PERMUTADS VIA OPERADIC CATEGORIES,
AND THE HIDDEN ASSOCIAHEDRON

MARTIN MARKL

Abstract. The present article exploits the fact that permutads (aka shuffle algebras) are algebras over a terminal operad in a certain operadic category $\text{Per}$. In the first, classical part we formulate and prove a claim envisaged by Loday and Ronco that the cellular chains of the permutohedra form the minimal model of the terminal permutad which is moreover, in the sense we define, self-dual and Koszul. In the second part we study Koszulity of $\text{Per}$-operads. Among other things we prove that the terminal $\text{Per}$-operad is Koszul self-dual. We then describe strongly homotopy permutads as algebras of its minimal model. Our paper shall advertise analogous future results valid in general operadic categories, and the prominent rôle of operadic (op)fibrations in the related theory.

Contents

Motivations and the main results 2

Part 1. Classical approach to permutads 3

1. Permutads – recollections. 3
2. Koszul duals of permutads 7
3. Koszulity of permutads 8

Part 2. Operadic category approach 12

4. Permutads as algebras over the terminal operad 12
5. Free $\text{Per}$-operads. 16
6. Minimal model of the terminal $\text{Per}$-operad and homotopy permutads 20
7. Koszul duals of $\text{Per}$-operads 24
8. Koszulity of $\text{Per}$-operads. 27

References 30

2010 Mathematics Subject Classification. 18D50 (Primary), 18D20, 18D10 (Secondary).
Key words and phrases. Operadic category, permutad, shuffle algebra, opfibration, Koszulity.
Supported by Praemium Academiae, grant GA ČR 18-07776S and RVO: 67985840.
Motivations and the main results

It is well known \([7, \text{Example 4.8}]\) that the cellular chain complex \(CC_* (\mathcal{K})\) of the Stasheff associahedron is the minimal model of the non-\(\Sigma\)-operad \(\text{Ass}\) for associative algebras. It moreover turns out that \(CC_* (\mathcal{K})\) is isomorphic to the bar construction of the Koszul dual operad \(\text{Ass}'\), proving that \(\text{Ass}\) is Koszul. Since \(\text{Ass}\) is the terminal non-\(\Sigma\)-operad, one may formulate the above observations concisely as the

**Fact.** *The cellular chain complex of the Stasheff’s associahedron is the minimal model of the terminal non-\(\Sigma\)-operad. This terminal operad is moreover quadratic and Koszul.*

The original humble aim of this note was to formulate and prove an analog of this Fact for permutads, using indications provided by J.-L. Loday and M. Ronco in \([4]\). We however decided to extend it to an advertisement for the theory of operadic categories developed in \([2]\) and \([1]\), using permutads as an excuse.

**Part 1**, independent of the theory of operadic categories, starts by recalling, following \([4]\), permutads and the related notions. We then introduce quadratic permutads and their Koszul duals and formulate the Koszulity property for quadratic permutads in terms of a suitably defined dual bar construction. We close Part 1 by a prermutadic analog of the Ginzburg-Kapranov power series test \([3, \text{Theorem 3.3.2}]\). The results of the first part, namely Proposition \(16\) and Theorem \(19\) combined with \([4, \text{Proposition 5.4}]\), give the following wished-for permutadic analog of the Fact above:

**Theorem A.** *The cellular chain complex of the permutohedron is the minimal model of the terminal permutad. This terminal permutad is quadratic Koszul.*

An important feature of the dual bar construction \(D(A)\) of a permutad \(A\) introduced in Definition \(15\) is that it is again a permutad. This self-duality is not automatic. For instance, the dual bar construction of a commutative associative algebra is a Lie algebra, the dual bar construction of a modular operad is a twisted modular operad, &c. Explaining this feature requires operadic categories, which are the subject of

**Part 2.** As shown in \([1, \S14.4]\), there exists an operadic category \(\text{Per}\) such that permutads are algebras, in the sense of \([2, \text{Definition 1.20}]\), over the terminal \(\text{Per}\)-operad \(1\text{Per}\); we formulate this statement with a slightly different proof as Proposition \(22\). The self-duality of the category of permutads follows from Theorem 14.4 of \([1]\) which says that the \(\text{Per}\)-operad \(1\text{Per}\) governing prermutads is binary quadratic and self-dual. We will analyze this phenomenon in the context of general operadic categories in our future work. We complement these results by proving:

---

1. Sundry definitions of the permutohedron are assembled in Appendix 2 to \([4]\).
2. \(i\)e. the bar construction applied to the linear dual.

[March 18, 2019]
Theorem B. The terminal $\text{Per}$-operad $1_{\text{Per}}$ is Koszul.

As in the case of ‘classical’ operads, Koszulity of $1_{\text{Per}}$ leads to an effective description of its minimal model $\mathcal{M}$ which we give in the proof of Theorem 24. Algebras over $\mathcal{M}$ are, due to the philosophy pioneered in [7], strongly homotopy permutads. Their explicit description in terms of operations and axioms is given in Proposition 32.

Where the associahedron hides? The present article shall also illustrate the potential of Grothendieck’s construction in operadic categories and the related discrete opfibrations [1, Subsection 5.2]. Let $\Delta_{\text{semi}}$ be the operadic category of finite ordinals and their order-preserving surjections; recall that $\Delta_{\text{semi}}$-operads are ordinary constant-free non-$\Sigma$-operads. It turns out that the operadic category $\text{Per}$ is the Grothendieck’s construction applied to a certain $\Delta_{\text{semi}}$-cooperad $C$. One thus has a discrete opfibration $\text{des} : \text{Per} \to \Delta_{\text{semi}}$.

A consequence is that the restrictions along $\text{des} : \text{Per} \to \Delta_{\text{semi}}$ of free non-$\Sigma$-operads are free $\text{Per}$-operads. In particular, the restriction $\text{des}^\ast(K)$ of Stasheff’s operad of the associahedra turns out to be the convex polyhedral realization of the minimal model $\mathcal{M}$ of the terminal $\text{Per}$-operad $1_{\text{Per}}$, cf. Remark 34.

In Theorem 44 we prove that the restriction along $\text{des} : \text{Per} \to \Delta_{\text{semi}}$ brings Koszul non-$\Sigma$ operads to Koszul $\text{Per}$-operads. This is a particular case of an important feature of opfibrations between operadic categories, and an advertisement for our future work.

Conventions. Our background monoidal category will be the category $\text{Vec}$ of differential graded, or dg for short, vector spaces $V = \bigoplus_{k \in \mathbb{Z}} V_k$ over a field $k$ of characteristic 0; the preferred degree of differentials will be $-1$. The linear duals are taken degree-wise, i.e. the degree $k$ component of the dual $V^\ast$ of a graded space above will be $\text{Vec}(V_k, k)$, $k \in \mathbb{Z}$.

If not stated otherwise, all algebra-like objects (monoidal categories, permutads) will be nonunital. Operadic categories, their operads and algebras over these operads were introduced in [2, §I.1]. The standard reference for ‘classical’ operads, quadratic duality and Koszulness is [8] or more recent [1] or [11].

Acknowledgment. I wish to express my gratitude to Vladimir Dotsenko for presenting permutads in a broader context to me.

Part 1. Classical approach to permutads

1. Permutads – recollections.

Permutads appear under various names in various disguises. The most concise definition uses the skeletal category $\text{Fin}$ of finite non-empty sets with objects the ordinals $n := \{1, \ldots, n\}$, $n \geq 1$, and its lluf subcategory $\text{Sur}$ of surjections. For any order-preserving
surjection $f : A \to B$ of finite ordered sets there exists a unique $\alpha : n \to k \in \text{Surj}$ in the commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow \cong & & \downarrow \cong \\
n & \xrightarrow{f} & k
\end{array}
\]

where the vertical arrows are order-preserving isomorphisms. In this sense, every order-preserving surjection $f : A \to B$ of finite ordered sets may be interpreted as a morphism in $\text{Surj}$. For $r : m \to n \in \text{Surj}$ and $i \in n$ we denote by $r^{-1}(i)$ the pullback

\[
\begin{array}{c}
r^{-1}(i) \\
\downarrow 1 \to i \\
n
\end{array}
\]

in $\text{Fin}$. Notice that $r^{-1}(i) = c$, with $c \geq 1$ the cardinality of the set-theoretic preimage of $i$ under $r$. Permutads live in the category $\text{Coll}$ of collections $A = \{ A(n) \in \text{Vec} \mid n \in \text{Fin} \}$, which we consider as a (non-unital) monoidal category with the product

\[
(A \otimes B)(n) := \bigoplus_{r \in \text{Surj}(n,2)} \{ A(r^{-1}(1)) \otimes A(r^{-1}(2)) \}.
\]

**Definition 1.** A permutad, also called a shuffle algebra, is a monoid for the monoidal product (2).

**Remark 2.** The term shuffle algebra comes from the fact that surjections $r$ in (2) are in one-to-one correspondence with $(n,n-i)$-unshuffles, $1 < i < n$ or, which is the same, with ordered decompositions of the set $\{1,\ldots,n\}$ into two disjoint non-empty subsets. The correspondence assigns to $r$ the subsets

\[
\{ j \mid r(j) = 1 \}, \{ j \mid r(j) = 2 \}
\]

of $\{1,\ldots,n\}$. The related unshuffle is then the unique permutation

\[
v_r : \{ j \mid r(j) = 1 \} \cup \{ j \mid r(j) = 2 \} \longrightarrow \{1,\ldots,n\}
\]

whose restriction to the subsets in (3) is order-preserving. For the future use we denote

\[
r_i = \text{card}\{ j \mid r(j) = i \}, \ i = 1,2, \ \text{and} \ \varepsilon(r) := \text{signum}(v_r) \in \{-1,+1\}.
\]

To make Definition 1 explicit, we introduce some notation. For surjections $t : m \to 2$ and $s : t^{-1}(2) \to 2$ we denote by $t(1 \otimes s) : m \to 3$ the surjection defined by the commutativity of the diagrams

\[
\begin{array}{ccc}
t^{-1}(1) & \longrightarrow & m \\
\downarrow \iota_1 & & \downarrow t(1 \otimes s) \\
1 & \longrightarrow & 3
\end{array} \quad \text{and} \quad \begin{array}{ccc}
t^{-1}(2) & \longrightarrow & m \\
\downarrow \iota_2 & & \downarrow t(1 \otimes s) \\
2 & \longrightarrow & 3
\end{array}
\]
where \( \nu_1(1) := 1, \nu_{2,3}(1) := 2 \) and \( \nu_{2,3}(2) := 3 \). Analogously we define, for surjections \( u : m \to 2 \) and \( v : u^{-1}(1) \to 2 \), the surjection \( u(v \otimes 1) : m \to 3 \); the details are left to readers. With these definitions at hand we may formulate the following biased

**Definition 3.** A permutad is a collection \( A \in \text{Coll} \) together with operations

\[
\diamond_r : A(r^{-1}(1)) \otimes A(r^{-1}(2)) \to A(n)
\]

defined for each surjection \( r \in \text{Surj}(n, 2) \). These operations shall satisfy

\[
\diamond_u (\diamond_v \otimes 1) = \diamond_t (1 \otimes \diamond_s)
\]

for each \( m \geq 1 \) and surjections \( u, t : m \to 2 \), \( s : t^{-1}(1) \to 2 \) and \( v : u^{-1}(2) \to 2 \) such that

\[
t(s \otimes 1) = u(1 \otimes v).
\]

Definition 3 suggests that permutads are close in nature to operads. One thus sometimes uses, as in \[4\], a shifted grading \( A_{n+1} := A(n) \), \( n \geq 1 \). The underlying collection \( A_2, A_3, \ldots \) of a permutad is then the same as the underlying collection of a 1-connected non-unital non-symmetric (non-\( \Sigma \)) operad.

Let us recall from \[4\] a monadic definition of permutads. For a collection \( B \in \text{Coll} \), define \( P(B) \in \text{Coll} \) by

\[
P(B) := \bigoplus_{k \geq 1} P^k(B),
\]

where

\[
P^k(B)(n) := \bigoplus_{r \in \text{Surj}(n, k)} \bigotimes_{i \in k} B(r^{-1}(i)).
\]

**Remark 4.** Assume that \( A \) is a permutad as in Definition 3. Since clearly \( P^1(A) \cong A \) and

\[
P^2(A) \cong \bigoplus_{r \in \text{Surj}(n, 2)} A(r^{-1}(1)) \otimes A(r^{-1}(2)),
\]

structure operations (5) assemble into one map

\[
\diamond : P^2(A) \longrightarrow P^1(A).
\]

As shown in \[4\], Section 1.6, there exists a natural morphism of collections

\[
\Gamma_B : P(P(B)) \longrightarrow P(B), \ B \in \text{Coll},
\]

which, together with the obvious inclusion \( \iota_B : B \hookrightarrow P(B) \), makes \( P(-) \) a monad in \( \text{Coll} \). One has the third

**Definition 5.** A permutad is an algebra for the monad \( P : \text{Coll} \to \text{Coll} \).

As a consequence of classical statements, see e.g. \[3\], Theorem VI.2.1], \( P(B) \) is the free permutad generated by \( B \).
Example 6. An important rôle will be featured by the permutad $\text{perAs}$ with $\text{perAs}(n) := \mathbb{k}$ for all $n \in \text{Fin}$, with all operations (3) the canonical isomorphisms $\mathbb{k} \otimes \mathbb{k} \cong \mathbb{k}$. It is the linearization of the terminal permutad in the category $\text{Set}$ of sets, so we will call it, being aware of slight abuse of terminology, the terminal permutad.

Example 7. The permutohedron $P_{n+1}$ is, for $n \geq 1$, a convex polytope whose faces are labelled by planar rooted trees with levels, see e.g. [13] or [4, Appendix 2]. Its cellular chain complex

$$CC_*(P) = \{ CC_*(P_{n+1}) | n \in \text{Fin} \}$$

is a permutad in the category of dg vector spaces whose underlying permutad is free, generated by the collection $B$ which has

(10) $$B(n) := \text{Span}(c_{n-1}) \cong \uparrow^{n-1} \mathbb{k}, \ n \geq 1,$$

the ground field $\mathbb{k}$ placed in degree $n-1$, cf. [3, Theorem 5.3]. In (10), $c_{n-1} := \uparrow^{n-1} 1$, $1 \in \mathbb{k}$, is the generator of $\uparrow^{n-1} \mathbb{k}$. Indeed, formula (7) gives

$$\mathbb{P}(B)(n) \cong \bigoplus_{1 \leq k \leq n} \bigoplus_{r \in \text{Surj}(n,k)} \text{Span}(c_{r_1-1} \otimes \cdots \otimes c_{r_k-1}) \cong \bigoplus_{1 \leq k \leq n} \bigoplus_{r \in \text{Surj}(n,k)} \uparrow^{n-k} \mathbb{k},$$

where $r_i$ is the cardinality of the set-theoretic preimage of $i \in \mathbb{k}$ via $r : n \rightarrow \mathbb{k}$. The set $\text{Surj}(n,k)$ is isomorphic to the set of ordered decompositions of $\{1, \ldots, n\}$ into $k$ disjoint non-empty subsets which is, in turn, isomorphic to the set of planar rooted trees with $n+1$ leaves and $k$ levels, see e.g. [3, §1.3]. Explicitly, the isomorphism $CC_*(P) \cong \mathbb{P}(B)$ identifies the ‘big’ $n$-dimensional cell of $P_n$ with the generator $c_n$ of $\mathbb{P}(B)$, $n \geq 0$. A formula for the boundary operator of $CC_*(P)$ written in terms of $\mathbb{P}(B)$ can be found in [4, §9.3]. We will return to the permutohedron in Proposition 16.

Example 8. Each permutad $A$ admits a permutadic suspension $\mathfrak{s}A$. Its underlying collection is $\mathfrak{s}A(n) := \uparrow^n A(n)$, $n \geq 1$, and the structure operations

$$\diamondsuit : \mathfrak{s}A(r^{-1}(1)) \otimes \mathfrak{s}A(r^{-1}(2)) \longrightarrow \mathfrak{s}A(n), \ r \in \text{Surj}(n,2),$$

are defined by

$$(\uparrow^{r_1} a_1) \diamondsuit (\uparrow^{r_2} a_2) := \varepsilon(r) \cdot (-1)^{r_2 (r_1 + \deg(a_1))} \uparrow^n (a_1 \diamond_r a_2),$$

in which $\diamond_r$ is the structure operation of $A$, $a_i \in A(r^{-1}(i))$, $i = 1, 2$, and $r_1, r_2$ and $\varepsilon(r)$ are as in (4).

---

3Quoting a Czech physicist: “Old men give only good examples, since they are unable to give bad ones.” [March 18, 2019]
2. Koszul duals of permutads

Let us start this section by recalling the following definition of \([4, \S 4.4]\).

**Definition 9.** A permutad \(A\) is *quadratic* if it is of the form \(P(B)/(S)\), where \((S)\) is the permutadic ideal generated by a subspace \(S \subset P^2(B)\). Such a quadratic permutad \(A\) is *binary*, if \(B(n) = 0\) for \(n \neq 1\).

**Example 10.** Let us denote, abusing the notation again, by \(P(\mu)\) the free permutad generated by one generator \(\mu \in P(\mu)(1)\). As stated in \([4, \text{Corollary 4.7}]\), the terminal permutad \(\text{per}_A s\) recalled in Example 6 is binary quadratic,\(^{11}\)

\[
\text{per}_A s \cong P(\mu)/(S),
\]

where \(S := \text{Span}\{\circ_{12}(\mu \otimes \mu) - \circ_{21}(\mu \otimes \mu)\}\), with \(\circ_{12} : 2 \to 2\) the identity and \(\circ_{21} : 2 \to 2\) the transposition \(1, 2 \mapsto 2, 1\). The verification of \((11)\) is simple. Formula \((7)\) immediately gives that \(P(\mu)(n) \sim = \text{Span}(\Sigma_n)\), the \(k\)-linear span of the symmetric group of \(n\) elements, while modding out by the ideal \((S)\) identifies any two permutations in \(\Sigma_n\) that differ by transposition of adjacent elements. Since transpositions act transitively, one gets \(P(\mu)/(S)(n) \cong k \cong \text{per}_A s(n), n \geq 1\), as expected.

Every binary quadratic permutad \(A\) as in Definition 9 possess its *Koszul dual* \(A^!\), which is a binary quadratic permutad defined as

\[
A^! := P(\uparrow B^*)/(S^\perp),
\]

where \(\uparrow B^*\) is the suspension of the component-wise linear dual of the generating collection \(B\), and \(S^\perp \subset P^2(\uparrow B^*)\) is the annihilator of \(S \subset P^2(B)\) with respect to the obvious natural degree \(-2\) pairing between

\[
P^2(\uparrow B^*) = \bigoplus_{r \in \text{Surj}(2, 2)} \uparrow B(1)^* \otimes \uparrow B(1)^* \cong \uparrow^2 \bigoplus_{\sigma \in \Sigma_2} B(1)^* \otimes B(1)^*
\]

and

\[
P^2(B) = \bigoplus_{r \in \text{Surj}(2, 2)} B(1) \otimes B(1) \cong \bigoplus_{\sigma \in \Sigma_2} B(1) \otimes B(1)
\]

**Proposition 11.** The terminal permutad \(\text{per}_A s\) is Koszul self-dual in the sense that its Koszul dual \(\text{per}_A s^!\) is isomorphic to its permutadic suspension \(\text{sper}_A s\). Explicitly,

\[
\text{per}_A s^!(n) \cong \uparrow^n k, n \geq 1,
\]

with the structure operation \((\text{[perm.tex]}\) the composition

\[
\text{per}_A s^!(r_1) \otimes \text{per}_A s^!(r_2) \cong \uparrow^{r_1} k \otimes \uparrow^{r_2} k \cong \uparrow^n k \cong \text{per}_A s^!(n)
\]

[perm.tex]
multiplied with $\varepsilon(r) \in \{-1, +1\}$, where $r_1$, $r_2$ and $\varepsilon(r)$ have the same meaning as in (4).

Proof. By the definition of the Koszul dual, one has

$$\text{per}\mathcal{A}^! \cong \mathbb{P}(\mu^!)/(S^\perp),$$

where $\mu^! \in \mathbb{P}(\mu^!)(1)$ is a degree 1 generator and $S^\perp$ the span

$$\text{Span}\{\diamond_1(\mu^! \otimes \mu^!) + \diamond_{(21)}(\mu^! \otimes \mu^!\})\}.$$

By formula (7),

$$\mathbb{P}(\mu^!)(n) \cong \uparrow^n \text{Span}(\Sigma_n),$$

while modding out by the ideal $(S^\perp)$ introduces the relation $\sigma' \sim -\sigma''$ for each $\sigma' \in \Sigma_n$ and $\sigma''$ obtained from $\sigma'$ by a transposition of two adjacent elements. Therefore the assignment

$$\uparrow^n k \ni \uparrow^n 1 \longmapsto \mathbb{1}_n \in \Sigma_n$$

leads to an isomorphism $\uparrow^n k \cong \mathbb{P}(\mu^!)/(S^\perp)(\mathbb{1})$. The advertised formula for the structure operations easily follows from the description of the operations in the free permutad $\mathbb{P}(\mu^!)$ given in [4, Section 1.6].

\[\square\]

3. Koszulity of permutads

We start this section with a permutadic version of the cobar construction and the related dual bar construction. We then establish that the dual bar construction of the Koszul dual $\text{per}\mathcal{A}^!$ of the terminal permutad $\text{per}\mathcal{A}$ is isomorphic to the cellular chain complex of the permutohedron. Finally, we prove that $\text{per}\mathcal{A}$ is Koszul and formulate a test for Koszulity of permutads. The first definition is a harmless formal dual of Definition 3.

Definition 12. A copermutad is a collection $C \in \text{Coll}$ together with operations

$$\delta_r : C(\mathbb{n}) \longrightarrow C(r^{-1}(1)) \otimes C(r^{-1}(2)), \quad r \in \text{Surj}(n, 2), \quad n \geq 2,$$

satisfying the obvious dual versions of axioms (3).

Example 13. Assume that $A$ is a permutad such that $A(\mathbb{n})$ is, for each $n \geq 1$, either finite-dimensional, or non-negatively or non-positively graded vector space of finite type. Then its component-wise linear dual $A^* = \{A^*(\mathbb{n})\}_{n \geq 1}$ is a copermutad. This is in particular satisfied if $A$ is binary quadratic as in Definition 9, with $B(1)$ finite-dimensional.

Definition 14. A degree $s$ derivation of a permutad $A$ is a degree $s$ linear map $\varsigma : A \rightarrow A$ of collections such that

$$\varsigma \diamond_r = \diamond_r(\varsigma \otimes \mathbb{1}) + \diamond_r(\mathbb{1} \otimes \varsigma), \quad r \in \text{Surj}(n, 2), \quad n \geq 2,$$

for the structure operations $\diamond_r$ in (5). In elements,

$$\varsigma(a_1 \diamond_r a_2) = \varsigma(a_1) \diamond_r a_2 + (-1)^{s \deg(a_1)} a_1 \diamond_r \varsigma(a_2), \quad a_i \in A(r^{-1}(i)), \quad i = 1, 2.$$
One easily verifies that each degree $s$ map of collections $\sigma : B \to \mathbb{P}(B)$ uniquely extends to a degree $s$ derivation $\zeta$ of the free permutad $\mathbb{P}(B)$ satisfying $\zeta|_B = \sigma$. Let $C$ be a copermutad. Notice that (12) assemble into one map

$$\delta : C \cong \mathbb{P}(C) \longrightarrow \mathbb{P}^2(C)$$

thus one has a degree $-1$ map $\sigma : \downarrow C \to \mathbb{P}^2(\downarrow C)$ defined as the composition

$$\sigma := \downarrow C \xrightarrow{\delta} \mathbb{P}^2(C) \xrightarrow{\cong} \uparrow^2 \mathbb{P}^2(\downarrow C) \xrightarrow{\downarrow} \mathbb{P}^2(\downarrow C) \hookrightarrow \mathbb{P}(\downarrow C)$$

with $\mathbb{P}^2(C) \xrightarrow{\cong} \uparrow^2 \mathbb{P}^2(\downarrow C)$ the obvious canonical isomorphism. Denote finally by $\partial_0$ the unique extension of $\sigma$ into a degree $-1$ derivation of $\mathbb{P}(\downarrow C)$. One may verify by direct calculation that $\partial_0^2 = 0$, thus the following definition makes sense.

**Definition 15.** The cobar construction of a copermutad $C$ is the differential graded (dg) permutad $\Omega(C) := (\mathbb{P}(\downarrow C), \partial_0)$. The dual bar construction of a permutad $A$ satisfying the assumptions of Example 13 is the dg permutad $D(A) := \Omega(A^\ast)$.

**Convention.** From now on we will assume that $A$ is a binary quadratic permutad as in Definition 9, with $B(1)$ finite dimensional.

Under the above assumption, one is allowed to define the Koszul dual copermutad $A^! := A^\ast$, the component-wise linear dual. With this notation, $D(A^!) = \Omega(A^!)$. As emphasized in [5], the copermutad $A^!$ is more fundamental than $A^1$. It can be defined directly, without the passage through the permutad $A^1$, using coideals in cofree copermutads, without any assumptions on the size of the generators of $A$. Given the applications we had in mind, we however decided not to use this more general definition and keep working with more intuitive ideals and free permutads.

**Proposition 16.** The dual bar construction $D(per\mathcal{A}s^!)$ of $per\mathcal{A}s^!$ is isomorphic to the dg permutad $CC_*(\mathbb{P})$ of cellular chains of the permutohedron.

**Proof.** The proof relies on the description of the permutad $per\mathcal{A}s^!$ given in Proposition 11. Denote by $e_n \in \downarrow per\mathcal{A}s^!(n)^\ast$, $n \geq 1$, the generator dual to $\uparrow^{n-1}1 \in \uparrow^{n-1} \mathbb{k} \cong \downarrow per\mathcal{A}s^!(n)$. Then $e_1, e_2, \ldots$ are the free permutadic generators of $D(per\mathcal{A}s^!)$. Using the description of the structure operations of $per\mathcal{A}s^!$ given in Proposition 11, one easily verifies that the differential $\partial_D$ in the dual bar construction $D(per\mathcal{A}s^!)$ is

$$\partial_D(e_n) = \sum_{r \in \text{Surj}(n,2)} (-1)^{r_1-1} \varepsilon(r) \cdot (e_{r_1} \circ_p e_{r_1}), \quad n \geq 1,$$

where $r_1, r_2$ and the sign $\varepsilon(r)$ are as in [11], and $\circ_p$ is the structure operation of the free permutad $\mathbb{P}(per\mathcal{A}s^!)$. 

[perm.tex]
Referring to the description of $CC_\ast(P)$ given in Example 7, we define an isomorphism $\xi : D(perAs^l) \cong CC_\ast(P)$ of free permutads by

$$\xi(e_n) := -c_{n-1}, \ n \geq 1,$$

with $c_{n-1}$ the generator in (10). Comparing (13) to the formula for the differentials of $c_n$’s in [3, §9.3] we realize that they match up to an overall minus sign. Therefore $\xi$ defined above commutes with the differentials, thus constitutes the required isomorphism of dg properads.

Let us proceed towards the Koszulity property of binary quadratic permutads in Definition 9. One starts from a monomorphism $\uparrow B \hookrightarrow A^!$ of collections defined as the composition

$$\uparrow B \hookrightarrow \mathbb{P}(\uparrow B) \to \mathbb{P}(\uparrow B)/(S^+) = A^!.$$ 

Its linear dual is a surjection $A^! \to \uparrow B$ which desuspends to a map $\pi : \downarrow A^! \to B$. The related twisting morphism $\downarrow A^! \to A$ is the composition

$$\downarrow A^! \to B \hookrightarrow \mathbb{P}(B) \to \mathbb{P}(B)/(S) = A.$$ 

It extends to a permutad morphism $\rho : \mathbb{P}(\downarrow A^!) \to A$ by the freeness of $\mathbb{P}(\downarrow A^!)$. One easily verifies:

**Proposition 17.** One has $\rho \circ \partial_D = 0$, therefore $\rho$ induces the canonical map

$$\text{can} : D(A^!) = (\mathbb{P}(\downarrow A^!), \partial_D) \longrightarrow (A, 0)$$

of dg permutads.

**Definition 18.** A binary quadratic permutad $A$ is **Koszul** if the canonical map (14) is a component-wise homology isomorphism.

If $A$ is concentrated in degree 0, clearly $H^0(D(A^!)) \cong A$, thus such an $A$ is Koszul if and only if $D(A^!)$ is acyclic in positive dimensions. This observation will be used in our proof of

**Theorem 19.** The terminal permutad $perAs$ is Koszul.

*Proof.* The statement follows from the isomorphism $D(perAs^l) \cong CC_\ast(P)$ established in Proposition 16 combined with the contractibility of the permutohedron which implies that $CC_\ast(P)$ is acyclic in positive dimensions.

Let us close this section by a Koszulity test in the spirit of the Ginzburg-Kapranov criterion for operads [3, Theorem 3.3.2]. For $A = \{A(n)\}_{n \geq 1} \in Coll$ with finite-dimensional pieces we define its generating power series as

$$f_A(t) := \sum_{n=1}^\infty \frac{\chi(A(n))}{n!},$$

in which $\chi$ denotes the Euler characteristic. One then has

---

We borrowed this terminology from [3, Chapter 6].
Proposition 20. Assume that $A$ is a binary quadratic permutad as in Definition 9 with $B(1)$ finite dimensional. If $A$ is Koszul, then its generating series $f_A$ and the series $f_{A'}$ of its Koszul dual are related by the functional equation

$$f_A(t) = \frac{-f_{A'}(t)}{1 + f_{A'}(t)}.$$  

Proof. Suppose that $M = \{M(n)\}_{n \geq 1}$ is a collection of graded vector spaces with finite-dimensional pieces. Simple combinatorics gives

$$\chi(P(M)(n)) = \sum_{s \geq 1} \sum_{k_1 + \cdots + k_s = n} \frac{n!}{k_1! \cdots k_s!} \chi(M(k_1)) \cdots \chi(M(k_s))$$

which is the same as

$$\frac{\chi(P(M)(n))}{n!} = \sum_{s \geq 1} \sum_{k_1 + \cdots + k_s = n} \frac{\chi(M(k_1)) \cdots \chi(M(k_s))}{k_1! \cdots k_s!}.$$  

Therefore the generating series of $M$ and that of the free permutad $P(M)$ are related by

$$f_{P(M)} = f_M + f_M^2 + f_M^3 + \cdots = f_M(1 - f_M)^{-1}.$$  

When $A$ is Koszul, one has $f_{D(A')} = f_A$, while we already know from the above calculations that

$$f_{D(A')} = f_{P(A')}(1 - f_{A'})^{-1} = -f_{A'}(1 + f_{A'})^{-1}.$$  

This finishes the proof. □

Example 21. One has

$$f_{\text{per}As}(t) = t + \frac{1}{2!} t^2 + \frac{1}{3!} t^3 + \cdots = e^t - 1$$

while

$$f_{\text{per}As'}(t) = -t + \frac{1}{2!} t^2 - \frac{1}{3!} t^3 + \cdots = e^{-t} - 1.$$  

Plugging it into (15) results in

$$e^t - 1 = \frac{-(e^{-t} - 1)}{1 + (e^{-t} - 1)} = \frac{1 - e^{-t}}{e^t} = e^t - 1$$

as expected.

Example 22. A ‘twisted’ version of the terminal permutad $\text{per}As$ of Example 10 is the quotient

$$\text{per}\tilde{A}s := P(\mu) / (\tilde{S})$$

where $\mu \in P(\mu)(1)$ is a degree 0 generator and $(\tilde{S})$ the permutadic ideal generated by

$$\bigtriangleup_1 (\mu \otimes \mu) + \bigtriangleup_{(21)} (\mu \otimes \mu).$$

Notice that $\text{per}\tilde{A}s$ equals the ‘parametrized associative permutad’ $q\text{-perm}As$ of [4, §4.5] taken with $q = -1$. Since $\text{per}\tilde{A}s$ is a permutadic version of the operad for antiassociative algebras
which serves as a standard example of a non-Koszul operad \[11\] Section 5, one would expect that $\text{per} \tilde{A}s$ is non-Koszul as well.

Surprisingly, it is not so. It turns out that the dg collections $\mathbb{D}(\text{per} \tilde{A}s^!)$ and $\mathbb{D}(\text{per} A s^!)$ are isomorphic, though they are not isomorphic as dg permutads. This however suffices for the acyclicity of $\mathbb{D}(\text{per} \tilde{A}s^!)$ in positive dimensions, and thus the Koszulity of $\text{per} \tilde{A}s$.

The isomorphism $\zeta : \mathbb{D}(\text{per} \tilde{A}s^!) \xrightarrow{\cong} \mathbb{D}(\text{per} A s^!)$ of dg collections is constructed as follows. As in the proof of Proposition $11$ we establish that $\text{per} \tilde{A}s^!(n) \cong \uparrow^n k$, $n \geq 1$, with the structure operations (5) the canonical isomorphisms $\uparrow^{r_1} k \otimes \uparrow^{r_2} k \cong \uparrow^n k$ without any additional sign factor. The calculation is actually even simpler than in the case of the untwisted $\text{per} A s$, since the relation induced by the ideal ($\tilde{S}^\perp_k$) does not involve any signs. We infer that $\mathbb{D}(\text{per} \tilde{A}s^!)$ is freely generated by degree $n-1$ generators $\tilde{e}_n$, $n \geq 1$, whose differential is given by a formula analogous to (13) but without the $\varepsilon(r)$-factor. The underlying permutad of the dg permutad $\mathbb{D}(\text{per} \tilde{A}s^!)$ is then described as

$$P(\downarrow \text{per} \tilde{A}s^!(n)) \cong \bigoplus_{1 \leq k \leq n} \bigoplus_{r \in \text{Surj}(n,k)} \text{Span}(\tilde{e}_{r_1} \otimes \cdots \otimes \tilde{e}_{r_k}), \quad n \geq 1,$$

where $r_i$ is the cardinality of the set-theoretic preimage of $i \in k$ via the map $r : n \to k$. We have a similar formula for the underlying permutad of $\mathbb{D}(\text{per} A s^!)$, namely

$$P(\downarrow \text{per} A s^!(n)) \cong \bigoplus_{1 \leq k \leq n} \bigoplus_{r \in \text{Surj}(n,k)} \text{Span}(e_{r_1} \otimes \cdots \otimes e_{r_k}), \quad n \geq 1.$$

The isomorphism $\zeta : \mathbb{D}(\text{per} \tilde{A}s^!) \xrightarrow{\cong} \mathbb{D}(\text{per} A s^!)$ is, under the above identifications, given by

$$\zeta(\tilde{e}_{r_1} \otimes \cdots \otimes \tilde{e}_{r_k}) := \varepsilon(r) \cdot (e_{r_1} \otimes \cdots \otimes e_{r_k}),$$

with $\varepsilon(r)$ the sign of the unshuffle associated to $r$. It is simple to verify that $\zeta$ defined this way commutes with the differentials of the dual bar constructions, so it is an isomorphism of dg collections. On the other hand, $\zeta$ is clearly not a morphism of properads.

**Part 2. Operadic category approach**

4. PERMUTADS AS ALGEBRAS OVER THE TERMINAL OPERAD

The fundamental feature of Batanin-Markl’s (BM) theory of operadic categories [2] is that the objects under study are viewed as algebras over (generalized) operads in a specific operadic category, cf. also the introduction to [1]. Thus, for instance, ordinary operads are algebras over the terminal operad $1_{RTr}$ in the operadic category $RTr$ of rooted trees. The operad $1_{RTr}$ is quadratic self-dual which, according to BM theory, implies that the bar
constructions of its algebras (i.e. operads) are algebras of the same type (i.e. operads) again. As we noticed in the introduction, the same is true for permutads.

Let us start by recalling, following [1, §14.4], the operadic category $\text{Per}$ that plays for permutads the same rôle as $\text{RTr}$ for ordinary operads. Objects of $\text{Per}$ are surjections $\alpha : n \to k \in \text{Surj}(n, k)$, $n \geq k \geq 1$, and morphisms $f : \alpha' \to \alpha''$ of $\text{Per}$ are diagrams

\[
\begin{array}{ccc}
\alpha' & \downarrow & \alpha'' \\
\alpha & \downarrow & \gamma \\
k' & \gamma & k''
\end{array}
\]

in which $\gamma$ is order-preserving (and necessarily a surjection).

The cardinality functor is defined by $|\alpha : n \to k| := k$. The $i$-th fiber $f^{-1}(i)$ of the morphism $\alpha$ is the surjection $(\gamma \alpha')^{-1}(i) \to \gamma^{-1}(i)$, $i \in k$. The only local terminal objects are the surjections

\[
U_n := \frac{n}{1}, \quad n \geq 1,
\]

which are also the chosen (trivial) ones. All quasi-bijections, and isomorphisms in general, are the identities.

Operadic category $\text{Per}$, as each operadic category, admits its terminal operad $1_{\text{Per}}$ with $1_{\text{per}}(\alpha) := k$ and all structure operations the identities. Its algebras are described in

**Proposition 23.** Algebras for the terminal $\text{Per}$-operad $1_{\text{Per}}$, in the sense of Definition 1.20 of [2], are permutads.

**Proof.** The statement is a part of [1, Theorem 14.4] whose proof uses an explicit presentation of $1_{\text{per}}$, but we will show directly that $1_{\text{per}}$-algebras are the same as algebras for the monad $P$ in Definition [3]. Let $f : \alpha' \to \alpha''$, $\alpha', \alpha'' \in \text{Surj}(n, k')$, $\alpha'' \in \text{Surj}(n, k'')$, be a morphism in $\text{Per}$ with fibers $f_1, \ldots, f_{k''}$. The crucial fact on which our proof is based is that

\[
\alpha' = (\alpha''; f_1, \ldots, f_{k''}),
\]

where $(-; -, \ldots, -)$ is the substitution introduced in [1, Section 1.2]. Formula (18) follows directly from definitions, as the reader may check easily.

Let us inspect what [2, Definition 1.20] gives in our case. The underlying spaces of an algebra over an operad in an operadic category $\mathcal{O}$ is indexed by the set $\pi_0(\mathcal{O})$ of its connected components, which is isomorphic to the set of chosen terminal objects. We identify

\[
\pi_0(\text{Per}) = \{1, 2, 3, \ldots\},
\]

\[\text{perm.tex}\]
thus $\mathbf{1}_{\text{Per}}$-algebras are collections $A = \{ A(n) \in \mathbf{Vec} \mid n \in \text{Fin} \} \in \text{Coll}$. By [2, Definition 1.20] again, the structure maps of an algebra $A$ over an operad $P$ in an operadic category $O$ are morphisms

$$\mu_T : P(T) \longrightarrow \mathbf{Vec}\left( \bigotimes_{c \in \pi_0(s(T))} A(c), A(\pi_0(T)) \right),$$

where $T \in O$, $\pi_0(s(T))$ is the subset of $\pi_0(O)$ formed by the connected components of $O$ to which the fibers of the identity map $\mathbb{1} : T \to T$ belong, and $\pi_0(T)$ is the connected component of $T$. In our case, $P$ is the constant $\text{Per}$-operad $\mathbf{1}_{\text{Per}}$ whose each piece equals $k$.

If $\alpha : n \twoheadrightarrow k \in \text{Per}$, clearly

$$\pi_0(s(\alpha)) = \{n_1, \ldots, n_k\}, \quad n_i := \alpha^{-1}(i), \quad 1 \leq i \leq k,$$

while $\pi_0(\alpha) = n$. The structure maps associated to such an $\alpha$ are therefore given by

$$m_{\alpha} := \mu_{\alpha}(1) : A(n_1) \otimes \cdots \otimes A(n_k) \longrightarrow A(n).$$

According to [2, Definition 1.20], the structure maps $[19]$ must assemble to a morphism $\mu : P \to \mathcal{E}nd^0_A$ of $P$ to the endomorphism operad of the collection $A$. Let us inspect what it means in our case.

First of all, $\mu : \mathbf{1}_{\text{Per}} \to \mathcal{E}nd^0_A$ must preserve operad units. This means that for each $\alpha$ which is chosen local terminal, i.e. $\alpha : n \twoheadrightarrow 1$ for some $n \geq 1$, the diagram

$$\begin{array}{ccc}
\mathbf{1}_{\text{Per}}(\alpha) & \xrightarrow{\mu_{\alpha}} & \mathcal{E}nd^0_A(\alpha) \\
\downarrow{\eta_\alpha} & & \downarrow{\eta} \\
\mathbb{1} & \xrightarrow{\eta_0} & \mathbb{1}_k,
\end{array}$$

in which $\eta_\alpha$ is the unit morphism of the endomorphism operad, commutes. This is the same as to require that, for $\alpha : n \twoheadrightarrow 1$, the structure map

$$m_{\alpha} : A(n) \longrightarrow A(n)$$

equals the identity. It thus bears no information, so we consider $m_{\alpha}$’s in $[20]$ only for $|\alpha| \geq 2$.

Next, we must verify that $\mu : \mathbf{1}_{\text{Per}} \to \mathcal{E}nd^0_A$ commutes with the operadic structure operations. Assume therefore that $f : \alpha' \to \alpha''$ is a morphism in $[16]$, with fibers $f_1, \ldots, f_{k''}$. Its $i$th fiber $f_i$, $1 \leq i \leq k''$, belongs to $\text{Surj}(n_i, l_i)$, where $l_i := \gamma^{-1}(i)$ and $n_i \geq 1$ are such that

$$n_1 + \cdots + n_{k''} = n.$$

Moreover,

$$\pi_0(s(f_i)) = \{n_i^1, \ldots, n_i^{l_i}\}, \quad 1 \leq i \leq k'',$$

with some $n_i^1, \ldots, n_i^{l_i} \geq 1$ such that

$$n_i^1 + \cdots + n_i^{l_i} = n_i, \quad 1 \leq i \leq k''.$$
while $\pi_0(f_i) = n_i$. Notice also that

\[ \pi_0(s(\alpha')) = \{n_1^1, \ldots, n_i^1, \ldots, n_k^{l_i}, \ldots, n_k^{l_{k'}}\} \]

and that

\[ \pi_0(s(\alpha'')) = \{n_1, \ldots, n_k^{l'}\}. \]

One easily finds that $\mu : 1^{\Per} \to \End^{\Per}_A$ commutes with the structure operations of $\Per$-operads if and only if, for each $f$ as above, the diagram

\[
\begin{array}{ccc}
\bigotimes_{1 \leq i \leq k''} 1^{\Per}(f_i) \otimes 1^{\Per}(\alpha'') & \xrightarrow{\gamma_f} & 1^{\Per}(\alpha') \\
\otimes \mu_{f_i} \otimes \mu_{\alpha''} & & \\
\bigotimes_{1 \leq i \leq k''} \text{Vec}(\bigotimes_{1 \leq i \leq k''} A(n_i^s), A(n_i)) \otimes \text{Vec}(\bigotimes_{1 \leq i \leq k''} A(n_l^i), A(n_l)) & \xrightarrow{\text{comp}} & \text{Vec}(\bigotimes_{1 \leq i \leq k''} A(n_l^s), A(n_l))
\end{array}
\]

in which $\gamma_f$ is the operatic composition in $1^{\per}$ and $\text{comp}$ the obvious composition of linear maps, commutes. Since, in our case,

\[
\bigotimes_{1 \leq i \leq k''} 1^{\Per}(f_i) \otimes 1^{\Per}(\alpha'') \cong 1^{\Per}(\alpha')
\]

and $\gamma_f$ is, under this identification, the identity, the commutativity of the above diagram is equivalent to the equation

\[
\text{comp}(\bigotimes_{1 \leq i \leq k''} m_{f_i} \otimes m_{\alpha''}) = m_{\alpha'}
\]

involving the structure maps (20). In other words, one requires that, whenever (18) is satisfied,

\[
m_{\alpha'} = m_{\alpha''}(m_{f_1}, \ldots, m_{f_{k'}})
\]

which is an equality of maps

\[
A(n_1^1) \otimes \cdots \otimes A(n_i^{l_i}) \otimes \cdots \otimes A(n_k^{l'}) \otimes \cdots \otimes A(n_k^{l_{k'}}) \rightarrow A(n).
\]

To finish the proof, it is enough to realize how the substitution (18) enters, in the proof of [1, Proposition 1.4], the definition of the map $\Gamma_E$ in (3). The fact that $\Per$-algebras are indeed the same as the structures described above will then be self-evident. 

\[\square\]

[perm.tex] [March 18, 2019]
5. Free \( \mathbf{Per} \)-operads.

This section is a preparation for the construction of the minimal model of the terminal \( \mathbf{Per} \)-operad given in Section 6 and for the introduction of Koszul duality and Koszulity of \( \mathbf{Per} \)-operads given in Sections 7 and 8. Free operads were, in the context of general operadic categories, addressed in [1, Section 10] to which we refer for the terminology used in the following sentences. Our situation is however simplified by the fact that all local terminal objects in \( \mathbf{Per} \) are the chosen terminal ones, thus unital \( \mathbf{Per} \)-operads are automatically strictly extended unital. Moreover, for our purposes it suffices to consider only 1-connected \( \mathbf{Per} \)-operads, i.e. to operads \( \mathcal{P} \) such that \( \mathcal{P}(\alpha) \cong k \) if \( |\alpha| = 1 \).

With the above in mind, we introduce the category \( \mathbf{Per} \text{-Col}_{1} \) whose objects are collections \( E = \{ E(\alpha) \mid \alpha \in \mathbf{Per} \} \) of graded vector spaces with \( E(\alpha) = 0 \) if \( |\alpha| = 1 \), and level-wise morphisms. One has the obvious forgetful functor

\[
\Box : \mathbf{Per} \text{-Oper}_{1} \rightarrow \mathbf{Per} \text{-Col}_{1}
\]

from the category of 1-connected unital \( \mathbf{Per} \)-operads given by

\[
\Box \mathcal{P}(\alpha) := \begin{cases} 
\mathcal{P}(\alpha) & \text{if } |\alpha| \geq 2, \text{ and} \\
0 & \text{otherwise.}
\end{cases}
\]

Theorem 10.11 of [1] guarantees that \( \Box \) admits a left adjoint \( \mathbb{F} : \mathbf{Per} \text{-Col}_{1} \rightarrow \mathbf{Per} \text{-Oper}_{1} \). We call \( \mathbb{F}(E) \) for \( E \in \mathbf{Per} \text{-Col}_{1} \) the free \( \mathbf{Per} \)-operad generated by a 1-connected \( \mathbf{Per} \)-collection \( E \).

We describe in detail \( \mathbb{F}(E) \) for generating collections of a particular form. The central rôles in this simplified construction will be played by the classical free non-\( \Sigma \) operads. This generality is sufficient for all concrete applications given in the rest of this paper. The construction of \( \mathbb{F}(E) \) for an arbitrary 1-connected \( \mathbf{Per} \)-collection \( E \) is sketched out at the end of this section.

Let \( \Delta_{\text{semi}} \) be the lluf subcategory of \( \mathbf{Fin} \) consisting of order-preserving surjections. It is an operadic category whose operads are the classical constant-free non-\( \Sigma \) (non-symmetric) operads [2, Example 1.15]. One has a strict operadic functor, in the sense of [2, p. 1635], \( \text{des} : \mathbf{Per} \rightarrow \Delta_{\text{semi}} \) given on objects by \( \text{des}(n \twoheadrightarrow k) := k \) while, for a morphism \( f \) in (16), one puts \( \text{des}(f) := \gamma \). According to general theory [2, p. 1639], each \( \Delta_{\text{semi}} \)-operad \( S \) determines, via the restriction along \( \text{des} \), a \( \mathbf{Per} \)-operad \( \text{des}^{*}(S) \) such that \( \text{des}^{*}(S)(\alpha) = S(\text{des}(\alpha)), \alpha \in \mathbf{Per} \).

Recall that the structure map \( m_{f} \) of a \( \mathbf{Per} \)-operad \( \mathcal{P} \) associated to a morphism \( f : \alpha' \rightarrow \alpha'' \) in \( \mathbf{Per} \) with fibers \( f_{1}, \ldots, f_{k''} \) is of the form

\[
m_{f} : \mathcal{P}(\alpha'') \otimes \mathcal{P}(f_{1}) \otimes \cdots \otimes \mathcal{P}(f_{k''}) \rightarrow \mathcal{P}(\alpha')
\]

\[\text{We use the convention pioneered in [11] and distinguish classical non-}\Sigma \text{ operads by underlining.}\]

[March 18, 2019]
For $\mathcal{P} = \text{des}^*(\mathcal{S})$, $m_f$ is determined by the structure operation $m_\gamma$ associated to $\gamma = \text{des}(f)$ via the commutativity of the diagram

$$
\begin{array}{ccc}
\text{des}^*(\mathcal{S})(\alpha'') & \otimes & \text{des}^*(\mathcal{S})(f_1) & \otimes & \cdots & \otimes & \text{des}^*(\mathcal{S})(f_{k''}) \\
m_f & \downarrow & S(k'') & \otimes & S(l_1) & \otimes & \cdots & \otimes & S(l_{k''}) \\
\text{des}^*(\mathcal{S})(\alpha') & \rightarrow & \text{des}^*(\mathcal{S})(\alpha) & \rightarrow & S(k').
\end{array}
$$

Let $E = \{E(\alpha)\}_{\alpha \in \text{Per}}$ be a collection of vector spaces such that

$$(22) \quad E(\alpha' : n' \to k') = E(\alpha'' : n'' \to k'') \quad \text{if} \quad k' = k''$$

and $E = \{E(k)\}_{k \geq 1} \in \text{Coll}$ be defined as

$$(23) \quad E(k) := E(\alpha : n \to k)$$

for an arbitrary $\alpha : n \to k$, $k \geq 1$. Let finally $F(E)$ be the free non-$\Sigma$ operad generated by a collection $E$ above, and $F(E) := \text{des}^*(F(E))$.

**Proposition 24.** The $\text{Per}$-operad $F(E) = \text{des}^*(F(E))$ is the free $\text{Per}$-operad generated by the collection $E$. It is naturally graded,

$$F(E)(\alpha) = \bigoplus_{s \geq 0} F_s(E)(\alpha), \alpha \in \text{Per}. $$

The grading is such that $F^1(E)(\alpha) = E(\alpha)$ for all $\alpha \in \text{Per}$, and

$$(24) \quad F^0(E)(\alpha : n \to k) = \begin{cases} k & \text{if } k = 1, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

**Proof.** The claim is immediately obvious from the explicit description of free operads given in [1, Section 10], but we give an independent proof. Recall from [1, §5.2] that a strict operadic functor $p : O \to P$ is a discrete opfibration if

(i) $p$ induces a surjection $\pi_0(O) \to \pi_0(P)$ and

(ii) for any morphism $f : T \to S$ in $P$ and any $t \in O$ such that $p(t) = T$ there exists a unique $\sigma : t \to s$ in $O$ such that $p(\sigma) = f$.

By dualizing [2, Theorem 2.4] one verifies that the restriction $p^* : \text{P-Oper} \to \text{O-Oper}$ between the associated categories of operads has a right adjoint $p_* : \text{O-Oper} \to \text{P-Oper}$ defined on objects by

$$p_*(\mathcal{O})(T) := \prod_{p(t) = T} \mathcal{O}(t), \mathcal{O} \in \text{O-Oper}, T \in \text{P}. $$

One thus has the adjunction

$$(25) \quad \text{O-Oper}(p^*(\mathcal{S}), \mathcal{O}) \cong \text{P-Oper}(\mathcal{S}, p_*(\mathcal{O})), \mathcal{S} \in \text{P-Oper}, \mathcal{O} \in \text{P-Oper}. $$
It is easy to check that des \( \mathcal{O} \rightarrow \Delta_{\text{semi}} \) is a discrete opfibration, therefore the adjunction \((\ref{eq:adjunction})\), with \( F(E) \) in place of \( S \) and a non-\( \Sigma \) constant-free operad \( O \) in place of \( \mathcal{O} \), gives

\[
\text{Per-Oper}(\mathcal{F}(E), \mathcal{O}) \cong \Delta_{\text{semi}}\text{-Oper}(\mathcal{F}(E), \text{des}\_\ast(\mathcal{O})).
\]

Invoking the fact that \( \mathcal{F}(E) \) is the free non-\( \Sigma \)-operad, one sees that the right hand side of the above isomorphism consists of families of linear maps

\[
\{E(k) \rightarrow \text{des}\_\ast(\mathcal{O})(k) \mid k \geq 1\}.
\]

Since, by definition, \( \text{des}\_\ast(\mathcal{O})(k) = \prod_{\alpha: n \rightarrow k} \mathcal{O}(\alpha) \), taking into account the definition \((\ref{eq:defE})\) of \( E \), one sees that the family in \((\ref{eq:family1})\) is the same as a family of linear maps

\[
\{E(\alpha) \rightarrow \mathcal{O}(\alpha) \mid \alpha \in \text{Per}\}.
\]

In other words, \( \text{Per}\)-operad maps \( \mathcal{F}(E) \rightarrow \mathcal{O} \) are in a natural one-to-one correspondence with families \((\ref{eq:family2})\). This makes the freeness of \( \mathcal{F}(E) \) obvious.

The free non-\( \Sigma \) operad \( \mathcal{F}(E) \) is naturally graded, with \( \mathcal{F}(E)(1) = E \) \([\text{page 1475}]\), and this grading clearly induces a grading of \( \mathcal{F}(E) = \text{des}\_\ast(\mathcal{F}(E)) \) having the requisite properties. This finishes the proof. \( \square \)

Let us point out that \((\ref{eq:restriction})\) characterizes collections \( E \in \text{Per-Coll}_1 \) that are the restrictions of some collection \( E' \in \Delta_{\text{semi}}\text{-Coll}_1 \) along \( \text{des} : \text{Per} \rightarrow \Delta_{\text{semi}} \), i.e. that are of the form \( E = \text{des}\_\ast(E') \) for some \( E' \).

**Remark 25.** As argued in \([1]\), Subsection 5.2], each discrete operadic opfibration \( p : \mathcal{O} \rightarrow \mathcal{P} \) is obtained from a certain \( \mathcal{P}\)-cooperad \( \mathcal{C} \) via the operadic Grothendieck construction. In our concrete case the corresponding \( \Delta_{\text{semi}}\)-cooperad \( \mathcal{C} \) is given by

\[
\mathcal{C}(k) := \coprod_{n \geq k} \text{Surj}(n, k), \quad k \geq 1.
\]

**General case.** Let \( E \) be an arbitrary 1-connected \( \text{Per}\)-collection. The constructions in Section 10 of \([1]\) specialize to the description of the free \( \text{Per}\)-operad \( \mathcal{F}(E) \) generated by \( E \) given below. We however need some notation.

For \( \alpha : n \rightarrow k \in \text{Per} \) and \( s \geq 1 \), let \( \text{Tr}^s(\alpha) \) be the set of planar rooted trees growing from the bottom up with \( s \) at least binary vertices, and leaves labeled from the left to right by the elements of the ordered set \( k \). We extend this definition by postulating \( \text{Tr}^0(\alpha) := \emptyset \) if \( |\alpha| > 1 \), while for \( |\alpha| = 1 \), \( \text{Tr}^0(\alpha) \) consists of a singular rooted tree with one leaf and no vertex.

Each vertex \( v \in \text{Vert}(\tau) \) of \( \tau \in \text{Tr}^s(\alpha) \) determines a segment \( k_v \subset k \) of those \( i \in k \) for which \( v \) lies on the path in \( \tau \) connecting the leaf labeled by \( i \) with the root. We then define \( n_v \). [March 18, 2019]
Figure 1. Constructing $\alpha_v : 7 \rightarrow 2$ out of $\alpha : 15 \rightarrow 9$, a tree $\tau \in \text{Tr}^5(\alpha)$ and $v \in \text{Vert}(\tau)$. The segment 9$\circlearrowleft$ is marked by the dashed oval, the elements of the set 15$\circlearrowleft$ by big punctured dots and the elements of $\text{In}(v)$ by two numbered balloons. The map $\alpha_v$ sends 2, 3, 4, 7 to 1 and 1, 5, 6 to 2.

to be the ordinal given by the pullback

$$
\begin{array}{ccc}
\alpha & \downarrow \\
\downarrow & & \downarrow \\
n \circlearrowleft & & n \\
\end{array}
$$

Consider finally the surjection

$$
(28) \quad \alpha_v : n_\circlearrowleft \longrightarrow \text{In}(v)
$$

to the set $\text{In}(v)$ of edges incoming to $v$ which sends $j \in n_\circlearrowleft$ to the unique edge in the path connecting $\alpha(j)$ to the root of $\tau$. The set $\text{In}(v)$ is ordered by the clockwise orientation of the plane, and (28) is order-preserving. We will thus interpret it as an object of $\text{Per}$, cf. the remark after diagram (1). We believe that Figure 1 makes these definitions clear. One puts

$$
\mathbb{F}(E)(\alpha) = \bigoplus_{s \geq 0} \mathbb{F}^s(E)(\alpha)
$$

with

$$
\mathbb{F}^s(E)(\alpha) := \bigoplus_{\tau \in \text{Tr}^s(\alpha)} \bigotimes_{v \in \text{Vert}(\tau)} E(\alpha_v).
$$

For each morphism $f : \alpha' \rightarrow \alpha''$ in $\text{Per}$ with fibers $f_1, \ldots, f_{k''}$, a straightforward calculation reveals the existence of the canonical isomorphism

$$
\mathbb{F}(\alpha'') \otimes \mathbb{F}(f_1) \otimes \cdots \otimes \mathbb{F}(f_{k''}) \xrightarrow{\cong} \mathbb{F}(\alpha')
$$
which we take as the structure operation \([21]\) of the operad \(\mathbb{F}(E)\). Notice that if \(E = \text{des}^s(E)\) for some \(E \in \Lambda_{\text{semi}}\text{-Coll}_1\), the above general construction coincides with the special one given in Proposition [24].

**Remark 26.** If the reader finds this remark confusing, he or she may safely ignore it. The \(\text{Per}\)-operad \(\mathbb{F}(E)\) described above is the operad associated to the free Markl’s \(\text{Per}\)-operad under the isomorphisms of the categories of ordinary and Markl’s operads stated in [1, Theorem 7.4] that holds due to the 1-connectivity assumption; cf. also the notes at the beginning of Section 8.

**Example 27.** It is easy to see that \(\mathbb{F}^0(E)\) is as in (24) and \(\mathbb{F}^1(E) \cong E\). To describe \(\mathbb{F}^2(E)\) we call, following [1, Definition 2.9], a map \(f : \alpha \to \beta \in \text{Per} \) elementary if all its fibers except precisely one, say the \(i\)th fiber \(F\), are trivial, i.e. the chosen terminal objects. We express this situation by writing \(F_{＞i} \alpha \xrightarrow{f} \beta\) or simply \(F \xrightarrow{f} \beta\) when \(i\) is understood. We leave as an easy exercise to verify that, with this notation,

\[
\mathbb{F}^2(E)(\alpha) = \bigoplus_{F_{＞i} \alpha \xrightarrow{f} \beta} E(\beta) \otimes E(F),
\]

where the summation runs over all elementary maps \(f : \alpha \to \beta\) with \(|\beta| \geq 2\).

### 6. Minimal model of the terminal \(\text{Per}\)-operad and homotopy permutads

**Definition 28.** A minimal model of an operad \(P \in \text{Per-Oper}_1\) is a differential graded (dg) \(\text{Per}\)-operad \(\mathcal{M} = (\mathcal{M}, \partial)\) together with a dg \(\text{Per}\)-operad morphism \(\rho : \mathcal{M} \to P\) such that

(i) the component \(\rho(\alpha) : (\mathcal{M}(\alpha), \partial) \to (P(\alpha), \partial = 0)\) of \(\rho\) is a homology isomorphism for each \(\alpha \in \text{Per}\), and

(ii) the underlying non-dg \(\text{Per}\)-operad of \(\mathcal{M}\) is free, and the differential \(\partial\) is quadratic with respect to the natural grading of \(\mathcal{M}\).

Notice that property (ii) implies the minimality, in the sense of [1, Theorem 2.1(i)], of the differential \(\partial\). Minimal models should be particular cofibrant replacements in a conjectural (semi)model structure on the category of \(\text{Per}\)-operads. For the purposes of applications it however suffices to realize that minimal models are ‘special’ cofibrant [8, Definition 17]. This already guarantees that their algebras are homotopy invariant concepts.

Let \(\mathcal{A}ss_\infty \to \mathcal{A}ss\) be the minimal model of the terminal non-Σ operad \(\mathcal{A}ss\) governing associative algebras. As we know from [1, Example 4.8], \(\mathcal{A}ss_\infty\) is generated by the collection \(E\) defined by

\[
E(k) := \text{Span}(\xi_{k-2}), \quad \deg(\xi_{k-2}) := k-2, \quad k \geq 2,
\]
with the differential acting on the generators by the formula
\[ \partial(\xi_r) = \sum (-1)^{(b+1)(i+1)+b} \cdot \xi_a \circ_i \xi_b, \quad r \geq 0, \]
where the summation runs over all \( a, b \geq 0 \) with \( a+b = r-1 \), \( 1 \leq i \leq a+2 \), and where \( \circ_i \) are the standard partial compositions in a classical operad. The dg operad morphism \( \mathfrak{a} : \mathcal{A}\mathcal{S}\mathcal{S}_\infty \to \mathcal{A}\mathcal{S}\mathcal{S} \) is given by
\[ \mathfrak{a}(\xi_r) := \begin{cases} \mu \in \mathcal{A}\mathcal{S}\mathcal{S}(2) & \text{if } r = 0, \\ 0 & \text{otherwise}, \end{cases} \]
where \( \mu \in \mathcal{A}\mathcal{S}\mathcal{S}(2) \) is the generator of \( \mathcal{A}\mathcal{S}\mathcal{S} \). Notice that \( 1_{\text{Per}} \cong \text{des}^*(\mathcal{A}\mathcal{S}\mathcal{S}) \). Define finally \( \mathcal{M} := \text{des}^*(\mathcal{A}\mathcal{S}\mathcal{S}_\infty) \), and \( \rho : \mathcal{M} \to 1_{\text{Per}} \) by \( \rho := \text{des}^*(\mathfrak{a}) \).

**Theorem 29.** The dg \( \text{Per} \)-operad map \( \rho : \mathcal{M} \to 1_{\text{Per}} \) defined above is a minimal model of the terminal operad \( 1_{\text{Per}} \).

**Proof.** The claim is almost obvious, but we still want to give some details, namely a formula for the differential \( \partial \) in \( \mathcal{M} \). Given \( \alpha : n \to k \in \text{Per} \), \( n \geq k \geq 1 \), one has, by definition,
\[ \mathcal{M}(\alpha) = \text{des}^*(\mathcal{A}\mathcal{S}\mathcal{S}_\infty)(\alpha) = \mathcal{A}\mathcal{S}\mathcal{S}_\infty(k) \]
while
\[ 1_{\text{Per}}(\alpha) \cong \text{des}^*(\mathcal{A}\mathcal{S}\mathcal{S})(\alpha) = \mathcal{A}\mathcal{S}\mathcal{S}(k). \]
Under these identifications, the component \( \rho(\alpha) : \mathcal{M}(\alpha) \to 1_{\text{Per}}(\alpha) \) of the map \( \rho \) equals
\[ \mathfrak{a}(k) : \mathcal{A}\mathcal{S}\mathcal{S}_\infty(k) \to \mathcal{A}\mathcal{S}\mathcal{S}(k), \]
which is a homology isomorphism since \( \mathcal{A}\mathcal{S}\mathcal{S}_\infty \) is the minimal model of \( \mathcal{A}\mathcal{S}\mathcal{S} \). As a non-dg \( \text{Per} \)-operad, \( \mathcal{M} \) is free, generated by the collection \( E \) defined by
\[ E(\alpha) := \text{Span}(\xi_{k-2}), \quad \text{for } \alpha : n \to k, \, k \geq 2, \]
where the \( \xi \)'s are the same as in (30).

Let us denote by \( \xi_\alpha \) the replica of \( \xi_{k-2} \) in \( E(\alpha) \) above. Our next task will be to describe \( \partial(\xi_\alpha) \in \mathcal{M}(\alpha) \). For natural numbers \( k, a, b \in \mathbb{N} \) such that
\[ k = a + b + 3 \geq 2 \]
we define the map \( \gamma_i^{a,b} : k \to a+2 \in \Delta_{\text{semi}} \) by the formula
\[ \gamma_i^{a,b}(j) := \begin{cases} j & \text{for } 1 \leq j \leq i-1, \\ i & \text{for } i \leq j \leq i+b+1, \text{ and} \\ j-b-1 & \text{for } i+b+2 \leq j \leq k. \end{cases} \]
Loosely speaking, \( \gamma_i^{a,b} \) shrinks the interval \( \{i, \ldots, i+b+1\} \subset k \) to \( \{i\} \).

Let \( \alpha : n \to k \in \text{Per} \). By the opfibration property of \( \text{des} : \text{Per} \to \Delta_{\text{semi}} \), there exists a unique \( \alpha_i^{a,b} : n \to a+2 \in \text{Per} \) and a unique morphism \( f_i^{a,b} : \alpha \to \alpha_i^{a,b} \) such that \( \text{des}(f_i^{a,b}) = \gamma_i^{a,b} \). Explicit formulas for \( \alpha_i^{a,b} \) and \( f_i^{a,b} \) can be given easily.
All fibers of $\gamma_{i}^{a,b}$ are trivial (i.e. unique surjections to $1$) except the $i$th one which we denote by $F_{i}^{a,b}$. Since $\mathcal{M}(U) \cong \mathbb{F}(E)(U) \cong k$ for trivial $U$’s as in (17), the structure map (21) for $\mathcal{M}$ corresponding to $\gamma_{i}^{a,b}$ reduces to the ‘partial composition’

$$\circ_{i} : \mathcal{M}(\alpha_{i}^{a,b}) \otimes \mathcal{M}(F_{i}^{a,b}) \to \mathcal{M}(\alpha).$$

The formula for the differential then reads

$$\partial(\xi_{\alpha}) = \sum ((-1)^{b+1}(i+1)+b) \cdot \xi_{\alpha_{i}^{a,b}} \circ_{i} \xi_{F_{i}^{a,b}},$$

where the summation runs over $k, a, b \in \mathbb{N}$ as in (31) and $\xi_{\alpha_{i}^{a,b}}$ (resp. $\xi_{F_{i}^{a,b}}$) are the replicas of $\xi_{a}$ (resp. $\xi_{b}$) in $E(\alpha_{i}^{a,b})$ (resp. in $E(F_{i}^{a,b})$). Formula (32) makes the quadraticity of the differential $\partial$ in $\mathcal{M}$ manifest.

Formula (32) can be written in a more intelligent way. Recall from Example 27 that $F \triangleright_{i} \alpha \xrightarrow{f} \beta$ expresses that $f$ is an elementary morphism whose only nontrivial fiber $F$ is the $i$th one. One then may rewrite (32) as

$$\partial(\xi_{\alpha}) = \sum_{F \triangleright_{i} \alpha \xrightarrow{f} \beta} ((-1)^{|F|+1}(i+1)+|F|) \cdot \xi_{\beta} \circ_{i} \xi_{F},$$

where the summation runs over all elementary maps $F \triangleright_{i} \alpha \xrightarrow{f} \beta$ such that $|\beta| \geq 2$. Following the philosophy of [7, Section 4], we formulate

**Remark 30.** Notice that to both (32) and (33) only partial compositions enter. This is because $\mathcal{M}$ is isomorphic to the dual bar construction of the Koszul dual of $1_{\text{Per}}$, cf. the notes at the beginning of Section 8 and Corollary 45.

**Definition 31.** A strongly homotopy permutad is an algebra for the minimal model $\mathcal{M}$ of $1_{\text{Per}}$.

Strongly homotopy permutads can be described directly via their structure operations:

**Proposition 32.** A strongly homotopy permutad is a collection $A = \{A(n)\}_{n \geq 1}$ of dg vector spaces together with structure maps

$$\pi_{\alpha} : A(n_{1}) \otimes \cdots \otimes A(n_{k}) \to A(n)$$

of degree $k-2$ defined for each $\alpha : n \to k, n \geq k \geq 2 \in \text{Per}$; here $n_{i} := \alpha^{-1}(i), 1 \leq i \leq k$. Moreover, for each such an $\alpha$, the equality

$$(P_{\alpha}) \quad \partial(\pi_{\alpha}) = \sum_{F \triangleright_{i} \alpha \xrightarrow{f} \beta} ((-1)^{|F|+1}(i+1)+|F|) \cdot \pi_{\beta} \circ_{i} \pi_{F}$$

is satisfied. Here the summation is the same as in (33), $\pi_{\beta} \circ_{i} \pi_{F}$ is the multilinear function obtained by inserting $\pi_{F}$ into the $i$th slot of $\pi_{\beta}$, and $\partial$ in the left hand side is the differential on the endomorphism complex induced by the differential of $\{A(n)\}_{n \geq 1}$.
Example 33. If $|\alpha| = 2$, the sum in $(P_\alpha)$ is empty, thus $\partial(\xi_\alpha) = 0$, so $\partial(\xi_\alpha)$ is a dg map. If $\alpha : m \to 3$, the sum in $(P_\alpha)$ has two terms, corresponding to the two possible order-preserving surjections $3 \to 2$. The associated elementary morphisms are

$$v \triangleright_1 \alpha \to u \quad \text{and} \quad s \triangleright_2 \alpha \to t,$$

where $u, v, s, t \in \text{Per}$ are as in Definition 3. Axiom $(P_\alpha)$ now takes the form

$$\partial(\pi_\alpha) = \pi_u \circ_1 \pi_v - \pi_t \circ_2 \pi_s = \pi_u (\pi_v \otimes \mathbb{1}) - \pi_t (\mathbb{1} \otimes \pi_s)$$

The degree 0 operations $\pi_r$ for $r \in \text{Per}$ with $|r| = 2$ are therefore of the same type as the operations $\diamond_r$ of Definition 3, but they satisfy the ‘associativity’ (3) only up to the homotopy $\pi_\alpha$. For $|\alpha| = 4$, $(P_\alpha)$ is a permutadic version of the Mac Lane’s pentagon.

Remark 34. Let $K = \{K_n\}_{n \geq 1}$ be the cellular topological non-$\Sigma$ operad of the Stasheff’s associahedra [11, II.1.6]. Then $M := \text{des}^*(K)$ is the cellular topological $\text{Per}$-operad such that the minimal model $\mathcal{M}$ of $1_{\text{Per}}$ is isomorphic to the associated cellular chain $\text{Per}$-operad $CC_*(M)$. This is where the ‘hidden associahedron’ of the title of this article hides.

Question 35. The classical recognition theorem [12] states that a connected topological space has a weak homotopy type of a based loop space if and only if it is a topological $K$-algebra (aka $A_\infty$-space). Does there exist an analogous statement for topological $M$-permutads?

As in the case of $A_\infty$-algebras, the left hand side of $(P_\alpha)$ can be absorbed into the right one by interpreting $\partial$ as a structure operation. This can be done by allowing $\pi_\alpha$’s also for $|\alpha| = 1$, i.e. for $\alpha$ the terminal object $U_n : n \to 1$, $n \geq 1$. The associated structure map then will be a degree $-1$ linear morphism

$$\pi_n := \pi_{U_n} : A(n) \longrightarrow A(n).$$

Further, we need to allow in the sum of $(P_\alpha)$ also trivial $F$ or $\beta$. The modified axiom reads

$$(P'_\alpha) \quad 0 = \sum_{F, \beta, \alpha} L_{\beta}(\partial(-1)^{|F|+1}(i+1) + |F|) \cdot \pi_{\beta} \circ_i \pi_F$$

For $\alpha = U_n$ it clearly gives $\pi_n^2 = 0$, thus $\pi_n$ is a degree $-1$ differential of $A(n)$. For a general $\alpha$, the left hand side of $(P_\alpha)$ is absorbed in the right hand side of $(P'_\alpha)$ in terms with $|F| = 1$ or $|\beta| = 1$. We leave the details as an exercise.
7. Koszul duals of Per-operads

A ‘classical’ operad is binary quadratic if it is generated by operations of arity 2, and its ideal of relations is generated by relations of arity 3. Each such an operad admits its Koszul, aka quadratic, dual, cf. [3, (2.1.9)] or [11, Definitions II.3.31 and II.3.37]. Similar notions exist for operads in a general operadic category [1, Section 11]. In the remaining two sections we will analyze the particular case of operads in Per. The floor plan is similar to that of the parallel theory for permutads presented in Sections 2 and 3, so we can afford to be more telegraphic. We start with

Definition 36. A Per-operad \( \mathcal{P} \) is binary quadratic if it is of the form \( \mathcal{P} \cong \mathcal{F}(E)/(R) \), where

(i) the generators \( E = \{E(\alpha)\}_{\alpha \in \text{Per}} \) are such that \( E(\alpha) = 0 \) if \( |\alpha| \neq 2 \), and

(ii) the generators \( R \) of the ideal of relations form a subcollection of \( \mathcal{F}^2(E) \).

In Definition 36, \( \mathcal{F}(E) \) is the free Per-operad generated by the collection \( E \). In concrete examples treated in the remainder of this section, \( E \) will always be the restriction \( \text{des}^*(E) \) of some \( E \) in \( \Delta_{\text{semi-Coll}}_1 \), thus it may be, by the virtue of Proposition 24, realized by the restriction \( \text{des}^*(\mathcal{F}(E)) \) of the free non-\( \Sigma \) operad \( \mathcal{F}(E) \). The following notion is however recalled for an arbitrary \( E \).

The Koszul dual \( \mathcal{P}^! \) of a binary quadratic Per-operad \( \mathcal{P} [1 \text{ Definition 11.3}] \) is the quotient

\[ \mathcal{P}^! := \mathcal{F}(\uparrow E^*)/(R^\perp), \]

where \( \uparrow E^* \) is the suspension of the component-wise linear dual of the generating Per-collection \( E \), and \( R^\perp \subset \mathcal{F}^2(\uparrow E^*) \) is the component-wise annihilator of \( R \subset \mathcal{F}^2(E) \) in the obvious degree \(-2\) pairing between

\[ \mathcal{F}^2(\uparrow E^*)(\alpha) = \bigoplus_{F \ni_\alpha \beta} \uparrow E(\beta)^* \otimes \uparrow E(F)^* \cong \bigoplus_{F \ni_\alpha \beta} E(\beta)^* \otimes E(F)^* \]

and

\[ \mathcal{F}^2(E)(\alpha) = \bigoplus_{F \ni_\alpha \beta} E(\beta) \otimes E(F); \]

here we use the explicit description of \( \mathcal{F}^2(-) \) given in Example 27. The following statement follows from [1, Proposition 14.4] but will present a proof that uses our explicit knowledge of the minimal model \( \mathfrak{M} \) of \( \mathbf{1}_{\text{Per}} \) acquired in Section 6.

Proposition 37. The operad \( \mathbf{1}_{\text{Per}} \) is binary quadratic. It is self-dual in the sense that the category of algebras over \( \mathbf{1}_{\text{Per}} \) is isomorphic to the category of permutads via the functor induced by the suspension of the underlying collection.
Proof. By Theorem 29, $1_{\text{Per}} \cong H_0(\mathcal{M})$. Since $\mathcal{M}$ is non-negatively homologically graded, its suboperad $Z_0(\mathcal{M})$ of degree 0 cycles equals the degree 0 piece $\mathcal{M}_0$ of $\mathcal{M}$. Clearly $\mathcal{M}_0 = \mathbb{F}(E)_0 \cong \mathbb{F}(E_0)$ where, by the definition of the generating collection $E$,

$$E_0(\alpha) = \begin{cases} \text{Span}(\xi_0) & \text{if } |\alpha| = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Likewise, $\mathcal{M}_1 = \mathbb{F}(E)_1$ consists of compositions of some number of elements of $E_0$ and precisely one element of $E_1$, where

$$E_1(\alpha) = \begin{cases} \text{Span}(\xi_1) & \text{if } |\alpha| = 3, \\ 0 & \text{otherwise.} \end{cases}$$

Thus $B_0(\mathcal{M}) = \text{Im}(\partial : \mathcal{M}_1 \to \mathcal{M}_0)$ equals the ideal generated by $\partial(\xi_\alpha)$, $|\alpha| = 3$, where $\xi_\alpha$ is the replica of $\xi_1$ in $E(\alpha)$. Using formula (33) for the differential, we conclude that

$$1_{\text{Per}} \cong \mathbb{F}(E_0)/\langle \xi_u \circ_1 \xi_v - \xi_t \circ_2 \xi_s \rangle,$$

where $u, v, s$ and $t$ have the same meaning as in Example 33. This is the required binary quadratic presentation of the terminal $\text{Per}$-operad. The natural pairing

$$\mathbb{F}^2(\uparrow E^*) \otimes \mathbb{F}^2(\uparrow E)^* \longrightarrow k$$

is given by

$$\langle \xi_a \circ_i \xi_b | \xi_c \circ_j \xi_d \rangle := \begin{cases} 1 \in k & \text{if } a = c, b = d \text{ and } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

In the above formula, the up-going arrow indicates the corresponding suspended generator. One thus immediately gets

$$1^{i}_{\text{Per}} \cong \mathbb{F}(\uparrow E_0)/\langle \xi_u \circ_1 \xi_v^\uparrow + \xi_t \circ_2 \xi_s^\uparrow \rangle.$$ 

As in the proof of Proposition 23 we verify that $1^{i}_{\text{Per}}$-algebras are collections $A = \{A(n)\}_{n \geq 1}$ equipped with degree +1 operations

$$\diamondsuit^i_r : A(r^{-1}(1)) \otimes A(r^{-1}(2)) \longrightarrow A(n)$$

satisfying the ‘anti-associativity’

$$\diamondsuit^i_u(\diamondsuit^i_v \otimes 1) + \diamondsuit^i_r(1 \otimes \diamondsuit^i_s) = 0.$$

Let $\uparrow A := \{\uparrow A(n)\}_{n \geq 1}$ be the component-wise suspension of the collection $A$. It is straightforward to see that the operations

$$\diamondsuit_r : \uparrow A(r^{-1}(1)) \otimes \uparrow A(r^{-1}(2)) \longrightarrow \uparrow A(r^{-1}(1)) \otimes A(r^{-1}(2)) \longrightarrow A(n) \longrightarrow \uparrow A(n)$$

make $\uparrow A$ a permutad. The correspondence $(A, \diamondsuit^i_r) \mapsto (\uparrow A, \diamondsuit_r)$ is clearly an isomorphism between the categories of $1^{i}_{\text{Per}}$-algebras and $1_{\text{Per}}$-algebras. One may in fact show that the ‘operadic desuspension’ $s^{-1} : 1_{\text{Per}} \to 1_{\text{Per}}$ defined by $s^{-1}(\alpha) := \uparrow |\alpha|^{-2}$ is an isomorphism of $\text{Per}$-operads. \[perm.tex\]
The following proposition will be formulated for any strict operadic functor [2, p. 1635] between arbitrary operadic categories, but the reader might as well consider only the case of des : Per → Δsemi.

**Proposition 38.** Let \( p : \emptyset \to \mathcal{P} \) be a strict operadic functor, \( \mathcal{P} \) a \( \mathcal{P} \)-operad and \( \mathcal{I} \subset \mathcal{P} \) an ideal. Then

(i) the restriction \( p^*(\mathcal{I}) \) is an ideal in the 0-operad \( p^*(\mathcal{P}) \).

(ii) If \( \mathcal{I} \) is generated by \( G \subset \mathcal{I} \), then \( p^*(\mathcal{I}) \) is generated by the restriction \( p^*(G) \). Finally,

(iii) the restriction \( p^*(\mathcal{P}/\mathcal{I}) \) of the quotient \( \mathcal{P}/\mathcal{I} \) is isomorphic to \( p^*(\mathcal{P})/p^*(\mathcal{I}) \).

**Proof.** Items (i) and (ii) easily follow from the basic properties of operads in operadic categories and their ideals. Let us establish item (iii). Since the restriction \( p^* \) is a functor \( \mathcal{P}-\text{Oper} \to \mathcal{O}-\text{Oper} \) between the categories of operads [2, p. 1639], applying it to the projection \( \pi : \mathcal{P} \twoheadrightarrow \mathcal{P}/\mathcal{I} \) leads to an operad morphism

\[
p^*(\pi) : p^*(\mathcal{P}) \to p^*(\mathcal{P}/\mathcal{I}),
\]

which is clearly surjective.

We want to prove that \( \text{Ker}(p^*(\pi)) \cong p^*(\mathcal{I}) \). Assume that \( u \in p^*(\mathcal{P})(t) \) for some \( t \in \emptyset \) is such that \( p^*(\pi)(u) = 0 \). By the definition of the restriction, \( p^*(\mathcal{P})(t) = \mathcal{P}(T) \) with \( T := p(t) \) and, likewise, \( p^*(\mathcal{P}/\mathcal{I})(t) = (\mathcal{P}/\mathcal{I})(T) = \mathcal{P}(T)/\mathcal{I}(T) \). Under this identification, \( p^*(\pi) \) acts as the projection \( \mathcal{P}(T) \to (\mathcal{P}/\mathcal{I})(T) = (\mathcal{P}(T)/\mathcal{I}(T)) \), so \( u \in \mathcal{I}(T) = p^*(\mathcal{I})(t) \). \( \square \)

The following corollary can be stated for any discrete opfibration \( p : \emptyset \to \mathcal{P} \), but for our purposes the case of des : Per → Δsemi for which we recalled all the relevant notions will be sufficient.

**Proposition 39.** Let \( \mathcal{P} \) be a binary quadratic non-Σ-operad. Then \( \text{des}^*(\mathcal{P}) \) is binary quadratic and, moreover, \( \text{des}^*(\mathcal{P})^! \cong \text{des}^*(\mathcal{P}^!) \)

**Proof.** Suppose that \( \mathcal{P} = \mathcal{F}(E)/(R) \) is a binary quadratic non-Σ operad. Then, by Proposition [2], \( \text{des}^*(\mathcal{F}(E)) \) is the free \( \text{Per} \)-operad \( \mathcal{F}(E) \) on the restriction \( E := \text{des}^*(E) \) thus, by Proposition [3],

\[
\text{des}^*(\mathcal{P}) \cong \text{des}^*(\mathcal{F}(E))/(\text{des}^*(R)) = \mathcal{F}(E)/(R),
\]

with \( R := \text{des}^*(R) \). It is clear from definitions that \( \mathcal{F}(E)/(R) \) is a binary quadratic presentation of the \( \text{Per} \)-operad \( \mathcal{P} := \text{des}^*(\mathcal{P}) \). The second part of the proposition follows from the canonical isomorphisms

\[ \text{des}^*(\uparrow E^*) \cong \uparrow (\text{des}^*(E))^* \quad \text{and} \quad \text{des}^*(R^+) \cong (\text{des}^*(R))^+ \]

which can be checked directly. \( \square \)
8. Koszulity of \textsf{Per}-operads.

The central object of this section will be the dual bar construction of a 1-connected \textsf{Per}-operad \( P \). To this end we need to represent it as a structure with quadratic operations. This can be done as follows. Recall that a map \( f : \alpha \to \beta \in \textsf{Per} \) in (16) is elementary if all its fibers except precisely one, say the \( i \)th fiber \( F \), are trivial, i.e. equal to some local terminal objects \( U_{n_s}, s \neq i \). This fact was recorded by \( F \triangleleft i \alpha \to \beta \). For each such an \( f \) we define the partial composition \( \circ_f : \mathcal{P}(\beta) \otimes \mathcal{P}(F) \to \mathcal{P}(\alpha) \) as the composite

\[
\mathcal{P}(\beta) \otimes \mathcal{P}(F) \xrightarrow{\cong} \mathcal{P}(\beta) \otimes \mathcal{P}(U_{n_1}) \otimes \cdots \otimes \mathcal{P}(F) \otimes \cdots \otimes \mathcal{P}(U_{n_{k'}}) \xrightarrow{m_f} \mathcal{P}(\alpha)
\]

in which \( m_f \) is the structure map (21) and the vertical map is the product of the identities and isomorphisms \( \mathcal{P}(U_{n_s}) \cong \mathcal{P}(F), s \neq i \), which follow from the 1-connectivity of \( \mathcal{P} \). The partial compositions satisfy appropriate axioms \cite[Definition 7.1]{[perm]} derived from the properties of \( m_f \)'s. The structure described above is Markl’s version of a \textsf{Per}-operad \( \mathcal{P} \). Under the 1-connectivity assumptions, the categories of \textsf{Per}-operads and Markl’s \textsf{Per}-operads coincide \cite[Theorem 7.4]{[perm]}.

A Markl’s \textsf{Per}-cooperad as a collection \( C = \{ C(\alpha) \}_{\alpha \in \textsf{Per}} \) with operations

\[
\Delta_f : C(\alpha) \longrightarrow C(\beta) \otimes C(F), \ F \triangleright \alpha \xrightarrow{f} \beta,
\]

satisfying the dual versions of Markl’s operads \cite[Definition 7.1]{[perm]}. As in Example \cite{[perm]}, under obvious finitary assumptions, the component-wise linear dual \( \mathcal{P}^\ast \) of a Markl’s \textsf{Per}-operad is a Markl’s \textsf{Per}-cooperad. We will need also a reduced version of \( C \) whose underlying \textsf{Per}-collection is defined by

\[
\overline{C}(\alpha) := \begin{cases} 
0 & \text{if } |\alpha| = 1, \\
C(\alpha) & \text{if } |\alpha| \geq 2,
\end{cases}
\]

and its structure operation

\[
\overline{\Delta}_f : \overline{C}(\alpha) \longrightarrow \overline{C}(\beta) \otimes \overline{C}(F)
\]

equals \( \Delta_f \) in (35) if \( |\beta| \geq 2 \) while for \( |\beta| = 1 \) it is the zero map

\[
\overline{C}(\alpha) \longrightarrow \overline{C}(\beta) \otimes \overline{C}(F) = 0.
\]

In fancy language, \( \overline{C} \) is the coaugmentation coideal in Markl’s cooperad \( C \). It follows from (29) that the individual structure operations (35) assemble into a single map

\[
\overline{\Delta} : \overline{C} \cong \mathbb{F}^1(\overline{C}) \to \mathbb{F}^2(\overline{C}).
\]
Definition 40. A degree $s$ linear map $\varpi : \mathcal{P} \to \mathcal{P}$ of Markl’s operad $\mathcal{P}$ if
\[ \varpi \circ f = \circ f(\varpi \otimes 1) + \circ f(1 \otimes \varpi), \]
for every elementary $F \triangleright \alpha \xrightarrow{f} \beta$ and the associated operation $\circ f : \mathcal{P}(\beta) \otimes \mathcal{P}(F) \to \mathcal{P}(\alpha)$.

One easily sees that, given a 1-connected $\text{Per}$-collection, each degree $s$ linear map of $\text{Per}$-collections $\zeta : \mathcal{E} \to \mathcal{F}(\mathcal{E})$ uniquely extends to a degree $s$ derivation $\varpi$ of the free $\text{Per}$-operad $\mathcal{F}(\mathcal{E})$. For a Markl’s $\text{Per}$-cooperad $\mathcal{C}$ one has a degree $-1$ map $\zeta : \downarrow \mathcal{C} \to \mathcal{F}^2(\downarrow \mathcal{C})$ of $\text{Per}$-collections defined as the composition
\[ \zeta := \downarrow \mathcal{C} \xrightarrow{\uparrow} \mathcal{C} \xrightarrow{\Delta} \mathcal{F}^2(\downarrow \mathcal{C}) \xrightarrow{\pi} \mathcal{F}^2(\downarrow \mathcal{C}) \]
where $\Delta$ is as in (38) and $\mathcal{F}^2(\downarrow \mathcal{C}) \xrightarrow{\pi} \mathcal{F}^2(\downarrow \mathcal{C})$ the obvious canonical isomorphism
\[ \mathcal{F}^2(\downarrow \mathcal{C}) \cong \bigoplus_{F > \alpha \downarrow \beta} \mathcal{C}(\beta) \otimes \mathcal{C}(F) \xrightarrow{\pi} \mathcal{F}^2(\downarrow \mathcal{C}) \cong \bigoplus_{F > \alpha \downarrow \beta} \downarrow \mathcal{C}(\beta) \otimes \downarrow \mathcal{C}(F) \cong \mathcal{F}^2(\downarrow \mathcal{C}). \]
Denote finally by $\partial_0$ the unique extension of $\zeta$ into a degree $-1$ derivation of $\mathcal{F}(\downarrow \mathcal{C})$. One easily verifies that $\partial_0^2 = 0$.

As we noticed at the beginning of this section, under the 1-connectivity assumption, operads and Markl’s operads are just different presentations of the same objects, which is true also for (Markl’s) cooperads. We will therefore make no difference between them. Having this in mind, we formulate

Definition 41. The cobar construction of a $\text{Per}$-cooperad $\mathcal{C}$ is the dg $\text{Per}$-operad $\Omega(\mathcal{C}) := (\mathcal{F}(\downarrow \mathcal{C}), \partial_0)$. The dual bar construction of a $\text{Per}$-operad $\mathcal{P}$ satisfying appropriate finitarity assumptions is the dg $\text{Per}$-operad $\mathcal{D}(\mathcal{P}) := \Omega(\mathcal{P})$.

To introduce the Koszulity of a binary quadratic $\text{Per}$-operad $\mathcal{P}$, one starts from an injection $\uparrow \mathcal{E} \hookrightarrow \mathcal{P}'$ of $\text{Per}$-collections defined as the composite
\[ \uparrow \mathcal{E} \hookrightarrow \mathcal{F}(\uparrow \mathcal{E}) \twoheadrightarrow \mathcal{F}(\uparrow \mathcal{E})/(R^\perp) = \mathcal{P}'. \]
Its linear dual $\mathcal{P}'^* \hookrightarrow \uparrow \mathcal{E}$ desuspends to a map $\pi : \downarrow \mathcal{P}'^* \to \mathcal{E}$. As for permutads, one has the related twisting morphism $\downarrow \mathcal{P}'^* \to \mathcal{P}$, which is the composition
\[ \downarrow \mathcal{P}'^* \xrightarrow{\pi} \mathcal{E} \hookrightarrow \mathcal{F}(\mathcal{E}) \twoheadrightarrow \mathcal{F}(\mathcal{E})/(R) = \mathcal{P}. \]
By the freeness of $\mathcal{F}(\downarrow \mathcal{P}'^*)$, it extends to a morphism $\rho : \mathcal{F}(\downarrow \mathcal{P}'^*) \to \mathcal{P}$ of dg $\text{Per}$-operads. One verifies by direct calculation:

**Proposition 42.** The morphism $\rho$ induces the canonical map
\[ \text{can} : \mathcal{D}(\mathcal{P}') = (\mathcal{F}(\downarrow \mathcal{P}'^*), \partial_0) \to (\mathcal{P}', 0) \]
of dg $\text{Per}$-operads.
Definition 43. A binary quadratic Per-operad $P$ is Koszul if the canonical map (39) is a component-wise homology isomorphism.

In the following theorem, $\Omega(C)$ denotes the cobar construction of a 1-connected classical non-$\Sigma$ cooperad $C$, i.e. a non-$\Sigma$ version of the construction in [3, Section 6.5.2], and $D(P) := \Omega(P^*)$ the dual bar construction of a 1-connected non-$\Sigma$ operad $P$ with appropriate finitary properties that make the dualization possible.

Theorem 44. Let $C$ and $P$ be as above, and $C := \des^*(C)$, $P := \des^*(P)$. Then

(i) the dg Per-operads $\Omega(C)$ and $\des^*(\Omega(C))$ are isomorphic, as they are
(ii) the dg Per-operads $D(P)$ and $\des^*(D(P))$.
(iii) Assume that $P$ is binary quadratic. Then the Per-operad $P$ is (binary quadratic) Koszul if and only if $P$ is one.

Proof. The restriction along $\des : \text{Per} \to \Delta_{\text{semi}}$ is a functor from the category of dg $\Delta_{\text{semi}}$-collections to the category of dg Per-collections. We already noticed that the restriction takes $\Delta_{\text{semi}}$-operads, i.e. non-$\Sigma$ operads, to Per-operads. The same is true for Markl’s operads and cooperads. The restriction also obviously commutes with the suspensions, component-wise linear duals and cohomology. Moreover, we established in Proposition 24 that, since $\des : \text{Per} \to \Delta_{\text{semi}}$ is a discrete opfibration, it brings free $\Delta_{\text{semi}}$-operads to free Per-operads. By Proposition 39, it takes binary quadratic operads to binary quadratic ones and commutes with the Koszul duals. These facts suffice to prove what we want.

Let us prove for instance (i). The underlying non-dg Per-operads $\des^*(\llbracket C \rrbracket)$ and $\llbracket C \rrbracket$ are isomorphic by Proposition 24. We must show that $\des^*(\partial_\Omega) = \partial_\Omega$. The differential $\partial_\Omega$ in $\Omega(C)$ is the unique extension of the composition

$$\zeta := \llbracket C \rrbracket \xrightarrow{\uparrow \llbracket C \rrbracket} \llbracket C \rrbracket \xrightarrow{\Delta} \llbracket C \rrbracket \xrightarrow{\sim} \llbracket C \rrbracket \xrightarrow{\partial_2} \llbracket C \rrbracket \xrightarrow{\partial_2} \llbracket C \rrbracket$$

whose constituents have, as we believe, obvious meanings, while $\partial_\Omega$ is the extension of the composition $\zeta : \llbracket C \rrbracket \to \llbracket C \rrbracket$ in (38). It follows from the above reasoning that $\zeta = \des^*(\zeta)$, therefore also their unique extensions $\partial_\Omega$ and $\des^*(\partial_\Omega)$ agree. The remaining parts of the theorem can be established analogously.

Corollary 45. The terminal Per-operad $1_{\text{Per}}$ is Koszul.

Proof. One way of proving the statement would be to identify $D(1_{\text{Per}})$ with the minimal model $\mathcal{M}$ described in Proposition 29. The corollary however follows from Theorem 44 since $1_{\text{Per}}$ is the restriction of the terminal non-$\Sigma$ operad $\As$ whose Koszulity is superclassical.

Notice that each 1-connected cooperad is coaugmented.
References

[1] M.A. Batanin and M. Markl. Koszul duality in operadic categories. Preprint arXiv:1812.02935, version 2, February 2019.

[2] M.A. Batanin and M. Markl. Operadic categories and duoidal Deligne’s conjecture. Adv. Math., 285:1630–1687, 2015.

[3] V. Ginzburg and M.M. Kapranov. Koszul duality for operads. Duke Math. J., 76(1):203–272, 1994.

[4] J.-L. Loday and M.O. Ronco. Permutads. J. Combin. Theory Ser. A, 120(2):340–365, 2013.

[5] J.-L. Loday and B. Vallette. Algebraic operads, volume 346 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer, Heidelberg, 2012.

[6] S. Mac Lane. Categories for the working mathematician. Springer-Verlag, New York, 1971. Graduate Texts in Mathematics, Vol. 5.

[7] M. Markl. Models for operads. Comm. Algebra, 24(4):1471–1500, 1996.

[8] M. Markl. Homotopy algebras are homotopy algebras. Forum Math., 16(1):129–160, 2004.

[9] M. Markl. Bipermutahedron and biassociahedron. J. of Homotopy and Related Structures, 10:205–238, 2015.

[10] M. Markl and E. Remm. (Non-)Koszulness of operads for n-ary algebras, galgalim and other curiosities. J. of Homotopy and Related Structures, 10:939–269, 2015.

[11] M. Markl, S. Shnider, and J.D. Stasheff. Operads in algebra, topology and physics, volume 96 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2002.

[12] J.D. Stasheff. Homotopy associativity of H-spaces, I. Trans. Amer. Math. Soc., 108:275–292, 1963.

[13] A. Tonks. Relating the associahedron and the permutohedron. In J.L. Loday, J.D Stasheff, and A.A. Voronov, editors, Operads: Proceedings of Renaissance Conferences, volume 202 of Contemporary Math., pages 33–36. Am. Math. Soc., 1997.