R-TORSION OF COMPACT ORIENTABLE SURFACES VIA PANTS DECOMPOSITION

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Abstract. Let \( \Sigma_{g,n} \) denote the compact orientable surface with genus \( g \geq 2 \) and boundary disjoint union of \( n \) circles. By using a particular pants-decomposition of \( \Sigma_{g,n} \), we obtain a formula that computes the Reidemeister torsion of \( \Sigma_{g,n} \) in terms of Reidemeister torsions of pairs of pants.

1. Introduction

The Reidemeister torsion (or R-torsion) was introduced by Reidemeister to classify 3 dimensional lens spaces [6]. This invariant was later generalized by Franz to other dimensions [2] and shown to be a topological invariant by Kirby-Siebenmann [3]. The R-torsion is also an invariant of the basis of the homology of a manifold [4]. Moreover, for compact orientable Riemannian manifolds the R-torsion is equal to the analytic torsion [1].

For a manifold \( M \) and an integer \( \eta \), we denote by \( h^\eta_M \) the basis of the homology \( H^\eta(M) = H^\eta(M; \mathbb{R}) \). Note that \( \Sigma_{2,0} \) is the double of \( \Sigma_{0,3} \). Let \( \Delta_{0,2}(\Sigma_{2,0}) \) be the matrix of the intersection pairing of \( \Sigma_{2,0} \) in the bases \( h^\Sigma_{2,0}, h^\Sigma_{2,0}, h^1_{\Sigma_{2,0}} = \{\omega_j\}_{i=1}^4 \) denote the Poincaré dual basis of \( H^1(\Sigma_{2,0}) \) corresponding to \( h^1_{\Sigma_{2,0}} \). We first prove the following theorem for the R-torsion of the pair of pants \( \Sigma_{0,3} \).

Theorem 1.0.1. For a given basis \( h^\Sigma_{0,3}, i = 0, 1 \), there is a basis \( h^\Sigma_{2,0}, \eta = 0, 1, 2 \) such that the following formula holds

\[
|T(\Sigma_{0,3}, \{h^\Sigma_{0,3}\}_0)| = \sqrt{\frac{|\det \Delta_{0,2}(\Sigma_{2,0})|}{|\det \varphi(h^1_{\Sigma_{2,0}}, \Gamma)|}},
\]

where \( \Gamma = \{\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4\} \) is the canonical basis for \( H_1(\Sigma_{2,0}) \), i.e. \( i = 1, 2, \Gamma_i \) intersects \( \Gamma_{i+2} \) once positively and does not intersect others, and \( \varphi(h^1_{\Sigma_{2,0}}, \Gamma) = [\int_{\Gamma} \omega_j] \) is the period matrix of \( h^1_{\Sigma_{2,0}} \) with respect to the basis \( \Gamma \).

By using the pants decomposition of \( \Sigma_{g,n} \) as in Figure 1, we prove the following theorem.

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Theorem 1.0.2. If $h_{g,n}^{\sum}$ is a given basis, $\eta = 0, 1$, then for each $\nu = 1, \ldots, 2g - 2 + n$ there exists a basis $h_{\eta}^{\sum_{0,3}}$ such that

$$|\mathbb{T}(\Sigma_{g,n}, \{h_{\eta}^{\sum_{g,n}}\}_0)| = \prod_{\nu=1}^{2g-2+n} |\mathbb{T}(\Sigma_{0,3}^{\nu}, \{h_{\eta}^{\sum_{0,3}}\}_0)|,$$

where $\Sigma_{0,3}^{\nu}$ is the pair of pants in the decomposition labelled by $\nu$.

2. R-torsion of a general chain complex

Let $C_s : 0 \to C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots \to C_1 \xrightarrow{\partial_1} C_0 \to 0$ be a chain complex of finite dimensional vector spaces over $\mathbb{R}$. Let $B_p(C_s) = \text{Im}(\partial_{p+1})$, $Z_p(C_s) = \text{Ker}(\partial_p)$, and $H_p(C_s) = Z_p(C_s)/B_p(C_s)$ denote the $p$-th homology of the chain complex $C_s$ for $p = 0, \ldots, n$. Then we have the following short exact sequences

(2.0.1) $0 \to Z_p(C_s) \xrightarrow{i} C_p(C_s) \xrightarrow{\partial_p} B_{p-1}(C_s) \to 0$,

(2.0.2) $0 \to B_p(C_s) \xrightarrow{j} Z_p(C_s) \xrightarrow{\varphi_p} H_p(C_s) \to 0$.

Here, $i$ and $\varphi_p$ are the inclusion and the natural projection, respectively. If we apply the Splitting Lemma to the above short exact sequences, then $C_p(C_s)$ can be expressed as the following direct sum

$$B_p(C_s) \oplus \ell_p(H_p(C_s)) \oplus s_p(B_{p-1}(C_s)).$$

Let $c_p$, $b_p$, and $h_p$ be respectively bases of $C_p(C_s)$, $B_p(C_s)$, and $H_p(C_s)$. Then we obtain a new basis $b_p \sqcup \ell_p(h_p) \sqcup s_p(b_{p-1})$ for $C_p(C_s)$.

Definition 2.0.1. The R-torsion of $C_s$ with respect to bases $\{c_p\}_0^n$, $\{h_p\}_0^n$ is defined by

$$\mathbb{T}(C_s, \{c_p\}_0^n, \{h_p\}_0^n) = \prod_{p=0}^{n} |b_p \sqcup \ell_p(h_p) \sqcup s_p(b_{p-1}), c_p|^{(-1)^{p+1}}.$$

Here, $|b_p \sqcup \ell_p(h_p) \sqcup s_p(b_{p-1}), c_p|$ is the determinant of the change-base-matrix from basis $c_p$ to $b_p \sqcup \ell_p(h_p) \sqcup s_p(b_{p-1})$ of $C_p(C_s)$.

The R-torsion of a general chain complex $C_s$ is an element of the dual of the vector space

$$\bigotimes_{p=0}^{n} (\det H_p(C_s))^{(-1)^{p}},$$

see [10] pp. 185 and [7] Thm. 2.0.6).

For a smooth $m$-manifold $M$ with a cell decomposition $K$, there is a chain complex

$$C_s(K) : 0 \to C_m(K) \xrightarrow{\partial_m} C_{m-1}(K) \to \cdots \to C_1(K) \xrightarrow{\partial_1} C_0(K) \to 0,$$

where $\partial_i$ is the usual boundary operator. The R-torsion of $M$ is defined as the R-torsion of its cellular chain complex $C_s(K)$ in the bases $\{c_i\}_0^m$ and $\{h_i\}_0^m$. Here, $c_i$ is the geometric basis for the $i$-cells $C_i(K)$, $i = 0, \ldots, m$. By [7] Lem. 2.0.5], the R-torsion of $M$ does not depend on the cell decomposition $K$. Thus, we write $\mathbb{T}(M, \{h_i\}_0^m)$ instead of $\mathbb{T}(C_s(K), \{c_i\}_0^m, \{h_i\}_0^m)$. For further details we refer to [7] [9].
Corollary 2.0.2. Let $Y = S^1 \times [-\epsilon, +\epsilon]$ be a cylinder with boundary circles $S^1 \times \{-\epsilon\}$ and $S^1 \times \{+\epsilon\}$, where $\epsilon > 0$. Let $h_i$ be a basis of $H_i(Y)$ for $i = 0, 1$. By Künneth formula, we have the isomorphisms: $C_i(Y) \cong C_i(S^1)$ and $H_i(Y) \cong H_i(S^1)$. Then [8, Thm. 3.5] gives the following result

$$|\mathbb{T}(Y, \{h_0, h_1\})| = |\mathbb{T}(S^1, \{[\varphi_0](h_0), [\varphi_1](h_1)\})| = 1.$$ 

3. Proofs of main results

Proof of Theorem 1.0.1 For any manifold $M$, let $C_*(M)$ denote the associated cellular chain complex.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Double of the pair of pants $\Sigma_{0,3}$.}
\end{figure}

Note that $\Sigma_{2,0}$ is the double of $\Sigma_{0,3}$. Let $\mathcal{B}$ be the intersection of the pairs of pants in $\Sigma_{2,0}$, so $\mathcal{B}$ is homeomorphic to the disjoint union of three circles, $S_1 \amalg S_2 \amalg S_3$. Then there is the natural short exact sequence of the chain complexes

\begin{equation}
0 \to C_*(\mathcal{B}) \to C_*(\Sigma_{0,3}) \oplus C_*(\Sigma_{0,3}) \to C_*(\Sigma_{2,0}) \to 0
\end{equation}

and the Mayer-Vietoris sequence associated to (3.0.1) is

\begin{equation}
\mathcal{H}_*: 0 \xrightarrow{\alpha} H_2(\Sigma_{2,0}) \xrightarrow{f} H_1(\mathcal{B}) \xrightarrow{\beta} H_1(\Sigma_{0,3}) \oplus H_1(\Sigma_{0,3}) \xrightarrow{h} H_1(\Sigma_{2,0}) \xrightarrow{g} H_0(\Sigma_{0,3}) \oplus H_0(\Sigma_{0,3}) \xrightarrow{k} H_0(\Sigma_{2,0}) \xrightarrow{\xi} 0.
\end{equation}

Let us denote by $C_p(\mathcal{H}_*)$ the vector spaces in (3.0.2) for $p = 0, \ldots, 6$ and consider the short exact sequences (2.0.1) and (2.0.2) for $\mathcal{H}_*$. Let us take the isomorphism $s_p : B_{p-1}(\mathcal{H}_*) \to s_p(B_{p-1}(\mathcal{H}_*))$ obtained by the First Isomorphism Theorem as a section of $C_p(\mathcal{H}_*) \to B_{p-1}(\mathcal{H}_*)$ for each $p$. By the exactness of $\mathcal{H}_*$, we get $Z_p(\mathcal{H}_*) = B_p(\mathcal{H}_*)$. Applying the Splitting Lemma to (2.0.2), we have

\begin{equation}
C_p(\mathcal{H}_*) = B_p(\mathcal{H}_*) \oplus s_p(B_{p-1}(\mathcal{H}_*)).
\end{equation}

Then the R-torsion of $\mathcal{H}_*$ with respect to basis $\{h_p\}_{0}^{n}$ is given as follows

$$\mathbb{T}(\mathcal{H}_*, \{h_p\}_{0}^{n}, \{0\}_{0}^{n}) = \prod_{p=0}^{n} [h_p', h_p]^{(-1)^{(p+1)}},$$

where $h_p' = b_p \amalg s_p(b_{p-1})$ for each $p$. In [4], Milnor proved that the R-torsion does not depend on bases $b_p$ and sections $s_p, \ell_p$. Therefore, we will choose a suitable bases $b_p$.
and sections \( s_p \) so that \( T(\mathcal{H}_s, \{h_p\}_{0}^{n}, \{0\}_{0}^{n}) = 1. \)

Let us consider the space \( C_0(\mathcal{H}_s) = H_0(\Sigma_{2,0}) \) in (3.0.3). Then \( \text{Im}(\ell) = 0 \) yields

\[
(3.0.4) \quad C_0(\mathcal{H}_s) = \text{Im}(k) \oplus s_0(\text{Im}(\ell)) = \text{Im}(k).
\]

Since \( \{(h_0^{\Sigma_{0,3}}, 0), (0, h_0^{\Sigma_{0,3}})\} \) is the basis of \( H_0(\Sigma_{0,3}) \oplus H_0(\Sigma_{0,3}) \),

\[
\{\alpha_{11}k(h_0^{\Sigma_{0,3}}, 0) + \alpha_{12}k(0, h_0^{\Sigma_{0,3}})\}
\]
can be taken as the basis \( \text{h}^{\text{Im}(k)} \) of \( \text{Im}(k) \), where \( (\alpha_{11}, \alpha_{12}) \) is a non-zero vector. By (3.0.3), \( \text{h}^{\text{Im}(k)} \) becomes the obtained basis \( h'_0 \) of \( C_0(\mathcal{H}_s) \). If we take the initial basis \( h_0 \) (namely, \( h_{0}^{\Sigma_{2,0}} \)) of \( C_0(\mathcal{H}_s) \) as \( h'_0 \), then

\[
(3.0.5) \quad [h'_0, h_0] = 1.
\]

If we use (3.0.3) for \( C_1(\mathcal{H}_s) = H_0(\Sigma_{0,3}) \oplus H_0(\Sigma_{0,3}) \), then we get

\[
(3.0.6) \quad C_1(\mathcal{H}_s) = \text{Im}(j) \oplus s_1(\text{Im}(k)).
\]

Note that the given basis \( h_1 \) of \( C_1(\mathcal{H}_s) \) is \( \{(h_0^{\Sigma_{0,3}}, 0), (0, h_0^{\Sigma_{0,3}})\} \). Since \( \text{Im}(j) \) is 1-dimensional subspace of 2-dimensional space \( C_1(\mathcal{H}_s) \), there is a non-zero vector \( (a_{21}, a_{22}) \) such that \( \{a_{21}(h_0^{\Sigma_{0,3}}, 0) + a_{22}(0, h_0^{\Sigma_{0,3}})\} \) is a basis of \( \text{Im}(j) \). In the previous step, the basis of \( \text{Im}(k) \) was chosen as \( \text{h}^{\text{Im}(k)} \) so

\[
s_1(\text{h}^{\text{Im}(k)}) = a_{11}(h_0^{\Sigma_{0,3}}, 0) + a_{12}(0, h_0^{\Sigma_{0,3}}).
\]

Then we obtain a non-singular \( 2 \times 2 \) matrix \( A = [a_{ij}] \) with entries in \( \mathbb{R} \). Let us choose the basis of \( \text{Im}(j) \) as

\[
\text{h}^{\text{Im}(j)} = \{-(\det A)^{-1}[a_{21}(h_0^{\Sigma_{0,3}}, 0) + a_{22}(0, h_0^{\Sigma_{0,3}})]\}.
\]

By (3.0.6), \( \{\text{h}^{\text{Im}(j)}, s_1(\text{h}^{\text{Im}(k)})\} \) becomes the obtained basis \( h'_1 \) of \( C_1(\mathcal{H}_s) \). Hence, we get

\[
(3.0.7) \quad [h'_1, h_1] = 1.
\]

Considering (3.0.3) for \( C_2(\mathcal{H}_s) = H_0(B) \), we obtain

\[
(3.0.8) \quad C_2(\mathcal{H}_s) = \text{Im}(i) \oplus s_2(\text{Im}(j)).
\]

Recall that \( \{h_{0}^{S_1}, h_{0}^{S_2}, h_{0}^{S_3}\} \) is the given basis \( h_2 \) of \( C_2(\mathcal{H}_s) \). Since \( \text{Im}(i) \) and \( s_2(\text{Im}(j)) \) are 2 and 1-dimensional subspaces of 3-dimensional space \( C_2(\mathcal{H}_s) \), there are non-zero vectors \( (b_{i1}, b_{i2}, b_{i3}), i = 1, 2, 3 \) such that \( \{\sum_{i=1}^{3} b_{ij} h_{0}^{S_i}\}_{j=1}^{2} \) is a basis of \( \text{Im}(i) \) and

\[
s_2(\text{h}^{\text{Im}(j)}) = \sum_{i=1}^{3} b_{ni} h_{0}^{S_i} \]

is a basis of \( s_2(\text{Im}(j)) \). Then \( 3 \times 3 \) real matrix \( B = [b_{ij}] \) is invertible. Let us choose the basis of \( \text{Im}(i) \) as follows

\[
\text{h}^{\text{Im}(i)} = \left\{ (\det B)^{-1} \sum_{i=1}^{3} b_{i1} h_{0}^{S_1}, \sum_{i=1}^{3} b_{i2} h_{0}^{S_2}, \sum_{i=1}^{3} b_{i3} h_{0}^{S_3} \right\}.
\]
By (3.0.8), \( \{ h^{im(i)}, s_4(h^{im(j)}) \} \) becomes the obtained basis \( h'_2 \) of \( C_2(H_*) \) and we have

\[
[h'_2, h_2] = 1.
\]

Using (3.0.3), \( C_3(H_*) = H_1(\Sigma) \) can be expressed as the following direct sum

\[
(3.0.10) \quad C_3(H_*) = \text{Im}(h) \oplus s_4(\text{Im}(i)).
\]

Note that the basis of \( H\Sigma \Sigma_{0,3} \) is \( \{(h_{1,1}^{\Sigma_3}, 0), (0, h_{1,1}^{\Sigma_3}), (h_{1,2}^{\Sigma_3}, 0), (0, h_{1,2}^{\Sigma_3}) \} \).

Since \( \text{Im}(h) \) is a 2-dimensional space, we can choose the basis of \( \text{Im}(h) \) as

\[
\begin{align*}
\text{Im}(h) = \{ & c_{11} h(h_{1,1}^{\Sigma_3}, 0) + c_{12} h(0, h_{1,1}^{\Sigma_3}) + c_{13} h(h_{1,2}^{\Sigma_3}, 0) + c_{14} h(0, h_{1,2}^{\Sigma_3}), \\
& c_{21} h(h_{1,1}^{\Sigma_3}, 0) + c_{22} h(h_{1,2}^{\Sigma_3}, 0) + c_{23} h(h_{1,2}^{\Sigma_3}, 0) + c_{24} h(0, h_{1,2}^{\Sigma_3}) \}.
\end{align*}
\]

Here, \( (c_{11}, c_{12}, c_{13}, c_{14}) \) is a non-zero vector for \( i = 1, 2, \) using (3.0.10), we have that \( \{ h^{im(h)}, s_3(h^{im(i)}) \} \) is the obtained basis \( h'_3 \) of \( C_3(H_*) \). If we take the initial basis \( h_3 \) (namely, \( h_{1,0} \)) of \( C_3(H_*) \) as \( h'_3 \), then we get

\[
(3.0.11) \quad [h'_3, h_3] = 1.
\]

If we consider (3.0.3) for \( C_4(H_*) = H_1(\Sigma) \oplus H_1(\Sigma) \), then we obtain

\[
(3.0.12) \quad C_4(H_*) = \text{Im}(g) \oplus s_4(\text{Im}(h)).
\]

Recall that \( \{(h_{1,1}^{\Sigma_3}, 0), (0, h_{1,1}^{\Sigma_3}), (h_{1,2}^{\Sigma_3}, 0), (0, h_{1,2}^{\Sigma_3}) \} \) is the given basis \( h_4 \) of \( C_4(H_*) \).

In the previous step, \( h^{im(h)} \) was chosen as the basis of \( \text{Im}(h) \) so

\[
s_4(h^{im(h)}) = \begin{cases} c_{11} (h_{1,1}^{\Sigma_3}, 0) + c_{12} (0, h_{1,1}^{\Sigma_3}) + c_{13} (h_{1,2}^{\Sigma_3}, 0) + c_{14} (0, h_{1,2}^{\Sigma_3}), \\
c_{21} (h_{1,1}^{\Sigma_3}, 0) + c_{22} (h_{1,2}^{\Sigma_3}, 0) + c_{23} (h_{1,2}^{\Sigma_3}, 0) + c_{24} (0, h_{1,2}^{\Sigma_3}) \end{cases}
\]

is a basis of \( s_4(\text{Im}(h)) \). As \( \text{Im}(g) \) is a 2-dimensional subspace of 4-dimensional space \( C_4(H_*) \), there are non-zero vectors \( (c_{i1}, c_{i2}, c_{i3}, c_{i4}) \), \( i = 3, 4 \) such that

\[
\begin{align*}
\{ & c_{31} (h_{1,1}^{\Sigma_3}, 0) + c_{32} (0, h_{1,1}^{\Sigma_3}) + c_{33} (h_{1,2}^{\Sigma_3}, 0) + c_{34} (0, h_{1,2}^{\Sigma_3}), \\
& c_{41} (h_{1,1}^{\Sigma_3}, 0) + c_{42} (h_{1,1}^{\Sigma_3}, 0) + c_{43} (h_{1,2}^{\Sigma_3}, 0) + c_{44} (0, h_{1,2}^{\Sigma_3}) \}
\end{align*}
\]

is a basis of \( \text{Im}(g) \) and \( C = [c_{ij}] \) is the non-singular \( 4 \times 4 \) real matrix. Thus, we can choose the basis of \( \text{Im}(g) \) as

\[
\text{Im}(g) = \begin{cases} (\det(C))^{-1} [c_{31} (h_{1,1}^{\Sigma_3}, 0) + c_{32} (0, h_{1,1}^{\Sigma_3}) + c_{33} (h_{1,2}^{\Sigma_3}, 0) + c_{34} (0, h_{1,2}^{\Sigma_3})], \\
c_{41} (h_{1,1}^{\Sigma_3}, 0) + c_{42} (0, h_{1,1}^{\Sigma_3}) + c_{43} (h_{1,2}^{\Sigma_3}, 0) + c_{44} (0, h_{1,2}^{\Sigma_3}) \end{cases}
\]

By (3.0.12), \( \{ h^{im(g)}, s_4(h^{im(h)}) \} \) becomes the obtained basis \( h'_4 \) of \( C_4(H_*) \) and the following equation holds

\[
(3.0.13) \quad [h'_4, h_4] = 1.
\]
Consider the space $C_5(H_*) = H_1(B)$, then (3.0.3) becomes
\[(3.0.14) \quad C_5(H_*) = \text{Im}(f) \oplus s_5(\text{Im}(g)).\]
Recall that the initial basis $h_5$ of $C_5(H_*)$ is \{h_{1}^{5}, h_{1}^{5}, h_{1}^{5}\}. Since Im($f$) and $s_5(\text{Im}(g))$ are respectively 1 and 2-dimensional subspaces of 3-dimensional space $C_5(H_*)$, there are non-zero vectors $(d_{1i}, d_{2i}, d_{3i})$, $i = 1, 2, 3$ such that \(\sum_{i=1}^{3} d_{1i} h_{1i}^{5}\) is a basis of Im($f$) and
\[s_5(\text{Im}(g)) = \left\{ \sum_{i=1}^{3} d_{2i} h_{1i}^{5}, \sum_{i=1}^{3} d_{3i} h_{1i}^{5} \right\}\]
is a basis of $s_5(\text{Im}(g))$. Then we get a non-singular 3 x 3 real matrix $D = [d_{ij}]$. Let us choose the basis of Im($f$) as
\[h_{\text{Im}(f)} = \left\{ (\det D)^{-1} \sum_{i=1}^{3} d_{1i} h_{1i}^{5} \right\}.\]
By (3.0.14), \(\{h_{\text{Im}(f)}, h_5(\text{Im}(g))\}\) becomes the obtained basis $h_5'$ of $C_5(H_*)$. Hence, we obtain
\[(3.0.15) \quad [h_5', h_5] = 1.\]
Finally, let us consider $C_6(H_*) = H_2(\Sigma_{2,0})$. Since Im($\alpha$) = 0, (3.0.3) becomes
\[(3.0.16) \quad C_6(H_*) = \text{Im}(\alpha) \oplus s_6(\text{Im}(f)) = s_6(\text{Im}(f)).\]
From (3.0.16) it follows that $s_6(\text{Im}(f))$ is the obtained basis $h_6'$ of $C_6(H_*)$. If we take the initial basis $h_6$ (namely, $h_{2}^{5,0}$) of $C_6(H_*)$ as $s_6(\text{Im}(f))$, then we have
\[(3.0.17) \quad [h_6', h_6] = 1.\]
If we combine (3.0.5), (3.0.7), (3.0.9), (3.0.11), (3.0.13), (3.0.15), and (3.0.17), then we get
\[(3.0.18) \quad T(H_*, \{h_p\}_0, \{0\}_0) = \prod_{p=0}^{6} [h_p', h_p]^{(-1)(p+1)} = 1.\]
Since the natural bases in (3.0.11) are compatible, [4] Thm. 3.2] yields
\[(3.0.19) \quad T(H_*, \{h_{\Sigma,3}^{0,3}\}_0)^2 = \prod_{j=1}^{3} T(S_{j}, \{h_{\Sigma,1}^{j}\}_0) T(H_*, \{h_{\Sigma,2,0}^{0,3}\}_0) T(H_*, \{h_p\}_0, \{0\}_0).\]
Considering [8] Thm. 3.5], (3.0.18], and (3.0.19], we obtain
\[(3.0.20) \quad |T(H_*, \{h_{\Sigma,3}^{0,3}\}_0)| = \sqrt{T(H_*, \{h_{\Sigma,2,0}^{0,3}\}_0)^2}.\]
By Poincaré Duality, Theorem 4.1 in [8] and (3.0.20], the main formula holds
\[|T(H_*, \{h_{\Sigma,3}^{0,3}\}_0)| = \frac{||\det \Delta_{0,2}(\Sigma_{2,0})||}{||\det \varphi(h_{\Sigma_{2,0}}^{1,1}, \Gamma)||}.\]
\[\square\]
A pants decomposition of $\Sigma_{g,n}$ is a finite collection of disjoint smoothly embedded circles cutting $\Sigma_{g,n}$ into pairs of pants $\Sigma_{0,3}$ and tori with one boundary circle $\Sigma_{1,1}$. The number of complementary components is $|\chi(\Sigma_{g,n})| = 2g - 2 + n$.

![Diagram of a pants decomposition](image)

**Figure 2.** Compact orientable surface $\Sigma_{g,n}$ with genus $g \geq 2$ and bordered by $n \geq 1$ circles.

**Proof of Theorem 1.0.2.** Consider the decomposition of $\Sigma_{g,n}$, as in Figure 1, obtained by cutting the surface along the circles in the following order

$S_1, \ldots, S_g, S_{g+1}, \ldots, S_{2g-3+n}$.

This decomposition consists of
- the torus $\Sigma_{1,1}'$ with boundary circle $S_\nu$, $\nu = 1, \ldots, g$,
- the pair of pants $\Sigma_{2g,3}'$ with boundaries $S_1, S_2, S_{g+1}$,
- the pair of pants $\Sigma_{2g,3}''$ with boundaries $S_{g+6}, S_{g+7}, S_{g+2}$, $\nu = 2, \ldots, g - 1$,
- the pair of pants $\Sigma_{2g,3}'''$ with boundaries $S_{g+6}, S_{g+7}, S_{g-6}, \nu = g, \ldots, g + n - 3$,
- the pair of pants $\Sigma_{2g,3}^0$ with boundaries $S_{2g-3, n-3}, S_{(n-1)}$, $S_{(n-2)}$.

Consider also the decomposition $\Sigma_{1,1}' = Y_\nu \cup Y_\nu'$ of $\Sigma_{0,3}$, $\nu = 1, \ldots, g$, where $Y_\nu$ is the cylinder $S_\nu \times [-\varepsilon, +\varepsilon]$ and $Y_\nu'$ is the pair of pants with boundaries $S_\nu \times \{-\varepsilon\}, S_\nu' \times \{\varepsilon\}$, for sufficiently small $\varepsilon > 0$.

**Case 1:** Consider the decomposition $\Sigma_{0,3} \cup \Sigma_{0,n-1}$ of $\Sigma_{0,n}$ for $n \geq 4$, where $\Sigma_{0,3}$ and $\Sigma_{0,n-1}$ are glued along the common boundary circle $S_1$. Then there is a short exact sequence of the chain complexes

$$0 \to C_*(S_1) \to C_*(\Sigma_{0,3}) \oplus C_*(\Sigma_{0,n-1}) \to C_*(\Sigma_{0,n}) \to 0$$

and the corresponding Mayer-Vietoris sequence $\mathcal{H}_*$. By using the arguments stated in the proof of Theorem 1.0.1 for the given bases $h_\eta^0$, $h_\eta^1$, $\eta = 0, 1$, there exist bases $h_{\Sigma_{0,3}}^0$ and $h_{\Sigma_{0,n-1}}^1$ such that the R-torsion of $\mathcal{H}_*$ in the corresponding bases is 1 and

$$\tau(\Sigma_{0,n}, \{h_{\eta}^{\Sigma_{0,n}}\}^1_0) = \tau(\Sigma_{0,3}, \{h_{\eta}^{\Sigma_{0,3}}\}^1_0) \tau(\Sigma_{0,n-1}, \{h_{\eta}^{\Sigma_{0,n-1}}\}^1_0) \tau(\Sigma_{1}, \{h_{\eta}^{\Sigma_{1}}\}^1_0)^{-1}.$$  

By [8, Thm. 3.5] and (3.0.21), we obtain

$$|\tau(\Sigma_{0,n}, \{h_{\eta}^{\Sigma_{0,n}}\}^1_0)| = |\tau(\Sigma_{0,3}, \{h_{\eta}^{\Sigma_{0,3}}\}^1_0)||\tau(\Sigma_{0,n-1}, \{h_{\eta}^{\Sigma_{0,n-1}}\}^1_0)|.$$
Applying (3.0.22) inductively, we get
\[ |T(\Sigma_{0,n}, \{h^\Sigma_{0,n,1,0}\})| = \prod_{\nu=1}^{n-2} |T(\Sigma_{0,3}, \{h^\Sigma_{0,3,1,0}\})|. \]

**Case 2**: For the decomposition \( \Sigma_{1,1} = Y \cup_{\partial Y} \Sigma_{0,3} \), where \( Y = S' \times [-\varepsilon, +\varepsilon] \), \( \partial Y = S' \times \{-\varepsilon\} \cup S' \times \{+\varepsilon\} \), and \( \Sigma_{0,3} \) is the pair of pants with boundaries \( S' \times \{-\varepsilon\}, S' \times \{+\varepsilon\} \), \( \Sigma \) for sufficiently small \( \varepsilon > 0 \), we have the following short exact sequence of the chain complexes
\[ (3.0.23) \quad 0 \to C_*(\Sigma_{0,3} \cap Y) \to C_*(\Sigma_{0,3}) \oplus C_*(Y) \to C_*(\Sigma_{1,1}) \to 0 \]
and the corresponding Mayer-Vietoris sequence \( H_* \). If we follow the arguments in the proof of Theorem 1.0.1 for the given bases \( h^\Sigma_{1,1} \) and \( h^S_{\eta} \), \( \eta = 0, 1 \), then we get the bases \( h^\Sigma_{0,3} \) and \( h^S_{\eta} \) such that the R-torsion of \( H_* \) in the corresponding bases equals to 1 and the formula is valid
\[ T(\Sigma_{1,1}, \{h^\Sigma_{1,1,0}\}) = T(\Sigma_{0,3}, \{h^\Sigma_{0,3,1,0}\}) T(Y, \{h^S_{\eta,0}\}) T(S', \{h^S_{\eta,1}\})^{-2}. \]

From [8] Thm. 3.5] and Corollary [2.0.2] it follows
\[ |T(\Sigma_{1,1}, \{h^\Sigma_{1,1,0}\})| = |T(\Sigma_{0,3}, \{h^\Sigma_{0,3,1,0}\})|. \]

**Case 3**: Let \( \Sigma_{g-1,1} \cup_{S} \Sigma_{1,1} \) be the decomposition of \( \Sigma_{g,0} \), \( g \geq 2 \), where \( \Sigma_{1,1} \) and \( \Sigma_{g-1,1} \) are glued along the common boundary circle \( S_0 \). By the decomposition, there exists the natural short exact sequence
\[ 0 \to C_*(S_0) \to C_*(\Sigma_{g-1,1}) \oplus C_*(\Sigma_{1,1}) \to C_*(\Sigma_{g,0}) \to 0 \]
and its corresponding Mayer-Vietoris sequence
\[ H_* : 0 \to H_2(\Sigma_{g,0}) \xrightarrow{\delta_2} H_1(S_0) \xrightarrow{j} H_1(\Sigma_{g-1,1}) \oplus H_1(\Sigma_{1,1}) \xrightarrow{\partial} H_1(\Sigma_{g,0}) \]
\[ \xrightarrow{\delta_0} H_0(\Sigma_{g,0}) \xrightarrow{i} H_0(\Sigma_{g-1,1}) \oplus H_0(\Sigma_{1,1}) \xrightarrow{j} H_0(\Sigma_{g,0}) \]
For the given bases \( h^\Sigma_{g,0} \) and \( h^S_{g,1} \) with the condition \( \delta_2(h^\Sigma_{g,0}) = h^S_{g,1}, \nu = 0, 1, 2, \eta = 0, 1 \), if we use the arguments stated in the proof of Theorem 1.0.1[1,0.1] then we obtain the bases \( h^\Sigma_{0,1} \) and \( h^\Sigma_{1,1} \) such that the R-torsion of \( H_* \) in the corresponding bases becomes 1 and the following formula holds
\[ T(\Sigma_{g,0}, \{h^\Sigma_{g,0,1,0}\}) = T(\Sigma_{g-1,1}, \{h^\Sigma_{g-1,1,0}\}) T(S_0, \{h^S_{g,1}\})^{-1}. \]

By [8] Thm. 3.5, we obtain
\[ |T(\Sigma_{g,0}, \{h^\Sigma_{g,0,1,0}\})| = |T(\Sigma_{g-1,1}, \{h^\Sigma_{g-1,1,0}\})| |T(S_0, \{h^S_{g,1}\})^{-1}|. \]

**Case 4**: Consider the decomposition \( \Sigma_{g,n} = \Sigma_{g-1,n+1} \cup_{S_1} \Sigma_{1,1} \) for \( g \geq 2, n \geq 1 \), where \( \Sigma_{1,1} \) and \( \Sigma_{g-1,n+1} \) are glued along the common boundary circle \( S_1 \). Then there is the natural short exact sequence of the chain complexes
\[ (3.0.24) \quad 0 \to C_*(S_1) \to C_*(\Sigma_{g-1,n+1}) \oplus C_*(\Sigma_{1,1}) \to C_*(\Sigma_{g,n}) \to 0, \]
and the corresponding Mayer-Vietoris sequence $\mathcal{H}_*$. Using the arguments in the proof of Theorem 1.0.1 for the given bases $h^{g,n}_\eta$ and $h^{g,1}_\eta$, $\eta = 0, 1$, we get the bases $h^{g-1,n+1}_\eta$ and $h^{g,1}_\eta$ such that the R-torsion of $\mathcal{H}_*$ in the corresponding bases is 1 and $T(\Sigma_{g,n}, \{h^{g,n}_\eta\}_0) = T(\Sigma_{g-1,n+1}, \{h^{g-1,n+1}_\eta\}_0) T(\Sigma_{1,1}, \{h^{g,1}_\eta\}_0) T(S_1, \{h^{g,1}_\eta\}_0)^{-1}$.

By [8] Thm. 3.5, the R-torsion of $\Sigma_{g,n}$ satisfies the following formula

$$|T(\Sigma_{g,n}, \{h^{g,n}_\eta\}_0)| = |T(\Sigma_{g-1,n+1}, \{h^{g-1,n+1}_\eta\}_0)| |T(\Sigma_{1,1}, \{h^{g,1}_\eta\}_0)|$$

Applying the Cases 1-4 inductively, we have the following R-torsion formula for the compact orientable surfaces $\Sigma_{g,n}$, $g \geq 2$, $n \geq 0$

$$|T(\Sigma_{g,n}, \{h^{g,n}_\eta\}_0)| = \prod_{\nu=1}^{2g-2+n} |T(\Sigma_{0,3}, \{h^{\nu,0}_\eta\}_0)|^2.$$

$\square$

4. Applications

4.1. Compact 3-manifolds with boundary. Let $N$ be a smooth compact orientable 3-manifold whose boundary consists of finitely many closed orientable surfaces $\partial N = \Sigma_{g_1,0} \sqcup \Sigma_{g_2,0} \sqcup \cdots \sqcup \Sigma_{g_n,0}$. Let $d(N)$ be the double of $N$. Consider the natural short exact sequence of the chain complexes

$$0 \to C_\ast(\partial N) \to C_\ast(N) \oplus C_\ast(d(N)) \to C_\ast(d(N)) \to 0$$

and the corresponding Mayer-Vietoris sequence $\mathcal{H}_*$. For the given bases $h^N_{\mu}$, $h^\partial N_{\nu}$, and $h^d(N) \eta$, $\nu = 0, 1, 2$, $\mu = 0, 1, 2, 3$, we will denote the corresponding basis of $\mathcal{H}_*$ by $h_n$, $n = 0, \ldots, 11$. As the bases in the sequence (4.1.1) are compatible, [8] Thm. 3.2 yields

$$T(N, \{h^N_{\mu}\}_0^2) = T(\partial N, \{h^\partial N_{\nu}\}_0^2) T(d(N), \{h^d(N)\}_0^3) T(\mathcal{H}_*, \{h_n\}_0^{11}).$$

By [8] Thm. 3.5 and (4.1.2), we have

$$|T(N, \{h^N_{\mu}\}_0^2)| = \sqrt{|T(\partial N, \{h^\partial N_{\nu}\}_0^2)||T(\mathcal{H}_*, \{h_n\}_0^{11})|}.$$

Note that $\partial N$ is equal to $\Sigma_{g_1,0} \sqcup \Sigma_{g_2,0} \sqcup \cdots \sqcup \Sigma_{g_n,0}$. By [8] Lem. 1.4, we get

$$|T(\partial N, \{h^\partial N_{\nu}\}_0^2)| = \prod_{i=1}^{m} |T(\Sigma_{g_i,0}, \{h^{g_i,0}_\nu\}_0^2)|.$$

For each $i = 1, \ldots, m$, consider the given basis $h^{g_i,0}_\nu$ for $\nu = 0, 1, 2$ and pants-decompositions $\{\Sigma^{j,i}_{0,3}\}_{j=1}^{2g_i-2}$ of $\Sigma_{g_i,0}$. By using Theorem 1.0.2, we obtain the basis $h^{g_i,0,j,i}_\eta$, $\eta = 0, 1$, $j = 1, \ldots, 2g_i - 2$ such that

$$|T(\partial N, \{h^\partial N_{\nu}\}_0^2)| = \prod_{i=1}^{m} \prod_{j=1}^{2g_i-2} |T(\Sigma^{j,i}_{0,3}, \{h^{g_i,0,j,i}_\eta\}_0)|.$$
Equations (4.1.4) and (4.1.5) yield the following formula

\[
|T(N, \{h^N_\mu\}_0^3)| = \prod_{i=1}^{m} \prod_{j=1}^{2g-2} |T(\Sigma^j_{0,3}, \{h^{\Sigma^j_{0,3}}_\eta\}_0^1)\| T(\mathcal{H}_*, \{h_i\}_0^1)|.
\]

Corollary 4.1.1. Let \(N\) be the handlebody of genus \(g \geq 2\). Clearly, the boundary \(\partial N\) of \(N\) is an orientable closed surface \(\Sigma_{g,0}\) and the double \(d(N)\) of \(N\) is equal to \(#(S \times S^2)\).

Then, we have the short exact sequence

(4.1.6) \[0 \to C_\ast(\Sigma_{g,0}) \to C_\ast(N) \oplus C_\ast(N) \to C_\ast(d(N)) \to 0\]

and the corresponding Mayer-Vietoris sequence \(\mathcal{H}_\ast\). For the given bases \(h^{d(N)}_\mu\) and \(h^N_\mu\) \(\mu = 0, \ldots, 3\), following the arguments above, there exists a basis \(h^{\Sigma_{g,0}}_i\); \(i = 0, 1, 2\) such that in the corresponding bases the R-torsion of \(\mathcal{H}_\ast\) is 1 and from [8, Thm. 3.5] it follows

\[|T(N, \{h^N_\mu\}_0^3)| = \sqrt{|T(\Sigma_{g,0}, \{h^{\Sigma_{g,0}}_i\}_0)|}.\]

Let us consider the pants-decomposition \(\{\Sigma^j_{0,3}\}_{j=1}^{2g-2}\) of \(\Sigma_{g,0}\). By Theorem 1.0.2, there exists the basis \(h^{\Sigma^j_{0,3}}_\eta\) for each \(j = 1, \ldots, 2g - 2, \eta = 0, 1\) and the formula holds

\[|T(N, \{h^N_\mu\}_0^3)| = \prod_{j=1}^{2g-2} |T(\Sigma^j_{0,3}, \{h^{\Sigma^j_{0,3}}_\eta\}_0)|.\]

4.2. Product of 2d-manifolds and compact 3-manifolds with boundary \(\Sigma_{g,0}\).

Let \(M\) be a smooth closed orientable 2d-manifold \((d \geq 1)\) and \(N\) an smooth compact orientable 3-manifold whose boundary consists of closed orientable surface \(\Sigma_{g,0}\) \((g \geq 2)\). Let \(X\) be the product manifold \(M \times N\) and \(d(X)\) denote the double of \(X\). Clearly, the boundary of \(X\) is \(M \times \Sigma_{g,0}\). Consider the natural short exact sequence of the chain complexes

(4.2.1) \[0 \to C_\ast(M \times \Sigma_{g,0}) \to C_\ast(X) \oplus C_\ast(X) \to C_\ast(d(X)) \to 0\]

and the Mayer-Vietoris sequence \(\mathcal{H}_\ast\) corresponding to (4.2.1). Let \(h^X_i, h^{d(X)}_\nu, h^M_i, \) and \(h^{\Sigma_{g,0}}_\ell\) be given bases for \(i = 0, \cdots, 2d + 3\), \(k = 0, \ldots, 2d\), \(\ell = 0, 1, 2\). Let \(h^M_{\nu \times \Sigma_{g,0}}\) denote the basis \(\oplus h^M_i \oplus h^{\Sigma_{g,0}}_\nu\) of \(H_\nu(M \times \Sigma_{g,0})\), \(\nu = 0, \ldots, 2d + 2\). For \(n = 0, \ldots, 8d + 11\), let \(h_n\) be the corresponding basis of \(\mathcal{H}_\ast\). Let \(\{\Sigma^j_{0,3}\}_{j=1}^{2g-2}\) be the pants-decomposition of \(\Sigma_{g,0}\). Since the bases in the sequence (4.2.1) are compatible and [8] Lem. 1.4, we obtain

(4.2.2) \[T(X, \{h^X_i\}_0^{2d+3}) = T(M \times \Sigma_{g,0}, \{h^M_{\nu \times \Sigma_{g,0}}\}_0^{2d+2}) T(d(X), \{h^{d(X)}_\nu\}_0^{2d+3})
\times T(\mathcal{H}_*, \{h_n\}_0^{8d+11}).\]

From [8, Thm. 3.5] and (4.2.2) it follows that

(4.2.3) \[|T(X, \{h^X_i\}_0^{2d+3})| = |T(M \times \Sigma_{g,0}, \{h^M_{\nu \times \Sigma_{g,0}}\}_0^{2d+2})|^{1/2} |T(\mathcal{H}_*, \{h_n\}_0^{8d+11})|^{1/2}.\]

By [5, Thm. 3.1], the R-torsion of \(M \times \Sigma_{g,0}\) satisfies the equality

(4.2.4) \[|T(M \times \Sigma_{g,0}, \{h^M_{\nu \times \Sigma_{g,0}}\}_0^{2d+2})| = |T(M, \{h^M_\kappa\}_0^{2d})| \chi(\Sigma_{g,0}) |T(\Sigma_{g,0}, \{h^{\Sigma_{g,0}}_\ell\}_0^2)| \chi(M).\]
Here, $\chi$ is the Euler characteristic. Then equations (4.2.3) and (4.2.4) yield
\[
|\mathcal{T}(X, \{h_i^X\}_{0}^{2d+3})| = |\mathcal{T}(M, \{h_k^M\}_{0}^{2d})|^{\chi(\Sigma_g, 0)/2} |\mathcal{T}(\Sigma_g, 0, \{h_{\xi}^{\Sigma_g, 0}\}_{0}^{2})|^{\chi(M)/2} \\
\times |\mathcal{T}(\mathcal{H}, \{h_n\}_{0}^{8d+11})|^{1/2}.
\] (4.2.5)

Since $\{\Sigma_j^{g, 0}\}_{j=1}^{2g-2}$ is the pants-decomposition of $\Sigma_g, 0$ as in Theorem 1.0.2, there exists a basis $h_{\eta_0}^{\Sigma_j^{0, 3}}$ of $H_{\eta}(\Sigma_j^{0, 3}), j = 1, \ldots, 2g - 2, \eta = 0, 1$ so that
\[
|\mathcal{T}(\Sigma_g, 0, \{h_{\xi}^{\Sigma_g, 0}\}_{0}^{2})| = \prod_{j=1}^{2g-2} |\mathcal{T}(\Sigma_j^{0, 3}, \{h_{\eta}^{\Sigma_j^{0, 3}}\}_{0}^{1})|.
\] (4.2.6)

Equations (4.2.5) and (4.2.6) yield
\[
|\mathcal{T}(X, \{h_i^X\}_{0}^{2d+3})| = \prod_{j=1}^{2g-2} |\mathcal{T}(\Sigma_j^{0, 3}, \{h_{\eta}^{\Sigma_j^{0, 3}}\}_{0}^{1})|^{\chi(\Sigma_g, 0)/2} |\mathcal{T}(M, \{h_k^M\}_{0}^{2d})|^{\chi(M)/2} \\
\times |\mathcal{T}(\mathcal{H}, \{h_n\}_{0}^{8d+11})|^{1/2}.
\]

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