Exact Solution of the Six-Vertex Model on a Random Lattice

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We solve exactly the 6-vertex model on a dynamical random lattice, using its representation as a large $N$ matrix model. The model describes a gas of dense nonintersecting oriented loops coupled to the local curvature defects on the lattice. The model can be mapped to the $c = 1$ string theory, compactified at some length depending on the vertex coupling. We give explicit expression for the disk amplitude and evaluate the fractal dimension of its boundary, the average number of loops and the dimensions of the vortex operators, which vary continuously with the vertex coupling.

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1. Introduction

The correspondence between the critical phenomena on flat and random lattices has been very useful in various problems related to random geometry. Especially interesting from this point of view are the $O(n)$ model\cite{1,2,3,4,5} and the $q$-state Potts model\cite{6,7,8,9}, which allow a neat geometrical interpretation in terms of a gas of loops or clusters with fugacity depending on the continuous parameter $n$ or $q$. Considered on a flat lattice, these statistical models can be mapped onto the 6-vertex model\cite{10}, whose infrared behavior is that of a compactified boson. The continuous parameters $n$ and $q$ are then related to the compactification radius. The mapping\cite{11}, known as Coulomb gas picture, also prescribes to install a background electric charge “at infinity”.

On an irregular lattice, the background electric charge, which is in fact coupled to the local curvature, should be distributed all over the lattice, which can be achieved only by introducing nonlocal operators at all points with curvature. The mapping does not hold any more, which means that, when considered on a random lattice, the 6-vertex model describes a new type of critical phenomena, in which the local curvature of the lattice is involved.

More concretely, let us consider the representation\cite{11} of the 6-vertex model as a gas of oriented loops. The fugacity of each loop is given by a phase factor, the phase being proportional to the geodesic curvature along the loop. On the flat lattice, the geodesic curvature is always $\pm 2\pi$ and all loops have the same fugacity. On the contrary, on a lattice with defects, the geodesic curvature can take any value between $\pi/2$ and $\infty$, and the fugacity of the loop depends on the local curvature of the lattice. In this way, the 6-vertex model on a random lattice allows to study extended objects coupled to the local curvature, unlike all previously considered statistical models.

The 6-vertex model on a random lattice is also interesting from the string theory point of view, because it gives a dual definition of the compactified $c = 1$ string, in which the vortices can be constructed explicitly as local operators. As such, it was considered first by P. Ginsparg\cite{12}, who formulated the corresponding matrix model. The relation between the 6-vertex model on a random lattice and the $c = 1$ string theory has been further elucidated by S. Dalley\cite{13}, who in particular identified the compactification radius as a function of the vertex coupling.

In this paper we present the exact solution of the 6v model defined on a planar random lattice of any finite size. The solution is obtained using the equivalence with a special hermitian matrix model, which is a generalization of the $O(2)$ matrix model\cite{1}.
We will calculate the disk amplitude and investigate its scaling behavior. We will evaluate the fractal dimension of the boundary of the random surface as well as the average number of loops per unit area. We will also calculate the dimensions of the $m$-vortex operators ($m \geq 1$), which in the loop gas representation are the sources of $m$ equally oriented vortex lines. Our results agree with the correspondence between the 6-vertex model on a random surface and the compactified $c = 1$ string theory, where vortices represent discontinuities on the world sheet.

The paper is organized as follows. In Sect. 2 we remind the definition of the 6v model on a flat and random lattices as well as the correspondence with a compactified boson, respectively with the compactified $c = 1$ string theory. The model defined on a random lattice depends on two coupling constants: the vertex coupling $\lambda \in [0, 1]$ and the "cosmological constant" $1/T$ coupled to the number of squares of the lattice. Then we formulate the partition function of the model as a matrix integral, making one step further than the original definition by P. Ginsparg [12]. In Sect. 3 we formulate and solve the saddle point equations for the resolvent of the random matrix for all allowed values of the two couplings. In Sect. 4 we analyze the the scaling limit of the solution near the critical coupling where the size of the random lattice explodes. In Sect. 5 we give a discussion and a summary of the results. The three appendices contain the proof of the exact correspondence between the 6v matrix model and the compactified $c = 1$ string theory (A), the technical details of the calculation (B), and the solution of the $O(2)$ matrix model obtained as the $\lambda \to 0$ limit of our general solution (C).

While this manuscript was being prepared, we received the interesting paper by Paul Zinn-Justin [14], in which some of our results concerning the critical behavior are derived using a different technique.

2. The 6-vertex model on a dynamical random lattice

2.1. Review of the 6-vertex model on a flat square lattice: F-model, loop gas, vortices, mapping to a compactified boson.

The degrees of freedom of the 6-vertex model on a square lattice [10] represent arrows associated with the links of the lattice. The admitted configurations of arrows do not contain sources and sinks, so that there is an equal number of arrows that begin and end at each vertex. The partition function is defined as the sum over all arrow configurations, with Boltzmann factor, which is a product of the statistical weights associated with the
vertices of the lattice. On the dual lattice, the arrows go across the edges and the vertices are associated with the squares. There are 6 possible vertices, which can be divided into two types depicted in Fig.1, the vertices of each type being related by $\pi/2$-rotations.

![Fig.1: The two types of vertices of the 6v model.](image)

For a special value of the spectral parameter the weights of the vertices of each class have the same Boltzmann weight. This specialization of the 6v model is known as the $F$-model. The weights of the two distinct vertices (a) and (b) of the F-model can be parametrized as

\[
(a) \quad a = b = 1, \quad (b) \quad c = 2 \cos(\pi \lambda)
\]

where the parameter $\lambda$ will be assumed to belong to the interval $[0,1]$. Following the standard argument of Baxter [10], we decompose the vertices into couples of line segments making left or right turn as is shown in Fig.2. The weights (2.1) are then obtained by assigning a Boltzmann factor $e^{\pm i\lambda/2}$ to each line segment, where the sign is positive (negative) if the line makes a left (right) turn.

![Fig.2: Decomposing the vertices as pairs of line segments.](image)

The sum over vertex configurations on each surface can be thus put in the form of a sum over fully packed closed nonintersecting loops. The Boltzmann weight of each loop $\mathcal{L}$ is the product of all phase factors along it,

\[
\prod_{\mathcal{L}} e^{\pm i\pi \lambda} = e^{i\lambda \Gamma(\mathcal{L})},
\]
where $\Gamma$ is the geodesic curvature along the loop

$$\Gamma(L) = \frac{\pi}{2} \left( \# \text{ left turns} - \# \text{ right turns} \right). \quad (2.3)$$

By the Gauss-Bonnet formula, on the flat lattice $\Gamma = \pm 4$ and the weight of each loop (after summing over both orientations) is $2 \cos(2\pi \lambda)$. In the 6-vertex model one can introduce vortex operators of vorticity $m$, representing sources of $m$ equally oriented lines. On the flat lattice $m$ is restricted to be a multiple of 4. The 8-vertex model represents a deformation of the 6-vertex model obtained by adding vortices with $m = \pm 4$ with finite fugacity $d$.

The 6-vertex model can be rewritten as a solid-on-solid (SOS) model by assigning a variable $\varphi \in \beta \mathbb{Z}$ to the sites of the dual lattice in such a way that the adjacent heights differ by $\pm \beta/4$. The oriented loops are then interpreted as domain walls of height $\pm \beta/4$. The $m$-vortex operator creates a discontinuity $m\beta/4$ at a site.

At large distances the height variable renormalizes to a free massless boson compactified on a circle of length $\beta = 2\pi R$. If we choose the scale so that the duality transformation acts as

$$R \to \frac{1}{R} \quad \text{or} \quad \beta \to \frac{4\pi^2}{\beta} \quad (2.4)$$

then the conformal dimensions of the magnetic operators are

$$\Delta_{\text{vortex}}^m = \bar{\Delta}_{\text{vortex}}^m = \frac{(m\beta/2\pi)^2}{4}. \quad (2.5)$$

Comparing the singularity of the free energy obtained from the exact solution of the 8-vertex model with the conformal dimension $\Delta_{\text{vortex}}^1$, one finds

$$\beta = \pi \sqrt{1 - 2\lambda}. \quad (2.6)$$

The Kosterlitz-Thouless point is at $\beta = \pi$, where the lowest vortex operator ($m = 4$) becomes marginal.

2.2. The 6-vertex model on a dynamical random lattice: loops, vortices, mapping to the compactified $c = 1$ string theory

Since the weights of the $F$-model are invariant with respect to $\pi/2$-rotations, the model can be considered on any irregular square lattice $\mathcal{S}$. The partition function can be again reformulated as a sum over loop configurations (Fig.3), each loop $L$ weighted by the geodesic curvature (2.3)

$$\mathcal{F}(\mathcal{S}) = \sum_{\text{loop configurations}} \prod_{\text{loops } L} e^{i\lambda \Gamma(L)}. \quad (2.7)$$
In this case the geodesic curvature (2.3) depends on the geometry of the lattice through the gaussian curvature (the total deficit angle) \( \mathcal{R}(\mathcal{L}) \) enclosed by the loop \( \mathcal{L} \)

\[
\Gamma(\mathcal{L}) = 2\pi - \mathcal{R}(\mathcal{L}).
\]  

Therefore \( \Gamma(\mathcal{L}) \) can be any integer (positive or negative) times \( \pi/2 \).

Fig.3: A piece of the random lattice covered by oriented loops.

We define the partition function \( \mathcal{F}(T, \lambda, N) \) of the F model on a dynamical random lattice as the average of the partition function (2.7) in the ensemble of all connected abstract lattices obtained by gluing squares along their edges. The partition function depends on the “cosmological constant” \( T \) coupled to the area (\# squares) \( A(S) \) and the “string coupling constant” \( 1/N \) coupled to the genus \( g(S) \) of the lattice:

\[
\mathcal{F}(T, N, \lambda) = \sum_S N^{2-2g}T^{-A} \mathcal{Z}(S).
\]  

(2.9)

It is natural to expect that the critical behavior of this statistical model is described by a compactified boson interacting with a Liouville field. Indeed, as it was shown in [13], the exponential \( \mathcal{Z} = e^\mathcal{F} \) can be identified with the partition function of a \( c = 1 \) string theory compactified at length \( \beta = 2\pi R \), related to the vertex coupling as

\[
R = \frac{1-\lambda}{2} \quad \text{or} \quad \beta = \pi(1-\lambda).
\]  

(2.10)
Here the scale is fixed again by adopting the convention (2.4), according to which the self-dual radius is \( R = 1 \) \((\beta = 2\pi)\). For completeness, we give a short prove of the correspondence in Appendix A. Note that this is not the same correspondence as on the flat lattice and there is no reason that it should be the same.

A vortex with vorticity \( m \) again represents a source of \( m \) equally oriented lines meeting at a point. However, since the lowest vortex charge is \( m = 1 \), an \( m \)-vortex operator creates a discontinuity \( m\beta \) on the world sheet. It is necessarily accompanied by a conical singularity with curvature \( \mathcal{R} = \frac{\pi}{4}(4 - m) \).

The scaling dimension of an \( m \)-vortex operator is obtained from the corresponding flat dimension \( \frac{1}{2} \) (2.3) by the KPZ rule \([15]\) and reads

\[
\delta_{\text{vertex}}^m = \frac{m\beta/2\pi}{2} = m\frac{1 - \lambda}{4}.
\] (2.11)

The scaling dimensions of vortex operators with \( m = 1, 2, 3, 4 \) are always smaller than one, which means that the interval \( 0 < \beta < \pi \) is deep in the vortex plasma phase. However, this is not a problem because in our model the fugacities of all vortices are strictly zero.

### 3. The 6v matrix model

#### 3.1. From a complex matrix integral to a Coulomb gas system

Ginsparg noticed in \([12]\) that the exponential of the partition function (2.9) can be identified with the perturbative expansion of the following matrix integral

\[
Z_N \equiv e^{\mathcal{F}(T,N,\lambda)} = \frac{\int dX dX^\dagger \exp \left( N\text{tr}[-\sqrt{T}XX^\dagger + X^2X^\dagger - \cos \beta (X^\dagger X)^2] \right)}{\int dX dX^\dagger e^{-\sqrt{T}\text{tr}XX^\dagger}}
\] (3.1)

where the integration goes over all \( N \times N \) complex matrices \( X \).

At the point \( \beta = \pi \) \((\lambda = 0)\), this matrix integral describes the dense phase of the \( O(2) \) model on a random triangulation \([1]\), or equivalently the compactified \( c = 1 \) string at the Kosterlitz-Thouless point \( \beta = \pi \). The correspondence with the compactified string for any \( \beta \) has been worked out by S. Dalley \([13]\). For reader’s convenience we give the proof in Appendix A.

\footnote{Of course the length \( \beta \) is now given by (2.10) and not by (2.6).}
The vortex operators are introduced by adding a source term to the matrix potential, namely

$$S_{\text{vortices}} = N \sum_{m \geq 1} \frac{d_m}{m} (\text{tr}X^m + \text{tr}X^\dagger m). \quad (3.2)$$

The model (3.1) in presence of vertices (the random-surface analog of the 8-vertex model) has been solved in two particular cases: $\lambda = 0$, $d_1$ and $d_2 \neq 0$ [13] and $c = 0$, $d_4 = b$ [16].

The complex matrix integral (3.1) becomes gaussian at the price of introducing an auxiliary hermitian matrix $A$:

$$Z_N \sim \int dA dX^\dagger dX \ e^{W(X,A)}, \quad (3.3)$$

$$W(X, A) = N \text{tr} \left( -\frac{1}{2} A^2 + iAX^\dagger e^{-i\beta/2} - iAX^\dagger X e^{i\beta/2} - \sqrt{T}X^\dagger X \right).$$

The result of the integration over $X$ is a matrix integral which can be viewed as a $\beta \neq \pi$ deformation of the O(2) matrix model

$$Z_N \sim \int dA \ e^{-\frac{1}{2}N\text{tr}A^2} \left| \frac{e^{-N\text{tr}A^2}}{\det \left[ \sqrt{T} \mathbb{1} \otimes \mathbb{1} - ie^{-i\beta/2} A \otimes \mathbb{1} + ie^{i\beta/2} \mathbb{1} \otimes A \right]} \right|. \quad (3.4)$$

The integrand depends only on the eigenvalues $a_1, ..., a_N$ of the matrix $A$. After a linear change of variables, we write the partition function as

$$Z_N \sim \int \prod_{i=1}^N \frac{da_i}{a_i} e^{-NV(a_i)} \left| \prod_{i \neq j} \frac{a_i - a_j}{e^{i\beta/2 a_i} - e^{-i\beta/2 a_j}} \right| \quad (3.5)$$

where

$$V(a) = \frac{T}{2} \left( a^2 + \frac{a}{\sin \frac{\beta}{2}} \right). \quad (3.6)$$

In the following we will use the more convenient exponential parametrization

$$a = -e^\phi \quad (3.7)$$

in which the partition function (3.5) looks like

$$Z_N \sim \int_{-\infty}^\infty \prod_{i=1}^N \ d\phi_i \ e^{-N\tilde{V}(\phi_i)} \prod_{i \neq j} \frac{\sinh \frac{1}{2}(\phi_i - \phi_j)}{\sinh \frac{1}{2}(\phi_i - \phi_j + i\beta)} \quad (3.8)$$

where

$$\tilde{V}(\phi) \equiv V(-e^\phi). \quad (3.9)$$

Note that when $N$ is finite, the partition functions (3.8) and (3.8) do not coincide, because the parametrization (3.7) assumes that $a$ is negative. However, the two partition functions lead to the same $1/N$ expansion of the free energy and the observables.
3.2. The saddle point equations as difference equations for the resolvent

We are interested in lattices with spherical topology, and therefore by the large $N$ limit of the integral (3.8). In this limit the spectral density

$$\rho(\phi) = \frac{1}{N} \sum_{i=1}^{N} \delta(\phi - \phi_i) \quad (3.10)$$

can be considered as a classical function, supported by some real interval $[b, a]$, where $b < a$. It is determined by the saddle point equation

$$\frac{d\hat{V}(\phi)}{d\phi} = \int_{b}^{a} d\phi' \left( \frac{\rho(\phi')}{\tanh\left( \frac{\phi - \phi'}{2} \right)} - \frac{1}{2} \frac{\rho(\phi')}{\tanh\left( \frac{\phi - \phi' + i\beta}{2} \right)} - \frac{1}{2} \frac{\rho(\phi')}{\tanh\left( \frac{\phi - \phi' - i\beta}{2} \right)} \right). \quad (3.11)$$

For a time being we will consider a general potential of the form

$$\hat{V}(\phi) = \sum_{n} t_n e^{n\phi} \quad (3.12)$$

and later will specify

$$t_1 = -\frac{T}{2 \sin \frac{\beta}{2}}, \quad t_2 = \frac{T}{2}. \quad (3.13)$$

Following the same logic as in [17], we will introduce the complex variable\(^2\)

$$z = \phi + it, \quad (3.14)$$

and formulate the saddle point equation (3.11) as a set of conditions on the resolvent

$$W(z) = \frac{1}{2} \int_{b}^{a} \frac{d\phi \rho(\phi)}{\tanh\left( \frac{z - \phi}{2} \right)}, \quad (3.15)$$

which is a holomorphic function on the cylinder $z + 2\pi i \equiv z$ cut along the interval $[b, a]$. The coefficients of the expansion of $W(z)$ at infinity

$$W(z) = \frac{1}{2} + W_1 e^{-z} + W_2 e^{-2z} + ... \quad (3.16)$$

\(^2\) In the context of the 2D string theory, the complex $z$-plane represents the target space of the string. In terms of the $c = 1$ noncritical string, the real and imaginary parts of $z$ are interpreted correspondingly as the Liouville and target-space directions. See, for example, the argument given in [18].

8
are equal to the moments of the spectral density

\[ W_n = \int_b^a d\phi \rho(\phi) e^{n\phi}. \]  \hspace{1cm} (3.17)

The spectral density is equal to the discontinuity of the resolvent along the cut

\[ W(\phi + i0) - W(\phi - i0) = -2\pi i \rho(\phi) \]  \hspace{1cm} (3.18)

and the integral equation \( \text{(3.11)} \) implies the following functional equation for the resolvent

\[ W(\phi + i0) + W(\phi - i0) - W(\phi + i\beta) - W(\phi - i\beta) = \frac{d\tilde{V}(\phi)}{d\phi}, \]  \hspace{1cm} (3.19)

where it is assumed that \( \phi \in [b, a] \).

The normalization condition for the density can be written also as an integral along a contour \( \mathcal{C} \) surrounding the interval \( [b, a] \):

\[ 1 = \int_b^a d\phi \rho(\phi) = \oint_{\mathcal{C}} \frac{dz}{2\pi i} W(z). \]  \hspace{1cm} (3.20)

3.3. Geometrical meaning of the saddle point equations

Let us introduce the new potential

\[ U(z) = \sum_n t_n \frac{e^{nz}}{2 \sin n\beta}, \]  \hspace{1cm} (3.21)

related to the old one by

\[ \tilde{V}(z) = \frac{U(z + i\beta/2) - U(z - i\beta/2)}{i}, \]  \hspace{1cm} (3.22)

and consider the meromorphic function

\[ J(z) = i[W(z + i\beta/2) - W(z - i\beta/2)] + U'(z). \]  \hspace{1cm} (3.23)

The function \( J(z) \) is completely determined by the following four conditions:

(1)– it is periodic with period \( 2i\pi \): \( J(z + i\pi) = J(z - i\pi) \);
(2)– it has two cuts \( \{ \text{Re} z \in [b, a], \text{Im} z = \pm i\beta/2 \} \) in the strip \( |\text{Im} z| \leq \pi \), and satisfies along the cuts a two-term functional equation

\[ J(\phi + i\beta/2 \pm i0) = J(\phi - i\beta/2 \mp i0), \quad \phi \in [b, a]; \]  \hspace{1cm} (3.24)

9
its series expansion at $z \to +\infty$ is

\[
J(z) = \frac{t_2}{\sin \beta} e^{2z} + \frac{t_1}{2 \sin \beta/2} e^z + 2 \sin(\beta/2) W_1 e^{-z} + 2 \sin \beta W_2 e^{-2z} + \ldots
\]

where $W_n$ are the moments of the spectral density defined by (3.17).

(3)– it satisfies the normalization condition (3.20): if $C_1$ is a contour surrounding the upper cut, then

\[
\oint_{C_1} \frac{dz}{2\pi} J(z) = 1.
\]

(3.25)

The first two conditions mean that the function $J(z)$ is a single valued analytic function on the torus obtained from the strip $|\text{Im} z| \leq \pi$ by cutting it along the intervals $\{\text{Re} z \in [b, a], \text{Im} z = \pm i\beta/2\}$, then identifying its two edges ($\text{Im} z = \pm \pi$) as well as the upper edge of the first cut with the lower edge of the other one. The two main cycles of this torus are homotopic to the contour $C_1$ surrounding the upper cut, and the contour $C_2$ connecting the two cuts.

The variable $J$ is periodic along both cycles while the variable $z$ is periodic along the cycle $C_1$ and has discontinuity $i\beta$ along the cycle $C_2$. Therefore, if we find a canonical parametrization

\[
z = z(u), \quad J = \zeta(u)
\]

for which the contours $C_1$ and $C_2$ are the images of the periods $\omega_1$ and $\omega_2$ of a flat rectangle, then the functions $z(u)$ and $\zeta(u)$ will satisfy the following periodicity conditions

\[
z(u + \omega_1) = z(u), \quad z(u + \omega_2) = z + i\beta,
\]

\[
\zeta(u + \omega_1) = \zeta(u), \quad \zeta(u + \omega_2) = \zeta(u).
\]

(3.28)

Therefore, once the elliptic parametrization is found, the problem is essentially resolved. The parameters of the solution are fixed by the normalization condition for the spectral density and the behavior of the function $J(z)$ at $z \to \infty$, which is determined by the potential $U(z)$.

Let $u_\infty$ be the point in the periodic rectangle corresponding to $z \to \infty$. The map is assumed to be regular, which means that $e^{-z(u)}$ can be Taylor expanded at $u = u_\infty$. Therefore $e^{z(u)} \sim (u - u_\infty)^{-1}$ and the third condition (3.25) says that the function $\zeta(u)$ has a second-order pole at $u = u_\infty$ and fixes the singular part of its Laurent expansion in $u - u_\infty$.

3 Our method is a natural generalization of the method applied by J. Hoppe, V. Kazakov, the author and N. Nekrasov to resolve technically similar problems [19,20], and which was originally proposed by J. Hoppe [21].
3.4. Explicit solution in the elliptic parametrization

We will identify the two periods of the torus with the standard quarter-periods of the
Jacobi elliptic functions $\omega_1 = 2K$ and $\omega_2 = 2iK'$. The $z$-cylinder with the two cuts is
parametrized by the rectangle

$$|\text{Re}u| \leq K, |\text{Im}u| \leq K'.$$  \hspace{1cm} (3.29)$$

The function $\zeta(z)$ is periodic with periods $2K$ and $2iK'$ and its only singularity in
the rectangle (3.29) is a double pole at $u = u_\infty$. The residue of this pole is zero, since
otherwise there would be another pole somewhere in the rectangle. The generic function
with such properties (the Weierstrass elliptic function) depends on two constants $A$ and $B$

$$\zeta(u) = A + B \frac{1}{\text{sn}^2(u - u_\infty)}.$$  \hspace{1cm} (3.30)$$

The function $z(u) - \frac{a+b}{2}$ is antisymmetric in $u$, which means that $e^{z(u)}$ has one simple pole
at $u = u_\infty$ and one simple zero at $u = -u_\infty$. It is therefore natural to look for it as a ratio
of Jacobi theta functions. One can easily check that the function

$$e^{z(u)} = e^{\frac{a+b}{2}} \frac{H(u_\infty + u)}{H(u_\infty - u)}$$  \hspace{1cm} (3.31)$$
satisfies the periodicity condition (3.28) if the parameter $u_\infty$ is related to the period $2K$
by

$$u_\infty = \frac{2\pi - \beta}{2\pi} K = \frac{1 + \lambda}{2} K.$$  \hspace{1cm} (3.32)$$

The function (3.31) gives the desired regular map between the strip with two cuts in the
$z$ plane and the rectangle (3.29), as is shown in Fig. 4.

The two horizontal edges of the $u$-rectangle are mapped to the two cuts in the $z$-strip.
Its two vertical edges are mapped to the line made of the segments $[i\frac{\beta}{2}, i\pi]$ and $[-i\pi, -i\frac{\beta}{2}]$.
The function $z(u)$ along the upper cut is given by

$$z(\pm iK' + y) = \frac{a + b}{2} \pm i\frac{\beta}{2} + \ln \frac{\Theta(u_\infty + y)}{\Theta(u_\infty - y)}.$$  \hspace{1cm} (3.33)$$

The endpoints of the cuts are the four branch points of the inverse map $u = u(z)$, deter-
mined by the condition $dz/du = 0$. The symmetry of the map (3.31) allows to determine
the four branch points through a single parameter $u_b \in [0, K]$:

$$a \pm i\beta/2 = z(u_b \pm iK'), \quad b \pm i\beta/2 = z(-u_b \pm iK').$$  \hspace{1cm} (3.34)$$
The parameter $u_b$ is determined from

$$Z(u_\infty + u_b) + Z(u_\infty - u_b) = 0,$$

(3.35)

where $Z(u) = \frac{\Theta'(u)}{\Theta(u)} = E(u) - K u$ is the Jacobi Zeta function [22]. In particular, if $\beta = \pi$, then $u_\infty = u_b$. Finally, the length of the cut is given, according to (3.34), by

$$a - b = \ln \frac{\Theta(u_\infty + u_b)}{\Theta(u_\infty - u_b)}.$$  

(3.36)

Note the following symmetry, of the solution, which reflects the symmetry $\beta \to 2\pi - \beta$ (or $\lambda \to -\lambda$) of the original matrix integral

$$\beta \to 2\pi - \beta,$$

$$z(u) \to -z(K - u),$$

$$\zeta(u) \to -\zeta(K - u).$$

Fig.4: The domains of the variables $z$ and $u$.

Now we have to fix, using the asymptotics (3.25) and the normalization condition (3.26), the four parameters of the solution, namely
the coordinate of the middle point of the cut \( \frac{a+b}{2} \),
the coefficients \( A \) and \( B \),
the elliptic nome \( k^2 \), which determines the two periods \( K \) and \( K' \).

The asymptotics at infinity gives three independent conditions, which together with
the normalization determine completely the four parameters. A relatively simple calculation (see Appendix B) gives
\[
e^{\frac{a+b}{2}} = \frac{\cot \frac{\beta}{2}}{2} \frac{H'(0)}{H'(2u_\infty)},
\]
\[
B = \frac{\pi \lambda T}{T^*} \frac{H^2(2u_\infty)}{H'^2(2u_\infty)},
\]
\[
T^* \lambda T = \frac{(K - E)[Z(2u_\infty) + \text{cn}(2u_\infty)] - K \frac{Z(2u_\infty)}{\text{sn}(2u_\infty)}}{[Z(2u_\infty) + \frac{\text{cn}(2u_\infty)}{\text{sn}(2u_\infty})]^2}.
\]

Here we introduced the critical value of \( T \)
\[
T^* = 16\pi \lambda \frac{\sin^3 \frac{\beta}{2}}{\cos \frac{\beta}{2}}
\]
for which \( K'/K \to 0 \). The latter coincides with the critical coupling \( b^* = 1/T^* \) obtained
by P. Zinn-Justin [14].

4. The scaling limit

4.1. Explicit form of the solution in the scaling limit

We would like to explore the scaling behavior near the critical point \( T \to T^* \), where
\( e^b \to 0 \), which in our parametrization corresponds to the limit \( K \to \infty, K' \to \pi/2 \). In this
limit the solution can be expanded in the dual modular parameter
\[
q = e^{-\pi K/K'}.
\]
The expansion takes simpler form if we rescale slightly the variable \( u \) to
\[
v = \frac{\pi}{2K'} u
\]
which belongs to the domain \(|\text{Im}v| \leq \frac{\pi}{2}, |\text{Re}v| \leq \frac{1}{2} \ln \frac{1}{q^*} \).
Our solution (3.30)–(3.31) expands as a series in \( q \) as follows:

\[
\zeta(u) = A - B \frac{E'}{K'} - B \frac{\pi^2}{4K'^2} \left[ \frac{1}{\sinh^2(v - v_\infty)} + \sum_{n=1}^{\infty} 8n \frac{q^{2n} \cosh 2n(v - v_\infty)}{1 - q^{2n}} \right]
\] (4.3)

\[
z(v) = \ln \left( \frac{a + b}{2} - (1 + \lambda)v + \ln \frac{\sinh(v_\infty + v)}{\sinh(v_\infty - v)} + 4 \sum_{n=1}^{\infty} \frac{q^{2n} \sinh 2nv_\infty \sinh 2nv}{n} \right),
\] (4.4)

where

\[
v_\infty = \frac{1 + \lambda}{2} \ln \frac{1}{q}.
\]

In order to find the \( q \to 0 \) asymptotics of (4.3) and (4.4), we need the explicit expressions for the parameters \( T, a, b, B \) as functions of \( q \). In the following we will use the symbol “\( \approx \)” for “equal up to subleading terms in \( q \)”. From (3.36), (3.37), (3.38) and (3.39) we get

\[
\Lambda \equiv \frac{T - T^*}{T^*} \approx \frac{1 - \lambda}{\lambda} q^{1-\lambda} \ln \frac{1}{q}
\]

\[
e^{\frac{a+b}{2}} \approx \frac{\tan \frac{\pi \lambda}{4}}{q^{1-\lambda^2}}
\]

\[
e^{\frac{a-b}{2}} \approx \left(1 + \frac{1+\lambda}{2} (1 - \lambda) \frac{1-\lambda}{2} \right)^{\frac{1-\lambda}{4}} q^{\frac{1-\lambda^2}{4}}
\]

\[
B \approx \frac{T}{T^*} \frac{\pi \lambda}{(\lambda + q^{1-\lambda})^2}
\]

(4.5)

We see that indeed the position of the left cut tends to \(-\infty\)

\[
e^b \sim q^{\frac{1-\lambda^2}{2}} \to 0,
\] (4.6)

while the right cut has a finite limit at \( q \to 0 \)

\[
e^a \approx \frac{\tan \frac{\pi \lambda}{2}}{8 \lambda} \left(1 + \lambda\right)^{\frac{1+\lambda}{2}} \left(1 - \lambda\right)^{\frac{1-\lambda}{2}}.
\] (4.7)

Now we are ready to determine the scaling behavior of the solution (4.3)–(4.4). The interesting scaling behavior is associated with the vicinity of the left pair of branch points of \( J(z) \), that is at

\[
b \pm i \frac{\pi}{2} = z \left( -v_b \pm i \frac{\pi}{2} \right).
\]

It is therefore convenient to introduce a new parameter \( \tau \)

\[
\tau = v + v_b,
\] (4.8)
such that the two left branch points occur at $\tau = \pm \frac{\pi}{2}$. From (3.35) we get

$$v_b \approx v_\infty - \frac{1}{2} \ln \frac{1 + \lambda}{1 - \lambda}. \quad (4.9)$$

and the point $+\infty$ of the $z$-plane corresponds to

$$\tau_\infty \approx 2v_\infty - \frac{1}{2} \ln \frac{1 + \lambda}{1 - \lambda}.$$  

Assuming that $|\tau| \ll \tau_\infty$ and neglecting the subleading terms we get

$$\zeta(\tau) \approx \frac{4\pi}{\lambda} \left( q^{1+\lambda} \frac{1 - \lambda}{1 + \lambda} e^{-2\tau} + q^{1-\lambda} \frac{1 + \lambda}{1 - \lambda} e^{2\tau} \right) \quad (4.10)$$

$$e^{z(\tau)} \approx C \left( (1 + \lambda) e^{(1-\lambda)\tau} - (1 - \lambda) e^{-(1+\lambda)\tau} \right) \quad (4.11)$$

where

$$C \approx \frac{\tan(\pi\lambda/2)}{4\lambda} \frac{q^{1-\lambda^2}}{(1 + \lambda) \frac{1}{1+\lambda} (1 - \lambda) \frac{1}{1-\lambda}}. \quad (4.12)$$

4.2. The scaling behavior of the spectral density. Critical exponents associated with the entropy of the loops

Let us first check that the “string susceptibility” exponent $\gamma_{\text{str}}$ is zero for all $\lambda$. Consider the first momentum of the spectral density, whose singular behavior is that of a sphere with one puncture

$$W_1 \sim \frac{\partial \ln Z}{\partial \Lambda} \sim \text{constant} + \text{(another constant)} \Lambda^{1-\gamma_{\text{str}}}.$$  

The singular behavior is that of the coefficient $B$, which is given by the third equation (4.5): $B - B^* \sim q^{1-\lambda}$. On the other hand, by the first equation (4.5)

$$\frac{q^{1-\lambda}}{\lambda} \approx \frac{\Lambda}{\ln \frac{1}{\Lambda}} \quad (4.13)$$

which implies that indeed $\gamma_{\text{str}} = 0$ and the free energy has logarithmic singularity

$$\ln Z \sim \frac{\Lambda^2}{\ln \frac{1}{\Lambda}}.$$  

Then we look for the scaling behavior of the spectral density and the resolvent. The spectral density is related to the function $\zeta(v)$ by

$$\rho(z) = \frac{\zeta(v + i\pi/2) + \zeta(u - i\pi/2)}{2\pi} \quad (4.14)$$
and the resolvent (3.13) is obtained by inverting the relation (3.23)

\[ W(i\pi + \tau) = W(i\pi - \tau) = \frac{\tau}{\pi} \left[ \zeta \left( \tau + \frac{i\pi}{2} \right) - \zeta \left( -\tau - \frac{i\pi}{2} \right) \right]. \]  

(4.15)

It is easy to see that the function \( W(z) \) has only one cut on the physical sheet (the strip \(|\text{Im} z| \leq \pi\)), which goes along the interval \( z \geq b \). On the other sheets there is an infinite number of cuts (at least for irrational \( \lambda \)), placed at \( \text{Im} z = i\pi n \pm \lambda, n \in \mathbb{Z} \).

At distances \( e^z \gg e^b \), the relation between \( z \) and \( \zeta \) has two branches

\[ \zeta_{\pm}(z) \sim q^{\frac{1+\lambda}{1+\lambda}} \exp \left( \frac{2z}{1+\lambda} \right) \]  

(4.16)

associated with the asymptotics at \( e^\tau \gg 1 \). It is the smaller exponent which is the relevant one, but in the limit \( \lambda \to 0 \) the two exponents coalesce and produce the observed logarithmic behavior of the spectral density the \( O(2) \) model. Correspondingly, the \( \tau \)-factor in the resolvent (3.13) becomes \( \tau^2 \) in this limit (see eq. (3.41) of [23]). The limit \( \lambda \to 0 \) of the solution is worked out in Appendix C.

Now let us find the scaling of an important observable: the typical length \( L \) of a loop on the world sheet. It is characterized by the critical exponent \( \nu \) [24]

\[ L \sim A^{\frac{1}{2\nu}}. \]  

(4.17)

A related quantity is the average number \( \mathcal{N} \) of loops on a worldsheet of area \( A \). Since \( A \) is the total length of all loops,

\[ \mathcal{N} = \frac{A}{L} \sim A^{\frac{2\nu-1}{2\nu}}. \]  

(4.18)

To follow the analysis of [24] and [23], it is convenient to return to the original variable \( a = e^\phi \) and consider the density

\[ \rho(a) = e^{-\phi} \rho(\phi) \sim a^{\frac{1-\lambda}{1+\lambda}} \]  

(4.19)

and the resolvent \( W(w) = \langle \text{Tr} \frac{1}{w-A} \rangle \) as a function of the complex variable \( w = -e^z \)

\[ W(w) = -e^{-z} W(z) \sim w^{\frac{1-\lambda}{1+\lambda}} \ln w. \]  

(4.20)

The resolvent \( W(w) \) has a cut along the interval \( -\infty < w < -M \), where

\[ M = e^b \sim q^{\frac{1-\lambda^2}{2}} \sim \left( \frac{\Lambda}{\ln \Lambda} \right)^{\frac{1+\lambda}{2}} \]  

(4.21)
gives the scale for the length of the boundary. Since $A \sim \Lambda$ and $L \sim 1/M$ (for detailed arguments see [23]), we conclude that

$$\nu = \frac{1}{1 + \lambda}. \quad (4.22)$$

and the number of loops grows with the area of the world sheet as

$$\mathcal{N} \sim A^{\frac{1-\lambda}{2}}. \quad (4.23)$$

Let us also mention that the exponent $\nu$ determines also the fractal dimension of the boundary of the disk amplitude, which is equal to $\frac{1}{2\nu}$.

Finally, let us calculate the dimensions of the $m$-vortex operators representing sources of $m$ equally oriented lines. The correlation function of two such operators is obtained by calculating the "watermelon" configuration made of $m$ nonintersecting lines relating the two points, as the one shown in Fig.5. The singular behavior of such an amplitude is related to the scaling dimension $\delta_m^{\text{vertex}}$ of the $m$-vortex operator as

$$\chi_m \sim \Lambda^{2\delta_m^{\text{vertex}} - \gamma_{\text{str}}}. \quad (4.24)$$

Fig.5: A line configuration for the correlator of two 3-vortices.

Proceeding in the same way as in Sect.4 of ref. [24], one can extract the scaling dimension of the $m$-vortex operator from the scaling behavior of the spectral density $\rho(a)$ (eq. (4.19)). The amplitude corresponding to a configuration with $m$ open lines is given (up to the phase factors $e^{i\lambda\Gamma}$ associated with the $m$ lines, which can be neglected) by the convolutive integral in eq. (4.28) of ref. [24], and has the following singular behavior

$$\chi_m \sim M^{\frac{1+\lambda}{1-\lambda}m} \sim \Lambda^{\frac{1+\lambda}{1-\lambda}m}. \quad (4.25)$$

Compared it with (4.24), we find the same dimensions (2.11)

$$\delta_m^{\text{vertex}} = m \frac{1-\lambda}{4}, \quad (4.26)$$

which confirms the interpretation of the $m$-vortex operator as a discontinuity $m\beta$ on the world sheet.
5. Summary and discussion

In this paper we gave the exact solution of the 6-vertex model on a dynamical random lattice with spherical topology. The model describes a gas of densely packed loops whose weights are coupled to the local curvature of the lattice. We have calculated the partition function of the disk with given length $L$ and area $A$. For any value of the vertex coupling $\lambda$, the critical singularity of the free energy corresponds to a theory with central charge $c = 1$, but the fractal dimension of the boundary of the disk changes continuously from $1/2$ to $1$. The limiting value is achieved at $\lambda = 1$, where the boundary has the same dimension as the bulk. This reminds what happens in the dense phase of the $O(n)$ model, where at $n \to 0$ the world sheet is all eaten by the boundary. Indeed, the exponent $\frac{1-\lambda}{2}$ for the growth of the number of loops vanishes when $\lambda \to 1$, which means that in this limit the effective fugacity of the loops renormalizes to zero due to the fluctuations of the curvature. We also calculated the scaling dimensions of the $m$-vortex operators, representing sources of $m$ equally oriented open lines on the world sheet.

All our results agree with the interpretation of the 6-vertex model as a special realization of the compactified $c = 1$ string theory. From the string theory point of view the model is interesting with the possibility to treat very explicitly the vortex operators, which are difficult to construct in the standard realizations of the $c = 1$ string.

The 6-vertex matrix model describes the $c = 1$ string theory compactified at length $\beta < \pi$, which is less than $1/4$ of the Kosterlitz-Thouless length $\beta = 4\pi$, associated with the most relevant vortex operator ($m = 1$). In order to describe a theory compactified at length up to $p\pi$, it is sufficient to consider a chain of $p$ coupled matrices. The diagonalization can be performed in a similar way and one obtains a chain of integral equations of the type (3.11), which seems to be exactly solvable as well.

The loop gas formulation of the $c = 1$ string theory places a bridge between the description with continuous target space via matrix quantum mechanics [25] and the matrix models [26] for the strings with discrete target space (the ADE models on a random lattice) [27], which describe all minimal conformal theories coupled to $2d$ gravity. In particular, it is clear why the string diagram technique based on a world sheet surgery [23] works only for strings with discrete target space. A discrete target space without loops, i.e., a Dynkin graph of ADE type, can (and should) be immersed in the continuum by placing its adjacent points at distance $\beta = \pi$, for which the curvature defects do not produce phase factors for the domain walls. In the case of the $c = 1$ string compactified at length $\beta$, this diagram technique works only at the point $\beta = 1$ (the $O(2)$ model).
The nonperturbative equivalence with the \( c = 1 \) string theory means that the 6\( v \) matrix model is exactly solvable in the so called double scaling limit \[28\]. The integrability of this model follows also directly from the fact that the partition function (3.8) is a \( \tau \)-function of the KP hierarchy. Using the scaling properties of the partition function integral, one can convert the KP equation into an ordinary differential equation, as it has been done in ref. \[19\].

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Appendix A. Equivalence to the \( c = 1 \) compactified string

Let us integrate with respect to the angular variables in the original matrix integral (3.1). We decompose

\[
XX\dagger = UXU^{-1}
\]

with \( U \in SU(N) \) and \( x = \text{diag}(x_1, \ldots, x_N) \), and perform the \( U(N) \) integration with the help of the Harish Chandra-Itzykson-Zuber formula. Using the Cauchy identity, the integrand can be written as a determinant, and the partition function takes the form

\[
Z_N = \int_0^\infty dx_1 \ldots dx_N \det_{ij} K(x_i, x_j) \tag{A.1}
\]

with

\[
K(x, x') = \exp \left( -\frac{1}{2} N \sqrt{T} (x + x') + Nx x' - \frac{1}{2} N (x^2 + x'^2) \cos \beta \right) \tag{A.2}
\]

We get rid of the linear term by a linear change in the integration variable

\[
y = \sqrt{N} \sin \beta x - y_0, \quad y_0 = \frac{\sqrt{NT} \sin \beta}{\sin^2 \frac{\beta}{2}} = 2\sqrt{2NT/T^*}, \tag{A.3}
\]

after which the kernel becomes identical (up to normalization) to that of the inverse oscillator

\[
K(y, y') = \frac{1}{\sqrt{2\pi \sin \beta}} e^{-\frac{2yy' - \cos \beta (y^2 + y'^2)}{2\sin \beta}} = \langle y | e^{\beta (\frac{1}{2} \partial_y^2 + \frac{1}{2} y^2)} | y' \rangle, \tag{A.4}
\]
and the integration is to be done in the semi-infinite interval

\[-y_0 < y < \infty.\]

Hence the partition function (A.1) describes a system of \( N \) nonrelativistic fermions in inverse oscillator potential, stabilized by a wall at distance \( y_0 \) from the origin\(^4\), at temperature \( \beta \) (see Fig.6). The wall should be sufficiently far, so that the level of the Fermi sea is below the top of the potential. The critical distance is

\[ y_0^* = 2\sqrt{2(\pi - \beta)N}, \quad (A.5) \]

which corresponds to \( T = T^* \). Then the level of the Fermi sea reaches the top of the inverse gaussian potential. The string cosmological constant \( \mu \), which measures the deviation from the critical point, is proportional to \((y_0^* - y_0)^2\).

![Fig.6 : Fermi sea for the inverse gaussian potential](image)

The system of \( N \) fermions on a circle of length \( \beta \) describes the singlet sector of the Matrix Quantum Mechanics (MQM)\(^{29}\) and gives a non-perturbative realization of the \( c = 1 \) string theory compactified at length \( \beta \). The partition function of this system, calculated in ref.\(^{30}\), is invariant under the duality transformation

\[ \frac{\beta}{2\pi} \rightarrow \frac{2\pi}{\beta}. \]

The equivalence with the compactified at euclidean time interval \( \beta \) is consistent only in the interval \( 0 < \beta < \beta_{KT} \), where \( \beta_{KT} = \pi \) is the Kosterlitz-Thouless point. Near this point the the cutoff wall approaches the top of the inverse gaussian potential and there is no (metastable) ground state in the large \( N \) limit.

\(^4\) The restriction \( y > -y_0 \) is in fact imposed only on the trace, but it leads to a rapid exponential decay of the wave function of the fermions when \( y < -y_0 \).
Appendix B. Fixing the parameters of the solution

B.1. The three conditions following from the asymptotics at $z \to \infty$

Using the explicit form (3.31) of $z(u)$, we rewrite (3.25) as a series expansion in $u - u_\infty$:

$$
\zeta(z) = \frac{t_2 e^{a+b}}{\sin \beta} \left[ \frac{H(2u_\infty)}{H'(0)} \right]^2 \frac{1}{(u_\infty - u)^2} - \left[ \frac{t_1 e^{a+b}}{2 \sin \frac{\beta}{2}} \frac{H(2u_\infty)}{H'(0)} + 2 \frac{e^{a+b}t_2}{\sin \beta} \frac{H(2u_\infty)H'(2u_\infty)}{[H'(0)]^2} \right] \frac{1}{u_\infty - u} + C + \mathcal{O}(u_\infty - u),
$$

(B.1)

where

$$
t_1 = -\frac{T}{2 \sin \frac{\beta}{2}}, \quad t_2 = \frac{T}{2}
$$

(B.2)

and the constant $C$ is certain function of the elliptic modulus, whose explicit form will not be needed. This is to be compared with the first three terms in the Laurent expansion of (3.30)

$$
\zeta(u) = \frac{B}{(u_\infty - u)^2} + 0 + A + \frac{1 + k^2}{3} B + \mathcal{O}(u_\infty - u),
$$

(B.3)

which gives

$$
B = \cot(\beta/2) \frac{t_1^2}{8t_2} \frac{H^2(2u_\infty)}{H'^2(2u_\infty)}
$$

and

$$
e^{a+b} = \frac{\cot \frac{\beta}{2}}{2} \frac{H'(0)}{H'(2u_\infty)}
$$

(B.4)

and

$$
A = C - \frac{1 + k^2}{3} B.
$$

B.2. The normalization condition

The normalization (3.20) means that the integral of $J(z)$ along the contour $C_1$ surrounding its upper cut is equal to one. In terms of the variable $u$, this integral takes the form

$$
\frac{1}{2\pi} \int_{-K+iK'}^{K+iK'} du' \zeta(u') = 1.
$$

(B.5)
Writing (using the standard notation $Z(u) = \frac{\Theta'(u)}{\Theta(u)}$)
\[
z'(y + iK') = Z(u_\infty + y) + Z(u_\infty - y)
\]
\[
\zeta(y + iK') = A + Bk^2\text{sn}^2(u_\infty - y),
\]
we arrive at the following integral
\[
1 = \frac{1}{2\pi} \int_{-K}^{K} dy[Z(u_\infty + y) + Z(u_\infty - y)][A + Bk^2\text{sn}^2(u_\infty - y)].
\] (B.6)
With the help of the addition formula [31], 17.4.35, we write it in the form
\[
1 = \frac{Bk^2}{2\pi} [Z(2u_\infty)I_1 + \text{sn}(2u_\infty)I_2]
\]
where
\[
I_1 = \int_{-K}^{K} dv \text{sn}^2 v = \frac{2(K - E)}{k^2}
\]
\[
I_2 = k^2 \int_{-K}^{K} dv \text{sn}(2u_\infty - v) \text{sn}^3 v.
\]
The second integral can be decomposed as
\[
I_2 = \text{cndn}(2u_\infty) \left[ I_1 - \tilde{I}_2 \right],
\]
where
\[
\tilde{I}_2 = \int_{-K}^{K} dv \frac{\text{sn}^2 v}{1 - k^2\text{sn}^2(2u_\infty)\text{sn}^2 v}
\]
is a standard elliptic integral [22], 435.02:
\[
\tilde{I}_2 = \frac{2K Z(2u_\infty)}{k^2\text{sn}cndn(2u_\infty)}.
\]
Substituting this in (B.6) yields
\[
\frac{2\pi}{B} = 2(K - E)[Z(2u_\infty) + \text{cndn}(2u_\infty)] - 2K \frac{Z(2u_\infty)}{\text{sn}(2u_\infty)}. \tag{B.7}
\]

Appendix C. The special case $\beta = \pi$.

Here we check that the solution of the $O(2)$ matrix model with quadratic potential is obtained as the limit $\beta \to \pi$ (or $\lambda \to 0$) of our general solution (3.30)-(3.31). This limit is quite subtle and does not commute with the scaling limit $q \to 0$. In this limit we have
\[
T^* = 32, \quad \frac{T - T^*}{T} = \frac{k'^2K^2}{E^2}, \quad e^a = \frac{\pi}{8E}, \quad e^b = k'e^a. \tag{C.1}
\]
At $\lambda = 0$ we have more symmetry because the special points of the map coincide with the quarter of the period, $u_{\infty} = u_b = K/2$, and the function $e^{z(u)}$ becomes a Jacobi elliptic function

$$e^{z(u)} = -i e^a \text{dn} \left( u - \frac{K}{2} - iK' \right) = e^a \frac{\text{cn}}{\text{sn}} \left( u - \frac{K}{2} \right).$$  \hfill (C.2)

The function $\zeta(u)$ is analytic in $z$ at $\lambda = 0$

$$\zeta(u) = A + \frac{B}{\text{sn}^2(u - K/2)} = A + B e^{-(a+b)} e^{2z(u)} \hfill (C.3)$$

but the coefficient $B$ tends to infinity as $1/\lambda$

$$B = \frac{\pi T/T^*}{\lambda E^2}. \hfill (C.4)$$

Therefore, in order to take the limit $\lambda \to 0$, we should first subtract the diverging analytic piece of $\zeta$. The finite, nonanalytic piece is then given by

$$\zeta(u) = \left( B\lambda \frac{d}{d\lambda} \frac{1}{\text{sn}^2(u - K/2 - \lambda K/2)} \right)_{\lambda = 0} \hfill (C.5)$$

where

$$\frac{d}{d\lambda} \equiv \frac{\partial}{\partial \lambda} + \left( \frac{\partial u}{\partial \lambda} \right)_z \frac{\partial}{\partial u}. \hfill$$

Further we write

$$\left( \frac{d}{d\lambda} \frac{1}{\text{sn}^2(u - K/2 - \lambda K/2)} \right)_{\lambda = 0} = (K - 2(\partial_\lambda u)_z) \left[ \text{cn} \frac{\text{dn}}{\text{sn}^2} \right] \left( u - \frac{K}{2} \right).$$

$$K - 2(\partial_\lambda u)_z = 2K \left[ \frac{H'}{H} \text{scn} \right] \left( u + \frac{K}{2} \right)$$

Since $k' \left[ \frac{\text{sn}}{\text{cn}} \right] (u + K/2) = -e^{-a}$, we can write $\zeta(u)$ as

$$\zeta(u) = -\frac{KT}{\pi} e^{2z(u)} \frac{\partial}{\partial u} \ln H \left( u + \frac{K}{2} \right). \hfill (C.6)$$

The corresponding spectral density

$$\rho(z) = \frac{\zeta(u + iK') + \zeta(u - iK')}{2\pi}$$

coincides with the one obtained by solving directly the saddle point equations for the O(2) model [32].
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