Quantum conditional operations

Alessandro Bisio, Michele Dall’Arno, and Paolo Perinotti

1 Quit group, Dipartimento di Fisica, Università degli studi di Pavia, via Bassi 6, 27100 Pavia, Italy
2 Istituto Nazionale di Fisica Nucleare, Gruppo IV, via Bassi 6, 27100 Pavia, Italy
3 Centre for Quantum Technologies, National University of Singapore, 3 Science Drive 2, 117543 Singapore, Republic of Singapore

An essential element of classical computation is the “if-then” construct, that accepts a control bit and an arbitrary gate, and provides conditional execution of the gate depending on the value of the controlling bit. On the other hand, quantum theory prevents the existence of an analogous universal construct accepting a control qubit and an arbitrary quantum gate as its input. Nevertheless, there are controllable sets of quantum gates for which such a construct exists. Here we provide a necessary and sufficient condition for a set of unitary transformations to be controllable, and we give a complete characterization of controllable sets in the two dimensional case. This result reveals an interesting connection between the problem of controllability and the problem of extracting information from an unknown quantum gate while using it.

I. INTRODUCTION

One of the key features of any programming language is conditional statements, that run an arbitrary gate depending on the value of a controlling variable. The Boolean “if-then” construct is fundamental to break code sequentiality, it allows the implementation of conditional loops and it prevents recursion from being infinite.

Also in quantum computation controlled gates play a crucial role. This is the case, e.g. for the ubiquitous Controlled-NOT (C-NOT), which is the pillar of most quantum algorithms (a remarkable example is the Shor’s algorithm, that relies on controlled routines for period-finding). The crucial difference between classical and quantum controlled gates is that the latter allow for the control qubit to be in a superposition of states. This fact leads to a further, radical difference with respect to classical computation. Indeed, while the requirement that the gate is a variable of—a opposed to being hard-coded into—the “if-then” construct can be trivially implemented in the classical world, it turns out to be impossible in the quantum one, as first noticed by Kitaev more than two decades ago. As it was recently shown, quantum theory prevents the implementation of a universal quantum “if-then” construct, namely one that can control the entire set of unitary gates, that is, no black-box transformation can map \( U \) to controlled-\( U \) for any unitary \( U \). This is a serious limitation, as it implies that one should provide a different implementation of the “if-then” construct for each controlled gate in the algorithm.

On the other hand, there exist sets of unitary transformations for which the implementation of a control is possible. Simple examples are: i) the sets of jointly perfectly discriminable unitaries; ii) the sets such that, for a given state \( |\psi\rangle \), one has \( U_i |\psi\rangle \propto U_j |\psi\rangle \) for all the unitaries in the set. The latter example corresponds to the setting of Refs. [7,8], where \( |\psi\rangle \) is trivially the vacuum state, and of Refs. [9,10], and shares similarities with that of Ref. [11]. It is also straightforward to notice that any set made of two unitary transformations \( \{U,V\} \) is always controllable, since if \( |\psi\rangle \) is an eigenstate of \( U^\dagger V \) then \( U |\psi\rangle \propto V |\psi\rangle \).

Despite its relevance and the recent interest in the problem, the pivotal question remains unanswered: what are the sets of gates that quantum theory allows to be controlled through a conditional statement?

Here, we answer this question by providing a necessary and sufficient condition for a set of quantum gates to be controllable by a quantum “if-then” clause. Surprisingly, our result unveils a connection between the controllability problem and the task of extracting information from an unknown quantum gate while using it.

II. FORMALIZATION

In operational terms, a general reversible quantum gate \( U \) consists of a black box transforming its input (left wire) into its output (right wire). The action of \( U \) on state \( \rho \) is represented by a unitary operator \( U \), i.e. \( U(\rho) = U\rho U^\dagger \), \( U^\dagger U = UU^\dagger = I \) (where \( I \) denotes the identity matrix). We say that the unitary operator \( U \) is a representative of the gate \( U \). Clearly, a gate \( U \) admits several representatative unitaries, differing by a physically irrelevant global phase \( e^{i\theta} \). In this work we use the term unitary when referring to a particular representative of a gate.

For some particular choice of representative \( U \), the controlled gate \( C-U \) acts on the system as the identity operator \( I \) if the control qubit is initialized in state \( |0\rangle \), while it performs \( U \) if the control qubit is in state \( |1\rangle \), namely

\[
U = I \otimes |0\rangle \langle 0| + U \otimes |1\rangle \langle 1| = U \oplus I.
\]
Since a phase in front of $U$ in Eq. (1) is local rather than global, different choices of the representative for the same gate $U$ clearly correspond to physically inequivalent controlled gates.

To address the problem in the most general case, we must consider a generic map that transforms the gate $U$ into its controlled version $C.U$. From the theory of quantum combs [12–14], it is well known that the most general transformation allowed by quantum theory on a quantum gate $U$ is realized by inserting $U$ in a quantum circuit board. Explicitly, if a map which transforms $U$ into $C.U$ exists, it corresponds to the following circuit (further details are given in the Methods section):

\[
\begin{array}{c}
A \quad U \quad B
\end{array} = \begin{array}{c}
0 \quad U \quad |\psi_U\rangle
\end{array}, \quad (2)
\]

in formula $B(U \otimes I)A(I \otimes |0\rangle) = (U \otimes I) \otimes |\psi_U\rangle$, where 0 denotes the preparation of an ancillary ready state $|0\rangle$, $A$ and $B$ denote unitary transformations, $U$ is a representative of $U$, and $|\psi_U\rangle$ is a state that depends on $U$. It is important to remark that the dimension of the Hilbert space in the circuit in Eq. (2) is always bounded [12–15].

The freedom in the choice of the representative, leading to many inequivalent controlled gates, requires to split the formulation of the controllability problem into two sub-problems. The first one regards controllability of a set of representatives

**Definition 1.** Given a set $S$ of gates and a set $R$ of representatives of $S$, we say that $S$ is controllable with representatives $R$ if there exists $A$ and $B$ such that Eq. (2) holds for any $U \in R$.

The second question regards the existence of such a set of representatives, as follows.

**Definition 2.** We say that a set $S$ of gates is controllable if there exists a set $R$ of representatives of $S$ such that $S$ is controllable with representatives $R$.

### III. INFORMATION WITHOUT DISTURBANCE

Equation (2) makes it manifest that, along with the desired task of controlling $U$, some side information about the unknown unitary $U$ can in principle be stored in state $|\psi_U\rangle$. The study of this side information plays a fundamental role in our analysis of controllability. To make this explicit, consider the case in which the control state in Eq. (2) is set to $|1\rangle$. We have then

\[
\begin{array}{c}
A \quad U \quad B
\end{array} = \begin{array}{c}
1 \quad U
\end{array}, \quad (3)
\]

Equation (3) is an instance of the information-disturbance trade-off problem in estimating a quantum transformation [10]. Let us suppose that we are provided with a black box implementing a single use of an unknown transformation $U$ belonging to a given set $R$. On one hand, one wants to identify the unknown transformation $U$, while on the other hand one is interested in applying the black box on a variable input state. In general these two tasks are incompatible, and there is a trade-off between the amount of information that can be obtained about a black box and the disturbance caused on its action, with an exception: when the black boxes can be jointly discriminated without error [15–19], and then also reproduced. The circuit in Eq. (3) fixes a scenario in which the unknown unitary must be left unperturbed.

Although we defer the details to the Methods section, it is important to notice here that without loss of generality one can take the states $|\psi_U\rangle$ and $|\psi_V\rangle$ corresponding to unitaries $U$ and $V$ in Eq. (3) to be either proportional or orthogonal. Indeed, since the linear span $U$ of maps $U$ is a finite dimensional space, and the circuit in Eq. (3) acts linearly on maps $U$, also the span of the states $|\psi_U\rangle$ is finite dimensional, with a dimension bounded by $\dim U$. Now, for every pair of unitaries $U$ and $V$, the amount of information provided by the circuit in Eq. (3) is a decreasing function of $|\langle \psi_U | \psi_V \rangle|$. Suppose that for a given pair $U, V$ the circuit providing maximum information about the pair $U, V$ has $0 < \alpha := |\langle \psi_U | \psi_V \rangle| < 1$. Then by applying the same circuit twice one has $|\psi_U^{(2)}\rangle = |\psi_U\rangle \otimes |\psi_U\rangle$, and $|\langle \psi_U^{(2)} | \psi_V^{(2)} \rangle| = \alpha^2 < \alpha$, while using an ancillary system having the same dimension as the initial one. This implies that the hypothesis of optimality of $\alpha$ is absurd. One must then have either $\alpha = 0$ or $\alpha = 1$. The argument can be repeated for every pair $U, V$.

This observation motivates the following definition.

**Definition 3** (Markable set of unitaries). Let $R$ be a set of unitaries, and let $P := \{P_n\}$ be a partition of $R$. We say that $R$ is $P$-markable, if there exist unitaries $C$ and $D$ such that, for any $n$, any $U \in P_n$, and some orthonormal set $\{|n\rangle\}$ one has

\[
\begin{array}{c}
C \quad U \quad D
\end{array} = \begin{array}{c}
U \quad |\alpha\rangle
\end{array}. \quad (4)
\]

Similarly, we say that a set $S$ of gates is $P$-markable if there exists a set $R$ of representatives of $S$ which is $P$-markable.
For the sake of precision, we made a distinction between the notions of a markable set of gates and a markable set of unitaries. However, this distinction is not substantial: if $R$ is a set of representatives of $S$ and $R$ is $P$-markable, then any other set $R'$ of representatives of $S$ is $P$-markable.

It is worth making some easy considerations: i) a necessary condition for $R$ to be $P$-markable is that $S$ is made of unitaries that are jointly perfectly discriminable from any $V \in P_m$, $n \neq m$, and ii) a sufficient condition for $R$ to be $P$-markable is that $S$ is made of unitaries that are jointly perfectly discriminable (in this case $S$ is $P$-markable for any partition $P$).

As we will prove later, none of these conditions is both necessary and sufficient.

As proved in the Methods section, another simple, yet important, property is the following one.

**Lemma 1** (Uniqueness of the minimal markable partition). For any set $R$ of unitaries, there exists a unique minimal partition $P$ such that $R$ is $P$-markable and $R$ is not $P'$-markable for any refinement $P'$ of $P$.

A relevant feature of the information-disturbance problem in Eq. (3) is that information about the unknown $U$ is available only after $U$ has been applied to the input state. A more restrictive scenario is the one in which the outcome of the estimation is available before we apply the unitary and it is described by the circuit

$$\chi U C D = n U^{-1}. \quad (5)$$

where the index $n$ labels the element of a partition of a set of unitaries $R$ is the minimal partition of a set of unitaries $P_m$ such that $R$ is $P$-markable and $R$ is not $P'$-markable for any refinement $P'$ of $P$.

This result is equivalent to the following statement: if $\{P_n\}$ is the minimal partition of a set of unitaries $R$ such that Eq. (4) holds, then all the unitaries in a subset $P_n$ are proportional (i.e. they must represent the same gate). Let us suppose that $P_{n'}$ contains $k$ unitaries $\{U_i\}$, $i = 1, \ldots, k$. The $\{U_i\}$ cannot be jointly perfectly discriminable, otherwise an iteration of the procedure would refine $\{P_n\}$ which is minimal by hypothesis. Then, if $|\chi\rangle$ is the state in Eq. (3) (without loss of generality, $|\chi\rangle$ can be assumed to be pure [13]), there must exist $U_i, U_j \in P_{n'}$ such that $\langle \chi| U_i^\dagger U_j \otimes I |\chi\rangle \neq 0$. Eq. (5) implies $(I \otimes D)(C \otimes I)(I \otimes U_i \otimes I) |\psi\rangle = |n\rangle \otimes U_i |\psi\rangle$ for any $|\psi\rangle$, and by taking the scalar product with $i \neq j$ we easily obtain $\langle \chi | (U_i^\dagger U_j \otimes I) |\chi\rangle = \langle \psi | U_i^\dagger U_j |\psi\rangle$. From $\langle \chi | U_i^\dagger U_j \otimes I |\chi\rangle \neq 0$, we have $U_i \propto U_j$.

The problem of deriving the minimal partition $P$ such that a given set $R$ of unitaries is $P$-markable is difficult in general. Our main result on the markability of unitaries is the following full characterization of the sets of markable unitaries of a qubit.

**Proposition 1** (Markable sets of qubit unitaries). A set $R$ of qubit unitaries is $P$-markable with respect to the non-trivial bipartition $P := \{P_0, P_1\}$ if and only if $\text{span}(P_0) \cap \text{span}(P_1) = \{0\}$ and $i)$ either both $\text{span}(P_0)$ and $\text{span}(P_1)$ are at most two-dimensional, or ii) $R$ is jointly discriminable.

While the proof of necessity is rather technical and is therefore deferred to the Methods section, it is relevant to provide here a constructive proof of sufficiency. Suppose without loss of generality that $\text{span}(P_0) \subseteq \text{span}(\{I, \sigma_x\})$ and $\text{span}(P_1) \subseteq \text{span}(\{\sigma_x, \sigma_y\})$. By the circuit

$$0 \quad U \quad = \quad U \quad n, \quad (6)$$

one can then easily check that $R$ is $P$-markable.

**Proposition 4** suggests a simple procedure to determine the existence of a bipartition $P$ such that a set $R$ of qubit unitaries is $P$-markable: i) diagonalize an arbitrary $U \in R$, $U \not\propto I$; ii) check whether the unitaries in $R$ are either diagonal or off-diagonal in the eigenbasis of $U$; iii) if this is not the case, repeat step (ii) for the unitaries in $U^\dagger R$. If the set is markable, the minimal partition will be a refinement of the partition $\{P_0, P_1\}$ corresponding to diagonal and off-diagonal elements, respectively. If neither step ii) nor step iii) provide a partition, the set is not markable. Further refinements are possible if and only if the set $R$ is either made of either three or four jointly discriminable unitaries. To verify this condition, both $P_0$ and the set $P_1' := U_0^{(i)'}P_1$ of diagonal unitaries must split into a subset proportional to the identity and a trace-less one. If so, the splittings provide the minimal partition, otherwise $\{P_0, P_1\}$ is minimal.

**Proposition 4** implies that: i) joint discriminability is not necessary for $R$ to be markable [contrarily to the case in Eq. (5)]; and ii) the existence of a bipartition $P := \{P_0, P_1\}$ such that any unitary in $P_0$ is perfectly discriminable from any unitary in $P_1$ is not sufficient for $P$-markability.

**IV. CONTROLLABILITY**

Thanks to these preliminary considerations, we are now ready to state our main result on controllability.

**Proposition 2** (Necessary and sufficient condition for controllability). Let $R$ be a set of unitary operators and $P := \{P_n\}$ be the minimal partition of $R$ such that $R$ is $P$-markable. Then $R$ is controllable if and only if there exists a vector $|\psi\rangle$ such that, for any $n$ and any $U, V \in P_n$, we have

$$V^\dagger U |\psi\rangle = |\psi\rangle. \quad (7)$$
While the proof of necessity is rather technical and is therefore deferred to the Methods section, it is relevant to provide here a constructive proof of sufficiency. Let $C$ and $D$ be the unitaries that realize the circuit in Eq. (5) for the minimal partition $P := \{P_n\}$ such that $R$ is $P$-markable. Then, one can verify that the following circuit controls the set $R$:

\[ \begin{array}{c}
\psi \\
S & C & U & D & T \\
0 & & & & \\
\end{array} \xrightarrow{U} \begin{array}{c}
\psi \\
S & & & & \\
0 & & & & \\
\end{array} \]

where $S$ is the swap operator $(S |a\rangle |b\rangle := |b\rangle |a\rangle)$ and we defined $T := \sum_n V_n^\dagger \otimes |n\rangle \langle n|$ where $V_n$ is any unitary in $P_n$. Indeed, after the use of $U$ in the circuit, the classical index $n$ (encoded in the lower output wire of $D$) is available.

As proved in the Methods section, as a trivial consequence of Prop. 2 we have

**Corollary 1.** Let $S$ be a set of gates and $P := \{P_n\}$ be the unique minimal partition of $S$ such that $S$ is $P$-markable. Then $S$ is controllable if and only if, for any choice of the representative set $R$, there exists a vector $|\psi\rangle$ such that, for any $n$ and any $U, V \in P_n$, we have

\[ V^\dagger U |\psi\rangle \propto |\psi\rangle. \]

The necessary and sufficient condition for controllability of Proposition 2 requires knowledge of the minimal partition $P$ such that $R$ is $P$-markable—which is usually difficult to obtain. However, as proved in the Methods section, Propositions 1 and 2 allow for the following complete characterization of controllable sets of qubit unitaries.

**Corollary 2** (Controllable sets of qubit unitaries). A set $R$ of qubit unitaries is controllable if and only if it is non trivially markable or it is commuting.

When considering controllable sets of qubit gate, the circuit in Eq. (6) simplifies as follows:

\[ \begin{array}{c}
0 \\
U \\
\end{array} \xrightarrow{U} \begin{array}{c}
n \\
\end{array} \]

In this case a single ancillary qubit is sufficient.

As a trivial consequence of Corollary 2 we have that the set $R$ of all gates is not controllable, since e.g. the set proposed in Ref. 2 $R := \{W_1, W_2, W_3\}$, with $W_1 = (\sigma_x + \sigma_y)/\sqrt{2}$, $W_2 = (\sigma_y + \sigma_z)/\sqrt{2}$, and $W_3 = (\sigma_z + \sigma_x)/\sqrt{2}$ does not fulfill the hypothesis of Corollary 2.

### V. CONCLUSIONS

In this work we explored under what conditions can a set of quantum conditional statements be implemented. We derived a necessary and sufficient condition for a set of quantum gates to be controllable along with a complete characterization of the controllable sets of qubit unitaries. These results show an intimate relation between controllability and the task of marking the unitaries, i.e. classifying them while applying them to an unknown input state. We completely solved the markability problem for two-dimensional unitaries through the circuit of Eq. (5), which could be considered for experimental implementation using e.g. the technology of Ref. 20. The problem of finding general markability conditions in higher dimension remains open.

### VI. ACKNOWLEDGEMENTS

We thank M. Sedlak for many useful discussions in the early stage of this work. M. D. is supported by Singapore Ministry of Education Academic Research Fund Tier 3 (Grant No. MOE2012-T3-1-009).

**Appendix A: Quantum circuit boards**

Let us consider four finite dimensional Hilbert spaces $\mathcal{H}_j, j = 0, \ldots, 3$ with dimensions $d_i := \dim(\mathcal{H}_i)$. Any map that transforms an input channel $I_{in} : \mathcal{B}(\mathcal{H}_1) \rightarrow \mathcal{B}(\mathcal{H}_2)$ into an output channel $I_{out} : \mathcal{B}(\mathcal{H}_0) \rightarrow \mathcal{B}(\mathcal{H}_3)$ can be realized by inserting the input channel into a quantum circuit board as follows:

\[ \begin{array}{c}
0 \\
X_1 \\
1 \\
T_{in} \\
2 \\
A \\
X_2 \\
3 \\
T_{out} \\
\end{array} \xrightarrow{I_{in}} \begin{array}{c}
0 \\
X_1 \\
1 \\
T_{in} \\
2 \\
A \\
X_2 \\
3 \\
T_{out} \\
\end{array} \]

where $\mathcal{H}_A$ is an ancillary Hilbert space and $X_1 : \mathcal{B}(\mathcal{H}_0) \rightarrow \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_A)$ and $X_2 : \mathcal{B}(\mathcal{H}_2 \otimes \mathcal{H}_A) \rightarrow \mathcal{B}(\mathcal{H}_3)$ are quantum channels. This result, along with many other properties of quantum circuit boards, is well known and is the subject of many publications (see e.g. 12, 13, 15).

In particular it is known that any quantum circuit board, as in Eq. (A1), corresponds to a positive operator (called quantum comb) $R \in \mathcal{B}(\mathcal{H}_0 \otimes \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3)$ subject to linear constraints. Obviously, as far as the Hilbert spaces $\mathcal{H}_i$ are finite dimensional, the set of all the admissible quantum circuit board is a compact set.

Also, the quantum channels which realize the quantum circuit board can be dilated to unitary channels acting on a larger Hilbert space $\mathcal{H}_B$ whose dimension $d_B$ satisfies $d_B \leq d_0 d_1 d_2 d_3$. Then, for each quantum circuit board as in Eq. (A1), there exist two unitary operators $A_1, A_2 \in$
\( \mathcal{B}(\mathcal{H}_B), \mathcal{H}_B = \mathcal{H}_0 \otimes \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3 \) such that

\[
\begin{array}{ccc}
A_1 & T_{in} & B_1 \\
0 & & I
\end{array} = T_{out}, \tag{A2}
\]

where \(|0\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3\) and \(I\) denotes the trace on \(\mathcal{H}_0 \otimes \mathcal{H}_1 \otimes \mathcal{H}_2\).

Appendix B: Information without disturbance

Let \(\{U_i\}\) be a (possibly infinite) set of \(SU(d)\) unitary operators. Each of them corresponds to a unitary channel \(U_i : \mathcal{B}(\mathcal{H}_1) \to \mathcal{B}(\mathcal{H}_2), U_i(\rho) = U_i \rho U_i^\dagger, \) \(d_1 = d_2 = d\). Let us consider a quantum circuit board such that

\[
\begin{array}{ccc}
Y_1 & U_i & Y_2 \\
E & & E
\end{array} = U_i |\psi_i \rangle \langle \psi_i| , \tag{B1}
\]

for all \(i\) and for some set of pure states \(\{|\psi_i\rangle\}\). Whichever the set \(\{|\psi_i\rangle\}\) is, the linearity of the circuit in Eq. \(\text{B1}\) implies that \(\dim(\text{span}\{|\psi_i\rangle\}) \leq \dim(\text{span}\{U_i\}) \leq d^2\).

Then we can consider the quantum circuit board in which the Hilbert space \(\mathcal{H}_E\), which carries the states \(\{|\psi_i\rangle\}\), is encoded into a \(d^2\)-dimensional Hilbert space. The resulting circuit board corresponds to a quantum comb \(R \in \mathcal{B}(\mathcal{H}^{\otimes 6})\).

Let us denote with \(M\) the set of the quantum circuit boards which obey Eq. \(\text{B1}\) for some set of pure states. We have the following result.

**Lemma 2.** The set \(M\) is compact.

*Proof.* The set \(M\) corresponds to the set defined as

\[
\begin{align*}
R \in \mathcal{B}(\mathcal{H}^{\otimes 6}) \text{ is a quantum comb,} \\
\text{Tr}_E[(|U_i^*| R |U_i^*\rangle)] = |U_i\rangle \langle U_i|, \\
\text{Tr}_{0,3}[|U_i^*| R |U_i^*\rangle]^2 = d^2 \text{ Tr}_{0,3}[|U_i^*| R |U_i^*\rangle],
\end{align*}
\]

where the last two equalities translate Eq. \(\text{B1}\) in terms of the operator \(R (|U_i\rangle)\langle U_i|\) is the Choi operator of the unitary channel \(U_i\), with the notation \(|A\rangle := \sum_{m,n} a_{m,n} |m\rangle |n\rangle\) for an operator \(A := \sum_{m,n} a_{m,n} |m\rangle \langle n|\). Eq. \(\text{B2}\) defines a closed subset of the compact set \(\{R \in \mathcal{B}(\mathcal{H}^{\otimes 6})\} \) is a quantum comb and hence it defines a compact set. \(\square\)

We also have the following result.

**Lemma 3.** Let \(R \in M\) be a quantum circuit obeying Eq. \(\text{B1}\) and let \(R^{(2)}\) denote the application of \(R\) twice, i.e.

\[
\begin{array}{ccc}
Y_1 & U_i & Y_2 \\
E & & E
\end{array} = U_i |\psi_i \rangle \langle \psi_i| . \tag{B3}
\]

We have that \(R^{(2)} \in M\).

*Proof.* Clearly we have \(\dim(\text{span}\{|\psi_i\rangle\}) = \dim(\text{span}\{|\psi_i\rangle |\psi_i\rangle\})\). We can encode the Hilbert space \(\mathcal{H}_E \otimes \mathcal{H}_F\) which carries the states \(\{|\psi_i\rangle |\psi_i\rangle\}\) into a \(d^2\)-dimensional Hilbert space. Then the resulting circuit board corresponds to a quantum comb in \(M\). \(\square\)

The quantum circuit board in Eq. \(\text{A1}\), while leaving unaffected the transformation \(U_i\), extracts some side information about \(U_i\), which is stored in the state \(|\psi_i\rangle\). Now, for every pair of unitaries \(U\) and \(V\), the amount of information provided by the circuit in Eq. \(\text{A1}\) can be defined as an arbitrary non-negative decreasing function \(f(\alpha_{U,V}) := |\langle \psi_U | \psi_V \rangle|\) for any pair \(U,V\).

Let now \(R^{opt} \in M\) be the optimal quantum circuit board which achieves the maximum value of \(f\). Such a \(R^{opt}\) must exists since \(M\) is compact as proved in Lemma 2. Suppose that for a given pair \(U,V\), \(R^{opt}\) is such that \(0 < \alpha_{U,V} := |\langle \psi_U | \psi_V \rangle| < 1\). Let us now consider \(R^{opt}(2)\) which is the circuit which corresponds to the application of \(R^{opt}\) twice as in Eq. \(\text{B3}\). As we proved in Lemma 3, \(R^{opt}(2)\) is an element of \(M\). The quantum circuit board \(R^{opt}(2)\) gives, for any pair \(U,V\), \(|\psi^{(2)}_{U,V} = |\psi_U\rangle \otimes |\psi_V\rangle\) and \(|\langle \psi^{(2)}_{U,V}| \psi^{(2)}_{U,V}\rangle| = \alpha^{opt}_{U,V} < \alpha_{U,V}\).

This implies that the hypothesis of optimality of \(R^{opt}\) is absurd. One must then have either \(\alpha_{U,V} = 0\) or \(\alpha_{U,V} = 1\). Since the argument can be repeated for every pair \(U,V\), one has that the optimal circuit board \(R^{opt}\) exists and is such that \(\alpha_{U,V} = 0, 1\) for all \(U,V\). As discussed in the previous section \(R^{opt}\) can be obtained by a pair of unitary operators \(A^{opt}_1, A^{opt}_2\), i.e.

\[
\begin{array}{ccc}
0 & A^{opt}_1 & 0 \\
0 & & 0
\end{array} = U_i |\psi_i \rangle \langle \psi_i| ,
\]

where \(|\langle \psi_i | \psi_j\rangle| = 0, 1\) for any \(i\) and \(j\), which trivially implies also

\[
\begin{array}{ccc}
0 & A^{opt}_1 & 0 \\
0 & & 0
\end{array} = U_i |\psi_i \rangle \langle \psi_i| ,
\]

where \(|\langle \phi_i | \phi_j\rangle| = 0, 1\) for any \(i\) and \(j\).

We can now prove Lemma 1 that we report here for the reader’s convenience.

**Lemma 4** (Uniqueness of the minimal markable partition). For any set \(R\) of unitaries, there exists a unique minimal partition \(P\) such that \(R\) is \(P\)-markable and \(R\) is not \(P'\)-markable for any refinement \(P'\) of \(P\).

*Proof.* The existence is proved by considering the trivial partition. To prove uniqueness, let \(P^{(0)}\) and \(P^{(1)}\) be two
different minimal partitions. By subsequently applying the circuit in Eq. (3) for $P^{(0)}$ and $P^{(1)}$, one proves that $R$ is $P'$-markable, with $P'$ the refinement of $P^{(0)}$ and $P^{(1)}$. □

We are now in a position to prove our main result about the markability of unitaries (given in Proposition 1 and reported here for the reader’s convenience), that is a complete characterization of the set of markable unitaries of a qubit.

**Proposition 3** (Markable sets of qubit unitaries). A set $R$ of qubit unitaries is $P$-markable with respect to the non-trivial bipartition $P := \{P_0, P_1\}$ if and only if $\dim(\text{span}(P_0) \cap \text{span}(P_1)) = 0$ and i) either both $\text{span}(P_0)$ and $\text{span}(P_1)$ are at most two-dimensional, or ii) $R$ is jointly discriminable.

**Proof.** First we prove necessity. The case ii) is trivial. Let us consider a bipartition $P := \{P_0, P_1\}$ such that $R$ is $P$-markable, and let us denote by $U_n^{(i)}$ the $n$-th element of $P_i$. The result relies on the fact that two qubit unitaries $U, V$ are perfectly discriminable if $U^\dagger V$ is traceless, which by Eq. (1) implies $\text{Tr}[U_n^{(i)} U_n^{(j)}] = 0$ for $i \neq j$. Without loss of generality we suppose $I \in P_0$ (otherwise, we can consider the set $R' := \{U_n^{(0)} U_n^{(1)}\}$ which is $P$-markable if and only if the set $R$ is). We show that $R$ is not $P$-markable for any partition $P := \{P_0, P_1\}$ such that $\dim(\text{span}(P_i)) = 3$. Since we must have $\dim(\text{span}(P_0)) = 1$, we have $\text{span}(P_0) = \text{span}(I)$ and $\text{span}(P_1) = \text{span}(\sigma_x, \sigma_y, \sigma_z)$, where $\text{span}(T)$ denotes the complex span of $T$, and $\sigma_i$ are the Pauli matrices. First we consider the case in which $P$ is the minimal partition. Then from Eq. (1), defining $T_i := D(\sigma_i \otimes I) C(I \otimes [0])$, we must have $T_0 = I \otimes [0]$, and $T_1 = \sigma_i \otimes [1]$, from which one derives the contradiction $\sigma_0 = iT_1^T T_2 = T_0^T T_0 = 0$. Let us now suppose that $P$ is not the minimal partition and all the unitaries in the set are not jointly perfectly discriminable. Then, the minimal partition $P'$ must be such that $P' := \{P_0, P_1, P_2\}$ with $\dim(\text{span}(P_0)) = \dim(\text{span}(P_1)) = 1$ and $\dim(\text{span}(P_2)) = 2$. Without loss of generality we can suppose $\text{span}(P_0) = \text{span}(I)$, $\text{span}(P_1) = \text{span}(\sigma_z)$ and $\text{span}(P_2) = \text{span}(\sigma_x, \sigma_y)$. Then from Eq. (1), we must have $T_0 = I \otimes [0]$, $T_2 = \sigma_z \otimes [1]$ and $T_1 = \sigma_i \otimes [2]$ for $i = x, y, z$ from which we obtain the contradiction $\sigma_z = iT_1^T T_2 = T_0^T T_0 = 0$.

To prove sufficiency, without loss of generality that $\text{span}(P_0) \subseteq \text{span}\{I, \sigma_z\}$ and $\text{span}(P_1) \subseteq \text{span}\{\sigma_x, \sigma_y\}$. By the circuit

\[
\begin{array}{c}
U \circ \sigma_x \\
\hline
0 \quad n
\end{array}
\]

one can then easily check that $R$ is $P$-markable. □

**Appendix C: Controllability**

We are now in a position to prove Proposition 2 that we report here for the reader’s convenience.

**Proposition 4** (Necessary and sufficient condition for controllability). Let $R$ be a set of unitary operators and $P := \{P_n\}$ be the minimal partition of $R$ such that $R$ is $P$-markable. Then $R$ is controllable if and only if there exists a vector $|\psi\rangle$ such that, for any $n$ and any $U, V \in P_n$, we have

\[
V^\dagger U |\psi\rangle = |\psi\rangle.
\]

**Proof.** First, let us assume that $R$ is controllable. Then there exists unitaries $A$ and $B$ such that

\[
B(I \otimes U)A(0) \otimes I = |\phi_n\rangle \otimes (I \otimes U)
\]

holds. By the optimality argument used to justify definition 3 it is not restrictive to take $|\phi_n\rangle = |n\rangle$ for some orthonormal basis $\{|n\rangle\}$. Let us consider a vector $|\chi\rangle$ such that $(I \otimes U) |\chi\rangle = |\chi\rangle$. From Eq. (C2) we have $(I \otimes U) |\Psi\rangle = |\Phi_n\rangle$ where we defined $|\Psi\rangle := A(|0\rangle \otimes |\chi\rangle)$ and $|\Phi_n\rangle := B(|n\rangle \otimes |\chi\rangle)$. Since the previous identity holds for any $U \in R$ for some $n$, there exists a partition $P'$ of $R$ such that $P'_n$ collects all the $U$ that satisfy equation (C2) with the same $n$. Equation (C2) can then be rewritten as $(I \otimes V^\dagger U) |\Psi\rangle = |\Psi\rangle$ for any $U, V \in P_n$. Let us now consider an expansion $|\Psi\rangle := \sum_{a,b} c_{a,b} |a\rangle \otimes |b\rangle$ (\{|a\rangle\} and \{|b\rangle\} are orthonormal basis) and let us fix $a'$ such that $c_{a',a} \neq 0$ for some $b$. By multiplying both sides of $(I \otimes V^\dagger U) |\Psi\rangle = |\Psi\rangle$ by $|a'\rangle \otimes I$ we recover Eq. (C1) with $|\psi\rangle := (|a'\rangle \otimes I) |\Psi\rangle$. We then proved that controllability implies the existence of a partition $P'$ such that Eq. (C1) holds in each $P'_n$. The same condition holds for the minimal partition $P$ since it is a refinement of $P'$. We then proved necessity.

The proof of sufficiency is as follows. Let $C$ and $D$ be the unitaries that realize the circuit (2) for the minimal partition $P := P_n$ such that $R$ is $P$-markable. Then, one can verify that the following circuit controls the set $R$:

\[
\begin{array}{c}
S \quad C \quad U \quad D \quad S \quad T \\
\hline
\psi \quad \omega \quad \omega \quad \omega \quad \psi
\end{array}
\]

where $S$ is the swap operator $(S |a\rangle \otimes |b\rangle = |b\rangle \otimes |a\rangle)$ and we defined $T := \sum_n V_n^\dagger \otimes |n\rangle \langle n|$ where $V_n$ is any unitary in $P_n$. Indeed, after the use of $U$ in the circuit, the classical index $n$ (encoded in the lower output wire of $D$) is available. □

**Corollary 3** (Necessary and sufficient condition for controllability). Let $S$ be a set of gates and $P := \{P_n\}$ be the
unique minimal partition of $S$ such that $S$ is $P$-markable. Then $S$ is controllable if and only if, for any choice of the representative set $R$ there exists a vector $|\psi\rangle$ such that, for any $n$ and any $U, V \in P_n$, we have

$$V^\dagger U |\psi\rangle \propto |\psi\rangle.$$  \hspace{1cm} (C3)

Proof. The conditions of Corollary 1 are met by a set of representatives if and only if they are met by any other set of representatives. Given a set of representatives and a vector such that Eq. (C3) holds, it is straightforward to find a set of representatives such that Eq. (C1) holds. \hfill \Box

Corollary 4 (Controllable sets of qubit unitaries). A set $R$ of qubit unitaries is controllable if and only if it is non-trivially markable or it is commuting.

Proof. If the set $R$ is markable, the conditions of Proposition 2 are met since the elements of $P_0$ commute, while multiplying two off diagonal matrices in $P_1$ provides a diagonal one. If the minimal partition of $R$ is trivial, then controllability is equivalent to commutativity of $R$. \hfill \Box

[1] M. A. Nielsen and I. L. Chuang, *Quantum computation and quantum information* (Cambridge university press, 2010).
[2] P. W. Shor, in *Proceedings of the 35th Annual Symposium on Foundations of Computer Science* (IEEE, 1994) pp. 124-134.
[3] A. Y. Kitaev, Quantum measurements and the Abelian stabilizer problem. Preprint at [http://arxiv.org/abs/quant-ph/9511026] (1995).
[4] J. Thompson, M. Gu, K. Modi, and V. Vedral, *Quantum Computing with black-box subroutines*, arXiv:1310.2927.
[5] A. Soeda, Limitations on quantum subroutine designing due to the linear structure of quantum operators, Talk at international conference on quantum information (IC-QIT) (2013).
[6] M. Araújo, A. Feix, F. Costa, and Č. Brukner, Quantum circuits cannot control unknown operations, New Journal of Physics 16, 093026 (2014).
[7] X.-Q. Zhou, T. C. Ralph, P. Kalasuwan, M. Zhang, A. Peruzzo, B. P. Lanyon, J. L. O’Brien, Adding control to arbitrary unknown quantum operations, Nature Communication 2, 413 (2011).
[8] X.-Q. Zhou, P. Kalasuwan, T. C. Ralph, J. L. O’Brien, Calculating unknown eigenvalues with a quantum algorithm, Nature Photonics 7, 223 (2013).
[9] N. Friis, V. Dunjko, W. Dur, and H. J. Briegel, Implementing quantum control for unknown subroutines, Phys. Rev. A 89, 030303 (2014).
[10] N. Friis, A. A. Melnikov, G. Kirchmair, and H. J. Briegel, Coherent controlization using superconducting qubits, Sci. Rep. 5, 18036 (2015).
[11] M. Oszmaniec, A. Grudka, M. Horodecki, and A. Wójtik, Creating a superposition of unknown quantum states, Phys. Rev. Lett. 116, 110403 (2016).
[12] G. Chiribella, G. M. D’Ariano, and P. Perinotti, Quantum circuit architecture, Phys. Rev. Lett. 101, 060401 (2008).
[13] G. Chiribella, G. M. D’Ariano, and P. Perinotti, Theoretical framework for quantum networks, Phys. Rev. A 80, 022339 (2009).
[14] A. Bisio, G. Chiribella, G. M. D’Ariano, and P. Perinotti, Quantum networks: general theory and applications, Acta Physica Slovaca 61, 273 (2011).
[15] A. Bisio, G. M. D’Ariano, P. Perinotti, and G. Chiribella, Minimal computational-space implementation of multiround quantum protocols, Phys. Rev. A 83, 022325 (2011).
[16] A. Bisio, G. Chiribella, G. M. D’Ariano, and P. Perinotti, Information-disturbance tradeoff in estimating a unitary transformation, Phys. Rev. A 82, 062305 (2010).
[17] A. Acín, Statistical distinguishability between unitary operations, Phys. Rev. Lett. 87, 177901 (2001).
[18] G. M. D’Ariano, P. Lo Presti, and M. G. A. Paris, Improved discrimination of unitary transformations by entangled probes, Journal of Optics B: Quantum and Semi-classical Optics 4, S273 (2002).
[19] R. Duan, Y. Feng, and M. Ying, Entanglement is not necessary for perfect discrimination between unitary operations, Phys. Rev. Lett. 98, 100503 (2007).
[20] J. L. O’Brien, G. J. Pryde, A. G. White, T. C. Ralph, and D. Branning, Demonstration of an all-optical quantum controlled-NOT gate, Nature 426, 264 (2003).