CASTELNUOVO-MUMFORD REGULARITY OF ASSOCIATED GRADED MODULES IN DIMENSION ONE

LE XUAN DUNG
Department of Natural Sciences, Hong Duc University
307 Le Lai, Thanh Hoa, Vietnam
E-mail: lxdung27@gmail.com

ABSTRACT. An upper bound for the Castelnuovo-Mumford regularity of the associated graded module of an one-dimension module is given in term of its Hilbert coefficients. It is also investigated when the bound is attained.

1. INTRODUCTION

Let $A$ be a local ring with the maximal ideal $\mathfrak{m}$ and $M$ be a finitely generated $A$-module of dimension $r$. Denote by $G_I(M) = \bigoplus_{n \geq 0} I^n M/I^{n+1} M$ the associated graded module of $M$ with respect to an $\mathfrak{m}$-primary ideal $I$. Because of the importance of the Castelnuovo-Mumford regularity $\text{reg}(G_I(M))$ of $G_I(M)$, it is of interest to bound it in terms of other invariants of $M$. Rossi, Trung and Valla [13] and Linh [12] gave bounds on $\text{reg}(G_I(M))$ in terms of the so-called extended degree of $M$ with respect to $I$.

On the other hand, Trivedi [15] and Brodmann and Sharp [1] gave bounds for so-called Castelnuovo-Mumford regularity at and above level 1 in terms of the Hilbert coefficients $e_0(I, M), e_1(I, M), ..., e_{r-1}(I, M)$. Using also $e_r(I, M)$, Dung and Hoa [5] can bound $\text{reg}(G_I(M))$. Bounds in [11, 13, 15] are very big: they are exponential functions of $r!$. In the general case, it follows from [10, Lemma 11 and Proposition 12] that in the worst case $\text{reg}(G_I(M))$ must be a double exponential function of $r$. However one may hope that in small dimensions, one can give small and sharp bounds.

The aim of this paper is to give a sharp bound in dimension one case.

Theorem Let $M$ be an one-dimensional module. Let $b$ be the maximal integer such that $IM \subseteq \mathfrak{m}^b M$. Then

$$\text{reg}(G_I(M)) \leq \left( e_0 - b + 2 \right) - e_1 - 1.$$ 

We also give characterizations for the case that the bound is attained (see Theorem 3.3 and Theorem 3.5).
The paper is divided into two sections. In Section 1 we start with a few preliminary results on bounding the Castelnuovo-Mumford regularity of graded modules of dimension at most one. Then we apply these results to derive a bound on $\text{reg}(G_t(M))$ provided $M$ is a Cohen-Macaulay module of dimension one (Proposition 2.9). Using a relation between Hilbert coefficients $e_0(I, M)$ and $e_1(I, M)$ given in Rossi-Valla\[14\] we can deduce and prove the above bound.

In Section 2 we give characterizations for the case that the bound is attained. The characterization is given in terms of the Hilbert-Poincare series (Theorem 3.5). In the case of Cohen-Macaulay modules there is also a characterization in terms of the Hilbert coefficients (Theorem 3.3).

2. An upper bound

Let $R = \oplus_{n \geq 0} R_n$ be a Noetherian standard graded ring over a local Artinian ring $(R_0, m_0)$. Since tensoring with $(R_0/m_0)(x)$ doesn’t change invariants considered in this paper, without loss of generality we may always assume that $R_0/m_0$ is an infinite field. Let $E$ be a finitely generated graded module of dimension $r$.

First let us recall some notation. For $0 \leq i \leq r$, put

$$a_i(E) = \sup \{ n \mid H_{R_+}^i(E)_n \neq 0 \},$$

where $R_+ = \oplus_{n > 0} R_n$. The Castelnuovo-Mumford regularity of $E$ is defined by

$$\text{reg}(E) = \max \{ a_i(E) + i \mid 0 \leq i \leq r \},$$

and the Castelnuovo-Mumford regularity of $E$ at and above level 1, is defined by

$$\text{reg}^1(E) = \max \{ a_i(E) + i \mid 1 \leq i \leq r \}.$$

We denote the Hilbert function $\ell_{R_0}(E_t)$ and the Hilbert polynomial of $E$ by $h_E(t)$ and $p_E(t)$, respectively. Writing $p_E(t)$ in the form:

$$p_E(t) = \sum_{i=0}^{r-1} (-1)^i e_i(E) \binom{t + r - 1 - i}{r - 1 - i},$$

we call the numbers $e_i(E)$ Hilbert coefficients of $E$.

We know that $h_E(t) = p_E(t)$ for all $t \gg 0$. The postulation number of a finitely generated graded $R$-module $E$ is defined as the number

$$p(E) := \max \{ t \mid h_E(t) \neq p_E(t) \}.$$

This number can be read off from the Hilbert-Poincare series of $E$, which is defined as follows:

$$HP_E(z) = \sum_{i \geq 0} h_E(i) z^i.$$

The following result is well-known, see e. g. [2, Lemma 4.1.7 and Proposition 4.1.12].

**Lemma 2.1.** There is a polynomial $Q_E(z) \in \mathbb{Z}[z]$ such that $Q_E(1) \neq 0$ and

$$HP_E(z) = \frac{Q_E(z)}{(1 - z)^r}.$$  

Moreover, $p(E) = \deg(Q_E(z)) - r$.  

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Remark 2.2. From the Grothendieck-Serre formula

\( h_E(t) - p_E(t) = \sum_{i=0}^{r} (-1)^i \ell(H^i_{R_1}(E)t), \)

we easily get

(i) If \( \text{depth}(E) = 0 \) then \( p(E) \leq \text{reg}(E) \);
(ii) If \( \text{depth}(E) > 0 \) then \( p(E) \leq \text{reg}(E) - 1 \).

Let \( d(E) \) denote the maximal generating degree of \( E \). For short, we also write \( d := d(E) \) and \( e := e_0(E) \). The proof of the following result is similar to that of \([9, \text{Lemma } 4.2]\].

Lemma 2.3. Assume that \( \dim(E) = 0 \). Let \( q := \sum_{i \leq d} \ell(E_i) \). Then

(i) \( \text{reg}(E) \leq d + e - q \).
(ii) The following conditions are equivalent:
   (a) \( \text{reg}(E) = d + e - q \);
   (b) \( h_E(t) = \begin{cases} 1 & \text{if } d + 1 \leq t \leq d + e - q, \\ 0 & \text{if } t \geq d + e - q + 1. \end{cases} \)

Proof. (i) Let \( m = \text{reg}(E) \). Since \( \dim(E) = 0 \), \( m = \max\{t \mid E_t \neq 0\} \). Note that \( E_i \neq 0 \) for all \( d \leq i \leq m \). Hence

\( m \leq d + \ell(E_{d+1} \oplus \cdots \oplus E_m) = d + \ell(E) - q = d + e - q. \)

(ii) By \([2]\) it is clear that \( m = d + e - q \) if only if \( \ell(E_i) = 1 \) for all \( d + 1 \leq i \leq m \). Since \( \ell(E_i) = 0 \) for all \( i > m \), we get the equivalence of (a) and (b). \( \square \)

Lemma 2.4. Let \( E \) be an one-dimensional Cohen-Macaulay module. Let \( z \in R_1 \) be an \( E \)-regular element and \( \rho := \ell(E_{d(E/zE)}) \). Then

(i) \( \text{reg}(E) \leq d + e - \rho \).
(ii) The following conditions are equivalent:
   (a) \( \text{reg}(E) = d + e - \rho \);
   (b) \( h_E(t) = \begin{cases} t - d + \rho & \text{if } d + 1 \leq t \leq d + e - \rho - 1, \\ e & \text{if } t \geq d + e - \rho; \end{cases} \)
   (c) \( p(E) = d + e - \rho - 1 \).

Proof. (i) Since \( z \in R_1 \) is an \( E \)-regular element, we have \( e(E/zE) = e, \ \text{reg}(E/zE) = \text{reg}(E) \) and

\( \sum_{i \leq t} h_{E/zE}(i) = h_E(t). \)

In particular \( \sum_{i \leq d(E/zE)} h_{E/zE}(i) = \rho \). Note that \( d(E/zE) \leq d(E) \). By Lemma 2.3

\( \text{reg}(E/zE) \leq d(E/zE) + e(E/zE) - \sum_{i \leq d(E/zE)} h_{E/zE}(i) \leq d + e - \rho. \)

Hence \( \text{reg}(E) \leq d + e - \rho \).

(ii) (a) \( \implies \) (b): If \( \text{reg}(E) = d + e - \rho \), then from (1) it implies that \( d(E/zE) = d(E) \) and \( \text{reg}(E/zE) = d(E/zE) + e - \rho \). Using Lemma 2.3 (ii) and (3) we get (b).

(b) \( \implies \) (c) follows from Lemma 2.1.
(c) \implies (a): Suppose that \( p(E) = d + e - \rho - 1 \). By Remark 2.2 (ii), we get \( d + e - \rho \leq \text{reg}(E) \). Since \( \text{reg}(E) \leq d + e - \rho \), we get \( \text{reg}(E) = d + e - \rho \). \( \square \)

Let \((A, \mathfrak{m})\) be a Noetherian local ring with an infinite residue field \( K := A/\mathfrak{m} \) and \( M \) a finitely generated \( A \)-module of dimension \( r \). In this paper, we always assume that \( I \) is an \( \mathfrak{m} \)-primary ideal. The associated graded module of \( M \) with respect to \( I \) is defined by

\[
G_I(M) = \bigoplus_{n \geq 0} I^n M / I^{n+1} M.
\]

This is a module over the associated graded ring \( G = \bigoplus_{n \geq 0} I^n / I^{n+1} \). Let \( HP_{I,M}(z) := H_{G_I(M)}(z) \). We call \( H_{I,M}(n) = \ell(M/I^{n+1} M) \) the Hilbert-Samuel function of \( M \) w.r.t. \( I \). This function agrees with a polynomial - called Hilbert-Samuel polynomial and denote by \( P_{I,M}(n) \) - for \( n \gg 0 \). If we write

\[
P_{I,M}(n) = e_0(I, M) \binom{n + r}{r} - e_1(I, M) \binom{n + r - 1}{r - 1} + \cdots + (-1)^r e_r(I, M),
\]

then the integers \( e_i := e_i(I, M) \) for all \( i = 1, \ldots, r \) are called Hilbert coefficients of \( M \) with respect to \( I \) (see [14, Section 1]).

Assume that \( M \) is a Cohen-Macaulay module. Kirby and Mehran [11] were able to show that \( e_1 \leq \binom{e_0 - b + 1}{2} \). This result was improved by Rossi-Valla as follows:

**Lemma 2.5.** [14, Proposition 2.8] Let \( M \) be an one-dimensional Cohen-Macaulay module. Let \( b \) be a positive integer such that \( IM \subseteq \mathfrak{m}^b M \). Then

(i) \( e_1 \leq \binom{e_0 - b + 1}{2} \).

(ii) The following conditions are equivalent:

(a) \( e_1 = \binom{e_0 - b + 1}{2} \);

(b) \( HP_{I,M}(z) = \frac{b + \sum_{i=1}^{e_0 - b} z^i}{1 - z} \).

Note that (ii) is not formulated in [14, Proposition 2.8], but it immediately follows from the proof of that result.

**Remark 2.6.** (i) If \( e_1 = \binom{e_0 - b + 1}{2} \), then from (i) of the above lemma it follows that \( b = \max\{t \mid IM \subseteq \mathfrak{m}^t M\} \).

(ii) If \( I = \mathfrak{m} \) and \( M = A \) then \( b = 1 \) and Lemma 2.5 (ii) is [11, Proposition 2.13] (see also [8, Theorem 3.1]).

Set \( G_I(M) := G_I(M) / H^0_{G_+}(G_I(M)) \).

**Lemma 2.7.** Let \( M \) be an one-dimensional module. Let \( b \) be a positive integer such that \( IM \subseteq \mathfrak{m}^b M \). Then

\[
h_{G_I(M)}(0) \geq b.
\]

**Proof.** Note that

\[
[H^0_{G_+}(G_I(M))]_0 = \frac{T_0}{IM},
\]

where \( T_0 = \cup_{n > 0} (I^{n+1} M : I^n) = I^{t+1} M : I^t \subseteq M \) for some \( t \gg 0 \). Set \( \widetilde{M} := M / T_0 \).

Suppose that

\[
\mathfrak{m}^{b-1} \widetilde{M} = \mathfrak{m}^b \widetilde{M}.
\]
By Nakayama’s Lemma, this gives $m^{b-1}M = 0$. Consequently, $m^{b-1}M \subseteq T_0$. Then $m^{b-1}I^{+1}M \subseteq I^{+1}M$. Since $IM \subseteq m^{b}M$,

$$I^{+1}M \subseteq m^{b}I^{+1}M \subseteq mI^{+1}M \subseteq I^{+1}M.$$  

Hence $I^{+1}M = mI^{+1}M$. By Nakayama’s Lemma, we get $I^{+1}M = 0$. This implies $\dim(M) \leq 0$, a contradiction. Therefore $m^{b-1}M \neq m^{b}M$ and we obtain a strict chain of submodules of $\widetilde{M}$:

$$\widetilde{M} \supsetneq m\widetilde{M} \supsetneq \cdots \supsetneq m^{b}\widetilde{M}.$$  

Since $G_{I}(M)_{0} = M/T_0 = \widetilde{M}$, we must have

$$h_{G_{I}(M)}(0) = \ell(\widetilde{M}) \geq \ell(\widetilde{M}/m^{b}\widetilde{M}) \geq b.$$  

\[\square\]

The following result was given by L. T. Hoa [9, Theorem 5.2] for rings, but its proof also holds for modules.

**Lemma 2.8.** Let $M$ be a module of positive depth. Then

$$a_0(G_{I}(M)) \leq a_1(G_{I}(M)) - 1.$$  

We are now in the position to show the main results of this section.

**Proposition 2.9.** Let $M$ be an one-dimensional Cohen-Macaulay module. Let $b$ be the maximal integer such that $IM \subseteq m^{b}M$. Then

$$\operatorname{reg}(G_{I}(M)) \leq e_0 - b.$$  

**Proof.** Since $\operatorname{depth}(M) > 0$, by Lemma 2.8,

$$\operatorname{reg}(G_{I}(M)) = \operatorname{reg}^1(G_{I}(M)) = \operatorname{reg}(\overline{G_{I}(M)}).$$  

Since $\dim(M) = 1$, $\overline{G_{I}(M)}$ is a Cohen-Macaulay module. Assume that $x^* \in I/I^{2}$ is an $\overline{G_{I}(M)}$-regular element. Since $G_{I}(M)$ is generated in degree 0, $d(G_{I}(M)/x^*\overline{G_{I}(M)}) = 0$. Applying Lemma 2.4 (i) and Lemma 2.8 we get

$$\operatorname{reg}(\overline{G_{I}(M)}) \leq e_0 - h_{G_{I}(M)}(0) \leq e_0 - b.$$  

\[\square\]

**Remark 2.10.** Under the above assumption, Linh [12, Corollary 4.5 (ii)] showed that $\operatorname{reg}(G_{I}(M)) \leq e_0 - 1$. Hence, if $b > 1$ the above result improves Linh’s bound.

The following result was formulated in the introduction.

**Theorem 2.11.** Let $M$ be an one-dimensional module. Let $b$ be the maximal integer such that $IM \subseteq m^{b}M$. Then

$$\operatorname{reg}(G_{I}(M)) \leq \left(\frac{e_0 - b + 2}{2}\right) - e_1 - 1.$$  

**Proof.** Let $L := H^0_m(M)$. By [12, Lemma 4.3] (or see [4, Lemma 1.9]) we have

$$\operatorname{reg}(G_{I}(M)) \leq \operatorname{reg}(\overline{G_{I}(M)}) + \ell(L).$$
By [14] Proposition 2.3, it implies that \( \ell(L) = \tau_1 - e_1 \) and \( \tau_0 = e_0 \), where \( \tau_0 := e_0(I, \overline{M}) \) and \( \tau_1 := e_1(I, \overline{M}) \). Note that \( IM \subseteq \mathfrak{m}^bM \) implies \( I\overline{M} \subseteq \mathfrak{m}^b\overline{M} \). Since \( \overline{M} \) is a Cohen-Macaulay module, Lemma 2.5 (i) says that \( \tau_1 \leq \binom{e_0 - b + 1}{2} \). This gives
\[
\ell(L) \leq \left( \frac{e_0 - b + 1}{2} \right) - e_1.
\]
By Proposition 2.9 \( \text{reg}(G_I(\overline{M})) \leq e_0 - b \). Hence,
\[
\text{reg}(G_I(M)) \leq e_0 - b + \left( \frac{e_0 - b + 1}{2} \right) - e_1 = \left( \frac{e_0 - b + 2}{2} \right) - e_1 - 1.
\]

3. Extremal case

In this section, let \((A, \mathfrak{m})\) be a Noetherian local ring with an infinite residue field and \( I \) an \( \mathfrak{m} \)-primary ideal. Let \( M \) be a finite generated \( A \)-module. The aim of this section is to give characterizations for the case that the bound in Theorem 2.11 is attained.

From the Grothendieck-Serre formula [1] and Lemma 2.8 we immediately get

**Lemma 3.1.** Let \( M \) be a module of positive depth. Then
\[
\text{p}(G_I(M)) \leq \text{reg}(G_I(M)) - 1.
\]

**Lemma 3.2.** Let \( M \) be a module with \( \text{dim}(M) \geq 1 \). Let \( L := H^0_\mathfrak{m}(M) \) and \( \overline{M} := M/L \). Then the following conditions are equivalent:

(i) \( \text{reg}(G_I(M)) = \text{reg}(G_I(\overline{M})) + \ell(L) \);
(ii) \( HP_K(z) = \sum_{n \geq \text{reg}(G_I(\overline{M}))+\ell(L)+1} z^n \), where \( K = \oplus_{n \geq 0} \frac{L^{n+1}M + L^nM}{L^{n+1}M} \).

**Proof.** Note that \( \ell(K) = \ell(L) \). If \( L = 0 \), there is nothing to prove. Assume that \( L \neq 0 \).

(ii) \( \Rightarrow \) (i): Since \( \text{dim}(K) = 0 \), \( \text{reg}(K) = \text{reg}(G_I(\overline{M}))+\ell(L) \). From the short exact sequence
\[
0 \longrightarrow K \longrightarrow G_I(M) \longrightarrow G_I(\overline{M}) \longrightarrow 0,
\]
we see that (see e. g., [6] Corollary 20.19 (d))
\[
\text{reg}(G_I(M)) = \max\{\text{reg}(K), \text{reg}(G_I(\overline{M}))\} = \text{reg}(G_I(\overline{M}))+\ell(L).
\]

(i) \( \Rightarrow \) (ii): Set \( a = \text{reg}(G_I(\overline{M})) \) and \( m = \max\{t \mid K_t \neq 0\} \). It was proved in [4] Lemma 1.9] that \( K_t \neq 0 \) for all \( a + 1 \leq t \leq m \). By (6),
\[
a + \ell(L) = \text{reg}(G_I(\overline{M}))+\ell(L) = \text{reg}(G_I(M)) \leq \max\{a, m\}
\]
\[
= a + \max\{0, m-a\} \leq a + \ell(K) = a + \ell(L).
\]
This implies \( \ell(K_t) = m - a \), and so \( \ell(K_t) = 1 \) for \( a + 1 \leq t \leq m \) and \( \ell(K_t) = 0 \) for all other values.

Now we can state and prove the main results of this section. It is interesting to mention that the equality \( e_1 = \binom{e_0 - b + 1}{2} \) implies the Cohen-Macaulayness of \( G_I(M) \). This implication was shown in [7] Proposition 2.13 for the case \( I = \mathfrak{m} \) and \( M = A \).
Theorem 3.3. Let $M$ be an one-dimensional Cohen-Macaulay module. Let $b$ be a positive integer such that $IM \subseteq m^b M$. Then the following conditions are equivalent:

(i) $\operatorname{reg}(G_I(M)) = \left(\frac{e_0 - b + 2}{2}\right) - e_1 - 1$;

(ii) $HP_{I,M}(z) = \frac{b + \sum_{i=1}^{e_0 - b - 1} z^i}{1-z}$;

(iii) $e_1 = \left(\frac{e_0 - b + 1}{2}\right)$;

(iv) $\operatorname{reg}(G_I(M)) = e_0 - b$ and $G_I(M)$ is a Cohen-Macaulay module.

Moreover, if one of the above conditions holds then $b = \max\{t \mid IM \subseteq m^t M\}$.

Proof. The last statement follows from Remark 2.6 (i).

(ii) $\iff$ (iii) is the Rossi-Valla result (see Lemma 2.5 (ii)).

(i) $\implies$ (iii): Since $M$ is a Cohen-Macaulay module, by Proposition 2.9, $\operatorname{reg}(G_I(M)) \leq e_0 - b$. Hence $\left(\frac{e_0 - b + 2}{2}\right) - e_1 - 1 \leq e_0 - b$, or equivalently $e_1 \geq \left(\frac{e_0 - b + 1}{2}\right)$. By Lemma 2.5 (i), we then get $e_1 = \left(\frac{e_0 - b + 1}{2}\right)$.

(ii) $\implies$ (iv): Note that we always have $e_0 \geq b$. By Remark 2.5 (ii), $p(G_I(M)) = e_0 - b - 1$. Applying Lemma 3.1, we get $e_0 - b \leq \operatorname{reg}(G_I(M))$. Combining with Proposition 2.9, we can conclude that $\operatorname{reg}(G_I(M)) = e_0 - b$.

By Lemma 2.8, $\operatorname{reg}(G_I(M)) = \operatorname{reg}(\overline{G_I(M)}) = e_0 - b$. From (5), we get $h_{\overline{G_I(M)}}(0) = b$. By Lemma 2.4 (ii) this implies

$$h_{\overline{G_I(M)}}(t) = \begin{cases} b & \text{if } t = 0, \\ t + b & \text{if } 1 \leq t \leq e_0 - b - 1, \\ e_0 & \text{if } t \geq e_0 - b. \end{cases}$$

Consequently,

$$HP_{\overline{G_I(M)}}(z) = \frac{b + \sum_{i=1}^{e_0 - b} z^i}{1-z} = HP_{I,M}(z).$$

Hence $G_I(M) = \overline{G_I(M)}$ and $G_I(M)$ is a Cohen-Macaulay module.

(iv) $\implies$ (i): Since $G_I(M)$ is a Cohen-Macaulay module and $\operatorname{reg}(G_I(M)) = e_0 - b$, using Lemma 2.4 (ii) we get $HP_{I,M}(z) = \frac{b + \sum_{i=1}^{e_0 - b} z^i}{1-z}$. By virtue of the equivalence of (ii) and (iii), it follows that $\left(\frac{e_0 - b + 1}{2}\right) = e_1$. Therefore

$$\operatorname{reg}(G_I(M)) = e - b = e - b + \left(\frac{e_0 - b + 1}{2}\right) - e_1 = \left(\frac{e_0 - b + 2}{2}\right) - e_1 - 1.$$

The following example shows that the assumption $G_I(M)$ being a Cohen-Macaulay module in (iv) of the above theorem is essential.

Example 3.4. Let $A = k[[t^3, t^4, t^5]] \cong k[[x, y, z]]/(x^3 - yz, xz - y^2, x^2 y - z^2)$, where $k$ is a field. Let $I = (t^3, t^4)$. We have $b = 1$. Using CoCoA package [3], we can compute $HP_{I,A}(z) = \frac{2 + z^2}{1-z}$. Hence $e_0 = 3$, $e_1 = 2$. By Lemma 2.1, $p(G_I(A)) = 1$. Using also Lemma 3.1, we then get $\operatorname{reg}(G_I(A)) \geq 2$. By Proposition 2.9, $\operatorname{reg}(G_I(A)) \leq e_0 - b = 2$, which yields $\operatorname{reg}(G_I(A)) = e_0 - b = 2$. However $2 = e_1 \neq \left(\frac{e_0 - b + 1}{2}\right) = 3$. Note that, by Theorem 3.3, $G_I(A)$ in this example cannot be a Cohen-Macaulay ring.

Theorem 3.5. Let $M$ be an one-dimensional module and $\operatorname{depth}(M) = 0$. Let $b$ be a positive integer such that $IM \subseteq m^b M$. Then the following conditions are equivalent:
(i) \( \text{reg}(G_I(M)) = \left(\frac{e_0 - b + 2}{2}\right) - e_1 - 1; \)

(ii) \( HP_{I,M}(z) = \frac{b + \sum_{i=1}^{e_0 - b + 1} z^i - \left(\frac{e_0 - b + 2}{2}\right) - e_1}{1 - z}. \)

Moreover, if one of the above condition holds then \( b = \max\{t \mid IM \subseteq m^t M\} \).

Proof. (i) \( \implies \) (ii): For simplicity we set \( \overline{M} := M/L, \overline{e}_0 := e_0(I, \overline{M}) = e_0 \) and \( \overline{e}_1 := e_1(I, \overline{M}), \) where \( L := H^0_m(M). \) Analyzing the proof of Theorem 2.11 we see that the condition (i) implies

\[
\text{reg}(G_I(\overline{M})) = e_0 - b,
\]

\[
\overline{e}_1 = \left(\frac{e_0 - b + 1}{2}\right),
\]

and

\[
\text{reg}(G_I(M)) = \text{reg}(G_I(\overline{M})) + \ell(L).
\]

By [14, Proposition 2.3] and (8) we get

\[
\ell(L) = \overline{e}_1 - e_1 = \left(\frac{e_0 - b + 1}{2}\right) - e_1.
\]

Using Lemma 3.2, the equality (9) implies

\[
HP_K(z) = \sum_{\text{reg}(G_I(\overline{M})) + 1}^{\text{reg}(G_I(M)) + \ell(L)} z^i.
\]

where \( K = \bigoplus_{n \geq 0} \frac{I^{n+1}M + L \cap I^n M}{I^{n+1}M}. \) Combining (7), (8) and (11) we get

\[
HP_K(z) = \sum_{e_0 - b + 1}^{(e_0 - b + 2) - e_1 - 1} z^i.
\]

Using Lemma 2.5(ii) and (7) we have

\[
HP_{I,M}(z) = \frac{b + \sum_{i=1}^{e_0 - b} z^i}{1 - z}. \]

Hence, using the short exact sequence (6) we conclude that

\[
HP_{I,M}(z) = HP_{I,M} + HP_K(z)
\]

\[
= \frac{b + \sum_{i=1}^{e_0 - b} z^i}{1 - z} + \sum_{e_0 - b + 1}^{(e_0 - b + 2) - e_1 - 1} z^i
\]

\[
= \frac{b + \sum_{i=1}^{e_0 - b + 1} z^i - z^{(e_0 - b + 2) - e_1}}{1 - z}.
\]

(ii) \( \implies \) (i): By Lemma 2.5 (i), \( (e_0 - b + 2) - e_1 > e_0 - b + 1. \) Hence, by Lemma 2.1 \( p(G_I(M)) = \left(\frac{e_0 - b + 2}{2}\right) - e_1 - 1. \) By Remark 2.2 (i), we obtain

\[
\left(\frac{e_0 - b + 2}{2}\right) - e_1 - 1 \leq \text{reg}(G_I(M)).
\]

Combining with Theorem 2.11 we then get \( \text{reg}(G_I(M)) = \left(\frac{e_0 - b + 2}{2}\right) - e_1 - 1. \)
For the last statement we see that in this case the equality (8) holds. By Remark 2.6 (i), $IM \nsubseteq m^{b+1}M$. This implies $IM \nsubseteq m^{b+1}M$ and $b = \max\{ t \mid IM \subseteq m^t M\}$. □

There are many examples of one-dimension Cohen-Macaulay rings $(A, m)$ and $m$-primary ideals such that $e_1 = \binom{e_0 - b + 1}{2}$. Hence, by Theorem 3.3, the upper bound $\binom{e_0 - b + 2}{2} - e_1 - 1$ is sharp. The following example shows that this bound is also attained in the non-Cohen-Macaulay case.

**Example 3.6.** Let $A = k[[x, y]]/(x^s y^{u+v}, x^{s+1} y^{v})$, where $s, u, v \in \mathbb{N}$ and $v > 0$. Then $G_m(A) \cong k[x, y]/(x^s y^{u+v}, x^{s+1} y^{v})$ and $b = 1$. It is easy to see that

$$HP_{m,A}(z) = \sum_{i=0}^{s+u} z^i - z^{s+u+v},$$

$e_0 = s + u, e_1 = \frac{(s + u)(s + u - 1)}{2} - v$, and reg($G_m(A)$) = $s + u + v - 1$. These equalities show that all conditions in Theorem 3.5 hold.

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