Fred C. Adams · Anthony M. Bloch

Hill’s Equation with Random Forcing Parameters: Determination of Growth Rates through Random Matrices

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Abstract This paper derives expressions for the growth rates for the random $2 \times 2$ matrices that result from solutions to the random Hill’s equation. The parameters that appear in Hill’s equation include the forcing strength $q_k$ and oscillation frequency $\lambda_k$. The development of the solutions to this periodic differential equation can be described by a discrete map, where the matrix elements are given by the principal solutions for each cycle. Variations in the $(q_k, \lambda_k)$ lead to matrix elements that vary from cycle to cycle. This paper presents an analysis of the growth rates including cases where all of the cycles are highly unstable, where some cycles are near the stability border, and where the map would be stable in the absence of fluctuations. For all of these regimes, we provide expressions for the growth rates of the matrices that describe the solutions.

Keywords Hill’s equation, Random matrices, Lyapunov exponents

1 Introduction

This paper considers the growth rates for Hill’s equation with parameters that vary from cycle to cycle. In this context, Hill’s equation takes the form

$$\frac{d^2 y}{dt^2} + [\lambda_k + q_k \dot{Q}(t)]y = 0,$$

(1)

F. C. Adams
Michigan Center for Theoretical Physics
Physics Department, University of Michigan, Ann Arbor, MI 48109
E-mail: fca@umich.edu

A. M. Bloch
Michigan Center for Theoretical Physics
Mathematics Department, University of Michigan, Ann Arbor, MI 48109
E-mail: abloch@umich.edu
where the barrier shape function $\hat{Q}(t)$ is periodic, so that $\hat{Q}(t + \Delta \tau) = \hat{Q}(t)$, where $\Delta \tau$ is the period. Here we take $\Delta \tau = \pi$, and the function $\hat{Q}$ is normalized so that $\int_0^{\Delta \tau} \hat{Q} dt = 1$. The forcing strength parameters $q_k$ are a set of independent identically distributed (i.i.d.) random variables that take on a new value every cycle (where the index $k$ labels the cycle). The parameters $\lambda_k$, which determine the oscillation frequency in the absence of forcing, also vary from cycle to cycle (and are i.i.d.). In principal, the cycle interval $\Delta \tau$ could also vary; however, this generalized case can be reduced to the problem of equation (1) through an appropriate re-scaling of the other parameters (see Theorem 1 of [AB]).

Hill’s equations [HI] with constant values of the parameters have been well studied and arise in a wide variety of applications [MW]. The introduction of parameters that sample a distribution of values is thus a natural generalization of this classic problem. Here we refer to the case with constant parameters as the “classical regime” of the general case.

For this class of periodic differential equations, the transformation that maps the coefficients of the principal solutions from one cycle to the next takes the form

$$M_k = \begin{bmatrix} h_k & (h_k^2 - 1)/g_k \\ g_k & h_k \end{bmatrix},$$

where the subscript denotes the cycle. The matrix elements are defined by $h_k = y_1(\pi)$ and $g_k = \dot{y}_1(\pi)$ for the $k$th cycle, where $y_1$ and $y_2$ are the principal solutions for that cycle. Note that the matrix has only two independent elements rather than four: Since the Wronskian of the original differential equation (1) is unity, the determinant of the matrix map must be unity, and this constraint eliminates one of the independent elements. In addition, this paper specializes to the case where the periodic functions $\hat{Q}(t)$ are symmetric about the midpoint of the period, so that $y_1(\pi) = \dot{y}_2(\pi)$, which eliminates a second independent element [MW]; this symmetry applies to the applications that motivated this work.

For transformation matrices $M_k$ of the form (2), the eigenvalues $\lambda_k$ can be used to classify the matrix types [LR]. The characteristic polynomial has the form

$$\lambda^2 - 2h_k \lambda + 1 = 0.$$  

This equation allows for three classes of eigenvalues $\lambda_k$: For $|h_k| > 1$, the eigenvalues are real and have the same sign, and the transformation matrix is hyperbolic symplectic; we denote this regime as classically unstable. When $|h_k| < 1$, the eigenvalues are complex and the matrix is elliptic; this regime is denoted as classically stable. The remaining possibility is for $|h_k| = 1$, which leads to degenerate eigenvalues equal to either $+1$ or $-1$; these matrices are parabolic and are stable under multiplication.

This paper studies the multiplication of infinite strings of random matrices of the form (2), i.e., the product of $N$ such matrices in the limit $N \to \infty$. The problem of finding growth rates for infinite products of matrices with random elements was formulated over four decades ago [FU, FK], where existence results were given. We recall the key result here for convenience:
For a $k \times k$ matrix $A$ with real or complex entries, let $||A||$ denote the Frobenius norm.

**Theorem (FK):** Let $X^1, X^2, X^3, \ldots$ form a metrically transitive stationary stochastic process with values in the set of $k \times k$ matrices. Suppose $\log^+ ||X^1||$ exists, where $\log^+ t = \max(\log t, 0)$, then the limit $\lim_{N \to \infty} ||X^N X^{N-1} \cdots X^1||$ exists.

Determination of the growth rates are thus carried out in the limit of large $N$, and all probabilistic limits given here are meant almost surely.

A great deal of subsequent work has studied differential equations of the form (1) and the growth rates of the corresponding random matrices [CL, PF, LGP]. In spite of this progress, there are relatively few examples that provide explicit expressions for the growth rates. The goal of this paper is relatively modest: It provides (what we believe to be) new analytic expressions for the growth rates of random matrices of the form (2). These expressions are derived for various regimes of parameter space, as described below.

The outline of this paper is as follows: Section 2 reviews the astrophysical background that led us to this topic. Section 3 considers matrix multiplication for the case where the solutions are unstable in the classical regime. Section 4 develops approximations for this regime and provides some numerical verification. Section 5 considers matrix multiplication in the regime where the solutions are classically stable. In this case, the transformation matrices $M_k$ correspond to elliptical rotations and matrix multiplication is stable in the absence of fluctuations; random variations in the matrix elements render the solutions unstable. The paper concludes (in Section 6) with a brief summary of the results.

### 2 Astrophysical Background

The motivation for considering random Hill’s equations arose in studies of orbit problems in astrophysics [AK]. When an orbit starts in the principal plane of a triaxial, extended mass distribution (such as a dark matter halo), the motion is unstable to perturbations in the perpendicular direction. The development of the instability is described by a random Hill’s equation with the form given by equation (1).

To illustrate this type of behavior, consider an extended mass distribution with a density profile of the form

$$
\rho = \frac{\rho_0}{m} \quad \text{with} \quad m^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2},
$$

where $\rho_0$ is a density scale. This form arises in many different astrophysical contexts, including dark matter halos, galactic bulges, and young embedded star clusters. The density field is thus constant on ellipsoids, where, without loss of generality, $a > b > c > 0$. For this density profile, one can find analytic forms for both the gravitational potential and the force terms [AK]. From these results, one can determine the orbital motion for a test particle moving in the potential resulting from the triaxial density distribution of equation (4). When the orbit begins in any of the three principal planes, the motion is
generally unstable to perturbations in the perpendicular direction [AB, AK]. For example, for an orbit initially confined to the \(x-z\) plane, the amplitude of the \(y\) coordinate will (usually) grow exponentially with time. In the limit of small \(|y| \ll 1\), the equation of motion for the perpendicular coordinate simplifies to the form

\[
\frac{d^2 y}{dt^2} + \omega_y^2 y = 0 \quad \text{where} \quad \omega_y^2 = \frac{4/b}{\sqrt{c^2x^2 + a^2z^2 + b\sqrt{x^2 + z^2}}}. \tag{5}
\]

The time evolution of the coordinates \((x, z)\) is determined by the orbit in the original \(x-z\) plane. Since the orbital motion is nearly periodic, the \([x(t), z(t)]\) dependence of \(\omega_y^2\) represents a nearly periodic forcing term. The forcing strengths, and hence the parameters \(q_k\) appearing in Hill's equation (1), are determined by the inner turning points of the orbit (with appropriate weighting from the axis parameters \([a, b, c]\)). Since the orbits are usually chaotic, the distance of closest approach, and hence the strength \(q_k\) of the forcing, varies from cycle to cycle. The outer turning points of the orbit provide a minimum value of \(\omega_y^2\), which defines the unforced oscillation frequency \(\lambda_k\) appearing in Hill's equation. As a result, the quantity \(\omega_y^2\) can be written in the form

\[
\omega_y^2 = \lambda_k + Q_k(t), \tag{6}
\]

where the index \(k\) counts the number of orbit crossings. The shapes of the functions \(Q_k\) are nearly the same, so that one can write \(Q_k = q_k \hat{Q}(t)\), where \(\hat{Q}(t)\) is periodic. The chaotic orbit in the original plane leads to different values of \(\lambda_k\) and \(q_k\) for each crossing. The equation of motion (5) for the \(y\) coordinate thus takes the form of Hill's equation (1), where the period, forcing strength, and oscillation frequency vary from cycle to cycle.

3 Matrix Multiplication for the Classically Unstable Regime

The goal of this work is to find growth rates for solutions of the differential equation (1). These growth rates are determined by multiplication of the random matrices \(\mathcal{M}_k\) (from equation (2)) that connect solutions from cycle to cycle. These transformation matrices can also be written in the form

\[
\mathcal{M}_k = h_k B_k \quad \text{where} \quad B_k = \begin{bmatrix} 1 & x_k \phi_k \\ 1/x_k & 1 \end{bmatrix}, \tag{7}
\]

where \(x_k = h_k/g_k\) and \(\phi_k = 1 - 1/h_k^2\). By virtue of our assumption on the variables \((q_k, \lambda_k)\), the matrices \(\mathcal{M}_k\) form a sequence of i.i.d. matrices. In this section, we consider the problem of matrix multiplication with matrices of the form (7). We specialize to the case where the solutions are unstable in the classical regime so that \(|h_k| \geq 1\) and to the case where \(x_k > 0\). We also assume that the \(h_k\), \(x_k\), and \(1/x_k\) have finite means. With the matrices written in the form (7), the highly unstable regime considered in [AB] can be defined as follows:
**Definition:** Given that solutions to Hill’s equation (1) are determined by transformation matrices of the form (7), the **highly unstable regime** is defined by setting $\phi_k = 1$. This specification thus defines a restricted problem.

We remark that the above regime applies when the matrix elements $|h_k| \gg 1$, which occurs for forcing strength parameters $q_k \gg 1$ [AB2].

The growth rates for Hill’s equation (1) are determined by the growth rates for matrix multiplication of the full set of matrices $M_k$. For a given matrix product, denoted here as $M^{(N)}$, the growth rate $\gamma$ is determined by

$$
\gamma = \lim_{N \to \infty} \frac{1}{N} \log ||M^{(N)}||, \quad (8)
$$

where the result is independent of the choice of norm $|| \cdot ||$. We note that the growth rate is called the top or largest Lyapunov exponent.

Equation (7) separates the growth rate for this problem into two parts. Let the expectation value of a sequence $X_k$ be denoted by

$$
\langle X_k \rangle = \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} X_k
$$

Then the first part $\gamma_h$ of the growth rate is given by

$$
\gamma_h = \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \log |h_k| = \langle \log |h_k| \rangle. \quad (9)
$$

We limit our discussion to distributions of the $h_k$ for which this limit is finite. The remaining part of the growth rate is determined by matrix multiplication of the $B_k$. Note that the original differential equation (1) is defined on a time interval $0 \leq t \leq \pi$, so that the definition of its growth rate includes a factor of $\pi$ [MW], whereas the growth rate for matrix multiplication (8) generally does not [FK]. Ignoring these normalization issues, this paper focuses on the calculation of the growth rates for the matrices $M_k$ and $B_k$.

The product of $N$ matrices of type $B_k$ can be written in the form

$$
B^{(N)} \equiv \prod_{k=1}^{N} B_k = \left[ \frac{\Sigma_{11}}{(1/x_1)\Sigma_{21}} \quad \frac{x_1\Sigma_{12}}{\Sigma_{22}} \right], \quad (10)
$$

where the first equality defines notation and where

$$
\Sigma_{11} = \sum_{j=1}^{2^{N-1}} r_j a_j, \quad \Sigma_{12} = \sum_{j=1}^{2^{N-1}} r_j b_j,
\Sigma_{21} = \sum_{j=1}^{2^{N-1}} \frac{1}{r_j} c_j, \quad \Sigma_{22} = \sum_{j=1}^{2^{N-1}} \frac{1}{r_j} d_j. \quad (11)
$$

Here, the variables $r_j$ are products of ratios of the form

$$
r_j = \frac{x_{\mu_1} x_{\mu_2} \cdots x_{\mu_n}}{x_{\nu_1} x_{\nu_2} \cdots x_{\nu_n}}. \quad (12)
$$
The indices are confined to the range $1 \leq \mu_i, \nu_i \leq N$. The additional factors $a_j, b_j, c_j, d_j$ are products of the variables $\phi_j$, and can be written in the form

$$a_j = \prod_{k=1}^{N} \phi_k^{p_k} \quad \text{where} \quad p_k = 0 \text{ or } 1.$$  \hspace{1cm} (13)

**Result 1:** For the case where $|h_k| > 1$ for all cycles, and in the limit of large $N$, the eigenvalue of the product matrix is given by the formula

$$\lambda = \Sigma_{11} + \Sigma_{22} + O(h^{-2N}) ,$$  \hspace{1cm} (14)

where each of these quantities should be labeled at the $N$th iteration.

**Proof:** The characteristic equation of the product matrix of equation (10) takes the form

$$\lambda^2 - \lambda(\Sigma_{11} + \Sigma_{22}) + \Sigma_{11}\Sigma_{22} - \Sigma_{12}\Sigma_{21} = 0 .$$  \hspace{1cm} (15)

The final term is the determinant of the product matrix, and this determinant is given by the product of the individual matrices, so that

$$\Sigma_{11}\Sigma_{22} - \Sigma_{12}\Sigma_{21} = \prod_{k=1}^{N} (1 - \phi_k) = \prod_{k=1}^{N} \frac{1}{h_k} .$$  \hspace{1cm} (16)

Given that $|h_k| > 1 \forall k$, this term vanishes in the limit $N \to \infty$. As a result, the growing eigenvalue of the characteristic equation (15) simplifies to the form $\lambda = \Sigma_{11} + \Sigma_{22}$. □

**Result 2:** The four sums that specify the matrix elements of the product matrix are not independent. In particular, for the case where $|h_k| > 1$ and in the limit $N \to \infty$, the ratios of the matrix elements approach the form

$$\frac{\Sigma_{12}}{\Sigma_{11}} = \frac{\Sigma_{22}}{\Sigma_{21}} = \text{constant} \equiv f .$$  \hspace{1cm} (17)

**Proof:** As shown above, the determinant of the product matrix vanishes in the limit $N \to \infty$, so that in the limit

$$\Sigma_{11}\Sigma_{22} = \Sigma_{12}\Sigma_{21} .$$  \hspace{1cm} (18)

The result implied by the first equality of equation (17) follows immediately.

Further, one can show by direct construction that if the relation of equation (17) holds, then the relation is preserved under matrix multiplication. Let the product matrix after $N$ cycles have the form

$$B^{(N)} = \begin{bmatrix} \Sigma_T & f x_1 \Sigma_T \\ (1/x_1) \Sigma_B & f \Sigma_B \end{bmatrix} ,$$  \hspace{1cm} (19)

where $f$ is the constant in equation (17). Then the matrix takes the following from after the next cycle:

$$B^{(N+1)} = \begin{bmatrix} \Sigma_T + (x/x_1) \phi \Sigma_B & x_1 f (\Sigma_T + (x/x_1) \phi \Sigma_B) \\ (1/x_1) (\Sigma_B + (x_1/x) \Sigma_T) & f (\Sigma_B + (x_1/x) \Sigma_T) \end{bmatrix} .$$  \hspace{1cm} (20)
so that the left-right symmetry relation is conserved. □

In the above proof we have adopted notation that is used throughout this paper: The subscript ‘1’ denotes the values of the parameters (e.g., $x_1$) for the first cycle in the series. Since the results of this problem can be written in terms of this starting value, these initial values play a recurring role. The subscript ‘$N$’ denotes the values of the parameters (e.g., $x_N$) appropriate for the $N$th cycle of the series. In iteration formulae, however, we use unsubscripted variables (e.g., $x$) for the next $(N+1)$st cycle.

**Result 3:** In the highly unstable regime, the ratio of $\Sigma_T$ to $\Sigma_B$ has the form:

$$\frac{\Sigma_T}{\Sigma_B} = \frac{x}{x_1}.$$  \hfill (21)

**Proof:** From our previous results (see equation [19] of [AB]), the product matrix after $N$ cycles has the form given by equation (19) with $f = 1$ (in the highly unstable regime). After one additional multiplication, we obtain the form given by equation (20) with $f = 1$. We thus find

$$\frac{\Sigma_T^{(N+1)}}{\Sigma_B^{(N+1)}} = \frac{\Sigma_T^{(N)} + (x/x_1)\Sigma_B^{(N)}}{\Sigma_B^{(N)} + (x_1/x)\Sigma_T^{(N)}} = \frac{x}{x_1}.$$  \hfill (22)

For each cycle the ratio $x/x_1$ has a different value, so that no limit is reached as $N \to \infty$. However, the ratio at any given finite cycle obeys equation (21).

□

To derive an expression for the growth rate for matrix multiplication, we first define

$$S \equiv \Sigma_{11} + \Sigma_{22}.$$  \hfill (23)

As shown in the proof of Result 1, the eigenvalue of the product matrix approaches $S$, as defined above, in the limit $N \to \infty$. By construction, the iteration formula for $S$ takes the form

$$S^{(N+1)} = S^{(N)} \left[ 1 + \frac{(x/x_1)\phi \Sigma_{21}^{(N)} + (x_1/x)\Sigma_{12}^{(N)}}{\Sigma_{11}^{(N)} + \Sigma_{22}^{(N)}} \right].$$  \hfill (24)

Using the definition of $f$, $\Sigma_T$, and $\Sigma_B$, this expression can be simplified to the form

$$S^{(N+1)} = S^{(N)} \left[ 1 + \frac{(x/x_1)\phi \Sigma_B^{(N)} + (x_1/x)f \Sigma_T^{(N)}}{\Sigma_T^{(N)} + f \Sigma_B^{(N)}} \right].$$  \hfill (25)

**Result 4:** In the highly unstable regime the iteration formula for the eigenvalue reduces to the form

$$S^{(N+1)} = S^{(N)} \left[ 1 + \frac{xN}{x} \right].$$  \hfill (26)

This result agrees with that of Theorem 2 from [AB].
Proof: In the highly unstable regime \( \phi = 1, f = 1 \), and equation (21) holds for the ratio of \( \Sigma_T/\Sigma_B \). The iteration formula of equation (25) thus reduces to

\[
S^{(N+1)} = S^{(N)} \left[ 1 + \frac{(x/x_1) + (x_N/x)}{1 + x_N/x_1} \right] = S^{(N)} \left[ 1 + \frac{x_N}{x} \right] \left[ \frac{x_1 + x}{x_1 + x_N} \right]. \tag{27}
\]

Since the starting value \( x_1 \) is fixed, the second factor in square brackets approaches unity in the limit \( N \to \infty \), i.e.,

\[
\lim_{N \to \infty} \prod_{k=1}^{N} \left[ \frac{x_1 + x_{k+1}}{x_1 + x_k} \right] = 1. \tag{28}
\]

The expression of equation (27) thus reduces to that of equation (26).

Motivated by the result of equation (21) for the highly unstable regime, we write the ratio of matrix elements for the general case in the form

\[
\frac{\Sigma_T^{(N)}}{\Sigma_B^{(N)}} = \frac{x_N}{x_1} \alpha_N, \tag{29}
\]

so that

\[
S^{(N+1)} = S^{(N)} \left[ 1 + \frac{(x/x_1)\phi + (x_N/x)f\alpha_N}{f + \alpha_N(x_N/x_1)} \right] = F_N S^{(N)}, \tag{30}
\]

where the second equality defines \( F_N \). The parameter \( \alpha_N \) incorporates the correction due to the matrices not being in the highly unstable regime. Note that \( f \) approaches a constant value (from Result 2) and \( x_1 \) is a constant (by definition). The iteration factor \( F_N \) can be rewritten in the form

\[
F_N = \left[ 1 + \frac{x^2_1 \phi + b\alpha_N x_N}{x_1(1 + \alpha_N x_N)} \right] \text{ where } b \equiv fx_1. \tag{31}
\]

**Theorem 1:** The growth rate for matrix multiplication, with products of the general form defined through equation (10), is given by

\[
\gamma = \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \log \left[ 1 + \frac{x_k^2 \phi_k + \alpha_{k-1} x_{k-1}}{x_k(1 + \alpha_{k-1} x_{k-1})} \right], \tag{32}
\]

where the \( \alpha_k \) are determined through the iteration formula

\[
\alpha_k = \frac{x_k \phi_k + x_{k-1} \alpha_{k-1}}{x_k + x_{k-1} \alpha_{k-1}}. \tag{33}
\]

Proof: Note that existence of the required limit holds by the Theorem of FK. Equations (29) – (31) show that the growth rate is given by

\[
\gamma = \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \log F_k = \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \log \left[ 1 + \frac{x_k^2 \phi_k + b\alpha_{k-1} x_{k-1}}{x_k(1 + \alpha_{k-1} x_{k-1})} \right], \tag{34}
\]
where this form is exact, provided that the $\alpha_k$ are properly specified. This issue is addressed below. To complete the proof, we must also show that the growth rate is independent of the value of $b$, so that we can set $b = 1$ in the above formula. The derivative of the growth rate with respect to the parameter $b$ takes the form

$$\frac{d \gamma}{db} = \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \frac{1}{F_k} \frac{dF_k}{db},$$  \quad (35)$$

which can be evaluated to take the form

$$\frac{d \gamma}{db} = \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \frac{(\alpha_{k-1}x_{k-1})^2 - x_k^2 \phi_k}{(b + \alpha_{k-1}x_{k-1}) [x_k(b + \alpha_{k-1}x_{k-1}) + x_k^2 \phi_k + b\alpha_{k-1}x_{k-1}]}.$$  \quad (36)$$

This expression vanishes in the limit.

To show that the $\alpha_k$ are given by equation (33), we start with the result of matrix multiplication from equation (20) and use the definition of $\alpha_k$ from equation (29); these two results imply that

$$\alpha_{k+1} = \frac{x_1}{x_{k+1}} \frac{\Sigma_T^{(k+1)}}{\Sigma_B^{(k+1)}} = \frac{x_1}{x_{k+1}} \frac{\Sigma_T^{(k)}}{\Sigma_B^{(k)}} + \frac{(x_{k+1}/x_1)\phi_{k+1}}{\Sigma_B^{(k)}} \frac{\Sigma_T^{(k)}}{\Sigma_B^{(k)}} + ((x_{k+1}/x_1)\Sigma_T^{(k)}}.$$

We can then eliminate the factors of $\Sigma_T$ and $\Sigma_B$ by again using the definition of $\alpha_k$ from equation (29), and thus obtain

$$\alpha_{k+1} = \frac{x_1}{x_{k+1}} \frac{(x_{k+1}/x_1)\alpha_k + (x_{k+1}/x_1)\phi_{k+1}}{1 + (x_1/x_{k+1})\alpha_k} = \frac{x_k\alpha_k + x_{k+1}\phi_{k+1}}{x_{k+1} + x_k\alpha_k}.$$  \quad (37)$$

After re-labeling the indices, we obtain equation (33). \Box

4 Approximations for the Classically Unstable Regime

For classically unstable matrices with $|h_k| > 1$, Theorem 1 provides an exact expression for the growth rate. Since the formulae are complicated, this section presents simpler but approximate expressions for the growth rates for the case where $\phi_k$ are small (Theorem 2) and where the differences $1 - \phi_k$ are small (Theorem 3). We also present two heuristic approximations for the growth rates for the general problem.

**Theorem 2:** In the regime where the variables $\phi_k$ are small, $\phi_k x_k \ll 1 \forall k$, the growth rate for the matrix $B_k$ tends in the limit of large $N$ to the form

$$\gamma = \log \left(1 + [(1/x_k)(x_k\phi_k)]^{1/2}\right) + O((x_k\phi_k)).$$  \quad (39)$$

**Proof:** We first break up the matrix into two parts so that $B_k = I + A_k$, where $I$ is the identity matrix and where

$$A_k = \begin{bmatrix} 0 & x_k\phi_k \\ 1/x_k & 0 \end{bmatrix} = \begin{bmatrix} 0 & \eta_k \\ y_k & 0 \end{bmatrix}.$$  \quad (40)$$
Note that the second equality defines \( \eta_k = x_k \phi_k \) and \( y_k = 1/x_k \). We first show (by induction) that repeated multiplications of the matrices \( A_k \) lead to products with simple forms. The products of even numbers \( N = 2\ell \) of matrices \( A_k \) produce diagonal matrices of the form

\[
A^{(N)} = A^{(2\ell)} = \prod_{k=1}^{N} A_k = \begin{bmatrix} P_\ell^A & 0 \\ 0 & P_\ell^B \end{bmatrix},
\]

where the products \( P_\ell \) are defined by

\[
P_\ell^A = \prod_{i=1}^{\ell} (\eta_{2i}) (y_{2i-1}) \quad \text{and} \quad P_\ell^B = \prod_{i=1}^{\ell} (\eta_{2i-1}) (y_{2i}).
\]

Similarly, the product of odd numbers \( N = 2\ell + 1 \) of matrices \( A_k \) produce off-diagonal matrices of the form

\[
A^{(N)} = A^{(2\ell+1)} = \prod_{k=1}^{N} A_k = \begin{bmatrix} 0 & Q_\ell^B \eta_1 \\ Q_\ell^A y_1 & 0 \end{bmatrix},
\]

where the products \( Q_\ell \) are defined analogously to the \( P_\ell \). The product of \( N \) matrices \( B_k \) can then be written in the form

\[
B^{(N)} = \prod_{k=1}^{N} B_k = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \eta_1 \\ \Sigma_{21} y_1 & \Sigma_{22} \end{bmatrix}.
\]

Without loss of generality, let \( N = 2\ell \) be even. Then the matrix elements are given by

\[
\Sigma_{11} = \sum_{\ell=0}^{N/2} \sum_{j=1}^{C_{2\ell}^N} (P_\ell^A)_j, \quad \Sigma_{22} = \sum_{\ell=0}^{N/2} \sum_{j=1}^{C_{2\ell}^N} (P_\ell^B)_j,
\]

\[
\Sigma_{12} = \sum_{\ell=0}^{N/2-1} \sum_{j=1}^{C_{2\ell+1}^{N+1}} (Q_\ell^B)_j, \quad \Sigma_{21} = \sum_{\ell=0}^{N/2-1} \sum_{j=1}^{C_{2\ell+1}^{N+1}} (Q_\ell^A)_j,
\]

where \( C^N_k \) is the binomial coefficient and where the subscripts on the \( P_\ell \) and \( Q_\ell \) denote different realizations of the products.

The eigenvalue \( \Lambda_N \) of the product matrix at the \( N \)th iteration is given by its characteristic equation, which has the solution

\[
\Lambda_N = \frac{1}{2} \left\{ \Sigma_{11} + \Sigma_{22} + \left[ (\Sigma_{12} - \Sigma_{21})^2 + 4 \Sigma_{12} \Sigma_{21} \eta_1 y_1 \right]^{1/2} \right\}.
\]

In the limit of large \( N \), we can make the approximation that \( \Sigma_{11} \approx \Sigma_{22} \) and \( \Sigma_{12} \approx \Sigma_{21} \), so that the expression for the eigenvalue takes the form

\[
\Lambda_N = \Sigma_{11} + \Sigma_{12} [\eta_1 y_1]^{1/2} = \sum_{\ell=0}^{N/2} \sum_{j=1}^{C_{2\ell}^N} (P_\ell^A)_j + \sum_{\ell=0}^{N/2-1} \sum_{j=1}^{C_{2\ell+1}^{N+1}} (Q_\ell^B)_j [\eta_1 y_1]^{1/2}.
\]
In the limit of large $N$, all the binomial coefficients are large except for the first and last one. We can thus rewrite the above equation in the form

$$A_N = \sum_{\ell=0}^{N/2} C_{2\ell}^N \left( \langle P_{\ell}^A \rangle + \varepsilon_{\ell} \right) + \sum_{\ell=0}^{N/2-1} C_{2\ell+1}^N \left( \langle Q_{\ell}^B \rangle + \varepsilon_{\ell} \right) [\eta_1 y_1]^{1/2}. \quad (48)$$

If the realizations of the products $(P_{\ell})_j$ were independent, the error terms $\varepsilon_{\ell}$ would vanish in the limit. However, for a given $N$, the sums contain $C_{2\ell}^N$ terms, and $C_{2\ell}^N > N$ in general, so all of the terms in the sum cannot be independent. We then write the products $\langle P_{\ell}^A \rangle$ and $\langle Q_{\ell}^B \rangle$ in the form

$$\langle P_{\ell}^A \rangle + \varepsilon_{\ell} = \langle \eta_j \rangle^\ell \langle y_j \rangle^\ell (1 + \varepsilon_{\ell}), \quad (49)$$

and similarly for $\langle Q_{\ell}^B \rangle$. This form is exact if one uses the proper expressions for the $\varepsilon_{\ell}$. Using this result, the expression for the eigenvalue $A_N$ becomes

$$A_N = \sum_{\ell=0}^{N/2} C_{2\ell}^N \langle \eta_j \rangle^\ell \langle y_j \rangle^\ell (1 + \varepsilon_{\ell})^{\ell/2} + \sum_{\ell=0}^{N/2-1} C_{2\ell+1}^N \langle \eta_j \rangle^\ell \langle y_j \rangle^\ell (1 + \varepsilon_{\ell})^{\ell/2} [\eta_1 y_1]^{1/2}, \quad (50)$$

which takes the form

$$A_N = \sum_{k=0}^{N} C_k^N \langle \eta_j \rangle^k \langle y_j \rangle^k (1 + \varepsilon_{k})^{k/2}. \quad (51)$$

If we expand this result, we find that

$$A_N = 1 + N \langle \eta_j \rangle^{1/2} \langle y_j \rangle^{1/2} (1 + \varepsilon_1)^{1/2} + C_2^N \langle \eta_j \rangle \langle y_j \rangle (1 + \varepsilon_2) + \ldots \quad (52)$$

Further, by performing an exact treatment of the first order expansion [AB2] we find that $\varepsilon_1 = 0$. This finding allows us to write the product in the form

$$A_N = \left[ 1 + \langle \eta_j \rangle^{1/2} \langle y_j \rangle^{1/2} + O(\eta_j) \right]^N. \quad (53)$$

The growth rate thus becomes

$$\gamma = \log \left[ 1 + \langle \eta_j \rangle^{1/2} \langle y_j \rangle^{1/2} + O(\eta_j) \right]. \quad (54)$$

This last expression is valid provided that $\eta_j \ll 1 \: \forall \: j \: \Box$

Note that to consistent order, we can replace the limiting form of equation (39) with the equivalent, simpler function

$$\gamma \to [(1/x_0)(\eta_0)]^{1/2}. \quad (55)$$

Figure 1 illustrates how well the approximation of Theorem 2 works. For the sake of definiteness, the variables $x_k$ are log-uniformly distributed with $\log_{10} x_k \in [-2, 2]$. The $\phi_k$ obey the relation $\phi_k = a_k \xi_k$, where $\xi_k$ is a uniformly distributed random variable over the interval $[0, 1]$. As shown by the figure, the limiting form of equation (50) provides an excellent description of the calculated growth rate for sufficiently small $\phi_k$. 
Fig. 1 Growth rates for small $\phi_k$. The variables $\phi_k$ are determined through the relation $\phi_k = a_\phi \xi_k$, where $\xi_k$ is uniformly distributed on $[0,1]$. The solid curve shows the growth rate $\gamma$ calculated directly from matrix multiplication as a function of the amplitude $a_\phi$. The dashed curve shows the estimate $\gamma_2$ for the growth rate from Theorem 2. The dotted curve shows the difference $\Delta \gamma = \gamma_2 - \gamma$. Note that $\gamma \propto \sqrt{a_\phi}$ whereas $\Delta \gamma \propto a_\phi$.

Next we consider the case where the correction factors $\phi_k$ are close to unity. In this case the variables $(1 - \phi_k) \ll 1$, and we can expand to leading order in $(1 - \phi_k)$. This procedure leads to the following result:

**Theorem 3:** Let $\gamma_0$ be the growth rate for the highly unstable regime where $\phi_k = 1$. For small perturbations about this limiting case, the growth rate takes the form $\gamma = \gamma_0 - \delta \gamma$, where

$$
\delta \gamma = \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \frac{(1 - \phi_k)x_{k}^2}{(x_{k+1} + x_k)(x_k + x_{k-1})} + \mathcal{O}\left(\langle x_{k}^2(1 - \phi_k)^2 \rangle\right). \quad (56)
$$
Proof: We again break up the matrix into two parts,

\[ B_k = C_k - \epsilon_k Z \quad \text{with} \quad Z \equiv \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \tag{57} \]

where here \( \epsilon_k \equiv x_k (1 - \phi_k) \) and \( C_k \) is the matrix appropriate for the highly unstable regime. Note that \( Z \) does not depend on the index \( k \). Here we work to first order in the small parameter \( \epsilon_k \). After \( N \) cycles, the product matrix takes the form

\[ B_k^{(N)} = \prod_{k=1}^{N} B_k = C_k^{(N)} - \sum_{k=1}^{N} \epsilon_k P_k^N + O(\epsilon_k^2), \tag{58} \]

where the partial product matrices \( P_k^N \) are given by

\[ P_k^N = \left\{ \prod_{j=k+1}^{N} C_j \right\} Z \left\{ \prod_{j=1}^{k-1} C_j \right\}. \tag{59} \]

We ignore the case where the \( Z \) factors appear on the ends – this effect is \( O(1/N) \) and vanishes in the limit. The products of the \( C_k \) matrices can be written in the form

\[ C_k^{(N)} = \Sigma J^N \begin{bmatrix} 1/\alpha_N & x_1/\alpha_N \\ x_1/\alpha_N & 1/\alpha_N \end{bmatrix} \quad \text{where} \quad \Sigma J^N = \prod_{j=2}^{N} \left( 1 + \frac{x_j}{x_{j-1}} \right), \tag{60} \]

where these results follow from previous work [AB]. As a result, the matrices \( P_k^N \) can be evaluated:

\[ P_k^N = \frac{x_k \Sigma J^N}{(x_k + x_{k+1})(x_{k-1} + x_k)} \begin{bmatrix} 1/\alpha_N & x_1/\alpha_N \\ x_1/\alpha_N & 1/\alpha_N \end{bmatrix} = \frac{x_k}{(x_k + x_{k+1})(x_{k-1} + x_k)} C_k^{(N)}. \tag{61} \]

The product matrix \( B_k^{(N)} \), given by equation (58) to leading order, can now be written in the form

\[ B_k^{(N)} = C_k^{N} \left[ 1 - \sum_{k=1}^{N} \frac{(1 - \phi_k)x_k^2}{(x_k + x_{k+1})(x_{k-1} + x_k)} \right]. \tag{62} \]

The first factor is the product of the matrices for the highly unstable regime. Since the second factor is a function (not a matrix) its contribution to the growth rate is independent of the first factor and represents a correction to the growth rate of the form

\[ \delta \gamma = \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \frac{(1 - \phi_k)x_k^2}{(x_k + x_{k+1})(x_{k-1} + x_k)} + O(\epsilon_k^2), \tag{63} \]

where the equalities hold to leading order. This correction to the growth rate has the form given by equation (56). □
Fig. 2 Growth rates for $\phi_k$ near unity. The variables $\phi_k$ are determined through the relation $\phi_k = 1 - A_\phi \xi_k$, where $\xi_k$ is uniformly distributed on [0,1]. The solid curve shows the quantity $\delta \gamma = \gamma_0 - \gamma$, where $\gamma$ is the growth rate calculated from matrix multiplication and $\gamma_0$ is the growth rate for the highly unstable regime ($\phi_k = 1$ for all $k$). The dashed curve shows the estimate $(\delta \gamma)_3 = (\gamma_0 - \gamma)_3$ for the difference in growth rate calculated from Theorem 3. The dotted curve shows the error $\Delta = (\delta \gamma)_3 - \delta \gamma$. Note that $\delta \gamma \propto A\phi$ whereas the error term $\Delta \propto (A\phi)^2$.

Figure 2 shows the growth rate for small departures from the highly unstable regime. The correction factors are taken to have the form $\phi_k = 1 - A_\phi \xi_k$, where $\xi_k$ is a uniformly distributed random variable over the interval [0,1]. The highly unstable regime corresponds to $A_\phi \to 0$. The figure shows the growth rate calculated from direct matrix multiplication (solid curve) and the approximation from Theorem 3 (dashed curve) plotted as a function of the amplitude $A_\phi$. Both curves plot the difference $\gamma_0 - \gamma$, where $\gamma_0$ is the growth rate for the highly unstable regime (where the $\phi_k = 1$).

Since the general case is quite complicated it is useful to have a good working approximation for the case where one is not in one of the two regimes.
φk small or near unity. Toward this end, we first show that the values of αk have a limited range:

**Result 5:** The variables αk are confined to the range φmin ≤ αk ≤ 1, where φmin is the minimum value of φk.

**Proof:** We can rewrite the iteration formula (33) for αk in the alternate form

$$\alpha_k = \frac{\phi_k + \beta_k}{1 + \beta_k}, \quad (64)$$

where we have defined the composite random variable $\beta_k \equiv \alpha_k - 1 \frac{x_k}{x_k + 1}$. In the present context, $0 \leq \beta_k < \infty$, and we can show that

$$\frac{d\alpha_k}{d\beta_k} > 0 \quad (65)$$

for all values of $\beta_k$. In the limit $\beta_k \to \infty$, $\alpha_k \to 1$, whereas in the limit $\beta_k \to 0$, $\alpha_k \to \phi$. Hence $\phi \leq \alpha_k \leq 1$ for all cycles. But $\phi \geq \phi_{\text{min}}$, by definition, so that $\phi_{\text{min}} \leq \alpha_k \leq 1$.

**Approximation 1:** As a first heuristic approximation, we replace the full iteration expression of equation (33) for $\alpha_k$ with the following simplified form

$$\alpha_{k+1} = \frac{x\phi + x_k}{x + x_k}, \quad (66)$$

i.e., we use $\alpha_k = 1$ as an approximation for the previous value [keep in mind that $x$ is the value at the ($k+1$)th cycle]. Using equation (66) to evaluate $\alpha_k$ in the iteration formula for $F_k$, we obtain a working approximation for the growth rate. Notice that $\alpha_k$ appears in the iteration formula for $F_k$, so that we must use equation (66) evaluated at $k$ rather than $k+1$. As a result, the iteration factor $F_k$ involves the random variables $x_k$ from three cycles, or, equivalently (since the $x_k$ are i.i.d.) three separate samplings of the variables. We change notation so that $x_{j1}, x_{j2}, x_{j3}$ denote the three independent samplings of the random variables $x_k$. Similarly, let $\phi_{j1}, \phi_{j2}$ denote two independent samplings of the $\phi_k$. The iteration formula for this approximation can then be written in the form

$$F_j = 1 + \frac{x_{j1}^2 \phi_{j1} (x_{j2} + x_{j3}) + x_{j2} (x_{j2} \phi_{j2} + x_{j3})}{x_{j1} [(x_{j2} + x_{j3}) + x_{j2} (x_{j2} \phi_{j2} + x_{j3})]}, \quad (67)$$

The growth rate for matrix multiplication can then be approximated by

$$\gamma = \lim_{N \to \infty} \frac{1}{N} \sum_{j=1}^{N} \log F_j, \quad (68)$$

where $F_j$ is given by equation (67). As a consistency check, for the restricted problem where the $\phi_{jn} = 1$, the iteration factor $F_j$ reduces to that appropriate for the highly unstable regime (see equation [26]).
Approximation 2: To derive a second approximation for the growth rate, we need a better approximation for the $\alpha_k$. If the values of $x_k$ and $\phi_k$ were constant, then the $\alpha_k$ would approach a constant value given by

$$\alpha_k = \frac{1}{2} \left\{ (1 - x_k/x_{k-1}) + [(1 - x_k/x_{k-1})^2 + 4(x_k/x_{k-1})\phi_k]^{1/2} \right\}. \quad (69)$$

Even though the $x_k$ and $\phi_k$ are not constant, and the $\alpha_k$ vary, we can use equation (69) as an approximation to specify the values of $\alpha_k$ appearing in the exact formula of equation (32) for the growth rate. After using this form to specify the $\alpha_k$, and relabeling the indices, the iteration factor takes the form

$$F_k = 1 + \frac{x_k^2 \phi_k x_k^2 + x_k^2 \left\{ (x_k^3 - x_k^2) + [(x_k^3 - x_k^2)^2 + 4x_k^2 x_k^3 \phi_k]^{1/2} \right\}}{x_k \left\{ 2x_k^3 + x_k^2 \left\{ (x_k^3 - x_k^2) + [(x_k^3 - x_k^2)^2 + 4x_k^2 x_k^3 \phi_k]^{1/2} \right\} \right\}}. \quad (70)$$

In the case $\phi_{jn} = 1$, the iteration factor of equation (70) reduces to the expression for the highly unstable regime (Result 4).

Figure 3 shows how well these two approximation schemes work. The $\phi_k$ variables are chosen from the expression $\phi_k = 1 - A\phi \xi_k$, where $\xi_k$ is a random variable uniformly sampled from the interval $0 \leq \xi_k \leq 1$ and where $A\phi$ sets the amplitude of the departures of the $\phi_k$ from unity. The growth rate is shown as a function of the amplitude.

In [AB], we derived a bound on the difference between the growth rate for the general case $\gamma$ (considered here) and the growth rate in the highly unstable regime $\gamma_0$, i.e.,

$$\gamma_0 - \gamma \leq \frac{1}{2} \langle \log \phi_k \rangle. \quad (71)$$

This bound is shown as the dashed curve in Figure 3. The true growth rates fall comfortably between this lower bound and the growth rate for the highly unstable regime (where the latter provides an upper bound).

Thus far, this paper has focused on the regime where the transformation matrices are classically unstable. Before considering classically stable matrix multiplication in the next section, we note the following result that applies at the transition between the two regimes:

**Result 6:** Consider the matrix transformation that maps the principal solutions from one cycle to the next. When the matrix elements $g_k = \dot{y}_1(\pi)$ vanish, then the remaining matrix elements are $h_k = y_1(\pi) = \pm 1$. The transformation matrix $M_{g0}$ for this case is stable under multiplication.

(The proof is a simple explicit computation.)

5 Elliptical Rotations and the Classically Stable Regime

When the principal solutions $h_k$ appearing in the discrete map of equation (2) are less than unity, matrix multiplication is stable for the case of constant parameters. In the case of interest, however, the parameters in Hill’s equation
Fig. 3 Validity of approximations of equation (63) and equation (70) as a function of the deviation of $\phi_k$ from unity. The upper solid line shows the growth rate for matrix multiplication in the highly unstable regime where $\phi_k = 1$. The lower solid curve shows the growth rate for the case where $\phi_k = 1 - A_k \xi_k$, where $\xi_k$ is a uniformly distributed random variable $0 \leq \xi_k \leq 1$. The dotted curve shows the estimate for growth rate calculated from equation (63) using the same sampling of the $\phi_k$ variables; similarly, the dot-dashed curve shows the approximation of equation (70). Notice that both of these approximations are almost identical to the actual result. The dashed curve shows the lower limit to the growth rate derived in [AB].

and the matrices vary from cycle to cycle. This section considers the case where the $|h_k| \leq 1$, but vary from cycle to cycle, and show that instability results. In this regime, the discrete map takes the form of an elliptical rotation matrix [LR] as described below. We thus find the growth rates for elliptical rotation matrices for the case where the matrix elements vary from cycle to cycle.
Definition: An elliptical rotation matrix is defined to be
\[
E(\theta; L) \equiv \begin{bmatrix} \cos \theta & -L \sin \theta \\ (1/L) \sin \theta & \cos \theta \end{bmatrix}.
\] (72)

These matrices have the following properties:

The product of elliptical rotation matrices with the same value of \( L \) produces another elliptical rotation matrix, also with the same \( L \),
\[
E(\theta_1; L)E(\theta_2; L) = E([\theta_1 + \theta_2]; L).
\] (73)

As a result, the elliptical rotation matrices form a group.

For fixed \( L \), matrix multiplication is stable. Specifically, the eigenvalues of the product of \( N \) matrices (with fixed \( L \)) have the form
\[
\lambda = \exp \left[ \pm i \sum_{j=1}^{N} \theta_j \right] = \exp \left[ \pm i \theta_N \right],
\] (74)

where \( \theta_N \) is the angle corresponding to the group element produced after \( N \) matrix multiplications.

Result 7: When an individual cycle of Hill’s equation is stable, specifically when \( |h_k| \leq 1 \), the full transformation matrix \( M_k \) takes the form of an elliptical rotation.

Proof: Since \( |h_k| \leq 1 \), we can define an angle \( \theta_k \) such that \( h_k = \cos \theta_k \). The full matrix \( M_k \) given by equation (2) then takes the form
\[
M_k = \begin{bmatrix} \cos \theta_k & -(\sin^2 \theta_k)/g_k \\ g_k & \cos \theta_k \end{bmatrix} = \begin{bmatrix} \cos \theta_k & -L_k \sin \theta_k \\ (1/L_k) \sin \theta_k & \cos \theta_k \end{bmatrix} = E_k(\theta_k; L_k),
\] (75)

where we have defined \( L_k = (\sin \theta_k)/g_k \). As before, we can factor out the \( \cos \theta_k = h_k \) and write the matrix in the form
\[
M_k = \cos \theta_k \begin{bmatrix} 1 & x_k \phi_k \\ 1/\sqrt{x_k} & 1 \end{bmatrix} = \cos \theta_k B_k,
\] (76)

where
\[
x_k = L_k/\tan \theta_k \quad \text{and} \quad \phi_k = -\tan^2 \theta_k.
\] (77)

Equation (77) thus specifies the transformation between the random variables \((x_k, \phi_k)\) appearing in the original transformation matrix and the random variables \((\theta_k, L_k)\) in the corresponding elliptical rotation matrix. Note that the values of \( \phi_k \) are strictly negative in this formulation. Otherwise, the matrix \( B_k \) has the same form as in equation (7).

If we let \( \gamma_B \) be the growth rate for matrix \( B_k \), then the growth rate \( \gamma_M \) for the full matrix \( M_k \) takes the form
\[
\gamma_M = \gamma_B + \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \log[\cos \theta_k].
\] (78)
The exact growth rate for the matrix $B_k$ (see equation \[76\]) is given by Theorem 1. In particular, equations (32) and (33) remain valid for negative values of the $\phi_k$ and can be used to calculate the growth rate.

**Result 8:** For an elliptical rotation matrix with constant angle $\theta$ and random $L_k$, the growth rate for matrix multiplication vanishes in the two limits $h = \cos \theta \to 0$ and $h = \cos \theta \to 1$.

**Proof:** In the limit $h \to 1$ we have $\sin \theta = 0$, and the elliptical rotation matrix becomes the identity matrix. As a result, the growth rate vanishes.

In the other case where $h \to 0$, $\sin \theta = 1$, and the matrix takes the form

$$E_k \to E_{0k} = \begin{bmatrix} 0 & -L_k \\ 1/L_k & 0 \end{bmatrix}. \quad (79)$$

In this case, for even numbers of matrix multiplications, say $N = 2n$, the product matrix takes the form

$$E^{(N)}_{0k} = \prod_{k=1}^{N} E_{0k} = (-1)^n \begin{bmatrix} P_A^n & 0 \\ 0 & P_B^n \end{bmatrix}, \quad (80)$$

where the matrix elements are given by the products

$$P_A^n = \prod_{k=1}^{n} \frac{L_{2k}}{L_{2k-1}} \quad \text{and} \quad P_B^n = \prod_{k=1}^{n} \frac{L_{2k-1}}{L_{2k}}. \quad (81)$$

The eigenvalues of the product matrix are given by $\lambda = P_A^n$ and $\lambda = P_B^n$. For odd $N = 2n + 1$, the eigenvalue $|\lambda| = (P_A^n P_B^n)^{1/2}$. In either case, in the limit of large $N$, the growth rate for matrix multiplication takes the form

$$\gamma = \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \log \left( \frac{L_{2k}}{L_{2k-1}} \right) = \langle \log L_{2k} \rangle - \langle \log L_{2k-1} \rangle = 0. \quad (82)$$

The final equality holds because the $L_k$ are independent. □

Elliptical rotation matrices are unstable under multiplication when their parameters vary from cycle to cycle:

**Theorem 4:** Consider an elliptical rotation matrix with variable angle $\theta_k$ and symmetric fluctuations of the $L_k$ parameter about its mean value $L_0$. The variations are thus written in the form $L_k = L_0(1 + \eta_k)$, where the odd moments $\langle \eta_k^{2n+1} \rangle = 0$ for all integers $n$. For small fluctuations $|\eta_k| < 1$, the growth $\gamma$ rate for matrix multiplication takes the form

$$\gamma = \frac{1}{2} \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \log \left( \cos^2 \theta_k + \sin^2 \theta_k \left( \frac{1}{1 + \eta_k} \right) \right) + O(\eta_k^4). \quad (83)$$

**Proof:** We first break up the matrix into two parts so that

$$E_k = I \cos \theta_k + \sin \theta_k Z_k, \quad (84)$$
where $I$ is the identity matrix and where
\[
Z_k = \begin{bmatrix} 0 & -L_k \\ 1/L_k & 0 \end{bmatrix}.
\] (85)

The product of $N$ matrices $E_k$ becomes
\[
E^{(N)} = \prod_{k=1}^{N} E_k = \sum_{\ell=0}^{N} \left( \prod_{i=1}^{\ell} \cos \theta_i \right)_{k} \left( \prod_{j=1}^{\ell} Z_j \sin \theta_j \right)_{k},
\] (86)

where the subscripts on the products denote different realizations. The products of even numbers $\ell = 2n$ of matrices $Z_k$ produce diagonal matrices of the form
\[
Z^{(\ell)} = Z^{(2n)} = \prod_{k=1}^{n} Z_{2k} Z_{2k-1} = (-1)^n \begin{bmatrix} P_n^A & 0 \\ 0 & P_n^B \end{bmatrix},
\] (87)

where the matrix elements $P_n^A$ and $P_n^B$ are given by equation (81). Similarly, the product of odd numbers $\ell = 2n + 1$ of matrices $Z_k$ produce off-diagonal matrices of the form
\[
Z^{(\ell)} = Z^{(2n+1)} = \left\{ \prod_{k=1}^{n} Z_{2k+1} Z_{2k} \right\} Z_1 = (-1)^n \begin{bmatrix} 0 & -P_n^A L_1 \\ P_n^B / L_1 & 0 \end{bmatrix},
\] (88)

where the $P_n$ are defined previously. Next we write the expectation values of these products in the form
\[
\langle P_n \rangle = \left\langle \prod_{j=1}^{n} \frac{L_{2j}}{L_{2j-1}} \right\rangle = \left\langle \prod_{j=1}^{n} \frac{1 + \eta_{2j}}{1 + \eta_{2j-1}} \right\rangle = \left\langle \frac{1}{1 + \eta_j} \right\rangle^n \equiv R^n.
\] (89)

This expression holds because the odd powers of the $\eta_j$ vanish in the mean, and the samples of the different $\eta$'s are independent.

The eigenvalue $A_N$ of the product matrix at the $N$th iteration can be written in terms of its matrix elements, i.e.,
\[
A_N = \sigma_{11} + \sigma_{22}.
\] (90)

Without loss of generality, let $N = 2K$ be even. The matrix elements $\sigma_{11} = \sigma_{22} = \sigma$ are given by
\[
\sigma = \sum_{m=0}^{K} \sum_{k=1}^{2m} C^{2m}_{2m} \left( \prod_{i=1}^{2m-2k} \cos \theta_i \right)_{k} \left( \prod_{i=1}^{2m} \sin \theta_i \right)_{k} (-1)^m R^m,
\] (91)

where $C^{2m}_{2m}$ is the binomial coefficient and where we have used equation (89). This expression for $\sigma$ contains the even terms of a binomial expansion. We can thus write the eigenvalue in the form
\[
A_N = \prod_{k=1}^{N} \left[ \cos \theta_k + i \sin \theta_k R^{1/2} \right]_{k} + \prod_{k=1}^{N} \left[ \cos \theta_k - i \sin \theta_k R^{1/2} \right]_{k}.
\] (92)
Next we define
\[ A_k \equiv \left[ \cos^2 \theta_k + \sin^2 \theta_k R \right]^{1/2} \quad \text{and} \quad \tan \alpha_k \equiv R^{-1/2} \tan \theta_k . \quad (93) \]
The eigenvalue takes the form
\[ \Lambda_N = 2 \left( \prod_{k=1}^{N} A_k \right) \cos \left( \sum_{k=1}^{N} \alpha_k \right) , \quad (94) \]
and the corresponding growth rate becomes
\[ \gamma = \frac{1}{2} \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \log \left[ \cos^2 \theta_k + \sin^2 \theta_k R \right] . \quad (95) \]
Using the definition of \( R \), we obtain the result of Theorem 4. The order of the error term follows by comparing equation (95) with the leading order expansion [AB2].

In the regime of small \( \eta_k \ll 1 \), the expression for the growth rate reduces to the form
\[ \gamma = \frac{1}{2} \left\langle \sin^2 \theta_k \right\rangle \left\langle \eta_k^2 \right\rangle . \quad (96) \]
This section shows that instability does not require a finite threshold for the amplitude of the fluctuations in \( L_k \). Nonzero amplitude leads to instability with growth rate \( \gamma \propto \left\langle \eta_k^2 \right\rangle \). Variations in the original parameters \( (\lambda_k, q_k) \) of Hill’s equation lead to fluctuations in the principal solutions \( (h_k, g_k) \); fluctuations in the \( (h_k, g_k) \) lead to variations in the \( L_k \) and hence growth. As a result, Hill’s equation with random forcing terms is generically unstable. One notable exception occurs when the \( h_k = 0 \) or \( h_k = 1 \) (Result 8).

6 Conclusion

This paper provides expressions for the growth rates for the random 2×2 matrices that result from solutions to the random Hill’s equation (1). Theorem 1 gives an exact expression for the growth rate. Theorems 2 and 3 provide approximate growth rates for the regimes where the variables \( \phi_k \) are small, and close to unity, respectively. Additional approximations for are given in Section 4. When Hill’s equation is classically stable, the discrete map that governs the solutions has the form of an elliptical rotation matrix (equ. [7]). With fixed elements, such matrices are stable under multiplication; variations in the \( L_k \) parameter lead to instability. For small symmetric fluctuations of the length parameter \( L_k \), the growth rate is given by Theorem 4.

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