POLISH ALGEBRAS, SHY FROM FREEDOM

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§0 Introduction

§1 Metric groups and metric models

[We give basic definitions and some relations.]

§2 Semi-metric groups: on automorphism groups of uncountable structures

[We prepare the ground to treating the automorphism group of a structure of cardinality strong limit of countable cofinality, e.g. $\beth_\omega$; this is the “semi”. We also consider replacing “the automorphism group of ...” by other derived structures.]

§3 Compactness of metric algebras

[The main lemma gives a sufficient condition for solvability of a set of equations of some form. We then deduce sufficient conditions for non-freeness.]

§4 Conclusion

[We show that Polish groups are not free, also even the semi-metric version suffice. We then derive the conclusion on automorphism groups.]

§5 Quite free but not free abelian groups

[We show that for every $n$ there is a very explicit definition of an abelian group (Borel and even $F_\sigma$) which is free if the continuum is at least $\aleph_{n+1}$. This applies to group and large family of varieties (families of algebra defined by a set of equations). We note that this works for subgroups of $\mathbb{Z}^\omega$.]

§6 Beginning of stability theory

[We prove some basic results.]
§ 0 Introduction

Our first motivation was the question: can a countable structure have an automorphism group, which a free uncountable group? This is answered negatively in [Sh 744].

This was a well known problem in group theory at least in England (David Evans in a meeting in Durham 1987) and we thank Simon Thomas for telling us about it. Independently in descriptive set theory, Howard and Kechris [BeKe96] ask if there is an uncountable free Polish group, i.e. which is on a complete separable metric space. A related result (before [Sh 744]) was gotten by Solecki [3] who proved that the group of automorphisms of a countable structure cannot be an uncountable free abelian group. Having the problem arise independently supported the feeling that it is a natural problem.

The idea of the proof in [Sh 744] was to prove that such a group has some strong algebraic completeness or compactness, more specifically for any sequence \( \langle d_n : n < \omega \rangle \) of elements of the group converging to the identity many countable sets of equations are solvable. This is parallel in some sense to Hensel lemma for the \( p \)-adics, and seem to me interesting in its own right.

Lecturing in a conference in Rutgers, February 2001, I was asked whether I am really speaking on Polish groups. We can prove this using a more restrictive condition on the set of equations. Parallel theorems, e.g. holds for semi groups and for metric algebras, e.g. with non isolated unit (\( e \) is a unit means \( \{e\} \) is a subalgebra). Here we do the general case.

More specifically we prove (see Conclusion 4.2, 4.1(1)).

**0.1 Theorem.** 1) There is no Polish group which as a group is free and uncountable.

2) Slightly more generally, assume

(a) \( G \) is a metric space

(b) \( G \) is a group with continuous \( xy, x^{-1} \)

(c) \( G \) is complete

(d) the density of \( G \) is \( < |G| \).

Then \( G \) is not free.

**0.2 Thesis:** If \( G \) is a Polish algebra satisfying one of the compactness conditions defined below, then it is in fact large in the sense of lots of sets of equations has a solution.
A reader interested just in this theorem can read just §3.

0.3 Question: What are the restrictions on Aut(\(A\)) for uncountable structures \(A\)?

We also prove that if \(A\) is a structure of cardinality \(\mu\), \(\mu\) is strong limit of cofinality \(\aleph_0\) (e.g. \(\beth_\omega\)) and the automorphism group of \(A\) is of cardinality \(>\mu\), then it is far from being free; this does not follow directly from 0.2(2) as the natural metric considered here does not satisfy all the conditions.

In [Sh 744] this is proved in the special case where \(A = \bigcup_n P^A_n\) satisfying \(|P^A_n| < \mu\).

\[
\ast \ast \ast
\]

Note: An arbitrary subgroup e.g. of the symmetric group of size \(\aleph_1\) can consistently be made into an automorphism group by Just, Shelah, Thomas [JShT 654]. So \(\beth_\omega\) is more interesting from this point of view.

Ideas regarding Aut(\(A\)), \(\|A\| = \beth_\omega\): Let the set of elements of \(A\) be \(\beth_\omega\). For \(f, g \in\) Aut(\(A\)) let

\[
d(f, g) = \sup\{2^{-n} : f \upharpoonright \beth_n \neq g \upharpoonright \beth_n \text{ or } f^{-1} \upharpoonright \beth_n \neq g^{-1} \upharpoonright \beth_n\}.
\]

Again a complete metric space and a Polish group. But we need to prove that there is a non-trivial convergent sequence.

This space is not necessarily of density \(\leq \beth_\omega\). A way out is to define

\[
d_1(f, g) = \sup\{2^{-n} : (\exists \alpha < \beth_n) \bigvee_m (f(\alpha) < \beth_m \neq \equiv g(\alpha) < \beth_m
\]

or \(f(\alpha) \neq g(\alpha)\) both \(< \beth_n\)

or similarly with \(g^{-1}, g^{-1}\)\}.

The purpose of introducing \(d_1\) is to decrease the density. Now “\(d_1(f, g) < 2^{-n}\)” is an e.g. relation with \(\leq 2^{2\beth_n}\) classes (rather than \((\beth_\omega)^{2\beth_n}\) classes). This is a complete metric space with density \(\leq \beth_\omega\). However, there are problems with “\((d(f_1, g_1), d(f_2, g_2) < 2^{-n} \Rightarrow d(f_1 \circ f_2, g_1 \circ g_2) < 2^{-n+1}\)”.

So we start the proof with \(d_1\), find a non trivial converging sequence \(\langle f_n : n < \omega \rangle\) in \(d_1\), prove it converges pointwise, replace \((\beth_n : u < \omega)\) by \(\langle A_n : n < \omega \rangle\) and get a sequence converging in \(d\).

0.4 Question: Is there a model theory of Polish spaces?
Naturally we would like to develop a parallel to classification theory (see [Sh:c]). A natural test problem is to generalize “Morley theorem = Los conjecture”. But we only have one model so what is the meaning? Well, we may change the universe. If we deal with abelian groups (or any variety) it is probably more natural to ask when is such (Borel) algebra free.

0.5 Example: If $\mathbb{P}$ is adding $(2^{\aleph_0})^+\text{-Cohen}$ subsets of $\omega$ then

$$(\mathbb{C})^\mathbb{V} \text{ and } (\mathbb{C})^\mathbb{V}[G]$$

are both algebraically closed fields of characteristic 0 which are not isomorphic (as they have different cardinalities).

So we restrict ourselves to forcing $\mathbb{P}_1 \leq \mathbb{P}_2$ such that

$$(2^{\aleph_0})^\mathbb{V}[\mathbb{P}_1] = (2^{\aleph_0})^\mathbb{V}[\mathbb{P}_1]$$

and compare the Polish models in $\mathbb{V}^{\mathbb{P}_1}, \mathbb{V}^{\mathbb{P}_2}$. We may restrict our forcing notions to c.c.c. or whatever...

0.6 Example: Under any such interpretation

(a) $\mathbb{C}$ = the field of complex numbers is categorical

(b) $\mathbb{R}$ = the field of the reals is not

(by adding $2^{\aleph_0}$ many Cohen reals).

(Why? Trivially: $\mathbb{R}^\mathbb{V}[\mathbb{P}_2]$ is complete in $\mathbb{V}[\mathbb{P}_2]$ while $\mathbb{R}^\mathbb{V}[\mathbb{P}_1]$ in $\mathbb{V}[\mathbb{P}_2]$ is not complete but there are less trivial reasons).

0.7 Conjecture: We have a dychotomy, i.e. either the model is similar to categorical theories, or there are “many complicated models”.

0.8 Thesis: Classification theory for such models resemble more the case of $\mathbb{L}_{\omega_1,\omega}$ than the first order.

See [Sh:h]; as support for this thesis we prove:

0.9 Theorem. There is an $F_{\sigma}$ abelian group (i.e. a $F_{\sigma}$-definition, in fact an explicit definition) such that $\mathbb{V} \models "G \text{ is a free abelian group}"$ iff $\mathbb{V} \models 2^{\aleph_0} < \aleph_{736}$.

Comments: In the context of the previous theorem we cannot do better than $F_{\sigma}$, but we may hope for some other example which is not a group or categoricity is not because of freeness.
0.10 Conclusion: Freeness (of an $F_\sigma$-abelian group) can stop at $\aleph_n$ (any $n$).

A connection with the model theories is that by Hart-Shelah [HaSh 323] such things can also occur in $L_{\omega_1, \omega}$ whereas (by [Sh 87a], [Sh 87b] Theorem) if $\bigwedge_n (2^{\aleph_n} < 2^{\aleph_{n+1}})$ and $\psi \in L_{\omega_1, \omega}$, categorical in every $\aleph_n$, then $\psi$ is categorical in every $\lambda$. See more in [ShVi 648].

The parallels here are still open.

This casts some light on the thesis that non-first order logics are “more distant” from the “so-called” mainstream mathematics.

Returning to stability theory per-se we have the modest:

0.11 Theorem. For “$\aleph_0$-stable Borel models” the theorem on the existence of indiscernibles can be generalized.

We may consider another version of the interpretation of “categoricity”. Of course, we can use more liberal than $L(A_2, r)$ or restrict the $A_i$’s further (as in the forcing version).

0.12 Definition. 1) We say that $A$ is categorical in $\lambda \leq 2^{\aleph_0}$ if: for some real $r$: for every $A_1, A_2 \subseteq \lambda$ the models $A^{L[A_1, r]}, A^{L[A_2, r]}$ are isomorphic (in $V$).

2) For a class $\mathcal{R}$ of forcing notions and cardinal $\lambda$ such that for at least one $P \in \mathcal{R}, \models “2^{\aleph_0} \geq \lambda”$, we have in $V^P$: the structure $A$ is categorical in $\lambda$ in the sense of part (1).

Comparing Definition 0.12(1) with the forcing version we lose when $V = L$, as it says nothing, we gain as (when $2^{\aleph_0} > \aleph_1$) we do not have to go outside the universe. Maybe best is categorical in $\lambda$ in $V^P$ for every c.c.c. forcing notion $P$ making $2^{\aleph_0} \geq \lambda$.

Note also that it may be advisable to restrict ourselves to the case $\lambda$ is regular as we certainly like to avoid the possibility $(2^{\aleph_0})^{L[A_1, r]} = \lambda < (2^{\aleph_0})^{L[A_2, r]}$ (see on this [Sh:g, VII]).

Of course, any reasonable definition of unstability implies non-categoricity: if we have many types we should have a perfect set of them, hence adding Cohen subsets of $\omega$ adds more types realized. If we add $\{\eta_i : i < 2^{\aleph_0}\}$ Cohen reals for every $A \subseteq 2^{\aleph_0}, \langle M^{[\eta_i : i \in A]} : A \subseteq 2^{\aleph_0} \rangle$ are non-isomorphic over the countable set of parameters, if we get $2^{2^{\aleph_0}}$ non-isomorphic models, we can forget the parameters and retain our “richness in models”.

Lately Blass asks on definable abelian subgroups of $\mathbb{Z}^\omega$, answers are derived for this from [Sh 402] and §5. We may be more humble than in 0.4.
0.13 **Question:** Is there model theory for equational theories, stressing free algebras?
The material in §1 - §5 (except some generalizations) was presented in a course in Rutgers, Sept. - Oct 2001 and I thank the audience for their comments. We shall continue elsewhere.

0.14 **Notation.** 1) Let $\omega$ denote the set of natural numbers, and let $x < \omega$ mean “$x$ is a natural number”.
2) Let $a, b, c, d$ denote members of $G$ (a group).
3) Let $\bar{d}$ denote a finite sequence $\langle d_n : n < n^* \rangle$, and similarly in other cases.
4) Let $k, \ell, m, n, i, j, r, s, t$ denote natural numbers.
5) Let $\varepsilon, \zeta, \xi$ denote reals $> 0$.

0.15 **Definition.** 1) A group word is a sequence $x_1^{r_1}x_2^{r_2}\ldots x_k^{r_k}$ where the $x_\ell$ are variables or elements of a group and $n_\ell \in \mathbb{Z}$ for $\ell = 1, \ldots, k$.
2) The word is reduced if $n_\ell \neq 0, x_\ell \neq x_{\ell+1}$.
3) The length of a word $w = x_1^{n_1}x_2^{n_2}\ldots x_k^{n_k}$ is $\sum_{\ell=1}^k |n_\ell|$.
4) A group term $w(x_1, \ldots, x_n)$ is a word of the form $x_1^{r_1}x_2^{r_2}\ldots x_k^{r_k}$ with $i_\ell \in \{1, \ldots, k\}, r_\ell \in \mathbb{Z}$ (actually $r_\ell \in \{1, \ldots, -1\}$ suffice). For a group $G$ and $a_1, \ldots, a_k \in G$, the meaning of $b = w(a_1, \ldots, a_k) \in G$ should be clear.
§1 METRIC GROUPS AND METRIC MODELS

We first define [semi]-metric [complete] group, and give a natural major example: automorphism groups. The natural example of semi-metric group is the semi-group of endomorphism of a countable structure.

1.1 Definition. \((G, \varnothing)\) is called a metric group if:

(a) \(G\) is a group
(b) \(G\) is a metric space for the metric \(\varnothing\)
(c) the functions \(xy, x^{-1}\) are continuous.

2) \((G, \varnothing)\) is a metric semi-group when

(a) \(G\) is a semi-group
(b) \(G\) is a metric space for the metric \(\varnothing\)
(c) the function \(xy\) is continuous (there is no \(x^2\) as \(G\) is just a semi-group).

3) Saying \((G, \varnothing)\) is complete, means complete as a metric space.

1.2 Notation: 1) For a metric group \(M\) the metric is denoted by \(d_M\) and the unit is denoted by \(e_M\) and the group by \(G_M\). When no confusion arises “\(G\) is a metric group” means \((G, d_G)\) is a metric group.

2) Similarly for semi-groups.

Now we define cases closer to automorphism groups, in those cases the proof is very similar to the one in [Sh 744].

1.3 Definition. 1) \(G\) is a specially metric group if:

(a) \(G\) is a metric group
(b) for every \(\zeta, \varepsilon \in \mathbb{R}^+\) there is \(\xi \in \mathbb{R}^+\) such that: if \(x_1, x_2, y_1, y_2 \in \{x : d_G(x, e_G) < \varepsilon\}\) then \(d_G(x_1, x_2) < \xi \land d(y_1, y_2) < \xi\) implies \(d_G(x_1 y_1, x_2 y_2) < \zeta \land d(x_1^{-1}, x_2^{-1}) < \zeta\); this is a kind of uniform continuity (inside the \(\varepsilon\)-neighborhood of \(e_G\); this is harder if we increase \(\varepsilon\) and/or decrease \(\zeta\))
(c) for arbitrarily small \(\zeta \in \mathbb{R}^+\) the set \(\{a \in G : d(a, e_G) < \zeta\}\) is a subgroup of \(G\).

2) We say \(\bar{\zeta} = \langle \zeta_n : n < \omega \rangle\) is strongly O.K. for \(G\) if:

(a) \(\zeta_n \in \mathbb{R}^+\)
(b) $\zeta_n$ satisfies clause (γ) of part (2), i.e. \( \{ a \in G : \delta(a, e_G) < \zeta_n \} \) is a subgroup of $G$

(c) $\zeta_{n+1} \leq \zeta_n$ and $0 = \inf \{ \zeta_n : n < \omega \}$

(d) if $x_1, x_2, y_1, y_2 \in \{ a \in G : \delta_G(a, e_G) < \zeta_0 \}$ and $\delta(x_1, x_2) < \zeta_{n+1}$, $\delta_G(y_1, y_2) < \zeta_{n+1}$ and $r(1), r(2) \in \{ 1, -1 \}$ then $\delta_G(x_1^{r(1)} y_1^{r(2)}, x_2^{r(1)} y_2^{r(2)}) < \zeta_n$.

3) We say $G$ is specially $^+\text{metric}$ group if in part (1) we have (α), (β) and (γ)$^+$ for every $\zeta \in \mathbb{R}^+$ the set $\{ a \in G : \delta(a, e_G) < \zeta \}$ is a subgroup of $G$.

4) We define similarly for semi groups omitting the operation $x^{-1}$ (this means omitting “$\delta(x_1^{-1}, x_2^{-1}) < \zeta$” in clause (β) of (1), and demanding $r(1), r(2) = 1$ in clause (d) of part (2).

1.4 Observation. 1) If $G$ is a special metric group then there is a sequence $\bar{\zeta}$ which is strongly O.K. for $G$.

2) We can in clause (d) of 1.3(2) above omit $r(1), r(2)$ and conclude only “$\delta(x_1 y_1, x_2 y_2) < \zeta_n$ and $\delta(x_1^{-1}, x_2^{-1}) < \zeta_n$”. This causes just slight changes in the computations of length in the proof or replacing $\bar{\zeta}$ by a suitable subsequence.

3) Every specially $^+$ metric group is a specially metric group.

4) Parts (1), (3) holds for semi groups, too.

Proof. Easy.

1.5 Definition. 1) Assume $\mathbb{A}$ is a countable structure with automorphism group $G = \text{Aut}(\mathbb{A})$ and for notational simplicity its set of elements is $\omega$, the set of natural numbers (and, of course, it is infinite, otherwise trivial).

We define a metric $\delta = \delta^\mathbb{A} = \delta^\text{aut}\mathbb{A}$ on $G$ by

$$\delta(f, g) = \sup \{ 2^{-n} : f(n) \neq g(n) \text{ or } f^{-1}(n) \neq g^{-1}(n) \}.$$  

Let $\text{Aut}_\mathbb{A} = (\text{Aut}(\mathbb{A}), \delta^\text{aut}\mathbb{A})$, but we may write $G^\mathbb{A}$ or $G^\text{aut}_\mathbb{A}$.

2) Assume $\mathbb{A}$ is a countable (infinite) structure, let $\text{End}(\mathbb{A})$ be the semi-group of endomorphisms of $\mathbb{A}$; assume for simplicity that its set of elements is $\omega$ and let $\delta^\text{end}\mathbb{A}$ be the following metric on $\text{End}(\mathbb{A})$

$$\delta(f, g) = \sup \{ 2^{-n} : f(n) \neq g(n) \}.$$  

Let $\text{End}_\mathbb{A}$ be $(\text{End}(\mathbb{A}), \delta^\text{end}\mathbb{A})$, we may write $G^\text{end}_\mathbb{A}$.

3) If in part (2) we restrict ourselves to monomorphisms, we write $\text{Mon}(\mathbb{A}), \text{Mon}_\mathbb{A}, \delta^\text{mon}_\mathbb{A}$.  

1.6 Claim. : For $\mathbb{A}$ as above:
1) $(\text{Aut}_{\mathbb{A}}, d^\text{aut}_{\mathbb{A}})$ is a complete separable specially $^+$ metric group.
2) $(\text{End}(\mathbb{A}), d^\text{end}_{\mathbb{A}})$ and $(\text{Mono}(\mathbb{A}), d^\text{mono}_{\mathbb{A}})$ are complete separable specially $^+$ metric semi-groups.

Proof. Easy.

We may think of a more general context.

1.7 Definition. 1) We say $a$ is a metric $\tau$-model ($\tau = \tau_a$ is a vocabulary, that is a set of function symbols and predicates; in the main case we say $\tau$-algebra when $\tau$ has no predicates only functions) if

(a) $a$ is a metric space with metric $\mathcal{d}_a$
(b) $M_a = M(a)$ is a model or an algebra with universe $|M_a|$, (of course with a set of elements the same as the set of points of the metric space), with vocabulary $\tau = \tau_a$
(c) if $F \in \tau_a$ is (an $n$-place) function symbol, then $F^{M(a)}$ is (an $n$-place) continuous function from $M_a$ to $M_a$ (for $\mathcal{d}_a$, i.e. by the topology which the metric $\mathcal{d}_G$ induces, of course)
(d) if $R \in \tau_a$ is an $n$-place predicate, then $R^a = R^{M(a)}$ is a closed subset of $M^n_a = n(|M_a|)$.

2) We say $a$ is unitary if some $e \in \tau_a$ is a unit of $a$ which means that $e$ an individual constant and $\{e_a\}$ is closed under $F^{M(a)}$ for $F \in \tau_G$ and $R \in \tau_G \Rightarrow \langle e, \ldots \rangle \in R^a$.
3) We say $a$ is complete if $(|M_a|, \mathcal{d}_a)$ is a complete metric space.
4) We replace “unitary” by “specially-unitary” above if we add:

(*) for every $\zeta \in \mathbb{R}^+$, the set $\{a \in M_a : \mathcal{d}_a(a, e_a) < \zeta\}$ is a subalgebra of $\mathbb{A}$.

5) We replace unitary by “specially-unitary” if some $\bar{\zeta}$ witness it which means:

(a) $\bar{\zeta}$ is a decreasing sequence of positive reals with limit zero
(b) for every $F \in \tau_a$ for some $n = n(F, a)$ we have for every $m \in [n, \omega)$ the set $\{a \in M_a : \mathcal{d}_a(a, e_a) < \zeta_m\}$ is closed under $F^{M(a)}$.

6) We add the adjective partial if we allow $F^{M(a)}$ to be a partial function, so in clause (c) of part (1) means now:

(c) if $F \in \tau_a = \tau(M_a)$ is an $n$-place function symbol then the set $\{(F^{M(a)}(a_1, \ldots, a_n) : a_1, \ldots, a_n \in M_a$ and $F^{M(a)}(a_1, \ldots, a_n)$ is well defined} is a closed subset of $n+1(M_a)$.
7) We say $a$ is specially-unitary if some $\zeta$ witnesses it which means (a),(b) from part (5) and

$$
\text{(d) for any } F \in \tau_a \text{ a } k\text{-place function for every } m \geq n(F,a) \text{ we also have:}
$$

if $d_a(x_\ell, y_\ell) < \zeta_{m+1}$ for $\ell = 1, \ldots, k$ and $d_a(x_\ell, e_a) < \zeta_{m+1}, d_a(y_\ell, e_a) < \zeta_{m+1}$ then $d_a(F^M(a)(x_1, \ldots, x_k), F^M(a)(y_1, \ldots, y_\ell)) < \zeta_m$.
§2 Semi metric groups: on automorphism groups of uncountable structures

It seems natural investigating the automorphism groups of a model $A$ say of cardinality, e.g. $\beth_\omega$, intending to put in a framework where we shall be able to prove it is not a free group of cardinality $> \beth_\omega$. Now choose $\bar{A} = \langle A_n : n < \omega \rangle$ such that, e.g. the universe of $A$ is $\beth_\omega$ and $A_n = \beth_n$. For such $\bar{A}$ there is a natural metric on $\text{Aut}(\bar{A})$ under which it is a complete metric group, but usually of too big density; there is another natural metric on $\text{Aut}(\bar{A})$ under which it is a complete metric space with density $\beth_\omega$ but multiplication is not continuous and Cauchi sequences may not converge to any point. To get the desired results we use “semi-metric” defined in 2.1, which combine the two metrics; in other words we weaken the completeness demand; hopefully this will have other applications as well, e.g. also for Borel groups. We also look at some generalizations: replacing $\text{Aut}(\bar{A})$ by other derived structures.

The reader may skip this section if not interested in the results concerning $\beth_\omega$. One way to present what we are doing is

2.1 Definition. 1) We say $G$ is an indirectly complete metric group if:

(a) $G$ is a group
(b) $G$ is a metric space under $d_G$
(c) if $\bar{c} = \langle c_n : n < \omega \rangle$ satisfies $d_G(c_n, c_{n+1}) < 1/2^n$, then letting $d_n = c_n^{-1}c_{2n+1}$ and $\bar{d} = \langle d_n : n < \omega \rangle$ we have

\[ (*) \] for some metric $d' = d'_{G, \bar{d}}$, under $d'$, $G$ is a metric group, and $\bar{d}$ converges to some $c$ under $d'$, see below.

1A) We say $G$ is an indirectly complete metric group and is defined similarly only in $(*)$ we replace “metric” by “complete metric”.
1B) Similarly for add another adjective or several of them.
2) Similarly for semi-groups and for algebras (and models, see Definition 1.7) with one change: in $(*)$ of clause (c) we let $d_n = c_n$.

2.2 Discussion: 1) In 2.1(1), $(*)$, clearly $c = e_G$. So $G$ in Definition 2.1(1) is not necessarily a metric group, i.e. product and inverse are not necessarily continuous. Also note that a metric group which is a complete metric space is not necessarily an indirectly metric group.
2) For groups $G$ in 2.1(1), clause (c), we can replace “$d(c_n, c_{n+1}) < 1/2^n$” by
\( \langle c_{2n}^{1}, c_{2n+1} : n < \omega \rangle \) converge to \( e_G \) with no significant difference in the results.

3) There are some other variants which can serve as well: we can just demand “\( \bar{c} \) is a Cauchy sequence” and/or add “\( \bar{c} \) converge to \( c' \)”, and/or in the conclusion say “some \( \omega \)-subsequence \( \bar{d}' \) of \( \bar{d} \) converge to some \( d' \).

4) Similarly in 2.1(2).

2.3 Definition. Assume \( A \) is a structure.

1) \( \bar{A} \) is an \( \omega \)-representation of \( A \) if \( \bar{A} = \langle A_n : n < \omega \rangle, A_n \subseteq A_{n+1} \) for \( n < \omega \) and \( \cup\{A_n : n < \omega \} \) is the universe of \( A \).

2) For every \( \omega \)-representation \( \bar{A} \) of \( A \) let \( \text{Aut}(\bar{A}) = \{ f \in \text{Aut}(\bar{A}) : \text{for every } n < \omega \text{ for some } m < \omega \text{ we have } (\forall x \in A_n)(f(x) \in A_m \& f^{-1}(x) \in A_m) \} \).

3) If \( \bar{A} \) is an \( \omega \)-representation of \( A \) and \( G = \text{Aut}(\bar{A}) \) then we define \( \delta = \delta_{\bar{A}, \bar{A}} = \delta_{\text{aut}}^{\bar{A}} \), a metric\(^1\) on \( G \) by

\[
\delta(f, g) = \sup\{2^{-n} : \text{there is } a \in A_n \text{ such that for some } (f', g') \in \{(f, g), (f^{-1}, g^{-1})\} \text{ one of the following possibilities holds}
\]

\( (a) \) for some \( m < \omega \) we have \( f'(a) \in A_m \Leftrightarrow g'(a) \notin A_m \),

\( (b) \) \( f'(a) \neq g'(a) \) are in \( A_n \).

4) If \( \bar{A} \) is an \( \omega \)-representation of \( A \) and \( G = \text{Aut}(\bar{A}) \) then we define \( \delta' = \delta'_{\bar{A}, \bar{A}} \) a metric on \( G \) by \( \delta'(f, g) = \sup\{2^{-n} : f \restriction A_n = g \restriction A_n \text{ and } f^{-1} \restriction A_n = g^{-1} \restriction A_n \} \).

2.4 Claim. Assume \( \bar{A} \) is an \( \omega \)-representation of an infinite structure \( \bar{A} \).

1) \( (\text{Aut}(\bar{A}), \delta_{\bar{A}, \bar{A}}) \) is an indirectly complete metric group with density \( \leq 2^{\aleph_0} + \Sigma\{2^{\|A_n\|} : n < \omega \} \); in fact it is a complete metric space (but in general not a metric group).

2) \( (\text{Aut}(\bar{A}), \delta'_{\bar{A}, \bar{A}}) \) is a complete metric group of density \( \leq \Sigma\{\|\bar{A}\|^{|A_n|} : n < \omega \} \); so if each \( A_n \) is finite the density is \( \leq \aleph_0 \).

3) If the universe of \( \bar{A} \) is \( \omega \) and \( A_n = n = \{0, \ldots, n-1\} \) then \( \bar{A} = \langle A_n : n < \omega \rangle \) is an \( \omega \)-representation of \( \bar{A} \) with each \( A_n \) finite and \( \delta'_{\bar{A}, \bar{A}} = \delta_{\text{aut}}^{\bar{A}} \) from Definition 1.5(1) and under it \( \text{Aut}(\bar{A}) \) is a complete separable specially metric group.

Proof. 1) Let \( \delta = \delta_{\bar{A}, \bar{A}} \). First we show that

\(^{1}\) this is proved in 2.4
Now we note that:

sequence $\langle A \rangle$

We may wonder whether $(\text{Aut}(A))$ is complete, i.e. whether every $\mathcal{D}$-Cauchi sequence $\langle f_n : n < \omega \rangle$ in $G$, $\mathcal{D}$-converge to some $f \in G$.

Before answering we prove a weaker substitute (in fact, it is the one we shall use for proving the indirect completeness below).

We say that $\langle f_n : n < \omega \rangle$ weakly converge to $f$ if (they are in $G$, or are just permutations of $A$ and) for every $\alpha \in A$ the sequence $\langle f_n(a) : n < \omega \rangle$ is eventually constant and moreover is eventually equal to $f(a)$.

Now we note that:

$(\star)_1 \mathcal{D}$ is a metric (even ultrametric).

[Why? Clearly, for $f, g \in \text{Aut}(A)$ we have $\mathcal{D}(f, g)$ is a non-negative real and $\mathcal{D}$ is symmetric (i.e. $\mathcal{D}(f, g) = \mathcal{D}(g, f)$) and $\mathcal{D}(f, g) = 0 \iff f = g$. Mainly we should prove that $\mathcal{D}(f_1, f_3) \leq \mathcal{D}(f_1, f_2) + \mathcal{D}(f_2, f_3)$. Now if $f_1 = f_2 \lor f_2 = f_3$ this is obvious, and also if $\mathcal{D}(f_1, f_2) = 1 \lor \mathcal{D}(f_1, f_2) = 1$ this is obvious (as $\mathcal{D}(f_1, f_3) \leq 1$) so assume $\mathcal{D}(f_\ell, f_{\ell+1}) = 2^{-n(\ell)}$ and $n(\ell) > 0$ for $\ell = 1, 2$. So if $n < n(1)$ and $n < n(2)$ and $m < \omega$ then we have, for every $a \in A_n$:

(i) $f_1(a) \in A_m \iff f_2(a) \in A_m$.

[Why? As $n < n(1)$ and $\mathcal{D}(f_1, f_2) = 2^{-n(1)}$.]

(ii) $f_2(a) \in A_m \iff f_3(a) \in A_m$.

[Why? As $n < n(2)$, $\mathcal{D}(f_2, f_3) = 2^{-n(2)}$.]

hence together

(iii) $f_1(a) \in A_m \iff f_3(a) \in A_m$.

Similarly for $f_1^{-1}, f_2^{-1}, f_3^{-1}$ so

(iv) $f_1^{-1}(a) \in A_m \iff f_3^{-1}(a) \in A_m$.

(v) if $f_1(a) \in A_n$ (equivalently $f_3(a) \in A_n$) then $f_1(a) = f_3(a)$.

[Why? First also $f_2(a) \in A_n$ by (i). Second $f_1(a) = f_2(a)$ (as $n < n(1)$ and $\mathcal{D}(f_1, f_2) = 2^{-n(1)}$) and third $f_2(a) = f_3(a)$ (as $n < n(2)$ and $\mathcal{D}(f_2, f_3) = 2^{-n(2)}$) hence together $f_1(a) = f_3(a)$.

(vi) Similarly $f_1^{-1}(a) \in A_n \iff f_3^{-1}(a) \in A_n \implies f_1^{-1}(a) = f_3^{-1}(a)$.

Together (and there is $n$ such that $n < n(1), n < n(2)$) this gives $\mathcal{D}(f_1, f_3) \leq 2^{\text{Min}\{n(1), n(2)\}} = \text{Max}\{\mathcal{D}(f_1, f_2), \mathcal{D}(f_2, f_3)\}$. So $(\star)_1$ holds.]

$(\star)_2 \mathcal{D}(f^{-1}, g^{-1}) = \mathcal{D}(f, g)$

[Why? Read the definition of $\mathcal{D}$ in 2.3(3).]

$(\star)_3 G$ as a metric space under $\mathcal{D}_G$ has density $\leq \Sigma\{2^{|A_\omega|} : n < \omega\}$

[Why? Just look at the definition using easy cardinal arithmetic.]

We may wonder whether $(\text{Aut}(A), \mathcal{D})$ is complete, i.e. whether every $\mathcal{D}$-Cauchi sequence $\langle f_n : n < \omega \rangle$ in $G$, $\mathcal{D}$-converge to some $f \in G$. 

Before answering we prove a weaker substitute (in fact, it is the one we shall use for proving the indirect completeness below).

We say that $\langle f_n : n < \omega \rangle$ weakly converge to $f$ if (they are in $G$, or are just permutations of $A$ and) for every $\alpha \in A$ the sequence $\langle f_n(a) : n < \omega \rangle$ is eventually constant and moreover is eventually equal to $f(a)$.
(*)₄ if \( \langle f_n : n < \omega \rangle \) is a \( \mathfrak{d} \)-Cauchi sequence, then it weakly converge, i.e. for every \( a \in A \), \( \langle f_n(a) : n < \omega \rangle \) is eventually constant and \( \langle f_n^{-1}(a) : n < \omega \rangle \) is eventually constant so the limit \( f \) is a well defined permutation of \( A \), moreover belongs to \( G = \text{Aut}(A) \).

[Why? Let \( x \in A \) so for some \( n(1) < \omega \) we have \( x \in A_{n(1)} \). As \( \langle f_n : n < \omega \rangle \) is a \( \mathfrak{d} \)-Cauchi sequence for some \( n(2) < \omega \) we have \( n \geq n(2) \Rightarrow \mathfrak{d}(f_n, f_{n(2)}) < 2^{-n(1)} \) hence \( n \geq n(2) \Rightarrow \wedge_m f_n(x) \in A_m \equiv f_{n(2)}(x) \in A_m \) by the definition of \( \mathfrak{d} \). Let \( m(1) \) be such that \( f_{n(2)}(x) \in A_{m(1)} \), and let \( n(3) \geq n(2) \) be such that \( n \geq n(3) \Rightarrow \mathfrak{d}(f_n, f_{n(3)}) < 2^{-m(1)} \) so necessarily, again by the definition of the metric, the sequence \( \langle f_n(x) : n = n(2), n(2) + 1, \ldots \rangle \) is constant; and call this value \( f(x) \). Similarly concerning \( f^{-1}(x) \).]

Note that the above argument shows that “if \( \bar{f} \) do \( \mathfrak{d} \)-converge to \( f \) then \( \bar{f} \) weakly converges to some \( f \)”, also it is interesting to note (though not explicitly used)

\[ (*)₅ \text{ assume } w(x_1, \ldots, x_n) \text{ is a group term and } f^k_n \in G \text{ for } k \in \{1, \ldots, n\}, k < \omega \text{ and } \langle f^k_n : k < \omega \rangle \text{ does } \mathfrak{d} \text{-converge to } g \ell \text{ (or just weakly converge to } g \ell \text{) for } k \in \{1, \ldots, k\} \text{ and we let } f_k = w(f^1_n, \ldots, f^n_n) \text{ for } k < \omega \text{ then } \langle f_k : k < \omega \rangle \text{ weakly converge to } g =: w(g_1, \ldots, g_n). \]

Very nice, but multiplication\(^2\) is not continuous in general for this metric, \( \mathfrak{d} \). We also have (though not actually used)

\[ (*)₆ \text{ (Aut}(A), \mathfrak{d}) \text{ is a complete metric space}. \]

[Why? Let \( \bar{f} = \langle f_n : n < \omega \rangle \) be a \( \mathfrak{d} \)-Cauchi sequence, by (*)₄ there is \( f \in \text{Aut}(A) \) to which \( \bar{f} \) weakly converges. Now let \( n(*) < \omega \) be given so for some \( n(1) < \omega \) we have: \( n \geq n(1) \Rightarrow \mathfrak{d}(f_n, f_{n(1)}) < 2^{-n(*)} \) and we shall prove \( n \geq n(1) = \mathfrak{d}(f, f_n) < 2^{-n(*)} \). So for each \( c \in A_{n(*)} \) we shall check clauses (a),(b) in the definition of \( \mathfrak{d} \) in Definition 2.3(3). First for every \( m < \omega \) we have \( n \geq n(1) \Rightarrow f_n(c) \in A_m \equiv f_{n(1)}(c) \in A_m \), but \( \langle f_n(c) : n < \omega \rangle \) is eventually constantly \( f(c) \) hence \( f(c) \in A_m \equiv f_{n(1)}(c) \in A_m \). This takes care of clause (a).

As for clause (b), similarly \( n \geq n(1) \wedge \{f_n(a), f_{n(1)}(a)\} \subseteq A_n \Rightarrow f_n(a) = f_{n(1)}(a) \) hence \( n \geq n(1) \wedge \{f(a), f_{n(1)}(a)\} \subseteq A_n \Rightarrow f(a) = f_{n(1)}(a) \). The same holds for \( \langle f_n^{-1} : n < \omega \rangle, f^{-1} \) so we are done.\]

\(^2\)E.g. let \( A \) be a trivial structure (i.e. with the empty vocabulary so \( G \) is the group of permutations of \( A \)) and \( b_n \neq c_n \in A_{n+1} \setminus A_n \) and \( a_n \in A_0 \) be pairwise distinct (for \( n < \omega \)). Let \( h_k \) exchange \( a_n, c_n \) if \( n < k \) and is the identity otherwise. Let \( f \) interchange \( a_n, b_n \) if \( n < \omega \) and is the identity otherwise. Let \( g_k \) interchange \( b_n, c_n \) if \( n < k \) and is the identity otherwise. Now \( \langle g_k : k < \omega \rangle \) is a Cauchy sequence and so has a limit \( g \), but \( h_k = f g_k f \) and \( \langle h_k : k < \omega \rangle \) is not a Cauchy sequence as \( \mathfrak{d}(h_{k_1}, h_{k_1}) = 1 \) if \( k_1 < k_2 \) as witnessed by \( a = a_{k_1} \).
But we have to prove that \((\text{Aut}(A), \vartheta)\) is an indirectly complete semi-metric group. The only clause left is (c) of Definition 2.1. So assume \(g_n \in G, \vartheta(g_n, g_{n+1}) < 1/2^n\), hence by \((*)_4 + (*)_6\) the sequence \(\langle g_n : n < \omega \rangle\) converge to some \(g \in G\) by the metric and also weakly converge to \(g\). Let \(f_n = g^{-1}_{2n}g_{2n+1}\), easily \(\langle f_n : n < \omega \rangle\) weakly converge to \(e_G = \text{id}_A\), let \(f = e_G\); so it suffices to find a metric \(\vartheta'\) such that \((G, \vartheta')\) is a complete specially metric group in which \(\langle f_n : n < \omega \rangle\) converge to \(f\).

We prove this assuming just

\[ \overline{\vartheta} = \langle f_n : n < \omega \rangle \] is an \(\omega\)-sequence of members of \(G\) which weakly converge.

Let \(B_n = B_f^n = \{ a \in A : a \in A_n \text{ and for every } m \in (n, \omega) \text{ we have } f_m(a) = f_n(a) \& f_m^{-1}(a) = f_n^{-1}(a) \}\). Clearly \(\overline{B} = \langle B_f^n : n < \omega \rangle\) is an increasing \(\omega\)-sequence of subsets of \(\mathbb{A}\) with union (the universe of) \(\mathbb{A}\). Recall that \(\vartheta' =: \vartheta'_{\mathbb{A}, \overline{B}}\) was defined by

\[ \vartheta'(f, g) = \inf \{2^{-n} : f \upharpoonright B_n \neq g \upharpoonright B_n \text{ or } f^{-1} \upharpoonright B_n \neq g^{-1} \upharpoonright B_n \}. \]

Now by parts (2),(3)

\[ (*)_6 \text{ the group } G \text{ with } \vartheta' \text{ is a complete metric group} \]

and obviously

\[ (*)_7 \vartheta(f_n, f_m) < \text{Max} \{2^{-n}, 2^{-m}\} \text{ and } \vartheta(f_n, f) \leq 2^{-n} \]

hence

\[ (*)_8 \langle f_n : n < \omega \rangle \text{ converge to } f \text{ by } \vartheta'. \]

So by \((*)_6 + (*)_8\) we are done.

(E.g. Why “specially” (in part (1) which we are proving)? As for each \(n < \omega, G_n = \{ f \in G : \vartheta'(f, e_G) < 2^{-n} \}\) is a subgroup of \(G\); the other requirements also are just like the proof of 1.6.)

2),3) Left to the reader. \(\square_{2.4}\)

2.5 Discussion We may consider cases like the endomorphism semi group of a structure and the endomorphism ring of an abelian group. The beautiful terms below are as in [Sh 61].

We may consider deriving from \(\mathbb{A}\) other structures (in addition to the automorphism group, endomorphism and monomorphism semi-group); see also 2.14.
2.6 Definition. Let $A$ be a structure.
1) A term $\sigma(x_1, \ldots, x_n)$ in the vocabulary of $M$ is called $A$-beautiful if for every function symbol $F$ of $\tau_A$, say $m$-place, the equations

$$
\sigma(F(x_1^1, x_1^2, \ldots, x_m^1), F(x_1^2, x_2^2, \ldots, x_m^2), \ldots, F(x_1^n, x_2^n, \ldots, x_m^n)) = F(\sigma(x_1^1, x_2^2, \ldots, x_n^n), \sigma(x_1^1, x_2^2, \ldots, x_n^n), \ldots, \sigma(x_1^1, x_2^2, \ldots, x_n^n))
$$

are satisfied by $A$.

3) If $\bar{A}$ is an $\omega$-representation of $A$ then we let $d_{\bar{A}}$, $\bar{A}$, $d_{\bar{A}}'$, $\bar{A}$, both two-place functions from $\text{End}(A)$ to $R^{\geq 0}$ be defined as follows:

$$
d_{\bar{A}}(f, g) = \sup\{2^{-n} : (\forall x \in A_n)(f(x) \in A_n \lor g(x) \in A_n \Rightarrow f(x) = g(x))\} \quad \text{and} \quad \sup\{2^{-n} : f \upharpoonright A_n = g \upharpoonright A_n\},
$$

$$
d_{\bar{A}}'(f, g) = \sup\{2^{-n} : f \upharpoonright A_n = g \upharpoonright A_n\}.
$$

4) $\text{End}_+^+(A) = \text{End}_+^+(A)$ is the structure with:

- universe $\text{End}(A)$,
- functions:
  - composition $(o)$, a two-place function
  - $e$, the identity of $A$, an individual constant
  - $F_{\bar{A}, \sigma}$, for every $A$-beautiful term $\sigma(x_1, \ldots, x_n)$, an $n$-place function, where
    
    $$
f = F_{A, \sigma}(f_1, \ldots, f_n)
    $$

    is defined by

    $$
f(x) = \sigma(f_1(x), \ldots, f_n(x)).
    $$

5) $\text{Aut}_+^+(A) = \text{Aut}_+^+(A)$ (or $\text{Mono}_+^+(A) = \text{Mono}_+^+(A)$) is defined similarly restricting ourselves to beautiful terms $\sigma(x_1, \ldots, x_n)$ which maps $\text{Aut}_A$ to $\text{Aut}_A$ (or $\text{Mono}_A$ to $\text{Mono}_A$).

2.7 Claim. For any structure $\mathfrak{A}$:
1) For any $\mathfrak{A}$-beautiful term $\sigma(x_1, \ldots, x_n)$ the function $F_{\mathfrak{A}, \sigma}^\text{end}$ is a full function from $\text{End}(A)$ into $\text{End}(A)$.
2) $\text{End}_+^+$ is a full structure (i.e. the functions are full not (strictly) partial).
2.8 Claim. Assume

(a) $A$ is a structure
(b) $\bar{A}$ is an $\omega$-representation of $A$
(c) $\mathfrak{d} = \mathfrak{d}^{\text{end}}_{A, \bar{A}}$ and $\mathfrak{d}' = \mathfrak{d}^{\text{end}}'_{A, \bar{A}}$.

1) $(\text{End}_{A}, \mathfrak{d})$ is an indirectly complete metric algebra (which is a semi group).
2) $(\text{Mono}_{A}, \mathfrak{d} |_{\text{Mono}_{A}})$ is an indirectly complete metric algebra which is a semi group.
3) $(\text{End}_{A}, \mathfrak{d}')$ is an indirectly complete metric algebra.
4) $(\text{Mono}_{A}, \mathfrak{d}' |_{\text{Mono}_{A}}, \bar{A})$ is an indirectly completed metric algebra.
5) In parts (3) + (4) the density is $\leq \sum_{n<\omega} |A_n|$ and in parts (1) + (2) the density is $\leq \sum_{n<\omega} 2^{|A_n|} + 2^{\aleph_0}$.
6) Assume that for every beautiful $F \in \tau_{A}$, for every $n$ large enough the set $A_n$ is closed under $F$. Then $(\text{End}_{A}^{\pm}, \mathfrak{d})$ is an indirectly complete metric algebra and $(\text{End}_{A}^{\pm}, \mathfrak{d}')$ is an indirectly complete metric algebra.

Proof. Like 2.4.

We may wonder:

2.9 Question: Can we have an uncountable Polish algebra $G$ (so $\tau_G$ is finite or just countable) which is free for some variety?

Of course, if $\tau_G$ is empty this holds; obviously we may discard many.

2.10 Example. The answer to 2.9 is yes.

Proof. Note that if $G$ is the vector space over a countable field $F$ with basis $\langle x_\eta : \eta \in \omega^2 \rangle$, it is a metric space with countable density, i.e. is Polish, as we can define $\mathfrak{d}(x, y) = \|x - y\|$ where $\|0\| = 0$ and for $x = \sum \{q_{\eta, x} x_\eta : \eta \in \omega^2\} \neq 0$ (so $\{\eta \in \omega^2 : q_{\eta, x} \neq 0\}$ is finite and, of course, $\eta \in \omega^2 \Rightarrow q_{\eta, x} \in F$) we let $\|x\|$ be $2^{-n}$ where $n$ is minimal such that for some $\nu \in n^2$ we have $0 \neq \Sigma q_{n(\eta)} : \nu < \eta \in \omega^2$ and $q_{n(\eta)} \neq 0$.

Clearly $G$ is a metric space; it has density $\aleph_0 + |F|$, and the addition and subtraction are continuous, and it is separable for $F$ countable. So, for countable $F$, the completion $\hat{G}$ is an (additive) Polish group and vector space. Moreover, if $F = \mathbb{Z}/p\mathbb{Z}$ for $p$ a prime, $\hat{G}$ is the free group of the appropriate variety. $\square_{2.9}$
2.11 Remark. Another metric on the same space is: for \( x \in G \) let \( n(x) = \min\{n: q_{\eta,x} \neq 0 \land q_{\nu,x} \neq 0 \land \eta \neq \nu \} \) and let \( \varrho(x) \) be:

**Case 1**: \( n(x) \neq n(y) \) or \( (\exists \eta, \nu \in \omega)[n(x) = n(y) \land q_{\eta,x} \neq 0 \land q_{\nu,y} \neq 0 \land q_{\eta,x} \neq q_{\eta,y}] \).

Then \( \varrho(x, y) = 2 \).

**Case 2**: Otherwise

\[
\varrho(x, y) = \sup\{2^{-n}: n \geq n(x) \text{ and } (\forall \eta, \nu \in \omega)[n(x) = n(y) \land q_{\eta,x} \neq 0 \land q_{\nu,y} \neq 0 \rightarrow \ell g(\eta \cap \nu) \geq n]\}.
\]

Now \( G \) under \( \varrho \) is a complete metric space, but it is not a metric group.

\[
\ast \quad \ast \quad \ast
\]

Closely related to semi metric (see Definition 2.1, but not enough for our theorems) is:

2.12 Definition. 1) We say \( a = (M, \varrho, U) \) is a metric-topological algebra if:

(a) \( M \) is an algebra
(b) \( \varrho \) is a metric (on the universe of \( M \))
(c) \( U \) is a Hausdorff topology (i.e. the family of open sets) on the universe of \( M \)
   such that
   (d) the operations of \( M \) are continuous by \( \varrho \) and by \( U \)
   (e) every open \( \varrho \)-ball i.e. set of the form \( \{a: \varrho(a, a_0) < \zeta\} \), is open also by the topology \( U \).

2) We say \( a = (M, \varrho, U) \) is complete if every \( \varrho \)-Cauchi sequence converge to some point of \( M \) by the topology \( U \) though not necessarily by \( \varrho \).

2.13 Claim. Assume

(a) \( \mathcal{A} \) is a structure
(b) \( \bar{A} \) is an \( \omega \)-representation of \( \mathcal{A} \)
(c) \( G = \text{Aut}(\mathcal{A}) \)
(d) $\mathfrak{d} = \mathfrak{d}_{\lambda, \lambda}$, see Definition 2.3

(e) $\mathbf{U}$ is the topology on $G$ such that a neighborhood basis for $f \in G$ is \{\(U_{f,X} : X \subseteq A \text{ finite}\) where \(u_{f,X} = \{g \in G : f \upharpoonright X = g \upharpoonright x \text{ and } f^{-1} \upharpoonright X = g^{-1} \upharpoonright X\}\).

Then $(G, \mathfrak{d}, \mathbf{U})$ is a complete metric-topological algebra.

Proof. Included in the proof of Claim 2.4. \qed_{2.13}

2.14 Discussion: 1) We can generally use topology instead of metric. What is the gain?

2) Instead of automorphisms we can consider a universal Horn Theory $T$ in a vocabulary $\tau^+ = \tau^+_T$ extending $\tau_\lambda$, e.g. $\tau^+ = \tau_\lambda \cup \{F^*\}$, $F^*$ a function symbol with arity $n^*$. So

$$\text{Exp}_T(A) =: \{A^+ : A^+ \text{ a } \tau^+\text{-expansion of } A \text{ and is a model of } T\},$$

if e.g. $\tau^+ \setminus \tau_\lambda = \{F\}$, we may replace $\text{Exp}_T(A)$ by $\{F^{\lambda^+} : A^+ \in \text{Exp}_T(A)\}$.

We may define

$\text{beautiful}(T, A) = \{\bar{\sigma}(\bar{x}) : \sigma \text{ a } \tau_\lambda\text{-term, } \bar{x} = (x_1, \ldots, x_k) \text{ and if } A^+_{\ell} \in \text{Exp}_T(A) \text{ for } \ell = 1, \ldots, k \text{ and } F \in \tau^+ \setminus \tau_\lambda \text{ we define the arity}(F)\text{-place function } F^*_\sigma \text{ from } A \text{ to } A \text{ by } F^*_\sigma(\bar{a}) = \sigma(F^{\lambda_1}(\bar{a}), \ldots, F^{\lambda_k}(\bar{a})) \text{ and let } A^+ = (A, F^*_\sigma)_{F \in \tau^+ \setminus \tau_\lambda} \text{ then } A^+ \text{ is a model of } T\}$,

we can consider more complicated operations. So $(\text{Exp}_T(A), H_{\vec{\sigma}(\vec{x})})_{\vec{\sigma}(\vec{x}) \in \text{beautiful}(A, T)}$ is a generalization of $\text{Aut}(A)$ where $H_{\vec{\sigma}(\vec{x})}(A^+_1, \ldots, A^+_k) = A^+$ is defined as above.
§3 Compactness of metric algebras

Note that below if $u_n = \{t_n\} = \{n\}$ we may write $x_n$ instead of $\bar{x}_n$, $n$ instead of $t \in u_n$ and $d_n$ instead of $\bar{d}_n$.

3.1 The completeness Lemma. Assume $a$ is a Polish algebra $M = M_a$ (so with countable vocabulary) such that

(a) $\langle u_n : n < \omega \rangle$ is a sequence of pairwise disjoint non-empty finite sets

(b) $\bar{x}_n = \langle x_t : t \in u_n \rangle$

(c) $\bar{\sigma}_n(\bar{x}_{n+1}) = \langle \sigma_{n,t}(\bar{x}_{n+1}) : t \in u_n \rangle$ is a sequence of $\tau_M$-terms, possibly with parameters (from $M_a$) so $\bar{\sigma}_n(\bar{d}) = \langle \sigma_{n,t}(\bar{d}) : t \in u_n \rangle$ for any $\bar{d} = \langle d_s : s \in u_{n+1} \rangle, d_s \in M$; if $a$ is a Polish group, the $\sigma_n$ are so called words

(d) $\zeta = \langle \zeta_n : n < \omega \rangle$ is a sequence of positive reals converging to 0

(e) $\bar{d}_{n+1} = \langle d_{n+1,t} : t \in u_{n+1} \rangle$ with each $d_{n+1,t}$ an element of $M$ such that if $\bar{d}_{n+1} = \langle d'_{n+1,t} : t \in u_{n+1} \rangle$ is of distance $< \zeta_{n+1}$ from $\bar{d}_{n+1}$, (that is $d'_{n+1,t} \in \text{Ball}_G(d_{n+1,t}, \zeta_{n+1})$ for each $t \in u_{n+1}$), then $\bar{\sigma}_n(\bar{d}'_{n+1}) \in \text{Ball}(\bar{d}_n, \zeta_n)$ which means: $t \in u_n \Rightarrow \bar{\sigma}_{n,t}(\ldots, d'_{n+1,s}, \ldots)_{s \in u_{n+1}} \in \text{Ball}(d_{n,t}, \zeta_n)$

(f) for every $n < \omega$ and a position real $\varepsilon$ there is $m > n$ such that

(*) if $d'_{m,t} \in \text{Ball}(d_{m,t}, \zeta_n)$ for every $t \in u_m$ then the distance between

$\bar{\sigma}_n(\bar{\sigma}_{n-1}(\ldots, \bar{\sigma}_{m-1}(d'_{m}), \ldots), \bar{\sigma}_n(\bar{\sigma}_{n+1}(\ldots, \bar{\sigma}_{m-1}(d_{m})), \ldots)$ is $< \varepsilon$.

Then there are $d'^*_{n,t} \in M$ for $n < \omega, t \in u_n$ which solves the set of equations

$$d'^*_{n,t} = \sigma_{n+1}(d'_{n+1,s})_{s \in u_{n+1}}$$

and satisfies

$$d'^*_{n,t} \in \text{Ball}_G(d_{n+1}, \zeta_n).$$

3.2 Remark. 1) In “special” versions we have $\bar{d}_n = \bar{\sigma}_n(\bar{d}_{n+1})$ (and in [Sh 744] we have $d_n = \sigma_n(d_{n+1})$) but here there is no “the true solution which we perturb”.

2) Condition (e) in Lemma 3.1 says that if in large $n$ we perturb $\bar{d}_n$ with error $< \zeta_n$ and compute down by the $\bar{\sigma}$’s we still get a reasonable $\bar{d}_k$ for every $k < n$ but not necessarily a very good one.

Proof. For every $k$ we shall define $\langle c^k_{n,t} : t \in u_n, n < \omega \rangle$, a sequence of elements of the algebra.
First, if \( n \geq k \) let \( c_{n,t}^k = d_{n,t} \). Second, we define \( \bar{c}_n^k = \langle c_{n,t}^k : t \in u_n \rangle \) by downward induction on \( n \leq k \).

\( n = k \) by the first case.

\( n < k \) let \( \bar{c}_n^k = \bar{\sigma}_n(\bar{c}_{n+1}^k) \).

Next we show

\((*)_1\) \( c_{n,t}^k \in \text{Ball}(d_{n,t}, \zeta_n) \).

[Why? If \( n \geq k \) this is trivial as \( c_{n,t}^k = d_{n,t} \). If \( n \leq k \) by downward induction on \( n \), using condition (e) of the assumptions.]

\((*)_2\) for every positive real \( \varepsilon > 0 \) and \( n < \omega \), there is \( m > n \) such that if \( k \geq m \) then \( t \in u_n \Rightarrow c_{n,t}^k \in \text{Ball}(c_{n,t}^m, \varepsilon) \).

[Why? Given \( n < \omega \) and \( \varepsilon > 0 \) choose \( m \) as in clause (f) of the assumption. Let \( k \geq m \). By \((*)_1\), \( t \in u_m \Rightarrow c_{m,t}^k \in \text{Ball}(d_{n,t}, \zeta_m) \). By the way the \( c_{n,t}^k \) were defined for \( t \in u_n \) we have \( \bar{c}_n^k = \bar{\sigma}_n(\bar{\sigma}_{n+1}(\ldots, \bar{\sigma}_{n-1}(\bar{d}_m)\ldots)) \) and similarly \( \bar{c}_m^m = \bar{\sigma}_n(\bar{\sigma}_{n+1}(\ldots, \bar{\sigma}_{m-1}(\bar{d}_m)\ldots)) \).

Condition (f) i.e. the choice of \( m \) tells us that the desired results holds.]

So, for each \( n \) and \( t \in u_n \) the sequence \( \langle c_{n,t}^k : k < \omega \rangle \) is a Cauchy sequence by \((*)_2\); hence it converges to some \( c_{n,t} \in M \). Now

\( \bar{x} \) the sequence \( \langle c_{n,t} : n < \omega, t \in u_n \rangle \) forms a solution: for every \( n < \omega \) and \( t \in u_n \) the equation \( c_{n,t}^k = \sigma_{n,t}(\ldots, c_{n+1,s}, \ldots) \) is satisfied whenever \( n > k \) hence in the limit \( c_{n,t} = \sigma_{n,t}(\ldots, c_{n+1,s}, \ldots)_{s \in u_{n+1}} \).

Recall about groups

3.3 Fact: A free group is torsion free and the group is not divisible, in fact, every element \( c \) has at most one \( n \)-th root for each \( n = 1, 2, \ldots \) and has no root for every large enough \( n \) except when \( c \) is the unit.

3.4 Fact: Every countable subgroup at a free group \( G \) is contained in a countable subgroup which is a retract of \( G \).

We now give a criterion to show non-freeness. We could use \( \bar{x} \) instead of \( x \), of course.

3.5 Claim. 1)

(a) \( a \) is a complete metric algebra, \( M = M_a \) with unit \( e_M \)

(b) \( B \subseteq M \) is countable with \( e_M \) belonging to the closure of \( B \setminus \{ e_a \} \)

(c) \( \Xi \) is a set of terms of the form \( \sigma(x, \bar{y}) \)
(d) if $\sigma(x, \bar{y}) \in \Xi$ and $\bar{b} \in B$ and $c \in G$ then $\{x \in M : c = \sigma(x, \bar{b})\}$ is finite (or at most a singleton)

(e) for every finite $A \subseteq M$ (or $A \subseteq M$ a singleton) and $\zeta$ a positive real there are a sequence $\bar{b}$ from $B$ and term $\sigma(x, \bar{y}) \in \Xi$ such that

- $\sigma(e, \bar{b}) \in \text{Ball}(e, \zeta)$
- $\sigma(c, \bar{b}) \notin A$ for every $c \in M$.

Then no countable subalgebra of $M$ containing $B$ is a retract (in the algebraic sense) of $M$. Hence $M$ is not free for any variety.

2) We can omit $e$, i.e. omit clause (b) and the last phrase of clause (a) and change clause (e) to

\((e)'\) for any finite $A \subseteq M$ (or $A \subseteq M$ a singleton) and real $\zeta > 0$ and $d \in M$ there is a term $\sigma(x, \bar{y}) \in \Xi$ and sequence $\bar{b} \in \ell g(\bar{y})B$ and element $d' \in M$ such that

- $\sigma^M(d', \bar{b}) \in \text{Ball}(d, \zeta)$
- for no $c \in M$ do we have $\sigma^M(c, \bar{b}) \in A$.

3.6 Remark. We can similarly phrase sufficient conditions for “$M$ is unstable in $\aleph_0$” [for quantifier free formulas, see §6].

Proof. 1) Like the proof of part (2) below except that we add to $(\ast)$:

- $(\eta) e = e_M$.

2) We rely on 3.1.

Assume toward contradiction that $M$ is a countable reduct of $M$ which includes $B$, so we can choose $h^*$, a homomorphism from $M$ onto $M$ which extends $id_M$. Let $\langle a_n : n < \omega \rangle$ list $M$. Let $u_n = \{n\}$. We choose $\bar{b}_n$ and $\sigma_n(x, \bar{y}_n)$ and $\zeta_n$ by induction on $n$ such that

- $(\ast)(\alpha) \sigma_n(x, \bar{y}_n) \in \Xi$
- $(\beta) \bar{b}_n$ a sequence from $M$ of length $\ell g(\bar{y}_n)$
- $(\gamma) \zeta_n$ a positive real, $\zeta_{n+1} < \zeta_n/2$ and $\zeta_{n+1, \ell}$ is a positive real $< \zeta_n$
- $(\delta) e_n \in M$
Let us carry the induction, in stage $n$ we choose $e_n, \zeta_n$ and $\sigma_{n-1}(x, \bar{b}_{n-1})$ if $n > 0$.

**Case 1:** $n = 0$.

This is straightforward.

**Case 2:** $n = k + 1$.

Let $D$ be the set of $\bar{c} = \langle c_m : m \leq k \rangle$ which satisfies

$$\Xi_k(i) \quad m < k \Rightarrow c_m = \sigma_m(c_{m+1}, h^*(\bar{b}_m))$$

$$(ii) \quad c_0 = a_k.$$ 

We can prove by induction on $m \leq k$ that \{ $c_m : \bar{c} \in D$ $\}$ is finite, and let $A = \{ c_k : \bar{c} \in D \}$. By clause (e)’ of the assumptions (see 3.5(2)), there are $r < \omega$, $\sigma = \sigma(x, y_0, \ldots, y_{r-1}) \in \Xi$ and $\bar{b} \in r(M_a)$ and $d'$ as there. We let $\bar{d}_k = \bar{b}, \sigma_k = \sigma, \bar{y}_k = \langle y_0, \ldots, y_{r-1} \rangle, e_n = d'$.

Lastly, we should choose $\zeta_n \in \mathbb{R}^+$. There are several demands but each holds for every small enough $\zeta > 0$, more exactly one for clause $(\varepsilon)$ and for each $m < n$, one for clause $(\zeta)$.

Having carried the induction, clearly 3.1 apply hence there is a solution $\langle d^*_n : n < \omega \rangle$, that is $M_a \models d^*_m = \sigma_m(d_{m+1}^*, \bar{b}_m)$ for $m < \omega$. But $h^*$ is a homomorphism from $M_a$ into $M$ so $\langle h^*(d^*_n) : n < \omega \rangle$ satisfies all the equations in $\Xi_k$ hence by our choice in stage $n = k + 1, h^*(d^*_0) \neq a_k$. As this holds for every $k$ and $\{ a_k : k < \omega \}$ list the elements of $M$ we are done. 

**3.7 Remark.** 1) If we phrase algebraic compactness, it is preserved by taking reducts.

2) In a reasonable variant we can replace “$M$ countable” by $\| M \| < \text{cov(meagre)}$; we’ll return to this elsewhere.

3) We can change the demand on $\Xi$: at most one solution in clause $(e)$, $A$ a singleton in clause $(f)$.

4) This suffices for groups.
§4 Conclusions

4.1 Conclusion

1) If \((G, \mathfrak{d})\) is a complete metric group of density \(< |G|\), then:

(a) \(G\) is not free,
(b) if \(G\) is \(\aleph_1\)-free then for some countable \(A \subseteq G\), there is no countable reduct \(B, M\) of \(M_a\) including \(A\).

2) It suffices that \(G\) is an indirectly complete metric group and as a metric space it is of density \(< |G|\).

3) Instead “density” \(< |G|\)” it is enough to assume that the topology induces by the metric is not discrete.

Proof. 1), 2) Easy. Let \(\mu = \text{density}(G)\) and \(y_i \in G\) be pairwise distinct for \(i < \mu^+\). Without loss of generality \(y_i \notin \langle \{y_j : j < i\} \rangle_G\). So for some increasing sequence \(\langle i_n : n < \omega \rangle\) the sequence \(\langle y_{i,n} : n < \omega \rangle\) is a Cauchi sequence.

For part (1), by completeness it converges say to \(y^*\), the convergence is for \(\mathfrak{d}_G\). Now \(\langle b_n : n < \omega \rangle =: \langle (y^*)^{-1} y_{i,2n+1} : n < \omega \rangle\) converges to \(e_G\), the members are pairwise distinct so without loss of generality \(\neq e\).

However for part (2) we know that some \((G, \mathfrak{d}')\) equal to \((G, \mathfrak{d})\) as a group but with a different metric; is a complete metric group with an \(\omega\)-sequence of members of \(G\setminus \{e_G\}\) converging to \(e_G\).

Let \(\Xi = \{x^m y_1 : m < \omega\}\) and \(B = \{b_n : n < \omega\}\). Now we shall apply 3.5. In the assumptions, clauses (a)-(c) are obvious. As for clause (d) we are using: equations of the form \(x^m a' = a''\) has at most one solution in \(G\), see 3.3. We are left with clause (e), so we are given a real \(\zeta > 0\) and a finite set \(A \subseteq G\) (in fact, a singleton is enough). We can choose \(b \in B \setminus A \setminus \{e_G\}\) of distance \(< \zeta\) from \(e_G\). Let \(\sigma(x, y) = xy\) and \(\bar{b} = \langle b \rangle\) where \(n < \omega\) is the minimal \(n > 1\) such that \([a \in A \Rightarrow ab^{-1} \text{ has no } n\text{-th root}]\). This is possible, see Fact 3.3 so \(\sigma(e_M, \bar{b}) = b \in \text{Ball}_G(e_M, \zeta)\) as required in subclause (a) of clause (e) and \(\sigma(e_M, \bar{b}) = b \notin A\) as required in subclause (b) of clause (e) so by 3.5 we are done.

3) Choose \(\langle y_n : n < \omega \rangle\) converging to some \(y^*\) such that \(\langle y_n : n < \omega \rangle \setminus \langle y^* \rangle\) is with no repetitions, possible on \((G, \mathfrak{d})\) is not discrete. Now continue as above. \(\square_{4.1}\)

In particular

4.2 Conclusion: There is no free uncountable Polish group.

4.3 Claim. (1) In the proof of Proposition 4.1(b) we do not use all the strength of “\(G\) is free”. E.g. if \((G, \mathfrak{d})\) is a complete metric group then (a) \(\Rightarrow \neg (b)\) where:
(a) for some group words \( w_n(x_1, \ldots, x_{r_n}) \) for \( n < \omega \) possibly with parameters in \( G \) we have

(α) there are \( y_i \in G \) for \( i < \mu^+ \) (where \( \mu = \text{density}(G) \)) such that \( i < j \Rightarrow y_i \neq y_j \)

(β) for some \( k \), for every \( b, a_2, \ldots, a_{2n} \in G \) the set \( \{ a_1 \in G : G \models w_n(a_1, \ldots, a_{r_n}) = b \} \) has at most \( k \) members

(γ) for every real \( \zeta > 0 \), finite \( A \subseteq G \setminus \{ e_G \} \) and an infinite set \( B \subseteq G \) such that \( e_G \) belongs to its closure, there are \( n < \omega \) and \( b_2, \ldots, b_{r_n} \in B \) such that \( w_r(e_\mu, b_2, \ldots) \in \text{Ball}(e_G, \zeta) \) and \( A \) is disjoint to \( \{ w_r(c, b_r, \ldots) : c \in G \} \)

(b) if \( X \) is a countable subset of \( G \), then there is a countable subgroup \( H \) of \( G \) which includes \( X \) and is a reduct of \( G \), that is there is a projection from \( G \) onto \( H \).

2) The uncountable free abelian group falls under this criterion, in fact, any uncountable strongly \( \aleph_1 \)-free abelian group also satisfies this criterion.

3) In part (1) we can weaken (b) to

\( (b)^- \) \( G \) is strongly \( \aleph_1 \)-free

or just:

\( (b)^-- \) for every countable \( X \subseteq G \) there is a countable subgroup \( H \) of \( G \), \( X \subseteq H \) such that: if \( H \subseteq H' \subseteq G \), \( H' \) countable then \( H \) is a reduct of \( H' \), i.e. there is a projection from \( H' \) onto \( H \).

Proof. The same as the proof of 4.1.

4.4 Remark. 1) We may consider for a metric space a group rank: the objects being finite approximation to the system of elements we actually use \( \langle \sigma_n(d_{n+1}, b_n) : n < \omega \rangle \) in 3.1 (or in 3.5).

2) The results above confirms the thesis that the compactness conditions say that \( G \) is “large”, “rich”.

3) Note that we can expand \( M_a \) by individual constants, equivalently consider terms with parameters.

4) In the applications of 4.3, we do not actually use \( r > 2 \).

5) Concerning semi groups we intend to say it in a continuation.

6) We may consider assumption “some \( h : M_A \to N \) is a homomorphisms onto
7) We may consider just \( \|N\| < 2^{\aleph_0} \), so have to split into two so we get \( \omega^2 \) cases among which at least one “succeeds”.

4.5 Conclusion: 1) Assume \( \mathbb{A} \) is a countable structure. Then \( \text{Aut}(\mathbb{A}) \), the group of automorphisms of \( \mathbb{A} \), is not a free uncountable group, in fact it satisfies the conclusions of 4.1, 4.3.

2) Assume \( \mathbb{A} \) is a structure of cardinality \( \lambda \) and \( \lambda = \mu = \sum_\omega \) or more generally \( \lambda = \Sigma\{\lambda_n : n < \omega\}, 2^{\lambda_n} < 2^{\lambda_{n+1}}, \mu = \Sigma\{2^{\lambda_n} : n < \omega\} < 2^\lambda \). Then \( \text{Aut}(\mathbb{A}) \) cannot be free of cardinality \( > \mu \), in fact, it satisfies the conclusions of 4.1, 4.3.

Proof. 1) By 1.6, \( \text{Aut}(\mathbb{A}) \) is a Polish group and apply 4.2.

2) Without loss of generality the universe of \( \mathbb{A} \) is \( \lambda \), using \( \bar{\mathbb{A}} = \langle \lambda_n : n < \omega \rangle \) we know by 2.4(1) that \( (\text{Aut}(\mathbb{A}), d_{\bar{\mathbb{A}}} \bar{\mathbb{A}}) \) is a complete semi-metric group and apply 4.1(2). \( \square_{4.5} \)

4.6 Claim. For complete specially metric groups the proof of [Sh 744] works, similarly for algebras.
§5 Quite free but not free abelian groups

If uncountable Polish groups are not free, we may look at wider classes: $F_\sigma$, Borel analytic, projective $L[\mathbb{R}]$.

5.1 Question: 1) Is the freeness of a reasonably definable abelian group absolute? 2) For which cardinals $\lambda$ does $\lambda$-freeness imply freeness (or $\lambda^+$-freeness) for nicely definable abelian groups, in particular for $\lambda = \aleph_\omega$? 3) Similarly for other varieties (or any case when “free” is definable like universal Horn theory).

This is connected also to [Sh 402] whose original aim was a question of Marker “are there non-free Whitehead Borel Abelian groups”. But already in [Sh 402] it seems to me the basic question is to clarify freeness in such groups; that is, question 5.1 above.

Blass asked about definable subgroups of $\mathbb{Z}^{\omega}$ (see question 5.10): by [Sh 402] and the construction here we quite resolve this.

Recall that [Sh 402] analyze $\aleph_1$-free abelian groups which are $\Sigma^1_1$ or so. A natural dividing line was suggested; the complicated half was proved to be not Whitehead, and at least for me is an analog to not $\aleph_0$-stable. The low half is $\aleph_2$-free. So under CH we were done, but what if $2^{\aleph_0} > \aleph_1$? Are they also free? This was left open by [Sh 402].

We shed some light by giving an example (an $F_\sigma$ one) showing that the non-CH case in [Sh 402] is a real problem. This resolves the original problem: it is consistent that there are non-free Whitehead groups, this is derived in 5.13. But what about the further question, e.g. 5.1(2)? The examples seem to indicate (at least to me) that the picture in [Sh 87a], [Sh 87b] is the right one here, connecting theories of $\psi \in L_{\omega_1,\omega}$ with $\Sigma^1_1$-models. Also related are [EM2], [MkSh 366] on almost freeness for varieties, and see [EM] on abelian groups. In particular we conjecture “every $\aleph_\omega$-free Borel group is free”.

We shall use freely the well known theorem saying

\[ \text{a subgroup of a free abelian group is a free abelian group.} \]

5.2 Definition. For $k(*) < \omega$ we define an abelian group $G = G_{k(*)}$ and is generated by \{\(x_{m,\bar{\eta},\nu}: m \leq k(*)\) and $\nu \in \omega^2$ and $\bar{\eta} = \langle \eta_\ell : \ell \leq k(*), \ell \neq m \rangle$ where $\eta_\ell \in \omega^2\} \cup \{y_{\bar{\eta},n}: n < \omega \text{ and } \bar{\eta} = \langle \eta_\ell : \ell \leq k(*) \rangle \text{ where } \eta_\ell \in \omega^2\}$ freely except the equations:

\[
\begin{align*}
\sum \{x_{m,\bar{\eta},\nu} + y_{\bar{\eta},n+1} - y_{\bar{\eta},n} & : m \leq k(*) \text{ and } \bar{\eta} = \bar{\eta} \restriction \{m' \leq k(*) : m' \neq m \} \text{ and } \\
\} & \text{ and } \nu = \eta_m \restriction n}.\end{align*}
\]
(Note that if \( m_1 < m_2 \leq k(*) \) then \( \bar{\eta}_{m_1} \neq \bar{\eta}_{m_2} \) having different index sets).

**Explanation.** A canonical example of a non-free group is \((\mathbb{Q}, +)\). Other examples are related to it after we divide by something. The \( y \)'s here play that role of providing (hidden) copies of \( \mathbb{Q} \). What about \( x \)'s? For each \( \bar{\eta} \in \Lambda \) we use \( m \leq k(*) \) to give \( \langle y_{\bar{\eta},n} : n < \omega \rangle \), \( k(*) \) “chances”, “opportunities” to avoid having \((\mathbb{Q}, +)\) as a quotient, one for each cardinal \( \leq \aleph \). More specifically, if \( H \subseteq G \) is the subgroup which is generated by \( X = \{ x_{m,\bar{\eta},\nu} : m \neq m(*) \} \) and \( \nu \) is a function from \( \{ \ell \leq k(*) : \ell \neq m \} \) to \( \omega \) and \( \nu \in \omega^2 \), still in \( G/H \) the \( \{ y_{\bar{\eta},n} : n < \omega \} \) does not generate a copy of \( \mathbb{Q} \), as witnessed by \( \{ x_{m(*)},\bar{\eta}_{m(*)},\bar{\eta}_{m(*)} \} \).

**5.3 Claim.** The abelian group \( G_{k(*)} \) is a Borel group, even an \( F_\sigma \)-one that is the set of elements and the graphs of \( + \) and the function \( x \mapsto -x \) (i.e. \( \{(x, y, z) : G_{k(*)} \models "x + y = z"\} \) hence also \( \{(x, -x) : x \in G_{k(*)}\} \) are \( F_\sigma \)-sets; hence Borel.

**Proof.** Let \( \text{cd} \) be a one-to-one function from the set of finite sequences of natural numbers onto the set of natural numbers and we define:

\[
\oplus_1 \begin{align*}
(a) & \quad \text{code}(x_{m,\bar{\eta},\nu}) = \langle \text{cd}(\langle m, \text{cd}(\nu), \ldots, \eta_\ell(i), \ldots \rangle_{\ell \leq k(*)}, \ell \neq m) : i < \omega \rangle \\
& \quad \text{in } \omega \omega \text{ and let } \mathcal{X} = \{ \text{code}(x_{n,\bar{\eta},\nu}) : (n, \bar{\eta}, \nu) \text{ as in Definition 5.2} \} \\
(b) & \quad \text{code}(y_{\bar{\eta},n}) = \langle \text{cd}(n, \ldots, \eta_\ell(i))_{\ell \leq n(*)} : i < \omega \rangle \\
& \quad \text{and } \mathcal{Y} = \{ \text{code}(y_{\bar{\eta},n}) : (\bar{\eta}, n) \text{ as in Definition 5.2} \} \\
(c) & \quad \text{for a sequence } \bar{\alpha} = \langle a_\ell : \ell < n \rangle \text{ of integers let } \rho_{\bar{\alpha}} = \langle \text{sign}(a_\ell) : \ell < n \rangle, \text{sign}(a_\ell) \text{ is 0,1,2 if } a_\ell \text{ is negative, zero, positive respectively.}
\end{align*}
\]

We say \( \nu \) represents \( x \in G_{k(*)} \) as witnessed by \( \langle (z_\ell, a_\ell, m) : \ell < n \rangle \) when:

\[
\oplus_2 \begin{align*}
(a) & \quad G \models x = \sum_{\ell < n} a_\ell z_\ell, \\
(b) & \quad z_\ell \in \{ x_{n,\bar{\eta},\nu} : (\eta, \bar{\eta}, \nu) \text{ as in Definition 5.2} \} \cup \{ y_{\bar{\eta},m} : (\bar{\eta}, m) \text{ as in Definition 5.2} \} \\
(c) & \quad \langle z_\ell : \ell < n \rangle \text{ is without repetitions} \\
(d) & \quad \langle \text{cd}(z_\ell) \upharpoonright m : \ell < n \rangle \text{ are pairwise distinct} \\
(e) & \quad \text{if } \langle (z_\ell', a_\ell', m') : \ell < n' \rangle \text{ satisfies clauses (a)-(d), then } m \leq m' \\
(f) & \quad \text{if } n = 0 \text{ then } m = 0 \\
(g) & \quad \text{cd}(z_0) <_{\text{lex}} \text{cd}(z_1) <_{\text{lex}} \cdots \\
(h) & \quad \nu = \langle \text{cd}(\langle n \rangle)^\text{sign}(\bar{\alpha}) \langle |a_\ell| : \ell < n \rangle^\text{cd}(z_\ell(i)) : \ell < k(*) \rangle : i < \omega \rangle.
\end{align*}
\]
Now for $n < \omega, \bar{a} = \langle a_\ell : \ell < n \rangle \in \omega^n Z, i < \omega$ and $\bar{\varrho} = \langle \varrho_\ell : \ell < n \rangle \in \omega^n (i Z)$ is $<_{\text{lex}}$-increasing hence without repetitions (and if $n = 0$ we let $i = 0$) we let

$$Z_{\bar{a}, \bar{\varrho}} = \{ \nu : \nu \text{ represent some } x \in G_{k(*)} \text{ as witnessed by}$$

$$(z_\ell, a_\ell) : \ell < n \} \text{ and } \text{cd}(z_\ell) = \varrho_\ell \text{ for } \ell < n \}.$$

Let $\mathcal{Y}$ be the set of such pairs $(\bar{a}, \bar{\varrho})$

\[(*)_1 \ (Z_{\bar{a}, \bar{\varrho}} : (\bar{a}, \bar{\varrho}) \in \mathcal{Y}) \text{ is a sequence of pairwise disjoint closed subsets of } \omega^\omega \]

\[(*)_2 \text{ every member of } G \text{ is represented by one and only one member of } Z := \bigcup (Z_{\bar{a}, \bar{\varrho}} : (\bar{a}, \bar{\varrho}) \in \mathcal{Y}).\]

[Why? For any $i < n$ clearly $\{ x_m, \bar{\eta}, \nu : (m, \bar{\eta}, \nu) \text{ as in Definition 5.2} \} \cup \{ y_{\bar{\eta}, i} : \bar{\eta} \in \Lambda \}$ generates freely a subgroup $G'_{k(*)_i}$ of $G_{k(*)}$ such that the quotient $G_{k(*)}/G'_{k(*)_i}$ is torsion. The rest should be clear, too.]

\[(*)_3 \mathcal{U} = \{(\nu_1, \nu_2, \nu_3) : \nu_\ell \text{ represent } x_\ell \in G_{k(*)} \text{ for } \ell = 1, 2, 3 \text{ and } G_{k(*)} \models \]

\[\text{“}x_1 + x_2 = x_3 \text{”} \} \text{ is the graph of a two-place function} \]

\[(*)_4 \text{ for any } (\bar{a}_\ell, \bar{\varrho}_\ell) \in \mathcal{Y} \text{ for } \ell = 1, 2, 3 \text{ the set } \{ (\nu_1, \nu_2, \nu_3) \in \mathcal{U} : \nu_\ell \in Z_{\bar{a}_\ell, \bar{\varrho}_\ell} \text{ for} \]

\[\ell = 1, 2, 3 \} \text{ is a closed set.} \]

Clearly we are done. $\square_{5.3}$

As a warm up we note:

**5.4 Claim.** $G_{k(*)}$ is an $\aleph_1$-free abelian group.

**Proof.** Let $U \subseteq \omega^2$ be countable (and infinite) and define $G'_U$ like $G$ restricting ourselves to $\eta_\ell \in U$; by the L"{o}wenheim-Skolem argument it suffices to prove that $G'_U$ is a free abelian group. List $k(*)^{+1} U$ without repetitions as $\langle \bar{\eta}_t : t < \omega \rangle$, and choose $s_t < \omega$ such that $[r < t \text{ & } \bar{\eta}_r \upharpoonright k(*) = \bar{\eta}_t \upharpoonright k(*) \Rightarrow \emptyset = \{ \eta_{t, k(*)} \upharpoonright \ell : \ell \in [s_t, \omega) \} \cap \{ \eta_{r, k(*)} \upharpoonright \ell : \ell \in [s_r, \omega) \}].$

Let

$$Y_1 = \{ x_m, \bar{\eta}, \nu : m < k(*) \text{, } \bar{\eta} \in k(*)^{+1} \setminus \{ m \} U \text{ and } \nu \in \omega^{>2} \}$$

$$Y_2 = \left\{ x_m, \bar{\eta}, \nu : m = k(*) \text{, } \bar{\eta} \in k(*) U \text{ and for no } t < \omega \text{ do we have} \right. \]

$$\bar{\eta} = \bar{\eta}_t \upharpoonright k(*) \text{ & } \nu \in \{ \eta_{t, k(*)} \upharpoonright \ell : s_t \leq \ell < \omega \} \}$$
\[ Y_3 = \{ y_{\bar{\eta}_t,n} : t < \omega \text{ and } n \in [s_t, \omega) \}. \]

Now

\[ (*)_1 \ Y_1 \cup Y_2 \cup Y_3 \text{ generates } G'_U. \]

[Why? Let \( G' \) be the subgroup of \( G'_U \) which \( Y_1 \cup Y_2 \cup Y_3 \) generates. First we prove by induction on \( n < \omega \) that for \( \bar{\eta} \in k(*)U \) and \( \nu \in n2 \) we have \( x_{k(*)},\bar{\eta},\nu \in G' \). If \( x_{k(*)},\bar{\eta},\nu \in Y_2 \) this is clear; otherwise, by the definition of \( Y_2 \) for some \( \ell < \omega \) and \( t < \omega \) such that \( \ell \geq s_t \) we have \( \bar{\eta} = \bar{\eta}_t \upharpoonright k(*) \), \( \nu = \eta_t,k(*) \upharpoonright \ell \).

Now

(a) \( y_{\bar{\eta}_t,\ell+1}, y_{\bar{\eta}_t,\ell} \) are in \( Y_3 \subseteq G' \).

Hence by the equation \( \boxdot_{\bar{\eta},n} \) in Definition 5.2, clearly \( x_{k(*)},\bar{\eta},\nu \in G' \). So as \( Y_1 \subseteq G' \subseteq G'_U \), all the generators of the form \( x_{m,\bar{\eta},\nu} \) with each \( \eta_\ell \in U \) are in \( G' \). Also we have

(b) \( x_{m,\bar{\eta}_t,\{i \leq k(*),i \neq m\},\nu} \) belong to \( Y_1 \subseteq G' \) if \( m < k(*) \).

Now for each \( t < \omega \) we prove that all the generators \( y_{\bar{\eta}_t,n} \) are in \( G' \). If \( n \geq s_t \) then clearly \( y_{\bar{\eta}_t,n} \in Y_3 \subseteq G' \). So it suffices to prove this for \( n \leq s_t \) by downward induction on \( n \); for \( n = s_t \) by an earlier sentence, for \( n < s_t \) by \( \boxdot_{\bar{\eta},n} \). The other generators are in this subgroup so we are done.]

\[ (*)_2 \ Y_1 \cup Y_2 \cup Y_3 \text{ generates } G'_U \text{ freely.} \]

[Why? Translate the equations.

Alternatively, let \( \langle z_\alpha : \alpha < \alpha(*) \rangle \) list the set of generators of \( G'_U \) without repetition such that for some increasing continuous \( \langle \alpha_i : i \leq \omega + \omega \rangle \) we have \( \alpha_0 = 0, \alpha_{\omega + \omega} = \alpha(*) \) and

(a) \( \{ z_\alpha : \alpha < \alpha_1 \} = Y_1 \cup Y_2 \cup Y_3 \)
(b) \( \{ z_\alpha : \alpha \in [\alpha_1+n,\alpha_1+n+1) \} = \{ x_{k(*)},\bar{\eta},\nu \in G'_U : x_{k(*)},\bar{\eta},\nu \notin Y_2 \}
\text{ and } \ell g(\nu) = n \}
(c) \( \{ z_\alpha : \alpha \in [\alpha_\omega+r,\alpha_\omega+r+1) \} = \{ y_{\bar{\eta}_t,n} : t < \omega, n < s_t \text{ and } r = s_t - n \}. \)

Now the proof above shows that:

/go/ there is a one-to-one function from the set \( \Xi \) of equations defining \( G'_U \) onto \( [\alpha_1, \alpha_\omega) \) such that:

if the equation \( \varphi \) is mapped to the ordinal \( \alpha \) then: if \( z_\beta \) appears in the equation then \( \beta \leq \alpha \) and \( z_\alpha \) appears in the equation and its coefficient is 1 or \(-1\).
This clearly suffices. \(\square_{5.4}\)

Now systematically

**5.5 Definition.** 1) For \(U \subseteq \omega^2\) let \(G_U\) be the subgroup of \(G\) generated by 
\[Y_U = \{y_{\bar{\eta},n} : \bar{\eta} \in \nu \} \cup \{x_{m,\bar{\eta},\nu} : m \leq k(*)\ \text{and} \ \bar{\eta} \in (k(*)+1)\setminus\{m\}(U)\ \text{and} \ \nu \in \omega^2\}.\] Let \(G_U^+\) be the divisible hull of \(G_U\) and \(G^+ = G_{1(\omega^2)}^+\).
2) For \(U \subseteq \omega^2\) and finite \(u \subseteq \omega^2\) let \(G_{U,u}\) be the subgroup\(^3\) of \(G\) generated by 
\[\cup \{G_{U \cup \{\eta\}} : \eta \in u\};\] and for \(\bar{\eta} \in (k(*)+U\) let \(G_{U,\bar{\eta}}\) be the subgroup of \(G\) generated by 
\[\cup \{G_{U \cup \{\eta \in k < \ell g(\bar{\eta}) \ \text{and} \ k \neq \ell\} : \ell < \ell g(\bar{\eta})\}\}.\]
3) For \(U \subseteq \omega^2\) let \(\Xi_U = \{\text{the equation } \Xi_{\bar{\eta},n} : \bar{\eta} \in k(*)+1(U)\ \text{and} \ n < \omega\}.\) Let
\[\Xi_{U,u} = \cup \{\Xi_{U \cup u} : \beta \in u\}.\]

**5.6 Claim.** 0) If \(U_1 \subseteq U_2 \subseteq \omega^2\) then \(G_{U_1}^+ \subseteq G_{U_2}^+ \subseteq G^+\).
1) For any \(n(*) < \omega\), the abelian group \(G_{U}^+\) (which is a vector space over \(\mathbb{Q}\)), has the basis
\[Y_{n(*)} = \{y_{\bar{\eta},n(*)} : \bar{\eta} \in k(*)+1(U)\} \cup \{x_{m,\bar{\eta},\nu} : m \leq k(*)\ \text{and} \ \bar{\eta} \in (k(*)+1)\setminus\{m\}(U)\ \text{and} \ \nu \in \omega^2\}.\]
2) For \(U \subseteq \omega^2\) the abelian group \(G_U\) is generated by \(Y_U\) freely (as an abelian group) except the set \(\Xi_U\) of equations.
3) If \(U_m \subseteq \omega^2\) for \(m < m(*)\) then the subgroup \(G_{U_0} + \ldots + G_{U_{m(*)-1}}\) of \(G\) is generated by \(Y_{U_0} \cup Y_{U_1} \cup \ldots \cup Y_{U_{m(*)-1}}\) freely (as an abelian group) except the equations in \(\Xi_{U_0} \cup \Xi_{U_1} \cup \ldots \cup \Xi_{U_{m(*)-1}}\) provided that
\[\forall \ell \leq k(*) \exists m < m(*) \exists \{\eta_0, \ldots, \eta_k\} \subseteq U_m\]
then for some \(m < m(*)\) we have \(\{\eta_0, \ldots, \eta_k\} \subseteq U_m\).
4) If \(U_\ell = U \setminus U_\ell\) for \(\ell < m(*)\) and \(\langle U_\ell : \ell < m(*)\rangle\) are pairwise disjoint then \(\otimes\) holds.
5) \(G_{U,u} \subseteq G_{U \cup u}\) if \(U \subseteq \omega^2\) and \(u \subseteq \omega^2\setminus U\); moreover \(G_{U,u} \subseteq G_{U \cup u}\) if \(u \subseteq \omega^2\).
6) If \(\langle U_\alpha : \alpha < \alpha(*)\rangle\) is \(\sqsubseteq\)-increasing continuous then also \(\langle G_{U_\alpha} : \alpha < \alpha(*)\rangle\) is 
\(\sqsubseteq\)-increasing continuous.
7) If \(U_1 \subseteq U_2 \subseteq U \subseteq \omega^2\setminus U\) is finite, \(|u| < k(*)\) and \(U_2 \setminus U_1 = \{\eta\}\) and 
\(\nu = u \cup \{\eta\}\) then \((G_{U_u} + G_{U_2 U_u}) / (G_{U_u} + G_{U_1 U_u})\) is isomorphic to \(G_{U_U U_v} / G_{U_1,v}\).
8) If \(U \subseteq \omega^2\) and \(u \subseteq \omega^2\setminus U\) has \(\leq k(*)\) members then \((G_{U_u} + G_u) / G_{U,u}\) is isomorphic to \(G_u / G_{\emptyset,u}\).

**Proof.** 0), 1) Obvious. 2),3,4) Follows.

\(^3\)note that if \(u = \{\eta\}\) then \(G_{U,u} = G_U\)
5) First, $G_{U,u} \subseteq G_{U\cup u}$ follows by the definition. Second, we deal with proving $G_{U,u} \subseteq_{pr} G_{U\cup u}$. So let $|u| = m(*) + 1$ and $\langle \eta_\ell : \ell \leq m(*) \rangle$ list $u$, necessarily with no repetitions and let $U_\ell = U \cup (u \{\eta_\ell\})$ (so $G_{U,u} = G_{U_0} + \ldots + G_{U_{m(*)}}$) and assume $z \in G_{U\cup u}$, $a \in \mathbb{Z}\setminus\{0\}$ and $az$ belongs to $G_{U_0} + \ldots + G_{U_{m(*)}}$ so it has the form $\Sigma\{b_i x_{m_\ell, i, \eta_\ell, \nu_\ell} : i < i(*)\} + \Sigma\{c_j y_{\rho_j, n_\ell, \nu_\ell} : j < j(*)\}$ with $b_i, c_j \in \mathbb{Z}$ and $\tilde{\eta}_\ell, \tilde{\rho}_j$ are (finite) sequences of members of $U_{\ell(i)}, U_{k(j)}$ respectively and are as required in Definition 5.2 where $\ell(i), k(j) < m(*)$.

Now similarly as $z \in G_{U\cup u}$, we can find $z = \Sigma\{b_i' x_{m_\ell', i, \eta_\ell', \nu_\ell'} : i < i(*)\} + \Sigma\{c_j' y_{\rho_j, n_\ell, \nu_\ell} : j < j(*)\}$.

By the equations in Definition 5.2 without loss of generality for some $n(*)$ we have: $i < i(*) \Rightarrow n_i = n(*)$ and $i < i(*) \Rightarrow n'_i = n(*)$. Also without loss of generality in each of the sequences $\langle (m_\ell, \eta_\ell, \nu_\ell) : i < i(*)\rangle, \langle \rho_j : j < j(*)\rangle$ is with no repetitions, and also in $\langle (m_\ell', \eta_\ell', \nu_\ell') : i < i(*)\rangle, \langle \rho'_j : j < j(*)\rangle$ there is no repetition (for $\langle \rho_j : j < j(*)\rangle$ and $\langle \rho'_j : j < j(*)\rangle$) we use $n_i = n(*)$, $n'_i = n(*)$. Together

$$
\oplus \Sigma\{b_i x_{m_\ell, i, \eta_\ell, \nu_\ell} : i < i(*)\} + \Sigma\{c_j y_{\rho_j, n(*)} : j < j(*)\} = \Sigma\{ab_i' x_{m_\ell', i, \eta_\ell', \nu_\ell'} : i < i(*)\} + \Sigma\{ac_j'y_{\rho'_j, n(*)} : j < j(*)\}.
$$

Now this equation holds in $G_{U\cup u}$ hence is $G$ and even in $G^+$. By part (1) and the “no repetitions” after possible permuting we get $i(*) = i'(*)$, $j(*) = j'(*)$, $(m_\ell, \eta_\ell, \nu_\ell) = (m_\ell', \eta_\ell', \nu_\ell')$, $b_i = ab_i'$ for $i < i(*)$, $\rho_j = \rho'_j$ for $j < j(*)$, $c_j = ac_j'$ for $j < j(*)$. But this proves that $\{x_{m_\ell', \eta_\ell', \nu_\ell} : i < i'(*)\} \cup \{y_{\rho'_j, n(*)} : j < j'(*)\} \subseteq G_{U,u}$ hence $z \in G_{U,u}$ as required.

Third, the proof of $G_{U\cup u} \subseteq_{pr} G$ is similar.

6) Easy.

7) Clearly $U_1 \cup v = U_2 \cup u$ hence $G_{U_1 \cup u} \subseteq G_{U_1 \cup v} = G_{U_2 \cup u}$ hence $G_{U,u} + G_{U_1 \cup u} = G_{U,u} + G_{U_2 \cup u}$ is a subgroup of $G_{U,u} + G_{U_2 \cup u}$, so the first quotient makes sense.

Hence by the isomorphism theorem $(G_{U,u}+G_{U_2 \cup u})/(G_{U,u}+G_{U_1 \cup u})$ is isomorphic to $G_{U_2 \cup u}/(G_{U_2 \cup u} \cap (G_{U,u}+G_{U_1 \cup u}))$. Now $G_{U_1,v} \subseteq G_{U_1 \cup v} = G_{U_2 \cup u}$ and $G_{U_1,v} = \Sigma\{G_{U_1,v \setminus \{v\}} : v \in v\} = \Sigma\{G_{U_1 \cup v \setminus \{v\}} : v \in v\} + G_{U_1,(v \setminus \{\eta\})} \subseteq G_{U,u} + G_{U_1,u}$. Together $G_{U_1,v}$ is included in their intersection, i.e. $G_{U_2 \cup u} \cap (G_{U,u}+G_{U_1,u})$ include $G_{U_1,v}$ and using part (1) both has the same divisible hull inside $G^+$. But $G_{U_1,v}$ is a pure subgroup of $G$ by part (5) hence of $G_{U_1 \cup v}$. Hence necessarily $G_{U_1 \cup u} \cap (G_{U,u} + G_{U_1,u}) = G_{U_1,u}, v$ so as $G_{U_2 \cup u} \subseteq G_{U_1 \cup v}$ we are done.

8) The proof is similar to the proof of part (7). Note that $G_{U,u} \subseteq G_{U,u} + G_u$ hence the first quotient makes sense. So by an isomorphism theorem $(G_{U,u} + G_u)/G_{U,u}$ is isomorphic to $G_u/(G_{U,u} \cap G_u)$. Now $G_{U, u} \cap G_u$ includes $G_{\emptyset,u}$ and using part (1) both has the same divisible hull inside $G^+$. But $G_{\emptyset,u}$ is a pure subgroup of $G_u$ by part (5). So necessarily $G_{U,u} \cap G_u = G_{\emptyset,u}$, so $G_u/(G_{U,u} \cap G_u) = G_u/G_{\emptyset,u}$, so we are done.
Discussion: For the reader’s convenience we write what the group $G_{k(*)}$ is for the case $k(*) = 0$. So, omitting constant indexes and replacing sequences of length one by the unique entry we get that it is generated by $y_{\eta,n}$ (for $\eta \in \omega^{+2}$, $n < \omega$) and $x_\nu$ (for $\nu \in \omega^{>2}$) freely as an abelian group except the equations $(n!)y_{\eta,n+1} = y_{\eta,n} + x_{\eta|n}$. Note that if $K$ is the countable subgroup generated by $\{x_\nu : \nu \in \omega^{>2}\}$ then $G/K$ is a divisible group of cardinality continuum hence $G$ is not free. So $G$ is $\aleph_1$-free but not free.

Now we have the main proof

5.7 Main Claim. 1) The abelian group $G_{U \cup u}/G_{U,u}$ is free if $U \subseteq \omega^2, u \subseteq \omega^2 \setminus U$ and $1 \leq |u| \leq k \leq k(*)$ and $|U| \leq \aleph_{k(*)-k}$.
2) If $U \subseteq \omega^2$ and $|U| \leq \aleph_{k(*)}$, then $G_{U}$ is free.

Proof. 1) We prove this by induction on $|U|$; without loss of generality $|u| = k$ as also $k' = |u|$ satisfies the requirements.

Case 1: $U$ is countable.

So let $\{\nu_\ell^* : \ell < k\}$ list $u$ be with no repetitions, now if $k = 0$, i.e. $u = \emptyset$ then $G_{U \cup u} = G_{U} = G_{U,u}$ so the conclusion is trivial. Hence we assume $u \neq \emptyset$, and let $u_\ell := u \setminus \{\nu_\ell^*\}$ for $\ell < k$.

Let $\langle \bar{\eta}_t : t < t^* \leq \omega \rangle$ list with no repetitions the set $\Lambda_{U,u} := \{\bar{\eta} \in k(*)+1(U \cup u) :$ for no $\ell < k$ does $\bar{\eta} \in k(*)+1(U \cup u_\ell)\}$. Now comes a crucial point: let $t < t^*$, for each $\ell < k$ for some $r_{t,\ell} \leq k(*)$ we have $\eta_{t,r_{t,\ell}} = \nu_\ell^*$ by the definition of $\Lambda_{U,u}$, so $|\{r_{t,\ell} : \ell < k\}| = k < k(*)+1$ hence for some $m_t \leq k(*)$ we have $\ell < k \Rightarrow r_{t,\ell} \neq m_t$ so for each $\ell < k$ the sequence $\bar{\eta}_t \upharpoonright (k(*)+1\{m_t\})$ is not from $\{(\rho_s : s \leq k(*)$ and $s \neq m_t) : \rho_s \in \omega(U \cup u_\ell)\}$ for every $s \leq k(*)$ such that $s \neq m_t$.

For each $t < t^*$ we define $S(t) = \{m \leq k(*) : \eta_{t,m} \leq k(*) \& s \neq m \}$ is included in $U \cup u_\ell$ for no $\ell \leq k$. So $m_t \in S(t) \subseteq \{0,\ldots,k(*)\}$ and $m \in S(t) \Rightarrow \bar{\eta}_t \upharpoonright \{j \leq k(*) : j \neq m\} \notin (U \cup u_\ell)$ for every $\ell \leq k$. For $m \leq k(*)$ let $\bar{\eta}'_{t,m} := \bar{\eta}_t \upharpoonright \{j \leq k(*) : j \neq m\}$ and $\bar{\eta}'_t := \bar{\eta}'_{t,m_t}$. Now we can choose $s_t < \omega$ by induction on $t$ such that

- if $t_1 < t, m \leq k(*)$ and $\bar{\eta}'_{t_1,m} = \bar{\eta}'_{t,m}$, then $\eta_{t,m} \upharpoonright s_t \notin \{\eta_{t_1,m} \upharpoonright \ell : \ell < \omega\}$.

Let $Y^* = \{x_{m,\bar{\eta},\nu} \in G_{U \cup u} : x_{m,\bar{\eta},\nu} \notin G_{U \cup u_\ell} \text{ for } \ell < k\} \cup \{y_{\bar{\eta},n} \in G_{U \cup u} : y_{\bar{\eta},n} \notin G_{U \cup u_\ell} \text{ for } \ell < k\}$. Let

$$Y_1 = \{x_{m,\bar{\eta},\nu} \in Y^* : \text{for no } t < t^* \text{ do we have } m = m_t \& \bar{\eta} = \bar{\eta}'_t\}.$$
Now the desired conclusion follows from $\varnothing$ suffices. For $x \in Y^*$ we are done, hence assume $\neg x$. Also $m = m_t$ and $\bar{\eta} = \bar{\eta}_t$. As $x \notin Y_2$, clearly for some $t$ the above we have $\eta_{t.m_t} \nmid s_t \leq \nu < \eta_t$. Hence by Definition 5.2 the equation $\bar{x}_{\bar{r},n}$ from Definition 5.2 holds, now $y_{\bar{r},n}, y_{\bar{r},n+1} \in G'_{U_u}$. So in order to deduce from the equation $x = x_{m,\bar{\eta},\nu}$, it suffices to show that $x_{j,\bar{\eta}_{t,j},\eta_{t,j}} \in n \in G'_{U_u}$ for each $j \leq k(*)$, $j \neq m_t$. But each such $x_{j,\bar{\eta}_{t,j},\eta_{t,j}} \in G'_{U_u}$ as it belongs to $Y_1 \cup Y_2$.

[Why? Otherwise necessarily for some $r < t^*$ we have $j = m_r, \bar{\eta}_{t,j} = \bar{\eta}_{t,m_r}$ and $\eta_{r,m_r} \nmid s_r \leq \eta_t \nmid n$ and as said above $n \geq s_t$. Clearly $r \neq t$ as $m_r = j \neq m_t$, now as $\bar{\eta}_{t,m_t} = \bar{\eta}_{t,m_r}$ and $\bar{\eta}_t \neq \bar{\eta}_r$ (as $t \neq r$) clearly $\eta_{r,m_r} \neq \eta_{r,m_r}$. Also $\neg (r < t)$ by (*) above applied with $r, t$ here standing for $t_1, t$ there as $\eta_{r,m_r} \nmid s_r \leq \eta_{t,j} \nmid n$. Lastly for if $t < r$, again (*) applied with $t, r$ here standing for $t_1, t$ there as $n \geq m_t$ gives contradiction.] So indeed $x \in G'_{U_u}$.

Second consider $y = y_{\bar{\eta},n} \in G'_{U_u}$, if $y \notin Y^*$ then $y \in Y_2 \subseteq G'_{U_u}$, so assume $y \in Y^*$. If $y \in Y_3$ we are done, so assume $y \notin Y_3$, so for some $t, \bar{\eta} = \bar{\eta}_t$ and $n < s_t$. We prove by downward induction on $s \leq s_t$ that $y_{\bar{\eta},s} \in G'_{U_u}$, this clearly suffices. For $s = s_t$ we have $y_{\bar{\eta},s} \in Y_3 \subseteq G'_{U_u}$, and if $y_{\bar{\eta},s+1} \in G'_{U_u}$ use the equation $\bar{x}_{\bar{r},s}$ from 5.2, in the equation $y_{\bar{\eta},s+1} \in G'_{U_u}$ and the $x$’s appearing in the equation belong to $G'_{U_u}$ by the earlier part of the proof (of (*)$_1$) so necessarily $y_{\bar{\eta},s} \in G'_{U_u}$, so we are done.

Proof of (*)$_2$. We rewrite the equations in the new variables recalling that $G_{U_u}$ is generated by the relevant variables freely except the equations of $\bar{x}_{\bar{r},n}$ from Definition 5.2. After rewriting, all the equations disappear.
Case 2: \( U \) is uncountable.

As \( \aleph_1 \leq |U| \leq \aleph_{k(*)-k} \), necessarily \( k < k(*) \).

Let \( U = \{ \rho_\alpha : \alpha < \mu \} \) where \( \mu = |U| \), list \( U \) with no repetitions. Now for each \( \alpha \leq |U| \) let \( U_\alpha := \{ \rho_\beta : \beta < \alpha \} \), \( u_\alpha = u \cup \{ \rho_\alpha \} \). Now

\( \odot_1 \langle (G_{U,u} + G_{U_\alpha \cup u})/G_{U,u} : \alpha < |U| \rangle \) is an increasing continuous sequence of subgroups of \( G/G_{U,u} \)

[Why? By 5.6(6).]

\( \odot_2 G_{U,u} + G_{U_0 \cup u}/G_{U,u} \) is free.

[Why? This is \( (G_{U,u} + G_{0 \cup u})/G_{U,u} = (G_{U,u} + G_u)/G_{U,u} \) which by 5.6(8) is isomorphic to \( G_u/G_\emptyset, u \) which is free by Case 1.]

Hence it suffices to prove that for each \( \alpha < |U| \) the group \( (G_{U,u} + G_{U_\alpha +1 \cup u})/(G_{U,u} + G_{U_\alpha \cup u}) \) is free. But easily

\( \odot_3 \) this group is isomorphic to \( G_{U_\alpha \cup u_\alpha}/G_{U_\alpha, u_\alpha} \).

[Why? By 5.6(7) with \( U_\alpha, U_{\alpha+1}, U, \rho_\alpha, u \) here standing for \( U_1, U_2, U, \eta, u \) there.]

\( \odot_4 G_{U_\alpha \cup u_\alpha}/G_{U_\alpha, u_\alpha} \) is free.

[Why? By the induction hypothesis, as \( \aleph_0 + |U_\alpha| < |U| \leq \aleph_{k(*)-(k+1)} \) and \( |u_\alpha| = k + 1 \leq k(*) \).]

2) If \( k(*) = 0 \) just use 5.4, so assume \( k(*) \geq 1 \). Now the proof is similar to (but easier than) the proof of case (2) inside the proof of part (1) above.

\[ \square \]

5.8 Claim. If \( U \subseteq \omega_2 \) and \( |U| \geq \aleph_{k(*)+1} \) then \( G_U \) is not free.

Proof. Assume toward contradiction that \( G_U \) is free and let \( \chi \) be large enough; for notational simplicity assume \( |U| = \aleph_{k(*)+1} \). O.K. as a subgroup of a free abelian group is a free abelian group. We choose \( N_\ell \) by downward induction on \( \ell \leq k(*) \) such that

(a) \( N_\ell \) is an elementary submodel\(^4\) of \( (\mathcal{H}(\chi), \in, <^*_\chi) \)

(b) \( ||N_\ell|| = |N_\ell \cap \aleph_{k(*)}| = \aleph_\ell \) and \( \aleph_\ell + 1 \subseteq N_\ell \)

(c) \( G_U \in N_\ell \) and \( N_{\ell+1}, \ldots, N_{k(*)} \in N_\ell \).

\(^4\) \( \mathcal{H}(\chi) \) is \( \{ x : \text{the transitive closure of } x \text{ has cardinality } < \chi \} \) and \( <^*_\chi \) is a well ordering of \( \mathcal{H}(\chi) \)
Let \( G_{\ell} = G_U \cap N_{\ell} \), a subgroup of \( G_U \). Now

\((*)_0\) \( G_U / \{ \Sigma \{ G_{\ell} : \ell \leq k(*) \} \} \) is a free (abelian) group

[easy or see [Sh 52], that is:

as \( G_U \) is free we can prove by induction on \( k \leq k(*) + 1 \) then \( G / \{ \Sigma \{ G_{\ell} : \ell < k \} \} \) is free, for \( k = 0 \) this is the assumption toward contradiction, the induction step is by Ax VIII in [Sh 52] for abelian groups and for \( k = k(*) + 1 \) we get the desired conclusion.]

Now

\((*)_1\) letting \( U^1_{\ell} \) be \( U \) for \( \ell = k(*) + 1 \) and \( \bigcap_{m=\ell}^{k(*)} (N_m \cap U) \) for \( \ell \leq k(*) \); we have:

\( U^1_{\ell} \) has cardinality \( \aleph_{\ell} \) for \( \ell \leq k(*) + 1 \)

[Why? By downward induction on \( \ell \). For \( \ell = k(*) + 1 \) this holds by an assumption. For \( \ell = k(*) \) this holds by clause (b). For \( \ell < k(*) \) this holds by the choice of \( N_{\ell} \) as the set \( \bigcap_{m=\ell+1}^{k(*)} (N_m \cap U) \) has cardinality \( \aleph_{\ell+1} \geq \aleph_{\ell} \) and belong to \( N_{\ell} \) and clause (b) above.]

\((*)_2\) \( U^2_{\ell} =: U^1_{\ell+1} \setminus (N_{\ell} \cap U) \) has cardinality \( \aleph_{\ell+1} \) for \( \ell \leq k(*) \)

[Why? As \( |U^1_{\ell+1}| = \aleph_{\ell+1} > \aleph_{\ell} = ||N_{\ell}|| \geq |N_{\ell} \cap U| \).]

\((*)_3\) for \( m < \ell \leq k(*) \) the set \( U^3_{m,\ell} =: U^2_{\ell} \cap \bigcap_{r=m}^{\ell-1} N_r \) has cardinality \( \aleph_m \)

[Why? By downward induction on \( m \). For \( m = \ell - 1 \) as \( U^2_{\ell} \in N_m \) and \( |U^2_{\ell}| = \aleph_{\ell+1} \) and clause (b). For \( m < \ell \) similarly.]

Now for \( \ell = 0 \) choose \( \eta^*_0 \in U^2_{\ell} \), possible by \((*)_2\) above. Then for \( \ell > 0, \ell \leq k(*) \) choose \( \eta^*_\ell \in U^3_{0,\ell} \). This is possible by \((*)_3\). So clearly

\((*)_4\) \( \eta^*_\ell \in U \) and \( \eta^*_\ell \in N_{m} \cap U \leftrightarrow \ell \neq m \) for \( \ell, m \leq k(*) \).

[Why? If \( \ell = 0 \), then by its choice, \( \eta^*_0 \in U^2_{\ell} \), hence by the definition of \( U^2_{\ell} \) in \((*)_2\) we have \( \eta^*_0 \notin N_{\ell} \), and \( \eta^*_\ell \in U^1_{\ell+1} \) hence \( \eta^*_\ell \in N_{\ell+1} \cap \ldots \cap N_{k(*)} \) by \((*)_1\) so \((*)_4\) holds for \( \ell = 0 \). If \( \ell > 0 \) then by its choice, \( \eta^*_\ell \in U^3_{0,\ell} \) but \( U^3_{m,\ell} \subseteq U^2_{\ell} \)

by \((*)_3\) so \( \eta^*_\ell \in U^2_{\ell} \) hence as before \( \eta^*_\ell \in N_{\ell+1} \cap \ldots \cap N_{k(*)} \) and \( \eta^*_\ell \notin N_{\ell} \).

Also by \((*)_3\) we have \( \eta^*_\ell \in \bigcap_{r=0}^{\ell-1} N_r \) so \((*)_4\) really holds.]

Let \( \bar{\eta}^* = \{ \eta^*_\ell : \ell \leq k(*) \} \) and let \( G' \) be the subgroup of \( G_U \) generated by \( \{ x_{m,\bar{\eta},\nu} : m \leq k(*) \) and \( \bar{\eta} \in k(*)+1 \setminus (m) U \) and \( \nu \in \omega > 2 \} \cup \{ y_{\bar{\eta},n} : \bar{\eta} \in k(*)+1 U \) but \( \bar{\eta} \neq \bar{\eta}^* \) and
$n < \omega \}$. Easily $G_\ell \subseteq G'$ recalling $G_\ell = N_\ell \cap G_U$ hence $\Sigma\{G_\ell : \ell \leq k(*)\} \subseteq G'$, but $y_{\bar{\eta}, 0} \notin G'$ hence

\[(*)_5 \ y_{\bar{\eta}, 0} \notin \sum\{G_\ell : \ell \leq k(*)\}.\]

But for every $n$

\[(*)_6 \ \bar{n}! y_{\bar{\eta}, n+1} - y_{\bar{\eta}, n} = \sum\{x_{m, \bar{\eta}^* | (k(*)+1\setminus\{m\}), \eta_{m,n}^* | n : m \leq k(*)\} \in \sum\{G_\ell : \ell \leq k(*)\}.\]

[Why? $x_{m, \bar{\eta}^* | (k(*)+1\setminus\{m\}), \eta_{m,n}^* | n \in G_m$ as $\bar{\eta}^* \upharpoonright (k(*)) + 1\setminus\{m\} \in N_m$ by $(*)_4$.]

We can conclude that in $G_U/\sum\{G_\ell : \ell \leq k(*)\}$, the element $y_{\bar{\eta}, 0} + \sum\{G_\ell : \ell \leq k(*)\}$ is not zero (by $(*)_5$) but is divisible by every natural number by $(*)_6$.

This contradicts $(*)_0$ so we are done. \[\Box_{5.8}\]

5.9 Conclusion. $G_{k(*)}$ is a Borel and even $F_\sigma$ abelian group which is $\aleph_{k(*)+1}$-free but if $2^{\aleph_0} \geq \aleph_{k(*)+1}$ is not free and even not $\aleph_{k(*)+2}$-free.

Proof. $G_{k(*)}$ is Borel and $F_\sigma$ by 5.3, it is $\aleph_{k(*)+1}$-free by 5.7 and if $2^{\aleph_0} \geq \aleph_{k(*)+1}$ it is not $\aleph_{k(*)+2}$-free by 5.8. \[\Box_{5.9}\]

Blass asks

5.10 Question: Suppose (a) + (b) below, does it follow that forcing with $Q$ add reals?

(a) $G$ is a Borel definition of an uncountable abelian subgroup of $\omega\mathbb{Z}$ (the Specker group) which is not free

(b) the forcing $Q$ satisfies $\mathbb{V} \models "G^V$ is free"."

Now

5.11 Fact: For just Borel abelian group $G$: if CH, then the answer to 5.10 is yes, if not CH then the answer is not for $Q = \text{Levy}(\aleph_1, 2^{\aleph_0})$.

Proof. First, assume CH holds and $G$ is as in (a) of 5.10; (or just defined absolutely enough such that $G^V$ is a subgroup of $G^V_Q$ for any forcing notion $Q$ and is still not free). Then by [Sh 402] the group $G^V$ is non-free in some strong way such that no forcing not collapsing $2^{\aleph_0}$ to $\aleph_0$ can make it free (that is, for some countable $G_0 \subseteq G^V$, $G_0^V / G$ contains the direct sum of $2^{\aleph_0}$ finite rank non-free abelian groups).
This is a strong yes answer. On the other hand, if $2^\aleph_0 > \aleph_1$ we can find such group: for $k(*) \geq 1$, our $G_{k(*)}$ if $\aleph_1 < \aleph_{k(*)} + 1 \leq 2^\aleph_0$, see below, is a strong negative answer. So together this gives answers to a question of Blass.

5.12 Corollary. 1) The group $G_{k(*)}$ is embeddable into $\omega\mathbb{Z}$, even purely.

2) Hence forcing which does not add bounded subsets to $\aleph_{k(*)}$ can make it free (i.e. Levy($\aleph_\ell, 2^{\aleph_0}$) if $\ell \leq k(*)$ while if our universe satisfies $2^\aleph_0 > \aleph_{k(*)}$ it is not free there).

Proof. 1) For every $n < \omega$ we define a function $f_n$ from $Y$ to $G_{k(*)}$ where $Y$ is the set of generators of $G_{k(*)}$, i.e.

$$Y = \{y_{\bar{\eta}, n+1} : n < \omega, \bar{\eta} \in k(*)^1(\omega^2)\} \cup \{x_{m, \bar{\eta}_m, \nu} : m \leq k(*),$$

$$\bar{\eta} \in \{\ell : \ell \leq k(*), \ell \neq n\} (\omega^2) \text{ and } \nu \in \omega^2\}.$$  

First define a function $h_n$: for $\eta \in \omega^2, g_n(\eta)$ is a sequence of length $\ell g(\eta)$ and

$$(h_n(\eta))(\ell) = \begin{cases} 
\eta(\ell) & \text{if } \ell < n \land \ell < \ell g(\eta) \\
0 & \text{if } \ell \geq n \land \ell < \ell g(\eta) 
\end{cases}.$$  

For $\bar{\eta} = \langle \eta_\ell : \ell \in u \rangle \in u(\omega^2)$ we let $h_n(\bar{\eta}) = \langle f_n(\eta_\ell) : \ell \in u \rangle$.

Lastly, let

$$f_n(y_{\bar{\eta}, n+1}) = y_{h_n(\bar{\eta}), n+1}$$

$$f_n(x_{m, \bar{\eta}_m, \nu}) = x_{m, h_n(\bar{\eta}_m), h_n(\nu)}.$$  

Does $f_n$ induce a homomorphism from $G_{k(*)}$ into $G_{k(*)}$? For this it is enough to check that for every one of the relations from Definition 5.2, its $f_n$-image is satisfied in $G_{k(*)}$, but this is obvious as it is mapped to another one of the equations in the definition of $G_{k(*)}$: the equation in $\boxtimes_{\eta, m}$ is mapped to the equation in $\boxtimes_{g_n(\bar{\eta}), m}$.

So $f_n$ extends to an endomorphism $\hat{f}_n$ of $G_{k(*)}$. Easily

$\diamond$ if $L \subseteq G_{k(*)}$ is a finite rank subgroup (so free) then for $n$ large enough $\hat{f} \upharpoonright L$ is one to one.
Now the range of $\hat{f}_n$ is clearly countable hence free, say is $\bigoplus_{\ell<\omega} \mathbb{Z} z_{n,\ell}$. Hence for some homomorphisms $g_{n,\ell}$ from $\text{Range}(f_n)$ to $\mathbb{Z}$ for $\ell < \omega$ we have

$$z \in \text{Rang}(\hat{f}_n) \Rightarrow z = \Sigma \{g_{n,\ell}(z) z_{n,\ell} : \ell < \omega\}$$

where $g_{n,\ell}(z) = 0$ for every $\ell$ large enough.

Let $f_{n,\ell} = g_{n,\ell} \circ \hat{f}_n \in \text{Hom}(G_{k(\ast)},\mathbb{Z})$. Those homomorphisms give, by renaming the $f_{n,\ell}$'s, an embedding of $G_{k(\ast)}$ into $\omega \mathbb{Z}$. Looking at the construction, it is a pure one.

2) By 5.7.

\[5.13\] Claim. Assume $\text{MA} + 2^{\aleph_0} > \aleph_2$.

If $k(\ast) > 2$ then $G = G_{k(\ast)}$ is a Whitehead Borel (abelian) group.

Proof. By 5.3 we know that $G_{k(\ast)}$ is a Borel group. Let $\langle \eta_\alpha : \alpha < 2^{\aleph_0} \rangle$ list $\omega 2$ with no repetitions and $\mathcal{U}_\alpha = \{\beta : \beta < \alpha\}$.

So $\langle \mathcal{U}_\alpha : \alpha < 2^{\aleph_0} \rangle$ be $\subseteq$-increasing continuous with union $\omega 2$ such that $\mathcal{U}_0 = \emptyset, |\mathcal{U}_\alpha| \leq |\alpha|$; and let $H_\alpha := G_{\mathcal{U}_\alpha}$, see Definition 5.5(1). So $\langle H_\alpha : \alpha < 2^{\aleph_0} \rangle$ is a $\subseteq$-increasing continuous sequence of subgroups of $G$ with union $G$. For $\alpha < 2^{\aleph_0}$, letting $u_\alpha = \{u_\alpha\}$ recalling Definition 5.5 we have $G_{\mathcal{U}_\alpha \cup u_\alpha} = G_{\mathcal{U}_{\alpha+1}} = H_{\alpha+1}$ and $G_{U_\alpha, u_\alpha} = G_{\mathcal{U}_\alpha} = H_\alpha$, hence $H_{\alpha+1}/H_\alpha = G_{\mathcal{U}_{\alpha+1}/G_{\mathcal{U}_\alpha, u_\alpha}}$ and by 5.7(1) the latter group is $\aleph_2$-free so $H_{\alpha+1}/H_\alpha$ is $\aleph_2$-free. As MA holds and $|H_{\alpha+1}/H_\alpha| < 2^{\aleph_0}$ and $H_{\alpha+1}/H_\alpha$ is $\aleph_2$-free we know that it is a Whitehead group.

As $H_\alpha$ is $\subseteq$-increasing continuous, $H_0 = \{0\}$ and each $H_{\alpha+1}/H_\alpha$ is a Whitehead group, it follows that $U\{H_\alpha : \alpha < 2^{\aleph_0}\}$ is a Whitehead group, which means that $G$ is as required.
§6 Beginning of stability theory

We may consider the dividing line for abelian groups from [Sh 402] and try to generalize it for any simply defined (e.g. \( \Sigma^1_1 \) or Borel) model. We deal with having two possibilities, in the high, complicated side we get a parallel of non \( \aleph_0 \)-stability; in the low side we have a rank. But even for minimal formulas, the example in §5 shows that we are far from being done, still we may be able to say something on the structure.

We may consider also ranks parallel to the ones for superstable theories. Note that there are two kinds of definability we are considering: the model theoretic one and the set theoretic one. See more in [Sh:F562].

6.1 Convention. If not said otherwise, \( \mathfrak{A} \) will be a structure with countable vocabulary and its set of elements is a set of reals.

6.2 Definition. 1) For a structure \( \mathfrak{A} \), an \( \mathfrak{A} \)-formula \( \varphi \) is a formula in the vocabulary of \( \mathfrak{A} \) with finitely many free variables, writing \( \varphi = \varphi(\bar{x}) \) means that \( \bar{x} \) is a finite sequence of variables with no repetitions including the free variables of \( \varphi \). We did not specify the logic; we may assume it is \( \subseteq L_{\omega_1,\omega} \) or even \( L_{\omega_1,\omega}(Q) \) where \( Q \) is the quantifier “there are uncountably many”.

2) \( \Delta \) denotes a set of such formulas and \( \bar{\varphi} \) a pair \( (\varphi_0(\bar{x}), \varphi_1(\bar{x})) \) of formulas so \( \bar{\varphi} \) is a \( \Delta \)-pair if \( \varphi_0, \varphi_1 \in \Delta \).

3) We say \( \varphi \) (or \( \Delta \) or \( \bar{\varphi} \)) is \( \Sigma^1_1 \) (or \( \Sigma^1_2 \) or \( \Delta^1_0 \) (= Borel)) if they are so as set theoretic formulas.

6.3 Definition. 1) We say \( (\mathfrak{A}, \Delta) \) is a \( \Sigma^1_1 \)-candidate when:

(a) \( \mathfrak{A} \) is a \( \Sigma^1_1 \)-model

(b) \( \Delta \) is a countable set of \( \mathfrak{A} \)-formulas which, are in the set theory sense, \( \Sigma^1_1 \) (we identify \( \varphi \) and \( \neg\neg \varphi \)).

We can replace being \( \Sigma^1_1 \) by \( \Sigma^1_2 \), etc., (naturally we need enough absoluteness); if we replace it by \( \Gamma \) we write \( \Gamma \)-candidate. If \( \Gamma \) does not appear we mean it is \( \Sigma^1_1 \) or understood from the content normal.

2) If \( (\mathfrak{A}, \Delta) \) is a candidate we say \( \mathfrak{A} \) is locally \( (\aleph_0, \Delta) \)-stable (or \( (\mathfrak{A}, \Delta) \) is \( \aleph_0 \)-stable), but we may omit “locally”; \textit{when} \( \Delta \) is a countable set of \( \mathfrak{A} \)-formulas and for \( \chi \) large enough and \( x \in H(\chi) \), for every countable \( N \prec (H(\chi), \in, \subset^*) \) to which \( x \) belongs and \( \bar{a} \in m^N \mathfrak{A} \) where \( m < \omega \) the following weak definability condition on \( tp_\Delta(\bar{a}, N \cap \mathfrak{A}, \mathfrak{A}) \) holds:
In Definition 6.3(2), the demand "\(N\) is an \(\mathbb{A}\)" can be omitted.

Moreover for any pregiven \(n < \omega\) such that:

\(\theta \in \Delta\) is \(\mathbb{A}\)-unstable (or \(\mathbb{A}\)-stationary).

For some function \(\varphi \in \mathbb{A}\) for \(\nu \in \omega > 2\) such that:

\(\varphi = (\varphi_0(\bar{x}, \bar{b}), \varphi_1(\bar{x}, \bar{b})) \in \Phi_{m}(\mathbb{A}, \Delta) \cap N\) and \(\ell < 2\) and \(\mathbb{A} \models \varphi_\ell(a, b)\) then \(\ell = c(\varphi)\).

3) We say that \((\mathbb{A}, \Delta)\) is \(\mathcal{N}_0\)-unstable (or \(\mathbb{A}\) is \((\mathcal{N}_0, \Delta)\)-unstable) if: there are \(\bar{a}_\eta \in m\mathbb{A}\) for \(\eta \in \omega^2\) and \(\varphi_{\nu, 0}(\bar{x}, \bar{y}_\nu) \in \Delta\) and \(\varphi_{\nu, 1}(\bar{x}, \bar{y}_\nu) \in \Delta\) and \(\bar{b}_\nu \in \ell g(\bar{y})\mathbb{A}\) for \(\nu \in \omega > 2\) such that:

\(a) \quad \mathbb{A} \models \neg(\exists \bar{x})(\varphi_{\nu, 0}(\bar{x}, \bar{b}_\nu) \& \varphi_{\nu, 1}(\bar{x}, \bar{b}_\nu))\)

\(b) \quad \text{if } \nu < \eta_0, \nu < \eta_1, n = \ell g(\nu)\) and \(\eta_0(n) = 0, \eta_1(n) = 1\) then \(\mathbb{A} \models \varphi_{\nu, 0}(\bar{a}_{\eta_0}, \bar{b}_\nu) \land \varphi_{\nu, 1}(\bar{a}_{\eta_1}, \bar{b}_\nu)\).

There are obvious absoluteness results (for \(\varphi \in \Phi_{m}(\mathbb{A}, \Delta)\), \((\mathbb{A}, \Delta)\) is \(\mathcal{N}_0\)-unstable and stable).

6.4 Observation. 1) If \(\Delta\) is closed under negation then in Definition 6.3(2) we have

\((*)' \quad \text{for some } c \in N \text{ we have: } \varphi(\bar{x}, \bar{y}) \in \Delta \& \bar{b} \in \ell g(\bar{y})\mathbb{A} \& \bar{b} \in N \text{ implies}
\((**)') \quad \mathbb{A} \models \varphi(a, \bar{b}) \iff c(\varphi(\bar{x}, \bar{b})) = 1.\)

2) In Definition 6.3(2) we can fix \(x = (\mathbb{A}, \Delta)\) and omit \(\prec^*\), at the expense of larger \(\chi\).

Proof. Straight.

6.5 The End-Extension Indiscernibility existence lemma. Assume \((\mathbb{A}, \Delta)\) is an \(\mathcal{N}_0\)-stable candidate.

1) In Definition 6.3(2), the demand "\(N\) is countable" can be omitted.

2) Assume \(\Delta\) is closed under negation and permuting the variables, \(m < \omega, \bar{a}_\alpha \in m\mathbb{A}\) for \(\alpha < \lambda\) and \(\mathcal{N}_0 < \lambda = \text{cf}(\lambda)\) and \(S \subseteq \lambda\) is stationary and \(A \subseteq \mathbb{A}\) has cardinality \(\prec^*\). Then for some stationary \(S' \subseteq S\) the sequence \(\langle \bar{a}_\alpha : \alpha \in S' \rangle\) is a \(\Delta\)-end extension indiscernible sequence over \(A\) in \(\mathbb{A}\) (see Definition 6.6(4),(5) below).

3) Moreover for any pregiven \(n < \omega\) we can find stationary \(S' \subseteq S\) such that \(\langle \bar{a}_\alpha : \alpha \in S' \rangle\) is \((\Delta, n)\)-end extension indiscernible over \(A\) in \(\mathbb{A}\).

4) We can find a club \(E\) of \(\lambda\) and regressive function \(f_n\) on \(S \cap E\) for \(n < \omega\) such that:

\[(i) \quad \text{if } \alpha, \beta \in S \cap E \text{ then } f_{n+1}(\alpha) = f_{n+1}(\beta) \Rightarrow f_n(\alpha) = f_n(\beta)\]
(ii) if \( n < \omega \) and \( \gamma < \lambda \), then the sequence \( \langle a_\alpha : \alpha \in S \cap E, f_n(\alpha) = \gamma \rangle \) is \((\Delta, n)\)-end extension indiscernible over \( A \)

(ii) moreover, if \( n < \omega \) and \( \beta, \gamma < \lambda \) then \( \langle a_\alpha : \alpha \in S \cap E \setminus \beta \ & f_n(\alpha) = \gamma \rangle \) is \((\Delta, n)\)-end extension indiscernible over \( A \cup \{a_\gamma : \gamma < \beta \} \).

Remark. Similar to [Sh:c, III.4,23,pg.120-1], but before proving we define:

6.6 Definition. 1) Let \( (\mathfrak{A}, \Delta) \) be a candidate. We say “\( \mathfrak{A} \) has \((\lambda, \Delta)\)-order” when:

\((*)_\lambda \) for some \( m(*) < \omega \) and \( \varphi(\bar{x}, \bar{y}) \in \Phi_{m(*)}^{\mathfrak{A}, \Delta} \) with \( \ell g(\bar{x}) = \ell g(\bar{y}) \) linear orders
some \( J \subseteq m(*)\mathfrak{A} \) of cardinality \( \lambda \), see part (2) for definition.

2) We say \( \bar{\varphi}(\bar{x}, \bar{y}) \) linear orders \( I \subseteq m(\lambda)\mathfrak{A} \) if for some \( \langle a_t : t \in I \rangle \) we have:

(a) \( I = \langle a_t : t \in I \rangle \)
(b) \( I \) is a linear order
(c) \( \bar{\varphi} = (\varphi_0(\bar{x}, \bar{y}), \varphi_1(\bar{x}, \bar{y})) \) and contradictory in \( \mathfrak{A} \)
(d) if \( s < t \) then \( \mathfrak{A} \models \varphi_0(a_s, a_t) \land \varphi_1[\hat{a}_t, \hat{a}_s] \).

3) For a linear order \( I \) (e.g. a set of ordinals), we say \( \langle a_t : t \in J \rangle \) is a \( \Delta \)-end-extension indiscernible (sequence over \( A \)) if for any \( n < \omega \) and \( t_0 < s_0 < \ldots < s_{n-1} < t \), the sequences \( a_{t_0} \ldots a_{t_{n-1}} \hat{a}_{t_{n-1}} \) and \( a_{s_0} \ldots a_{s_{n-1}} \hat{a}_{s_{n-1}} \) realizes the same \( \Delta \)-type (over \( A \)) in \( \mathfrak{A} \).

4) We say that \( \langle a_t : t \in J \rangle \) is \((\Delta, n_0, n_1)\)-end-extension indiscernible over \( A \) in \( \mathfrak{A} \) when:

(a) \( J \) a linear order for some \( m, \bar{a}_t \in m\mathfrak{A}, A \subseteq \mathfrak{A} \)
(b) if \( \langle r_\ell : \ell < n_0 \rangle, \langle s_\ell : \ell < n_1 \rangle, \langle t_\ell : \ell < n_1 \rangle \) are \( <_J \)-increasing sequences, \( r_{n_0} \vdash \hat{a}_{r_0} \ldots \hat{a}_{r_{n_0-1}} \hat{a}_{s_0} \ldots \hat{a}_{s_{n_1-1}} \) and \( a_{t_0} \ldots a_{t_{n_1-1}} \hat{a}_{t_{n_1-1}} \) realizes the same \( \Delta \)-type over \( A \) in \( \mathfrak{A} \)
(c) if \( J \) has a last element we allow to decrease \( n_0 \) and/or \( n_1 \).

5) If we omit \( n_0 \) this means for every \( n_0 \), (so “\( \Delta \)-end extension...” means \((\Delta, 1)\)-end extension.

Proof of 6.5. 1) Let \( N^* < (\mathcal{H}(\chi), \in, <^*_{\chi}) \) be such that \( \mathfrak{A}, \Delta \in N^* \). Now for every countable \( N < N^* \) to which \( (\mathfrak{A}, \Delta) \) belongs there is \( c_N \in N \) as mentioned in the definition 6.4(2). Hence by normality of the club filter on \([N^*]^{\aleph_0}\), the family of countable subsets of \( N^* \), for some \( c^* \) the set \( N = \{N : N < N^* \) is countable and
\( c_N = c^* \) is a stationary subset of \([N^*]^{\aleph_0}\), so \( c^* \) can serve for \( N \).

2) Let \( \langle N_\alpha : \alpha < \lambda \rangle \) be an increasing continuous sequence of elementary submodels of \( (\mathcal{M}(\lambda), \in, \in^\lambda) \) to which \( \mathfrak{A} \) belongs, such that \( \|N_\alpha\| < \lambda \), \( N_\alpha \cap \lambda = \lambda \) and \( \alpha \subseteq N_\alpha \) and \( \langle a_\alpha : \alpha < \lambda \rangle \in N_0 \) (hence \( a_\alpha \in N_{\alpha+1} \)). For each \( \alpha \in S \), applying 6.5(1) to \( N_\alpha, a_\alpha \) we get \( c_\alpha \in N_\alpha \) as in Definition 6.3(2). So for some \( c^* \) and some stationary subsets of \( S' \subseteq S \) of \( \lambda \) we have \( \alpha \in S' \Rightarrow c_\alpha = c^* \). Now \( \Delta \)-end extension indiscernibility follows.

3) We prove this by induction on \( n \):

\[ \mathfrak{I}^n \] for all \( m < \omega \) a stationary \( S \subseteq \lambda, a_\alpha \in m\mathfrak{A} \) for \( \alpha < \lambda \)
there is a stationary \( S' \subseteq S \) such that:
if \( \beta < \lambda, \alpha_\ell \in S', \alpha_\ell'' \in S' \) for \( \ell < n \) and \( \beta \leq \alpha_0' < \alpha_1' < \ldots \) and \( \beta \leq \alpha''_0 < \alpha''_1 < \ldots \) then \( \bar{a}_{\alpha_0'} \ldots \bar{a}_{\alpha_n'} \) realizes the same type over \( A \cup \{\bar{a}_\gamma : \gamma < \beta\} \).

For \( n = 0 \) the demand is empty so \( S' = S \) is as required. For \( n = 1 \) apply part (2). For \( n + 1 > 1 \) by the induction hypothesis we can find stationary \( S_1 \subseteq S \) as required in \( \mathfrak{I}^n \). For each \( \alpha < \lambda \) we can choose \( \beta_{\alpha, \ell} = \beta(\alpha, \ell) \) for \( \ell < n \) such that \( \alpha = \beta_{\alpha,0} < \beta_{\alpha,1} < \ldots < \beta_{\alpha,n} \) and \( 0 < \ell < n \Rightarrow \beta_{\alpha, \ell} \in S_1 \). Let \( \bar{a}_\alpha^* = \bar{a}_{\beta_{\alpha,0}} \ldots \bar{a}_{\beta_{\alpha,n}} \) so \( \bar{a}_\alpha^* \in m(n+1)\mathfrak{A} \) and apply the induction hypothesis to \( m \times (n+1), S_1, \langle \bar{a}_\alpha^* : \alpha < \lambda \rangle \)
getting a stationary \( S_2 \subseteq S_1 \) as required in \( \mathfrak{I}^n \).

We claim that \( S_2 \) is as required. So assume \( \beta \leq \alpha_0' < \ldots < \alpha_n' < \lambda, \beta \) and \( \beta \leq \alpha_0'' < \ldots < \alpha_n'' < \lambda \) and \( \alpha_\ell', \alpha_\ell'' \in S_2 \). Now

(i) \( a_{\alpha_0'} \ldots \bar{a}_{\alpha_n'} \) and \( \bar{a}_{\alpha_0'} \ldots \bar{a}_{\bar{a}_{\beta(\alpha_0',1)}} \ldots \bar{a}_{\beta(\alpha_0',n)} \) realizes the same \( \Delta \)-type over \( A \cup \{\bar{a}_\gamma : \gamma < \beta\} \) in \( \mathfrak{A} \)
[why? as \( \beta(\alpha_0', \ell) \in S_1, \alpha_\ell' \in S_2 \subseteq S_1 \) and the choice of \( S_1 \)]

(ii) \( \bar{a}_{\alpha_0'} \ldots \bar{a}_{\beta(\alpha_0',1)} \ldots \bar{a}_{\beta(\alpha_0',n)} \) is equal to \( \bar{a}_{\alpha_0'} \)
[why? by the choice of \( \bar{a}_{\alpha_0} \)]

(iii) \( \bar{a}_{\alpha_0'}, \bar{a}_{\alpha_0''} \) realizes the same \( \Delta \)-type over \( A \cup \{\bar{a}_\gamma : \gamma < \beta\} \) hence over \( A \cup \{\bar{a}_\gamma : \gamma < \beta\} \)
[why? by the choice of \( S_2 \)].
Similarly

(iv) \( \bar{a}_{\alpha_0'} \) is equal to \( \bar{a}_{\alpha_0'} \ldots \bar{a}_{\beta(\alpha_0',1)} \ldots \bar{a}_{\beta(\alpha_0',n)} \)

(v) \( \bar{a}_{\alpha_0'} \ldots \bar{a}_{\beta(\alpha_0',1)} \ldots \bar{a}_{\beta(\alpha_0',n)} \) and \( \bar{a}_{\alpha_0''} \bar{a}_{\alpha_0''} \ldots \bar{a}_{\alpha_n''} \) realizes the same \( \Delta \)-type over \( A \cup \{\bar{a}_\gamma : \gamma < \beta\} \).

By (i)-(v) the set \( S_2 \) is as required in \( \mathfrak{I}^{n+1} \).

4) The proofs of parts (2), (3) actually give this. \( \square_{6.5} \)
6.7 The order/unstability lemma. Assume that

\( (a, \Delta) \) is a candidate

(b) \( \varphi_0(\bar{x}, \bar{y}), \varphi_1(\bar{x}, \bar{y}) \in \Delta \) are contradictory in \( a \)

(c) \( J \) is a linear order of cardinality \( \lambda \)

(d) \( \lambda \) we have \( \bar{a}_t \in m^a \) for \( t \in J \) satisfies \( a \models \varphi_0[\bar{a}_s, \bar{a}_t] \) & \( \varphi_1[\bar{a}_t, \bar{a}_s] \) whenever \( s < t \)

\( \exists 2 \lambda \geq \aleph_1 \) or \( J \) is uncountable with density \( \mu < |J| \).

Then \( (a, \Delta) \) is \( \aleph_0 \)-unstable; even more specifically the demand in Definition 6.3(3) holds with \( \varphi_{\nu,0} = \varphi_0, \varphi_{\nu,1} = \varphi_1 \).

6.8 Question: What can \( \{ \lambda : a \text{ has a } (\Delta, \lambda)\text{-order} \} \) be?

We first prove a claim from which we can derive the lemma.

6.9 Claim. Assume

(a) \( (a, \Delta) \) is a \( \Sigma^1_{\ell(\ast)} \)-candidate, \( \ell(\ast) \in \{1,2\} \) and \( m < \omega, \Phi = \Phi^m_{(a, \Delta)} \) or just \( \Phi \subseteq \Phi^m_{(a, \Delta)} \)

(b) \( P = \{ P_\alpha : \alpha < \omega_{\ell(\ast)} \} \)

(c) \( P_\alpha \) is a non-empty family of subsets of \( m^a \)

(d) if \( \alpha < \beta < \omega_{\ell(\ast)} \) and \( B \in P_\beta \) then for some \( B_0, B_1 \in P_\alpha \) and pair \( (\varphi_0(\bar{x}, \bar{b}), \varphi_1(\bar{x}, \bar{b})) \in \Phi \) we have \( \ell < 2 \) & \( \bar{a} \in B_\ell \Rightarrow a \models \varphi_\ell[\bar{a}, \bar{b}] \)

(e) if \( B \in P_\beta \) and \( \alpha < \beta < \omega_1 \) and \( F \) is a function with domain \( B \) and countable range, then there is \( B' \in P_\alpha \) such that \( B' \subseteq B \) and \( F \upharpoonright B' \) is constant

(f) if \( \ell(\ast) = 2 \) we then in clause (e), on \( \text{Rang}(F) \) we demand just \( |\text{Rang}(F)| \leq \aleph_1 \).

Then \( (a, \Delta) \) is \( \aleph_0 \)-unstable.

Proof of 6.7 from 6.9.

Let \( \Phi = \{ (\varphi_0(\bar{x}, \bar{y}), \varphi_1(\bar{x}, \bar{y})) \} \) and for \( \alpha < \omega_{\ell(\ast)} \) let

\[ P_\alpha = \{ I : I \subseteq m^a \text{ is linearly ordered by } \bar{\varphi} \text{ and has cardinality } \geq \aleph_\alpha \} \]

This should be clear. \( \square \)
Proof of 6.9. For each $\varphi(\bar{x}) \in \Delta$ as $\{\bar{a} \in \ell g(\bar{x}) \mathfrak{A} : \mathfrak{A} \models \varphi(\bar{a})\}$ is a $\Sigma^1_{\ell(*)}$-set and
let $\{\bar{a} \in \ell g(\bar{x}) \mathfrak{A} : \mathfrak{A} \models \varphi(\bar{a})\} = \{\bar{a} : \text{for some } \alpha < \omega_{\ell(*)} \text{ and } \nu \in \omega, (\bar{a}, \nu) \in C_{\varphi, \alpha}\}$
where for each $\alpha < \omega_{\ell(*)} - 1$ we have $C_{\varphi, \alpha}$ closed subset of $(\ell g(\bar{x}) + 1)(\omega_\nu)$. We can find
$F_0, F_1$ such that if $\varphi(\bar{x}) \in \Delta$ and $\mathfrak{A} \models \varphi(\bar{a})$ then $F^0_{\varphi}(\bar{a}) < \omega_{\ell(*)} - 1$ and $F^1_{\varphi}(\bar{a}) \in \omega$ witnessing this. For notational simplicity and without loss of generality
let $W = \{w : w \subseteq \omega > 2 \text{ is a front hence finite}\}$.

For $w \in W$ and $n < \omega$ let $Q_{n,w}$ be the family of objects $\mathbf{r} = (n, \bar{a}, \bar{u}, \varphi) = (n^I, \Gamma^I, \ldots)$ such that:

\begin{align*}
(*)_{n,w} \text{ for unboundedly many } \alpha < \omega_{\ell(*)} & \text{ we can find witness (or } \alpha\text{-witness) } \mathbf{y} = \\
& = (\langle \bar{a}_\ell : \ell < n \rangle, \langle B_\rho : \rho \in w \rangle) \text{ which means:} \\
(a) & \bar{a}_\ell = \langle (u^0_\rho, u^1_\rho) : \rho \in w \rangle \text{ and } w \Rightarrow u^0_\rho, u^1_\rho \subseteq n \text{ and } \varphi = \langle \varphi^\ell : \ell < n \rangle, \varphi^\ell = \langle \varphi^0_\ell(x, \bar{y}_\ell), \varphi^1_\ell(x, \bar{y}_\ell) \rangle \in \Phi \\
(b) & \bar{a}_\ell \in \ell g(\bar{y}) \mathfrak{A} \\
(c) & B_\rho \in \mathfrak{P}_\alpha \\
(d) & \text{if } \rho \in w, b \in B_\rho \text{ and } \ell < n \text{ then } \langle \varphi^0_\ell(\bar{x}, \bar{y}), \varphi^1_\ell(\bar{x}, \bar{y}) \rangle \in \Phi, \ell g(\bar{x}) = m, \ell g(\bar{y}) \text{ arbitrary (but finite) and } \\
& \ell \in u^0_\rho \Rightarrow \mathfrak{A} \models \varphi^0_\ell[b, a_\ell] \\
& \ell \in u^1_\ell \Rightarrow \mathfrak{A} \models \varphi^1_\ell[b, a_\ell] \\
(e) & \text{if } \nu \neq \rho \text{ are from } w \text{ then } (u^0_\rho \cap u^1_\nu \neq \emptyset) \lor (u^0_\nu \cap u^1_\rho \neq \emptyset) \\
(f) & \bar{v} = \langle v^\ell_\rho : \rho \in w, i \in \{0, 1\} \text{ and } \ell \in u^i_\rho \rangle \\
(g) & \text{if } b \in B_\rho, i \in \{0, 1\}, \ell \in u^i_\rho \text{ then } F^0_{\varphi^i_\ell}(b, a_\ell) = \alpha^i_{\rho, \ell}, (F^1_{\varphi^i_\ell}(b, a_\ell)) \upharpoonright n = \\
& v^i_{\rho, \ell}.
\end{align*}

Clearly

\begin{align*}
(*)_1 & \text{ } Q_{0,\{<>\}} \neq \emptyset. \\
& \text{[Why? Let } \mathbf{r} = (0, \langle <> \rangle, \langle <> \rangle, \langle <> \rangle, \langle <> \rangle) \text{ and if } \alpha < \omega_{\ell(*)} \text{ choose } I \in \mathfrak{P}_\alpha \text{ we let } \mathbf{B}_{<>} = I \\
(*)_2 & \text{ if } \mathbf{r} \in Q_{n,w} \text{ and for } \rho \in w, F^I_\rho \text{ is an } (n + 1)\text{-place function with domain } \mathfrak{A} \\
& \text{and range } \subseteq \omega_{\ell(*)} - 1 \text{ or just countable range, then there is } \langle y^I_\alpha : \alpha < \omega_{\ell(*)} \rangle \\
& \text{such that } y^I_\alpha \text{ is an } \alpha\text{-witness for } \mathbf{r} \in Q_{n,w} \text{ and } \langle F^I_\rho(\bar{a}^\rho, \bar{b}^\rho) : \rho \in w \rangle \text{ is the same for all } b^\rho \in B^\rho_{\varphi^I_\ell}, \alpha < \omega_{\ell(*)} \text{ where } \ell g(\bar{a}) = n \\
& \text{[why? as } \mathbf{r}_1 \in Q_{n,\omega} \text{ we know that for some unbounded } Y \subseteq \omega_{\ell(*)} \text{ for each}
\end{align*}
Together it is not hard to prove the non \( \aleph_0 \)-unstability (as in [Sh 522]).. \( \square \)

6.10 Remark. 1) This claim can be generalized replacing \( \aleph_0 \) by \( \mu \), strong limit singular of cofinality \( \aleph_0 \).

* * *

6.11 Definition. 1) \( \text{tp}_\Delta(\bar{a},A,\mathfrak{A}) = \{\varphi(\bar{x},\bar{b}) : \varphi(\bar{x},\bar{y}) \in \Delta \text{ and } \bar{b} \in \ell g(\bar{y})(A) \text{ and } \mathfrak{A} \models \varphi(\bar{a},\bar{b})\} \)

2) \( \Phi^{pr,m}_{\mathfrak{A},\Delta,A} = \{\langle \varphi_0(\bar{x},\bar{b}),\varphi_1(\bar{x},\bar{b}) \rangle : \varphi_0(\bar{x},\bar{y}),\varphi_1(\bar{x},\bar{y}) \text{ belongs to } \Delta \text{ and } \bar{b} \in \ell g(\bar{y})A \text{ and } \bar{x} = (x_\ell : \ell < m) \text{ and } \mathfrak{A} \models \neg(\exists \bar{x})[\varphi_0(\bar{x},\bar{b}) \& \varphi_1(\bar{x},\bar{b})]\} \)

where \( A \subseteq \mathfrak{A}, \Delta \text{ a set of } \mathfrak{A}\)-formulas, and so

\( \Phi^{pr}_{\mathfrak{A},\Delta} = \{\langle \varphi_0(\bar{x},\bar{y}),\varphi_1(\bar{x},\bar{y}) \rangle : \varphi_0,\varphi_1 \in \Delta, \mathfrak{A} \models \neg(\exists \bar{x})[\varphi_0(\bar{x},\bar{y}) \& \varphi_1(\bar{x},\bar{y})]\} \).

3) \( \text{S}^m_\Delta(A,\mathfrak{A}) = \{\text{tp}_\Delta(\bar{a},A,\mathfrak{A}) : \bar{a} \in m\mathfrak{A}\} \) where \( A \subseteq \mathfrak{A} \) and \( \Delta \text{ a set of } \text{L}(r_\mathfrak{A})\)-formulas

6.12 Definition. 1) We say \( (\mathfrak{A},\Delta) \) is \( (\mu,\Delta,\lambda)\)-unstable if there are \( M \subseteq \mathfrak{A}, m < \omega \) and \( \{\bar{a}_\alpha : \alpha < \lambda\} \) such that:

(a) \( \bar{a}_\alpha \in m\mathfrak{A} \)
(b) if $\alpha \neq \beta$ are $< \lambda$ then for some $(\varphi_0(\bar{x}, \bar{b}), \varphi_1(\bar{x}, \bar{b})) \in \Phi_{m,pr}^{m,pr}$ (see Definition 6.11 below) we have $\varphi_0(\bar{x}, \bar{b}) \in \text{tp}_\Delta(\bar{a}_\alpha, M, \mathfrak{A})$ and $\varphi_1(\bar{x}, \bar{b}) \in \text{tp}_\Delta(\bar{a}_\beta, M, \mathfrak{A})$

(c) $\|M\| \leq \mu$.

1A) Let $\mathfrak{A}$ be $(\aleph_0, \Delta, \text{per})$-unstable mean that $(\mathfrak{A}, \Delta)$ is $\aleph_0$-unstable; here per stands for perfect.

2) We add “weakly” if we weaken clause (b) to

$$(b^-) \text{ tp}_\Delta(\bar{a}_\eta, M, \mathfrak{A}) \neq \text{ tp}_\Delta(\bar{a}_\nu, M, \mathfrak{A}) \text{ for } \eta \neq \nu \text{ from } X$$ (so if $\Delta$ is closed under negation there is no difference); in part (1), $X = \lambda$ and in part (2), $X = \omega$.

3) We use $(\mu_0, \Delta, x, Q)$ where $Q$ is a forcing notion if the example is found in $V^Q$ such that usually $M$ is in $V$ and we add an additional possibility if $x = \text{per}^V$ then $M \in V$ and $X = (\omega^2)^V$ (here per stands for perfect).

4) We may replace “a forcing notion $Q$” by a family $K$ of forcing notions (e.g. the family of c.c.c. ones) meaning: for at least one of them.

5) We replace stable by unstable for the negation.

---

6.13 Observation: 1) If $\Delta$ is closed under negation, then $\mathfrak{A}$ is weakly $(\aleph_0, \Delta, \lambda)$-unstable iff $\mathfrak{A}$ is $(\aleph_0, \Delta, \lambda)$-unstable.

6.14 Definition. Let $(\mathfrak{A}, \Delta)$ be a $\Sigma^1_{\ell(*)}$-candidate where $\ell(*) \in \{1, 2\}$. For $m < \omega$ and $B \subseteq m\mathfrak{A}$ we define $\text{rk}^{\ell(*)}(B) = \text{rk}^{\ell(*)}(B, \Delta, \mathfrak{A})$, an ordinal or infty or $-1$ by defining for any ordinal $\alpha$ when $\text{rk}^{\ell(*)}(B) \geq \alpha$ by induction on $\alpha$.

**Case 1:** $\alpha = 0$.  
$\text{rk}^{\ell(*)}(B) \geq \alpha$ iff $B \neq \emptyset$.

**Case 2:** $\alpha$ limit.  
$\text{rk}^{\ell(*)}(B) \geq \alpha$ iff $\text{rk}^{\ell(*)}(B) \geq \beta$ for every $\beta < \alpha$.

**Case 3:** $\alpha = \beta + 1$.  
$\text{rk}^{\ell(*)}(B) \geq \alpha$ iff (a) + (b) holds where

(a) if $B = \cup\{B_i : i < \kappa_{\ell(*)} - 1\}$ then for some $i$ we have $\text{rk}^{\ell(*)}(B_i) \geq \beta$

(b) we can find $\bar{a}(\bar{x}, \bar{b}) \in \Phi_{m,\Delta}^m$ and $B_0, B_1 \subseteq B$ such that $\text{rk}^{\ell(*)}(B_i) \geq \beta$ and $\bar{a} \in B_{\ell} \Rightarrow \mathfrak{A} \models \varphi_\ell(\bar{a}, \bar{b})$ for $\ell = 0, 1$. 
6.15 Observation: Assume $(\mathfrak{A}, \Delta)$ is $\aleph_{\ell(*)}$-candidate, $\ell(*) \in \{1, 2\}$.

1) If $\alpha \leq \beta$ are ordinals and $\text{rk}^{\ell(*)}(B) \geq \beta$ then $\text{rk}^{\ell(*)}(B) \geq \alpha$.
2) $\text{rk}^{\ell(*)}(B) \in \text{Ord} \cup \{-1, \infty\}$ is well defined (for $B \subseteq m\mathfrak{A}$).
3) If $B_1 \subseteq B_2 \subseteq \mathfrak{A}$ then $\text{rk}^{\ell(*)}(B_1) \leq \text{rk}^{\ell(*)}(B_2)$.

Proof. Trivial.

6.16 Claim. The following are equivalent if $2^{\aleph_0} \geq \aleph_{\ell(*)}$, $(\mathfrak{A}, \Delta)$ is a $\Sigma^1_{\ell(*)}$-candidate:

(a) $\text{rk}^{\ell(*)}(m\mathfrak{A}) \geq \omega_{\ell(*)}$
(b) $\mathfrak{A}$ is $(\aleph_0, \Delta)$-unstable
(c) $\mathfrak{A}$ is $(\aleph_0, \Delta, \aleph_{\ell(*)})$-unstable
(d) $\text{rk}^{\ell(*)}(\mathfrak{A}) = \infty$.

Proof. (a) $\Rightarrow$ (b).
Let $\mathcal{P}_\alpha = \{B \subseteq m\mathfrak{A} : \text{rk}^{\ell(*)}(B) \geq \alpha\}$ and apply 6.9.

(b) $\Rightarrow$ (c): Trivial.

(c) $\Rightarrow$ (d):
Let $A \subseteq \mathfrak{A}$ be countable and $\{\bar{a}_\alpha : \alpha < \aleph_{\ell(*)}\} \subseteq m\mathfrak{A}$ exemplifies that $\mathfrak{A}$ is $(\aleph_0, \Delta, \aleph_{\ell(*)})$-unstable.
Without loss of generality

(*) if $\bar{b} \subseteq A, \varphi(\bar{x}, \bar{y}) \in \Delta$ and $\{\alpha < \aleph_{\ell(*)} : \mathfrak{A} \models \varphi(\bar{a}_\alpha, \bar{b})\}$ is bounded then it is empty.

Now let $\mathcal{P} = \{\{\bar{a}_\alpha : \alpha \in S\} : S \subseteq \aleph_{\ell(*)}$ is unbounded. Now we can prove by induction on $\alpha$ that $B \in \mathcal{P} \Rightarrow \text{rk}^{\ell(*)}(B) \geq \alpha$.\hspace{1cm} \Box_{6.16}

(d) $\Rightarrow$ (a): Trivial.

6.17 Definition. If $p$ is a $(\Delta_1, m)$-type in over $A$ in $\mathfrak{A}$ (i.e. a set of formulas $\varphi(\bar{x}, \bar{a})$ with $\varphi(\bar{x}, \bar{y}) \in \Delta_1, \bar{a} \subseteq A$), we let

$$\text{rk}^{\ell(*)}(p, \Delta, \mathfrak{A}) = \text{Min}\{\text{rk}^{\ell(*)} \cap \text{rk}^{\ell(*)}(m\mathfrak{A}, \bar{b}_\ell, \Delta, \mathfrak{A}) : n < \omega$$

and $\varphi(\bar{x}, \bar{b}_\ell) \in p$ for $\ell < n$.\hspace{1cm}
6.18 Observation 1) If \( p \subseteq q \) (or just \( q \vdash p \)) are \((\Delta, m)\)-types in \( \mathfrak{A} \) then \( \text{rk}^{\ell(*)}(q, \Delta, \mathfrak{A}) \leq \text{rk}^{\ell(*)}(p, \Delta, \mathfrak{A}) \).

2) If \( q \) is a \((\Delta, m)\)-type in \( \mathfrak{A} \) then for some finite \( p \subseteq q \) we have

\[
\text{rk}^{\ell(*)}(q, \Delta, \mathfrak{A}) = \text{rk}^{\ell(*)}(p, \Delta, \mathfrak{A})
\]

hence

\[
p \subseteq r \subseteq q \Rightarrow \text{rk}^{\ell(*)}(r, \Delta, \mathfrak{A}) = \text{rk}^{\ell(*)}(p, \Delta, \mathfrak{A}).
\]

6.19 Claim. 1) In 6.16 we can add

\[
(e) \ (\mathfrak{A}, \Delta) \text{ is not } \aleph_0\text{-stable}
\]

\[
(f) \ for \ some \ \mu < \lambda \ the \ pair \ (\mathfrak{A}, \Delta) \ is \ (\mu, \Delta, \lambda)\text{-unstable and } \aleph_\ell(*) < \lambda.
\]

Proof. \( \neg(e) \Rightarrow \neg(c) \).

Let \( M < \langle \mathcal{H}(\chi), \in, \langle^* \rangle \rangle \) be countable such that \( x \in M \) for suitable \( x \) and \( m < \omega \).

For every \( \bar{a} \in {}^m \mathfrak{A} \) there is a function \( \mathbf{c}_{\bar{a}} \in M \) from \( \Phi^m_{\mathfrak{A}, \Delta} \) to \( \{0, 1\} \) as in Definition 6.3. So if \( \bar{a}_i \in {}^m \mathfrak{A} \) for \( i < \omega_{\ell(*)} \) then for some \( i < j < \omega_{\ell(*)} \) we have \( \mathbf{c}_{\bar{a}_i} = \mathbf{c}_{\bar{a}_j} \) because \( M \) is countable. So clearly \( (c) \) fails \( \bar{v} \).

\( (e) \Rightarrow (c) \).

Fix \( \langle \mathcal{H}(\chi_0), \in, \langle^* \rangle \rangle \) and let

\[
\mathcal{I}_0 = \{ M < \langle \mathcal{H}(\chi_0), \in, \langle^* \rangle \rangle : \mathfrak{A} \in M \text{ and } \|M\| = \aleph_{\ell(*)-1} \text{ and } \omega_{\ell(*)-1} + 1 \subseteq M \}.
\]

For \( m < \omega \) and \( \mathbf{I} \subseteq {}^m \mathfrak{A} \) let \( \mathcal{I} = \mathcal{I}[\mathbf{I}] \) be the family of \( \mathcal{I} \subseteq \mathcal{I}_0 \) such that: we can find \( \langle F_x, \mathbf{c}_x : x \in \mathcal{H}(\chi) \rangle \) (a witness) such that:

\[
(\alpha) \ \mathbf{c}_x : \Phi^m_{\mathfrak{A}, \Delta} \to \{0, 1\}
\]

\[
(\beta) \ F_x : \omega > (\mathcal{H}(\chi)) \to \mathcal{H}(\chi)
\]

\[
(\gamma) \ if \ M \in S \ is \ closed \ under \ F_x \ for \ x \in M \ then \ for \ every \ \bar{a} \in \mathbf{I} \ for \ some \ y \in M, \mathbf{c}_y \ is \ a \ witness \ for \ tp(\bar{a}_M, M \cap \mathfrak{A}, \mathfrak{A}).
\]
Clearly \( \mathcal{J}_1 \) is a normal ideal on \( \mathcal{J}_0 \). Also if \( m < \omega \Rightarrow S_0 \in \mathcal{J}^m[A] \) then increasing \( \chi \) we get the desired result. Toward contradiction assume that \( m < \omega \) and \( \mathcal{J} \notin \mathcal{J}^m[A] \) and let \( \mathcal{P} \) (i.e. \( \mathcal{P}_\alpha = \mathcal{P} \) for \( \alpha < \omega_{\ell(\ast)} \)) be the family of \( \mathbf{I} \subseteq m[A] \) such that \( \mathcal{J}_0 \notin \mathcal{J}_1 \).

We now finish by 6.9 once we prove 

\[ \text{if } \mathbf{I} \in \mathcal{P} \text{ then for some } \varphi(\bar{x}, \bar{b}) \in \Phi_{\mathbf{I}, \Delta}^m \text{ for each } \ell < 2 \text{ the set } I^\ell \varphi(x, \bar{b}) \] 

\[ \{ \bar{a} \in \mathbb{A} : \mathbb{A} \models \varphi(\bar{a}, \bar{b}) \} \] 

belong to \( \mathcal{P} \).

If not, for every \( \varphi(\bar{x}, \bar{b}) \in \Phi_{\mathbf{I}, \Delta}^m \) there is \( \ell = c[\varphi(\bar{x}, \bar{b})] < 2 \) and \( \langle (F^\varphi_x(\bar{x}, \bar{b}), c^\varphi_x(\bar{x}, \bar{b})) : x \in \mathcal{H}(\chi) \rangle \) witnessing \( \mathcal{J}_0 \in \mathcal{J}[\mathbf{I}_\ell^\varphi] \).

Define \((F_y, c_y)\) for \( y \in \mathcal{H}(\chi) \) by: if \( y = \langle x, \varphi(\bar{x}, \bar{b}) \rangle \) then \( F_y = F^\varphi_x(\bar{x}, \bar{b}), c_y = c^\varphi_x(\bar{x}, \bar{b}) \), otherwise \( c \).

Clearly we can find \( M \in \mathcal{J}_0 \) such that

\[ \bigoplus_1 \text{ if } \varphi(\bar{x}, \bar{b}) \in \Phi_{\mathbf{I}, \Delta}^m \cap N \text{ and } x \in M \text{ then } M \text{ is closed under } F^\varphi_x(\bar{x}, \bar{b}) \]

\[ \bigoplus_2 \text{ for some } \bar{a} \in m[A], \text{ no } c_y, y \in M \text{ defines } \text{tp}_\Delta(\bar{a}, M \cap A, A) \].

But \( c \) does it! So we are done.

\[(f) \Rightarrow (d). \]

Like \((c) \Rightarrow (d)\).

\[(c) \Rightarrow (f). \]

Just use \( \mu = \aleph_0 \). \( \square \)

6.20 Claim. Assume that \((\mathbb{A}, \Delta)\) is a \( \Sigma^1_{\ell(\ast)} \)-candidate, \( \ell(\ast) \in \{1, 2\} \) and is \( \mu \)-stable.

For some \( \xi < \omega_1 \) we have: if \( \lambda \geq \mu, m < \omega, A \subseteq \mathbb{A}, |A| \leq \lambda \) and \( \bar{a}_\alpha \in m[A] \) for \( \alpha < \lambda^{+\xi} \) then for some \( S \subseteq \lambda^{+\xi} \) of cardinality \( \lambda \), the sequence \( \langle \bar{a}_\alpha : \alpha \in S \rangle \) is \( \Delta \)-indiscernible over \( A \) in \( \mathbb{A} \).

Remark. See more in [Sh:F562].

Proof. Assume not. For \( \xi < \omega_1 \) let

\[ \mathcal{P}_\xi = \{ \langle \bar{a}_\alpha : \alpha < \lambda^{+\xi} \rangle : \text{for some } \lambda \geq \mu, \text{ for no } S \subseteq \lambda^{+\xi} \text{ of cardinality } \lambda \text{ is } \langle \bar{a}_\alpha : \alpha \in S \rangle \text{ is } \Delta \text{-indiscernible over } A \text{ in } \mathbb{A} \}. \]

The point is:
if \( \lambda^+ \xi \) is regular, \( \xi > 0, A \subseteq \mathfrak{A}, |A| \leq \lambda, a_\alpha \in m \mathfrak{A} \) for \( \alpha < \lambda^+ \xi \) and \( S \subseteq \lambda^+ \xi \) is stationary then (a) or (b) where

(a) for some club \( E \) of \( \lambda, \langle a_\alpha : \alpha \in S \cap E \rangle \) is \( \Delta \)-indiscernible over \( A \) in \( \mathfrak{A} \)

(b) for some \( m < \omega \) and club \( E^*_m \) of \( \lambda^+ \xi \) we have

\[ (b)_m(i) \] \( \langle a_\alpha : \alpha \in S \cap E^*_m \rangle \) is \( (\Delta, m) \)-end extension indiscernible

\[ (ii) \] for no club \( E' \subseteq E^*_m \) of \( \lambda^+ \xi \) is \( \langle a_\alpha : \alpha \in S \cap E \rangle \) a sequence which is \( (\Delta, m + 1) \)-end extension indiscernible.

Clearly clause (a) is impossible by our present assumptions so let \( E^*_m, m \) be as in clause (b). By claim 6.5(4) there is a club \( E \) of \( \lambda \) and \( \langle f_n : n < \omega \rangle \) as there and let \( S^*_\gamma = \{ \alpha \in S : f_{m+1}(\alpha) = \gamma \} \), so \( \alpha > \gamma, \mathcal{P}_{m+1} = \{ \gamma : S^*_\gamma \) is stationary. Without loss of generality \( E^* \subseteq E \) and \( \gamma \notin S_{m+1} \Rightarrow S^*_\gamma = \emptyset \). Without loss of generality \( f_{m+1} \) is as in claim 6.21 below.

So by \( (b)_m(ii) \) clearly \( \Gamma_{m+1} \) is not a singleton (and it cannot be empty), so we clearly have finished. \( \Box \)

6.21 Claim. Let \( A, \langle a_\alpha : \alpha < \lambda \rangle, \langle f_n : n < \omega \rangle \) be as in 6.5(4). Then without loss of generality (possibly shrinking \( E \) and changing the \( f_n \)'s) we can add

\[ (iii) \] if \( m < \omega \) and \( \gamma_1 \neq \gamma_2 \) are in \( \text{Rang}(f_{n+1}) \) but \( f_{n+1}(\alpha_1) = \gamma_1 \land f_{n+1}(\alpha_2) = \gamma_2 \Rightarrow f_n(\gamma_1) = f_n(\gamma_2) \) letting \( S = \{ \alpha : f_n(\alpha) = f_n(\gamma_1) = f_n(\gamma_2) \} \) and \( \beta = \text{Min}(S \cap E \backslash (\gamma_1 + 1) \backslash (\gamma_2 + 1)) \)

then for some formula \( \varphi(\bar{x}_0, \ldots, \bar{x}_n) \) with parameters from \( A \cup \{ a_\gamma : \gamma < \beta \} \) such that:

\[ (*) \] if \( i < 2, \alpha'_0 < \ldots < \alpha'_{n+1} \) are from \( S \cap E(\ell \leq n)(\exists \alpha)(f_n(\alpha) = f_n(\alpha'_\ell) \land f_{n+1}(\alpha) = \gamma_i) \) and \( f(\alpha'_0) = \gamma_i \) then \( \mathfrak{A} \models \varphi(\bar{a}_{\alpha'_0}, \ldots, \bar{a}_{\alpha'_{n+1}}) \iff i = 0 \).

Proof. Easy.
Glossary

§0 Introduction

Theorem 0.1: No Polish group
Thesis 0.2: Polish algebras are large
Question 0.3: What can be Aut(A), A uncountable
Question 0.4: Is there model theory of Polish algebras
Example 0.5: Adding many Cohens
Example 0.6: The complex field, the real field
Conjecture 0.7: There is a dichotomy
Thesis 0.8: Classification theory of such structures exists
Theorem 0.9: There is a $F_\sigma$ abelian groups with complicated categoricity behaviour
Conclusion 0.10: Categoricity can stop at $\aleph_n$
Theorem 0.11: Indiscernibles exist
Definition 0.12: Categoricity
Categoricity Question 0.13: Is there such a classification theory for equational theories
Notation 0.14:
Definition 0.15: group words

§1 Metric groups and metric models

Definition 1.1: metric group, metric semigroups
Notation 1.2: For metric group $M, d_M$ is the metric, $e_M$ the unit, $G_M$ the group
Definition 1.3: specially (metric group), specially $^+$, $\zeta$ is strongly O.K.
Observation 1.4: basic properties
Definition 1.5: automorphism of countable structures, endomorphism semi group, monomorphism semi group
Claim 1.6: the above are separable metric groups semi groups
Definition 1.7: $a$ is a metric algebra; unitary, complete; specially($^+$) unitary; partial

§2 Semi-metric groups: automorphism groups of uncountable structures

Definition 2.1: 1) $G$ is a complete/special metric group.
2) Similarly for semi-group
Discussion 2.2: 1) Note that in 2.1 we do not necessarily have metric groups.
2)-4) Variants.
Definition 2.3: The sequence $\bar{A}$ is an $\omega$-representation of $A$, and related metrics
Claim 2.4: when $A, \bar{A}$ gives a [semi] complete/specially$^+$ metric group of
automorphisms (or semi group of endomorphisms or semi group
of monomorphisms).
Discussion 2.5: On variants of 2.4
Definition 2.6: $A$-beautiful term and some distance functions depending
on a representation
Claim 2.7: beautiful terms induce operations on endomorphism semi-group
Claim 2.8: how nice is the derived metric algebra from auto/endo/mono
semi-groups
Question 2.9: can an uncountable Polish algebra be free for some variety?
Observation 2.10: example answering the question
Remark 2.11: another metric
Definition 2.12: $a = (M, \delta, U)$ is a metric topological algebra
Claim 2.13: sufficient condition for being complete metric topological algebra
Discussion 2.14: 1) Replacing metric by a topology.
2) Replacing automorphisms by expansion to models of a universal Horn theory.

§3 Compactness of metric algebras
The completeness Lemma 3.1: give sufficient conditions for solvability of a set of
equations in a Polish algebra.
Remark 3.2: Explaining 3.1.
Fact 3.3: Recall free group is torsion free with no non-trivial element divisible.
Fact 3.4: Recall another consequence of freeness.
Claim 3.5: sufficient conditions for complete metric algebra to be far from free.
Remark 3.6: On stable variants on the theorems (e.g. $\|M_a\| < \text{cov(meagre)}$ instead
$M$ countable).
Remark 3.7: On variants.

§4 Conclusions
Conclusion 4.1: if $(G, \delta)$ is a complete metric space of density $< |G|$ then $G$ is
similar to free; semi-complete; is enough; not discrete is enough.
Conclusion 4.2: There is no free uncountable Polish group.
Claim 4.3: Strengthening the “non-free” replacing free.
Remark 4.4: On related ranks; this conclusion confirms the complicatedness thesis.

Conclusion 4.5: On $\text{Aut}(A)$.

Claim 4.6: when the proof of [Sh 744] works

§5 Quite free but not free abelian groups

Question 5.1: 1) Is the “freeness of a (definable) abelian group” absolute?

2),3) Variants.

Definition 5.2: of $G_{k(*)}$

Claim 5.4: $G_{k(*)}$ is $\aleph_1$-free

Definition 5.5: 1),2) the subgroups $G_{U}, G_{U,u}$.

3) The set of equations $\xi_{U_1}, \Xi_{U,u}$

Claim 5.6: How $G_{U}$ is generated

Main Claim 5.7: 1) $G_{U\cup u}/G_{U,u}$ is free if $|u| \leq k, |U| \leq \aleph_k$.

2) $G_{U\cup u}/G_{U,u}$ is free if $|U| \leq \aleph_{k(*)-|u|}$

Claim 5.8: $G_{U}$ is not free if $|U| \geq \aleph_{k(*)+1}$

Claim 5.9: $G_{k(*)}$ is a Borel (even $F_{\sigma}$) abelian group

$\rightarrow$

scite{6.6A} undefined

Conclusion 5.10: Does appropriate $\mathbb{Q}$ necessarily add reals

Fact 5.11: Information from [Sh 402].

Corollary 5.12: 1) $G_{k(*)}$ purely embeddable into $\omega\mathbb{Z}$.

2) Forcing making it free.

§6 Beginning of stability theory

Convention 6.1: $\tau_{A}$ countable, members of $\mathfrak{A}$ are reals

Definition 6.2: $\mathfrak{A}$-formula, pairs of formulas and set $\Delta$ of pairs

Definition 6.3: $(\mathfrak{A}, \Delta)$ a candidate, stability

Observation 6.4: Basic facts.

Claim 6.5: From stability to the existence of indiscernibles

Definition 6.6: $\mathfrak{A}$ has $(\lambda, \Delta)$-order

Claim 6.7: From order to unstability

Question 6.8: What can be $\{\lambda : \mathfrak{A} \text{ has } (\Delta, \lambda)-\text{order}\}$?

Claim 6.9: Sufficient conditions for being unstable (i.e. having a perfect set of
pairwise explicitly contradictory type)

Remark 6.10: Replacing $\aleph$ by, e.g. $\beth_\omega$

Comment: nonstable is unstable

Definition 6.11: $tp_\Delta(\bar{a}, A, \mathfrak{A})$, $\Phi^{pr,m}_{\kappa,\Delta,A}$ and $S^m_\Delta(A, \mathfrak{A})$

Definition 6.12: $(A, \Delta)$ is $(\mu, \Delta, \lambda)$-unstable

Observation 6.13: Weakly stable/unstable

Definition 6.14: of $rk(B)$

Observation 6.15: Properties of $rk$

Claim 6.16: Equivalences to rank being infinite

Definition 6.17: $rk$ in more cases

Subclaim 6.18: properties of $rk$

Claim 6.19: More cases of equivalence in 6.16

Claim 6.20: Existence of indiscernible

Claim 6.21: helping 6.20
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