HILL’S EQUATION WITH SMALL FLUCTUATIONS:
CYCLE TO CYCLE VARIATIONS AND STOCHASTIC PROCESSES

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Abstract

Hill’s equations arise in a wide variety of physical problems, and are specified by a natural frequency, a periodic forcing function, and a forcing strength parameter. This classic problem is generalized here in two ways: [A] to Random Hill’s equations which allow the forcing strength $q_k$, the oscillation frequency $\lambda_k$, and the period $(\Delta \tau)_k$ of the forcing function to vary from cycle to cycle, and [B] to Stochastic Hill’s equations which contain (at least) one additional term that is a stochastic process $\xi$. This paper considers both random and stochastic Hill’s equations with small parameter variations, so that $p_k = q_k - \langle q_k \rangle$, $\ell_k = \lambda_k - \langle \lambda_k \rangle$, and $\xi$ are all $O(\epsilon)$, where $\epsilon \ll 1$. We show that random Hill’s equations and stochastic Hill’s equations have the same growth rates when the parameter variations $p_k$ and $\ell_k$ obey certain constraints given in terms of the moments of $\xi$. For random Hill’s equations, the growth rates for the solutions are given by the growth rates of a matrix transformation, under matrix multiplication, where the matrix elements vary from cycle to cycle. Unlike classic Hill’s equations where the parameter space (the $\lambda-q$ plane) displays bands of stable solutions interlaced with bands of unstable solutions, random Hill’s equations are generically unstable. We find analytic approximations for the growth rates of the instability; for the regime where Hill’s equation is classically stable, and the parameter variations are small, the growth rate $\gamma = O(p_k^2) = O(\epsilon^2)$. Using the relationship between the $(\ell_k, p_k)$ and the $\xi$, this result for $\gamma$ can be used to find growth rates for stochastic Hill’s equations.
I. INTRODUCTION

Hill’s equation is a second order periodic differential equation that arises in many physical problems [22]. In addition to its original application for lunar orbits [15], Hill’s equation describes celestial dynamics [7,8,21], orbit instabilities in dark matter halos [5], particle production at the end of the inflationary epoch in the early universe [17,18,19], the motion of a jogger’s ponytail [16], and many other physical systems. The goal of this paper is to consider the addition of random perturbations to Hill’s equation and thereby generalize our current understanding. Toward this end, we adopt the following definitions:

Definition 1.1: A Classical Hill’s equation has the form

\[
\frac{d^2 y}{dt^2} + [\lambda + q\dot{Q}(t)]y = 0,
\]  

(1)

where the parameters \((\lambda, q)\) are constant and where the barrier shape function \(\dot{Q}(t)\) is periodic, so that \(\dot{Q}(t + \Delta \tau) = \dot{Q}(t)\), where \(\Delta \tau\) is the period. For the sake of definiteness we take \(\Delta \tau = \pi\) and normalize the barrier shape function so that \(\int_0^\pi \dot{Q}(t) dt = 1\).

Definition 1.2: A Hill’s Equation is said to be Cycle-to-Cycle Random (CC Random) if the differential equation is deterministic over each interval of the periodic function \(\dot{Q}\), but the values of the forcing strength \(q_k\) and/or oscillation frequency \(\lambda_k\) take on different values for each periodic interval. The CC Random Hill’s equation has the form

\[
\frac{d^2 y}{dt^2} + [\lambda_k + q_k\dot{Q}(t)]y = 0,
\]  

(2)

where the parameters \((\lambda_k, q_k)\) vary from cycle to cycle, and are drawn independently from well-defined distributions. The index \(k\) labels the cycle.

In this work the period \(\Delta \tau\) is considered fixed; one can show that cycle to cycle variations in \(\Delta \tau\) can be scaled out of the problem and included in the variations of the \((\lambda_k, q_k)\) by changing their distributions accordingly (see Theorem 1 of Ref. [2]).

Periodic differential equations in this class can be described by a discrete mapping of the coefficients of the principal solutions from one cycle to the next. The transformation matrix takes the form

\[
\mathcal{M}_k = \begin{bmatrix}
    h_k & (h_k^2 - 1)/g_k \\
    g_k & h_k \\
\end{bmatrix},
\]  

(3)

where the subscript denotes the cycle. The matrix elements for the \(kth\) cycle are given by

\[
h_k = y_1(\pi) \quad \text{and} \quad g_k = \dot{y}_1(\pi),
\]  

(4)

where \(y_1\) and \(y_2\) are the principal solutions for that cycle. The index \(k\) indicates that the quantities \((\lambda_k, q_k)\), and hence the solutions \((h_k, g_k)\), vary from cycle to cycle. Throughout
this work, the random variables are taken to be independent and identically distributed (iid).

Note that the matrix in equation (3) has only two independent elements (not four). Since the Wronskian of the original differential equation (2) is unity, the determinant of the matrix map (3) must also be unity, and this constraint eliminates one independent element. In addition, this paper specializes to the case where the periodic functions $\hat{Q}(t)$ are symmetric about the midpoint of the period. This property implies that $y_1(\pi) = \dot{y}_2(\pi) = h_k$, which eliminates a second independent matrix element [1,22]. These two constraints imply that $y_2(\pi) = (h_k^2 - 1)/g_k$, resulting in the form for the matrix given by equation (3).

The growth rates for Hill’s equation (2) are determined by the growth rates for matrix multiplication of the matrices $\mathcal{M}_k$ given by equation (3). Here we denote the product of $N$ such matrices as $\mathcal{M}^{(N)}$, and the growth rate $\gamma$ is defined by

$$\gamma = \lim_{N \to \infty} \frac{1}{N} \log ||\mathcal{M}^{(N)}||. \quad (5)$$

This result is independent of the choice of the norm $|| \cdot ||$, and the limit exists almost surely, as shown in previous work [12,13,20].

**Definition 1.3:** A Stochastic Hill’s Equation is a Hill’s equation with the inclusion of one or more additional terms that is a stochastic process. In this paper, the stochastic Hill’s equation is written as a Langevin equation. For example, one can write stochastic Hill’s equations in the form

$$\frac{d^2y}{dt^2} + [\lambda + q\hat{Q}(t)]y = \xi, \quad \text{(6)}$$

or

$$\frac{d^2y}{dt^2} + [\lambda + (q + \xi)\hat{Q}(t)]y = 0, \quad \text{(7)}$$

where $\xi$ is a stochastic process and $(\lambda, q)$ are constant.

Note that in the stochastic Hill’s equation, we can evaluate the stochastic term $\xi$ using the calculus of either Stratonovich or Ito. For purposes of this work, however, we only require that the process $\xi$ has zero mean and finite variance.

The goal of this paper is to study both the CC Random Hill’s equation and the Stochastic Hill’s equation in the limit where the parameter variations and/or the stochastic terms are small. For the CC Random Hill’s equation, the parameters are rewritten in the form

$$\lambda_k = \lambda + \ell_k \quad \text{and} \quad q_k = q + p_k, \quad \text{(8)}$$

so that $(\lambda, q)$ are constant whereas the $\ell_k$ and $p_k$ are small and are allowed to vary from cycle to cycle. The distributions of the $\ell_k$ and the $p_k$ are assumed to have zero mean $\langle \ell_k \rangle = 0 = \langle p_k \rangle$ and finite second moments $\langle \ell_k^2 \rangle$ and $\langle p_k^2 \rangle$. 

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For both the CC Random and the stochastic Hill’s equation, the solution over a given cycle (period of the function \( \hat{Q}(t) \)) can be expanded such that

\[
y(t) = y_0(t) + y_k(t),
\]

(9)

where \( y_0 \) obeys the Classical Hill’s equation (11), and is the same for every cycle. The correction term \( y_k(t) \) is, in general, different for every cycle, due to the parameter variations \( (\ell_k, p_k) \) for the CC Random Hill’s equation and due to different realizations of the stochastic process \( \xi \) for the Stochastic Hill’s equation (different samplings of the process).

Next we introduce an ordering parameter \( \epsilon \ll 1 \) such that

\[
y_k = \mathcal{O}(\epsilon), \quad \ell_k = \mathcal{O}(\epsilon), \quad p_k = \mathcal{O}(\epsilon), \quad \text{and} \quad \xi = \mathcal{O}(\epsilon).
\]

(10)

The results of this paper are correct to leading order in \( \epsilon \).

**Remark:** The subscripts ‘1’ and ‘2’ on the function \( y(t) \) denote the two principal solutions of any Hill’s equation for a given cycle, i.e., for given values of the parameters. The subscripts ‘0’ and ‘\( k \)’ on \( y(t) \) denote the leading order and first order parts (in \( \epsilon \)) of the function, respectively (see equation (9)).

To place this work in context, we note that a number of previous studies have considered Mathieu’s equation, a particular type of Hill’s equation, with stochastic forcing [25,26]. These papers show that stochastic fluctuations can lead to instability in otherwise stable systems. Other related work includes random operators in the Schrödinger wave equation in quantum mechanics and population variations in biomathematics [6,10,23], and stability studies of the quasi-periodic Mathieu equation [28]. This body of existing work does not, in general, provide closed-form expressions for the growth rates of the differential equations. Our previous work has studied the CC Random Hill’s equation in the regime where the forcing strengths are large and the solutions are highly unstable [2], for the limit where the periodic functions \( \hat{Q}(t) \) become (periodic) delta functions [3], and for more general cases [4]; here we find closed-form expressions for the growth rates of corresponding random matrices (given by equation (3)). Examples of such analytic results are rare in the literature – the classic papers [9,12,13] show the existence of growth rates, but present few examples (see also Ref. [24]).

The main goal of this present work is to build a bridge between the Stochastic Hill’s equation and the CC Random Hill’s equation. This paper focuses on the regime of small fluctuations as defined through equation (10). We derive the conditions necessary for the two types of Hill’s equations to have the same growth rates (Theorem 2.3). We also find analytic expressions for the matrix elements (Theorem 2.1) and the growth rates...
(Theorem 2.2) for CC Random Hill’s equations. The combination of these three results provides analytic expressions for the growth rates for Stochastic Hill’s equations. Section III presents a special case of the problem where the natural oscillation frequency \( \lambda \) is constant and the forcing parameters \( q_k \) are small but fluctuating from cycle to cycle. The paper concludes in Section IV with a summary and short discussion of the results; representative applications are described in the appendices.

II. HILL’S EQUATION WITH SMALL FLUCTUATIONS

In this section, we consider the case where the parameters of Hill’s equation are arbitrary, but the variations from cycle to cycle and/or the amplitude of the stochastic process are small.

**Theorem 2.1:** Consider the CC Random Hill’s equation (2) with the ordering of equation (10). Let \( h \) and \( g \) denote the matrix elements of the map (3), and let \( h_0 \) and \( g_0 \) denote the matrix elements in the absence of fluctuations \( (\ell_k = 0 = p_k) \). Provided that the zeroth order matrix elements \( h_0 \) and \( g_0 \) are nonvanishing, the matrix elements \( h \) and \( g \) are given by

\[
h = h_0 - \ell_k X - p_k Y + \mathcal{O}(\epsilon^2) \quad \text{and} \quad g = g_0 - \ell_k W - p_k Z + \mathcal{O}(\epsilon^2),
\]

where \( X, Y, W, Z \) are integral quantities that are constant from cycle to cycle:

\[
X = \frac{1}{2g_0} \int_0^{\pi} \left( (h_0^2 - 1) y_{01}^2 - g_0^2 y_{02}^2 \right) dt, \quad Y = \frac{1}{2g_0} \int_0^{\pi} \hat{Q}(t) \left( (h_0^2 - 1) y_{01}^2 - g_0^2 y_{02}^2 \right) dt,
\]

\[
W = \frac{1}{2h_0} \int_0^{\pi} \left( (h_0^2 + 1) y_{01}^2 - g_0^2 y_{02}^2 \right) dt, \quad Z = \frac{1}{2h_0} \int_0^{\pi} \hat{Q}(t) \left( (h_0^2 + 1) y_{01}^2 - g_0^2 y_{02}^2 \right) dt.
\]

**Proof:** By definition, the leading order part of the solution \( y_0 \) obeys the Classical Hill’s equation in the form of equation (11), where \( \lambda \) and \( q \) are constant. To all orders (in \( \epsilon \)), and for both principle solutions, the correction \( y_k \) obeys a differential equation of the form

\[
\frac{d^2 y_k}{dt^2} + \left( \lambda + \ell_k + (q + p_k) \hat{Q} \right) y_k + \left[ \ell_k + p_k \hat{Q} \right] y_0 = 0,
\]

where \( y_0 \) is the zeroeth order part of the principal solution. To leading order in \( \epsilon \), this equation reduces to the form

\[
\frac{d^2 y_k}{dt^2} + \left[ \lambda + q \hat{Q} \right] y_k + \left[ \ell_k + p_k \hat{Q} \right] y_0 + \mathcal{O}(\epsilon^2) = 0.
\]

For the remainder of this argument, we neglect terms that are \( \mathcal{O}(\epsilon^2) \), so that the results are correct (only) to first order in \( \epsilon \). After multiplying equation (14) by \( y_0 \) and integrating
over one cycle, the expression becomes
\[
\int_0^\pi \frac{d^2 y_k}{dt^2} y_0 dt + \int_0^\pi \left[ \lambda + q \dot{Q} \right] y_0 y_k dt + \int_0^\pi \left[ \ell_k + p_k \dot{Q} \right] y_0^2 dt = 0 , \tag{15}
\]
where we ignore higher order terms. By integrating the first term by parts, applying the boundary conditions at \( t = 0 \), and using the fact that \( y_0 \) satisfies the zeroth order equation, we obtain
\[
y_0(\pi) y_k(\pi) - y_0(\pi) y_k(\pi) + \ell_k \int_0^\pi y_0^2 dt + p_k \int_0^\pi \dot{Q} y_0^2 dt = 0 . \tag{16}
\]
This equation holds for both cases, i.e., where \( y = y_0 + y_k \) is the first or the second principal solution. Using the definitions of the matrix elements, we obtain two equations for two unknowns (note that we suppress the index \( k \) as convenient to simplify the notation)
\[
h_0 g - g_0 h + K_1 = 0 , \tag{17}
\]
and
\[
\frac{h_0^2 - 1}{g_0} h - h_0 \frac{h_0^2 - 1}{g_0} + K_2 = 0 , \tag{18}
\]
where the integrals \( K_1 \) and \( K_2 \) are defined by
\[
K_1 \equiv \int_0^\pi \left[ \ell_k + p_k \dot{Q} \right] y_0^2 dt \quad \text{and} \quad K_2 \equiv \int_0^\pi \left[ \ell_k + p_k \dot{Q} \right] y_0^2 dt . \tag{19}
\]
The matrix element \( h \) is thus given by the quadratic equation
\[
h^2 + h \left[ \frac{h_0^2 - 1}{g_0} K_1 - g_0 K_2 \right] + K_1 K_2 - h_0^2 = 0 , \tag{20}
\]
which has solution
\[
h = \pm \left[ h_0^2 - K_1 K_2 + \frac{1}{4} \left( \frac{h_0^2 - 1}{g_0} K_1 - g_0 K_2 \right)^2 \right]^{1/2} - \frac{1}{2} \left( \frac{h_0^2 - 1}{g_0} K_1 - g_0 K_2 \right) . \tag{21}
\]
We choose the + solution so that \( h \to h_0 \) in the limit \( \ell_k, p_k \to 0 \). This expression can be simplified by keeping only the leading order terms in \( \epsilon \), i.e.,
\[
h = h_0 - \frac{1}{2} \left( \frac{h_0^2 - 1}{g_0} K_1 - g_0 K_2 \right) + \mathcal{O}(\epsilon^2) . \tag{22}
\]
A similar result can be obtained for \( g \), i.e.,
\[
g = g_0 - \frac{1}{2h_0} \left[ (h_0^2 + 1) K_1 - g_0^2 K_2 \right] + \mathcal{O}(\epsilon^2) . \tag{23}
\]
Since we want to isolate the dependence of the results on the parameters \( \ell_k \) and \( p_k \) that vary from cycle to cycle, the matrix elements can be rewritten to take the forms given in equations (11) and (12). •
Keep in mind that only the \((\ell_k, p_k)\) vary from cycle to cycle. As a result, the parameters \(X, Y, W, Z\) are constant, i.e., the same for all cycles.

Alternatively, we can define the moment integrals
\[
I_j = \int_0^\pi y_{0j}^2 dt \quad \text{and} \quad J_j = \int_0^\pi \hat{Q}(t)y_{0j}^2 dt, \tag{24}
\]
where \(j = 1, 2\), and can write the parameters \(X, Y, W, Z\) in the form
\[
X = \frac{1}{2g_0} \left[ (h_0^2 - 1) I_1 - g_0^2 J_2 \right], \quad Y = \frac{1}{2g_0} \left[ (h_0^2 - 1) J_1 - g_0^2 J_2 \right],
\]
\[
W = \frac{1}{2h_0} \left[ (h_0^2 + 1) I_1 - g_0^2 J_2 \right], \quad \text{and} \quad Z = \frac{1}{2h_0} \left[ (h_0^2 + 1) J_1 - g_0^2 J_2 \right]. \tag{25}
\]

**Remark:** Note that this perturbation scheme fails in the limit where the leading order principal solutions vanish, i.e., when either \(h_0 \to 0\) or \(g_0 \to 0\).

The following results consider the case where \(|h_0| < 1\) so that the solutions are classically stable (i.e., stable in the absence of fluctuations in the parameter values). For the case of nonzero parameter fluctuations, however, the solutions grow exponentially.

**Theorem 2.2:** For a CC Random Hill’s equation in the regime where the leading order matrix element \(|h_0| < 1\), the solutions are stable in the limit \(\ell_k, p_k \to 0\). In general, the solutions are unstable with growth rate
\[
\gamma \approx \frac{1}{2} \left(1 - h_0^2\right) \left\{ \left[ \frac{h_0}{1 - h_0} X + \frac{1}{g_0} W \right]^2 \langle \ell_k^2 \rangle + \left[ \frac{h_0}{1 - h_0} Y + \frac{1}{g_0} Z \right]^2 \langle p_k^2 \rangle \right\} + O\left(\epsilon^3\right), \tag{26}
\]
where the angular brackets denote averages over the parameter distributions, and where the integral quantities \(X, Y, Z, W\) are defined by equations (12).

**Proof:** In the limit \(\ell_k, p_k \to 0\), the matrix element \(h \to h_0\). Since \(|h_0| < 1\), the solutions are stable [22]. For nonzero fluctuations, the transfer matrix \([3]\) takes the form of an elliptical rotation matrix \([4]\), which can be written in the form
\[
\mathcal{M}_k = \mathcal{E}(\theta_k, L_k) = \begin{bmatrix} \cos \theta_k & -L_k \sin \theta_k \\ (1/L_k) \sin \theta_k & \cos \theta_k \end{bmatrix}, \tag{27}
\]
where \(\cos \theta_k = h_k\) and \(L_k = (\sin \theta_k)/g_k\). Theorem 4 of Ref. [4] shows that the growth rate for multiplication of elliptical rotation matrices of the form (27), and hence the growth rate for the CC Random Hill’s equation under consideration, can be written
\[
\gamma = \frac{1}{2} \langle \sin^2 \theta_k \rangle \langle \eta_k^2 \rangle + O\left(\epsilon^3\right), \tag{28}
\]
where the perturbation \(\eta_k\) is defined by \(L_k = L_0(1 + \eta_k)\).
The length parameter $L_k$ of the elliptical rotation is given by

$$L_k = \frac{(1 - h_k^2)^{1/2}}{g_k}.$$  \hspace{1cm} (29)

Using the expressions for the matrix elements from equation (11) in the definition (29) of $L_k$, and keeping only the leading order terms in $\epsilon$, we find

$$L_k = \frac{(1 - h_0^2)^{1/2}}{g_0} \left\{ 1 + \frac{h_0}{1 - h_0^2} [\ell_k X + p_k Y] + \frac{1}{g_0} [\ell_k W + p_k Z] \right\} + \mathcal{O} (\epsilon^2),$$  \hspace{1cm} (30)

where $X, Y, Z, W$ are the integral quantities defined by equations (12) or (25). We can thus write the length parameter $L_k$ in the form

$$L_k = L_0 (1 + \eta_k),$$  \hspace{1cm} (31)

where $L_0 = \frac{(1 - h_0^2)^{1/2}}{g_0}$ and

$$\eta_k = \frac{h_0}{1 - h_0^2} [\ell_k X + p_k Y] + \frac{1}{g_0} [\ell_k W + p_k Z].$$

Since we have defined the parameters $\ell_k$ and $p_k$ (which vary from cycle to cycle) to have zero mean, it follows that $\langle \eta_k \rangle = 0$. The quantity that appears in the growth rate from equation (28) is the expectation value $\langle \sin^2 \theta_k \rangle \langle \eta_k^2 \rangle$. The perturbation $\eta_k = \mathcal{O}(\epsilon)$, whereas to leading order $\sin^2 \theta_k = 1 - h_0^2 + \mathcal{O}(\epsilon)$. As a result, the growth rate can be written in the form

$$\gamma = \frac{1}{2} (1 - h_0^2) \left\langle \left( \ell_k \left[ \frac{h_0}{1 - h_0^2} X + \frac{1}{g_0} W \right] + p_k \left[ \frac{h_0}{1 - h_0^2} Y + \frac{1}{g_0} Z \right] \right)^2 \right\rangle + \mathcal{O} (\epsilon^3).$$  \hspace{1cm} (32)

The random variables $\ell_k$ and $p_k$ are independently distributed, so that averages over the distributions can be rewritten in the form

$$\left\langle \left( \ell_k \left[ \frac{h_0}{1 - h_0^2} X + \frac{1}{g_0} W \right] + p_k \left[ \frac{h_0}{1 - h_0^2} Y + \frac{1}{g_0} Z \right] \right)^2 \right\rangle =$$

$$\left[ \frac{h_0}{1 - h_0^2} X + \frac{1}{g_0} W \right]^2 \langle \ell_k^2 \rangle + \left[ \frac{h_0}{1 - h_0^2} Y + \frac{1}{g_0} Z \right]^2 \langle p_k^2 \rangle.$$  \hspace{1cm} (33)

The growth rate $\gamma$ thus has the form given by equation (26). 

**Theorem 2.3:** Consider a Stochastic Hill’s equation of the form (6) and a CC Random Hill’s equation of the form (2), where the parameters have the forms $\lambda + \ell_k$ and $q + p_k$. Both the parameter variations $(\ell_k, p_k)$ and the stochastic process $\xi$ are $\mathcal{O}(\epsilon)$. To leading order in $\epsilon$, the growth rates of the solutions of the two types of differential equations are equivalent when the parameter variations $(\ell_k, p_k)$ are chosen according to

$$\ell_k = \frac{J_2 \Xi_1 - J_1 \Xi_2}{I_1 J_2 - I_2 J_1} \quad \text{and} \quad p_k = \frac{I_1 \Xi_2 - I_2 \Xi_1}{I_1 J_2 - I_2 J_1},$$  \hspace{1cm} (34)
where the integrals \( I_j \) and \( J_j \) are defined by equation (24), and where the \( \Xi_j \) are integrals of the stochastic process defined by

\[
\Xi_1 \equiv \int_0^\pi y_{01}\xi dt \quad \text{and} \quad \Xi_2 \equiv \int_0^\pi y_{02}\xi dt .
\]  

(35)

The growth rates \( \gamma \) are given by equation (26).

**Proof:** The growth rates of both types of differential equations can be written in terms of the growth rates of the transfer matrix (under matrix multiplication). The growth rates will be the same if the matrix elements are the same (see equation (3)).

The Stochastic Hill’s equation (6) has a solution of the form given by equation (9), where \( y_0(t) \) is the solution to equation (6) for the case where the right hand side vanishes. We consider the stochastic process \( \xi \) and the correction \( y_k \) to be \( O(\epsilon) \). To consistent order, \( y_k(t) \) must obey the equation

\[
\frac{d^2y_k}{dt^2} + [\lambda + q\hat{Q}(t)]y_k = \xi ,
\]  

(36)

If we multiply the above equation by \( y_0(t) \) and then integrate over one period, from \( t = 0 \) to \( t = \pi \), we obtain the result

\[
y_0(\pi)\dot{y}_k(\pi) - \dot{y}_0(\pi)y_k(\pi) = \int_0^\pi y_0\xi dt .
\]  

(37)

Now compare this result to that obtained from the CC Random Hill’s equation where the parameters vary from cycle to cycle,

\[
y_0(\pi)\dot{y}_k(\pi) - \dot{y}_0(\pi)y_k(\pi) + \ell_k \int_0^\pi y_0^2 dt + p_k \int_0^\pi y_0^2\hat{Q}dt = O(\epsilon^2) .
\]  

(38)

The matrix elements of the transfer matrices will be the same if equations (37) and (38) are equal. But we obtain equations of these forms for both principle solutions, so this requirement results in two coupled equations

\[
\ell_k I_1 + p_k J_1 = \Xi_1 \quad \text{and} \quad \ell_k I_2 + p_k J_2 = \Xi_2 ,
\]  

(39)

where where \( I_j \) and \( J_j \) are the moments defined in equation (24) and the \( \Xi_j \) are moments of the stochastic process defined through equations (35).

Equations (39) provide two equations for two unknowns, and can be solved to find the perturbative quantities \( \ell_k \) and \( p_k \). We thus find the constraint of equation (34) as claimed. The moment integrals \( I_j \) and \( J_j \) are defined by the solutions to the classical Hill’s equation and are the same for all cycles. On the other hand, the moments \( \Xi_j \) of the stochastic process will, in general, have different values for each realization (over each cycle labeled
by the index \( k \). For a given cycle, equations (34) thus define the values of the \( \ell_k \) and \( p_k \) that make the matrix elements for the CC Random Hill’s equation that same as those for the Stochastic Hill’s equation. This procedure thus results in distributions for the \( \ell_k \) and \( p_k \). Finally, with the distributions of the (equivalent) \( \ell_k \) and \( p_k \) specified, the growth rate \( \gamma \) is given by equation (26).

**Corollary:** Consider a stochastic Hill’s equation in the alternate form

\[
\frac{d^2 y}{dt^2} + [\lambda + (q + \xi) \hat{Q}(t)]y = 0.
\]

(40)

The solutions to this differential equation will have the same growth rate as that of the CC Random Hill’s equation when the parameter variations \( \ell_k, p_k \) are chosen according to the constraints of equation (34), where the moments \( \Xi_j \) have the alternate form

\[
\Xi_1 = -\int_0^\pi \hat{Q}y_0^2 \xi dt \quad \text{and} \quad \Xi_2 = -\int_0^\pi \hat{Q}y_0^2 \xi dt.
\]

(41)

**Proof:** The proof is analogous to that of the previous result.

**Remark:** In the limit where the stochastic process \( \xi \) has a correlation time \( \tau_c \to 0 \), the integrals \( \Xi_j \) in the numerators of equations (34) vanish. As a result, the equivalent perturbations vanish,

\[
\lim_{\tau_c \to 0} \ell_k = 0 \quad \text{and} \quad \lim_{\tau_c \to 0} p_k = 0.
\]

(42)

In this limit, the growth rate of either differential equation will be the same as that of the corresponding Classical Hill’s equation.

**III. HILL’S EQUATION WITH SMALL FORCING PARAMETERS**

This section considers the CC Random Hill’s equation (2) in the limit where the natural oscillation frequency \( \lambda \) is constant and the forcing parameter \( q_k = \mathcal{O}(\epsilon) \) where \( \epsilon \ll 1 \). This application is thus a special case of the general problem considered in the previous section.

We first find solutions for the matrix elements \( h = y_1(\pi) \) and \( g = \dot{y}_1(\pi) \). The solutions can be expanded in orders of \( q_k = \mathcal{O}(\epsilon) \), where the zeroth order solutions are given by

\[
y_{01}(t) = \cos \sqrt{\lambda} t, \quad \text{and} \quad y_{02}(t) = \frac{1}{\sqrt{\lambda}} \sin \sqrt{\lambda} t.
\]

(43)

The first order parts of the solutions obey the equation

\[
\frac{d^2 y_k}{dt^2} + \lambda y_k + q_k \hat{Q}y_0 = \mathcal{O}(\epsilon^2),
\]

(44)

where the parameter \( q_k = \mathcal{O}(\epsilon) \). Next we multiply by \( y_0 \) and integrate over the interval \([0, \pi]\). After integrating by parts twice, the expression becomes

\[
[y_0 \dot{y}_k - \dot{y}_0 y_k]_0^\pi + q_k \int_0^\pi \hat{Q}y_0^2 dt = \mathcal{O}(\epsilon^2).
\]

(45)
Note that this form holds for the perturbations for both the first and second principal solutions. Since the zeroth order solutions (given by equation (43)) satisfy the boundary conditions, the perturbations \( y_k \) and their derivatives \( \dot{y}_k \) must vanish at \( t = 0 \). The solution thus becomes

\[
y_{0j}(\pi)\dot{y}_{kj}(\pi) - \dot{y}_{0j}(\pi)y_{kj}(\pi) + q_k J_j = 0,
\]

where the index \( j = 1,2 \) determines the principal solution and the moment integrals \( J_j \) are defined by equation (24). The matrix elements \( h \) and \( g \) are given by

\[
h = y_1(\pi) = \cos \varphi + y_{k1}(\pi) \quad \text{and} \quad g = \dot{y}_1(\pi) = -\sqrt{\lambda} \sin \varphi + \dot{y}_{k1}(\pi),
\]

where we have defined \( \varphi \equiv \sqrt{\lambda} \pi \) to simplify the zeroth order parts. Using these forms in the two equations (46), we find two expressions for the two unknown matrix elements \( h \) and \( g \):

\[
g \cos \varphi + h \sqrt{\lambda} \sin \varphi + q_k J_1 = O(\epsilon^2),
\]

and

\[
h \frac{\sin \varphi}{\sqrt{\lambda}} - \frac{h^2 - 1}{g} \cos \varphi + q_k J_2 = O(\epsilon^2).
\]

The matrix elements are thus given by

\[
h = \cos \varphi - \frac{q_k}{2\sqrt{\lambda}} \sin \varphi + O(\epsilon^2),
\]

and

\[
g = -\sqrt{\lambda} \sin \varphi + \frac{q_k}{\cos \varphi} \left( \frac{1}{2} \sin^2 \varphi - J_1 \right) + O(\epsilon^2).
\]

Note that the expression for \( h \) is the same as the full solution for \( h \) in the limiting case where the barrier shape \( \hat{Q} \) is a delta function [3]. In this same limit, the integral \( J_1 = \cos^2(\varphi/2) \), and the expression for \( g \) also gives the exact result. Notice also that the results of Theorem 2.1 reduce to equations (50) and (51) in the limit \( \ell_k \to 0 \) and \( q + p_k \to q_k \) (where equations 43 specify the zeroth order solutions).

Figure 1 illustrates the fidelity of the first order approximation scheme. The figure shows the matrix element \( h \) as a function of \( \lambda \) using the exact (numerically determined) form and the first order expression from equation (50). For this illustration, the parameter \( q_k \) has constant value \( q_k = 1/2 \). Note that the first order expression provides a good estimate. Smaller \( q_k \) values will result in an even better approximation.
Figure 1: Matrix element $h$ as a function of natural oscillation frequency $\lambda$ for Hill’s equation with small forcing parameter $q_k = 1/2$. Solid curve shows numerically determined result. Dotted curve shows the result for the approximation of equation (50), which is correct to leading order in $q_k$. Dashed curve shows the result for the approximation of equation (52), which includes some higher order terms (see text).
For completeness, we note the following: Although the approximation (from equation (45)) ignores terms that are second order in $q_k$, after we make that approximation we can find the corresponding solution keeping all terms, i.e., we don’t have to get rid of the remaining second order terms. We thus find the generalized solutions

$$h = -\frac{q_k}{2\sqrt{\lambda}} \sin \varphi \pm \left[ \cos^2 \varphi + \frac{q_k^2}{2\lambda} \sin^2 \varphi - q_k^2 J_1 J_2 \right]^{1/2},$$

(52)

and

$$g = \frac{1}{\cos \varphi} \left\{ -q_k J_1 + \frac{q_k^2}{2} \sin^2 \varphi \mp \sqrt{\lambda} \sin \varphi \left[ \cos^2 \varphi + \frac{q_k^2}{2\lambda} \sin^2 \varphi - q_k^2 J_1 J_2 \right]^{1/2} \right\},$$

(53)

where the signs are chosen so that the expressions reduce to those of equations (50) and (51) when the $q_k^2$ terms are ignored.

We now consider the growth rates. The transformation matrix of equation (3) can be written in the form of an elliptical rotation matrix, as in equation (27). The length parameter of the rotation can be written in the form $L_k = L_0(1 + \eta_k)$, where the $\eta_k$ vary from cycle to cycle. For the case of symmetric variations, where $\langle \eta_k \rangle = 0$, the growth rate takes the form

$$\gamma = \log \left[ 1 + \frac{1}{2} \langle \eta_k^2 \rangle \langle \sin \theta_k^2 \rangle \right],$$

(54)

where $\theta_k$ is the angle of the elliptical rotation matrix (see [4] and Theorem 2.2).

Using the matrix elements found above, and dropping the subscripts on the $h$ and $g$, we find

$$\cos \theta_k = h = \cos \varphi - \frac{q_k}{2\sqrt{\lambda}} \sin \varphi + \mathcal{O}(\epsilon^2),$$

(55)

so that

$$\theta_k = \varphi + \frac{q_k}{2\sqrt{\lambda}} + \mathcal{O}(\epsilon^2),$$

(56)

and hence

$$\sin \theta_k = (1 - h^2)^{1/2} = \sin \varphi + \frac{q_k}{2\sqrt{\lambda}} \cos \varphi + \mathcal{O}(\epsilon^2).$$

(57)

The length parameter $L_k$ of the elliptical rotation matrix can then be written in the form

$$L_k = \frac{\sin \theta_k}{g} = -\frac{1}{\sqrt{\lambda}} \left[ 1 + \frac{1 - 2J_1}{2\lambda} \frac{q_k}{\cos \varphi \sin \varphi} \right] \equiv -\frac{1}{\sqrt{\lambda}} \left[ 1 + \eta_k \right] + \mathcal{O}(\epsilon^2),$$

(58)

where the final equality in equation (58) defines the perturbation $\eta_k$. Combining with the above results we find the following Theorem:
Theorem 3.1: The growth rate for the CC Random Hill’s equation in the limit of constant \( \lambda \) and small \( q = q_k = \mathcal{O}(\epsilon) \) is given by
\[
\gamma = \log \left[ 1 + \frac{\langle q_k^2 \rangle}{8\lambda} \left( \frac{2J_1 - 1}{\cos \varphi} \right)^2 \right] \approx \frac{\langle q_k^2 \rangle}{8\lambda} \left( \frac{2J_1 - 1}{\cos \varphi} \right)^2 + \mathcal{O}(\epsilon^2). \tag{59}
\]

Note that this form is valid for the case of symmetric variations of the length parameter, i.e., where the distribution of the \( \eta_k \) is symmetric w.r.t. zero.

Notice also that the expression for the growth rate is not defined when \( \cos \varphi \to 0 \). We can define the quantity \( J \) such that
\[
J = \frac{2J_1 - 1}{\cos \varphi} = \frac{1}{\cos(\sqrt{\lambda}\pi)} \int_0^\pi \cos(2\sqrt{\lambda}t)\dot{Q}dt. \tag{60}
\]

When \( \cos \varphi \to 0, \sqrt{\lambda} \to (2k+1)/2 \), and both the numerator and denominator of the above expression vanish. To evaluate the quantity \( J \), we can use L’Hopital’s rule, which implies that
\[
J = \pm \frac{2}{\pi} \int_0^\pi t \sin[(2k + 1)t]\dot{Q}dt, \tag{61}
\]
where the \((+(-))\) sign arises for \( k \) even(odd).

Figure 2 illustrates how well this approximation scheme works. For the sake of definiteness, the \( q_k \) are allowed to vary over a uniform distribution with amplitude \( A_q \). We expect the above results for the growth rate to be exact in the limit of small fluctuations, i.e., where the fluctuation amplitude \( A_q \to 0 \). The figure shows the growth rates as a function of the amplitude \( A_q \) for constant \( \lambda = 1/2 \). Here the matrix elements are calculated using the first order expressions from equations (50) and (51). The solid curve shows an estimate of the growth rate determined directly from matrix multiplication; note that the growth rate converges very slowly at small amplitudes, so the plot contains errors due to incomplete sampling. The dotted curve shows the growth rate calculated from equation (54), where the variations \( \langle \eta_k^2 \rangle \) are found by numerical sampling. The dashed curve shows the growth rate from equation (59), which represents the main result of this section. Finally, even though the expressions derived here are correct only to first order in \( q_k \), we can include the second order terms (in \( q_k \)) for purposes of determining the \( \eta_k \) and hence the growth rate; for this case, the result is shown as the dot-dashed curve. All four of the curves are coincident for sufficiently small amplitudes \( A_q \). The analytic expression of equation (59) represents the crudest approximation (dashed curve), and converges the slowest.

Corollary: Consider a CC Random Hill’s equation in the limit of small (but finite) forcing strength \( |q_k| \ll 1 \), with constant oscillation parameter \( \lambda \), and where \( \dot{Q}(t) \) is a function. In the limit \( \lambda \to \infty \), the growth rate \( \gamma \to 0 \).
Proof: In this limit, the growth rate is given by equation (59). For nonzero fluctuations \( \langle q_k^2 \rangle > 0 \), the growth rate vanishes if and only if the integral \( J_1 = 1/2 \), i.e.,

\[
J_1 = \int_0^{\pi} dt \dot{Q}(t) \cos^2(\sqrt{\lambda} t) = 1/2.
\]

Using trigonometric identities, this expression can be written in the form

\[
\int_0^{\pi} dt \dot{Q}(t) \cos(2\sqrt{\lambda} t) = 0.
\]

In the limit \( \lambda \to \infty \), this integral vanishes, and hence \( \gamma \to 0 \).

Remark: The above argument works provided that \( \dot{Q}(t) \) is a function. For the limiting case where \( \dot{Q} \) is a delta-function (and hence a distribution), the integral of equation (63) does not necessarily vanish; nonetheless, the growth rate \( \gamma \propto 1/\lambda \) in this limit [3], so that \( \gamma \to 0 \) as \( \lambda \to \infty \).

For example, let \( \dot{Q} = (2/\pi) \sin^2 t \). Then the quantity \( 2J_1 - 1 \) becomes

\[
2J_1 - 1 = -\frac{\sin 2\varphi}{2\pi \sqrt{\lambda(\lambda - 1)}},
\]

which vanishes as \( \lambda \to \infty \).

The general trend of decreasing growth rate \( \gamma \) with increasing \( \lambda \) is shown in Figure 3 for three choices for the barrier function. The various curves show the growth rates for \( \dot{Q}(t) = (8/3\pi) \sin^4 t \) (solid), \( \dot{Q}(t) = (2/\pi) \sin^2 t \) (dashed), and \( \dot{Q}(t) = \delta([t] - \pi/2) \) (dotted), where we have included the normalization constants, and where \([t]\) denotes that the variable \( t \) is to be evaluated mod-\( \pi \). The amplitude of the \( q_k \) fluctuations are the same for all three cases.

IV. CONCLUSION

This paper has generalized the Classical Hill’s equation (1) to include random elements, where this treatment focuses on the case of small fluctuations with amplitude \( \mathcal{O}(\epsilon) \). This generalization can take two distinct, but related, forms: The parameters of the CC Random Hill’s equation (2) vary from cycle to cycle, whereas the Stochastic Hill’s equation (6) or (7) includes a stochastic process.

Theorem 2.3 shows that the growth rates of these two types of (generalized) Hill’s equations are the same when (for a given stochastic process \( \xi \)) the parameter variations of the CC Random Hill’s equation are chosen according to equation (34). Theorem 2.2 provides an approximation for the growth rate, correct to \( \mathcal{O}(\epsilon^2) \), for the CC Random Hill’s equation (and hence for the equivalent Stochastic Hill’s equation). These growth rates
Figure 2: Growth rates for random Hill’s equation in the limit of small random forcing parameter $q_k$, plotted as a function of the amplitude $A_q$ of the fluctuations (for $\lambda = 1/2$). The solid curve shows the result from direct matrix multiplication; dotted curve shows the result from equation (54), where the $\eta_k$ are sampled numerically; the dashed curve shows the result from equation (59); the dot-dashed curve shows the growth rate calculated by including higher order terms (in $q_k$).
Figure 3: Growth rates for random Hill’s equation as a function of $\lambda$, where the fluctuation amplitude $\langle q_k^2 \rangle = 1$. The three curves show the result for $\hat{Q} = \delta([t] - \pi/2)$ (dotted), $\hat{Q} = (3/8\pi) \sin^4 t$ (solid), and for $\hat{Q} = (2/\pi) \sin^2 t$ (dashed).
\[ \gamma \sim \langle p_k^2 \rangle \] are nonzero except for isolated cases (see Figure 3). As a result, for the CC Random Hill’s equation, the (generalized) plane of parameters has no regions of stability. This result is not due to the fluctuations moving the matrix elements into the unstable regions of parameter space. One obtains nonzero growth rates even when all of the matrix elements are classically stable. In contrast, the Stochastic Hill’s equation can be stable, where stability requires that the moments \( \Xi_j \) of the stochastic process vanish (see equations 35 and 41). Stability occurs in the limit where the correlation time \( \tau_c \rightarrow 0 \).

The results of this paper can be used in a wide variety of applications. As one example, orbit instabilities in extended mass distributions, such as dark matter halos, lead to a CC Random Hill’s equation, as shown in Appendix A (see also Refs. [5,12]). As another example, the reheating problem at the end of inflation naturally leads to a Stochastic Hill’s equation, as shown in Appendix B (see also Refs. [17,18,27]), and the growth rates can be calculated using Theorems 2.2 and 2.3.

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APPENDIX A:
RANDOM HILL’S EQUATION FROM ASTROPHYSICAL ORBITS

Orbits in extended mass distributions are subject to an instability [5] that can be described by a CC Random Hill’s equation (2). As one example, the density profile \( \rho(\varpi) \) of a dark matter halo has the form

\[ \rho(\varpi) = \rho_0 \frac{F(\varpi)}{\varpi}, \tag{A1} \]

where \( \rho_0 \) is a density scale and the variable \( \varpi \) is written in terms of Cartesian \((x, y, z)\) coordinates through the relation

\[ \varpi^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}, \tag{A2} \]

where (without loss of generality) \( a > b > c > 0 \). The density field is thus constant on ellipsoids. The function \( F(\varpi) \) approaches unity in the limit \( \varpi \rightarrow 0 \) so that the density...
profile approaches the form $\rho \sim 1/\omega$. For this regime, one can find analytic forms for both the potential and the force terms [5].

When an orbiting body is initially confined to any of the three principal planes, the motion can be unstable to perturbations in the perpendicular direction. Consider an orbit initially confined to the $x-z$ plane, with a small perturbation in the perpendicular $\hat{y}$ direction. In the limit $|y| \ll 1$, the equation of motion for the $y$-coordinate takes the form

$$\frac{d^2y}{dt^2} + \Omega_y^2 y = 0 \quad \text{where} \quad \Omega_y^2 = \frac{4/b}{\sqrt{c^2 x^2 + a^2 z^2 + b \sqrt{x^2 + z^2}}}.$$  \hspace{1cm} (A3)

In this context, the time evolution of the coordinates $(x, z)$ is determined by the original orbit and the motion is nearly periodic. As a result, the $[x(t), z(t)]$ dependence of the parameter $\Omega_y^2$ provides a (nearly) periodic forcing term. The orbit has outer turning points which define a minimum value for $\Omega_y^2$, which in turn defines the natural oscillation frequency $\lambda_k$. The function $\Omega_y^2$ defined above can be written in the form

$$\Omega_y^2 = \lambda_k + Q_k(t),$$  \hspace{1cm} (A4)

where the index $k$ counts the number of orbit crossings, and the chaotic orbit in the original plane leads to different $\lambda_k$ and $Q_k(t)$ for each crossing. The shape of the functions $Q_k(t)$ are nearly the same, so that one can write $Q_k(t) = q_k \tilde{Q}(t)$, where the forcing strength parameters $q_k$ vary from cycle to cycle. These forcing strengths $q_k$ are determined by the inner turning points of the orbit (weighted by the axis parameters $[a, b, c]$). Given the expansion of equation (A4), the equation of motion (A3) for the perpendicular coordinate becomes a CC Random Hill’s equation, with the form of equation (2).

**APPENDIX B:**

**STOCHASTIC HILL’S EQUATION FROM REHEATING IN INFLATION**

In the inflationary universe paradigm [14], the accelerated expansion of the universe is driven by the vacuum energy associated with a scalar field $\varphi$ (or fields). During the phase of accelerated expansion, the energy density of the universe itself decreases exponentially and the cosmos becomes increasingly empty. This epoch is thought to take place when the universe is extremely young, with typical time scales of $\sim 10^{-36}$ sec [19]. In order for the inflationary epoch to solve the cosmological issues it was designed to alleviate, the end of inflation must include a mechanism to refill the universe with energy [19]. This process is called reheating or preheating.

During the reheating epoch, the equation of motion for the inflation field displays oscillatory behavior about the minimum of its potential. In order for the universe to
become filled with energy (reheat), the inflation field $\varphi$ must couple to matter or radiation fields. One simple type of interaction arises from a coupling term in the Lagrangian of the form

$$\mathcal{L}_{\text{int}} = g \varphi \chi^2,$$

where $\chi$ is another scalar field that represents matter (or radiation) and where the coupling constant $g$ sets the interaction strength. The field $\chi$ can be expanded in terms of its Fourier modes $\chi_\ell$ since these quantities evolve independently. The resulting equation of motion for the matter field modes $\chi_\ell$ then takes the form

$$\frac{d^2 \chi_\ell}{dt^2} + \left[ \Omega^2_\ell + Q_\ell(t) + \xi \right] \chi_\ell = 0,$$

where $Q_\ell(t)$ is a periodic function (given by oscillatory behavior of the inflation field) and $\xi$ is a noise term that can be described by a stochastic process. Keep in mind that the index $\ell$ refers to the Fourier mode. In the absence of fluctuations, the modes $\chi_\ell$ of the matter fields obey a Classical Hill’s equation [11], which can be subject to parametric instability [17,18]. However, the noise perturbations $\xi$ [17,18,27] convert the reheating equation (B2) into a stochastic Hill’s equation. As shown by Theorem 2.3, the growth rates for this stochastic Hill’s equation are the same as an equivalent CC Random Hill’s equation, where the conditions for equivalence are given by equation (35); the growth rates can thus be calculated according to Theorem 2.2.
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