Varying Newton constant and black hole quantization

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The thermodynamics of black holes is discussed for the case, when the Newton constant $G$ is not a constant, but is the thermodynamic variable. This gives for the first law of the Schwarzschild black hole thermodynamics: $dS_{\text{BH}} = -A\,dK + \frac{dM}{G_{\text{grav}}}$, where the gravitational coupling $K = 1/4G$, $M$ is the black hole mass, $A$ is the area of horizon, and $T_{\text{BH}}$ is Hawking temperature. From this first law it follows that $M^2/K$ is the adiabatic invariant, which according to Bekenstein should be quantized. The entropy of the Kerr black hole is expressed in terms of two integers, angular momentum $J$ and the radial quantum number $N$, which comes from the quantization of the dimensionless adiabatic parameter $M^2/K$. It can be also expressed in terms of square roots of integer numbers similar to that in string theory. The area of the black hole horizon does not have such property, because it contains the non-universal function $K$. For the Kerr-Newmann black holes there are at least two additional quantum numbers: the electric charge $q$ in terms of the electric charge of electrons, and the number $\nu$, which reflects the quantization of the fine structure constant as another adiabatic invariant.

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I. INTRODUCTION

Bekenstein\textsuperscript{1} proposed that the horizon area $A$ is an adiabatic invariant and thus can be quantized according to the Ehrenfest principle that classical adiabatic invariants may correspond to observables with discrete spectrum. Here we consider the quantization under condition when the gravitational coupling is represented by a scalar thermodynamic function. This modifies the first law of the black hole thermodynamics and leads to the alternative adiabatic invariant, which prescribes different quantization rules.

II. MODIFICATION OF THE FIRST LAW OF BLACK HOLE THERMODYNAMICS

The Einstein-Hilbert gravitational action is

$$S_{\text{grav}} = \frac{1}{4\pi} \int d^3x \sqrt{-g} K \mathcal{R},$$

where $\mathcal{R}$ is the scalar curvature, and we choose the gravitational coupling $K = 1/(4G) = 1/(4R_{\text{Planck}}^2)$. In the modified gravity theories, such as the scalar-tensor and $f(R)$ theories (see e.g.\textsuperscript{2} with references therein), the effective Newton “constant” $G$ can be space-time dependent, and thus is not fundamental.

Here we assume that the variables, which enter in Einstein action – the scalar Riemann curvature $R$ and the gravitational coupling constant $K$ – are the local thermodynamic variables, which are similar to temperature, pressure, chemical potential, number density, etc. All the thermodynamical characteristics of the macroscopic matter are the functions of the Riemann curvature.\textsuperscript{3} Correspondingly the gravitational coupling constant $K$ in front of the scalar curvature in Einstein action becomes the thermodynamic quantity.\textsuperscript{4} The application of this thermodynamics to the global object, such as black hole, suggests that the variable $K$ (actually its asymptotic value at infinity) should enter the thermodynamic laws for the black hole.

In terms of this coupling $K$ the Hawking temperature of Schwarzschild black hole and its Bekenstein entropy are:

$$T_{\text{BH}} = \frac{K}{2\pi M}, \quad S_{\text{BH}} = \frac{\pi M^2}{K}.$$  \hspace{1cm} (2)

Then, using the black hole area $A = \pi M^2/K^2$; the gravitational coupling $K = 1/4G$; the Hawking temperature $T_{\text{BH}} = M/2AK = K/2\pi M$; and the black hole entropy $S_{\text{BH}} = AK$, one obtains:

$$dS_{\text{BH}} = d(AK) = \pi d(M^2/K) = -\pi \frac{M^2}{K^2} dK + 2\pi \frac{M}{K} dM = -A\,dK + \frac{dM}{T_{\text{BH}}}. \hspace{1cm} (3)$$

This suggests the following modification of the first law of black hole thermodynamics in case if $K$ is a thermodynamic variable:

$$dS_{\text{BH}} = -A\,dK + \frac{dM}{T_{\text{BH}}}. \hspace{1cm} (4)$$

The variable $AT_{\text{BH}}$ and the variable $K$ are conjugate thermodynamic variables. In general, the variable $K$ depends on space, but in the black hole thermodynamics one uses the asymptotic values of the parameters at infinity.
III. ADIABATIC CHANGE OF $K$ AND ADIABATIC INVARIANT

Let us change $K$ and $M$ adiabatically, i.e. at constant entropy. Then the equation $dS_{BH} = 0$ gives

$$\frac{dM}{dK} = AM_{BH} = \frac{M}{2K}. \quad (5)$$

This shows that $M^2/K = \text{const}$ is the adiabatic invariant for spherical neutral black hole, and thus according to the Bekenstein conjecture, it can be quantized in quantum mechanics.

$$\frac{M^2}{K} = aN . \quad (6)$$

Here $N$ is integer, and $a$ is some fundamental dimensionless parameter of order unity.

Note that strictly speaking, the black hole area $A = S_{BH}/K = \pi M^2/K^2$ is not quantized, since it is not dimensionless, and thus the prefactor cannot be fundamental. However, the entropy is dimensionless and thus can be expressed as some dimensionless function of integers. The simplest quantization follows from Bohr quantization rule in Eq.(6), which gives

$$S_{BH}(N) = \pi \frac{M^2}{K} = \pi aN . \quad (7)$$

IV. KERR BLACK HOLE WITH TWO QUANTUM NUMBERS

Let us now estimate the parameter $a$. For that let us consider the Kerr black hole, which contains another integer quantum number, the angular momentum quantum number $J$. Entropy of the rotating electrically neutral black hole is:

$$S_{BH}(M, J, K) = \frac{\pi}{2} \left( \frac{M^2}{K} + \sqrt{\frac{M^4}{K^2} - 16J^2} \right) . \quad (8)$$

It shows that the quantity $M^2/K$ remains to be the adiabatic invariant. Then using Eq.(6) for this invariant one obtains that the entropy of rotating Kerr BH depends on two quantum numbers, the angular momentum quantum number $J$ and the radial quantum number $N$:

$$S_{BH}(N, J) = \frac{\pi}{2} \left( aN + \sqrt{a^2N^2 - 16J^2} \right) . \quad (9)$$

For extremal Kerr BH, for which $aN = J$ and $S_+ = S_-$, one has the universal result

$$S_{BH}(J) = 2\pi J , \quad (10)$$

which is well known, see e.g. Refs.\textsuperscript{7,8}.

Eq.(9) can be applicable to the extremal black hole only in case if $aN/4$ is integer, i.e. if the prefactor $a = 4k$. With the simplest choice $k = 1$ one obtains the following quantization of the black hole entropy:

$$S_{BH}(N, J) = 2\pi \left( N + \sqrt{N^2 - J^2} \right) = \pi \left( \sqrt{N_1 + \sqrt{N^2 - J^2}} \right)^2$$

$$\quad = \pi \left( \sqrt{N_1 + \sqrt{N_2}} \right)^2 \quad (12)$$

where $N_1 = N + J$ and $N_2 = N - J \geq 0$.

V. $A$ AND $K$ AS CANONICALLY CONJUGATE VARIABLES

Quantization in Eq.(6) with $a = 4$ can be obtained assuming that $A$ and $K/4$ are canonically conjugate variables and applying Bohr quantization rule:

$$\frac{1}{4} \int_{M=\text{const}} dA = \frac{\pi}{4} \int \frac{dM}{2} = \frac{\pi}{2} M^2 K = 2\pi N . \quad (13)$$

VI. APPLICATION TO SCHWARZSCHILD BLACK HOLE

This can be now applied to the Schwarzschild black hole, i.e. with $J = 0$. For $k = 1$ its entropy is:

$$S_{BH}(N) = 4\pi N , \quad (14)$$

The mass of the Schwarzschild black hole:

$$M = \frac{\sqrt{N}}{l_{\text{Planck}}} . \quad (15)$$

Eq.(14) reproduces Eq.(44) in Ref.\textsuperscript{2}.

VII. KERR-NEUMANN BLACK HOLE

For charged BH another quantum number should enter: the integer or fractional electric charge $q$ in terms of the electric charge of electron. However, the electric charge appears together with the fine structure constant $\alpha$ in combination $\alpha q^2$:

$$S_{BH}(M, J, K, q, \alpha) = \frac{\pi}{2} \left( \frac{M^2}{K} - 2\alpha q^2 + \sqrt{\frac{M^4}{K^2} - 4\alpha q^2 \frac{M^2}{K} - 16J^2} \right) . \quad (16)$$

In Standard Model, $\alpha$ is not a universal parameter, since the inverse fine structure "constant", $1/\alpha$, is logarithmic function which depends both on UV and IR cut-off. The universality can be restored only in case if $\alpha$ has quantized values, which is rather problematic.\textsuperscript{10} However, the quantization can be possible if $\alpha$ serves as adiabatic invariant for quantum vacuum. Earlier it was suggested
that there is connection between the fine structure constant and the gravitational coupling $K$, see Refs.\textsuperscript{11–16}. In the phenomenological theory of the quantum vacuum,\textsuperscript{2} both $K$ and $\alpha$ are the functions of the dynamical variables, which describe the vacuum.

Let us consider the quantum vacuum of charged Dirac fields with mass $m$ and electric charges $q_a$, then in the logarithmic approximation the running coupling constant has the following dependence on the ultraviolet and infrared cut-off parameters, $K$ and $m^2$ corespondingly:

$$\frac{1}{\alpha} = \frac{1}{6\pi} \ln \frac{K}{m^2}. \quad (17)$$

Here $a$ marks the sum over fermions.

The equation (17) together with Eq. (10) for entropy suggests that the parameter $K/m^2$ is the adiabatic invariant of the quantum vacuum, which may have quantized values in quantum mechanics, $K/m^2 \sim \nu$. It automatically becomes the adiabatic invariant for the black hole together with the parameter $M^2/K \sim N$. The square root of the product of these parameters gives the number of particle of mass $m$, which participated in formation of the black hole, $\sqrt{\nu N} = \sqrt{M^2/m^2} = N_m$. Thus the black hole entropy depends on several integer numbers, $N, J, q, \nu$, and the other possible fermionic numbers:

$$S_{BH}(N, J, q, \nu) = 2\pi \left( N - \frac{1}{2} \alpha(\nu)q^2 + \sqrt{N^2 - \alpha(\nu)q^2 N - J^2} \right). \quad (18)$$

For massless fermions, instead of fermionic mass the Hawking temperature enters, and one has

$$\frac{1}{\alpha} = \frac{1}{6\pi} \sum_a q_a^2 \ln \frac{M^2}{K} = \frac{\sum_a q_a^2}{6\pi} \ln N. \quad (19)$$

\section*{VIII. DISCUSSION}

For varying gravitational coupling $K$, one has an alternative quantization scheme for the black hole. Instead of the area of the horizon $A$, the entropy $S = KA$ should be considered, since according to Ted Jacobson\textsuperscript{12} it does not change under renormalization of $K$. While $K$ and $A$ are dimensionful and cannot be quantized, the entropy is dimensionless and thus can be quantized in terms of some the fundamental numbers.

Here we suggest that the entropy of Kerr black hole is quantized in terms of square roots of integer numbers in Eq. (12), which is similar to linear combinations of square roots of positive integers discussed in string theory applied to extremal and near-extremal black holes.\textsuperscript{8,18} The Kerr-Newmann black holes adds at least two quantum numbers: the electric charge $q$ in terms of the electric charge of electrons, and the number $\nu$, which reflects the suggested quantization of the fine structure constant as adiabatic parameter.

There are the open problems. We assumed that Eq. (6) for quantization of adiabatic invariant $M^2/K$ in terms of radial quantum number $N$ does not depend on the other quantum numbers, such as $J$ in the Kerr black hole. However, there is connection between these two quantum numbers, because of the constraint $N \geq J$. The constraint can be removed if we use different quantization rule for the adiabatic invariant $M^2/K$ in terms of two numbers, such as:

$$\frac{M^2}{K} = 4\sqrt{N^2 + J^2}. \quad (20)$$

Then instead of Eq. (11) one would have

$$S_{BH}(N, J) = 2\pi \left( N + \sqrt{N^2 + J^2} \right), \quad (21)$$

which is now applicable for any $N, J \geq 0$. However, it is not clear, whether this can be applicable for small $N$ and $J$, because thermodynamics is valid only for macroscopic black holes, with $N, J \gg 1$, where semiclassical approximation works.

Anyway, all these speculations with quantum black holes should be supported by microscopic theory\textsuperscript{21} and the Bekenstein idea on the role of adiabatic invariants in quantization of the black hole requires further consideration.

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