I review the progress made in the calculation of the electroweak penguins $Q_{7,8}$ in the framework of the Hadronic Approximation to large-$N_c$ QCD, and give an update of results.

A nonzero value of $\epsilon'/\epsilon$ signals the existence of direct CP violation in the decay $K \to \pi\pi$ and is a measure of the amount of CP violation in the Standard Model. At present the world average is given by

$$\frac{\epsilon'}{\epsilon} = (16.6 \pm 1.6) \times 10^{-4}.$$  

To compute this number in the SM has become a major theoretical challenge. Roughly, $\epsilon'/\epsilon$ is dominated by the difference of $K \to \pi\pi$ matrix elements of the $Q_6$ and $Q_8$ operators,

$$Q_{\{6,8\}} = - \sum_{q=u,d,s}^{\{8,12\} \ e_q} \ (\bar{s}_L q_R) (\bar{q}_R d_L),$$

where the factors $e_q = (2e/3, -e/3, -e/3)$ are the quark electric charges. These factors make a profound difference between $Q_6$ and $Q_8$ at the meson level as they change their chiral properties making $Q_8$ an operator which starts at $\mathcal{O}(p^0)$, whereas $Q_6$ only starts at $\mathcal{O}(p^2)$. This makes $Q_8$ a simpler object to deal with and allows a connection to the well-known problem of the $\pi^+ - \pi^0$ electromagnetic mass difference.

Strong interactions make these operators change with the renormalization scale and mix with others so that, strictly speaking, they can never be considered in isolation. In particular $Q_8$ mixes with $Q_7$,

$$Q_7 = 6 \sum_{q=u,d,s} e_q \ (\bar{s}_L \gamma^\mu d_L) (\bar{q}_R \gamma_\mu q_R).$$

This apparent complication will actually be very helpful in the calculation of $Q_8$. 
The main effect is due to the fact that $Q_7$ and $Q_8$ contribute to the operator

$$\text{Tr} \left( U \lambda^{(23)} L U^\dagger Q_R \right),$$

where $(\lambda^{(32)})_{ij} = \delta_{i3}\delta_{j2}$ and $Q_R = \text{diag}(2/3, -1/3, -1/3)$. This operator is a “flavor-rotated” version of the operator responsible for the $\pi^+ - \pi^0$ electromagnetic mass difference, which is a problem that has been reasonably understood for a long time. This relationship is the underlying reason why both the physics of the $\pi^+ - \pi^0$ mass difference and the bosonization of $Q_{7,8}$ into the operator (4) are governed by one and the same Green’s function, which turns out to be the analog of the vacuum polarization but between the left- and right-handed currents $\overline{q}_{L,R} \gamma^\mu q_{L,R}$. Furthermore, $Q_7$ can be looked upon as due to the exchange of an imaginary “photon” with couplings to the quarks which are nondiagonal in flavor and whose “propagator” is unity instead of the usual $1/q^2$. One then obtains that

$$Q_7 = 6 < O_1(\mu) > \text{Tr} \left( U \lambda^{(23)} U^\dagger Q_R \right)^\dagger,$$

with

$$< O_1(\mu) > = < \overline{s} L \gamma^\mu d_L \overline{d} R \gamma^\mu s_R > = \frac{1}{6} \left( -3i g_{\mu\nu} \int \frac{d^4q}{(2\pi)^4} \Pi_{LR}^{\mu\nu}(q) \right) \overline{m}_S,$$

where current conservation implies $\Pi_{LR}^{\mu\nu}(q) = (q^\mu q^\nu - q^2 g^{\mu\nu}) \Pi_{LR}(q^2)$, since we are in the chiral limit. Because our imaginary photon has unity as its propagator, this integral is divergent and has to be regularized and renormalized. The subscript $\overline{m}_S$ refers to this fact. Eq. (3) shows an example of how a coupling constant in the effective meson Lagrangian (i.e. $< O_1(\mu) >$) is related to integrals over euclidean momentum of QCD Green’s functions. This is also true in general.

However, in order to really compute $< O_1 >$ one still needs to know the function $\Pi_{LR}(q^2)$ for all values of $q^2$ which is not known. The large-$N_c$ limit of QCD simplifies the problem because it determines the analytic structure of this function: it has to be meromorphic. Therefore it can only have poles (and no cut), which in physical terms correspond to the meson states. However this is still not enough as the number of poles is infinite and, since the solution to large-$N_c$ QCD has not been found, the location of these poles and their residues are unknown. It is at this point that a further approximation has to be made. This approximation, which we name “The Hadronic Approximation” to large-$N_c$ QCD, is defined by keeping only a finite number of resonances, whose residues and masses are fixed by matching to the first few terms of both the chiral and the OPE expansions of $\Pi_{LR}(Q^2)$. This is called in mathematics a rational approximant and is an interpolating function, ratio of two polynomials, which by construction has the same low- and high-$Q^2$ behavior as the full $\Pi_{LR}(Q^2)$ of large-$N_c$ QCD. This approximation rationalizes and systematizes old phenomenological approaches such as Vector Meson Dominance; incorporating chiral symmetry (to control long distances) and the OPE (to control short distances). It is also an improvement over approaches where only the OPE constraints were considered, such as QCD sum rules, because it also uses chiral symmetry at long distances.

An analysis of Aleph data shows that just the pion and one vector and one axial-vector states do a pretty good job provided their masses and decay constants fulfill the above-mentioned long and short-distance constraints. I urge the reader to take a look at the curves in Fig. 1 of

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*a*In the past we used the term “Minimal Hadronic Approximation” for cases in which only the leading term in the OPE expansion was used in the matching.

*b*Notice that this is possible because $\Pi_{LR}(Q^2)$ has no parton-model log $Q^2$; it would not be possible in the case of the vector-vector correlator, for instance.
the first paper in Ref. 9. In this case one obtains the remarkably simple expression

$$-Q^2 \Pi_{LR}(Q^2) = \frac{f_0^2 M_V^2 M_A^2}{(Q^2 + M_V^2)(Q^2 + M_A^2)},$$

(7)

where $f_0 = 87 \pm 3$ MeV, $M_V = 748 \pm 29$ MeV and $M_V^2/M_A^2 \simeq 0.50 \pm 0.06$. These values do not have to be exactly equal to the physical ones although it is natural to expect that, if the Hadronic Approximation and large-$N_c$ work, they have to be close to each other.

Having the function $\Pi_{LR}(Q^2)$, one can plug it in Eq. (8) and compute the integral. The result is

$$\langle O_1 \rangle (\mu) = -\frac{3}{32\pi^2} f_0^2 M_V^4 \log \left[ \frac{M_V^2}{M_A^2} \left( \frac{\Lambda^2}{M_V^2} \right)^{\frac{M_A^2}{M_V^2} - 1} \right],$$

(8)

with $\Lambda^2 \equiv \mu^2 \exp(1/3 + \kappa)$, where $\kappa = -1/2(3/2)$ in NDR(HV) schemes, respectively. The control over the renormalization scheme ($\gamma$, vanishing operators, etc...) is the result of the OPE constraints used in the construction of the Hadronic Approximation.\textsuperscript{9} It is obvious that knowing $\langle O_1 \rangle$ is tantamount to knowing any matrix element like, e.g., $\langle \pi \pi | Q_7 | K \rangle$: one only expands the $U$ in Eq. (8).

The calculation of the matrix elements of $Q_8$ is a bit more tricky because it is the product of two scalar-pseudoscalar densities -see Eq. (8). The previous analogy with an imaginary “photon”\textsuperscript{10} still works and leads to a result similar to (6) but involving these scalar-pseudoscalar densities, to wit \textsuperscript{11}

$$Q_8 = -12 \langle O_2(\mu) \rangle \text{ Tr } \left( U \lambda_{L}^{(23)} U^\dagger \right),$$

(9)

where

$$\langle O_2(\mu) \rangle = \langle \bar{s}_L s_R \bar{d}_R d_L \rangle$$

$$= \frac{1}{4} \langle \bar{\psi} \psi \rangle \frac{2}{\Lambda_{MS}} + \left( \int \frac{d^D q}{(2\pi)^D} \int d^4 x e^{iqx} \langle 0 | T \left[ d_L d_R(x) \bar{s}_R s_L(0) \right] | 0 \rangle \right)_{MS}$$

(10)

Taking the large-$N_c$ limit selects the quark condensate in the previous equation. However, there is some evidence -although circumstantial\textsuperscript{12} that this limit in the scalar-pseudoscalar sector may have somewhat large subleading corrections. This is why in our original paper in Ref. 9 we decided to take a short detour. The operator $Q_7$ mixes through the renormalization group into $Q_8$; this is tantamount to saying that $\langle O_2 \rangle$ controls the large-$Q^2$ fall-off of $\Pi_{LR}(Q^2)$. Consequently, by expanding Eq. (8) and selecting the coefficient in $1/Q^6$, one obtains

$$-Q^6 \Pi_{LR}(Q^2) \xrightarrow{Q^2 \rightarrow \infty} \frac{f_0^2 M_V^2 M_A^2}{16\pi \alpha_s} \left( 1 + \frac{\alpha_s}{\pi} \right) \langle O_2(\mu) \rangle + \cdots$$

(11)

where $\xi = (25/8,21/8)$ in the (NDR, HV) schemes\textsuperscript{10}, and the ellipses stand for numerically negligible terms\textsuperscript{12}. Inputting the value of $\alpha_s(2 \text{ GeV}) \simeq 0.33 \pm 0.04$ one easily obtains $\langle O_2(2 \text{ GeV}) \rangle$ from matching the two asymptotic behaviors in Eq. (11). Saturation of this value for $\langle O_2(2 \text{ GeV}) \rangle$ by the quark condensate in Eq. (10) leads to $\langle \bar{\psi} \psi \rangle_{MS}(2 \text{ GeV}) \simeq -(300 \pm 20 \text{ MeV})^3$, where the error is a combined estimate of unknown $1/N_c$ and $\alpha_s$ corrections.

At any rate, having the value of $\langle O_2(2 \text{ GeV}) \rangle$ one can easily compute any $K - \pi$ matrix element through Eq. (8). Table 1 shows our results together with those obtained by other

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\footnotesize

\textsuperscript{9}This “photon” is now even more strange: it doesn’t have spin one!.

\textsuperscript{10}At the time we wrote our paper the term proportional to $\xi$ was not known.
Table 1: Summary of results for $M_{\tau,8} \equiv \langle \pi \pi \rangle_{\tau=2}|K^0 \rangle$ (2 GeV), in units of GeV$^3$.

| Refs. | $M_7$(NDR) | $M_7$(HV) | $M_8$(NDR) | $M_8$(HV) |
|-------|-------------|-------------|-------------|-------------|
| (This work) Knecht et al. | 0.11 ± 0.03 | 0.67 ± 0.20 | 2.34 ± 0.73 | 2.52 ± 0.79 |
| Narison | 0.17 ± 0.05 | 1.4 ± 0.3 | | |
| Cirigliano et al. | 0.16 ± 0.10 | 0.49 ± 0.07 | 2.22 ± 0.67 | 2.46 ± 0.70 |
| Maltman et al. | 0.21 ± 0.03 | 0.46 ± 0.08 | 1.65 ± 0.45 | 1.84 ± 0.46 |
| Bijnens et al. | 0.24 ± 0.03 | 0.37 ± 0.08 | 1.2 ± 0.9 | 1.3 ± 0.9 |
| Battacharya et al. | 0.32 ± 0.06 | 1.2 ± 0.2 | | |
| Donini et al. | 0.11 ± 0.04 | 0.18 ± 0.06 | 0.51 ± 0.10 | 0.62 ± 0.12 |

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Acknowledgments

I would like to emphasize that the Hadronic Approximation to large-$N_C$ QCD is not limited to simple operators like $Q_{7,8}$ only, but is a general framework which has also been successfully used in a variety of other problems: $B_K \rightarrow \pi^0 \rightarrow e^+e^-$ and $\eta \rightarrow \mu^+\mu^-$, hadronic contributions to $g - 2 \equiv \epsilon \equiv \eta$, etc...; even in those cases when the underlying QCD Green’s function could not be related to any experimental data – which are the most –. More work is in progress.

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