HEAD AND TAIL SPEEDS OF MEAN CURVATURE FLOW WITH FORCING

HONGWEI GAO AND INWON KIM

Abstract. In this paper, we investigate the large time behavior of interfaces moving with motion law
\( V = -\kappa + g(x) \), where \( g \) is positive, Lipschitz and \( \mathbb{Z}^n \)-periodic. It turns out that the behavior of the
interface can be characterized by its head and tail speed, which depends continuously on its overall
direction of propagation \( \nu \). If head speed equals tail speed at a given direction \( \nu \), the interface has a
unique large-scale speed in that direction. In general the interface develops linearly growing “long fingers”
in the direction where the equality breaks down. We discuss these results in both general setting and in
laminar setting, where further results are obtained due to regularity properties of the flow.

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1. Introduction

We consider the evolution of domains \((\Omega_\varepsilon(t))_{t \geq 0}\) in \(\mathbb{R}^n\), where \(\Gamma_\varepsilon(t) := \partial \Omega_\varepsilon(t)\) moves with the (outward)
normal velocity

\[
V = -\varepsilon \kappa + g(x/\varepsilon) \quad \text{on} \quad \Gamma_\varepsilon(t).
\]
Here \( \kappa \) denotes the mean curvature of \( \Gamma_\varepsilon(t) \), with positive sign when \( \Omega_\varepsilon(t) \) is convex, \( g \) is a \( \mathbb{Z}^n \)-periodic function in \( \mathbb{R}^n \). Note that \( \Gamma_\varepsilon(t) \) is a zoomed-out version of \( \Gamma_1(t) \) with scaling \( (x,t) \to (\varepsilon x, \varepsilon t) \). The oscillation in the forcing term \( g \) will be reflected in the oscillatory behavior of \( \Gamma_\varepsilon \).

We are interested in the asymptotic behavior of \( \Gamma_\varepsilon \) as \( \varepsilon \to 0 \), or equivalently, the large-scale behavior of \( \Gamma_1 \). \( \Gamma_\varepsilon \) may go through topological changes and other singularities as it intersects with the oscillatory forcing. Thus the evolution (1.1) must be understood in a weak sense, while for our purpose the weak notation should still be able to describe the pointwise behavior of the solution. To this end we work with viscosity solution \( u^\varepsilon \) of the corresponding level set equation (here \( \hat{p} = \frac{p}{|p|} \) for \( p \in \mathbb{R}^n \setminus \{0\} \)) with \( \Gamma_\varepsilon = \{u^\varepsilon = 0\} \),

\[
u^\varepsilon = \mathcal{F}(\varepsilon D^2 u^\varepsilon, Du^\varepsilon, x/\varepsilon) := \varepsilon \operatorname{tr} \left\{ D^2 u^\varepsilon \left( I - \hat{D} u^\varepsilon \otimes \hat{D} u^\varepsilon \right) \right\} + g (x/\varepsilon) \left| Du^\varepsilon \right| \quad \text{in } \mathbb{R}^n \times (0, \infty), \tag{1.2}\]

which is a degenerate viscous Hamilton-Jacobi equation. We say homogenization occurs when \( u^\varepsilon \) converges to a homogenization profile as \( \varepsilon \to 0 \). If not we say homogenization fails. The study of (1.2) as \( \varepsilon \to 0 \) has attracted much attention in the past decade, for instance see [19], [11], [4], [6], [5], [1] and the references therein.

We investigate the case of positive, Lipschitz continuous \( g \), where (1.1) is most well understood. i.e., there exist \( m_0, M_0, L_0 > 0 \), such that

\[egin{cases}
g(x) : \mathbb{R}^n \to [m_0, M_0] \text{ is a } \mathbb{Z}^n \text{-periodic Lipschitz continuous function}
\quad \text{and } |g(x) - g(y)| \leq L_0 |x - y|, \quad (x, y) \in \mathbb{R}^n \times \mathbb{R}^n. 
\end{cases} \tag{H}
\]

In this setting Lions and Souganidis [19] showed homogenization results with the condition \( |Dg|_g < \frac{1}{n+1} \). This condition amounts to ensuring the existence of Lipschitz continuous solution \( v \) of the corresponding cell problem (see below for further discussion of the cell problem), to ensure the existence of plane-like solutions of the form \( u^\varepsilon \sim u_0 + \varepsilon v + o(\varepsilon) \), where \( u_0 \) is a linear profile. Here the regularity of \( v \) is central to obtain Lipschitz continuity of the homogenized front velocity. For two space dimensions Caffarelli and Monneau [5] shows that homogenization always occurs, with continuous homogenized velocity. The main step here is to show the existence of a bounded solution of the cell problem, by a geometric argument that is particular to two dimensions. In general homogenization may fail when the oscillation of \( g \) grows large, as we will discuss in the paper (sect. 8.2). In three or higher dimensions, [5] gave an example in the laminar setting \( g(x) = g(x'), x = (x', x_n) \), where the oscillation of \( \Gamma_\varepsilon \) grows linearly as \( \varepsilon \to 0 \).

Even when homogenization fails and \( \Gamma_\varepsilon \) does not approach an asymptotic profile, it is still reasonable to expect its head and tail speeds to homogenize, as the front propagates through the periodic media. Our goal is to describe this behavior of \( \Gamma_\varepsilon \) as \( \varepsilon \to 0 \) in general setting. As stated below, these speeds \( \bar{s} \) and \( \underline{s} \) only depend on the asymptotic direction of propagation \( \nu \).

**Theorem 1.1** (Proposition 5.14, 5.21, 5.22, Theorem 7.7). Let \( S^{n-1} \) denote the set of unit vectors in \( \mathbb{R}^n \). Then there exist two functions \( \bar{s}, \underline{s} : S^{n-1} \to [m_0, M_0] \) with the following properties:

(a) \( \bar{s} \) and \( \underline{s} \) are continuous in \( S^{n-1} \) and \( \bar{s} > \underline{s} \). In particular if \( \bar{s} > \underline{s} \) at a direction \( \nu_0 \), then the same holds for \( \nu \) sufficiently close to \( \nu_0 \).

(b) Let \( \nu \in S^{n-1} \) and \( u^\varepsilon \) solve (1.2) with its initial data \( u^\varepsilon(x, 0) = -(x - x_0) \cdot \nu \) for some \( x_0 \in \mathbb{R}^n \). Then in micro-scale (for \( \varepsilon = 1 \))

\[
\bar{s}(\nu) = \lim_{t \to \infty} \sup_{t} \left\{ \frac{x \cdot \nu : u^1(x, t) = 0} {t} \right\}, \quad \underline{s}(\nu) = \lim_{t \to \infty} \inf_{t} \left\{ \frac{x \cdot \nu : u^1(x, t) = 0} {t} \right\},
\]

and in macro-scale

\[
\lim_{\varepsilon \to 0} \sup_{\varepsilon} u^\varepsilon(x, t) = -(x - x_0) \cdot \nu + \bar{s}(\nu) t, \quad \lim_{\varepsilon \to 0} \inf_{\varepsilon} u^\varepsilon(x, t) = -(x - x_0) \cdot \nu + \underline{s}(\nu) t.
\]

In particular, when \( \bar{s}(\nu) > \underline{s}(\nu) \), the set \( \{u^\varepsilon(\cdot, t) = 0\} \) oscillates by unit size as \( \varepsilon \to 0 \) and thus homogenization fails.

Above results state that the head speed \( \bar{s} \) and the tail speed \( \underline{s} \) provide a comprehensive description of the asymptotic behavior in the limit \( \varepsilon \to 0 \) for the motion law (1.1) in all scenarios. Let us mention that if, in addition, we have local regularity properties in micro-scale \( \varepsilon = 1 \), our approach would yield
the existence of localized “pulsating travelling waves” with speeds $\bar{s}$ and $\underline{s}$. This is indeed the case in the laminar setting discussed below.

For solutions with general initial data, it is more difficult to pinpoint the precise location of the heads and tails of the front in the asymptotic limit $\varepsilon \to 0$. However the following holds, which provides in particular the optimal upper and lower bound for the propagation of solutions with general geometry.

**Theorem 1.2** (Proposition 6.1, 6.2, 6.3, Corollary 7.8). Let $u^\varepsilon$ solve (1.2) with initial data $u_0$ that are uniformly continuous in $\mathbb{R}^n$. Then $u^\varepsilon := \lim_{\varepsilon \to 0} \sup u^\varepsilon$ is a viscosity subsolution of $u_i = \bar{s}(-\bar{D}u)\vert Du\vert$. Similarly $u_\ast := \lim_{\varepsilon \to 0} \inf u^\varepsilon$ is a viscosity supersolution of $u_i = \underline{s}(-\bar{D}u)\vert Du\vert$.

In particular

- Let us consider a collection of points and directions $A = \{(x_i, \nu_i)\} \subset \mathbb{R}^n \times \mathbb{S}^{n-1}$, and define the associated convex sets $E(t) := \inf_{(x_i, \nu_i) \in A} \{(x - x_i) \cdot \nu_i \leq \bar{s}(\nu)\vert t\}$. If initially $\{u_0 = 0\} \subset E(0)$, then $\{u^\varepsilon(\cdot, t) = 0\} \subset E(t)$.
- If $s = \bar{s} = \underline{s}$, then $u^\varepsilon$ uniformly converges to $u$, the unique viscosity solution of $u_i = s(-\bar{D}u)\vert Du\vert$ with initial data $u_0$.

Stronger statements are available in the Laminar setting, when $g(x) = g(x')$ for $x = (x', x_n)$. For $\varepsilon = 1$, if we start from a Lipschitz and periodic graph $\Gamma_0 = \{(x, x_n) : x_n = U^\varepsilon(x)\}$ that is bounded, then we can show that $\Gamma_1$ stays as a graph and moreover remains as $C^{1,\alpha}$ hypersurface in space, locally uniformly for all large times (see Proposition 8.2). This is sufficient regularity for $\Gamma_1$ to yield the following results.

**Theorem 1.3** (Theorem 8.4, 8.5). Let $g(x) = g(x')$ for $x = (x', x_n)$. Suppose that $\bar{s}(e_n) > \underline{s}(e_n)$. Then there are disjoint, open, non-empty sets $E_1, E_2$ in $\mathbb{R}^{n-1}$ and functions $U_1 : E_1 \to (-\infty, 0], U_2 : E_2 \to [0, \infty)$ such that the following is true:

1. The sets $E_1 \times (-\infty, \infty)$ are stationary solutions of (1.1).
2. $U_1 \to -\infty$ as $x \to \partial E_1$ and $U_2 \to +\infty$ as $x \to \partial E_2$.
3. The surfaces $\Gamma_i := \{x_n = U_i(x') + s_i\}$, $i = 1, 2$, satisfy (1.1) with $\varepsilon = 1$, away from the “obstacle” $\{x_n = s_i\}$. (Here $s_1 = \bar{s}(e_n)$ and $s_2 = \underline{s}(e_n)$)
4. $\Gamma_1$ and $\Gamma_2$ are respectively a subsolution and a supersolution of (1.1) with $\varepsilon = 1$.

For $\nu \neq e_n$ a parallel argument should lead to the existence of pulsating traveling waves away from the obstacles, but we do not pursue this.

Our results accompanies that of Cesaroni and Novaga [6], where variational methods were adopted to yield the existence of the maximal traveling wave in the above laminar setting. While our approach allows to describe travelling waves both at maximal and minimal speed, we only recover partial travelling waves away from their highest and lowest positions, as described in (c). In fact in the scenario where there exists multiple localized travelling waves at the same asymptotic speed, let’s say $\bar{s}(e_n)$, our method appears to capture the most external profile of these waves.

In laminar setting, when the oscillation of $g$, $M_0 = m_0$, is smaller than a dimensional constant, [6] shows the existence of global traveling wave solution with a unique speed $\bar{s}(e_n) = \underline{s}(e_n)$, which provides the large-time behavior of graph solutions in the direction of $e_n$. When the oscillation of $g$ is allowed to be large, it is not hard to generate examples of $\bar{s}(e_n) > \underline{s}(e_n)$ following that of [5]. We briefly discuss this in section 8.2.

**Main challenges and new ingredients**

The central difficulty in obtaining these results is the lack of regularity of the solutions, which comes naturally with the general scenario. In aforementioned literature regarding homogenization of (1.1), one starts with an Ansatz $u^\varepsilon(x, t) = u^0(x, t) + \varepsilon v(x/t) + o(\varepsilon)$, where $v$ solves a cell problem given by the limit profile $u^0$, which is, for (1.2), a linear profile $x \cdot \nu - st$. The idea is then to look for $s = s(\nu)$ for which there exists a $\mathbb{Z}^n$-periodic solution $v$ of the cell problem

$$\mathcal{F}(D^2v, \nu + Dv, y) = s \text{ in } [0, 1]^n,$$

where $\mathcal{F}$ is as given in (1.2). The existence of such $v$ is central in establishing homogenization results.
In our setting this approach fails to apply for two reasons. First, in our general settings, there may be no global limit profile for $u^\epsilon$, let alone an asymptotic planar profile. Indeed our goal is to look for profiles of limit supremum and limit infimum of $u^\epsilon$, as stated above. To study these partial limits, we will introduce “obstacle cell problems”, which amounts to looking for the maximal subsolution and minimal supersolution of a “cell problem”. Second, our “cell problem” is not the standard cell problem in the sense that the corresponding solutions are not periodic if $\nu$ is irrational. This necessitates formulation of the problem in a bounded domain instead, generating an “approximate” sub- and super-cell problem (Definition 2.10).

The obstacle approach was first introduced by Caffarelli, Souganidis and Wang [3] for random homogenization of uniformly elliptic PDEs, and later adopted by Kim [15, 16] and Pozár [21] for free boundary problems. In both of these results the common feature is that there are no standard cell problems one can expect to solve, either due to the non-periodic environment or non-periodic evolution of the free boundaries. This corresponds to our second difficulty described above. However in all of the aforementioned results homogenization is expected to hold: indeed the obstacle solutions in these settings turn out to be asymptotically regular. Our contribution in this paper is thus introducing a “cell problem” type approach for a problem where homogenization is not expected to occur in general, or more precisely when large-scale regularity is missing for the $\epsilon$-solutions.

Roughly speaking the obstacle solutions solve (1.1) with the constraint for the solutions to be below or above the planar obstacle $x \cdot \nu - st$. For instance $\mathcal{S}(\nu)$ is then obtained as the largest speed for which the solutions put below the obstacle stay close to it, which is what is expected for the head speed of an oscillatory interface. We observe that, when $\nu$ is irrational i.e. if $\nu \notin \mathbb{R}^n$, this approach has the advantage of introducing a fine-scale dynamic recurrence property to the problem (Proposition 4.4), which compensates for the lack of regularity properties to study its large-scale behavior. A more precise form of this observation is formulated in the local comparison (Proposition 4.7), which is an important new ingredient in our analysis. This theorem, of independent interest, localizes obstacle solutions of the curvature flow (1.1) which are only continuous. Such localization procedure is central in showing qualitative properties of the head and tail speeds, such as linear detachment, continuity and fingering (see e.g. Propositions 5.11, 5.14, 5.21, 7.4).

Our framework is rather general, and we expect that it could be used to study other geometric flows where homogenization does not always hold. In particular we plan to pursue the case when $g$ changes sign, where there is an added feature of a trapping zone, where $u^\epsilon$ converges to its initial data as $\epsilon \to 0$. See [4] for illuminating discussions of this phenomena. Technically speaking there are added challenges. For instance when $g$ is positive, $u^\epsilon$ with affine initial data turns out to be monotone increasing in time. This adds additional stability in the evolution which is useful in our analysis. Still at the heuristic level our approach should apply to this case. In particular we believe that Theorem ?? should still apply to the general, sign-changing $g$.

Outline of the paper

We start with formulation of obstacle solutions in Section 2, with their properties. In particular the recurrence property mentioned above is given as the Birkhoff property in Section 2.2. In Section 3 we introduce a local perturbation of solutions that was inspired from its usage in free boundary problems (see [2] and [8]). Section 4 proves local comparison principle in terms of the obstacle semi-solutions with irrational directions. To show this, we use the discrepancy results in Section 4.1 to show that the Birkhoff property leads to a fine-scale recurrence property for irrational directions. Then we prove the local comparison principle (Proposition 4.7 in Section 4.2), using this property as well as the local perturbation introduced in Section 3. Similar results are available in [15, 16, 21], however in our problem neither large scale regularity nor perturbation parameters exist. Both of these facts lead to significant challenges in the proof. In Section 5 we define $\bar{s}$ and $\bar{s}$ based on the detachment of solutions from the obstacles (Definition 5.2 - 5.3 in Section 5.2), and use approximation by irrational directions to show continuity of these functions at all directions, based both local comparison (Proposition 5.20) and a blow-up argument using global solutions (Proposition 5.21). Section 6 and 7 contains the proof our main results, Theorem 1.1 and Theorem 1.2. Lastly Section 8 discusses the Laminar case, where Theorem 1.3 is proved. We finish with Section 8.2 where some scenarios are discussed under which homogenization fails.
2. Obstacle problems

In this section, we introduce the obstacle problem associated to the forced mean curvature flow (1.2) with \( \varepsilon = 1 \). In later sections, it allows us to analyze the homogenization in each direction independently. The role an obstacle problem plays here is similar to that of the usual cell problem in homogenization problems. Therefore, the obstacle problem here can be regarded as a variant version of the cell problem.

2.1. Setup. Let us denote by \( \mathcal{F} \) the operator regarding space derivatives in the equation (1.2) with \( \varepsilon = 1 \):

\[
\mathcal{F}(D^2u, Du, x) := \text{tr} \left\{ D^2u \left( I - \frac{Du}{|Du|} \otimes \frac{Du}{|Du|} \right) \right\} + g(x)|Du|.
\]

**Definition 2.1** (c.f. [5]). Let \( \mathcal{S}^n \) be the set of all \( n \times n \) symmetric matrices and denote \( \mathcal{D}_0 := \mathcal{S}^n \times (\mathbb{R}^n \setminus \{0\}) \times \mathbb{R}^n \). We define for all \((X, p, x) \in \mathcal{S}^n \times \mathbb{R}^n \times \mathbb{R}^n\):

\[
\mathcal{F}^*(X, p, x) := \limsup_{\eta \to 0} \left\{ \mathcal{F}(Y, q, y) \mid (Y, q, y) \in \mathcal{D}_0, |X - Y|, |p - q|, |x - y| \leq \eta \right\},
\]

\[
\mathcal{F}_*(X, p, x) := \liminf_{\eta \to 0} \left\{ \mathcal{F}(Y, q, y) \mid (Y, q, y) \in \mathcal{D}_0, |X - Y|, |p - q|, |x - y| \leq \eta \right\}.
\]

In particular, we have \( \mathcal{F}^*(x, p, x) = \mathcal{F}_*(x, p, x) = \mathcal{F}(x, p, x) \) for \((x, p, x) \in \mathcal{D}_0\).

**Definition 2.2** (c.f. [10, 5]). Let \( \Omega \subseteq \mathbb{R}^n \times (-\infty, \infty) \) and \( u(x,t) \in \text{USC}(\Omega) \), the space of upper semicontinuous functions over \( \Omega \). Then \( u(x, t) \) is called a *viscosity subsolution* in \( \Omega \), which is denoted as follows

\[
u_t \leq \mathcal{F}(D^2u, Du, x), \quad (x, t) \in \Omega,
\]

if for any \((x_0, t_0) \in \Omega, r > 0\) and \( \phi(x, t) \in C^{2,1}(B_r(x_0, t_0)) \), such that

\[
u(x, t) \leq \phi(x, t) \text{ in } B_r(x_0, t_0) \quad \text{and} \quad \nu(x_0, t_0) = \phi(x_0, t_0),
\]

then

\[
\phi(x_0, t_0) \leq \mathcal{F}^*(D^2\phi(x_0, t_0), D\phi(x_0, t_0), x_0).
\]

**Definition 2.3.** Let \( \Omega \subseteq \mathbb{R}^n \times (-\infty, \infty) \) and \( u(x,t) \in \text{USC}(\Omega) \). Then \( u(x,t) \) is called a *pseudo viscosity subsolution* in \( \Omega \), if for any \((x_0, t_0) \in \Omega, r > 0\) and \( \phi(x, t) \in C^{2,1}(B_r(x_0, t_0)) \), such that

\[
u(x, t) \leq \phi(x, t) \text{ on } B_r(x_0, t_0), \quad \nu(x_0, t_0) = \phi(x_0, t_0) \quad \text{and} \quad |D\phi(x_0, t_0)| > 0,
\]

then

\[
\phi(x_0, t_0) \leq \mathcal{F}^*(D^2\phi(x_0, t_0), D\phi(x_0, t_0), x_0).
\]

**Definition 2.4** (c.f. [10, 5]). Let \( \Omega \subseteq \mathbb{R}^n \times (-\infty, \infty) \) and \( v(x,t) \in \text{LSC}(\Omega) \), the space of lower semicontinuous functions. Then \( u(x, t) \) is called a *viscosity supersolution* in \( \Omega \), which is denoted as follows

\[
u_t \geq \mathcal{F}(D^2v, Dv, x), \quad (x, t) \in \Omega,
\]

if for any \((x_0, t_0) \in \Omega, r > 0\) and \( \psi(x, t) \in C^{2,1}(B_r(x_0, t_0)) \), such that

\[
v(x, t) \geq \psi(x, t) \text{ in } B_r(x_0, t_0) \quad \text{and} \quad \nu(x_0, t_0) = \psi(x_0, t_0),
\]

then

\[
\psi(x_0, t_0) \geq \mathcal{F}_*(D^2\psi(x_0, t_0), D\psi(x_0, t_0), x_0).
\]

**Definition 2.5.** Let \( \Omega \subseteq \mathbb{R}^n \times (-\infty, \infty) \) and \( v(x,t) \in \text{LSC}(\Omega) \). Then \( u(x, t) \) is called a *pseudo viscosity supersolution* in \( \Omega \), if for any \((x_0, t_0) \in \Omega, r > 0\) and \( \psi(x, t) \in C^{2,1}(B_r(x_0, t_0)) \), such that

\[
v(x, t) \geq \psi(x, t) \text{ in } B_r(x_0, t_0), \quad \nu(x_0, t_0) = \psi(x_0, t_0) \quad \text{and} \quad |D\psi(x_0, t_0)| > 0,
\]

then

\[
\psi(x_0, t_0) \geq \mathcal{F}_*(D^2\psi(x_0, t_0), D\psi(x_0, t_0), x_0).
\]

**Definition 2.6** (c.f. [10, 5]). Let \( \Omega \subseteq \mathbb{R}^n \times (-\infty, \infty) \) and \( u(x,t) : \Omega \to \mathbb{R} \). Then \( u(x, t) \) is called a *viscosity solution* if \( u^*(x, t) \) is a viscosity subsolution and \( u_* (x, t) \) is a viscosity supersolution, where

\[
u^*(x, t) := \limsup_{(y, \tau) \to (x, t)} u(y, \tau) \quad \text{and} \quad u_* (x, t) := \liminf_{(y, \tau) \to (x, t)} u(y, \tau).
\]

It is well-known that for any \( \varepsilon > 0 \) and \( u_0(x) \in \text{UC}(\mathbb{R}^n) \), the equation (1.2) has a unique continuous viscosity solution.
**Proposition 2.1** (Comparison principle, see [5]). Let us consider \( \Omega = \hat{\Omega} \times (0, T) \) with \( T > 0 \), where \( \hat{\Omega} \subseteq \mathbb{R}^n \). Assume that either \( \hat{\Omega} = \mathbb{R}^n \) or \( \hat{\Omega} \) is a bounded open subset of \( \mathbb{R}^n \), assume that \( u(x, t) \) is a viscosity subsolution of (2.2) and \( v(x, t) \) is a viscosity supersolution of (2.3) such that

\[
\limsup_{\delta \to 0} \left\{ u(x, t) - v(y, 0) \mid |x - y| \leq \delta \right\} \leq 0, \quad \text{if } \hat{\Omega} = \mathbb{R}^n
\]

\[
u \leq v \quad \text{on} \quad \partial_p \left( \hat{\Omega} \times (0, T) \right), \quad \text{if } \hat{\Omega} \text{ is bounded},
\]

then

\[
u(x, t) \leq u(x, t), \quad (x, t) \in \Omega.
\]

**Definition 2.7.** Let us denote some frequently used sets throughout the paper.

\[
\mathbb{D} := S^{n-1} \times (0, \infty) \times \mathbb{R} \quad \text{(i)}
\]

\[
\mathbb{E} := \left\{ (\nu, q, s) \in S^{n-1} \times (\mathbb{R}^n \setminus \{0\}) \times [m_0, M_0] \mid \nu = -\frac{q}{|q|} \right\} \quad \text{(ii)}
\]

\[
\mathbb{F} := \left\{ (r, \varphi) \mid \varphi : [0, \infty) \to (0, \infty), \quad \varphi(x) : \mathbb{R}^n \to (0, \infty) \right\} \quad \text{(iii)}
\]

\[
\mathbb{A} := \left\{ (\nu, R, \mathscr{R}, q, s) \in S^{n-1} \times (0, \infty) \times \mathbb{R} \times (\mathbb{R}^n \setminus \{0\}) \times [m_0, M_0] \mid \nu = -\frac{q}{|q|} \right\} \quad \text{(iv)}
\]

**Definition 2.8.** Fix any \( d := (\nu, R, \mathscr{R}) \in \mathbb{D} \), denote by \( \mathcal{C}(t) \) the \( \nu \) directional cylinder within radius \( R \) and expanding/shrinking speed \( \mathcal{R} \) at time \( t \) i.e.,

\[
\mathcal{C}_d(t) := \left\{ x \in \mathbb{R}^n \mid |x - (x \cdot \nu) \nu| < R + \mathcal{R} t \right\},
\]

where \( R + \mathcal{R} t > 0 \). Let us also denote the whole space-time domain by that

\[
\mathcal{C}_d := \left\{ (x, t) \in \mathbb{R}^n \times [0, \infty) \mid x \in \mathcal{C}_d(t), \ R + \mathcal{R} t > 0 \right\}. \quad \text{(2.4)}
\]

In particular, let \( (x, r, \nu) \in \mathbb{R}^n \times (0, \infty) \times S^{n-1} \) and denote a static region as follows,

\[
\Omega(x, r; \nu) := \left\{ y \in \mathbb{R}^n \mid |(y - x) - ((y - x) \cdot \nu) \nu| \leq r \right\}. \quad \text{(2.5)}
\]

**Definition 2.9.** Fix any \( e := (\nu, q, s) \in \mathbb{E} \), we denote by \( \mathcal{O}_e(x, t) \) the obstacle function with slope \( q \) and speed \( s \) in the \( \nu \) direction. To be more precise,

\[
\mathcal{O}_e(x, t) := x \cdot q + st|q|, \quad \text{for} \quad x \in \mathbb{R}^n \text{ and } t \geq 0. \quad \text{(2.6)}
\]

**Remark 2.2.** Let \( e = (\nu, q, s) \in \mathbb{E} \), then the zero level set of \( \mathcal{O}_e(x, t) \) is a hyperplane moving with speed \( s \) in the normal direction \( \nu \).

**Definition 2.10.** Fix any \( a := (\nu, R, \mathcal{R}, q, s) \in \mathbb{A} \), then set \( d := (\nu, R, \mathcal{R}) \in \mathbb{D} \) and \( e := (\nu, q, s) \in \mathbb{E} \), let us denote by \( \mathcal{F}_a \) (resp. \( \mathcal{L}_a \)) the set of all subsolutions (resp. supersolutions) in \( \mathcal{C}_d \) that is bounded from above (resp. below) by \( \mathcal{O}_e(x, t) \). i.e.,

\[
\mathcal{F}_a := \left\{ (x, t) \in \text{USC}(\mathcal{C}_d) \mid u_t \leq \mathcal{F} (D^2 u, Du, x), \ u(x, t) \leq \mathcal{O}_e(x, t) \right\},
\]

\[
\mathcal{L}_a := \left\{ (x, t) \in \text{LSC}(\mathcal{C}_d) \mid u_t \geq \mathcal{F} (D^2 u, Du, x), \ u(x, t) \geq \mathcal{O}_e(x, t) \right\}.
\]

Let us also denote the obstacle subsolution/supersolution as follows.

\[
\mathcal{U}_a(x, t) := \left( \sup \left\{ u(x, t) \mid u \in \mathcal{F}_a \right\} \right)^* \quad \text{and} \quad \mathcal{U}_a(x, t) := \left( \inf \left\{ u(x, t) \mid u \in \mathcal{L}_a \right\} \right)^*.
\]

2.2. Properties.

**Lemma 2.3.** Fix any \( a \in \mathbb{A} \), then

\[
\mathcal{U}_a(x, t) \in \mathcal{F}_a \quad \text{and} \quad \mathcal{U}_a(x, t) \in \mathcal{L}_a.
\]

**Proof.** If follows from the definition of viscosity sub/super-solution (c.f. [10]).
2.2.1. **Coincidence on the boundary.** The following Lemma shows that the obstacle subsolution coincides with the obstacle if the domain is not shrinking.

**Lemma 2.4.** Fix \( a := (\nu, \mathcal{R}, q, s) \in \mathcal{A} \) with \( \mathcal{R} \geq 0 \), then set \( d := (\nu, \mathcal{R}) \in \mathcal{D} \) and \( e := (\nu, q, s) \in \mathcal{E} \), then
\[
\mathcal{U}_a(x, t) = \mathcal{O}_e(x, t), \quad (x, t) \in \{ (y, \tau) | y \in \partial \mathcal{C}_d(\tau), \ y \cdot \nu = s\tau \}.
\]

**Proof.** Let us denote the set of admissible normal directions:
\[
\overline{M}_a := \{ \mu \in \mathbb{S}^{n-1} | \mu \cdot \nu = \sigma \}, \quad \text{where} \quad \sigma := m_0 s + \mathcal{R} \sqrt{\mathcal{R}^2 + s^2} \geq 0.
\]
Then for any \( \mu \in \overline{M}_a \), let us define the moving hyperplane
\[
\nabla \mu(x, t) := -\frac{|q|}{\sigma} \left( x \cdot \mu - m_0 t + R \sqrt{1 - \sigma^2} \right),
\]
and a specific subsolution \( \nabla_a(x, t) := \sup_{\mu \in \overline{M}_a} \nabla \mu(x, t) \in \mathcal{F}_a \).

Based on the above construction, we have that
\[
\nabla_a(x, t) = \mathcal{O}_e(x, t), \quad (x, t) \in \{ (y, \tau) | y \in \partial \mathcal{C}_d(\tau), \ y \cdot \nu = s\tau \}.
\]
The result follows from the ordering relation \( \nabla_a(x, t) \leq \mathcal{U}_a(x, t) \leq \mathcal{O}_e(x, t) \).

In a similar manner, the next Lemma says that if the domain’s expanding speed is large enough, the obstacle supersolution matches the obstacle on the boundary.

**Lemma 2.5.** Fix \( a := (\nu, \mathcal{R}, q, s) \in \mathcal{A} \) with \( \mathcal{R} \geq \sqrt{M_0^2 - s^2} \), then set \( d := (\nu, \mathcal{R}) \in \mathcal{D} \) and \( e := (\nu, q, s) \in \mathcal{E} \), then
\[
\underline{U}_a(x, t) = \mathcal{O}_e(x, t), \quad (x, t) \in \{ (y, \tau) | y \in \partial \mathcal{C}_d(\tau), \ y \cdot \nu = s\tau \}.
\]

**Proof.** Let us denote the set of admissible normal directions
\[
\underline{M}_a := \{ \mu \in \mathbb{S}^{n-1} | \nu \cdot \mu = \sigma \}, \quad \text{where} \quad \sigma := \frac{s}{\sqrt{\mathcal{R}^2 + s^2}} > 0.
\]
Then for any \( \mu \in \underline{M}_a \), let us define the moving hyperplane
\[
\nabla \mu(x, t) := \left\{ x \cdot \mu - \left[ \frac{R \mathcal{R} \sigma}{s} + \sqrt{\mathcal{R}^2 + s^2 t} \right] \right\}. (-|q|) \in \mathcal{Z}_a,
\]
and a specific supersolution \( \nabla_a(x, t) \) in \( \mathcal{C}_d \) as below:
\[
\nabla_a(x, t) := \inf_{\mu \in \underline{M}_a} \nabla \mu(x, t) \in \mathcal{Z}_a.
\]
Based on the above construction, we have that
\[
\nabla_a(x, t) = \mathcal{O}_e(x, t), \quad (x, t) \in \{ (y, \tau) | y \in \partial \mathcal{C}_d(\tau), \ y \cdot \nu = s\tau \},
\]
the result follows from the ordering relation \( \mathcal{O}_e(x, t) \leq \underline{U}_a(x, t) \leq \nabla_a(x, t) \).

2.2.2. **The Birkhoff properties.** The Birkhoff property describes the monotonicity of a specific obstacle sub/super-solution with respect to time, under certain integer vector shift. The monotonicity depends on two aspects: (i) subsolution or supersolution; (ii) expanding domain or shrinking domain. Let us discuss each of them respectively.

In the expanding domain, the obstacle sub/super-solution tends to keep away from the obstacle as time evolves. Therefore, the obstacle subsolution (resp. supersolution) shows a decreasing (resp. an increasing) pattern.
Proposition 2.6. Fix $a := (v, R, A, q, s) \in A$ with $A \geq 0$, then set $d := (v, R) \in D$ and $e := (v, q, s) \in E$. Let $\Delta t > 0$ and $\Delta z \in \mathbb{Z}^n$, such that
\[
0 < s \Delta t \leq \Delta z \cdot v \quad \text{and} \quad A \Delta t \geq |\Delta z - (\Delta z \cdot v)\nu|,
\]
then
\[
\mathcal{U}_a(x + \Delta z, t + \Delta t) \leq \mathcal{U}_a(x, t), \quad (x, t) \in C_d.
\]
Proof. By the choice of $\Delta t$ and $\Delta z$, $(x, t) \in C_d$ indicates $(x + \Delta z, t + \Delta t) \in C_d$. Moreover, $\mathcal{U}_a(x + \Delta z, t + \Delta t) \leq O_e(x, t)$, for any $(x, t) \in C_d$. Because $\mathcal{U}_a(x, t) \in \mathcal{F}_a$ and $\Delta z \in \mathbb{Z}^n$, $\mathcal{U}_a([\cdot + \Delta z, \cdot + \Delta t])_{C_d} \in \mathcal{F}_a$. Hence the maximality of $\mathcal{U}_a(x, t)$ from the Definition 2.10 implies that $\mathcal{U}_a(x + \Delta z, t + \Delta t) \leq \mathcal{U}_a(x, t)$, for any $(x, t) \in C_d$.

Proposition 2.7. Fix $a := (v, R, A, q, s) \in A$ with $A \geq 0$, then set $d := (v, R) \in D$ and $e := (v, q, s) \in E$. Let $\Delta t > 0$ and $\Delta z \in \mathbb{Z}^n$, such that
\[
s \Delta t \geq \Delta z \cdot v > 0 \quad \text{and} \quad A \Delta t \geq |\Delta z - (\Delta z \cdot v)\nu|,
\]
then
\[
\mathcal{U}_a(x, t) \leq \mathcal{U}_a(x + \Delta z, t + \Delta t), \quad (x, t) \in C_d.
\]
Proof. By the choice of $\Delta t$ and $\Delta z$, if we have $(x, t) \in C_d$, so does $(x + \Delta z, t + \Delta t)$. In addition, $O_e(x, t) \leq \mathcal{U}_a(x + \Delta z, t + \Delta t)$, for any $(x, t) \in C_d$. Since $\mathcal{U}_a(x, t) \in \mathcal{F}_a$ and $\Delta z \in \mathbb{Z}^n$, $\mathcal{U}_a([\cdot + \Delta z, \cdot + \Delta t])_{C_d} \in \mathcal{F}_a$. The minimality of $\mathcal{U}_a(\cdot, \cdot)$ from the Definition 2.10 implies that $\mathcal{U}_a(x, t) \leq \mathcal{U}_a(x + \Delta z, t + \Delta t)$, for any $(x, t) \in C_d$.

Next, let us investigate the case of static domains, i.e., $A = 0$. In the following two propositions, we shall compare the sub/super-solutions in two different static domains. It turns out that the larger the domain is, the further the sub/super-solutions stay away from the associated obstacles.

Proposition 2.8. Fix $a_i := (v, R_i, 0, q, s) \in A$, where $i = 1, 2$ and $0 < R_1 < R_2 < \infty$, then set $d_i := (v, R_i, 0) \in D$, $i = 1, 2$ and $e := (v, q, s) \in E$. Let $\Delta t \geq 0$ and $\Delta z \in \mathbb{Z}^n$, such that
\[
0 < s \Delta t \leq \Delta z \cdot v \quad \text{and} \quad R_2 - R_1 \geq |\Delta z - (\Delta z \cdot v)\nu|,
\]
then
\[
\mathcal{U}_{a_2}(x + \Delta z, t + \Delta t) \leq \mathcal{U}_{a_1}(x, t), \quad (x, t) \in C_{d_1}.
\]
Proof. By the choice of $R_1$, $R_2$, $\Delta t$ and $\Delta z$, $(x, t) \in C_{d_1}$ indicates $(x + \Delta z, t + \Delta t) \in C_{d_2}$. Moreover, $\mathcal{U}_{a_2}(x + \Delta z, t + \Delta t) \leq O_e(x, t)$, for any $(x, t) \in C_{d_1}$. Because $\mathcal{U}_{a_2}(\cdot, \cdot) \in \mathcal{F}_{a_2}$ and $\Delta z \in \mathbb{Z}^n$, $\mathcal{U}_{a_2}([\cdot + \Delta z, \cdot + \Delta t])_{C_{d_1}} \in \mathcal{F}_{a_1}$. Hence the maximality of $\mathcal{U}_{a_1}(x, t)$ from Definition 2.10 implies that $\mathcal{U}_{a_2}(x + \Delta z, t + \Delta t) \leq \mathcal{U}_{a_1}(x, t)$, for any $(x, t) \in C_{d_1}$.

Proposition 2.9. Fix $a_i := (v, R_i, 0, q, s) \in A$, where $i = 1, 2$ and $0 < R_1 < R_2 < \infty$, then set $d_i := (v, R_i, 0) \in D$, $i = 1, 2$ and $e := (v, q, s) \in E$. Let $\Delta t \geq 0$ and $\Delta z \in \mathbb{Z}^n$, such that
\[
s \Delta t \geq \Delta z \cdot v > 0 \quad \text{and} \quad R_2 - R_1 \geq |\Delta z - (\Delta z \cdot v)\nu|,
\]
then
\[
\mathcal{U}_{a_1}(x, t) \leq \mathcal{U}_{a_2}(x + \Delta z, t + \Delta t), \quad (x, t) \in C_{d_1}.
\]
Proof. By the choice of $R_1$, $R_2$, $\Delta t$ and $\Delta z$, if we have $(x, t) \in C_{d_1}$, then $(x + \Delta z, t + \Delta t) \in C_{d_2}$. In addition, $O_e(x, t) \leq \mathcal{U}_{a_2}(x + \Delta z, t + \Delta t)$, for any $(x, t) \in C_{d_1}$. Since $\mathcal{U}_{a_2}(x, t) \in \mathcal{F}_{a_2}$ and $\Delta z \in \mathbb{Z}^n$, $\mathcal{U}_{a_2}([\cdot + \Delta z, \cdot + \Delta t])_{C_{d_1}} \in \mathcal{F}_{a_2}$. Therefore, the minimality of $\mathcal{U}_{a_1}(\cdot, \cdot)$ from Definition 2.10 implies that $\mathcal{U}_{a_1}(x, t) \leq \mathcal{U}_{a_2}(x + \Delta z, t + \Delta t)$, for any $(x, t) \in C_{d_1}$.

Finally, in the case of shrinking domains, we have the monotonicity with an opposite direction. i.e., as time passes by, the obstacle sub/super-solutions tend to stay closer to the associated obstacle.
Proposition 2.10. Fix $a := (\nu, R, s, q, s) \in A$ with $R < 0$, then set $d := (\nu, R, s) \in D$ and $e := (\nu, q, s) \in E$. Let $\Delta t > 0$ and $\Delta z \in \mathbb{Z}^n$, such that

$$m_0 \Delta t \geq \Delta z \cdot \nu > 0 \quad \text{and} \quad (-R) \Delta t \geq |\Delta z - (\Delta z \cdot \nu) \nu|,$$

then

$$U_a(x - \Delta z, t) \leq U_a(x, t + \Delta t), \quad x \in C_d(t + \Delta t).$$

Proof. Since $m_0 \leq s \leq M_0$, the function $-|q|(x \cdot \nu - m_0 t)$ is a subsolution in $C_d$. The choice of $\Delta t$ and $\Delta z$ indicates that $U_a(x - \Delta z, 0) \leq U_a(x, \Delta t)$, for any $x \in C_d(\Delta t)$. It also implies that $U_a(x - \Delta z, t) \leq O_c(x, t + \Delta t)$, for any $x \in C_d(t + \Delta t)$. Because $\Delta z \in \mathbb{Z}^n$, $U_a(x - \Delta z, t)$ is a subsolution bounded from above by $O_c(x, t + \Delta t)$, in $\hat{C}_d := \{(x, t) \in \mathbb{R}^n \times (0, \infty) \mid x \in C_d(t + \Delta t)\}$, so does $\max \{U_a(x - \Delta z, t), U_a(x, t + \Delta t)\}$. By the maximality of $U_a$ from Definition 2.10, we conclude that

$$\max \{U_a(x - \Delta z, t), U_a(x, t + \Delta t)\} \leq U_a(x, t + \Delta t), \quad (x, t) \in \hat{C}_d.$$

Equivalently,

$$U_a(x - \Delta z, t) \leq U_a(x, t + \Delta t), \quad x \in C_d(t + \Delta t).$$

□

Proposition 2.11. Fix $a := (\nu, R, s, q, s) \in A$ with $R < 0$, then set $d := (\nu, R, s) \in D$ and $e := (\nu, q, s) \in E$. Let $\Delta t > 0$ and $\Delta z \in \mathbb{Z}^n$, such that

$$\Delta z \cdot \nu \geq M \Delta t \geq 0 \quad \text{and} \quad (-R) \Delta t \geq |\Delta z - (\Delta z \cdot \nu) \nu|,$$

then

$$U_a(x - \Delta z, t) \geq U_a(x, t + \Delta t), \quad x \in C_d(t + \Delta t).$$

Proof. Since $m_0 \leq s \leq M_0$, the function $-|q|(x \cdot \nu - M_0 t)$ is a supersolution in $C_d$. The choice of $\Delta t$ and $\Delta z$ indicates that $U_a(x - \Delta z, 0) \geq U_a(x, \Delta t)$, for any $x \in C_d(\Delta t)$. It also implies that $U_a(x - \Delta z, t) \geq O_c(x, t + \Delta t)$, for any $x \in C_d(t + \Delta t)$. Because $\Delta z \in \mathbb{Z}^n$, $U_a(x - \Delta z, t)$ is a supersolution bounded from below by $O_c(x, t + \Delta t)$, in $\hat{C}_d := \{(x, t) \in \mathbb{R}^n \times (0, \infty) \mid x \in C_d(t + \Delta t)\}$, so does $\min \{U_a(x - \Delta z, t), U_a(x, t + \Delta t)\}$. By the minimality of $U_a$ from Definition 2.10, we conclude that

$$\min \{U_a(x - \Delta z, t), U_a(x, t + \Delta t)\} \geq U_a(x, t + \Delta t), \quad (x, t) \in \hat{C}_d.$$

Equivalently,

$$U_a(x - \Delta z, t) \geq U_a(x, t + \Delta t), \quad x \in C_d(t + \Delta t).$$

□

Remark 2.12. In the previous Propositions 2.6, 2.7, 2.8, 2.9, 2.10, 2.11, the space shift $\Delta z \in \mathbb{Z}^n$ is only due to the periodicity of $g(x)$. In the laminar case, i.e., $g(x) = g(x')$ with $x = (x', x_n)$, it suffices to have $\Delta z = (\Delta z', \Delta z_n)$ with $\Delta z' \in \mathbb{Z}^{n-1}$ and $\Delta z_n \in \mathbb{R}$.

3. Inf-convolution

3.1. Concepts and properties.

Definition 3.1. Let $h := (r(t), \varphi(x)) \in F$ and $u(x, t) : U \to \mathbb{R}$ be a function defined in a space time domain $U \subseteq \mathbb{R}^n \times (0, \infty)$. The $h$ inf-convolution of $u(x, t)$, denoted by $u^h(\cdot, t)$, is defined as follows.

$$u^h(\cdot, t) := \inf_{y \in B_r(t) \varphi(x)} u(y, t), \quad (B_r(t) \varphi(x)) \subseteq U.$$

Here $B_r(t) \varphi(x) := \{y : y - x \leq r(t) \varphi(x)\}$. 

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Proposition 3.2. Let \( f(x), g(x) : \Omega \to \mathbb{R} \), where \( \Omega \) is a subset of \( \mathbb{R}^n \), for any \( \mu \in \mathbb{R} \), let us denote the sublevel set and superlevel set of the function \( f \) in \( \Omega \) as follows.

\[
L^-_\mu(f;\Omega) := \{ x \in \Omega | f(x) \leq \mu \}, \quad L^+_\mu(f;\Omega) := \{ x \in \Omega | f(x) \geq \mu \}.
\]

For later convenience, we also denote the sub/super level-set based ordering relation as below:

\[
f \prec_{(\Omega,\mu)} g, \text{ if } (i) \ f < g, (ii) \ L^+_\mu(f;\Omega) \cap L^-_\mu(g;\Omega) = \emptyset, (iii) \ \inf \{ x \cdot \nu | x \in L^+_\mu(f;\Omega) \} = -\infty, \sup \{ x \cdot \nu | x \in L^-_\mu(g;\Omega) \} = +\infty
\]

The next proposition shows that any sublevel (resp. superlevel) set of the \( (r(t), \varphi(x)) \) inf-convolution of a lower semicontinuous function has a \( r(t)\varphi(x) \) interior (resp. exterior) ball condition. That is to say, the inf-convolution makes the sublevel (resp. superlevel) set more regular from one direction.

**Proposition 3.1.** Fix \( \mu \in \mathbb{R}, h := (r(t), \sigma) \in \mathbb{F} \), where \( \sigma > 0 \) is a constant and \( u(x,t) \in \text{LSC}(U) \), where \( U \subseteq \mathbb{R}^n \times (0,\infty) \) is a space time domain. Let \( u^h(x,t) \) be the \( h \) inf-convolution through Definition 3.1. Assume \( (x,t) \) satisfies the following (i)-(iii),

(i) \( x \in \partial \{ w \in \mathbb{R}^n | (w,t) \in U, \ u^h(w,t) \leq \mu \} \);
(ii) \( (B_{r(t)}(x),t) \subseteq U \);
(iii) \( u^h(x,t) = \mu \).

Then there exists \( (y,t) \in U \), such that

\[ |y - x| = r(t) \sigma \quad \text{and} \quad u^h(z,t) \leq \mu, \ \text{if} \ z \in B_{r(t)\sigma}(y). \]

**Proof.** By the choice of \( x \), we get that \( \inf_{y \in B_{r(t)\sigma}(x)} u(y,t) = \mu \). Since \( u \in \text{LSC}(U) \), there exists \( y \in B_{r(t)\sigma}(x) \), such that \( u(y,t) = \mu \). Suppose \( |y - x| < r(t) \sigma \), then \( u^h(\cdot,t) \leq \mu \) in a neighborhood of \( x \), which contradicts to the choice of \( x \). Hence \( |y - x| = r(t) \sigma \). Moreover, \( u(\cdot,t) \geq \mu \) in the interior of \( B_{r(t)\sigma}(x) \). Next, let \( z \in B_{r(t)\sigma}(x) \), then \( y \in B_{r(t)\sigma}(z) \), therefore, \( u^h(z,t) \leq u(y,t) = \mu \).

Based on the construction of a super barrier as follows, we show that the superlevel set of an obstacle subsolution propagates with a finite speed.

**Proposition 3.2.** Fix \( (\delta,\mu,t_0,C) \in (0,\infty) \times \mathbb{R} \times (0,\infty) \times (1,\infty) \), \( a := (\nu, \mathcal{A}, q, s) \in \mathcal{H} \) and then set \( d := (\nu, R, \mathcal{A}) \in \mathcal{D} \) and \( h := (\delta,1) \in \mathbb{F} \). Assume that

(i) \( \nu(x,t) \in \mathcal{A}, \ u(x,t) \in \mathcal{A} \);
(ii) \( \Omega \) be a domain such that \( (\Omega + B_\delta(0),t) \subseteq C_d, t_0 - \Delta t \leq t \leq t_0; \)
(iii) \( v^h(x,t) : \Omega \times [t_0 - \Delta t, t_0] \to \mathbb{R} \) is defined through Definition 3.1;
(iv) \( 0 < \Delta t < \frac{C^2}{(n-1)^2 + (C-1)\delta + M_0} \) and \( u(x,t_0 - \Delta t) \prec_{(\Omega,\mu)} v^h(x,t_0) \).

Then

\[ u(x,t_0) \prec_{(\Omega,\mu)} v^h(x,t_0), \ \text{where} \ \hat{h} := \left( \frac{1}{C} \right) \delta, 1 \right) \in \mathbb{F}. \]

**Proof.** For any \( y \in L^-_\mu(\nu(\cdot,t_0);\Omega) \), let us define the superbarrier

\[ G_y(x,t) := \begin{cases} \mu, & x \in B_{r(t)}(y) \\ +\infty, & x \in \Omega \setminus B_{r(t)}(y) \end{cases}, \]

where

\[ r(t) := \delta - (t - t_0 + \Delta t) \cdot \left( \frac{(n-1)C}{(C-1)\delta + M_0} \right). \]

Here \( r(t) \) is chosen such that \( r(t_0 - \Delta t) = \delta \) and \( r(t_0) = \frac{1}{C} \delta \). In addition, \( r'(t) = -\left( \frac{(n-1)C}{(C-1)\delta + M_0} \right) \) guarantees that \( G_y(x,t) \) is a supersolution. i.e.,

\[
\partial_t G_y \geq \mathcal{F} \left( D^2 G_y, DG_y, x \right), \quad (x,t) \in \Omega \times (t_0 - \Delta t, t_0).
\]
On the other hand, let us consider the function

$$H_u(x, t) := \begin{cases} \mu, & u(x, t) \geq \mu \\ -\infty, & u(x, t) < \mu. \end{cases}$$

Since the operator $\mathcal{F}(\cdot, \cdot, \cdot)$ is geometric (c.f. [5]), it is clear that $H_u(x, t)$ is a subsolution. i.e.,

$$\partial_t H_u \leq \mathcal{F}(D^2 H_u, DH_u, x), \quad (x, t) \in \overline{\Omega} \times (t_0 - \Delta t, t_0).$$

Then Proposition 3.1 and an application of the usual comparison principle (c.f. Proposition 2.1), restricted to certain bounded domain if necessary, shows that

$$H_u(x, t) < G_y(x, t), \quad (x, t) \in \overline{\Omega} \times [t_0 - \Delta t, t_0].$$

In particular, it is true that

$$H_u(x, t_0) < G_y(x, t_0), \quad x \in \Omega.$$

Hence

$$L^+_{\mu}(u(\cdot, t_0); \Omega) \cap L^-_{\mu}(G_y(\cdot, t_0); \Omega) = \emptyset, \quad y \in L^-_{\mu}(v(\cdot, t_0); \Omega).$$

Notice that

$$\bigcup_{y \in L^-_{\mu}(v(\cdot, t_0); \Omega)} L^-_{\mu}(G_y(\cdot, t_0); \Omega) = L^-_{\mu}(v(\cdot, t_0); \Omega) + B_{1-\delta} = L^-_{\mu}(v^h(-\cdot, t_0); \Omega).$$

Based on Definition 3.2, the conclusion follows immediately. \(\square\)

3.2. Evolution law. The coming proposition shows that if we choose $h \in \mathbb{F}$ in an appropriate way, the $h$ inf-convolution of a (pseudo) supersolution is still a (pseudo) supersolution. This plays an important role, in later sections, in proving the local comparison principle.

**Proposition 3.3.** Let $h := (r(t), \varphi(x)) \in \mathbb{F}$ and assume the following (i)-(iv).

(i) $|r(t)D\varphi(x)| < 1$;

(ii) $r(t)$ and $\varphi(x)$ satisfy the differential inequality as follows.

$$r'(t) + \left(\frac{(n + 1)\|D^2\varphi\|_{\infty} + M_0|D\varphi(x)|}{\varphi(x)} + L_0\right) r(t) + \frac{|D\varphi(x)|^2 r(t)}{(1 - r(t)|D\varphi(x)|)^2 \varphi^2(x)} \leq 0 \quad (3.1)$$

where $M_0, L_0$ are from $(H)$;

(iii) Let $T > 0$, $\Omega \subseteq \mathbb{R}^n$ be an open set and $u(x, t) : \Omega \times (0, T) \to \mathbb{R}$ be a pseudo viscosity supersolution (c.f. Definition 2.5);

(iv) Denote the space domain $\Omega^h$ as follows:

$$\Omega^h := \left\{ x \in \Omega \mid \text{dist}(x, \Omega^c) > \sup_{(x, t) \in \Omega \times (0, T)} r(t) \varphi(x) \right\}.$$ 

Then $u^h(x, t) : \Omega^h \times (0, T) \to \mathbb{R}$ is also a pseudo viscosity supersolution.

**Proof.** Fix $(x_0, t_0) \in \Omega \times (0, T)$, $\phi \in C^{2,1}(\Omega \times (0, T))$ with (a) and (b) as below.

(a) $|D\phi(x_0, t_0)| > 0$;

(b) $u^h(x_0, t_0) - \phi(x_0, t_0) \leq u^h(x, t) - \phi(x, t), (x, t) \in \Omega \times (0, T)$.

Let us assume for simplicity that $u^h(x_0, t_0) = \phi(x_0, t_0) = \mu$, without loss of generality, we can take $\mu = 0$. The case of the general $\mu$ level set can be argued similarly. By the above (a), we have that $x_0 \in \partial \{ w \in \mathbb{R}^n \mid w \in \Omega, \phi(w, t_0) > 0 \}$. Similar to the proof of Proposition 3.1, there exists $y_0 \in \Omega$, such that

$$|y_0 - x_0| = r(t_0)\varphi(x_0) \quad \text{and} \quad u(y_0, t_0) = 0.$$ 

Denote the orthonormal basis of $\mathbb{R}^n$ by $e_1, e_2, \cdots, e_n$, such that $y_0 - x_0 = r(t_0)\varphi(x_0)e_1$.

**Step 1.** Let us define $\psi(y, t)$, which touches $u(y, t)$ from below at $(y_0, t_0)$, by the relation:

$$\phi(x, t) = \psi(x + r(t)\varphi(x)e_1, t).$$
Since $|r(t)D\varphi(x)| < 1$, we have $\det J \neq 0$, where $J$ is the Jacobian matrix associated to the map $x \mapsto x + r(t)\varphi(x)e_1$. i.e.,
\[
J = \begin{pmatrix}
1 + r(t)\varphi_{x_1}(x) & r(t)\varphi_{x_2}(x) & \cdots & r(t)\varphi_{x_n}(x) \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{pmatrix}.
\]
By the inverse function theorem, $\psi(y,t)$ exists in a neighborhood of $(y_0,t_0)$.

Step 2. By the choice of $\psi(y,t)$, let us set the (smooth) zero level set as follows
\[
\Gamma_\psi(t) := \partial \{y \in \Omega | \psi(y,t) > 0\},
\]
and we define the signed distance function $d^\psi(y,t)$ as below (c.f. [13]).
\[
d^\psi(y,t) := \begin{cases}
\text{dist} \{y, \Gamma_\psi(t)\}, & y \in \Gamma_\psi^+, \quad y \in \Omega \setminus \{z \in \Omega | \psi(z,t) > 0\} \\
0, & y \in \Gamma_\psi(t) \\
-\text{dist} \{y, \Gamma_\psi(t)\}, & y \in \Omega^-, \quad y \in \Omega \setminus \{z \in \Omega | \psi(z,t) < 0\}.
\end{cases}
\]
Then $d^\psi$ is well defined and smooth in a neighborhood of $(y_0,t_0)$. In general, although $d^\psi$ may not touch $u(y,t)$ at $(y_0,t_0)$ from below, the zero level set of $d^\psi$ indeed touches that of $u(y,t)$ from below. Since $|D\phi(x_0,t_0)| > 0$, then $|D\psi(y_0,t_0)| > 0$, the viscosity inequality satisfied by $\psi$ is equivalent to that
\[
\frac{\psi_t}{|D\psi|}(y_0,t_0) \geq -\kappa(y_0) + g(y_0),
\]
where $\kappa(y_0)$ denotes the mean curvature of the hypersurface $\{y \in \Omega | u(y,t_0) = 0\}$ at $y_0$. Since $|D\psi(y_0,t_0)| > 0$, this is equivalent to the same inequality with $\psi$ replaced by $d^\psi$, i.e.,
\[
d^\psi_t(y_0,t_0) \geq \mathcal{F}(D^2d^\psi(y_0,t_0), Dd^\psi(y_0,t_0), y_0),
\]
which can be rewritten as
\[
d^\psi_t(y_0,t_0) - [(\Delta - \Delta_{\infty})d^\psi(y_0,t_0) - g(y_0)]Dd^\psi(y_0,t_0) \geq 0,
\]
where $\Delta$ is the usual Laplace operator and $\Delta_{\infty}$ is defined as follows
\[
\Delta_{\infty}d^\psi := \frac{1}{|Dd^\psi|^2} \sum_{1 \leq i,j \leq n} \frac{\partial d^\psi}{\partial y_i} \cdot \frac{\partial^2 d^\psi}{\partial y_i \partial y_j} \cdot \frac{\partial d^\psi}{\partial y_j}.
\]
which is called the (normalized) infinity Laplace operator (c.f. [9, 20], etc.). It denotes the second derivative of \( d^\psi \) in the direction of \( \frac{Dd^\psi}{|Dd^\psi|} \).

Step 3. Let us introduce the function \( d^{#\phi}(x, t) \) as follows, which is well defined and smooth in a neighborhood of \( (x_0, t_0) \):

\[
d^{#\phi}(x, t) := d^\psi (x + r(t)\varphi(x)e_1, t). \tag{3.3}
\]

It is clear that the zero level set of \( d^{#\phi} \) coincides with that of \( \phi \), at least in a neighborhood of \( (x_0, t_0) \). However, it is not necessary the case that \( d^{#\phi} \) matches the signed distance function \( d^\phi \), associated to the test function \( \phi \). Our aim in the following steps is: by equation (3.2) and relation (3.3), we derive the evolution law of the zero level set of \( d^{#\phi} \) (i.e., the zero level set of \( \phi \)) at \( (x_0, t_0) \). From which, we get the viscosity inequality satisfied by the test function \( \phi \).

Step 4. Since \( d^\psi(y, t) \) (resp. \( d^{#\phi}(x, t) \)) is smooth around \( (y_0, t_0) \) (resp. \( (x_0, t_0) \)), we are allowed to differentiate it in the classical sense. In fact, we have the following properties (c.f. [13]) regarding the derivatives of \( d^\psi \).

1. \( |Dd^\psi(y, t_0)| = 1 \) in a neighborhood of \( y_0 \);
2. \( \frac{\partial d^\psi}{\partial y_i}(y_0, t_0) = -1 \) and \( \frac{\partial d^\psi}{\partial y_k}(y_0, t_0) = 0, \ k \neq 1 \);
3. \( \frac{\partial^2 d^\psi}{\partial y_k \partial y_l}(y_0, t_0) = 0 \), i.e., \( \Delta_\infty d^\psi(y_0, t_0) = 0 \).

The above (1) implies that \( \left( \frac{\partial}{\partial y_i} |Dd^\psi|^2 \right)(y_0, t_0) = 0 \), which is equivalent to that

\[
\sum_{i=1}^{n} \left( \frac{\partial^2 d^\psi}{\partial y_i \partial y_k} \cdot \frac{\partial d^\psi}{\partial y_i} \right)(y_0, t_0) = 0, \quad k = 1, \ldots, n.
\]

Then by (2), we have for any \( k \) that \( \frac{\partial^2 d^\psi}{\partial y_k \partial y_i}(y_0, t_0) = 0 \). A direct differentiation in (3.3) implies that

\[
\frac{\partial d^{#\phi}}{\partial x_k}(x_0, t_0) = \frac{\partial d^\psi}{\partial y_i}(y_0, t_0) + \frac{\partial d^\psi}{\partial y_k}(y_0, t_0) \cdot r(t_0) \frac{\partial \varphi}{\partial x_k}(x_0), \quad 1 \leq k \leq n,
\]

and then

\[
\frac{\partial^2 d^{#\phi}}{\partial x_k}(x_0, t_0) = \frac{\partial^2 d^\psi}{\partial y_k}(y_0, t_0) + 2 \frac{\partial^2 d^\psi}{\partial y_1 \partial y_k}(y_0, t_0) \cdot r(t_0) \frac{\partial \varphi}{\partial x_k}(x_0) + \frac{\partial^2 d^\psi}{\partial y_1}(y_0, t_0) \cdot r^2(t_0) \left( \frac{\partial \varphi}{\partial x_k}(x_0) \right)^2 + \frac{\partial d^\psi}{\partial y_1} \cdot r(t_0) \frac{\partial^2 \varphi}{\partial x_k}(x_0).
\]

Hence,

\[
\Delta d^{#\phi}(x_0, t_0) = \Delta d^\psi(y_0, t_0) - r(t_0) \Delta \varphi(x_0).
\]

Next, let us explore the evolution law of \( d^{#\phi}(x, t) \) through the following relation.

\[
d_t^{#\phi}(x_0, t_0) = d_t^\psi(y_0, t_0) + \frac{\partial d^\psi}{\partial y_i}(y_0, t_0) \cdot \varphi(x_0)r'(t_0)
\geq \Delta d^\psi(y_0, t_0) + g(y_0)|Dd^\psi(y_0, t_0)| - \varphi(x_0)r'(t_0)
= (\Delta - \Delta_\infty) d^{#\phi}(x_0, t_0) + g(x_0)|Dd^{#\phi}(x_0, t_0)|
= (-\kappa^{#\phi}(x_0) + g(x_0)) |Dd^{#\phi}(x_0, t_0)| + \frac{\Delta d^{#\phi}(x_0, t_0) + r(t_0) \Delta \varphi(x_0)}{\varepsilon_1} + \frac{\Delta d^{#\phi}(x_0, t_0) + r(t_0) \Delta \varphi(x_0)}{\varepsilon_2} + \frac{g(y_0) - g(x_0) |Dd^{#\phi}(x_0, t_0)|}{\varepsilon_3},
\]

where \( \kappa^{#\phi}(x_0) \) is the mean curvature of the set \( \{x \in \Omega | d^{#\phi}(x, t_0) = 0\} \) at \( x_0 \).

Step 5. Keep the choice of \( e_1 \), let us select \( e_2, \ldots, e_n \), so that \( D \varphi(x_0) = \alpha e_1 + \beta e_2 \) for two numbers \( \alpha \)}
and $\beta$. Recall the formula of $\Delta_\infty d^{\#\phi}(x_0, t_0)$ derived in (9.1) and $\Gamma^\psi(t_0)$ has an interior $r(t_0)\varphi(x_0)$ ball condition at $y_0$, which implies that

$$\frac{\partial^2 d}{\partial y_2^2}(y_0, t_0) \geq -\frac{1}{r(t_0)\varphi(x_0)}.$$  

Then, we can estimate the error term $\varepsilon_2$ as follows,

$$\varepsilon_2 := \Delta_\infty d^{\#\phi}(x_0, t_0) + r(t_0)\Delta \varphi(x_0)$$

$$\geq \frac{(r(t_0)\beta)^2}{(1 + r(t_0)\alpha)^2 + (r(t_0)\beta)^2} \cdot \frac{-1}{r(t_0)\varphi(x_0)} + r(t_0)\Delta \varphi(x_0)$$

$$- \frac{(1 + r(t_0)\alpha)^2 r(t_0)}{(1 + r(t_0)\alpha)^2 + (r(t_0)\beta)^2} \frac{\partial^2 \varphi}{\partial x_1^2}(x_0)$$

$$- \frac{(r(t_0)\beta)^2 r(t_0)}{(1 + r(t_0)\alpha)^2 + (r(t_0)\beta)^2} \frac{\partial^2 \varphi}{\partial x_2^2}(x_0)$$

$$\geq - \frac{r(t_0)|D\varphi(x_0)|^2}{(1 - r(t_0)|D\varphi(x_0)|^2) \varphi(x_0)} - (n + 1)r(t_0)||D^2 \varphi||_\infty.$$  

And then the term $\varepsilon_3$ as below:

$$\varepsilon_3 := g(y_0) - g(x_0)|Dd^{\#\phi}(x_0, t_0)|$$

$$= \left( g(y_0) - g(x_0) \right) \left( 1 - |Dd^{\#\phi}(x_0, t_0)| \right)$$

$$\geq -L_0r(t_0)|\varphi(x_0)| - M_0r(t_0)|D\varphi(x_0)|,$$

where $L_0$ and $M_0$ are from the hypothesis (H). The assumption (3.1) implies that $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 \geq 0$. Therefore,

$$d^{\#\phi}(x_0, t_0) \geq (-k^{\#\phi}(x_0) + g(x_0)) |Dd^{\#\phi}(x_0, t_0)|,$$

which is equivalent to (since $|D\phi(x_0, t_0)| > 0$) the inequality as follows, as desired:

$$\phi_t(x_0, t_0) \geq \mathcal{F}(D^2\phi(x_0, t_0), D\phi(x_0, t_0), x_0).$$

\[\square\]

4. Local comparison principle

4.1. Discrepancy and the lattice points. In this part, based on the language of discrepancy, we investigate the existence of certain lattice points that are arbitrarily close to a given hyperplane with irrational normal direction $\nu \in S^{n-1}\setminus \mathbb{R}Z^n$. Let us first recall some definitions and Lemmas.

**Definition 4.1 ([18, 7, 14]).** A bounded sequence $\{x_\ell\}_{\ell \geq 1} \subseteq [a, b]$ is said to be equidistributed on the interval $[a, b]$ if for any $[c, d] \subseteq [a, b]$, we have that

$$\lim_{\ell \to \infty} \frac{\{x_\ell\}_{i=1}^\ell \cap [c, d]}{\ell} = \frac{d - c}{b - a}.$$

**Definition 4.2.** A sequence $\{x_\ell\}_{\ell \geq 1}$ is said to be equidistributed modulo 1 if $\{x_\ell - [x_\ell]\}_{\ell \geq 1}$ is equidistributed in the interval $[0, 1]$.

**Lemma 4.1** (Weyl’s equidistribution theorem). Let $x \in \mathbb{R}\setminus \mathbb{Q}$, then $\{\ell x\}_{\ell \geq 1}$ is equidistributed modulo 1.

**Definition 4.3** ([18, 7, 14]). Let $\{x_n\}_{n \geq 1}$ be a sequence in $\mathbb{R}$. For a subset $E \subseteq [0, 1]$, let $A(E; N)$ denote the number of points $\{x_\ell\}_{i=1}^N$ that lie in $E$.

(a) The sequence $\{x_\ell\}_{\ell \geq 1}$ is said to be uniformly distributed modulo 1 in $\mathbb{R}$ if

$$\lim_{N \to \infty} \frac{A(E; N)}{N} = \mathcal{L}(E),$$

for all $E = [a, b] \subseteq [0, 1]$. Here $\mathcal{L}$ denotes the Lebesgue measure in $\mathbb{R}$.  


For \( x \in [0, 1] \), we define the discrepancy
\[
D_N(x) := \sup_{E = [a, b] \subseteq [0, 1]} \left| \frac{A(E; N)}{N} - \mathcal{L}(E) \right|,
\]
where \( A(E; N) \) is defined with the sequence \( \{ \ell x \}_{\ell \geq 1} \) modulo 1.

For \( x \in [0, 1] \), we define the modified discrepancy
\[
D^*_N(x) := \sup_{0 < a \leq 1} \left| \frac{A([0, a); N)}{N} - a \right|,
\]
where \( A(E; N) \) is defined with the sequence \( \{ \ell x \}_{\ell \geq 1} \) modulo 1.

**Lemma 4.2** ([18, 14]). Let us list some properties of the discrepancies as follows.

(i) The discrepancies \( D_N \) and \( D^*_N \) are equivalent up to constants:
\[
D^*_N \leq D_N \leq 2D^*_N.
\]

(ii) Let \( x_1 \leq x_2 \leq \cdots \leq x_N \) be in \([0, 1)\), then
\[
D^*_N = 1/2N + \max_{i=1,\ldots,N} \left| x_i - \frac{2i - 1}{2N} \right|.
\]

(iii) Fix any \( x \in \mathbb{R} \setminus \mathbb{Q} \), the modified discrepancy function \( D^*_N(\cdot) \) is continuous in a neighborhood of \( x \).

**Definition 4.4** ([14]). Fix any \( \nu = (\nu_1, \ldots, \nu_n) \in S^{n-1} \), let \( m_j(\nu) \), \( 1 \leq j \leq n \), be numbers such that \( m_j(\nu)|\nu|_{\ell^\infty} = |\nu_j| \), where \( |\nu|_{\ell^\infty} := \max_{1 \leq i \leq n} |\nu_i| \). Let us define the number \( \omega_{\nu}(N) \) as follows:
\[
\omega_{\nu}(N) := 2 \min_{1 \leq j \leq n} D^*_N(m_j(\nu)), \quad N > 1.
\]

Let us denote by \( H_{\nu} \) the hyperplane with \( \nu \in S^{n-1} \) as its normal direction. i.e.,
\[
H_{\nu} := \{ x \in \mathbb{R}^n | x \cdot \nu = 0 \}.
\]

**Lemma 4.3** ([18]). Fix \( \nu \in S^{n-1} \), then

(i) If \( \nu \in S^{n-1} \cap \mathbb{R}^n \), then \( \omega_{\nu}(\cdot) \in (0, 1) \) has a positive lower bound;

(ii) If \( \nu \in S^{n-1} \setminus \mathbb{R}^n \), then \( \lim_{N \to \infty} \omega_{\nu}(N) = 0 \).

**Proposition 4.4.** For any \( \nu \in S^{n-1} \setminus \mathbb{R}^n \) and \( 0 < \delta < 1 \), there exists \( R_0(\nu, \delta) > 0 \), such that the following statement holds: fix any \( R \geq R_0(\nu, \delta) \) and \( x_0 \in S^{n-1} \cap H_{\nu} \), there exists \( z_0 \in \mathbb{R}^n \), such that (i)-(iii) as follows are satisfied.

(i) \( \frac{\delta}{3} < z_0 \cdot \nu < \delta \); (ii) \( |z_0 - 2x_0| < \frac{R}{3} \); (iii) \( z_0 - x_0 \in \mathbb{Z}^n \).

Figure 2. A lattice point that is close to \( H_{\nu} \).
Proof. Since \( \nu \) is an irrational direction, by Lemma 4.3, there exists a positive integer \( N \) such that
\[
0 < \omega_\nu(N) < \frac{\delta}{3|\nu|_\infty}.
\]
Let us assume without loss of generality that
\[
\nu_n = |\nu|_\infty \quad \text{and} \quad \omega_\nu(N) = 2D_\nu^*(m_j(\nu)), \quad \text{where} \quad m_j(\nu) \in \mathbb{R} \setminus \mathbb{Q}.
\]
Pick \( s \in (0,1)^n \) such that \( -x_0 = s \mod \mathbb{Z}^n \), then
\[
s = \left( s - \frac{s \cdot \nu}{\nu_n} e_n \right) + \frac{s \cdot \nu}{\nu_n} e_n, \quad \text{where} \quad \left( s - \frac{s \cdot \nu}{\nu_n} e_n \right) \in H_\nu.
\]
Let us only consider the case \( |\nu_j| = -\nu_j \) since the other case can be analyzed similarly. Then the sequence of vectors \( k(m_j(\nu)e_n + e_j), 1 \leq k \leq N \) lie on \( H_\nu \). Because \( 0 < D_\nu(m_j(\nu)) \leq \omega_\nu(N) < \frac{\delta}{3|\nu|_\infty} \), there exists \( 1 \leq k_0 \leq N \), such that if \( \frac{2\delta}{3|\nu|_\infty} < \frac{s \cdot \nu}{\nu_n} \), then take \( k_0m_j(\nu) - [k_0m_j(\nu)] \in \left( \frac{2\delta}{3|\nu|_\infty} \right) \equiv \left( \frac{s \cdot \nu}{\nu_n} \right) \).
\[
\frac{\delta}{3|\nu|_\infty} < b := \left( 1 + \frac{s \cdot \nu}{\nu_n} \right) \left( k_0m_j(\nu) - [k_0m_j(\nu)] \right) < 3\omega_\nu(N) < \delta.
\]
And if \( 0 < \frac{s \cdot \nu}{\nu_n} < \frac{\delta}{3|\nu|_\infty} \), then take \( k_0m_j(\nu) - [k_0m_j(\nu)] \in \left( 1 - \frac{2\delta}{3|\nu|_\infty}, 1 - \frac{\delta}{3|\nu|_\infty} \right) \).
\[
\frac{\delta}{3|\nu|_\infty} < b := \left( 1 + \frac{s \cdot \nu}{\nu_n} \right) \left( k_0m_j(\nu) - [k_0m_j(\nu)] \right) < 3\omega_\nu(N) < \delta.
\]
Let us denote \( y_0 \) and \( z_0 \) as follows to guarantee (i) and (ii).
\[
y_0 := 2x_0 + k_0(m_j(\nu)e_n + e_j) + \left( s - \frac{s \cdot \nu}{\nu_n} e_n \right) \in H_\nu \quad \text{and} \quad z_0 := y_0 + be_n.
\]
Finally, let us estimate \( |z_0 - 2x_0| \),
\[
|z_0 - 2x_0| \leq |z_0 - y_0| + |y_0 - 2x_0| < 2 + 2N + (1 + \sqrt{n}) = 2N + \sqrt{n} + 3.
\]
Hence, let us set \( R_\nu(\nu, \delta) := 6N + 3\sqrt{n} + 9 \), where \( N \) is picked such that \( 0 < \omega_\nu(N) < \frac{\delta}{3|\nu|_\infty} \). And (iii) follows immediately. \( \square \)

Remark 4.5. The above arguments are motivated by similar proofs in [7, 14].

Remark 4.6. Since \( D_\nu^*(\cdot) \) is continuous in a neighborhood of any \( x \in \mathbb{R} \setminus \mathbb{Q} \), \( \omega_\nu(N) \) is also continuous in an \( \mathbb{S}^{n-1} \)-neighborhood of any \( \nu \in \mathbb{S}^{n-1} \setminus \mathbb{R}\mathbb{Z}^n \). Therefore, the point \( z_0 = z_0(\nu) \), which is characterized as above, depends on \( \nu \) continuously in a neighborhood of \( \nu \).

4.2. The local comparison principle.

Definition 4.5. The triplet \( (\nu, T, R) \in \mathbb{S}^{n-1} \times (0, \infty) \times \left( \frac{\sqrt{n}}{2}, \infty \right) \) is called comparison consistent if the following property holds: there exists \( 0 < \delta < \delta(T) \), such that for any \( x_0 \in \partial \Omega(0, R; \nu) \) (see (2.5)), we can find \( z_0 \), such that the following (i)-(iii) hold:
\[
(i) \quad \frac{\delta}{3} < (z_0 - x_0) \cdot \nu < \delta; \quad (ii) \quad |z_0 - 2x_0| < \frac{R}{3}; \quad (iii) \quad z_0 - x_0 \in \mathbb{Z}^n,
\]
here \( \delta(T) \) is defined as follows.
\[
\delta(T) := \frac{M_0m_0}{M_0 - m_0} \cdot \frac{\gamma^2(T)}{\left( \sqrt{M_0\gamma(T) + (n - 1)} + \sqrt{n - 1} \right)^2}, \quad \text{with} \quad \gamma(t) := \frac{1}{2} e^{-2L_0t}.
\]
Proposition 4.7 (Local comparison principle). Fix any \( m_0 \leq s_1 < s_2 \leq M_0, q \in \mathbb{R}^n \setminus \{0\} \) and \( \xi_0 \in \text{arg min} |\xi| \). Let \((\nu,T,R)\) be a comparison consistent triplet (c.f. Definition 4.5), such that \( R > \frac{12(3n+M_0+27)}{L_0} \), then

\[
\bar{U}_{a_2}(x,t) < \underline{U}_{a_1}(x-\xi_0,t), \quad \text{if} \quad x \in \Omega(0,R;\nu), \quad 0 \leq t < T,
\]

where \( a_i = (\nu,R,q,s_i), i = 1,2, R = \frac{4M_0R}{\delta} \), and \( \Omega(0,R;\nu) \) is defined in (2.5).

Proof. Let us denote \( d := (\nu,R) \) and define functions

\[
\gamma(t) = \frac{1}{2} e^{-2\lambda t} \quad \text{and} \quad U(x,t) := \underline{U}_{a_1}(x-\xi_0,t)
\]

The choice of \( \xi_0 \) and the location of the obstacle implies that (c.f. Definition 2.8)

\[
\bar{U}_{a_2}(x,t) < U(x,t), \quad x \in C_d \cap (C_d + \xi_0), \quad 0 \leq t < \frac{1}{s_2 - s_2}.
\]

In order to establish the desired result, we assume on the contrary \( \sup \{ t > 0 | \bar{U}_{a_2}(x,t) < U(x,t), \quad x \in \Omega(0,R;\nu) \} \leq T \) and derive a contradiction through the following steps.

Step 1. Let \( h := (\gamma(t),1) \in \mathbb{F} \) and \( U^{h_-(x,t)} \) be the \( h \) inf-convolution of \( U(x,t) \) by Definition 3.1. Then the above assumption (4.1) shows that

\[
\frac{1}{2(s_2 - s_1)} \leq t_0 := \sup \{ t > 0 | \bar{U}_{a_2}(x,t) < U^{h_-(x,t)}, \quad x \in \Omega(0,R;\nu) \} \leq T.
\]

Assume \( x_0 \) is the first crossing point between \( \bar{U}_{a_2} \) and \( U^{h_-(x,t)} \), more precisely,

\[
\bar{U}_{a_2}(x_0,t_0) = U^{h_-(x_0,t_0)}.
\]

By applying Proposition 2.1 to \( \bar{U}_{a_2}(\cdot,t) \) and \( U^{h_-(x,t)} \) in \( \Omega(0,R;\nu) \) (we can only consider a bounded subdomain if necessary), the maximum of \( \bar{U}_{a_2}(x,t) - U^{h_-(x,t)} \) is obtained on \( \partial \Omega(0,R;\nu) \). Since \( h \) is \( x \)-independent, \( x_0 \in \partial \Omega(0,R;\nu) \). Therefore, \( \bar{U}_{a_2}(x_0,t_0) = U^{h_-(x_0,t_0)} = \mu \) for some \( \mu \in \mathbb{R} \).

Step 2. By the choice of \((\nu,T,R)\), we can find \( 0 < \delta < \delta(T) \). Moreover, there exists a constant \( C > 1 \), such that

\[
\delta = \left( \frac{M_0m_0}{M_0 - m_0} \right) \cdot \frac{(C - 1)\gamma^2(T)}{(n-1)C^2 + C(C - 1)\gamma(T)M_0}.
\]

Moreover, there exists \( z_0 \in \mathbb{R}^n \) with the following (i)-(iii) satisfied.

\[
(i) \quad \frac{\delta}{3} < (z_0 - x_0) \cdot \nu < \delta; \quad (ii) \quad |z_0 - 2x_0| < \frac{R}{3}; \quad (iii) \quad z_0 - x_0 \in \mathbb{Z}^n.
\]

Step 3. Let us investigate the ordering relation between \( \bar{U}_{a_2}(x,t) \) and \( U^{h_-(x,t)} \) in the cylinder domain \( \Omega(x_0,\frac{2R}{3};\nu) \). Let us choose

\[
\Delta z := z_0 - x_0; \quad \sigma := |\Delta z \cdot \nu|; \quad \Delta t := \frac{\sigma}{s_1} - \frac{\sigma}{s_2}; \quad \mathcal{R} := \frac{4M_0R}{\delta}.
\]

Then Proposition 2.6 indicates that

\[
\bar{U}_{a_2}(x,t) \leq \bar{U}_{a_2}\left(x - \Delta z, t - \frac{\sigma}{s_2}\right), \quad x \in \Omega(x_0,\frac{2R}{3};\nu), \quad \frac{\sigma}{s_2} \leq t \leq t_0 - \Delta t.
\]

Because of the inclusion \( \Omega(x_0,\frac{2R}{3};\nu) + \Delta z \subseteq \Omega(0,R;\nu) \) and (4.2), we have that

\[
\bar{U}_{a_2}\left(y, t - \frac{\sigma}{s_2}\right) < U^{h_-(y,t - \frac{\sigma}{s_2})}, \quad y \in \Omega(0,R;\nu), \quad \frac{\sigma}{s_2} \leq t \leq t_0 - \Delta t.
\]

The Proposition 2.7 applied to the supersolution \( \bar{U}_{a_1} \) with above shift \( \Delta z \) shows that

\[
\bar{U}_{a_1}\left(y, t - \frac{\sigma}{s_2}\right) \leq \bar{U}_{a_1}\left(y + \Delta z, t + \frac{\sigma}{s_1} - \frac{\sigma}{s_2}\right)
\]

\[
= \bar{U}_{a_1}\left(y + \Delta z, t + \Delta t\right), \quad y \in \Omega(0,R;\nu), \quad \frac{\sigma}{s_2} \leq t \leq t_0 - \Delta t.
\]
Then apply the $\xi_0$ shift and the $h$ inf-convolution to both sides, we conclude that
\[
U^{h-} \left( y, t - \frac{\sigma}{s_2} \right) \leq U^{h-} \left( y + \Delta z, t + \Delta t \right), \quad y \in \Omega \left( 0, R; \nu \right), \quad \frac{\sigma}{s_2} \leq t \leq t_0 - \Delta t.
\] (4.8)

A combination of (4.6), (4.7) and (4.8) gives the relation
\[
\bar{U}_{a_2}(x, t) < U^{h-} (x, t + \Delta t), \quad x \in \Omega \left( x_0, \frac{R}{3}; \nu \right), \quad \frac{\sigma}{s_2} < t < t_0 - \Delta t.
\]

In particular,
\[
\bar{U}_{a_2}(x, t_0 - \Delta t) < U^{h-} (x, t_0), \quad x \in \Omega \left( x_0, \frac{R}{3}; \nu \right).
\]

By the choice of $\Delta t$ through (4.2), (4.3), (4.4) and (4.5),
\[
0 < \Delta t < \frac{(C - 1)\gamma^2(t_0)}{(n - 1)C^2 + C(C - 1)\gamma(t_0)M_0}.
\]

Then Proposition 3.2 implies that
\[
\bar{U}_{a_2}(x, t_0) < U^{h-} (x, t_0), \quad x \in \Omega \left( x_0, \frac{R}{3}; \nu \right), \quad \text{where } \hat{h} := \left( \left( 1 - \frac{1}{C} \right) \gamma(t), 1 \right).
\]

Step 4. Let us construct $\varphi(x) : \Omega \left( x_0, \frac{R}{3}; \nu \right) \to \mathbb{R}$ as follows:
\[
\varphi(x) := -\frac{9(1 + C)}{2CR^2} \left| (x - x_0)^\top \right|^2 + \frac{1}{2} \left( 3 - \frac{1}{C} \right),
\]

where $(x - x_0)^\top := (x - x_0) - ((x - x_0) \cdot \nu) \nu$. Then $\varphi(x)$ satisfies that
\[
\begin{align*}
\varphi \left| \left( (x-x_0)^\top = 0 \right) \right. &= \frac{1}{2} \left( 3 - \frac{1}{C} \right) > 1, \\
\varphi \left| \left( (x-x_0)^\top = \frac{2}{3} \right) \right. &= 1 - \frac{1}{C},
\end{align*}
\]

and
\[
\begin{align*}
\left\| D\varphi \right\|_\infty &\leq \frac{6}{\pi}, \\
\left\| D^2\varphi \right\|_\infty &\leq \frac{18}{\pi^2}.
\end{align*}
\]

Because $R \geq \max \left\{ 6, \frac{12(3n+M_0+27)}{L_0} \right\}$, then
\[
\gamma'(t) + \left( \frac{(n + 1) \left\| D^2\varphi \right\|_\infty + M_0 |D\varphi(x)|}{\varphi(x)} + L_0 \right) \gamma(t) + \frac{|D\varphi(x)|^2 \gamma(t)}{(1 - \gamma(t)|D\varphi(x)|)^2 \varphi(x)} \leq 0.
\]

Then by Proposition 3.3, $U^{(\gamma, \varphi)}_\gamma(x, t)$ is a pseudo viscosity supersolution. Based on the finite speed of propagation regarding the subsolution (c.f. Proposition 3.2), we have for some $\tau > 0$ that
\[
\bar{U}_{a_2}(x, t) < U^{(\gamma, \varphi)}_\gamma(x, t), \quad \frac{1}{2(M_0 - m_0)} \leq t \leq t_0 + \tau, \quad x \in \partial \Omega \left( x_0, \frac{R}{3}; \nu \right).
\] (4.9)

Note that the following strict ordering relation holds.
\[
\bar{U}_{a_2} \left( x, \frac{1}{2(M_0 - m_0)} \right) < U^{(\gamma, \varphi)}_\gamma \left( x, \frac{1}{2(M_0 - m_0)} \right), \quad x \in \Omega \left( x_0, \frac{R}{3}; \nu \right).
\] (4.10)

Recall that $x_0$ is the first touching point and $\bar{U}_{a_2}(x_0, t_0) = U^{h-} (x_0, t_0) = \mu$, which is a contradiction due to Proposition 9.1.

\[
\square
\]

5. HEAD AND TAIL SPEEDS

5.1. Detachment.

**Definition 5.1.** Let $a = (\nu, R, \mathcal{A}, q, s) \in \mathcal{A}$ with $\mathcal{A} \geq 0$ and set $e := (\nu, q, s) \in \mathcal{E}$.

1. Let $\mu \in \mathbb{R}$, the obstacle subsolution $\bar{U}_a(x, t)$ (resp. supersolution $U_a(x, t)$) detaches from the associated obstacle $O_e(x, t)$ at the $\mu$ level set if there exist $r > \frac{\sqrt{n}}{2}$ and $T > 0$, such that (c.f. Definition 3.2)
\[
\bar{U}_a(., t) \prec_{(\Omega(0, r; \nu), \mu)} O_e(., t), \quad t > T; \quad (\text{resp. } O_e(., t) \prec_{(\Omega(0, r; \nu), \mu)} \bar{U}_a(., t), \quad t > T).
\]
(2) The obstacle subsolution $\mathbb{U}_a(x,t)$ (resp. supersolution $\mathbb{U}_a(x,t)$) detaches from the associated obstacle $O_e(x,t)$ if for any $\mu \in \mathbb{R}$, there exists $T = T(\mu) > 0$, such that

\[
\mathbb{U}_a(\cdot,t) \prec_{(\Omega(0,r^\mu),\mu)} O_e(\cdot,t), \quad t > T(\mu); \quad (\text{resp.} \quad O_e(\cdot,t) \prec_{(\Omega(0,r^\mu),\mu)} \mathbb{U}_a(\cdot,t), \quad t > T(\mu)).
\]

(3) The obstacle subsolution $\mathbb{U}_a(x,t)$ (resp. supersolution $\mathbb{U}_a(x,t)$) detaches uniformly from the associated obstacle $O_e(x,t)$ if there exist $r > \frac{1}{n}$ and $T > 0$, such that

\[
\mathbb{U}_a(\cdot,t) \prec_{(\Omega(0,r^\mu),\mu)} O_e(\cdot,t), \quad t > T \quad \text{for any} \quad \mu \in \mathbb{R}
\]

(\text{resp.} \quad O_e(\cdot,t) \prec_{(\Omega(0,r^\mu),\mu)} \mathbb{U}_a(\cdot,t), \quad t > T \quad \text{for any} \quad \mu \in \mathbb{R}).

**Lemma 5.1.** Let $a := (\nu, R, A, q, s) \in \mathbb{A}$ with $R \geq 0$, and then set $d := (\nu, q, s) \in \mathbb{E}$. Assume that $\mathbb{U}_a$ (resp. $\mathbb{U}_a$) (uniformly) detaches from $O_e$. Then for any number $\zeta > 0$, $\mathbb{U}_a$ (resp. $\mathbb{U}_a$) (uniformly) detaches from $O_e$, where

\[
\hat{a} := (\nu, R, A, \zeta q, s) \quad \text{and} \quad \hat{e} := (\nu, \zeta q, s).
\]

**Proof.** It follows from Definition 5.1 and the facts as follows.

\[
\mathbb{U}_a(x,t) = \zeta \mathbb{U}_a(x,t), \quad \mathbb{U}_a(x,t) = \zeta \mathbb{U}_a(x,t), \quad O_e(x,t) = \zeta O_e(x,t).
\]

\[\square\]

**Proposition 5.2.** Let $a := (\nu, R, A, q, s) \in \mathbb{A}$, then set $d := (\nu, R, A) \in \mathbb{D}$ and $e := (\nu, q, s) \in \mathbb{E}$, for any space time domain $\Sigma \subseteq C_d$ (see (2.4)), we have that

\begin{enumerate}[(i)]
  \item If $\mathbb{U}_a(x,t) < O_e(x,t)$, for $(x,t) \in \Sigma$, then $(\mathbb{U}_a)_s$ is a supersolution in $\Sigma$;
  \item If $O_e(x,t) < \mathbb{U}_a(x,t)$, for $(x,t) \in \Sigma$, then $(\mathbb{U}_a)_s$ is a subsolution in $\Sigma$.
\end{enumerate}

**Proof.** Let us only prove (i) since (ii) can be established by parallel arguments. Let $\phi(x,t)$ be a $C^{2,1}$ function on $\Sigma$, assume that $(\mathbb{U}_a)_s(x,t) - \phi(x,t)$ obtains its local minimum at $(x_0, t_0) \in \Sigma$, without loss of generality, we can assume that $(\mathbb{U}_a)_s(x_0, t_0) = \phi(x_0, t_0)$. To prove that $(\mathbb{U}_a)_s$ is a viscosity supersolution in $\Sigma$, let us assume on the contrary that

\[
\phi_t(x_0, t_0) < \mathbb{F}_s(D^2 \phi(x_0, t_0), D\phi(x_0, t_0), x_0).
\]

Then by the continuity of $\phi_t$, $D\phi$, $D^2\phi$ and the lower semi continuity of $\mathbb{F}_s$, there exists a neighborhood of $(x_0, t_0)$ (let us still denote it by $\Sigma$), where the above strict inequality holds. By subtracting a multiple of $|x-x_0|^4 + (t-t_0)^2$ from $\phi(x,t)$ if necessary (this does not change the above inequality), we can assume that $(\mathbb{U}_a)_s(x,t) - \phi(x,t)$ has a strict minimum at $(x_0, t_0)$, over $\Sigma$. Then there exists $\delta > 0$, such that

\begin{enumerate}[(a)]
  \item $\phi(x,t) + \delta < O_e(x,t)$, for $(x,t) \in \Sigma$;
  \item $\phi(x,t) + \delta < (\mathbb{U}_a)_s(x,t)$, for $(x,t) \in \partial \Sigma$.
\end{enumerate}

Then we can define the function as follows,

\[
\hat{\phi}(x,t) := \max \{ \mathbb{U}_a(x,t), \phi(x,t) + \delta \},
\]

which is also a viscosity subsolution in $C_{\mathbb{D}}$. On the other hand, there exists $(\hat{x}, \hat{t})$ arbitrarily close to $(x_0, t_0)$, such that $\mathbb{U}_a(\hat{x}, \hat{t}) < \phi(\hat{x}, \hat{t}) + \delta$. Therefore, $\hat{\phi}(\hat{x}, \hat{t}) > \mathbb{U}_a(\hat{x}, \hat{t})$, which contradicts to the maximality of $\mathbb{U}_a$. \[\square\]

### 5.2. Irrational directions.

#### 5.2.1. Basic facts of head/tail speed.

**Definition 5.2.** Let $\nu \in \mathbb{S}^{n-1} \setminus \mathbb{R}^n$, the head speed in $\nu$ direction, denoted by $\mathbb{F}(\nu)$, is defined as the smallest number, such that: for any $\delta > 0$, there exists $R > 0$, such that $\mathbb{U}_a(x,t)$ detaches (c.f. Definition 5.1) from $O_e(x,t)$, where

\[
a := (\nu, R, 0, q, \mathbb{F}(\nu) + \delta) \in \mathbb{A}, \quad e := (\nu, q, \mathbb{F}(\nu) + \delta) \in \mathbb{E}.
\]
Definition 5.3. Let $\nu \in S^{n-1} \setminus \mathbb{R} \mathbb{Z}^n$, the tail speed in $\nu$ direction, denoted by $\underline{s}(\nu)$, is defined as the largest number, such that: for any $\delta > 0$, there exists $R > 0$, such that $\overline{U}_a(x,t)$ detaches (c.f. Definition 5.1) from $O_e(x,t)$, where

$$a := (\nu, R, 0, q, \underline{s}(\nu) - \delta) \in \mathbb{A}, \quad e := (\nu, q, \underline{s}(\nu) - \delta) \in \mathbb{E}.$$ 

Remark 5.3. By Lemma 5.1, the head speed $\overline{s}(\nu)$ and the tail speed $\underline{s}(\nu)$ are independent of $q$, therefore, they are both well-defined.

Lemma 5.4. Fix any $\nu \in S^{n-1} \setminus \mathbb{R} \mathbb{Z}^n$, then $m_0 \leq \overline{s}(\nu), \underline{s}(\nu) \leq M_0$.

Proof. Let us only prove $m_0 \leq \overline{s}(\nu) \leq M_0$, since a parallel argument applies to $\underline{s}(\nu)$. Fix any $R, q$ and $s < m_0$ (resp. $s > M_0$), such that $a := (\nu, R, 0, q, s) \in \mathbb{A}$, then set $e := (\nu, q, s) \in \mathbb{E}$. Then $O_e(x,t)$ is a subsolution (resp. supersolution) and $\overline{U}_a(x,t) = O_e(x,t)$ (resp. $\underline{U}_a(x,t) = O_e(x,t)$). In this case, there is no detachment between $\overline{U}_a$ (resp. $\underline{U}_a$) and $O_e$, hence $\overline{s}(\nu) \geq m_0$ (resp. $\underline{s}(\nu) \leq M_0$). \hfill $\Box$

The next Lemma says that in the case of expanding domain, i.e., $\mathcal{R} > 0$, if the detachment happens at certain level set, then for any $r > 0$, the sub and super solution shall be away from the obstacle in $\Omega(0; r; \nu)$, after certain amount of time. i.e., the detachment expands as time evolves.

Lemma 5.5. Let $a := (\nu, R, \mathcal{R}, q, s) \in \mathbb{A}$ with $\mathcal{R} > 0$, $\mu \in \mathbb{R}$ and then set $e := (\nu, q, s) \in \mathbb{E}$, if $\overline{U}_a(x,t)$ (resp. $\underline{U}_a(x,t)$) detaches from $O_e(x,t)$ in the $\mu$ level set (c.f. Definition 5.1), then for any $r > 0$, there exists $T := T(\mu, s, \mathcal{R}) > 0$, such that

$$\overline{U}_a(\cdot, t) \prec_{(\Omega(0, r, \mu), \nu)} O_e(\cdot, t), \quad (\text{resp. } \underline{U}_a(\cdot, t) \prec_{(\Omega(0, r, \mu), \nu)} O_e(\cdot, t)), \quad t > T.$$ 

Proof. Let us only prove the case associated to the obstacle subsolution $\overline{U}_a$, since the other case can be argued in a similar way. By Definition 5.1, there exist $T_0 > \sqrt{\mathcal{R}^n}$ and $T_0 = T_0(\mu) > 0$, such that

$$\overline{U}_a(\cdot, t) \prec_{(\Omega(0, r, \mu), \nu)} O_e(\cdot, t), \quad t > T_0.$$ 

Denote

$$K := \Omega(0, r_0; \nu) \cap \{ x \in \mathbb{R}^n | sT_0 \leq x \cdot \nu \leq sT_0 + \sqrt{n} \}.$$ 

For any $x \in \Omega(0; r; \nu) \setminus \Omega(0, r_0; \nu)$, there exists $B \in \mathbb{R}$, such that

$$\hat{x} := x + B \nu \in \left\{ y \in \mathbb{R}^n | y \cdot \nu > s \left( T_0 + \frac{\sqrt{n}}{s} + \frac{r + r_0}{\mathcal{R}} \right) \right\}.$$ 

Then there exists $\hat{\omega} \in K$, such that

$$\Delta z := \hat{x} - \hat{\omega} \in \mathbb{Z}^n \quad \text{and} \quad x - \Delta z \in \Omega(0, r_0; \nu).$$ 

Set $\Delta t := \frac{\Delta z \cdot \nu}{s}$, then $\Delta z$ and $\Delta t$ satisfy the assumptions in Proposition 2.6. Let us take $T(\mu, s, \mathcal{R}) := T_0 + \Delta t$, then we have for any $t > T$ that

$$\overline{U}_a(x,t) \leq \overline{U}_a(x - \Delta z, t - \Delta t) \prec_{(\Omega(0, r_0; \mu), \nu)} O_e(\cdot, t - \Delta t) = O_e(x, t).$$ 

\hfill $\Box$

The next Lemma is a static version of Lemma 5.5. i.e., we can expect that the sub and supersolution be away from its obstacle in $\Omega(0; r; \nu)$, for any $r > 0$, as long as the obstacle sub and supersolution is initially defined in $\Omega(0; R; \nu)$ with sufficiently large $R$.

Lemma 5.6. Fix $\nu \in S^{n-1} \setminus \mathbb{R} \mathbb{Z}^n$, $\delta > 0$ and set $s := \overline{s}(\nu) + \delta$. Then for any $\mu \in \mathbb{R}$ and $r > 0$, there exist $R = R(\nu, \mu, \delta, r) > 0$ and $T = T(\nu, \mu, \delta) > 0$, such that

$$\overline{U}_a(\cdot, t) \prec_{(\Omega(0, r, \mu), \nu)} O_e(\cdot, t), \quad t > T,$$

where

$$a := (\nu, R, 0, q, s) \in \mathbb{A} \quad \text{and} \quad e := (\nu, q, s) \in \mathbb{E}.$$
Proof. As $s > \pi(\nu)$, there exist three numbers
\[ r_0 > \frac{\sqrt{n}}{2}, \quad R_0 = R_0(\nu, \mu, \delta) > 0 \quad \text{and} \quad T_0 = T_0(\nu, \mu, \delta) > 0, \]
such that (where $a_0 := (\nu, R_0, 0, q, s) \in \mathbb{A}$)
\[ \mathcal{U}_{a_0}(\cdot, t) \prec_{(\Omega(0, r_0; \nu), \mu)} O_\nu(\cdot, t), \quad t > T_0. \]  
(5.1)
Let us take $R$ and $T$, such that
\[ R > R_0 + r + r_0 \quad \text{and} \quad T > T_0 + \frac{2\sqrt{n}}{m_0}. \]
For any $x \in \Omega(0, r; \nu)$, let us choose $A \in \mathbb{R}$, such that
\[ \hat{x} \cdot \nu - T_0 s \in (\sqrt{n}, 2\sqrt{n}) \quad \text{where} \quad \hat{x} = x + A \nu. \]
Then there exists $\hat{x} \in \Omega(0, r_0)$, such that
\[ \hat{x} \cdot \nu - T_0 s \in (0, \sqrt{n}) \quad \text{and} \quad \Delta z := \hat{x} - \hat{x} \in \mathbb{Z}^n. \]
Let us set $\Delta t := \frac{\Delta z \cdot \nu}{\hat{x} \cdot \nu}$, then by Proposition 2.8, we conclude that
\[ \mathcal{U}_a(x, t) = \bar{\mathcal{U}}_{a}(x - \Delta z + \Delta z; t - \Delta t + \Delta t) \leq \mathcal{U}_{a_0}(x - \Delta z, t - \Delta t), \quad x \in \Omega(0, R; \nu). \]
(5.1)
Then for any $t > T$, we have that $t - \Delta t > T_0$, hence
\[ \mathcal{U}_a(\cdot, t) \prec_{(\Omega(0, r; \nu), \mu)} O_\nu(\cdot, t), \quad t > T, \]
where we have used (5.1) and the fact that $O_\nu(x - \Delta z; t - \Delta t) = O_\nu(x, t)$.

\[ \square \]

**Lemma 5.7.** Fix $\nu \in \mathbb{S}^{n-1} \setminus \mathbb{R} \mathbb{Z}^n$, $\delta > 0$, and set $s := \pi(\nu) - \delta$. Then for any $\mu \in \mathbb{R}$ and $r > 0$, there exist $R = R(\nu, \mu, \delta, r) > 0$ and $T = T(\nu, \mu, \delta) > 0$, such that
\[ O_\nu(\cdot, t) \prec_{(\Omega(0, r; \nu), \mu)} \mathcal{U}_a(\cdot, t), \quad t > T, \]
where
\[ a := (\nu, R, 0, q, s) \in \mathbb{A} \quad \text{and} \quad e := (\nu, q, s) \in \mathbb{E}. \]

**Proof.** It is similar to the above Lemma 5.6. Instead of using Proposition 2.8, Proposition 2.9 should be applied. Hence, we omit the details here. \[ \square \]

**5.2.2. The second version of local comparison principles.** Recall that in Proposition 4.7, we established an ordering relation between an obstacle subsolution with a fast obstacle speed and an obstacle supersolution with a slow obstacle speed. In this part, we shall compare two obstacle subsolutions with the obstacle speed over its tail speed. In this case, the obstacle subsolution detaches from its obstacle, so it is actually a supersolution. In order to apply the Birkhoff property with a right monotonicity, we shall require a shrinking domain.

**Proposition 5.8.** Let $(\nu, T, R)$ be a comparison consistent triplet (c.f. Definition 4.5), fix $\mu \in \mathbb{R}$ and $s_i := \pi(\nu) + \delta_i$, $i = 1, 2$, where $\delta_2 > \delta_1 > 0$ are two fixed numbers. Then for any $r > \max \left\{ R, R_0, \frac{12(3n + M_0 + 27)}{\hat{L}_a} \right\}$,
\[ T > 0, \quad \text{where} \quad R_0 \text{ is the radius such that the detachment occurs for the following} \mathcal{U}_{a_i}, i = 1, 2. \]
Then there exist $A > 0$ (independent of $r$ and $T$), $R_i$ and $\mathcal{I}_i > 0$, $i = 1, 2$, such that the following (i) and (ii) hold.
\[ (i) \quad R_1 \geq R_2 + (\mathcal{I}_1 + \mathcal{I}_2) T; \]
\[ (ii) \quad \mathcal{U}_{a_2}(\cdot, t) \prec_{(\Omega(0, r; \nu), \mu)} \mathcal{U}_{a_1}(\cdot - \xi A, t), \quad 0 \leq t \leq T, \]
where
\[ a_1 := (\nu, R_1, -\mathcal{I}, q, s_1), \quad a_2 := (\nu, R_2, \mathcal{I}, q, s_2) \in \mathbb{A} \quad \text{and} \quad \xi A \in \arg \min_{\xi \in \mathbb{E}, \xi \nu > A} |\xi|.$
Proof. The key idea of the argument is similar to that of Proposition \ref{prop:mu-Minkowski}. Since there are several modifications in both the Proposition and the proof, we still provide the details as follows. Because $s_i > \tau(\nu)$, there exist $T_0 > 0$ and $R_0 > 2r$, such that

$$\mathbb{U}_{b_i}(\cdot, t) \prec_{(\Omega(0, t, r, \nu), \mu)} O_{\nu}(\cdot, t), \ t > T_0, \quad \text{where} \quad b_i := (\nu, R_0, 0, q, s_i), \ i = 1, 2.$$ 

Let us set $A := (s_2 - m_0)T_0 + 1$ and $R_2 > R_0$, then

$$\mathbb{U}_{a_2}(\cdot, t) \prec_{(\Omega(0, 2r, r, \nu), \mu)} \left(\mathbb{U}_{a_1}\right)_*(x - \xi_A, t), \quad 0 \leq t \leq T_0.$$ 

Recall the Proposition \ref{prop:zero-order} (with \(\Delta z = 0 \) and \(\Delta t = 0\)), we get that

$$\mathbb{U}_{i_1}(\cdot, t) \prec_{(\Omega(0, 2r, r, \nu), \mu)} \mathbb{U}_{b_i}(\cdot, t) \prec_{(\Omega(0, 2r, r, \nu), \mu)} O_{\nu}(\cdot, t), \quad T_0 < t < \infty, \quad i = 1, 2.$$ 

Let us set

$$Z(x, t) := \begin{cases} 1, & x \in L^+_{\mu} \left(\mathbb{U}_{a_2}(\cdot, t); \Omega(0, 2r; \nu)\right), \\ -1, & x \in \Omega(0, 2r; \nu) \setminus L^+_{\mu} \left(\mathbb{U}_{a_2}(\cdot, t); \Omega(0, 2r; \nu)\right), \\ 0, & x \in L^+_{\mu} \left(\mathbb{U}_{a_1}\right)_* (\cdot - \xi_A, t); \Omega(0, 2r; \nu). \end{cases}$$ 

$$Y(x, t) := \begin{cases} 2, & x \in \Omega(0, 2r; \nu) \setminus L^+_{\mu} \left(\mathbb{U}_{a_1}\right)_* (\cdot - \xi_A, t); \Omega(0, 2r; \nu), \\ 0, & x \in L^+_{\mu} \left(\mathbb{U}_{a_1}\right)_* (\cdot - \xi_A, t); \Omega(0, 2r; \nu). \end{cases}$$

Because $\mathbb{U}_{a_2}(x, t)$ detach from the obstacle $O_{\nu}(x, t)$ in the $\mu$ level set, by upper semicontinuity of $\mathbb{U}_{a_2}(x, t)$, Proposition \ref{prop:regularity}, the operator $\mathcal{F}(\cdot, \cdot, \cdot)$ is geometric and the fact that $\xi_A \in \mathbb{Z}^n$, we can conclude that $Y(x, t)$ is a viscosity supersolution for $x \in \Omega(0, 2r; \nu)$ and $t > T_0$. In order to prove the Proposition, we assume on the contrary as follows, where $U(\cdot, t) := \left(\mathbb{U}_{a_1}\right)_* (\cdot - \xi_A, t)$:

$$\sup \{t > 0 | \mathbb{U}_{a_2}(\cdot, t) \prec_{(\Omega(0, r, r, \nu), \mu)} U(\cdot, t)\} \leq T. \quad (5.2)$$

Step 1. Let us define (recall Definition \ref{def:gamma})

$$\gamma(t) := \frac{1}{2} e^{-2L_0 t} \quad \text{and} \quad h := (\gamma(t), 1) \in \mathbb{F}.$$ 

Let $U^h - (x, t)$ be the $h$ inf-convolution of $U(x, t)$ by Definition \ref{def:inf-convolution}. Then the above assumption \ref{eq:gamma-bound} implies that

$$T_0 < t_0 := \sup \{t > 0 | \mathbb{U}_{a_2}(\cdot, t) \prec_{(\Omega(0, r, r, \nu), \mu)} U^h - (\cdot, t)\} \leq T. \quad (5.3)$$

Assume $x_0$ is the first touching point between the $\mu$ superlevel set of $\mathbb{U}_{a_2}$ and the $\mu$ sublevel set of $U^h -$ in $\Omega(0, r; \nu)$. i.e.,

$$\mathbb{U}_{a_2}(x_0, t_0) = U^h - (x_0, t_0) = \mu \quad \text{and} \quad Z(x_0, t_0) = 1 > 0 = Y^h - (x_0, t_0).$$

By applying the comparison principle (i.e., Proposition \ref{prop:comparison}) to $Z$ and $Y$ in $\Omega(0, r; \nu)$, the maximum difference of $Z(x, t) - Y(x, t)$ is achieved on $\partial \Omega(0, r; \nu)$. As $h$ does not depend on the space variable, we must have that $x_0 \in \partial \Omega(0, r; \nu)$.

Step 2. By the choice of $r$, there exists $z_0 \in \mathbb{R}^n$ with the following (i)-(iii) hold:

$$\frac{\delta}{3} < (z_0 - x_0) \cdot \nu < \delta; \quad (ii) \ |z_0 - 2x_0| < \frac{r}{3}; \quad (iii) \ z_0 - x_0 \in \mathbb{Z}^n,$$ 

where $0 < \delta < \delta(T)$ with $\delta(T)$ defined in Definition \ref{def:gamma}. Therefore, there exists $C > 1$, such that $\delta$ can be written as follows:

$$\delta := \left(\frac{M_0 m_0}{M_0 - m_0}\right) \cdot \frac{(C - 1)\gamma^2(T)}{(n - 1)C^2 + C(C - 1)\gamma(T)M_0} > 0. \quad (5.5)$$

Step 3. Let us investigate the ordering relation between the $\mu$ superlevel set of $\mathbb{U}_{a_2}(x, t)$ and the $\mu$ sublevel set of $U^h - (x, t)$ in $\Omega(x_0, \frac{r}{3}; \nu)$. Let us introduce the notations

$$\Delta z := z_0 - x_0; \quad \sigma := |\Delta z \cdot \nu|; \quad \Delta t := \frac{\sigma}{m_0} - \frac{\sigma}{s_2}; \quad \mathcal{R}_1 = \mathcal{R}_2 := \frac{4M_0 R}{\delta}. \quad (5.6)$$

Then Proposition \ref{prop:ordering} indicates that

$$\mathbb{U}_{a_2}(x, t) \leq \mathbb{U}_{a_2} \left(x - \Delta z, t - \frac{\sigma}{s_2}\right), \quad x \in \Omega(x_0, \frac{r}{3}; \nu), \quad \frac{\sigma}{s_2} \leq t \leq t_0 - \Delta t. \quad (5.7)$$
Because of the inclusion $\Omega(0, \frac{r}{3}; \nu) + \Delta z \subseteq \Omega(0, r; \nu)$ and (5.3), we have that
\[
\bar{U}_{a_2} \left( \cdot, t - \frac{\sigma}{s_2} \right) \preceq (\Omega(0, r; \nu), \mu) U^{h-} \left( \cdot, t - \frac{\sigma}{s_2} \right), \quad \frac{\sigma}{s_2} \leq t \leq t_0 - \Delta t.
\] (5.8)

The Proposition 2.10 applied to the subsolution $\bar{U}_{a_1}$ with above shift $\Delta z$ shows that
\[
\bar{U}_{a_1} \left( y, t - \frac{\sigma}{s_2} \right) \leq \bar{U}_{a_1} \left( y + \Delta z, t + \frac{\sigma}{\mu_0} - \frac{\sigma}{s_2} \right) = \bar{U}_{a_1} \left( y + \Delta z, t + \Delta t \right), \quad y \in \Omega(0, r; \nu), \quad \frac{\sigma}{s_2} \leq t \leq t_0 - \Delta t.
\]

Then apply the $\xi_A$ shift and the $h$ inf-convolution to both sides, we conclude that
\[
U^{h-} \left( y, t - \frac{\sigma}{s_2} \right) \leq U^{h-} \left( y + \Delta z, t + \Delta t \right), \quad y \in \Omega(0, r; \nu), \quad \frac{\sigma}{s_2} \leq t \leq t_0 - \Delta t.
\] (5.9)

A combination of (5.7), (5.8) and (5.9) gives the relation
\[
\bar{U}_{a_2} \left( \cdot, t \right) \preceq (\Omega(x_0, \frac{r}{3}; \nu), \mu) U^{h-} \left( \cdot, t + \Delta t \right), \quad \frac{\sigma}{s_2} < t < t_0 - \Delta t.
\]

In particular,
\[
\bar{U}_{a_2} \left( \cdot, t_0 - \Delta t \right) \preceq (\Omega(x_0, \frac{r}{3}; \nu), \mu) U^{h-} \left( \cdot, t_0 \right).
\]

By the choice of $\Delta t$ through (5.3), (5.5), (5.4) and (5.6), we have that
\[
0 < \Delta t < \frac{(C - 1)\gamma^2(t_0)}{(n - 1)C^2 + C(C - 1)\gamma(t_0)\mu_0}.
\]

Then the Proposition 3.2 implies that
\[
\bar{U}_{a_2} \left( \cdot, t_0 \right) \preceq (\Omega(x_0, \frac{r}{3}; \nu), \mu) U^{h-} \left( \cdot, t_0 \right), \quad \text{where} \quad \hat{h} := \left( \left(1 - \frac{1}{C} \right) \gamma(t), 1 \right) \in \mathcal{F}.
\]

Step 4. Let us construct $\varphi(x) : \Omega \left( x_0, \frac{r}{3}; \nu \right) \rightarrow (0, \infty)$ as follows:
\[
\varphi(x) := -\frac{9(1 + C)}{2Cr^2} \vert (x - x_0) \vert^2 + \frac{1}{2} \left( 3 - \frac{1}{C} \right),
\]
where $(x - x_0)^\top := (x - x_0) - ((x - x_0) \cdot \nu) \nu$. Then $\varphi(x)$ satisfies that
\[
\begin{align*}
\varphi \left( \mathbf{x} \vert (x - x_0)^\top = 0 \right) &= \frac{1}{2} \left( 3 - \frac{1}{C} \right) > 1, \\
\varphi \left( \mathbf{x} \vert (x - x_0)^\top = \frac{r}{3} \right) &= 1 - \frac{1}{C},
\end{align*}
\]
and
\[
\begin{align*}
\|D\varphi\|_{\infty} &\leq \frac{6}{7}r, \\
\|D^2\varphi\|_{\infty} &\leq \frac{18}{7}.
\end{align*}
\]
Then, because $r \geq \max \left\{ 6, \frac{12(3n + M_0 + 27)}{L_0} \right\}$, we have that
\[
\gamma'(t) + \left( \frac{(n + 1)\|D^2\varphi\|_{\infty}}{\varphi(x)} + M_0\|D\varphi(x)\|_{\infty} \right) \gamma(t) + \frac{|D\varphi(x)|^2\gamma(t)}{(1 - \gamma(t)|D\varphi(x)|)^2|\varphi^2(x)|} \leq 0.
\]

Then, by Proposition 3.3, $U^{(\gamma, \varphi)}$ is a pseudo viscosity supersolution in $\Omega \left( x_0, \frac{r}{3}; \nu \right)$. Based on the finite speed of propagation regarding the subsolution (c.f. Proposition 3.2), there exists $\tau > 0$ such that
\[
\bar{U}_{a_2} \left( \cdot, t \right) \preceq (\partial \Omega(x_0, \frac{r}{3}; \nu), \mu) U^{(\gamma, \varphi)} \left( \cdot, t \right), \quad T_0 \leq t \leq t_0 + \tau.
\] (5.10)

Note that the following strict ordering relation holds.
\[
\bar{U}_{a_2} \left( \cdot, T_0 \right) \preceq (\Omega(x_0, \frac{r}{3}; \nu), \mu) U^{(\gamma, \varphi)} \left( \cdot, T_0 \right).
\] (5.11)

Let us introduce the function
\[
W(x, t) := \begin{cases} 2, & x \in \Omega \left( x_0, \frac{r}{3}; \nu \right) \setminus L_{\mu} \left( U^{(\gamma, \varphi)} \left( \cdot, t \right) ; \Omega \left( x_0, \frac{r}{3}; \nu \right) \right), \\
0, & L_{\mu} \left( U^{(\gamma, \varphi)} \left( \cdot, t \right) ; \Omega \left( x_0, \frac{r}{3}; \nu \right) \right).
\end{cases}
\]
Since $\mathcal{F} \left( \cdot, \cdot, \cdot \right)$ is geometric, $W(x, t)$ is a pseudo viscosity supersolution in $\Omega(x_0, \frac{r}{3}; \nu)$.

Step 5. Based on observations of (5.10), (5.11), we have equivalently that
\[ Z(x, t) < W(x, t), \quad (x, t) \in \left( \Omega \left( x_0, \frac{r}{3}; \nu \right) \times \left[ T_0 \right) \cup \left( \partial \Omega \left( x_0, \frac{r}{3}; \nu \right) \times [T_0, t_0 + \tau] \right). \]
Then we apply Proposition 9.1 to the above $Z$ and $W$ and conclude that
\[ Z(x,t) < W(x,t), \quad T_0 \leq t \leq t_0 + \tau, \quad x \in \Omega \left( x_0, \frac{r}{3}; \nu \right). \]
On the other hand, recall that $h = (\gamma(t), 1)$ and consider the facts
\[
\overline{U_a}(x_0, t_0) = U^-(x_0, t_0) = \mu \quad \text{and} \quad \varphi(x, x - x_0) = \frac{1}{2} \left( 3 - \frac{1}{C} \right) > 1.
\]
It follows that $Z(x_0, t_0) = 1 > 0 = W(x_0, t_0)$, which is a contradiction.

**Proposition 5.10.** Let $(\nu, T, R)$ be a comparison consistent triplet (c.f. Definition 4.5), fix $\mu \in \mathbb{R}$ and $s_i := \pi(\nu) - \delta_i$, $i = 1, 2$, where $\frac{m_0}{2} > \delta_2 > \delta_1 > 0$ are two fixed numbers. Then for any $r > \max \left\{ \frac{R}{12(3n+M_0+2T)}, T/3 \right\}$, $T > 0$, there exist $A > 0$ (independent of $r$ and $T$), $R_i$ and $s_i > 0$, $i = 1, 2$, such that the following (i) and (ii) hold.

(i) $R_1 \geq R_2 + (\mathcal{R}_1 + \mathcal{R}_2)T$;

(ii) $\overline{U_a}(\cdot, t) \prec_{(0(0, r, x), \mu)} O_a(\cdot + \xi_A, t), \quad 0 \leq t \leq T$;

where
\[
a_1 := (\nu, R_1, -\mathcal{R}_1, q, s_1), \quad a_2 := (\nu, R_2, \mathcal{R}_2, q, s_2) \in \mathbb{A} \quad \text{and} \quad \xi_A \in \arg\min_{\xi \in \mathbb{Z}^n, \xi, \nu > A} \|\xi\|.
\]

**Proof.** The idea of proof is similar to Proposition 5.8, in view of Proposition 3.2, 3.3, 4.4, 5.2, etc. The difference is that, this time, we use the Proposition 2.7, 2.9, 2.11, instead of Proposition 2.6, 2.8, 2.10. Note that $\frac{m_0}{2} \leq s_2 < s_1 \leq M_0$, the speeds have positive bounds, the arguments of Proposition 5.8 or Proposition 4.7 still apply, merely with $m_0$ replaced by $\frac{m_0}{2}$. Thus, we omit the details here. \hfill \square

5.2.3. The detachment lemma.

**Proposition 5.10 (Detachment Property).** Let $(\nu, T, r)$ be a comparison triplet (c.f. Definition 4.5), fix $\sigma > 0$ and set $s := \pi(\nu) + \sigma$, then for any $\mu \in \mathbb{R}$, there exists $R > 0$ and $B := B(\nu, \sigma) > 0$, such that the following statement holds.
\[
\overline{U_a} \left( \cdot - \left( \frac{1}{2} \nu \right) t - B \right) \prec_{(0(0, r, x), \mu)} O_e(\cdot, t), \quad 0 \leq t \leq T,
\]
where
\[
a := (\nu, R, 0, q, s) \in \mathbb{A} \quad \text{and} \quad e := (\nu, q,s) \in \mathbb{E}.
\]
Moreover, there exists a constant $C > 0$ independent of $\nu$ and $T$ such that $B = CT_0(\nu, \sigma) + C$, where $T_0(\nu, \sigma)$ is the time after which the $\mu$ level set of $\overline{U_a}$ detaches from $O_e$ in $\Omega(0, 2r; \nu)$.

**Proof.** Let us set
\[
s_1 := \pi(\nu) + \frac{\sigma}{2} \quad \text{and} \quad s_2 := \pi(\nu) + \sigma.
\]
Then by Proposition 5.8, there are numbers
\[
A := A(\nu, \sigma) > 0, \quad R_i := R_i(\nu, \sigma, r, T), \quad R_i := R_i(\nu, \sigma, r, T) > 0, \quad i = 1, 2,
\]
such that the following (i) and (ii) hold:

(i) $R_1 \geq R_2 + (\mathcal{R}_1 + \mathcal{R}_2)T$;

(ii) $\overline{U_a}(\cdot, t) \prec_{(0(0, r, x), \mu)} O_a(\cdot - \xi_A, t), \quad 0 \leq t \leq T$,

where
\[
a_1 := (\nu, R_1, -\mathcal{R}_1, q, s_1), \quad a_2 := (\nu, R_2, \mathcal{R}_2, q, s_2) \in \mathbb{A} \quad \text{and} \quad \xi_A \in \arg\min_{\xi \in \mathbb{Z}^n, \xi, \nu > A} \|\xi\|.
\]
Clearly, $A < \xi_A \cdot \nu < A + \sqrt{n}$. Let us also take
\[
B(\nu, \sigma) := A(\nu, \sigma) + \sqrt{n} \quad \text{and} \quad R := R_1(\nu, \sigma, r, T),
\]
Then we denote $\mu$ independent of $t$. Proposition 2.8 indicates that $\mu(t) = 0$. Let us consider the general level set, there exists $\Delta > 0$, such that the following statement holds. For any $x \in \Omega(0, r; \nu)$, then for any $\nu, \sigma > 0$, such that for any $\nu, \sigma > 0$, there exists $R > 0$ and $B := B(\nu, \sigma) > 0$, and the following statement holds.

$$
O_\varepsilon(t) \prec_{(\Omega(0, r; \nu), \mu)} \mathbb{U}_a \left( \cdot + \left( \frac{1}{2} \sigma t - B \right) \nu, t \right), \quad 0 \leq t \leq T,
$$

where

$$
a \varepsilon := (\nu, q, s) \in A \quad \text{and} \quad e \varepsilon := (\nu, q, s) \in E.
$$

Moreover, there exists a constant $C > 0$ independent of $\nu$ and $T$ such that $B = CT_0(\nu, \sigma) + C$, where $T_0(\nu, \sigma)$ is the time after which the $\mu$ level set of $\mathbb{U}_a$ detaches from $O_\varepsilon$ in $\Omega(0, 2r; \nu)$.

**Proof.** It is similar to that of Proposition 5.10. The difference is that instead of using Proposition 2.8, 5.8, we use Proposition 2.9, 5.9. Hence, we do not repeat the details any more. \hfill \Box

Next, let us show that if a speed is strictly larger (resp. strictly smaller) than the head (resp. tail) speed in an irrational direction $\nu$, then all superlevel (resp. sublevel) sets of the obstacle subsolution (resp. supersolution) detaches linearly and uniformly from its obstacle.

**Proposition 5.12.** Let $\nu \in \mathbb{S}^{n-1} \setminus \mathbb{R}^n$, assume that $s > \pi(\nu)$. Then there exists $\delta > 0$ and $B := B(\nu, \delta) > 0$, such that the following statement holds. For any $\mu \in \mathbb{R}$, $r > 0$ and $T > 0$, there exists $R > 0$, independent of $\mu$, such that

$$
\mathbb{U}_a \left( \cdot - (\delta t - B) \nu, t \right) \prec_{(\Omega(0, r; \nu), \mu)} O_\varepsilon(t), \quad 0 \leq t \leq T,
$$

where

$$
a := (\nu, R, 0, q, s) \in A \quad \text{and} \quad e := (\nu, q, s) \in E.
$$

**Proof.** Based on Proposition 5.10, there exists $\delta > 0$ and $B_0 := B_0(\nu, \delta) > 0$, such that for any $T > 0$, there exists $R_0 > r + \sqrt{n}$, with the following statement holds.

$$
\mathbb{U}_{a_0} \left( \cdot - (\delta t - B_0) \nu, t \right) \prec_{(\Omega(0, r + \sqrt{n}; \nu), 0)} O_\varepsilon(t), \quad 0 \leq t \leq T,
$$

where

$$
a_0 := (\nu, R_0, 0, q, s) \in A \quad \text{and} \quad e := (\nu, q, s) \in E.
$$

Let us consider the general $\mu$ level set, there exists $\Delta z \in \mathbb{Z}^n$, such that

$$
-\frac{\mu}{|q|} \leq \Delta z \cdot \nu \leq -\frac{\mu}{|q|} + \sqrt{n} \quad \text{and} \quad |\Delta z - (\Delta z \cdot \nu) \nu| \leq \sqrt{n}.
$$

Then we denote

$$
U(x, t) := \mathbb{U}_{a_0} (x - \Delta z, t) + \mu.
$$
Then $U(x,t)$ is the largest subsolution that is in $\Omega(0, R_0; \nu) + \Delta z$ and bounded from above by $O(x, t) := O_e(x - \Delta z, t) + \mu$, which is no less than $O_e(x, t)$. Moreover,

$$U(\cdot - (\delta t - B_0) \nu, t) \prec_{(\Omega(0, r; \nu), \mu)} O(x, t), \quad 0 \leq t \leq T.$$ 

Now, let us take $R := R_0 + \sqrt{n}$, then the associated obstacle subsolution $\overline{U}_n(x, t)$, restricted to $\Omega(0, R_0; \nu) + \Delta z$ (which includes $\Omega(0, r; \nu)$), is no larger than $U(x, t)$. Therefore,

$$\overline{U}_n(\cdot - (\delta t - B) \nu, t) \prec_{(\Omega(0, r; \nu), \mu)} O_e(\cdot, t), \quad 0 \leq t \leq T,$$

where $B = B_0 + \sqrt{n}$. \hfill \qed

**Proposition 5.13.** Let $\nu \in S^{n-1} \setminus \mathbb{R} \mathbb{Z}^n$, assume that $0 < s < \bar{g}(\nu)$. Then there exists $\delta > 0$ and $B := B(\nu, \delta) > 0$, such that the following statement holds. For any $\mu \in \mathbb{R}$, $r > 0$ and $T > 0$, there exists $R > 0$, independent of $\mu$, such that

$$O_e(\cdot, t) \prec_{(\Omega(0, r; \nu), \mu)} \overline{U}_n(\cdot - (\delta t - B) \nu, t), \quad 0 \leq t \leq T,$$

where $$a := (\nu, r, 0, q, s) \in \mathbb{A} \quad \text{and} \quad e := (\nu, q, s) \in \mathbb{E}.$$ 

**Proof.** It follows the same idea as Proposition 5.12, we omit the details here. \hfill \qed

5.2.4. The ordering relation. Our goal of introducing the head/tail speed is to model the highest/lowest speed of the level set of a real solution. The next Proposition shows that the head speed is indeed no less than the tail speed, at least in each irrational direction.

**Proposition 5.14** (Ordering relation). Fix any $\nu \in S^{n-1} \setminus \mathbb{R} \mathbb{Z}^n$, then $\bar{g}(\nu) \leq \bar{\pi}(\nu)$.

**Proof.** Let us assume on the contrary that $\theta := \frac{\bar{g}(\nu) - \bar{\pi}(\nu)}{3} > 0$ and then we derive a contradiction through the following steps.

**Step 1.** For any $\mu \in \mathbb{R}$, we set

$$s_1 := \bar{\pi}(\nu) + \theta \quad \text{and} \quad s_2 := \bar{g}(\nu) - \theta.$$ 

Then there exist $R_0 = R_0(\nu) > r_0 = r_0(\nu) > \frac{\sqrt{n}}{2}$ and $T_0 = T_0(\nu) > 0$, such that

$$\begin{cases}
\overline{U}_{a_1}(\cdot, t) \prec_{(\Omega(0, r_0; \nu), \mu)} O_{c_1}(\cdot, t), \\
O_{c_2}(\cdot, t) \prec_{(\Omega(0, r_0; \nu), \mu)} \overline{U}_{a_2}(\cdot, t),
\end{cases} \quad t > T_0,$$

where

$$a_i := (\nu, R_0, 0, q, s_i) \in \mathbb{A} \quad \text{and} \quad e_i := (\nu, q, s_i) \in \mathbb{E}, \quad i = 1, 2.$$ 

Let us denote

$$A := (M_0 - m_0) T_0 + 2\sqrt{n}, \quad \xi_A \in \arg \min_{\xi \in \mathbb{Z}^n} |\xi| \quad \text{and} \quad T := \frac{(M_0 - m_0) T_0 + 3\sqrt{n}}{\theta}.$$ 

For a fixed $C > 1$, we set a small positive number as follows, where $\gamma(t) := \frac{1}{2} e^{-2L_0 t}$.

$$\delta := \left( \frac{M_0 m_0}{M_0 - m_0} \right) \cdot \left( \frac{(C - 1) \gamma^2(T)}{(n - 1) C^2 + C(C - 1) \gamma(T) M_0} \right) > 0. \quad (5.12)$$

Denote by $R_0(\nu, \delta)$ the number defined in Proposition 4.4, associated to the above $(\nu, \delta)$. And then denote

$$R := \max \left\{ R_0(\nu), R_0(\nu, \delta), \frac{12(3n + M_0 + 27)}{L_0} \right\}.$$ 

The Proposition 4.4 indicates that for any $\hat{x}$ with $|\hat{x} - (\hat{x} \cdot \nu)\nu| = R$, there exists $\hat{z}$, such that

(i) $0 < (\hat{x} - \hat{z}) \cdot \nu < \delta$; \quad (ii) $|\hat{z} - 2\hat{x}| < \frac{R}{3}$; \quad (iii) $\Delta \hat{z} := \hat{z} - \hat{x} \in \mathbb{Z}^n$.

Since there are finite such integer vectors $\Delta \hat{z}$, we can set

$$\sigma := \min \left\{ |\Delta \hat{z} \cdot \nu| |\hat{x} - (\hat{x} \cdot \nu)\nu| = R \right\} > 0 \quad \text{and} \quad \mathcal{R} := 2 \left( \frac{1}{\sigma} + 1 \right) M_0 R.$$
Consider
\[ b_i := (\nu, R_i, -\mathcal{R}, q_i), \quad \text{where} \quad R_i > R + \mathcal{R} + \sqrt{n}, \quad i = 1, 2. \]
And denote \( U(x,t) := (\mathcal{U}_{b_i})_*(x - \xi_A, t) \), then by Lemma 5.6, we have that
\[ (\mathcal{U}_{b_2})^*(\cdot,t) \prec_{(\Omega(0, R; \nu), \mu)} U(\cdot,t), \quad 0 \leq t \leq T_0 + \frac{\sqrt{n}}{\theta}. \]
Then for any \( t > T_0 + \frac{\sqrt{n}}{\theta} \), we have that
\[ O_{c_2}(\cdot,t) \prec_{(\Omega(0, R; \nu), \mu)} (\mathcal{U}_{b_2})^*(\cdot,t) \quad \text{and} \quad (\mathcal{U}_{b_1})^*(\cdot,t) \prec_{(\Omega(0, R; \nu), \mu)} O_{c_1}(\cdot,t). \]
Because \( \theta \cdot T > |\xi_A \cdot \nu| \), we conclude that
\[ T_0 + \frac{\sqrt{n}}{\theta} < \sup \left\{ t > 0 \mid (\mathcal{U}_{b_2})^*(\cdot,t) \prec_{(\Omega(0, R; \nu), \mu)} U(\cdot,t) \right\} < T. \]
Let us set
\[ Z(x,t) := \begin{cases} \mu, & x \in L^+_\mu \left( (\mathcal{U}_{b_2})^*(\cdot,t); \Omega(0, \frac{4R}{3}; \nu) \right), \\ \mu - 1, & x \in \Omega \left( 0, \frac{4R}{3}; \nu \right) \setminus L^+_\mu \left( (\mathcal{U}_{b_2})^*(\cdot,t); \Omega(0, \frac{4R}{3}; \nu) \right), \end{cases} \]
\[ Y(x,t) := \begin{cases} \mu, & x \in \Omega \left( 0, \frac{4R}{3}; \nu \right) \setminus L^-_\mu \left( (\mathcal{U}_{b_1})^*(\cdot,t); \Omega(0, \frac{4R}{3}; \nu) \right), \\ \mu - 1, & x \in L^-_\mu \left( (\mathcal{U}_{b_1})^*(\cdot,t); \Omega(0, \frac{4R}{3}; \nu) \right). \end{cases} \]
Because \( \mathcal{U}_{b_2}(x,t) \) detaches from the obstacle \( O_{c_2}(x,t) \) in the \( \mu \) level set, by lower semicontinuity of \( \mathcal{U}_{b_2}(x,t) \), Proposition 5.2, the operator \( \mathcal{F}(\cdot, \cdot, \cdot) \) is geometric and the fact that \( \xi_A \in \mathbb{Z}^n \), we can conclude that \( Z(x,t) \) is a viscosity subsolution for \( x \in \Omega \left( 0, \frac{4R}{3}; \nu \right) \) and \( t > T_0 + \frac{\sqrt{n}}{\theta} \). Similarly, we have that \( Y(x,t) \) is a viscosity supersolution in the same space time domain.

Step 2. Let \( h := (\gamma(t), 1) \in \mathbb{F} \) and \( U^h(\cdot,t) \) be the \( h \) inf-convolution of \( U(\cdot,t) \) by Definition 3.1. Then the choice of \( A \) and \( \xi_A \) indicates that
\[ T_0 + \frac{\sqrt{n}}{\theta} < t_0 := \sup \left\{ t > 0 \mid (\mathcal{U}_{b_2})^*(\cdot,t) \prec_{(\Omega(0, R; \nu), \mu)} U^h(\cdot,t) \right\} < T. \]
Assume \( x_0 \) is the first crossing point between (the \( \mu \) superlevel set of) \( (\mathcal{U}_{b_2})^* \) and (the \( \mu \) sublevel set of) \( U^h \) over \( \Omega(0, R; \nu) \). i.e.,
\[ (\mathcal{U}_{b_2})^*(x_0, t_0) = Z(x_0, t_0) = Y^h(x_0, t_0) = U^h(x_0, t_0) = \mu. \]
Then by applying Proposition 2.1 to \( \mathcal{U}_{b_2}(\cdot,t) \) and \( U(\cdot,t) \) in \( \Omega(0, R; \nu) \), the maximum of \( (\mathcal{U}_{b_2})^*(x,t) - U(x,t) \) over \( \Omega(0, R; \nu) \) is obtained on \( \partial \Omega(0, R; \nu) \). Since \( h \) is \( x \)-independent, \( x_0 \in \partial \Omega(0, R; \nu) \). As a result of Proposition 4.4, there exists \( z_0 \in \mathbb{R}^n \), such that
\[ (i) \quad \frac{\delta}{\beta} < (z_0 - x_0) \cdot \nu < \delta; \quad (ii) \quad |z_0 - 2x_0| < \frac{R}{\beta}; \quad (iii) \quad \Delta z := z_0 - x_0 \in \mathbb{Z}^n. \]

Step 3. Let us denote
\[ \sigma_0 := |\Delta z \cdot \nu| \quad \text{and} \quad \Delta t := \frac{\sigma_0}{m_0} - \frac{\sigma_0}{M_0}. \]
It is clear that \( \sigma_0 \geq \sigma \). Then by Proposition 2.11, we have the ordering relation
\[ \mathcal{U}_{b_2}(x, t) \leq \mathcal{U}_{b_2} \left( x - \Delta z, t - \frac{\sigma_0}{M_0} \right), \quad x \in \Omega \left( x_0, \frac{R}{\beta}; \nu \right), \quad \frac{\sigma_0}{M_0} \leq t \leq t_0 - \Delta t. \]
Because of the inclusion \( \Omega \left( x_0, \frac{R}{\beta}; \nu \right) + \Delta z \subseteq \Omega(0, R; \nu) \) and (5.13), we have that
\[ (\mathcal{U}_{b_2})^* \left( \cdot, t - \frac{\sigma_0}{M_0} \right) \prec_{(\Omega(0, R; \nu), \mu)} U^h \left( \cdot, t - \frac{\sigma_0}{M_0} \right), \quad \frac{\sigma_0}{M_0} \leq t \leq t_0 - \Delta t. \]
Based on the observations of (5.18) and (5.19), let us apply the proof of Proposition 9.1 to the

\[ U_{b_i} \left( y, t - \frac{\sigma_0}{M_0} \right) \leq U_{b_i} \left( y + \Delta z, t + \frac{\sigma_0}{m_0} - \frac{\sigma_0}{M_0} \right) \]

\[ = U_{b_i} \left( y + \Delta z, t + \Delta t \right), \quad y \in \Omega(0, R; \nu), \quad \frac{\sigma_0}{M_0} \leq t \leq t_0 - \Delta t. \]

Then apply the \( \xi_A \) shift and the \( h \) inf-convolution to both sides, we conclude that

\[ U^{h-} \left( y, t - \frac{\sigma_0}{M_0} \right) \leq U^{h-} \left( y + \Delta z, t + \Delta t \right), \quad y \in \Omega(0, R; \nu), \quad \frac{\sigma_0}{M_0} \leq t \leq t_0 - \Delta t. \]  

(5.17)

A combination of (5.15), (5.16) and (5.17) gives the following relation:

\[ (U_{b_2})^* (\cdot, t) \prec (\alpha(x_0, \frac{2}{3}, \nu), \mu) \quad U^{h-} (\cdot, t + \Delta t), \quad \frac{\sigma_0}{M_0} \leq t \leq t_0 - \Delta t. \]

In particular, we have that

\[ (U_{b_2})^* (\cdot, t_0 - \Delta t) \prec (\alpha(x_0, \frac{2}{3}, \nu), \mu) \quad U^{h-} (\cdot, t_0). \]

Then apply the \( \xi \) in\( \uparrow \)v\( \downarrow \)ction to both sides, we conclude that

\[ 0 < \Delta t \leq \frac{(C-1)\gamma^2(\tau)}{(n-1)C^2 + C(C-1)\gamma(t_0)M_0}. \]

Then the Proposition 3.2 implies that

\[ (U_{b_2})^* (\cdot, t_0) \prec (\alpha(x_0, \frac{2}{3}, \nu), \mu) \quad U^{\hat{h}-} (\cdot, t_0), \quad \text{where} \quad \hat{h} := \left( \left( 1 - \frac{1}{C} \right) \gamma(t), 1 \right). \]

Step 4. Let us still construct \( \varphi(x) : \Omega \left( x_0, \frac{R}{3}; \nu \right) \rightarrow \mathbb{R} \) as follows:

\[ \varphi(x) := \frac{9(1 + C)}{2C^2 R^2} \left( (x - x_0)^\top \right)^2 + \frac{1}{2} \left( 3 - \frac{1}{C} \right), \]

where \( (x - x_0)^\top := (x - x_0) - ((x - x_0) \cdot \nu) \nu. \) Then \( \varphi(x) \) satisfies that

\[ \varphi \left\{ x \mid (x - x_0)^\top = 0 \right\} = 1 = \varphi \left\{ x \mid (x - x_0)^\top > 0 \right\} = \frac{1}{2}, \quad \text{and} \quad \left\{ \| D\varphi \|_\infty \leq \frac{6}{R}, \right. \]

\[ \left. \| D^2\varphi \|_\infty \leq \frac{18}{R^2} \right. \]

Then, because \( R \geq \max \left\{ \frac{6, \frac{12(3n + M_0 + 27)}{L_0}}{\frac{12(3n + M_0 + 27)}{L_0}} \right\}, \) we have that

\[ \gamma(t) + \left( \frac{(n + 1)\| D^2\varphi \|_\infty}{\varphi(x)} + \frac{M_0|D\varphi(x)|}{\varphi(x)} + L_0 \right) \gamma(t) + \frac{|D\varphi(x)|^2\gamma(t)}{(1 - \gamma(t)|D\varphi(x)|)^2} \varphi^2(x) \leq 0. \]

Then, based on the finite speed of propagation regarding the subsolution (c.f. Proposition 3.2), we have for some \( \tau > 0 \) that

\[ (U_{b_2})^* (\cdot, t) \prec (\beta\alpha(x_0, \frac{2}{3}, \nu), \mu) \quad U^{(\gamma, \nu)^-} (\cdot, t), \quad T_0 + \frac{\sqrt{n}}{\theta} \leq t \leq t_0 + \tau. \]  

(5.18)

Note that the following strict ordering relation holds.

\[ (U_{b_2})^* (\cdot, T_0 + \frac{\sqrt{n}}{\theta}) \prec (\alpha(x_0, \frac{2}{3}, \nu), \mu) \quad U^{(\gamma, \nu)^-} (\cdot, T_0 + \frac{\sqrt{n}}{\theta}). \]  

(5.19)

Let us introduce the function

\[ W(x, t) := \begin{cases} \mu, & x \in \Omega \left( x_0, \frac{R}{3}; \nu \right) \setminus L_0^- \left( U^{(\gamma, \nu)^-} (\cdot, t); \Omega \left( x_0, \frac{R}{3}; \nu \right) \right), \\ \mu - 1, & x \in L_0^- \left( U^{(\gamma, \nu)^-} (\cdot, t); \Omega \left( x_0, \frac{R}{3}; \nu \right) \right). \end{cases} \]

Step 5. Based on the observations of (5.18) and (5.19), let us apply the proof of Proposition 9.1 to the above \( Z(x, t) \) and \( W(x, t) \), and we can conclude that

\[ Z(x, t) \leq W(x, t), \quad T_0 + \frac{\sqrt{n}}{\theta} \leq t \leq t_0 + \tau, \quad x \in \Omega \left( x_0, \frac{R}{3}; \nu \right). \]
On the other hand, recall that \( h = (\gamma(t), 1) \) and consider the facts
\[
(U_{\partial B})^* (x_0, t_0) = U_{h_0}(x_0, t_0) = \mu \quad \text{and} \quad \varphi_{\{x_0, t_0\}} = \frac{1}{2} \left( 3 - \frac{1}{C} \right) > 1.
\]
It follows that \( Z(x_0, t_0) = \mu > \mu - 1 = W(x_0, t_0) \), which is a contradiction. \( \square \)

5.3. General directions.

5.3.1. The extension of the head/tail speed. Up to now, we have defined the head/tail speed in all irrational directions \( \nu \in S^{n-1} \setminus \mathbb{R}^n \), in which the local comparison principle (Proposition 4.7) and the detachment property (Proposition 5.10) hold. In order to study the homogenization, it is necessary to extend the concept to all directions \( \nu \in S^{n-1} \). In particular, let us define the head/tail speed in rational directions, it turns out that the detachment property is the essential ingredient of the concept.

**Definition 5.4.** Fix \( \vartheta \in S^{n-1} \), a number \( s \) is called sub-strict-detached (resp. super-strict-detached) with respect to \( \vartheta \) if the following holds: There exists \( \delta > 0, B := B(\vartheta, \delta) > 0 \), such that for any \( \mu \in \mathbb{R}, r > 0, \) \( q \in \mathbb{R}^n \setminus \{0\} \) and \( T > 0 \), there exists \( R := R(\vartheta, \mu, \delta, q, s) > 0 \), such that we have the following relation, where \( a := (\vartheta, R, 0, q, s) \in \mathbb{R} \).

\[
\mathcal{U}_a ((- \delta t - B) \vartheta, t) \prec (\Omega(0, T, q), \mu) \quad \mathcal{O}_a (\cdot, t), \quad 0 \leq t \leq T
\]

(resp. \( \mathcal{O}_a (\cdot, t) \prec (\Omega(0, T, q), \mu) \mathcal{U}_a ((+ \delta t - B) \vartheta, t), \quad 0 \leq t \leq T \)).

**Proposition 5.15.** Fix \( \nu \in S^{n-1} \setminus \mathbb{R}^n \), then any \( s \in (\vartheta, \infty) \) is sub-strict-detached; any \( s \in (0, \vartheta) \) is super-strict-detached. Moreover, we have the following expression of head/tail speed.

\[
\pi(\nu) := \inf \left\{ s > 0 | s \text{ is sub-strict-detached with respect to } \nu \right\},
\]

\[
\mathcal{S}(\nu) := \sup \left\{ s > 0 | s \text{ is super-strict-detached with respect to } \nu \right\}.
\]

**Proof.** It is an immediate corollary of Proposition 5.10, 5.11. \( \square \)

**Definition 5.5.** Let \( \vartheta \in S^{n-1} \cap \mathbb{R}^n \), the head speed (resp. tail speed) in \( \vartheta \), denoted by \( \pi(\vartheta) \) (resp. \( \mathcal{S}(\vartheta) \)), is defined as follows.

\[
\pi(\vartheta) := \inf \left\{ s > 0 | s \text{ is sub-strict-detached with respect to } \vartheta \right\},
\]

\[
\mathcal{S}(\vartheta) := \sup \left\{ s > 0 | s \text{ is sup-strict-detached with respect to } \vartheta \right\}.
\]

5.3.2. An equivalent description.

**Definition 5.6.** Let \( \vartheta \in S^{n-1} \), the global head (resp. tail) speed in \( \vartheta \) direction, denoted by \( \underline{\pi}(\vartheta) \) (resp. \( \underline{\mathcal{S}}(\vartheta) \)), is defined as the smallest (resp. largest) number, such that: for any \( \delta > 0, \mathcal{U}_{a_{\infty}}(x, t) \) (resp. \( \mathcal{O}_{a_{\infty}}(x, t) \)) detaches (c.f. Definition 5.1) from \( \mathcal{O}_{a_{\infty}}(x, t) \) (resp. \( \mathcal{O}_{e_{\infty}}(x, t) \)) in \( \Omega(0, \gamma, \vartheta) \) for some \( \gamma > \sqrt{\frac{\pi}{C}} \), where

\[
a_{\infty} := (\vartheta, 0, 0, q, \underline{\pi}(\vartheta) + \delta) \in \mathbb{A}, \quad \epsilon_{\infty} := (\nu, q, \underline{\pi}(\vartheta) + \delta) \in \mathbb{E},
\]

\[
\text{(resp. } \quad a_{\infty} := (\vartheta, 0, 0, q, \underline{\mathcal{S}}(\vartheta) - \delta) \in \mathbb{A}, \quad \epsilon_{\infty} := (\nu, q, \underline{\mathcal{S}}(\vartheta) - \delta) \in \mathbb{E}).
\]

**Proposition 5.16 (Equivalence).** Fix any \( \vartheta \in S^{n-1} \), then

\[
\pi(\vartheta) = \underline{\pi}(\vartheta) \quad \text{and} \quad \mathcal{S}(\vartheta) = \underline{\mathcal{S}}(\vartheta).
\]

**Proof.** Let us only prove the equality regarding the head speed, the other one follows by a similar pattern.

Step 1. Under the same obstacle speed, the global obstacle subsolution is clearly less or equal to the obstacle subsolution associated to domain with finite radius. Therefore, we have that \( \pi(\vartheta) \geq \underline{\pi}(\vartheta) \).

Step 2. Let us assume on the contrary that \( \pi(\vartheta) = \underline{\pi}(\vartheta) + \gamma \) for some \( \gamma > 0 \). Set

\[
\delta := \frac{\gamma}{2}, \quad a := (\vartheta, 0, 0, q, \underline{\pi}(\vartheta) + \delta) \in \mathbb{A} \quad \text{and} \quad a_{\ell} := (\vartheta, 0, 0, q, \pi(\vartheta) - \delta) \in \mathbb{A}.
\]

Define

\[
\mathcal{U}(x, t) := \limsup_{n \to 0} \left\{ \mathcal{U}_{a_n}(s, t) \right\}.
\]
Therefore, due to Proposition 5.17 (Detachment Property), we have that
\[ U_\infty(x, t) \leq \overline{U}_{a\infty}(x, t), \quad (x, t) \in \mathbb{R}^n \times (0, \infty). \] (5.20)

Let us now consider the set \( \mathcal{T}_\ell \subseteq \Omega \left( 0, \frac{\sqrt{n}}{2}; \vartheta \right) \times (0, \infty) \), on which \( \overline{U}_{a\ell}(x, t) \) touches \( O_{e\infty}(x, t) \). To be more precise,
\[ \mathcal{T}_\ell := \left\{ (x, t) \in \Omega \left( 0, \frac{\sqrt{n}}{2}; \vartheta \right) \times (0, \infty) \left| \overline{U}_{a\ell}(x, t) = O_{e\infty}(x, t) \right. \right\}. \]

Since \( \overline{U}_{a\ell}(x, t) \) is upper semicontinuous, \( \mathcal{T}_\ell \), \( \ell \in \mathbb{N} \) are all closed sets. Moreover, we have that (c.f. an independent Lemma 5.19)
\[ \mathcal{T}_\ell \cap \{ x \in \mathbb{R}^n \mid \alpha \leq x \cdot \vartheta \leq \alpha + \sqrt{n} \} \neq \emptyset, \text{ for any } \alpha \geq 0. \]

Let us also set
\[ \mathcal{T}_\infty := \left\{ (x, t) \in \Omega \left( 0, \frac{\sqrt{n}}{2}; \vartheta \right) \times (0, \infty) \left| U_\infty(x, t) = O_{e\infty}(x, t) \right. \right\}. \]

Then by the definition of \( U_\infty \), we have that \( \mathcal{T}_\infty = \cap_{\ell=1}^\infty \mathcal{T}_\ell \). Accordingly,
\[ \mathcal{T}_\infty \cap \{ x \in \mathbb{R}^n \mid \alpha \leq x \cdot \vartheta \leq \alpha + \sqrt{n} \} \neq \emptyset, \text{ for any } \alpha \geq 0, \]

which means that \( U_\infty(x, t) \) does not detach from \( O_{e\infty}(x, t) \), neither does \( \overline{U}_{a\infty}(x, t) \) (this is due to (5.20)).

Then, \( \mathcal{T}_\infty \subseteq \mathcal{T}_\ell \), which contradicts to the definition of \( \mathcal{T}_\ell \).

**Proposition 5.17** (Detachment Property). Fix \( \nu \in \mathbb{S}^{n-1} \setminus \mathbb{R}^n, q \in \mathbb{R}^n \setminus \{0\} \) and \( \mu \in \mathbb{R} \), then for any \( \delta > 0 \), there exists a number \( B := B(\nu, \delta) > 0 \), such that
\[ \overline{U}_{a\infty} \left( \cdot - \left( \frac{\delta}{2} t - B \right) \nu, t \right) \prec_{(\mathbb{R}^n, \mu)} O_{e\infty}(\cdot, t), \quad t \geq 0, \]
where
\[ a\infty := (\nu, \infty, 0, q, s\infty(\nu) + \delta) \in \mathcal{A} \quad \text{and} \quad e\infty := (\nu, q, s\infty(\nu) + \delta) \in \mathcal{E}. \]

**Proof.** For any \( T > 0 \) and \( r > \sqrt{n} \), by Proposition 5.10, there exist \( R = R(\nu, T, r, \delta) > 0 \) and \( B = B(\nu, \delta) > 0 \), such that
\[ \overline{U}_a \left( \cdot - \left( \frac{\delta}{2} t - B \right) \nu, t \right) \prec_{(\mathcal{O}(0, r, \nu, \mu))} O_{e\infty}(\cdot, t), \quad 0 \leq t \leq T. \]

Then since \( \overline{U}_{a\infty} \subseteq \overline{U}_a \), we have for all \( t \geq 0 \) (because \( a\infty \) is independent of \( T \)) that
\[ \overline{U}_{a\infty} \left( \cdot - \left( \frac{\delta}{2} t - B \right) \nu, t \right) \prec_{(\mathcal{O}(0, r, \nu, \mu))} O_{e\infty}(\cdot, t). \]

Let us state a Birkhoff property for the global obstacle subsolution \( U_{a\infty}(x, t) \). For any \( \Delta z \in \mathbb{Z}^n \), such that \( \Delta z \cdot \nu > 0 \), let \( \Delta t := \Delta z \cdot \nu / s\infty(\nu) + \delta \), then by the maximality of \( \overline{U}_{a\infty} \), we get that
\[ \overline{U}_{a\infty}(x + \Delta z, t + \Delta t) \leq \overline{U}_{a\infty}(x, t), \quad (x, t) \in \mathbb{R}^n \times (0, \infty). \]

Due to \( r > \sqrt{n} \), for any \( x \in \mathbb{R}^n \), there exists \( \Delta z \in \mathbb{Z}^n \), such that \( |(x + \Delta z) - \nu| < \sqrt{n}, \Delta z \cdot \nu > 0 \) and \( \Delta t := \frac{\Delta z \cdot \nu}{s\infty(\nu) + \delta} < \frac{\sqrt{n}}{m_0} \). Consider any \( t > \frac{\sqrt{n}}{m_0} \), we have that
\[ \overline{U}_{a\infty}(x, t) \leq \overline{U}_{a\infty}(x - \Delta z, t - \Delta t) \prec_{(\mathcal{O}(0, r, \nu, \mu))} O_{e\infty}(x - \Delta z, t - \Delta t) \]
\[ = O_{e\infty} \left( x + \left[ \frac{\delta}{2} t - (B + \frac{\delta}{2} \Delta t) \right] \nu, t - \Delta t \right). \]

Therefore,
\[ \overline{U}_{a\infty} \left( \cdot - \left[ \frac{\delta}{2} t - (B + \frac{\delta}{2} \Delta t) \right] \nu, t \right) \prec_{(\mathbb{R}^n, \mu)} O_{e\infty}(\cdot, t). \]
By taking $\hat{B} = B + \frac{\delta \sqrt{n}}{2m_0}$, we have the desired result.

Even though we do not have detachment property for obstacle sub/super solutions associated to rational directions in a cylinder with finite radius, we have similar result for the global obstacle sub/super solutions.

**Proposition 5.18** (Detachment Property). Fix $\vartheta \in S^{n-1} \cap \mathbb{R}Z^n$, $q \in \mathbb{R}^n \setminus \{0\}$ and $\mu \in \mathbb{R}$, then for any $\delta > 0$, there exists a number $B := B(\vartheta, \delta) > 0$, such that

$$\bar{U}_a^\infty \left( \cdot - \left( \frac{\delta}{2} t - B \right) \vartheta, t \right) \prec_{(\mathbb{R}^n, \mu)} O_\varphi(\cdot, t), \quad t \geq 0,$$

where

$$a^\infty := (\vartheta, \infty, 0, q, \bar{\pi}(\vartheta) + \delta) \in A \quad \text{and} \quad e^\infty := (\vartheta, q, \bar{\pi}^\infty + \delta) \in E.$$

**Proof.** Let us also set

$$\hat{a}^\infty := \left( \vartheta, \infty, 0, q, \bar{\pi}(\vartheta) + \delta \right) \quad \text{and} \quad \hat{e}^\infty := \left( \vartheta, q, \bar{\pi}^\infty + \delta \right) \in E.$$

By the definition of $\bar{\pi}^\infty(\vartheta)$, there exist $r > \frac{\sqrt{n}}{2}$ and $t_0 := t_0(\vartheta, \delta) > 0$, such that

$$\bar{U}_a^\infty(\cdot, t) \prec_{(\Omega(0, r, \vartheta), \mu)} O_\varphi(\cdot, t) \quad \text{and} \quad \bar{U}_{\hat{a}}^\infty(\cdot, t) \prec_{(\Omega(0, r, \vartheta), \mu)} O_{\hat{\varphi}}(\cdot, t), \quad t \geq t_0.$$ Because $\vartheta \in S^{n-1} \cap \mathbb{R}Z^n$, both $\bar{U}_a^\infty(\cdot, t)$ and $\bar{U}_{\hat{a}}^\infty(\cdot, t)$ have periodic structure (through not necessarily $\mathbb{Z}^n$ periodic), therefore (chose a larger $t_0$ if necessary),

$$\bar{U}_a^\infty(\cdot, t) \prec_{(\mathbb{R}^n, \mu)} O_{\varphi}(\cdot, t) \quad \text{and} \quad \bar{U}_{\hat{a}}^\infty(\cdot, t) \prec_{(\mathbb{R}^n, \mu)} O_{\hat{\varphi}}(\cdot, t), \quad t \geq t_0.$$ Again since $\vartheta \in S^{n-1} \cap \mathbb{R}Z^n$, there exists $B \geq (M_0 - m_0)t_0$, such that $B\vartheta \in \mathbb{Z}^n$, then

$$\bar{U}_a^\infty(\cdot + B\vartheta, t) \prec_{(\mathbb{R}^n, \mu)} \bar{U}_{\hat{a}}^\infty(\cdot, t), \quad 0 \leq t \leq t_0.$$ However, for any $t > t_0$, both $\bar{U}_a^\infty(x + B\vartheta, t)$ and $\bar{U}_{\hat{a}}^\infty(x, t)$ are globle solutions, therefore, the comparison principle implies the above ordering relation for $t > t_0$. As a combination, we conclude that

$$\bar{U}_a^\infty(\cdot + B\vartheta, t) \prec_{(\mathbb{R}^n, \mu)} \bar{U}_{\hat{a}}^\infty(\cdot, t), \quad t \geq 0.$$ Finally, the desired result follows (for any $t \geq 0$) as

$$\bar{U}_a^\infty \left( \cdot - \left( \frac{\delta}{2} t - B \right) \vartheta, t \right) \prec_{(\mathbb{R}^n, \mu)} \bar{U}_{\hat{a}}^\infty \left( \cdot - \left( \frac{\delta}{2} t \vartheta, t \right) \right) \leq O_{\hat{\varphi}} \left( \cdot - \left( \frac{\delta}{2} t \vartheta, t \right) \right) = O_{\varphi}(\cdot, t).$$

\[ \square \]

5.4. **Continuity and ordering.**

5.4.1. **The semicontinuity.**

**Lemma 5.19.** Fix $\mu \in \mathbb{R}$, $a := (\nu, R, 0, q, s) \in A$ with $R > \frac{\sqrt{n}}{2}$ and set $e := (\nu, q, s) \in E$, if there exists $r > \frac{\sqrt{n}}{2}$ and $T_0 > 0$, such that

$$\bar{U}_a(\cdot, t) \prec_{(\Omega(0, r, \nu), \mu)} O_e(\cdot, t), \quad T_0 \leq t \leq T_0 + \frac{\sqrt{n}}{s}$$

(resp. $O_e(\cdot, t) \prec_{(\Omega(0, \nu), \mu)} \bar{U}_a(\cdot, t), \quad T_0 \leq t \leq T_0 + \frac{\sqrt{n}}{s}$). Then we have for $\hat{a} := (\nu, R + \sqrt{n}, 0, q, s)$ that

$$\bar{U}_{\hat{a}}(\cdot, t) \prec_{(\Omega(0, r, \nu), \mu)} O_e(\cdot, t)$$

(resp. $O_e(\cdot, t) \prec_{(\Omega(0, \nu), \mu)} \bar{U}_{\hat{a}}(\cdot, t)$), $t \geq T_0$, i.e., $\bar{U}_{\hat{a}}(x, t)$ (resp. $\bar{U}_a(x, t)$) detaches from $O_e(x, t)$ at the $\mu$ level set.
Proof. By Proposition 2.8, we have that
\[ \overline{U}_a(\cdot,t) \preceq \overline{U}_a(\cdot,t) \prec (\Omega(0,r;\nu),\mu) \ O_e(\cdot,t), \quad T_0 \leq t \leq T_0 + \frac{\sqrt{n}}{s}. \]
We only need to prove the following ordering relation.
\[ \overline{U}_a(\cdot,t) \prec (\Omega(0,r;\nu),\mu) \ O_e(\cdot,t), \quad t > T_0 + \frac{\sqrt{n}}{s}. \]
For any \( x_0 \in \Omega(0,r;\nu) \) and \( t_0 > T_0 + \frac{\sqrt{n}}{s} \), then there exists \( \Delta z \in \mathbb{Z}^n \), such that
\[ (t_0 - T_0) s - \sqrt{n} \leq \Delta z \cdot \nu \leq (t_0 - T_0) s \quad \text{and} \quad x_0 - \Delta z \in \Omega(0,r;\nu). \]
Let us set \( \Delta t := \frac{\Delta z \cdot \nu}{s} \), then
\[ \overline{U}_a(x_0,t_0) \preceq \overline{U}_a(x_0 - \Delta z,t_0 - \Delta t) \preceq \overline{U}_a(x_0 - \Delta z,t_0 - \Delta t) < O_e(x_0 - \Delta z,t_0 - \Delta t) = O_e(x_0,t_0). \]
Since the above \((x_0,t_0)\) is arbitrary, the desired result follows. \( \square \)

**Proposition 5.20** (semicontinuity). The head (resp. tail) speed \( \overline{s}(\nu) \) (resp. \( s(\nu) \)) : \( S^{n-1} \to [m_0,M_0] \) is upper (resp. lower) semicontinuous.

\[ \begin{array}{c}
\nu \quad \nu_0 \\
\hline
0 \quad R_0 \\
\hline
0 \quad R_0 - r_0 \\
\hline
0 \quad sT_0 + 2\sqrt{n} \\
\hline
0 \\
\end{array} \]

**Figure 3.** The upper semicontinuity of \( \overline{s}(\nu) : S^{n-1} \to [m_0,M_0] \)

Proof. Fix any \( \nu_0 \in S^{n-1}, \delta > 0 \) and set \( s := \overline{s}(\nu_0) + \delta \). To prove the upper semicontinuity of \( \overline{s}(\nu_0) \), it suffices to show the statement: there exists a neighborhood \( \mathcal{N}(\nu_0) \) of \( \nu_0 \) in \( S^{n-1} \), such that for any \( \nu \in \mathcal{N}(\nu_0) \), there exists \( R > 0 \), such that \( \overline{U}_a(x,t) \) detaches from \( O_e(x,t) \), where
\[ a := (\nu,R,0,q,s) \quad \text{and} \quad e := (\nu,q,s) \in \mathbb{E}. \]
For the simplicity of the notation, we shall only prove the detachment at the zero level set. The case of general \( \mu \) level set can be established similarly.

**Step 1.** As \( s > \overline{s}(\nu_0) \), there exist \( R_0 > r_0 > 2\sqrt{n} \) and \( T_0 > 0 \), such that
\[ \overline{U}_{a_0}(\cdot,t) \prec (\Omega(0,r_0;\nu_0),\mu) \ O_{e_0}(\cdot,t), \quad t \geq T_0, \]
where
\[ a_0 := (\nu_0,R_0,0,q,s) \in \mathbb{A} \quad \text{and} \quad e_0 := (\nu_0,q,s) \in \mathbb{E}. \]
By Proposition 5.10, we can adjust \( R_0 \) and \( T_0 \) if necessary, such that
\[ \text{dist} \left( L_0^+ (\overline{U}_{a_0}(\cdot,t);\Omega(0,r_0;\nu_0)), L_0^- (O_{e_0}(\cdot,t);\Omega(0,r_0;\nu_0)) \right) > 2\sqrt{n}, \tag{5.21} \]
for any \( t \in \left[ T_0, T_0 + \frac{2\sqrt{n}}{s} \right] \).

Step 2. Let us define a set of angles

\[
\Theta := \left\{ \theta \in \left( 0, \frac{T}{2} \right) \right\} \left\{ r_0 \cos \theta - (sT_0 + 2\sqrt{n}) \sin \theta > 2\sqrt{n} \right. \\
\left. R_0 \sin \theta \cos \theta + (sT_0 + 2\sqrt{n}) \sin^2 \theta < \frac{\sqrt{n}}{4} \right\}.
\]

(5.22)

Then we consider the neighborhood of direction \( \nu_0 \).

\[ \mathcal{N}(\nu_0) := \left\{ \nu \in S^{n-1} \left| \theta \in \Theta, \text{ where } \theta \text{ is the angle between } \nu \text{ and } \nu_0 \right. \right\}. \]

Our aim is to construct a larger cylinder of the form \( \Omega(0, R; \nu) \) that includes the interesting part of the above cylinder, i.e.,

\[ \{ x \in \Omega(0, R_1; \nu_0) \mid 0 \leq x \cdot \nu_0 \leq sT_0 + 2\sqrt{n} \}. \]

Let us take \( R \) and \( r \) as follows, where \( \nu \in \mathcal{N}(\nu_0) \).

\[ R_1 := R_0 \cos \theta + (sT_0 + 2\sqrt{n}) \sin \theta, \quad r_1 := r_0 \cos \theta - (sT_0 + 2\sqrt{n}) \sin \theta. \]

The cylinder and the obstacle are as follows.

\[ \mathcal{C} := \Omega(-R_1 \tan \theta, \nu, R_1; \nu), \quad \mathcal{O}(x, t) := [(x \cdot \nu) + R_1 \sin \theta - st](-|q|). \]

Then we have properties

\[ \{ x \in \Omega(0, R_0; \nu_0) \mid 0 \leq x \cdot \nu_0 \leq sT_0 + 2\sqrt{n} \} \subseteq \Omega(0, R_1; \nu), \quad \mathcal{O}(x, t) \prec \Omega(0, R_0; \nu_0). \]

Similar to Definition 2.10, let us consider the set of subsolutions bounded from above by \( \mathcal{O}(x, t) \).

\[ \mathcal{F} := \left\{ u \in \text{USC}(\Omega(0, R_1; \nu) \times (0, \infty)) \mid u_t \leq \mathcal{F} (D^2 u, Du, x), \ u(x, t) \leq \mathcal{O}(x, t) \right\}. \]

And in particular, we have the associated largest subsolution \( \mathcal{U}(x, t) \) as follows.

\[ \mathcal{U}(x, t) := \left( \sup \{ u(x, t) \mid u \in \mathcal{F} \} \right)^+. \]

Step 3. Since \( \mathcal{U}(x, t) \) is a subsolution in \( \Omega(0, R; \nu) \), so is \( Z(x, t) \) defined as follows:

\[ Z(x, t) := \begin{cases} 0, & x \in L^+_0(\mathcal{U}(\cdot, t); \Omega(-R_1 \tan \theta \nu, R_1; \nu)), \\
-\infty, & x \in \Omega(-R_1 \tan \theta \nu, R_1; \nu) \setminus L^+_0(\mathcal{U}(\cdot, t); \Omega(-R_1 \tan \theta \nu, R_1; \nu)). \end{cases} \]

Consider the modified super zero level set \( \mathcal{L}^+_0 \) in \( \Omega(0, R_0; \nu_0) \)

\[ \mathcal{L}^+_0 := \left\{ x \in \Omega(0, R_0; \nu_0) \mid x \cdot \nu \leq -R_1 \tan \theta + \frac{m_0 t}{\cos \theta} \right\} \cup \left\{ x \in \Omega(0, R_0; \nu_0) \mid \mathcal{U}(x, t) \geq 0 \right\}. \]

Then we can define the modified characteristic function.

\[ \mathcal{F}(x, t) := \begin{cases} 0, & x \in \mathcal{L}^+_0, \\
-\infty, & x \in \Omega(0, R_0; \nu_0) \setminus \mathcal{L}^+_0 \in \mathcal{F}_a. \end{cases} \]

Therefore, by the maximality of \( \mathcal{U}_{a_0}(x, t) \), we conclude that

\[ \mathcal{F}(x, t) \leq \mathcal{F}_{a_0}(x, t), \quad x \in \Omega(0, R_0; \nu_0), \quad 0 \leq t \leq T_0 + \frac{2\sqrt{n}}{s}. \]

In particular, we have that

\[ \mathcal{F}(\cdot, t) \prec (\Omega(0, R_0; \nu_0), 0) \mathcal{U}_{a_0}(\cdot, t), \quad 0 \leq t \leq T_0 + \frac{2\sqrt{n}}{s}. \]

From (5.21) and (5.22), we conclude for any \( T_0 \leq t \leq T_0 + \frac{2\sqrt{n}}{s} \) that

\[ \text{dist} (\mathcal{L}^+_0, L^-_0(\mathcal{O}(\cdot, t); \Omega(0, r_0; \nu_0))) \geq \text{dist} (L^+_0(\mathcal{U}_{a_0}(\cdot, t); \Omega(0, r_0; \nu_0)), L^-_0(\mathcal{O}(\cdot, t); \Omega(0, r_0; \nu_0))) \geq \text{dist} (L^+_0(\mathcal{U}_{a_0}(\cdot, t); \Omega(0, r_0; \nu_0)), L^-_0(\mathcal{O}_{a_0}(\cdot, t); \Omega(0, r_0; \nu_0)) - 2R_1 \tan \theta > \frac{3\sqrt{n}}{2} \]
Step 4. Now let us set
\[ R = R_1 + \sqrt{n}, \quad r := r_1 - \sqrt{n} \quad \text{and} \quad \xi_0 := \arg \min_{\xi \in \mathbb{R}^n, \xi \nu \geq R \tan \theta} |\xi|. \]
Then \(0 < \xi_0 \cdot \nu < \frac{3\sqrt{n}}{2}\), we shall compare the standard obstacle subsolution \(\bar{U}_a(x,t)\) and \(U(x,t)\) at the zero level set in \(\Omega(0,r;\nu)\). Therefore,
\[
\text{dist} \left( L_0^+ \left( \bar{U}_a(\cdot,t); \Omega(0,r;\nu) \right), L_0^- (O_e(\cdot,t); \Omega(0,r;\nu)) \right) \\
\geq \text{dist} \left( L_0^+ (U(\cdot - \xi_0,t); \Omega(0,r;\nu)), L_0^- (O(\cdot - \xi_0,t); \Omega(0,r;\nu)) \right) \\
> \frac{3\sqrt{n}}{2} - \xi_0 \cdot \nu > 0,
\]
\[ T_0 \leq t \leq T_0 + \frac{2\sqrt{n}}{s}. \]
By Lemma 5.19, we then conclude that \(\bar{U}_a(x,t)\) detaches from \(O_e(x,t)\). Hence,
\[ \bar{\pi}(\nu_0) + \delta \geq \overline{\pi}(\nu), \quad \text{for any} \quad \nu \in \mathcal{N}(\nu_0). \]
In other words, \(\overline{\pi}(\nu) : \mathbb{S}^{n-1} \to [m_0, M_0]\) is upper semicontinuous.
Similarly, we can show that \(\bar{\pi}(\nu) : \mathbb{S}^{n-1} \to [m_0, M_0]\) is lower semicontinuous. \(\square\)

5.4.2. The continuity of head and tail speeds.

**Proposition 5.21** (Continuity). The functions \(\bar{\pi}(\nu), \bar{\pi}(\nu) : \mathbb{S}^{n-1} \to [m_0, M_0]\) are both continuous.

**Proof.** Let us only prove the continuity of \(\bar{\pi}\), since the case of \(\bar{\pi}\) can be argued similarly. By Proposition 5.20, it suffice to show that for any \(\vartheta \in \mathbb{S}^{n-1}\) and \(\nu_\vartheta \in \mathbb{S}^{n-1}\), such that \(\nu_\vartheta \to \vartheta\), then \(\liminf_{t \to \infty} \bar{\pi}(\nu_t) \geq \bar{\pi}(\vartheta)\). Assume this is not true, then according to Proposition 5.20 and Proposition 5.16, we have that (up to a subsequence if necessary)
\[ \bar{\pi}\left( \liminf_{t \to \infty} (\nu_t) \right) = \bar{\pi}(\vartheta) = \lim_{t \to \infty} \bar{\pi}(\nu_t) + \delta = \lim_{t \to \infty} \bar{\pi}^\infty(\nu_t) + \delta, \quad \text{with} \quad \delta > 0. \]
Fix \(0 < \sigma < \frac{\delta}{3}\) and \(\mathfrak{s} := \bar{\pi}(\vartheta) + \sigma\). Then from Proposition 5.17, we have (\(\mu \in \mathbb{R}\)) that
\[
\bar{U}_a^\infty \left( \cdot - \left( \frac{s - \bar{\pi}^\infty(\nu_t)}{2} - \hat{B}_t \right) \nu_t, t \right) \prec_{(\mathbb{R}^n,\mu)} O_{e_\mathfrak{s}}(\cdot,t), \quad t > \sqrt{n} m_0,
\]
where
\[
a_\mathfrak{s} := (\nu_t, \infty, 0, q, s) \in \mathcal{A} \quad \text{and} \quad e_\mathfrak{s} := (\nu_t, q, s) \in \mathbb{E}.
\]
Then similar to the argument of Proposition 5.20, we have the upper semicontinuity of the detachment time with respect to the direction. Then we have that \(\limsup_{t \to \infty} \hat{B}_t \leq \hat{B} < \infty\) for some number \(\hat{B}\). Then, we have that
\[
\liminf_{t \to \infty} \left( \frac{s - \bar{\pi}^\infty(\nu_t)}{2} - \hat{B}_t \right) \geq \frac{\delta + \sigma}{2} t - \hat{B} > 3\sigma t - \hat{B}.
\]
Let us set \(T_0 := \frac{2(\hat{B} + 1)}{\delta + \sigma}\), recalling Proposition 5.2, there exists \(\ell_0\), such that for any \(t > T_0\) and any \(\ell > \ell_0\), \(\bar{U}_a^\infty(x,t)\) is a global solution. Denote
\[
U_*(x,t) := \liminf_{\eta \to 0} \left\{ \left( \bar{U}_a^\infty \right)_+ (y,s) \left| y - x \right| + \left| s - t \right| + \frac{1}{\ell} < \eta \right\}, \quad t > T_0,
\]
which is a global supersolution detached from (at each \(\mu\) level set) the obstacle by at least \(3\sigma \hat{B} - \hat{B}\). Next, we consider the obstacle subsolution
\[ \bar{U}_a^\infty(x,t), \quad \text{with} \quad \mathfrak{a} := (\vartheta, \infty, 0, q, s) \in \mathcal{A}. \]
Then by comparison principle (c.f. Proposition 2.1), we have that
\[
\bar{U}_a^\infty(x,t) - U_*(x,t) \leq \sup_{y \in \mathbb{R}^n} \left( \bar{U}_a^\infty(y,T_0) - U_*(y,T_0) \right) \leq sT_0 |q| < \infty, \quad t > T_0.
\]
On the other hand, due to the ordering relation
\[
U_*(x,t) - O_{e^\infty}(x,t) \leq - \left( 3\sigma t - \hat{B} \right) |q|, \quad e^\infty := (\vartheta, q, s) \in \mathbb{E}, \quad t > T_0.
\]
We then have
\[ U_a(x, t) - O_e(x, t) \leq -\left(3\sigma t - \hat{B}\right)|q| + sT_0|q|, \quad t > T_0, \]
which implies that
\[ U_a(\cdot - \left(3\sigma t - \hat{B} - sT_0\right)\vartheta, t) \prec_{(\mathbb{R}^n, \mu)} O_e(\cdot, t), \quad t > T_0. \]

Finally, let us consider \( \hat{\vartheta}^\infty \) and \( \hat{\vartheta}^\infty \) with \( \hat{\vartheta} = \sigma - 2\sigma = \vartheta^\infty(\vartheta) - \sigma \), then
\[ U_{\hat{\vartheta}^\infty} \left(\cdot - \left(\sigma t - \hat{B} - sT_0\right)\hat{\vartheta}, t\right) \leq U_{\hat{\vartheta}^\infty} \left(\cdot - \left(\sigma t - \hat{B} - sT_0\right)\vartheta, t\right) \prec_{(\mathbb{R}^n, \mu)} O_{\vartheta^\infty}(\cdot + 2\sigma t) = O_{\vartheta^\infty}(\cdot, t). \]
By Definition 5.6, we must have that \( \vartheta^\infty(\vartheta) \leq \hat{\vartheta} \), which is a contradiction.

**5.4.3. The ordering relation in all directions.**

**Proposition 5.22 (Ordering).** For any \( \nu \in \mathbb{S}^{n-1} \), we have \( M_0 \geq \vartheta(\nu) \geq \vartheta(\nu) \geq m_0 \).

**Proof.** From Proposition 5.14, we already have the ordering relation for all irrational directions. Then according to the upper semicontinuity of \( \vartheta(\cdot) \) and the lower semicontinuity of \( \vartheta(\cdot) \), we can prove the relation for all rational directions. More precisely, let \( \vartheta \in \mathbb{S}^{n-1} \cap \mathbb{R}^n \) and let \( \{\nu_k\}_{k \geq 1} \subseteq \mathbb{S}^{n-1} \setminus \mathbb{R}^n \) such that \( \lim_{k \to 0} |\nu_k - \vartheta| = 0 \), then
\[ \vartheta(\vartheta) \geq \limsup_{k \to \infty} \vartheta(\nu_k) \geq \limsup_{k \to \infty} \vartheta(\nu_k) \geq \liminf_{k \to \infty} \vartheta(\nu_k) \geq \vartheta(\vartheta) \]

\[ \square \]

**6. Homogenization**

**Definition 6.1.** For \( 0 < \varepsilon < 1 \), let \( u^\varepsilon(x, t) \) be the solution in (1.2), for any \( (x, t) \in \mathbb{R}^n \times (0, \infty) \), let us denote the upper and lower half relaxed limits:
\[ u^*(x, t) := \limsup_{\eta \to 0} \{ u^\varepsilon(y, s)||y - x| + |s - t| + \varepsilon < \eta \}, \]
\[ u_*(x, t) := \liminf_{\eta \to 0} \{ u^\varepsilon(y, s)||y - x| + |s - t| + \varepsilon < \eta \}. \]

**Proposition 6.1.** Let \( \nu \in \mathbb{S}^{n-1} \) and \( \vartheta(\nu) \) be the head speed in the \( \nu \) direction. Let \( \phi(x, t) \) be a \( C^{2,1} \) function, assume \( u^*(x, t) - \phi(x, t) \) obtains a strict local maximum at \( (x_0, t_0) \in \mathbb{R}^n \times (0, \infty) \), denote \( q_0 := D\phi(x_0, t_0) \), then
\[ \phi(x(t), 0) \leq \vartheta(\nu)||D\phi(x_0, t_0)||, \quad q_0 \neq 0, \quad \nu := -\frac{q_0}{|q_0|}, \]
\[ \phi(x(t), 0) \leq 0, \quad q_0 = 0. \]

**Proof.** Case 1: \( D\phi(x_0, t_0) \neq 0 \). To prove the statement, let us derive a contradiction from the following contrary hypothesis. Assume the existence of \( \delta > 0 \), such that
\[ \phi(x, t) > (\vartheta(\nu) + 3\delta)||D\phi(x, t)||, \quad (x, t) \in U(\delta) := C(x_0, \delta; \nu) \times (t_0 - \delta, t_0 + \delta), \]
where
\[ C(x_0, \delta; \nu) := \{ x \in \mathbb{R}^n | ((x - x_0) \cdot \nu)| < \delta, \quad |(x - x_0) \cdot \nu| < 2\delta \}. \]
We can choose \( \delta \) so small that
\[ y \in \{ x \in \mathbb{C}(x_0, \delta; \nu) \mid \nu x^*(x_0, t_0) = 0 \text{ or } \phi(x_0, t_0) = 0 \}, \quad (y - x_0) \cdot \nu < \delta. \]
Let us also assume without loss of generality that \( u^*(x_0, t_0) = 0 \). Then there exists \( \{((x, t))_{0 < \varepsilon < 1} \subseteq U(\delta) \), such that \( u^\varepsilon(x, t) - \phi(x, t) \) obtains a strict local maximum in \( U(\delta) \) at \( (x, t) \). Moreover,
\[ \lim_{\varepsilon \to 0} (|u^\varepsilon(x^\varepsilon, t_0) - u^*(x_0, t_0)| + |x^\varepsilon - x_0| + |t - t_0|) = 0. \]
Apply perturbations if necessary, suppose that in \( U(\delta) \), \( \phi(x, t) \) has only linear term in \( t \) and quadratic terms in \( x \). Denote \( \phi(x_0, t_0) = \hat{s}||D\phi(x_0, t_0)|| \), where we have \( \hat{s} > \vartheta(\nu) + 3\delta \). Then the following holds (where \( A := \frac{D^2\phi(x_0, t_0)}{2} \)):
\[ \phi(x, t) = |q_0|\hat{s}(t - t_0) + q_0 \cdot (x - x_0) + (x - x_0) \cdot A(x - x_0)^T, \quad (x, t) \in U(\delta). \]
Let us also denote $q_{\varepsilon} := D\phi(x_{\varepsilon}, t_{\varepsilon})$ for small $\varepsilon$ and $(x, t) \in U(\delta)$, then
\[
\phi(x, t) - \phi(x_{\varepsilon}, t_{\varepsilon}) = \phi_t(x_{\varepsilon}, t_{\varepsilon})(t - t_{\varepsilon}) + D\phi(x_{\varepsilon}, t_{\varepsilon}) \cdot (x - x_{\varepsilon}) + A(x - x_{\varepsilon})^T \leq |q_{\varepsilon}| \delta(t - t_{\varepsilon}) + q_{\varepsilon} \cdot (x - x_{\varepsilon}) + |q_{\varepsilon}| |x - x_{\varepsilon}| + \|A\|_{\infty} |x - x_{\varepsilon}|^2.
\]

Let $B := B(\nu, 3\delta)$ be the constant from Proposition 5.10. We shall select a small constant $h$ as follows,
\[
h := H\varepsilon \quad \text{with} \quad H := \frac{B + 2\sqrt{m} + 1}{\delta} + R,
\]
where $R := R(\nu, 3\delta, \sqrt{m}, \frac{B + 2\sqrt{m} + 1}{\delta})$ is the radius from Proposition 5.10, associated to the time range $0 \leq t \leq \frac{B + 2\sqrt{m} + 1}{\delta}$. Then we have that $0 < \varepsilon R < h$. Next, we shall shift $(x_{\varepsilon}, t_{\varepsilon})$ backwards as follows:
\[
y_{\varepsilon} := x_{\varepsilon} - h\tilde{s}\nu + \left(\frac{\|A\|_{\infty} h^2 + |q_{\varepsilon} - q_{\varepsilon}| h}{|q_{\varepsilon}|} + \sqrt{n\varepsilon}\right) \nu,
y_{\varepsilon} \in \arg \min_{x \in S^n} |x - y_{\varepsilon}|, \quad \tau_{\varepsilon} := t_{\varepsilon} - h.
\]
A direct calculation shows that
\[
\phi(x, t) - \phi(x_{\varepsilon}, t_{\varepsilon}) < |q_{\varepsilon}|(\delta(t - \tau_{\varepsilon}) - (x - y_{\varepsilon}) \cdot \nu), \quad (x, t) \in U(h).
\]
Moreover, we have the estimates
\[
\text{dist}(y_{\varepsilon} + h\tilde{s}\nu, x_{\varepsilon}) \leq |y_{\varepsilon} - \tilde{y}_{\varepsilon}| + |\tilde{y}_{\varepsilon} + h\tilde{s}\nu - x_{\varepsilon}| \leq 2\sqrt{n}\varepsilon + \frac{\|A\|_{\infty} H^2 + |q_{\varepsilon} - q_{\varepsilon}| h}{|q_{\varepsilon}|}.
\]
Let us consider $\varepsilon$ so small that
\[
\frac{\|A\|_{\infty} H^2 + |q_{\varepsilon} - q_{\varepsilon}| H}{|q_{\varepsilon}|} < 1 \quad \text{and} \quad 0 < h < \delta.
\]
Then on one hand, we have that (by rescaling $(x, t)$ to $(\varepsilon x, \varepsilon t)$ in Proposition 5.10)
\[
\text{dist}(y_{\varepsilon} + h\tilde{s}\nu, x_{\varepsilon}) < \delta h - B\varepsilon.
\]
On the other hand, based on the above calculations, we get (when $0 < \varepsilon \ll 1$) that
\[
u^*(x, t) - \nu^*(x_{\varepsilon}, t_{\varepsilon}) < |q_{\varepsilon}|(\delta(t - \tau_{\varepsilon}) - (x - y_{\varepsilon}) \cdot \nu), \quad (x, t) \in U(h).
\]
Because at the moment $t_{\varepsilon}$, the center of zero level set of the obstacle is $y_{\varepsilon} + h\tilde{s}\nu$. We shall shift the above relation by $(-y_{\varepsilon}, -\tau_{\varepsilon})$ and rescale it to the unit scale, then apply the Proposition 5.10 with the time range $0 \leq t \leq H$, finally, we scale it back to the $\varepsilon$ scale and shift it by $(y_{\varepsilon}, \tau_{\varepsilon})$, this process indicates that
\[
\text{dist}(y_{\varepsilon} + h\tilde{s}\nu, x_{\varepsilon}) > (1.5\delta H - B)\varepsilon > \delta h - B\varepsilon,
\]
which is the desired contradiction.

**Case 2:** $D\phi(x_0, t_0) = 0$. Let us assume on the contrary that $\phi_t(x_0, t_0) > 0$. Since $\nu^*(x, t) - \phi(x, t)$ has a strict local maximum at $(x_0, t_0)$, there exist small numbers $r, \sigma > 0$, and the following hold, where $V_r(x_0, t_0) := B_r(x_0) \times (t_0 - r, t_0 + r)$.
\[
V_r(x_0, t_0) \subseteq \mathbb{R}^n \times (0, \infty), \quad \max_{V_r} (\nu^* - \phi) < \max_{V_r} (\nu^* - \phi), \quad (6.1)
\]
\[
\min_{(x, t) \in V_r} \phi_t(x, t) > \sigma, \quad \sup_{(x, t) \in V_r} |D\phi(x, t)| < \frac{\sigma}{2M_0}.
\]
Since $\nu \in S^{n-1}$, therefore, if $\varepsilon$ is small, we have that
\[
\sup_{\nu \in S^{n-1}} \varepsilon \text{tr} \left\{ \epsilon^2 D^2 \phi(I - \nu \otimes \nu) \right\} < \frac{\sigma}{2}.
\]
Hence
\[
\mathcal{F}^\varepsilon \left( \frac{\epsilon^2 D^2 \phi(x, t)}{\nu^*(x, t)}, \frac{\epsilon^2 D \phi(x, t)}{\nu^*(x, t)} \right) < \sigma < \phi_t(x, t), \quad (x, t) \in V_r(x_0, t_0),
\]
which means that $\phi(x, t)$ is a (classical) supersolution of (1.2), then the Proposition 2.1 indicates that
\[
\max_{V_r} (\nu^* - \phi(x, t)) \leq \max_{\partial V_r} (\nu^* - \phi(x, t)).
\]
Let us apply the upper half relaxed limit operator (Definition 6.1) on both sides and derive that
\[ \max_{V_r} (u^*(x,t) - \phi(x,t)) \leq \max_{\partial_r V_r} (u^*(x,t) - \phi(x,t)), \]
which contradicts (6.1).

**Proposition 6.2.** Let \( \nu \in \mathbb{S}^{n-1} \) and \( s(\nu) \) be the tail speed in the \( \nu \) direction. Let \( \psi(x,t) \) be a \( C^{2,1} \) function, assume \( u_*(x,t) - \psi(x,t) \) obtains a strict local minimum at \( (x_0,t_0) \in \mathbb{R}^n \times (0,\infty) \), denote \( q_0 := D\psi(x_0,t_0) \), then
\[
\begin{cases}
\psi_t(x_0,t_0) \geq s(\nu)|D\psi(x_0,t_0)|, & q_0 \neq 0, \quad \nu := -\frac{q_0}{|q_0|}, \\
\psi_t(x_0,t_0) \geq 0, & q_0 = 0.
\end{cases}
\]

**Proof.** It is similar to that of Proposition 6.1, we omit it here. \( \square \)

**Definition 6.2.** Consider the equation \((E)\) as follows, where
\[
\begin{align*}
u & = \nabla \left( -\frac{1}{|Du|} |Du| \right), \quad (x,t) \in \mathbb{R}^n \times (0,\infty), \\
u(x,0) = u_0(x), & \quad x \in \mathbb{R}^n.
\end{align*}
\]

(a) Let \( s(\cdot) = \bar{s}(\cdot) \), an upper semicontinuous function \( u(x,t) : \mathbb{R}^n \times (0,\infty) \to \mathbb{R} \) is called a viscosity subsolution of \((E)\), if the following hold.
(i) Let \( \phi(x,t) \) be a \( C^{2,1} \) function, assume \( u(x,t) - \phi(x,t) \) obtains a local maximum at \( (x_0,t_0) \in \mathbb{R}^n \times (0,\infty) \), denote \( q_0 := D\phi(x_0,t_0) \), then
\[
\begin{cases}
\phi_t(x_0,t_0) \leq \bar{s}(\nu)|D\phi(x_0,t_0)|, & q_0 \neq 0, \quad \nu := -\frac{q_0}{|q_0|}, \\
\phi_t(x_0,t_0) \leq 0, & q_0 = 0.
\end{cases}
\]
(ii) \( u(x,0) \leq u_0(x) \), \( x \in \mathbb{R}^n \).
(b) Let \( s(\cdot) = \underline{s}(\cdot) \), a lower semicontinuous function \( v(x,t) : \mathbb{R}^n \times (0,\infty) \to \mathbb{R} \) is called a viscosity supersolution of \((E)\), if the following hold.
(i) Let \( \psi(x,t) \) be a \( C^{2,1} \) function, assume \( v(x,t) - \psi(x,t) \) obtains a local minimum at \( (x_0,t_0) \in \mathbb{R}^n \times (0,\infty) \), denote \( q_0 := D\psi(x_0,t_0) \), then
\[
\begin{cases}
\psi_t(x_0,t_0) \geq s(\nu)|D\psi(x_0,t_0)|, & q_0 \neq 0, \quad \nu := -\frac{q_0}{|q_0|}, \\
\psi_t(x_0,t_0) \geq 0, & q_0 = 0.
\end{cases}
\]
(ii) \( v(x,0) \geq u_0(x) \), \( x \in \mathbb{R}^n \).
(c) If \( s(\cdot) = \bar{s}(\cdot) = \underline{s}(\cdot) \), then a continuous function \( w(x,t) \) is called a viscosity solution of \((E)\) if \( w(x,0) = u_0(x) \), and that \( w \) is both a viscosity subsolution and a viscosity supersolution of \((E)\).

**Proposition 6.3.** If \( \bar{s}(\cdot) \equiv \underline{s}(\cdot) \), we denote it by \( s(\cdot) \). Let \( u^*(x,t) \) be the unique viscosity solution of \((E)\), then \( u^*(x,t) \) converges locally uniformly, as \( \varepsilon \to 0 \), to a continuous function \( \bar{u}(x,t) \) in \( \mathbb{R}^n \times (0,\infty) \), which is the unique viscosity solution of \((E)\).

**Proof.** The uniqueness, if \( u(x,t) \) is a solution of \((E)\), then \( w(x,t) := e^{-t}u(x,t) \) is a solution of the following equation, which has a unique solution.
\[
\begin{cases}
w_t + s \left( -\frac{Dw}{|Dw|} \right) |Dw|, & (x,t) \in \mathbb{R}^n \times (0,\infty), \\
w(x,0) = u_0(x), & x \in \mathbb{R}^n.
\end{cases}
\]
Therefore, the equation \((E)\) has a unique solution. On the other hand, by Proposition 6.1 and the Proposition 6.2, we have that \( u^*(x,t) \leq u_*(x,t), \ (x,t) \in \mathbb{R}^n \times (0,\infty) \). Clearly, by Definition 6.1, we have \( u_*(x,t) \leq u^*(x,t) \). Therefore, \( u_*(x,t) = u^*(x,t) \), let us denote it by \( \bar{u}(x,t) \). By Definition 6.1 again, we have that
\[ \lim_{\varepsilon \to 0} u^*(x,t) = \bar{u}(x,t) \text{ locally uniformly in } \mathbb{R}^n \times (0,\infty). \] \( \square \)
7. Nonhomogenization

In this section, we study the case that the head speed is not identically equal to the tail speed. i.e., there exists \( \nu_0 \in S^{n-1} \), with \( \bar{g}(\nu_0) < \bar{\pi}(\nu_0) \). It turns out that in this case we can find “long fingers”, growing linearly in time in the \( \nu_0 \) direction, in certain level set of the real solution.

7.1. An ordering relation.

**Definition 7.1.** For any \( q \in \mathbb{R}^n \), let \( u := u(x,t; q) \) and \( u^\ast := u^\ast(x,t; q) \) be the unique solution of the following equation (7.1) and (7.2) (c.f. (2.1)), respectively.

\[
\begin{cases}
 u_t = \mathcal{F}(D^2 u, Du, x), & (x,t) \in \mathbb{R}^n \times (0,\infty), \\
 u(x,0;q) := q \cdot x, & x \in \mathbb{R}^n.
\end{cases}
\]  

(7.1)

\[
\begin{cases}
 u_t^\ast = \mathcal{F}(\varepsilon D^2 u^\ast, Du^\ast, x), & (x,t) \in \mathbb{R}^n \times (0,\infty), \\
 u^\ast(x,0;q) := q \cdot x, & x \in \mathbb{R}^n.
\end{cases}
\]  

(7.2)

**Lemma 7.1.** Let \( a := (\nu,R,0,q,\bar{\pi}(\nu) + \sigma) \in \mathbb{A} \) (resp. \( a := (\nu,R,0,q,\bar{g}(\nu) - \sigma) \in \mathbb{A} \)) with \( \sigma > 0 \), then there exists \( A = A(\nu,\sigma) > 0 \), with \( \xi_\nu \in \arg \min_{\xi \in \mathbb{R}^n} |\xi| \), we have for any \( t \geq 0 \) that

\[
 u(\cdot,t;q) \prec_{(\Omega(0,R;\nu),\mu)} \mathcal{U}_a(\cdot - \xi_\nu, t) \quad \text{(resp. } \mathcal{U}_a(\cdot + \xi_\nu, t) \prec_{(\Omega(0,R;\nu),\mu)} u(\cdot,t;q)).
\]

*Proof.* Because sub-strict-detachment (c.f. Definition 5.4) implies uniform detachment, without loss of generality, we can take \( \mu = 0 \). By Lemma 5.6, for any \( r > 0 \), there exists \( R := R(\nu,\sigma) > 0 \) and \( T = T(\nu,\sigma) > 0 \), such that for any \( t > T \), we have

\[
 \mathcal{U}_a(\cdot,t) \prec_{(\Omega(0,r;\nu),0)} O_e(\cdot,t), \quad t > T, \quad \text{with } e := (\nu,q,\bar{\pi}(\nu) + \sigma) \in \mathbb{E}.
\]

Moreover, we have that

\[
 u(\cdot,t;q) \prec_{(\Omega(0,r;\nu),0)} O_{(\nu,q,m_0+M_0)}(\cdot,t), \quad 0 \leq t \leq T + 1.
\]

(7.3)

Since the above \( T \) is independent of \( r \), we have \( r \to \infty \) if we send \( R \to \infty \). Let us set \( A := (m_0 + M_0)(T(\nu,\sigma) + 1) + 1 \) and denote \( U_{\infty} \) as follows, which is a supersolution as \( \mathcal{U}_a(x,t) \) is a solution in \( \Omega(0,R;\nu) \):

\[
 U_{\infty}(x,t) := \lim_{R \to \infty} \mathcal{U}_a(x - \xi_\nu, t) = \inf_{R > 0} \mathcal{U}_a(x - \xi_\nu, t).
\]

By the above choice of \( T \) and the comparison principle, we get that

\[
 u(\cdot,t;q) \prec_{(\Omega(0,\infty;\nu),0)} U_{\infty}(\cdot,t), \quad t \geq T.
\]

Recalling (7.3), we conclude that

\[
 u(\cdot,t;q) \prec_{(\Omega(0,\infty;\nu),0)} U_{\infty}(\cdot,t), \quad 0 \leq t < \infty.
\]

Then the desired result is valid due to the following inequality.

\[
 U_{\infty}(x,t) \leq \mathcal{U}_a(x - \xi_\nu, t), \quad (x,t) \in \Omega(0,R;\nu) \times [0,\infty).
\]

\[\square\]

7.2. A closeness property. If the obstacle speed is below (resp. above) the head (resp. tail) speed, it is necessary to describe the closeness of the obstacle subsolution (resp. supersolution) to the associated obstacle function.

7.2.1. Irrational directions. If \( \nu \in S^{n-1} \setminus \mathbb{R}^n \), the Proposition 5.10 (resp. the Proposition 5.11) shows that the detachment is equivalent to sub-strict-detached (super-strict-detached) obstacle speed. Therefore, if the obstacle speed is strictly smaller (resp. larger) than the head speed (resp. tail speed), the obstacle subsolution (resp. supersolution) touches the obstacle very frequently. In addition, the Birkhoff properties indicate repeated pattern of this kind of touching.

**Lemma 7.2.** Let \( a := (\nu,R,0,q,s) \in \mathbb{A} \) and set \( e := (\nu,q,s) \in \mathbb{E} \), where \( \nu \in S^{n-1} \setminus \mathbb{R}^n \) and \( s := \bar{\pi}(\nu) - \sigma \) (resp. \( s := \bar{g}(\nu) + \sigma \)) with \( \sigma > 0 \). Then for any \( R > r > \sqrt{\frac{m}{2}}, \mu \in \mathbb{R} \) and \( T > 0 \), there exists \( (x,t) \in \Omega(0,r;\nu) \times (T,\infty) \), such that

\[
 \mathcal{U}_a(x,t) = O_e(x,t) = \mu \quad \text{(resp. } \mathcal{U}_a(x,t) = O_e(x,t) = \mu).
\]

[The rest of the text is not transcribed since it's not relevant to the question.]
Proof. Since $s < \pi(\nu)$ (resp. $s > \tilde{g}(\nu)$), the Lemma is the negation of detachment. \qed

**Proposition 7.3.** Let $a := (\nu, R, 0, q, s) \in \mathfrak{A}$, where $\nu \in S^{n-1}\\setminus\mathbb{R}Z^n$, $R > \sqrt{\pi}$, $0 < s < \pi(\nu)$ (resp. $s > \tilde{g}(\nu)$) and set $e = (\nu, q, s) \in E$. Then for any $\mu \in \mathbb{R}$ and $h > 0$, there exists $\xi \in [0, 1]^n$, such that $U_a(x_1, t_1) = O_e(x_1, t_1) = \mu$ (resp. $\underline{U}_a(x_1, t_1) = O_e(x_1, t_1) = \mu$) at any $(x_1, t_1)$ satisfying

\[ x_1 \in \Omega(0, R; \nu) \cap (\xi + \mathbb{Z}^n), \quad x_1 \cdot \nu + \frac{\mu}{q} \in [0, h] \quad \text{and} \quad t_1 = \frac{x_1 \cdot \nu}{s}. \]

Proof. Because $\nu \in S^{n-1}\\setminus\mathbb{R}Z^n$ and $0 < s < \pi(\nu)$, the detachment does not happen at the $\mu$ level set. Then by Lemma 7.2, with the above $R$ and $T := \frac{h}{s}$, there exist $x_0 \in \Omega(0, R; \nu)$ and $t_0 > T$, such that (here $b := (\nu, 3R, 0, q, s) \in \mathfrak{A}$)

\[ \underline{U}_b(x_0, t_0) = O_e(x_0, t_0) = \mu. \]

Let us take $\xi \in [0, 1]^n$ with $x_0 - \xi \in \mathbb{Z}^n$. Consider any above $(x_1, t_1)$, then set $\Delta z_1$ and $\Delta t_1$ as follows. It suffices to prove that $\underline{U}_a(x_1, t_1) = O_e(x_1, t_1) = \mu$.

\[ \Delta z_1 := x_0 - x_1 \quad \text{and} \quad \Delta t_1 := \frac{\Delta z_1 \cdot \nu}{s}. \]

Because $U(x, t) := \underline{U}_b(x + \Delta z_1, t + \Delta t_1)$, restricted to $\Omega(0, R; \nu)$, is a subsolution bounded from above by $O_e(x, t)$, the maximality of $\underline{U}_a(x, t)$ implies that $U(x, t) \leq \underline{U}_a(x, t)$ with $x \in \Omega(0, R; \nu), \ t \geq 0$.

Finally, the result from the inequality

\[ \mu = \underline{U}_b(x_0, t_0) = U(x_1, t_1) \leq \underline{U}_a(x_1, t_1) \leq O_e(x_1, t_1) = O_e(x_0, t_0) = \mu. \]

\qed

**Proposition 7.4.** Fix any $\mu \in \mathbb{R}$ and assume (i)-(iii) as follows.

(i) $\nu \in S^{n-1}\\setminus\mathbb{R}Z^n$ and $q = -|q|\nu \in \mathbb{R}^n \setminus \{0\}$;
(ii) $0 < \sigma < \min \{\pi(\nu) - \frac{\pi}{2}, \tilde{g}(\nu)\}$;
(iii) $u(x, t; q)$ is the unique solution of (7.1).

Then there exists $C := C(\nu, \sigma) > 0$, such that for any $x_0 \in \mathbb{R}^n$ and $r > \sqrt{\pi}$, we have (a) and (b) as follows:

(a) There is a sequence of numbers $\{t_k\}_{k \geq 1}$ (resp. $\{\tau_k\}_{k \geq 1}$), such that

\[ \lim_{k \to \infty} t_k = \infty \quad \text{and} \quad 0 < t_{k+1} - t_k \leq \frac{1}{\tilde{g}(\nu) - \sigma}, \]

\[ \text{(resp.} \quad \lim_{k \to \infty} \tau_k = \infty \quad \text{and} \quad 0 < \tau_{k+1} - \tau_k \leq \frac{1}{\tilde{g}(\nu) + \sigma}). \]

(b) For each $k$, there exists $u(x_k, t_k; q)$ such that $u(x_k, t_k; q) = \mu$ and $x_k \cdot \nu + \frac{\mu}{q} \in [0, r]$ and

\[ C(\nu, \sigma) \leq t_k \leq C(\nu, \sigma) + (2\nu + \sigma) t_k. \]

Proof. We only consider $(x_0, \mu) = (0, 0)$, the case of general $(x_0, \mu) \in \mathbb{R}^n \times \mathbb{R}$ can be argued similarly. Let us set $s_1 := \tilde{g}(\nu) - \sigma$ and $s_2 := -\frac{\pi}{\tilde{g}(\nu) - \sigma}$. It suffices to prove a finite time version of the statement. i.e., for any $T > 0$, there exist $0 \leq t_{T, 1} < t_{T, 2} \leq \cdots < t_{T, k} \leq t_{T, k+1} \leq \cdots < T$, such that $0 < t_{T, k+1} - t_{T, k} \leq \frac{1}{s(\nu) - \sigma}$ and (b) holds. Then we take $\{t_k\}_{k=1}^{T} := \{t_{t_i}\}$. By Proposition 4.7, there exist $R > \mathfrak{S} > 0$, such that for $0 \leq t \leq T$, we have

\[ \underline{U}_a(x + \xi_0, t) \leq \underline{U}_a(x, t), \quad \text{where} \quad a_i := (\nu, R, \mathfrak{S}, q, s_i), \quad \xi_0 \in \arg \min_{\xi \in \mathbb{Z}^n} |\xi|. \]

Let us take $\hat{R} := R + \mathfrak{S}T$ and set $b := (\nu, \hat{R}, 0, q, s_1)$, then

\[ \underline{U}_a(x, t) \leq \underline{U}_b(x, t), \quad \text{for} \quad x \in \Omega(0, r; \nu) \quad \text{and} \quad 0 \leq t \leq T. \]
According to Lemma 7.1, there exists $A := A(\nu, \sigma) > 0$, such that
$$\mathcal{U}_b(\nu, A, t) \supseteq (\Omega(0, r; \nu), 0) u(\nu, t; q), \quad \text{where } t \geq 0, \quad \xi_A \subseteq \arg \min_{\xi \in \mathbb{R}^n, \xi \sigma \neq A} |\xi|.$$

Finally, we have for $0 \leq t \leq T$ that
$$\mathcal{U}_b(\nu, \xi_A \sigma A, t) \supseteq (\Omega(0, r; \nu), 0) u(\nu, t; q).$$

Set $C(\nu, \sigma) := |\xi_0 + \xi_A\sigma|$ and the Proposition 7.3 indicates the existence of $\{x_k\}_{k \geq 1}$ is a set of points in $\Omega(0, r; \nu)$ that are relative integers to each other. Since $r > \frac{\sqrt{n}}{2}$, we can choose $\{x_k\}_{k \geq 1}$ such that $|(x_{k+1} - x_k) \cdot \nu| \leq 1$. Then set $T_{t,k} := \frac{|x_k, \nu|}{(\overline{s} \sigma) - \sigma}$ and so $0 < t_{T_{t,k}} < t_{T_{t,k}} \leq \frac{1}{(\overline{s} \sigma) - \sigma}$. The statement of (b) follows from (7.4).

The other ordering relation (the ‘resp.’) can be proved similarly.

**Proposition 7.5.** Let $\nu \in S^{n-1} \setminus \mathbb{R}^n$ and $u(x, t; q)$ be the unique solution of (7.1), where $q = -|q|\nu \in \mathbb{R}^n \setminus \{0\}$. Assume $\overline{s}(\nu) > \underline{s}(\nu)$, then for any $0 < \sigma \leq \overline{s}(\nu) - \underline{s}(\nu)$, there exist constant $K := K(\nu, \sigma)$, such that the following statement holds: for any $(z_0, \mu, r, t) \in \mathbb{R}^n \times \mathbb{R} \times \left(\frac{\sqrt{n}}{2}, \infty\right) \times \left(\frac{1}{m_0}, \infty\right)$, there exist $x, y \in \Omega(z_0, r; \nu)$, such that
$$u(x, t; q) = u(y, t; q) = \mu \quad \text{and} \quad \left\{\begin{array}{l}
x \cdot \nu > (\overline{s}(\nu) - \sigma) t - \frac{\mu}{|q|} - K, \\
y \cdot \nu < (\underline{s}(\nu) + \sigma) t - \frac{\mu}{|q|} + K.
\end{array}\right.$$

**Proof.** Let us consider $(\mu, z_0) = (0, 0)$ and the general $(\mu, z_0)$ can be argued similarly. By Proposition 7.4, there exist $C_1 = C_1(\nu, \sigma) > 0$, $t_i > 0$ (with $0 < t_{i+1} - t_i \leq \frac{1}{(\overline{s} \sigma) - \sigma}$) and $\hat{x}_i \in \Omega(0, r; \nu)$, such that
$$u(\hat{x}_i, t_i; q) = 0 \quad \text{and} \quad \hat{x}_i \cdot \nu > (\overline{s}(\nu) - \sigma) t_i - C_1(\nu, \sigma).$$

Similarly, there exist $C_2 := C_2(\nu, \sigma) > 0$, $\tau_j > 0$ (with $0 < \tau_{j+1} - \tau_j \leq \frac{1}{(\overline{s} \sigma) + \sigma}$) and $\hat{y}_j \in \Omega(0, r; \nu)$, such that
$$u(\hat{y}_j, \tau_j; q) = 0 \quad \text{and} \quad \hat{y}_j \cdot \nu < (\underline{s}(\nu) + \sigma) \tau_j - C_2(\nu, \sigma).$$

Because $u(x, t; q)$ is increasing in time (c.f. Proposition 5.1 [5]). Then for any $t > \frac{1}{m_0}$, there exist $x, \hat{x}_i \in \Omega(0, r; \nu)$, $0 \leq t - t_i \leq \frac{1}{(\overline{s} \sigma) - \sigma}$ with $u(x, t; q) = 0$ and
$$x \cdot \nu > \hat{x}_i \cdot \nu > (\overline{s}(\nu) - \sigma) t_i - \frac{\mu}{|q|} - C_1(\nu, \sigma) \geq (\overline{s}(\nu) - \sigma) t - \frac{\mu}{|q|} - C_1(\nu, \sigma) - 1.$$

Similarly, there exist $y, \hat{y}_j \in \Omega(0, r; \nu)$, $0 \leq \tau_j - t \leq \frac{1}{(\underline{s} \sigma) + \sigma}$ with $u(y, t; q) = 0$ and
$$y \cdot \nu < \hat{y}_j \cdot \nu < (\overline{s}(\nu) + \sigma) \tau_j - \frac{\mu}{|q|} + C_2(\nu, \sigma) \leq (\underline{s}(\nu) + \sigma) t - \frac{\mu}{|q|} + C_2(\nu, \sigma) + 1.$$

Thus the desired result follows once we set $K(\nu, \sigma) := C_1(\nu, \sigma) + C_2(\nu, \sigma) + 2$.

7.2.2. **Rational directions.**

**Proposition 7.6.** Let $\vartheta \in S^{n-1} \cap \mathbb{R}^n$, $q = -|q|\vartheta \in \mathbb{R}^n \setminus \{0\}$ and $u(x, t; q)$ be the unique solution of (7.1). Assume $\overline{s}(\nu) > \underline{s}(\nu)$ and fix $0 < \sigma \leq \overline{s}(\vartheta) - \underline{s}(\vartheta)$, then there exists $r = r(\vartheta) > \sqrt{n}$, such that for any $\mu \in \mathbb{R}$ and $z_0 \in \mathbb{R}^n$, there exist $x, y \in \Omega(z_0, r; \vartheta)$, such that $u(x, t; q) = u(y, t; q) = \mu$ and
$$u(x, t; q) = u(y, t; q) = \mu \quad \text{and} \quad \left\{\begin{array}{l}
x \cdot \vartheta > (\overline{s}(\vartheta) - \sigma) t - \frac{\mu}{|q|} - \sqrt{n}, \\
y \cdot \vartheta < (\overline{s}(\vartheta) + \sigma) t - \frac{\mu}{|q|} + \sqrt{n}.
\end{array}\right.$$

**Proof.** Because $\vartheta \in S^{n-1} \cap \mathbb{R}^n$, then there exists $r = r(\vartheta) > \sqrt{n}$, such that there exists $w_0 \in \mathbb{R}^n \setminus \{0\}$, with $w_0 \cdot \vartheta = 0$ and $|w_0| \leq r(\vartheta) - \sqrt{n}$. By Proposition 5.16, $\overline{s}(\vartheta) = s_\infty$ and $\underline{s} = s_\infty$. Recall Definition 5.6 and denote
$$a^- := (\vartheta, \infty, 0, q, s_\infty - \sigma) \quad \text{and} \quad e^- := (\vartheta, q, s_\infty - \sigma),$$

$$a^+ := (\vartheta, \infty, 0, q, s_\infty + \sigma) \quad \text{and} \quad e^+ := (\vartheta, q, s_\infty + \sigma).$$

Clearly, we have that
$$\mathcal{U}_b(x, t) \leq u(x, t; q) \leq \mathcal{U}_b(x, t), \quad (x, t) \in \mathbb{R}^n \times [0, \infty).$$
By the choice of \( w_0 \) and the uniqueness of both \( \overline{U}_a(x, t) \) and \( \underline{U}_{a_\infty}(x, t) \), then

\[
\overline{U}_a(x, t) = \overline{U}_{a_\infty}(x + w_0, t) \quad \text{and} \quad \underline{U}_{a_\infty}(x, t) = \underline{U}_{a_\infty}(x + w_0, t).
\]

Since the detachment does not happen to \( \overline{U}_a(x, t) \) or \( \underline{U}_{a_\infty}(x, t) \), by Lemma 5.19, there are sequences \( \{(t_i, x_i)\}_{i=1}^\infty \) and \( \{(	au_j, y_j)\}_{j=1}^\infty \), such that

\[
\lim_{i \to \infty} t_i = \infty, \quad 0 < t_{i+1} - t_i \leq \frac{\sqrt{n}}{\overline{\sigma}(\hat{\theta}) - \sigma}, \quad |x_i - (x_i \cdot \hat{\theta})\hat{\theta}| < r(\hat{\theta}) \quad \text{and} \quad \overline{U}_a(x_i, t_i) = \mu,
\]

\[
\lim_{j \to \infty} \tau_j = \infty, \quad 0 < \tau_{j+1} - \tau_j \leq \frac{\sqrt{n}}{\underline{\sigma}(\hat{\theta}) + \sigma}, \quad |y_j - (y_j \cdot \hat{\theta})\hat{\theta}| < r(\hat{\theta}) \quad \text{and} \quad \underline{U}_{a_\infty}(y_j, \tau_j) = \mu.
\]

Because \( u(x, t; q) \) is increasing in time (c.f. Proposition 5.1 [5]). Then for any \( t > 0 \), there exist \( x, x_i \in \Omega(0, r; \hat{\theta}), 0 \leq t - t_i \leq \frac{\sqrt{n}}{\overline{\sigma}(\hat{\theta}) - \sigma} \), with \( u(x, t; q) = \mu \) and

\[
x \cdot \hat{\theta} \geq x_i \cdot \hat{\theta} > (\overline{\sigma}(\hat{\theta}) - \sigma) t_i - \frac{\mu}{|q|} \geq (\overline{\sigma}(\hat{\theta}) - \sigma) t - \frac{\mu}{|q|} - \sqrt{n}.
\]

Similarly, there exist \( y, y_j \in \Omega(0, r; \hat{\theta}), 0 \leq \tau_j - t \leq \frac{\sqrt{n}}{\underline{\sigma}(\hat{\theta}) + \sigma} \), with \( u(y, t; q) = 0 \) and

\[
y \cdot \hat{\theta} \leq y_j \cdot \hat{\theta} < (\underline{\sigma}(\hat{\theta}) + \sigma) \tau_j - \frac{\mu}{|q|} \leq (\underline{\sigma}(\hat{\theta}) + \sigma) t - \frac{\mu}{|q|} + \sqrt{n}.
\]

\[\square\]

### 7.2.3. The description of head/tail speed in macro and micro scales.

**Theorem 7.7.** For any \( \nu \in \mathbb{S}^{n-1}, x_0, z_0 \in \mathbb{R}^n \) and \( \mu \in \mathbb{R} \), let \( u(x, t) \) and \( u^\nu(x, t) \) be the unique solution of (1.2), such that \( u^\nu(x, 0) = -(x - x_0) \cdot \nu \). Then there exists \( r = r(\nu) > 0 \), such that in micro-scale

\[
\overline{\sigma}(\nu) = \lim_{t \to 0} \sup \left\{ x \cdot \nu \mid x \in \Omega(z_0, r; \nu), u^1(x, t) = \mu \right\}, \quad \underline{\sigma}(\nu) = \lim_{t \to 0} \inf \left\{ x \cdot \nu \mid x \in \Omega(z_0, r; \nu), u^1(x, t) = \mu \right\},
\]

and in macro-scale (for \( \varepsilon = 1 \))

\[
\lim_{t \to 0} \sup_{\varepsilon} u^\nu(x, t) = -(x - x_0) \cdot \nu + \overline{\sigma}(\nu) t \quad \text{and} \quad \lim_{t \to 0} \inf_{\varepsilon} u^\nu(x, t) = -(x - x_0) \cdot \nu + \underline{\sigma}(\nu) t.
\]

**Proof.** Based on comparison principle, we can consider without loss of generality that \( x_0 = 0 \). From Lemma 7.1, Proposition 7.5 and Proposition 7.6, there exists \( r = r(\nu) > 0 \), such that for any \( z_0 \in \mathbb{R}^n \) and \( 0 < \sigma \ll 1 \), there exists \( K = K(\nu, \sigma) \), such that

\[
\left| \sup \left\{ x \cdot \nu \mid x \in \Omega(z_0, r; \nu), u^1(x, t) = \mu \right\} - (\overline{\sigma}(\nu) - \sigma) t + \mu \right| \leq K,
\]

\[
\left| \inf \left\{ x \cdot \nu \mid x \in \Omega(z_0, r; \nu), u^1(x, t) = \mu \right\} - (\underline{\sigma}(\nu) + \sigma) t + \mu \right| \leq K.
\]

Let us divide both side by \( t \) and let \( t \to \infty \), consider that \( \sigma \) can be arbitrarily small, we get the result for the micro-scale case. Now, we consider the macro-scale case, the uniqueness of the solution for (1.2) indicates that \( u^\nu(x, t) = \varepsilon u(\varepsilon \frac{x}{\varepsilon}, \frac{t}{\varepsilon}) \), then

\[
\left| \sup \left\{ x \cdot \nu \mid x \in \Omega(z_0, \varepsilon r; \nu), u^\nu(x, t) = \mu \right\} - (\overline{\sigma}(\nu) - \sigma) \frac{t}{\varepsilon} + \mu \right| \leq K
\]

\[
\left| \inf \left\{ x \cdot \nu \mid x \in \Omega(z_0, \varepsilon r; \nu), u^\nu(x, t) = \mu \right\} - (\underline{\sigma}(\nu) + \sigma) \frac{t}{\varepsilon} + \mu \right| \leq K,
\]

Taking \( \varepsilon \to 0 \) and then send \( \sigma \to 0 \), we have that

\[
\lim_{\varepsilon \to 0} \sup_{\varepsilon} u^\nu(x, t) = \overline{\sigma}(\nu) t - \mu \quad \text{and} \quad \lim_{\varepsilon \to 0} \inf_{\varepsilon} u^\nu(x, t) = \underline{\sigma}(\nu) t - \mu,
\]

which gives the result in macro-scale case. \[\square\]
Corollary 7.8. Let \( u^c \) be the solution of (1.2) with \( u^c(x, 0) = u_0(x) \), a uniformly continuous function in \( \mathbb{R}^n \). Let \( \mathcal{A} := \{ (x_i, \nu_i) \} \subseteq \mathbb{R}^n \times \mathbb{S}^{n-1} \) be a collection of points and directions, and define the associated convex sets
\[
\mathcal{E}(t) := \inf_{(x_i, \nu_i) \in \mathcal{A}} \{ (x - x_i) \cdot \nu_i \leq \pi(\nu_i)t \}.
\]
Then (c.f. Definition 6.1) if initially \( \{ u_0(x) \geq 0 \} \subseteq \mathcal{E}(0) \), then \( \{ u^c(\cdot, t) \geq 0 \} \subseteq \mathcal{E}(t) \).

Proof. It follows from the comparison principle and the Theorem 7.7. \( \square \)

8. Laminar forcing term

In the laminar case, i.e., \( g(x) = g(x', 0) \) with \( x = (x', x_n) \). Throughout this section, let us abuse the notation and denote the forcing term by \( g(y) \) with \( y = x' \in \mathbb{R}^{n-1} \), where \( n \geq 3 \). The zero level set is now a graph, i.e., \( \{ x_n = u(y,t) \} \), where \( u(y,t) \), with initial graph \( u_0(y) \), solves the equation as follows.
\[
\begin{cases}
 u_t = \sqrt{[Du]^2 + 1} \operatorname{div} \left( \frac{Du}{\sqrt{[Du]^2 + 1}} \right) + g \sqrt{[Du]^2 + 1}, & (y,t) \in \mathbb{R}^d \times (0, \infty), \\
 u(y,0) = u_0(y), & y \in \mathbb{R}^{n-1}.
\end{cases}
\]

(8.1)

8.1. Travelling wave sub and super solutions with head and tail speeds. In this subsection, we assume that the homogenization associated to (8.1) fails, i.e., \( \pi(e_n) > s(e_n) \). Fix any \( s \in [m_0, M_0] \) and denote \( a^{\infty} := (e_n, \infty, 0, -e_n, s) \in \mathcal{A} \). Let us consider \( \{ x \in \mathbb{R}^n | U^s_{a^{\infty}}(x,t) = 0 \} \) (resp. \( \{ x \in \mathbb{R}^n | U^{\# a^{\infty}}(x,t) = 0 \} \)), which is also a graph \( \{ x \in \mathbb{R}^n | x_n = U^s(x',t) \} \) (resp. \( \{ x \in \mathbb{R}^n | x_n = U^{\#}(x',t) \} \)). Clearly, \( U^s(y,t) \) (resp. \( U^{\#}(y,t) \)) is a subsolution (resp. supersolution) of (8.1) with \( U^s(y,0) = 0 \) (resp. \( U^{\#}(y,0) = 0 \)) and \( U^s(y,t) \leq s \) (resp. \( U^{\#}(y,t) \geq s \)). The uniqueness also implies that \( U^s(\cdot, t) \) (resp. \( U^{\#}(\cdot, t) \)) is \( \mathbb{Z}^{n-1} \)-periodic. In the following discussion, we shall denote without confusion that \( \pi = \pi(e_n) \) (resp. \( s = s(e_n) \)).

Definition 8.1. For any \( s \in [m_0, M_0] \), denote by \( T^+(s) \) (resp. \( T_-(s) \)) the time after which \( U^s(y,t) \) (resp. \( U^{\#}(y,t) \)) detaches from the obstacle \( \{ x \in \mathbb{R}^n | x_n = st \} \) totally by 1. i.e.,
\[
T^+(s) := \inf \left\{ t \geq 0 | U^s(y,t) < st - 1, \quad y \in \mathbb{R}^{n-1} \right\},
\]
\[
T_-(s) := \inf \left\{ t \geq 0 | U^{\#}(y,t) > st + 1, \quad y \in \mathbb{R}^{n-1} \right\}.
\]

Lemma 8.1. \( T^+(s) : [\pi, M_0] \) is a right continuous function, such that \( T^+(s) \geq \frac{1}{s-\pi} \). Similarly, \( T_-(s) : [m_0, s] \) is a left continuous function, such that \( T_-(s) \geq \frac{1}{s-\pi} \).

Proof. Step 1. By the definition of \( \pi(e_n) \) and that \( U^s(\cdot, t) \) is \( \mathbb{Z}^{n-1} \)-periodic, the function \( T^+(s) \) is well-defined. Fix any sequence \( s_t \rightarrow s^+ > \pi \), then \( T^+(y,t) \geq \tilde{U}^s(y,t) \), and therefore \( \lim_{t \rightarrow \infty} \tilde{U}^{s_t}(y,t) \geq \tilde{U}^s(y,t) \). On the other hand, \( \lim_{t \rightarrow \infty} \sup \tilde{U}^{s_t}(y,t) \), as a subsolution bounded from above by \( x_n = st \), should be bounded by \( \tilde{U}^s(y,t) \). Therefore, \( \lim_{t \rightarrow \infty} \tilde{U}^{s_t}(y,t) = \tilde{U}^s(y,t) \) uniformly. Then \( \lim_{t \rightarrow \infty} T(s_t) = T(s) \).

Step 2. Let us now choose \( s_j \rightarrow \pi^- \), then \( \lim_{j \rightarrow \infty} T^+(y,t) \geq \lim \sup \tilde{U}^{s_j}(y,t) \), therefore, \( \max_{y \in \mathbb{R}^{n-1}} \{ \tilde{U}^s(y,t) - \pi t \} = 0 \), for any \( t \geq 0 \). Then for the sequence \( s_t \), we have that
\[
\max_{y \in \mathbb{R}^{n-1}} \{ \tilde{U}^{s_t}(y,t) - st \} \leq \max_{y \in \mathbb{R}^{n-1}} \{ \tilde{U}^s(y,t) - st \} \leq (s_t - \pi)t.
\]
Hence, \( T^+(s) \geq \frac{1}{s-\pi} \).

Step 3. Similarly, we can prove the results associated to \( T_-(s) \).

\( \square \)

Proposition 8.2 (Proposition 4.4 of [6]). For all \( t > 0 \), assume the function \( w(\cdot, t) \) satisfies in the viscosity sense that
\[
\lambda \leq -\div \left( \frac{Dw(x,t)}{\sqrt{[Dw(x,t)]^2 + 1}} \right) \leq \Lambda, \quad t \in I \subseteq \mathbb{R},
\]
for two fixed numbers \( \lambda, \Lambda \). Then \( w(\cdot, t) \) are of class \( C^{1, \alpha} \), for all \( \alpha \in (0, 1) \), uniformly in \( t \in I \).
Corollary 8.3. Fix any $\tau > 0$ and $s \in [m_0, M_0]$, then $\overline{U}^s(\cdot, t)$ and $\underline{U}_s(\cdot, t)$ are of $C^{1,\alpha}$ for all $\alpha \in (0, 1)$, uniformly in $[\tau, \infty)$.

Proof. The standard comparison principle yields

$$\overline{U}^s(y, t) - s \Delta t \leq U^s(y, t - \Delta t) \quad \text{and} \quad U^s(y, t) + m_0 \Delta t \leq U^s(y, t + \Delta t),$$

and thus

$$m_0 \leq \partial_t \overline{U}^s(y, t) \leq s \leq M_0, \quad (y, t) \in \mathbb{R}^{n-1} \times (0, \infty). \quad (8.2)$$

Recall Proposition 5.2, $\overline{U}^s(x, t)$ is a solution of the equation (8.1) away the obstacle $\pi t$, therefore we have on this set that

$$m_0 - M_0 \leq -\operatorname{dive} \left( \frac{D \overline{U}^s(x, t)}{\sqrt{|D \overline{U}^s(x, t)|^2 + 1}} \right) = -\partial_t \overline{U}^s \left( \frac{D \overline{U}^s(x, t)}{\sqrt{|D \overline{U}^s(x, t)|^2 + 1}} \right) + g \leq 2M_0.$$

On the obstacle, since $\overline{U}^s(\cdot, t)$ is touched from above by a hyperplane, we have in viscosity sense that

$$0 \leq -\operatorname{dive} \left( \frac{D \overline{U}^s(x, t)}{\sqrt{|D \overline{U}^s(x, t)|^2 + 1}} \right) \leq 2M_0.$$

Therefore, the Proposition the regularity of front in laminar case applies. Similarly, we also have the regularity for $\underline{U}_s(\cdot, t)$. \hfill \Box

Theorem 8.4. If $\pi(e_n) > 2(e_n)$, then there is an open, nonempty set $E^\infty \subset \mathbb{T}^{n-1}$ and functions $U^\infty(y) : E^\infty \to (-\infty, 0)$, such that the following are true:

(a) The function $U^\infty(y) + \pi t$ is a viscosity subsolution of (8.1);
(b) $U^\infty(y) \to -\infty$ as $y \to \partial E^\infty$;
(c) The function $U^\infty(y) + \pi t$ is a solution of (8.1) away from $x_n = \pi t$;
(d) The set $\partial E^\infty \times (-\infty, \infty)$ is a stationary solution of (1.2) with $\varepsilon = 1$.

Proof. Fix $K > 0$, let $s_\ell := \pi + \frac{1}{\ell^2}$ and $t_\ell := \ell$, then $T(s_\ell) \geq \ell^2 \geq t_\ell$. Next, we define the following function, which is spatially $\mathbb{Z}^{n-1}$-periodic.

$$\hat{U}^\ell(y, t) := \overline{U}^{s_\ell}(y, t + t_\ell) - s_\ell t_\ell, \quad (y, t) \in \mathbb{R}^{n-1} \times [-K, 0]. \quad (8.3)$$

Hence the highest point of $\hat{U}^\ell$ is bounded by $-1$ and $-M_0K$. By a comparison between $\overline{U}^{s_\ell}$ and $\underline{U}_s$, the lowest point of $\hat{U}^\ell$ is bounded from above by $-(s_\ell - 2)t_\ell$. Furthermore, based on Corollary 8.3, one can show that the hypersurface $\hat{U}^\ell(y, t)$ is spatially $C^{1,\alpha}$ hypersurface in $\mathbb{T}^{n-1}$, uniformly for $t > 0$. Let us define the set $E_{t, K}$ whose measure is neither 0 nor 1, due to $\pi(e_n) > 2(e_n)$.

$$E_{t, K} := \left\{ y \in \mathbb{T}^{n-1} \big| \hat{U}^\ell(y, 0) > -2M_0K \right\} \quad \text{and} \quad E_K := \liminf_{t \to \infty} E_{t, K}.$$

The regularity of the hypersurface along with the fact that $\hat{U}^\ell(y, 0)$ have uniformly bounded maximum over $\mathbb{T}^{n-1}$ implies that $E_M$ contains a unit neighborhood of some point in $\mathbb{T}^{n-1}$. Let us now define

$$U^\infty(y, t) := \lim_{\ell \to \infty} \sup_{t \in (-\infty, 0]} \hat{U}^\ell(y, t), \quad (y, t) \in E_M \times [-M, 0].$$

The limit is uniform due to Arzela- Ascoli theorem. Now let us define

$$E^\infty := \cup_{M \to 0} E_M.$$

By the Birkhoff property (c.f. Proposition 2.6 and Remark 2.12), then

$$\overline{U}^{s_\ell}(y, t + k) \leq \overline{U}^{s_\ell}(y, t) + s_\ell k, \quad \text{for any} \quad (y, t) \in \mathbb{R}^{n-1} \times (0, \infty), \quad k > 0.$$

And then

$$\hat{U}^\ell(y, t + k) \leq \hat{U}^\ell(y, t) + s_\ell k, \quad \text{for any} \quad (y, t) \in \mathbb{R}^{n-1} \times (0, \infty), \quad k > 0.$$

Hence

$$U^\infty(y, t + k) \leq U^\infty(y, t) + s_\ell k, \quad \text{for any} \quad (y, t) \in E^\infty \times (-\infty, 0), \quad k > 0.$$
on the other hand, by the choice of \((s_\ell, t_\ell)\), we have that \(\lim_{\ell \to \infty}(s_{\ell+1} - s_\ell)t_{\ell+1} = 0\). By the ordering relation \(\bar{U}^{s_{\ell+1}} \leq \bar{U}^{s_\ell}\) and the Birkhoff property, we have that
\[
\bar{U}^{s_{\ell+1}}(y, t + t_{\ell+1}) \leq \bar{U}^{s_\ell}(y, t + t_\ell + k) \leq \bar{U}^{s_\ell}(y, t + t_\ell + k) + s_\ell(t_{\ell+1} - t_\ell) = s_\ell k.
\]
Or, equivalently,
\[
\bar{U}^{s_{\ell+1}}(y, t) + (s_{\ell+1} - s_\ell)t_{\ell+1} + s_\ell k \leq \bar{U}^t(y, t + k).
\]
Sending \(\ell \to \infty\), we get the other inequality
\[
U^\infty(y, t) + s_\ell k \leq U^\infty(y, t + k), \quad \text{for any } (y, t) \in E^\infty \times (-\infty, 0), \; k > 0.
\]
Thus we can define \(U^\infty(y) := U^\infty(y, 0)\) and have that \(U^\infty(y, t) = U^\infty(y) + \pi t\) is a travelling wave viscosity subsolution over \(E^\infty\). Shift \(U^\infty(y)\) by its maximum value if necessary, we have \(\max_y U^\infty(y, 0) = 0\), and thus (a) is established. The part (b) follows from the definition of \(U^\infty(y, t)\). The part (c) is basically a restatement of Proposition 5.2. Let us now consider the part (d). Fix any \(y^* \in \partial E^\infty\) and let \(E^\infty \ni y \to y^*\), then \(U^\infty(y^*_i) < 0\) as \(i \to \infty\). Plugging \(U^\infty(y^*) + \pi t\) into (8.1), we have that
\[
\frac{s}{\sqrt{[DU^\infty(y^*)]^2 + 1}} - \text{div} \left( \frac{DU^\infty(y^*)}{\sqrt{[DU^\infty(y^*)]^2 + 1}} \right) - g(y) = 0, \quad (y, t) \in E^\infty \times (-\infty, \infty).
\]
Because \(\lim_{i \to \infty} [DU^\infty(y^*_i)] = \infty\), let us send \(i \to \infty\) and get (in viscosity sense) \(g(y^*) = \kappa(y^*)\), which is the curvature of \(\partial E^\infty\) at \(y^*\).

Proposition 8.5. If \(\varpi(e_n) > \varphi(e_n)\), then there is an open, nonempty set \(E^\infty \subset \mathbb{T}^{n-1}\) and functions \(U^\infty(y) : E^\infty \to [0, \infty)\), such that the following are true:

(a) The function \(U^\infty(y) + \pi t\) is a viscosity supersolution of (8.1);
(b) \(U^\infty(y) \to +\infty\) as \(y \to \partial E^\infty\);
(c) The function \(U^\infty(y) + \pi t\) is a solution of (8.1) away from \(x_n = \pi t\);
(d) The set \(\partial E^\infty \times (-\infty, \infty)\) is a stationary solution of (1.2) with \(\varepsilon = 1\).

Proof. It is parallel to that of Proposition 8.5, we omit it here.

8.2. More discussion of travelling wave sub/super solution. In this part, we investigate the properties of the laminar forcing term \(g(y)\) that could induce the failing of the homogenization. i.e., the existence of travelling wave subsolution with head speed \(\varpi\) and travelling wave supersolution with tail speed \(\varphi\), such that \(0 < \varphi < \varpi < \infty\). The idea is partially motivated by the example by [5].

Let us consider \(0 < r_1 < r_2 < \frac{1}{2}\) so that \(B(y_1, r_1)\) and \(\mathbb{T}^{n-1} \setminus B(y_2, r_2)\) are two disjoint sets in \(\mathbb{T}^{n-1}\) for some \(y_1, y_2\). Define a decreasing function \(\zeta : (0, r_1) \to \mathbb{R}\) such that \(\zeta(r_1) = -\infty\), and an increasing function \(\eta : (r_2, \infty) \to \mathbb{R}\) such that \(\eta(r_2) = -\infty\) and \(\eta(r) \equiv 0\) if \(r \geq R\) for some \(r_2 < R < \frac{1}{2}\). Let us next choose two sets \(E_1 := B(y_1, r_1)\) and \(E_2 := \mathbb{T}^{n-1} \setminus B(y_2, r_2)\), and construct travelling wave sub and supersolutions \(U_1 : E_1 \to (-\infty, 0)\) and \(U_2 : E_2 \to (0, \infty)\) by
\[
U_1(y) := \int_0^r \zeta(r)dr, \quad r := |y - y_1| \quad \text{and} \quad U_2(y) := \int_r^\infty \eta(r)d\tau, \quad r := |y - y_2|.
\]
Then \(U_1(y) + \pi t\) is a subsolution of (8.1) if
\[
g(y) \geq \frac{\varpi}{\sqrt{\zeta^2(r) + 1}} - \left( \frac{\zeta(r)}{\sqrt{\zeta^2(r) + 1}} + \frac{n - 2}{r} \cdot \frac{\zeta(r)}{\sqrt{\zeta^2(r) + 1}} \right) \quad \text{with} \quad r = |y - y_1| \in [0, r_1]. \tag{8.4}
\]
Similarly, \(U_2(y) + \pi t\) is a supersolution of (8.1) if
\[
g(y) \leq \frac{\varphi}{\sqrt{\eta^2(r) + 1}} - \left( \frac{\eta(r)}{\sqrt{\eta^2(r) + 1}} + \frac{n - 2}{r} \cdot \frac{\eta(r)}{\sqrt{\eta^2(r) + 1}} \right) \quad \text{with} \quad r = |y - y_2| \in (r_2, R]. \tag{8.5}
\]
Let us choose \(\zeta(r) := \frac{r}{r_1 - r}\) and \(\eta(r) := \min[\frac{r - R}{r_2 - r}, 0]\). Then (8.4) is written as
\[
g(y) \geq \frac{\varpi(r_1 - r)}{\sqrt{r^2 + (r_1 - r)^2}^2} + \frac{r_1(r_1 - r)}{[r^2 + (r_1 - r)^2]^2} + \frac{n - 2}{\sqrt{r^2 + (r_1 - r)^2}^2}, \quad y \in E_1,
\]
which is satisfied if we define \( \overline{\sigma} \) as

\[
\overline{\sigma} := \min_{y \in E_1} g(y) - \frac{\sqrt{2n}}{r_1}.
\]  \hfill (8.6)

Next, (8.5) is written as

\[
g(y) \leq J := \frac{g(r - r_2)}{\sqrt{(R - r)^2 + (r - r_2)^2}} - \frac{(R-r_2)(r-r_2)}{\sqrt{(R - r)^2 + (r - r_2)^2}} - \frac{(n-2)(R-r)}{r\sqrt{(R - r)^2 + (r - r_2)^2}}.
\]

We will show that

\[
\underline{s} := \frac{2}{R - r_2} + \sigma \text{ with } \sigma > 0
\]

satisfies (8.5) if \( \max_{E_2} g(y) < \min \{\sigma, n - 2\} \). This is because

\[
J \geq \frac{r - r_2}{R - r_2} \sigma + \frac{R - r}{R - r_2} (n - 2) > \min \{\sigma, n - 2\} \text{ for } r_2 < r < R.
\]

We have shown the corollary

**Corollary 8.6.** Homogenization fails if, for \( 0 < r_1 < r_2 < R < \frac{1}{2} \), \( E_1 \) and \( E_2 \) are disjoint, and there exists \( \sigma > 0 \) such that

\[
0 < \sigma < \min_{y \in E_1} g(y) - \left( \frac{\sqrt{2n}}{r_1} + \frac{2}{R - r_2} \right) \quad \text{and} \quad \max_{y \in E_2} g(y) < \min \{\sigma, n - 2\}.
\]  \hfill (8.8)

**Proof.** It remains to observe that if first condition holds, then \( \bar{s} \) and \( \underline{s} \) given in (8.6) and (8.7) satisfy \( 0 < \underline{s} < \bar{s} \). \hfill \Box

9. Appendix

9.1. Some calculations. In this subsection, we carry out calculations regarding two functions such that

\[
\tilde{d}(x) = d\left(x + r\varphi(x)e_1\right) \quad \text{for} \quad x \in U,
\]

where \( r \) is a constant and \( \varphi(x) \) is a positive smooth functions defined in some region \( U \subseteq \mathbb{R}^n \). Let us choose \( \{e_1, e_2, \ldots, e_n\} \) as the orthonormal coordinate system for \( \mathbb{R}^n \), fix \( x_0 \in U \) and denote \( y_0 = x_0 + r\varphi(x_0)e_1 \). Moreover, we assume that \( D\varphi(x_0) = \alpha e_1 + \beta e_2 \), where \( \alpha, \beta \) are two fixed real numbers. Furthermore, let us also assume the following:

(i) \( |D\tilde{d}(y)| = 1 \) in a neighborhood of \( y_0 \);

(ii) \( \frac{\partial d}{\partial y_1}(y_0) = -1 \) and \( \frac{\partial d}{\partial y_k}(y_0) = 0, \ k \neq 1 \);

(iii) \( \frac{\partial^2 d}{\partial y_k^2}(y_0) = 0, k = 1, 2, \ldots, n \).

Our goal is to compute the term \( \Delta_{\nabla}\tilde{d}(x_0), \) i.e., the second derivative of \( \tilde{d} \) in the gradient direction of \( \tilde{d} \) at the point \( x_0 \). First, we have that

\[
D\tilde{d}(x_0) = (-1 - r\alpha) e_1 + (-r\beta) e_2.
\]

Then the normal derivative operator of \( \tilde{d} \) at \( x_0 \) writes

\[
\frac{\partial}{\partial n} = \frac{-1 - r\alpha}{\sqrt{(1 + r\alpha)^2 + (r\beta)^2}} \frac{\partial}{\partial x_1} + \frac{-r\beta}{\sqrt{(1 + r\alpha)^2 + (r\beta)^2}} \frac{\partial}{\partial x_2}.
\]

The 1st order derivative in the normal direction is

\[
\frac{\partial \tilde{d}}{\partial n} = \frac{-1 - r\alpha}{\sqrt{(1 + r\alpha)^2 + (r\beta)^2}} \left( \frac{\partial d}{\partial y_1} + r \frac{\partial d}{\partial y_1} \frac{\partial \varphi}{\partial x_1} \right) + \frac{-r\beta}{\sqrt{(1 + r\alpha)^2 + (r\beta)^2}} \left( \frac{\partial d}{\partial y_2} + r \frac{\partial d}{\partial y_1} \frac{\partial \varphi}{\partial x_2} \right).
\]

Then the 2nd order directional derivative becomes

\[
\frac{\partial}{\partial n} \left( \frac{\partial \tilde{d}}{\partial n} \right) = \frac{-1 - r\alpha}{\sqrt{(1 + r\alpha)^2 + (r\beta)^2}} \left[ \frac{\partial}{\partial n} \left( \frac{\partial d}{\partial y_1} + r \frac{\partial d}{\partial y_1} \frac{\partial \varphi}{\partial x_1} \right) \right] + \frac{-r\beta}{\sqrt{(1 + r\alpha)^2 + (r\beta)^2}} \left[ \frac{\partial}{\partial n} \left( \frac{\partial d}{\partial y_2} + r \frac{\partial d}{\partial y_1} \frac{\partial \varphi}{\partial x_2} \right) \right].
\]
9.1.1. The term $A$. The term $A$ is the following,

$$
A = \frac{-1 - r\alpha}{\sqrt{(1 + r\alpha)^2 + (r\beta)^2}} \left[ \frac{\partial}{\partial x_1} \left( \frac{\partial d}{\partial y_1} + r \frac{\partial d}{\partial y_1} \frac{\partial \phi}{\partial x_1} \right) \right] + \frac{-r\beta}{\sqrt{(1 + r\alpha)^2 + (r\beta)^2}} \left[ \frac{\partial}{\partial x_2} \left( \frac{\partial d}{\partial y_1} + r \frac{\partial d}{\partial y_1} \frac{\partial \phi}{\partial x_2} \right) \right],
$$

where

$$
A_1 = \frac{\partial}{\partial x_1} \left( \frac{\partial d}{\partial y_1} \right) + r \frac{\partial}{\partial x_1} \left( \frac{\partial d}{\partial y_1} \frac{\partial \phi}{\partial x_1} \right) = \frac{\partial^2 d}{\partial y_1^2} \left( 1 + r \frac{\partial \phi}{\partial x_1} \right) + r \frac{\partial^2 d}{\partial y_1^2} \frac{\partial \phi}{\partial x_1} + r \frac{\partial d}{\partial y_1} \frac{\partial^2 \phi}{\partial x_1^2} = \frac{\partial^2 \phi}{\partial x_1^2} (x_0)
$$

and

$$
A_2 = \frac{\partial}{\partial x_2} \left( \frac{\partial d}{\partial y_1} \right) + r \frac{\partial}{\partial x_2} \left( \frac{\partial d}{\partial y_1} \frac{\partial \phi}{\partial x_2} \right) = \frac{\partial^2 d}{\partial y_2 \partial y_1} \frac{\partial \phi}{\partial x_2} + r \left( \frac{\partial^2 d}{\partial y_2 \partial y_1} \frac{\partial \phi}{\partial x_2} + r \frac{\partial d}{\partial y_1} \frac{\partial^2 \phi}{\partial x_2 \partial x_2} \right) \frac{\partial \phi}{\partial x_1} + r \frac{\partial d}{\partial y_1} \frac{\partial^2 \phi}{\partial x_1 \

So

$$
A = \frac{(1 + r\alpha) r}{\sqrt{(1 + r\alpha)^2 + (r\beta)^2}} \cdot \frac{\partial^2 \phi}{\partial x_1^2} (x_0) + \frac{r\beta}{\sqrt{(1 + r\alpha)^2 + (r\beta)^2}} \cdot \frac{\partial^2 \phi}{\partial x_2 \partial x_1} (x_0).
$$

9.1.2. The term $B$. The term $B$ is the following:

$$
B = \frac{-1 - r\alpha}{\sqrt{(1 + r\alpha)^2 + (r\beta)^2}} \left[ \frac{\partial}{\partial x_1} \left( \frac{\partial d}{\partial y_2} + r \frac{\partial d}{\partial y_2} \frac{\partial \phi}{\partial x_1} \right) \right] + \frac{-r\beta}{\sqrt{(1 + r\alpha)^2 + (r\beta)^2}} \left[ \frac{\partial}{\partial x_2} \left( \frac{\partial d}{\partial y_2} + r \frac{\partial d}{\partial y_2} \frac{\partial \phi}{\partial x_2} \right) \right],
$$

where

$$
B_1 = \frac{\partial}{\partial x_1} \left( \frac{\partial d}{\partial y_2} \right) + r \frac{\partial}{\partial x_1} \left( \frac{\partial d}{\partial y_2} \frac{\partial \phi}{\partial x_1} \right) = \frac{\partial^2 d}{\partial y_1 \partial y_2} \left( 1 + r \frac{\partial \phi}{\partial x_1} \right) + r \frac{\partial^2 d}{\partial y_1 \partial y_2} \frac{\partial \phi}{\partial x_1} + r \frac{\partial d}{\partial y_1} \frac{\partial^2 \phi}{\partial x_1 \partial x_2} = \frac{\partial^2 \phi}{\partial x_1 \partial x_2} (x_0)
$$

and

$$
B_2 = \frac{\partial}{\partial x_2} \left( \frac{\partial d}{\partial y_2} \right) + r \frac{\partial}{\partial x_2} \left( \frac{\partial d}{\partial y_1} \frac{\partial \phi}{\partial x_2} \right) = \left( \frac{\partial^2 d}{\partial y_2^2} + r \frac{\partial^2 d}{\partial y_1 \partial y_2} \frac{\partial \phi}{\partial x_2} + r \frac{\partial^2 d}{\partial y_1 \partial y_2} \frac{\partial \phi}{\partial x_2} + r \frac{\partial d}{\partial y_1} \frac{\partial^2 \phi}{\partial x_2^2} \right) + r \frac{\partial d}{\partial y_1} \frac{\partial^2 \phi}{\partial x_2^2} = \frac{\partial^2 \phi}{\partial x_2^2} (y_0) - r \frac{\partial^2 \phi}{\partial x_2^2} (x_0).
$$

So

$$
B = \frac{(1 + r\alpha) r}{\sqrt{(1 + r\alpha)^2 + (r\beta)^2}} \cdot \frac{\partial^2 \phi}{\partial x_1 \partial x_2} (x_0) + \frac{-r\beta}{\sqrt{(1 + r\alpha)^2 + (r\beta)^2}} \left( \frac{\partial^2 d}{\partial y_2^2} (y_0) - r \frac{\partial^2 \phi}{\partial x_2^2} (x_0) \right).
$$
Finally, we have the conclusion as follows.

\[
\frac{\partial}{\partial n} \left( \frac{\partial d}{\partial m} \right) (x_0) = \frac{(r\beta)^2}{(1 + r\alpha)^2 + (r\beta)^2} \cdot \frac{\partial^2 d}{\partial y_2^2}(y_0) - \frac{(1 + r\alpha)^2 r}{(1 + r\alpha)^2 + (r\beta)^2} \cdot \frac{\partial^2 \varphi}{\partial x_1^2}
\]

\[
- \frac{2(1 + r\alpha)(r\beta)^r}{(1 + r\alpha)^2 + (r\beta)^2} \cdot \frac{\partial^2 \varphi}{\partial x_1 \partial x_2} - \frac{(r\beta)^2 r}{(1 + r\alpha)^2 + (r\beta)^2} \cdot \frac{\partial^2 \varphi}{\partial x_2^2}.
\]

(9.1)

9.2. A comparison principle. In this subsection, we prove a variant version of the comparison principle regarding a pseudo viscosity subsolution and a pseudo viscosity supersolution. The idea here is partially motivated by [5] and [17].

Proposition 9.1. Fix \( \nu \in \mathbb{S}^{n-1}, \ x_0 \in \mathbb{R}^n, \ R > 0, \ 0 \leq \alpha < \beta < \infty. \) Let \( U(x,t), V(x,t) : \Omega(x_0, R; \nu) \times [\alpha, \beta] \to \mathbb{R} \) satisfy the following (i), (ii) and (iii):

(i) \( U(x,t) \) is a pseudo viscosity subsolution (c.f. Definition 2.3);
(ii) \( V(x,t) \) is a pseudo viscosity supersolution (c.f. Definition 2.5);
(iii) \( U(x,t) < V(x,t), \ if \ (x,t) \in (\Omega(x_0, R; \nu) \times \{\alpha\}) \cup (\partial \Omega(x_0, R; \nu) \times [\alpha, \beta]), \)

then

\[ U(x,t) < V(x,t), \ (x,t) \in \Omega(x_0, R; \nu) \times (\alpha, \beta). \]

Proof. Let us assume on the contrary that

\[
\sup_{(x,t) \in \Omega(x_0, R; \nu) \times (\alpha, \beta)} (U(x,t) - V(x,t)) \geq 0,
\]

(9.2)

without loss of generality, we can assume the existence of the \( \tau_0 \in (\alpha, \beta), \) such that \( U(x_0, \tau_0) = \mu \geq V(x_0, \tau_0). \) Then we consider two characteristic functions.

\[
Z(x,t) := \begin{cases} 
1, & x \in \Omega(x_0, R; \nu), U(x,t) \geq \mu, \\
0, & x \in \Omega(x_0, R; \nu), U(x,t) < \mu,
\end{cases}
\]

\[
W(x,t) := \begin{cases} 
1, & x \in \Omega(x_0, R; \nu), V(x,t) > \mu, \\
0, & x \in \Omega(x_0, R; \nu), V(x,t) \leq \mu.
\end{cases}
\]

Then \( Z(x_0, \tau_0) = 1 > 0 = W(x_0, \tau_0). \) Let us choose

\[
t_0 := \min \{ \alpha \leq t \leq \beta | Z(x,t) = 1 \geq 0 = W(x,t) \ for \ some \ x \in \Omega(0, R; \nu) \}.
\]

Then by the assumptions, we have \( t_0 \in (\alpha, \beta) \) and \( Z(x,t_0) = 1 > 0 = W(x,t_0) \) for some \( x \notin \partial \Omega(0, R; \nu). \)

It is well-known that \( Z(x,t) \) (resp. \( W(x,t) \)) is a pseudo viscosity subsolution (resp. pseudo viscosity supersolution) of the forced mean curvature flow. Let us also set

\[
z(x,t) := e^{-t}Z(x,t), \quad w(x,t) := e^{-t}W(x,t), \quad \tilde{\mathcal{F}}(X,p,x,t) := e^{-t}\mathcal{F}(e^tX,e^tp,x).
\]

Then we have inequalities in the pseudo viscosity sense:

\[
z_t + z \leq \tilde{\mathcal{F}} \left( D^2z, Dz, x, t \right), \quad (x,t) \in \Omega(x_0, R; \nu) \times (\alpha, \beta),
\]

\[
w_t + w \geq \tilde{\mathcal{F}} \left( D^2w, Dw, x, t \right), \quad (x,t) \in \Omega(x_0, R; \nu) \times (\alpha, \beta).
\]

The assumption (9.2) indicates that

\[
\sup_{(x,t) \in \Omega(x_0, R; \nu) \times [\alpha, t_0]} (z(x,t) - w(x,t)) \geq e^{-t_0} > 0.
\]

Next, for any \( \varepsilon, \eta > 0, \) let us consider the function.

\[
\Phi_{\varepsilon, \eta}(x,y,t) := z(x,t) - w(y,t) - \frac{|x - y|^4}{4\varepsilon} - \frac{\eta}{t_0 - t},
\]

where \( (x,y,t) \in \Omega(x_0, R; \nu) \times \Omega(x_0, R; \nu) \times (\alpha, t_0). \) Since \( \Phi_{\varepsilon, \eta}(x,y,t) \) is bounded by 1 from above and upper semicontinuous, there exists \( (x_{\varepsilon, \eta}, y_{\varepsilon, \eta}, t_{\varepsilon, \eta}) \), at which \( \Phi_{\varepsilon, \eta}(\cdot, \cdot, \cdot) \) achieves its finite maximum. Clearly,

\[
0 < t_{\varepsilon, \eta} < t_0 - O(\eta) \quad and \quad |x_{\varepsilon, \eta} - y_{\varepsilon, \eta}| = O \left( \varepsilon^{\frac{1}{4}} \right).
\]
Even though $\Omega(x_0, R; \nu)$ is unbounded, we can actually find above $x_{\varepsilon, \eta}$ and $y_{\varepsilon, \eta}$ in the bounded domain 
\{ $x \in \Omega(x_0, R; \nu)$ $|$ $0 \leq (x-x_0) \cdot \nu \leq M_0 T$ \}. Up to extracting a subsequence, we have as $(\varepsilon, \eta) \to (0, 0)$
\[
\Phi_{\varepsilon, \eta}(x_{\varepsilon, \eta}, y_{\varepsilon, \eta}, t_{\varepsilon, \eta}) \to e^{-t_0} > 0.
\]
Then the fact
\[
\Phi_{2\varepsilon, \frac{\eta}{2}, \frac{\eta}{2}, x_{\varepsilon, \eta}, y_{\varepsilon, \eta}, t_{\varepsilon, \eta}} \geq \Phi_{\varepsilon, \eta}(x_{\varepsilon, \eta}, y_{\varepsilon, \eta}, t_{\varepsilon, \eta})
\]
implies that (up to a subsequence)
\[
\lim_{(\varepsilon, \eta) \to (0, 0)} \frac{|x_{\varepsilon, \eta} - y_{\varepsilon, \eta}|^4}{\varepsilon} = 0, \quad \lim_{(\varepsilon, \eta) \to (0, 0)} \frac{\eta}{t_0 - t_{\varepsilon, \eta}} = 0.
\]
Then we have that
\[
x_{\varepsilon, \eta} \neq y_{\varepsilon, \eta}, \quad z(x_{\varepsilon, \eta}, t_{\varepsilon, \eta}) = e^{-t_{\varepsilon, \eta}} \quad \text{and} \quad w(y_{\varepsilon, \eta}, t_{\varepsilon, \eta}) = 0.
\]
If for a subsequence of $(\varepsilon, \eta) \to (0, 0)$, $(x_{\varepsilon, \eta}, y_{\varepsilon, \eta}) \in (\partial \Omega(x_0, R; \nu) \times \Omega(x_0, R; \nu)) \cup (\Omega(x_0, R; \nu) \times \partial \Omega(x_0, R; \nu))$, then we have a contradiction as follows:
\[
e^{-t_0} \leq \sup_{(x,t) \in \Omega(x_0,R;\nu) \times [\alpha, t_0]} (z(x,t) - w(x,t)) \leq \lim_{\varepsilon \to 0} \sup_{(x,t) \in \Omega(x_0,R;\nu) \times [\alpha, t_0]} \left( z(x,t) - w(y,t) - \frac{|x-y|^4}{4\varepsilon} \right) \leq \lim_{\varepsilon \to 0} \sup_{y \to 0} \Phi_{\varepsilon, \eta}(x_{\varepsilon, \eta}, y_{\varepsilon, \eta}, t_{\varepsilon, \eta}) \leq 0.
\]
If both of $x_{\varepsilon, \eta}$ and $y_{\varepsilon, \eta}$ are interior points of $\Omega(x_0, R; \nu)$, we can derive two viscosity inequalities associated to $z(x,t)$ and $w(x,t)$, respectively. Let us denote
\[
\Psi(x, y, t) := \frac{|x-y|^4}{4\varepsilon} + \frac{\eta}{t_0 - t}.
\]
By the Theorem 8.3 in [10], we deduce that for any $\lambda > 0$, there exists
\[
b_1, D_x \Psi(x_{\varepsilon, \eta}, y_{\varepsilon, \eta}, t_{\varepsilon, \eta}), X \in \mathcal{F}_\psi^2 \alpha z(x_{\varepsilon, \eta}, t_{\varepsilon, \eta}),
\]
and
\[
b_2, -D_y \Psi(x_{\varepsilon, \eta}, y_{\varepsilon, \eta}, t_{\varepsilon, \eta}), Y \in \mathcal{F}_\psi^2 \omega w(y_{\varepsilon, \eta}, t_{\varepsilon, \eta}),
\]
such that
\[
b_1 - b_2 = \Psi(t(x_{\varepsilon, \eta}, y_{\varepsilon, \eta}, t_{\varepsilon, \eta}),
\]
and
\[
- \left( \frac{1}{\lambda} + ||A|| \right) \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq A + \lambda A^2.
\]
where $A = D^2 \Psi(x_{\varepsilon, \eta}, y_{\varepsilon, \eta}, t_{\varepsilon, \eta})$ and $||A|| := \sup_{||\xi||=1} (\alpha \xi, \xi)$. Because $x_{\varepsilon, \eta} \neq y_{\varepsilon, \eta}$, we have that $|D_x \Psi(x_{\varepsilon, \eta}, y_{\varepsilon, \eta})| > 0$ and $|D_y \Psi(x_{\varepsilon, \eta}, y_{\varepsilon, \eta})| > 0$. Since $z(x,t)$ is a pseudo viscosity subsolution and $w(x,t)$ is a pseudo viscosity supersolution, then the following viscosity inequalities appear.
\[
b_1 + z(x_{\varepsilon, \eta}, t_{\varepsilon, \eta}) \leq \mathcal{F}(X, D_x \Psi(x_{\varepsilon, \eta}, y_{\varepsilon, \eta}, t_{\varepsilon, \eta}), x_{\varepsilon, \eta}, t_{\varepsilon, \eta}),
\]
and
\[
b_2 + w(x_{\varepsilon, \eta}, t_{\varepsilon, \eta}) \geq \mathcal{F}(Y, -D_y \Psi(x_{\varepsilon, \eta}, y_{\varepsilon, \eta}, t_{\varepsilon, \eta}), y_{\varepsilon, \eta}, t_{\varepsilon, \eta}).
\]
similar to (9.4) and (9.5). Then when \( \varepsilon, \eta \) are sufficiently small, taking the difference of the above two inequalities gives

\[
0 < \frac{e^{-t_0}}{2} \leq \frac{\eta}{(t_0 - t_{\varepsilon, \eta})^2} + z(x_{\varepsilon, \eta}, t_{\varepsilon, \eta}) - w(y_{\varepsilon, \eta}, t_{\varepsilon, \eta}) \leq \tilde{F}(X, p, x_{\varepsilon, \eta}, t_{\varepsilon, \eta}) - \tilde{F}(Y, p, y_{\varepsilon, \eta}, t_{\varepsilon, \eta})
\]

where

\[
p = \delta(x_{\varepsilon, \eta} - y_{\varepsilon, \eta}), \quad \delta = \frac{|x_{\varepsilon, \eta} - y_{\varepsilon, \eta}|^2}{\varepsilon}
\]

Since we always have \( x_{\varepsilon, \eta} \neq y_{\varepsilon, \eta} \), by setting \( \tilde{p} := \frac{p}{|p|} \), we have that (Since \( E^2 = 2E \), we have that \( A^2 \leq 18\delta^2E \))

\[
0 \leq A = \delta \left( I + 2\tilde{p} \otimes \tilde{p} \right) - I \leq 3\delta E \text{ with } E = \left( \begin{array}{cc} I & -I \\ -I & I \end{array} \right).
\]

Because \( \|A\| = 6\delta \), by setting \( \lambda = \frac{1}{3\delta} \) in (9.3), we get that

\[
-9\delta \left( \begin{array}{cc} I & 0 \\ 0 & I \end{array} \right) \leq \left( \begin{array}{cc} X & 0 \\ 0 & -Y \end{array} \right) \leq 9\delta \left( \begin{array}{cc} I & -I \\ -I & I \end{array} \right).
\]

For any \( \xi \in \mathbb{R}^n \), let us multiply the above inequalities on the left by \( (\xi, \xi) \) and on the right by \( (\xi, \xi)^T \), then

\[
\xi(X - Y)\xi^T \leq 0, \quad \text{i.e.,} \quad X \leq Y.
\]

Finally, we get the following contradiction, where \( L_0 \) is from (H).

\[
0 < \frac{e^{-t_0}}{2} \leq \tilde{F}(X, p, x_{\varepsilon, \eta}, t_{\varepsilon, \eta}) - \tilde{F}(Y, p, y_{\varepsilon, \eta}, t_{\varepsilon, \eta}) = e^{-t_{\varepsilon, \eta}}\tilde{F}(e^{t_{\varepsilon, \eta}}X, e^{t_{\varepsilon, \eta}}p, x_{\varepsilon, \eta}) - e^{-t_{\varepsilon, \eta}}\tilde{F}(e^{t_{\varepsilon, \eta}}Y, e^{t_{\varepsilon, \eta}}p, y_{\varepsilon, \eta}) = tr \{ (X - Y)(I - \tilde{p} \otimes \tilde{p}) \} + (g(x_{\varepsilon, \eta}) - g(y_{\varepsilon, \eta})) |p| \leq 0 + L_0 |x_{\varepsilon, \eta} - y_{\varepsilon, \eta}| \rightarrow 0, \quad \text{as} \quad (\varepsilon, \eta) \rightarrow (0, 0).
\]

Therefore, we must have \( U(x, t) \leq V(x, t) \) for any \( (x, t) \in \Omega(x_0, R; n) \times (\alpha, \beta) \).

\[\square\]

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Department of Mathematics, University of California, Los Angeles, CA, 90095, USA
E-mail address: hwgao@math.ucla.edu

Department of Mathematics, University of California, Los Angeles, CA, 90095, USA
E-mail address: ikim@math.ucla.edu