INVERSION FORMULAE FOR SIEGEL TRANSFORMS

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Abstract. Let \( n \in \mathbb{Z}_{\geq 3} \) be given. We prove Lebesgue-almost everywhere pointwise inversion formulae for the Siegel transforms in the geometry of numbers. These inversion formulae are quite general; for instance, they are valid for the Siegel transforms of any even and compactly supported Borel measurable function \( f : \mathbb{R}^n \to \mathbb{R} \) that belongs to \( L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \).

Notation 1.1. Let \( n \in \mathbb{Z}_{\geq 3} \) be given. We regard elements of \( \mathbb{R}^n \) as column vectors. Let \( G := \text{SL}_n(\mathbb{R}) \) and \( \Gamma := \text{SL}_n(\mathbb{Z}) \). Let \( e_1 \) denote the first standard basis vector in \( \mathbb{R}^n \). Let \( H \) denote the stabilizer in \( G \) of \( e_1 \) when \( G \) acts on \( \mathbb{R}^n \) in the usual fashion. Set \( \Gamma_{\infty} := \Gamma \cap H \). We identify \( G/H \) with \( \mathbb{R}^n_{\neq 0} := (\mathbb{R}^n \setminus \{0\}) \) via the bijective correspondence \( gH \leftrightarrow ge_1 \). We identify \( X := G/\Gamma \) with the space of all covolume one full-rank lattices in \( \mathbb{R}^n \) via the bijective correspondence \( g\Gamma \leftrightarrow g\mathbb{Z}^n \). Let \( m \) denote the Lebesgue measure on \( \mathbb{R}^n \). Let \( \mu_G \) denote the Haar measure on \( G \) that is normalized so that any fundamental domain for \( X \) in \( G \) has \( \mu_G \)-measure equal to one. Let \( \mu_X \) denote the unique \( G \)-invariant Radon probability measure on \( X \). Let \( \mu_G/H \) denote the \( G \)-invariant Radon measure on \( G/H \) that is obtained by pushing forward the measure \( m \) on \( \mathbb{R}^n_{\neq 0} \) by the aforementioned bijective correspondence \( \mathbb{R}^n_{\neq 0} \to G/H \), which is a homeomorphism. Let \( \mu_{G/\Gamma_{\infty}} \) denote the counting measure on \( G/\Gamma_{\infty} \). Let \( \mu_{G/\Gamma_{\infty}} \) denote the unique \( G \)-invariant Radon measure on \( G/\Gamma_{\infty} \) for which the usual unfolding formula holds for the triple \( (\mu_{G/\Gamma_{\infty}}, \mu_X, \mu_{\Gamma/\Gamma_{\infty}}) \). Let \( \mu_{H/\Gamma_{\infty}} \) denote the unique \( H \)-invariant Radon measure on \( H/\Gamma_{\infty} \) for which the unfolding formula holds for the triple \( (\mu_{G/\Gamma_{\infty}}, \mu_G/H, \mu_{H/\Gamma_{\infty}}) \). Let \( \zeta \) denote the Euler–Riemann zeta function. It is then a result of C. L. Siegel that \( \mu_{H/\Gamma_{\infty}}(H/\Gamma_{\infty}) = (\zeta(n))^{-1} \); see [6]; see also [8] or [7] Proposition 1.4.2 for the proof of A. Weil, which uses Poisson summation, of Siegel’s result. Finally, for any \( \Lambda \in X \), we write \( \Lambda_{\text{pr}} \) to denote the set of all primitive points of \( \Lambda \); an arbitrary point \( v \in \Lambda \) is said to be primitive if the following is true: for any \( k \in \mathbb{Z}_{\geq 1} \) and any \( w \in \Lambda \) for which \( v = kw \), we have \( k = 1 \) and \( v = w \). Notice that for any \( g \in G \), we have \( (g\mathbb{Z}^n)_{\text{pr}} = g(\mathbb{Z}^n_{\text{pr}}) \). For each \( p \in \{1, +\infty\} \subset \mathbb{R} \), we adopt the notational convention that the elements of any \( L^p \) space are always real-valued. As usual, we say that a function \( f : \mathbb{R}^n \to \mathbb{R} \) is even (respectively, odd) if for every \( z \in \mathbb{R}^n \), we have \( f(z) = f(-z) \) (respectively, \( f(z) = -f(-z) \)).

Remark 1.2. Throughout this note, we identify any two mappings that are equal almost everywhere; similarly, we regard any mapping that is defined almost everywhere as being defined everywhere.

Definition 1.3. Let \( \mathcal{F} \) denote the set of all Borel measurable functions \( f : \mathbb{R}^n \to \mathbb{R} \) that belong to \( L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \) and that satisfy the following condition: for each \( g \in G \), 
\[
\sum_{v \in \mathbb{Z}^n_{\text{pr}}} |f(g(v))| < +\infty.
\]

Let \( \mathcal{F}_{\text{even}} \) denote the set of all even elements of \( \mathcal{F} \). Note that each of \( \mathcal{F} \) and \( \mathcal{F}_{\text{even}} \) is a real vector space.
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Second, we may identify $\Gamma/\Gamma_\infty$ with $\mathbb{Z}_n^{pr}$ via the bijective correspondence $\gamma \Gamma_\infty \leftrightarrow \gamma e_1$. We may thus regard $\hat{f}$ as the pseudo-Eisenstein series given by

$$g\Gamma \mapsto \sum_{[\gamma] \in \Gamma/\Gamma_\infty} f(g\gamma) = \int_{\Gamma/\Gamma_\infty} f(g\gamma) d\mu_{\Gamma/\Gamma_\infty}(\gamma).$$

Thanks to the preceding characterization of the primitive Siegel transform as a pseudo-Eisenstein series, one may regard the primitive Siegel transform as a particular example of a Radon transform for homogeneous spaces in duality, per the terminology of integral geometry in the fashion of S. Helgason: see [1, Chapter II, §§1–2] and, especially, [1, Chapter II, §4, Example H]. In the field of integral geometry, an important problem is to determine whether a given Radon transform is injective; if the transform is injective, then one seeks to prove an inversion formula: see, for instance, [1, Chapter II, §2, (iii)].

Our first objective in this note is to establish that the restriction of the primitive Siegel transform to $\mathcal{F}_\text{even}$ is injective and then prove a Lebesgue-almost everywhere pointwise inversion formula for it. To this end, we shall use well-known mean and inner product formulae for the primitive Siegel transform. We note here that the primitive Siegel transform of every odd function in $\mathcal{F}$ is identically zero.

In the seminal paper [3], Siegel proved the following theorem, usually known as the Siegel mean value theorem.

**Theorem 1.5.** [3] Let $f \in \mathcal{F}$. Then

$$\int_X \hat{f} \, d\mu_X = \frac{1}{\zeta(n)} \int_{\mathbb{R}^n} f \, dm.$$

In the paper [5], C.A. Rogers proved a theorem that immediately yields the following inner product formula as a special case.

**Theorem 1.6.** [5] Theorem 5\textsuperscript{1}

Let $f_1, f_2 \in \mathcal{F}$. Then

$$\int_X \hat{f}_1 \hat{f}_2 \, d\mu_X = (\zeta(n))^{-2} \left( \int_{\mathbb{R}^n} f_1 \, dm \right) \left( \int_{\mathbb{R}^n} f_2 \, dm \right) + (\zeta(n))^{-1} \int_{\mathbb{R}^n} (f_1(z) [f_2(z) + f_2(-z)]) \, dm(z).$$

If we assume further that $f_2$ is even, then

$$\int_X \hat{f}_1 \hat{f}_2 \, d\mu_X = (\zeta(n))^{-2} \left( \int_{\mathbb{R}^n} f_1 \, dm \right) \left( \int_{\mathbb{R}^n} f_2 \, dm \right) + 2(\zeta(n))^{-1} \int_{\mathbb{R}^n} f_1 f_2 \, dm.$$

**Remark 1.7.** Let $f_1, f_2 \in \mathcal{F}$ be given; then $\int_X |\hat{f}_1 \hat{f}_2| \, d\mu_X < +\infty$. This follows by noting that

$\{ |f_1|, |f_2| \} \subset \mathcal{F}$ and $\int_X |\hat{f}_1 \hat{f}_2| \, d\mu_X \leq \int_X |\hat{f}_1| |\hat{f}_2| \, d\mu_X$, applying Theorem 1.6 to $|f_1|$ and $|f_2|$, and then using the Cauchy-Schwarz inequality to show

$$\int_{\mathbb{R}^n} |f_1(z)| (|f_2(z)| + |f_2(-z)|) \, dm(z) \leq 2 \|f_1\|_2 \|f_2\|_2 < +\infty.$$

\textsuperscript{1}Rogers incorrectly stated that [5, Theorem 5] is also valid for functions defined on $\mathbb{R}^2$; for a correct analogue of Rogers’s theorem in the setting of $\mathbb{R}^2$, see [3, Theorem 1 and Remark 0.8] and [2, Proposition 2.10]. See also [1, §2].
Notice that Theorem 1.6 and the foregoing discussion imply the following: for any \( f_3, f_4 \in \mathcal{F} \) with \( f_3 = f_4 \) Lebesgue-almost everywhere, we have \( \| \hat{f}_3 - \hat{f}_4 \|_2 = 0 \) and thus have \( \hat{f}_3 = \hat{f}_4 \) \( \mu_X \)-almost everywhere. It is therefore permissible to identify the primitive Siegel transforms of elements of \( \mathcal{F} \) that are equal Lebesgue-almost everywhere.

**Definition 1.8.** Define \( S : \mathcal{F}_{\text{even}} \to L^2(X) \) by \( S(f) := \hat{f} \). Remark 1.7 implies that the \( \mathbb{R} \)-linear transformation \( S \) is well-defined.

**Proposition 1.9.** The \( \mathbb{R} \)-linear transformation \( S : \mathcal{F}_{\text{even}} \to L^2(X) \) is injective. Let \( \mathcal{T} : \text{Im}(S) \to \mathcal{F}_{\text{even}} \) denote the left inverse of \( S \). We then have \( \| \mathcal{T} \|_{\text{op}} = \left( \frac{\zeta(n)}{2} \right)^{1/2} \), where \( \| \mathcal{T} \|_{\text{op}} \) denotes the operator norm of \( \mathcal{T} \) with respect to the respective \( L^2 \) norms on \( L^2(X) \) and on \( L^2(\mathbb{R}^n) \).

**Proof.** Let \( f \in \mathcal{F}_{\text{even}} \). Theorem 1.6 implies
\[
\| S(f) \|_2^2 = \frac{1}{\zeta(n)} \int_{\mathbb{R}^n} f \, dm + \frac{2}{\zeta(n)} \| f \|_2^2 \geq \frac{2}{\zeta(n)} \| f \|_2^2.
\]
Hence, the kernel of \( S \) is equal to zero. A simple manipulation now yields \( \| f \|_2 \leq \left( \frac{\zeta(n)}{2} \right)^{1/2} \| S(f) \|_2 \).

Finally, it is easy to explicitly construct a compactly supported function \( f_0 \in \mathcal{F}_{\text{even}} \) with \( \int_{\mathbb{R}^n} f_0 \, dm = 0 \) and \( \| f_0 \|_2 > 0 \); we then have \( 0 < \| f_0 \|_2 = \left( \frac{\zeta(n)}{2} \right)^{1/2} \| S(f_0) \|_2 \).

Let us now introduce the transform that is dual to the primitive Siegel transform.

**Definition 1.10.** Given any \( \varphi \in L^2(X) \), we define \( \tilde{\varphi} : G/H \left( \cong \mathbb{R}^n_{\neq 0} \right) \to \mathbb{R} \), the dual primitive Siegel transform of \( \varphi \), by
\[
\tilde{\varphi}(gH) := \int_{H/\Gamma_{\infty}} \varphi(gh) \, d\mu_{H/\Gamma_{\infty}}(h).
\]

**Remark 1.11.** It is necessary to confirm that the dual primitive Siegel transform of \( \varphi \in L^2(X) \) is well-defined. Let \( \varphi \in L^2(X) \). Let \( t : \mathbb{R}^n \to \mathbb{R} \) be an arbitrary continuous function with compact support. By Remark 1.7, we have \( \hat{t} \in L^2(X) \). The Cauchy-Schwarz inequality then implies \( \hat{t} \varphi \in L^1(X) \). We then have the valid calculations
\[
\int_X \hat{t} \varphi \, d\mu_X = \int_{G/\Gamma_{\infty}} t(g) \varphi(g) \, d\mu_{G/\Gamma_{\infty}}(g) = \int_{G/H} t \varphi \, d\mu_{G/H}.
\]
We thus conclude \( \tilde{\varphi} : G/H \to \mathbb{R} \) is well-defined.

**Definition 1.12.** Given any \( \varphi \in L^2(X) \), we define \( \mathcal{L}(\varphi) : G/H \left( \cong \mathbb{R}^n_{\neq 0} \right) \to \mathbb{R} \), the primitive Lebesgue transform of \( \varphi \), by
\[
\mathcal{L}(\varphi)(gH) := \frac{1}{2} \left[ \zeta(n) \tilde{\varphi}(gH) - \left( \int_X \varphi \, d\mu_X \right) \right].
\]
This definition makes sense because \( L^2(X) \subseteq L^1(X) \).

**Remark 1.13.** For any \( g \in G \), notice that \( \{ gh \Gamma : h \in H \} \) is precisely equal to the set of all elements of \( X \) that contain \( gh \in G/H \left( \cong \mathbb{R}^n_{\neq 0} \right) \) as a primitive point. Equivalently, the homogeneous space \( H/\Gamma_{\infty} \cong (HG)/\Gamma \) may be identified with the set of all elements of \( X \) that contain \( \text{id}_G H = e_1 \) as a primitive point. Notice also that \( \zeta(n) \mu_{H/\Gamma_{\infty}} \) is the unique \( H \)-invariant Radon probability measure on \( H/\Gamma_{\infty} \). Thus, we may interpret \( \zeta(n) \tilde{\varphi}(gH) \) in Definition 1.12 as the mean value of \( \varphi \), with
respect to the natural probability measure, on the set of all elements of $X$ that contain $gH$ as a primitive point. The probabilistic interpretation of $\int_X \varphi d\mu_X$ is obvious.

We then have the following inversion theorem, whose proof will utilize the Siegel mean value theorem and the Rogers inner product formula.

**Theorem 1.14.** For any $f \in F_\text{even}$, we have $f = \mathcal{L}(S(f))$.

**Proof.** Let $f \in F_\text{even}$, Set $\varphi := \hat{f} = S(f)$. Let $t : \mathbb{R}^n \to \mathbb{R}$ be an arbitrary continuous function with compact support. Using Theorem 1.6 and Theorem 1.5, we have

\[
\int_{G/H} t f d\mu_{G/H} = \frac{\zeta(n)}{2} \left[ \left( \int_X \hat{t} \hat{f} d\mu_X \right) - (\zeta(n))^{-1} \left( \int_{G/H} t d\mu_{G/H} \right) \left( \int_X \hat{f} d\mu_X \right) \right]
\]

\[
= \frac{\zeta(n)}{2} \left[ \left( \int_X \hat{t} \hat{f} d\mu_X \right) - \left( \int_{G/H} t d\mu_{G/H} \right) \left( \int_X \hat{f} d\mu_X \right) \right]
\]

\[
= \frac{1}{2} \left[ \zeta(n) \left( \int_{G/H} t d\mu_{G/H} \right) - \left( \int_{G/H} t d\mu_{G/H} \right) \left( \int_X \hat{f} d\mu_X \right) \right]
\]

\[
= \frac{1}{2} \left[ \zeta(n) \left( \int_{G/H} t \varphi d\mu_{G/H} \right) - \left( \int_{G/H} t d\mu_{G/H} \right) \left( \int_X \varphi d\mu_X \right) \right]
\]

\[
= \frac{1}{2} \left[ \zeta(n) \left( \int_{G/H} t \varphi d\mu_{G/H} \right) - \left( \int_{G/H} t d\mu_{G/H} \right) \left( \int_X \varphi d\mu_X \right) \right]
\]

\[
= \frac{1}{2} \int_{G/H} t \mathcal{L}(S(f)) d\mu_{G/H}.
\]

We thus have $\int_{G/H} t f d\mu_{G/H} = \int_{G/H} t \mathcal{L}(S(f)) d\mu_{G/H}$. Since $t : \mathbb{R}^n \to \mathbb{R}$ is an arbitrary continuous function with compact support, it follows $f = \mathcal{L}(S(f))$. \(\square\)

We shall soon introduce the full Siegel transform (as opposed to the primitive one that has been considered thus far) and prove an analogue of Theorem 1.14 for it.

**Definition 1.15.** Let $\mathcal{G}$ denote the set of all Borel measurable functions $f : \mathbb{R}^n \to \mathbb{R}$ that belong to $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ and that satisfy the following conditions:

- for each $g \in G$, we have $\sum_{u \in \mathbb{Z}^n} |f(g(u))| < +\infty$;

- for each $z \in \mathbb{R}^n_{\neq 0}$, we have $\sum_{\ell \in \mathbb{Z}_{\geq 1}} |f(\ell k z)| < +\infty$, where the index $\ell$ ranges over the set of all squarefree integers in $\mathbb{Z}_{\geq 1}$.

Let $\mathcal{G}_\text{even}$ denote the set of all even elements of $\mathcal{G}$. Note that each of $\mathcal{G}$ and $\mathcal{G}_\text{even}$ is a real vector space.
For any \( f \in \mathcal{G} \), we define \( f_\infty : \mathbb{R}^n \rightarrow \mathbb{R} \) by

\[
f_\infty(z) = \begin{cases} \sum_{k=1}^{+\infty} f(kz) & \text{if } z \neq 0, \\ f(0) & \text{if } z = 0. \end{cases}
\]

(This definition makes sense because \( 1 \in \mathbb{Z}_{\geq 1} \) is a squarefree integer.)

For any \( f \in \mathcal{G} \), we have \( f_\infty \in \mathcal{F} \) because of the following:

\[
\|f_\infty\|_1 \leq \sum_{k=1}^{+\infty} \|f(k\bullet)\|_1 = \sum_{k=1}^{+\infty} k^{-n} \|f\|_1 = \zeta(n) \|f\|_1 < +\infty;
\]

\[
\|f_\infty\|_2 \leq \sum_{k=1}^{+\infty} \|f(k\bullet)\|_2 = \sum_{k=1}^{+\infty} k^{-n/2} \|f\|_2 = \zeta(n/2) \|f\|_2 < +\infty;
\]

and for any \( g \in G \),

\[
\sum_{\nu \in \mathbb{Z}_n^2} |f_\infty(g(\nu))| \leq \sum_{\nu \in \mathbb{Z}_n^2} \sum_{k=1}^{+\infty} |f(g(k\nu))| = -|f(0)| + \sum_{\nu \in \mathbb{Z}_n} |f(g(\nu))| < +\infty.
\]

For any \( f \in \mathcal{G}_{\text{even}} \), we clearly have \( f_\infty \in \mathcal{F}_{\text{even}} \).

**Definition 1.16.** Given any \( f \in \mathcal{G} \), we define \( \tilde{f} : X \rightarrow \mathbb{R} \), the **full Siegel transform** of \( f \), by

\[
\tilde{f}(\Lambda) := \sum_{w \in (\Lambda \setminus \{0\})} f(w).
\]

Notice that for each \( f \in \mathcal{G} \), we have \( f_\infty \in \mathcal{F} \) and \( \tilde{f} = \tilde{f}_\infty \). Notice also that the full Siegel transform of every odd function in \( \mathcal{G} \) is identically zero. We now establish that the restriction of the full Siegel transform to \( \mathcal{G}_{\text{even}} \) is injective and then prove a Lebesgue-almost everywhere pointwise inversion formula for it.

**Theorem 1.17.** The \( \mathbb{R} \)-linear transformation \( S_{\text{full}} : \mathcal{G}_{\text{even}} \rightarrow L^2(X) \) given by \( S_{\text{full}}(f) := \tilde{f} \) is well-defined and injective. Let \( \mu_M \) denote the Möbius function. Then for any \( f \in \mathcal{G}_{\text{even}} \), the following is true: for Lebesgue-almost every \( z \in \mathbb{R}^n \), we have

\[
f(z) = \sum_{k=1}^{+\infty} \mu_M(k) \mathcal{L}(S_{\text{full}}(f))(kz);
\]

the infinite series on the right-hand side is absolutely convergent.

**Proof.** Let \( f_0 \in \mathcal{G} \). Then \( (f_0)_\infty \in \mathcal{F} \) and \( \tilde{f}_0 = (\tilde{f}_0)_\infty \). Remark 1.7 then implies that \( S_{\text{full}} \) is well-defined. Möbius inversion implies that for each \( z \in \mathbb{R}^n_{\neq 0} \), we have \( f_0(z) = \sum_{k=1}^{+\infty} \mu_M(k)(f_0)_\infty(kz) \).

Notice that this infinite series is absolutely convergent by the definition of \( \mathcal{G} \). In particular, it follows that the mapping \( \mathcal{G}_{\text{even}} \rightarrow \mathcal{F}_{\text{even}} \) given by \( f \mapsto f_\infty \) is injective. Appealing to Proposition 1.9, we conclude that \( S_{\text{full}} \) is injective. The foregoing discussion and Theorem 1.14 then imply the desired result. \( \square \)

**Remark 1.18.** Let us note that \( \mathcal{G} \) contains the set of all compactly supported Borel measurable functions \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) that belong to \( L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \). Thus, the inversion formulae in Theorems 1.14 and 1.17 are quite general.

**Acknowledgements**

The author would like to thank Dmitry Kleinbock, Jayadev Athreya, Uri Shapira, Paul Garrett, Seungki Kim, Omer Offen, Keith Merrill, Anish Ghosh, and Fulton Gonzalez.
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