Three examples of quantum dynamics on the half-line with smooth bouncing

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Abstract

This article is an introductory presentation of the quantization of the half-plane based on affine coherent states (ACS). The half-plane is viewed as the phase space for the dynamics of a positive physical quantity evolving with time, and its affine symmetry is preserved due to the covariance of this type of quantization. We promote the interest of such a procedure for transforming a classical model into a quantum one, since the singularity at the origin is systematically removed, and the arbitrariness of boundary conditions can be easily overcome. We explain some important mathematical aspects of the method. Three elementary examples of applications are presented, the quantum breathing of a massive sphere, the quantum smooth bouncing of a charged sphere, and a smooth bouncing of “dust” sphere as a simple model of quantum Newtonian cosmology.

Keywords: Integral quantization, Half-plane, Affine symmetry, Coherent states, Quantum smooth bouncing

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1. Introduction

The basic, or so-called canonical, quantization procedure [1] for the motion of a particle on the line consists in transforming pairs of canonical variables \((q, p)\) in the corresponding phase space \(\mathbb{R}^2\) into a non-commuting pair of self-adjoint operators in some Hilbert space, e.g. the space of square integrable complex-valued functions on the line,

\[(q, p) \mapsto (Q, P); \quad [Q, P] = i\hbar I; \quad Q\psi(x) = x\psi(x); \quad P\psi(x) = -i\hbar \frac{\partial}{\partial x}\psi(x). \] (1.1)
The procedure is extended to the quantization of classical observables \( f(q, p) \),

\[
f(q, p) \mapsto f(Q, P) \mapsto (\text{Sym})f(Q, P),
\]

where Sym stands for a certain choice of symmetrisation of the operator-valued function \( f(Q, P) \). Besides the above ordering ambiguity, the procedure immediately raises deep questions about its domain of validity. What about singular \( f \), e.g. the phase or angle function \( \arctan(p/q) \)? What about other phase space geometries which are limited by impassable boundaries? Despite their elementary aspects, these singular geometries leave open many questions both on mathematical and physical levels, irrespective of the variety of quantization methods \([2, 3, 4]\). Indeed, most of the latter, despite their aesthetic mathematical content, are too demanding for models to be quantized.

This article is precisely devoted to one of the most elementary examples of such geometries, namely the half-plane \( \{(q, p) \mid q > 0, \ p \in \mathbb{R}\} \) corresponding to a motion on the positive half-line, the origin \( x = 0 \) being viewed as an inaccessible singularity. It is deemed a pedagogical introduction to affine covariant integral quantization of functions on the half-plane and its applications. This procedure has been introduced recently \([5, 6]\) for providing smooth solutions to singularity problems in early quantum cosmology (see \([7]\) and most recent references therein). It is consistent with the phase space symmetries of the system, and carries the name of the group that represents such symmetries.

Let us explain more about the term “affine covariant integral quantization”. The adjective “affine” refers to the group of symmetries of the half-plane combining translations and dilations. This symmetry is different from the translational symmetry of the plane on which is based the familiar canonical or Weyl-Heisenberg quantization \([5]\). “Covariant” means that the quantization map intertwines classical (geometric operation) and quantum (unitary transformations) symmetries. Integral means that we use all resources of integral calculus, in order to implement the method when we apply it to singular functions, or distributions, for which the integral calculus is an essential ingredient.

Classical physics is rich in one-dimensional models with law determining the time behavior of a positive dynamical physical quantity, like the position \( x(t) \) of a particle moving on the positive half-line \( \mathbb{R}^*_+ = \{ x \in \mathbb{R} \mid x > 0 \} \), its kinetic energy, a length \( l(t) \), the radius \( r(t) \) of a sphere, and many other examples, like in optomechanics the distance between a fixed mirror and a moveable mirror, a small vibrating element that forms one of the end mirrors of a Fabry-Perot cavity \([8]\), or like those involving the dynamics of the Hubble scale factor in Cosmology (see Chapter 1 in \([9]\)). In each of these cases, the origin or the value \( x = 0 \) is considered as a classically
impenetrable barrier. Due to the restriction $x > 0$, this barrier is more than a simple singularity. It is purely geometrical and it should not be confused with a Dirac potential at the origin, $V(x) = k\delta(x)$, or a singular potential like $V(x) = k/x$ (Kepler-Coulomb on the half-line), or others, for which the position $x = 0$ is supposed to be accessible. On the classical level, such a geometric singularity $x = 0$ is, in principle, attainable at the price to deal with infinite quantities, like an infinite acceleration in the case of the reflection of the particle. In each case of such dynamical models, the corresponding phase space, i.e. the set of initial positions and momenta for any motion on the half-line, is the positive half-plane $\mathbb{R}_+ \times \mathbb{R} = \{(q, p) \mid q > 0, p \in \mathbb{R}\}$. This geometry has a nice group structure, and this will be the rationale backing our quantization method based on affine coherent states (ACS) or, equivalently, wavelets [10]. ACS quantization is a particular approach pertaining to covariant affine integral quantization [11] (see also the recent extension to the motion in the punctured plane [12]).

The organisation of the paper is as follows. In Section 2 we present the geometry of the half-plane and its particular symmetry which underlies a group structure, namely the affine group of transformations “$ax + b$” of the real line. This group has two unitary irreducible representations only, and we use one of them to build our affine coherent states, similarly to the continuous wavelet construction in signal analysis [10]. Section 3 is devoted to the ACS quantization with its principal definition and implementation formulas in Subsection 3.1, while the subsequent ACS mean values or semi-classical portraits of operators are presented in Subsection 3.2. In order to illustrate our method with simple models, we give in Section 4 a brief survey of Lagrangian and Hamiltonian mechanics appropriate to the motion of the half-line and its phase space, the necessary formalism for implementing ACS quantization of Hamiltonian dynamics in Section 5. Three simple and illuminating examples are then presented, in Section 6 with a “breathing” massive sphere, in Section 7 with a “bouncing” charged sphere, and in Section 8 with a bouncing “dust” sphere as an elementary model of Newtonian cosmology, the latter one being the most developed in terms of dynamical evolution. Our results are discussed in Section 9, where we also list some future perspectives.

Since this article is intended to be a pedagogical initiation to the ACS quantization, we start with the basics of the procedure, progressively evolving into the applications in form of examples cited above. As expected, the quantum correction in the semi-classical approach eliminates the singularity at the origin, creating the bounce mentioned above. Appendices are devoted to the most elaborate part of the mathematical formalism. Thus, the reader is expected to go through the main text without serious difficulty, and to revisit the appendices if there are any doubts about
2. The affine group and its representation $U$

The half-plane can be viewed as the phase space for the (time) evolution of a positive physical quantity, for instance the position of a particle moving in the half-line. Let the upper half-plane $\Pi^+: = \{(q,p) \mid q > 0, p \in \mathbb{R}\} \simeq \mathbb{R}_+^* \times \mathbb{R}$ be equipped with the uniform measure $dq \, dp$. Together with the multiplication

$$(q,p)(q_0,p_0) = \left(q q_0, \frac{p_0}{q} + p\right), \quad q \in \mathbb{R}_+^*, \ p \in \mathbb{R},$$ (2.1)

the unity $(1,0)$ and the inverse

$$(q,p)^{-1} = \left(\frac{1}{q}, -qp\right),$$ (2.2)

$\Pi^+$ is viewed as the affine group $\text{Aff}^+(\mathbb{R})$ of the real line, i.e., the two-parameter group of transformations of the line defined by

$$\mathbb{R} \ni x \mapsto (q,p) \cdot x = \frac{x}{q} + p.$$ (2.3)

We have chosen the standard (Liouville) phase space measure $dq \, dp$ because it is invariant with respect to the left action of the affine group on itself

$$\text{Aff}^+(\mathbb{R}) \ni (q,p) \mapsto (q_0,p_0)(q,p) = (q',p'), \quad dq' \, dp' = dq \, dp.$$ (2.4)

Note that if we instead consider the right action $(q,p) \mapsto (q,p)(q_0,p_0)$, the corresponding invariant measure would be $dq \, dp/q$.

The affine group $\text{Aff}^+(\mathbb{R})$ has two non-equivalent unitary irreducible representations (UIR) $U_\pm$ [13, 14] (see Appendix A for a concise explanation about this terminology). Both are square integrable, i.e. $\int_{\Pi^+} dq \, dp |\langle \phi | U_\pm(q,p) \phi \rangle|^2 < \infty$ for all $\phi$ in a dense subset of the Hilbert space carrying the representation $U_\pm$, and this is the rationale behind continuous wavelet analysis [15]. Without loss of generality, only the UIR $U_+ \equiv U$ is concerned from now on. This representation is realized in the Hilbert space $L^2(\mathbb{R}_+^*, dx)$ as

$$(U(q,p)\psi)(x) = e^{ipx} \psi \left(\frac{x}{q}\right).$$ (2.5)

The above Hilbert space is actually the Fourier image of functions on the line which can be extended analytically to the upper half-plane (Hardy space [16]).
3. Quantization with affine coherent states (ACS)

3.1. ACS quantization

Let us implement the affine integral covariant quantization, which is described in Appendix C in its generality, by restricting the method to the specific case of rank-one density operator or projector \( \rho = |\psi\rangle \langle \psi| \), where \( \psi \) is a unit-norm state and also square integrable on \( \mathbb{R}_+^* \) equipped with the measure \( dx/x \).

\[
\psi \in L^2(\mathbb{R}_+^*, dx) \cap L^2(\mathbb{R}_+^*, dx/x).
\]  

(3.1)

This \( \psi \) is also called “fiducial vector” or “wavelet”.

Now, we recall and extend a set of results already given in previous works [6]. The action of the UIR \( U \) produces all affine coherent states (ACS), i.e. wavelets, defined as

\[
|q,p\rangle_\psi := U(q,p)|\psi\rangle.
\]  

(3.2)

In the sequel we simplify the notation as \( |q,p\rangle_\psi = |q,p\rangle \), unless we need to specify the fiducial vector. The unit norm states (3.2) are not orthogonal. Their overlap is given by the Fourier transform of functions with support on the half-line

\[
\langle q,p|q',p'\rangle = \int_0^\infty dx \, e^{i(p'-p)x} \overline{\psi_q(x)} \psi_{q'}(x),
\]  

(3.3)

with \( \psi_q(x) := \frac{1}{\sqrt{q}} \psi\left(\frac{x}{q}\right) \) is obtained from \( \psi \) by unitary dilation. The affine coherent states (3.2) satisfy the resolution of identity in the Hilbert space \( L^2(\mathbb{R}_+^*, dx) \),

\[
\int_{\Pi_+} |q,p\rangle \langle q,p| \frac{dp}{2\pi c_{-1}} = I,
\]  

(3.4)

where

\[
c_\gamma(\psi) := \int_0^\infty \frac{dx}{x^{2+\gamma}} |\psi(x)|^2.
\]  

(3.5)

A detailed proof of the crucial identity (3.4) is given in Appendix B. Thus, a necessary condition to have (3.4) true is that \( c_{-1}(\psi) < \infty \), which implies \( \psi(0) = 0 \), a well-known requirement in wavelet analysis, and which explains the initial request on \( \psi \) to be square integrable with respect to \( dx/x \). Actually (3.4) is the illustration of a general result derived from the irreducibility and square-integrability of the UIR \( U \), and the application of Schur’s Lemma [17] (see Appendix A.1).

In the sequel we will often simplify the notation as \( c_{\gamma}(\psi) = c_{\gamma} \), unless we need to specify the fiducial vector.
The ACS quantization reads as the map that transforms a function (or distribution) on the phase space into an operator in $L^2(\mathbb{R}_+^*, dx)$:

$$f(q, p) \mapsto A_f = \int_{\Pi_+} f(q, p) |q, p\rangle \langle q, p| \frac{dq dp}{2\pi c_{-1}}. \quad (3.6)$$

This map is covariant with respect to the unitary affine action $U$:

$$U(q_0, p_0) A_f U^\dagger(q_0, p_0) = A_{U(q_0, p_0)} f, \quad (3.7)$$

with

$$(\mathcal{U}(q_0, p_0) f)(q, p) = f((q_0, p_0)^{-1}(q, p)) = f\left(\frac{q}{q_0}, q_0(p - p_0)\right), \quad (3.8)$$

$\mathcal{U}$ being the left regular representation of the affine group when $f \in L^2(\Pi_+, dq dp)$. The symmetry property (3.7) means, and this is certainly the cornerstone of the method, that no point in the phase space $\Pi_+$ is privileged. Precisely, the choice of the origin $(1, 0) \in \Pi_+$ for the affine geometry of $\Pi_+$ is totally arbitrary, and this is reflected in the unitary map (3.7). From now on, our choice of fiducial vector in (3.2) is restricted to real-valued functions, to simplify. Formulas derived with a complex fiducial vector are slightly more involved, but their physical content is not changed.

One interesting feature of the map (3.6) lies in the quantization of the phase space point $(q_0, p_0)$ described by the Dirac peak

$$\delta(q - q_0) \delta(p - p_0) \equiv \delta(q_0, p_0)(q, p).$$

Its quantum counterpart is the ACS projector

$$\delta(q_0, p_0)(q, p) \mapsto A_{\delta(q_0, p_0)} = \frac{|q_0, p_0\rangle \langle q_0, p_0|}{2\pi c_{-1}}. \quad (3.9)$$

The deep meaning of this expression will be explained in the part devoted to semi-classical portraits ($\sim$ lower symbols).

Next, we obtain from the general formulas proven in Appendix B the affine quantum versions of the following elementary functions.

$$A_p = -i \frac{\partial}{\partial x} = P, \quad A_{q^\beta} = \frac{c_{\beta-1}}{c_{-1}} Q^\beta, \quad Q f(x) = x f(x). \quad (3.10)$$

The multiplication operator $Q$ is (essentially) self-adjoint with spectrum equal to the positive half-line. Its spectral decomposition reads as

$$Q = \int_{0_+}^{+\infty} \lambda dE_Q(\lambda), \quad dE_Q(\lambda) \equiv |\lambda\rangle \langle \lambda| d\lambda \quad \lambda > 0, \quad (3.11)$$
where \( \langle x|\lambda \rangle = \delta_\lambda(x) \). On the other hand, the operator \( P \) is symmetric but has no self-adjoint extension [18], as it is expected from the canonical commutation rule, which holds here up to an irrelevant multiplicative constant,

\[
[Q, P] = i \frac{c_0}{c_{-1}} I. \tag{3.12}
\]

Indeed, we know that the latter holds true with a pair of self-adjoint operators if both have the whole real line as a spectrum, contrarily to the present case where the operator \( Q \) is semi-bounded. If the presence of the constant factor \( c_0/c_{-1} \) is considered as a problem, it is always possible to make it equal to 1 through a specific choice of the fiducial vector \( \psi \), or by rescaling the ACS as \( |q,p \rangle \mapsto |\kappa q,p \rangle \) with \( \kappa = c_0/c_{-1} \), as explained in Appendix B.

The quantization of the product \( qp \) yields:

\[
A_{qp} = \frac{c_0}{c_{-1}} \frac{QP + PQ}{2} = \frac{c_0}{c_{-1}} D, \tag{3.13}
\]

where \( D \) is the dilation generator. As one of the two generators (with \( Q \)) of the UIR \( U \) of the affine group, it is essentially self-adjoint. The quantization of the kinetic energy (up to a factor) of the particle gives

\[
A_{p^2} = P^2 + \frac{K_\psi}{Q^2}, \quad K_\psi := \int_0^\infty (\psi'(x))^2 x \frac{dx}{c_{-1}} > 0. \tag{3.14}
\]

Therefore, ACS quantization prevents a quantum free particle moving on the positive line from reaching the origin. It is well known that the operator \( P^2 = -d^2/dx^2 \) in \( L^2(\mathbb{R}^*_+, dx) \) is not essentially self-adjoint, whereas the above regularized operator, defined on the domain of smooth function of compact support, is essentially self-adjoint for \( K_\psi \geq 3/4 \) [18]. Thus, quantum dynamics of the free motion is unique with a suitable choice of the fiducial vector or of the rescaling the parameter \( q \) of the wavelet. We should insist with Reed and Simon in [18], p. 145, that the existence of a continuous set of self-adjoint extensions for the operator \( P^2 \) alone corresponds to the existence of different physics for this problem. They are distinguished by boundary conditions at the origin, \( \psi'(0) + a \psi(0) = 0 \) for finite real \( a \), and \( \psi(0) = 0 \) for \( a = \infty \), which are imposed on functions \( \psi \) in their respective extension domains. The physical interpretation of these conditions lies in a dependent change of phase for functions behaving like incoming and outgoing plane waves near the origin where they reflect. No such ambiguity exists with our approach as soon as the factor \( K_\psi \) is adjusted to a value \( \geq 3/4 \). Moreover, as we illustrate below with our examples, the reflection at the origin is replaced by a smooth bouncing resulting from the centrifugal potential \( K_\psi q^{-2} \).
3.2. Semi-classical portraits

By semi-classical portraits of quantum states and observables we mean represen-
tations of these objects as functions on the classical phase space [3]. In the present
context, the quantum states and their dynamics have phase space representations
through their ACS or lower symbols. Thus the ACS symbol of $|\phi\rangle$ is defined as

$$\Phi(q, p) = \frac{\langle q, p|\phi \rangle}{\sqrt{2\pi}} ,$$

(3.15)

with the associated probability distribution on phase space, resulting from the reso-
lution of the identity and given by

$$\rho_{\phi}(q, p) = \frac{1}{c-1} |\Phi(q, p)|^2 .$$

(3.16)

Having at our disposal the (energy) eigenstates of some quantum Hamiltonian $\hat{H}$,
for instance the affine quantized $A_H$ of a classical Hamiltonian $H(q, p)$, it is particu-
larly instructive to compute (and draw) the time evolution of the distribution (3.16)
defined as

$$\rho_{\phi}(q, p, t) := \frac{1}{2\pi c-1} |\langle q, p|e^{-i\hat{H}t}|\phi \rangle|^2 .$$

(3.17)

As explained in Appendix C, the quantization map $f \mapsto A_f$ is completed with a
semi-classical portrait encapsulated by the lower symbol $\tilde{f}(q, p)$ of the operator $A_f$.
This new function is defined as the ACS expected value of $A_f$

$$\tilde{f}(q, p) = \langle q, p| A_f |q, p \rangle .$$

(3.18)

The explicit form of (3.18) is given in (B.25). It amounts to calculate the local average
value of the original $f(q, p)$ with respect to the probability distribution (3.16) with
$|\phi\rangle = |q', p'\rangle$.

As a first example, let us calculate with real $\psi$ the lower symbol of the Dirac
delta localised at $(q_0, p_0)$. We find

$$\tilde{\delta}_{(q_0, p_0)} = \frac{1}{2\pi c-1} \left| \int_0^\infty dx \ e^{-i(p-p_0)x} \ \psi \left( \frac{x}{q_0} \right) \ \psi \left( \frac{x}{q_0} \right) \right|^2 . $$

(3.19)

Thus we get a new probability distribution on the phase space, centred at $(q_0, p_0)$,
which regularises the original Dirac probability distribution. In Figure (1) is shown
the shape of this regularized delta at the origin, with the following choice of rapidly
decreasing fiducial function

\[
\psi_\nu(x) = \left(\frac{\nu}{\pi}\right)^{1/4} \frac{1}{\sqrt{x}} \exp \left[-\frac{\nu}{2} \left(\ln x - \frac{3}{4\nu}\right)^2\right].
\] (3.20)

The above real function, which is nothing but the square root of a Gaussian distri-
bution on the real line with variable \( y = \ln x \), centered at \( y = 3/4\nu \) \((x = e^{\frac{3}{4\nu}})\), and
with variance \( 1/\nu \), verifies \( c_{-2}(\psi_\nu) = 1 \), \( c_0(\psi_\nu) = c_{-1}(\psi_\nu) \), and more generally

\[
c_\gamma(\psi_\nu) = \exp \left[\frac{(\gamma + 2)(\gamma - 1)}{4\nu}\right].
\]

As \( \nu \to \infty \), the function (3.20) approaches a Dirac peak. More precisely, it is shown

in Fig. (2) that as \( \nu \) grows, this function smoothly concentrates around \( \delta(x-1) \), which
is the position eigendistribution for \( x = 1 \). Conversely, as \( \nu \) goes to 0, (3.20) tends
to 0, which illustrates the total lack of information about the \( x \) position. Through
these features, one can understand one of the aspects of ACS quantization, which is
to smear the classical system variables.

Other examples, which are particularly relevant to the content of the present
paper and whose calculations are developed in Appendix B, are given below.

Lower symbol of powers of \( q \)

It is given with the same power up to a constant factor

\[
q^\beta \mapsto \tilde{q}^\beta = \frac{c_{\beta-1}c_{-\beta-2}}{c_{-1}} q^\beta.
\] (3.21)
Figure 2: Fiducial function (3.20) for different $\nu$. As $\nu$ grows, it approximates to the Dirac delta.

Lower symbols of momentum, kinetic energy, and product $qp$

Calculated with real $\psi$, they read respectively

$$p \mapsto \hat{p} = p; \quad (3.22)$$

$$p^2 \mapsto \hat{p}^2 = p^2 + \frac{c(\psi)}{q^2}, \quad c(\psi) = \int_0^\infty (\psi'(x))^2 \left(1 + \frac{c_0}{c_{-1}}x\right) dx; \quad (3.23)$$

$$qp \mapsto \hat{q}p = \frac{c_0c_{-3}}{c_{-1}}qp. \quad (3.24)$$

4. A short reminder of Lagrangian and Hamiltonian formalism

The motion on the half-line of a particle of mass $m$ is described by the Lagrangian:

$$\mathcal{L}(q, \dot{q}, t) = \frac{m\ddot{q}^2}{2} - V(q), \quad (4.1)$$

where $q > 0$ and $V$ is the potential. The corresponding Lagrange equation reads

$$0 = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = m\ddot{q} + V'(q). \quad (4.2)$$

One passes to the Hamiltonian formalism through the momentum

$$p := \frac{\partial \mathcal{L}}{\partial \dot{q}} = m\dot{q}, \quad (4.3)$$
and the corresponding Hamiltonian

$$H = p\dot{q} - L = \frac{p^2}{2m} + V(q). \quad (4.4)$$

In the sequel, we will be interested in potentials of the form of superposition of powers of $q$

$$V(q) = \int_{-\infty}^{+\infty} w(\beta) q^\beta \, d\beta, \quad (4.5)$$

where the weight function, usually with bounded support, can be extended to a distribution.

Due to the time independence of the Hamiltonian, energy $= H$ is conserved and phase space trajectories on the half-plane are determined by initial conditions $(q_0, p_0)$ as

$$\frac{p^2}{2m} + V(q) = E := \frac{p_0^2}{2m} + V(q_0). \quad (4.6)$$

The flow along them is described by Hamilton equations

$$\dot{q} = \{q, H\} = \frac{p}{m}, \quad \dot{p} = -\{p, H\} = -V'(q). \quad (4.7)$$

5. ACS quantization of dynamics on half-line

Having in hand the formulas established in the previous section, it is straightforward to establish the quantum version of the Hamiltonian (4.4)-(4.5):

$$A_H = \frac{P^2}{2m} + \frac{K_\psi}{2Q^2} + A_V, \quad (5.1)$$

with

$$A_V = \int_{-\infty}^{+\infty} w(\beta) \frac{c_{\beta-1}}{c_{-1}} Q^\beta \, d\beta. \quad (5.2)$$

In the sequel, we suppose that the weight function $w(\beta)$ and the fiducial vector are chosen such that the quantum Hamiltonian $A_H$ is essentially self-adjoint: quantum dynamics does not depend on a choice of boundary conditions at the origin of the half-line.

Concerning our choice of fiducial vector, we consider two other options, in addition to our previous choice (3.20). The most immediate is to pick one of the elements
of the well-known orthonormal basis of $L^2(\mathbb{R}^*_+, dx)$ built from Laguerre polynomials [19],
\[ e^{(\alpha)}_n(x) := \sqrt{\frac{n!}{(n+\alpha)!}} e^{-\frac{x}{2}} x^{\frac{\alpha}{2}} L_n^{(\alpha)}(x), \quad \int_0^\infty e^{(\alpha)}_n(x) e^{(\alpha)}_{n'}(x)dx = \delta_{nn'}, \quad (5.3) \]
where $\alpha > -1$ is a free parameter, and $(n+\alpha)! = \Gamma(n+\alpha+1)$. Actually, since we wish to work with functions which, with a certain number of their derivatives, vanish at the origin, the parameter $\alpha$ should be imposed to be larger than some $\alpha_0 > 0$. On the other hand, for a general $n$, the expression of the constants $c_\gamma$ appears quite involved [20]:
\[ c_\gamma(e^{(\alpha)}_n) = \frac{\Gamma(\alpha - \gamma - 1)}{\Gamma(\alpha + 1)} \frac{1}{n!} \frac{d^n}{dh^n} \left. \frac{2F_1\left(\frac{a-\gamma-1}{2}, \frac{a-\gamma}{2}; \alpha + 1; \frac{4h}{(1+h)^2}\right)}{(1+h)^{\alpha-\gamma-1}(1-h)^{\gamma+2}} \right|_{h=0}, \quad (5.4) \]
which is valid for $\alpha > \gamma + 1$.

The second option is the normalised function in $L^2(\mathbb{R}^*_+, dx)$ [6]:
\[ \psi(x) = \psi^{\mu,\xi}(x) = \frac{1}{\sqrt{2\pi K_0(\nu)}} e^{-\xi x/\nu} \left( \frac{\xi x}{\nu} \right)^{\frac{\nu-1}{2}}, \quad (5.5) \]
with $\nu > 0$ and $\xi > 0$. Here and in the following, $K_r(z)$ denotes the modified Bessel functions [19]. Actually, we only deal with ratios of such functions throughout.

Whence we adopt the convenient notation
\[ \xi_{rs} = \xi_{rs}(\nu) = \frac{K_r(\nu)}{K_s(\nu)} = \frac{1}{\xi_{sr}}. \quad (5.6) \]

One attractive feature of such a notation is that $\xi_{rs}(\nu) \sim 1$ as $\nu \to \infty$ (we recall that the asymptotic behavior at large argument $\nu$ is $K_r(\nu) \sim e^{-\nu} \sqrt{\pi/2\nu}$, whereas at small $\nu \ll \sqrt{r+1}$, $K_r(\nu) \sim (1/2) \Gamma(r)(2/\nu)^r$ for $r > 0$ and $K_0(\nu) \sim -\ln(\nu/2) - \gamma$).

We notice that $\psi^{\mu,\xi}(x)$ falls off with all its derivatives at the origin and at the infinity. Normalization constant and other integrals involving the function $\psi^{\mu,\xi}$ are easily obtained thanks to the formula [20]
\[ \int_0^\infty x^{a-1} e^{-cx-b/x}dx = 2 \left( \frac{b}{c} \right)^{a/2} K_a(2\sqrt{bc}), \quad (5.7) \]
\[ \forall a, b, c \in \mathbb{C}, \Re(b) > 0, \Re(c) > 0. \]

With such a fiducial vector the integrals $c_\gamma$ read as
\[ c_\gamma(\psi^{\mu,\xi}) = \xi_{\gamma+2}^{\frac{\gamma+2}{2}} \frac{K_{-\gamma-2}(\nu)}{K_0(\nu)} = \xi_{\gamma+2}^{\frac{\gamma+2}{2}} \xi_{-\gamma-2,0}. \quad (5.8) \]
With these fiducial functions, we have two free parameters $\xi$ and $\nu$ (besides the scaling parameter $\kappa$). Hence some freedom is left to us to give ratios $c_\gamma/c_{\gamma'}$ the value we wish, an opportunity we use in the first example (next Section).

6. First example: quantum breathing of a massive sphere

Let us consider an isotropic medium with constant mass density $\rho_0$. This implies that the ball of center $O$ and radius $q$ has a total mass equal to

$$M(q) = \frac{4\pi}{3} \rho_0 q^3. \quad (6.1)$$

Gauss theorem allows to determine easily the gravitational vector field acting on a test mass at the surface of the ball. Hence, the Newton equation for a test particle, mass $m$, at the surface of the sphere of radius $q$ reads as

$$m\ddot{q} = -\frac{G m M(q)}{q^2} = -m \frac{4\pi G}{3} \rho_0 q \equiv -k q, \quad (6.2)$$

where $G$ is the universal gravitational constant. The Hamiltonian is the same as the one for the half-harmonic oscillator [21], that is, whose the motion is restricted to the half-line. A physical interpretation of this could be a spring that can be stretched from its equilibrium position but not compressed. With the choice $m = 1 \text{ kg}$,

$$H = \frac{p^2}{2} + \frac{k}{2} q^2, \quad p = \dot{q} \quad q > 0. \quad (6.3)$$

An example of phase space trajectory, a truncated circle, is given in Figure 3a.

According to (5.1), the ACS quantization of this classical dynamics yields the quantum Hamiltonian

$$A_H = \frac{p^2}{2} + \frac{\hbar^2}{2} K_\psi + \frac{k}{2} c_{\gamma} c_{\gamma-1} Q^2, \quad (6.4)$$

in which the presence of the Planck constant is due to the fact that we have to take into account the physical dimensions of the phase space variables $(q,p)$ and consistently replace in (3.6) the measure $dq \, dp$ by $dq \, dp/\hbar$. Passing to the lower symbol of the equation (6.4) through formulas given in (3.21) and (3.23) at constant energy $A_H = E$ yields the semi-classical correction to (6.3)

$$E = \frac{p^2}{2} + \frac{\hbar^2}{2} c(\psi) + \frac{k}{2} c_{\gamma} c_{\gamma-4} q^2 \equiv \frac{\tilde{p}^2}{2} + \frac{\tilde{K}}{2} q^2 + \frac{\tilde{k}}{2} q^2, \quad (6.5)$$
Figure 3: Figure (3a) is an example of phase space trajectory in the positive half-plane defined by the equation $E = \frac{p^2}{2} + \frac{kq^2}{2}$ with $E = 2$ and $k = 1$. The reflection at the origin produces the momentum discontinuity $-p_0 \mapsto p_0$. Figure (3b) is an example of ACS semiclassical regularised phase space trajectory in the positive half-plane defined by the equation (6.5) with $E = 2$, $\tilde{k} = 1$, and $\tilde{K} = 1$. The latter choices for $\tilde{K}$ and $\tilde{k}$ are easily made possible thanks to a suitable fixing of parameters of the fiducial vector, as was stressed at the end of the previous section. The classical reflection has become a smooth bouncing near the origin.

The presence of the centrifugal potential in equation (6.5), of purely quantum origin, allows to eliminate the singularity due to the reflection by creating a smooth bouncing as it is illustrated by Figures (3b).

Note that there is a modification of the oscillator strength $k$ which becomes $\tilde{k}$. If one considers this fact as a problem, the “renormalised” $\tilde{k}$ can be made arbitrarily close to $k$ by choosing in a suitable way the parameters present in the expression of the fiducial $\psi$. For instance, with the choice of fiducial (5.5), we have $\tilde{k} = \xi^4 \xi_{30} \xi_{2-1}$ and with $\xi = 1$, the product $\xi_{30} \xi_{2-1}$ becomes rapidly closer to 1, as shown in the Figure (4). On the other hand, one could decide that what is measured is not $k$, which pertains to the classical model, viewed as incomplete because “classical”, but rather the “effective” $\tilde{k}$, viewed as more “realistic” since we suppose that the quantum model supersedes the classical one. This might open a debate analogous to that one arising from the distinction between bare mass and dressed or effective mass in Quantum Field Theory.

The eigenvalues $E_n$ and eigenfunctions $\phi_n$ of equation (6.4) in its operator form
can be found by solving the eigenvalue equation
\[
\frac{1}{2} \left( -\hbar^2 \frac{\partial^2}{\partial x^2} + \frac{\hbar^2 K}{x^2} + \tilde{k} x^2 \right) \phi_n = E_n \phi_n. \tag{6.6}
\]

Defining the quantities
\[
\mu = \frac{1}{2} \sqrt{1 + 4K}, \quad \lambda = \frac{1}{2\hbar^2} \left( \frac{k^2}{2} \right)^{\frac{3}{4}}, \tag{6.7}
\]
the solutions are a combination of exponentials and associated Laguerre polynomials as in the following
\[
\phi_n(x) = 2^\frac{3}{2}(\mu+1)x^{(\mu+\frac{1}{2})}e^{-\lambda x^2} L_n^\mu (2\lambda x^2), \tag{6.8}
\]
with \( n \in \mathbb{N} \), and the eigenvalues are given by
\[
E_n = 2\hbar^3 \lambda (2n + \mu + 2). \tag{6.9}
\]

A similar model was analysed in full analytical and numerical details in the article [6] devoted to the study of gravitational singularities in the case of the Robertson-Walker metric coupled to a perfect fluid.

7. Second example: quantum bouncing of charged sphere

Let us consider an isotropic negatively charged insulating medium whose the density of charge varies as \( 1/q \) at distance \( q \) (don’t confuse with a charge!) of the
symmetry center $O$. This means that the ball of center $O$ and radius $q$ has a total charge equal to

$$\Omega = -k_s q^2, \quad k_s > 0. \quad (7.1)$$

The Newton equation for a test positive unit charge and unit mass, at distance $q$ of the center, reads as

$$\ddot{q} = \frac{\Omega}{4\pi\varepsilon_0 q^2} = -k_s \frac{4\pi\varepsilon_0}{4\pi\varepsilon_0} \equiv -k, \quad (7.2)$$

where the radial force or electric field acting on the test charge is determined by using the Gauss theorem. The corresponding Hamiltonian is given by

$$H = \frac{p^2}{2} - kq, \quad (7.3)$$

which corresponds to the weight function $w(\beta) \propto -\delta(\beta - 1)$ in (4.5). An example of phase space trajectory, a truncated parabola, is given in Figure (5a).

![Figure 5:](image)

(a) Classical trajectory
(b) Semi-classical trajectory

Figure 5: Figure (5a) is an example of phase space trajectory in the positive half-plane defined by the equation $E = p^2/2 - kq$ with $E = 1 = k$. Figure (5b) is an example of semiclassical phase space trajectory in the positive half-plane defined by the equation (7.5) with $E = 1 = \tilde{k} = \tilde{K}$.

According to (5.1), the ACS quantization of this classical dynamics yields the quantum Hamiltonian

$$A_H = \frac{P^2}{2} + \frac{\hbar^2}{2} \frac{K_\psi}{Q^2} - k \frac{c_0}{c_{-1}} Q, \quad (7.4)$$
in which the insertion of $\hbar$ has same justification as for (6.5). Passing to the lower symbol of the equation (7.4), through formulas given in (3.21) and (3.23) at constant energy $A_{\mathcal{H}} = E$ yields the semi-classical correction to (7.3)

$$E = \frac{p^2}{2} + \frac{\hbar^2}{2} \frac{c(\psi)}{q^2} - k \frac{c_0}{c_{-1}} \frac{q}{q} \equiv \frac{p^2}{2} + \frac{\tilde{K}}{q^2} - \tilde{k} q .$$  

(7.5)

Again, the presence of the centrifugal potential, of purely quantum origin, allows to eliminate the singularity by creating a smooth bouncing as it is illustrated by Figure (5b). Nevertheless, for this case, we cannot obtain analytical solutions for the eigenvalue equation derived from (7.4) in its operator form

$$\frac{1}{2} \left( -\hbar^2 \partial_x^2 + \hbar^2 K_{\psi} x^2 - k \frac{c_0}{c_{-1}} x \right) \phi_n = E_n \phi_n ,$$

(7.6)

and, therefore, the time evolution of a state could be only calculated numerically.

8. Dust in cosmology

In our third example, the most expanded one in this paper, we deal with the simple model of dynamics of dust in Newtonian cosmology that is presented by Mukhanov in Chapter 1 of his book [9]. We consider a sphere of radius $q(t)$ in an infinite, expanding, homogeneous and isotropic universe filled with dust, i.e. a matter with negligible pressure compared with its energy density. Newtonian gravity is applicable in the case of weak gravity and not too large radius. Also using Gauss theorem, one ignores the gravitational effect on a particle within the sphere due to the matter outside the sphere, a feature which can be also justified within the framework of general relativity (Jebsen-Birkhoff theorem [22]). Therefore, the Newton equation applied to a probe mass $m$ located at the surface of the sphere reads

$$m \ddot{q} = - \frac{G m M}{q^2} ,$$

(8.1)

where $M$ is the time-independent mass of the sphere. Deleting the probe mass, the corresponding Hamiltonian is Kepler-like,

$$H = \frac{p^2}{2} - \frac{k}{q} , \quad k = G M .$$

(8.2)

According to (5.1), the ACS quantization of this classical model gives the quantum Hamiltonian:

$$A_{\mathcal{H}} = \frac{P^2}{2} + \frac{\hbar^2}{2} \frac{K_{\psi}}{Q^2} - \frac{1}{c_{-1}} \frac{k}{Q} .$$

(8.3)
Applying our general formulas (B.27) and (B.29), we obtain its lower symbol at constant energy $A_{H} = E$

$$E = \frac{p^2}{2} + \frac{\hbar^2}{2} \frac{c(\psi)}{q^2} - \frac{k}{q} \equiv \frac{p^2}{2} + \frac{\tilde{K}}{q^2} - \frac{k}{q}.$$  \hspace{1cm} (8.4)

with $c(\psi)$ defined in (3.23). It is the semi-classical correction to (8.2). Note that in this case, there is no “renormalisation” of the classical gravitational coupling $k$.

The spectrum of the operator $A_{H}$ is analogous to the spectrum of the Hydrogen atom obtained from the resolution of the radial Schrödinger equation with non-zero angular momentum (of course, there is no degeneracy in the present model). Hence, we have to distinguish between pure point spectrum corresponding to the bound states and the continuous spectrum corresponding to the scattering states. In the present example, bound states describe a pulsing or breathing dust sphere while scattering states correspond to a bouncing without recollapse.

The eigenvalues $E_n$ and eigenfunctions $\phi_n$ of equation (8.3) in the case of the bound states are given by

$$\frac{1}{2} \left( -\hbar^2 \frac{\partial^2}{\partial x^2} + \frac{\hbar^2 K_\psi}{x^2} - \frac{2}{c-1} \frac{GM}{x} \right) \phi_n = E_n \phi_n. \hspace{1cm} (8.5)$$

Redefining the parameters as

$$\kappa_n^2 = -\frac{2E_n}{\hbar^2}; \hspace{0.5cm} \alpha = \frac{1}{2} + \frac{1}{2} \sqrt{1 + 4K_\psi}, \hspace{1cm} (8.6)$$

the square-integrable solutions to this equation are:

$$\phi_n(x) = N(n, \alpha) e^{-\kappa_n x} (\kappa_n x)^\alpha L_n^{(2\alpha-1)}(2\kappa_n x), \hspace{1cm} (8.7)$$

with $n \in \mathbb{N}$, and $N(n, \alpha)$ is the normalization factor given by the expression:

$$N(n, \alpha) = 2^\alpha \kappa_n \sqrt{n! \Gamma(2\alpha + n)}. \hspace{1cm} (8.8)$$

The eigenvalues of Equation (8.5) are given by

$$\kappa_n = \frac{GM}{\hbar^2 c-1 (n + \alpha)} \Rightarrow E_n = -\frac{G^2 M^2}{2\hbar^2 (c-1)^2 (n + \alpha)^2}. \hspace{1cm} (8.9)$$

With this we can find the time evolution distribution function (3.17) by choosing the normalized fiducial vector $|\psi\rangle$ as

$$\psi(x) = \frac{9}{\sqrt{6}} e^{-\frac{3x}{2}}. \hspace{1cm} (8.10)$$
such as the constants $K_{\psi}$ and $c_{-1}$ are respectively $3/4$ (so $\alpha = 3/2$) and $1$. With this choice, the semiclassical expression of the energy (8.4) reads

$$E = \frac{p^2}{2} + \frac{9}{8} \frac{1}{q^{\alpha}} - \frac{GM}{q}.$$  \hfill (8.11)

Let us choose as an initial state a coherent state $|\phi\rangle = |q_0, p_0\rangle$. The time-dependent probability distribution on the phase space reads:

$$\rho_\phi(q, p, t) = \rho_{q_0, p_0}(q, p, t) = \frac{1}{2\pi} |\langle q_0, p_0 | e^{-iA_H t} | q_0, p_0 \rangle|^2. \hfill (8.12)$$

In order to get a qualitative idea of this distribution, we project the initial state onto the finite-dimensional subspace $H_{n_{\text{max}}}$ spanned by the orthonormal set of bound states $\{|\phi_n\rangle\}_{0 \leq n \leq n_{\text{max}}}$. The coefficients $c_n(q_0, p_0) := \langle \phi_n | q_0, p_0 \rangle$ are given (for a general $\alpha$) by

$$c_n(q_0, p_0) = \frac{9}{2^{5/2} \sqrt{6}} \frac{\Gamma(\alpha + 5/2)}{\Gamma(2\alpha)} \sqrt{\frac{\Gamma(2\alpha + n)}{(n + \alpha) (n!)^3}} \left(\kappa_n q_0\right)^{\alpha + 1/2} \times \left(\frac{4}{2\kappa_n q_0 + 3 - 2iq_0 p_0}\right)^{\alpha + 1/2} \, _2F_1\left(-n, \frac{\alpha + 5}{2}; \frac{2\alpha}{2\kappa_n q_0 + 3 - 2iq_0 p_0}\right), \hfill (8.14)$$

where $\, _2F_1(-n; b; c; z)$ is a Gauss hypergeometric polynomial of degree $n$. Hence, we can calculate the time evolution (8.12) for $\alpha = 3/2$ by choosing as an initial state a specific $(q_0, p_0)$:

$$\rho_\phi(q, p, t) = \frac{1}{2\pi} \left| \sum_{n=0}^{n_{\text{max}}} c_n(q, p) c_n(q_0, p_0) e^{-i\frac{E_n}{\hbar} t} \right|^2. \hfill (8.15)$$

We recall that the Hamiltonian $A_H$ involved in (8.12) has a continuous spectrum. Therefore the above expression holds as a good approximation only if the initial state $|q_0, p_0\rangle$ can be essentially represented as a linear combination of bound states. In fact, this condition depends on the choice of $(q_0, p_0)$. A numerical check based on norm convergence yields $\lim_{n_{\text{max}} \to \infty} 2\pi \rho_\phi(q_0, p_0, t = 0) \simeq 1$. We present in Figure (6) an example of this dynamical behavior with initial state.
Figure 6: Phase space representation of the quantum dynamical behavior of an initial coherent state $|q_0 = 4, p_0 = 0\rangle$. We choose $G = h = 1$ and $M = 2$. On each figure the thick curve represents the semi-classical trajectory due to (8.11) for these particular values, while increasing values of the density $\rho(q, p)$ are encoded by colors from blue to red. The different figures show the evolution of the density $\rho(q, p)$. Time is increasing from the top left to the bottom right.

taken at $q_0 = 4$ and $p_0 = 0$. As expected, the peak of the probability density evolves over the classical trajectory.

In the above calculations, we used the formulae [19]

$$\int_0^\infty du \, e^{-u} \, u^\gamma \left( L_n^{(\gamma - 1)}(u) \right)^2 = (2n + \gamma) \, n! \, \Gamma(\gamma + n) \, , \quad (8.16)$$

and

$$\int_0^\infty du \, e^{-su} \, u^\gamma \, L_n^{(\delta)}(u) = \frac{\Gamma(\gamma + 1) \, \Gamma(\delta + n + 1)}{n! \, \Gamma(\delta + 1)} \, s^{-\gamma - 1} \, _2F_1 \left( -n, \gamma + 1; \delta + 1; \frac{1}{s} \right) \, , \quad (8.17)$$

which is valid for $\text{Re} \, s > 0 \, , \, \text{Re} \, \gamma > -1$.

9. Conclusion

We have presented an integral quantization method for the dynamics on the positive half-line. It is based on the affine symmetry of the corresponding half-plane
phase space and the related coherent states. The procedure rests upon the resolution of the identity by these states which can be identified with wavelet families in Signal Analysis. This method of quantization differs from the canonical one, \( q \mapsto Q, \ p \mapsto P, \) and \( f(q,p) \mapsto \text{Sym}[f(Q,P)] \). Indeed, canonical quantization (and its more elaborate versions) is based on the translational symmetry of the whole plane viewed as the phase space for the motion on the whole line. This is a crucial point which should be always seriously considered in any quantization procedure. Would have we adopted the canonical quantization for the motion on the half-line, we would have never obtained the repulsive centrifugal potential responsible for the regularization of the singularity at the origin of the half-line and for the smooth bouncing of dynamical processes. This fact, which lies at the heart of our results, is illustrated in the present paper with three illuminating, although quite elementary, examples, the breathing sphere, the bouncing charged sphere, and dust sphere in cosmology. More elaborate applications of the method are found in recent works related with quantum cosmology [6].

Another important issue of affine ACS quantization is a systematic rescaling (renormalization?) of quantities depending on the position \( q \). This rescaling can be adjusted at will with an original rescaling in the definition of the ACS or/and with the arbitrariness left to us in the choice of the fiducial vector. As a matter of fact, experiments or observations determine the constants appearing in the expression of physical quantities, and if these experiments/observations are worked out within the framework of quantum models, their observed numerical values should be consistently inserted in the quantum model, forgetting the classical one. The challenge is now to detect at the scale of our laboratories, for instance with the optomechanical model mentioned in the introduction, the appearance of more or less smooth bouncing when are involved in the formalism positive physical quantities.

Appendix A.

A brief review of group transformations and representations

A transformation of a set \( S \) is a one-to-one mapping of \( S \) onto itself. A group \( G \) is realized as a transformation group of a set \( S \) if to each \( g \in G \), there is associated a transformation \( s \mapsto g \cdot s \) of \( S \) where for any two elements \( g_1 \) and \( g_2 \) of \( G \) and \( s \in S \), we have \((g_1g_2) \cdot s = g_1 \cdot (g_2 \cdot s)\). The set \( S \) is then called a \( G \)-space. A transformation group is transitive on \( S \) if, for each \( s_1 \) and \( s_2 \) in \( S \), there is a \( g \in G \) such that \( s_2 = g \cdot s_1 \). In that case, the set \( S \) is called a homogeneous \( G \)-space.

A (linear) representation of a group \( G \) is a continuous function \( g \mapsto T(g) \) which takes values in the group of nonsingular continuous linear transformations of a vector.
space $\mathcal{V}$, and which satisfies the functional equation

$$T(g_1g_2) = T(g_1)T(g_2) \quad \text{and} \quad T(e) = I,$$

where $I$ is the identity operator in $V$ and $e$ is the identity element of $G$. It follows that $T(g^{-1}) = (T(g))^{-1}$. That is, $T(g)$ is a homomorphism of $G$ into the group of nonsingular continuous linear transformations of $V$.

A representation is unitary if the linear operators $T(g)$ are unitary with respect to the inner product $\langle \cdot | \cdot \rangle$ on $V$. That is, $\langle T(g)v_1|T(g)v_2 \rangle = \langle v_1|v_2 \rangle$ for all vectors $v_1$, $v_2$ in $V$. A representation is irreducible if there is no non-trivial subspace $V_0 \subset V$ such that for all vectors $v_o \in V_0$, $T(g)v_o$ is in $V_0$ for all $g \in G$. That is, there is no non trivial subspace $V_0$ of $V$ which is invariant under the operators $T(g)$. An important property of unitary irreducible representations (UIR) of a group is the content of Schur’s Lemma [17]:

**Proposition Appendix A.1. [Schur’s Lemma]** Let $T$ and $T'$ be unitary, irreducible representations of $G$ in $V$ and $V'$, respectively. If $S$ is a bounded linear map of $V \rightarrow V'$ such that

$$STx = T'_xS, \quad \forall x \in G,$$

then, either $S$ is an isomorphism of the spaces $V$ and $V'$, i.e., $T \cong T'$, or $S = 0$. Moreover, if $V = V'$, then $S$ is a multiple of the identity, $S = cI$.

**Appendix B.**

**Affine coherent states quantization: details**

**Resolution of the identity by affine coherent states**

Let us introduce the operator $B$ such as

$$B = \int_0^\infty \int_{-\infty}^\infty |q,p\rangle \langle q,p| \frac{dqdp}{2\pi}.$$  \hfill (B.1)

Then, for arbitrary functions $\phi_1, \phi_2 \in \mathcal{H}$ we have

$$\langle \phi_1|B|\phi_2 \rangle = \frac{1}{2\pi} \int_0^\infty dq \int_{-\infty}^\infty dp \langle \phi_1|q,p\rangle \langle q,p|\phi_2 \rangle.$$  \hfill (B.2)

Using (2.5), we obtain

$$\langle \phi_1|B|\phi_2 \rangle = \int_0^\infty \int_{-\infty}^\infty dqdp \int_0^\infty \int_{-\infty}^\infty dx_1dx_2 e^{ip(x_1-x_2)} \overline{\phi_1(x_1)} \psi \left( \frac{x_1}{q} \right) \phi_2(x_2) \psi \left( \frac{x_2}{q} \right).$$

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Since \( \int_\mathbb{R} dp e^{ip(x_1-x_2)} = 2\pi \delta(x_1-x_2) \), the integration over \( p \) and then over \( x_2 \) gives

\[
\langle \phi_1 | B | \phi_2 \rangle = \int_0^\infty \frac{dq}{q} \int_0^\infty dx \overline{\phi_1(x)} \phi_2(x) \left| \psi \left( \frac{x}{q} \right) \right|^2.
\]  

(B.3)

Changing the coordinate \( q \mapsto q' = x/q \), we have

\[
\langle \phi_1 | B | \phi_2 \rangle = \int_0^\infty \frac{dq'}{q'} \left| \psi(q') \right|^2 \langle \phi_1 | \phi_2 \rangle = c_{-1} \langle \phi_1 | \phi_2 \rangle,
\]  

(B.4)

where \( c_{-1} \) is a constant given by equation (3.5). This result is a direct consequence of Schur’s Lemma Appendix A.1. Therefore, the operator \( B \) is proportional to the identity:

\[
B = c_{-1} I, \quad c_\gamma = c_\gamma(\psi) := \int_0^\infty \left| \psi(x) \right|^2 \frac{dx}{x^{2+\gamma}}.
\]  

(B.5)

Dilating the fiducial vector

Let us now explore the possibilities offered by unitary dilations acting on the fiducial vector \( \psi \) and defining the family

\[
\psi_\kappa(x) := (U(\kappa, 0) \psi)(x) = \frac{1}{\sqrt{\kappa}} \psi \left( \frac{x}{\kappa} \right), \quad \kappa > 0.
\]  

(B.6)

We easily check (with the notation (3.2)) that

\[
|q,p\rangle_\psi = |\kappa q,p\rangle_\psi,
\]  

(B.7)

\[
c_\gamma(\psi_\kappa) = \frac{1}{\kappa^{2+\gamma}} c_\gamma(\psi) \equiv c_\gamma^{(\kappa)}.
\]  

(B.8)

Let us consider the quantization map based upon the resolution of the identity obeyed by the ACS \( |q,p\rangle_\psi \),

\[
f(q,p) \leftrightarrow A_f^{(\kappa)} = \int_0^\infty \int_{-\infty}^\infty f(q,p)|q,p\rangle_\psi \langle q,p| \psi_\kappa \psi_\kappa \langle q,p| \frac{dq dp}{2\pi c_{-1}(\psi_\kappa)}.
\]  

(B.9)

The change of variable \( \kappa q \mapsto q \) yields the relation between \( A_f^{(\kappa)} \) and \( A_f = A_f^{(1)} \)

\[
A_f^{(\kappa)} = A_f^{(\kappa)}, \quad f(\kappa)(q,p) := f \left( \frac{q}{\kappa}, p \right).
\]  

(B.10)

Note that this “dilation covariance” is different of the covariance property (3.7). This gives us an extra degree of freedom besides the choice of the fiducial vector \( |\psi\rangle \).
\( A_f^{(\kappa)} \) as an integral operator

Given two elements \( \phi_1, \phi_2 \in \mathcal{H} \), let us determine the corresponding transition matrix element \( \langle \phi_1 | A_f^{(\kappa)} | \phi_2 \rangle \) of \( A_f^{(\kappa)} \). We obtain from the change of variable \( q \mapsto x_1/q \)

\[
\langle \phi_1 | A_f^{(\kappa)} | \phi_2 \rangle = \int_0^\infty \int_0^\infty dx_1 dx_2 \overline{\phi_1(x_1)} A_f^{(k)}(x_1, x_2) \phi_2(x_2), \quad (B.11)
\]

where

\[
A_f^{(k)}(x_1, x_2) = \frac{1}{\sqrt{2\pi c_{-1}(\psi)}} \int_0^\infty dq \frac{q}{\kappa q} F_p \left( \frac{x_1}{\kappa q}, x_2 - x_1 \right) \psi(q) \psi \left( \frac{x_2 q}{x_1} \right), \quad (B.12)
\]

where \( F_p \) stands for the partial inverse Fourier transform

\[
F_p(q, x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dpe^{-ipx} f(q, p). \quad (B.13)
\]

Hence, the above (B.11) allows to view \( A_f^{(\kappa)} \) as the integral operator in \( L^2(\mathbb{R}^*_+, dx) \)

\[
\left( A_f^{(\kappa)} \phi \right)(x) = \int_0^\infty dx' A_f^{(\kappa)}(x, x') \phi(x'), \quad (B.14)
\]

with integral kernel \( A_f^{(\kappa)}(x, x') \) given in (B.12).

For example, if we have a function that depends only on \( q \), \( f(q, p) = u(q) \), the partial Fourier transform of \( f^{(\kappa)}(q, p) \) is

\[
F_p \left( \frac{x_1}{\kappa q}, x_2 - x_1 \right) = \sqrt{2\pi} \delta(x_2 - x_1) u \left( \frac{x_1}{\kappa q} \right).
\]

Then the kernel (B.12) reads as

\[
A_f^{(\kappa)}(x_1, x_2) = \frac{1}{c_{-1}} \delta(x_2 - x_1) (|\psi|^2 \ast_{\text{aff}} u) \left( \frac{x_1}{\kappa} \right), \quad (B.15)
\]

where we have introduced the (commutative) convolution \( \ast_{\text{aff}} \),

\[
(f_1 \ast_{\text{aff}} f_2)(x) = \int_0^\infty dq \frac{f_1(q)f_2 \left( \frac{x}{q} \right)}{q} = (f_2 \ast_{\text{aff}} f_1)(x). \quad (B.16)
\]

(B.15) simply means that \( A_f^{(\kappa)} \) is the multiplication operator

\[
A_f^{(\kappa)} = (|\psi|^2 \ast_{\text{aff}} u) \left( \frac{Q}{\kappa} \right), \quad (B.17)
\]

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and $Q$ is the multiplication operator, $Q\phi(x) = x\phi(x)$. For the important case $u(q) = q^\beta$, the operator $A_{q^\beta}^{\kappa}$ assumes the simple expression

$$A_{q^\beta}^{\kappa} = \frac{c_{\beta - 1}(\psi) Q^\beta}{c_{-1}(\psi) \kappa^\beta}, \quad (B.18)$$

The other important particular case holds when the function $f$ depends on $p$ only, $f(q,p) = v(p)$. Then the integral kernel is independent of $\kappa$ and becomes

$$A_{\psi}^{\kappa}(x_1, x_2) = A_{\psi}(x_1, x_2) = \frac{1}{\sqrt{2\pi c_{-1}(\psi)}} \hat{v}(x_2 - x_1)(\psi *_{\text{aff}} \tilde{\psi}) \left( \frac{x_1}{x_2} \right), \quad (B.19)$$

where $\hat{v}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ipx} v(p) \, dp$ is the Fourier transform of $v$ and $\tilde{\psi}(x) := \psi(1/x)$. Hence, to $v(p) = p^n$, $n \in \mathbb{N}$, corresponds the operator

$$A_{p^n}^{\kappa} = A_{p^n} = \frac{1}{c_{-1}(\psi)} \sum_{m=0}^{n} \binom{n}{m} c_{m-n-1}^{(-1)^{n-m}} P^n \frac{(-1)^{n-m}}{Q^{n-m}} P^m, \quad P = -i\frac{\partial}{\partial x}, \quad (B.20)$$

where we have introduced the convenient notation which extends (B.5),

$$c_{\gamma}^{(m)}(\psi) := \int_{0}^{\infty} \frac{dx}{x^{\gamma+2}} \psi(x) \frac{\psi^{(m)}(x)}{\overline{\psi^{(m)}}(x)}, \quad c_{\gamma}^{(0)}(\psi) = c_{\gamma}(\psi). \quad (B.21)$$

Applied to the lowest powers (relevant to this paper), (B.21) yields the expressions

$$A_p = P - \frac{c_{-2}^{(1)}(\psi)}{c_{-1}(\psi)} \frac{i}{Q}, \quad A_{p^2} = P^2 - \frac{c_{-2}^{(1)}(\psi) 2i}{c_{-1}(\psi)} \frac{1}{Q} = P - \frac{c_{-2}^{(1)}(\psi)}{c_{-1}(\psi)} \frac{1}{Q^2}. \quad (B.22)$$

When $\psi$ is real, we have (from integration by part and boundary values) $c_{-2}^{(1)}(\psi) = 0$ and $c_{-3}^{(2)}(\psi) = -\int_{0}^{\infty} dx \, x \, (\psi'(x))^2$. Hence, (B.22) reduces to

$$A_p = P, \quad A_{p^2} = P^2 + \frac{K_{\psi}}{Q^2}, \quad K_{\psi} = \frac{1}{c_{-1}(\psi)} \int_{0}^{\infty} dx \, x \, (\psi'(x))^2 > 0. \quad (B.23)$$

**Lower symbols**

We now give details about the calculation of lower symbols introduced in (3.18)

$$\hat{f}(q,p) = \langle q,p | A_f | q,p \rangle. \quad (B.24)$$
Supposing that inverting the order of the integrals is legitimate here, we obtain

$$\tilde{f}(q,p) = \frac{1}{\sqrt{2\pi c_{-1}(\psi)}} \int_{0}^{\infty} \frac{dq'}{qq'} \int_{0}^{\infty} dx \int_{0}^{\infty} dx' \left[ e^{ip(x-x')} F_p(q', x-x') \right] \psi \left( \frac{x}{q} \right) \overline{\psi} \left( \frac{x'}{q'} \right) \psi \left( \frac{x'}{q} \right) \overline{\psi} \left( \frac{x'}{q} \right), \quad (B.25)$$

where $F_p$ is given by (B.13).

For a function depending on $q$ only, $f(q,p) = u(q)$, this integral does not depend on $p$ and is expressed in terms of the inner product in $L^2(\mathbb{R}^*_+, dx)$ and the affine convolution as

$$\tilde{u}(q) = \frac{1}{c_{-1}(\psi)} \left\langle \frac{1}{q} |\psi \left( \frac{\cdot}{q} \right)|^2 |u_{aff} \psi|^2 \right\rangle. \quad (B.26)$$

Applied to powers of $q$, this formula simplifies to

$$\tilde{q}^\beta = c_{\beta-1}(\psi) c_{\beta-2}(\psi) c_{-1}(\psi) q^\beta. \quad (B.27)$$

And, applied to nonnegative integer powers of $p$, (B.25) leads to the polynomial expansion in powers of $p$ with coefficients which are proportional to inverse powers of $q$:

$$\tilde{p}^n = \frac{1}{c_{-1}(\psi)} \sum_{0 \leq m+m' \leq n} \frac{n!}{m!m!(n-m-m')!} c_{-m'-1}(\psi) c_{m'-2}(\psi) (-i)^{m+m'} \frac{p^{n-m-m'}}{q^{m+m'}}, \quad (B.28)$$

where the constants $c_{(m)}(\psi)$ were introduced in (B.21). For $n = 1$ and $n = 2$, and with real $\psi$, this formula simplifies to

$$\tilde{p} = p; \quad (B.29)$$

$$\tilde{p}^2 = p^2 - \left( c_{(2)}(\psi) + \frac{c_{(2)}(\psi) c_0(\psi)}{c_{-1}(\psi)} \right) \frac{1}{q^2} = p^2 + c(\psi) \frac{1}{q^2}, \quad (B.30)$$

where $c(\psi)$ was defined in (3.23).

**Appendix C.**

**Covariant integral quantizations**

Lie group representations [17] offers a wide range of possibilities for implementing integral quantization(s). Let $G$ be a Lie group with left Haar measure $d\mu(g)$, i.e.
\( d\mu(g_0g) = d\mu(g) \) for all \( g_0 \in G \), and let \( g \mapsto U(g) \) be a unitary irreducible representation (UIR) of \( G \) in a Hilbert space \( \mathcal{H} \). Consider a bounded operator \( M \) on \( \mathcal{H} \) and suppose that the operator

\[
R := \int_G M(g) \, d\mu(g) \, , \quad M(g) := U(g) M U^\dagger(g) \, ,
\]

is defined in a weak sense, i.e.

\[
\langle \phi_1 | R \phi_2 \rangle = \int_G \langle \phi_1 | M \phi_2 \rangle (g) \, d\mu(g) \, ,
\]

for all \( \phi_1, \phi_2 \) in a dense subset of \( \mathcal{H} \). From the left invariance of \( d\mu(g) \) we have

\[
U(g_0) R U^\dagger(g_0) = \int_G M(g_0g) \, d\mu(g) = R \, ,
\]

so \( R \) commutes with all operators \( U(g) \), \( g \in G \). Thus, from Schur’s Lemma in the (Appendix A.1), \( R = c_M I \) with

\[
c_M = \int_G \text{tr} (\rho_0 M(g)) \, d\mu(g) \, ,
\]

where the unit trace positive operator \( \rho_0 \) is chosen in order to make the integral converge. This family of operators provides the resolution of the identity on \( \mathcal{H} \).

\[
\int_G M(g) \, d\nu(g) = I , \quad d\nu(g) := \frac{d\mu(g)}{c_M}
\]

and the subsequent quantization of complex-valued functions (or distributions, if well-defined) on \( G \)

\[
f \mapsto A_f = \int_G M(g) \, f(g) \, d\nu(g) \, .
\]

This linear map, function \( \mapsto \) operator in \( \mathcal{H} \), is covariant in the sense that

\[
U(g) A_f U^\dagger(g) = A_{U(g)f} \, .
\]

where \((U(g)f)(g') := f(g^{-1}g')\). In the case when \( f \in L^2(G, d\mu(g)) \), the latter is the regular unitary representation.

A semi-classical analysis [23, 24] of the operator \( A_f \) can be implemented through the study of new functions, denoted by \( \tilde{f} \), on \( X \). They are a generalisation of objects called lower symbols by Lieb [25] and covariant symbols by Berezin [26]. Suppose
that $M$ is a density, i.e. non-negative unit-trace, operator $M = \rho$ on $\mathcal{H}$. Then the operators $\rho(g)$ are also densities, and this allows to build the function $\tilde{f}(g)$ as

$$\tilde{f}(g) \equiv \tilde{A}_f := \int_G \text{tr}(\rho(g) \rho(g')) f(g')d\nu(g').$$  \hfill (C.8)

The map $f \mapsto \tilde{f}$ is a generalization of the Berezin or heat kernel transform on $G$ [27].

Let us illustrate the above procedure with the case of square integrable UIR’s and rank one $\rho$. For a square-integrable UIR $U$ for which $|\psi\rangle$ is an admissible unit vector, i.e.,

$$c(\psi) := \int_G d\mu(g) |\langle \psi | U(g) |\psi\rangle|^2 < \infty,$$  \hfill (C.9)

the resolution of the identity is obeyed by the coherent states $|\psi_g\rangle = U(g) |\psi\rangle$, in a generalized sense, for the group $G$:

$$\int_G \rho(g)d\nu(g) = I \quad d\nu(g) = \frac{d\mu(g)}{c(\psi)}, \quad \rho(g) = |\psi_g\rangle \langle \psi_g|.$$  \hfill (C.10)

Choosing as $M$ a density operator $\rho$, as we did in this case, has multiple advantages, peculiarly in regard to probabilistic aspects both on classical and quantum levels [28].

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References

References

[1] P. A. M. Dirac Principles of Quantum Mechanics, USA: Oxford University Press, 1982. 1

[2] S T Ali and M Engliš, Quantization methods: a guide for physicists and analysts, Rev. Math. Phys. 17 (2005) 391. 1

[3] C.K. Zachos, D.B. Fairlie, and T.L. Curtright, Quantum Mechanics in Phase Space, World Scientific, Singapore (2005). 1, 3, 2

29
[4] T.L Curtright, D.B Fairlie, and C.K Zachos, A Concise Treatise on Quantum Mechanics in Phase Space, World Scientific, Singapore (2016). 1

[5] H. Bergeron and J.-P. Gazeau, Integral quantizations with two basic examples, Annals of Physics (NY), 344 (2014) 43. arXiv:1308.2348 [quant-ph, math-ph] 1

[6] H. Bergeron, A Dapor, J.-P Gazeau and P Mal'kiewicz, “Smooth big bounce from affine quantization”, Phys. Rev. D 89 (2014) 083522; arXiv:1305.0653 [gr-qc]. 1, 3.1, 5, 6, 9

[7] H. Bergeron, E. Czuchry, J.-P. Gazeau, and P. Mal'kiewicz, Nonadiabatic bounce and an inflationary phase in the quantum mixmaster universe, Phys. Rev D 93 (2016)124053; arXiv:1511.05790 [gr-qc]. 1

[8] M. Aspelmeyer, T. J. Kippenberg, and F. Marquardt, Cavity optomechanics, Rev. Mod. Phys., 86 (2014) 1391. 1

[9] V. Mukhanov, Physical Foundations of Cosmology, Cambridge Un. Press 2005. 1, 8

[10] S.T. Ali, J.-P. Antoine, and J.-P. Gazeau, Coherent States, Wavelets and their Generalizations 2d edition, Theoretical and Mathematical Physics, Springer, New York, 2014. 1

[11] J.-P. Gazeau and R. Murenzi, Covariant Affine Integral Quantization(s), J. Math. Phys. 57 (2016) 052102; arXiv:1512.08274 [quant-ph] 1

[12] J.-P. Gazeau, T. Koide and R. Murenzi, More quantum centrifugal effect in rotating frame, EPL 118 (2017) 50004. 1

[13] I. M. Gel’fand and M.A. Naîmark, “Unitary representations of the group of linear transformations of the straight line”, Dokl. Akad. Nauk SSSR 55 (1947) 567. 2

[14] E. W. Aslaksen and J. R. Klauder, “Unitary Representations of the Affine Group”, J. Math. Phys. 15 (1968) 206; “Continuous Representation Theory Using the Affine Group”, J. Math. Phys. 10 (1969) 2267. 2

[15] J.-M. Combes, A. Grossmann and Ph. Tchamitchian (eds.), Wavelets, Time-Frequency Methods and Phase Space (Proc. Marseille 1987), Springer-Verlag, Berlin, 1989; 2nd ed. 1990. 2
[16] L. Grafakos, *Modern Fourier Analysis* (Graduate Texts in Mathematics), 2nd ed. Springer 2009. 2

[17] A. O. Barut and R. Rączka, *Theory of Group Representations and Applications*, PWN, Warszawa, 1977. 3.1, Appendix A, Appendix C

[18] M. Reed and B. Simon, *Methods of Modern Mathematical Physics, II. Fourier Analysis, Self-Adjointness* Volume 2, Academic Press, New York, 1975. 3.1, 3.1

[19] Wilhelm Magnus, Fritz Oberhettinger, and Raj Pal Soni. *Formulas and Theorems for the Special Functions of Mathematical Physics*. Springer-Verlag, Berlin, Heidelberg and New York, 1966. 5, 5, 8

[20] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, edited by A. Jeffrey and D. Zwillinger, Academic Press, New York, 7th edition, 2007. 5, 5

[21] D. J. Griffiths *Introduction to Quantum Mechanics*, 2nd Edition; Pearson Education, 2005. Problem 2.42. 6

[22] J. L. Synge, Relativity: the General Theory, North-Holland Publishing Company, Amsterdam, 1960. 8

[23] J. R. Klauder, Enhanced Quantization: A Primer, *J. Phys. A: Math. Theor.* 45 (2012) 285304; arXiv:1204.2870. Appendix C

[24] J.R. Klauder, *Enhanced Quantization Particles, Fields & Gravity*, World Scientific, 2015. Appendix C

[25] E.H. Lieb, The classical limit of quantum spin systems, *Commun. Math. Phys.* 31 (1973) 327. Appendix C

[26] F.A. Berezin, Quantization, *Math. USSR Izvestija* 8 (1974) 1109; General concept of quantization, *Commun. Math. Phys.* 40 (1975) 153. Appendix C

[27] B.C. Hall, The range of the heat operator. In Jay Jorgenson and Lynne Walling, editors, The ubiquitous heat kernel, *Contemp. Math.* 398, 203, Providence, R.I., Am. Math. Soc 2006. Appendix C

[28] B. Heller and J.-P. Gazeau, Positive-Operator Valued Measure (POVM) Quantization, *Axioms* 4 (2015) 1; doi:10.3390/axioms4010001 Appendix C