Edge magic and bimagic harmonious labeling of ladder graphs

M. Regees\(^1\), L. Merrit Anisha\(^2\)* and T. Nicholas \(^3\)

Abstract

A graph \(G = (V, E)\) with \(p\) vertices and \(q\) edges is said to be edge magic harmonious if there exists a bijection \(f : V \cup E \to \{1, 2, 3, \ldots, p+q\}\) such that for each edge \(xy\) in \(E(G)\), the value of \([(f(x) + f(y)) \mod q + f(xy)]\) is equal to the constant \(k\), called magic constant. A bijection \(f : V \cup E \to \{1, 2, 3, \ldots, p+q\}\) is called an edge bimagic harmonious labeling if \([(f(x) + f(y)) \mod q + f(xy)] = k_1\) or \(k_2\) for each edge \(xy\) in \(E(G)\), where \(k_1\) and \(k_2\) are two distinct magic constants. A graph \(G\) is said to be edge bimagic harmonious, if it admits an edge bimagic harmonious labeling. Here we prove that the ladder, double ladder are edge bimagic harmonious graphs and circular ladder, triangular ladder are edge magic and bimagic harmonious graphs.

Keywords

Graph, Bijection, Harmonious, Magic labeling, Bimagic labeling, Ladder, Circular ladder, Triangular ladder, Double ladder.

AMS Subject Classification

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\(^1\)Department of Mathematics, Malanka Catholic College, Mariagiri-629153, Tamil Nadu, India.  
\(^2\)Research Scholar, Department of Mathematics, St.Jude’s College, Thoothoor-629176, Tamil Nadu, India.  
\(^3\)Department of Mathematics, St. Jude’s College, Thoothoor-629176, Tamil Nadu, India.  
*Corresponding author: merritanisha@gmail.com [Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli-627012, Tamil Nadu, India.]

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1. Introduction

All graphs in this paper are finite and undirected with \(p\) vertices and \(q\) edges without loops or parallel edges. The graph labeling was introduced by Rosa in 1960 [7]. Magic labeling was introduced by Sedlacek [9]. In 1970 Kotzig and Rosa [8] defined a magic valuation of a graph. In 1996, Ringel and Llado [3] called this labeling as edge magic. Edge bimagic labeling of graphs was introduced by Babujee [1] in 2004. Harmonious labeling naturally arose in the study by Graham and Sloane [4]. Dushyant Tanna [2] introduce some harmonious labeling techniques. For more annotations, we utilize dynamic survey of graph labeling by Gallian [7]. Here, we introduce the concept of edge magic and bimagic harmonious labeling of graphs and proved that ladder, double ladder are edge bimagic harmonious graphs and circular ladder, triangular ladder are edge magic and bimagic harmonious graphs.

Definition 1.1. [6] The Cartesian product graph \(G_1 \times G_2\) of two graphs \(G_1\) and \(G_2\) is defined to be the graph whose vertex set is \(V_1 \times V_2\), that is every vertex of \(G_1 \times G_2\) is an ordered pair \((u, v)\), where \(u \in V_1\) and \(v \in V_2\) and two distinct vertices \((u, v)\) and \((x, y)\) are adjacent in \(G_1 \times G_2\) if either \(u = x\) and \(vy \in E(G_2)\) or \(v = y\) and \(ux \in E(G_1)\). \(P_n \times K_2\) is called a ladder.

Definition 1.2. [5] A circular ladder graph is defined as the cartesian product \(C_n \times K_2\) where \(K_2\) is the complete graph on two vertices and \(C_n\) is the cycle graph on \(n\) vertices.

Definition 1.3. [6] A triangular ladder \(TL_n, n \geq 2\), is a graph obtained from the ladder \(P_n \times K_2\) by adding the edges \(u_iv_{i+1}\) for \(1 \leq i \leq n-1\).

Definition 1.4. [6] The double ladder \(L_n\) is the graph \(P_n \times P_3\) with vertex set \(V = \{u_i/1 \leq i \leq n\} \cup \{v_i/1 \leq i \leq n\}\)
Theorem 2.6. The ladder \( L_n \) admits an edge bimagic harmonious labeling for all \( n > 2 \).

**Proof.** Let \( V(L_n) = \{ u_i, v_i / 1 \leq i \leq n \} \) and \( E(L_n) = \{ u_i u_{i+1}, v_i v_{i+1} / 1 \leq i \leq n - 1 \} \cup \{ u_i v_i / 1 \leq i \leq n - 1 \}. \) Then the graph \( L_n \) has \( 2n \) vertices and \( 3n - 2 \) edges.

Case 1: \( n \) is odd

Define a bijection \( f : V \cup E \to \{ 1, 2, 3, \cdots, 5n - 2 \} \) such that

\[
\begin{align*}
  f(u_i) &= i, 1 \leq i \leq n \\
  f(v_1) &= 2n \\
  f(v_i) &= n + i - 1, 2 \leq i \leq n \\
  f(u_i u_{i+1}) &= 5n - 2i - 3, 1 \leq i \leq n - 1 \\
  f(v_1 v_2) &= 4n - 3 \\
  f(v_i v_{i+1}) &= 3n - 2i - 1, 2 \leq i \leq n - 1 \\
  f(u_1 v_1) &= 3n - 3 \\
  f(u_i v_i) &= 4n - 2i - 1, 2 \leq i \leq n - 1 \\
  f(u_n v_n) &= 5n - 3 \\
\end{align*}
\]

For the edges \( u_i u_{i+1}, 1 \leq i \leq n - 1 \).

\[
\begin{align*}
  [(f(u_i) + f(u_{i+1})) \mod (q) + f(u_i u_{i+1})] \\
  &= [(i + i + 1) \mod (3n - 2) + 5n - 2i - 3] \\
  &= [(2i + 1) + 5n - 2i - 3] = 5n - 2 = k_1 \text{ (say)} \\
\end{align*}
\]

For the edge \( v_1 v_2 \)

\[
\begin{align*}
  [(f(v_1) + f(v_2)) \mod (q) + f(v_1 v_2)] \\
  &= [(2n + n + 1) \mod (3n - 2) + 4n - 3] \\
  &= [3 + 4n - 3] = 4n = k_2 \text{ (say)} \\
\end{align*}
\]

For the edges \( v_i v_{i+1}, 2 \leq i \leq n - \frac{3}{2} \).

\[
\begin{align*}
  [(f(v_i) + f(v_{i+1})) \mod (q) + f(v_i v_{i+1})] \\
  &= [((n + i - 1) + (n + i)) \mod (3n - 2) + 3n - 2i - 1] \\
  &= [(2n + 2i - 1) + 3n - 2i - 1] \\
  &= 5n - 2 = k_1 \\
\end{align*}
\]

For the edges \( v_i v_{i+1}, \frac{n - 1}{2} \leq i \leq n - 1 \).

\[
\begin{align*}
  [(f(v_i) + f(v_{i+1})) \mod (q) + f(v_i v_{i+1})] \\
  &= [((n + i - 1) + (n + i)) \mod (3n - 2) + 6n - 2i - 3] \\
  &= [2i - n + 1 + 6n - 2i - 3] \\
  &= 5n - 2 = k_1 \\
\end{align*}
\]

For the edge \( u_1 v_1 \)

\[
\begin{align*}
  [(f(u_1) + f(v_1)) \mod (q) + f(u_1 v_1)] \\
  &= [(1 + 2n) \mod (3n - 2) + 3n - 3] \\
  &= [(2n + 1) + 3n - 3] = 5n - 2 = k_1 \\
\end{align*}
\]

For the edges \( u_i v_i, 2 \leq i \leq n - 1 \).

\[
\begin{align*}
  [(f(u_i) + f(v_i)) \mod (q) + f(u_i v_i)] \\
  &= [(i + (n + i - 1)) \mod (3n - 2) + 4n - 2i - 1] \\
  &= [(n + 2i - 1) + 4n - 2i - 1] = 5n - 2 = k_1 \\
\end{align*}
\]
For the edge $u_nv_n$

\[
[(f(u_n) + f(v_n)) \mod q + f(u_nv_n)]
\]
\[
= [(n+2n-1) \mod (3n-2) + 5n-3]
\]
\[
= [1 + 5n-3] = 5n-2 = k_1
\]

Here, the edge labels are distinct and there exist two magic constants for each edge $xy \in E$. $[(f(x) + f(y)) \mod q + f(xy)]$ yields any one of the magic constants $k_1 = 5n-2$ and $k_2 = 4n$. Therefore, the ladder $L_n$ admits an edge bimagic harmonious labeling for odd $n > 2$.

**Case 2:** $n$ is even

Define a bijection $f : V \cup E \rightarrow \{1, 2, 3, \ldots, 5n-2\}$ such that

\[
f(u_i) = i, \quad 1 \leq i \leq n-1
\]
\[
f(v_i) = n + i, \quad 1 \leq i \leq n-1
\]
\[
f(u_{u_{i+1}}) = 5n-2i-4, \quad 1 \leq i \leq n-1
\]
\[
f(v_{v_{i+1}}) = 3n-2i-3, \quad 1 \leq i \leq \frac{n-4}{2}
\]
\[
f(v_{v_{i+1}}) = 6n-2i-5, \quad \frac{n-2}{2} \leq i \leq n-1
\]
\[
f(u_{v_i}) = 4n-2i-3, \quad 1 \leq i \leq \frac{n}{2}
\]
\[
f(u_{v_i}) = 4n-2i-2, \quad \frac{n+2}{2} \leq i \leq n-2
\]
\[
f(u_{v_i}) = 7n-2i-4, \quad n-1 \leq i \leq n
\]

For the edges $u_{u_{i+1}}, 1 \leq i \leq n-1$

\[
[(f(u_i) + f(u_{i+1})) \mod q + f(u_{u_{i+1}})]
\]
\[
= [(i + i + 1) \mod (3n-2) + 5n-2i-4]
\]
\[
= [(2i+1) + 5n-2i-4] = 5n-3 = k_1 \text{ (say)}
\]

For the edges $v_{v_{i+1}}, 1 \leq i \leq \frac{n-4}{2}$

\[
[(f(v_i) + f(v_{i+1})) \mod q + f(v_{v_{i+1}})]
\]
\[
= [((n + i) + (n + i + 1)) \mod (3n-2) + 3n-2i-3]
\]
\[
= [(2n+2i+1) + 3n-2i-3]
\]
\[
= 5n-2 = k_2 \text{ (say)}
\]

For the edges $v_{v_{i+1}}, \frac{n-2}{2} \leq i \leq n-1$

\[
[(f(v_i) + f(v_{i+1})) \mod q + f(v_{v_{i+1}})]
\]
\[
= [((n + i) + (n + i + 1)) \mod (3n-2) + 6n-2i-5]
\]
\[
= [(2i-n+3) + 6n-2i-5] = 5n-2 = k_2
\]

For the edges $u_{v_i}, n-1 \leq i \leq n$

\[
[(f(u_i) + f(v_i)) \mod q + f(u_{v_i})]
\]
\[
= [(i + n + i) \mod (3n-2) + 4n-2i-3]
\]
\[
= [(n+2i) + 4n-2i-3] = 5n-2 = k_1
\]

For the edges $u_{v_i}, \frac{n+2}{2} \leq i \leq n$

\[
[(f(u_i) + f(v_i)) \mod q + f(u_{v_i})]
\]
\[
= [(i + (n + i)) \mod (3n-2) + 4n-2i-2]
\]
\[
= [(n+2i) + 4n-2i-2] = 5n-2 = k_2
\]

Here, the edge labels are distinct and there exist two magic constants for each edge $xy \in E$. $[(f(x) + f(y)) \mod q + f(xy)]$ yields any one of the magic constants $k_1 = 5n-2$ and $k_2 = 4n$. Therefore, the ladder $L_n$ admits an edge bimagic harmonious labeling for even $n > 2$. From cases (1) and (2), ladder $L_n$ admits an edge bimagic harmonious labeling for all $n > 2$.

**Corollary 2.7.** The ladder $L_n$ admits a super edge bimagic harmonious labeling for all $n > 2$.

**Proof.** We prove that the ladder $L_n$ admits an edge bimagic harmonious labeling for all $n > 2$. The labeling given in the proof of Theorem 2.6, the vertices get labels $1, 2, 3, \ldots, 2n$. Since the ladder graph has $2n$ vertices and the $2n$ vertices have labels $1, 2, 3, \ldots, 2n$ for odd and even $n > 2$, the ladder graph $L_n$ is a super edge bimagic harmonious for all $n > 2$.

**Example 2.8.** Bimagic harmonious labeling of $L_9$ and $L_{10}$ are given in figure 3 and figure 4.

**Figure 3.** ladder $L_9$ with $k_1 = 43$ and $k_2 = 36$.

**Figure 4.** ladder $L_{10}$ with $k_1 = 47$ and $k_2 = 48$.

**Theorem 2.9.** The circular ladder $CL_n$ admits an edge magic harmonious labeling for odd $n$. 

![Circular ladder CL_n](image-url)
Proof. Let $V(CL_n) = \{u_i, v_i/1 \leq i \leq n\}$ and 
$E(CL_n) = \{u_iu_{i+1}, v_iv_{i+1}/1 \leq i \leq n-1\}$ 
$\cup \{u_iv_i/1 \leq i \leq n\} \cup \{u_iu_i, v_iv_i\}$. Then the graph $CL_n$ has
2n vertices and 3n edges. 
Define a bijection $f: V \cup E \rightarrow \{1, 2, 3, \ldots, 5n\}$ such that 

$f(u_i) = i, 1 \leq i \leq n$
$f(v_i) = 2n$
$f(u_iu_{i+1}) = 5n - 2i - 1, 1 \leq i \leq n - 1$
$f(u_iu_1) = 4n - 1$
$f(v_1v_2) = 5n - 1$
$f(v_iv_{i+1}) = 3n - 2i + 1, 2 \leq i \leq n - 1$
$f(v_iv_{i+1}) = 6n - 2i + 1, \frac{n+1}{2} \leq i \leq n - 1$
$f(v_nv_1) = 4n + 1$
$f(u_1v_1) = 3n - 1$
$f(u_iv_i) = 4n - 2i + 1, 2 \leq i \leq n$

For the edges $u_iu_{i+1}, 1 \leq i \leq n - 1$

$[(f(u_i) + f(u_{i+1})) \mod (q) + f(u_iu_{i+1})]$
$= [(i + i + 1) \mod (3n) + 5n - 2i - 1]$
$= [2i + 1 + 5n - 2i - 1] = 5n = k (say)$

For the edge $u_iv_i$

$[(f(u_i) + f(v_i)) \mod (q) + f(u_iv_i)]$
$= [(i + (n + i - 1)) \mod (3n) + 4n - 2i + 1]$
$= [(n + 2i - 1) + 4n - 2i + 1] = 5n = k$

For the edges $u_iv_i, 2 \leq i \leq n$

$[(f(u_i) + f(v_i)) \mod (q) + f(u_iv_i)]$
$= [(i + (n - i - 1)) \mod (3n) + 4n - 2i + 1]$
$= [(n - i + 1 + 4n - 2i + 1] = 5n = k$

For the edges $v_iv_{i+1}, 2 \leq i \leq \frac{n - 1}{2}$

$[(f(v_i) + f(v_{i+1})) \mod (q) + f(v_iv_{i+1})]$
$= [(2n + n + 1) \mod (3n) + 5n - 1]$
$= [1 + 5n - 1] = 5n = k$

For the edges $v_iv_{i+1}, \frac{n + 1}{2} \leq i \leq n - 1$

$[(f(v_i) + f(v_{i+1})) \mod (q) + f(v_iv_{i+1})]$
$= [(2 - n - 1) + 6n - 2i + 1] = 5n = k$

For the edge $v_nv_1$

$[(f(v_n) + f(v_1)) \mod (q) + f(v_nv_1)]$
$= [(2n - 1 + 2n) \mod (3n) + 4n + 1]$
$= [(n - 1) + 4n + 1] = 5n = k$

Here, the edge labels are distinct and there exist a magic constant for each edge $xy \in E, [(f(x) + f(y))(mod q) + f(xy)]$
yields the magic constant $k = 5n$. Therefore, the circular ladder $CL_n$ admits an edge magic harmonious labeling for odd $n$. 

Corollary 2.10. The circular ladder $CL_n$ admits a super edge magic harmonious labeling for odd $n$.

Proof. We have that the circular ladder $CL_n$ admits an edge magic harmonious labeling for odd $n$. The labeling given in the proof of Theorem 2.9, the vertices get labels $1, 2, 3, \ldots, 2n$. Since the circular ladder graph has $2n$ vertices and the $2n$ vertices have labels $1, 2, 3, \ldots, 2n$ for odd $n$, the circular ladder graph $CL_n$ is a super edge magic harmonious for odd $n$. □

Example 2.11. Magic harmonious labeling of $CL_{11}$ is given in figure 5.

![Figure 5. Circular ladder $CL_{11}$ with $k = 55$.](image)

Theorem 2.12. The circular ladder $CL_n$ admits an edge bimagic harmonious labeling for all $n$. 

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Theorem 2.12. The circular ladder $CL_n$ admits an edge bimagic harmonious labeling for all $n$. 

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For the edge $v_nv_1$

$[(f(v_n) + f(v_1)) \mod (q) + f(v_nv_1)]$
$= [(2n - 1 + 2n) \mod (3n) + 4n + 1]$
$= [(n - 1) + 4n + 1] = 5n = k$

For the edge $u_1v_1$

$[(f(u_1) + f(v_1)) \mod (q) + f(u_1v_1)]$
$= [(1 + 2n) \mod (3n) + 3n - 1]$
$= [(2n + 3n - 1)] = 5n = k$

For the edges $u_iv_i, 2 \leq i \leq n$

$[(f(u_i) + f(v_i)) \mod (q) + f(u_iv_i)]$
$= [(i + (n + i - 1)) \mod (3n) + 4n - 2i + 1]$
$= [(n + 2i - 1) + 4n - 2i + 1] = 5n = k$
Proof. Let $V(CL_n) = \{u_i, v_i/1 \leq i \leq n\}$ and $E(CL_n) = \{u_iu_{i+1}, v_iv_{i+1}/1 \leq i \leq n-1\} \cup \{u_nv_i/1 \leq i \leq n\} \cup \{u_1u_n, v_1v_n\}$. Then the graph $CL_n$ has $2n$ vertices and $3n$ edges.

Case 1: $n$ is odd
Define a bijection $f : V \cup E \rightarrow \{1, 2, 3, \ldots, 5n\}$ such that

$$f(u_i) = i, 1 \leq i \leq n$$
$$f(v_1) = 2n$$
$$f(v_i) = n + i - 1, 2 \leq i \leq n$$
$$f(u_iu_{i+1}) = 3n - 2i + 1, 1 \leq i \leq \frac{n-1}{2}$$
$$f(u_iu_{i+1}) = 6n - 2i + 1, \frac{n+1}{2} \leq i \leq n - 1$$
$$f(u_iu_1) = 2n + 1$$
$$f(v_1v_2) = 3n + 1$$
$$f(v_iv_{i+1}) = 4n - 2i + 3, 2 \leq i \leq n - 1$$
$$f(v_nv_1) = 2n + 3$$
$$f(u_1v_1) = 4n + 1$$
$$f(u_iv_i) = 5n - 2i + 3, 2 \leq i \leq n$$

For the edges $u_iu_{i+1}, 1 \leq i \leq \frac{n-1}{2}$

$$[(f(u_i) + f(u_{i+1})) \mod (q) + f(u_iu_{i+1})]$$
$$= [(i + i + 1) \mod (3n) + 3n - 2i + 1]$$
$$= [(2i + 1) + 3n - 2i + 1] = 3n + 2 = k_1 \text{ (say)}$$

For the edges $u_iu_{i+1}, \frac{n+1}{2} \leq i \leq n - 1$

$$[(f(u_i) + f(u_{i+1})) \mod (q) + f(u_iu_{i+1})]$$
$$= [(i + i + 1) \mod (3n) + 6n - 2i + 1]$$
$$= [(2i + 1) + 6n - 2i + 1] = 6n + 2 = k_2 \text{ (say)}$$

For the edge $u_nv_1$

$$[(f(u_n) + f(u_1)) \mod (q) + f(u_nv_1)]$$
$$= [(n + 1) \mod (3n) + 2n + 1]$$
$$= [(n + 1) + 2n + 1] = 3n + 2 = k_1 \text{ (say)}$$

For the edge $v_1v_2$

$$[(f(v_1) + f(v_2)) \mod (q) + f(v_1v_2)]$$
$$= [(2n + n + 1) \mod (3n) + 3n + 1]$$
$$= [1 + 3n + 1] = 3n + 2 = k_1$$

For the edges $v_iv_{i+1}, 2 \leq i \leq \frac{n-1}{2}$

$$[(f(v_i) + f(v_{i+1})) \mod (q) + f(v_iv_{i+1})]$$
$$= [(i + i + 1) \mod (3n) + 4n - 2i + 3]$$
$$= [(2n + 2i - 1) + 4n - 2i + 3]$$
$$= 6n + 2 = k_2$$

For the edges $v_iv_{i+1}, \frac{n+1}{2} \leq i \leq n - 1$

$$[(f(v_i) + f(v_{i+1})) \mod (q) + f(v_iv_{i+1})]$$
$$= [(i + i + 1) + (n + i) \mod (3n) + 4n - 2i + 3]$$
$$= [(2n + 2i - 1) + 4n - 2i + 3]$$
$$= 6n + 2 = k_2$$

For the edge $v_nv_1$

$$[(f(v_n) + f(v_1)) \mod (q) + f(v_nv_1)]$$
$$= [(2i - n + 1) + 4n - 2i + 3]$$
$$= 3n + 2 = k_1$$

For the edge $u_1v_1$

$$[(f(u_1) + f(v_1)) \mod (q) + f(u_1v_1)]$$
$$= [(1 + 2n) \mod (3n) + 4n + 1]$$
$$= [2n + 1 + 4n + 1] = 6n + 2 = k_2$$

For the edges $u_iv_i, 2 \leq i \leq n$

$$[(f(u_i) + f(v_i)) \mod (q) + f(u_iv_i)]$$
$$= [(i + (n + i - 1)) \mod (3n) + 5n - 2i + 3]$$
$$= [(2i + n - 1) + 5n - 2i + 3]$$
$$= 6n + 2 = k_2$$

Here, the edge labels are distinct and there exist two magic constants for each edge $xy \in E, [(f(x) + f(y)) \mod q + f(xy)]$ yields one of the magic constant $k_1 = 3n + 2$ and $k_2 = 6n + 2$. Therefore, the circular ladder $CL_n$ admits an edge magic harmonious labeling for odd $n$.

Case 2: $n$ is even
Define a bijection $f : V \cup E \rightarrow \{1, 2, 3, \ldots, 5n\}$ such that

$$f(u_i) = i, 1 \leq i \leq n$$
$$f(v_1) = n + 1, 1 \leq i \leq n$$
$$f(u_iv_{i+1}) = 5n - 2i - 1, 1 \leq i \leq \frac{n}{2}$$
$$f(u_iu_1) = 5n - 2i - 2, \frac{n+2}{2} \leq i \leq n - 1$$
$$f(u_nv_1) = 4n - 2$$
$$f(v_1v_2) = 3n - 2i - 1, 1 \leq i \leq \frac{n-2}{2}$$
$$f(v_nv_{n+1}) = 5n - 1$$
$$f(v_iv_{i+1}) = 6n - 2i - 2, \frac{n+2}{2} \leq i \leq n - 1$$
$$f(v_1v_2) = 5n - 2$$
$$f(u_1v_1) = 4n - 2i - 1, 1 \leq i \leq \frac{n}{2}$$
$$f(u_nv_1) = 4n - 2i, \frac{n+2}{2} \leq i \leq n - 1$$
$$f(u_nv_n) = 5n$$
For the edges $u_iu_{i+1}$, $1 \leq i \leq \frac{n}{2}$
\[ [(f(u_i) + f(u_{i+1})) \mod (q) + f(u_iu_{i+1})] \\
= [(i + i + 1) \mod (3n) + 5n - 2i - 1] \\
= [(2i + 1) + 5n - 2i - 1] = 5n = k_1 \text{ (say)} \]

For the edges $u_iu_{i+1}$, $\frac{n+2}{2} \leq i \leq n - 1$
\[ [(f(u_i) + f(u_{i+1})) \mod (q) + f(u_iu_{i+1})] \\
= [(i + (n + i)) \mod (3n) + 4n - 2i] \\
= [(2i + n) + 4n - 2i] = 5n = k_1 \]

For the edge $u_iv_i$
\[ [(f(u_n) + f(v_i)) \mod (q) + f(u_nv_i)] \\
= [(n + ((n + i))) \mod (3n) + 5n - 1] \\
= [5n - 1 = k_2 \text{ (say)}] \]

For the edges $v_iu_{i+1}$, $1 \leq i \leq \frac{n}{2}$
\[ [(f(v_i) + f(v_{i+1})) \mod (q) + f(v_iv_{i+1})] \\
= [(i + (n + i + 1)) \mod (3n) + 3n - 2i - 1] \\
= [(2n + 2i + 1) + 3n - 2i - 1] = 5n = k_1 \]

For the edges $v_iu_{i+1}$, $\frac{n+2}{2} \leq i \leq n - 1$
\[ [(f(v_i) + f(v_{i+1})) \mod (q) + f(v_iv_{i+1})] \\
= [(i + (n + i)) \mod (3n) + 6n - 2i - 2] \\
= [5n - 1 = k_2] \]

For the edge $v_nv_i$
\[ [(f(v_n) + f(v_i)) \mod (q) + f(v_nv_i)] \\
= [(n + (n + i)) \mod (3n) + 6n - 2i - 2] \\
= [1 + 5n - 1] \\
= 5n = k_1 \]

Here, the edge labels are distinct and there exist two magic constants for each edge $xy \in E, [(f(x) + f(y))(\mod q) + f(xy)]$ yields any one of the magic constant $k_1 = 5n$ and $k_2 = 5n - 1$. Therefore, the circular ladder $CL_n$ admits an edge bimagic harmonious labeling for even $n$.

From cases (1) and (2), circular ladder $CL_n$ admits an edge bimagic harmonious labeling for all $n$.

**Corollary 2.13.** The circular ladder $CL_n$ admits a super edge bimagic harmonious labeling for all $n$.

**Proof.** We proven that the circular ladder $CL_n$ admits an edge bimagic harmonious labeling for all $n$. The labeling given in the proof of Theorem 2.12, the vertices get labels $1, 2, 3, \ldots, 2n$. Since the circular ladder graph has $2n$ vertices and the $2n$ vertices have labels $1, 2, 3, \ldots, 2n$ for odd and even $n$, the circular ladder graph $CL_n$ is a super edge bimagic harmonious for all $n$.

**Example 2.14.** Bimagic harmonious labeling of $CL_{11}$ and $CL_{12}$ are given in figure 6 and figure 7.

![Figure 6. Circular ladder $CL_{11}$ with $k_1 = 35$ and $k_2 = 68.$](image)
Theorem 2.15. The triangular ladder \( TL_n \) admits an edge magic harmonious labeling for all \( n \).

Proof. Let \( V(TL_n) = \{u_i, v_i/1 \leq i \leq n\} \) and \( E(TL_n) = \{u_iu_{i+1}, v_iv_{i+1}/1 \leq i \leq n-1\} \cup \{u_iv_i/1 \leq i \leq n\} \cup \{u_iv_{i+1}/1 \leq i \leq n-1\} \). Then the graph \( TL_n \) has 2n vertices and 4n - 3 edges.

Define a bijection \( f : V \cup E \rightarrow \{1, 2, 3, \ldots, 6n - 3\} \) such that
\[
\begin{align*}
  f(u_i) &= 2i - 1, \ 1 \leq i \leq n \\
  f(v_i) &= 2i, \ 1 \leq i \leq n \\
  f(u_iu_{i+1}) &= 6n - 4i - 3, \ 1 \leq i \leq n - 1 \\
  f(v_iv_{i+1}) &= 6n - 4i - 5, \ 1 \leq i \leq n - 2 \\
  f(v_{i-1}v_i) &= 6n - 4 \\
  f(u_iv_i) &= 6n - 4i - 2, \ 1 \leq i \leq n - 1 \\
  f(u_iv_n) &= 6n - 5 \\
  f(u_{i+1}v_n) &= 6n - 4i - 4, \ 1 \leq i \leq n - 2 \\
  f(u_nv_{n-1}) &= 6n - 3
\end{align*}
\]

For the edges \( u_iu_{i+1}, 1 \leq i \leq n - 1 \)
\[
[(f(u_i) + f(u_{i+1})) \mod (q) + f(u_iu_{i+1})] \\
= [((2i - 1) + (2i + 1)) \mod (4n - 3) + 6n - 4i - 3] \\
= [4i + 6n - 4i - 3] = 6n - 3 = k \text{(say)}
\]

For the edges \( v_iv_{i+1}, 1 \leq i \leq n - 2 \)
\[
[(f(v_i) + f(v_{i+1})) \mod (q) + f(v_iv_{i+1})] \\
= [((2i) + (2i + 2)) \mod (4n - 3) + 6n - 4i - 5] \\
= [(4i + 2) + 6n - 4i - 5] = 6n - 3 = k
\]

For the edge \( v_{n-1}v_n \)
\[
[(f(v_{n-1}) + f(v_n)) \mod (q) + f(v_{n-1}v_n)] \\
= [((2n - 2) + 2n) \mod (4n - 3) + 6n - 4] \\
= [1 + 6n - 4] = 6n - 3 = k
\]

For the edges \( u_iv_i, 1 \leq i \leq n - 1 \)
\[
[(f(u_i) + f(v_i)) \mod (q) + f(u_iv_i)] \\
= [((2i - 1) + 2i) \mod (4n - 3) + 6n - 4i - 2] \\
= [(4i - 1) + 6n - 4i - 2] = 6n - 3 = k
\]

For the edge \( u_nv_n \)
\[
[(f(u_n) + f(v_n)) \mod (q) + f(u_nv_n)] \\
= [((2n - 1) + 2n) \mod (4n - 3) + 6n - 5] \\
= [2 + 6n - 5] = 6n - 3 = k
\]

For the edges \( u_iv_{i+1}, 1 \leq i \leq n - 2 \)
\[
[(f(u_i) + f(v_{i+1})) \mod (q) + f(u_iv_{i+1})] \\
= [((2i - 1) + (2i + 2)) \mod (4n - 3) + 6n - 4i - 4] \\
= [(4i + 1) + 6n - 4i - 4] = 6n - 3 = k
\]

Here, the edge labels are distinct and there exist a magic constant for each edge \( xy \in E \), \( [(f(x) + f(y)) \mod q + f(xy)] \) yields the magic constant \( k = 6n - 3 \). Therefore, the triangular ladder \( TL_n \) admits an edge magic harmonious labeling for all \( n \).

Corollary 2.16. The triangular ladder \( TL_n \) admits a super edge magic harmonious labeling for all \( n \).

Proof. We proven that the triangular ladder \( TL_n \) admits an edge magic harmonious labeling for all \( n \). The labeling given in the proof of Theorem 2.15, the vertices get labels \( 1, 2, 3, \ldots, 2n \). Since the triangular ladder graph has 2n vertices and the 2n vertices have labels \( 1, 2, 3, \ldots, 2n \) for odd and even \( n \), the triangular ladder graph \( TL_n \) is a super edge magic harmonious for all \( n \).

Example 2.17. Magic harmonious labeling of \( TL_{10} \) is given in figure 8.

Figure 7. Circular ladder \( CL_{12} \) with \( k_1 = 60 \) and \( k_2 = 59 \).

Figure 8. Triangular ladder \( TL_{10} \) with \( k = 57 \).

Theorem 2.18. The triangular ladder \( TL_n \) admits an edge bimagic harmonious labeling for all \( n \).
Theorem 2.21. The double ladder $P_n \times P_3$ admits an edge bicritical harmonious labeling for odd n.

Proof. Let $V((P_n \times P_3)) = \{u_i, v_i, w_i/1 \leq i \leq n\}$ and $E((P_n \times P_3)) = \{u_iu_{i+1}, v_iv_{i+1}, w_iw_{i+1}/1 \leq i \leq n-1\} \cup \{u_iv_i/1 \leq i \leq n\} \cup \{w_iw_{i+1}/1 \leq i \leq n-1\}$. Then the graph $P_n \times P_3$ has $3n$ vertices and $5n-3$ edges.

Define a bijection $f : V \cup E \rightarrow \{1, 2, 3, \ldots, 8n-3\}$ such that

For the edge $u_{n-1}v_n$

\[
[(f(u_{n-1}) + f(v_n)) \mod (q) + f(u_{n-1}v_n)]
\]

= $[(2n - 3) + (2n + 3)]$

= $0 + 2n + 3 = 2n + 3 = k_2$

Here, the edge labels are distinct and there exist two magic constants for each edge $xy \in E$, $[(f(x) + f(y))(mod \ q) + f(xy)]$ yields one of the magic constant $k_1 = 6n$ and $k_2 = 2n + 3$. Therefore, the double ladder $P_n \times P_3$ admits an edge bicritical harmonious labeling for all $n$.

Corollary 2.19. The triangular ladder $T L_n$ admits a super edge bicritical harmonious labeling for all $n$.

Proof. We proven that the triangular ladder $T L_n$ admits an edge bicritical harmonious labeling for all $n$. The labeling given in the proof of Theorem 2.18, the vertices get labels $1, 2, 3, \ldots, 2n$. Since the triangular ladder graph has $2n$ vertices and the $2n$ vertices have labels $1, 2, 3, \ldots, 2n$ for odd and even $n$, the triangular ladder graph $T L_n$ is a super edge bicritical harmonious for all $n$.

Example 2.20. Bicritical harmonious labeling of $T L_{10}$ is given in figure 9.

\[\text{Figure 9. Triangular ladder } T L_{10} \text{ with } k_1 = 60 \text{ and } k_2 = 23.\]
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For the edge $v_1v_2$
\[
[(f(v_1) + f(v_2)) \mod (q) + f(v_1v_2)] = [(2n + n + 1) \mod (5n - 3) + 5n - 5] = [3n + 1 + 5n - 5] = 8n - 4 = k_1
\]

For the edges $v_iv_{i+1}, 2 \leq i \leq n-1$,
\[
[(f(v_i) + f(v_{i+1})) \mod (q) + f(v_iv_{i+1})] = [(n + i - 1) + (n + i)] \mod (5n - 3) + 5n - 2i - 3] = [2n + 2i - 1 + 6n - 2i - 3] = 8n - 4 = k_1
\]

For the edge $u_1v_1$
\[
[(f(u_1) + f(v_1)) \mod (q) + f(u_1v_1)] = [(1 + 2n) \mod (5n - 3) + 6n - 5] = [2n + 1 + 6n - 5] = 8n - 4 = k_1
\]

For the edges $u_iv_i, 2 \leq i \leq n$
\[
[(f(u_i) + f(v_i)) \mod (q) + f(u_iv_i)] = [(i + (n + i - 1)] \mod (5n - 3) + 7n - 2i - 3] = [(n + 2i - 1 + 7n - 2i - 3] = 8n - 4 = k_1
\]

For the edge $v_1w_1$ for the graph $n = 3$
\[
[(f(v_1) + f(w_1)) \mod (q) + f(v_1w_1)] = [[(2n) + (2n + 1)] \mod (5n - 3) + 7n - 1] = [(n - 2) + 7n - 1] = 8n - 3 = k_2
\]

For the edge $v_1w_1$ for the graph $n > 3$
\[
[(f(v_1) + f(w_1)) \mod (q) + f(v_1w_1)] = [[(2n) + (2n + 1)] \mod (5n - 3) + 4n - 4] = [4n + 1 + 4n - 4] = 8n - 3 = k_2
\]

For the edges $w_iw_{i+1}, 2 \leq i \leq n - 1$
\[
[(f(w_i) + f(w_{i+1})) \mod (q) + f(w_iw_{i+1})] = [(2n + i) + (2n + i + 1)] \mod (5n - 3) + 4n - 2i - 4] = [4n + 2i + 1 + 4n - 2i - 4] = 8n - 4 = k_1
\]

For the edges $w_iw_{i+1}, \frac{n - 3}{2} \leq i \leq n - 1$
\[
[(f(w_i) + f(w_{i+1})) \mod (q) + f(w_iw_{i+1})] = [(2n + i) + (2n + i + 1)] \mod (5n - 3) + 9n - 2i - 7] = [2i - n + 4 + 9n - 2i - 7] = 8n - 3 = k_2
\]

For the edge $u_1v_1$
\[
[(f(u_1) + f(v_1)) \mod (q) + f(u_1v_1)] = [(1 + 2n) \mod (5n - 3) + 6n - 5] = [2n + 1 + 6n - 5] = 8n - 4 = k_1
\]

For the edges $u_iv_i, 2 \leq i \leq n$
\[
[(f(u_i) + f(v_i)) \mod (q) + f(u_iv_i)] = [(i + (n + i - 1)] \mod (5n - 3) + 7n - 2i - 3] = [(n + 2i - 1 + 7n - 2i - 3] = 8n - 4 = k_1
\]

Here, the edge labels are distinct and there exist two magic constants for each edge $xy \in E$, $((f(x) + f(y)) \mod (q) + f(xy))$ yields any one of the magic constant $k_1 = 8n - 4$ and $k_2 = 8n - 3$. Therefore, the double ladder $P_n \times P_3$ admits an edge bimagic harmonious labeling for odd $n$.

**Corollary 2.22.** The double ladder $P_n \times P_3$ admits a super edge bimagic harmonious labeling for odd $n$.

**Proof.** We proven that the double ladder $P_n \times P_3$ admits an edge bimagic harmonious labeling for odd $n$. The labeling given in the proof of Theorem 2.21, the vertices get labels $1, 2, 3, \ldots, 3n$. Since the double ladder graph has $3n$ vertices and the $3n$ vertices have labels $1, 2, 3, \ldots, 3n$ for odd $n$, the double ladder graph $P_n \times P_3$ is a super edge bimagic harmonious for odd $n$. 

\[\]
Example 2.23. Bimagic harmonious labeling of $P_9 \times P_3$ is given in figure 10.

Figure 10. Double ladder $P_9 \times P_3$ with $k_1 = 68$ and $k_2 = 69$.

3. Conclusion

Here we proven that the ladder $L_n$, double ladder $P_n \times P_3$ are edge bimagic harmonious graphs and circular ladder $CL_n$, triangular ladder $TL_n$ are edge magic and bimagic harmonious graphs.

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