Darboux and binary Darboux transformations for discrete integrable systems I. Discrete potential KdV equation

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Abstract
The Hirota–Miwa equation can be written in ‘nonlinear’ form in two ways: the discrete KP equation and, by using a compatible continuous variable, the discrete potential KP equation. For both systems, we consider the Darboux and binary Darboux transformations, expressed in terms of the continuous variable, and obtain exact solutions in Wronskian and Grammian form. We discuss reductions of both systems to the discrete KdV and discrete potential KdV equation, respectively, and exploit this connection to find the Darboux and binary Darboux transformations and exact solutions of these equations.

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1. Introduction
The Hirota–Miwa equation [1, 2] is the three-dimensional discrete integrable system

$$(a_1 - a_2)\tau_{12}\tau_3 + (a_2 - a_3)\tau_{23}\tau_1 + (a_3 - a_1)\tau_{31}\tau_2 = 0, \quad (1.1)$$

where lattice parameters $a_k$ are constants, $k = 1, 2, 3$, and for $\tau = \tau(n_1, n_2, n_3)$ each subscript $i$ denotes a forward shift in the corresponding discrete variable $n_i$. There are many papers on the Hirota–Miwa equation describing a number of important results. These include finding its exact solutions by direct methods and Bäcklund/Darboux transformations [3–7]; its extension to nonautonomous [8–11] and noncommutative forms [12, 13], and reductions to lower dimensional discrete integrable systems [14–16]. In this paper, we also discuss in detail the reduction to the discrete Korteweg de Vries (dKdV) equation [17] and the discrete potential KdV (dpKdV) equation [18]. We exploit these connections to get the Lax pairs, and
the Darboux and binary Darboux transformations of these systems in a natural way. The dpKdV equation is also known as the H1 equation in the Adler–Bobenko–Suris (ABS) classification [19]. Its integrability is understood in the sense of the multidimensional consistency property [20, 21], and gives a Lax pair directly. However, this Lax pair is not suitable for the application of classical Darboux transformations [22, 23]. So, one motivation for this work is to obtain a Lax pair through reduction of the linear system of the Hirota–Miwa equation. This paper is the first of a planned series which will explore the equations in the ABS list, their Lax pairs and Darboux transformations as reductions of the Hirota–Miwa equation.

The Hirota–Miwa equation (1.1) can also be written in terms of ‘nonlinear’ variables rather than \( \tau \)-function in two distinct ways [12, 13], when using variables \( u^i := \tau_{ij} / \tau \), and the linear system
\[
a_i \phi_i - a_j \phi_j = (a_i - a_j) u^i \phi, \quad 1 \leq i < j \leq 3,
\]
where for \( \phi = \phi(n_1, n_2, n_3) \) each subscript \( i \) denotes a forward shift in the corresponding discrete variable \( n_i \). This linear system (1.2) is compatible if and only if
\[
(a_1 - a_2) u^{12} + (a_2 - a_3) u^{13} + (a_3 - a_1) u^{13} = 0, \quad (1.3a)
\]
\[
(u^i)_k = (a_k) u^i. \quad (1.3b)
\]
Each of the variables \( u^i \) relates to the solution \( u = (\log \tau)_{xx} \) of the KP equation in the continuum limit, so we call (1.3a) the discrete KP (dKP) equation. Note that when one uses the formula \( u^i = \tau_{ij} / \tau \), (1.3a) gives (1.1) and (1.3b) is satisfied identically. A second way is to suppose \( u^i = (v_i - v_j + (a_i - a_j)) / (a_i - a_j) \), where \( v = (\log \tau)_x \). This ansatz solves (1.3a) exactly and (1.3b) becomes the discrete potential KP (dpKP) equation (see also [24]).

The outline of this paper is as follows. In section 2, we recall important definitions and properties of the Hirota–Miwa equation. In particular, we write the Hirota–Miwa equation in ‘nonlinear’ form in two ways: the discrete KP equation and, by using a compatible continuous variable, the dpKP equation. For both equations, we give two different associated linear systems and their corresponding auxiliary linear systems in differential–difference form. So their Darboux and binary Darboux transformations are given in differential, rather than the more usual difference form [5, 6]. The differential form uses the first continuous flow \( x \) of the KP hierarchy which is compatible with the discrete flows in the Hirota–Miwa equation. These transformations are used to derive exact solutions in Wronskian and Grammian form, respectively. In section 3, we discuss the 2-reduction of the Hirota–Miwa equation, and in particular, the dKP to the dKdV equation and of the dpKP to the dpKdV equation (H1 in the ABS classification [19]). Then, by taking appropriate reductions of the results in section 2, we derive the Lax pairs for the dKdV and dpKdV equations, and their Darboux and binary Darboux transformations and exact solutions.

2. Hirota–Miwa equation

2.1. Wronskian and Grammian solutions of the Hirota–Miwa equation

The Hirota–Miwa equation (1.1) is the compatibility condition for the linear system [5, 12]
\[
a_i \phi_i - a_j \phi_j = (a_i - a_j) \frac{\tau_{ij}}{\tau} \phi, \quad (2.1)
\]
for \( 1 \leq i < j \leq 3 \). It is invariant with respect to the reversal of all lattice directions \( n_i \rightarrow -n_i \). On the other hand, the linear system (2.1) does not have such invariance and the reflections \( n_i \rightarrow -n_i \) acting on (2.1) give a new linear system
\[
a_i \psi_i - a_j \psi_j = (a_i - a_j) \frac{\tau_{ij}}{\tau} \psi, \quad (2.2)
\]
for all $1 \leq i < j \leq 3$. The subscript $\tilde{j}$ denotes a backward shift with respect to $n_i$, for example, $\psi_{\tilde{j}} := \psi(n_i - 1, n_2, n_3)$. In [5, 12], the difference form of the Darboux and binary Darboux transformations were derived for the Hirota–Miwa equation (1.1) and these were used to construct exact solutions in the form of Casoratian and discrete Grammian determinants. Here, we will express the Darboux and binary Darboux transformations in differential form instead, using the lowest order continuous flow $x$ of the KP hierarchy, and then the solutions obtained will be expressed as Wronskian and (continuous) Grammian determinants. The differential–difference linear equations for $\phi$ and $\psi$ are

$$\phi_x = a_i \phi_{\tilde{i}} + \left( \frac{\tau_x}{\tau} \right)_i - \frac{\tau_x}{\tau} - a_i \phi,$$  \hspace{1cm} (2.3)

and

$$\psi_x = -a_i \psi_{\tilde{i}} + \left( \frac{\tau_x}{\tau} \right)_j - \frac{\tau_x}{\tau} + a_i \psi,$$  \hspace{1cm} (2.4)

where the subscript $x$ denotes the derivative, with $\tau$ satisfying the semi-discrete KP equation

$$(a_i - a_j) (\tau_x \tau - \tau_x \tau_j) + \tau_x \tau_j - \tau_x \tau_{\tilde{j}} = 0.$$  \hspace{1cm} (2.5)

It is straightforward to check that (2.3) and (2.4) are compatible with (2.1) and (2.2). Note that the reflection symmetry which relates (2.1) and (2.2) may be extended by adding $x \rightarrow -x$ to relate (2.3) and (2.4).

The basic Darboux transformation for the Hirota–Miwa equation is stated in the following proposition.

**Proposition 2.1.** Let $\theta$ be a non-zero solution of the linear system (2.1) and (2.3) for some $\tau$. Then the transformations

$$\phi \rightarrow \tilde{\phi} = a_i (\phi_i - \theta_i \theta^{-1} \phi) = \phi_{\tilde{i}} - \theta_i \theta^{-1} \phi, \quad \tau \rightarrow \tilde{\tau} = \theta \tau,$$  \hspace{1cm} (2.6)

leave (2.1) and (2.3) invariant, where $i = 1, 2, 3$.

The proof is a straightforward computation. Note that there are four expressions for $\tilde{\phi}$ in the Darboux transformation (2.6). These are equivalent because of the linear equations (2.1) and (2.3).

Next we write down the formulae for iterated Darboux transformations, which give solutions in Wronskian and Casoratian determinant form. The Wronskian determinant is the determinant of the $N \times N$ matrix with columns $\Theta^{(j)} = \Theta^{(j)}(x, y, t)$, for $j = 0, 1, \ldots, N - 1$, where $\Theta^{(0)} = (\theta_1(x, y, t), \theta_2(x, y, t), \ldots, \theta_N(x, y, t))^T$ and $\Theta^{(j)} = \partial / \partial x^{j} \Theta^{(0)}$. It is written as

$$W(\theta_1, \theta_2, \ldots, \theta_N) = |\Theta^{(0)}, \Theta^{(1)}, \ldots, \Theta^{(N-1)}|,$$

or in a more compact notation \[25\]

$$W(\theta_1, \theta_2, \ldots, \theta_N) = |N - 1|.$$

The Casoratian determinant can be seen as a discrete analogue of the Wronskian determinant. It is the determinant of the $N \times N$ matrix with columns $\Theta^x = \Theta^x(n_1, n_2, n_3)$, for $j = 0, 1, \ldots, N - 1$, where $\Theta^x = (\theta_1(n_1, n_2, n_3), \theta_2(n_1, n_2, n_3), \ldots, \theta_N(n_1, n_2, n_3))^T$, and $\Theta^{(j)}$ is defined by the forward shifts, i.e., $\Theta^{(j)} = T^n_j(\Theta^{(0)}) = \Theta^{(0)}(n_1 + j, n_2, n_3)$. It is written as

$$C(\theta_1, \theta_2, \ldots, \theta_N) = |\Theta^{(0)}, \Theta^{(1)}, \ldots, \Theta^{(N-1)}| = |0, 1, \ldots, N - 1| = |N - 1|.$$  \hspace{1cm} (2.7)

We can also use $\Theta^{(j)}$, which is defined by the backward shifts, to replace the $\Theta^{(j)}$, where $\Theta^{(j)} = T^n_{\tilde{j}}(\Theta^{(0)}) = \Theta^{(0)}(n_1 - j, n_2, n_3)$.

Below, we use the subscript $[i]$ to designate that the shifts of the Casoratian determinant are with respect to the variable $n_i$. For example, the Casoratian determinant in (2.7) could be denoted as $C_{(i)}(\theta_1, \theta_2, \ldots, \theta_N)$. 

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Proposition 2.2. Let $\theta_1, \theta_2, \ldots, \theta_N$ be non-zero, independent solutions of the linear system (2.1) and (2.3) for some $\tau$. Then $N$ applications of the above Darboux transformations give the transformations

$$\phi \rightarrow \phi[N] = \frac{a_i^N C_{i_0}(\theta_1, \theta_2, \ldots, \theta_N, \phi)}{C_{i_0}(\theta_1, \theta_2, \ldots, \theta_N)} = \frac{W(\theta_1, \theta_2, \ldots, \theta_N, \phi)}{W(\theta_1, \theta_2, \ldots, \theta_N)},$$  \hspace{1cm} (2.8)

and

$$\tau \rightarrow \tau[N] = a_i^{-N \tau} C_{i_0}(\theta_1, \theta_2, \ldots, \theta_N) \tau = W(\theta_1, \theta_2, \ldots, \theta_N) \tau,$$  \hspace{1cm} (2.9)

which leave (2.1) and (2.3) invariant. Here $C_{i_0}$ denotes the Casoratian determinant in forward shifts with respect to the discrete variable $n_i$, for $i = 1, 2, 3$.

For example, by using the results (2.9), the $N$-soliton solutions of the Hirota–Miwa equation can be expressed in both Casoratian and Wronskian form in terms of the eigenfunctions,

$$\theta_k(n_1, n_2, n_3) = \exp^{3N} \prod_{i=1}^{3} \left( 1 + \frac{p_i}{a_i} \right)^{n_i} + \exp^{3N} \prod_{i=1}^{3} \left( 1 + \frac{q_i}{a_i} \right)^{n_i}.$$  \hspace{1cm} (2.10)

Here $\theta_k, k = 1, 2, \ldots, N$, is obtained from (2.1) and (2.3), by choosing the trivial solution $\tau = 1$.

Now we can apply the reflections $n_i \rightarrow -n_i$ and $x \rightarrow -x$ to the above results to deduce corresponding results for the second linear system (2.2) and (2.4).

Proposition 2.3. Let $\rho$ be a non-zero solution of the linear system (2.2) and (2.4) for some $\tau$. Then the transformations

$$\psi \rightarrow \tilde{\psi} = a_i(\psi - \rho \tau^{-1} \psi) = \psi - \rho x^{-1} \psi, \quad \tau \rightarrow \tilde{\tau} = \rho \tau,$$  \hspace{1cm} (2.11)

leave (2.2) and (2.4) invariant, for all $i = 1, 2, 3$.

In the statement of this proposition the linearity of (2.2) and (2.4) allows us to omit a minus sign.

Proposition 2.4. Let $\rho_1, \rho_2, \ldots, \rho_N$ be non-zero, independent solutions of the linear system (2.2) and (2.4) for some $\tau$. Then $N$ applications of the above Darboux transformations give the transformations

$$\psi \rightarrow \psi[N] = \frac{a_i^N C_{\rho_0}(\rho_1, \rho_2, \ldots, \rho_N, \psi)}{C_{\rho_0}(\rho_1, \rho_2, \ldots, \rho_N)} = \frac{W(\rho_1, \rho_2, \ldots, \rho_N, \psi)}{W(\rho_1, \rho_2, \ldots, \rho_N)},$$  \hspace{1cm} (2.12)

and

$$\tau \rightarrow \tau[N] = a_i^{-N \tau} C_{\rho_0}(\rho_1, \rho_2, \ldots, \rho_N) \tau = W(\rho_1, \rho_2, \ldots, \rho_N) \tau,$$  \hspace{1cm} (2.13)

which leave (2.2) and (2.4) invariant. Here $C_{i_0}$ denotes the Casoratian determinant in backward shifts with respect to the discrete variable $n_i$, for $i = 1, 2, 3$.

The $N$-soliton solutions of the Hirota–Miwa equation are given by (2.13). From the linear system (2.2) and (2.4), if we choose the seed solution as $\tau = 1$, then the Casoratian and Wronskian determinants are defined by eigenfunctions,

$$\rho_k(n_1, n_2, n_3) = \exp^{-N p_k} \prod_{i=1}^{3} \left( 1 + \frac{p_i}{a_i} \right)^{-n_i} + \exp^{-N q_k} \prod_{i=1}^{3} \left( 1 + \frac{q_i}{a_i} \right)^{-n_i},$$  \hspace{1cm} (2.14)

for $k = 1, 2, \ldots, N$. 

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To construct a binary Darboux transformation, we introduce the potential \( \omega = \omega(\phi, \psi) \), defined by the relations
\[
\Delta_1 \omega(\phi, \psi) = \phi \psi_i, \quad (2.15)
\]
\[
\omega_2(\phi, \psi) = \phi \psi, \quad (2.16)
\]
where \( \Delta_1 = a_i(T_n - 1) \) and \( T_n \) is the forward shift operator in variable \( n_i \), for \( i = 1, 2, 3 \). If \( \phi \) and \( \psi \) satisfy the linear systems (2.1), (2.3) and (2.2), (2.4), respectively, then (2.15) and (2.16) are compatible. So the potential \( \omega \) is well-defined.

The following proposition gives the binary Darboux transformation of the Hirota–Miwa equation.

**Proposition 2.5.** Suppose \( \theta \) and \( \phi \) are non-zero solutions of the linear system (2.1) and (2.3), \( \rho \) and \( \psi \) are non-zero solutions of the linear system (2.2) and (2.4), then the transformations
\[
\phi \rightarrow \hat{\phi} = \phi - \theta \omega(\theta, \rho)^{-1} \omega(\phi, \rho), \quad (2.17)
\]
\[
\psi \rightarrow \hat{\psi} = \psi - \rho \omega(\theta, \rho)^{-1} \omega(\theta, \psi), \quad (2.18)
\]
leave (2.1), (2.3) and (2.2), (2.4) invariant, respectively, with \( \tau \) changing to
\[
\hat{\tau} = \omega(\theta, \rho) \tau. \quad (2.19)
\]

The \( N \)-fold iteration of the binary Darboux transformation is as follows.

**Proposition 2.6.** Let \( \theta_1, \ldots, \theta_N \) and \( \rho_1, \ldots, \rho_N \) satisfy the linear systems (2.1), (2.3) and (2.2), (2.4), respectively. Define \( N \)-vectors \( \Theta = (\theta_1, \ldots, \theta_N)^T \) and \( P = (\rho_1, \ldots, \rho_N)^T \). Then \( N \) applications of the binary Darboux transformation give the transformations
\[
\phi \rightarrow \phi[N] = \left[ \Omega(\Theta, P) \phi \right] \Theta \left[ \Omega(\Theta, P) \right]^{-1}, \quad \psi \rightarrow \psi[N] = \left[ \Omega(\Theta, P) \psi \right] P \left[ \Omega(\Theta, P) \right]^{-1}, \quad (2.20)
\]
which leave (2.1), (2.3) and (2.2), (2.4) invariant, respectively, with \( \tau \) changing to
\[
\tau[N] = |\Omega(\Theta, P)| \tau. \quad (2.21)
\]

Here \( \Omega(\Theta, P) = (\omega(\theta_j, \rho_i))_{i,j=1,...,N} \) is an \( N \times N \) matrix, \( \Omega(\phi, P) = (\omega(\phi, \rho_j))_{j=1,...,N} \) and \( \Omega(\psi, \Theta) = (\omega(\theta_i, \psi))_{i=1,...,N} \) are \( N \)-row vectors.

The proofs of those above propositions are straightforward computation, so we do not give the details. The reader is also referred to the papers [5, 6, 12].

### 2.2. The discrete potential KP equation

By introducing a potential \( v(n_1, n_2, n_3; x) := \tau_i / \tau \), the semi-discrete equation (2.5) gives the relation
\[
u^{ij} = \frac{\tau_j x}{\tau_i} \frac{v_j - v_i + (a_i - a_j)}{a_i - a_j}, \quad 1 \leq i < j \leq 3, \quad (2.22)
\]
for \( v \) each subscript \( i \) denotes a forward shift in the corresponding discrete variable \( n_i \). So (1.3a) is satisfied identically, and (1.3b) becomes
\[
\frac{v_2 - v_1 + a_1 - a_2}{(v_2 - v_1 + a_1 - a_2)_1} = \frac{v_3 - v_1 + a_1 - a_3}{(v_3 - v_1 + a_1 - a_3)_1} = \frac{v_3 - v_2 + a_2 - a_3}{(v_3 - v_2 + a_2 - a_3)_1}. \quad (2.23a)
\]
The equation (2.23a) can be written in two other equivalent forms as either
\[(v_2 - v_1 + a_1 - a_2) (v_{23} - v_{12} + a_1 - a_3) = (v_3 - v_1 + a_1 - a_3) (v_{23} - v_{13} + a_1 - a_2),\]
(2.23b)
or
\[(a_3 + v_{12}) (a_1 - a_2 + v_2 - v_1) + (a_2 + v_{13}) (a_3 - a_1 + v_1 - v_3) + (a_1 + v_{23}) (a_2 - a_3 + v_3 - v_2) = 0.\]
(2.23c)

Equation (2.23) is called the dpKP equation. It was first found in [24] as the nonlinear superposition of solutions to the potential KP equation related by the B"acklund transformations [24]. The trivial solution of the dpKP equation (2.23) could be \(v = c\), where \(c\) is an arbitrary constant.

The linear systems associated with the dpKP equation, obtained by using relations (2.1) and (2.2), together with (2.22), are
\[a_i \phi_i - a_j \phi_j = (v_j - v_i + a_i - a_j) \phi,\]
(2.24)
and
\[a_i \psi_i - a_j \psi_j = (v_j - v_i + a_i - a_j) \psi,\]
(2.25)
where \(1 \leq i < j \leq 3\). Its corresponding differential–difference linear systems are (2.3) and (2.4) with \(\tau_i/\tau = v\).

Together with the differential–difference linear system (2.3), the Darboux transformation of the linear system (2.24) gives a new solution of the dpKP equation
\[\tilde{v} = (\log (\theta \tau)) x = v + (\log \theta)_x,\]
where \(\theta\) is a non-zero solution of (2.24) and (2.3). More generally, \(N\)-fold iteration gives the Wronskian solution
\[v[N] = v + (\log W(\theta_1, \theta_2, \ldots, \theta_N))_x,\]
where \(\theta_k, k = 1, 2, \ldots, N\) are the non-zero independent solutions of (2.24) and (2.3).

The binary Darboux transformation gives a new solution of the dpKP equation
\[\tilde{v} = v + (\log \omega(\theta, \rho))_x,\]
where \(\omega(\theta, \rho)\) is defined by (2.15) and (2.16), and \(\theta, \rho\) are non-zero solutions of (2.24), (2.3) and (2.25), respectively. Its \(N\)-fold iteration gives the Grammian solution
\[v[N] = v + (\log |\Omega(\Theta, \Phi)|)_x,\]
where the \(N \times N\) matrix \(\Omega(\Theta, \Phi) = (\omega(\theta_i, \rho_j))_{i,j=1,...,N}\), \(\omega(\theta_i, \rho_j)\) is defined by (2.15) and (2.16), and \(\theta_i, \rho_j\) are non-zero solutions of (2.24), (2.3) and (2.25), (2.4), respectively.

3. Reductions to the dKdV and dpKdV equations

3.1. The dKdV equation to the dKdV equation

The discrete KdV (dKdV) equation is a 2-reduction of the dKP equation (1.3a). In the reduction, it is necessary that one takes \(a_2 + a_3 = 0\), since the solutions satisfy the reduction condition \(\tau_{23} = \tau\), cf [3, 10]. The reduction condition for the eigenfunction of the linear system (2.1) and (2.3) is \(\phi_{23} = (1 - \lambda^2) \phi\), where the form of the coefficient is chosen for its correspondence with the discrete one-dimensional Schr"{o}dinger equation [26, 27]. To realise this reduction, we make the ansatz
\[\tau(n_1, n_2, n_3) = \tau(n_1, n_2 - n_3)\]
(3.1)
\[\phi(n_1, n_2, n_3) = (1 - \lambda^2)^{n_3} \phi(n_1, n_2 - n_3).\]
(3.2)
Proposition 3.1. Suppose $\theta$ is a non-zero solution of the linear system (3.6) and (3.8) for some $u$ and $\tau$, together with $u = \tau_1 \tau/\tau_2 \tau$, then the transformations

\[ \phi \rightarrow \tilde{\phi} = a_i (\phi_i - \theta_i \theta^{-1} \phi) = \phi_i - \theta_i \theta^{-1} \phi, \quad i = 1, 2, \]

and

\[ \tau \rightarrow \tilde{\tau} = \theta \tau, \quad u \rightarrow \tilde{u} = \theta_1 \theta_2 \theta^{-1} u, \]

leave (3.6) and (3.8) invariant.

Now taking the reduction conditions, the dKP equation (1.3a) can be written as either

\[ (a_2 - a_1) \frac{\tau_2 \tau}{t_1 t_2} + (a_2 + a_1) \frac{\tau_2 \tau}{t_1 t_2} = 2a_2 \frac{\tau}{t_1 t_2}, \tag{3.3a} \]

or

\[ (a_2 - a_1) \frac{\tau_3 \tau}{t_1 \tau_2} + (a_2 + a_1) \frac{\tau_3 \tau}{t_1 \tau_2} = 2a_2 \frac{\tau_1}{t_1} \frac{\tau_1}{t_2}, \tag{3.3b} \]

We then express the above two equations in terms of nonlinear variable

\[ u(n_1, n_2) := \frac{\tau_1 \tau_3}{t_1 t_3} = \frac{\tau_1 \tau_2}{t_1 \tau_2}, \tag{3.4} \]

By eliminating the tau-function parts on the right-hand sides in both (3.3a) and (3.3b), we obtain

\[ \frac{1}{u_1} - \frac{1}{u_2} = \frac{a_1 - a_2}{a_1 + a_2} (u_{12} - u), \tag{3.5} \]

which is the discrete KdV equation, first found by Hirota [17].

From the linear system (2.1) and using the 2-reduction, we get the linear system of the dKdV equation

\[ a_1 \phi_1 - a_2 \phi_2 = (a_1 - a_2) u_2 \phi, \tag{3.6a} \]

\[ a_1 \phi_1 + a_2 (1 - \lambda^2) \phi_2 = (a_1 + a_2) \frac{1}{u} \phi. \tag{3.6b} \]

Note here that, by eliminating the $\phi_1$ in these two equations, we have

\[ a_2 \phi_2 + a_2 (1 - \lambda^2) \phi_2 = (a_2 - a_1) u_2 + (a_2 + a_1) \frac{1}{u} \phi, \]

which is a discrete one-dimensional Schrödinger equation [26, 27].

The dKdV equation (3.5) is also invariant with respect to the reflections $n_i \rightarrow -n_i$, for $i = 1, 2$. So applying the reflections to the system (3.6) gives a new linear system of the dKdV equation

\[ a_1 \psi_1 - a_2 \psi_2 = (a_1 - a_2) u_2 \psi, \tag{3.7a} \]

\[ a_1 \psi_1 + a_2 (1 - \lambda^2) \psi_2 = (a_1 + a_2) \frac{1}{u} \psi. \tag{3.7b} \]

This system could also be obtained from the linear system (2.2), by using the 2-reduction $\tau_1 \tau_2 = \tau$, and $\psi_2 = (1 - \lambda^2) \psi$ with $a_2 + a_1 = 0$.

For the linear differential–difference equations (2.3) and (2.4), the 2-reduction does not affect the $n_1$- or $n_2$- parts, but the $n_3$-parts become

\[ \phi_i = -a_2 (1 - \lambda^2) \phi_2 + \left( \frac{\tau_1}{\tau} \frac{\tau_1}{\tau} + a_2 \right) \phi, \tag{3.8} \]

\[ \psi_i = a_2 (1 - \lambda^2) \psi_2 + \left( \frac{\tau_1}{\tau} \frac{\tau_1}{\tau} - a_2 \right) \psi. \tag{3.9} \]

The fundamental Darboux transformation for the dKdV equation is
Here again we use the 2-reduction, 3.2. The dpKP equation to the dpKdV equation using the idea that the Darboux transformation of the dKP equation in the proposition 2.1 also works for the dKdV equation after taking the 2-reduction. From the linear system (3.6), \( \phi \) and \( \theta \) are its solutions, so we have
\[
a_i(\phi_i - \theta_i \theta^{-1} \phi) = -a_2(1 - \lambda^2)(\phi_2 - \theta_x \theta^{-1} \phi),
\]
for \( i = 1, 2 \). On the other hand, the 2-reduction gives
\[
a_3(\phi_3 - \theta_x \theta^{-1} \phi) = -a_2(1 - \lambda^2)(\phi_2 - \theta_x \theta^{-1} \phi).
\]
(3.12)

For the potential \( u \), we have
\[
\tilde{u} = \frac{\tau_1 \tau_2 \tau'}{\tau_1 \tau_2 ^2} = u \frac{\theta_1}{\theta_2} \theta.
\]
(3.11)

So together with the transformation in (2.6), it means that after taking 2-reduction, the transformations (3.10) leave the linear system (3.6) and (3.8) invariant. \( \Box \)

Similarly, by using the reflections \( n_i \to -n_i \) and \( x \to -x \), or the 2-reduction of the Darboux transformation of the dKP equation in proposition 2.3, we get the Darboux transformation of the linear system (3.7) and (3.9).

Proposition 3.2. Suppose \( \rho \) is a non-zero solution of the linear system (3.7) and (3.9) for some \( u \) and \( \tau \), together with \( u = \tau_\tau \tau_2 / \tau_2 \tau_\tau \), then the transformations
\[
\psi \to \tilde{\psi} = a_i(\psi_i - \rho \rho^{-1} \psi) = \psi_x - \rho_x \rho^{-1} \psi, \quad i = 1, 2,
\]
and
\[
\tau \to \tilde{\tau} = \rho \tau, \quad u \to \tilde{u} = u \frac{\rho \rho_2}{\rho_2 \rho},
\]
(3.13b)

leave (3.7) and (3.9) invariant.

Similarly, in the light of the reductions, the binary Darboux transformations of the dKdV equation can also be obtained directly from the one in proposition 2.5. Thus the results of the \( N \)-applications of the Darboux and binary Darboux transformations for the dKdV equation can be gotten from the ones in propositions 2.2, 2.4 and 2.6. For this reason, we will not go to talk about them in detail here.

3.2. The dpKP equation to the dpKdV equation

Here again we use the 2-reduction, \( v_{23} = v \) and \( a_2 + a_3 = 0 \). Now by introducing the potential \( v(n_1, n_2; x) := \tau_{\tau} / \tau \) into (2.22), we get
\[
(a_1 - a_2)u_2 = v_2 - v_1 + a_1 - a_2, \quad (3.14a)
\]
\[
(a_1 + a_2)u = v_2 - v_1 + a_1 + a_2. \quad (3.14b)
\]

By eliminating the potential \( u \) from (3.14), we obtain
\[
(v_2 - v_1 + a_1 - a_2)(v_2 - v_1 + a_1 + a_2) = a_1^2 - a_2^2. \quad (3.15)
\]

This is the dpKdV equation [18], and we see that its solution can be written as \( v = (\log \tau)_x \). The dpKdV equation could also be obtained by the permutability property of the Bäcklund transformations of the continuous potential KdV equation, cf [18]. Taking the potential transformation \( v \to v + a_1 n_1 + a_2 n_2 + \gamma \), where \( \gamma \) is an arbitrary constant, (3.15) becomes
\[
(v_2 - v_1)(v_2 - v_1) = a_1^2 - a_2^2, \quad (3.16)
\]
which is called the H1 equation [19].
By taking the 2-reduction, the linear system of the dpKP equation (2.24) and (2.25) give the Lax pairs of the dpKdV equation (3.15),

\[ a_1\phi_1 - a_2\phi_2 = (v_2 - v_1 + a_1 - a_2)\phi, \]
\[ a_1\phi_1 + a_2(1 - \lambda^2)\phi_2 = (v_2 - v_1 + a_1 + a_2)\phi, \]

and with reflections \( n_i \to -n_i \), this also gives

\[ a_1\psi_1 - a_2\psi_2 = (v_2 - \tau + a_1 - a_2)\psi, \]
\[ a_1\psi_1 + a_2(1 - \lambda^2)\psi_2 = (v_2 - \tau + a_1 + a_2)\psi. \]

Their corresponding differential–difference linear systems are (2.3) and (2.4) with \( \tau_i/\tau = v \), for \( i = 1, 2 \), respectively. Note here that the Lax pair (3.17) is not the same as but can be transformed into the one derived from the multidimensional consistency property, cf [28–30], by using appropriate parameters transformations, such as \( r = a_2\lambda \).

As we showed for the dpKP equation, the Darboux transformation gives the new solution of the dpKdV equation in differential form,

\[ \tilde{v} = v + (\log \theta)_x, \]

where \( \theta \) is the non-zero solution of (3.17) and (2.3). Its \( N \)-fold iteration gives the solution in Wronskian form

\[ v[N] = v + (\log W(\theta_1, \theta_2, \ldots, \theta_N))_x, \]

where \( \theta_k \), for \( k = 1, 2, \ldots, N \), are the non-zero solutions of (3.17) and (2.3).

The soliton solutions of the dpKdV equation (3.15) can be obtained from (3.19). From the linear system (3.17) and (2.3), by choosing the seed solution \( v = 0 \), the eigenfunctions of the \( \tau \)-function in Wronskian determinant are

\[ \theta_k(n_1, n_2) = e^{\alpha_k^1 n_1} e^{\alpha_k^2 n_2} e^{\beta n_1} e^{\gamma n_2}, \]

\[ \theta_k(n_1, n_2) = e^{\alpha_k^1 n_1} e^{\alpha_k^2 n_2} e^{\beta n_1} e^{\gamma n_2}, \]

for \( k = 1, 2, \ldots, N \). This result coincides with the one given by Hietarinta and Zhang [29]. In their paper, they employed \( f = |N - 1| \) and \( g = |N - 2, N| \), which were given as Casoratian determinants with respect to an auxiliary discrete variable. This auxiliary variable is compatible with the original independent variables, \( n_1 \) and \( n_2 \), but is external to the system. We will denote this auxiliary variable to be \( n_4 \) here. Then for an arbitrary constant \( c, v = \tau + c \) satisfies the dpKdV equation (3.15). \( f \) and \( g \) in Casoratian determinants in compact form are

\[ f = |N - 1| = [\phi, T_{n_4}(\phi), T_{n_4}^2(\phi), \ldots, T_{n_4}^{N-1}(\phi)], \]

and

\[ g = |N - 2, N| = [\phi, T_{n_4}(\phi), T_{n_4}^2(\phi), \ldots, T_{n_4}^{N-2}(\phi), T_{n_4}^N(\phi)], \]

where \( \phi = \phi(n_1, n_2, n_4) \) and \( T_{n_4}(\phi) = \phi(n_1, n_2, n_4 + j, x), j = 0, 1, \ldots, N \). Similarly, \( \tau \) and \( \tau_x \) in Wronskian determinants are

\[ \tau = |N - 1| = [\phi, \partial_x \phi, \partial_x^2 \phi, \ldots, \partial_x^{N-1} \phi], \]

and

\[ \tau_x = |N - 2, N| = [\phi, \partial_x \phi, \partial_x^2 \phi, \ldots, \partial_x^{N-2} \phi, \partial_x^N \phi]. \]

So for the linear systems (3.17) and (2.3) with \( \tau_x/\tau = v \), we could also introduce a virtual discrete variable \( n_4 \), say, which is another compatible discrete flow, with parameter \( a_4 \). If we choose the seed solution of the dpKdV equation \( v = 0 \), then the differential–difference linear system (2.3), with respect to the discrete variable \( n_4 \), gives the entries of the above Wronskian and Casoratian determinants satisfying the relations

\[ \phi_x = a_4(T_{n_4} - 1)(\phi), \]
and
\[ \partial^N_{x} \phi = a_4^{-N}(T_{a_4} - 1)^N(\phi). \]
So we have
\[ \tau = a_4^{-N+1} f, \quad \tau_x = a_4^{-N+1} (g - N f). \]
By taking \( a_4 = 1 \), the soliton solution of the dpKdV equation is
\[ v = (\log \tau)_x = \frac{\tau_x}{\tau} = \frac{g}{f} - N. \]
So if the soliton solution of the dpKdV equation is expressed in Wronskian determinant, by using a compatible continuous variable, then it needs to employ \( \tau \) and its derivative \( \tau_x \). If it is expressed in Casoratian determinant, through a virtual discrete variable \( n_4 \), then it needs to employ \( f \) and \( g \). But there is no direct relation between \( f \) and \( g \).

The binary Darboux transformation gives the exact solution of the dpKdV equation,
\[ \hat{v} = v + (\log \omega(\theta, \rho))_x, \]
where \( \omega(\theta, \rho) \) is defined by (2.15) and (2.16), and \( \theta, \rho \) are non-zero solutions of (3.17), (2.3) and (3.18), (2.4), respectively, with \( \tau_x/\tau = v \). Its \( N \)-fold iteration gives the exact solution in Grammian form,
\[ v[N] = v + (\log |\Omega(\Theta, P)|)_x, \]
where the \( N \times N \) matrix \( \Omega(\Theta, P) = (\omega(\theta_i, \rho_j))_{i,j=1,...,N}, \) \( \omega(\theta, \rho) \) is defined by (2.15) and (2.16), and \( \theta_i, \rho_j \) are non-zero solutions of (3.17), (2.3) and (3.18), (2.4), respectively, with \( \tau_x/\tau = v \).

4. Conclusions

In this paper, we have revisited the Darboux and binary transformations of the Hirota–Miwa equation given in [5, 6] not only in difference form in a more general case, including the lattice parameters \( a_4 \), but also in differential form [22, 23]. These allow one to obtain solutions of the Hirota–Miwa equation in both Casoratian and Wronskian forms, and as discrete and continuous Grammians as well. It is straightforward to obtain corresponding results for reductions of the Hirota–Miwa equation. In this paper we do this for the dKdV and dpKdV equations. The Lax pair obtained by reductions are not the same as the ones given by the multidimensional consistency property, and allow the application of the classical Darboux transformations [22, 23]. We find that the Hirota–Miwa equation can be written in ‘nonlinear’ form in two distinct ways: as the discrete KP equation in terms of the variable \( u^{ij} = \tau_j \tau_i \) and, by using a compatible continuous variable \( x \), as the discrete potential KP (dpKP) equation in variable \( u^{ij} = (v_i - v_j + (a_i - a_j))/(a_i - a_j) \), where \( v = (\log \tau)_x \). This leads us to see clearly the relationship between the dKP and dpKP equations, which is similar to the relationship \( u = v_1 \) in the continuous case. Thus we understand better the form of the Casoratian solutions to the H1 equation in [29], and it will be helpful in future to deal with other members in the ABS list, such as the H2 equation.

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