THREE CHARACTERIZATIONS OF A SELF-SIMILAR APERIODIC 2-DIMENSIONAL SUBSHIFT

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Abstract. The goal of this chapter is to illustrate a generalization of the Fibonacci word to the case of 2-dimensional configurations on $\mathbb{Z}^2$. More precisely, we consider a particular subshift of $\mathcal{A}^{\mathbb{Z}^2}$ on the alphabet $\mathcal{A} = \{0,\ldots,18\}$ for which we give three characterizations: as the subshift $X_\phi$ generated by a 2-dimensional morphism $\phi$ defined on $\mathcal{A}$; as the Wang shift $\Omega_\mathcal{U}$ defined by a set $\mathcal{U}$ of 19 Wang tiles; as the symbolic dynamical system $X_{\mathcal{P}_\mathcal{U}, R_\mathcal{U}}$ representing the orbits under some $\mathbb{Z}^2$-action $R_\mathcal{U}$ defined by rotations on $\mathbb{T}^2$ and coded by some topological partition $\mathcal{P}_\mathcal{U}$ of $\mathbb{T}^2$ into 19 polygonal atoms. We prove their equality $X_\phi = \Omega_\mathcal{U} = X_{\mathcal{P}_\mathcal{U}, R_\mathcal{U}}$ by showing they are self-similar with respect to the substitution $\phi$.

This chapter provides a transversal reading of results divided into four different articles obtained through the study of the Jeandel-Rao Wang shift. It gathers in one place the methods introduced to desubstitute Wang shifts and to desubstitute codings of $\mathbb{Z}^2$-actions by focussing on a simple 2-dimensional self-similar subshift. Algorithms to find marker tiles and compute the Rauzy induction of $\mathbb{Z}^2$-rotations are provided as well as the SageMath code to reproduce the computations.

1. Introduction

The rule $s : a \mapsto ab, b \mapsto a$ defines a morphism on the monoid $\{a,b\}^\ast$. The successive application of this morphism on the letter $a$ defines longer and longer words covering the negative and non-negative integers:

\[
\begin{align*}
& s^0(a) = a \\
& s^1(a) = s^0(a) b \\
& s^2(a) = s^1(a) a \\
& s^3(a) = s^2(a) ba \\
& s^4(a) = s^3(a) aba \\
& s^5(a) = s^4(a) abaababaab
\end{align*}
\]

The letters that change from line to line are underlined. It is an interesting exercise to show that at the limit, we obtain $\lim_{n \to \infty} s^{2n}(a) | s^{2n}(a) = \tilde{F}ba | F$ and $\lim_{n \to \infty} s^{2n+1}(a) | s^{2n+1}(a) = \tilde{F}ab | F$ where $F$ is the well-known right-infinite Fibonacci word $\left[\text{Ber80}\right]$. The rule $s$ can be seen as a substitution that we may apply on the biinfinite words $x = \tilde{F}ba | F$ and $y = \tilde{F}ab | F$ and we observe that $s(x) = y$ and $s(y) = x$. Thus $s^2(x) = x$ and $s^2(y) = y$ and we say that...

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$x$ and $y$ are fixed points of $s^2$. The set of finite words that appear in $x = s^2(x)$ defines a language

$$\mathcal{L}_s = \{\varepsilon, a, b, aa, ab, ba, aba, baa, bab, aaba, \ldots\} \subset \{a, b\}^*$$

and a subshift

$$\mathcal{X}_s = \{u \in \{a, b\}^\mathbb{Z}: \text{the finite words that appear in } u \text{ are in } \mathcal{L}_s\}.$$ 

The subshift $\mathcal{X}_s$ contains $x$, $y$, all shifts of $x$ and $y$, and much more. Indeed, $\mathcal{X}_s$ is a Sturmian shift which is an uncountable set. The reader will find detailed information on Sturmian sequences in [Lot02, Chapter 2] and [Fog02, Chapter 6].

It is known since the early work of Morse and Hedlund in [MH40] and Coven and Hedlund in [CH73] that the 1-dimensional subshift $\mathcal{X}_s$, being a Sturmian subshift, has many equivalent characterizations:

- as the subshift generated by the 1-dimensional substitution $s$;
- as the subshift on $\{a, b\}$ having exactly $n + 1$ factors of length $n$ and such that the ratio of the frequency of the two letters is $\frac{1 + \sqrt{5}}{2}$;
- as the symbolic representation of a rotation on the 1-dimensional torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ through a partition into two intervals.

The goal of this chapter is to illustrate a generalization of the above to the case of 2-dimensional configurations on $\mathbb{Z}^2$. More precisely, we consider a particular subshift of $[0, 18]^{\mathbb{Z}^2}$ for which we give three characterizations:

- as the subshift $\mathcal{X}_\phi$ generated by the 2-dimensional morphism $\phi$ defined on the alphabet $\mathcal{A} = [0, 18]$ by the rule
  \[
  \phi : \quad [0, 18] \to [0, 18]^{\mathbb{Z}^2}
  \]
  \[
  \begin{cases}
  0 \mapsto (17), & 1 \mapsto (16), & 2 \mapsto (15, 11), & 3 \mapsto (13, 9), \\
  4 \mapsto (17, 8), & 5 \mapsto (16, 8), & 6 \mapsto (15, 8), & 7 \mapsto (14, 8), \\
  8 \mapsto \begin{pmatrix} 14 \end{pmatrix}, & 9 \mapsto \begin{pmatrix} 17 \end{pmatrix}, & 10 \mapsto \begin{pmatrix} 3 \\ 16 \end{pmatrix}, & 11 \mapsto \begin{pmatrix} 2 \\ 14 \end{pmatrix}, \\
  12 \mapsto \begin{pmatrix} 15 \\ 11 \end{pmatrix}, & 13 \mapsto \begin{pmatrix} 6 \\ 1 \end{pmatrix}, & 14 \mapsto \begin{pmatrix} 7 \\ 1 \end{pmatrix}, & 15 \mapsto \begin{pmatrix} 6 \\ 12 \end{pmatrix}, \\
  16 \mapsto \begin{pmatrix} 5 \\ 18 \end{pmatrix}, & 17 \mapsto \begin{pmatrix} 4 \\ 13 \end{pmatrix}, & 18 \mapsto \begin{pmatrix} 2 \\ 14 \end{pmatrix}.
  \end{cases}
  \]

- as the Wang shift $\Omega_U$, that is, the set of valid configuration $\mathbb{Z}^2 \to [0, 18]$ of the plane, defined by the following set $U$ of 19 Wang tiles:

|   |   |   |   |   |   |   |   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| O | J | 0 | F | O | I | J | 0 | F | O | I | J | 0 | F |
| H | F | L | 1 | F | 2 | J | P | M | F | 3 | D | K | P |
| 7 | H | P | 8 | H | 9 | G | L | C | G | L |
| 11 | H | 11 | A | O | L | 11 | 11 | A | P | G | 12 | E | P |
| P | G | 14 | K | P | I | 15 | I | K | K | A | 17 | I | K |
| P | B | 16 | M | P | I | 18 | C | P | I | 18 | C | P |
| B | 16 | M | A | 17 | I | K | P | G | 14 | K | P | I | 15 | I | K |

- as the Wang shift $\Omega_U$, that is, the set of valid configuration $\mathbb{Z}^2 \to [0, 18]$ of the plane, defined by the following set $U$ of 19 Wang tiles:
• as the symbolic dynamical system $\mathcal{X}_{\mathcal{P}_U,R_U}$ representing the orbits under the $\mathbb{Z}^2$-action $R_{U}$ defined by rotations on $\mathbb{T}^2$ and coded by the topological partition $\mathcal{P}_U$ of $\mathbb{T}^2$:

\[ R_{U}^n(x) = x + \varphi^{-2} n \]

The reader may observe that while increasing the dimension from 1 to 2, we replaced the second characterization of the Fibonacci word based on the factor complexity by the notion of Wang shift or more generally subshift of finite type (SFT). It may seem counter-intuitive since 1-dimensional SFT always contain a periodic configuration [LM95], but this is no longer holds in higher dimension [Ber66].

We show that $\Omega_U$ and $\mathcal{X}_{\mathcal{P}_U,R_U}$ are self-similar. The tools used in the proofs are completely different in each case: based on the notion of marker tiles in the former case and on Rauzy induction of $\mathbb{Z}^2$-rotations in the latter. It turns out that the 2-dimensional morphism describing the self-similarities is the above morphism $\phi$ in both cases.

**Theorem 1.1.** Let $\phi : [0,18] \rightarrow [0,18]^2$ be defined in Equation (1). The following hold:

(i) The Wang shift $\Omega_U \subset [0,18]^{\mathbb{Z}^2}$ is self-similar satisfying $\Omega_U = \phi(\Omega_U)^\sigma$,

(ii) The subshift $\mathcal{X}_{\mathcal{P}_U,R_U} \subset [0,18]^{\mathbb{Z}^2}$ is self-similar satisfying $\mathcal{X}_{\mathcal{P}_U,R_U} = \phi(\mathcal{X}_{\mathcal{P}_U,R_U})^\sigma$.

The equality of the three subshifts follows as $\mathcal{X}_\phi$ is the unique nonempty subshift $X \subset [0,18]^{\mathbb{Z}^2}$ such that $X = \overline{\phi(X)}^\sigma$ (see Exercise 3.12).

**Corollary 1.2.** The three subshifts are equal: $\mathcal{X}_\phi = \Omega_U = \mathcal{X}_{\mathcal{P}_U,R_U}$.

The 2-dimensional subshift $\Omega_U$ was introduced in [Lab19a] and discovered during the study of the substitutive structure [Lab19b] of the Jeandel-Rao Wang shift [JR15]. Its description as the coding of a toral $\mathbb{Z}^2$-action was presented in [Lab20a] and its substitutive structure was further developed in [Lab20c]. This chapter provides a transversal reading of results divided in four different articles and gathers the methods introduced by focussing on the self-similar subshift hidden in the Jeandel-Rao Wang shift. Thus we avoid the difficulty raised by the Jeandel-Rao Wang shift itself which is not a minimal subshift, has a long preperiod in its substitutive description and needs the definition of other tools including the shear-conjugacy.

**Structure of the chapter.** Section 2 gathers preliminary notions on topological dynamical systems, subshifts and shifts of finite type and $d$-dimensional languages. In Section 3 we define a 2-dimensional self-similar subshift $\mathcal{X}_\phi$ from a 2-dimensional substitution $\phi$ defined on 19-letter alphabet. We show that $\mathcal{X}_\phi$ is aperiodic. In Section 4 we introduce a Wang shift $\Omega_U$ defined from a set $\mathcal{U}$ of 19 Wang tiles and we show using the notion of marker tiles that it is self-similar and $\Omega_U = \mathcal{X}_\phi$. In Section 5 we introduce a 2-dimensional subshift defined as the
symbolic representation of a toral $\mathbb{Z}^2$-rotation using a partition of $\mathbb{R}^2/\mathbb{Z}^2$ into 19 polygons. We show that it is also self-similar and equal to $X_0$.

Algorithms to find marker tiles and compute the Rauzy induction of $\mathbb{Z}^2$-rotations are provided as well as the SageMath code to reproduce the computations.

2. Preliminaries

2.1. Topological dynamical systems. Most of the notions introduced here can be found in [Wal82]. A dynamical system is a triple $(X, G, T)$, where $X$ is a topological space, $G$ is a topological group and $T$ is a continuous function $G \times X \to X$ defining a left action of $G$ on $X$: if $x \in X$, $e$ is the identity element of $G$ and $g, h \in G$, then using additive notation for the operation in $G$ we have $T(e, x) = x$ and $T(g + h, x) = T(g, T(h, x))$. In other words, if one denotes the transformation $x \mapsto T(g, x)$ by $T^g$, then $T^{g + h} = T^g T^h$. In this work, we consider the Abelian group $G = \mathbb{Z} \times \mathbb{Z}$.

If $Y \subset X$, let $\overline{Y}$ denote the topological closure of $Y$ and let $T(Y) := \cup_{g \in G} T^g(Y)$ denote the $T$-closure of $Y$. A subset $Y \subset X$ is $T$-invariant if $T(Y) = Y$. A dynamical system $(X, G, T)$ is called minimal if $X$ does not contain any nonempty, proper, closed $T$-invariant subset. The left action of $G$ on $X$ is free if $g = e$ whenever there exists $x \in X$ such that $T^g(x) = x$.

Let $(X, G, T)$ and $(Y, G, S)$ be two dynamical systems with the same topological group $G$. A homomorphism $\theta : (X, G, T) \to (Y, G, S)$ is a continuous function $\theta : X \to Y$ satisfying the commuting property that $S^g \circ \theta = \theta \circ T^g$ for every $g \in G$. A homomorphism $\theta : (X, G, T) \to (Y, G, S)$ is called an embedding if it is one-to-one, a factor map if it is onto, and a topological conjugacy if it is both one-to-one and onto and its inverse map is continuous. If $\theta : (X, G, T) \to (Y, G, S)$ is a factor map, then $(Y, G, S)$ is called a factor of $(X, G, T)$ and $(X, G, T)$ is called an extension of $(Y, G, S)$. Two subshifts are topologically conjugate if there is a topological conjugacy between them.

Let $B$ be the Borel $\sigma$-algebra of subsets of $X$. The set of all $T$-invariant probability measures of a dynamical system $(X, G, T)$ is denoted by $\mathcal{M}^T(X)$. An invariant probability measure on $X$ is called ergodic if for every set $B \in B$ such that $T^g(B) = B$ for all $g \in G$, we have that $B$ has either zero or full measure. A dynamical system $(X, G, T)$ is uniquely ergodic if it has only one invariant probability measure, i.e., $|\mathcal{M}^T(X)| = 1$. A dynamical system $(X, G, T)$ is said strictly ergodic if it is uniquely ergodic and minimal.

A measure-preserving dynamical system is defined as a system $(X, G, T, \mu, B)$, where $\mu$ is a probability measure defined on the $\sigma$-algebra $B$, and $T^g : X \to X$ is a measurable map which preserves the measure $\mu$ for all $g \in G$, that is, $\mu(T^g(B)) = \mu(B)$ for all $B \in B$. The measure $\mu$ is said to be $T$-invariant. In what follows, $B$ is always the Borel $\sigma$-algebra of subsets of $X$, so we omit $B$ and write $(X, G, T, \mu)$ when it is clear from the context.

Let $(X, G, T, \mu, B)$ and $(X', G, T', \mu', B')$ be two measure-preserving dynamical systems. We say that the two systems are isomorphic if there exist measurable sets $X_0 \subset X$ and $X_0' \subset X'$ of full measure (i.e., $\mu(X \setminus X_0) = 0$ and $\mu'(X' \setminus X_0') = 0$) with $T^g(X_0) \subset X_0$, $T'^g(X_0') \subset X_0'$ for all $g \in G$ and there exists a map $\phi : X_0 \to X_0'$, called an isomorphism, that is one-to-one and onto and such that for all $A \in B'(X_0')$,

- $\phi^{-1}(A) \in B(X_0)$,
- $\mu(\phi^{-1}(A)) = \mu'(A)$, and
- $\phi \circ T^g(x) = T'^g \circ \phi(x)$ for all $x \in X_0$ and $g \in G$. 

The role of the set $X_0$ is to make precise the fact that the properties of the isomorphism need to hold only on a set of full measure.

2.2. Subshifts and shifts of finite type. We follow the notation of [Sch01]. Let $\mathcal{A}$ be a finite set, $d \geq 1$, and let $\mathcal{A}^d_z$ be the set of all maps $x : \mathbb{Z}^d \to \mathcal{A}$, endowed with the prodiscrete topology. An element $x \in \mathcal{A}^d_z$ is called configuration and we write it as $x = (x_m) = (x_m : m \in \mathbb{Z}^d)$, where $x_m \in \mathcal{A}$ denotes the value of $x$ at $m$. The topology on $\mathcal{A}^d_z$ is compatible with the metric defined for all configurations $x, x' \in \mathcal{A}^d_z$ by $\text{dist}(x, x') = 2^{-\min\{|n| : x_m \neq x'_m\}}$ where $|n| = |n_1| + \cdots + |n_d|$. The shift action $\sigma : n \mapsto \sigma^n$ of $\mathbb{Z}^d$ on $\mathcal{A}^d_z$ is defined by

$$ (\sigma^n(x))_m = x_{m+n} $$

for every $x = (x_m) \in \mathcal{A}^d_z$ and $n \in \mathbb{Z}^d$. If $X \subset \mathcal{A}^d_z$, let $\overline{X}$ denote the topological closure of $X$ and let $\overline{X} : = \{\sigma^n(x) \mid x \in X, \; n \in \mathbb{Z}^d\}$ denote the shift-closure of $X$. A subset $X \subset \mathcal{A}^d_z$ is shift-invariant if $\overline{X} = X$. A closed, shift-invariant subset $X \subset \mathcal{A}^d_z$ is a subshift. If $X \subset \mathcal{A}^d_z$ is a subshift we write $\sigma = \sigma^X$ for the restriction of the shift action (2) to $X$. When $X$ is a subshift, the triple $(X, \mathbb{Z}^d, \sigma)$ is a dynamical system and the notions presented in the previous section hold.

A configuration $x \in X$ is periodic if there is a nonzero vector $n \in \mathbb{Z}^d \setminus \{0\}$ such that $x = \sigma^n(x)$ and otherwise it is said nonperiodic. We say that a nonempty subshift $X$ is aperiodic if the shift action $\sigma$ on $X$ is free.

For any subset $S \subset \mathbb{Z}^d$ let $\pi_S : \mathcal{A}^d_z \to \mathcal{A}^S$ denote the projection map which restricts every $x \in \mathcal{A}^d_z$ to $S$. A pattern is a function $p \in \mathcal{A}^S$ for some finite subset $S \subset \mathbb{Z}^d$. To every pattern $p \in \mathcal{A}^S$ corresponds a subset $\pi_S^{-1}(p) \subset \mathcal{A}^d_z$ called cylinder. A subshift $X \subset \mathcal{A}^d_z$ is a shift of finite type (SFT) if there exists a finite set $\mathcal{F}$ of forbidden patterns such that

$$ X = \{x \in \mathcal{A}^d_z \mid \pi_S \circ \sigma^n(x) \notin \mathcal{F} \text{ for every } n \in \mathbb{Z}^d \text{ and } S \subset \mathbb{Z}^d\}. $$

In this case, we write $X = \text{SFT}(\mathcal{F})$. In this article, we consider shifts of finite type on $\mathbb{Z} \times \mathbb{Z}$, that is, the case $d = 2$.

2.3. d-dimensional word. In this section, we recall the definition of $d$-dimensional word that appeared in [CKR10] and we keep the notation $u \circ^j v$ they proposed for the concatenation.

We denote by $\{e_k | 1 \leq k \leq d\}$ the canonical basis of $\mathbb{Z}^d$ where $d \geq 1$ is an integer. If $i \leq j$ are integers, then $[i, j]$ denotes the interval of integers $\{i, i+1, \ldots, j\}$. Let $n = (n_1, \ldots, n_d) \in \mathbb{N}^d$ and $\mathcal{A}$ be an alphabet. We denote by $\mathcal{A}^n$ the set of functions

$$ u : [0, n_1 - 1] \times \cdots \times [0, n_d - 1] \to \mathcal{A}. $$

An element $u \in \mathcal{A}^n$ is called a $d$-dimensional word of shape $n = (n_1, \ldots, n_d) \in \mathbb{N}^d$ on the alphabet $\mathcal{A}$. We use the notation $\text{shape}(u) = n$ when necessary. The set of all finite $d$-dimensional words is $\mathcal{A}^* = \{\mathcal{A}^n \mid n \in \mathbb{N}^d\}$. A $d$-dimensional word of shape $e_k + \sum_{i=1}^d e_i$ is called a domino in the direction $e_k$. When the context is clear, we write $\mathcal{A}$ instead of $\mathcal{A}^{(1, \ldots, 1)}$. When $d = 2$, we represent a $d$-dimensional word $u$ of shape $(n_1, n_2)$ as a matrix with Cartesian coordinates:

$$ u = \begin{pmatrix} u_{0,n_2 - 1} & \cdots & u_{n_1 - 1, n_2 - 1} \\ \cdots & \cdots & \cdots \\ u_{0,0} & \cdots & u_{n_1 - 1, 0} \end{pmatrix}. $$
Let $n, m \in \mathbb{N}^d$ and $u \in \mathcal{A}^n$ and $v \in \mathcal{A}^m$. If there exists an index $i$ such that $n_j = m_j$ for all $j \in \{1, \ldots, d\} \setminus \{i\}$, then the concatenation of $u$ and $v$ in the direction $e_i$ is defined: it is the $d$-dimensional word $u \odot^i v$ of shape $(n_1, \ldots, n_{i-1}, n_i + m_i, n_{i+1}, \ldots, n_d) \in \mathbb{N}^d$ given as
\[
(u \odot^i v)(a) = \begin{cases} 
  u(a) & \text{if } 0 \leq a_i < n_i, \\
  v(a - n_i e_i) & \text{if } n_i \leq a_i < n_i + m_i.
\end{cases}
\]

If the shapes $n$ and $m$ are not equal except at index $i$, we say that the concatenation of $u \in \mathcal{A}^n$ and $v \in \mathcal{A}^m$ in the direction $e_i$ is not defined. The following equation illustrates the concatenation of words in the direction $e_2$ when $d = 2$:
\[
\begin{pmatrix} 4 & 5 \\ 10 & 5 \end{pmatrix} \odot^2 \begin{pmatrix} 3 & 10 \\ 9 & 9 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 10 \\ 9 & 9 \\ 4 & 5 \\ 10 & 5 \end{pmatrix}
\]
in the direction $e_1$ when $d = 2$:
\[
\begin{pmatrix} 2 & 8 & 7 \\ 1 & 1 & 0 \\ 6 & 6 & 7 \\ 7 & 4 & 3 \end{pmatrix} \odot^1 \begin{pmatrix} 3 & 10 \\ 9 & 9 \\ 0 & 0 \\ 4 & 5 \\ 10 & 5 \end{pmatrix} = \begin{pmatrix} 2 & 8 & 7 & 3 & 10 \\ 7 & 3 & 9 & 9 & 9 \\ 1 & 1 & 0 & 0 & 0 \\ 6 & 6 & 7 & 4 & 5 \\ 7 & 4 & 3 & 10 & 5 \end{pmatrix}.
\]

Let $n, m \in \mathbb{N}^d$ and $u \in \mathcal{A}^n$ and $v \in \mathcal{A}^m$. We say that $u$ occurs in $v$ at position $p \in \mathbb{N}^d$ if $v$ is large enough, i.e., $m - p - n \in \mathbb{N}^d$ and
\[
v(a + p) = u(a)
\]
for all $a = (a_1, \ldots, a_d) \in \mathbb{N}^d$ such that $0 \leq a_i < n_i$ with $1 \leq i \leq d$. If $u$ occurs in $v$ at some position, then we say that $u$ is a $d$-dimensional subword or factor of $v$.

2.4. $d$-dimensional language. A subset $L \subseteq \mathcal{A}^d$ is called a $d$-dimensional language. The factorial closure of a language $L$ is
\[
\mathcal{L}^\text{Fact} = \{ u \in \mathcal{A}^d \mid u \text{ is a } d\text{-dimensional subword of some } v \in L \}.
\]
A language $L$ is factorial if $\mathcal{L}^\text{Fact} = L$. All languages considered in this contribution are factorial. Given a configuration $x \in \mathcal{A}^d$, the language $\mathcal{L}(x)$ defined by $x$ is
\[
\mathcal{L}(x) = \{ u \in \mathcal{A}^d \mid u \text{ is a } d\text{-dimensional subword of } x \}.
\]
The language of a subshift $X \subseteq \mathcal{A}^d$ is $\mathcal{L}_X = \bigcup_{x \in X} \mathcal{L}(x)$. Conversely, given a factorial language $L \subseteq \mathcal{A}^d$ we define the subshift
\[
\mathcal{X}_L = \{ x \in \mathcal{A}^d \mid \mathcal{L}(x) \subseteq L \}.
\]
A $d$-dimensional subword $u \in \mathcal{A}^d$ is allowed in a subshift $X \subseteq \mathcal{A}^d$ if $u \in \mathcal{L}_X$ and it is forbidden in $X$ if $u \notin \mathcal{L}_X$. A language $L \subseteq \mathcal{A}^d$ is forbidden in a subshift $X \subseteq \mathcal{A}^d$ if $L \cap \mathcal{L}_X = \emptyset$. 
2.5. **d-dimensional morphisms.** In this section, we generalize the definition of \( d \)-dimensional morphisms \([\text{CKR10}]\) to the case where the domain and codomain are different as for \( S \)-adic systems \([\text{BD14}]\).

Let \( \mathcal{A} \) and \( \mathcal{B} \) be two alphabets. Let \( L \subseteq \mathcal{A}^d \) be a factorial language. A function \( \omega : L \rightarrow \mathcal{B}^d \) is a **\( d \)-dimensional morphism** if for every \( i \) with \( 1 \leq i \leq d \), and every \( u, v \in L \) such that \( u \odot^i v \in L \) is defined we have that the concatenation \( \omega(u) \odot^i \omega(v) \) in direction \( e_i \) is defined and

\[
\omega(u \odot^i v) = \omega(u) \odot^i \omega(v).
\]

Note that the left-hand side of the equation is defined since \( u \odot^i v \) belongs to the domain of \( \omega \). A \( d \)-dimensional morphism \( L \rightarrow \mathcal{B}^d \) is thus completely defined from the image of the letters in \( \mathcal{A} \), so we sometimes denote a \( d \)-dimensional morphism as a rule \( \mathcal{A} \rightarrow \mathcal{B}^d \) when the language \( L \) is unspecified.

The next lemma can be deduced from the definition. It says that when \( d \geq 2 \) every \( d \)-dimensional morphism defined on the whole set \( L = \mathcal{A}^d \) is uniform. We say that a \( d \)-dimensional morphism \( \omega : L \rightarrow \mathcal{B}^d \) is **uniform** if there exists a shape \( n \in \mathbb{N}^d \) such that \( \omega(a) \in \mathcal{B}^n \) for every letter \( a \in \mathcal{A} \). These are called block-substitutions in \([\text{Fra18}]\).

**Lemma 2.1.** If \( d \geq 2 \), every \( d \)-dimensional morphism \( \omega : \mathcal{A}^d \rightarrow \mathcal{B}^d \) is uniform.

Therefore, to consider non-uniform \( d \)-dimensional morphisms when \( d \geq 2 \), we need to restrict the domain to a strict subset \( L \subseteq \mathcal{A}^d \). In \([\text{CKR10}]\) and \([\text{Moz89}, \text{p.144}]\), they consider the case \( \mathcal{A} = \mathcal{B} \) and they restrict the domain of \( d \)-dimensional morphisms to the language they generate.

Given a language \( L \subseteq \mathcal{A}^d \) of \( d \)-dimensional words and a \( d \)-dimensional morphism \( \omega : L \rightarrow \mathcal{B}^d \), we define the image of the language \( L \) under \( \omega \) as the language

\[
\overline{\omega(L)}^\text{Fact} = \{ u \in \mathcal{B}^d \mid u \text{ is a } d\text{-dimensional subword of } \omega(v) \text{ with } v \in L \} \subseteq \mathcal{B}^d.
\]

Let \( L \subseteq \mathcal{A}^d \) be a factorial language and \( \mathcal{X}_L \subseteq \mathcal{A}^{\mathbb{Z}^d} \) be the subshift generated by \( L \). A \( d \)-dimensional morphism \( \omega : L \rightarrow \mathcal{B}^d \) can be extended to a continuous map \( \omega : \mathcal{X}_L \rightarrow \mathcal{B}^{\mathbb{Z}^d} \) in such a way that the origin of \( \omega(x) \) is at zero position in the word \( \omega(x_0) \) for all \( x \in \mathcal{X}_L \). More precisely, the image under \( \omega \) of the configuration \( x \in \mathcal{X}_L \) is

\[
\omega(x) = \lim_{n \rightarrow \infty} \sigma^n \omega((\sigma^{-n} x)[0,2n_{11}]) \in \mathcal{B}^{\mathbb{Z}^d}
\]

where \( 1 = (1, \ldots, 1) \in \mathbb{Z}^d \), \( f(n) = \text{shape} \left( \omega \left( (\sigma^{-n} x)[0,n_{11}] \right) \right) \) for all \( n \in \mathbb{N} \) and \([0,n_{11}] = [0,n_1-1] \times \cdots \times [0,n_d-1] \).

In general, the closure under the shift of the image of a subshift \( X \subseteq \mathcal{A}^{\mathbb{Z}^d} \) under \( \omega \) is the subshift

\[
\overline{\omega(X)}^d = \{ \sigma^k \omega(x) \in \mathcal{B}^{\mathbb{Z}^d} \mid k \in \mathbb{Z}^d, x \in X \} \subseteq \mathcal{B}^{\mathbb{Z}^d}.
\]

Now we show that \( d \)-dimensional morphisms preserve minimality of subshifts.
Lemma 2.2. Let \( \omega : X \to \mathcal{B}^{zd} \) be a d-dimensional morphism for some \( X \subseteq \mathcal{A}^{zd} \). If \( X \) is a minimal subshift, then \( \omega(X)^\omega \) is a minimal subshift.

Proof. Let \( \varnothing \neq Z \subseteq \omega(X)^\omega \) be a closed shift-invariant subset. We want to show that \( \omega(X)^\omega \subset Z \). Let \( u \in \omega(X)^\omega \). Thus \( u = \sigma^k \omega(x) \) for some \( k \in \mathbb{Z}^d \) and \( x \in X \). Since \( Z \neq \varnothing \), there exists \( z \in Z \). Thus \( z = \sigma^k \omega(x') \) for some \( k' \in \mathbb{Z}^d \) and \( x' \in X \). Since \( X \) is minimal, there exists a sequence \( (k_n)_{n \in \mathbb{N}} \), \( k_n \in \mathbb{Z}^d \), such that \( x = \lim_{n \to \infty} \sigma^{k_n} x' \). For some other sequence \( (h_n)_{n \in \mathbb{N}} \), \( h_n \in \mathbb{Z}^d \), we have

\[
\begin{align*}
u &= \sigma^k \omega(x) = \sigma^k \omega \left( \lim_{n \to \infty} \sigma^{k_n} x' \right) = \sigma^k \lim_{n \to \infty} \sigma^{h_n} \omega (x') = \lim_{n \to \infty} \sigma^{k+h_n-k'} z. 
\end{align*}
\]

Since \( Z \) is closed and shift-invariant, \( u \in Z \). \( \square \)

The 2-dimensional morphism \( \phi \) defined in the next exercise plays a prominent role in this chapter.

Exercise 2.2

We consider the 2-dimensional morphism \( \phi \) defined in Equation (1) on the alphabet \( \mathcal{A} = [0, 18] \). Compute \( \phi(u) \) for each 2-dimensional word \( u \) below

\[
\begin{pmatrix} 13 \end{pmatrix}, \quad \begin{pmatrix} 16, 8 \end{pmatrix}, \quad \begin{pmatrix} 6 \\ 11 \end{pmatrix}, \quad \begin{pmatrix} 7 \\ 13 \end{pmatrix}
\]

except one for which the image \( \phi(u) \) is not well-defined.

3. Aperiodic self-similar subshifts

3.1. Self-similar subshifts. In this section, we consider languages and subshifts defined from morphisms leading to self-similar structures. In this situation, the domain and codomain of morphisms are defined over the same alphabet. Formally, we consider the case of \( d \)-dimensional morphisms \( \mathcal{A} \to \mathcal{B}^{zd} \) where \( \mathcal{A} = \mathcal{B} \).

The definition of self-similarity depends on the notion of expansiveness. It avoids the presence of lower-dimensional self-similar structure by having expansion in all directions.

Definition 3.1. We say that a \( d \)-dimensional morphism \( \omega : \mathcal{A} \to \mathcal{A}^{sd} \) is expansive if for every \( a \in \mathcal{A} \) and \( K \in \mathbb{N} \), there exists \( m \in \mathbb{N} \) such that

\[
\min(\text{SHAPE}(\omega^m(a))) > K.
\]

Definition 3.2. A subshift \( X \subseteq \mathcal{A}^{zd} \) is self-similar if there exists an expansive \( d \)-dimensional morphism \( \omega : \mathcal{A} \to \mathcal{A}^{sd} \) such that \( X = \overline{\omega(X)^\omega} \).

Respectively, a language \( L \subseteq \mathcal{A}^d \) is self-similar if there exists an expansive \( d \)-dimensional morphism \( \omega : \mathcal{A} \to \mathcal{A}^d \) such that \( L = \overline{\omega(L)^\text{Fact}} \).

Self-similar languages and subshifts can be constructed by iterative application of a morphism \( \omega \) starting with the letters. The language \( \mathcal{L}_\omega \) defined by an expansive \( d \)-dimensional morphism \( \omega : \mathcal{A} \to \mathcal{A}^{sd} \) is

\[
\mathcal{L}_\omega = \{ u \in \mathcal{A}^{sd} \mid u \text{ is a } d\text{-dimensional subword of } \omega^n(a) \text{ for some } a \in \mathcal{A} \text{ and } n \in \mathbb{N} \}.
\]

It satisfies \( \mathcal{L}_\omega = \overline{\omega(\mathcal{L}_\omega)^\text{Fact}} \) and thus is self-similar. The substitutive shift \( \mathcal{X}_\omega = \mathcal{X}_{\mathcal{L}_\omega} \) defined from the language of \( \omega \) is a self-similar subshift since \( \mathcal{X}_\omega = \overline{\omega(\mathcal{X}_\omega)^\omega} \) holds.
Let $\phi : [0,18] \to [0,18]^2$ be the 2-dimensional morphism defined in Exercise 2.2. At Figure 1, we compute the sequence of 2-dimensional words $(\phi^n(14))_{n\in\mathbb{N}}$ for the first values of $n \in \{0,1,2,3,4,5,6\}$. The gray rectangles surround patterns seen two steps before in the application of $\phi$. The limit $\lim_{n \to \infty} \phi^{2n}(14)$ defines a configuration of the positive quadrant $\mathbb{N}^2$ and similarly for the limit $\lim_{n \to \infty} \phi^{2n+1}(14)$.

**Figure 1.** Building a configuration of the positive quadrant with $\phi$. We compute $(\phi^n(14))_{n\in\mathbb{N}}$ for the first values of $n \in \{0,1,2,3,4,5,6\}$. The gray rectangles surround patterns seen two steps before in the application of $\phi$. The limit $\lim_{n \to \infty} \phi^{2n}(14)$ defines a configuration of the positive quadrant $\mathbb{N}^2$ and similarly for the limit $\lim_{n \to \infty} \phi^{2n+1}(14)$. 

Let $\phi : [0,18] \to [0,18]^2$ be the 2-dimensional morphism defined in Exercise 2.2. At Figure 1, we compute the sequence of 2-dimensional words $(\phi^n(14))_{n\in\mathbb{N}}$ for the first values of $n$. Since 14 appears in the lower left corner of $\phi^2(14)$, the limit $\lim_{n \to \infty} \phi^{2n}(14)$ is well-defined and it defines a configuration of the positive quadrant $\mathbb{N}^2$. 
This procedure can be done in each of the four quadrants. At Figure 2, we compute the sequence of 2-dimensional words \( (\phi^n(\frac{9}{1}, \frac{14}{6}))_{n \in \mathbb{N}} \) for the first values of \( n \in \{0, 1, 2, 3, 4, 5\} \). The gray rectangles surround patterns seen two steps before in the application of \( \phi \). The limit \( \lim_{n \to \infty} \phi^{2n}(\frac{9}{1}, \frac{14}{6}) \) defines a configuration of \( \mathbb{Z}^2 \) and similarly for the limit \( \lim_{n \to \infty} \phi^{2n+1}(\frac{9}{1}, \frac{14}{6}) \).

**Figure 2.** Building a configuration of \( \mathbb{Z}^2 \) with \( \phi \). We compute \((\phi^n(\frac{9}{1}, \frac{14}{6}))_{n \in \mathbb{N}}\) for the first values of \( n \in \{0, 1, 2, 3, 4, 5\} \). The gray rectangles surround patterns seen two step before in the application of \( \phi \). The limit \( \lim_{n \to \infty} \phi^{2n}(\frac{9}{1}, \frac{14}{6}) \) defines a configuration of \( \mathbb{Z}^2 \) and similarly for the limit \( \lim_{n \to \infty} \phi^{2n+1}(\frac{9}{1}, \frac{14}{6}) \).
are well-defined and define two configurations of $\mathbb{Z}^2$. They satisfy $\phi(x) = y$ and $\phi(y) = x$. This implies that the configurations $x$ and $y$ are fixed points of $\phi^2$ since $\phi^2(x) = x$ and $\phi^2(y) = y$.

**Exercise 3.1**
Prove that $\phi$ defined in Exercise 2.2 is expansive.

**Exercise 3.2**
Prove that $x_\phi \neq \emptyset$.

**Exercise 3.3**
The language of horizontal and vertical dominoes that we see in Figure 1 and Figure 2 obtained from the morphism $\phi$ are

$$H = \left\{ \left( \begin{array}{c} 0 \ 3 \\ 1 \ 2 \\ 1 \ 3 \\ 1 \ 6 \\ 2 \ 0 \\ 2 \ 4 \\ 3 \ 7 \\ 4 \ 1 \\ 5 \ 1 \end{array} \right), \left( \begin{array}{c} 6 \ 1 \\ 6 \ 5 \\ 7 \ 1 \\ 8 \ 16 \\ 9 \ 14 \\ 10 \ 12 \\ 10 \ 14 \\ 11 \ 17 \end{array} \right), \left( \begin{array}{c} 12 \ 9 \\ 13 \ 9 \\ 14 \ 8 \\ 14 \ 11 \\ 14 \ 13 \\ 14 \ 18 \\ 15 \ 8 \end{array} \right), \left( \begin{array}{c} 15 \ 11 \\ 16 \ 8 \\ 16 \ 13 \\ 17 \ 8 \\ 17 \ 13 \end{array} \right) \right\}$$

and

$$V = \left\{ \left( \begin{array}{c} 8 \ 0 \\ 1 \ 1 \\ 1 \ 6 \\ 3 \ 4 \\ 5 \ 6 \\ 10 \ 11 \\ 16 \ 17 \end{array} \right), \left( \begin{array}{c} 15 \ 8 \\ 9 \ 9 \\ 11 \ 11 \\ 10 \ 10 \\ 9 \ 11 \\ 11 \ 12 \\ 11 \ 13 \end{array} \right), \left( \begin{array}{c} 4 \ 13 \\ 7 \ 13 \\ 18 \ 13 \\ 14 \ 14 \\ 12 \ 14 \\ 7 \ 15 \\ 13 \ 15 \end{array} \right), \left( \begin{array}{c} 3 \ 16 \\ 14 \ 16 \\ 17 \ 17 \\ 14 \ 17 \\ 14 \ 18 \end{array} \right) \right\}$$

Prove that $A \circ^1 A \cap L_\phi = H$ and $A \circ^2 A \cap L_\phi = V$ where $A = [0, 18]$ is the alphabet on which the morphism $\phi$ is defined.

**Exercise 3.4**
List the 50 elements of the set $A^{(2, 2)} \cap L_\phi$.

**Exercise 3.5**
Describe the 8 periodic points of $\phi$, i.e. the configurations $x \in A^{\mathbb{Z}^2}$ such that $\phi^k(x) = x$ for some $k \geq 1$.

### 3.2. $d$-dimensional recognizability

The definition of recognizability dates back to the work of Host, Queffelec and Mossé [Mos92]. The definition introduced below is based on work of Berthé et al. [BSTY18] on the recognizability in the case of $S$-adic systems where more than one substitution is involved. The results proved here can be found in [Lab19a] and [Lab19b].

**Definition 3.3** (recognizable). Let $X \subseteq A^{\mathbb{Z}^d}$ and $\omega : X \rightarrow B^{\mathbb{Z}^d}$ be a $d$-dimensional morphism. If $y \in \omega(X)^\omega$, i.e., $y = \sigma^k \omega(x)$ for some $x \in X$ and $k \in \mathbb{Z}^d$, where $\sigma$ is the $d$-dimensional shift map, we say that $(k, x)$ is a $\omega$-representation of $y$. We say that it is centered if $y_0$ lies inside of the image of $x_0$, i.e., if $0 \leq k < \text{SHAPE}(\omega(x_0))$ coordinate-wise. We say that $\omega$ is recognizable in $X \subseteq A^{\mathbb{Z}^d}$ if each $y \in B^{\mathbb{Z}^d}$ has at most one centered $\omega$-representation $(k, x)$ with $x \in X$. 

Lemma 3.4. Let $\omega : X \rightarrow Y$ be some $d$-dimensional morphism between two subshifts $X$ and $Y$.

(1) If $Y$ is aperiodic, then $X$ is aperiodic.

(2) If $X$ is aperiodic and $\omega$ is recognizable in $X$, then $\overline{\omega(X)}^\sigma$ is aperiodic.

Proof. If $X$ contains a periodic configuration $x$, then $\omega(x) \in Y$ is periodic.

(ii) Let $y \in \overline{\omega(X)}^\sigma$. Then, there exist $k \in \mathbb{Z}^d$ and $x \in X$ such that $(k, x)$ is a centered $\omega$-representation of $y$, i.e., $y = \sigma^k \omega(x)$. Suppose by contradiction that $y$ has a nontrivial period $p \in \mathbb{Z}^d \setminus \{0\}$. Since $y = \sigma^p y = \sigma^{p+k} \omega(x)$, we have that $(p + k, x)$ is a $\omega$-representation of $y$. Since $\omega$ is recognizable, this representation is not centered. Therefore there exists $q \in \mathbb{Z}^d \setminus \{0\}$ such that $y_0$ lies in the image of $x_q = (\sigma^q x)_0$. Therefore there exists $k' \in \mathbb{Z}^d$ such that $(k', \sigma^q x)$ is a centered $\omega$-representation of $y$. Since $\omega$ is recognizable, we conclude that $k = k'$ and $x = \sigma^q x$. Then $x \in X$ is periodic which is a contradiction. \qed

In general, $\omega(X)$ is not closed under the shift which implies that $\omega$ is not onto. This motivates the following definition.

Definition 3.5. Let $X, Y$ be two subshifts and $\omega : X \rightarrow Y$ be a $d$-dimensional morphism. If $Y = \overline{\omega(X)}^\sigma$, then we say that $\omega$ is onto up to a shift.

Exercise 3.6

Let $\phi : [0, 18] \rightarrow [0, 18]^2$ be the morphism defined in Exercise 2.2. Find periodic configurations $x, y \in [0, 18]^{\mathbb{Z}^2}$ and $k \in \mathbb{Z}^2$ such that

- $(k, x)$ is a $\phi$-representation of $y$,
- $(k, x)$ is a centered $\phi$-representation of $y$,
- $(k, x)$ is a $\phi$-representation of $y$ which is not centered.

Exercise 3.7

Does there exist a configuration $y \in [0, 18]^{\mathbb{Z}^2}$ that has more than one centered $\phi$-representation $(k, x)$ with $x \in [0, 18]^{\mathbb{Z}^2}$?

3.3. Aperiodicity, primitivity and minimality of self-similar subshifts. The next proposition is well-known, see [Sol98, Mos92], who showed that recognizability and aperiodicity are equivalent for primitive substitutive sequences. We state and prove only one direction (the easy one) of the equivalence which does not need the notion of primitivity.

Proposition 3.6. Let $\omega : A \rightarrow A^{\mathbb{Z}^d}$ be an expansive $d$-dimensional morphism. If $X \subseteq A^{\mathbb{Z}^d}$ is a self-similar subshift such that $\overline{\omega(X)}^\sigma = X$ and $\omega$ is recognizable in $X$, then $X$ is aperiodic.

Proof. Suppose that there exists a periodic configuration $y \in X$ with period $p \in \mathbb{Z}^d \setminus \{(0, 0)\}$ satisfying $\sigma^p y = y$. Since $\omega$ is expansive, let $m \in \mathbb{N}$ such that the shape of the image of every letter $a \in A$ by $\omega^m$ is large enough, that is, $\text{shape}(\omega^m(a)) \geq p$ for every letter $a \in A$. By hypothesis, every $y \in X$ has a $\omega$-representation. Recursively, there exists a $\omega^m$-representation $(k, x)$ of $y$ satisfying $y = \sigma^k \omega^m(x)$. We may assume that it is centered since $X$ is shift-invariant. By definition of centered representation, for every $u \in \mathbb{Z}^d$ such that $0 \leq u < \text{shape}(\omega^m(x_0))$, $(u, x)$ is a centered $\omega^m$-representation of $\sigma^u \omega^m(x) = \sigma^{u-k} y$. By the choice of $m$, there exists $u \in \mathbb{Z}^d$ such that $0 \leq u < \text{shape}(\omega^m(x_0))$ and $0 \leq u + p < \text{shape}(\omega^m(x_0))$. 

Therefore \((u, x)\) is a centered \(\omega^m\)-representation of \(\sigma^u \omega^m(x) = \sigma^{u-k}y\) and \((u + p, x)\) is a centered \(\omega^m\)-representation of \(\sigma^{u+p} \omega^m(x) = \sigma^{u+p-k}y = \sigma^{u-k} \sigma^p y = \sigma^{u-k}y\). Therefore, \(\omega^m\) is not recognizable which implies that \(\omega\) is not recognizable which is a contradiction. We conclude that there is no periodic configuration \(y \in X\).

Substitutive shift obtained from expansive and primitive morphisms are interesting for their properties. As in the one-dimensional case, we say that \(\omega\) is primitive if there exists \(m \in \mathbb{N}\) such that for every \(a, b \in \mathcal{A}\) the letter \(b\) occurs in \(\omega^m(a)\).

**Lemma 3.7.** Let \(\omega : \mathcal{A} \to \mathcal{A}^d\) be an expansive and primitive \(d\)-dimensional morphism. Then \(\mathcal{X}_\omega\) is minimal.

**Proof.** The substitutive shift of \(\omega\) is well-defined since \(\omega\) is expansive and it is minimal since \(\omega\) is primitive using standard arguments \cite[§5.2]{Que10}.

**Lemma 3.8.** Let \(\omega : \mathcal{A} \to \mathcal{A}^d\) be an expansive and primitive \(d\)-dimensional morphism. Let \(X \subseteq \mathcal{A}^\mathbb{Z}^d\) be a nonempty subshift such that \(X = \omega(X)^\sigma\). Then \(\mathcal{X}_\omega \subseteq X\).

**Proof.** The language of \(X\) is also self-similar satisfying \(\mathcal{L}(X) = \overline{\omega(\mathcal{L}(X))}^{\text{Fact}}\). Recursively, \(\mathcal{L}(X) = \overline{\omega^m(\mathcal{L}(X))}^{\text{Fact}}\) for every \(m \geq 1\). Thus there exists a letter \(a \in \mathcal{A}\) such that for all \(m \geq 1\), the \(d\)-dimensional word \(\omega^m(a)\) is in the language \(\mathcal{L}(X)\). From the primitivity of \(\omega\), there exists \(m \geq 1\) such that \(\omega^m(a)\) contains an occurrence of every letter of the alphabet \(\mathcal{A}\). Therefore every letter is in \(\mathcal{L}(X)\) and the \(d\)-dimensional word \(\omega^m(a)\) is in the language \(\mathcal{L}(X)\) for all letter \(a \in \mathcal{A}\) and all \(m \geq 1\). So we conclude that \(\mathcal{L}_\omega \subseteq \mathcal{L}(X)\) and \(\mathcal{X}_\omega \subseteq X\).

From Lemma 3.7 and Lemma 3.8, when \(\omega\) is expansive and primitive, then \(\mathcal{X}_\omega\) is the smallest nonempty subshift \(X \subseteq \mathcal{A}^\mathbb{Z}^d\) satisfying \(X = \omega(X)^\sigma\). The next result provides sufficient conditions for the unicity of the subshift \(X\) satisfying \(X = \omega(X)^\sigma\).

Let \(G_\omega = (V, E)\) be the directed graph whose vertices are \(V = \mathcal{A}^{(2,...,2)}\) and whose edges are \(\{(u, v) \in V \times V \mid v\) is a factor of \(\omega(u)\}\). Let \(\text{SEEDS}(\omega) \subset V\) be the set of vertices that belong to a cycle of \(G_\omega\).

**Lemma 3.9.** Let \(\omega : \mathcal{A} \to \mathcal{A}^d\) be an expansive and primitive \(d\)-dimensional morphism. If \(\text{SEEDS}(\omega) \subset \mathcal{L}_\omega\), then \(\mathcal{X}_\omega\) is the unique nonempty subshift such that \(X = \omega(X)^\sigma\).

**Proof.** Assume that \(X = \overline{\omega(X)}^\sigma\) for some \(\emptyset \neq X \subseteq \mathcal{A}^\mathbb{Z}^d\). From Lemma 3.8, we have \(\mathcal{X}_\omega \subseteq X\).

Let \(z \in \mathcal{L}(X)\). Since \(\omega\) is expansive, let \(m \in \mathbb{N}\) such that the image of every letter \(a \in \mathcal{A}\) by \(\omega^m\) is larger than \(z\), that is, \(\text{SHAPE}(\omega^m(a)) \geq \text{SHAPE}(z)\) for all \(a \in \mathcal{A}\). We have \(z \in \mathcal{L}(\omega^m(\mathcal{L}(X)))\). By the choice of \(m\), \(z\) can not overlap more than two blocks \(\omega^m(a)\) in the same direction. Then, there exists a word \(u \in \mathcal{L}(X)\) of shape \((2, \ldots, 2)\) such that \(z\) is a subword of \(\omega^m(u)\). We have \(u \in V\). Since \(u \in \mathcal{L}(X)\) and \(X\) is self-similar, there exists an sequence \((u_k)_{k \in \mathbb{N}}\) with \(u_k \in V\) for all \(k \in \mathbb{N}\) such that

\[
\cdots \to u_{k+1} \to u_k \to \cdots \to u_1 \to u_0 = u
\]

is a left-infinite path in the graph \(G_\omega\). Since \(V\) is finite, there exists some \(k, k' \in \mathbb{N}\) with \(k < k'\) such that \(u_k = u_{k'}\). Thus \(u_k \in \text{SEEDS}(\omega)\) and \(u\) is a subword of \(\omega^k(u_k)\). From the hypothesis, we have \(\text{SEEDS}(\omega) \subset \mathcal{L}_\omega\) so that \(u_k \in \mathcal{L}_\omega\). Since \(\omega\) is primitive, there exists \(\ell\) such that \(u_k\) is a subword of \(\omega^\ell(a)\) for every \(a \in \mathcal{A}\). Therefore, \(z\) is a subword of \(\omega^{m+k+\ell}(a)\) for every \(a \in \mathcal{A}\). Then \(z \in \mathcal{L}_\omega\) and \(\mathcal{L}(X) \subseteq \mathcal{L}_\omega\). Thus \(X \subseteq \mathcal{X}_\omega\). \(\Box\)
Exercise 3.8

Let $\phi : [0, 18] \to [0, 18]^{\ast^2}$ be the morphism defined in Exercise 2.2. Prove that $\phi$ is recognizable in $X_{\phi}$.

Exercise 3.9

Prove that $X_{\phi}$ is aperiodic.

Exercise 3.10

Prove that $\phi$ is primitive.

Exercise 3.11

Prove that $X_{\phi}$ is minimal.

Exercise 3.12

Prove that there exists a unique nonempty subshift $X \subset [0, 18]^{\ast^2}$ such that $X = \phi(X)^\sigma$. Conclude that $X = X_{\phi}$.

3.4. Markers. The next lemma provides a sufficient condition for recognizability of $d$-dimensional morphism. It is weak in the sense that it applies to morphisms whose images of letters are letters or a domino in a given direction, but it has shown to be very useful in the study of the Jeandel-Rao Wang shift [Lab19b].

Lemma 3.10. Let $d \geq 1$ and $i$ such that $1 \leq i \leq d$. $\omega : B \to A^{Z^d}$ be a $d$-dimensional morphism such that the image of letters are letters or dominoes in the direction $e_i$. If $\omega|_B$ is injective and there exists a subset $M \subset A$ such that

\begin{align}
\omega(B) &\subseteq (A \setminus M) \cup \left((A \setminus M) \odot^i M\right), \\
\text{or} \quad \omega(B) &\subseteq (A \setminus M) \cup \left(M \odot^i (A \setminus M)\right),
\end{align}

then $\omega$ is recognizable in $B^{Z^d}$.

Informally, the letters $m \in M$ have the role of markers: when we see them in the image under $\omega$, we know that they must appear as the left (resp. right) part of a domino. A formal definition of markers is given below (Definition 3.11).

Proof. Let $(k, x)$ and $(k', x')$ be two centered $\omega$-representations of $y \in A^{Z^d}$ with $k, k' \in Z^d$ and $x, x' \in B^{Z^d}$. We want to show that they are equal.

Since the image of a letter under $\omega$ is a letter or a domino in the direction $e_i$, then $k, k' \in \{0, e_i\}$. If $y_0 \in M$, then $y_0$ appears as the left or right part of a domino and thus $k = k' = e_i$ if Equation (4) holds or $k = k' = 0$ if Equation (5) holds.

Suppose now that $y_0 \in A \setminus M$. If Equation (4) holds, then $k = k' = 0$. Suppose that Equation (5) holds. By contradiction, suppose that $k \neq k'$ and assume without lost of generality that $k = 0$ and $k' = e_i$. This means that $\omega(x_0') = y_{-e_i} \odot^i y_0$ is a domino in
the direction \( e_i \). Since \( y_0 \in A \setminus M \), we must have that \( y_{-e_i} \in M \) is a left marker. This is impossible as \( \omega(x_{-e_i}) = y_{-e_i} \in A \setminus M \) or \( \omega(x_{-e_i}) = y_{-2e_i} \cap y_{-e_i} \in M \cap^i (A \setminus M) \). Therefore, we must have \( k = k' \) and \( \omega(x) = \omega(x') \).

Suppose by contradiction that \( x \neq x' \). Let \( a = (a_1, \ldots, a_d) \in \mathbb{Z}^d \) be some minimal vector with respect to \( \|a\|_\infty \) such that \( x_a \neq x'_a \). Injectivity of \( \omega \) implies that \( \omega(x_a) \) and \( \omega(x'_a) \) must have different shapes. Suppose without lost of generality that \( \omega(x_a) \in a \) and \( \omega(x'_a) = b \cap^i c \in A \cap^i A \). We need to consider two cases: \( a_i \geq 0 \) and \( a_i < 0 \).

Suppose \( a_i \geq 0 \). We must have that Equation (5) holds. We have \( \omega(x_a) = b \in A \setminus M \) and \( c \in M \). But then \( \omega(x_{a+e_i}) = c \) or \( \omega(x_{a+e_i}) = c \cap^i d \) for some \( c \in A \setminus M \) and \( d \in A \) which is a contradiction.

Suppose \( a_i < 0 \). We must have that Equation (5) holds. We have \( \omega(x_a) = c \in A \setminus M \) and \( b \in M \). But then \( \omega(x_{a-e_i}) = b \) or \( \omega(x_{a-e_i}) = d \cap^i b \) for some \( b \in A \setminus M \) and \( d \in A \) which is a contradiction. We conclude that \( x = x' \). \( \square \)

We now define the notion of markers for subshifts \( X \subset A^{\mathbb{Z}^d} \) and prove that its presence allows to desubstitute uniquely the configurations in \( X \) using a \( d \)-dimensional morphism. Originally, those results were proved for \( d = 2 \) in order to desubstitute configurations from Wang shifts, see [Lab19a] and [Lab19b]. It turns out that the notion of markers is more general and the results holds in general subshifts \( X \subset A^{\mathbb{Z}^d} \).

Recall that if \( w : \mathbb{Z}^d \to A \) is a configuration and \( a \in A \) is a letter, then \( w^{-1}(a) \subset \mathbb{Z}^d \) is the set of positions where the letter \( a \) appears in \( w \).

**Definition 3.11.** Let \( A \) be an alphabet and \( X \subset A^{\mathbb{Z}^d} \) be a subshift. A nonempty subset \( M \subset A \) is called **markers in the direction** \( e_k \), with \( k \in \{1, \ldots, d\} \), if positions of the letters of \( M \) in any configuration are nonadjacent \( (d - 1) \)-dimensional layers orthogonal to \( e_i \), that is, for all configuration \( w \in X \) there exists \( P \subset \mathbb{Z} \) such that the positions of the markers satisfy
\[
w^{-1}(M) = Pe_k + \sum_{i \neq k} Ze_i \quad \text{with} \quad 1 \notin P - P.
\]

Note that it follows from the definition that a subset of markers is a proper subset of \( A \) as the case \( M = A \) is impossible.

Proving that a subset \( M \subset A \) is a subset of markers uses very local observations, namely the set of dominoes in the language of the subshift. It leads to the following criteria.

**Lemma 3.12.** Let \( A \) be an alphabet and \( X \subset A^{\mathbb{Z}^d} \) be a subshift. A nonempty subset \( M \subset A \) is a subset of markers in the direction \( e_k \) if and only if
\[
M \cap^k M, \quad M \cap^i (A \setminus M), \quad (A \setminus M) \cap^i M
\]
are forbidden in \( X \) for every \( i \in \{1, \ldots, d\} \setminus \{k\} \).

**Proof.** Suppose that \( M \subset A \) is a subset of markers in the direction \( e_k \). For any configuration \( w \in X \), there exists \( P \subset \mathbb{Z} \) such that \( w^{-1}(M) = Pe_k + \sum_{i \neq k} Z e_i \) with \( 1 \notin P - P \). In any configuration \( w \in X \) such that \( w(p) \in M \), then \( w(p \pm e_i) \in M \) also belongs to \( M \) for every \( i \neq k \). Therefore, \( M \cap^i (A \setminus M) \) and \( (A \setminus M) \cap^i M \) are forbidden in \( X \) for every \( i \neq k \). Moreover, the fact that \( 1 \notin P - P \) implies that \( M \cap^k M \) is forbidden in \( X \).

Conversely, suppose that \( M \cap^k M, \ M \cap^i (A \setminus M) \) and \( (A \setminus M) \cap^i M \) are forbidden in \( X \) for every \( i \neq k \). The last two conditions implies that in any configuration \( w \in X \) such that \( w(p) \in M \), then \( w(p \pm e_i) \in M \) also belongs to \( M \) for every \( i \neq k \). Therefore letters in \( M \) appears as complete layers in \( w \), that is, \( w^{-1}(M) = Pe_k + \sum_{i \neq k} Z e_i \) for some \( P \subset \mathbb{Z} \).
Since \( M \odot^k M \) is forbidden in \( X \), it means that the layers are nonadjacent, or equivalently, \( 1 \notin P - P \). We conclude that \( M \) is a subset of markers in the direction \( e_k \). □

The presence of markers allows to desubstitute uniquely the configurations of a subshift. There is even a choice to be made in the construction of the substitution. We may construct the substitution in such a way that the markers are on the left or on the right in the image of letters that are dominoes in the direction \( e_k \). We make this distinction in the statement of the following result which was stated in the context of Wang shifts in [Lab19a] and [Lab19b].

**Theorem 3.13.** Let \( A \) be an alphabet and \( X \subset A^{\mathbb{Z}^d} \) be a subshift. If there exists a subset \( M \subset A \) of markers in the direction \( e_i \in \{e_1, \ldots, e_d\} \), then

(i) (markers on the right) there exists an alphabet \( B_R \), a subshift \( Y \subset B_R^{\mathbb{Z}^d} \) and a 2-dimensional morphism \( \omega_R : Y \to X \) such that

\[
\omega_R(B_R) \subseteq (A \setminus M) \cup ((A \setminus M) \odot^i M)
\]

which is recognizable and onto up to a shift and

(ii) (markers on the left) there exists an alphabet \( B_L \), a subshift \( Y \subset B_L^{\mathbb{Z}^d} \) and a 2-dimensional morphism \( \omega_L : Y \to X \) such that

\[
\omega_L(B_L) \subseteq (A \setminus M) \cup (M \odot^i (A \setminus M))
\]

which is recognizable and onto up to a shift.

**Proof.** We do only the proof of (i) when the markers are on the right, since one case can be deduced from the other using symmetry.

Since \( X \) is a subshift, there exists a language \( F \subset A^* \) such that \( X \) is the set of configurations of \( A^{\mathbb{Z}^d} \) without any occurrence of pattern from \( F \). Notice that since \( M \) is a set of markers in the direction \( e_i \), we may assume \( M \odot^i M \subset F \). Let \( P \subset A \odot^i A \) and \( Q \subset A \) be the following sets:

\[
P = ((A \setminus M) \odot^i M) \setminus F,
\]

\[
Q = \{ u \in A \setminus M \mid \text{there exists } v \in A \setminus M \text{ such that } u \odot^i v \notin F \}.
\]

We choose some ordering of their elements with indices starting from zero:

\[
P = \{ p_0, \ldots, p_{|P|-1} \},
\]

\[
Q = \{ q_0, \ldots, q_{|Q|-1} \}.
\]

We construct the alphabet \( B = \{0, 1, \ldots, |Q| + |P| - 1\} \) and define the rule \( \omega \) by

\[
\omega : B \to A^{\mathbb{Z}^d}
\]

\[
j \mapsto \begin{cases} q_j & \text{if } 0 \leq j < |Q|, \\ p_{j-|Q|} & \text{if } |Q| \leq j < |Q| + |P|. \end{cases}
\]

We want to show that \( \omega \) extends to a map from a set of configurations to \( X \) which is onto up to a shift. Let \( x \in X \) be a configuration which can be seen as a function \( x : \mathbb{Z}^d \to A \). Consider the set \( x^{-1}(M) \subset \mathbb{Z}^d \) of positions of markers in \( x \). From the definition of markers in the direction \( e_i \), markers appear in nonadjacent hyperplane orthogonal to \( e_i \) in the configuration \( x \). Formally, there exists a set \( H \subset \mathbb{Z} \) such that \( x^{-1}(M) = \mathbb{Z}^{i-1} \times H \times \mathbb{Z}^{d-i} \) and \( 1 \notin H - H \). Since \( 1 \notin H - H \), there exists a strictly increasing sequence \( (a_k)_{k \in \mathbb{Z}} \) such
that $\mathbb{Z} \setminus H = \{a_k \mid k \in \mathbb{Z}\}$. We assume that $a_0 = 0$ if $0 \in \mathbb{Z} \setminus H$ and $a_0 = -1$ if $0 \in H$ which makes the sequence $(a_k)_{k \in \mathbb{Z}}$ uniquely defined.

In order to define the preimage of $x$ under $\omega$, it is convenient to represent $x$ fiber by fiber. For every $m \in \mathbb{Z}^{d-1}$, let $x_m : \mathbb{Z} \to A$, be the sequence such that

$$x(n_1, \ldots, n_{i-1}, n_i, n_{i+1}, \ldots, n_d) = x(n_{i-1}, n_{i+1}, \ldots, n_d)(n_i)$$

for every $n = (n_1, \ldots, n_{i-1}, n_i, n_{i+1}, \ldots, n_d) \in \mathbb{Z}^d$. For every $m \in \mathbb{Z}^{d-1}$, let $y_m : \mathbb{Z} \to B$ be defined as

$$y_m : \mathbb{Z} \to B$$

$$k \mapsto \begin{cases} j & \text{if } a_k + 1 \in H \text{ and } x_m(a_k) = q_i, \\ j & \text{if } a_k + 1 \notin H \text{ and } x_m(a_k) \circ^i x_m(a_k + 1) = p_{j-\lvert Q \rvert}, \end{cases}$$

The function $y_m$ is well-defined. Indeed, let $k \in A$ and $m \in \mathbb{Z}^{d-1}$. If $a_k + 1 \in H$, then $x_m(a_k + 1) \in A \setminus M$ and $y_m(k) = x_m(a_k) \in Q$. Also if $a_k + 1 \notin H$, then $x_m(a_k) \in A \setminus M$ and $x_m(a_k + 1) \in M$. Since $x \in X$, then $x_m(a_k) \circ^i x_m(a_k + 1) \notin F$ and therefore $x_m(a_k) \circ^i x_m(a_k + 1) \in P$. We define the configuration $y : \mathbb{Z}^d \to B$ by its fibers constructed above

$$y(n_1, \ldots, n_{i-1}, n_i, n_{i+1}, \ldots, n_d) = y(n_{i-1}, n_{i+1}, \ldots, n_d)(n_i)$$

for every $n = (n_1, \ldots, n_{i-1}, n_i, n_{i+1}, \ldots, n_d) \in \mathbb{Z}^d$.

We may now finish the proof of surjectivity. If $0 \in H$, then the configuration $x$ is exactly the image under $\omega$ of the configuration $y \in B^{\mathbb{Z}^d}$ that we constructed: $x = \omega(y)$. If $0 \notin H$, then the configuration $x$ is a shift of the image under $\omega$ of the configuration $y \in B^{\mathbb{Z}^d}$: $x = \sigma^e \omega(y)$. Let

$$Y = \omega^{-1}(X) = \{y \in B^{\mathbb{Z}^d} \mid \omega(y) \in X\}$$

making $\omega : Y \to X$ a continuous map which is onto up to a shift. Notice that the subset $Y$ is closed since $\omega$ is continuous and $X$ is closed. Moreover, $Y$ is shift-invariant since for all $y \in Y$ and $n \in \mathbb{Z}^d$ there exists $k \in \mathbb{Z}^d$ such that $\omega(\sigma^n(y)) = \sigma^k(\omega(y))$ meaning that $\sigma^n(y) \in Y$. Thus $Y$ is a subshift.

The function $\omega$ is of the form

$$\omega(B) \subseteq (A \setminus M) \cup ((A \setminus M) \circ^i M)$$

and its restriction on $B$ is injective by construction. Therefore, we conclude from Lemma 3.10 that $\omega$ is recognizable in $B^{\mathbb{Z}^d}$. \qed

Remark that if $X$ is an effective subshift one may also show that $Y$ is an effective subshift. Moreover if $X$ is a Wang shift, then $Y$ is a Wang shift and the Wang tiles defining $Y$ can be obtained from the Wang tiles defining $X$ together with some fusion operation. This is what was done in [Lab19a] and [Lab19b].

**Exercise 3.13**

Using the value of $H$ and $V$ from Exercise 3.3 prove that $\{0, 1, 2, 3, 4, 5, 6, 7\}$ is a subset of markers for the direction $e_2$ in the subshift $X_0$. 

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THREE CHARACTERIZATIONS OF A SELF-SIMILAR APERIODIC 2-DIMENSIONAL SUBSHIFT 17
Exercise 3.14

Using the value of $H$ and $V$ from Exercise 3.3, prove that $\{0, 1, 8, 9, 10, 11\}$, $\{2, 3, 6, 12, 14, 16, 17\}$, $\{4, 5, 7, 13, 15, 18\}$ are subsets of markers for the direction $e_1$ in the subshift $\mathcal{X}_\phi$.

Exercise 3.15

Using $M = \{0, 1, 2, 3, 4, 5, 6, 7\}$ as subset of markers for the direction $e_2$, find an alphabet $B$, a subshift $Y \subset B^{\mathbb{Z}^2}$ and a 2-dimensional morphism $\alpha : B \to [0, 18]^{*^2}$ such that

$$\alpha(B) \subseteq ([0, 18] \setminus M) \cup \left(([0, 18] \setminus M) \circ 2 M\right)$$

which extends to a recognizable continuous map $\alpha : Y \to \mathcal{X}_\phi$ which is onto up to a shift.

Exercise 3.16

Use one of the subsets $M$ of markers for the direction $e_1$ found in Exercise 3.14 to find an alphabet $C$, a subshift $Y' \subset C^{\mathbb{Z}^2}$ and a 2-dimensional morphism $\xi : C \to [0, 18]^{*^2}$ such that

$$\xi(C) \subseteq ([0, 18] \setminus M) \cup \left(([0, 18] \setminus M) \circ 1 M\right)$$

which extends to a recognizable continuous map $\xi : Y' \to \mathcal{X}_\phi$ which is onto up to a shift.

4. Wang shifts

A Wang tile is a tuple of four colors $(a, b, c, d) \in I \times J \times I \times J$ where $I$ is a finite set of vertical colors and $J$ is a finite set of horizontal colors, see [Wan61, Rob71]. A Wang tile is represented as a unit square with colored edges:

$$\begin{array}{c}
  a \\
  b \\
  c \\
  d \\
\end{array}$$

For each Wang tile $\tau = (a, b, c, d)$, let $\text{RIGHT}(\tau) = a$, $\text{TOP}(\tau) = b$, $\text{LEFT}(\tau) = c$, $\text{BOTTOM}(\tau) = d$ denote respectively the colors of the right, top, left and bottom edges of $\tau$.

Let $T = \{t_0, \ldots, t_{m-1}\}$ be a set of Wang tiles as the one shown in Figure 3. A configuration $x : \mathbb{Z}^2 \to \{0, \ldots, m - 1\}$ is valid with respect to $T$ if it assigns a tile in $T$ to each position.
of \( \mathbb{Z}^2 \) so that contiguous edges of adjacent tiles have the same color, that is,

\[
\begin{align*}
\text{RIGHT}(t_{x(n)}) & = \text{LEFT}(t_{x(n+e_1)}) \\
\text{TOP}(t_{x(n)}) & = \text{BOTTOM}(t_{x(n+e_2)})
\end{align*}
\]

for every \( n \in \mathbb{Z}^2 \) where \( e_1 = (1, 0) \) and \( e_2 = (0, 1) \). A finite pattern which is valid with respect to \( \mathcal{U} \) is shown in Figure 4.

Let \( \Omega_\mathcal{T} \subset \{0, \ldots, m-1\}^{\mathbb{Z}^2} \) denote the set of all valid configurations with respect to \( \mathcal{T} \), called the Wang shift of \( \mathcal{T} \). Together with the shift action \( \sigma \) of \( \mathbb{Z}^2 \), \( \Omega_\mathcal{T} \) is a SFT of the form \( \sigma(\Omega_\mathcal{T}) \), since there exists a finite set of forbidden patterns made of all horizontal and vertical dominoes of two tiles that do not share an edge of the same color. This definition of Wang shifts allows to use the concepts of languages, 2-dimensional morphisms, recognizability introduced in the previous sections.

A configuration \( x \in \Omega_\mathcal{T} \) is periodic if there exists \( n \in \mathbb{Z}^2 \setminus \{0\} \) such that \( x = \sigma^n(x) \). A set of Wang tiles \( \mathcal{T} \) is periodic if there exists a periodic configuration \( x \in \Omega_\mathcal{T} \). Originally, Wang thought that every set of Wang tiles \( \mathcal{T} \) is periodic as soon as \( \Omega_\mathcal{T} \) is nonempty [Wan61]. This statement is equivalent to the existence of an algorithm solving the domino problem, that is, taking as input a set of Wang tiles and returning yes or no whether there exists a valid configuration with these tiles. Berger, a student of Wang, later proved that the domino problem is undecidable and he also provided a first example of an aperiodic set of Wang tiles [Ber66]. A set of Wang tiles \( \mathcal{T} \) is aperiodic if the Wang shift \( \Omega_\mathcal{T} \) is a nonempty aperiodic subshift. This means that in general one can not decide the emptiness of a Wang shift \( \Omega_\mathcal{T} \). This illustrates that the behavior of \( d \)-dimensional SFTs when \( d \geq 2 \) is much different than the one-dimensional case where emptiness of a SFT is decidable [LM95]. Note that another important difference between \( d = 1 \) and \( d \geq 2 \) is expressed in terms of the possible values of entropy of \( d \)-dimensional SFTs, see [HM10].

The goal of this section is to prove Theorem 1.1 (i), i.e., that the Wang shift \( \Omega_{\mathcal{U}} \subset \{0, \ldots, 18\}^{\mathbb{Z}^2} \) defined by the set of Wang tiles \( \mathcal{U} \) shown in Figure 3 is self-similar where the self-similarity is given by the 2-dimensional morphism \( \phi \) defined in Equation (1).
Exercise 4.1

Find a valid $7 \times 7$ tiling with the set $\mathcal{U}$ of Wang tiles.

4.1. Markers in the context of Wang tilings. A tiling with the tiles from the set $\mathcal{U}$ is shown in Figure 5. It illustrates that there exists a subset $M \subset \mathcal{U}$ of tiles such that each horizontal row of tiles in the tiling is using either tiles from $M$ or from $\mathcal{U} \setminus M$. Moreover, the horizontal lines using tiles from $M$ are nonadjacent. If these conditions are satisfied for all configurations in $\Omega_\mathcal{U}$, then $M$ is a subset of markers in the direction $e_2$.

![Figure 5](image-url)

**Figure 5.** A $5 \times 10$ tiling with tiles from the set $\mathcal{U}$. The tiles labeled from 0 to 7 (shown with yellow background) are marker tiles for the direction $e_2$ in the Wang shift $\Omega_\mathcal{U}$ since they always appear on nonadjacent rows.

In this section we propose an algorithm to find and prove that a subset of tiles is a subset of markers in a Wang shift. We use Lemma 3.12 which provides a way to prove that a subset of Wang tiles is a subset of markers and searching for them. To use it in the context of Wang shifts and more generally in the context of SFTs, we need the following definition.

**Definition 4.1** (surrounding of radius $r$). Let $X = SFT(\mathcal{F}) \subset \mathcal{A}^{\mathbb{Z}^2}$ be a shift of finite type for some finite set $\mathcal{F}$ of forbidden patterns. A 2-dimensional word $u \in \mathcal{A}^{n}$, with $n = (n_1, n_2) \in \mathbb{N}^2$, admits a **surrounding of radius** $r \in \mathbb{N}$ if there exists $w \in \mathcal{A}^{n+2(r,r)}$ such that $u$ occurs in $w$ at position $(r,r)$ and $w$ contains no occurrences of forbidden patterns from $\mathcal{F}$. 
If a word admits a surrounding of radius \( r \in \mathbb{N} \), it does not mean it is in the language of the SFT. But if it admits no surrounding of radius \( r \) for some \( r \in \mathbb{N} \), then for sure it is not in the language of the SFT. We state the following lemma in the context of Wang tiles.

**Lemma 4.2.** Let \( \mathcal{T} \) be a set of Wang tiles and \( u \in \mathcal{T}^n \) be a rectangular pattern seen as a 2-dimensional word with \( n = (n_1, n_2) \in \mathbb{N}^2 \). If \( u \) is allowed in \( \Omega_\mathcal{T} \), then for every \( r \in \mathbb{N} \) the word \( u \) has a \( \mathcal{T} \)-surrounding of radius \( r \).

Equivalently the lemma says that if there exists \( r \in \mathbb{N} \) such that \( u \) has no \( \mathcal{T} \)-surrounding of radius \( r \), then \( u \) is forbidden in \( \Omega_\mathcal{T} \) and this is how we use Lemma 4.2 to find markers. We propose Algorithm 1 to compute markers from a Wang tile set and a chosen surrounding radius to bound the computations. If the algorithm finds nothing, then maybe there is no markers or maybe one should try again after increasing the surrounding radius. We prove in the next lemma that if the output is nonempty, it contains a subset of markers.

**Algorithm 1** Find markers. If no markers are found, one should try increasing the radius \( r \).

**Precondition:** \( \mathcal{T} \) is a set of Wang tiles; \( i \in \{1, 2\} \) is a direction \( \mathbf{e}_i \); \( r \in \mathbb{N} \) is some radius.

1: function \textsc{FindMarkers}(\( \mathcal{T}, \mathbf{e}_i, r \))
2: \( j \leftarrow 3 - i \)
3: \( D_j \leftarrow \left\{ (u, v) \in \mathcal{T}^2 \mid \text{domino } u \odot^j v \text{ admits a } \mathcal{T} \text{-surrounding of radius } r \right\} \)
4: \( U \leftarrow \left\{ \{u\} \mid u \in \mathcal{T} \right\} \quad \triangleright \text{Suggestion: use a union-find data structure} \)
5: for all \( (u, v) \in D_j \) do
6: Merge the sets containing \( u \) and \( v \) in the partition \( U \).
7: \( D_i \leftarrow \left\{ (u, v) \in \mathcal{T}^2 \mid \text{domino } u \odot^i v \text{ admits a } \mathcal{T} \text{-surrounding of radius } r \right\} \)
8: return \{set \( M \) in the partition \( U \mid (M \times M) \cap D_i = \varnothing \)\}

**Postcondition:** The output contains zero, one or more subsets of markers in the direction \( \mathbf{e}_i \).

**Lemma 4.3.** If there exists \( r \in \mathbb{N} \) and \( i \in \{1, 2\} \) such that the output of \textsc{FindMarkers}(\( \mathcal{T}, \mathbf{e}_i, r \)) contains a set \( M \), then \( M \subset \mathcal{T} \) is a subset of markers in the direction \( \mathbf{e}_i \).

**Proof.** Suppose that \( i = 2 \), the case \( i = 1 \) being similar. The output set \( M \) is nonempty since it was created from the union of nonempty sets (see lines 4-6 in Algorithm 1). Using Lemma 4.2, lines 3 to 6 implies that \( M \odot^1 (\mathcal{T} \setminus M) \) and \( (\mathcal{T} \setminus M) \odot^1 M \) are forbidden in \( \Omega_\mathcal{T} \). The lines 7 and 8 implies that \( M \odot^2 M \) is forbidden in \( \Omega_\mathcal{T} \). Then we deduce from Lemma 3.12 that \( M \subset \mathcal{T} \) is a subset of markers in the direction \( \mathbf{e}_i \). \( \square \)

We believe that if a set of Wang tiles \( \mathcal{T} \) has a subset of markers in the direction \( \mathbf{e}_i \) then there exists a surrounding radius \( r \in \mathbb{N} \) such that \( \textsc{FindMarkers}(\mathcal{T}, \mathbf{e}_i, r) \) outputs this set of markers. The fact that there is no upper bound for the surrounding radius is related to the undecidability of the domino problem. In practice, in the study of Jeandel-Rao tilings done in \cite{Lab19b}, a surrounding radius of 1, 2 or 3 was enough.
Using the set $\mathcal{U}$ of Wang tiles defined in Figure 3, compute the sets of horizontal and vertical dominoes that admit a $\mathcal{U}$-surrounding of radius 2:

$$D_1 = \{(u, v) \in \mathcal{U}^2 \mid u \odot^1 v \text{ admits a } \mathcal{U}\text{-surrounding of radius 2}\},$$

$$D_2 = \{(u, v) \in \mathcal{U}^2 \mid u \odot^2 v \text{ admits a } \mathcal{U}\text{-surrounding of radius 2}\}.$$

Exercise 4.3

Use the function $\text{FindMarkers}$ defined in Algorithm 1 with a surrounding radius 2 to show that the subset $M = \{0, F, O, J, O, 1, F, O, H, L, 2, J, M, F, P, 3, D, M, F, K, 4, H, P, J, P, 5, H, P, H, N, 6, H, K, F, P, 7, H, K, D, P\}$ of $\mathcal{U}$ is a subset of markers for the direction $e_2$ in $\Omega_\mathcal{U}$.

4.2. **Fusion of Wang tiles.** Recall that a magma is a set $\mathcal{I}$ equipped with a binary operation $\cdot$ such that for all $a, b \in \mathcal{I}$, the result of the operation $a \cdot b$ is also in $\mathcal{I}$. If the operation $\cdot$ is associative and has an identity, then $\mathcal{I}$ is a monoid and the operation $\cdot$ can be omitted and represented as concatenation. But, in the general context of fusion of Wang tiles defined below where $\mathcal{I}$ is the set of horizontal or vertical colors, we cannot assume that the operation $\cdot$ is associative.

The fusion operation on Wang tiles is defined on pair of tiles sharing an edge in a tiling according to Equations 7 and 8. Let $(\mathcal{I}, \cdot)$ and $(\mathcal{J}, \cdot)$ be two magmas and let $\{A, C, Y, W\} \subset \mathcal{I}$ be some vertical colors and $\{B, D, X, Z\} \subset \mathcal{J}$ be some horizontal colors. We define two binary operations $\sqcap$ and $\square$ on Wang tiles as

\[
\begin{bmatrix}
B & C \\
A & D
\end{bmatrix} \sqcap \begin{bmatrix}
X & Y \\
W & Z
\end{bmatrix} = \begin{bmatrix}
B \cdot X \\
C \cdot W \\
D \cdot Z
\end{bmatrix}
\text{ if } A = Y
\]

and

\[
\begin{bmatrix}
C & A \\
B & D
\end{bmatrix} \square \begin{bmatrix}
X & Y \\
W & Z
\end{bmatrix} = \begin{bmatrix}
X \\
\hat{Y} \\
\hat{W} \\
\square Z
\end{bmatrix}
\text{ if } B = Z.
\]

If $A \neq Y$, the operation $\sqcap$ is not defined. Similarly, if $B \neq Z$, the operation $\square$ is not defined. For the Wang tiles considered in this contribution, the operation $\cdot$ is associative so we always denote it implicitly by concatenation of colors.

In what follows, we propose algorithms and results that works for both operations $\sqcap$ and $\square$. It is thus desirable to have a common notation to denote both, so we define

$$u \square^1 v = u \sqcap v \quad \text{and} \quad u \square^2 v = u \square v.$$}

If $u \square^i v$ is defined for some $i \in \{1, 2\}$, it means that tiles $u$ and $v$ can appear at position $n$ and $n + e_i$ in a tiling for some $n \in \mathbb{Z}^d$. For each $i \in \{1, 2\}$, one can define a new set of tiles from two sets $\mathcal{T}$ and $\mathcal{S}$ of Wang tiles as

$$\mathcal{T} \square^i \mathcal{S} = \{u \square^i v \text{ defined } \mid u \in \mathcal{T}, v \in \mathcal{S}\}.$$
Exercise 4.4

Using the set $D_2$ of dominoes that admits a $\mathcal{U}$-surrounding of radius 2 computed in Exercise 4.2 and the subset $M \subset \mathcal{U}$ of markers for the direction $e_2$ in $\Omega_\mathcal{U}$ computed in Exercise 4.3 compute the set of fusion tiles:

$$\{u \oplus v \mid (u,v) \in D_2 \text{ and } v \in M\}.$$  

What is the meaning of this set?

4.3. Desubstitution of Wang shifts. In Theorem 3.13 we proved that the presence of markers allows to desubstitute uniquely the configurations of a subshift on $\mathbb{Z}^d$. In case of Wang shifts, we show in this section that the preimage is also a Wang shift and we may construct the new set of Wang tiles using the fusion operation defined in the previous section. We also propose an algorithm to find the desubstitution of Wang shifts when there exists a subset of marker tiles.

![Figure 6. A $5 \times 10$ tiling with tiles from the set $\mathcal{U}$ is shown on the left. The tiles labeled from 0 to 6 (shown with yellow background) are marker tiles for the direction $e_2$ since they appear on nonadjacent rows. It can be desubstituted as a $5 \times 6$ pattern with tiles from the set $\mathcal{V}$ using a substitution $\alpha_0$. Each marker tile (yellow background) is glued with its below tile (green background) to form a new Wang tile (blue background) using the fusion operation $\boxplus$. The remaining tiles are kept the same (green background) but get new indices in the set $\mathcal{V}$. This process is uniquely defined since the substitution $\alpha_0$ is recognizable.](image)
Before stating the result, let us see how the markers allows to desubstitute tilings. In Figure [6] we observe that markers $M$ computed in Exercise [4.3] appear as nonadjacent rows in the Wang shift $\Omega_U$. Therefore the row above (and below) some row of markers is made of nonmarker tiles. Let us consider the row below. The idea is to collapse that row with the row of markers just above. Each tile is being collapsed with the above marker tile using the fusion of tiles. The set of tiles that we obtain through this process is exactly the set computed in Exercise [4.4].

Therefore to build some configuration in $\Omega_U$, it is sufficient to build a tiling with another set $V$ of Wang tiles obtained from the set $U$ after removing the markers and adding the tiles obtained from the fusion operation. One may also remove the tile which always appear below of a marker tile. One may recover some configuration in $\Omega_U$ by applying a 2-dimensional morphism $\alpha_0 : V \to U^2$ which replaces the merged tiles by their associated equivalent vertical dominoes and keeps the remaining tiles invariant, see Figure [6]. It turns out that this decomposition is unique. The creation of the set $V$ from $U$ gives the intuition on the construction of Algorithm [2] which follows the same recipe and takes any set of Wang tiles with markers as input.

We now state the result that if a set of Wang tiles $T$ has a subset of marker tiles, then there exists another set $S$ of Wang tiles and a nontrivial recognizable 2-dimensional morphism $\Omega_S \to \Omega_T$ that is onto up to a shift. Thus, every Wang tiling by $T$ is up to a shift the image under a nontrivial 2-dimensional morphism $\omega$ of a unique Wang tiling in $\Omega_S$. The 2-dimensional morphism is essentially 1-dimensional.

**Theorem 4.4.** [Lab19a, Lab19b] Let $T$ be a set of Wang tiles and let $\Omega_T$ be its Wang shift. If there exists a subset $M \subset T$ of markers in the direction $e_i \in \{e_1, e_2\}$, then

1. there exists a set of Wang tiles $S_R$ and a 2-dimensional morphism $\omega_R : \Omega_S \to \Omega_T$ such that
   \[
   \omega_R(S_R) \subseteq (T \setminus M) \cup (T \setminus M) \circ \omega \quad M
   \]
   which is recognizable and onto up to a shift and
2. there exists a set of Wang tiles $S_L$ and a 2-dimensional morphism $\omega_L : \Omega_S \to \Omega_T$ such that
   \[
   \omega_L(S_L) \subseteq (T \setminus M) \cup (T \setminus M) \circ \omega
   \]
   which recognizable and onto up to a shift.

There exists a surrounding radius $r \in \mathbb{N}$ such that $\omega_R$ and $\omega_L$ are computed using Algorithm [2].

**Proof.** The existence of the recognizable 2-dimensional morphism which is onto up to a shift was done in Theorem [3.13]. We only need to prove that the preimage of $\Omega_T$ is a Wang shift. The proof of this fact can be found in [Lab19a] and [Lab19b]. It follows the line of Algorithm [2]. \qed

In the definition of $\omega$ in Algorithm [2] given two Wang tiles $u$ and $v$ such that $u \varnothing^i v$ is defined for $i \in \{1, 2\}$, the map
\[
 u \varnothing^i v \mapsto u \odot^i v
\]
can be seen as a decomposition of Wang tiles:
Algorithm 2 Find a recognizable desubstitution of $\Omega_T$ from markers

Precondition: $T$ is a set of Wang tiles; $M \subset T$ is a subset of markers; $i \in \{1, 2\}$ is a direction $e_i$; $r \in \mathbb{N}$ is a surrounding radius; $s \in \{$LEFT, RIGHT$\}$ determines whether the image of merged tiles is in $M \odot^i (T \setminus M)$ (markers on the left) or in $(T \setminus M) \odot^i M$ (markers on the right).

1: function FindSubstitution($T$, $M$, $i$, $r$, $s$)
2: $D \leftarrow \{(u, v) \in T^2 \mid \text{domino } u \odot^i v \text{ admits a } T \text{-surrounding of radius } r\}$
3: if $s =$ LEFT then
4: $P \leftarrow \{(u, v) \in D \mid u \in M \text{ and } v \in T \setminus M\}$
5: $K \leftarrow \{v \in T \setminus M \mid \text{there exists } u \in T \setminus M \text{ such that } (u, v) \in D\}$
6: else if $s =$ RIGHT then
7: $P \leftarrow \{(u, v) \in D \mid u \in T \setminus M \text{ and } v \in M\}$
8: $K \leftarrow \{u \in T \setminus M \mid \text{there exists } v \in T \setminus M \text{ such that } (u, v) \in D\}$
9: $K \leftarrow \text{Sort}(K)$, $P \leftarrow \text{Sort}(P)$ \>$\triangleright$ lexicographically on the indices of tiles
10: $S \leftarrow K \cup \{u \odot^i v : (u, v) \in P\}$ \>$\triangleright$ defines uniquely indices of tiles in $S$ from 0 to $|S| - 1$.
11: return $S, \omega : \Omega_S \rightarrow \Omega_T : \begin{cases} u \odot^i v \mapsto u \odot^i v & \text{if } (u, v) \in P \\ u \mapsto u & \text{if } u \in K. \end{cases}$

Postcondition: $S$ is a set of Wang tiles; $\omega : \Omega_S \rightarrow \Omega_T$ is recognizable and onto up to a shift.

whether $i = 1$ or $i = 2$ and where $A = Y$ and $B = Z$. The reader may wonder how the substitution decides the color $A$ (color $B$ if $i = 2$) from its input tiles. The answer is that Algorithm 2 is performing a desubstitution. Therefore the two tiles sharing the vertical side with letter $A$ are known from the start and the algorithm just creates a new tile $(W, BX, C, DZ)$ and claims that it will always get replaced by the two tiles with shared edge with color $A$.

**Exercise 4.5**

Using the function FindSubstitution defined in Algorithm 2 with the subset $M \subset U$ of markers for the direction $e_2$ computed in Exercise 4.3 construct a set of tiles $V$ and a recognizable 2-dimensional morphism $\alpha_0 : \Omega_V \rightarrow \Omega_U$ which is onto up to a shift and such that $\alpha_0(u) \in U \setminus M$ or $\alpha_0(u) \in (U \setminus M) \odot^2 M$.

4.4. Self-similarity of the Wang shift $\Omega_U$ defined by 19 tiles. We prove that $\Omega_U$ is self-similar by executing the function FindMarkers on the set of Wang tiles $U$ followed by FindSubstitution and repeating this process until we obtain a set of Wang tiles which is equivalent to the original one (two steps are needed). Each time FindMarkers finds at least one subset of markers using a surrounding radius of size at most 2. Thus using Theorem 4.4 we find a desubstitution of tilings with FindSubstitution. The proof is done in SageMath [Sag20] using slabbe optional package [Lab20b].

The following result was shown in [Lab19a] where it was also proved that $\Omega_U$ is aperiodic and minimal. In [Lab19b], it was later shown that the set $\Omega_U$ describes the internal self-similar structure hidden in Jeandel-Rao aperiodic tilings [JR15]. One of its consequence is
that the Wang shift $\Omega_\ell$ provides another description for the substitutive subshift $\mathcal{X}_\phi$, see Exercise 4.7.

**Proof of Theorem 1.1 (i).** In this proof, we show there exist sets of Wang tiles $\mathcal{V}$ and $\mathcal{W}$ together with their associated Wang shifts $\Omega_\mathcal{V}$ and $\Omega_\mathcal{W}$ and there exist two recognizable 2-dimensional morphisms $\alpha_0$ and $\alpha_1$ and a bijection $\alpha_2$:

$$\Omega_\ell \xrightarrow{\alpha_0} \Omega_\mathcal{V} \xrightarrow{\alpha_1} \Omega_\mathcal{W} \xrightarrow{\alpha_2} \Omega_\ell$$

that are onto up to a shift, i.e., $\alpha_0(\Omega_\mathcal{V})^\sigma = \Omega_\ell$, $\alpha_1(\Omega_\mathcal{W})^\sigma = \Omega_\mathcal{V}$ and $\alpha_2(\Omega_\ell) = \Omega_\mathcal{W}$.

First we define the set $\mathcal{U}$ of Wang tiles in SageMath:

```python
sage: from slabbe import WangTileSet
sage: tiles = ["FOJO", "FOHL", "JMFP", "DMFK", "HPJP", "HPHN", "HKFP", ....: "HKDP", "BOIO", "GLEO", "GLCL", "ALIO", "EPGP", "EPIP", "IPGK", ....: "IPIK", "IKBM", "IKAK", "CNIP"]
sage: U = WangTileSet([tuple(tile) for tile in tiles])
```

![Wang tiles set](image)

We desubstitute $\mathcal{U}$ with the set $\{0, 1, 2, 3, 4, 5, 6, 7\}$ of markers in the direction $e_2$:

```python
sage: U.find_markers(i=2, radius=2, solver="dancing_links")
[[0, 1, 2, 3, 4, 5, 6, 7]]
sage: M = [0, 1, 2, 3, 4, 5, 6, 7]
sage: V, alpha0 = U.find_substitution(M, i=2, radius=2, solver="dancing_links")
```

We obtain $\alpha_0 : \Omega_\mathcal{V} \rightarrow \Omega_\ell$ given as a rule of the form

$$\alpha_0 : [0,20] \rightarrow [0,18]^2$$

$$\begin{align*}
0 &\mapsto (8), \quad 1 \mapsto (9), \quad 2 \mapsto (11), \quad 3 \mapsto (13), \\
4 &\mapsto (14), \quad 5 \mapsto (15), \quad 6 \mapsto (16), \quad 7 \mapsto (17), \\
8 &\mapsto \begin{pmatrix} 0 \\ 8 \end{pmatrix}, \quad 9 \mapsto \begin{pmatrix} 1 \\ 9 \end{pmatrix}, \quad 10 \mapsto \begin{pmatrix} 1 \\ 10 \end{pmatrix}, \quad 11 \mapsto \begin{pmatrix} 1 \\ 11 \end{pmatrix}, \\
12 &\mapsto \begin{pmatrix} 6 \\ 12 \end{pmatrix}, \quad 13 \mapsto \begin{pmatrix} 4 \\ 13 \end{pmatrix}, \quad 14 \mapsto \begin{pmatrix} 7 \\ 13 \end{pmatrix}, \quad 15 \mapsto \begin{pmatrix} 2 \\ 14 \end{pmatrix}, \\
16 &\mapsto \begin{pmatrix} 6 \\ 14 \end{pmatrix}, \quad 17 \mapsto \begin{pmatrix} 7 \\ 15 \end{pmatrix}, \quad 18 \mapsto \begin{pmatrix} 3 \\ 16 \end{pmatrix}, \quad 19 \mapsto \begin{pmatrix} 3 \\ 17 \end{pmatrix}, \\
20 &\mapsto \begin{pmatrix} 5 \\ 18 \end{pmatrix}.
\end{align*}$$
and the set $\mathcal{V}$ of 21 Wang tiles

$$\mathcal{V} = \begin{cases} 
\begin{array}{|c|c|}
\hline
O & I \ 0 \\
I & O \\
L & E \ 1 \\
I & A \\
P & E \ 1 \\
G & I \ 4 \\
P & I \ 5 \\
K & B \ 6 \\
\hline
A & 7 \\
I & 8 \\
O & A \\
I & G \\
O & L \\
O & A \\
I & O \\
I & O \\
K & P \\
P & I \\
\hline
\end{array}
\end{cases}.$$  

We desubstitute $\mathcal{V}$ with the set $\{0, 1, 2, 8, 9, 10, 11\}$ of markers in the direction $e_1$:

```
sage: V.find_markers(i=1, radius=1, solver="dancing_links")
[[0, 1, 2, 8, 9, 10, 11],
 [3, 5, 13, 14, 17, 20],
 [4, 6, 7, 12, 15, 16, 18, 19]]
sage: M = [0, 1, 2, 8, 9, 10, 11]
sage: W, alpha1 = V.find_substitution(M, i=1, radius=1, solver="dancing_links")
```

We obtain $\alpha_1 : \Omega_{\mathcal{V}} \to \Omega_{\mathcal{V}}$ given as a rule of the form

$\alpha_1 : [0, 18] \to [0, 20]^2$

$$
\begin{align*}
0 & \mapsto (6), \\
1 & \mapsto (7), \\
2 & \mapsto (15), \\
3 & \mapsto (16), \\
4 & \mapsto (18), \\
5 & \mapsto (19), \\
6 & \mapsto (3, 1), \\
7 & \mapsto (4, 0), \\
8 & \mapsto (5, 0), \\
9 & \mapsto (5, 2), \\
10 & \mapsto (6, 0), \\
11 & \mapsto (7, 0).
\end{align*}$$

and the set $\mathcal{W}$ of 19 Wang tiles

$$\mathcal{W} = \begin{cases} 
\begin{array}{|c|c|c|c|c|c|c|c|c|}
\hline
K & B & 0 & 1 & M & A & 1 & 1 & K \\
G & 0 & P & O & K & O & G & 1 & K \\
L & I & B & 0 & G & I & A & K & B \\
L & I & 9 & A & K & B & 0 & G & I \\
L & I & 1 & B & 0 & G & I & A & K \\
L & I & 1 & B & 0 & G & I & A & K \\
L & I & 9 & A & K & B & 0 & G & I \\
L & I & 9 & A & K & B & 0 & G & I \\
\hline
K & O & 1 & 4 & 0 & P & Q & G & O \\
K & O & 1 & 4 & 0 & P & Q & G & O \\
K & O & 1 & 4 & 0 & P & Q & G & O \\
K & O & 1 & 4 & 0 & P & Q & G & O \\
K & O & 1 & 4 & 0 & P & Q & G & O \\
P & Q & G & O & K & O & 1 & 4 & 0 \\
\hline
\end{array}
\end{cases}.$$  

It turns out that $\mathcal{U}$ and $\mathcal{W}$ are equivalent:

```
sage: W.is_equivalent(U)
True
```

The bijection $\text{vert}$ between the vertical colors, the bijection $\text{horiz}$ between the horizontal colors and bijection $\alpha_2$ from $\mathcal{U}$ to $\mathcal{W}$ is computed as follows:

```
sage: _, vert, horiz, alpha2 = U.is_equivalent(W, certificate=True)
sage: vert
{'A': 'IJ', 'B': 'IH', 'C': 'BE', 'D': 'G', 'E': 'AF', 'F': 'I', 'G': 'ID', 'H': 'B', 'I': 'GF', 'J': 'A'}
sage: horiz
{'K': 'PO', 'L': 'M', 'M': 'PL', 'N': 'MO', 'O': 'K', 'P': 'KO'}
```
We obtain the morphism \( \alpha_2 : \Omega_U \to \Omega_W \) given as a rule of the form

\[
\begin{align*}
0 \mapsto (1), & \quad 1 \mapsto (0), \quad 2 \mapsto (9), \quad 3 \mapsto (6), \\
4 \mapsto (11), & \quad 5 \mapsto (10), \quad 6 \mapsto (8), \quad 7 \mapsto (7), \\
8 \mapsto (3), & \quad 9 \mapsto (5), \quad 10 \mapsto (4), \quad 11 \mapsto (2), \\
12 \mapsto (17), & \quad 13 \mapsto (16), \quad 14 \mapsto (14), \quad 15 \mapsto (12), \\
16 \mapsto (18), & \quad 17 \mapsto (13), \quad 18 \mapsto (15).
\end{align*}
\]

We may check that \( \alpha_0 \circ \alpha_1 \circ \alpha_2 = \phi \):

```
sage: from slabbe import Substitution2d
sage: Phi = Substitution2d({0: [17], 1: [16], 2: [15], [11]}, ....: 3: [[13],[9]], 4: [[17],[8]], 5: [[16],[8]], 6: [[15],[8]], ....: 7: [[14],[8]], 8: [[14,6],[17,3]], 9: [[16,3]], ....: 11: [[14,2]], 12: [[15,7],[11,1]], 13: [[14,6],[11,1]], ....: 14: [[13,7],[9,1]], 15: [[12,6],[9,1]], 16: [[18,5],[10,1]], ....: 17: [[13,4],[9,1]], 18: [[14,2],[8,0]])
sage: alpha0 * alpha1 * alpha2 == Phi
True
```

We conclude that \( \Omega_U = \alpha_0(\Omega_V)'' = \alpha_0 \alpha_1(\Omega_W)'' = \alpha_0 \alpha_1 \alpha_2(\Omega_U)'' = \phi(\Omega_U)'' \).

In the proof, we used Knuth’s dancing links algorithm \cite{Knu00} because it is faster at this particular task than the MILP solver Gurobi \cite{GO20} or the SAT solvers Glucose \cite{AS18} as we can see below:

```
sage: %
CPU times: user 3.34 s, sys: 0 ns, total: 3.34 s
Wall time: 3.34 s
[[0, 1, 2, 3, 4, 5, 6, 7]]
sage: %
CPU times: user 12.4 s, sys: 572 ms, total: 13 s
Wall time: 13 s
[[0, 1, 2, 3, 4, 5, 6, 7]]
sage: %
CPU times: user 50.6 s, sys: 2.53 s, total: 53.1 s
Wall time: 2 min 10s
[[0, 1, 2, 3, 4, 5, 6, 7]]
```

Note that for other tasks like finding a valid tiling an \( n \times n \) square with Wang tiles, the Glucose SAT solver \cite{AS18} based on MiniSAT \cite{SE05} is faster \cite{Lab18} than Knuth’s dancing links algorithm or MILP solvers.

**Exercise 4.6**

Using the value of \( H \) and \( V \) from Exercise 3.3 prove that \( \Omega_U \neq \emptyset \).

**Exercise 4.7**

Building on Exercise 3.12 and Exercise 4.6 prove that \( \Omega_U = X_\phi \).
Exercise 4.8

Prove the self-similarity of $\Omega_U$ by using first markers in the direction $e_1$ in the set $U$ and then using markers in the direction $e_2$.

5. CODING OF TORAL $\mathbb{Z}^2$-ROTATIONS

In this section, we consider the dynamical system $(\mathbb{T}^2, \mathbb{Z}^2, R_U)$ defined on the 2-dimensional torus $\mathbb{T}^2 = (\mathbb{R}/\mathbb{Z})^2$ by the continuous $\mathbb{Z}^2$-action

$$R_U : \mathbb{Z}^2 \times \mathbb{T}^2 \to \mathbb{T}^2$$

$$(n, x) \mapsto x + \varphi^{-2}n$$

where $\varphi = \frac{1+\sqrt{5}}{2}$. We define a symbolic representation of that dynamical system using a well-chosen partition $\mathcal{P}_U$ of $\mathbb{T}^2$ that was introduced in [Lab20a] where it was proved to be a Markov partition for the dynamical system $(\mathbb{T}^2, \mathbb{Z}^2, R_U)$. We use the index $U$ to be consistent with the notations used in [Lab20a]. As illustrated in Figure 7, the partition $\mathcal{P}_U$ can be defined from the following 8 segments in $\mathbb{R}^2$:

$$(1, \varphi^2) \to (0, \varphi^2) \to (\varphi, 0) \to (\varphi, 1),$$
$$(1, 1) \to (0, 1) \to (1, 0) \to (1, 1),$$

$$\left(1 - \frac{1}{\varphi^2}, 2\right) \to \left(1 + \frac{1}{\varphi^2}, 1\right) \to \left(1 + \frac{1}{\varphi^2}, 2\right).$$

The translations of the 8 segments under the group of translation $\mathbb{Z}^2$ splits the torus $\mathbb{T}$ into 19 polygonal regions indexed with integers from the set $[0, 18]$. The coding by the partition $\mathcal{P}_U$ of the orbit of some starting point in $\mathbb{T}^2$ by the $\mathbb{Z}^2$-action of $R_U$ defines a configuration $w \in [0, 18]^{\mathbb{Z}^2}$, see Figure 8. The topological closure of the set of all such configurations is
the symbolic dynamical system $\mathcal{X}_{P_U, R_U}$ corresponding to $\mathcal{P}_U, R_U$ (see Lemma 5.6). It turns out that $\mathcal{X}_{P_U, R_U}$ is a subshift as it is also closed under the shift $\sigma$ by integer translations.

The goal of the next sections is to prove that the symbolic dynamical system $\mathcal{X}_{P_U, R_U}$ is self-similar where the self-similarity is given by the 2-dimensional morphism $\phi$ defined in Equation (1) (Theorem 1.1 (ii)).


Exercise 5.1

Figure 8 provides the construction of a 2-dimensional word of shape (6,8). Using the same construction, extend that pattern by one unit in all directions to obtain a word of shape (8,10).

5.1. Toral \( \mathbb{Z}^2 \)-rotations and polygon exchange transformations (PETs). Let \( \Gamma \) be a lattice in \( \mathbb{R}^2 \), i.e., a discrete subgroup of the additive group \( \mathbb{R}^2 \) with 2 linearly independent generators. This defines a 2-dimensional torus \( T = \mathbb{R}^2 / \Gamma \). By analogy with the rotation \( x \mapsto x + \alpha \) on the circle \( \mathbb{R} / \mathbb{Z} \) for some \( \alpha \in \mathbb{R} / \mathbb{Z} \), we use the terminology of rotation to denote the following \( \mathbb{Z}^2 \)-action defined on a 2-dimensional torus.

Definition 5.1. For some \( \alpha, \beta \in T \), we consider the dynamical system \((T, \mathbb{Z}^2, R)\) where \( R: \mathbb{Z}^2 \times T \to T \) is the continuous \( \mathbb{Z}^2 \)-action on \( T \) defined by

\[
R^\mathbf{n}(x) := R(n, x) = x + n_1 \alpha + n_2 \beta
\]

for every \( n = (n_1, n_2) \in \mathbb{Z}^2 \). We say that the \( \mathbb{Z}^2 \)-action \( R \) is a toral \( \mathbb{Z}^2 \)-rotation or a \( \mathbb{Z}^2 \)-rotation on \( T \) which defines a dynamical system \((T, \mathbb{Z}^2, R)\).

When the \( \mathbb{Z}^2 \)-action \( R \) is a \( \mathbb{Z}^2 \)-rotation on the torus \( T \), the maps \( R^e_1 \) and \( R^e_2 \) can be seen as polygon exchange transformations [Hoo13, Sch14] on a fundamental domain of \( T \).

Definition 5.2. [AKY19] Let \( X \) be a polygon together with two topological partitions of \( X \) into polygons

\[
X = \bigcup_{k=0}^N P_k = \bigcup_{k=0}^N Q_k
\]

such that for each \( k \), \( P_k \) and \( Q_k \) are translation equivalent, i.e., there exists \( v_k \in \mathbb{R}^2 \) such that \( P_k = Q_k + v_k \). A polygon exchange transformation (PET) is the piecewise translation on \( X \) defined for \( x \in P_k \) by \( T(x) = x + v_k \). The map is not defined for points \( x \in \bigcup_{k=0}^N \partial P_k \).

The fact that a rotation on a circle can be seen as an exchange of two intervals is well-known as noticed for example in [Rau77]. It generalizes in higher dimension where a generic translation on a \( d \)-dimensional torus is a polyhedron exchange transformation defined by the exchange of at most \( 2^d \) pieces on a fundamental domain having for shape a \( n \)-dimensional parallelepiped. We state a 2-dimensional version of this lemma restricted to the case of rectangular fundamental domain because we use this connection several times in the following sections to prove that induced \( \mathbb{Z}^2 \)-actions are again \( \mathbb{Z}^2 \)-rotations on a torus.

Lemma 5.3. Let \( \Gamma = \ell_1 \mathbb{Z} \times \ell_2 \mathbb{Z} \) be a lattice in \( \mathbb{R}^2 \) and its rectangular fundamental domain \( D = [0, \ell_1) \times [0, \ell_2) \). For every \( \alpha = (\alpha_1, \alpha_2) \in D \), the dynamical system \((\mathbb{R}^2 / \Gamma, \mathbb{Z}, x \mapsto x + \alpha)\) is measurably conjugate to the dynamical system \((D, \mathbb{Z}, T)\) where \( T: D \to D \) is the polygon exchange transformation shown in Figure 9.

Proof. It follows from the fact that toral rotations and such polygon exchange transformations are the Cartesian product of circle rotations and exchange of two intervals. \( \Box \)
The closures \( \{ \) \( \) definition.

To points in Definition 5.4. Hence, using standard arguments \([\text{LM}95, \text{Prop. 1.3.4}]\) there is a

\( L \) rotation \( R T \)=

5.2. **Symbolic dynamical systems for toral \( Z^2 \)-rotations.** Let \( \Gamma \) be a lattice in \( \mathbb{R}^2 \) and \( T = \mathbb{R}^2/\Gamma \) be a 2-dimensional torus. Let \( (T, Z^2, R) \) be the dynamical system given by a \( Z^2 \)-rotation \( R \) on \( T \). For some \( A \) finite set, a **topological partition** of \( T \) is a finite collection \( \{ P_a \}_{a \in A} \) of disjoint open sets \( P_a \subset T \) such that \( T = \bigcup_{a \in A} P_a \). If \( S \subset \mathbb{Z}^2 \) is a finite set, we say that a pattern \( w \in A^S \) is **allowed** for \( P, R \) if

\[
\bigcap_{k \in S} R^{-k}(P_{w_k}) \neq \emptyset.
\]

Let \( \mathcal{L}_{P, R} \) be the collection of all allowed patterns for \( P, R \). It can be checked that \( \mathcal{L}_{P, R} \) is the language of a subshift. Hence, using standard arguments \([\text{LM}95, \text{Prop. 1.3.4}]\), there is a unique subshift \( X_{P, R} \subset A^{Z^2} \) whose language is \( \mathcal{L}_{P, R} \).

**Definition 5.4.** We call \( X_{P, R} \) the **symbolic dynamical system** corresponding to \( P, R \).

For each \( w \in X_{P, R} \subset A^{Z^2} \) and \( n \geq 0 \) there is a corresponding nonempty open set

\[
D_n(w) = \bigcap_{\|k\| \leq n} R^{-k}(P_{w_k}) \subseteq M.
\]

The closures \( D_n(w) \) of these sets are compact and decrease with \( n \), so that \( D_0(w) \supseteq D_1(w) \supseteq D_2(w) \supseteq \ldots \). It follows that \( \cap_{n=0}^{\infty} D_n(w) \neq \emptyset \). In order for points in \( X_{P, R} \) to correspond to points in \( M \), this intersection should contain only one point. This leads to the following definition.

A topological partition \( P \) of \( M \) gives a **symbolic representation** of \( (M, Z^2, R) \) if for every \( w \in X_{P, R} \) the intersection \( \cap_{n=0}^{\infty} D_n(w) \) consists of exactly one point \( m \in M \). We call \( w \) a **symbolic representation** of \( m \).

**Definition 5.5.** A topological partition \( P \) of \( M \) is a **Markov partition** for \( (M, Z^2, R) \) if

- \( P \) gives a symbolic representation of \( (M, Z^2, R) \) and
- \( X_{P, R} \) is a shift of finite type (SFT).

The set

\[
\Delta_{P, R} := \bigcup_{n \in \mathbb{Z}^2} R^n \left( \bigcup_{a \in A} \partial P_a \right) \subset T
\]
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is the set of points whose orbit under the \( \mathbb{Z}^2 \)-action of \( R \) intersect the boundary of the topological partition \( \mathcal{P} = \{ P_a \}_{a \in A} \). From Baire Category Theorem [LM95, Theorem 6.1.24], the set \( T \setminus \Delta_{\mathcal{P},R} \) is dense in \( T \).

A topological partition \( \mathcal{P} = \{ P_a \}_{a \in A} \) of \( T = \mathbb{R}^2 / \Gamma \) is associated to a coding map

\[
\text{Code} : \ T \setminus (\bigcup_{a \in A} \partial P_a) \to A \\
\ x \mapsto a \quad \text{if and only if} \quad x \in P_a.
\]

For every starting point \( x \in T \setminus \Delta_{\mathcal{P},R} \), the coding of its orbit under the \( \mathbb{Z}^2 \)-action of \( R \) is a configuration \( \text{Config}^{P,R}_x \in \mathcal{A}^{\mathbb{Z}^2} \) defined by

\[
\text{Config}^{P,R}_x(n) = \text{Code}(R^n(x)).
\]

Lemma 5.6. The symbolic dynamical system \( \mathcal{X}_{\mathcal{P},R} \) corresponding to \( \mathcal{P},R \) is the topological closure of the set of configurations:

\[
\mathcal{X}_{\mathcal{P},R} = \left\{ \text{Config}^{P,R}_x \mid x \in T \setminus \Delta_{\mathcal{P},R} \right\}.
\]

Proof. (\( \supseteq \)) Let \( x \in T \setminus \Delta_{\mathcal{P},R} \). The patterns appearing in the configuration \( \text{Config}^{P,R}_x \) are in \( \mathcal{L}_{\mathcal{P},R} \). Thus \( \text{Config}^{P,R}_x \in \mathcal{X}_{\mathcal{P},R} \). The topological closure of such configurations is in \( \mathcal{X}_{\mathcal{P},R} \) since \( \mathcal{X}_{\mathcal{P},R} \) is topologically closed.

(\( \subseteq \)) Let \( w \in \mathcal{A}^S \) be a pattern with finite support \( S \subset \mathbb{Z}^2 \) appearing in \( \mathcal{X}_{\mathcal{P},R} \). Then \( w \in \mathcal{L}_{\mathcal{P},R} \) and from Equation (9) there exists \( x \in T \setminus \Delta_{\mathcal{P},R} \) such that \( x \in \bigcap_{k \in S} R^{-k}(P_{w_k}) \). The pattern \( w \) appears in the configuration \( \text{Config}^{P,R}_x \). Any configuration in \( \mathcal{X}_{\mathcal{P},R} \) is the limit of a sequence \( (w_n)_{n \in \mathbb{N}} \) of patterns covering a ball of radius \( n \) around the origin, thus equal to some limit \( \lim_{n \to \infty} \text{Config}^{P,R}_{x_n} \) with \( x_n \in T \setminus \Delta_{\mathcal{P},R} \) for every \( n \in \mathbb{N} \). \( \square \)

Lemma 5.7. [Lab20a] Let \( \mathcal{P} \) give a symbolic representation of the dynamical system \( (T, \mathbb{Z}^2, R) \).

Then

(1) if \( (T, \mathbb{Z}^2, R) \) is minimal, then \( (\mathcal{X}_{\mathcal{P},R}, \mathbb{Z}^2, \sigma) \) is minimal,

(2) if \( R \) is a free \( \mathbb{Z}^2 \)-action on \( T \), then \( \mathcal{X}_{\mathcal{P},R} \) aperiodic.

**Exercise 5.3**

Prove that \( \mathcal{P}_U \) gives a symbolic representation of the dynamical system \( (T^2, \mathbb{Z}^2, R_U) \).

**Exercise 5.4**

Prove that \( (\mathcal{X}_{\mathcal{P},R_U}, \mathbb{Z}^2, \sigma) \) is minimal and aperiodic.

5.3. **Induced \( \mathbb{Z}^2 \)-actions.** Renormalization schemes also known as Rauzy induction were originally defined for dynamical system including interval exchange transformations (IET) [Rau79]. A natural way to generalize it to higher dimension is to consider polygon exchange transformations [Hoo13,AKY19] or even polytope exchange transformations [Sch14,Sch11] where only one map is considered. But more dimensions also allows to induce two or more (commuting) maps at the same time.

In this section, we define the induction of \( \mathbb{Z}^2 \)-actions on a sub-domain. We consider the torus \( T = \mathbb{R}^2 / \Gamma \) where \( \Gamma \) is a lattice in \( \mathbb{R}^2 \). Let \( (T, \mathbb{Z}^2, R) \) be a minimal dynamical system
given by a $\mathbb{Z}^2$-action $R$ on $T$. For every $n \in \mathbb{Z}^2$, the toral translation $R^n$ can be seen as polygon exchange transformations on a fundamental domain of $T$.

The set of return times of $x \in T$ to $W \subset T$ under the $\mathbb{Z}^2$-action $R$ is the subset of $\mathbb{Z} \times \mathbb{Z}$ defined as:

$$\delta_W(x) = \{n \in \mathbb{Z} \times \mathbb{Z} \mid R^n(x) \in W\}.$$ 

**Definition 5.8.** Let $W \subset T$. We say that the $\mathbb{Z}^2$-action $R$ is **Cartesian on** $W$ if the set of return times $\delta_W(x)$ can be expressed as a Cartesian product, that is, for all $x \in T$ there exist two strictly increasing sequences $r_1^{(1)}, r_2^{(2)} : \mathbb{Z} \to \mathbb{Z}$ such that

$$\delta_W(x) = r_1^{(1)}(\mathbb{Z}) \times r_2^{(2)}(\mathbb{Z}).$$

We always assume that the sequences are shifted in such a way that

$$r_1^{(i)}(n) \geq 0 \iff n \geq 0 \quad \text{for } i \in \{1, 2\}.$$ 

In particular, if $x \in W$, then $(0, 0) \in \delta_W(x)$, so that $r_1^{(1)}(0) = r_2^{(2)}(0) = 0$.

When the $\mathbb{Z}^2$-action $R$ is Cartesian on $W \subset T$, we say that the tuple

$$(r_1^{(1)}(1), r_2^{(2)}(1))$$

is the first return time of a starting point $x \in T$ to $W \subset T$ under the action $R$. When the $\mathbb{Z}^2$-action $R$ is Cartesian on $W \subset T$, we may consider its return map on $W$ and we prove in the next lemma that this induces a $\mathbb{Z}^2$-action on $W$.

**Lemma 5.9.** If the $\mathbb{Z}^2$-action $R$ is Cartesian on $W \subset T$, then the map $\hat{R}|_W : \mathbb{Z}^2 \times W \to W$ defined by

$$(\hat{R}|_W)^n(x) := \hat{R}|_W(n, x) = R^{(r_1^{(1)}(n_1), r_2^{(2)}(n_2))}(x)$$

for every $n = (n_1, n_2) \in \mathbb{Z}^2$ is a well-defined $\mathbb{Z}^2$-action.

We say that $\hat{R}|_W$ is the **induced** $\mathbb{Z}^2$-action of the $\mathbb{Z}^2$-action $R$ on $W$.

**Proof.** Let $x \in W$. We have that

$$\hat{R}|_W(0, x) = R^{(r_1^{(1)}(0), r_2^{(2)}(0))}(x) = R^{(0, 0)}(x) = x.$$ 

Firstly, using

$$r_1^{(i)}(k + n) = r_1^{(i)}(n) + r_1^{(i)}(k),$$

and skipping few details, we get

$$\hat{R}|_W(ke_i + ne_i, x) = \hat{R}|_W(k, \hat{R}|_W(ne_i, x)).$$

Secondly, using the fact that

$$r_1^{(1)}(k_1) = r_1^{(1)}(k_1)$$

whenever $y = R^{(0, r_2^{(2)}(k_1))}(x) = \hat{R}|_W(k_2e_2, x)$, we obtain

$$\hat{R}|_W(k, x) = R^{(r_1^{(1)}(k_1), r_2^{(2)}(k_2))}(x) = R^{(r_1^{(1)}(k_1), 0)}R^{(0, r_2^{(2)}(k_2))}(x)
= R^{(r_1^{(1)}(k_1), 0)}\hat{R}|_W(k_2e_2, x) = \hat{R}|_W(k_1e_1, \hat{R}|_W(k_2e_2, x)).$$
Therefore, for every \( k, n \in \mathbb{Z}^2 \), we have
\[
(\tilde{R}|_W)^{k+n}(x) = (\tilde{R}|_W)^{(k_1+n_1)e_1}(\tilde{R}|_W)^{(k_2+n_2)e_2}(x) \\
= (\tilde{R}|_W)^{k_1e_1}(\tilde{R}|_W)^{n_1e_1}(\tilde{R}|_W)^{k_2e_2}(x) \\
= (\tilde{R}|_W)^{k_1e_1}(\tilde{R}|_W)^{(n_1,k_2)}(\tilde{R}|_W)^{n_2e_2}(x) \\
= (\tilde{R}|_W)^{k_1e_1}(\tilde{R}|_W)^{k_2e_2}(\tilde{R}|_W)^{n_1e_1}(\tilde{R}|_W)^{n_2e_2}(x) \\
= (\tilde{R}|_W)^{k}(\tilde{R}|_W)^{n}(x),
\]
which shows that \( \tilde{R}|_W \) is a \( \mathbb{Z}^2 \)-action on \( W \). \( \square \)

A consequence of the lemma is that the induced \( \mathbb{Z}^2 \)-action \( \tilde{R}|_W \) is generated by two commutative maps
\[
(\tilde{R}|_W)^{e_1}(x) = R^{(r^{(1)}_1,0)}(x) \quad \text{and} \quad (\tilde{R}|_W)^{e_2}(x) = R^{(0,r^{(2)}_2)}(x)
\]
which are the first return maps of \( R^{e_1} \) and \( R^{e_2} \) to \( W \):
\[
(\tilde{R}|_W)^{e_1}(x) = \tilde{R}^{e_1}|_W(x) \quad \text{and} \quad (\tilde{R}|_W)^{e_2}(x) = \tilde{R}^{e_2}|_W(x).
\]

Recall that the \textbf{first return map} \( \tilde{T}|_W \) of a dynamical system \((X, T)\) maps a point \( x \in W \subset X \) to the first point in the forward orbit of \( T \) lying in \( W \), i.e.
\[
\tilde{T}|_W(x) = T^{r(x)}(x) \quad \text{where} \quad r(x) = \min\{k \in \mathbb{Z}_{>0} : T^k(x) \in W\}.
\]

From Poincaré’s recurrence theorem, if \( \mu \) is a finite \( T \)-invariant measure on \( X \), then the first return map \( \tilde{T}|_W \) is well defined for \( \mu \)-almost all \( x \in W \). When \( T \) is a translation on a torus, if the subset \( W \) is open, then the first return map is well-defined for every point \( x \in W \). Moreover if \( W \) is a polygon, then the first return map \( \tilde{T}|_W \) is a polygon exchange transformation. An algorithm to compute the induced transformation \( \tilde{T}|_W = \tilde{R}^{e_1}|_W \) of the sub-action \( R^{e_1} \) is provided in Section 5.5.

\begin{exercise}
Recall that \( R_{\mu}(n, x) = x + \varphi^{-2}n \) is a \( \mathbb{Z}^2 \)-action defined on \( \mathbb{T}^2 \) where \( \varphi = \frac{1+\sqrt{5}}{2} \). Let \( W_0 = (0, 1) \times (0, \varphi^{-1}) + \mathbb{Z}^2 \) be a subset of \( \mathbb{T}^2 \).

- Prove that the action \( R_{\mu} \) is Cartesian on \( W_0 \).
- Prove that \( \tilde{R}_{\mu}|_{W_0} : \mathbb{Z}^2 \times W_0 \to W_0 \) is a well-defined induced \( \mathbb{Z}^2 \)-action.
- Describe \( \tilde{R}^{e_1}_{\mu}|_{W_0} \) and \( \tilde{R}^{e_2}_{\mu}|_{W_0} \) as polygon exchange transformations on \( W_0 \).
- Describe \( \tilde{R}_{\mu}|_{W_0} \) as a toral \( \mathbb{Z}^2 \)-rotation on \( \mathbb{R}^2 / \Gamma_1 \) with \( \Gamma_1 = \mathbb{Z} \times (\varphi^{-1}\mathbb{Z}) \).
\end{exercise}

\textbf{5.4. Toral partitions induced by toral \( \mathbb{Z}^2 \)-rotations.} For IETs, the interval on which we define the Rauzy induction is usually given by one of the atom of the partition which defines the IET itself. In our setting, it is not the case. The partition that we use carries more information than the natural partition which allows to define the \( \mathbb{Z}^2 \)-rotation \( R \) as a pair of polygon exchange transformations. The partition is a refinement of the natural partition involving well-chosen sloped lines.

Let \( \Gamma \) be a lattice in \( \mathbb{R}^2 \) and \( T = \mathbb{R}^2 / \Gamma \) be a 2-dimensional torus. Let \((T, \mathbb{Z}^2, R)\) be the dynamical system given by a \( \mathbb{Z}^2 \)-rotation \( R \) on \( T \). Assuming the \( \mathbb{Z}^2 \)-rotation \( R \) is Cartesian
on a window $W \subset T$, then there exist two strictly increasing sequences $r_x^{(1)}, r_x^{(2)} : \mathbb{Z} \to \mathbb{Z}$ are such that
\[(r, s) = \left(r_x^{(1)}(1), r_x^{(2)}(1)\right)\]
is the first return time of a starting point $x \in T$ to the window $W$ under the action $R$, see Equation \[10\]. It allows to define the return word map as
\[
\text{ReturnWord} : \quad W \to A^2
\]
\[
x \mapsto \begin{pmatrix}
\text{Code}(R^{(0,s-1)}x) & \ldots & \text{Code}(R^{(r-1,s-1)}x) \\
\vdots & \ddots & \vdots \\
\text{Code}(R^{(0,0)}x) & \ldots & \text{Code}(R^{(r-1,0)}x)
\end{pmatrix},
\]
where $r, s \geq 1$ both obviously depend on $x$.

The image $L = \text{ReturnWord}(W) \subset A^2$ is a language called the set of return words. When the return time to $W$ is bounded, the set of return words $L$ is finite. Let $L = \{w_b\}_{b \in B}$ be an enumeration of $L$ for some finite set $B$.

**Remark 5.10.** The way the enumeration of $L$ is done influences the substitutions which is obtained afterward. To obtain a canonical ordering when the words are 1-dimensional, we use the total order $(L, \prec)$ defined by $u \prec v$ if $|u| < |v|$ or $|u| = |v|$ and $u <_{\text{lex}} v$. See Line \[11\] of Algorithm \[3\].

The induced partition of $P$ by the action of $R$ on the sub-domain $W$ is a topological partition of $W$ defined as the set of preimage sets under ReturnWord:
\[
\hat{P}|_W = \{\text{ReturnWord}^{-1}(w_b)\}_{b \in B}.
\]
This yields the induced coding on $W$
\[
\text{Code}|_W : \quad W \to B
\]
\[
y \mapsto b \quad \text{if and only if} \quad y \in \text{ReturnWord}^{-1}(w_b).
\]
A natural substitution comes out of this induction procedure:
\[(11)\]
\[
\omega : \quad B \to A^2
\]
\[
b \mapsto w_b.
\]
The partition $\hat{P}|_W$ of $W$ can be effectively computed by the refinement of the partition $P$ with translated copies of the sub-domain $W$ under the action of $R$. In Section \[5.5\] we propose Algorithm \[3\] to compute the induced partition $\hat{P}|_W$ and substitution $\omega$. An implementation of it in SageMath is provided in the optional package \texttt{slabbe} \[Lab20b\] and is used below on an example. The next result shows that the coding of the orbit under the $\mathbb{Z}^2$-rotation $R$ is the image under the 2-dimensional substitution $\omega$ of the coding of the orbit under the $\mathbb{Z}^2$-action $\hat{R}|_W$.

**Lemma 5.11.** If the $\mathbb{Z}^2$-action $R$ is Cartesian on a window $W \subset T$, then $\omega$ is a 2-dimensional morphism, and for every $x \in W$ we have
\[
\text{Config}^{P,R}_x = \omega \left(\text{Config}^{\hat{P}|_W,\hat{R}|_W}_x\right).
\]

**Proof.** Let $x \in W$. By hypothesis, there exists $P, Q \subset \mathbb{Z}$ such that the set of returns times satisfies $\delta_W(x) = P \times Q$ and we may write $P = \{r_i\}_{i \in \mathbb{Z}}$ and $Q = \{s_j\}_{j \in \mathbb{Z}}$ as increasing sequences such that $r_0 = s_0 = 0$. Therefore, $\text{Config}^{P,R}_x$ may be decomposed into a lattice
of rectangular blocks. More precisely, for every \(i, j \in \mathbb{Z}\), the following block is the image of a letter under \(\omega\):

\[
\begin{pmatrix}
\text{CODE}(R^{(r_i, s_j+1-1)}x) & \cdots & \text{CODE}(R^{(r_i+1, s_j+1-1)}x) \\
\vdots & \ddots & \vdots \\
\text{CODE}(R^{(r_i, s_j)}) & \cdots & \text{CODE}(R^{(r_i+1, s_j)})
\end{pmatrix} = \text{RETURNWORD}(R^{(r_i, s_j)}x) = w_{b_{ij}} = \omega(b_{ij})
\]

for some letter \(b_{ij} \in \mathcal{B}\). Moreover,

\[
b_{ij} = \text{CODE}_W(R^{(r_i, s_j)}x) = \text{CODE}_W((\tilde{R}|_W)^{(i,j)}x).
\]

Since the adjacent blocks have matching dimensions, for every \(i, j \in \mathbb{Z}\), the following concatenations

\[
\omega \left( b_{ij} \odot^1 b_{(i+1)j} \right) = \omega \left( b_{ij} \right) \odot^1 \omega \left( b_{(i+1)j} \right) \quad \text{and} \quad \omega \left( b_{ij} \odot^2 b_{(i+1)} \right) = \omega \left( b_{ij} \right) \odot^2 \omega \left( b_{i(j+1)} \right)
\]

are well defined. Thus \(\omega\) is a 2-dimensional morphism on the set \(\{\text{CONFIG}_x^{\tilde{R}|_W, \tilde{R}|_W} \mid x \in W\}\) and we have

\[
\text{CONFIG}_x^{P, R} = \omega \left( \text{CONFIG}_x^{\tilde{P}|_W, \tilde{R}|_W} \right)
\]

which ends the proof. Note that the domain of \(\omega\) can be extended to its topological closure.

\[\square\]

**Proposition 5.12.** Let \(P\) be a topological partition of \(T\). If the \(\mathbb{Z}^2\)-action \(R\) is Cartesian on a window \(W \subset T\), then \(X_{P, R} = \overline{\omega(X_{P|_W, \tilde{R}|_W})}\).

**Proof.** Let

\[
Y = \left\{ \text{CONFIG}_x^{P, R} \mid x \in T \right\} \quad \text{and} \quad Z = \left\{ \text{CONFIG}_x^{\tilde{P}|_W, \tilde{R}|_W} \mid x \in W \right\},
\]

\((\supseteq)\). Let \(x \in W\). From Lemma 5.11, \(\omega \left( \text{CONFIG}_x^{\tilde{P}|_W, \tilde{R}|_W} \right) = \text{CONFIG}_x^{P, R}\) with \(x \in W \subset T\).

\((\subseteq)\). Let \(x \in T\). There exists \(k_1, k_2 \in \mathbb{N}\) such that \(x' = R^{-(k_1, k_2)}(x) \in W\). Therefore, we have \(x = R^{(k_1, k_2)}(x')\) where \(0 \leq k_1 < r(x')\) and \(0 \leq k_2 < s(x')\). Thus the shift \(k = (k_1, k_2) \in \mathbb{Z}^2\) is bounded by the maximal return time of \(R^{e_1}\) and \(R^{e_2}\) to \(W\). We have

\[
\text{CONFIG}_x^{P, R} = \text{CONFIG}_x^{P, R_{x'} \omega} = \sigma^k \circ \text{CONFIG}_x^{P, R}
\]

where we used Lemma 5.11 with \(x' \in W\). We conclude that \(Y = \overline{\omega(Z)}\). The result follows from Lemma 5.6 by taking the topological closure on both sides. \[\square\]
Recall that $R_u(n, x) = x + \varphi^{-2}n$ is a $\mathbb{Z}^2$-action defined on $\mathbb{T}^2$ where $\varphi = \frac{1+\sqrt{5}}{2}$. Let $\mathcal{P}_u$ be the polygonal partition of $\mathbb{T}^2$ shown in Figure 8 with indices in the set $\mathcal{A}_0 = [0, 18]$. Let $W_0 = (0, 1) \times (0, \varphi^{-1}) + \mathbb{Z}^2$ be a subset of $\mathbb{T}^2$.

- For each starting point in $W_0$ compute the first return time to $W_0$ under the $\mathbb{Z}^2$-action $R_u$.
- Describe the map $\text{Code} : \mathbb{T}^2 \to \mathcal{A}_0$ obtained from $\mathcal{P}_u$.
- Compute the return word map $\text{ReturnWord} : W_0 \to \mathcal{A}_0^2$.
- Compute the set of return words $\mathcal{L} = \{w_b\}_{b \in \mathcal{A}_1}$ for some alphabet $\mathcal{A}_1$ of size 21.
- Compute the induced partition $\mathcal{P}_u|_{W_0}$ of $\mathcal{P}_u$ by the action of $R_u$ on the sub-domain $W_0$.
- Compute the natural substitution $\beta_0 : \mathcal{A}_1 \to \mathcal{A}_0$ that comes out of this induction procedure.
- Let $p = (0.1357, 0.2938) \in W_0$ and consider the 2-dimensional word of shape $(6, 8)$ shown in Figure 8 which is a subword of $\text{Config}_{\mathcal{P}_u,R_u}$. Using that subword, verify the equality $\text{Config}_{\mathcal{P}_u,R_u} = \beta_0\left(\text{Config}_{\mathcal{P}_u|_{W_0},R_u|_{W_0}}\right)$ given in Lemma 5.11 using the induced $\mathbb{Z}^2$-action $R_u|_{W_0}$ computed in Exercise 5.5.

5.5. Algorithms. In this section, we provide two algorithms to compute the induced partition and induced transformation of a polytope exchange transformation on a sub-domain. More precisely, Algorithm 3 computes the induced partition $\mathcal{P}|_W$ of a partition $\mathcal{P}$ of $D$ by a polytope exchange transformation $T : D \to D$ on a sub-domain $W \subset D$. It also computes the substitution $\omega$ allowing to express configuration coded by the partition $\mathcal{P}$ as the image of configurations coded by the induced partition $\mathcal{P}|_W$. Algorithm 4 computes the induced transformation $\mathcal{T}|_W$ of a polytope exchange transformation $T : D \to D$ on a sub-domain $W \subset D$. We first present the algorithm computing the induced partition since the other one can be deduced from it.

Algorithms are written for domain $D \subset \mathbb{R}^d$, with $d \geq 1$, as they work in arbitrary dimension. All polytopes manipulated in the algorithms are assumed to be open so that the intersection of two polytopes is nonempty if and only if it of positive volume and convex so that they can be represented as a set of linear inequalities. It is possible to deal with polytope partitions where some polytopes are not convex, but it is to the user responsibility to split the non-convex atoms into many convex polytopes. Partitions are represented as a list of pairs $(a, p)$ where $a$ is the index associated to the convex polytope $p$. The same index can be used for different polytopes.

We assume that the polytope exchange transformation $T : D \to D$ is defined on domain $D \subset \mathbb{R}^d$ which is a convex polytope and that the sub-domain $W$ on which the induction is being done is such that $W = D \cap H_v$ where $v \in \mathbb{R}^{d+1}$ and $H_v$ is a half-space given by the part of $\mathbb{R}^d$ on one side of a hyperplane:

$$H_v = \left\{ x \in \mathbb{R}^d \mid v_0 + \sum_{i=1}^d v_i x_i \geq 0 \right\}.$$
Algorithm 3 Compute the induced partition \( \hat{\mathcal{P}}|_W \) and substitution \( \omega \) associated to the induced transformation \( \hat{T}|_W \) (we use it when \( T = R^{e_i} \) for some \( i \).

**Precondition:** \( T \) is a polytope exchange transformation (PET) on a convex domain \( D \subset \mathbb{R}^d \) and \( \mathcal{G} \) is a partition of \( D \) into convex polytopes such that the restriction of \( T \) on each atom of \( \mathcal{G} \) is continuous; \( v \in \mathbb{R}^{d+1} \) defines a half space \( H_v = \{ x \in \mathbb{R}^d \mid v_0 + \sum_{i=1}^d v_ix_i \geq 0 \} \) such that \( D \cap H_v = W; \mathcal{P} \) is a list of pairs \( (a,p) \) such that \( \{ p \mid (a,p) \in \mathcal{P} \} \) is a partition of \( D \) into convex polytopes indexed by the alphabet \( \mathcal{A} = \{ a \mid (a,p) \in \mathcal{P} \} \).

1: function INDUCEDPARTITION\((T, v, \mathcal{P})\)  
2: \( Q \leftarrow \{ (\varepsilon, W) \} \quad \triangleright \varepsilon \in \mathcal{A}^* \) is the empty word and \( W = D \cap H_v \)  
3: \( \mathcal{K} \leftarrow T(\mathcal{P} \land \mathcal{G}) = \{ (a, T(p \land g)) \mid (a,p) \in \mathcal{P} \) and \( g \in \mathcal{G} \) and \( p \land g \neq \emptyset \} \)  
4: \( S \leftarrow \text{EMPTYLIST}() \)  
5: while \( Q \) not empty do  
6: \( Q \leftarrow T^{-1}(Q \land \mathcal{K}) = \{ (au, T^{-1}(q \land k)) \mid (u, q) \in Q \) and \( (a, k) \in \mathcal{K} \) and \( q \land k \neq \emptyset \} \)  
7: \( S \leftarrow S \cup \{ (u, q \land H_v) \mid (u, q) \in Q \) and \( q \land H_v \neq \emptyset \} \)  
8: \( Q \leftarrow Q \land (\mathbb{R}^d \setminus H_v) = \{ (u, q \cap (\mathbb{R}^d \setminus H_v)) \mid (u, q) \in Q \) and \( q \cap (\mathbb{R}^d \setminus H_v) \neq \emptyset \} \)  
9: \( \mathcal{L} \leftarrow \{ u \in \mathcal{A}^* \mid (u, q) \in S \} \quad \triangleright \) the set of return words  
10: \( \mathcal{B} \leftarrow \{ 0, 1, \ldots, \# \mathcal{L} - 1 \} \quad \triangleright \) the new alphabet  
11: \( \omega \leftarrow \text{bijection} \mathcal{B} \rightarrow \mathcal{L} \quad \triangleright \) such that \( i < j \) if and only if \( \omega(i) < \omega(j) \)  
12: \( \mathcal{P}' \leftarrow \{ (\omega^{-1}(u), q) \mid (u, q) \in S \} \quad \triangleright \) the induced partition labeled by \( \mathcal{B} \)  
13: return \((\mathcal{P}', \omega)\)

**Postcondition:** \( \{ q \mid (b, q) \in \mathcal{P}' \} = \hat{\mathcal{P}}|_W \) is the induced partition of \( W \) into convex polytopes; \( \mathcal{P}' \) is a list of pairs \( (b, q) \) such that \( \text{CODE}|_W(x) = b \) for every \( x \in q; \) the map \( \omega : \mathcal{B} \rightarrow \mathcal{A}^* \) extends to a morphism of monoid \( \omega : \mathcal{B}^* \rightarrow \mathcal{A}^* \) satisfying the following equation for one-dimensional subshifts: \( X_{\mathcal{P},T} = \omega \left( X_{\hat{\mathcal{P}},\hat{T}} \right) \); if \( T = R^{e_i} \) for some \( i \) and the \( \mathbb{Z}^2 \)-action \( R \) is Cartesian on the window \( W \), then \( \omega : \mathcal{B} \rightarrow \mathcal{A}^* \) defines a \( d \)-dimensional morphism in the direction \( e_i \); satisfying \( X_{\mathcal{P},R} = \omega \left( X_{\hat{\mathcal{P}},\hat{R}} \right) \).

The reason is that in the algorithms, polytopes are intersected sometimes with \( H_v \) and sometimes with the complement \( \mathbb{R}^d \setminus H_v \), that is, both sides of the hyperplane given by \( v \) and we want the result to be convex in both cases. Inducing on more general polytope \( W \) must be done in many steps (once for each inequalities defining \( W \)).

When considering \( \mathbb{Z}^d \)-action \( R \) on a polytope \( D \subset \mathbb{R}^d \) that is Cartesian on a window \( W \subset D \), it is possible to compute the induced \( \mathbb{Z}^d \)-action \( \hat{R}|_W \) by considering each subaction \( \hat{R}^{e_i}|_W \) individually for \( i \in \{ 1, \ldots, d \} \). In the next sections, we use the algorithms when the return times to \( W \subset D \) under \( R^{e_i} \) is 1 for all \( x \in W \) and for every direction \( e_j \) except for some \( j = i \). In that case, the set of return words \( \mathcal{L} = \text{RETURNWORD}(W) \) is a set of one-dimensional words in the direction \( e_i \) for some \( i \) and the substitution \( \omega \) is a \( d \)-dimensional morphism.

Algorithm 3 and Algorithm 4 are implemented in the module on polyhedron exchange transformations of the optional package slabbe \cite{Lab20b} for SageMath \cite{Sag20}. 

The reason is that in the algorithms, polytopes are intersected sometimes with \( H_v \) and sometimes with the complement \( \mathbb{R}^d \setminus H_v \), that is, both sides of the hyperplane given by \( v \) and we want the result to be convex in both cases. Inducing on more general polytope \( W \) must be done in many steps (once for each inequalities defining \( W \)).
Algorithm 4 Compute the induced transformation $\hat{T}|_W$

Precondition: $T$ is a polytope exchange transformation (PET) on a convex domain $D \subset \mathbb{R}^d$ given as a pair $(\mathcal{P}, h)$ where $\mathcal{P}$ is a list of pairs $(a, p)$ such that $\{p \mid (a, p) \in \mathcal{P}\}$ is a partition of $D$ into convex polytopes, $\mathcal{A} = \{a \mid (a, p) \in \mathcal{P}\}$ is some alphabet and $h : \mathcal{A} \to \mathbb{R}^d$ is a map such that $(a, p) \in \mathcal{P}$ implies that $T(x) = x + h(a)$ for all $x \in p$, the map $h$ extends to a morphism of monoids $h : \mathcal{A}^* \to \mathbb{R}^d$ satisfying $h(u \cdot v) = h(u) + h(v)$; $v \in \mathbb{R}^{d+1}$ defines a half space $H_v = \{x \in \mathbb{R}^d \mid v_0 + \sum_{i=1}^d v_i x_i \geq 0\}$ such that $D \setminus H_v = W$.

1: function InducedTransformation($T$, $v$) 
2: $(\mathcal{P}, h) \leftarrow T$
3: $(\mathcal{P}', \omega) \leftarrow \text{InducedPartition}(T, v, \mathcal{P})$
4: $T' \leftarrow (\mathcal{P}', h \circ \omega)$ 
   $\triangleright$ the induced transformation
5: return $(T', \omega)$

Postcondition: $T'$ is a PET equal to the induced transformation $\hat{T}|_W$ given as a pair $(\mathcal{P}', h \circ \omega)$ where $\{q \mid (b, q) \in \mathcal{P}'\}$ is a partition of $W$ into convex polytopes, $\mathcal{B} = \{b \mid (b, q) \in \mathcal{P}'\}$ is some alphabet, $h \circ \omega : \mathcal{B}^* \to \mathbb{R}^d$ is a morphism of monoids such that $(b, q) \in \mathcal{P}'$ implies that $T'(x) = x + h \circ \omega(b)$ for all $x \in q$; $\omega : \mathcal{B}^* \to \mathcal{A}^*$ is a morphism of monoid satisfying the following equation for one-dimensional subshifts: $X_{\mathcal{P}, T} = \omega \left( X_{\mathcal{P}|_W, \hat{T}|_W} \right)$.

Exercise 5.7 Using the implementation of Algorithm 4 in SageMath, confirm the computations made by hand in Exercise 5.5. More precisely, describe $R_{\mathcal{U}}^{e_1}|_W$ and $R_{\mathcal{U}}^{e_2}|_W$ as polygon exchange transformations on $W_0$.

Exercise 5.8 Using the implementation of Algorithm 3 in SageMath, confirm the computations made by hand in Exercise 5.6.

The answers to the above two exercises and the code to reproduce the computations in SageMath are available in the next section in the proof of Theorem 1.1 (ii).

5.6. Self-similarity of the subshift $X_{\mathcal{P}_U, R_U}$. We induce the topological partition $\mathcal{P}_U$ until the process loops. We need two induction steps before obtaining a topological partition which is self-induced.

The proof contains SageMath code using the slabbe optional package [Lab20b] to reproduce the computation of the induced partitions and 2-dimensional morphisms.

Proof of Theorem 1.1 (ii). First, we define the golden mean $\phi$ as an element of a number field defined by a quadratic polynomial which is more efficient when doing arithmetic operations and comparisons. We also import the necessary functions.

sage: z = polygen(QQ, "z")
sage: K.<phi> = NumberField(z**2-z-1, "phi", embedding=RR(1.6))
sage: from slabbe import PolyhedronExchangeTransformation as PET
sage: from slabbe.arXiv_1903_06137 import self_similar_19_atoms_partition
The proof uses Proposition 5.12 two times to induce both the vertical and horizontal actions, starting with the vertical action. We begin with the lattice $\Gamma_0 = \mathbb{Z}^2$, the partition $\mathcal{P}_U$, the coding map $\text{Code}_0 : \mathbb{R}^2 / \Gamma_0 \to A_0$, the alphabet $A_0 = [0, 18]$ and $\mathbb{Z}^2$-action $R_U$ defined on $\mathbb{T}^2$ as shown below.

We consider the window $W_0 = (0, 1) \times (0, \varphi^{-1}) + \Gamma_0$ as a subset of $\mathbb{R}^2 / \Gamma_0$. The action $R_U$ is Cartesian on $W_0$. Thus from Lemma 5.9, $R_1 := R_U|_{W_0} : \mathbb{Z}^2 \times W_0 \to W_0$ is a well-defined $\mathbb{Z}^2$-action. From Lemma 5.3, the $\mathbb{Z}^2$-action $R_1$ can be seen as toral translation on $\mathbb{R}^2 / \Gamma_1$ with $\Gamma_1 = \mathbb{Z} \times (\varphi^{-1}\mathbb{Z})$, see Exercise 5.5. Let $\mathcal{P}_1 = \mathcal{P}_U|_{W_0}$ be the induced partition. From Proposition 5.12, then $\mathcal{X}_{\mathcal{P}_U,R_U} = \beta_0(\mathcal{X}_{\mathcal{P}_1,R_1})$. The partition $\mathcal{P}_1$, the action $R_1$ and substitution $\beta_0$ are given below with alphabet $A_1 = [0, 20]$.

We consider the window $W_0 = (0, 1) \times (0, \varphi^{-1}) + \Gamma_0$ as a subset of $\mathbb{R}^2 / \Gamma_0$. The action $R_U$ is Cartesian on $W_0$. Thus from Lemma 5.9, $R_1 := R_U|_{W_0} : \mathbb{Z}^2 \times W_0 \to W_0$ is a well-defined $\mathbb{Z}^2$-action. From Lemma 5.3, the $\mathbb{Z}^2$-action $R_1$ can be seen as toral translation on $\mathbb{R}^2 / \Gamma_1$ with $\Gamma_1 = \mathbb{Z} \times (\varphi^{-1}\mathbb{Z})$, see Exercise 5.5. Let $\mathcal{P}_1 = \mathcal{P}_U|_{W_0}$ be the induced partition. From Proposition 5.12, then $\mathcal{X}_{\mathcal{P}_U,R_U} = \beta_0(\mathcal{X}_{\mathcal{P}_1,R_1})$. The partition $\mathcal{P}_1$, the action $R_1$ and substitution $\beta_0$ are given below with alphabet $A_1 = [0, 20]$.

$$
\beta_0 : A_1 \to A_0^2
\begin{align*}
0 &\mapsto (8), & 1 &\mapsto (9), & 2 &\mapsto (11), & 3 &\mapsto (13), \\
4 &\mapsto (14), & 5 &\mapsto (15), & 6 &\mapsto (16), & 7 &\mapsto (17), \\
8 &\mapsto (0, 8), & 9 &\mapsto (1, 9), & 10 &\mapsto (1, 10), & 11 &\mapsto (1, 11), \\
12 &\mapsto (6, 12), & 13 &\mapsto (4, 13), & 14 &\mapsto (7, 13), & 15 &\mapsto (2, 14), \\
16 &\mapsto (6, 14), & 17 &\mapsto (7, 15), & 18 &\mapsto (3, 16), & 19 &\mapsto (3, 17), \\
20 &\mapsto (5, 18).
\end{align*}
$$
We consider the window $W_1 = (0, \varphi^{-1}) \times (0, \varphi^{-1}) + \Gamma_1$ as a subset of $\mathbb{R}^2/\Gamma_1$. The action $R_1$ is Cartesian on $W_1$. Thus from Lemma 5.9, $R_2 := \tilde{R}_1|_{W_1} : \mathbb{Z}^2 \times W_1 \to W_1$ is a well-defined $\mathbb{Z}^2$-action. From Lemma 5.3, the $\mathbb{Z}^2$-action $R_2$ can be seen as toral translation on $\mathbb{R}^2/\Gamma_2$ with $\Gamma_2 = (\varphi^{-1}\mathbb{Z}) \times (\varphi^{-1}\mathbb{Z})$. Let $\mathcal{P}_2 = \tilde{\mathcal{P}}_1|_{W_1}$ be the induced partition. From Proposition 5.12, then $\mathcal{X}_{{\mathcal{P}_1}, R_1} = \beta_1(\mathcal{X}_{{\mathcal{P}_2}, R_2})$. The partition $\mathcal{P}_2$, the action $R_2$ and substitution $\beta_1$ are given below with alphabet $\mathcal{A}_2 = [0, 18]$:

\[
\beta_1 : \mathcal{A}_2 \to \mathcal{A}_1^2
\]

\[
\begin{align*}
0 &\mapsto (6), \quad 1 \mapsto (7), \quad 2 \mapsto (15), \quad 3 \mapsto (16), \\
4 &\mapsto (18), \quad 5 \mapsto (19), \quad 6 \mapsto (3, 1), \quad 7 \mapsto (4, 0), \\
8 &\mapsto (5, 0), \quad 9 \mapsto (5, 2), \quad 10 \mapsto (6, 0), \quad 11 \mapsto (7, 0), \\
12 &\mapsto (12, 9), \quad 13 \mapsto (13, 9), \quad 14 \mapsto (14, 9), \quad 15 \mapsto (15, 8), \\
16 &\mapsto (16, 11), 17 \mapsto (17, 11), 18 \mapsto (20, 10).
\end{align*}
\]

Now it is appropriate to rescale the partition $\mathcal{P}_2$ by the factor $-\varphi$. Doing so, the new obtained action $R_2'$ is the same as two steps before, that is, $R_2'$ on $\mathbb{R}^2/\mathbb{Z}^2$. More formally, let $h : (\mathbb{R}/\varphi^{-1}\mathbb{Z})^2 \to (\mathbb{R}/\mathbb{Z})^2$ be the homeomorphism defined by $h(x) = -\varphi x$. We define $\mathcal{P}'_2 = h(\mathcal{P}_2)$, $\text{CODE}'_2 = \text{CODE}_2 \circ h^{-1}$, $(R'_2)^n = h \circ (R_2)^n \circ h^{-1}$ as shown below:

\[
R_2^n(x) = x - \varphi^{-3}n
\]
We observe that the scaled partition $P'_2$ is the same as $P_U$ up to a permutation $\beta_2$ of the indices of the atoms in such a way that $\beta_2 \circ \text{CODE}_0 = \text{CODE}_2$. The partition $P_U$, the action $R_U$ and substitution $\beta_2 : A_2 \to A_0$ are given below.

By construction, the following diagrams commute:

Using the above commutative properties, for every $y \in \mathbb{R}^2/\Gamma_2$ and $n \in \mathbb{Z}^2$, we have

Thus $X_{P'_2,R_2} = \beta_2(X_{P_U,R_U})$. We may check that $\beta_0 \circ \beta_1 \circ \beta_2 = \phi$:
sage: from slabbe import Substitution2d
sage: Phi = Substitution2d(  
....:  [0: [[17]], 1: [[16]], 2: [[15], [11]],  
....:  3: [[13], [9]], 4: [[17], [8]], 5: [[16], [8]], 6: [[15], [8]],  
....:  7: [[14], [8]], 8: [[14, 6]], 9: [[17, 3]], 10: [[16, 3]],  
....:  11: [[14, 2]], 12: [[15, 7], [11, 1]], 13: [[14, 6], [11, 1]],  
....:  14: [[13, 7], [9, 1]], 15: [[12, 6], [9, 1]], 16: [[18, 5], [10, 1]],  
....:  17: [[13, 4], [9, 1]], 18: [[14, 2], [8, 0]])

sage: beta0 * beta1 * beta2 == Phi
True

We conclude that

\[
X_{P_{U,R_{U}}} = \beta_0 (X_{P_{1,R_{1}}})^\sigma = \beta_0 \beta_1 (X_{P_{2,R_{2}}})^\sigma = \beta_0 \beta_1 \beta_2 (X_{P_{U,R_{U}}})^\sigma = \phi (X_{P_{U,R_{U}}})^\sigma.
\]

Exercise 5.9

Building on Exercise 3.12 prove that \(X_{P_{U,R_{U}}} = X_\phi\).

Exercise 5.10

Prove the self-similarity of \(X_{P_{U,R_{U}}}\) by doing the induction first horizontally with \(R_{U}^{e_1}\), and then vertically with \(R_{U}^{e_2}\). Compare with the result of Exercise 4.8. See also Exercise 3.16.

6. Conclusion

In Section 3, we defined the 2-dimensional subshift \(X_\phi\) from some 2-dimensional morphism \(\phi\). We showed that \(X_\phi\) is the unique nonempty self-similar subshift \(X \subset [0, 18]^2\) such that \(X = \overline{\phi(X)}\).

In Section 4, we proved that \(\Omega_\mathcal{U} = \overline{\phi(\Omega_\mathcal{U})}\) using the desubstitution of Wang shifts using marker tiles. In Section 5, we proved that \(X_{P_{U,R_{U}}} = \overline{\phi(X_{P_{U,R_{U}}})}\) using induction of \(\mathbb{Z}^2\)-rotations. Thus, we have the equality

\[
X_\phi = \Omega_\mathcal{U} = X_{P_{U,R_{U}}}.
\]

providing three different characterizations of the same aperiodic 2-dimensional subshift.

The link between the subshifts \(\Omega_\mathcal{U}\) and \(X_{P_{U,R_{U}}}\) can be explained directly without the 2-dimensional morphism \(\phi\) by the existence of a factor map from \((\Omega_\mathcal{U}, \mathbb{Z}^2, \sigma)\) to \((\mathbb{T}^2, \mathbb{Z}^2, R_{U})\). It turns out that the factor is also an isomorphism of strictly ergodic measure-preserving dynamical systems.

**Theorem 6.1.** [Lab20a] The Wang shift \(\Omega_\mathcal{U}\) has the following properties:

1. the subshift \(X_\phi = \Omega_\mathcal{U} = X_{P_{U,R_{U}}}\) is minimal and aperiodic,
2. \(P_\mathcal{U}\) is a Markov partition for the dynamical system \((\mathbb{T}, \mathbb{Z}^2, R_{U})\),
3. \((\mathbb{T}, \mathbb{Z}^2, R_{U})\) is the maximal equicontinuous factor of \((\Omega_\mathcal{U}, \mathbb{Z}^2, \sigma)\),
4. the set of fiber cardinalities of the factor map \(\Omega_\mathcal{U} \to \mathbb{T}\) is \(\{1, 2, 8\}\),
5. \((\Omega_\mathcal{U}, \mathbb{Z}^2, \sigma)\) is strictly ergodic and the measure-preserving dynamical system \((\Omega_\mathcal{U}, \mathbb{Z}^2, \sigma, \nu)\) is isomorphic to \((\mathbb{T}, \mathbb{Z}^2, R_{U}, \lambda)\) where \(\nu\) is the unique shift-invariant probability measure on \(\Omega_\mathcal{U}\) and \(\lambda\) is the Haar measure on \(\mathbb{T}\).
Moreover, there exists a 4-to-2 cut and project scheme such that the set of occurrences of any pattern in $\Omega_4$ is a model set.

**Theorem 6.2.** \([\text{Lab20a}]\) There exists a cut and project scheme such that for every configuration $w \in \Omega_4$, the set $Q \subseteq \mathbb{Z}^2$ of occurrences of a pattern in $w$ is a regular model set. If $w$ is a generic (resp. singular) configuration, then $Q$ is a generic (resp. singular) model set.

We refer the reader to \([\text{Lab20a}]\) where the proofs of Theorem 6.1 and Theorem 6.2 can be found.

**Exercise 6.1**
Using Definition 5.5, prove that the topological partition $\mathcal{P}_4$ of $\mathbb{T}^2$ is a Markov partition for the dynamical system $(\mathbb{T}^2, \mathbb{Z}^2, R_4)$.

**Exercise 6.2**
Using SageMath, verify that the equalities $\beta_0 = \alpha_0$, $\beta_1 = \alpha_1$ and $\beta_2 = \alpha_2$ hold. As a consequence, what is the relation between $\Omega_\mathcal{V}$ and $\mathcal{X}_{\mathcal{P}_1, R_1}$? What is the relation between $\Omega_\mathcal{W}$ and $\mathcal{X}_{\mathcal{P}_2, R_2}$?

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