APPLICATIONS OF BRAIDED ENDMORPHISMS FROM CONFORMAL INCLUSIONS

FENG XU

ABSTRACT. We give three applications of general theory about braided endomorphisms from conformal inclusions developed previously by us. The first is an example of subfactors associated with conformal inclusion whose dual fusion ring is non-commutative. In the second application we show that the Kac-Wakimoto hypothesis about certain relations between branching rules and S-matrices, which has existed for almost a decade, is not true in at least three examples. Finally we show that the fusion rings of subfactors associated with the conformal inclusions \( SU(n)(n+2) \subset SU(n(n+1)/2) \) and \( SU(n+2) \subset SU((n+1)(n+2)/2) \) are canonically isomorphic using a version of level-rank duality.

§0. Introduction

Let \( G_k \subset H \) be a conformal inclusion with both \( G \) and \( H \) being semisimple compact Lie groups, and \( k \) the Dykin index of the inclusion (cf. [GNO] or [KW]). We shall use \( i \) (resp. \( \lambda \)) to denote the irreducible projective positive energy representation of loop group \( LH \) (resp. \( LG \)) at level 1 (resp. \( k \)) (cf. [PS]). Denote by \( b_{i\lambda} \) the branching coefficients, i.e., when restricting to \( LG \), \( i \) decomposes as \( \sum_{\lambda} b_{i\lambda} \lambda \). Denote by \( S_{ij} \) (resp. \( S_{\lambda\mu} \)) the S-matrices of \( LH \) (resp. \( LG \)) at level 1 (resp. \( k \)) (cf. [Kac2]). A general hypothesis of Kac-Wakimoto (cf. [KW]) in this case states that \( S_{\lambda\mu} S_{ij} \geq 0 \) whenever \( b_{i\lambda} b_{j\mu} \neq 0 \). This hypothesis has been checked to be correct in many cases.

Although there are explicit formula for \( S \) matrices (cf. [Kac2]), it is in general very difficult to calculate \( S \) matrices in the high level case. For this reason there is no general theoretical approach to Kac-Wakimoto hypothesis except case by case study.

In this paper we will argue that the general theory developed in [X1], which is motivated by subfactor theory, is particularly useful in studying the "Kac-Wakimoto..."
hypothesis type” question. We find three counter examples to Kac-Wakimoto hypothesis, and in the course of studying such questions, we also find answers to some of the questions naturally arised in [X1] and [X3].

Let us describe the content of this paper in more details.

In §2 we recall some of the material in [X1] to set up notations. In §2.1 we define sectors and what we mean by fusion ring and dual fusion ring associated with a sector. In §2.2 we recall the braided endomorphisms from conformal inclusions and summarize their properties in Th.2.2 and Th.2.3. Both theorems are proved in [X1].

§3 and §4 contain the main results of this paper which are proved by using the results stated in §2.

In §3, we first notice a curious inequality as the first application of results in [X1]. This inequality can be stated entirely in terms of S-matrices and branching coefficients, and it seems to be hard to prove it by other means. Prop.3.1 is a slightly generalized version of (1) of Th.3.9 in [X1] about the spectrum of certain ring which arises naturally in [X1]. Cor.3.2 shows that the framework of [X1] is almost tailor made to attack the Kac-Wakimoto hypothesis. Prop.3.3 allows one to obtain information about the non-commutativity of certain elements in a ring defined in [X1] from the branching rules.

Th.3.4 is the main result of §3 which follows from applying Cor.3.2 and Prop.3.3 to a special case of conformal inclusion $SU(3)_9 \subset E_6$. In (1) of Th.3.4 we show that certain ring $A_\rho$ which appears natural in [X1] is not generated by the descendants of the braided endomorphisms.

In (2) of Th.3.4 we show that the dual fusion ring of the Jones-Wassermann subfactor associated with $SU(3)_9 \subset E_6$ is non-commutative by exhibiting explicitly non-commutative relations. Notice it is in general very difficult to obtain detailed information about the dual fusion ring and this is the first example of conformal inclusions which shows non-commutative dual fusion ring.

In (3) of Th.3.4 we show the same conformal inclusion gives a counter example to the Kac-Wakimoto hypothesis. After discovering the counter example in (3) of Th.3.4 we find two more counter examples where $S$ matrices can be computed explicitly by elementary computations, and they are given at the end of §3.

In §4.1 we give a version of level-rank duality which follows naturally from [X1], and this is summarized in Th.4.1. The coincidence of certain fusion coefficients in the fusion rings associated with $SU(m)_n$ and $SU(n)_m$ are already noticed before, but Th.4.1 gives an embedding of these two fusion rings in one ring where they can be compared. By using Th.4.1, we show in Th.4.2 that the fusion rings of the Jones-Wassermann subfactors associated with the conformal inclusions $SU(n)(n+2) \subset SU(n(n+1)/2)$ and $SU(n+2)_n \subset SU((n+1)(n+2)/2)$ are canonically isomorphic. This fact was first noticed in an example in [X3].

In §5, we give our conclusions and questions.

2.1. Sectors. Let $M$ be a properly infinite factor and $\text{End}(M)$ the semigroup of the unit preserving endomorphism of $M$. In this paper $M$ will always be the unique hyperfinite $III_1$ factors. Let $\text{Sect}(M)$ denote the quotient of $\text{End}(M)$ modulo
unitary equivalence in \( M \). We shall denote by \([\rho]\) the image of \( \rho \in \text{End}(M) \) in \( \text{Sect}(M) \).

It follows from [L3] and [L4] that \( \text{Sect}(M) \), with \( M \) a properly infinite von Neumann algebra, is endowed with a natural involution \( \theta \to \overline{\theta} \) that commutes with all natural operations; moreover, \( \text{Sect}(M) \) is naturally a semiring.

Suppose \( \rho \in \text{End}(M) \) is given together with a normal faithful conditional expectation \( \epsilon : M \to \rho(M) \). We define a number \( d_\epsilon \) (possibly \( \infty \)) such that (cf. [PP]):
\[
d_\epsilon^{-2} := \max\{ \lambda \in [0, +\infty) | \epsilon(m_+) \geq \lambda m_+, \forall m_+ \in M_+ \}
\].

Now assume \( \rho \in \text{End}(M) \) is given together with a normal faithful conditional expectation \( \epsilon : M \to \rho(M) \), and assume \( d_\epsilon < +\infty \). We define
\[
d = \min\{d_\epsilon\}
\].

\( d \) is called the statistical dimension of \( \rho \). It is clear from the definition that the statistical dimension of \( \rho \) depends only on the unitary equivalence classes of \( \rho \). The properties of the statistical dimension can be found in [L1], [L3] and [L4].

Denote by \( \text{Sect}_0(M) \) those elements of \( \text{Sect}(M) \) with finite statistical dimensions. For \( \lambda, \mu \in \text{Sect}_0(M) \), let \( \text{Hom}(\lambda, \mu) \) denote the space of intertwiners from \( \lambda \) to \( \mu \), i.e. \( a \in \text{Hom}(\lambda, \mu) \) iff \( a\lambda(x) = \mu(x)a \) for any \( x \in M \). \( \text{Hom}(\lambda, \mu) \) is a finite dimensional vector space and we use \( \langle \lambda, \mu \rangle \) to denote the dimension of this space. \( \langle \lambda, \mu \rangle \) depends only on \([\lambda]\) and \([\mu]\). Moreover we have \( \langle \nu\lambda, \mu \rangle = \langle \nu, \mu\lambda \rangle \) which follows from Frobenius duality (see [L2] or [Y]). We will also use the following notations: if \( \mu \) is a subsector of \( \lambda \), we will write \( \mu \prec \lambda \) or \( \lambda \succ \mu \). A sector is said to be irreducible if it has only one subsector.

For an endomorphism \( \rho \in \text{End}(M) \), the fusion ring (resp. the dual fusion ring) associated with the inclusion \( \rho(M) \subset M \) is defined to be the ring generated by the irreducible descendants of \( \rho \bar{\rho} \) (resp. \( \bar{\rho}\rho \)). The origin of such notions comes from subfactor theory (cf. [DB]).

### 2.2. Braided endomorphisms from conformal inclusions.

In this paper, we shall restrict our attention to the following conformal inclusions (cf. [PZ] or [GNO]):

- \( SU(2)_{10} \subset Spin(5) \), \( SU(2)_{28} \subset G_2 \),
- \( SU(3)_5 \subset SU(6) \), \( SU(3)_9 \subset E_6 \), \( SU(3)_{21} \subset E_7 \);
- \( (A_8)_1 \subset (E_8) \);

and five infinite series:

\[
SU(N)_{N-2} \subset SU\left(\frac{N(N-1)}{2}\right), \quad N \geq 4; \quad (a)
\]

\[
SU(N)_{N+2} \subset SU\left(\frac{N(N+1)}{2}\right), \quad (b)
\]

\[
SU(N)_{2N} \subset Spin(4N^2-1), \quad N \geq 2; \quad (c)
\]

\[
SU(2N+1)_{2N+1} \subset Spin(4N(N+1)); \quad (d)
\]

\[
SU(M)_N \times SU(N)_M \subset SU(NM). \quad (e)
\]
These cover all the maximal conformal inclusions of the form $G = SU(N) \times SU(M) \subset H$ with $H$ being a simple simply connected group.

The aim of this section is to recall some of the results in [X1] which will be used in §3 and §4. For the proofs and unexplained terminology, we safely refer the reader to [X1]. For the representation theory of Loop groups, we refer the reader to [PS].

Denote by $\lambda$ a positive energy projective representation of $LG$ at the level (Dykin index given above). $\lambda$ is not necessarily irreducible but we assume that it is a finite direct sum of irreducible representations. By [W2], each $\lambda$ naturally gives a sector, denoted by the same letter $\lambda \in \text{Sect}(M)$, where $M$ is the unique hyperfinite $III_1$ factor. Such sectors generate a finite dimensional ring denoted by $Gr(C_k)$ with $k$ indicating the level. A basis of $Gr(C_k)$ is given by all the irreducible positive energy projective representation, denoted by $\mu_i$'s, of $LG$ of fixed level. The structure constants $N_{\mu_1\mu_2}^{\mu_3}$ are given by:

$$
\mu_1\mu_2 = \sum_{\mu_3} N_{\mu_1\mu_2}^{\mu_3} \mu_3
$$

, and it is known (cf. Cor.1 in Chapter 34 of [W2] and P.288 of [Kac2]) that $N_{\mu_1\mu_2}^{\mu_3}$ are determined uniquely by $S$-matrices of $LG$.

Let $\gamma_i := \sum_i b_{i\lambda} \lambda$. We shall use 1 to denote the vacuum representation of $LH$. It is shown (cf. (1) of Prop.2.8 in [X1]) that there are sectors $\rho, \sigma_i \in \text{Sect}(M)$ such that:

$$
\rho \sigma_i \bar{\rho} = \gamma_i
$$

. Notice that $\sigma_i$ are in one-to-one correspondence with the irreducible representations $i$ of $LH$ and they are irreducible sectors, generating a finite dimensional ring. The structure constants $N_{ij}^k$ are given by :

$$
\sigma_i \sigma_j = \sum_k N_{ij}^k \sigma_k
$$

, and $N_{ij}^k$ are uniquely determined the $S$ matrices of $LH$ (cf. the paragraph after Th.1.6. in [X1]).

The subfactor $\rho(M) \subset M$ is called the Jones-Wassermann subfactor associated with conformal inclusions. For more detailed discussions on the Jones-Wassermann subfactor, we refer the reader to [X1].

The crucial observation in [X1] is the following: for each $\lambda$, there exists a sector denoted$^1$ by $a_{\lambda}$, such that the following theorem is true (cf. Th.3.1, Cor.3.2 and Th.3.3 of [X1]):

**Theorem 2.2.** (1) The map $\lambda \to a_{\lambda}$ is a ring homomorphism;

(2) $\rho a_{\lambda} = \lambda \rho, a_{\lambda} \bar{\rho} = \bar{\rho} \lambda$;


---

$^1$In the notations of [X1] it should be denoted by $[a_{\lambda}]$ to emphasize that it is a sector rather than an endomorphism, but we omit $[,]$ for simplicity.
\((3)\). \(\langle \rho a_\lambda, \rho a_\mu \rangle = \langle a_\lambda, a_\mu \rangle = \langle a_\lambda \bar{\rho}, a_\mu \bar{\rho} \rangle;\)
\((4)\). \(\langle \rho a_\lambda, \rho \sigma_i \rangle = \langle a_\lambda, \sigma_i \rangle = \langle a_\lambda \bar{\rho}, \sigma_i \bar{\rho} \rangle;\)
\((5)\). \((3)\) (resp. \((4)\)) remains valid if \(a_\lambda, a_\mu\) (resp. \(a_\lambda\)) is replaced by any of its subsectors.

Proof. (1) to (4) follows from Th.3.1, Cor.3.2 and Th.3.3 of [X1], so we just have to show (5). Let \(b\) (resp. \(c\)) be subsectors of \(a_\lambda\) (resp. \(a_\mu\)), then we have:
\[\langle \rho b, \rho c \rangle \geq \langle b, c \rangle\]
But if:
\[\langle \rho b, \rho c \rangle > \langle b, c \rangle\]
we would have
\[\langle \rho a_\lambda, \rho a_\mu \rangle > \langle a_\lambda, a_\mu \rangle\]
contradicting (3). The remaining cases are proved similarly.

Q.E.D.

Perhaps the most surprising part is (3) and (4) of Th.2.2. Notice one obviously has \(\langle \rho a_\lambda, \rho a_\mu \rangle \geq \langle a_\lambda, a_\mu \rangle\), the nontrivial part is the equality which is proved in [X1] by locality considerations of [LR]. Combined with Frobenius duality, Th.3.3 puts a powerful constraint on the subsectors of \(a_\lambda\).

To make a connection with the branching coefficients introduced in §1, we have:
\[\langle a_\lambda, \sigma_j \rangle = \langle a_\lambda \bar{\rho}, \sigma_j \bar{\rho} \rangle = \langle \bar{\rho} \lambda, \sigma_j \bar{\rho} \rangle = \langle \lambda, \rho \sigma_j \bar{\rho} \rangle = \langle \lambda, \gamma_j \rangle = b_{j\lambda}\]
where we have used (3) and (2) of Th.2.2 in the first and the second step, Frobenius duality in the third step, the equation \(\rho \sigma_j \bar{\rho} = \gamma_j\) in the fourth step and definitions of \(\gamma_j\) in the last step.

It is shown (cf. §3.4 [X1]) there is another sector \(\tilde{a}_\lambda\) with exactly the same properties as \(a_\lambda\) in Th.2.4. It is associated to the choice of under-crossing of braiding in the definition and it is known that \(\tilde{a}_\lambda\) is in general different from \(a_\lambda\) (cf. Lemma 3.2 of [X1]).

Define \(C_\rho\) (resp. \(\tilde{C}_\rho\)) to be the complex finite dimensional ring generated by the irreducible subsectors of \(a_\lambda\) (resp. \(\tilde{a}_\lambda\)) for all \(\lambda\) of fixed level. The paring \(\langle,\rangle\) introduced in §2.1 extends by linearly in the first variable and conjugate linearly in the second variable to a positive definite form on \(C_\rho\). Notice the conjugation \(b \rightarrow \bar{b}\) in §2.1 extends conjugate linearly to the elements in \(C_\rho\) such that the Frobenius duality holds.

Define \(A_\rho\) to be the finite dimensional ring generated by the irreducible subsectors of \(\bar{\rho} \lambda \rho\) for all \(\lambda\) of fixed level. Notice that \(C_\rho \subset A_\rho\). The commutativity of certain elements in these rings is investigated and they are summarized in the following theorem (cf. Th.3.6, Lemma 3.3 of [X1])
Theorem 2.3. Let $b$ be any subsector of $A(a_{\mu})$ where $A$ is an arbitrary polynomial in $a_{\mu}$, $\mu \in Gr(C_k)$, then $a_{\lambda}b = ba_{\lambda}$ for any $\lambda \in Gr(C_k)$;

(2) Let $c$ be any subsector of $\bar{p}p$, then $a_{\lambda}c = ca_{\lambda}$ for any $\lambda \in Gr(C_k)$;

(3) Let $x$ and $y$ be subsectors of $\bar{a}_{\lambda}$ and $a_{\mu}$ respectively. Then $xy = yx$.

§3. AN EXAMPLE OF CONFORMAL INCLUSION

We preserve the set up of §2.2. We shall denote the set of irreducible sectors of $a_{\lambda}$ for all $\lambda$ of fixed level by $V$. Notice $\sigma_i \in V$, and these are refered to as ”special nodes” in §3.4 of [X1]. Let:

$$a_{\lambda}a = \sum_{b \in V} V_{ab}^\lambda b$$

, where $V_{ab}^\lambda$ are nonnegative integers. Denote by $V^\lambda$ the matrix such that $(V^\lambda)_a^b = V_{ab}^\lambda$. By (1) of Th.2.2 $V_{a\lambda}^{\mu_1}V_{a\sigma_i}^{\mu_2} = \sum_{\mu_3} N_{\mu_1\mu_2}^{\mu_3} V_{a\sigma_i}^{\mu_3}$, so we have:

$$\sum_{\mu_3} N_{\mu_1\mu_2}^{\mu_3} V_{a\sigma_i}^{\mu_3} = \sum_{a} V_{1a}^{\mu_1} V_{a\sigma_i}^{\mu_2} \geq \sum_{j} V_{1a}^{\mu_1} V_{1\sigma_j}^{\mu_2}$$

, where 1 denotes the identity sector. Recall that $V_{1\sigma_i}^{\mu_3} = b_{i\mu_3}$, $V_{1\sigma_j}^{\mu_3} = b_{j\mu_1}$, and

$$V_{\sigma_j\sigma_i}^{\mu_2} = \langle a_{\mu_2}\sigma_j, \sigma_i \rangle = \langle a_{\mu_2}, \sigma_j\sigma_i \rangle$$

$$= \langle a_{\mu_2}, \sum_k N_k^{\mu_2} \bar{a}_{\sigma_j}\sigma_k \rangle$$

$$= \sum_k N_k^{\mu_2} \langle a_{\mu_2}, \sigma_k \rangle$$

$$= \sum_k N_k^{\mu_2} b_{k\mu_2}$$

Putting the above together, we have derived the following inequality:

$$\sum_{\mu_3} N_{\mu_1\mu_2}^{\mu_3} b_{i\mu_3} \geq \sum_{k,j} N_k^{\mu_2} b_{k\mu_2} b_{j\mu_1}$$

Since $N_{\mu_1\mu_2}^{\mu_3}$ (resp.$N_k^{\mu_2}$) are determined uniquely by S-matrices of $LG$ (resp. $LH$) by so called Verlinde formula, the above inequality is a relation between branching rules and S-matrices, and we don’t know if one can prove it by the usual representation theoretical approach, i.e., by using the results of [KW] and [Kac2].

Define matrix $N_c$ by $N_c^{ab} = \langle ca, b \rangle$ for $a, b, c \in V$. Then $V^\lambda = \sum_c V_{1c}^\lambda N_c$. Since $[a_{\lambda}] = [a_{\lambda}], [\sigma_j a_{\lambda}] = [a_{\lambda}\sigma_j], V^\lambda, N_{\sigma_j}$ are commuting normal matrices, so they can be simultaneously diagonalized. Recall the irreducible representations of $Gr(C_k)$ are given by

$$\lambda \to \frac{S_{\lambda \mu}}{S_{1\mu}}.$$
Assume $V^\lambda_{ab} = \sum_{\mu,i,s \in (Exp)} \frac{S_{\mu \lambda}}{S_{1 \mu}} \cdot \psi_{a}^{(\mu,i,s)} \psi_{b}^{(\mu,i,s)^*}$ where $\psi_{a}^{(\mu,i,s)}$ are normalized orthogonal eigenvectors of $V^\lambda$ (resp. $N_{\sigma i}$) with eigenvalue $\frac{S_{\mu \lambda}}{S_{1 \mu}}$ (resp. $\frac{S_{\sigma i}}{S_{1 i}}$). $(Exp)$ is a set of $\mu, i, s$’s and $s$ is an index indicating the multiplicity of $\mu, i$. Recall if a representation is denoted by 1, it will always be the vacuum representation.

**Proposition 3.1.** $(\delta, k, s) \in (Exp)$ if and only if $b_{k \delta} > 0$. Moreover, there is a choice of eigenvectors such that $\psi_{1}^{(\delta, k, s)} > 0$ for any $(\delta, k, s) \in (Exp)$.

**Proof:** Since $b_{j \lambda} = V_{1 \sigma j, \lambda}$, we have:

$$b_{j \lambda} = \sum_{(\mu, i, s) \in (Exp)} \frac{S_{\mu \lambda} S_{ji}}{S_{1 \mu} S_{1 i}} |\psi_{1}^{(\mu,i,s)}|^2$$

, where we have also used

$$\psi_{1}^{(\mu,i,s)} = \frac{S_{\sigma i}}{S_{1 i}} \psi_{1}^{(\mu,i,s)}$$

(cf. (4) of Th.3.9 in [X1]). By using the following equation (cf. [KW]):

$$\sum_{j} S_{kj} b_{j \lambda} = \sum_{\mu} b_{k \mu} S_{\mu \lambda}$$

we obtain:

$$\sum_{(\mu, s) \in Exp(k)} \frac{S_{\mu \lambda}}{S_{1 \mu} S_{1 k}} \frac{1}{S_{1 k}} |\psi_{1}^{(\mu,k,s)}|^2 = \sum_{\mu} b_{k \mu} S_{\mu \lambda}$$

, where $(\mu, s) \in Exp(k)$ means $k$ is fixed and $(\mu, k, s) \in (Exp)$ . Multiply both sides by $S_{\lambda \delta}$ and summed over $\lambda$, using the fact that $S$ matrices are unitary (cf. [Kac2]), we get:

$$\sum_{s \in Exp(\delta, k)} \frac{1}{S_{1 \delta} S_{1 k}} |\psi_{1}^{(\delta,k,s)}|^2 = b_{k \delta}$$

(1)

, where $s \in Exp(\delta, k)$ means $\delta, k$ are fixed and $(\delta, k, s) \in (Exp)$. It follows immediately that if $b_{k \delta} > 0$, then $(k, \delta, s) \in (Exp)$ for some $s$.

Let $(\delta, k, 1), \ldots (\delta, k, p)$ be the subset in $(Exp)$ with $(\delta, k)$ fixed. By Th.2.2 and the comments after it $\sigma_i$ appears as subsectors of some $\bar{a}_\mu$, so by Th.2.3 $\bar{a}$ commutes with $a_\lambda$ and $\sigma_i$, therefore $\bar{a}$ preserves the subspace spanned by vectors $\psi_{1}^{(\delta,k,1)}, \ldots \psi_{1}^{(\delta,k,p)}$. Note for any $a \in V$, we have:

$$\psi_{a}^{(\delta,k,s)} = \langle \psi_{a}^{(\delta,k,s)}, a \rangle = \langle N_{\bar{a}} \psi_{1}^{(\delta,k,s)}, 1 \rangle$$

$$= \sum_{t} N_{a s}^{t} \psi_{1}^{(\delta,k,t)}$$

where $N_{a s}^{t} = \langle N_{\bar{a}} \psi_{1}^{(\delta,k,s)}, \psi_{1}^{(\delta,k,t)} \rangle$. 


It follows that if $(\delta, k, s) \in (Exp)$, then $\psi_a^{(\delta, k, s)} \neq 0$ for some $a \in V$ which implies $\psi_1^{(\delta, k, t)} \neq 0$ for some $t$. By equation (1) this implies $b_{k\delta} > 0$.

Let $(\delta, k, 1), \ldots, (\delta, k, p)$ be the subset in $(Exp)$ with $(\delta, k)$ fixed. It follows from (1) that we can always make a gauge choice such that $\psi_1^{(\delta, k, 1)} = \ldots = \psi_1^{(\delta, k, p)} > 0$. Q.E.D.

We remark that the existence of the choice of eigenvectors in Prop.3.1 is postulated as (2) of ix in [PZ].

Recall the Kac-Wakimoto hypothesis as stated in the beginning of the introduction. It follows immediately from Prop.3.1. that:

**Corollary 3.2.** Kac-Wakimoto hypothesis is true if and only if for any $\lambda$ with $b_{j\lambda} \neq 0$, $V^\lambda N_{\sigma j}$ is semi-positive definite, i.e. $\langle a_\lambda \sigma_j x, x \rangle \geq 0$ for any $x \in C_\rho$.  

Cor.3.2 is very effective in verifying that Kac-Wakimoto hypothesis is true in many cases. The strategy is to write $a_\lambda \sigma_j = \sum b \bar{b}$ by using the information on the ring structure of $C_\rho$ which can be effectively determined by Th.2.2 in many cases (cf. examples in [X1]). We have done so in examples of §4.1 and example 1 and 4 in [X1], and series (e) of §2.2.

In trying to prove Kac-Wakimoto hypothesis by using Cor.3.2, we find the following proposition:

**Proposition 3.3.** (1) If $\sigma_j \bar{\rho} \rho = \bar{\rho} \rho \sigma_j$, then $a_{\gamma_j} = a_{\gamma_1} \sigma_j$;

(2) $a_{\gamma_j} = a_{\gamma_1} \sigma_j$ for all $j$ if and only if the following holds: if for any $i, \mu$ with $b_{i\mu} > 0$, we have $b_{j\mu} = 0$ for $j \neq i$.

**Proof:** (1): It is sufficient to show that

$$\langle a_{\gamma_j}, b \rangle = \langle a_{\gamma_1} \sigma_j, b \rangle$$

for any $b \in V$. We have:

$$\langle a_{\gamma_j}, b \rangle = \langle a_{\gamma_j} \bar{\rho}, b \bar{\rho} \rangle$$

$$= \langle a_{\gamma_j} \bar{\rho} \rho, b \rangle$$

$$= \langle \rho \gamma_j \rho, b \rangle$$

$$= \langle \bar{\rho} \rho \sigma_j \bar{\rho}, b \rangle$$

where we have used (5) of Th.2.2 in the first step, Frobenius duality in the second step and (2) of Th.2.2 in the third step and the identity at the beginning of §3 in the last step.

On the other hand, we have:

$$\langle a_{\gamma} \sigma_j, b \rangle = \langle a_{\gamma} \sigma_j \bar{\rho}, b \bar{\rho} \rangle$$

$$= \langle a_{\gamma} \sigma_j \bar{\rho} \rho, b \rangle$$

$$= \langle \sigma_j a_{\gamma} \rho, b \rangle$$

$$= \langle \sigma_j \bar{\rho} \rho \bar{\rho}, b \rangle$$

$$= \langle \bar{\rho} \rho \sigma_j \bar{\rho}, b \rangle$$
where the intermediate steps are similar to the previous one and in the last step we have used the condition $\sigma_j \bar{\rho} = \bar{\rho} \rho \sigma_j$.

(2) Notice that $a_{\gamma_j} = a_{\gamma_1} \sigma_j$ iff $V^{\gamma_j} = V^{\gamma} N_{\sigma_j}$. It follows from Prop.3.1 that: $a_{\gamma_j} = a_{\gamma_1} \sigma_j$ for all $j$ iff for any $i, \mu$ with $b_{i\mu} \neq 0$, we have

$$\sum_{\lambda} b_{j\lambda} S_{\lambda \mu} = \sum_{\lambda} b_{1\lambda} S_{\lambda \mu} S_{ji} S_{1i} \quad (2)$$

. Use the fact $\sum_{\lambda} b_{j\lambda} S_{\lambda \mu} = \sum_{k} S_{jk} b_{k\mu}$, (2) is equivalent to:

$$\sum_{k} S_{jk} b_{k\mu} = \sum_{k} S_{1k} b_{k\mu} \frac{S_{ji}}{S_{1i}}$$

. Multiply both sides of the above equation by $\bar{S}_{jl}$ and summed over $j$, it is easy to see that (2) is equivalent to the statement that $b_{l\mu} = 0$ for any $l \neq i$. Q.E.D.

Now we are ready to apply Cor.3.2 and Prop.3.3 to an example, i.e., the conformal inclusion $SU(3)_9 \subset E_6$ (cf. example 2 in §4 of [X1]). Denote by $H_0, H_1, H_2$ the level 1 irreducible representations of $LE_6$ ($H_1$ is the vacuum representation) and label the dominant weights of $SU(3)$ by $(\lambda_1, \lambda_2)$ with $\lambda_1 \geq \lambda_2 \geq 0$, we have the following decompositions:

$$H_0 = H_{(4,2)} + H_{(7,2)} + H_{(7,5)}$$
$$H_2 = H_{(4,2)} + H_{(7,2)} + H_{(7,5)}$$
$$H_1 = H_{(0,0)} + H_{(9,0)} + H_{(9,9)} + H_{(8,4)} + H_{(5,1)} + H_{(5,4)}$$

. By using similar calculations as in example 2 in §4 of [X1] and the computation of statistical dimensions, one easily obtains the following identities:

$$a^2_{(2,1)} = 6a_{(2,1)} + \sigma_0 + \sigma_1 + \sigma_2 \quad (3)$$
$$a_{(4,2)} = 2a_{(2,1)} + \sigma_0 + \sigma_2 \quad (4)$$
$$a_{(5,1)} = 2a_{(2,1)} + \sigma_1 \quad (5)$$

. In fact, $d_{a_{(2,1)}} = 3 + 2\sqrt{3}, d_{a_{(4,2)}} = 8 + 4\sqrt{3}$ and the above identities are proved by first showing that the left side contains the right side using Th.2.2 and then by comparing the statistical dimensions. Now we are ready to prove the following theorem:

**Theorem 3.4.** For the conformal inclusion $SU(3)_9 \subset E_6$, we have:

(1) The ring $A_\rho$ is not generated by the subsectors of $a_\lambda$ and $\tilde{a}_\lambda$ for any dominant weights $\lambda$ of $SU(3)$ at level 9;

(2) The dual fusion ring of the Jones-Wassermann subfactor associated with $SU(3)_9 \subset E_6$ is not commutative;

(3) The Kac-Wakimoto hypothesis is not true.

**Proof:** (1). It follows immediately from the decompositions given above
and Prop.3.3 that we must have:

\[ \sigma_j \bar{\rho} \rho \neq \bar{\rho} \sigma_j \]

for some \( \sigma_j \). Notice that \( \sigma_j \) are irreducible subsector of both \( a_\mu \) and \( \tilde{a}_\mu \) for some \( \lambda \) by (4) of Th.2.2. It follows from (3) of Th.2.3 that \( \sigma_j \) commutes with all the descendants of \( a_\lambda \) and \( \tilde{a}_\lambda \) for any dominant weights \( \lambda \) of \( SU(3) \) at level 9. Therefore \( \bar{\rho} \rho \in A_\rho \) must contain some irreducible sector which don’t appear as the descendants of \( a_\lambda \) and \( \tilde{a}_\lambda \) for any dominant weights \( \lambda \) of \( SU(3) \) at level 9.

(2). Notice that \( \bar{\rho} \rho \bar{\rho} = \hat{\rho} a_\gamma \succ a_\gamma \succ a_{(5,1)} \), remember that the dual fusion ring is generated by the irreducible subsectors of \( \bar{\rho} \rho \), so \( a_{(5,1)} \) appears as the subsectors in the dual fusion ring. It follows from equations (3) and (5) that all \( \sigma_j \) appear as irreducible subsectors of the dual fusion ring. Since as shown in (1) that

\[ \sigma_j \bar{\rho} \rho \neq \bar{\rho} \sigma_j \]

for some \( \sigma_j \), it follows that the dual fusion ring is not commutative;

(3). By using equation (4) and \( \sigma_0 \sigma_2 = \sigma_1 \) we get:

\[ \sigma_0 a_{(4,2)} = 2a_{(2,1)} + \sigma_2 + \sigma_1 \]

But \( a_{(2,1)} = a_{(2,1)}, \sigma_1 = \sigma_1, \sigma_2 = \sigma_0 \), it follows that the matrix \( V^{(4,2)} N_{\sigma_0} \) is not even Hermitian. By Cor.3.2, this implies that Kac-Wakimoto hypothesis is not true in our example.

Q.E.D.

Despite the fact that (3) of Th.3.4 is the first counter-example we find by using Cor.3.2, it is not easy to exhibit the S-matrix explicitly. We then find the following two examples (cf. Ex. 0 and 5 in [X1]) where the S-matrix are much simpler and can be computed explicitly. However, the computation of S-matrix is not the way we find such examples. What we did is to compute the ring structure as in the proof of (3) in Th.3.4 which is much simpler (cf. Ex. 0 and 5 in [X1]) in the next two examples, and by using Cor.3.2. We shall give only the S-matrix in the next two examples.

Example 2: (cf. Ex.0 in [X1]) Take the conformal inclusion \( SU(3)_3 \subset SO(8) \). Let us label the weights of \( SU(3) \) as \( (\lambda_1, \lambda_2) \) with \( \lambda_1 \geq \lambda_2 \). The convention is that \((0,0)\) label the vacuum or the trivial representation. Let \( a = (2,1) \). Let \( v \) denote the vector representation of SO(8). By simple calculations one finds:

\[ S_{aa} = -1/2, S_{vv} = 1/2. \]

So \( S_{aa} S_{vv} = -1/4 < 0 \). But it is easy to see that \( b_{av} = 1 \).

Example 3: (cf. Ex.5 in [X1]) Take the conformal inclusion \( SU(4)_2 \subset SU(6) \). The level 1 weights of \( SU(6) \) are in one-to-one correspondence with \( \mathbb{Z}_6 = \{ \omega^i, i = 0,1,...5 \} \). Let \( \mu = (1,1,0) \). One checks easily that \( S_{\mu \mu} = 1/\sqrt{6}, S_{\omega \omega} = 1/\sqrt{6} \exp(2\pi i/6) \).

It follows that \( S_{\mu \mu} \bar{S}_{\omega \omega} = 1/6 \exp(2\pi i/6) \). But \( b_{\omega \mu} = 1 \). So this is another counter example.

The example in Th.3.4 appears in [Kac3] where the authors claim that the Kac-Wakimoto hypothesis has been checked to be true. Our theorem shows that this
is not the case. However, this doesn’t affect the main purposes of [Kac3] which is the computation of the branching coefficients. What we have proved is that the algorithm in [Kac3] is not true in general. But in specific examples, including all three counter examples above, the branching coefficients are easily determined by considerations of [KW].

§4. Level-Rank duality

Level-Rank duality has been explained by different methods in [GW], [Sa] and [Ts]. The one that is close in spirit to our approach is [Ts].

4.1. A conformal inclusion. We shall be interested in the following conformal inclusion:

\[ L(SU(m)_n \times SU(n)_m) \subset L SU(nm) \]

In the classification of conformal inclusions in [GNO], the above conformal inclusion corresponds to the Grassmanian \( SU(m+n)/SU(n) \times SU(m) \times U(1) \).

Let \( \Lambda_0 \) be the vacuum representation of \( L SU(nm) \) on Hilbert space \( H^0 \). The decomposition of \( \Lambda_0 \) under \( L(SU(m) \times SU(n)) \) is known, see, e.g. [Itz]. To describe such a decomposition, let us prepare some notation. We shall use \( \dot{S} \) to denote the \( S \)-matrices of \( SU(m) \), and \( \ddot{S} \) to denote the \( S \)-matrices of \( SU(n) \). The level \( n \) (resp. \( m \)) weight of \( L SU(m) \) (resp. \( L SU(n) \)) will be denoted by \( \dot{\lambda} \) (resp. \( \ddot{\lambda} \)).

We start by describing \( \dot{P}^n_+ \) (resp. \( \ddot{P}^m_+ \)), i.e. the highest weights of level \( n \) of \( L SU(m) \) (resp. level \( m \) of \( L SU(n) \)).

\( \dot{P}^n_+ \) is the set of weights

\[ \dot{\lambda} = \tilde{k}_0 \dot{\Lambda}_0 + \tilde{k}_1 \dot{\Lambda}_1 + \cdots + \tilde{k}_{m-1} \dot{\Lambda}_{m-1} \]

where \( \tilde{k}_i \) are non-negative integers such that

\[ \sum_{i=0}^{m-1} \tilde{k}_i = n \]

and \( \dot{\Lambda}_i = \dot{\Lambda}_0 + \dot{\omega}_i \), \( 1 \leq i \leq m-1 \), where \( \dot{\omega}_i \) are the fundamental weights of \( SU(m) \).

Instead of \( \dot{\lambda} \) it will be more convenient to use

\[ \dot{\lambda} + \dot{\rho} = \sum_{i=0}^{m-1} k_i \dot{\Lambda}_i \]

with \( k_i = \tilde{k}_i + 1 \) and \( \sum_{i=0}^{m-1} k_i = m + n \). Due to the cyclic symmetry of the extended Dykin diagram of \( SU(m) \), the group \( \mathbb{Z}_m \) acts on \( \dot{P}^n_+ \) by

\[ \dot{\Lambda}_i \rightarrow \dot{\Lambda}_{(i+\dot{\mu}) \mod m}, \quad \dot{\mu} \in \mathbb{Z}_m. \]
Let $\Omega_{m,n} = \hat{P}^n_+ / \mathbb{Z}_m$. Then there is a natural bijection between $\Omega_{m,n}$ and $\Omega_{n,m}$ (see §2 of [Itz]). The proof given in [Itz] is very clear and let us repeat it here since it will be important later on. The idea is to draw a circle and divide it into equal length. To each partition $\sum_{0 \leq i \leq n-1} k_i = m + n$ there corresponds a ”slicing of the pie” into $m$ successive parts with angles $2\pi k_i/(m+n)$, drawn with solid lines. We choose this slicing to be clockwise. The complementary slicing in broken lines (The lines which are not solid) defines a partition of $m$ into $m$ (The lines which are not solid) defines a partition of $m$ into $n$ successive parts, $\sum_{0 \leq i \leq n-1} t_i = m + n$. We choose the later slicing to be counterclockwise, and it is easy to see that such a slicing corresponds uniquely to an element of $\Omega_{n,m}$.

We shall parameterize the bijection by a map

$$\beta : \hat{P}^n_+ \to \hat{P}^m_+$$

as follows. Set

$$r_j = \sum_{i=j}^{m} k_i, \quad 1 \leq j \leq m$$

where $k_m \equiv k_0$. The sequence $(r_1, \ldots, r_m)$ is decreasing, $m + n = r_1 > r_2 > \cdots > r_m \geq 1$. Take the complementary sequence $(\bar{r}_1, \bar{r}_2, \ldots, \bar{r}_n)$ in $\{1, 2, \ldots, m+n\}$ with $\bar{r}_1 > \bar{r}_2 > \cdots > \bar{r}_n$. Put

$$S_j = m + n + \bar{r}_n - \bar{r}_{n-j+1}, \quad 1 \leq j \leq n.$$ 

Then $m + n = s_1 > s_2 > \cdots > s_n \geq 1$. The map $\beta$ is defined by

$$(r_1, \ldots, r_m) \to (s_1, \ldots, s_n).$$

The following lemma summarizes what we will use. For the proof, see Th.1 of [Itz].

**Lemma 4.1.** Let $\hat{Q}$ be the root lattice of $SU(m)$, $\hat{\Lambda}_i$, $0 \leq i \leq m-1$ its fundamental weights and $\hat{Q}_i = (\hat{Q} + \hat{\Lambda}_i) \cap \hat{P}^n_+$. Let $\Lambda \in \mathbb{Z}_{mn}$ denote a level 1 highest weight of $SU(mn)$ and $\hat{\lambda} \in \hat{Q}_{\Lambda_{\text{mod}mn}^n}$. Then there exists a unique $\hat{\lambda} \hat{\beta}(\hat{\lambda})$ for some unique $\mu \in \mathbb{Z}_n$ such that $H_{\hat{\Lambda}} \otimes H_{\hat{\lambda}}$ appears once and only once in $H^\Lambda$. The map $\hat{\lambda} \to \hat{\lambda} = \mu \hat{\beta}(\hat{\lambda})$ is one-to-one. Moreover, $H^\Lambda$, as representations of $L(SU(m) \times SU(n))$, is a direct sum of all such $H_{\hat{\lambda}} \otimes H_{\hat{\lambda}}$.

**Proof.** By Th.1 of [Itz], only the fact that the map $\hat{\lambda} \to \hat{\lambda} = \mu \hat{\beta}(\hat{\lambda})$ is one-to-one needs to be proved. It follows by the proof of Th.1 in [Itz] and the complete symmetry between $\hat{\lambda}$ and $\hat{\lambda}$ that Th.1 of [Itz] remains true with $\hat{\lambda}$ and $\hat{\lambda}$ exchanged. This implies the bijection. Q.E.D.

We shall denote by $b(\Lambda, \hat{\lambda} \otimes \hat{\lambda})$ the multiplicity of $H_{\hat{\lambda}} \otimes H_{\hat{\lambda}}$ in $H^\Lambda$ and $\Lambda_0$ the vacuum representation. Let $\rho$ be the sector such that $\rho(\hat{M}) \subset \hat{M}$ (M is the unique hyperfinite III$_1$ factor) is the Jones-Wassermann subfactor associated with the conformal inclusion $L(SU(m) \times SU(n))_m \subset L SU(nm)$. Then $\rho \bar{\rho} = \gamma_{\Lambda_0} := \sum \lambda b(\Lambda_0, \hat{\lambda} \otimes \hat{\lambda}) \hat{\lambda} \otimes \hat{\lambda}$, where $\hat{\lambda} \otimes \hat{\lambda}$ is the sector corresponding to the representation $H_{\hat{\lambda}} H_{\hat{\lambda}}$ of $L(SU(m) \times SU(n))$. As before, we shall reserve 1 to denote the identity or the vacuum sector. We will be interested in the ring homomorphism $\hat{\lambda} \otimes \hat{\lambda} \to a_{\hat{\lambda} \otimes \hat{\lambda}}$. The proof of Theorem 2.2, 2.3 applies to the present case without modifications and we have:
Theorem 4.1. (1). The ring homomorphisms \( \hat{\lambda} \to a_{\lambda \otimes 1}, \bar{\lambda} \to a_{1 \otimes \bar{\lambda}} \) are embeddings;

(2). We have \( a_{\hat{\lambda} \otimes 1} = \sigma_\Lambda a_{1 \otimes \bar{\lambda}} \) where \( \Lambda \) and \( \hat{\lambda} = \mu \beta(\bar{\lambda}) \) are as in Lemma 4.1.

Proof. (1). It is sufficient to show that \( a_{\hat{\lambda} \otimes 1}, a_{1 \otimes \bar{\lambda}} \) are irreducible sectors. By (2) of Th.2.2 and Frobenius duality we have:

\[
\langle a_{\hat{\lambda} \otimes 1}, a_{\hat{\lambda} \otimes 1} \rangle \leq \langle a_{\hat{\lambda} \otimes 1}, a_{\hat{\lambda} \otimes 1} \rangle = \langle \hat{\rho} \hat{\lambda} \otimes 1, \hat{\rho} \hat{\lambda} \otimes 1 \rangle = \langle \rho \rho, \hat{\lambda} \hat{\lambda} \otimes 1 \rangle = 1
\]

where in the last step we have used the fact that \( b(\Lambda_0, \hat{\lambda}, 1) = 1 \) iff \( \hat{\lambda} = 1 \) which follows from Lemma 4.1. The proof that \( a_{1 \otimes \bar{\lambda}} \) is irreducible is similar.

(2). By a similar proof as in (1) we have that both \( a_{\hat{\lambda} \otimes 1} \) and \( \sigma_\Lambda a_{1 \otimes \bar{\lambda}} \) are irreducible sectors. So to prove (2) it is sufficient to show \( \langle a_{\hat{\lambda} \otimes 1}, \sigma_\Lambda a_{1 \otimes \bar{\lambda}} \rangle = 1 \). By (3), (4) of Th.2.2 and Frobenius duality we have:

\[
\langle a_{\hat{\lambda} \otimes 1}, \sigma_\Lambda a_{1 \otimes \bar{\lambda}} \rangle = \langle a_{\hat{\lambda} \otimes 1} a_{1 \otimes \bar{\lambda}}, \sigma_\Lambda \rangle = \langle a_{\hat{\lambda} \otimes \bar{\lambda}}, \sigma_\Lambda \rangle = \langle \hat{\lambda} \hat{\lambda}, \gamma_\Lambda \rangle = 1
\]

where in the last step we have used Lemma 4.1.

Q.E.D.

For another application of the conformal inclusions considered in this section, see [X4].

4.2. Two series of subfactors. Let \( \hat{\rho} \) (resp. \( \bar{\rho} \)) be the sectors corresponding to the Jones-Wassermann subfactors associated with the conformal inclusions \( SU((n+2)_n \subset SU((n+1)(n+2))/2) \) (resp. \( SU((n)_n \subset SU((n+1)/2)) \)). It is observed in [X3] that in the case \( n = 3 \), the two subfactors are closely related. The aim of this section is to show that this is true in general.

We shall be using the setup of §4.1 with \( m = n + 2 \). Let \( \hat{\rho} \hat{\rho} = \hat{\gamma} \) (resp. \( \bar{\rho} \bar{\rho} = \bar{\gamma} \)). Then \( \hat{\gamma} = \sum \hat{\lambda} b(\Lambda'_0, \hat{\lambda}) \hat{\lambda} \) (resp. \( \bar{\gamma} = \sum \bar{\lambda} b(\Lambda'_0, \bar{\lambda}) \bar{\lambda} \)), where \( b(\Lambda'_0, \hat{\lambda}) \) (resp. \( b(\Lambda'_0, \bar{\lambda}) \)) are branching coefficients and \( \Lambda'_0 \) (resp. \( \Lambda'_0 \)) is the vacuum representation of \( SU((n+1)(n+2))/2 \) (resp. \( SU((n+1)/2)) \). Fortunately the branching coefficients are worked out in [LL]. The result is the following (cf. Th.1.2 and Th.2.1 of [LL]): Define the group \( (\mathbb{Z}_2)^m_{\text{even}} := \{ (s_{11}, ..., s_{1m}) \in (\mathbb{Z}_2)^m : \# \{ i, s_{1i} = -1 \} \in 2\mathbb{N} \} \). Define \( c(s_1) := \sum_i s_{1i} = 1 \). Let \( \mu_1 \) (resp. \( \mu_2 \)) be the generator of the center of \( SU(m) \) (resp. \( SU(n) \)). Let \( \{ a_1, ..., a_m \} = \{ s_{1i}(m-i), i = 1, ..., m \} \) with \( a_1, ..., a_m \) in decreasing order. Define a weight\(^2\) \( \sigma_{s_1}(\rho_1) := (2m-2 + a_m - a_1) \hat{\lambda}_0 + \ldots \)

\( ^2 \)This is to be compared to the similar expressions in §2 of [LL]. The weights in this paper differ from the weights of [LL] by the Weyl vector according to our convention in §4.1.
\[ \sum_{1 \leq i \leq m-1} (a_i - a_{i+1}) \hat{\gamma}_i. \] Similarly define \( s_2 \in (\mathbb{Z}_2)^n \) to be \( s_2 = (s_{21}, \ldots, s_{2n}) \). Let \( \{b_1, \ldots, b_n\} = \{s_{2i} + (n - 1 - i), i = 1, \ldots, n\} \) with \( b_1, \ldots, b_n \) in decreasing order. Define a weight \( \sigma_{s_2} (\rho_1) := (2n + 2 + b_n - b_1) \Lambda_0 + \sum_{1 \leq i \leq n-1} (b_i - b_{i+1}) \hat{\gamma}_i. \)

Then it follows from Th.2.1 [LL] that \( b(\Lambda'_0, \lambda) = 1 \) iff

\[ \hat{\lambda} = \mu_{2k_1}(\sigma_{s_1}(\rho_1)) \]

, with \( 0 \equiv c(s_1) + 2k_1(m-1) \mod m(m-1)/2 \). Notice in the notation of Lemma 4.1, such a \( \hat{\lambda} \) belongs to \( \hat{Q}_0 \), and there exists a unique \( \hat{\lambda} = \mu \beta(\hat{\lambda}) \) such that \( b(\Lambda_0, \hat{\lambda}, \bar{\lambda}) = 1 \). We claim that such a \( \hat{\lambda} \) appears in the sequence \( \{s_i\} \) and either \( \sigma \) or \( \rho \) can not have both \( \sigma \) and \( \rho \) in decreasing order. It is easy to see that there is a unique \( s_2 \in (\mathbb{Z}_2)^n \) such that \( \{b_1, \ldots, b_n\} = \{s_{2i} + (n - 1 - i), i = 1, 2, \ldots, n\} \) to be the complement sequence of \( \{a_1, \ldots, a_m\} \) in \( 0, 1, \ldots, m - 1 \) in decreasing order. Define \( b_1, \ldots, b_n \) to be \( -b_n, \ldots, -b_1 \), in decreasing order. It is then easy to see that there exists a \( k \) such that \( \beta(\sigma_{s_1}(\rho_1)) = \mu_2(\sigma_{s_2}(s_2(\rho_2))) \).

It follow that \( \mu \beta(\hat{\lambda}) = \hat{\lambda} = \mu^{k_2}(\sigma_{s_2}(s_2(\rho_2))) \) for some \( k_2 \in \mathbb{Z} \). So we have (cf. Page 9 of [LL]):

\[ b(\Lambda', \hat{\lambda}) = 1 \]

for some weights \( \Lambda' \) of \( SU(n(n+1))/2 \). Let us show that \( \Lambda' \) is the vacuum representation \( \Lambda'_0 \). Since \( b(\Lambda', \hat{\lambda}) = 1, b(\Lambda'_0, \hat{\lambda}) = 1, b(\Lambda_0, \hat{\lambda}, \bar{\lambda}) = 1 \), it follow from [KW] that: \( h_{\Lambda'} - h_{\hat{\lambda}} \in \mathbb{Z}, h_{\hat{\lambda}} + h_{\bar{\lambda}} \in \mathbb{Z}, \) and \( h_{\bar{\lambda}} \in \mathbb{Z} \), where \( h_a \) is the conformal anomaly of the weight \( a \). So we have:

\[ h_{\Lambda'} \in \mathbb{Z} \]

, and it follows from (1.19) of [LL] that \( \Lambda' \) is the vacuum representation \( \Lambda'_0 \).

To summarize, we have shown that if \( b(\Lambda'_0, \hat{\lambda}) = 1 \) and \( \hat{\lambda} = \mu \beta(\hat{\lambda}) \) as in Lemma 4.1, then \( b(\Lambda'_0, \hat{\lambda}) = 1 \). Notice \( d_{\lambda} = d_{a\lambda, 0} = d_{a1, 0} = d_{\hat{\lambda}} \) by Th.4.1, where \( d_a \) is the statistical dimension of sector \( a \). Since \( \hat{\lambda} \rightarrow \mu \beta(\hat{\lambda}) \) is one to one, we have proved
\[
\sum_{\lambda} b(\Lambda_0', \lambda) d_\lambda = \sum_{\lambda} b(\Lambda_0', \mu \beta(\lambda)) d_{\mu \beta(\lambda)} \\
\leq \sum_{\lambda} b(\Lambda_0', \bar{\lambda}) d_{\bar{\lambda}}
\]

Now if we start with $\bar{\lambda}$ with $b(\Lambda_0', \bar{\lambda}) = 1$ and go through the previous arguments, we obtain the reverse inequality above, therefore proving that the above inequality is actually an equality. So we have shown $\lambda \to \mu \beta(\lambda)$ is a one-to-one and onto map between the irreducible subsectors of $\rho \bar{\rho}$ and $\bar{\rho} \bar{\rho}$. Since $b(\Lambda_0', \lambda) = b(\Lambda_0', \bar{\lambda})$, the map $\phi : \lambda \to \mu \beta(\lambda)$ is a one-to-one and onto map between the irreducible subsectors of $\rho \bar{\rho}$ and $\bar{\rho} \bar{\rho}$. By (2) of Th.4.1, $a_{\lambda \otimes 1} = a_{1 \otimes \mu \beta(\lambda)}$. It follows from (1) of Th.4.1 that the map $\phi$ is really a ring isomorphism. Since the fusion ring of $\rho$ (resp. $\bar{\rho}$) is generated by irreducible subsectors of $\rho \bar{\rho}$ (resp. $\bar{\rho} \bar{\rho}$), we have proved the following theorem:

**Theorem 4.2.** The fusion ring of the Jones-Wassermann subfactors associated with the conformal inclusions $SU(n+2) \subset SU((n+1)(n+2)/2)$ and $SU(n)_{n+2} \subset SU((n+1)n/2)$ are canonically isomorphic via $\phi$ defined above.

§5. Conclusions and questions

In this paper we have given applications of the general theory developed in [X1]. The following questions arise naturally from our approach:

(1) In all the counter examples we have about Kac-Wakimoto hypothesis, there are always multiplicities, i.e., there is a $\mu$ such that $b_{i \mu} \neq 0, b_{j \mu} \neq 0$ for some $i \neq j$. It is not clear to us if there are examples in the multiplicity-free case;

(2) Let $s(\mu, i)$ denote the multiplicity of $(\mu, i)$ in (Exp), is it true that $s(\mu, i) = b_{\mu, i}$? If this is true, it will imply that the ring $C_\rho$ in the conformal inclusions considered in §4.2 is commutative. This question seems to be related to the $M$-algebra in [PZ];

(3) In §4.2 we have studied the fusion ring of the Jones-Wassermann subfactors associated with two series of conformal inclusions. It will be interesting to see if the ring $C_\rho$ of these conformal inclusions is related in simple way;

(4) The conformal inclusions considered in §4.2 is related to the compact Hermitian symmetric spaces $Sp(n)/U(n)$ and $SO(2n)/U(n)$. It will be interesting to see if the ring $C_\rho$ is related to the quantum cohomology ring of those Hermitian symmetric spaces.

References

[L1] R. Longo, *Minimal index and braided subfactors*, J. Funct. Anal., 109, 98-112 (1992).

[L2] R. Longo, *Duality for Hopf algebras and for subfactors*, I, Comm. Math. Phys., 159, 133-150 (1994).
[L3] R. Longo, *Index of subfactors and statistics of quantum fields*, I, Comm. Math. Phys., 126, 217-247 (1989).

[L4] R. Longo, *Index of subfactors and statistics of quantum fields*, II, Comm. Math. Phys., 130, 285-309 (1990).

[L5] R. Longo, Proceedings of International Congress of Mathematicians, 1281-1291 (1994).

[LR] R. Longo and K.-H. Rehren, *Nets of subfactors*, Rev. Math. Phys., 7, 567-597 (1995).

[Po] S. Popa, *Classification of subfactors and of their endomorphisms*, CBMS Lecture Notes Series, 86.

[PP] M. Pimsner and S. Popa, *Entropy and index for subfactors*, Ann. Éc. Norm. Sup. 19, 57-106 (1986).

[W1] A. Wassermann, *Operator algebras and Conformal field theories III*, to appear.

[W2] A. Wassermann, Proceedings of International Congress of Mathematicians, 966-979 (1994).

[PZ] V. B. Petkova and J.-B. Zuber, *From CFT to graphs*, hep-th-9510198.

[Fran] P. Di Francesco and J.-B. Zuber, *Integrable lattice models associated with SU(N)*, Nucl. Phys. B, 338, 602-623 (1990).

[Ka1] Y. Kawahigashi, *Classification of paragroup actions on subfactors*, Publ. RIMS, Kyoto Univ., 31 481-517 (1995).

[GHJ] F. M. Goodman, P. de la Harpe and V. Jones, *Towers of algebras and Coxeter graphs*, MSRI publication, no. 14.

[PS] A. Pressley and G. Segal, *Loop groups*, Clarendon Press, 1986.

[GNO] P. Goddard, W. Nahm and D. Olive, *Symmetric spaces, Sugawara’s energy momentum tensor in two dimensions and free fermions*, Phy. Lett., 160B, 111-116 (1985)

[KW] V. G. Kac and M. Wakimoto, *Modular and conformal invariance constraints in representation theory of affine algebras*, Advances in Math., 70, 156-234 (1988).

[Kac2] V. G. Kac, *Infinite dimensional algebras*, 3rd Edition, Cambridge University Press, 1990.

[Kac3] V. G. Kac and M. Sanielevici, *Decompositions of representations of exceptional affine algebras with respect to conformal subalgebras*, Phys.Rev.D 37, 2231-2237 (1988).

[X1] F. Xu, *New braided endomorphisms from conformal inclusions*, Comm. Math. Phys., (1997) (in press)

[X2] F. Xu, *Generalized Goodman-Harper-Jones construction of subfactors*, I, Comm. Math. Phys., 184, 475-491 (1997).

[X3] F. Xu, *Generalized Goodman-Harpe-Jones construction of subfactors*, II, Comm. Math. Phys., 184, 493-508 (1997).

[X4] F. Xu, *Jones-Wassermann subfactors for Disconnected Intervals*, q-alg 9704003.

[Y] S. Yamagami, *A note on Ocneanu’s approach to Jones index theory*, Inter-
[GW] F. M. Goodman and H. Wenzl, *Littlewood Richardson coefficients for Hecke algebras at roots of unity*, Adv. Math., 1990.

[S] H. Saleur, *Level-rank duality*, Nucl. Phys. B., 363, 177-192 (1992).

[Ts] T. Nakanishi and A. Tsuchiya, *Level-rank duality of WZW models in conformal field theory*, Comm. Math. Phys., 144, 351-372 (1992).

[DB] D. Bisch, *On the structure of finite depth subfactors*, Algebraic Methods in Operator Theory, Birkhäuser, 175-194.