Non-Renormalization Theorems in Non-Renormalizable Theories

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Abstract

A perturbative non-renormalization theorem is presented that applies to general supersymmetric theories, including non-renormalizable theories in which the $\int d^2\theta$ integrand is an arbitrary gauge-invariant function $F(\Phi, W)$ of the chiral superfields $\Phi$ and gauge field-strength superfields $W$, and the $\int d^4\theta$-integrand is restricted only by gauge invariance. In the Wilsonian Lagrangian, $F(\Phi, W)$ is unrenormalized except for the one-loop renormalization of the gauge coupling parameter, and Fayet–Iliopoulos terms can be renormalized only by one-loop graphs, which cancel if the sum of the $U(1)$ charges of the chiral superfields vanishes. One consequence of this theorem is that in non-renormalizable as well as renormalizable theories, in the absence of Fayet–Iliopoulos terms supersymmetry will be unbroken to all orders if the bare superpotential has a stationary point.

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The remarkable absence of various radiative corrections in supersymmetric theories was first shown using supergraph techniques.\textsuperscript{1} Later Seiberg introduced a simple and powerful new approach to this problem,\textsuperscript{2} and used it to prove non-renormalization theorems in various special cases. This paper will use a generalized version of the Seiberg approach to give a proof of the perturbative non-renormalization theorems, that applies not only to the usual renormalizable theories but also to general non-renormalizable theories. These have a much richer set of couplings, involving terms of arbitrary order in the gauge superfield, that are shown to be unrenormalized.

This is of some importance in model building. As Witten\textsuperscript{3} pointed out long ago, it is the perturbative non-renormalization theorems that, by limiting the radiative corrections responsible for supersymmetry breaking to exponentially small non-perturbative terms, offer a hope that supersymmetry might solve the hierarchy problem. But the theory with which we have deal below the Planck or grand-unification scales is surely an effective quantum field theory, which contains non-renormalizable as well as renormalizable terms. Thus, in relying on supersymmetry to solve the hierarchy problem, we had better make sure that the non-renormalization theorems apply to non-renormalizable as well as to renormalizable theories.

We consider a general supersymmetric theory involving left-chiral superfields $\Phi_n$, their right-chiral adjoints $\Phi^*_n$, the matrix gauge superfield $V$, and their derivatives. The general supersymmetric action has a Lagrangian den-
\[ L = \int d^2 \theta_L d^2 \theta_R \left[ (\Phi^\dagger e^{-V} \Phi) + G(\Phi, \Phi^\dagger, V, \mathcal{D} \cdots) \right] + 2 \text{Re} \int d^2 \theta_L \left[ \frac{\tau}{8\pi i} \sum_{\alpha\beta} \epsilon_{\alpha\beta} \text{Tr} W_\alpha W_\beta + F(\Phi, W) \right], \] (1)

where \( G \) and \( F \) are general gauge-invariant functions of the arguments shown; ‘\( \mathcal{D} \cdots \)’ denotes a dependence of \( G \) on superderivatives (or spacetime derivatives) of the other arguments; \( W_\alpha \) is the usual gauge-covariant matrix left-chiral gauge superfield formed from \( V \); \( \alpha \) and \( \beta \) are two-component spinor indices with \( \epsilon_{\alpha\beta} \) antisymmetric; and \( \tau \) is the usual complex gauge coupling parameter

\[ \tau = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi}. \] (2)

The function \( F(\Phi, W) \) must be holomorphic in the left-chiral superfields \( \Phi_n \) and \( W \), and may not depend on their superderivatives or spacetime derivatives because, as well known, any term in \( F(\Phi, W) \) that did depend on these derivatives could be replaced with a gauge-invariant contribution to \( G \). The terms \( (\Phi^\dagger e^{-V} \Phi) \) and \( (\tau/8\pi i) \text{Tr} (W^T e W) \) in Eq. (1) could have been included in \( G \) and \( F \), respectively; they are displayed here to identify the zeroth-order kinematic terms that serve as a starting point for perturbation theory.

The first perturbative non-renormalization theorem to be proved here states that the ‘Wilsonian’ effective Lagrangian density \( L_\lambda \) (which, with an ultraviolet cut-off \( \lambda \), yields the same results as the original Lagrangian density
(1)) takes the form
\[ \mathcal{L}_\lambda = \int d^2\theta_L d^2\theta_R \left[ (\Phi^i e^{-V} \Phi) + G_\lambda (\Phi, \Phi^\dagger, V, \mathcal{D} \ldots) \right] + 2 \Re \int d^2\theta_L \left[ \frac{\tau_\lambda}{8\pi i} \sum_{\alpha\beta} \epsilon_{\alpha\beta} \text{Tr} W_\alpha W_\beta + F_\lambda (\Phi, W) \right], \tag{3} \]
where \( G_\lambda \) is some new function of the displayed variables, and \( \tau_\lambda \) is a one-loop renormalized coupling parameter. For instance, for a simple gauge group we have
\[ \tau_\lambda = \tau + i \left( \frac{3C_1 - C_2}{2\pi} \right) \ln(\lambda/\Lambda). \tag{4} \]
Here \( \Lambda \) is an integration constant, and \( C_1 \) and \( C_2 \) are the Casimir constants of the gauge and left-chiral superfields, defined by
\[ \sum_{CD} C_{ACD} C_{BCD} = C_1 \delta_{AB}, \quad \text{Tr} \{ t_A t_B \} = C_2 \delta_{AB}, \tag{5} \]
where \( C_{ABC} \) are the structure constants, and \( t_A \) are the matrices representing the gauge algebra on the left-chiral superfields. Not only is the superpotential \( F(\Phi, 0) \) not renormalized — the whole \( W \)-dependent integrand of the \( \int d^2\theta_L \) integral is not renormalized, except for a one-loop renormalization of the gauge coupling constant.

Here is the proof. Assuming that the cut-off respects supersymmetry and gauge invariance, these symmetries require the Wilsonian effective Lagrangian to take the same general form as Eq. (1):
\[ \mathcal{L}_\lambda = \int d^2\theta_L d^2\theta_R \left[ (\Phi^i e^{-V} \Phi) + G_\lambda (\Phi, \Phi^\dagger, V, \mathcal{D} \ldots) \right] + 2 \Re \int d^2\theta_L \left[ \frac{\tau}{8\pi i} \sum_{\alpha\beta} \epsilon_{\alpha\beta} \text{Tr} W_\alpha W_\beta + F_\lambda (\Phi, W) \right], \tag{6} \]
where $G_\lambda$ and $F_\lambda$ are again general gauge-invariant functions of the arguments shown, with $G_\lambda$ Hermitian. (Since the functions $F_\lambda$ and $G_\lambda$ are not yet otherwise restricted, there is no loss of generality in extracting the terms shown explicitly in Eq. (6) from them.) Following Seiberg,\textsuperscript{2} we regard $\tau$ and the coefficients $f_r$ of the various terms in $F(\Phi, W)$ as independent external left-chiral superfields that happen to have constant scalar components and no spinor or auxiliary components, and that should also appear among the arguments of $F_\lambda$ and (along with their adjoints) in $G_\lambda$.

To deal with the non-renormalizable part of the $\int d^4\theta$ integral, we now also regard the coefficients of the various terms in the real function $G(\Phi, \Phi^\dagger, V, D \cdots)$ as \textit{real} external superfields, which also have only constant scalar components. It is tempting to say that because these real superfields are non-chiral, they cannot appear in the integrand of the $\int d^2\theta_L$ integral in $L_\lambda$, so that $F_\lambda$ may be analyzed as if $G$ were not present. This would be too hasty, because any real superfield $P$ with a positive scalar component may be expressed in terms of a left-chiral superfield $Z$, as

$$P = Z^* Z \exp(V_P) ,$$  \hfill (7)

where $V_P$ has the form of a $U(1)$ gauge superfield in any fixed gauge. (Note that $\ln P \to \ln P - \ln Z - \ln Z^*$ is a generalized gauge transformation.) Eq. (7) is invariant under a phase transformation $Z \to Z e^{i\alpha}$, which if unbroken would prevent the left-chiral superfield $Z$ from appearing in $F_\lambda$. This symmetry is actually violated by non-perturbative effects. For instance, if we modify the
usual renormalizable kinematic term for a multiplet of left-chiral superfields to read \[ \int d^2\theta_L d^2\theta_R Z^* Z (\Phi^* e^{-V} \Phi) \], then since this depends only on \( Z\Phi \), the transformation \( Z \rightarrow Ze^{i\alpha} \), \( \Phi \rightarrow \Phi \) has the same anomaly as the transformation \( \Phi \rightarrow \Phi e^{i\alpha} \), \( Z \rightarrow Z \). This anomaly leads to a breakdown of this symmetry, which allows \( Z \) to appear in non-perturbative corrections to the superpotential. Indeed, if it were not for this breakdown of the symmetry under \( Z \rightarrow Ze^{i\alpha} \), there could be no non-perturbative corrections to the superpotential, because the kinematic term is invariant under a non-anomalous transformation \( Z \rightarrow Ze^{i\beta} \), \( \Phi \rightarrow \Phi e^{-i\beta} \), which would prevent the generation of a non-perturbative term in the superpotential that depends on \( \Phi \) but not \( Z \). Here we are considering only perturbation theory, so \( Z \) cannot appear in \( F_\lambda \), and by the same reasoning neither can any of the real superfields that appear as coefficients of the terms in \( G \). Thus \( G \) can have no effect on \( F_\lambda \).

With \( G \) ignored, the perturbation theory based on the Lagrangian density (1) has two symmetries that restrict the dependence of \( F_\lambda \) on \( \tau \) and on the parameters \( f_r \) in \( F(\Phi,W) \). The first symmetry is conservation of an \( R \) quantum number: \( \theta_L \) and \( \theta_R \) have \( R = +1 \) and \( R = -1 \); \( \tau \) and the \( \Phi_m \) and \( V \) have \( R = 0 \); and the coefficients \( f_r \) of all terms in \( F(\Phi,W) \) with \( r \) factors of \( W_\alpha \) have \( R = 2 - r \). (Since \( W_\alpha \) involves two superderivatives of \( V \) with respect to \( \theta_R \) and one with respect to \( \theta_L \), it has \( R = +1 \). Also, in accordance with the usual rules for integration over Grassman parameters, integration of a function over \( \theta_L \) lowers its \( R \) value by two units.) This symmetry requires
the function $F_{\lambda}(\Phi, W, f, \tau)$ to have $R = 2$, like $F(\Phi, W)$. The other symmetry is invariance under translation of $\tau$ by an arbitrary real number $\xi$:

$$\tau \rightarrow \tau + \xi ,$$

which leaves the action invariant because $\text{Im} \int d^2 \theta_L \sum_{\alpha \beta} \epsilon_{\alpha \beta} \text{Tr} W_\alpha W_\beta$ is a spacetime derivative. This tells us that $F_{\lambda}$ is independent of $\tau$, except for a possible term proportional to the $W W$ term in $F$:

$$F_{\lambda}(\Phi, W, f, \tau) = c_{\lambda} \tau \sum_{\alpha \beta} \epsilon_{\alpha \beta} \text{Tr} W_\alpha W_\beta + H_{\lambda}(\Phi, W, f) ,$$

with $c_{\lambda}$ a real function of $\lambda$.

To use this information, let us consider how many powers of $\tau$ are contributed to $F_{\lambda}$ by each graph. Suppose a graph has $E_V$ external left-handed gaugino lines and any number of external $\Phi$-lines; $I_V$ internal $V$-lines and any number of internal $\Phi$-lines; $A_m$ pure gauge vertices with $m \geq 3$ $V$-lines, arising from the $W W$ term in Eq. (1); $B_{mr}$ vertices with $m \geq r$ $V$-lines and any number of $\Phi$-lines, arising from the terms in $F(\Phi, W)$ with $r$ factors of $W$; and $C_m$ vertices with two $\Phi$-lines and $m \geq 1$ $V$-lines, arising from the $\Phi^\dagger e^{-V} \Phi$ term in Eq. (1). (By ‘$\Phi$-lines’ and ‘$V$-lines’ are meant lines of the component fields of left-chiral or gauge superfields, respectively; these are ordinary Feynman graphs, not supergraphs.) These numbers are related by

$$2I_V + E_V = \sum_{m \geq 3} m A_m + \sum_r \sum_{m \geq r} m B_{mr} + \sum_{m \geq 1} m C_m .$$

Also, since we have specified that all external $V$ lines are for gauginos, this graph can only contribute to a term in $\mathcal{L}_{\lambda}$ with $E_V$ factors of $W$, so it must
be proportional to a product of \( f_r \) factors with total \( R \)-value \( 2 - E_V \), and so

\[
\sum_r \sum_{m \geq r} (2 - r)B_{mr} = 2 - E_V.
\]  \hspace{1cm} (11)

Using this to eliminate \( E_V \) in Eq. (10), the number of factors of \( \tau \) contributed by such a graph is then

\[
N_\tau = \sum_{m \geq 3} A_m - I_V
\]

\[= 1 - \frac{1}{2} \left[ \sum_{m \geq 3} (m - 2)A_m + \sum_r \sum_{m \geq r} (2 - r + m)B_{mr} + \sum_{m \geq 1} mC_m \right]. \hspace{1cm} (12)
\]

Each of the \( A_s, B_s, \) and \( C_s \) in the square brackets in Eq. (12) has a positive-definite coefficient, so there is a limit to the number of vertices of each type that can contribute to the \( \tau \)-independent function \( H_\lambda \). To have \( N_\tau = 0 \), we can have \( A_3 = 2 \) and all other \( A_s, B_s, \) and \( C_s \) zero, or \( A_4 = 1 \) and all other \( A_s, B_s, \) and \( C_s \) zero, which give the one-gauge-loop contributions proportional to \( C_1 \) in Eq. (4); or \( B_{mr} = 1 \) for some \( r \) and \( m = r \), and all other \( A_s, B_s, \) and \( C_s \) zero, which add up to the one-vertex tree contribution \( F(\Phi, W) \) to the function \( H_\lambda(\Phi, W) \) in Eq. (9); or \( C_1 = 2 \) and all other \( A_s, B_s, \) and \( C_s \) zero, or \( C_2 = 2 \) and all other \( A_s, B_s, \) and \( C_s \) zero, which give the one-\( \Phi \)-loop contributions proportional to \( C_2 \) in Eq. (4). (Graphs with \( A_3 = C_1 = 1 \) and all other \( A_s, B_s, \) and \( C_s \) zero are one-particle-reducible, and therefore do not contribute to \( L_\lambda \).) Finally, Eq. (12) shows that there are no graphs at all with \( N_\tau = 1 \), so the constant \( c_\lambda \) in Eq. (9) vanishes, completing the proof of the non-renormalization theorem (3).
In theories where the gauge group has a $U(1)$ factor, there is also one term in $G_\lambda$ which is subject to a non-renormalization theorem. As pointed out by Fayet and Iliopoulos, although a $U(1)$ gauge superfield $V_1$ is not gauge invariant, $\int d^4x \int d^4\theta V_1$ is gauge invariant as well as supersymmetric, so we can include a term $\xi V_1$ in $G$. By detailed calculation, Fischler et al. showed that the constant $\xi$ receives corrections for renormalizable theories only from one-loop diagrams, and that these corrections vanish if the sum of the $U(1)$ charges of the left-chiral superfields vanish. Using the Seiberg trick of regarding coupling parameters as the scalar components of external superfields, it is easy to give a very simple proof of this result, which applies also in non-renormalizable theories, and even non-perturbatively. The point is, that a term $\int d^4x \int d^4\theta S V_1$ in $G_\lambda$ is not gauge-invariant if $S$ depends in a non-trivial way on any superfields, including the external superfields $\tau$ or $f$, or those appearing as coefficients of the non-renormalizable terms in $G$. There is just one graph that can make a correction to $\xi$ that is independent of all coupling constants: it is the one-loop tadpole graph, in which an external $V_1$ line interacts with left-chiral superfields through the term $(\Phi^\dagger \exp(-V)\Phi)$ in Eq. (1). This graph vanishes if the sum of the $U(1)$ charges of the left-chiral superfields vanish. This condition is necessary (unless the $U(1)$ gauge symmetry is spontaneously broken) to avoid gravitational anomalies that would violate the conservation of the $U(1)$ current.

What good are the non-renormalization theorems, when so little is known
about the structure of the function $G_\lambda(\Phi, \Phi^\dagger, V, D \cdots)$? Fortunately, in the absence of Fayet–Iliopoulos terms, it turns out that only the bare superpotential matters in deciding if supersymmetry is spontaneously broken: if the superpotential $F(\Phi, 0)$ allows solutions of the equations $\partial F(\Phi, 0)/\partial \Phi_n = 0$ then supersymmetry is not broken in any finite order of perturbation theory.

To test this, we must examine Lorentz-invariant field configurations, in which the $\Phi_n$ have only constant scalar components $\phi_n$ and constant auxiliary auxiliary components $F_n$, while (in Wess–Zumino gauge) the coefficients $V_A$ of the gauge generators $t_A$ in the matrix gauge superfield $V$ have only auxiliary components $D_A$. Supersymmetry is unbroken if there are values of $\phi_n$ for which $\mathcal{L}_\lambda$ has no terms of first order in $F_n$ or $D_A$, in which case there is sure to be an equilibrium solution with $F_n = D_A = 0$. In the absence of Fayet–Iliopoulos terms, this requires that for all $A$

$$\sum_{nm} \frac{\partial K_\lambda(\phi, \phi^*)}{\partial \phi_n^*} (t_A)_{mn} \phi_m^* = 0,$$

and for all $n$

$$\frac{\partial F(\phi, 0)}{\partial \phi_n} = 0,$$

where the effective Kahler potential $K_\lambda(\phi, \phi^*)$ is

$$K(\phi, \phi^*) = \left(\phi^\dagger \phi\right) + G_\lambda(\phi, \phi^*, 0, 0 \cdots)$$

with $G_\lambda(\phi, \phi^*, 0, 0 \cdots)$ obtained from $G_\lambda$ by setting the gauge superfield and all superderivatives equal to zero. (With superderivatives required to vanish by Lorentz invariance, the only dependence of $G_\lambda$ on $V$ is a factor $\exp(-V)$
following every factor $\Phi^\dagger$.) If there is any solution $\phi^{(0)}$ of Eq. (14), then the
gauge symmetry tells us that there is a continuum of such solutions, with $\phi_n$
replaced with
\[
\phi_n(\xi) = \left[\exp\left(i \sum_A t_A \xi_A\right)\right]_{nm} \phi_m^{(0)}
\]  
(16)
where (since $F$ depends only on $\phi$, not $\phi^*$), the $\xi_A$ are an arbitrary set of
complex parameters. If $K_\lambda(\phi, \phi^*)$ has a stationary point anywhere on the
surface $\phi = \phi(\xi)$, then at that point
\[
0 = \sum_{nm} \frac{\partial K_\lambda(\phi, \phi^*)}{\partial \phi_n} (t_A)_{nm} \phi_m \delta \xi_A
- \sum_{nm} \frac{\partial K_\lambda(\phi, \phi^*)}{\partial \phi^*_n} (t_A)_{nm} \phi_m^* \delta \xi^*_A.
\]  
(17)
Since this must be satisfied for all infinitesimal complex $\delta \xi_A$, the coefficients of
both $\delta \xi_A$ and $\delta \xi^*_A$ must both vanish, and therefore Eq. (13) as well as Eq. (14)
is satisfied at this point. Thus the existence of a stationary point of $K_\lambda(\phi, \phi^*)$
on the surface $\phi = \phi(\xi)$ would imply that supersymmetry is unbroken to
all orders of perturbation theory. The zeroth-order Kahler potential $(\phi^\dagger \phi)$
is bounded below and goes to infinity as $\phi \to \infty$, so it certainly has a
minimum on the surface $\phi = \phi(\xi)$, where of course it is stationary. At this
minimum there are flat directions: ordinary global gauge transformations
$\delta \phi = i \sum_A \delta \xi_A t_A \phi$ with $\xi_A$ real. But these are also flat directions for the
perturbation $G_\lambda(\phi, \phi^*, 0, 0)$. Thus there is still a local minimum of $K_\lambda$ on the
surface $\phi = \phi(\xi)$ for any perturbation $G_\lambda(\phi, \phi^*, 0, 0)$ in at least a finite range,
and thus to all orders in whatever couplings appear in $G_\lambda(\phi, \phi^*, 0, 0)$. We see
that in such a theory supersymmetry can only be broken by non-perturbative
effects, which can naturally lead to a solution of the hierarchy problem.
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