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The sum of digits of polynomial values in arithmetic progressions

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Abstract

Let \( q, m \geq 2 \) be integers with \((m, q-1) = 1\). Denote by \( s_q(n) \) the sum of digits of \( n \) in the \( q \)-ary digital expansion. Further let \( p(x) \in \mathbb{Z}[x] \) be a polynomial of degree \( h \geq 3 \) with \( p(\mathbb{N}) \subset \mathbb{N} \). We show that there exist \( C = C(q, m, p) > 0 \) and \( N_0 = N_0(q, m, p) \geq 1 \), such that for all \( g \in \mathbb{Z} \) and all \( N \geq N_0 \),

\[
\{0 \leq n < N : s_q(p(n)) \equiv g \mod m\} \geq CN^{4/(3h+1)}.
\]

This is an improvement over the general lower bound given by Dar-tyge and Tenenbaum (2006), which is \( CN^{2/h!} \).

1 Introduction

Let \( q, m \geq 2 \) be integers and denote by \( s_q(n) \) the sum of digits of \( n \) in the \( q \)-ary digital expansion of integers. In 1967/68, Gelfond [1] proved that for nonnegative integers \( a_1, a_0 \) with \( a_1 \neq 0 \), the sequence \( (s_q(a_1n + a_0))_{n \in \mathbb{N}} \) is well distributed in arithmetic progressions \( \mod m \), provided \((m, q-1) = 1\). At the end of his paper, he posed the problem of finding the distribution of \( s_q \) in arithmetic progressions where the argument is restricted to values of polynomials of degree \( \geq 2 \). Recently, Mauduit and Rivat [8] answered Gelfond’s question in the case of squares.

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Theorem 1.1 (Mauduit & Rivat (2009)). For any $q, m \geq 2$ there exists $\sigma_{q,m} > 0$ such that for any $g \in \mathbb{Z}$, as $N \to \infty$,
\[
\#\{0 \leq n < N : s_q(n^2) \equiv g \mod m\} = \frac{N}{m} Q(g, d) + O_{q,m}(N^{1-\sigma_{q,m}}),
\]
where $d = (m, q - 1)$ and
\[
Q(g, d) = \#\{0 \leq n < d : n^2 \equiv g \mod d\}.
\]

The proof can be adapted to values of general quadratic polynomial instead of squares. We refer the reader to [7] and [8] for detailed references and further historical remarks. The case of polynomials of higher degree remains elusive so far. The Fourier-analytic approach, as put forward in [7] and [8], seems not to yield results of the above strength. In a recent paper, Drmota, Mauduit and Rivat [4] applied the Fourier-analytic method to show that well distribution in arithmetic progressions is obtained whenever $q$ is sufficiently large.

In the sequel, and unless otherwise stated, we write
\[
p(x) = a_h x^h + \cdots + a_0
\]
for an arbitrary, but fixed polynomial $p(x) \in \mathbb{Z}[x]$ of degree $h \geq 3$ with $p(\mathbb{N}) \subset \mathbb{N}$.

Theorem 1.2 (Drmota, Mauduit & Rivat (2011)). Let
\[
q \geq \exp(67 h^3 (\log h)^2)
\]
be a sufficiently large prime number and suppose $(a_h, q) = 1$. Then there exists $\sigma_{q,m} > 0$ such that for any $g \in \mathbb{Z}$, as $N \to \infty$,
\[
\#\{0 \leq n < N : s_q(p(n)) \equiv g \mod m\} = \frac{N}{m} Q^*(g, d) + O_{q,m,p}(N^{1-\sigma_{q,m}}),
\]
where $d = (m, q - 1)$ and
\[
Q^*(g, d) = \#\{0 \leq n < d : p(n) \equiv g \mod d\}.
\]

It seems impossible to even find a single “nice” polynomial of degree 3, say, that allows to conclude for well distribution in arithmetic progressions for small bases, let alone that the binary case $q = 2$ is an emblematic case. Another line of attack to Gelfond’s problem is to find lower bounds that are valid for all $q \geq 2$. Dartyge and Tenenbaum [3] provided such a general lower bound by a method of descent on the degree of the polynomial and the estimations obtained in [2].
Theorem 1.3 (Dartyge & Tenenbaum (2006)). Let \( q, m \geq 2 \) with \( (m, q - 1) = 1 \). Then there exist \( C = C(q, m, p) > 0 \) and \( N_0 = N_0(q, m, p) \geq 1 \), such that for all \( g \in \mathbb{Z} \) and all \( N \geq N_0 \),

\[
\#\{0 \leq n < N : s_q(p(n)) \equiv g \pmod{m}\} \geq CN^{2/h!}.
\]

The aim of the present work is to improve this lower bound for all \( h \geq 3 \). More importantly, we get a substantial improvement of the bound as a function of \( h \). The main result is as follows.\(^1\)

Theorem 1.4. Let \( q, m \geq 2 \) with \( (m, q - 1) = 1 \). Then there exist \( C = C(q, m, p) > 0 \) and \( N_0 = N_0(q, m, p) \geq 1 \), such that for all \( g \in \mathbb{Z} \) and all \( N \geq N_0 \),

\[
\#\{0 \leq n < N : s_q(p(n)) \equiv g \pmod{m}\} \geq CN^{4/(3h+1)}.
\]

Moreover, for monomials \( p(x) = x^h, h \geq 3 \), we can take

\[
N_0 = q^{3(2h+m)} \left(2hq^2 (6q)^h\right)^{3h+1},
\]

\[
C = \left(16hq^5 (6q)^h \cdot q^{(24h+12m)/(3h+1)}\right)^{-1}.
\]

The proof is inspired from the constructions used in [5] and [6] that were helpful in the proof of a conjecture of Stolarsky [9] concerning the pointwise distribution of \( s_q(p(n)) \) versus \( s_q(n) \). As a drawback of the method of proof, however, it seems impossible to completely eliminate the dependency on \( h \) in the lower bound.

\section{Proof of Theorem 1.4}

Consider the polynomial

\[
t(x) = m_3x^3 + m_2x^2 - m_1x + m_0,
\]

where the parameters \( m_0, m_1, m_2, m_3 \) are positive real numbers that will be chosen later on in a suitable way. For all integers \( l \geq 1 \) we write

\[
T_l(x) = t(x)^l = \sum_{i=0}^{3l} c_i x^i
\]

\(^1\)Gelfond’s work and Theorem 1.1 give precise answers for linear and quadratic polynomials, so we do not include the cases \( h = 1, 2 \) in our statement though our approach works without change.
Sum of digits of polynomial values

to denote its $l$-th power. (For the sake of simplicity we omit to mark the
dependency on $l$ of the coefficients $c_i$.) The following technical result is the
key in the proof of Theorem 1.4. It shows that, within a certain degree of
uniformity in the parameters $m_i$, all coefficients but one of $T_l(x)$ are positive.

Lemma 2.1. For all integers $q \geq 2$, $l \geq 1$ and $m_0, m_1, m_2, m_3 \in \mathbb{R}^+$ with
\begin{align*}
1 \leq m_0, m_2, m_3 < q, \quad 0 < m_1 < l^{-1}(6q)^{-l}
\end{align*}
we have that $c_i > 0$ for $i = 0, 2, 3, \ldots, 3l$ and $c_i < 0$ for $i = 1$. Moreover,
for all $i$,
\begin{align*}
|c_i| \leq (4q)^l. \tag{2.3}
\end{align*}

Proof. The coefficients of $T_l(x)$ in (2.2) are clearly bounded above in abso-
lute value by the corresponding coefficients of the polynomial $(q x^3 + q x^2 +
q x + q)^l$. Since the sum of all coefficients of this polynomial is $(4q)^l$ and all
coefficients are positive, each individual coefficient is bounded by $(4q)^l$.
This proves (2.3). We now show the first part. To begin with, observe that
$c_0 = m_0^l > 0$ and $c_1 = -lm_0^{l-1}$ which is negative for all $m_1 > 0$. Suppose
now that $2 \leq i \leq 3l$ and consider the coefficient of $x^i$ in
\begin{align*}
T_l(x) = (m_3 x^3 + m_2 x^2 + m_0)^l + r(x), \tag{2.4}
\end{align*}
where
\begin{align*}
r(x) &= \sum_{j=1}^{l} \binom{l}{j} (-m_1 x)^j (m_3 x^3 + m_2 x^2 + m_0)^{l-j} \\
&= \sum_{j=1}^{3l-2} \binom{l}{j} d_j x^j.
\end{align*}

First, consider the first summand in (2.4). Since $m_0, m_2, m_3 \geq 1$ the coeffi-
cient of $x^i$ in the expansion of $(m_3 x^3 + m_2 x^2 + m_0)^l$ is $\geq 1$. Note also that
all the powers $x^2, x^3, \ldots, x^{3l}$ appear in the expansion of this term due to the
fact that every $i \geq 2$ allows at least one representation as $i = 3i_1 + 2i_2$ with
non-negative integers $i_1, i_2$. We now want to show that for sufficiently small
$m_1 > 0$ the coefficient of $x^i$ in the first summand in (2.4) is dominant. To
this end, we assume $m_1 < 1$ so that $m_1 > m_1^j$ for $2 \leq j \leq l$. Using $\binom{l}{j} < 2^l$
and a similar reasoning as above we get that
\begin{align*}
|d_j| < l^2 m_1 (3q)^l = l (6q)^l m_1, \quad 1 \leq j \leq 3l - 2.
\end{align*}
This means that if $m_1 < l^{-1}(6q)^{-l}$ then the powers $x^2, \ldots, x^{3l}$ in the poly-
nomial $T_l(x)$ indeed have positive coefficients. This finishes the proof. \qed
To proceed we recall the following splitting formulas for $s_q$ which are simple consequences of the $q$-additivity of the function $s_q$ (see [5] for the proofs).

**Proposition 2.2.** For $1 \leq b < q^k$ and $a, k \geq 1$, we have

\[
\begin{align*}
  s_q(aq^k + b) &= s_q(a) + s_q(b), \\
  s_q(aq^k - b) &= s_q(a - 1) + k(q - 1) - s_q(b - 1).
\end{align*}
\]

We now turn to the proof of Theorem 1.4. To clarify the construction we consider first the simpler case of monomials,

\[p(x) = x^h, \quad h \geq 1.\]

(We here include the cases $h = 1$ and $h = 2$ because we will need them to deal with general polynomials with linear and quadratic terms.) Let $u \geq 1$ and multiply $t(x)$ in (2.1) by $q^{u-1}$. Lemma 2.1 then shows that for all integers $m_0, m_1, m_2, m_3$ with

\[
q^{u-1} \leq m_0, m_2, m_3 < q^u, \quad 1 \leq m_1 < q^u/(hq(6q)^h),
\]

the polynomial $T_h(x) = (t(x))^h = p(t(x))$ has all positive (integral) coefficients with the only exception of the coefficient of $x^1$ which is negative. Let $u$ be an integer such that

\[q^u \geq 2hq(6q)^h\]  \hspace{1cm} (2.6)

and let $k \in \mathbb{Z}$ be such that

\[k > hu + 2h.\]  \hspace{1cm} (2.7)

For all $u$ with (2.6) the interval for $m_1$ in (2.5) is non-empty. Furthermore, relation (2.7) implies by (2.3) that

\[q^k > q^{hu} \cdot q^{2h} \geq (4q^u)^h > |c_i|, \quad \text{for all } i = 0, 1, \ldots, 3h,
\]

where $c_i$ here denotes the coefficient of $x^i$ in $T_h(x)$. Roughly speaking, the use of a large power of $q$ (i.e. $q^k$ with $k$ that satisfies (2.7)) is motivated by the simple wish to split the digital structure of the $h$-power according to Proposition 2.2. By doing so, we avoid to have to deal with carries when adding terms in the expansion in base $q$ since the appearing terms will not interfere. We also remark that this is the point where we get the dependency of $h$ in the lower bound of Theorem 1.4.
Now, by $c_2, |c_1| \geq 1$ and the successive use of Proposition 2.2 we get

$$s_q(t(q^k)^h) = s_q\left(\sum_{i=3}^{3h} c_i q^{i-1}k + c_2 q^k - |c_1| q^k + c_0\right)$$

$$= s_q\left(\sum_{i=3}^{3h} c_i q^{i-1}k + c_2 q^k - |c_1|\right) + s_q(c_0)$$

$$= s_q\left(\sum_{i=3}^{3h} c_i q^{i-3}k\right) + s_q(c_2 - 1) + k(q - 1) - s_q(|c_1| - 1) + s_q(c_0)$$

$$= \sum_{i=3}^{3h} s_q(c_i) + s_q(c_2 - 1) + k(q - 1) - s_q(|c_1| - 1) + s_q(c_0)$$

$$= k(q - 1) + M,$$  \hspace{1cm} (2.8)

where we write

$$M = \sum_{i=3}^{3h} s_q(c_i) + s_q(c_2 - 1) - s_q(|c_1| - 1) + s_q(c_0).$$

Note that $M$ is an integer that depends (in some rather obscure way) on the quantities $m_0, m_1, m_2, m_3$. Once we fix a quadruple $(m_0, m_1, m_2, m_3)$ in the ranges (2.5), the quantity $M$ does not depend on $k$ and is constant whenever $k$ satisfies (2.7). We now exploit the appearance of the single summand $k(q - 1)$ in (2.8). Since by assumption $(m, q - 1) = 1$, we find that

$$s_q(t(q^k)^h), \quad \text{for } k = hu + 2h + 1, \ hu + 2h + 2, \ldots, hu + 2h + m,$$  \hspace{1cm} (2.9)

runs through a complete set of residues mod $m$. Hence, in any case, we hit a fixed arithmetic progression mod $m$ (which might be altered by $M$) for some $k$ with $hu + 2h + 1 \leq k \leq hu + 2h + m$.

Summing up, for $u$ with (2.6) and by (2.5) we find at least

$$\left(q^u - q^{u-1}\right)^3\left(\frac{q^u}{(hq(6q)^h)} - 1\right) \geq \left(\frac{1 - 1/q}{2hq (6q)^h}\right)^3 q^{4u}$$  \hspace{1cm} (2.10)

integers $n$ that in turn by (2.1), (2.5), (2.7) and (2.9) are all smaller than

$$q^u \cdot q^{3(hu+2h+m)} = q^{3(2h+m)} \cdot q^{u(3h+1)}$$

and satisfy $s_q(n^h) \equiv g \mod m$ for fixed $g$ and $m$. By our construction and by choosing $k > hu + 2h > u$ all these integers are distinct. We denote

$$N_0 = N_0(q, m, p) = q^{3(2h+m)} \cdot q^{u(3h+1)},$$
where
\[ u_0 = \left\lceil \log_q \left( 2hq(6q)^h \right) \right\rceil \leq \log_q \left( 2hq^2(6q)^h \right). \]

Then for all \( N \geq N_0 \) we find \( u \geq u_0 \) with
\[ q^{3(2h+m)} \cdot q^{u(3h+1)} \leq N < q^{3(2h+m)} \cdot q^{(u+1)(3h+1)}. \]

By (2.10) and (2.11), and using \((1 - 1/q)^3 \geq 1/8\) for \( q \geq 2\), we find at least
\[ \frac{(1 - 1/q)^3}{2hq(6q)^h} q^{4u} \geq \left( 16hq^5 (6q)^h \cdot q^{(24h+12m)/(3h+1)} \right)^{-1} N^{4/(3h+1)} \]

integers \( n \) with \( 0 \leq n < N \) and \( s_q(n^h) \equiv g \mod m \). We therefore get the statement of Theorem 1.4 for the case of monomials \( p(x) = x^h \) with \( h \geq 3 \). The estimates are also valid for \( h = 1 \) and \( h = 2 \).

The general case of a polynomial \( p(x) = a_h x^h + \cdots + a_0 \) of degree \( h \geq 3 \) (or, more generally, of degree \( h \geq 1 \)) follows easily from what we have already proven. Without loss of generality we may assume that all coefficients \( a_i, 0 \leq i \leq h \), are positive, since otherwise there exists \( e = e(p) \) depending only on \( p \) such that \( p(x + e) \) has all positive coefficients. Note that a finite translation can be dealt with choosing \( C \) and \( N_0 \) appropriately in the statement. Since Lemma 2.1 holds for all \( l \geq 1 \) and all negative coefficients are found at the same power \( x^1 \), we have that the polynomial \( p(t(x)) \) has again all positive coefficients but one where the negative coefficient again corresponds to the power \( x^1 \). It is then sufficient to suppose that
\[ k > hu + 2h + \log_q \max_{0 \leq i \leq h} a_i \]
in order to split the digital structure of \( p(t(q^k)) \). In fact, this implies that
\[ q^k > \left( \max_{0 \leq i \leq h} a_i \right) \cdot (4q^u)^h, \]
and exactly the same reasoning as before yields \( \gg_{q,p} q^{4u} \) distinct positive integers that are \( \ll_{q,m,p} q^{u(3h+1)} \) and satisfy \( s_q(p(n)) \equiv g \mod m \). This completes the proof of Theorem 1.4.

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