B.7)

From that, (B.3) follows. □

B.2. Charged Components Decomposition. By induction on the scale \(j\), the polymer activities \(K_{0,j}(\varphi, \text{X})\) are invariant under the global translations \(\varphi_y \rightarrow \varphi_y + \frac{2\mu_m}{\alpha}\) for any \(m \in \mathbb{Z}\). Define the function of real variable \(F(t) := K_{0,j}(\varphi + t, \text{X})\), which is smooth and periodic of period \(2\pi/\alpha\). Expanding \(F(t)\) in (absolutely convergent) Fourier series and setting \(t = 0\), one obtains the first of (4.26) with charged components

\[
\hat{K}_{0,j}(q, \varphi, \text{X}) := \frac{\alpha}{2\pi} \int_0^{2\pi} ds \; K_{0,j}(\varphi + s, \text{X})e^{-iqs}. 
\]

Besides, since \(G_j(\varphi, \text{X})\) only depends upon the derivatives of \(\varphi\),

\[
||\hat{K}_{0,j}(q, \varphi, \text{X})||_{h,T_j(\varphi,\text{X})} \leq ||K_{0,j}(\text{X})||_{h,T_j(\text{X})} G_j(\varphi, \text{X}),
\]

which proves (4.29). To obtain the other two of (4.26), one can verify by inspection of (5.20) and (5.30) and inductively that \(e^{-iq\sigma_{\psi}\varphi}K_{1,j}(\varphi, \text{X}, x, \sigma)\) and \(e^{-\pi\sigma_{\psi}\varphi}K_{1,j}^t(\varphi, \text{X}, x, \sigma)\) are invariant under the transformation \(\varphi_y \rightarrow \varphi_y + \frac{2\mu_m}{\alpha}\) for any \(m \in \mathbb{Z}\). Therefore the charged components in these cases are

\[
\hat{K}_{1,j}(q, \varphi, \text{X}, x, \sigma) := \frac{\alpha}{2\pi} \int_0^{2\pi} ds \; K_{1,j}(\varphi + s, \text{X}, x, \sigma)e^{-iqs},
\]

\[
\hat{K}_{1,j}^t(q, \varphi, \text{X}, x, \sigma) := \frac{\alpha}{2\pi} \int_0^{2\pi} ds \; K_{1,j}(\varphi + s, \text{X}, x, \sigma)e^{-iqs}.
\]

Again it is not difficult to see that

\[
||\hat{K}_{1,j}(q, \varphi, \text{X}, x, \sigma)||_{h,T_j(\varphi,\text{X})} \leq ||K_{1,j}(\text{X}, x, \sigma)||_{h,T_j(\text{X})} G_j(\varphi, \text{X}),
\]

\[
||\hat{K}_{1,j}^t(q, \varphi, \text{X}, x, \sigma)||_{h,T_j(\varphi,\text{X})} \leq ||K_{1,j}^t(\text{X}, x, \sigma)||_{h,T_j(\text{X})} G_j(\varphi, \text{X}),
\]

which proves (4.30). The proof of (4.27) and (4.31) follows from similar arguments.

B.3. Proof of the first dimensional bound. Here we prove Theorem 4.5, which provides the first type of dimensional bound. We begin with setting up some notations. Consider the Gaussian expectation \(E_j\) with covariance \(\Gamma_j\) and also the Gaussian expectation \(E_I\) with covariance \(I = (\delta_{ij})\). Decompose \(\Gamma_j\) as \(\Gamma_j = g_j \circ g_j\) and call \((g_j f)_x := \sum_{y \in A} g_j(x-y)f_y\) and likewise for \((\Gamma_j f)_x\). Consider an integrable charge \(p\) activity \(F(\varphi) \equiv F(\varphi, \text{X})\). Under the imaginary translation \(\zeta_x \rightarrow \zeta_x + i(g_j f)_x\) where \(f\) is any test function with finite support,

\[
E_j[F(\varphi)] = E_I[F(\varphi' + (g_j \zeta)) = e^{\frac{i}{2}(f,\Gamma_j f)} E_j \left[ e^{-i(\zeta, f)} F(\varphi + i(\Gamma_j f)) \right].
\]

(The measure \(E_I\) is involved in the identity to avoid to make the imaginary translation in a *degenerate* Gaussian measure, as in principle \(E_j\) could be.) Now use the identity \(F(\varphi) = e^{i\alpha \varphi_\theta} F(\varphi - \vartheta)\) for any constant complex field \(\vartheta\); calling \(\psi_x := (\Gamma_j f)_x\) and, for \(x_0 \in \text{X}\), setting \(\delta_x := (\Gamma_j f)_x - (\Gamma_j f)_{x_0}\), we have

\[
E_j \left[ F(\varphi) \right] = e^{\frac{i}{2}(f,\Gamma_j f) - \alpha p(\delta_{x_0}, \Gamma_j f)} E_j \left[ e^{-i(\zeta, f)} F(\varphi + i\delta_x) \right] \quad \text{(B.11)}
\]
where \((\delta_{x_0})_x := \delta_{x,x_0}\). In order to minimize the prefactor in the r.h.s. of (B.11), one can set \(f_x = \alpha p\delta_{x,x_0}\). However, the size of such an \(f_x\) grows in \(p\), and this conflicts with the assumption of finite radius of analyticity for all the activities \(F(\varphi)\). To avoid this problem, we consider two cases:
1. if \(|p| \leq 1\), we make the optimal choice \(f_x = \alpha p\delta_{x,x_0}\)
2. if \(|p| > 1\), we follow [Dimock and Hurd, 2000] and set \(f_x = \alpha \text{sgn}(p)\delta_{x,x_0}\) (for \(\text{sgn}(x) := x/|x|\)).

Therefore, from (B.11) we obtain
\[
\|E_j [F(\varphi)]\|_{h,T_{j+1}(\varphi',X)} \leq e^{-d(p)\frac{2}{3}L}E_j \left[\|F(\varphi + i\delta\varphi)\|_{h,T_{j+1}(\varphi',X)}\right] \tag{B.12}
\]
where \(d(p) := p^2\) for \(|p| \leq 1\) and \(d(p) := 2|p| - 1\) otherwise. Note that according to definition (4.7) for any value of \(p\) we have
\[
\Delta := \|\delta\varphi\|_{c^2_j(X)} \leq \frac{h}{2}.
\]

Now consider the expectation on the r.h.s. of (B.12); and set \(p := 5L^{-1}\), \(H_x := \zeta_x + i\delta\varphi_x\). Since \(\|e^{ip\varphi_0}\|_{h,T_{j+1}(\varphi',X)}\) is less than \(e^{\rho|p|}\alpha\) (which is \(L\)-independent), by (B.1), (B.2), (B.4), and for \(L\) so large that \(\rho \leq \frac{3}{2}\), (hence \(\rho\h + \Delta \leq h\))
\[
\|F(\varphi + i\delta\varphi)\|_{h,T_{j+1}(\varphi',X)} \leq e^{\rho|p|\alpha}\|F(\xi + H)\|_{h,T_j(\xi,X)} \tag{B.13}
\]
Finally, (4.34) is obtained by plugging (B.13) into (B.12) and using (4.16) for the integration \(E_j\). This completes the proof of Theorem 4.5.

B.4. Proof of the second dimensional bound. We want to prove Theorem 4.6, which gives the second dimensional bound. From (B.12) and the inequality \(\|e^{ip\varphi_0}\|_{h,T_{j+1}(\varphi',X)} \leq e^{\rho|p|\alpha}\), we find
\[
\|\text{Rem}_m E_j[F(\varphi)]\|_{h,T_{j+1}(\varphi',X)} \leq e^{-d(p)\frac{2}{3}L}E_j \left[\|F(\varphi + i\delta\varphi)\|_{h,T_{j+1}(\varphi',X)}\right] \tag{B.14}
\]
where \(H_x := \zeta_x + i\delta\varphi_x\). As in the previous proof, \(\Delta := \|\delta\varphi\|_{c^2_j(X)} \leq \frac{h}{2}\) and \(\rho := 5L^{-1}\) is small for large enough \(L\). Now use (B.1), (B.3) and (B.2) to obtain (the definition of the seminorm \(\|\cdot\|_{\rho h,T_{j+1}(\varphi',X)}\) is in Lemma B.3)
\[
\|\text{Rem}_m F(\delta\varphi' + H)\|_{h,T_{j+1}(\varphi',X)} \leq \|\text{Rem}_m F(\xi + H)\|_{\rho h,T_j(\xi,X)} \tag{B.15}
\]
\[
\leq 2 \left(1 + \rho h\right)^{-1}\|\xi\|_{c^2_j(X)} \sup_{t \in [0,1]} \|F(t\xi + H)\|_{\rho h,T_{j+1}(t\xi,X)} \tag{B.16}
\]
\[
\leq 2 \left(1 + \rho h\right)^{-1}\|\xi\|_{c^2_j(X)} (2\rho)^{m+1} \sup_{t \in [0,1]} \|F(t\xi + H)\|_{\rho h,T_j(t\xi,X)} \tag{B.17}
\]
\[
\leq 2 \left( 1 + (\rho h)^{-1} \| \xi \|_{C^2_j(X)} \right)^{m+1} (2\rho)^{m+1} \sup_{t \in [0,1]} \left\| F(t\xi + \zeta) \right\|_{H^{\Delta,T_j(t\xi,X)}_{\zeta=\delta\phi'}}^{m+1} \quad (B.15)
\]

To obtain the third line we used that \( \| \cdot \|_{sh,T_j^2(\varphi,X)} \leq s^{m+1} \| \cdot \|_{h,T_j(\varphi,X)} \). As \( X \in S_j \),

\[
L \| \delta\varphi' \|_{C^2_j(X)} \leq C \max_{p=1,2} \| \nabla^{p+1}_j \varphi' \|_{L^\infty(X^*)}.
\]  

(B.16)

Besides, since \( G_j \) depends upon the derivatives of the fields, \( G_j(t\delta\varphi' + \zeta, X) = G_j(t\varphi' + \zeta, X) \). Therefore

\[
\| \operatorname{Rem} F(\delta\varphi' + H) \|_{h,T_{j+1}(\varphi',X)} \leq C^{m+1} L^{-(m+1)} \left\| F \right\|_{h,T_j(X)} \left( 1 + \max_{p=1,2} \| \nabla^{p+1}_j \varphi' \|_{L^\infty(X^*)} \right)^{m+1} \times \sup_{t \in [0,1]} G_j(t\varphi' + \zeta, X). \]

(B.17)

Finally, Theorem 4.6 is proven once (B.17) is plugged into (B.14) and last part of Lemma 4.2 is used for the integration \( E_j \).

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