Each $n$-by-$n$ matrix with $n > 1$ is a sum of 5 coninvolutory matrices

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Abstract

An $n \times n$ complex matrix $A$ is called coninvolutory if $\bar{A}A = I_n$ and skew-coninvolutory if $\bar{A}A = -I_n$ (which implies that $n$ is even). We prove that each matrix of size $n \times n$ with $n > 1$ is a sum of 5 coninvolutory matrices and each matrix of size $2m \times 2m$ is a sum of 5 skew-coninvolutory matrices.

We also prove that each square complex matrix is a sum of a coninvolutory matrix and a condiagonalizable matrix. A matrix $M$ is called condiagonalizable if $M = \bar{S}^{-1}DS$ in which $S$ is nonsingular and $D$ is diagonal.

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1. Introduction

An $n \times n$ complex matrix $A$ is called coninvolutory if $\bar{A}A = I_n$ and skew-coninvolutory if $\bar{A}A = -I_n$ (and so $n$ is even since $\det(\bar{A}A) \geq 0$). We prove that each matrix of size $n \times n$ with $n \geq 2$ is a sum of 5 coninvolutory matrices and each matrix of size $2m \times 2m$ is a sum of 5 skew-coninvolutory matrices.
These results are somewhat unexpected since the set of matrices that are sums of involutory matrices is very restricted. Indeed, if $A^2 = I_n$ and $J$ is the Jordan form of $A$, then $J^2 = I_n$, $J = \text{diag}(1, \ldots, 1, -1, \ldots, -1)$, and so $\text{trace}(A) = \text{trace}(J)$ is an integer. Thus, if a matrix is a sum of involutory matrices, then its trace is an integer. Wu [7, Corollary 3] and Spiegel [5, Theorem 5] prove that an $n \times n$ matrix can be decomposed into a sum of involutory matrices if and only if its trace is an integer being even if $n$ is even.

We also prove that each square complex matrix is a sum of a coninvolutory matrix and a condiagonalizable matrix. A matrix is condiagonalizable if it can be written in the form $S^{-1}DS$ in which $S$ is nonsingular and $D$ is diagonal; the set of condiagonalizable matrices is described in [2, Theorem 4.6.11].

Similar problems are discussed in Wu’s survey [8]. Wu [8] shows that each matrix is a sum of unitary matrices and discusses the number of summands (see also [3]). Wu [7] establishes that $M$ is a sum of idempotent matrices if and only if $\text{trace}(M)$ is an integer and $\text{trace}(M) \geq \text{rank}(M)$. Rabanovich [4] proves that every square complex matrix is a linear combination of three idempotent matrices. Abara, Merino, and Paras [1] study coninvolutory and skew-coninvolutory matrices.

2. Each matrix is a sum of a coninvolutory matrix and a condiagonalizable matrix

Two matrices $A$ and $B$ over a field $\mathbb{F}$ are similar (or, more accurately, $\mathbb{F}$-similar) if there exists a nonsingular matrix $S$ over $\mathbb{F}$ such that $S^{-1}AS = B$. A matrix $A$ is diagonalizable if it is similar to a diagonal matrix. Two complex matrices $A$ and $B$ are consimilar if there exists a nonsingular matrix $S$ such that $S^{-1}AS = B$; a canonical form under consimilarity is given in [2, Theorem 4.6.12]. A complex matrix $A$ is real-condiagonalizable if it is consimilar to a diagonal real matrix.

By the statement (b) of the following theorem, each square complex matrix is a sum of two condiagonalizable matrices, one of which may be taken to be coninvolutory.

Theorem 1. (a) Each square matrix over an infinite field is a sum of an involutory matrix and a diagonalizable matrix.

(b) Each square complex matrix is a sum of a coninvolutory matrix and a real-condiagonalizable matrix.
Each square complex matrix is consimilar to $I_n + D$, in which $D$ is a real-condiagonalizable matrix.

Each square complex matrix is consimilar to $C + D$, in which $C$ is coninvolutory and $D$ is a diagonal real matrix.

Proof. The theorem is trivial for $1 \times 1$ matrices.

Let $\mathbb{F}$ be any field. The *companion matrix of a polynomial* 

$$f(x) = x^m - a_1x^{m-1} - \cdots - a_m \in \mathbb{F}[x]$$

is the matrix

$$F(f) := \begin{bmatrix} 0 & 0 & a_m \\ 1 & \ddots & \vdots \\ \vdots & 0 & a_2 \\ 0 & 1 & a_1 \end{bmatrix} \in \mathbb{F}^{n \times m};$$

its characteristic polynomial is $f(x)$. By [6, Section 12.5],

each $A \in \mathbb{F}^{n \times n}$ is $\mathbb{F}$-similar to a direct sum of companion matrices whose characteristic polynomials are powers of prime polynomials; this direct sum is uniquely determined by $A$, up to permutations of summands.

Moreover,

if $f, g \in \mathbb{F}[x]$ are relatively prime, then $F(f) \oplus F(g)$ is $\mathbb{F}$-similar to $F(fg)$.

(a) Let $A$ be a matrix of size $n \times n$ with $n \geq 1$ over an infinite field $\mathbb{F}$. It is similar to a direct sum of companion matrices:

$$SAS^{-1} = B = F_1 \oplus \cdots \oplus F_t,$$

$S$ is nonsingular.

If $B = C + D$ is the sum of an involutory matrix $C$ and a diagonalizable matrix $D$, then $A = S^{-1}CS + S^{-1}DS$ is also the sum of an involutory matrix and a diagonalizable matrix. Thus, it suffices to prove the statement (a) for $B$. Moreover, it suffices to prove it for an arbitrary companion matrix $\mathbb{F}$. 

3
Each matrix
\[
G = \begin{bmatrix}
1 & 0 & b_m \\
\vdots & \ddots & \vdots \\
0 & \cdots & 1 & b_2 \\
0 & \cdots & 0 & -1
\end{bmatrix} \in \mathbb{F}^{m \times m}
\]
is involutory. Changing \(b_2, \ldots, b_m\), we get
\[
F(f) - G + I_m = \begin{bmatrix}
0 & 0 & c_m \\
1 & \ddots & \vdots \\
\vdots & \ddots & 0 & c_2 \\
0 & \cdots & 0 & a_1 + 2
\end{bmatrix}
\]
with arbitrary \(c_2, \ldots, c_m \in \mathbb{F}\). For each pairwise unequal \(\lambda_1, \ldots, \lambda_m \in \mathbb{F}\) such that \(\lambda_1 + \cdots + \lambda_m = a_1 + 2 = \text{trace}(F(f) - G + I_m)\), we can take \(G\) such that the characteristic polynomial of \(F(f) - G + I_m\) is equal to
\[
x^m - (a_1 + 2)x^{m-1} - c_2x^{m-2} - \cdots - c_m = (x - \lambda_1) \cdots (x - \lambda_m).
\]
Thus,
\[
F(f) - G + I_m \text{ is } \mathbb{F}\text{-similar to } \text{diag}(\lambda_1, \ldots, \lambda_m),
\]
and so the matrix \(F(f) - G\) is diagonalizable.

(b) Let us prove the statement (b) for \(A \in \mathbb{C}^{n \times n}\) with \(n > 1\). By [2, Corollary 4.6.15],
\[
\text{each square complex matrix is consimilar to a real matrix},
\]
hence \(A = \tilde{S}^{-1}BS\) for some \(B \in \mathbb{R}^{n \times n}\) and nonsingular \(S \in \mathbb{C}^{n \times n}\). By the statement (a), \(B = C + D\), in which \(C \in \mathbb{R}^{n \times n}\) is involutory and \(D \in \mathbb{R}^{n \times n}\) is real-diagonalizable. Then \(D = R^{-1}ER\), in which \(R \in \mathbb{R}^{n \times n}\) is nonsingular and \(E \in \mathbb{R}^{n \times n}\) is diagonal. Thus, \(A = \tilde{S}^{-1}CS + (RS)^{-1}E(RS)\) is a sum of a coninvolutory matrix and a real-condiagonalizable matrix.

(c) Let \(A \in \mathbb{C}^{n \times n}\) with \(n > 1\). By (b), \(A = C + D\), in which \(C\) is coninvolutory and \(D\) is real-condiagonalizable. By [2, Lemma 4.6.9], \(C\) is coninvolutory if and only if there exists a nonsingular \(S\) such that \(C = \tilde{S}^{-1}S\) (that is, \(C\) is consimilar to the identity). Then \(\tilde{S}AS^{-1} = I_n + SDS^{-1}\), in which \(\tilde{S}DS^{-1}\) is real-condiagonalizable.

(d) This statement follows from (b). \(\square\)
Corollary 2. Each \( m \times m \) companion matrix with \( m \geq 2 \) is \( \mathbb{F} \)-similar to \( G + \text{diag}(\mu_1, \ldots, \mu_m) \), in which \( G \) is involutory and \( \mu_1, \ldots, \mu_m \in \mathbb{F} \) are arbitrary pairwise unequal numbers such that \( \mu_1 + \cdots + \mu_m = a_1 + 2 - m \).

We get this corollary from (1) by taking \( \text{diag}(\mu_1, \ldots, \mu_m) := \text{diag}(\lambda_1, \ldots, \lambda_m) - I \).

3. Each \( n \times n \) matrix with \( n > 1 \) is a sum of 5 coninvolutory matrices

Theorem 3. Each \( n \times n \) complex matrix with \( n \geq 2 \) is a sum of 4 coninvolutory matrices if \( n = 2 \) and 5 coninvolutory matrices if \( n \geq 2 \).

Proof. Let us prove the theorem for \( M \in \mathbb{C}^{n \times n} \). By (5), \( M = \bar{S}^{-1}AS \) for some \( A \in \mathbb{R}^{n \times n} \) and a nonsingular \( S \). If \( A = C_1 + \cdots + C_k \) is a sum of coninvolutory matrices, then \( M = \bar{S}^{-1}C_1S + \cdots + \bar{S}^{-1}C_kS \) is also a sum of coninvolutory matrices.

Thus, it suffices to prove Theorem 3 for \( A \in \mathbb{R}^{n \times n} \).

Case 1: \( n = 2 \). By [2, Theorem 3.4.1.5], each \( 2 \times 2 \) real matrix is \( \mathbb{R} \)-similar to one of the matrices

\[
\begin{bmatrix}
a & 0 \\
0 & b
\end{bmatrix}, \quad
\begin{bmatrix}
a & 1 \\
0 & a
\end{bmatrix}, \quad
\begin{bmatrix}
a & b \\
-b & a
\end{bmatrix} \quad (b > 0), \quad a, b \in \mathbb{R}.
\] (6)

(i) The first matrix is a sum of 4 coninvolutory matrices since it is represented in the form

\[
\begin{bmatrix}
a & 0 \\
0 & b
\end{bmatrix} = \begin{bmatrix}
(a - b)/2 & 0 \\
0 & (a + b)/2
\end{bmatrix} + \begin{bmatrix}
(a + b)/2 & 0 \\
0 & (a + b)/2
\end{bmatrix}
\] and each summand is a sum of two coninvolutory matrices because

\[
\begin{bmatrix}
2c & 0 \\
0 & -2c
\end{bmatrix} = \begin{bmatrix}
c & 1 \\
(1 - c^2) & -c
\end{bmatrix} + \begin{bmatrix}
c & -1 \\
(1 - c^2) & -c
\end{bmatrix}
\] and

\[
\begin{bmatrix}
2c & 0 \\
0 & 2c
\end{bmatrix} = \begin{bmatrix}
c & i \\
(1 - c^2)i & c
\end{bmatrix} + \begin{bmatrix}
c & -i \\
(1 - c^2)i & c
\end{bmatrix}
\] (7)
are sums of two coninvolutory matrices for all \( c \in \mathbb{R} \).
(ii) The second matrix is a sum of 4 coninvolutory matrices since
\[
\begin{bmatrix}
a & 1 \\
0 & a
\end{bmatrix}
= \begin{bmatrix}
a & 0 \\
0 & a
\end{bmatrix}
+ \begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix}
\]
and each summand is a sum of two coninvolutory matrices: the first due to [7] and the second due to
\[
\begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix}
= \begin{bmatrix}
1 & 1 \\
0 & -1
\end{bmatrix}
+ \begin{bmatrix}
-1 & 0 \\
0 & 1
\end{bmatrix}.
\]

(iii) The third matrix is a sum of 4 coninvolutory matrices since
\[
\begin{bmatrix}
a & b \\
-b & a
\end{bmatrix}
= \begin{bmatrix}
a & 0 \\
0 & a
\end{bmatrix}
+ \begin{bmatrix}
0 & b \\
-b & 0
\end{bmatrix}
\]
and each summand is a sum of two coninvolutory matrices due to [7] and
\[
\begin{bmatrix}
0 & b \\
-b & 0
\end{bmatrix}
= \begin{bmatrix}
1 & b \\
0 & -1
\end{bmatrix}
+ \begin{bmatrix}
-1 & 0 \\
0 & 1
\end{bmatrix}.
\]

Thus, each 2 \times 2 matrix A is a sum of 4 coninvolutory matrices. Applying this statement to \( A - I_2 \), we get that \( A = I_2 + (A - I_2) \) is also a sum of 5 coninvolutory matrices.

Case 2: \( n \) is even. By Theorem [1(d)], A is consimilar to \( C + D \), where C is coninvolutory and D is a diagonal real matrix, which proves Theorem [3] in this case due to Case 1 since D is a direct sum of 2 \times 2 matrices.

Case 3: \( n \) is odd. By [2], A is \( \mathbb{R} \)-similar to a direct sum
\[
B = F(f_1) \oplus \cdots \oplus F(f_t), \quad f_i(x) = x^{m_i} - a_{i1}x^{m_i-1} - \cdots - a_{im} \in \mathbb{R}[x]. \quad (8)
\]
We can suppose that \( m_1 > 1 \). Indeed, if \( m_i > 1 \) for some \( i \), then we interchange \( F(f_1) \) and \( F(f_i) \). Let \( m_1 = \cdots = m_t = 1 \) and let \( a_{11} \neq 0 \) (if \( B = 0 \), then \( B = I + (-I) \) is the sum of involutory matrices). If \( a_{11} = a_{21}, \) then we replace \( a_{11} \) by \( -a_{11} \) using the consimilarity of \([a_{11}]\) and \([-a_{11}]\). By [3], \( F(f_1) \oplus F(f_2) = [a_{11}] \oplus [a_{21}] \) is \( \mathbb{R} \)-similar to \( F((x-a_{11})(x-a_{21})) \).

We obtain B of the form \( F(f_1) \oplus C \) with \( m_1 > 1 \). By Corollary [2], \( F(f_1) \) is \( \mathbb{R} \)-similar to \( G + \text{diag}(\mu_1, \ldots, \mu_{m_1}) \), in which G is a real involutory matrix and \( \mu_1, \ldots, \mu_{m_1} \in \mathbb{R} \) are arbitrary pairwise unequal numbers such that \( \mu_1 + \cdots + \mu_{m_1} = a_{11} + 2 - m_1 \).
We take \( \mu_1 = 2 \) (and then \( \mu_2 = -2 \)) if \( f_1(x) = x^2 - a_{12} \). We take \( \mu_1 = 0 \) if \( f_1(x) \neq x^2 - a_{12} \). Applying Theorem 3(d) to the other direct summands \( F(f_2), \ldots, F(f_t) \), we find that \( B \) is \( \mathbb{R} \)-similar to

\[
\begin{bmatrix}
G & 0 \\
0 & C
\end{bmatrix} + \begin{bmatrix}
\mu_1 & 0 \\
0 & D
\end{bmatrix},
\]

in which the first summand is coninvolutory and the second is a diagonal real matrix. By Case 1,

\[ D = C_1 + C_2 + C_3 + C_4, \]

in which \( C_1, C_2, C_3, C_4 \) are coninvolutory matrices. Then

\[
\begin{bmatrix}
\mu_1 & 0 \\
0 & D
\end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & C_1 \end{bmatrix} + \begin{bmatrix} \mu_1 - 1 & 0 \\ 0 & C_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & C_3 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & C_4 \end{bmatrix}
\]

is a sum of 4 coninvolutory matrices. \( \square \)

4. Each \( 2m \times 2m \) matrix is a sum of 5 skew-coninvolutory matrices

We recall that an \( n \times n \) complex matrix \( A \) is called skew-coninvolutory if \( \bar{A}A = -I_n \) (and so \( n \) is even since \( \det(\bar{A}A) \geq 0 \)).

**Theorem 4.** Each \( 2m \times 2m \) complex matrix is a sum of at most 5 skew-coninvolutory matrices.

**Proof.** Let us prove the theorem for \( A \in \mathbb{C}^{2m \times 2m} \). If \( A = \bar{S}^{-1}BS \) and \( B = C_1 + \cdots + C_k \) is a sum of skew-coninvolutory matrices, then \( A = \bar{S}^{-1}C_1S + \cdots + \bar{S}^{-1}C_kS \) is a sum of skew-coninvolutory matrices too. Thus, it suffices to prove the theorem for any matrix that is consimilar to \( A \).

By [2, Theorem 4.6.12], each square complex matrix is consimilar to a direct sum, uniquely determined up to permutation of summands, of matrices of the following two types:

\[
J_n(\lambda) := \begin{bmatrix}
\lambda & 1 & & 0 \\
& \lambda & & \\
& & \ddots & \\
0 & & & \lambda
\end{bmatrix} \quad (n \text{-by-} n, \ \lambda \in \mathbb{R}, \ \lambda \geq 0) \tag{9}
\]

and

\[
H_{2m}(\mu) := \begin{bmatrix}
0 & I_n \\
J_n(\mu) & 0
\end{bmatrix} \quad (\mu \in \mathbb{C}, \ \mu < 0 \text{ if } \mu \in \mathbb{R}) \tag{10}
\]
Thus, we suppose that $A$ is a direct sum of matrices of these types.

**Case 1: $A$ is diagonal.** Then $A$ is a sum of 4 skew-coninvolutory matrices since $A$ is a direct sum of $m$ real diagonal 2-by-2 matrices and each real diagonal 2-by-2 matrix is represented in the form

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} (a-b)/2 & 0 \\ 0 & -(a-b)/2 \end{bmatrix} + \begin{bmatrix} (a+b)/2 & 0 \\ 0 & (a+b)/2 \end{bmatrix}$$

in which each summand is a sum of two skew-coninvolutory matrices because

$$\begin{bmatrix} 2c & 0 \\ 0 & -2c \end{bmatrix} = \begin{bmatrix} c & -1 \\ (1+c^2) & -c \end{bmatrix} + \begin{bmatrix} c & 1 \\ -(1+c^2) & -c \end{bmatrix}$$

and

$$\begin{bmatrix} 2c & 0 \\ 0 & 2c \end{bmatrix} = \begin{bmatrix} c & -i \\ (1+c^2)i & c \end{bmatrix} + \begin{bmatrix} c & i \\ -(1+c^2)i & c \end{bmatrix}$$

(11)

are sums of two skew-coninvolutory matrices for all $c \in \mathbb{R}$.

**Case 2: $A$ is a direct sum of matrices of type [9].** Then it has the form

$$A = \begin{bmatrix} \lambda_1 & \varepsilon_1 & 0 \\ & \lambda_2 & \ddots \\ & & \ddots & \varepsilon_{2m-1} \\ & & & \lambda_{2m} \end{bmatrix}$$

in which all $\lambda_i \geq 0$ and all $\varepsilon_i \in \{0, 1\}$.

Represent $A$ in the form $A = C + D$, in which

$$C := \begin{bmatrix} c_1 \\ -1 + c_1^2 & 1-c_1 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} c_m \\ -1 + c_m^2 & 1-c_m \end{bmatrix}, \quad \text{all } c_i \in \mathbb{R},$$

is a skew-coninvolutory matrix. Let us show that $c_1, \ldots, c_m$ can be chosen such that all eigenvalues of $D$ are distinct real numbers.

The matrix $D$ is upper block-triangular with the diagonal blocks

$$D_1 := \begin{bmatrix} \lambda_1 - c_1 & \varepsilon_1 - 1 \\ 1 - c_1^2 & \lambda_2 + c_1 \end{bmatrix}, \ldots, D_m := \begin{bmatrix} \lambda_{2m-1} - c_m & \varepsilon_{2m-1} - 1 \\ 1 - c_m^2 & \lambda_{2m} + c_m \end{bmatrix}.$$
Let \( c_1, \ldots, c_{k-1} \) have been chosen such that the eigenvalues of \( D_1, \ldots, D_{k-1} \) are distinct real numbers \( \nu_1, \ldots, \nu_{2k-2} \). Depending on \( \varepsilon_{2k-1} \in \{0, 1\} \), the matrix \( D_k \) is

\[
\begin{bmatrix}
\lambda_{2k-1} - c_k & -1 \\
1 - c_k^2 & \lambda_{2k} + c_k
\end{bmatrix}
\quad \text{or} \quad
\begin{bmatrix}
\lambda_{2k} - c_k & 0 \\
1 - c_k^2 & \lambda_{2k} + c_k
\end{bmatrix}.
\]  

(12)

- Let \( D_k \) be the first matrix in (12). Its characteristic polynomial is

\[
\chi_k(x) = x^2 - \text{trace}(D_k)x + \det(D_k)
= x^2 - (\lambda_{2k-1} + \lambda_{2k})x + (\lambda_{2k-1} - c_k)(\lambda_{2k} + c_k) + 1 - c_k^2.
\]

Its discriminant is

\[
\Delta_k = (\lambda_{2k-1} + \lambda_{2k})^2 - 4[\lambda_{2k-1}\lambda_{2k} + (\lambda_{2k-1} - \lambda_{2k})c_k - 2c_k^2 + 1]
= (\lambda_{2k-1} - \lambda_{2k})^2 + 4(-\lambda_{2k-1} + \lambda_{2k})c_k + 8c_k^2 - 4.
\]

For a sufficiently large \( c_k \), \( \Delta_k > 0 \) and so the roots of \( \chi_k(x) \) are some distinct real numbers \( \nu_{2k-1} \) and \( \nu_{2k} \). Since

\[\nu_{2k-1} + \nu_{2k} = \text{trace}(D_k) = \lambda_{2k-1} + \lambda_{2k},\]

we have

\[\det(D_k) = \nu_{2k-1}\nu_{2k} = \nu_{2k-1}(\lambda_{2k-1} + \lambda_{2k} - \nu_{2k-1})
= (\lambda_{2k-1} + \lambda_{2k} - \nu_{2k})\nu_{2k}.
\]

Taking \( c_k \) such that

\[\det(D_k) \neq \nu_i(\lambda_{2k-1} + \lambda_{2k} - \nu_i) \quad \text{for all} \ i = 1, \ldots, 2k-2,\]

we get \( \nu_{2k-1} \) and \( \nu_{2k} \) that are not equal to \( \nu_1, \ldots, \nu_{2k-2} \).

- Let \( D_k \) be the second matrix in (12). Then its eigenvalues are \( \lambda_{2k} - c_k \) and \( \lambda_{2k} + c_k \). We choose a nonzero real \( c_k \) such that these eigenvalues are not equal to \( \nu_1, \ldots, \nu_{2k-2} \).

We have constructed the real skew-coninvolutory matrix \( C \) such that \( A = C + D \), in which \( D \) is a real matrix with distinct eigenvalues \( \nu_1, \ldots, \nu_{2m} \in \mathbb{R} \). Since \( D \) is \( \mathbb{R} \)-similar to a diagonal matrix and by Case 1, \( D \) is a sum of 4 skew-coninvolutory matrices.
Case 3: A is a direct sum of matrices of types (9) and (10). Due to Case 2, it suffices to prove that each matrix $H_{2m}(\mu)$ is a sum of 5 skew-coninvolutory matrices. Write

$$
\begin{bmatrix}
0 & I_n \\
J_n(\mu) & 0
\end{bmatrix} = \begin{bmatrix}
0 & I_n \\
-I_n & 0
\end{bmatrix} + \begin{bmatrix}
0 & 0 \\
J_n(\mu) + I_n & 0
\end{bmatrix}.
$$

The first summand is a skew-coninvolutory matrix, and so we need to proof that the second summand is a sum of 4 skew-coninvolutory matrices. By \[1\], there exists a nonsingular $S$ such that $B := \bar{S}^{-1}(J_n(\mu) + I_n)S$ is a real matrix. Then the second summand is consimilar to a real matrix:

$$
\begin{bmatrix}
S^{-1} & 0 \\
0 & S^{-1}
\end{bmatrix} \begin{bmatrix}
0 & 0 \\
J_n(\mu) + I_n & 0
\end{bmatrix} \begin{bmatrix}
S & 0 \\
0 & S
\end{bmatrix} = \begin{bmatrix}
0 & 0 \\
B & 0
\end{bmatrix},
$$

which is the sum of two coninvolutory matrices:

$$
\begin{bmatrix}
0 & 0 \\
B & 0
\end{bmatrix} = \begin{bmatrix}
I_n & 0 \\
0 & -I_n
\end{bmatrix} + \begin{bmatrix}
-I_n & 0 \\
0 & I_n
\end{bmatrix}. \tag{13}
$$

By \[2\] Lemma 4.6.9, each coninvolutory matrix is consimilar to the identity matrix. Hence, each summand in (13) is consimilar to $I_{2n}$, which is a sum of two skew-coninvolutory matrices due to (11). Thus, the matrix (13) is a sum of 4 skew-coninvolutory matrices.

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