Linear tree codes and the problem of explicit constructions

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Abstract

We reduce the problem of constructing asymptotically good tree codes to the construction of triangular totally positive matrices over fields with polynomially many elements. We show a connection of this problem to Birkhoff interpolation in finite fields.

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1 Introduction

Tree codes, in the sense we are going to use in this paper, were introduced by L.J. Schulman in 1993. He showed that asymptotically good tree codes can be used in efficient interactive communication protocols and proved by a probabilistic argument that such tree codes exist [8, 9]. He posed as an open problem to give an explicit effectively computable construction of them. Efficiently constructible tree codes would be very useful in designing robust interactive protocols. The field has attracted a lot of attention in recent years, however this central problem still remains open. A possible solution may be the construction of Moore and Schulman [5] found recently. Their construction provides asymptotically good tree codes if a certain number-theoretical conjecture, introduced in their paper, is true. The conjecture is inspired by some well-known results about exponential sums and is supported by numerical evidence.

In this paper we propose a different approach to this problem. We study generator and parity check matrices of linear codes and reduce the problem to constructing triangular totally nonsingular matrices over fields of polynomial size. A lower triangular matrix $M$ is called triangular totally nonsingular if every square submatrix of $M$ whose diagonal is entirely in the lower triangle is nonsingular. Explicit examples of such matrices are known over the field real numbers, and these include matrices with integral elements. One can also show

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that triangular totally nonsingular matrices exist over finite fields of exponential size. The question whether they exist over finite fields of polynomial size (or at least subexponential size) is open. Since totally nonsingular matrices (i.e., matrices whose all square submatrices are nonsingular) do exist over fields of linear size, we conjecture that there exist triangular totally nonsingular matrices over fields of polynomial size.

In this way we may be reducing the problem of constructing tree codes to a more difficult problem. But since the concept of triangular totally nonsingular matrices is very natural, the problem of constructing such matrices over small fields is of independent interest. We also hope that due to this connection we will be able draw attention of the linear algebra community to this important open problem in coding theory.

Here is a brief overview of the paper. In Section 1 we define linear codes and prove some basic facts about them. Some facts in this section are well-known, or well-known in some form. In particular, the existence asymptotically good linear tree codes was first proved by Schulman. In Section 2 we observe that one can concatenate a tree code with a constant size alphabet and input length \( \log n \) with a tree code with an alphabet of polynomial size and input length \( n \) in order to obtain a tree code with a constant size alphabet and input length \( O(n \log n) \). Since the “short” tree code can be found by brute force search in polynomial time, it suffices to construct in polynomial time an asymptotically good tree code with an alphabet of polynomial size in order to get a polynomial time construction of asymptotically good tree codes. This is also a well-known fact and is included for the sake of completeness.

In the main part of the paper we focus on linear tree codes of rate \( 1/2 \). In Section 4 we give a characterization of parity check matrices of linear tree codes with a given minimum distance. In Section 5 we introduce MDS linear tree codes. We show that an MDS linear tree code of rate \( 1/2 \) is determined by a triangular totally nonsingular matrix. Since the minimum distance of rate \( 1/2 \) MDS tree codes is greater than \( 1/2 \), in order to solve the construction problem, it suffices to construct triangular totally nonsingular matrices over fields of polynomial size. We discuss some approaches to the problem of constructing such matrices in Section 6. In the last section we show a connection between MDS linear tree codes and the Birkhoff interpolation problem.

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2 Basic concepts and facts

We will assume that the reader is familiar with the basic concepts and results from the theory of block codes. (The reader can find missing definitions, e.g., in [1].)

A tree code of input length \( n \) with finite alphabets \( \Pi \) and \( \Sigma \) is a mapping \( c : \Pi^n \to \Sigma^n \) of the form

\[
c(x_1, \ldots, x_n) = (c_1(x_1), c_2(x_1, x_2), \ldots, c_n(x_1, \ldots, x_n))
\]
where \( c_i : \Pi^i \to \Sigma \) and
\[
(x_1, \ldots, x_i) \mapsto (c_1(x_1), \ldots, c_i(x_1, \ldots, x_i))
\]
is a one-to-one mapping for every \( i = 1, \ldots, n \). Hence \( c \) induces an isomorphism of the tree of the input words onto the tree of output words, the code words of \( c \).

A natural way to define tree codes is to define them as mappings of infinite sequences to infinite sequences, i.e., \( c : \Pi^\omega \to \Sigma^\omega \). This would somewhat complicate the relations to the concepts in linear algebra that we want to use, so we prefer the definition with finite strings, although most of the concepts and results presented here can easily be translated to the infinite setting.

Let \( c \) be a tree code of input length \( n \). Let \( C \) be the set of the code words, i.e., the range of the function \( c \). Then the minimum relative distance of the tree code \( c \), denoted by \( \delta(c) \), is the minimum over all \( 0 \leq k < l \leq n \), \( u \in \Sigma^k \), \( v, v' \in \Sigma^{l-k} \), \( w, w' \in \Sigma^{n-l} \), \( (u, v, w), (u, v', w') \in C \), \( v_1 \neq v'_1 \) of the quantity
\[
\frac{\text{dist}(v, v')}{l - k},
\]
where \( \text{dist}(x, y) \) denotes the Hamming distance and \( v_1 \) and \( v'_1 \) are the first elements of the strings \( v \) and \( v' \).

The rate of the tree code is
\[
\rho(c) = \frac{\log |\Pi|}{\log |\Sigma|}.
\]

**Definition 1** A tree code \( c \) is linear, if \( \Pi \) and \( \Sigma \) are finitely dimensional vector spaces over a finite field \( F \) and \( c \) is a linear mapping.

It should be noted that convolutional codes are special instances of linear tree codes, but they are not interesting for us, because their minimum relative distance, as defined above, is very small.

In this paper we will focus on the codes where \( \Pi \) is the field \( F \) and \( \Sigma = F^d \). In this case, the rate of a linear code is the inverse of the dimension of \( \Sigma \), i.e., \( \rho(c) = 1/d \).

As in linear block codes, the minimum relative distance is characterized by the minimum weight of nonzero code words: the minimum relative distance of a linear tree code \( c \) is the minimum over all \( 0 \leq k < l \leq n \), \( v \in \Sigma^{l-k} \), \( w \in \Sigma^{n-l} \), \((\bar{0}^k, v, w) \in C \), \( v_1 \neq \bar{0} \) of
\[
\frac{\text{wt}_\Sigma(v)}{l - k},
\]
where \( \bar{0}^k \) is the zero vector in \( \Sigma^k \) and \( \text{wt}_\Sigma \) denotes the Hamming weight with respect to the alphabet \( \Sigma \). Note that it is also natural to consider the Hamming weight with respect to \( F \). So we define \( \tilde{\delta}(c) \) as the minimum of
\[
\frac{\text{wt}_F(v)}{d(l - k)}
\]
and focus on this quantity in the rest of this paper. Clearly \( \tilde{\delta}(c) \leq \delta(c) \).
Theorem 2.1 Let $n \geq 1$, $q = |F|$, $r = q^d = |\Sigma|$ and $0 < \delta < \frac{r-1}{r}$ such that
\[
\log_r 2q + H_r(\delta) \leq 1. \tag{2}
\]
Then there exists a linear code $c : F^n \rightarrow \Sigma^n$ with $\delta(c) > \delta$. Moreover, if $q, r$ and $\delta$ are fixed, and $\delta$ is rational, then such codes can be constructed for every $n$ in time $2^{O(n)}$.

Remarks
1. In the theorem, $H_r$ denotes the $r$-entropy function defined by
\[
H_r(x) = x \log_r (r - 1) - x \log_r x - (1 - x) \log_r (1 - x).
\]
2. Peczarski \[6\] proved that for every prime power $q$, there exist codes with relative distance $1/2$ and rate $1/(2 + \lfloor \log_q 4 \rfloor)$. This is better than the bound in the theorem above for $\delta = 1/2$.
3. Note that there exists $\delta > 0$ such that for every $q > 2$, there exist tree codes with rate $1/2$ (i.e., $d = 2$) and minimum relative distance $\geq \delta$. We do not know if binary (i.e., $q = 2$) tree codes with rate $1/2$ can have asymptotically positive minimum relative distance. (For binary block codes, this is not possible.)

Proof. [essentially, Schulman’s] Let $\Sigma = F^d$; thus $r = q^d$. Suppose $q, r$ and $\delta$ satisfy the inequality (2) above. We will prove the existence by induction. For $n = 1$, take the repetition code. Now suppose we have such a code $c$ for $n$ and want to construct a code $c' : F^{n+1} \rightarrow \Sigma^{n+1}$. We take $v \in \Sigma^n$ random and put
\[
c'(x_0, x_1, \ldots, x_n) := (x_0 \bar{1}^d, c(x_1, \ldots, x_n) + x_0 v).
\]
Here we denote by $\bar{1}^d$ the vector in $F^d$ whose all the $d$ coordinates are 1.

The minimum weight condition is satisfied in the case when $x_0 = 0$ by the induction assumption. Thus we only need to satisfy the condition for $x_0 \neq 0$ by a suitable choice of $v$. As above, let $C$ denote the range of $c$. Let $C_{[\delta dk]}$ denote the projection of the vectors of $C$ on the first $\delta dk$ coordinates.

Lemma 2.2 Let $1 \leq k \leq n$. Let $v \in \Sigma^k = F^{dk}$ be a uniformly randomly chosen vector. Let $\langle C_{[\delta dk]} \cup \{v\} \rangle$ be the span of $C_{[\delta dk]}$ and the vector $v$. Then the probability that $\langle C_{[\delta dk]} \cup \{v\} \rangle$ contains a nonzero vector $u$ with weight $wt_F(u) \leq \delta dk$ is at most $2^{-k}$.

Proof. Since by the induction assumption, there is no $u$ in $C_{[\delta dk]}$ whose weight is $\leq \delta dk$, such a vector must be a linear combination $aw + bv$ where $w \in C_{[\delta dk]}$ and $b \neq 0$. So $\text{dist}(v, -ab^{-1}w) \leq \delta dk$. The number of vectors whose distance from $C$ is $\leq \delta dk$ is estimated by $q^k$, the cardinality of $C$, times the size of a ball of radius $\delta dk$ in $F^{dk}$, which we can bound using the entropy function by $r^{H_r(\delta)k}$. Thus the probability is at most $2^{-k}$, if
\[
q^k r^{H_r(\delta)k} / r^k \leq 2^{-k},
\]
which is equivalent to (2). \[\blacksquare\]
Now we can finish the proof of the existence of the tree code. The probability that \( \tilde{\delta}(c') \leq \delta \) is at most the probability that, for some \( k \), \( \langle C|_{dk} \cup \{v\} \rangle \) contains a nonzero vector \( u \) with weight \( \text{wt}_F(u) \leq \delta dk \), which is, according to the lemma, at most \( \sum_{k=1}^{n} 2^{-k} < 1 \).

We now estimate the number of operations that are needed to find such a code. For every \( k = 1, \ldots, n-1 \), we have to search \( r^k \) vectors and we have to determine their distances from \( r^k \) vectors of the code from the previous round. Thus we have to consider \( \sum_{k=1}^{n-1} r^{2k} < r^{2n} \) cases, each of which takes polynomial time. Thus the time is \( 2^{O(n)} \).

The generator matrix of a tree code \( c \) is defined in the same way as for ordinary codes. Let \( e^n_i \) denote the vectors of the standard basis of \( F^n \), i.e., vectors

\[
(1,0,\ldots,0), (0,1,\ldots,0), \ldots, (0,0,\ldots,1).
\]

The generator matrix of \( c \) is the \( n \times dn \) matrix whose rows are vectors \( c(e^n_i) \). It is a block upper triangular matrix where the blocks are \( 1 \times d \) submatrices and the blocks on the main diagonal are nonzero vectors of \( F^d \), because the mappings \( [1] \) are one-to-one.

We define cyclic tree codes as the linear tree codes that satisfy

\[
v \in C \Rightarrow \bar{0}v|_{[d(n-1)]} \in C.
\]

This means that with every code word \( v \), the code contains a word that is obtained by adding \( d \) zeros at the beginning of \( v \) and deleting the last \( d \) coordinates. (In this particular case it would be better to use infinite sequences instead of finite ones.) We observe that if \( c \) is a cyclic tree code with the space \( C \) of the code words, there exists a cyclic tree code with the same code words whose generator matrix is block-Toeplitz. Indeed, define the generator matrix of \( c' \) as the shifts of \( c(e_1) \). Formally, put

\[
c'(e^n_i) := (\bar{0}, \ldots, \bar{0}, c(e^n_1)|_{[n-i+1]})
\]

for \( i = 1, \ldots, n \).

Note that convolutional codes (see, e.g., \([10]\)) are, essentially, a special case of cyclic linear tree codes. To this end we must consider linear tree codes of the form \( c : \Pi^d \rightarrow \Sigma^\omega \). Then \( c \) is a convolutional code if it is generated by a vector \( c(e^n_1) \) that has only a finite number of nonzero entries. Obviously, such a code cannot be asymptotically good.

We will now give a slightly different proof of the existence of good linear tree codes with the additional property of cyclicity. Note that in this proof we need only a linear number of random bits.

**Proof.** Let \( v_2, \ldots, v_n \in \Sigma \) be chosen uniformly randomly and independent. Thus \( (v_2, \ldots, v_n) \) is a random vector from \( \Sigma^{n-1} \). Let \( T := T(\bar{1}^d, v_2, \ldots, v_n) \) be the upper block triangular Toeplitz matrix with the first row equal to \( \bar{1}^d, v_2, \ldots, v_n \). Since \( T \) is Toeplitz, we only need to ensure the condition about the number of nonzero elements in nonzero vectors for vectors with the first block nonzero. Let \( 1 \leq k \leq n \) and let \( T(\bar{1}^d, v_2, \ldots, v_k) \) be the submatrix of \( T \).
determined by the first \( k \) rows and the first \( dk \) columns. We will estimate the probability that for a linear combination of the rows in which the first row has nonzero coefficient is a vector \( u \) with \( \leq \delta dk \) nonzero coordinates. The vector \( u \) can be expressed, using matrix multiplication, as

\[
\mathbf{u} = (a_1, \ldots, a_k) \mathbf{T}(\bar{1}^d, v_2, \ldots, v_k),
\]

(3)

where \( a_1 \neq 0 \). The vector \( u \) has the form \((a_1 \bar{1}^d, u_2, \ldots, u_k)\), \( u_i \in \Sigma \). Let \( a_1 \neq 0, a_2, \ldots, a_k \) be fixed and view \( v_2, \ldots, v_k \) as variables. Then (3) defines a linear mapping from \( F^{d(k-1)} \) to itself. Due to the form of the matrix \( \mathbf{T}(\bar{1}^d, v_2, \ldots, v_k) \) and the fact that \( a_1 \neq 0 \), the mapping is onto, hence the vector \((u_2, \ldots, u_k)\) is uniformly distributed. Thus we can use the Chernoff bound, or the bound by the entropy function, to estimate the probability for a fixed linear combination. Then use the union bound to estimate the probability that such a linear combination exists. The rest is the same computation as in the first proof.

Parity-check matrices for linear tree codes are defined in the same way as for ordinary codes: their row vectors are the vectors of some basis of the dual space to the space of the code words \( C \). (Thus parity-check matrices uniquely determine \( C \), but, in general, not the function \( c \).) We now describe a normal form of the parity-check matrices of linear tree codes.

**Proposition 2.3 (Normal Form)** Every linear tree code \( c : F^n \rightarrow F^{dn} \) has a parity-check matrix of the following form:

- lower block triangular matrix with blocks of dimensions \((d-1) \times d\) and with blocks on the main diagonal of full rank \( d-1 \).

Vice versa, any matrix satisfying the condition above is a parity-check matrix of a linear tree code \( c : F^n \rightarrow F^{dn} \).

**Proof.** Let \( M \) be a parity-check matrix of a tree code. We will transform \( M \) into the form described above using row operations, i.e., we will use Gaussian elimination to rows.

The matrix \( M \) has dimensions \((d-1)n \times dn\) because its rows span a vector space dual to \( C \) and \( C \) has dimension \( n \). The basic property of the matrix is:

\( (*) \) for every \( 1 \leq k \leq n \) the last \( dk \) columns of \( M \) span a vector space of dimension \((d-1)k\).

To prove \( (*) \), consider the matrix \( M' \) consisting of the last \( dk \) columns. Let \( L \) be the row space of \( M' \). The dual space \( L^\perp \) is the space of all vectors \( v \in F^{dk} \) such that \((\bar{0}^{d(n-k)}, v) \in C \). Its dimension is at least \( k \), because it contains the projections of \( k \) linearly independent vectors \( c(e^n_i) \), \( i = (n-k) + 1, \ldots, n \). It also is at most \( k \), because every linear combination of generating vectors that contains some \( c(e^n_i) \), \( i \leq n-k \), with a nonzero coefficient has a nonzero coordinate outside of the last \( dk \) positions. Thus, indeed, the dimension of \( L \) is \((d-1)k\).

We start the elimination process with the last \( d \) columns. Since the rank of this matrix is \( d-1 \), we can eliminate all rows of this \( d \times (d-1)n \) matrix except for \( d-1 \) ones that form a
basis of the row space. We permute the rows so that these $d - 1$ rows are at the bottom. Now consider the submatrix $M'$ with the first $(d - 1)(n - 1)$ rows and the first $d(n - 1)$ columns of the transformed parity-check matrix and the submatrix $N'$ of the generator matrix $N$ of $C$ with the first $n - 1$ rows and the first $d(n - 1)$ columns. The matrix $M'$ has full rank, because $M$ has it, and the rows of $N'$ are orthogonal to the rows of $N'$. Hence $M'$ is a parity-check matrix of the code defined by $N'$. So we can assume as the induction hypothesis that $M'$ can be transform into a normal form. Thus $M$ has been transformed into a normal form.

Now we prove the opposite direction. Let $M$ be a matrix satisfying the condition of the proposition (in fact, we will be using the property (*) that follows from it). We will construct a generator matrix of a tree code $c$ starting from the last row of the matrix and going upwards. Let $v \in \Sigma$ be a nonzero vector that is orthogonal to the row space of the submatrix of $M$ consisting of the last $d$ columns. We define $c(e_n)$ to be $v$ preceded with $d(n - 1)$ zeros. Suppose we already have $c(e_{n-k+1}), \ldots, c(e_n)$. We take any vector $u \in F^{d(k+1)}$ that is orthogonal to the row space of the submatrix of $M$ consisting of the last $d(k + 1)$ columns and is independent of the vectors $c(e_{n-k+1}), \ldots, c(e_n)$ restricted to the last $d(k + 1)$ coordinates. The vector $u$ must have some nonzero on the first $d$ coordinates, because $c(e_{n-k+1}), \ldots, c(e_n)$ span the space dual to the row space of $M$ restricted to the last $dk$ columns. Then we define $c(e_{n-k-1})$ to be $u$ preceded with $d(n - k - 1)$ zeros.

3 From a large alphabet to a small one

It is well-known that it suffices to construct an asymptotically good tree code whose input and output alphabets have polynomial sizes in order to construct an asymptotically good tree code with finite alphabets. The resulting construction is not quite explicit, because it relies on the construction of small tree codes by brute-force search, but it can produce the code in polynomial time. We present this reduction for the sake of completeness and also in order to check that it works for linear codes. For simplicity, we will restrict ourselves to the binary input alphabet and finite fields of characteristic 2.

**Proposition 3.1** Let $b, d$ and $\delta > 0$ be constants, then there exist constants $d'$ and $\delta' > 0$ such that the following is true. Suppose a generator (or parity-check) matrix, of a linear tree code $c : F_{2^b}^n \to F_{2^{dn}}^{d}$ is given, where $\ell \leq \log_2 n$ and the minimum relative distance of $c$ is $\delta$. Then one can construct in polynomial time a generator matrix of a binary linear tree code $c' : F_2^{n'} \to F_2^{d'n'}$ where $n' = \ell(n + 1)$ and the minimum relative distance of $c'$ is $\delta'$.

**Proof.** The basic idea is simple: we will replace the symbols of the large code $c$ by bit-strings of a binary code $a$ of logarithmic length. The short code $a$ can be found in polynomial time in $n$ by Theorem 2.1 because its length is logarithmic. Here is the construction in more detail.

Let $c : F_{2^b}^n \to F_{2^{dn}}$ be given by its generator matrix. Let $a : F_2^{d'} \to F_{2^{d'}\ell}$ be a cyclic linear tree code with minimum distance $\delta'' > 0$\footnote{The use of cyclic codes is not essential, but it simplify the construction.} By Theorem 2.1, we can pick constants...
and \( d'' > 0 \) (independent on \( \ell \)), and construct the generator matrix of such a tree code \( a \) in polynomial time in \( n \). Further we need a good block code \( f : F_2^{d'\ell} \to F_2^{d''\ell} \). So \( d^* \) and its minimum relative distance \( \epsilon > 0 \) are further constants. (Explicit polynomial time constructions of such block codes are well-known.) We may assume w.l.o.g. that \( d'' = d^* \).

The generator matrix of \( c' \) is defined as follows. We set \( d' := 2d'' \) (because we want to use odd bits for the code words of \( a \) and even bits for the code words of \( f \)). We need to define \( c'(e_i^n) \) for \( i = 1, \ldots, n' \). Let \( i \) be given and let \( i = \ell k + j \) where \( 0 \leq k \leq n \) and \( 1 \leq j \leq \ell \).

1. On the odd bits of \( c'(e_i) \), we put the string \( a(e_i^1) \), where \( e_i^1 = 10^{\ell-1} \), so that it starts at the \( d'(i-1) + 1 \)-st bit (the first bit on which \( c'(e_i) \) should be nonzero). We fill the rest of the odd bits by zeros. Recall that in the cyclic tree code \( a \) all vectors \( a(e_{i+1}^1) \) are obtained from \( a(e_i^1) \) by shifting and truncating. Here we truncate it only if \( k = n \) and \( j > 1 \).

2. On the even bits, if \( k < n \), we put \( c(e_k^\ell) \) encoded by \( f \) shifted by \( d'\ell \). If \( k = n \), we put zeros everywhere.

We will now estimate the minimum weight of segments of the code words of \( c' \). Let \( 0 \leq i < j \leq n' \) be given and suppose that \( v \) is an input word in which the first nonzero element is on the coordinate \( i + 1 \).

If \( j \leq i + \ell \), then there are at least \( \delta''d'/(j - i) \) nonzero elements among the odd elements in the interval in \( c'(v) \) corresponding to the interval \( (i, j] \) in \( v \) because the vector restricted to odd coordinates in this interval is a code word of the tree code \( a \). Hence the relative weight is at least \( \delta''/2 \), and if \( j \leq i + 2\ell \), the relative weight is at least \( \delta''/4 \).

Now suppose that \( j > i + 2\ell \). Then there is at least one entire \( \ell \)-block between \( i \) and \( j \). Suppose there are \( k \) such \( \ell \)-blocks. They correspond to \( k \) consecutive elements of a code word of \( c \) in which the first element is nonzero. Hence there are at least \( \delta k \) nonzero elements among them. Using the code \( f \), they are encoded in \( c'(v) \) to a string with at least \( \epsilon\delta d'\ell k \) nonzero elements. Since entire blocks cover at least \( 1/3 \) of the interval, this ensures positive relative minimum distance at least \( \epsilon\delta/3 \) on even bits, hence \( \epsilon\delta/6 \) on all bits.

It is possible that an explicit construction of tree codes is found where the fields have polynomial size, but their characteristic increases with \( n \). E.g., the fields could be prime fields with the prime \( p \) larger than \( n \). Then the above construction cannot be used to construct a linear tree code over a constant size field. However, essentially the same construction yields a general tree code. But when talking about polynomial time constructions of general tree codes, we have to be more specific about what this means. We will say that a family of tree codes is constructible in polynomial time if there is a polynomial time algorithm that for every tree code \( c : \Pi^n \to \Sigma^n \) in the family and every given input word \( x \in \Pi^n \) computes \( c(x) \). We assume that an encoding of the alphabets \( \Pi \) and \( \Sigma \) by binary strings is given.

**Proposition 3.2** Let \( b, d \) and \( \delta > 0 \) be constants, then there exist constants \( d' \) and \( \delta' > 0 \) such that the following is true. Suppose a family of polynomial time constructible tree codes
C is given such that for every code \( c : \Pi^n \rightarrow \Sigma^n \) in the family, \( q = |\Pi| \leq n^b \), \( \Sigma = \Pi^d \) and the minimum relative distance of \( c \) is \( \delta \). Then there exists a family \( \mathcal{C}' \) of polynomial time constructible tree codes such that for every tree code \( c : \Pi^n \rightarrow \Sigma^n \), \( |\Pi| \leq n^b \) in \( \mathcal{C} \), there is a tree code \( c' : \{0,1\}^{n'} \rightarrow \{0,1\}^{d'n'} \) where \( n' = \lfloor \log_2 q \rfloor (n + 1) \) and the minimum relative distance of \( c' \) is \( \delta' \).

Proof-sketch. Choose a one-to-one mapping from \( \{0,1\}^{\lfloor \log_2 q \rfloor} \) into \( \Sigma \). Then proceed in the same way as in the proof above with the only difference that now we have to define every code word instead of the generators.

4 A characterization of the minimum distance

We will characterize the minimum distance the tree codes defined by parity-check matrices in normal forms. For the sake of simplicity, we will assume that the rate of the tree codes is \( 1/2 \) (i.e., \( c : F^n \rightarrow F^{2n} \)). We will use the following standard notation. Given a matrix \( M \) and indices of rows \( i_1 < \cdots < i_\ell \) and columns \( j_1 < \cdots < j_k \),

\[
M[i_1, \ldots, i_\ell | j_1, \ldots, j_k]
\]
denotes the submatrix of \( M \) determined by these rows and columns.

**Proposition 4.1** Let \( M \) be an \( n \times 2n \) parity-check matrix of a linear tree code \( c \) in a normal form. Then \( \tilde{\delta}(c) \) is the least \( \delta > 0 \) such that there are \( 0 \leq k < \ell \leq n \) and \( t \) indices \( 2k < j_1 < \cdots < j_t \leq 2\ell \), \( j_1 \leq 2k + 2 \) such that

1. \( t \leq 2\delta(\ell - k) \), and
2. in \( M[k+1, k+2, \ldots, \ell | j_1, \ldots, j_t] \) the first column is a linear combination of the other columns.

**Proof.** Let \( v \) be a nonzero code word of the code for which the minimum distance is attained. Let \( j_1 \) be the first coordinate of \( v \) that is nonzero and let \( 2k < j_1 \leq 2k + 2 \), \( k < \ell \leq n \) and \( j_1, \ldots, j_t \) be all the nonzero coordinates of \( v \) between \( 2k + 1 \) and \( 2\ell \) such that

\[
\tilde{\delta}(c) = \frac{t}{2(\ell - k)}.
\]

Since \( M \) is a parity-check matrix of the code, the sum of columns of \( M \) with weights \( v_t \) must be a zero vector. Note the following two facts. First, the columns \( 2k+1, 2k+2, \ldots, 2n \) have zeros on the rows \( 1, \ldots, k \). Second, the columns \( 2\ell+1, 2\ell+2, \ldots, 2n \) have zeros on the rows \( 1, \ldots, 2\ell \). From this, we get condition 2. Hence \( \tilde{\delta}(c) \) is at least the minimum \( \delta \) that satisfies the conditions of the lemma.

To show that it is at most \( \delta \), suppose that \( 2k < j_1 < \cdots < j_t \leq 2\ell \), \( j_1 \leq 2k + 2 \) are such that the two conditions are satisfied. Let \( 1, \alpha_2, \ldots, \alpha_t \) be the weights of a linear combination
that makes the zero vector from the columns of $M[k+1, k+2, \ldots, \ell | j_1, \ldots, j_t]$. We will show that there is a code word $v$ that has zeros before the coordinate $j_1$, it has 1 on it, and all nonzeros between $2k+1$ and $2\ell$ are on coordinates $j_1, \ldots, j_t$. It is clear what are the coordinates of $v$ up to $2\ell$. This guarantees that the vector $Mv$ has zeros on all coordinates 1, \ldots, $\ell$, no matter how we define $v$ on the remaining coordinates $2\ell+1, \ldots, 2n$. Now we observe that the matrix $M[\ell+1, \ldots, n | 2\ell+1, \ldots, 2n]$ has full rank, so a suitable choice of the coordinates $2\ell+1, \ldots, 2n$ will make the product $Mv^\perp$ zero vector. Hence

$$\tilde{\delta}(c) \leq \frac{t}{2(\ell-k)} \leq \delta.$$ 

\section{MDS tree codes}

In this section we define tree codes that correspond to MDS block codes and prove two characterizations of them. Again, for the sake of simplicity, we define it only for rate 1/2 codes. First we prove a general upper bound on the relative distance of linear tree codes of rate 1/2 that corresponds to the Singleton bound for block codes. (As in the theory of block codes, this bound holds true also for nonlinear tree codes and similar bounds can be proven for other rates.)

\begin{proposition}
For every tree code $c : F^n \to F^{2n}$, $\tilde{\delta}(c) \leq \frac{n+1}{2n}$.
\end{proposition}

\begin{proof}
Let $M = (m_{ij})_{i,j}$ be a parity check matrix in a normal form. If $m_{11} = 0$ or $m_{12} = 0$ we can construct a code word whose second, respectively, first coordinate is zero. Hence $\delta(c) \leq \frac{1}{2} \leq \frac{n+1}{2n}$.

So suppose that $m_{11} \neq 0$ and $m_{12} \neq 0$. Let $3 \leq j_2 \leq 4 \ldots 2n - 1 \leq j_n \leq 2n$ be indices of columns such that $M_{i,j_i} \neq 0$. We know that such columns exist by Proposition 2.3. Then the first column of $M$ is a linear combination of columns $2, j_2, j_3, \ldots, j_n$. Hence there is a code word whose first coordinate is nonzero and it has at most $n+1$ nonzero coordinates. Thus $\delta(c) \leq \frac{n+1}{2n}$.

The tree codes that meet the bound of Proposition 5.1 naturally correspond to MDS block codes and therefore we make the following definition.

\begin{definition}
A linear tree code $c : F^n \to F^{2n}$ will be called an MDS tree code if $\tilde{\delta}(c) = \frac{n+1}{2n}$.
\end{definition}

By Proposition 5.1 the condition $\tilde{\delta}(c) = \frac{n+1}{2n}$ is equivalent to $\tilde{\delta}(c) > \frac{1}{2}$.

\begin{proposition}
Let $M$ be a parity-check matrix of a linear tree code $c : F^n \to F^{2n}$ and let $M$ be in a normal form. Then $c$ is an MDS tree code if and only if for every $n$-tuple $1 \leq j_1 < \cdots < j_n \leq 2n$ satisfying

$$j_1 \leq 2, j_2 \leq 4, \ldots, j_n \leq 2n,$$

the columns $j_1, \ldots, j_n$ are linearly independent.
\end{proposition}
Proof. First we show that the condition in the proposition implies the following formally stronger condition:

(ξ) for every \(0 \leq \ell < \ell + t \leq n\) and \(2\ell < j_1 < \cdots < j_t\), where \(j_1 \leq 2(\ell + 1), \ldots, j_t \leq 2(\ell + t)\), the matrix \(M[\ell + 1, \ldots, \ell + t \mid j_1, \ldots, j_t]\) is nonsingular.

Indeed, given \(j_1 < \ldots < j_t\) satisfying the general condition, we can add \(\ell\) elements before it and \(n - \ell - t\) elements after it so that the resulting \(n\)-tuple satisfies the condition of the proposition. Let \(N\) be the matrix consisting of these \(n\) columns of \(M\). The matrix \(N\) has the following block structure consisting of square matrices

\[
\begin{pmatrix}
T_1 & 0 & 0 \\
A & M^* & 0 \\
B & C & T_2
\end{pmatrix}
\]

where \(M^* = M[\ell + 1, \ldots, \ell + t \mid j_1, \ldots, j_t]\) and \(T_1, T_2\) are lower triangular. Since \(N\) is nonsingular, \(M^*\) must also be nonsingular.

Now suppose \(M\) satisfies condition (ξ). Arguing by contradiction, suppose that \(\tilde{\delta}(c) \leq \frac{1}{2}\). By Proposition 4.1 we have \(0 \leq \ell < k \leq n\) and \(t\) indices \(2\ell < j_1 < \cdots < j_t \leq 2k\), \(j_1 \leq 2\ell + 2\), \(t \leq k - \ell\) such that in \(M[\ell + 1, \ell + 2, \ldots, k \mid j_1, \ldots, j_t]\) the first column is a linear combination of the other columns. (We are using the fact that \(\lfloor \frac{1}{2}(2k - j_1 + 1) \rfloor = k - \ell\).) Let \(s\) be the maximal element \(1 \leq s \leq t\) such that for all \(1 \leq r \leq s\), the inequality \(j_r \leq 2(\ell + r)\) is true. If \(s < t\), then \(j_{s+1} > 2(\ell + s + 1)\). Hence any column \(j_r\), for \(r > s\), has zeros in rows \(\ell + 1, \ldots, \ell + s\). Thus the fact that the first column of \(M[\ell + 1, \ell + 2, \ldots, k \mid j_1, \ldots, j_t]\) is a linear combination of the others implies that the same holds true for \(M[\ell + 1, \ell + 2, \ldots, \ell + s \mid j_1, \ldots, j_s]\). But this is impossible, because this matrix is nonsingular according to (ξ).

To prove the opposite implication, suppose that we have an \(n\)-tuple \(1 \leq j_1 < \cdots < j_n \leq 2n\) satisfying \(j_1 \leq 2, j_2 \leq 4, \ldots, j_n \leq 2n\) such that the columns \(j_1, \ldots, j_n\) are linearly dependent. Suppose that for some \(\ell\), the column \(j_{\ell + 1}\) is a linear combination of columns \(j_{\ell + 2}, \ldots, j_n\). Then in \(M[\ell + 1, \ell + 2, \ldots, n \mid j_{\ell + 1}, \ldots, j_n]\) the first column is a linear combination of the other columns, which violates condition 2 of Proposition 4.1.

If \(M = (m_{i,j})_{i,j}\) is a parity check matrix in a normal form of a code of rate 1/2, we have either \(m_{i,2i-1} \neq 0\) or \(m_{i,2i} \neq 0\) for every \(1 \leq i \leq n\). Since permuting columns \(2i - 1\) and \(2i\) does not change the relative distance, we can always assume w.l.o.g. that

(η) all entries \(m_{i,2i}, i = 1, \ldots, n\), are nonzero.

Let \(M\) be in a normal form and suppose that it satisfies (η). Using row operations we can eliminate all nonzero entries in even columns, except for \(m_{i,2i}\). Then we can multiply the rows to get \(m_{i,2i} = 1\). The resulting matrix consists of a lower triangular matrix interleaved with the identity matrix \(I_n\). We will characterize these triangular matrices of MDS tree codes.

A matrix \(M\) is called totally nonsingular if every square submatrix of \(M\) is nonsingular. A triangular matrix of dimension \(n \geq 2\) cannot be totally nonsingular because in a totally
nonsingular matrix every element is nonzero. However, there is a natural modification that does make sense for triangular matrices.

**Definition 3** An \( n \times n \) lower triangular matrix \( L \) is called triangular totally nonsingular if for every \( 1 \leq s \leq n \) and every \( 1 \leq i_1 < \cdots < i_s \leq n \), \( 1 \leq j_1 < \cdots < j_s \leq n \) such that \( j_1 \leq i_1, \ldots, j_s \leq i_s \), the submatrix \( L[i_1, \ldots, i_s]_{j_1, \ldots, j_s} \) is nonsingular.

Roughly speaking, \( L \) is triangular totally nonsingular if it is triangular and every square submatrix of \( L \) that can be nonsingular, is nonsingular. Upper triangular totally nonsingular matrices are defined by reversing the inequalities between \( i_s \) and \( j_s \).

**Theorem 5.3** Suppose that a parity check matrix of linear tree code \( c : F^n \rightarrow F^{2n} \) has the form of a lower triangular matrix \( T \) interleaved with the identity matrix \( I_n \). Then \( c \) is an MDS tree code if and only if \( T \) is triangular totally nonsingular.

Let us note that a similar fact for MDS codes is well-known (namely, the statement with totally nonsingular matrices instead of triangular totally nonsingular matrices).

**Proof.** Let \( j_1 < \cdots j_p \) be some columns of \( T \) and \( k_1 < \cdots < k_q \) some columns of \( I_n \) where \( p + q = n \). Consider the determinant of the matrix formed by these columns. Observe that each nonzero term in the formula for this determinant must choose elements with coordinates \((k_1, k_1), \ldots, (k_q, k_q)\) from \( I_n \) because these are the only nonzero elements in these columns. This implies that the determinant is equal, up to the sign, to

\[
\det(T[i_1, \ldots, i_p | j_1, \ldots, j_p]),
\]

where \( \{i_1, \ldots, i_p\} = [1, n] \setminus \{k_1, \ldots, k_q\} \), \( i_1 < \cdots < i_q \). (Note that we are now indexing columns by numbers from 1 to \( n \) in both matrices \( T \) and \( I_n \).) Hence to prove the theorem it suffices to show that the condition (4) of Proposition 5.2 on the indices of columns that should be independent is equivalent to the condition on the indices of rows and columns of submatrices that should be nonsingular in a triangular nonsingular matrix.

First we note that the condition (4) translates to the following

\[\text{for all } s \leq p + q, \quad |\{j_1, \ldots, j_p\} \cap [1, s]| + |\{k_1, \ldots, k_q\} \cap [1, s]| \geq s. \quad (5)\]

Since

\[|\{k_1, \ldots, k_q\} \cap [1, s]| = s - |\{i_1, \ldots, i_p\} \cap [1, s]|,
\]

condition (5) is equivalent to

\[|\{j_1, \ldots, j_p\} \cap [1, s]| \geq |\{i_1, \ldots, i_p\} \cap [1, s]|. \quad (6)\]

This inequality is satisfied for all \( s \leq p + q \) if and only if it is satisfied for all \( s = i_1, \ldots, i_p \). But for \( s = i_r \), the inequality (6) is equivalent to the simple condition that \( j_r \leq i_r \), which is the condition required in the definition of triangular totally nonsingular matrices.

---

\[\text{See Ch.11, §4, Theorem 8 in [4].} \]
We note that we get a similar characterization of generator matrices of MDS linear tree codes of rate 1/2.

**Corollary 5.4** Suppose that a linear tree code $c : F^n \rightarrow F^{2n}$ has a parity check matrix satisfying condition ($\eta$). Then $c$ is an MDS tree code if and only if it has a generator matrix $N$ whose form is an upper triangular totally nonsingular matrix $S$ interleaved with $-I_n$ (minus the identity matrix).

**Proof.** Let $T$ be a lower triangular totally nonsingular matrix. Let $T$ interleaved with $I_n$ be a parity check matrix of an MDS tree code $c : F^n \rightarrow F^{2n}$. Then $N$ constructed from $(T^{-1})^T$ and $-I_n$ is, clearly, a generator matrix that generates the code words of $c$. By Jacobi’s equality, $T^{-1}$ is triangular totally nonsingular.

The proof of the opposite direction is essentially the same and we leave it to the reader.

6 Triangular totally nonsingular matrices

By Proposition 5.2 and Theorem 5.3, the problem of constructing an asymptotically good tree code reduces to the problem of constructing a triangular totally nonsingular matrix over a field of polynomial size. We are not able to construct such matrices and, in fact, we are even not able to prove that they exist.

**Problem 1** Do there exist triangular totally nonsingular matrices over fields with polynomially many elements? If they do, construct them explicitly.

According to Theorem 5.3, the problem is equivalent to the question whether there exist linear MDS tree codes over fields with polynomially many elements. We believe that in order to prove that such matrices (and such codes) exist, one has to define them explicitly. In this section we will discuss some approaches to this problem.

First we observe that triangular totally nonsingular matrices exist in fields of every characteristic. A simple way of proving this fact is to take a lower triangular matrix whose entries on and below the main diagonal are algebraically independent over a field of a given characteristic. Below is a slightly more explicit example.

**Lemma 6.1** Let $x$ be an indeterminate and let $W_n(x) = (w_{ij})_{i,j=1}^n$ be the lower triangular $n \times n$ matrix defined by

$$w_{ij} = x^{(n-i+j-1)^2}$$

for $i \geq j$, and $w_{ij} = 0$ otherwise. Then for every $1 \leq i_1 < \cdots < i_s \leq n$, $1 \leq j_1 < \cdots < j_s \leq n$ such that $j_1 \leq i_1, \ldots, j_s \leq i_s$, the determinant

$$\det(W_n(x)[i_1, \ldots, i_s | j_1, \ldots, j_s])$$

is a nonzero polynomial in every characteristic.
Proof. We will show that the monomial of the highest degree occurs exactly once in the formula defining the determinant. We will use induction. For \( s = 1 \), it is trivial. Suppose that \( s > 1 \) and let \( 1 \leq i_1 < \cdots < i_s \leq n, 1 \leq j_1 < \cdots j_s \leq n \) such that \( j_1 \leq i_1, \ldots, j_s \leq i_s \) be given. Denote by \( M := W_n(x)[i_1, \ldots, i_s, j_1, \ldots, j_s] \).

First we need to prove an auxiliary fact. We will call \((i, j)\) an extremal position in the matrix \( M \) if

1. \( i \in \{i_1, \ldots, i_s\}, j \in \{j_1, \ldots, j_s\}, w_{ij} \neq 0 \) and,
2. for every \((i', j')\), if \( i' \in \{i_1, \ldots, i_s\}, j' \in \{j_1, \ldots, j_s\} \) \( i' \leq i, j' \geq j \) and \((i, j) \neq (i', j')\), then \( w_{i'j'} = 0 \).

We will show that the monomial with the highest degree must contain all \( w_{ij} \) where \((i, j)\) is extremal.

Suppose that \((i, j)\) is an extremal position and some nonzero monomial \( m \) does not contain \( w_{ij} \). Then \( m \) must contain some elements \( w_{ij'} \) and \( w_{i'j} \) where \( j' < j \) and \( i' > i \). Observe that

\[
w_{ij}w_{ij'} = x^{(n-i+j'-1)^2 + (n-i'+j-1)^2} \quad \text{and} \quad w_{ij}w_{i'j'} = x^{(n-i+j-1)^2 + (n-i'+j'-1)^2}.
\]

The difference between the second and the first exponent is

\[
-2ij - 2i'j' + 2ij' + 2i'j = 2(i' - i)(j - j') > 0.
\]

Hence we get a monomial of higher degree if we replace \( w_{ij}w_{ij'} \) by \( w_{ij}w_{i'j'} \). This establishes the fact.

Now we can finish the proof. Delete from \( M \) all rows and columns whose indices occur in the extremal position. The remaining matrix has a unique monomial \( m' \) of the maximal degree. The unique monomial of the highest degree of \( M \) is obtained from \( m' \) by multiplying it by \( w_{ij} \) for all extremal positions \((i, j)\).

Given a prime \( p \) and a number \( n \geq 1 \), we can take an irreducible polynomial \( f(x) \) over \( \mathbb{F}_p \) of degree higher than the degrees of the determinants of the square submatrices of \( W_n(x) \). Then the matrix is triangular totally nonsingular over the field \( \mathbb{F}_p[x]/(f(x)) \). The size of this field is exponential, because the degree of \( f(x) \) is polynomial in \( n \).

While we do not know the answer to the problem above, constructions of totally nonsingular matrices over fields of polynomial size are known. Here are some examples.

Let \( F \) be an arbitrary field and let \( a_1, \ldots, a_m, b_1, \ldots, b_n \in F \) be such that \( a_i \neq b_j \) for all \( i, j \). The matrix \( \left( \frac{1}{a_i-b_j} \right)_{i,j} \) is called a Cauchy matrix. If all \( a_i \) and all \( b_j \) are distinct elements, then the matrix is nonsingular. Since every submatrix of a Cauchy matrix is a Cauchy matrix, the condition also implies that the Cauchy matrix with distinct elements \( a_1, \ldots, a_m, b_1, \ldots, b_n \) is totally nonsingular. Thus given a field with at least \( 2n \) elements, we are able to construct a totally nonsingular matrix of the dimension \( n \). A special case of a Cauchy matrix is the Hilbert matrix \( \left( \frac{1}{i+j-1} \right)_{i,j} \).
More generally, we call a matrix of the form \( \begin{pmatrix} g_i h_j \\ a_i - b_j \end{pmatrix} \), \( g_i, h_j \in F \) a Cauchy-like matrix. Such a matrix is totally nonsingular if and only if all the elements \( a_1, \ldots, a_m, b_1, \ldots, b_n \) are distinct and the elements \( g_1, \ldots, g_m, h_1, \ldots, h_n \) are nonzero. One special case is the Singleton matrices which are matrices of the form \( \begin{pmatrix} \frac{1}{1-a^i+b^j} \end{pmatrix} \), where the order of the element \( a \) is larger than \( 2n - 2 \).

One can also construct a totally nonsingular matrix from a parity-check matrix of an MDS block code. MDS block codes are usually constructed from Vandermonde matrices. We will explain the construction of a totally nonsingular matrix on this special case. We denote by \( V_m(x_1, \ldots, x_n) \) the Vandermonde matrix \( (x_i^j)_{i=0,\ldots,m-1,j=1,\ldots,n} \). Consider the Vandermonde matrix \( V_m(a_1, \ldots, a_m, b_1, \ldots, b_n) \) where all the elements \( a_1, \ldots, a_m, b_1, \ldots, b_n \) are distinct. If we diagonalize the first \( m \) columns using row operations, then the submatrix consisting of the last \( n \) columns becomes a totally nonsingular matrix. (This follows from the fact that any set of \( m \) columns is independent using the argument used in the proof of Theorem 5.3.) The diagonalization can be represented as multiplying \( V_m(a_1, \ldots, a_m, b_1, \ldots, b_n) \) by \( V_m(a_1, \ldots, a_m)^{-1} \) from the left. So this means that the matrix

\[
V_m(a_1, \ldots, a_m)^{-1}V_m(b_1, \ldots, b_n)
\]

is totally nonsingular. One can check that this matrix is also Cauchy-like. (See, e.g., [3], page 159; note that the notation used there is different.)

The theory of totally positive matrices is another source of totally nonsingular matrices. A matrix over \( \mathbb{R} \) is totally positive if all its square submatrices have a positive determinant. An example of a totally positive matrix is the Pascal matrix \( P_n := \binom{i+j}{j}^n_{i,j=0} \). The elements of this matrix are integers, so we can consider it over any field. Hence if \( p \) is a sufficiently large prime, then \( P_n \) is totally nonsingular in \( \mathbb{F}_p \), but we do not know if \( p \) can be subexponential. The same problem arises for other examples of totally positive matrices with rational coefficients, and triangular totally positive matrices as well—we do not know if we can find a polynomially large prime for which these matrices are still totally nonsingular.

Another important example of a totally positive matrix is the Vandermonde matrix \( V_n(a_1, \ldots, a_n) \) with \( 0 < a_1 < \cdots < a_n \). A general Vandermonde matrix is not always totally nonsingular.

Triangular totally positive matrices are defined in a similar way as triangular totally nonsingular matrices. The most interesting fact for us is that total positivity is preserved by \( LU \) decompositions. We quote the following result of Cryer [1], see also [2], Corollary 2.4.2.

**Theorem 6.2** A matrix \( M \) is totally positive if and only if it has an \( LU \) factorization in which the terms \( L \) and \( U \) are both triangular totally positive.

Recall that the terms in an \( LU \) decomposition are unique up to multiplication by diagonal nonsingular matrices. Hence if \( M \) is totally positive, then \( L \) and \( U \) are both triangular totally nonsingular for any \( LU \) factorization. (The factors that are totally positive are those whose all entries are nonnegative.)
This theorem suggests to construct triangular totally nonsingular matrices from totally nonsingular matrices using an LU decomposition. Unfortunately, the statement corresponding to the theorem above is not true for totally nonsingular matrices.

**Fact 1** There exists a $4 \times 4$ matrix over rational numbers which is totally nonsingular, but its L-factor is not triangular totally nonsingular.

**Proof.** Suppose that we use Gaussian elimination to transform a totally nonsingular $4 \times 4$ matrix to its L-factor and we arrive at the following situation:

$$
\begin{pmatrix}
1 & 0 & 0 & 0 \\
1 & 2 & 0 & 0 \\
2 & 3 & 2 & * \\
3 & 4 & 3 & *
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 0 & 0 & 0 \\
1 & 2 & 0 & 0 \\
2 & 3 & 2 & * \\
3 & 4 & 3 & *
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
2 & 3 & 2 & 0 \\
3 & 4 & 3 & *
\end{pmatrix}
$$

The entries denoted by * will be determined later. The last matrix is not triangular totally nonsingular, because it contains \( \begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix} \) as a submatrix. So we only need to show that the first matrix can be obtained from a totally nonsingular matrix by elimination in the first row. Equivalently, we need to show that there exists a totally nonsingular matrix $M$ of the form

$$
\begin{pmatrix}
1 & a & b & c \\
1 & 2+a & 2+b & d \\
2 & 3+2a & 5+2b & e \\
3 & 4+3a & 7+3b & f
\end{pmatrix}
$$

Let $N$ be the $3 \times 3$ submatrix in the left lower corner of the first matrix in the chain above. We will use the fact that $N$ is totally nonsingular, which is easy to verify.

We first show that one can choose $a$ and $b$ so that the first three columns form a totally nonsingular matrix $M'$. Let $A$ be a submatrix of $M'$. If $A$ contains the first column, then it is nonsingular, because $N$ is. If $A$ is a $2 \times 2$ submatrix in the second two columns, we can transform it, using a row operation, into a matrix in which $a$ and $b$ only appear in the first row and there are nonzero elements in the second. Thus $A$ is nonsingular provided that a certain nontrivial linear function in $a$ and $b$ does not vanish. Hence there is a finite number of nontrivial linear equations such that if we pick $a$ and $b$ so that none is satisfied, then $M'$ is totally nonsingular.

Let $a$ and $b$ be fixed so that $M'$ is totally nonsingular. Now consider a submatrix $A$ of $M$ that contains the last column. If $A$ is singular, then a certain linear function in $c, d, e, f$ must vanish. This function is nontrivial, because its coefficients are subdeterminants of $M'$, possibly with negative signs, or it is just one of the variables $c, d, e, f$. Thus, again, there is a choice of $c, d, e$ and $f$ that makes all these functions nonzero and hence all these matrices $A$ nonsingular.

On the positive side, we can prove the following simple fact, which is, however, not sufficient for constructing good tree codes.
Proposition 6.3 Let $F$ be an arbitrary field and let $M$ be an $n \times n$ totally nonsingular matrix over $F$. Let $M = LU$ be an LU-factorization. Then for every $1 \leq k \leq n$, and $1 \leq j \leq i_1 < \cdots < i_k \leq n$, the matrix

$$L[i_1, \ldots, i_k | j, j+1, \ldots, j+k-1]$$

is nonsingular.

Proof. An $L$ factor of $M$ can be obtained by Gaussian elimination using column operations. Hence every matrix of the form $L[i_1, \ldots, i_k | 1, \ldots, k]$ is a matrix obtained from $M[i_1, \ldots, i_k | 1, \ldots, k]$ using column operations. Thus every such matrix is nonsingular.

Now consider a submatrix $L[i_1, \ldots, i_k | j, j+1, \ldots, j+k-1]$, where $1 \leq j \leq i_1 < \cdots < i_k \leq n$. Extend this matrix to $L[1, \ldots, j-1, i_1, \ldots, i_k | 1, 2, \ldots, j+k-1]$, which is nonsingular by the previous observation. Also the matrix $L[1, \ldots, j-1 | 1, \ldots, j-1]$ is nonsingular. Since $L[1, \ldots, j-1 | j, j+1, \ldots, j+k-1]$ is a zero matrix, this implies that $L[i_1, \ldots, i_k | j, j+1, \ldots, j+k-1]$ is nonsingular.

\[
\text{Proposition 6.3} \quad \text{Let } F \text{ be an arbitrary field and let } M \text{ be an } n \times n \text{ totally nonsingular matrix over } F. \text{ Let } M = LU \text{ be an } LU\text{-factorization. Then for every } 1 \leq k \leq n, \text{ and } 1 \leq j \leq i_1 < \cdots < i_k \leq n, \text{ the matrix } \\
L[i_1, \ldots, i_k | j, j+1, \ldots, j+k-1] \\
is nonsingular. \\
\text{Proof.} \text{ An } L \text{ factor of } M \text{ can be obtained by Gaussian elimination using column operations. Hence every matrix of the form } L[i_1, \ldots, i_k | 1, \ldots, k] \text{ is a matrix obtained from } M[i_1, \ldots, i_k | 1, \ldots, k] \text{ using column operations. Thus every such matrix is nonsingular.} \\
\text{Now consider a submatrix } L[i_1, \ldots, i_k | j, j+1, \ldots, j+k-1], \text{ where } 1 \leq j \leq i_1 < \cdots < i_k \leq n. \text{ Extend this matrix to } L[1, \ldots, j-1, i_1, \ldots, i_k | 1, 2, \ldots, j+k-1], \text{ which is nonsingular by the previous observation. Also the matrix } L[1, \ldots, j-1 | 1, \ldots, j-1] \text{ is nonsingular. Since } L[1, \ldots, j-1 | j, j+1, \ldots, j+k-1] \text{ is a zero matrix, this implies that } L[i_1, \ldots, i_k | j, j+1, \ldots, j+k-1] \text{ is nonsingular.} \\
\]

7 Birkhoff interpolation

MDS block codes are constructed from Read-Solomon codes. These codes are based on Vandermonde matrices and Lagrange interpolation. A natural question then is whether there are similar concepts connected with MDS tree codes. In this section we will argue that the problem corresponding to Lagrange interpolation is Birkhoff interpolation.

Birkhoff interpolation is the following problem. Given distinct complex numbers $a_1, \ldots, a_m$, integers $0 \leq i_{1,0} < \cdots < i_{1,j_1}, \ldots, 0 \leq i_{m,0} < \cdots < i_{m,j_m}$, and arbitrary complex numbers $A_{10}, \ldots, A_{1j_1}, \ldots, A_{m0}, \ldots, A_{mjm}$, find a polynomial $f(x)$ of degree $m \leq n-1+j_1+\cdots+j_m$ such that

$$f^{(i_{1,0})}(a_1) = A_{10}, \quad \ldots \quad f^{(i_{1,j_1})}(a_1) = A_{1j_1},$$

$$\vdots$$

$$f^{(i_{m,0})}(a_m) = A_{m0}, \quad \ldots \quad f^{(i_{m,j_m})}(a_m) = A_{mjm},$$

where $f^{(t)}$ denotes the $t$th derivative of $f$.

The special case in which $i_{k,l} = l$ for all $k, l$, called Hermite interpolation, has always a unique solution. However in general, special conditions for the numbers $i_{k,l}$ and $a_k$ must be imposed if we want to have a solution for every choice of $A_{10}, \ldots, A_{1j_1}, \ldots, A_{m0}, \ldots, A_{mjm}$. We are interested in the special case of $m = 2$, which was solved by Pólya [7].

Theorem 7.1 Let $a, b \in \mathbb{C}$, let $0 \leq j_1 < \ldots < j_p$ and $0 \leq k_1 < \cdots < k_q$ be integers and let $A_1, \ldots, A_p, B_1, \ldots, B_q \in \mathbb{C}$. Suppose that $a \neq b$ and the integers satisfy the following condition (already stated in Section 6)

$$|\{j_1, \ldots, j_p\} \cap [1, s]| + |\{k_1, \ldots, k_q\} \cap [1, s]| \geq s. \quad (5)$$

Then there exists a unique polynomial $f(x)$ of degree $n \leq p + q - 1$ such that $f^{(j_s)}(a) = A_s$ for all $s = 1, \ldots, p$ and $f^{(k_t)}(b) = B_t$ for all $t = 1, \ldots, q$.
This theorem is stated for the field of complex numbers, but it holds true also for fields of characteristic $r > 0$, in particular for prime fields $\mathbb{F}_r$, if $r$ is sufficiently large. For those primes $r$ for which it is true, one can construct an MDS tree code. Unfortunately we only know that for primes exponentially big in $p + q$.

To see the connection of MDS tree codes to Birkhoff interpolation, consider the matrix of the linear equations that one needs to solve in order to find the interpolating polynomial. For $n \geq 1$, let $M(x, y)$ be the $(n + 1) \times 2(n + 1)$ matrix with entries $M_{i,2j} = (x^i)^{(j)}$ and $M_{i,2j+1} = (y^i)^{(j)}$ (the derivatives of terms $x^i$ and $y^i$) where $i, j = 0, \ldots, n$. So our matrix is

\[
M(x, y) = \begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
x & y & 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\
x^2 & y^2 & 2x & 2y & 2 & 2 & \cdots & 0 & 0 \\
x^n & y^n & nx^{n-1} & ny^{n-1} & n(n-1)x^{n-2} & n(n-1)y^{n-2} & \cdots & n! & n!
\end{pmatrix}
\]

Let $f(x) = c_n x^n + \cdots + c_0$ be a polynomial. Then

\[(c_0, \ldots, c_n)M(x, y) = (f(x), f(y), f'(x), f'(y), \ldots, f^{(n)}(x), f^{(n)}(y)).\]

Theorem 7.1 tells us for which submatrices of $M(a, b)$, $a, b \in \mathbb{C}$, $a \neq b$, the interpolation problem has a solution, hence which sets of columns of $M(a, b)$ are independent. Recall that in the proof of Proposition 5.2 we observed that (5) is equivalent to (4) of that proposition. Hence we have:

**Proposition 7.2** Let $F$ be a field, $a, b \in F$, $a \neq b$, and $n$ a positive integer. Then Pólya’s interpolation theorem (Theorem 7.1) holds true for every $p$ and $q$ such that $p + q \leq n$ if and only if $M(a, b)$ is a parity check matrix of an MDS tree code.

In order to get an idea for which fields the theorem can be true, we will sketch a proof of Theorem 7.1. Since any pair of distinct elements of $F$ can be mapped to any other pair, we can w.l.o.g. assume that $a = 1$ and $b = 0$. Next we divide each column $2j$ and $2j + 1$ by $j!$. (In other words, we are replacing standard derivatives by Hasse derivatives.)

In the resulting matrix the even columns are the matrix $L_n := \left( \binom{x}{y} \right)_{i,j}$, where the binomial coefficients are defined to be zero for $j > i$, and the odd columns form the identity matrix. One can easily check that $L_n$ is an $L$-factor of an $LU$ factorization of the Pascal matrix $P_n$. (This can be shown by applying the binomial formula to the equality $(x + y)^{i+j} = (x + y)^i(x + y)^j$.) Since $P_n$ is totally positive, $L_n$ is triangular totally positive by Theorem 6.2. This implies that condition (5) suffices for the solubility of Birkhoff interpolation.

\[^{3}\text{Here we are exceptionally numbering rows and columns starting with zero.}\]
So the problem boils down to the question, for which primes $r$, the matrix $L_n$ is triangular totally nonsingular over the field $\mathbb{F}_r$. This seems to be a very difficult problem, so we do not dare to conjecture that $r$ may be of polynomial size. A more promising approach is to study the cases of Birkhoff interpolation that are solvable in fields of polynomial size and see if they suffice to ensure a positive minimum distance of the corresponding tree codes.

References

[1] C.W. Cryer: The $LU$-factorization of totally positive matrices. Linear Algebra and Appl., 7:83-92, 1973.

[2] S.M. Fallat and C.R. Johnson: Totally Nonnegative Matrices. Princeton Univ. Press, 2011.

[3] M. Fiedler: Special matrices and their applications in numerical mathematics. Martinus Nijhoff Publishers, 1986.

[4] F.J. MacWilliams and N.J.A. Sloane: The Theory of Error-Correcting Codes, North-Holland, 1977.

[5] C. Moore and L.J. Schulman: Tree codes and a conjecture on exponential sums. arXiv:1308.6007v1

[6] M. Peczarski: An improvement of the tree code construction. Information Processing Letters 99:92-95, 2006.

[7] G. Pólya (1931): Bemerkung zur Interpolation und zur Naherungstheorie der Balkenbiegung. Journal of Applied Mathematics and Mechanics 11: 445-449, 1931.

[8] L.J. Schulman: Deterministic Coding for Interactive Communication. In Proc. 25th Annual Symp. on Theory of Computing, 747-756, 1993.

[9] L.J. Schulman: Coding for Interactive Communication. IEEE Transactions on Information Theory, 42(6):1745-1756, 1996.

[10] J.H. van Lint: Introduction to Coding Theory. Springer-Verlag, 1982.