The analogue Hawking effect in rotating polygonal hydraulic jumps

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Rotation of non-circular hydraulic jumps is a recent experimental observation that lacks a theory based on first principles. Here we furnish a basic theory of this phenomenon founded on the shallow-water model of the circular hydraulic jump. The breaking of the axial symmetry morphs the circular jump into a polygonal state. Variations on this state rotate the polygon in the azimuthal direction. The dependence of the rotational frequency on the flow rate and on the number of polygon vertices agrees with known experimental results. We also predict how the rotational frequency varies with viscosity. Finally, we establish a correspondence between the rotating polygonal structure and the Hawking effect in an analogue white hole. The rotational frequency of the polygons affords a direct estimate of the frequency of the thermal Hawking radiation.

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In low-dimensional flows, the hydraulic jump is associated with an abrupt increase in the height of a flowing liquid [1]. Along a standing circular locus, this feature is known as the two-dimensional circular hydraulic jump [2,3]. Its geometry maintains a symmetry about a reference axis that passes normally through the point of the origin of the flow on a flat plane. To develop a cogent mathematical theory, the flow is viewed as a shallow layer of liquid [2] confined to a horizontal plane, diverging radially outwards from a source point. Thereafter, applying the boundary-layer approximation in the shallow flow, viscosity is invoked [2,4], along with all other complications of the nonlinear Navier-Stokes equation. A compelling evidence in favour of viscosity comes from an experimentally verified formula of the jump radius, scaled in terms of viscosity [2,5]. Various physical interpretations of this scaling law have been forwarded [2,3].

Circular hydraulic jumps appear in Type-I and Type-II states [6], of which the latter is formed by increasing the height of the flow in the post-jump region. Type-II jumps have a wider jump region with a surface eddy all along its circular rim like a floating torus [6]. When water is replaced with a liquid of much greater viscosity, the Type-II state spontaneously breaks the axial symmetry of the circular state, resulting in a front that has a polygonal geometry [6,9]. This transition has visibly temporal features because a linear instability arises from the initially circular state, and leads to the formation of a non-axisymmetric polygonal structure, whose dependence on the azimuthal angle is periodic [9]. Within the widened jump region, the transport of liquid also becomes azimuthal, although prior to it, in the pre-jump region, the flow is radial [9]. Clearly, the breaking of axial symmetry is localized only at the jump radius or thereabouts. All of these features of the polygonal jump have been known well for two decades, but in addition, as reported very recently, the polygonal jump undergoes a rotation [10]. This new phenomenon has been named the “rotational hydraulic jump” [10,24].

Thus far we have provided a brief narrative of the role that viscosity plays in the multifarious properties of the two-dimensional hydraulic jump. Now we consider the jump from a different perspective [3,11–14], according to which the position of the standing jump is a boundary where the steady radial velocity, \(v_0(r)\), equals the local speed of long-wavelength surface gravity waves, \(gh_0\), with \(h_0(r)\) being the steady local height of the flow layer. This boundary is like a standing horizon, segregating the supercritical and the subcritical regions of the flow, with criticality referring to the condition under which the speed of the bulk flow matches the speed of surface gravity waves [3]. This entire point of view lays strong emphasis on the advective and pressure terms in the Navier-Stokes equation, to the neglect of the viscous term. The radius of the horizon (which is also the jump), \(r_J\), is defined by the critical condition, \(v_0(r_J) = gh_0(r_J)\), and sets a spatial limit for the transmission of information. As the equatorial flow proceeds from its point of origin, its radial velocity, \(v_0 > \sqrt{gh_0}\), but viscosity and the radial geometry slow down the flow. When the critical condition is met, both the jump and the horizon occur simultaneously [3]. In the supercritical part of the jump, where \(v_0 > \sqrt{gh_0}\), gravity waves (as carriers of information) cannot travel upstream against the bulk flow, and hence every point in the supercritical region remains uninformed about the fate of the flow downstream. This state of affairs prevails everywhere within the jump, and so it acts as an impenetrable barrier against the percolation of any information from the outside to the inside. The horizon implied by the critical condition has been amply demonstrated [15], but the horizon by itself is inadequate to explain why a jump should coincide with it, a point that has been qualitatively addressed and appreciated for long [11]. Concisely stated, the horizon is a necessary condition for the jump but not a sufficient one, and so without reference to anything regarding the jump, the horizon is just an analogue of a white hole. In this study we show that the formation of a polygonal jump and
its observed rotation \[10\] are the natural outcomes of breaking the axial symmetry of the circular horizon of the analogue white hole. The breaking of the axial symmetry, localized near the horizon, is due to the analogue surface gravity. The rotation has a persuasive similarity with the Hawking effect, which is also due to the analogue surface gravity.

The mathematical description of the two-dimensional flow is most succinctly framed in the cylindrical coordinate system, \((r, \phi, z)\), and by tailoring the Navier-Stokes equation accordingly \[11\]. Our analysis starts with the steady height-integrated Navier-Stokes equation of a shallow-water radial base flow \[2\].

\[
\frac{d v_0}{dr} + \frac{g}{h_0} \frac{dh_0}{dr} = -\nu \frac{v_0}{h_0} \tag{1}
\]

and the steady height-integrated equation of continuity,

\[
\frac{1}{r} \frac{d}{dr} \left( r v_0 h_0 \right) = 0 \tag{2}
\]

in its differential form. The integral form of Eq. (2) is \(r v_0 h_0 = Q/2\pi\), where \(Q\), a constant, is the steady volumetric flow rate. The subscript “0” in the following equations stands for steady radially varying quantities, of which the velocity of the flow, \(v_0(r)\), has been obtained in the shallow-water theory by vertically averaging the radial component of the velocity across the height of the flow. The boundary conditions used for the averaging are that velocities vanish at \(z = 0\) (the no-slip condition), and vertical gradients of velocities vanish on the free surface of the flow (the no-stress condition) \[2\], \[14\], \[16\]. These boundary conditions are applied under the standard assumption that while the vertical velocity is much small compared with the radial velocity, the vertical variation of the radial velocity (through the shallow layer of water) is much greater than its radial variation \[2\]. In this manner all dependence on the \(z\)-coordinate is averaged out.

Now, about the steady radial background flow, as implied by Eqs. \[1\] and \[2\], we develop a time-dependent azimuthal perturbation scheme prescribed by \(h(t, r, \phi) = h_0(r) + h'(t, r, \phi), v_r(t, r, \phi) = v_0(r) + v_{r}'(t, r, \phi)\) and \(v_{\phi}(t, r, \phi) = 0 + v_{\phi}'(t, r, \phi)\). All primed quantities are time-dependent perturbations in both the radial and azimuthal coordinates, with \(v_{r}'\) and \(v_{\phi}'\) being perturbations on the radial and azimuthal velocity components, respectively. We have designed the azimuthal flow to be entirely a perturbative effect without any presence in the steady background flow. Our next step is to define a new variable, \(f = r h v_r\), whose steady value, \(f_0 = r h_0 v_0\), is a constant, as we can see from Eq. (2). Within a multiplicative factor, owed to the two-dimensional geometry of the system, \(f_0\) gives the conserved volumetric flow rate under steady conditions. So \(f'\) is a perturbation on this constant background volumetric flow rate. Linearizing in \(f'\) under the formula, \(f = f_0 + f'\), gives \(f' = r (h_0 v_{r}' + v_0 h')\). A similar linearization of the general time-dependent continuity equation, bearing both radial and azimuthal variations \[1\],

leads to

\[
\frac{\partial h'}{\partial t} = \frac{1}{r} \frac{\partial f'}{\partial r} - h_0 \frac{\partial v_\phi}{r} \frac{\partial \phi}{r} \tag{3}
\]

Likewise, from the radial and azimuthal components of the Navier-Stokes equation, as expressed in cylindrical coordinates \[1\], we derive

\[
\frac{\partial v_r'}{\partial t} + \frac{\partial}{\partial r} \left( v_0 v_r' \right) + \frac{g h'}{r} = 0 \tag{4}
\]

and

\[
\frac{\partial v_{\phi}'}{\partial t} + v_0 \left( \frac{\partial v_{\phi}'}{\partial r} + \frac{v_{\phi}'}{r} \right) + \frac{g h'}{r} \frac{\partial \phi}{\partial \phi} = 0, \tag{5}
\]

respectively. We stress here that in extracting Eqs. \(4\) and \(5\), we have ignored all product terms of viscosity and the primed quantities. Therefore, in our treatment, viscosity makes its mark through the zero-order stationary quantities, \(v_0\) and \(h_0\), as Eqs. \(1\) and \(2\) indicate. In all first-order equations involving the primed quantities, Eqs. \(3\), \(4\) and \(5\), viscosity does not appear explicitly, and exerts its influence implicitly through the zero-order coefficients. This is a satisfactory approximation, to the extent that our principal concern is the effect of azimuthal variations on the axially symmetric radial flow, and the breaking of the axial symmetry therefrom.

We use Eq. \(5\) to substitute \(\partial h'/\partial t\) in the first-order partial time derivative of \(f'\), whereby we also obtain

\[
\frac{\partial v_r'}{\partial t} = \frac{v_0}{f_0} \frac{\partial f'}{\partial r} + \frac{v_0^2}{f_0} \frac{\partial f'}{\partial r} + h_0 v_0^2 \frac{\partial v_{\phi}}{\partial \phi}. \tag{6}
\]

Collectively, Eqs. \(3\) and \(6\) present a closed set of conditions that express \(h'\) and \(v_{\phi}'\) as a linear combination of \(f'\) and \(v_\phi\). At this stage we require two independent mathematical conditions on which we can impose Eqs. \(3\) and \(6\). Such conditions are readily supplied by the perturbations of the radial and azimuthal dynamics, as shown in Eqs. \(4\) and \(5\), respectively. We take the second-order partial time derivative of these two coupled equations, and on them we apply the conditions provided by Eqs. \(3\) and \(6\), along with the second-order partial time derivative of Eq. \(6\). This long algebraic exercise ultimately delivers two equations that are second-order in time. They are

\[
\frac{\partial}{\partial t} \left( v_0 \frac{\partial f'}{\partial t} \right) + \frac{\partial}{\partial r} \left( v_0^2 \frac{\partial f'}{\partial r} \right) + \frac{\partial}{\partial r} \left( v_0^2 \frac{\partial f'}{\partial r} \right)
\]

\[
- \frac{\partial}{\partial r} \left[ v_0 h_0 (v_0^2 - gh_0) \frac{\partial v_{\phi}}{\partial \phi} \right], \tag{7}
\]

derived from the perturbation of the radial dynamics, as given by Eq. \(4\), and

\[
\frac{\partial^2 v_{\phi}}{\partial t^2} + \frac{v_0}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v_{\phi}}{\partial r} \right) - \frac{gh_0 v_0^2}{r^2} = \frac{g}{r^2} \frac{\partial}{\partial \phi} \left( \frac{\partial f'}{\partial r} \right), \tag{8}
\]
derived likewise from the perturbation of the azimuthal dynamics, as given by Eq. (5). The two foregoing equations, linearized and coupled, form a set of wave equations in $f'$ and $v_{\phi}$. A familiar wave equation in $f'$ only, pertaining just to the radial dynamics, is obtained when all the $\phi$-derivatives on the right hand side of Eq. (7) are made to vanish [3]. Thereafter, the expression on the left hand side of Eq. (7) is rendered compactly as $\partial_{\alpha}(f^{00} \partial_{\beta}f') = 0$, in which the Greek indices run from 0 to 1, with 0 standing for $t$ and 1 standing for $r$. This simplification establishes an acoustic metric and an acoustic horizon in the physical problem of the two-dimensional hydraulic jump, the details of which have been discussed in a previous work [3]. Of course, we must remember that this reasoning is valid only for an inviscid fluid in a potential flow.

In our present approach, the steady background flow is affected by viscosity, while the first-order perturbation has an azimuthal component, which we consider to be a major advancement. It cannot be ignored at the jump front, as far as the formation of polygonal structures is concerned.

We anticipate solutions that are separable in $t$, $r$ and $\phi$, for the two linearly coupled wave equations, given by Eqs. (7) and (8). In keeping with this stipulation, we set down the azimuthal component, which we consider to be a major advancement. It cannot be ignored at the jump front, as far as the formation of polygonal structures is concerned.

Marginal stability (signalling the onset of a possible instability) extracts another root of $\omega = 0$ from the cubic equation, something that is possible only when $\Gamma_0 = 0$, and which in turn leads to a critical value of $m = m_c$. From the vanishing of the real part of $\Gamma_0$ we get

$$m_c^2 = -\frac{r_j}{gh_0} \frac{d}{dr} \left( \frac{v_0^2}{\Theta} - gh_0 \right) \left( 1 + \frac{r_j}{\psi} \frac{dv}{dr} \right),$$

and similarly the vanishing of the imaginary part of $\Gamma_0$ gives

$$m_c^2 = \frac{v_0^2 r_j^2}{2gh_0} \left( \frac{d\Theta}{dr} \right)^2 \left( \frac{dv}{dr} \right)^{-1} \frac{d}{dr} \left( \frac{v_0^2}{\Theta} - gh_0 \right),$$

which determines $d\Theta/dr$ at the horizon.

Having obtained $m_c$ from the condition of marginal stability, we now cause a slight perturbation, $m = m_c + \Delta m$ (with $\Delta m/m_c \ll 1$), to effect a very small change in $\omega$ from $\omega = 0$. Smallness of the value of $\omega$ allows us to ignore its higher orders in the cubic equation and retain only $\Gamma_1 \omega + \Gamma_0 \approx 0$. Hence, $-\omega \approx \frac{\Gamma_0}{\Gamma_1}$. On linearizing in $\Delta m$, we get

$$\Gamma_0 = \frac{2m_c gh_0 v_0}{r_j^2} \left( \frac{d\Theta}{dr} - i \frac{2}{v_0} \frac{dv}{dr} \right) \Delta m,$$

which while we write $\Gamma_1 = P + iQ$, in which, aided by some simplifications due to Eqs. (11) and (12).

$$P = v_0^2 \left( \frac{d\Theta}{dr} \right)^2 \left[ 2 + \frac{d}{dr} \left( \frac{gh_0}{v_0^2} \right) \right],$$

$$Q = -2 \frac{d\Theta}{dr} \left( \frac{2v_0^2}{r_j} - gh_0 \right) + 2v_0^2 \frac{r_j}{\psi} \left( 1 + \frac{r_j}{\psi} \right).$$

For convenience we write $-\omega = \gamma \Delta m$, from which we get

$$\gamma = \frac{2m_c v_0 gh_0}{r_j^2 (P^2 + Q^2)} \times \left[ \left( \frac{P d\Theta}{v_0} - 2Q \frac{dv}{v_0} \right) - i \left( Q \frac{d\Theta}{v_0} + 2P \frac{dv}{v_0} \right) \right],$$

whose form is best read as $\gamma = R(\gamma) + i\Im(\gamma)$. Now, the phase of the wave solution is $\exp[-i\omega t + i\Theta(r) + i\phi \Theta(r)]$, from which we extract only the time-dependent part and recast it as $\exp[iR(\gamma) \Delta m t] \times \exp[-\Im(\gamma) \Delta m t]$. The conclusion we draw is that $R(\gamma)$ causes the wave to travel along the $\phi$ coordinate (the azimuthal direction), and $\Im(\gamma)$, depending on its sign, causes either a growth or a decay in the amplitude of the travelling wave. To examine the latter feature, we look at Eqs. (11) and (12), both of which give $m_c^2$ as a perfect square. As such, $m_c$ will have two roots of the same value but opposite signs. Since $\Im(\gamma)$ depends on $m_c$, as Eq. (14) shows, it will similarly carry both signs, with the signs of all other quantities in Eq. (14) arguably being fixed. If $\Im(\gamma) > 0$, then stability will be achieved only by $\Delta m > 0$, i.e. if $m > m_c$. This is to say that a polygon, so formed, will be stable only if the number of its vertices is above a threshold given by $m_c$. The opposite argument applies if $\Im(\gamma) < 0$, because in this
case stability is ensured by $m < m_c$, with there being an upper limit to the vertices of a stable polygon.

Recent experiments by Teymourtash and Mokhlesi [10] have shown unstable polygons to undergo a rotational behaviour. We reproduce this feature theoretically with the help of $\Omega(\gamma)$, in which the sign of $m_c$ controls the clockwise or the anticlockwise direction of the rotation. The angular frequency of the rotation is $\Omega_{\text{rot}} = |\Omega(\gamma)|\Delta m|$. In an unstable situation where $m < m_c$, small values of m yield high values of $|\Delta m|$, and so $\Omega_{\text{rot}} \propto |m_c - m|$. This linear decay of the angular frequency with increasing number of vertices matches experimental results [10]. To find how $\Omega_{\text{rot}}$ depends on the flow rate, $Q$, we first note from Eq. (11) that $\Omega_{\text{rot}} \sim v_0(r_1)/r_1$, in which $r_1 \sim Q^{1/8}\nu^{-3/8}g^{-1/8}$, a well-known scaling result [2] that is derived by equating the dynamic time scale of the steady radial outflow with the time scale of viscous dissipation [3]. The steady background quantities depend on viscosity, and at the jump, where $v_0(r_1) = \sqrt{gh_0(r_1)}$, the flow height, $h_0(r_1)$, is scaled by combining the aforementioned scaling of $r_1$ with the integral solution of Eq. (2). This results in $h_0(r_1) \sim Q^{1/4}\nu^{1/4}g^{-1/4}$ [2], with which we get the scale, $\Omega_{\text{rot}} \sim Q^{-1/2}\nu^{1/2}g^{1/2}$. Evidently, $\Omega_{\text{rot}}$ decreases with increasing flow rate, $Q$, something that has been observed by Teymourtash and Mokhlesi [10] in their experiments. Our theoretical treatment is, therefore, well in accord with two experimental findings of Teymourtash and Mokhlesi [10], namely, the two ways for $\Omega_{\text{rot}}$ to decay — with increasing number of polygon vertices and with increasing flow rate. Beyond these two established facts, we make a prediction, based on $\Omega_{\text{rot}} \propto \nu^{1/2}$, that the angular velocity of the rotating polygons will increase with increasing kinematic viscosity.

Our most crucial claim is that the breaking of the axial symmetry has a connection with the Hawking radiation in an acoustic white hole. If the right hand side of Eq. (7) were to vanish, the left hand side will bring forth the symmetric metric of an analogue white hole [3, 11, 14], a point of view that is applicable to a steady radial outflow. The $\phi$-dependent terms on the right hand side of Eq. (7) break the axial symmetry of the steady radial flow, and at the horizon, where $v_0^2 = gh_0$, the flow becomes azimuthal. The analogue “surface gravity” at the horizon $\Omega_{\text{rot}} = |\Omega(\gamma)|\Delta m|$, as given is as

$$\Omega_{\text{rot}} = |\Omega(\gamma)|\Delta m|.$$