The Problem of Large-N Phase Transition
in Kazakov–Migdal Model of Induced QCD

S. Khokhlachev
Cybernetics Council, Academy of Science
Vavilov st. 40, 117333 Moscow, Russia

and

Yu. Makeenko
Institute of Theoretical and Experimental Physics
B.Cheremuskinskaya 25, 117259 Moscow, Russia

Abstract

We study the lattice gauge model proposed recently by Kazakov and Migdal for inducing QCD. We discuss an extra local $Z_N$ which is a symmetry of the model and propose of how to construct observables. We discuss the role of the large-$N$ phase transition which should occur before the one associated with the continuum limit in order that the model describes continuum QCD. We formulate the mean field approach to study the large-$N$ phase transition for an arbitrary potential and show that no first order phase transition occurs for the quadratic potential.
1 Introduction

Recently Kazakov and Migdal \[1\] have proposed a very interesting lattice gauge model for inducing QCD. The model is defined by the partition function

$$Z = \int \prod_{x,\mu} dU_\mu(x) \prod_x d\Phi(x) e^{\sum_x N \text{tr} \left( -V[\Phi(x)] + \sum_\mu \Phi(x)U_\mu(x)\Phi(x+\mu)U_\mu^\dagger(x) \right)}$$

(1.1)

where the field $\Phi(x)$ takes values in the adjoint representation of the gauge group $SU(N)$ and the link variable $U_\mu(x)$ belongs to the group. The potential $V[\Phi]$ can be expanded at small lattice spacing $a$ in the power series

$$V[\Phi] = m_0^2 \Phi^2 + \lambda \Phi^4 + \ldots$$

(1.2)

with $m_0$ being the bare mass of the field $\Phi$. A strong coupling solution to this model has been constructed by Migdal \[2\] at $N = \infty$ and then extended to next orders of the $1/N$-expansion \[3\].

To obtain the solution two assumptions have been made. The first one is that the critical point $m_0^2 = D$ (where $D$ is the dimensionality of euclidean space-time), which separates the strong coupling region, is associated with continuum QCD. The second one is that the path integral over $\Phi(x)$ is saturated by a single $x$-independent configuration — the master field.

On our mind along solving the Kazakov–Migdal model, one should better understand qualitative properties of the model like how to construct observables or how to recover the area law in the continuum. As usual the question of which phase transitions occur in the lattice system is of special importance. The knowledge in advance of the phase structure might help to understand what kind of the distribution of eigenvalues of the matrices $\Phi_{ij}(x)$ would correspond to a physical solution. In this paper we apply a standard lattice gauge theory technique to study phase transitions in the Kazakov–Migdal model.

A subtle point with the model (1.1) is that it possesses an extra local $Z_N$ symmetry

$$U_\mu(x) \rightarrow Z_\mu(x)U_\mu(x).$$

(1.3)

The important role of this symmetry in the Kazakov–Migdal model has been pointed out recently by Kogan, Semenoff and Weiss \[4\]. These authors argued that due to the $Z_N$-symmetry the model undergoes a phase transition which was assumed to occur simultaneously with the transition from strong to weak coupling solution. This phase transition separates two phases which differ by the nature of confinement — the so-called local confinement versus usual area law.

The main subject of the present paper is a study of properties of the Kazakov–Migdal model which come from the presence of the $Z_N$-symmetry. To calculate observables, in particularly the string tension, we propose a general procedure based on the Wilson loops in the adjoint representation $SU(N)$ which are invariant under the $Z_N$-symmetry and
can be expressed entirely in terms of averages over $\Phi(x)$. We show that, being proper normalized to be of order 1 in the naive continuum limit, these quantities are $\sim 1/N^2$ in the strong coupling expansion. For the quadratic potential $V[\Phi]$, we look for the corresponding critical point at $N = \infty$ using the mean field method. We show that no phase transition occurs in $D = 4$ up to the point $m_0^2 = D$ where the model becomes unstable.

We discuss that, if the large-$N$ phase transition occurs before the one associated with the continuum limit, a consistent picture of inducing continuum QCD by the Kazakov–Migdal model emerges. In this case one gets below the large-$N$ phase transition a theory with usual confinement (area law) rather than local confinement as in each order of the strong coupling expansion. We formulate the mean field approach to study the large-$N$ phase transition for the case of an arbitrary $V[\Phi]$.

We speculate that the phase transition associated with the continuum limit should be identified with the one separating confinement and Higgs phases and discuss the general scenario of inducing QCD by the Kazakov–Migdal model. We point out that the mean field analyses of the Higgs phase transition is reduced to a one-matrix problem which is solvable for cubic and logarithmic potentials.

2 $Z_N$ and observables

The local $Z_N$ symmetry has far-reaching consequences for the model (1.1). First of all, the average of any non-invariant quantity like the Wilson loop in the fundamental representation vanishes except for a loop passing the same contour back and forth (i.e. with vanishing minimal area). Moreover, this property holds independently of how many phase transitions the system undergoes under the way to the continuum since the local symmetry can not be broken spontaneously. Therefore, the average of the fundamental Wilson loop always vanishes.

To obtain the continuum Wilson loops from the Kazakov–Migdal model, we propose to consider averages of the adjoint Wilson loops

$$W_A(C) = \left\langle \frac{1}{N^2} \left( |\text{tr} U(C)|^2 - 1 \right) \right\rangle$$

(2.1)

which are invariant under the $Z_N$. We use the normalization that $W_A \sim 1$ in the naive continuum limit.

For the sake of simplicity let us postpone for a moment discussing properties of the adjoint Wilson loop for the model (1.1) and review some results [5] for the case of the 3-rd kind.

2 Since the transformation can be done independently on each link of the lattice, such a symmetry was called in Ref.[6] that of the 3-rd kind.
pure adjoint single-plaquette lattice action

\[ S_A = -\frac{\beta_A}{2} \sum_\square |\text{tr} U(\square)|^2 \tag{2.2} \]

where \( \beta_A \sim 1 \) as \( N \to \infty \) in order to have a nontrivial large-\( N \) limit. Due to the factorization at large \( N \), the following formula holds

\[ W_A(C) = \left( \frac{1}{N} \text{tr} U(C) \right)^2 + O(N^{-2}) \tag{2.3} \]

where the average of the Wilson loop in the fundamental representation on the r.h.s. should be calculated for the Wilson action

\[ S_{\text{Wilson}} = -N\bar{\beta} \sum_\square \Re \text{tr} U(\square) \tag{2.4} \]

with the coupling \( \bar{\beta} \) given by the self-consistency equation

\[ \bar{\beta} = \beta_A W(\square; \bar{\beta}) \tag{2.5} \]

where \( W(\square; \bar{\beta}) \) is the plaquette average for the Wilson action (with the coupling being \( \bar{\beta} \)). When Eq. (2.5) possesses a nontrivial solution which is valid in the weak coupling region so that asymptotically

\[ \bar{\beta} \to \beta_A - \frac{1}{4} \quad \text{as} \quad \beta_A \to \infty, \tag{2.6} \]

one sees from Eq. (2.3) that the adjoint Wilson loop display the area law at \( N = \infty \) with the adjoint string tension being

\[ K_A = 2K_{\text{Fundamental}}. \tag{2.7} \]

The perimeter law which is expected for the adjoint Wilson loop at finite \( N \) enters the term of order \( O(N^{-2}) \).

Let us now return to the model (1.1). Our idea is to define at \( N = \infty \) the ‘fundamental’ Wilson loop, which enters, say, the correlator of electromagnetic currents represented via sum over paths or determines the string tension, by Eq. (2.3) taking the square root of the adjoint Wilson loop. This procedure is unambiguous since the imaginary part of the fundamental Wilson loop never shows up to the leading order of \( 1/N \)-expansion. It is crucial for this procedure the large-\( N \) phase transition to occur \textit{before} the continuum limit sets in. Only in this case the induced continuum theory would possess normal area law while otherwise one gets local confinement.

Thus, we can define the set of QCD observables at the kinematical level despite the \( Z_N \)-symmetry while this procedure works only at \( N = \infty \). It is a dynamical question whether the large-\( N \) phase transition makes this construction sensible.
3 Large-$N$ phase transition

Let us start again from the single-plaquette action (2.2). Since the only solution of Eq. (2.3) in the strong coupling expansion is $\bar{\beta} = 0$, the adjoint Wilson loop $W_A(C)$ vanishes in this region according to Eq. (2.3). One can directly verify this result estimating the order in $1/N$ of the strong coupling expansion which gives $W_A(C) \sim 1/N^2$ for the action (2.2). Therefore, as we concluded in Ref. [5], there should be a first order phase transition for the action (2.2) at some value $\beta_A^* < 2$.

This phase transition was observed by the Monte–Carlo simulations for $N = 2 \div 6$ [6] with the critical value of $\beta_A^*$ growing from 1.6 for $N = 3$ to 1.8 for $N = 6$ what is close to the estimate $\beta_A^* < 2$. More detailed studies of Eq. (2.3) were performed in Ref. [7]. An uncertainty of the value of $\beta_A^*$ is related, in particular, to an ambiguity of criterion for the first order phase transition at large $N$. The standard criterion stating that the phase transition occurs when free energies of two phases coincide (the ‘Maxwell rule’) may not work at $N = \infty$ because the free energy itself is $\sim N^2$ and a barrier which separates the two phases becomes infinite in this case. An alternative criterion says that the phase transition is associated with the point where a metastable weak coupling solution terminates. Using the ‘Maxwell rule’, Samuel obtained $\beta_A^* = 1.54$ [7]. Analogous studies of this phase transition using mean field / variational technique [3] yields $\beta_A^* = 2.8$ while the criterion based on terminating the metastable region gives $\beta_A^* = 1.7$ in better agreement with the Monte–Carlo data. The large-$N$ phase transition is not related to breaking of a symmetry and was shown to be associated with dynamics of $Z_N$-monopoles which are condensed in the strong coupling phase [9].

An occurrence of the analogous phase transition for the Kazakov–Migdal model has been advocated by Kogan, Semenoff and Weiss [4]. We support the consideration of this paper by the following argument. Let us estimate the order in $1/N$ of the adjoint Wilson loops (2.1) in the strong coupling expansion of the model (1.1) with the quadratic potential $V[\Phi]$ using the induced action [1]

$$S_{ind}[U] = -\frac{1}{2} \sum_{\Gamma} \frac{|\text{tr} U(\Gamma) |^2}{l(\Gamma)m_0^2(l(\Gamma))}. \quad (3.1)$$

It is easy to see now using the standard lattice technique that $W_A(C) \sim 1/N^2$ in the strong coupling expansion while it is of order 1 in the naive continuum limit. This estimate is quite similar to the one discussed in the previous section for the case of the singleplaquette adjoint action. Thus, we conclude the model undergoes a first order phase transition with decreasing $m_0^2$.

Let us now estimate the location of this phase transition. Following the scenario of Ref. [4], we start from the induced action (3.1) and decrease $m_0^2$. A very interesting question is whether the large-$N$ phase transition occurs before the point $m_0^2 = D$ after which the quadratic action in Eq. (1.1) is not bounded from below. The point is that the action (3.1) is not, say, the classical action for an external field problem. One should inte-
grate $\exp(-S_{\text{ind}}[U])$ over $U_\mu(x)$ with the Haar measure according to the definition (1.1).

As $m_0^2 \to \infty$, typical configurations of $U_\mu(x)$ are uniformly distributed on the group being independent on each link of the lattice. This disordering suppresses the contribution of long loops to the sum over paths on the r.h.s. of Eq. (3.1) — the longer the loop the stronger the suppression. In usual lattice gauge theory the $U$-matrices become ordered under the way to the continuum limit. The ordering is, however, only for distances smaller than the correlation length while for large distances the disorder is needed for confinement. This ordering occurs either gradually or is enhanced by the presence of a first-order phase transition. It is, however, a dynamical question whether this phase transition occurs in the region $m_0^2 \geq D$.

A simplest logical possibility for the model (1.1) might be that the large-$N$ phase transition occurs at some value of $m_0^2 > D$ which is large enough in order that the action (3.1) can be approximated by the single-plaquette term, i.e. coincides with the action (2.2) with

$$\beta_A = \frac{1}{4m_0^8}.$$ (3.2)

This would mean that the Kazakov–Migdal model were induce a lattice gauge theory with the action (2.2). Substituting the above numerical value $\beta_0^* \approx 2$, one would get $m_0^2 \approx 0.6$ which is too small. The next terms on the r.h.s. of Eq. (3.1) are essential for such $m_0^2$ so that this situation is excluded by dynamics.

### 4 The mean field analyses

To study the large-$N$ phase transition, we apply to our problem the mean field method, which usually works pretty well for first order phase transitions, quite analogously it was applied to the single-plaquette adjoint action in Ref. [8].

To construct the mean field, we use the variational approach which was advocated in the context of modern field theory by Sakita [10]. Let us introduce the trial partition function

$$Z_0 = \int \prod_{x,\mu} dU_\mu(x) e^{\frac{b}{4} \sum_{x,\mu} |\text{tr} U_\mu(x)|^2}$$ (4.1)

which is a product of one-link integrals. We have chosen for them a simplest form which possesses the $Z_N$ symmetry. The Jensen’s inequality yields then the following bound on the partition function (1.1):

$$Z \geq Z_0 \int \prod_x d\Phi(x) e^{-\sum_x N \text{tr} V[\Phi(x)] + \left( \sum_{x,\mu} \left( N \text{tr} \Phi(x) U_\mu(x) \Phi(x+\mu) U_\mu^\dagger(x) - \frac{b}{4} |\text{tr} U_\mu(x)|^2 \right) \right)_0},$$ (4.2)

where $\langle \ldots \rangle_0$ means averaging w.r.t. the trial action. Since the exponent contains the sum
of one-link averages, it can be expressed via the one-matrix integral

$$
\eta^2 = \frac{\int dU e^{\frac{1}{N} \text{tr} U^2} \frac{1}{N} \text{tr} U^2}{\int dU e^{\frac{1}{N} \text{tr} U^2}},
$$

(4.3)

by the formula

$$
\langle \text{tr} \Phi(x) U_\mu(x) \Phi(x + \mu) U^T_\mu(x) \rangle_0 = \eta^2 \text{tr} \Phi(x) \Phi(x + \mu).
$$

(4.4)

The idea of the variational mean field method is to fix $b_A$ from the condition that the trial ansatz (4.1) would give the best approximation to $Z$ in the given class. Calculating the derivative w.r.t. $b_A$ and taking into account that $\eta$ depends on $b_A$ according to Eq. (4.3), one finds the maximum of the r.h.s. of Eq. (4.2) at

$$
b_A = \frac{\int \prod_x d\Phi(x) e^{\sum_x N \text{tr} \left( -V[\Phi(x)] + \eta^2 \sum_\mu \Phi(x) \Phi(x + \mu) \right)} \frac{1}{N} \text{tr} \Phi(0) \Phi(0 + \mu)}{\int \prod_x d\Phi(x) e^{\sum_x N \text{tr} \left( -V[\Phi(x)] + \eta^2 \sum_\mu \Phi(x) \Phi(x + \mu) \right)}}.
$$

(4.5)

This is the final formula that relates $b_A$ to the potential $V[\Phi]$ providing the r.h.s. of Eq. (4.3) is known as a function of $b_A$. Its standard interpretation is that one obtains the one matrix integral on the r.h.s. of Eq. (4.3) replacing the matrix $[U_\mu(x)]_{ij}$ in the partition function (1.1) by the mean field value $\eta \delta_{ij}$ on each link of the lattice except the given one $(0, 0 + \mu)$ while Eq. (4.3) gives a self-consistency condition at this link.

The one-matrix integral on the r.h.s. of Eq. (1.3) was first calculated by Chen, Tan and Zheng [8] (see also Ref. [7]). In the weak coupling region where $b_A > 2$ or $1/2 \leq \eta \leq 1$, the result can be represented as

$$
2(\eta - \eta^2) = \frac{1}{b_A}.
$$

(4.6)

We have written the self-consistency condition in the form which would be convenient to study the phase transition.

Let us consider the case of the quadratic potential $V[\Phi]$ when the mean field analyses is drastically simplified. The gaussian integral on the r.h.s. of Eq. (1.5) can easily be calculated to give

$$
b_A = \frac{1}{\eta^2} \int_0^\infty d\alpha \ e^{-\frac{\alpha m^2_0}{\eta^2}} \ I_0^{D-1}(\alpha) \ I_1(\alpha)
$$

(4.7)

where $I(\alpha)$ are modified Bessel function.

Eqs. (4.3), (4.7) can be analyzed similarly to those of Ref. [8]. The r.h.s. of Eq. (4.7) multiplied by $\eta^2$ monotonically increases with decreasing $m^2_0/\eta^2$ with maximal value $\approx 0.1$ at $m^2_0/\eta^2 = 4$ and then diverges. This value is too small to provide a solution to Eq. (4.6) with $\eta > 1/2$ as it should be for the weak coupling solution.
We conclude, therefore, that for the quadratic potential $V[\Phi]$ there is no first order large-$N$ phase transition for $m_0^2 \geq 4$ when the model is well-defined. Notice that we exclude a possibility that the large-$N$ phase transition occurs exactly at $m_0^2 = 4$. According to Eq. (4.6) this would occur if $b_A$ were $\geq 1/2$ at this point which is not the case.

Let us mention that the gaussian integral over $\Phi$ on the r.h.s. of Eq. (4.5) for the quadratic $V[\Phi]$ can be represented in the $1/m_0^2$ expansion as the sum over paths:

$$b_A = \frac{1}{m_0^2} \sum_{\Gamma, x, x+\mu} \left( \frac{\eta}{m_0} \right)^{2l(\Gamma)}$$

where the sum goes over the open loops with the endpoints $x$ and $x + \mu$. This sum is not exactly the same as what would appear if the mean field method were applied directly to the action (3.1). The point is that if a link is passed back and forth, it does not contribute to the r.h.s. of Eq. (3.1) because $U$ is unitary while it is taken into account on the r.h.s. of Eq. (4.8). In particularly first few terms of the series expansions of r.h.s. of Eq. (4.7) in $D = 4$ read

$$b_A \eta^2 = \frac{1}{2} \left( \frac{\eta}{m_0} \right)^4 + \frac{21}{8} \left( \frac{\eta}{m_0} \right)^8 + 20 \left( \frac{\eta}{m_0} \right)^{12} \left( \frac{\eta}{m_0} \right)^{16} + \frac{124047}{64} \left( \frac{\eta}{m_0} \right)^{20}$$

$$+ \frac{1397319}{64} \left( \frac{\eta}{m_0} \right)^{24} + \frac{4148859}{16} \left( \frac{\eta}{m_0} \right)^{28} + O \left( \left( \frac{\eta}{m_0} \right)^{32} \right)$$

which differs to order $O(1/m_0^8)$ from what one would obtain from Eq. (3.2). All the extra terms are, however, positive so that $b_A$ can be considered as an upper limit which is enough for our purposes.

5 Discussion

The main conclusion from the fact that the large-$N$ phase transition does not occur for the model (1.1) with the quadratic potential $V[\Phi]$ is that it remains in the strong coupling phase with local confinement and can not induce QCD. This phase is pretty trivial: the averages of all the Wilson loop vanish at $N = \infty$ except of those with vanishing minimal area, in contrast to the strong coupling expansion of the lattice gauge theory with the Wilson action\(^3\). Therefore, one should incorporate the self-interaction of $\Phi$ and look for the large-$N$ phase transition. The mean field method which leads in the case of an arbitrary potential $V[\Phi]$ to Eqs. (4.5) and (4.6) can be used to estimate the location of the large-$N$ phase transition.

As we discussed already, the large-$N$ phase transition should occur before the one associated with the continuum limit in order that the Kazakov–Migdal model induce normally confining QCD. One might tempt to relate the latter phase transition with the

\(^3\)The existence of a master field for the Wilson action in the strong coupling region was first advocated by Kazakov, Kozhamkulov and Migdal [11].
one separating confining and Higgs phases which always occurs in gauge theories. A continuum theory with confinement can be usually obtained by approaching this phase transition from above while approaching from below one gets the deconfining Higgs phase. If one accepts this scenario, the main problem with inducing QCD by the Kazakov–Migdal model is to study whether the large-$N$ phase transition is indeed separated from the Higgs one. If the two phase transitions coincide, this would mean that one passes from the phase with local confinement directly to the Higgs phase while the phase with normal confinement were missing. Monte–Carlo simulations of the Kazakov–Migdal model might help to answer this question.

The mean field method could be useful to study the Higgs phase transition as well. In the Higgs phase one would get a condensate of the $\Phi$-field, $\Phi^*$, with a nonsymmetric distribution of eigenvalues of $\Phi^*$ which violates the $SU(N)$ quite similarly, say, to the Higgs phase transition in the Georgi–Glashow model. One should add then one more self-consistency condition to determine $\Phi^*$. Substituting $\Phi(x)$ by an (independent on $x$) mean-field value $\Phi^*$ for all sites of the lattice except given one, we get the self-consistency condition

$$\frac{1}{N} \text{tr} \frac{1}{\lambda - \Phi^*} = \frac{\int d\Phi e^{N \text{tr} (-V[\Phi] + D\eta^2 \Phi \Phi^*)} \frac{1}{N} \text{tr} \frac{1}{\lambda - \Phi^*}}{\int d\Phi e^{N \text{tr} (-V[\Phi] + D\eta^2 \Phi \Phi^*)}}. \quad (5.1)$$

This condition means that the saddle-point configuration of the integral over $\Phi$ coincides with $\Phi^*$ and $\lambda$ plays the role of a spectral parameter.

The matrix integral in Eq. (5.1) coincides with the partition function of the hermitian one-matrix model in an external field, $\Phi^*$. While its solution for a quartic potential $V[\Phi]$, which has been taken into account in Ref. [2], is not yet known, this model was explicitly solved in genus zero (the $N = \infty$ limit) for a cubic potential [12] and for a logarithmic potential [13]. We hope the obtained results could be useful in this context.

Acknowledgments

Yu.M. thanks A.Migdal, A.Morozov, G.Semenoff and N.Weiss for e-mail correspondences.

Added note

When this paper was being prepared for publication, there appeared more papers [14] on the Kazakov–Migdal model. The results by Gross agree with ours for the quadratic potential while the Monte–Carlo study by Gocksch and Shen seems to indicate that for $N = 2$ the ‘large-$N$’ phase transition coincides with the Higgs one.
References

[1] V.A.Kazakov and A.A.Migdal, *Induced QCD at large N*, Paris / Princeton preprint LPTENS-92/15 / PUPT-1322 (June, 1992)

[2] A.A.Migdal, *Exact solution of induced lattice gauge theory at large N*, Princeton preprint PUPT-1323 (June, 1992)

[3] A.A.Migdal, *1/N expansion and particle spectrum in induced QCD*, Princeton preprint PUPT-1332 (July, 1992)

[4] I.I.Kogan, G.W.Semenoff and N.Weiss, *Induced QCD and hidden local Z_N symmetry*, UBC preprint UBCTP-92-022 (June, 1992)

[5] S.B.Khokhlov and Yu.M.Makeenko, *Phys. Lett. 101B* (1981) 403; ZhETF 80 (1981) 448 (Sov. Phys. JETP 53 (1981) 228)

[6] I.G.Holliday and A.Schwimmer, *Phys. Lett. 101B* (1981) 327;
   J.Greensite and B.Lautrup, *Phys. Rev. Lett* 47 (1981) 9;
   G.Bhanot, *Phys. Lett. 108B* (1982) 337;
   M.Creutz and K.J.M.Moriarty, *Nucl. Phys. B210[FS6] 50*

[7] Yu.M.Makeenko and M.I.Polikarpov, *Nucl. Phys. B205[FS5] (1982) 386;
   S.Samuel, *Phys. Lett. 112B* (1982) 237, 122B (1983) 287

[8] J.Greensite and B.Lautrup, *Phys. Lett. 104B* (1981) 41;
   P.Cvitanović, J.Greensite and B.Lautrup, *Phys. Lett. 105B* (1981) 197;
   T.-L.Chen, C.-I Tan and X.-T.Zheng, *Phys. Lett. 109B* (1982) 383; *Phys. Rev. D26* (1982) 2843;
   M.C.Ogilvie and A.Horowitz, *Nucl.Phys. B215* (1983) 249

[9] I.G.Holliday and A.Schwimmer, *Phys. Lett. 102B* (1981) 337;
   R.C.Brower, D.A.Kessler and H.Levine, *Nucl. Phys. B205[FS5] (1982) 77;
   L.Caneschi, I.G.Holliday and A.Schwimmer, *Nucl. Phys. B200[FS4] (1982) 409

[10] B.Sakita, in *Proc. of Workshop on Nonperturbative Studies in Quantum Chromodynamics*, eds. A.Jevicki and C.-I Tan, Brown-HET-457, 1981, p.235

[11] V.A.Kazakov, T.A.Kozhamkulov and A.A.Migdal, *Pis’ma v ZhETF 41* (1985) 442; *Nucl.Phys. B285[FS19] (1987) 760

[12] V.A.Kazakov and I.K.Kostov, unpublished, as cited in I.K.Kostov, *Nucl. Phys. B Proc. Suppl. A10* (1989) 295;
   D.J.Gross and M.J.Newman, *Phys.Lett.*, 266B (1991) 291;
   Yu.Makeenko and G.Semenoff, *Mod.Phys.Lett.*, A6 (1991) 3455

[13] L.Chekhtov and Yu.Makeenko, *Phys. Lett. 278B* (1992) 271; *Mod. Phys. Lett. 7* (1992) 1223

[14] M.Caselle, A.D.’Adda and S.Panzeri, *Exact solution of D=1 Kazakov-Migdal induced gauge theory*, Turin preprint DFTT 38/92 (July, 1992);
   A.Gocksch and Y.Shen, *The phase diagram of the N = 2 Kazakov-Migdal model*, BNL preprint (July, 1992);
   D.Gross, *Some remarks about induced QCD*, Princeton preprint PUPT-1335 (August, 1992)