New positivity bounds on polarized parton distributions in multicolored QCD

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We derive new positivity bounds on spin-dependent parton distributions in multicolored QCD. They are stronger than Soffer inequality. We check that the new inequalities are stable under one-loop DGLAP evolution to higher normalization points.

1. In spite of the vast phenomenological data providing information about the quark and gluon distributions in nucleons we are still far from the complete knowledge of all twist-two distributions, especially of the spin-dependent ones. Therefore various positivity bounds on parton distributions are very helpful. In particular, an important role is played by Soffer inequality [1] that constrains the transversity quark distribution which is not measured yet. At first sight the set of already known positivity bounds for twist-two distributions is complete and no enhancement can be derived on general grounds without solving the dynamics of QCD. However, if one takes QCD in the limit of large number of colors $N_c$ then it is possible to obtain new stronger inequalities.

It is well known that in the large $N_c$ limit [2] the baryons are described by mean field (Hartree) equations [3], which can be written in terms of bilocal meson fields with all possible spin and isospin quantum numbers. Although the corresponding effective bilocal meson action is not known, a number of results have been obtained for the large $N_c$ baryons without specifying the dynamics. Usually these results use the spin-flavor symmetry [4] of the solution of the Hartree equation describing the nucleon and $\Delta$ resonance: this solution is invariant under simultaneous space, spin and isospin rotations (but not invariant under a pure flavor rotation). Owing to this spin-flavor symmetry the baryon states (described by this mean field solution) appear as "rotational excitations" with spin $S$ and isospin $T$ connected by constraint $S = T$ [4]. In particular, this approach allows to describe nucleon ($S = T = 1/2$) and $\Delta$ resonance ($S = T = 3/2$).

This picture of the large $N_c$ baryons is extensively used in the Skyrme model [4,5] and various chiral quark-soliton models [6]. But some general model-independent results have been extracted for the large $N_c$ baryons using only the spin-flavor symmetry [4]. This includes, for example, the identities for the mass splitting of baryons and the large $N_c$ classification of spin and isospin structures of various form factors. In this paper we study quark distribution functions relying only on the large $N_c$ limit and on the spin-flavor symmetry of the large $N_c$ mean field solution.

We first present our main result — the new positivity bounds which can be considered as a serious enhancement of the Soffer inequality and then give the derivation and check the stability of the bounds under one-loop the DGLAP evolution.

2. Let us first formulate the new positivity bound. At finite number of QCD colors the unpolarized ($q_f$), longitudinal polarized ($\Delta_L q_f$) and transverse polarized ($\Delta_T q_f$) quark (antiquark) distributions are known to be constrained by the following set of inequalities

\[
\begin{align*}
|\Delta_L q_f| &\leq q_f, \\
|\Delta_T q_f| &\leq q_f, \\
q_f + \Delta_L q_f &\geq 2|\Delta_T q_f|.
\end{align*}
\]

The last inequality is known as Soffer inequality [1]. These inequalities hold for each quark flavor $f$. In the leading order of the large $N_c$ limit we have for the $u$ and $d$ distribution functions of the proton [10,11]:

\[
q_u = q_d, \quad \Delta_L q_u = -\Delta_L q_d, \quad \Delta_T q_u = -\Delta_T q_d.
\]

Therefore the set of usual inequalities (1) takes the following form in the large $N_c$ limit

\[
\begin{align*}
|\Delta_L q_u| &\leq q_u, \\
|\Delta_T q_u| &\leq q_u, \\
q_u - |\Delta_L q_u| &\geq 2|\Delta_T q_u|.
\end{align*}
\]

Note that in the last inequality (3) we have the absolute value of $\Delta_L q_u$ in contrast to Soffer inequality at finite $N_c$ [1]. Actually we combined two Soffer inequalities for $f = u, d$ [1] and we made use of the large $N_c$ relation $\Delta_L q_u = -\Delta_L q_d$ (see eq. (2)).

In this paper we shall derive the following enhancement of inequalities (3)

\[
\begin{align*}
\frac{1}{3} q_u + \Delta_L q_u &\geq 2|\Delta_T q_u|, \\
\frac{1}{3} q_u - \Delta_L q_u &\geq 0.
\end{align*}
\]

Let us introduce compact notations

\[
L = \frac{\Delta_L q_u}{q_u} = \frac{\Delta_L q_d}{q_d} + O \left( \frac{1}{N_c} \right),
\]

\[
T = \frac{\Delta_T q_u}{q_u} = -\frac{\Delta_T q_d}{q_d} + O \left( \frac{1}{N_c} \right).
\]

Then our new result (3) becomes

\[
\begin{align*}
\frac{1}{3} + L &\geq 2|T|, \\
\frac{1}{3} - L &\geq 0.
\end{align*}
\]
whereas the old inequalities in the limit $N_c \to \infty$ result in
\[
|L| \leq 1, \\
|T| \leq 1, \\
|L + 2T| \leq 1.
\]

Graphically the old and new regions of the allowed values in the $L, T$ plane is shown in Fig. 1.

![Fig. 1. Soffer inequality at large $N_c$.](image)

3. Now we turn to the derivation of our main result. In order to consider the parton distributions in the large $N_c$ limit we start from the quark correlator $\langle B_2|\psi(x_2)\bar{\psi}(x_1)|B_1 \rangle$. The Hartree picture of the large $N_c$ nucleon described in leads to the following form of this correlation function
\[
\langle B_1, p_1|\psi_{s_1, f_1}(x_1, x_1)\rangle \psi_{s_2, f_2}(x_2, x_2)|B_2, p_2 \rangle = \\
N_c \int d^3\!x e^{i(p_2 - p_1)\cdot x} \int dR D_B^+_1(R) D_{B_1}^+(R) \\
\times R_{f_2 f_1} F_{s_2 f_2, s_1 f_1} (x_1^0 - x_2^0, x_1 - x_2) (R^{-1}) f_1^* f_1.
\]

Here $F_{s_2 f_2, s_1 f_1}(x_1^0 - x_2^0, x_1, x_2)$ is the quark correlation function $\langle \psi^{+}(x_1)\bar{\psi}(x_2) \rangle$ in the background field of the solution to the Hartree equation, $s_i$ are Dirac spinor indices and $f$ are $SU(2)$ isospin indices. The $x^0 - y^0$ time dependence expresses the fact that we deal with a static solution. Solution $F_{s_2 f_2, s_1 f_1}(x_1^0 - x_2^0, x_1, x_2)$ also violates the $SU(2)$ flavor and the 3-space translation invariance. Therefore in eq. we used plane waves $e^{ip \cdot x}$ and the rotator wave functions $D_B(R)$ depending on $SU(2)$ matrix $R$ in order to construct baryon states with given spin, isospin and momentum.

Since the exact form of the effective Hartree Hamiltonian corresponding to the large $N_c$ limit is not known we do not know the function $F_{s_2 f_2, s_1 f_1}(x_1^0 - x_2^0, x_1, x_2)$ either. However, following we assume that this solution has the spin-flavor symmetry:
\[
SU(2)_{\text{spin}} \otimes SU(2)_{\text{flavor}} \subset SU(2)_{\text{flavor}} \otimes SO(3, 1)_{\text{Lorentz}}.
\]

The transformations of $SU(2)_{\text{spin}} \otimes SU(2)_{\text{flavor}}$ perform a simultaneous rotation both in the isospin and in the usual 3D space. The invariance of the mean field solution under $SU(2)_{\text{spin}} \otimes SU(2)_{\text{flavor}}$ means that
\[
S_{s_2 f_2, s_1 f_1}(R) R_{f_2 f_1} (R^{-1}) f_1^* f_1 = S_{s_2 f_2, s_1 f_1}(R^{-1}) \\
\times F_{s_2 f_2, s_1 f_1} (x_1^0 - x_2^0, O(R)x_1, O(R)x_2)
\]
\[
= F_{s_2 f_2, s_1 f_1} (x_1^0 - x_2^0, x_1, x_2).
\]

Here we use notation $S_{s_2 f_2, s_1 f_1}(R)$ for spin rotations and $O(R)$ for space rotations associated with isospin matrix $R$.

The quark distribution functions can be expressed through the correlation function in the standard way. Using the spin-flavor symmetry and performing the integration over $R$ in we easily arrives at the large $N_c$ relations. Moreover, using expression one can show that any quark parton distribution function $f_i(x)$ can be represented in the large $N_c$ limit as follows
\[
f_i(x) = \text{Sp} \left[ \Gamma_i \rho(x) \right].
\]

Here $\Gamma_i$ and $\rho(x)$ are matrices with $SU(2)$ flavor indices and Dirac spin indices so that we deal with the trace of $8 \times 8$ matrices ($4_{\text{spin}} \times 2_{\text{isospin}} = 8$). Matrix $\rho(x)$ can be expressed through function $F_{s_2 f_2, s_1 f_1}$ as follows:
\[
\rho(x) = \frac{N_c}{2\pi} \int dx^0 \int d^3x_1 \int d^3x_2 \int d^3p \frac{e^{ip \cdot (x_1 - x_2)}}{2} \\
\times e^{-ix^0(xMN - p)} \frac{1 + \gamma^0 \gamma^3}{2} F_{0^+, x_1, x_2} \frac{1 + \gamma^0 \gamma^3}{2}.
\]

Matrix $\rho(x)$ is determined by the dynamics of the large $N_c$ QCD and is not known. But $\rho(x)$ is universal for all twist-two parton distributions.

Matrices $\Gamma_i$ in eq. are determined by the specific type of the distribution function $f_i(x)$. Due to the large $N_c$ constraints we can restrict to three independent distribution functions $f_i(x)$ listed in the table below with the corresponding $8 \times 8$ matrices $\Gamma_i$

| $i$ | $f_i$ | $\Gamma_i$ |
|-----|-------|------------|
| $O$ | $q_u + q_d$ | $1$ |
| $L$ | $\Delta_L q_u - \Delta_L q_d$ | $\frac{1}{2} \gamma^5 \gamma_3$ |
| $T$ | $\Delta_T q_u - \Delta_T q_d$ | $\frac{1}{2} \gamma^5 \gamma_1 \gamma_3$ |

Although matrix $\rho(x)$ can be found only if one solves the large $N_c$ QCD, one can establish its general properties:
1) $\rho$ is a hermitean matrix
\[ \rho^+ = \rho, \] (13)
positive in the matrix sense
\[ \rho \geq 0, \] (14)
2) $\rho$ lives in the subspace of the projector $(1 + \gamma^0\gamma^3)/2$
\[ \rho\gamma^0\gamma^3 = \gamma^0\gamma^3 \rho = \rho, \] (15)
3) $\rho$ commutes with $i\gamma^1\gamma^2 + \tau^3$ due to the spin-flavor symmetry \([10]\)
\[ [\rho, (i\gamma^1\gamma^2 + \tau^3)] = 0. \] (16)

One could have an impression that our statements violate the Lorentz and isotopic invariance. But actually everything is correct: the mean field solution for the large $N_c$ nucleon has the mixed spin-flavor invariance \([9]\) but due to the choice of the third axis for the boost of the infinite momentum frame we are left only with the axial spin-isospin rotations. As a result our eqs. \([13,16]\) are invariant only with respect to the $U(1)$ axial spin-isospin rotations. However, we stress that the final results for the parton distributions are certainly both Lorentz and isospin invariant.

Using the properties \([13,16]\) of matrix $\rho(x)$ and discrete $P, C, T$ symmetries one can obtain the following representation for it
\[ \rho = \frac{1 + \gamma^0\gamma^3}{2} \left[ c_11 + c_2\gamma^5\tau^3 + c_3\gamma^5(\gamma^1\tau^1 + \gamma^2\tau^2) \right] \] (17)
with some coefficients $c_i$. Matrices $\gamma^0\gamma^3$, $\gamma^5\gamma^3$, $\gamma^5\gamma^1\tau^1$, $\gamma^5\gamma^2\tau^2$ appearing in this equation commute with each other and have eigenvalues $\pm 1$. Therefore they can be diagonalized simultaneously. Restricting the consideration to the 4-dimensional subspace of the projector $(1 + \gamma^0\gamma^3)/2$ (see \([17]\)) we can diagonalize $\rho$ as follows
\[ \rho = \text{diag}(c_1 - c_2 + 2c_3, c_1 + c_2, c_1 + c_2, c_1 - c_2 - 2c_3). \] (18)
The requirement of positivity \([14]\) leads to the following constraints on coefficients $c_i$
\[ c_1 - c_2 \geq 2|c_3|, \] (19)
\[ c_1 + c_2 \geq 0. \] (20)

Inserting the explicit representation for $\rho$ \([17]\) into the general expression for parton distributions \([11]\) we immediately express the parton distributions $f_i(x)$ through the coefficients $c_i$, and the constraints \([19\), \(20]\) on these coefficients together with \([6]\) lead to bounds \([4]\) on parton distributions.

4. Now we would like to check that our new positivity bounds are preserved by the one-loop evolution to higher normalization points $\mu$ with DGLAP kernels taken in the leading order of the large $N_c$ limit. The derivation follows the ideas of papers \([12,13]\).

In the leading order of the large $N_c$ limit the one-loop evolution of the quark distributions is not affected by the gluon distribution and the evolution equations for quark distributions listed in the above table take the form
\[ \frac{\mu d}{d\mu} f_O = P_{||} \otimes f_O, \]
\[ \frac{\mu d}{d\mu} f_L = P_{||} \otimes f_L, \]
\[ \frac{\mu d}{d\mu} f_T = P_T \otimes f_T, \] (21)
where the singlet unpolarized quark distribution $f_O$ and the longitudinally polarized isovector distribution $f_L$ have the same one-loop DGLAP kernel $P_{||}$.

We want to show that if for some numbers $\alpha, \beta$ at a given normalization $\mu_1$ the positivity of functions $f_O$ and $f_L$ is preserved by the one-loop evolution to higher normalization points $\mu_2$ then we shall see that our inequality
\[ \alpha f_L \leq f_O, \quad \alpha f_L + \beta f_T \leq f_O \] (22)
then the evolution to higher normalization points preserves these inequalities. Indeed, introducing functions
\[ f_\alpha = f_O - \alpha f_L, \quad f_{\alpha,\beta} = f_O - \alpha f_L - \beta f_T, \] (23)
we can rewrite the evolution equations as follows
\[ \frac{\mu d}{d\mu} \left( f_{\alpha,\beta} \right) = \left( P_T - P_{||} \right) \otimes \left( f_{\alpha,\beta} \right). \] (24)
Here all kernels are positive \([14]\):
\[ P_T > 0, \quad P_{||} - P_T > 0, \quad P_{||} > 0, \] (25)
so that the positivity of functions $F_{\alpha,\beta}$ and $F_\alpha$ is preserved during the evolution to larger $\mu$ (note that the subtraction terms in the evolution equations do not violate this positivity \([13]\)). This completes the proof of the evolution stability of inequalities \([12]\) for arbitrary coefficients $\alpha, \beta$. In particular, we can choose for $\alpha, \beta$ the coefficients appearing in our bounds \([6]\). Thus the evolution stability of inequalities \([6]\) is established.

5. Comparing our new bounds with the phenomenological data one should keep in mind that the allowed regions shown in Fig. \([4]\) correspond to the leading order of the large $N_c$ expansion and that the $1/N_c$ corrections can cause the phenomenological distribution functions to exceed the “triangle boundary” of Fig. \([4]\). Indeed, if we take the GRV \([13]\) parametrization for $q_u + q_d$ and GRSV \([16]\) for $\Delta_L q_u - \Delta_L q_d$ then we shall see that our inequality
\[ \frac{|\Delta L q_u - \Delta L q_d|}{q_u + q_d} \leq \frac{1}{3} \quad \text{(large } N_c) \]  

(26)

is essentially modified by the $1/N_c$ corrections:

\[ \max_x \frac{|\Delta L q_u - \Delta L q_d|}{q_u + q_d} \sim 0.6. \quad \text{(GRSV)} \]  

(27)

Actually this deviation from our large $N_c$ bounds agrees with the fact that in chiral quark soliton models the $1/N_c$ corrections to $g_A$ are known to be large [17].

The large $N_c$ inequalities derived in this paper also hold for antiquark distributions. Since our knowledge about $\Delta L \bar{q}_u - \Delta L \bar{q}_d$ and $\Delta T \bar{q}_u - \Delta T \bar{q}_d$ is still scare the new bounds can be helpful in pinning down the polarized antiquark distributions (note that model estimates indicate that $1/N_c$ corrections to polarized antiquark distributions are rather small [18]). The knowledge of polarized antiquark distributions is important for the analysis of data on semi-inclusive and Drell-Yan processes with polarized nucleons.

Apart from pure phenomenological applications we think that the new large $N_c$ bounds can be used as a consistency test for various models of polarized parton distributions.

We are grateful to A.V. Efremov, K. Goeke, V.Yu. Petrov, A. Shuvaev and O. Teryaev for inspiring discussions. M.V.P. is thankful to Barcelona University for hospitality where a part of this work has been done. This work was supported in parts by RFBR grant 96-15-96764, DFG and BMFB.

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