On Janowski functions associated with \((n, m)\)-symmetrical functions

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1. Introduction

Let \(\mathcal{H}(U)\) be the space of all analytic functions in the open unit disk \(U := \{z \in \mathbb{C} : |z| < 1\}\), and let \(\mathcal{A}\) denote the class of functions \(f \in \mathcal{H}(U)\) normalized with \(f(0) = f'(0) - 1 = 0\). Denote by \(\Omega\) for the class of \(\text{Schwarz functions}\), that is

\[
\Omega := \{w \in \mathcal{A}, \ w(0) = 0, |w(z)| < 1, z \in U\}. \tag{1}
\]

For \(f\) and \(g\) are two analytic functions in \(U\), we say that the function \(f\) is \textit{subordinate} to the function \(g\) in \(U\), if there exists a function \(w \in \Omega\), such that \(f(z) = g(w(z))\) for all \(z \in U\), and we denote this by \(f \prec g\). Furthermore, if \(g\) is univalent in \(U\), then the subordination is equivalent to \(f(0) = g(0)\) and \(f(U) \subset g(U)\). (see, for details, [1])

Using the notation of the subordination, let define the class \(\mathcal{P}\) of \textit{functions with positive real parts} (or \textit{Carathéodory functions}) in \(U\):

\[
\mathcal{P} = \{p \in \mathcal{H}(U) : p(0) = 1, \ \text{Re} \ p(z) > 0 \ \text{for all} \ z \in U\}.
\]

Definition 1.1 ([2]): Let \(\mathcal{P}\) denote the class of functions \(p \in \mathcal{H}(U)\) satisfying \(p(0) = 1\) and \(\text{Re} \ p(z) > 0\) for all \(z \in U\).

From the above-mentioned reasons, it follows that any function \(p \in \mathcal{P}\) has the representation \(p(z) = (1 + w(z))/(1 - w(z))\), for some \(w \in \Omega\).

The class of functions with positive real part plays a significant role in complex function theory. Its significance can be seen from the fact that all simple subclasses of the class of univalent functions have been defined by using the concept of the class of functions with positive real part like the classes \(S^n, C, S\) which are respectively the class of starlike, convex functions and the class of starlike functions with respect to symmetric points, etc., have been defined by using the class \(\mathcal{P}\).

Definition 1.2: (i) Like in [3], let \(\mathcal{P}[X, Y, \gamma]\), with \(-1 \leq Y < X \leq 1\), denote the class functions \(p \in \mathcal{H}(U)\) that satisfy the subordination condition \(p(z) < (1 + Xz)/(1 + Yz)\).

(ii) The class of \textit{generalized Janowski type functions} \(\mathcal{P}[X, Y, \gamma]\) was introduced in [4] as follows: for arbitrary fixed numbers \(X, Y, \gamma\), with \(-1 \leq Y < X \leq 1\), \(0 \leq \gamma < 1\),

\[
p \in \mathcal{P}[X, Y, \gamma] \iff p(z) < \frac{1 + [(1 - \gamma)X + \gamma Y]z}{1 + Yz}.
\]

(iii) Like in [4], a function \(f \in \mathcal{A}\), then

\[
f \in S^n[X, Y, \gamma] \iff \frac{zf'(z)}{f(z)} \in \mathcal{P}[X, Y, \gamma].
\]

Definition 1.3: (i) For the positive integer number \(m\), a domain \(D \subseteq \mathbb{C}\) is said to be \textit{\(m\)-fold symmetric domain}, if a rotation of \(D\) about the origin through and angle \(2\pi/m\) carries \(D\) onto itself.

(ii) A function \(f : D \to \mathbb{C}\), where \(D\) is a \(m\)-fold symmetric domain, is said to be \textit{\(m\)-fold symmetrical function}.
if, \( f(e^{2\pi i/m}z) = e^{2\pi i/m}f(z) \), \( z \in D \). The set of all \( m \)-fold symmetrical denoted by \( S^m \).

The theory of \((n, m)\)-symmetrical functions for \((n = 0, 1, \ldots, m - 1)\) and \((m = 1, 2, \ldots)\) is a generalization of the notion of odd, even, \(m\)-symmetrical functions.

**Definition 1.4:** Let \( \varepsilon = e^{2\pi i/m} \), \( n = 0, 1, \ldots, m - 1 \), where \( m \in \mathbb{N} \) and \( m \geq 2 \).

A function \( f : D \to \mathbb{C} \), where \( D \) is a \( m \)-fold symmetric domain, is called \((n, m)\)-symmetrical function if \( f(\varepsilon z) = e^{n\varepsilon}f(z) \), \( z \in D \).

Denoted be \( S^{(n,m)} \), if the following class:

\[
\{ \text{symmetric points.} \}
\]

The following decomposition theorem holds:

**Theorem 1.1:** Let \( D \) be a \( m \)-fold symmetric domain, then for every function \( f : D \to \mathbb{C} \), can be written in the form

\[
f(z) = \sum_{n=0}^{m-1} f_{n,m}(z), \quad z \in D,
\]

and this partition is unique sequence of \((n, m)\)-symmetrical functions.

\[
f_{n,m}(z) = \frac{1}{m} \sum_{i=0}^{m-1} e^{-\varepsilon i n} f(e^{i\varepsilon}z), \quad z \in D,
\]

\[(m = 2, 3, \ldots; n = 0, 1, 2, \ldots, m - 1). \quad (2)\]

Al Sarari and Latha [6] introduced the classes \( S^{(n,m)}[X, Y] \) and \( K^{(n,m)}[X, Y] \) which are the classes of Janowski type functions with respect to \((n, m)\)-symmetrical points.

By using the theory of \((n, m)\)-symmetrical functions and the generalized Janowski type functions, we will define the following class:

**Definition 1.5:** A function \( f \in A \) is said to belongs to the class \( S^{(n,m)}[X, Y, \gamma] \), with \(-1 \leq Y < X \leq 1\) and \( 0 \leq \gamma < 1 \), if

\[
\left| \frac{zf'(z)}{f_{n,m}(z)} - 1 \right| < \left| (1 - \gamma)X + Y \right| \frac{zf'(z)}{f_{n,m}(z)}, \quad z \in U.
\]

where the function \( f_{n,m}(z) \) is defined by (2).

**Remark 1.1:** By applying the definition of the subordination we can easily obtain that the equivalent condition for a function \( f \) belonging to the class \( S^{(n,m)}[X, Y, \gamma] \), with \(-1 \leq Y < X \leq 1\) and \( 0 \leq \gamma < 1 \), is

\[
\left| \frac{zf'(z)}{f_{n,m}(z)} - 1 \right| < \left| (1 - \gamma)X + Y \right| \frac{zf'(z)}{f_{n,m}(z)}.
\]

Note that special values of \( n, m, \gamma, X \) and \( Y \) yield the following classes that have been previously introduced by different authors:

(i) \( S^{(n,m)}[X, Y, 0] = S^{(n,m)}[X, Y] \) is the class studied by Al Sarari and Latha [6].

(ii) \( S^{(1,m)}[1, -1, 0] = S_m^m(1, -1) \) is the class introduced by Sakaguchi [7].

(iii) \( S^{(1,m)}[X, Y, 0] = K^{(m)}[X, Y] \) is the class was introduced by Ohang and Younjae in [8].

The following lemmas are important to proof our results:

**Lemma 1.1 ([4, Corollary 1]):** For a function \( f \in S^*[X, Y, \gamma] \), then

\[
f(z) = \left\{ \begin{array}{ll}
\z \exp \left[ (1 - \gamma)Xw(z) \right], & \text{if } Y = 0, \\
\z (1 + Yw(z) \left( (1 - \gamma)(X - Y) \right)^{1/r}, & \text{if } Y \neq 0,
\end{array} \right.
\]

for some \( w \in \Omega, \) where \( \Omega \) was defined by (1).

**Lemma 1.2 ([9, Lemma 2.3]):** For \( p \in P[X, Y, \gamma] \), then

\[
1 - \frac{[(1 - \gamma)X + \gamma Y]r}{1 - Yr} \leq |p(z)| \leq 1 + \frac{[(1 - \gamma)X + \gamma Y]r}{1 + Yr}, \quad |z| \leq r < 1.
\]

2. Main results

**Theorem 2.1:** If \( f \in S^{(n,m)}[X, Y, \gamma] \), then

\[
f_{n,m}(z) = \left\{ \begin{array}{ll}
\z \exp[(1 - \gamma)Yw(z)], & \text{if } Y = 0, \\
\z (1 + Yw(z) \left( (1 - \gamma)(X - Y) \right)^{1/r}, & \text{if } Y \neq 0,
\end{array} \right.
\]

where \( f_{n,m}(z) \) is defined by (2), and for some \( w \in \Omega \).

**Proof:** Let \( f \in S^{(n,m)}[X, Y, \gamma] \), we can get

\[
\frac{zf'(z)}{f_{n,m}(z)} < \frac{1 + [(1 - \gamma)X + \gamma Y]z}{1 + Yz}.
\]

Replacing \( z \) by \( e^{\nu}z \) in (4),

\[
\frac{e^{\nu}zf(e^{\nu}z)}{f_{n,m}(e^{\nu}z)} < \frac{1 + [(1 - \gamma)X + \gamma Y]e^{\nu}z}{1 + Ye^{\nu}z} \leq \frac{1 + [(1 - \gamma)X + \gamma Y]z}{1 + Yz},
\]

hence

\[
\frac{e^{\nu}zf(e^{\nu}z)}{f_{n,m}(z)} < \frac{1 + [(1 - \gamma)X + \gamma Y]z}{1 + Yz},
\]

Letting \( \nu = 0, 1, \ldots, m - 1 \) in (5), and since \( P[X, Y, \gamma] \) is a convex set, we deduce that

\[
\frac{1}{m} \sum_{\nu=0}^{m-1} \frac{e^{\nu}zf(e^{\nu}z)}{f_{n,m}(z)} < \frac{1 + [(1 - \gamma)X + \gamma Y]z}{1 + Yz},
\]
or equivalently
\[
\frac{zf'_\mu(z)}{f_{n,m,\mu}(z)} < \frac{1 + [(1 - \gamma)X + \gamma Y]z}{1 + Yz},
\]
that is \(f_{n,m} \in \mathcal{S}^n[X, Y, \gamma]\), and by Lemma 1.1 we finally obtain our result.

\[\text{Theorem 2.2: For } f \in \mathcal{S}^{(n,m)}[X, Y, \gamma], \text{ with } -1 \leq Y < X \leq 1 \text{ and } 0 \leq \gamma < 1. \text{ Then,} \]
\[
f(z) = \begin{cases} 
\int_0^1 [(1 - \gamma)X + \gamma Y] \bar{w}(\xi) \exp[(1 - \gamma)Xw(\xi)] \, d\xi' & \text{if } Y = 0, \\
\int_0^1 [(1 - \gamma)X + \gamma Y] \bar{w}(\xi) \exp[(1 - \gamma)Xw(\xi)] \, d\xi' & \text{if } Y \neq 0,
\end{cases}
\]
for some \(\bar{w}, w \in \Omega\).

\[\text{Proof: Supposing that } f \in \mathcal{S}^{(n,m)}[X, Y, \gamma], \text{ then there exists a function } \bar{w} \in \Omega, \text{ such that} \]
\[
\frac{zf'(z)}{f_{n,m}(z)} = \frac{1 + [(1 - \gamma)X + \gamma Y] \bar{w}(z)}{1 + Y \bar{w}(z)}, \quad z \in \mathbb{U}.
\]
Combining the above relation with Theorem 2.1, we have
\[
f'(z) = \begin{cases} 
[1 + (1 - \gamma)X \bar{w}(z)] & \text{if } Y = 0, \\
1 + [(1 - \gamma)X + \gamma Y] \bar{w}(z) & \text{if } Y \neq 0,
\end{cases}
\]
and integrating the above relations we obtain our result.

\[\text{Theorem 2.3: Let } f \in \mathcal{S}^{(n,m)}[X, Y, \gamma] \text{ and } f_\mu(z) := \mu f(z) + (1 - \mu)z, \text{ with } 0 < \mu < 1. \text{ Then,} \]
(i) \(f_\mu \in \mathcal{S}^{(n,m)}[X, 0, \gamma]\), if \(Y = 0\);
(ii) \(f_\mu \in \mathcal{S}^{(n,m)}[X, Y, \gamma]\), for \(|z| < 1/Y \sin(Y/((1 - \gamma)X + \gamma Y)(\pi/2)), \text{ if } Y > 0\);
(iii) \(f_\mu \in \mathcal{S}^{(n,m)}[X, Y, \gamma]\), for \(|z| < 1/Y \sin(Y/((1 - \gamma)X + \gamma Y)(\pi/2)), \text{ if } Y < 0\).

\[\text{Proof: Since } f \in \mathcal{S}^{(n,m)}[X, Y, \gamma], \text{ then} \]
\[
\frac{zf'(z)}{f_{n,m}(z)} < \frac{1 + [(1 - \gamma)X + \gamma Y]z}{1 + Yz}, \text{ with} \]
\[
f_{n,m,\mu}(z) = \sum_{\nu=0}^{m-1} \epsilon^{-\nu}f_\mu(e^{\nu}z).
\]
Thus,
\[
f_{n,m,\mu}(z) = \mu f_{n,m}(z) + (1 - \mu)z, \quad zf'_\mu(z) = (1 - \mu)z + \mu zf'(z),
\]
hence
\[
\frac{zf'_\mu(z)}{f_{n,m,\mu}(z)} = \frac{(1 - \mu)z + \mu zf'(z)}{(1 - \mu)z + \mu}.
\]
(i) For \(Y = 0\) it is sufficient to show that
\[
\frac{(1 - \mu)z + \mu zf'(z)}{(1 - \mu)z + \mu} < (1 - \gamma)X, \ z \in \mathbb{U}.
\]
From \(f \in \mathcal{S}^{(n,m)}[X, Y, \gamma]\) we have
\[
\frac{zf'(z)}{f_{n,m}(z)} < 1 + (1 - \gamma)Xz
\]
which implies
\[
\frac{zf'(z)}{f_{n,m}(z)} < (1 - \gamma)X, \ z \in \mathbb{U},
\]
and according to (3)
\[
\frac{f_{n,m}(z)}{z} < \exp((1 - \gamma)Xz),
\]
there exist a Schwarz function \(w \in \Omega\) such that
\[
\frac{f_{n,m}(z)}{z} = \exp((1 - \gamma)Xw(z)), \ z \in \mathbb{U}.
\]
Thus,
\[
\frac{(1 - \mu)z + \mu zf'(z)}{(1 - \mu)z + \mu} < 1
\]
and using the fact that \(|w(z)| < 1\) for all \(z \in \mathbb{U}\), we may easily prove that
\[
\frac{(1 - \mu)z + \mu zf'(z)}{(1 - \mu)z + \mu} < 1
\]
From the above two inequalities, it follows
\[
\frac{(1 - \mu)z + \mu zf'(z)}{(1 - \mu)z + \mu} < 1
\]
and consequently, from (6) we obtain
\[
\frac{zf'_\mu(z)}{f_{n,m,\mu}(z)} < 1 + (1 - \gamma)Xz,
\]
that is \(f_\mu \in \mathcal{S}^{(n,m)}[X, 0, \gamma]\).
(ii) For \( Y \neq 0 \), we need to determine the value \( r_\ast \in (0, 1) \), such that

\[
\left| \frac{(1 - \mu) \frac{z}{f_{n,m}(z)} + \mu \frac{zf'(z)}{f_{n,m}(z)}}{(1 - \mu) \frac{z}{f_{n,m}(z)} + \mu} - 1 \right| < \left| \frac{(|(1 - \gamma)X + \gamma Y| - Y) \left( \frac{1}{\mu} - 1 \right) \frac{z}{f_{n,m}(z)} + [(1 - \gamma)X + \gamma Y - Y \frac{zf'(z)}{f_{n,m}(z)}]}{(1 - \gamma)X - Y} \right|, \tag{8}
\]

whenever \( |z| < r_\ast \), which is equivalent to

\[
\left| \frac{zf'(z)}{f_{n,m}(z)} - 1 \right| < \left| [(1 - \gamma)X + \gamma Y - Y \frac{zf'(z)}{f_{n,m}(z)}]^{\frac{1}{\mu} - 1} \frac{z}{f_{n,m}(z)} \right| + [(1 - \gamma)X + \gamma Y - Y \frac{zf'(z)}{f_{n,m}(z)}], \tag{9}
\]

For \( |z| < r_\ast \). According to the Remark 1.1 and the definition of the subordination we have that

\[ f \in S^{(n,m)}[X, Y, \gamma], \] for \( |z| < r_\ast \),

is equivalent to

\[
\left| \frac{zf'(z)}{f_{n,m}(z)} - 1 \right| < \left| [(1 - \gamma)X + \gamma Y - Y \frac{zf'(z)}{f_{n,m}(z)}]^{\frac{1}{\mu} - 1} \frac{z}{f_{n,m}(z)} \right|, \tag{9}
\]

Next, we will prove that

\[
\left| \arg \left( \frac{z}{f_{n,m}(z)} \right) - \arg \left( \frac{zf'(z)}{f_{n,m}(z)} \right) \right| < \frac{\pi}{2}, \quad z \in U, \tag{10}
\]

implies

\[
\left| [(1 - \gamma)X + \gamma Y - Y \frac{zf'(z)}{f_{n,m}(z)}]^{\frac{1}{\mu} - 1} \frac{z}{f_{n,m}(z)} \right| + [(1 - \gamma)X + \gamma Y - Y \frac{zf'(z)}{f_{n,m}(z)}], \quad z \in U. \tag{11}
\]

Since \( f \in S^{(n,m)}[X, Y, \gamma] \), from Definition 1.5 and Theorem 2.1, it follows that there exist the functions

\[ w, \tilde{w} \in \Omega, \] such that

\[
\left| (X - Y)(1 - \gamma) \right| \frac{1}{1 + Yw(z)} < \left| (X - Y)(1 - \gamma) \right| \frac{1}{1 + Yw(z)} \left( \frac{1}{\mu} - 1 \right) \times (1 + \tilde{w}(z)) \left( \frac{1}{\mu} - 1 \right), \quad z \in U,
\]

which represents (11).
that
\[
\frac{zf'(z)}{f_{n,m}(z)} = 1 + \frac{[(1 - \gamma)X + \gamma Y]w(z)}{1 + Yw(z)}, \quad z \in \mathbb{U},
\]
therefore
\[
\left| \arg \left( \frac{(1 - \gamma)X + \gamma Y - Y \frac{zf'(z)}{f_{n,m}(z)}}{f_{n,m}(z)} \right) \right| = \left| \arg \left( \frac{(1 - \gamma)X + \gamma Y}{1 + Yw(z)} \right) \right| \leq \left| \arg \left( (X - Y)(1 - \gamma) \right) \right| + \left| \arg \left( 1 + Yw(z) \right) \right| \leq \arcsin \left( |Y|r, \ |z| \right) \leq r < 1.
\]

From (12) and (13) we easily deduce that
\[
\left| \arg \left( \frac{Z}{f_{n,m}(z)} \right) - \arg \left( \frac{[(1 - \gamma)X + \gamma Y - Y \frac{zf'(z)}{f_{n,m}(z)}]}{f_{n,m}(z)} \right) \right| \leq \left| \arg \left( \frac{Z}{f_{n,m}(z)} \right) \right| + \left| \arg \left( \frac{[(1 - \gamma)X + \gamma Y - Y \frac{zf'(z)}{f_{n,m}(z)}]}{f_{n,m}(z)} \right) \right| \leq \arcsin \left( |Y|r, \ |(X - Y)(1 - \gamma)\right) \arcsin \left( Yr, \ \frac{\pi}{2} \right), \ |z| \leq r.<
\]

where \( r_< \) is given like in the assumptions (ii) and (iii) of Theorem 1.1.

The next distortion and covering theorems for the class \( S^{(n,m)}[X, Y, \gamma] \) holds:

**Theorem 2.4:** If \( f \in S^{(n,m)}[X, Y, \gamma] \), then
\[
g(r) = \left\lfloor f'(z) \right\rfloor \leq q(r),
\]
where
\[
g(r) = \begin{cases} 
1 - (1 - \gamma)Xr & \text{if } Y = 0, \\
\exp \left[ (1 - \gamma)Xr \right] & \text{if } Y \neq 0,
\end{cases}
\]
\[
q(r) = \begin{cases} 
1 + (1 - \gamma)Xr & \text{if } Y = 0, \\
\exp \left[ (1 - \gamma)Xr \right] & \text{if } Y \neq 0,
\end{cases}
\]
and \( |z| \leq r < 1 \).

**Proof:** Let \( f \in S^{(n,m)}[X, Y, \gamma] \), according to Theorem 2.1 we need to distinguish the next two cases:

(i) If \( Y \neq 0 \), then there exists Schwarz functions, \( w \in \Omega \) such that \( f_{n,m}(z) := z(1 + Yw(z))^{(1 - \gamma)(X - Y)}/(X - Y) \), and by Lemma 1.2 we get
\[
1 - \frac{|Y + (1 - \gamma)X|}{1 - Y} |Yw(z) + 1|^{(1 - \gamma)(X - Y)}/(X - Y) \leq |f'(z)| \leq 1 + |Y + (1 - \gamma)X| |Yw(z) + 1|^{(1 - \gamma)(X - Y)}/(X - Y),
\]
\[
|z| \leq r < 1.
\]
Since \( w \in \Omega \), we have
\[
1 - |Y|r \leq |Yw(z) + 1| \leq |Y|r + 1, \quad |z| \leq r < 1.
\]

Case 1. For \( Y > 0 \). By using the fact that \(-1 \leq Y < X \leq 1 \) and \( 0 \leq \gamma < 1 \), we have
\[
(1 - |Y|r)^{(X - Y)(1 - \gamma)/Y} \leq |Yw(z) + 1|^{(X - Y)(1 - \gamma)/Y} \leq ((Y + 1)^{(X - Y)(1 - \gamma)/Y}, \ |z| \leq r < 1,
\]
and from (16) we obtain
\[
1 - [\gamma Y + (1 - \gamma)X]r \leq \frac{1 - [\gamma Y + (1 - \gamma)X]r}{1 + Yr} (|Y|r + 1)(X-r^{(1-\gamma)})^{\gamma},
\]
\[|z| \leq r < 1. \tag{17}\]

Case 2. If \(Y < 0\), from the fact that \(-1 \leq Y < X \leq 1\) and \(0 \leq \gamma < 1\), we have
\[
1 - [\gamma Y + (1 - \gamma)X]r \geq \frac{1 - [\gamma Y + (1 - \gamma)X]r}{1 + Yr} (|Y|r + 1)(X-r^{(1-\gamma)})^{\gamma},
\]
\[|z| \leq r < 1. \tag{18}\]

Now, by combining (17) and (18), we get
\[
1 - [(1 - \gamma)X + \gamma Y]r \leq \frac{1 - [(1 - \gamma)X + \gamma Y]r}{1 - Yr} (|Y|r + 1)(X-r^{(1-\gamma)})^{\gamma},
\]
\[|z| \leq r < 1. \tag{19}\]

(iii) If \(Y = 0\), there exists Schwarz functions, \(w \in \Omega\) such that \(f_{r,m}(z) = w \exp((1 - \gamma)Xw(z))\), and therefore
\[
1 - (1 - \gamma)Xr \left|\exp\left[(1 - \gamma)Xw(z)\right]\right| \leq |f'(z)|
\]
\[\leq \left[1 - (1 - \gamma)Xr + 1\right] \left|\exp\left[(1 - \gamma)Xw(z)\right]\right|, \tag{20}\]
for \(|z| \leq r < 1.\) Since
\[
\left|\exp\left[(1 - \gamma)Xw(z)\right]\right| = \left|\exp\left[(1 - \gamma)X\text{Re}w(z)\right]\right|, z \in \mathbb{U}.
\]
By a similar way as in the previous case, we get
\[
\exp\left[-(1 - \gamma)Xr\right] \leq \left|\exp\left[(1 - \gamma)Xw(z)\right]\right| \leq \exp\left[-(1 - \gamma)Xr\right], \quad |z| \leq r < 1.
\]
Thus, (20) yield to
\[
1 - (1 - \gamma)Xr \exp\left[-(1 - \gamma)Xr\right] \leq |f'(z)|
\]
\[\leq \left[1 - (1 - \gamma)Xr + 1\right] \exp\left[-(1 - \gamma)Xr\right], \quad |z| \leq r < 1.
\]
Therefore, for \(|z| \leq r < 1.\) That is the complete proof of our theorem.

\textbf{Theorem 2.5:} If \(f \in S^{(n,m)}[X, Y, \gamma]\), then
\[
|f(z)| \leq \begin{cases}
\int_0^\gamma \left[1 + (1 - \gamma)X\rho\right] \exp\left[-(1 - \gamma)X\rho\right] d\rho, & \text{if } Y = 0, \\
\int_0^\gamma \frac{1 + [\gamma Y + (1 - \gamma)X]\rho}{1 + Y\rho} (1 + Y\rho)^{(X-Y)(1-\gamma)/Y} d\rho, & \text{if } Y \neq 0,
\end{cases}
\]
\[|z| \leq r < 1.\]

\textbf{Proof:} Integrated the function \(f'\) along the close segment connecting the origin with an arbitrary \(z \in U\), since any point of this segment is of the form \(z = \rho e^{i\theta}\), with \(\rho \in [0, r]\), where \(\theta = \arg z\) and \(r = |z|\), we get
\[
f(z) = \int_0^\gamma f' (\zeta) d\zeta, z = re^{i\theta},
\]

hence
\[
|f(z)| \leq \int_0^\gamma \left|f' (\rho e^{i\theta})\right| e^{i\theta} d\rho \\ \leq \int_0^\gamma \left|f' (\rho e^{i\theta})\right| e^{i\theta} d\rho \\ \leq \int_0^\gamma \left[1 + [(1 - \gamma)X + \gamma Y] \rho\right] \exp\left[-(1 - \gamma)X\rho\right] d\rho,
\]
\[|z| \leq r < 1.\]

Using this inequality and the right-hand side inequalities of Theorem 2.4, we need to discuss the next two cases:

(i) If \(Y \neq 0\), then
\[
|f(z)| \leq \int_0^\gamma \left|f' (\rho e^{i\theta})\right| e^{i\theta} d\rho \\ \leq \int_0^\gamma \left[1 + (1 - \gamma)X + \gamma Y\rho\right] \exp\left[-(1 - \gamma)X\rho\right] d\rho,
\]
that is
\[
|f(z)| \leq \int_0^\gamma \left[1 + [(1 - \gamma)X + \gamma Y] \rho\right] \exp\left[-(1 - \gamma)X\rho\right] d\rho, \quad |z| \leq r < 1.
\]

(ii) If \(Y = 0\), then
\[
|f(z)| \leq \int_0^\gamma \left|f' (\rho e^{i\theta})\right| e^{i\theta} d\rho \\ \leq \int_0^\gamma \left[1 + (1 - \gamma)X\rho\right] \exp\left[-(1 - \gamma)X\rho\right] d\rho,
\]
that is
\[
|f(z)| \leq \int_0^\gamma \left[1 + (1 - \gamma)X\rho\right] \exp\left[-(1 - \gamma)X\rho\right] d\rho, \quad |z| \leq r < 1.
\]

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\textbf{References}

[1] Miller S, Mocanu PT. 2000. “Differential Subordinations, Theory and Applications,” Series of Monographs and Textbooks in Pure and Applied Mathematics, Vol. 225, Marcel Dekker Inc., New York.
[2] Duren PL. Univalent functions. New York: Springer-Verlag; 1983.
[3] Janowski W. Some extremal problems for certain families of analytic functions I. Ann Polon Math. 1973;28(3): 297–326.
[4] Polatoglu Y, Bolcal M, Şen A, et al. A study on the generalization of Janowski functions in the unit disc. Acta Math Acad Paedagog Nyházi (NS). 2006;22: 27–31.
[5] Liczberski P, Połubiński J. On \((j, k)\)-symmetrical functions. Math Bohem. 1995;120(1):13–28.
[6] Al Sarari F, Latha S. A few results on functions that are Janowski starlike related to \((j, k)\)-symmetric points. Octo Math Maga. 2013;21(2):556–563.
[7] Sakaguchi K. On a certain univalent mapping. J Math Soc Jpn. 1959;11(1):72–75.
[8] Kwon O, Sim Y. A certain subclass of Janowski type functions associated with \(k\)-symmetric points. Commun Kor Math Soc. 2013;28(1):143–154.
[9] Hussain S, Arif M, Nawaz Malik S. Higher order close-to-convex functions associated with Attiya-Srivastava operator. Bull Iran Math Soc. 2014;40(4):911–920.