Vortices and polynomials: non-uniqueness of the Adler–Moser polynomials for the Tkachenko equation

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Abstract

Stationary and translating relative equilibria of point vortices in the plane are studied. It is shown that stationary equilibria of any system containing point vortices with arbitrary choice of circulations can be described with the help of the Tkachenko equation. It is also obtained that translating relative equilibria of point vortices with arbitrary circulations can be constructed using a generalization of the Tkachenko equation. Roots of any pair of polynomials solving the Tkachenko equation and the generalized Tkachenko equation are proved to give positions of point vortices in stationary and translating relative equilibria accordingly. These results are valid even if the polynomials in a pair have multiple or common roots. It is obtained that the Adler–Moser polynomial provides non-unique polynomial solutions of the Tkachenko equation. It is shown that the generalized Tkachenko equation possesses polynomial solutions with degrees that are not triangular numbers.

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(Some figures may appear in colour only in the online journal)

1. Introduction

The problem of finding stationary and relative equilibrium solutions to Helmholtz’s equations describing motion of point vortices in the plane has attracted much attention in recent years [1–12]. The so-called polynomial method is often used while studying this problem [2]. According to this method, one introduces polynomials with roots at vortex positions. For example, it was found by Stieltjes that the generating polynomial of $N$ identical point vortices in relative equilibrium on a line is essentially the $N$th Hermite polynomial. Considering stationary equilibrium of point vortices with equal in absolute value circulations leads to an ordinary differential equation first discovered by Tkachenko [13]. It is well known that the Adler–Moser polynomials give polynomial solutions to the Tkachenko equation. Originally,
these polynomials arose in the theory of one of the most well-known soliton equations, the Korteweg–de Vries equation [16]. This fact provides a remarkable and rather unexpected connection between the dynamics of point vortices and the theory of integrable partial differential equations [17]. Not long ago, analogous connections between stationary equilibria of point vortices with circulations $\Gamma, -2\Gamma$ and rational solutions of the Sawada–Kotera and the Kaup–Kupershmidt equations were established [18].

In this paper, we study stationary and translating relative equilibria of multi-vortex systems with circulations $\Gamma_1, \ldots, \Gamma_N$. Our aim is to show that the stationary case can be described with the help of the Tkachenko equation and the translating case is reducible to a generalization of the Tkachenko equation. We prove that roots of any pair of polynomials satisfying these equations give positions of vortices in equilibrium. We show that this statement holds even if the polynomials in a pair have multiple or common roots.

Burchnall and Chaundy proposed a hypothesis that the Adler–Moser polynomials give unique polynomial solutions to the Tkachenko equation [19]. It turns out that this assertion is valid if pairs of polynomials satisfying the Tkachenko equation have neither multiple nor common roots inside a pair. However, some articles contain mistakes and inaccurate statements [3, 6, 12, 20] as if this hypothesis is a proven fact in the general case. In this work, we clarify this misbelief. As a consequence of our results, we obtain that the Adler–Moser polynomials provide non-unique polynomial solutions to the Tkachenko equation. We find alternative polynomial solutions of this equation. A similar situation takes place with the generalized Tkachenko equation. Sometimes it is supposed that the generalized Tkachenko equation possesses polynomial solutions only with degrees being triangular numbers [1, 12]. We show that this assertion is not valid.

This paper is organized as follows. In section 2, we consider stationary equilibria of point vortices with arbitrary choice of circulations. We derive an ordinary differential equation satisfied by generating polynomials of arrangements and transform this equation to the Tkachenko equation. In section 3, we construct transformations relating the Tkachenko equation and the equation describing stationary equilibria of vortices with circulations $\Gamma$ and $-\mu \Gamma$. Section 4 is devoted to the translating case. We find an ordinary differential equation satisfied by generating polynomials of the vortices and reduce the resulting equation to the generalized Tkachenko equation. In section 5, we study properties of the Tkachenko equation. Our approach allows one to construct the polynomial solutions of the Tkachenko equation not included into the sequence of the Adler–Moser polynomials. Examples are given in section 5. Properties of the generalized Tkachenko equation are considered in section 6. We present an example of polynomials solving the generalized Tkachenko equation and possessing non-triangular degrees.

2. Stationary equilibria of point vortices with an arbitrary choice of circulations

The motion of M point vortices with circulations (or strengths) $\Gamma_k$ at positions $z_k, k = 1, \ldots, M$, is described by the following system of differential equations:

$$\frac{dz_k}{dt} = \frac{1}{2\pi i} \sum_{m=1}^{M} \frac{\Gamma_m}{z_k - z_m}, \quad k = 1, \ldots, M.$$  \hspace{1cm} (1)

The prime in this expression means that the case $m = k$ is excluded and the symbol * stands for complex conjugation. In this section, we shall deal with stationary vortex configurations. The case of point vortices possessing equal in absolute value circulations has been intensively studied since the work of Tkachenko [13]. It is a well-known fact that equilibrium positions of point vortices with equal in absolute value circulations are given by roots of polynomials that
satisfy the Tkachenko equation and do not have multiple and common roots (see, for example, [1–3, 6, 12]). In this paper, we study the problem of finding stationary equilibrium positions of point vortices with arbitrary choice of circulations.

Let us consider a system of \( M \) point vortices with the circulations \( \Gamma_1, \ldots, \Gamma_N \). We set \( \frac{dz^*_j}{dt} = 0 \) in the equations of motion (1) and subdivide the vortices into groups according to the values of circulations. Suppose vortices at positions \( a_1^{(j)}, \ldots, a_d^{(j)} \) have circulations \( \Gamma_j, \ j = 1, \ldots, N \). This yields \( M = l_1 + \cdots + l_N, \ M \geq N \). A convenient tool for analyzing such a situation is to introduce polynomials with roots at the positions of vortices [2]. In our case, we have

\[
P_j(z) = \prod_{i=1}^{l_j} (z - a_i^{(j)}), \quad j = 1, \ldots, N. \tag{2}
\]

Thus, we see that roots of the polynomial \( P_j(z) \) give positions of vortices with circulation \( \Gamma_j \). Note that the polynomials \( P_j(z), j = 1, \ldots, N \), do not have multiple and common roots. From equations (1), we find the system of algebraic relations

\[
\sum_{j=1}^{N} \sum_{i=1}^{l_j} \Gamma_j \left( a_k^{(j)} - a_i^{(j)} \right) = 0, \quad i_0 = 1, \ldots, l_{j_0}, \quad j_0 = 1, \ldots, N, \tag{3}
\]

where the case \((j_0, i_0) = (j, i)\) is excluded. Using properties of the logarithmic derivative, we obtain the following equalities:

\[
P_{j,z} = P_j \sum_{i=1}^{l_j} \frac{1}{z - a_i^{(j)}}, \quad P_{j,z} = 2P_j \sum_{i=1}^{l_j} \sum_{k=1}^{l_j} \frac{1}{(z - a_i^{(j)})(a_k^{(j)} - a_i^{(j)})}. \tag{4}
\]

Now, let \( z \) tend to one of the roots of the polynomial \( P_{j,0}(z) \). Calculating the limit \( z \to a_0^{(j)} \) in the expression for \( P_{j,0}(z) \) yields

\[
P_{j,0}(a_0^{(j)}) = 2P_{j,0}(a_0^{(j)}) \sum_{i=1}^{l_j} \frac{1}{a_k^{(j)} - a_i^{(j)}}. \tag{5}
\]

Using equalities (3) and (4), we obtain the conditions

\[
\Gamma_j P_{j,0}(a_0^{(j)}) \prod_{i=1}^{N} P_i(z) = -2 \sum_{j=1, j \neq j_0}^{N} \Gamma_j P_{j,z}(a_0^{(j)}) P_{j,0}(a_0^{(j)}), \quad i_0 = 1, \ldots, l_{j_0}, \quad j_0 = 1, \ldots, N, \tag{6}
\]

which are valid for any root \( a_0^{(j)} \) of the polynomial \( P_{j,0}(z) \). Furthermore, on using (5) and (6), we see that the polynomial

\[
\prod_{i=1}^{N} P_i(z) \left\{ \sum_{j=1}^{N} \Gamma_j \left( \frac{P_{j,z}}{P_j} \right) + 2 \sum_{k < j} \Gamma_k \Gamma_j \frac{P_{k,z} P_{j,0} P_{j,z}}{P_k P_j} \right\}, \tag{7}
\]

being of degree \( M - 2 \) possesses \( M \) roots \( a_i^{(j)}, i = 1, \ldots, l_j, j = 1, \ldots, N \). Thus, this polynomial identically equals zero. Consequently, the generating polynomials \( P_j(z), j = 1, \ldots, N \), of the arrangements described above satisfy the differential correlation (see also [8])

\[
\prod_{i=1}^{N} P_i(z) \left\{ \sum_{j=1}^{N} \Gamma_j \left( \frac{P_{j,z}}{P_j} \right) + 2 \sum_{k < j} \Gamma_k \Gamma_j \frac{P_{k,z} P_{j,0} P_{j,z}}{P_k P_j} \right\} = 0. \tag{8}
\]

The reverse result is also valid. If \( N \) polynomials \( P_j(z), j = 1, \ldots, N \), with no common and multiple roots satisfy correlation (8), then the roots of these polynomials give positions of vortices with circulations \( \Gamma_j, j = 1, \ldots, N \), in stationary equilibrium.
Equation (8) is invariant under the transformation $z \mapsto \alpha z + \beta$ with $\alpha$ and $\beta$ being complex numbers and $\alpha \neq 0$. This transformation converts any equilibrium arrangement into another equilibrium arrangement. We shall regard all such arrangements as equivalent.

We note that finding the Laurent series in a neighborhood of infinity for the expression in (7) and setting the highest order coefficient to zero yields a necessary condition of an equilibrium in the arrangements of vortices specified above to exist. This condition is the following:

$$\sum_{j=1}^{N} L_j \Gamma_j^2 - \left( \sum_{j=1}^{N} L_j \right)^2 = 0. \quad (9)$$

Furthermore, we define new functions $\tilde{P}(z)$ and $\tilde{Q}(z)$ according to the rules

$$\tilde{P}(z) = \prod_{j=1}^{N} \frac{P_j(z)^{\nu_j(n_j+1)}}{P_j(z)}, \quad \tilde{Q}(z) = \prod_{j=1}^{N} \frac{P_j(z)^{\nu_j(n_j+1)}}{P_j(z)}. \quad (10)$$

By straightforward computations, we see that the equalities

$$\frac{d^2}{dz^2} \ln \{\tilde{P}(z)\tilde{Q}(z)\} = \sum_{j=1}^{N} \Gamma_j^2 \left( \frac{P_j\tilde{\psi}(z)}{P_j} - \frac{\tilde{P}_j\tilde{\psi}(z)}{P_j} \right),$$

$$\frac{d}{dz} \ln \left\{ \frac{\tilde{P}(z)}{\tilde{Q}(z)} \right\} = \sum_{j=1}^{N} \Gamma_j \left( \frac{P_j\tilde{\psi}(z)}{P_j} \right), \quad (11)$$

are valid. Substituting expressions (11) into equation (8), we obtain the following differential equation for the functions $\tilde{P}(z)$ and $\tilde{Q}(z)$:

$$\frac{d^2}{dz^2} \ln \{\tilde{P}(z)\tilde{Q}(z)\} + \left( \frac{d}{dz} \ln \left\{ \frac{\tilde{P}(z)}{\tilde{Q}(z)} \right\} \right)^2 = 0. \quad (12)$$

Rewriting this relation in a more convenient form yields

$$\tilde{P}_z\tilde{Q} - 2\tilde{P}\tilde{Q}_z + \tilde{P}\tilde{Q}_z = 0. \quad (13)$$

Along with this we see that substituting equalities (10) into expression (13), we return back to equation (8).

Equation (13) was studied in detail by Burchnall and Chaundy [19]. In the framework of the vortex theory, equation (13) is known as the Tkachenko equation [2, 13–15]. It is a remarkable fact that this equation appears not only in the vortex theory but also in the theory of the Korteweg–de Vries equation [16].

Consequently, any solution of equation (13) in the form (10) describes stationary equilibrium of $M$ point vortices with circulations $\Gamma_j$, $j = 1, \ldots, N$. Suppose we have found such a solution; then, a vortex with circulation $\Gamma_j$ is situated at the point $z = z_0$ whenever the function $\tilde{P}(z)$ has a ‘root’ of ‘multiplicity’ $\Gamma_j(\Gamma_j + 1)/2$ at the point $z = z_0$ and the function $\tilde{Q}(z)$ has a ‘root’ of ‘multiplicity’ $\Gamma_j(\Gamma_j - 1)/2$ at the point $z = z_0$. The circulation is calculated as the difference of the corresponding ‘multiplicities’. Here and in what follows, we state that the function $f(z)$ has a ‘root’ of ‘multiplicity’ $k$ at the point $z = z_0$ if $f(z) = (z - z_0)^k \psi(z)$, with $\psi(z)$ being an analytic function (possibly multivalued) in a neighborhood of $z_0$ and $\psi(z_0) \neq 0$. If $k \in \mathbb{N}$, then the point $z = z_0$ is a root of the function $f(z)$ in the usual sense.

Furthermore, let us suppose that all the circulations $\Gamma_j$, $j = 1, \ldots, N$, are multiples of the basic circulation $\Gamma$, $\Gamma \neq 0$. In other words, we consider the situation $\Gamma_j = n_j \Gamma$, $j = 1, \ldots, N$. In this case, we see that the multiplier $\Gamma^2$ can be removed from equation (8). Then, defining the functions

$$\tilde{P}(z) = \prod_{j=1}^{N} P_j^{\nu_j(n_j+1)}(z), \quad \tilde{Q}(z) = \prod_{j=1}^{N} P_j^{\nu_j(n_j+1)}(z), \quad (14)$$
we again arrive at the Tkachenko equation (13). If the parameters \( n_j, j = 1, \ldots, N \), are all integers, then \( \tilde{P}(z) \) and \( \tilde{Q}(z) \) in (14) are polynomials. In the case \( n_j \in \mathbb{Z} \), we obtain that stationary equilibria of \( M \) point vortices with circulations \( \Gamma_j = n_j \Gamma, j = 1, \ldots, N \), can be described with the help of polynomials that solve equation (13) and can be presented in the form (14). If the polynomial \( \tilde{P}(z) \) has a root of multiplicity \( n_j(n_j + 1)/2 \) at the point \( z = z_0 \) and the polynomial \( \tilde{Q}(z) \) has a root of multiplicity \( n_j(n_j - 1)/2 \) at the point \( z = z_0 \), then a vortex with circulation \( n_j \Gamma \) is situated at the point \( z = z_0 \).

Properties of equation (13) and some applications of our results will be discussed in sections 3 and 5.

3. Transformations relating the Tkachenko equation and the equation describing stationary equilibria of vortices with circulations \( \Gamma \) and \( -\mu \Gamma \)

In this section, our aim is to establish a connection between the Tkachenko equation (13) and the following differential equation:

\[
S_{zz}T - 2\mu S z T_z + \mu^2 S T_{zz} = 0. \tag{15}
\]

The polynomials \( S(z) \) and \( T(z) \), which satisfy equation (15), describe stationary equilibria of vortices with circulations \( \Gamma \) and \( -\mu \Gamma, \mu > 0 \), provided that these polynomials do not have multiple and common roots. Roots of the polynomial \( S(z) \) give positions of vortices with circulation \( \Gamma \) and roots of the polynomial \( T(z) \) give positions of vortices with circulation \( -\mu \Gamma \). Equation (15) can be obtained from relation (8), where we set \( \Gamma_1 = \Gamma, \Gamma_2 = -\mu \Gamma \), \( P_1(z) = S(z), P_2(z) = T(z) \). In the case \( \mu = 2 \) this equation possesses interesting applications. Polynomial solutions of equation (15) with \( \mu = 2 \) can be used to construct rational solutions of the Sawada–Kotera and the Kaup–Kupershmidt equations [18]. The Sawada–Kotera and the Kaup–Kupershmidt equations belong to the class of integrable partial differential equations as well as the Korteweg–de Vries equation.

Let us consider formula (14) under the conditions \( P_1(z) = S(z), P_2(z) = T(z), n_1 = 1, n_2 = -\mu \). We see that the transformations

\[
\tilde{P}(z) = S(z)T(z)^{\frac{n_1+1}{n_1}}, \quad \tilde{Q}(z) = T(z)^{\frac{n_1+1}{n_1}}, \quad S(z) = \tilde{P}(z)\tilde{Q}(z)^{-\frac{1}{n_1}}, \quad T(z) = \tilde{Q}(z)^{\frac{1}{n_1}}. \tag{16}
\]

relate the solutions of equations (13) and (15). Indeed, the transformation from \((S,T)\) to \((\tilde{P}, \tilde{Q})\) can be obtained by means of expression (14). The reverse transformation is a direct consequence of the former.

Furthermore, we return to multi-vortex systems that we have studied at the beginning of section 2. Substituting expression (10) into the second pair of equalities in (16), we obtain the relation

\[
S(z) = \prod_{j=1}^{N} P_j^{\Gamma_j(z_0+1)/2}(z), \quad T(z) = \prod_{j=1}^{N} P_j^{\Gamma_j(z_0+1)/2}(z). \tag{17}
\]

Thus, we obtain that stationary equilibria of point vortices with circulations \( \Gamma_j, j = 1, \ldots, N \) can be described with the help of equation (15). Suppose we have found a solution of equation (15) in the form (17), then a vortex with circulation \( \Gamma_j \) is situated at the point \( z = z_0 \) whenever the point \( z = z_0 \) is a ‘root’ of the function \( S(z) \) with the ‘multiplicity’ \( \Gamma_j/(\Gamma_j+\mu)/(\mu+1) \) and the point \( z = z_0 \) is a ‘root’ of the function \( T(z) \) with the ‘multiplicity’ \( \Gamma_j/(\Gamma_j-1)/[\mu(\mu+1)] \). In order to calculate the circulation of the vortex at the point \( z = z_0 \), we take the ‘multiplicity’ of the ‘root’ \( z = z_0 \) of the function \( S(z) \) and subtract \( \mu \) multiplied by the ‘multiplicity’ of the ‘root’ \( z = z_0 \) of the function \( T(z) \).
Concluding this section, we note that for vortex systems with circulations being multiples of the basic circulation $\Gamma$, $\Gamma \neq 0$, the following relation is valid:

$$S(z) = \prod_{j=1}^{N} P_{j}\left(\frac{z^{\lambda}}{\Gamma_{j}}\right), \quad T(z) = \prod_{j=1}^{N} P_{j}\left(\frac{z^{\lambda}}{\Gamma_{j}}\right).$$

(18)

Consequently, the stationary equilibria of point vortices with circulations $\Gamma_{j} = n_{j}\Gamma$, $j = 1, \ldots, N$, can be constructed with the help of equation (15) and expression (18).

4. Translating equilibria of point vortices with an arbitrary choice of circulations

In this section, we study the system of $M$ point vortices moving uniformly with equal velocities. This motion is usually referred to as translating equilibrium of point vortices. Setting $\frac{d\zeta}{dt} = v^{*}$, with $v$ being a constant $v \overset{\text{def}}{=} \lambda/(2\pi i)$, $\lambda \neq 0$, in equations of motion (1), we obtain

$$\sum_{j=1}^{M} \frac{\Gamma_{j}}{z_{k} - s_{m}} - \lambda = 0, \quad k = 1, \ldots, M.$$  

(19)

Again we subdivide the vortices into groups according to the values of circulations $\Gamma_{j}$, $j = 1, \ldots, N$. By $a_{j}^{(i)}$, $a_{j}^{(j)}$, $j = 1, \ldots, N$, we denote instantaneous positions of the vortices. We rewrite relations (19) as follows:

$$\sum_{j=1}^{N} \sum_{i=1}^{l_{j}} \frac{\Gamma_{j}}{a_{j}^{(i)} - a_{i}^{(j)}} - \lambda = 0, \quad i_{0} = 1, \ldots, l_{j}, \quad j_{0} = 1, \ldots, N.$$  

(20)

Introducing the polynomials $P_{j}(z)$, $j = 1, \ldots, N$, with no multiple and common roots as given in formula (2), we obtain the correlations

$$\Gamma_{j_{0}} \frac{P_{j_{0}, z^{*}}(a_{j_{0}}^{(j_{0})})}{P_{j_{0}, z}(a_{j_{0}}^{(j_{0})})} = -2 \sum_{j=1, j \neq j_{0}}^{N} \Gamma_{j} P_{j, z^{*}}(a_{j_{0}}^{(j_{0})}) P_{j, z}(a_{j_{0}}^{(j_{0})}) + 2\lambda, \quad i_{0} = 1, \ldots, l_{j}, \quad j_{0} = 1, \ldots, N.$$  

(21)

valid for any root $a_{i_{0}}^{(j_{0})}$ of the polynomial $P_{j_{0}}(z)$. Let us consider the polynomial

$$\prod_{i=1}^{N} P_{i}(z) \left\{ \sum_{j=1}^{N} \frac{\Gamma_{j}^{2} P_{j, z^{*}} P_{j} - 2 \sum_{k<j} \Gamma_{k} \frac{P_{k, z}}{P_{j}} \frac{P_{j, z}}{P_{k}} - 2\lambda \sum_{j=1}^{N} \Gamma_{j} \frac{P_{j, z}}{P_{j}}}{P_{j}} \right\}.$$  

(22)

This polynomial is of degree $M - 1$ and possesses $M$ roots $a_{i}^{(j)}$, $i = 1, \ldots, l_{j}$, $j = 1, \ldots, N$. Thus, this polynomial identically equals zero. Consequently, the generating polynomials $P_{j}(z)$, $j = 1, \ldots, N$, of the arrangements described above satisfy the differential correlation (see also [8])

$$\prod_{i=1}^{N} P_{i}(z) \left\{ \sum_{j=1}^{N} \frac{\Gamma_{j}^{2} P_{j, z^{*}} P_{j} + 2 \sum_{k<j} \Gamma_{k} \frac{P_{k, z}}{P_{j}} \frac{P_{j, z}}{P_{k}} - 2\lambda \sum_{j=1}^{N} \Gamma_{j} \frac{P_{j, z}}{P_{j}}}{P_{j}} \right\} = 0.$$  

(23)

The reverse result is also valid. If $N$ polynomials $P_{j}(z)$, $j = 1, \ldots, N$, with no common and multiple roots satisfy correlation (23), then the roots of these polynomials give instantaneous positions of vortices with circulations $\Gamma_{j}$, $j = 1, \ldots, N$, in translating equilibrium. Finding the Laurent series in a neighborhood of infinity for expression (22) and setting the highest order coefficient to zero, we obtain the following necessary condition:

$$\sum_{j=1}^{N} l_{j} \Gamma_{j} = 0$$  

(24)
for a translating equilibrium to exist. Using expressions (11), we see that the transformation (10) reduces equation (23) to the equation
\[
\frac{d^2}{dz^2} \ln[\tilde{P}(z)\tilde{Q}(z)] + \left( \frac{d}{dz} \ln \left\{ \frac{\tilde{P}(z)}{\tilde{Q}(z)} \right\} \right)^2 - 2\lambda \frac{d}{dz} \ln \left\{ \frac{\tilde{P}(z)}{\tilde{Q}(z)} \right\} = 0.
\] (25)
Rewriting this relation in more convenient form yields the equation
\[
\tilde{P}_z\tilde{Q} - 2\tilde{P}\tilde{Q}_z + \tilde{P}\tilde{Q}_{zz} - 2\lambda(\tilde{P}\tilde{Q} - \tilde{P}\tilde{Q}_z) = 0,
\] (26)
which is a generalization of the Tkachenko equation (13). Consequently, we see that translating equilibria of \(M\) point vortices with circulations \(\Gamma_j, j = 1, \ldots, N\), can be described with the help of equation (26).

5. Properties of the Tkachenko equation

In this section, we shall study properties of polynomials that solve equations (13). Let a pair of polynomials \(\tilde{P}(z), \tilde{Q}(z)\) satisfy equation (13). Balancing the highest order terms in equation (13), we see that the degrees of the polynomials \(\tilde{P}(z), \tilde{Q}(z)\) are two successive triangular numbers. In other words, \(\deg \tilde{P}(z) = m(m + 1)/2\) and \(\deg \tilde{Q}(z) = m(m - 1)/2, m \in \mathbb{Z}\). Furthermore, let us prove the following theorem.

**Theorem 1.** Any pair of polynomials \(\tilde{P}(z), \tilde{Q}(z)\) solving the Tkachenko equation (13) gives an arrangement of point vortices in stationary equilibrium.

**Proof.** Let a pair of polynomials \(\tilde{P}(z), \tilde{Q}(z)\) satisfy the Tkachenko equation (13). In addition, suppose that a point \(z = z_0\) is a root of at least one of the polynomials \(\tilde{P}(z)\) and \(\tilde{Q}(z)\). We substitute the expressions
\[
\tilde{P}(z) = (z - z_0)^r h(z), \quad \tilde{Q}(z) = (z - z_0)^s g(z), \quad r, s \in \mathbb{N} \cup \{0\},
\] (27)
where \(h(z)\) and \(g(z)\) are the polynomials, such that \(h(z_0) \neq 0, g(z_0) \neq 0\), into equation (13) and find the Taylor series in a neighborhood of the point \(z_0\) of the resulting relation. Setting to zero the coefficient at \((z - z_0)^{r+s-2}\), we obtain the algebraic equation
\[
(r - s)^2 = r + s.
\] (28)
Thus, we obtain the system
\[
r - s = l, \quad r + s = l^2, \quad l \in \mathbb{Z}.
\] (29)
Solving this system yields
\[
r = \frac{l(l + 1)}{2}, \quad s = \frac{l(l - 1)}{2}, \quad l \in \mathbb{Z}.
\] (30)
Analyzing expressions (30), we make sure that the following statements are valid.

(i) If the point \(z_0\) is a multiple root of one of the polynomials \(\tilde{P}(z)\) and \(\tilde{Q}(z)\), then it is also a root of another polynomial and the multiplicities of the root \(z_0\) for the polynomials \(\tilde{P}(z)\) and \(\tilde{Q}(z)\) are two successive triangular numbers.

(ii) If the point \(z_0\) is a common root of the polynomials \(\tilde{P}(z)\) and \(\tilde{Q}(z)\), then it is a multiple root of at least one of them and the multiplicities of the root \(z_0\) for the polynomials \(\tilde{P}(z)\) and \(\tilde{Q}(z)\) are two successive triangular numbers.
Consequently, any polynomial solution of the Tkachenko equation (13) can always be presented in the form (14) with \( n_j \in \mathbb{Z}, j = 1, \ldots, N. \)

It follows from theorem 1 that the roots of any polynomial solution of equation (13) provide positions of point vortices in stationary equilibrium. The vortex circulations can be calculated as the differences of corresponding multiplicities of the roots (see section 2).

The Adler–Moser polynomials provide a sequence of polynomial solutions to the Tkachenko equation [16, 19]. These polynomials can be constructed with the help of the following recurrence relation [19]:

\[
V_{k+1}(z) = V_{k-1}(z) \int \sigma_k \frac{V_k(z)}{V_{k-1}(z)} \, dz, \quad k \in \mathbb{N},
\]

(31)

The sequence begins with \( V_0(z) = 1, V_1(z) = z + c_1. \) Usually, the parameter \( c_1 \) is taken as zero. This choice results from the invariance of the Tkachenko equation under the transformation \( z \mapsto az + \beta. \) At each step, an arbitrary constant \( c_{k+1} \) is introduced in (31) and the parameter \( \sigma_{k+1} \) is chosen in such a way that the corresponding polynomial is monic. Another way to find the Adler–Moser polynomials in the explicit form is to use the Darboux transformations of the operator \( d^2/dz^2 \) [16, 19]. As a result, the Wronskian representation arises. Some other recurrence relations satisfied by these polynomials are given in [16, 21, 22]. Two successive Adler–Moser polynomials solve the Tkachenko equation, i.e. we may set \( \tilde{P}(z) = V_{k+1}(z) \) and \( \tilde{Q}(z) = V_k(z) \) in (13). As a consequence of our results (see section 2), we can obtain new solutions of equation (13). Suppose a pair \( P(z), Q(z) \) provides a solution of the Tkachenko equation; then, the functions

\[
\tilde{P}(z) = P^{\frac{dP}{d\tau}}(z)Q^{\frac{d\tau}{dz}}(z), \quad \tilde{Q}(z) = P^{\frac{dP}{d\tau}}(z)Q^{\frac{d\tau}{dz}}(z)
\]

(32)

also satisfy equation (13). In particular, we obtain that the functions

\[
\tilde{P}_k^{(1)}(z) = V_k^{\frac{dP}{d\tau}}(z)V_k^{\frac{d\tau}{dz}}(z), \quad \tilde{Q}_k^{(1)}(z) = V_k^{\frac{dQ}{d\tau}}(z)V_k^{\frac{d\tau}{dz}}(z), \quad k \in \mathbb{N} \cup \{0\},
\]

(33)

solve equation (13). If \( \Gamma \in \mathbb{Z}, \) then \( \tilde{P}_k^{(1)}(z) \) and \( \tilde{Q}_k^{(1)}(z) \) in (33) are the polynomials. Furthermore, let us consider an example. Suppose we study stationary equilibria of \( l_1 \) point vortices with circulation \( \Gamma_1 = \Gamma \) and \( l_2 \) point vortices with circulation \( \Gamma_2 = -2\Gamma. \) In our situation, we have only two groups of vortices and we set \( P_1(z) = S(z) \) and \( P_2(z) = T(z) \) (see (8) or (15) with \( \mu = 2 \)). The polynomials \( S(z) \) and \( T(z) \) describing these configurations satisfy the equation

\[
S_{zT} - 4S_{Tz} + 4ST_{zT} = 0.
\]

(34)

Note that the vortices with circulation \( \Gamma \) are situated at the roots of the polynomial \( S(z) \) and the vortices with circulation \( -2\Gamma \) are situated at the roots of the polynomial \( T(z). \) In new variables

\[
\tilde{P}(z) = S(z)T(z), \quad \tilde{Q}(z) = T^3(z),
\]

(35)

we rewrite equation (34) in the form (13). In order to obtain relation (35), we use expression (16) with \( \mu = 2. \) The polynomial solutions of equation (34) were studied in [18, 23–25]. Two neighboring polynomials from the sequences [18]

\[
S_{k+1}(z) = S_{k-1}(z) \int \gamma_{k+1} T_k^2(z) S_k^2(z) \, dz, \quad k \in \mathbb{N},
\]

\[
T_{k+1}(z) = T_{k-1}(z) \int \delta_{k+1} S_k(z) T_k^2(z) \, dz, \quad k \in \mathbb{N},
\]

(36)

i.e. \( S_k, T_k \) and \( S_{k+1}, T_{k+1} \), provide polynomial solutions of equation (34). The ‘initial conditions’ for these sequences are \( S_0(z) = 1, S_1(z) = z + s_1, T_0(z) = 1, T_1(z) = z + t_1. \) Again we can
Table 1. Polynomials from the sequences in (37).

| Polynomial | Form |
|------------|------|
| $\tilde{P}_0^{(2)}(z)$ | $z$ |
| $\tilde{P}_0^{(3)}(z)$ | $z^3 + t_2 z$ |
| $\tilde{P}_2^{(2)}(z)$ | $z^{10} + t_2 z^6 - 3 s t z^5 + s t z^4 - 45 z^2$ |
| $\tilde{Q}_0^{(2)}(z)$ | $z^3$ |
| $\tilde{Q}_0^{(3)}(z)$ | $z^3 + 3 t_2 z^4 + 3 t_2 z^3 + z^2$ |
| $\tilde{P}_2^{(3)}(z)$ | $z^{15} + 3 z^{11} - 12 s t z^{10} + 3 t_2 z^9 - 24 s t z^8$ |
| $\tilde{Q}_2^{(3)}(z)$ | $z^6 + 3 s t z^2 + 3 t_2 z^2 + t_2^2$ |

set $s_1 = 0$, $t_1 = 0$. Calculating the indefinite integrals in expressions (36), we introduce the integration constant $s_{k+1}$ in the first one and $t_{k+1}$ in the second. For convenience, the parameters $\gamma_{k+1}$, $\delta_{k+1}$ can be chosen in such a way that all the polynomials are monic. For more details and derivation of formula (36), see [18]. With the help of expression (16) with $\mu = 2$, we obtain that the Tkachenko equation admits the polynomial solutions of the form

$$
\tilde{P}_k(z) = S_k(z) T_{k-1}(z), \quad \tilde{Q}_k(z) = T_{k+1}(z), \quad k \in \mathbb{N} \cup \{0\},
$$

with $S_k(z)$, $T_{k+1}(z)$ given by (36); see also table 1.

For example, the following polynomials satisfy equation (34):

$$
S_3(z) = z^8 + 28 t_2 z^6 + 14 t_2^2 z^4 + 28 t_2^3 z^2 + s t z - 64 z^2, \quad T_2(z) = z^2 + t_2.
$$

In this expression, $t_2$ and $s_3$ are arbitrary constants. Substituting expressions (38) into relations given in (37), we see that the polynomials

$$
\tilde{P}_3(z) = z^{10} + \frac{33}{2} t_2 z^8 + \frac{98}{3} t_2^2 z^6 + 42 t_2^3 z^4 + s t z^3 + 21 t_2^4 z^2 + s t z - 7 t_2^5,
$$

$$
\tilde{Q}_3(z) = z^6 + 3 s t z^2 + 3 t_2 z^2 + t_2^2
$$

solve equation (13). Note that these polynomials are not included in the sequence of Adler–Moser polynomials. Indeed the Adler–Moser polynomials of the corresponding degrees are the following:

$$
V_3(z) = z^{10} + 15 c_2 z^7 + 7 c_2 z^5 + c_2 z^3 - 35 c_2 c_3 z^2 + 175 c_2^3 z + c_4 c_2 - \frac{7}{2} c_2^3,
$$

$$
V_5(z) = z^6 + 5 c_2 z^3 + c_2 z^3 - 5 c_2 z^3,
$$

where $c_2$, $c_3$, $c_4$ are the arbitrary constants. The roots of two neighboring polynomials from the sequences in (36) give positions of vortices with circulations $\Gamma$ (roots of $S_k(z)$ or $S_{k+1}(z)$) and $-\Gamma$ (roots of $T_k(z)$) in stationary equilibrium whenever the corresponding polynomials do not have multiple and common roots. Similarly, if the polynomials $V_{k+1}(z)$, $V_k(z)$ (see (31)) do not possess multiple and common roots, then their roots provide positions of vortices with circulations $\Gamma$ (roots of $V_{k+1}(z)$) and $-\Gamma$ (roots of $V_k(z)$) in stationary equilibrium. Several examples are plotted in figure 1.

Sometimes it is supposed that polynomials from the sequences in (36) provide unique polynomial solutions of equation (34) accurate to the freedom discussed above [23]. Let us consider an example and show that it is not the case. The polynomials

$$
S(z) = z^5 (z + 6) (z^2 + 9 z + 21 d^2), \quad T(z) = z^5 (z + 7 d)^2
$$

satisfy equation (34). Studying the structure of these polynomials with the help of the results of section 3 (see expression (17) with $\mu = 2$), we obtain that they describe equilibria of four
Figure 1. Plots of vortex positions in stationary equilibrium described by (a) polynomials (38) with $s_2 = 1$, $t_2 = 10$, $s_3 = 1$ and (b) polynomials (40) with $c_2 = 10$, $c_3 = 1$, $c_4 = 1$. Circles denote vortices with circulation $\Gamma_1$ and squares denote vortices with circulations $-2\Gamma_1$ in (a) and $-\Gamma_1$ in (b).

point vortices ($\delta \neq 0$) in the following arrangement:

$$
\begin{align*}
\Gamma_1 &= -5, \quad a_1^{(1)} = 0, \\
\Gamma_2 &= -3, \quad a_1^{(2)} = -7\delta, \\
\Gamma_3 &= 1, \quad a_1^{(3)} = \left( -\frac{9}{2} + \frac{\sqrt{3}}{2}i \right) \delta, \quad a_1^{(4)} = \left( -\frac{9}{2} - \frac{\sqrt{3}}{2}i \right) \delta.
\end{align*}
$$

We would like to mention that application of relation (18) gives the values of circulations in the form $\Gamma_1 = -5\Gamma$, $\Gamma_2 = -3\Gamma$, $\Gamma_3 = \Gamma$. The polynomials of the corresponding degrees from the sequences in (36) are given by

$$
\begin{align*}
S_3(z) &= z^8 + \frac{28}{3}t_2z^6 + 14t_2^2z^4 + 28t_2^3z^2 + s_3z - 7t_2^4, \\
T_4(z) &= z^7 + 7t_2z^5 + 35t_2^2z^3 + 7t_2^3z^2 - 35t_2^4z - \frac{5}{2}s_3 + t_4t_2,
\end{align*}
$$

where $t_2$, $s_3$, $t_4$ are arbitrary constants. We see that the polynomials given by (41) cannot be included in these sequences. Thus, we have constructed an alternative polynomial solution of equation (34). Note that while comparing the polynomials, one should take into account the invariant transformation $z \mapsto az + b$.

6. Properties of the generalized Tkachenko equation

Now, let us study the generalized Tkachenko equation (26). If $\tilde{P}(z)$ and $\tilde{Q}(z)$ in equation (26) are polynomials, then they have equal degrees. This fact can be obtained balancing the highest order terms in equation (26). A theorem similar to theorem 1 can be proved for polynomial solutions of equation (26).

**Theorem 2.** Any pair of polynomials $\tilde{P}(z)$, $\tilde{Q}(z)$ solving the generalized Tkachenko equation (26) gives an arrangement of point vortices in translating relative equilibrium.
The proof of theorem 2 is carried out by analogy with theorem 1.

Thus, we conclude that the roots of any polynomial solution of equation (26) provide positions of point vortices in translating relative equilibrium. The vortex circulations can be found as the difference of corresponding multiplicities of the roots (see sections 2, 4). It is well known that equation (26) possesses polynomial solutions given by the Adler–Moser and the modified Adler–Moser polynomials (see [12]). Let us note that the parameter $\lambda$ can be removed from equation (26) if we introduce the new variable $\xi = \lambda z$.

Frequently, it is stated that polynomial solutions of equation (26) have triangular degrees. Let us consider an example and show that this statement is not valid. Suppose we study positions for multi-vortex systems with circulations $/Gamma_1$ and $/Gamma_2$. In this paper, we have studied the problem of finding stationary and translating equilibrium generalizations of the Tkachenko equation. We hope that our results will be useful in further studying these equations.

Finally, we would like to mention that our approach (expressions (10) and (14), theorem 1) allows one to find additional polynomial solutions of the Tkachenko equation and the generalized Tkachenko equation (26). Suppose $\Gamma$ is integer: $\Gamma = m$, $m \in \mathbb{Z}$. The polynomials

$$P(z) = z + \frac{1}{\lambda_0}, \quad Q(z) = z$$

provide a solution of equation (44), which is in fact the generalized Tkachenko equation with the parameter $\lambda$ replaced by the parameter $\lambda_0$. Using formula (45), we see that the polynomials

$$P(z; \lambda) = P^{\frac{m}{\lambda_0}}(z; \lambda_0) Q^{\frac{m}{\lambda_0}}(z; \lambda_0), \quad \lambda_0 = \frac{\lambda}{\Gamma},$$

$$Q(z; \lambda) = P^{\frac{m}{\lambda_0}}(z; \lambda_0) Q^{\frac{m}{\lambda_0}}(z; \lambda_0), \quad \lambda_0 = \frac{\lambda}{\Gamma},$$

solve the generalized Tkachenko equation (26). Thus, for any solution $P(z; \lambda_0), Q(z; \lambda_0)$ of equation (44), transformation (45) gives a solution of the generalized Tkachenko equation (26). Suppose $\Gamma$ is integer: $\Gamma = m$, $m \in \mathbb{Z}$. The polynomials

$$P(z) = z + \frac{1}{\lambda_0}, \quad Q(z) = z$$

provide a solution of equation (44), which is in fact the generalized Tkachenko equation with the parameter $\lambda$ replaced by the parameter $\lambda_0$. Using formula (45), we see that the polynomials

$$P(z) = z^{\frac{m}{\lambda_0}} \left( z + \frac{m}{\lambda_0} \right)^{\frac{m}{\lambda_0}-1}, \quad Q(z) = z^{\frac{m}{\lambda_0}} \left( z + \frac{m}{\lambda_0} \right)^{\frac{m}{\lambda_0}-1}$$

solve the generalized Tkachenko equation (26). Let us take $m = 2$. In this case, the degree of the polynomials $P(z)$ and $Q(z)$ in (47) equals 4, and 4 is not a triangular number.

Finally, we would like to mention that our approach (expressions (10) and (14), theorem 1) allows one to find additional polynomial solutions of the Tkachenko equation and the generalized Tkachenko equation. We hope that our results will be useful in further studying these equations.

7. Conclusion

In this paper, we have studied the problem of finding stationary and translating equilibrium positions for multi-vortex systems with circulations $\Gamma_1, \ldots, \Gamma_N$. We have obtained that stationary equilibria of point vortices with an arbitrary choice of circulations can be described with the help of the Tkachenko equation, while translating relative equilibria of point vortices with arbitrary circulations can be constructed using a generalization of the Tkachenko equation. We have proved that roots of any pair of polynomials solving the Tkachenko equation and the generalized Tkachenko equation give positions of point vortices in stationary and translating relative equilibrium accordingly. These results remain valid even if the polynomials inside a pair possesses multiple or common roots.
It is often believed that the Adler–Moser polynomials provide unique polynomial solutions of the Tkachenko equation. We have shown that this statement does not hold. Usually, it is supposed that the polynomial solutions of the generalized Tkachenko equation have degrees that are triangular numbers. We have found alternative polynomial solutions of the latter equation, i.e. polynomial solutions with non-triangular degrees.

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