SMALL DENOMINATORS AND LARGE NUMERATORS OF QUASIPERIODIC SCHRÖDINGER OPERATORS

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ABSTRACT. We initiate an approach to simultaneously treat numerators and denominators of Green’s functions arising from quasi-periodic Schrödinger operators, which in particular allows us to study completely resonant phases of the almost Mathieu operator.

Let \((H_{\lambda,\alpha,\theta}u)(n) = u(n + 1) + u(n - 1) + 2\lambda \cos 2\pi(\theta + n\alpha)u(n)\) be the almost Mathieu operator on \(\ell^2(\mathbb{Z})\), where \(\lambda, \alpha, \theta \in \mathbb{R}\). Let

\[
\beta(\alpha) = \limsup_{k \to \infty} \frac{-\ln||k\alpha||_{\mathbb{R}/\mathbb{Z}}}{|k|}.
\]

We prove that for any \(\theta\) with \(2\theta \in \alpha\mathbb{Z} + \mathbb{Z}\), \(H_{\lambda,\alpha,\theta}\) satisfies Anderson localization if \(|\lambda| > e^{2\beta(\alpha)}\). This confirms a conjecture of Avila and Jitomirskaya [The Ten Martini Problem. Ann. of Math. (2) 170 (2009), no. 1, 303–342] and a particular case of a conjecture of Jitomirskaya [Almost everything about the almost Mathieu operator. II. XIth International Congress of Mathematical Physics (Paris, 1994), 373–382, Int. Press, Cambridge, MA, 1995].

1. INTRODUCTION

In this paper, we study one-dimensional quasiperiodic Schrödinger operators \(H = H_{\lambda v,\alpha,\theta}\) defined on \(\ell^2(\mathbb{Z})\):

\[
(H_{\lambda v,\alpha,\theta}u)(n) = u(n + 1) + u(n - 1) + \lambda v(\theta + n\alpha)u(n),
\]

where \(v : \mathbb{R}/\mathbb{Z} \to \mathbb{R}\) is the potential, \(\lambda\) is the coupling constant, \(\alpha \in \mathbb{R}\setminus\mathbb{Q}\) is the frequency, and \(\theta \in \mathbb{R}\) is the phase. One of the most studied examples in both mathematics and physics is the almost Mathieu operator (denoted by \(H_{\lambda,\alpha,\theta}\)), where \(v(\theta) = 2\cos(2\pi\theta)\).

There are two small denominator problems arising from quasi-periodic Schrödinger operators. In the regime of small coupling constants, the operator is close to the free discrete Schrödinger operator and thus, it is natural to establish the reducibility to a constant cocycle by writing down the eigen-equation \(Hu = Eu\) as a dynamical system on \((\mathbb{R}/\mathbb{Z}, \mathbb{R}^2)\):

\[
(\theta, w) \to (\theta + \alpha, A(\theta)w),
\]
where $A(\theta) = \begin{pmatrix} E - \lambda v(\theta) & -1 \\ 1 & 0 \end{pmatrix}$ is referred to as Schrödinger cocycle of (1).

In the regime of large coupling constants, the operator can be viewed as a perturbation of a purely diagonal matrix with dense eigenvalues. In both regimes, spectral theory of quasi-periodic operators has seen significant progress through earlier perturbative methods [16–21, 51], and then non-perturbative methods [4, 11, 12, 14, 22, 23, 27, 39, 40]. We refer readers to [28, 36, 47, 49, 52] and references therein for more details.

In this paper, we are interested in the regime of large coupling constants. In this regime, small denominator problems essentially become problems of dealing with resonances coming from phases and frequencies [12–14, 21, 23, 34, 35, 40]. By avoiding the resonance (treating the small denominator) that usually is achieved by imposing arithmetic conditions on phases and frequencies, it is expected that the operator exhibits Anderson localization (pure point spectrum with exponentially decaying eigenfunctions). Recently, several remarkable sharp arithmetic transitions between singular continuous spectrum and pure point spectrum were obtained [33–35]. However, all localization proofs [5, 6, 11, 12, 14, 31, 35, 38–40, 43–45] simply bound the numerators of Green’s functions of the quasi-periodic Schrödinger operators above by the Lyapunov exponent of the corresponding dynamical system [2].

The main goal of this paper is to initiate an approach to simultaneously treat numerators and denominators in the study of resonances arising from analytic quasi-periodic Schrödinger operators. As a starting point, we focus on a particular case, namely, completely resonant phases for the almost Mathieu operator, in which phase resonances and frequency resonances overlap. As we were finalizing this paper we learned of a preprint [26] where the study of the numerators also plays an important role, but in a completely different setting of unbounded potentials and the so-called anti-resonances.

We say a phase $\theta \in \mathbb{R}$ is completely resonant with respect to a frequency $\alpha$ if $2\theta \in \alpha \mathbb{Z} + \mathbb{Z}$. One of the motivations to study completely resonant phases is their critical roles in the spectral theory of quasi-periodic Schrödinger operators, for instance, spectral gap edges [30, 48]. Another motivation comes from a conjecture of Avila and Jitomirskaya in [5] and Jitomirskaya [29].

\textbf{Conjecture 1:} Avila and Jitomirskaya [5] conjectured that the almost Mathieu operator $H_{\lambda, \alpha, \theta}$ satisfies Anderson localization if $\ln |\lambda| > 2\beta(\alpha)$ and $2\theta \in \alpha \mathbb{Z} + \mathbb{Z}$, where

$$\beta(\alpha) = \limsup_{k \to \infty} \frac{\ln \|k\alpha\|_{\mathbb{R}/\mathbb{Z}}}{|k|},$$

and $\|x\|_{\mathbb{R}/\mathbb{Z}} = \text{dist}(x, \mathbb{Z})$.

Conjecture 1 is also a particular case of a conjecture of Jitomirskaya [29]. In this paper, we prove Conjecture 1 as it is.
Theorem 1.1. Assume that $\alpha$ satisfies $\beta(\alpha) < \infty$. Then the almost Mathieu operator $H_{\lambda,\alpha,\theta}$ satisfies Anderson localization if $2\theta \in \alpha \mathbb{Z} + \mathbb{Z}$ and $\ln |\lambda| > 2\beta(\alpha)$. Moreover, if $\phi$ is an eigenfunction, that is $H_{\lambda,\alpha,\theta} \phi = E \phi$ with $\phi \in \ell^2(\mathbb{Z})$, we have

$$\limsup_{k \to \infty} \frac{\ln(\phi^2(k) + \phi^2(k-1))}{2|k|} \leq - (\ln |\lambda| - 2\beta(\alpha)).$$

Before going into the historical results attempting to solve Conjecture 1, we introduce the phase and frequency resonances in the supercritical regime (this is also called the positive Lyapunov exponent regime). Define

$$\delta(\alpha, \theta) = \limsup_{k \to \infty} -\frac{\ln ||2\theta + k\alpha||_{\mathbb{R}/\mathbb{Z}}}{|k|}.$$

When $\lambda \to \infty$, after a rescaling, $H_{\lambda,\alpha,\theta}$ is a perturbation of a diagonal matrix that has well localized eigenfunctions. The diagonal entries (also eigenvalues) \(\{2\lambda \cos(2\pi(\theta + k\alpha))\}_{k \in \mathbb{Z}}\) are dense and the distance between any two entries $2\lambda \cos(2\pi(\theta + k_1\alpha))$ and $2\lambda \cos(2\pi(\theta + k_2\alpha))$ is

$$|2\lambda \cos(2\pi(\theta + k_1\alpha)) - 2\lambda \cos(2\pi(\theta + k_2\alpha))|$$

$$= 4|\lambda \sin(\pi(2\theta + (k_1 + k_2)\alpha))\sin(\pi(k_1 - k_2)\alpha)|$$

$$\simeq |\lambda| ||2\theta + (k_1 + k_2)\alpha||_{\mathbb{R}/\mathbb{Z}}||(k_1 - k_2)\alpha||_{\mathbb{R}/\mathbb{Z}}.$$

By approximating to the infinite dimensional matrix $H_{\lambda,\alpha,\theta}$ from its finite dimensional cut off, the resonances $||2\theta + k\alpha||_{\mathbb{R}/\mathbb{Z}}$ (referred to as phase resonances) and $||k\alpha||_{\mathbb{R}/\mathbb{Z}}$ (referred to as frequency resonances) appear. The strength of frequency and phase resonances are essentially quantified by (3) and (4).

With recent developments, the almost Mathieu operator undergoes a clear spectral transition (called metal-insulator transition or first transition line) when $|\lambda|$ changes from small to large.

**First transition line:**

- if $|\lambda| < 1$, $H_{\lambda,\alpha,\theta}$ has purely absolutely continuous spectrum for every $\alpha$ and $\theta$ [1, 2, 6, 25, 40].
- if $|\lambda| = 1$, $H_{\lambda,\alpha,\theta}$ has purely singular continuous spectrum for any irrational $\alpha$ and $\theta$ [7, 31].
- if $|\lambda| > 1$, $H_{\lambda,\alpha,\theta}$ has Anderson localization for $\beta(\alpha) = \delta(\alpha, \theta) = 0$ [32, 40].

Understanding the first transition $|\lambda| = 1$ is now clear by Avila’s global theory [2]. When $|\lambda| < 1$ (subcritical regime) the Lyapunov exponent on the spectrum is 0 even after the complexification of the phase. When $|\lambda| = 1$ (critical regime), the Lyapunov exponent on the spectrum is 0 and it changes to positive after the complexification. When $|\lambda| > 1$ (supercritical regime), the Lyapunov exponent is positive and equals $\ln |\lambda|$ on the spectrum.

The spectral theory in the supercritical regime is more delicate. By Kotani theory, $H$ does not have any absolutely continuous spectrum in the supercritical
It is believed that the spectral type of $H$ (singular continuous spectrum or pure point spectrum/localization) depends on the competition between the Lyapunov exponent and resonances associated with (3) and (4). Such observations go back to [9, 24, 37, 39, 40].

Second transition line in the frequency:
- if $\ln |\lambda| > \beta(\alpha)$, $H_{\lambda,\alpha,\theta}$ has Anderson localization for $\delta(\alpha, \theta) = 0$ [34].
- if $0 < \ln |\lambda| < \beta(\alpha)$, $H_{\lambda,\alpha,\theta}$ has purely singular continuous spectrum [8].

Second transition line in the phase:
- if $\ln |\lambda| > \delta(\alpha, \theta)$, $H_{\lambda,\alpha,\theta}$ has Anderson localization for $\beta(\alpha) = 0$ [35].
- if $\ln |\lambda| < \delta(\alpha, \theta)$, $H_{\lambda,\alpha,\theta}$ has purely singular continuous spectrum [35].

The second transition line states that when the Lyapunov exponent beats frequency/phase resonances, the operator exhibits localization. In the regime of localization, Jitomirskaya and the author also determined exact exponential asymptotics of eigenfunctions and corresponding transfer matrices. Moreover, the (reflective) hierarchical structure of eigenfunctions was discovered in [34, 35].

Let us turn back to completely resonant phases. For completely resonant phases, $\delta(\alpha, \theta) = \beta(\alpha)$. Thus both phase resonances and frequency resonances appear and have the same strength. Conjecture 1 states that when the Lyapunov exponent beats phase and frequency resonances, the operator has Anderson localization. It is natural to think that the proof of Conjecture 1 could follow from a combination of the second transition line in the phase and the frequency. However, the completely resonant phase brings many new challenges, in particular the completely resonant phenomenon. The original arguments of Jitomirskaya [40] do not work for completely resonant phases directly. In [32], Jitomirskaya-Koslover-Schulteis found a trick to fix the gap by shrinking the size of intervals around 0 (referred to as the “shrinking scale” technique). This allows them to prove Conjecture 1 when $\beta(\alpha) = 0$. The proof in [6, 46] implies that $H_{\lambda,\alpha,\theta}$ has Anderson localization if $\ln |\lambda| > C\beta(\alpha)$, where $C$ is an absolute constant. In [3], Avila and Jitomirskaya proved that when $\delta(\alpha, \theta) = 0$ and $\ln |\lambda| > \frac{16}{9} \beta(\alpha)$, $H_{\lambda,\alpha,\theta}$ has Anderson localization, which is a key step to solve the ten Martini problem. This approach has been pushed to the limit $\ln |\lambda| > \frac{3}{2} \beta(\alpha)$ by the author and Yuan [44]. By a combination of arguments in Jitomirskaya-Koslover-Schulteis [32], Avila-Jitomirskaya [3] and Liu-Yuan [44], the author and Yuan established the Anderson localization for completely resonant phases when $\ln |\lambda| > 7\beta(\alpha)$ [45].

With localization proofs in [34, 35] and methods in [32, 45], we could obtain the Anderson localization for $\ln |\lambda| > 4\beta(\alpha)$ in Conjecture 1, where 4 is the non-trivial technical limit in such an approach since we have to shrink the scale to avoid the complete resonance, doubling the numerical number. In [43], the author noticed that for completely resonant phases, the phase resonance and frequency resonance are not symmetric, namely, the frequency resonance only
happens at sites $k = jq_n, j \in \mathbb{Z}\{0\}$ and the phase resonance happens at both $k = jq_n + \frac{1}{2}q_n$ and $k = jq_n, j \in \mathbb{Z}$ ($\frac{q_n}{q_n}$ is the continued fraction approximations to $\alpha$). See Sections 4 and 5 for more details. So, instead of using the Lagrange interpolation uniformly, the author treated the Lagrange interpolation individually during the process of finding Green’s functions without “small denominators”, which allows him to push the numerical number 4 to 3 [43].

As we mentioned earlier, all localization proofs in previous literature [5, 6, 32, 43–45] are devoted to treating small denominators (establishing the lower bound of the denominators of Green’s functions) and simply bound numerators by the rate of the Lyapunov exponent. To the best of our knowledge, this paper is the first time to analyze numerators of Green’s functions arising from quasi-periodic operators. Moreover, we expand approaches in [5, 6, 32, 43–45] to deal with denominators in many directions and add several significant ingredients to treat the numerators and denominators simultaneously.

For completely resonant phases, the most challenging case comes from resonant sites $k = jq_n, j \in \mathbb{Z}\{0\}$ and $k = jq_n + \frac{1}{2}q_n, j \in \mathbb{Z}$. The Gordon type argument is an approach to show the absence of eigenvalues based on repetitions of potentials governed by frequency resonances [8, 9, 24, 33, 34]. We develop the Gordon type argument to establish sharp bounds of the numerators of Green’s functions around resonant sites $jq_n$. However, such an argument does not work for resonant sites $jq_n + \frac{1}{2}q_n$ (only coming from phase resonances) since the repetition of potentials only appears at sites $jq_n$ by the continued fraction expansion approximation. We regard norms of generalized eigenfunctions at resonant sites $k = jq_n, q_n + \frac{1}{2}q_n$ and numerators of Green’s functions around resonant sites $jq_n + \frac{1}{2}q_n$ as variables. Our first step is to establish inequalities among those variables. By solving those inequalities, we obtain the relations among resonant sites. Then the exponential decay at the resonant sites follows from standard iterations.

Finally, we want to highlight another technical novelty in our proof. The Gordon type argument works along with the transfer matrices, which slightly differs from the denominators and numerators of Green’s functions. We introduce the notation $P_{[x_1, x_2]}$ to represent the denominator of the Green’s function restricting to the interval $[x_1, x_2]$. The advantage of the notation $P_{[x_1, x_2]}$ is that it inherits information from the transfer matrix from sites $x_1$ to $x_2$. This could simplify the localization proof in [5, 6, 32, 43–45].

The rest of this paper is organized as follows. We list the definitions and standard facts in Section 2. In Section 3 we provide several technical lemmas. Sections 4 and 5 are devoted to treating resonances. In Section 6 we complete the proof.
2. Some notations and known facts

Let $H$ be an operator on $\ell^2(\mathbb{Z})$. We say $\phi$ is a generalized eigenfunction corresponding to the generalized eigenvalue $E$ if

$$H\phi = E\phi, \quad |\phi(k)| \leq \hat{C}(1 + |k|).$$

By Shnol’s theorem [10], in order to prove Anderson localization of $H$, we only need to show that every generalized eigenfunction is in fact an exponentially decaying eigenfunction. To be more precise, there exists some constant $c > 0$ such that

$$|\phi(k)| \leq e^{-c|k|} \text{ for large } k.$$

For simplicity, we assume $\hat{C} = 1$ in (5).

From now on, we always assume $\phi$ is a generalized eigenfunction of $H_{\lambda, \alpha, \theta}$ and $E$ is the corresponding generalized eigenvalue. Without loss of generality assume $\phi(0) = 1$. It is well known that every generalized eigenvalue must be in the spectrum, namely $E \in \Sigma_{\lambda, \alpha}$, where $\Sigma_{\lambda, \alpha}$ is the spectrum of $H_{\lambda, \alpha, \theta}$ (the spectrum does not depend on $\theta$). Our goal is to show that there exists some constant $c > 0$ such that for large $k$,

$$|\phi(k)| \leq e^{-c|k|}.$$

For any $x_1, x_2 \in \mathbb{Z}$ with $x_1 < x_2$, denote by

$$P_{[x_1, x_2]}(\lambda, \alpha, \theta, E) = \det(R_{[x_1, x_2]}(H_{\lambda, \alpha, \theta} - E)R_{[x_1, x_2]}),$$

where $R_{[x_1, x_2]}$ is the restriction on $[x_1, x_2]$. Let us denote

$$P_k(\lambda, \alpha, \theta, E) = \det(R_{[0, k-1]}(H_{\lambda, \alpha, \theta} - E)R_{[0, k-1]}).$$

When there is no ambiguity, we drop the dependence of parameters $E, \lambda, \alpha$ or $\theta$. Clearly,

$$P_{[x_1, x_2]}(\theta) = P_k(\theta + x_1\alpha),$$

where $k = x_2 - x_1 + 1$.

Let

$$A_k(\theta) = \prod_{j=k-1}^{0} A(\theta + j\alpha) = A(\theta + (k-1)\alpha)A(\theta + (k-2)\alpha)\cdots A(\theta)$$

and

$$A_{-k}(\theta) = A_k^{-1}(\theta - k\alpha)$$

for $k \geq 1$, where $A(\theta) = \begin{pmatrix} E - 2\lambda \cos 2\pi \theta & -1 \\ 1 & 0 \end{pmatrix}$. $A_k$ is called the ($k$-step) transfer matrix.
By the definition, for any \( k \in \mathbb{Z}_+ \), \( m \in \mathbb{Z} \), one has

\[
\begin{pmatrix}
\phi(k + m) \\
\phi(k + m - 1)
\end{pmatrix} = A_k(\theta + m\alpha) \begin{pmatrix}
\phi(m) \\
\phi(m - 1)
\end{pmatrix}.
\]

It is easy to check that for \( k \in \mathbb{Z}_+ \),

\[
A_k(\theta) = \begin{pmatrix}
P_k(\theta) & -P_{k-1}(\theta + \alpha) \\
-P_{k-1}(\theta) & -P_{k-2}(\theta + \alpha)
\end{pmatrix}.
\]

The Lyapunov exponent is given by

\[
L(E) = \lim_{k \to \infty} \frac{1}{k} \int_{\mathbb{R}/\mathbb{Z}} \ln \| A_k(\theta) \| d\theta.
\]

The Lyapunov exponent can be computed precisely for \( E \) in the spectrum of \( H_{\lambda,\alpha,\theta} \).

**Lemma 2.1.** [15] For \( E \in \Sigma_{\lambda,\alpha} \) and \(|\lambda| > 1\), we have \( L(E) = \ln |\lambda| \).

In the following, denote by \( L := \ln |\lambda| \).

By upper semicontinuity and unique ergodicity, one has

\[
L = \lim_{k \to \infty} \sup_{\theta \in \mathbb{R}/\mathbb{Z}} \frac{1}{k} \ln \| A_k(\theta) \|.
\]

Therefore, for any \( \varepsilon > 0 \),

\[
\| A_k(\theta) \| \leq e^{(L+\varepsilon)k},
\]

when \( k \) is large enough (independent of \( \theta \)).

By (10) and (13), one has for large \( k \),

\[
|P_k(\theta)| \leq e^{(L+\varepsilon)k},
\]

and hence

\[
|P_{[x_1,x_2]}(\theta)| \leq e^{(L+\varepsilon)|x_2-x_1|}.
\]

By Cramer’s rule (see p.15, [11] for example) for any \( y \in [x_1,x_2] \), one has

\[
|G_{[x_1,x_2]}(x_1,y)| = \frac{P_{y+1,x_2}}{P_{[x_1,x_2]}} ,
\]

\[
|G_{[x_1,x_2]}(y,x_2)| = \frac{P_{[x_1,y-1]}}{P_{[x_1,x_2]}}.
\]
It is easy to check that (p. 61, [11]) for any \( y \in [x_1, x_2] \),
\[
\phi(y) = -G_{[x_1,x_2]}(x_1, y)\phi(x_1 - 1) - G_{[x_1,x_2]}(y, x_2)\phi(x_2 + 1).
\]

Denote by \( x'_1 = x_1 - 1 \) and \( x'_2 = x_2 + 1 \).

By (17), (18) and (19), one has that for any \( y \in [x_1, x_2] \),
\[
|\phi(y)| \leq |P_{[x_1,x_2]}||\phi(x'_1)| + |P_{[x_1,x_2]}||\phi(x'_2)|.
\]

Given a set \( \{\theta_1, \ldots, \theta_{k+1}\} \), the lagrange Interpolation terms \( \text{Lag}_m \), \( m = 1, 2, \ldots, k + 1 \), are defined by
\[
\text{Lag}_m = \ln \max_{x \in [-1, 1]} \prod_{j=1, j \neq m}^{k+1} \frac{|x - \cos 2\pi \theta_j|}{|\cos 2\pi \theta_m - \cos 2\pi \theta_j|}.
\]

The following lemma is another form of Lemma 9.3 in [3], which has been reformulated in [43, Lemma 2.3].

\textbf{Lemma 2.2.} [43, Lemma 2.3] Given a set \( \{\theta_1, \ldots, \theta_{k+1}\} \), there exists some \( \theta_m \) in \( \{\theta_1, \ldots, \theta_{k+1}\} \) such that
\[
|P_k(\theta_m - \frac{k-1}{2}\alpha)| \geq e^{kL - \text{Lag}_m}.
\]

In the following, we always assume
- \( \varepsilon > 0 \) is an arbitrarily small constant and it may change even in the same equation.
- \( C \) is a large constant (depends on \( \lambda \) and \( \alpha \)) and it may change even in the same equation.
- \( n \) is large enough which depends on all parameters and constants.

Recall that \( \frac{p_n}{q_n} \) is the continued fraction approximations to \( \alpha \). By the definition of \( \beta(\alpha) \), one has that
\[
\beta = \beta(\alpha) = \limsup_{n \to \infty} \frac{\ln q_{n+1}}{q_n}.
\]

Let \( b_n = 10^{-5}q_n \). For any \( \ell \in \mathbb{Z} \), let
\[
r^{\varepsilon,n}_\ell = \sup_{|r| \leq 10\varepsilon} |\phi(\ell q_n + rq_n)|,
\]
and
\[
r^{\varepsilon,n}_{\ell + \frac{1}{2}} = \sup_{|r| \leq 10\varepsilon} |\phi(\ell q_n + \lfloor \frac{q_n}{2} \rfloor + rq_n)|,
\]
where \( \lfloor x \rfloor \) is the largest integer that is less than or equal to \( x \).

\textbf{Lemma 2.3.} [43, Lemma 2.5] Assume that \( |\lambda| > 1 \) and \( 2\theta \in \alpha\mathbb{Z} + \mathbb{Z} \). Suppose \( k \in [\ell q_n, (\ell + \frac{1}{2})q_n] \) or \( k \in [(\ell + \frac{1}{2})q_n, (\ell + 1)q_n] \) with \( 0 \leq |\ell| \leq 100\frac{b_{n+1}}{q_n} + 100 \), and \( \text{dist}(k, q_n\mathbb{Z} + \frac{q_n}{2}\mathbb{Z}) \geq 10\varepsilon q_n \). Let \( d_t = |k - tq_n| \) for \( t \in \{\ell, \ell + \frac{1}{2}, \ell + 1\} \). Then for sufficiently large \( n \), we have that
• when \( k \in [(\ell + \frac{1}{2})q_n, (\ell + 1)q_n] \),

\[
|\phi(k)| \leq r_{\ell+\frac{1}{2}}^{\varepsilon,n} \exp\{-L\varepsilon(d_{\ell+\frac{1}{2}} - 3\varepsilon q_n)\} + r_{\ell+1}^{\varepsilon,n} \exp\{-L\varepsilon(d_{\ell+1} - 3\varepsilon q_n)\};
\]

(22) \( \frac{25}{27} \) by (20), one has

• when \( k \in [(\ell + \frac{1}{2})q_n, (\ell + 1)q_n] \),

\[
|\phi(k)| \leq r_{\ell+\frac{1}{2}}^{\varepsilon,n} \exp\{-L\varepsilon(d_{\ell+\frac{1}{2}} - 3\varepsilon q_n)\} + r_{\ell+1}^{\varepsilon,n} \exp\{-L\varepsilon(d_{\ell+1} - 3\varepsilon q_n)\}.
\]

(23) \( \frac{25}{27} \) \( \Rightarrow \) \( \frac{25}{27} \)

A similar version of Lemma 2.3 firstly appears in [34, Lemma 4.1].

3. Technical preparations

Without loss of generality, we assume \( \alpha \in (0, 1) \) and \( \lambda > e^{2\beta\langle\alpha\rangle} \). Since \( 2\theta \in \alpha \mathbb{Z} + \mathbb{Z} \), essentially we only need to study \( \theta \in \left\{ -\frac{\alpha}{2}, -\frac{\alpha}{2} - \frac{1}{2}, 0, -\frac{1}{2} \right\} \) by shifting the operator from \( H_{\lambda, \alpha, \theta} \) to \( H_{\lambda, \alpha, \theta \pm \alpha} \). For this reason, we always assume \( \theta \in \left\{ -\frac{\alpha}{2}, -\frac{\alpha}{2} - \frac{1}{2}, 0, -\frac{1}{2} \right\} \) in the following arguments.

For simplicity, we drop superscripts \( n \) and \( \varepsilon \) from \( r_{j}^{\varepsilon,n} \) and \( r_{j+\frac{1}{2}}^{\varepsilon,n} \), since \( n \) and \( \varepsilon \) will be fixed.

**Lemma 3.1.** Let \( 0 \leq x \leq \frac{q_n}{4} \). Assume that \( p_1 \) satisfies \( |p_1 - \frac{1}{2} q_n| \leq 20\varepsilon q_n \) and \( p_2 \) satisfies \( |p_2 - q_n| \leq 20 \varepsilon q_n \). Then we have

\[
|P_{[-x, p_1]}| \leq e^{L\varepsilon q_n - x + C\varepsilon q_n},
\]

(24) \( \frac{25}{27} \) \( \Rightarrow \) \( \frac{25}{27} \)

\[
|P_{[-p_1, x]}| \leq e^{L\varepsilon q_n - x + C\varepsilon q_n},
\]

(25) \( \frac{25}{27} \) \( \Rightarrow \) \( \frac{25}{27} \)

and

\[
|P_{[-x, p_2]}| \leq e^{L(q_n - x) + C\varepsilon q_n}.
\]

(26) \( \frac{25}{27} \) \( \Rightarrow \) \( \frac{25}{27} \)

Proof. We are going to prove (24) first. Let \( I = [x_1, x_2] \), where \( x_1 = -x \) and \( x_2 = p_1 \).

By Lemma 2.3, one has

\[
|\phi(x_1')| \leq r_0 e^{-Lx + C\varepsilon q_n} + r_{-\frac{1}{2}} e^{-L\left(\frac{q_n}{4} - x\right) + C\varepsilon q_n}.
\]

(27) \( \frac{25}{27} \) \( \Rightarrow \) \( \frac{25}{27} \)

By (20), one has

\[
|\phi(0)| \leq |P_{[-x, p_1]}|^{-1}(|P_{[-x, -1]}||\phi(x_2')| + |P_{[1, p_1]}||\phi(x_1')|).
\]

(28) \( \frac{25}{27} \) \( \Rightarrow \) \( \frac{25}{27} \)

By (15), (27) and (28), one has

\[
|\phi(0)| \leq |P_{[-x, p_1]}|^{-1} e^{C\varepsilon q_n} \left(e^{L\left(\frac{q_n}{4} - x\right)} r_0 + e^{Lx} r_{-\frac{1}{2}} + e^{Lx} r_{\frac{1}{2}} \right).
\]

(29) \( \frac{25}{27} \) \( \Rightarrow \) \( \frac{25}{27} \)

Since \( \phi(0) = 1 \) and \( |\phi(k)| \leq 1 + |k| \), by (29), we have that

\[
|P_{[-x, p_1]}| \leq e^{L\varepsilon q_n - x + C\varepsilon q_n}.
\]

(30) \( \frac{25}{27} \) \( \Rightarrow \) \( \frac{25}{27} \)

We are going to prove (25) (24). In this case, we only need to set \( I = [x_1, x_2] \), where \( x_1 = -p_1 \) and \( x_2 = x \) (\( x_1 = -x \) and \( x_2 = p_2 \)), and repeat the proof of (24). □
Lemma 3.2. \cite{56} Let $A^1, A^2, \ldots, A^n$ and $B^1, B^2, \ldots, B^n$ be $2 \times 2$ matrices with $\| \prod_{m=0}^{j-1} A^{k+m} \| \leq De^{dj}$ for some constant $D$ and $d$. Then

$$\|(A^n + B^n) \cdots (A^1 + B^1) - A^n \cdots A^1\| \leq De^{dn}(\prod_{j=1}^{n}(1 + De^{-d}\|B^j\|) - 1).$$

Lemma 3.3. Assume that $0 < k \leq 10q_n$ and $0 < j \leq Cq_n + 1$. Then for large enough $k$, we have

$$\|A_k(\theta) - A_k(\theta + jq_n\alpha)\| \leq e^{(L+\varepsilon)k} \frac{j}{q_n+1}.\tag{30}$$

In particular,

$$|P_k(\theta) - P_k(\theta + jq_n\alpha)| \leq e^{(L+\varepsilon)k} \frac{j}{q_n+1}.\tag{31}$$

Proof. By the Diophantine approximation (see (115) in the Appendix), we have

$$\|q_n\alpha\|_{\mathbb{R}/\mathbb{Z}} \leq \frac{1}{q_n+1},$$

and hence

$$\|jq_n\alpha\|_{\mathbb{R}/\mathbb{Z}} \leq \frac{j}{q_n+1}.$$

This implies

$$\|A(\theta + jq_n\alpha) - A(\theta)\| \leq C j \frac{q_n+1}{q_n+1}.\tag{33}$$

Applying (13) and Lemma 3.2, one has

$$\|A_k(\theta + jq_n\alpha) - A_k(\theta)\| \leq e^{(L+\varepsilon)k}(1 + C j \frac{q_n+1}{q_n+1})^k - 1).\tag{32}$$

Using the fact $|e^y - 1| \leq ye^y$ for $y > 0$, we obtain

$$k(1 + C j \frac{q_n+1}{q_n+1})^k \leq \frac{kC j}{q_n+1},$$

and

$$\leq C \frac{k j}{q_n+1}.$$

Combining this with (32) completes the proof. \hfill \Box

4. Resonance I: sites $jq_n + \frac{q_n}{2}$, $j \in \mathbb{Z}$

In this section, we deal with resonances arising from sites $jq_n + \frac{q_n}{2}$, $j \in \mathbb{Z}$. Denote by

$$\beta_j = \frac{\ln q_n+1 - \ln(|j| + 1)}{q_n}.$$  \tag{33}

We are going to prove
Theorem 4.1. Let $|j| \leq 2^{b_{n+1}/q_n} + 10$. Then we have

$$r_{j+\frac{1}{2}} \leq \exp\left\{-\frac{1}{2}(L - 2\beta_j - C\varepsilon)q_n\right\}(r_j + r_{j+1}).$$

Proof. Take $\phi(jq_n + [\frac{q_n}{2}] + rq_n)$ with $|r| \leq 10\varepsilon$ into consideration. Denote by $p = jq_n + [\frac{q_n}{2}] + rq_n$. Without loss of generality assume $j \geq 0$. Let $n_0$ be the least positive integer such that $q_n - n_0 \leq \frac{1}{8} - 2\varepsilon q_n$. Let $s$ be the largest positive integer such that $sq_n - n_0 \leq (\frac{1}{8} - 2\varepsilon)q_n$. By the fact $(s + 1)q_n - n_0 \geq (\frac{1}{8} - 2\varepsilon)q_n$, one has

and

$$s \geq \varepsilon$$

and

$$\left(\frac{1}{8} - 3\varepsilon\right)q_n \leq sq_n - n_0 \leq (\frac{1}{8} - 2\varepsilon)q_n.$$

Set $I_1, I_2 \subset \mathbb{Z}$ as follows

$$I_1 = [-2sq_n - n_0, -1],$$
$$I_2 = [jq_n + [\frac{q_n}{2}] - 2sq_n - n_0, jq_n + [\frac{q_n}{2}] + 2sq_n - n_0 - 1],$$

and let $\theta_m = \theta + m\alpha$ for $m \in I_1 \cup I_2$. The set $\{\theta_m\}_{m \in I_1 \cup I_2}$ consists of $6sq_n - n_0$ elements. Let $k = 6sq_n - n_0 - 1$.

By modifying the proof of [3], Lemma 9.9 and [45], Lemma 4.1 (or Appendices in [34] and [43]), we can prove the claim (Claim 1): for any $m \in I_2$,

$$\text{Lag}_m \leq q_n(\beta_j + \varepsilon),$$

and for any $m \in I_1$,

$$\text{Lag}_m \leq q_n\varepsilon.$$

For convenience, we include the proof in the Appendix.

By Lemma 2.2, there exists some $j_0 \in I_2$ such that

$$|P_k(\theta_{j_0} - \frac{k - 1}{2}\alpha)| \geq e^{kL - (\beta_j + \varepsilon)q_n}.$$
By (38) and (38), one has that when $j_0 \in I_2$, 
\begin{equation}
|P_{[x_1,x_2]}(\theta)| = |P_k(\theta_{j_0} - \frac{k - 1}{2} \alpha)| \geq e^{kL - (\beta_j + \varepsilon)q_n}.
\end{equation}

Recall that $x'_1 = x_1 - 1$ and $x'_2 = x_2 + 1$. By (20), (41) and (35), one has that when $j_0 \in I_2$, 
\begin{align*}
|\phi(p)| & \leq e^{-kL + \beta_jq_n + \varepsilon q_n}(|P_{[x_1,p-1]}| |\phi(x'_2)| + |P_{[p+1,x_2]}| |\phi(x'_1)|) \\
& \leq e^{-\frac{1}{2}Lq_n + \beta_jq_n + C\varepsilon q_n}(|P_{[x_1,p-1]}| |\phi(x'_2)| + |P_{[p+1,x_2]}| |\phi(x'_1)|).
\end{align*}

Clearly, 
\begin{equation*}
I_1 \subset [-\frac{q_n}{4}, 0]
\end{equation*}
and 
\begin{equation*}
I_2 \subset [j_0q_n + \frac{q_n}{4}, j_0q_n + \frac{3q_n}{4}].
\end{equation*}

**Case 1:** $j_0 \in \left[j_0q_n + \frac{3}{8}q_n, j_0q_n + \frac{5}{8}q_n\right] \cap I_2$

In this case, by (35) and (10), one has that 
\begin{equation}
x_1 \in \left[j_0q_n + 6\varepsilon q_n, j_0q_n + \frac{1}{4}q_n + 9\varepsilon q_n + 1\right],
\end{equation}
x_2 \in \left[j_0q_n + \frac{3}{4}q_n - 9\varepsilon q_n - 1, j_0q_n + q_n - 6\varepsilon q_n\right].

In order to make the following arguments neat, we are not going to make the difference between $a \in \mathbb{Z}$ and $a' \in \mathbb{Z}$ if $|a - a'| \leq 50\varepsilon q_n$. For example, instead of using (43), we simply write 
\begin{equation}
x_1 \in \left[j_0q_n, j_0q_n + \frac{1}{4}q_n\right],
x_2 \in \left[j_0q_n + \frac{3}{4}q_n, j_0q_n + q_n\right].
\end{equation}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1.png}
\caption{Fig.1}
\end{figure}

In Fig.1, $x_1$ and $x_2$ locate at the red part. $j_0$ locates at the blue part. The numbers above are the sites after deducting $j_0q_n$.

By (15), one has 
\begin{equation}
|P_{[x_1,p-1]}| \leq e^{(L+\varepsilon)|p-x_1|},
\end{equation}
and 
\begin{equation}
|P_{[p+1,x_2]}| \leq e^{(L+\varepsilon)|x_2-p|}.
\end{equation}

By (42), (45) and (46), one has that 
\begin{equation}
|\phi(p)| \leq \sum_{i=1,2} e^{(\beta_j + C\varepsilon)q_n} |\phi(x'_i)| e^{-|p-x_i|L}.
\end{equation}
By Lemma 2.3, one has
\[|\phi(x_1')| \leq r_je^{-L(x_1-jqn)+C\varepsilon qn} + r_{j+\frac{1}{2}}e^{-L(jqn+\frac{3}{4}x_1)+C\varepsilon qn}\]
and
\[|\phi(x_2')| \leq r_{j+\frac{1}{2}}e^{-L(x_2-jqn-\frac{3}{4}x_2)+C\varepsilon qn} + r_{j+1}e^{-L(jqn+q-n-x_2)+C\varepsilon qn}.
\]
By (47), (48), (49) and the fact that \(|x_1 - p| \geq \frac{3a}{4} - C\varepsilon q_n, i = 1, 2,\) we have
\[r_{j+\frac{1}{2}} \leq e^{-\frac{1}{2}(L-2\beta_j-C\varepsilon)q_n}r_j + e^{-\frac{1}{2}(L-2\beta_j-C\varepsilon)q_n}r_{j+1} + e^{-\frac{1}{2}(L-2\beta_j-C\varepsilon)q_n}r_{j+\frac{1}{2}}.
\]
Since \(\varepsilon\) is small and \(L > 2\beta \geq 2\beta_j - \varepsilon,\) one has that
\[e^{-\frac{1}{2}(L-2\beta_j-C\varepsilon)q_n}r_{j+\frac{1}{2}} \leq \frac{1}{2}r_{j+\frac{1}{2}}.
\]
Therefore (50) becomes
\[r_{j+\frac{1}{2}} \leq \exp\{-\frac{1}{2}(L-2\beta_j-C\varepsilon)q_n\}(r_j + r_{j+1}).
\]

**Case 2:** \(j_0 \in [jq_n + \frac{1}{4}q_n, jq_n + \frac{3}{8}q_n] \cap I_2\)

Let \(x = jq_n + \frac{3}{8}q_n - j_0.\) Therefore, \(0 \leq x \leq \frac{1}{8}q_n,\)

\[x_1 = jq_n - x \in [jq_n - \frac{1}{8}q_n, jq_n]
\]
and
\[x_2 = jq_n + \frac{3}{4}q_n - x \in [jq_n + \frac{5}{8}q_n, jq_n + \frac{3}{4}q_n].
\]

In Fig.2, \(x_1\) and \(x_2\) locate at the red part. \(j_0\) locates at the blue part. The numbers above are the sites after deducting \(jq_n.\)

By Lemma 2.3, one has that
\[|\phi(x_1')| \leq r_je^{-Lx+C\varepsilon qn} + r_{j-\frac{1}{2}}e^{-L(\frac{3}{4}x)-C\varepsilon qn},\]
and
\[|\phi(x_2')| \leq r_{j+\frac{1}{2}}e^{-L(\frac{3}{4}x-\frac{3}{4})+C\varepsilon qn} + r_{j+1}e^{-L(\frac{3}{4}+x)+C\varepsilon qn}.
\]
By (16) and (52), one has
\[
e^{-\frac{3}{4}Lq_n + \beta_j q_n + C\varepsilon q_n} |P_{[p+1,x_2]}| |\phi(x'_1)|
\leq e^{-\frac{4}{4}Lq_n + \beta_j q_n + C\varepsilon q_n} e^{L(\frac{3q_n}{q_n} - x)} (r_j e^{-Lx} + r_j - \frac{1}{4} e^{-L(\frac{3q_n}{q_n} - x)})
\leq e^{-\frac{4}{4}Lq_n - 2Lx + \beta_j q_n + C\varepsilon q_n} r_j + e^{-Lq_n + \beta_j q_n + C\varepsilon q_n} r_j - \frac{1}{2}
\]
(54)

By (16), one has
(55)

\[
r_j - \frac{1}{2} \leq e^{C\varepsilon q_n + \frac{5q_n}{q_n}L} r_j.
\]

By (54) and (55), one has

(56)
\[
e^{-\frac{4}{4}Lq_n + \beta_j q_n + C\varepsilon q_n} |P_{[p+1,x_2]}| |\phi(x'_1)| \leq e^{-\frac{4}{4}(L - 2\beta_j - C\varepsilon) q_n} r_j.
\]

By Lemma 3.1

(57)
\[
|P_{[x_1 - q_n, p - 1 - jq_n]}| \leq e^{C\varepsilon q_n} e^{L(\frac{1}{2}q_n - x)}.
\]

By Lemma 3.3 and (57), one has

(58)
\[
|P_{[x_1, p - 1]}| \leq |P_{[x_1, p - 1]} - P_{[x_1 - j q_n, p - 1 - j q_n]}| + |P_{[x_1 - j q_n, p - 1 - j q_n]}|
\leq e^{C\varepsilon q_n} e^{L(\frac{1}{2}q_n - x)} + e^{C\varepsilon q_n} e^{L|x_1 - p|} e^{-\beta_j q_n}
\leq e^{C\varepsilon q_n} e^{L(\frac{1}{2}q_n - x)} + e^{C\varepsilon q_n} e^{L(\frac{1}{2}q_n + x) - \beta_j q_n}.
\]

By (53) and (58), one has

(59)
\[
e^{-\frac{3}{4}Lq_n + \beta_j q_n + C\varepsilon q_n} |P_{[x_1, p - 1]}| |\phi(x'_2)|
\leq e^{-\frac{4}{4}Lq_n + \beta_j q_n + C\varepsilon q_n} (r_j + \frac{1}{2} e^{-L(\frac{3q_n}{q_n} - x)} + r_j + 1 e^{-L(\frac{3q_n}{q_n} + x)})
\times (e^{L(\frac{1}{2}q_n - x)} + e^{L(\frac{1}{2}q_n + x) - \beta_j q_n})
\leq e^{C\varepsilon q_n} (e^{-(L - 2\beta_j) \frac{3q_n}{q_n}} + e^{-(L - 2\beta_j) \frac{1}{2} - Lx}) r_j + \frac{1}{2}
\leq e^{C\varepsilon q_n} (e^{-(L - 2\beta_j) \frac{3q_n}{q_n}} + e^{-(L - 2\beta_j) \frac{1}{2} - Lx} + e^{-L(\frac{3q_n}{q_n})}) r_j + r_j + 1
\]

where the last inequality holds by the fact that \(0 \leq x \leq \frac{q_n}{8}\).

By (42), (56) and (59), one has

(60)
\[
r_j + \frac{1}{2} \leq e^{-\frac{3}{4}(L - 2\beta_j - C\varepsilon) q_n} r_j + e^{-\frac{3}{4}(L - 2\beta_j - C\varepsilon) q_n} r_j + 1
\]
\[+ e^{C\varepsilon q_n} (e^{-(L - 2\beta_j) \frac{3q_n}{q_n}} + e^{-(L - 2\beta_j) \frac{1}{2} - Lx}) r_j + \frac{1}{2}.
\]
Since $\varepsilon$ is small and $L > 2\beta \geq 2\beta_j - \varepsilon$, one has that

$$e^{C\varepsilon q_n}(e^{-(L-2\beta_j)\frac{2}{q_n}} + e^{-L\frac{2}{q_n}})r_{j+\frac{1}{2}} \leq \frac{1}{2}r_{j+\frac{1}{2}}.$$ 

Therefore, (60) becomes

$$r_{j+\frac{1}{2}} \leq \exp\{-\frac{1}{2}(L-2\beta_j - C\varepsilon)q_n\}(r_j + r_{j+1}).$$

**Case 3:** $j_0 \in [jq_n + \frac{5}{8}q_n, jq_n + \frac{3}{8}q_n] \cap I_2$

Let $x = j_0 - jq_n - \frac{5}{8}q_n$. Therefore, $0 \leq x \leq \frac{1}{8}q_n$, $x_1 = jq_n + \frac{4}{9} + x$ and $x_2 = jq_n + q_n + x$.

By Lemmas 3.1 and 3.3 one has

$$|P_{[p+1, x_2]}| \leq |P_{[p+1-jq_n-q_n, x_2-jq_n-q_n]}| + e^{C\varepsilon q_n}e^{L(\frac{1}{2}q_n+x)} - \beta_j q_n
\leq e^{C\varepsilon q_n}e^{L(\frac{1}{2}q_n-x)} + e^{C\varepsilon q_n}e^{L(\frac{1}{2}q_n+x)} - \beta_j q_n
\leq C\varepsilon q_n
\leq 2(\frac{1}{2})r_{j+\frac{1}{2}}.$$ (62)

Replacing (58) with (62) and following the proof of Case 2, we also have that

$$r_{j+\frac{1}{2}} \leq \exp\{-\frac{1}{2}(L-2\beta_j - C\varepsilon)q_n\}(r_j + r_{j+1}).$$

If $j_0 \in I_1$, one has that

$$x_1 \in \left[-\frac{5}{8}q_n, -\frac{3}{8}q_n\right], \quad x_2 \in \left[\frac{1}{8}q_n, \frac{3}{8}q_n\right].$$

By (15), one has

$$|P_{[x_1-1]}| \leq e^{L|x_1|+C\varepsilon q_n}, \quad |P_{[1,x_2]}| \leq e^{L|x_2|+C\varepsilon q_n}.$$ (64)

By (20), (30), (33), (34) and (31), one has

$$|\phi(0)| \leq e^{-kL+\varepsilon q_n}(|P_{[x_1-1]}|\phi(x'_2)| + |P_{[1,x_2]}||\phi(x'_1)|)
\leq e^{-\frac{3}{4}Lq_n+C\varepsilon q_n}(e^{L|x_1|} + e^{L|x_2|})
\leq \frac{1}{2}.$$ (65)

This is impossible since $\phi(0) = 1$. Therefore, we must have $j_0 \in I_2$ and Theorem 4.1 follows from (51), (61) and (63).

\[\square\]

**5. Resonance II: sites jq_n, j \in \mathbb{Z}\{0\}**

In this section, we deal with resonances arising from sites $jq_n, j \in \mathbb{Z}\{0\}$.

**Theorem 5.1.** Assume $j \in \mathbb{Z}$ satisfies $0 < |j| \leq 2\frac{b_{n+1}}{q_n} + 10$. Then we have

$$r_j \leq \exp\{-\frac{1}{2}(L-2\beta_j - C\varepsilon)q_n\}(r_{j+\frac{1}{2}} + r_{j-\frac{1}{2}}) + \exp\{-\frac{1}{2}(L-2\beta_j - C\varepsilon)q_n\}(r_{j+1} + r_{j-1}).$$
Proof. Take $\phi(jq_n + rq_n)$ with $|r| \leq 10\varepsilon$ into consideration. Denote by $p = jq_n + rq_n$. Without loss of generality assume $j > 0$. Let $n_0$ be the least positive integer such that

$$q_{n-n_0} \leq \frac{\varepsilon}{2}(\frac{1}{4} - 2\varepsilon)q_n.$$ 

Let $s$ be the largest positive integer such that $sq_{n-n_0} \leq (\frac{1}{4} - 2\varepsilon)q_n$. By the fact that $(s+1)q_{n-n_0} \geq (\frac{1}{4} - 2\varepsilon)q_n$, one has

$$s \geq \frac{1}{\varepsilon}$$

and

$$\frac{1}{4} - 3\varepsilon)q_n \leq sq_{n-n_0} \leq (\frac{1}{4} - 2\varepsilon)q_n.$$ 

Set $I_1, I_2 \subset \mathbb{Z}$ as follows

$$I_1 = [-2sq_{n-n_0} - 1, -1],$$

$$I_2 = [jq_n - 2sq_{n-n_0}, jq_n + 2sq_{n-n_0} - 1],$$

and let $\theta_m = \theta + m\alpha$ for $m \in I_1 \cup I_2$. The set $\{\theta_m\}_{m \in I_1 \cup I_2}$ consists of $6sq_{n-n_0}$ elements. Let $k = 6sq_{n-n_0} - 1$.

By modifying the proof of \cite{3}, Lemma 9.9 and \cite{45}, Lemma 4.1 (or Appendices in \cite{34} and \cite{43}), we can prove the claim (Claim 2): for any $m \in I_1 \cup I_2$, one has $\text{Lag}a_m \leq 2q_n(\beta_j + \varepsilon)$. We also give the proof in the Appendix.

Applying Lemma \ref{22} there exists some $j_0$ with $j_0 \in I_1 \cup I_2$ such that

$$|P_k(\theta_{j_0} - \frac{k - 1}{2}\alpha)| \geq e^{kL-2(\beta_j + \varepsilon)q_n}.$$ 

Let

$$I = [j_0 - 3sq_{n-n_0} + 1, j_0 + 3sq_{n-n_0} + 1] = [x_1, x_2].$$

By (20), (67), and (66), one has

- I: $j_0 \in I_1$. Then for any $y_1$ with $|y_1| \leq 10\varepsilon q_n$,

  $$|\phi(y_1)| \leq e^{-kL+2\beta_j q_n+\varepsilon q_n}(|P_{[x_1,y_1-1]}|\phi(x'_2)| + |P_{[y_1+1,x_2]}|\phi(x'_1)|)$$

  $$\leq e^{-\frac{3}{2}Lq_n + 2\beta_j q_n + C\varepsilon q_n}(|P_{[x_1,y_1-1]}|\phi(x'_2)| + |P_{[y_1+1,x_2]}|\phi(x'_1)|).$$

- II: $j_0 \in I_2$. Then for any $y_2$ with $|y_2 - jq_n| \leq 10\varepsilon q_n$,

  $$|\phi(y_2)| \leq e^{-\frac{3}{2}Lq_n + 2\beta_j q_n + C\varepsilon q_n}(|P_{[x_1,y_2-1]}|\phi(x'_2)| + |P_{[y_2+1,x_2]}|\phi(x'_1)|).$$

Clearly,

$$I_1 \subset \left[-\frac{q_n}{2}, 0\right]$$

and

$$I_2 \subset jq_n - \frac{q_n}{2}, jq_n + \frac{q_n}{2}.$$
**Case 1:** \( j_0 \in [jq_n - \frac{1}{4}q_n, jq_n] \cap I_2 \)

Let \( x = j_0 - (jq_n - \frac{1}{4}q_n) \). Therefore, \( 0 \leq x \leq \frac{1}{4}q_n \) and

\[
x_1 = jq_n - q_n + x \in [jq_n - q_n, jq_n - \frac{3}{4}q_n]
\]

and

\[
x_2 = jq_n + \frac{1}{2}q_n + x \in [jq_n + \frac{1}{2}q_n, jq_n + \frac{3}{4}q_n].
\]

In Fig.3, \( x_1 \) and \( x_2 \) locate at the red part. \( j_0 \) locates at the blue part. The numbers above are the sites after deducting \( jq_n \).

By Lemma \( 2.3 \) one has that

\[
|\phi(x'_1)| \leq r_{j-1} e^{-Lx+C\epsilon q_n} + r_{j-\frac{1}{2}} e^{-L(\frac{3q_n}{4}-x)+C\epsilon q_n}
\]

and

\[
|\phi(x'_2)| \leq r_{j+\frac{1}{2}} e^{-Lx+C\epsilon q_n} + r_{j+1} e^{-L(\frac{3q_n}{2}-x)+C\epsilon q_n}.
\]

By (15), one has

\[
|P_{[x_1, p-1]}| \leq e^{L(q_n-x)+C\epsilon q_n},
\]

and

\[
|P_{[p+1, x_2]}| \leq e^{L(\frac{3q_n}{2}+x)+C\epsilon q_n}.
\]
By (70)-(74), one has that
\[
|\phi(p)| \leq e^{-\frac{3}{2}Lq_n+2\beta_jq_n+C\varepsilon q_n} |P_{[x_1,p-1]}| (r_{j+\frac{1}{2}} e^{-Lx} + r_{j+1} e^{-L(\frac{4q}{2}-x)}) \\
+ e^{-\frac{3}{2}Lq_n+2\beta_jq_n+C\varepsilon q_n} |P_{[p+1,x_2]}| (r_{j-1} e^{-Lx} + r_{j-\frac{1}{2}} e^{-L(\frac{4q}{2}-x)}) \\
\leq e^{-\frac{3}{2}Lq_n+2\beta_jq_n+C\varepsilon q_n} (|P_{[x_1,p-1]}| r_{j+\frac{1}{2}} e^{-Lx} + e^L(q_n-x) r_{j+1} e^{-L(\frac{4q}{2}-x)}) \\
+ e^{-\frac{3}{2}Lq_n+2\beta_jq_n+C\varepsilon q_n} e^{L(\frac{4q}{2}+x)} (r_{j-1} e^{-Lx} + r_{j-\frac{1}{2}} e^{-L(\frac{4q}{2}-x)}) \\
\leq e^{-Lq_n+2\beta_jq_n+C\varepsilon q_n} (r_{j+1} + r_{j-1}) + e^{-\frac{3}{2}Lq_n+2\beta_jq_n+2Lx+C\varepsilon q_n} r_{j-\frac{1}{2}} \\
+ e^{-\frac{3}{2}Lq_n-Lx+2\beta_jq_n+C\varepsilon q_n} |P_{[x_1,p-1]}| r_{j+\frac{1}{2}}.
\]
(75)

where the last inequality holds by the fact that \(0 \leq x \leq \frac{1}{4} q_n\).

The term in (75) decays, so we are going to bound (76).

Clearly, \(x_1 + q_n \in [jq_n, jq_n + \frac{1}{4} q_n]\). By Lemma 2.3, one has that
\[
|\phi(x') + q_n| \leq r_{j} e^{-Lx+C\varepsilon q_n} + r_{j+\frac{1}{2}} e^{-L(\frac{4q}{2}-x)+C\varepsilon q_n}.
\]
(77)

By (20), (15) and (77), one has for any \(y\) with \(|y - jq_n - \frac{1}{2} q_n| \leq 10 \varepsilon q_n\),
\[
|\phi(y)| \leq |P_{[x_1+q_n,p-1+q_n]}|^{-1} e^{L\frac{4q}{2}+C\varepsilon q_n} (r_{j} e^{-Lx} + r_{j+\frac{1}{2}} e^{-L(\frac{4q}{2}-x)}) \\
+ |P_{[x_1+q_n,p-1+q_n]}|^{-1} e^{L(\frac{4q}{2}-x+C\varepsilon q_n)} r_{j+1}.
\]

This implies
\[
r_{j+\frac{1}{2}} \leq |P_{[x_1+q_n,p-1+q_n]}|^{-1} e^{L\frac{4q}{2}+C\varepsilon q_n} (r_{j} e^{-Lx} + r_{j+\frac{1}{2}} e^{-L(\frac{4q}{2}-x)}) \\
+ |P_{[x_1+q_n,p-1+q_n]}|^{-1} e^{L(\frac{4q}{2}-x+C\varepsilon q_n)} r_{j+1}.
\]
Therefore, we have
\[ r_{j+\frac{1}{2}} |P_{[x_1+q_n,p-1+q_n]}| \leq e^{C\varepsilon q_n (r_j e^{L(x_{\frac{1}{2}}-x)} + r_{j+\frac{1}{2}} e^{Lx} + e^{L(x_{\frac{1}{2}}-x)} r_{j+1})}. \]
(78)

By Lemma 3.3, one has
\[ |P_{[x_1+q_n,p-1+q_n]} - P_{[x_1,p-1]}| \leq e^{(L+C\varepsilon)q_n - Lx - \beta q_n}. \]
(79)

By (78) and (79), one has
\[ |P_{[x_1,p-1]}| r_{j+\frac{1}{2}} \leq e^{L(x_{\frac{1}{2}}-x) + C\varepsilon q_n (r_j + r_{j+1}) + e^{C\varepsilon q_n (e^{Lx} + e^{(L-\beta)q_n - Lx}) r_{j+\frac{1}{2}}}}. \]
(80)

Therefore,
\[ |P_{[x_1,p-1]}| r_{j+\frac{1}{2}} e^{-\frac{1}{2}Lq_n - Lx + 2\beta j q_n + C\varepsilon q_n} \leq e^{C\varepsilon q_n (e^{-(-L-2\beta)\frac{1}{2}q_n - 2Lx} + e^{-\frac{1}{2}Lq_n + 2\beta j q_n + C\varepsilon q_n}) r_{j+\frac{1}{2}}} \]
\[ + e^{-Lq_n + 2\beta j q_n - 2Lx + C\varepsilon q_n} (r_j + r_{j+1}) \leq e^{C\varepsilon q_n (r_j e^{-(-L-2\beta)q_n} + r_{j+\frac{1}{2}} e^{-(L-2\beta)q_n}) + r_{j+1} e^{-(L-2\beta)q_n}}. \]
(81)

By (81), (75) and (76), one has
\[ |\phi(p)| \leq e^{-(L-2\beta)q_n + C\varepsilon q_n (r_j - 1 + r_{j-\frac{1}{2}} + r_j + r_{j+1}) + r_{j+\frac{1}{2}}} e^{-(L-2\beta j - C\varepsilon) \frac{q_n}{3}}. \]
(82)

Therefore,
\[ r_j \leq e^{-(L-2\beta)q_n + C\varepsilon q_n (r_j - 1 + r_{j-\frac{1}{2}} + r_j + r_{j+1}) + r_{j+\frac{1}{2}}} e^{-(L-2\beta j - C\varepsilon) \frac{q_n}{3}}. \]
(83)

By the fact that \( L > 2\beta \geq 2\beta_j - \varepsilon \), we have
\[ e^{-(L-2\beta)q_n + C\varepsilon q_n r_j \leq \frac{1}{2} r_j} \]
and hence
\[ r_j \leq e^{-(L-2\beta j - C\varepsilon) q_n (r_j - 1 + r_{j-\frac{1}{2}} + r_j + r_{j+1}) + r_{j+\frac{1}{2}}} e^{-(L-2\beta j - C\varepsilon) \frac{q_n}{2}}. \]
(84)

**Case 2:** \( j_0 \in [j q_n, j q_n + \frac{1}{4} q_n] \cap I_2 \)
In this case, by the similar proof of (84), we have
\[ r_j \leq e^{-(L-2\beta j - C\varepsilon) q_n (r_j + r_{j-\frac{1}{2}} + r_j + r_{j+1}) + r_{j+\frac{1}{2}}} e^{-(L-2\beta j - C\varepsilon) \frac{q_n}{2}}. \]
(85)

**Case 3:** \( j_0 \in [j q_n - \frac{1}{2} q_n, j q_n - \frac{1}{4} q_n] \cap I_2 \)
In this case, let \( x = j_0 - (j q_n - \frac{1}{2} q_n) \). It is easy to see that \( 0 \leq x \leq \frac{q_n}{4} \),
\[ x_1 = j q_n - \frac{5 q_n}{4} + x \in [j q_n - \frac{5}{4} q_n, j q_n - q_n] \]
and
\[ x_2 = j q_n + \frac{1}{4} q_n + x \in [j q_n + \frac{q_n}{4}, j q_n + \frac{q_n}{2}] \]
We finish the estimate of (89). Now we are in the position to bound (90).

By (96), one has

\[ |P_{[x_1-(j-1)q_n,p-1-(j-1)q_n]}| \leq e^{L(\frac{3q_n}{4}+x)+C\varepsilon q_n}. \]

By Lemma 3.3 and (95), we have

\[ |P_{[x_1,p-1]}| \leq e^{L(\frac{3q_n}{4}+x)+C\varepsilon q_n} + e^{L(\frac{5q_n}{4}-x)-\beta_j q_n+C\varepsilon q_n}. \]

By (96), one has

\[
e^{-\frac{3}{2}Lq_n+2\beta_j q_n} e^{-L(\frac{2q_n}{4}+x)} |P_{[x_1,p-1]}| r_j \]
\[
\leq e^{-\frac{3}{2}Lq_n+2\beta_j q_n+C\varepsilon q_n} (e^{L(\frac{3q_n}{4}+x)} + e^{L(\frac{5q_n}{4}-x)-\beta_j q_n}) e^{-L(\frac{2q_n}{4}+x)} r_j \]
\[
\leq e^{-(L-2\beta_j)q_n+C\varepsilon q_n} r_j + e^{-(L-2\beta_j)^2 q_n - 2Lx + C\varepsilon q_n} r_j \]
\[
\leq e^{-(L-2\beta_j)^2 q_n + C\varepsilon q_n} r_j. \tag{97}
\]

This implies that the first term in (90) decays. We are going to bound the remaining term in (90):

\[
e^{-\frac{3}{2}Lq_n+2\beta_j q_n} e^{-L(\frac{2q_n}{4}+x)} |P_{[x_1,p-1]}| r_j + \frac{1}{2}.
\]

By Lemma 3.3 one has

\[ |P_{[x_1+q_n,p-1+q_n]} - P_{[x_1,p-1]}| \leq e^{L(\frac{2q_n}{4}+x)-\beta_j q_n+C\varepsilon q_n}. \tag{98} \]

By (98), we have

\[
e^{-\frac{3}{2}Lq_n+2\beta_j q_n} e^{-L(\frac{2q_n}{4}+x)} |P_{[x_1,p-1]}| r_j + \frac{1}{2} \]
\[
\leq e^{-\frac{3}{2}Lq_n+2\beta_j q_n} |P_{[x_1+q_n,p-1+q_n]}| e^{-L(\frac{2q_n}{4}+x)} r_j + \frac{1}{2} + e^{L(\frac{2q_n}{4}+x)-\beta_j q_n+C\varepsilon q_n} e^{-\frac{3}{2}Lq_n+2\beta_j q_n} e^{-L(\frac{2q_n}{4}+x)} r_j + \frac{1}{2} \]
\[
\leq e^{-\frac{3}{2}Lq_n+2\beta_j q_n} e^{-L(\frac{2q_n}{4}+x)} |P_{[x_1+q_n,p-1+q_n]}| r_j + \frac{1}{2} + e^{-(L-2\beta_j-C\varepsilon)q_n} r_j + \frac{1}{2}. \tag{99}
\]

The second term in (99) decays, so we are going to bound the first term in (99):

\[
e^{-\frac{3}{2}Lq_n+2\beta_j q_n} e^{-L(\frac{2q_n}{4}+x)} |P_{[x_1+q_n,p-1+q_n]}| r_j + \frac{1}{2}. \]

Clearly, \(x_1 + q_n \in \lfloor \hat{q}_n - \frac{1}{4} q_n, \hat{q}_n \rfloor \). By Lemma 2.3 one has that

\[ |\phi(x_1' + q_n)| \leq r_{j} e^{-L(\frac{q_n}{4}+x)+C\varepsilon q_n} + r_{j} - \frac{1}{2} e^{-L(\frac{q_n}{4}+x)+C\varepsilon q_n}. \tag{100} \]
By (20), (15) and (101), one has for any $y$ with $|y - j\frac{n}{4} - \frac{1}{2}q_n| \leq 10\varepsilon q_n$,

\[
|\phi(y)| \leq |P_{[x_1 + q_n, p+1 + q_n]}^{-1} P_{[y_1 + q_n, p+1 + q_n]}^{-1} (r_j e^{-L(p_j - x) + C\varepsilon q_n} + r_{j+1} e^{-L(p_{j+1} - x) + C\varepsilon q_n})
+ |P_{[x_1 + q_n, p+q_n]}^{-1} P_{[x_1 + q_n, y_1 + q_n]}^{-1} |r_{j+1}
\leq |P_{[x_1 + q_n, p+1 + q_n]}^{-1} P_{[x_1 + q_n, y_1 + q_n]}^{-1} | \leq 10^2 e^{\frac{3}{2} L q_n - L x + C\varepsilon q_n} (r_j e^{-L x} + r_{j+1} e^{-L x}) + e^{\frac{3}{2} L q_n - L x + C\varepsilon q_n} r_{j+1}.
\]

(101) If $r_{j+1} \leq e^{\frac{3}{2} L q_n + 2\beta j q_n + C\varepsilon q_n} r_j$ and $0 \leq x \leq \frac{q_n}{4}$.

By (102), one has

\[
e^{-\frac{3}{2} L q_n + 2\beta j q_n} |P_{[x_1 + q_n, p+1 + q_n]}^{-1} P_{[y_1 + q_n, p+1 + q_n]}^{-1} | r_{j+1}
\leq e^{-\frac{3}{2} L q_n + 2\beta j q_n + 2L x + C\varepsilon q_n} r_j + e^{-\frac{3}{2} L q_n + 2\beta j q_n + C\varepsilon q_n} r_{j+1}
\leq e^{-L q_n + 2\beta j q_n + C\varepsilon q_n} r_j, \quad \text{where the last inequality holds by the fact that } r_j \leq e^{\frac{3}{2} L q_n + 2\beta j q_n + C\varepsilon q_n} r_j \text{ and } 0 \leq x \leq \frac{q_n}{4}.
\]

By (99) and (103), one has

\[
e^{-3/2 L q_n + 2\beta j q_n} |P_{[x_1 + q_n]}^{-1} | e^{-L(p_j - x)} r_{j+1}
\leq e^{-L q_n + 2\beta j q_n + C\varepsilon q_n} (r_{j+1} + r_j) + e^{-(L-2\beta_j - C\varepsilon) q_n/2} r_{j+1}.
\]

(104) By (88), (91), (97) and (104),

\[
r_j \leq e^{-L(2\beta_j - C\varepsilon) q_n} (r_{j-1} + r_{j+1}) + e^{-(L-2\beta_j) q_n/2 + C\varepsilon q_n} r_{j+1}
+ (e^{-L q_n + 2\beta j q_n + C\varepsilon q_n} + e^{-\frac{1}{2} L q_n + \beta j q_n + C\varepsilon q_n}) r_j.
\]

(105) Since

\[
(e^{-L q_n + 2\beta j q_n + C\varepsilon q_n} + e^{-\frac{1}{2} L q_n + \beta j q_n + C\varepsilon q_n}) r_j \leq \frac{1}{2} r_j,
\]
by (105), one has

\[(106) \quad r_j \leq e^{-L - 3j - C\varepsilon} q_n (r_{j-1} + r_{j+1}) + e^{-L - 3j - C\varepsilon} \frac{q_n}{2} r_{j+1} .\]

**Case 4:** \( j_0 \in [j q_n + \frac{1}{4} q_n, j q_n + \frac{1}{2} q_n] \cap I_2 \)

In this case, following the proof of Case 3, we have

\[(107) \quad r_j \leq e^{-L - 3j - C\varepsilon} q_n (r_{j-1} + r_{j+1}) + e^{-L - 3j - C\varepsilon} \frac{q_n}{2} r_{j+1} .\]

If \( j_0 \in I_1 \), then (84), (85), (106) and (107) hold for \( j = 0 \). Therefore, we have

\[|\phi(0)| \leq \frac{1}{2} .\]

This is in contradiction with \( \phi(0) = 1 \). Therefore we must have \( j_0 \in I_2 \) and (65) follows from (84), (85), (106) and (107).

\[\square\]

6. **Proof of Theorem 1.1**

Once we have Theorems 4.1 and 5.1 at hand, Theorem 1.1 follows from standard iterations. See [34, 43] for example. For the convenience, we include a proof here.

**Proof of Theorem 1.1.** Without loss of generality, we only bound \( \phi(k) \) with \( k > 0 \). Let \( n \) be such that \( b_n \leq k < b_{n+1} \). Clearly, \( \beta_j \leq \beta + \varepsilon \). By Theorems 4.1 and 5.1, we have for any \( j \) with \( 1 \leq j \leq 2 b_n + 10 \),

\[(108) \quad r_{j-\frac{1}{2}} \leq \exp\left\{-\frac{1}{2} (L - 2\beta - C\varepsilon) q_n \right\} \max\{r_{j-1}, r_j\},\]

and

\[(109) \quad r_j \leq \max_{t \in O} \{\exp\{-|t|(L - 2\beta - C\varepsilon) q_n\} r_{j+t}\},\]

where \( O = \{\pm 1, \pm \frac{1}{2}\} \).

Suppose \( 1 \leq \ell \leq \frac{b_n + 1}{q_n} + 4 \). Let \( j = \ell \) in (109) and (108), and iterate \( 2\ell \) times or until \( j \leq 1 \), we obtain

\[(110) \quad r_\ell \leq (2\ell + 2) q_n \exp\{- (L - 2\beta - C\varepsilon) \ell q_n\},\]

and

\[(111) \quad r_{\ell-\frac{1}{2}} \leq (2\ell + 2) q_n \exp\{- (L - 2\beta - C\varepsilon) (\ell - \frac{1}{2}) q_n\} .\]

Notice that we have used the fact that \( |r_j| \leq (j + 1) q_n \) and \( |r_{j-\frac{1}{2}}| \leq (j + 1) q_n \).

**Case 1:** \( k \geq \frac{q_n}{4} \) and \( \text{dist}(k, q_n \mathbb{Z} + \frac{q_n}{2} \mathbb{Z}) \leq 10\varepsilon q_n \).

In this case, applying (110) and (111), one has

\[(112) \quad |\phi(k)|, |\phi(k - 1)| \leq \exp\{- (L - 2\beta - C\varepsilon) k\} .\]

**Case 2:** others
Applying Lemma 2.3 with sufficiently small $\varepsilon$, and by (110) and (111), one also has

\[(113) \quad |\phi(k)|, |\phi(k - 1)| \leq \exp\{- (L - 2\beta - C\varepsilon) k\}.
\]

We finish the proof. \hfill \Box

\section*{Appendix A. Proof of Claims 1 and 2}

Let $\frac{p_n}{q_n}$ be the continued fraction approximations to $\alpha$, then

\[(114) \quad \forall 1 \leq k < q_{n+1}, \text{ dist}(k\alpha, \mathbb{Z}) \geq |q_n\alpha - p_n|,
\]

and

\[(115) \quad \frac{1}{2q_{n+1}} \leq |q_n\alpha - p_n| \leq \frac{1}{q_{n+1}}.
\]

**Proof of Claim 1.** By the construction of $I_1$ and $I_2$ in Claim 1, (114) and (115), we have that

- for any $m \in I_1$,

\[(116) \quad \min_{\ell \in I_1 \cup I_2} \min_{m} \ln |\sin \pi (2\theta + (\ell + m)\alpha)| \geq -C \ln q_n,
\]

and

\[(117) \quad \min_{\ell \in I_1 \cup I_2} \min_{m} \ln |\sin \pi (\ell - m)\alpha| \geq -C \ln q_n;
\]

- for any $m \in I_2$,

\[(118) \quad \min_{\ell \in I_1 \cup I_2} \ln |\sin \pi (2\theta + (\ell + m)\alpha)|_{\mathbb{R}/\mathbb{Z}} \geq -\beta_j q_n - C \ln q_n,
\]

and

\[(119) \quad \min_{\ell \in I_1 \cup I_2} \min_{m} \ln |\sin \pi (\ell - m)\alpha| \geq -C \ln q_n.
\]

We should mention that, for each $m \in I_2$, there is at most one $\ell \in I_1 \cup I_2$ such that the lower bound of (118) can be achieved.

Once we have (116)-(119) at hand, by the standard arguments (e.g. Appendices in [34, 43]), we have that for any $m \in I_1$,

\[\text{Lag}_m \leq \varepsilon q_n\]

and for any $m \in I_2$,

\[\text{Lag}_m \leq \beta_j q_n + \varepsilon q_n.\]

\hfill \Box
Proof of Claim 2. By the construction of $I_1$ and $I_2$ in Claim 2, (114) and (115), we have that for $m \in I_1 \cup [jq_n - 2sq_{n-n_0}, jq_n - 1]$, 
\begin{equation}
\min_{\ell \in I_1 \cup I_2} \ln |\sin \pi (2\theta + (\ell + m)\alpha)| \geq \beta_j q_n - C \ln q_n,
\end{equation}
and 
\begin{equation}
\min_{\ell \neq m, \ell \in I_1 \cup I_2} \ln |\sin \pi (\ell - m)\alpha| \geq \beta_j q_n - C \ln q_n.
\end{equation}

We should mention that, for each $m \in I_1 \cup [jq_n - 2sq_{n-n_0}, jq_n - 1]$, there is at most one $\ell \in I_1 \cup I_2$ such that the lower bound of (120) or (121) can be achieved.

We also have that for $m \in [jq_n, jq_n + 2sq_{n-n_0} - 1]$, 
\begin{equation}
\min_{\ell \in I_1 \cup I_2} \ln |\sin \pi (2\theta + (\ell + m)\alpha)| \geq \beta_j q_n - C \ln q_n,
\end{equation}
and 
\begin{equation}
\min_{\ell \neq m, \ell \in I_1 \cup I_2} \ln |\sin \pi (\ell - m)\alpha| \geq -C \ln q_n.
\end{equation}

Moreover, for each $m \in [jq_n, jq_n + 2sq_{n-n_0} - 1]$, there is at most two $\ell \in I_1 \cup I_2$ such that the lower bound of (122). Once we have (120)-(123) at hand, by the standard arguments (e.g. Appendices in [34, 43]), we have that for any $m \in I_1 \cup I_2$, 
\[ \mathrm{Lag}_m \leq 2\beta_j q_n + \varepsilon q_n. \]

\[ \square \]

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