Supersymmetric higher-derivative actions in ten and eleven dimensions, the associated superalgebras and their formulation in superspace

Kasper Peeters$^1$, Pierre Vanhove$^2$ and Anders Westerberg$^3$

$^1$ CERN  
TH-division  
1211 Geneva 23  
Switzerland  

$^2$ SPT  
Orme des Merisiers  
CEA/Saclay  
91191 Gif-sur-Yvette Cedex  
France

$^3$ NORDITA  
Blegdamsvej 17  
DK-2100 Copenhagen Ø  
Denmark

k.peeters, p.vanhove@damtp.cam.ac.uk, a.westerberg@nordita.dk

Abstract

Higher-derivative terms in the string and M-theory effective actions are strongly constrained by supersymmetry. Using a mixture of techniques, involving both string-amplitude calculations and an analysis of supersymmetry requirements, we determine the supersymmetric completion of the $R^4$ action in eleven dimensions to second order in the fermions, in a form compact enough for explicit further calculations. Using these results, we obtain the modifications to the field transformation rules and determine the resulting field-dependent modifications to the coefficients in the supersymmetry algebra. We then make the link to the superspace formulation of the theory and discuss the mechanism by which higher-derivative interactions lead to modifications to the supertorsion constraints. For the particular interactions under discussion we find that no such modifications are induced.
1 Introduction

The effective field theory actions describing the dynamics of the massless modes of the various string theories contain, in addition to the well-known supergravity terms, an infinite number of higher-derivative corrections. These terms encode the dynamics of the massive modes and the extended character of strings. For the first few orders in the string slope $\alpha'$ and the string coupling $g_s$, many of these terms are known explicitly from either sigma-model anomaly computations or from string amplitudes. The best-known one is perhaps the Green-Schwarz Lorentz anomaly cancellation term [1] in the heterotic string effective action, appearing at genus one:

$$S_{GS} = \int B \wedge t_8 R^4.$$  

(1.1)

The $t_8$ tensor (whose precise definition will be recalled below) contracts the eight flat indices of the Riemann tensors. As was first pointed out by Vafa and Witten [2], this term arises also in the non-chiral type IIA theory, where a duality argument relates it to the five-brane Lorentz...
anomaly (see Duff et al. \cite{3}). Another higher-derivative term which has received considerable
attention is the famous
\[ S_{R^4} = \int d^{10} x \, e \, t_5 t_8 R^4. \] (1.2)
This term arises in all string theories, and not only at one-loop order but also at tree level,
when multiplied with the appropriate power of the dilaton\(^1\). Upon compactification to four
dimensions, the action (1.2) is responsible for corrections to the hypermultiplet geometry; see
for instance Strominger \cite{6} and Antoniadis et al. \cite{7}. Moreover, both (1.1) and (1.2) contribute
to the equations of motion and are therefore expected to modify the supergravity brane solutions.
In addition, several other explicit higher-derivative terms are known or conjectured to
exist by duality arguments (see e.g. Berkovits and Vafa \cite{8}).

Although one can in principle determine higher-derivative terms through explicit string
scattering calculations, it is expected that their structure is strongly constrained by the sym-
metries of string theory. In particular, suitable linear combinations of the bosonic terms
discussed above should be part of a full supersymmetric action when amended with quadratic
and higher-order fermion terms. Ultimately one may perhaps hope that for every string the-
ory (i.e. for every string model with a given field content) and for every genus (i.e. for every
given power of the dilaton), supersymmetry will be enough to fix the structure of the action
at the corresponding order of the string loop expansion. One can argue (see also section 2.1)
that supersymmetry does not mix the different string loop orders. However, in some cases
additional symmetries may be enough to bridge even the genus gap; an explicit example is
the \(SL(2,\mathbb{Z})\) symmetry of type IIB string theory, as shown by Green and Sethi \cite{9}.

Supersymmetry on the component level often provides us with an elegant underlying ex-
planation of the rather complicated form of higher-derivative actions (the present paper will
exhibit many examples of this fact). But if we are interested in world-volume actions for
branes embedded in target-space supergravity backgrounds, it is necessary to also understand
how one can describe such background theories in their superspace formulations. Indeed,
supersymmetric brane actions are known only in formulations with manifest target-space su-
persymmetry. One expects, based on previous experience with string models, that there will
be a close link between world-volume quantum effects and target-space corrections, although
such a link is obviously much more difficult to establish given the complications that arise in
quantising kappa-symmetric actions.

The relation between the two formalisms is easiest to understand when one realises that
superspace provides a geometric rationale for the algebraic structure of the component-space
theory. Therefore, one should focus on possible modifications to the supersymmetry algebra
induced by higher-derivative terms in the action. In a component language, one finds that,
when higher-derivative terms are included, one also has to modify the supersymmetry trans-
formation rules of the fields. In other words, supersymmetry for these actions means that the
invariance of the action is expressed by
\[ \left( \delta_0 + \sum_n (\alpha')^n \delta_n \right) \left( S_0 + \sum_n (\alpha')^n S_n \right) = 0. \] (1.3)
Due to these modifications, the field-dependent coefficients of the supersymmetry algebra can
pick up corrections as well. As a result, the geometrical structure of superspace is modified,\(^1\)

\(^1\)The results of Bern et al. \cite{4} and Dunbar et al. \cite{5} show that it does not get corrections beyond one loop.
There are, however, non-perturbative contributions weighted with an appropriate power of the dilaton.
which is reflected in modified supertorsion constraints. We will return to these issues in more
detail later; it suffices to say at this point that our ultimate goal is to derive the modifications
to the supersymmetry transformation rules and use these to make contact with a manifestly
supersymmetric formulation of the theory.

The task of constructing supersymmetric string effective actions using supersymmetry as
the only input is horrendously complicated. Already in pure gauge theories without gravity
such an approach is rather difficult. The sub-leading terms were constructed by Metsaev and
Rakhmanov [10] and later extended to the non-abelian case by Bergshoeff et al. [11]. Much
less is known about the higher-derivative supergravity invariants. Explicit results are only
known for the heterotic effective action, while there are some expectations about the form of
higher-derivative invariants for other theories (based on string calculations) which, however,
have so far defied a supersymmetry analysis.

It turns out that the heterotic theory admits two distinct types of higher-derivative super-invariants. The first type is related to the shift of the Neveu-Schwarz three-form field strength
by the Lorentz Chern-Simons term as required by anomaly cancellation. In order to restore
supersymmetry, which is broken by this mechanism, additional terms are required at genus
zero. The $\alpha' R^2$ terms were first analysed by Romans and Warner [12], but there are corrections
at any order in the string slope, of which the $(\alpha')^3 R^4$ ones have been computed explicitly
by Bergshoeff and de Roo [13]. Of course, none of these terms are expected to play a role in
the other string theories or in eleven dimensions. The second set of superinvariants, which are
known to be relevant for the other string theories as well (by virtue of the sigma-model computa-
tions of Grisaru et al. [14] and several string-amplitude calculations) has $(\alpha')^3 R^4$ terms
at the lowest order. These invariants have been studied in an impressive paper by de Roo
et al. [15] and reported in more detail in the thesis by Suelmann [16]. These authors were able
to pin down the exact form (up to and including fermion bilinears) of three actions, each of
which is separately invariant under supersymmetry within the limitations of their analysis.

These limitations arise as follows. The supergravity theories in ten dimensions involve
many fields, so one often restricts to checking supersymmetry only with respect to a subset
of the transformation rules. Furthermore, the four-fermi terms of higher-derivative actions
are extremely difficult to compute and are therefore in practice always ignored. Because of
these restrictions, the analysis of de Roo et al. [15] has provided us with three candidate
building blocks for invariants, for which invariance has been checked, but only for a subset of
the full transformation rules. In addition, they constructed a generalisation of the Yang-Mills
invariant which includes the couplings of lowest order in the gravity fields. Their results can
be transcribed in a form adapted to string theory, in which case the bosonic terms of the
gravitational invariants are expressed as (the precise form of the index contractions is not
important at this point)

\begin{align}
I_X &= t_8 t_8 R^4 + \frac{1}{2} \varepsilon_{10} t_8 B R^4 + O(\alpha') , \\
I_{Y_1} &= (t_8 + \frac{1}{2} \varepsilon_{10} B)(\mathrm{tr} R^2)^2 + 4 HR^2 DR + O(\alpha') , \\
I_{Y_2} &= (t_8 + \frac{1}{2} \varepsilon_{10} B) \mathrm{tr} R^4 + HR^2 DR + O(\alpha') , \\
I_Z &= -\varepsilon_{10} \varepsilon_{10} R^4 + 4 \varepsilon_{10} t_8 B R^4 + O(\alpha') .
\end{align}

(The first three invariants are related via the identity $I_X = 24 I_{Y_2} - 6 I_{Y_1}$, and so they comprise
only two linearly invariant combinations.) The terms of higher order in the string slope involve
higher powers of the fields as well as derivatives thereof. Only one particular linear combination
of the above invariants is completely independent of the anti-symmetric tensor field, namely
Because of gauge invariance for the $B$-field, this is the only invariant that can appear in string theory at arbitrary loop order, and in particular at tree level.

The fact that these particular combinations can be made supersymmetric is, as expected, consistent with explicit calculations of the heterotic string effective action. The bosonic parts of the tree-level contributions have been evaluated by Cai and Nuñez [17] and Gross and Sloan [18] (three- and four-point amplitudes) while the bosonic one-loop terms were computed by Sakai and Tani (19), Abe et al. [20, 21] and Ellis et al. [22] (four-point amplitudes) and Lerche et al. [23] (five-point amplitudes involving the Neveu-Schwarz tensor field). Ignoring normalisation factors, the result reads

$$L_{\text{heterotic}}\bigg|_{(\alpha')^3} = e^{-2\phi_4} \left( I_{BdR} + I_X - \frac{1}{8} I_Z \right) + \left( I_{FR} + I_X \right). \quad (1.5)$$

The term $I_{BdR}$ is the $(\alpha')^3$ piece of the invariant constructed by Bergshoeff and de Roo [13], and $I_{FR}$ is the invariant of de Roo et al. [15] which describes the coupling to the Yang-Mills fields. When the transformations of the dilaton and the tensor field are fully taken into account, the linear combinations appearing above are expected to be completely fixed by supersymmetry, although this has to our knowledge never been checked. Because the relative normalisation (suppressed above) between the tree-level and one-loop terms is known to contain a transcendental $\zeta(3)$ factor, it is impossible to get further constraints from supersymmetry that relate these two contributions.

The heterotic invariants discussed above seem to have some relevance for the other theories as well. The absence of Yang-Mills fields makes these actions considerably simpler (when higher-rank form fields are excluded). The four-point amplitudes for the IIA theory were computed by Green and Schwarz [24, 25] and Kiritsis and Pioline [26]; we already commented on the five-point analysis by Vafa and Witten [2]. The result is that

$$L_{\text{IIA}}\bigg|_{(\alpha')^3} = e^{-2\phi_4} \left( I_X - \frac{1}{8} I_Z \right) + \left( I_X + \frac{1}{8} I_Z \right). \quad (1.6)$$

Alternatively, the sigma-model results of Grisaru et al. [14] can be used to determine this action. The type IIB theory [27, 28, 29] can, due to its modular invariance, be written in terms of a single linear combination of invariants at any loop order, multiplied with an overall factor in terms of the complexified coupling constant $\Omega = \mathcal{C}^{(0)} + i e^{-\phi}$ [30, 9],

$$L_{\text{IIB}}\bigg|_{(\alpha')^3} = f(\Omega, \bar{\Omega}) \left( I_X - \frac{1}{8} I_Z \right). \quad (1.7)$$

It should be stressed that none of these type-II results have any solid backing from a supersymmetry analysis; the bosonic terms have been computed directly in string theory and their fermionic completion is as of yet unknown.

The above summarises the current knowledge about the form of the higher-derivative actions in string theory. Given the considerable technical difficulties that arise when one wants to analyse the supersymmetric completions of these actions, and the even bigger obstacles one faces when deriving the required modifications to the supersymmetry transformation rules, we will in this paper follow a slightly different path. Using information from various sources, namely string amplitudes, supersymmetry requirements as well as an interesting parallel between super-Maxwell theory and supergravity, we set out to systematically analyse string-based higher-derivative supergravity theories. One of our main goals is to obtain enough information to construct the superinvariant in eleven dimensions, write down the full set of
modifications to the supersymmetry transformation rules and determine the implications of these modifications for the field-dependent coefficients in the supersymmetry algebra. Using the latter, it will then be possible to study the interplay of kappa symmetry with higher-derivative target space and world-volume corrections. However, there are many steps involved before we reach this goal.

In the first part of this paper we re-analyse the higher-derivative $F^4$ invariants for the simpler super-Maxwell model and the coupling to an external supergravity background. We show how the various fermionic terms arise from string-amplitude calculations and use supersymmetry to fix their relative normalisation. This provides a very compact form of the results reported in the thesis by Suelmann [16].

We then show, using observations made earlier by Bellucci and Gates [31] and Bergshoeff and de Roo [13], how the super-Maxwell results can be used to bootstrap the construction of the higher-derivative $I_X$ invariant for $N = 1$ supergravity. Since all our terms will be organised, right from the beginning, in a form that is adapted to string theory, we will again find a very compact form of the fermionic bilinears and the modified transformation rules. More importantly, since one can check superinvariance of this action by hand, it becomes feasible to try to understand the way in which generalisations to the type IIA and IIB models can be implemented.

We will leave such considerations for later and instead focus on the extension to eleven dimensions. This poses additional problems. First of all, supersymmetry is highly dimension dependent, so it is not at all clear whether the analysis of de Roo et al. [15] can be extended in a straightforward way. Another puzzle is the appearance of the $t_8$ tensors. In string theory, they arise from the integration of eight world-sheet fermion zero modes in the even spin-structure sector, while the $\varepsilon_{10}$ comes from the odd spin structure. As this can happen for left- and right-movers separately, the structure of the contractions in (1.4) is very natural from a string point of view. Clearly, such a split is not expected to survive in eleven dimensions. Our analysis of the fermionic terms sheds considerable light on this issue. As we shall see, the part of the action bilinear in the fermions does not exhibit a complete $t_8 t_8$ structure, even in ten dimensions. In addition, it will become clear that there is an inherent dimension dependence in these terms which, when lifted to eleven dimensions, forces us to give up many of the nice $t_8$ tensors in exchange for more complicated couplings. One of the main results of our paper, namely the very compact form of the eleven-dimensional superinvariant (in the absence of gauge fields) is exhibited in (3.17). The new tensorial structures found there may perhaps be explained by looking at a manifestly covariant eleven-dimensional superparticle or membrane loop calculation using the vertex operators recently constructed by Dasgupta et al. [32], although we have not yet attempted to do so.

With these results at hand, we are able to return to our original motivation for this project. In section 3.4 and 3.5 we compute the supersymmetry algebra generated by the modified transformation rules. In general, the supersymmetry algebra of supergravity theories takes the form

$$[\delta_1^{\text{susy}}, \delta_2^{\text{susy}}] = \delta_{\text{translation}} + \delta_{\text{susy}} + \delta_{\text{gauge}} + \delta_{\text{Lorentz}}.$$  

The coefficients on the right-hand side are, however, not necessarily simply functions of the supersymmetry parameters $\epsilon_1$ and $\epsilon_2$ appearing on the left-hand side; rather, they can (in an on-shell formulation without auxiliary fields) depend explicitly on the supergravity fields. With the $\alpha'$ corrections in place, each of these coefficients can pick up correction terms as
well. Focussing on the translation part, we have for instance

$$[\bar{\epsilon}_1 Q, \bar{\epsilon}_2 Q] e^{r}_\mu = \xi^\nu \partial_\nu e^{r}_\mu + \cdots ,$$

(1.9)

where the translation parameter is $\xi^\nu = 2(\bar{\epsilon}_1 \Gamma^\nu \epsilon_2) + \cdots$, the dots indicating field-dependent $\alpha'$ correction terms. In the superspace formulation, the algebra is generated by supercovariant derivatives,

$$\{D_a, D_b\} = T_{ab}^{\ r} D_r .$$

(1.10)

In other words, the structure of the bundle tangent to the supermanifold reflects the component-field supersymmetry algebra, and the link is provided by constraints on the superspace torsions. Consequently, corrections to the algebra are in direct correspondence with modifications to the classical superspace supergravity constraints. Indeed, as has been shown by Howe [33], such corrections are expected to be necessary in order to describe modifications of the eleven-dimensional supergravity theory. Since our analysis provides us with the full set of modified field transformation rules, we can derive the required superspace torsion constraints.

Within the limitations of our analysis (we do not consider variations proportional to the gauge field), we find that there are, however, no corrections to the supersymmetry algebra arising from the lifted $I_X$ action. Consequently, the modifications to the superspace torsion constraints anticipated by Howe [33] and Cederwall et al. [34, 35] will have to follow from a more elaborate analysis involving also gauge-field dependent terms in the superinvariant. In the last part of this paper we discuss possible explanations for and implications of this result. Our techniques can be used to include gauge-field terms in the analysis as well, and work along these directions is in progress.

For reference, we have included an appendix on the first- and second-order formulations of the $N = 1$ supergravity theories in ten and eleven dimensions. Most of it is not new, but the reader may find it helpful to have the entire derivation of these standard actions spelled out in one place, together with an explanation of the origin of various normalisation factors which have been quite crucial in our analysis. A second appendix collects details on the expansion of the $t_8$ tensor, a number of useful gamma-matrix identities and the conventions used in this paper.

## 2 First step: stringy construction of the $F^4$ action

### 2.1 Effective actions and field redefinitions

As the first step in our construction of the supersymmetric $(\alpha')^3$ corrections to the supergravity actions, we will tackle a related problem, namely the construction of the $(\alpha')^2$ corrections to the super-Maxwell action in ten dimensions. This invariant has been constructed before by Metsaev and Rakhmanov [10], and the non-abelian case (which we will not address) was subsequently worked out by Bergshoeff et al. [11].\footnote{We should also mention that a supersymmetric version of the ten-dimensional Born-Infeld action has been constructed by Aganagic et al. [36] by gauge-fixing the kappa-symmetric D9-brane action.} We will also discuss the coupling to supergravity which was worked out by Suelmann [16]. However, as we will be using string input, we are able to rederive these results in a much simpler way, and at the same time this approach allows us to see how to generalise the invariants to the supergravity $W^4$ actions. In
the present section we discuss a number of general issues, while the following two sections are concerned with the string analysis and the supersymmetry constraints respectively.

The field content of the on-shell super-Maxwell theory consists of an abelian vector $A_\mu$ and a negative-chirality Majorana-Weyl spinor $\chi$. We also consider the interaction with the vielbein $e^r_\mu$, the negative-chirality Majorana-Weyl gravitino $\psi_\mu$ and the two-form $B_{\mu\nu}$ of the supergravity multiplet. At the lowest order, the super-Maxwell action coupled to background supergravity is given by (we have normalised the fields in such a way as to make comparison with the supergravity calculations using the conventions of appendix A simple) 

$$S_{F^2} = \int d^{10}x \, e \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - 8 \bar{\chi} \hat{D}(\omega) \chi + 2 \bar{\chi} \Gamma^\mu \Gamma^{\nu\rho} \psi_\mu F_{\nu\rho} \right]$$ (2.1)

(the coupling of the gauge field to the gaugino is identically zero in the abelian case due to the fact that the gaugino is Majorana). This action is invariant under local supersymmetry, under which the fields transform according to the following rules:

$$\delta A_\mu = -4 \bar{\epsilon} \Gamma_\mu \chi,$$

$$\delta \chi = \frac{1}{8} \Gamma^{\mu\nu} \epsilon F_{\mu\nu},$$

$$\delta F_{\mu\nu} = -8 \left( D_{[\mu}(\omega) \bar{\epsilon} \right) \Gamma_{\nu]} \chi + 8 \bar{\epsilon} \Gamma_{[\mu} D_{\nu]} (\omega) \chi.$$

(2.2)

We have not displayed the gravitational part of the action (see appendix A for details and references). The only coupling there consists of the usual shift of the field strength $H_{\mu\nu\rho}$ of the two-form $B_{\mu\nu}$ by the Yang-Mills Chern-Simons form. The contribution of that term to the equation of motion is irrelevant in the variations of the terms that we consider below. However, in order for local supersymmetry to work out we still have to consider the transformations of the supergravity multiplet fields:

$$\delta e^r_\mu = 2 \bar{\epsilon} \Gamma^r \psi_\mu,$$

$$\delta \psi_\mu = D_\mu (\omega) \epsilon + \cdots,$$

$$\delta B_{\mu\nu} = 2 \bar{\epsilon} \Gamma_{[\mu} \psi_{\nu]}.$$

(2.3)

The dots represent terms proportional to the field strength $H_{\mu\nu\rho}$ of the two-form $B_{\mu\nu}$ which again will not be needed for our calculation below. We have suppressed the dilaton as it only transforms into the dilatino and we do not consider variations proportional to the latter. The transformation rules above are given in the string frame.

The equations of motion that follow from the above action are

$$\mathcal{E}(A)^\mu = \frac{1}{e} \frac{\delta S_{F^2}}{\delta A_\mu} = D_\lambda F^{\lambda\mu} - 4 D_\nu \left( \bar{\chi} \Gamma^\lambda \Gamma^{\nu\rho} \psi_\lambda \right),$$

(2.4a)

$$\mathcal{E}(\bar{\chi}) = \frac{1}{e} \frac{\delta S_{F^2}}{\delta \bar{\chi}} = 16 \Gamma^\mu \left( D_\mu (\omega) \chi - \frac{1}{8} \Gamma^{rs} \psi_\mu F_{rs} \right) =: 16 \Gamma^\mu \tilde{\chi}_\mu.$$

(2.4b)

The derivative on the gaugino is not supercovariant, i.e. it picks up a derivative of the supersymmetry parameter $\epsilon$, in contrast to the hatted object appearing in the equation of motion for the gaugino. In the higher-derivative action, the identity

$$\hat{D}(\omega) \chi = \frac{1}{8} \Gamma^\mu \Gamma^{rs} \psi_\mu F_{rs} + \frac{1}{16} \mathcal{E}(\bar{\chi})$$

(2.5)
can be used to trade derivatives on the gaugino for terms proportional to the lowest-order equations of motion plus terms proportional to the gravitino, and will be used repeatedly below. We stress that it is an expression which is non-linear in the fields, and therefore, when used in the amplitudes, can produce a five-point interaction from a four-point string amplitude.

There is considerable ambiguity in the structure of the on-shell higher-derivative effective action, due to the fact that field redefinitions can be used to remove higher-order $\alpha'$ terms proportional to the field equations; see for instance Tseytlin [37] for an overview of this problem from various calculational points of view. As the field equations involve both gauge fields as well as fermions, we have to be very careful about the field-redefinition freedom. If the action at higher order in some coupling constant $g$ contains terms proportional to the equations of motion obtained from the lowest-order action, as in

$$S[\phi] = S_0[\phi] + g \int \! dx \frac{\delta S_0[\phi]}{\delta \phi(x)} R(\phi(x), \partial \phi(x)),$$

these can be removed by a field redefinition:

$$\phi \rightarrow \phi - g R(\phi, \partial \phi) \quad \Rightarrow \quad S[\phi] \rightarrow S_0[\phi] + O(g^2).$$

Similarly, higher-order terms in the variation of the action under some transformation $\delta_{\lambda} \phi$ which are proportional to the lowest-order equations of motion,

$$\delta_{\lambda} S[\phi] = V_0[\phi] + g \int \! dx \frac{\delta S_0[\phi]}{\delta \phi(x)} V_1(\phi(x), \partial \phi(x); \lambda),$$

can be removed by a modification of the field transformation rules:

$$\delta_{\lambda} \phi \rightarrow \delta_{\lambda} \phi - g V_1(\phi, \partial \phi; \lambda) \quad \Rightarrow \quad \delta_{\lambda} S[\phi] \rightarrow V_0[\phi] + O(g^2).$$

In addition to these two mechanisms, we will also encounter terms that are zero in a specific gauge. In particular, terms proportional to the spin-1/2 part of the gravitino or terms involving the de Donder gauge condition of the graviton are identically zero in string theory. Such terms can, however, not be removed by a procedure of the type sketched above, and they have to be kept for the supersymmetric completion.

We should finally comment on the field redefinition freedom involving rescaling by powers of the dilaton (see also appendix A). In (2.3) we wrote transformation rules which do not involve the dilaton on the right-hand side. The existence of a particular frame in which the transformations take such a simple form is related to the scale invariance of the heterotic (see the work of Kallosh [38] and Kallosh and Nilsson [39]) and type II supergravity theories. In this frame—the string frame—the classical supersymmetry transformations only involve derivatives of the dilaton (as was observed by Bergshoeff and de Roo [13]). Since the string genus expansion is ordered by powers of exponentials of the dilaton, classical supersymmetry thus does not mix string loop orders. Moreover, the higher-order modifications to the supersymmetry transformations that we derive in this paper do not contain the dilaton, and thus also respect the genus expansion.

### 2.2 Tensor structures from string amplitudes

In order to get a covariant result, which is by far the most convenient starting point for the supersymmetry analysis, we employ the covariant formalism to compute the string amplitudes.
Moreover, the tensor structures arise here in a very simple way from the operator products of world-sheet fermions, whereas a manifestly space-time supersymmetric approach would require complicated fermion zero-mode integrals to be performed. We will comment further on such alternative approaches in the conclusions.

There are few detailed accounts of string amplitudes with external fermions in the literature (for exceptions see Green and Schwarz [24, 25] for a light-cone approach and Atick and Sen [40] and Pasquinucci and Petrini [41, 42] for covariant calculations). Let us therefore first review some of the formalism and present the required technical details. For other texts on this subject the reader is referred to the book by Lüst and Theisen [43] and the review by D’Hoker and Phong [44]. Some of the subtleties have, however, only been discussed in the papers cited below.

In order to find the higher-derivative action we analyse string amplitudes with two external fermions. The relevant vertex operators are given by

\[
\begin{align*}
V^{(0)}_A(k) &= \int d^2z A_\mu :\left(i\partial X^\mu + \frac{\sqrt{2}}{2} k \cdot \Psi \Psi^\mu\right) e^{ik \cdot X} : , \\
V^{(-1/2)}_\chi(k) &= \int d^2z \bar{\chi} :S_L^+ e^{-\phi/2} e^{ik \cdot X} :, \\
V^{(-1/2)}_\psi(k) &= \int d^2z \bar{\psi} :S_L^+ e^{-\phi/2} \bar{\partial}X^\nu e^{ik \cdot X} :, \\
V^{(-1)}_B(k) &= \int d^2z B_{\mu \nu} : e^{-\phi} \Psi^\mu \bar{\partial}X^\nu e^{ik \cdot X} :,
\end{align*}
\]

(2.10)

for the abelian gauge field, the abelian gaugino, the gravitino and two-form gauge potential, respectively (as we will not determine precise normalisations of our amplitudes, we have ignored any overall factors in the vertex operators listed above). The right-moving sector carrying the abelian gauge degree of freedom has been omitted. The world-sheet bosons are denoted by \(X^\mu\), while \(\Psi^\mu\) are the world-sheet fermions and \(\phi\) is a bosonic field representing the ghosts [45]. The spin field \(S_L^+\) has positive space-time chirality and arises purely from world-sheet fermions of one chirality, associated to the left-moving sector after imposing the equations of motion.

All of the vertex operators above are taken in their canonical ghost picture, as deduced by linearisation of the string action. The ghost charges are balanced by inserting a sufficient number of copies of the picture-changing operator

\[
Y(w) := T_F e^\phi(w) = i \sqrt{\frac{2}{\alpha'}} \partial X^\mu \Psi_\mu e^\phi(w)
\]

(2.11)

at arbitrary points in the correlators (in the odd spin-structure sector, one always gets at least one power of the picture-changing operator from integration over the odd supermoduli). The end result can be shown to be independent of these arbitrary locations (see Friedan et al. [45], Verlinde and Verlinde [46, 47] and in particular also section 3.2 of Pasquinucci and Roland [48] for an explicit example). One may be tempted to use the vertex operators in different pictures by taking the insertion points of \(Y(w)\) to coincide with those of the vertex operators. As was shown by Green and Seiberg [49], this requires very careful treatment of terms in the vertex operators that are proportional to the world-sheet equations of motion. These extra pieces have been interpreted as vertex operators for the \(N = 1\) auxiliary fields of the gauge multiplets [50]. This phenomenon, specific to \(N = 1\) superstring models (the heterotic and
the open string), does not occur for type II superstrings. Since the origin of the tensorial structures for the type II invariants are the same up to modifications we will explain later, we adopt a more symmetric procedure, similar to the computations of Atick and Sen [40].

In order to compute the operator product expansions, we need the following building blocks:

\[ \Psi_\mu(z)\Psi_\nu(w) = \frac{\eta_{\mu\nu}}{z-w} + \text{finite}, \]  
\[ X^\mu(z, \bar{z})X^\nu(w, \bar{w}) = -\frac{1}{2} \alpha' \eta^{\mu\nu} \ln |z-w| + \text{finite}, \]  
\[ \partial X^\mu(z, \bar{z})\partial X^\nu(w, \bar{w}) = \frac{\pi \alpha'}{2} \eta^{\mu\nu} \delta^2(z-w) + \text{finite}, \]  
\[ X^\mu(z)e^{ik\cdot X} = -\frac{1}{2} i \alpha' k^\mu e^{ik\cdot X} \ln(z) + \text{finite}. \]

For the ghost we have

\[ e^{q_1\phi(z)}e^{q_2\phi(0)} = \frac{1}{z^{q_1+q_2}}e^{(q_1+q_2)\phi(z)} + \text{finite}. \]

The operator products of the spin operators are slightly more complicated, as we need some finite parts as well. One finds

\[ S^+(z, \bar{z}) \otimes S^+(0) = 2^{-4} \sum_{n=1,3,\ldots} \frac{1}{n!}(P^+ C\Gamma_{\mu_1\cdots\mu_n} P^+)^\dagger :\Psi_{\mu_1} \cdots \Psi_{\mu_n} : z^{-5/4+n/2}, \]
\[ S^+(z, \bar{z}) \otimes S^-(0) = 2^{-4} \sum_{n=0,2,4,\ldots} \frac{1}{n!}(P^+ C\Gamma_{\mu_1\cdots\mu_n} P^-)^\dagger :\Psi_{\mu_1} \cdots \Psi_{\mu_n} : z^{-5/4+n/2}, \]
\[ S^\pm(z, \bar{z})\Psi^\mu(0) = (C\Gamma^\mu)S^\mp z^{-1/2} - i(C\Gamma^\mu) :S^\mp \Psi^\mu : z^{1/2} + \text{finite}. \]

The objects \( P^\pm \) are projectors onto positive and negative space-time chirality. These identities hold for both left- and right-handed world-sheet sectors. We refer to appendix B for further details about our conventions.

There are two strong constraints that have to be satisfied for a particular vertex operator product to give a non-vanishing expectation value: the fermionic \( \Psi \) zero-mode integrals (when present) have to be saturated and the total ghost number has to cancel the ghost charge of the vacuum (or in other words the ghost zero modes have to be saturated as well). When integrating over the world-sheet fermions \( \Psi \) there are ten fermionic zero-mode integrals for odd spin structure. This implies that in this sector the integral picks out the part of the vertex operator product that depends on ten \( \Psi \)s. The result is an \( \varepsilon_{10} \) tensorial structure. For the even spin structure there are no restrictions on the number of world-sheet fermions. An integral over eight \( \Psi \)s with anti-periodic boundary conditions can be rewritten (through the usual bosonisation and refermionisation procedure) in terms of an integral over a space-time fermion \( \theta \) with periodic boundary conditions. The resulting tensorial structure corresponds to the \( t_8 \) tensor. For smaller numbers of fermions, other tensorial structures will appear. Concerning the ghost number cancellation, recall that the Riemann-Roch theorem yields \( 2g - 2 \) for the difference between the number of \( \beta \) zero modes minus the number of \( \gamma \) zero modes. For our one-loop considerations, we therefore have to consider vertex operator products with total ghost-charge zero. We refer the reader to section 12.6 of Polchinski [51] and to Verlinde and Verlinde [47] for more details.
even spin structure:

\[
\begin{align*}
V_A^{(0)} & \quad V_A^{(0)} & \quad V_A^{(0)} & \quad V_A^{(0)} \\
:k\Psi\Psi: & \quad :k\Psi\Psi: & \quad :k\Psi\Psi: & \quad :k\Psi\Psi: \\
\end{align*}
\]

\[
\begin{align*}
T_F & \quad V_A^{(-1/2)} & \quad V_A^{(-1/2)} & \quad V_A^{(0)} \\
\partial X\Psi & \quad :k\Psi\Psi: & \quad :k\Psi\Psi: & \quad :k\Psi\Psi: \\
\end{align*}
\]

\[
\begin{align*}
T_F & \quad V_A^{(-1/2)} & \quad V_B^{(-1/2)} & \quad V_A^{(0)} & \quad V_A^{(0)} \\
\partial X\Psi & \quad S^+ & \quad \Gamma^{[1]} S^+ \phi^2 & \quad :k\Psi\Psi: & \quad :k\Psi\Psi: \\
\partial X\nu & \quad :k\Psi\Psi: & \quad :k\Psi\Psi: & \quad :k\Psi\Psi: \\
\end{align*}
\]

odd spin structure:

\[
\begin{align*}
V_\chi^{(-1/2)} & \quad V_\chi^{(-1/2)} & \quad T_F & \quad V_A^{(0)} & \quad V_A^{(0)} \\
S^+ & \quad \Gamma^{[1]} \phi^3 & \quad :k\Psi\Psi: & \quad :k\Psi\Psi: \\
\partial X\Psi & \quad :k\Psi\Psi: & \quad :k\Psi\Psi: & \quad :k\Psi\Psi: \\
\partial X\nu & \quad :k\Psi\Psi: & \quad :k\Psi\Psi: & \quad :k\Psi\Psi: \\
\end{align*}
\]

\[
\begin{align*}
T_F & \quad V_\phi^{(-1/2)} & \quad V_\phi^{(-1/2)} & \quad V_A^{(0)} & \quad V_A^{(0)} \\
\partial X\Psi & \quad S^+ & \quad \Gamma^{[1]} S^+ \phi^2 & \quad :k\Psi\Psi: & \quad :k\Psi\Psi: \\
\partial X\nu & \quad :k\Psi\Psi: & \quad :k\Psi\Psi: & \quad :k\Psi\Psi: \\
\end{align*}
\]

\[
\begin{align*}
T_F & \quad V_\nu^{(-1/2)} & \quad V_\nu^{(-1/2)} & \quad V_A^{(0)} & \quad V_A^{(0)} \\
\partial X\Psi & \quad S^+ & \quad \Gamma^{[1]} \phi^3 & \quad :k\Psi\Psi: & \quad :k\Psi\Psi: \\
\partial X\nu & \quad :k\Psi\Psi: & \quad :k\Psi\Psi: & \quad :k\Psi\Psi: \\
\end{align*}
\]

\[
\begin{align*}
T_F & \quad V_\nu^{(-1/2)} & \quad V_\nu^{(-1/2)} & \quad V_A^{(0)} & \quad V_A^{(0)} \\
\partial X\Psi & \quad S^+ & \quad \Gamma^{[1]} S^+ \phi^2 & \quad :k\Psi\Psi: & \quad :k\Psi\Psi: \\
\partial X\nu & \quad :k\Psi\Psi: & \quad :k\Psi\Psi: & \quad :k\Psi\Psi: \\
\end{align*}
\]

\[
\begin{align*}
T_F & \quad V_\nu^{(-1/2)} & \quad V_\nu^{(-1/2)} & \quad V_A^{(0)} & \quad V_A^{(0)} \\
\partial X\Psi & \quad S^+ & \quad \Gamma^{[1]} \phi^3 & \quad :k\Psi\Psi: & \quad :k\Psi\Psi: \\
\partial X\nu & \quad :k\Psi\Psi: & \quad :k\Psi\Psi: & \quad :k\Psi\Psi: \\
\end{align*}
\]

Table 1: Vertex operator products for the heterotic string leading to the supersymmetric completion of the $F^4$ effective action. The top row shows the vertex operators, while the second and third row exhibit the relevant fields in the left- and right-moving sector respectively. The given products are those relevant for one-loop amplitudes and Abelian gauge group; the first column shows the relevant terms from each vertex operator in the product, while the second column shows which terms in the effective action are generated by the amplitude on the left. The spin structure from which the various terms originate has been indicated. The gauge part and the ghost factors, being trivial, have been omitted. Plane-wave exponentials are only displayed whenever they are non-trivially contracted. The precise index structure of the terms in the last column is exhibited in (2.22).
The first corrections to the action (2.1) come in at order \((\alpha')^2\) relative to the classical action. As the \(F^4\) term arises from a four-point function in string theory, one may expect that the fermion bilinears which are related to it by linear supersymmetry can also be obtained by considering four-point functions only. This turns out to be true when considering only global supersymmetry invariance, but local supersymmetry forces us to consider both four- and five-point functions (see in this context also the work by Grisaru et al. [52, 53] which discusses a related but not quite identical issue). Table 1 lists all the various operator products that occur. Many of them are straightforward to obtain so we will here just comment on the more subtle ones.

In the even spin-structure sector, there is a term which receives contributions from only six world-sheet fermions. Only the charge conjugation matrix is kept in the product of the spin field with a world-sheet fermion, and using the first term of the third identity in (2.17) one obtains

\[
S^+ \otimes S^+ \Psi^\mu \Psi^{\nu_1} \Psi^{\nu_2} \Psi^{\nu_3} \Psi^{\nu_4} \rightarrow \Gamma^\nu \times \eta^{\nu_1} \eta^{\nu_2} \eta^{\nu_3} \eta^{\nu_4}.
\]  

(2.18)

The \(\partial X^\mu\) from the picture-changing operator is contracted with the plane-wave factors of the fermion vertex operators. Since we are only interested in this leading contribution the remaining correlation function between the plane-wave factors is approximated to one

\[
\left\langle \prod_{i=1}^{4} e^{i(k_i) \cdot X} \right\rangle \simeq 1 + \mathcal{O}(k^2).
\]

(2.19)

This contribution has the correct (anti-)symmetry in the labels of the external (fermions) bosons.

Another tricky term in the even spin-structure sector is the one on the second line of table 1. Here one has to keep, for each of the operator products of a spin field with a world-sheet fermion, the term \((\bar{C} \Gamma^m) \Psi^m\). The result is

\[
S^+ \otimes S^+ \Psi^\mu \Psi^{\nu_1} \Psi^{\nu_2} \Psi^{\nu_3} \Psi^{\nu_4} \rightarrow \Gamma^\nu \Gamma^\rho \Gamma^\lambda \times t^8_{\nu \rho \lambda \nu_1 \nu_2 \nu_3 \nu_4}.
\]

(2.20)

The fermions \(\Psi\) have been contracted in the usual way [51] to obtain the \(t_8\)-structure, while the plane-wave part has been treated as before. This contribution to the amplitude is already symmetric in the labels of the external gauge bosons, and the anti-symmetry in the labels of the external gaugini forbids the appearance of a \(\Gamma^\nu_{\rho \lambda}\) term. The final form of this contribution is therefore

\[
t^{(r)}_8 (\bar{\chi} [\Gamma_{r_1 r_2} \Gamma_{r_3} - \Gamma_{r_3} \Gamma_{r_1 r_2}] D_{r_4} \chi) \ F_{r_4 r_5} \ F_{r_7 r_8}.
\]

(2.21)

The other products are straightforward. In the next section we will use supersymmetry to determine the precise coefficients of these terms.

### 2.3 Supersymmetry of the higher-derivative effective action

The analysis in the previous section has provided us with all the distinct tensorial structures of the fermion bilinears appearing in the supersymmetric completion of the \(F^4\) action. Since we did not compute the full amplitudes, we have not obtained the normalisations of these terms. However, the normalisations can easily be fixed using supersymmetry. As we will show
in detail in this section, the correct combination of terms is given by the following action:

\[
S_{F^4} = \frac{(\alpha')^2}{32} \int d^{10} x \left[ \frac{1}{6} e t_8^{(r)} F_{r_1 r_2} \cdots F_{r_7 r_8} + \frac{1}{12} \varepsilon_{10}^{(r)} B_{r_1 r_2} F_{r_3 r_4} \cdots F_{r_9 r_{10}} 
\right.
\]

\[
- \frac{32}{5} e t_8^{(r)} \eta_{r_2 r_3} (\bar{\chi} \Gamma_{r_1} \chi_{r_4}) F_{r_5 r_6} F_{r_7 r_8} 
\]

\[
+ \frac{1232}{5} e (\bar{\chi} \Gamma_{r_1} \chi_{r_2}) F_{r_1 m} F_{m r_2} 
\]

\[
- \frac{16}{5 \varepsilon_{10}} (\bar{\chi} \Gamma_{r_1 \cdots r_7} \chi_{r_8}) F_{r_7 r_8} F_{r_9 r_{10}} 
\]

\[
+ \frac{16}{3} e t_8^{(r)} (\bar{\psi}_{r_1} \Gamma_{r_2} \chi) F_{r_3 r_4} F_{r_5 r_6} F_{r_7 r_8} 
\]

\[
+ \frac{8}{3} e (\bar{\psi}_m \Gamma^{m r_1 \cdots r_6} \chi) F_{r_1 r_2} \cdots F_{r_5 r_6}. 
\]

Here \( t_8^{(r)} := t_8^{r_1 \cdots r_8} \) and \( \varepsilon_{10}^{(r)} := \varepsilon^{r_1 \cdots r_{10}} \). The pure gauge-field terms in this action taken together with \((2.1)\) agree with the expansion of the Born-Infeld action to the corresponding order (and we used this correspondence to fix the overall normalisation of \((2.22)\)). Our result is minimal in the sense that we have not included any terms proportional to the lowest-order equations of motion. Indeed, this is perhaps the most natural thing to do, given our on-shell string results. Anyhow, such terms can always be eliminated by field redefinitions.

The action above displays several surprising features. The most striking one is perhaps the explicit dependence on the spacetime dimension arising from the contraction of \( t_8^{(r)} \) indices in the term on the second line (the tensor \( \eta_{r_2 r_3} \) in this term originates from the gamma matrix product in \((2.21)\)). The dimension dependence makes it clear already at this point that some of the stringy \( t_8^{(r)} \) structure will be lost when going to eleven dimensions; we will address this point in more detail later. Secondly, one observes that there is a term which does not have the \( t_8^{(r)} \) at all, due to the fact that it only involves six world-sheet fermion zero modes.

In order to compare the result \((2.22)\) to equation \((C.9)\) of Suelmann [16], one has use the gaugino equation of motion \((2.5)\) wherever possible in order to produce additional gravitino terms. In this way, one arrives at the following Lagrangian (from here on we will write just \( D \) instead of \( D(\omega) \)):

\[
(\alpha')^{-2} \mathcal{L}_{\Gamma^0} = - \frac{1}{32} e ((F^2)^2 - 4F^4) 
\]

\[
+ \frac{1}{384} \varepsilon_{12}^{r_1 r_2} B_{t_1 t_2} F_{r_1 r_2} \cdots F_{r_7 r_8} , 
\]

\[
(\alpha')^{-2} \mathcal{L}_{\Gamma^1} = - 4 e (\bar{\chi} \Gamma^{r_1} D^{r_2} \chi) F_{r_1 r_2}^2 
\]

\[
+ \frac{1}{4} e (\bar{\psi}^{r_1} \Gamma^{r_2} \chi) F_{r_1 r_2} F^2 
\]

\[
- 3 e (\bar{\psi}^{r_1} \Gamma^{r_2} \chi) F_{r_1 r_2}^3 , 
\]

\[
(\alpha')^{-2} \mathcal{L}_{\Gamma^3} = + 2 e (\bar{\chi} \Gamma^{r_1 r_2} D^{r_3} \chi) F_{r_1 r_2} F_{r_3 r_4} 
\]

\[
- \frac{1}{8} e (\bar{\psi}_m \Gamma^{m r_1 r_2} \chi) F_{r_1 r_2} F^2 
\]

\[
+ \frac{1}{2} e (\bar{\psi}_m \Gamma^{m r_1 r_2} \chi) F_{r_1 r_2}^3 
\]

\[
+ e (\bar{\psi}_4 \Gamma^{r_1 r_2 r_3} \chi) F_{r_1 r_2} F_{r_3 r_4}^2 , 
\]

\[
(\alpha')^{-2} \mathcal{L}_{\Gamma^5} = + \frac{1}{8} e (\bar{\psi}_6 \Gamma^{r_1 \cdots r_5} \chi) F_{r_1 r_2} F_{r_3 r_4} F_{r_5 r_6} , 
\]

\[
(\alpha')^{-2} \mathcal{L}_{\Gamma^7} = + \frac{1}{48} e (\bar{\psi}_m \Gamma^{m r_1 \cdots r_6} \chi) F_{r_1 r_2} F_{r_3 r_4} F_{r_5 r_6} . 
\]
The terms $\mathcal{L}_{\Gamma[1]}^2$ and $\mathcal{L}_{\Gamma[1]}^3$ receive contributions from both the four- and five-point functions; the coefficients arise as $\frac{1}{4} = -\frac{3}{4} + 1$ and $-3 = -1 - 4$ respectively.

We should comment on the fact that we dropped the terms proportional to the equations of motion when going from (2.22) to (2.23). The underlying idea is that string theory cannot give us any information about such terms, so from this perspective the two actions are equally good. When we discuss supersymmetry of the action, and the required modifications to the transformation rules, we will always start from (2.23). The transformation rules of the fields which appear in (2.22) pick up additional modifications due to the fact that they are related to the fields in (2.23) by a field redefinition.

Let us now exhibit supersymmetry in detail. For pedagogical purposes, we write the variation of the above action in a form which makes it easy to read off the global supersymmetry transformation rules, we will always start from (2.23). The transformation rules of the fields from both the four- and five-point functions; the coefficients arise as $\frac{1}{4} = -\frac{3}{4} + 1$ and $-3 = -1 - 4$ respectively.

We should stress that in the above variation, it never happened that we got a dimension-dependent factor (which could in principle occur when contracting gamma matrices). This
The fact will be very important later on in the supergravity generalisation.

The $G$ terms in (2.24a) are zero in global susy by using the gaugino equation of motion; note that their sum can be recast in $t_8$ form as

$$G_1 + G_2 = -\frac{i}{48} e^{(r)} (\bar{\chi} \Gamma^m \Gamma_{rrs} \epsilon) D_m (F_{r_1 r_2} \cdots F_{r_3 r_6}).$$

The $Z_i$ terms are zero identically in global susy. The above shows that the action without gravitini is invariant under global susy.

For local susy, we are now left with the following terms in the variation of terms coming from the four-point functions:

$$\delta L_{\text{four-point}} = + \frac{1}{2} e (\bar{\epsilon} \Gamma^{r_1 r_2 r_3} \chi) D_{r_1} (F^3_{r_2 r_3}) \quad X_6$$

$$+ e (\bar{\epsilon} \Gamma^{r_3} \chi) D^{r_2} F^3_{r_3 r_2}, \quad X_2$$

$$- \frac{1}{8} e (\bar{\epsilon} \Gamma^{r_1 r_2 r_3} \chi) D_{r_1} (F_{r_2 r_3} F^2) \quad X_7$$

$$- \frac{1}{4} e (\bar{\epsilon} \Gamma^{r_1} \chi) D^{r_2} (F_{r_1 r_2} F^2), \quad X_1$$

$$+ e (\bar{\epsilon} \Gamma^{r_3 r_4} D^{r_2} \chi) F_{r_3 r_4} F^2_{r_1 r_2} \quad X_{11}$$

$$+ e (\bar{\epsilon} \Gamma^{r_3 r_4} \chi) D^{r_2} (F_{r_3 r_4} F_{r_1 r_2}^{r_2}) \quad X_8$$

$$+ 2 e (\bar{\epsilon} \Gamma^{r_4} D^{r_2} \chi) F_{r_4 r_2}^{r_3} \quad X_{12}$$

$$+ 2 e (\bar{\epsilon} \Gamma^{r_4} \chi) D^{r_2} (F_{r_4 r_2}^{r_3}) \quad X_2$$

$$+ \frac{1}{8} e (\bar{\epsilon} \Gamma^{r_1 \cdots r_5} \chi) D^{r_6} (F_{r_1 r_2} \cdots F_{r_5 r_6}). \quad X_5$$

The four different blocks come from $G_1$, $G_2$, $Z_1$ and $Z_2$ respectively. These terms will have to cancel against the variation of the terms derived from a five-point calculation in string theory, namely the terms proportional to the gravitino. For those five-point terms, we integrate away from the supersymmetry parameter and use the identity (2.5) whenever possible. The upshot is:

$$(\alpha')^{-2} \delta L^2_{\Gamma^{|i|}} = \frac{1}{4} e (\bar{\epsilon} \Gamma^{r_1} D^{r_2} \chi) F_{r_1 r_2}^{r} F^2 \quad Y_1$$

$$+ \frac{1}{4} e (\bar{\epsilon} \Gamma^{r_1} \chi) D^{r_2} F_{r_1 r_2}^{r} F^2 \quad X_1$$

$$+ \frac{1}{32} e (\bar{\psi}^{r_1} \Gamma^{r_2 r_3 r_4} \epsilon) F_{r_1 r_2}^{r} F_{r_3 r_4} F^2 \quad X_{12}$$

$$+ \frac{1}{16} e (\bar{\psi}^{r_1} \Gamma^{r_2} \epsilon) F_{r_1 r_2}^{r} F^2 \quad (2.24c)$$

$$(\alpha')^{-2} \delta L^3_{\Gamma^{|i|}} = - 3 e (\bar{\epsilon} \Gamma^{r_1} D^{r_2} \chi) F_{r_1 r_2}^{r} F^3 \quad X_{12}$$

$$- 3 e (\bar{\epsilon} \Gamma^{r_1} \chi) D^{r_2} F_{r_1 r_2}^{r} F^3 \quad X_2$$

$$- \frac{3}{8} e (\bar{\psi}^{r_1} \Gamma^{r_2 r_3 r_4} \epsilon) F_{r_1 r_2}^{r} F_{r_3 r_4} \quad X_{12}$$

$$- \frac{3}{4} e (\bar{\psi}^{r_1} \Gamma^{r_2} \epsilon) F_{r_1 r_2}^{r} F^4 \quad X_{12}$$

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\[(\alpha')^{-2} \delta L^2_{\Gamma[3]} = + \frac{1}{64} e (\bar{\epsilon} \Gamma^r \bar{r}_2 \Gamma^{r_3} r_4 \psi_m) F_{r_1 r_2} F_{r_3 r_4} F^2 - \frac{1}{64} e (\epsilon \leftrightarrow \psi_m) Y_1 \]
\[\quad + \frac{1}{32} e (\bar{\epsilon} \Gamma^r \bar{r}_2 \mathcal{E}(\bar{\chi})) F_{r_1 r_2} F^2 \]
\[\quad + \frac{1}{32} e (\bar{\psi} r_2 \Gamma^r \bar{r}_1 \Gamma^{r_3} r_4 \epsilon) F_{r_1 r_2} F_{r_3 r_4} F^2 \]
\[\quad - \frac{i}{4} e (\bar{\epsilon} \Gamma^r \bar{r}_2 \chi) F_{r_1 r_2} F^2 \]
\[\quad + \frac{1}{8} e (\bar{\epsilon} \Gamma^{m \bar{r}_1} r_2 \chi) D_m (F_{r_1 r_2} F^2) X_7 \]

\[(\alpha')^{-2} \delta L^3_{\Gamma[3]} = - \frac{1}{16} e (\bar{\epsilon} \Gamma^r \bar{r}_2 \Gamma^{r_3} r_4 \psi_m) F_{r_3 r_4} F_{r_1 r_2} F^3 + \frac{1}{16} e (\epsilon \leftrightarrow \psi_m) X_12 \]
\[\quad - \frac{1}{32} e (\bar{\epsilon} \Gamma^{r_3} r_2 \mathcal{E}(\bar{\chi})) F_{r_1 r_2} F^3 \]
\[\quad - \frac{i}{8} e (\bar{\psi} r_2 \Gamma^r \bar{r}_1 \Gamma^{r_4} r_3 \epsilon) F_{r_1 r_2} F_{r_3 r_4} \]
\[\quad + e (\bar{\epsilon} \Gamma^r \bar{r}_2 \chi) F_{r_1 r_2} F_{r_3 r_4} F^3 \]
\[\quad - \frac{i}{8} e (\bar{\epsilon} \Gamma^{r_2} r_3 \chi) D_r F_{r_2 r_3} F_{r_1 r_2} F_{r_3 r_4} X_6 \]
\[\quad + \frac{i}{8} e (\bar{\psi} r_2 \Gamma^r \bar{r}_1 \Gamma^{r_3} r_4 \epsilon) F_{r_1 r_2} F_{r_3 r_4} F_{r_3 r_6} X_8 \]

\[(\alpha')^{-2} \delta L^4_{\Gamma[3]} = - e (\bar{\epsilon} \Gamma^r \bar{r}_2 \Gamma^{r_3} r_4 \chi) F_{r_1 r_2} F_{r_1 r_4} F^2 \]
\[\quad - e (\bar{\epsilon} \Gamma^{r_3} r_2 \chi) F_{r_1 r_2} F^4 (F_{r_1 r_2} F_{r_3 r_4} F_{r_1 r_5} F_{r_1 r_6} F_{r_1 r_7}) Y_11 \]
\[\quad - \frac{i}{8} e (\bar{\psi} r_2 \Gamma^r \bar{r}_1 \Gamma^{r_6} \epsilon) F_{r_1 r_2} F_{r_1 r_5} F_{r_3 r_6} Y_3 \]
\[\quad - \frac{i}{8} e (\bar{\psi} r_2 \Gamma^r \bar{r}_1 \Gamma^{r_5} \epsilon) D^r (F_{r_1 r_2} F_{r_1 r_3} F_{r_4 r_5}) X_5 \]
\[\quad - \frac{i}{64} e (\bar{\psi} m \Gamma^r \bar{r}_1 \Gamma^{r_8} \epsilon) F_{mr_1} F_{r_2 r_3} F_{r_4 r_5} F_{r_5 r_7} Y_4 \]

\[(\alpha')^{-2} \delta L^7_{\Gamma[7]} = - \frac{1}{384} e (\bar{\epsilon} \Gamma^r \bar{r}_2 \Gamma^{r_3} \Gamma^{r_7} \epsilon) F_{r_1 r_2} \cdots F_{r_7 r_8} + \frac{1}{384} e (\epsilon \leftrightarrow \psi_m) \]
\[\quad - \frac{1}{768} e (\bar{\epsilon} \Gamma^r \bar{r}_2 \mathcal{E}(\bar{\chi})) F_{r_1 r_2} \cdots F_{r_5 r_6} \]
\[\quad + \frac{1}{64} e (\bar{\psi} m \Gamma^r \bar{r}_1 \Gamma^{r_8} \epsilon) F_{mr_1} F_{r_2 r_3} \cdots F_{r_6 r_7} \]
\[\quad + \frac{1}{8} e (\bar{\epsilon} \Gamma^{r_7} r_8 \chi) F_{r_1 r_2} \cdots F_{r_5 r_6} Y_3 \]

The last step consists of showing that the $\bar{\psi} \epsilon$ terms cancel, which makes contact with the remaining unused transformation rule, namely the one of the vielbein. Expanding all the gamma-matrix products in the expression above, we obtain in this final step the variations

\[(\alpha')^{-2} \delta L^1_{\Gamma[0]} \overset{\delta \epsilon}{\rightarrow} - \frac{1}{16} e (\bar{\epsilon} \Gamma^m \psi_m)((F^2)^2 - 4 F^4) Y_{12} \]
\[\quad + \frac{1}{2} e (\bar{\epsilon} \Gamma^r \psi_{r_2}) F^2_{r_1 r_2} F^2 \]
\[\quad - 2 e (\bar{\psi} \Gamma^r \psi_{r_2}) F^4_{r_1 r_2} Y_{11} \]
\[\quad + \frac{1}{192} e (\bar{\epsilon} \Gamma^{r_7} \psi_{r_9}) F_{r_1 r_2} \cdots F_{r_7 r_8} B \]

\[(\alpha')^{-2} \delta L^2_{\Gamma[0]} \rightarrow + \frac{1}{32} e (\bar{\psi} r_1 \Gamma^{r_7} \epsilon) F_{r_1 r_2} F_{r_3 r_4} F^2 Y_5 \]
\[\quad + \frac{1}{16} e (\bar{\psi} r_1 \Gamma^r \epsilon) F_{r_1 r_2} F^2 \]

\[(\alpha')^{-2} \delta L^3_{\Gamma[1]} \rightarrow - \frac{3}{8} e (\bar{\psi} r_1 \Gamma^{r_7} \epsilon) F^3_{r_1 r_2} F_{r_3 r_4} Y_6 \]
\[\quad + \frac{3}{4} e (\bar{\psi} r_1 \Gamma^r \epsilon) F^4_{r_1 r_2} Y_{11} \]

(2.24d)
\[ (\alpha')^{-2} \delta L^2_{\Gamma^{[3]}_3} \rightarrow + \frac{1}{32} \epsilon (\epsilon \Gamma_{r_1 \cdots r_5} \psi_{r_5}) F_{r_1 r_2} F_{r_3 r_4} F^2 \]

\[ + \frac{1}{32} \epsilon (\epsilon \Gamma_{r_1 r_2 r_3} \psi^{r_4}) F_{r_1 r_2} F_{r_3 r_4} F^2 \]

\[ - \frac{3}{16} \epsilon (\epsilon \Gamma_{r_1} \psi_{r_2}) F^2_{r_1 r_2} F^2 \]

\[ + \frac{1}{16} \epsilon (\epsilon \Gamma^m \psi_m)(F^2)^2 \]

\[ (\alpha')^{-2} \delta L^3_{\Gamma^{[3]}_3} \rightarrow - \frac{1}{8} \epsilon (\epsilon \Gamma_{r_1 \cdots r_5} \psi_{r_5}) F_{r_1 r_2} F_{r_3 r_4} F^3 \]

\[ - \frac{1}{8} \epsilon (\epsilon \Gamma_{r_1 r_2 r_3} \psi^{r_4}) F_{r_1 r_2} F_{r_3 r_4} F^3 \]

\[ + \frac{3}{4} \epsilon (\epsilon \Gamma_{r_1} \psi_{r_2}) F^4_{r_1 r_2} \]

\[ - \frac{1}{4} \epsilon (\epsilon \Gamma^m \psi_m) F^4 \]

\[ (\alpha')^{-2} \delta L^4_{\Gamma^{[3]}_3} \rightarrow - \frac{1}{8} \epsilon (\epsilon \Gamma_{r_1 \cdots r_5} \psi^{r_6}) F_{r_1 r_2} F_{r_3 r_4} F^2 \]

\[ - \frac{1}{4} \epsilon (\epsilon \Gamma_{r_1 r_2 r_3} \psi^{r_4}) F_{r_1 r_2} F_{r_3 r_4} F^3 \]

\[ + \frac{1}{2} \epsilon (\epsilon \Gamma_{r_1} \psi_{r_2}) F^4_{r_1 r_2} \]

\[ - \frac{1}{4} \epsilon (\epsilon \Gamma^m \psi_m) F^2 \]

\[ (\alpha')^{-2} \delta L^7_{\Gamma^{[7]}_7} \rightarrow - \frac{1}{102} \epsilon (\epsilon \Gamma_{r_1 \cdots r_9} \psi_{r_9}) F_{r_1 r_2} \cdots F_{r_7 r_8} \]

\[ - \frac{1}{32} \epsilon (\epsilon \Gamma_{r_1 \cdots r_9} \psi_{r_5}) F_{r_1 r_2} F_{r_3 r_4} F^2 \]

\[ + \frac{1}{8} \epsilon (\epsilon \Gamma_{r_1 \cdots r_9} \psi_{r_5}) F_{r_1 r_2} F_{r_3 r_4} F^3 \]

\[ + \frac{1}{8} \epsilon (\epsilon \Gamma_{r_1 \cdots r_9} \psi^{r_6}) F_{r_1 r_2} F_{r_3 r_4} F^2 \]

All terms cancel which proves supersymmetry of the action (2.22).

If we had not started with the input from string theory, but had tried to obtain the fermionic terms directly using supersymmetry, things would have been much more difficult. While it is definitely possible to find the higher-order $\Gamma$ terms in (2.23), the lower-order terms are hard to guess due to the fact that there is no $t^{(r)}_{8}$ tensor present. On the other hand, obtaining the unexpanded form of the action (2.22) is also complicated, as it relies on the subtle mixing between terms with four and terms with five powers of the fields. Altogether, we have found the string input crucial to understand the supersymmetric completion, even though supersymmetry has played an essential role in finding the ‘non-standard’ five-point contact terms.

### 2.4 Modifications to the transformation rules

We have so far not discussed the terms in the variation that are proportional to the equations of motion. As was explained in section 2.1, these terms can all be absorbed by modifications to the field transformation rules at order $(\alpha')^2$. For the globally supersymmetric action, one obtains from (2.24a) that the new transformation rules are

\[ \delta A_{\mu} = -4 \epsilon \Gamma^\mu \chi - (\alpha')^2 \left[ \epsilon (\epsilon \Gamma^\mu \chi) F^2 - (\epsilon \Gamma^m \chi) F^2_{m\mu} - \frac{1}{8} (\epsilon \Gamma_{r_1 \cdots r_4} \mu \chi) F_{r_1 r_2} F_{r_3 r_4} \right], \]

\[ \delta \chi = \frac{1}{8} \Gamma^{\mu \nu} \epsilon F_{\mu \nu} + \frac{1}{108} (\alpha')^2 \left[ (t^{(r)}_{8} \Gamma_{r_7 r_8} \epsilon - \Gamma^{r_1 \cdots r_6}_6 \epsilon) F_{r_1 r_2} F_{r_3 r_4} F_{r_5 r_6} \right]. \]
Using these modified transformation rules one can now verify that the modified gaugino equation of motion is again supercovariant. This equation can be read off from the action (2.23) in a straightforward way and one indeed finds that

$$\left(\delta^{(a')2} + \delta^{(a')2}\right) \frac{\delta(S_{F^2} + S_{F^4})}{\delta \chi} = \text{independent of } D\epsilon.$$  

(2.26)

This constitutes an independent check of our results and is required for the equations of motion to sit in a supermultiplet.

We can now compute the commutator of two supersymmetry transformations (though only on the gauge field; the commutator on the gaugino requires knowledge about higher-order fermi terms in the modified transformation rules). It turns out that the structure coefficients of the algebra are unmodified as compared to the lowest-order ones:

$$\left[\delta^{(a')0} + \delta^{(a')2}, \delta^{(a')0} + \delta^{(a')2}\right] A_\mu = \left[\delta^{(a')0}, \delta^{(a')0}\right] A_\mu + O\left((a')^4\right).$$  

(2.27)

Note that all of the modifications in (2.25) are needed to make the $(a')^2$ contributions cancel. The above result was obtained previously by Metsaev and Rakhmanov [10] and generalized to the non-Abelian $t_8 \text{tr}(F^4)$ action by Bergshoeff et al. [11] with the same conclusions. When we extend their result to include the coupling to the gravitino background, (2.24a) no longer produces equation-of-motion terms for the gaugino (because one no longer performs a partial integration of the $G_1 + G_2$ terms). However, one finds that the same modifications to the gaugino transformation rules are now obtained from the equation-of-motion terms in (2.24c) instead. In order to determine the algebra, one would of course need to have information also about the gravity sector which have not addressed so far.

Before we proceed to the analysis of the supergravity invariants, let us note that the fact that the structure coefficients of the supersymmetry algebra are unchanged for super-Maxwell is compatible with the observation that standard superspace (i.e. superspace for which the canonical dimension-zero torsion constraint $T_{abr} = 2 (\Gamma^r)_{ab}$ is imposed) is sufficient to describe the super-Born-Infeld action. This is most explicit in the construction of the $N = 1$ action in four dimensions, which can be done in a power-series expansion in the standard vector superfield; see in particular section 3 of Bagger and Galperin [54]. Of course, an analogous formulation in terms of an unconstrained superfield does not exist in ten dimensions, but our results nevertheless show that the superspace geometry does not have to be changed in this case either.

Let us now turn to the supergravity case.

3 Second step: completion of the $W^4$ action

3.1 From super-Maxwell to supergravity

In the previous section we derived the compact form (2.22) of the leading higher-derivative string correction to the super-Maxwell action in ten dimensions, including the fermion bilinears required for supersymmetry and the coupling to the supergravity background. As should be clear from the lengthy supersymmetry analysis, the information we used from string scattering amplitudes was crucial to enable us to organise the terms in a systematic way. Such additional string input is also helpful in finding the supersymmetric completion of the higher-derivative corrections to the supergravity action, although for this case we fortunately do not have to start
the calculations from scratch. In fact, with the string-inspired higher-derivative super-Maxwell action (2.22) at hand, it turns out that no additional string analysis is actually needed to arrive at the supersymmetric completion of the invariant $I_X$ due to the existence of a very close formal similarity between the super-Maxwell supersymmetry structure and its supergravity analogue. As a consequence, most results derived in the previous section can be mapped in a straightforward way to the gravity case, with only a few easy-to-handle exceptions; the details of the procedure are given below. Once the construction of the supergravity invariant for the $N = 1$ case is completed, we will comment on extensions to the type II theories and then show explicitly how to lift the analysis to eleven dimensions.

The main ingredient in the construction of the supergravity invariant for the $N = 1$ model in ten dimensions is the strong parallel between the super-Maxwell field transformation rules (2.2) and the on-shell, lowest-order supergravity ones obtained from (A.6) and (A.28), which we redisplay here for the convenience of the reader in a form that highlights the similarities:

$$\delta \chi = \frac{1}{8} \Gamma^{\mu\nu} \epsilon F_{\mu\nu}, \quad \delta \psi_{rs} = \frac{1}{8} \Gamma^{\mu\nu} \epsilon R_{\mu\nu rs} + \cdots,$$

$$\delta F_{\mu\nu} = -8 D_{[\mu} (\bar{\epsilon} \Gamma_{\nu]} \chi), \quad \delta R_{\mu\nu rs} = -4 D_{[\mu} (\bar{\epsilon} \Gamma_{\nu]} \psi^{rs} - 2 \bar{\epsilon} \Gamma^{[r} \psi^{s]}_{\nu]}) + \cdots. \quad (3.1)$$

(In addition, the transformation laws for the vielbein, the gravitino and the two-form potential as given in (2.3) apply in both cases.) Crucial for the correspondence is also the fact that the super-Maxwell identity (2.5) has a direct analogue in (A.35) on the supergravity side:

$$\bar{D} \chi = \frac{1}{8} \Gamma^m \Gamma^{\mu\nu} \psi_m F_{\mu\nu} + \cdots, \quad \bar{D} \psi_{rs} = \frac{1}{8} \Gamma^m \Gamma^{\mu\nu} \psi_m R_{\mu\nu rs} + \cdots. \quad (3.2)$$

In both (3.1) and (3.2) the dots indicate terms of higher order in the fermionic fields and/or terms proportional to the lowest-order equations of motion. For later use, it is important to observe that the supergravity results quoted above are dimension independent, essentially because of the fact that we did not include the dilaton or any gauge-field dependent terms. More information can be found in appendix A.

In the transition from the Maxwell case to gravity, one is led by the above equations to make the tentative substitutions

$$F_{r_1 r_2} \rightarrow R_{r_1 r_2 s_1 s_2}, \quad \chi \rightarrow \psi_{s_1 s_2}, \quad D_r \chi \rightarrow D_r \psi_{s_1 s_2}, \quad (3.3)$$

in the action (2.22), while inserting at the same time an additional $\frac{1}{8}^{(s)}$ tensor to saturate the extra vector indices introduced in the process. The origin of this ‘trick’—which indeed turns out to be most useful in spite of the imperfect match between the transformation rules for the gauge-field strength and the Riemann tensor—becomes evident when one compares the string amplitudes involving the super-Maxwell multiplet with those involving the supergravity fields, as we do in the next subsection.

However, let us first explain in more detail how the differences between the super-Maxwell and the supergravity cases arise from a supersymmetry perspective. One source for these differences is the mismatch between the gauge-potential and the spin-connection transformation rules, carrying over to the transformation rules for the corresponding field strengths (in
other words, the map \( F_{r_1r_2} \rightarrow R_{r_1r_2s_1s_2} \) does not commute with supersymmetry). Rewriting, trivially, the Riemann tensor transformation rule in (3.1) as

\[
\delta R_{r_1r_2}^{s_1s_2} = - 8 D_{[r_1}(\bar{\epsilon} \Gamma_{r_2]} \psi^{s_1s_2})
\]

\[
+ 4 D_{[r_1}(\bar{\epsilon} \Gamma_{r_2]} \psi^{s_1s_2} + 2 \bar{\epsilon} \Gamma^{[s_1} \psi^{s_2]}_{r_2]} r_2]) + \ldots,
\]

the first line is what the transformation rule would have looked like, had the naive matching with the super-Maxwell case been sufficient. Putting it differently, given the mapping (3.3) and the accompanying introduction of \( t^{(s)}_8 \) applied to the higher-derivative super-Maxwell action (2.22), this part of the transformation acting on the \( R^4 \) term thus generated will conspire with the variations of all the remaining terms in the ‘naive’ action to produce a vanishing result. We are then faced with the puzzle that there is nothing left to cancel the contribution from the second line of (3.4) to the variation of the \( R^4 \) term, which after partial integration can be written as:

\[
6 \cdot 32 (\alpha')^{-3} \Delta := 48 e t^{(r)}_8 t^{(s)}_8 (\bar{\epsilon} \Gamma_{s_7} \psi_{s_7s_8} - 2 \bar{\epsilon} \Gamma_{s_s} \psi_{s_7s_7}) (D_{r_8} R_{r_1r_2s_1s_2}) R_{r_3r_4s_3s_4} R_{r_5r_6s_5s_6}.
\]

A key observation that points towards the resolution of this puzzle, is that the second term above exhibits non-trivial mixing between the left-moving \( r \)-indices and the right-moving \( s \)-indices. This fact, in turn, leads us to examine the effects of such mixing on the naive gravitino bilinears; indeed, using the cyclic identity (A.34) on the gravitino curvature, one can show that

\[
\delta \left[ \alpha_1 e t^{(r)}_8 t^{(s)}_8 (\bar{\psi}_{r_7} \Gamma_{r_8} \psi_{s_7s_8}) R_{r_1r_2s_1s_2} \cdots R_{r_5r_6s_5s_6} + 2 \alpha_2 (r_8 \leftrightarrow s_7) \right]
\]

\[
= + \frac{1}{8} (\alpha_1 + \alpha_2) e t^{(r)}_8 t^{(s)}_8 (\bar{\psi}_{r_7} \Gamma_{r_8}^{mn}) \bar{e} R_{r_1r_2s_1s_2} \cdots R_{r_5r_6s_5s_6}
\]

\[
+ \frac{1}{8} (\alpha_1 + \alpha_2) e t^{(r)}_8 t^{(s)}_8 (\bar{\psi}_{r_7} \Gamma_{r_8}^{mn}) \bar{e} R_{r_1r_2s_1s_2} \cdots R_{r_5r_6s_5s_6}
\]

\[
+ (\alpha_1 + \alpha_2) e t^{(r)}_8 t^{(s)}_8 (\bar{\psi}_{r_7} \Gamma_{r_8}^{mn}) \bar{e} R_{r_1r_2s_1s_2} \cdots R_{r_5r_6s_5s_6}
\]

\[
+ 3 \alpha_1 e t^{(r)}_8 t^{(s)}_8 (\bar{\psi}_{r_7} \Gamma_{r_8}^{mn}) \bar{e} R_{r_1r_2s_1s_2} \cdots R_{r_5r_6s_5s_6}
\]

\[
- 6 \alpha_2 e t^{(r)}_8 t^{(s)}_8 (\bar{\psi}_{r_7} \Gamma_{r_8}^{mn}) \bar{e} R_{r_1r_2s_1s_2} \cdots R_{r_5r_6s_5s_6}.
\]

Comparison with (3.5) immediately shows that adding to the naive action terms of the above kind with relative weight such that \( \alpha_1 + \alpha_2 = 0 \) produces a variation of precisely the kind that we are looking to cancel. In particular, choosing the coefficients of these additional terms as

\[
\alpha_1 = - \frac{1}{32} (\alpha')^3 \frac{8}{3} \text{ and } 2 \alpha_2 = \frac{1}{32} (\alpha')^3 \frac{16}{3},
\]

respectively, leads to the supersymmetric completion of the \( R^4 \) invariant.

However, before we are ready to down write down the result, we need to address the second point on which the super-Maxwell and the supergravity cases differ. Since it is more natural from a string theory perspective, we would like to present the action in a form in which all terms proportional to the lowest-order equations of motion have been subtracted. This amounts to making the substitution

\[
R_{mn}^{pq} \rightarrow W_{mn}^{pq} - \frac{16}{d - 2} \delta_{[m}^{[p} (\bar{\psi}_{r_7} \Gamma_{r_8}^{[r]} \psi_{s_7s_8]q} - \bar{\psi}_{r_7}^{[r} \Gamma_{s_7s_8]} \psi_{[r]}^{s_7s_8])
\]

wherever the Riemann tensor occurs in the action. Here we again made use of the results (A.31) and (A.32) of appendix A. From (3.7) it immediately follows that the extraction of equation-of-motion terms is, with one exception, achieved by simply replacing the Riemann tensor by
the Weyl tensor. The sole exception is the $R^4$ term, from which a new term bilinear in the gravitino appears (strictly speaking one would also get a new gravitino bilinear from the $BR^4$ term, but that one does not contribute to any of the variations we considered, so it falls outside the scope of our analysis). This term, given on the second line of the action below, lacks a partner on the super-Maxwell side. Let us also mention that when subtracting the equation-of-motion terms we implicitly used the fact that the supersymmetry variation of these terms is again proportional to the equations of motion (see section 2.3 of Bergshoeff et al. [55] for more on this issue).

Finally, we are in a position to write down the supersymmetric completion of the $N = 1$ invariant $I_X$ in ten dimensions:

$$S_X = \frac{(\alpha')^3}{32} \int d^{10}x \left[ \frac{1}{6} \epsilon^t t_8^{(r)} t_8^{(s)} W_{r_1 r_2 s_1 s_2} \cdots W_{r_7 r_8 s_7 s_8} + \frac{1}{12} \epsilon^t t_8^{(r)} B_{r_9 r_{10}} W_{r_1 r_2 s_1 s_2} \cdots W_{r_7 r_8 s_7 s_8} ight]$$

Notice the term on the penultimate line, which introduces left-right mixing in the action. Note also that the coefficient of the term on the preceding line is different from the naive super-Maxwell value.

An alternative, perhaps more elegant way to resolve the discrepancies between the super-Maxwell transformation rules and those of the supergravity multiplet can be found in the literature (see e.g. Cai and Nunez [17], Gross and Sloan [18], Bellucci and Gates [31] and Bergshoeff and de Roo [13]), although it is strictly bound to ten dimensions (which is why we did not adopt it above). It consists of using a spin connection with $H$-torsion, which is defined in the string frame as

$$\Omega_{\mu r s} = \omega_{\mu r s} + \hat{H}_{\mu r s} = \omega_{\mu r s} + H_{\mu r s} - 3 \bar{\psi}^{\mu} \Gamma_{r s} \psi_s .$$

While one usually considers this shift to incorporate the effects of the field strength $H$ in the action, the fermionic terms are also important. The above combination, in contrast to the spin connection itself, transforms in a nice way under supersymmetry:

$$\delta \Omega_{\mu r s} = -4 \epsilon \Gamma_{\mu} \bar{\psi}^{r s} .$$

It is easy to see that by using $W(\Omega)$ instead of $W(\omega)$ in the bosonic part of the action, one generates upon expansion precisely the fermionic terms with left-right mixed indices (the term linear in the field strength $H$ drops out) and one obtains the naive generalisation of the super-Maxwell action. The above argument does, unfortunately, not have an obvious generalisation that applies in arbitrary dimensions.
In analogy with the discussion of the super-Maxwell action, let us expand the $t_8^{(s)}$ on the left-moving side to arrive at the analogue of (2.23) (all indices have been lowered for esthetical reasons):

$$(\alpha')^{-3} L_{[0]} = -\frac{1}{32} e t_8^{(s)} ((W^2)^2 - 4W^4)$$

$$+ \frac{1}{384} e t_8^{(s)} B_{11} t_2 W_{r_1 r_2 s_1 s_2} \cdots W_{r_7 r_8 s_7 s_8}.$$ 

$$(\alpha')^{-3} L_{[1]} = -4e t_8^{(s)} (\bar{\psi}_{s_1 s_2} \Gamma_{r_1} D_{r_2} \psi_{s_3 s_4}) W_{r_1 r_3 s_5 s_6} W_{r_3 r_2 s_7 s_8}$$

$$- \frac{1}{4} e t_8^{(s)} (\bar{\psi}_{r_1} \Gamma_{r_2} \psi_{s_7 s_8}) W_{r_1 r_2 s_1 s_2} W_{m s s_3 s_4} W_{m s s_5 s_6}$$

$$- e t_8^{(s)} (\bar{\psi}_{r_1} \Gamma_{s_7} \psi_{s_8}) W_{r_1 m s_1 s_2} W_{m s s_3 s_4} W_{n r_2 s_5 s_6}$$

$$+ e t_8^{(s)} (\bar{\psi}_{s_1} \Gamma_{s_7} \psi_{s_8}) W_{r_1 r_2 s_1 s_2} W_{m s s_3 s_4} W_{m s s_5 s_6}$$

$$- 4e t_8^{(s)} (\bar{\psi}_{r_1} \Gamma_{s_7} \psi_{s_8}) W_{r_1 m s_1 s_2} W_{m s s_3 s_4} W_{n r_2 s_5 s_6}$$

$$+ \frac{1}{4} e t_8^{(s)} (\bar{\psi}_{m} \Gamma_{n} \psi_{m s_8}) W_{p q s_1 s_2} W_{q p s_3 s_4} W_{n s_7 s_5 s_6} (\frac{d-2}{2})$$

$$- e t_8^{(s)} (\bar{\psi}_{m} \Gamma_{n} \psi_{m s_8}) W_{n p s_1 s_2} W_{p q s_3 s_4} W_{q s_7 s_5 s_6} (\frac{-8}{d-2})$$

$$+ 2e t_8^{(s)} (\bar{\psi}_{s_5 s_6} \Gamma_{r_1 r_2 r_3} D_{r_4} \psi_{s_7 s_8}) W_{r_1 r_2 s_1 s_2} W_{r_3 r_4 s_3 s_4}$$

$$- \frac{1}{8} e t_8^{(s)} (\bar{\psi}_{m} \Gamma_{m r_1 r_2} \psi_{s_7 s_8}) W_{r_1 r_2 s_1 s_2} W_{p n s_3 s_4} W_{n p s_5 s_6}$$

$$+ \frac{1}{8} e t_8^{(s)} (\bar{\psi}_{m} \Gamma_{m r_1 r_2} \psi_{s_7 s_8}) W_{r_1 p s_1 s_2} W_{p n s_3 s_4} W_{n r_2 s_5 s_6}$$

$$+ e t_8^{(s)} (\bar{\psi}_{m} \Gamma_{r_1 r_2 r_3} \psi_{s_7 s_8}) W_{r_1 r_2 s_1 s_2} W_{m n s_3 s_4} W_{n r_3 s_5 s_6}.$$ 

$$(\alpha')^{-3} L_{[5]} = + \frac{1}{8} e t_8^{(s)} (\bar{\psi}_{r_6} \Gamma_{r_1 \cdots r_5} \psi_{s_7 s_8}) W_{r_1 r_2 s_1 s_2} W_{r_3 r_4 s_3 s_4} W_{r_5 r_6 s_5 s_6};$$

$$(\alpha')^{-3} L_{[7]} = + \frac{1}{48} e t_8^{(s)} (\bar{\psi}_{m} \Gamma_{r_1 \cdots r_6} \psi_{s_7 s_8}) W_{r_1 r_2 s_1 s_2} W_{r_3 r_4 s_3 s_4} W_{r_5 r_s s_6 s_5 s_6}.$$ 

Written in this form, all direct dependence of the action on the space-time dimension can be made explicit, as we indeed have done in the rightmost column. This makes it suitable for lifting to eleven dimensions, an issue we will return to in section 3.3. Next we will, however, discuss the string-theory origin of the terms in (3.8).

### 3.2 String-amplitude calculations and extensions to type II theories

We have intentionally not started with a string analysis for the gravity case, as there are several additional trickly elements there as compared to the super-Maxwell situation. Nevertheless, the action (3.8) can be understood also from a string point of view, as we will explain in the present section. In addition, this allows us to make a few comments about possible generalisations of the superinvariant to the type IIA and IIB theories, although we are at this moment not able to give a detailed discussion of those cases. Readers who are more interested in the lifting procedure to eleven dimensions can skip this section and continue with section 3.3.

Let us first discuss the vertex operators for the gravity case. We will focus on the type IIA/IIB theories as those are slightly more complicated, but the heterotic results can be extracted without too much difficulty. Again, because the explicit forms of vertex operators...
in various pictures are scattered in the literature, we list here the vertex operators which which we need. In the NS⊗NS sector we have the graviton, dilaton and two-form operators, combined into

\[ V_g^{(0,0)}(k) = \frac{1}{2} \int d^2 \zeta \zeta_{\mu \nu} (i\partial X^\mu + \frac{\alpha'}{2} k \cdot \Psi \tilde{\Psi}^\mu) \times (i\partial X^\nu + \frac{\alpha'}{2} k \cdot \tilde{\Psi} \Psi^\nu) e^{ik \cdot X}, \]

\[ V_g^{(-1,-1)}(k) = \frac{1}{2} \int d^2 \zeta \zeta_{\mu \nu} \Psi^\mu \tilde{\Psi}^\nu e^{-\phi} e^{ik \cdot X}. \]

The \( \zeta_{\mu \nu} \) denotes the polarisation, while \( X^\mu \) and \( \Psi^\mu \) are the usual world-sheet bosons and fermions. In the R⊗NS sector we have the operators for the gravitino,

\[ V_\psi^{(-1/2,0)}(k) = \frac{1}{2} \int d^2 \bar{\psi}_\nu :S e^{-\phi/2} (i\partial X^\mu + \frac{\alpha'}{2} k \cdot \Psi \tilde{\Psi}^\nu) e^{ik \cdot X}, \]

\[ V_\psi^{(1/2,0)}(k) = \frac{1}{2} \int d^2 \bar{\psi}_\nu \Gamma_\mu :S e^{\phi/2} (i\partial X^\mu + \frac{\alpha'}{2} k \cdot \Psi \Psi^\mu) (i\partial X^\nu + \frac{\alpha'}{2} k \cdot \tilde{\Psi} \tilde{\Psi}^\nu) e^{ik \cdot X}. \]

As with the super-Maxwell vertex operators (2.10), we have ignored any overall normalisation factors. In order for these vertex operators to have the right conformal dimension to be primary fields, the polarisation tensors have to satisfy the conditions \( \Gamma^\mu \psi_\mu = 0 \) as well as \( k^\mu \zeta_{\mu \nu} = 0 \).

Using these operators, one arrives at table 2. Observe that there is now one more way to produce a five-point function in the odd/odd spin-structure sector. As shown in table 2, in the odd/odd spin structure one graviton-vertex operator has to be taken in the \((-1, -1)\) ghost picture, say the fifth state, while the other four are in the \((0, 0)\) ghost picture. The resulting tensorial structure is

\[ \xi^{\mu \nu \lambda \rho \sigma}_{10} \times \eta_{\mu \nu} \times \zeta^{(5)}_{\kappa \lambda} \times \prod_{i=1}^{4} \epsilon^{(i)}_{s_1 s_2 s_3 s_4} \zeta^{(i)}_{r_1 r_2 s_1 s_2}.
\]

Here, the \( \eta_{\mu \nu} \) tensor arises from the contraction of the \( \partial X^\mu \) with \( \bar{\partial} X^\nu \) from the supercurrent. Furthermore, \( \zeta^{(5)} \) is the polarisation of the fifth graviton and is seen as the fluctuation of the metric around the flat background \( g_{\kappa \lambda} \sim \eta_{\kappa \lambda} + \kappa_{10} \zeta_{\kappa \lambda} \). Therefore, (3.16) is the linearised version of \( \xi^{\mu \nu \lambda \rho \sigma}_{10} \times \eta_{10} \), \( g_{\mu \nu} \), \( \kappa_{10} R^4 \). We should remark that the double epsilon term does not seem to appear in the heterotic theory, as the relevant operator product does not exist. An alternative way to check the absence of this term is to compute the relative normalisation of the \( ts R^4 \) and \( \xi_{10} t_{s} B R^4 \) terms along the lines of Lerche et al. [23, 56], and verify that it corresponds to the combination appearing in the \( I_X \) invariant.

### 3.3 Lifting to eleven dimensions

After the string interlude of the previous section, let us get back to the field theory construction of supersymmetric invariants and discuss how to lift the results obtained in section 3.1 to eleven dimensions. As we shall see, this is essentially straightforward at the level of the purely gravitational terms (i.e. including just the graviton and gravitino) and the anomaly cancellation term.

The supersymmetry transformation rules which we have used so far are shared between the ten- and eleven-dimensional theories (differences only come in when the gauge-field and dilaton parts of the theory are considered). Therefore, the supersymmetry analysis performed in section 3.1 remains valid provided one takes proper care of explicit dimension dependence
even/even spin structure:

\[
\begin{array}{c}
V_{g}^{(0,0)} \\
: \cdot kΨ: : \cdot kΨ: : \cdot kΨ: : \cdot kΨ: : \cdot kΨ: \\
\end{array}
\]

\[
\begin{array}{c}
T_{F} \\
V_{g}^{(0,0)} \quad V_{g}^{(-1/2,0)} \\
\cdot kΨ: : \cdot kΨ: : \cdot kΨ: : \cdot kΨ: : \cdot kΨ: \\
\end{array}
\]

\[
\begin{array}{c}
\partial XΨ \\
: \cdot kΨ: : \cdot kΨ: : \cdot kΨ: : \cdot kΨ: : \cdot kΨ: \\
\end{array}
\]

\[
\begin{array}{c}
S^{+} \quad \Gamma_{[1]} S^{+} \phi^{2} \\
\cdot kΨ: : \cdot kΨ: : \cdot kΨ: : \cdot kΨ: : \cdot kΨ: \\
\end{array}
\]

\[
\begin{array}{c}
\partial X \\
: \cdot kΨ: : \cdot kΨ: : \cdot kΨ: : \cdot kΨ: : \cdot kΨ: \\
\end{array}
\]

odd/even spin structure:

\[
\begin{array}{c}
V_{g}^{(-1/2,0)} \\
: \cdot kΨ: : \cdot kΨ: : \cdot kΨ: : \cdot kΨ: : \cdot kΨ: \\
\end{array}
\]

\[
\begin{array}{c}
T_{F} \\
V_{g}^{(0,0)} \quad V_{g}^{(0,0)} \quad V_{g}^{(0,0)} \\
: \cdot kΨ: : \cdot kΨ: : \cdot kΨ: : \cdot kΨ: : \cdot kΨ: \\
\end{array}
\]

\[
\begin{array}{c}
\partial XΨ \\
: \cdot kΨ: : \cdot kΨ: : \cdot kΨ: : \cdot kΨ: : \cdot kΨ: \\
\end{array}
\]

\[
\begin{array}{c}
S^{+} \quad \Gamma_{[1]} S^{+} \phi^{2} \\
: \cdot kΨ: : \cdot kΨ: : \cdot kΨ: : \cdot kΨ: : \cdot kΨ: \\
\end{array}
\]

\[
\begin{array}{c}
\partial X \\
: \cdot kΨ: : \cdot kΨ: : \cdot kΨ: : \cdot kΨ: : \cdot kΨ: \\
\end{array}
\]

odd/odd spin structure:

\[
\begin{array}{c}
T_{F} \\
V_{g}^{(-1,-1)} \quad V_{g}^{(0,0)} \quad V_{g}^{(0,0)} \\
: \cdot kΨ: : \cdot kΨ: : \cdot kΨ: : \cdot kΨ: : \cdot kΨ: \\
\end{array}
\]

\[
\begin{array}{c}
\partial XΨ \\
: \cdot kΨ: : \cdot kΨ: : \cdot kΨ: : \cdot kΨ: : \cdot kΨ: \\
\end{array}
\]

\[
\begin{array}{c}
\partial X \\
: \cdot kΨ: : \cdot kΨ: : \cdot kΨ: : \cdot kΨ: : \cdot kΨ: \\
\end{array}
\]

Table 2: The vertex operator products for the IIA string leading to the supersymmetric completion of the \(t_{g}sW^{4}\) effective action. In the heterotic model, the \(t_{g}\) tensors on the right-moving side arise from \((\partial X e^{ikX})^{4}\) instead of \((kΨΦ)_{g}^{4}\), while the odd/odd term is absent.
arising from contracted Kronecker deltas. As we have already remarked below (2.24a), one never encounters such explicit dimension factors in the variation starting from the super-Maxwell $F^4$ action (2.22) (there are, for instance, no contracted delta symbols arising from the expansion of gamma-matrix products).

A first attempt at constructing the eleven-dimensional version of the invariant could be to try to lift the compact form (3.8). However, it is easy to see that such a naive approach is bound to fail. The reason is that there is a term in (3.8) with index contractions within a single $t_8$ tensor, hiding an explicit dependence on the spacetime dimension (see (B.12)). To be precise, the third line of (3.8) contains the tensor $\eta_{r_3r_3}$, stemming from a product $[\Gamma^{[2]},\Gamma^{[1]}]$ in the string amplitude analysis (cf. (2.21)). When verifying supersymmetry invariance (which, as we have seen in section 2.3, was done on the expanded action (3.11)), these trace terms come out with the right coefficients only in ten dimensions. Moreover, it is not difficult to see that it is impossible to write down a generalised $t_8$ tensor in eleven dimensions with the same symmetry properties as (B.12) that reproduces these coefficients.

Having made this observation, the remedy suggests itself immediately: instead of trying to preserve the full $t_8$ structure of the ten-dimensional action, we should lift the expanded action (3.11); this action contains left-over $t_8$ tensors of the right-moving sector, but does not involve any Kronecker delta contractions. The only dimension dependence of this action is, as we have already alluded to, related to the subtraction of the equation-of-motion terms (cf. the discussion below (3.7)). For these terms, however, the dimension dependence cancels in the variation, as should be clear from their origin. As a result, the lifting procedure is essentially trivial in the gravitational sector, including the replacement of the two-form in ten dimensions with the three-form gauge field in eleven dimensions.

The net result reads (again with all indices lowered)

$$(\alpha'_M)^{-3}L_{[0]} = + \frac{1}{192} e t_8^{(r_8)} W_{r_1r_2s_1s_2} W_{r_3r_4s_3s_4} W_{r_5r_6s_5s_6} W_{r_7r_8s_7s_8}
+ \frac{1}{(48)^2} e t_8^{(t_8)} B_{t_1t_2t_3} W_{r_1r_2s_1s_2} \cdots W_{r_7r_8s_7s_8},$$

$$(\alpha'_M)^{-3}L_{[1]} = - 4 e t_8^{(s)} (\bar{\psi}_{s_1s_2} \Gamma_{r_1} D_{r_2} \psi_{s_3s_4}) W_{r_1s_5s_6} W_{r_2s_7s_8}
- \frac{1}{4} e t_8^{(s)} (\bar{\psi}_{r_1s_2} \Gamma_{s_3s_4}) W_{r_1r_2s_1s_2} W_{s_5s_6} W_{s_7s_8},$$

$$= - e t_8^{(s)} (\bar{\psi}_{r_1s_2} \Gamma_{s_3s_4}) W_{r_1s_5s_6} W_{s_7s_8},$$

$$+ e t_8^{(s)} (\bar{\psi}_{r_1s_2} \Gamma_{s_3s_4}) W_{r_1s_5s_6} W_{s_7s_8},$$

$$= -4 e t_8^{(s)} (\bar{\psi}_{r_1s_2} \Gamma_{s_3s_4}) W_{r_1s_5s_6} W_{s_7s_8},$$

$$+ \frac{2}{5} e t_8^{(s)} (\bar{\psi}_{m}s_5s_6) W_{pqs_1s_2} W_{ps_3s_4} W_{nqrs_7s_8},$$

$$- \frac{8}{9} e t_8^{(s)} (\bar{\psi}_{m}s_5s_6) W_{npqrs_7s_8};$$

26
when corrections at higher order in $\alpha' M$ to the transformations on the left-hand side are taken into account, the parameters on the right-hand side can receive field-dependent modifications according to their dependence on the fields involved.

**3.4 Superspace constraints from modified supersymmetry rules**

In contrast to the higher-derivative actions, which as we have seen share many structural features between the super-Maxwell and the supergravity models, the modifications to the transformation rules are rather different in the two cases. This is due to the fact that the formal analogy that led us to the substitution trick described in section 3.1 does not extend to the equations of motion, as is evident from a comparison of (2.4) and (A.29). For this reason, the conclusion that the supersymmetry algebra does not receive any modifications when the super-Maxwell $(\alpha')^2$ corrections are taken into account (see section 2.4) cannot simply be taken over to the supergravity case. Instead, these modifications have to be recomputed from scratch. However, before we do so in the next section, let us explain which of the modifications are actually of interest from a superspace point of view.

In order to identify the relevant modifications to the transformation rules, we need to examine the structure of the supersymmetry algebra. Specifically, for the Cremmer-Julia-Scherk action, the commutator of two supersymmetry transformations has the schematic structure (the $S$ tensor is defined in (A.26))

\[
[\delta_{\epsilon_1}, \delta_{\epsilon_2}] = \delta^{\text{translation}}(2 \bar{\epsilon}_2 \Gamma^\nu \epsilon_1) + \delta^{\text{susy}}(-2 \bar{\epsilon}_2 \Gamma^\nu \epsilon_1 \psi_\nu) + \delta^{\text{gauge}}(-4 \bar{\epsilon}_2 \Gamma^\sigma \epsilon_1 B_{\sigma\nu\rho} - 2 \bar{\epsilon}_2 \Gamma_{\nu\rho} \epsilon_1) + \delta^{\text{Lorentz}}(2 \bar{\epsilon}_2 \Gamma^\nu \epsilon_1 \hat{\omega}_\nu r^i + 4 \bar{\epsilon}_2 S^r_{\nu\rho\kappa} \epsilon_1 \hat{H}_{\nu\rho\kappa}) \tag{3.18}
\]

When corrections at higher order in $\alpha' M$ to the transformations on the left-hand side are taken into account, the parameters on the right-hand side can receive field-dependent modifications...
at the corresponding order. In particular, we will focus on the corrections to the translation parameter for reasons that will become clear below.

Since our analysis allows us to compute the modifications to the transformation rules only to lowest order in the fermions, and since we have not considered variations proportional to the gauge field, we are only able to compute the algebra on the vielbein. For this case, the translation parameter can be identified by focusing on the part

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}] e_\mu^r = \partial_\mu \xi^r e_\nu^r + \cdots$$

(3.19)

of the $\alpha'_M$-corrected version of (3.18). The expansion of the left-hand side in powers of $\alpha'_M$ reads

$$\left( \frac{1}{2} \left[ \delta^\alpha_1 (\alpha'_M)^0, \delta^\alpha_2 (\alpha'_M)^0 \right] + \delta^\alpha_1 (\alpha'_M)^0 \delta^\alpha_2 (\alpha'_M)^3 - \delta^\alpha_2 (\alpha'_M)^3 \delta^\alpha_1 (\alpha'_M)^0 - (\epsilon_1 \leftrightarrow \epsilon_2) \right) e_\mu^r + \mathcal{O}\left((\alpha'_M)^0\right).$$

(3.20)

Inserting the lowest-order transformation rules, the corrections at order $(\alpha'_M)^3$ can thus be written as

$$\delta^\alpha_2 (\alpha'_M)^3 e_\mu^r \bigg|_{\psi_{\lambda} \rightarrow D_{\lambda} \epsilon_1} - 2\epsilon_1 \Gamma^r \delta^\alpha_2 (\alpha'_M)^3 \psi_{\mu} - (\epsilon_1 \leftrightarrow \epsilon_2).$$

(3.21)

All terms in this result which have the $\mu$-index sitting on either a gamma matrix or a Weyl tensor can be interpreted as modifications to the local Lorentz transformation parameter on the right-hand side of the supersymmetry algebra. Instead, we only have to look for terms which have the $\mu$-index sitting on a (covariant) derivative.\(^3\) Looking at (3.21), these terms can arise in three ways: from terms in $\delta^\alpha_2 (\alpha'_M)^3 e_\mu^r$ which have the $\mu$-index on the gravitino, from terms in this transformation rule which are proportional to $\psi(2)$ and have the $\mu$-index on the derivative, and finally from terms in $\delta^\alpha_2 (\alpha'_M)^3 \psi_{\mu}$ where the $\mu$-index sits on the derivative.

Restricting to these three classes of terms, the analysis already becomes much simpler. However, even among the terms that fall under one of these categories, not all correspond to non-trivial modifications of the translation parameter. We shall explain how to isolate those who do shortly, but first we need to make contact with superspace.

Starting from our component-field results, the link to superspace is made as follows (see also Peeters [57]). When embedding the component theory in superspace, one makes a gauge choice for the lowest components of the superfields. The most natural one is

$$E_{\mu}^r = e_{\mu}^r + \mathcal{O}(\theta), \quad E_{\nu}^a = \psi_{\nu}^a + \mathcal{O}(\theta), \quad E_{r}^a = \delta_{r}^a + \mathcal{O}(\theta), \quad E_{a}^r = \mathcal{O}(\theta),$$

(3.22)

implying

$$E_{\mu}^a = \delta_{\mu}^a, \quad E_{a}^\nu = \mathcal{O}(\theta), \quad E_{a}^r = \mathcal{O}(\theta).$$

The torsion is defined in terms of the supervielbeine and superconnections by

$$T_{AB}^C = (-)^{M(B+N)} E_{A}^{M} E_{B}^{N} \left( \mathcal{D}_{M} E_{N}^{C} - (-)^{MN} \mathcal{D}_{N} E_{M}^{C} \right),$$

(3.23)

where the local Lorentz covariant derivatives are defined as

$$\mathcal{D}_{M} E_{N}^{r} = \partial_{M} E_{N}^{r} + \Omega_{M}^{r}s E_{N}^{s}, \quad \mathcal{D}_{M} E_{N}^{a} = \partial_{M} E_{N}^{a} + \frac{1}{4}(-)^{bN} \Omega_{Mr}(\Gamma^{rs})^{a}_{b} E_{N}^{b}.$$  

(3.24)

\(^3\)Of course, there could in principle be subtleties involving the Ricci cyclic identity so one has to be careful to also check for terms containing $D_{W_{\mu}}$; however, it turns out that no such terms appear.
Imposing the gauge (3.22), the dimension-zero torsion component in (3.23) reduces at $\theta = 0$ to

$$T_{ab}^r \bigg|_{\text{gauge of (3.22)}} = -2 \delta_a^\alpha \delta_b^\beta \partial_{(\alpha} E_{\beta)}^r .$$

It now only remains to find out how the modified supersymmetry algebra determines the order-$\theta$ term of the vielbein component appearing in (3.25).

This component of the supervielbein can be found by using a procedure called gauge completion (used previously in e.g. Cremmer and Ferrara [58] and de Wit et al. [59] to determine the theta expansion of the superfields for the standard Cremmer-Julia-Scherk theory). This procedure relates the transformation rules of the component fields (on the left-hand side of the equation below) to the transformation rules of the superfields (on the right-hand side). Explicitly, one has

$$\delta_{\text{components}} E_M^A = \Xi^N \partial_N E_M^A + \partial_M \Xi^N E_N^A + \frac{1}{2} (-)^{BM} (\Lambda^r s X_{rs})^A_B E_M^B ;$$

where $\frac{1}{2} (\Lambda^r s X_{rs})^m_n = \lambda^m_n$ and $\frac{1}{2} (\Lambda^r s X_{rs})^\alpha_\beta = \frac{1}{2} \lambda^r s (T_{rs})^a_b$ while the spinor-vector components vanish. Writing out the summations in bosonic and fermionic parts, one first finds that the $\partial_\mu \epsilon$ pieces cancel, as expected. Denoting the remaining part of the gravitino component transformation rule by $\hat{\delta}$, one obtains the following characterisation of the order-$\theta$ components:

$$\begin{align*}
(\hat{\delta}) \psi^a_\mu &= E^a_\mu \bigg|_{\theta} , \\
(\hat{\delta}) e^r_\mu &= E^r_\mu \bigg|_{\theta} , \\
\Xi^\beta &\bigg|_{\theta} = -\Xi^\nu \bigg|_{\theta} \psi^\beta_\nu , \\
e^r_\nu \partial_\beta E^\nu_\alpha \bigg|_{\theta} &= -\partial_\alpha \Xi^\nu \bigg|_{\theta} e^r_\nu 
\end{align*}$$

(3.27)

(here $(\hat{\delta})$ in the first two lines denote the respective supersymmetry transformations with the supersymmetry parameter $e^a \alpha$ substituted by $\theta^a$). From the fourth line one observes that the problem of determining the dimension-zero constraint is now reduced to finding the superspace translation parameter $\Xi^\nu$. This parameter is defined in superspace by the relation

$$\Xi^\nu_3 \bigg|_{\theta=0} = e^a_2 \partial_\alpha \Xi^\nu_2 \bigg|_{\theta} - e^a_1 \partial_\alpha \Xi^\nu_1 \bigg|_{\theta} .$$

(3.28)

Whenever the left-hand side picks up $\alpha'_M$ corrections, the above equation implies that these corrections will also be visible in the first-order $\theta$ component of the transformation parameter $\Xi^\nu$. Hence, we find that knowledge about the supercharge commutator—represented by $\Xi^\nu_3$—is sufficient to fix the dimension-zero component $T_{ab}^r$ of the supertorsion. This component is of particular importance in the superspace formulation of eleven-dimensional supergravity, where the constraint imposed on it completely determines the dynamics of the theory via the superspace Bianchi identities; we shall have reason to come back to this point in the following.

Although our primary interest, for the reason just discussed, lies in determining the dimension-zero component $T_{ab}^r$, we should mention that our gauge-completion approach also allows us to find explicit manifestations of the interdependence of the various torsion components expressed by the superspace Bianchi identities. For instance, in the chosen gauge we obtain

$$T_{a_r}^s = \delta_a^\alpha e^r_\mu \partial_\alpha E^s_\mu \bigg|_{\theta} - \psi_r^b T_{a_b}^s .$$

(3.29)
The second equation in (3.27) then allows us to write

\[ T_{ar}^s = e_r^\mu (\delta e_\mu^s)_a - \psi_r^b T_{ab}^s, \quad (3.30) \]

where \( \tilde{\delta} \) is defined by \( \delta e_\mu^s = e^a (\tilde{\delta} e_\mu^s)_a \). For this dimension-1/2 component of the torsion we thus see that \( \alpha_M' \) corrections will result as a consequence of such corrections to the vielbein transformation rule as well as to the dimension-zero torsion constraint.

Before we turn to the issue of how the torsion constraints are affected by the inclusion of the higher-derivative action (3.17), we need to discuss how to identify corrections that cannot be set to zero by field redefinitions. For example, not all potential corrections to the dimension-zero torsion component \( T_{ab}^r \) are non-trivial, as explained in Cederwall et al. [34, 35] (see also references therein). In eleven dimensions, the most general expression for this constraint reads

\[ T_{ab}^r = 2 \left( (C \Gamma^r_{1})_{ab} X^r_{r_1} + \frac{1}{2!}(C \Gamma^r_{1r_2})_{ab} X^r_{r_1r_2} + \frac{1}{5!}(C \Gamma^r_{1...r_5})_{ab} X^r_{r_1...r_5} \right). \quad (3.31) \]

As was argued in [34, 35], the interesting physics is contained in the coefficients \( X^r_{r_1r_2} \) and \( X^r_{r_1...r_5} \). At this stage, the former is not relevant for us as it is necessarily zero within our gauge-field independent analysis. As far as the latter coefficient is concerned, the only SO(10,1)-irreducible component contained therein which cannot be redefined away is the ‘fish-hook’ one, \( \bigotimes \), of dimension 4290; the two other possibilities—the antisymmetric tensors \( \bigotimes \) and \( \bigotimes \)—can be made to vanish by a superfield redefinition, which in our language corresponds to an appropriate change of the gauge choice (3.22) for the supervielbein. Indeed, the symmetrised product of three Weyl-tensor representations \( 1144 \) (\( \bigotimes \)) does contain \( 4290 \), so \( (\alpha_M')^3 W^3 \) corrections could in principle appear.

The upshot of this section is that, in order to find non-trivial modifications to the superspace constraints in the algebra which involve a factor \( \epsilon_2 \Gamma^{[5]} \epsilon_1 \). All other superspace modifications are of lesser importance and can be ignored.

### 3.5 Absence of corrections to the supersymmetry algebra

Although our method will eventually produce the modifications to the super torsion constraints along the lines sketched in the previous section, we are unfortunately rather severely restricted by the fact that we have not yet included the gauge-field terms in the superinvariant. As we will show in this section, there are no non-trivial modifications to the supersymmetry algebra when one only considers the part of the superinvariant given by (3.17). We will comment on possible reasons for and implications of the absence of these corrections. However, let us begin by discussing in general terms which variations are responsible for the equation-of-motion terms, in particular those of the kind that would give rise to modifications of the supersymmetry algebra.

As far as the vielbein equation-of-motion terms are concerned, they can, broadly speaking, be generated in two different ways: either by the split of a Riemann tensor in Weyl-tensor plus Ricci parts, or from additional, explicit Ricci terms that appear in the \( \mathcal{D}\psi_{(2)} \) identity (A.35) and the \( \mathcal{D} \cdot \psi_{(2)} \) identity (A.36) as well as in the contracted Bianchi identity \( D_{[\mu \rho]r^s} e_r^\mu = 0 \) (the Ricci terms should in all cases be substituted by their on-shell expressions given in (A.31)–(A.32)). We shall return to the role of the contracted Bianchi identity in the variation of the action shortly and instead first discuss the \( \mathcal{D}\psi_{(2)} \) and \( \mathcal{D} \cdot \psi_{(2)} \) identities, which are also the only
sources for gravitino equation-of-motion terms. Upon insertion of (A.31)–(A.32) they read:

\[ D\psi_{\mu\nu} = \frac{1}{8} \Gamma^m \Gamma^{rs} W_{\mu\nu rs} \psi_m \]

\[ + \frac{1}{(d-1)(d-2)} \Gamma_{m\mu\nu} \psi^m \mathcal{E}(e)_\lambda \psi^\lambda - \frac{1}{d-2} \Gamma_{m[\mu} \psi^{m] \mathcal{E}(e)_{\nu]} \psi^\lambda \]

\[ + \frac{1}{d-2} \left( \Gamma_{[\mu} \psi^m + (d-3) \Gamma^m_{\mu[\nu} \right) \mathcal{E}(e)_{\nu] m} + \frac{d-3}{(d-1)(d-2)} \Gamma_{\mu[\nu} \mathcal{E}(e)_{\nu] \lambda} \]

\[ + D_{[\mu} \mathcal{E}(\bar{\psi})_{\nu]} + \mathcal{O}(H^2) + \mathcal{O}(\psi^3) , \]

\[ D'_{\nu} \psi_{\mu} = \frac{1}{8} \Gamma^{rs} \psi_{\nu} W_{\mu\nu rs} \]

\[ - \frac{1}{2(d-2)} \Gamma_{\mu\nu} \psi^r \mathcal{E}(e)_{r\nu} + \frac{d-3}{2(d-2)} \Gamma^{rs} \psi_r \mathcal{E}(e)_{\mu s} - \frac{1}{2} \psi^r \mathcal{E}(e)_{\nu \mu r} \]

\[ - \frac{d-3}{2(d-1)(d-2)} \Gamma_{\mu\nu} \psi^r \mathcal{E}(e)_\lambda \]

\[ + \frac{1}{4} D_{[\mu} (\Gamma \cdot \mathcal{E}(\bar{\psi})) - \frac{1}{2} \bar{\psi} \mathcal{E}(\bar{\psi})_{\mu} + \mathcal{O}(H^2) + \mathcal{O}(\psi^3) . \]

(3.32)

(3.33)

Note that in both of the above identities some of the vielbein equation-of-motion terms come from expanded Riemann tensors (cf. (A.35)–(A.36)). Additionally, further vielbein equation-of-motion terms appear after expansion of the Riemann tensor in the supersymmetry transformation rule for the gravitino curvature (A.6):

\[ \delta \psi_{\mu\nu} = \frac{1}{8} W_{\mu\nu mn} \Gamma^{mn} \epsilon + \frac{1}{d-2} \left[ \Gamma^m_{[\mu} \epsilon \mathcal{E}(e)_{\nu] m} + \frac{1}{d-1} \Gamma_{\mu\nu} \epsilon \mathcal{E}(e)_\lambda \right] + \mathcal{O}(F^2) + \mathcal{O}(\psi^3) . \]

(3.34)

In the above equations one has to be careful with the index order on \( \mathcal{E}(e) \); the second index corresponds to the curved index of the vielbein (cf. (A.29a)).

A quick inspection of the variation of the eleven-dimensional action (3.17) shows that, when we use (3.32)–(3.34) to isolate the equation-of-motion terms, there is a plethora of terms which have to be cancelled by modifying the supersymmetry transformation rules. For the gravitino one can collect these modifications in a reasonably compact form (see (3.36) below), but the result for the vielbein has an order of magnitude more terms. Fortunately, as explained in the previous section the interesting physics is contained in a limited subset of these terms, so we do not have to consider them all.

With the identities (3.32) and (3.33) at hand, the full set of gravitino equation-of-motion terms in the variation of the action (3.17) is readily determined to be\(^4\)

\[ (\alpha')^{-3} \delta \mathcal{L} \bigg|_{\mathcal{E}(\bar{\psi})} = \mathcal{E}(\bar{\psi})_{[\mu} \epsilon t^{(s)} \left[ \right. \right.

\[ - \frac{1}{24} \tilde{P}_{\mu s} \Gamma_{\nu_1 \cdots \nu_6} D_{\nu_7} ( \epsilon W_{\nu_1 \nu_2 s_{1} s_{2}} \cdots W_{s_{5} s_{6} s_{8}} ) \]

\[ + \frac{1}{24} t^{(r)} \tilde{P}_{\mu s} \Gamma_{\tau_7 \tau_8} D_{\nu_7} ( \epsilon W_{\nu_1 \nu_2 s_{1} s_{2}} \cdots W_{s_{5} s_{6} s_{8}} ) \]

\[ + \frac{1}{3(d-2)} t^{(r)} \tilde{P}_{\mu s} \Gamma_{\tau_7 \tau_8} D_{\nu_7} ( \epsilon W_{\nu_1 \nu_2 s_{1} s_{2}} \cdots W_{s_{5} s_{6} s_{8}} ) \]

\[ - \frac{1}{6(d-2)} t^{(r)} \tilde{P}_{\mu m} \Gamma_{\tau_7 \tau_8} D_{s_8} ( \epsilon W_{\nu_1 \nu_2 s_{1} s_{2}} \cdots W_{s_{5} s_{6} s_{8}} ) \]

\[ + \frac{1}{3(d-2)} t^{(r)} \left( \tilde{P}_{\mu s} \epsilon \eta_{s_{7} s_{8}} \epsilon - \frac{1}{d-2} \tilde{P}_{\mu m} \eta_{s_{7} s_{8}} \eta_{s_{6} s_{8}} \epsilon \right) D_{m} ( \epsilon W_{\nu_1 \nu_2 s_{1} s_{2}} \cdots W_{s_{5} s_{6} s_{8}} ) \].

\[ \right] . \]

\(^{4}\)Although we are discussing the eleven-dimensional case, here and in the following we keep the spacetime dimension \( d \) as a parameter as this helps us to keep track of the origin of various terms.
In computing the above result one has to vary the $W^4 - \psi^2$ action and cannot get away with a simple variation of the $R^4$ action only, since the two differ by equation-of-motion terms and the variations will similarly differ by such terms. The complete compensating gravitino transformation rule at order $(\alpha'_M)^3$ readily follows:

\[
\delta \psi_\mu = D_\mu \epsilon + (\alpha'_M)^3 t^{(s)}_8 \left[ \frac{1}{24} \tilde{P}_{\mu s} \Gamma_{r_1 \cdots r_6} D_{s7}(\epsilon W_{r_1 r_2 s1 s2} \cdots W_{r_7 r_6 s5 s6}) \right.
- \frac{1}{24} t^{(r)}_8 \tilde{P}_{\mu s} \Gamma_{r_7 r_8} D_{s7}(\epsilon W_{r_1 r_2 s1 s2} \cdots W_{r_7 r_6 s5 s6})
- \frac{1}{3(d-2)} t^{(r)}_8 \tilde{P}_{\mu s} \Gamma_m \Gamma_{r_7 r_8} \eta_{s7} D_m(\epsilon W_{r_1 r_2 s1 s2} \cdots W_{r_7 r_6 s5 s6})
+ \frac{1}{3(d-2)} t^{(r)}_8 \eta_{r_8 s7} \eta_{r_7 m} \tilde{P}_{\mu s} + \frac{1}{d-2} \tilde{P}_{\mu m} \epsilon D_m(W_{r_1 r_2 s1 s2} \cdots W_{r_5 r_6 s5 s6}) \left. \right]
\]

(3.36)

where we used a shorthand notation for (see (A.33))

\[
\tilde{P}_{mn} := \frac{1}{2(d-2)}(\Gamma_{mn} - (d-3)\eta_{mn}) .
\]

The above result can in principle be checked in an independent way by verifying that the modified equation of motion for the gravitino is indeed supercovariant under the transformation rule (3.36), but this turns out to be considerably more difficult than the analogous check which we performed in section 2.4 for the super-Maxwell theory (consequently, we have only verified supercovariance for the highest-order gamma-matrix terms).

Recall from the previous section that the $\delta (\alpha'_M)^3 \psi_\mu$ terms that modify the supersymmetry algebra are those where the $\mu$-index sits on the covariant derivative. Although there are such terms contained in the third and fifth lines of (3.36), none of these are proportional to either $\Gamma^4$ or $\Gamma^6$, which are the only ones that would contribute to the $\Gamma^5$ component of the translation parameter. Hence, (3.36) does not lead to any non-trivial modification to the supersymmetry algebra.

Turning to the vielbein equation-of-motion terms, we again focus on terms that modify the $\Gamma^5$ component of the translation parameter. As discussed in the previous section, these correspond to terms in $\delta (\alpha'_M)^3 e^{\mu r}$ that either have the $\mu$-index on the gravitino or contain $\psi^{(2)}$ and have the $\mu$-index on a covariant derivative. Only terms of the former kind can potentially be generated by the equation-of-motion terms in (3.32)–(3.34). There is indeed such a term, namely

\[
(\alpha'_M)^{-3} \delta \mathcal{L}_{\Gamma^7} \rightarrow -\frac{1}{4(d-2)} \epsilon t^{(s)}_8 (\epsilon \Gamma^{r_1 \cdots r_5} \psi_\mu) \delta^{r_6}_{s7} W_{r_1 r_2 s1 s2} \cdots W_{r_5 r_6 s5 s6} \mathcal{E}(e)^{s\mu} .
\]

(3.38)

Cancellation of this term requires the transformation rule to be modified according to

\[
\delta e^{\mu r} \bigg|_{\text{from (3.38)}} = +\frac{(\alpha'_M)^3}{4(d-2)} t^{(s)}_8 (\epsilon \Gamma^{r_1 \cdots r_5} \psi_\mu) \delta^{r_6}_{s7} W_{r_1 r_2 s1 s2} \cdots W_{r_5 r_6 s5 s6} ;
\]

(3.39)

which for the supersymmetry algebra, in turn, implies the contribution

\[
[\delta_{\epsilon_1}, \delta_{\epsilon_2}] e^{\mu r} \bigg|_{\text{from (3.38)}} = +\frac{(\alpha'_M)^3}{4(d-2)} t^{(s)}_8 D_\mu (\epsilon_2 \Gamma^{r_1 \cdots r_5} \epsilon_1) \delta^{r_6}_{s7} W_{r_1 r_2 s1 s2} W_{r_3 r_4 s3 s4} W_{r_5 r_6 s5 s6} .
\]

(3.40)
We are not done yet, however; in the cases where the super-Maxwell variations produce terms proportional to $\mathcal{E}(A)^m = D_n \Gamma^{mn} + \cdots$, the analogous Weyl-tensor terms in the supergravity action give equation-of-motion terms for the vielbein. That this is so can be seen by inserting the decomposition (B.6) of the Riemann tensor in the contracted Bianchi identity $D_{[\mu} R_{\nu]rr}^s \epsilon_r \epsilon^r = 0$, and then substituting the on-shell expressions (A.31)–(A.32) for the Ricci terms. Neglecting all higher-order terms and keeping only the equation-of-motion terms of interest one is left with

$$D^n W \eta_{mnrs} = \frac{4}{d - 2} D_\mu \mathcal{E}(e)_{[s} \eta_{r]m} + \cdots.$$  \hspace{1cm} (3.41)

Instead of scanning the variation of the eleven-dimensional $W^4$ action (3.17) for terms of this kind, we can, alternatively, make use of the $\mathcal{E}(A)$ terms in the variation of the super-Maxwell $F^4$ action displayed in (2.24a)–(2.24c).\(^5\) In this way we obtain the term

$$(\alpha'_M)^3 \delta L^{1}_{\Gamma[3]} \rightarrow + \frac{1}{8} \epsilon t^{(s)}_8 (\epsilon \Gamma^{r_1 \cdots r_5} \psi_{s_7 s_8}) W_{r_1 r_2 s_1 s_2} W_{r_3 r_4 s_3 s_4} D^n W_{r_5 r_6 s_5 s_6},$$  \hspace{1cm} (3.42)

which arises from the fourth line of $\delta L^{1}_{\Gamma[3]}$ in (2.24a) after mapping to the supergravity side. Using (3.41), the modification to the transformation rule is then found to be

$$\delta e^r_{\mu} \bigg|_{\text{from (3.42)}} = + \frac{(\alpha'_M)^3}{2(d - 2)} t^{(s)}_8 D_\mu (\epsilon \Gamma^{r_1 \cdots r_5} \psi_{s_7 s_8}) W_{r_1 r_2 s_1 s_2} W_{r_3 r_4 s_3 s_4} \delta^{r_5 r_6}_{s_5 s_6}. \hspace{1cm} (3.43)$$

Such a non-supercovariant term is not very elegant (we will get back to this shortly) but we can still use it to compute the supersymmetry algebra. The modification (3.43) produces only a $\Gamma^{[5]}$ term because of the anti-symmetrisation on $\epsilon_1$ and $\epsilon_2$. The result is

$$\left[ \delta_{\epsilon_1}, \delta_{\epsilon_2} \right] e^r_{\mu} \bigg|_{\text{from (3.42)}} = - \frac{(\alpha'_M)^3}{4(d - 2)} t^{(s)}_8 D_\mu (\epsilon_2 \Gamma^{r_1 \cdots r_5} \epsilon_1) W_{r_1 r_2 s_1 s_2} W_{r_3 r_4 s_3 s_4} W_{r_5 r_6 s_5 s_6} \delta^{r_5 r_6}_{s_5 s_6}. \hspace{1cm} (3.44)$$

Finally, adding (3.40) and (3.44) we arrive at the conclusion that the $\Gamma^{[5]}$ part of the translation parameter in the supersymmetry algebra, and thereby also the $\Gamma^{[5]}$ part of the torsion component $T_{ab}^r$, does not receive any corrections proportional to $(\alpha'_M)^3 W^3$.

In order to check the correctness of this rather unexpected result, we can rederive it in a different way. Namely, since the non-trivial information encoded in the supersymmetry algebra (and the superspace constraints) should not be affected by field redefinitions in the higher-derivative action, we should obtain the same vanishing result for non-trivial $\Gamma^{[5]}$ corrections by using the supersymmetric $R^4$ action instead of the $W^4$ action (3.17) (recall the discussion around (3.7)). As the former is obtained from the latter simply by replacing the Weyl tensor with the Riemann tensor and removing the last two terms of $\mathcal{L}_{\Gamma^{[3]}}$, we can to a large extent draw on the above analysis of the $W^4$ case. The main difference is that in the variation of the $R^4$ action no vielbein equation-of-motion terms arise from splitting Riemann tensors into Weyl and Ricci parts, since such splitting is not required. But the equation-of-motion terms generating the modifications (3.39) and (3.43) are both of precisely this type, as can be seen from the factor $(d - 2)^{-1}$ in their coefficients (cf. (A.35)–(A.36)). Hence, neither of these modifications occurs in the variation of the $R^4$ action, and since no new $\Gamma^{[5]}$ corrections to

\(^5\)Although the left-right mixing and the replacement $R \rightarrow W$ on the gravity side lead to modifications as compared to the super-Maxwell $F^4$ Lagrangian (2.23), it is easy to see that these do not affect the terms can potentially contribute to the $\Gamma^{[5]}$ part of the translation parameter.
the transformation rules are generated, we can indeed again conclude that the supersymmetry algebra is not modified at this level.

At this point, it is interesting to notice that the situation is slightly different for the dimension-1/2 component \( T_{\alpha} \). For this case, \((3.29)\) shows that there is a direct relation between the transformation rule of the vielbein and the torsion component. However, the gauge choice \((3.22)\) is not appropriate in case the vielbein does not transform supercovariantly. The field transformation that brought us from the \( R^4 \) action to the \( W^4 \) action has introduced such non-supercovariant terms, as one can see from \((3.43)\). We will not go into details here, but it suffices to say that we can only reliably compute the dimension-1/2 component of the torsion, under the assumptions of section 3.4, for the \( R^4 \) action. The modifications induced by \((3.39)\) and \((3.43)\) are then absent, but this does not constitute the complete result at this level; other equation-of-motion terms which are not relevant for the discussion of the supertranslation algebra (and therefore have not been discussed here) do nevertheless contribute to the dimension-1/2 torsion. However, we shall not concern ourselves with a systematic analysis of these terms here, but instead return to discuss the dimension-zero torsion.

As hinted at above, the vanishing \((\alpha')^3 W^3\) correction to the supersymmetry algebra is quite surprising in light of Howe’s result [33] that by imposing only the standard constraint \( T_{\alpha\beta} = 2 (\Gamma^\alpha)_{\alpha\beta} \) on the supertorsion in eleven-dimensional superspace, the equations of motion for classical Cremmer-Julia-Scherk supergravity follow upon solving the superspace Bianchi identities. Hence, any non-trivial correction to the latter theory requires that the constraint on this supertorsion component be modified in a non-trivial manner. We are thus led to the conclusion that these non-trivial corrections must be due solely to the gauge-field terms that we have not yet included in the \( t_st_s W^4 \) superinvariant. Moreover—and more remarkably—when the gauge-field strength \( H_4 \) is set to zero, our result, in conjunction with Howe’s, seems to tell us that the dynamics encoded in the action \((3.17)\) is equivalent to the dynamics of the CJS theory for configurations with vanishing gauge-field strength \(^6\). We should, however, point out that a complete analysis of the action at order \((\alpha')^3\) is likely to require the inclusion of the \( \epsilon \epsilon W^4 \) term (together with its associated fermi bilinears). Its presence can be deduced by lifting the one-loop term in the IIA action \((1.6)\), and it is conceivable that this term is also required once one includes the gauge field in the supersymmetry analysis. At present we do, however, not know whether this purely gravitational term will produce non-trivial modifications to the algebra; it seems more likely that such corrections will again arise from the gauge-field terms once they are included.

To the best of our knowledge, the only other computation of a supergravity commutator (where corrections could be expected) in the presence of higher-derivative terms that is available in the literature was done in the type IIB paper by Green and Sethi [9]. These authors considered the algebra on the dilatino. As their main result required only the information of the terms independent of the Riemann tensor, they only had to take into account the modified dilatino transformation rule. As a consequence their calculation did not reveal any modifications to the supertranslation parameter either.

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\(^6\)Some gauge-field dependent contributions to the bosonic part of the invariant were obtained by Deser and Seminara [60, 61] from an analysis of supergravity four-point functions at tree-level. However, their method is neither complete (higher-point functions are necessary as well), nor does it produce a result which exhibits directly the \( t_s \) tensorial structures.
4 Discussion, conclusions and outlook

In this paper we have presented the first part of a systematic derivation of the modifications to the superspace torsion constraints due to higher-order $\alpha'$ string effects. In contrast to existing approaches, our analysis is based on the component formalism of supergravity. Using string-amplitude calculations in combination with supersymmetry requirements, we were able to find a compact form for the superinvariant associated to the anomaly cancellation term in ten dimensions. In addition, we derived the modifications to the supersymmetry transformation rules. In the sector which we considered, which is largely independent of the gauge field, the invariant could be lifted to eleven dimensions. The field-dependent parameters in the supersymmetry algebra were shown to receive no corrections from this ‘purely gravitational’ part of the theory.

Given the complexity of higher-derivative supergravity actions, it is most encouraging that the use of string-amplitude information has enabled us to reduce the supersymmetry analysis to a problem that can be worked out by hand. As a result, we now have a much better understanding of the tensorial structure of the fermion bilinears. In addition, the compact form of our result has enabled us to lift our results to eleven dimensions, in spite of the fact that it was based on string input.

The next step of our programme is obviously to extend the analysis to cover the gauge-field sector in our analysis and also to include the $\epsilon_\xi W^4$ term. Since we have found that no non-trivial purely gravitational modifications to the dimension-zero supertorsion constraint are generated by the higher-derivative interactions we have derived so far, while the superspace analysis of Howe [33] proves that any non-trivial M-theory corrections must show up precisely in this torsion component, it is clear that either one (or both) should be responsible for these corrections. The strong link between the presence of the gauge field and the structure of eleven-dimensional superspace at the quantum level is perhaps not so unexpected, given the central roles the membrane and the five-brane—both supported by a non-vanishing gauge-field strength—play in M-theory. In addition, there are several reasons to expect the presence of the $\epsilon\epsilon W^4$ term in the action. First of all it is obtained by lifting the IIA action. But more importantly, absence of corrections to the torsion constraints generated by this term would imply, through Howe’s analysis [33], that the dynamics of the purely gravitational theory (i.e. setting the gauge field to zero) is equivalent to that of the Cremmer-Julia-Scherk theory. It is not clear how such a rather strong conclusion would fit into the various duality conjectures.

The presence of the $\epsilon\epsilon W^4$ term in the higher-derivative eleven-dimensional action can in principle be studied via our string theory analysis. As our approach is based on one-loop amplitudes, and because the heterotic string does not exhibit this particular bosonic term at one loop, it is, however, necessary to first extend our analysis to the maximally extended supergravity theories in ten dimensions. Treatment of the gauge-field sector using our methods is also possible, although one expects that the lifting procedure to eleven dimensions will be more complicated. But perhaps the rather compact form of our invariant will make it possible to analyse these gauge-field dependent terms using only supersymmetry. Work on these issues is in progress. Once the gauge-field terms have been included in the higher-derivative action as well as in the transformation rules, it also becomes of interest to study $\alpha'_M$-corrected M-brane supergravity solutions as well as applications to compactification problems.

On the sideline, many other interesting questions have appeared. One of them is to understand whether a superparticle (or supermembrane) vertex operator analysis (in the space-time supersymmetric formalism) is able to reproduce the tensorial structures of the fermionic bilinears in our eleven-dimensional action (3.17). This, however, requires complicated zero-mode
integrals to be performed (an alternative way to state this problem is that one has to calculate contractions of a sixteen-dimensional spinorial epsilon tensor with a number of gamma matrices, which is hard except for special cases like the contraction that leads to the $t_8 t_8$ structure).

The main goal of this programme, however, remains to understand how the higher-derivative modifications to the target-space theory are related to similar corrections of world-volume theories of branes. For instance, the known bosonic higher-derivative gravitational corrections to the Born-Infeld part of the D-brane actions (as derived by Bachas et al. [62]) and the ones correcting the Wess-Zumino term (see e.g. Green et al. [63] and Cheung and Yin [64]) have so far not been incorporated in a kappa-symmetric framework, generalising the actions of Cederwall et al. [65, 66], Bergshoeff and Townsend [67] and Aganagic et al. [36]. Intuitively one expects that the gravitational corrections, together with the modified torsion constraints and perhaps a modified form of the kappa-symmetry projector, conspire to yield again a kappa-symmetric action. The superembedding formalism seems to be a very promising tool with which to address this question.

Acknowledgements

We have benefitted from discussions with Eugene Cremmer, Stanley Deser, Jim Gates, Michael Green, Renata Kallosh, Hermann Nicolai, Jan Plefka, Sebastian Silva, Per Sundell, Paul Townsend, Andrew Waldron, Niclas Wyllard and Marija Zamaklar. In addition, we would like to thank the theory group at Chalmers University in Gothenburg, and in particular Martin Cederwall, Ulf Gran, Mikkel Nielsen and Bengt Nilsson, for an inspiring week of discussions, for sharing with us an early draft of [34] and for very useful remarks concerning our superspace results. Finally, we thank Mees de Roo for many behind-the-scenes comments about [15] and for providing us with details about their computer calculation.

P.V. was partially supported by the TMR contract ERB FMRXCT 96-0012.


A Supergravities in first- and second-order formalism

A.1 Normalisation issues

This appendix contains various details about the standard, non-extended supergravity theories in ten and eleven dimensions (though some of the results apply to other supergravity theories as well). Our main goal is to explain the origins of various normalisation factors, to elaborate on the presence of the dilaton and to explain some of the subtleties one encounters when dealing with the transformation rule of the spin connection. None of the results are new, although few accounts of higher-dimensional supergravity in the first-order formulation have appeared previously in the literature (exceptions are the papers by Castellani et al. [68] and Julia and Silva [69]). As we do not need higher-order fermi terms in the main text, they have been omitted here for the sake of brevity. The original references for the theories discussed here are Cremmer et al. [70] (eleven dimensions) and Chamseddine [71] and Bergshoeff et al. [55] (ten dimensions).

The gravity supermultiplets for the theories under consideration consist first of all of a graviton (described by the vielbein $e_\mu^r$), a gravitino ($\psi_\mu$) and a bosonic gauge field (which we will denote by $B_{\mu_1\cdots\mu_{n-1}}$). In ten dimensions, there is in addition a dilaton ($\phi$) and dilatino ($\lambda$). We define the bosonic and fermionic field strengths as

$$H_{\mu_1\cdots\mu_n} = n \partial_{[\mu_1} B_{\mu_2\cdots\mu_n]} ,$$

$$\psi_{\mu\nu} = D_{[\mu} \psi_{\nu]} ,$$

(A.1)

Note the perhaps somewhat unusual normalisation of the gravitino curvature.

Let us first discuss some normalisation issues. In eleven dimensions the kinetic terms are unique up to normalisations. In ten dimensions, in contrast, the presence of the dilaton forces us to make a choice of Weyl scaling. In the present section we eliminate this choice by requiring that the kinetic terms for the fermions are diagonal and we will comment on the other possibilities later. With this choice, our normalisations are fixed by

$$S_{\text{graviton}} = -\frac{1}{4\kappa_d^2} \int d^d x \, e \, R(\omega) ,$$

$$S_{\text{gravitino}} = -\frac{1}{\kappa_d^2} \int d^d x \, e \, \bar{\psi}_\mu \Gamma^{\mu
u\rho} \psi_{\nu\rho} ,$$

$$S_{\text{gauge field}} = -\frac{1}{2\kappa_d^2} \int d^d x \, e \, \frac{1}{n!} \phi^{-p} H_{\mu_1\cdots\mu_n} H^{\mu_1\cdots\mu_n} .$$

(A.2)

In eleven dimensions $n = 4$, while in the ten-dimensional theory $n = 3$. The power $p$ of the dilaton in the last line depends on the dimension and the rank of the gauge field, and is completely fixed by supersymmetry once we have fixed the normalisation of the dilaton kinetic term. For this field (and its superpartner) we use the actions

$$S_{\text{dilaton}} = -\frac{1}{2\kappa_d^2} \int d^d x \, e \, (\phi^{-1} \partial_\mu \phi)^2 ,$$

$$S_{\text{dilatino}} = -\frac{1}{\kappa_d^2} \int d^d x \, e \, \bar{\lambda} \Gamma^\mu D_\mu \lambda .$$

(A.3)

With this choice, it turns out that $p = 2$ for the three-form theory (see below). The spinors are all minimal, i.e. in eleven dimensions the gravitino is Majorana while in ten dimensions
the gravitino and the dilatino are both Majorana and Weyl (though of opposite chirality). In addition to the above terms, the complete non-linear actions will contain three-point vertices coupling the fermions, the gauge field and the dilaton. Finally, there will of course be higher-order fermi terms, and in eleven dimensions also Chern-Simons couplings of the gauge fields; we will ignore the former but comment on the way in which the latter arise at the end of this section.

The structure of the supersymmetry transformation rules of the various fields can be determined by analysing the kinetic terms given above (see also Townsend and van Nieuwenhuizen [72]). Let us first state the result, which can be summarised by the general form

\begin{align}
\delta e_\mu^r &= 2 \tilde{e} \Gamma^r \psi_\mu, \\
\delta \psi_\mu &= D_\mu \epsilon + N_\psi \phi^{-p/2} (T^{\sigma_1 \cdots \sigma_n}_\mu \epsilon) \hat{H}_{\sigma_1 \cdots \sigma_n}, \\
\delta B_{\mu_1 \cdots \mu_{n-1}} &= (n - 1) N_\psi \phi^{p/2} (\tilde{\psi}_{[\mu_1} \Gamma_{\mu_2 \cdots \mu_{n-1}] \epsilon) - N_\lambda \phi^{p/2} (\tilde{\lambda} \Gamma_{\mu_1 \cdots \mu_{n-1}} \epsilon), \\
\phi^{-1} \delta \phi &= \sqrt{2} (\epsilon \lambda), \\
\delta \lambda &= \frac{1}{\sqrt{2}} (\Gamma^\mu \epsilon) \phi^{-1} \phi_\mu - \frac{1}{2 \sqrt{2}} N_\lambda \phi^{-p/2} (\Gamma^{\sigma_1 \cdots \sigma_n} \epsilon) \hat{H}_{\sigma_1 \cdots \sigma_n}.
\end{align}

We should stress that these rules are not the ones which we use in the main text; instead, a super-Weyl rescaling to the string frame has been used there. More details are given in the next section. The spin connection appearing e.g. in the graviton kinetic action can be viewed as either an independent field (first-order formalism) or as a composite one defined in terms of the vielbein (second-order formalism). In between the first- and second-order formalisms there is also the so-called 1.5-order formulation, where the spin-connection is treated as an independent field for the purpose of the supersymmetry variation only. These issues will be discussed in more detail below.

Given the transformation rules, one can define so-called supercovariant objects, which are by definition such that they transform without any derivatives of the supersymmetry parameter. Notationally, we distinguish fields with this property by hats. They are readily constructed with the result:

\begin{align}
\hat{\phi}_\mu &= \partial_\mu \phi - \sqrt{2} \lambda \psi_\mu \phi, \\
\hat{\lambda}_\mu &= D_\mu \lambda - \frac{1}{\sqrt{2}} (\Gamma^\nu \psi_\nu) \phi^{-1} \phi_\mu + \frac{1}{2 \sqrt{2}} N_\lambda \phi^{-p/2} (\Gamma^{\sigma_1 \cdots \sigma_n} \psi_\mu) \hat{H}_{\sigma_1 \cdots \sigma_n}, \\
\hat{\psi}_{\mu \nu} &= D_{[\mu} \psi_{\nu]} + N_\psi \phi^{-p/2} T_{[\mu} \hat{H}_{\nu]}, \\
\hat{H}_{\sigma_1 \cdots \sigma_n} &= n \partial_{[\sigma_1} B_{\sigma_2 \cdots \sigma_n]} - \frac{1}{2} n (n - 1) N_\psi \phi^{p/2} (\tilde{\psi}_{[\sigma_2} \Gamma_{\sigma_3 \cdots \sigma_n] \psi_{\sigma_1]} \\
&\quad + n N_\lambda \phi^{p/2} (\tilde{\lambda} \Gamma_{\sigma_1 \cdots \sigma_n} \psi_{\sigma_1})).
\end{align}

Let us now explain how to obtain the gauge-field dependent coefficients in (A.4) (the other ones are easily fixed). The first thing to do is to determine the structure of the three-point vertices in the action. The easiest way to achieve this is to demand that the fermion equations of motion are supercovariant. Using the definitions (A.5) one then writes down these couplings immediately. The next step is to fix the structure of the tensors \( T \). This can be done by focussing on variations proportional to a derivative of the gauge field strength, and observing that terms with a \( \Gamma^{[n]} \) matrix only come from this tensor and should therefore
vanish by themselves. Once this tensor is known, the normalisation factors $N_\psi$ and $N_\lambda$ follow by looking at for instance the $H^2$ terms in the variation, or by computing the algebra on the bosonic fields. Finally, the exponent $p$ can be fixed by considering variations proportional to one power of the gauge-field strength and a derivative of the dilaton.

To illustrate this procedure, let us discuss the invariance of the eleven-dimensional action in some more detail. The three-point vertices can all be incorporated in the action by just taking the supercovariant gravitino curvature in (A.2) (such a simple substitution does no longer work in ten dimensions, but the rest of the logic is the same). This object, and the supercovariant gauge-field strength, transform as

$$\delta \hat{\psi}_{\mu \nu} = \frac{1}{8} R_{\mu \nu mn} \Gamma^{mn} \epsilon + N_\psi D_\mu (T_{\nu} \cdot \hat{H}) \epsilon + N_\lambda^2 \eta^{\mu \nu} \cdot \hat{H} T_{\nu} \cdot \hat{H} \epsilon + O(\epsilon^2),$$  

(A.6)

$$\delta \hat{H}_{\mu \nu \rho \sigma} = -12 N_\psi \epsilon \Gamma_{[\mu \nu \psi_{\rho \sigma]} - 12 N_\psi \epsilon T_{[\mu \nu} \hat{\Gamma}_{\rho \sigma]} + O(\epsilon^3).$$  

(A.7)

Employing $\delta e = e e_\nu^\mu \delta e_\nu^\rho$ as well as $\partial_\mu e = e e_\lambda^\mu \delta e_\lambda^n$, one finds that the three kinetic terms transform as

$$\kappa_d^2 \delta S_{\text{graviton}} = e \left(R_{\mu \nu} - \frac{1}{2} e_{\mu}^\nu R \right) \hat{e}^{\mu \nu} \psi_\mu + \frac{1}{2} e \left(\frac{1}{2} T_{mn}^\nu - T_{m\lambda}^\lambda \epsilon_{e}^\nu \right) \delta \omega_{mn}.$$  

(A.8a)

$$\kappa_d^2 \delta S_{\text{gravitino}} = -e \left(R_{\mu \nu} - \frac{1}{2} e_{\mu}^\nu R \right) \hat{e}^{\mu \nu} \psi_\mu + \frac{1}{2} e \epsilon \Gamma_{\mu \nu}^{\mu \nu} \psi_\mu \psi_\rho R_{\nu \rho}$$

$$- e \epsilon \left(T_{\mu \lambda}^\nu - T_{\mu \lambda}^\lambda e_{\lambda}^\nu \right) \hat{\psi}_{\nu \rho}$$

$$+ e N_\psi \epsilon \Gamma_{\mu \nu}^{\mu \nu} D_\mu \left(T_{\nu} \cdot \hat{H} \psi_\rho \right) - e N_\psi \epsilon T_{\nu} \cdot \hat{H} \Gamma^{\mu \nu} \psi_\nu$$

$$- e N_\psi \hat{\psi}_\mu \Gamma^{\mu \nu} \psi_\nu D_\nu \left(T_{\rho} \cdot \hat{H} \right) - e \frac{1}{4} \hat{\psi}_\mu \Gamma^{\mu \nu} \Gamma^{\nu \rho} \psi_\rho \delta \omega_{\nu \rho}$$

$$+ e N_\psi^2 \epsilon T_{\mu} \cdot \hat{H} \Gamma^{\mu \nu} T_{\nu} \cdot \hat{H} \psi_\rho + O(\epsilon^3).$$  

(A.8b)

$$\kappa_d^2 \delta S_{\text{gauge field}} = -\frac{1}{48} e e \epsilon \Gamma_{\mu}^{\nu} \psi_\mu H^2 + \frac{1}{4} e \epsilon \Gamma_{\mu}^{\nu} \psi_\nu H_{\mu \lambda \sigma} H_{\nu}^{\lambda \sigma} - \frac{1}{2} e N_\psi \epsilon g_{\mu \lambda} \partial_\mu H_{\lambda \nu \rho \sigma} \epsilon \Gamma_{\nu \rho} \psi_\sigma$$

$$+ \frac{1}{2} e N_\psi \epsilon \Gamma_{\nu \rho} \psi_\sigma H_{\mu \rho \sigma} \left(T_{\mu \nu} - T_{\mu \lambda} \delta_\lambda^\nu \right) + \frac{1}{4} e N_\psi \epsilon \Gamma_{\nu \rho} \psi_\sigma H_{\mu \rho \lambda} T_{\mu \sigma}$$

$$- \frac{1}{2} e N_\psi \epsilon \Gamma_{s_1 s_2 s_3 s_4} D_\mu (e_{s_1}^\lambda \cdots e_{s_4}^\lambda) e^{s_1 \mu} H_{\lambda_1 \cdots \lambda_4}.$$  

(A.8c)

We have kept $\delta \omega$ as well as the terms proportional to the torsion $^7 T_{\mu \nu}^\tau = 2 D_{[\mu} e_{\nu]}^\tau$ for the discussion in section A.3. The first terms to focus on are those with a derivative on the gauge-field strength. They only come with a $\Gamma_{[n-2]}$ matrix in (A.8c), and by comparison with (A.8b) one finds

$$\mathcal{T}_{\mu}^{\sigma_1 \cdots \sigma_n} = \frac{n - 1}{2(d - 2)n!} \left( \Gamma_{\mu}^{\sigma_1 \cdots \sigma_n} - \frac{n}{n - 1} (d - n - 1) \delta_\mu^{[\sigma_1} \Gamma^{\sigma_2 \cdots \sigma_n]} \right).$$  

(A.10)

---

$^7$ A useful relation, valid under the integral, which isolates the appearance of the torsion terms, is

$$e \delta \left(e_{\mu} e_{\nu} R_{\mu \nu \rho \sigma} \right) S_{\rho \sigma}^{\nu \mu} = 4 e \left(\epsilon \Gamma_{\mu}^{\rho} \psi_\nu \right) R_{\rho}^{\mu \nu \sigma} S_{\sigma \tau}^{\rho \mu}$$

$$+ 2 \left(\frac{1}{2} T_{\rho}^{\tau \mu \nu} + T_{\rho}^{\lambda \nu} S_{\tau}^{\rho \mu} \right) \delta \omega_{\nu \rho \sigma}$$

$$- 2 e \left(D_{\mu} S_{\sigma \tau}^{\nu \rho} \right) e_{\nu} e_{\rho} \delta \omega_{\nu \rho}.$$  

(A.9)

This can be used in higher-derivative actions, but of course also reproduces the variation of the Einstein-Hilbert term after insertion of $S_{\rho \sigma}^{\nu \mu} = \frac{1}{2} (\eta_{\nu \mu} \eta_{\rho \sigma} - \eta_{\rho \nu} \eta_{\mu \sigma})$ (which satisfies $D_{\mu} S_{\rho \sigma}^{\nu \mu} = 0$).
The cancellation now arises due to the fact that relative normalisation between the two terms leads to
\[
\Gamma_{\mu
u}^r T^r_{\sigma_1 \cdots \sigma_n} = -\frac{1}{2 \cdot n!} \left( \Gamma_{\mu
u}^\alpha \sigma_1 \cdots \sigma_n + n(n-1) \delta_{\mu}^{[\alpha} \delta_{\nu]}^{\sigma_2} \Gamma_{\sigma_3 \cdots \sigma_n} \right) .
\]  
(A.11)

When inserted in the relevant terms in (A.8b), one finds that the first gamma produces a Bianchi identity on the gauge-field strength, while the second one exactly cancels the contribution from (A.8c). By computing the algebra on the two bosonic fields or by analysing the variations proportional to \( H^2 \), one obtains \( N_\psi = 1 \).

The Chern-Simons term in the eleven-dimensional action arises because of the fact that there is a \( \bar{\psi} \Gamma^{[\alpha} T^\beta_{\gamma \cdots} \psi \) term left over after all other terms with two powers of the gauge-field strength have been cancelled. By dualising the gamma matrix one can get rid of this variation by adding the term
\[
S_{\text{Chern-Simons}} = -\frac{2}{(12)^4 \kappa_{11}^2} \int d^{11}x \, \varepsilon^{\mu_1 \cdots \mu_{11}} B_{\mu_1 \mu_2 \mu_3} H_{\mu_4 \cdots \mu_7} H_\mu \psi_{\mu_8 \cdots \mu_{11}} .
\]  
(A.12)

The story is very similar in the ten-dimensional case, where one finds the same tensorial structure for \( T \) and in addition obtains that \( N_\lambda = 1/\sqrt{2} \) and \( p = 2 \).

### A.2 Super-Weyl transformations and the string frame

In ten dimensions we have the freedom of going to a different frame by rescaling the vielbein by a power of the dilaton. In order to maintain the canonical transformation rule of the vielbein, an accompanying redefinition of the fermion fields is necessary. Starting from the transformation rules (A.4) and performing a rescaling
\[
e_\mu^r = \phi^{-\alpha} e_\mu^r, \quad \lambda = \phi^{-\beta} \lambda, \quad \psi_\mu = \phi^{-\beta} \psi_\mu - \gamma e_\mu^r \Gamma_r \lambda, \quad \epsilon = \phi^{-\beta} \epsilon ,
\]  
(A.13)

(thes second column ensures that the first terms in the dilatino and gravitino transformation rules remain independent of the dilaton) then one finds that the new vielbein transforms as
\[
\delta e_\mu^r = 2 \phi^{\alpha-2\beta} (\tilde{e} \Gamma^r \tilde{\psi}_\mu) + e_\mu^r (\tilde{\epsilon} \lambda) \phi^\alpha \left( \sqrt{2} \alpha - 2 \gamma \right) - 2 \gamma \phi^{\alpha-2\beta} (\tilde{e} \Gamma_r \tilde{\lambda}) e_\mu^s .
\]  
(A.14)

Imposing that this is identical to the transformation rule in the original frame requires that the two remaining parameters are expressed in terms of \( \alpha \) as
\[
\beta = \frac{1}{2} \alpha, \quad \gamma = \frac{1}{\sqrt{2}} \alpha .
\]  
(A.15)

In addition, there are some changes in the supersymmetry transformation rules. Firstly, a Lorentz transformation has to be added in order to accommodate the last term in (A.14), and addition there are changes to the fermionic transformation rules. Focussing on the \( n = 3 \) case (we drop the tildes from now on) one obtains:
\[
\delta e_\mu^r = 2 e \Gamma^r \psi_\mu \sqrt{2} \alpha (e \Gamma^r \lambda) e_\mu^s ,
\]  
(A.16a)
\[
\delta \psi_\mu = D_\mu \epsilon + \phi^q \left( T_\mu \Gamma^{\sigma_1 \cdots \sigma_3} \right) \hat{H}_{\sigma_1 \cdots \sigma_3} - \alpha_{\Gamma_\mu} \phi^q \left( \Gamma_\mu \Gamma^{\sigma_1 \cdots \sigma_3} \epsilon \right) \hat{H}_{\sigma_1 \cdots \sigma_3} ,
\]  
(A.16b)
\[
\delta B_{\mu_1 \mu_2} = 2 \phi^{-q} \left( \psi_{[\mu_1} \Gamma_{\mu_2]} \epsilon \right) + \frac{(2 \alpha - 1)}{\sqrt{2}} \phi^{-q} \left( \Lambda \Gamma_{\mu_1 \mu_2} \epsilon \right) ,
\]  
(A.16c)
\[
\hat{\phi}^{-1} \delta \phi = \sqrt{2} (\tilde{\epsilon} \lambda) ,
\]  
(A.16d)
\[
\delta \lambda = \frac{1}{\sqrt{2}} (\Gamma^r \epsilon) \phi^{-1} \hat{\phi} - \frac{1}{2 \sqrt{2} n!} \phi^q \left( \Gamma^{\sigma_1 \cdots \sigma_3} \epsilon \right) \hat{H}_{\sigma_1 \cdots \sigma_3} ,
\]  
(A.16e)
(here \( q = \alpha(n - 1) - p/2 \)). The rules above of course include higher order fermi terms which we have suppressed.

Under these super-Weyl rescalings, the kinetic terms for the graviton, the gravitino, the dilaton and the dilatino all pick up a dilaton prefactor \( \phi^{-(d-2)\alpha} \). The gauge-field kinetic term instead will get an overall factor \( \phi^{(2n-d)\alpha-p} \) where \( p \) is the factor that was already present. The frame in which all transformation rules are independent of the dilaton, namely for which \( \alpha = p/(2n - 2) \), is called the “string frame”. In this frame one in addition observes that all kinetic terms have the same dilaton prefactor, while the transformation rule of the gauge field becomes independent of the dilatino and the gravitino rule simplifies drastically. We refer the reader to the work of Kallosh [38] and Kallosh and Nilsson [39] (the \( N = 1 \) case) and Bellucci et al. [73] (the extended supergravities) for more information.

These results can of course be formulated in superspace as well. Following Gates [74], Howe [75] and in particular section 6 of Gates and Vashakidze [76], one finds that the transformed supervielbeine are given by

\[
\tilde{E}_M^A = \begin{pmatrix}
\Phi e^{\beta \Phi} \left( E_{\alpha a}^a - f_r a E_{\alpha r}^a \right) & e^{\alpha \Phi} E_{\alpha r}^a \\
\Phi e^{\beta \Phi} \left( E_{\mu}^a - f_r a E_{\mu r}^a \right) & e^{\alpha \Phi} E_{\mu r}^a
\end{pmatrix},
\]

(A.17)

Observe that the components \( E_{\mu r}^a \) and \( E_{\alpha r}^a \) (which are the only components appearing in e.g. the string world-sheet action) transform in a simple way. The \( f_r a \) field is a superfield and has a non-trivial expansion in powers of theta.

### A.3 Transformation of the spin connection in eleven dimensions

We have not yet discussed how the terms proportional to the torsion and the variation of the spin connection in (A.8a)–(A.8c) are cancelled. In the second-order formalism the torsion is of higher order in the fermions, while the transformation rule of the spin connection follows from that of the vielbein. In contrast, the first-order formalism keeps both of these as independent objects. The 1.5-order formalism, somewhere in between the previous two, applies only to the classical supergravity theories as it makes use of the fact that the equation of motion of the spin connection is algebraic. As a result, this object does not have to be varied in the action as it only leads to terms proportional to its defining equation. We will exhibit in detail how the differences arise in the case of the eleven-dimensional theory.

We first observe that the fully anti-symmetrised Riemann tensor in (A.8b) can be rewritten using

\[
R_{[\mu
\nu\rho]}^s = -D_{[\mu} T_{\nu\rho]}^s.
\]

(A.18)

The only candidate \( \delta \omega \) term that can cancel this variation is the second line of (A.8a) with a non-supercovariant term in \( \delta \omega \) (the first term in (A.19) below). Part of the transformation rule for the spin connection should therefore be

\[
\delta_1 \omega^m \nu = D_{\phi} \left( \epsilon \Gamma_{\nu} \phi^{m\nu} \psi_p \right) + 4 \epsilon \Gamma^{mpn} \psi_{p} - \frac{4}{d-2} \epsilon \Gamma^{rs} \psi_{r} \psi_{n} + O(\epsilon^3),
\]

(A.19)

(for the cancellation of the torsion terms, there is a third possible candidate in the variation with a \( \Gamma^{[3]} \), namely \( \epsilon \Gamma_{\nu} \psi_{[p} \psi_{m]} \), but it turns out that this term is not needed). However, this
is not the complete story as we have remaining terms coming from the third and fourth line of (A.8b) as well as the second line of (A.8c). Taken together, they are

\[
\text{remaining} = -e \left( \bar{\psi}_\rho \Gamma^{\mu\nu\rho} T_\lambda \cdot \hat{H} \epsilon - (\epsilon \leftrightarrow \psi_\rho) \right) T_{\mu\nu}^\lambda \times \frac{1}{12} e \epsilon \Gamma^{\mu\nu\lambda\sigma_2\sigma_3\sigma_4} T_{\mu\lambda} \cdot \delta_{\sigma_1\sigma_2} \psi_\rho \\
+ \frac{1}{5} e \epsilon \Gamma_{\nu\rho} \psi_\sigma H^{\mu\nu\rho} \left( T_{\mu\nu}^\lambda - T_{\mu\lambda}^\lambda \delta_\kappa^{\nu} \right) + \frac{1}{8} e \epsilon \Gamma_{\nu\rho} \psi_\sigma H^{\mu\nu\rho} T_{\mu\lambda}^\sigma .
\] (A.20)

The first two terms can be reduced further, as the anti-symmetric combination picks out only the $\Gamma^{[6]}$ and $\Gamma^{[2]}$ pieces. The fact that (A.20) is non-zero implies that we need additional terms in $\delta \omega$.

Just adding $\epsilon \psi H$ terms to the transformation rule of the spin connection does not, however, make the action invariant. The only way out of this problem is to add an additional term to the action, which vanishes when the torsion is taken on-shell. Such an addition has also appeared in Castellani et al. [68] and Julia and Silva [69]. The guiding principle to find this action will be to make (A.19) supercovariant. This requires

\[
\frac{\delta \omega_{\nu}^mn}{\kappa_{11}} = -\left( D_{\phi} \tilde{e} \right) \Gamma_{\nu}^m \delta_{\sigma_1} \cdot \bar{\psi}_\lambda \Gamma^{\nu\sigma_1\sigma_2\sigma_3\sigma_4} T_{\mu\lambda} \cdot \hat{H} \psi_\lambda . \tag{A.21}
\]

The first term produces a variation proportional to the derivative of the supersymmetry parameter. The appropriate term in the action to cancel this variation is

\[
S_T = \frac{1}{\kappa_{11}^2} \int d^{11} x \frac{1}{8} e \left( T_{\mu\nu} - \text{“value of} \ T_{\mu\nu} \text{on-shell”} \right) \bar{\psi}_\lambda \Gamma_{\nu}^\lambda \delta_{\sigma_1} \cdot \bar{\psi}_\lambda \Gamma^{\nu\sigma_1\sigma_2\sigma_3\sigma_4} T_{\mu\lambda} \cdot \hat{H} \psi_\lambda . \tag{A.22}
\]

The second term in (A.21) now produces additional terms proportional to the three-form field strength and so does the variation of the gravitini in $\delta S_T$. The sum of these, even though it involves a contraction of $\Gamma^{[6]}$ with $T$, is rather simple:

\[
\kappa_{11}^2 \left( \delta S_{\text{CJS}} + \delta S_T \right) = \frac{1}{4} e T_{\mu\nu}^\lambda \delta^{\mu\nu}_{\sigma_1} \cdot \bar{\psi}_\lambda \Gamma^{\nu\sigma_1\sigma_2\sigma_3\sigma_4} T_{\mu\lambda} \cdot \hat{H} \psi_\lambda + \left( \epsilon \leftrightarrow \psi_\sigma \right). \tag{A.23}
\]

Adding (A.23) to the terms (A.20) which were still remaining, and working out the $\Gamma^{[6]}$ and $\Gamma^{[2]}$ terms in the gamma products (this is a bit tedious but can be done more easily by making use of (B.23)), one finds that many terms cancel and one obtains the following rather elegant result

\[
\kappa_{11}^2 \delta S = -\frac{1}{72} \left( \bar{\psi}_\lambda \Gamma_{\mu\nu}^{\sigma_1\sigma_2\sigma_3\sigma_4} + 24 \bar{\psi}_\lambda \Gamma^{\sigma_1\sigma_2\sigma_3\sigma_4} \delta^{\mu\nu} \cdot \bar{\psi} \epsilon \right) \left( \frac{1}{2} T_{\mu\nu}^\lambda - T_{\mu\lambda}^\nu \delta^\rho_\lambda \right) . \tag{A.24}
\]

These can be cancelled by one final addition to the transformation of the spin connection,

\[
\delta^3 \omega_{\nu}^mn = -4 e \tilde{\epsilon} S^{\sigma_1\sigma_2\sigma_3\sigma_4}_{\mu\nu} H_{\sigma_1\sigma_2\sigma_3} \psi_\nu , \tag{A.25}
\]

where we have defined the tensor $S$ as

\[
S^{\sigma_1\sigma_2\sigma_3\sigma_4}_{\mu\nu} = \frac{1}{288} \left( \Gamma^{\mu\sigma_1\sigma_2\sigma_3\sigma_4}_{\nu\sigma_1\sigma_2} + 24 \eta^{\mu\nu}_{\sigma_1\sigma_2\sigma_3\sigma_4} \right) . \tag{A.26}
\]

It also happens to be the symmetric part of the product of a single $\Gamma$ with $T$, but it is as of yet unclear how to find $S$ in the above calculation in such an elegant way (it probably requires rewriting of the transformation rule of the three-form, as that is the only place where $T$ is not yet present manifestly).
Summarising, the action (A.2) (plus higher-order fermi terms) is the first-order form of eleven-dimensional supergravity provided we add $S_T$ as given in (A.22). It is invariant under the transformations (A.4) together with the transformation rule of the spin connection

$$
\delta \omega_{\nu}^{mn} = \epsilon \Gamma_{\nu}^m \phi_{\rho mn} \hat{\psi}_{\rho} + 4 \epsilon \Gamma^{mn p} \hat{\psi}_{\rho p} - \frac{4}{3} \epsilon \Gamma^{rs [m} \phi_{\rho s] n} - 4 \epsilon \delta S^{mn} \cdot \hat{H} \psi_{\nu} + O(\epsilon \psi^3) + O(T) .
$$

(A.27)

(The additional torsion term can be determined by considering the variation of $S_T$). The first two terms above can be rewritten in such a way that the relation to the second-order formalism becomes more clear. Using $\Gamma^{[5]} = \Gamma^{[2]} \Gamma^{[3]} - \Gamma^{[3]} \eta$ and the corresponding expansion of $\Gamma^{[3]}$, one finds

$$
\delta \omega_{\nu}^{mn} = -\frac{1}{2} \epsilon \Gamma^{mn} \mathcal{E}(\bar{\psi})_{\nu} + (1 + \frac{2}{3}) \epsilon \mathcal{E}(\bar{\psi})'[\epsilon e_{\nu}] + 2 \epsilon \Gamma_{\nu}^{[m} \mathcal{E}(\bar{\psi})^{n]} - 2 \epsilon \Gamma^{mn} \mathcal{E}(\bar{\psi})_{\nu}

- 2 \epsilon \Gamma_{\nu} \hat{S}^{mn} + 4 \epsilon \Gamma^{[m} \hat{\psi}_{\rho n]} - 4 \epsilon \delta S^{mn} \cdot \hat{H} \psi_{\nu} + O(\epsilon \psi^3) + O(T) .

$$

(A.28)

The capital $\mathcal{E}(\bar{\psi})$ symbols on the first line are proportional to the gravitino equation of motion (see (A.29b) and (A.33) below). The gauge-field independent terms of this transformation rule also apply to the ten-dimensional theory.

### A.4 Equations of motion and other identities

The equations of motion associated to the eleven-dimensional action discussed in the previous sections are as follows ($n = 4$ in the following):

$$
\mathcal{E}(e)_{\nu}^{\mu} := \frac{\kappa_d^2}{e} \frac{\delta S_d}{\delta e_{\mu}} = \frac{1}{2} \left( R_{\nu}^{\mu} - \frac{1}{2} e_{\nu}^{\mu} R \right) + \frac{1}{2 m} \left( 2n (H^2)_{\nu}^{\mu} - e_{\nu}^{\mu} H^2 \right)

+ 3 \left( \bar{\psi}_{[\nu} \Gamma_{\rho \mu \lambda} \psi_{\nu]} - \frac{1}{3} e_{\nu}^{\lambda} \bar{\psi}_{\lambda} \Gamma_{\rho \mu} \psi_{\nu} \right) + O(\psi^2 H) + O(\psi^4) ,

$$

(A.29a)

$$
\mathcal{E}(\bar{\psi})^{\mu} := \frac{\kappa_d^2}{e} \frac{\delta S_d}{\delta \psi_{\mu}} = 2 \Gamma_{\mu \rho \nu} \bar{\psi}_{\rho \nu} ,

$$

(A.29b)

$$
\mathcal{E}(A)^{\sigma_1 \cdots \sigma_{n-1}} := \frac{\kappa_d^2}{e} \frac{\delta S_d}{\delta A_{\sigma_1 \cdots \sigma_{n-1}}} = \frac{1}{e} \partial_{\kappa} \left( e H^{\kappa \sigma_1 \cdots \sigma_{n-1}} \right) + 4 D_{\kappa} \left[ \bar{\psi}_{\rho} \Gamma_{\rho \mu \lambda} T_{\nu}^{\kappa \mu \rho} \psi_{\lambda} \right]

- \frac{1}{(n!)^2} e^{\sigma_1 \cdots \sigma_{n-1} 1 \cdots \lambda_8} H_{\lambda_1 \cdots \lambda_4} H_{\lambda_5 \cdots \lambda_8} ,

$$

(A.29c)

$$
\mathcal{E}(\omega)_{mn}^{\nu} := \frac{\kappa_d^2}{e} \frac{\delta S_d}{\delta \omega_{\nu}^{mn}} = \frac{1}{2} \left( \frac{1}{2} T_{mn}^{\nu} - T_{m \lambda} \lambda^{\nu} \right) - \bar{\psi}_{\lambda} \Gamma_{\lambda} \psi_{[m} e_{n]}^{\nu} - \frac{1}{2} \bar{\psi}_{m} \Gamma^{\nu} \psi_{n} .

$$

(A.29d)

Here, $n$ is the field strength form degree. In the equation of motion of the spin connection, the $\Gamma^{[5]}$ part cancels between the variation of the gravitino kinetic term and the variation of the extra part $S_T$ (see (A.22)) that was added to the action.

At this point one can make contact between the first-order results and the well-known second order formulations. Using the equation of motion for the spin connection one determines the on-shell value of the torsion,

$$
T_{\mu \nu}^{\lambda} = 2 \bar{\psi}_{[\mu} \Gamma^{\lambda} \psi_{\nu]} .

$$

(A.30)
The transformation rule of the spin connection in the second-order formalism is obtained by using the equation of motion for the gravitino and inserting that in the first-order transformation rule.

The Ricci tensor and scalar can now be expressed as

\[
R = \frac{4}{d-2} \left[ -\mathcal{E}(e)_{\lambda} - \frac{1}{2n!} (d-2n) H^2 - (d-3) \bar{\psi}_\mu \Gamma^{\mu\rho\nu} \psi_{\nu\rho} \right] + \mathcal{O}(\psi^2 H) + \mathcal{O}(\psi^4), \quad (A.31)
\]

\[
R_{\mu} = 2 \mathcal{E}(e)_{\mu} - \frac{2}{d-2} \mathcal{E}(e)_{\lambda} e_{\mu} + \frac{2}{(n-1)!} (H^2)_{\mu} - \frac{n-1}{d-2} \frac{2}{n!} H^2 e_{\mu}
\]

\[- \frac{d-1}{d-2} \bar{\psi}_{\lambda} \Gamma^{\lambda\mu\nu} \psi_{\nu\rho} e_{\mu} + 6 \bar{\psi}_{\mid\Gamma^{\mu\rho\nu} \psi_{\nu\rho]} + \mathcal{O}(\psi^2 H) + \mathcal{O}(\psi^4). \quad (A.32)
\]

The gravitino equation of motion will also often appear in a slightly different form, obtained by multiplying the field equation with a non-singular operator,

\[
\bar{\mathcal{E}}(\bar{\psi})_{\lambda} \equiv \frac{1}{2(d-2)} \left( \Gamma_{\lambda\mu} - (d-3) \eta_{\lambda\mu} \right) \mathcal{E}(\tilde{\psi})_{\mu} = 2 \Gamma^\rho \tilde{\psi}_{\rho\lambda}, \quad (A.33)
\]

where the operator can be shown to satisfy \( (\Gamma_{\lambda\mu} - (d-3) \eta_{\lambda\mu}) (\Gamma^{\mu\nu} - \eta^{\mu\nu}) = 2(d-2) \eta_{\lambda\nu}. \)

We will encounter this singly contracted gravitino curvature several times in the variation of the higher-order corrections to the action.

We also need a few more identities to get rid of covariant derivatives on gravitino curvatures. In particular, using

\[
D_{[\lambda} \psi_{\nu\lambda]} = \frac{1}{8} \Gamma_{mn} \psi_{[\mu \Gamma_{\nu} \lambda]}^{mn}, \quad (A.34)
\]

one derives that

\[
\bar{D} \psi_{\nu\lambda} = D_{[\lambda} \bar{\mathcal{E}}(\bar{\psi})_{\mid\lambda] + \frac{1}{2} \Gamma_{s} \psi_{[\nu R_{\lambda]s}^{s} + \frac{1}{8} \Gamma_{m}^{rs} \psi_{m R_{\nu\lambda]s} + \mathcal{O}(\psi^3) + \mathcal{O}(T \psi). \quad (A.35)
\]

Further multiplication with a single gamma matrix can be used to derive

\[
D^\nu \psi_{\lambda\nu} = -\frac{1}{2} \bar{D} \bar{\mathcal{E}}(\bar{\psi})_{\lambda} + \frac{1}{4} D_{\lambda} (\Gamma \cdot \bar{\mathcal{E}}(\bar{\psi})) - \frac{1}{8} R^\nu_{\lambda rs} \Gamma^{rs} \psi_{\nu} + \frac{1}{4} R_{\lambda} (\Gamma^{rs} \psi_{s} - \psi^r) + \frac{1}{8} R \psi_{\lambda}. \quad (A.36)
\]

The equations in the present section can be used in the ten-dimensional theory as well, as long as one restricts to the sector of the theory which is independent of the gauge field, dilaton and dilatino.

## B Conventions, notation and some \( \Gamma \) algebra

### B.1 Indices and signs

We use a ‘mostly plus’ metric \( \eta = (-, +, +, \ldots, +) \). Our index conventions are as exhibited in the following table:

| spinor | vector | super |
|--------|--------|-------|
| curved | \( \alpha, \beta, \ldots \) | \( \mu, \nu, \ldots \) |
| flat   | \( a, b, \ldots \)       | \( r, s, \ldots \) |
|        | \( M, N, \ldots \)       | \( A, B, \ldots \) |
We denote the charge conjugation matrix by $C$ and always write it explicitly wherever it appears.

The sign conventions we use for the Riemann tensor and its contractions are

\[ R(\omega)_{\mu\nu mn} = \partial_\mu \omega_{\nu mn} + \omega_{\mu np} \omega_{\nu pq} (\mu \leftrightarrow \nu) , \quad (B.1) \]

\[ R(\omega)_{\mu\nu} = R_{\mu\lambda \nu} , \quad R(\omega) = R_{\mu}^\mu . \quad (B.2) \]

The covariant derivative associated to $\omega$ acts on Lorentz vector and spinor indices according to

\[ D(\omega)_\mu V^m = \partial_\mu V^m + \omega_{\mu m} V^n , \quad (B.3) \]

\[ D(\omega)_\mu \epsilon = \partial_\mu \epsilon + \frac{1}{4} \omega_{\mu rs} \Gamma_{rs} \epsilon . \quad (B.4) \]

With these conventions one obtains

\[ [D_\mu , D_\nu] \epsilon = \frac{1}{4} R_{\mu\nu rs} \Gamma_{rs} \epsilon . \quad (B.5) \]

The decomposition of the Riemann tensor in terms of the Weyl tensor, the Ricci tensor (or the irreducible traceless Ricci tensor $S_{mn} = R_{mn} - d^{-1} \delta_{mn} R$) and the Ricci scalar is given by

\[ R_{mn}^{pq} = W_{mn}^{pq} - \frac{4}{d-2} \delta_{[m}^{[p} R_{n]q]} + \frac{2}{(d-1)(d-2)} R \delta_{m}^{[p} \delta_{n]}^{q]} \]

\[ = W_{mn}^{pq} - \frac{4}{d-2} \delta_{[m}^{[p} S_{n]q]} - \frac{2}{d(d-1)} R \delta_{m}^{[p} \delta_{n]}^{q]} . \quad (B.6) \]

From the former expansion the following operator that projects onto the Weyl part of the Riemann tensor is readily derived:

\[ P_{m_1 m_2 m_3 m_4}^{n_1 n_2 n_3 n_4} = \delta_{m_1 m_2}^{n_1 n_2} \delta_{m_3 m_4}^{n_3 n_4} + \frac{4}{d-2} \delta_{m_1 m_2}^{n_1 n_2} \delta_{m_3 m_4}^{n_3 n_4} + \frac{2}{(d-1)(d-2)} \delta_{m_1 m_2}^{n_3 m_4} \delta_{m_3 m_4}^{n_1 n_2} . \quad (B.7) \]

Here we used the notation $\delta_{p_1 \ldots p_k}^{q_1 \ldots q_k} := \delta_{p_1}^{[q_1} \ldots \delta_{p_k]}^{q_k}$ with antisymmetrisation with unit weight ($\delta_{p}^{q}$ should be read as $\eta_{pq}$ where appropriate). As appropriate, the operator (B.7) obeys the property $P^2 = P$, is traceless on the $m$-indices and satisfies

\[ P_{m_1 m_2 m_3 m_4}^{n_1 n_2 n_3 n_4} R_{n_1 n_2 n_3 n_4} = W_{m_1 m_2 m_3 m_4} . \quad (B.8) \]

### B.2 Riemann tensor polynomials and the $t_8$ tensor

For reference, we present in this section the expressions for the various higher-derivative invariants in terms of the tensor polynomial normal forms as analysed by Fulling et al. [77]. Our choice of basis has a maximal number of invariants with no mixed-type index contractions (double indices are summed over using a metric and traces are on Lorentz indices):

\[ R_{41} = \text{tr}(R_{\mu\nu} R_{\rho\sigma} R_{\sigma\mu} R_{\rho\nu}) = A_6 , \]

\[ R_{42} = \text{tr}(R_{\mu\nu} R_{\rho\sigma} R_{\sigma\nu} R_{\rho\mu}) , \]

\[ R_{43} = \text{tr}(R_{\mu\sigma} R_{\rho\nu} \text{tr}(R_{\mu\nu} R_{\rho\sigma})) = A_3 , \]

\[ R_{44} = \text{tr}(R_{\mu\sigma} R_{\rho\nu} \text{tr}(R_{\rho\sigma} R_{\mu\nu})) = A_1 , \]

\[ R_{45} = \text{tr}(R_{\mu\nu} R_{\rho\sigma}) \text{tr}(R_{\rho\nu} R_{\sigma\mu}) = A_2 , \]

\[ R_{46} = \text{tr}(R_{\mu\rho} R_{\sigma\nu}) \text{tr}(R_{\mu\sigma} R_{\rho\nu}) = A_4 , \]

\[ R_{47} = R_{mn}^{[mm} R_{pq}^{pp} R_{rs}^{rs} R_{tu}^{tu}] = \frac{1}{2 \cdot 8 !} Z . \quad (B.9) \]
The ‘$A_i$’ and ‘$Z$’ symbols are (up to an overall normalisation $2 \cdot 8!$ of the latter) the notation of de Roo et al. [15] and Fulling et al. [77], who do not employ $R_{42}$ but instead use

$$R^{pqrs} R_{pq} t_r u w R_{uvw} = A_5, \quad R^{pqrs} R_{pq} t_r R_{t s w} R_{uvw} = A_7. \quad (B.10)$$

The relation between their choice and ours is given by

$$A_7 - A_5 = R_{42} - \frac{1}{4} R_{46}, \quad (B.11)$$

which can be verified by repeated use of the cyclic Ricci identity.

In string theory it is more useful to employ the specific tensorial structures that appear in string-amplitude calculations. The $t_8$ tensor has four index pairs and is defined as

$$t_8^{m_1 m_2 n_1 n_2 p_1 p_2 q_1 q_2} = -2 \left( \delta^{m_1 n_2} \delta^{m_2 n_1} \delta^{p_1 q_2} \delta^{p_2 q_1} + \delta^{n_1 p_2} \delta^{n_2 p_1} \delta^{m_1 q_2} \delta^{m_2 q_1} + \delta^{m_1 p_2} \delta^{m_2 p_1} \delta^{q_1 q_2} \delta^{n_2 n_1} \right)$$

$$+ 8 \left( \delta^{m_1 q_2} \delta^{m_2 n_1} \delta^{n_2 p_1} \delta^{p_2 q_1} + \delta^{n_1 q_2} \delta^{m_2 p_1} \delta^{p_2 m_1} \delta^{q_2 n_1} + \delta^{m_1 n_2} \delta^{m_2 p_1} \delta^{p_2 q_1} \delta^{n_1 q_2} \right)$$

+ anti-symmetrisation of each index pair, with total weight one.

The following special case (for an anti-symmetric tensor $N$) is useful:

$$t_8^{r_1 \cdots r_8} M_{r_1 r_2} N_{r_3 r_4} \cdots N_{r_7 r_8} = -6 M_{t_1 t_2} N_{t_3 t_4} N_{m n} N_{m n} + 24 M_{t_1 t_2} N_{t_3 t_4} N_{mn} N_{nt_1}. \quad (B.13)$$

The invariants formed from the $t_8$ tensor and the usual $\varepsilon_{10}$ can be reduced to the seven fundamental invariants of (B.9) as follows:

$$X := t_8 t_8 R^4 = 192 R_{41} + 384 R_{42} + 24 R_{43} + 12 R_{44} - 192 R_{45} - 96 R_{46},$$

$$\frac{1}{8} Z := -\frac{1}{8} \varepsilon_{10} \varepsilon_{10} R^4 = 192 R_{41} + 384 R_{42} + 24 R_{43} + 12 R_{44} - 192 R_{45} + 96 R_{46}$$

$$- 768 A_7 + \text{Ricci terms}. \quad (B.14)$$

This leads to

$$X - \frac{1}{8} Z = -192 R_{46} + 768 A_7 + \text{Ricci terms}. \quad (B.15)$$

In addition, the two Yang-Mills-like invariants are

$$t_8 Y_2 := t_8 \text{tr} R^4 = 8 R_{41} + 16 R_{42} - 4 R_{45} - 2 R_{46} + \cdots,$$

$$t_8 Y_1 := t_8 (\text{tr} R^2)^2 = -4 R_{43} - 2 R_{44} + 16 R_{45} + 8 R_{46} + \cdots. \quad (B.16)$$

Moreover, there is the relation

$$t_8 X_8 = t_8 t_8 R^4 = 24 t_8 \text{tr} R^4 - 6 t_8 (\text{tr} R^2)^2. \quad (B.17)$$

We should stress that the $t_8 t_8$ tensor does not automatically project on the Weyl part of the Riemann tensor, as can easily be verified by e.g. computing $t_8 t_8 R^4$ taking only the Ricci parts in the expansion (B.6) to be non-zero. In a similar way, trace terms appear in the $t_{16}$ tensor of Green et al. [78]. So far, the presence of these trace terms has not played any role in the literature because they are proportional to the equations of motion when the gauge fields are ignored. For clarity we will always write Weyl tensors explicitly.

In this context, let us also mention that for the $N = 1$ case in ten dimensions a super-symmetric extension of $t_8 t_8 R^4$ written as an integral over the sixteen spinorial superspace.
coordinates has been given by Nilsson and Tollstén [79] and Kallosh [80]. Indeed, we have verified by an explicit covariant calculation that the trace-free version of (B.15) can be obtained as

\[ X - \frac{1}{8} Z |_{R \to W} = \frac{3}{2} \int d^{16} \theta (\theta C \Gamma^{mn\rho \sigma} \theta C \Gamma^{\rho \sigma} R_{mn})^4, \]  

(B.18)

The \( \theta \)-contracted Riemann tensor above—appearing at fourth order in the \( \theta \) expansion of the scalar superfield of \( N = 1 \) supergravity in ten dimensions, as formulated by Nilsson [81]—is manifestly free from trace terms. This is an immediate consequence of the group-theoretical fact that the fully anti-symmetrised product of four \( \text{SO}(1,9) \) spinor representations \( 16 \) contains the irrep \( 770 \) \((8)\), but neither \( 54 \) \((\overline{8})\) nor \( 1 \) \([79]\). Indeed, we have verified that the trace-free version of (B.15) can be obtained by an explicit covariant calculation that the trace-free version of (B.15) can be obtained as

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### B.3 Some useful \( \Gamma \)-matrix identities

The matrices \( \Gamma_\mu (\mu = 0 \cdots , d - 1) \) are taken to satisfy the Clifford algebra

\[ \{ \Gamma_\rho , \Gamma_\sigma \} = 2 \eta_{\rho \sigma} \]  

(B.19)

In \( d = 10 \), the matrix \( \Gamma^# := \Gamma^0 \Gamma^1 \cdots \Gamma^9 \) squares to the identity and anti-commutes with \( \Gamma^\mu \), and can therefore be used to define the Weyl projectors \( P_\pm = \frac{1}{2} (1 \pm \Gamma^#) \).

From (B.19) we can derive the commutators

\[
\begin{align*}
[\Gamma^r s , \Gamma^d] &= 4 \Gamma^{[r} \delta^{s]}_d , \\
[\Gamma^r s , \Gamma_{tu}] &= 8 \Gamma^{[r} [u \delta^{s]}_t ] , \\
[\Gamma^r s , \Gamma_{tuv}] &= 12 \Gamma^{[r} [u \delta^{s]}_t [v].
\end{align*}
\]

(B.20)

These, in turn, immediately lead to the following commutators of covariant derivatives and gamma matrices with curved indices:

\[
[D_\mu , \Gamma^{\nu_1 \cdots \nu_n}] = n \partial_\mu \epsilon_{r}^{[\nu_1} \Gamma^{\nu_2 \cdots \nu_n]} + \frac{1}{2} \delta_{\mu}^{r s} \Gamma_{r s} , \Gamma^{\nu_1 \cdots \nu_n} ,
\]

(B.21)

The following commutators and anti-commutators of higher products of gamma matrices are also useful:

\[
\begin{align*}
[\Gamma^{r_s \cdots r_5} , \Gamma_{s_1 s_2 s_3}] &= 2 \Gamma^{r_s \cdots r_5} \sigma_{s_1 s_2 s_3} - 240 \Gamma^{r_1 r_2 r_3} \delta_{s_1}^{r_4} \delta_{s_2}^{r_5} [s_3] , \\
\{ \Gamma^{r_s \cdots r_5} , \Gamma_{s_1 s_2 s_3} \} &= 30 \Gamma^{r_1 r_2 r_3} \delta_{s_1}^{s_2 s_3} - 120 \Gamma^{r_1 r_2} \delta_{s_1}^{r_4} \delta_{s_2}^{r_5} [s_3] .
\end{align*}
\]

(B.22)

Furthermore, we have the contraction identity

\[
\delta^p_q \Gamma^{r_1 \cdots r_n} \Gamma_{s_1 s_2 \cdots s_1} = \sum_{k=0}^{\min(m,n)} (d - m - n + k) \frac{m! n!}{k! (m - k)! (n - k)!} \Gamma^{r_1 \cdots r_{n-k}} \delta_{s_{m-k-s_1}}^{s_{m-k+s_1}} \delta_{s_{m-k+1+s_2}}^{s_{m-k+1-s_2}} .
\]

(B.23)
Finally, we have made frequent use of the famous fermi flip property
\[(\psi_1 C \Gamma^\mu_1 \cdots \mu_n \psi_2) = (-1)^{n(n+1)/2} (\psi_2 C \Gamma^\mu_1 \cdots \mu_n \psi_1),\] (B.24)
in this form valid for arbitrary spinors; for Majorana spinors we can of course write \(\psi^T C = \bar{\psi}\).

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