Rotational self-friction problem of elastic rods

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Abstract

The aim of this paper is to extend the modeling of a hyperelastic rod undergoing large displacements with tangential self-friction to their modeling with rotational self-friction. The discontinuity of contact force into a contact region not known in advance with taking into account the effects of friction in this problem type leads to serious difficulties in modelization, mathematical and numerical analysis. In this paper, we present an accurate modeling of rotational and tangential self-friction with Coulomb’s law and describe also an augmented Lagrangian method to present a weak variational formulation approach of this problem. We then use the minimization method of the total energy to present an existence result of solution for the nonlinear penalized formulation. Finally, we give the linearization and the finite-element discretization of the weak variational formulation that can be useful for a numerical implementation.

Keywords Cosserat rods · Contact distance · Self-friction · Augmented Lagrangian method · Finite elements

Mathematics Subject Classification 7400 · 7410

1 Introduction

Modeling beam to beam contact, beam to surface contact or self-contact in an elastic rod with or without friction has practical applications in many fields of industrial engineering and biomechanics, such as: the pipelines-soil interaction and drill-string dynamics in oil industries [14, 38], beams-slab interaction in civil engineering and subsea cables installation in marine
industries [24, 34, 39]. This modeling has also attracted the interest in biomedical applications in live sciences, such as: the use of stents to repair the endovascular [21, 30], the simulation of Minimally Invasive Surgery in which guidewires are inserted through the patient tissue with limited visual aids [9, 10] and the study of deformations and supercoiling phenomena of fragments of the DNA molecule where this study and the mechanical explanation of the phenomenon can help produce and/or improve drugs for genetic diseases [16, 27]. Many problems that remain to be studied for a good understanding of the supercoiling process of DNA fragments, which are based on the modeling and numerical treatment of frictional or frictionless self-contact in elastic rods.

Contact problems seem to be among difficult to be treated both analytically and numerically. However, the techniques that take into account unilateral contact conditions are well advanced with the remarkable progress of computing machines. These techniques which have solved these problems, are still need to be examined in return, especially the problem of self-contact. The unilateral contact problem with non-local Coulomb friction has been studied by Renard [33]. For static and dynamic unilateral contact problems with Coulomb friction, where the existence and/or uniqueness of solutions for discretized problems have been proved, we can refer to [6, 13]. In [29], Nečas et al. used the fixed point theorems to prove the existence of solutions of unilateral contact problem in the static state using a local Coulomb friction law.

Key roles in studying the self-contact problem in an elastic rod or rod to rod contact are how these structures react when they suffer contact forces. A new three-stages contact searching algorithm based on bounding volume hierarchies and orthogonal projections for a frictional self-contact problem between rope fibers in the large deformations case has been developed by Peng et al. [31]. Béal and Touzani in [4] studied the frictionless contact problem between two rods in large displacements, where the existence of a static solution has been proved. The frictionless self-contact problem in elastic rods has been studied by Chamekh et al. [7] and recently by Bozorgmehri et al. [2], who gave an existence result based on a nonlinear minimization problem and numerical examples based on the complementarity problem (CP), the linear complementarity problem (LCP), the penalty and augmented Lagrangian approaches.

The modeling of elastic rods with tangential self-friction was studied by Chamekh et al. [8] and well detailed in the thesis of Latrach [25]. The present work expands this modeling to the rotational self-friction case, where the great unexpected difficulty (theoretically) consists in proving the existence of solutions to the theoretical problem of finding the equilibrium configurations like the tangential self-friction case, let us note that sometimes the mathematical tools at our disposal cannot solve some problems because of their strong nonlinearity like this example where the mathematical models used in themselves are nonlinear and lead to systems of nonlinear equations.

1.1 Assumptions and notations used

Throughout this report, we recall that the antisymmetric tensor \( a \times = \text{skew}(a) \) associated with the vector \( a \in \mathbb{R}^3 \) is defined by \( a \times b = a \times b \) for all \( b \in \mathbb{R}^3 \), and

\[
P_B(\tau)[a] = \begin{cases} \frac{a}{\|a\|} & \text{if } \|a\| < \tau, \\ \tau & \text{otherwise}, \end{cases}
\]

is the projection of the vector \( a \in \mathbb{R}^3 \) onto the centered ball at the origin and with a radius \( \tau \). For any function \( G \) differentiation with respect to the curvilinear abscissa is denoted with \( \partial \).

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\[ \dot{G'} = \frac{\partial G}{\partial s} \] and with respect to the time is denoted with dot \( \dot{G} = \frac{\partial G}{\partial t} \). Time discretization is based on a backward Euler approximation of the last notation which reads, see [28]

\[
G(s) = \frac{G(s) - G(s_0)}{\Delta t}
\] (1.1)

2 Cosserat rod

2.1 A 3-D curved Cosserat rod

We make use in this paper the geometrically exact theory of rods which is largely based on the model introduced by Cosserat brothers [5]. This model is used in the abstract framework proposed by Antman [1] and developed in [32, 35]. More recently, Chamekh et al. [8] has treated the frictional self-contact problem of elastic rods in the case of tangential friction only. We will extend this study to the rotational self-friction case.

Let \( \mathcal{R} \) an unclosed elastic rod with the length \( l \) defined by a regular set of cross sections \( \{A(s); s \in [0; l]\} \) and fixed on the spatial basis \( (e_i)_i \). The cross sections are circular of uniform diameter \( 2\varepsilon \). In [8], the authors presented the different configurations of the rod. Within this framework, the deformed configuration, under the action of external efforts, can be completely determined by the position vector \( r \in \mathbb{R}^3 \) and the orthogonal tensor \( R = d_i \otimes e_i \in SO(3) \), with \( SO(3) \) is the group of rotation matrices in \( \mathbb{R}^3 \). The orthogonal tensor \( R \) is the matrix which associates with the spatial basis \( (e_i)_i \) the material basis \( (d_i)_i \). The position vector \( r_0 \) and the tensor \( R_0 = d_{0i} \otimes e_i \) are associated with the stress-free configuration, see Fig. 1. The Kinematics of deformation of a Cosserat elastic rod is detailed in the thesis of Latrach [25].

2.2 Strains Measures and mechanical interpretation

The rotation tensor \( R \) may be expressed by the well-known Euler-Rodrigues formula

\[
R = I_3 + \sin(\theta)\omega \times + [1 - \cos(\theta)](\omega \times)^2,
\] (2.1)

in which \( \theta = \theta \omega \) is the Euler rotation vector. We use the Taylor series for \( \sin(\theta) \) and \( \cos(\theta) \) and the relations \((\omega \times)^2n+1 = (-1)^n\omega \times \) and \((\omega \times)^2n = (-1)^{n-1}(\omega \times)^2 \), we obtain the exponential form of \( R \)

\[
R = I_3 + \frac{[\theta \omega \times]}{1!} + \frac{[\theta \omega \times]^2}{2!} + \frac{[\theta \omega \times]^3}{3!} + \cdots = \exp[\theta \times].
\] (2.2)

The vector \( \theta' \) is exactly the vector \( u \) defined in [8], i.e. \( \theta' \times d_i = u \times d_i = d'_i \). Following [1], we introduce the measures of deformation:

\[
v = R^T r' \quad \text{and} \quad u = R^T \theta'.
\] (2.3)

With \( v \) is a translational deformations vector, which its components \( v_3 \) measure the elongation and \( v_1 \) and \( v_2 \) measure the shear deformations, respectively, in the \( d_1 \) and \( d_2 \) directions.
components of vector $\mathbf{u}$, are $u_1$ and $u_2$ that measure the total flexion of the rod, respectively, in the planes $(d_2, d_3)$ and $(d_1, d_3)$ and $u_3$ that is a measure of the total physical twist of the rod around the vector $d_3$.

3 Self-contact with Coulomb friction

Our aim is to describe the two types of self-friction in an elastic rod such as the rotational self-friction. In [7], the authors have undertaken a rigorous mathematical analysis of the frictionless self-contact problem. Recently, we have presented the tangential self-friction problem in [8].

3.1 Self-contact problem

The maximum curvature condition of the unclosed rod expressing the fact that for each curvilinear abscissa $s \in [0, l]$, there exists a subset $\mathcal{I}_s$ of $[0, l]$ of the points likely to come into contact with $s$:

$$\mathcal{I}_s = \begin{cases} [s + \delta, l] & \text{if } s \in [0, \varepsilon], \\ [0, s - \delta] \cup [s + \delta, l] & \text{if } s \in [\varepsilon, l - \varepsilon], \\ [0, s - \delta] & \text{if } s \in [l - \varepsilon, l], \end{cases}$$

where $\delta$ is a strictly positive constant which is selected according to the rigidity and the regular size of the cross-sections of the rod. Thereby, $\mathcal{I}_s^c = [0, l] \setminus \mathcal{I}_s$ defines the set of points very close and which cannot come into contact with $s$ (see Fig. 2). In the deformed configuration, as the curve $\mathbf{r}([0, l])$ is continuous and the subset $\mathcal{I}_s$ is compact, the research of the point of contact with $\mathbf{r}(s)$ can result in finding the orthogonal projection $\mathbf{r}(\bar{s}) \in \mathbf{r}(\mathcal{I}_s)$.
of \( r(s) \). We can therefore write
\[
  r'(\bar{s}) \cdot (r(s) - r(\bar{s})) = 0. 
\]
We can also define \( \bar{s} \) by the arclength of the closest point to \( s \), i.e. \( \bar{s} = \arg\min_{\sigma \in \mathcal{I}_s} \| r(s) - r(\sigma) \| \).

The evident self-impenetrability condition expressing the fact the volumes cannot inter-penetrate themselves in a kinematically and physically admissible configuration of the rod. Consequently, for a given configuration of the rod we define a \textit{gap function} with their imposed constraint
\[
  d_r(s, \bar{s}) = 2\varepsilon - \| r(s) - r(\bar{s}) \| \leq 0, \quad \forall s \in [0, l]. 
\]

The true self-contact distance must be dependent on both the position vector \( r(s) \) and the rotation tensor \( R(s) \) because it must be defined with respect the cross-section boundary \( A(s) \) and the lateral surface of \( \{ A(\sigma), \quad \sigma \in \mathcal{I}_s \} \). The determination of the true distance of contact could be quite complicated. Thus, \( d_r \) is defined as an approximation to the true self-contact distance (\textit{gap function}) such that the associated \textit{gap vector} is given by
\[
  D_r(s, \bar{s}) = -d_r(s, \bar{s}) N(s, \bar{s}). 
\]

According to (3.1), \( N(s, \bar{s}) \) is the unit normal vector orthogonal to \( T(s, \bar{s}) \) the tangent vector to the center line of the rod at arclength \( \bar{s} \):
\[
  N(s, \bar{s}) = \frac{r(s) - r(\bar{s})}{\| r(s) - r(\bar{s}) \|}, \quad \text{and} \quad T(s, \bar{s}) = \frac{r'(\bar{s})}{\| r'(\bar{s}) \|}. 
\]

### 3.2 Modeling of tangential and rotational self-friction

The constraint of the self-impenetrability in (3.2) can be expressed in terms of a contact force. Under the assumption of frictional self-contact, this force must be decomposed to a normal pressure and two frictional efforts (tangential and rotational forces). In this paper, the Cosserat model is based on the central curve of the rod, whereas the contact force must be mechanically defined as a force applied by the point \( r(\bar{s}) \) on a point of a compact 3D-neighborhood of \( r(s) \) defined by the friction cone, see Fig. 2.

\[
  f_c(s) = f_N(s)N(s, \bar{s}) + f_T(s) + f_R(s). 
\]

With the internal production of this force and under the action of external forces \( f_{\text{ext}} \) and external torques \( t_{\text{ext}} \), the local form of the balance equations for a rod can be written in the following way:
\[
  n' + f_{\text{ext}} + f_N N + f_T = 0 \quad \text{and} \quad m' + r' \times n + t_{\text{ext}} + f_R = 0, 
\]
where \( n \) and \( m \) denote respectively resultant force (of the material with \( \sigma > s \) on the material with \( \sigma < s \)) and resultant moments acting across the cross-section \( A(r(s)) \).

The rod model studied in this paper is supposed to be made of a hyperelastic material, i.e. there exists an elastic energy density \( \mathcal{W}(\cdot, u, v) \) such that:
\[
  n = R \frac{\partial \mathcal{W}(\cdot, u, v)}{\partial v} \quad \text{and} \quad m = R \frac{\partial \mathcal{W}(\cdot, u, v)}{\partial u}. 
\]
The elastic energy density \( W(s, u, v) \) is supposed to obey the convexity and coercivity hypotheses with respect to its last two arguments.

The constraint of the self-impenetrability is gained by the following Kuhn-Tucker conditions:

\[
d_r(s, \bar{s}) \leq 0, \quad f_N(s) \geq 0 \quad \text{and} \quad d_r(s, \bar{s}) f_N(s) = 0.
\] (3.6)

In what follows, we assume that the sliding is persistent, i.e. if \( d_r(s, \bar{s}) = 0 \), then \( \dot{d}_r(s, \bar{s}) = 0 \).

To model the self-friction, it is necessary to determine the tangential and rotational sliding velocities which will make it possible to evaluate the pressure of the frictional efforts. The rotational sliding must be carried out on the cross-section boundary \( A(\bar{s}) \) and around the \( d_3(\bar{s}) \)-axis while the tangential sliding must be carried out on the plane tangent and orthogonal to the cross-section \( A(\bar{s}) \). The tangential sliding velocity is given by (see [8]):

\[
g_r(s, \bar{s}) = \dot{r}(s) - \dot{r}(\bar{s}) + 2\varepsilon \dot{N}(s, \bar{s}).
\] (3.7)

The tangential sticking can be defined as follows:

- if \( g_r(s, \bar{s}) = 0 \) then the curve is a tangentially self-sticking;
- if \( g_r(s, \bar{s}) \neq 0 \) then the curve is a tangentially self-sliding.
By differentiation with respect to time in (2.2) we obtain
\[ \dot{d}_i(s) = \dot{\theta}(s) \times d_i(s), \]
\[ \dot{d}_i(\bar{s}) = \left[ \dot{\theta}(\bar{s}) + \ddot{\theta}'(\bar{s}) \right] \times d_i(\bar{s}). \]

Then, we consider the following rotational sliding velocity
\[ \dot{g}_R(s, \bar{s}) = \dot{\theta}(s) - \dot{\theta}(\bar{s}) - \ddot{\theta}'(\bar{s}). \] (3.8)

The rotational sticking can be defined as follows:
- if \( \dot{g}_R(s, \bar{s}) = 0 \), rotation along the \( d_3(\bar{s}) \)-axis is not possible.
- if \( \dot{g}_R(s, \bar{s}) \neq 0 \), can be rotated along the \( d_3(\bar{s}) \)-axis.

### 3.3 Coulomb’s law

The simplest (in appearance) is Trisca’s law, has been successfully implanted in many works, such as proving the existence and uniqueness of solutions for bilateral contact problems using the variational formulations [11]. But, the conditions of Trisca friction do not really make sense in greater deformations case because the self-penetration is possible and they also impose a little friction even before self-contact, which is obviously not at all physically. Thus, friction is generally taken into account using Coulomb’s law, that is relevant for a broad range of applications. In the following, one will count on Coulomb’s law to discuss the problem of self-friction in elastic rod:

\[ \| f_T(s) \| < \gamma_T f_N(s) \] if \( g_R(s, \bar{s}) = 0 \), \( (3.9a) \)
\[ f_T(s) = \gamma_T f_N(s) \frac{g_R(s, \bar{s})}{\| g_R(s, \bar{s}) \|} \] otherwise, \( (3.9b) \)
\[ \| f_R(s) \| < \gamma_R f_N(s) \] if \( g_R(s, \bar{s}) = 0 \), \( (3.9c) \)
\[ f_R(s) = \gamma_R f_N(s) \frac{g_R(s, \bar{s})}{\| g_R(s, \bar{s}) \|} \] otherwise. \( (3.9d) \)

Here, \( \gamma_T \) and \( \gamma_R \) are respectively the tangential and rotational friction coefficients for the master point which are generally constant depending on the reality of the materials and the environments considered (surface states, temperature, tangential sliding speed, ...). The inequalities and equality (3.6) express the condition of non-self-penetration of matter and the fact that the normal contact stress is of compression which must be zero as well as the tangential and rotational stress if there is no self-contact (3.9a, 3.9c). The tangential and rotational efforts of Coulomb friction cannot exceed the positive thresholds given by \( \gamma_T f_N \) and \( \gamma_R f_N \) which is translated into the friction conditions (3.9) by the fact that if the tangential or rotational stress of the contact force reaches the given threshold, then there is a tangential or rotational self-sliding, else there is a tangential or rotational self-sticking. In other words, the tangential and rotational friction cones, respectively defined by the angles \( \alpha_T = \arctan(\gamma_T) \) and \( \alpha_R = \arctan(\gamma_R) \), decide whether there is a slip or not, see Fig. 2.

### 4 Principle of virtual work of the self-friction problem

We couldn’t offer an expression of the total energy of an elastic rod subjected to external forces in the presence of friction because the frictional Coulomb’s law does not admit of
potential. For this reason, we will directly present a weak variational formulation approach associated with equations (3.4) describing the equilibrium states of a hyperelastic rod. In this approach, we will adopt the augmented Lagrangian method and the approximations used in [8] to present a global approximation of the principle of virtual work, taking into account the constraint of self-impenetrability:

\[
\mathcal{G}(r, \theta; \delta r, \delta \theta) = \int_0^l \left\{ \begin{array}{l}
    \mathbf{n}(s) \cdot (\delta \mathbf{r}'(s) - \delta \theta(s) \times \mathbf{r}'(s)) + \mathbf{m}(s) \cdot \delta \theta'(s) \\
    - \int_0^l \{ f_{\text{ext}}(s) \cdot \delta \mathbf{r}(s) + t_{\text{ext}} \cdot \delta \theta(s) \} \, ds \\
    + \int_0^l \{ (f_{\nu N}(s) \mathbf{N}(s, \bar{s}) + f_{\nu T}(s)) \cdot (\delta \mathbf{r}(\bar{s}) - \delta \mathbf{r}(s)) \} \, ds \\
    + \int_0^l \{ f_{\nu R}(s) \cdot (\delta \theta(\bar{s}) - \delta \theta(s)) \} \, ds
\end{array} \right\} \, ds = 0.
\]

(4.1)

The first two integral parts represent the virtual work of internal forces and external loads, and the last two integral parts represent an approach of virtual work of self-friction force given by the following triple:

\[
\begin{align*}
    f_{\nu N}(s) &= \frac{1}{\mu_N} \left[ d_r(s, \bar{s}) + \mu_N \lambda_n^N(s) \right] +, \quad [x]_+ = \frac{1}{2}(x + |x|), \\
    f_{\nu T}(s) &= \frac{1}{\mu_T} \left[ g_T(s, \bar{s}) + \mu_T \lambda_n^T(s) \right], \quad B_T = B(\mu_T \gamma_T f_{\nu N}), \\
    f_{\nu R}(s) &= \frac{1}{\mu_R} \left[ g_R(s, \bar{s}) + \mu_R \lambda_n^R(s) \right], \quad B_R = B(\mu_R \gamma_R f_{\nu N}),
\end{align*}
\]

(4.2a–c)

where \( \mu_N, \mu_T \) and \( \mu_R \) are the normal, tangential and rotational penalties parameters, and the iterative approaches \( \lambda_n^N(s), \lambda_n^T(s) \) and \( \lambda_n^R(s) \) are updated according to iterations used in [8]. Note that the penalty method corresponds to the choice \( \lambda_n^N = \lambda_n^T = \lambda_n^R = 0 \) in (4.2). Since the work of Laursen and Simo [26], it has been customary to neglect this term containing the Dirac term. A fact that has been justified numerically.

Thereby, the problem of finding the equilibrium configurations satisfying the equilibrium equations (3.4) leads to the following nonlinear penalized problem

\[
\left\{ \begin{array}{l}
    \text{Find } (r, \theta) \in C \text{ such that } \\
    \mathcal{G}(r, \theta; \delta r, \delta \theta) = 0, \quad \forall (\delta r, \delta \theta),
\end{array} \right.
\]

the set \( C \) being given by

\[
C = \left\{ (r, \theta) \in H^1([0, l]; \mathbb{R}^3 \times \mathbb{R}^3), \quad r(0) = 0, \quad \theta(0) = \theta^0, \quad \theta(l) = \theta^l \right\}.
\]

**Remark 41** As in [8], the self-friction force given by (4.2) is equivalent to the Kuhn-Tucker and the self-friction conditions (3.6) and (3.9). Then, the augmented Lagrangian method reassures us about the self-contact conditions.
5 Existence of solution for the penalized formulation

Penalty methods provide an alternative approach to constrained minimization problems which avoid the necessity of introducing the additional unknowns corresponding to Lagrange multipliers [37]. The solution of the contact problem by the penalty method may induce numerical oscillations.

If the external force \( f_{\text{ext}} \) is independent of \( r \), then the problem of finding the equilibrium configurations that satisfy the equilibrium equations (3.4) with the penalized frictional self-contact force (4.2) and the constitutive laws (3.5) leads to the next nonlinear approximated minimization problem

\[
\begin{align*}
\mathcal{J}_\mu^n(r_n^\mu, \theta_n^\mu) &\leq \mathcal{J}_\mu^n(q, \varphi) \quad \forall (q, \varphi) \in \mathcal{C}, \\
\text{Find} \quad (r_n^\mu, \theta_n^\mu) \in \mathcal{C} \quad \text{such that} \\
\end{align*}
\]

where (4.1) is the necessary condition for minimizing the above total energy with the same normal, tangential and rotational penalties parameters (they are equal to \( \mu \)).

In this paper, for the sake of simplicity, we shall assume that the rod is not subject to distributed torque \( t_{\text{ext}} = 0 \). Then, the energy expression \( \mathcal{J}_\mu^n \) for an elastic rod in balance is given by

\[
\mathcal{J}_\mu^n(r, \theta) = \int_0^l \left[ \mathcal{W}(s, v, u) - f_{\text{ext}} \cdot r - (\lambda^N_N N^e + \lambda^T_T \cdot r - \lambda^R_R \cdot \theta) \right] ds,
\]

where \( \lambda^N_N N^e, \lambda^T_T \) and \( \lambda^R_R \) are the Lagrange multipliers given in (4.2) with the next regularizations to ensure that the self-contact force fields belong to the space \( L^2([0, l]; \mathbb{R}^3) \):

\[
N^e = \begin{cases} 
\frac{r(s) - r(\bar{s})}{2\varepsilon} & \text{if } \|r(s) - r(\bar{s})\| \leq 2\varepsilon, \\
\frac{r(s) - r(\bar{s})}{\|r(s) - r(\bar{s})\|} & \text{otherwise},
\end{cases}
\]

if \( \|g_r + \mu\lambda^T_T\| \geq \mu \gamma_T \lambda^N_N \):

\[
\lambda^{n,\mu}_T = \begin{cases} 
\gamma_T \lambda^N_N & \text{if } \|g_r + \mu\lambda^T_T\| \leq \mu, \\
\gamma_T \lambda^N_N & \text{otherwise},
\end{cases}
\]

if \( \|g_R + \mu\lambda^R_R\| \geq \mu \gamma_R \lambda^N_N \):

\[
\lambda^{n,\mu}_R = \begin{cases} 
\gamma_R \lambda^N_N & \text{if } \|g_R + \mu\lambda^R_R\| \leq \mu, \\
\gamma_R \lambda^N_N & \text{otherwise},
\end{cases}
\]

The function \( \mathcal{W} \) that appeared in (5.2) is a typical example for the elastic stored energy functional: \( \mathcal{W}(\cdot, u, v) = \frac{1}{2}w^T A w \), where \( A(s) \in M_{6}(\mathbb{R}) \) is a coercive, symmetric and positive-definite matrix and \( w^T = (u^T, v^T) \).

**Theorem 5.1** Let us assume that the external force \( f_{\text{ext}} \) is in \( L^2([0, l]; \mathbb{R}^3) \), then for each \( n \in \mathbb{N} \) and \( \mu \in (0, 1] \), the problem (5.1) with the energy functional (5.2) has at least one solution.

It turned out that the minimization technique is equivalent to finding the solution of variational inequalities/equalities associate to the balance equations. This fact motivated us to cite the work of Glowinski et al. [15] and Hlaváček et al. [17], they developed the auxiliary principle technique to study the existence of the solution of variational inequalities. This technique
deals with the auxiliary variational inequality problem and shows that the solution of the auxiliary problem is the solution of the variational inequality problem, which is equivalent to finding the minimum of the functional associated with the auxiliary variational inequality problem.

**Proof** The proof of the theorem is based on the generalized Weierstrass theorem [23]. That is to say, must be proved on the one hand that the energy functional (5.2) is weakly lower semicontinuous and coercive, and on the other hand that the set $C$ is sequentially weakly closed in $H^1([0, l]; \mathbb{R}^3 \times \mathbb{R}^3)$. To avoid complications caused superlinear and to be extremely fast and reliable in numerical practice. For more details on the semismooth Newton method, see [19]. In this section, to avoid complications caused

6 Linearization of virtual work equation

In [22], the authors presented a modeling of the contact problem with a given friction. This modeling leads to a non-differentiable minimization problem due to the non-differentiable contact law. They applied an infinite-dimensional semi-smooth Newton’s algorithm method to the regularized problem with Tresca friction. The method is shown to converge locally superlinear and to be extremely fast and reliable in numerical practice. For more details on the semismooth Newton method, see [19]. In this section, to avoid complications caused
by this lack of differentiability, we shall work with a regularized version of the penalized frictional self-contact force defined in the next by the functions $H$, $F_T$ and $F_R$.

To solve the nonlinear problem (4.3), we propose the Newton-Raphson method. This method is based on the linearization of each term of the formulation (4.1) to solve the following linearized problem

$$
G(r^k, R^k; \delta r, \delta \theta) + \Delta G(r^k, R^k; \delta r, \delta \theta; \eta, \vartheta) = 0,
$$

where the symbol $\Delta$ denotes the directional derivative defined by

$$
\Delta G(r^k, R^k; \delta r, \delta \theta; \eta, \vartheta) := \frac{d}{d\epsilon} \bigg|_{\epsilon=0} G(r^k + \epsilon \eta, \exp(\epsilon \theta^\times) R^k; \delta r, \delta \theta).
$$

The Newton update scheme of the displacement $r$ is additive $r^{k+1} = r^k + \eta$ but, to updating the rotation $R$ we use a multiplicative update $R^{k+1} = \exp(\vartheta^\times) R^k$.

The linearization of the virtual work of internal forces and moments (the first two integral parts in (4.1)) is presented by Chamekh et al. [7] of a rod not subject to distributed torque, i.e. $t_{\text{ext}} = 0$. The linearizations of the virtual work associated with normal force and the tangential effort (the third integral part in (4.1)) are respectively already made in [7] and [8].

In the next, we highlight more on the linearization of the virtual work of rotational self-friction force given by the fourth integral part in (4.1).

6.1 Linearization of the virtual work of rotational self-friction force

In view of (4.1), the virtual work of self-friction force is given by

$$
G_c(r, \theta; \delta r, \delta \theta) = \int_0^l \left( f_{\text{HN}}^\mu N + f_T^\mu T \right) \cdot \left( \frac{\delta r(s) - \delta r(s)}{\delta \theta(s) - \delta \theta(s)} \right) \, ds.
$$

Then, the directional derivative $\Delta G_c(r, \theta; \delta r, \delta \theta; \eta, \vartheta)$ is given by

$$
\Delta G_c = \int_0^l \left( \frac{\Delta f_{\text{HN}}^\mu N + \Delta f_T^\mu T \Delta N + \Delta f_T^\mu T}{\Delta s} \right) \cdot \left( \frac{\delta r(s) - \delta r(s)}{\delta \theta(s) - \delta \theta(s)} \right) \, ds.
$$

The directional derivatives $f_{\text{HN}}^\mu N$, $\Delta f_T^\mu T$, $\Delta N$ and $\Delta \bar{s}$ were calculated in [8], with notations $\bar{h}(s) = h(s) - h(\bar{s})$ and $\bar{h} = h(\bar{s})$ for any vector-valued function $h(\cdot)$, are given by

$$
\Delta f_{\text{HN}}^\mu N = -\frac{1}{\mu_N} H(d_r + \mu_N \lambda_N^\mu) \bar{\eta} \cdot N,
$$

$$
\Delta f_T^\mu T = \frac{F_T(g_r + \mu_T \lambda_T^\mu)}{\mu_T \Delta t} \left( \bar{\eta} - \Delta \bar{s} \bar{r}' + 2\varepsilon \Delta N \right),
$$

$$
\Delta N = \frac{(I - N \otimes N) \bar{\eta} - \Delta \bar{s} \bar{r}'}{2\varepsilon - d_r},
$$

and, it follows from the directional derivative of (3.1) that the next equality gives an expression for $\Delta \bar{s}$

$$
\left[ \Vert \bar{r}' \Vert^2 - \bar{r}'' \cdot \bar{r} \right] \Delta \bar{s} = \bar{\eta} \cdot \bar{r}' + \bar{r} \cdot \bar{\eta}'.
$$
The Heaviside function \( H(x) \) in (6.4a) is equal to 1 if \( x > 0 \) and is equal to 0 if \( x < 0 \). The derivative is not defined at \( x = 0 \), a fact which we recognize but generally ignore in practice by taking \( H(x) \big|_{x=0} = 1 \). The function \( F_T \) in (6.4b) is defined by

\[
F_T(x) = \begin{cases} 
1 & \text{if } \|x\| \leq \mu_T \gamma_T \nu_H, \\
0 & \text{otherwise},
\end{cases}
\]

The similar function

\[
F_R(x) = \begin{cases} 
1 & \text{if } \|x\| \leq \mu_R \gamma_R \nu_H, \\
0 & \text{otherwise},
\end{cases}
\]

will appear in the directional derivative of \( f^{\mu_R}_R \) as follows

\[
\Delta f^{\mu_R}_R = \Delta \left\{ \frac{1}{\mu_R} P_{\mu_R}^{n+1}(g_R + \mu_R \lambda^{n}_R) \right\} = \frac{1}{\mu_R} \frac{\partial P_{\mu_R}^{n+1}(g_R + \mu_R \lambda^{n}_R)}{\partial g_R} \Delta g_R = \frac{1}{\mu_R} F_R(g_R + \mu_R \lambda^{n}_R) \Delta g_R. \tag{6.6}
\]

In view of the Euler approximation (1.1), the directional derivative of the rotational sliding velocity is given by

\[
\Delta g_R = \frac{1}{\Delta t} \left[ \tilde{\theta} - 2 \Delta \tilde{\dot{\theta}}' - (\tilde{s} - \tilde{s}_0) \left( \Delta \tilde{\theta}'' + \tilde{\dot{\theta}}' \right) \right]. \tag{6.7}
\]

Upon substitution of (6.4), (6.5), (6.6) and (6.7) into (6.3), the final form of the directional derivative \( \Delta G_c(r, \theta; \delta r, \delta \theta; \eta, \dot{\theta}) \) is given by

\[
\Delta G_c = \int_0^T \left[ \frac{H(d_r + \mu_N \lambda^n_R)}{\mu_N(2 \varepsilon - d_r)} Q_N + \frac{F_T(g_r + \mu_T \lambda^n_T)}{\mu_T \Delta t(2 \varepsilon - d_r)} \tilde{\eta} \cdot (2 \varepsilon N \otimes N - d_r I) \delta r \\
+ \frac{F_R(g_R + \mu_R \lambda^n_R)}{\Delta t \mu_R} ((\tilde{s} - \tilde{s}_0) \tilde{\theta}'' \cdot \delta \theta - \tilde{\dot{\theta}}' \cdot \delta \theta) \right] ds, \tag{6.8}
\]

with \( Q_N \) and \( Q_T \) are described respectively in [7] and [8], and \( Q_R \) is given by

\[
Q_R = \Omega \left( \tilde{\eta} \right) \left( \Xi_R \tilde{\eta}' \otimes (\tilde{\theta}' + \frac{\tilde{s} - \tilde{s}_0}{2} \tilde{\theta}'') \tilde{\eta}' \otimes f^{\mu_R}_R \right) \left( \delta \theta \right)
\]

where

\[
\Xi_R = \frac{2 F_R(g_R + \mu_R \lambda^n_R)}{\mu_R \Delta t}, \quad \text{and} \quad \Omega = \frac{1}{\| \tilde{\eta}' \|^2 - \tilde{\eta}' \cdot \tilde{\eta}}.
\]

### 7 Finite-element implementation

In [12], the authors present a new numerical implementation by mixed finite-elements for solving contact problems with Coulomb friction that is based on the realization of a sequence of variational inequalities describing the reduced problems. This implementation showing that the resulting algorithm may be much faster than the original fixed point method and
its efficiency is even comparable with the solution of frictionless contact problems. They extended this study this study to a splitting type approach in [18]. In this paper, we use a finite-element method with penalty and augmented Lagrangian approaches to present a discretization of the frictional self-contact problem, which can be used for numerical implementation in future research. Due to its generality and richness, the finite-element method has been used with remarkable success in solving a wide variety of problems in engineering fields. This method consists in dividing the domain of the solution into a finite number of subdomains and using the variational formulations to build an approximation of the solution on each finite element.

Now we proceed to give a finite-element implementation of the linearized frictional self-contact problem (6.1). For this, we define a uniform subdivision of the interval [0, 1] into typical elements as follows

$$[0, 1] = \bigcup_{e=1}^{N} I_{h}^{e}, \quad h = \frac{l}{N}, \quad I_{h}^{e} = [s_{e-1}, s_{e}],$$

where $N \in \mathbb{N}$ is the number of elements. Accordingly, we can approximate a field variable $g$ (such as $r, \theta, \delta r, \delta \theta, \eta$ and $\vartheta$) with $g_{h}$ by using LaGrange cubic finite elements whose their restriction to a typical element $I_{h}^{e}$ is $g_{h}^{e}$. This field is interpolated using the Lagrangian shape function as follows

$$g_{h}^{e}(s) = \sum_{n=1}^{4} \phi_{h}^{e}(s) g_{h}^{e,n}, \quad \text{such that} \quad \phi_{h}^{e}(s) = \phi_{n} \left( \frac{s - s_{e-1}}{h} \right), \quad \forall s \in I_{h}^{e},$$

(7.1)

where $(\phi_{n}(\cdot))_{1 \leq n \leq 4}$ are the usual cubic interpolation functions on a reference element, and $g_{h}^{e,n}$ represent the nodal value at node $n$ of element $e$ of the field $g_{h}^{e}$. The discretizations of the first two integral parts in (4.1) concerning the virtual work of internal forces and external loads are well detailed in [7].

### 7.1 Discretization of the frictional self-contact energy

We denote by $N_{G} = \{s_{i}; \ i = 0, \ldots, N_{G}\}$ the set of total Gauss quadrature nodes on [0, 1]. For all $s_{i} \in N_{G}$ and $I_{h}^{e}$ the slave element that contains the slave node $s_{i}$, we denote by $I_{h}^{e}$ the corresponding master element that contains the master node $\tilde{s}_{i}$. For the elements of frictional self-contact, we introduce the following vectors of nodal values of $r$ and $\theta$:

$$r^{(v,e)} = (r_{v,1}, r_{v,2}, r_{v,3}, r_{v,4}, -r_{e,1}, -r_{e,2}, -r_{e,3}, -r_{e,4})^{T},$$

$$\theta^{(v,e)} = (\theta_{v,1}, \theta_{v,2}, \theta_{v,3}, \theta_{v,4}, -\theta_{e,1}, -\theta_{e,2}, -\theta_{e,3}, -\theta_{e,4})^{T}.$$  

By substitution of the finite-element approximations defined at the beginning of this section, the discretized form of the contact energy (6.2) writes

$$G_{c} = \sum_{v=1}^{N} G_{c}^{v} = -\sum_{v=1}^{N} \int_{I_{h}^{e}} \left( f_{N}^{H} N_{V}^{V} + f_{R}^{\mu} R_{T}^{\mu} \right) \cdot \left( \sum_{n=1}^{4} \phi_{n}^{v} \delta r_{h}^{v,n} - \sum_{n=1}^{4} \phi_{n}^{e} \delta r_{h}^{e,n} \right) ds.$$  

For what follows, by introducing $\Psi_{N}^{(v,e)}, \Psi_{T}^{(v,e)}$ and $\Psi_{R}^{(v,e)}$ the discretizations of the normal, tangential and rotational self-contact forces defined by
\[
\psi_N^{(v,e)} = \begin{bmatrix}
\phi_1 f_{\mu N}^{T} N^v \\
\phi_2 f_{\mu N}^{T} N^v \\
\phi_3 f_{\mu N}^{T} N^v \\
\phi_4 f_{\mu N}^{T} N^v \\
-\frac{\gamma}{\phi_1} f_{\mu N}^{T} N^v \\
-\frac{\gamma}{\phi_2} f_{\mu N}^{T} N^v \\
-\frac{\gamma}{\phi_3} f_{\mu N}^{T} N^v \\
-\frac{\gamma}{\phi_4} f_{\mu N}^{T} N^v
\end{bmatrix},
\psi_T^{(v,e)} = \begin{bmatrix}
\phi_1 f_{\mu T}^{T} T^v \\
\phi_2 f_{\mu T}^{T} T^v \\
\phi_3 f_{\mu T}^{T} T^v \\
\phi_4 f_{\mu T}^{T} T^v \\
-\frac{\gamma}{\phi_1} f_{\mu T}^{T} T^v \\
-\frac{\gamma}{\phi_2} f_{\mu T}^{T} T^v \\
-\frac{\gamma}{\phi_3} f_{\mu T}^{T} T^v \\
-\frac{\gamma}{\phi_4} f_{\mu T}^{T} T^v
\end{bmatrix},
\psi_R^{(v,e)} = \begin{bmatrix}
\phi_1 f_{\mu R}^{T} R^v \\
\phi_2 f_{\mu R}^{T} R^v \\
\phi_3 f_{\mu R}^{T} R^v \\
\phi_4 f_{\mu R}^{T} R^v \\
-\frac{\gamma}{\phi_1} f_{\mu R}^{T} R^v \\
-\frac{\gamma}{\phi_2} f_{\mu R}^{T} R^v \\
-\frac{\gamma}{\phi_3} f_{\mu R}^{T} R^v \\
-\frac{\gamma}{\phi_4} f_{\mu R}^{T} R^v
\end{bmatrix},
\]

and by using a Gaussian quadrature rule with three integration points for approximating the integral over \(I_h^v\), we can write

\[
\mathcal{G}_c \simeq \sum_{v=1}^{N} \sum_{i=1}^{3} \left( F_{NT}^{v,i} \right)^T \left( \delta r^{(v,e_i)} \right) \cdot \left( \delta \theta^{(v,e_i)} \right),
\]

where \(\left( F_{NT}^{v,i}, F_{R}^{v,i} \right)^T = -\omega_i \left( \psi_N^{(v,e_i)} + \psi_T^{(v,e_i)} + \psi_R^{(v,e_i)} \right)^T\) represents the discretized frictional self-contact forces applied to element \(v\), and \(\omega_i\) are the corresponding Gauss integration weights.

We similarly use the finite-element approximation (7.1) and the Gaussian quadrature rule with three integration points for evaluating the integrals in the expression (6.8) to obtain their following discretized form:

\[
\Delta \mathcal{G}_c = \sum_{v=1}^{N} \Delta \mathcal{G}_c^{(v,e)}
\]

\[
\simeq \sum_{v=1}^{N} \sum_{i=1}^{3} \omega_i \left( (\Delta r^{(v,e_i)} \right)^T K_{NT}^{v,i} (\delta r^{(v,e_i)}) + (\Delta \theta^{(v,e_i)})^T \left( K_{R1}^{v,i} K_{R2}^{v,i} \right) \delta \theta^{(v,e_i)}),
\]

where \(K_{NT}^{v,i}\) and \(K_{R1}^{v,i}, K_{R2}^{v,i}\) are the frictional self-contact element stiffness matrices given as follows

\[
K_{NT}^{v,i} = \frac{H \left( d_r^{(v,e)} + \frac{\mu N \eta \gamma_{v,e}}{\mu N (2\varepsilon - d_r^{(v,e)})} \right)}{\mu N (2\varepsilon - d_r^{(v,e)})} (\Sigma_1^{v,e})^T \Gamma_{NT}^{v,e} \Sigma_1^{v,e} + (\Sigma_1^{v,e})^T \Gamma_{T}^{v,e} \Sigma_1^{v,e},
\]

\[
(K_{R1}^{v,i}, K_{R2}^{v,i}) = \left( \Sigma_1^{v,e})^T \Gamma_{R1}^{v,e} \Sigma_1^{v,e}, (\Sigma_1^{v,e})^T \Gamma_{R2}^{v,e} \Sigma_1^{v,e} \right),
\]

where

\[
\Gamma_{NT}^{v,e} = \left[ \begin{array}{cccc}
\Gamma_{N,11}^{v,e} & \Gamma_{N,12}^{v,e} \\
\Gamma_{N,12}^{v,e} & \Gamma_{N,22}^{v,e}
\end{array} \right], \quad \Gamma_{T}^{v,e} = \left[ \begin{array}{cccc}
\Gamma_{T,11}^{v,e} & \Gamma_{T,12}^{v,e} \\
\Gamma_{T,12}^{v,e} & \Gamma_{T,22}^{v,e}
\end{array} \right], \quad \Gamma_{R1}^{v,e} = \left[ \begin{array}{cccc}
\Gamma_{R1,11}^{v,e} & \Gamma_{R1,12}^{v,e} \\
\Gamma_{R1,12}^{v,e} & \Gamma_{R1,22}^{v,e}
\end{array} \right],
\]

\[
\Gamma_{R2}^{v,e} = \left[ \begin{array}{cccc}
0 & 0 \\
0 & 0
\end{array} \right], \quad \Sigma_1^{v,e} = \left[ \begin{array}{cccc}
\Phi_1 & -\Phi_e \\
0 & 0
\end{array} \right], \quad \Sigma_2^{v,e} = \left[ \begin{array}{cccc}
\Phi_1 & 0 \\
0 & \Phi_e
\end{array} \right],
\]

\[
\Phi_e = \left[ \Phi_1 I \Phi_2 I \Phi_3 I \Phi_4 I \right],
\]

with

\[
\Gamma_{N,11}^{v,e} = \frac{d_r^{(v,e)}}{\| \tilde{r}^{e'} \|^2 - \tilde{r}^{e''} \cdot (r^e - \tilde{r}^e) + 2\varepsilon N^v \otimes N^v - d_r^e I},
\]
It should be noted that the stiffness matrices are symmetric, whereas the stiffness matrix $K^{v,i}_{NT}$ is nonsymmetric. In the stability analysis, which is based on resolving an eigenvalue problem, it is highly desirable to have a method to symmetrize the stiffness matrix $K^{v,i}_{R1}$ as well as the geometric part of the tangential element stiffness matrix given by Chamekh et al. [7]. For more details on the symmetrization technique, see [20, 36].

8 Conclusions

One of the most important points in these types of problems is the existence result for the theoretical formulation. Unfortunately, the self-friction problem in elastic rods is strongly nonlinear due to the coupling between the displacement and rotation. As in the tangential self-friction case, the existence and uniqueness of the solution remain a major challenge in the rotational self-friction case. We have used the augmented Lagrangian method to present a weak nonlinear variational formulation approach and have given an existence theorem for the penalized problem. Newton-Raphson method is proposed to relax the problem of this nonlinearity. In addition, linearization and discretization by finite-elements are well developed, which may facilitate the numerical implementation in future research.
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Declarations

Conflict of interest  The authors declare that they have no conflict of interest regarding the publication of this article.

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