Exact computation of the Special geometry for Calabi–Yau hypersurfaces of Fermat type.

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Abstract

We continue to develop our method for effectively computating the special Kähler geometry on the moduli space of Calabi–Yau manifolds. We generalize it to all polynomial deformations of Fermat hypersurfaces.

1 Introduction.

When the Superstring theory is compactified on a Calabi–Yau (CY) threefold $X$, the low-energy effective theory is defined in terms of the Special Kähler geometry of the CY moduli space.

It is known that the Kähler potential is given by the logarithm of the holomorphic volume of CY manifold $\Omega_{\phi}$:

$$G(\phi)_{\alpha\bar{\beta}} = \partial_{\alpha} \overline{\partial}_{\bar{\beta}} K(\phi, \bar{\phi}),$$

$$e^{-K(\phi)} = \int_{\Omega_{\phi}} \Omega \wedge \bar{\Omega},$$

(1)

This can be rewritten in terms of periods of $\Omega$ as

$$\omega_{\mu}(\phi) := \int_{q_{\mu}} \Omega, \quad q_{\mu} \in H_{3}(X, \mathbb{R}),$$

$$e^{-K} = \omega_{\mu}(\phi) C_{\mu\nu} \omega_{\nu}(\phi),$$

(2)

where $C_{\mu\nu} = [q_{\mu}] \cap [q_{\nu}]$ is an intersection matrix of 3-cycles. The Kähler metric on the moduli space is also called as the Weil–Petersson metric or $t^* t$ metric and is closely related to the Zamolodchikov metric. Apart from its own interest, the knowledge of its explicit form is useful in various contexts. In particular it enters the vacua equation in the moduli stabilization problem and the holomorphic anomaly equation for the higher genus B-model and allows computing distances in the moduli space. For instance, it was recently used to check the Refined Swampland Distance Conjecture in

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This metric was first found for the Quintic threefold in [6]. There were then results in the one-modulus cases in [9] and in some 2-moduli cases [10, 11]. As far as we know, there have not been many advances apart from these until recently. Another interesting approach to computing the Special geometry was proposed in [12] based on mirror symmetry and localization computations in 2-dimensional models [13, 14].

We lately suggested a new much simpler method for computing the Kähler metric for a large class of CY defined as hypersurfaces in weighted projective spaces [15, 16, 17]. This method uses the correspondence between the middle cohomology of CY manifolds and the invariant Frobenius algebra associated with the potential $W$ defining the given CY manifold. This correspondence is realized by the oscillatory integral representation for the periods of the holomorphic CY 3-form. Having established this correspondence we obtain an efficient method for computing the special geometry on the moduli space.

If a CY manifold $X$ is realized by a quasi-homogeneous polynomial $W(x)$ in a weighted projective space $\mathbb{P}^d_{k_1,\ldots,k_5}$, then a subgroup of the cohomology group $H^{1,0}(X) \oplus H^{2,1}(X) \oplus H^{1,2}(X) \oplus H^{0,3}(X)$, the complex conjugation $\ast$ of differential forms and Poincare pairing $\langle \cdot, \cdot \rangle$ is isomorphic to the invariant Milnor ring $R^Q$ defined by $W(x)$ with a Hodge decomposition given by the monomial weight grading, antiholomorphic involution $M$ and the residue pairing $\eta$. From this fact, we obtain the formula for the Kähler potential $K(\phi)$

$$e^{-K(\phi)} = \sigma_\mu^\ast(\phi) \eta_{\mu\lambda} M_{\lambda\nu} \overline{\sigma_\nu(\phi)},$$

where $\sigma_\mu^\ast(\phi)$ are periods computed as oscillatory integrals, $\eta_{\mu\nu}$ is a residue pairing in the Milnor ring and $M_{\mu\nu}$ is the antiholomorphic involution of the ring $R^Q$. The three ingredients $\sigma_\mu(\phi)$, $\eta_{\mu\nu}$ and $M_{\mu\nu}$ can all be efficiently computed.

For CY manifolds defined as a Fermat hypersurface, the computation is not more difficult than in the Quintic threefold case. Here we compute this case and find the real structure (see below)

Section 2 of the paper is a short exposition of the methods developed in [15, 16, 17] and section 3 is a specification and computation for the Fermat case.

2 CY as the hypersurface in the weighted projective space

Let

$$\mathbb{P}^d_{k_1,\ldots,k_5} = \{(x_1 : \cdots : x_5) | (x_1 : \cdots : x_5) \simeq (\lambda^{k_1} x_1 : \cdots : \lambda^{k_5} x_5)\}$$

denote the weighted projective space and

$$X = \{x_1, \ldots, x_5 \in \mathbb{P}^d_{k_1,\ldots,k_5} | W_0(x) = 0\}$$

be a transverse hypersurface for some quasi-homogeneous polynomial $W_0(x)$,

$$W_0(\lambda^{k_1} x_i) = \lambda^d W_0(x_i)$$

and

$$\deg W_0(x) = d = \sum_{i=1}^5 k_i.$$  

(7)

The last relation ensures that $X$ is a CY manifold. The (polynomial part of the) moduli space of complex structures of $X$ is then given by homogeneous polynomial deformations of this singularity up to coordinate transformations:

$$W(x, \phi) = W_0(x) + \sum_{s} \phi_s c_s(x),$$

(8)

where $c_s(x)$ are monomials in $x$ of the same degree $d$. We use the index $s$ for monomials and periods in this range. The holomorphic 3-form $\Omega$ is given as a residue of a 5-form in the underlying affine space $\mathbb{C}^5$:

$$\Omega = \frac{x_5 dx_1 \wedge dx_2 \wedge dx_3}{\partial W(x)/\partial x_4}.$$  

1Although these spaces in general are orbifolds, we work with them as if they were smooth. For our purposes this does not lead to any problems in computing the integrals over 3-cycles.
We define period integrals or periods of $\Omega$ needed for our goal:

$$\omega_\mu(\phi) := \int_{q_\mu} \Omega, \quad q_\mu \in H_0(X, \mathbb{R}),$$

where $H^3(X)$ has a natural Hodge structure $H^3(X) = \oplus_{k=0}^3 H^{3-k,k}(X)$,

$$\dim H^{3,0}(X) = \dim H^{0,3}(X) = 1, \quad \dim H^{2,1}(X) = \dim H^{1,2}(X) = h^{2,1}.$$

The Poincaré pairing can be written in terms of integrals over some cycles $q_\mu$ as

$$\eta(\chi_a, \chi_b) = \int_X \chi_a \wedge \chi_b = \int_{q_\mu} \chi_a \cdot C_{\mu\nu} \int_{q_\nu} \chi_b$$

and is invariant under complex conjugation $(p, q)$-forms. Here, $C_{\mu\nu} = [q_\mu] \cap [q_\nu]$ is the intersection matrix of 3-cycles.

### 2.1 Q-invariant Milnor ring.

On the other hand, the polynomial $W_0(x)$, considered as a singularity, defines a Milnor ring $R_0$. Let $Q = \mathbb{Z}_2$ denote a symmetry group $Q$ acting on $\mathbb{C}^5$ diagonally as $\lambda \cdot (x_1, \cdots, x_5) = (\lambda x_1, \cdots, \lambda x_5)$ for $\lambda^d = 1$. This action is trivial on $\mathbb{P}^d(k_1, \cdots, k_5)$ and, moreover, preserves $W(x, \phi)$. We consider the $Q$-invariant part of the Milnor ring

$$R^Q = \left( \frac{\mathbb{C}[x_1, \cdots, x_5]}{\text{Jac}(W_0)} \right)^Q, \quad \text{Jac}(W_0) = \langle \partial_i W_0 \rangle^Q_{i=1}.$$

The subring $R^Q$ becomes a Frobenius ring if it is endowed with the pairing

$$\eta(e_\alpha, e_\beta) = \text{Res} e_\alpha(x)e_\beta(x) dx \prod_{i=1}^5 \partial_i W_0(x).$$

The Hodge decomposition of $R^Q$ corresponds with to the quasi-homogeneity degrees 0, $d, 2d, 3d$ of its components

$$R^Q = (R^Q)^0 \oplus (R^Q)^1 \oplus (R^Q)^2 \oplus (R^Q)^3,$$

where $\dim(R^Q)^0 = \dim(R^Q)^1 = 1, \quad \dim(R^Q)^1 = \dim(R^Q)^2 = h_{2,1},$ and $\dim R^Q = \dim H^3(X).$ Here $h_{2,1}$ denotes the number of deformations of complex structure which can be represented as polynomial deformations in the ambient projective space. In particular, $(R^Q)^3 = \langle e_\mu(x) \rangle$, where we introduced the notation

$$e_\mu(x) = \det \partial_i \partial_j W_0(x).$$

### 2.2 Q-invariant cohomology $H^*_D((\mathbb{C}^5)^\text{inv})$ and the oscillatory integrals.

In the next step, we define two differentials $D_\pm$,

$$D_\pm = e^{\mp W_0} d e^{\pm W_0} = d \pm d W_0 \wedge, \quad (D_\pm)^2 = 0,$$

and two groups of $Q$-invariant cohomology $H^*_D((\mathbb{C}^5)^\text{inv})$. As vector spaces, they are isomorphic to $R^Q$. Choosing some basis $\{e_\mu(x)\}$ in the ring $R^Q$ we can write a basis of $H^*_D((\mathbb{C}^5)^\text{inv})$ as $\{e_\mu(x) d^5 x\}$. These groups inherit the grading degree and Hodge structure from $R^Q$. These cohomology groups are naturally subgroups of the middle cohomology group $\in H^3(X)$ ($\mathbb{19})$. This isomorphism, defined below, maps the Hodge decomposition components of $H^*_D((\mathbb{C}^5)^\text{inv})$ spanned by $e_\mu(x) d^5 x$ with $e_\mu(x) \in (R^Q)^8$ to the corresponding components $H^{3-\eta \cdot q}(X)$. It also sends the Poincare pairing on the differential forms on $X$ to the invariant ring $R^Q$ pairing $\eta$. Having $H^*_D((\mathbb{C}^5)^\text{inv})$, we define their dual homology group, i.e., the $Q$-invariant relative homology groups $\mathcal{H}^*_D := H_2(\mathbb{C}^5, \text{Re} W_0 = L \to \pm \infty)Q$ as a quotient of the relative homology group $H_2((\mathbb{C}^5, \text{Re} W_0 = L \to \pm \infty)$. For this, we define the pairing via oscillatory integrals

$$(e_\mu(x) d^5 x, Q^\perp_{\eta \cdot q}^+) := \int_{Q^\perp_{\eta \cdot q}^+} e_\mu(x) e^{\mp W(x)} d^5 x.$$
$H^2_\Delta(C^5)_Q$. The crucial fact in what follows is that $R^Q$ and $H^3(X)$ and all their additional structures are isomorphic to each other. First, there exists an isomorphism $S$ of cycles for each $\phi$. This gives

$$S(Q^+_{i\nu}) = q_{i\nu}, \quad Q^+_{i\nu} \in H^{2+i-Q}_i, \quad q_{i\nu} \in H_3(X,\mathbb{Z}). \quad (13)$$

The isomorphism is defined by the oscilatory integrals as follows. Let $\{q_{i\nu}\}$ is a basis of $H_3(X,\mathbb{Z})$, then the basis $Q^+_{i\nu}$ of $H^{2+i-Q}_i$ can be chosen in such a way that the integrals over the corresponding cycles of these bases are equal

$$\int_{q_{i\nu}} \Omega_{\phi} = \int_{Q^+_{i\nu}} e^{\pi W(x,\phi)} d^5 x. \quad (14)$$

### 2.3 $H^3(X)$ versus $H^5_\Delta(C^5)_{\text{inv}}$ correspondence.

Having an isomorphism between $H_3(X)$ and $H^{2+i-Q}_i$, we define the isomorphism between the two cohomology groups $H^3(X)$ and $H^5_\Delta(C^5)_{\text{inv}}$ also using oscilatory integrals. We take the basis of cycles $q_{i\nu} \in H_3(X)$ and the corresponding basis of cycles $Q^+_{i\nu} \in H^{2+i-Q}_i$ at $\phi = 0$. The form $\chi_{i\nu} \in H^1(X)$ then corresponds to the form $e_{i\nu}(x) d^5 x \in H^5_\Delta(C^5)_{\text{inv}}$ iff

$$\int_{q_{i\nu}} \chi_{i\nu} = \int_{Q^+_{i\nu}} e_{i\nu}(x) e^{\pi W(x,\phi)} d^5 x \quad (15)$$

for all pairs $\{q_{i\nu}, Q^+_{i\nu}\}$. Thus these two forms are isomorphic if they have equal coordinates (i.e., periods) in some isomorphic bases. This isomorphism preserves the Hodge filtration i.e., the elements $(R^Q)_{\leq k\text{d}}$ are mapped to $F^k H^3(X) := \oplus_{i \leq k} H^{3-i,i}(X)$. This can be seen by differentiating \( (14) \) with respect to the deformation parameters $\phi$. The $k$th derivative of the RHS belongs to $\oplus_{i \leq k} H^{3-k-i,i}(X)$ by Kodaira’s lemma or Griffiths transversality while the $k$th derivative of the LHS belongs to $R^Q_{\leq k\text{d}}$. As is seen below, for Fermat hypersurfaces, the isomorphism also preserves the decomposition (therefore $k$th derivative of the RHS belongs to $H^{3-k-k}(X)$). The intersection matrices of the cycles $q_{i\nu} \cap q_{r\sigma}$ and $Q^+_{i\nu} \cap Q^+_{r\sigma}$ coincide, as we now show. It follows from the coincidence of the pairings of the differential forms $\in H^3(X)$ and of the corresponding elements $\in R^Q$.

We rewrite the Poincaré pairing of $\chi_{i\nu}$ and $\chi_{j\sigma}$ in $H^3(X)$,

$$\langle \chi_{i\nu}, \chi_{j\sigma} \rangle := \int_X \chi_{i\nu} \wedge \chi_{j\sigma} \quad (16)$$

as the bilinear expression of periods,

$$\langle \chi_{i\nu}, \chi_{j\sigma} \rangle = \int_{q_{i\nu}} \chi_{i\nu} C_{i\nu j\sigma} \int_{q_{j\sigma}} \chi_{j\sigma}, \quad (17)$$

where $C_{i\nu j\sigma} = q_{i\nu} \cap q_{j\sigma}$ is the intersection matrix of the cycles. On the other hand, the residue pairing $\eta(e_{i\nu}, e_{j\sigma})$ in the ring $R^Q$ can be written in terms of the periods as explained in \( [20,3] \), also see \( [21] \) as

$$\eta(e_{i\nu}, e_{j\sigma}) = \int_{Q^+_{i\nu}} e_{i\nu} e^{W_0(x)} d^5 x \hat{C}_{i\nu j\sigma} \int_{Q^+_{j\sigma}} e_{j\sigma} e^{W_0(x)} d^5 x, \quad (18)$$

where $\hat{C}_{i\nu j\sigma} = Q^+_{i\nu} \cap Q^+_{j\sigma}$. Taking into account the equality \( (15) \) and the equality of the pairings

$$\langle \chi_{i\nu}, \chi_{j\sigma} \rangle = \eta(e_{i\nu}, e_{j\sigma}) \quad (19)$$

we obtain the relation $\hat{C}_{i\nu j\sigma} = C_{i\nu j\sigma}$. The relation \( (18) \) will be used below for expressing the intersection matrix $C_{i\nu j\sigma}$ in terms of the $R^Q$ pairing.

### 2.4 Anti-involution $M$ on $R^Q$

The same isomorphism allows defining an anti-Involution $M$ on $R^Q$ and on the $Q$-invariant cohomology $H^5_\Delta(C^5)_{\text{inv}}$, that corresponds to a complex conjugation $\ast$ on the differential forms in $H^3(X)$. Let the form $\phi_{i\nu} \in H^3(X)$ correspond to $\{e_{i\nu}(x)\} \in R^Q$ under the isomorphism $S$, and let

$$\ast \phi_{i\nu} = M_{i\nu \sigma} \phi_{j\sigma}.$$

Then $R^Q$ inherits this involution. For the basis $\{e_{i\nu}(x)\}$, the antiholomorphic operation $\ast$ is

$$\ast e_{i\nu}(x) = M_{i\nu \sigma} e_{j\sigma}(x). \quad (20)$$
Because $(*)^2 = I$, it follows from this definition that $MM = I$.

We introduce the convenient basis $\Gamma^\pm_\mu \in H^\pm_\mu (\mathbb{C})_{\nu \nu}$ dual to the basis $\{e_\mu(x)\}$ such that

$$ (\Gamma^\pm_\mu, e_\nu(x) d^5 x) = \int_{\Gamma^\pm_\mu} e_\nu(x) e^T W_0(x) d^5 x = \delta_{\mu \nu}. \tag{21} $$

This definition induces the antiholomorphic operation $*$ on $\Gamma^\pm_\mu$

$$ * \Gamma^\pm_\mu = M_{\mu \nu} \Gamma^\pm_\nu, $$

and hence

$$ \langle * \Gamma_\mu, e_\nu(x) d^5 x \rangle = \langle \Gamma_\mu, e_\nu(x) d^5 x \rangle. \tag{22} $$

The cycles $\Gamma^\pm_\mu$ belong to the homology group $2\mathbb{H}^{\pm, Q}$, therefore they are linear combinations of some geometric cycles with complex coefficients. If we define $T$ as a transition matrix from cycles $\Gamma^\pm_\mu$ to an arbitrary real basis of cycles, for example, Lefschetz thimbles $L^\pm_\mu = *L^\pm_\mu$, then we have

$$ L^\pm_\mu = T_{\mu \nu} \Gamma^\pm_\nu. $$

Comparing this relation with

$$ * \Gamma^\pm_\mu = M_{\mu \nu} \Gamma^\pm_\nu, $$

we obtain the expression for $M$ in terms of $T$,

$$ M = T^{-1} \tilde{T}. $$

Obviously $M$ is independent from the choice of real cycles. From the definition of the cycles $\Gamma^\pm_\mu$, we obtain the useful relation for computing $T_{\mu \nu}$ and $M_{\mu \nu}$ (as is seen below)

$$ T_{\mu \nu} = \int_{L^\pm_\mu} e_\nu(x) e^T W_0(x) d^5 x. $$

### 2.5 Deriving the main formula for Kähler potential

The expression for the pairing on the ring $R^Q$ in terms of periods is

$$ \eta(e_\mu, e_\nu) = \int_{L^\pm_\mu} e_\mu e^{-W_0(x)} d^5 x \ C_{ab} \int_{L^\pm_\nu} e_\nu e^{W_0(x)} d^5 x = T_{\mu \nu} C_{ab} T_{ba}. \tag{23} $$

We have also the formula for $K(\phi)$

$$ e^{-K} = \omega_0^+(\phi) C_{ab} \omega^+_0 (\phi) $$

with

$$ \omega^+_0 (\phi) = \int_{L^\pm_0} e^{T W(x, \phi)} d^5 x = T_{0 \nu} \sigma^+_0 (\phi), $$

where the periods $\sigma^+_0 (\phi)$ are integrals over cycles $\Gamma^+_0$

$$ \sigma^+_0 (\phi) = \int_{\Gamma^+_0} e^{T W(x, \phi)} d^5 x. $$

Eliminating the matrix $C_{ab}$ from these relations we obtain

$$ e^{-K(\phi)} = \sum_{\mu, \nu, \lambda} \sigma_+^\mu (\phi) \eta_{\lambda \nu} M_{\lambda \mu} \sigma^+_\nu (\phi). \tag{24} $$

### 3 Fermat threefolds

Let $X$ be a Fermat CY $X = \{x_1, \ldots, x_5 \in \mathbb{P}^{k_1, \ldots, k_5}; |W(x, \phi) = 0\}$,

$$ W(x, \phi) = \sum_{i=1}^{5} x_i^{d_i} + \sum_{s=1}^{\#poly} \phi_s e_s(x), \quad d = \sum_{i=1}^{5} k_i, $$

where $d_i$ are positive integers. The monomials $e_s(x) = e_{(s_1, \ldots, s_5)}(x) := \prod_{i=1}^{5} x_i^{s_i}$ correspond to deformations of the complex structure of $X$. Their weights are equal to $d$, $\sum_{i=1}^{5} k_i s_i = d$, and each variable $x_i$ has a nonnegative integer power $s_i \leq d_i - 2$. The number of such monomials is denoted $h^{poly}_{21}$ and is less than or equal to the Hodge number $h_{21}$, and $\dim H_{21}(X) = 2h_{21} + 2$ (which can be verified from the combinatorics of the corresponding weighted projective space).
3.1 Q-invariant ring and phase symmetry.

The Milnor ring $R_0$ of the Fermat polynomial $W_0(x)$ is generated as a vector space by monomials $e_\mu(x) = \prod x_i^{\mu_i}$, where each nonnegative variable $\mu_i$ is less than $\frac{d}{k_i} - 1$, and \( \dim R_0 = \prod (\frac{d}{k_i} - 1) \).

$R^Q$ is multiplicatively generated by the monomials $e_\mu(x)$ of weight $d$, which correspond to the deformations of the complex structure of $X$. More precisely, $R^Q$ consists of elements of degree $0, d, 2d$ and $3d$, and the dimensions of the corresponding subspaces are $1, h_{31}^Q, h_{21}^Q$, and $1$. This degree grading defines a Hodge structure on $R^Q$. As mentioned above $R^Q$ is isomorphic to a subgroup of $H^3(X)$. This isomorphism sends the degree filtration to the Hodge filtration on $H^3(X)$ [10]. Fermat polynomials have a nice property that there is a bigger symmetry group $\prod Z_{d/k_i}$ that diagonally acts on $\mathbb{C}^d$: $\alpha \cdot (x_1, \ldots, x_d) = (\alpha^{k_1}{x_1}, \ldots, \alpha^{k_d}{x_d})$, $\alpha^{k_i} = 1$. This action preserves $W_0 = \sum_{i=1}^d x_i^{\frac{d}{k_i}}$.

In particular, the quantum symmetry group $Q = Z_2$ is the diagonal subgroup of $\prod Z_{d/k_i}$. The monomial basis $\{e_\mu(x) = e_\mu(x_1, \ldots, x_d) = \prod x_i^{\mu_i}\}$ of $R^Q$ is an eigenbasis of the phase symmetry group action on each $e_\mu(x)$ has a unique weight. We can extend the phase symmetry action to the parameter space $\{\phi_{s}\}_{s=1}^{\infty} \mathbb{P}$ such that $W(x, \phi)$ is invariant under this action. That is, if $\alpha \cdot e_\mu(x) = \lambda_{\mu}e_\mu(x)$ for some root of unity $\lambda_{\mu}$, then we must define $\alpha \cdot \phi_s := \lambda_{\mu}s \phi_s$. In particular, the equations $W(x, \phi) = 0$ and $W(x, \alpha \cdot \phi) = 0$ define the same CY manifold because the action of $\alpha$ can be undone by a coordinate transformation of the variables $x_i$. This means that for Fermat polynomials $W_0$, the point $\phi = 0$ is an orbifold point in the CY moduli space. Such a symmetry allows simplifying the computations significantly.

The phase symmetry group action obviously preserves the Hodge decomposition. The complex conjugation acts on $H^3(X)$ such that $H^{2,1}(X) = H^{1,2}(X)$, in particular, $H^{2,1}(X) = H^{1,2}(X)$. Through the isomorphism between $R^Q$ and $H^3(X)$, the complex conjugation also acts on the elements of the ring $R^Q$.

3.2 Oscillatory representation and computing the periods.

We introduce the special bases $\Gamma_{\mu}^\pm$ in the homology groups $H^3_{\mathbb{R}}(\mathbb{C}^d)$ by requiring their duality to the bases in $H^3_{\mathbb{R}}(\mathbb{C}^d)$, m.s.:

$$\int_{\Gamma_{\mu}^\pm} e_\mu(x)e^{\mp W_0(x)} d^3x = \delta_{\mu\nu}$$

(25)

with the corresponding periods $\sigma_\mu^\pm(\phi) := \int_{\Gamma_{\mu}^\pm} e_\mu(x)e^{\mp W(x, \phi)} d^3x$, $\sigma_\mu^\mp(\phi) := \sigma_\mu^\pm(\phi)$. These periods are eigenfunctions of the phase symmetry group action.

To explicitly compute $\sigma_\mu^\pm(\phi)$, following [22] [15] we first expand the exponent in the integral regarding the terms in $W(x, \phi) = W_0(x) + \sum_s \phi_se_s(x)$, which are proportional $e_\mu(x)$, as a perturbation. We then obtain $\sigma_\mu^\pm(\phi) = \sum_m \int_{\Gamma_{\mu}^\pm} \prod_s e_s(x)^{m_s} e^{\mp W_0(x)} d^3x \left( \prod_s (\pm \phi_s)^{m_s} \right)$, (27)

where $m := \{m_s\}_s$, $m_s \geq 0$, denotes a multi-index of powers of $\phi_s$ in the above expansion. Because $\sigma_\mu^\pm(\phi) = (-1)^{\mu_\nu/2} \sigma_\nu^\pm(\phi)$, we focus on $\sigma_\mu^+(\phi) := \sigma_\mu^+(\phi)$.

For each of the summands, the form $\prod_s e_s(x)^{m_s} d^3x$ belongs to $H^3_{\mathbb{R}}(\mathbb{C}^d)$, because it is killed by $D_+ \equiv Q$-invariant. The oscillatory integrals of $D_+ \equiv Q$-exact terms are zero, and therefore

$$\int_{\Gamma_{\mu}^+} e^{-W_0(x)} P(x) d^3x = \int_{\Gamma_{\mu}^+} e^{-W_0(x)} (P(x) d^3x + D_+ U)$$

(28)

for any polynomial $P(x)$ and any polynomial $4$–form $U$.

We set $m_s \nu_i = \nu_i + \frac{k_i}{d}$, $\nu_i < \frac{d}{k_i}$, for later convenience.

To compute $\int_{\Gamma_{\mu}^+} e^{-W_0(x)} \prod_i x_i^{\nu_i + \frac{k_i}{d}} d^3x$, (29)

we use the above trick with $\prod_i x_i^{\nu_i + \frac{k_i}{d}} d^3x = (-1)^{\frac{k_i}{d}} \prod_i x_i^{\nu_i + \frac{k_i}{d}} d^3x + D_+ U$. (30)
where

\[ U = \frac{k_1}{d} x_1^{\nu_i + 1 + (n_1 - 1) \frac{d}{k_1}} \prod_{i > 1} x_i^{\nu_i + n_i \frac{d}{k_i}} \, dx_2 \wedge \cdots \wedge dx_5. \]  

(31)

We continue this procedure by induction with respect to all \( n_i \). We can write the final result compactly using Pochhammer’s symbols:

\[ \prod_i x_i^{\nu_i + n_i \frac{d}{k_i}} \, dx_i^5 \, x_i^{\nu_i + n_i \frac{d}{k_i}} = (-1)^n \prod_i \Gamma(\nu_i + 1) \prod_i x_i^{\nu_i + n_i \frac{d}{k_i}} \, dx_i^5, \nu_i < \frac{d}{k_i}. \]  

(32)

where \((a)_n = \Gamma(a + n)/\Gamma(a)\).

If any \( \nu_i = d/k_i - 1 \), then the differential form is exact, and the integral is zero. Otherwise, the RHS of the equation is proportional to \( e^{\nu_i(x)} \), and we can use the definition of \( \Gamma + \mu \):

\[ \int_{\Gamma^+_{\mu}} e^{\nu_i(x)} e^{-W_0(x)} \, dx^5 = \delta_{\mu\nu}. \]  

(33)

Doing in this way and integrating over \( \Gamma^+_{\mu} \) we obtain the explicit expression for the periods

\[ \sigma_{\mu}(\phi) = \sum_{n_i \geq 0} \Gamma(\nu_i + 1) \prod_i x_i^{\nu_i + n_i \frac{d}{k_i}} \sum_{m \in \Sigma_n} \frac{\delta_{s_i} m_s}{m_s!}, \]  

(34)

where

\[ \Sigma_n = \{ m_s \mid \sum_s m_s s_i = \mu_i + \frac{d}{k_i} n_i \}. \]  

(35)

### 3.3 Computing the antiholomorphic involution \( M_{\mu\nu} \)

We now want to compute the antiholomorphic involution \( M_{\mu\nu} \). For this, we use its connection with the transition matrix \( T \). We must first choose an real basis of cycles. We choose Lefschetz thimbles \( L_{\mu}^{\pm} \) as the basis of such cycles. The matrix \( T \) can then be found ast the transition matrix that connects the cycles \( \Gamma^{\pm}_{\mu} \) and Lefschetz thimbles \( L_{\mu}^{\pm} \):

\[ \Gamma^{\pm}_{\mu} = (T^{-1})_{\mu \nu} L_{\nu}^{\pm}. \]

It follows that the transition matrix \( T_{\mu\nu} \) is just given by the integral

\[ T_{\mu\nu} = \int_{L_{\mu}^{\pm}} e^{\nu_i(x)} e^{-W_0(x)} \, dx^5. \]

After computing this integral, we obtain the matrix \( M \) from the formula

\[ M = T^{-1} \tilde{T}. \]

The Lefschetz thimbles \( L_{\mu}^{\pm} \) are products of one-dimensional cycles \( C_{\mu_i} \),

\[ L_{\mu}^{\pm} = \prod_{i=1}^5 C_{\mu_i}, \]

and \( C_{\mu_i} = \tilde{\rho}_{\mu_i} \cdot C_i \) with \( \rho_i = e^{-2\pi i k_i/d} \). This definition of the one-dimensional cycle \( C_{\alpha_i} \) means that this cycle is the path in the \( x_i \) plane obtained by the operation \( \tilde{\rho}^{\pm} \) of rotating counterclockwise through an angle \( \frac{2\pi k_i}{d} \) from the basic path \( C_i \) depicted on the figure.

By construction, \( L_{\mu}^{\pm} \) are the steepest descent/ascent cycles for \( \text{Re} W_0 \). We now compute \( T_{\alpha \mu} \) explicitly

\[ T_{\alpha \mu} = \int_{L_{\mu}^{\pm}} e^{\nu_i} e^{-W_0} \, dx^5 = \rho^{(\alpha, \beta)} A(\mu), \]  

(36)
where $A_\mu$ is a product of five gamma integrals,

$$A_\mu = \prod_i \Gamma\left(\frac{k_i}{d}\right) \Gamma\left(\frac{k_i(\mu_i + 1)}{d}\right).$$

Then

$$T_{\mu\alpha}^{-1} = B(\mu)[\bar{\rho}(\bar{\rho} + 1)] - 1,$$

$$B(\mu) = \prod_i \Gamma\left(\frac{k_i(\mu_i + 1)}{d}\right),$$

$$M_{\mu\nu} = (T^{-1}T)_{\mu\nu} = \prod_i \gamma\left(\frac{k_i(\mu_i + 1)}{d}\right) \delta_{\mu,\rho-\nu},$$

$$\gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)}.$$  

3.4 Kähler potential for the moduli space of Fermat threefolds.

Substituting the explicit expressions for the periods $\sigma_\mu$, the pairing $\eta_{\mu\nu}$, and the anti-involution $M$ in the above expression for the Kähler potential on the moduli space, we obtain

$$e^{-K(\phi)} = \sum_\mu (-1)^{\deg(\mu)/d} \prod_i \gamma\left(\frac{k_i(\mu_i + 1)}{d}\right) |\sigma_\mu(\phi)|^2,$$

where

$$\sigma_\mu(\phi) = \sum_{n_1, \ldots, n_5 \geq 0} \prod_{i=1}^5 \Gamma\left(\frac{n_i + 1}{d}\right) \sum_{m \in \Sigma_n} \prod_{s} \phi_{\frac{m_s}{n_s}} \gamma_{\frac{m_s}{n_s}},$$

$$0 \leq \mu_i \leq \frac{d}{k_i} - 2, \quad \sum_{i=1}^5 \mu_i = 0, d, 2d, 3d,$$

$$\gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)}, \quad \Sigma_n = \{m_s \mid s \sum_{i=1}^5 m_i s_i = \mu_i + \frac{d}{k_i} n_i\}.$$  

4 Conclusion.

We have applied our method to the CY hypersurfaces given as zero sections of Fermat polynomials in weighted projective spaces. We use a better way to compute the real (and even integral structure) for the periods compared with our previous work.

While preparing this paper we learned that the periods and their integral structure were basically computed in a different language and different setting in the mathematical literature [23, 24].

The possible application of the method, used in this paper, which should be done, is the computation for the invertible singularities of the Berglund–Hubsch type [25, 26].

It would be also interesting to know connections with the other points of moduli spaces, that is our method gives Special Geometry metric as a power series around orbifold points of the moduli spaces, which correspond to nonsingular CY manifolds. However, there are many interesting points in the moduli space of CY varieties, many of which are singular such as maximal unipotent monodromy points (mirror to large volume points describing Gromov-Witten theory of the mirror manifold) or conifold points which are the simplest degenerations of the CY manifold.

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