Entanglement-assisted capacity of constrained channels

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Abstract

In this paper we fill a gap in previous works by proving the conjectured formula for the classical entanglement-assisted capacity of quantum channel with additive constraint (such as Bosonic Gaussian channel). The main tools are the coding theorem for classical-quantum constrained channels and a finite dimensional approximation of the input density operators for entanglement-assisted capacity. We also give sufficient conditions under which suprema in the capacity formulas are achieved.

The formula for the entanglement-assisted capacity of a noisy quantum channel, expressing it as the maximum of mutual information over input states, was obtained in [2], [3] for channels in finite dimensional Hilbert space. Alternative proof was given in [4]. In [3], [5] the appropriately modified formula was also applied to quantum Gaussian channels which are the most important example of constrained channels in infinite dimensions. In this paper we fill the gap in previous works by proving the conjectured formula [6]. The main tools are the coding theorem for classical-quantum constrained channels and a finite dimensional approximation of the input density operators for entanglement-assisted capacity. We also give sufficient conditions under which suprema in the capacity formulas are achieved.

1. We first consider the case of classical-quantum (c-q) channel with infinite alphabet $\mathcal{X} = \{x\}$. For every $x$ let $S_x$ be a density operator in a Hilbert space $\mathcal{H}$ (in general, infinite dimensional) with finite von Neumann entropy $H(S_x)$. The c-q channel is given by the mapping $x \rightarrow S_x$.

Let $f(x)$ be a nonnegative nonconstant function defined on the input alphabet. Passing to block coding, we put the additive constraint onto the input words $w = (x_1, \ldots, x_n)$ by asking

$$f(x_1) + \ldots + f(x_n) \leq nE,$$

where $E$ is a positive constant. The classical capacity of such channels was defined and computed in [3] under a condition of uniform boundedness of the
entropies $H(S_x)$. This condition is not suitable for our purpose here, and by using almost the same argument we can prove

**Proposition 1.** Denote by $\mathcal{P}$ the class of finite input distributions $\pi = \{\pi_x\}$ on $\mathcal{X}$ satisfying

$$\sum_x \pi_x f(x) \leq E. \quad (2)$$

We assume $\mathcal{P}$ is nonempty and impose the following condition onto the channel:

$$\sup_{\pi \in \mathcal{P}} H \left( \sum_x \pi_x S_x \right) < \infty. \quad (3)$$

The classical capacity of the channel $x \rightarrow S_x$ under the constraint (1) is finite and given by

$$C = \sup_{\pi \in \mathcal{P}} \left[ H \left( \sum_x \pi_x S_x \right) - \sum_x \pi_x H(S_x) \right]. \quad (4)$$

Let the input alphabet $\mathcal{X}$ be a locally compact subset of a separable metric space (e. g. a closed finite dimensional manifold, or a discrete countable set, in which case the integrals below should be understood as sums). Consider the channel given by weakly continuous mapping $x \rightarrow S_x$ from the input alphabet $\mathcal{X}$ to the set of density operators in $\mathcal{H}$ (the weak continuity means continuity of all matrix elements $\langle \psi | S_x | \phi \rangle; \psi, \phi \in \mathcal{H}$). Note that according to [10] weak convergence in the set of density operators is equivalent to the trace norm convergence. For arbitrary Borel measure $\pi$ on $\mathcal{X}$ we define

$$\bar{S}_\pi = \int_X S_x \pi(dx). \quad (5)$$

Because of the continuity of the function $S_x$ the integral is well defined and represents a density operator in $\mathcal{H}$. Assuming that $H(\bar{S}_\pi) < \infty$, we have

$$H(\bar{S}_\pi) - \int_X H(S_x) \pi(dx) = \int_X H(S_x; \bar{S}_\pi) \pi(dx) \geq 0, \quad (6)$$

where the functions $H(S_x), H(S_x; \bar{S}_\pi)$ (where denotes the quantum relative entropy) are nonnegative and lower semicontinuous [3], and hence the integrals are well defined. We assume that the function $f$ is Borel and consider the set $\mathcal{P}^B$ of Borel probability measures $\pi$ on $\mathcal{X}$ satisfying

$$\int_X f(x) \pi(dx) \leq E. \quad (7)$$

**Proposition 2.** Let the function $f$ be lower semicontinuous and tend to infinity at infinity and let there exist a selfadjoint operator $F$ satisfying

$$\text{Tr} \exp(-\beta F) < \infty \quad \text{for all} \quad \beta > 0, \quad (8)$$
such that
\[ f(x) \geq \text{Tr} S_x F, \quad x \in \mathcal{X}. \tag{9} \]

Then \( C \) is finite and
\[ C = \max_{\pi \in \mathcal{P}} \left[ H(\bar{S}_\pi) - \int_{\mathcal{X}} H(S_x) \pi(dx) \right]. \tag{10} \]

**Proof.** The condition (3) implies that the spectrum of \( F \) is bounded from below; for simplicity we assume that \( F \geq 0 \) but the general case can be reduced to that one. Then the right hand side of (9) is defined as in (13) below. Denoting
\[ S_\beta = \left[ \text{Tr} \exp (-\beta F) \right]^{-1} \exp (-\beta F), \]
we have
\[ \beta \text{Tr} \bar{S}_\pi F - H(\bar{S}_\pi) = H(\bar{S}_\pi; S_\beta) - \log \text{Tr} \exp (-\beta F), \tag{11} \]
whence, by using (9),
\[ H(\bar{S}_\pi) \leq \beta \text{Tr} \bar{S}_\pi F + \log \text{Tr} \exp (-\beta F) \leq \beta E + \log \text{Tr} \exp (-\beta F), \]
hence the condition (3) is fulfilled, \( C \) is finite and equal to (4). Under the assumptions that the mapping \( x \to S_x \) is weakly continuous, the function \( f \) is lower semicontinuous and the condition (3) holds, it follows from Proposition 2 of [6] that
\[ C = \sup_{\pi \in \mathcal{P}} \left[ H(\bar{S}_\pi) - \int_{\mathcal{X}} H(S_x) \pi(dx) \right], \tag{12} \]
and we wish to prove that the supremum is attained.

In the set of all Borel probability measures on \( \mathcal{X} \) we consider the topology of weak convergence: the sequence \( \pi^{(l)}(dx) \) weakly converges to \( \pi(dx) \) if
\[ \int_{\mathcal{X}} g(x) \pi^{(l)}(dx) \to \int_{\mathcal{X}} g(x) \pi(dx) \]
for all bounded continuous functions \( g \) on \( \mathcal{X} \). Then one can show that the set \( \mathcal{P}^B \) is compact, by using the general criterion [11]: a subset \( \mathcal{P}' \) of Borel probability measures on \( \mathcal{X} \) is weakly relatively compact iff for any \( \varepsilon > 0 \) there is a compact \( \mathcal{K} \subset \mathcal{X} \) such that \( \pi(\mathcal{X} \setminus \mathcal{K}) \leq \varepsilon \) for all \( \pi \in \mathcal{P}' \). Then \( \pi(\mathcal{X} \setminus \mathcal{K}) \leq E / \inf_{x \in \mathcal{X} \setminus \mathcal{K}} f(x) \) for \( \pi \in \mathcal{P}^B \), which can be made arbitrarily small.

The map \( \pi \to \bar{S}_\pi \) is continuous in the weak operator topology, and hence, in the trace norm topology. By using this fact we can prove that the function in the squared brackets of (12) is upper semicontinuous and hence attains its maximum on \( \mathcal{P}^B \). Consider the first term in the formula (3). The quantum entropy is lower semicontinuous, hence the function \( \pi \to H(\bar{S}_\pi) \) is lower semicontinuous. Let us show that it is upper semicontinuous and hence continuous on the set \( \mathcal{P}^B \). By (11), (9) we have
\[ H(\bar{S}_\pi) \geq \lim_{n \to \infty} \sup_{\pi} H(\bar{S}_{\pi^n}) - \beta \lim_{n \to \infty} \sup_{\pi} \text{Tr} \bar{S}_{\pi^n} F \geq \lim_{n \to \infty} \sup_{\pi} H(\bar{S}_{\pi^n}) - \beta E, \]
for arbitrary sequence \( \{\pi^n\} \in \mathcal{P}^B \) weakly converging to \( \pi \). Letting \( \beta \to 0 \), we get the upper semicontinuity.

The second term in (iii) is upper semicontinuous as the greatest lower bound of continuous functions \( \pi \to -\int g(x)\pi(dx) \), where \( g \) varies over bounded continuous functions satisfying \( 0 \leq g(x) \leq H(S_x), \quad x \in X \). Hence (iii) is upper semicontinuous and the statement follows. QED

2. Now let \( \Phi \) be a (quantum-quantum) channel in a Hilbert space \( \mathcal{H} \), i.e. a trace-preserving completely positive map on trace-class operators in \( \mathcal{H} \). We wish to define the capacity of this channel under additive constraint at the input of the channel. Let \( F \) be positive selfadjoint nonconstant (i.e. not a multiple of the identity), in general unbounded operator in \( \mathcal{H} \), representing observable the mean value of which is to be constrained (e.g. energy of the system). For arbitrary density operator \( S \) with the spectral decomposition \( S = \sum_{j=1}^{\infty} \lambda_j |e_j\rangle\langle e_j| \) we define

\[
\text{Tr}SF := \sum_{j=1}^{\infty} \lambda_j ||\sqrt{F}e_j||^2 \leq +\infty,
\]

assuming \( ||\sqrt{F}e_j|| = +\infty \) if \( e_j \) is not in the domain of \( \sqrt{F} \). We impose the analog of the condition (3):

\[
\sup_{S: \text{Tr}SF \leq E} H(\Phi[S]) < \infty,
\]

where \( E \) is a positive constant.

For the channel \( \Phi^\otimes n \) in \( \mathcal{H}^\otimes n \) the corresponding observable is

\[
F^{(n)} = F \otimes \cdots \otimes I + \cdots + I \otimes \cdots \otimes F.
\]

We want the input states \( S^{(n)} \) of the channel \( \Phi^\otimes n \) satisfy the additive constraint

\[
\text{Tr}S^{(n)}F^{(n)} \leq nE.
\]

Note that (14) implies similar property of the channel \( \Phi^\otimes n \):

\[
\sup_{S^{(n)}: \text{Tr}S^{(n)}F^{(n)} \leq nE} H(\Phi^\otimes n[S^{(n)}]) < \infty.
\]

Indeed, by subadditivity of quantum entropy with respect to tensor products,

\[
H(\Phi^\otimes n[S^{(n)}]) \leq \sum_{k=1}^{n} H(\Phi[S^{(n)}_k]),
\]

where \( S^{(n)}_k \) is the \( k \)-th partial state of \( S^{(n)} \). Also by concavity of the entropy

\[
\sum_{k=1}^{n} H(\Phi[S^{(n)}_k]) \leq nH(\Phi[S^{(n)}]),
\]
where $\bar{S}(n) = \frac{1}{n} \sum_{k=1}^{n} S_k^{(n)}$. The inequality (15) can be rewritten as

$$\frac{1}{n} \sum_{k=1}^{n} \text{Tr} S_k^{(n)} F = \text{Tr} \bar{S}(n) F \leq E,$$

which implies that

$$\sup_{S(n): \text{Tr} S(n) F(n) \leq nE} H(\Phi^{\otimes n}[S(n)]) \leq n \sup_{S: \text{Tr} S F \leq E} H(\Phi[S]).$$

**Definition.** We call by code $(\Sigma^{(n)}, M^{(n)})$ of length $n$ and of size $N$ the collection $\Sigma^{(n)} = \{S_w^{(n)}; w = 1, \ldots, N\}$ of states satisfying (14), with an observable $M^{(n)} = \{M_j^{(n)}; j = 0, 1, \ldots, N\}$ in $\mathcal{H}^{\otimes n}$. The error probability for the code is

$$P_e(\Sigma^{(n)}, M^{(n)}) = \max_{w=1,\ldots,N} \left\{ 1 - \text{Tr} \Phi^{\otimes n}[S_w^{(n)}|M_w^{(n)}] \right\},$$

and the minimal error probability over all codes of the length $n$ and the size $N$ is denoted $p_e(n, N)$. The classical capacity $C(\Phi)$ is the least upper bound of the rates $R$ for which $\liminf_{n \to \infty} p_e(n, 2^{nR}) = 0$.

Let us denote by $\mathcal{S}^{(n)}$ the set of states in $\mathcal{H}^{\otimes n}$ satisfying (15), and by $\mathcal{P}^{(n)}$ the collection of couples $(\pi^{(n)}, \Sigma^{(n)})$, where $\pi_w^{(n)}$ are probabilities for the states $S_w^{(n)}$, satisfying

$$\sum_{w=1}^{N} \pi_w^{(n)} \text{Tr} S_w^{(n)} F^{(n)} \leq nE. \quad (17)$$

If a probability distribution $\pi^{(n)} = \{\pi_w^{(n)}\}$ on the input codewords $S_w^{(n)}$ is given, then using the transition probability $p(j|w) = \text{Tr} \Phi^{\otimes n} [S_w^{(n)}|M_j^{(n)}]$ we can find the joint distribution of input and output, compute the Shannon information $I_n(\pi^{(n)}, \Sigma^{(n)}, M^{(n)})$, and define the quantity

$$\bar{C}^{(n)}(\Phi) = \sup_{(\pi^{(n)}, \Sigma^{(n)}) \in \mathcal{P}^{(n)}} \left[ H\left( \sum_w \pi_w^{(n)} \Phi^{\otimes n}[S_w^{(n)}] \right) - \sum_w \pi_w^{(n)} H\left( \Phi^{\otimes n}[S_w^{(n)}] \right) \right].$$

If $\Sigma^{(n)} \subset \mathcal{S}^{(n)}$, then $(\pi^{(n)}, \Sigma^{(n)}) \in \mathcal{P}^{(n)}$, and

$$I_n(\pi^{(n)}, \Sigma^{(n)}, M^{(n)}) \leq \bar{C}^{(n)}(\Phi), \quad (18)$$

by the quantum entropy bound $[5]$.

**Proposition 3.** Let the channel $\Phi$ satisfy the condition (15). Then the classical capacity of this channel under the constraint (15) is finite and equals to

$$C(\Phi) = \lim_{n \to \infty} \frac{1}{n} \sup_{\pi^{(n)}, \Sigma^{(n)} \subset \mathcal{S}^{(n)}, M^{(n)}} I_n(\pi^{(n)}, \Sigma^{(n)}, M^{(n)}) \quad (19)$$

and

$$= \lim_{n \to \infty} \frac{1}{n} \bar{C}^{(n)}(\Phi). \quad (20)$$
Proof. Relation (19) follows from the classical coding theorem. Inequality \( \leq \) in (20) follows then from (18). Let us show that
\[
C(\Phi) \geq \lim_{n \to \infty} \frac{1}{n} C(n) \equiv \bar{C}(\Phi).
\]
(21)

Take \( R < \bar{C}(\Phi) \), then we can choose \( n_0 \), probability distribution \( \pi^{(n_0)} \) and collection of states \( \Sigma^{(n_0)} = \{S_w^{(n_0)}\} \) in \( H^{\otimes n_0} \) such that \( (\pi^{(n_0)}, \Sigma^{(n_0)}) \in \mathcal{P}^{(n_0)} \) and
\[
n_0 R < H \left( \sum_w \pi^{(n_0)} \Phi^{\otimes n_0} [S_w^{(n_0)}] \right) - \sum_w \pi^{(n_0)} H \left( \Phi^{\otimes n_0} [S_w^{(n_0)}] \right).
\]
(22)

Consider the c-q channel \( \tilde{\Phi} \) in \( H^{\otimes n_0} \) given by the formula
\[
\tilde{\Phi}[S] = \sum_w \Phi^{\otimes n_0} [S_w] \langle e_w | S e_w \rangle,
\]
and define the constraint function for this channel as \( f(w) = \text{Tr} S_w^{(n_0)} F^{(n_0)} \). The condition (16) implies
\[
\sup_{\pi} H \left( \sum_w \pi w \Phi^{\otimes n_0} [S_w] \right) < \infty,
\]
where the supremum is over the probability distributions \( \pi \), satisfying
\[
\sum_w \pi w f(w) \leq n_0 E.
\]
(23)

that is, the condition (18). By the Proposition 1, the capacity of \( \tilde{\Phi} \) is
\[
C(\tilde{\Phi}) = \sup_{\pi} \left\{ H \left( \sum_w \pi w \Phi^{\otimes n_0} [S_w^{(n_0)}] \right) - \sum_w \pi w H \left( \Phi^{\otimes n_0} [S_w^{(n_0)}] \right) \right\},
\]
where the states are fixed and the supremum is over the probability distributions \( \pi \), satisfying (23). By (22) this is greater than \( n_0 R \). Denoting \( \tilde{p}_e(n, N) \) the minimal error probability for \( \tilde{\Phi} \), we have
\[
\tilde{p}_e(n, 2^{(n_0)R}) \leq \tilde{p}_e(n, 2^{n(n_0)R}),
\]
(24)
since every code of size \( N \) for \( \tilde{\Phi} \) is also code of the same size for \( \Phi \). Indeed, if \( \tilde{w} = (w_1, \ldots, w_n) \) is a codeword for \( \tilde{\Phi} \), it satisfies the constraint \( f(w_1) + \cdots + f(w_n) \leq n n_0 E \). Defining the state \( S_{\tilde{w}}^{(n_0)} = S_{w_1}^{(n_0)} \otimes \cdots \otimes S_{w_n}^{(n_0)} \), we see that this is equivalent to \( \text{Tr} S_{\tilde{w}}^{(n_0)} F^{(n_0)} \leq n n_0 E \), that is to the constraint (16) for the q-q channel \( \Phi^{\otimes n_0} \). Thus having chosen \( R < \bar{C}(\Phi) \), we can make the right and hence the left hand side of (24) tend to zero as \( n \to \infty \). This proves (21). QED

These estimates rise questions, to which there is no answer at present. One may ask whether the additivity \( C(n)(\Phi) = n C(1)(\Phi) \) holds, in which case
This question looks even harder than the still unsettled additivity problem in the case of unconstrained inputs (see [7] for comments on this problem). The quantity

\[ \bar{C}(1)(\Phi) = \sup_{\sum_i \pi_i \Tr S_i F \leq E} \left[ H \left( \sum_i \pi_i \Phi[S_i] \right) - \sum_i \pi_i H(\Phi[S_i]) \right] \]  

(25)

looks tractable, although even for the simplest quantum Gaussian channel (39) there is only a natural conjecture about its value and the solution of the maximization problem (see Subsection 12.6.1 of [5]).

3. Let us now turn to the entanglement-assisted capacity. Consider the following protocol of the classical information transmission through the channel \( \Phi \). Systems \( A \) and \( B \) share an entangled (pure) state \( S_{AB} \). We assume that the amount of entanglement is unlimited but finite i.e. \( H(S_A) = H(S_B) < \infty \). \( A \) does some encoding \( i \to E_i \) depending on a classical signal \( i \) with probabilities \( \pi_i \) and sends its part of this shared state through the channel \( \Phi \) to \( B \). Thus \( B \) gets the states \( (\Phi \otimes \Id_B)[S_i] \), where \( S_i = (\mathcal{E}_i \otimes \Id_B)[S_{AB}] \) with probabilities \( \pi_i \) and \( B \) is trying to extract the maximum classical information by doing measurements on these states. Now to enable block coding, all this picture should be applied to the channel \( \Phi \otimes n \).

Then the signal states \( S^{(n)}_w \) transmitted through the channel \( \Phi \otimes n \otimes \Id^n_B \) have the special form

\[ S^{(n)}_w = (\mathcal{E}^{(n)}_w \otimes \Id^n_B)[S^{(n)}_{AB}] , \]  

(26)

where \( S^{(n)}_{AB} \) is the pure entangled state for \( n \) copies of the system \( AB \), satisfying the condition \( H(S^{(n)}_{AB}) < \infty \), and \( w \to \mathcal{E}^{(n)}_w \) are the encodings for \( n \) copies of the system \( A \). We impose the constraint (15) onto the input states of the channel \( \Phi \otimes n \), which is equivalent to similar constraint for the channel \( \Phi \otimes n \otimes \Id^n_B \) with the constraint operators \( F^{(n)}_{AB} = F^{(n)} \otimes \Id^n_B \). We denote by \( \mathcal{P}^{(n)}_{AB} \) the collection of couples \( (\pi^{(n)}, \Sigma^{(n)}) \), where \( \pi^{(n)} = \{ \pi^{(n)}_w \} \) is the probability distribution and \( \Sigma^{(n)} = \{ S^{(n)}_w \} \) is the collection of states of the form (26) satisfying the constraint (15) with the operators \( F^{(n)}_{AB} \). The classical capacity of this protocol will be called \textit{entanglement-assisted classical capacity} \( C_{ea}(\Phi) \) of the channel \( \Phi \) under the constraint (15).

Let \( S \) be a density operator such that both \( H(S) \) and \( H(\Phi(S)) \) are finite, then the \textit{quantum mutual information} is

\[ I(S, \Phi) = H(S) + H(\Phi(S)) - H(S; \Phi) \]  

(27)

where \( H(S; \Phi) \) is the entropy exchange (see e.g. [5]). If the constraint operator \( F \) satisfies (38), then \( H(S) \) is finite for all \( S \) satisfying \( \Tr SF \leq E \). Indeed, we have

\[ \beta \Tr SF - H(S) = H(S; S\beta) - \log \Tr \exp(-\beta F) , \]  

(28)
hence
\[ H(S) \leq \beta E + \log \text{Tr} \exp (-\beta F). \]  

**Proposition 4.** Let \( \Phi \) be a channel satisfying the condition (14) with the operator \( F \) satisfying (8), then its entanglement-assisted classical capacity under the constraint (15) is finite and equals to
\[ C_{ea}(\Phi) = \sup_{S: \text{Tr}SF \leq E} I(S, \Phi). \]  

**Proof.** By a modification of the proof of Proposition 2, we have
\[ C_{ea}(\Phi) = \lim_{n \to \infty} \frac{1}{n} C_{ea}^{(n)}(\Phi), \]  
where
\[ C_{ea}^{(n)}(\Phi) = \sup_{(\pi^{(n)}, \Sigma^{(n)}) \in \mathcal{P}_{AB}^{(n)}} \left[ H\left( \sum_{w=1}^{N} \pi_{w}^{(n)} (\Phi \otimes \text{Id}_{B}^{\otimes n}) [S_{w}^{(n)}] \right) \right. \]
\[ \left. - \sum_{w=1}^{N} \pi_{w}^{(n)} H\left( (\Phi \otimes \text{Id}_{B}^{\otimes n}) [S_{w}^{(n)}] \right) \right]. \]  

Note that all terms in squared brackets are finite because of the assumed finiteness of the entropy \( H(S_{w}^{(n)}) \) and (14).

We first prove the inequality \( \leq \) in (30). By using (31), (32) and the inequality (17) from [4] we obtain
\[ C_{ea}(\Phi) \leq \lim_{n \to \infty} \frac{1}{n} \sup_{(\pi^{(n)}, \Sigma^{(n)}) \in \mathcal{P}_{AB}^{(n)}} I\left( \sum_{w=1}^{N} \pi_{w}^{(n)} \text{Tr}_{B}S_{w}^{(n)}, \Phi \otimes \text{Id}_{B}^{\otimes n} \right). \]

The right hand side is less than or equal to
\[ \sup_{S^{(n)}: \text{Tr}S^{(n)}F^{(n)} \leq nE} I\left( S^{(n)}, \Phi \otimes \text{Id}_{B}^{\otimes n} \right) \equiv \tilde{I}_{n}(\Phi). \]

But the sequence \( \tilde{I}_{n}(\Phi) \) is additive; it is sufficient to prove only
\[ \tilde{I}_{n}(\Phi) \leq nI_{1}(\Phi). \]  

Indeed, by subadditivity of quantum mutual information,
\[ I\left( S^{(n)}, \Phi \otimes \text{Id}_{B}^{\otimes n} \right) \leq \sum_{j=1}^{n} I\left( S_{j}^{(n)}, \Phi \right), \]
where \( S_{j}^{(n)} \) are the partial states, and by concavity,
\[ \sum_{j=1}^{n} I\left( S_{j}^{(n)}, \Phi \right) \leq n \sum_{j=1}^{n} I\left( \frac{1}{n} \sum_{j=1}^{n} S_{j}^{(n)}, \Phi \right). \]
But \(\text{Tr} S^{(n)} F^{(n)} \leq nE\) is equivalent to \(\text{Tr} \left( \frac{1}{n} \sum_{j=1}^{n} S_j^{(n)} \right) F \leq E\), hence follows. Thus

\[
C_{ca}(\Phi) \leq \sup_{S: \text{Tr} SF \leq E} I(S, \Phi). \tag{34}
\]

The proof of the converse inequality is based on the expression \(32\) and the specific encoding protocol from \(3\), \(4\).

Since \(F\) is nonconstant operator, the image of the convex set of all density operators under the map \(S \rightarrow \text{Tr} SF\) is an interval. Assume first that \(E\) is not the minimal eigenvalue of \(F\). Then there exist a real number \(E' < E\) and a density operator \(S\) in \(\mathcal{H}_A\) such that \(\text{Tr} SF \leq E'\). Let \(S = \sum_{j=1}^{\infty} \lambda_j |e_j\rangle \langle e_j|\) be its spectral decomposition, and define \(S_d = \sum_{j=1}^{d} \lambda_j |e_j\rangle \langle e_j|\), where \(\lambda_j = \left( \sum_{k=1}^{d} \lambda_k \right)^{-1} \lambda_j\). Then \(\|S - S_d\|_1 \rightarrow 0\) as \(d \rightarrow \infty\), where \(\|\cdot\|_1\) is the trace norm. Denote \(f(j) = \|\sqrt{F} e_j\|^2\), then

\[
\text{Tr} S_d F = \sum_{j=1}^{d} \lambda_j f(j) = E' + \varepsilon_d,
\]

where \(\varepsilon_d \rightarrow 0\) as \(d \rightarrow \infty\). Now consider the density operator \(S_d^{(n)}\), denote by \(P^{n,\delta}\) its strongly typical projector \(4\) and let \(d_{n,\delta} = \dim P^{n,\delta} = \frac{n^\delta}{d}\). Due to the strong typicality, we have similarly to the estimate at the bottom of p. 4329 in \(4\)

\[
\left| \text{Tr} \left( S_d^{(n)} - S_d^{(n)} \right) F^{(n)} \right| \leq n\delta \max \{f(j); j = 1, \ldots, d\},
\]

whence

\[
\text{Tr} S_d^{(n)} F^{(n)} \leq \text{Tr} S_d^{(n)} F^{(n)} + n\delta \max \{f(j); j = 1, \ldots, d\} = n(E' + \varepsilon_d + \delta \max \{f(j); j = 1, \ldots, d\}).
\]

For every \(d\) large enough one can find \(\delta_0\) such that the right hand side is \(\leq nE\) for \(\delta \leq \delta_0\). Then using the expression \(32\) and the aforementioned encoding protocol, we can prove similarly to \(3\) or to (7) in \(4\):

\[
C_{ca}(\Phi) \geq I\left( S_d^{(n)}, \Phi^{(n)} \right).
\]

Indeed, take the classical signal to be transmitted as \(w = (\alpha, \beta); \alpha, \beta = 1, \ldots, d_{n,\delta}\) with equal probabilities \(\pi_w = 1/d\), the maximally entangled state \(S_{AB} = |\psi_{AB}\rangle \langle \psi_{AB}|\) and the unitary encodings \(E_A^{w}[S] = W_{\alpha\beta} S W_{\alpha\beta}^*\) (see the proof of Theorem in \(4\)). Such an encoding satisfies the input constraint because

\[
\sum_w \pi_w E_A^{w}[S_{AB}] = S_d^{(n)} \otimes S_d^{(n)}.
\]

Thus for this protocol the condition \((\pi^{(n)}, \Sigma^{(n)}) \in \mathcal{P}_{AB}\) in \(32\) is equivalent to \(\text{Tr} S_d^{(n)} F^{(n)} \leq nE\).
Passing to the limit \( n \to \infty, \delta \to 0 \), and using the approximation argument from [4] we obtain

\[
C_{ea}(\Phi) = \lim_{n \to \infty} \frac{1}{n} C_{ea}^{(n)}(\Phi) \geq I(S_d, \Phi),
\]

where Tr\( S_d F = E' + \varepsilon_d \leq E \). Finally, we pass to the limit \( d \to \infty \) and show that

\[
\liminf_{d \to \infty} I(S_d, \Phi) \geq I(S, \Phi), \tag{35}
\]

To see it, we represent the mutual information as quantum relative entropy

\[
I(S, \Phi) = H((\Phi \otimes \text{Id}_R) |\psi\rangle \langle \psi| ; \Phi [S] \otimes S), \tag{36}
\]

where \(|\psi\rangle\langle \psi|\) is a purification for \( S \), \( R \) is a purifying system, and similarly for \( I(S_d, \Phi) \). If \(|\psi\rangle = \sum_{j=1}^{\infty} \sqrt{\lambda_j} |e_j\rangle \otimes |e_j\rangle \), then

\[
|\psi_d\rangle = \sum_{j=1}^{d} \sqrt{\tilde{\lambda}_j} |e_j\rangle \otimes |e_j\rangle
\]
is a purification for \( S_d \). We have \(||\psi\rangle - |\psi_d\rangle|| \to 0\), and hence

\[
|||\psi\rangle \langle \psi| - |\psi_d\rangle \langle \psi_d||| \to 0 \quad \text{as} \quad d \to \infty,
\]

therefore (35) follows from the lower semicontinuity of the relative entropy [8]. Thus we obtain

\[
C_{ea}(\Phi) \geq I(S, \Phi),
\]

where \( S \) is an arbitrary density operator with Tr\( SF < E \). This is easily extended to operators with Tr\( SF \leq E \) by approximating them with the operators \( S_\varepsilon = (1 - \varepsilon)S + \varepsilon|e\rangle \langle e| \), where \( e \) is chosen such that \(|e| F |e\rangle < E\).

In case \( E \) is the minimal eigenvalue of \( F \), the condition Tr\( SF \leq E \) amounts to the fact that the support of \( S \) is contained in the spectral projection of \( F \) corresponding to this minimal eigenvalue. The condition [3] implies that the eigenvalues of \( F \) have finite multiplicity. Thus the support of \( S \) is fixed finite dimensional subspace and we can take \( S_d = S \). Then we can repeat the above argument with the equality Tr\( SF = E \) holding at each step. To sum up, we have established

\[
C_{ea}(\Phi) \geq \sup_{S : \text{Tr} SF \leq E} I(S, \Phi),
\]

and thus the equality in (30). QED

Now we investigate the question when the supremum in the right hand side of (30) is achieved.

**Lemma.** Let the spectrum of operator \( F \) consist of eigenvalues \( f_n \) of finite multiplicity and \( \lim_{n \to \infty} f_n = +\infty \), then the set \( \mathcal{S}_E := \{ S : \text{Tr} SF \leq E \} \) is compact.
Proof. Without loss of generality we assume that \( f_n \) is monotonously increasing and denote by \( P_n \) the finite dimensional projection onto the eigenspace corresponding to the first \( n \) eigenvalues, then \( P_n \uparrow I \). By a general criterion, a weakly closed subset \( \mathcal{S}' \) of density operators is weakly compact if and only if for every \( \varepsilon > 0 \) there is a finite dimensional projection \( P \) such that \( \text{Tr} S(I - P) \leq \varepsilon \) for all \( S \in \mathcal{S}' \), see §III.9 of [9]. But according to [10], the weak convergence of density operators is equivalent to their trace norm convergence. Since \( f_{n+1}(I - P_n) \leq F \), we have \( \text{Tr} S(I - P_n) \leq f_{n+1} \text{Tr} SF \leq f_{n+1} E \) for \( S \in \mathcal{S}_E \), whence the lemma follows. QED

Notice that condition (8) implies that \( F \) satisfies the condition of the lemma.

Proposition 5. Let the constraint operator \( F \) satisfy the condition (8), and let there exist a selfadjoint operator \( \tilde{F} \) satisfying (8) such that \( \Phi^* [\tilde{F}] \leq F \), where \( \Phi^* \) is the dual channel. Then

\[
C_{ca}(\Phi) = \max_{S: \text{Tr} SF \leq E} I(S, \Phi).
\] (37)

Proof. We shall treat separately each term in the formula (27). Notice that quantum entropy is lower semicontinuous. Since the entropy exchange can be represented as \( H(S; \Phi) = H(\Phi_E[S]) \), where \( \Phi_E \) is a channel from the system space \( H_A \) to the environment space \( H_E \), it is also lower semicontinuous and thus the last term in (27) is upper semicontinuous. Concerning the first term, it is upper semicontinuous and hence continuous on the set \( \mathcal{S}_E = \{ S : \text{Tr} SF \leq E \} \) if the constraint operator \( F \) satisfies (8). The proof goes as follows: we have

\[
\beta \text{Tr} S_n F - H(S_n) = H(S_n; S_\beta) - \log \text{Tr} \exp(-\beta F),
\] (38)

and similarly for \( S \) instead of \( S_n \). By using lower semicontinuity of the relative entropy, we obtain

\[
H(S) \geq \lim \sup_{n \to \infty} H(S_n) - \beta \lim \sup_{n \to \infty} \text{Tr} S_n F.
\]

For \( S_n \in \mathcal{S}_E \) the last term is \( \geq -\beta E \), which can be made arbitrarily small.

We can apply similar argument to the second term in (27) under the assumption that there exists a selfadjoint operator \( \tilde{F} \) satisfying (8) and such that \( \Phi^* [\tilde{F}] \leq F \); the relation (38) is then replaced with

\[
\beta \text{Tr} S_n F - H(\Phi[S_n]) \geq H(\Phi[S_n]; \tilde{S}_\beta) - \log \text{Tr} \exp(-\beta \tilde{F}),
\]

where \( \tilde{S}_\beta = \left[ \text{Tr} \exp(-\beta \tilde{F}) \right]^{-1} \exp(-\beta \tilde{F}) \), and the proof goes in a similar way. Moreover, this assumption implies also that the condition (14) and hence (30) holds. Indeed, denoting \( \Phi[S] = S' \) and using (27) we have

\[
\sup_{S: \text{Tr} SF \leq E} H(\Phi[S]) \leq \sup_{S': \text{Tr} SF \leq E} H(S') \leq \beta E + \log \text{Tr} \exp(-\beta \tilde{F}).
\]
This set of conditions ensuring that the supremum in $C_{ea}(\Phi)$ is achieved, is fulfilled for example in the case where $\Phi$ is a Bosonic Gaussian channel and $F$ is positive quadratic polynomial in canonical variables, e. g. energy operator $\mathbb{F}$. The simplest Gaussian channel ”quantum signal plus classical noise” is described in the Heisenberg picture by the equation:

$$a \rightarrow a + \xi,$$  

(39)

where $a$ is the annihilation operator of the mode, and $\xi$ is the classical complex Gaussian random variable with zero mean and the variance $N$ (the mean photon number in the noise). The constraint is $\text{Tr} S a^\dagger a \leq E$, where $S$ is the density operator of the signal $a$. The gain of entanglement assistance $G = C_{ea}(\Phi)/C^{(1)}(\Phi)$ was computed in [3], [5]. In particular, when the signal mean photon number $E$ tends to zero while $N > 0$,

$$C^{(1)}(\Phi) \sim E \log \left( \frac{N+1}{N} \right), \quad C_{ea}(\Phi) \sim -E \log E/(N+1),$$

and $G$ tends to infinity as $-\log E$.

In this paper we were interested in the situation where all the entropy terms entering the expressions for the capacities are finite, which was ensured by the conditions (14), (8). Taking this as an approximation, one can obtain in the general case expressions involving only relative entropy and thus unambiguously defined with values in the range $[0, +\infty)$. For unassisted capacities cf. [6], [11]; $C_{ea}(\Phi)$ will be given by (30), where $I(S, \Phi)$ is defined as in (36).

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