Vibration of a continuous beam excited by a moving mass and experimental validation

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Abstract. The work presented in this paper deals with the vibration of a continuous slender beam excited by a mass moving at various speeds along it. An experimental model is designed and set up to study this problem. This model, which consists of a four-span continuous beam traversed by a moving mass at a constant speed, is used to build a theoretical model for the supporting structure. A series of tests designed to assess the accuracy of the model are carried out. The final section of the paper is dedicated to the numerical and experimental results and discussion.

1. Introduction
The problem of an MDOF system travelling along a flexible structure has long been studied in relation to the vehicle-bridge interaction dynamics. The supporting structure has commonly been modelled as a beam whereas the moving structure is modelled with different degrees of complexity in order to take into account various geometric and dynamic characteristics. Although this type of interaction is a complex phenomenon, which is influenced by many parameters as mentioned in [1], there might be some limiting cases where the interest rests mainly in the dynamics of the supporting structure and the flexibility of the moving structure can be neglected. In many of these cases, the moving structure can be assimilated with a moving mass with an important reduction in the number of parameters that may influence the model.

Although a lot of work has been dedicated to the theoretical study of this problem, yet few studies were concerned with any experimental investigation. Many of the experimental results previously mentioned in the literature focused on the direct validation of the model by comparing it with the experimental results obtained when a real vehicle moves along a bridge [2, 3]. To the authors’ best knowledge, few references in literature were dedicated to studying small-scale models. In [4], an experimental model consisting of a single-span aluminium beam subjected to a moving mass train was used to assess the capabilities of a FEM formulation to predict the dynamic response of a beam under a moving continuous load. Bilello [5] used a scaled model to assess the dynamic response of a mass moving along an aluminium beam and came to the conclusion that the measured response is basically due to the first beam mode. A great body of work was carried out in studying the inverse problem of contact force identification using the experimental data, for example, in a series of papers by Zhu and Law [6-8].

The object of the following study is the experimental investigation of the vibration of a multiple-span continuous beam excited by a rigid body modelled as a mass moving at constant speed over it.
2. Experimental set up

The moving load experiment uses a 90 mm diameter steel ball rolling along a multi-span steel strip 3.64 m long, 100 mm wide and 3.12 mm thick. Each of the four spans is 910 mm long. A schematic of the test setup is shown in Fig. 1. The first roadway pivot is fixed in the longitudinal, transverse and vertical directions. The remaining roadway pivots allow free movement in the longitudinal direction but are fixed in the transverse and vertical directions. All pivots are free to rotate, their pivot centres being at the roadway thickness centre.

The ball is released from some point on the acceleration ramp and enters the first span of the roadway. As the ball proceeds along the roadway the vertical flexing of the roadway is measured by the eight laser displacement transducers.

The ball is kept on the roadway centre by a ‘U’ shaped plastic track glued to the roadway surface. At the end of its run, the ball is caught in a sand trap.

Laser beams across the roadway at ball centre height are positioned above each roadway pivot. As the ball breaks each laser beam a timing signal is sent to the data logger, with the first one being used as a trigger to start data logging. Time domain data is saved from the laser beam switches and from the eight laser displacement transducers for the period that the ball is on the roadway and for some seconds following to record the roadway vibration decay.

The data logger is an LMS Scadas III system using LMS TestLab software. All nine channels used are recorded simultaneously.

![Figure 1. Schematic arrangement of moving load experiment](image)

3. Theoretical Model

3.1. Assumptions and free vibration response

The supporting structure presented in Fig. 1 is modelled under the assumptions: the moving mass travels on the central axis of the structure only without exciting any torsional vibration, the structure has a linear elastic behaviour with small deflections, the effects of shear deformation and rotary inertia are neglected as well as the effect of the initial sag, the contact between the two structures is assumed smooth without friction. The assumption made about the supports is that they restrain all degrees of freedom of the beam except the rotation in the vertical plane. The model proposed is a four-span continuous beam. The flexural rigidity $EI$ and mass per unit length $\rho A$ are assumed constant on the whole length of the beam $L$. An experimentally determined frequency-dependent damping denoted by the letter $c$ will be considered.

The dynamic behaviour of an Euler-Bernoulli beam oriented along the $x$-axis when crossed by a mass $m$ moving from left to right at constant speed $v$ is governed by equation (1):

$$EI \frac{\partial^4 W}{\partial x^4}(x,t) + \rho A \frac{\partial^2 W}{\partial t^2}(x,t) + \rho Ac \frac{\partial W}{\partial t}(x,t) = -m(\ddot{W}(vt,t) + g) \delta(x- vt)$$

where $\delta$ is Dirac delta function and the over-dots represents the total derivative with respect to time $t$. 

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The transverse deflection \( w(x, t) \) of the vibrating continuous beam will be approximated as a finite series expansion of the form:

\[
w(x, t) = \sum_{k=1}^{N} \psi_k(x)q_k(t)
\]

The modal shape function corresponding to mode \( k = 1 \) to \( N \) restrained to each span for a multi-span beam with transverse displacement constrained at the supports can be written as:

\[
\psi_k(x) = A_j \sin(\beta_k x) + C_j \sinh(\beta_k x) + B_j (\cos(\beta_k x) - \cosh(\beta_k x))
\]

where coordinate \( x \) varies from zero to the span length \( L_s \). The coefficients \( A_j, B_j \) and \( C_j \) for each span \( j = 1 \) to \( 4 \) are to be determined by imposing continuity conditions for displacements, slopes and moments at the supports [9]. For different continuous beam configurations, the modal shape functions are tabulated or presented in a detailed form in many references [9, 10]. A slight inconvenience worth mentioning when using modal beam functions is that they contain hyperbolic functions whose accuracy is poor when the terms \( \beta_k x \) or \( \beta_k L_s \) increases and special attention should be paid to this [11, 12]. Different approaches to deal with a continuous beam were presented in [13, 14].

Generally, any commercial software with symbolic capabilities is good enough to solve the four equations with all nine remaining constraints both for frequencies and for coefficients. For this study, some capabilities of Mathematica to simplify or reduce parts of a symbolic expression were used. The frequency equation determined for the case of a four-span beam contains three factors:

\[
\sin(\beta L_s)
\]

\[
\cos(\beta L_s)^2 + 2 \frac{\sin(\beta L_s)}{\sinh(\beta L_s)} + \coth(\beta L_s)^2 \sin(\beta L_s)^2 - 2 \coth(\beta L_s) \sin(2\beta L_s)
\]

which when equated with zero give three sets of solutions:
- \( \beta_k L_s = k\pi \) (exact solution) which corresponds to the modes 1+4\( k \) which are asymmetric about all three support points (first row of Figure 2);
- \( \beta_k L_s \approx k\pi + \frac{\pi}{12} \) and \( \beta_k L_s \approx k\pi + \frac{5\pi}{12} \) which correspond to modes 2+4\( k \) and 4+4\( k \) respectively with symmetry about the midpoint (second and forth rows of Figure 2);
- \( \beta_k L_s \approx k\pi + \frac{\pi}{4} \) that correspond to modes 3+4\( k \) which are anti-symmetric about the midpoint and symmetric about the first and third supports (third row of Figure 2).

The natural frequencies can be further calculated from:

\[
f_k = \frac{1}{2\pi} \beta_k^2 \sqrt{\frac{EI}{\rho A}}.
\]

The experimentally and theoretically determined natural frequencies of the four-span beam are shown in Table 1 and the first four mode shapes in Fig. 2. In order to obtain a better accuracy of the theoretical frequency values, the beam’s thickness, which, for a rectangular cross section beam, controls the flexural stiffness to unit mass ratio, is modified to 3.045 mm.
A brief analysis of this result shows that the continuous beam model considered can predict the free vibration response of the experimental structure with a high degree of accuracy.

Table 1. Natural frequencies of the four-span continuous beam

| Mode | Experimental frequency (Hz) | Theoretical frequency (Hz) | Relative Error [%] | Modal damping (%) |
|------|-----------------------------|-----------------------------|--------------------|-------------------|
| 1    | 8.37                        | 8.41                        | 0.44               | 2.29              |
| 2    | 9.80                        | 9.81                        | 0.10               | 3.07              |
| 3    | 13.25                       | 13.13                       | 0.91               | 1.58              |
| 4    | 17.21                       | 16.97                       | 1.39               | 1.16              |
| 5    | 33.38                       | 33.63                       | 0.75               | 0.77              |
| 6    | 36.55                       | 36.50                       | 0.14               | 0.89              |
| 7    | 42.78                       | 42.56                       | 0.51               | 0.56              |
| 8    | 49.57                       | 49.10                       | 0.95               | 0.51              |

Mode 1

Mode 2

Mode 3
Figure 2. Mode shapes. continuous line: – theoretically determined, dots : experimental results

3.2. Theoretical Model for the interaction of two structures
The beam’s differential equation (1) can be solved by replacing the deflection function \( w(x, t) \) with the series expansion shown in (2). This results in a linear system of ordinary differential equations with variable coefficients whose matrix form can be written as:

\[
(M + \Delta M(vt))\ddot{q}(t) + (D + \Delta D(vt))\dot{q}(t) + (K + \Delta K(vt))q(t) = -m g\psi(vt)
\]

where the constant matrices \( M, D, \) and \( K \) elements are defined as:

\[
M_{kk} = \int_0^L \psi_k^2(x)dx, \quad D_{kk} = c(f_i)M_{kk}, \quad K_{kk} = \beta_k^4M_{kk};
\]

\[
M_{ki} = 0, \quad D_{ki} = 0 \quad K_{ki} = 0 (k \neq i)
\]

with \( c(f_i) \) the frequency-dependent damping mentioned in section 3.1, and \( i, k = 1, 2, \ldots N; \) and

\[
\Delta M_{ki}(vt) = m \psi_k(vt)\psi_i(vt), \quad \Delta D_{ki}(vt) = 2m \psi_k(vt)\psi_i'(vt),
\]

\[
\Delta K_{ki}(vt) = m v^2 \psi_k(vt)\psi_i''(vt).
\]

In equation (6), the mode shape functions and the modal coordinates are assembled into two vectors:

\[
\psi(x) = (\psi_1(x) \quad \psi_2(x) \quad \ldots \quad \psi_N(x))^T, \quad q(t) = (q_1(t) \quad q_2(t) \quad \ldots \quad q_N(t))^T.
\]

4. Numerical Results
A series of tests are run for different speeds (ranged from 1.8 m/s to 3.7 m/s) of the moving mass. The speed shows very small variations along the beam’s length. Therefore it is assumed constant all over the beam.
Figure 3. Time history of beam deflection (mm) versus \( vt \) at four different points along the beam (- - experiment, - - theory)

These tests are used to compare the theoretical results obtained using the method presented in section 3 with the experimental results. The time history of the beam deflection (mm) at \( v=3.32 \) m/s at four points (at approximately \( 2L/3 \), \( 5L/3 \), \( 7L/3 \), \( 11L/3 \)) is shown in Figure 3. For this model, the weight of the beam and the initial sag together with their influence on the results have been neglected in relation with the dynamic effects induced by the moving mass. The mass exits the beam at \( 1L \) and the beam’s oscillations continue freely, from \( 1L \) to above \( 2L \).

The graphical comparison of the figures shows that the model slightly underestimates the experimental results. There are also noticeable oscillations of the experimentally obtained signals that are “cut” by the theoretical model (the early history in the bottom left graph of Fig. 3), and this is not due to the number of modes considered for approximation. In fact, the theoretically obtained deflections show a fast convergence when the number of modes \( N > 8 \). In this case, 16 modes are considered.

It is also visible that the experimentally measured signal shown in Figure 3, presents a beat-like phenomenon whose nodes, around \( 1L \), \( 3L/2 \) and \( 2L \) are poorly captured by the theoretical model.
The accuracy of the results decreases with the increase in the moving speed and the results shown in Figure 3 presents one of the worst matches obtained.

The beam deflections (in mm) at time 0.38 and 0.68 are shown in Figure 4, along with the mass position at that time (marked by a small red circle). The experimental results are taken at eight equally spaced points along the beam. In this figure, the experimental results are replaced by a spline-interpolated curve shown in dashed line. The curve fits exactly through the discrete experimental data points, but it may show a slightly different slope at the entry point (0L) and exit point (1L). The support positions are marked by ticks on the horizontal axis.

There are some discrepancies between the experimental data and the numerical results although these differences are less than 1 mm. The contact force theoretically estimated (solution approximated with 20 modes) shows only positive values, which suggests that there is permanent contact between the supporting structure and the moving mass.

**Figure 4.** Beam’s deflection at $t = 0.38$ and $0.68$s
A method that might be thought to be simpler in some respects to deal with the problem of a continuous beams subjected to a moving structure is presented in [15] where the beam is considered to have a single span and the supports are introduced as elastic constraints whose elasticity is increased to a very high value. That method presents the advantage of dealing with the modes corresponding to only one-span beam. The results in terms of displacements obtained with both methods are similar.

Within the speed range tested, no separation between the moving ball and the beam structure is found. It is believed that higher moving speeds should cause separation with a subsequent change of the dynamic response [15-19]. The limit speed that allows separation, based on the theoretical model is determined to be 3.9 m/s. Further tests at higher moving speeds are now underway. When the stationary structure has many spans or it is complicated structure, the numerical (the finite element method) and analytical combined method [20] should be used.

5. Conclusions

The main object of this paper is to study the possibility of this model to represent the dynamic effects that occur when a four-span continuous beam is excited by a moving mass. Both experimentally obtained time history of the beam deflection and beam deflection shape at different fixed time instants are compared with the theoretical results. Although there are some discrepancies, their correlation is reasonably good. Free vibration of the continuous beam is also studied. The method proposed shows a good capability in representing the mode shapes and the natural frequencies experimentally obtained. It can be concluded that the theoretical model is more accurate at low speeds and the displacement response converges when the number of modes is greater than 8. The contact force variation determined based on the theoretical model shows that no separation occurs in spite of the high flexibility of the beam and the high travelling speed.

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