On approximation of Ginzburg-Landau minimizers by $\mathbb{S}^1$-valued maps in domains with vanishingly small holes

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Abstract: We consider a two-dimensional Ginzburg-Landau problem on an arbitrary domain with a finite number of vanishingly small circular holes. A special choice of scaling relation between the material and geometric parameters (Ginzburg-Landau parameter vs hole radius) is motivated by a recently discovered phenomenon of vortex phase separation in superconducting composites. We show that, for each hole, the degrees of minimizers of the Ginzburg-Landau problems in the classes of $\mathbb{S}^1$-valued and $C^2$-valued maps, respectively, are the same. The presence of two parameters that are widely separated on a logarithmic scale constitutes the principal difficulty of the analysis that is based on energy decomposition techniques.

1 Introduction

The present study is motivated by the pinning phenomenon in type-II superconducting composites. Type-II superconductors are characterized by vanishing resistivity and complete expulsion of magnetic fields from the bulk of the material at sufficiently low temperatures. When the magnitude $h_{\text{ext}}$ of an external magnetic field exceeds a certain threshold, the field begins to penetrate the superconductor along isolated vortex lines that may move, resulting in energy dissipation. This motion and related energy losses can be inhibited by pinning the lines to impurities or holes in a superconducting composite. Understanding the role of imperfections in a superconductor can thus be used to design more efficient superconducting materials. In what follows, we will consider a cylindrical superconducting sample containing rod-like inclusions or columnar defects elongated along the axis of the cylinder, so that the sample can be represented by its cross-section $\Omega \subset \mathbb{R}^2$.

Then the vortex lines penetrate each cross-section at isolated points, called vortices.

Superconductivity is typically modeled within the framework of the Ginzburg-Landau theory [11] in terms of an order-parameter $u \in \mathbb{C}$ and the vector potential of the induced magnetic field $A \in \mathbb{R}^2$. The appearance and behavior of vortices for the minimizers of the Ginzburg-Landau functional

$$GL^\varepsilon[u, A] = \frac{1}{2} \int_{\Omega} |(\nabla - iA)u|^2 \, dx + \frac{1}{4\varepsilon^2} \int_{\Omega} (1 - |u|^2)^2 \, dx + \frac{1}{2} \int_{\Omega} (\text{curl} A - h_{\text{ext}})^2 \, dx$$

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have been studied, in particular, in [16, 18] where the existence of two critical magnetic fields, $H_{c1}$ and $H_{c2}$, was established rigorously for simply-connected domain when $\varepsilon > 0$ is small. When the external magnetic field is weak ($h_{\text{ext}} < H_{c1}$) it is completely expelled from the bulk semiconductor (Meissner effect) and there are no vortices. When the field strength is ramped up from $H_{c1}$ to $H_{c2}$, the magnetic field penetrates the superconductor through an increasing number of isolated vortices while the superconductivity is destroyed everywhere, once the field exceeds $H_{c2}$.

The pinning phenomenon that we consider in this paper is observed in non-simply-connected domains with holes that may or may not contain another material. If a hole "pins" a vortex, the order parameter $u$ has a nonzero winding number on the boundary of the hole. We refer to this object as a hole vortex. Note that degrees of the hole vortices increase along with the strength of the external magnetic field. This situation is in contrast with the regular bulk vortices that have degree $\pm 1$ and increase in number as the field becomes stronger.

An alternative way to model the impurities is to consider a potential term $(a(x) - |u|^2)^2$ where $a(x)$ varies throughout the sample. It was proven in [9] that the impurities corresponding to the weakest superconductivity (where $a(x)$ is minimal) pin the vortices first. This model was studied further in [1] and [4] to demonstrate the existence of nontrivial pinning patterns and in [2] to investigate the breakdown of pinning in an increasing external magnetic field, among other issues. A composite consisting of two superconducting samples with different critical temperatures was considered in [5, 14] where nucleation of vortices near the interface was shown to occur.

In our model we consider a superconductor with holes, similar to the setup in [3]. In that work, the authors considered the asymptotic limits of minimizers of $GL^\varepsilon$ as $\varepsilon \to 0$ and determined that holes act as pinning sites gaining nonzero degree for moderate but bounded magnetic fields. For magnetic fields below the threshold of order $|\ln \varepsilon|$ the degree of the order parameter on the holes continues to grow without bound, however beyond the critical field strength, the pinning breaks down and vortices appear in the interior of the superconductor. Since the contribution to the energy from the hole vortices has a logarithmic dependence on the diameter of the holes, the hole size can be used as an additional small parameter to enforce a finite degree of the hole vortex in the limit of small $\varepsilon$. The domain with finitely many shrinking (pinning) subdomains with weakened superconductivity was considered in [10] in the case of the simplified Ginzburg-Landau functional. The model with a potential term $(a(x) - |u|^2)^2$ with piecewise constant $a(x)$ was used to enforce pinning and it was observed that the vortices are localized within pinning domains and converge to their centers.

The problem considered in this work was inspired by the result in [6] where a periodic lattice of vanishingly small holes was considered. The main interest was in the regime when the radii of the holes were exponentially small compared to the period $a$ of the lattice; both of these parameters were assumed to converge to zero along with $\varepsilon$. Using homogenization-type arguments, it was shown in [6] that in the limit of $\varepsilon \to 0$ and when the external magnetic field of order $O(a^{-2})$, the minimizers can be characterized by nested subdomains of constant vorticity. The physical nature of this result was discussed in [12]. The analysis in [6] relies on a conjecture that for small $\varepsilon$, the degrees of the hole vortices are the same for both $C$- and $S^1$-valued maps. The principal aim of the present paper is to establish the validity of this conjecture in the case of finitely many vanishingly small holes.

Our approach builds on that of [3], combined with the appropriately chosen lower bounds on the energy and the ball construction method [7], [13], [15]-[19].

The paper is organized as follows. Section 2 contains the formulation of the problem, as well as the main result described in Theorem 1. In Section 3, we prove that the minimizers in the class
of $S^1$-valued maps are characterized by the unique set of integer degrees on the holes. In Section 4, we use the approach, similar to that in [3], to express the energy of a $\mathbb{C}$-valued minimizer as the sum of the energy of the $S^1$-valued minimizer and the remainder terms. Compared to [3], additional complications arise in the analysis due to the fact that the radius of the holes is not fixed in the present work. In particular, because of the presence of another small parameter, we use a different ball construction method that incorporates both the Ginzburg-Landau parameter $\varepsilon$ and the holes radius $\delta$. In Section 5 we show that the minimizes cannot have vortices with nonzero degrees outside of the holes. This section also provides sharp energy estimates that allow us to prove the main theorem. Finally, in Section 6, the equality of degrees is established based on the estimates obtained in the previous section.

2 Main results

Let $B(x_0, R) \subset \mathbb{R}^2$ denote a disk of radius $R$ centered at $x_0$. Let $\Omega$ be an arbitrary smooth, bounded, simply connected domain and suppose that $\omega_j^\delta = B(a_j^\delta, \delta) \subset \Omega$, $j = 1, \ldots, N$ represent the holes in $\Omega$, where $a_j$ is the center of the hole $j = 1, \ldots, N$ and $\delta \ll 1$ is its radius. We introduce the perforated domain

$$\Omega_\delta = \Omega \setminus \bigcup_{j=1}^N \omega_j^\delta$$

and consider the Ginzburg-Landau functional

$$GL^\delta_5[u, A] = \frac{1}{2} \int_{\Omega_\delta} |(\nabla - iA) u|^2 \, dx + \frac{1}{4\varepsilon^2} \int_{\Omega_\delta} (1 - |u|^2)^2 \, dx + \frac{1}{2} \int_{\Omega} (\text{curl } A - h_{\text{ext}})^2 \, dx. \quad (3)$$

The domain $\Omega_\delta$ represents a cross-section of a superconducting sample. Here $u : \Omega_\delta \to \mathbb{C}$ is an order parameter, $A : \Omega \to \mathbb{R}^2$ is a vector potential of the induced magnetic field, and $h_{\text{ext}}$ is the magnitude of the external magnetic field. By $\varepsilon$ we denote the inverse of the Ginzburg-Landau parameter that determines the radius of a typical vortex core. In what follows, we will assume that the cores radii are much smaller than the radius of the holes $\omega_j^\delta$.

The functional $GL^\delta_5[u, A]$ is gauge-invariant, i.e., for any $\varphi \in H^2(\Omega, \mathbb{R})$ and any admissible pair $(u, A)$, the equality $GL^\delta_5[u, A] = GL^\delta_5[u e^{i\varphi}, A + \nabla \varphi]$ always holds. This degeneracy can be eliminated by imposing the Coulomb gauge, that is requiring that

$$A \in H(\Omega, \mathbb{R}^2) := \{ a \in H^1(\Omega, \mathbb{R}^2) \mid \text{div } a = 0 \text{ in } \Omega, \ a \cdot \nu = 0 \text{ on } \partial \Omega \}, \quad (4)$$

where $\nu$ is an outward unit normal vector to $\partial \Omega$. We will fix the Coulomb gauge throughout the rest of this work.

We consider the minimizers of the two variational problems

$$(u_\delta^\varepsilon, A_\delta^\varepsilon) := \arg \min \left\{ GL^\delta_5[u, A] \mid u \in H^1(\Omega_\delta; \mathbb{C}), A \in H(\Omega; \mathbb{R}^2) \right\}, \quad (5)$$

and

$$(u_\delta, A_\delta) := \arg \min \left\{ GL^\delta_5[u, A] \mid u \in H^1(\Omega_\delta; S^1), A \in H(\Omega; \mathbb{R}^2) \right\}. \quad (6)$$

Note that, trivially,

$$(u_\delta, A_\delta) := \arg \min \left\{ GL^\delta_5[u, A] \mid u \in H^1(\Omega_\delta; S^1), A \in H(\Omega; \mathbb{R}^2) \right\}. \quad (7)$$
where

\[
GL_\delta[u,A] = \frac{1}{2} \int_{\Omega_\delta} |\nabla u - iA u|^2 \, dx + \frac{1}{2} \int_{\Omega} (\text{curl} A - h_{\text{ext}})^2 \, dx.
\]  

(8)

For any hole center \(a^j\), \(j = 1, \ldots, N\) and \(R > 0\), let \(\gamma_R^j = \partial B(a^j,R)\) be a circle of radius \(R\) centered at \(a^j\). In what follows we make a frequent use of the following

**Definition 1.** Given a \(u \in H^1(\Omega_\delta, \mathbb{C})\) and \(a^j, j = 1, \ldots, N\), suppose there exists an \(R = \delta + o(\delta)\) such that the winding number \(d = \text{deg} \left( \frac{u}{|u|}, \gamma_R^j \right) \neq 0\). Then \(u\) is said to have a hole vortex of the degree \(d\) inside \(\omega_j^\delta\).

The existence of \(\gamma_R^j\) is established in the Theorem 1 and they are specified using the results of Theorem 3. Hole vortices may exist inside \(\omega_j^\delta\) for the minimizers of both (5) and (7) and our principal goal is to prove that the respective degrees of the hole vortices arising in both problems coincide for the same external magnetic field as long as the parameter \(\delta\) is sufficiently small. This result implies that the non-linear potential term can be effectively replaced by the constraint \(|u| = 1\) when one is interested in studying the distribution of degrees of the hole vortices for the minimizer of the problem (5).

The main result of this work is the following theorem.

**Theorem 1.** Assume that the parameters \(\varepsilon\) and \(\delta\) satisfy

\[
|\log \varepsilon| \gg |\log \delta|.
\]

Suppose

\[
\sigma \in \mathbb{R}_+ \setminus \Sigma
\]

where \(\Sigma\) is a discrete set described below. Let

\[
h_{\text{ext}} = \sigma |\log \delta|
\]

and \((u^\varepsilon_\delta, A^\varepsilon_\delta)\) and \((u_\delta, A_\delta)\) be defined by (5) and (7), respectively.

Then, for a sufficiently small \(\delta\), there exists an \(R_\delta \in [\delta, \delta + \delta^2]\) such that

(i) \(D^j_\delta = \text{deg} \left( u^\varepsilon_\delta, \gamma_R^j \right)\) coincide for all \(j = 1 \ldots N\) when \(D^j_{\delta,\varepsilon}\) are defined, e.g. when \(u^\varepsilon_\delta \neq 0\) on \(\gamma_R^j\);

(ii) the degrees of the hole vortices \(D^j_{\delta,\varepsilon} = \text{deg} \left( \frac{u^\varepsilon_\delta}{|u^\varepsilon_\delta|}, \gamma_R^j \right)\);

for any \(R \geq R_\delta\) for which \(\gamma_R^j = \partial B(a^j,R), j = 1 \ldots N\) are mutually disjoint and do not intersect \(\partial \Omega\).

**Remark 1.** The set \(\Sigma\) includes the appropriately scaled values of the external field at which the degree of one of the hole vortices increments by one, i.e. from \(d\) to \(d + 1\). At these threshold field strengths, the leading order approximation of the energy is the same for both degrees \(d\) and \(d + 1\) and the degrees of the hole vortices of minimizers \(u^\varepsilon_\delta\) and \(u_\delta\) cannot be determined uniquely. The set \(\Sigma\) is described as follows:

\[
\Sigma = \bigcup_{j=1}^N \Sigma_j \quad \text{where} \quad \Sigma_j = \left\{ \sigma > 0 \mid \sigma \left( 1 - \xi_0(a^j) \right) \in \mathbb{Z} + \frac{1}{2} \right\}
\]

(12)
consists of the threshold field values for the hole $j = 1 \ldots N$ and the function $\xi_0$ solves the boundary value problem
\[
\begin{cases}
-\Delta \xi_0 + \xi_0 = 0 & \text{in } \Omega, \\
\xi_0 = 1 & \text{on } \partial \Omega.
\end{cases}
\] (13)

**Remark 2.** Notice that, since $u_\delta(x) \in S^1$, there are no vortices outside of the holes and thus
\[
D^j_\delta = \deg (u_\delta, \gamma^j_\delta) = \deg \left( u_\delta, \partial \omega^j_\delta \right)
\] (14)
for all $j = 1 \ldots N$.

**Remark 3.** As we will show in Section 5, although the external magnetic field satisfying the bound (11) is strong enough to generate hole vortices, it is too weak for vortices to appear inside the bulk superconductor $\Omega_\delta$, away from the boundary $\partial \Omega$.

We prove Theorem 1 in two steps. First, we consider minimizers $(u_\delta D, A_\delta D)$ of the variational problem (8) in the class of $S^1$-valued maps with the prescribed degrees, $\deg(u, \partial \omega^j_\delta) = D^j$, $j = 1 \ldots N$, by setting
\[
(u_\delta D, A_\delta D) := \arg \min \left\{ GL_\delta [u, A] \mid u \in H^1(\Omega_\delta; S^1), A \in H(\Omega; \mathbb{R}^2), \deg(u, \partial \omega^j_\delta) = D^j \right\}.
\] (15)
Then the degrees $D^j_\delta$ of the map $u_\delta$ minimize the energy
\[
l_\delta(D) := GL_\delta [u_\delta D, A_\delta D]
\] (16)
where $D = (D^1, \ldots, D^N)$. It turns out that the function $l_\delta(D)$ is a quadratic polynomial in $D^1, \ldots, D^N$. Its minimum is attained at one of the integer points adjacent to the vertex of paraboloid $l_\delta(T)$ with $T \in \mathbb{R}^N$. We enforce the condition (10) to ensure that such minimizing integer point is unique.

We then express a minimizer $(u_\delta^*, A_\delta^*)$ of $GL_\delta^* [u, A]$ as a sum of $(u_\delta, A_\delta)$ and an appropriate correction term and consider a corresponding energy decomposition in the spirit of the approach in [3] for finite-size holes. The analysis relies principally on the techniques developed in [3] and the ball construction method [19]. Compared to [3], new challenges arise due to the presence of the second small parameter that require additional estimates and sharper energy bounds.

### 3 \textit{S}^1\text{-valued problem}

The main goal of this section is to establish the relation between the energy of the minimizer $(u_\delta D, A_\delta D)$ and the degrees $D$ of the hole vortices corresponding to $u_\delta D$. We approximate the minimizer $(u_\delta D, A_\delta D)$, calculate its energy $l_\delta(D) = GL_\delta [u_\delta D, A_\delta D]$, and find the minimizing degrees $D_\delta = (D^1_\delta, \ldots, D^N_\delta)$. We prove the following theorem.

**Theorem 2.** Let $(u_\delta D, A_\delta D)$ be a minimizer of (15) with the prescribed degrees $D \in \mathbb{Z}^N$ on the holes. Then the Ginzburg-Landau energy $GL_\delta [u_\delta D, A_\delta D]$, expressed as a function of $D$, takes the following form:
\[
l_\delta(D) = \pi \sum_{j=1}^{N} \left[ (D^j)^2 - 2\sigma \left( 1 - \xi_0(a^j) \right) D^j \right] |\log \delta| + C|\log \delta|^2 + |D|^2 O(1)
\] (17)
where $\xi_0$ solves the boundary value problem (13), $C = O(1)$, and $|D| = \max_j |D^j|$.

**Proof.** The main idea of the proof is to approximate the induced magnetic field $h_{\delta D} = \text{curl} A_{\delta D}$ as a sum of functions that depend on external magnetic field and the prescribed degrees on the holes, respectively. First, prescribe the degrees of the order parameter
\[
\text{deg}(u, \partial \omega_j^\delta) = D_j, \quad j = 1 \ldots N
\]  
(18)
and write down the Euler-Lagrange equation for (8) in terms of the induced magnetic field $h = \text{curl} A$ with the corresponding boundary conditions:
\[
\begin{cases}
-\Delta h + h = 0, & \text{in } \Omega, \\
h = h_{\text{ext}}, & \text{on } \partial \Omega, \\
h = H_j, & \text{in } \omega_j^\delta, \quad j = 1 \ldots N, \\
-\int_{\partial \omega_j^\delta} \frac{\partial h}{\partial n} ds = 2\pi D_j - \int_{\omega_j^\delta} h dx, & j = 1 \ldots N.
\end{cases}
\]  
(19)
The constants $H_j$ are a priori unknown and are defined through the solution $h_{\delta D} = h_{\delta}(D)$ of (19) where $D = (D^1, \ldots, D^N)$ is the vector of the prescribed degrees. The energy (8) of the minimizer $(u_{\delta D}, A_{\delta D})$ can be expressed in terms of $h_{\delta D}$:
\[
GL_{\delta}[u_{\delta D}, A_{\delta D}] = GL_{\delta}[h_{\delta D}] = \frac{1}{2} \int_{\Omega_{\delta}} |\nabla h_{\delta D}|^2 dx + \frac{1}{2} \int_{\Omega} (h_{\delta D} - h_{\text{ext}})^2 dx.
\]  
(20)
Decompose the solution of (19) $h_{\delta D}$ into
\[
h_{\delta D} = h_1 + h_2 + h_3,
\]  
(21)
where $h_1$ captures the influence of the external field $h_{\text{ext}}$, $h_2$ takes into account the hole vortices, and $h_3$ is the remainder. More precisely,
\[
h_1 = h_{\text{ext}}\xi_0.
\]  
(22)
where $\xi_0$ solves the boundary value problem (13) in the domain $\Omega$ with no holes:
\[
\begin{cases}
-\Delta \xi_0 + \xi_0 = 0 & \text{in } \Omega, \\
\xi_0 = 1 & \text{on } \partial \Omega.
\end{cases}
\]  
(23)
The function $h_2$ is defined by
\[
h_2(x) = \sum_{j=1}^N D^j \theta_j(x) \phi_j(x)
\]  
(24)
where $D^j$ are as in (18). Here
\[
\theta_j(x) = \theta(x - a^j), \quad j = 1, \ldots, N
\]
and $\theta$ is a truncated modified Bessel function of the second kind
\[
\theta(x) = \begin{cases}
K_0(\delta), & |x| \leq \delta, \\
K_0(|x|), & |x| > \delta.
\end{cases}
\]  
(25)
The cutoff function \( \phi_j(x) = \phi(x - a^j) \in C^\infty(\mathbb{R}^2) \) satisfies

\[
\phi(x) = \begin{cases} 
1, & |x| \leq R/4, \\
0, & |x| \geq R/2, 
\end{cases}
\]  

with \( R \) being defined as the largest radius for which \( B(a^j, R), j = 1 \ldots N \) intersect neither each other nor the boundary \( \partial \Omega \). Here the choice of \( K_0(|x|) \) is motivated by the fact that it is a fundamental solution of the equation \(-\Delta u + u = 2\pi \delta(x)\) in \( \mathbb{R}^2 \). Note that \( h_2 \) solves the following problem:

\[
\begin{aligned}
\begin{cases}
-\Delta h_2 + h_2 &= \sum_{j=1}^N D^j \left(-\Delta + I\right) (\theta_j \phi_j), \\
h_2 &= 0, \\
h_2 &= D^j K_0(\delta), \\
-\int_{\partial \omega_3^j} \frac{\partial h_2}{\partial \nu} ds &= 2\pi D^j - D^j K_0(\delta)|\omega_3^j| + D^j O(\delta^2),
\end{cases}
& \text{in } \Omega_{\delta},
\end{aligned}
\]  

on \( \partial \Omega \),

on \( \partial \omega_3^j, \ j = 1 \ldots N, \)

Since for each \( j = 1, \ldots, N \) the function \( f_j(x) := [-\Delta + I] (\theta_j \phi_j) \) is nonzero only inside the annular region \( T_j := B(a^j, R/2) \setminus B(a^j, R/4) \) that does not intersect any of the holes, the functions \( f_j, \ j = 1, \ldots, N \) are smooth and finite. Thus, for every \( j = 1, \ldots, N \), the function \( h_2 \) has the degree \( D^j \) on the hole \( \omega_3^j \) and \( \theta_j \phi_j \) is constant on \( \omega_3^j \) and decays to zero on \( \partial B(a^j, R/2) \).

Next, we show that the contribution of the remainder \( h_3 = h - h_1 - h_2 \) to the energy is small, hence the interaction between the hole vortices contributes a negligible amount to the energy. This provides a justification for treating each hole vortex as being independent from the other hole vortices.

We deduce the boundary value problem for \( h_3 \) from the original problem (19), the problem (13) for \( h_1 = h_{\text{ext}} \xi_0 \), and the expression (24) for \( h_2 \) to obtain:

\[
\begin{aligned}
\begin{cases}
-\Delta h_3 + h_3 &= -\sum_{j=1}^N D^j f_j(x), \\
h_3 &= 0, \\
h_3 &= \bar{H}_j - h_{\text{ext}}(\xi_0(x) - \xi_0(a^j)), \\
-\int_{\partial \omega_3^j} \frac{\partial h_3}{\partial \nu} ds &= -\bar{H}_j|\omega_3^j| + D^j O(\delta^2) + O(\delta^3 \log \delta),
\end{cases}
& \text{in } \Omega_{\delta},
\end{aligned}
\]  

on \( \partial \Omega \),

on \( \partial \omega_3^j, j = 1 \ldots N, \)

where \( \bar{H}_j = H_j - h_{\text{ext}} \xi_0(a^j) - D^j K_0(\delta) \) are the unknown constants. The next lemma establishes the necessary estimates for \( h_3 \).

**Lemma 1.** The solution \( h_3 \) of (28) satisfies the following estimates:

\[
\begin{align}
\|h_3\|_{L^\infty(\Omega)} &\leq C_1 \delta \log \delta^2 + C_2|D|, \\
\|\nabla h_3\|_{L^\infty(\Omega)} &\leq C_1 |\log \delta|^2 + C_2|D| |\log \delta|, \\
\left| \frac{\partial h_3}{\partial \nu} \right| &\leq C_1 |\log \delta| + C_2|D| \text{ on } \partial \omega_3^j \text{ for all } j = 1 \ldots N.
\end{align}
\]  

**Proof.** We begin by splitting (28) into several subproblems. First, let \( \eta = \sum_{j=1}^N D^j \eta_j \) be a solution of the nonhomogeneous equation in (28), where \( \eta_j \) solves

\[
\begin{aligned}
\begin{cases}
-\Delta \eta_j + \eta_j &= -[-\Delta + I] (\theta_j \phi_j) 1_{T_j}, \\
\eta_j &= 0,
\end{cases}
& \text{in } \Omega, \\
& \text{on } \partial \Omega,
\end{aligned}
\]  

where \( \bar{H}_j = H_j - h_{\text{ext}} \xi_0(a^j) - D^j K_0(\delta) \) are the unknown constants. The next lemma establishes the necessary estimates for \( h_3 \).
for every \( j = 1, \ldots, N \). Here \( \eta_j, j = 1, \ldots, N \) are smooth and do not depend on \( \delta \). Next, introduce \( \eta_0 \) that both solves the homogeneous equation and satisfies the conditions on \( \partial \omega_j^\delta \) in (28) to give

\[
\begin{align*}
-\Delta \eta_0 + \eta_0 &= 0, & \text{in } \Omega_\delta, \\
\eta_0 &= 0, & \text{on } \partial \Omega, \\
\eta_0 &= -h_{\text{ext}}(\xi(x) - \xi(a^j)) - (\eta(x) - \eta(a^j)), & \text{on } \partial \omega_j^\delta, & j = 1 \ldots N.
\end{align*}
\]  

(33)

Note that, by the Maximum Principle,

\[
\|\eta_0\|_{L^\infty} \leq C\delta(|\log \delta| + \max_j |D^j|).
\]  

(34)

Lemma 6 provides the estimate on the gradient of \( \eta_0 \) of the form

\[
\|\nabla \eta_0\|_{L^\infty} \leq C(|\log \delta| + \max_j |D^j|).
\]  

(35)

The remainder \( \zeta = h_3 - \sum_{j=0}^N \eta_j \) solves the following system:

\[
\begin{align*}
-\Delta \zeta + \zeta &= 0, & \text{in } \Omega_\delta, \\
\zeta &= 0, & \text{on } \partial \Omega, \\
\zeta &= c_j, & \text{on } \partial \omega_j^\delta, & j = 1 \ldots N, \\
-\int_{\partial \omega_j^\delta} \frac{\partial \zeta}{\partial \nu} d\sigma &= -|\omega_j^\delta| c_j + A_j^\delta, & j = 1 \ldots N,
\end{align*}
\]  

(36)

where \( c_j = \bar{H}_j - \eta(a^j) \) are unknown constants and \( A_j^\delta = |D\omega(a^j) + O(\delta \log \delta) \) is an error. The first three equations in (36) set up the boundary value problem for \( \zeta \) with the unknown boundary values \( c_j \). The fourth line in (36) gives the system of \( N \) equations for \( N \) unknowns \( c_j \). Since the boundary value problem for \( \zeta \) is linear, we start with the estimates for the basis functions \( \zeta_i \) that solve the problem

\[
\begin{align*}
-\Delta \zeta_i + \zeta_i &= 0, & \text{in } \Omega_\delta, \\
\zeta_i &= 0, & \text{on } \partial \Omega, \\
\zeta_i &= \delta_{ij}, & \text{on } \partial \omega_j^\delta, & j = 1 \ldots N,
\end{align*}
\]  

(37)

for every \( i = 1, \ldots, N \). Then, using representation \( \zeta = \sum_i c_i \zeta_i \), we will solve the linear system for \( c_i \).

We use the method of sub- and supersolutions to get estimates for \( \zeta_i \). By the Maximum Principle, we have that \( 0 \leq \zeta_i \leq 1 \) for every \( i = 1, \ldots, N \). In the case of a radially symmetric domain with one hole at the center, the solutions of (37) are the modified Bessel functions. We show that they provide a good approximation for \( \zeta_i \). First, fix \( i \in 1 \ldots N \) and construct a supersolution for \( \zeta_i \). Take \( R_{\text{max}} > 0 \) such that \( \Omega \in B(a^i, R_{\text{max}}) \) and set

\[
\zeta_{i}^{\text{sup}} = \frac{K_0\left(\frac{|x-a^i|}{R_{\text{max}}}\right)}{K_0\left(\frac{\delta}{R_{\text{max}}}\right)},
\]  

(38)

The function \( \zeta_{i}^{\text{sup}} \) is strictly positive in \( \Omega_\delta \), equals 1 on \( \partial \omega_j^\delta \), and has \( [-\Delta + I] \zeta_{i}^{\text{sup}} = 0 \). Therefore
it satisfies
\[
\begin{align*}
-\Delta \zeta_i^{\text{sup}} + \zeta_i^{\text{sup}} &= 0 & \text{in } \Omega_i, \\
\zeta_i^{\text{sup}} &> 0 & \text{on } \partial \Omega, \\
\zeta_i^{\text{sup}} &= 1 & \text{in } \omega^i_\delta, \\
\zeta_i^{\text{sup}} &> 0 & \text{in } \omega^j_\delta, \ j \neq i, \ j = 1 \ldots N,
\end{align*}
\]
and is thus a supersolution. This yields the bound
\[
0 \leq \zeta_i \leq \zeta_i^{\text{sup}} \quad \text{in } \Omega, \quad i = 1 \ldots N.
\]

Next, we construct a subsolution. Take \( R_{\text{min}} > 0 \) such that \( B(a^i, 2R_{\text{min}}) \in \Omega_\delta \) for every \( i = 1 \ldots N \) and set
\[
\zeta_i^{\text{sub}} = \frac{K_0 \left( \frac{|x-a^i|}{R_{\text{min}}} \right)}{K_0 \left( \frac{\delta}{R_{\text{min}}} \right)}
\]
The Bessel function is a fundamental solution of \([-\Delta + I] u = \delta(x)\) and it is decreasing, therefore \( \zeta_i^{\text{sub}} \) is negative outside \( B(a^i, R_{\text{min}}) \). Thus it satisfies
\[
\begin{align*}
-\Delta \zeta_i^{\text{sub}} + \zeta_i^{\text{sub}} &= 0 & \text{in } \Omega_\delta, \\
\zeta_i^{\text{sub}} &< 0 & \text{on } \partial \Omega, \\
\zeta_i^{\text{sub}} &= 1 & \text{in } \omega^i_\delta, \\
\zeta_i^{\text{sub}} &< 0 & \text{in } \omega^j_\delta, \ j \neq i, \ j = 1 \ldots N,
\end{align*}
\]
and is thus a subsolution. This, together with (40), implies that
\[
\max(0, \zeta_i^{\text{sub}}) \leq \zeta_i \leq \zeta_i^{\text{sup}},
\]
for every \( i = 1 \ldots N \), giving a very sharp description of the behavior of \( \zeta_i \) near \( i \)th hole. Note that, for \( x \in \partial \omega^i_\delta \), we have
\[
\frac{L_1}{\delta \log \delta} \leq \frac{\partial \zeta_i^{\text{sub}}}{\partial \nu}(x) \leq \frac{L_2}{\delta \log \delta}
\]
with \( L_1, L_2 > 0 \), therefore
\[
\frac{\partial \zeta_i}{\partial \nu}(x) \sim \frac{1}{\delta \log \delta} \quad \text{on } \partial \omega^i_\delta.
\]
To estimate the normal derivative of \( \zeta_i \) on \( \partial \omega^j_\delta \) for \( j \neq i \) we need a better supersolution that captures the appropriate Dirichlet boundary conditions. Outside of \( B(a^i', R_{\text{min}}) \), we have
\[
|\zeta_i(x)| \leq \frac{K_0 \left( \frac{R_{\text{min}}}{R_{\text{max}}} \right)}{K_0 \left( \frac{\delta}{R_{\text{max}}} \right)} \leq C_R |\log \delta|^{-1}.
\]
Construct \( \zeta_{ij}^{\text{sup}} \) that solves the following conditions:
\[
\begin{align*}
-\Delta \zeta_{ij}^{\text{sup}} + \zeta_{ij}^{\text{sup}} &= 0 & \text{in } B(a^j, R_{\text{min}}) \setminus B(a^j, \delta), \\
\zeta_{ij}^{\text{sup}} &= C_R |\log \delta|^{-1} & \text{on } \partial B(a^j, R_{\text{min}}), \\
\zeta_{ij}^{\text{sup}} &= 0 & \text{on } \partial \omega^j_\delta.
\end{align*}
\]
This problem is radially symmetric in \( B(a^j, R_{\text{min}}) \setminus \overline{B(a^j, \delta)} \). The function
\[
\zeta_{ij}^{\text{sup}} = C_1 I_0(r) + C_2 K_0(r), \quad r = |x - a^j|
\] (48)
with
\[
C_1 \sim -|\log \delta|^{-1} \quad \text{and} \quad C_2 \sim |\log \delta|^{-2}.
\] (49)
satisfies (47) because the modified Bessel functions \( I_0 \) and \( K_0 \) behave as 1 and \( -\log r \), respectively, near the origin. Therefore
\[
0 \leq \frac{\partial \zeta_{ij}}{\partial \nu} \leq \frac{\partial \zeta_{ij}^{\text{sup}}}{\partial \nu} = \frac{C_{ij}}{\delta |\log \delta|^2} \quad \text{on} \quad \partial \omega^j_{\delta}.
\] (50)
As a result
\[
\int_{\partial \omega^j_{\delta}} \left| \frac{\partial \zeta_{ij}}{\partial \nu} \right| \, ds \leq \frac{C}{|\log \delta|^2},
\] (51)
for all \( i \neq j \). Combining the estimates on the behavior of \( \zeta_i \) on \( \partial \omega^i_{\delta} \) in (45) with (51) and estimating the constants \( c_i \) using the fourth equation in (36) we find:
\[
\pi \delta^2 |c_i| + |A^i| \geq \int_{\partial \omega^i_{\delta}} \left| \frac{\partial \zeta_i}{\partial \nu} \right| \, ds \geq |c_i| \int_{\partial \omega^i_{\delta}} \left| \frac{\partial \zeta_i^{\text{sup}}}{\partial \nu} \right| \, ds - \sum_{j \neq i} |c_j| \int_{\partial \omega^j_{\delta}} \left| \frac{\partial \zeta_j}{\partial \nu} \right| \, ds
\]
\[
\geq |c_i| \frac{C_1}{|\log \delta|} - \sum_{j \neq i} |c_j| \frac{C_2}{|\log \delta|^2}
\] (52)
or
\[
|c_i| \left( \frac{C_1}{|\log \delta|} - \pi \delta^2 \right) - \sum_{j \neq i} |c_j| \frac{C_2}{|\log \delta|^2} \leq |A^i|,
\] (53)
with some positive \( C_1, C_2 > 0 \) for all \( i = 1 \ldots N \). The coefficient matrix is a small perturbation of the identity matrix, up to the factor \( C_1 |\log \delta|^{-1} \). This allows us to conclude that
\[
|c_i| \leq |D|O(\delta \log \delta) + O(\delta \log^2 \delta)
\] (54)
for all \( i = 1 \ldots N \). Let
\[
c_i = \max_j |c_j|.
\] (55)
Then
\[
|c_i| \leq |A^i| \left( \frac{C_1}{|\log \delta|} - \pi \delta^2 - (N - 1) \frac{C_2}{|\log \delta|^2} \right)^{-1} \leq |D|O(\delta \log \delta) + O(\delta \log^2 \delta),
\] (56)
hence
\[
\|\zeta\|_{L^\infty(\Omega_{\delta})} \leq \sum_j |c_j| \leq C_1 |D| |\log \delta| + C_2 |D| |\log \delta|^2.
\] (57)
The statement of the lemma for
\[
h_3 = \eta_0 + \sum_{j=1}^N D^j \eta_j + \sum_{j=1}^N c_j \zeta_j
\] (58)
then follows once we combine the estimates above.
Proof of Theorem 2, continued. We are now able to find the asymptotics for the energy \( l_\delta(D) = GL_\delta[h_{SD}] \):

\[
l_\delta(D) = GL_\delta[h_1 + h_2 + h_3]
\]

\[
= \frac{1}{2} \int_{\Omega_\delta} |\nabla h_1|^2 dx + \frac{1}{2} \int_{\Omega_\delta} |\nabla h_2|^2 dx + \frac{1}{2} \int_{\Omega_\delta} |\nabla h_3|^2 dx
\]

\[
+ \frac{1}{2} \int_{\Omega} (h_1 - h_{ext})^2 dx + \frac{1}{2} \int_{\Omega} h_2^2 dx + \frac{1}{2} \int_{\Omega} h_3^2 dx
\]

\[
+ \int_{\Omega_\delta} \left[ \nabla (h_1 - h_{ext}) \cdot \nabla \hat{h} + (h_1 - h_{ext}) \hat{h} \right] dx + \int_{\Omega_\delta} |\nabla h_2 \cdot \nabla h_3 + h_2 h_3| dx
\]

\[
+ |D|^2 O(\delta^2 \log \delta^3) + O(\delta^2 \log \delta^3),
\]

where \( \hat{h} = h_2 + h_3 \) and the integrals over holes \( \omega^j_\delta \) are the source of the error. Next, we estimate each term in (59). The terms that involve \( h_1 \) only do not depend on the degrees of the hole vortices and thus they do not play a role in the minimization of \( l_\delta(D) \):

\[
\int_{\Omega_\delta} |\nabla h_1|^2 dx + \frac{1}{2} \int_{\Omega} (h_1 - h_{ext})^2 dx = h_{ext}^2 \frac{1}{2} \int_{\Omega_\delta} |\nabla \xi_0|^2 dx + h_{ext}^2 \frac{1}{2} \int_{\Omega} (1 - \xi_0)^2 dx
\]

\[
= O(\log \delta^2). \quad (60)
\]

The gradient of \( h_2 \) gives the main quadratic term:

\[
\int_{\Omega_\delta} |\nabla h_2|^2 dx = \frac{1}{2} \sum_{j=1}^{N} (D^j)^2 \int_{T_j} |\nabla (\theta_j(x) \phi_j(x))|^2 dx
\]

\[
= \pi \sum_{j=1}^{N} (D^j)^2 \left[ \int_{\delta}^{R/4} |K_0(r)^2| r dr + \int_{R/4}^{R} \left| \frac{d}{dr} (K_0(r) \phi(r)) \right|^2 rdr \right]
\]

\[
= \pi \sum_{j=1}^{N} (D^j)^2 \left[ \int_{\delta}^{R/4} \frac{1}{r} + O(\log r) \right] r dr + O(1) \quad (61)
\]

\[
= \pi \sum_{j=1}^{N} (D^j)^2 |\log \delta| + |D|^2 O(1). \quad (62)
\]

The \( L^2 \)-norm of \( h_2 \) is much smaller, indeed:

\[
\frac{1}{2} \int_{\Omega} h_2^2 dx = \pi \sum_{j=1}^{N} (D^j)^2 \int_{0}^{R/2} |\theta_j \phi|^2 r dr = |D|^2 O(1). \quad (63)
\]

We now estimate the integral involving \( \hat{h} \) that gives the linear terms in terms of the degrees. Note that, since \( h_{SD} \) and \( h_1 \) solve the homogeneous equation \([-\Delta + I] h = 0\), then so does their difference
\[ \hat{h} = h_{\delta D} - h_1: \]

\[
\langle h_1 - h_{ext}, \hat{h} \rangle_{H^1(\Omega_s)} = \int_{\Omega_s} (h_1 - h_{ext}) (-\Delta \hat{h} + \hat{h}) \, dx - \int_{\partial \Omega_s} (h_1 - h_{ext}) \frac{\partial \hat{h}}{\partial \nu} \, ds \\
= \sum_{j=1}^{N} \int_{\partial \omega_j} (h_1 - h_{ext}) \frac{\partial (h_2 + h_3)}{\partial \nu} \, ds \\
= \sum_{j=1}^{N} \int_{\partial \omega_j} (h_1 - h_{ext}) \left[ D (\frac{1}{\delta} + O(\delta \log \delta)) + O(\log \delta) + |D|O(1) \right] \, ds \\
= \sum_{j=1}^{N} D (h_1(a_j) - h_{ext}) \, 2 \pi \delta \cdot \frac{1}{\delta} + O(\delta \log \delta)^2 + |D|O(\delta \log \delta) \\
= -2 \pi \sigma |\log \delta| \sum_{j=1}^{N} D (1 - \xi_0(a_j)) + O(\delta \log \delta)^2 + |D|O(\delta \log \delta), \quad (64)
\]

where use the notation \( \langle u, v \rangle_{H^1} = \int [\nabla u \cdot \nabla v + uv] \, dx \). The other terms in (59) are small and are estimated using integration by parts:

\[
\| h_3 \|^2_{H^1(\Omega_s)} = \int_{\Omega_s} h_3 (-\Delta h_3 + h_3) \, dx - \int_{\partial \Omega_s} h_3 \frac{\partial h_3}{\partial \nu} \, ds = \sum_{j=1}^{N} \int_{\partial \omega_j} h_3 \frac{\partial h_3}{\partial \nu} \, ds \\
= C \delta \left( C_1 \delta |\log \delta|^2 + C_2 |D| \right) \left( C_1 |\log \delta| + C_2 |D| \right) \\
= O(\delta^2 |\log \delta|^3) + |D|^2O(\delta |\log \delta|) \quad (65)
\]

\[
\langle h_2, h_3 \rangle_{H^1(\Omega_s)} = \int_{\Omega_s} h_2 (-\Delta h_3 + h_3) \, dx - \int_{\partial \Omega_s} h_2 \frac{\partial h_3}{\partial \nu} \, ds = \sum_{j=1}^{N} \int_{\partial \omega_j} h_2 \frac{\partial h_3}{\partial \nu} \, ds \\
= \sum_{j=1}^{N} 2 \pi \delta D (K_0(\delta) (C_1 |\log \delta| + C_2 |D|) \\
= |D|^2O(\delta |\log \delta|^2) \quad (66)
\]

Combining all of the above estimates, we obtain the asymptotic expansion (17).

**Corollary 1.** The leading part of the energy \( l_\delta(Z) \) is a sum of \( N \) one-dimensional parabolas with the vertices at

\[ Z_j = \sigma(1 - \xi_0(a_j)) \in \mathbb{R}. \quad (67) \]

Since the degrees are integer-valued, the minimizing degrees \( D^j \) are the integers, closest to \( Z_j \):

\[ D^j = \left\lfloor \sigma(1 - \xi_0(a_j)) \right\rfloor, \quad (68) \]

where \( \lfloor x \rfloor \) denotes the integer nearest to \( x \).
4 Energy Decomposition

Since \((u_{\delta D}, A_{\delta D})\) is an admissible pair for the problem (5), we can use the representation of \(S^1\)-valued energy (17) with \(D = 0\) to obtain an upper bound

\[
GL_\delta^\varepsilon[u_\delta^\varepsilon, A_\delta^\varepsilon] \leq GL_\delta^\varepsilon[u_{\delta 0}, A_{\delta 0}] \leq C|\log \delta|^2
\]

(69)

on the energy of the minimizer of (5). In order to obtain a matching lower energy bound, we need to localize the regions of the domain where the magnitude of the order parameter is small. To this end, we use the following theorem.

**Theorem 3** (Ball Construction Method [19]). For any \(\alpha \in (0, 1)\) there exists \(\varepsilon_0(\alpha) > 0\) such that, for any \(\varepsilon < \varepsilon_0\), if \((u, A)\) is a configuration such that \(GL_\delta^\varepsilon[u, A] < \varepsilon^{\alpha-1}\), where \(\varepsilon\) is an inverse of the Ginzburg-Landau parameter, the following holds.

For any \(1 > \rho > C\varepsilon^{\alpha/2}\), where \(C\) is a universal constant, there exists a finite collection of disjoint closed balls \(\mathcal{B} = \{B_i = B(b^i, r_i)\}_{i \in \mathcal{I}}\) such that

1. \(r(\mathcal{B}) = \rho\) where \(r(\mathcal{B}) = \sum_{i \in \mathcal{I}} r(B_i)\).
2. Letting \(V = \Omega_\delta \cap \cup_{i \in \mathcal{I}} B_i\),
   \[
   \left\{ x \in \Omega_\delta \mid |u(x)| - 1 \geq \varepsilon^{\alpha/4} \right\} \subset V.
   \]
3. Writing \(d_i = \deg(u, \partial B_i)\), if \(B_i \subset \Omega_\delta\) and \(d_i = 0\) otherwise,
   \[
   \frac{1}{2} \int_V \left[ |\nabla_A u|^2 + \rho^2 |\text{curl} A|^2 + \frac{1}{2\varepsilon^2} (1 - |u|^2)^2 \right] dx \geq \pi d \left( \log \frac{\rho}{d\varepsilon} - C \right),
   \]
   \[
   \text{where } d = \sum_{i \in \mathcal{I}} |d_i| \text{ is assumed to be nonzero and } C \text{ is a universal constant.}
   \]
4. There exists a universal constant \(C\) such that
   \[
   d \leq C \frac{GL_\delta^\varepsilon[u, A]}{\alpha |\log \varepsilon|}.
   \]

We consider now a domain with \(N\) holes \(\omega_\delta^j = B(a^j, \delta)\) so that \(\Omega_\delta = \Omega \setminus \cup_{j=1}^N \omega_\delta^j\). Set \(\alpha = 1/2\) and \(\rho = \delta^\alpha/2\) in the ball construction method. Assume that \(\varepsilon\) is small enough so that \(|u(x)| > 1 - \theta\) on \(\Omega_\delta \cap (\cup_{i \in \mathcal{I}} B_i)\). The parameter \(\theta\) will be chosen later, in Section 6.

**Lemma 2.** Let \((u_\delta^\varepsilon, A_\delta^\varepsilon)\) be a minimizer of the problem (5). Then the following energy decomposition holds:

\[
GL_\delta^\varepsilon[u_\delta^\varepsilon, A_\delta^\varepsilon] = GL_\delta[u_{\delta D}, A_{\delta D}] + F_\delta[v, B] - \int_{\Omega_\delta} \nabla h_{\delta D} \cdot \text{Im} \nabla v dx + o(1)
\]

(73)

where \(u_\delta^\varepsilon = v_{\delta D}, A_\delta^\varepsilon = A_{\delta D} + B, h_{\delta D} = \text{curl} A_{\delta D}\) and

\[
F_\delta[v, B] = \frac{1}{2} \int_{\Omega_\delta} \left( |(\nabla - iB)v|^2 + \frac{1}{2\varepsilon^2} (1 - |v|^2)^2 \right) dx + \frac{1}{2} \int_{\Omega} (\text{curl} B)^2 dx.
\]

(74)

Here \((u_{\delta D}, A_{\delta D})\) is the minimizer of the \(S^1\)-valued problem (15) with the prescribed degrees \(D\).
Proof. Using the representation (20) of Ginzburg-Landau functional in terms of \( h_{SD} \), note that the pair \((u_{SD}, A_{SD})\) satisfies the following equation

\[
\nabla^2 h_{SD} = -\text{Im} \left( \tau_{SD} \nabla u_{SD} - i A_{SD} \right)
\]

outside of the holes. We start the proof with representing \( GL_{\delta}[u_{\delta}, A_{\delta}] \) as a sum of three terms:

\[
GL_{\delta}[u_{\delta}, A_{\delta}] = I_1 + I_2 + I_3,
\]

where

\[
I_1 = \frac{1}{2} \int_{\Omega} |\nabla u_{\delta}^\prime - i A_{\delta}^\prime u_{\delta}^\prime|^2 \, dx, \quad I_2 = \frac{1}{4 \varepsilon^2} \int_{\Omega} (1 - |u_{\delta}|^2)^2 \, dx, \quad I_3 = \frac{1}{2} \int_{\Omega} (\text{curl} A_{\delta}^\prime - h_{ext})^2 \, dx.
\]

Observe that \( |u_{\delta}^\prime| = |v| \) as \( u_{\delta}^\prime = v u_{SD} \) and \( |u_{SD}| = 1 \). Hence we can rewrite \( I_2 \) as

\[
I_2 = \frac{1}{4 \varepsilon^2} \int_{\Omega} (1 - |u_{SD}|^2)^2 \, dx = \frac{1}{2} \int_{\Omega} (1 - |v|^2)^2 \, dx,
\]

giving us the second term in the definition of \( F_{\delta}[v, B] \). Now rewrite \( I_3 \):

\[
I_3 = \frac{1}{2} \int_{\Omega} (\text{curl} A_{\delta}^\prime - h_{ext})^2 \, dx
= \frac{1}{2} \int_{\Omega} (h_{SD} - h_{ext})^2 \, dx + \frac{1}{2} \int_{\Omega} (\text{curl} B)^2 \, dx + \int_{\Omega} \text{curl} B \cdot (h_{SD} - h_{ext}) \, dx
\]

Here, the first term is a part of \( GL_{\delta}[u_{SD}, A_{SD}] \) and the second term is a part of \( F_{\delta}[v, B] \). The last term will eventually cancel with a component of \( I_1 \). To this end,

\[
|\nabla u_{\delta}^\prime - i A_{\delta}^\prime u_{\delta}^\prime|^2 = |v (\nabla u_{SD} - i A_{SD} u_{SD}) + u_{SD} (\nabla v - i Bv)|^2
= |v|^2 |\nabla u_{SD} - i A_{SD} u_{SD}|^2 + |u_{SD}|^2 |\nabla v - i Bv|^2 + 2 \text{Re} \left( \tau_{SD} (\nabla u_{SD} - i A_{SD} u_{SD}) \cdot (\nabla v + i Bv) \right)
= |\nabla v - i Bv|^2 + |v|^2 |\nabla h_{SD}|^2 + 2 |v|^2 \nabla \cdot h_{SD} \cdot B - 2 \nabla \cdot h_{SD} \cdot \text{Im} (\tau \nabla v)
\]

The first term in (80) contributes to \( F_{\delta}[v, B] \). The last term is included in the right hand side of the decomposition. The sum of two other terms has the form \( |v|^2 \cdot R(x) \), where

\[
R(x) = |\nabla h_{SD}|^2 + 2 \nabla \cdot h_{SD} \cdot B
\]

Now add and subtract \( \frac{1}{2} \int_{\Omega_{ext}} R(x) \, dx \) to the energy \( GL_{\delta}[u_{\delta}, A_{\delta}] \). The first term \( \frac{1}{2} \int_{\Omega_{ext}} |\nabla h_{SD}|^2 \, dx \) is a part of \( GL_{\delta}[u_{SD}, A_{SD}] \). Using integration by parts we prove that the second term \( \int_{\Omega_{ext}} \nabla \cdot h_{SD} \cdot B \, dx \)
indeed cancels with the last term in the representation (79) of $I_3$ as alluded to above:

$$\int_{\Omega_\delta} \nabla^\perp h_{\delta D} \cdot B \, dx = \int_{\Omega_\delta} \nabla^\perp (h_{\delta D} - h_{\text{ext}}) \cdot B \, dx$$

$$= \int_{\partial \Omega_\delta} (h_{\delta} - h_{\text{ext}}) B \cdot \tau \, dS - \int_{\Omega_\delta} (h_{\delta D} - h_{\text{ext}}) \nabla^\perp \cdot B \, dx$$

$$= - \sum_{j=1}^{N} (h_{\delta D} - h_{\text{ext}}) |_{\partial B(a^j, \delta)} \int_{\partial B(a^j, \delta)} B \cdot \tau \, dS - \int_{\Omega_\delta} (h_{\delta D} - h_{\text{ext}}) \text{curl} \, B \, dx$$

$$= - \sum_{j=1}^{N} (h_{\delta D} - h_{\text{ext}}) |_{\partial B(a^j, \delta)} \int_{B(a^j, \delta)} \text{curl} B \, dS - \int_{\Omega_\delta} (h_{\delta D} - h_{\text{ext}}) \text{curl} \, B \, dx$$

$$= - \int_{\Omega} (h_{\delta D} - h_{\text{ext}}) \text{curl} B \, dx.$$  (81)

Here we used the facts that $h_{\delta D} = h_{\text{ext}}$ on the boundary $\partial \Omega$ and $h_{\delta D} = \text{const}$ in $B(a^j, \delta)$ that follow from the equation for $h_{\delta D}$.

Adding up the results above gives:

$$GL_{\delta}^{\varepsilon}[u_{\delta}, A_{\delta}] = GL_{\delta}^{\varepsilon}[u_{\delta D}, A_{\delta D}] + F_{\delta}^{\varepsilon}[v, B]$$

$$- \int_{\Omega_\delta} \nabla^\perp h_{\delta D} \cdot \text{Im} \nabla v \, dx + \int_{\Omega_\delta} (1 - |v|^2) R(x) \, dx + o(1)$$  (82)

The remaining task is to show that

$$I = \int_{\Omega_\delta} (1 - |v|^2) R(x) \, dx$$

go to zero as $\delta \to 0$. Hölder’s inequality implies that

$$|I| \leq \|1 - |v|^2\|_{L^2(\Omega_\delta)} \cdot \left(2\|\nabla h_{\delta D}\|_{L^4(\Omega_\delta)}^2 + \|B\|_{L^4(\Omega_\delta)}^2\right).$$  (83)

The first multiplier in this expression is less then $M\varepsilon \log \delta$ when $\delta \to 0$ because of the a priori estimate on the energy. Using the relation between $\varepsilon$ and $\delta$

$$|\log \varepsilon| \gg |\log \delta|,$$  (84)

we show that $\varepsilon$ is sufficiently small to compensate for the growth of the other terms.

The function $h_{\delta D}$ is described in Theorem 2 and because of Lemma 6 it satisfies the estimate

$$\|\nabla h_{\delta D}\|_{L^4(\Omega_\delta)}^2 \leq C|\log \delta|^2 \delta^2.$$  (85)

In order to estimate $\|B\|_{L^4(\Omega_\delta)}$, recall that $\text{div} A_{\delta} = 0$ due to the gauge invariance. Then by the Poincaré’s lemma $A_{\delta}$ has a potential, i.e. there exists $\Pi_{\delta}$ such that $\nabla^\perp \Pi_{\delta} = A_{\delta}$. Substituting this into $h_{\delta} = \text{curl} A_{\delta}$, we obtain the equality $\Delta \Pi_{\delta} = h_{\delta}$. The function $\Pi_{\delta}$ is a potential so we are able to make it zero on the boundary $\partial \Omega$. From the theory of elliptic operators and the a priori energy estimate, we obtain

$$\|\Pi_{\delta}\|_{H^2(\Omega)}^2 \leq \|h_{\delta}\|_{L^2(\Omega)}^2 \leq C|\log \delta|^2.$$  (86)
Since the embedding $H^1(\Omega) \subset L^4(\Omega)$ is continuous we have
$$\|A^\varepsilon_\delta\|_{L^4(\Omega)} \leq C\|\Pi^\varepsilon_\delta\|_{H^2(\Omega)} \leq C\log \delta.$$  

The same estimate holds for $A^\delta_D$. Using the decomposition $A^\varepsilon_\delta = B + A^\delta_D$ we obtain this estimate for $B$:
$$\|B\|_{L^4(\Omega_\delta)} \leq C\|\Pi^\varepsilon_\delta\|_{H^2(\Omega)} \leq C\log \delta^2.$$ 

Combining all estimates obtained in this section, we conclude that
$$|I| \leq C\varepsilon \log \delta \left( \frac{|\log \delta|^2}{\delta^2} + |\log \delta|^2 \right).$$ 

The condition $|\log \varepsilon| \gg |\log \delta|$ implies that $\varepsilon$ is much smaller than any power of $\delta$, therefore $I$ goes to zero as $\delta \to 0$ that completes the proof.  

5 Absence of Bulk Vortices

In this section we further analyze the energy decomposition (73). The energy of the unconstrained solution is minimal, hence
$$GL^\varepsilon_\delta[u^\varepsilon_\delta, A^\varepsilon_\delta] \leq GL^\delta_\delta[u^\delta_D, A^\delta_D],$$

and using (73) we have
$$F^\delta_\delta[v, B] \leq \int_{\Omega_\delta} \nabla^\perp h^\delta_D \cdot \text{Im} \nabla v \, dx + o(1).$$

First, we derive an upper bound for the integral term in (88) and thus for the energy $F^\delta_\delta$. We start with a simple fact that will also be used later on.

**Proposition 1.** Given a sufficiently smooth domain $S \subset \mathbb{R}^2$ and any $R \in L^2(S, \mathbb{R})$, $P \in H^1(S, \mathbb{R}^2)$, and $v \in H^1(S, \mathbb{C})$ such that $|v| \leq 1$ a.e. $x \in S$, we have that

\[
\left| \int_S R(x) \cdot \text{Im} \nabla v \, dx \right| \leq \left| \int_S R(x) \cdot (\text{Im} \nabla(v - iP)v + P|v|^2) \, dx \right| \\
\leq \|R\|_{L^2(S)} \cdot \left( \|\nabla(v - iP)v\|_{L^2(S)} + \|P\|_{L^2(S)} \right) 
\]

We are now in the position to state and prove

**Lemma 3.** The following estimates hold:

- $$F^\delta_\delta[v, B] \leq |\log \delta|^2,$$  

- $$\left| \int_{\Omega_\delta} \nabla^\perp h^\delta_D \cdot \text{Im} \nabla v \, dx \right| \leq |\log \delta|^2.$$

**Proof.** Use (89) and Poincaré inequality to estimate the integral term in (88):
\[
\left| \int_{\Omega_\delta} \nabla^\perp h^\delta_D \cdot \text{Im} \nabla v \, dx \right| \leq \|\nabla h^\delta_D\|_{L^2(\Omega_\delta)} \cdot \left( \|\nabla(v - iP)v\|_{L^2(\Omega_\delta)} + C_\Omega \|\text{curl } B\|_{L^2(\Omega)} \right) \\
\leq \frac{1}{2\alpha} \|\nabla h^\delta_D\|_{L^2(\Omega_\delta)}^2 + \frac{\alpha}{2} \left( \|\nabla(v - iP)v\|_{L^2(\Omega_\delta)}^2 + C_\Omega^2 \|\text{curl } B\|_{L^2(\Omega)}^2 \right) \\
\leq O(|\log \delta|^2) + \frac{1}{2} F^\delta_\delta[v, B] 
\]
where \( \alpha = \min(1, C^{-2}_\Omega) \). Here we have used the standard fact that \( |u_\delta| \leq 1 \) and, therefore, \( |v| \leq 1 \) a.e. \( x \in \Omega_\delta \).

Combining the inequality (92) with (88) gives

\[
F_\delta[v, B] \leq O(\| \log \delta \|^2).
\]  

(93)

The estimates (92) and (93) imply (91).

The bound (93) allows us to apply the ball construction method to \( F_\delta \). Theorem 3 gives the following lower bound on the energy inside “bad” disks:

\[
F_\delta[v, B; B_i] \geq \pi |d_i| \left( \log \frac{d_i^2}{|d_i|} - C \right) \quad \text{for every } i \in \mathcal{I}.
\]  

(94)

Here \( F_\delta[v, B; B_i] \) is the energy \( F_\delta[v, B] \) where first two integrals are taken over the domain \( B_i = B(b^i, r_i) \). To continue working with (88) we prove the following lemma.

**Lemma 4.** The following representation holds:

\[
\int_{\Omega_\delta} \nabla h_\delta D : \text{Im } \nabla v dx = 2\pi \sum_{i \in \mathcal{I}_1} (h_{\text{ext}} - h_\delta D(b^i))d_i + 2\pi \sum_{j=1}^N D_j^i(h_{\text{ext}} - H_R^j) + O(1)
\]  

(95)

where \( D_j^i = \text{deg}(v, \gamma_j^i) = D_j^i - D^j \), the circular curves \( \gamma_j^i = \partial B(a^j, R) \) enclose \( \omega_j^i \) with \( R = \delta + O(\delta^2) \), the quantities \( H_R^j = D^j K_0(R) + h_{\text{ext}} \xi_0(a^j) \), and \( \mathcal{I}_1 \) includes only the balls that are proper subsets of \( \Omega_\delta \setminus \bigcup_{j=1}^N \omega_j^i \) and do not intersect the boundary \( \partial \Omega_\delta \).

**Proof.** We divide the domain \( \Omega_\delta \) into three disjoint parts:

\[
\Omega_\delta = S \cup V \cup G,
\]  

(96)

where \( S = \bigcup_{j=1}^N S_j \) consists of the annuli between \( \partial \omega_j^i \) and \( \gamma_j^i \), the set \( V = [(\cup_{i \in \mathcal{I}} B_i) \setminus S] \bigcap \Omega_\delta \) consists of the “bad” disks, and \( G \) corresponds to the remainder of the set \( \Omega_\delta \).

Consider the subdomains \( S, V, \) and \( G \) separately. The balls \( B_i \)—as well as stripes \( S_j \)—are very small so that

\[
\int_{V \cup S} \nabla h_\delta D : \text{Im } \nabla v dx \leq \text{meas } (V \cup S)^{1/4} \cdot \| \nabla h_\delta D \|_{L^4(V \cup S)} \\
\cdot (\| (\nabla - iB)v \|_{L^2(V \cup S)} + \| B \|_{L^2(V \cup S)}) \\
\leq C \delta^{3/4} \cdot | \log \delta | \cdot | \log \delta | = o(1).
\]  

(97)

Introduce the function \( w = v/|v| \). Then

\[
\int_G \nabla h_\delta D : \text{Im } \nabla v dx = \int_G \nabla h_\delta D : \text{Im } \nabla w dx + \int_G \nabla h_\delta D : (\text{Im } \overline{\nabla v} - \text{Im } \overline{\nabla w}) dx \\
= I_1 + I_2.
\]  

(98)

To estimate the second integral, use the following:

\[
\text{Im } \overline{\nabla v} - \text{Im } \overline{\nabla w} = \text{Im } (\overline{|v|} (|w| \nabla |v| + |v| \nabla w) - \overline{w} \nabla w) \\
= \text{Im } (|v| \nabla |v| + (|v|^2 - 1) \overline{w} \nabla w) = (|v|^2 - 1) \text{Im } \overline{w} \nabla w
\]  

(99)
and
\[ |\nabla v|^2 = |v|^2 |\nabla w|^2 + |\nabla v|| \geq (1 - \theta)^2 |\nabla w|^2 \geq \frac{1}{4} |\nabla w|^2 \] (100)
since by Theorem 3 we have \(|v| \geq 1 - \theta\) outside \(B_i\). The function \(v\) admits the same estimate as \(u_3\). Add and subtract \(iBv\) to get
\[
\frac{1}{2} \|\nabla v\|^2_{L^2(G)} = \frac{1}{2} \int_G |\nabla v|^2 \, dx \leq \int_G ((\nabla - iB)v|^2 + |v|^2 |B|^2) \, dx \\
\leq \int_{\Omega} (|\nabla - iB)v|^2 \, dx + C_\Omega \int_{\Omega} |\text{curl} B|^2 \, dx \leq C|\log \delta|^2.
\] (101)
This leads to the following estimate:
\[
|I_2| \leq \int_G \nabla \cdot (|v|^2 - 1) \text{Im} \varpi w \, dx \\
\leq \|\nabla \delta \|_{L^\infty(G)} \cdot \int_G (|v|^2 - 1) \cdot |\nabla w| \, dx \\
\leq C\delta^{-1} \cdot \|\nabla \delta\|_{L^2(G)} \cdot \|\nabla v\|_{L^2(G)} \\
\leq C\delta^{-1} \cdot |v|^2 - 1 \text{Im} \varpi w \, dx
\] (102)
due to (9).

Now rewrite the integral \(I_1\). Integrating by parts, we obtain:
\[
I_1 = \int_G \nabla \cdot (\delta \delta D - h_{\text{ext}}) \cdot \text{Im} \varpi w \, dx = - \int_G (\delta \delta D - h_{\text{ext}}) \nabla \cdot \text{Im} \varpi w \, dx \\
+ \int_{\partial \Omega} (\delta \delta D - h_{\text{ext}}) \text{Im} \varpi w \cdot \tau \, ds - \int_{\partial V} (\delta \delta D - h_{\text{ext}}) \text{Im} \varpi w \cdot \tau \, ds \\
- \int_{\bigcup \gamma_j} (\delta \delta D - h_{\text{ext}}) \text{Im} \varpi w \cdot \tau \, ds
\] (103)
where \(I_{1i} = \int_{\partial V_i} (\delta \delta D - h_{\text{ext}}) \text{Im} \varpi w \cdot \tau \, ds \) and \(V_i = B_i \cap \Omega_3\). The term \(\nabla \cdot \text{Im} \varpi w = \text{curl} \Phi = 0\), where \(\Phi\) is a phase of \(w\), disappears.

Since the curves \(\gamma_j\) are small, we can approximate \(\delta \delta D\) by a constant \(H_{R_j}\) to conclude that
\[
\int_{\gamma_j} (\delta \delta D - h_{\text{ext}}) \text{Im} \varpi w \cdot \tau \, ds = 2\pi D_j (\delta \delta D - h_{\text{ext}}) + \int_{\gamma_j} (\delta \delta D - H_{R_j}^2) \text{Im} \varpi w \cdot \tau \, ds.
\]
Set \(H_{R_j} = h_{\text{ext}} \xi_0(a^2) + D_j K_0(R)\). Using the decomposition (21) of \(\delta \delta D\), we get
\[
|h_{\delta \delta D}(x) - H_{R_j}^2| \leq h_{\text{ext}}|\xi_0(x) - \xi_0(a^2)| + |h_3(x)| \leq C_1|\log \delta|^2 + C_2|D|
\] (104)
for $x \in \gamma_i^L$. This yields

$$\left| \int_{\gamma_i} (h_{SD} - H^l_D) \text{Im} \nabla w \cdot \tau \, ds \right| \leq (C_1 \delta \log \delta)^2 + C_2 |D|) \cdot D_v^l = O(1). \quad (105)$$

As a result we estimate that

$$I_1 = - \sum_{i \in I} I_{1i} - \sum_{j=1}^N 2\pi D_v^l (H^l_R - h_{ext}) + O(1). \quad (106)$$

We now consider two cases. First, suppose that the set $\mathcal{I}_1 \subset \mathcal{I}$ is such that $B_1 \subset \Omega_{\delta} \setminus S$ for $i \in \mathcal{I}_1$. We estimate the integrals $I_{1i}$ in a similar way as we did for the hole vortices. Approximate $h_{SD}(x)$ by a constant value in the center of $B_i$:

$$I_{1i} = \int_{\partial V_i} (h_{SD} - h_{SD}(b^i)) \text{Im} \nabla w \cdot \tau \, ds + \int_{\partial V_i} (h_{SD}(b^i) - h_{ext}) \text{Im} \nabla w \cdot \tau \, ds = J_{1i} + J_{2i} \quad (107)$$

Second integral directly gives the degree $d_i$ of the possible bulk vortex:

$$J_{2i} = 2\pi d_i (h_{SD}(b^i) - h_{ext}). \quad (108)$$

To estimate $J_{1i}$ we introduce the subdomains $U_i = V_i \cap \{x \mid |v(x)| \leq 1/2\}$ so that their boundaries are the level sets of $v$. We add and subtract the integral over $\partial U_i$:

$$\sum_{i \in \mathcal{I}} J_{1i} = J_1 + J_2, \quad (109)$$

where

$$J_1 = \int_{U_i \mathcal{I}_1 V \cup U_i} (h_{SD} - h_{SD}(b^i)) \text{Im} \nabla w \cdot \tau \, ds, \quad (110)$$

$$J_2 = \int_{\partial U_i} (h_{SD} - h_{SD}(b^i)) \text{Im} \nabla w \cdot \tau \, ds - \int_{\partial U_i} (h_{SD} - h_{SD}(b^i)) \text{Im} \nabla w \cdot \tau \, ds$$

$$= \int_{U_i \mathcal{I}_1 V \cup U_i} \nabla \cdot [(h_{SD} - h_{SD}(b^i)) \text{Im} \nabla w] \, dx = \int_{U_i \mathcal{I}_1 (V \cup U_i)} \nabla^\perp h_{SD} \cdot \text{Im} \nabla w \, dx, \quad (111)$$

since $\nabla^\perp \cdot \text{Im} \nabla w = 0$. The term $J_2$ is small:

$$|J_2| \leq \text{meas} (\mathcal{I})^{1/2} \cdot \|\nabla^\perp h_{SD}\|_{L^\infty(\mathcal{I})} \cdot 2\|\nabla v\|_{L^2(\mathcal{I})} \leq O(\delta^2) \cdot O\left(\frac{1}{\delta}\right) \cdot O(|\log \delta|) = o(1). \quad (112)$$

To estimate $J_1$, note, that $|v| = 1/2$ on $\partial U_i$ so that $\nabla w \cdot \tau = 2\nabla v \cdot \tau$ on $\partial U_i$ and:

$$J_1 = \int_{\partial U_i V \cup U_i} (h_{SD} - h_{SD}(b^i)) \text{Im} \nabla w \cdot \tau \, ds = 4 \int_{\partial U_i V \cup U_i} (h_{SD} - h_{SD}(b^i)) \text{Im} \nabla v \cdot \tau \, ds$$

$$= 4 \int_{U_i \mathcal{I}_1 U_i} \nabla^\perp h_{SD} \cdot \text{Im} \nabla v \, dx + 4 \int_{U_i \mathcal{I}_1 U_i} (h_{SD} - h_{SD}(b^i)) \text{Im} (\nabla^\perp \nabla) \, dx = L_1 + L_2.
The first integral $L_1$ admits the same estimate as in (112). To estimate $L_2$ note that

$$|\text{Im} (\nabla \cdot \nabla v)| \leq |\nabla \cdot \nabla v| = |\nabla v|^2.$$  \hspace{1cm} (113)

Then

$$|L_2| \leq 4 \sum_{i \in \mathcal{I}_1} \|h_{\delta D} - h_{\delta D}(b_i')\|_{L^\infty(U_i)} \cdot \|\nabla v\|_{L^2(\Omega)}^2$$

$$\leq 4 \sum_{i \in \mathcal{I}_1} \|\nabla h_{\delta D}\|_{L^\infty(U_i)} \cdot r_i \cdot |\log \delta|^2 \leq O\left(\frac{1}{\delta^2}\right) \cdot |\log \delta|^2 = o(1).$$ \hspace{1cm} (114)

Thus all integrals $L_1$, $L_2$, and therefore $J_1$, $J_2$, and $J_3$ are small. The only ingredient left to consider is the set $\mathcal{I}_2$ consisting of the balls that intersect the boundary $\partial \Omega$. Here the estimates are very similar to those on the balls from $\mathcal{I}_1$ if we recall the boundary condition $h_{\delta D} = h_{\text{ext}}$ on $\partial \Omega$:

$$\sum_{i \in \mathcal{I}_2} I_{1i} = \int_{\cup_{i \in \mathcal{I}_2} \partial U_i} (h_{\delta D} - h_{\text{ext}}) \text{Im} \nabla w \cdot \tau ds$$

$$= 4 \int_{\cup_{i \in \mathcal{I}_2} \partial U_i} (h_{\delta D} - h_{\text{ext}}) \text{Im} \nabla v \cdot \tau ds + \int_{\cup_{i \in \mathcal{I}_2} (V_i \setminus U_i)} \nabla \cdot h_{\delta D} \cdot \text{Im} \nabla w dx$$

$$= 4 \int_{\cup_{i \in \mathcal{I}_2} U_i} \nabla \cdot (h_{\delta D} - h_{\text{ext}}) \cdot \text{Im} \nabla v dx + 4 \int_{\cup_{i \in \mathcal{I}_2} U_i} (h_{\delta D} - h_{\text{ext}}) \text{Im} (\nabla \cdot \nabla v) dx + o(1)$$

$$= o(1).$$ \hspace{1cm} (115)

The external magnetic field here plays the same role as $h_{\delta D}(b_i')$ in (114), that is:

$$|h_{\delta D}(x) - h_{\text{ext}}| \leq \|\nabla h_{\delta D}\|_{L^\infty(\Omega)} \cdot 2r_i \leq O(\delta)$$ \hspace{1cm} (116)

in $B_i$ for $B_i \cap \partial \Omega \neq \emptyset$ because $h_{\delta D} = h_{\text{ext}}$ on $\partial \Omega$.

Combining the estimates we obtain

$$\sum_{i \in \mathcal{I}} I_{1i} = \sum_{i \in \mathcal{I}_1} \sum_{i \in \mathcal{I}_2} 2\pi d_i (h_{\delta D}(b_i') - h_{\text{ext}}) + o(1),$$ \hspace{1cm} (117)

thus concluding the proof. \hspace{1cm} $\square$

Putting together (88), (94), and (95) we get

$$F_5[v, B; G] + \pi d \left(\log \frac{\delta^2}{dx} - C\right) \leq 2\pi \sum_{i \in \mathcal{I}} (h_{\text{ext}} - h_{\delta D}(b_i')) d_i + 2\pi \sum_{j=1}^N D_j^v (h_{\text{ext}} - H_j),$$ \hspace{1cm} (118)

where $d = \sum_{i \in \mathcal{I}} |d_i|$ as before. This inequality holds under the assumption that $d$ is nonzero. If, on the other hand, $d$ equals zero, the term $\pi d \left(\log \frac{\delta^2}{dx} - C\right)$ should be dropped.

In the following lemma we obtain the lower bound for $F_5$ that allows us to show that there are no bulk vortices, i.e, $d_i = 0$. 

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Lemma 5. There exists a $\delta_0 > 0$ such that, for any $\delta \leq \delta_0$, there are no bulk vortices inside the domain $\Omega \setminus \overline{S}$. Moreover, there exist an $\alpha > 1$ and an $\delta \ll R' \ll 1$ such that the following inequality holds:

$$
\sum_{j=1}^{N} \left[ \pi (1 - \theta)^2 (|\log \delta| - |\log R'| + O(\delta))(D_j^i)^2 - 2\pi D_j^i (h_{ext} - H_j^2) \right] \leq O(1).
$$

(119)

Proof. Fix $\alpha > 1$ and consider two cases:

1. $\sum_{j=1}^{N} |D_j^i| \leq \alpha \sum_{i \in \mathcal{G}} |d_i|$. The leading term in (118) is $\pi d \log \varepsilon$ on the left hand side and it cannot be bounded by the right hand side if $d \neq 0$ because the leading term there is of order $d \cdot O(|\log \delta|)$. Therefore $d = 0$, and there are no bulk vortices and all $D_v = 0$.

2. $\sum_{j=1}^{N} |D_j^i| > \alpha \sum_{i \in \mathcal{G}} |d_i|$. We need an additional lower bound on the energy $F_\delta[v; B; G]$.

To estimate $F_\delta[v; B; G]$, we integrate over circles $\gamma_j^i = \partial B(a_j, r)$ around the holes $\omega_j^i$ with $r > R$. If $|u| \neq 0$ on $\gamma_j^i$ for some $r > R$, we can define the degree on $\gamma_j^i$ via

$$
D_j^i = \deg(u, \gamma_j^i) = \deg(v, \gamma_j^i)
$$

(120)

Denote

$$
\mathcal{R} = \{ r \in (R, R_{max}) : |u| > 1 - \theta \text{ on } \gamma_j^i \text{ for all } j = 1 \ldots N \},
$$

(121)

where $\theta$ is specified in the Ball Construction Method and $R_{max}$ plays the same role as in Lemma 1, i.e., it is the maximal radius $r$ such that $B(a_j, r)$ are disjoint and do not intersect $\partial \Omega$. The total degree on $\partial \Omega$ is the sum of the degrees of all vortices. Since $D_v^i = D_v^i$ by definition of $D_v^i$, we have

$$
\sum_{j=1}^{N} |D_j^i| \geq \sum_{j=1}^{N} |D_j^i| - \sum_{i \in \mathcal{G}} |d_i| \geq \frac{\alpha - 1}{\alpha} \sum_{j=1}^{N} |D_j^i|.
$$

(122)

Using the definition of the degree and the Divergence Theorem for $r \in \mathcal{R}$ we get

$$
2\pi D_j^i - \int_{\gamma_j^i} \text{curl} B \, dx = \int_{\gamma_j^i} \nabla \Phi \cdot \tau - B \cdot \tau dS = \int_{\gamma_j^i} (\nabla \Phi - B) \cdot \tau dS
$$

(123)

or

$$
2\pi D_j^i = \int_{\gamma_j^i} (\nabla \Phi - B) \cdot \tau dS + \int_{B_j^i} \text{curl} B \, dx = I_1(r) + I_2(r)
$$

(124)

for any $j = 1 \ldots N$. Here $B_j^i = B(a_j, r)$ and $v = |v|e^{i\Phi}$. The following estimates

$$
I_1^2 \leq \text{meas}(\gamma_j^i) \int_{\gamma_j^i} |\nabla \Phi - B|^2 dS \leq 2\pi r \int_{\gamma_j^i} \frac{|(\nabla - iB)v|^2}{|v|^2} dS \leq \frac{2\pi r}{(1 - \theta)^2} \int_{\gamma_j^i} |(\nabla - iB)v|^2 dS,
$$

(125)

$$
I_2^2 \leq \text{meas}(B_j^i) \int_{B_j^i} |\text{curl} B|^2 dx \leq C_1 |\log \delta|^2 r^2,
$$

(126)

hold since $|v| > 1 - \theta$ by the Ball Construction Method. Further

$$
4\pi^2 (D_j^i)^2 = (I_1(r) + I_2(r))^2
$$

$$
\leq \frac{2\pi r}{(1 - \theta)^2} \int_{\gamma_j^i} |(\nabla - iB)v|^2 dS + 2C_1 |\log \delta|^2 r^2 \cdot I_1 + C_1 |\log \delta|^2 r^2
$$

(127)
for \( r \in \mathfrak{R} \). Now, divide both sides of (127) by \( r \) and integrate outside of the “bad” disks from \( R \) to ~

\[
4\pi^2 \int_{(R,R') \cap \mathfrak{R}} \frac{(D_j)^2}{r} \, dr \leq \frac{2\pi}{(1-\theta)^2} \int_{(R,R') \cap \mathfrak{R}} \gamma_j^r |(\nabla - iB)v|^2 \, dSdr
\]

\[
+ 2C_1|\log \delta|^2 \cdot \int_{(R,R') \cap \mathfrak{R}} I_1 rdr + C_1 |\log \delta|^2 \frac{r^2}{2} \bigg|_{R'}^R
\]

\[
\leq \frac{4\pi}{(1-\theta)^2} F_\delta[v, B; B'_R] + \frac{C_1}{2}|\log \delta|^2 R'^2
\]

\[
+ 2C_1|\log \delta|^2 \cdot R' \cdot \sqrt{\pi R'^2} \cdot \left( \int_{(R,R') \cap \mathfrak{R}} \int_{\gamma_r^R} |(\nabla - iB)v|^2 \, dSdr \right)^{1/2}
\]

\[
\leq \frac{4\pi}{(1-\theta)^2} F_\delta[v, B; K'] + \frac{C_1}{2}|\log \delta|^2 R'^2 + \frac{C_3(1-\theta)}{4\pi}|\log \delta|^3 R'^2,
\]

(128)

since \(|v| > 1 - \theta\) by the definition of \( \mathfrak{R} \). Here \( R' \ll R_{\text{max}} \) that will be prescribed later on and \( K^j \) is a union of concentric rings around \( j \)th hole:

\[
K^j = \bigcup_{r \in (R,R') \cap \mathfrak{R}} \gamma_j^r = \bigcup_{r \in (R,R') \cap \mathfrak{R}} \partial B(a^j, r). \tag{129}
\]

Notice that all \( K^j \) are disjoint since \( R' \ll R_{\text{max}} \) and \( K^j \subset G \) for all \( j = 1 \ldots N \).

In order to obtain the lower bound for \( F_\delta \) we divide both sides in (128) by \( 4\pi/(1-\theta)^2 \):

\[
\pi(1-\theta)^2 \int_{(R,R') \cap \mathfrak{R}} \frac{(D_j)^2}{r} \, dr \leq F_\delta[v, B; K'] + \frac{C_1(1-\theta)^2}{8\pi}|\log \delta|^2 R'^2 + \frac{C_3(1-\theta)}{4\pi}|\log \delta|^3 R'^2
\]

(130)

We can choose

\[
R' = C\zeta^{1/2}|\log \delta|^{-2} \gg R
\]

(131)

and an appropriate constant \( C \) such that for \( \zeta = |\log \delta|^{-1} = o(1) \) the sum of last two terms in (130) is less than \( \zeta \) for small \( \delta \). Notice, that \( \text{meas}((R,R') \setminus \mathfrak{R}) < \delta^2 \) by the Ball Construction Method and \( R \leq \delta + \delta^2 \). Therefore

\[
\sum_{j=1}^N \int_{(R,R') \cap \mathfrak{R}} \frac{(D_j)^2}{r} \, dr \geq \frac{(\alpha - 1)^2}{\alpha^2} \frac{1}{N} \sum_{j=1}^N |D_j|^2 \log r \bigg|_{\delta+2\delta^2}^{R'}
\]

\[
\geq \frac{(\alpha - 1)^2}{\alpha^2} \frac{1}{N} \sum_{j=1}^N |D_j|^2 (|\log \delta| - |\log R'| + O(\delta)). \tag{132}
\]

Thus we can combine (130) and (132) to express the lower bound for \( F_\delta[v, B; G] \) in terms of the additional degrees \( D_j \):

\[
F_\delta[v, B; G] \geq \sum_{j=1}^N F_\delta[v, B; K^j]
\]

\[
\geq \pi(1-\theta)^2 \frac{(\alpha - 1)^2}{\alpha^2} \frac{1}{N} \sum_{j=1}^N |D_j|^2 (|\log \delta| - |\log R'| + O(\delta)) - \zeta
\]

(133)
Substituting $\zeta = |\log \delta|^{-1}$ and combining (133) with (118), we get
\[
\sum_{j=1}^{N} \left( \frac{1}{N} \pi (1 - \theta)^2 \frac{(\alpha - 1)^2}{\alpha^2} (|\log \delta| - |\log R'| + O(\delta))(D_v^j)^2 - 2\pi (h_{\text{ext}} - H_{R}^j)D_v^j \right) \\
\leq -\pi \sum_{i \in \mathcal{I}_1} |d_i| (|\log \epsilon| - 2|\log \delta| + |\log d| - C) + 2\pi \sum_{i \in \mathcal{I}_1} (h_{\text{ext}} - h_{\delta}(b_i))d_i + O(1).
\]
(134)

Compare the order of the leading terms in (134):
\[
\sum_{j=1}^{N} \left( A|\log \delta|(D_v^j)^2 - O(|\log \delta|)D_v^j \right) \leq -d|\log \epsilon| + O(1)
\]
(135)
with $A > 0$. The left hand side of (135) is a sum of quadratic functions in $D_v^j$ with positive leading coefficients:
\[
q_j(D_v^j) = A|\log \delta|(D_v^j)^2 - O(|\log \delta|)D_v^j.
\]
(136)
The values of parabolas $q_j$ are bounded from below by the values at their vertices
\[
t^j = \frac{O(|\log \delta|)}{2A|\log \delta|} = O(1),
\]
(137)
that are themselves bounded. Therefore
\[
-d|\log \epsilon| + o(1) \geq \sum_{j=1}^{N} q_j(D_v^j) \geq \sum_{j=1}^{N} q_j(t_j) = O(|\log \delta|).
\]
(138)

Since $|\log \epsilon| \gg |\log \delta|$, the inequality (138) can hold only if $d = 0$, i.e. there are no bulk vortices. This, in turn, implies that $D_v^j = D_v^j$ and the inequality (122) is no longer needed. It simplifies the lower bound (132) and yields the desired inequality. \(\square\)

### 6 Proof of Theorem 1: Equality of Degrees

**Proof.** To finish the proof of Theorem 1 we need to show that all $D_v^j = 0$. We start with the quadratic inequality for $D_v^j$ obtained in Lemma 5:
\[
\sum_{j=1}^{N} \left[ \pi (1 - \theta)^2 (|\log \delta| - |\log R'| + O(\delta))(D_v^j)^2 - 2\pi D_v^j(h_{\text{ext}} - H_{R}^j) \right] \leq O(1),
\]
(139)
where $H_{R}^j = h_{\text{ext}}\xi_0(a^j) + D^j K_0(R)$. This inequality has the same structure as the quadratic functional in $S^1$-valued case: there are no mixed terms $D_v^j D_v^j$. Therefore we can find zeros for each $j = 1 \ldots N$ separately.

Fix $1 \leq j \leq N$. Clearly, $D_v^j = 0$ is one of two roots of
\[
\pi (1 - \theta)^2 (|\log \delta| - |\log R'| + O(\delta))(D_v^j)^2 - 2\pi D_v^j(h_{\text{ext}} - H_{R}^j) = 0.
\]
(140)
Since $K_0(R) = |\log \delta| + O(1)$ and
\[ D^j = \left\lfloor \sigma \left( 1 - \xi_0(a^j) \right) \right\rfloor, \tag{141} \]
we can calculate the coefficient for the linear term in (139):
\[ -2\pi(h_{\text{ext}} - H^j) = -2\pi(\sigma|\log \delta| - \sigma|\log \delta|\xi_0(a^j) - \left\lfloor \sigma \left( 1 - \xi_0(a^j) \right) \right\rfloor |\log \delta|) + O(1) \]
\[ = -2\pi|\log \delta|\left( \sigma \left( 1 - \xi_0(a^j) \right) - \left\lfloor \sigma \left( 1 - \xi_0(a^j) \right) \right\rfloor \right) + O(1). \tag{142} \]
Since $\left\lfloor \cdot \right\rfloor$ is the nearest integer, we have
\[ \left| \sigma \left( 1 - \xi_0(a^j) \right) - \left\lfloor \sigma \left( 1 - \xi_0(a^j) \right) \right\rfloor \right| \leq \frac{1}{2} - \xi, \tag{143} \]
assuming the uniqueness condition (10) and taking
\[ \xi = \min_{j=1,\ldots,N} \text{dist} \left( \sigma \left( 1 - \xi_0(a^j) \right), \mathbb{Z} + \frac{1}{2} \right) > 0. \tag{144} \]
The second zero of (140) can be estimated as follows:
\[ |t_j| = \left| \frac{-2\pi(\sigma \left( 1 - \xi_0(a^j) \right) - \left\lfloor \sigma \left( 1 - \xi_0(a^j) \right) \right\rfloor)}{\pi(1 - \theta)^2 + o(1)} \right| + o(1) < \frac{1 - 2\xi}{(1 - \theta)^2} + o(1). \tag{145} \]
Having $\xi$ fixed and $\delta < \delta_0$ sufficiently small, we can always take $\theta > 0$ small enough to make sure that $|t_j| < 1 - \xi$.
Since $D^j_\nu$ can take only integer values, if at least one $D^j_\nu$ is nonzero, the left hand side of (139) becomes strictly positive of order $O(\log \delta)$. This contradiction finishes the proof of main theorem yielding
\[ D^j_\nu = 0 \text{ or } D^j_{\delta,\epsilon} = D^j \tag{146} \]
for all $j = 1 \ldots N$.

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A Appendix. Gradient estimate

Lemma 6. Let \( u \) solve the Poisson equation with Dirichlet boundary conditions in \( \Omega_\delta = \Omega \setminus \bigcup_{j=1}^N \omega_j \) with \( \omega_j = B(a_j, \delta) \), that is

\[
\begin{align*}
-\Delta u &= f & \text{in } \Omega_\delta, \\
u &= g & \text{on } \partial \Omega, \\
u &= g_j & \text{on } \partial \omega_j, 
\end{align*}
\]

(147)

where \( g \) and \( g_j \) are smooth functions that are defined in the whole of \( \Omega_\delta \). Then

\[
\| \nabla u \|_{L^\infty(\Omega_\delta)} \leq C \left( \frac{1}{\delta} \| u \|_{L^\infty(\Omega_\delta)} + \| f \|_{L^\infty(\Omega_\delta)} + \| \Delta g \|_{L^\infty(\Omega_\delta)} + \delta \sum_{j=1}^N \| \Delta g_j \|_{L^\infty(\Omega_\delta)} \right). 
\]

(148)

Proof. The proof is based on lemmas A.1 and A.2 from [8]. Consider the three cases: the point \( x_0 \in \Omega_\delta \) is far from the boundaries of \( \partial \Omega_\delta \), it is close to \( \partial \Omega \), and it is close to \( \partial \omega_j \) for some \( j = 1 \ldots N \). The first case when \( x_0 \in K \subset \subset \Omega_\delta \) is resolved in Lemma A.1 [8] and the second case, when \( x_0 \) is close to \( \partial \Omega \), can be deduced from Lemma A.2 using \( \tilde{u} = u - g \). The results of both lemmas can be merged together in the following estimate:

\[
|\nabla u(x_0)| \leq C \left( \| u \|_{L^\infty} + \| f \|_{L^\infty} + \| \Delta g \|_{L^\infty} \right) \quad \text{a.e.} 
\]

(149)

when \( \text{dist} (x_0, \partial \omega_j) > m > 0 \) with some fixed \( m \) independent of \( \delta \).

The third case is specific to our setting. Let \( x_0 \) be close to one of the holes: \( \text{dist} (x_0, \partial \omega_j) \leq m \) for some \( j = 1 \ldots N \). Without loss of generality assume \( a_j = 0 \). We introduce the new spatial variable \( y = \frac{x}{\delta} \) to rescale the domain so that the \( \omega_j \) becomes \( B(0, 1) \) and \( x_0 \) becomes \( y_0 \). The Poisson equation in new coordinates becomes

\[
-\Delta_y u = \delta^2 f. 
\]

(150)

If \( \text{dist} (y_0, \partial B(0, 1)) > m \), we apply Lemma A.1 from [8] again. It gives us the estimate for \( |\nabla_y u(y_0)| \):

\[
|\nabla_y u(y_0)| \leq C \left( \| u \|_{L^\infty} + \delta^2 \| f \|_{L^\infty} \right) 
\]

(151)

that in turn implies the estimate for \( |\nabla_x u(x_0)| \):

\[
|\nabla_x u(x_0)| = \frac{1}{\delta} |\nabla_y u(y_0)| \leq C \| u \|_{L^\infty(\Omega_\delta)} + C \delta \| f \|_{L^\infty(\Omega_\delta)}. 
\]

(152)

Finally, we apply Lemma A.2 to \( \tilde{u}_j = u - g_j \) that satisfies the problem

\[
\begin{align*}
-\Delta_y \tilde{u}_j &= \delta^2 f + \Delta_y g_j & \text{in } B(0, 1+m) \setminus B(0, 1), \\
\tilde{u}_j &= h_j & \text{on } \partial B(0, 2+m), \\
\tilde{u}_j &= 0 & \text{on } \partial B(0, 1).
\end{align*}
\]

(153)

where \( h_j(y) = u(y) - g_j(y) \). Since the proof of Lemma A.2 uses only local estimates and \( y_0 \) is far from the \( \partial B(0, 2+m) \), the function \( h_j \) does not play a role for the estimate of \( |\nabla_y u(y_0)| \). It yields the estimate

\[
|\nabla_y u(y_0)| \leq C \left( \| u \|_{L^\infty} + \delta^2 \| f \|_{L^\infty} + \| \Delta y g_j \|_{L^\infty} \right). 
\]

(154)
Going back to \( x \) we obtain

\[
|\nabla_x u(x_0)| \leq \frac{C}{\delta} \|u\|_{L^\infty} + C\delta (\|f\|_{L^\infty} + \|\Delta_x g\|_{L^\infty}).
\] (155)

Merging all the estimates we finish the proof. \qed