EXACT VERIFICATION OF THE STRONG BSD CONJECTURE FOR SOME ABSOLUTELY SIMPLE ABELIAN SURFACES

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ABSTRACT. Let $X$ be one of the 28 Atkin-Lehner quotients of a curve $X_0(N)$ such that $X$ has genus 2 and its Jacobian variety $J$ is absolutely simple. We show that the Shafarevich-Tate group $\Sha(J/\Q)$ is trivial. This verifies the strong BSD conjecture for $J$.

1. Introduction

Let $A$ be an abelian variety over $\Q$ and assume that its $L$-series $L(A,s)$ admits an analytic continuation to the whole complex plane. The weak BSD conjecture (or BSD rank conjecture) predicts that the Mordell-Weil rank $r = \text{rk}_A(\Q)$ of $A$ equals the analytic rank $r_{an} = \text{ord}_{s=1} L(A,s)$. The strong BSD conjecture asserts that the Shafarevich-Tate group $\Sha(A/\Q)$ is finite and that its order equals the “analytic order of Sha”:

$$\#\Sha(A/\Q)_{an} := \frac{\#A(\Q)_{tors} \cdot \#A^\vee(\Q)_{ tors}}{\prod_v c_v} \cdot \frac{L^*(A,1)}{\Omega_A \text{Reg}_{A/\Q}}.$$ (1)

Here $A^\vee$ is the dual abelian variety, $A(\Q)_{tors}$ denotes the torsion subgroup of $A(\Q)$, the product $\prod_v c_v$ runs over all finite places of $\Q$ and $c_v$ is the Tamagawa number of $A$ at $v$, $L^*(A,1)$ is the leading coefficient of the Taylor expansion of $L(A,s)$ at $s = 1$, and $\Omega_A$ and $\text{Reg}_{A/\Q}$ denote the volume of $A(\R)$ and the regulator of $A(\Q)$, respectively.

If $A$ is modular in the sense that $A$ is an isogeny factor of the Jacobian $J_0(N)$ of the modular curve $X_0(N)$ for some $N$, then the analytic continuation of $L(A,s)$ is known. If $A$ is in addition absolutely simple, then $A$ is associated (up to isogeny) to a Galois orbit of size $\dim(A)$ of newforms of weight 2 and level $N$, such that $L(A,s)$ is the product of $L(f,s)$ with $f$ running through these newforms. Such an abelian variety has real multiplication: its endomorphism ring (over $\Q$ and over $\Q$) is an order in a totally real number field of degree $\dim(A)$. If, furthermore, $\text{ord}_{s=1} L(f,s) \in \{0,1\}$ for one (equivalently, all) such $f$, then the weak BSD conjecture holds for $A$; see [12].

All elliptic curves over $\Q$ arise as one-dimensional modular abelian varieties [25, 20, 2] such that $N$ is the conductor of $A$. For all elliptic curves of (analytic) rank $\leq 1$ and $N < 5000$, the strong BSD conjecture has been verified [9, 14, 6].

In this note, we consider certain absolutely simple abelian surfaces and show that strong BSD holds for them. One class of such surfaces arises as the Jacobians

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of quotients $X$ of $X_0(N)$ by a group of Atkin-Lehner operators. Hasegawa [11] has determined the complete list of such $X$ of genus 2; 28 of them have absolutely simple Jacobian $J$. For most of these Jacobians (and those of further curves taken from [24]), it has been numerically verified in [8, 22] that $\#\Sha(J/Q)_\text{an}$ is very close to an integer, which equals $\#\Sha(J/Q)[2]$ (= 1 in the cases considered here). We complete the verification of strong BSD for these Jacobians by showing that $\#\Sha(J/Q)_\text{an}$ is indeed an integer and $\Sha(J/Q)$ is trivial.

2. METHODS AND ALGORITHMS

In the following, we denote the abelian surface under consideration by $A$; it is an absolutely simple isogeny quotient of $J_0(N)$, defined over $\mathbb{Q}$. We frequently use the fact that $A$ can be obtained as the Jacobian variety of a curve $X$ of genus 2. The algorithms described below have been implemented in Magma [1].

Recall that a Heegner discriminant for $A$ is a fundamental discriminant $D < 0$ such that for $K = \mathbb{Q}(\sqrt{D})$, the analytic rank of $A/K$ equals $\dim A = 2$ and all prime divisors of $N$ split in $K$. Heegner discriminants exist by [3, 23]. Since Magma can determine whether $\text{ord}_p l(f, s)$ is 0, 1, or larger (for a newform $f$ as considered here), we can easily find one or several Heegner discriminants for $A$.

Associated to each Heegner discriminant $D$ is a Heegner point $y_D \in A(K)$, unique up to sign and adding a torsion point. In particular, the Heegner index $I_D = (A(K) : \text{End}(A) \cdot y_D)$ is well-defined.

Recall that $\mathcal{O} = \text{End}(A) = \text{End}_\mathbb{Q}(A)$ is an order in a real quadratic field. In all cases considered here, $\mathcal{O}$ is a maximal order and a principal ideal domain. For each prime ideal $p$ of $\mathcal{O}$, we have the residual Galois representation $\rho_p : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{Aut}(A[p]) \cong \text{GL}_2(\mathbb{F}_p)$, where $\mathbb{F}_p = \mathcal{O}/p$ denotes the residue class field.

We can use Magma’s functionality for 2-descent on hyperelliptic Jacobians based on [19] to determine $\text{III}(A/Q)[2]$. In all cases considered here, this group is trivial, which implies that $\text{III}(A/Q)[2^\infty] = 0$. (In fact, this had already been done in [8] for most of the curves.) It is therefore sufficient to consider the $p$-primary parts of $\text{III}(A/Q)$ for odd $p$.

**Theorem 1.** Let $A$ be an abelian variety of $\text{GL}_2$-type over $\mathbb{Q}$. Assume that $\text{ord}_{s=1} l(f, s) \in \{0, 1\}$ for one (equivalently, all) newform associated to $A$.

1. If the level $N$ of $A$ is square-free, $\text{ord}_p(\#\Sha(A/Q)_\text{an}) = \text{ord}_p(\#\Sha(A/Q))$ for all rational primes $p \neq 2$ such that $\rho_p$ is irreducible for all $p | p$.
2. If there exists a polarization $\lambda : A \to A^\vee$, $\text{III}(A/Q)[p] = 0$ for all prime ideals $p | p \neq 2$ such that $\rho_p$ is irreducible and $p$ does not divide $\deg \lambda$, and, for some Heegner field $K$ with Heegner discriminant $D$, $I_D$ and the order of the groups $H^1(K_v^\text{nr}[K_v, A])$ with $v$ running through the places of $K$.

**Proof.** (1) is [4, Theorems C and D]. (2) is an explicit version of [12].

We have implemented the following algorithms.

1. **Image of the residual Galois representations.** Extending the algorithm described in [7], which determines a finite small superset of the primes $p$ with $\rho_p$ reducible in the case that $\text{End}(A) = \mathbb{Z}$, we obtain a finite small superset of the prime ideals $p$ of $\mathcal{O}$ such that $\rho_p$ is reducible. Building upon this and [5], we can also check whether $\rho_p$ has maximal possible image $\text{GL}_2(\mathbb{F}_p)\text{det}^{-1}\mathbb{F}_p^\times$.\]
The irreducibility of \( \rho_p \) for all \( p \mid p \) is the crucial hypothesis in [4, Theorems C and D], and in [12].

(2) **Computation of the Heegner index.** We can compute the height of a Heegner point using the main theorem of [10]. By enumerating all points of that approximate height using [15], we can identify the Heegner point \( y_D \in A(K) \) as a \( \mathbb{Q} \)-point on \( A \), or on the quadratic twist \( A^K \), depending on the analytic rank of \( A/\mathbb{Q} \). An alternative implementation uses the \( j \)-invariant morphism \( X_0(N) \to X_0(1) \) and takes the preimages of the \( j \)-invariants belonging to elliptic curves with CM by the order of discriminant \( D \). A variant of this is based on approximating \( q \)-expansions of cusp forms analytically and finding the Heegner point as an algebraic approximation.

(3) **Determination of the (geometric) endomorphism ring of \( A/\mathbb{Q} \) and its action on the Mordell-Weil group \( A(\mathbb{Q}) \).** Given the Heegner point \( y_D \), this can be used to compute the Heegner index \( I_D \).

We can also compute the kernel of a given endomorphism as an abstract Gal(\( \mathbb{Q} \)) module together with explicit generators in \( A(\mathbb{Q}) \). We apply this to find the characters corresponding to the constituents of \( \rho_p \) when the representation is reducible.

(4) **Analytic order of \( \text{III} \).** If the \( L \)-rank \( \ord_{s=1} L(f,s) \) of \( A/\mathbb{Q} \) is zero, then we can compute \( \# \text{III}(A/K) \text{an} \) exactly as a rational number using modular symbols via Magma’s \text{L Ratio} function, which gives \( L(A,1)/\Omega_A^{-1} \in \mathbb{Q}_{>0} \), together with (1), since \( \#A(\mathbb{Q}) \text{tors} = \#A^\vee(\mathbb{Q}) \text{tors} \) and the Tamagawa numbers \( c_v \) are known.

When the \( L \)-rank is 1, we can compute the analytic order of \( \text{III} \) from \( \# \text{III}(A/K) \text{an} = \# \text{III}(A/\mathbb{Q}) \text{an} \cdot \# \text{III}(A^K/\mathbb{Q}) \text{an} \cdot 2^{(\text{bounded exponent})} \) and the formula

\[
\# \text{III}(A/K) \text{an} = \frac{\#A(K) \text{tors} \#A^\vee(K) \text{tors}}{c_2^2 u_K^2 \prod_p c_p(A/\mathbb{Q})^2} \cdot \frac{\|\omega_f\|^2 \|\omega_{f^*}\|^2 \cdot \hat{h}(y_D,f) \hat{h}(y_D,f^*) \text{disc} \mathcal{O}}{\Omega_A/K \cdot \text{Reg}_{A/K}}
\]

deduced from [10]; here, the last two factors are integral. In the computation of \( \# \text{III}(A^K/\mathbb{Q}) \text{an} \), we use van Bommel’s code to compute the Tamagawa numbers of \( A/\mathbb{Q} \) and \( A^K/\mathbb{Q} \) and the real period of \( A^K/\mathbb{Q} \). In the cases where his code did not succeed, we used another Heegner discriminant.

(5) **Isogeny descent.** In the cases when \( p \) is odd and \( \rho_p \) is reducible, we determined characters \( \chi_1 \) and \( \chi_2 \) such that

\[
\rho_p \cong \begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix};
\]

see (3) above. We then compute upper bounds for the \( \mathbb{F}_p \)-dimensions of the two Selmer groups associated to the corresponding two isogenies of degree \( p \) whose composition is multiplication by a generator \( \pi \) of \( p \) on \( A \); see [16]. From this, we deduce an upper bound for the dimension of the \( \pi \)-Selmer group of \( A \), which, in the cases considered here, is always \( \leq 1 \). Using the known finiteness of \( \text{III}(A/\mathbb{Q}) \), which implies that \( \text{III}(A/\mathbb{Q})[p] \) has even dimension, this shows that \( \text{III}(A/\mathbb{Q})[p] = 0 \).

(6) **Computation of the \( p \)-adic \( L \)-function.** We can also compute the \( p \)-adic \( L \)-functions of newforms of weight 2, trivial character and arbitrary coefficient ring for \( p^2 \nmid N \). Computing \( \ord_p \mathcal{L}_p^\ast(f,0) \) and using the known results [18, 17] about the \( GL_2 \) Iwasawa Main Conjecture (IMC) with the hypotheses...
Our results are summarized in Figure 1. The first column gives the genus 2 curve \( X \) as a quotient of \( X_0(N) \) by a subgroup of the Atkin-Lehner involutions. We denote the Atkin-Lehner involution associated to a divisor \( d \) of \( N \) such that \( d \) and \( N/d \) are coprime by \( w_d \). We write \( X_0(N)^\pm \) for \( X_0(N)/w_N \) and \( X_0(N)^* \) for the quotient of \( X_0(N) \) by the full group of Atkin-Lehner operators. We are considering the Jacobian \( A \) of \( X \).

The second column gives the algebraic rank of \( A/\mathbb{Q} \), which is equal to its analytic rank by the combination of the main results of [10] and [12].

| \( X \)      | \( r \) | \( \mathcal{O} \) | \#\( \mathrm{III}_{\mathrm{an}} \) | \( \rho_p \) red. | \( c \)   | \( (D, I_D) \) | \#\( \mathrm{III} \) |
|-------------|---------|-----------------|----------------|-----------------|-------|----------------|-----------------|
| \( X_0(23) \) | 0       | \( \sqrt{5} \) | 1              | 111             | 11    | \((-7, 11)\) | \( 11^0 \)      |
| \( X_0(29) \) | 0       | \( \sqrt{2} \) | 1              | 71              | 7     | \((-7, 7)\)   | \( 7^0 \)       |
| \( X_0(31) \) | 0       | \( \sqrt{5} \) | 1              | \( \sqrt{5} \) | 5     | \((-11, 5)\) | \( 5^0 \)       |
| \( X_0(35)/w_7 \) | 0      | \( \sqrt{17} \) | 1              | 21              | 1     | \((-19, 1)\) | 1               |
| \( X_0(39)/w_{13} \) | 0     | \( \sqrt{3} \) | 1              | \( \sqrt{2}, 71 \) | 7     | \((-23, 7)\) | \( 7^0 \)       |
| \( X_0(67)^* \) | 2      | \( \sqrt{5} \) | 1              | 1               | \((-7, 1)\) | 1               |
| \( X_0(73)^* \) | 2      | \( \sqrt{5} \) | 1              | 1               | \((-19, 1)\) | 1               |
| \( X_0(85)^* \) | 2      | \( \sqrt{2} \) | 1              | \( \sqrt{2} \) | 1     | \((-19, 1)\) | 1               |
| \( X_0(87)/w_{29} \) | 0    | \( \sqrt{5} \) | 1              | \( \sqrt{5} \) | 5     | \((-23, 5)\) | \( 5^0 \)       |
| \( X_0(93)^* \) | 2      | \( \sqrt{5} \) | 1              | 1               | \((-11, 1)\) | 1               |
| \( X_0(103)^* \) | 2      | \( \sqrt{5} \) | 1              | \((-11, 1)\) | 1     | \((-11, 1)\) | \( 1^0 \)       |
| \( X_0(107)^* \) | 2      | \( \sqrt{5} \) | 1              | \((-7, 1)\) | 1     | \((-11, 1)\) | 1               |
| \( X_0(115)^* \) | 2      | \( \sqrt{5} \) | 1              | \((-11, 1)\) | 1     | \((-11, 1)\) | \( 5^0 \)       |
| \( X_0(125)^+ \) | 2      | \( \sqrt{5} \) | 1              | \( \sqrt{5} \) | 1     | \((-11, 1)\) | \( 5^0 \)       |
| \( X_0(133)^* \) | 2      | \( \sqrt{5} \) | 1              | \((-31, 1)\) | 1     | \((-31, 1)\) | \( 1^0 \)       |
| \( X_0(147)^* \) | 2      | \( \sqrt{2} \) | 1              | \( \sqrt{2}, 71 \) | 7     | \((-47, 1)\) | \( 7^0 \)       |
| \( X_0(161)^* \) | 2      | \( \sqrt{5} \) | 1              | \(-19, 1\) | 1     | \((-131, 1)\) | \( 1^0 \)       |
| \( X_0(165)^* \) | 2      | \( \sqrt{3} \) | 1              | \( \sqrt{2} \) | 1     | \((-15, 1)\) | \( 1^0 \)       |
| \( X_0(167)^+ \) | 2      | \( \sqrt{5} \) | 1              | \(-11, 1\) | 1     | \((-11, 1)\) | \( 1^0 \)       |
| \( X_0(177)^* \) | 2      | \( \sqrt{5} \) | 1              | \((-7, 1)\) | 1     | \((-7, 1)\) | \( 1^0 \)       |
| \( X_0(191)^+ \) | 2      | \( \sqrt{5} \) | 1              | \(-31, 1\) | 1     | \((-31, 1)\) | \( 1^0 \)       |
| \( X_0(205)^* \) | 2      | \( \sqrt{5} \) | 1              | \(-31, 1\) | 1     | \((-31, 1)\) | \( 1^0 \)       |
| \( X_0(209)^* \) | 2      | \( \sqrt{2} \) | 1              | \((-51, 1)\) | 1     | \((-51, 1)\) | \( 1^0 \)       |
| \( X_0(213)^* \) | 2      | \( \sqrt{5} \) | 1              | \(-11, 1\) | 1     | \((-11, 1)\) | \( 1^0 \)       |
| \( X_0(221)^* \) | 2      | \( \sqrt{5} \) | 1              | \(-35, 1\) | 1     | \((-35, 1)\) | \( 1^0 \)       |
| \( X_0(287)^* \) | 2      | \( \sqrt{5} \) | 1              | \(-31, 1\) | 1     | \((-31, 1)\) | \( 1^0 \)       |
| \( X_0(299)^* \) | 2      | \( \sqrt{5} \) | 1              | \(-43, 1\) | 1     | \((-43, 1)\) | \( 1^0 \)       |
| \( X_0(357)^* \) | 2      | \( \sqrt{2} \) | 1              | \(-47, 1\) | 1     | \((-47, 1)\) | \( 1^0 \)       |

Figure 1. BSD data for the absolutely simple modular Jacobians of Atkin-Lehner quotients of \( X_0(N) \).

that \( \rho_p \) is irreducible and there is a \( q \parallel N \) with \( \rho_p \) ramified at \( q \neq p \) gives us information about the \( p^\infty \)-Selmer group.

3. Results

Our results are summarized in Figure 1. The first column gives the genus 2 curve \( X \) as a quotient of \( X_0(N) \) by a subgroup of the Atkin-Lehner involutions. We denote the Atkin-Lehner involution associated to a divisor \( d \) of \( N \) such that \( d \) and \( N/d \) are coprime by \( w_d \). We write \( X_0(N)^\pm \) for \( X_0(N)/w_N \) and \( X_0(N)^* \) for the quotient of \( X_0(N) \) by the full group of Atkin-Lehner operators. We are considering the Jacobian \( A \) of \( X \).

The second column gives the algebraic rank of \( A/\mathbb{Q} \), which is equal to its analytic rank by the combination of the main results of [10] and [12].
The third column specifies $O$ as the maximal order in the number field obtained by adjoining the given square root to $\mathbb{Q}$.

The fourth column gives the analytic order of the Shafarevich-Tate group of $A$, defined as in the introduction. For the surfaces of $L$-rank 1, the intermediate results of our computation are contained in Figure 2.

The fifth column specifies the prime ideals $p$ of $O$ such that $\rho_p$ is reducible. The notation $p_1$ means that $p$ is split in $O$ and $\rho_p$ is reducible for exactly one $p \mid p$. If $p$ is ramified in $O$, we write $\sqrt{p}$ for the unique prime ideal $p \mid p$.

The sixth column gives the odd part of $\text{lcm}_p c_p(A/\mathbb{Q})$, which can be obtained from the LMFDB [21].

The seventh column gives a Heegner discriminant $D$ for $A$ together with the odd part of the Heegner index $I_D$. Our computation confirms that the Tamagawa product divides the Heegner index.

The last column contains the order of the Shafarevich-Tate group of $A/\mathbb{Q}$. An entry 1 means that it follows immediately from the previous columns, the computation of $\text{Sel}_2(A/\mathbb{Q})$ and Theorem 1 that all $p$-primary components of $\text{III}(A/\mathbb{Q})$ vanish.

### Table

| $X$            | $D_K$ | $\#\text{III}(A^K/\mathbb{Q})_{\text{an}}$ | $\#\text{III}(A/K)_{\text{an}}$ | $\#\text{III}(A/\mathbb{Q})_{\text{an}}$ |
|---------------|------|---------------------------------|---------------------------------|---------------------------------|
| $X_0(67)^+$   | -7   | 4                              | 1                               | 1                               |
| $X_0(73)^+$   | -19  | 4                              | 1                               | 1                               |
| $X_0(85)^*$   | -11  | 1                              | 1                               | 1                               |
| $X_0(93)^*$   | -11  | 4                              | 1                               | 1                               |
| $X_0(103)^+$  | -7   | 4                              | 1                               | 1                               |
| $X_0(107)^+$  | -11  | 1                              | 1                               | 1                               |
| $X_0(115)^*$  | -11  | 4                              | 1                               | 1                               |
| $X_0(125)^+$  | -31  | 4                              | 1                               | 1                               |
| $X_0(147)^*$  | -47  | 4                              | 1                               | 1                               |
| $X_0(161)^*$  | -19  | 1                              | 1                               | 1                               |
| $X_0(165)^*$  | -131 | 16                             | 4                               | 1                               |
| $X_0(167)^+$  | -15  | 4                              | 1                               | 1                               |
| $X_0(177)^*$  | -11  | 4                              | 1                               | 1                               |
| $X_0(191)^+$  | -7   | 4                              | 1                               | 1                               |
| $X_0(205)^*$  | -31  | 4                              | 1                               | 1                               |
| $X_0(209)^*$  | -79  | 2                              | 1                               | 1                               |
| $X_0(213)^*$  | -11  | 4                              | 1                               | 1                               |
| $X_0(221)^*$  | -35  | 4                              | 1                               | 1                               |
| $X_0(287)^*$  | -21  | 4                              | 1                               | 1                               |
| $X_0(299)^*$  | -43  | 4                              | 1                               | 1                               |
| $X_0(357)^*$  | -47  | 2                              | 1                               | 1                               |

**Figure 2.** Analytic order of $\text{III}$ for the curves of $L$-rank 1. (A * means that we used a different Heegner discriminant than in Figure 1 in the case where van Bommel’s TamagawaNumber did not succeed.)
Otherwise, the order of $\text{III}(A/\mathbb{Q})$ is given as a product of powers of the odd primes $p$ such that some $\rho_p$ with $p \mid p$ is reducible or $p$ divides $c \cdot I_p$. (In each of these cases, there is exactly one such $p$.) We have to justify that the exponents are all zero. In the first three rows we use [13] to show that for the reducible odd $p$ has $\text{III}(J_0(p)/\mathbb{Q})[p] = 0$; this is a consequence of these prime ideals being Eisenstein primes.

In the remaining cases, we used the approach described in item (5) in Section 2. For the rows with $A = \text{Jac}(X_0(39)/w_{13})$ and $A = \text{Jac}(X_0(87)/w_{29})$, one has non-split short exact sequences of Galois modules

$$0 \to \mathbb{Z}/p \to A[p] \to \mu_p \to 1$$

with $p = 7$ and 5, respectively. For the only two non-semistable abelian surfaces we found the following isomorphism and exact sequence.

$$J_0(125)^7[\sqrt{5}] \cong \mu_5^2 \oplus \mu_5^3$$

$$1 \to \mu_7^4 \to \text{Jac}(X_0(147)^*)[p] \to \mu_7^3 \to 1$$

In all cases, we find that $\text{III}(A/\mathbb{Q})[p] = 0$. Note that for the $p \mid 7$ for which $\rho_p$ is irreducible, $\text{Jac}(X_0(147)^*)[p] = 0$ follows from [12] because $L_{-43}$ is not divisible by 7. In the case of the square-free levels $N = 23, 29, 39$, we computed that the $p$-adic $L$-function is a unit for the $p \mid p$ with $\rho_p$ irreducible, so we can conclude that $\text{Sel}_p(A/\mathbb{Q}) = 0$ and hence $\#\text{III}(A/\mathbb{Q})[p] = 0$ from the known cases of the $\text{GL}_2$ IMC.

Note that our computation shows that in these cases, the image of $\rho_p \sim$ is maximal, so it contains $\text{SL}_2(\mathbb{Z}_p)$. This implies that the IMC holds integrally.

Details will be presented in a forthcoming article, where plan also to extend our computations to cover some two-dimensional absolutely simple isogeny factors of $J_0(N)$ that are not Jacobians of quotients of $X_0(N)$ by Atkin-Lehner involutions.

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