Theoretical status of the $B_c$ meson in the shifted $l$-expansion technique

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Abstract

In the framework of phenomenological static and QCD-motivated model potentials for heavy quarkonium, we compute the $\overline{b}c$ mass spectrum as well as its 1S hyperfine splitting using the recently introduced shifted $l$-expansion technique. We also predict the leptonic constant $f_{B_c}$ of the lightest pseudoscalar $B_c$ and $f_{B_c^*}$ of the vector $B_c^*$ states taking into account the one-loop and two-loop QCD corrections. Further, we use the scaling relation to predict the leptonic constant of the $nS$-states of the $\overline{b}c$ system. For the sake of comparison, we use the same fitting parameters of our previous model potentials. From our results we conclude that shifted $l$-expansion method has the same accuracy, convergence and status as our previous work.

Keywords: $B_c$ meson; mass spectrum; leptonic constant; hyperfine splittings; heavy quarkonium.

PACS numbers: 03.65.Ge, 12.39.Jh, 13.30.Gd

1. INTRODUCTION

The spectrum and properties of the $\overline{b}c$ quarkonium system have been calculated various times in the past in the framework of heavy quarkonium theory [1]. Moreover, the recent
discovery [2] of $B_c$ meson (the lowest pseudoscalar $^1S_0$ state of the $B_c$ system) opens up new theoretical interest in the subject [1,3-8]. The Collider Detector at Fermilab (CDF) Collaboration quotes $M_{B_c} = 6.40^{+0.39}_{-0.13} \text{ GeV}$ [2]. This state should be one of a number of states lying below the threshold for emission of B and D mesons. Further, such states are very stable in comparison with their counterparts in charmonium ($\bar{c}c$) and bottomonium ($\bar{b}b$) systems. A particularly interesting quantity should be the hyperfine splitting that as for $\bar{c}c$ case seems to be sensitive to relativistic and subleading corrections in the strong coupling constant $\alpha_s$. For the above reasons it seems worthwhile to give a detailed account of the Schrödinger energies for $\bar{c}c$, $\bar{b}b$ and $\bar{b}c$ meson systems below the continuum threshold.

Because of the success of the nonrelativistic potential model and the flavour independence of the $q_1q_2$ potential, we choose a wide class of phenomenological and a QCD-motivated potentials by insisting upon strict flavor-independence of its parameters. We also use a potential model that includes running coupling constant effects in both the spherically symmetric potential and the spin-dependent potentials to give a simultaneous account of the properties of the $\bar{c}c$, $\bar{b}b$ and $\bar{b}c$ heavy quarkonium systems. Since one would expect the average values of the momentum transfer in the various quark-antiquark states to be different, some variation in the values of the strong coupling constant and the normalization scale in the spin-dependent potential should be expected. To minimize the role of flavor-dependence, we use consistent values for the coupling constant and universal QCD renormalization scale for each of the levels in a given system.

Recently, Kwong and Rosner [7] predicted the masses of the lowest vector and pseudoscalar states of the $\bar{b}c$ system using an empirical mass formula and a logarithmic potential. Eichten and Quigg [1] gave a more comprehensive account of the energies and decays of the $B_c$ system that was based on the QCD-motivated potential of Buchmüller and Tye [9]. Gershtein et al. [8] also computed the energies and decays of the $B_c$ system using the QCD sum-rule calculations. Baldicchi and Prosperi [6] have fitted the entire light-heavy quarkonium spectrum and computed the $\bar{b}c$ spectrum based on an effective mass operator with full relativistic kinematics. Fulcher [4] extended the treatment of the spin-dependent
potentials to the full radiative one-loop level using the renormalization scheme developed by Gupta and Radford [10]. Ebert et al. [1] investigated the $B_c$ meson masses and decays in the relativistic quark model. Very recently, we reproduced the $\tau c$, $\tau b$ and $\tau c$ spectroscopy by applying the shifted large-$N$ expansion technique (SLNET) on the nonrelativistic and relativistic wave equations using a group of static and improved QCD motivated potentials [11].

Encouraged by the success of SLNET application on heavy quarkonium systems [11,12], we extend the previous works in [11] by applying the shifted $l$- expansion technique (SLET) [13] on the Schrödinger equation to reproduce the $\tau c$ spectroscopy. The SLET has been recently introduced to solve mathematically the two and three-dimensional Schrödinger equation for spherically and cylindrically symmetric potentials [13]. Further, the KG and Dirac equations with radially symmetric Lorentz vector and Lorentz scalar potentials have also been solved [13]. We anticipate that, by working SLET, we would be able to obtain a better understanding on the status, convergence and accuracy of this method among the other methods [11]. We also consider it as a complementary investigation to our previous works [11]. Further, our results will enable us to check clearly that SLET is just a parallel perturbative expansion method, i.e., $l$-expansion procedure in a similar manner to $k$-expansion procedure.

The contents of this article are as follow: in section 2, we present the solution of the Schrödinger equation using the SLET for the the non-self conjugate $Qq$ meson mass spectrum. In section 3 we briefly present the potentials used. In section 4 we give the first-loop and second-loop correction of the $B_c$ leptonic decay constant. Finally, a discussion and conclusion appear in section 5.

2. SCHRÖDINGER WAVE EQUATION

In previous papers [11,12] we have applied the shifted $1/N$ expansion technique (SLNET) to solve nonrelativistic and relativistic wave equations. The method starts by writing
the original wave equation in an N-dimensional space which is sufficiently large and using expansion $1/k$ as a perturbation parameter [14]. Here $k = N + 2l - a$, $N$ being the number of spatial dimensions of interest, $l$ the angular quantum number, and $a$ is a suitable shift as an additional degree of freedom and is responsible for speeding up the convergence of the resulting energy series. In this work another method called the shifted $l$-expansion technique (SLET) which is simply consists of using $1/l$ as an expansion parameter, where $l = l - a$, $l$ is an angular quantum number, $a$ is a suitable shift which is mainly introduced to avoid the trivial case $l = 0$. The choice of $a$ is physically motivated so that the next to the leading energy eigenvalue series vanish as in SLNET. The method does not require writing the $N$-dimensional form of the wave equation and we expand it directly through the quantum number $l$ involved in the problem. This method seems more flexible and simpler in treatment and has a quite different mathematical expansion than SLNET. Like SLNET, SLET is also a pseudoperturbative technique. The radial part of the Schrödinger equation for an arbitrary spherically symmetric potential $V(r)$ (in units $\hbar = 1$)

$$\left\{- \frac{1}{4\mu} \frac{d^2}{dr^2} + \frac{l(l+1)}{4\mu r^2} + V(r)\right\} u(r) = E_{n,l} u(r), \tag{1}$$

where $\mu = (m_q m_Q)/(m_q + m_Q)$ is the reduced mass for the two constituent interacting particles and $E_{n,l}$ denotes the Schrödinger binding energy. Equation (1) can be rewritten as

$$\left\{- \frac{1}{4\mu} \frac{d^2}{dr^2} + \frac{\tau^2 + (2a + 1) \tau + a (a + 1)}{4\mu r^2} + V(r)\right\} u(r) = E_{n,l} u(r), \tag{2}$$

where $\tau = l - a$ with $a$ representing a proper shift to be calculated later. It is clear from Eq. (2) and our previous work [11] [cf. Eq. (1) in hep-ph/0303182] that $\tau$ is as much as $k/2$ of Imbo et al. [14]. We follow the shifted $l$-expansion method [13] by defining

$$V(y(r_0)) = \sum_{m=0}^{\infty} \left( \frac{d^m V(r_0)}{dr_0^m} \right) (r_0 y)^m \frac{m! Q}{(m-4)/2} \tau^{-(m-4)/2}, \tag{3}$$

and also the energy eigenvalue expansion [11,12]

$$E_{n,l} = \sum_{m=0}^{\infty} \frac{\tau^{2-m}}{Q} E_m. \tag{4}$$
Here \( y = \frac{T^{1/2}(r/r_0 - 1)}{l} \) with \( r_0 \) is an arbitrary point where the Taylor expansions is being performed about and \( Q \) is a scale to be set equal to \( T^2 \) at the end of these calculations. Inserting Eqs. (3) and (4) into Eq. (2) yields

\[
\left[ -\frac{1}{4\mu} \frac{d^2}{dy^2} + \frac{1}{4\mu} \left( T + (2a + 1) + \frac{a(a + 1)}{T} \right) \sum_{m=0}^{\infty} \frac{(-1)^m(m + 1)y^m}{T^{m/2}} \right.
\]

\[
+ \frac{r_0^2}{Q} \sum_{m=0}^{\infty} \left( \frac{d^m V(r_0)}{dr_0^m} \right) \frac{(r_0 y)^m}{m!} T^{(2-m)/2} \right] \chi_{n_r}(y) = \xi_{n_r} \chi_{n_r}(y). \tag{5}
\]

Hence the final analytic expression for the \( 1/l \) expansion of the energy eigenvalues appropriate to the Schrödinger particle is [13]

\[
\xi_{n_r} = \frac{r_0^2}{Q} \sum_{m=0}^{\infty} T^{(1-m)} E_m. \tag{6}
\]

Now we formulate the SLET for the nonrelativistic motion of spinless particle bound in spherically symmetric potential \( V(r) \). Equation (6) is exactly of the type of Schrödinger-like equation for the one dimensional anharmonic-oscillator and has been investigated for spherically symmetric potential by Imbo et al. [14]:

\[
\xi_{n_r} = T \left[ \frac{1}{4\mu} + \frac{r_0^2 V(r_0)}{Q} \right] + \left[ (n_r + \frac{1}{2}) \omega + \frac{(2a + 1)}{4\mu} \right]
\]

\[
+ \frac{1}{T} \left[ \frac{a(a + 1)}{4\mu} + \gamma^{(1)} \right] + \frac{\gamma^{(2)}}{T^2} + O \left( \frac{1}{T^3} \right), \tag{7}
\]

where the expressions \( \gamma^{(1)} \) and \( \gamma^{(2)} \) are given explicitly in Appendix A. Thus, comparing Eq. (6) with Eq. (7) gives

\[
E_0 = V(r_0) + \frac{Q}{4\mu r_0^2}, \tag{8}
\]

\[
E_1 = \frac{Q}{r_0^2} \left[ (n_r + \frac{1}{2}) \omega + \frac{(2a + 1)}{4\mu} \right], \tag{9}
\]

\[
E_2 = \frac{Q}{r_0^2} \left[ \frac{a(a + 1)}{4\mu} + \gamma^{(1)} \right], \tag{10}
\]

and
\[ E_3 = \frac{Q}{r_0^2} \gamma^{(2)}. \]  

(11)

The quantity \( r_0 \) is chosen so as to minimize the leading term, \( E_0 \), that is, [12]

\[ \frac{dE_0}{dr_0} = 0 \quad \text{and} \quad \frac{d^2E_0}{dr_0^2} > 0, \]  

(12)

which yields the relation

\[ Q = 2\mu r_0^3 V'(r_0). \]  

(13)

Further, to solve for the shifting parameter \( a \), the next contribution to the energy eigenvalues is chosen to vanish [11-14], i.e., \( E_1 = 0 \), which provides smaller contributions for the higher-order corrections in (4) compared to the leading term contribution (8). It implies that the energy states are being calculated by considering only the leading term \( E_0 \), the second-order, \( E_2 \) and the third-order, \( E_3 \) corrections. Therefore, the shifting parameter is determined via

\[ a = - \frac{[1 + 2\mu(2n_r + 1)\omega]}{2}, \]  

(14)

with

\[ \omega = \frac{1}{2\mu} \left[ 3 + \frac{r_0 V''(r_0)}{V'(r_0)} \right]^{1/2}. \]  

(15)

Thus, the Schrödinger binding energy (4) to the third order is

\[ E_{n,l} = V(r_0) + \frac{1}{r_0^2} \left[ \frac{a(a + 1) + Q}{4\mu} + \gamma^{(1)} + \frac{\gamma^{(2)}}{l} + O \left( \frac{1}{l^2} \right) \right]. \]  

(16)

Further, setting \( \tilde{l} = \sqrt{Q} \) which rescales the potential, we derive an analytic expression that satisfies \( r_0 \) as

\[ 2l + \left[ 1 + (2n_r + 1) \left[ 3 + \frac{r_0 V''(r_0)}{V'(r_0)} \right]^{1/2} \right] = 2 \left[ 2\mu r_0^3 V'(r_0) \right]^{1/2}, \]  

(17)

where \( n_r = n - 1 \) is the radial quantum number. Once \( r_0 \) is being found through Eq. (17) for any arbitrary state, the determination of the binding energy for the \( \overline{Q}q \) system becomes relatively simple and straightforward. Finally, the Schrödinger binding mass can be determined via
\[ M(\overline{Q}q) = m_q + m_Q + 2E_{n,l}. \] (18)

It is being found that for a fixed \( n \), the computed energies become more accurate as \( l \) increases [11-14]. This is expected since the expansion parameter \( 1/\ell \) becomes smaller as \( l \) becomes larger since the parameter \( \ell \) is proportional to \( n \) and appears in the denominator in higher-order correction.

### 3. SOME MODEL POTENTIALS

The \( \overline{\tau}c \) system that we investigate is often considered as nonrelativistic system [1] and consequently our treatment is based upon Schrödinger equation with a Hamiltonian [1,4]

\[ H_o = -\nabla^2 + V(r) + V_{SD}, \] (19)

where we have supplemented our nonrelativistic Hamiltonian with the standard spin-dependent terms [1,11,15]

\[ V_{SD} \rightarrow V_{SS} = \frac{32\pi\alpha_s}{9m_qm_Q}(s_1\cdot s_2)\delta^3(r). \] (20)

Here, the spin dependent potential is simply a spin-spin part [1,15] that would enable us to make some preliminary calculations of the energies of the lowest two S-states of the \( \overline{\tau}c \) system. The potential parameters in this section are all strictly flavor-independent and fitted to the low-lying energy levels of \( \tau c \) and \( \overline{b}b \) systems. Like most authors (cf. [1]), we determine the coupling constant \( \alpha_s(m_c^2)^1 \) from the well measured hyperfine splitting for the 1S(\( \overline{c}c \)) state [16]

\[ \Delta E_{HF}(1S, \text{exp}) = M_{J/\psi} - M_{\eta_c} = 117.2 \pm 1.5 \text{ MeV}, \] (21)

and for the 2S(\( \overline{c}c \)) state [16-18]

\(^1\text{Due to the lack of any experimental splitting data on the } B_c \text{ meson, as there is one established state, we have fitted the coupling constant to reproduce the available } c\overline{c} \text{ splittings.}\)
\[ \Delta E_{\text{HF}}(2S, \text{exp}) = M_{\psi'} - M_{\eta_c} = 32 \pm 14 \text{ MeV}, \]  

(22)

for each desired potential to produce the center-of-gravity (cog) of the \( \overline{M}_\psi(1S) \) value. The numerical value of \( \alpha_s \) is found to be dependent on the potential form and also be compatible with the other measurements \([1,3,4,6-8]\). Therefore, the IS-state hyperfine splitting \([1,11,15]\) is given by\(^2\)

\[ \Delta E_{\text{HF}} = \frac{8\alpha_s}{9m_cm_b} |R_{1S}(0)|^2, \]

(23)

with the radial wave function at the origin is determined via \([15]\)

\[ |R_{1S}(0)|^2 = 2\mu \left\langle \frac{dV(r)}{dr} \right\rangle. \]

(24)

Hence, the total mass of the low-lying pseudoscalar \( B_c \) meson is \([11]\)

\[ M_{B_c}(0^-) = m_c + m_b + 2E_{1,0} - 3\Delta E_{\text{HF}}/4, \]

(25)

and for the vector \( B_c^* \) meson

\[ M_{B_c^*}(1^-) = m_c + m_b + 2E_{1,0} + \Delta E_{\text{HF}}/4. \]

(26)

Hence, the square-mass difference can be simply found as

\[ \Delta M^2 = M_{B_c^*}(1^-) - M_{B_c}(0^-) = 2\Delta E_{\text{HF}} [m_c + m_b + 2E_{1,0} - \Delta E_{\text{HF}}/4]. \]

(27)

The perturbative part of such a quantity was evaluated at the lowest order in \( \alpha_s \). Baldicchi and Prosperi \([6]\) used the standard running QCD coupling expression

\[ \alpha_s(Q) = \frac{4\pi}{(11 - \frac{2}{3}n_f)\ln \left( \frac{Q^2}{\Lambda^2} \right)}. \]

(28)

\(^2\)To the moment, the only measured splitting of nS–levels is that of \( \eta_c \) and \( J/\psi \), which allows us to evaluate the so-called SAD using \( \overline{M}_\psi(1S) = (3M_{J/\psi} + M_{\eta_c})/4 \) and also \( \overline{M}(\text{nS}) = M_V(\text{nS}) - (M_{J/\psi} - M_{\eta_c})/4n \) \([15,19]\).
with \( n_f = 4 \) and \( \Lambda = 0.2 \) GeV cut at a maximum value \( \alpha_s(0) = 0.35 \), to give the right \( J/\psi - \eta_c \) splitting (21) and to treat properly the infrared region [6]. Further, Brambilla and Vairo [3] took in their perturbative analysis \( 0.26 \leq \alpha_s(\mu = 2 \) GeV) \( \leq 0.30 \). After the observation of the \( \eta_c(2S) \) meson [17], Badalian and Bakker [18] determination of the coupling constant \( \alpha_{HF}(\mu_1,1S) \approx 0.335 \) is rather large with \( \mu_1 \approx \frac{1}{2} M(J/\psi) = 1.55 \) GeV and \( \alpha_{HF}(\mu_2,2S) \approx 0.18 \) with \( \mu_2 \approx 2M(\psi') = 7.4 \) GeV which implies the very large value for the renormalization scale. They [20] also used \( \alpha_s(\mu = 0.92 \) GeV) \( \approx 0.36 \) for all states, but the splittings do practically not change if \( \alpha_s(\mu = 1.48 \) GeV) \( = 0.30 \) is taken. This result shows that the strong coupling constant depends on the renormalization scale and it changes from one state to another [18]. Further, Motyka and Zalewski [21] found \( \alpha_s(m_c^2) = 0.3376 \) and from which they calculated \( \alpha_s(m_b^2) = 0.2064 \) and \( \alpha_s(4\mu_b^2) = 0.2742 \). Therefore, it is quite clear that the coupling constant is dependent on the quarkonium system.

The commonly used potentials are of two types: (i) pure phenomenological and (ii) partly phenomenological, but motivated by both perturbative and non-perturbative QCD at short and long distances. In this work we use the following types of potentials:

**A. Static potentials**

It is seen that the most phenomenological potentials in Eq. (19) may be gathered up in general form [21,22]:

\[
V(r) = -ar^{-\alpha} + br^\beta + c \quad 0 \leq \alpha, \beta \leq 1, \quad a, b \geq 0.
\]

where \( c \) may be of either sign. The mixed powerlaw (29) comprises more than ten potentials given by Refs. [21,22]. It seems quite reasonable that the short range behaviour of the quarkonium potential is less singular than \( -r^{-1} \) and the confining potential does not rise quickly as \( r \) due to the screening effects of quark pair creation. Therefore, the effective quarkonium potential consist of two terms, one of which, \( V_v(r) = -ar^{-\alpha} \), transforms like a time-component of a Lorentz 4-vector and the other, \( V_s(r) = br^\beta + c \), like a Lorentz scalar. We limit our study to the \( \alpha = \beta = \nu \) case
\[ V(r) = -ar^{-\nu} + br^\nu + c \quad 0 \leq \nu \leq 1, \quad a,b \geq 0. \]  

proposed by Lichtenberg et al. [23]. Like most authors [1,4,11,23], we consider a class of static potential which give reasonable accounts for the $\overline{c}c$ and $\overline{b}b$ spectra. This comprises a wide class of potentials presented explicitly in our previous works [11] some are QCD-motivated potentials like Cornell ($\nu = 1$), and other typical pure phenomenological potentials like Song-Lin ($\nu = 1/2$), Turin ($\nu = 3/4$), Logarithmic ($\nu \to 0$) which are all belonging to the class (30) and Martin ($\alpha = 0, \beta = 0.1$) belonging to the class (29). The motivation of this choice is that all of these potentials lie very close together in the range of distances $0.1 \leq r \leq 1$, which is the characteristic interval of $\overline{c}c$ and $\overline{b}b$ spectra.

B. QCD-motivated potentials

1. Igi-Ono potential

Buchmüller and Tye [9] proposed a potential which is consisting of two parts, at short distances the two-loop perturbative calculation of the interquark one-gloun exchange [24]:

\[ V_{\text{OGE}}^{(n_f=4)}(r) = -\frac{16\pi}{25} \frac{1}{rf(r)} \left[ 1 - \frac{462}{625} \frac{\ln r f(r)}{f(r)} + \frac{2\gamma_E + \frac{53}{75}}{f(r)} \right], \]

with

\[ f(r) = \ln \left[ \frac{1}{r^2 \Lambda_{MS}^2} + b \right], \]

where $n_f = 4$ is the number of flavors with mass below $\mu$ and $\gamma_E = 0.5772$ is the Euler’s number. Moreover, at long distances the interquark potential grows linearly leading to confinement as

\[ V_L(r) = ar. \]

Therefore, the Igi-Ono potential is [24]

\[ V^{(n_f=4)}(r) = V_{\text{OGE}}^{(n_f=4)} + ar + d e^{-gr}, \]
where the term $d r e^{-gr}$ in (34) is added to interpolate smoothly between the two parts and to adjust the intermediate range behavior by which the range of $\Lambda_{\overline{\text{MS}}}$ is extended keeping linearly rising confining potential. Numerical calculations show that potential is good for $\Lambda_{\overline{\text{MS}}}$ in the range 100-500 MeV keeping a good fit to the $\overline{c}c$ and $\overline{b}b$ spectra. The QCD coupling constant $\alpha_s$ in (20) is defined in the Gupta-Radford (GR) renormalization scheme [10]

$$\alpha_s = \frac{6\pi}{(33 - 2n_f) \ln \left( \frac{\mu}{\Lambda_{\overline{\text{GR}}}} \right)},$$  

(35)

where the scale parameter in the Gupta-Radford (GR) renormalization scheme $\Lambda_{\overline{\text{GR}}}$ [10] is related to $\Lambda_{\overline{\text{MS}}}$ by

$$\Lambda_{\overline{\text{GR}}} = \Lambda_{\overline{\text{MS}}} \exp \left[ \frac{49 - 10n_f/3}{2(33 - 2n_f)} \right].$$

(36)

Thereby, the three types of this potential are displayed in [11].

2. **Improved Chen-Kuang potential**

Chen and Kuang [25] proposed two improved potential models so that the parameters therein all vary explicitly with $\Lambda_{\overline{\text{MS}}}$ so that these parameters can only be given numerically for several values of $\Lambda_{\overline{\text{MS}}}$. Such potentials have the natural QCD interpretation and explicit $\Lambda_{\overline{\text{MS}}}$ dependence both for giving clear link between QCD and experiment and for convenience in practical calculation for a given value of $\Lambda_{\overline{\text{MS}}}$. It has the general form

$$V^{(n_f=4)}(r) = kr - \frac{16\pi}{25} \frac{1}{rf(r)} \left[ 1 - \frac{462}{625} \frac{\ln f(r)}{f(r)} + \frac{2\gamma_E + \frac{53}{75}}{f(r)} \right],$$

(37)

where the string tension is related to Regge slope by $k = \frac{1}{2\alpha' \alpha}$. The function $f(r)$ in (37) is

$$f(r) = \ln \left[ \frac{1}{\Lambda_{\overline{\text{MS}}}r} + 5.10 - A(r) \right]^2,$$

(38)

with

$$A(r) = \left[ 1 - \frac{\Lambda_{\overline{\text{MS}}}^2}{4\Lambda_{\overline{\text{MS}}}r} \right] \frac{1 - \exp \left\{ - \left[ 15 \left( \frac{\Lambda_{\overline{\text{MS}}}^2}{4\Lambda_{\overline{\text{MS}}}r} - 1 \right) \Lambda_{\overline{\text{MS}}} r \right]^2 \right\}}{\Lambda_{\overline{\text{MS}}} r}. $$

(39)
The scale parameter $\Lambda_{\overline{MS}}^I$ is very close to the value of $\Lambda_{\overline{MS}}$ determined from the two-photon processes and is also close to the world-averaged value of $\Lambda_{\overline{MS}}$. The fitted values of its parameters are displayed in Ref. [11].

4. LEPTONIC CONSTANT OF THE $B_c$-MESON

The study of the heavy quarkonium system has played a vital role in the development of the QCD. Some of the earliest applications of perturbative QCD were calculations of the decay rates of charmonium [26]. These calculations were based on the assumption that, in the nonrelativistic (NR) limit, the decay rate factors into a short-distance (SD) perturbative part associated with the annihilation of the heavy quark and antiquark and a long-distance (LD) part associated with the quarkonium wavefunction. Calculations of the annihilation decay rates of heavy quarkonium have recently been placed on a solid theoretical foundation by Bodwin et al. [27]. Using NRQCD [28] to separate the SD and LD effects, Bodwin et al. derived a general factorization formula for the inclusive annihilation decay rates of heavy quarkonium. The SD factors in the factorization formula can be calculated using pQCD [19], and the LD factors are defined rigorously in terms of the matrix elements of NRQCD that can be estimated using lattice calculations [5]. It applies equally well to S-wave, P-wave, and higher orbital-angular-momentum states, and it can be used to incorporate relativistic corrections to the decay rates.

In the NRQCD [28] approximation for the heavy quarks, the calculation of the leptonic decay constant for the heavy quarkonium with the two-loop accuracy requires the matching of NRQCD currents with corresponding full-QCD axial-vector currents [29]

$$J^\lambda|_{\text{NRQCD}} = -\chi_b^\dagger \psi_c v^\lambda \quad \text{and} \quad J^\lambda|_{\text{QCD}} = \bar{b} \gamma^\lambda \gamma_5 c,$$

(40)

where $b$ and $c$ are the relativistic bottom and charm fields, respectively, $\chi_b^\dagger$ and $\psi_c$ are the NR spinors of anti-bottom and charm and $v^\lambda$ is the four-velocity of heavy quarkonium. The NRQCD lagrangian describing the $B_c$-meson bound state dynamics is [30]
where $\mathcal{L}_{\text{light}}$ is the relativistic lagrangian for gluons and light quarks. The two-component spinor field $\psi_c$ annihilates charm quarks, while $\chi_b$ creates bottom anti-quarks. The relative velocity $v$ of heavy quarks inside the $B_c$-meson provides a small parameter that can be used as a nonperturbative expansion parameter. To express the decay constant $f_{B_c}$ in terms of NRQCD matrix elements we express $J^\lambda |_{\text{QCD}}$ in terms of NRQCD fields $\psi_c$ and $\chi_b$. The $\lambda = 0$ current-component contributes to the matrix element and consequently the $J^\lambda |_{\text{QCD}}$ has the following operator expansion

$$\langle 0 | \bar{b} \gamma^\lambda \gamma_5 c | B_c(P) \rangle = i f_{B_c} P^\lambda,$$

where $|B_c(P)\rangle$ is the state of the $B_c$-meson with four-momentum $P$. Only the $\lambda = 0$ component contributes to the matrix element (42) in the rest frame of the $B_c$-meson. It has the standard covariant normalization

$$\frac{1}{(2\pi)^3} \int \psi^\ast_{B_c}(p') \psi_{B_c}(p) d^3p = 2E \delta^{(3)}(p' - p),$$

and its phase has been chosen so that $f_{B_c}$ is real and positive. Hence, the matching yields

$$\bar{b} \gamma^0 \gamma_5 c = K_0 \chi_b^\dagger \psi_c + K_2 (D\chi_b)^\dagger D\psi_c + \cdots,$$

where $K_0 = K_0(m_c, m_b)$ and $K_2 = K_2(m_c, m_b)$ are Wilson SD coefficients. They can be determined by matching perturbative calculations of the matrix element $\langle 0 | \bar{b} \gamma^0 \gamma_5 c | B_c \rangle$, a contribution is mostly coming up from the first term in

$$\langle 0 | \bar{b} \gamma^0 \gamma_5 c | B_c \rangle \bigg|_{\text{QCD}} = \left[ K_0 \langle 0 | \chi_b^\dagger \psi_c | B_c \rangle \bigg|_{\text{NRQCD}} + K_2 \langle 0 | (D\chi_b)^\dagger D\psi_c | B_c \rangle \bigg|_{\text{NRQCD}} \right] + \cdots,$$

where the matrix element on the left side of (45) is taken between the vacuum and the state $|B_c\rangle$. Hence, equation (45) can be estimated as:

$$\left| \langle 0 | \chi_b^\dagger \psi_c | B_c \rangle \right|^2 \simeq \frac{3M_{B_c}}{\pi} |R_{1S}(0)|^2.$$

Onishchenko and Veretin [30] calculated the matrix elements on both sides of equation (45) up to $\alpha_s^2$ order. In one-loop calculation, the SD-coefficients are
$K_0 = 1 \text{ and } K_2 = -\frac{1}{8\mu^2}$, \hfill (47)

with $\mu$ defined after Eq. (1). Further, Braaten and Fleming in their work [31] calculated the perturbation correction to $K_0$ up to order $\alpha_s$ (one-loop correction) as

$$K_0 = 1 + c_1 \frac{\alpha_s(\mu)}{\pi},$$ \hfill (48)

with $c_1$ being calculated in Ref. [31] as

$$c_1 = -\left[2 - \frac{m_b - m_c}{m_b + m_c} \ln \frac{m_b}{m_c}\right].$$ \hfill (49)

Finally, the leptonic decay constant for the one-loop calculations is

$$f_{B_c}^{(1{-}\text{loop})} = \left[1 - \frac{\alpha_s(\mu)}{\pi} \left[2 - \frac{m_b - m_c}{m_b + m_c} \ln \frac{m_b}{m_c}\right]\right] f_{B_c}^{\text{NR}},$$ \hfill (50)

where the NR leptonic constant [32] is

$$f_{B_c}^{\text{NR}} = \sqrt{\frac{3}{\pi M_{B_c}}} |R_{1S}(0)|$$ \hfill (51)

and $\mu$ is any scale of order $m_b$ or $m_c$ of the running coupling constant. On the other hand, the calculations of two-loop correction in the case of vector current and equal quark masses was done in Ref. [33]. Further, Onishchenko and Veretin [30] extended the work of Ref. [33] into the non-equal mass case. They found an expression for the two-loop QCD corrections to $B_c$-meson leptonic constant given by

$$K_0(\alpha_s, M/\mu) = 1 + c_1(M/\mu) \frac{\alpha_s(M)}{\pi} + c_2(M/\mu) \left(\frac{\alpha_s(M)}{\pi}\right)^2 + \cdots,$$ \hfill (52)

where $c_2(M/\mu)$ is the two-loop matching coefficient and with $c_{1,2}$ are explicitly given in Eq. (49) and (cf. Ref. [30]; Eqs. (16)-(20) therein), respectively. In the case of $B_c$-meson and pole quark masses ($m_b = 4.8\text{ GeV}, m_c = 1.65\text{ GeV}$), they found

$$f_{B_c}^{(2{-}\text{loop})} = \left[1 - 1.48 \left(\frac{\alpha_s(m_b)}{\pi}\right) - 24.24 \left(\frac{\alpha_s(m_b)}{\pi}\right)^2\right] f_{B_c}^{\text{NR}}.$$ \hfill (53)

Therefore, the two-loop corrections are large and constitute nearly 100% of one-loop correction as stated in Ref. [30], where $\mu$ is chosen as the scale of the running coupling constant.
5. DISCUSSION AND CONCLUSION

We use Eq. (21) to determine the position of the charmonium center-of-gravity $\overline{M}_\psi(1S)$ mass spectrum. Further, we fix the coupling constant $\alpha_s(m_c)$ for each potential. For simplicity we neglect the variation of $\alpha_s$ with momentum in (28) to have a common spectra for all states and scale the splitting of $\bar{b}c$ and $\bar{b}\bar{b}$ from the charmonium value in (21). The consideration of the variation of the effective Coulomb interaction constant becomes especially essential for the $\Upsilon$ particle, for which $\alpha_s(\Upsilon) \neq \alpha_s(\psi)$.\footnote{Kiselev et al. [34] have taken into account that $\Delta M_\Upsilon(1S) = \frac{\alpha_s(\Upsilon)}{\alpha_s(\psi)} \Delta M_\psi(1S)$ with $\alpha_s(\Upsilon)/\alpha_s(\psi) \simeq 3/4$. Further, Motyka and Zalewiski [21] also found $\frac{\alpha_s(m_c^2)}{\alpha_s(m_b^2)} \simeq 11/18.$} We follow our previous works [11] to calculate the corresponding low-lying center-of-gravity $\overline{M}_\Upsilon(1S)$ and consequently the low-lying $\overline{M}_{Bc}(1S)$.

Table 1 reports our prediction for the Schrödinger mass spectrum of the four lowest $\bar{b}c$ S-states together with the first three P- and D-states below their strong decay threshold for different static potentials. Since the model is spin independent and as the energies of some singlet states of quarkonium families have not been measured [11,19,23], a theoretical estimates of these unknown levels introduces uncertainty into the calculated SAD [11]. Our results in Table 1 for the $B_c$ and $B^*_c$ meson masses are in a pretty good agreement with the other authors [1,4,7,11]. Thus, it is clear that the Song-Lin and Turin potentials are the most preferred interactions. Here, we report the range of the strong coupling constant at the $m_c$ scale we take in our analysis $0.1985 \leq \alpha_s(m_c^2) \leq 0.320$ for all types of potentials and $0.220 \leq \alpha_s(m_c^2) \leq 0.320$ for the class of static potentials [11]. We point out a different choice of the potential would in general lead to a different value of the wave function at the origin and to a different determination of $\alpha_s(m_c^2)$ from the same hyperfine splitting. Further, our predictions to the $\bar{b}c$ masses of the lowest S-wave (singlet and triplet) together with the other estimations by many authors are given in Table 2. Moreover, in Table 3, we also estimate the radial wave function of the low-lying state of the $\bar{b}c$ system, so that

$$|R_{1S}(0)| = 1.280 - 1.540 \text{ GeV}^{3/2},$$  \hspace{1cm} (54)
for the group of static potentials. Further, we present our calculations for the NR leptonic constant $f_{B_c}^{NR} = 466^{+19}_{-25}$ MeV and $f_{B_c^*}^{NR} = 463^{+19}_{-24}$ MeV as an estimation of the potential models without the matching [4,19]. Our results are compared with those of other works [1,35,36]. Our calculation for the one-loop correction:

$$f_{B_c}^{(1\text{-loop})} = 408^{+16}_{-14} \text{ MeV and } f_{B_c^*}^{(1\text{-loop})} = 405^{+17}_{-14} \text{ MeV},$$

and for two-loop correction:

$$f_{B_c}^{(2\text{-loop})} = 315^{+16}_{-51} \text{ MeV and } f_{B_c^*}^{(2\text{-loop})} = 313^{+26}_{-51} \text{ MeV}.$$  

(55)

(56)

So, our numerical value for $f_{B_c}^{NR}$ agrees with the estimates obtained in the framework of the lattice QCD result [5], $f_{B_c}^{NR} = 440 \pm 20$ MeV, QCD sum rules [37], potential models [1,4,19], and from the scaling relation [34]. It indicates that the one-loop matching [33] provides the magnitude of correction of nearly 12%. Further, the most recent calculation [29] in the heavy quark potential in the static limit of QCD with the one-loop matching is

$$f_{B_c}^{(1\text{-loop})} = 400 \pm 15 \text{ MeV}.$$  

(57)

Therefore, in contrast to the discussion given in [29], we see that the difference is not crucially large in our estimation to one-loop value in the $B_c$-meson. On the other hand, our final result of the two-loop calculations is

$$f_{B_c}^{(2\text{-loop})} = 315^{+26}_{-50} \text{ MeV},$$

the larger error value in (58) is due to the strongest running coupling constant in Cornell potential. Moreover, Motyka and Zalewiski [21] also found $f_{B_c}^{(1\text{-loop})} = 435$ MeV for the ground state of $\bar{b}c$ quarkonium.

In the potential model, we note that slightly different additive constants is permitted to bring up data to its center-of-gravity value. However, with no additive constant to the Cornell potential [38], we notice that the smaller mass values for the composing quarks of the meson leads to a rise in the values of the potential parameters which in turn produces a notable lower value for the leptonic constant.
Our predictions for the $\bar{b}c$ mass spectrum for the Igi-Ono potential (type I and II) are given in Table 4. Moreover, the singlet and triplet masses together with the hyperfine splittings predicted for the two types of this potential are also reported in Table 5. We, hereby, tested acceptable parameters for $\Lambda_{\overline{MS}}$ from 100 to 500 MeV for the type I and II potentials. Small discrepancies between our prediction and SAD experiment [11,16,19,23] can be seen for higher states and such discrepancies are probably seen for any potential model and it might be related to the threshold effects or quark-gluon mixings. The fitted set of parameters for the Igi-Ono potential (type III) [11] are also tested in our method with $b = 19$ and ($\Lambda_{\overline{MS}} = 300$ MeV and also 390 MeV) and then $b = 16.3$ and $\Lambda_{\overline{MS}} = 300$ MeV which seems to be more convenient than $\Lambda_{\overline{MS}} = 500$ MeV used by other authors [8]. Results of this study are also presented in Table 6. We see that the quark masses $m_c$ and $m_b$ are sensitive to the variation of $\Lambda_{\overline{MS}}$. Therefore, as $\Lambda_{\overline{MS}}$ increases the contribution of the potential (cf. e.g., Eqs. (31) and (32)) and consequently the binding energy $E_{n,l}$ term decreases which leads to an increase in the constituent quark masses of the convenient meson, cf. Eq. (18).

In this model, we see that the experimental $\bar{b}c$ splittings can be reproduced for $\Lambda_{\overline{MS}} \sim 300$ MeV in type I, $\Lambda_{\overline{MS}} \sim 400$ MeV in type II (cf. Table 5) and $\Lambda_{\overline{MS}} \sim 300$ MeV in type III (cf. Table 6). We also predicted the splittings to several MeV with the other formalisms (c.f., Table 1 of [6]).

In Table 6, we also find that $m_c$ and $m_b$ are insensitive to the variation of $\Lambda_{\overline{MS}}$ for this Chen-Kuang potential. This is consistent with the conventional idea that, for heavy quarks, the constituent quark mass is close to the current quark mass which is $\Lambda_{\overline{MS}}$ independent. Numerical calculations show that this potential is insensitive to $\Lambda_{\overline{MS}}$ in the range from 100 to 300 MeV and as $\Lambda_{\overline{MS}}$ increases, the potential becomes more sensitive for the 1S-state only. The theoretically calculated $n^1S_0$ and $n^3S_1$ hyperfine splittings for the $B_c$ meson in the Chen-Kuang potential are also listed in Table 6. They are considerably smaller than the

\[\text{Table 3 of Ref. [11].}\]
corresponding calculated values $\Delta_{1S}(\bar{b}c) = 76$ MeV, and $\Delta_{2S}(\bar{b}c) = 42$ MeV predicted by the quadratic formalism of Ref. [6]. Moreover, Chen-Kuang [25] calculated $\Delta_{1S}(\bar{b}c) = 49.9$ MeV, and $\Delta_{2S}(\bar{b}c) = 29.4$ MeV for their potential with $\Lambda_{\overline{MS}} = 200$ MeV in which the last splitting is almost constant as $\Lambda_{\overline{MS}}$ increases. Our theoretical calculation for $\Delta_{1S}(\bar{b}c) = 68$ MeV, and $\Delta_{2S}(\bar{b}c) = 35$ MeV for the Chen-Kuang potential with $\Lambda_{\overline{MS}}$ runs from 100 into 375 MeV. We also find $\Delta_{1S}(\bar{b}c) = 67$ MeV, and $\Delta_{2S}(\bar{b}c) = 33$ MeV for the Igi-Ono potential with $\Lambda_{\overline{MS}} = 300$ MeV and $b = 16.3$. This model has the following features: (1) The present potential predicts smaller calculated $\Delta_{1S}$ and $\Delta_{2S}$ than the other potentials do for $\bar{b}c$ system and our calculated $\Delta_{1S}$ and $\Delta_{2S}$ do not depend on $\Lambda_{\overline{MS}}$ more sensitively (2) The theoretical $\bar{b}c$ splitting can be reproduced for the preferred $\Lambda_{\overline{MS}}$ in the range 100 – 300 MeV. Furthermore, in Table 5, for instance, we may choose $f_{B_c}^{NR} = 420$ MeV with $\Lambda_{\overline{MS}} = 300$ MeV, for the type I, and $f_{B_c}^{NR} = 396$ MeV with $\Lambda_{\overline{MS}} = 300$ MeV, for the type II. The result on $f_{B_c}$ is within the errors given by the other authors [5,19,29]. Further, for the CK potential, we present the decay constant in Table 6.

The scaling relation (SR) for the S-wave heavy quarkonia has the form [34]

$$\frac{f_n^2}{M_n(\bar{b}c)} \left(\frac{M_n(\bar{b}c)}{M_1(\bar{b}c)}\right)^2 \left(\frac{m_c + m_b}{4\mu}\right) = \frac{d}{n},$$  

(59)

where $m_c$ and $m_b$ are the masses of heavy quarks composing the $B_c$-meson, $\mu$ is the reduced mass of quarks, and $d$ is a constant independent of both the quark flavors and the level number $n$. The value of $d$ is determined by the splitting between the 2S and 1S levels or the average kinetic energy of heavy quarks, which is independent of the quark flavors and $n$ with the accuracy accepted. The accuracy depends on the heavy quark masses and it is discussed in detail [34]. The parameter value in Eq. (59), $d \approx 55$ MeV, can be extracted from the experimentally known leptonic constants of $\psi$ and $\Upsilon$. So, from Table 1, the SR gives for the 1S-level

$$f_{B_c}^{(SR)} \approx 444^{+6}_{-23} \text{ MeV},$$  

(60)

for all static potentials used. Kiselev [29,34] estimated $f_{B_c} = 400 \pm 45$ MeV and $f_{B_c}^{(SR)} = 385 \pm 25$ MeV, Narison [39] found $f_{B_c}^{(SR)} = 400 \pm 25$ MeV.
Overmore, we present the leptonic constants for the excited $nS$-levels of the $\bar{b}c$ in Table 7. We see that our prediction $f^{(\text{SR})}_{B_c(2S)} = 300 \pm 15 \text{ MeV}$ is in good agreement with the ones predicted by Kiselev et al. [19], $f^{(\text{SR})}_{B_c(2S)} = 280 \pm 50 \text{ MeV}$ for the $2S$-level in the $\bar{b}c$ system. In Figure 1, we plot the calculated values of $B_c$ leptonic constants using Eq. (59) for different potential models together with the calculated values of the excited $nS$-states using [34]

$$f_{n_2} = \sqrt{\frac{n_1}{n_2}} f_{n_1}. \quad (61)$$

The conclusion can be drawn from the Figure 1, that the approximated values of the excited $nS$-states agree well with the simple scaling relation (SR) derived from QCD sum rules for the state density. It is clear that the estimates obtained from the potential model and SR are in good agreement to several MeV as in Figure 1. Therefore, the difference between the leptonic constants for the pseudoscalar and vector $1S$-states is caused by the spin-dependent corrections, which are small. Numerically, we get $|f_{B_c^*} - f_{B_c}| / f_{B_c^*} < 1\%$. For the heavy quarkonia, the QCD sum rule approximation, provides that the $f_P$ and $f_V$ values for the pseudoscalar and vector states leptonic constant is practically independent of the total spin of quarks, so that

$$f_{V,n} \simeq f_{P,n} = f_n. \quad (62)$$

Our numerical approximation for the decay constants of the pseudoscalar and vector states in Tables 3, 5, and 6 is a confirmation to the last formula (62).

In this paper, we have developed the SLET in the treatment of the $\bar{b}c$ system using a wide class of static and QCD-motivated potentials. For such quarkonium potentials the method simply predicts the results of [11]. In this context, in reproducing the SAD, we used the same fitted parameters of [11] for the sake of comparison. Further, we demonstrate to the reader that the SLET method generates exactly the same numerical energy spectrum for $c\bar{c}$, $b\bar{b}$, and $c\bar{b}$ results as in the SLNET. This refutes the claims of [13] that this method is a reformation to SLNET and has a wider domain of applicability. Clearly, the method is simply an alternative parallel mathematical pseudoperturbative expansion technique having the same accuracy of [11,12,14].
ACKNOWLEDGMENTS

The author would like to thank Prof. Ramazan Sever (METU) for his valuable discussions. He also thanks his family members for their continuous encouragement and patience during this work.

Appendix A: SLET Parameters for the Schrödinger Equation:

Here, we list the analytic expressions of $\gamma^{(1)}$, $\gamma^{(2)}$, $\varepsilon_i$ and $\delta_j$ for the Schrödinger equation:

$$
\gamma^{(1)} = \left[(1 + 2n_r)\bar{\varepsilon}_2 + 3(1 + 2n_r + 2n_r^2)\bar{\varepsilon}_4\right]
- \omega^{-1} \left[\bar{\varepsilon}_1^2 + 6(1 + 2n_r)\varepsilon_1\varepsilon_3 + (11 + 30n_r + 30n_r^2)\varepsilon_3^2\right],
$$

$$
\gamma^{(2)} = \left[(1 + 2n_r)\bar{\delta}_2 + 3(1 + 2n_r + 2n_r^2)\bar{\delta}_4 + 5(3 + 8n_r + 6n_r^2 + 4n_r^3)\bar{\delta}_6\right]
- \omega^{-1} \left[\bar{\delta}_1^2 + 12(1 + 2n_r + 2n_r^2)\varepsilon_1\varepsilon_4 + 2\varepsilon_1\bar{\delta}_1\right]
+ 2(21 + 59n_r + 51n_r^2 + 34n_r^3)\varepsilon_1^2 + 6(1 + 2n_r)\varepsilon_1\bar{\delta}_3
+ 30(1 + 2n_r + 2n_r^2)\varepsilon_1\bar{\delta}_5 + 2(11 + 30n_r + 30n_r^2)\varepsilon_3\bar{\delta}_3
+ 10(13 + 40n_r + 42n_r^2 + 28n_r^3)\bar{\varepsilon}_3\bar{\delta}_5 + 6(1 + 2n_r)\varepsilon_3\bar{\delta}_1\right]
+ \omega^{-2} \left[4\bar{\varepsilon}_2^2\varepsilon_2 + 36(1 + 2n_r)\varepsilon_1\varepsilon_2\varepsilon_3 + 8(11 + 30n_r + 30n_r^2)\varepsilon_2\varepsilon_3^2\right]
+ 24(1 + 2n_r)\bar{\varepsilon}_1^2\varepsilon_4 + 8(31 + 78n_r + 78n_r^2)\varepsilon_1\varepsilon_3\varepsilon_4
+ 12(57 + 189n_r + 225n_r^2 + 150n_r^3)\varepsilon_3^2\bar{\varepsilon}_4\right]
- \omega^{-3} \left[8\bar{\varepsilon}_3^2\varepsilon_3 + 108(1 + 2n_r)\varepsilon_1^2\varepsilon_3^2 + 48(11 + 30n_r + 30n_r^2)\varepsilon_1\varepsilon_3^3\right]
+ 30(31 + 109n_r + 141n_r^2 + 94n_r^3)\varepsilon_3^4\right],
$$

where

$$
\bar{\varepsilon}_i = \frac{\varepsilon_i}{(4\mu\omega)^{1/2}}, \quad i = 1, 2, 3, 4.
$$

and

$$
\bar{\epsilon}_i = \frac{\epsilon_i}{(4\mu\omega)^{1/2}}, \quad i = 1, 2, 3, 4.
$$
\[ \delta_j = \frac{\delta_j}{(4\mu\omega)^{j/2}}, \quad j = 1, 2, 3, 4, 5, 6. \]  
(A4)

\[ \varepsilon_1 = -\frac{(2a + 1)}{2\mu}, \quad \varepsilon_2 = \frac{3(2a + 1)}{4\mu}, \]  
(A5)

\[ \varepsilon_3 = -\frac{1}{\mu} + \frac{r_0^5 V''(r_0)}{6Q}; \quad \varepsilon_4 = \frac{5}{4\mu} + \frac{r_0^6 V'''(r_0)}{24Q} \]  
(A6)

\[ \delta_1 = -\frac{a(a + 1)}{2\mu}; \quad \delta_2 = \frac{3a(a + 1)}{4\mu}, \]  
(A7)

\[ \delta_3 = -\frac{(2a + 1)}{\mu}; \quad \delta_4 = \frac{5(2a + 1)}{4\mu}, \]  
(A8)

\[ \delta_5 = -\frac{3}{2\mu} + \frac{r_0^7 V''''(r_0)}{120Q}; \quad \delta_6 = \frac{7}{4\mu} + \frac{r_0^8 V'''''(r_0)}{720Q}. \]  
(A9)
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FIGURES

Fig. 1. The $nS$-levels leptonic constant of the $B_c$ system calculated in different static potential models using SR.
Table 1. The $\bar{b}c$ masses and hyperfine splittings ($\Delta_{nS}$) calculated in different static potentials (in MeV).

| States | Refs. 1,6 | Cornell | Song-Lin | Turin | Martin | Logarithmic |
|--------|-----------|---------|----------|-------|--------|-------------|
| $\alpha_s(m_c^2)$ = |          | 0.320   | 0.263    | 0.286 | 0.251  | 0.220       |
| $m_c\ (GeV)$ = |          | 1.840   | 1.820    | 1.790 | 1.800  | 1.500       |
| $m_b\ (GeV)$ = |          | 5.232   | 5.199    | 5.171 | 5.174  | 4.905       |
| $M(\bar{b}c)$ |          |         |          |       |        |             |
| $1S$ | 6315   | 6315    | 6306     | 6307  | 6301   | 6317        |
| $1^3S_1$ | 6334   | 6335    | 6325     | 6326  | 6319   | 6334        |
| $1^1S_0$ | 6258   | 6252    | 6249     | 6249  | 6247   | 6266        |
| $\Delta_{1S}^a$ | 77     | 83.5    | 76.1     | 76.7  | 71.6   | 68.0         |
| $2S$ | 6873   | 6888    | 6875     | 6880  | 6892   | 6903        |
| $2^3S_1$ | 6883   | 6897    | 6884     | 6889  | 6902   | 6911        |
| $2^1S_0$ | 6841   | 6860    | 6850     | 6852  | 6865   | 6879        |
| $\Delta_{2S}$ | 42     | 37.9    | 34.0     | 36.5  | 36.7   | 31.3        |
| $3S$ | 7246   | 7271    | 7209     | 7246  | 7236   | 7225        |
| $4S$ | 7587   | 7455    | 7535     | 7483  | 7448   |             |
| $1P$ | 6772   | 6743    | 6733     | 6731  | 6730   | 6754        |
| $2P$ | 7154   | 7138    | 7104     | 7123  | 7125   | 7127        |
| $3P$ | 7464   | 7371    | 7428     | 7398  | 7375   |             |
| $1D$ | 7043   | 7003    | 6998     | 6998  | 7011   | 7027        |
| $2D$ | 7367   | 7340    | 7284     | 7320  | 7311   | 7301        |
| $3D$ | 7636   | 7510    | 7588     | 7536  | 7502   |             |

$a\Delta_{nS} = M(n^3S_1) - M(n^1S_0).$
Table 2. The predicted $\bar{t}c$ masses of the lowest S-wave and its theoretically calculated splittings compared with the other authors (in MeV).

| Work                        | $M_{Bc}(1^1S_0)^{a}$ | $M_{Bc}(1^3S_1)$ | $\Delta_{1S}$ |
|-----------------------------|----------------------|------------------|---------------|
| Eichten et al. [1]          | 6258 ± 20            |                  |               |
| Colangelo and Fazio [3]     | 6280                 | 6350             |               |
| Baker et al. [40]           | 6287                 | 6372             |               |
| Roncaglia et al. [40]       |                      | 6320 ± 10        |               |
| Godfrey et al. [1]          | 6270                 | 6340             |               |
| Bagan et al. [1,40]         | 6255 ± 20            | 6330 ± 20        |               |
| Brambilla et al. [3]        |                      | 6326$^{+29}_{-9}$ | 60$^b$        |
| Baldicchi et al. [6]        | 6194 ~ 6292          | 6284 ~ 6357      | 65 $\leq \Delta_{1S} \leq$ 90 |
| SLET$^c$                    | 6253$^{+13}_{-6}$   | 6328$^{+7}_{-9}$ | 68 $\leq \Delta_{1S} \leq$ 83 |
| SLET$^d$                    | 6258$^{+8}_{-11}$   | 6333$^{+2}_{-14}$|               |

$^a$The experimental mass of the singlet state is given in [2].

$^b$We remark that we have calculated splittings from the single t and triplet states.

$^c$Averaging over the five values in Table 1.

$^d$We treat Eichten and Quigg’s results in the same manner (e.g., [1]).
Table 3. The characteristics of the radial wave function at the origin $|R_{1S}(0)|^2$ (in GeV$^3$), NR, one-loop and two-loop corrections to pseudoscalar and vector decay constants of the low-lying $B_c$ meson (the accuracy is 5%) calculated in different static potential models (in MeV).

| Quantity          | Cornell | Song-Lin | Turin | Martin | Logarithmic | GKLT[35] | EFG[1] | JW[36] |
|-------------------|---------|----------|-------|--------|-------------|----------|--------|--------|
| $|\psi_{1S}(0)|^2$ | 0.112   | 0.123    | 0.111 | 0.119  | 0.102       |          |        |        |
| $|R_{1S}(0)|^2$     | 1.413   | 1.54     | 1.397 | 1.495  | 1.28        |          |        |        |
| $f_{B_c}^{(NR)}$  | 464.5   | 485.1    | 462.0 | 478.0  | 441.7       | 460±60   | 433    | 420±13 |
| $f_{B_c}^{(NR)}$  | 461.5   | 482.2    | 459.2 | 475.3  | 439.3       | 460±60   | 503    |        |
| $f_{B_c}^{(1-loop)}$ | 393.6$^a$ | 424.4 | 399.6 | 421.2  | 399.3       |          |        |        |
| $f_{B_c}^{(2-loop)}$ | 264.1$^b$ | 333.0 | 296.6 | 339.1  | 340.9       |          |        |        |
| $f_{B_c}^{(1-loop)}$ | 391.0   | 421.9    | 397.1 | 418.8  | 397.2       |          |        |        |
| $f_{B_c}^{(2-loop)}$ | 262.3   | 331.0    | 294.8 | 337.2  | 339.0       |          |        |        |

$^a$First loop SD Wilson coefficient for all potentials, $K_0 = 0.85 - 0.90$.

$^b$Second loop SD Wilson coefficient for all potentials, $K_0 = 0.57 - 0.77$. 
Table 4. The $\bar{b}c$ mass spectra predicted for various $\Lambda_{\overline{MS}}$ using Igi-Ono (type I and II) potential (in MeV).

| States | $[6,24]$ | $100$ | $200$ | $300$ | $400$ | $500$ |
|--------|----------|-------|-------|-------|-------|-------|
| $b = 20^a$ | $\alpha_s =$ | 0.1985 | 0.217 | 0.238 | 0.250 | 0.262 |
| $1S$ | | 6327 | 6329 | 6318 | 6310 | 6316 | 6327 |
| $2S$ | | 6906 | 6915 | 6904 | 6881 | 6880 | 6901 |
| $3S$ | | 7246 | 7264 | 7242 | 7244 | 7241 | 7252 |
| $4S$ | | 7508 | 7522 | 7545 | 7542 | 7552 | 7552 |
| $1P$ | | 6754 | 6755 | 6744 | 6733 | 6732 | 6742 |
| $2P$ | | 7154 | 7144 | 7131 | 7125 | 7122 | 7134 |
| $1D$ | | 7028 | 7029 | 7017 | 7004 | 7000 | 7010 |
| $2D$ | | 7367 | 7334 | 7327 | 7327 | 7323 | 7333 |

| $b = 5^b$ | $\alpha_s =$ | 0.1985 | 0.227 | 0.230 | 0.2405 |
| $1S$ | | 6327 | 6331 | 6324 | 6316 | 6307 |
| $2S$ | | 6906 | 6914 | 6898 | 6910$^c$ | 6918 |
| $3S$ | | 7246 | 7258 | 7277 | 7236 | 7201$^c$ |
| $4S$ | | 7508 | 7522 | 7545 | 7500 |
| $1P$ | | 6754 | 6756 | 6743 | 6737 | 6730 |
| $2P$ | | 7154 | 7142 | 7138 | 7134 | 7120 |
| $1D$ | | 7028 | 7029 | 7015 | 7012$^c$ | 7007 |
| $2D$ | | 7367 | 7335 | 7323$^c$ | 7314 | 7316 |

$^a_{c_0 = -0.022}$ to $-0.031$ MeV.

$^b_{c_0 = -0.019}$ to $-0.026$ MeV.

$^c$Carried out to the second correction order.
Table 5. The $\bar{b}c$ mass spectrum, splittings and leptonic constant predicted for various $\Lambda_{\overline{MS}}$ using Igi-Ono (type I and II) potential (in MeV).

| States  | $\Lambda_{\overline{MS}}$ |
|---------|-----------------|
|         | 100  | 200  | 300  | 400  | 500  |
| Type I  |      |      |      |      |      |
| $1^3S_1$| 6343 | 6334 | 6327 | 6334 | 6344 |
| $1^1S_0$| 6287 | 6272 | 6259 | 6263 | 6274 |
| $\Delta_{1S}$| 56.3 | 62.0 | 68.3 | 71.1 | 69.8 |
| $|R_{1S}(0)|^2$| 0.826| 1.005| 1.156| 1.19 | 1.114 |
| $f_{B_c}^{NR}$| 354.1| 391.1| 420.0| 426.0| 411.7 |
| $f_{B_c}^{(1-loop)}$| 328.1| 356.5| 376.8| 379.2| 364.4 |
| $f_{B_c}^{(2-loop)}$| 290.0| 306.1| 311.7| 306.5| 287.1 |
| $f_{B_c}^{NR}$| 352.6| 389.2| 417.7| 423.6| 409.4 |
| $f_{B_c}^{(1-loop)}$| 326.7| 354.8| 374.7| 377.1| 362.4 |
| $f_{B_c}^{(2-loop)}$| 288.7| 304.6| 310.0| 304.7| 285.6 |
| Type II |      |      |      |      |      |
| $1^3S_1$| 6345 | 6340 | 6331 | 6323 |
| $1^1S_0$| 6288 | 6279 | 6269 | 6259 |
| $\Delta_{1S}$| 56.7 | 60.6 | 61.8 | 64.4 |
| $|R_{1S}(0)|^2$| 0.819| 0.891| 1.03 | 1.204 |
| $f_{B_c}^{NR}$| 352.7| 368.2| 396.0| 428.6 |
| $f_{B_c}^{(1-loop)}$| 327.1| 334.9| 357.4| 382.1 |
| $f_{B_c}^{(2-loop)}$| 289.1| 283.0| 300.1| 314.4 |
| $f_{B_c}^{NR}$| 351.2| 366.4| 394.1| 426.4 |
| $f_{B_c}^{(1-loop)}$| 325.6| 333.3| 355.7| 380.1 |
| $f_{B_c}^{(2-loop)}$| 287.8| 281.7| 298.6| 312.8 |
Table 6. The $b\bar{c}$ mass spectrum, splittings and leptonic constant predicted for various $\Lambda_{\overline{MS}}$ using Igi-Ono (type III) and Chen-Kuang potentials (in MeV).

| State | IO (III) | CK |
|-------|----------|----|
| $b = \Lambda_{\overline{MS}}$ | 16.3 300 | 5.1 350 |
| $\alpha_s = 0.250$ | 0.2505 0.2205 | 0.270 0.270 |
| $1S$ | 6309 6309 | 6297 6324 |
| $2S$ | 6880 6870 | 6877 6880 |
| $3S$ | 7247 7236 | 7254 7258 |
| $4S$ | 7553 7541 | 7563 7570 |
| $1P$ | 6725 6721 | 6737 6723 |
| $2P$ | 7124 7114 | 7135 7127 |
| $3P$ | 7441 7429 | 7452 7452 |
| $1D$ | 6997 6990 | 7013 6993 |
| $2D$ | 7328 7317 | 7341 7332 |
| $3D$ | 7613 7599 | 7624 7625 |
| $1^3S_1$ | 6326 6327 | 6315 6341 |
| $1^1S_0$ | 6259 6258 | 6243 6273 |
| $\Delta_{1S}$ | 67.3 68.6 | 72.6 67.8 |
| $|R_{1S}(0)|^2$ | 1.115 1.119 | 1.339 1.017 |
| $f_{B_{c}}^{NR}$ | 412.4 413.2 | 452.6 393.5 |
| $f_{B_{c}}^{(1-\text{loop})}$ | 367.1 367.2 | 408.0 347.5 |
| $f_{B_{c}}^{(2-\text{loop})}$ | 296.4 294.0 | 345.4 269.0 |
| $f_{B_{c}}^{NR}$ | 410.2 411.0 | 450.0 391.4 |
| $f_{B_{c}}^{(1-\text{loop})}$ | 365.2 365.2 | 405.7 345.6 |
| $f_{B_{c}}^{(2-\text{loop})}$ | 294.8 292.4 | 343.4 267.6 |
Table 7. The $nS$-levels leptonic constant of the $\bar{b}c$ system, calculated in different static potential models (the accuracy is $3 - 7\%$), in MeV, using the SR.

| Quantity | Cornell | Song-Lin | Turin | Martin | Logarithmic |
|----------|---------|----------|-------|--------|-------------|
| $f_{1S}$ | 449.6   | 450.4    | 448.0 | 448.8  | 420.9       |
| $f_{2S}$ | 305.8   | 305.0    | 303.3 | 303.5  | 284.7       |
| $f_{3S}$ | 243.0   | 243.2    | 241.3 | 241.8  | 227.2       |
| $f_{4S}$ | 206.0   | 207.1    | 204.9 | 205.9  | 193.8       |