Quantum Hadrondynamics in Two Dimensions

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ABSTRACT

A nonlocal and nonlinear theory of hadrons, equivalent to the color singlet sector two dimensional QCD, is constructed. The phase space space of this theory is an infinite dimensional Grassmannian. The baryon number of QCD corresponds to a topological invariant (‘virtual rank’) of the Grassmannian. It is shown that the hadron theory has topological solitons corresponding to the baryons of QCD. $\frac{1}{N_c}$ plays the role of $\hbar$ in this theory; $N_c$ must be an integer for topological reasons. We also describe the quantization of a toy model with a finite dimensional Grassmannian as the phase space. In an appendix, we show that the usual Hartree–Fock theory of atomic and condensed matter physics has a natural formulation in terms of infinite dimensional Grassmannians.
1. Introduction

Quantum Chromodynamics (QCD) is the generally accepted theory of strong interactions. It describes the strongly interacting particles (hadrons) in terms of their constituents, the quarks and gluons. However the quarks and gluons themselves are not directly observable; only their color singlet bound states (hadrons) exist as isolated particles. A challenging problem of particle physics is to construct a theory of strong interactions directly in terms of hadrons. There are indications that such a theory exists and is a sort of string theory \[1\], although a complete understanding is not available yet.

In this paper we will present such a theory of hadrons (Quantum Hadrondynamics, QHD) in two spatial space–time dimensions. It will be shown that the color singlet sector of two dimensional QCD (2DQCD) is equivalent to QHD for all energy scales and number of colors. At low energies the theory tends to a nonlinear sigma model. For large \(N_c\) it tends to a classical theory; the quantum fluctuations in QHD are of order \(\frac{1}{N_c}\). This classical hadron theory however has no relation to classical chromodynamics; it is equivalent instead to the large \(N_c\) limit of quantum chromodynamics.

QHD can be viewed as an extension of the ideas of Berezin \[2\], \[3\] to QCD. There is also some previous work by Kikkawa \[4\] in the same spirit, although our theory differs in some important aspects. One main point clarified by our work is the origin of baryon number as a homotopic invariant in QHD. For this, as well as other reasons, an understanding of the quadratic constraints satisfied by the field variable is crucial. The mathematical framework necessary is the theory of infinite dimensional Grassmannians as formulated in the book by Pressley and Segal \[5\]. A less explicit summary of this work was given in \[6\].

The main distinguishing feature of QHD is that it is not a local quantum field theory. The field variable depends on a pair of space–time points separated by a null distance.
Also, the theory is highly nonlinear, the phase space of the theory being a curved manifold (the infinite dimensional Grassmannian). We will first describe the classical limit of QHD, which is equivalent to large $N_c$ QCD. Small oscillations around the vacuum describe mesons and can be described by a linear approximation to large $N_c$ QCD constructed in the work of ’t Hooft [7]. Our approach will construct the complete theory, which has interactions at every possible order, since the field variable takes values in a curved manifold. It is important to note that the infinite $N_c$ limit of QCD is equivalent to an interacting, highly nonlinear, theory. It is only if we further restrict ourselves to the small oscillations around the vacuum that it becomes a theory of free mesons. If large $N_c$ QCD were a free field theory as is occasionally claimed there would be no soliton solutions, which are necessary to describe baryons as argued by Witten [8]. That baryons should be solitons of a theory whose small oscillations are mesons was proposed independently by Skyrme [9] before QCD. It is now known that this idea is consistent with QCD [10]. QHD has a homotopically conserved quantum number which can be identified with baryon number. There are static solutions to our theory carrying this quantum number, which do look like baryons [11].

Our classical hadron theory is equivalent to the sum over planar diagrams of 2DQCD. The field variable $M(x, y)$ of our theory is the ‘master field’; we will obtain its equations in closed form. It would have been too hard to obtain by actually summing diagrams, since there are vortices at every order. It is only the geometrical understanding of the phase space as a Grassmannian that makes the formulation of the theory possible.

Quantization of our hadron theory corresponds to studying QCD at finite $N_c$. In the hadron theory, $N_c$ plays the role of $\frac{1}{\hbar}$; it is required to be a positive integer for homotopic reasons. The solution of the hadron theory in the semiclassical approximation corresponds to the $\frac{1}{N_c}$ expansion of 2DQCD.
2. Classical Dynamics on Finite Dimensional Grassmannians

2.1. Parametrizations of the Grassmannian

We are eventually interested in the infinite dimensional Grassmanian, which is the phase space of classical hadron dynamics. However, the particular infinite dimensional Grassmannian (the ‘restricted’ Grassmannian of G. Segal [5]) we need, is very similar to the finite dimensional counterparts. It is useful therefore to study the analogous finite dimensional spaces first. This can be viewed as a ‘regularization’ of our theory.

For a positive integer $M$ we will define the Grassmannian $Gr_M$ to be the set of all $M \times M$ hermitean matrices satisfying a quadratic constraint:

$$Gr_M = \{ \Phi | \Phi^\dagger = \Phi; \Phi^2 = 1 \}.$$

The eigenvalues of $\Phi$ will then be $\pm 1$. Each point $\Phi \in Gr_M$ picks out a subspace of $C^M$, the eigenspace of $\Phi$ with eigenvalue $-1$. Using this correspondence the Grassmannian can be viewed as the set of subspaces of $C^M$. This is the conventional definition [12].

The trace $\text{tr} \Phi$ is an integer:

$$\text{tr} \Phi = M - 2m$$

where $m$ is the number of eigenvalues equal to $-1$. Thus $\text{tr} \Phi$ is invariant under a continuous deformation of $\Phi$. In fact $Gr_M$ is a union of components labelled by $m = 1, \cdots M - 1$. The case $m = 0$ ($m = M$) is trivial, containing just one point $\Phi = 1(\Phi = -1)$.

$$Gr_M = \bigcup_{m=0}^{M} Gr_{m,M}.$$
To see this, note that any hermitean matrix can be diagonalized by a unitary transformation. There will be precisely \( m \) eigenvalues equal to \(-1\) and \( M - m \) equal to \(+1\). Thus, for each \( \Phi \in Gr_{m,M} \), there is a \( g \in U(M) \) such that

\[
\Phi = g\epsilon g^\dagger
\]

(5)

where

\[
\epsilon = \begin{pmatrix}
-1_{m \times m} & 0 \\
0 & 1_{(M-m) \times (M-m)}
\end{pmatrix}.
\]

(6)

Both \( g \) and \( gh \) correspond to the same \( \Phi \) if \( h \) commutes with \( \epsilon \). The subgroup of elements \( U(M) \) that commutes with \( \epsilon \) is \( U(m) \times U(M-m) \), consisting of unitary matrices that are block diagonal:

\[
h = \begin{pmatrix}
h_1 & 0 \\
0 & h_2
\end{pmatrix}.
\]

(7)

Thus there is a one–one correspondence between \( \Phi \in Gr_{m,M} \) and the coset space \( U(M)/U(m) \times U(M-m) \).

We see now that each component \( Gr_{m,M} \) is a connected, compact manifold of dimension \( 2m(M-m) \). Also the map \( \Phi \rightarrow -\Phi \) is a diffeomorphism of \( Gr_{m,M} \) to \( Gr_{M-m,M} \). If \( M = 2m \), this is a map of \( Gr_{m,M} \) to itself. The group \( U(M) \) acts transitively on each connected component \( Gr_{m,M} \) by the action

\[
\Phi \rightarrow g\Phi g^\dagger.
\]

(8)

The simplest special case of a Grassmannian is

\[
Gr_{1,2} = U(2)/U(1) \times U(1)SU(2)/U(1) = S^2.
\]

(9)

In fact the description of the general Grassmannian as the set of matrices satisfying a quadratic condition is similar to that of \( S^2 \) as the set of vectors of unit length. It is useful to keep this simple example in mind as we develop the general theory. More generally,
is the complex projective space $CP^{M-1}$ each point of which is a one dimensional subspace in $C^M$.

A vector field on the Grassmannian can be thought of as a matrix valued function $V(\Phi)$ satisfying

$$[V(\Phi), \Phi]_+ := V(\Phi)\Phi + \Phi V(\Phi) = 0. \quad (10)$$

(If $\Phi = \epsilon$, this means that $V$ is off-diagonal $V = \begin{pmatrix} 0 & v \\ v^\dagger & 0 \end{pmatrix}$.) Any such matrix valued function is of the form

$$V(\Phi) = [\Phi, U(\Phi)] \quad (11)$$

for some other function $U(\Phi)$. However, $U$ is not uniquely determined by $V$: the transformation $U(\Phi) \to U(\Phi) + [\Phi, \Lambda(\Phi)]_+$ will leave $V(\Phi)$ unchanged. Thus $U(\Phi)$ is a sort of ‘potential’ for $V(\Phi)$, since the commutator with $\epsilon$ is like a differentiation. (It satisfies the formal rules for cyclic cohomology; see [13]). If $V(\Phi)$ is a vector field, we have also the identity

$$[\Phi, [\Phi, V(\Phi)] = 4V(\Phi). \quad (12)$$

2.2. Coordinates on the Grassmannian

It is sometimes convenient to solve the constraint on $\Phi$ by introducing explicit coordinates. Just as for $S^2$, one needs several coordinate charts to cover all of the Grassmannian.

Consider first the case of $Gr_{1,2} = CP^1$. The one dimensional subspace picked out by $\Phi$ consists of vectors of type $\lambda Z^i$, $i = 1, 2$. The vector $Z^i$ can be chosen to have unit length, so that it forms an ‘orthonormal basis’ in this one dimensional subspace. Since $Z^i$ is an eigenvector of $\Phi$ with eigenvalue $-1$ and the other eigenvalue is $1$,

$$\Phi_\Phi^i = \delta^i_{\delta^j} - 2Z^*_i Z^j. \quad (13)$$
Note that $Z^i \to Z^i h$ will leave $\Phi$ invariant if $h \in U(1)$. The point $\Phi = \epsilon$ corresponds to $Z^i = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. More generally, we can choose the unit vector $Z^i$ in the form

$$Z = \frac{1}{\sqrt{1 + \phi^* \phi}} \begin{pmatrix} 1 \\ \phi^* \end{pmatrix}$$

(14)

$\phi$ being some complex number. The magnitude of the first entry is fixed by the length of $Z^i$ and its phase can be chosen as above by an appropriate choice of $h$. This gives,

$$\Phi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{2}{1 + \phi \phi^*} \begin{pmatrix} 1 & \phi \\ \phi^* & \phi^* \phi \end{pmatrix}.$$  

(15)

One can check easily that $\Phi^2 = 1$ and tr$\Phi = 0$. This coordinate system overs an open neighborhood of $\epsilon$, but it breaks down at $\Phi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Another co–ordinate system, based on the choice

$$Z^i = \frac{1}{\sqrt{1 + \phi'^* \phi'}} \begin{pmatrix} \phi' \\ 1 \end{pmatrix}$$

(16)

and

$$\Phi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{2}{1 + \phi' \phi'^*} \begin{pmatrix} \phi' \phi'^* & \phi'^* \\ \phi' & 1 \end{pmatrix}.$$  

(17)

is well–defined in a neighborhood of this point. The transformation that links these systems is

$$\phi' = \frac{1}{\phi}.$$  

(18)

Since the space can be covered by coordinate charts related by holomorphic(analytic) coordinate transformations, we see that $Gr_{1,2}$ is a one dimensional complex manifold.

The above construction can be generalized to an arbitrary Grassmannian $Gr_{m,M}$. We will write

$$\Phi^j_i = \delta^j_i - 2Z^*_{ia} Z^{ja}.$$  

(19)

where $Z^{ia}$ for $a = 1, \cdots m$ is an orthonormal basis for the eigenspace of $\Phi$ with eigenvalues $-1$. $Z$ can be regarded as an $M \times m$ matrix and the orthonormality relation is

$$Z^\dagger Z = 1.$$  

(20)
In matrix notation, $\Phi = 1 - 2Z Z^\dagger$. $Z$ and $Z h$ give the same $\Phi$ if $h \in U(m)$. We can choose $Z$ to be

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \left[ 1 + \phi \phi^\dagger \right]^{-1/2} \quad (21)$$

where $\phi$ is an $m \times (M - m)$ complex matrix. (The 1 in the above equation stands for an $m \times m$ identity matrix. $1 + \phi \phi^\dagger$ is a positive invertible $m \times m$ matrix so it has a well-defined $-1/2$ power.) This gives

$$\Phi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - 2 \begin{pmatrix} (1 + \phi \phi^\dagger)^{-1} & (1 + \phi \phi^\dagger)^{-1} \phi \\ \phi^\dagger (1 + \phi \phi^\dagger)^{-1} & \phi^\dagger (1 + \phi \phi^\dagger)^{-1} \phi \end{pmatrix} \quad (22)$$

One can check explicitly that $\Phi^2 = 1$, $\text{tr} \Phi = M - 2m$.

The action of $U(M)$ on $\Phi$, $\Phi \rightarrow g \Phi g^\dagger$ can be seen to be the transformation

$$\phi \rightarrow g \circ \phi = (c + d \phi)(a + b \phi)^{-1} \quad (23)$$

where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Again this coordinate system breaks down at the points $\Phi = \pi \epsilon \pi^{-1}$, $\pi$ being a permutation that interchanges a negative with a positive eigenvalue of $\epsilon$. (Permutation of a pair of eigenvectors of $\epsilon$ is a unitary transformation in $C^M$). If we define,

$$\phi_{\pi} = \pi \circ \phi \quad (24)$$

the new coordinate will remain well-defined even at $\Phi = \pi \epsilon \pi^{-1}$. Thus we can cover all of $Gr_{m,M}$ by $\binom{M}{m}$ coordinate charts. (This is the number of permutations that actually change $\epsilon$). Again, since the co-ordinate transformations are analytic functions, this shows that $Gr_{m,M}$ is a complex manifold of dimension $m(M - m)$.

2.3. The Symplectic Structure on the Grassmannian

We will be interested in physical systems for which the Grassmannian is a Phase space. Since it is compact, it cannot be the cotangent bundle of any configuration space. It is not
possible to decompose the dynamical variables into configuration space and momentum space variables. A canonical formalism is still possible if we have a symplectic form, i.e., a closed, nondegenerate 2-form. Since each connected component is a homogenous space, it is natural to look for one that is invariant under the action of $U(M)$. In fact there is a unique (upto overall constant) homogenous symplectic form. In terms our descrption of the Grassmannian in terms of hermitean matrices,

$$\omega = -\frac{i}{8} \text{tr} \Phi d\Phi d\Phi.$$  \hspace{1cm} (25)

The normalization is chosen so that the integral of $\omega$ over a two–sphere embedded in the Grassmannian is an integer multiple of $2\pi$. This will be convenient later.

The form $\omega$ is obviously invariant under the transformation $\Phi \rightarrow g\Phi g^\dagger$ where $g$ is a constant unitary matrix. To see that $\omega$ is closed, note that

$$d\omega = -\frac{i}{8} \text{tr}(d\Phi)^3 = -\frac{i}{8} \text{tr}(d\Phi)^3 \Phi^2$$  \hspace{1cm} (26)

since $\Phi^2 = 1$. By differentiating this constraint,

$$\Phi d\Phi + d\Phi \Phi = 0.$$  \hspace{1cm} (27)

Now,

$$d\omega = -\frac{i}{8} \text{tr}(d\Phi)^3 \Phi = \frac{i}{8} \text{tr}(d\Phi)^3 \Phi^2 = -d\omega$$  \hspace{1cm} (28)

so that it is zero. Since $\omega$ is homogenous, it is enough to verify nondegeneracy at one point in each connected component, say $\Phi = \epsilon \in Gr_{m,M}$. A tangent vector $U$ at this point is a hermitean matrix satisfying

$$[\epsilon, U]_+ = 0;$$  \hspace{1cm} (29)

i.e., of the form

$$U = \begin{pmatrix} 0 & u \\ u^\dagger & 0 \end{pmatrix}.$$  \hspace{1cm} (30)
Then,

$$
\omega(U, V) = -\frac{i}{8} \text{tr}[U, V] = -\frac{i}{4} \text{tr}[u^\dagger v - v^\dagger u].
$$

(31)

If \( \omega(U, V) = 0 \) for all \( V \) implies that \( u = 0; \) \( \omega \) is nondegenerate. Thus \( \omega \) is a sympletic form on \( Gr_{m, M} \).

In fact what we have is a special case of the coadjoint orbit construction of homogenous symplectic manifolds due to Kirillov. The space of hermitean matrices is the Lie algebra of the unitary group. Since the trace is an invariant inner product, we can identify vector space with its dual. The adjoint (identified with coadjoint) action is \( \Phi \to g\Phi g^\dagger \). The Grassmannian \( Gr_{m, M} \) is just the orbit of \( \epsilon \).

This suggests a generalization of our construction to more general coset spaces, ‘flag manifolds’. Let \( \xi \) be some hermitean matrix with eigenvalues \( x_\alpha \) with degeneracy \( m_\alpha \) for \( \alpha = 1, \cdots d \). Clearly \( \sum_{\alpha=1}^{d} m_\alpha = M \). The orbit of \( \xi \) is the set of all elements related to it by an action of \( U(M) \):

$$
O_\xi = \{ \Phi = g\xi g^\dagger | G \in U(M) \}. \tag{32}
$$

This is a connected component in the flag manifold

$$
Fl(M) = \{ \Phi | \Phi^\dagger = \Phi; \prod_{\alpha=1}^{d} (\Phi - x_\alpha) = 0 \}. \tag{33}
$$

The particular connected component containing \( \xi \) is picked out by putting enough constraints on the traces to determine the multiplicities:

$$
O_\xi = \{ \Phi | \Phi^\dagger = \Phi; \prod_{\alpha=1}^{d} (\Phi - x_\alpha); \text{tr}\Phi = \sum m_\alpha x_\alpha; \cdots; \text{tr}\Phi^{d-1} = \sum m_\alpha x_\alpha^{d-1} \} \tag{34}
$$

It is enough to have \( d - 1 \) trace constraints since \( \sum m_\alpha = M \). As a homogenous space, \( O_\xi = U(M)/U(m_1) \times \cdots \times U(m_d) \). The homogenous sympletic form defined by Kirillov’s procedure is just

$$
\omega = -\frac{i}{8} \text{tr}\Phi (d\Phi)^2. \tag{35}
$$
It is clearly invariant under the action of $U(M)$. At the point $\xi$,

$$\omega(U, V) = -\frac{i}{8} \text{tr}[U, V]$$  \hspace{1cm} (36)

which agrees with Kirillov’s definition \[14\] upto a constant. It should be interesting to consider the generalization of the hadron theory whose phase space is an infinite dimensional flag manifold; however, it would no longer be equivalent to 2DQCD for fermionic matter fields. Perhaps there is some more general quantum field theory that is related to it.

2. 4. Poisson Brackets

We recall some basic facts of classical mechanics to define the terminology. \[15\]. A symplectic form $\omega$ has an inverse $\omega^{-1}$ which is an antisymmetric contravariant tensor. Acting on a pair of one- - forms, it will produce a function. In terms of coordinates, $\omega = \omega_{\mu\nu} dx^\mu dx^\nu$ and $\omega^{-1} = \omega^{\mu\nu} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\nu}$, $\omega^{\mu\nu}$ being the inverse of the matrix $\omega_{\mu\nu}$.

Observables of a classical dynamical system are smooth functions on a symplectic manifold, its phase space. Given a pair of functions $f_1, f_2$, their Poisson Bracket is the function defined by

$$\{f_1, f_2\} = -\omega^{-1}(df_1, df_2).$$  \hspace{1cm} (37)

This bracket is antisymmetric and satisfies the Jacobi identity if $\omega$ is closed. The Poisson algebra of a complete set of observables defines the symplectic structure uniquely.

To each smooth function $f$ their corresponds a vector field, defined by

$$\omega(V_f, \cdot) = df.$$  \hspace{1cm} (38)

$V_f$ can be viewed as the infinitesimal canonical transformation generated by $f$. Then,

$$\{f_1, f_2\} = \mathcal{L}_{V_{f_2}} f_1 = \omega(V_{f_1}, V_{f_2}).$$  \hspace{1cm} (39)
A classical dynamical system is defined by a manifold, a symplectic form $\omega$ and a function $H$, the Hamiltonian. The time evolution of the system is given by the integral curves of the vector field $V_H$. Functions that have zero Poisson brackets with $H$ are constant under time evolution.

In our case the matrix elements of $\Phi$ form a complete set of observables. Of course they are not all independent, being related by the constraints on $\Phi$. It is convenient to think of the symplectic structure in terms of the Poisson algebra of these functions. For a constant hermitean matrix $u$ define the function

$$f_u = -\frac{1}{2} \text{tr} \Phi u. \quad (40)$$

We can show that

$$\{f_u, f_v\} = f_{-[u,v]}. \quad (41)$$

Let us first find the symplectic vector field $V_u$ associated to $f_u$. Let us think of the vector field as a matrix valued function $V_u(\Phi)$ satisfying $\Phi V_u(\Phi) + V_u(\Phi) \Phi = 0$. Then,

$$\omega(V_u, \cdot) = df_u \Rightarrow \text{tr} \left( \Phi V_u(\Phi), d\Phi \right) = -\frac{1}{2} \text{tr} d\Phi u \quad (42)$$

That is,

$$[\Phi, V_u(\Phi)] = -4iu. \quad (43)$$

Using the identity $[\Phi, [\Phi, V_u(\Phi)]] = 4V_u(\Phi)$, we get

$$V_u = -i[u, \Phi]. \quad (44)$$

This is just the infinitesimal action of $U(M)$ on $Gr(M)$. Then,

$$\{f_u, f_v\}(\Phi) = -\mathcal{L}_{V_u} f_v(\Phi) = f_{-[u,v]}(\Phi). \quad (45)$$

Now, let $e^i_j$ be the Weyl matrices, $(e^i_j)_k = \delta^i_k \delta^j_l$. Then, the matrix elements of $\Phi$ are given by $\Phi^i_j = \text{tr} \Phi e^i_j$. Thus the Poisson algebra of the matrix elements can be written as

$$\{\Phi^i_j, \Phi^k_l\} = -2i(\Phi^i_j \delta^k_l - \Phi^k_l \delta^i_j). \quad (46)$$
The Poisson algebra of a dynamical system whose phase space is the Grassmannian can be defined by the above relations on its generators along with the constraints $\Phi^i_j \Phi^j_k = \delta^i_k$.

The functions $f_u$ generate the infinitesimal action of the Unitary group; they are the moment maps on the coadjoint orbit. The dynamical system with one of these as hamiltonian (energy) is not very interesting, since it corresponds upon quantization to a free fermion system (see below). The hamiltonians of interest are quadratic functions of the $\Phi$:

$$H(\Phi) = [-\text{tr} h \Phi + \frac{1}{2} \text{tr} \Phi \hat{G}(\Phi)]$$  \hspace{1cm} (47)

(The constant factor is chosen for later convenience). Here $h$ is some constant hermitean matrix. $\hat{G}$ is a positive symmetric linear operator on the vector space of Hermitean matrices. In terms of components,

$$(\hat{G} \Phi)^i_j = G^{ik}_j \Phi^k_i$$  \hspace{1cm} (48)

with

$$G^{ik}_j = G^{ki}_j = G^{*jli}$$  \hspace{1cm} (49)

and $G^{ik}_{jl} \xi^i_j \xi^l_k \geq 0$. If $G^{ik}_{jl} = B^i_j \delta^k_l + \delta^i_j B^k_l$, the quadratic function will be a constant on each connected component (depends only on the trace of $\Phi$). Similarly, if $G^{ik}_{jl} = C^i_l \delta^k_j + \delta^i_l C^k_j$ it will also be a constant. Therefore the tensor $\hat{G}$ can be chosen to be traceless in all the indices at the cost of changing the hamiltonian by a constant, which will not affect the equations of motion. ($h$ can also be chosen traceless). The pair of irreducible tensors $h$ and $G$ defines a ‘quadratic function’ on the Grassmannian. Quadratic functions on Grassmannians have been studied before, \[16\] but the ones we are interested in do not seem to have a natural geometrical meaning. Any $G$ can be expanded in terms of its eigenvectors, $G^{ij}_{kl} = \sum_a \gamma_a s^i_k s^j_a$, $\gamma_a$ being positive numbers. The particular tensors $\hat{G}$ that arise in our theory have the Weyl matrices as eigenvectors. It would be interesting to obtain an algebraic or characterization of these tensors, independent of the connection to fermionic systems.
The extrema of the Hamiltonian function (static solutions of the equations of motion) on the Grassmannians are of special interest. A variation that preserves the constraint $\Phi^2 = 1$ is of the form $\delta \Phi = [\Phi, U]$, where $U$ is an arbitrary anti-hermitean matrix. Thus the condition for an extremum is

$$[h - \hat{G}(\Phi), \Phi] = 0. \quad (50)$$

If $\hat{G} = 0$, the solutions are of the form $\Phi = \sum_i \epsilon_i u_i \otimes u_i^\dagger$, with $\epsilon_i = \pm 1$ and $u_i$ being the eigenvectors of $h$. In each connected component, the minimum is corresponds to the choice $\epsilon_i = -1(+1)$ for the lowest $m$ eigenvalues of $h$. Even the case $\hat{G} = 0$ corresponds to a highly nonlinear classical system, since the phase space is a curved manifold. The minimum in $Gr_{m,M}$ has the physical meaning of the ground state of a free fermionic system, with $m$ fermions. The operator $N = \frac{1}{2}(1 - \Phi)$ (which satisfies $N^2 = N$) has as its eigenvalues the occupation numbers of the fermionic system. The additional (quadratic) term represents interactions between the fermions; this can of course, distort the ground state. The configuration $\Phi_m$ with the minimum energy in each connected component $Gr_{m,M}$ is the semi-classical approximation to the ground state of the fermionic system in the expansion we will introduce soon.

The time evolution of our system will be determined by the ordinary differential equations

$$i \frac{d\Phi}{2 \, dt} = [h - \hat{G}(\Phi), \Phi]. \quad (51)$$

Small oscillations around the minimum $\Phi_0$ in $Gr_{m,M}$ are of physical interest; they correspond to charge density waves (mesons) of the fermionic system. (A small fluctuation will grow exponentially in time if we expand around a critical point that is not a minimum.) The equation for small fluctuations is

$$i \frac{d\delta \Phi}{2 \, dt} = [h - \hat{G}(\Phi_0), \delta \Phi] + [\Phi_0, \hat{G}(\delta \Phi)]. \quad (52)$$

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In other words, the small oscillations satisfy \( \frac{d\delta \Phi}{dt} = \hat{K}(\delta \Phi) \), \( K \) being the linear operator on the tangent space of the Grassmannian at \( \Phi_0 \),

\[
\hat{K} = \text{ad} (h - \hat{G}(\Phi_0)) + \text{ad} \Phi_0 \hat{G}.
\] (53)

2. 5. Action of the Grassmannian system

By a straightforward application of the usual point of view on multivalued actions, we can construct an action principle that defines the classical system on the Grassmannian completely.

In Hamiltonian mechanics, the symplectic form is usually expressed in canonical coordinates, \( \omega = dq^i \wedge dp_i \). Given a path in phase space \( c(t) = (q^i(t), p_i(t)) \), its action is defined to be

\[
S = \int_c p_i dq^i - \int_c H(p(t), q(t))dt
\] (54)

the subscript indicating that integrals are evaluated along the path \( c \). The path along which this is stationary is the classical trajectory. If the curve \( c \) is closed, this can also be expressed as

\[
S = \int_s \omega - \int_{c=\partial s} H(p(t), q(t))dt
\] (55)

where \( s \) is a surface in the phase space whose boundary is \( c \). (We will be interested only in situations where such a curve exists). The latter point of view is suited to situations where the symplectic form is closed but not exact.

This gives us an action principle on the Grassmannian defining our system:

\[
S = -\frac{i}{8} \int \text{tr} \tilde{\Phi}(s, t) \frac{\partial \tilde{\Phi}}{\partial t} \frac{\partial \tilde{\Phi}}{\partial s} dt \wedge ds - \int \text{tr}[-h\Phi(t) + \frac{1}{2} \Phi(t) \hat{G}(\Phi(t))] dt.
\] (56)

Here \( \Phi(t) \) is a periodic function of \( t \) describing a closed curve on the Grassmannian. Each component of the Grassmannian is simply connected, and \( \tilde{\Phi}(s, t) \) is a deformation of this
curve to a point (i.e., it is a surface whose boundary is $\Phi(t)$). Since $\omega$ is closed, a continuous deformation of the surface itself will not change the value of the action. However, if it is changed by a closed surface, the value of the action can jump by a constant. Since $\omega$ has been normalized such that its integral over any closed 2–surface is an integer multiple of $2\pi$, the change of $S$ will also be an integer multiple of $2\pi$. This has no effect on the classical theory, but does affect the quantum theory. In order that the quantum path integral

$$\int \mathcal{D}\Phi e^{-\frac{i}{\hbar}S}$$

have a single valued integrand, $\frac{1}{\hbar}$ must be an integer. This is the integer denoted by $N_c$ below. We will see that the same restriction arises in the canonical quantization method.

3. Quantization on the Grassmannian

We have been studying a classical system whose phase space is the Grassmannian with the hamiltonian

$$H = \text{tr}[ -\hbar \Phi + \frac{1}{2} \Phi \hat{G}(\Phi) ].$$

(58)

We will now show that the quantization of this system yields a system of fermions with a hamiltonian that is quartic in the fermion operators. In this sense, the Grassmannian system we described earlier is the classical limit of a fermionic system. The classical limit of the fermionic system corresponds to the one in which the occupation numbers go to infinity.

There are several different routes to constructing the quantum theory. The simplest is the algebraic (or canonical) quantization.

In the algebraic point of view, the classical system is defined by the (i) Poisson brackets of a complete set of observables

$$\{ \Phi^i_j, \Phi^k_l \} = (\Phi^i_j \delta^k_l - \Phi^k_l \delta^i_j),$$

(59)
(ii) the constraints satisfied by them,
\[ \Phi_j^i \Phi_k^j = \delta_k^i \]  
(60)

and (iii) the hamiltonian
\[ H = \text{tr}[ -h \Phi + \frac{1}{2} \Phi \hat{G}(\Phi) ]. \]  
(61)

One way to pass to the quantum theory is to associate to each classical observable \( f \) (function on the phase space) a self–adjoint operator \( \hat{f} \) on a quantum Hilbert space such that
\[ [\hat{f}_1, \hat{f}_2] = i\hbar \{ f_1, f_2 \}. \]  
(62)

In conventional canonical quantization, one thinks of the canonical coordinates as the basic observables and require them to satisfy the Heisenberg relations. For more complicated operators, it is possible to preserve the connection of Poisson brackets to commutators only up to higher order terms in \( \hbar \). The real parameter \( \hbar \) measures how much the quantum theory deviates from the classical theory.

It is not a good idea to express our theory in terms of canonical coordinates, as such coordinates would be singular somewhere in the phase space. Instead, we will take the Poisson brackets of the \( \Phi_j^i \) as the analogues of the canonical commutation relations. Quantization then consists of finding a representation of the above algebra in terms of operators \( \hat{\Phi}_j^i \) on a complex Hilbert space satisfying
\[ [\hat{\Phi}_j^i, \hat{\Phi}_l^k] = -i\hbar (\hat{\Phi}_j^i \delta_l^k - \hat{\Phi}_l^k \delta_j^i). \]  
(63)

This means that the operators \( \frac{\hat{\Phi}}{\hbar} \) provide a representation of the unitary Lie algebra. We can assume this representation is irreducible since otherwise, the whole theory will split up into a sum of irreducible components that do not interact with each other. The real and imaginary parts of \( \Phi_j^i \) are observables, so that the corresponding operators must satisfy
the hermiticity condition
\[ \hat{\Phi}_j^\dagger i = \hat{\Phi}^j_i \] (64)

analogous to the classical condition \( \Phi^*_j i = \Phi^j_i \). This condition requires the representation of the Lie algebra to be unitary.

Unitary irreducible representations of the Lie algebra \( U(M) \) are well known. A basis in any irreducible representation is given by the simultaneous eigenvectors of \( \frac{1}{\hbar} \Phi^i_i \) (no sum on \( i \)),
\[ \frac{1}{\hbar} \Phi^i_i w > = w_i |w> . \] (65)

In a unitary representation, these weights \( w_i \) are integers. There is a unique lowest weight vector, satisfying
\[ \hat{\Phi}^i_j |w> = 0 \text{ for } i > j \] (66)

Also the lowest weight is a nondecreasing sequence \( w_1 \leq w_2 \cdots w_M \). There is in fact a one–one correspondence between such sequences and unitary irreducible representations of \( U(M) \).

At this point our quantization scheme is ambiguous since there are many irreducible representations. (The usual canonical quantization is more unique since the Heisenberg algebra has only one unitary irreducible representation). Now recall the constraints \( \Phi^i_j \Phi^j_k = \delta^i_k ; \Phi^i_i = M - 2m \) of the classical theory. If we require that these be true at least in the sense of an expectation value on the lowest weight state,
\[ w |\hat{\Phi}^j_j |w> < w |\hat{\Phi}^j_k |w> = \delta^i_k ; < w |\Phi^i_i |w> = M - 2m \] (67)

we will get
\[ \hbar^2 w_i^2 = 1 \hbar \sum_i w_i = M - 2m. \] (68)

This implies first of all that \( \hbar \) is the inverse of an integer, \( \hbar = \frac{1}{N_c} \). (This integer is the number of ‘colors’; the reason for this terminology will become clear in a minute).
Furthermore, it determines the lowest weight vector in terms of this integer:

\[ w_1 = -N_c \cdots w_m = -N_c, \quad w_{m+1} = N_c \cdots \]  

(69)

Thus each connected component of the phase space corresponds to an irreducible representation of the unitary Lie algebra. The representation is unique up to the choice of the integer \( N_c \), whose inverse has the meaning of \( \hbar \). In particular, the large \( N_c \) limit is the classical limit of the theory.

This representation of \( U(M) \) can be described more explicitly in terms of fermionic operators. Define a set of operators satisfying the anti-commutation relations

\[
[a^{\dagger a}_i, a^{\dagger b}_j]_+ = \delta^i_j \delta^a_b, \quad [a^{\dagger a}_i, a^{\dagger b}_j]_+ = 0, \quad [a^{\dagger a}_i, a^{\dagger b}_j]_+ = 0
\]

(70)

where \( i, j = 1, \cdots M \), and \( a, b = 1, \cdots N_c \). Then, there is a representation of this algebra on a fermionic Fock space \( \mathcal{F} \) of dimension \( 2^{MN_c} \). The bilinears

\[
\Phi^i_j = \frac{1}{N_c} \sum [a^{\dagger a}_i, a^{\dagger b}_j]
\]

(71)

provide a representation of our algebra of \( \Phi \)'s on this Fock space. However, as it stands this representation is reducible. For example, the operators

\[
Q^a_b = -\frac{1}{2}[a^{\dagger a}_i, a^{\dagger b}_i] + \frac{1}{2M} \delta^a_b [a^{\dagger b}_i, a^{\dagger b}_i]
\]

(72)

which generate a \( SU(N_c) \) algebra commute with the \( \Phi^i_j \). To get an irreducible representation, we must look at the subspace of vectors annihilated by these operators.

This \( SU(N_c) \) symmetry is called ‘color’ symmetry, since it has to do with the number of copies of otherwise identical fermions. It has no meaning within our theory, and it is natural that all our states be invariant under color. We emphasize that \( Q^a_b \) does not generate an ordinary symmetry that commutes with just the hamiltonian; it is a gauge symmetry, that commutes with all the observables of our theory.
The fermion number operator $a_{ia}^\dagger a^ia$ also commutes with the $\Phi^i_j$ and so must be fixed to get an irreducible representation. If we restrict ourselves to the states of zero ‘color’ and of fermion number $mN_c$,

$$F_{m0} = \{ |\psi >\in F|Q^a_0|\psi > = 0; \frac{1}{N_c}a_{ia}^\dagger a^ia|\psi > = m| > \}$$

we get an irreducible representation. The lowest weight state is the state in which the states labelled by $i = 1, \cdots m$ are occupied, while the remaining ones are unoccupied

$$a_{ia}^\dagger |w > = 0 \text{ for } i \leq m; \quad a_{ia}|w > = 0 \text{ for } i > m.$$ 

The lowest weight can be calculated:

$$\Phi^i_j|w > = \frac{1}{N_c} \sum_a [a^ia_{ia}, a_{ia}]|w > = \mp |w > \text{ for } i \leq m \text{ or } i > m$$

agreeing with the earlier result. Thus the color singlet fermionic states do indeed provide the particular irreducible representation we need.

We can now show that the constraints are true as operator equations, in the Hilbert space $F_{m0}$:

$$\hat{\Phi}^i_j \hat{\Phi}^j_k = \delta^i_k, \quad \text{tr}\Phi^i_i = M - 2m.$$ 

The linear constraint is trivial to see. As for the quadratic constraint, if we expand

$$\hat{\Phi}^i_j \hat{\Phi}^j_k = \frac{1}{N^2} \{ [a^{ja}, a_j^\dagger] [a^{jb}, a_{kb}^\dagger]\}$$

it will be clear that every term can be written as a bilinear or as a bilinear multiplied by $a_{ja}^\dagger a^{jb}$ on the left or the right. Now we can use the condition of color invariance to reduce the latter terms also to bilinears. Thus we see that

$$\hat{\Phi}^i_j \hat{\Phi}^j_k = a\delta^i_k + b\Phi^i_k$$
where \( a, b \) are real numbers that may depend on \( m \). We can determine them by taking the matrix elements of the equation in the lowest weight state. If we put \( i = k \leq m \), the expectation value of the R.H.S. is \( a - b \). For the L.H.S.,

\[
\frac{1}{N_c^2} \langle w | [a^{ia}, a_\dagger^{ja}][a^{jb}, a_\dagger^{ib}] | w \rangle = \frac{1}{N_c^2} \sum_{j \leq m} \langle w | a_\dagger^{ja} a^{ia} a_\dagger^{ib} a^{jb} | w \rangle
\]

\[
= \frac{1}{N_c^2} || a_\dagger^{ib} a^{ib} | w \rangle ||^2 = \frac{N_c^2}{N_c^2} = 1
\]

so that \( a - b = 1 \). If we consider \( i = k > m \), we will get \( a + b = 1 \). Thus \( a = 1 \) and \( b = 0 \), proving our identity.

Thus we have a representation of the commutation relations that satisfies the constraint. To complete the construction of the quantum system, we find the hamiltonian operator,

\[
\hat{H} = -\hbar^2 \hat{\Phi}_i^j + \frac{1}{2} G^{ij}_{kl} \hat{\Phi}_i^k \hat{\Phi}_j^l.
\] (78)

This describes a system of fermions interacting through a two body ‘potential’. \( G^{ij}_{kl} \) is the scattering amplitude in Born approximation for a pair of fermions. When the interaction is strong, it becomes very difficult to solve the system in this language. It is simpler to solve it in the \( \frac{1}{N_c} \) expansion, which corresponds to the semi-classical approximation of the Grassmannian model.

The quantum fluctuations in the observables (real and imaginary parts of \( \hat{\Phi}_j^i \)) go to zero in the large \( N_c \) limit. This is clear from the fact that the commutators of the \( \hat{\Phi}_j^i \) vanish as \( \frac{1}{N_c} \) and the Heisenberg uncertainty relation

\[
(\Delta A)^2 (\Delta B)^2 \geq \frac{1}{4} | \langle \psi | [A, B] | \psi \rangle |^2
\] (79)

for a pair of hermitean operators. For finite \( N_c \), one can consider the states in \( \mathcal{F}_{m0} \) that minimize the uncertainty of measuring the various components of \( \hat{\Phi}_j^i \). These would be the analogue of minimum uncertainty wave packets of quantum mechanics (‘coherent states’).
One invariant measure [3] of this uncertainty is

$$\Delta C_2 = \langle \psi | \hat{\Phi}_j^i \hat{\Phi}_i^j | \psi \rangle - \langle \psi | \hat{\Phi}_j^j | \psi \rangle \langle \hat{\Phi}_i^i | \psi \rangle.$$ (80)

The states that minimize this are the lowest weight states, i.e., satisfy $\hat{\Phi}_j^i | \psi \rangle = 0$ for some choice of basis in the Lie algebra. In fact the set of minimum uncertainty states is just the orbit of a particular lowest weight state under the action of $U(M)$. The isotropy group of a lowest weight state is $U(m) \times U(M - m)$ so that this orbit is a Grassmannian. We have here an embedding of the Grassmannian into the Hilbert space $\mathcal{F}_{m0}$. This extends to a holomorphic embedding of $Gr(M)$ to the projective space $P(\mathcal{F}_{m0})$. The points on this submanifold of $\mathcal{F}_{m0}$ form an overcomplete basis; i.e., any state can be written as a linear combination of the points on the orbit of the lowest weight state. For $N_c = 1$ (when the condition of color invariance is trivial) this is the standard Plucker embedding of the Grassmannian into the projective space $P(C^{(M)}_{\psi})$.

To summarize, the quantum Hilbert space of the dynamical system on the Grassmannian is the color singlet sector of a fermionic system. $\frac{1}{N_c}$ plays the role of $\hbar$ in this quantization. A Hamiltonian on $Gr(M)$ that is quadratic in $\Phi$ leads to a fermionic system with four point (two–body) interactions. The points of the Grassmannian correspond to coherent states in the Hilbert space; in the large $N_c$ limit, the quantum fluctuations disappear and we recover the classical model on the Grassmannian.

4. Hadrons in two dimensions

4.1. The Grassmannian $Gr_1$

We will first give a reasonably precise definition of the phase space of our theory, which will be an infinite rank Grassmannian [5]. There is a homotopy invariant (called ‘virtual rank’) that plays an important role (baryon number) in our theory. Without a precise definition it would not be possible to see that there is such an invariant. We don’t need to
maintain this level of mathematical rigor for other parts of the paper, since the answers do not depend critically on topological subtleties. It might be useful to consult a standard reference on operator ideals (e.g. [17]) before reading this section; the summary given below is very brief and not very pedagogical.

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{H} \pm$ a pair of infinite dimensional orthogonal subspaces such that $\mathcal{H} = \mathcal{H} - \oplus \mathcal{H} +$. Define the self-adjoint operator $\epsilon$ which has eigenvalues $\pm 1$ on $\mathcal{H} \pm$. The ‘restricted Unitary group’ $U_1$ is defined by

$$U_1 = \{ g | g^\dagger g = 1; [\epsilon, g] \in \mathcal{I}_2 \}$$

(81) $\mathcal{I}_2$ being the space of Hilbert-Schmidt operators. (An operator $A$ is Hilbert-Schmidt if $A^\dagger A$ is trace class: $\text{tr} A^\dagger A < \infty$). Thus the elements of $U_1$ are unitary and do not mix the subspaces $\mathcal{H} \pm$ ‘too much’. This group is of interest in quantum field theory [5] because the bilinears of a $1 + 1$ dimensional fermionic field theory form a representation of its Lie algebra. There, $\mathcal{H}$ is the one particle Hilbert space; $\mathcal{H} \pm$ are the positive (negative) eigenspaces of the Dirac hamiltonian. Thus $\epsilon$ is the sign of the first quantized energy operator. In general, we would split the one particle Hilbert space into $\mathcal{H} -$, the states with energy less than the Fermi energy and $\mathcal{H} +$, the states with more than Fermi energy. In nonrelativistic quantum mechanics the dimension of $\mathcal{H} -$ is finite; in relativistic quantum mechanics, this space is infinite dimensional.

We will be interested in the infinite rank Grassmannian

$$Gr_1 = U_1(\mathcal{H})/U(\mathcal{H}-) \times U(\mathcal{H}+)$$

(82) $U(\mathcal{H} \pm)$ is the subgroup of operators that leave the negative (positive) energy states invariant. (Equivalently, $U(\mathcal{H}-) \times U(\mathcal{H}+)$ is the subgroup that commutes with $\epsilon$). It can be shown that this is an infinite dimensional manifold modelled on the Hilbert space $\mathcal{I}_2(\mathcal{H}-, \mathcal{H}+)\) of Hilbert-Schmidt operators from $\mathcal{H}-$ to $\mathcal{H}+$. We will now define two dimensional hadronic theory as a classical dynamical system with $Gr_1$ as the phase space.
As before the Grassmannian can be parametrized by operators $\Phi$:

$$Gr_1 = \{ \Phi | \Phi \dagger = \Phi; \Phi^2 = 1; [\epsilon, \Phi] \in \mathcal{I}_2 \}.$$ (83)

It is clear that any such operator can be reduced to $\epsilon$ by a unitary transformation: $\Phi = g\epsilon g\dagger$. (The convergence condition on $\Phi$ will ensure that $g$ itself satisfies the convergence condition of $U_1$.) But $g$ is ambiguous up to a right multiplication by $h \in U(\mathcal{H}_-) \times U(\mathcal{H}_+)$. Hence there is a 1-1 correspondence between $\Phi$ and points of the coset space $Gr_1$. The point $\Phi = \epsilon$ will be the ‘vacuum’ configuration of the theory so in fact it is more convenient to introduce the ‘normal ordered’ variable $M = \Phi - \epsilon$. Thus, we parametrize the Grassmannian by

$$Gr_1 = \{ M | M \dagger = M; M^2 + \epsilon M + M\epsilon = 0; \text{tr}[\epsilon, M]^2 < \infty \}. (84)$$

If we decompose $M$ into a $2 \times 2$ matrix with respect to the splitting $\mathcal{H} = \mathcal{H}_- \oplus \mathcal{H}_+$

$$M = \begin{pmatrix} \alpha & \beta \\ \beta^\dagger & \delta \end{pmatrix},$$

the convergence condition just says that $\beta$ is Hilbert–Schmidt. The quadratic constraint now implies that $\alpha$ and $\delta$ are trace–class. To see this write $M = g\epsilon g\dagger - \epsilon$ where, $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Here $b, c$ are Hilbert–Schmidt and $a$ and $d$ are only bounded. Then, $\alpha = -aa\dagger + bb\dagger + 1$ etc. Using the unitarity of $g$, $(aa\dagger + cc\dagger = 1)$, we see that $\alpha = bb\dagger + cc\dagger$ which is trace class. This technical remark will be useful in defining the topological invariant called ‘virtual rank’ below.

The group $U_1$ has an infinite number of connected components labelled by an integer. To see this, consider $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ decomposed into blocks as before; $a: \mathcal{H}_- \to \mathcal{H}_-, b: \mathcal{H}_+ \to \mathcal{H}_-, c: \mathcal{H}_- \to \mathcal{H}_+, d: \mathcal{H}_+ \to \mathcal{H}_+$. Since $g$ is invertible and $b, c$ compact, it follows that, $a$ and $d$ are Fredholm operators of opposite index.

We digress a little to give some basic definitions and results of operator theory. There are many standard references where results are stated more precisely e.g., [17]. An operator is compact if there is a sequence of finite rank operators that converges to it in
the operator norm. Hilbert–Schmidt and trace class operators are in particular compact. Not all bounded operators are compact: for example the identity is not. An operator is Fredholm if it is invertible modulo the addition of a compact operator. The kernel of an operator \( a \) is the subspace of all \( |\psi> \) such that \( a|\psi> = 0 \). The co–kernel is the set of \( |\psi> \) such that \( a^\dagger|\psi> = 0 \). The dimension of the kernel is \textit{not} a continuous function of \( a \). However, the index of \( a \) which is the dimension of its kernel minus that of its cokernel is invariant under continuous deformation of \( a \).

Returning to our context, the index of the submatrix \( a \) of \( g \) is the net number of states in \( \mathcal{H}_- \) which are pushed out to \( \mathcal{H}_+ \) by \( g \). The index of \( a \) is invariant under a continuous deformation of \( g \). The index of \( d \) is just minus that of \( a \): as we change \( g \), if a certain number of states cross from \( \mathcal{H}_- \) to \( \mathcal{H}_+ \), an equal number will cross in the opposite direction. Two elements of \( U_1 \) with the same index are connected by a continuous path: \( U_1 \) is the disjoint union of connected components labelled by the index.

It is easy to give an example of a unitary operator \( g \) with non– zero index for \( a \). Suppose we label an orthonormal basis of \( \mathcal{H} \) by integers, with \( e_n \) for \( n \leq 0 \) (\( n > 0 \)) spanning \( \mathcal{H}_-(\mathcal{H}_+) \). An element of the connected component of \( U_1 \) with index \( k \) is the shift operator \( \sigma_k \)

\[
\sigma_k e_n = e_{n+k}.
\]

Note that if we had not imposed the condition that \([\epsilon, g]\) is Hilbert–Schmidt, the Unitary group would have been contractible; there would be no such topological invariant. We will see that this topological invariant has the meaning of baryon number in our hadronic theory, which is an essential feature. More generally \( \pi_{2m}(U_1) = \mathbb{Z} \) for even homotopy groups and \( \pi_{2m+1}(U_1) = 0 \) for odd homotopy groups.

The Grassmannian is also a disjoint union of connected components labelled by an integer. This was already true for finite rank Grassmannians. In fact \(-\frac{1}{2}\text{tr}(\Phi - 1)\) is the
dimension of the subspace in which \( \Phi \) has eigenvalue \(-1\) (the rank of the Grassmannian).

For our present case this would diverge. We can get a convergent quantity by subtracting the rank of the vacuum \( \Phi = \epsilon \); i.e., consider \(-\frac{1}{2} \text{tr}(\Phi - \epsilon) = -\frac{1}{2} \text{tr}M\). This is the ‘virtual rank’ of the Grassmannian, the difference between the number of negative eigenvalues of \( \Phi \) and those of \( \epsilon \). (A technical remark: we can define \( \text{tr}M \) to be \( \text{tr}\alpha + \text{tr}\delta \). Here \( M = \begin{pmatrix} \alpha & \beta \\ \beta^{\dagger} & \delta \end{pmatrix} \) in a basis in which \( \epsilon = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \). We showed earlier that \( \alpha \) and \( \delta \) are trace class. This indirect definition of \( \text{tr}M \) is necessary since \( M \) itself is not trace class.)

In fact if \( M = g\epsilon g^{\dagger} - \epsilon \), it is easy to see that \(-\frac{1}{2} \text{tr}M = \text{index}(a)\). (For example, if \( g = \sigma_1 \) the shift operator, \((e_n, \sigma_1\epsilon\sigma_1 - \epsilon e_n) = \text{sgn}(n - 1) - \text{sgn}(n)\). Then \( \text{tr}M = -2 \) and the index of \( a \) is 1.)

It is best to think of \( \mathcal{H} \) as the space \( L^2(R; C) \) of complex valued functions on the real line. In this parametrization, \( \epsilon \) is the sign of the momentum operator,

\[
\epsilon = \int [dk] e^{ikx} \text{sgn}(k). \tag{86}
\]

Here,

\[
[dk] = \frac{dk}{2\pi}. \tag{87}
\]

We can also describe it by its integral kernel in position space

\[
\epsilon(x, y) = \frac{i}{2\pi} \mathcal{P}\left(\frac{1}{x-y}\right); \tag{88}
\]

This is, \( i \) times the well-known Hilbert transform operator. The Grassmannian can be parametrized by the integral kernel \( M(x, y) \) of the operator \( M \). The condition of self-adjointness is just

\[
M^*(x, y) = M(y, x). \tag{89}
\]

The quadratic constraint is the nonlinear integral equation

\[
\int dy [M(x, y)M(y, z) + \epsilon(x, y)M(y, z) + M(x, y)\epsilon(y, z)] = 0. \tag{90}
\]
The convergence condition is best understood in terms of the Fourier transform

$$
\tilde{M}(p, q) = \int e^{ipx - iqy} M(x, y) dxdy.
$$

(91)

The condition is that the off–diagonal component of \( \tilde{M} \) with \( p \) and \( q \) having opposite sign has finite norm:

$$
\int_{p>0} [dp] \int_{q<0} [dq] |\tilde{M}(p, q)|^2 < \infty.
$$

(92)

The topological invariant (‘virtual rank’) of the Grassmannian is then

$$
w(M) = -\frac{1}{2} \int dx M(x, x)
$$

(93)

which we will be shown to have the physical meaning baryon number.

4. 2. Symplectic structure and Poisson Brackets

We saw in the finite rank case, that the Grassmannian is a symplectic manifold with \( \omega = -\frac{i}{8} \text{tr} \Phi (d\Phi)^2 \). The analogue in infinite dimensions is just \( \omega = -\frac{i}{8} \text{tr}(M + \epsilon)(dM)^2 \). A tangent vector at the point \( M = 0 \) is a matrix of the form \( U = \begin{pmatrix} 0 & u \\ u^\dagger & 0 \end{pmatrix} \) with \( \text{tr} u^\dagger u < \infty \). It is straightforward to see that \( \omega(U, V) = -\frac{1}{4} \text{tr}(u^\dagger v - v^\dagger u) \). This shows that \( \omega \) exists and is non–degenerate at the vacuum. That \( \omega \) exists and is non–degenerate at an arbitrary point follows from its homogeneity (see below). The proof that \( d\omega = 0 \) goes through as before.

The transformation \( \Phi \rightarrow g\Phi g^\dagger \) or \( M \rightarrow gMg^\dagger + g[\epsilon, g^\dagger] \) leaves the symplectic form invariant. From general principles, there must be functions that generate the corresponding infinitesimal canonical transformations, \( V_u = i[u, \epsilon + M] \). (The matrix \( u = \begin{pmatrix} a & b \\ b^\dagger & d \end{pmatrix} \) must have diagonal elements \( a, b \) bounded and off–diagonal elements \( c, d \in \mathbb{I}_2 \) in order that \( e^{iu} \) be in \( U_1 \).) That is, there must be a function \( f_u \) such that \( \omega(V_u, .) = df_u \). In the finite dimensional case, this is just \( f_u(\Phi) = \text{tr} u \Phi \). In the finite dimensional case, these are just linear functions of \( \Phi \). For our present case, again \( \text{tr} u \Phi \) will diverge; we must subtract
the vacuum contribution and consider instead \( f_u(M) = -\frac{1}{2} \text{tr} u M = -\frac{1}{2} \text{tr} (\Phi - \epsilon) \). (It is easy to check that the trace in \( f_u(M) \) does indeed converge.) But we will pay a price for this vacuum subtraction. The Poisson brackets of the \( f_u \) will no longer be that of the Lie algebra of \( U_1 \): there will be a non–trivial central extension.

The computation that verifies that \( f_u(M) \) generates \( V_u \) is straightforward:

\[
\omega(V_u, \cdot) = -\frac{i}{8} \text{tr}(\epsilon + M)[V_u(M), dM] = -\frac{i}{8} \text{tr}[V_u, \epsilon + M]dM
\]

\[
= \frac{1}{8} \text{tr}[\epsilon + M, [\epsilon + M, u]]dM = -\frac{1}{2} \text{tr} u dM = df_u.
\]

Now we get

\[
\{f_u, f_v\}(M) = -\mathcal{L}_{V_u} f_v(M) = -i \text{tr}[u, \epsilon + M] v = f_{(-i)u,v}(M) + \frac{i}{2} \text{tr}[\epsilon, u] v. \tag{94}
\]

This is the central extension of the Lie algebra of \( U_1 \). We cannot remove the central term by redefining \( f_u(M) \to f_u(M) + \text{tr} \epsilon u \) since the second term does not converge in general. This nontrivial extension of the unitary Lie algebra was found in a different context by Kac and Petersen. In the co–ordinate basis, these Poisson brackets are,

\[
\frac{i}{2} \{M(x, y), M(z, u)\} = \delta(y - z)[M(x, u) + \epsilon(x - u)] - \delta(x - u)[M(z, y) + \epsilon(z - y)]. \tag{95}
\]

This may be verified by multiplying both sides by \( u(y, x)v(u, z) \) and integrating over \( x, y, z, u \) to get the earlier Poisson bracket relation.

Occasionally the above Lie algebra is also called the \( W_\infty \) algebra. This algebra arises in the matrix model approach to string theory. Our theory of hadrons is a nonlinear matrix model, the dynamical variable being \( M(x, y) \). We believe that in fact our theory is a string field theory in two target space dimensions written in light cone gauge. Finding an ungauge fixed (manifestly reparametrization invariant form) of this theory is an extremely interesting problem. It is already known that the linear approximation to our theory (which is described by ’t Hooft’s integral equation) is equivalent to the free Nambu string in two target space dimensions \[18\]
The Poisson brackets above along with the constraints can be thought of as the definition of our phase space. The interpretation in terms of the Grassmannian shows how natural these Poisson brackets are, given the constraints. Conversely, given the above Poisson brackets, we can look for homogenous symplectic manifolds on which the Lie group acts. These are given by the co–adjoint orbits, of which the Grassmannian is one of the simplest. The other co–adjoint orbits do not describe fermion fields upon quantization.

4.3. The Hamiltonian

We will now describe two dimensional classical hadron theory. The most convenient co–ordinate system of Minkowsky space for our purpose is the one with metric $ds^2 = du(du + 2dx)$. In terms of conventional space–time co–ordinates $ds^2 = (dx^0)^2 - (dx^1)^2$ and $u = x^0 - x^1, x = x^1$. The surface $u = \text{constant}$ is a null line and $x$ is a co–ordinate in this null direction. The vector $\frac{\partial}{\partial u}$ (for fixed $x$) is just the same as the time–like vector $\frac{\partial}{\partial x^0}$. Thus momentum component $p_u$ is just energy while the $x$ component of momentum (which we will call just $p$) is $p = p_0 + p_1$. The mass shell condition for a particle of mass $\mu$ is

$$2p_up - p^2 = \mu^2 \Rightarrow p_u = \frac{1}{2}[p + \frac{\mu^2}{p}]. \quad (96)$$

The Lorentz transformation is, in this co–ordinate system,

$$u \rightarrow e^\theta u \quad x \rightarrow -\sinh \theta u + e^{-\theta} x \quad (97)$$

where $\theta$ is the rapidity. From the invariance of $p_u du + p dx$, we get

$$p \rightarrow e^\theta p \quad p_u \rightarrow e^{-\theta} p_u + \sinh \theta p. \quad (98)$$

The phase space will be given by the initial conditions on the surface $u = 0$; the dynamics will be given by the hamiltonian function $H$ which will generate translation in $u$. There should also be functions $L$ and $P$ generating Lorentz tranformations and translations, satisfying the Poincare algebra:

$$[P,L] = 0 \quad [L,P] = P \quad [L,H] = -H + P. \quad (99)$$
A dynamical system is completely specified by the phase space (a manifold along with a symplectic form) and a Hamiltonian. We will define two-dimensional hadron theory by choosing \((Gr_1, \omega)\) as the phase space. Thus our dynamical variable is a function of two points \(M(x, y)\) satisfying the constraint

\[
M^*(x, y) = M(y, x), \quad \int dy[M(x, y)M(y, z) + \epsilon(x, y)M(y, z) + M(x, y)\epsilon(y, z)] = 0. \quad (100)
\]

This specifies the initial condition at \(u = 0\). The value at any later value of \(u\) is given by solving the classical equations of motion

\[
\frac{\partial M(x, y)}{\partial u} = \{H, M(x, y)\}. \quad (101)
\]

We choose the Hamiltonian to be

\[
H = \int dxdyh(x - y)M(x, y) - \frac{1}{2} \tilde{g}^2 \int dxdyG(x - y)M(x, y)M(y, x). \quad (102)
\]

The operators \(h\) and \(G\) are defined by

\[
h = \frac{1}{2}(p + \mu^2), \quad G = \frac{1}{p^2}, \quad (103)
\]

where \(p = -i \frac{d}{dx}\). The theory depends on the parameters \(\tilde{g}\) and \(\mu\) with the dimensions of mass. We will show later that \(\mu\) is related to the quark mass and \(\tilde{g}\) to the gauge coupling constant of 2DQCD. The integral kernels of the operator are

\[
h(x - y) = \frac{1}{2}[-i\delta'(x - y) + i \frac{2}{2} \text{sgn}(x - y)], \quad G(x - y) = -\frac{1}{2}|x|. \quad (104)
\]

The choice of \(h\) as the kinetic energy operator is motivated by the earlier discussion of the mass shell condition.

Although the field variable is bilocal, the theory we have just defined is a Poincare invariant field theory. Translation invariance in the \(x\) direction is obvious. Under Lorentz transformations, the field transforms as

\[
M(x, y) \rightarrow e^\theta M(e^\theta x, e^\theta y). \quad (105)
\]
It is straightforward to check that \( H \) and \( P = \text{tr}pM \) transform as the components of a Lorentz vector under this transformation. Once we assume the form \((\cdot)\) for the hamiltonian, specific expressions for \( h \) and \( G \) are required by Lorentz invariance alone. It should be possible to show by an argument based on string theory that \((\cdot)\) is the only possible form for the hamiltonian.

The equations of motion that follow from this hamiltonian are,

\[
\frac{i}{2} \frac{\partial M(x, y)}{\partial u} = \frac{i}{2} \{H, M(x, y)\} = \int dz [h(x - z)M(z, y) - M(x, z)h(z, y)]
+ \bar{g}^2 \int dz [G(y - z)\epsilon(x, z)M(z, y) - G(z - x)\epsilon(z, y)M(x, z)]
+ \bar{g}^2 \int dz M(x, z)M(z, y)[G(y - z) - G(z - x)].
\]

By construction, the quadratic constraints are preserved under time evolution.

The solution \( M = 0 \) is the vacuum solution. Small oscillations around this are described by the linear approximation to the above equation. These will describe mesons. There are also stationary solutions which are very far from this vacuum solution, which describe baryons. The quantity \( B = -\frac{1}{2} \text{tr}M \), is an integer valued topological invariant and hence is conserved by time evolution. The lowest energy solution with \( B = 1 \) should be identified with the baryon. Linearizing (105) around this stationary solution will describe meson–baryon scattering; the bound states in this linearized equation will describe excited states of the baryon (analogues of \( \Delta, N^* \) etc.).

4. 4. Linear Approximation

Let us now study the linearization of the above theory around the vacuum \( M = 0 \). We must linearize not only the equations of motion but also the constraints. If we drop the second order term, the constraint becomes,

\[
\int dy [\epsilon(x, y)M(y, z) + M(x, y)\epsilon(y, z)] = 0.
\]

(106)
Since our equations are translation invariant, it is better to use the Fourier transformed variable
\[ \tilde{M}(p, q) = \int dx dy e^{i(-px + qy)}M(x, y). \] (107)

The constraint becomes:
\[ [\text{sgn } (p) + \text{sgn } (q)]\tilde{M}(p, q) = 0. \] (108)

In addition the hermiticity condition becomes
\[ \tilde{M}^*(p, q) = M(q, p). \] (109)

This means that our field operator has only off-diagonal components; i.e., those with opposite signs for \( p \) and \( q \). (Of course, this is just the statement that \( M \) is tangential to \( Gr_1 \) at the origin). The translation, \( M(x, y) \rightarrow M(x + a, y + a) \), and
\[ \tilde{M}(p, q) \rightarrow e^{i(-qa + pa)}\tilde{M}(p, q). \] (110)

Thus \( P = p - q \) has the meaning of total momentum. Due to the hermiticity condition, it is sufficient to consider the case \( p > 0 \) and \( q < 0 \), so that the total momentum is positive. Define now the dimensionless variable \( \xi = p/P \); then \( \xi \) varies in the range \( 0 \leq \xi \leq 1 \). We can use \( P \) and \( \xi \) as our independent variables. Set
\[ \chi(P, x) = P\tilde{M}(\xi P, (1 - \xi)P). \] (111)

The linearized equations of motion is, in position space,
\[ i\partial M(x, y)/\partial u = \int dz [h(x - z)M(z, y) - M(x, z)h(z, y)] + \tilde{g}^2 \int dz [G(y - z)\epsilon(x, z)M(z, y) - G(z - x)\epsilon(z, y)M(x, z)]. \]

In momentum space this becomes,
\[ i\partial \tilde{M}(p, q)/\partial u = [h(p) - h(q)]\tilde{M}(p, q) + \tilde{g}^2[\text{sgn } (p) - \text{sgn } (q)]\int [dr]/r^2 \tilde{M}(p - r, q - r). \] (112)
It is clearly natural to assume that the \( u \) dependence is exponential

\[
\tilde{M}(p, q, u) = e^{-iP_u u} \tilde{M}(p, q)
\]

. In terms of the variable \( \chi(P, \xi) \) we get the eigenvalue equation

\[
M^2 \chi = \left[ \frac{\mu^2}{\xi} + \frac{\mu^2}{1-\xi} \right] \chi(\xi) + 4\tilde{g}^2 \int_0^1 \frac{d\xi'}{(\xi - \xi')^2} \chi(\xi')
\]

(113)

where \( M^2 = 2P_u P - P^2 \) is the mass\(^2\) of the meson. This is precisely ’t Hooft’s integral equation. He has shown that all solutions are bound states (discrete eigenvalues) and that spectrum is asymptotically \( M^2_n \sim n\tilde{g}^2 \).

4. 5. The baryon solution

If ours is to be a complete theory of hadrons, it must contain baryons as well as mesons. From Witten’s [8] arguments one should expect baryons to arise in a Hartree approximation. On the other hand, in the Skyrme model, they arise as solitons in a theory whose small oscillations are mesons. Normally, this meson theory is only known approximately. However, we have an exact theory of hadrons in two dimensions. This gives us an opportunity to study the baryon within this exact theory. Also, we will get a description of a topological soliton in a bilocal field theory. Since we should expect our bilocal theory to be a string field theory, we will also get a glimpse of how a topological soliton looks like in string theory.

We already showed that there is a topologically conserved quantum number \( w(M) = -\frac{1}{2} \text{tr} M \) in our theory. It takes integer values. (It will be shown later that this indeed corresponds to baryon number of 2DQCD.) We will look for a configuration that minimizes the energy, and with \( w(M) = 1 \). Unlike the vacuum solution, it will have non–zero energy: it is a topological soliton. Upon quantization, its mass will seen to be \( O(N_c) \), since \( \frac{1}{N_c} \) has the meaning of \( \hbar \) in the quantum theory. We will see that the equation satisfied by soliton
field configuration has a natural interpretation in terms of a Hartree approximation for the quark wavefunction.

Our first task is to produce an ansatz that has virtual rank one. It will be much more convenient to deal with functions of one variable rather two, so we will seek a separable ansatz of the form

\[ M_1(x, y) = -2\psi(x)\psi^*(y). \] (114)

In order to have virtual rank one we must require the function \( \psi \) to be normalized:

\[ \int dx |\psi(x)|^2 = 1. \] (115)

Now let us see the condition on \( \psi \) implied by the quadratic constraint, \( M_1^2 + \epsilon M_1 + M_1 \epsilon = 0 \).

It is clear that \( \psi(x)\psi^*(y) \) is the kernel of a projection operator, so that \( M_1^2 = -2M_1 \). Thus we get \( \epsilon M_1 + M_1 \epsilon = 2M_1 \). In terms of \( \psi \) this is just the condition \( \epsilon \psi = \psi \). That is, \( \psi \) is an element of \( \mathcal{H}_+ \), the space with eigenvalue +1 for \( \epsilon \):

\[ \int \epsilon(x - y)\psi(y)dy = \psi(x). \] (116)

Thus our ansatz \( M_1 \) is indeed an element of \( Gr_1 \) with virtual rank one. (It is trivial to check that \([\epsilon, M]\) is a rank one operator, so that the convergence condition \( \text{tr}|[\epsilon, M]|^2 < \infty \) is satisfied).

In fact \( M_1 \) is unchanged under the change of phase \( \psi(x) \to e^{i\theta} \psi(x) \). The ansatz gives a map from the projective space \( \mathcal{P}(\mathcal{H}_+) \) to \( Gr_1 \). We will see that \( \mathcal{H}_- \) is the set of negative energy states for the quarks, which are occupied in the ground state of 2DQCD. To produce a baryon, we must put \( N_c \) quarks in one positive energy state as well. In the ground state, all these quarks will have the same wavefunction, apart from a color factor which is completely anti–symmetric. This is the meaning of \( \psi \) in the quark model. Note that the condition that \( \psi \) is a positive energy state arises naturally in the above formalism. It will be crucial for the stability of the soliton.
The energy of this configuration is obtained by substituting the ansatz into the hamiltonian \( H = \text{tr} M = (\psi, h\psi) \). The first term in the hamiltonian becomes \( \text{tr} hM = (\psi, h\psi) \) or, \( \int \psi^*(x)h(x-y)\psi(y)dxdy \). Noting also that \( M(x,y)M(y,x) = |\psi(x)|^2|\psi(y)|^2 \) we get

\[
H(\psi) = \int \psi^*(x)h(x-y)\psi(y)dxdy - \frac{1}{2}\tilde{g}^2 \int G(x-y)|\psi(x)|^2|\psi(y)|^2dxdy. \tag{117}
\]

It is now clear that the first term is just the kinetic energy of the quark wavefunction. The second term is the potential energy due to a Coulomb field generated by the quarks. There is an attractive linear potential between the quarks. This is exactly what we would have obtained in a Hartree approximation. There are \( N_c \) quarks in a baryon, whose color indices must anti-symmetrized to get a singlet state. The quarks being fermions, the wavefunction must be symmetric in the remaining (position and flavor) indices. Thus effectively they are bosons and the ground state wavefunction of all the quarks will be the same. The second term in the energy then describes the interaction with the mean field.

The condition \( \epsilon\psi = \psi \) ensures that the kinetic energy \( (\psi, h\psi) \) is positive. \( h = \frac{1}{2}[p + \mu^2] \) is not in general positive; \( \mathcal{H}_+ \) is the subspace in which it is positive. If it had not been for the constraint that \( \psi \in \mathcal{H}_+ \), the energy would not be bounded below and our soliton would have been unstable. The constraint is best understood in momentum space. If we define the Fourier transform

\[
\tilde{\psi}(p) = \int \psi(x)e^{-ipx}dx \tag{118}
\]

we see that it has support only for positive \( p \). In terms of position space, \( \psi(x) \) must be a function that can be analytically continued to the upper half of the complex plane.

Thus we must minimize the energy over all configurations \( \tilde{\psi}(p) \) satisfying

\[
\tilde{\psi}(p) = 0 \text{ for } p < 0 \int_0^\infty |\psi(p)|^2dp = 1. \tag{119}
\]

The Kinetic energy is simple to understand in momentum space while the potential energy is simpler in position space. The energy is

\[
H(\tilde{\psi}) = \frac{1}{2} \int_0^\infty [p + \mu^2]|\tilde{\psi}(p)|^2dp + \frac{1}{2}\tilde{g}^2 \int V(x)|\psi(x)|^2dx. \tag{120}
\]
Here $V(x)$ is the solution to the differential equation

$$V''(x) = |\psi(x)|^2 \tag{121}$$

with the boundary conditions $V(0) = 0$ and $V'(0) = 0$. This variational problem can be reduced to a nonlinear integral eigenvalue problem. We have not been able to solve this problem analytically. The problem was solved by numerically minimizing the energy in Ref. [11].

Here we will find the qualitative behaviour of the soliton using a simple variational approximation. An ansatz that satisfies the positivity of energy and the normalization condition is $\tilde{\psi}(p) = Npe^{-pa}$, where $a > 0$ is a variational parameter. (The even simpler possibility $\tilde{\psi}(p) = Ne^{-pa}$ has to discarded because it has infinite potential energy). Then, $\psi(x) = N\frac{e^{-pa}}{\pi(x+ia)^2}$. Note that this has a double pole on the lower half plane, but is analytic in the upper half plane. The normalization condition is $\frac{N^2}{2\pi} = 4a^3$. The kinetic energy integral is easily done:

$$\frac{1}{2} \int \left[p + \frac{\mu^2}{p}\right]|\tilde{\psi}(p)|^2 [dp] = \frac{1}{2} \left[\frac{3}{2a} + \mu^2 a\right]. \tag{122}$$

The differential equation

$$V''(x) = |\psi(x)|^2 = \frac{2a^3}{\pi} \frac{1}{(x^2 + a^2)^2} \tag{123}$$

has solution

$$V(x) = \frac{1}{\pi} x \arctan \frac{x}{a}. \tag{124}$$

Note that $V(x) \sim \frac{1}{2} |x|$ as $|x| \to \infty$, as expected. Thus the potential energy becomes

$$\frac{1}{2} \tilde{g}^2 \int V(x) |\psi(x)|^2 dx = \frac{1}{2} I \tilde{g}^2 a \tag{125}$$

where

$$I = \frac{4}{\pi} \int_0^\infty \frac{x \arctan x}{1 + x^2} dx = 2. \tag{126}$$
Thus we estimate the energy of the soliton to be

\[ H(a) = \frac{1}{2} \left[ \frac{3}{2a} + \frac{\mu_{\text{eff}}^2}{a} \right] \] (127)

where the quark mass \( \mu \) is replaced by the effective mass:

\[ \mu_{\text{eff}}^2 = \mu^2 + 2g^2. \] (128)

As the mass of the quark goes to zero, the lightest meson also has a very small mass \([7]\). But we see that even as the mass of the quark goes to zero, the mass of the baryon remains finite! The effective mass of the quark in the baryon is heavier by an amount proportional to the gauge coupling constant. That is, the ‘constituent quark mass’ is of the order of the gauge coupling constant, even as the current quark mass goes to zero. We have just established a phenomenon analogous to chiral symmetry breaking in two dimensions. (Of course the chiral symmetry itself cannot break spontaneously in two dimensions; the meson of small mass also has small coupling, so that it is not a Goldstone boson.)

In the quantum theory, there is an overall factor of \( N_c = \frac{1}{\hbar} \) in front of the action (and hence the hamiltonian) so that the mass of the baryon is \( N_c \) times the minimum of \( H \) as determined above. Thus our estimate \( \mu_{\text{eff}} \) is the mass of the baryon divided by \( N_c \), which can be thought of as the constituent quark mass. In the case of mesons, the energy of the vacuum configurations are zero so that the masses of the mesons come from small oscillations around the vacuum. These are therefore \( \hbar = \frac{1}{N_c} \) order smaller than the baryon mass, hence are \( O(N_c^0) \).

5. Quantization of Hadron Theory

As in the finite dimensional case, we can find an action principle for QHD. The quantum path integral would be

\[ \int \mathcal{D}\Phi e^{-\frac{i}{\hbar}S} \] (129)
where
\[ S = \int \omega \left( \frac{\partial \tilde{M}}{\partial t}, \frac{\partial \tilde{M}}{\partial s} \right) dt \wedge ds - \int H(M(t)) dt. \] (130)

Again, \( \tilde{M} \) is an extension of \( M \) to a function of two variables \( t, s \) such that the boundary value is the closed curve \( M(t) \). As before, \( H^2(Gr_1) = \mathbb{Z} \) and the integral of \( \omega \) on a generator of \( H^2(Gr_1) \) is \( 2\pi \). Thus \( \frac{1}{\hbar} = N_c \) must be an integer in order that \( e^{-i\bar{\hbar}S} \) be single valued. Since at the moment it is difficult to define such path integrals rigorously, we will follow instead a canonical (algebraic) point of view. The definition of the path integral can be accomplished by exploiting localization formulae.

We can now quantize the hadron theory by algebraic (canonical) methods. We would convert the Poisson brackets into commutation relations:
\[ [\hat{f}_u, \hat{f}_v] = i\hbar (\hat{f}_{-i[u,v]} + \text{tr}[\epsilon, u]v). \] (131)

Equivalently, we look for operator–valued distributions \( \hat{M}(x, y) \) satisfying,
\[ [\hat{M}(x, y), \hat{M}(z, u)] = -i\hbar (\delta(y - z)[\hat{M}(x, u) + \epsilon(x, u)] - \delta(u - x)[\hat{M}(z, y) + \epsilon(z, y)]). \]

These will provide a representation for the Lie algebra of \( U_1 \) on some space \( \mathcal{F} \). (Each matrix element \( \hat{M}(x, y) \) of \( M \) must itself be an operator on \( \mathcal{F} \).)

The representation we pick must be unitary:
\[ \hat{M}(x, y) \dagger = \hat{M}(y, x) \] (132)
and irreducible. (If the \( \hat{M}(x, y) \) are to be a complete set of observables, the only operators that commute with them must be multiples of the identity; but then the representation is irreducible.) Also the quadratic constraint must be satisfied at least up to terms that vanish as \( \hbar \to 0 \).

Furthermore, the hamiltonian must become a well–defined (self–adjoint) operator that is bounded below. Even for the simplest case of the hamiltonian with \( \tilde{g} = 0 \) this imposes
a nontrivial constraint on the representation. It is best to discuss this case in momentum picture, so we define \( \hat{M}(p, q) = \int dx dy \hat{M}(x, y)e^{-ipx + qy} \). The commutation relations become,

\[
[\hat{M}(p, q), \hat{M}(r, s)] = -i\hbar [2\pi \delta(q - r)[\hat{M}(p, s) + \text{sgn}(p)\delta(p - q)] - 2\pi \delta(s - p)[\hat{M}(r, q) + \text{sgn}(r)\delta(r - s)]].
\]

It is easy to recognize this as a representation of the central extension of the unitary Lie algebra, if we write this in terms of \( \hat{\hat{M}} \):

\[
\left[ \frac{\hat{\hat{M}}(p, q)}{\hbar}, \frac{\hat{\hat{M}}(r, s)}{\hbar} \right] = -i \left[ 2\pi \delta(q - r)\left( \frac{\hat{\hat{M}}(p, s)}{\hbar} + \frac{1}{\hbar} \text{sgn}(p)\delta(p - q) \right) - 2\pi \delta(s - p)\left( \frac{\hat{\hat{M}}(r, q)}{\hbar} + \frac{1}{\hbar} \text{sgn}(r)\delta(r - s) \right) \right].
\]

Note that the central terms are proportional to \( \frac{1}{\hbar} \).

The hamiltonian is \( \hat{H}_0 = \int [dp] h(p) \hat{\hat{M}}(p, p) \) In the case of interest to us, the operator \( h \) has an infinite number of eigenvectors with a negative eigenvalue: elements of \( \mathcal{H}^- \). It is clear that acting with \( \hat{M}(p, q) \) on any state will add \( p - q \) to its momentum. By operating with \( \hat{M}(p, q) \) with \( p < q \) repeatedly, we may be able to construct states of arbitrarily negative momentum (and hence energy). The only way to avoid this catastrophe is to eventually arrive at a state (‘vacuum’) \( |0 > \in \mathcal{F} \) such that

\( \hat{M}(p, q)|0 > = 0 \) for \( p < q \)

(133)

It is clear that \( \hat{M}(p, q) \) for \( p < q \) can be thought of as the negative roots of the Lie algebra of \( U_1 \): we have just shown that the representation of \( \mathcal{F} \) must be a lowest weight representation. If the representation is to be irreducible, the lowest weight vector must be unique. Also, every vector in \( \mathcal{F} \) can be written as linear combination of the vectors \( \hat{M}(p_1, q_1 \cdots \hat{M}(p_r, q_r)|0 > \) obtained from the vacuum by positive roots: \( p_1 > q_1 \cdots p_r > q_r \). The vacuum itself is an eigenvector of the diagonal generators:

\[
\hat{\hat{M}}(p, p)|0 > = w(p)|p >
\]

(134)
(To make the theory mathematically rigorous, we can suppose that space is a circle so that the momenta take on just discrete values. At the end the radius of this circle can be taken to infinity.) Lowest weight unitary representations are classified by the eigenvalues of the diagonal elements (lowest weight) [19]. If this representation of the Lie algebra can be exponentiated to one of the Lie group, the weights \( w(p) \) must be integers. Moreover, \( \frac{1}{\hbar} \) is an integer \( N_c \) for the central term to be exponentiated. (In fact the condition that there be a positive invariant inner product already requires \( w(p), N_c \) to be integers). The quantization of \( N_c \) is analogous to the quantization of the level number in the classification of lowest weight unitary representations of the affine Kac–Moody algebras. Indeed, we saw that \( \frac{1}{\hbar} \) appears only as a coefficient of the central term of the Lie algebra.

Now we are interested in representations in which the quadratic constraint on \( M \) are preserved in some sense. As in the finite dimensional case, we could simply impose the quadratic condition on the expectation value of \( \hat{M} \) on the lowest weight state. This will lead to the condition

\[
[hw(p) + \text{sgn}(p)]^2 = 1.
\] (135)

If we restrict to a topological sector where the baryon number is \( B \) we have the linear constraint,

\[
-\frac{1}{2} \sum_p \langle w|\hat{M}(p,p)|w\rangle = B \Rightarrow \sum_p [hw(p)] = B.
\] (136)

These conditions uniquely determine the lowest weight \( w(p) \) in terms of \( N_c \) and \( B \):

\[
w(p) = N_c[\text{sgn}(p-p_F) - \text{sgn}(p)]
\] (137)

the ‘Fermi momentum’ \( p_F \) being fixed to be \( p_F = B \).

This particular representation of the unitary Lie algebra can be written in terms of fermionic variables, just as we did in the finite dimensional case. Introduce the fermionic fields satisfying the canonical anti-commutation relations:

\[
[\chi^a(x), \chi^b_\dagger(y)]_+ = \delta^a_b \delta(x-y), \quad [\chi^a(x), \chi^b(y)]_+ = 0, \quad [\chi^a_\dagger(x), \chi^b_\dagger(y)]_+ = 0.
\] (138)
Here \( a, b = 1 \cdots N_c \) is the ‘color’ index. Then, we define the lowest weight state (vacuum of the free field theory) by

\[
\hat{\chi}^a(p)|0> = 0, \text{ for } p > p_F, \quad \hat{\chi}^\dagger_a(p)|0> = 0 \text{ for } p \leq p_F, \tag{139}
\]

where \( \hat{\chi} \) is the Fourier transform of \( \chi \). We can now verify that

\[
\hat{M}(x, y) = \frac{1}{N_c} : \hat{\chi}^\dagger_a(x) \chi^a(y) :
\]

satisfies the commutation relations above. The central terms arise from the fact that \( \hat{M} \) has to be normal ordered with respect to the above lowest weight state.

This representation of the unitary Lie algebra on the fermionic Fock space is reducible; in order to get an irreducible representation, we must impose the conditions

\[
Q^b_a |\psi> = 0 \tag{141}
\]
on all the physical states, \( Q^b_a = \int [ : \chi^\dagger_a(x) \chi^a(x) - \frac{1}{N_c} \delta^a_b \chi^c(x) \chi^c(x) ] dx \) being the ‘color charge’. On each subspace of color singlet states of fixed baryon number, our algebra has an irreducible representation. In terms of the fermion fields, baryon number is

\[
B = \frac{1}{N_c} \int : \chi^\dagger_a(x) \chi^a(x) : dx. \tag{142}
\]

Now we can express the hamiltonian in terms of \( \hat{M} \) and hence \( \chi \).

\[
H = \int dx dy h(x - y) \hat{M}(x, y) - \frac{1}{2} \bar{g}^2 \int dx dy G(x - y) \hat{M}(x, y) \hat{M}(y, x). \tag{143}
\]

As before, the operators \( h \) and \( G \) are defined by

\[
h = \frac{1}{2} (p + \frac{\mu^2}{p}); \quad G = \frac{1}{p^2}, \tag{144}
\]

where \( p = -i \frac{d}{dx} \). We have constructed the representation and the normal ordering of \( \hat{M} \) such that the first term is finite. The interaction term involving \( G(x - y) \) is also now finite.
If the coupling constant had been dimensionless as in the Thirring model, this would have been true only after a coupling constant renormalization.

Next we will show that the above quantization of the hadron theory is equivalent to two dimensional QCD. More precisely, we will show that it is equivalent to the color singlet sector of QCD. As part of this we will see that $N_c$ corresponds to the number of colors and $B$ to the baryon number of QCD.

We will find it convenient to formulate QCD in the light–cone type co–ordinates defined earlier:

$$ds^2 = du(du + 2dx), \quad u = x^0 - x^1, x = x^1. \quad (145)$$

The action of 2DQCD is

$$S = \int \left\{ -\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu b} + \bar{q}_a [\gamma^\mu (-i\delta^a_b \partial_\mu - gA_{\mu b}^a) - m\delta^a_b] q^b \right\} d^2x \quad (146)$$

We are considering the theory with just one flavor and $SU(N_c)$ colors. Thus $A_{\mu b}^a$ are traceless hermitean matrices, with $a, b = 1 \cdots N_c$. The generalization to several flavors is quite straightforward. The variables $q, \bar{q}$ are to be viewed as anti–commuting (Grassmann–valued) in a path integral approach.

The Dirac matrices satisfy the relations

$$(\gamma^u)^2 = 0, \quad (\gamma^x)^2 = -1, \quad [\gamma^u, \gamma^x]_+ = 2. \quad (147)$$

A convenient representation is

$$\gamma^u = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}, \quad \gamma^x = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (148)$$

It will will be convenient label the components of the quark fields as follows,

$$q = \frac{1}{\sqrt{2}} \begin{pmatrix} \chi \\ \eta \end{pmatrix}. \quad (149)$$
Then,
\[ \tilde{q} = q^\dagger \gamma^0 = \frac{1}{\sqrt{2}} \begin{pmatrix} \eta^\dagger & \chi^\dagger \end{pmatrix}. \]  
(150)

As for the gauge fields, it is convenient to choose the light–cone gauge, \( A_x = 0 \) and denote the remaining component by \( A_u = A \). With these choices, the action of the theory becomes,
\[ S = \int \left\{ -\frac{1}{2} \text{tr}(\partial_x A)^2 + \chi^\dagger \left[ -i \partial_u - g A \right] \chi + \frac{1}{2} \left[ -i \eta \partial_x \eta + i \chi^\dagger \partial_x \chi \right] + \frac{1}{2} m \left[ \chi^\dagger \eta + \eta^\dagger \chi \right] \right\} dx du. \]  
(151)

Now it is clear that the fields \( \eta \) and \( A \) do not propagate; their equations of motion can be used to eliminate them in favor of \( \chi \):
\[ -i \partial_x \eta + m \chi = 0, \quad \partial_x^2 A^a + g \chi^b \chi^a = 0. \]  
(152)

After this elimination, the action becomes
\[ S = -i \int \chi^a \partial_u \chi^a dx du - \int \chi^a \partial_u \chi^a dx du + \frac{g^2}{2} \int G(x - y) \rho^a_b(x) \rho^b_a(y) dx dy du. \]  
(153)

Here
\[ \rho^a_b(x) = \chi^\dagger_b(x) \chi^a(x) - \frac{1}{N_c} \delta^a_b \chi^\dagger_c(x) \chi^c(x) \]  
(154)
is the color charge density. Also, \( h \) and \( G \) are defined as before. We already begin to see the sort of expressions that appear in QHD.

The first term in the action implies that in canonical quantization, \( \chi \) and \( \chi^\dagger \) are conjugate; the remaining terms determine the Hamiltonian. The commutation relations are
\[ [\tilde{\chi}^a(x), \tilde{\chi}^b_\dagger(y)]_+ = \delta^a_b \delta(x - y), \quad [\chi^a(x), \chi^b(y)]_+ = 0, \quad [\chi^a_\dagger(x), \chi^b_\dagger(y)]_+ = 0. \]  
(155)

The Fock representation of these relations can now be constructed, based on the naive vacuum state
\[ \tilde{\chi}^a(p)|0 >= 0, \quad \text{for } p > 0, \quad \chi^a_\dagger(p)|0 >= 0 \quad \text{for } p \leq 0. \]  
(156)
The hamiltonian is

\[ H = \int :\chi_a^\dagger h \chi^a : dx + \frac{g^2}{2} \int G(x - y) : \rho^a_b(x) \rho^b_a(y) : dxdy. \] (157)

A normal ordering with respect to the above vacuum is necessary to make the hamiltonian of the quantum theory well-defined. It is now possible to write the hamiltonian in terms of the color singlet operator

\[ \hat{M}(x, y) = \frac{1}{N_c} :\chi_a^\dagger(x) \chi^a(y) :. \] (158)

We have already seen that this operator satisfies the commutation relations of QHD. Thus we see the operator $\hat{M}(x, y)$ is a quark–antiquark bilinear. We see that $N_c$ is the number of colors and $B = \int \hat{M}(x, x)$ the baryon number of QCD.

The kinetic energy term is already of the form $tr hM$. The interaction term also can be written in terms of $\hat{M}$ if we use the Fierz identity [11]. This will lead to precisely the hamiltonian of QHD. Since the calculation has been done elsewhere [6], [11], [4], we will not carry it out here. Thus we see that the theory we have been studying is equivalent to the color singlet sector of QCD. The parameters $\mu, \tilde{g}$ are related to the quark mass $m$ and gauge coupling constant $g$ by the formulae,

\[ \mu^2 = m^2 - \frac{g^2 N_c}{4\pi}, \quad \tilde{g}^2 = g^2 N_c. \] (159)

It is to be noted that although the parameter $m$ in the 2DQCD lagrangian is traditionally called the quark mass, the quark is not asymptotic particle in the theory and has no well–defined mass. If the scattering of color singlet particles produce only color singlet particles, the S–matrix of QHD will be unitary. This would amount to a proof of confinement of QCD. It seems likely that such a proof can be made to all orders in the $\frac{1}{N_c}$ expansion.

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Appendix: Hartree–Fock Theory and Grassmannians

1. Slater Determinants and Grassmannians

In this appendix we will show the analogue of our approach to 2DQCD, in electronic physics is just the Hartree–Fock approximation. It is hoped that our point of view in terms of Grassmannians to Hartree–Fock theory is of some use in atomic and condensed matter physics.

Let $\mathcal{H} = L^2(R^3, C^2)$ be the one particle Hilbert space of electrons; we will think of its elements as complex valued functions $u(x, \sigma)$ of position $x \in R^3$ and spin $\sigma = \pm 1$. The Hilbert space of an $m$-electron system is the exterior power $\Lambda^m(\mathcal{H})$. A wavefunction in this space is a completely antisymmetric function $\psi(x_1, \ldots, x_n)$. The hamiltonian of the $m$ electron system is the following operator on $\Lambda^m(\mathcal{H})$:

$$H = \sum_{a=1}^{m} [-\nabla^2_a + V(x_a)] + \sum_{1 \leq a < b \leq m} G(x_a - x_b).$$

(160)

For an atom, $V(x) = -\frac{Z}{|x|}$ describing the Coulomb attraction to the nucleus and $G(x - y) = \frac{1}{|x - y|}$. (We are using units with $\hbar^2 = 1$ and $e = 1$.) The ground state is the minimum of the expectation value of the hamiltonian over all states of unit norm:

$$E_0 = \inf_{||\psi||=1} (\psi, H\psi).$$

(161)

If $||\psi|| = 1$, the expectation value of $H$ can be written as

$$\langle \psi | H | \psi \rangle = \text{tr}_1 \rho_1 + \text{tr}_2 \rho_2$$

(162)

where $h = -\nabla^2 + V$ and the ‘one–particle density matrix’ $\rho_1$ is a selfadjoint operator $\mathcal{H}$ with kernel

$$\rho_1(x, y) = m \int \psi^*(x, x_2, \ldots, x_m) \psi(y, x_2, \ldots, x_m) dx_2 \cdots dx_m.$$
Also $G_2$ and $\rho_2$ are selfadjoint operators on the two particle Hilbert space $\Lambda^2(\mathcal{H})$ with kernels

$$
\rho_2(x, y; z, u) = \left(\frac{m}{2}\right) \int \psi^*(x, y, x_3, \cdots x_m) \psi(z, x_3, \cdots, x_m) dx_3 \cdots dx_m \quad (164)
$$

$$
G_2(x, y; z, u) = \frac{1}{2} [\delta(x - z)\delta(y - u) - \delta(x - u)\delta(y - z)] G(x - y). \quad (165)
$$

The one and two particle density matrices are normalized as below:

$$
\text{tr} \rho_1 = \int \rho_1(x, x) dx = m, \quad \text{tr} \rho_2 = \int \rho_2(x, y; x, y) dxdy = \left(\frac{m}{2}\right). \quad (166)
$$

The problem of finding the ground state of this system is very hard. The idea of the Hartree–Fock theory is to approximate the wavefunction by a Slater determinant

$$
\psi(x_1 \cdots x_m) = \frac{1}{\sqrt{m!}} \begin{vmatrix}
    u_1(x_1) & \cdots & u_1(x_m) \\
    \vdots & \ddots & \vdots \\
    u_m(x_1) & \cdots & u_m(x_m)
\end{vmatrix}. \quad (167)
$$

The one particle wavefunctions $u_a \in \mathcal{H}$ are orthonormal $(u_a, u_b) = \delta_{ab}$. A change $u_a(x) \to g_{ab} u_b(x)$ will change the Slater determinant only by a phase $\det g$, so it doesn’t change the physical state represented by it.

The Hartree–Fock approximation to the ground state energy is the minimum over all choices of $u$ of the expectation value $E^{HF}(u)$ of the hamiltonian in this state:

$$
E_0^{HF} = \inf_{(u_a, u_b) = \delta_{ab}} E^{HF}(u). \quad (168)
$$

Clearly, $E_0 \leq E_0^{HF}$. $E^{HF}$ can be worked out to be:

$$
E^{HF}(u) = \sum_{a=1}^{m} \int dx \int dxdy u_a^*(x) [-\nabla^2 + V(x)] u_a(x) + \\
\sum_{a < b} \int dxdy G(x - y) \frac{1}{\sqrt{2}} [u_a^*(x)u_b^*(y) - u_b^*(x)u_a^*(y)] \frac{1}{\sqrt{2}} [u_a(x)u_b(y) - u_b(x)u_a(y)].
$$
However, it is somewhat unnatural to think of the $u_a(x)$ as the variational parameters of Hartree–Fock theory; a change $u_a(x) \rightarrow g_a^b u_b(x)$ by a unitary matrix does not affect the multiparticle state. In fact the larger $m$ is, the larger the number of such spurious degrees of freedom in $u_a$. In quantum field theory, $m$ is infinity, this is particularly awkward. We will therefore develop a new point of view in terms of Grassmannians which has a natural generalization to quantum field theory. Perhaps this point of view is useful even in atomic physics.

To get a more intrinsic form for $E^{HF}$, let us compute $\rho_1$ and $\rho_2$ for this ansatz. We will get

$$
\rho_1(x,y) = P(x,y); \quad \rho_2(x,y; z, u) = \frac{1}{2} [P(x,z)P(y,u) - P(x,u)P(y,z)]
$$

where

$$
P(x,y) = \sum_{a=1}^{m} u^*_a(x) u_a(y).
$$

$P$ is the projection operator to the $m$–dimensional subspace spanned by $u_a$. The expression for $\rho_2$ in terms of $P$ can be written as

$$
\rho_2 = P \wedge P
$$

where the wedge product of two operators $A, B$ on $\mathcal{H}$ is the operator on $\Lambda^2(\mathcal{H})$

$$
(A \wedge B)(u,v) = (Au) \wedge (Bv).
$$

Thus we get

$$
E^{HF}(P) = \text{tr} h P + \text{tr} G_2 P \wedge P.
$$

In coordinate basis,

$$
E^{HF}(P) = \int dx \left[ \frac{\partial^2 P(x,y)}{\partial x \partial y} \right]_{x=y} + V(x) P(x,x) + \\
\frac{1}{2} \int dxdy G(x-y) [P(x,x)P(y,y) - P(x,y)P(y,x)]
$$
The first term represents the kinetic energy and the second term the potential energy due to the nucleus. The third term is the energy of Coulomb repulsion of the electrons (‘direct energy’). The last term is the ‘exchange energy’ due to the antisymmetry of the wavefunctions. (The fact that Hartree–Fock theory can be expressed in this way is well known [20]). The Hartree–Fock ground state energy is the minimum of $E^{HF}(P)$ over all projection operators of rank $m$:

$$E_{0}^{HF} = \inf_{P^2=P, \text{tr}P=m} E^{HF}(P). \quad (174)$$

The set of all projection of operators of finite rank defines an infinite dimensional analogue of the Grassmannian

$$Gr(\mathcal{H}) = \{P|P^\dagger = P; P^2 = P; P \text{ is finite rank}\}. \quad (175)$$

Then $\text{tr}P$ exists and is an integer; the Grassmannian is a union of connected components labelled by the trace; each connected component is the homogenous space

$$Gr_m(\mathcal{H}) = U(\mathcal{H})/U(m) \times U(\mathcal{H}) \quad (176)$$

just as the finite dimensional case. $Gr_m(\mathcal{H})$ is an infinite dimensional manifold. Each point in the Grassmannian represents an $m$ dimensional subspace of $\mathcal{H}$, the eigenspace of $P$ with eigenvalue 1. There is an embedding of $Gr_m(\mathcal{H})$ into $\mathcal{P}(\Lambda^m \mathcal{H})$, (generalization of the Plucker embedding) which is precisely the meaning of the Slater determinant. To construct this analogue of the Plucker embedding, pick an orthonormal basis in the subspace of $P$:

$$P = \sum_{a} u_a \otimes u_a^\dagger \quad (177)$$

and define the corresponding point in $\Lambda^m \mathcal{H}$ to be

$$u_1 \wedge u_2 \wedge \cdots w_m. \quad (178)$$

* We can require $P$ to be compact rather than finite rank; any compact projection operator is also finite rank.
Again, this vector changes by the phase det $g$ as we change the orthonormal basis. The Slater determinant is precisely this wedge product written in the position basis. There is a one–one correspondence between states of the Slater determinant type and points on the Grassmannian; there are no longer any spurious variables in the problem. The ground state just corresponds to the minimum of $E_{HF}(P)$ on the Grassmannian.

This point of view can be used perhaps to prove the existence of extrema (critical points) for the Hartree–Fock energy. We make a digression to point out a connection to Morse theory. $Gr_m(\mathcal{H})$ is the well known model for the Classifying Space [21] of $U(m)$. So its homotopy type is particularly simple. Also, the cohomology of $Gr_m(\mathcal{H})$ has a simple description; any element can be written as a linear combination of wedge products of the forms

$$\omega_{2i} = \text{tr}P(dP)^{2i} \quad \text{for} \quad i = 1, 2 \cdots \leq m. \quad (179)$$

Furthermore there are no relations among these generators. (There would be some relations among these generators if were considering instead the Grassmannian of a finite dimensional vector space.) The odd cohomology groups vanish. The number of generators of $H^{2k}$ is equal to the number of ways $k$ written as the sum of the numbers $1, 2, \cdots m$. Thus the generating function of the Betti numbers (‘Poincare polynomial’) is

$$P(t) := \sum_{p=0}^{\infty} \dim H^p t^p = \prod_{i=0}^{m} \frac{1}{1 - t^{2i}} \quad (180)$$

If $h$ and $G_2$ are reasonable operators on $\mathcal{H}$ and $\Lambda^2 \mathcal{H}$, $\mathcal{E}$ will be a differentiable function on $Gr_m(\mathcal{H})$. Then Morse theory would guarantee the existence of extrema for $\mathcal{E}$. The minimum would be the Hartree–Fock approximation to the ground state; the other extrema of finite index will represent excited states. This approach is not possible if we think of the HF energy as a function of $u_a$. For, the space of such orthonormal frames on $\mathcal{H}$ (the Stiefel manifold of $\mathcal{H}$) is contractible; Morse theory would be trivial. There should also be, in general, a continuum in the case of atomic physics, corresponding to the scattering states.
One can enclose the system in a large box so that all states are discrete and then the energy function will only have isolated critical points.

The weak Morse inequalities say that the number of critical points of index $p$, $N_p$, is greater than or equal to $\dim H^p$. Thus $N_{2k}$ is greater than or equal to the number of ways in which $k$ can be written as a sum of the numbers $\{1, \cdots, m\}$. One can see that this agrees with the counting based on the interpretation of the excited states of an atom in terms of electron–hole pairs. The index is the number of directions in which the energy of a state decreases, which is related to the number of states with energy less than the given state.

2. Second Quantized approach to Hartree–Fock Theory

It is clear that mean field theory is a sort of semi–classical approximation to the atomic system: the quantum fluctuation are small. So there must be some sense in which Hartree–Fock theory describes a classical theory whose quantization gives the atomic hamiltonian. However this cannot be the conventional classical theory of particles moving in a Coulomb potential, since the atom has no stable ground state in that approach. There must be some other classical theory, with a stable ground state, whose quantization also leads to the atomic hamiltonian. We will show that this is a system whose phase space is the Grassmannian, and that time–dependent Hartree–Fock theory is equivalent to hamiltonian dynamics on this space.

To begin with let us rewrite the problem in second quantized language. Define the creation–annihilation operators associated to the Hilbert space $\mathcal{H}$:

$$[a^\dagger(x), a(y)]_+ = \delta(x - y); \quad [a^\dagger(x), a^\dagger(y)]_+ = 0; \quad [a(x), a(y)]_+ = 0. \quad (181)$$

There is a representation of this algebra on the Fock space $\mathcal{F} = \bigoplus_{m=0}^{\infty} \Lambda^m(\mathcal{H})$. The vacuum
is the state containing no electrons

\[ a(x)|0 >= 0. \]  \hspace{1cm} (182)

The number operator is

\[ N = \int a^\dagger(x)a(x)dx \]  \hspace{1cm} (183)

and its eigenspace with eigenvalue \( m \) is \( \Lambda^m(\mathcal{H}) \). The elements of \( \Lambda^m(\mathcal{H}) \) are of the form

\[ |\psi >= \int \psi(x_1, \ldots, x_m)a^\dagger(x_1) \cdots a^\dagger(x_m)dx_1 \cdots dx_m|0 >. \]  \hspace{1cm} (184)

In this language, the Hamiltonian is the operator on \( \mathcal{F} \)

\[ \hat{H} = \int a^\dagger(x)ha(x)dx + \int dx dy G(x - y)a^\dagger(x)a^\dagger(y)a(y)a(x). \]  \hspace{1cm} (185)

The relation to the earlier language is,

\[ \hat{H}|\psi >= |H\psi >. \]  \hspace{1cm} (186)

Now suppose we consider a generalization of this atomic physics problem where each electron has another quantum number (‘color’) \( \alpha = 1, \ldots, N_c \). We have the operators satisfying

\[ [a^\dagger_\alpha(x), a_\beta(y)]_+ = \delta_\beta^\alpha \delta(x - y); \quad [a^\dagger_\alpha(x), a^\dagger_\beta(y)]_+ = 0; \quad [a_\alpha(x), a_\beta(y)]_+ = 0. \]  \hspace{1cm} (187)

There is again a representation of this on the Fock space. But we will allow as physical states only those annihilated by

\[ Q_\beta^\alpha = \int dx [a^\dagger_\alpha(x)a_\beta(x) - \frac{1}{N_c}a^\dagger_\gamma(x)a_\gamma(x)]. \]  \hspace{1cm} (188)

These operators generate an \( SU(N_c) \) symmetry. Let us denote this ‘color invariant’ subspace by \( \mathcal{F}_0 \):

\[ |\psi > \in \mathcal{F}_0 \iff Q_\beta^\alpha|\psi >= 0. \]  \hspace{1cm} (189)
Generalize the hamiltonian to the operator,

\[
\hat{H} = \frac{1}{N_c} \int a^\dagger(x) h a_\alpha(x) \, dx + \frac{1}{2N_c^2} \int dx dy G(x - y) \nonumber
\]

\[
[a^\dagger(x) a^\dagger(y) a_\beta(y) a_\alpha(x) + a^\dagger(x) a^\dagger(y) a_\alpha(y) a_\beta(x)].
\]

This is a generalization of the hamiltonian familiar from atomic physics. It is clear that if \(N_c = 1\), the color singlet condition on the states becomes trivial and the hamiltonian reduces to the previous one. This theory of colored fermions is of physical interest only for \(N_c = 1\). Yet, we will show that Hartree–Fock (mean field) approximation is the large \(N_c\) limit of this theory. Expansion in powers of \(1/N_c\) will yield the expansions around mean field theory. Although \(N_c = 1\) in the physical case, we know that this \(1/N_c\) expansion is in fact a good approximation method in atomic physics. Thus we will see that the conventional mean field theory can be understood in terms of a ‘replica trick’.

To see this, introduce the operators

\[
\hat{P}(x, y) = \frac{1}{N_c} a^\dagger_i(x) a_i(y).
\]

They satisfy the commutation relations

\[
[\hat{P}(x, y), \hat{P}(z, u)] = \frac{1}{N_c} [\delta(y - z) \hat{P}(x, u) - \delta(u - x) \hat{P}(z, y)]
\]

of an infinite dimensional unitary Lie algebra.\footnote{Strictly speaking we should regard the Lie algebra as consisting of the smeared objects \(\int dx dy \hat{P}(x, y) K(x, y)\) where \(K\) is the Kernel of a \textit{compact} operator. Then there will be no divergence problems in the representation theory of this algebra.} The trace \(\int \hat{P}(x, x) \, dx\) is the number operator divieigenvalues of this operator are positive integers; let \(\mathcal{F}_{0m}\) be the set of all states with eigenvalue \(m\).

The operators \(\hat{P}(x, y)\) forms a complete system of observables on \(\mathcal{F}_{0m}\). i.e., any operator that commutes with \(\hat{P}(x, y)\) for all \(x, y\) is a multiple of the identity. This follows
from Schur’s lemma if we can see that the representation of the above Unitary algebra on \( \mathcal{F}_{0m} \) is irreducible. In fact the irreducible representations of the Lie algebra of compact selfadjoint operators (and the corresponding group of unitary operators that differ from the identity by a compact operator) has been worked out by Kirillov. The representations are classified by Young tableaux, just as in the finite dimensional representation theory. Now it is straightforward to verify that the representation on \( \mathcal{F}_{0m} \) is the one with a rectangular Young diagram with \( N_c \) columns and \( m \) rows.

Although the \( \hat{P}(x,y) \) form a complete set of observables on \( \mathcal{F}_{0m} \), they are not all independent of each other; they satisfy a quadratic constraint. To see this, first define the normal ordered product of two \( \hat{P} \)'s:

\[
: \hat{P}(x,z) \hat{P}(y,u) := \hat{P}(x,z) \hat{P}(y,u) - \frac{1}{N_c} \delta(y-z) \hat{P}(x,u).
\]

(192)

In terms of the creation–annihilation operators, this means that all the creation operators stand to the left of the annihilation operators:

\[
: \hat{P}(x,z) \hat{P}(y,u) := -\frac{1}{N_c^2} a^{\dagger \alpha}(x) a^{\dagger \beta}(y) a_{\alpha}(z) a_{\beta}(u).
\]

(193)

The normal ordered product will have finite matrix elements in the limit \( z \to y \).

Between a pair of elements \(|\psi\rangle, |\psi'\rangle \) of \( \mathcal{F}_{0m} \), these operators satisfy the identity

\[
<\psi'| \int : \hat{P}(x,y) \hat{P}(y,u) : dy |\psi> = <\psi'| \left( 1 - \frac{m}{N_c} \right) \hat{P}(x,u) |\psi>.
\]

(194)

The proof is to note that

\[
<\psi'| \int : \hat{P}(x,y) \hat{P}(y,u) dy : |\psi> = \frac{1}{N_c} \int <\psi'| a^{\dagger \alpha}(x) a^{\dagger \beta}(y) a_{\alpha}(y) a_{\beta}(u) |\psi> dy
\]

\[
= -\frac{1}{N_c} \int <\psi'| a^{\dagger \alpha}(x) a_{\beta}(u) a^{\dagger \beta}(y) a_{\alpha}(y) |\psi> dy
\]

\[
+ \frac{1}{N_c} \delta^\beta_\beta <\psi'| a^{\dagger \alpha}(x) a_{\alpha}(u) |\psi>
\]

\[
= \left( 1 - \frac{m}{N_c} \right) <\psi'| \hat{P}(x,u) |\psi>.
\]
We have used
\[
\int dy a^\dagger_\gamma(y)a_\gamma(y)\psi > = -\frac{1}{N_c} \delta^\beta_\alpha \int dy a^\dagger_\gamma(y)a_\gamma(y)\psi > = m\delta^\beta_\alpha \psi >
\] (195)
for \( \psi \in \mathcal{F}_{0m} \).

Now we can reformulate the theory entirely in terms of the color singlet variables \( \hat{P} \).

The hamiltonian is
\[
\hat{H} = \int dx \left( \frac{\partial^2 \hat{P}(x,y)}{\partial x \partial y} \right)_{x=y} + V(x)\hat{P}(x,x) + \\
\frac{1}{2} \int dxdy G(x-y) [\hat{P}(x,x)\hat{P}(y,y) - :\hat{P}(x,y)\hat{P}(y,x):].
\]

The normal ordering is necessary to avoid self–energy terms.

Now we see that with our definitions, \( N_c \) appears only as a coefficient of the commutation relations of the \( \hat{P}(x,y) \). Thus \( \frac{1}{N_c} \) plays the role of \( \hbar \); it determines the uncertainty in measuring the different components of \( \hat{P}(x,y) \) simultaneously. There are certain states in the space \( \mathcal{F}_{0m} \) that minimize this uncertainty; they are the analogue of the minimum uncertainty wavepackets (coherent states) of the usual canonical quantization. An invariant measure of the uncertainty ( or the size of quantum fluctuations) is
\[
\int [<\psi|:\hat{P}(x,y)\hat{P}(y,x):|\psi> - <\psi|\hat{P}(x,y)|\psi><\psi|\hat{P}(y,x)|\psi>]dxdy.
\] (196)

The states that minimize are the highest weight states of some basis. Explicitly, they are of the form
\[
|u> = \int dx_1 \cdots dx_m u_1(x_1) \cdots u_m(x_m) B^\dagger(x_1) \cdots B^\dagger(x_m)|0>
\] (197)
where
\[
B^\dagger(x) = a^\dagger_1(x) \cdots a^\dagger_{N_c}(x).
\] (198)

Clearly \( B^\dagger(x) \) createSince the quantum fluctuation around the expectation values vanish in these states, they will have a good classical limit.
In the limit $N_c = 1$ these are precisely the states represented by the Slater determinant. More generally, they describe an embedding of the Grassmannian $Gr_m(\mathcal{H})$ into the projective space $\mathcal{P}(\mathcal{F}_0^m)$, generalizing the Plucker embedding. We can now calculate the expectation value of $\hat{P}$:

$$< u | \hat{P}(x, y) | u > = \sum_a u^*_a(x) u_a(y) := P(x, y).$$

(199)

This is just the projection operator to the subspace spanned by the $u_a$. Furthermore, the expectation value of the Hamiltonian is

$$< u | \hat{H} | u > = \int dx \left[ \left( \frac{\partial^2 P(x, y)}{\partial x \partial y} \right)_{x=y} + V(x) P(x, x) \right] + \frac{1}{2} \int dxdy G(x-y) [P(x,x)P(y,y) - P(x,y)P(y,x)].$$

Note that so far we have not made any approximations: these are the exact expectation values. Since these Slater–type states are not the exact eigenfunctions of $\hat{H}$, time evolution will take them into more complicated states.

However, in the large $N_c$ (semiclassical) limit the deviations will be small and we will be able to describe the system completely in terms of the classical variable $P(x, y)$. The classical phase space is the set of all projection operators of rank $m$; i.e., the Grassmannian $Gr_m(\mathcal{H})$. The commutation relations of the variables $\hat{P}$ tend to Poisson brackets of the classical variables:

$$\{P(x,y), P(z, u)\} = i[\delta(y-z)P(x,u) - \delta(u,x)P(z,y)].$$

(200)

These are precisely the ones that follow from the standard symplectic form on the Grassmannian:

$$\omega = -\frac{i}{8} \text{tr} P(dP)^2$$

(201)

Thus we see that the limit $N_c \to \infty$ of our generalized atomic system is a classical
dynamical system with $Gr_m(\mathcal{H})$ as phase space, $\omega$ as symplectic form, and
\[
E^{HF}(P) = \int dx \left( \frac{\partial^2 P(x,y)}{\partial x \partial y} \right)_{x=y} + V(x)P(x,x) + \frac{1}{2} \int dxdy G(x-y) [P(x,x)P(y,y) - P(x,y)P(y,x)]
\]
as the hamiltonian. In particular, the ground state is determined by the minimum of $E^{HF}$. Also the excited states of the system correspond to stationary points of higher index, as anticipated earlier.

Our derivation shows that time dependent Schrodinger equation tends in the limit $N_c \to \infty$ to the equations
\[
\frac{dP(x,y)}{dt} = \{P(x,y), E^{HF}(P)\}
\]
as the equations of motion. This could be an interesting approach to calculating the scattering amplitudes of atoms by other atoms. The small oscillations around a stationary point will have some characteristic frequencies. These will determine the correction to order $\frac{1}{N_c}$ of the energy levels. It is clearly possible to develop a systematic semiclassical expansion in $\frac{1}{N_c}$.

Thus we have a classical system whose quantization gives the atomic hamiltonian. In fact, quantization can be performed by finding an irreducible representation of the algebra above. The parameter $N_c$ (effectively $\frac{1}{\hbar}$) must be a positive integer in order that a highest weight representation exist. The physically interesting value happens to be $N_c = 1$. The success of Hartree–Fock theory shows that even in this extreme case, the $\frac{1}{N_c}$ expansion gives reliable results.

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