Self-force on a scalar particle in a class of wormhole spacetimes

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We consider the self-energy and the self-force for scalar massive and massless particles at rest in the wormhole space-time. We develop a general approach to obtain the self-force and apply it to the two specific profiles of the wormhole throat, namely, with singular and with smooth curvature. We found that the self-force changes its sign at the point where nonminimal coupling $\xi = 1/8$ (for massless case) and it tends to infinity for specific values of $\xi$. It may be attractive as well repulsive depending on the profile of the throat. For massless particle and minimal coupling case the electromagnetic results are recovered.

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I. INTRODUCTION

The wormholes are topological bridges which connect different universes or different parts of the same universe. These kind of tunnels in space-time have been appeared in different contexts of physics: in the analysis of black hole backgrounds \cite{1,2}, as an idea to construct ”charge without charge” and ”mass without mass” \cite{3,4}, and as a possibility for time machine \cite{5,6}. The wormholes require an amount of exotic matter which breaks energy conditions of a solution of Einstein equations. There are many exotic ideas how to produce exotic matter and how much exotic matter one needs to make possible existence of wormhole. Carefull discussion of wormhole’s geometry and physics may be found in the Visser book \cite{7}, in the review by Lobo \cite{8}, as well as in paper \cite{9} related with astrophysical implementation of wormholes. Among interesting publications we can mention Ref. \cite{10} which analysed numerically the process of the passage of a radiation pulse through a wormhole and the subsequent evolution of the wormhole. It was shown that the wormhole is unstable and it is transformed into a spacetime with horizon. The analysis was made for normal as well as exotic matter pulse formed by scalar fields.

It is well known that a particle in curved spacetimes may interact with the gravitational background by specific interaction due to self-force \cite{11}. The origin of this self-force is associated with nonlocal structure of the field, the source of which the particle is. The self-force may be the unique gravitational interaction on a particle as it happens for particles in the space-time of a cosmic string \cite{12}. In this case there is no gravitational interaction particle with cosmic string, but nevertheless there exist the repulsive self-interaction force. In contrast to standard self-interaction Dirac-Lorentz force \cite{13}, the self-interaction force in curved space-times depends on all history of the particle and it is usually non-zero even for a particle at rest. For a particle at rest it may be found as coincidence limit of the renormalized Green function \cite{14}. A detailed discussion of the self-force maybe found in reviews \cite{15,16}.

In a recent paper \cite{17} the self-force for an electromagnetically charged particle at rest in the static wormhole background was analyzed in detail. The general expression for self-energy for arbitrary profile of the throat was obtained. It was shown that the particle is attracted by the wormhole and this effect may has astrophysical applications. For a specific profile of the throat the result was confirmed by Linet in Ref. \cite{18} using a different approach. There is also another approach for this question which was developed by Krasnikov in recent publication \cite{19} for a specific profile of the throat. The difference of results connects with understanding the self-force itself. The self-force for scalar particle reveals peculiarities \cite{20,21,22,23} due to nonminimal coupling of the scalar and the gravitational fields. For example, in Schwarzschild spacetime the self-force on a particle at rest is zero for minimal coupling \cite{20}. In the present paper we analyse in detail the self-force on a scalar particle at rest in the background of wormholes. We found that the self-force has crucial dependence on the nonminimal coupling constant $\xi$: it is zero for $\xi = 1/8$ in massless case and it tends to infinity for specific values of $\xi$.

The organization of this paper is as follows. In Sec. II we develop our approach and consider the origin for the divergence of self-energy for specific values of $\xi$, from the point of view quantum mechanics. In Sec. III we consider the self-energy and self-force for massive and massless cases for two specific profiles and for a general profile of the throat. Section IV is devoted to the discussion of the results. Throughout this paper we use units $c = G = 1$.

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II. APPROACH

Firstly let us say some words about the background under consideration. We use the following line element of the spherically symmetric wormhole space-time

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2),$$

where the profile function $r(\rho)$ describes the shape of the throat and $d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$. The variables are defined in the following ranges: $t, \rho \in \mathbb{R}$, $\theta \in [0, \pi]$ and $\varphi \in [0, 2\pi]$. The radius of the throat is defined by $a = r(0)$ and $r'(0) = 0$. The nonzero components of the Ricci tensor and scalar curvature read:

$$R^p_p = -\frac{2r''}{r},$$
$$R^\varphi_\varphi = -\frac{1 + r'^2 + rr''}{r^2},$$
$$R = -\frac{2(1 + r'^2 + 2rr'')}{r^2}.$$

The three dimensional section corresponding to constant time of this space-time is conformally flat. Indeed, let us consider the 3D flat space in spherical coordinates $\tilde{r}, \tilde{\theta}, \tilde{\varphi}$. Thus, we can write

$$dl^2_{fl} = d\tilde{r}^2 + \tilde{r}^2 d\Omega^2.$$

Let us choose a new radial coordinate $\rho$ by the relation $\tilde{r} = r(\rho)e^{\sigma(\rho)}$ with

$$\sigma = \pm \int_0^\rho \frac{dx}{r(x)} - \ln r(\rho).$$

Using this coordinate system we obtain

$$dl^2_{fl} = e^{2\sigma}(d\rho^2 + r(\rho)^2 d\Omega^2) = e^{2\sigma}dl^2_{wh}.$$ 

Therefore $g^{wh}_{ik} = e^{-2\sigma}g^{fl}_{ik}$ and the 3D section is conformally flat. For this reason it is expected \[24\] that the self-force is zero for $\xi = 1/8$ and $m = 0$ as will be shown by manifest calculations. More information about the wormhole’s space-time may be found in book \[7\] and review \[8\]. Some figures for specific profile of throat are in Ref. \[25\], and the physics in this spacetime is discussed in Ref. \[26\].

Let us consider a massive scalar field, $\phi$, with scalar source, $j$, which lives in the wormhole background with non-conformal coupling $\xi$. The action consists of two part, the first one is for the field itself and the second one describes the interaction of the source, a scalar charge $e$, with the field and is given by

$$S = -\frac{1}{8\pi} \int (\phi_{,\mu}\phi^{,\mu} + \xi R\phi^2 + m^2\phi^2)\sqrt{-g}d^4x + \int j\phi\sqrt{-g}d^4x.$$

The variation of this action with respect to the metric gives the energy momentum tensor of the field with contribution due to the interaction field with charge

$$T_{\mu\nu} = j\phi g_{\mu\nu} + \frac{1}{4\pi} \left( \phi_{,\mu}\phi_{,\nu} - \frac{1}{2} g_{\mu\nu}\phi_{,\sigma}\phi^{,\sigma} - \frac{1}{2} m^2 g_{\mu\nu}\phi^2 \right) + \frac{\xi}{4\pi} \left( G_{\mu\nu}\phi^2 + g_{\mu\nu}\Box\phi^2 - \phi^2\phi_{,\mu\nu} \right),$$

while the variation with respect to the field gives the equation of motion

$$(\Box - \xi R - m^2)\phi = -4\pi j,$$

with scalar current

$$j(x) = e \int \delta^{(4)}(x - x_p(r)) \frac{d\tau}{\sqrt{-g}}.$$

We consider only the case in which the particle is at rest at the point $x_p$ in the wormhole space-time. This means that there is no dependence on the time and the equation of motion for the field has the following form

$$(\triangle - m^2 - \xi R)\phi(x) = -4\pi j = -\frac{4\pi e}{\sqrt{-g}} \delta^{(3)}(x - x_p).$$
From the general point of view the full energy of particle reads

\[ E = - \int T_{\mu\nu} \xi^\mu d\Sigma', \]  

where \( \Sigma \) is 3-surface with \( \xi^\mu \) as a normal. The spacetime under consideration possesses the time-like Killing vector \( \xi^\mu = \delta_0^\mu \). Thus choosing the hypersurface of constant time we obtain

\[ E = - \int T_{00} \sqrt{-g} d^3x, \]  

where for the static case under consideration we have

\[ T_{00} = -j \phi + \frac{1}{8\pi} (\phi,\phi) + (\xi R + m^2)\phi^2 - \frac{\xi}{2\pi} (\phi,\phi) + \phi \Delta \phi. \]  

Integrating by part and taking into account the equation of motion (10) we get

\[ E = \frac{1}{2} \int j \phi \sqrt{-g} d^3x = \frac{1}{2} \int j(x)G(x,x')j(x')\sqrt{-g(x')\sqrt{-g(x)}}d^3xd^3x' = \frac{e^2}{2} G(x_p,x_p). \]

The Green’s function \( G(x,x') \) which we need for calculation self-force is divergent in the coincidence limit, \( x' \to x \). Several methods to obtain a finite result are known. The most simple way is to consider total mass as a sum of observed finite mass and an infinite electromagnetic contribution. Usually this procedure is called ”classical renormalization” because there is no Planck constant in the divergent term. Dirac [13] suggested to consider radiative Green’s function to calculate the self-force. Since the radiative Green’s function is the difference between retarded and advanced Green’s functions, singular contribution cancels out and we obtain a finite result. There is also an axiomatic approach suggested by Quinn and Wald [27]. In the framework of this approach one obtains finite expression by using a ”comparison” axiom. This approach was used in Refs. [28, 29] for a specific space-time.

There is a problem with renormalization for massive uncharged particle (see Ref. [30]). The self-force for charged particle is obtained by classical renormalization of particle’s mass, \( m \to m + Ae^2 \), where \( A \) is some infinite constant. Therefore for massive and uncharged particle we have to renormalize mass by using the same mass of particle. To solve this problem in the Ref. [30] has suggested to consider particle as Schwarzschild solution in external gravitational field. In this way it is possible to obtain finite result in lower power of mass. More rigorous derivation has been done in Ref. [31].

We will use a general approach to renormalization in curved space-time [32], which means subtraction of the first terms from DeWitt-Schwinger asymptotic expansion of the Green’s function. In general there are two kinds of divergences in this expansion, namely, pole and logarithmic ones [33]. In three-dimensional case which we are interested in there is only pole divergence, while the logarithmic term is absent. The singular part of the Green’s function, which must be subtracted, has the following form (in 3D case)

\[ G^{\text{sing}} = \frac{1}{4\pi} \left\{ \frac{\Delta^{1/2}}{\sqrt{2\sigma}} + m \right\}, \]

where \( \sigma \) is half of the square of geodesic distance and \( \Delta \) is the DeWitt-Morette determinant. If we take the coincidence limit for angular variables then these quantities are easily calculated using the metric [11]:

\[ \sigma = (\rho - \rho')^2/2 \] and \( \Delta = 1 \). Thus to carry out renormalization we should subtract the singular part of the Green’s function, which has the following form:

\[ G^{\text{sing}} = \frac{1}{4\pi} \frac{1}{|\rho - \rho'|} + \frac{m}{4\pi}. \]

This approach was used many times in different curved backgrounds (e.g. see [10]). Therefore the self-energy \( U \) for particle at rest at the point \( x_p \) has the following form

\[ U(x_p) = \frac{e^2}{2} G^{\text{reg}}(x_p,x_p) = \frac{e^2}{2} \lim_{x' \to x_p} \left\{ G(x_p,x') - G^{\text{sing}}(x_p,x') \right\}. \]  

For simplification of notations we will denote hereinafter the position of the particle by \( x = x_p \).

In order to find the self-energy we have to calculate the 3D Green function which obeys the equation

\[ (\Delta - m^2 - \xi R)G(x;x') = -\frac{\delta^{(3)}(x-x')}{\sqrt{-g}}, \]  

where \( \delta^{(3)} \) is the coincidence limit for angular variables then these quantities are easily calculated using the metric (1):

\[ = \frac{1}{2\pi} (\phi,\phi) + (\xi R + m^2)\phi^2 - \frac{\xi}{2\pi} (\phi,\phi) + \phi \Delta \phi. \]
and then adopt the renormalization procedure. Due to spherical symmetry we can expand the Green function using spherical functions as

\[
G(x; x') = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y_{l,m}^*(\Omega)Y_{l,m}(\Omega) g_l(\rho, \rho') = \sum_{l=0}^{\infty} \frac{2l+1}{4\pi} P_l(\cos \gamma) g_l(\rho, \rho').
\]  

(16)

Here \(P_l\) are the Legendre polynomials and \(\cos \gamma = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\varphi - \varphi')\).

The radial Green function, \(g_l(\rho, \rho')\), obeys the following equation

\[
g''_l + \frac{2r'}{r} g'_l - \left( m^2 + \frac{l(l+1)}{r^2} + \xi R \right) g_l = -\frac{\delta(\rho - \rho')}{r^2}.
\]  

(17)

Differently from the electromagnetic case \[17\] we have a contribution which arises from the non-minimal coupling, even for massless case.

Now, we adopt the approach developed in Ref. \[17\] with some modifications. We represent the Green function in the following form

\[
g_l = \theta(\rho - \rho')\Psi_2(\rho)\Psi_1(\rho') + \theta(\rho' - \rho)\Psi_2(\rho')\Psi_1(\rho),
\]  

(18)

where \(\Psi_1\) and \(\Psi_2\) are independent solutions of the corresponding homogeneous equation

\[
\Psi'' + \frac{2r'}{r} \Psi' - \left( m^2 + \frac{l(l+1)}{r^2} + \xi R \right) \Psi = 0,
\]  

(19)

with boundary conditions

\[
\lim_{\rho \to \infty} \Psi_2 = 0, \quad \lim_{\rho \to \infty} \Psi_1 \neq 0,
\]  

(20)

and Wronskian condition

\[
\Psi_1'\Psi_2 - \Psi_2'\Psi_1 = -\frac{1}{r^2}.
\]  

(21)

It is worthy calling attention to the fact that if we change the function \(\Psi = \Phi/r\) in Eq. (19), we obtain

\[
\Phi'' + \left( -m^2 - \frac{l(l+1)}{r^2} - \xi R - \frac{r''}{r} \right) \Phi = 0.
\]  

(22)

From the quantum mechanical point of view, Eq. (22) describes a quantum particle in the potential

\[
V = \xi R + \frac{r''}{r}
\]  

(23)

Let us consider this potential far from the wormhole’s throat, \(\rho \gg a\). The behavior of the potential crucially depends on the profile function \(r(\rho)\). We divide the profiles of the throat in two large classes depending on its behavior at infinity. The first class may be called as wormhole without parameter of the throat’s length. In this case the expansion of the profile function over \(\rho \to \infty\) has a polynomial form

\[
r = \rho + \sum_{k=n}^{\infty} b_k \rho^{-k},
\]  

(24)

which starts from \(\rho^{-n}\) with some integer \(n \geq 0\). The potential far from the wormhole’s throat has the following expansion

\[
\xi R + \frac{r''}{r} = \frac{4n(\xi_c - \xi)}{\rho^{n+3}}b_n + \cdots
\]  

(25)

where \(\xi_c = (n+1)/4n\). The point \(\xi_c\) is crucial because at this point the potential changes its sign. The smaller \(n\), the greater the critical value of \(\xi\) and vice-versa. The case \(n = 0\) in some sense corresponds to the following “weak” dependence of the profile of the throat on \(\rho\) at infinity

\[
r(\rho) = \rho + a + \frac{b}{(\ln \rho)^n}
\]  

(26)
with \( n > 0 \). Indeed, in this case there is no critical value of \( \xi \). This kind of expansion corresponds to \( n \to 0 \) in the above case and \( \xi_c \to \infty \).

Another kind of wormholes have a dimensional parameter, \( \tau \), which describes the throat’s length. In this case the expansion for \( \rho \to \infty \) has the following form

\[
 r(\rho) = \rho + a + c_n \rho^n e^{-\rho / \tau}, \quad n \geq 0, 
\]

and the space-time becomes flat exponentially. The expansion of the potential starts from the following term

\[
 \xi R + \frac{\rho''}{\rho} = c_n \frac{1 - 4\xi}{\tau^2} \rho^{n-1} e^{-\rho / \tau} + \ldots .
\]

Therefore the critical value of \( \xi \) is 1/4 and it does not depend on \( n \). Therefore, we claim the following statement: The critical value for the first kind of throat is \( \xi_c = (n + 1)/4n \) if \( r - \rho \sim \rho^{-n} \), and for second kind is \( \xi_c = 1/4 \). Obviously the delta function is a short range potential and it belongs to the second kind of wormhole as we will see later. In the mixed case the main role is played by the polynomial part of the expansion.

Let us consider specific examples. For \( r = \sqrt{\rho^2 + a^2} \) we have the first kind of throat and \( n = 1, \ b_1 = 1/2, \) the critical value of \( \xi_c \) is 1/2. For the profiles

\[
 r = \rho \coth \frac{\rho}{\tau} + a - \tau, \\
 r = \rho \tanh \frac{\rho}{\tau} + a,
\]

we have correspondingly \( n = 1, b = 1/2 \) and \( n = 1, b = -1/2 \) and therefore the critical value \( \xi_c = 1/4 \). The case of a singular potential is a limiting case of shortest length of the throat and it belongs to the second case. Therefore, we can expect peculiarities for \( \xi \approx \xi_c \). Below we will see them in manifest forms.

Let us consider the case of singular scalar curvature separately. The point is that for the throat profile \( r = |\rho| + a \), the scalar curvature reads

\[
 R = -\frac{8}{a} \delta(\rho)
\]

and we have the following radial equation

\[
 \Psi'' + \frac{2r'}{r} \Psi' - \left( m^2 + \frac{l(l + 1)}{r^2} - \frac{8\xi}{a} \delta(\rho) \right) \Psi = 0.
\]

Integrating this equation around \( \rho = 0 \) we obtain the following matching conditions at the throat

\[
 \Psi(+0) - \Psi(-0) = 0, \\
 \Psi'(+0) - \Psi'(-0) = -\frac{8\xi}{a} \Psi(+0).
\]

Let us now represent the solutions \( \Psi \) as linear combinations of two independent solutions in each domain of the wormhole spacetime, ”+” and ”–”, which correspond to the signs of \( \rho \). In each domain we find two independent solutions \( \phi_1^\pm \) and \( \phi_2^\pm \) with the condition that \( \phi_2^+ \) falls down for \( \rho \to \infty \) and the Wronskian condition

\[
 W(\phi_1^+, \phi_2^+) = \frac{A_+}{r^2},
\]

where \( A_\pm \) are constants. Therefore, we have in general

\[
 \Psi_1 = \begin{cases} 
 \frac{\alpha_1^+ \phi_1^+ + \beta_1^+ \phi_2^+}{\alpha_1^+ - \beta_1^+}, & \rho > 0 \\
 \frac{\alpha_1^- \phi_1^+ + \beta_1^- \phi_2^+}{\alpha_1^- - \beta_1^-}, & \rho < 0,
\end{cases}
\]

\[
 \Psi_2 = \begin{cases} 
 \frac{\alpha_2^+ \phi_1^+ + \beta_2^+ \phi_2^+}{\alpha_2^+ - \beta_2^+}, & \rho > 0 \\
 \frac{\alpha_2^- \phi_1^+ + \beta_2^- \phi_2^+}{\alpha_2^- - \beta_2^-}, & \rho < 0,
\end{cases}
\]

The Wronskian condition implies the following constraints on the coefficients:

\[
 \alpha_1^+ \beta_2^+ - \beta_1^+ \alpha_2^+ = -\frac{1}{A_+}.
\]

Taking into account the boundary conditions (31), then the solutions \( \Psi_1 \) and \( \Psi_2 \) turns into the following forms

\[
 \Psi_1 = \alpha_1^+ \phi_1^+ + \beta_1^+ \phi_2^+,
\]

\[
 \Psi_2 = \alpha_2^- \phi_1^+ + \beta_2^- \phi_2^+.
\]
\[ \Psi_2 = \alpha_+^2 \phi_1^2 + \beta_+^2 \phi_2^2, \]

where

\[
\begin{align*}
\phi_1^1 &= \left\{ \phi_1^+ \left\{ W(\phi_1^+, \phi_2^+) - \frac{8 \xi \phi_1^1 \phi_2^1}{a W(\phi_1^+, \phi_2^+)} \right\}_0 + \phi_2^+ \left\{ W(\phi_1^+, \phi_2^+) - \frac{8 \xi \phi_1^1 \phi_2^1}{a W(\phi_1^+, \phi_2^+)} \right\}_0, \rho > 0, \\
& \quad \phi_1^1 = \left\{ \phi_1^- \left\{ W(\phi_1^-, \phi_2^-) + \frac{8 \xi \phi_1^1 \phi_2^1}{a W(\phi_1^-, \phi_2^-)} \right\}_0 + \phi_2^- \left\{ W(\phi_1^-, \phi_2^-) + \frac{8 \xi \phi_1^1 \phi_2^1}{a W(\phi_1^-, \phi_2^-)} \right\}_0, \rho < 0, \\
\phi_2^2 &= \left\{ \phi_1^+ \left\{ W(\phi_1^+, \phi_2^+) - \frac{8 \xi \phi_1^1 \phi_2^1}{a W(\phi_1^+, \phi_2^+)} \right\}_0 + \phi_2^+ \left\{ W(\phi_1^+, \phi_2^+) - \frac{8 \xi \phi_1^1 \phi_2^1}{a W(\phi_1^+, \phi_2^+)} \right\}_0, \rho > 0, \\
& \quad \phi_2^2 = \left\{ \phi_1^- \left\{ W(\phi_1^-, \phi_2^-) + \frac{8 \xi \phi_1^1 \phi_2^1}{a W(\phi_1^-, \phi_2^-)} \right\}_0 + \phi_2^- \left\{ W(\phi_1^-, \phi_2^-) + \frac{8 \xi \phi_1^1 \phi_2^1}{a W(\phi_1^-, \phi_2^-)} \right\}_0, \rho < 0.
\end{align*}
\]

In order to satisfy the boundary conditions for \( \Psi \), namely \( \lim_{\rho \to -\infty} \Psi_2 = 0 \), we have to take \( \alpha_+^2 = 0 \). For the second solution there is no additional condition to leave only one constant. In the space-time without wormhole we have the additional condition at the point \( \rho = 0 \) and therefore, the functions must be finite. Here we have no origin, there is a bridge starting from some distance of throat. For this reason we have to consider both possibilities. Let us consider the specific solution for a second solution for \( \alpha_+^1 = 0 \). It means that we consider the function which is symmetric to \( \Psi_2 \), and tends to zero in the mirror spacetime for \( \rho \to -\infty \). These solutions have the following form:

\[
\Psi_1 = \beta_+^1 \phi_2^1,
\]

\[
\Psi_2 = \beta_+^2 \phi_2^2.
\]

In what follows we will consider the symmetric profile of the throat \( r(-\rho) = r(\rho) \). Taking into account the above relations we obtain the radial Green's function in the following form

1. \( \rho > \rho' > 0 \)

\[
g_1^{(1)}(\rho, \rho') = -\frac{1}{A_+} \phi_2^2(\rho)\phi_1^1(\rho') + \frac{1}{A_+} \frac{W_+(\phi_1^1, \phi_2^1)}{W_+(\phi_1^2, \phi_2^2)} + \frac{8 \xi \phi_1^1 \phi_2^1}{a W_+(\phi_1^2, \phi_2^2)} \phi_2^2(\rho')\phi_1^1(\rho)
\]

2. \( 0 < \rho < \rho' \)

\[
g_1^{(2)}(\rho, \rho') = g_1^{(1)}(\rho', \rho)
\]

3. \( \rho < \rho' \) and \( \rho' > 0, \rho < 0 \)

\[
g_1^{(3)}(\rho, \rho') = -\frac{1}{A_+} \frac{W(\phi_1^1, \phi_2^1)}{W_+(\phi_1^2, \phi_2^2) + \frac{8 \xi \phi_1^1 \phi_2^1}{a \phi_2^2}} \phi_2^2(\rho')\phi_1^1(-\rho)
\]

4. \( \rho > \rho' \) and \( \rho' < 0, \rho > 0 \)

\[
g_1^{(4)}(\rho, \rho') = g_1^{(3)}(\rho', \rho')
\]

5. \( \rho' < \rho < 0 \)

\[
g_1^{(5)}(\rho, \rho') = g_1^{(1)}(-\rho, -\rho')
\]

6. \( \rho < \rho' < 0 \)

\[
g_1^{(6)}(\rho, \rho') = g_1^{(5)}(\rho', \rho)
\]

In fact, we have to write out only \( g_1^{(1)} \) and \( g_1^{(3)} \) in manifest form.

The second kind of solutions has the following form

\[
\Psi_1 = \alpha_+^2 \phi_1^2,
\]

\[
\Psi_2 = \beta_+^2 \phi_2^2.
\]
In this case the Green functions are given by

1. \( \rho > \rho' > 0 \)

\[
g_1^{(1)}(\rho, \rho') = -\frac{1}{A_+} \phi_+^2(\rho) \phi_+^1(\rho') + \frac{1}{A_+} \left. W_+\left(\phi_+^1, \phi_+^2\right) + \frac{8}{a} \phi_+^1 \phi_+^2 \right|_0 \phi_+^2(\rho') \phi_+^2(\rho) 
\]

(36a)

2. \( 0 < \rho < \rho' \)

\[
g_1^{(2)}(\rho, \rho') = g_1^{(1)}(\rho', \rho) 
\]

(36b)

3. \( \rho < \rho' > 0 \), \( \rho < 0 \)

\[
g_1^{(3)}(\rho, \rho') = -\frac{1}{A_+} \left. W_+\left(\phi_+^1, \phi_+^2\right) + \frac{8}{a} \phi_+^1 \phi_+^2 \right|_0 \phi_+^2(\rho') \phi_+^1(\rho) 
\]

(36c)

4. \( \rho > \rho' \) and \( \rho' < 0 \), \( \rho > 0 \)

\[
g_1^{(4)}(\rho, \rho') = g_1^{(3)}(\rho', \rho') 
\]

(36d)

5. \( \rho' < \rho < 0 \)

\[
g_1^{(5)}(\rho, \rho') = \frac{1}{A_+} \phi_+^1(-\rho') \phi_+^2(-\rho) - \frac{1}{A_+} \left. W_+\left(\phi_+^1, \phi_+^2\right) + \frac{8}{a} \phi_+^1 \phi_+^2 \right|_0 \phi_+^1(-\rho') \phi_+^1(-\rho) 
\]

(36e)

6. \( \rho < \rho' < 0 \)

\[
g_1^{(6)}(\rho, \rho') = g_1^{(5)}(\rho', \rho) 
\]

(36f)

For smooth background we have to set \( \xi = 0 \) in the above formulas except for the differential equation for the radial Green function.

From relation (36e) we observe that the second solution gives a divergent Green function in the domain \( \rho < 0 \) because the function \( g_1^{(5)} \) contains multiplication of functions which tends to infinity for great \( \rho \). For this reason we have to throw away this solution and consider them as unphysical and thus, the Green function is uniquely defined.

### III. SELF-ENERGY AND SELF-FORCE

Let us firstly call attention to the fact that for scalar particles it is possible to solve the problem concerning the determination of the self-force and self-energy in a closed form, only for an infinitely short throat and for the profile \( \sqrt{\rho^2 + a^2} \), in massless case. For arbitrary profile there is a problem to solve this problem for the zero mode \( l = 0 \). Otherwise, for the electromagnetic case, this problem can be solved in a closed form for arbitrary profile.

#### A. Profile \( r = |\rho| + a \)

Let us first of all consider the simplest profile of the throat given by \( r = |\rho| + a \).

1. **Massless case**

In this case we may use the same solutions as the one in Ref. [17]. We have the equation

\[
\phi'' + \frac{2r'}{r} \phi' - \frac{l(l+1)}{r^2} \phi = 0, 
\]

which has the following solutions

\[
\phi_+^1 = (a \pm \rho)^l, \\
\phi_+^2 = a^{2l+1}(a \pm \rho)^{-l-1}. 
\]
Taking into account the above formulas we obtain the following expression for the Green function

\[(2l + 1)g_1^{(1)} = \frac{r^l}{r^{l+1}} - \frac{a^{2l+1}}{r^{l+1}r''^{l+1}} \frac{1 - 8\xi}{2(l + 1) - 8\xi},\]

\[(2l + 1)g_1^{(3)} = \frac{a^{2l+1}}{r^{l+1}r''^{l+1}} \frac{1 - 8\xi}{2(l + 1) - 8\xi}.
\]

We use now the first expression above in the Eq. (10) with the coincidence limit for the angular variables, namely, \(\gamma = 0\), and thus we have

\[G(\rho; \rho') = \sum_{l=0}^{\infty} \frac{2l + 1}{4\pi} g_l(\rho, \rho'),\]

where \(\rho'\) is the observation point and \(\rho\) is the position of the particle. It is very easy to make the summation over \(l\). Doing this, we arrive at the following results

\[G(\rho, \rho') = \frac{1}{4\pi \rho - \rho'} - \frac{a(1 - 8\xi)}{8\pi r^{l'} r'} \Phi\left(\frac{a^2}{r^{l'}}, 1, 1 - 4\xi\right),\]

\[U(\rho) = -\frac{ae^2(1 - 8\xi)}{4\pi^2} \Phi\left(\frac{a^2}{\rho}, 1, 1 - 4\xi\right)\]

for Green function and self-energy, respectively. The first term in the Green function is the DeWitt-Schwinger expansion and we subtracted this term in accordance with the discussion in the end of last section. The definition and properties of function \(\Phi\),

\[\Phi\left(\frac{a^2}{\rho}, 1, 1 - 4\xi\right) = \sum_{n=0}^{\infty} (1 - 4\xi + n)^{-1} \left(\frac{a}{\rho}\right)^{2n},\]

may be found in Ref. [34].

The limiting cases \(\rho \to \infty\) and \(\rho \to 0\) gives us the following results

\[\lim_{\rho \to \infty} U = -\frac{ae^2}{4\pi^2} \frac{1 - 8\xi}{1 - 4\xi},\]

\[\lim_{\rho \to 0} U = \frac{e^2(1 - 8\xi)}{4a}(\ln \frac{2\rho}{a} + \gamma + \Psi(1 - 4\xi)).\]

For \(\xi = 1/8\) we obtain zero as should be the case due to conformal flatness of the 3D section of constant time. We observe that according with the discussion above the divergence for \(\xi = 1/4\) appears. Furthermore we observe appearance infinite number poles at points \(\xi = (n + 1)/4\). The appearance of problems with the delta-like potential was noted in literature [35, 36, 37, 38]. In the limit of minimal coupling we recover the result for the electromagnetic field given by

\[U = \frac{e^2}{4a} \ln \left(1 - \frac{a^2}{\rho^2}\right).\]

Indeed, in the limit \(\xi \to 0\) and for \(m = 0\) the equation for scalar field (10) coincides with the equation for component \(A_0\) of vector field and the formula for calculation the self-energy (14) coincides with that for electromagnetic field (see Ref. [17]). Therefore, considering the \(e\) as an electric charge we obtain that in this limit we have to recover the electromagnetic case.

For the second kind solution given by [36c] we obtain the following expression

\[(2l + 1)g_1^{(5)} = -\frac{r^l}{r^{l+1}} + \frac{2l + 1 - 8\xi}{1 - 8\xi} \frac{r^l r''}{a^{2l+1}}.\]

We observe that the series over \(l\) is divergent because \(r \geq a\) and \(r' \geq a\). Therefore, as noted above we have throw away this solution due to the fact that it is unphysical.

2. Massive case

In this case the radial equation turns to

\[\phi'' + \frac{2r'}{r} \phi' - \left(\frac{l(l+1)}{r^2} + m^2\right) \phi = 0\]
In general we observe that divergences appear as solutions of the following equation

\[ \phi(x) = 0, \]

where \( x = m(a + \rho) \) and \( \nu = l + 1/2 \). Using these solutions we obtain the Green function

\[
g^{(1)}_l = \frac{K_\nu(m r)I_\nu(m r')}{\sqrt{rr'}} - \frac{ma(I_\nu K'_\nu + I'_\nu K_\nu) + (8 \xi - 1)I_\nu K_\nu}{2maK_\nu K'_\nu + (8 \xi - 1)K^2_\nu} \bigg|_{ma} K_\nu(m r)K_\nu(m r'),
\]

\[
g^{(3)}_l = -\frac{1}{2maK_\nu K'_\nu + (8 \xi - 1)K^2_\nu} \bigg|_{ma} K_\nu(m r)K_\nu(m r'),
\]

where \( r = |\rho| + a, \ r' = |\rho'| + a \).

Using the addition theorem for Bessel function \( [39] \) we obtain the following formula \((r > r')\)

\[
\frac{1}{\sqrt{rr'}} \sum_{l=0}^{\infty} (2l + 1)I_\nu(m r')K_\nu(m r) = \frac{1}{r - r'}e^{-m(r-r')},
\]

and consequently the first term in \( g^{(1)}_l \) represents the standard Yukawa contribution. The second term cannot be represented in a closed form and we have to calculate it numerically. After renormalization we get the expression \( \rho > 0 \)

\[
G_{\text{ren}}(\rho, \rho) = -\frac{1}{2\pi} \sum_{l=0}^{\infty} \frac{ma(I_\nu K'_\nu + I'_\nu K_\nu) + (8 \xi - 1)I_\nu K_\nu}{2maK_\nu K'_\nu + (8 \xi - 1)K^2_\nu} \bigg|_{ma} K^2_\nu(m r) r,
\]

\[
U(\rho) = 2\pi e^2 G_{\text{ren}}(\rho, \rho).
\]

At the beginning we discussed the divergence at \( \xi = 1/4 \). Let us consider in manifest form the first term \((l = 0)\) in the renormalized radial Green function

\[
g^{(1)}_{l,\text{ren}} = -\frac{ma(I_\nu K'_\nu + I'_\nu K_\nu) + (8 \xi - 1)I_\nu K_\nu}{2maK_\nu K'_\nu + (8 \xi - 1)K^2_\nu} \bigg|_{ma} K^2_\nu(m r),
\]

It has the following forms

\[
g^{(1)}_{0,\text{ren}} = \frac{-e^{-2m(r-a)}(3 - 8 \xi) + e^{-2mr}(3 - 8 \xi + 4am)}{4mr^2(1 + am - 4 \xi)},
\]

\[
g^{(1)}_{0,\text{ren}} \big|_{m=0} = -\frac{a}{2r^2} \frac{1 - 8 \xi}{1 - 4 \xi},
\]

for massive and massless cases, respectively. We observe that there is no singularity for massive case at point \( \xi = 1/4 \). Indeed, for \( \xi = 1/4 \) we have

\[
g^{(1)}_{0,\text{ren}} = \frac{-e^{-2m(r-a)} + e^{-2mr}(1 + 4am)}{4m^2a r^2}.
\]

But this expression blows up for massless case because the expansion gives us

\[
g^{(1)}_{0,\text{ren}} = \frac{1}{2mr^2} + \ldots
\]

Therefore in the massive case the expression is no longer singular at point \( \xi = 1/4 \). The singularity appears at point \( 1/4 + ma/4 \). The next term with \( l = 1 \) will show a singularity at point \( 1/2 + (ma)^2/4(1 + ma) \) and so on. In general we observe that divergences appear as solutions of the following equation

\[
2xK'_\nu(x) + (8 \xi - 1)K_\nu(x) = 0.
\]

For massless case \((x = 0)\) there is general solution of this equation, \( \xi_n = (n + 1)/4 \), where \( \nu = n + 1/2 \), and we obtain infinite number of poles. For massive case we have infinite number of solutions, too, but it is impossible obtain solution in closed form for arbitrary \( n \). From above consideration we may conclude that nonzero mass of field gives correction for \( \xi_n \) obtained for massless case.
Let us consider the convergence of the series for the Green function, that is, we have to consider the expressions for $\nu \to \infty$ and fixed $r$. With this aim we use the uniform expansion for the Bessel function [40] which is valid for great index. We suppose that $mr/\nu$ and $ma/\nu$ are constants and use the uniform expansion for functions $I_{\nu}(\nu z)$ and $K_{\nu}(\nu z)$. We make expansion for $\nu q_{1,ren}^{(1)}$ over $\nu^{-1} \to 0$ and then make the expansion of the obtained expression over $\nu \to \infty$ because each term of the expansion depends on $\nu$ through $mr/\nu$ and $ma/\nu$. Doing this, we obtain the following result

$$
\nu q_{1}^{(1)} = \frac{(a)^{\nu}}{r} \left\{ -\frac{\zeta}{4\nu^2} + \frac{\zeta}{8\nu^2} \left[ 2m^2 (r^2 - a^2) + 4a^2 m^2 \right] - \frac{1}{16\nu^3} \left[ 4a^2 m^2 + 2m^2 (a^4 m^2 - r^2 - m^2 a^2 - a^2 (1 + 2m^2 a^2)) \zeta + 2m^2 (r^2 - a^2) \zeta^2 + \zeta^3 \right] + \ldots \right\},
$$

where $\zeta = 1 - 8\xi$. We observe that for $r > a$ the series is always convergent. For $r = a$ the series is still convergent only for $\xi = 1/8$. Therefore, the energy at the throat, $r = a$, is divergent for any case, except for $\xi = 1/8$. The numerical simulations of the self-energy are shown in Fig. 1. We note that the massive field will produce a self-force which is localized close to the throat. It falls down exponentially fast as $e^{-mr}$ far from the throat. This behavior is in agreement with Linet result [41].

**B. Profile $r = \sqrt{\rho^2 + a^2}$**

Because the mass of a field leads to suppression of the self-force by factor $e^{-mr}$ we consider the massless case, $m = 0$. The radial equation is written as

$$
\phi'' + \frac{2\rho}{(\rho^2 + a^2)} \phi' - \left( \frac{l(l+1)}{\rho^2 + a^2} - \frac{2\xi a^2}{(\rho^2 + a^2)^2} \right) \phi = 0,
$$

which has two linearly independent solutions ($\mu = \sqrt{2\xi}$), namely, $\phi_{1}^\mu$ and $\phi_{2}^\mu$, given by

$$
\phi_{1}^\mu = c_1^+ P_{l}^\mu(z),
\phi_{2}^\mu = c_2^+ Q_{l}^\mu(z),
$$

with Wronskian

$$
W(\phi_{1}^\mu, \phi_{2}^\mu) = i c_1^+ c_2^+ a \frac{\rho}{\rho^2 + a^2} e^{i\pi\mu}.
$$

Here $P_{l}^\mu$ and $Q_{l}^\mu$ are the Legendre functions of the first and second kinds, and $z = i\rho/a$. Taking into account the above relations we obtain the following expression for the Green function

$$
G(\rho, \rho') = \frac{1}{4\pi a} \sum_{l=0}^{\infty} (2l + 1) \left\{ i e^{-i\pi\mu} P_{l}^{-\mu}(z')Q_{l}^\mu(z) + \frac{e^{i\pi\mu}}{\pi} Q_{l}^{-\mu}(z')P_{l}^\mu(z) \right\}
$$
\[ G(\rho, \rho') = \frac{1}{4\pi a} \frac{\sqrt{p'}}{(p - p')^{1/4}(1 + x'^2)^{1/4}}. \]
Thus, we obtain the following equation for $S$:

$$S' + S'' + \frac{2r'}{r} S' - \nu^2 - 1/4 - \xi R = 0,$$

(61)

where $\nu = l + 1/2$. The next step is to expand $S$ in the following power series in $\nu$:

$$S = \sum_{n=-1}^{\infty} \nu^{-n} S_n.$$

(62)

As noted in Ref. [17], we have to consider the term with $l = 0$ separately. In this case the equation (59) simplifies considerably ($\varphi$ stands here for the zero mode only):

$$\varphi'' + \frac{2r'}{r} \varphi' - \xi R \varphi = 0.$$

The general solution of the above equation for our specific profile reads

$$\varphi = C_1 \cos(\mu \arctan \frac{\rho}{a}) + C_2 \sin(\mu \arctan \frac{\rho}{a}).$$

(63)

Thus, we have the following solutions

$$\varphi^1_+ = \frac{2 \tan \frac{\pi \mu}{2}}{\pi \mu} \left( k_1 \cos(\mu \arctan \frac{\rho}{a}) + k_2 \sin(\mu \arctan \frac{\rho}{a}) \right),$$

$$\varphi^2_+ = \frac{\pi}{2} \left\{ \cos(\mu \arctan \frac{\rho}{a}) - \cot \frac{\pi \mu}{2} \sin(\mu \arctan \frac{\rho}{a}) \right\} = \frac{\pi}{2 \sin \frac{\pi \mu}{2}} \sin \mu \left( \int_{\rho}^{\infty} \frac{d\rho}{\rho^2} \right),$$

(64)

with Wronskian

$$W(\varphi^1_+, \varphi^2_+) = -\frac{a}{r^2} (k_1 + k_2 \tan \frac{\pi \mu}{2}).$$

(65)

Therefore we obtain expression

$$g^{(1)}_0 = \frac{\cos(2 \mu \arctan \frac{\rho}{a}) - \cos(\pi \mu)}{2 a \mu \sin \pi \mu}$$

(66)

which does not depend on $k_1$ and $k_2$.

Now we consider terms with $l > 0$. Substitution of (62) into (61) yields the set of expressions for the functions $S_n$. General solution of the first four equations of this chain reads

$$S'_{-1} = \pm \frac{1}{r},$$

$$S'_0 = -\frac{r'}{2r} = -\frac{1}{2} (\ln r)',$$

$$S'_1 = \pm \frac{\zeta}{4} \left[ r'' + \frac{r'^2}{2r} - \frac{1}{2r} \right],$$

$$S'_2 = -\frac{\zeta}{8} \left[ r^{(3)}_{rr} + 2 r' r'' \right] = -\frac{\zeta}{8} (r r'' + \frac{1}{2} r'^2),$$

$$S'_3 = \frac{\zeta}{16} \left[ r^{(4)} r'^2 + 2 r'^2 r + 4 r' r^{(3)} r + 2 r'^2 r'' \right] - \frac{\zeta^2}{128 r} \left[ r'^2 + 2 r r'' - 1 \right]^2,$$

$$S'_4 = \frac{\zeta^2}{32} \left[ r'^2 + 2 r r'' - 1 \right] \left[ 2 r'^2 r'' + r^{(3)} \right]$$

$$+ \frac{\zeta}{32} \left[ -2 r'' r^{(3)} - 10 r r'' r^{(3)} r + 8 r'^2 + 7 r^{(4)} \right] r' - r^2 \left( 8 r'^2 r^{(3)} + r^{(5)} \right)$$

$$- \frac{\zeta}{128} \left[ r'^4 + 2 (2 r'' - 1) r'^2 + 4 r'^2 r'' (r'' - 1) \right] + \frac{\zeta}{32} \left[ 2 r'^2 r'' + 4 r r^{(3)} r' + r \left( 2 r'^2 + r^{(4)} \right) \right].$$

We observe that (i) all $S'_k$ with $k \geq 1$ are proportional to $\zeta = 1 - 8 \xi$, and (ii) all $S'_{2k}$ are full derivative as in the case of $\xi = 0$. 
Now we are in position to calculate the Green function. First of all we calculate the Wronskian and found \(A_+\):
\[
A_+ = W(\phi^1_+, \phi^2_+) r^2 = e^{S^+ + S^n_+ (S^t + S^{t+1})} r^2. \tag{67}
\]
By using the formulas above we obtain
\[
g^{(1)}_1(\rho, \rho') = \frac{e^{-\nu f^1_2 \frac{\rho}{\rho'}}}{2\nu \sqrt{r(\rho)r(\rho')}} e^{-\sum_{n=1}^{\infty} -n(S^+_n(\rho) - S^+_n(\rho'))},
\]
Then, we change summation over \(n\) and \(l\) and get the following result
\[
\sum_{l=0}^{\infty} 2\nu \frac{2\nu}{4\pi} g_l(\rho, \rho') = \frac{1}{4\pi} \frac{1}{\sqrt{r(\rho)r(\rho')}} \sum_{k=0}^{\infty} f_k(b) j_k(\rho, \rho') + \frac{1}{4\pi} g_0(\rho, \rho'),
\]
where
\[
\frac{f_0(b)}{\sqrt{r(\rho)r(\rho')}} = \frac{1}{\rho - \rho'} - \frac{1}{r} + O(\rho - \rho'),
\]
\[
\frac{f_1(b)}{\sqrt{r(\rho)r(\rho')}} = -\ln \frac{\rho - \rho'}{4r} - \frac{2}{r} + O(\rho - \rho'),
\]
\[
\frac{f_k(b)}{\sqrt{r(\rho)r(\rho')}} = \frac{1}{r} \zeta_H(k, \frac{3}{2}) + O(\rho - \rho').
\]
The functions \(j_k\) may be found by using simple code in package "Mathematica". There is no general form of these coefficients for arbitrary index but for numerical calculations we need only for some first ones (see Ref. [17]). The first four coefficients have the following form in coincidence limit
\[
j_0(\rho, \rho') = 1,
\]
\[
j_1(\rho, \rho') = -\zeta \int_{\rho'}^{\rho} \frac{-1 + r'^2 + 2rr''}{8r} dr' = -\zeta \frac{-1 + r'^2 + 2rr''}{8r} (\rho - \rho') + O((\rho - \rho')^2),
\]
\[
j_2(\rho, \rho') = -\zeta \frac{-1 + r'^2 + 2rr''}{8},
\]
\[
j_4(\rho, \rho') = \frac{3\zeta^2}{128} (r'^2 + 2rr'' - 1)^2 - \frac{r\zeta}{16} \left( 2r''r^2 + 4rr(3)r' + r \right) \left( 2r'r^2 + rr(4) \right).
\]
Therefore we obtain
\[
\sum_{l=0}^{\infty} 2\nu \frac{2\nu}{4\pi} g_l(\rho, \rho') = \frac{1}{4\pi} \frac{1}{\rho - \rho'} - \frac{1}{r} + \frac{1}{r} \sum_{k=1}^{\infty} \zeta_H(2k, \frac{3}{2}) j_{2k}(\rho, \rho') + \frac{\cos(2\mu \arctan \frac{\rho}{\rho')} - \cos(\pi \mu)}{2\mu \sin \pi \mu},
\]
where each term \(j_{2k}\) is proportional to \(\zeta = 1 - 8\xi\). Here \(\zeta_H(s, p)\) is the Hurwitz zeta function (see, for example [34]). After regularization we arrive at the following formula for the self-energy \((\mu = \sqrt{2\xi})\)
\[
U(\rho) = \frac{e^2}{2} \left[ -\frac{1}{r} + \frac{1}{r} \sum_{k=1}^{\infty} \zeta_H(2k, \frac{3}{2}) j_{2k}(\rho, \rho') + \frac{\cos(2\mu \arctan \frac{\rho}{\rho')} - \cos(\pi \mu)}{2\mu \sin \pi \mu} \right]. \tag{69}
\]
As expected it is zero for \(\xi = 1/8\) and it is divergent for \(\xi = 1/2\). Far from the throat we obtain
\[
U \approx -\frac{e^2}{2} \frac{a\mu}{2 \rho^2 \tan \pi \mu}. \tag{70}
\]
By numerical analysis it is enough to take into account only two terms of the series above. In fact we use half of sum the first two terms. The numerical simulations are reproduced in Fig. 2.

For arbitrary profile of the wormhole we have the following formula
\[
U(\rho) = \frac{e^2}{2} \left[ -\frac{1}{r} + \frac{1}{r} \sum_{k=1}^{\infty} \zeta_H(2k, \frac{3}{2}) j_{2k}(\rho, \rho') + g^{(1)}(\rho) \right], \tag{71}
\]
FIG. 2: The numerical simulation of the self-force on a massless scalar field for profile $r = \sqrt{\rho^2 + a^2}$ for different parameters from $\xi = 0$ (thick line) up to $\xi = \frac{1}{10}$ (thin line). For $\xi = \frac{1}{8}$ it is zero and for $\xi = \frac{1}{2}$ it tends to infinity.

with the same $j_{2k}$ as above and

$$g_0^{(1)}(\rho, \rho) = -\frac{1}{A_+} \varphi_+^2(\rho) \varphi_+^1(\rho) + \frac{1}{2A_+} \left( \frac{\varphi_+^1}{\varphi_+^2} + \frac{\varphi_+^1}{\varphi_+^2} \right) \varphi_+^2(\rho),$$

(72)

where $A_+ = W_+(\varphi_+^1, \varphi_+^2)\rho^2(\rho)$. The functions $\varphi_+^{1,2}$ are the solutions of the equation

$$\varphi'' + \frac{2r'}{r} \varphi' - \xi R \varphi = 0.$$

(73)

Unfortunately, differently from the electromagnetic field case, there is no general solution of this equation for arbitrary $\xi$ and $r$. For $\xi = 1/8$ it is easy to find a general solution of this equation by using the conformal flatness of the equation. They read

$$\varphi^1 = \frac{1}{\sqrt{r}} e^{f r/\varphi_+^1}, \quad \varphi^2 = \frac{1}{\sqrt{r}} e^{-f r/\varphi_+^1},$$

(74)

with Wronskian $W(\varphi^1, \varphi^2) = -1/r^2$. For $\xi \neq 1/8$ we may only make conclusion about behavior of the self-force far from the wormhole’s throat. Indeed, changing function by the relation $\varphi = w/r$ we obtain

$$w'' - \left( \frac{r''}{r} + \xi R \right) w = 0.$$

(75)

Therefore, for great distance we have simple equation $w'' = 0$, with two solutions

$$w_1 = c_2 \rho, \quad w_2 = c_1.$$

(76)

The Wronskian corresponding solutions is

$$W(\varphi_1, \varphi_2) = -\frac{c_1 c_2}{r^2}.$$

(77)

It is not difficult to show that the solutions with next corrections are

$$\varphi_1 = c_1 \left( 1 + \frac{h_1}{\rho} + O(\rho^{-2}) \right),$$

$$\varphi_2 = c_2 \left( 1 + O(\rho^{-2}) \right),$$

(78)

and therefore we obtain for great $\rho$

$$U \approx -\frac{e^2}{2\rho^2} \left( h_1 + \frac{c_2}{2c_1} \left( \frac{\varphi_1}{\varphi_2} + \frac{\varphi_1'}{\varphi_2'} \right) \right).$$

(79)

But we can not make any conclusion about sign of these expression.
IV. DISCUSSION AND CONCLUSION

In this paper we considered in details the self-interaction on a scalar particle at rest in the wormhole space-time with non-minimal coupling with curvature. The main peculiarities of the self-force on a scalar field are (i) mass of field and (ii) nonminimal coupling $\xi$. We consider a particle at rest and for this reason all equations become effectively three dimensional, because they touch only spacial part of wormhole space-time which is conformally flat. For this reason for $\xi = 1/8$ and massless field we expect that the self-force is zero \cite{24}. Our calculations confirm this result, the self-force zero indeed in all considered above special examples. For $\xi < 1/8$ the scalar particle is attracted to the wormhole and for $\xi > 1/8$ the particle is repelled by the wormhole. For $\xi = 1/8$ the self-force is zero. In the space-time of a black hole \cite{20} one has different behavior of the self-energy, in which case it is proportional to $\xi$ and the self-force is zero for minimal coupling $\xi = 0$.

The self-force for scalar massless particles reveals peculiarity for specific values of the nonminimal coupling $\xi$. The energy has a simple pole, $(\xi - \xi_c)^{-1}$, at this point. Because the self-energy is defined in terms of the three dimensional Green function we may analyse this pole by using analogy with the scattering theory \cite{42}. The combination $V = \xi R + r''/r$ plays the role of a potential for the wave function in non-relativistic quantum mechanics and all information about boundary or scattering states is encoded in the Green function. This potential tends to a constant at the wormhole’s throat, $\rho = 0$, and it falls down to zero far from the throat. The critical value depends on the shape of the throat. If the space-time far from the throat differs from Minkowski spacetime as $b_n \rho^{-n}$, then the critical value $\xi_c = (n + 1)/4n$ and the potential, $V = nb_n(\xi_c - \xi)\rho^{-n-3}$ changes its sign in this point. If the space-time goes to Minkowski spacetime exponentially fast as $c_n \rho^n e^{-\rho/r}$ then the critical value $\xi_c = 1/4$ and the potential, $V = 4c_n(\xi_c - \xi)\tau^{-2}\rho^{n-1}e^{-\rho/\tau}$, also changes its sign at this point. This kind of wormholes possesses a dimensional parameter, $\tau$, which may be regarded as length of the throat. The profile $r = |\rho| + a$ gives singular, delta like potential concentrated at throat which has no longer possess the length of the throat. This kind of profile belongs to the second type of throat due to localization potential close to the throat. We obtain the same critical value $1/4$ in this case by manifest calculations (see Eq. (38)).

For wormhole with profile $r(\rho) = \sqrt{\rho^2 + a^2}$ the Green function for zero mode, $l = 0$, is expressed in terms of two solutions of radial equation

$$\Phi'' - \frac{a^2(1 - \mu^2)}{(\rho^2 + a^2)^2}\Phi = 0,$$

with $\mu^2 = 2\xi$. This equation describes the particle in the potential $V = -a^2(1 - \mu^2)/(|\rho^2 + a^2)^2$ in one dimension. It is well-known \cite{42} that the Green function has the poles for energy of boundary states. The point $\mu = 1(\xi = 1/2)$ is critical because the potential changes its sign and boundary states appears or disappears.

The mass of the field gives additional factor $e^{-m\rho}$ and leads to localization of the self-force close to the wormhole’s throat inside a sphere with the Compton wavelength radius $m^{-1}$ (see Fig. 1). The self-force reveals singularity too but the point depends on the mass of the field. For example, for a simple profile of the throat, $r = |\rho| + a$, the singularity appears at the point $\xi_c = 1/4 + ma/4$ \cite{15}, where $a$ is radius of the throat.

We developed a procedure and found general expression (71) for the self-force for general profile of the throat. But differently from the electromagnetic field case there is no general solution for zero mode $\phi_{(3)}^{(3)}$ \cite{22} in terms of profile function $r$. We would like to note that this relation contains two independent solutions of homogeneous radial equation for zero mode at observation point as well as at the origin, for $\rho = 0$. Because the fall down function $\varphi^2$ and its derive appears at the denominator we have to use irregular solution for this function in terms of the scattering theory (see, for example \cite{43}). We observe that zero mode gives main contribution to self-force and it depends on the global structure of space-time which is in agreement with consideration the self-force in black hole background \cite{44, 15}. For example, for profile function $r = \sqrt{\rho^2 + a^2}$ we observe that zero mode solutions given by Eq. (61) contains the integral over profile function and therefore is defined by global structure of the space-time. The numerical evaluations (see Fig. 2) for this profile show that the self force as expected changes its sign at the point $\xi = 1/8$: for $\xi < 1/8$ it is attractive and for $\xi < 1/8$ it is repulsive. For $\xi \to 1/2$ the self-force reveals singularity as a simple pole $(\xi - 1/2)^{-1}$ and therefore tends to infinity. The self-force has extrema for $\rho \approx a$ and it is zero at origin. Far from the string the self-energy is given by Eq. (70). Therefore we may say the same conclusion, the scalar particle will be concentrated at the throat for $\xi < 1/8$ as for electromagnetic field case \cite{17}.

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