A new approach for the calculation of falling droplets from a cylindrical glass capillary based on force balance and velocity

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Abstract. This paper presents a new simple analytical method to estimate the properties of falling droplets without solving complex differential equations. The derivation starts from the balance of forces and uses Newton’s second law and the equations of motion to calculate the volume of growing and detaching droplets and the time between two successive droplets falling out of a thin cylindrical capillary of borosilicate glass. In this specific case the reservoir is located above the capillary and the hydrostatic pressure of the fluid level leads to drop formation times about one second. In the second part of this paper experimental results are presented to validate the introduced calculation method. It is shown that the new approach describes the measuring results within a deviation of ±6.2%. The third part of the paper sums up the advantages of the new approach and an outlook is given on how the research on this topic will be continued.

1. Introduction
In today’s microfluidic hanging and falling droplets are widely used, for example in medical dosing devices. These are used to measure small amounts of liquids for specific applications, for example in scientific laboratories for the preparation of different substances or in the clinical dosage of medication in infusions.
Tate [1] was the first to investigate the volume of falling droplets. He presented an approach starting from the balance of surface force and gravitational force to calculate the maximum mass of a falling droplet. Rayleigh [2] and Harkins & Brown [3] investigated the weight of falling droplets in their experiments. They discovered that the droplets are smaller than proposed by [1] and they expanded Tate’s approach by experimental correction factors based on the specific capillary length. The aim of the approaches presented in [1], [2] and [3] was to estimate the maximum droplet size under quasi-static conditions.
More detailed approaches based on differential equations are presented by Chesters, Eggers & Dupont and Clanet & Lasheras. Chesters [4] derived several equations for different parts of the shape of a droplet which enables him to plot different states in the droplet formation process. Eggers & Dupont [5] used the Navier-Stokes-Equations in cylindrical coordinates to simulate the shape of growing, necking and detaching droplets numerically and Clanert & Lasheras [6] focused on the transition from dripping to jetting. The approaches [4], [5] and [6] are on the one hand much more accurate and on the other hand much more complicated than those presented in [1], [2] and [3].
Comprehensively none of the above presented approaches provides an easy and approximately accurate method to calculate falling droplets. This gap will be closed with this paper by presenting a new simple method based on the balance of forces acting on the hanging droplet to estimate the volume and the time between two successive falling droplets.

2. The new calculation approach
Starting from Tate’s ideas [1] the balance of the forces acting on the droplet leads to an approach which can be divided up in three phases:

(i) In the first phase the upwards directed forces overcome the downwards directed forces and the droplet is hanging on the bottom of the capillary. The maximum size of a hanging droplet is reached in the equilibrium of forces.

(ii) In the second phase the downwards directed forces overcome the upwards directed forces. The resulting force accelerates the droplet in the direction of the gravitational constant and finally leads to the detachment in Phase 3. As long as the velocity of the droplet is slower than the velocity of the fluid streaming out of the capillary and into the droplet the growth of the droplet goes on, which is already described by Neumann & Seeliger [7].

(iii) If the velocity of the droplet overcomes the velocity of the fluid flow no more fluid can reach the drop and the falling drop has reached its maximum size. The fluid streaming out of the capillary until the final detachment of the droplet is already used to build a new droplet.

2.1. Phase 1
The force acting upwards on a hanging droplet is the surface force

\[ F_{sf} = 2 \cdot \pi \cdot R \cdot \sigma \]  

where \( R \) is the outer radius of the cylindrical capillary according to the observations presented in figure 2 and \( \sigma \) the surface tension. While the droplet grows hanging at the capillary, its shape changes from spherical to the typical shape of a hanging droplet and the surface force increases until a cylindrical transition from the capillary to the droplet is reached. This transition point represents a threshold which has to be overcome before the droplet detaches. Therefore the surface force is assumed to be maximum in equation (1).

The first force acting downwards on a hanging droplet is the gravity

\[ F_g = \rho \cdot V \cdot g \]  

where \( \rho \) is the density, \( V \) the volume of the hanging droplet and \( g \) the gravitational constant.

The second force is the pressure force

\[ F_p = p \cdot A_{cap} \]  

due to the pressure \( p \) in the droplet acting on the plane surface of the capillary

\[ A_{cap} = \pi \cdot (R^2 - r^2) \]  

which is covered by the droplet, with \( R \) being the outer radius of the capillary and \( r \) being the inner one. The capillary is assumed to be hydrophilic and the edge between the plane surface and the curved surface of the capillary is assumed to act as an geometrical stop for the fluid. Due to its immobility the capillary responds to this force by an opposite directed force which is considered here as a force acting downwards on a hanging droplet. Compared to [1] the pressure force is added in this approach and explains the smaller droplets detected by [2] and [3]. For
further simplification the pressure \( p \) inside the droplet is assumed to be constant over the droplet formation process though it changes with the increasing droplet volume according to the law of Young-Laplace [8]. To calculate the droplet properties an average pressure is used, computed with the maximum droplet pressure at the given capillary and the pressure in the droplet at the equilibrium of forces, assuming the shape of the droplet to be spherical. According to this assumptions the droplet volume is the only parameter changing with time and the equilibrium of forces can be used to calculate the droplet volume \( V_1 \) in the equilibrium state by an iterative procedure.

\[
V_1 = \frac{F_{sf} - F_p}{\rho \cdot g} = \frac{2 \cdot \pi \cdot R \cdot \sigma - p \cdot A_{cap}}{\rho \cdot g} \tag{5}
\]

Using the volumetric flow rate \( \dot{V}_0 \) into the droplet, which can be calculated by the law of Hagen-Poiseuille [9], the time \( t_1 \) needed to reach the equilibrium state starting from a completely filled capillary can be calculated.

\[
t_1 = \frac{V_1}{V_0} \tag{6}
\]

### 2.2. Phase 2

Phase 2 starts from the equilibrium state in which the droplet has already reached the volume \( V_1 \). Due to the ongoing flow into the droplets its volume continues increasing and the downwards directed forces starts to overcome the surface force. According to Newton’s second law the resulting acceleration

\[
a = \frac{dv}{dt} = \frac{F_g + F_p - F_{sf}}{m} \tag{7}
\]

of the droplet in the direction of the gravitational constant can be calculated with the mass \( m \) of the hanging droplet. The integration of equation (7) leads to the droplet velocity \( v \) depending on the time \( t \).

\[
v = g \cdot t - \frac{F_{sf} - F_p}{\rho \cdot V_0} \cdot \ln \left(1 + \frac{t}{t_1}\right) \tag{8}
\]

Computing the intersection of \( v \) with the inlet velocity \( v_0 \) leads to the time \( t_2 \) which describes the time elapsing in phase 2. \( t_2 \) enables the calculation of the volume \( V_2 \) for which the droplet volume is increasing in phase 2.

\[
V_2 = \dot{V}_0 \cdot t_2 \tag{9}
\]

### 2.3. Phase 3

Phase 3 deals with the final detachment of the droplet. As already described no further increase of the droplet volume has to be expected. An appraisal for the necking time \( t_3 \) is given by Eggers [10] and modified with experimental results by Clanet & Lasheras [6]. Using Eggers’ original approach

\[
t_3 = \sqrt{\frac{r^3 \cdot \rho}{\sigma}} \tag{10}
\]

with \( r \) being the radius of the cylindrical fluid jet or the neck of a droplet before the constriction begins, \( \rho \) being the density and \( \sigma \) being the surface tension, leads to a necking time about several milliseconds.

### 2.4. Summary

Summarizing the presented approach leads to the droplet volume

\[
V_{droplet} = V_1 + V_2 \tag{11}
\]
which consists of the volume $V_1$ for which the droplet is increasing in phase 1 and of the volume $V_2$ for which the droplet is increasing in the second phase.

The time

$$T = t_1 + t_2 + t_3$$

is needed to form the first droplet until its final detachment out of a completely filled capillary. To calculate the time between two droplets it has to be mentioned that the fluid streaming through the capillary in phase 3 already contributes to the volume of the next droplet. Because the calculation of $t_1$ starts from a completely filled capillary $t_3$ has to be subtracted from $t_1$ to calculate the time $\Delta T$ between two successive droplets.

$$\Delta T = (t_1 - t_3) + t_2 + t_3 = t_1 + t_2$$

3. Validation of the calculation method

3.1. Experimental setup

To validate the new calculation approach a measurement setup was created using a RaspberryPi microcontroller and a RaspberryPi camera module. This setup allows to record the droplet formation and detachment with up to 118 frames per second and a resolution of 640 x 480 pixels, slightly reduced because of the internal processing of the images by the RaspberryPi microcontroller. The sample with the capillary to test and the reservoir located above the capillary are fixed by a tripod clamp and illuminated by a LED with the power of 1 W. The RaspberryPi microcontroller is programmed using python and OpenCV and allows an automated evaluation of the recorded droplets using a graphical user interface. The setup is shown in figure 1, an exemplary droplet of water near its point of detachment is shown in figure 2.

Figure 1. The figure shows the measurement setup with the camera module in the blue box on the left, the cylindrical capillary and the spot of the LED on a diffuser on the right.

Figure 2. Photography of a droplet hanging on a cylindrical capillary near its point of detachment (Phase 3).

The experimental setup consists of a cylindrical capillary of borosilicate glass with an outer diameter of 880 $\mu$m, an inner diameter of 440 $\mu$m and a length of 20 mm. Different fluid levels, measured over the bottom of the capillary are used to vary the hydrostatic pressure acting on the flow through the capillary and setting the volumetric flow rate $\dot{V}_0$ due to the law of Hagen-Poiseuille [9]. In scope of this work four different levels between $h = 37.5$ mm and $h = 63$ mm with an accuracy of $\pm 0.25$ mm have been used, each with 17 to 70 analyzed droplets to satisfy statistical methods. The fluid used in the experiments is degassed water. Due to the significant change of the properties of water in the range of room temperature [11], the temperature of the fluid was monitored and the obtained information were used to correct the properties of water.
used in the calculations by linear interpolation according to the temperature at the time of the experiments.

Although the frame rate and the resolution of the camera are low compared to an industrial high-speed-camera, the described measurement setup provides accurate results. Inaccuracies can occur for example due to the required vertical orientation of the sample which has to be done manually by the operator. This has to be mentioned while interpreting the measurement values.

### 3.2. Results and discussion

Figure 3 and 4 show the results of the measurements and the calculations (dashed line). The error indicators at the measured values show the standard deviation of the measurements. The accuracy of the presented calculation is represented by the maximum deviation between the calculated and the measured values of ±6.2%.

![Figure 3](image3.png)

**Figure 3.** The figure sums up the measured droplet volumes for different fluid levels in the reservoir and compares them to the calculated values (dashed line).

![Figure 4](image4.png)

**Figure 4.** The figure compares the measured and calculated time between two successive droplets for different fluid levels in the reservoir. It shows the decreasing time needed to form a droplet with increasing hydrostatic pressure acting on the capillary.

Figure 3 and 4 validate the presented calculation approach. The accuracy of the measurement setup is shown by the standard deviations which are with a maximum value of 7% of the average values small compared to the average values.

The measurement values fit to a ±6.2% belt around the calculation results. Merely the values for the levels \( h = 37.5 \text{ mm} \) and \( h = 45.5 \text{ mm} \) in figure 3 differ from the calculated values for...
these fluid levels. This could be explained by the difficulties which occurred while recording the measurement values: The software analyzing the recorded pictures like the one shown in figure 2 has to walk through several steps to divide the capillary (the part of the picture which is not changing from one picture to the next) of the droplet. Due to the slow growth of the droplet the recognition of the capillary is partially incorrect and therefore the measurement results are also incorrect. The second thing to be mentioned regarding figure 3 is the trend shown by the curves. While the calculated values increase with the fluid level, the measured values stagnate at a fixed level. A final explanation for this phenomena can not yet be given, it might be explained by inaccuracies of the measurement or the calculation but further investigations are required.

4. Conclusions
In this paper a new approach was presented to estimate the droplet volume and the time between two successive droplets. The approach is validated by experimental results and corresponds to the accuracy of ±6.2%. This shows that it is in principle possible to estimate the droplet volume and the time between two successive droplets of dynamic droplet formation processes with little effort and acceptable accuracy without using complex differential equations.

5. Outlook: additional investigations done and further research topics
Though many insights have been gained so far, there are lots of topics which has to be investigated. There are some phenomena in figure 3 which could not be explained so far and which have to be explained in further investigations. Changing the geometric parameters of the capillary will change the droplet formation process as well as changing the capillary material or the fluid used to form the droplets, for example due to hydrophilic or hydrophobic behavior of the material-fluid-combination. Using glycerol in the same capillaries as described in this work, for example, leads to droplet formation times in the scale of minutes.

In addition to the described additional investigations an extension of the range of fluid levels which leads to different flow rates into the droplet has to be investigated to get the validity area of the presented approach. To complete the validation of the presented approach numerical simulations have to be performed, for example using the CFD-tool OpenFOAM.

Finally it has to be mentioned that Neumann & Seeliger [7] detected a maximum droplet size in their experiments where the droplet volume stops increasing with increasing flow rate into the droplet and starts decreasing again due to a further increase of the flow rate. This effect is not yet considered in the presented approach and has to be taken into account in future investigations, maybe being responsible for the phenomena in figure 3.

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