Analytical Solutions of the Nonlinear Time-Fractional Coupled Boussinesq-Burger Equations Using Laplace Residual Power Series Technique

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Abstract: In this paper, we present the series solutions of the nonlinear time-fractional coupled Boussinesq-Burger equations (T-FCB-BEs) using Laplace-residual power series (L-RPS) technique in the sense of Caputo fractional derivative (C-FD). To assert the efficiency, simplicity, performance, and reliability of our proposed method, an attractive and interesting numerical example is tested analytically and graphically. In addition, our obtained results show that this algorithm is compatible and accurate for investigating the fractional-order solutions of engineering and physical applications. Finally, Mathematica software 14 is applied to compute the numerical and graphical results.

Keywords: Caputo operator; Coupled Boussinesq-Burger equation; Laplace transform (LT); residual power series (RPS) method

1. Introduction

In the past twenty years, partial fractional differential equations (P-FDEs) have been motivated due to their various applications in several fields of science such as fluid and layer flows, multi-energy groups of neutron diffusion processes, neutral and multi pantograph systems, dynamic and hyperbolic systems, statistical mechanics model, material sciences and engineering [1–20]. These important phenomena and applications are well described by P-FDEs. The nonlocal property is the most significant advantage of using P-FDEs in diverse mathematical modeling.

The main advantage of using fractional derivatives with an arbitrary order is that they are flexible more than classical derivatives and also they are not-local. The two famous and important fractional derivatives in applications are: The Riemann-Liouville FD (R-L-FD) and C-FD [1–20]. The relationship between the R-L and the C-FDs are very closed since the R-L-FD can be converted to the C-FD under some regularity assumptions of the function. In P-FDEs, the time-fractional derivatives are commonly defined using the C-FDs. The main reason lies in that the P-FDEs in R-L sense needs initial conditions containing the limit values of R-L-FD at the origin of time \( t = 0 \), whose physical meanings are not very clear. While in P-FDEs via C-FD, the initial conditions are given in integer-orders, whose physical meanings are very clear [9–17].

In most cases, exact solutions do not exist for many Partial differential equations (PDEs), therefore, several numerical methods are created and applied to get the approximate series solutions for such P-FDEs such as the homotopy analysis, asymptotic and perturbation methods [1–3,5], variational iteration and Adomian decomposition methods [2,4,8], LT and differential transform techniques, RPS method [9–14] and L-RPS method [15–19].

In 1870, Boussinesq [21] introduced the Boussinesq equation to describe the motions of waves in shallow water and then it was used in many physics and engineering wave phenomena [22–27]. In 1915, Bateman presented Burger’s equation [28] which describes
several phenomena in physics and engineering such as acoustic and shock waves [29], stochastic processes [30], and gas dynamics [31–33]. There are several techniques and methods [8,11,34–41] were applied by researchers to obtain the approximate solutions to Burger’s equations. One of the most interesting mathematical models is the (generalized) Boussinesq-Burger’s equation (B-BEs) which describes the propagation waves of shallow water in the behaviors of fluids flow [3,42–46]. This equation was solved analytically and numerically by different techniques. For example, Gupta et al. [3] obtained the soliton solutions of B-BEs based on the optimal homotopy perturbation and asymptotic methods; Rady and Khalfallah [45] presented the periodic wave and multiple soliton solutions for B-BEs by applying Jacobi elliptic method; Wang et al. [46] presented type of solutions and interaction behaviors of the solitons and Lax pair for the B-BEs; Zhang et al. [44] introduced some new solutions of the generalized B-BEs using the modified mapping method; and Chen and Li [43] established some new soliton solutions of soliton B-BEs by applying Darboux transformation.

The well-known nonlinear time T-CB-BEs are given by [46–48]:

\[
\begin{align*}
    & u_t(x,t) + 2u(x,t)u_x(x,t) - \frac{1}{2}w_x(x,t) = 0, \\
    & w_t(x,t) - \frac{1}{2}u_{xxx}(x,t) + 2(u(x,t)w(x,t))_x = 0,
\end{align*}
\]

where \( x \) is the normalized space, \( t \) is the time, \( u(x,t) \) is the horizontal velocity and \( w(x,t) \) is the height of the water surface above the horizontal level.

Some methods were used to solve this coupled such as Lax pair and Bäcklund transformation technique [46], exp-function method [47] and reduced differential transform method [48]. Finally, the generalized T-FCB-BEs can be formulated as [49,50]:

\[
\begin{align*}
    & D_\beta^t u(x,t) + w_x(x,t) + u(x,t)u_x(x,t) = 0, \\
    & D_\beta^t w(x,t) + (u(x,t)w(x,t))_x + u_{xxx}(x,t) = 0,
\end{align*}
\]

subject to:

\[
    u(x,0) = f(x), w(x,0) = g(x),
\]

where \( 0 < \beta \leq 1, x \in \mathbb{I}, t \geq 0, f(x), g(x) \) are analytic functions, and \( u(x,t), w(x,t) \) are unknown real-valued functions to be solved.

2. Materials and Methods

There are few methods were used to solve this system such as fractional decomposition method with the definition of Caputo fractional derivative [49] and by applying first integral method with the definitions of Riemann-Liouville fractional and local conformable derivatives [50].

The main aim of our work is to employ L-RPS method for obtaining the fractional-order series solutions to the T-FCB-BEs as in Equations (1) and (2). The proposed method is a new efficient method, and it provides the solution in a rapidly convergent series which yields the solution in a closed form. The L-RPS method combines two power full methods (Laplace transform and RPS methods) for getting the series solution for the system of F-PDEs. In L-RPS method, few calculations are needed to get the series coefficients compared with RPS method since it is determined by employing the concept of limit not the fractional derivative as in RPS technique. The methodology of our proposed method (L-RPS) will be introduced with detail in Section 4. Mathematica software 14 is used to compute the numerical and graphical results.

The novelty of this work is shown in the proposed method chosen to solve the target problem. L-RPS method is a strong method that provides the solution in a rapidly convergent series, and we illustrate that in the results, in which we don’t need many terms to
get a good approximate solution. Moreover, this method does not need the linearization, discretization, or fractional differentiation like other numerical methods.

The rest of this present paper is arranged as follows: Basic definitions and basic idea of L-RPS method with convergence analysis are introduced in Section 3. The methodology of the proposed method is explaining in Section 4. An attractive application with graphical results are given and discusses in Section 5 to confirm the efficiency and reliability of our technique. Finally, Section 6 concludes the output of the whole paper.

3. Basic Concepts on Fractional and Laplace Operators

This section reviews some definitions and theorems for the fractional operators and the LT [1–19] which are essential in constructing the L-RPS solutions for the nonlinear T-FCB-BEs as in Equations (1) and (2).

Definition 1. The C-FD of \( u(x, t) \) of order \( \beta > 0 \) is defined as:

\[
\mathcal{D}_t^\beta y(x, t) = \int_t^{m-\beta} \mathcal{D}_t^m u(x, t), \quad m - 1 < \beta < m, \quad m \in \mathbb{N}, \quad x \in K, \quad t > 0,
\]

where \( K \) is a given interval and

\[
\mathcal{D}_t^\beta u(x, t) = \begin{cases} 
\frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} u(x, \tau) \, d\tau, & t > \tau > 0 \\
u(x, t), & \beta = 0
\end{cases}
\]
is the time R-L fractional integral of order \( \beta > 0 \).

Most important and useful properties of fractional operators can be summarized as below [1–19]:

Lemma 1. For \( \mu > -1, c \in \mathbb{R}, m - 1 < \beta \leq m, \) and \( t \geq 0, \) we have:

(i) \( \mathcal{D}_t^\beta c = 0. \)

(ii) \( \mathcal{D}_t^\beta t^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu+1-\beta)} t^{\mu-\beta}. \)

(iii) \( \mathcal{D}_t^\beta [\mathcal{D}_t^\beta u(x, t)] = u(x, t). \)

Definition 2. Let \( u(x, t) \) be a piecewise continuous function (PCF) on \( K \times [0, \infty) \) and of exponential order (EO) \( \delta. \) Then the LT of \( u(x, t) \) is given by:

\[
U(x, s) = \mathcal{L}[u(x, t)] := \int_0^\infty e^{-st} u(x, t) \, dt, \quad s > \delta,
\]

and the inverse LT of \( U(x, s) \) is:

\[
u(x, t) = \mathcal{L}^{-1}[U(x, s)] := \int_{z=\delta}^{z+\infty} e^{zt} U(x, s) \, ds, \quad z = \text{Re}(s) > z_0.
\]

Lemma 2. If \( u(x, t) \) and \( w(x, t) \) are PCFs on \( K \times [0, \infty) \) and of EOs \( \delta_1 \) and \( \delta_2, \) respectively, where \( \delta_1 < \delta_2. \) Considering \( U(x, s) = \mathcal{L}[u(x, t)], W(x, s) = \mathcal{L}[w(x, t)], \) and \( a, b \in \mathbb{R}, \) then:

(i) \( \mathcal{L}[au(x, t) + bw(x, t)] = aU(x, s) + bW(x, s), \quad x \in K, s > \delta_1. \)

(ii) \( \mathcal{L}^{-1}[aU(x, s) + bW(x, s)] = au(x, t) + bw(x, t), \quad x \in K, t \geq 0. \)

(iii) \( \lim_{s \to \delta_1^+} sU(x, s) = u(x, 0), \) \( x \in K. \)

(iv) \( \mathcal{L}[\mathcal{D}_t^\beta u(x, t)] = s^\beta U(x, s) - \sum_{k=0}^{m-1} s^{\beta-k-1} \mathcal{D}_t^k u(x, 0), \quad m - 1 < \beta \leq m. \)
(v) \(\mathcal{L}\left[D_t^\beta u(x, t)\right] = s^{\beta}U(x, s) - \sum_{k=0}^{n-1} s^{(n-k)\beta-1} D_t^k u(x, 0), 0 < \beta < 1,\)

where \(D_t^\beta = D_t^\beta \mathcal{L} D_t^\beta \ldots D_t^\beta \) (n -times).

**Theorem 1.** [15] Let \(u(x, t)\) be a PCF on \(K \times [0, \infty)\) of EO \(\delta\) and \(U(x, s) = \mathcal{L}[u(x, t)].\) Then

\[ U(x, s) = \sum_{n=0}^{\infty} \frac{f_n(x)}{s^{n\beta+1}}, 0 < \beta \leq 1, \ x \in K, \ s > \delta. \quad (3) \]

Then \(f_n(x) = D_t^{n\beta} u(x, 0), n = 0, 1, 2, \ldots.\)

The convergence conditions of the fractional expansion in Equation (3) are demonstrated in the following theorem.

**Theorem 2.** [15] If \(\left|s \mathcal{L}\left[D_t^{(n+1)\beta} u(x, t)\right]\right| \leq \mathcal{M}(x), \) on \(K \times (\delta, d]: 0 < \beta \leq 1.\) Then the reminder \(R_n(x, s)\) of the fractional expansion in Equation (3) satisfies the following inequality:

\[ |R_n(x, s)| \leq \frac{\mathcal{M}(x)}{s^{(n+1)\beta+1}}, \ x \in K, \ \delta < s \leq d. \quad (4) \]

**4. Constructing the L-RPS Solutions for Nonlinear T-FCB-BEs**

The main objective of this section is to construct a solitary solution to the nonlinear T-FCB-BEs using the L-RPS method. This method can be applied to solve nonlinear P-FDEs, while the LT fails to solve nonlinear equations without using power series technique. The main idea of L-RPS method focuses on the power series method to obtain a solution to the given nonlinear FDE in the Laplace space, and this requires an appropriate expansion that represents the solutions in final version. Moreover, we apply this section a new technique in detail to find the expansion coefficients.

Consider the nonlinear T-FCB-BEs as given in Equations (1) and (2) in Section 1. Now, operating the LT of both equations in Equation (1) to get:

\[ \mathcal{L}\left[D_t^\beta u(x, t)\right] + \mathcal{L}[w_x(x, t)] = \mathcal{L}[u(x, t)u_x(x, t)], \]
\[ \mathcal{L}\left[D_t^\beta w(x, t)\right] = \mathcal{L}[y(x, t)w(x, t)], \]

Applying Lemma 2 and using Equation (2), then the coupled equations in Equation (1) can be written as:

\[ s^\beta U(x, s) - s^{\beta-1} f(x) + \mathcal{L}\left[w_x(x, t)\right] + \mathcal{L}\left[U(x, s)\right] = 0, \]
\[ s^\beta W(x, s) - s^{\beta-1} g(x) + \mathcal{L}\left[w_x(x, t)\right] + \mathcal{L}\left[W(x, s)\right] = 0. \quad (6) \]

From Equation (6), we obtain:

\[ U(x, s) = \frac{f(x)}{s^\beta} + \frac{W(x, s)}{s^\beta} + \frac{\beta}{s^\beta} \mathcal{L}\left[U(x, s)\right] = 0, \]
\[ W(x, s) = \frac{g(x)}{s^\beta} + \frac{W_x(x, s)}{s^\beta} + \frac{\beta}{s^\beta} \mathcal{L}\left[W(x, s)\right] = 0. \quad (7) \]

The system in Equation (7) represents a nonlinear system of PDEs that contains derivatives relative \(x.\) Now, according to the L-RPS and using the facts: \(\lim_{s \to 0} sU(x, s) = u(x, 0)\) and \(\lim_{s \to 0} sW(x, s) = w(x, 0),\) then the \(k\)th truncated series of \(U(x, s)\) and \(W(x, s)\) in Equation (7) can be written as:

\[ U_k(x, s) = \frac{f(x)}{s} + \sum_{n=1}^{k} \frac{f_n(x)}{s^{n\beta+1}}, \ x \in I, \ s > \delta \geq 0, \quad (8) \]
In the next step, we define the Laplace-residual functions (L-RFs) of the coupled equations in Equation (7) to find the unknown coefficients of the series in Equations (8) and (9):

\begin{align}
LRes(U(x,s)) &= U(x,s) - \frac{f(x)}{s} + \frac{W(x,s)}{\Gamma(1+\beta)} s^{\beta-1} \bigg[ L^{-1} \bigg[ \mathfrak{L}[U(x,s)] \bigg] \bigg], \\
LRes(W(x,s)) &= W(x,s) - \frac{g(x)}{s} + \frac{U(x,s)}{\Gamma(1+\beta)} s^{\beta-1} \bigg[ L^{-1} \bigg[ \mathfrak{L}[W(x,s)] \bigg] \bigg].
\end{align}

(10)

and the kth L-RFs are:

\begin{align}
LRes_k(U(x,s)) &= U_k(x,s) - \frac{f(x)}{s} + \frac{W_k(x,s)}{\Gamma(1+\beta)} s^{\beta-1} \bigg[ L^{-1} \bigg[ \mathfrak{L}[U_k(x,s)] \bigg] \bigg], \\
LRes_k(W(x,s)) &= W_k(x,s) - \frac{g(x)}{s} + \frac{U_k(x,s)}{\Gamma(1+\beta)} s^{\beta-1} \bigg[ L^{-1} \bigg[ \mathfrak{L}[W_k(x,s)] \bigg] \bigg].
\end{align}

(11)

Since, \( LRes(U(x,s)) = 0 \), \( LRes(W(x,s)) = 0 \), we have \( s^{k\beta+1} LRes(U(x,s)) = 0 \), \( s^{k\beta+1} LRes(W(x,s)) = 0 \).

Therefore,

\[ \lim_{s \to \infty} \left( s^{k\beta+1} LRes_k(U(x,s)) \right) = 0, \quad \lim_{s \to \infty} \left( s^{k\beta+1} LRes_k(U(x,s)) \right) = 0 \text{ for } k = 0, 1, 2, \ldots \]

(12)

To find \( f_1(x) \) and \( g_1(x) \) in Equation (11), we substitute \( U_1(x,s) = \frac{f(x)}{s} + \frac{f(x)}{\Gamma(1+\beta)} s^{\beta-1} \) and \( W_1(x,s) = \frac{g(x)}{s} + \frac{g(x)}{\Gamma(1+\beta)} s^{\beta-1} \) in the first L-RFs to get:

\begin{align}
LRes_1(U(x,s)) &= \frac{f(x)}{s} + \frac{f(x)}{\Gamma(1+\beta)} s^{\beta-1} \bigg[ L^{-1} \bigg[ \mathfrak{L}[U(x,s)] \bigg] \bigg] \\
&= \frac{1}{\Gamma(1+\beta)} \left( f_1(x) + f(x) f'(x) + g'(x) \right) + \frac{1}{\Gamma(1+\beta)} \left( f_1(x) + f(x) f'(x) + g'(x) \right).
\end{align}

(13)

\begin{align}
LRes_1(W(x,s)) &= \frac{g(x)}{s} + \frac{g(x)}{\Gamma(1+\beta)} s^{\beta-1} \bigg[ L^{-1} \bigg[ \mathfrak{L}[W(x,s)] \bigg] \bigg] \\
&= \frac{1}{\Gamma(1+\beta)} \left( g_1(x) + g(x) f'(x) + f(x) g'(x) \right) + \frac{1}{\Gamma(1+\beta)} \left( g_1(x) f'(x) + f(x) g'(x) \right).
\end{align}

(14)

Next, by solving: \( \lim_{s \to \infty} s^{\beta+1} LRes_1(U(x,s)) = 0 \), \( \lim_{s \to \infty} s^{\beta+1} LRes_1(U(x,s)) = 0 \), one can get:

\[ f_1(x) = -\left( f(x) f'(x) + g'(x) \right) g_1(x) = -\left( g(x) f'(x) + f(x) g'(x) + f'(3) \right). \]

Thus, the first Laplace series solution (LSS) of the system in Equations (8) and (9) can be written as:

\begin{align}
U_1(x,s) &= \frac{f(x)}{s} + \frac{-(f(x) f'(x) + g'(x))}{s^{\beta+1}}, \\
W_1(x,s) &= \frac{g(x)}{s} + \frac{-(g(x) f'(x) + f(x) g'(x) + f'(3))}{s^{\beta+1}}.
\end{align}

(15)

To find the second LSS of system in Equations (8) and (9), substitute \( U_2(x,s) = \frac{f(x)}{s} + \frac{f(x)}{\Gamma(1+\beta)} s^{\beta-1} \) and \( W_2(x,s) = \frac{g(x)}{s} + \frac{g(x)}{\Gamma(1+\beta)} s^{\beta-1} \) into the second LRF \( LRes_2(U(x,s)) \), \( LRes_2(W(x,s)) \) as:
\[ \text{LRes}_2(U(x,s)) = \frac{1}{s^{3\beta+1}} \left( f_1(x) + f(x)f'(x) + g'(x) + \frac{1}{s^{3\beta+1}} \left( f_2(x) + f(x)f'_1(x) + f(x)f'_1(x) + g_1'(x) \right) \right) + \frac{1}{s^{3\beta+1}} \left( f_2(x)f'(x) + \frac{f_1(1+2\beta)f_2(x)g(x) + f(x)f_1'(x)}{1+\beta} + \frac{f_1(1+2\beta)f_2(x)g(x) + f(x)f_1'(x)}{1+\beta} + f(x)f_2'(x) + g_2'(x) \right) + \frac{1}{s^{3\beta+1}} \left( \frac{\Gamma(1+3\beta)f_2(x)f'_1(x)}{1+\beta} + \frac{\Gamma(1+3\beta)f_2(x)f'_1(x)}{1+\beta} \right) \right), \] (16)

\[ \text{LRes}_2(W(x,s)) = \frac{1}{s^{3\beta+1}} \left( g_2(x) + g(x)f'(x) + f(x)g'(x) + f'(x) \right) + \frac{1}{s^{3\beta+1}} \left( g_2(x) + g(x)f'(x) + f(x)g'(x) + f(x)g_1'(x) + f'(x) \right) + \frac{1}{s^{3\beta+1}} \left( g_2(x) + g(x)f'(x) + f(x)g'(x) + f(x)g_2'(x) + f(x)g_2'(x) + f'(x) \right) + \frac{1}{s^{3\beta+1}} \left( \frac{f_1(1+2\beta)g_2(x)f'_1(x)}{1+\beta} + \frac{f_1(1+2\beta)g_2(x)f'_1(x)}{1+\beta} \right), \] (16)

Thus, \( f_2(x) \) and \( g_2(x) \) can be obtained by substituting the values of \( f_2(x) \) and \( g_1(x) \) into Equation (16), then multiplying both sides of the new equation by \( s^{2\beta+1} \) and taking the limit as \( s \to \infty \) to get:

\[ f_2(x) = -(f_1(x)f'_1(x) + f(x)f_1'(x) + g_1'(x)), \] (17)

Again, to find out the second LSS of the system in Equations (8) and (9), substitute \( U_3(x,s) = \frac{f_1(x)}{s} + \frac{f_2(x)}{s^{3\beta+1}} + \frac{f_3(x)}{s^{3\beta+1}} + \frac{f_1(x)}{s} \) and \( W_3(x,s) = \frac{g_2(x)}{s} \) into the second L-RF: \( \text{LRes}_3(U(x,s)), \text{LRes}_3(W(x,s)) \) to get:

\[ \text{LRes}_3(U(x,s)) = \frac{1}{s^{3\beta+1}} \left( f_1(x) + f(x)f'(x) + g'(x) + \frac{1}{s^{3\beta+1}} \left( f_2(x) + f(x)f'_1(x) + f(x)f'_1(x) + g_1'(x) \right) \right) + \frac{1}{s^{3\beta+1}} \left( f_2(x)f'(x) + \frac{f_1(1+2\beta)f_2(x)g(x) + f(x)f_1'(x)}{1+\beta} + \frac{f_1(1+2\beta)f_2(x)g(x) + f(x)f_1'(x)}{1+\beta} + f(x)f_2'(x) + g_2'(x) \right) + \frac{1}{s^{3\beta+1}} \left( \frac{\Gamma(1+3\beta)f_2(x)f'_1(x)}{1+\beta} + \frac{\Gamma(1+3\beta)f_2(x)f'_1(x)}{1+\beta} \right), \] (18)

\[ \text{LRes}_3(W(x,s)) = \frac{1}{s^{3\beta+1}} \left( g_3(x) + g(x)f'(x) + f(x)g'(x) + f'(x) \right) + \frac{1}{s^{3\beta+1}} \left( g_3(x) + g(x)f'(x) + f(x)g'(x) + f(x)g_1'(x) + f'(x) \right) + \frac{1}{s^{3\beta+1}} \left( g_3(x) + g(x)f'(x) + f(x)g'(x) + f(x)g_2'(x) + f(x)g_2'(x) + f'(x) \right) + \frac{1}{s^{3\beta+1}} \left( \frac{f_1(1+2\beta)g_3(x)f'_1(x)}{1+\beta} + \frac{f_1(1+2\beta)g_3(x)f'_1(x)}{1+\beta} \right), \]

Thus, \( f_3(x) \) and \( g_3(x) \) can be obtained by substituting the values of \( f_1(x) \), \( f_2(x) \), \( g_1(x) \) and \( g_2(x) \) into the coupled equations in Equation (18), then multiplying both sides of the new equations by \( s^{3\beta+1} \) and taking the limit as \( s \to \infty \) to get:
\[ f_2(x) = -\left( f_2(x)f'(x) + f(x)f_2'(x) + g(x) + \frac{f(1+2\beta)(x)f'(x)}{I(1+\beta)} \right), \]
\[ g_2(x) = -\left( g_2(x)f'(x) + f(x)g_2'(x) + f(x)g(x) + \frac{f(1+3\beta)(x)g'(x)}{I(1+\beta)} \right) \frac{f(1+3\beta)(x)}{I(1+\beta)}. \]

If we continue in the same manner, substituting the kth truncated series \( U_k(x, s) \), \( W_k(x, s) \) into the kth L-RFS \( LRes_k(U(x, s)) \), \( LRes_k(W(x, s)) \). By multiplying the resulting new equations by \( s^k \beta + 1 \) and taking the limit as \( s \to \infty \), \( f_k+1(x) \), \( g_k+1(x) \) for \( k \geq 2 \), then we obtain the following recurrence relation:

\[ f_{k+1}(x) = -\left( f_k(x)f(x) + g_k(x)' + \sum_{i+j=k} \frac{r(f_i(x)g_i(x))'(1+\beta)}{r(1+\beta)} \right), \quad i, j \in \mathbb{Z}^+. \]

Now, the series solution of the system in Equation (7) is given by:

\[ U(x, s) = \sum_{s=0}^{\infty} \frac{U_s(x)}{s!}, \quad W(x, s) = \sum_{s=0}^{\infty} \frac{W_s(x)}{s!} \]

So, the series solution of the nonlinear T-FCB-BEs in Equations (1) and (2) can be obtained by transforming the above solution into the original space by using the inverse LT. Therefore, the L-RPS solution of the system in Equation (1) is given by:

\[ (x, t) = f(x) + \frac{-(f(x)f'(x) + g'(x)')}{I(\beta+1)} + \frac{f(x)f'_i(x)}{I(2\beta+1)} + \sum_{k=3}^{\infty} \frac{f(x)I(\beta)}{I(k\beta+1)}, \quad t \geq 0, \quad x \in I. \]

5. Application with Graphical Result

In this Section, we give an attractive and interesting example with incuding graphical resukts to assert the efficiency and simplicity our proposed method in Section 4.

Application 1. Consider the following nonlinear T-FCB-BEs:

\[ D_x^\beta u(x, t) + w_x(x, t) + u(x, t)u_a(x, t) = 0, D_x^\beta w(x, t) + (u(x, t)w(x, t))_x + u_{xxx}(x, t) = 0. \]

with the initial conditions:

\[ u(x, 0) = 1 + \tan h \left( \frac{x}{2} \right), \quad w(x, 0) = \frac{1}{2} - \frac{1}{2}\tan h^2 \left( \frac{x}{2} \right). \]

Comparing Equations (23) and (24) with Equations (1) and (2), respectively, then we find that

\[ f(x) = 1 + \tan h \left( \frac{x}{2} \right), \quad g(x) = \frac{1}{2} - \frac{1}{2}\tan h^2 \left( \frac{x}{2} \right). \]

Therefore, according to the discussion and obtained results in Section 4, the L-RPS solution of the system in Equation (23) is given by
\[ u(x,t) = f(x) + \frac{-(f(x)f'(x)+g'(x))t}{t(\beta+1)} - \frac{(f(x)f'(x)+f(x)f'_1(x)+g_1'(x))t^2}{t(\beta+1)} + \sum_{k=3}^{\infty} \frac{f_k(x) t^k}{k!(\beta+1)}, t \geq 0, x \in I. \]

\[ w(x,t) = g(x) + \frac{-(g(x)f'(x)+f(x)g'(x)+f_1'(x))t}{t(\beta+1)} - \frac{(g_1(x)f'(x)+f_1(x)g_1'(x)+f_2'(x))t^2}{t(\beta+1)} + \sum_{k=3}^{\infty} \frac{g_k(x) t^k}{k!(\beta+1)}, t \geq 0, x \in I. \] (25)

Now, Equation (20) produces the series coefficients as follow:

\[ f_0(x) = 1 + \tanh \left( \frac{x}{\beta} \right). \]
\[ g_0(x) = \frac{1}{2} - \frac{1}{2} \tanh^2 \left( \frac{x}{\beta} \right). \]
\[ f_1(x) = -\frac{1}{2} \text{sech}^2 \left( \frac{x}{\beta} \right). \]
\[ g_1(x) = -\frac{1}{2} \text{sech}^2 \left( \frac{x}{\beta} \right) + \frac{1}{2} \text{sech}^4 \left( \frac{x}{\beta} \right) + \frac{1}{2} \text{sech}^2 \left( \frac{x}{\beta} \right) \tanh \left( \frac{x}{\beta} \right) + \frac{1}{4} \text{sech}^2 \left( \frac{x}{\beta} \right) \tanh^2 \left( \frac{x}{\beta} \right) = 4 \text{csch}^3(x) \sinh^4 \left( \frac{x}{\beta} \right). \]
\[ f_2(x) = -\frac{1}{2} \text{sech}^2 \left( \frac{x}{\beta} \right) \tanh \left( \frac{x}{\beta} \right) + \frac{1}{2} \text{sech}^4 \left( \frac{x}{\beta} \right) \tanh \left( \frac{x}{\beta} \right) + \frac{1}{4} \text{sech}^2 \left( \frac{x}{\beta} \right) \tanh^2 \left( \frac{x}{\beta} \right) = 8 \text{csch}^3(x) \sinh^4 \left( \frac{x}{\beta} \right). \]

Continue in the same manner to get:

\[ u(x,t) = 1 + \tanh \left( \frac{x}{\beta} \right) - \frac{\text{sech}^2 \left( \frac{x}{\beta} \right) t^2}{2 \Gamma(\beta+1)} + \frac{8 \text{csch}^3(x) \sinh^4 \left( \frac{x}{\beta} \right) t^2}{\Gamma(2\beta+1)} + \cdots = 1 + \tanh \left( \frac{x - \frac{\beta}{\beta}}{2} \right), \] (26)

\[ w(x,t) = \frac{1}{2} - \frac{1}{2} \tanh^2 \left( \frac{x}{\beta} \right) + \frac{4 \text{csch}^3(x) \sinh^4 \left( \frac{x}{\beta} \right) t^2}{\Gamma(2\beta+1)} + \frac{(\cosh(x) - 2) \sinh^4 \left( \frac{x}{\beta} \right) t^2}{4 \Gamma(2\beta+1)} + \cdots = \frac{1}{2} - \frac{1}{2} \tanh^2 \left( \frac{x - \frac{\beta}{\beta}}{2} \right). \]

Figures 1 and 2 below show the graphical results for the 5th-approximate L-RPS solutions \( u_5(x,t) \) and \( w_5(x,t) \), respectively, of Equations (23) and (24) at different values of \( \beta \).

![Figure 1](image1.png)  
(a) \hspace{1cm}  
![Figure 2](image2.png)  
(b) \hspace{1cm}  
![Figure 3](image3.png)  
(c)

**Figure 1.** The surface graph of the 3D plots of the 5th-approximate L-RPS solution \( u_5(x,t) \) in Equations (23) and (24) at: (a) \( \beta = 0.6 \) (b) \( \beta = 0.75 \) (c) \( \beta = 0.9 \).
6. Conclusions

We have employed an attractive L-RPS method for solving system of nonlinear T-FCB-BEs. The proposed method is a new efficient method, and it provides the solution in a rapidly convergent series which yields the solution in a closed form. That is, few calculations are needed in L-RPS method to get the series coefficients compared with RPS method since it is determined by employing the concept of limit not the fractional derivative as in RPS method. The L-RPS method will open the door for solving many complicated nonlinear F-PDEs in future studies, since it can be easily employed for creating the exact and approximate solutions of many physical and engineering phenomena depend on F-PDEs such as the nonlinear KdV-Burger, parabolic and mKdV space-time F-PDEs. Moreover, there is a newly proposed fractional derivative definition which is called the “Abu-Shady-Kaabar fractional derivative” and recently introduced by Abu-Shady and Kaabar [51]. This definition obtains the same results of C-FD in a very simple way which is more efficient for solving many nonlinear FDEs, see, [51–53]. In the future, we attend to solve some new attractive modeling scientific phenomena via L-RPS method using Abu-Shady-Kaabar fractional derivative [51–54]. Mathematica software 14 is used to compute the numerical and graphical results.

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