DUAL PAIRS AND CONTRAGREEDIENTS OF IRREDUCIBLE REPRESENTATIONS

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Abstract. Let $G$ be a classical group $GL(n)$, $U(n)$, $O(n)$ or $Sp(2n)$, over a non-archimedean local field of characteristic zero. It is well known that the contragredient of an irreducible admissible smooth representation of $G$ is isomorphic to a twist of it by an automorphism of $G$. We prove that similar results hold for double covers of $G$ which occur in the study of local theta correspondences.

1. Introduction and the results

Fix a non-archimedean local field $k$ of characteristic zero. We introduce the following notations in order to treat the four series of classical groups $GL(n)$, $U(n)$, $O(n)$ and $Sp(2n)$ simultaneously. Let $A$ be a $k$-algebra and $\tau$ be a $k$-algebra involution of $A$ so that

$$(A, \tau) = \begin{cases} 
(k \times k, \text{the nontrivial automorphism}), \\
(a \text{ quadratic field extension of } k, \text{the nontrivial automorphism}), \text{ or} \\
(k, \text{the trivial automorphism}). 
\end{cases}$$

Let $\epsilon = \pm 1$ and let $E$ be an $\epsilon$-hermitian $A$-module, namely it is a free $A$-module of finite rank, equipped with a non-degenerate $k$-bilinear map

$$\langle \cdot, \cdot \rangle_E : E \times E \rightarrow A$$

satisfying

$$\langle u, v \rangle_E = \epsilon \langle v, u \rangle_E^\tau, \quad \langle au, v \rangle_E = a \langle u, v \rangle_E, \quad a \in A, u, v \in E.$$

Denote by $U(E)$ the group of all $A$-module automorphisms of $E$ which preserve the form $\langle \cdot, \cdot \rangle_E$. Depending on $A$ and $\epsilon$, it is a general linear group, unitary group, orthogonal group or symplectic group.

Following Moeglin-Vigneras-Waldspurger ([MVW Proposition 4.1.2]), we extend $U(E)$ to a larger group, which is denoted by $\hat{U}(E)$, and consisting of pairs $(g, \delta) \in$

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$GL_k(E) \times \{\pm 1\}$ such that either
\[ \delta = 1 \quad \text{and} \quad g \in U(E), \]
or
\[ \begin{cases} 
\delta = -1, \\
g(au) = a^\delta g(u), \quad a \in A, \ u \in E, \quad \text{and} \\
\langle gu, gv \rangle_E = \langle v, u \rangle_E, \quad u, v \in E.
\end{cases} \]

Clearly $\tilde{U}(E)$ contains $U(E)$ as a subgroup of index two.

In general, if $\pi$ is a representation of a group $H$, and $g$ is an element of a group which acts on $H$ as automorphisms, we define the twist $\pi^g$ to be the representation of $H$ which has the same underlying space as that of $\pi$, and whose action is given by
\[ \pi^g(h) := \pi(g.h), \quad h \in H. \]

If $\tilde{H}$ is a group containing $H$ as a subgroup of index two, we always let it act on $H$ by conjugations:
\[ \text{Ad} : \tilde{H} \times H \to H, \quad (\tilde{g}, x) \mapsto \text{Ad}_\tilde{g}(x) := \tilde{g}x\tilde{g}^{-1}. \]

It is a classical result in linear algebra that for any $\tilde{g} \in \tilde{U}(E) \setminus U(E)$ and any $x \in U(E)$, $\tilde{g}x\tilde{g}^{-1}$ is conjugate to $x^{-1}$ inside $U(E)$. Then by the localization principle of Bernstein and Zelevinsky ([BZ, Theorem 6.9 and Theorem 6.15 A]), for every conjugation invariant generalized function $f$ on $U(E)$, and every $\tilde{g} \in \tilde{U}(E) \setminus U(E)$, we have that
\[ f(\tilde{g}x\tilde{g}^{-1}) = f(x^{-1}) \]
as generalized functions on $U(E)$. For the usual notion of generalized functions, see [Sun, Section 2], for example. We get the following well know result by (1) and by considering characters of irreducible admissible smooth representations (which are conjugation invariant generalized functions).

**Theorem 1.1. ([MVW, Theorem 4.II.1])** Let $\tilde{g} \in \tilde{U}(E) \setminus U(E)$, and let $\pi$ be an irreducible admissible smooth representation of $U(E)$. Then $\pi^\vee$ is isomorphic to $\pi^g$.

Here and as usual, we use “$\vee$” to indicate the contragredient of an admissible smooth representation of a totally disconnected locally compact group.

If $E$ is a symplectic space, i.e., if $\varepsilon = -1$ and $A = k$, then $\tilde{Sp}(E) := \tilde{U}(E)$ equals to the subgroup of $GSp(E)$ with similitudes $\pm 1$. Denote by
\[ 1 \to \{\pm 1\} \to \tilde{Sp}(E) \to Sp(E) \to 1\]
the metaplectic cover of the symplectic group $\text{Sp}(E)$. It is shown in [MVW, Page 36] that there is a unique continuous action

$$ (3) \quad \tilde{\text{Ad}} : \tilde{\text{Sp}}(E) \times \tilde{\text{Sp}}(E) \to \tilde{\text{Sp}}(E) $$

of $\tilde{\text{Sp}}(E)$ on $\tilde{\text{Sp}}(E)$ as group automorphisms which lifts the adjoint action

$$ \text{Ad} : \text{Sp}(E) \times \text{Sp}(E) \to \text{Sp}(E) $$

and leaves the central element $-1 \in \tilde{\text{Sp}}(E)$ fixed.

We first extend Theorem 1.1 to the case of metaplectic groups:

**Theorem 1.2.** Assume that $E$ is a symplectic space. Let $\tilde{g} \in \tilde{\text{Sp}}(E) \setminus \text{Sp}(E)$, and let $\pi$ be a genuine irreducible admissible smooth representation of $\tilde{\text{Sp}}(E)$. Then $\pi^\vee$ is isomorphic to $\pi^{\tilde{g}}$.

Here and henceforth, “genuine” means that the central element $-1 \in \tilde{\text{Sp}}(E)$ acts via the scalar multiplication by $-1$.

**Remarks.** (a) Under the hypothesis that the character of $\pi$ is a locally integrable function, Theorem 1.2 is proved in [MVW, Theorem 4.II.2]. (b) Harish-Chandra proves locally integrability of irreducible characters for $p$-adic linear reductive groups. But metaplectic groups are not included in his setting. (c) The proofs of both Theorem 1.1 and Theorem 1.2 do not depend on locally integrability of irreducible characters.

Now we consider dual pairs. Write $\epsilon' := -\epsilon$, and let $(E', \langle , \rangle_{E'})$ be an $\epsilon'$-hermitian $A$-module. Then

$$ E := E \otimes_A E' $$

is a skew-hermitian $A$-module under the form

$$ \langle u \otimes u', v \otimes v' \rangle_E := \langle u, v \rangle_E \langle u', v' \rangle_{E'} . $$

Write $E_k := E$, viewed as a $k$-symplectic space under the form

$$ \langle u, v \rangle_{E_k} := \text{tr}_{A/k}(\langle u, v \rangle_E) . $$

Put

$$ G := U(E), \quad \tilde{G} := \tilde{U}(E), \quad G' := U(E'), \quad \tilde{G}' := \tilde{U}(E') . $$

The group $G$ obviously maps to the symplectic group $\text{Sp}(E_k)$. Denote the fiber product

$$ \tilde{G} := \tilde{\text{Sp}}(E_k) \times_{\text{Sp}(E_k)} G, $$

which is a double cover of $G$. 

In what follows, we define an action
\[ \tilde{\text{Ad}} : \tilde{G} \times \tilde{G} \to \tilde{G} \]
which lifts the adjoint action
\[ \text{Ad} : \tilde{G} \times G \to G \]
and fixes the central element \(-1 \in \tilde{G}\). Let \( \tilde{g} = (g, \delta) \in \tilde{G} \). Choose an arbitrary element \((g', \delta) \in \tilde{G}'\). Then
\[ \tilde{g} := (g \otimes g', \delta) \in \tilde{\text{Sp}}(E_k), \]
and the automorphism
\[ \tilde{\text{Ad}}_{\tilde{g}} \times \text{Ad}_{\tilde{g}} : \tilde{\text{Sp}}(E_k) \times G \to \tilde{\text{Sp}}(E_k) \times G \]
leaves the subgroup \( \tilde{G} \) stable. It restricts to an automorphism
\[ \tilde{\text{Ad}}_{\tilde{g}} : \tilde{G} \to \tilde{G} \]
which is independent of the choice of \( g' \). We obtain (4) by gluing (6) for all \( \tilde{g} \in \tilde{G} \).

We unify and generalize Theorem 1.1 and Theorem 1.2 as follows:

**Theorem 1.3.** Let \( \tilde{g} \in \tilde{G} \setminus G \), and let \( \pi \) be a genuine irreducible admissible smooth representation of \( \tilde{G} \). Then \( \pi'^{\vee} \) is isomorphic to \( \pi'^{\tilde{g}} \).

Theorem [1.3] is specified to Theorem [1.1] when \( E' = 0 \), and is specified to Theorem [1.2] when \( E' = A = k \) and \( e' = 1 \).

Theorem [1.3] has the following consequence, which is known to experts (up to a proof of Theorem [1.2]). But as far as the author knows, no proof of it in full generality was written down in the literature.

**Theorem 1.4.** Denote by \( \omega_\psi \) the smooth oscillator representation of \( \tilde{\text{Sp}}(E_k) \) corresponding to a non-trivial character \( \psi \) of \( k \). Then for all genuine irreducible admissible smooth representation \( \pi \) of \( \tilde{G} \), and \( \pi' \) of \( \tilde{G}' \), the equality
\[ \dim \text{Hom}_{G \times G'}(\omega_\psi \otimes \pi \otimes \pi', \mathbb{C}) = \dim \text{Hom}_{G \times G'}(\omega_\psi^\vee \otimes \pi^\vee \otimes \pi'^\vee, \mathbb{C}) \]
holds.

Here \( \tilde{G}' := \tilde{\text{Sp}}(E_k) \times_{\text{Sp}(E_k)} G' \) is a double cover of \( G' \). Note that both \( \omega_\psi \otimes \pi \otimes \pi' \) and \( \omega_\psi^\vee \otimes \pi^\vee \otimes \pi'^\vee \), which are originally representations of \( \tilde{G} \times \tilde{G}' \), descend to representations of \( G \times G' \).

**Remarks.** (a) In a sequential paper ([LST]), Theorem 1.4 will be used to prove multiplicity preservations in theta correspondences (for all residue characteristics), i.e., the dimension in Theorem 1.4 is at most one. This is the main reason for the author to provide a detailed proof of Theorem 1.4 in this note.
(b) The archimedean analogs of Theorem 1.3 and Theorem 1.4 are proved by T. Przebinda in [Pr]. His method is different from ours. (He uses the Langlands classification.)

(c) As showed in [Pr], assuming Howe duality conjecture, Theorem 1.4 implies that theta lifting maps hermitian representations to hermitian representations.

2. A generalization of Theorem 1.2

2.1. Skew hermitian modules and Jacobi groups. Assume that $\epsilon = -1$. As in the last section, $E$ is an $\epsilon$-hermitian $A$-module, and $E_k := E$ is a symplectic space under the form

$$\langle u, v \rangle_{E_k} := \text{tr}_{A/k}(\langle u, v \rangle_{E}).$$

Denote by

$$H(E) := E_k \times k$$

the Heisenberg group associated to $E_k$, whose multiplication is given by

$$(u, t)(u', t') := (u + u', t + t' + \langle u, u' \rangle_{E_k}).$$

The group $\tilde{U}(E)$ acts on $H(E)$ as group automorphisms by

$$(g, \delta).(u, t) := (gu, \delta t).$$

It defines a semidirect product

$$\tilde{J}(E) := \tilde{U}(E) \ltimes H(E),$$

which contains

$$J(E) := U(E) \ltimes H(E)$$

as a subgroup of index two.

The results of this note depend heavily on following

**Lemma 2.1.** ([Sun, Theorem D]) Let $f$ be a generalized function on $J(E)$. If it is invariant under conjugations by $U(E)$, i.e.,

$$f(gxg^{-1}) = f(x), \quad \text{for all } g \in U(E),$$

then

$$f(\tilde{g}x\tilde{g}^{-1}) = f(x^{-1}), \quad \text{for all } \tilde{g} \in \tilde{U}(E) \setminus U(E).$$

Actually, we only need the following lemma, which is much weaker than Lemma 2.1

**Lemma 2.2.** Let $f$ be a conjugation invariant generalized function on $J(E)$. Then

$$f(\tilde{g}x\tilde{g}^{-1}) = f(x^{-1}), \quad \text{for all } \tilde{g} \in \tilde{J}(E) \setminus J(E).$$

Lemma 2.2 has the following consequence.
Proposition 2.3. Let \( \tilde{g} \in \tilde{J}(E) \setminus J(E) \), and let \( \pi \) be an irreducible admissible smooth representation of \( J(E) \). Then \( \pi^\vee \) is isomorphic to \( \pi^\tilde{g} \).

Proof. Denote by \( f \) the character of \( \pi \), which is thus a conjugation invariant generalized function on \( J(E) \). Therefore

\[
f(\tilde{g}x\tilde{g}^{-1}) = f(x^{-1})
\]

by Lemma 2.2. The left hand side of (8) is the character of \( \pi^\tilde{g} \), and the right hand side is the character of \( \pi^\vee \). Therefore \( \pi^\tilde{g} \) and \( \pi^\vee \) have the same character, and they are thus isomorphic to each other. \(\square\)

2.2. Metaplectic groups and a generalization of Theorem 1.2. We continue with the notation of Section 2.1. Denote by \( \tilde{\U}(E) := \tilde{\Sp}(E_k) \times_{\Sp(E_k)} U(E) \) the double cover of \( U(E) \) induced by the metaplectic cover

\[
1 \to \{ \pm 1 \} \to \tilde{\Sp}(E_k) \to \Sp(E_k) \to 1.
\]

As in (4), we have an action

\[
\tilde{\Ad} : \tilde{\U}(E) \times \tilde{\U}(E) \to \tilde{\U}(E).
\]

Theorem 1.2 is one of the three cases (corresponding to the three cases of \( A \)) of the following theorem.

Theorem 2.4. Assume that \( \epsilon = -1 \). Let \( \tilde{g} \in \tilde{\U}(E) \setminus U(E) \), and let \( \pi \) be a genuine irreducible admissible smooth representation of \( \tilde{\U}(E) \). Then \( \pi^\vee \) is isomorphic to \( \pi^\tilde{g} \).

Proof. Denote by \( \omega_\psi \) the smooth oscillator representation of \( \tilde{\Sp}(E_k) \ltimes H(E) \) corresponding to a nontrivial character \( \psi \) of \( k \). Up to isomorphism, this is the only genuine smooth representation which, as a representation of \( H(E) \), is irreducible and has central character \( \psi \).

Both \( \omega_\psi \) and \( \pi \) are viewed as smooth representations of \( \tilde{J}(E) := \tilde{\U}(E) \ltimes H(E) \), via the restriction and the inflation, respectively. The tensor product \( \omega_\psi \otimes \pi \) descends to an irreducible admissible smooth representation of \( J(E) \) (Sun, Lemma 5.3).

The actions of \( \tilde{\U}(E) \) on \( \tilde{\U}(E) \), \( U(E) \) and \( H(E) \) induce its actions on the semidirect products \( \tilde{J}(E) \) and \( J(E) \). By Proposition 2.3 as irreducible admissible smooth representations of \( J(E) \),

\[
(\omega_\psi \otimes \pi)^\tilde{g} \cong (\omega_\psi \otimes \pi)^\vee,
\]
or the same,

\[
\omega_\psi^\tilde{g} \otimes \pi^\tilde{g} \cong \omega_\psi^\vee \otimes \pi^\vee.
\]
Note that
\[ \omega_g \cong \omega \]
as smooth representations of \( \tilde{J}(E) \). (This is a special case of Lemma 3.3 of the next section.) Therefore
\[
\omega \otimes \pi \cong \omega \otimes \pi.
\]
As in the proof of [Sun, Lemma 5.3], we have that
\[
\pi \cong \text{Hom}_H(\omega, \omega \otimes \pi).
\]
Here the right hand side carries the action of \( \tilde{U}(E) \) given by
\[
(g, \phi)(v) := g(\phi(g^{-1}v)),
\]
where
\[
\tilde{g} \in \tilde{U}(E), \quad \phi \in \text{Hom}_H(\omega, \omega \otimes \pi), \quad v \in \omega,
\]
and \( g \) is the image of \( \tilde{g} \) under the covering map \( \tilde{U}(E) \to U(E) \). Similarly,
\[
\pi \cong \text{Hom}_H(\omega, \omega \otimes \pi).
\]
We finish the proof by combining (11), (12) and (13). □

3. Proofs of Theorem 1.3 and Theorem 1.4

3.1. Proof of Theorem 1.3 for symplectic groups. Now we return to use the notation of Section 1. First assume that \( A = k \) and \( \epsilon = -1 \). Then \( G \) is a symplectic group and is thus perfect, i.e., \( G \) equals to its own commutator group. Consequently, there is only one action of \( \tilde{G} \) on \( \tilde{G} \) which lifts the adjoint action and fixes the central element \(-1 \in \tilde{G}. \) There are two cases.

Case 1: The covering map \( \tilde{G} \to G \) splits. Then \( \tilde{G} = G \times \{ \pm 1 \} \), and Theorem 1.3 is one case of Theorem 1.1.

Case 2: The covering map \( \tilde{G} \to G \) does not split. Then \( G = \tilde{Sp}(E) \) ([Moo, Theorem 10.4]), and Theorem 1.3 is one case of Theorem 1.2.

3.2. Proof of Theorem 1.3 when \( A \neq k \). Assume that \( A \neq k \). Then \( U(E) \) is a general linear group or a unitary group.

Lemma 3.1. There exists a genuine character on \( \tilde{U}(E) \).

Proof. It is well known that the exact sequence
\[
1 \to C^\times \to (\tilde{Sp}(E_k) \times C^\times) / \text{diag}(\{ \pm 1 \}) \to \text{Sp}(E_k) \to 1
\]
splits continuously over $U(E)$ (this is trivial for general linear groups, and for unitary groups, see [Ku, Proposition 4.1] or [HKS, Section 1]). Write $\iota$ for such a splitting and write $p : \tilde{U}(E) \rightarrow U(E)$ for the covering map. Then
\[ x \in \tilde{U}(E) \mapsto x^{-1} \iota(p(x)) \in \mathbb{C}^\times \]
is a genuine character. \qed

**Lemma 3.2.** There exists a genuine character $\chi$ of $\tilde{G}$ such that $\chi^\hat{g} = \chi^{-1}$ for all $\hat{g} \in \tilde{G} \setminus G$.

**Proof.** As in Section 1, let
\[ \hat{g} = (g, -1) \in \tilde{G} \setminus G, \quad (g', -1) \in \tilde{G}' \setminus G', \]
and write
\[ \hat{g} := (g \otimes g', -1) \in \tilde{U}(E) \setminus U(E). \]
It is obvious that the diagram
\[
\begin{array}{ccc}
\tilde{U}(E) & \xrightarrow{\text{Ad}_{\hat{g}}} & \tilde{U}(E) \\
\uparrow & & \uparrow \\
\tilde{G} & \xrightarrow{\text{Ad}_{\hat{g}}} & \tilde{G}
\end{array}
\]
commutes.

Take a character $\chi_E$ as in Lemma 3.1 and denote by $\chi$ its restriction to $\tilde{G}$. Then
\[ \begin{align*}
\chi^\hat{g} &= (\chi_E|_{\tilde{G}})^\hat{g} \\
&= (\chi^\hat{g}_E|_{\tilde{G}}) \quad \text{by commutativity of (14)} \\
&= (\chi^{-1}_E|_{\tilde{G}}) \quad \text{by Theorem 2.4} \\
&= \chi^{-1}.
\end{align*} \]
\qed

Fix $\chi$ as in Lemma 3.2. Let $\hat{g} \in \tilde{G} \setminus G$, and let $\pi$ be a genuine irreducible admissible smooth representation of $\tilde{G}$. Then $\pi \otimes \chi$ descends to an irreducible admissible smooth representation of $G$. By Theorem 1.1,
\[ (\pi \otimes \chi)^{\hat{g}} \cong (\pi \otimes \chi)^{\hat{g}^\vee}, \]
or equivalently,
\[ \pi^\hat{g} \otimes \chi^\hat{g} \cong \pi^{\hat{g}^\vee} \otimes \chi^{-1}, \]
Therefore, $\pi^\hat{g} \cong \pi^{\hat{g}^\vee}$ since $\chi^\hat{g} = \chi^{-1}$. This proves Theorem 1.3 when $A \neq k$. 

3.3. **Proof of Theorem 1.3 for orthogonal groups.** Assume that $A = k$ and $e = 1$, i.e., $G$ is an orthogonal group. In what follows, we show that Lemma 3.2 still holds in this case. Fix a complete polarization

$$E' = E'_+ \oplus E'_-$$

of the symplectic space $E'$. Then

$$E = E_+ \oplus E_-$$

is a complete polarization of the symplectic space $E$, where $E_{\pm} := E \otimes E'_{\pm}$. Depending on this polarization, we define a skew-hermitian $k \times k$-module $E'$ as follows. As an abelian group, $E' = E$. The scalar multiplication is given by

$$(ae_1 + be_2)(u + v) := au + bv, \quad a, b \in k, \quad u \in E_+, \quad v \in E_-,$$

where

$$e_1 := (1, 0) \quad \text{and} \quad e_2 := (0, 1)$$

are the two idempotent elements of $k \times k$. The skew-hermitian form is given by

$$\langle u_+ + u_-, v_+ + v_- \rangle_{E'} := \langle u_+, v_+ \rangle_E e_1 + \langle u_-, v_+ \rangle_E e_2,$$

where $u_+, v_+ \in E_+, \quad u_-, v_- \in E_-.$

Let $\hat{g} = (g, -1) \in \hat{G} \setminus G$. Choose an element $(g', -1) \in \hat{G}' \setminus G'$ such that

$$g'(E'_+) = E'_- \quad \text{and} \quad g'(E'_-) = E'_+.$$

Then

$$\tilde{g} := (g \otimes g', -1) \in \tilde{U}(E') \setminus U(E'),$$

and the diagram

$$\begin{array}{ccc}
\tilde{U}(E') & \xrightarrow{\tilde{A}d_{\tilde{g}}} & \tilde{U}(E') \\
\uparrow & & \uparrow \\
\tilde{G} & \xrightarrow{\tilde{A}d_{\tilde{g}}} & \tilde{G}
\end{array}$$

commutes.

Take a genuine character $\chi_{E'}$ of $\tilde{U}(E')$ as in Lemma 3.1 and denote by $\chi$ its restriction to $\tilde{G}$. Then as in the proof of Lemma 3.2, we show that $\chi$ fulfills the requirement of Lemma 3.2. Now we argue as in the end of the last subsection, and prove Theorem 1.3 for orthogonal groups.

The proof of Theorem 1.3 is now complete by combining this subsection with the last two subsections.
3.4. **Proof of Theorem 1.4.** The group

\[ \tilde{G} := \tilde{G} \times_{\{\pm 1\}} \tilde{G}^\prime = \{(g, g', \delta) \mid (g, \delta) \in \tilde{G}, (g', \delta) \in \tilde{G}^\prime\} \]

contains

\[ G := G \times G' \]

as a subgroup of index two. Define a homomorphism

\[ \xi : \tilde{G} \to \tilde{Sp}(E_k), \quad (g, g', \delta) \mapsto (g \otimes g', \delta). \]

By using the covering map

\[ \tilde{G} \times \tilde{G}^\prime \to G = G \times G' \]

and the map

\[ \xi|_G : G \to \tilde{Sp}(E_k), \]

we form the semidirect product

\[ (\tilde{G} \times \tilde{G}^\prime) \rtimes H(E) \]

as in Section 2. Let \( \tilde{G} \) act on \((\tilde{G} \times \tilde{G}^\prime) \rtimes H(E)\) as group automorphisms by

\[ \tilde{g}.(x, y, z) := (\tilde{Ad}_g(x), \tilde{Ad}_{g'}(y), \xi(\tilde{g}).z), \]

where

\[ \tilde{g} = (g, g', \delta), \quad \tilde{g} = (g, \delta), \quad \tilde{g}' = (g', \delta), \]

and the last term of the right hand side of (15) is defined as in (7).

Let \( \omega, \pi \) and \( \pi' \) be as in Theorem 1.4

**Lemma 3.3.** View \( \omega_\psi \) as an admissible smooth representation of \((\tilde{G} \times \tilde{G}^\prime) \rtimes H(E)\) (via the restriction). Then for every \( \tilde{g} \in \tilde{G} \setminus G \), we have

\[ \omega_\psi \cong \omega_{\tilde{g}_\psi}. \]

**Proof.** Recall that the group \( \tilde{Sp}(E_k) \) acts on \( \tilde{Sp}(E_k) \rtimes H(E) \) diagonally through its action on the two factors. We have

\[ \omega_\psi \cong \omega_{\xi(\tilde{g})} \]

as smooth oscillator representations of \( \tilde{Sp}(E_k) \rtimes H(E) \), since both corresponding to the character \( \psi^{-1} \). We prove the lemma by restricting both sides of (16) to the group \((\tilde{G} \times \tilde{G}^\prime) \rtimes H(E)\). \( \square \)

**Lemma 3.4.** View \( \pi \) and \( \pi' \) as admissible smooth representations of \((\tilde{G} \times \tilde{G}^\prime) \rtimes H(E)\) (via the inflations). Then for every \( \tilde{g} \in \tilde{G} \setminus G \), we have

\[ \pi \cong \pi_{\tilde{g}} \quad \text{and} \quad \pi' \cong \pi'_{\tilde{g}}. \]

(17)
Proof. Write
\[ \check{\mathfrak{g}} = (g, g', -1), \quad \text{and} \quad \check{\mathfrak{g}} = (g, -1). \]

By Theorem 1.3
\[ \pi^\vee \cong \pi^\check{\mathfrak{g}} \]
as irreducible admissible smooth representations of \( \widetilde{G} \). Pull back this isomorphism to the group \((\widetilde{G} \times \widetilde{G}') \ltimes H(E)\), we obtain the first isomorphism of (17). The second isomorphism follows similarly. \( \Box \)

Lemma 3.5. For every \( \check{\mathfrak{g}} \in \check{\mathcal{G}} \setminus G \), we have
\[ (18) \quad \omega_\psi \otimes \pi^\vee \otimes \pi'^\vee \cong (\omega_\psi \otimes \pi \otimes \pi')^{\check{\mathfrak{g}}} \]
as smooth representations of \((\widetilde{G} \times \widetilde{G}') \ltimes H(E)\).

Proof. This is a combination of Lemma 3.3 and Lemma 3.4. \( \Box \)

Fix an element \( \check{\mathfrak{g}} \in \check{\mathcal{G}} \setminus G \). Since the action of \( \check{\mathfrak{g}} \) stabilizes the subgroup \( \widetilde{G} \times \widetilde{G}' \) of \((\widetilde{G} \times \widetilde{G}') \ltimes H(E)\), we have
\[ (19) \quad \text{Hom}_{\widetilde{G} \times \widetilde{G}'}((\omega_\psi \otimes \pi \otimes \pi')^{\check{\mathfrak{g}}}, \mathbb{C}) = \text{Hom}_{\widetilde{G} \times \widetilde{G}'}((\omega_\psi \otimes \pi \otimes \pi')^{\check{\mathfrak{g}}}, \mathbb{C}). \]

Now Theorem 1.4 is a consequence of (18) and (19).

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