A note on modified Hermite matrix polynomials

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Abstract

The main aim of this paper is to investigate the modified Hermite matrix polynomials $M_{H_n}(\zeta_1, \lambda; A)$ by finding some important results such as generating functions, recurrence relations, Rodrigues formula, orthogonality conditions, expansion formula, integrals, fractional integrals, fractional derivatives and some other properties.

Keywords: Gamma matrix function, hypergeometric matrix function, three term matrix recurrence relation, modified Hermite matrix differential equation, modified Hermite matrix polynomials, orthogonal matrix polynomials.

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1. Introduction and preliminaries

An opening explanation to the class of Hermite matrix polynomials $H_n(\zeta_1; A)$ and some basic properties have been considered in \cite{2, 4–7, 10, 11, 13, 22, 23}. A comprehensive view on Hermite matrix polynomials are given in \cite{12, 26–29}. Jódar and co-authors established the classical families of Hermite matrix polynomials $H_n(\zeta_1; A)$ defined by

$$
\exp(\zeta_1 z \sqrt{2 A} - z^2 I) = \sum_{n \geq 0} \frac{H_n(\zeta_1; A)}{n!} z^n,
$$

where

$$
H_n(\zeta_1; A) = \sum_{k=0}^{n} \frac{(-1)^k n! \left(\zeta_1 \sqrt{2 A}\right)^{n-2k}}{k! (n-2k)!},
$$

where $A$ is a $+$ve stable matrix in the complex space $\mathbb{C}^{N \times N}$ of all square matrices of common order $N$ which appear as a finite series solution of second order matrix differential equation $y'' - \zeta_1 A y' + n A y = \ast$

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0, for a matrix \( \mathcal{A} \) in \( \mathbb{C}^{N \times N} \) whose eigenvalues are all in the right open half plane. It has been given by Defez and Jódar [1, 4] the matrices \( \mathcal{A}(k, n) \) and \( \mathcal{B}(k, n) \) in \( \mathbb{C}^{N \times N} \) where \( n \geq 0, k \geq 0 \), the following relations are satisfied,

\[
\sum_{n \geq 0} \sum_{k \geq 0} \mathcal{A}(k, n) = \sum_{n \geq 0} \sum_{k=0}^{[n/2]} \mathcal{A}(k, n-2k),
\]

and

\[
\sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} \mathcal{B}(k, n) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \mathcal{B}(k, n-k). \tag{1.1}
\]

Similarly, we can write

\[
\sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} \mathcal{A}(k, n) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \mathcal{A}(k, n+2k),
\]

\[
\sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} \mathcal{B}(k, n) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \mathcal{B}(k, n-k),
\]

\[
\sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} \mathcal{A}(k, n) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \mathcal{A}(k, n+k).
\]

Also, if \( \mathcal{A} + nI \) is invertible for every integer \( n \geq 0 \), then form [9]. The matrix version of the Pochhammer symbol is

\[
(\mathcal{A})_n = \Gamma(\mathcal{A} + nI)\Gamma(\mathcal{A})^{-1},
\]

\[
(\mathcal{A})_n = \begin{cases} 1, & \text{if } n = 0 \\ \mathcal{A}(\mathcal{A} + 1)(\mathcal{A} + 2I) \cdots (\mathcal{A} + (n - 1)I), & \text{if } n = 1, 2, \ldots \end{cases} \tag{1.2}
\]

From (1.2), it is easy to calculate that

\[
(\mathcal{A})_{n-k} = (-1)^k (\mathcal{A})_n (1 - \mathcal{A} - nI)_k^{-1}; \quad 0 \leq k \leq n.
\]

From [25, pp-58], one can obtain

\[
\frac{(-1)^k k!}{(n-k)!} I = \frac{(-n)_k k!}{n!} I = \frac{(-nI)_k}{n!}; \quad 0 \leq k \leq n,
\]

if \( \mathcal{A} + nI, \mathcal{B} + nI \), and \( \mathcal{A} + \mathcal{B} + nI \) all are invertible

\[
\beta(\mathcal{A}, \mathcal{B}) = \frac{\Gamma(\mathcal{A})\Gamma(\mathcal{B})}{\Gamma(\mathcal{A} + \mathcal{B})}; \quad 0 \leq k \leq n,
\]

where \( \beta(\mathcal{A}, \mathcal{B}) \) denotes beta matrix function for the pair \( \mathcal{A}, \mathcal{B} \). The hypergeometric matrix function \( F(\mathcal{A}, \mathcal{B}; \mathcal{C}; z) \) has been given in the form [9, 14] for matrix \( \mathcal{A}, \mathcal{B}, \) and \( \mathcal{C} \) in \( \mathbb{C}^{N \times N} \) such that \( \mathcal{C} + nI \) is invertible for all integer \( n \geq 0 \),

\[
F(\mathcal{A}, \mathcal{B}; \mathcal{C}; z) = \sum_{n \geq 0} \frac{(\mathcal{A})_n (\mathcal{B})_n}{(\mathcal{C})_n n!} z^n,
\]

it converges for \( |z| < 1 \), for any matrix \( \mathcal{A} \) in \( \mathbb{C}^{N \times N} \) we will make use of the following relation due to [9],

\[
(1 - \zeta_1)^{-\mathcal{A}} = \sum_{n \geq 0} \frac{(\mathcal{A})_n \zeta_1^n}{n!}; \quad |\zeta_1| < 1.
\]

For more details about various type of polynomials and its properties, one can be referred to [15, 16, 18–20, 24, 31].
3. Generating function for $M_{\mathcal{H}}$ \(M(2.1)\) does not change, so 

Using equation (2.1), we have

$$
M_{\mathcal{H}}n(\zeta_1; \lambda; A) = \sum_{n=0}^{\infty} \frac{M_{\mathcal{H}}n(\zeta_1; \lambda; A)J^n}{n!}, \quad \lambda > 0, \quad \lambda \neq 1. 
$$

(2.1)

Using equation (2.1), we have

$$
M_{\mathcal{H}}n(\zeta_1; \lambda; A) = \sum_{k=0}^{[n/2]} \frac{(-1)^k n! (\zeta_1 \sqrt{2A})^n - 2k \ln(\lambda)^n - k}{k!(n - 2k)!},
$$

for \(\lambda = e\), it reduces to Hermite Matrix polynomials \(\mathcal{H}(\zeta_1, A)\) which have been considered by Jódar’s and Company [7]. In equation (2.1), if we replace \(\zeta_1\) by \(-\zeta_1\) and \(3\) by \(-3\) left hand side of the equation (2.1) does not change, so

$$
M_{\mathcal{H}}n(\zeta_1; \lambda; A) = (-1)^n M_{\mathcal{H}}n(\zeta_1; \lambda; A).
$$

It shows that \(M_{\mathcal{H}}n(\zeta_1; \lambda; A)\) is an odd function of \(\zeta_1\) for odd \(n\), an even function of \(\zeta_1\) for even \(n\). Also

$$
M_{\mathcal{H}}2n_2(0; \lambda; A) = (-1)^n 2^n \left(\frac{1}{2}\right) n! \ln(\lambda)^n, \quad M_{\mathcal{H}}2n_2(0; \lambda; A) = 0,
$$

and

$$
M_{\mathcal{H}}2n_2+1(0; \lambda; A) = (-1)^n 2^n \left(\frac{3}{2}\right) n! \sqrt{2A} (\ln(\lambda)^{n+1}, \quad M_{\mathcal{H}}2n_2(0; \lambda; A) = 0.
$$

The first few modified Hermite matrix polynomials are

$$
M_{\mathcal{H}}0(\zeta_1; \lambda; A) = I,
$$

$$
M_{\mathcal{H}}1(\zeta_1; \lambda; A) = \zeta_1 \sqrt{2A} (\ln(\lambda)),
$$

$$
M_{\mathcal{H}}2(\zeta_1; \lambda; A) = (\zeta_1 \sqrt{2A})^2 (\ln(\lambda))^2 - 2(\ln(\lambda))I,
$$

$$
M_{\mathcal{H}}3(\zeta_1; \lambda; A) = (\zeta_1 \sqrt{2A})^3 (\ln(\lambda))^3 - 6(\zeta_1 \sqrt{2A}) (\ln(\lambda))^2,
$$

$$
M_{\mathcal{H}}4(\zeta_1; \lambda; A) = (\zeta_1 \sqrt{2A})^4 (\ln(\lambda))^4 - 12(\zeta_1 \sqrt{2A})^2 (\ln(\lambda))^3 + 12 \ln(\lambda)I,
$$

$$
M_{\mathcal{H}}5(\zeta_1; \lambda; A) = (\zeta_1 \sqrt{2A})^5 (\ln(\lambda))^5 - 20(\zeta_1 \sqrt{2A})^3 (\ln(\lambda))^4 + 60(\zeta_1 \sqrt{2A}) (\ln(\lambda))^3,
$$

$$
M_{\mathcal{H}}6(\zeta_1; \lambda; A) = (\zeta_1 \sqrt{2A})^6 (\ln(\lambda))^6 - 30(\zeta_1 \sqrt{2A})^4 (\ln(\lambda))^5 + 180(\zeta_1 \sqrt{2A})^2 (\ln(\lambda))^4 - 120(\ln(\lambda))^3 I.
$$

3. Generating function for \(M_{\mathcal{H}}n(\zeta_1; \lambda; A)\)

In this section, we give the following generating functions.

Brafman type generating function

$$
\sum_{n=0}^{\infty} \frac{(c)nM_{\mathcal{H}}n(\zeta_1; \lambda; A)J^n}{n!} = (1 - \zeta_1 \sqrt{2A} \ln(\lambda))^{-c} _2F_0 \left[ \frac{\zeta_1 \sqrt{2A}}{2}, \frac{\zeta_1 + 1}{2}; \frac{-4\sqrt{2A} \ln(\lambda)}{(1 - \zeta_1 \sqrt{2A} \ln(\lambda))^2} \right].
$$

(3.1)
Bilinear generating function

\[ \sum_{n=0}^{\infty} \frac{M_{\mathcal{H}}(\zeta_1; \lambda; \mathcal{A}) M_{\mathcal{H}}(\zeta_2; \lambda; \mathcal{A}) \beta^n}{n!} = (1 - 4\beta^2 \ln \lambda)^{-\frac{1}{2}} \lambda \left( \frac{2\alpha \beta \ln(\zeta_2 - \zeta_1 \ln \lambda - \beta \lambda)}{1 - 4\beta^2 \ln \lambda} \right). \]  

(3.2)

Other generating functions:

Some other generating functions are as follows:

\[ \sum_{n=0}^{\infty} \frac{M_{\mathcal{H}}(\zeta_1; \lambda; \mathcal{A}) \beta^n}{n!} = \lambda^{\zeta_1 \sqrt{2\alpha} - \zeta_1} M_{\mathcal{H}}(\zeta_1 - (2\alpha^{-1})^{\frac{1}{2}} \lambda; \mathcal{A}), \]  

(3.3)

\[ \sum_{n=0}^{\infty} \frac{2F_0[-n, c; -\zeta_1 \mathcal{H} n(\zeta_1; \lambda; \mathcal{A}) \beta^n}{n!} = \lambda^{\zeta_1 \sqrt{2\alpha} - \zeta_1} \left( 1 + \zeta_1 \sqrt{2\alpha} (\zeta_1 - \sqrt{2\alpha^{-1}}) \right)^{-c} \times 2F_0 \left[ \frac{c, c+1/2; 1}{\lambda^{-2} \ln \lambda \left( \zeta_1 - \sqrt{2\alpha^{-1}} \right)^2} \right], \]  

(3.4)

\[ \sum_{n=0}^{\infty} \frac{(-1)^n M_{\mathcal{H}}(2n)(\zeta_1; \lambda; \mathcal{A}) \beta^n}{(2n)!} = \lambda^2 \cos(\zeta_1 \ln \lambda \sqrt{2\alpha \beta}), \]  

(3.5)

Proof of (3.1).

\[ \sum_{n=0}^{\infty} \frac{(c)_{n \alpha} M_{\mathcal{H}}(\zeta_1; \lambda; \mathcal{A}) \beta^n}{n!} = \sum_{n=0}^{\infty} \left[ \frac{(-1)^k (c)_{n-k}(\zeta_1 \sqrt{2\alpha \beta})^{n-2k}(\ln \lambda)^{n-k}}{k!(n-2k)!} \right], \]  

Hence the proof of (3.1).

\[ \sum_{n=0}^{\infty} \frac{M_{\mathcal{H}}(\zeta_1; \lambda; \mathcal{A}) M_{\mathcal{H}}(\zeta_2; \lambda; \mathcal{A}) \beta^n}{n!} = \sum_{n=0}^{\infty} \left[ \frac{(-1)^k (\zeta_1 \sqrt{2\alpha \beta})^{n-2k}(\ln \lambda)^{n-k} M_{\mathcal{H}}(\zeta_2; \lambda; \mathcal{A}) \beta^n}{k!(n-2k)!} \right]. \]  

Proof of (3.2).
Hence, we obtain the result of (3.2).

**Proof of (3.3).**

\[
\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} M_{n+k}(\zeta_1; \lambda; \mathcal{A})_3^n V^k = \sum_{n=0}^{\infty} \sum_{k=0}^{n} M_{n}(\zeta_1; \lambda; \mathcal{A})_3^{n-k} V^k \frac{n!}{(n-k)!k!}
\]

\[
= \sum_{n=0}^{\infty} M_{n}(\zeta_1; \lambda; \mathcal{A})_3^k \left(1 - 4\zeta_1^2 \ln \lambda^2\right)^{-\frac{1}{2}} \lambda^2 \zeta_1 \zeta_3 \ln \lambda - 2\zeta_1^2 \zeta_3^2 \ln \lambda - 2\zeta_2 \zeta_3 \ln \lambda \left(1 - 4\zeta_1^2 \ln \lambda^2\right)^{-\frac{1}{2}} \lambda
\]

\[
= \lambda \zeta_1 \zeta_3 \sqrt{\mathcal{A}} - z^2 I^n M_{n+k}(\zeta_1 - (2\mathcal{A} - 1)^{\frac{1}{2}} \lambda; \mathcal{A})V^n \frac{n!}{k!}.
\]

By equating the coefficients of \( V^k \), we have,

\[
\sum_{n=0}^{\infty} M_{n+k}(\zeta_1; \lambda; \mathcal{A})_3^n \frac{n!}{k!} = \lambda \zeta_1 \zeta_3 \sqrt{\mathcal{A}} - z^2 I^n M_{n+k}(\zeta_1 - (2\mathcal{A} - 1)^{\frac{1}{2}} \lambda; \mathcal{A})V^n\frac{n!}{k!},
\]

therefore the result.

**Proof of (3.4).** When applying equation (3.3) to any known generating relation and obtain a new result.

\[
\sum_{k=0}^{\infty} \frac{(c)_{k} M_{k}(\zeta_1 - (2\mathcal{A} - 1)^{\frac{1}{2}} \lambda; \mathcal{A})(-3)^y k^k}{k!} = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{(c)_{k} \lambda^{-\zeta_1 \zeta_3 \sqrt{\mathcal{A}} + z^2 I^n M_{n+k}(\zeta_1; \lambda; \mathcal{A})(-3)^y k^k}{n!k!}
\]

\[
= \lambda^{-\zeta_1 \zeta_3 \sqrt{\mathcal{A}} + z^2 I^n M_{n+k}(\zeta_1; \lambda; \mathcal{A})3^n k!(n-k)!}
\]

\[
= \lambda^{-\zeta_1 \zeta_3 \sqrt{\mathcal{A}} + z^2 I^n M_{n+k}(\zeta_1; \lambda; \mathcal{A})3^n n!}
\]

\[
= \frac{2^{1/2}(2\mathcal{A} - 1)^{-c}(\zeta_1 - \sqrt{2\mathcal{A} - 1})^{-c}}{n!}
\]

using (3.1), then

\[
\sum_{n=0}^{\infty} \frac{2^{1/2}(2\mathcal{A} - 1)^{-c}(\zeta_1 - \sqrt{2\mathcal{A} - 1})^{-c}}{n!} = \lambda \zeta_1 \zeta_3 \sqrt{\mathcal{A}} - z^2 I^n M_{n+k}(\zeta_1; \lambda; \mathcal{A})3^n n!.
\]
Let \( y \) yields the recurrence relation
\[
z_n = \sum_{k=0}^{n} \frac{(-1)^{n+k} (\zeta_1 \sqrt{2\alpha} )^{2n-2k} (\ln \lambda)^{2n-k} \lambda^n}{k!(2n-2k)!}.
\]
Hence proved.

Proof of (3.5).
\[
\sum_{n=0}^{\infty} \frac{(-1)^n M H_{2n}(\zeta_1; \lambda; \alpha) \lambda^n}{(2n)!} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^{n+k} (\zeta_1 \sqrt{2\alpha} )^{2n-2k} (\ln \lambda)^{2n-k} \lambda^n}{k!(2n-2k)!}.
\]
\[
= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^{n+k} (\zeta_1 \sqrt{2\alpha} )^{2n-2k} (\ln \lambda)^{2n-k} \lambda^n}{k!(2n)!}.
\]
\[
= \lambda^3 \cos(\zeta_1 \ln \sqrt{2\alpha}).
\]
Similarly, we can prove the other results.

4. Recurrence relations of \( M H_n(\zeta_1; \lambda; \alpha) \)

The generating relation of \( M H_n(\zeta_1; \lambda; \alpha) \) is
\[
\lambda(\zeta_1 \sqrt{2\alpha}) = \frac{\sum_{n=0}^{\infty} M H_n(\zeta_1; \lambda; \alpha) \lambda^n}{n!}, \quad \lambda > 0, \quad \lambda \neq 1.
\]
Let
\[
F = \lambda(\zeta_1 \sqrt{2\alpha} - \lambda^2),
\]
then
\[
\frac{\partial F}{\partial \zeta_1} = \lambda(\zeta_1 \sqrt{2\alpha} - \lambda^2) \ln \lambda, \quad (4.1)
\]
\[
\frac{\partial F}{\partial \alpha} = \lambda(\zeta_1 \sqrt{2\alpha} - \lambda^2) (\zeta_1 \sqrt{2\alpha} - 2\lambda^2) \ln \lambda. \quad (4.2)
\]
Multiplying equation (4.1) by \((\zeta_1 \sqrt{2\alpha} - 2\lambda^2)\) and equation (4.2) by \((\zeta_1 \sqrt{2\alpha})\) and subtracting
\[
\sum_{n=0}^{\infty} \zeta_1 \sqrt{2\alpha} M H_n(\zeta_1; \lambda; \alpha) \lambda^n - \sum_{n=0}^{\infty} 2 M H_n(\zeta_1; \lambda; \alpha) \lambda^n - \sum_{n=0}^{\infty} n \sqrt{2\alpha} M H_n(\zeta_1; \lambda; \alpha) \lambda^n = 0.
\]
Comparing the coefficient of \( \zeta_1^k \) yields the recurrence relation
\[
\zeta_1 \sqrt{2\alpha} M H_n(\zeta_1; \lambda; \alpha) - 2n M H_{n-1}(\zeta_1; \lambda; \alpha) = n \sqrt{2\alpha} M H_n(\zeta_1; \lambda; \alpha). \quad (4.3)
\]
Using \( \frac{\partial F}{\partial \zeta_1} \),
\[
\sum_{n=0}^{\infty} M H_n(\zeta_1; \lambda; \alpha) \lambda^n = \sum_{n=0}^{\infty} \ln \sqrt{2\alpha} M H_{n-1}(\zeta_1; \lambda; \alpha) \lambda^n \lambda = \sum_{n=0}^{\infty} n \ln \sqrt{2\alpha} M H_{n-1}(\zeta_1; \lambda; \alpha) \lambda^n \lambda,
\]
yields the recurrence relation
\[
M H_n(\zeta_1; \lambda; \alpha) = n \ln \sqrt{2\alpha} M H_{n-1}(\zeta_1; \lambda; \alpha). \quad (4.4)
\]
By equation (4.4) and (4.3) we have the recurrence relation
\[ 2 \mathcal{M} \ln \lambda \mathcal{M} n^{-1} (\zeta; \lambda; w) - 2(n-1)\sqrt{2w} \ln \lambda \mathcal{M} n^{-2} (\zeta; \lambda; w) = \sqrt{2w} \mathcal{M} n^2. \] (4.5)

Replacing \( n \) by \( n+1 \) in (4.5), to obtain the three terms recurrence relations in the form
\[ \mathcal{M} (\zeta; \lambda; w) = \sqrt{2w} \ln \lambda \mathcal{M} n (\zeta; \lambda; w) - 2(n) \ln \lambda \mathcal{M} n^{-1} (\zeta; \lambda; w). \] (4.6)

Also,
\[ \mathcal{M} (\zeta; \lambda; w) = n(n-1)(\ln \lambda \sqrt{2w})^2 \mathcal{M} n^{-2} (\zeta; \lambda; w), \] (4.7)

using (4.5) in (4.7) yields the modified Hermite matrix differential equation
\[ \mathcal{M} (\zeta; \lambda; w) - \mathcal{M} (\zeta; \lambda; w) = \mathcal{M} (\zeta; \lambda; w) + n \mathcal{M} n (\zeta; \lambda; w) = 0. \] (4.8)

5. Rodrigues formula for \( \mathcal{M} n (\zeta; \lambda; w) \)

The Rodrigues formula for the modified Hermite matrix polynomials \( \mathcal{M} n (\zeta; \lambda; w) \) is given by the following relation,
\[ \mathcal{M} n (\zeta; \lambda; w) = (-1)^n \left( \frac{w}{2} \right)^\frac{n}{2} \mathcal{M} n (\zeta; \lambda; w) = \frac{\lambda}{\mathcal{M} n (\zeta; \lambda; w)} \frac{\lambda}{n!}. \] (5.1)

The proof of (5.1) is as
\[ \lambda (\zeta; \sqrt{2w})^{-j^2} = \sum_{n=0}^{\infty} \mathcal{M} n (\zeta; \lambda; w) \frac{j^n}{n!}. \]

Using the Maclaurin’s theorem
\[ \mathcal{M} n (\zeta; \lambda; w) = \left[ \frac{d^n}{d^2} \lambda (\zeta; \sqrt{2w})^{-j^2} \right]_{j=0} = \left( \frac{\lambda}{2w} \right) \left[ \frac{d^n}{d^2} \lambda \lambda (\zeta; \sqrt{2w})^{-j^2} \right]_{j=0}. \]

Now put \( \zeta = 1 - \sqrt{2w} = w \),
\[ = \left( \frac{\lambda}{2w} \right) (-1)^n \left( \frac{w}{2} \right)^\frac{n}{2} \left[ \frac{d^n}{d^2} \lambda \lambda (\zeta; \sqrt{2w})^{-j^2} \right]_{w=\zeta}, \]

when \( w = \zeta \), where \( \frac{d}{d^2} \lambda = D \),
\[ \mathcal{M} n (\zeta; \lambda; w) = (-1)^n \left( \frac{w}{2} \right)^\frac{n}{2} \mathcal{M} n (\zeta; \lambda; w) = \frac{\lambda}{\mathcal{M} n (\zeta; \lambda; w)} \frac{\lambda}{n!}. \]

Hence the proof of Rodrigues formula.

6. Hypergeometric form of \( \mathcal{M} n (\zeta; \lambda; w) \)

\[ \mathcal{M} n (\zeta; \lambda; w) = \sum_{k=0}^{[n/2]} \frac{(-1)^k n!(\zeta; \sqrt{2w})^{n-2k}(\ln \lambda)^{n-k}}{k!(n-2k)!}, \]
\[ = \sum_{k=0}^{[n/2]} \frac{(-1)^k (-n)^{2k}(\zeta; \sqrt{2w})^{n-2k}(\ln \lambda)^{n-k}}{k!}, \]
\[ = (\zeta; \sqrt{2w} \ln \lambda)^{n} F_0 \left[ \begin{array}{c} -\frac{n}{2}, -\frac{n+1}{2}, -\frac{n}{2} \in \mathcal{M} \end{array} \right]. \]
7. Orthogonality of $M \mathcal{H}_n(\zeta_1; \lambda; \mathcal{A})$

The differential equation of $M \mathcal{H}_n(\zeta_1; \lambda; \mathcal{A})$ is given by equation (4.8),

$$M \mathcal{H}_n''(\zeta_1; \lambda; \mathcal{A}) - \zeta_1 \mathcal{A} \ln M \mathcal{H}_n'(\zeta_1; \lambda; \mathcal{A}) + n \mathcal{A} \ln M \mathcal{H}_n(\zeta_1; \lambda; \mathcal{A}) = 0,$$

which may be written as

$$\left( \lambda - \frac{\zeta_1 \mathcal{A}}{2} M \mathcal{H}_n'(\zeta_1; \lambda; \mathcal{A}) \right)' + n \mathcal{A} \ln \lambda \left( \lambda - \frac{\zeta_1 \mathcal{A}}{2} \right) M \mathcal{H}_n(\zeta_1; \lambda; \mathcal{A}) = 0. \quad (7.1)$$

Along with above equation (7.1) write

$$\left( \lambda - \frac{\zeta_1 \mathcal{A}}{2} M \mathcal{H}_m(\zeta_1; \lambda; \mathcal{A}) \right)' + m \mathcal{A} \ln \lambda \left( \lambda - \frac{\zeta_1 \mathcal{A}}{2} \right) M \mathcal{H}_m(\zeta_1; \lambda; \mathcal{A}) = 0. \quad (7.2)$$

Combining (7.1) and (7.2), we obtain

$$(n - m) \mathcal{A} \ln \lambda \left( \lambda - \frac{\zeta_1 \mathcal{A}}{2} \right) M \mathcal{H}_n(\zeta_1; \lambda; \mathcal{A}) M \mathcal{H}_m(\zeta_1; \lambda; \mathcal{A}) = \left[ \lambda - \frac{\zeta_1 \mathcal{A}}{2} (M \mathcal{H}_n(\zeta_1; \lambda; \mathcal{A}) M \mathcal{H}_m(\zeta_1; \lambda; \mathcal{A}) - M \mathcal{H}_n'(\zeta_1; \lambda; \mathcal{A}) M \mathcal{H}_m'(\zeta_1; \lambda; \mathcal{A})) \right].$$

It follows that

$$(n - m) \mathcal{A} \ln \lambda \int_a^b \left( \lambda - \frac{\zeta_1 \mathcal{A}}{2} \right) M \mathcal{H}_n(\zeta_1; \lambda; \mathcal{A}) M \mathcal{H}_m(\zeta_1; \lambda; \mathcal{A}) d\zeta_1 = \left[ \lambda - \frac{\zeta_1 \mathcal{A}}{2} (M \mathcal{H}_n(\zeta_1; \lambda; \mathcal{A}) M \mathcal{H}_m(\zeta_1; \lambda; \mathcal{A}) - M \mathcal{H}_n'(\zeta_1; \lambda; \mathcal{A}) M \mathcal{H}_m'(\zeta_1; \lambda; \mathcal{A})) \right]_{a}^{b}.$$

Since the product of any polynomials in $\zeta_1$ by $\lambda - \frac{\zeta_1 \mathcal{A}}{2}$ as $\zeta_1 \to \infty$ or $\zeta_1 \to -\infty$, we may conclude that

$$\int_{-\infty}^{\infty} \left( \lambda - \frac{\zeta_1 \mathcal{A}}{2} \right) M \mathcal{H}_n(\zeta_1; \lambda; \mathcal{A}) M \mathcal{H}_m(\zeta_1; \lambda; \mathcal{A}) d\zeta_1 = 0; \quad \text{for} \quad m \neq n.$$

Now for $m = n$ we have

$$\lambda (\zeta_1^2 - \lambda^2) = \sum_{m=0}^{\infty} \frac{M \mathcal{H}_m(\zeta_1; \lambda; \mathcal{A}) \lambda^m}{m!}.$$

$$\lambda (\zeta_1^2 + \lambda^2) = \sum_{m=0}^{\infty} \sum_{k=0}^{m} \frac{M \mathcal{H}_m(\zeta_1; \lambda; \mathcal{A}) \lambda^m \mathcal{H}_k(\zeta_1; \lambda; \mathcal{A}) \lambda^k}{m!}.$$

$$\int_{-\infty}^{\infty} \lambda (\zeta_1^2 - \lambda^2) d\zeta_1 = \sum_{m=0}^{\infty} \sum_{k=0}^{m} \frac{\lambda^m \mathcal{H}_m(\zeta_1; \lambda; \mathcal{A}) \lambda^k \mathcal{H}_k(\zeta_1; \lambda; \mathcal{A})}{(m-k)!} d\zeta_1.$$

$$\int_{-\infty}^{\infty} \lambda (\zeta_1^2 + \lambda^2) d\zeta_1 = \sum_{m=0}^{\infty} \sum_{k=0}^{m} \frac{\lambda^m \mathcal{H}_m(\zeta_1; \lambda; \mathcal{A}) \lambda^k \mathcal{H}_k(\zeta_1; \lambda; \mathcal{A})}{(m-k)!} d\zeta_1.$$
\[
\sqrt{2\pi e^{-1} \ln \lambda} \sum_{n=0}^{\infty} \frac{2^n (\ln \lambda)^n \beta^{2n}}{n!} = \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \lambda^{-\frac{c_{12}}{2}} M \mathcal{H}_n^{2}(\zeta_1; \lambda; \alpha) \beta^{2n} d\zeta_1.
\]

So
\[
\int_{-\infty}^{\infty} \lambda^{-\frac{c_{12}}{2}} M \mathcal{H}_n^{2}(\zeta_1; \lambda; \alpha) d\zeta_1 = 2^n (\ln \lambda)^n n! \sqrt{\frac{2\pi e^{-1}}{\ln \lambda}}.
\]

Hence, the orthogonality condition for Modified Hermite Matrix polynomials \( M \mathcal{H}_n(\zeta_1; \lambda; \alpha) \) is as follows
\[
\int_{-\infty}^{\infty} \lambda^{-\frac{c_{12}}{2}} M \mathcal{H}_n(\zeta_1; \lambda; \alpha) M \mathcal{H}_m(\zeta_1; \lambda; \alpha) d\zeta_1 = \begin{cases} 
0, & \text{if } m \neq n \\
2^n (\ln \lambda)^n n! \sqrt{\frac{2\pi e^{-1}}{\ln \lambda}}, & \text{if } m = n.
\end{cases}
\]

**Theorem 7.1.** For the modified Hermite matrix polynomials \( M \mathcal{H}_n(\zeta_1; \lambda; \alpha) \),
\begin{enumerate}
\item[(a)]
\[
\int_{-\infty}^{\infty} \lambda^{-\frac{c_{12}}{2}} \zeta_1^k M \mathcal{H}_n(\zeta_1; \lambda; \alpha) d\zeta_1 = 0; \quad \text{for } k = 0, 1, 2, \ldots (n - 1);
\]
\item[(b)] the zeros of \( M \mathcal{H}_n(\zeta_1; \lambda; \alpha) \) are real and distinct;
\item[(c)] the Christoffel-Darboux formula of summation is
\[
\sum_{k=0}^{n} \frac{M \mathcal{H}_k(\zeta_1; \lambda; \alpha) M \mathcal{H}_k(\zeta_1; \lambda; \alpha)}{k! 2^k (\ln \lambda)^k} = \frac{M \mathcal{H}_{n+1}(\zeta_1; \lambda; \alpha) M \mathcal{H}_n(\zeta_1; \lambda; \alpha) - M \mathcal{H}_n(\zeta_1; \lambda; \alpha) M \mathcal{H}_{n+1}(\zeta_1; \lambda; \alpha)}{n! 2^n (\ln \lambda)^{n+1}} (\sqrt{2\alpha})^{-1}.
\]
\end{enumerate}

**Proof.** The proof of parts (a) and (b) are straightforward. Hence we omit the details.

Proof of (c): Using the equation (4.6)
\[
M \mathcal{H}_{k+1}(\zeta_1; \lambda; \alpha) = \sqrt{2\alpha} \zeta_1 \ln \lambda M \mathcal{H}_k(\zeta_1; \lambda; \alpha) - 2(k) \ln \lambda M \mathcal{H}_{k-1}(\zeta_1; \lambda; \alpha).
\]

Now, multiplying by \( \frac{M \mathcal{H}_k(\zeta_1; \lambda; \alpha)}{k! 2^k (\ln \lambda)^k} \), yield the equation
\[
\frac{M \mathcal{H}_{k+1}(\zeta_1; \lambda; \alpha) M \mathcal{H}_k(\zeta_1; \lambda; \alpha)}{k! 2^k+1} = \sqrt{2\alpha} \zeta_1 \ln \lambda \frac{M \mathcal{H}_k(\zeta_1; \lambda; \alpha) M \mathcal{H}_k(\zeta_2; \lambda; \alpha)}{k! 2^k+1} - 2(k) \ln \lambda \frac{M \mathcal{H}_{k-1}(\zeta_1; \lambda; \alpha) M \mathcal{H}_k(\zeta_2; \lambda; \alpha)}{k! 2^k+1}.
\]

Interchanging \( \zeta_1 \) and \( \zeta_2 \), yield the equation
\[
\frac{M \mathcal{H}_{k+1}(\zeta_2; \lambda; \alpha) M \mathcal{H}_k(\zeta_1; \lambda; \alpha)}{k! 2^k+1} = \sqrt{2\alpha} \zeta_2 \ln \lambda \frac{M \mathcal{H}_k(\zeta_2; \lambda; \alpha) M \mathcal{H}_k(\zeta_1; \lambda; \alpha)}{k! 2^k+1} - 2(k) \ln \lambda \frac{M \mathcal{H}_{k-1}(\zeta_2; \lambda; \alpha) M \mathcal{H}_k(\zeta_1; \lambda; \alpha)}{k! 2^k+1}.
\]

Subtracting the equation (7.3) from (7.4) and putting \( k = 0, 1, 2, \ldots, n \), yields the result.

**8. Expansion of polynomials**

Any polynomials can be expended in a series of modified Hermite matrix polynomials and the coefficients can be determined as
\[
\zeta_1^n = \sum_{k=0}^{[n/2]} \frac{n! (2\alpha)^{-n/2} M \mathcal{H}_{n-2k}(\zeta_1; \lambda; \alpha) (\ln \lambda)^{k-n}}{k!(n-2k)!}.
\]
Expansion of the Legendre matrix polynomials in a series of Legendre matrix polynomials is
\[ P_n(\zeta_1, a) = \sum_{k=0}^{[n/2]} (-1)^k \left(\frac{1}{2}\right)_{n-k} \frac{\ln \ln (2k-n)_M H_{n-2k}(\zeta_1; a)}{k!(n-2k)!} {}_2F_0 \left[ -k, 1; 2 + n - 2k; -\frac{1}{\ln \lambda} \right]. \]

Expansion of the modified Hermite matrix polynomials in a series of Legendre matrix polynomials is
\[ M H_n(\zeta_1; a) = \sum_{k=0}^{[n/2]} (-1)^k \ln(2n-4k+1)(\ln \lambda)^{n-k} P_{n-2k}(\zeta_1, a) \frac{1}{2} {}_1F_1 \left[ -k; 3; n - 2k; \ln \lambda \right]. \]

Expansion of modified Hermite matrix polynomials in a series of Laguerre matrix polynomials is
\[ M H_n(\zeta_1; a) = (a^2 + I) n (\sqrt{2a^2} \ln \lambda)^n \sum_{s=0}^{\infty} (-1)^s[(a + I)_s]^{-1} \times {}_2F_2 \left[ \frac{(n-s)_1}{a^2 + 1}, \frac{(n-s-1)_1}{a^2 (n-1)_1}; -\frac{1}{a^2 \ln \lambda} \right] I_{s, s}^{(2)}(\zeta_1). \]

**Proof of (8.2).** Consider the series of Legendre matrix polynomials form [17, 30, 33],
\[ \sum_{n=0}^{\infty} P_n(\zeta_1, a) x^n = \sum_{n=0}^{[n/2]} (-1)^k \left(\frac{1}{2}\right)_{n-k} (\ln \lambda)^{n-2k} \frac{n!}{k!(n-2k)!} \int_{\zeta_1}. \]

Using equation (8.1), we may write
\[ \sum_{n=0}^{\infty} P_n(\zeta_1, a) x^n = \sum_{n=0}^{[n/2]} \sum_{k=0}^{[n/2]} \frac{(-1)^k \left(\frac{1}{2}\right)_{n-k} M H_{n-2k}(\zeta_1; a) (\ln \lambda)^s - n_3^{n+2k}}{k!(n-2k)!} \int_{\zeta_1}. \]

Using the equation (1.1),
\[ \sum_{n=0}^{\infty} P_n(\zeta_1, a) x^n = \sum_{n=0}^{[n/2]} \sum_{k=0}^{[n/2]} \frac{(-1)^k \left(\frac{1}{2}\right)_{n-k} M H_{n-2k}(\zeta_1; a) (\ln \lambda)^s - n_3^{n+2k}}{k!(n-2k)!} {}_2F_1 \left[ -k, 1; 2 + n - k; -\frac{1}{\ln \lambda} \right]. \]

Hence, the final result is
\[ P_n(\zeta_1, a) = \sum_{k=0}^{[n/2]} \frac{(-1)^k \left(\frac{1}{2}\right)_{n-k} (\ln \lambda)^{2k-n} M H_{n-2k}(\zeta_1; a)}{k!(n-2k)!} {}_2F_1 \left[ -k, 1; 2 + n - k; -\frac{1}{\ln \lambda} \right]. \]

**Proof of (8.3).** Now, from [8, 17, 30] we have
\[ \left(\frac{\zeta_1 \sqrt{2a^2}}{n!} \right)^n = \sum_{k=0}^{[n/2]} \frac{(2n-4k+1) P_{n-2k}(\zeta_1, a)}{k! \left(\frac{3}{2}\right)_{n-k} \int_{\zeta_1}. \]

\[ \left(\frac{1}{n!} \right)^n = \sum_{k=0}^{[n/2]} \frac{(-1)^k \left(\frac{1}{2}\right)_{n-k} (\ln \lambda)^{2k-n} M H_{n-2k}(\zeta_1; a)}{k!(n-2k)!} {}_2F_1 \left[ -k, 1; 2 + n - k; -\frac{1}{\ln \lambda} \right]. \]
Using this in the series
\[
\sum_{n=0}^{\infty} \frac{M \mathcal{H}_n (\zeta_1; \lambda; \omega)}{n!} z^n = \sum_{n=0}^{[n/2]} \frac{(-1)^k (\zeta_1 \sqrt{2 \omega})^{n-k} (\ln \lambda)^{n-k-1}}{k!(n-2k)!}
\]
we get the other results.

9. Integral representation

Several integral involving modified Hermite matrix polynomials \( M \mathcal{H}_n (\zeta_1; \lambda; \omega) \) are as follows.

\[
\mathcal{P}_n (\zeta_1, \omega) = \frac{2}{n!} \sqrt{\frac{\ln \lambda}{\pi}} \int_{0}^{\infty} \lambda^{-3/2} \lambda^{n-1} M \mathcal{H}_n (\zeta_1; \lambda; \omega) d\lambda,
\]

\[
\int_{-\infty}^{\infty} \lambda^{-3/2} M \mathcal{H}_n (\zeta_1; \lambda; \omega) d\lambda = 2n! \sqrt{\frac{\ln \lambda}{\pi}} \frac{\left( \frac{n}{2} \right) \left( \frac{3}{2} \right)^s \ln \lambda^{n+s}}{1 + 2s - 2k}
\]

\[
\int_{0}^{\zeta_1} \lambda^{-3/2} M \mathcal{H}_n (\zeta_1; \lambda; \omega) d\lambda = \frac{2\sqrt{\omega}^{-2k} \left( \frac{\omega}{2} \right) (-1/2) \left( \frac{1}{2} \right) k \left( \frac{3}{2} \right)^s \ln \lambda^{n+s}}{1 + 2s - 2k}
\]

\[
\int_{-\infty}^{\infty} \lambda^{-3/2} M \mathcal{H}_n (\zeta_1; \lambda; \omega) d\lambda = \frac{(2n)! \sqrt{\pi} \left( \frac{\omega}{2} \right)^{-1/2} (\ln \lambda)^{-1/2} (\zeta_1 - 1)^n}{n!}
\]

\[
\int_{-\infty}^{\infty} \lambda^{-3/2} M \mathcal{H}_n (\zeta_1; \lambda; \omega) d\lambda = \frac{(2n+1)! \sqrt{\pi} \ln \lambda^{-1/2} (\ln \lambda)^{-1/2} (\zeta_1 - 1)^n}{n!}
\]

\[
\int_{-\infty}^{\infty} \lambda^{-3/2} M \mathcal{H}_n (\sqrt{2} \zeta_1; \lambda; \omega) d\lambda = \frac{(2n)! \sqrt{\pi} \left( \frac{\omega}{2} \right)^{-1/2} (\ln \lambda)^{-1/2}}{n!}
\]

\[
\int_{-\infty}^{\infty} \lambda^{-3/2} M \mathcal{H}_n (\zeta_1; \lambda; \omega) d\lambda = 0.
\]

\[
\lambda^{-3/2} = \frac{2 \sqrt{\ln \lambda}}{\sqrt{\pi}} \int_{0}^{\infty} \lambda^{-3/2} \cos (\zeta_1 \ln \lambda \sqrt{2 \omega}) d\lambda.
\]

\[
M \mathcal{H}_2 (\zeta_1; \lambda; \omega) = \frac{(-1)^n 2^{2n+1} (\ln \lambda)^{2n+1/2}}{\sqrt{\pi}} \int_{0}^{\infty} \lambda^{-3/2} 2^n \cos (\zeta_1 \ln \lambda \sqrt{2 \omega}) d\lambda.
\]

\[
M \mathcal{H}_2 (\zeta_1; \lambda; \omega) = \frac{(-1)^n 2^{2n+2} (\ln \lambda)^{2n+1}}{\sqrt{\pi}} \int_{0}^{\infty} \lambda^{-3/2} 2^{n+1} \sin (\zeta_1 \ln \lambda \sqrt{2 \omega}) d\lambda.
\]
the following fractional integrals and fractional derivatives for modified Hermite matrix polynomials

Where

\[ L_n^{(\alpha)}(\bar{\zeta}_1) \] is the Laguerre matrix polynomials see [21].

\section{10. Fractional integrals and derivatives of \( M_n^{(\zeta_1; \lambda; \alpha)} \)}

By using the definitions of fractional integrals and fractional derivatives given in [32], we have obtained the following fractional integrals and fractional derivatives for modified Hermite matrix polynomials \( M_n^{(\zeta_1; \lambda; \alpha)} \) given as,

\[
I^\mu_{\{M_n^{(\zeta_1; \lambda; \alpha)}\}} = \frac{M_n^{(\zeta_1; \lambda; \alpha)}(\bar{\zeta}_1; \alpha, \lambda)}{(1 + n)^\mu(\sqrt{2\alpha} \ln \lambda)^\mu},
\]

the Riemann-Liouville left side fractional integral:

\[
a I^\alpha_{\zeta_1} \{M_n^{(\zeta_1; \lambda; \alpha)}\} = \frac{M_n^{(\zeta_1; \lambda; \alpha)}(\bar{\zeta}_1 - a; \alpha, \lambda)}{(1 + n)^\alpha(\sqrt{2\alpha} \ln \lambda)^\alpha},
\]

the Riemann-Liouville right side fractional integral:

\[
\zeta_1 I^\alpha_{\zeta} \{M_n^{(\zeta_1; \lambda; \alpha)}\} = \frac{M_n^{(\zeta_1; \lambda; \alpha)}(c - \zeta_1; \alpha, \lambda)}{(1 + n)^\alpha(\sqrt{2\alpha} \ln \lambda)^\alpha},
\]

the Weyl integral of \( M_n^{(\zeta_1; \lambda; \alpha)} \) of order \( \alpha \):

\[
\zeta_1 W^\alpha_{\infty} \{M_n^{(\zeta_1; \lambda; \alpha)}\} = \frac{(-1)^\alpha M_n^{(\zeta_1; \lambda; \alpha)}(\bar{\zeta}_1; \alpha, \lambda)}{(1 + n)^\alpha(\sqrt{2\alpha} \ln \lambda)^\alpha},
\]

the Erdelyi-Kober operator of first kind for \( M_n^{(\zeta_1; \lambda; \alpha)} \):

\[
I[\alpha, \eta, M_n^{(\zeta_1; \lambda; \alpha)}] = \frac{(\sqrt{2\alpha} \zeta_1 \ln \lambda)^n}{(1 + n + \eta)^\alpha} \binom{\Delta(2, -n), \Delta(2, -\alpha - n - \eta)}{\Delta(2, -n - \eta),} \binom{-2\alpha^{\delta - 1}}{\zeta_1^\delta \ln \lambda},
\]

the Erdelyi-Kober operator of second kind for \( M_n^{(\zeta_1; \lambda; \alpha)} \):

\[
I[\alpha, \eta, M_n^{(\zeta_1; \lambda; \alpha)}] = \frac{(\sqrt{2\alpha} \zeta_1 \ln \lambda)^n}{(1 + n + \eta)^\alpha} \binom{\Delta(2, -n + \eta), \Delta(2, -n + \eta + 1)}{\Delta(2, -n + \eta + \alpha),} \binom{-2\alpha^{\delta - 1}}{\zeta_1^\delta \ln \lambda},
\]

the Saigo integral operator of first kind:

\[
I^{\alpha, \beta, \eta}_{\{M_n^{(\zeta_1; \lambda; \alpha)}\}} = \frac{(\sqrt{2\alpha} \ln \lambda)^n(1 + n)\Gamma(1 + n + \eta - \beta)\zeta_1^{n - \beta}}{\Gamma(1 + n - \beta)\Gamma(1 + n + \eta + \alpha)} \binom{\Delta(2, -n + \beta), \Delta(2, -n - \eta - \alpha)}{\Delta(2, -n - \eta + \beta),} \binom{-2\alpha^{\delta - 1}}{\zeta_1^\delta \ln \lambda},
\]

Where \( L_n^{(\alpha)}(\bar{\zeta}_1) \) is the Laguerre matrix polynomials see [21].

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the Saigo integral operator of second kind:

\[ I^{\alpha,\beta,n}_{0} M_{\mathcal{H}} n(\zeta_1;\lambda; \mathcal{A}) = \frac{(\sqrt{2s^2} \ln \lambda)^n \Gamma(1-n)\Gamma(1-n+\eta-\beta)\zeta_1^{n-1}}{\Gamma(1-n-\beta)\Gamma(1-n+\eta+\alpha)} \]

\[ \times \, _{6}F_{4} \left[ \begin{array}{c} \Delta(2,-n), \Delta(2,-n+1), \Delta(2,1-n-\beta) \\
\Delta(2,-n-\beta+1), \Delta(2,1-n+\alpha+\eta); \\
\end{array} \right. \frac{2_{s-1}^{2}}{\zeta_1^{n+1}} \right] , \]

the left sided Riemann-Liouville fractional derivative of order \( \alpha \):

\[ aD_{c}^{\alpha} M_{\mathcal{H}} n(\zeta_1-\alpha;\lambda; \mathcal{A}) = \frac{(\sqrt{2s^2} \ln \lambda)^{\alpha} \Gamma(1+n)}{\Gamma(1+n-\alpha)} M_{\mathcal{H}} n-\alpha(\zeta_1-\alpha;\lambda; \mathcal{A}), \]

the Right sided Riemann-Liouville fractional derivative of order \( \alpha \):

\[ \zeta_1 D_{c}^{\alpha} M_{\mathcal{H}} n(c-\zeta_1;\lambda; \mathcal{A}) = \frac{(\sqrt{2s^2} \ln \lambda)^{\alpha} \Gamma(1+n)}{\Gamma(1+n-\alpha)} M_{\mathcal{H}} n-\alpha(c-\zeta_1;\lambda; \mathcal{A}), \]

the Weyl fractional derivative of order \( \alpha \):

\[ \zeta_1 D_{\infty}^{\alpha} M_{\mathcal{H}} n(\zeta_1;\lambda; \mathcal{A}) = (-n)^{\alpha}(\sqrt{2s^2} \ln \lambda)^{\alpha} M_{\mathcal{H}} n-\alpha(\zeta_1;\lambda; \mathcal{A}). \]

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