Entanglement generation between a charge qubit and its bosonic environment during pure dephasing - dependence on environment size

Tymoteusz Salamon
Department of Theoretical Physics, Faculty of Fundamental Problems of Technology, Wrocław University of Science and Technology, 50-370 Wrocław, Poland

Katarzyna Roszak
Department of Theoretical Physics, Faculty of Fundamental Problems of Technology, Wrocław University of Science and Technology, 50-370 Wrocław, Poland
(Dated: September 5, 2017)

We study entanglement generated between a charge qubit and a bosonic bath due to their joint evolution which leads to pure dephasing of the qubit. We tune the parameters of the interaction, so that the decoherence is quantitatively independent of the number of bosonic modes taken into account and investigate, how the entanglement generated depends on the size of the environment. A second parameter of interest is the mixedness of the initial state of the environment which is controlled by temperature. We show analytically that for a pure initial state of the environment, entanglement does not depend on environment size. For mixed initial states of the environment, the generated entanglement decreases with the increase of environment size. This effect is stronger for larger temperatures, when the environment is initially more mixed, but in the limit of an infinitely large environment, no entanglement is created at any finite temperature.

I. INTRODUCTION

Decoherence due to the interaction of a qubit with its environment can often be modeled by classical noise with a good deal of accuracy. This is especially true with respect to pure dephasing, i.e., when decoherence does not disturb the occupations of the qubit, but affects only its coherence (the off-diagonal elements of the density matrix which are responsible for quantum behavior). Such loss of qubit coherence can always be mapped by a random unitary channel acting on the system, where the channel describes the interaction with a fictitious classical environment [1] (see Ref. [2] for examples of constructions of such fictitious classical environments for different types of open quantum systems). The existence of such a mapping does not invalidate the importance of qubit-environment entanglement in pure-dephasing scenarios, since the presence of entanglement in the system will influence the evolution of the system in general, e.g., it changes the state of the environment pre- and post-measurement [3]. Furthermore, if the study of a quantum system is not limited to its preparation, allowing it to evolve freely, and then measurement, but also involves some manipulation of the system, such as performing gates [4, 5], coherence maximizing schemes [6, 7], etc. then the presence of system-environment entanglement will influence the end result and can be highly relevant.

Hence, it is at least in principle possible to have different qubit-environment setups, in which the pure dephasing of the qubit is qualitatively and quantitatively the same, but the origin of the dephasing is different, since it can, but does not have to be the result of entanglement generation. If the whole system is always in a pure state (for an evolution described by a Hamiltonian this is equivalent to the initial states of the qubit and environment being pure) pure dephasing is unambiguously related to entanglement [10, 11]. In the case of mixed states, the relation between qubit coherence and qubit-environment entanglement is much more ambiguous and although entanglement not accompanied by dephasing is not possible, dephasing without entanglement is [12]. In fact, the latter situation is often realized in real systems, especially in the case of large environments, high temperatures, or noise resulting from e.g., fluctuating semi-classical fields [13–16]. The distinction between entangling and non-entangling evolutions is not trivial in itself, since the non-entangling case is not limited to random unitary evolutions [17, 21], and a straightforward criterion for the generation of qubit-environment entanglement during pure dephasing has only recently been found [8].

We study the amount of qubit-environment entanglement generated during the joint evolution of a charge qubit interacting with a bath of phonons as an example of a realistic system in which the qubit undergoes strictly pure dephasing which is always accompanied by the creation of entanglement (with the exception of only the infinite-temperature situation) [3]. The system is particularly convenient, because not only can the decoherence-curves be reproduced using an arbitrary number of phonon modes (for short enough times and large enough temperatures), but the results can be obtained in a semi-analytical fashion, which simplifies changing the number of phonon modes and later interpretation of the results. Furthermore, we control the initial level of mixedness of the environment by setting the temperature of the phonon bath.

The correlation build-up between a system and its bosonic environment have thus far been studied in the context of its relation towards non-Markovian dynamics [22], decoherence (especially for mixed initial qubit
The conclusions are given in Sec. VI.

We find that the dependence of generated entanglement on the temperature shows monotonously decreasing behavior which is steep above a physically well motivated threshold temperature. The temperature dependence is non-trivial even above this temperature, and for reasonably low temperatures displays exponential decay, while for high temperatures the dependence becomes proportional to $1/T^3$. On the other hand, the dependence on the number of phonon modes is much more complex. For pure states, the amount of entanglement generated does not depend on the number of phonon modes at all. Yet for any mixed environmental state (finite temperature) there is a pronounced dependence on the number of phonon modes (on the size of the environment) of $1/n$ character (where $n$ is the number of modes). The maximum entanglement generated throughout the evolution always decreases when the environment becomes larger, even though the decoherence curve is unaffected in the studied scenario for high enough temperatures. Furthermore, this decrease is steeper when the temperature is higher (the initial state is more mixed), but when the number of phonon modes approaches infinity (the continuous case), all finite-temperature entanglement vanishes.

The article is organized as follows. In Sec. II we introduce the system under study, the Hamiltonian describing this system, and the full qubit-environment evolution resulting from this Hamiltonian. We furthermore describe, how the respective strengths of the bosonic modes are determined, so that the decoherence at short times and large enough temperatures is qualitatively and quantitatively the same independently of the size of the environment. Sec. III contains a brief description of Negativity, which is the entanglement measure which is later used to quantify qubit-environment entanglement. In Sec. IV the dependence of the purity of the whole system (which is constant throughout the evolution) on the initial state of the environment is determined, especially on the temperature and size of the environment. Sec. V contains the results pertaining to the dependence of the generated entanglement on temperature and consequently, on the degree of mixedness of the initial state of the environment, as well as the results concerning the effect of environment size on entanglement, when the characteristics of the resulting qubit decoherence remain unchanged. The conclusions are given in Sec. VI.

II. THE SYSTEM AND THE EVOLUTION

The system under study consists of a charge qubit interacting with phonons. The Hamiltonian of this system is

$$H = \epsilon |1\rangle \langle 1| + \sum_k \hbar \omega_k b_k^\dagger b_k + |1\rangle \langle 1| \sum_k (f_k^* b_k + f_k b_k^\dagger),$$

where the first term describes the energy of the qubit ($\epsilon$ is the energy difference between the qubit states $|0\rangle$ and $|1\rangle$ in the absence of phonons), the second term is the Hamiltonian of the free phonon subsystem and the third term describes their interaction. Here, $\omega_k$ is the frequency of the phonon mode with the wave vector $k$ and $b_k^\dagger$, $b_k$ are phonon creation and annihilation operators corresponding to mode $k$ and $f_k$ are coupling constants.

The Hamiltonian can be diagonalized exactly using the Weyl operator method (see Ref. 28 for details; the same results can be obtained using a different approach 29, 30). For a product initial state of the system and the environment, $\sigma(0) = |\psi\rangle\langle\psi| \otimes R(0)$, where the qubit state is pure, $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, and the environment is at thermal equilibrium,

$$R(0) = \frac{e^{-\frac{\beta e^{-i\omega_k \omega t} \sum_k \hbar \omega_k b_k^\dagger b_k}{\hbar}}}{\text{Tr}[e^{-\frac{\beta e^{-i\omega_k \omega t} \sum_k \hbar \omega_k b_k^\dagger b_k}{\hbar}]},$$

where $k_B$ is the Bolzmann constant and $T$ is the temperature, the joint qubit-environment density matrix evolves according to

$$\hat{\sigma}(t) = \begin{pmatrix} |\alpha|^2 \hat{R}(0) & \alpha \beta^* e^{-i\omega_k \omega t} \hat{R}(0) \hat{u}(t) \hat{R}(0) \hat{u}^\dagger(t) \\ \alpha^* \beta e^{i\omega_k \omega t} \hat{u}(t) \hat{R}(0) & |\beta|^2 \hat{R}(0) \hat{u}(t) \hat{R}(0) \hat{u}^\dagger(t) \end{pmatrix}. \tag{3}$$

Here, the matrix is written in the basis of the qubit states $|0\rangle$ and $|1\rangle$, while the degrees of freedom of the environment are contained in the density matrix $\hat{R}(0)$ and time-evolution operators acting only on the environment, $\hat{u}(t)$. The evolution operators can be found following Ref. 28, and are given by

$$\hat{u}(t) = \exp \left\{ \sum_k \left( \frac{f_k^*}{\omega_k} (1 - e^{i\omega_k \omega t}) b_k^\dagger - \frac{f_k}{\omega_k} (1 - e^{i\omega_k \omega t}) b_k \right) \right\} \times \exp \left\{ i \sum_k \frac{f_k^2}{(\omega_k)^2} \sin \omega_k t \right\}. \tag{4}$$

In order to obtain the density matrix of the qubit alone, a trace over the degrees of freedom of the environment needs to be performed, $\hat{\rho}(t) = \text{Tr}_E \hat{\sigma}$. This yields a density matrix with time-independent occupations and coherence which undergo decay governed by the function

$$|\langle \hat{u}(t) \rangle| = \exp \left\{ -\sum_k \left( \frac{f_k}{\omega_k} \right)^2 (1 - \cos \omega_k t) (2n_k + 1) \right\}. \tag{5}$$
where $n_k = 1/(e^{\hbar \omega_k / k_B T} - 1)$ is the Bose-Einstein distribution.

If, as in our case, the quantity of interest is qubit-environment entanglement and not just the coherence of the qubit, the time-evolution of the full system density matrix is needed. This can be found by acting with the evolution operator given by eq. (4) on the initial density matrix of the environment. The density matrix of the whole system $\hat{\sigma}$ can be divided into four parts with respect to the way that the evolution operator acts on the density matrix of the environment, which correspond to the $|0\rangle$ qubit state occupation (for which the environment remains unaffected $\hat{R}_{00}(t) = \hat{R}(0)$), the $|1\rangle$ qubit state occupation (for which $\hat{R}_{11}(t) = \hat{u}(t)\hat{R}(0)\hat{u}^\dagger(t)$), and the two qubit coherences (with $\hat{R}_{01}(t) = \hat{R}(0)\hat{u}^\dagger(t)$ when the qubit density matrix element corresponds to $|0\rangle|1\rangle$ and $\hat{R}_{10}(t) = \hat{u}(t)\hat{R}(0)$ when the qubit density matrix element corresponds to $|1\rangle|0\rangle$).

Since the initial density matrix of the environment is a product of density matrices for each boson mode $\hat{R}(0) = \bigotimes_k \hat{R}^k(0)$ and so is the evolution operator at all times $\hat{u}(t) = \bigotimes_k \hat{u}^k(t)$, each matrix $\hat{R}_{ij}(t)$ ($i, j = 0, 1$) also has product form with respect to the different boson modes at all times, so each boson mode can be treated separately. This is convenient, since every $\hat{R}^k_{ij}(t)$ matrix is in principle of infinite dimension (the number of phonons in each mode can be arbitrarily large; the actual distribution of states for a single phonon mode is governed by the temperature and the qubit-phonon coupling) and a reasonable cut-off needs to be implemented to keep the density matrix $\sigma$ manageable without the loss of physical meaning. Note, that although the $\hat{R}_{ij}(t)$ matrices corresponding to the diagonal elements of the qubit density matrix are density matrices themselves, this is not always true for the $\hat{R}_{ij}(t)$ matrices with $i \neq j$ (which is a first indicator of qubit-environment entanglement).

It can be shown that the evolution of any state of $m$ phonons in mode $k$ is given by

$$|m(t)\rangle_k = \hat{u}^k(t)|m\rangle_k = \sum_{p=-m}^{\infty} \left( \frac{f_k}{\hbar \omega_k} \right)^p \sqrt{\frac{m!}{(m+p)!}} \times L_m^{(p)} \left( \frac{f_k}{\hbar \omega_k} \right)^2 |m+p\rangle_k,$$

where $L_m^{(p)}(x)$ is a generalized Laguerre polynomial. Given the initial state of the environment, eq. (5) is sufficient to find the time evolution of the whole system-environment density matrix $\hat{\sigma}$, since

$$\hat{R}_{11}(t) = \hat{u}(t)\hat{R}(0)\hat{u}^\dagger(t) = \bigotimes_k \left( \sum_{m_k=0}^{\infty} c_{m_k}|m(t)\rangle_k \langle m| \right),$$

$$\hat{R}_{10}(t) = \hat{u}(t)\hat{R}(0) = \bigotimes_k \left( \sum_{m_k=0}^{\infty} c_{m_k}|m(t)\rangle_k \langle m| \right),$$

$$\hat{R}_{01}(t) = \hat{R}(0)\hat{u}^\dagger(t) = \hat{R}_{10}^\dagger(t).$$

Here, the initial occupations of each state $|m\rangle_k$ are found for a given temperature using eq. (2).

$$c_{m_k} = e^{-\hbar \omega_k / k_B T} m_k \left( 1 - e^{-\hbar \omega_k / k_B T} \right).$$

A. Excitonic quantum dot qubits

The exciton-phonon interaction constants used in the calculations correspond to excitonic qubits confined in quantum dots \cite{28, 31, 33}, where qubit state $|0\rangle$ corresponds to an empty dot, while state $|1\rangle$ denotes an exciton in its ground state confined in the dot. They are given by

$$f_k = (\sigma_e - \sigma_h) \sqrt{\frac{\hbar k}{2\rho V_N c}} \int_{-\infty}^{\infty} d^3r \psi^*(r) e^{-i k \cdot r} \psi(r),$$

describing the deformation potential coupling, which is the dominating decoherence mechanism for excitons \cite{33}. Hence, $\omega_k = ck$, where $c$ is the speed of longitudinal sound and the phonon-bath is super-Ohmic. Here $g$ is the crystal density, $V_N$ unit cell volume, and $\sigma_e, h$ are deformation potential constants for electrons and holes respectively. The exciton wave function $\psi(r)$ is modeled as a product of two identical single-particle wave functions $\psi(r_e)$ and $\psi(r_h)$, corresponding to the electron and hole, respectively.

The parameters used in the calculations correspond to small self-assembled InAs/GaAs quantum dots, which are additionally assumed to be isotropic (for the sake of simplicity when limiting the number of phonon modes and with little loss of realism, when the evolution of coherence is found). The single particle wave functions $\psi(r)$ are modeled by Gaussians with 3 nm width in all directions. The deformation potential difference is $\sigma_e - \sigma_h = 9.5$ eV, the crystal density is $\rho = 5300$ kg/m$^3$, and the speed of longitudinal sound is $c = 5150$ m/s. The unit cell volume for GaAs is $V_N = 0.18$ nm$^3$ (note, that this volume does not enter into the decoherence function, but is relevant, when individual elements of the system-environment density matrix needed to evaluate entanglement are found).

B. Discretization

Typically for realistic systems, the number of phonon modes is very large and the summation over $k$ can be substituted by integration, which in spherical coordinates yields

$$\sum_k \to \frac{V}{(2\pi)^3} \int_0^{2\pi} d\phi \int_0^\pi \sin \theta d\theta \int_{-\infty}^{\infty} k^2 dk.$$

If the studied system has spherical symmetry, as do the quantum dots, the parameters of which are used in the
calculations, the integration over the angles can be performed analytically. In the following, when we study qubit-environment entanglement and qubit decoherence due to the interaction with an environment which supports only a limited number of phonon modes, we do not differentiate the modes with respect to their direction, only with respect to the length of the wave vector. This means that we consider a simplified scenario, where phonon modes are averaged over all directions.

The actual discretization of the phonon modes is done only on the level of the length of the wave vector $k$. A minimum and maximum wave vector length is, somewhat arbitrarily, chosen, so that a large enough range of $k$ is considered to account for different phonon modes with values of the function $|f_k/\hbar \omega_k|^2$ which are large enough to be relevant for both pure dephasing and entanglement generation. In the following they are always set to $k_{\min} = 0.001 \text{ nm}^{-1}$ and $k_{\max} = 0.9 \text{ nm}^{-1}$. For a given number of phonon modes $n$, the range $[k_{\min}, k_{\max}+k_{\min}]$ is evenly divided, and only wave vectors of lengths $k_i = (i - 1)\Delta k + k_{\min}$, with $\Delta k = k_{\max}/(n - 1)$ and $i = 1, 2, ..., n$, are taken into account. The slight off-set by of the range of $k$ by $k_{\min}$ allows for the decoherence of the qubit in the continuous case (when an infinite number of phonon modes is taken into account) to be well approximated by only a few phonon modes for a wide range of temperatures as seen below.

The decay of qubit coherence for a quantum dot interacting with an environment for which only a few discrete lengths of phonon wave vectors are allowed (in which only a few phonon modes are present) is plotted in Fig. 1 for $T = 6 \text{ K}$ (note that the coherence on the plot is limited from below by 0.7 and not by the minimum value of the degree of coherence, 0, for clarity). The initial drop of coherence which is present in the continuous case, and which is in the continuous case followed by a slight rise in coherence, after which the coherence stabilizes at some finite value due to the super-Ohmic nature of the environment (hence the term “partial pure dephasing”; see Ref. [28] for details) is reproduced very well already when only three phonon modes are present in the environment. For larger boson mode numbers $n$, the longer-time features of decoherence are also reproduced for a finite time, and the refocusing of the qubit (which is the result of the whole qubit-environment density matrix returning to its initial state in the course of its unitary evolution) is further and further delayed in time with increasing number of phonon modes.

In fact, it is easy to reproduce continuous short-time behavior of the coherence even with small phonon mode numbers as long as the initial temperature is above some threshold, which depends on the phonon energy spectrum, and is here around 3 K (this also obviously depends on the number of phonon modes and the threshold is higher for higher mode numbers). Below this temperature, the approximation gets progressively worse, if the choice of phonon modes remains unchanged and spans the whole energy range where the spectral density is reasonably large. This is because the temperature enters into the calculation only through the initial state of the environment (which is at thermal equilibrium) and, if only a few phonon modes with well separated energies are allowed, then below some temperature practically no phonons will be initially excited. This can be remedied by a redistribution of the phonon modes to lower energies, but since we wish to compare qubit-environment entanglement evolutions, when environments with different numbers of boson modes lead to the same dynamics of the qubit alone, but with a set system under study for a given number of modes, we simply restrict ourselves to higher temperatures. This means that for a set number of phonon modes the studied system is always qualitatively the same (and even though the initial state of the system depends on temperature, the types of phonons which can be excited remain unchanged).

![FIG. 1. Evolution of the degree of qubit coherence at $T = 6 \text{ K}$ for different numbers of phonon modes: $n = 3$ - dashed blue line, $n = 5$ - dotted red line, $n = 7$ - dashed-dotted green line, $n = 100$ - solid black line.](image)

The measure of entanglement which is most convenient (easiest to compute) in the context of quantifying entanglement between a qubit (a small quantum system) and its environment (a large quantum system) is Negativity [34, 35] (or equivalently logarithmic Negativity [36]). The measure is based on the PPT criterion of separability [37, 38], which does not detect bound entanglement [39, 40]. Fortunately, in the case of the evolution of a qubit initially in a pure state interacting with an arbitrary environment due to an interaction which can only lead to pure dephasing of the qubit, bound entanglement is never formed [3, 11], so non-zero Negativity is a good criterion for the presence of entanglement in the system and the value of Negativity unambiguously indicates the amount of said entanglement.

Negativity can be defined as the absolute sum of the
negative eigenvalues of the density matrix of the whole system after a partial transposition with respect to one of the two potentially entangled subsystems has been performed (it does not depend on which of the subsystems is chosen for the partial transposition),

\[ N(\hat{\sigma}) = \sum_i \frac{|\lambda_i| - \lambda_i}{2}, \]

where \( \lambda_i \) are the eigenvalues of \( \hat{\sigma} \), and \( \Gamma_A \) denotes partial transposition with respect to system \( A = Q, E \) (qubit or environment). In the case of the studied system, it is particularly simple to perform partial transposition with respect to the qubit, as it is sufficient to exchange the off-diagonal terms in eq. (3) to get the desired partially transposed state.

**IV. PURITY**

An important factor for the amount of entanglement generated between the qubit and the environment is the initial purity of the state of the whole system. Note, that since the qubit-environment evolution is unitary, the purity does not change with time, since

\[ P(\hat{\sigma}(t)) = Tr \hat{\sigma}^2(t) = Tr \left[ \hat{U}(t)\hat{\sigma}(0)\hat{U}^\dagger(t)\hat{U}(t)\hat{\sigma}(0)\hat{U}^\dagger(t) \right] = Tr \hat{\sigma}^2(0) = P(\hat{\sigma}(0)). \]

Taking into account that the initial state of the studied system is a product state and the state of the qubit subsystem is pure, we have

\[ P(\hat{\sigma}(0)) = P(\hat{\rho}(0))P(\hat{R}(0)) = P(\hat{R}(0)), \]

so the purity of the system only depends on the initial purity of the density matrix of the environment. Furthermore, this initial density matrix is a product of the thermal-equilibrium density matrices for each phonon mode, so the purity is a product of the purities of the state of each mode, \( P(\hat{R}(0)) = \prod_k P(\hat{R}^k(0)) \). The purity of the initial state of mode \( k \) is easily found from eq. (2) and is given by

\[ P(\hat{R}^k(0)) = \left( 1 - e^{-\frac{\omega_k}{k_B T}} \right)^2 \left( 1 - e^{-2\frac{\omega_k}{k_B T}} \right). \]

Since \( e^{-\frac{\omega_k}{k_B T}} \) tends to one with growing temperature more slowly than \( e^{-2\frac{\omega_k}{k_B T}} \), the numerator in eq. (12) tends to zero much faster than the denominator, and the purity of the initial state of a given phonon mode is a decreasing function of temperature (which reaches zero for infinite temperature, since the dimension of the Hilbert space of each phonon mode is infinite). Consequently, the purity of the whole environment for a set choice and number of phonon modes is always a decreasing function of temperature as well.

Less obviously, for the system under study the purity is also a decreasing function of the number of phonon modes. For a given number of modes \( n, n \) wave vectors are taken into account which are evenly distributed throughout a set wave vector lengths \( k \) where the coupling constants are most relevant as explained in Sec. II B, and the purity of their initial state is a product of the corresponding single-phonon-mode purities. The energy of each phonon mode is proportional to its wave vector length, \( \omega_k = ck \), so the single-phonon-mode purity is an increasing function of \( k \) for any finite temperature. If the number of phonon modes is increased by one, each phonon mode is substituted by one with smaller wave vector length \( k \) (with the exception of the two phonon modes limiting the range of \( k \)), and hence, of lesser purity, and an additional phonon mode with a longer wave vector is taken into account (since now we are dealing with \( n + 1 \) phonon modes evenly distributed over the same range). Consequently, the product of the new \( n + 1 \) purities, which yields the purity of the initial state of the environment for an increased number of phonon modes, must be smaller for any finite temperature than the purity for \( n \) modes. The exception is the zero-temperature case, for which the purity of the initial environment is always equal to one, since the initial state is pure, and the infinite-temperature case, when the purity is always equal to zero. Hence, although for every number of phonon modes the purity is a decreasing function of temperature ranging from one to zero, the decrease is faster, if \( n \) is larger.

**V. RESULTS: QUBIT-ENVIRONMENT ENTANGLEMENT GENERATION**

**A. Temperature dependence**

The time-evolution of entanglement for the initial qubit state with \( \alpha = \beta = 1/\sqrt{2} \) (equal superposition) quantified by Negativity is plotted in Fig. (2) for \( n = 10 \) boson modes at three different temperatures (the temperature increases from top to bottom of the figure). The relatively large number of boson modes guarantees that the evolution of the qubit state due to the exciton-phonon interaction not only reproduces the fast initial decay of qubit coherence which occurs in the first two picoseconds after the creation of the excitonic superposition for a continuous environment, but also gives a reasonable approximation of the coherence plateau for the next four picoseconds (up to slight oscillations which are absent when an infinite number of phonon modes is taken into account). The temperatures in the plot start at 6 K, well above the threshold temperature, so phonon modes which are evenly distributed over the range of relevant coupling constants \( f_k \) reproduce the dynamics of decoherence for short times well. The plots in Fig. (2) capture a single cycle of qubit-environment evolution, so at the right end of the plots, the density matrix of the whole system
returns to its initial state, and then the evolution is repeated (this is an unavoidable feature of systems with discrete spectra).

![Figure 2](image.png)

**FIG. 2.** Entanglement evolution for \( n = 10 \) boson modes at \( T = 6 \) K (upper panel), \( T = 9 \) K (middle panel), and \( T = 12 \) K (lower panel).

As can be seen, changing the temperature does not change the qualitative features of the evolution of Negativity, but entanglement (at any given time) is a decreasing function of temperature. Contrarily, decoherence increases with temperature, so for higher temperatures the effect of the environment on the qubit is larger (leading to stronger pure dephasing), but this is due to a buildup of classical qubit-environment correlations, since the amount of entanglement generated between the two subsystems decreases with decreasing purity of the state.

In Fig. 3 the dependence of the maximum Negativity reached during the pure dephasing evolution (Negativity reached at the first maximum which corresponds to the initial strong loss of coherence) is plotted as a function of temperature for a choice of three different numbers of boson modes. Below around \( T = 2 \) K, the maximum Negativity stabilizes at an almost fixed value. This is because the initial density matrix of the environment below this temperature becomes a very weak function of temperature, since only phonon modes with high energies compared to \( k_B T \) are taken into account (if there are only a few phonon modes allowed in the system). The density matrix of the environment is then almost in the pure state \( \hat{R}(0) \approx |0\rangle\langle 0| \), and the resulting qubit decoherence is no longer a good approximation of the continuous case. Note that in such situations, the plateau in Negativity is strictly related to the discrete nature of the phonon energy spectrum. An agreement between continuous and few-phonon-mode decoherence could be reached also for low temperatures, but this would require changing \( k_{\text{max}} \) and would qualitatively change the system under study (which we want to avoid). If the necessary redistribution of the phonon modes taken into account (to account for decoherence well) were made, the plateau in low-temperature negativity would not be observed.

At higher temperatures, maximum Negativity decreases strongly with temperature, regardless of the number of phonon modes taken into account, although the actual amount of entanglement in the system depends strongly on \( n \). The shapes of the Negativity curves plotted in Fig. 3 roughly resemble the temperature dependence of the purity, which is found using eq. \((12)\) with appropriate values of wave vectors \( k \), meaning that the dependence in the shown temperature range is predominantly exponential decay. At high temperatures (for nanostructures, meaning far outside the 20 K range of Fig. 3), the decay is dominated by terms proportional to \( 1/T^3 \). The fitted dependence of Negativity on temperature is presented at the end of Sec. (V C), since the dependence on temperature is convoluted with the dependence on environment size and cannot be considered separately.

Note that the trade-off temperature behavior, which is characteristic for the build up of correlations in the studied system \([42]\) and which results from the decrease of purity with temperature accompanied by an increase of the overall effect of the environment on the qubit, is not present here, as only the purity of the system state is relevant for the generation of entanglement, as long as the system-environment interaction is capable of entangling the two subsystems. Contrarily, this type of trade-off behavior has been reported for boson-boson system-environment ensembles \([27]\).

**B. Dependence on environment size - pure initial state**

In the case of a pure initial state of the environment (at zero temperature in the case of the studied system, so there are initially no phonons), the joint evolution of the system and the environment remains pure and entangle-
ment at any time can be evaluated in a straightforward manner using the von Neumann entropy of one of the entangled subsystems (such von Neumann entropy is the unique entanglement measure for pure states). The measure is defined as

\[ E(\ket{\psi(t)}) = -\frac{1}{\ln 2} \text{Tr} (\rho(t) \ln \rho(t)), \] (13)

where \( \ket{\psi(t)} \) is the pure system-environment state and \( \rho(t) = \text{Tr}_E \ket{\psi(t)} \bra{\psi(t)} \) is the density matrix of the qubit at time \( t \) (obtained by tracing out the environment). The entanglement measure in eq. (13) is normalized to yield unity for maximally entangled states. The same result would be obtained when tracing out the qubit degrees of freedom instead of the environmental degrees of freedom, but the small dimensionality of the qubit makes this way much more convenient.

Let us denote the pure initial state of the environment as \( \ket{R_0} \). Then qubit-environment state at time \( t \) is given by

\[ |\psi(t)\rangle = \alpha \ket{0} \otimes |R_0\rangle + \beta e^{i\epsilon t/\hbar} \ket{1} \otimes \hat{u}(t) |R_0\rangle, \] (14)

and \( \hat{u}(t) \) is given by eq. (4). The density matrix of the qubit is now of the form

\[ \rho(t) = \begin{pmatrix} |\alpha|^2 & \alpha^* \beta e^{-i\epsilon t/\hbar} u(t) \\ \alpha \beta e^{i\epsilon t/\hbar} u^*(t) & |\beta|^2 \end{pmatrix}, \] (15)

where \( u(t) = \langle R_0 | \hat{u}(t) | R_0 \rangle \) and the absolute value of the function \( u(t) \) constitutes the degree of coherence retained in the qubit system at a given time (it is given by eq. (5) with \( T = 0 \)).

The entanglement measure of eq. (13) can be calculated using eq. (15) which yields

\[ E(\ket{\psi(t)}) = \frac{1}{\ln 2} \left[ \frac{1 + \sqrt{\Delta(t)}}{2} \ln \frac{1 + \sqrt{\Delta(t)}}{2} + \frac{1 - \sqrt{\Delta(t)}}{2} \ln \frac{1 - \sqrt{\Delta(t)}}{2} \right], \] (16)

with \( \Delta(t) = 1 - 4|\alpha|^2 |\beta|^2 + |\alpha|^2 |\beta|^2 |u(t)|^2 \). Note that the von Neumann entropy during pure dephasing depends only on the degree of coherence \( |u(t)| \). This means that, if two qubits lose the same amount of coherence, they must be entangled with the environment to the same degree, regardless of the number of boson modes which constitute the environment. Hence, for pure initial environmental states, the amount of entanglement generated during evolution does not depend on the size of the environment, as long as the degree of coherence at a given time does not depend on its size.

A similar analysis can be performed using Negativity as the measure of pure state entanglement (Negativity does not converge to von Neumann entropy for pure states contrarily to most entanglement measures). It is then fairly straightforward to show that entanglement does not depend on the size of the environment, but only on the degree of coherence \( u(t) \), taking into account the fact that Negativity in the studied system does not depend on the phase relations between different components of the density matrix \( \hat{\sigma} \). What is not straightforward is obtaining the explicit relation between Negativity and decoherence, and therefore the von Neumann entropy was used in the analysis above.

C. Dependence on environment size

![FIG. 4. Entanglement evolution for three different numbers of boson modes (n = 6 - blue solid line, n = 8 - red dashed line, n = 10 - green dotted line) at temperature T = 6 K. Vertical line indicates the first time at which entanglement is maximized.](image)

![FIG. 5. Maximal entanglement as a function of the number of boson modes for different temperatures. The points correspond to numerical data (blue dots - 6 K, red triangles - 9 K, green squares - 12 K), while the lines depict the fitting function and are color-coded in the same way.](image)
account becomes very important. In Fig. [4] the evolution of Negativity for the equal superposition initial state of the qubit (α = β = 1/√2) is plotted at T = 6 K for different numbers of boson modes n. Note that the temperature is high enough, so that the initial drop of qubit coherence is always the same as in the continuous case, while the plateau is reproduced for some time after the drop up to small oscillations (and this time is longer for larger n). This means that the three curves in Fig. [4] correspond to the same qubit decoherence curves at short times (up to roughly 5 ps here). Obviously, the amount of entanglement generated during these decoherence processes is not the same, as both the maximum values and the values at the plateau decrease with increasing number of boson modes. As the temperature dependence, this is related to the purity of the whole system during the evolution, but the temperature also affects the decoherence curves, while the number of phonon modes (when they are chosen as outlined in Sec. [1]) does not.

The dependence of maximum Negativity as a function of the number of boson modes taken into account is plotted in Fig. [5] with points for three different temperatures (well above the threshold value). The decrease of Negativity with growing n is rather steep for the temperatures shown and this steepness increases when the temperature grows. This corresponds to the fact that at zero temperature, entanglement does not depend on the size of the environment, while at infinite temperature no entanglement is generated between the qubit and the environment is generated at all [3]. Furthermore, for any finite temperature entanglement approaches zero with growing n according to a function proportional to 1/n, and for a continuous environment, no entanglement is generated in the system. Although separability is reached more slowly at lower temperatures it is reached nonetheless for large enough values of n (technically, zero-Negativity is only obtained for n = ∞, but for high enough n the values of Negativity will be so small that such entanglement will no longer be detectable and will have practically no effect on the properties of the system).

Fitting of the curves displayed in Fig. [4] for temperatures above the threshold temperature and points displayed in Fig. [5] allows to find the dependence of maximum Negativity on temperature and environment size. The dependence on size exhibits good ~ 1/n behavior. The temperature dependence, on the other hand, shows strong exponential decay for low temperatures and, while increasing temperature, a 1/T^3 dependence becomes dominant. Furthermore, the temperature and size dependencies are convoluted, so they cannot be represented as a simple product of temperature-dependent and size-dependent functions. A reasonable fit is obtained using the function

\[ N_{\text{max}}(n, T) \approx e^{-\alpha T \frac{A}{T(n - BT + CT^2 + D)}} \]

where the fitting parameters are given by α = 0.0857, A = 3.51, B = 0.4674, C = 0.01865, and D = 2.57. The curves obtained using eq. [17] are also plotted on Fig. [5], showing a very good n-dependence for different temperatures, especially for higher numbers of boson modes (n ≥ 3).

VI. CONCLUSION

We have studied the generation of entanglement quantified by Negativity between a charge qubit and its bosonic environment during evolution which leads to pure dephasing of the qubit. In particular, we studied and excitonic qubit confined in a quantum dot in the presence of a super-Ohmic phonon bath, but the results could be easily extended to other charge qubits undergoing similar decoherence processes. The quantity of interest was the dependence of the amount of generated entanglement on the size of the environment (the number of boson modes taken into account) in the situation, when the evolution of the qubit alone does not depend on environment size (for short enough times and high enough temperatures, such a situation is easily obtained). We have found that although for pure states entanglement does not depend on the system size (and the amount of generated entanglement for pure dephasing is an explicit function of the degree of qubit coherence), for finite temperatures Negativity is a decreasing function of environment size proportional to 1/n and there is no entanglement generated for a continuous bosonic environment regardless of the temperature (as long as T ≠ 0).

The temperature, which governs the initial mixedness of the environment, and consequently the mixedness of the whole system throughout its unitary evolution, similarly governs entanglement generated between the system and environment. This means that for higher temperatures, the state of the whole system is more mixed, so less entanglement is generated. In fact, for reasonably low temperatures, the temperature-dependence of the maximum Negativity reached during the joint evolution is almost exponential and roughly resembles the temperature-dependence of the purity. This dependence is obviously monotonously decreasing and does not display the trade-off resulting from the fact that the effect of the environment on the qubit is stronger at higher temperatures while purity is decreased.

[1] Julius Helm and Walter T. Strunz, “Quantum decoherence of two qubits,” Phys. Rev. A 80, 042108 (2009).
[2] Daniel Crow and Robert Joynt, “Classical simulation of quantum dephasing and depolarizing noise,” Phys. Rev. A 89, 042123 (2014).
[3] Katarzyna Roszak and Łukasz Cywiński, “Charac-
terization and measurement of qubit-environment-entanglement generation during pure dephasing," Phys. Rev. A 92, 032310 (2015).

[4] T. Calarco, A. Datta, P. Fedichev, E. Pazú, and P. Zoller, “Spin-based all-optical quantum computation with quantum dots: understanding and suppressing decoherence,” Phys. Rev. A 68, 012310 (2003).

[5] Sophia E. Economou, L. J. Sham, Yanwen Wu, and D. G. Steel, “Proposal for optical u(1) rotations of electron spin trapped in a quantum dot,” Phys. Rev. B 74, 205415 (2006).

[6] Sophia E. Economou and T. L. Reinecke, “Theory of fast optical spin rotation in a quantum dot based on geometric phases and trapped states,” Phys. Rev. Lett. 99, 217401 (2007).

[7] Lorenza Viola and Seth Lloyd, “Dynamical suppression of decoherence in two-state quantum systems,” Phys. Rev. A 58, 2733 (1998).

[8] Lorenza Viola and Emmanuel Knill, “Robust dynamical decoupling of quantum systems with bounded controls,” Phys. Rev. Lett. 90, 037901 (2003).

[9] Katarzyna Roszak, Radim Filip, and Tomáš Novotný, “Characterization and measurement of qubit-environment-entanglement generation during pure dephasing,” Sci. Rep. 5, 9796 (2015).

[10] Wojciech Hubert Żurek, “Decoherence, einselection, and the quantum origins of the classical,” Rev. Mod. Phys. 75, 715 (2003).

[11] Klaus Hornberger, “Introduction to decoherence theory,” Lect. Notes Phys. 678, 221 (2000).

[12] Jens Eisert and Martin B. Plenio, “Quantum and classical correlations in quantum brownian motion,” Phys. Rev. Lett. 89, 137902 (2002).

[13] I. A. Merkulov, A. L. Efros, and M. Rosen, “Electron spin relaxation by nuclei in semiconductor quantum dots,” Phys. Rev. B 65, 205309 (2002).

[14] W. A. Coish and D. Loss, “Hyperfine interaction in a quantum dot: Non-Markovian electron spin dynamics,” Phys. Rev. B 70, 195340 (2004).

[15] L. Cywiński, W. M. Witzel, and S. Das Sarma, “Pure quantum dephasing of a solid-state electron spin qubit in a large nuclear spin bath coupled by long-range hyperfine-mediated interactions,” Phys. Rev. B 79, 245314 (2009).

[16] L. Cywiński, W. M. Witzel, and S. Das Sarma, “Electron Spin Dephasing due to Hyperfine Interactions with a Nuclear Spin Bath,” Phys. Rev. Lett. 102, 057601 (2009).

[17] Julius Helm and Walter T. Strunz, “Decoherence and entanglement dynamics in fluctuating fields,” Phys. Rev. A 81, 042314 (2010).

[18] Jaroslav Novotný, Gernot Alber, and Igor Jex, “Asymptotic evolution of random unitary operations,” Central European Journal of Physics 8, 1001–1014 (2010).

[19] R. Lo Franco, B. Bellomo, E. Andersson, and G. Compagno, “Revival of quantum correlations without system-environment back-action,” Phys. Rev. A 85, 032318 (2012).

[20] Gang-Qin Liu, Xin-Yu Pan, Zhan-Feng Jiang, Nan Zhao, and Ren-Bao Liu, “Controllable effects of quantum fluctuations on spin free-induction decay at room temperature,” Sci. Rep. 2, 432 (2012).

[21] Dariusz Chruściński and Filip A. Wudarski, “Non-Markovian random unitary qubit dynamics,” Physics Letters A 377, 1425 – 1429 (2013).

[22] A. Pernice, J. Helm, and W. T. Strunz, “System-environment correlations and non-Markovian dynamics,” Journal of Physics B: Atomic, Molecular and Optical Physics 45, 15–74 (2012).

[23] A. Pernice and W. T. Strunz, “Decoherence and the nature of system-environment correlations,” Phys. Rev. A 84, 062121 (2011).

[24] Jonas Maziero and Fábio M. Zimmer, “Genuine multipartite system-environment correlations in decoherent dynamics,” Phys. Rev. A 86, 042121 (2012).

[25] A. C. S. Costa, M. W. Beims, and W. T. Strunz, “System-environment correlations for dephasing two-qubit states coupled to thermal baths,” Phys. Rev. A 93, 052316 (2016).

[26] Jens Eisert and Martin B. Plenio, “Quantum and classical correlations in quantum brownian motion,” Phys. Rev. Lett. 89, 137902 (2002).

[27] Stefanie Hilt and Eric Lutz, “System-bath entanglement in quantum thermodynamics,” Phys. Rev. A 79, 010101 (2009).

[28] K. Roszak and P. Machnikowski, “Which path” decoherence in quantum dot experiments,” Phys. Lett. A 351, 251–256 (2006).

[29] Gerardo A. Paz-Silva, Seung-Woo Lee, Todd J. Green, and Lorenza Viola, “Dynamical decoupling sequences for multi-qubit dephasing suppression and long-time quantum memory,” New Journal of Physics 18, 073020 (2016).

[30] Gerardo A Paz-Silva, Leigh M. Norris, and Lorenza Viola, “Multiquit spectroscopy of gaussian quantum noise,” (2016), arXiv:1609.01792 [quant-ph].

[31] B. Krummheuer, V. M. Axt, and T. Kuhn, “Theory of pure dephasing and the resulting absorption line shape in semiconductor quantum dots,” Phys. Rev. B 65, 195313–1–12 (2002).

[32] A. Vagov, V. M. Axt, and T. Kuhn, “Impact of pure dephasing on the nonlinear optical response of single quantum dots and dot ensembles,” Phys. Rev. B 67, 115338 (2003).

[33] A. Vagov, V. M. Axt, T. Kuhn, W. Langbein, P. Borri, and U. Woggon, “Nonmonotonous temperature dependence of the initial decoherence in quantum dots,” Phys. Rev. B 70, 201305(R)–1–4 (2004).

[34] G. Vidal and R. F. Werner, “Computable measure of entanglement,” Phys. Rev. A 65, 032314 (2002).

[35] J. Lee, M.S. Kim, Y.J. Park, and S. Lee, “Partial teleportation of entanglement in a noisy environment,” J. Mod. Opt. 47, 2157 (2000).

[36] M. B. Plenio, “Logarithmic negativity: A full entanglement monotone that is not convex,” Phys. Rev. Lett. 95, 090503 (2005).

[37] Asher Peres, “Separability criterion for density matrices,” Phys. Rev. Lett. 77, 1413–1415 (1996).

[38] Michał Horodecki, Paweł Horodecki, and Ryszard Horodecki, “Separability of mixed states: necessary and sufficient conditions,” Phys. Lett. A 223, 1–8 (1996).

[39] Paweł Horodecki, “Separability criterion and inseparable mixed states with positive partial transposition,” Phys. Lett. A 232, 333–339 (1997).

[40] Michał Horodecki, Paweł Horodecki, and Ryszard Horodecki, “Mixed-state entanglement and distilla-
tion: Is there a “bound” entanglement in nature?”

[41] Paweł Horodecki, Maciej Lewenstein, Guifrè Vidal, and Ignacio Cirac, “Operational criterion and constructive checks for the separability of low-rank density matrices,” Phys. Rev. A 62, 032310 (2000)

[42] Jan Krzywda and Katarzyna Roszak, “Phonon-mediated generation of quantum correlations between quantum dot qubits,” Sci. Rep. 6, 23753 (2016).