ON THE IMPROVED INTERIOR REGULARITY OF A BOUNDARY VALUE PROBLEM MODELLING THE DISPLACEMENT OF A LINEARLY ELASTIC ELLIPTIC MEMBRANE SHELL SUBJECT TO AN OBSTACLE

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(Communicated by Irena Lasiecka)

Abstract. In this paper we show that the solution of an obstacle problem for linearly elastic elliptic membrane shells enjoys higher differentiability properties in the interior of the domain where it is defined.

1. Introduction. In this paper we study the regularity of the solution of an obstacle problem for linearly elastic elliptic membrane shells, which was obtained as a result of a rigorous asymptotic analysis in the papers [13, 14]. Since the solution of this second order problem is uniquely determined, the problem in object is formulated as a set of variational inequalities posed over a non-empty, closed, and convex subset of a Sobolev space.

The study of the augmentation of regularity of solutions for boundary value problems modelled via elliptic equations began between the end of the Fifties and the early Sixties, when Agmon, Douglis & Nirenberg published the two pioneering papers [1] and [2] about the regularity properties of solutions of elliptic systems near the boundary of the integration domain.

The augmentation of regularity for solutions of variational inequalities was first addressed by Frehse in the early Seventies [19, 20]. In the late Seventies and early Eighties, Caffarelli and his collaborators published the two papers [4, 5], where they proved that the solution of an obstacle problem for the biharmonic operator (cf., e.g., Section 6.7 of [9]) could not be too regular. It was recently established in [24] that the solution of an obstacle problem for linearly elastic shallow shells enjoys higher regularity properties in the interior of the domain where it is defined.

To our best knowledge, there is no record in the literature treating the augmentation of regularity of the solution of second order variational inequalities in the

2020 Mathematics Subject Classification. Primary: 74B05, 74G40; Secondary: 39A06.

Key words and phrases. Elliptic membrane shell, obstacle problems, elliptic variational inequalities, improved regularity, Second order problems.

The author is greatly indebted to Professor Philippe G. Ciarlet for his encouragement and guidance.

The author would like to express his sincere gratitude to the Anonymous Referee for the proposed suggestions and improvements.

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case where one such solution is a vector field and the constraint defining the non-empty, closed, and convex subset of the Sobolev space where the solution is sought is expressed in terms of all of the three components of the displacement vector field. The purpose of this paper is exactly to remedy this situation.

This paper is divided into five sections (including this one). In section 2 we present some background and notation. In section 3 we recall the formulation and the properties of a three-dimensional obstacle problem for a “general” linearly elastic shell. It is worth mentioning that this three-dimensional problem is the starting point for deriving the variational formulation of the two-dimensional problem, whose solution regularity is the object of interest of this paper. In section 4 we rigorously state the two-dimensional problem we are interested in studying. Finally, in section 5, we show that the solution of the problem announced in section 4 enjoys higher differentiability properties in the interior of the domain where it is defined.

We will see that, differently from [19] and [24], the fact that the constraint is defined in terms of all of the three components of the displacement vector field introduces new challenges for defining a suitable test function; a test function that will be used to recover uniform estimates for the first order finite difference quotient. We will see that the choice of the test function we will use to recover the desired conclusion is justified by the “density property” established in [13, 14] and also used in [15, 16] in the context of a the justification of Koiter’s model for a linearly elastic elliptic membrane shell subject to an obstacle.

2. Background and notation. For a complete overview about the classical notions of differential geometry used in this paper, see, e.g. [7] or [8].

Greek indices, except \( \varepsilon \), take their values in the set \( \{1, 2\} \), while Latin indices, except when they are used for ordering sequences, take their values in the set \( \{1, 2, 3\} \), and the summation convention with respect to repeated indices is used jointly with these two rules. As a model of the three-dimensional “physical” space \( \mathbb{R}^3 \), we take a real three-dimensional affine Euclidean space, i.e., a set in which a point \( O \in \mathbb{R}^3 \) has been chosen as the origin and with which a real three-dimensional Euclidean space, denoted \( E^3 \), is associated. We equip \( E^3 \) with an orthonormal basis consisting of three vectors \( e^i \), with components \( e^i_j = \delta^i_j \).

The definition of \( \mathbb{R}^3 \) as an affine Euclidean space means that with any point \( x \in \mathbb{R}^3 \) is associated an uniquely determined vector \( O x \in E^3 \). The origin \( O \in \mathbb{R}^3 \) and the orthonormal vectors \( e^i \in E^3 \) together constitute a Cartesian frame in \( \mathbb{R}^3 \) and the three components \( x_i \) of the vector \( O x \) over the basis formed by \( e^i \) are called the Cartesian coordinates of \( x \in \mathbb{R}^3 \), or the Cartesian components of \( O x \in E^3 \). Once a Cartesian frame has been chosen, any point \( x \in \mathbb{R}^3 \) may be thus identified with the vector \( O x = x_i e^i \in E^3 \). As a result, a set in \( \mathbb{R}^3 \) can be identified with a “physical” body in the Euclidean space \( E^3 \). The Euclidean inner product and the vector product of \( u, v \in E^3 \) are respectively denoted by \( u \cdot v \) and \( u \wedge v \); the Euclidean norm of \( u \in E^3 \) is denoted by \( |u| \). The notation \( \delta^i_j \) designates the Kronecker symbol.

Given an open subset \( \Omega \) of \( \mathbb{R}^n \), where \( n \geq 1 \), we denote the usual Lebesgue and Sobolev spaces by \( L^2(\Omega) \), \( L^1_{loc}(\Omega) \), \( H^1(\Omega) \), \( H^1_0(\Omega) \), \( H^1_{loc}(\Omega) \), and the notation \( D(\Omega) \) designates the space of all functions that are infinitely differentiable over \( \Omega \) and have compact supports in \( \Omega \). We denote \( ||| \cdot |||_X \) the norm in a normed vector space \( X \). Spaces of vector-valued functions are written in boldface. The Euclidean norm of any point \( x \in \Omega \) is denoted by \(|x|\).
The boundary $\Gamma$ of an open subset $\Omega$ in $\mathbb{R}^n$ is said to be Lipschitz-continuous if the following conditions are satisfied (cf., e.g., Section 1.18 of [9]): Given an integer $s \geq 1$, there exist constants $\alpha_1 > 0$ and $L > 0$, a finite number of local coordinate systems, with coordinates

$$\phi'_r = (\phi'_1, \ldots, \phi'_{n-1}) \in \mathbb{R}^{n-1} \quad \text{and} \quad \phi_r = \phi'_n, \ 1 \leq r \leq s,$$

sets

$$\tilde{\omega}_r := \{ \phi_r \in \mathbb{R}^{n-1}; |\phi_r| < \alpha_1 \}, \quad 1 \leq r \leq s,$$

and corresponding functions

$$\tilde{\theta}_r : \tilde{\omega}_r \rightarrow \mathbb{R}, \quad 1 \leq r \leq s,$$

such that

$$\Gamma = \bigcup_{r=1}^{s} \{ (\phi'_r, \phi_r); \phi'_r \in \tilde{\omega}_r \ \text{and} \ \phi_r = \tilde{\theta}_r(\phi'_r) \},$$

and

$$|\tilde{\theta}_r(\phi'_r) - \tilde{\theta}_r(\nu'_r)| \leq L|\phi'_r - \nu'_r|, \quad \text{for all} \ \phi'_r, \nu'_r \in \tilde{\omega}_r, \ \text{and all} \ 1 \leq r \leq s.$$

We observe that the second last formula takes into account overlapping local charts, while the last set of inequalities expresses the Lipschitz continuity of the mappings $\tilde{\theta}_r$.

An open set $\Omega$ is said to be locally on the same side of its boundary $\Gamma$ if, in addition, there exists a constant $\alpha_2 > 0$ such that

$$\{(\phi'_r, \phi_r); \phi'_r \in \tilde{\omega}_r \ \text{and} \ \tilde{\theta}_r(\phi'_r) < \phi_r < \tilde{\theta}_r(\phi'_r) + \alpha_2 \} \subset \Omega, \quad \text{for all} \ 1 \leq r \leq s,$$

$$\{(\phi'_r, \phi_r); \phi'_r \in \tilde{\omega}_r \ \text{and} \ \tilde{\theta}_r(\phi'_r) - \alpha_2 < \phi_r < \tilde{\theta}_r(\phi'_r) \} \subset \mathbb{R}^n \setminus \overline{\Omega}, \quad \text{for all} \ 1 \leq r \leq s.$$

A domain in $\mathbb{R}^n$ is a bounded and connected open subset $\Omega$ of $\mathbb{R}^n$, whose boundary $\partial \Omega$ is Lipschitz-continuous, the set $\Omega$ being locally on a single side of $\partial \Omega$.

Let $\omega$ be a domain in $\mathbb{R}^2$ with boundary $\gamma := \partial \omega$, and let $\omega_1 \subset \omega$. The special notation $\omega_1 \subset \subset \omega$ means that $\overline{\omega_1} \subset \omega$ and $\text{dist}(\gamma, \partial \omega_1) > 0$. Let $y = (y_n)$ denote a generic point in $\omega$, and let $\partial_{y} := \partial/\partial y_n$. A mapping $\theta \in C^1(\overline{\omega}; \mathbb{R}^3)$ is said to be an immersion if the two vectors

$$a_\alpha(y) := \partial_{y} \theta(y)$$

are linearly independent at each point $y \in \overline{\omega}$. Then the set $\theta(\overline{\omega})$ is a surface in $\mathbb{R}^3$, equipped with $y_1, y_2$ as its curvilinear coordinates. Given any point $y \in \overline{\omega}$, the linear combinations of the vectors $a_\alpha(y)$ span the tangent plane to the surface $\theta(\overline{\omega})$ at the point $\theta(y)$, the unit vector

$$a_3(y) := \frac{a_1(y) \wedge a_2(y)}{|a_1(y) \wedge a_2(y)|}$$

is orthogonal to $\theta(\overline{\omega})$ at the point $\theta(y)$, the three vectors $a_i(y)$ form the covariant basis at the point $\theta(y)$, and the three vectors $a^i(y)$ defined by the relations

$$a^i(y) \cdot a_j(y) = \delta^i_j$$

form the contravariant basis at $\theta(y)$; note that the vectors $a^\beta(y)$ also span the tangent plane to $\theta(\overline{\omega})$ at $\theta(y)$ and that $a^3(y) = a_3(y)$.

The first fundamental form of the surface $\theta(\overline{\omega})$ is then defined by means of its covariant components

$$a_{\alpha \beta} := a_\alpha \cdot a_\beta = a_{\beta \alpha} \in C^0(\overline{\omega}),$$
or by means of its contravariant components

\[ a^{\alpha\beta} := a^\alpha \cdot a^\beta = a^{\beta\alpha} \in C^0(\overline{\omega}). \]

Note that the symmetric matrix field \( (a^{\alpha\beta}) \) is then the inverse of the matrix field \( (a_{\alpha\beta}) \), that \( a^\beta = a^{\alpha\beta} a_\alpha \), and \( a_\alpha = a_{\alpha\beta} a^\beta \), and that the area element along \( \theta(\overline{\omega}) \) is given at each point \( \theta(y), y \in \overline{\omega} \), by \( \sqrt{a(y)} \, dy \), where

\[ a := \det(a_{\alpha\beta}) \in C^0(\overline{\omega}). \]

Given an immersion \( \theta \in C^2(\overline{\omega}; \mathbb{E}^3) \), the second fundamental form of the surface \( \theta(\overline{\omega}) \) is defined by means of its covariant components

\[ b_{\alpha\beta} := \partial_\alpha a_\beta \cdot a_3 = -a_\beta \cdot \partial_\alpha a_3 = b_{3\alpha} \in C^0(\overline{\omega}), \]

or by means of its mixed components

\[ b_\alpha^\beta := a^{\beta\sigma} b_{\alpha\sigma} \in C^0(\overline{\omega}), \]

and the Christoffel symbols associated with the immersion \( \theta \) are defined by

\[ \Gamma^\beta_{\alpha\gamma} := \partial_\alpha a_\beta \cdot a^\gamma = \Gamma^\beta_{\gamma\alpha} \in C^0(\overline{\omega}). \]

The Gaussian curvature at each point \( \theta(y), y \in \overline{\omega}, \) of the surface \( \theta(\overline{\omega}) \) is defined by

\[ \kappa(y) := \frac{\det(b_{\alpha\beta}(y))}{\det(a_{\alpha\beta}(y))} = \det\left(\frac{b_\alpha^\beta(y)}{a^{\alpha\beta}(y)}\right). \]

Observe that the denominator in the above relation does not vanish since \( \theta \) is assumed to be an immersion. Note that the Gaussian curvature \( \kappa(y) \) at the point \( \theta(y) \) is also equal to the product of the two principal curvatures at this point.

Given an immersion \( \theta \in C^2(\overline{\omega}; \mathbb{E}^3) \) and a vector field \( \eta = (\eta_i) \in C^1(\overline{\omega}; \mathbb{R}^3) \), the vector field

\[ \tilde{\eta} := \eta_i a^i \]

may be viewed as the displacement field of the surface \( \theta(\overline{\omega}) \), thus defined by means of its covariant components \( \eta_i \) over the vectors \( a^i \) of the contravariant bases along the surface. If the norms \( \|\eta_i\|_{C^1(\overline{\omega})} \) are small enough, the mapping \( (\theta + \eta_i a^i) \in C^1(\overline{\omega}; \mathbb{E}^3) \) is also an immersion, so that the set \( (\theta + \eta_i a^i)(\overline{\omega}) \) is again a surface in \( \mathbb{E}^3 \), equipped with the same curvilinear coordinates as those of the surface \( \theta(\overline{\omega}) \) and is called the deformed surface corresponding to the displacement field \( \tilde{\eta} = \eta_i a^i \).

It is thus possible to define the first fundamental form of the deformed surface in terms of its covariant components by

\[ a_{\alpha\beta}(\eta) := (a_\alpha + \partial_\alpha \tilde{\eta}) \cdot (a_\beta + \partial_\beta \tilde{\eta}) = a_{\alpha\beta} + a_\alpha \cdot \partial_\beta \tilde{\eta} + \partial_\alpha \tilde{\eta} \cdot a_\beta + \partial_\alpha \tilde{\eta} \cdot \partial_\beta \tilde{\eta}. \]

The linear part with respect to \( \tilde{\eta} \) in the difference \( (a_{\alpha\beta}(\eta) - a_{\alpha\beta})/2 \) is called the linearised change of metric, or strain, tensor associated with the displacement field \( \eta_i a^i \); the covariant components of which are thus defined by

\[ \gamma_{\alpha\beta}(\eta) := \frac{1}{2} (a_\alpha \cdot \partial_\beta \tilde{\eta} + \partial_\alpha \tilde{\eta} \cdot a_\beta) = \frac{1}{2} (\partial_\beta \eta_\alpha + \partial_\alpha \eta_\beta) - \Gamma^\gamma_{\alpha\beta} \eta_\gamma - b_{\alpha\beta} \eta_3 = \gamma_{\beta\alpha}(\eta). \]

In this paper, we shall consider a specific class of surfaces, according to the following definition: Let \( \omega \) be a domain in \( \mathbb{R}^2 \). Then a surface \( \theta(\overline{\omega}) \) defined by means of an immersion \( \theta \in C^2(\overline{\omega}; \mathbb{E}^3) \) is said to be elliptic if its Gaussian curvature is everywhere \( > 0 \) in \( \overline{\omega}, \) or equivalently, if there exists a constant \( \kappa_0 \) such that

\[ 0 < \kappa_0 \leq \kappa(y), \] for all \( y \in \overline{\omega}. \]
It turns out that, when an elliptic surface is subjected to a displacement field \( \eta_i a^i \) whose tangential covariant components \( \eta_\alpha \) vanish on the entire boundary of the domain \( \omega \), the following inequality holds. Note that the components of the displacement fields and linearised change of metric tensors appearing in the next theorem are no longer assumed to be continuously differentiable functions; they are instead to be understood in a generalised sense, since they now belong to \textit{ad hoc} Lebesgue or Sobolev spaces.

**Theorem 1.** Let \( \omega \) be a domain in \( \mathbb{R}^2 \) and let an immersion \( \theta \in C^3(\overline{\omega}; \mathbb{E}^3) \) be given such that the surface \( \theta(\overline{\omega}) \) is elliptic. Define the space

\[
V_M(\omega) := H^1_0(\omega) \times H^1_0(\omega) \times L^2(\omega).
\]

Then, there exists a constant \( c_0 = c_0(\omega, \theta) > 0 \) such that

\[
\left\{ \sum_\alpha \| \eta_\alpha \|^2_{{H^1(\omega)}} + \| \eta \|^2_{{L^2(\omega)}} \right\}^{1/2} \leq c_0 \left\{ \sum_{\alpha,\beta} \| \gamma_{\alpha\beta}(\eta) \|^2_{{L^2(\omega)}} \right\}^{1/2}
\]

for all \( \eta = (\eta_i) \in V_M(\omega). \)

The above inequality, which is due to \cite{12} and \cite{17} (see also Theorem 2.7-3 of \cite{7}), constitutes an example of a Korn’s inequality on a surface, in the sense that it provides an estimate of an appropriate norm of a displacement field defined on a surface in terms of an appropriate norm of a specific “measure of strain” (here, the linearised change of metric tensor) corresponding to the displacement field considered.

3. The three-dimensional obstacle problem for a “general” linearly elastic shell. Let \( \omega \) be a domain in \( \mathbb{R}^2 \), let \( \gamma := \partial \omega \), and let \( \gamma_0 \) be a non-empty relatively open subset of \( \gamma \). For each \( \varepsilon > 0 \), we define the sets

\[
\Omega^\varepsilon = \omega \times ]-\varepsilon, \varepsilon[, \text{ and } \Gamma_0^\varepsilon := \gamma_0 \times ]-\varepsilon, \varepsilon[, \]

we let \( x^\varepsilon = (x_1^\varepsilon) \) designate a generic point in the set \( \overline{\Omega^\varepsilon} \), and we let \( \partial_\alpha^\varepsilon := \partial/\partial x_\alpha^\varepsilon \).

Hence we also have \( x_\alpha^\varepsilon = y_\alpha \) and \( \partial_\alpha^\varepsilon = \partial_\alpha \).

Given an immersion \( \theta \in C^3(\overline{\omega}; \mathbb{E}^3) \) and \( \varepsilon > 0 \), consider a \textit{shell with middle surface} \( \theta(\overline{\omega}) \) and with \textit{constant thickness} \( 2\varepsilon \). This means that the \textit{reference configuration} of the shell is the set \( \Theta(\overline{\Omega})(\overline{\Omega}) \), where the mapping \( \Theta : \overline{\Omega} \to \mathbb{E}^3 \) is defined by

\[
\Theta(x^\varepsilon) := \theta(y) + x_3^\varepsilon a^3(y) \text{ at each point } x^\varepsilon = (y, x_3^\varepsilon) \in \overline{\Omega^\varepsilon}.
\]

One can then show (cf., e.g., Theorem 3.1-1 of \cite{7}) that, if \( \varepsilon > 0 \) is small enough, such a mapping \( \Theta \in C^2(\overline{\Omega}; \mathbb{E}^3) \) is an \textit{immersion}, in the sense that the three vectors \( g_i^\varepsilon(x^\varepsilon) := \partial_i^\varepsilon \Theta(x^\varepsilon) \)

are linearly independent at each point \( x^\varepsilon \in \overline{\Omega^\varepsilon} \); these vectors then constitute the \textit{covariant basis} at the point \( \Theta(x^\varepsilon) \), while the three vectors \( g^j(x^\varepsilon) \) defined by the relations

\[
g^{j\varepsilon}(x^\varepsilon) = \delta^j_i
\]

constitute the \textit{contravariant basis} at the same point. It will be implicitly assumed in the sequel that \( \varepsilon > 0 \) is small enough so that \( \Theta : \overline{\Omega} \to \mathbb{E}^3 \) is an immersion.

One then defines the \textit{metric tensor associated with the immersion} \( \Theta \) by means of its \textit{covariant components}

\[
g_{ij}^\varepsilon := g_i^\varepsilon \cdot g_j^\varepsilon \in C^1(\overline{\Omega^\varepsilon}),
\]
or by means of its contravariant components

\[ g^{ij,\varepsilon} := g^{i\varepsilon} \cdot g^{j\varepsilon} \in C^1(\Omega^\varepsilon). \]

Note that the symmetric matrix field \( g^{ij,\varepsilon} \) is then the inverse of the matrix field \( (g_{ij}^\varepsilon) \), that \( g^{ij,\varepsilon} = g^{ij,\varepsilon} g_{ij}^\varepsilon \) and \( g_{ij}^\varepsilon = g_{ij,\varepsilon} g^{\varepsilon,ij} \), and that the volume element in \( \Theta(\Omega^\varepsilon) \) is given at each point \( \Theta(x^\varepsilon), x^\varepsilon \in \Omega^\varepsilon \), by \( \sqrt{g^\varepsilon(x^\varepsilon)} \, dx^\varepsilon \), where

\[ g^\varepsilon := \text{det}(g_{ij}^\varepsilon) \in C^1(\Omega^\varepsilon). \]

One also defines the Christoffel symbols associated with the immersion \( \Theta \) by

\[ \Gamma_{ij}^{p,\varepsilon} := \partial_i g_j^p \cdot g^{p,\varepsilon} = \Gamma_{j}^{p,\varepsilon} \in C^0(\Omega^\varepsilon). \]

Note that \( \Gamma_{00}^{3,\varepsilon} = \Gamma_{33}^{3,\varepsilon} = 0 \).

Given a vector field \( v^\varepsilon = (v_i^\varepsilon) \in C^1(\Omega^\varepsilon; \mathbb{R}^3) \), the associated vector field

\[ \tilde{v}^\varepsilon := v_i^\varepsilon g^{i\varepsilon} \]

can be viewed as a displacement field of the reference configuration \( \Theta(\Omega^\varepsilon) \) of the shell, thus defined by means of its covariant components \( v_i^\varepsilon \) over the vectors \( g^{i\varepsilon} \) of the contravariant bases in the reference configuration.

If the norms \( ||v_i^\varepsilon||_{C^1(\Omega^\varepsilon)} \) are small enough, the mapping \( \Theta + v_i^\varepsilon g^{i\varepsilon} \) is also an immersion, so that one can also define the metric tensor of the deformed configuration \( \Theta + v_i^\varepsilon g^{i\varepsilon}(\Omega^\varepsilon) \) by means of its covariant components

\[ g_{ij}^\varepsilon(v^\varepsilon) := (g_{ij}^\varepsilon + \partial_i^\varepsilon \tilde{v}^\varepsilon \cdot (g_{ij}^\varepsilon + \partial_j^\varepsilon \tilde{v}^\varepsilon)) = g_{ij}^\varepsilon + g_{ij}^\varepsilon \cdot \partial_i^\varepsilon \tilde{v}^\varepsilon + \partial_j^\varepsilon \tilde{v}^\varepsilon \cdot g_{ij}^\varepsilon + \partial_i^\varepsilon \tilde{v}^\varepsilon \cdot \partial_j^\varepsilon \tilde{v}^\varepsilon. \]

The linear part with respect to \( \tilde{v}^\varepsilon \) in the difference \((g_{ij}^\varepsilon(v^\varepsilon) - g_{ij}^\varepsilon)/2 \) is then called the linearised strain tensor associated with the displacement field \( v_i^\varepsilon g^{i\varepsilon} \), the covariant components of which are thus defined by

\[ e^\varepsilon_{ij}(v^\varepsilon) := \frac{1}{2} (g_{ij}^\varepsilon \cdot \partial_i^\varepsilon \tilde{v}^\varepsilon + \partial_j^\varepsilon \tilde{v}^\varepsilon \cdot g_{ij}^\varepsilon) = \frac{1}{2} (\partial_j^\varepsilon v_i^\varepsilon + \partial_i^\varepsilon v_j^\varepsilon) = \Gamma_{ij}^{\varepsilon,\varepsilon} v_i^\varepsilon - e^\varepsilon_{ij}(v^\varepsilon). \]

The functions \( e^\varepsilon_{ij}(v^\varepsilon) \) are called the linearised strains in curvilinear coordinates associated with the displacement field \( v_i^\varepsilon g^{i\varepsilon} \).

We assume throughout this paper that, for each \( \varepsilon > 0 \), the reference configuration \( \Theta(\Omega^\varepsilon) \) of the shell is a natural state (i.e., stress-free) and that the material constituting the shell is homogeneous, isotropic, and linearly elastic. The behaviour of such an elastic material is thus entirely governed by its two Lamé constants \( \lambda \geq 0 \) and \( \mu > 0 \) (for details, see, e.g., Section 3.8 of [6]).

We will also assume that the shell is subjected to applied body forces whose density per unit volume is defined by means of its covariant components \( f_{i,\varepsilon} \in L^2(\Omega^\varepsilon) \), and to a homogeneous boundary condition of place along the portion \( \Gamma_0^\varepsilon \) of its lateral face (i.e., the displacement vanishes on \( \Gamma_0^\varepsilon \)).

In this paper, we consider a specific obstacle problem for such a shell, in the sense that the shell is also subjected to a confinement condition, expressing that any admissible deformed configuration remains in a half-space of the form

\[ \mathbb{H} := \{ O \Omega \in \mathbb{R}^3; \, O \Omega \cdot q \geq 0 \}, \]

where \( q \in \mathbb{R}^3 \) is a non-zero vector given once and for all. In other words, any admissible displacement field must satisfy

\[ (\Theta(x^\varepsilon) + v_i^\varepsilon(x^\varepsilon) g^{i\varepsilon}(x^\varepsilon)) \cdot q \geq 0 \]
for all \( x^\epsilon \in \overline{\Omega^\epsilon} \), or possibly only for almost all (a.a. in what follows) \( x^\epsilon \in \Omega^\epsilon \) when the covariant components \( v_i^\epsilon \) are required to belong to the Sobolev space \( H^1(\Omega^\epsilon) \) as in Theorem 2 below.

We will of course assume that the reference configuration satisfies the confinement condition, i.e., that

\[
\Theta(\overline{\Omega^\epsilon}) \subset \mathbb{H}.
\]

It is to be emphasised that the above confinement condition considerably departs from the usual Signorini condition favoured by most authors, who usually require that only the points of the undeformed and deformed “lower face” \( \omega \times \{-\epsilon\} \) of the reference configuration satisfy the confinement condition (see, e.g., [21], [22], [23], [27]). Clearly, the confinement condition considered in the present paper is more physically realistic, since a Signorini condition imposed only on the lower face of the reference configuration does not prevent – at least “mathematically” – other points of the deformed reference configuration to “cross” the plane \( \{Ox \in \mathbb{E}^3: Ox \cdot q = 0\} \) and then to end up on the “other side” of this plane. The mathematical models characterised by the confinement condition introduced beforehand, confinement condition which is also considered in the seminal paper [21] in a different geometrical framework, do not take any traction forces into account. Indeed, there could be no traction forces applied to the portion of the three-dimensional shell boundary that engages contact with the obstacle. Besides, friction is not considered in the context of this analysis.

Unlike the classical Signorini condition, the confinement condition here considered is more suitable in the context of multi-scales multi-bodies problems like, for instance, the study of the motion of the human heart valves, conducted by Quar-teroni and his associates in [25, 26, 28] and the references therein.

Such a confinement condition renders the study of this problem considerably more difficult, however, as the constraint now bears on a vector field, the displacement vector field of the reference configuration, instead of on only a single component of this field.

The mathematical modelling of such an obstacle problem for a linearly elastic shell is then clear; since, apart from the confinement condition, the rest, i.e., the function space and the expression of the quadratic energy \( J^\epsilon \), is classical (see, e.g. [7]). More specifically, let

\[
A^{ijkt,\epsilon} := \lambda g^{ij,\epsilon} g^{kt,\epsilon} + \mu (g^{ik,\epsilon} g^{jt,\epsilon} + g^{it,\epsilon} g^{jk,\epsilon}) = A^{ijkl,\epsilon} = A^{klji,\epsilon}
\]

denote the contravariant components of the elasticity tensor of the elastic material constituting the shell. Then the unknown of the problem, which is the vector field \( u^\epsilon = (u_i^\epsilon) \) where the functions \( u_i^\epsilon : \overline{\Omega^\epsilon} \rightarrow \mathbb{R} \) are the three covariant components of the unknown “three-dimensional” displacement vector field \( u_i^\epsilon g^{j,\epsilon} \) of the reference configuration of the shell, should minimise the energy \( J^\epsilon : H^1(\Omega^\epsilon) \rightarrow \mathbb{R} \) defined by

\[
J^\epsilon(v^\epsilon) := \frac{1}{2} \int_{\Omega^\epsilon} A^{ijkl,\epsilon} e_i^\epsilon e_j^\epsilon (v^\epsilon) e_k^\epsilon e_l^\epsilon (v^\epsilon) \sqrt{g^\epsilon} \, dx^\epsilon - \int_{\Omega^\epsilon} f^i,\epsilon v_i^\epsilon \sqrt{g^\epsilon} \, dx^\epsilon
\]

for each \( v^\epsilon = (v_i^\epsilon) \in H^1(\Omega^\epsilon) \) over the set of admissible displacements defined by:

\[
U(\Omega^\epsilon) := \{v^\epsilon = (v_i^\epsilon) \in H^1(\Omega^\epsilon); \; v^\epsilon = 0 \text{ on } \Gamma_0^\epsilon, \quad (\Theta(x^\epsilon) + v_i^\epsilon(x^\epsilon) g^{j,\epsilon}(x^\epsilon)) \cdot q \geq 0 \text{ for a.a. } x^\epsilon \in \Omega^\epsilon \}.
\]
The solution to this minimisation problem exists and is unique, and it can be also characterised as the unique solution of a set of appropriate variational inequalities (cf., Theorem 2.1 of [14]).

**Theorem 2.** The quadratic minimisation problem: Find a vector field $u^\varepsilon \in U(\Omega^\varepsilon)$ such that
\[
J^\varepsilon(u^\varepsilon) = \inf_{v^\varepsilon \in U(\Omega^\varepsilon)} J^\varepsilon(v^\varepsilon)
\]
has one and only one solution. Besides, the vector field $u^\varepsilon$ is also the unique solution of the variational problem $P(\Omega^\varepsilon)$: Find $u^\varepsilon \in U(\Omega^\varepsilon)$ that satisfies the following variational inequalities:
\[
\int_{\Omega^\varepsilon} A^{ijkl\varepsilon} e_{kjl\varepsilon}(u^\varepsilon) \left( e_{i\varepsilon}^l(u^\varepsilon) - e_{i\varepsilon}^l(u^\varepsilon) \right) \sqrt{g^\varepsilon} \, dx^\varepsilon \geq \int_{\Omega^\varepsilon} f^{i\varepsilon}(v^\varepsilon_i - u^\varepsilon_i) \sqrt{g^\varepsilon} \, dx^\varepsilon
\]
for all $v^\varepsilon = (v^\varepsilon_i) \in U(\Omega^\varepsilon)$. 

Since $\theta(\omega) \subset \Theta(\Omega^\varepsilon)$, it evidently follows that $\theta(y) \cdot q \geq 0$ for all $y \in \omega$. But in fact, a stronger property holds (cf., Lemma 2.1 of [14]):

**Lemma 1.** Let $\omega$ be a domain in $\mathbb{R}^2$, let $\theta \in C^1(\omega; \mathbb{E}^3)$ be an immersion, let $q \in \mathbb{E}^3$ be a non-zero vector, and let $\varepsilon > 0$. Then the inclusion
\[
\Theta(\Omega^\varepsilon) \subset \mathbb{H} = \{ x \in \mathbb{E}^3; \, Ox \cdot q \geq 0 \}
\]
implies that
\[
\min_{y \in \omega} (\theta(y) \cdot q) > 0.
\]

Clearly, the assumed inclusion $\Theta(\Omega^\varepsilon) \subset \mathbb{H}$ implies that $\min_{y \in \omega} (\theta(y) \cdot q)$ depends on $\varepsilon > 0$. In order to identify the two-dimensional problem that constitutes the object of interest of this paper via a rigorous asymptotic analysis, we need to assume that $\min_{y \in \omega} (\theta(y) \cdot q)$ is strictly positive and independent of $\varepsilon > 0$.

### 4. The scaled three-dimensional problem for a family of linearly elastic elliptic membrane shells.

In section 3, we considered an obstacle problem for “general” linearly elastic shells. From now on, we will restrict ourselves to a specific class of shells, according to the following definition (proposed in [11]; see also [7]).

Consider a linearly elastic shell, subjected to the various assumptions set forth in section 3. Such a shell is said to be a linearly elastic elliptic membrane shell (from now on simply membrane shell) if the following two additional assumptions are satisfied: first, $\gamma_0 = \gamma$, i.e., the homogeneous boundary condition of place is imposed over the entire lateral face $\gamma \times [-\varepsilon, \varepsilon]$ of the shell, and second, its middle surface $\theta(\omega)$ is elliptic, according to the definition given in section 2.

In this paper, we consider the obstacle problem (as defined in section 3) for a family of membrane shells, all sharing the same middle surface and whose thickness $2\varepsilon > 0$ is considered as a “small” parameter approaching zero. In order to conduct an asymptotic analysis on the three-dimensional model as the thickness $\varepsilon \to 0$, we resorted to a (by now standard) methodology first proposed in [10]: To begin with, we “scale” each problem $P(\Omega^\varepsilon), \varepsilon > 0$, over a fixed domain $\Omega$, using appropriate scalings on the unknowns and assumptions on the data.

More specifically, let
\[
\Omega := \omega \times [-1, 1],
\]
let \( x = (x_i) \) denote a generic point in the set \( \overline{\Omega} \), and let \( \partial_i := \partial / \partial x_i \). With each point \( x = (x_i) \in \overline{\Omega} \), we associate the point \( x^\varepsilon = (x^\varepsilon_i) \) defined by
\[
x^\varepsilon_i := x_\alpha = y_\alpha \quad \text{and} \quad x^\varepsilon_i := \varepsilon x_3,
\]
so that \( \partial^\varepsilon_i = \partial_\alpha \) and \( \partial^\varepsilon_i = \varepsilon^{-1} \partial_3 \). To the unknown \( u^\varepsilon = (u^\varepsilon_i) \) and to the vector fields \( v^\varepsilon = (v^\varepsilon_i) \) appearing in the formulation of the problem \( P(\Omega^\varepsilon) \) corresponding to a membrane shell, we then associate the scaled unknown \( u(\varepsilon) = (u_i(\varepsilon)) \) and the scaled vector fields \( v = (v_i) \) by letting
\[
u_i(x) := u^\varepsilon_i(x^\varepsilon) \quad \text{and} \quad v_i(x) := v^\varepsilon_i(x^\varepsilon)
\]
at each \( x \in \overline{\Omega} \). Finally, we assume that there exist functions \( f^i \in L^2(\Omega) \) independent on \( \varepsilon \) such that the following assumptions on the data hold
\[
f^{i,\varepsilon}(x^\varepsilon) = f^i(x) \quad \text{at each} \quad x \in \Omega.
\]

Note that the independence on \( \varepsilon \) of the Lamé constants assumed in section 3 in the formulation of problem \( P(\Omega^\varepsilon) \) implicitly constituted another assumption on the data.

The variational problem \( P(\varepsilon; \Omega) \) defined in the next theorem will constitute the point of departure of the asymptotic analysis performed in [14].

**Theorem 3.** For each \( \varepsilon > 0 \), define the set
\[
U(\varepsilon; \Omega) := \{ v = (v_i) \in H^1(\Omega); \ v = 0 \ \text{on} \ \gamma \times ]-1, 1[ \}, \quad
\]
\[
(\theta(y) + \varepsilon x_3 a_3(y) + v_i(x)g^i(\varepsilon)(x)) \cdot q \geq 0 \ \text{for a.a.} \ x = (y, x_3) \in \Omega,
\]
where
\[
g^i(\varepsilon)(x) := g^{i,\varepsilon}(x^\varepsilon) \quad \text{at each} \quad x \in \Omega.
\]

Then the scaled unknown of the variational problem \( P(\Omega^\varepsilon) \) is the unique solution of the variational problem \( P(\varepsilon; \Omega) \): Find \( u(\varepsilon) \in U(\varepsilon; \Omega) \) that satisfies the following variational inequalities:
\[
\int_\Omega A^{ijkl}(\varepsilon)e_{kll}(\varepsilon; u(\varepsilon)) \left( e_{ij\|i}(\varepsilon; v) - e_{ij\|j}(\varepsilon; u(\varepsilon)) \right) \sqrt{g(\varepsilon)} \, dx
\]
\[
\geq \int_\Omega f^i(v_i - u_i(\varepsilon)) \sqrt{g(\varepsilon)} \, dx \quad \text{for all} \quad v \in U(\varepsilon; \Omega),
\]
where
\[
g(\varepsilon)(x) := g^\varepsilon(x^\varepsilon) \quad \text{and} \quad A^{ijkl}(\varepsilon)(x) := A^{ijkl,\varepsilon}(x^\varepsilon) \quad \text{at each} \quad x \in \overline{\Omega},
\]
\[
e_{\alpha\|\beta}(\varepsilon; v) := \frac{1}{2} (\partial_\beta v_\alpha + \partial_\alpha v_\beta) - \Gamma^k_{\alpha\beta}(\varepsilon)v_k = e_{\beta\|\alpha}(\varepsilon; v),
\]
\[
e_{3\|\alpha}(\varepsilon; v) := \frac{1}{2} \left( \varepsilon^{-1} \partial_3 v_\alpha + \partial_\alpha v_3 \right) - \Gamma^\sigma_{\alpha3}(\varepsilon)v_\sigma = e_{\alpha\|3}(\varepsilon; v),
\]
\[
e_{3\|3}(\varepsilon; v) := \varepsilon \partial_3 v_3,
\]
where
\[
\Gamma^p_{ij}(\varepsilon)(x) := \Gamma^{p,\varepsilon}_{ij}(x^\varepsilon) \quad \text{at each} \quad x \in \overline{\Omega}.
\]

The problem we are interested in is derived as a result of the rigorous asymptotic analysis conducted in Theorem 4.1 of [14].
Theorem 4. Let \( \omega \) be a domain in \( \mathbb{R}^2 \), let \( \mathbf{\theta} \in C^3(\mathbb{R}; \mathbb{R}^3) \) be an immersion such that the surface \( \mathbf{\theta}(\mathbb{R}) \) is elliptic (cf. section 2), and let \( q \in \mathbb{E}^3 \) be a non-zero vector. Define the space and sets

\[
\begin{align*}
V_M(\omega) & := H_0^1(\omega) \times H_0^1(\omega) \times L^2(\omega), \\
U_M(\omega) & := \{ \eta = (\eta_i) \in H_0^1(\omega) \times H_0^1(\omega) \times L^2(\omega); \ \ (\mathbf{\theta}(y) + \eta_i(y)\mathbf{a}(y)) \cdot q \geq 0 \text{ for a.a. } y \in \omega \}, \\
\tilde{U}_M(\omega) & := \{ \eta = (\eta_i) \in H_0^1(\omega) \times H_0^1(\omega) \times H_0^1(\omega); \ \ (\mathbf{\theta}(y) + \eta_i(y)\mathbf{a}(y)) \cdot q \geq 0 \text{ for a.a. } y \in \omega \},
\end{align*}
\]

and assume that the immersion \( \mathbf{\theta} \) is such that

\[
\tilde{d} := \min_{y \in \mathbb{R}} (\mathbf{\theta}(y) \cdot q) > 0,
\]

is independent of \( \varepsilon \), and assume that the following “density property” holds:

\( \tilde{U}_M(\omega) \) is dense in \( U_M(\omega) \) with respect to the norm of \( \| \cdot \|_{H^1(\omega) \times H^1(\omega) \times L^2(\omega)} \).

Let there be given a family of membrane shells with the same middle surface \( \mathbf{\theta}(\mathbb{R}) \) and thickness \( 2\varepsilon > 0 \), and let

\[
\begin{align*}
u(\varepsilon) = (u_i(\varepsilon)) \in U(\varepsilon; \Omega) := \{ \mathbf{v} = (u_i) \in H^1(\Omega); & \ \mathbf{v} = 0 \text{ on } \gamma \times ]-1,1[, \\
(\mathbf{\theta}(y) + \varepsilon x_3\mathbf{a}(y) + u_i(x)\mathbf{g}(\varepsilon)(x)) \cdot q & \geq 0 \text{ for a.a. } x = (y, x_3) \in \Omega \}
\end{align*}
\]

denote for each \( \varepsilon > 0 \) the unique solution of the corresponding problem \( \mathcal{P}(\varepsilon; \Omega) \) introduced in Theorem 3.

Then there exist functions \( u_\alpha \in H^1(\Omega) \) independent of the variable \( x_3 \) and satisfying

\[
u_\alpha = 0 \text{ on } \gamma \times ]-1,1[,
\]

and there exists a function \( u_3 \in L^2(\Omega) \) independent of the variable \( x_3 \), such that

\[
u_\alpha(\varepsilon) \to u_\alpha \text{ in } H^1(\Omega) \text{ and } u_3(\varepsilon) \to u_3 \text{ in } L^2(\Omega).
\]

Define the average

\[
\overline{\mathbf{u}} = (\overline{u}_i) := \frac{1}{2} \int_{-1}^1 \mathbf{u} \, dx_3 \in V_M(\omega).
\]

Then

\[
\overline{\mathbf{u}} = \zeta,
\]

where \( \zeta \) is the unique solution of the two-dimensional variational problem \( \mathcal{P}_M(\omega) \): Find \( \zeta \in U_M(\omega) \) that satisfies the following variational inequalities

\[
\int_\omega a^{\alpha\beta\sigma\tau} \gamma_{\alpha\beta}(\zeta) \gamma_{\sigma\tau}(\eta - \zeta) \sqrt{\mathbf{a}} \, dy \geq \int_\omega p^i(\eta_i - \zeta_i) \sqrt{\mathbf{a}} \, dy \quad \text{for all } \eta = (\eta_i) \in U_M(\omega),
\]

where

\[
a^{\alpha\beta\sigma\tau} := \frac{4\lambda \mu}{\lambda + 2\mu} a^{\alpha\beta} a^{\sigma\tau} + 2\mu (a^{\alpha\sigma} a^{\beta\tau} + a^{\alpha\tau} a^{\beta\sigma}) \quad \text{and} \quad p^i := \int_{-1}^1 f^i \, dx_3.
\]

\[\square\]
Critical to establish the convergence of the family \((u_\varepsilon)_{\varepsilon>0}\) is the “density property” assumed there, which asserts that the set \(U_M(\omega)\) is dense in the set \(U_M(\omega)\) with respect to the norm \(\| \cdot \|_{H^1(\omega)\times H^1(\omega)\times L^2(\omega)}\). The same “density property” is used to provide a justification, via a rigorous asymptotic analysis, of Koiter’s model for membrane shells subject to an obstacle (cf. [15], [16]). We hereby recall a sufficient geometric condition ensuring the assumed “density property” (cf. Theorem 5.1 of [14]).

**Theorem 5.** Let \(\theta \in C^2(\mathbb{R}; \mathbb{R}^3)\) be an immersion with the following property: There exists a non-zero vector \(q \in \mathbb{R}^3\) such that

\[
\min_{y \in \omega} (\theta(y) \cdot q) > 0 \text{ and } \min_{y \in \omega} (a_3(y) \cdot q) > 0.
\]

Define the sets

\[
U_M(\omega) := \{ \eta = (\eta_i) \in H^1_0(\omega) \times H^1_0(\omega) \times L^2(\omega); \theta(y) + \eta_i(y)a_i(y) \cdot q \geq 0 \text{ for a.a. } y \in \omega \},
\]

\[
U_M(\omega) \cap D(\omega) := \{ \eta = (\eta_i) \in D(\omega) \times D(\omega) \times D(\omega); \theta(y) + \eta_i(y)a_i(y) \cdot q \geq 0 \text{ for a.a. } y \in \omega \}.
\]

Then the set \(U_M(\omega) \cap D(\omega)\) is dense in the set \(U_M(\omega)\) with respect to the norm \(\| \cdot \|_{H^1(\omega)\times H^1(\omega)\times L^2(\omega)}\).

Examples of elliptic membrane shells satisfying the “density property” thus include those whose middle surface is a portion of an ellipsoid that is strictly contained in one of the open half-spaces that contain two of its main axes (see, e.g., that drawn in Figure 4.1-1 in [7]), the boundary of the half-space coinciding with the obstacle in this case.

As a final step, we de-scale Problem \(\mathcal{P}_M(\omega)\) and we get the following variational formulation (cf. Theorem 4.2 of [14]).

**Problem** \(\mathcal{P}^\varepsilon_M(\omega)\). Find \(\zeta^\varepsilon = (\zeta_i^\varepsilon) \in U_M(\omega)\) satisfying the following variational inequalities:

\[
\varepsilon \int_\omega \alpha^{\beta\gamma\rho\tau} \gamma_{\sigma\tau}(\zeta^\varepsilon) \gamma_{\alpha\beta}(\eta - \zeta^\varepsilon) \sqrt{a} \, dy \geq \int_\omega p^{i\varepsilon}(\eta_i - \zeta_i^\varepsilon) \sqrt{a} \, dy,
\]

for all \(\eta = (\eta_i) \in U_M(\omega)\), where \(p^{i\varepsilon} := \int_\varepsilon^\varepsilon f_i^\varepsilon \, dx_3^\varepsilon\).

By virtue of the Korn inequality recalled in Theorem 1, it results that Problem \(\mathcal{P}^\varepsilon_M(\omega)\) admits a unique solution.

The main objective of this paper is to show that the solution of the Problem \(\mathcal{P}^\varepsilon_M(\omega)\), derived from the asymptotic analysis conducted in Theorem 4, is also of class \(H^1_{loc}(\omega) \times H^2_{loc}(\omega) \times H^1_{loc}(\omega)\). We will see that a crucial role for establishing the desired higher regularity property will be played by the assumed “density property”.

5. **Augmentation of interior regularity of the solution of Problem** \(\mathcal{P}^\varepsilon_M(\omega)\). Let \(\omega_0 \subset \omega\) and \(\omega_1 \subset \omega\) be such that

\[
\omega_1 \subset\subset \omega_0 \subset\subset \omega.
\]

By the definition of the symbol \(\subset\subset\) in (1), we obtain that the quantity

\[
d := \frac{1}{2} \min\{\text{dist}(\partial\omega_1, \partial\omega_0), \text{dist}(\partial\omega_0, \gamma)\}
\]
is, actually, strictly greater than zero. Let $\varphi_1 \in \mathcal{D}(\omega)$ be such that

$$\text{supp } \varphi_1 \subset \omega_1 \text{ and } 0 \leq \varphi_1 \leq 1.$$ 

In the rest of the paper, the immersion $\theta$ modelling the middle surface of the membrane shell under consideration is assumed to be of class $C^3(\omega; \mathbb{R}^3)$.

Denote by $D_{\rho h}$ the first order (forward) finite difference quotient of either a function or a vector field in the canonical direction $e_\rho$ of $\mathbb{R}^2$ and with increment size $0 < h < d$ sufficiently small. We can regard the first order (forward) finite difference quotient of a function as a linear operator defined as follows:

$$D_{\rho h} : L^2(\omega) \rightarrow L^2(\omega_0).$$

The first order finite difference quotient of a function $\xi$ in the canonical direction $e_\rho$ of $\mathbb{R}^2$ and with increment size $0 < h < d$ is defined by:

$$D_{\rho h}\xi(y) := \frac{\xi(y + he_\rho) - \xi(y)}{h},$$

for all (or, possibly, a.a.) $y \in \omega$ such that $(y + he_\rho) \in \omega$.

The first order finite difference quotient of a vector field $\xi = (\xi_\alpha)$ in the canonical direction $e_\rho$ of $\mathbb{R}^2$ and with increment size $0 < h < d$ is defined by

$$D_{\rho h}\xi(y) := \frac{\xi(y + he_\rho) - \xi(y)}{h},$$

or, equivalently,

$$D_{\rho h}\xi(y) = (D_{\rho h}\xi_\alpha(y)).$$

Similarly, we can show that the first order (forward) finite difference quotient of a vector field is a linear operator from $L^2(\omega)$ to $L^2(\omega_0)$.

We define the second order finite difference quotient of a function $\xi$ in the canonical direction $e_\rho$ of $\mathbb{R}^2$ and with increment size $0 < h < d$ by

$$\delta_{\rho h}\xi(y) := \frac{\xi(y + he_\rho) - 2\xi(y) + \xi(y - he_\rho)}{h^2},$$

for all (or, possibly, a.a.) $y \in \omega$ such that $(y \pm he_\rho) \in \omega$.

The second order finite difference quotient of a vector field $\xi = (\xi_\alpha)$ in the canonical direction $e_\rho$ of $\mathbb{R}^2$ and with increment size $0 < h < d$ is defined by

$$\delta_{\rho h}\xi(y) := \left(\frac{\xi_\alpha(y + he_\rho) - 2\xi_\alpha(y) + \xi_\alpha(y - he_\rho)}{h^2}\right),$$

for all (or, possibly, a.a.) $y \in \omega$ such that $(y \pm he_\rho) \in \omega$.

Note in passing that the second order finite difference quotient of a function $\xi$ can be expressed in terms of the first order finite difference quotient via the following identity:

$$\delta_{\rho h}\xi = D_{-\rho h}D_{\rho h}\xi. \quad (3)$$

Similarly, the second order finite difference quotient of a vector field $\xi$ can be expressed in terms of the first order finite difference quotient via the following identity:

$$\delta_{\rho h}\xi = D_{-\rho h}D_{\rho h}\xi. \quad (4)$$

Before proving the main result of this section, we briefly outline the main steps we are going to implement for showing that the solution $\xi^r$ of Problem $P_{\rho h}(\omega)$ is also of class $H^2_{\text{loc}}(\omega) \times H^2_{\text{loc}}(\omega) \times H^1_{\text{loc}}(\omega)$. The main difficulty in doing so amounts to constructing an admissible test vector field from which it is possible to infer the desired augmented regularity property.
Our strategy consists in locally perturbing the solution $\zeta^\varepsilon$ by a vector field with compact support in a neighbourhood of $\omega$, compact support which is sufficiently far away from the boundary $\gamma$.

In order to implement this local perturbation strategy, we have to first fix a neighbourhood $U_1 \subset \omega$ sufficiently far away from the boundary $\gamma$, and consider a smooth function $\varphi$ whose support is compact in $U_1$.

Second, we multiply the smooth function $\varphi$ by a suitable coefficient $\rho > 0$, $\varrho = O(h^2)$ and a suitable second order finite difference quotient. The purpose of the positive coefficient $\rho$ is to ensure that the geometrical constraint appearing in the set $U_M(\omega)$ is satisfied in the selected neighbourhood $U_1$. The argument of the second order difference quotient is the product between the smooth function $\varphi$ and a smooth approximation of the solution. The approximation by smooth functions requirement cannot be dropped since the transverse component of the solution, namely $\zeta^\varepsilon$, is a priori only of class $L^2(\omega)$. This regularity is in general too low for working with a second order finite difference quotient.

Thanks to the assumed "density property", we are in a position to consider an approximation by smooth functions of the solution $\zeta^\varepsilon$, approximations which all belong to the non-empty, closed and convex subset $U_M(\omega)$.

In the proof of the main result of the paper (Theorem 6), we thus end up resorting to a vector field $\tilde{\zeta}^{\varepsilon,k} \in H^1(U_1) \times H^1(U_1) \times L^2(U_1)$ satisfying the main geometrical constraint, and which is defined in terms of $\rho$, $\varepsilon$ as well as one of the sequences $\{\zeta^{\varepsilon,k}\}_{k \geq 1} \subset U_M(\omega) \cap \mathcal{D}(\omega)$ that converges to the solution $\zeta^\varepsilon$ in $H^1(U_1) \times H^1(U_1) \times L^2(U_1)$ as $k \to \infty$.

Finally, using Theorem 1 and classical integration-by-parts formulae for finite difference quotients (cf., e.g., [18]), we aim to show that the finite difference quotient $\|D_{\rho h}(\varphi \zeta^\varepsilon)\|_{H^1(U_1) \times H^1(U_1) \times L^2(U_1)}$ is uniformly bounded with respect to $h$, where the presence of the smooth function is needed to insure the local validity of the result.

To begin with, we establish two abstract preparatory lemmas.

**Lemma 2.** Let $\omega$, $\omega_0$ and $\omega_1$ be subsets of $\mathbb{R}^2$ as in (1). For each $y \in \omega$, let $a^i(y)$ denote a vector of the contravariant basis at the point $\theta(y)$ of the surface $\theta(\omega)$.

Assume that $\vartheta \in C^1(\omega; \mathbb{R}^3)$ is such that $\vartheta \cdot q$ concave on $\omega_0$. Let $\eta = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times L^2(\omega)$ be such that $(\vartheta(y) + \eta_i(y)a^i(y)) \cdot q \geq 0$ for a.a. $y \in \omega_0$.

Then, for each $0 < h < \delta$ and all $0 < q < h^2/2$, the vector field $\eta_{\rho}((\eta_{\rho, i})) \in L^2(\omega_1)$ defined in a way such that

$$\eta_{\rho, j}a^j := \eta JA^i + \varphi \varphi_1(\varphi A^i),$$

is such that $(\vartheta(y) + \eta_{\rho, i}(y)a^i(y)) \cdot q \geq 0$ for a.a. $y \in \omega_1$.

**Proof.** For a.a. $y \in \omega_1$ we have that

$$\eta_{\rho, j}(y)a^j(y) = \eta_i(y)a^i(y) + \varphi \varphi_1(y)\delta_{\rho h}(\eta_i A^i)(y)$$

$$= \eta_i(y)a^i(y) + \varphi \varphi_1(y)\eta_i(y + \rho e_p)A^i(y + \rho e_p) - 2\eta_i(y)A^i(y) + \eta_i(y - \rho e_p)A^i(y - \rho e_p)$$

$$= \frac{\varphi}{\rho h} \varphi_1(y)(\eta_i a^i)(y + \rho e_p) + \frac{\varphi}{\rho h} \varphi_1(y)(\eta_i a^i)(y - \rho e_p).$$

By virtue of the properties of $\varphi_1$ and $\varphi$, the functions $c_1(y) = c_{-1}(y) := gh^{-2}\varphi_1(y)$ and $c_0(y) := 1 - 2gh^{-2}\varphi_1(y)$ are non-negative in $\omega$. Combining these properties with the assumption that $(\vartheta + \eta A^i) \cdot q \geq 0$ almost everywhere in $\omega_0$ gives:

$$\eta_{\rho, j}(y)a^j(y) \cdot q \geq -c_1(y)\vartheta(y + \rho e_p) \cdot q - c_0(y)\vartheta(y) \cdot q - c_{-1}(y)\vartheta(y - \rho e_p) \cdot q$$
\[ = - \left( \vartheta(y) + \varphi_1(y) \frac{\vartheta(y + h e_\rho) - 2 \vartheta(y) + \vartheta(y - h e_\rho)}{h^2} \right) \cdot q, \]

for a.a. \( y \in \omega_1 \).

The assumed concavity of \( \vartheta \cdot q \) in \( \omega_0 \) gives

\[ \delta_{\rho h} \vartheta(y) \cdot q = \left( \frac{\vartheta(y + h e_\rho) - 2 \vartheta(y) + \vartheta(y - h e_\rho)}{h^2} \right) \cdot q < 0, \]

for all \( 0 < h < d \) and all \( y \in \omega_1 \). Recalling that \( \text{supp} \, \varphi_1 \subseteq \omega_1 \), we derive \((\vartheta(y) + \eta_{\theta,i}(y) a^i(y)) \cdot q \geq 0\) for a.a. \( y \in \omega_1 \), as it was to be proved.

We observe that the concavity of the function \( \vartheta \cdot q \) has to be assumed over \( \omega_0 \) since the points of the form \((y \pm h e_\rho)\), with \( y \in \omega_1 \), may lie outside \( \omega_0 \).

The next result is a direct application of Lemma 2 and of the “density property” (cf. (6) below).

**Corollary 1.** Let \( \omega, \omega_0 \) and \( \omega_1 \) be subsets of \( \mathbb{R}^2 \) as in (1). For each \( y \in \omega \), let \( a^i(y) \) denote a vector of the contravariant basis at the point \( \vartheta(y) \) of the surface \( \vartheta(\omega) \).

Assume that \( \vartheta \in C^1(\overline{\omega}; \mathbb{R}^3) \) is such that \( \vartheta \cdot q \) concave on \( \omega_0 \). Let \( \eta = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times L^2(\omega) \) be such that \((\vartheta(y) + \eta_i(y) a^i(y)) \cdot q \geq 0\) for a.a. \( y \in \omega_0 \). Let \( 0 < h < d \) be given, and let \( 0 < \varphi < h^2/2 \). In correspondence of one such number \( h \), define the number \( \epsilon = \epsilon(h) > 0 \) by

\[ 0 < \epsilon \leq h^2 \left( \inf_{\rho \in (1,2]} \min_{y \in \omega} (\delta_{\rho h}(\vartheta \cdot q)) \right) \cdot \frac{2 \max \{ \| a^i \cdot q \|_{C^0(\omega)} : 1 \leq i \leq 3 \}}{\epsilon}. \quad (5) \]

Assume that in correspondence of such a number \( \epsilon \) there exists an element \( \xi = (\xi_i) \in C^1(\overline{\omega}) \) such that \((\vartheta(y) + \xi_i(y) a^i(y)) \cdot q \geq 0\) for all \( y \in \omega_0 \) and such that

\[ |\xi_i(y) - \eta_i(y)| \leq \frac{\epsilon}{3} \quad \text{for a.a.} \ y \in \omega_1 \text{ and all } 1 \leq i \leq 3. \quad (6) \]

Then, the vector field \( \tilde{\eta}_\theta \) defined in a way such that

\[ \tilde{\eta}_{\theta,i} a^i := \eta_i a^i + \varphi_1 \delta_{\rho h}(\xi_i a^i), \]

is such that \( \tilde{\eta}_\theta = (\tilde{\eta}_{\theta,i}) \in H^1(\omega) \times H^1(\omega) \times L^2(\omega) \) and \((\vartheta(y) + \tilde{\eta}_{\theta,i}(y) a^i(y)) \cdot q \geq 0\) for a.a. \( y \in \omega_1 \).

**Proof.** In the same spirit as Lemma 2, we have that for a.a. \( y \in \omega_1 \)

\[ \tilde{\eta}_{\theta,i}(y) a^i(y) \cdot q \geq \eta_i(y) a^i(y) \cdot q - \frac{\varphi_1(y)}{h^2} (\vartheta(y + h e_\rho) + \vartheta(y - h e_\rho)) \cdot q \]

\[ = \frac{2 \varphi_1(y)}{h^2} \xi_i(y) a^i(y) \cdot q \geq \eta_i(y) a^i(y) \cdot q - \frac{\varphi_1(y)}{h^2} (\vartheta(y + h e_\rho) + \vartheta(y - h e_\rho)) \cdot q \]

\[ = \frac{2 \varphi_1(y)}{h^2} \max \{ \| a^i \cdot q \|_{C^0(\omega)} : 1 \leq i \leq 3 \} \sum_{i=1}^{3} |\xi_i(y) - \eta_i(y)| \]

\[ = \frac{2 \varphi_1(y)}{h^2} \eta_i(y) a^i(y) \cdot q \]

\[ = \left( 1 - \frac{2 \varphi_1(y)}{h^2} \right) \eta_i(y) a^i(y) \cdot q - \frac{\varphi_1(y)}{h^2} (\vartheta(y + h e_\rho) + \vartheta(y - h e_\rho)) \cdot q \]

\[ \geq \left( 1 - \frac{2 \varphi_1(y)}{h^2} \right) \vartheta(y) \cdot q - \frac{\varphi_1(y)}{h^2} (\vartheta(y + h e_\rho) + \vartheta(y - h e_\rho)) \cdot q \]
$$- \frac{2 \varphi_1(y)}{h^2} \max \{ \| \alpha^j \cdot q \|_{\mathbb{C}^0(\overline{\omega})}; 1 \leq j \leq 3 \} \sum_{i=1}^{3} | \xi_i(y) - \eta_i(y) |$$

$$\geq - \vartheta(y) \cdot q + \varphi_1(y) \delta_{\rho h} (- \vartheta(y) \cdot q) - \frac{2 \varphi_1(y)}{h^2} \max \{ \| \alpha^j \cdot q \|_{\mathbb{C}^0(\overline{\omega})}; 1 \leq j \leq 3 \} \epsilon$$

$$\geq - \vartheta(y) \cdot q + \varphi_1(y) \delta_{\rho h} (- \vartheta(y) \cdot q)$$

$$- \varphi_1(y) \left( \inf_{h > 0} \min_{y \in \overline{\omega}} (\delta_{\rho h} (- \vartheta(y) \cdot q)) \right) \geq - \vartheta(y) \cdot q,$$

where the second last inequality derives from the definition of $\epsilon$ in (5) and the assumed concavity of $\vartheta \cdot q$ over the set $\omega_0$. \hfill $\Box$

We observe that the number $\epsilon$ defined in (5) depends on $h$, and is such that $\epsilon = O(h^2) - \text{big \textordfntilde Oh in Landau notation.}$

For treating the case where the concavity assumption fails, we need the following auxiliary result.

**Lemma 3.** Let the function $\vartheta = (\vartheta_1, \vartheta_2, \vartheta_3) \in C^1(\overline{\omega}; \mathbb{R}^3)$ be such that $\vartheta \cdot q > 0$ in $\overline{\omega}$. Then, for every $y_0 = (y_{0,1}, y_{0,2}) \in \omega$, there exists a neighbourhood $U$ of $y_0$ and numbers $B \in \mathbb{R}$, $B_0 > 0$, $r > 0$ such that the function $(- \vartheta \cdot q + B) \tilde{g}$ is convex in $U$, where

$$\tilde{g}(y) := 1 - \frac{1}{2} \prod_{\rho=1}^{2} \exp(ry_{\rho} - ry_{0,\rho}) \quad \text{for all } y = (y_1, y_2) \in \overline{\omega}.$$

Moreover, it results $\tilde{g}(y) \geq B_0$, for all $y \in U$.

**Proof.** Fix $y_0 \in \omega$. Owing to the fact that $\vartheta \in C^1(\overline{\omega}; \mathbb{R}^3)$ and $\vartheta \cdot q > 0$ in $\overline{\omega}$, we can find numbers $B \in \mathbb{R}$ and $T > 0$, and a neighbourhood $U_0$ of $y_0$ such that

$$- \vartheta(y) \cdot q + B \leq - T < 0 \quad \text{for all } y \in U_0.$$

For all $y = (y_1, y_2) \in \overline{\omega}$, define the function:

$$\Pi(y) := \prod_{\rho=1}^{2} \exp(ry_{\rho} - ry_{0,\rho}).$$

We recall that a function is convex if and only if it is convex along any lines that intersect the function domain (cf., e.g., page 67 of [3]). In other words, checking the convexity of the function $(- \vartheta \cdot q + B) \tilde{g}$ in $U$ amounts to checking that, for all $v = (v_1, v_2) \in \mathbb{R}^2$ and all $y = (y_1, y_2) \in U$, the function

$$H(t) := (- \vartheta(y + tv) \cdot q + B) \tilde{g}(y + tv), \quad t \in \mathbb{R}, (y + tv) = (y_1 + tv_1, y_2 + tv_2) \in U,$$

is convex. Let us fix an arbitrary point $y \in U$, a vector $v = (v_1, v_2) \in \mathbb{R}^2$ and a scalar $t$ with the aforementioned properties. By direct computation, we have that

$$\frac{dH}{dt} (t) = \left[ - (\nabla \vartheta_1(y + tv)) \cdot v_1 - (\nabla \vartheta_2(y + tv)) \cdot v_2 - (\nabla \vartheta_3(y + tv)) \cdot v_3 \right] \tilde{g}(y + tv)$$

$$+ \left[ - \vartheta(y + tv) \cdot q + B \right] \frac{d}{dt} (\tilde{g}(y + tv)),$$

and that

$$\frac{d^2H}{dt^2} (t) = \left[ - v^T (\nabla^2 \vartheta_1(y + tv) + \nabla^2 \vartheta_2(y + tv) + \nabla^2 \vartheta_3(y + tv)) v \right] \tilde{g}(y + tv)$$

$$+ 2 \left[ - \nabla \vartheta_1(y + tv) \cdot v_1 - \nabla \vartheta_2(y + tv) \cdot v_2 - \nabla \vartheta_3(y + tv) \cdot v_3 \right] \frac{d}{dt} (\tilde{g}(y + tv))$$

$$+ \left[ - \vartheta(y + tv) \cdot q + B \right] \frac{d^2}{dt^2} (\tilde{g}(y + tv)).$$
By the definition of $\tilde{g}$, we have that
\[
\frac{d\tilde{g}}{dt}(y + tv) = \nabla \tilde{g}(y + tv) \cdot v = -\frac{1}{2} \frac{d\Pi}{dt}(y + tv) = -\frac{r}{2}(v_1 + v_2)\Pi(y + tv),
\]
and that
\[
\frac{d^2\tilde{g}}{dt^2}(y + tv) = v^T \nabla^2 \tilde{g}(y + tv)v = -\frac{r^2}{2}(v_1 + v_2)^2\Pi(y + tv).
\]

Thanks to the global uniform boundedness of the first and second derivative of $\vartheta \cdot q$ in $\mathcal{W}$, the properties of the numbers $B$ and $T$, and the positiveness of the function $\Pi$ we derive that there exists a positive number $M$ such that:
\[
\frac{1}{\Pi} \frac{d^2 H}{dt^2}(t) \geq -M \left| \frac{1}{\Pi} - \frac{1}{2} \right| - rM + \frac{T}{2}(v_1 + v_2)^2 r^2.
\]

We observe that, for $r$ sufficiently large,
\[
p_1(r) := \frac{T}{2}(v_1 + v_2)^2 r^2 - Mr - M > 0.
\]

Let us choose a neighbourhood $\mathcal{U}$ of $y_0$ such that $\mathcal{U} \subset \subset \mathcal{U}_0$ and
\[
|y_\rho - y_{0,\rho}| \leq \frac{1}{2r} \ln \frac{3}{2} \quad \text{for all } y \in \mathcal{U} \text{ and all } \rho \in \{1, 2\}. \quad (7)
\]

On the one hand, it is immediate to see that, by the monotonicity of the exponential and (7),
\[
0 < \Pi(y) \leq \prod_{\rho=1}^{2} \exp\left(\ln \sqrt{\frac{3}{2}}\right) = \frac{3}{2} \quad \text{for all } y \in \mathcal{U},
\]
so that we have
\[
0 < \Pi(y) \leq \frac{3}{2} \quad \text{for all } y \in \mathcal{U}. \quad (8)
\]

On the other hand, again by (7), we have that
\[
\frac{1}{\Pi(y)} - \frac{1}{2} = \prod_{\rho=1}^{2} \exp(-r(y_\rho - y_{0,\rho})) - \frac{1}{2} \leq 1 \quad \text{for all } y \in \mathcal{U}, \quad (9)
\]
and, by virtue of (8), we have that
\[
\frac{1}{\Pi(y)} - \frac{1}{2} \geq \frac{2}{3} - \frac{1}{2} > 0 \quad \text{for all } y \in \mathcal{U}. \quad (10)
\]

In conclusion, putting (9) and (10) together, we have that
\[
\left| \frac{1}{\Pi(y)} - \frac{1}{2} \right| \leq 1 \quad \text{for all } y \in \mathcal{U}. \quad (11)
\]

An application of (11) immediately gives
\[
\frac{1}{\Pi} \frac{d^2 H}{dt^2}(t) \geq -M \left| \frac{1}{\Pi} - \frac{1}{2} \right| - Mr + \frac{T}{2}(v_1 + v_2)^2 r^2 \geq p_1(r) \geq 0.
\]

By virtue of (8), we also obtain $\tilde{g}(y) \geq 1 - \Pi/2 \geq 1/4$, for all $y \in \mathcal{U}$. Letting $B_0 := 1/4$ completes the proof. \hfill \Box
Let us observe that the assumption $\vartheta \cdot q > 0$ in $\omega$ in Lemma 3 is related to the result stated in Lemma 1.

In the next preparatory lemma, we prove some convergence properties of the operator $D_{ph}$ for functions. As a consequence of Theorem 6.6-4 of [9] we have that

$$\bar{C}^1(\bar{\omega}) \subset H^1(\omega).$$

**Lemma 4.** Let $\{v_k\}_{k \geq 1}$ be a sequence in $\bar{C}^1(\bar{\omega})$ that converges to a certain element $v \in H^1(\omega)$ with respect to the norm $\| \cdot \|_{H^1(\omega)}$. Then, we have that for all $0 < h < d$ and all $\rho \in \{1, 2\}$,

$$D_{ph}v \in H^1(\omega_0)$$

and $D_{ph}v_k \to D_{ph}v$ in $H^1(\omega_0)$ as $k \to \infty$. \hspace{1cm} (12)

**Proof.** Fix $0 < h < d$ and fix $\rho \in \{1, 2\}$. Clearly, by definition of $D_{ph}$ we have that $D_{ph}v \in L^2(\omega_0)$. Let us show that $D_{ph}$ is continuous. A direct computation gives

$$\int_{\omega_0} |D_{ph}v_k(y) - D_{ph}v(y)|^2 \, dy$$

$$\leq \frac{4}{h^2} \|v_k - v\|^2_{L^2(\omega)} \to 0 \quad \text{as} \quad k \to \infty.$$

We now show that $D_{ph}v \in H^1(\omega_0)$. For each $\beta \in \{1, 2\}$ and all $\psi \in D(\omega_0)$, we have that

$$\int_{\omega_0} D_{ph}v(y) \partial_\beta \psi(y) \, dy$$

$$= \frac{1}{h} \int_{\omega_0} v(y + h\varrho) \partial_\beta \psi(y) \, dy - \frac{1}{h} \int_{\omega_0} v(y) \partial_\beta \psi(y) \, dy$$

$$= \frac{1}{h} \int_{\omega_0 + h\varrho} v(z) \partial_\beta \psi(z - h\varrho) \, dz - \frac{1}{h} \int_{\omega_0} v(y) \partial_\beta \psi(y) \, dy$$

$$= \int_{\omega_0} v(y) \partial_\beta(D_{ph}\psi)(y) \, dy$$

$$= \int_{\omega_0} \partial_\beta v(y)D_{ph}\psi(y) \, dy$$

$$= \int_{\omega_0} (D_{ph}(\partial_\beta v))(y)\psi(y) \, dy,$$

where the second equality takes into account the change of variables $z = y + h\varrho$, the third equality holds for $0 < h < d$, being $\psi \in D(\omega_0)$, the fourth equality is owing to the fact that $v \in H^1(\omega_0)$ and, finally, the fifth equality is owing to the properties of $D_{ph}$. Since, by definition of $D_{ph}$, we have that $D_{ph}(\partial_\beta v) \in L^2(\omega_0)$, we then obtain that

$$D_{ph}v \in H^1(\omega_0) \quad \text{with} \quad \partial_\beta(D_{ph}v) = D_{ph}(\partial_\beta v) \in L^2(\omega_0).$$

Finally, we observe that the assumed convergence $\|v_k - v\|_{H^1(\omega)} \to 0$ as $k \to \infty$ gives:

$$\int_{\omega_0} |\partial_\beta(D_{ph}v_k) - \partial_\beta(D_{ph}v)|^2 \, dy = \int_{\omega_0} |D_{ph}(\partial_\beta(v_k - v))|^2 \, dy$$
thus showing that
\[ C \in \mathcal{C}_h > \frac{\epsilon}{K} \]

Thus, the number \( 0 \leq \epsilon \) such that \( \epsilon = \epsilon_0 \) in the same spirit as (5) of Corollary 1, i.e.,

\[ \|
\frac{\partial C}{\partial \beta}(D_{ph}v_k) - \frac{\partial C}{\partial \beta}(D_{ph}v)\|_{L^2(\omega)} \to 0 \quad \text{as } k \to \infty, \]

and this completes the proof.

As a remark, we observe that the convergence in (12) is to be read as follows.

Let \( h > 0 \) be given. For each \( \epsilon > 0 \) (in general, independent of \( h \)) there exists an integer \( K = K(h, \rho, \epsilon) \geq 1 \) such that for all \( k \geq K \) it results

\[ \|D_{ph}v_k - D_{ph}v\|_{H^1(\omega_0)} < \epsilon. \]

We are now ready to prove the main result of this paper.

**Theorem 6.** Assume that the “density property” stated in Theorem 4, namely,

\( \mathcal{U}_M(\omega) \) is dense in \( U_M(\omega) \) with respect to the norm of \( \| \cdot \|_{H^1(\omega) \times H^1(\omega) \times L^2(\omega)} \) holds. Assume also that the vector \( f^\alpha = (f^{i, \alpha}) \) defining the applied body force density is of class \( H^1(\Omega^\alpha) \).

Then, the solution \( \zeta^\alpha \) of Problem \( P_M^\alpha(\omega) \) is of class \( V_M(\omega) \cap H^2_{loc}(\omega) \times H^2_{loc}(\omega) \times H^1_{loc}(\omega) \).

**Proof.** For sake of clarity we break the proof into two steps, numbered (i) and (ii).

Fix \( \phi \in \mathcal{D}(\mathcal{U}) \) such that \( supp \varphi \subset \subset \mathcal{U}_1 \subset \subset \mathcal{U} \), for some neighbourhood \( \mathcal{U}_1 \) of \( y_0 \), such that \( 0 \leq \varphi \leq 1 \), and such that \( \varphi \equiv 1 \) in a compact strict subset of its support. Following the notation of Corollary 1, here we let \( \omega_0 := \mathcal{U}_1 \) and \( \omega_1 := \mathcal{U}_1 \). In what follows, the number \( 0 < h < d \) is fixed, where the number \( d \) has been defined in (2).

(i) **Construction of a candidate test field.** Let \( 0 < \rho < h^2/2 \). Let \( y_0 \in \omega \) and let \( \mathcal{U} \) be the neighbourhood of \( y_0 \) constructed in Lemma 3 in correspondence of the immersion \( \theta \) which is, by assumption, of class \( C^3(\overline{\omega}; \mathbb{R}^3) \). Let \( B \in \mathbb{R} \) and the function \( \tilde{g} \) be defined as in Lemma 3. The vector field

\[ \tilde{\theta} := -\left(-\theta + \frac{Bq}{|q|^2}\right) \tilde{g}, \]

is of class \( C^3(\overline{\omega}; \mathbb{R}^3) \) and, by Lemma 3, it is such that \( \tilde{\theta} \cdot q \) is concave in \( \mathcal{U} \). Without loss of generality, we can assume that \( \mathcal{U}_1 \) is a domain; otherwise, we take a polygon \( Q \) such that \( supp \varphi \subset \subset Q \subset \subset \mathcal{U}_1 \) and we rename it \( \mathcal{U}_1 \) without any loss of rigour.

Fix \( \epsilon = \epsilon(h) > 0 \) in the same spirit as (5) of Corollary 1, i.e.,

\[ 0 < \epsilon \leq h^2 \left( \inf_{\rho \in (1,2]} \min_{y \in \mathcal{U}_1} \frac{\delta_{ph}(-\tilde{\theta} \cdot q)}{2 \max\{\|a^i \cdot q\|_{C^0(\overline{\omega})} : 1 \leq i \leq 3\}} \right), \]

and observe that, in the notation of Corollary 1, here we have let \( \theta := \tilde{\theta} \).
Since, by Lemma 3, we have \( \tilde{g} \geq B_0 > 0 \) in \( \mathcal{U} \), and since \( \zeta^\epsilon \in U_M(\omega) \), writing \( q \) in terms of the contravariant basis gives
\[
\left( \zeta^\epsilon (y) a^i(y) + \frac{B q}{|q|} \right) \tilde{g}(y) \cdot q
\]
\[
= \left[ \left( \zeta^\epsilon (y) + \frac{B q \cdot a^i(y)}{|(q \cdot a^i(y))a^i(y)|^2} \right) a^i(y) \cdot (q \cdot a^i(y))a^i(y) \right] \tilde{g}(y)
\]
\[
\geq (-\theta(y) \cdot q + B) \tilde{g}(y) = -\theta(y) \cdot q, \quad \text{for a.a. } y \in \omega.
\]

Let \( \{\zeta_i^\epsilon\}_{i \geq 1} \subset U_M(\omega) \cap \mathcal{D}(\omega) \) be one of the sequences generated by the “density property”. As a result of the assumed “density property”, there exists a positive integer \( k_0 = k_0(\epsilon, \mathcal{U}) \) for which
\[
\|\zeta_i^\epsilon - \zeta_i^{\epsilon - k}\|_{H^1(\omega) \times H^1(\omega) \times L^2(\omega)} \leq \frac{\epsilon}{3 \max \{\tilde{g}(y); y \in \mathcal{U}\}},
\]
\[
|\zeta_{i-1}^\epsilon - \zeta_i^\epsilon| \leq \frac{\epsilon}{3 \max \{\tilde{g}(y); y \in \mathcal{U}\}}, \quad \text{for a.a. } y \in \omega \text{ and all } 1 \leq i \leq 3,
\]
for all \( k \geq k_0 \).

Besides, in correspondence of any integer \( k \geq k_0 \), the assumed “density property” and the same computations yielding (14) give
\[
-\tilde{\theta}(y) \cdot q \leq \left( \zeta_i^{\epsilon - k} (y) + B q \frac{a^i(y)}{|q|^2} \right) a^i(y)\tilde{g}(y) \cdot q, \quad \text{for all } y \in \omega.
\]

Again, with respect to the notation of Corollary 1, we hereby let
\[
\xi_i := \left( \zeta_i^{\epsilon - k} + B q \frac{a^i(y)}{|q|^2} \right) \tilde{g}, \quad \text{so that } \xi \in C^1(\omega),
\]
\[
\eta_i := \left( \zeta_i^\epsilon + B q \frac{a^i(y)}{|q|^2} \right) \tilde{g}, \quad \text{so that } \eta \in H^1(\omega) \times H^1(\omega) \times L^2(\omega),
\]
and we observe that a straightforward computation gives
\[
|\xi_i(y) - \eta_i(y)| \leq \left( \max \{\tilde{g}(y); y \in \mathcal{U}\} \right)|\zeta_i^{\epsilon - k}(y) - \zeta_i^\epsilon(y)| \leq \frac{\epsilon}{3}, \quad \text{for all } 1 \leq i \leq 3 \text{ and a.a. } y \in \omega.
\]

Thanks to the concavity of \( \tilde{\theta} \cdot q \) in \( \mathcal{U} \), the definition \( \epsilon \) in (13), and (14)–(16), letting \( \varphi_1 = \varphi^2 \) in the notation of Corollary 1, we obtain (once again up to re-writing \( q \) in terms of the contravariant basis) by Corollary 1,
\[
-\tilde{\theta}(y) \cdot q \leq \left( \zeta_i^\epsilon (y) a^i(y) + \frac{B q}{|q|^2} \right) \tilde{g}(y) \cdot q
\]
\[
+ \varphi \varphi^2(y) a^i(y) \left[ \left( \zeta_i^{\epsilon - k}(y) a^i(y) + \frac{B q}{|q|^2} \right) \tilde{g}(y) \right] \cdot q,
\]
for a.a. \( y \in \mathcal{U}_i \). Dividing (17) by \( \tilde{g} \) and then subtracting \( B \) from each member of (17) gives
\[
-\theta(y) \cdot q \leq \zeta_i^\epsilon (y) a^i(y) \cdot q
\]
\[
+ \varphi(\tilde{g}(y))^{-1} \varphi^2(y) a^i(y) \left[ \left( \zeta_i^{\epsilon - k}(y) a^i(y) + \frac{B q}{|q|^2} \right) \tilde{g}(y) \right] \cdot q \quad \text{for a.a. } y \in \mathcal{U}_i.
\]
We then define the vector field $\tilde{\zeta}^{\varepsilon,k}$ in a way such that
\[
\tilde{\zeta}^{\varepsilon,k}(y) a_i(y) = \zeta^\varepsilon_i(y) a_i(y)
+ \rho(\tilde{g}(y))^{-1} \varphi^2(y) \delta_{ph} \left[ \left( \zeta_i^{\varepsilon,k}(y) a_i(y) + \frac{B q}{|q|^2} \tilde{g}(y) \right) \right]
\text{for a.a. } y \in \mathcal{U}_1.
\]

A direct application of (18) and (19) gives that $\theta(y) \cdot q + \tilde{\zeta}^{\varepsilon,k}(y) a_i(y) \cdot q \geq 0$, for a.a. $y \in \mathcal{U}_1$. By virtue of the definition of the covariant and contravariant bases, we have that $\tilde{\zeta}^{\varepsilon,k}_\rho \in H^1(\mathcal{U}_1) \times H^1(\mathcal{U}_1) \times L^2(\mathcal{U}_1)$. 

(ii) The solution $\zeta^\varepsilon$ is of class $H^2_{\text{loc}}(\omega) \times H^2_{\text{loc}}(\omega) \times H^1_{\text{loc}}(\omega)$.

Since, by definition, we have that $\tilde{\zeta}^{\varepsilon,k}(y) = \zeta^\varepsilon(y)$ for a.a. $y \in \mathcal{U}_1 \setminus \text{supp } \varphi$, we are in a position to extend $\tilde{\zeta}^{\varepsilon,k}$ by $\zeta^\varepsilon$ outside $\mathcal{U}_1$; we denote this extension by $\tilde{\zeta}^{\varepsilon}_\rho$. Besides, by virtue of (i), such a vector field satisfies the geometrical constraint. As a result, we can infer that $\tilde{\zeta}^{\varepsilon}_\rho \in \mathcal{U}_M(\omega)$.

Observe that the assumption that the $\phi^{3,\varepsilon} \in H^1(\Omega^\varepsilon)$ implies, by Theorem 4.2-1(b) of [7], that the transverse component of the vector $\zeta^\phi = (p^{3,\varepsilon})$ appearing in the statement of Problem $P^{3,\varepsilon}_M(\omega)$ is of class $H^1(\omega)$.

Specialising $\eta = \tilde{\zeta}^{\varepsilon,k} \rho$ in Problem $P^{3,\varepsilon}_M(\omega)$ gives:
\[
- \varepsilon \int_{\mathcal{U}_1} a^{\alpha\beta \sigma \gamma} \gamma_{\alpha \beta} \left( \tilde{g}^{-1} \varphi^2 \delta_{ph} \left[ \left( \zeta_i^{\varepsilon,k} a_i + \frac{B q}{|q|^2} \tilde{g} \right) \cdot a_j \right] \right) \nabla_\alpha v(\mathcal{U}_1) \leq - \int_{\mathcal{U}_1} p^{3,\varepsilon} \left( \tilde{g}^{-1} \varphi^2 \delta_{ph} \left[ \left( \zeta_i^{\varepsilon,k} a_i + \frac{B q}{|q|^2} \tilde{g} \right) \cdot a_j \right] \right) \nabla_\alpha v(\mathcal{U}_1).
\]

Let us define the translation operator $E$ in the canonical direction $\epsilon_\rho \in \mathbb{R}^2$ and with increment size $0 < h < d$ for a smooth enough function $v : \mathcal{U}_1 \to \mathbb{R}$ by
\[
E_{ph}(y) := v(y + h \epsilon_\rho),
E_{-ph}(y) := v(y - h \epsilon_\rho).
\]

Moreover, the following identities can be easily checked out (cf. [19] and [24]):
\[
D_{ph}(vw) = (E_{ph}(w))(D_{ph}v) + v(D_{ph}w),
D_{-ph}(vw) = (E_{-ph}(w))(D_{-ph}v) + v(D_{-ph}w),
\]
\[
\delta_{ph}(vw) = w \delta_{ph}v + (D_{ph}w)(D_{ph}v) + (D_{-ph}w)(D_{-ph}v) + v \delta_{ph}w.
\]

Given two functions $f_1, f_2 \in L^2(\mathcal{U}_1)$, we say that $f_1 \preceq f_2$ if there exists a constant $C > 0$ independent of $h$ and eventually dependent on $\|p^\phi\|_{H^1(\omega)}$, $\tilde{g}$, its reciprocal $\tilde{g}^{-1}$, $\|\zeta^\phi\|_{H^1(\omega) \times H^1(\omega) \times L^2(\omega)}$, $\varphi$ and its derivatives such that:
\[
\int_{\mathcal{U}_1} f_1 \nabla \alpha v(\mathcal{U}_1) \leq \int_{\mathcal{U}_1} f_2 \nabla \alpha v(\mathcal{U}_1) + C(1 + \|D_{ph}(\varphi \zeta^{\varepsilon,k})\|_{H^1(\mathcal{U}_1) \times H^1(\mathcal{U}_1) \times L^2(\mathcal{U}_1)}).
\]

We first show that there exists a constant $C > 0$ independent of $h$ but eventually dependent on $\|p^\phi\|_{H^1(\omega)}$, $\tilde{g}$, its reciprocal $\tilde{g}^{-1}$, $\|\zeta^\phi\|_{H^1(\omega) \times H^1(\omega) \times L^2(\omega)}$, $\varphi$ and its derivatives such that:
\[
- \int_{\mathcal{U}_1} p^{3,\varepsilon} \left( \tilde{g}^{-1} \varphi^2 \delta_{ph} \left[ \left( \zeta_i^{\varepsilon,k} a_i + \frac{B q}{|q|^2} \tilde{g} \right) \cdot a_j \right] \right) \nabla_\alpha v(\mathcal{U}_1) \leq C(1 + \|D_{ph}(\varphi \zeta^{\varepsilon,k})\|_{H^1(\mathcal{U}_1) \times H^1(\mathcal{U}_1) \times L^2(\mathcal{U}_1)}).
An application of formulas (3), (21)–(23), Hölder’s inequality, the assumed boundedness of $\varphi$ and its derivatives, the fact that $p^{\varepsilon,\gamma} \in H^1(\omega)$, the smoothness of the immersion $\theta \in C^0(\overline{\omega}, E^1)$ (and so of the corresponding fundamental forms, viz. section 2), the integration-by-parts formula for the first order finite difference quotient $D_{\text{ph}}$ (cf., e.g., formula (16) in Section 6.3.1 of [18]) and the assumed “density property” (cf. Theorem 5) give:

- $\int_{\Omega} p^{\varepsilon,\gamma}(\tilde{g}^{-1} \varphi^{-2}) \delta_{\text{ph}} \left( \left( \zeta^{\varepsilon,\gamma}_{i,\alpha}a^i + \frac{Bq}{|q|^2} \right) g \right) \cdot a_j \sqrt{\gamma} \; dy$

- $= - \int_{\Omega} \int_{\mathcal{T}_1} \left( p^{\varepsilon,\gamma} \varphi^{-1} \varphi_{\alpha} \right) \cdot \delta_{\text{ph}} \left( \left( \zeta^{\varepsilon,\gamma}_{i,\alpha}a^i + \frac{Bq}{|q|^2} \right) g \right) - (D_{\text{ph}} g) D_{\text{ph}} \left( \left( \zeta^{\varepsilon,\gamma}_{i,\alpha}a^i + \frac{Bq}{|q|^2} \right) g \right) \; dy$

- $= \int_{\Omega} \int_{\mathcal{T}_1} \left( p^{\varepsilon,\gamma} \varphi^{-1} \varphi_{\alpha} \right) \cdot \delta_{\text{ph}} \left( \left( \zeta^{\varepsilon,\gamma}_{i,\alpha}a^i + \frac{Bq}{|q|^2} \right) g \right) \sqrt{\gamma} \; dy$

- $\leq C \left\{ \int p^{\varepsilon,\gamma} \varphi_{\alpha} L^2(\omega) \left\| D_{\text{ph}} \left( \zeta^{\varepsilon,\gamma}_{i,\alpha}a^i + \frac{Bq}{|q|^2} \right) g \right\| L^2(\Omega) \right\} + 2 \int p^{\varepsilon,\gamma} \varphi_{\alpha} L^2(\omega) \left\| D_{\text{ph}} \left( \zeta^{\varepsilon,\gamma}_{i,\alpha}a^i + \frac{Bq}{|q|^2} \right) g \right\| L^2(\Omega) + \left( \text{meas } \omega \right) \int p^{\varepsilon,\gamma} \varphi_{\alpha} L^2(\omega) \left\| \frac{Bq}{|q|^2} \right\| c^1(\gamma) \right\}$

- $\leq C \left\{ \int p^{\varepsilon,\gamma} \varphi_{\alpha} L^2(\omega) \left\| \delta_{\alpha} \right\| L^2(\Omega) + \left( \text{meas } \omega \right) \int p^{\varepsilon,\gamma} \varphi_{\alpha}(\nabla \varphi^{\varepsilon,\gamma}_{i,\alpha}) L^2(\Omega) \right\}$

- $\leq C \left\{ \left\| \nabla \varphi^{\varepsilon,\gamma}_{i,\alpha} \right\| L^2(\Omega) \right\}$

- $\leq C \left\{ \left\| \nabla \varphi^{\varepsilon,\gamma}_{i,\alpha} \right\| L^2(\Omega) \right\}$

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- $\leq C \left\{ \left\| \nabla \varphi^{\varepsilon,\gamma}_{i,\alpha} \right\| L^2(\Omega) \right\}$

for some constant $C > 0$ independent of $h$ but eventually dependent on $\| p^\varepsilon \| H^1(\omega)$, $\tilde{g}^\varepsilon$, its reciprocal $\tilde{g}^{-1}$, $\| \zeta^\varepsilon \| H^1(\omega) \times H^1(\omega) \times L^2(\omega)$, $\varphi$ and its derivatives as in (21). In the previous calculations, we used Hölder’s inequality to derive the first estimate, and we used (21) to derive the second estimate.

By Lemma 3 we know that the derivatives of $\tilde{g}$ and $\tilde{g}^{-1}$ are uniformly bounded in $\mathcal{U}$. As a result, we have that (20) and the computations above imply:

$$- a^{\alpha \beta \sigma \gamma} \gamma_{\alpha \beta} (\zeta^\varepsilon)^{\gamma \alpha \beta} \left( \left( \tilde{g}^{-1} \varphi^{-2} \delta_{\text{ph}} \left( \left( \zeta^{\varepsilon,\gamma}_{i,\alpha}a^i + \frac{Bq}{|q|^2} \right) g \right) \cdot a_j \right)^3_{j=1} \right) \lesssim 0. \quad (25)$$
Let us now use (21)–(23) for computing $\delta_{ph}(\zeta^{\varepsilon,k}\varphi)$. Define $A := -\varphi(D_{ph}\varphi)$ and observe that (23) gives:

$$
\varphi\delta_{ph}(\zeta^{\varepsilon,k}\varphi) + A = \varphi[\varphi\delta_{ph}\zeta^{\varepsilon,k} + \zeta^{\varepsilon,k}\delta_{ph}\varphi].
$$

The next step consists in showing that:

$$
-a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\varphi\zeta^{\varepsilon})\gamma_{\alpha\beta}(\delta_{ph}(\varphi\zeta^{\varepsilon,k})) \lesssim -a^{\alpha\beta\gamma\delta}\gamma_{\gamma\delta}(\varphi\delta_{ph}(\varphi\zeta^{\varepsilon,k})).
$$

Recalling the definition of the change of metric tensor components $\gamma_{\alpha\beta}$ (cf. section 2) and recalling that $\theta \in C^3(\overline{\Omega}; \mathbb{R}^3)$, we have that the integral

$$
\int_{U_1} -a^{\alpha\beta\gamma\delta}\gamma_{\gamma\delta}(\varphi\delta_{ph}(\varphi\zeta^{\varepsilon,k}))\sqrt{a} \, dy
$$

can be estimated by estimating the following main nine addends of its. In the evaluation of the following nine terms, the indices are assumed to be fixed, i.e., the summation rule with respect to repeated indices is not enforced in (28)–(36) below.

Overall, the strategy we resort to is the following: we take into accounts the addends of the linearised change of metric tensor and we apply Green’s formula and the integration-by-parts formula for finite difference quotients for suitably arranging the position of the compactly supported function $\varphi$.

First, thanks to an application of Green’s formula (cf., e.g., Theorem 6.6–7 of [9]), we estimate:

$$
\int_{U_1} -a^{\alpha\beta\sigma\tau}\partial_\sigma(\varphi\zeta^{\varepsilon})\partial_\beta(\delta_{ph}(\varphi\zeta^{\varepsilon,k}))\sqrt{a} \, dy
$$

$$
= \int_{U_1} -a^{\alpha\beta\sigma\tau}[\partial_\sigma(\varphi\zeta^{\varepsilon}) + \varphi\partial_\sigma(\partial_\tau(\varphi\zeta^{\varepsilon,k}))]\partial_\beta(\delta_{ph}(\varphi\zeta^{\varepsilon,k}))\sqrt{a} \, dy
$$

$$
= \int_{U_1} \partial_\beta(a^{\alpha\beta\sigma\tau}(\partial_\sigma(\varphi\zeta^{\varepsilon}))\delta_{ph}(\varphi\zeta^{\varepsilon,k}))\sqrt{a} \, dy
$$

$$
+ \int_{U_1} \partial_\beta[\partial_\sigma\delta_{ph}(\varphi\partial_\tau(\varphi\zeta^{\varepsilon,k}))]\sqrt{a} \, dy
$$

$$
\leq C\|\zeta^{\varepsilon}\|_{H^1(U_1) \times H^1(U_1)}\|D_{ph}(\varphi\zeta^{\varepsilon,k})\|_{H^1(U_1)}
$$

$$
+ \int_{U_1} -a^{\alpha\beta\gamma\delta}\gamma_{\gamma\delta}(\partial_\sigma(\varphi\zeta^{\varepsilon}))\partial_\beta(\delta_{ph}(\varphi\zeta^{\varepsilon,k}))\sqrt{a} \, dy
$$

$$
+ \int_{U_1} a^{\alpha\beta\gamma\delta}(\partial_\sigma(\varphi\zeta^{\varepsilon}))\delta_{ph}(\varphi\zeta^{\varepsilon,k})\sqrt{a} \, dy
$$

$$
\leq \int_{U_1} -a^{\alpha\beta\gamma\delta}(\partial_\sigma(\varphi\zeta^{\varepsilon}))\partial_\beta(\delta_{ph}(\varphi\zeta^{\varepsilon,k}))\sqrt{a} \, dy + C\|D_{ph}(\varphi\zeta^{\varepsilon,k})\|_{H^1(U_1)}.
$$

Second, we estimate:

$$
\int_{U_1} -a^{\alpha\beta\gamma\delta}(\theta_{\sigma\tau})(\varphi\delta_{ph}(\varphi\zeta^{\varepsilon}))\sqrt{a} \, dy
$$

$$
= -\int_{U_1} \partial_\beta(a^{\alpha\beta\gamma\delta}\gamma_{\gamma\delta}(\varphi\zeta^{\varepsilon}))\delta_{ph}(\varphi\zeta^{\varepsilon,k})\sqrt{a} \, dy
$$

$$
\leq C\|\zeta^{\varepsilon}\|_{H^1(U_1)}\|D_{ph}(\varphi\zeta^{\varepsilon,k})\|_{H^1(U_1)},
$$

where the equality holds as a consequence of Green’s formula.
Third, we estimate:

\[
\int_{U_1} -a^{\alpha \beta \sigma \tau} (b_{\alpha \beta} \varphi \zeta^\alpha) \partial_\beta (\delta_{\rho h} (\varphi \zeta^\alpha)) \sqrt{\alpha} \, dy = \int_{U_1} a^{\alpha \beta \sigma \tau} (b_{\alpha \beta} \zeta^\alpha) \partial_\beta (\varphi \delta_{\rho h} (\varphi \zeta^\alpha)) \sqrt{\alpha} \, dy - \int_{U_1} a^{\alpha \beta \sigma \tau} b_{\alpha \beta} \zeta^\alpha (\partial_\beta \varphi \delta_{\rho h} (\varphi \zeta^\alpha)) \sqrt{\alpha} \, dy \leq \int_{U_1} a^{\alpha \beta \sigma \tau} (b_{\alpha \beta} \zeta^\alpha) \partial_\beta (\varphi \delta_{\rho h} (\varphi \zeta^\alpha)) \sqrt{\alpha} \, dy + C \| \delta_{\rho h} (\varphi \zeta^\alpha) \|_{H^1(U_1)}.
\]

(30)

Fourth, we estimate:

\[
\int_{U_1} -a^{\alpha \beta \sigma \tau} (\partial_{\sigma \varphi} \zeta^\alpha) \Gamma_{\alpha \beta}^{-1} \partial_\rho (\varphi \zeta^\alpha) \sqrt{\alpha} \, dy = \int_{U_1} -a^{\alpha \beta \sigma \tau} (\partial_{\sigma \varphi} \zeta^\alpha) \Gamma_{\alpha \beta}^{-1} \partial_\rho (\varphi \zeta^\alpha) \sqrt{\alpha} \, dy + \int_{U_1} -a^{\alpha \beta \sigma \tau} (\partial_{\sigma \varphi} \zeta^\alpha) \Gamma_{\alpha \beta}^{-1} \partial_\rho (\varphi \zeta^\alpha) \sqrt{\alpha} \, dy \leq C \| \delta_{\rho h} (\varphi \zeta^\alpha) \|_{H^1(U_1)} + \int_{U_1} -a^{\alpha \beta \sigma \tau} (\partial_{\sigma \varphi} \zeta^\alpha) \Gamma_{\alpha \beta}^{-1} [\varphi \delta_{\rho h} (\varphi \zeta^\alpha)] \sqrt{\alpha} \, dy.
\]

(31)

Fifth, we straightforwardly observe that:

\[
\int_{U_1} -a^{\alpha \beta \sigma \tau} (\Gamma_{\alpha \beta} \zeta^\alpha \varphi) \Gamma_{\epsilon \kappa} \partial_\rho (\varphi \zeta^\alpha) \sqrt{\alpha} \, dy \leq C (1 + \| \delta_{\rho h} (\varphi \zeta^\alpha) \|_{H^1(U_1) \times H^1(U_1) \times L^2(U_1)}) + \int_{U_1} -a^{\alpha \beta \sigma \tau} (\Gamma_{\alpha \beta} \zeta^\alpha \varphi) \Gamma_{\epsilon \kappa} \partial_\rho (\varphi \zeta^\alpha) \sqrt{\alpha} \, dy.
\]

(32)

Sixth, we straightforwardly observe that:

\[
\int_{U_1} -a^{\alpha \beta \sigma \tau} (b_{\alpha \beta} \zeta^\alpha \varphi) \Gamma_{\epsilon \kappa} \partial_\rho (\varphi \zeta^\alpha) \sqrt{\alpha} \, dy \leq C (1 + \| \delta_{\rho h} (\varphi \zeta^\alpha) \|_{H^1(U_1) \times H^1(U_1) \times L^2(U_1)}) + \int_{U_1} -a^{\alpha \beta \sigma \tau} b_{\alpha \beta} \zeta^\alpha \Gamma_{\epsilon \kappa} \partial_\rho (\varphi \zeta^\alpha) \sqrt{\alpha} \, dy.
\]

(33)

Seventh, we straightforwardly observe that:

\[
\int_{U_1} -a^{\alpha \beta \sigma \tau} (\Gamma_{\alpha \beta} \zeta^\alpha \varphi) b_{\alpha \beta} \delta_{\rho h} (\varphi \zeta^\alpha) \sqrt{\alpha} \, dy \leq C (1 + \| \delta_{\rho h} (\varphi \zeta^\alpha) \|_{H^1(U_1) \times H^1(U_1) \times L^2(U_1)}) + \int_{U_1} -a^{\alpha \beta \sigma \tau} (\Gamma_{\alpha \beta} \zeta^\alpha \varphi) b_{\alpha \beta} \delta_{\rho h} (\varphi \zeta^\alpha) \sqrt{\alpha} \, dy.
\]

(34)
Eighth, we estimate:
\[ \int_{\mathcal{U}_1} -a^{\alpha\beta\sigma\tau} \partial_\sigma (\varphi \xi_z) b_{\alpha\beta} \partial_\rho (\xi^{\varepsilon,k}_3) \varphi \sqrt{a} \, dy \]

\[ = \int_{\mathcal{U}_1} -a^{\alpha\beta\sigma\tau} (\partial_\sigma \xi_z) b_{\alpha\beta} [\varphi \partial_\rho (\xi^{\varepsilon,k}_3) \varphi] \sqrt{a} \, dy \]

\[ + \int_{\mathcal{U}_1} -a^{\alpha\beta\sigma\tau} (\partial_\sigma \varphi) \xi_z b_{\alpha\beta} \partial_\rho (\xi^{\varepsilon,k}_3) \varphi \sqrt{a} \, dy \]

\[ = \int_{\mathcal{U}_1} -a^{\alpha\beta\sigma\tau} (\partial_\sigma \xi_z) b_{\alpha\beta} [\varphi \partial_\rho (\xi^{\varepsilon,k}_3) \varphi] \sqrt{a} \, dy \]

\[ + \int_{\mathcal{U}_1} D_{\rho h} (-a^{\alpha\beta\sigma\tau} (\partial_\sigma \varphi) \xi_z b_{\alpha\beta}) D_{\rho h} (\varphi \xi^{\varepsilon,k}_3) \sqrt{a} \, dy \]

\[ \leq \int_{\mathcal{U}_1} -a^{\alpha\beta\sigma\tau} (\partial_\sigma \xi_z) b_{\alpha\beta} [\varphi \partial_\rho (\xi^{\varepsilon,k}_3) \varphi] \sqrt{a} \, dy \]

\[ + C(1 + \| D_{\rho h} (\varphi \xi^{\varepsilon}) \|_{H^1(\mathcal{U}_1) \times H^1(\mathcal{U}_1) \times L^2(\mathcal{U}_1)}), \]

where in the last equality we used the integration-by-parts formula for finite difference quotients.

Ninth, and last, we straightforwardly observe that
\[ \int_{\mathcal{U}_1} -a^{\alpha\beta\sigma\tau} (b_{\alpha\beta} \xi^{\varepsilon,k}_3) \varphi \sqrt{a} \, dy \]

\[ = \int_{\mathcal{U}_1} -a^{\alpha\beta\sigma\tau} (b_{\alpha\beta} \xi_z) b_{\alpha\beta} [\varphi \partial_\rho (\xi^{\varepsilon,k}_3) \varphi] \sqrt{a} \, dy \]

\[ \leq C(1 + \| D_{\rho h} (\varphi \xi^{\varepsilon}) \|_{H^1(\mathcal{U}_1) \times H^1(\mathcal{U}_1) \times L^2(\mathcal{U}_1)}) \]

In conclusion, combining (28)–(36) together gives:
\[ -a^{\alpha\beta\sigma\tau} \gamma_{\sigma\tau} (\varphi \xi_z) \gamma_{\alpha\beta} (\partial_\rho (\varphi \xi^{\varepsilon,k}_3)) \lesssim -a^{\alpha\beta\sigma\tau} \gamma_{\sigma\tau} (\xi_z) \gamma_{\alpha\beta} (\partial_\rho (\varphi \xi^{\varepsilon,k}_3)). \]

Using (23), (26) in the left hand side of (25) and applying (37) gives
\[ -a^{\alpha\beta\sigma\tau} \gamma_{\sigma\tau} (\varphi \xi_z) \gamma_{\alpha\beta} (\partial_\rho (\varphi \xi^{\varepsilon,k}_3)) \]

\[ \lesssim -a^{\alpha\beta\sigma\tau} \gamma_{\sigma\tau} (\xi_z) \gamma_{\alpha\beta} (\partial_\rho (\varphi \xi^{\varepsilon,k}_3)) \]

\[ \lesssim -a^{\alpha\beta\sigma\tau} \gamma_{\sigma\tau} (\xi_z) \gamma_{\alpha\beta} (\tilde{g}^{-1} \varphi^2 [\delta_\rho \xi^{\varepsilon,k}_3 + (D_{\rho h} \tilde{g}) (D_{\rho h} \xi^{\varepsilon,k}_3) + (D_{-\rho h} \tilde{g}) (D_{-\rho h} \xi^{\varepsilon,k}_3)]) \]

\[ \lesssim -a^{\alpha\beta\sigma\tau} \gamma_{\sigma\tau} (\xi_z) \gamma_{\alpha\beta} \left( \tilde{g}^{-1} \varphi^2 \left[ \delta_\rho \xi^{\varepsilon,k}_3 + Bq \frac{|q|^2}{|q|^2} \cdot a_j \right] \right. \]

\[ + (D_{\rho h} \tilde{g}) \left( D_{\rho h} \left( \xi^{\varepsilon,k}_3 a_i + Bq \frac{|q|^2}{|q|^2} \cdot a_j \right) \right) + (D_{-\rho h} \tilde{g}) \left( D_{-\rho h} \left( \xi^{\varepsilon,k}_3 a_i + Bq \frac{|q|^2}{|q|^2} \cdot a_j \right) \right) \]

\[ + \left. \left( \xi^{\varepsilon,k}_3 + Bq \frac{|q|^2}{|q|^2} \cdot a_j \right) \right) \]

\[ \lesssim -a^{\alpha\beta\sigma\tau} \gamma_{\sigma\tau} (\xi_z) \gamma_{\alpha\beta} \left( \left( \tilde{g}^{-1} \varphi^2 \delta_\rho \left[ \left( \xi^{\varepsilon,k}_3 a_i + Bq \frac{|q|^2}{|q|^2} \cdot a_j \right) \right] \right)_{j=1}^3 \right), \]

where the latter term corresponds to the left hand side of (25).

To sum up, we have that
\[ -a^{\alpha\beta\sigma\tau} \gamma_{\sigma\tau} (\varphi \xi_z) \gamma_{\alpha\beta} (\partial_\rho (\varphi \xi^{\varepsilon,k}_3)) \lesssim 0, \]
which means that there exists a constant $C > 0$ independent of $h$ and eventually dependent on $\tilde{g}$, $\tilde{g}^{-1}$, $\|\mathcal{C}\|_{H^1(\omega) \times H^1(\omega) \times L^2(\omega)}$, $\varphi$ and its derivatives such that

$$-\varepsilon \int_{\Omega} a^{\alpha\beta\gamma\delta} \gamma_{\alpha\beta}(\varphi \mathcal{C}^{\delta}) \gamma_{\gamma\delta} (\delta_{\rho\kappa}(\varphi \mathcal{C}^{\rho})) \sqrt{a} \, dy \leq C(1 + \|D_{\rho\kappa}(\varphi \mathcal{C}^{\rho})\|_{H^1(\Omega_t) \times H^1(\Omega_t) \times L^2(\Omega_t)}).
$$

Writing the second order finite difference quotient $\delta_{\rho\kappa}$ in the form (3) and recalling the properties of $\text{supp} \, \varphi$, we are then in a position to apply the integration-by-parts formula for the first order finite difference quotient $D_{\rho\kappa}$ (cf., e.g., formula (16) in section 6.3.1 of [18]), thus getting:

$$\varepsilon \int_{\Omega} a^{\alpha\beta\gamma\delta} \gamma_{\alpha\beta}(D_{\rho\kappa}(\varphi \mathcal{C})) \gamma_{\gamma\delta} (D_{\rho\kappa}(\varphi \mathcal{C})) \sqrt{a} \, dy \leq C(1 + \|D_{\rho\kappa}(\varphi \mathcal{C})\|_{H^1(\Omega_t) \times H^1(\Omega_t) \times L^2(\Omega_t)}).
$$

A simple computation thus gives

$$\varepsilon \int_{\Omega} a^{\alpha\beta\gamma\delta} \gamma_{\alpha\beta}(D_{\rho\kappa}(\varphi \mathcal{C})) \gamma_{\gamma\delta} (D_{\rho\kappa}(\varphi \mathcal{C})) \sqrt{a} \, dy \leq C(1 + \|D_{\rho\kappa}(\varphi \mathcal{C})\|_{H^1(\Omega_t) \times H^1(\Omega_t) \times L^2(\Omega_t)})$$

$$+ \varepsilon \int_{\Omega} a^{\alpha\beta\gamma\delta} \gamma_{\alpha\beta}(D_{\rho\kappa}(D_{\rho\kappa}(\varphi \mathcal{C})) \gamma_{\gamma\delta} (D_{\rho\kappa}(\varphi \mathcal{C} - \mathcal{C})) \sqrt{a} \, dy \leq C(1 + \|D_{\rho\kappa}(\varphi \mathcal{C})\|_{H^1(\Omega_t) \times H^1(\Omega_t) \times L^2(\Omega_t)}) + C\|D_{\rho\kappa}(\varphi \mathcal{C} - \mathcal{C})\|_{H^1(\Omega_t) \times H^1(\Omega_t) \times L^2(\Omega_t)}$$

$$+ \varepsilon C \|D_{\rho\kappa}(\varphi \mathcal{C})\|_{H^1(\Omega_t) \times H^1(\Omega_t) \times L^2(\Omega_t)} \|D_{\rho\kappa}(\varphi \mathcal{C} - \mathcal{C})\|_{H^1(\Omega_t) \times H^1(\Omega_t) \times L^2(\Omega_t)}$$

$$\leq C(1 + \|D_{\rho\kappa}(\varphi \mathcal{C})\|_{H^1(\Omega_t) \times H^1(\Omega_t) \times L^2(\Omega_t)}) + \frac{C}{r} (C + \varepsilon C) \|\mathcal{C} - \mathcal{C}^\kappa\|_{H^1(\omega) \times H^1(\omega) \times L^2(\omega)}$$

$$\leq C(1 + \|D_{\rho\kappa}(\varphi \mathcal{C})\|_{H^1(\Omega_t) \times H^1(\Omega_t) \times L^2(\Omega_t)}) + \frac{C}{r} (C + \varepsilon C),$$

where the constant $C_1 > 0$ in second inequality is associated with the continuity of the bilinear form appearing in Problem $P^{\Delta}_M(\omega)$, the third inequality holds by the continuity of $D_{\rho\kappa}$ shown in Lemma 4 (up to taking a suitably large $k$ in step (i)), and the last inequality holds by the assumed “density property”.

Since, $\varepsilon = O(h^2)$, we have that the term $2h^{-1}(C + \varepsilon C)\varepsilon$ introduced in the above set of inequalities is uniformly bounded by a positive constant.

Therefore, up to renaming the bounding constant $\tilde{C}$, the following estimate holds:

$$\varepsilon \int_{\Omega} a^{\alpha\beta\gamma\delta} \gamma_{\alpha\beta}(D_{\rho\kappa}(\varphi \mathcal{C})) \gamma_{\gamma\delta} (D_{\rho\kappa}(\varphi \mathcal{C})) \sqrt{a} \, dy \leq \tilde{C}(1 + \|D_{\rho\kappa}(\varphi \mathcal{C})\|_{H^1(\Omega_t) \times H^1(\Omega_t) \times L^2(\Omega_t)}).$$

Since $\varphi \in C^3(\overline{\omega}; \mathbb{R}^3)$, we obtain that the estimates in Theorem 1 and the uniform positive definiteness of the two-dimensional fourth order elasticity tensor (cf., e.g., Theorem 3.3.2 of [7]) give the existence of a constant $c = c(\omega, \varphi, \mu) > 0$ such that

$$\varepsilon \int_{\Omega} a^{\alpha\beta\gamma\delta} \gamma_{\alpha\beta}(D_{\rho\kappa}(\varphi \mathcal{C})) \gamma_{\gamma\delta} (D_{\rho\kappa}(\varphi \mathcal{C})) \sqrt{a} \, dy \geq \frac{1}{c} \left\{ \sum_{\alpha} \|D_{\rho\kappa}(\varphi \mathcal{C})\|_{H^1(\Omega_t)}^2 + \|D_{\rho\kappa}(\varphi \mathcal{C})\|_{L^2(\Omega_t)}^2 \right\}$$

$$= \frac{1}{c} \|D_{\rho\kappa}(\varphi \mathcal{C})\|_{H^1(\Omega_t) \times H^1(\Omega_t) \times L^2(\Omega_t)}^2.$$
An application of Theorem 3 of Section 5.8.2 of [18] together with the fact that \( \varphi \equiv 1 \) in a compact strict subset of its support shows that \( \zeta \in H^2_{\text{loc}}(\omega) \times H^2_{\text{loc}}(\omega) \times H^1_{\text{loc}}(\omega) \), as it was to be proved.

As a final remark, we point out that the higher regularity for the applied body force density \( f^\varepsilon \) has been used in part (ii) of Theorem 6 in order to apply formulas (16) and (17) in section 6.3.1 of [18], and estimate the right hand side of (20) in terms of \( \| D_{\rho h}(\varphi \zeta_{\varepsilon,k}^3) \|_{L^2(U_1)} \). This, in general, would not possible if one considered a second order finite difference quotient as in [19] and [24].

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Received February 2021; 1st revision July 2021; 2nd revision August 2021; early access October 2021

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