Derived Categories of Fano Threefolds

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In memory of V.A. Iskovskikh

Abstract—We consider the structure of the derived categories of coherent sheaves on Fano threefolds with Picard number 1 and describe a strange relation between derived categories of different threefolds. In the appendix we discuss how the ring of algebraic cycles of a smooth projective variety is related to the Grothendieck group of its derived category.

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1. INTRODUCTION

A smooth proper connected algebraic variety $V$ is a Fano variety if the anticanonical class $-K_V$ on $V$ is ample. In dimension 1 the only Fano variety is the projective line $\mathbb{P}^1$. In dimension 2 the Fano varieties are known under the name of del Pezzo surfaces. There are ten deformation classes of these—the projective plane with up to eight blown-up points (in generic position) and the quadric.

An ambitious program of classification of Fano threefolds was initiated by G. Fano in the beginning of the 20th century and was mostly accomplished by V. Iskovskikh in 1979 [5]. The final touches were added by Mukai and Umemura in 1983 [14]. In higher dimensions only some pieces of the classification are known.

There are many interconnections between Fano varieties, which help in the classification problems. For example, a hyperplane section of a Fano $n$-fold $V$ is a Fano variety of dimension $n - 1$ if the anticanonical class of $V$ is sufficiently large. This is most helpful for the classification of Fano varieties with large anticanonical class. On the other hand, there are many birational transformations between different Fano varieties of the same dimension. This also is very useful. For example, the original approach of Fano developed by Iskovskikh was based on this kind of interconnections (the double projection from a line is one of the most important transformations).

The goal of the present paper is to indicate that there are interconnections between some Fano threefolds on a higher level, the level of derived categories. On the one hand, we consider a Fano threefold of index 2 and degree $d$, and on the other hand, a Fano threefold of index 1 and degree $4d + 2$. Then, in both derived categories, we find an exceptional pair of vector bundles and consider the arising semiorthogonal decompositions. The crucial observation is that the nontrivial components of these decompositions are equivalent.

Actually, this is a very rough formulation. It is well known that in general Fano varieties have nontrivial moduli spaces, so the components of the derived categories under consideration in general vary. So, to be more precise, one should say that there is a correspondence in the product of the moduli spaces of both types of Fano threefolds such that its points correspond to Fano threefolds with equivalent nontrivial components of derived categories. The study of the structure of this correspondence is an interesting question. We conjecture that this correspondence is dominant over both moduli spaces and give some speculations about its structure.

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Although it is not so easy to prove that this relation holds (actually, we can check this only for \(d = 3, 4, 5\), while the cases \(d = 1, 2\) are still open to question), it is much more difficult to understand why such a relation takes place. We believe that any progress in this direction will be very useful for understanding the structure of Fano varieties and for the classification in higher dimensions.

2. CLASSIFICATION OF FANO THREEFOLDS

An excellent modern survey of the classification of Fano varieties is given in [6]. Let us briefly recall those parts of the classification that are most important for us. We will work over an algebraically closed field \(k\) of zero characteristic.

The most important discrete invariant of a Fano variety \(V\) is its Picard lattice \(\text{Pic}\, V\), which is endowed with the intersection form and a distinguished element (the canonical class). In this paper we will be mostly concerned with the (most important) case \(\text{Pic}\, V = \mathbb{Z}\). In this case the distinguished element is represented by a positive integer \(i_V\) such that

\[ K_V = -i_V H, \]

where \(H\) is the positive generator of \(\text{Pic}\, V\), and the intersection form is completely determined by a positive integer \(d_V = H^{\dim V}\).

These invariants are known as the index and the degree of \(V\), respectively.

The most general result concerning the index is the following

**Theorem 2.1** [4]. If \(V\) is a Fano variety of index \(i_V\), then \(i_V \leq \dim V + 1\). Moreover,

- if \(i_V = \dim V + 1\), then \(V = \mathbb{P}^n\);
- if \(i_V = \dim V\), then \(V = Q^n \subset \mathbb{P}^{n+1}\).

In particular, for threefolds we have

**Corollary 2.2.** If \(V\) is a Fano threefold, then \(i_V \leq 4\). Moreover,

- if \(i_V = 4\), then \(V = \mathbb{P}^3\);
- if \(i_V = 3\), then \(V = Q^3 \subset \mathbb{P}^4\).

Fano threefolds of index 2 are also known as del Pezzo threefolds (since their hyperplane sections are del Pezzo surfaces).

**Theorem 2.3.** Let \(V\) be a Fano threefold with \(\text{Pic}\, V = \mathbb{Z}\) of index \(i_V = 2\) and of degree \(d_V\). Then

\[ 1 \leq d_V \leq 5 \]

and for each \(1 \leq d \leq 5\) there exists a unique deformation class of Fano threefolds \(Y_d\) with \(\text{Pic}\, Y_d = \mathbb{Z}\) of index 2 and of degree \(d\). They have the following explicit description:

- \(Y_5 = \text{Gr}(2, 5) \cap \mathbb{P}^6 \subset \mathbb{P}^9\) is a linear section of codimension 3 of the Grassmannian \(\text{Gr}(2, 5)\) in the Plücker embedding;
- \(Y_4 = Q \cap Q' \subset \mathbb{P}^5\) is an intersection of two 4-dimensional quadrics;
- \(Y_3 \subset \mathbb{P}^4\) is a cubic hypersurface;
- \(Y_2 \to \mathbb{P}^3\) is a double covering ramified in a quartic;
- \(Y_1\) is a hypersurface of degree 6 in the weighted projective space \(\mathbb{P}(3, 2, 1, 1, 1)\).
Now let $V$ be a Fano threefold of index 1. Its general anticanonical section $S \subset V$ is a K3 surface endowed with a polarization $H_S = H|_S$. It follows that

$$d_V = H^3 = H_S^2 = 2g_V - 2$$

for some integer $g_V \geq 2$, which is known as the genus of $V$.

**Theorem 2.4.** Let $V$ be a Fano threefold with $\text{Pic} \, V = \mathbb{Z}$ of index $i_V = 1$ and of genus $g_V$. Then

$$2 \leq g_V \leq 12, \quad g_V \neq 11,$$

and for each $g$ in this range there exists a unique deformation class of Fano threefolds $X_{2g-2}$ with $\text{Pic} \, X_{2g-2} = \mathbb{Z}$ of index 1 and of genus $g$. They have the following explicit description:

- $X_{22} \subset \mathbb{P}^{13}$ is the zero locus of a global section of the vector bundle $\Lambda^2 U^* \oplus \Lambda^3 U^* \oplus \Lambda^2 U^*$ on the Grassmannian $\text{Gr}(3,7)$, where $U$ denotes the tautological rank 3 bundle;
- $X_{18} = G_2 \text{Gr}(2,7) \cap \mathbb{P}^{11} \subset \mathbb{P}^{13}$, where $G_2 \text{Gr}(2,7)$ is the minimal compact homogeneous space for the simple algebraic group of type $G_2$, which can be realized as the zero locus of a global section of the vector bundle $U^\perp(1)$ on the Grassmannian $\text{Gr}(2,7)$;
- $X_{16} = \mathbb{P}^{10} \cap \text{SGr}(3,6) \subset \mathbb{P}^{13}$ is a linear section of codimension 3 of the symplectic Lagrangian Grassmannian $\text{SGr}(3,6)$ in the Plücker embedding;
- $X_{14} = \mathbb{P}^9 \cap \text{Gr}(2,6) \subset \mathbb{P}^{14}$ is a linear section of codimension 5 of the Grassmannian $\text{Gr}(2,6)$ in the Plücker embedding;
- $X_{12} = \mathbb{P}^8 \cap \text{OGr}_4(5,10) \subset \mathbb{P}^{15}$ is a linear section of codimension 7 of the connected component of the orthogonal Lagrangian Grassmannian $\text{OGr}_4(5,10)$ in the half-spinor embedding;
- $X_{10} = \mathbb{P}^7 \cap Q \cap \text{Gr}(2,5) \subset \mathbb{P}^9$ is a quadric section of a linear section of codimension 2 of the Grassmannian $\text{Gr}(2,5)$ in the Plücker embedding, or $X_{10} \rightarrow Y_5$ is a twofold covering ramified in a quadric;
- $X_8 = Q \cap Q' \cap Q'' \subset \mathbb{P}^6$ is an intersection of three 5-dimensional quadrics;
- $X_6 = Q \cap F_3 \subset \mathbb{P}^5$ is an intersection of a quadric and a cubic;
- $X_4 \subset \mathbb{P}^4$ is a quartic, or $X_4 \rightarrow Q$ is a double covering of a quartic $Q \subset \mathbb{P}^4$ ramified in the intersection of $Q$ with a quartic;
- $X_2 \rightarrow \mathbb{P}^3$ is a double covering ramified in a sextic.

We will also need the following result of S. Mukai.

**Theorem 2.5** [13]. Assume that the genus $g$ of a Fano threefold $X_{2g-2}$ can be represented as a product $g = r \cdot s$ of two integers. Then on $X_{2g-2}$ there exists a unique stable vector bundle $\mathcal{E}_r$ of rank $r$ with $c_1(\mathcal{E}_r) = -H$ and $c_2(\mathcal{E}_r) = \frac{1}{2}H^2 + (r-s)L$, where $L$ is the class of a line on $X_{2g-2}$. Moreover, $\mathcal{E}_r$ is exceptional and $H^\bullet(X, \mathcal{E}_r) = 0$.

The definition of exceptional bundles is given below (see Definition 3.2).

**Remark 2.6.** When applied to the Fano threefolds $X_{2g-2}$ with $g \geq 6$, this theorem gives vector bundles that are the restrictions of the dual tautological bundles from the corresponding Grassmannians.

3. RELATION OF DERIVED CATEGORIES

For an algebraic variety $V$ we denote by $\mathcal{D}^b(V)$ the bounded derived category of coherent sheaves on $V$. Recall that $\mathcal{D}^b(V)$ is triangulated. We will always assume that $V$ is smooth and projective.

**Definition 3.1** [2, 3]. A semiorthogonal decomposition of a triangulated category $\mathcal{T}$ is a sequence of full triangulated subcategories $\mathcal{C}_1, \ldots, \mathcal{C}_n$ in $\mathcal{T}$ such that $\text{Hom}_\mathcal{T}(\mathcal{C}_i, \mathcal{C}_j) = 0$ for $i > j$ and
for every object $T \in \mathcal{T}$ there exists a chain of morphisms $0 = T_n \rightarrow T_{n-1} \rightarrow \ldots \rightarrow T_1 \rightarrow T_0 = T$ such that the cone of the morphism $T_k \rightarrow T_{k-1}$ is contained in $C_k$ for each $k = 1, 2, \ldots, n$.

We will write $\mathcal{T} = \langle C_1, C_2, \ldots, C_n \rangle$ for a semiorthogonal decomposition of a triangulated category $\mathcal{T}$ with components $C_1, C_2, \ldots, C_n$.

**Definition 3.2** [1]. An object $F \in \mathcal{T}$ is called exceptional if $\text{Hom}(F, F) = k$ and $\text{Ext}^p(F, F) = 0$ for all $p \neq 0$. A collection of exceptional objects $(F_1, \ldots, F_m)$ is called exceptional if $\text{Ext}^p(F_i, F_k) = 0$ for all $l > k$ and all $p \in \mathbb{Z}$.

**Proposition 3.3** [3]. Any exceptional collection $F_1, \ldots, F_m$ in $\mathcal{D}^b(V)$ gives a semiorthogonal decomposition

$$\mathcal{D}^b(V) = \langle C, F_1, \ldots, F_m \rangle,$$

where $C = \langle F_1, \ldots, F_m \rangle = \{ F \in \mathcal{D}^b(V) \mid \text{Ext}^\bullet(F_k, F) = 0 \text{ for all } 1 \leq k \leq m \}$ and all the other components are the subcategories of $\mathcal{D}^b(V)$ generated by $F_k$ (each of these is equivalent to $\mathcal{D}^b(k)$, the derived category of $k$-vector spaces).

Now let $V$ be a Fano variety of index $i = i_V$. The following result is well known.

**Lemma 3.4.** Let $V$ be a Fano variety of index $i = i_V$. Then the collection $\mathcal{O}_V, \mathcal{O}_V(H), \ldots, \mathcal{O}_V((i - 1)H)$ in $\mathcal{D}^b(V)$ is exceptional.

**Proof.** We note that $H^p(V, \mathcal{O}_V(-kH)) = 0$ for $1 \leq k \leq i - 1$ and all $p$ by the Kodaira vanishing theorem; hence $\text{Ext}^p(\mathcal{O}_V(H), \mathcal{O}_V(kH)) = 0$ for $0 \leq k < l \leq i - 1$. Similarly, $H^p(V, \mathcal{O}_V) = 0$ for all $p > 0$ by the Kodaira vanishing theorem, while $H^0(V, \mathcal{O}_V) = k$ since $V$ is connected. Therefore, all line bundles on $V$ are exceptional. □

**Corollary 3.5.** For any Fano variety $V$ we have the following semiorthogonal decomposition:

$$\mathcal{D}^b(V) = \langle \mathcal{B}_V, \mathcal{O}_V, \mathcal{O}_V(H), \ldots, \mathcal{O}_V((i - 1)H) \rangle,$$

where $i = i_V$ is the index of $V$ and $\mathcal{B}_V = \{ F \in \mathcal{D}^b(V) \mid H^\bullet(V, F(-kH)) = 0 \text{ for all } 0 \leq k \leq i - 1 \}$.

In particular, for Fano threefolds $Y_d$ of index 2 we obtain a semiorthogonal decomposition

$$\mathcal{D}^b(Y_d) = \langle \mathcal{B}_{Y_d}, \mathcal{O}_V, \mathcal{O}_V(H) \rangle.$$

For Fano threefolds $X_{2g-2}$ of index 1 and even genus $g = 2t$ we consider the vector bundle $\mathcal{E}_2$ of rank 2 provided by Theorem 2.5. Applying this theorem, we deduce the following

**Lemma 3.6.** Let $X = X_{2g-2}$ be a Fano threefolds of index 1 and even genus $g = 2t$. Let $\mathcal{E} = \mathcal{E}_2$ be the vector bundle of rank 2 on $X$ constructed in Theorem 2.5. Then $(\mathcal{E}, \mathcal{O}_X)$ is an exceptional pair on $X$ and we have a semiorthogonal decomposition

$$\mathcal{D}^b(X_{2g-2}) = \langle \mathcal{A}_{X_{2g-2}}, \mathcal{E}, \mathcal{O}_{X_{2g-2}} \rangle,$$

where $\mathcal{A}_{X_{2g-2}} = \{ F \in \mathcal{D}^b(X_{2g-2}) \mid H^\bullet(X_{2g-2}, F) = \text{Ext}^\bullet(\mathcal{E}, F) = 0 \}$.

Now we can formulate the conjecture.

**Conjecture 3.7.** Let $\mathcal{MF}_d$ be the moduli spaces of Fano threefolds of index $i$ and degree $d$. Then there is a correspondence $Z_d \subset \mathcal{MF}_d \times \mathcal{MF}_{d+2}$ that is dominant over each factor and such that for any point $(Y_d, X_{d+2}) \in Z_d$ there is an equivalence of categories

$$\mathcal{A}_{X_{d+2}} \cong \mathcal{B}_{Y_d}.$$

The main support for this conjecture is provided by the following result.

**Theorem 3.8.** Conjecture 3.7 is true for $d = 3, 4, 5$. 

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A proof of Theorem 3.8 will be given in the next section. It consists of a case-by-case analysis
(see Corollaries 4.3, 4.6, and 4.9). Now we are going to compare the numerical Grothendieck groups
of the categories $A_{4d+2}$ and $B_{Y_d}$.

For a triangulated category $T$ we denote by $K_0(T)$ its Grothendieck group. It is endowed with
a bilinear Euler form

$$\chi([F],[G]) = \sum_p (-1)^p \dim \operatorname{Ext}^p(F,G).$$

The numerical Grothendieck group $K_0(T)_{\text{num}}$ is defined as the quotient $K_0(T)_{\text{num}} := K_0(T)/\ker \chi$. If $T = D^b(V)$, we write $K_0(V)$ and $K_0(V)_{\text{num}}$ instead of $K_0(D^b(V))$ and $K_0(D^b(V))_{\text{num}}$ for brevity.

**Proposition 3.9.** For all $1 \leq d \leq 5$ there is an isomorphism of numerical Grothendieck groups

$$K_0(A_{4d+2})_{\text{num}} \cong K_0(B_{Y_d})_{\text{num}}$$

compatible with the Euler bilinear forms.

**Proof.** A computation is based on the Riemann–Roch theorem. Note that by Corollary 5.8 (see
below) the Chern character map $\operatorname{ch}: K_0(X_{2g-2})_{\text{num}} \to \bigoplus_{p=0}^3 H^{2p}(X_{2g-2},\mathbb{Q})$ identifies the numerical Grothendieck group $K_0(X_{2g-2})_{\text{num}}$ with the lattice generated by the elements

$$\operatorname{ch}(\mathcal{O}_{X_{2g-2}}) = 1, \quad \operatorname{ch}(\mathcal{O}_H) = H - (g-1)L + \frac{g-1}{3}P, \quad \operatorname{ch}(\mathcal{O}_L) = L + \frac{1}{2}P, \quad \operatorname{ch}(\mathcal{O}_P) = P,$$

where $P$ is the class of a point. By the Riemann–Roch theorem the Euler form can be expressed as

$$\chi(u,v) = \chi_0(u^* \cap v),$$

where $u \mapsto u^*$ is the involution of $\bigoplus_{p=0}^3 H^{2p}(X_{2g-2},\mathbb{Q})$ given by the $(-1)^p$-multiplication on $H^{2p}(X_{2g-2},\mathbb{Q})$ and $\chi_0$ is given by the formula

$$\chi_0(x+yH+zL+wP) = x + \frac{g+11}{6}y + \frac{1}{2}z + w.$$

On the other hand, we have $\operatorname{ch}(\mathcal{O}_{X_{2g-2}}) = 1$, and using Theorem 2.5 it is easy to compute

$$\operatorname{ch}(\mathcal{E}) = 2 - H + \frac{g-4}{2}L - \frac{g-10}{12}P.$$

It follows that

$$K_0(A_{X_{2g-2}})_{\text{num}} = \left\langle \frac{1}{4} - H + \frac{g-4}{2}L - \frac{g-10}{12}P \right\rangle.$$

Computing the form $\chi$ on the base vectors, we conclude that

$$K_0(A_{X_{2g-2}})_{\text{num}} \cong \mathbb{Z}^2, \quad \chi_{A} = \begin{pmatrix} 1 - g/2 & -g/2 \\ 3 - g & 1 - g \end{pmatrix}$$

as a lattice with a bilinear form.

Similarly, by Corollary 5.8 the Chern character map $\operatorname{ch}: K_0(Y_d)_{\text{num}} \to \bigoplus_{p=0}^3 H^{2p}(Y_d,\mathbb{Q})$ identifies $K_0(Y_d)_{\text{num}}$ with the lattice generated by the elements

$$\operatorname{ch}(\mathcal{O}_{Y_d}) = 1, \quad \operatorname{ch}(\mathcal{O}_H) = H - \frac{d}{2}L + \frac{d}{6}P, \quad \operatorname{ch}(\mathcal{O}_L) = L, \quad \operatorname{ch}(\mathcal{O}_P) = P,$$
and it is easy to see that \( \chi_0 \) is given by the formula
\[
\chi_0(x + yH + zL + wP) = x + \frac{d + 3}{3}y + z + w.
\]

On the other hand, we have \( \text{ch}(\mathcal{O}_{X_d}) = 1 \) and \( \text{ch}(\mathcal{O}_{X_d}(H)) = 1 + H + \frac{d}{2}L + \frac{d}{6}P \), so it follows that
\[
K_0(\mathcal{B}_{Y_d})_{\text{num}} = \left\langle 1, 1 + H + \frac{d}{2}L + \frac{d}{6}P \right\rangle = \left\langle 1 - L, H - \frac{d}{2}L + \frac{d - 6}{6}P \right\rangle.
\]
Computing the form \( \chi \) on the base vectors, we conclude that
\[
K_0(\mathcal{B}_{Y_d})_{\text{num}} \cong \mathbb{Z}^2, \quad \chi_{\mathcal{B}} = \begin{pmatrix} -1 & -1 \\ 1 - d & -d \end{pmatrix}
\]
as a lattice with a bilinear form.

A direct check shows that for the map \( \mathbb{Z}^2 \to \mathbb{Z}^2 \) given by the matrix
\[
A = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}
\]
we have
\[
A^T \cdot \chi_{\mathcal{B}} \cdot A = \begin{pmatrix} -d & -1 - d \\ 1 - 2d & -1 - 2d \end{pmatrix},
\]
which coincides with the matrix of \( \chi_A \) for \( g = 2d + 2 \). Thus the map \( A \) gives a required isomorphism \( K_0(\mathcal{B}_{Y_d})_{\text{num}} \cong K_0(\mathcal{A}_{X_{4d+2}})_{\text{num}} \) compatible with the Euler forms. \( \square \)

4. CASES \( d \geq 3 \)

In this section we prove Theorem 3.8 by a case-by-case analysis.

4.1. The case \( d = 5 \). Recall that the del Pezzo threefold \( Y_5 \) of degree 5 is rigid, so that the moduli space \( \mathcal{M}_{\mathcal{F}_5}^1 \) is a point. On the contrary, Fano threefolds \( X_{22} \) of genus 12 have a 6-dimensional moduli space \( \mathcal{M}_{\mathcal{F}_{22}}^1 \). We will show that the correspondence \( Z_5 \subset \mathcal{M}_{\mathcal{F}_5}^1 \times \mathcal{M}_{\mathcal{F}_{22}}^1 \) is the whole product. In other words, we are going to show that for any \( X_{22} \) and for the unique \( Y_5 \) there is an equivalence \( \mathcal{A}_{X_{22}} \cong \mathcal{B}_{Y_5} \). For this purpose we give an explicit description of both categories in question.

For the \( X_{22} \) the description is based on the following result. Let \( \mathcal{E}_2 \) and \( \mathcal{E}_3 \) be the vector bundles of rank 2 and 3 on \( X = X_{22} \) provided by Theorem 2.5 for the factorizations \( g_X = 12 = 2 \times 6 = 3 \times 4 \). Let \( W = H^0(X, \mathcal{E}_3^*) \cong k^7 \), so that \( X \subset \text{Gr}(3, W) \).

**Theorem 4.1** [7, 8]. The bundles \( W/\mathcal{E}_3 \otimes \mathcal{O}_X(-1), \mathcal{E}_3^* \otimes \mathcal{O}_X(-1), \mathcal{E}_2, \) and \( \mathcal{O}_X \) form a full exceptional collection, so that
\[
\mathcal{D}^b(X_{22}) = \langle W/\mathcal{E}_3 \otimes \mathcal{O}_X(-1), \mathcal{E}_3^* \otimes \mathcal{O}_X(-1), \mathcal{E}_2, \mathcal{O}_X \rangle.
\]
Moreover, \( \text{Hom}(W/\mathcal{E}_3 \otimes \mathcal{O}_X(-1), \mathcal{E}_3^* \otimes \mathcal{O}_X(-1)) = k^7 \) and \( \text{Ext}^{\geq 0}(W/\mathcal{E}_3 \otimes \mathcal{O}_X(-1), \mathcal{E}_3^* \otimes \mathcal{O}_X(-1)) = 0 \), so that \( \mathcal{A}_{X_{22}} \cong \mathcal{D}^b(Q_3) \),
\[
where \quad Q_3 = \begin{pmatrix} \bullet & \bullet & \bullet \end{pmatrix} \text{ is the Kronecker quiver with three arrows.}
\]

On the other hand, for the \( Y_5 \) a description of the derived category was given by D. Orlov. Let \( \mathcal{U} \) be the restriction to \( Y = Y_5 \) of the tautological bundle from the Grassmannian \( \text{Gr}(2, 5) \). Let \( W' = H^0(Y, \mathcal{U}^*) \cong k^5 \), so that \( Y \subset \text{Gr}(2, W') \).
Theorem 4.2 [15]. The bundles $W'/U \otimes \mathcal{O}_Y(-1), U, \mathcal{O}_Y,$ and $\mathcal{O}_Y(1)$ form a full exceptional collection, so that

$$\mathcal{D}^b(Y_5) = (W'/U \otimes \mathcal{O}_Y(-1), U, \mathcal{O}_Y, \mathcal{O}_Y(1)).$$

Moreover, $\text{Hom}(W'/U \otimes \mathcal{O}_Y(-1), U) = k^3$ and $\text{Ext}^d(W'/U \otimes \mathcal{O}_Y(-1), U) = 0,$ so that $\mathcal{B}_{Y_5} \cong \mathcal{D}^b(Q_3).$

From these two results we immediately deduce the required equivalence.

Corollary 4.3. For any Fano threefold $X_{22}$ of genus 12 and for the unique del Pezzo threefold $Y_5$ of degree 5 there is an equivalence of categories $\mathcal{A}_{X_{22}} \cong \mathcal{D}^b(Q_3) \cong \mathcal{B}_{Y_5}.$

4.2. The case $d = 4.$ Recall that the moduli space $\mathcal{M}^2_4$ of del Pezzo threefolds $Y_4$ of degree 4 is isomorphic to the moduli space $\mathcal{M}_2$ of curves of genus 2. The isomorphism is constructed as follows. Let $Y = Y_4 = Q \cap Q' \subset \mathbb{P}^5.$ Consider the pencil of quadrics $\{Q_\lambda\}_{\lambda \in \mathbb{P}^1}$ generated by $Q$ and $Q'.$ If $Y$ is smooth, then the generic $Q_\lambda$ is smooth and there are precisely six distinct points $\lambda_1, \ldots, \lambda_6 \in \mathbb{P}^1$ for which the quadric $Q_\lambda$ is degenerate. Consider the twofold covering $C(Y) \rightarrow \mathbb{P}^1$ ramified at the points $\lambda_i.$ Then $C(Y)$ is a smooth curve of genus 2.

Theorem 4.4 [3, 11]. The map $\mathcal{M}^2_4 \rightarrow \mathcal{M}_2,$ $Y \mapsto C(Y)$ is an isomorphism. Moreover, there is an equivalence $\mathcal{B}_{X_4} \cong \mathcal{D}^b(C(Y_4)).$

Our goal is to show that $X_{18}$ threefolds behave in a similar fashion. Indeed, recall that by definition any $X = X_{18}$ is a linear section of codimension 2 in $\text{Gr}(2, 7).$ Let $\{X_\lambda\}_{\lambda \in \mathbb{P}^1}$ be the pencil of hyperplane sections of $\text{Gr}(2, 7)$ passing through $X.$ Since the projective dual of $\text{Gr}(2, 7)$ is a hypersurface of degree 6, it follows that there are precisely six distinct points $\lambda_1, \ldots, \lambda_6 \in \mathbb{P}^1$ for which $X_\lambda$ is singular. Consider the twofold covering $C(X) \rightarrow \mathbb{P}^1$ ramified at the points $\lambda_i.$ Then $C(X)$ is a smooth curve of genus 2. Thus we obtain a map $\mathcal{M}^2_{18} \rightarrow \mathcal{M}_2,$ $X \mapsto C(X).$

Theorem 4.5 [10]. There is an equivalence $\mathcal{B}_{X_{18}} \cong \mathcal{D}^b(C(X_{18})).$

Combining these results, we obtain the case $d = 4.$ Let $Z_4 \subset \mathcal{M}^2_4 \times \mathcal{M}^1_{18}$ be the graph of the morphism $\mathcal{M}^1_{18} \rightarrow \mathcal{M}_2 \cong \mathcal{M}^2_4.$

Corollary 4.6. For any pair of threefolds $(Y_4, X_{18}) \in Z_4 \subset \mathcal{M}^2_4 \times \mathcal{M}^1_{18}$ we have an equivalence of categories $\mathcal{A}_{X_{18}} \cong \mathcal{B}_{Y_4}.$

4.3. The case $d = 3.$ While in the previous two cases we were able to describe the categories under consideration explicitly, for $d = 3$ this is no longer possible. We can only prove an equivalence in this case.

Recall that by definition any $X = X_{14}$ is a linear section of codimension 5 in $\text{Gr}(2, 6).$ Let $W$ be a 6-dimensional vector space, so that $\text{Gr}(2, 6) = \text{Gr}(2, W) \subset \mathbb{P}(\Lambda^2 W).$ Then $X$ can be described by a 5-dimensional subspace $A \subset \Lambda^2 W^*$ or, equivalently, by an injective map $\alpha: A \rightarrow \Lambda^2 W,$ where $A$ is a fixed vector space of dimension 5. Let us denote the corresponding threefold $X_{14}$ by $X(\alpha).$

On the other hand, consider the whole space $\mathbb{P}(\Lambda^2 W^*),$ the space of skew-symmetric forms on $W.$ Consider the hypersurface therein consisting of degenerate skew-forms. It is well known that the equation of this hypersurface is given by the Pfaffian polynomial. It follows that it is a cubic hypersurface, which is denoted by $\text{Pf}(W)$ and is called the Pfaffian variety. Certainly, the Pfaffian variety is singular, its singular locus coincides with the set of all skew-forms of rank 2 on $W$, that is, with the Grassmannian $\text{Gr}(2, W^*) \subset \text{Pf}(W) \subset \mathbb{P}(\Lambda^2 W^*).$ However, the codimension of the singular locus in $\mathbb{P}(\Lambda^2 W^*)$ is $14 - 8 = 6,$ so for generic $\alpha$ the preimage of $\text{Pf}(W)$ in $\mathbb{P}(A)$ is a smooth cubic hypersurface, which we denote by $Y(\alpha).$ Further, associating with a degenerate skew-form on $W$ its kernel defines a rank 2 subbundle $K \subset W \otimes \mathcal{O}$ on the smooth locus of $\text{Pf}(W).$ Let $E(\alpha) = \alpha^* K \otimes \mathcal{O}_{Y(\alpha)}(1),$ where on the right-hand-side we consider $\alpha$ as a map $Y(\alpha) \rightarrow \text{Pf}(W)$ and $\mathcal{O}_{Y(\alpha)}(1)$ is the restriction of $\mathcal{O}_{\mathbb{P}(A)}(1).$
Theorem 4.7 [9, 12]. The map \( X(\alpha) \hookrightarrow (Y(\alpha), E(\alpha)) \) gives an isomorphism of the moduli space \( \mathcal{M}F_{14}^1 \) of Fano threefolds \( X_{14} \) and the moduli space \( \mathcal{M}FT_{3}^{2}(2) \) of pairs \((Y, E)\), where \( Y \) is a smooth cubic threefold and \( E \) is a stable vector bundle on \( Y \) of rank 2 with \( c_1(E) = 0, c_2(E) = 2L \), and \( H^1(Y, E(-1)) = 0 \). For every \( \alpha \) there is an equivalence of categories \( \mathcal{B}_{X_{14}(\alpha)} \cong \mathcal{A}_{Y_{3}(\alpha)} \).

Remark 4.8. The bundles \( E \) in the statement of the theorem are known as instanton bundles of charge 2.

Let \( Z_3 \subset \mathcal{M}F_{3}^{2} \times \mathcal{M}F_{14}^{1} \) be the graph of the morphism \( \mathcal{M}F_{14}^{1} \cong \mathcal{M}FT_{3}^{2}(2) \to \mathcal{M}F_{3}^{2} \).

Corollary 4.9. For any pair of threefolds \((Y_3, X_{14}) \in Z_3 \subset \mathcal{M}F_{3}^{2} \times \mathcal{M}F_{14}^{1} \) we have an equivalence of categories \( \mathcal{A}_{X_{14}} \cong \mathcal{B}_{Y_{3}} \).

4.4. Geometrical correspondences. Actually, the proof of Theorem 4.7 in [9] gives more than just an equivalence of categories. It also gives a geometrical correspondence between \( X(\alpha) \) and \( Y(\alpha) \).

Let \( \mathbb{P}(X(\alpha))(\mathcal{E}) \) be the projectivization of the exceptional rank 2 bundle \( \mathcal{E} \) on \( X(\alpha) \). Since \( \mathcal{E} \) is the restriction of the tautological bundle from the Grassmannian \( \text{Gr}(2, W) \), we have a canonical map \( \mathbb{P}(X(\alpha))(\mathcal{E}) \to \mathbb{P}(W) \). On the other hand, one can check that we have an isomorphism \( H^0(Y(\alpha), E(\alpha)^* \otimes \mathcal{O}_{Y(\alpha)}(1)) \cong W^* \); hence we also have a canonical map \( \mathbb{P}(Y(\alpha))(E(\alpha)) \to \mathbb{P}(W) \).

Theorem 4.10 [9]. The images of \( \mathbb{P}(X(\alpha))(\mathcal{E}) \) and \( \mathbb{P}(Y(\alpha))(E(\alpha)) \) in \( \mathbb{P}(W) \) coincide with a quartic hypersurface \( M \subset \mathbb{P}(W) \) singular along a curve \( C \subset M \) of genus 26. The maps \( \mathbb{P}(X(\alpha))(\mathcal{E}) \to M \) and \( \mathbb{P}(Y(\alpha))(E(\alpha)) \to M \) are small contractions and induce isomorphisms over the complement of \( C \). Moreover, the induced birational isomorphism \( \mathbb{P}(X(\alpha))(\mathcal{E}) \dasharrow \mathbb{P}(Y(\alpha))(E(\alpha)) \) is a flop.

Remark 4.11. The hypersurface \( M \subset \mathbb{P}(W) \) is known as the da Palatini quartic. The curve \( C \) parameterizes lines on \( X(\alpha) \) and at the same time jumping lines for \( E(\alpha) \) on \( Y(\alpha) \) (that is, lines \( L \subset Y(\alpha) \) for which \( E(\alpha)|_L \cong \mathcal{O}_L(1) \oplus \mathcal{O}_L(-1) \)).

We expect that some generalization of this result should hold for other values of \( d \). For example, let \( \mathcal{M}FB_{d+2}^{1}(t) \) be the moduli space of pairs \((X, F)\), where \( X \) is a Fano threefold of index 1 and degree \( 4d + 2 \) and \( F \) is a stable vector bundle on \( X \) of rank 2 with \( c_1(F) = -H \) and \( c_2(F) = (d + 2 + t)L \) (note that for \( t = 0 \) by Theorem 2.5 there is only one such bundle, the exceptional bundle \( \mathcal{E}_2 \)); hence \( \mathcal{M}FB_{d+2}^{1}(0) = \mathcal{M}F_{d+2}^{1} \). Using the Riemann–Roch theorem (see the proof of Proposition 3.9), one can check that for any \( d \), any \( t \), and any \((X, F) \in \mathcal{M}F_{d+2}^{1}(t) \) we have

\[
\dim H^0(X, F^*) = d + 3 - t,
\]

so that we have a map \( \mathbb{P}(X(F)) \to \mathbb{P}^{d+2-t} \). Moreover, since \( c_1(F^*) = H \) and \( c_2(F^*) = (d + 2 - t)L \), it follows that the degree of the image of \( \mathbb{P}(X(F)) \) in \( \mathbb{P}^{d+2-t} \) is

\[
\deg \mathbb{P}(X(F)) = c_1(F^*)^3 - 2c_1(F^*)c_2(F^*) = H^3 - 2(d + 2 - t)HL
\]

\[
= (4d + 2) - 2(d + 2 - t) = 2d - 2 + 2t.
\]

Similarly, let \( \mathcal{M}FT_{d}^{2}(k) \) be the moduli space of pairs \((Y, E)\), where \( Y \) is a Fano threefold of index 2 and degree \( d \) and \( E \) is an instanton bundle of charge \( k \) on \( Y \), that is, a stable vector bundle of rank 2 with \( c_1(E) = 0, c_2(E) = kL \), and \( H^1(Y, E(-1)) = 0 \) (see [9]). Using the Riemann–Roch theorem (see the proof of Proposition 3.9), one can check that for any \( d \), any \( k \), and any \((Y, E) \in \mathcal{M}FT_{d}^{2}(k) \) we have

\[
\dim H^0(Y, E^*(1)) = 2d - 2k + 4,
\]
so that we have a map $\mathbb{P}_Y(E) \to \mathbb{P}^{2d-2k+3}$. Moreover, since $c_1(E^*(1)) = 2H$ and $c_2(E(1)) = H^2 + kL$, it follows that the degree of the image of $\mathbb{P}_Y(E)$ in $\mathbb{P}^{2d-2k+3}$ is

$$\deg \mathbb{P}_Y(E) = c_1(E^*(1))^3 - 2c_1(E^*(1))c_2(E^*(1)) = 8H^3 - 4H(H^2 + kL) = 8d - 4(d + k) = 4d - 4k.$$  

Note that whenever $d + 1 = 2k - t$ the dimensions and the degrees coincide.

**Conjecture 4.12.** For each $1 \leq d \leq 5$ there are integers $k, t \geq 0$ satisfying $d + 1 = 2k - t$ for which there is an isomorphism $\xi: \mathcal{MFT}^2_d(k) \cong \mathcal{MF}^1_{4d+2}(t)$ such that for $(X, F) = (Y, E)$ there is an isomorphism $h: H^0(Y, E^*(1)) \cong H^0(X, F^*)$ and a birational isomorphism $\theta: \mathbb{P}_X(F) \to \mathbb{P}_Y(E)$ such that the diagram

$$\begin{array}{ccc}
\mathbb{P}_X(F) & \xrightarrow{-\theta} & \mathbb{P}_Y(E) \\
\downarrow & & \downarrow \\
\mathbb{P}(H^0(X, F^*)) & \xrightarrow{h} & \mathbb{P}(H^0(Y, E^*(1))^*)
\end{array}$$

commutes. Moreover, there is an equivalence $A_X \cong B_Y$; that is, $Z_d \subset \mathcal{MFT}^2_d \times \mathcal{MF}^1_{4d+2}$ is the image of the graph of the isomorphism $\xi: \mathcal{MFT}^2_d(k) \to \mathcal{MF}^1_{4d+2}(t)$.

**Remark 4.13.** For $d = 3$ by Theorem 4.10 we should take $k = 2$ and $t = 0$. For $d = 5$ we expect that $k = 4$ and $t = 2$ will work.

5. APPENDIX: THE GROTHENDIECK GROUP AND ALGEBRAIC CYCLES

Let $X$ be a smooth projective variety of dimension $n$. Let $A^p(X)$ denote the group of algebraic cycles on $X$ of codimension $p$ modulo rational equivalence. Let $A^\bullet(X) = \bigoplus_{p=0}^n A^p(X)$ be the Chow ring. Let $K_0(X)$ be the Grothendieck group of the category of coherent sheaves on $X$ (equivalently, of the derived category $\mathcal{D}^b(X)$). Consider the Chern character map $\text{ch}: K_0(X) \to A^\bullet(X) \otimes \mathbb{Q}$. It is well known that $\text{ch}$ induces an isomorphism of $\mathbb{Q}$-vector spaces $K_0(X) \otimes \mathbb{Q} \to A^\bullet(X) \otimes \mathbb{Q}$.

On the other hand, consider on both sides the numerical equivalence. Recall that an algebraic cycle $a \in A^p(X)$ is numerically equivalent to zero if it lies in the kernel of the bilinear intersection form:

$$A^\bullet(X) \otimes A^\bullet(X) \xrightarrow{\cap} A^\bullet(X) \xrightarrow{\text{pr}} A^n(X) \xrightarrow{\deg} \mathbb{Z};$$

in other words, if its intersection with any cycle in $A^{n-p}(X)$ is zero. Let $A^\bullet(X)_\text{num} = \bigoplus A^p(X)_\text{num}$ be the ring of algebraic cycles modulo numerical equivalence. Note that any torsion class in $A^\bullet(X)$ is numerically trivial; hence $A^\bullet(X)_\text{num}$ is torsion free.

Similarly, a class $v \in K_0(X)$ is numerically equivalent to zero if it lies in the kernel of the Euler bilinear form:

$$\chi: K_0(X) \otimes K_0(X) \to \mathbb{Z}, \quad \chi([F], [G]) = \sum_i (-1)^i \dim \text{Ext}^i(F, G).$$

Let $K_0(X)_\text{num} = K_0(X)/\text{Ker} \chi$ be the numerical Grothendieck group.

The Riemann–Roch formula shows that the kernel of the Euler form coincides with the preimage under the Chern character map of the subring of $A^\bullet(X) \otimes \mathbb{Q}$ consisting of numerically trivial algebraic cycles. It follows that $\text{ch}$ descends to a map $K_0(X)_\text{num} \to A^\bullet(X)_\text{num} \otimes \mathbb{Q}$, which we denote by $\text{ch}$ as well, and induces an isomorphism of $\mathbb{Q}$-vector spaces $K_0(X)_\text{num} \otimes \mathbb{Q} \cong A^\bullet(X)_\text{num} \otimes \mathbb{Q}$.
For any $p$-cycle $Z = \sum a_i S_i$ we define $[O_Z] := \sum a_i [O_{S_i}] \in K_0(X)$. Note that $[O_Z]$ really depends on the cycle $Z$, not only on its rational or numerical equivalence class. Slightly later we will show how one can get rid of this dependence (see Remark 5.3).

**Definition 5.1.** We will say that a smooth projective $n$-dimensional variety $X$ is $AK$-compatible if for any collection of cycles $Z_p^i$ on $X$, $0 \leq p \leq n$, $1 \leq i \leq m_p$, such that $\text{codim } Z_p^i = p$ and $\{Z_p^1, Z_p^2, \ldots, Z_p^{m_p}\}$ is a basis in $A^p(X)_{\text{num}}$ the classes $[O_{Z_p^i}]$, $0 \leq p \leq n$, $1 \leq i \leq m_p$, form a $\mathbb{Z}$-basis in $K_0(X)_{\text{num}}$.

If an algebraic variety $X$ is $AK$-compatible, then one can easily describe its numerical Grothendieck group by choosing some bases in the groups of algebraic cycles and considering their structure sheaves. Certainly, such a description may be useful in many cases. The goal of this section is to find some easily verifiable criterion for $AK$-compatibility.

We start with some preparations. Consider the following two filtrations on $K_0(X)_{\text{num}}$. The first one is induced by the codimension filtration on $A^\bullet(X)$:

$$F^p K_0(X)_{\text{num}} = \text{ch}^{-1}\left( \bigoplus_{q=p}^n A^q(X)_{\text{num}} \otimes \mathbb{Q} \right).$$

The second one is induced by the codimension of the support:

$$S^p K_0(X)_{\text{num}} = \langle [G] \mid \text{codim supp}(G) \geq p \rangle,$$

where $\langle \cdot \rangle$ stands for the linear span. We will call the filtration $F^\bullet$ the induced filtration and $S^\bullet$ the support filtration. Let $\text{gr}_{F}^p K_0(X)_{\text{num}}$ and $\text{gr}_{S}^p K_0(X)_{\text{num}}$ be the graded factors of these filtrations.

Note that for any $G \in D^b(X)$ with $\text{codim supp}(G) \geq p$ we have $\text{ch}(G) \in \bigoplus_{q \geq p} A^q(X)_{\text{num}} \otimes \mathbb{Q}$; hence $S^p K_0(X)_{\text{num}} \subseteq F^p K_0(X)_{\text{num}}$ and we have the following commutative diagram:

$$
\begin{array}{ccc}
S^p K_0(X)_{\text{num}} & \longrightarrow & F^p K_0(X)_{\text{num}} \\
\uparrow & \searrow \text{ch} & \uparrow \text{ch} \\
S^{p+1} K_0(X)_{\text{num}} & \longrightarrow & F^{p+1} K_0(X)_{\text{num}}
\end{array}
$$

Passing to the graded factors, we obtain a chain of maps

$$\text{gr}_{S}^p K_0(X)_{\text{num}} \overset{i_p}{\longrightarrow} \text{gr}_{F}^p K_0(X)_{\text{num}} \overset{\text{ch}_p}{\longrightarrow} A^p(X)_{\text{num}} \otimes \mathbb{Q}.$$

Here $\text{ch}_p$ is the $p$th coefficient of the Chern character and $i_p$ is the map induced by the identity map of $K_0(X)_{\text{num}}$. Note that it follows that $\text{ch}_p : \text{gr}_{F}^p K_0(X)_{\text{num}} \otimes \mathbb{Q} \rightarrow A^p(X)_{\text{num}} \otimes \mathbb{Q}$ is an isomorphism. Let us show that the inverse map is defined over $\mathbb{Z}$.

**Lemma 5.2.** There exists a linear map $O_p : A^p(X)_{\text{num}} \rightarrow \text{gr}_{F}^p K_0(X)_{\text{num}}$ that is inverse to $\text{ch}_p$. Moreover, $O_p(Z) = [O_Z] \mod F^{p+1} K_0(X)_{\text{num}}$.

**Proof.** For each $p$-cycle $Z$ on $X$ define $O_p(Z)$ as the image of the class of its structure sheaf in $\text{gr}_{F}^p K_0(X)_{\text{num}}$. Since

$$\text{ch}(O_Z) = Z + \text{terms of degree higher than } p,$$

we have $[O_Z] \in F^p K_0(X)_{\text{num}}$ and $\text{ch}_p(O_p(Z)) = Z$. Since $\text{ch}_p$ is injective, it follows that $O_p$ is correctly defined and $\text{ch}_p \circ O_p = \text{id}$. □

**Remark 5.3.** As we see from this lemma, the class of $[O_Z]$ in $\text{gr}_{F}^p K_0(X)_{\text{num}}$ only depends on the numerical class of $Z$. 

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Further, it is easy to see that for any coherent sheaf $G$ supported in codimension $p$ the $p$th coefficient of the Chern character is integer, $\text{ch}_p(G) \in A^p(X)_{\text{num}}$. Indeed, if $Z_1, \ldots, Z_m$ are codimension $p$ components of $\text{supp}G$ and $\ell_i$ is the length of $G$ at a generic point of $Z_i$, then $\text{ch}_p(G) = \sum \ell_i Z_i$. Moreover, the same argument shows that the map $\text{ch}_p: \text{gr}_S^p K_0(X)_{\text{num}} \to A^p(X)_{\text{num}}$ is surjective. It follows that we have the following commutative diagram:

\[
\begin{array}{ccc}
A^p(X)_{\text{num}} & \xrightarrow{\text{ch}_p} & A^p(X)_{\text{num}} \otimes \mathbb{Q} \\
\text{O}_p \downarrow & & \downarrow \text{ch}_p \\
\text{gr}_S^p K_0(X)_{\text{num}} & \xrightarrow{i_p} & \text{gr}_F^p K_0(X)_{\text{num}}
\end{array}
\]

**Proposition 5.4.** The following properties for a smooth projective variety $X$ are equivalent:

(i) $\text{O}_p$ is an isomorphism for all $0 \leq p \leq n$;

(ii) $X$ is AK-compatible;

(iii) $S^p K_0(X)_{\text{num}} = F^p K_0(X)_{\text{num}}$ for all $0 \leq p \leq n$;

(iv) $\text{ch}_p(\text{gr}_F^p K_0(X)_{\text{num}}) \subset A^p(X)_{\text{num}}$ for all $0 \leq p \leq n$.

**Proof.** (i) $\Rightarrow$ (ii): An evident induction argument shows that $\{[\mathcal{O}_{Z_q}]_{q \geq p}^{1 \leq i \leq m_q}\}$ is a basis in $F^p K_0(X)_{\text{num}}$.

(ii) $\Rightarrow$ (iii): Assume that $X$ is AK-compatible. Choose bases in all $A^p(X)_{\text{num}}$ as in Definition 5.1 and assume that $v = \sum a_{q}^i [\mathcal{O}_{Z_q}] \in F^p K_0(X)_{\text{num}}$ for some $a_{q}^i \in \mathbb{Z}$. Let $q$ be the minimal integer such that $a_{q}^i \neq 0$ for some $i$ and assume that $q < p$. Then, applying $\text{ch}_q$, we see that

\[
0 = \text{ch}_q(v) = \sum \alpha_{q}^i \text{ch}_q([\mathcal{O}_{Z_q}]) = \sum \alpha_{q}^i \text{ch}_q(\mathcal{O}_q(Z_q)) = \sum \alpha_{q}^i Z_q,
\]

which implies $\alpha_{q}^i = 0$ for all $i$. So, it follows that $q \geq p$; hence $\alpha_{q}^i = 0$ for all $q < p$. But then it is clear that $v \in S^p K_0(X)_{\text{num}}$. So we see that $F^p K_0(X)_{\text{num}} \subset S^p K_0(X)_{\text{num}}$.

(iii) $\Rightarrow$ (iv): If $S^p K_0(X)_{\text{num}} = F^p K_0(X)_{\text{num}}$, then $\text{gr}_S^p K_0(X)_{\text{num}} = \text{gr}_F^p K_0(X)_{\text{num}}$ for all $p$; hence $\text{ch}_p(\text{gr}_S^p K_0(X)_{\text{num}}) = \text{ch}_p(\text{gr}_F^p K_0(X)_{\text{num}}) \subset A^p(X)_{\text{num}}$.

(iv) $\Rightarrow$ (i): Since $\text{ch}_p$ and $\text{O}_p$ are mutually inverse isomorphisms of $\text{gr}_F^p K_0(X)_{\text{num}} \otimes \mathbb{Q}$ and $A^p(X)_{\text{num}} \otimes \mathbb{Q}$ and preserve the lattices $\text{gr}_F^p K_0(X)_{\text{num}}$ and $A^p(X)_{\text{num}}$, it follows that they induce isomorphisms of these lattices. $\square$

Our next goal is to give several sufficient conditions for AK-compatibility.

**Lemma 5.5.** For any smooth projective variety $X$ we have

\[
\text{ch}_p(\text{gr}_F^p K_0(X)_{\text{num}}) \subset A^p(X)_{\text{num}}
\]

for $p = 0, 1, 2$ and $p = n$.

**Proof.** Note that $\text{ch}_0(G)$ is the rank of $G$ and $\text{ch}_1(G) = c_1(G)$, which implies the claim for $p = 0$ and $p = 1$. For $p = 2$ we have $\text{ch}_2(G) = c_1(G)^2/2 - c_2(G)$, so if $c_1(G) = \text{ch}_1(G) = 0$, then $\text{ch}_2(G) = -c_2(G) \in A^2(X)_{\text{num}}$. Finally, if $G \in F^p K_0(X)_{\text{num}}$, then by the Riemann–Roch theorem we have $\text{ch}_n(X) = \chi(\mathcal{O}_X, G) \in \mathbb{Z}$, where we have identified $A^n(X)_{\text{num}}$ with $\mathbb{Z}$ via the degree map. This proves the claim for $p = n$. $\square$

**Corollary 5.6.** If $\dim X \leq 3$, then $X$ is AK-compatible.

Another approach to AK-compatibility is given by the following

**Lemma 5.7.** Assume that the intersection pairing $A^p(X)_{\text{num}} \otimes A^{n-p}(X)_{\text{num}} \to \mathbb{Z}$ induces an isomorphism $A^p(X)_{\text{num}} \to A^{n-p}(X)^*_{\text{num}}$. Then the map $\mathcal{O}_p: A^p(X)_{\text{num}} \to \text{gr}_F^p K_0(X)_{\text{num}}$ is an

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isomorphism. In particular, if the intersection pairing $A^p(X)_{\text{num}} \otimes A^{n-p}(X)_{\text{num}} \to \mathbb{Z}$ induces an isomorphism $A^p(X)_{\text{num}} \to A^{n-p}(X)_{\text{num}}^*$ for all $p$, then $X$ is AK-compatible.

**Proof.** Let $Z, W \subset X$ be subschemes of codimension $p$ and $n-p$, respectively. Then (†) and the Riemann–Roch theorem imply that

$$Z \cdot W = (-1)^p \chi(O_Z, O_W);$$

hence we have a commutative diagram

$$
\begin{array}{ccc}
A^p(X)_{\text{num}} & \xrightarrow{O_p} & \gr_F^p K_0(X)_{\text{num}} \\
\downarrow & & \downarrow (-1)^p \chi \\
A^{n-p}(X)_{\text{num}}^* & \xleftarrow{O_{n-p}^*} & \gr_F^{n-p} K_0(X)_{\text{num}}^*
\end{array}
$$

Note that all the maps are finite index embeddings. So, if the left vertical arrow is an isomorphism, then $O_p$ is also an isomorphism. □

The results of this section allow one to describe $K_0(X)_{\text{num}}$ for all Fano threefolds.

**Corollary 5.8.** Let $V$ be a Fano threefold with $\text{Pic} V \cong \mathbb{Z}$. Let $H$ be the generator of $\text{Pic} V$, $L$ a line on $V$, and $P$ a point on $V$. Then $K_0(X)_{\text{num}} = \langle [O_V], [O_H], [O_L], [O_P] \rangle$.

**Proof.** We can argue either by Corollary 5.6 or by Lemma 5.7 that $V$ is AK-compatible. Hence by the definition of AK-compatibility we obtain the required basis. □

**Remark 5.9.** One can combine the results of Lemmas 5.5 and 5.7 for the verification of AK-compatibility. In other words, if for an algebraic variety $X$ the conditions of Lemma 5.7 are true for all $3 \leq p \leq n-1$, then $X$ is AK-compatible. Indeed, it is easy to see from the proof of Proposition 5.4 that property $(iv_p)$ for each $p$ implies property $(i_p)$.

These considerations apply, e.g., to cubic fourfolds. Indeed, for $p = 3$ the conditions of Lemma 5.7 are true; hence the cubic fourfold is AK-compatible.

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