Jean Bourgain’s analytic partition of unity via holomorphic martingales

Paul F. X. Müller*
Division of Mathematics, California Institute of Technology
CA 91125, Pasadena.

and
Institut für Mathematik J. Kepler Universität
Linz, Austria.

24 September 1992

Abstract
Using stopping time arguments on holomorphic martingales we present a soft way of constructing J. Bourgain’s analytic partitions of unity. Applications to Marcinkiewicz interpolation in weighted Hardy spaces are discussed.

1 Introduction
In his 1984 Acta Mathematica paper Jean Bourgain derives new Banach space properties of $H^\infty$ and the disc algebra from the existence of the following analytic partition of unity:

**Theorem 1 (J. Bourgain)** Given $f$, a strictly positive integrable function on $T$ with $\int f(t)dt = 1$ and $0 < \delta < 1$ then, there exist functions $\tau_j, \gamma_j \in H^\infty(T)$ and positive numbers $c_i$ such that:

1. $\|\gamma_j\|_\infty < C$
2. $\sum |w_j| < C$
3. $|\tau_j|f < c_j$
4. $\sum c_j\|\tau_j\|_1 < \delta^{-C}$
5. $\int |1 - \sum \gamma_j \tau_j^2|f dt < \delta.$

*Supported by FFWF Pr.Nr. JP 90061
Here I wish to present a soft way to this construction which results from the use of **probabilistic tools** such as holomorphic martingales. I should like to point out here that a proof for the existence of analytic partitions of unity – much simpler than J. Bourgain’s – has been given recently by Serguei Kislyakov. See [K1] and [K2].

In [K3] S. Kislyakov derived J. Bourgain’s result on p-summing operators from the following weighted Marcinkiewicz decomposition.

**Theorem 2 (S. Kislyakov)** For any positive weight $b$ on $T$ there exists a weight $B \geq b$ and $\int B \, dt < C \int b \, dt$ so that for any $\lambda > 0$ and $f \in H^1(T, B)$ there exists $g \in H^\infty(T)$ and $h \in H^1(T, B)$ satisfying:

1. $f = g + h$
2. $\|g\|_\infty \leq \lambda$
3. $\int hB \, dt \leq C \int \{|f| > \lambda\} |f| \, B \, dt$.

Up to small perturbations we shall obtain a **stochastic version** of Kislyakov’s decomposition which allows us to prove the following:

**Theorem 3 (J. Bourgain)** For any 2-summing operator $S$ on the disc algebra and any $2 < q < \infty$ the $q$-summing norm satisfies the interpolation inequality

$$\pi_q(S) = C_q \pi_2(S) \frac{2}{q} \|S\|^{1 - \frac{q}{2}}.$$

A very elegant proof of this interpolation inequality has been given by Gilles Pisier who used vectorvalued $H^1$ spaces. See [P].

## 2 The main result

Holomorphic martingales were introduced by N. Varopoulos in [V]. They are stable under stopping times, and generalize analytic functions on the unit circle. This connection has lead to probabilistic proofs of several results in Analysis, including Carleson’s corona theorem [V], the existence of a logmodular Banach algebra having no analytic structure [C] and P.W. Jones’s interpolation theorems between $H^1$ and $H^\infty$ [M1,M2].

This paper is not self-contained! We freely use notations and definitions from [V] without further explanation.

**Theorem 4** Given $\Delta$, a strictly positive integrable function on $(\Omega, P)$ with $\int \Delta \, dP = 1$ and $0 < \delta < 1$, there exist functions $w_j, \theta_j \in H^\infty(\Omega)$ and positive numbers $c_i$ such that:

1. $\|\theta_j\|_\infty < C$
2. $\sum |w_j| < C$
3. $|w_j|\Delta < c_j$
4. $\sum c_j\|w_j\|_1 < \delta^{-\zeta}$
5. $\int |1 - \sum \theta_j w_j^2|\Delta dP < \delta$

Probability offers a **soft way of constructing** the functions $\theta_j$ so that the verification if (5) becomes much easier than in J. Bourgain’s proof. See [B, pp. 11, 12]. The probabilistic concept will be merged with analytic tools, such as Havin’s lemma, which we use in the following form, due to Bourgain:

**Theorem 5** *For every measurable subset $E$ of $\Omega$ and $0 < \epsilon < 1$ there exist functions $\alpha, \beta \in H^\infty(\Omega)$ such that:*

1. $|\alpha| + |\beta| \leq 1$
2. $|\alpha - \frac{1}{2}| < \epsilon$ on $E$.
3. $|\beta| < \epsilon$ on $E$.
4. $\|\alpha\|_1 < C|\log \epsilon|^2 P(E)$
5. $\|1 - \beta\|_2 < |\log \epsilon|^2 P(E)^{\frac{1}{2}}$.

**Proof of Theorem 4:** We shall first determine a new weight: Let $d$ be the outer function, so that $|d| = \Delta$ and put

$$A(\Delta) := sup_t|E(d|F_t)|$$

then we let

$$\Delta_1 = \sum_{n=0}^{\infty} A^n(\Delta)(C2)^{-n}$$

where $C$ is determined by Varopoulos’ inequality: For $d \in H^1(\Omega)$

$$\int sup_t|E(d|F_t)| < C\|d\|_1$$

Clearly, this construction gives,

1. $A(\Delta_1) < \Delta_1 3C$
2. $\Delta < C\Delta_1$
3. $\int \Delta_1 dP < C \int \Delta dP$. 

3
We next define holomorphic partitions of unity: Let $\Psi$ be the outer function so that $|\Psi| = \Delta_1$. Consider now the stopping times $\tau_0 = 0$ and

$$\tau_j := \inf\{t > \tau_{j-1} : |E(\Psi|\mathcal{F}_t)| > M_j\}$$

to define $\Psi_i := E(\Psi|\mathcal{F}_{\tau_j})$ and $d_j := \Psi_{j+1} - \Psi_j$, elements of $H^\infty(\Omega)$ for or which, obviously the identity

$$1 = \frac{E(\Psi)}{\Psi} + \sum_{j=0}^\infty \frac{d_j}{\Psi}$$

holds. The summands of the above expression will be our choice of $\theta_i$: Indeed we define $\theta_{-1} := \frac{E(\Psi)}{\Psi}$ and $\theta_i := \frac{d_i}{\Psi}$ for $i = 0, 1, 2, \ldots$. Obviously we obtain

$$\|\theta_i\|_\infty \leq C.$$

Havin’s lemma allows us to truncate the above partition of unity: We apply it to sets $E_i := \{\Psi^* > M^i\}$ and denote the resulting functions by $\alpha_i, \beta_i$. Then define for $i = -1, 0, 1, \ldots$

$$w_i := 5\alpha_0 \prod_{s=8}^\infty \beta_{i+s}$$

**Verification of property (5).**

We first eliminate the weight $\Delta$:

$$\int |1 - \sum_{i=-1}^\infty \theta_i w_i^2| \Delta dP = \int |\sum_{i=-1}^\infty \theta_i (1 - w_i^2)| \Delta dP \leq 2 \sum_{i=-1}^\infty \int |d_i (1 - w_i)| dP$$

Using the inequality

$$|1 - \prod z_i| \leq \sum |1 - z_i|$$

which holds for complex numbers in the closed unit disc, we get the following upper bound for the above sum of integrals:

$$\sum_{i} \int |d_i (1 - 5\alpha_i)| + \sum_{s>8} \int |d_i (1 - \beta_{i+s}^s)| dP$$

The martingale differences $d_i$ are supported on $E_i$ and bounded by $M^{i+1}$. Therefore we obtain a domination by:

$$\sum_{i} \int_{E_i} |(1 - 5\alpha_i) M^{i+1}| + \sum_{s>8} M^{i+1} \int_{E_i} |M^{i+1} (1 - \beta_{i+s}^s)| dP$$

Invoking the estimates from Havin’s Lemma and applying Cauchy-Schwarz’ inequality give the following estimates:

$$\epsilon MC + \log(\epsilon^{-1}) \sum_{s>8} \sum_{i=-1}^\infty M^{i+1} s P(E_i)^{\frac{1}{2}} P(E_{i+s})^{\frac{1}{2}}$$
Again by Cauchy-Schwarz we dominate the above sum by:

\[ \epsilon MC + \log(\epsilon^{-1}) \sum_{s > 8} M^{1 - \frac{s}{2}} s (\sum_{i=-1}^{\infty} M^i P(E_i))^{\frac{1}{2}} (\sum_{i=-1}^{\infty} M^{i+s} P(E_{i+s}))^{\frac{1}{2}} \leq \epsilon MC + \log(\epsilon^{-1}) \sum_{s > 8} M^{1 - \frac{s}{2}} s C \]

This is what we want if \( \epsilon \) is chosen of order \( M^{-2} \) and \( M := \delta^{-1} \).

Havin’s lemma, repeatedly applied, gives the following statements:

1. \( \sum |w_i| < C \)
2. \( |w_i|\Delta < M^{i+8} \)
3. \( \sum M^i \|w_i\|_1 \leq \sum M^i \|\alpha_i\|_1 \leq \sum M^i P(E_i) \|\log \epsilon\| \)

To finish the proof it is now enough to take \( c_i = M^i \)

3 Reduction of J. Bourgain’s partition of unity

To obtain Bourgain’s original result, we lift the density \( f \) from \( T \) to \( \Omega \) construct a new weight together with holomorphic partitions of unity there and project the solutions back to \( \Omega \). This is done by norm-one operators

\[ M : H^p(T) \to H^p(\Omega) \]

and

\[ N : H^p(\Omega) \to H^p(T) \]

so that \( Id = NM \), and \( N(M(f)F) = fN(F) \).

**Proof of theorem 1**: Apply Theorem 4 to the density \( \Delta := Mf \). Let

\[ g_i := N(\theta_i w_i^2) \]

We define \( \gamma_i \) to be the inner factor of \( g_i \) and put

\[ \tau_i := a_i \]

where \( a_i \) denotes the outer factor of \( g_i \). Using our main result it is easy to verify conditions 1) . . . . . . 5) of Bourgain’s theorem.
4 Truncating functions in weighted $H^p$

Here we combine stopping times and holomorphic partitions of unity to obtain Marcinkiewicz decomposition in weighted Hardy spaces. Although the next theorem looks terribly complicated, it simply states that up to a (reasonable) change of density and up to a small error, interpolation is possible in weighted Hardy spaces.

**Theorem 6** For any density $\Delta$ on $\Omega$ and $\delta > 0$ there exists $\phi \in H^\infty(\Omega)$ so that for any

1. $\|\phi\|_\infty < C$
2. $\Delta_1 > \Delta$ and $\int \Delta_1 dP < \delta^{-C} \int \Delta dP$
3. $\int |1 - \phi| \Delta dP < \delta \|\Delta\|_1$
4. For any $f \in H^q(\Omega, \Delta_1)$ and any $\lambda > 0$ there exists $g \in H^\infty(\Omega)$ and $h \in H^2(\Omega, \Delta_1)$ satisfying
   
   (a) $f \phi = g + h$
   
   (b) $\|g\|_\infty \leq \lambda$
   
   (c) $\int |h|^2 \Delta_1 dP < C_\phi \lambda^{2-q} \int |f|^q \Delta_1 dP$

Proof: Let $w_i \in H^\infty(\Omega)$ and $\theta_i \in H^\infty(\Omega)$ be given by Theorem 4. Then we define:

$\phi := \sum \theta_i w_i^2$

$\Delta_1 := \Delta + \sum c_i |w_i|$

$f_i = w_i f$

Now we use the stopping time

$\tau_j := \inf \{t : |E(f_j|\mathcal{F}_t)| > \lambda\}$

to define $g_j := E(f_j|\mathcal{F}_{\tau_j})$ and $h_j := f_j - g_j$. By the stability property of holomorphic martingales these functions are certainly holomorphic and satisfy

1. $\|g_j\|_\infty \leq \lambda$
2. $\int h_j dP \leq 2 \int_{\{|f_j| > \lambda\}} |f_j| dP$

Now, using partitions of unity we glue these partial solutions together

$g := \sum g_j w_j \theta_j$

and

$h := \sum h_j w_j \theta_j$
Then clearly
\[ g + h = \sum (g_j + h_j)v_j = f \phi \]

and
\[ \|g\|_\infty < \sup_j \|g_j\|_\infty \sum |w_j|_\infty < \lambda C \]

The estimate for \( \int |h|^2 \Delta_1 dP \) follows a well established pattern, which has been carefully presented in the central chapter of Wojtaszczyk’s book. See [W, Ch III.I].

Property 3) of theorem 4 implies that:
\[ \int |h|^2 \Delta_1 dP \leq C \sum \int |h_j|^2 |w_j|^2 \Delta_1 dP \]

The last sum can be estimated, using the interplay between the partitions and the density, by
\[ \sum c_j \int |h_j|^2 dP \leq \sum c_j \int_{\{|f_j^+| > \lambda\}} |f_j|^2 dP \]

Using Hölder’s inequality for conjugate indices \( r, s \) we estimate the above expression by
\[ (\sum c_j \|f_j\|^2_{2r})^{\frac{1}{r}} (\sum \lambda^{-2r} c_j \|f_j\|^2_{2r})^{\frac{1}{s}} \]

As the martingale maximal function is bounded in \( L^{2r}(\Omega, P) \) we obtain an upper bound proportional to
\[ (\int |f|^{2r} \sum c_j |w_j| dP)^{\frac{1}{r}} (\sum \lambda^{-2r} c_j \|f_j\|^2_{2r})^{\frac{1}{s}} \]

Specializing \( r = \frac{q}{2} \) this product is finally dominated by a constant, depending on \( q \) times
\[ \lambda^{2-q} \int |f|^q \Delta_1 dP. \]

5 Reduction of J. Bourgain’s interpolation inequality

For the 2- summing operator \( S \) there exists a positive probability measure on \( T \) so that
\[ \|Sx\| \leq \pi_2(S)(\int |x|d\mu)^{\frac{1}{2}} \quad \text{for } x \in A \]

Without loss of generality we may assume that \( \mu \) is absolutely continuous w.r.t. Lebesgue measure, i.e.,
\[ d\mu = f dt. \]
Consequently for \( b \in H^\infty(\Omega) \) the operator \( U = SN \) satisfies
\[
\|Ub\| \leq \pi_2(S)(\int |b|\Delta dP)^{\frac{2}{q}}
\]
where \( \Delta = Mf \).

**Proof of Theorem 3:** Let \( 0 < \delta < 1 \) be given. Theorem 6 applied to the density \( \Delta \) shows that \( U \) can be split into \( U = U_1 + R_1 \) so that
\[
\pi_q(U_1) \leq \delta^{-C} \pi_2(S)^{\frac{2}{q}}\|S\|^{1-\frac{2}{q}}
\]
and
\[
\pi_2(R_1) \leq \delta\pi_2(S),
\]
where \( U_1g = U(b\phi) \) and \( R_1b = U(b(1 - \phi)) \). Indeed fix \( b \in H^q(\Omega, \Delta_1) \) of norm one in that space. Then according to Theorem 6 for \( \lambda = \|S\|^{\frac{2}{q}}\pi_2(S)^{-\frac{2}{q}} \) we find a Marcinkiewicz decomposition of \( \phi b \) into
\[
\phi b = g + h.
\]
Therefore
\[
\|Ub\phi\| \leq \|Ug\| + \|Uh\| < \|S\|\lambda + \pi_2(S)\lambda^{1-\frac{2}{q}} \leq \|S\|^{1-\frac{2}{q}}\pi_2(S)^{\frac{2}{q}}.
\]
As for the error term we have
\[
\|U(b(1 - \phi))\| \leq \pi_2(S)(\int |b|^2|1 - \phi|^2\Delta dP)^{\frac{1}{q}}
\]
Using property 3), 4) and 5) of theorem 4 gives
\[
\int |1 - \phi|^2\Delta dP \leq C \int |1 - \phi|\Delta dP \leq C\delta \int \Delta dP
\]
and
\[
\int \Delta_1 dP \leq \delta^{-C} \int \Delta dP
\]
We therefore obtained the correct estimates for \( U_1 \) and \( R_1 \). To finish the proof of theorem 3, we now iterate the above decomposition and observe that \( S = UM \).

6 References

B1 J. Bourgain, New Banach space properties of the disc algebra and \( H^\infty \), Acta. Math. 152 (1984)

B2 . . . , Bilinear forms on \( H^\infty \) and bounded bianalytic functions, Trans.Amer. Math. Soc 286 (1984)

8
C K Carne, The algebra of bounded holomorphic martingales, J.F.Aanal. 45 (1982)

K1 S Kislyakov, Absolutely summing operators on the disc algebra, St.Petersburg Math.J 3 (1991)

K2 . . . , Extensions of (p,q) summing operator on the disc algebra with an appendix on Bourgain’s analytic partition, Preprint 1990.

K3 . . . , Truncating functions in weighted $H^p$ and two theorems of J. Bourgain, Preprint Uppsala University, 1989.

M1 P F X Müller, Holomorphic martingales and interpolation between Hardy spaces, J. d’Analyse Math. Jerusalem, to appear.

M2 . . . , Holomorphic martingales and interpolation between Hardy spaces: The complex method, preprint 1992.

P G Pisier, A simple proof of a theorem of J. Bourgain, preprint 1990

V N Varopoulos, Helson Szego Theorem $A_p$, functions for Brownian motion and several variables, J.F.Anal. 39 (1980)

W P Wojtaszczyk, Banach spaces for analysts, Cambridge Univ. Press 1991