EXISTENCE AND ASYMPTOTIC BEHAVIORS OF TRAVELING WAVES OF A MODIFIED VECTOR-DISEASE MODEL

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ABSTRACT. In this paper, we are concerned with the existence and asymptotic behavior of traveling wave fronts in a modified vector-disease model. We establish the existence of traveling wave solutions for the modified vector-disease model without delay, then explore the existence of traveling fronts for the model with a special local delay convolution kernel by employing the geometric singular perturbation theory and the linear chain trick. Finally, we deal with the local stability of the steady states, the existence and asymptotic behaviors of traveling wave solutions for the model with the convolution kernel of a special non-local delay.

1. Introduction. In the past decades, one has seen that vector-borne diseases have become major public health problems throughout the world. The spatial spread of newly introduced diseases is a subject of continuing interest to both theoreticians and empiricists. As an important research topic, a great number of papers are devoted to constructing different models to investigate the spread of the vector-disease, see [6, 33, 35, 20, 29, 28, 3, 11, 19] and the references therein.

Let us briefly recall some recent works on disease models. Ruan-Xiao [29] presented a host-vector model for a disease without immunity. The model is called the diffusive and time-delayed integro-differential equation:

\[ \frac{\partial u}{\partial t}(x, t) = d\Delta u(x, t) - au(x, t) + b[1 - u(x, t)]v(x, t), \]  

(1.1)

where \( d \) is the diffusion constant, \( a \) is the cure rate of the infected host, \( b \) is the host-vector contact rate, \( \Delta \) is the Laplacian operator, \( u(x, t) \) represents the normalized spatial density of infectious host at time \( t \) in \( x \) and \( v(x, t) \) represents the normalized spatial density of susceptible host at time \( t \) in \( x \). The stability of the steady states was studied by using the contacting-convex sets technique [28], and the existence of traveling wave solutions was established by using the linear chain trick and
the geometric singular perturbation method \cite{11, 19}. Zhao-Xiao \cite{36} showed that the asymptotic speed of equation (1.1) coincides with its minimal wave speed for monotone traveling waves. Lv-Wang \cite{24} studied the existence, uniqueness and asymptotic behavior of traveling wave fronts for equation (1.1) with the weak generic delay kernel:

\[ v(x, t) = (f * u)(x, t) = \int_{-\infty}^{t} \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi d(t - s)}} e^{-\frac{(x-y)^2}{4(t-s)}} e^{-\frac{t-s}{\tau}} u(y, s)dyds. \]

They explored the existence of traveling wave fronts by the geometric singular perturbation theory and obtained the asymptotic behavior and uniqueness of traveling wave fronts by using the standard asymptotic theory and the sliding method.

Wang-Li-Ruan \cite{31} considered the reaction-diffusion equation:

\[ \frac{\partial u}{\partial t}(x, t) = d\Delta u(x, t) + g(u(x, t), (h * S(u))(x, t)), \quad (1.2) \]

where \( h * S(u) \) denotes a convolution with respect to both spatial and temporal variables. They dealt with the existence of traveling wave fronts of equation (1.2) by the monotone iterations and a nonstandard ordering for the set of profiles of the corresponding wave system. They investigated the stability of traveling wave fronts, and showed that traveling wave fronts are unique up to translations. Li-Wang-Wu \cite{23} proved the existence of entire solutions which behave as two traveling wave solutions of equation (1.2) coming from both directions and established the existence of a heteroclinic orbit by using the monotone dynamical system theory. Li-Liu-Wang \cite{22} discussed entire solutions and the interaction of traveling wave fronts of the bistable reaction-advection-diffusion equation with infinite cylinders.

As we know, there exist quite a few powerful methods in studying the existence of traveling wave solutions such as the squeezing technique \cite{2, 26}, the Conley index theory \cite{14}, the degree theory \cite{13, 7}, the shooting method \cite{9}, the phase-plane technique \cite{30} and the monotone iteration technique \cite{32} etc. In particular, the geometric singular perturbation theory \cite{11} has received a great deal of interest and has been widely used in several scientific fields \cite{29, 1, 26, 27}. Li-Zhu \cite{21} studied limit cycles bifurcating from the limit periodic sets of the predator-prey systems with the response functions of Holling types, as well as the multiplicity of such limit cycles by applying the geometric singular perturbation theory developed by Dumortier-Roussarie etc \cite{8, 5}.

Another useful approach is to use the cross iteration method as well as the Schauder’s fixed point theorem to prove the existence of traveling wave solution
connecting two steady states by constructing a pair of upper-lower solutions [12, 25]. For example, Gan et al. [12] considered the infectious disease model

\[
\begin{align*}
\frac{\partial S(x, t)}{\partial t} &= D_S \frac{\partial^2 S(x, t)}{\partial x^2} + \Lambda - \mu S(x, t) - r I(x, t) S(x, t), \\
\frac{\partial I(x, t)}{\partial t} &= D_I \frac{\partial^2 I(x, t)}{\partial x^2} - \beta I(x, t) + \varepsilon \int_{-\infty}^{\infty} r I(y, t - \tau) S(y, t - \tau) f_{\alpha}(x - y) dy,
\end{align*}
\]

for \(t > 0\) and \(x \in \mathbb{R}\), where \(\varepsilon = e^{-\mu \tau}\) and \(f_{\alpha} = \frac{1}{\sqrt{4\pi \alpha}} e^{-\frac{x^2}{4\alpha}}\). Here \(S\) denotes the susceptible class, \(I\) denotes the infectious class, \(\mu\) is the natural death rate, \(r\) is the infection rate, \(\Lambda\) is the recruitment of the susceptible individuals, and \(\tau\) represents the latency length of the disease.

Motivated by the ideas and results described in [11, 19, 24, 21, 8, 5, 12] etc., the purpose of this study is to establish the existence and asymptotic behaviors of traveling wave solutions for the following equation:

\[
\frac{\partial u}{\partial t}(x,t) = \Delta u(x,t) - au(x,t) + b[1-u(x,t)](f * u)(x,t) + d[1-u(x,t)]u(x,t),
\]

where \(a\) is the cure rate of the infected host, \(b\) is the host-vector contact rate, \(d\) is the susceptible-infected host contact rate, \(\Delta\) is the Laplacian operator, and \(u(x,t)\) represents the normalized spatial density of infectious host at time \(t\) in \(x\). In the whole context, we consider three cases of \((f * u)(x,t)\):

(i) without delay:

\[(f * u)(x,t) = u(x,t);\]

(ii) with the local delay:

\[(f * u)(x,t) = \int_{-\infty}^{t} \frac{t - s}{\tau^2} e^{-\frac{(t-s)^2}{\tau^2}} u(x,s) ds;\]

(iii) and with the non-local delay:

\[(f * u)(x,t) = \int_{-\infty}^{t} \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi(t-s)}} e^{-\frac{(x-y)^2}{4(t-s)}} \frac{1}{\tau} e^{-\frac{(t-s)}{\tau}} u(y,s) dy ds.\]

The rest of this paper is organized as follows. In Section 2, we briefly introduce the geometric singular perturbation theory and present the existence of traveling wave solutions for the modified vector-disease model (1.4) without delay. In Section 3, we explore the existence of traveling wave fronts for equation (1.4) with the convolution kernel of a special local delay by employing the geometric singular perturbation theory and the linear chain trick. Section 4 is dedicated to traveling wave solutions of equation (1.4) with the convolution kernel of a special non-local delay. Local stability of the steady states is discussed through analyzing the corresponding characteristic equations. The existence of traveling wave solution connecting two steady states is proved by two methods: the geometric singular perturbation theory and the method of upper-lower solutions. In Section 5, the asymptotic behavior of traveling wave fronts is established by employing the qualitative theory of differential equations.

2. Preliminaries. In this section, let us introduce some basic results on invariant manifolds [11, 19].
Lemma 2.1 (Geometric Singular Perturbation Theorem). For the system
\[
\begin{aligned}
x'(t) &= f(x, y, \epsilon), \\
y'(t) &= e g(x, y, \epsilon), \\
\end{aligned}
\tag{2.5}
\]
where \(x \in \mathbb{R}^n\), \(y \in \mathbb{R}^l\) and \(\epsilon\) is a real parameter. Here \(f\) and \(g\) are \(C^\infty\) on the set \(V \times I\), where \(V \in \mathbb{R}^{n+l}\) and \(I\) is an open interval containing \(\epsilon = 0\). If when \(\epsilon = 0\), the system has a compact, normally hyperbolic manifold of the critical point \(M_0\), which is contained in the set \(\{f(x, y, 0) = 0\}\), then for any \(0 < r < +\infty\) and the sufficiently small \(\epsilon > 0\), there exists a manifold \(M_\epsilon\) such that
(I) it is locally invariant under the flow of system (2.5);
(II) it is \(C^r\) in \(x, y\) and \(\epsilon\);
(III) \(M_\epsilon = \{(x, y) : x = h^r(y)\}\) for the function \(h^r(y) \in C^r\) and \(y\) in some compact \(K\);
(IV) there exist locally invariant stable and unstable manifolds \(W^s(M_\epsilon)\) and \(W^u(M_\epsilon)\) that lie within \(O(\epsilon)\), and are diffeomorphic to \(W^s(M_0)\) and \(W^u(M_0)\), respectively.

Consider equation (1.4) with \((f * u)(x, t) = u(x, t)\), namely
\[
\frac{\partial u}{\partial t}(x, t) = \Delta u(x, t) - au(x, t) + b[1 - u(x, t)]|u(x, t) + d[1 - u(x, t)]|u(x, t).
\tag{2.6}
\]
Substituting \(u(x, t) = \phi(z)\) and \(z = x + ct\) \((c > 0)\) into equation (2.6), we get a second order ordinary differential equation
\[
\phi'' - c\phi' - a\phi + (b + d)(1 - \phi)\phi = 0,
\]
which is equivalent to
\[
\begin{aligned}
\phi' &= \psi, \\
\psi' &= c\psi + a\phi + (b + d)(\phi - 1)\phi,
\end{aligned}
\tag{2.7}
\]
where the prime denotes differentiation with respect to \(z\). System (2.7) has two equilibria \(E_1(0, 0)\) and \(E_2\left(1 - \frac{a}{b+d}, 0\right)\) when \(b + d \neq 0\). We have the following result regarding a traveling front solution of system (2.7).

Theorem 2.2. For system (2.7), assume that \(b + d > a\) and \(c \geq 2\sqrt{b + d - a}\). In the \((\phi, \psi)\) phase plane, there is a heteroclinic orbit connecting the two equilibria \(E_1\) and \(E_2\). The heteroclinic connection is confined to \(\psi > 0\) and the traveling wave \(\phi(z)\) is strictly monotonically increasing.

Proof. When \(c \geq 2\sqrt{b + d - a}\), we observe that \(E_1\) is a node and \(E_2\) is a saddle. To establish the existence of a heteroclinic orbit connecting the two equilibria for \(\psi > 0\), we show that, for a proper value of \(\lambda > 0\), the triangular set
\[
\Omega = \left\{(\phi, \psi) : 0 \leq \phi \leq 1 - \frac{a}{b + d}, \ 0 \leq \psi < \lambda \phi \right\}
\]
is negative invariant. Let \(f\) be the vector defined by the right-hand sides of system (2.7) and \(n\) be the inward normal vector on the boundary of \(\Omega\). When \(\psi = \lambda \phi\) and \(0 < \phi \leq 1 - \frac{a}{b + d}\), it gives
\[
f \cdot n = \left(\frac{\psi}{c\psi + a\phi + (b + d)(\phi - 1)\phi}\right) \cdot (\lambda, -1) |_{(\phi, \lambda \phi)}
= \lambda^2 \phi - c\lambda \phi - (b + d)\phi^2 + (b + d - a)\phi
\leq \phi[\lambda^2 - c\lambda + (b + d - a)].
\]
Apparently, $\lambda^2 - c\lambda + (b + d - a) = 0$ has two real positive roots with $0 < \lambda_1 \leq \lambda_2$ when $c \geq 2\sqrt{b + d - a}$. This implies that $f \cdot \nabla f \leq \phi|\lambda^2 - c\lambda + (b + d - a)| \leq 0$ when $0 < \lambda_1 \leq \lambda \leq \lambda_2$. Thus, one branch of the unstable manifold of $E_2$ enters the region $\Omega$ and joins $E_1$ to form a heteroclinic orbit.

3. **Existence of traveling wave fronts with the local delay.** In this section, we consider equation (1.4) with the strong local delay kernel by using the geometric singular perturbation theory and the linear chain trick. Namely, the convolution $f * u$ is defined by

$$ (f * u)(x, t) = \int_{-\infty}^{t} f(t - s)u(x, s)ds, $$

where the kernel $f : [0, +\infty) \to [0, +\infty)$ satisfies the normalization assumption:

$$ f(t) \geq 0 \text{ for all } t \geq 0 \text{ and } \int_{0}^{\infty} f(t)dt = 1. $$

It is notable that the normalization assumption on $f$ ensures that the uniform non-negative steady-state solutions: 0 and $1 - \frac{a}{b + d}$, are unaffected by the delay.

The kernel $f(t) = \frac{1}{\tau}e^{-\frac{t}{\tau}}$ and $f(t) = \frac{t}{\tau^2}e^{-\frac{t}{\tau^2}}$ are frequently used in the literature on delay differential equations. The first one is called the weak generic delay kernel and the second one is called the strong general delay kernel.

**Theorem 3.1.** Assume that $b + d > a$ and $c \geq 2\sqrt{b + d - a}$. Then for any sufficiently small $\tau > 0$, equation (1.4) with the strong local delay kernel has a traveling wave solution $u(x, t) = \phi(x + ct)$ connecting two equilibria $(0, 0)$ and $\left(1 - \frac{a}{b + d}, 0\right)$.

**Proof.** Rewrite equation (1.4) with the strong local delay kernel

$$ f(t) = \frac{t}{\tau^2}e^{-\frac{t}{\tau^2}}, \quad \tau > 0. $$

It becomes

$$ \frac{\partial u}{\partial t}(x, t) = \Delta u(x, t) - au(x, t) + b[1 - u(x, t)] \int_{-\infty}^{t} \frac{t - s}{\tau^2}e^{-\frac{t - s}{\tau^2}}u(x, s)ds + d[1 - u(x, t)]u(x, t). \quad (3.8) $$

A traveling wave front is a solution of the form $u(x, t) = \phi(z)$ and $z = x + ct$ ($c > 0$). Then $\phi(z)$ is monotone increasing and satisfies

$$ \phi'' - c\phi' - a\phi + b(1 - \phi)(f \ast \phi) + d(1 - \phi)\phi = 0, \quad (3.9) $$

with

$$ \phi(-\infty) = 0, \quad \phi(\infty) = 1 - \frac{a}{b + d}, $$

where

$$ f \ast \phi = \int_{0}^{\infty} \frac{t}{\tau^2}e^{-\frac{t}{\tau^2}}\phi(z - ct)dt. \quad (3.10) $$

If we define $\eta = (f \ast \phi)(z)$, differentiating it with respect to $z$ gives

$$ \frac{d\eta}{dz} = \frac{1}{ct}(\xi - \eta), \quad (3.11) $$
where
\[ \xi(z) = \int_0^\infty \frac{1}{\tau} e^{-\frac{t}{\tau}} \phi(z - ct) \, dt. \] (3.12)

Differentiating both side of (3.12) with respect to \( z \) yields
\[ \frac{d\xi}{dz} = \frac{1}{c\tau} (\phi - \xi). \] (3.13)

Let \( \phi' = \psi \). From (3.11)-(3.13), the traveling wave equation (3.9) is equivalent to
\[
\begin{align*}
\phi_z &= \psi, \\
\psi_z &= c\psi + a\phi + b(\phi - 1)\phi + d(\phi - 1)\phi, \\
c\tau\eta_z &= \xi - \eta, \\
c\tau\xi_z &= \phi - \xi.
\end{align*}
\] (3.14)

Note that as \( \tau \to 0 \), it is easy to see that \( \eta \to \phi \). So we arrive at the non-delay version of the model
\[
\begin{align*}
\phi' &= \psi, \\
\psi' &= c\psi + a\phi + b(\phi - 1)\phi + d(\phi - 1)\phi.
\end{align*}
\] (3.15)

By virtue of Theorem 2.2, system (3.15) has a traveling wave solution connecting \( E_1 \) and \( E_2 \). When \( \tau > 0 \), system (3.14) has two equilibria in the \((\phi, \psi, \eta, \xi)\) phase space:
\[
(\phi, \psi, \eta, \xi) = (0, 0, 0, 0), \quad (\phi, \psi, \eta, \xi) = \left( 1 - \frac{a}{b + d}, 0, 1 - \frac{a}{b + d}, 1 - \frac{a}{b + d} \right).
\]

When \( \tau = 0 \), system (3.14) does not define a dynamical system in \( \mathbb{R}^4 \). This may be overcome by the transformation \( z = \tau s \), under which the system becomes
\[
\begin{align*}
\phi_s &= \tau \psi, \\
\psi_s &= \tau [c\psi + a\phi + b(\phi - 1)\phi + d(\phi - 1)\phi], \\
c\tau\eta_s &= \xi - \eta, \\
c\tau\xi_s &= \phi - \xi.
\end{align*}
\] (3.16)

We refer to (3.14) as the slow system and (3.16) as the fast system. The two systems are equivalent to each other when \( \tau > 0 \). If \( \tau \) is set to zero in system (3.14), the flow of that system is confined to the set
\[ M_0 = \{ (\phi, \psi, \eta, \xi) \in \mathbb{R}^4, \ \eta = \xi, \ \xi = \phi \}, \]
which is a two-dimensional invariant manifold for system (3.14) with \( \varepsilon = 0 \). In order to obtain a two-dimensional invariant manifold for the sufficiently small \( \varepsilon > 0 \) by the geometric singular theory, it suffices to verify the normal hyperbolicity of \( M_0 \). The linearized matrix of system (3.16) restricted to \( M_0 \) is
\[
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & -\frac{1}{c} & 1 & \frac{1}{c} \\
\frac{1}{c} & 0 & 0 & -\frac{1}{c}
\end{pmatrix}.
\]

A straightforward calculation indicates that the matrix has four eigenvalues: \( 0, 0, -\frac{1}{c}, -\frac{1}{c} \) and the manifold \( M_0 \) is normally hyperbolic following the method
used in [13]. By the geometric singular perturbation theory, there exists a sub-
n manifold \( M_\tau \) of the perturbed system (3.14) of \( \mathbb{R}^4 \) for the sufficiently small \( \tau > 0 \),
which can be written as
\[
M_\tau = \{(\phi, \psi, \eta, \xi) \in \mathbb{R}^4, \eta = \xi + g(\phi, \psi, \tau), \xi = \phi + h(\phi, \psi, \tau)\},
\]
where the functions \( g \) and \( h \) satisfy
\[
g(u, \tilde{u}, 0) = h(u, \tilde{u}, 0) = 0.
\]
Taylor expansions about \( \tau \) for \( g \) and \( h \) are
\[
g(\phi, \psi, \tau) = \tau g_1(\phi, \psi) + \tau^2 g_2(\phi, \psi) + \cdots,
\]
\[
h(\phi, \psi, \tau) = \tau h_1(\phi, \psi) + \tau^2 h_2(\phi, \psi) + \cdots. \tag{3.17}
\]
Substituting \( \eta = \xi + g(\phi, \psi, \tau) \) and \( \xi = \phi + h(\phi, \psi, \tau) \) into the slow system (3.14),
we have
\[
c_\tau \left\{ \left(1 + \frac{\partial h}{\partial \phi} + \frac{\partial g}{\partial \phi}\right) \psi + \left(\frac{\partial h}{\partial \psi} + \frac{\partial g}{\partial \psi}\right)[c\psi + a\phi + b(\phi - 1)(\phi + h + g) + d(\phi - 1)\phi] \right\} = -g,
\]
\[
c_\tau \left\{ \left(1 + \frac{\partial h}{\partial \phi}\right) \psi + \frac{\partial h}{\partial \psi}[c\psi + a\phi + b(\phi - 1)(\phi + h + g) + d(\phi - 1)\phi] \right\} = -h. \tag{3.18}
\]
Combining (3.17) with (3.18) and equating coefficients of \( \varepsilon \) and \( \varepsilon^2 \), we have
\[
g_1 = -c\psi,
\]
\[
g_2 = 2\varepsilon^2[c\psi + a\phi + (b + d)(\phi - 1)\phi],
\]
\[
h_1 = -c\psi,
\]
\[
h_2 = \varepsilon^2[c\psi + a\phi + (b + d)(\phi - 1)\phi]. \tag{3.19}
\]
The slow system (3.14) restricted to \( M_\tau \) is given by
\[
\begin{aligned}
\phi' &= \psi, \\
\psi' &= c\psi + a\phi + b(\phi - 1)(\phi + h + g) + d(\phi - 1)\phi,
\end{aligned} \tag{3.20}
\]
where \( g \) and \( h \) are given by (3.17) and (3.19). System (3.20) has two equilibria points \((\phi, \psi) = (0, 0)\) and \(\left(1 - \frac{a}{b+d}, 0\right)\) for any sufficiently small \( \varepsilon > 0 \), and reduces
to the corresponding non-local system (3.15) when \( \varepsilon = 0 \).

We now show that there exists a heteroclinic orbit connecting the two equilibria \((0, 0)\) and \(\left(1 - \frac{a}{b+d}, 0\right)\), so equation (1.4) has a corresponding traveling wave solution.

Rewrite system (3.20) as
\[
\begin{aligned}
\phi' &= \psi, \\
\psi' &= \Phi(\phi, \psi, c, \tau),
\end{aligned} \tag{3.21}
\]
where \( \tau = \varepsilon^2 \) and \( \Phi(\phi, \psi, c, \tau) = c\psi + a\phi + (b + d)(\phi - 1)\phi \). We know that when \( \tau = 0 \), the traveling wavefront of (3.21) exists. In the \((\phi, \psi)\) phase plane, it can be characterized as the graph of some function \( w \). Namely, when \( \tau = 0 \),
\[\psi = w(\phi, c).\]
By the stable manifold theorem, for the sufficiently small \( \tau > 0 \), we can characterize the stable manifold at \( (1 - \frac{a}{b + d}, 0) \) as the graph of some function

\[
\psi = w_1(\phi, c, \tau),
\]
where \( w_1(1 - \frac{a}{b + d}, c, \tau) = 0 \). Furthermore, based on the continuous dependence of solution trajectories on parameters, the manifold crosses the line \( \phi = \frac{1}{2} \left( 1 - \frac{a}{b + d} \right) \) somewhere provided \( \tau \) is sufficiently small.

Similarly, let \( \psi = w_2(\phi, c, \tau) \) be the equation for the unstable manifold at the origin with \( w_2(0, c, \tau) = 0 \). It must cross the line \( \phi = \frac{1}{2} \left( 1 - \frac{a}{b + d} \right) \) somewhere for the sufficiently small \( \tau \). Hence, it has \( w_1(\phi, c, 0) = w_2(\phi, c, 0) = w(\phi, c) \).

For the unperturbed problem, we fix a value of \( c = c^* \geq 2\sqrt{b + d - a} \), so that the equation of corresponding front in the phase plane is \( \psi = w(\phi, c^*) \).

In order to show that a heteroclinic connection exists in the perturbed problem \( (\tau > 0) \), we only need to prove that there exists a value of \( c = c(\tau) \), near to \( c^* \), so that the manifolds \( w_1 \) and \( w_2 \) cross the line \( \phi = \frac{1}{2} \left( 1 - \frac{a}{b + d} \right) \) at a point. We need to use the implicit function theorem to prove that there exists a unique wave speed \( c = c(\tau) \). For this purpose, we construct the auxiliary function

\[
F(c, \tau) = w_1\left[ \frac{1}{2} \left( 1 - \frac{a}{b + d} \right), c, \tau \right] - w_2\left[ \frac{1}{2} \left( 1 - \frac{a}{b + d} \right), c, \tau \right].
\]

To verify \( \frac{\partial F}{\partial c} |_{(c^*, 0)} \neq 0 \), we start with systems (3.20) and (3.22):

\[
\frac{d\psi}{d\phi} = \frac{\psi'}{\phi'} = \frac{\Phi(\phi, \psi, c, \tau)}{\psi},
\]
and

\[
\frac{d}{d\phi} \left( \frac{\partial w_1(\phi, c^*, 0)}{\partial c} \right) = \frac{\partial}{\partial c} \left( \frac{dw_1(\phi, c, 0)}{d\phi} \right) \bigg|_{c = c^*} = \frac{\partial}{\partial c} \left( \frac{\Phi(\phi, w_1(\phi, c, 0), c, \tau)}{w_1(\phi, c, 0)} \right) \bigg|_{c = c^*} = \frac{\partial}{\partial c} \left( \frac{c + a + (b + d)(\phi - 1)}{w_1(\phi, c, 0)} \right) \bigg|_{c = c^*} = 1 - \frac{\phi(a + (b + d)(\phi - 1))}{w_1^2(\phi, c^*, 0)} \right]\frac{\partial w_1}{\partial c}(\phi, c^*, 0).
\]

Integrating this equation from \( \phi = \frac{1}{2} \left( 1 - \frac{a}{b + d} \right) \) to \( \phi = 1 - \frac{a}{b + d} \) with respect to \( \phi \), we have

\[
\frac{\partial w_1}{\partial c} \left[ \frac{1}{2} \left( 1 - \frac{a}{b + d} \right), c^*, 0 \right] = -\int_{\frac{1}{2} \left( 1 - \frac{a}{b + d} \right)}^{1 - \frac{a}{b + d}} \exp \left( \int_{1 - \frac{a}{b + d}}^{z} \frac{as + (b + d)(s - 1)s}{w^2(s, c^*)} ds \right) dz.
\]

(3.24)
Similarly, we have
\[
\frac{\partial w_2}{\partial c} \left[ \frac{1}{2} \left( 1 - \frac{a}{b + d} \right), c^*, 0 \right] = \int_0^{1 - \frac{a}{b + d}} \exp \left( \int_{1 - \frac{a}{b + d}}^z \frac{as + (b + d)(s - 1)s}{w^2(s, c^*)} ds \right) dz.
\]  
(3.25)

Using (3.24) and (3.25), we have
\[
\frac{\partial F}{\partial c}(c^*, 0) = \frac{\partial w_1}{\partial c} \left[ \frac{1}{2} \left( 1 - \frac{a}{b + d} \right), c^*, 0 \right] - \frac{\partial w_2}{\partial c} \left[ \frac{1}{2} \left( 1 - \frac{a}{b + d} \right), c^*, 0 \right]
\]
\[
= - \int_0^{1 - \frac{a}{b + d}} \exp \left( \int_{1 - \frac{a}{b + d}}^z \frac{as + (b + d)(s - 1)s}{w^2(s, c^*)} ds \right) dz < 0.
\]

This implies that equation (1.4) with the strong local delay kernel has a traveling wave solution \( u(x, t) = \phi(x + ct) \) connecting two equilibria \( (0, 0) \) and \( \left( 1 - \frac{a}{b + d}, 0 \right) \) for any sufficiently small \( \tau > 0 \).

4. **Local stability and asymptotic behavior.** In this section, we investigate the local stability of the steady states, the existence and the asymptotic behavior of traveling wave solutions for the model (1.4) with a special non-local delay convolution kernel. We discuss the local stability of the steady states to the modified vector-disease model by analyzing the corresponding characteristic equations. Then we use two methods to prove the existence of a traveling wave solution connecting two steady states: the geometric singular perturbation theory and the method of upper-lower solutions. Finally, asymptotic behaviors of traveling wave fronts are presented by using the qualitative theory of differential equations.

Define the convolution \( f \ast u \) as
\[
(f \ast u)(x, t) = \int_{-\infty}^t \int_{-\infty}^\infty f(x - y, t - s)u(y, s)dyds,
\]
where the kernel \( f : [0, +\infty) \to [0, +\infty) \) satisfies the following normalization assumption
\[
f(t) \geq 0 \text{ for all } t \geq 0 \text{ and } \int_{-\infty}^t \int_{-\infty}^\infty f(x, t)dxdt = 1.
\]

We observe that equations of various types can be derived from equation (1.4) by taking different delay kernels. For example, when we take the kernel to be \( f(x, t) = \delta(x)\delta(t) \), where \( \delta \) denotes the Dirac’s delta function, equation (1.4) becomes the corresponding undelayed perturbed equation
\[
\frac{\partial u}{\partial t}(x, t) = \Delta u(x, t) - au(x, t) + b[1 - u(x, t)]u(x, t) + d[1 - u(x, t)]u(x, t).
\]

While taking \( f(x, t) = \delta(x)\delta(t - \tau) \), one can see that equation (1.4) becomes the following equation with discrete delay
\[
\frac{\partial u}{\partial t}(x, t) = \Delta u(x, t) - au(x, t) + b[1 - u(x, t)]u(x, t - \tau) + d[1 - u(x, t)]u(x, t).
\]
We consider the two special kernels
\[
\begin{aligned}
f(x, t) &= \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} - \frac{1}{\tau} e^{-\frac{t}{\tau}}, \\
f(x, t) &= \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} - \frac{t}{\tau^2} e^{-\frac{t}{\tau}},
\end{aligned}
\]
where \(\tau > 0\) in each case. The first one is called the weak generic delay kernel and the second one is called the strong general delay kernel. The two kernels have been frequently used in the literature, for example, see [1, 17].

In this section, we consider equation (1.4) with the weak generic delay kernel. Define
\[
v(x, t) = (f \ast u)(x, t) = \int_{-\infty}^{t} \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi(t-s)}} e^{-\frac{(x-y)^2}{4(t-s)}} e^{-\frac{s}{\tau}} u(y, s) dy ds.
\]
A straightforward calculation gives
\[
v_t = v_{xx} + \frac{1}{\tau}(u - v).
\]
So equation (1.4) can be written as
\[
\begin{aligned}
u_t &= u_{xx} - au + b(1-u)v + d(1-u)u, \\
v_t &= v_{xx} + \frac{1}{\tau}(u - v).
\end{aligned}
\tag{4.26}
\]

4.1. Local stability. In this subsection, we consider the local stability of steady states to system (4.26) by analyzing the corresponding characteristic equations. It is easy to see that this system has two equilibria:
\[
E_1(0, 0) \text{ and } E_3 \left(1 - \frac{a}{b+d}, 1 - \frac{a}{b+d}\right).
\]
If \(a < b + d\), then \(1 - \frac{a}{b+d} > 0\), and \(E_3 \left(1 - \frac{a}{b+d}, 1 - \frac{a}{b+d}\right)\) exists. If \(a > b + d\), then \(1 - \frac{a}{b+d} < 0\), and \(E_3 \left(1 - \frac{a}{b+d}, 1 - \frac{a}{b+d}\right)\) is not feasible.

**Theorem 4.1.** For system (4.26), we have
(1) If \(a < b + d\), \(E_1(0, 0)\) is unstable; if \(a > b + d\), \(E_1(0, 0)\) is asymptotically stable.
(2) If \(a < b + d\), \(E_3 \left(1 - \frac{a}{b+d}, 1 - \frac{a}{b+d}\right)\) is asymptotically stable.

**Proof.** For any constant equilibrium \((u^*, v^*)\), the linearized system of (4.26) at \((u^*, v^*)\) is
\[
\begin{aligned}
u_t &= u_{xx} - au + bv - bu^*v - buv^* + du - 2du^*u, \\
v_t &= v_{xx} + \frac{1}{\tau}(u - v).
\end{aligned}
\]

Following [16, 10], we deduce that the non-trivial solutions of the system has the form
\[
\begin{pmatrix}
u(x, t) \\
v(x, t)
\end{pmatrix} = \begin{pmatrix} c_1 \\
c_2
\end{pmatrix} e^{\lambda + i\sigma x},
\]
if and only if
\[
\begin{vmatrix}
\lambda + \sigma^2 + a + bv^* - d + 2du^* & -b + bu^* \\
-\frac{1}{\tau} & \lambda + \sigma^2 + \frac{1}{\tau}
\end{vmatrix} = 0.
\]
That is,

\[(\lambda + \sigma^2 + a + bu^* - d + 2du^*) \left(\lambda + \sigma^2 + \frac{1}{\tau}\right) + \frac{1}{\tau}(-b + bu^*) = 0.\] (4.27)

Let \((u^*, v^*) = (0, 0)\) in (4.27). It has

\[(\lambda + \sigma^2)^2 + \left(a - d + \frac{1}{\tau}\right)(\lambda + \sigma^2) + \frac{a - b - d}{\tau} = 0.
\]

Define

\[f_1(\lambda + \sigma^2) = (\lambda + \sigma^2)^2 + \left(a - d + \frac{1}{\tau}\right)(\lambda + \sigma^2) + \frac{a - b - d}{\tau}.
\]

It is easy to see that

\[\Delta_1 = \left(a - d + \frac{1}{\tau}\right)^2 - \frac{4(a - b - d)}{\tau} = \left(a - d - \frac{1}{\tau}\right)^2 \geq 0,
\]

and

\[f_1(0) = \frac{a - b - d}{\tau}.
\]

If \(a < b + d\), then \(f_1(0) = \frac{a - b - d}{\tau} < 0\). There exists a \(\sigma\) such that equation (4.27) has a positive real root \(\lambda\), so \(E_1(0, 0)\) is unstable. If \(a > b + d\), then \(f_1(0) > 0\) and the axis of symmetry is \(A_1 = -\frac{a - d + \frac{1}{\tau}}{2} < 0\). Note that \(f_1(\lambda + \sigma^2)\) has no positive real root, so \(E_1(0, 0)\) is asymptotically stable.

Let \((u^*, v^*) = \left(1 - \frac{a}{b + d}, 1 - \frac{a}{b + d}\right)\) in (4.27). Then it follows that

\[(\lambda + \sigma^2)^2 + \left[a - d + \frac{1}{\tau} + (2d + b) \left(1 - \frac{a}{b + d}\right)\right](\lambda + \sigma^2) + \frac{b + d - a}{\tau} = 0.
\]

Define

\[f_2(\lambda + \sigma^2) = (\lambda + \sigma^2)^2 + \left[a - d + \frac{1}{\tau} + (2d + b) \left(1 - \frac{a}{b + d}\right)\right](\lambda + \sigma^2) + \frac{b + d - a}{\tau}.
\]

A direct calculation gives

\[\Delta_2 = \left[a - d + \frac{1}{\tau} + (2 + b)(1 - \frac{a}{b + d})\right]^2 - \frac{4(b + d - a)}{\tau}
\]

\[= \left(b + d - a - \frac{1}{\tau}\right)^2 \geq 0,
\]

and

\[f_2(0) = \frac{b + d - a}{\tau}.
\]
If \( a < b + d \), then \( f_2(0) > 0 \) and the axis of symmetry is

\[
A_2 = -\frac{1}{2} + a - d + (2d + b)(1 - \frac{a}{b + d})
\]
\[
= -\frac{b + d + (a - d)(b + d)\tau + (2d + b)(b + d - a)\tau}{2\tau(b + d)}
\]
\[
= \frac{ad\tau - \tau(b + d)^2 - (b + d)}{2\tau(b + d)}
\]
\[
< \frac{(b + d)d\tau - \tau(b + d)^2 - (b + d)}{2\tau(b + d)}
\]
\[
= -\frac{b\tau - 1}{2\tau} < 0.
\] (4.29)

This implies that \( f_2(\lambda + \sigma^2) \) has no positive real root. Consequently, \( E_3 \left(1 - \frac{a}{b + d}, 1 - \frac{a}{b + d}\right) \) is asymptotically stable.

4.2. Existence of traveling wave solutions. In this subsection, we explore the existence of traveling wave solutions of system (4.26) by using the geometric singular perturbation theory, the Fredholm theory and the linear chain trick.

**Theorem 4.2.** Assume that \( b + d > a \). For any sufficiently small \( \tau > 0 \), there exists a speed \( c \geq 2\sqrt{b + d} - a \) such that system (4.26) has a traveling wave solution \( u(x, t) = \phi(z) \) and \( v(x, t) = \psi(z) \) connecting two equilibria \((0, 0)\) and \( (1 - \frac{a}{b + d}, 1 - \frac{a}{b + d})\).

**Proof.** Assume that system (4.26) has the traveling wave solution of the form

\[
\begin{align*}
\phi' &= \phi_1, \\
\phi'' + c\phi' - a\phi + b(1 - \phi)\psi + d(1 - \phi)\phi &= 0, \\
c\psi' - \psi'' - \frac{1}{\tau}(\phi - \psi) &= 0,
\end{align*}
\] (4.30)

where \( \frac{d}{dz} \), under the boundary value conditions:

\[
\begin{align*}
\lim_{z \to -\infty} (\phi(z), \psi(z)) &= (0, 0), \\
\lim_{z \to +\infty} (\phi(z), \psi(z)) &= \left(1 - \frac{a}{b + d}, 1 - \frac{a}{b + d}\right).
\end{align*}
\]

Define \( \phi' = \phi_1 \) and \( \psi' = \psi_1 \). System (4.30) can be formulated as

\[
\begin{align*}
\phi' &= \phi_1, \\
\phi'' + c\phi_1 + a\phi + b(\phi - 1)\psi + d(\phi - 1)\phi &= 0, \\
\psi' &= \psi_1, \\
\psi'' + \frac{1}{\tau}(\phi - \psi) &= 0.
\end{align*}
\] (4.31)

Let \( \varepsilon = \sqrt{\tau} \) and

\[
\begin{align*}
u_1 &= \phi, \\
u_2 &= \phi_1, \\
v_1 &= \psi, \\
v_2 &= \varepsilon\psi_1.
\end{align*}
\]
Then system (4.31) can be cast into the standard form of a singular perturbation problem:

\[
\begin{align*}
\epsilon u_1' &= u_2, \\
u_2' &= cu_2 + au_1 + b(u_1 - 1)v_1 + d(u_1 - 1)u_1, \\
\epsilon v_1' &= v_2, \\
\epsilon v_2' &= c\epsilon v_2 - (u_1 - v_1),
\end{align*}
\]  

which is called a slow system. When \( \epsilon = 0 \), the system reduces to

\[
\begin{align*}
\epsilon u_1' &= u_2, \\
u_2' &= cu_2 + au_1 + b(u_1 - 1)v_1 + d(u_1 - 1)u_1, \\
\end{align*}
\]

which has a heteroclitic orbit connecting two equilibria \((0, 0)\) and \((1 - \frac{a}{b + d}, 0)\) from Theorem 1. In the case of \( \epsilon = 0 \), we may use the transformation \( \xi = \epsilon \eta \) under which system (4.32) becomes

\[
\begin{align*}
\dot u_1 &= \epsilon u_2, \\
\dot u_2 &= \epsilon [cu_2 + au_1 + b(u_1 - 1)v_1 + d(u_1 - 1)u_1], \\
\dot v_1 &= v_2, \\
\dot v_2 &= c\epsilon v_2 - (u_1 - v_1),
\end{align*}
\]  

where \( \dot u_1 \) denotes differentiation with respect to \( \eta \). System (4.34) is called a fast system. We know that the slow system (4.32) and the fast system (4.34) are equivalent to each other when \( \epsilon > 0 \).

Let \( \epsilon \to 0 \) in system (4.32). Then the flow of system (4.32) is confined to the set

\[ M_0 = \{(u_1, u_2, v_1, v_2) \in \mathbb{R}^4, \ v_2 = 0, \ v_1 = u_1 \} , \]

which is a two-dimensional invariant manifold for system (4.32) with \( \epsilon = 0 \). In order to obtain a two-dimensional invariant manifold for the sufficiently small \( \epsilon > 0 \) by the geometric singular theory, it suffices to verify the normal hyperbolicity of \( M_0 \). The linearized matrix of (4.34) restricted to \( M_0 \) is

\[
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
-1 & 0 & 1 & 0 \\
\end{pmatrix},
\]

which has four eigenvalues as 0, 0, 1 and -1. The number of the eigenvalues with zero real part is equal to \( \text{dim} M_0 \) and the other eigenvalues are hyperbolic, so the manifold \( M_0 \) is normally hyperbolic. By the geometric singular perturbation theory, there exists a sub-manifold \( M_\epsilon \) of the perturbed system (4.32) of \( \mathbb{R}^4 \) for the sufficiently small \( \epsilon > 0 \), which can be expressed as

\[ M_\epsilon = \{(u_1, u_2, v_1, v_2) \in \mathbb{R}^4, \ v_1 = u_1 + f(u_1, u_2, \epsilon), \ v_2 = g(u_1, u_2, \epsilon) \} , \]

where \( f \) and \( g \) smoothly depend on \( \epsilon \) and satisfy

\[ f(u_1, u_2, 0) = g(u_1, u_2, 0) = 0. \]

Thus, the functions \( f \) and \( g \) can be expanded into the form of a Taylor series about \( \epsilon \):

\[
\begin{align*}
f(u_1, u_2, \epsilon) &= \epsilon f_1 + \epsilon^2 f_2 + \cdots, \\
g(u_1, u_2, \epsilon) &= \epsilon g_1 + \epsilon^2 g_2 + \cdots. 
\end{align*}
\]  

(4.35)
Substituting $v_1 = u_1 + f$ and $v_2 = g$ into the slow system (4.32) gives
\[\varepsilon \left[ \frac{\partial f}{\partial u_1} u_2 + \frac{\partial f}{\partial u_2} (cu_2 + au_1 + b(u_1 - 1)(u_1 + f) + d(u_1 - 1)u_1) + u_2 \right] = g,\]
\[\varepsilon \left[ \frac{\partial g}{\partial u_1} u_2 + \frac{\partial g}{\partial u_2} (cu_2 + au_1 + b(u_1 - 1)(u_1 + f) + d(u_1 - 1)u_1) \right] = \varepsilon^2 g + f.\] (4.36)

Combining (4.35) with (4.36) and equating coefficients of $\varepsilon$ and $\varepsilon^2$, respectively, yields
\[f_1 = 0, \quad f_2 = cu_2 + au_1 + b(u_1 - 1) + d(u_1 - 1)u_1,\]
\[g_1 = u_2, \quad g_2 = 0.\]

So we have
\[f = \varepsilon^2 (cu_2 + au_1 + b(u_1 - 1) + d(u_1 - 1)u_1 + o(\varepsilon^2)),\]
\[g = \varepsilon u_2 + o(\varepsilon^2).\] (4.37)

The slow system (4.33) restricted to $M_\varepsilon$ is given by
\[\begin{cases}
  u_1' = u_2, \\
  u_2' = cu_2 + au_1 + b(u_1 - 1)(u_1 + f) + d(u_1 - 1)u_1,
\end{cases}\] (4.38)
where $f$ is given by (4.37). Obviously, system (4.38) reduces to (4.33) when $\varepsilon = 0$. It is easy to see that system (4.38) has two equilibria $(u_1, u_2) = (0, 0)$ and $(1 - \frac{a}{b + d}, 0)$ for any sufficiently small $\varepsilon > 0$. Now we show that there exists a heteroclinic orbit connecting the two equilibria, then equation (1.4) has a traveling wave solution connecting $(0, 0)$ and $(1 - \frac{a}{b + d}, 1 - \frac{a}{b + d})$.

Set
\[u_1 = u_0 + \varepsilon^2 \phi + \cdots, \quad u_2 = \bar{u}_0 + \varepsilon^2 \psi + \cdots.\]

Substituting $u_1$ and $u_2$ into (4.38) and equating the coefficients of $\varepsilon^2$ yields
\[\frac{d}{dz} \begin{pmatrix} \phi \\ \psi \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ b + d - a - 2bu_0 - 2du_0 & -c \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} 0 \\ bu_0 F(u_0, \bar{u}_0) \end{pmatrix},\] (4.39)
where $F(u_0, \bar{u}_0) = \bar{c}u_0 + au_0 + b(u_0 - 1) + d(u_0 - 1)u_0$ and this system has a solution satisfying $\phi, \psi \to 0$ as $z \to \pm \infty$.

Let $L^2$ denote the space of square integral functions with the inner product
\[\int_{-\infty}^{\infty} (x(z), y(z)) dz,\]
where $(\cdot, \cdot)$ represents the Euclidean inner product on $\mathbb{R}^2$. From the Fredholm theory, we know that system (4.39) has a solution if and only if
\[\int_{-\infty}^{\infty} \left[ x(z), \begin{pmatrix} 0 \\ bu_0 F(u_0, \bar{u}_0) \end{pmatrix} \right] dz < +\infty,\]
for all functions $x(z) \in \mathbb{R}^2$ in the kernel of the adjoint of operator $l$ defined by the left-hand side of (4.39). One can verify that the adjoint operator $l^*$ is given by
\[l^* = -\frac{d}{dz} + \begin{pmatrix} 0 & b + d - a - 2bu_0 - 2du_0 \\ -1 & -c \end{pmatrix},\] (4.40)
and thus to compute $Ker l^*$ we have to find all functions $x(z)$ satisfying
\[\frac{dx(z)}{dz} = \begin{pmatrix} 0 & b + d - a - 2bu_0 - 2du_0 \\ -1 & -c \end{pmatrix} x(z).\] (4.41)
The general solutions are very difficult to find because the matrix is nonconstant. However, here we are only looking for solutions satisfying \( x(\pm \infty) = 0 \). Recall that \( u_0(z) \) is the solution of the unperturbed problem and it tends to zero as \( z \to -\infty \).

Let \( z \to -\infty \) in (4.41). The matrix becomes a constant matrix, with the eigenvalue \( \lambda \) satisfying 
\[
\lambda^2 + c\lambda + b + d - a = 0,
\]
and, for \( c \geq 2\sqrt{b+d-a} \), it has both eigenvalues being real and negative. Hence, as \( z \to -\infty \), the solution of (4.41) other than the zero solution must decrease exponentially for small \( z \). So the only solution satisfying \( x(\pm \infty) = 0 \) is the zero solution. Then, the Fredholm orthogonality condition trivially holds and so solutions of (4.40) exist. Thus, a heteroclinic connection exists between the two non-negative equilibrium points \((0,0)\) and \((1-x,0)\) of system (4.31). Consequently, system (4.26) has traveling wavefronts connecting the two equilibria \((0,0)\) and \((1-x,0)\).

4.3. Existence of traveling wave solutions. In this subsection, we explore the existence of traveling wave solutions to system (4.26) by applying the Schauder’s fixed point theorem, the cross iteration scheme and the method of upper-lower solutions.

Assume that 
\[
u(x,t) = \phi(x + ct), \quad v(x,t) = \psi(x + ct),
\]
and denote the traveling wave coordinate \( x + ct \) still by \( t \), where \( \phi, \psi \in C^2(\mathbb{R}, \mathbb{R}^2) \) and \( c > 0 \) is a constant. Then after substitution we have 
\[
\begin{aligned}
\phi''(t) - c\phi'(t) + f_{c1}(\phi_t, \psi_t) &= 0, \\
\psi''(t) - c\psi'(t) + f_{c2}(\phi_t, \psi_t) &= 0,
\end{aligned}
\]
(4.42)
where
\[
\begin{aligned}
f_{c1}(\phi_t, \psi_t) &= -a\phi + b(1 - \phi)\psi + d(1 - \phi)\phi, \\
f_{c2}(\phi_t, \psi_t) &= \frac{1}{\tau}(\phi - \psi),
\end{aligned}
\]
(4.43)
under the boundary value conditions:
\[
\begin{aligned}
\lim_{t \to -\infty} \phi(t) &= 0, & \lim_{t \to +\infty} \phi(t) &= k, \\
\lim_{t \to -\infty} \psi(t) &= 0, & \lim_{t \to +\infty} \psi(t) &= k,
\end{aligned}
\]
(4.44)
with 
\[
k = 1 - \frac{a}{b+d}.
\]

We introduce the concept of the desirable upper and lower solutions of system (4.42) as follows.

**Definition 4.3.** A pair of continuous functions \( \Phi = (\overline{\phi}, \overline{\psi}) \) and \( \underline{\Phi} = (\underline{\phi}, \underline{\psi}) \) is called a pair of upper-lower solutions of system (4.42) if \( \Phi \) and \( \underline{\Phi} \) are twice differentiable almost everywhere in \( \mathbb{R} \) and essentially bounded on \( \mathbb{R} \), and there hold
\[
\begin{aligned}
\overline{\phi}''(t) - c\overline{\phi}'(t) + f_{c1}(\overline{\phi}_t, \overline{\psi}_t) &\leq 0, & \text{a.e. in } \mathbb{R}, \\
\overline{\psi}''(t) - c\overline{\psi}'(t) + f_{c2}(\overline{\phi}_t, \overline{\psi}_t) &\leq 0, & \text{a.e. in } \mathbb{R}, \\
\underline{\phi}''(t) - c\underline{\phi}'(t) + f_{c1}(\underline{\phi}_t, \underline{\psi}_t) &\geq 0, & \text{a.e. in } \mathbb{R}, \\
\underline{\psi}''(t) - c\underline{\psi}'(t) + f_{c2}(\underline{\phi}_t, \underline{\psi}_t) &\geq 0, & \text{a.e. in } \mathbb{R},
\end{aligned}
\]
(4.45)
Lemma 4.5.

\[ \frac{\phi''(t) - c \phi'(t) + f_{c1}(\phi_t, \psi_t)}{\psi''(t) - c \psi'(t) + f_{c2}(\psi_t, \psi_t)} \geq 0, \quad \text{a.e. in } \mathbb{R}, \]

(4.46)

Unlike the standard upper and lower solutions defined in [32], \( f_{c2} \) is evaluated in a cross iteration scheme given in (4.45) and (4.46), respectively.

Define

\[ M_1 = \max_{t \in \mathbb{R}} \phi(t), \quad M_2 = \max_{t \in \mathbb{R}} \psi(t), \]

\[ H_1(\phi, \psi) = f_{c1}(\phi_t, \psi_t) + \rho_1 \phi(t), \quad H_2(\phi, \psi) = f_{c2}(\phi_t, \psi_t) + \rho_2 \psi(t), \]

and

\[ \rho_1 \geq a + bM_2 + d, \quad \rho_2 \geq \frac{1}{\tau}. \]

Then, for

\( (\phi, \psi) \in C_K(\mathbb{R}, \mathbb{R}^2) := \{ (\phi, \psi) \in C(\mathbb{R}, \mathbb{R}^2) : (0, 0) \leq (\phi, \psi) \leq (k_1, k_2) \} \),

we define

\[ F = (F_1, F_2) : C_K(\mathbb{R}, \mathbb{R}^2) \rightarrow C(\mathbb{R}, \mathbb{R}^2), \]

where

\[ F_1(\phi, \psi)(t) = \frac{1}{\lambda_2 - \lambda_1} \left[ \int_{-\infty}^{t} e^{\lambda_1(t-s)} H_1(\phi, \psi)(s) ds + \int_{t}^{\infty} e^{\lambda_2(t-s)} H_1(\phi, \psi)(s) ds \right], \]

\[ F_2(\phi, \psi)(t) = \frac{1}{\lambda_4 - \lambda_3} \left[ \int_{-\infty}^{t} e^{\lambda_3(t-s)} H_2(\phi, \psi)(s) ds + \int_{t}^{\infty} e^{\lambda_4(t-s)} H_2(\phi, \psi)(s) ds \right], \]

with

\[ \lambda_1 = \frac{c - \sqrt{c^2 + 4\rho_1}}{2}, \quad \lambda_2 = \frac{c + \sqrt{c^2 + 4\rho_1}}{2}, \]

\[ \lambda_3 = \frac{c - \sqrt{c^2 + 4\rho_2}}{2}, \quad \lambda_4 = \frac{c + \sqrt{c^2 + 4\rho_2}}{2}. \]

We look for traveling wave solutions to the system in the following profile set

\[ \Gamma((\phi, \psi), (\phi, \psi)) = \left\{ \begin{array}{ll} (i) & (\phi(t), \psi(t)) \in C(\mathbb{R}, \mathbb{R}^2); \\
(ii) & (\phi(t) \leq \phi(t) \leq \phi(t), \psi(t) \leq \psi(t) \leq \psi(t) \leq \psi(t) \leq \psi(t) \leq \psi(t) \leq \psi(t) \leq \psi(t) \leq \psi(t) \leq \psi(t) \). \end{array} \right. \]

Following Lemmas 3.1-3.6 in [18], we can obtain the following two lemmas immediately.

**Lemma 4.4.** \( F(\Gamma((\phi, \psi), (\phi, \psi))) \subset \Gamma((\phi, \psi), (\phi, \psi)) \), where \( F = (F_1, F_2) \).

**Lemma 4.5.** \( F : \Gamma((\phi, \psi), (\phi, \psi)) \rightarrow \Gamma((\phi, \psi), (\phi, \psi)) \) is compact.

From Lemmas 4.4 and 4.5, we see that the problem of the existence of traveling wave solutions of system (4.26) is equivalent to the existence of a pair of upper-lower solutions \((\phi, \psi)\) and \((\phi, \psi)\) of system (4.42) which satisfy the following conditions:

\( (H1) \) \( (0, 0) \leq (\phi(t), \psi(t)) \leq (\phi(t), \psi(t)) \leq (M_1, M_2), \quad t \in \mathbb{R}, \)

\( (H2) \) \( \lim_{t \to -\infty} (\phi(t), \psi(t)) = (0, 0) \) and \( \lim_{t \to +\infty} (\phi(t), \psi(t)) = (k, k). \)

For the sake of simplicity, we choose \( \epsilon_i > 0 \) \( (i = 0, 1, 2, 3) \) such that

\[ \epsilon_1 - M_1 > \epsilon_0, \quad \epsilon_2 - M_2 > \epsilon_0, \quad \epsilon_3 - 1 > \epsilon_0. \]

Let \( c > c^* = 2\sqrt{b\epsilon_1 + d - a} \). There exists \( 0 < \lambda_0 \leq c \) such that

\[ \lambda_0^2 - c\lambda_0 - a + b\epsilon_1 + d \leq 0. \]
Using the above constants, we define the continuous functions $\Phi(t) = (\overline{\phi}(t), \overline{\psi}(t))$ and $\Psi(t) = (\overline{\phi}(t), \overline{\psi}(t))$ as follows

$$\overline{\phi}(t) = \begin{cases} e^{\lambda_0 t}, & \text{if } t \leq t_1, \\ k + e^{-\lambda t}, & \text{if } t > t_1, \end{cases}$$

$$\overline{\psi}(t) = \begin{cases} e^{\lambda_0 t}, & \text{if } t \leq t_2, \\ k + \epsilon_2 e^{-\lambda t}, & \text{if } t > t_2, \end{cases}$$

and

$$\phi(t) = \begin{cases} 0, & \text{if } t \leq t_3, \\ k - \epsilon_3 e^{-\lambda t}, & \text{if } t > t_3, \end{cases}$$

$$\psi(t) = \begin{cases} 0, & \text{if } t \leq t_4, \\ k - e^{-\lambda t}, & \text{if } t > t_4, \end{cases}$$

where $t_i$ ($i = 0, 1, 2, 3, 4$) satisfy $t_3 \geq t_4 \geq t_2 \geq t_1 \geq 0$ and $\lambda > 0$ is a constant to be determined. One can verify that $\Phi(t) = (\overline{\phi}(t), \overline{\psi}(t))$ and $\Psi(t) = (\phi(t), \psi(t))$ satisfy $(H_1)$ and $(H_2)$. We now prove that the continuous functions $(\overline{\phi}(t), \overline{\psi}(t))$ and $(\phi(t), \psi(t))$ are the upper-lower solutions of system (4.26), respectively.

**Theorem 4.6.** $\Phi(t) = (\overline{\phi}(t), \overline{\psi}(t))$ is an upper solution of system (4.26).

**Proof.** If $t \leq t_1$, let $\overline{\phi}(t) = e^{\lambda_0 t}$ and $\overline{\psi}(t) = \epsilon_1 e^{\lambda_0 t}$. Then we have

$$\overline{\phi}'(t) - c \overline{\phi}(t) - a \overline{\phi}(t) + b(1 - \overline{\phi}(t)) \overline{\psi}(t) + d(1 - \overline{\phi}(t)) \overline{\phi}(t)$$

$$\leq \lambda_0^2 e^{\lambda_0 t} - c \lambda_0 e^{\lambda_0 t} - a e^{\lambda_0 t} + b \epsilon_1 e^{\lambda_0 t} + d e^{\lambda_0 t}$$

$$= e^{\lambda_0 t} \left( \lambda_0^2 - c \lambda_0 - a + b \epsilon_1 + d \right) \leq 0.$$

If $t > t_1$, let $\overline{\phi}(t) = k + e^{-\lambda t}$. Then we get

$$\overline{\phi}'(t) - c \overline{\phi}(t) - a \overline{\phi}(t) + b(1 - \overline{\phi}(t)) \overline{\psi}(t) + d(1 - \overline{\phi}(t)) \overline{\phi}(t)$$

$$\leq \lambda^2 e^{-\lambda t} + c \lambda e^{-\lambda t} - a e^{-\lambda t} - ak + (dM_1 + bM_2)(1 - k - e^{-\lambda t})$$

$$= e^{-\lambda t} \left( \lambda^2 + c \lambda - a - dM_1 - bM_2 + [(dM_1 + bM_2)(1 - k) - ak] e^{-\lambda t} \right).$$

Let

$$I_1(\lambda) = \lambda^2 + c \lambda - a - dM_1 - bM_2 + [(dM_1 + bM_2)(1 - k) - ak] e^{-\lambda t}.$$ 

Then $I_1(0) = -a - ak - k(dM_1 + bM_2) < 0$, so there exists a $\lambda_1 > 0$ such that $I_1(\lambda) < 0$ for all $\lambda \in (0, \lambda_1)$.

If $t \leq t_2$, let $\overline{\psi}(t) = \epsilon_1 e^{\lambda_0 t}$. One has

$$\overline{\psi}'(t) - c \overline{\psi}(t) + \frac{1}{\tau} \Phi(t) - \overline{\psi}(t)$$

$$\leq \epsilon_1 \lambda_0^2 e^{\lambda_0 t} - c \epsilon_1 \lambda_0 e^{\lambda_0 t} + \frac{M_1}{\tau} - \frac{\epsilon_1}{\tau} e^{\lambda_0 t}$$

$$= e^{\lambda_0 t} \left( \epsilon_1 \lambda_0^2 - c \epsilon_1 \lambda_0 - \frac{\epsilon_1}{\tau} + \frac{M_1}{\tau} e^{\lambda_0 t} \right).$$

Let

$$I_2(\lambda) = \epsilon_1 \lambda_0^2 - c \epsilon_1 \lambda_0 - \frac{\epsilon_1}{\tau} + \frac{M_1}{\tau} e^{\lambda_0 t}.$$ 

Then $I_2(0) = -\frac{\epsilon_1}{\tau} + \frac{M_1}{\tau} < 0$ for $\epsilon_1 - M_1 > \epsilon_0$. So there exists a $\lambda_2 > 0$ such that $I_2(\lambda) < 0$ for all $\lambda \in (0, \lambda_2)$.

If $t > t_2$, let $\phi(t) = k + e^{-\lambda t}$ and $\psi(t) = k + \epsilon_2 e^{-\lambda t}$. We have

$$\overline{\psi}'(t) - c \overline{\psi}(t) + \frac{1}{\tau} \Phi(t) - \overline{\psi}(t)$$

$$= \epsilon_2 \lambda_2 e^{-\lambda t} + c \epsilon_2 \lambda e^{-\lambda t} + \frac{1}{\tau} e^{-\lambda t}$$

$$= e^{-\lambda t} \left( \epsilon_2 \lambda_2^2 + c \epsilon_2 \lambda + \frac{1}{\tau} - \epsilon_2 \right).$$

Let

$$I_3(\lambda) = \epsilon_2 \lambda_2^2 + c \epsilon_2 \lambda + \frac{1}{\tau} - \epsilon_2.$$
Then $I_3(0) = \frac{1 - \epsilon_2}{\epsilon_2 - 1} < 0$ for $\epsilon_2 - 1 > \epsilon_0$. So there exists a $\lambda^*_3 > 0$ such that $I_3(\lambda) < 0$ for all $\lambda \in (0, \lambda^*_3)$. Choosing $\lambda \in (0, \min(\lambda^*_1, \lambda^*_2, \lambda^*_3))$, we see that $(\overline{\phi}(t), \overline{\psi}(t))$ satisfies the definition of the upper solution.

\textbf{Theorem 4.7.} \(\Psi(t) = (\overline{\phi}(t), \overline{\psi}(t))\) is a lower solution of system (4.26).

\textbf{Proof.} If $t \leq t_3$, where $t_3$ is the same as in the last page, let $\overline{\phi}(t) = 0$. Then we have

$$\overline{\phi}''(t) - c\overline{\phi}'(t) - a\overline{\phi}(t) + b(1 - \overline{\phi}(t))\overline{\psi}(t) + d(1 - \overline{\phi}(t))\overline{\phi}(t) = b\overline{\psi}(t) \geq 0.$$ 

If $t > t_3$, let $\overline{\phi}(t) = k - \epsilon_3 e^{-\lambda t}$ and $\overline{\psi}(t) = k - e^{-\lambda t}$. A direct calculation gives

$$\overline{\phi}''(t) - c\overline{\phi}'(t) - a\overline{\phi}(t) + b(1 - \overline{\phi}(t))\overline{\psi}(t) + d(1 - \overline{\phi}(t))\overline{\phi}(t)$$

$$= -\lambda^2 \epsilon_3 e^{-\lambda t} - c\epsilon_3 e^{-\lambda t} - a(k - \epsilon_3 e^{-\lambda t})(bk - be^{-\lambda t} + dk - de_3 e^{-\lambda t})(1 - k + e^{-\lambda t})$$

$$= e^{-\lambda t}[-\epsilon_3 \lambda^2 - c\epsilon_3 \lambda + a\epsilon_3(1 - k) - ak + k(b + d)(1 - k)]$$

Let

$$I_4(\lambda) = -\epsilon_3 \lambda^2 - c\epsilon_3 \lambda + \frac{ab(\epsilon_3 - 1)}{b + d}.$$ 

Then $I_4(0) = \frac{ab(\epsilon_3 - 1)}{b + d} > 0$ for $\epsilon_3 - 1 > \epsilon_0$. So there exists a $\lambda^*_4 > 0$ such that $I_4(\lambda) > 0$ for all $\lambda \in (0, \lambda^*_4)$.

If $t \leq t_4$, let $\overline{\psi}(t) = 0$. Then one can see that

$$\overline{\psi}''(t) - c\overline{\psi}'(t) + \frac{1}{\tau} (\overline{\phi}(t) - \overline{\psi}(t)) = \frac{1}{\tau} \overline{\phi}_2 \geq 0.$$ 

If $t > t_4$, let $\overline{\psi}(t) = k - e^{-\lambda t}$. Then we have

$$\overline{\psi}''(t) - c\overline{\psi}'(t) + \frac{1}{\tau} (\overline{\phi}(t) - \overline{\psi}(t)) \geq -\lambda^2 e^{-\lambda t} - c\lambda e^{-\lambda t} - \frac{1}{\tau}(k - e^{-\lambda t})$$

$$= e^{-\lambda t} \left(-\lambda^2 - c\lambda + \frac{1}{\tau} \right) - \frac{k}{\tau}$$

$$= e^{-\lambda t} \left(-\lambda^2 - c\lambda + \frac{1}{\tau} \right) - \frac{k}{\tau} e^{-\lambda t}.$$ 

Let

$$I_5(\lambda) = -\lambda^2 - c\lambda + \frac{1}{\tau} - \frac{k}{\tau} e^{-\lambda t}.$$ 

Then $I_5(0) = \frac{1 - k}{\tau} - \frac{a}{b + d} > 0$ and there exists a $\lambda^*_5 > 0$ such that $I_5(\lambda) > 0$ for all $\lambda \in (0, \lambda^*_5)$. Choosing $\lambda \in (0, \min(\lambda^*_1, \lambda^*_2, \lambda^*_3))$, one can see that $(\overline{\phi}(t), \overline{\psi}(t))$ satisfies the definition of the lower solution.

\textbf{Theorem 4.8.} Assume that $b + d > a$. Then system (4.26) has a traveling wave solution connecting $E_1(0, 0)$ and $E_3 \left(1 - \frac{a}{b + d}, 1 - \frac{a}{b + d}\right)$.

\textbf{Proof.} From Lemmas 4.4 and 4.5, Theorems 4.6 and 4.7, as well as the Schauder’s fixed point theorem, we know that there exists a fixed point $(\overline{\phi}^*(t), \overline{\psi}^*(t))$ of $F$ in $\Gamma((\overline{\phi}, \overline{\psi}), (\overline{\phi}, \overline{\psi}))$, which gives a solution of system (4.42).
Let 

\[ \text{Proof.} \]

(5.51) and (5.56), respectively.

\[ \text{Asymptotic behaviors.} \]

Assume that 

\[ \text{Theorem 5.1.} \]

where 

\[ \text{as} \]

\[ \text{lim} \]

\[ t \to -\infty \]

\[ \lim_{t \to -\infty} (\phi^*(t), \psi^*(t)) = (0, 0), \]

\[ \lim_{t \to +\infty} (\phi^*(t), \psi^*(t)) = (k, k). \]

Consequently, the fixed point \((\phi^*(t), \psi^*(t))\) satisfies the asymptotic boundary conditions (4.44), and then there exists a traveling wave solution for system (4.26) connecting \(E_1(0, 0)\) and \(E_3 \left(1 - \frac{a}{b+d}, 1 - \frac{a}{b+d}\right)\).

5. Asymptotic behaviors.

**Theorem 5.1.** Assume that \(b + d > a\). For any sufficiently small \(\tau > 0\), there exist positive constants \(P_i\) and \(Q_i\) \((i = 1, 2)\) such that system (4.30) has a traveling wave front \(\Phi\) with the following asymptotic properties

\[ \Phi(\xi) = \begin{pmatrix} (P_1 + o(1))e^{\lambda \xi} \\ (P_2 + o(1))e^{\lambda \xi} \end{pmatrix}, \]

as \(\xi \to -\infty\), where \(\lambda\) may be \(\lambda_1, \lambda_3\) or \(\lambda_4\), and

\[ \Phi(\xi) = \begin{pmatrix} 1 - \frac{a}{b+d} - (Q_1 + o(1))e^{\Lambda \xi} \\ 1 - \frac{a}{b+d} - (Q_2 + o(1))e^{\Lambda \xi} \end{pmatrix}, \]

as \(\xi \to +\infty\), where \(\Lambda\) may be \(\Lambda_2\) or \(\Lambda_4\). Here \(\lambda_1, \lambda_3, \lambda_4\) and \(\Lambda_2, \Lambda_4\) are defined by (5.51) and (5.56), respectively.

**Proof.** Let \(\xi = x + ct\) and \(\Phi(\xi) = (\phi_0(\xi), \psi_0(\xi))^T\) be the traveling wave solution of system (4.30). Differentiating both sides of (4.30) with respect to \(\xi\) and denoting \(\Phi'(\xi) = (\phi_1(\xi), \psi_1(\xi))^T\), we have

\[ \begin{cases} 
\phi_0''(\xi) - c\phi_1'(\xi) - a\phi_1(\xi) + [b\psi_1(\xi) + d\phi_1(\xi)][1 - \phi_0(\xi)] \\
- [b\psi_0(\xi) + d\phi_0(\xi)]\phi_1(\xi) = 0, \\
\psi_1''(\xi) - c\psi_1'(\xi) + \frac{1}{\tau}(\phi_1(\xi) - \psi_1(\xi)) = 0.
\end{cases} \]  

As \(\xi \to -\infty\), the limiting system of (5.47) is

\[ \begin{cases} 
\phi_0''(\xi) - c\phi_1'(\xi) - a\phi_1(\xi) + b\psi_1(\xi) + d\phi_1(\xi) = 0, \\
\psi_1''(\xi) - c\psi_1'(\xi) + \frac{1}{\tau}(\phi_1(\xi) - \psi_1(\xi)) = 0.
\end{cases} \]  

By setting \(\phi_1' = \bar{\phi}_1\) and \(\psi_1' = \bar{\psi}_1\), we can rewrite (5.48) as a first order system of ordinary differential equations in the four components \((\phi_1, \bar{\phi}_1, \psi_1, \bar{\psi}_1)^T\) with a constant \(4 \times 4\) coefficient matrix. That is,

\[ Z' = PZ, \quad Z = (\phi_1, \bar{\phi}_1, \psi_1, \bar{\psi}_1)^T, \]  

where

\[ P = \begin{pmatrix} 0 & 1 & 0 & 0 \\
\frac{a - d}{\tau} & c & -b & 0 \\
0 & 0 & 0 & 1 \\
-\frac{1}{\tau} & 0 & \frac{1}{\tau} & c \end{pmatrix}. \]
The linearization of system (5.49) admits non-trivial solutions with the form

\[ (\phi_{1-}, \bar{\phi}_{1-}, \psi_{1-}, \bar{\psi}_{1-})^T = \sum_{i=1}^{4} c_i h_i e^{\lambda_i \xi}, \]  

(5.50)

where

\[ \lambda_1 = \frac{c + \sqrt{c^2 + 2\mu_1}}{2}, \quad \lambda_2 = \frac{c - \sqrt{c^2 + 2\mu_1}}{2}, \]  

\[ \lambda_3 = \frac{c + \sqrt{c^2 + 2\mu_2}}{2}, \quad \lambda_4 = \frac{c - \sqrt{c^2 + 2\mu_2}}{2}, \]  

(5.51)

\[ \mu_1 = a - d + 1 + \sqrt{(a - d + 1)^2 + \frac{4(b + d - a)}{\tau}} > 0, \]  

\[ \mu_2 = a - d + 1 - \sqrt{(a - d + 1)^2 + \frac{4(b + d - a)}{\tau}} < 0, \]  

and \( h_i \) (\( i = 1, 2, 3, 4 \)) are eigenvectors of the matrix \( P \) corresponding to the eigenvalues \( \lambda_i \), and \( \alpha_j \)'s are arbitrary constants.

Note that \((\phi_{1-}, \bar{\phi}_{1-}, \psi_{1-}, \bar{\psi}_{1-})^T \to (0, 0, 0, 0)^T \) as \( \xi \to -\infty \). It follows from (5.50) that \( c_2 = 0 \) and

\[ (\phi_{1-}, \bar{\phi}_{1-}, \psi_{1-}, \bar{\psi}_{1-})^T = c_1 h_1 e^{\lambda_1 \xi} + c_3 h_3 e^{\lambda_3 \xi} + c_4 h_4 e^{\lambda_4 \xi}. \]

So as \( \xi \to -\infty \), we can define the asymptotic behavior as:

\[
\begin{pmatrix}
\phi_1(\xi) \\
\phi_2(\xi)
\end{pmatrix} = \begin{pmatrix}
\alpha_1(m_1 + o(1))e^{\lambda_1 \xi} + \alpha_2(m_2 + o(1))e^{\lambda_2 \xi} + \alpha_3(m_3 + o(1))e^{\lambda_3 \xi} \\
\alpha_1(n_1 + o(1))e^{\lambda_1 \xi} + \alpha_2(n_2 + o(1))e^{\lambda_2 \xi} + \alpha_3(n_3 + o(1))e^{\lambda_3 \xi}
\end{pmatrix},
\]

(5.52)

where \( m_i, n_i \) (\( i = 1, 2, 3 \)) are constants, and \( \alpha_i \) (\( i = 1, 2, 3 \)) can not be zero simultaneously [4]. Consider the solution \( h_i e^{\lambda_i \xi} \) of system (5.49). If either the first or the third component of eigenvector \( h_i \) is zero, for the matrix \( P \) it implies that the other components are zero. So we can obtain that \( m_i \neq 0 \) and \( n_i \neq 0 \) (\( i = 1, 2, 3 \)).

Similarly, as \( \xi \to +\infty \), system (5.47) becomes

\[
\begin{aligned}
\phi_{1+}'(\xi) - c\phi_{1+}(\xi) - (b + d - \frac{ad}{b+d})\phi_{1+}(\xi) + \frac{ab}{b+d}\psi_{1+}(\xi) &= 0, \\
\psi_{1+}'(\xi) - c\psi_{1+}(\xi) + \frac{1}{\tau}(\phi_{1+}(\xi) - \psi_{1+}(\xi)) &= 0.
\end{aligned}
\]

(5.53)

By setting \( \phi_{1+}' = \bar{\phi}_{1-} \) and \( \psi_{1+}' = \bar{\psi}_{1-} \), we rewrite (5.48) as a first order system of ordinary differential equations in the four components \((\phi_{1+}, \bar{\phi}_{1+}, \psi_{1+}, \bar{\psi}_{1+})^T \) with a constant \( 4 \times 4 \) coefficient matrix. More precisely, system (5.53) becomes

\[Z' = QZ, \quad Z = (\phi_{1+}, \bar{\phi}_{1+}, \psi_{1+}, \bar{\psi}_{1+})^T, \]

(5.54)

where

\[
Q = \begin{pmatrix}
0 & 1 & 0 & 0 \\
b + d - \frac{ad}{b+d} & c - \frac{ab}{b+d} & 0 & 0 \\
0 & 0 & 0 & 1 \\
-\frac{1}{\tau} & 0 & \frac{1}{\tau} & c
\end{pmatrix}.
\]

(5.55)

So we can deduce that the general solutions of system (5.54) take the form

\[ (\phi_{1+}, \bar{\phi}_{1+}, \psi_{1+}, \bar{\psi}_{1+})^T = \sum_{i=1}^{4} d_i k_i e^{\lambda_i \xi}, \]

(5.55)
where
\[
\Lambda_1 = \frac{c + \sqrt{c^2 + 2\nu_1}}{2}, \quad \Lambda_2 = \frac{c - \sqrt{c^2 + 2\nu_1}}{2},
\]
\[
\Lambda_3 = \frac{c + \sqrt{c^2 + 2\nu_2}}{2}, \quad \Lambda_4 = \frac{c - \sqrt{c^2 + 2\nu_2}}{2},
\]
\[
\nu_1 = b + d + 1 - \frac{ad}{b + d} + \sqrt{(b + d + 1 - \frac{ad}{b + d})^2 + \frac{4(a - b - d)}{\tau}} > 0,
\]
\[
\nu_2 = b + d + 1 - \frac{ad}{b + d} - \sqrt{(b + d + 1 - \frac{ad}{b + d})^2 + \frac{4(a - b - d)}{\tau}} > 0,
\]
and \(k_i \ (i = 1, 2, 3, 4)\) are eigenvectors of the matrix \(Q\) corresponding to eigenvalues \(\Lambda_i\), and \(d'_i\)s are arbitrary constants.

Note that \(T\phi_1 + \omega_1 \psi_1 + \psi_1 \to \left(1 - \frac{a}{b + d}, 0, 1 - \frac{a}{b + d}, 0\right)^T\) as \(\xi \to +\infty\). It follows from system (5.55) that \(d_1 = d_3 = 0\), and
\[
\left(\phi_1, \phi_3, \psi_1, \psi_3\right)^T = d_2 k_2 e^{\Lambda_2 \xi} + d_4 k_4 e^{\Lambda_4 \xi}.
\]
So as \(\xi \to +\infty\), we can define the following asymptotic behavior as follows:
\[
\left(\phi_1(\xi), \phi_2(\xi)\right) = \left(\beta_1 \left(p_1 + o(1)\right) e^{\Lambda_1 \xi} + \beta_2 \left(p_2 + o(1)\right) e^{\Lambda_3 \xi}, \beta_1 \left(q_1 + o(1)\right) e^{\Lambda_2 \xi} + \beta_2 \left(q_2 + o(1)\right) e^{\Lambda_4 \xi}\right),
\]
where \(p_i, q_i \ (i = 1, 2)\) are constants, and \(\beta_1, \beta_2\) cannot be zero simultaneously. Similarly, in the case of \(\xi \to -\infty\), we can prove that \(p_i \neq 0\) and \(q_i \neq 0 \ (i = 1, 2)\).

Consequently, based on the above discussions, we complete the proof. □

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