On a model boundary value problem for Laplacian with frequently alternating type of boundary condition

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Abstract

Model two-dimensional singular perturbed eigenvalue problem for Laplacian with frequently alternating type of boundary condition is considered. Complete two-parametrical asymptotics for the eigenelements are constructed.
Introduction

Elliptic boundary value problems with frequently alternating type of boundary condition are mathematical models used in various applications. We briefly describe the formulation of these problems. In a given bounded domain with a smooth or a piecewise smooth boundary an elliptic equation is considered. On the boundary one selects a subset depending on a small parameter and consisting of a large number of disjoint parts. The measure of each part tends to zero as the small parameter tends to zero, while the number of these parts increases infinitely. On the subset described the Dirichlet boundary condition is imposed, whereas the Neumann boundary condition is imposed on the rest part of the boundary. There is a number of papers devoted to averaging of such problems (see, for instance, [1]–[4]). The main objective of these works was to describe limiting (homogenized) problems. The case of periodic alternating of boundary conditions was investigated in [2], [3], while the nonperiodic one was treated in [1], [4]. The main result of these works can be formulated as follows. The form of limiting problem (namely, type of boundary condition) depends of the relation between measures of parts of the boundary with different types of boundary conditions.

Further studying of the boundary value problems with frequently alternating boundary conditions was carried out in two directions. First direction consists in the estimates for degree of convergence under minimal number of restrictions to the structure of alternating of boundary conditions [2], [4]–[6]. Another direction in studying of these problems is a constructing the asymptotics expansions of solutions. Present paper develops exactly this direction.

In this paper we study a two-dimensional singular perturbed eigenvalue problem for Laplace operator in a unit circle $D$ with center at the origin. On the boundary of the circle $D$ we select a periodic subset $\gamma_\varepsilon$ consisting of $N$ disjoint arcs, length of each arc equals $2\varepsilon\eta$, where $N \gg 1$ is an integer number, $\varepsilon = 2N^{-1}$, $\eta = \eta(\varepsilon)$, $0 < \eta < \pi/2$. Each of these arcs can be obtained from an neighbouring one by rotation about the origin through the angle $\varepsilon\pi$ (cf. figure). On $\gamma_\varepsilon$ we impose the Dirichlet boundary condition and the Neumann boundary condition is considered on the rest part of the boundary. From [1]–[2] it follows that the main role in determination of limiting problem belongs to the limit $\lim_{\varepsilon \to 0} (\varepsilon \ln \eta(\varepsilon))^{-1} = -A$. If $A \geq 0$, then the limiting problem is either the Robin problem ($A > 0$) or the Neumann problem ($A = 0$). The assumption $\lim_{\varepsilon \to 0} (\varepsilon \ln \eta(\varepsilon))^{-1} = -A$ does not define the function $\eta(\varepsilon)$ uniquely; clear, it is equivalent to the equality $\eta(\varepsilon) = \exp\left(-\frac{1}{\varepsilon(A+\mu)}\right)$, where $\mu = \mu(\varepsilon)$ is an arbitrary function tending to zero as $\varepsilon \to 0$, and also, $A + \mu > 0$ for $\varepsilon > 0$. Thus, the problem studied contains actually two parameters, $\varepsilon$ and $\mu$. In paper [7] complete power (on $\varepsilon$) asymptotics for the eigenelements of the perturbed problem were constructed in the case of the Neumann limiting problem ($A = 0$) under an additional assumption $\mu(\varepsilon) = A_0 \varepsilon$, $A_0 = \text{const} > 0$.

In this paper we study the case of limiting Neumann or Robin problem ($A \geq 0$) without any additional assumptions for $\eta(\varepsilon)$. On the basis of the method
of matched asymptotics expansions [8], the method of composite expansions [9] and the multiscaled method [10] we obtain complete two-parametrical (on $\varepsilon$ and $\mu$) asymptotics for the eigenelements of the perturbed problem. Employing the asymptotics expansions for the eigenvalues, we prove that the perturbed problem has only simple and double eigenvalues, and we show criterion distinguishing these cases.

1. The problem and main results

Let $x = (x_1, x_2)$ be the Cartesian coordinates, $(r, \theta)$ be the associated polar coordinates, $\Gamma_\varepsilon = \partial D \setminus \gamma_\varepsilon$. Without loss of generality we may assume that the set $\gamma_\varepsilon$ is symmetric with respect to the axis $Ox_1$. We study singular perturbed eigenvalue problem

$$-\Delta \psi_\varepsilon = \lambda_\varepsilon \psi_\varepsilon, \quad x \in D,$$  

$$\psi_\varepsilon = 0, \quad x \in \gamma_\varepsilon, \quad \frac{\partial \psi_\varepsilon}{\partial r} = 0, \quad x \in \Gamma_\varepsilon.$$  

From [1] [2] it follows that in the case $A \geq 0$ the eigenelements of the perturbed problem converge to the eigenelements of the following limiting problem

$$-\Delta \psi_0 = \lambda_0 \psi_0, \quad x \in D, \quad \left( \frac{\partial}{\partial r} + A \right) \psi_0 = 0, \quad x \in \partial D.$$  

The eigenfunctions converge strongly in $L_2(D)$ and weakly in $H^1(D)$. Total multiplicity of the perturbed eigenvalues converging to a $p$-multiply eigenvalue equals $p$. 

Figure.
It is well known fact that the eigenvalues of the problem (1.3) coincide with the roots of the equation

$$\sqrt{\lambda_0} J_n' \left( \sqrt{\lambda_0} \right) + AJ_n \left( \sqrt{\lambda_0} \right) = 0,$$

where $J_n$ are Bessel functions of integer order $n \geq 0$, and associated eigenfunctions are defined by the equalities $\psi_0 = J_0(\sqrt{\lambda_0}r)$ (for $n = 0$) and $\psi_0^\pm = J_n(\sqrt{\lambda_0}r) \phi^\pm(n\theta)$ (for $n > 0$), $\phi^+ = \cos, \phi^- = \sin$.

Remark 1.1. It should be stressed that the problem (1.3) can have eigenvalues of various multiplicity, including multiplicity more than two. This situation takes place because for some values of $A$ there exists $\lambda_0$ being root of equation (1.4) for different $n$ simultaneously. The proof of existence of such $A$ is given in Appendix.

This paper is devoted to the proof of the following statement.

**Theorem 1.1.** Let $\lambda_0$ be a root of the equation (1.4) for $n \geq 0$. Then there exists an eigenvalue $\lambda_\varepsilon$ of the perturbed problem converging to $\lambda_0$ and satisfying asymptotics

$$\lambda_\varepsilon = \Lambda_0(\mu) + \sum_{i=3}^{M-1} \varepsilon^i \Lambda_i(\mu) + O(\varepsilon^M(A + \mu)),$$

for any $M \geq 3$, where $\Lambda_0(\mu)$ is the root of the equation

$$\sqrt{\Lambda_0} J_n' \left( \sqrt{\Lambda_0} \right) + (A + \mu)J_n \left( \sqrt{\Lambda_0} \right) = 0, \quad \Lambda_0(0) = \lambda_0,$$

$$\Lambda_3(\mu) = -\frac{\zeta(3)}{4} \frac{(A + \mu)^2 (\Lambda_0(\mu) + 2n^2) \Lambda_0(\mu)}{\Lambda_0(\mu) - n^2 + (A + \mu)^2},$$

$$\Lambda_4(\mu) = \frac{\pi^4}{5760} \frac{(A + \mu)^2 (8\Lambda_0(\mu) + 1) \Lambda_0(\mu)}{\Lambda_0(\mu) - n^2 + (A + \mu)^2}.$$

$\zeta(t)$ is the Riemann zeta function. The functions $\Lambda_i(\mu), i \geq 0$, are holomorphic on $\mu$; for $A = 0$ and $i \geq 3$ the representations $\Lambda_i(\mu) = \mu^2 \tilde{\Lambda}_i(\mu)$ hold, where $\tilde{\Lambda}_i(\mu)$ are holomorphic on $\mu$ functions. The eigenvalue $\lambda_\varepsilon$ is simple, if $n = 0$, and it is double, if $n > 0$. The asymptotics of the associated eigenfunctions have the form (2.32) for $n = 0$ and (3.1) for $n > 0$.

Remark 1.2. It is known ([11]) that for $n \geq 0$ the functions $J_n(t)$ and $J_n'(t)$ are positive at the points $t \in (0, n]$. For this reason, the least root of the equation (1.4) exceeds $n^2$, what and (1.5) imply the same for $\Lambda_0(\mu)$, i.e., the denominators in (1.7) are nonzero. If $A = n = \lambda_0 = 0$, then $\Lambda_0 > 0$ and $\mu > 0$, and the denominators in (1.7) are nonzero again.

Remark 1.3. It should be stressed that Theorem 1.1 can be applied to each eigenvalue of the perturbed problem. If $\lambda_0$ is a root of the equation (1.4) only for one value of $n$, then Theorem 1.1 implies immediately that only one perturbed eigenvalue converges to $\lambda_0$ and this perturbed eigenvalue is simple or double, if $\lambda_0$ is a...
root of the equation (1.4) for some values \( n = n_i, i = 1, \ldots, m, m \geq 2 \), then for this case below it will be shown (see Lemma 1.4) that asymptotic series (1.5)–(1.7) do not coincide for different \( n \), and for this reason, exactly \( m \) perturbed eigenvalues (that are simple or double) converge to \( \lambda_0 \) that have asymptotics (1.5)–(1.7) with \( n = n_i, i = 1, \ldots, m \).

This paper has the following structure. In two next sections we formally construct asymptotics for the eigenvalues converging to the roots of the equation (1.4). Also we formally construct the asymptotics for the associated eigenfunctions. We separate the cases \( n = 0 \) and \( n > 0 \), the former is considered in the second section, while the latter is studied in the third one. However, the results of the second and third section do not guarantee that the asymptotic series constructed formally are really asymptotics of the eigenelements of the perturbed problem. In the fourth section we carry out the justification of the asymptotics, i.e., we prove that the asymptotic series formally constructed do coincide with the asymptotics of the eigenelements of the perturbed problem. As it has been already mentioned in Remark 1.1 in Appendix we prove the existence of positive \( A \) for which there exists \( \lambda_0 \) being root of the equation (1.4) for different \( n \) simultaneously.

### 2. Formal construction of the asymptotics for the case \( n = 0 \)

In this section on the basis of the method of composite expansions and the method of matched asymptotic expansions we formally construct the asymptotics for an eigenvalue \( \lambda_\varepsilon \), converging to a root \( \lambda_0 \) of the equation (1.4) with \( n = 0 \), and also, the asymptotics for the associated eigenfunction \( \psi_\varepsilon \).

At first, we briefly describe the scheme of construction. We seek for the asymptotics of the eigenvalue as the series (1.5). It easily seen that the function

\[
\psi_{\varepsilon}^{ex}(x) = J_0 \left( \sqrt{\lambda_\varepsilon} r \right),
\]

is a solution of the equation (1.1) for each \( \lambda_\varepsilon \). At the same time, it does not satisfy boundary condition (1.2). In order to satisfy homogeneous Neumann boundary condition on \( \Gamma_\varepsilon \), using the method of composite expansions, we construct a boundary layer in the vicinity of the boundary of the circle \( D \). This layer is constructed in the form of the asymptotic series

\[
\psi_{\varepsilon}^{mid}(\xi) = \sum_{i=1}^{\infty} \varepsilon^i \psi_i(\xi, \mu),
\]

where \( \xi = (\xi_1, \xi_2) = (\theta \varepsilon^{-1}, (1-r)\varepsilon^{-1}) \) are ”scaled” variables. However, the employment of only the method of composite expansions does not allow to satisfy the homogeneous Dirichlet boundary condition on \( \Gamma_\varepsilon \) simultaneously. In order to obtain the homogeneous Dirichlet boundary condition, we apply the method of matched asymptotics expansions in a neighbourhood of the points \( x_m = (\cos \varepsilon \pi m, \sin \varepsilon \pi m) \),
We shall obtain the explicit formulae for these quantities. The relations (2.3), (2.5) are a recurrence system of boundary value problems for the functions \( v \).

The objective of this section is to determine the coefficients of the series (1.5), (2.1) and then pass to the polar coordinates what implies the equation (2.4) asymptotics. They play an auxiliary role in the proof of Theorem 2.1 which is used in justification of the asymptotics in the fourth section.

Let us proceed to construction. In accordance with the method of composite expansions we postulate the sum of the functions \( \psi_{\varepsilon}^{ex} \) and \( \psi_{\varepsilon}^{mid} \) to satisfy the homogeneous boundary condition everywhere on the boundary \( \partial D \) except the points \( x_k \), i.e.,

\[
\sqrt{\lambda_{\varepsilon}} J_0^i \left( \sqrt{\lambda_{\varepsilon}} \right) - \frac{1}{\varepsilon} \frac{\partial}{\partial \xi_2} \psi_{\varepsilon}^{mid} = 0, \quad \xi \in \Gamma^0,
\]

where \( \Gamma^0 \) is the axis \( O\xi_1 \) without points \( (\pi k; 0), k \in \mathbb{Z} \). Replacing \( \lambda_{\varepsilon}, \psi_{\varepsilon}^{mid} \) by the series (1.5), (2.1) in the equality obtained, expanding the first term in Taylor series with respect to \( \varepsilon \), and equaling to zero the coefficients of powers of \( \varepsilon \), we deduce boundary conditions for the functions \( v_i \):

\[
\begin{align*}
\frac{\partial v_i}{\partial \xi_2} &= \alpha_i, \quad \xi \in \Gamma^0, \quad \alpha_i = \alpha_i(\Lambda_0, \ldots, \Lambda_{i-1}), \\
\alpha_1 &= \sqrt{\Lambda_0} J_0 \left( \sqrt{\Lambda_0} \right), \quad \alpha_2 = \alpha_3 = 0, \\
\alpha_i &= -\frac{1}{2} J_0 \left( \sqrt{\Lambda_0} \right) \Lambda_{i-1} + f_i, \quad i \geq 4, \quad f_4 = f_5 = 0.
\end{align*}
\]

Here \( f_i = f_i(\Lambda_0, \ldots, \Lambda_{i-4}) \) are polynomials on variables \( \Lambda_1, \ldots, \Lambda_{i-4} \) with holomorphic on \( \Lambda_0 \) coefficients, moreover, \( f_i(\Lambda_0, 0, \ldots, 0) = 0 \). Let us deduce the equations for the functions \( v_i \). In order to do it, we substitute \( \psi_{\varepsilon}^{mid} \) and \( \lambda_{\varepsilon} \) in the equation (1.1), and then pass to the polar coordinates what implies the equation

\[
\left( r^2 \frac{\partial^2}{\partial r^2} + r \frac{\partial}{\partial r} + \frac{\partial^2}{\partial \theta^2} + r^2 \lambda_{\varepsilon} \right) \psi_{\varepsilon}^{mid} = 0.
\]

Replacing \( \lambda_{\varepsilon} \) and \( \psi_{\varepsilon}^{mid} \) by the series (1.5) and (2.1) in this equation, passing to the variables \( \xi \) and equaling to zero coefficients of powers of \( \varepsilon \), we can write

\[
\Delta_\xi v_i = F_i \equiv L_i(v_1, \ldots, v_{i-1}), \quad \xi_2 > 0,
\]

\[
L_i(v_1, \ldots, v_{i-1}) = \sum_{k=0}^{1} \sum_{j=1}^{2} a_{kj} \xi_2^{k+j-1} \frac{\partial^j}{\partial \xi_2^j} v_{i-k} + \sum_{k=0}^{2} a_k \xi_2^k \sum_{j=1}^{i-k-2} \Lambda_{i-j-k-2} v_j,
\]

where \( a_{11} = a_{12} = a_0 = a_2 = -1, \, a_{01} = 1, \, a_1 = a_{02} = 2, \, v_{-1} = v_0 = 0, \, \Lambda_1 = \Lambda_2 = 0 \).

The relations (2.3), (2.5) are a recurrence system of boundary value problems for
the functions $v_i$. According to the method of composite expansions, we are to seek its solutions exponentially decaying as $\xi_2 \to +\infty$. We shall obtain the explicit formulae for $v_i$; for this we use the following auxiliary statements.

We indicate by $\mathcal{V}$ the space of $\pi$-periodic on $\xi_1$ functions uniformly exponentially decaying as $\xi_2 \to +\infty$ together with all their derivatives, and belonging to $C^\infty (\{\xi : \xi_2 > 0\} \cup \Gamma^0)$. By $\mathcal{V}^+$ ($\mathcal{V}^-$) we denote the subset of $\mathcal{V}$ containing even (odd) on $\xi_1$ functions. We introduce the operators $\mathcal{A}_k$, $k \geq 0$ is an integer number; their action on a function $u \in \mathcal{V}$ reads as follows

$$\mathcal{A}_0[u](\xi) = u(\xi), \quad \mathcal{A}_k[u](\xi) = \int_{\xi_2}^{+\infty} t \mathcal{A}_{k-1}[u](\xi_1, t) \, dt.$$ 

By definition of the spaces $\mathcal{V}$, $\mathcal{V}^+$ and $\mathcal{V}^-$ and the definition of the operators $\mathcal{A}_k$, one can check that $\mathcal{A}_k : \mathcal{V} \to \mathcal{V}$ and $\mathcal{A}_k : \mathcal{V}^\pm \to \mathcal{V}^\pm$.

**Lemma 2.1.** For each $k \geq 0$ the equalities

$$\Delta_\xi \mathcal{A}_k[u] = -2k \mathcal{A}_{k-1}[u] + \mathcal{A}_k[\Delta_\xi u]$$

hold.

**Proof.** Clear, for each function $u \in \mathcal{V}$ we can write

$$\frac{\partial^{m_1+m_2}}{\partial^{m_1}_1 \partial^{m_2}_2} \int_{\xi_2}^{+\infty} u(\xi_1, t) \, dt = \int_{\xi_2}^{+\infty} \frac{\partial^{m_1+m_2}}{\partial^{m_1}_1 \partial^{m_2}_2} u(\xi_1, t) \, dt, \quad \xi_2 > 0,$$

where $m_1, m_2 \in \mathbb{Z}_+$, what yields

$$\Delta_\xi \mathcal{A}_0[u] = \Delta_\xi u, \quad \Delta_\xi \mathcal{A}_k[u] = \int_{\xi_2}^{+\infty} t \mathcal{A}_{k-1}[u](\xi_1, t) \, dt - 2\mathcal{A}_{k-1}[u].$$

Employing the equalities obtained by induction, it is easy to prove the lemma.

We set $\Pi = \{\xi : -\pi/2 < \xi_1 < \pi/2, \xi_2 > 0\}$, $(\rho, \vartheta)$ are the polar coordinates associated with the variables $\xi$.

**Lemma 2.2.** Let the function $F(\xi) \in \mathcal{V}^+$ has infinitely differentiable asymptotics

$$F(\xi) = \alpha \rho^{-1} \sin 3\vartheta + O(\ln \rho), \quad \rho \to 0,$$

and there exists a natural number $k$, such that $\Delta^k_\xi F \equiv 0$ for $\xi_2 > 0$. Then the function

$$v = -\sum_{j=1}^{k} \frac{1}{2^j j!} \mathcal{A}_j[\Delta_{\xi}^{j-1} F]$$

(2.6)
is a solution of the boundary value problem
\[
\Delta_\xi v = F, \quad \xi_2 > 0, \quad \frac{\partial v}{\partial \xi_2} = 0, \quad \xi \in \Gamma^0, \quad (2.7)
\]
belonging to \(H^1(\Pi) \cap V^+,\) and having infinitely differentiable asymptotics
\[
v(\xi) = v(0) + \frac{1}{2} \alpha \xi_2^3 \rho^{-2} + O(\rho^2 \ln \rho), \quad \rho \to 0. \quad (2.8)
\]

**Proof.** Since \(F \in V^+,\) then, obviously, \(\Delta^j_{\xi} F \in V^+,\) and, therefore, each term in the right hand side of (2.6) belongs to \(V^+,\) what implies \(v \in V^+.\) Let us check that the function \(v\) defined by the equality (2.6) is really a solution of the boundary value problem (2.7). Indeed, for each point \(\xi \in \Gamma^0\) we have
\[
\frac{\partial v}{\partial \xi_2} \bigg|_{\xi \in \Gamma^0} = \sum_{j=1}^{k} \frac{1}{2j!} (\xi_2 A_{j-1} [\Delta_{\xi}^{j-1} F]) \bigg|_{\xi \in \Gamma^0} = 0.
\]

For \(\xi \in \Pi,\) applying the Laplace operator to \(v,\) using Lemma 2.1, and employing the equality \(\Delta^k_{\xi} F \equiv 0,\) we get
\[
\Delta_{\xi} v = - \sum_{j=1}^{k} \frac{1}{2j!} A_j [\Delta^{j}_{\xi} F] + \sum_{j=1}^{k} \frac{2j}{2j!} A_{j-1} [\Delta^{j-1}_{\xi} F] = F.
\]

We proceed to the proof of the asymptotics (2.8). Let a function \(U(\xi) \in V^+\) have differentiable asymptotics
\[
U(\xi) = O \left( \rho^{-p} \ln^q \rho \right), \quad \rho \to 0, \quad p, q \in \mathbb{Z}, \quad p, q \geq 0. \quad (2.9)
\]
We set \(u(\xi) = A_1[U](\xi).\) As \(U \in V^+,\) then the representation
\[
u(\xi) = \int_{\xi_2}^{a} tU(\xi_1, t) \, dt + u_1(\xi), \quad (2.10)
\]
is true, where \(u_1 \in V^+ \cap C^\infty(\{\xi : \xi_2 \geq 0\}),\) \(a\) is a fixed sufficiently small number. It is obvious that
\[
u_1(\xi) = u_1(0) + O(\rho^2), \quad \rho \to 0. \quad (2.11)
\]
Now we replace the function \(U\) by its asymptotics (2.9) in (2.10). After that the integral in (2.10) can be calculated explicitly, from what and (2.11) it follows that
\[
u(\xi) = O \left( \ln^{q+1} \rho \right), \quad p = 2, \quad u(\xi) = O \left( \rho^{-p+2} \ln^q \rho \right), \quad p > 2. \quad (2.12)
\]
as ρ → 0. For p = 0, 1 one can see that
\[
\begin{align*}
\Delta F &= -8\alpha \rho^{-3} \sin 3\theta + O(\rho^{-2} \ln \rho), \\
\Delta^j F &= O(\rho^{-2j} \ln \rho), \quad j \geq 2,
\end{align*}
\]
which and (2.6), (2.9), (2.12), (2.13) and definition of the operators \(A_j\) imply the asymptotics (2.8). In view of latter and the inclusion \(v \in \mathcal{V}\) we conclude that \(v \in H^1(\Pi)\). The proof is complete.

Let \(X(\xi) = \text{Re} \ln \sin z + \ln 2 - \xi_2\), where \(z = \xi_1 + i\xi_2\) is a complex variable. By direct calculations we check that \(X \in \mathcal{V}^+\) is a harmonic function as \(\xi_2 > 0\), satisfying the boundary condition
\[
\frac{\partial X}{\partial \xi_2} = -1, \quad \xi \in \Gamma^0,
\]
and having differentiable asymptotics
\[
X(\xi) = \ln \rho + \ln 2 - \xi_2 + O(\rho^2), \quad \rho \to 0.
\]
(2.14)

The lemmas proved enable us to solve the system of the problems (2.3), (2.5).

**Lemma 2.3.** For each sequence \(\{\Lambda_i(\mu)\}_{i=0}^\infty\), \(\Lambda_1(\mu) = \Lambda_2(\mu) = 0\) there exist solutions of the boundary value problems (2.3), (2.5) defined by formula (2.6) with \(F = F_i, k = k_i\), where \(k_i\) are some natural numbers. For the functions \(v_i\) the representations
\[
\begin{align*}
v_i(\xi, \mu) &= \tilde{v}_i(\xi, \mu) - \alpha_i X(\xi), \\
\tilde{v}_i(\xi, \mu) &= \sum_{j=1}^{M_i} a_{ij}(\alpha_1, \Lambda_0, \ldots, \Lambda_{i-2}) v_{ij}(\xi),
\end{align*}
\]
hold, where \(v_{ij} \in H^1(\Pi) \cap \mathcal{V}^+\), \(a_{ij}\) are polynomials on \(\Lambda_1, \ldots, \Lambda_{i-2}\) with holomorphic on \(\Lambda_0\) and \(\alpha_1\) coefficients, \(a_{ij}(0, \Lambda_0, 0, \ldots, 0) = 0\). The equalities \(M_1 = 0, M_2 = 1,\)
\[ M_3 = 2, \quad M_4 = 3, \]
\[ a_{21} = a_{31} = a_{41} = -\alpha_1, \quad a_{32} = -\alpha_1 \Lambda_0, \quad a_{42} = \frac{\alpha_1 (6\Lambda_0 + 1)}{24}, \]
\[ a_{43} = \frac{\alpha_1 (8\Lambda_0 + 1)}{32}, \]
\[ v_{21} = \frac{1}{2} \xi^2 \frac{\partial X}{\partial \xi}, \quad v_{31} = \frac{1}{8} \xi^4 \frac{\partial^4 X}{\partial \xi^4} + \frac{1}{6} \xi^2 \frac{\partial X}{\partial \xi}, \]
\[ v_{32} = \frac{1}{2} \mathbf{A}_1[X], \quad v_{41} = \frac{1}{48} \xi^6 \frac{\partial^3 X}{\partial \xi^3} + \frac{2}{15} \xi^5 \frac{\partial^2 X}{\partial \xi^2} + \frac{1}{16} \xi^4 \frac{\partial X}{\partial \xi}, \]
\[ v_{42} = \xi^3 X, \quad v_{43} = \mathbf{A}_1[\xi_2 X] + \xi^2 \int_{\xi_2}^{+\infty} X(\xi_1, t) \, dt. \tag{2.16} \]

take place. The asymptotics
\[ v_i(\xi, \mu) = -\alpha_i (\ln \rho + \ln 2 - \xi_2) + \tilde{v}_i(0, \mu) - \frac{1}{2} \alpha_{i-1} \xi_2 \rho^{-2} + O(\rho^2 \ln \rho), \tag{2.17} \]
are correct as \( \rho \to 0 \), where \( \alpha_0 = 0 \).

**Proof.** The statement of the lemma for \( i = 1, \ldots, 4 \) and the equalities (2.16) are checked by direct calculations. For \( i \geq 5 \) we carry out the proof by induction. Let the lemma is valid for \( i < K \). Then, due to (2.5) and induction assumption we have the relation
\[ F_K = \sum_{j=1}^{K} a_{Kj} F_{Kj}, \]
where \( F_{Kj} \) satisfy to all assumptions of Lemma 2.2 and the functions \( a_{Kj} = a_{Kj}(\alpha_1, \Lambda_0, \ldots, \Lambda_{K-2}) \) posses all the properties described in the statement of the lemma being proved. Let \( v_{Kj} \) be the solutions of the problem (2.7) for \( F = F_{Kj} \) defined in accordance with (2.6). Then \( \tilde{v}_K \in H^1(\Pi) \cap V^+ \) is a solution of the equation (2.5) for \( i = K \), satisfying the homogeneous Neumann boundary condition on \( \Gamma^0 \). From this fact it follows that the function \( v_K \) defined in accordance with (2.15) is really a solution of the boundary value problem (2.3), (2.5) for \( i = K \). Clear, the function \( F_K \) satisfies the hypothesis of Lemma 2.2 and has the asymptotics
\[ F_K = -\alpha_{K-1} \rho^{-1} \sin 3\vartheta + O(\ln \rho), \quad \rho \to 0, \]
which and (2.8) imply
\[ \tilde{v}_i(\xi, \mu) = \tilde{v}_i(0, \mu) - \frac{1}{2} \alpha_{i-1} \xi_2 \rho^{-2} + O(\rho^2 \ln \rho), \quad \rho \to 0. \]
Combining the last equality with (2.14), (2.15), we obtain the asymptotics (2.17). The proof is complete.

As it follows from the definition of the functions \( v_j \), the sum of \( \psi_{ex}^i \) and \( \psi_{mid}^i \) does not satisfy homogeneous Dirichlet boundary condition on \( \gamma_\varepsilon \). Moreover, the
functions $v_j$ have logarithmic singularities at the points $x_k$. For this reason, we use the method of matched asymptotics expansions for the construction of the asymptotics for the eigenfunction in a neighbourhood of these points. We construct this asymptotics in the form of the series (2.2). The functions $v_j$, being periodic on $\xi_1$, it is sufficient to carry out the matching in the vicinity of the point $x_0 = (1, 0)$ and then to extend the results obtained for other points $x_k$.

We introduce the notation $\varsigma = \varsigma^0 = \xi \eta^{-1}$. Let us substitute the series (1.5) and (2.2) in (1.1), (1.2), and calculate after that the coefficients of the same powers of $\varepsilon$. As a result, we have the following problems for $w_{i,j}$:

\[ \Delta_{\varsigma} w_{i,0} = 0, \quad \varsigma > 0, \quad w_{i,0} = 0, \quad \varsigma \in \gamma^1, \quad \frac{\partial}{\partial \varsigma} w_{i,0} = 0, \quad \varsigma \in \Gamma^1, \quad (2.18) \]

\[ \Delta_{\varsigma} w_{i,1} = \left( \frac{\partial}{\partial \varsigma} + 2\varsigma \frac{\partial^2}{\partial \varsigma^2} \right) w_{i,0}, \quad \varsigma > 0, \]

\[ w_{i,1} = 0, \quad \varsigma \in \gamma^1, \quad \frac{\partial}{\partial \varsigma} w_{i,1} = 0, \quad \varsigma \in \Gamma^1, \quad (2.19) \]

where $\gamma^1$ is the interval $(-1, 1)$ in the axis $\xi_2 = 0$, and $\Gamma^1$ is the complement of $\gamma^1$ on the axis $O_{\varsigma}$. Next following the method of matched asymptotics expansions, we calculate the asymptotics as $|\varsigma| \to \infty$ for the functions $w_{i,j}$. We denote

\[ \lambda_{\varepsilon,K} = \Lambda_0(\mu) + \sum_{i=3}^K \varepsilon^i \Lambda_i(\mu), \quad \psi_{\varepsilon,K}^{ex}(x) = J_0\left( \sqrt{\lambda_{\varepsilon,K}}r \right), \]

\[ \psi_{\varepsilon,K}^{mid}(\xi) = \sum_{i=1}^{K+1} \varepsilon^i v_i(\xi, \mu), \quad \Psi_{\varepsilon,K}(x) = \psi_{\varepsilon,K}^{ex}(x) + \chi(1 - r)\psi_{\varepsilon,K}^{mid}(\xi), \]

where $\chi(t)$ is an infinitely differentiable cut-off function equal to one as $t < 1/3$ and to zero as $t > 1/2$. Expanding in Taylor series, we can write

\[ J_0\left( \sqrt{\lambda_{\varepsilon,K}}r \right) = \sum_{i=0}^K \varepsilon^i G_i(\Lambda_0, \ldots, \Lambda_i) + \varepsilon^{K+1} G^{(K)}_\varepsilon(\Lambda_0, \ldots, \Lambda_K) - \varepsilon \xi_2 \sqrt{\lambda_{\varepsilon,K}} J_0''\left( \sqrt{\lambda_{\varepsilon,K}} \right) + O\left( \varepsilon^2 \xi_2^2 \right), \quad (2.20) \]

\[ G_0 = J_0\left( \sqrt{\Lambda_0} \right), \quad G_1 = G_2 = 0, \]

\[ G_i = \frac{J_0'\left( \sqrt{\Lambda_0} \right)}{2\sqrt{\Lambda_0}} \Lambda_i + g_i, \quad i \geq 3, \quad g_3 = g_4 = 0, \quad (2.21) \]

where the functions $g_i = g_i(\Lambda_0, \ldots, \Lambda_{i-3})$ are polynomials with respect to $\Lambda_1, \ldots, \Lambda_{i-3}$ with holomorphic on $\Lambda_0$ coefficients, $g_i(\Lambda_0, 0, \ldots, 0) = 0$, $G^{(K)}_\varepsilon(\Lambda_0, 0, \ldots, 0) = 0$. From (2.17) and the
equality \( \ln \eta = -\frac{1}{\varepsilon(A+\mu)} \) it follows that

\[
v_i(\xi, \mu) = \frac{\alpha_i}{\varepsilon A + \mu} - \alpha_i (\ln |\varsigma| + \ln 2) + \alpha_i \xi_2 + \tilde{v}_i(0, \mu) - \frac{1}{2} \eta \alpha_i - \varsigma^3 |\varsigma|^{-2} + O \left( \eta^2 |\varsigma|^2 \ln |\varsigma| \right) \tag{2.22}
\]

as \( \eta^{1/2} < \rho < 2\eta^{1/2} \) (i.e., as \( \eta^{-1/2} < |\varsigma| < 2\eta^{-1/2} \)). We substitute (2.20) and (2.22) in the formula for \( \Psi_{\varepsilon,K} \). Then, as \( \eta^{-1/2} < |\varsigma| < 2\eta^{-1/2} \),

\[
\Psi_{\varepsilon,M}(x) = \sum_{i=0}^{K} \varepsilon^i W_{i,0}(\varsigma, \mu) + \varepsilon \sum_{i=1}^{K} \varepsilon^i W_{i,1}(\varsigma, \mu) + W_{\varepsilon}^{(K)}(x) + O \left( \varepsilon^2 |\varsigma|^2 \ln |\varsigma| \right),
\]

\[
W_{i,0}(\varsigma, \mu) = -\alpha_i (\ln |\varsigma| + \ln 2) + \frac{\alpha_i + 1}{A + \mu} + \tilde{v}_i(0, \mu) + G_i, \quad \tilde{v}_0 = 0, \tag{2.23}
\]

\[
W_{i,1}(\varsigma, \mu) = -\frac{1}{2} \alpha_i \varsigma^2 |\varsigma|^{-2}, \tag{2.24}
\]

\[
W_{\varepsilon}^{(K)}(x) = \varepsilon^{K+1} \left( -\alpha_K + (\ln |\varsigma| + \ln 2 - \xi_2) + G_{\varepsilon}^{(K)}(x) + \tilde{v}_{K+1}(0, \mu) \right) + \varepsilon b_K(\varepsilon) \xi_2,
\]

\[
b_K(\varepsilon) = \sum_{i=1}^{K} \varepsilon^{i-1} \alpha_i - \sqrt{\lambda_{\varepsilon,K}^{(i)}} \left( \sqrt{\lambda_{\varepsilon,K}^{(i)}} \right).
\]

Observe, in view of definition of the functions \( \alpha_i \), the quantity \( b_K(\varepsilon) \) is small, namely, \( b_K(\varepsilon) = O \left( \varepsilon^K \right) \). In accordance with the method of matched asymptotics expansions, we must find the solutions of (2.18), (2.19), satisfying the asymptotics

\[
w_{i,j}(\varsigma, \mu) = W_{i,j}(\varsigma, \mu) + o \left( |\varsigma|^j \right), \quad |\varsigma| \to \infty. \tag{2.25}
\]

We introduce the function \( Y(\varsigma) = \text{Re} \ln \left( y + \sqrt{y^2 - 1} \right) \), where \( y = \varsigma_1 + i\varsigma_2 \) is a complex variable. By definition, \( Y(\varsigma) \) is a solution of the problem (2.18) and has the asymptotics

\[
Y(\varsigma) = \ln |\varsigma| + \ln 2 + O \left( |\varsigma|^{-2} \right), \quad |\varsigma| \to \infty. \tag{2.26}
\]

From the properties of the function \( Y \), the asymptotics (2.28), (2.29), (2.20) and the problem (2.18) we deduce that

\[
w_{i,0} = -\alpha_i Y. \tag{2.27}
\]

Comparing the asymptotics for the function \( w_{i,0} \) implied by (2.26), (2.27) with the equalities (2.23), (2.25), we conclude that

\[
\tilde{v}_i(0, \mu) + \frac{\alpha_i + 1}{A + \mu} + G_i = 0. \tag{2.28}
\]

From the equality obtained for \( i = 0 \) and from (2.3), (2.21) it follows the equation (1.6) for \( \Lambda_0 \). The condition \( \Lambda_0(0) = \lambda_0 \) is obvious due to \( \lambda_\varepsilon \to \lambda_0 \). If \( \lambda_0 \neq 0 \), then
the holomorphy of $\Lambda_0$ on $\mu$ is the corollary to the implicit function theorem. If $\lambda_0 = 0$, then $A = 0$, and in this case the equation (1.6) has a solution of the form $\Lambda_0(\mu) = \mu \tilde{\Lambda}_0(\mu)$, where $\tilde{\Lambda}_0$ is a holomorphic on $\mu$ function, $\tilde{\Lambda}_0(0) = 2$. By Lemma 2.3 (see (2.15), (2.16)) we have: $\tilde{v}_1(0, \mu) = \tilde{v}_2(0, \mu) \equiv 0$. Employing these relations and the equalities $\alpha_2 = \alpha_3 = G_1 = G_2 = 0$ (see (2.3), (2.21)) one can check that the equality (2.28) holds for $i = 1, 2$. Let us consider the case $i \geq 3$. Substituting the formulae (2.4), (2.28) for $f_i$ and that the functions $\tilde{\alpha}_i$ we get (1.7) for $n = 1$. Let us prove that $\Lambda_i$ are holomorphic on $\mu$. Since $J_0 \left(\sqrt{\lambda_0}\right) \neq 0$, then the function $J_0 \left(\sqrt{\lambda_0(\mu)}\right)$ is holomorphic on $\mu$ and does not vanish for small $\mu \geq 0$. If $\lambda_0 \neq 0$, then the function $\Lambda_0(\mu) + (A + \mu)^2$ also does not vanish for $\mu \geq 0$. In the case $\lambda_0 = 0$ (here $A = 0$) the functions $\Lambda_0(\mu)$ and $(\Lambda_0(\mu) + \mu^2)$ have a zero of first order at the point $\mu = 0$, so, for all possible values of $\lambda_0$ and $A$ the quotient

$$\Lambda_i(\mu) = \frac{2\Lambda_0(\mu) \left(f_{i+1}(\mu) + (A + \mu) (\lambda_i(\mu) + \tilde{v}_i(0, \mu))\right)}{J_0(\sqrt{\lambda_0(\mu)}) (\Lambda_0(\mu) + (A + \mu)^2)}, \quad (2.29)$$

is a holomorphic function as $\mu \geq 0$. In view of the statement of Lemma 2.3 for the functions $a_{ij}$ and of the formula (2.15) for the function $\tilde{v}_i$, the function $\tilde{v}_i(0, \mu)$ is holomorphic on $\mu$, provided $\Lambda_0, \ldots, \Lambda_{i-1}$ are holomorphic on $\mu$. Using this fact and that the functions $f_{i+1}$ and $g_i$ are holomorphic on $\Lambda_0, \ldots, \Lambda_{i-3}$, one can easy prove by induction that $\Lambda_i$ are holomorphic on $\mu$.

We proceed to the case $A = 0$. For $i = 3, 4$ from (1.7) it follows that $\Lambda_i(\mu) = \mu^2 \tilde{\Lambda}_i(\mu)$, where $\tilde{\Lambda}_i(\mu)$ are holomorphic on $\mu$ functions. Let us show the same for $i \geq 5$. Suppose that it is true for $i < M$. Since the functions $f_{M+1}$, $g_M$, $\tilde{v}_M(0, \mu)$ are holomorphic on $\Lambda_0, \ldots, L_{M-1}$, $\alpha_1$ is holomorphic on $\Lambda_0$, $f_{M+1}(\Lambda_0, 0, \ldots, 0) = g_M(\Lambda_0, 0, \ldots, 0) = a_{M-1}(0, \Lambda_0, 0, \ldots, 0) = 0$, then $\tilde{f}_{M+1}(\mu) = \mu^2 \tilde{f}(M+1)(\mu)$, $\tilde{g}_M(\mu) = \mu^2 \tilde{g}(M)(\mu)$, $\tilde{v}_M(0, \mu) = \mu v_0(M)(\mu)$, where $\tilde{f}(M+1)(\mu)$, $\tilde{g}(M)(\mu)$, $\tilde{v}(M)(\mu)$ are holomorphic on $\mu$ functions. By this fact and (2.29) we arrive at the desired representations.
Let us determine the functions \( w_{i,1} \). By direct calculations we check that the solutions of the problems (2.19), satisfying asymptotics (2.24), (2.25), have the form
\[
w_{i,1} = \frac{1}{2} \sin^2 \frac{\partial}{\partial \varsigma^2} w_{i,0}. \tag{2.31}
\]

Thus, the formally constructed asymptotics for the eigenfunction looks as follows
\[
\psi_\varepsilon(x) = (\psi_{\varepsilon}^{ex}(x) + \chi(1 - r)\psi_{\varepsilon}^{mid}(\xi)) \chi_\varepsilon(x) + \sum_{m=0}^{N-1} \chi(|\varsigma|^2|g_\varepsilon^{1/2}) \psi_\varepsilon^{in}(\varsigma^m), \tag{2.32}
\]
\[
\chi_\varepsilon(x) = 1 - \sum_{m=0}^{N-1} \chi(|\varsigma|^2|g_\varepsilon^{1/2}).
\]

We introduce the notations
\[
\psi_{\varepsilon,K}(x) = \psi_{\varepsilon,K}(x) \chi_\varepsilon(x) + \sum_{m=0}^{N-1} \chi(|\varsigma|^2|g_\varepsilon^{1/2}) \psi_\varepsilon^{in,K}(\varsigma^m),
\]
\[
\psi_\varepsilon^{in,K}(\varsigma) = \sum_{i=1}^{K} \varepsilon^i (w_{i,0}(\varsigma, \mu) + \varepsilon \eta w_{i,1}(\varsigma, \mu)),
\]
\[
\tilde{\psi}_{\varepsilon,K}(x) = \psi_{\varepsilon,K}(x) - R_{\varepsilon,K}(x),
\]
\[
R_{\varepsilon,K}(x) = \chi(1 - r) (b_\varepsilon(\varepsilon)(1 - r) - \varepsilon^{K+1} \alpha_{K+1} X(\xi)) + \varepsilon^{K+1} G_\varepsilon^{(K)} + \varepsilon^{K+1} \tilde{\psi}_{K+1,1}(0, \mu) - \varepsilon^K \frac{\alpha_{K+1}}{A + \mu}.
\]

We set \( \| \cdot \| = \| \cdot \|_{L_2(D)} \).

**Theorem 2.1.** The functions \( \psi_{\varepsilon,K}, \tilde{\psi}_{\varepsilon,K} \in H^1(D) \cap C^\infty(D) \) converges to \( \psi_0 \) in \( L_2(D) \) as \( \varepsilon \to 0 \), \( \lambda_{\varepsilon,K} \) converges to \( \lambda_0 \), \( \| R_{\varepsilon,K} \| = O(\varepsilon^K(A + \mu)) \). The functions \( \tilde{\psi}_{\varepsilon,K} \) and \( \lambda_{\varepsilon,K} \) are the solutions of the problem
\[
- \Delta u_{\varepsilon} = \lambda u_{\varepsilon} + f, \quad x \in D, \quad u_{\varepsilon} = 0, \quad x \in \gamma_\varepsilon, \quad \frac{\partial u_{\varepsilon}}{\partial r} = 0, \quad x \in \Gamma_\varepsilon, \tag{2.33}
\]

with \( u_{\varepsilon} = \tilde{\psi}_{\varepsilon,K}, \lambda = \lambda_{\varepsilon,K}, f = f_{\varepsilon,K} \), where \( \| f_{\varepsilon,K} \| = O(\varepsilon^K(A + \mu)) \).

**Remark 2.1.** The expressions of the form \( O(\varepsilon^p(A + \mu)) \) in the statement of this theorem should be interpreted in the following way. For \( A > 0 \) it means \( O(\varepsilon^p) \), for \( A = 0 \) it does \( O(\varepsilon^p \mu) \).

**Proof.** The desired smoothness of \( \psi_{\varepsilon,K} \) and \( \tilde{\psi}_{\varepsilon,K} \) follows directly from the definition of these functions and the smoothness of the functions \( \psi_{\varepsilon,K}^{ex}, v_i \) and \( w_{i,j} \). It is obvious that \( \lambda_{\varepsilon,K} \to \lambda_0, \psi_{\varepsilon,K}, \tilde{\psi}_{\varepsilon,K} \to \psi_0 \) as \( \varepsilon \to 0 \). By definition and properties
of the quantities $\alpha_i$ we deduce that $b_k(\varepsilon) = O(\varepsilon^K(A + \mu))$, from what and the definition of the functions $G_\varepsilon^{(K)}$ and $\alpha_{K+1}$ and the smoothness of the function $\chi$ it follows that $\|R_{\varepsilon,K}\| = O(\varepsilon^{K+1}(A + \mu))$. 

Since the function $\chi_\varepsilon(x)$ equals zero in a small neighbourhood of the set $\gamma_\varepsilon$, and the functions $w_{ij}'$ vanish on $\gamma^1$, then the function $\tilde{\psi}_{\varepsilon,K}$ satisfies Dirichlet homogeneous boundary condition on $\gamma_\varepsilon$. By direct calculations we check that for $x \in \Gamma_\varepsilon$

$$\frac{\partial}{\partial r} \tilde{\psi}_{\varepsilon,K}(x) = \chi_\varepsilon(x) \frac{\partial}{\partial r} \Psi_{\varepsilon,K}(x) - \frac{\partial}{\partial r} R_{\varepsilon,K}(x) = \chi_\varepsilon(x) \left( \sqrt{\lambda_{\varepsilon,K} J_0'(\sqrt{\lambda_{\varepsilon,K}})} - \varepsilon i \frac{1}{\varepsilon} \frac{1}{\varepsilon^2} \sqrt{\lambda_{\varepsilon,K}} \right) - \sum_{i=1}^K \varepsilon i \frac{1}{\varepsilon} \frac{1}{\varepsilon^2} \sqrt{\lambda_{\varepsilon,K}} \right) + \varepsilon K \alpha_{K+1} + b_\varepsilon(\varepsilon) = 0.$$ 

Applying the operator $-(\Delta + \lambda_{\varepsilon,K})$ to the function $\tilde{\psi}_{\varepsilon,K}(x)$, we obtain that

$$f_{\varepsilon,K} = -\sum_{i=1}^5 f_{\varepsilon,K}^{(i)}, \quad \text{where}$$

$$f_{\varepsilon,K}^{(1)}(x) = - \chi_\varepsilon(x) (\Delta + \lambda_{\varepsilon,K}) R_{\varepsilon,K}(x),$$

$$f_{\varepsilon,K}^{(2)}(x) = \chi(1-r) \chi_\varepsilon(x) (\Delta + \lambda_{\varepsilon,K}) \overline{\psi}_{\varepsilon,K}^{\text{mid}}(\xi),$$

$$f_{\varepsilon,K}^{(3)}(x) = \tilde{\psi}_{\varepsilon,K}^{\text{mid}} \Delta \chi(1-r) + 2 \left( \nabla_x \chi(1-r), \nabla_x \overline{\tilde{\psi}}_{\varepsilon,K}^{\text{mid}}(\xi) \right),$$

$$f_{\varepsilon,K}^{(4)}(x) = \sum_{m=0}^{N-1} \chi(|\zeta^m| \eta^{1/2}) (\Delta + \lambda_{\varepsilon,K}) \psi_{\varepsilon,K}^{\text{in}}(\zeta^m),$$

$$f_{\varepsilon,K}^{(5)}(x) = \sum_{m=0}^{N-1} \Delta \chi(|\zeta^m| \eta^{1/2}) \psi_{\varepsilon,K}^{\text{mat}}(x) + 2 \left( \nabla_x \chi(|\zeta^m| \eta^{1/2}), \nabla_x \psi_{\varepsilon,K}^{\text{mat}}(x) \right),$$

$$\psi_{\varepsilon,K}^{\text{in}} = \psi_{\varepsilon,K}^{\text{in}} + \varepsilon K^{1/2} \alpha_{K+1} X,$$

$$\psi_{\varepsilon,K}^{\text{mat}} = \psi_{\varepsilon,K}^{\text{mat}} - \varepsilon b_\varepsilon(\varepsilon) \xi_2 + \varepsilon K^{1/2} G_\varepsilon^{(K)} + \varepsilon K^{1/2} \overline{\tilde{\psi}}_{\varepsilon,K}^{\text{in}}(0, \mu) - \varepsilon K^{1/2} \alpha_{K+1}.$$

Direct calculations yield $\| f_{\varepsilon,K}^{(1)} \| = O(\varepsilon^K(\varepsilon A + \mu))$. Due to the equations (2.3) the representation

$$(\Delta + \lambda_{\varepsilon,K}) \tilde{\psi}_{\varepsilon,K}^{\text{mid}}(\xi) = \varepsilon K^{-1/2} \sum_{j=0}^{K-1} F_{\varepsilon,K}^{(j)}(\xi, \mu),$$

holds, where $F_{\varepsilon,K}^{(j)}$ are explicitly calculated functions, and it easy to show that $F_{\varepsilon,K}^{(j)} \in V \cap L_2(\Pi), \| F_{\varepsilon,K}^{(j)} \| = O((A+\mu))$ as $\mu \to 0$, from what it follows that $\| f_{\varepsilon,K}^{(2)} \| = O(\varepsilon^{K+1/2}(A+\mu))$. By exponential decaying as $\xi_2 \to +\infty$ of the functions $v_i$ one can deduce that $\| f_{\varepsilon,K}^{(3)} \| = O(e^{-1/\varepsilon^q}(A+\mu))$, where $q$ is a some fixed number.
Bearing in mind the problems for the functions $w_{i,j}$, we see that
\[
\Delta \psi_{\varepsilon,K} = \frac{1}{r^2} \sum_{i=0}^{K} \varepsilon^i \left( \sum_{k=0}^{i-1} \sum_{j=1}^{2} a_{kj}s_2^{k+j-1} \frac{\partial}{\partial s_2} w_{k,j} + \varepsilon \eta s_2^2 \frac{\partial}{\partial s_2} \left( \frac{\partial}{\partial s_2} \right) w_{i,1} \right).
\]
Using the explicit formulae for the functions $w_{i,j}$ and the asymptotics (2.25), we obtain the equality
\[
\left\| f^{(4)}_{\varepsilon,K} \right\| = O \left( \varepsilon^{1/2} \eta A + \mu \right). \tag{2.24}
\]
In view of the matching carried out for $\eta < (\xi_1 - \pi m)^2 + \xi_2^2 < 4\eta$
\[
\psi_{\varepsilon,K}^{\text{mat}}(x) = O \left( \varepsilon (A + \mu) \rho^2 \ln \rho \right), \tag{2.34}
\]
from what it follows that
\[
\left\| f^{(5)}_{\varepsilon,K} \right\| = O \left( \eta^{1/5} \right). \tag{2.35}
\]
Observe that it is impossible to get (2.34) without introducing the functions $w_{i,1}$, i.e., it is impossible to attain the rapid decaying of the norm $\left\| f^{(5)}_{\varepsilon,K} \right\|$ as $\varepsilon \to 0$. This is the only reason for that the functions $w_{i,1}$ were employed. Collecting now the estimates for the functions $f^{(i)}_{\varepsilon,K}$, we arrive at the desired estimate for $\| f_{\varepsilon,K} \|$. The proof is complete.

3. Formal construction of the asymptotics for the case $n > 0$

In present section we shall formally construct the asymptotics for the eigenvalue $\lambda_{\varepsilon}$, converging to a root $\lambda_0$ of the equation (1.4) with $n > 0$, and we shall formally construct the asymptotics for the associated eigenfunctions $\psi_{\varepsilon}^\pm$. On the whole, the scheme of construction is similar to the case $n = 0$. The only (and not principal) distinction is the using of the multiscaled method.

The asymptotics for the eigenvalue is constructed in the form of the series (1.5), and we construct the asymptotics of the eigenfunctions $\psi_{\varepsilon}^\pm$ as the series
\[
\psi_{\varepsilon}^\pm(x) = \left( \psi_{\varepsilon}^{ex,\pm}(x) + \chi(1 - r) \psi_{\varepsilon}^{mid,\pm}(\xi, \theta) \right) \chi_{\varepsilon}(x) + \sum_{m=0}^{N-1} \chi \left( |s^m| \eta^{1/2} \right) \psi_{\varepsilon}^{in,\pm} (s^m, \mu), \tag{3.1}
\]
\[
\psi_{\varepsilon}^{ex,\pm}(x) = J_n \left( \sqrt{\lambda_{\varepsilon} r} \right) \phi^\pm(n\theta),
\]
\[
\psi_{\varepsilon}^{mid,\pm}(\xi, \theta) = \phi^\pm(n\theta) \sum_{i=1}^{\infty} \varepsilon^i v_i(\xi, \mu) \pm \phi^\mp(n\theta) \varepsilon \sum_{i=1}^{\infty} \varepsilon^i v^{ad}_i(\xi, \mu), \tag{3.2}
\]
\[
\psi_{\varepsilon}^{in,\pm}(s, \theta) = \phi^\pm(n\theta) \sum_{i=1}^{\infty} \varepsilon^i (w_{i,0}(s, \mu) + \varepsilon \eta w_{i,1}(s, \mu)) \pm \phi^\mp(n\theta) \varepsilon \sum_{i=0}^{\infty} \varepsilon^i w^{ad}_{i,1}(s, \mu). \tag{3.3}
\]
By analogy with the previous section, the functions $\psi_{\varepsilon}^{mid,\pm}$ are the boundary layers, we introduce them in order to attain the Neumann boundary condition
Employing the method of matched asymptotics expansions, we construct the asymptotics for the eigenfunctions $\psi_{\pm}^\varepsilon$ in the form of the series (3.3) in the vicinity of the points $x_m$ what allows us to get homogeneous Dirichlet boundary condition on $\gamma_\varepsilon$. Here the distinction from the case $n = 0$ is the appearance of the additional functions $v_{i,1}^{ad}$ and $w_{i,1}^{ad}$, and the using of multiscaled method. To the latter it corresponds the presence of the functions $\phi^{\pm}(n\theta)$ in (3.2), (3.3), the variable $\theta$ plays the role of "slow time".

The objective of this section is to determine the functions $\Lambda_i, v_i, v_{i,1}^{ad}, w_{i,1}^{ad}$, for which we shall obtain the explicit formulae.

We proceed to the construction. We postulate the sum of the functions $\psi_{\varepsilon, \pm}^{ex}$ and $\psi_{\varepsilon, \pm}^{mid}$ to satisfy homogeneous Neumann boundary condition everywhere on $\partial D$ except the points $x_k$, i.e.,

$$\sqrt{\lambda_\varepsilon} J_n'(\sqrt{\lambda_\varepsilon}) \phi^{\pm}(n\theta) - \frac{1}{\varepsilon} \frac{\partial}{\partial \xi_2} \psi_{\varepsilon, \pm}^{mid}(\xi, \theta) = 0, \quad \xi \in \Gamma_0.$$  

Replacing now $\lambda_\varepsilon$ and $\psi_{\varepsilon, \pm}^{mid}$ by the series (1.5) and (3.2), and calculating the coefficients of the powers of $\varepsilon$ separately for $\phi^+(n\theta)$ and $\phi^-(n\theta)$, we get the boundary conditions for the functions $v_i^{\pm}$:

$$\frac{\partial v_i}{\partial \xi_2} = \alpha_i, \quad \frac{\partial v_{i,1}^{ad}}{\partial \xi_2} = 0, \quad \xi \in \Gamma_0, \quad \alpha_i = \alpha_i(\Lambda_0, \ldots, \Lambda_{i-1}), \quad (3.4)$$

$$\alpha_1 = \sqrt{\Lambda_0} J_n'(\sqrt{\Lambda_0}), \quad \alpha_2 = \alpha_3 = 0,$$

$$\alpha_i = -\frac{J_n'(\sqrt{\Lambda_0}) (\Lambda_0 - n^2)}{2\Lambda_0} \Lambda_{i-1} + f_i, \quad i \geq 4, \quad f_4 = f_5 = 0, \quad (3.5)$$

where $f_i = f_i(\Lambda_0, \ldots, \Lambda_{i-4})$ are polynomials on $\Lambda_1, \ldots, \Lambda_{i-4}$ with holomorphic on $\Lambda_1$ coefficients, $f_i(\Lambda_0, 0, \ldots, 0) = 0$. Similarly to the way by which the equations (2.5) were obtained, we substitute (1.5) and (3.2) in (1.1) and calculate the coefficients of powers of $\varepsilon$ separately for $\phi^+(n\theta)$ and $\phi^-(n\theta)$. As a result, we deduce the equations for $v_i^{\pm}$:

$$\Delta_\varepsilon v_i = F_i \equiv L_i(v_1, \ldots, v_{i-1}) - n^2 v_{i-2} + 2n \frac{\partial v_{i-2}^{ad}}{\partial \xi_1}, \quad \xi_2 > 0,$$

$$\Delta_\varepsilon v_i^{ad} = F_i^{ad} \equiv L_i(v_1^{ad}, \ldots, v_{i-1}^{ad}) - n^2 v_{i-2}^{ad} - 2n \frac{\partial v_{i-1}^{ad}}{\partial \xi_1}, \quad \xi_2 > 0, \quad (3.6)$$

where $\Lambda_1 = \Lambda_2 = 0$, $v_i^{\pm} = v_i^{\pm, 0} = 0$. We seek the exponentially decaying as $\xi_2 \to +\infty$ solutions of the recurrence system of boundary value problems (3.4), (3.6).

By analogy with Lemma 2.2 one can prove the following statement.

**Lemma 3.1.** Let the function $F(\xi) \in V^-$ has infinitely differentiable asymptotics

$$F(\xi) = \alpha \rho^{-1} \cos \vartheta + O(\ln \rho), \quad \rho \to 0,$$
and there exists a natural number \( k \), such that \( \Delta^k_x F \equiv 0 \) for \( \xi_2 > 0 \). Then the function \( v \) defined in accordance with (2.6) is a solution of the boundary value problem (2.7), belonging to \( H^1(\Pi) \cap \mathcal{V}^- \), and having infinitely differentiable asymptotics
\[
v(\xi) = \frac{1}{2} \xi_1 \ln \rho + \tilde{\alpha}_1 + O(\rho^2 \ln \rho), \quad \rho \to 0,
\]
where \( \tilde{\alpha} \) is a some number.

Employing Lemmas 2.2 and 3.1 by analogy with Lemma 2.3, it is easy to prove the following lemma.

**Lemma 3.2.** For each sequence \( \{ \Lambda_i(\mu) \}_{i=0}^{\infty}, \Lambda_i(\mu) = \Lambda_j(\mu) = 0, \) there exist solutions of the boundary value problems (3.4), (3.6) defined by formula (2.6) with \( F = F_i, k = k_i \) and \( F = F_{iad}, k = k_i^{iad} \), where \( k_i, k_i^{iad} \) are some natural numbers. For the functions \( v_i \) and \( v_{iad}^\alpha \) the representations (2.15) with \( \alpha_i \) from (3.5) and
\[
v_{iad}^\alpha(\xi, \mu) = \sum_{j=1}^{M_{iad}} a_{ij}^{ad}(\alpha_1, \Lambda_0, \ldots, \Lambda_{i-2}) v_{i}^{ad}(\xi),
\]
hold, where \( v_{ij} \in H^1(\Pi) \cap \mathcal{V}^+, v_{iad}^\alpha \in H^1(\Pi) \cap \mathcal{V}^-, a_{ij}, a_{iad}^\alpha \) are polynomials on \( \Lambda_1, \ldots, \Lambda_{i-2} \) with holomorphic on \( \Lambda_0 \) and \( \alpha_1 \) coefficients, \( a_{ij}(0, \Lambda_0, 0, \ldots, 0) = a_{iad}^\alpha(0, \Lambda_0, 0, \ldots, 0) = 0 \). The equalities \( M_1 = 0, M_2 = M_1^{ad} = M_2^{ad} = 1, M_3 = 2, M_4 = 3, \)
\[
a_{21} = a_{31} = a_{41} = a_{11}^{ad} = a_{21}^{ad} = -\alpha_1, \quad a_{32} = -\alpha_1 (\Lambda_0 + 2n^2), \\
a_{42} = \frac{\alpha_1 (6\Lambda_0 + 1)}{24}, \quad a_{43} = \frac{\alpha_1 (8\Lambda_0 + 1)}{32}, \quad v_{21} = \frac{1}{2} \xi_2 \partial X \partial(\xi_2),
\]
\[
v_{11}^{ad} = -n A_1 \left[ \frac{\partial X}{\partial \xi_1} \right], \quad v_{31} = \frac{1}{8} \xi_2 \partial X \partial(\xi_2) + \frac{1}{6} \xi_2^3 \partial^2 X \partial(\xi_2) + \frac{n^2}{2} \xi_2^2 X, \\
v_{32} = \frac{1}{2} \bar{A}_1[X], \quad v_{41} = \frac{1}{48} \xi_2^6 \partial^3 X \partial(\xi_2) + \frac{2}{15} \xi_2^5 \partial^2 X \partial(\xi_2) + \frac{4n^2 + 1}{16} \xi_2^4 \partial X \partial(\xi_2),
\]
\[
v_{42} = \xi_2^3 X, \quad v_{43} = \bar{A}_1[\xi_2 X] + \xi_2^2 \int_{\xi_2}^{+\infty} X(\xi_1, t) \, dt, \quad v_{22}^{ad} = \frac{n}{2} \xi_2^2 \partial X \partial(\xi_1).
\]

**take place.** The asymptotics (2.17) with \( \alpha_i \) from (3.5) and
\[
v_i^{ad}(\xi, \mu) = -n \alpha_i \xi_1 \ln \rho + \tilde{\alpha}_i \xi_1 + O(\rho^2 \ln \rho),
\]
are correct as \( \rho \to 0 \). Here \( \alpha_0 = 0, \tilde{\alpha}_i = \bar{\alpha}_i (\alpha_1, \Lambda_0, \ldots, \Lambda_{i-2}) \) are polynomials on \( \Lambda_1, \ldots, \Lambda_{i-2} \) with holomorphic on \( \Lambda_0 \) and \( \alpha_1 \) coefficients, moreover, \( \bar{\alpha}_i(0, \Lambda_0, 0, \ldots, 0) = 0 \).
Similarly to the previous section, for construction of the asymptotics for the eigenfunctions \( \psi_\varepsilon^\pm \) in a neighbourhood of the points \( x_m \) we apply the method of matched asymptotics expansions. The asymptotics of the functions \( \psi_\varepsilon^\pm \) in a neighbourhood of the points \( x_m \) are constructed in the form of the series (3.8), doing this, we match the functions \( w_{i,j} \) with \( v_i \), whereas the functions \( w_{i,1}^{ad} \) are matched with \( v_i^{ad} \).

We substitute the series (1.5) and (3.3) in the problem (1.1), (1.2), pass to the variables \( \zeta \) and collect coefficients of powers of \( \varepsilon \) separately for \( \phi^+(n\theta) \) and \( \phi^-(n\theta) \). As a result, we get the boundary value problems (2.18) and (2.19) for the functions \( w_{i,j} \), and the following ones for the functions \( w_{i,1}^{ad} \):

\[
\Delta_i w_{i,1}^{ad} = 2n \frac{\partial}{\partial \zeta_1} w_{i,0}, \quad \zeta_2 > 0,
\]

\[
w_{i,0}^{ad} = 0, \quad \zeta \in \gamma^1, \quad \frac{\partial}{\partial \zeta_2} w_{i,1}^{ad} = 0, \quad \zeta \in \Gamma^1,
\]

where \( w_{0,0}^{ad} = 0 \). Let us deduce the asymptotics for \( w_{i,j} \) and \( w_{i,1}^{ad} \) as \( |\zeta| \to \infty \). We denote by \( \lambda_{\varepsilon, K} \) partial sum of (1.5),

\[
\Psi_{\varepsilon, K}^\pm (x) = \psi_{\varepsilon, K}^{ex, \pm} (x) + \chi (1-r) \psi_{\varepsilon, K}^{mid, \pm} (\xi), \quad \psi_{\varepsilon, K}^{ex, \pm} (x) = J_n \left( \sqrt{\lambda_{\varepsilon, K}} r \right) \phi^\pm (n\theta),
\]

\[
\psi_{\varepsilon, K}^{mid, \pm} (\xi) = \phi^\pm (n\theta) \sum_{i=1}^{K+1} \varepsilon^i v_i (\xi, \mu) \pm \phi^\mp (n\theta) \varepsilon \sum_{i=1}^{K} \varepsilon^i v_i^{ad} (\xi, \mu).
\]

It is easily seen that

\[
J_n \left( \sqrt{\lambda_{\varepsilon, K}} \right) = \sum_{i=0}^{K} \varepsilon^i G_i (\Lambda_0, \ldots, \Lambda_i) + \varepsilon^{K+1} G_{\varepsilon}^{(K)} (\Lambda_0, \ldots, \Lambda_K) - \varepsilon \xi_2 \sqrt{\lambda_{\varepsilon, K}} J_n' \left( \sqrt{\lambda_{\varepsilon, K}} \right) + O \left( \varepsilon^2 \xi_2^2 \right),
\]

\[
G_0 = J_n \left( \sqrt{\Lambda_0} \right), \quad G_1 = G_2 = 0,
\]

\[
G_i = \frac{J_n' \left( \sqrt{\Lambda_0} \right)}{2 \sqrt{\Lambda_0}} \Lambda_i + g_i, \quad i \geq 3, \quad g_3 = g_4 = 0,
\]

where the functions \( g_i = g_i (\Lambda_0, \ldots, \Lambda_{i-3}) \) are polynomials on \( \Lambda_1, \ldots, \Lambda_{i-3} \) with holomorphic on \( \Lambda_0 \) coefficients, \( g_1 (\Lambda_0, 0, \ldots, 0) = 0, G_{\varepsilon}^{(K)} \) is a bounded holomorphic on \( \Lambda_1, \ldots, \Lambda_K \) function, \( G_{\varepsilon}^{(K)} (\Lambda_0, 0, \ldots, 0) = 0 \). From the relations obtained, the asymptotics (2.17) and (3.8), and the equality \( \ln \eta = -\frac{1}{\varepsilon (A+\mu)} \) it follows that

\[
\Psi_{\varepsilon, M}^\pm (x) = \phi^\pm (n\theta) \left( \sum_{i=0}^{K} \varepsilon^i W_{i,0} (\zeta, \mu) + \varepsilon \eta \sum_{i=1}^{K} \varepsilon^i W_{i,1} (\zeta, \mu) + W_{\varepsilon}^{(K)} (x) \right) \pm \phi^\mp (n\theta) \varepsilon \eta \left( \sum_{i=0}^{K} \varepsilon^i W_{i,1}^{ad} (\zeta, \mu) + \varepsilon^{K+1} W_{K}^{ad} (\zeta, \mu) \right) + O \left( \varepsilon^2 |\zeta|^2 \ln |\zeta| \right),
\]
as \( \eta^{1/2} < \rho < 2\eta^{1/2} \), where \( W_{i,j}, W^{(K)}_{\varepsilon}(x) \) from (2.23), (2.24) with \( \alpha_i \) from (3.5),

\[
W_{i,1}^{ad}(\varsigma, \mu) = -n\alpha_i \varsigma_1 \ln|\varsigma| + \left( \tilde{\alpha}_i + \frac{n\alpha_{i+1}}{A + \mu} \right) \varsigma_1, \quad (3.11)
\]

\[
W_{i}^{ad}(\varsigma, \mu) = -\frac{n\alpha_{i+1}}{A + \mu} \varsigma_1,
\]

\[
W_{\varepsilon}^{(K)}(x) = \varepsilon^{K+1} \left( -\alpha_{K+1} (\ln|\varsigma| + \ln 2 - \xi_2) + G^{(K)}_{\varepsilon} + \tilde{v}_{K+1}(0, \mu) \right) + \varepsilon b_K(\varepsilon) \xi_2,
\]

\[
b_K(\varepsilon) = \sum_{i=1}^{K} \varepsilon^{i-1} \alpha_i - \sqrt{\lambda_{\varepsilon,K}} J'_\alpha \left( \sqrt{\lambda_{\varepsilon,K}} \right),
\]

\( \tilde{\alpha}_0 = 0 \). Following the method of matched asymptotics expansions, we must construct the solutions of the problems (2.18), (2.19) and (3.9) with asymptotics (2.25) and

\[
w_{i,1}^{ad}(\varsigma, \mu) = W_{i,1}^{ad}(\varsigma, \mu) + o(|\varsigma|), \quad |\varsigma| \to \infty. \quad (3.12)
\]

We define the functions \( w_{i,j} \) in accordance with (2.27) and (2.31), where \( \alpha_i \) from (3.5). Then in view of the definition of the function \( w_{i,0} \) and the asymptotics (2.23), (2.28) we deduce the equality (2.28), where \( \alpha_i \) from (3.5), \( \tilde{v}_i(0, \mu) \) from Lemma 3.2, \( G_i \) from (3.10). For \( i = 0 \), this equality becomes the equation (1.6) for \( \Lambda_0 \). Since for \( n > 0 \) the eigenvalue \( \lambda_0 \) is nonzero, the holomorphy of \( \Lambda_0 \) easily follows from the implicit function theorem. The equalities (2.28) hold for \( i = 1, 2 \), since by (3.5), (3.7), and (3.10) we have \( \alpha_2 = \alpha_3 = G_1 = G_2 = 0 \), \( \tilde{v}_1(0, \mu) = \tilde{v}_2(0, \mu) = 0 \). For \( i \geq 3 \), by analogy with the way by which (2.29) was obtained from (1.6), (2.28), (3.5) and (3.10), one can get the formulae for \( \Lambda_i \):

\[
\Lambda_i(\mu) = \frac{2\Lambda_0(\mu) \left( \tilde{f}_{i+1}(\mu) + (A + \mu) (\tilde{g}_{i}(\mu) + \tilde{v}_{i}(0, \mu)) \right)}{J_\alpha(\sqrt{\Lambda_0(\mu)}) (\Lambda_0(\mu) - n^2 + (A + \mu)^2)},
\]

\[
\tilde{f}_{i+1}(\mu) = f_{i+1} (\Lambda_0(\mu), \ldots, \Lambda_{i-3}(\mu)), \quad \tilde{g}_{i}(\mu) = g_{i} (\Lambda_0(\mu), \ldots, \Lambda_{i-3}(\mu)).
\]

Making \( i = 3, 4 \) in the formulae obtained and using the equalities (2.15), (2.30) and (3.7), we have (1.7) also for \( n > 0 \). Reproducing the arguments of the previous section, one can prove that \( \Lambda_i(\mu) \) are holomorphic on \( \mu \geq 0 \) functions satisfying the representations \( \Lambda_i(\mu) = \mu^2 \Lambda_i(\mu) \) for \( A = 0 \), where \( \Lambda_i(\mu) \) are holomorphic functions. Let us construct the functions \( w_{i,1}^{ad} \). It is easy to see, that the functions

\[
Y_1(\varsigma) = \text{Re} \sqrt{y^2 - 1}, \quad Y_2(\varsigma) = \frac{1}{2} (\varsigma_1 Y(\varsigma) - \ln 2 Y_1(\varsigma))
\]

are solutions of the boundary value problems

\[
\Delta Y_1 = 0, \quad \Delta Y_2 = \frac{\partial Y}{\partial \varsigma_1}, \quad \varsigma_2 > 0,
\]

\[
Y_j = 0, \quad \varsigma \in \gamma^1, \quad \frac{\partial Y_j}{\partial \varsigma_2} = 0, \quad \varsigma \in \Gamma^1, \quad j = 1, 2,
\]
and have asymptotics
\[ Y_1(\varsigma) = \varsigma_1 + O(|\varsigma|^{-1}), \quad Y_2(\varsigma) = \frac{1}{2} \varsigma_1 \ln |\varsigma| + O(|\varsigma|^{-1}), \quad |\varsigma| \to \infty. \]

By the properties of the functions \( Y_j \), the definition of the function \( w_{i,0} \), the problem (3.9) and the asymptotics (3.11), (3.12) we obtain that
\[ w_{i,1}^{ad} = -2n\alpha_i Y_2 + \left( \tilde{\alpha}_i + \frac{\alpha_{i+1}}{A + \mu} \right) Y_1. \]

We set
\[
\psi_{\epsilon,K}^\pm(x) = \Psi_{\epsilon,K}^\pm(x) \chi_{\epsilon}(x) + \sum_{m=0}^{N-1} \chi(|\varsigma| \eta^{1/2}) \psi_{\epsilon,K}^{\pm,n}(\varsigma^m),
\]
\[
\psi_{\epsilon,K}^{\pm,n}(\varsigma) = \phi^{\pm}(n\theta) \sum_{i=1}^{K} \varepsilon^i \left( w_{i,0}(\varsigma, \mu) + \varepsilon \eta w_{i,1}(\varsigma, \mu) \right) \pm \phi^{\mp}(n\theta) \varepsilon \eta \sum_{i=0}^{K} \varepsilon^i w_{i,1}^{ad}(\varsigma, \mu),
\]
\[
\tilde{\psi}_{\epsilon,K}^\pm(x) = \psi_{\epsilon,K}^\pm(x) - R_{\epsilon,K}^\pm(x),
\]
\[
R_{\epsilon,K}^\pm(x) = \chi_{\epsilon}(x) \phi^{\pm}(n\theta) \tilde{R}_{\epsilon,K}(x) \pm \chi_{\epsilon}(x) \phi^{\mp}(n\theta) \frac{n\alpha_{K+1}}{A + \mu} \sum_{m=0}^{N-1} \chi(|\varsigma| \eta^{1/2}) Y_1(\varsigma^m),
\]
\[
\tilde{R}_{\epsilon,K}(x) = \chi(1 - r) \left( b_{k}(\epsilon)(1 - r) - \varepsilon^{K+1} \alpha_{K+1} X(\xi) \right) + \varepsilon^{K+1} G_{\epsilon}^{(K)} + \varepsilon^{K+1} v_{K+1}(0, \mu) - \varepsilon^{K+1} \frac{\alpha_{K+1}}{A + \mu}.
\]

By analogy with Theorem 2.1 one can prove the following statement.

**Theorem 3.1.** The functions \( \psi_{\epsilon,K}^\pm, \tilde{\psi}_{\epsilon,K}^\pm \in H^1(D) \cap C^\infty(D) \) converge to \( \psi_{0}^\pm \) in \( L_2(D) \) as \( \epsilon \to 0 \), \( \lambda_{\epsilon,K} \) converges to \( \lambda_0 \), \( \| R_{\epsilon,K}^\pm \| = O(\epsilon^{K}(A + \mu)) \). The functions \( \tilde{\psi}_{\epsilon,K}^\pm \) and \( \lambda_{\epsilon,K} \) are the solutions of the problem (2.33) with \( u_\epsilon = \tilde{\psi}_{\epsilon,K}^\pm \), \( \lambda = \lambda_{\epsilon,K} \), \( f = f_{\epsilon,K}^\pm \), where \( \| f_{\epsilon,K}^\pm \| = O(\epsilon^{K}(A + \mu)) \).

**4. Justification of the asymptotics**

In this section we shall prove that asymptotic expansions formally constructed in two previous sections are really provide asymptotics for the eigenvalues of the problem (1.1), (1.2). In order to do it we shall employ the following statements.

**Lemma 4.1.** Let \( Q \) be any compact set in complex plane containing no eigenvalues of the limiting problem. Then for all \( \lambda \in Q \), \( f \in L_2(D) \) and sufficiently small \( \epsilon \) the problem (2.33) is uniquely solvable and for its solution the uniform on \( \epsilon, \mu, \lambda \) and \( f \) estimate
\[
\| u_\epsilon \|_1 \leq C \| f \|, \quad \text{(4.1)}
\]
holds, where $\| \cdot \|_1$ is the $H^1(D)$-norm. The function $u_\varepsilon$ converges to the solution of the problem

$$-\Delta u_0 = \lambda u_0 + f, \quad x \in D, \quad \left( \frac{\partial}{\partial r} + A \right) u_0 = 0, \quad x \in \partial D.$$  \hspace{1cm} (4.2)

uniformly on $\lambda$.

**Proof.** The solvability of the problem \((2.33)\) is obvious. Clear, in order to prove the uniqueness of its solution it is sufficient to prove the estimate \((4.1)\). We prove the latter by arguing by contradiction. Suppose that there exist sequences $\varepsilon_k \to \varepsilon_k^* \to 0$, $f_k$ and $\lambda_k$ such that for $\varepsilon = \varepsilon_k$, $f = f_k$, $\lambda = \lambda_k \in Q$ the inequality

$$\| u_{\varepsilon_k} \|_1 \geq k \| f_k \|.$$  \hspace{1cm} (4.3)

takes place. There is no loss of generality in assuming that $\| u_{\varepsilon} \|_1 = 1$. We multiply both sides of the equation in \((2.33)\) by $u_{\varepsilon}^*$ and integrate by part. Then we have an a priori uniform estimate

$$\| u_{\varepsilon_k} \|_1 \leq C (\| f \| + \| u_{\varepsilon_k} \|).$$

By this estimate, the equality $\| u_{\varepsilon} \|_1 = 1$, and \((4.3)\) we deduce

$$\| u_{\varepsilon_k} \|_1 \leq C, \quad \| f_k \| \to 0.$$  \hspace{1cm} (4.4)

From the assertions obtained and the theorem about the compact embedding of $H^1(D)$ in $L^2(D)$ it follows that there exists a subsequence of indexes $k$ (we indicate it by $k'$), such that $\lambda_{k'} \to \lambda_\ast \in Q$ and

$$u_{\varepsilon_{k'}} \to u_\ast \neq 0 \quad \text{weakly in } H^1(D) \text{ and strongly in } L^2(D).$$

In \([2]\) it was shown that for each function $V \in C^\infty(D)$ there exists a sequence of functions $V_{\varepsilon} \in H^1(D)$, vanishing on $\gamma_{\varepsilon}$, such that

$$V_{\varepsilon} \to V \quad \text{weakly in } H^1(D) \text{ and strongly in } L^2(D),$$

$$\int_D (\nabla V_{\varepsilon}, \nabla v_{\varepsilon}) \ dx \to \int_D (\nabla V, \nabla v) \ dx + \int_{\partial D} V v \ d\theta,$$  \hspace{1cm} (4.5)

where $v_{\varepsilon}$ is an arbitrary sequence of functions from $H^1(D)$, $v_{\varepsilon} = 0$ on $\gamma_{\varepsilon}$, $v_{\varepsilon}$ converges to $v \in H^1(D)$ strongly in $L^2(D)$ and weakly in $H^1(D)$. In view of \((2.33)\) we have the equality

$$\int_D (\nabla V_{k'}, \nabla u_{k'}) \ dx = \lambda_{k'} \int_D V_{k'} u_{k'} \ dx + \int_D f_{k'} u_{k'} \ dx,$$

passing in which to limit as $k' \to \infty$ and bearing in mind \((4.4), (4.5)\), we conclude that $u_\ast$ is a solution of the problem

$$-\Delta u_\ast = \lambda u_\ast, \quad x \in D, \quad \left( \frac{\partial}{\partial r} + A \right) u_\ast = 0, \quad x \in \partial D,$$
i.e., \( \lambda_* \in Q \) is an eigenvalue of the limiting problem, whereas by assumption the set \( Q \) does not contain the eigenvalues of the limiting problem, a contradiction. The estimate (4.1) is proved.

By similar arguments, employing (4.1) instead of (4.4), it easy to prove the convergence of the solution of the problem (2.33) with \( \lambda = \lambda(k) \to \lambda_* \) to the solution of the problem (4.2) with \( \lambda = \lambda_* \). From this fact and continuity on \( \lambda \) of \( u_0 \) it follows the uniform on \( \lambda \) convergence of \( u_\varepsilon \) to \( u_0 \). The proof is complete.

**Lemma 4.2.** Let \( \lambda_0 \) be a \( p \)-multiply eigenvalue of the limiting problem, \( \lambda_\varepsilon^{(j)} \), \( j = 1, \ldots, p \) be the eigenvalues of the perturbed problem, converging to \( \lambda_0 \), with multiplicity taken into account, \( \psi_\varepsilon^{(j)} \) be the associated eigenfunctions orthonormalized in \( L^2(D) \). Then for \( \lambda \) close to \( \lambda_0 \) for solution of the problem (2.33) the representation

\[
u_\varepsilon = \sum_{j=1}^{p} \frac{b_\varepsilon^{(j)}}{\lambda_\varepsilon^{(j)} - \lambda} \int_D \psi_\varepsilon^{(j)}(x) f(x) \, dx + \tilde{u}_\varepsilon, \tag{4.6}
\]

holds, where \( \tilde{u}_\varepsilon \) is a holomorphic (in \( L^2(D) \)-norm) on \( \lambda \) function, orthogonal in \( L^2(D) \) to all \( \psi_\varepsilon^{(j)} \). For \( u_\varepsilon \) uniform on \( \varepsilon, \mu, \lambda \) and \( f \) estimate

\[
\|\tilde{u}_\varepsilon\|_1 \leq C \|f\|. \tag{4.7}
\]

takes place.

**Proof.** It is known that the solution \( u_\varepsilon \) of the problem (2.33) is a meromorphic on \( \lambda \) function having only simple poles coinciding with the eigenvalues of the perturbed problem. Residua at these poles (eigenvalues) are the associated eigenfunctions of the perturbed problem. Since \( \lambda_\varepsilon^{(j)} \) converge to \( \lambda_0 \), then \( \lambda \), close to \( \lambda_0 \), are close to \( \lambda_\varepsilon^{(j)} \). For this reason, for the function \( u_\varepsilon \) the representation

\[
u_\varepsilon = \sum_{j=1}^{p} \frac{b_\varepsilon^{(j)}}{\lambda_\varepsilon^{(j)} - \lambda} \psi_\varepsilon^{(j)} + \tilde{u}_\varepsilon, \tag{4.8}
\]

is valid, where \( b_\varepsilon^{(j)} \) are some scalar coefficients, \( \tilde{u}_\varepsilon \) is a holomorphic on \( \lambda \) function. From the equation for \( u_\varepsilon \) it follows that

\[
(\lambda_\varepsilon^{(j)} - \lambda) \int_D \psi_\varepsilon^{(j)} u_\varepsilon \, dx = \int_D \psi_\varepsilon^{(j)} f \, dx.
\]

Substituting the formula (4.8) into this equality we obtain that

\[
b_\varepsilon^{(j)} + (\lambda_\varepsilon^{(j)} - \lambda) \int_D \psi_\varepsilon^{(j)} \tilde{u}_\varepsilon \, dx = \int_D \psi_\varepsilon^{(j)} f \, dx,
\]

from what by holomorphy of \( \tilde{u}_\varepsilon \) we deduce:

\[
b_\varepsilon^{(j)} = \int_D \psi_\varepsilon^{(j)} f \, dx, \quad \int_D \psi_\varepsilon^{(j)} \tilde{u}_\varepsilon \, dx = 0, \quad j = 1, \ldots, p.
\]
These relations and (1.8) imply (4.6).

Let us show the estimate (4.7). We indicate by $S(z, a)$ an open circle of radius $a$ in complex plane with center at the point $z$. We choose the number $\delta$ by the condition that the circle $S(\lambda_0, \delta)$ contains no eigenvalues of the limiting problem except $\lambda_0$. Then for all sufficiently small $\varepsilon$ each $\lambda^{(j)}_{\varepsilon}$ lies in the circle $S(\lambda_0, \delta/2)$. Therefore, by the representation (4.8) and Lemma 4.1 for $\lambda \in \partial S(\lambda_0, \delta)$ the uniform estimate

$$
\|u_\varepsilon\| = \left\| u_\varepsilon - \sum_{j=1}^{p} \frac{\psi^{(j)}_\varepsilon}{\lambda^{(j)}_\varepsilon - \lambda} \int_D \psi^{(j)}_\varepsilon f \, dx \right\|_1 \leq C\|f\| + \frac{2}{\delta} \sum_{j=1}^{p} \|\psi^{(j)}_\varepsilon\|_1 \|f\| \leq C\|f\|.
$$

is true. Since $\tilde{u}_\varepsilon$ is holomorphic on $\lambda$, then due to module maximum principle the last inequality holds also for $\lambda \in S(\lambda_0, \delta)$. The proof is complete.

**Lemma 4.3.** The eigenvalues of the perturbed problem have the asymptotics (1.5)–(1.7).

**Proof.** Let $\lambda_0$ be an eigenvalue of the problem (1.3) and be a root of the equation (1.4) for $n = n_i$, $i = 1, \ldots, m$, where $n_i$ are different. We suppose that $n_1 = 0$, $n_i > 0$, $i = 2, \ldots, m$. The cases $n_i > 0$, $i = 1, \ldots, m$, and $m = 1, n_1 = 0$, are proved in the similar way. The eigenfunctions associated with $\lambda_0$ have the form

$$
\psi^{(1)}_0(x) = J_0(\sqrt{\lambda_0}r),
$$

$$
\psi^{(2i-2)}_0(x) = J_{n_i}(\sqrt{\lambda_0}r) \phi^+(n_i\theta), \quad i = 2, \ldots, m,
$$

$$
\psi^{(2i-1)}_0(x) = J_{n_i}(\sqrt{\lambda_0}r) \phi^-(n_i\theta), \quad i = 2, \ldots, m.
$$

Similarly, we denote by $\psi^{(j)}_{\varepsilon,K}, \tilde{\psi}^{(j)}_{\varepsilon,K}, f^{(j)}_{\varepsilon,K}$ the functions $\tilde{\psi}_{\varepsilon,K}, \psi_{\varepsilon,K}, \tilde{\psi}_{\varepsilon,K}, \tilde{\psi}_{\varepsilon,K}, f_{\varepsilon,K}, f_{\varepsilon,K}$, constructed in second and third sections and associated with the indexes $n_i$. Let $\lambda^{(1)}_{\varepsilon,K} = \lambda_{\varepsilon,K}$, where $\lambda_{\varepsilon,K}$ was defined in the second section, $\lambda^{(2i-2)}_{\varepsilon,K} = \lambda^{(2i-1)}_{\varepsilon,K} = \lambda_{\varepsilon,K}$, where $\lambda_{\varepsilon,K}$ was defined in the third section and associated with the index $n_i$, $i = 2, \ldots, m$. Clear, the multiplicity of $\lambda_0$ equals $(2m - 1)$. Due to Theorems 2.1 and 3.1 and Lemma 4.2 for the functions $\tilde{\psi}^{(j)}_{\varepsilon,K}$ the representations

$$
\tilde{\psi}^{(j)}_{\varepsilon,K} = \sum_{k=1}^{2m-1} b_{\varepsilon}^{jk} \psi^{(k)}_{\varepsilon} + \tilde{u}^{(j)}_{\varepsilon},
$$

$$
b_{\varepsilon}^{jk} = \int_D \psi^{(k)}_{\varepsilon} \tilde{\psi}^{(j)}_{\varepsilon,K} \, dx = \frac{1}{\lambda^{(k)}_{\varepsilon,K} - \lambda^{(j)}_{\varepsilon,K}} \int_D \psi^{(k)}_{\varepsilon} f^{(j)}_{\varepsilon,K} \, dx,
$$

$$
\|\tilde{u}^{(j)}_{\varepsilon}\|_1 \leq C \left\| f^{(j)}_{\varepsilon,K} \right\| = O(\varepsilon^K(A + \mu))
$$

hold. Suppose that some of the eigenvalues $\lambda^{(j)}_{\varepsilon}$ do not satisfy the asymptotics (1.5)–(1.7), namely, uniform on $\varepsilon$ and $\mu$ estimate

$$
\left| \lambda^{(k)}_{\varepsilon} - \lambda^{(j)}_{\varepsilon,K} \right| \geq C\varepsilon^p(A + \mu), \quad j = 1, \ldots, 2m - 1, \quad k \in I,
$$

(4.10)
hold, where \( p \) is a some number independent on \( K \), \( I \) is a subset of the indexes, \( I \subseteq \{1, 2, \ldots, 2m - 1\} \). By (4.10) and the statement of Theorems 2.1 3.1 for the functions \( f^{(j)}_{\varepsilon,K} \), we deduce that for \( K \geq p + 1 \) the convergences \( b^{(j)}_{\varepsilon,k} \to 0 \), \( k \in I, \ j = 1, \ldots, 2m - 1 \) hold. From the definition of the functions \( b^{(j)}_{\varepsilon} \), the orthogonality of \( \psi^{(j)}_{\varepsilon} \) and the convergence \( \psi^{(j)}_{\varepsilon,K} \to q_{0}^{(j)} \) it follows that \( b^{(j)}_{\varepsilon,k} \) are bounded, so, there exists a subsequence \( \varepsilon' \to 0 \), for that \( b^{(j)}_{\varepsilon,k} \to b^{(j)}_{0,k} \), moreover, \( b^{(j)}_{0,k} = 0 \), if \( k \in I, \ j = 1, \ldots, 2m - 1 \). In view of (4.11) and Lemma 4.2 we have the equalities

\[
\int_{D} \psi^{(j)}_{\varepsilon',K} \psi^{(i)}_{\varepsilon',K} \, dx = \sum_{k=1}^{2m-1} b^{(j)}_{\varepsilon,k} b^{(i)}_{\varepsilon,k} + \int_{D} \tilde{u}^{(j)}_{\varepsilon',K} \tilde{u}^{(i)}_{\varepsilon',K} \, dx,
\]

passing to limit as \( \varepsilon' \to 0 \) in which and bearing in mind the convergences \( \psi^{(j)}_{\varepsilon',K} \to \psi^{(i)}_{0} \), and the estimate for the functions \( \tilde{u}^{(j)}_{\varepsilon',K} \), we get

\[
c_{ij} \delta_{jl} = \sum_{k=1}^{2m-1} b^{(j)}_{0,k} b^{(i)}_{0,k}, \quad c_{jj} \neq 0, \quad (4.11)
\]

where \( \delta_{jl} \) is the Kronecker delta. Let \( b^{(j)}_{0,k} \) be a vector with components \( b^{(j)}_{0,k} \), \( k = 1, \ldots, 2m - 1 \), \( j \notin I, \ j = 1, \ldots, 2m - 1 \). In view of (4.11) we have \( (2m - 1) \) nonzero orthogonal \( q \)-dimensional vectors \( b^{(i)}_{0} \), where \( q < 2m - 1 \). The contradiction obtained proves the lemma.

**Lemma 4.4.** Let \( \lambda^{(1)}_{\varepsilon} \) and \( \lambda^{(2)}_{\varepsilon} \) be eigenvalues of the problem (1.4), (1.2), having asymptotics (1.3)–(1.4), associated with indexes \( n \) and \( m \), \( n \neq m \). Then uniform on \( \varepsilon \) and \( \mu \) estimate

\[
|\lambda^{(1)}_{\varepsilon} - \lambda^{(2)}_{\varepsilon}| \geq C\varepsilon^{4}(A + \mu). \quad (4.12)
\]

**Proof.** If \( \lambda^{(i)}_{\varepsilon}, i = 1, 2, \) converge to different limiting eigenvalues, then the estimate (4.12) is obvious. So, we assume that \( \lambda^{(i)}_{\varepsilon} \) converge to a same eigenvalue \( \lambda_{0} \). First we consider the case \( A = 0 \). Then

\[
\lambda^{(1)}_{\varepsilon} = \lambda + \mu \frac{2\lambda}{\lambda + \varepsilon^{2}} + O(\mu(\mu + \varepsilon^{3})), \quad \lambda^{(2)}_{\varepsilon} = \lambda + \mu \frac{2\lambda}{\lambda + \varepsilon^{2}} + O(\mu(\mu + \varepsilon^{3})),
\]

\[
\lambda^{(1)}_{\varepsilon} - \lambda^{(2)}_{\varepsilon} = \mu \frac{2\lambda^{0}(n^{2} - m^{2})}{(\lambda^{0} - n^{2})(\lambda^{0} - m^{2})} + O(\mu(\mu + \varepsilon^{3})),
\]

from what it follows (4.12) for \( A = 0 \). We proceed to the case \( A > 0 \). If \( \varepsilon = o(\mu^{1/3}) \), then by, (4.10), we deduce

\[
\lambda^{(1)}_{\varepsilon} - \lambda^{(2)}_{\varepsilon} = \mu \frac{2\lambda^{0}(n^{2} - m^{2})}{(\lambda^{0} - n^{2} + A^{2})(\lambda^{0} - m^{2} + A^{2})} + O(\varepsilon^{3}),
\]
from what it follows (4.12) for $A > 0$, $\varepsilon = o(\mu^{1/3})$. If $\mu = O(\varepsilon^3)$, then

$$
\lambda^{(1)}_\varepsilon - \lambda^{(2)}_\varepsilon = -\varepsilon^3 A^2 \lambda_0 \zeta(3) \left( \frac{\lambda_0 + 2n^2}{\lambda_0 - n^2 + A^2} - \frac{\lambda_0 + 2m^2}{\lambda_0 - m^2 + A^2} \right) + O(\varepsilon^4 + \mu) = \\
= -\varepsilon^3 A^2 \lambda_0 \zeta(3)(2A^2 + 3\lambda_0)(n^2 - m^2) \left( \lambda_0 - n^2 + A^2 \right) \left( \lambda_0 - m^2 + A^2 \right) + O(\varepsilon^4 + \mu),
$$
i.e., the estimate (4.12) is true in this case, too. If $\mu = O(\varepsilon^3)$, then it easy to see that

$$
\lambda^{(1)}_\varepsilon - \lambda^{(2)}_\varepsilon = \frac{\lambda_0(n^2 - m^2)(8\mu - \varepsilon^3 A^2 \zeta(3)(2A^2 + 3\lambda_0))}{4(\lambda_0 - n^2 + A^2)(\lambda_0 - m^2 + A^2)} + \varepsilon^4 \frac{\pi^4 \lambda_0 (8\lambda_0 + 1)(n^2 - m^2)}{5760(\lambda_0 - n^2 + A^2)(\lambda_0 + 2m^2 - A^2)} + O(\varepsilon^5),
$$
The first term in the formula obtained being nonzero, the inequality (4.12) takes place. The first term being zero, the second term does not vanish and we arrive at (4.12) again. The proof is complete.

**Proof of Theorem 1.1.** Hereafter we employ the notations introduced in the proof of Lemma 4.3 and we only deal with the case considered there (proof of other cases is similar). Let us prove that an eigenvalue $\lambda^{(j)}_\varepsilon$ is simple if associated number $n_i$ equals zero and it is double if this number is positive. We consider the eigenvalues $\lambda^{(2p-2)}_\varepsilon$ and $\lambda^{(2p-1)}_\varepsilon$ associated with the same number $n_p > 0$. Due to Lemma 4.3, these eigenvalues have the same asymptotic expansions. Let us show that they are equal, too. Suppose that they are different. In view of Lemma 4.4, other eigenvalues of the perturbed problem converging to $\lambda_0$ have asymptotics distinct from the asymptotics for $\lambda^{(2p-2)}_\varepsilon$ and $\lambda^{(2p-1)}_\varepsilon$. For this reason, the assumption that $\lambda^{(2p-2)}_\varepsilon \neq \lambda^{(2p-1)}_\varepsilon$ means that they are simple. To prove that they are coincide is to prove that the eigenvalue $\lambda^{(2p-1)}_\varepsilon$ is double. We write the representations (4.9) for the functions $\psi^{(2p-2)}_{\varepsilon,K}$ and $\psi^{(2p-1)}_{\varepsilon,K}$:

$$
\tilde{\psi}^{(i)}_{\varepsilon,K} = \sum_{j=2p-2}^{2p-1} b^{ij}_\varepsilon \psi^{(j)}_{\varepsilon} + \tilde{u}^{(i)}_{\varepsilon}, \quad i = 2p - 2, 2p - 1,
$$

$$
\tilde{u}^{(i)}_{\varepsilon} = \tilde{u}^{(i)}_{\varepsilon} + \sum_{k=1}^{2m-1} \sum_{k \neq 2p-2} b^{ik}_\varepsilon \psi^{(k)}_{\varepsilon}. \quad (4.13)
$$

By Lemma 4.4 and the definition of the quantities $b^{ik}_\varepsilon$ we get that $b^{ik}_\varepsilon = O(\varepsilon^{K-4})$, $i = 2p - 2, 2p - 1$, $k = 1, \ldots, 2m - 1$, $k \neq 2p - 2, 2p - 1$. Since $\| \tilde{u}^{(i)}_{\varepsilon} \| = O(\varepsilon^{K}(A + \mu))$, then $\| \tilde{u}^{(i)}_{\varepsilon} \| \to 0$ as $\varepsilon \to 0$ if $K \geq 5$. In view of Theorem 3.1, we have the convergences $\tilde{\psi}^{(2p-2)}_{\varepsilon,K} \to J_{n_p} \left( \sqrt{\lambda_0 r} \right) \phi^+(n_p \theta)$, $\tilde{\psi}^{(2p-1)}_{\varepsilon,K} \to J_{n_p} \left( \sqrt{\lambda_0 r} \right) \phi^-(n_p \theta)$ as $\varepsilon \to 0$. So, there are two linear combinations of the eigenfunctions $\tilde{\psi}^{(2p-2)}_{\varepsilon}$ and
\( \psi_{\varepsilon}^{(2p-1)} \) converging to \( J_{n_p} \left( \sqrt{\lambda_0 r} \right) \phi^\pm (n_p \theta) \):

\[
\begin{align*}
\mathbf{c}_{\varepsilon}^{(1)} \psi_{\varepsilon}^{(2p-2)} + \mathbf{c}_{\varepsilon}^{(2)} \psi_{\varepsilon}^{(2p-2)} & \to J_{n_p} \left( \sqrt{\lambda_0 r} \right) \phi^+ (n_p \theta), \\
\mathbf{c}_{\varepsilon}^{(3)} \psi_{\varepsilon}^{(2p-2)} + \mathbf{c}_{\varepsilon}^{(4)} \psi_{\varepsilon}^{(2p-2)} & \to J_{n_p} \left( \sqrt{\lambda_0 r} \right) \phi^- (n_p \theta),
\end{align*}
\]  

(4.14)

\( \mathbf{c}_{\varepsilon}^{i} = b_{\varepsilon}^{2p-2,2p-2} \mathbf{c}_{\varepsilon}^{(2)} = b_{\varepsilon}^{2p-2,2p-1}, \mathbf{c}_{\varepsilon}^{(3)} = b_{\varepsilon}^{2p-1,2p-2}, \mathbf{c}_{\varepsilon}^{(4)} = b_{\varepsilon}^{2p-1,2p-1} \). We introduce the functions \( \psi_{\varepsilon}^{(i)}(r, \theta) = \psi_{\varepsilon}^{(i)}(r, \theta) \left( r, \theta + \left[ \frac{N}{4n_p} \right] \varepsilon \pi \right), \ i = 2p - 2, 2p - 1 \). Hence, \( \lambda_{\varepsilon}^{(i)} \) is the integral part of a number. One can see that \( \psi_{\varepsilon}^{(i)} \) are eigenfunctions of the perturbed problem associated with \( \lambda_{\varepsilon}^{(i)} \). By assumption, the eigenvalues \( \lambda_{\varepsilon}^{(i)}, i = 2p - 2, 2p - 1 \) are simple and the associated eigenfunctions are orthonormalized in \( L_2(D) \). Thus, \( \psi_{\varepsilon}^{(2p-2)} = \mathbf{c}_{\varepsilon}^{(5)} \psi_{\varepsilon}^{(2p-2)}, \ \psi_{\varepsilon}^{(2p-1)} = \mathbf{c}_{\varepsilon}^{(6)} \psi_{\varepsilon}^{(2p-1)}, \ |\mathbf{c}_{\varepsilon}^{(5)}| = |\mathbf{c}_{\varepsilon}^{(6)}| = 1 \). From these equalities and (4.14) we obtain that

\[
\begin{align*}
\mathbf{c}_{\varepsilon}^{(5)} \mathbf{c}_{\varepsilon}^{(1)} \psi_{\varepsilon}^{(2p-2)} + \mathbf{c}_{\varepsilon}^{(6)} \mathbf{c}_{\varepsilon}^{(2)} \psi_{\varepsilon}^{(2p-2)} & \to -J_{n_p} \left( \sqrt{\lambda_0 r} \right) \phi^- (n_p \theta), \\
\mathbf{c}_{\varepsilon}^{(5)} \mathbf{c}_{\varepsilon}^{(3)} \psi_{\varepsilon}^{(2p-2)} + \mathbf{c}_{\varepsilon}^{(6)} \mathbf{c}_{\varepsilon}^{(4)} \psi_{\varepsilon}^{(2p-2)} & \to J_{n_p} \left( \sqrt{\lambda_0 r} \right) \phi^+ (n_p \theta),
\end{align*}
\]  

(4.15)

Calculating scalar product (in \( L_2(D) \)) for the first relation in (4.14) and the second one in (4.15) and for the second relation in (4.14) and the first one in (4.15), we arrive at the convergences

\[
\begin{align*}
\mathbf{c}_{\varepsilon}^{(1)} \mathbf{c}_{\varepsilon}^{(3)} + \mathbf{c}_{\varepsilon}^{(2)} \mathbf{c}_{\varepsilon}^{(4)} & \to -\mathbf{c}, \\
\mathbf{c}_{\varepsilon}^{(1)} \mathbf{c}_{\varepsilon}^{(3)} + \mathbf{c}_{\varepsilon}^{(2)} \mathbf{c}_{\varepsilon}^{(4)} & \to \mathbf{c},
\end{align*}
\]

\( \mathbf{c} = ||J_{n_p} \left( \sqrt{\lambda_0 r} \right) \phi^+ (n_p \theta)||^2 = ||J_{n_p} \left( \sqrt{\lambda_0 r} \right) \phi^- (n_p \theta)||^2 \). The convergences obtained can not hold at the same time, hence, \( \lambda_{\varepsilon}^{(2p-2)} = \lambda_{\varepsilon}^{(2p-1)} \). So, if \( n_p > 0 \), then the associated eigenvalue \( \lambda_{\varepsilon}^{(2p-2)} = \lambda_{\varepsilon}^{(2p-1)} \) is double. The perturbed eigenvalue associated with the index \( n_p = 0 \) has the asymptotics distinct from the asymptotics associated with other indexes, what means that this eigenvalue is simple.

We proceed to the justification of the asymptotics of the perturbed eigenfunctions. Let \( n_p > 0 \). We set

\[
\Psi_{\varepsilon}^{(i)} = \sum_{j=2p-2}^{2p-1} b_{\varepsilon}^{j} \psi_{\varepsilon}^{(j)}, \quad i = 2p - 2, 2p - 1.
\]

It is obvious that \( \Psi_{\varepsilon}^{(i)} \) are eigenfunctions of the perturbed problem associated with the double eigenvalue \( \lambda_{\varepsilon}^{(2p-2)} \). Due to (4.13), the above estimates for \( u_{\varepsilon}^{(i)} \) and the convergences of \( \Psi_{\varepsilon}^{(i)} \) to \( \psi_{\varepsilon}^{(i)} \), we obtain that \( \Psi_{\varepsilon}^{(i)} \) converges to \( \psi_{\varepsilon}^{(i)} \). The assertions (4.13), estimates for \( u_{\varepsilon}^{(i)} \) and the statements of Theorems 2.1 and 3.1 for the functions \( R_{\varepsilon,K} \) imply the inequalities

\[
\left| \left| \Psi_{\varepsilon,K}^{(i)} - \Psi_{\varepsilon}^{(i)} \right| \right| \leq \left| \left| \Psi_{\varepsilon,K}^{(i)} - \Psi_{\varepsilon}^{(i)} \right| \right| + \left| \left| \Psi_{\varepsilon,K}^{(i)} - \Psi_{\varepsilon}^{(i)} \right| \right| \leq \left| \left| \Psi_{\varepsilon,K}^{(i)} - \Psi_{\varepsilon}^{(i)} \right| \right| + \left| \left| \Psi_{\varepsilon,K}^{(i)} - \Psi_{\varepsilon}^{(i)} \right| \right|.
\]
can easily prove the estimate $n$ for some functions $B$ which mean that the asymptotics for the eigenfunction associated with eigenvalue $\lambda_0^{(2p-2)} = \lambda^{(2p-1)}_0$ have the form (3.1). For the simple eigenvalue $\lambda_0^{(1)}$ we consider the associated eigenfunction

$$
\Psi_0^{(1)} = b_0^n \psi_0^{(1)},
$$

converging to $J_0(\sqrt{\lambda_0} t)$, and by analogy with the case of double eigenvalue one can easily prove the estimate

$$
\left\| \psi_{\varepsilon,K} - \psi_0^{(1)} \right\| = O \left( \varepsilon^{K-4} \right),
$$

which means that the asymptotics for the eigenfunction associated with $\lambda_0^{(1)}$, has the form (2.32). The proof of Theorem 1.1 is complete.

Appendix

Here we shall prove that for some positive $A$ there exists $\lambda_0$ being a root of the equation of the equation (1.4) for different $n$ simultaneously. We introduce the notations $f_{n,A}(t) = t J_n'(t) + A J_n(t)$, $n \in \mathbb{Z}_+$, $A \geq 0$, $t \in (0, +\infty)$. The zeroes of the function $f_{n,A}(t)$ for nonnegative $A$ are roots of the equation (1.1). Let us prove that there exist $A > 0$, $n$, $m$, $n \neq m$, for those the functions $f_{n,A}$ and $f_{m,A}$ have a common positive root. We set $F_{n,m}(t) = t (J_n'(t) J_m(t) - J_m'(t) J_n(t))$. Let some point $t = t_0$ be a root of the function $F_{n,m}$ and it is not a zero of the functions $J_n$ and $J_m$. Then it follows from the equality $t_0 (J_n'(t_0) J_m(t_0) - J_m'(t_0) J_n(t_0)) = 0$ that

$$
t_0 \frac{J_n'(t_0)}{J_n(t_0)} = t_0 \frac{J_m'(t_0)}{J_m(t_0)}.
$$

Let $B = t_0 J_n'(t_0)/J_n(t_0)$. If $B \leq 0$, then the point $t_0$ is a common root of the functions $f_{n,A}$ and $f_{m,A}$ as $A = -B$. Thus, if we find a root of the function $F_{n,m}$ for some $n$ and $m$ and check the inequality $t_0 J_n'(t_0)/J_n(t_0) < 0$, we shall get the statement being proved. We make $n = 6$, $m = 3$. Then $F_{6,3}(8) \approx -0.1673037488 < 0$, $F_{6,3}(9) \approx 0.0658220035 > 0$. Since $F_{6,3}$ is a smooth function, then there exists a zero of the function $F_{6,3}$ in the interval $(8,9)$, we denote it by $t_0$. Let $j_{p,q}$, $j_{p,q}'$ be positive roots of $J_p$ and $J_p'$ taken in ascending order: $j_{p,1} < j_{p,2} < \ldots$, $j_{p,1}' < j_{p,2}' < \ldots$. We have: $j_{3,1} \approx 6.380161896 < 8$, $j_{3,2} \approx 9.761023130 > 9$, $j_{6,1} \approx 9.936109524 > 9$, $j_{6,1}' \approx 7.501266145 < 8$. These equalities imply that there are no zeroes of the functions $J_3$ and $J_6$ in the interval $(8,9)$. The function $J_6(t)$ is positive for $t \in (0,j_{6,1})$, and $j_{6,1}' < 8 < j_{6,1}$. For this reason, the inequalities $J_6(t) > 0$, $J_6'(t) < 0$ are true as $t \in (8,9)$, from what we deduce that $t J_6'(t)/J_6(t) < 0$ as $t \in (8,9)$. Thus, there exists a zero $t_0$ of the function $F_{6,3}$ in the interval $(8,9)$, that is not a zero of the functions $J_3$ and $J_6$, moreover, $t_0 J_6'(t_0)/J_6(t_0) < 0$. Therefore, $t_0$ is a of the equation (1.4) for $n = 6$ and $n = 3$ with $A = -t_0 J_6'(t_0)/J_6(t_0) > 0$. 

\[ \leq \left\| \tilde{u}^{(i)}_{\varepsilon} \right\| + \left\| \tilde{\psi}_{\varepsilon,K}^{(i)} - \psi_{\varepsilon}^{(i)} \right\| = O \left( \varepsilon^{K-4} \right) \]
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