Global Convergence of Policy Gradient for Linear-Quadratic Mean-Field Control/Game in Continuous Time

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Abstract

Reinforcement learning is a powerful tool to learn the optimal policy of possibly multiple agents by interacting with the environment. As the number of agents grow to be very large, the system can be approximated by a mean-field problem. Therefore, it has motivated new research directions for mean-field control (MFC) and mean-field game (MFG). In this paper, we study the policy gradient method for the linear-quadratic mean-field control and game, where we assume each agent has identical linear state transitions and quadratic cost functions. While most of the recent works on policy gradient for MFC and MFG are based on discrete-time models, we focus on the continuous-time models where some analyzing techniques can be interesting to the readers. For both MFC and MFG, we provide policy gradient update and show that it converges to the optimal solution at a linear rate, which is verified by a synthetic simulation. For MFG, we also provide sufficient conditions for the existence and uniqueness of the Nash equilibrium.

Keywords: Reinforcement learning, Mean-field control/game, Continuous linear dynamics, Policy gradient.

1 Introduction

Reinforcement learning (RL) [45] has become a very powerful tool for learning the optimal policy of a complicated system, with many successful applications including playing games achieving potential superhuman performance, such as Atari [32], GO [42] [44], Poker [20] [34], multiplayer online video games Dota [35] and StarCraft [48], and more realistic real-world problems, such as robotic control [50], autonomous driving [40], and social dilemmas [10] [30] [24]. The above are just some illustrative examples. More generally, RL has been applied to design efficient algorithms for decision making to minimize the long-term expected overall cost through interacting with the environment sequentially.

On a separate line of research, the subject of the optimal control assumes knowledge of the system dynamics and the observed reward/cost function, and studies the existence and uniqueness

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of the optimal solution. Extensive literature extends this area from the most basic setting of linear-quadratic regulator problem [49, 5, 1] to zero-sum game [15, 52] and to multi-agent control/game [12, 35, 41, 39, 11]. However, the multi-agent control/game is typically computationally intractable for a large real-world problem, as the joint state and action spaces grow exponentially in the number of agents. Mean-field control/game proposed by [22, 23, 27, 28, 29] can be viewed as an approximation to the multi-agent control/game when the number of agents grows to infinity. In a mean-field control/game, each agent share the same cost function and state transition, which depend on other agents only through their aggregated mean effect. Consequently, each agent’s optimal policy only depend on its own state and the mean-field state of the population. This symmetry across all agents significantly simplifies the analysis. Mean-field control/game has already found a lot of meaningful applications such as power grids [31], swarm robots [16, 2] and financial systems [53, 21].

Although the traditional optimal control approach lays a solid foundation for theoretical analysis, it fails to adapt well to the modern situation where we may have a huge system or complicated environment to explore. Therefore, recent years have witnessed increased interest in applying the RL techniques to various optimal control settings. See [17, 51, 6, 13] for some examples. Specifically, this paper focuses on the RL technique of policy gradient [46, 25, 43], where we update the policy following the gradient of the cost function, and the setting of the linear-quadratic mean-field control/game (MFC/MFG), where we assume each agent has identical linear state transition and quadratic cost function. The MFC differs from the MFG in that the former allows all the agents to directly control the mean-field state and collaborate in order to maximize the social welfare together, while the latter can only allow each agent to make individual decision with a guess on the mean-field output, hoping to achieve the Nash equilibrium of the system. The paper aims to show that policy gradient methods can achieves a desired linear convergence for both MFC and MFG. We choose the model-based approach for simplicity following the traditional optimal control approach for better presentation of the theoretical results and algorithm. The corresponding model-free algorithm to estimate the gradient can be derived similar to for example [17, 9, 18].

Many of the recent stochastic mean-field control/game literature are based on the continuous-time models, e.g. [4, 7, 8], where the main focus is on characterizing the properties of the optimal solution through solving a pair of Hamilton-Jacobi-Bellman (HJB) and Fokker-Planck (FP) equations, rather than designing provably efficient learning algorithms. However, new developments on policy gradient algorithms for MFC and MFG are mainly based on discrete-time models, e.g. [14, 19, 9, 18]. One reason is that discrete-time models can be more straightforward to analyze. For example, [17] pioneered the techniques to show the theoretical global convergence of policy gradient for the classical linear-quadratic regulator (LQR) based on the discrete-time models. One contribution of the current paper is to extend those techniques to the setting of continuous-time stochastic models.

We organize the paper as follows. In Section 2 we review the continuous-time classical LQR problem and show that the policy gradient converges to the optimal solution at a linear rate, with techniques designed for analyzing continuous stochastic dynamics. In Section 3 we formulate the MFC problem and reveal that with some reparametrization, MFC can be readily transformed into a
LQR problem. The MFG however is more involved to study, so we present the drifted LQR problem first in Section 4 as an intermediate step towards analyzing policy gradient for MFG. In Section 5, we provide an algorithm for solving MFG which provably also enjoys the linear convergence rate. The algorithm naturally contains two update steps: for a given mean-field state, each agent seeks the best response by solving a drifted LQR problem; then to find the Nash equilibrium, we update the mean-field state assuming each agent follows the best strategy. We will define the Nash equilibrium more concretely and provide sufficient conditions for its existence and uniqueness in Section 5 as well. Finally, we conclude the paper with a simple simulation and some discussions in Section 6.

Notations. For a matrix $M$, we denote by $\|M\|_2$ (or $\|M\|_F$) the spectral and Frobenius norm, $\sigma_{\min}(M), \sigma_{\max}(M)$ its minimum and maximum singular value, and $\text{tr}(M)$ the trace of $M$ when $M$ is a square matrix. Let $\langle M, N \rangle = \text{tr}(M^T N)$. We use $\|\alpha\|_2$ (or $\|\alpha\|$) to represent the $\ell_2$-norm of a vector $\alpha$. For scalars $a_1, \ldots, a_n$, we denote by $\text{poly}(a_1, \ldots, a_n)$ the polynomial of $a_1, \ldots, a_n$.

2 Linear-Quadratic Regulator

As the simplest optimal control problem, linear quadratic regulator serves as a perfect baseline to examine the performance of reinforcement learning methods. Viewing LQR from the lens of Markov decision process (MDP), the state and action spaces are $X = \mathbb{R}^d$ and $U = \mathbb{R}^k$, respectively. The continuous-time state transition dynamics is specified as the SDE

$$dX_t = (AX_t + Bu_t)dt + DdW_t,$$

where $W_t$ is standard $d$-dimensional Brownian motion. We consider the infinite-horizon time-average cost that each agent aims to minimize

$$\limsup_{T \to \infty} \mathbb{E} \left[ \frac{1}{T} \int_0^T c(X_t, u_t)dt \right], \quad X_0 \sim \mu_0, \quad c(x, u) = x^T Q x + u^T R u,$$

where the initial state $X_0$ is assumed to be sampled from the initial distribution $\mu_0$. The $A \in \mathbb{R}^{d \times d}$, $B \in \mathbb{R}^{d \times k}$, $D \in \mathbb{R}^{d \times d}$, $Q \in \mathbb{R}^{d \times d}$, $R \in \mathbb{R}^{k \times k}$ are matrices of proper dimensions with $Q, R \succ 0$.

It is known that the optimal action are linear in the corresponding state \cite{1, 2}. Specifically, the optimal actions satisfy $u_t^* = -K^* X_t$ for all $t \geq 0$, where $K^* \in \mathbb{R}^{k \times d}$ can be written as $K^* = R^{-1} B^T P^*$, with $P^*$ being the solution to the continuous time algebraic Riccati equation

$$A^T P^* + P^* A - P^* B R^{-1} B^T P^* + Q = 0.$$  

2.1 Ergodic Cost and Relative Value Function

Inspired by the form of the optimal policy, we consider the general linear policy $u_t = -K X_t$, where $K \in \mathbb{R}^{k \times d}$ is the parameter to be optimized. The state dynamics becomes

$$dX_t = (A - BK)X_t dt + DdW_t,$$
Unless otherwise specified, we assume $A-BK$ is stable, that is the real parts of all the eigenvalues of $A-BK$ are negative. Denote the invariant distribution of (4) as $\rho_K$. It is a Gaussian distribution $N(0, \Sigma_K)$, where $\Sigma_K$ satisfies the continuous Lyapunov equation

$$(A - BK)\Sigma_K + \Sigma_K (A - BK)^\top + DD^\top = 0. \tag{5}$$

Then the associated ergodic cost and the relative value function can be expressed as

$$J(K) := \mathbb{E}_{X_t \sim \rho_K} [c(X_t, u_t)] = \mathbb{E}_{X_t \sim \rho_K} [X_t^\top (Q + K^\top RK)X_t] = \langle Q + K^\top RK, \Sigma_K \rangle. \tag{6}$$

$$V_K(x) := \mathbb{E} \left[ \int_0^\infty [c(X_t, u_t) - J(K)] dt \mid X_0 = x \right]. \tag{7}$$

Using dynamic programming, we have the Hamilton-Jacobi-Bellman (HJB) equation for $V_K(x)$

$$c(x, -Kx) - J(K) + \langle (A - BK)x, \nabla V_K(x) \rangle + \frac{1}{2} \langle \nabla^2 V_K(x), DD^\top \rangle = 0. \tag{8}$$

Assuming the ansatz $V_K(x) = x^\top P_K x + C_K$ with a symmetric $P_K$ and plugging it into (8), we need the following two equations to be valid at the same time

$$(A - BK)^\top P_K + P_K (A - BK) + Q + K^\top RK = 0, \tag{9}$$

$$J(K) = \langle P_K, DD^\top \rangle. \tag{10}$$

To see it is possible, we combine (5)(6)(9) and find

$$J(K) = \langle Q + K^\top RK, \Sigma_K \rangle = - \text{tr}[(A - BK)^\top P_K + P_K (A - BK)] \Sigma_K] = - \text{tr}[P_K (\Sigma_K (A - BK)^\top + (A - BK) \Sigma_K)] = \langle P_K, DD^\top \rangle.$$ 

Therefore if $A - BK$ is stable, there exists a well-defined $P_K$ satisfying (9)(10) simultaneously. Note that by definition $\mathbb{E}_{x \sim \rho_K} [V_K(x)] = 0$, so the constant term in $V_K(x)$ can be determined as

$$C_K = \mathbb{E}_{x \sim \rho_K} [x^\top P_K x] = \langle P_K, \mathbb{E}_{x \sim \rho_K} [xx^\top] \rangle = \langle P_K, \Sigma_K \rangle.$$

### 2.2 Policy Gradient and Convergence

To implement the gradient descent method on $J(K)$, with a fixed stepsize $\eta$, we follow $K \leftarrow K - \eta \nabla_K J(K)$. The following proposition gives out the explicit formula for $\nabla_K J(K)$.

**Proposition 1** *(Expression of the gradient).*

$$\nabla_K J(K) = 2(RK - B^\top P_K) \Sigma_K = 2E_K \Sigma_K, \tag{11}$$

where we define $E_K := RK - B^\top P_K$.

With the above explicit formula for policy gradient, we present an upper bound for $J(K) - J(K^*)$ below, which shows the cost function is gradient dominated [26]. This property is essential in establishing the linear convergence of policy gradient.
Lemma 2 (Gradient domination).

\[ J(K) - J(K^*) \leq \frac{\|\Sigma_K\|}{\sigma_{\min}(R)\sigma_{\min}^2(DD^\top)} \text{tr}(\nabla_K J(K)^\top \nabla_J J(K)). \] (12)

The following theorem is the main result for this section, revealing that policy gradient method for continuous-time LQR achieves linear convergence rate. Its proof, together with those for the above proposition and lemma can be found in Appendix B of the supplemental material.

Theorem 3 (Global convergence of model-based gradient descent). With an appropriate constant setting of the stepsize \( \eta \) in the form of \( \eta = \text{poly}\left(\sigma_{\min}(Q), \sigma_{\min}(DD^\top), \|B\|^{-1}, \|R\|^{-1}\right) \), and number of iterations

\[ N \geq \frac{\|\Sigma_K\|}{\eta\sigma_{\min}^2(DD^\top)\sigma_{\min}(R)} \log \frac{J(K_0) - J(K^*)}{\varepsilon}, \]

the iterates of gradient descent enjoys \( J(K_N) - J(K^*) \leq \varepsilon \). Comparing to Theorem 7 of [17] for the linear convergence of policy gradient for the discrete-time LQR, the results for the continuous case is simpler in that \( \eta \) does not depend on \( \|A\| \) and \( \sigma_{\min}(R) \).

3 Linear-Quadratic Mean-Field Control

Now we consider a linear-quadratic regulator with mean-field interactions

\[ dX_t = (AX_t + \bar{A}E_0[X_t] + Bu_t + \bar{B}E_0[u_t])dt + Dw_t + \bar{D}dW^0_t, \] (13)

in which \( W_t, W^0_t \) are the idiosyncratic and common noise modeled by two independent \( d \)-dimensional Brownian motions and \( E_0 \) denotes the conditional expectation given \( W^0_t \). The discrete version of the model has been considered in [9]. Note that [13] also contains a mean-field action term. The agent seeks for policy in terms of \( u_t = u(X_t, E_0[X_t]) \) to minimize the following infinite-horizon time-average cost

\[ \limsup_{T \to \infty} E \left[ \frac{1}{T} \int_0^T c(X_t, E_0[X_t], u_t, E_0[u_t])dt \right], \quad X_0 \sim \mu_0, \]

\[ c(x, \bar{x}, u, \bar{u}) = x^\top Qx + \bar{x}^\top \bar{Q}\bar{x} + u^\top Ru + \bar{u}^\top \bar{R}\bar{u}, \quad Q, \bar{Q}, R, \bar{R} > 0. \] (14)

3.1 Reparametrization

For this problem under some suitable conditions, one can prove the optimal control is a linear combination of \( X_t \) and \( E_0[X_t] \), see e.g. [5]. We can actually recast the original MFC problem into a LQR problem with a larger state space. Specifically, motivated by the form of the optimal policy, we consider the general linear policy

\[ u_t = -K(X_t - E_0[X_t]) - L\bar{E}_0[X_t], \] (15)
where $\theta = (K, L)$ are the two parameter matrices to be optimized. Denote by $Y_t^1 = X_t - E_0[X_t]$ and $Y_t^2 = E_0[X_t]$. An important observation is that, under the policy (15), the dynamics of these two processes are decoupled

\[
\begin{align*}
\mathrm{d}Y_t^1 &= (A - BK)Y_t^1 \mathrm{d}t + D\mathrm{d}W_t, \\
\mathrm{d}Y_t^2 &= (A + \bar{A} - (B + \bar{B})L)Y_t^2 \mathrm{d}t + \tilde{D}\mathrm{d}W_t^0.
\end{align*}
\]

Moreover, the running cost can also be written as a quadratic function of $(Y_t^1, Y_t^2)$. Therefore one can essentially optimize $K$ and $L$ similar to the LQR, and all the theoretical results should follow.

4 Drifted Linear-Quadratic Regulator

In this section, we extend the simplest linear SDE dynamics to include an intercept in the drift. This extension is going to be useful for MFG. The state transition dynamics considered in this section is

\[
\mathrm{d}X_t = (a + AX_t + Bu_t) \mathrm{d}t + D\mathrm{d}W_t.
\] (16)

The agent still aims to minimize the the same quadratic cost $c(x, u) = x^\top Q x + u^\top R u$.

4.1 Ergodic Cost

We again consider the general linear policy, but with an extra intercept, $u_t = -KX_t + b$, where $K \in \mathbb{R}^{k \times d}$ and $b \in \mathbb{R}^k$ are the parameters to be optimized. The state dynamics becomes

\[
\mathrm{d}X_t = ((A - BK)X_t + a + Bb) \mathrm{d}t + D\mathrm{d}W_t.
\] (17)

The invariant distribution $\rho_{K,b}$ of (17) is a Gaussian distribution $N(\mu_{K,b}, \Sigma_K)$, where $\mu_{K,b}$ satisfies $\mu_{K,b} = -(A - BK)^{-1}(a + Bb)$ and $\Sigma_K$ does not depend on $b$ and still satisfies the continuous Lyapunov equation $(A - BK)\Sigma_K + \Sigma_K(A - BK)^\top + DD^\top = 0$. The associated ergodic cost can be expressed as

\[
J(K, b) := \mathbb{E}_{X_t \sim \rho_{K,b}}[c(X_t, u_t)] = J_1(K) + J_2(K, b),
\] (18)

where $J_1(K)$ and $J_2(K, b)$ are defined as

\[
J_1(K) = \langle Q + K^\top RK, \Sigma_K \rangle = \langle P_K, DD^\top \rangle,
\]

\[
J_2(K, b) = \left( \begin{array}{c}
\mu_{K,b} \\
b
\end{array} \right)^\top \begin{pmatrix} Q + K^\top RK & -K^\top R \\
-RK & R
\end{pmatrix} \begin{pmatrix} \mu_{K,b} \\
b
\end{pmatrix} + \langle \mu_{K,b}, b \rangle.
\]

Here $J_1(K)$ is the the expected total cost in the regular LQR problem without intercept and $P_K$ is the solution of the continuous Lyapunov equation (9). Meanwhile, $J_2(K, b)$ corresponds the expected cost induced by the intercept drift.
4.2 Policy Gradient and Convergence

**Proposition 4** The optimal intercept \(b^K\) to minimize \(J_2(K, b)\) for any given \(K\) is that

\[
b^K = -(KQ^{-1}A^\top + R^{-1}B^\top)(AQ^{-1}A^\top + BR^{-1}B^\top)^{-1}a
\]  

(19)

Furthermore, \(J_2(K, b^K)\) takes the form of

\[
J_2(K, b^K) = a^\top(AQ^{-1}A^\top + BR^{-1}B^\top)^{-1}a
\]  

(20)

which is independent of \(K\).

Since \(\min_b J_2(K, b)\) does not depend on \(K\), it holds that the optimal \(K^*\) can be obtained by minimizing \(J_1(K)\) similar to the case of no intercept, that is, updating \(K\) following the gradient direction \(\nabla_K J_1(K)\). So the optimal \(K^*\) does not depend on the intercept \(a\) at all. Once we have the optimal \(K^*\), the optimal \(b^* = b^{K^*}\) is obtained by plugging in \(K^*\) in (40). From Proposition 1 we know \(\nabla_K J(K) = \nabla_K J_1(K) = 2(RK - B^\top P_K)\Sigma_K\).

Define \(\mu^K\) to be the mean of the invariant density corresponding to \(u_t = -KX_t + b^K\). Then \(\mu^K = -(A - BK)^{-1}(a + Bh^K) = -Q^{-1}A^\top(AQ^{-1}A^\top + BR^{-1}B^\top)^{-1}a\), which does not depend on \(K\). The state dynamics can be written as

\[
d(X_t - \mu^K) = (A - BK)(X_t - \mu^K)dt + DdW_t.
\]  

(21)

And the cost function \(J(K) = J(K, b^K) = J_1(K) + a^\top(AQ^{-1}A^\top + BR^{-1}B^\top)^{-1}a\). This means we can directly apply convergence theorem of the policy gradient for regular LQR to \(X_t - \mu^K\). We relegate all the proofs to Appendix C of the supplemental material.

**Theorem 5** (Global convergence for drifted LQR). With the stepsize \(\eta\) in the same form as Theorem 3 and the number of iterations

\[
N \geq \eta \frac{\|\Sigma_{K^*}\|}{\sigma_{\min}^2(DD^\top)\sigma_{\min}(R)} \frac{\log J_1(K_0) - J_1(K^*)}{\epsilon},
\]

if we follow \(b^K = -(KQ^{-1}A^\top + R^{-1}B^\top)(AQ^{-1}A^\top + BR^{-1}B^\top)^{-1}a\), we have \(J(K_N, b^{K_N}) - J(K^*, b^*) \leq \epsilon\). Furthermore,

\[
\|K_N - K^*\|_F \leq \sigma_{\min}^{-1/2}(R)\sigma_{\min}^{-1/2}(DD^\top)\sqrt{\epsilon}, \quad \|b^{K_N} - b^*\|_2 \leq C_b(a)\sigma_{\min}^{-1/2}(R)\sigma_{\min}^{-1/2}(DD^\top)\sqrt{\epsilon},
\]

where \(C_b(a) = \|Q^{-1}A^\top(AQ^{-1}A^\top + BR^{-1}B^\top)^{-1}a\|_2\) is a constant depending on the intercept \(a\).

5 Linear-Quadratic Mean-Field Game

The linear-quadratic MFG has the same dynamics in (13) and cost function (14) as the MFC problem. But the key difference is that MFC allows all the agents to conduct the control together, whereas in MFG each agent has to optimize its own objective assuming a guess of the mean-field state/action. Therefore, the ultimate goal of studying MFG is to see if multiple agents can reach
a Nash equilibrium, where given the mean-field state/action, the policy of each agent is optimal and given all the agents carry out the optimal policy, we recover exactly the same mean-field state/action.

So the idea of policy gradient for MFG is straightforward: for any given mean-field state/action, we update policy by following the gradient and then with the updated policy we update the mean-field state/action. We will provide sufficient conditions for the existence and uniqueness of Nash equilibrium and show that policy gradient can converge to the Nash equilibrium in linear rate.

To that end, we need to study the linear quadratic control problem for any given mean-field state $\mu$ and mean-field action $\pi$: 

$$
\begin{align*}
    dX_t &= (AX_t + \dot{A}\mu_x + Bu_t + \dot{B}\mu_u)dt + DdW_t + \dot{D}dW_t^0, \\
    c(X_t, u_t) &= X_t^TQX_t + u_t^T\dot{R}u_t + \mu_x^T\dot{Q}\mu_x + \mu_u^T\dot{R}\mu_u, \\
    J_{(\mu,\pi)}(\mu) &= \limsup_{T \to \infty} \mathbb{E} \left[ \frac{1}{T} \int_0^T c(X_t, u_t)dt \right], \quad X_0 \sim \mu_0,
\end{align*}
$$

where $u_t$ is the action vector generated by playing policy $\pi$. Define $\mu = (\mu_x^T, \mu_u^T)^T \in \mathbb{R}^{d+k}$. We hope to find an optimal policy $\pi^*_\mu = \inf_{\pi \in \Pi} J_{\mu}(\pi)$. This is clearly a drifted LQR problem with an intercept $\dot{A}\mu_x + \dot{B}\mu_u$ in the drift. As in the drifted LQR, we consider the class of linear policies with an intercept, that is, 

$$
\Pi = \{ \pi(x) = -Kx + b : K \in \mathbb{R}^{k \times d}, b \in \mathbb{R}^k \}.
$$

Hence it suffices to find the optimal policy $\pi^*_\mu$ within $\Pi$.

Now, we introduce the definition of the Nash equilibrium [37]. The Nash equilibrium is obtained if we can find a pair $(\pi^*, \mu^*)$, such that the policy $\pi^*$ is optimal for each agent when the mean-field state is $\mu^*$, while all the agents following the policy $\pi^*$ generate the mean-field state $\mu^*$ as $t \to \infty$. To present its formal definition, we define $\Lambda_1(\mu)$ as the optimal policy in $\Pi$ given the mean-field state $\mu$, and define $\Lambda_2(\mu, \pi)$ as the mean-field state generated by the policy $\pi$ given the current mean-field state $\mu$ as $t \to \infty$.

**Definition 6 (Nash Equilibrium Pair).** The pair $(\mu^*, \pi^*) \in \mathbb{R}^d \times \Pi$ constitutes a Nash equilibrium pair of (22) if it satisfies $\pi^* = \Lambda_1(\mu^*)$ and $\mu^* = \Lambda_2(\mu, \pi^*)$. Here $\mu^*$ is called the Nash mean-field state/action and $\pi^*$ is called the Nash policy.

### 5.1 Existence and Uniqueness of Nash Equilibrium

Let us first rewrite (22) as follows:

$$
\begin{align*}
    dX_t &= (\tilde{a}_\mu + AX_t + Bu_t)dt + \tilde{D}d\tilde{W}_t, \quad c(X_t, u_t) = X_t^TQX_t + u_t^T\tilde{R}u_t + \tilde{C}_\mu,
\end{align*}
$$

where $\tilde{a}_\mu = A\mu_x + B\mu_u$ is the intercept in the drift term, $\tilde{D} = (D, \tilde{D}) \in \mathbb{R}^{d \times 2d}$ is an expanded matrix, $\tilde{W}_t = (W_t^T, W_t^{0T})^T \in \mathbb{R}^{2d}$ is $2d$-dimensional Brownian motion, $\tilde{C}_\mu = \mu_x^T\tilde{Q}\mu_x + \mu_u^T\tilde{R}\mu_u$ is a constant. So this is exactly the drifted LQR problem we considered in [16] with the same quadratic cost function ignoring the constant term.
Therefore, for the mapping $\pi^\mu_\mu = \Lambda_1(\mu)$, from (40) in Proposition 4, we know $\pi^\mu_\mu(x) = -K^*x + b^*_\mu$ where
\begin{equation}
  b^*_\mu = -(K^*Q^{-1}A^T + R^{-1}B^T)(AQ^{-1}A^T + BR^{-1}B^T)^{-1}\tilde{a}_\mu.  \tag{25}
\end{equation}

Note that $K^*$ is fixed for all $\mu$. For the mapping $\mu_{\text{new}} = \Lambda_2(\mu, \pi) = (\mu^\text{new}_x, \mu^\text{new}_u)^T$ where $\pi(x) = -K_\pi x + b_\pi$, we see the new mean of the mean-field state/action should be
\begin{align*}
  \mu_{\text{new},x} &= -(A - BK_\pi)^{-1}(Bb_\pi + \tilde{a}_\mu), \tag{26} \\
  \mu_{\text{new},u} &= b_\pi + K_\pi(A - BK_\pi)^{-1}(Bb_\pi + \tilde{a}_\mu). \tag{27}
\end{align*}

With the more detailed formulas for the mapping $\Lambda_1$ and $\Lambda_2$, we then establish the existence and uniqueness of the Nash equilibrium. The following regularity conditions are required.

**Assumption 7** We assume the following conditions hold.

(i) The continuous-time Riccati equation $A^TP^* + P^*A - P^*BR^{-1}B^TP^* + Q = 0$ admits a unique symmetric positive definite solution $P^*$.

(ii) The optimal $K^* = R^{-1}B^TP^*$. It holds that $L_0 = L_1L_3 + L_2 < 1$, where
\begin{align*}
  L_1 &= \left\|K^*Q^{-1}A^T + R^{-1}B^T\right\| \max\left\{\|\Gamma^{-1}A\|, \|\Gamma^{-1}B\|\right\}, \tag{28} \\
  L_2 &= \max\left\{\|\Delta_A\| + \|K^*\Delta_A\|, \|\Delta_B\| + \|K^*\Delta_B\|\right\}, \tag{29} \\
  L_3 &= \left\|(A - BK^*)^{-1}B\right\| + \|I + K^*(A - BK^*)^{-1}B\|, \tag{30}
\end{align*}
where $\Gamma = AQ^{-1}A^T + BR^{-1}B^T$, $\Delta_A = (A - BK^*)^{-1}A$, $\Delta_B = (A - BK^*)^{-1}B$.

**Proposition 8** *(Existence and Uniqueness of Nash Equilibrium).* Under Assumption 7, the operator $\Lambda(\cdot) = \Lambda_2(\cdot, \Lambda_1(\cdot))$ is $L_0$-Lipschitz, where $L_0$ is given in Assumption 7. Moreover, there exists a unique Nash equilibrium pair $(\mu^*, \pi^*)$ of the MFG.

**5.2 Policy Gradient Algorithm and Convergence**

To achieve the Nash equilibrium, the natural algorithm is that (i) for any given mean-field state/action $\mu_s$, we solve the drifted LQR problem in (22) until reasonably accuracy by policy gradient update, say $J_{\mu_s}(\pi_{s+1}) - J_{\mu_s}(\pi^*_{\mu_s}) \leq \varepsilon_s$ where $\pi^*_{\mu_s} = \Lambda_1(\mu_s)$ and $\varepsilon_s$ will be determined later; (ii) with the given $\pi_{s+1}$, we update the mean-field state/action $\mu_{s+1}$ by $\mu_{s+1} = \Lambda_2(\mu_s, \pi_{s+1})$ where the detailed formulas for $\Lambda_2(\cdot, \cdot)$ are provided in (26) (27). We summarize the above procedure in Algorithm 1.

We have the following theorem to show the linear convergence of Algorithm 1 to the MFG Nash equilibrium. The proof is deferred to Appendix D in the supplementary material.

**Theorem 9** *(Convergence of Algorithm 1).* For a sufficiently small tolerance $0 < \varepsilon < 1$, we choose the number of iterations $S$ in Algorithm 1 such that
\begin{equation}
  S \geq \frac{\log(2\|\mu_0 - \mu^*\|_2 \cdot \varepsilon^{-1})}{\log(1/L_0)}. \tag{31}
\end{equation}
Algorithm 1 Policy Gradient for Mean-Field Game

**Input:** Total number of iterations $S$, stepsize $\eta$, number of iterations $N_s$ for each policy update;

Initial mean-field state/action $\mu_0$, initial policy $\pi_0$ with parameters $K_{\pi_0}$ and $b_{\pi_0}$.

**Output:** Pair $(\pi_S, \mu_S)$.

1: for $s = 0, 1, \ldots, S - 1$ do
2: Policy Update:
3: $K^0 = K_{\pi_s}; \hat{\mu}_s \leftarrow \tilde{A}\mu_{s,x} + \tilde{B}\mu_{s,u};$
4: for $n = 0, 1, \ldots, N_s - 1$ do
5: $K^{n+1} \leftarrow K^n - 2\eta(RK^n - B^TP_{K^n})\Sigma K^n;$
6: end for
7: $K_{\pi_{s+1}} \leftarrow K^{N_s};$
8: $b_{\pi_{s+1}} = -(K_{\pi_{s+1}}Q^{-1}A^T + R^{-1}B^T)(AQ^{-1}A^T + BR^{-1}B^T)^{-1}\hat{\alpha}_s;$
9: $\pi_{s+1}(x) = -K_{\pi_{s+1}}x + b_{\pi_{s+1}};$
10: Mean-Field State/Action Update:
11: $\mu_{s+1,x} \leftarrow -(A - BK_{\pi_{s+1}})^{-1}(Bb_{\pi_{s+1}} + \hat{\alpha}_s);$ 
12: $\mu_{s+1,u} \leftarrow b_{\pi_{s+1}} + K_{\pi_{s+1}}(A - BK_{\pi_{s+1}})^{-1}(Bb_{\pi_{s+1}} + \hat{\alpha}_s);$ 
13: end for

For any $s = 0, 1, \ldots, S - 1$, define

$$\epsilon_s = \min \left\{ 2^{-2}\|B\|^2_2 \|A - BK^*\|^{-1}_2 \|2^2, C_b(\mu_s)^{-2}\epsilon^2, \right.$$

$$2^{-2s-4}(L_3C_b(\mu_s) + 2C_K(\mu_s))^{-2}\epsilon^2, \epsilon^2 \} \cdot \sigma_{\min}(R)\sigma_{\min}(DD^T),$$

(32)

where $C_b(\mu_s) = \|Q^{-1}A^T(AQ^{-1}A^T + BR^{-1}B^T)^{-1}\hat{\alpha}_s\|_2$ and $C_K(\mu_s) = (\|\hat{\alpha}_s\|_2 + (1 + L_1\|\mu_s\|_2)\|B\|_2) \cdot (\|(A - BK^*)^{-1}\|_2 + (1 + \|K^*\|_2))(A - BK^*)^{-1}\|\Sigma\|_2^2 \|B\|_2)$. In the $s$-th policy update, we choose the stepsize $\eta$ as in Theorem 3 and number of iterations

$$N_s \geq \frac{\|\Sigma K^*\|_2}{\eta\sigma_{\min}(DD^T)\sigma_{\min}(R)} \frac{J_{\mu_{s+1}}(K_{\pi_s}) - J_{\mu_{s+1}}(K^*)}{\epsilon_s},$$

such that $J_{\mu_{s}}(K_{\pi_{s+1}}, b_{\pi_{s+1}}) - J_{\mu_{s}}(K^*, b^*) \leq \epsilon_s$ where $K^*, b^*$ are parameters of the optimal policy $\pi^*_s = \Lambda_1(\mu_s)$ generated from mean-field state/action $\mu_s$, $J_{\mu_s}(K, b) = J_{\mu_s}(\pi)$ is defined in the drifted MFG problem [22], and $J_{\mu_{s+1}}(K_{\pi_s})$ is defined in [18] corresponding to $J_{\mu_{s}}(K, b)$. Then it holds that

$$\|\mu_s - \mu^*\|_2 \leq \epsilon, \quad \|K_{\pi_s} - K^*\|_F \leq \epsilon, \quad \|b_{\pi_s} - b^*\|_2 \leq (1 + L_1)\epsilon.$$ 

(33)

Here $\mu^*$ is the Nash mean-field state/action, $K_{\pi_S}, b_{\pi_S}$ are parameters of the final output policy $\pi_S$, and $K^*, b^*$ are the parameteris of the Nash policy $\pi^* = \Lambda_1(\mu^*)$.

Theorem 9 shows the linear convergence of the proposed Algorithm 1. This confirms that for the continuous-time MFG, policy gradient can achieve the ideal linear convergence performance in finding Nash equilibrium. This lays an important theoretical foundations for applying modern reinforcement learning techniques to the general continuous mean-field games.
6 Simulation and Conclusion

The paper aims to focus on the policy gradient method for the continuous-time MFC and MFG under the same framework. Specifically, we provide the linear convergence of the policy gradient algorithm for each problem setting. Although the paper is theory oriented, we demonstrate the theory through a simple simulation in Appendix [A] of the supplementary material and comment more on the comparison of MFC and MFG. The key observation is that MFG accumulates a larger total cost compared to MFC, although Nash equilibrium has been reached. In MFG, obviously agents have no control over the mean-field state and do not access $\bar{Q}, \bar{R}$ at all.

A key limitation of the current work is that all the results are model-based, although the corresponding model-free algorithm to approximate the policy gradient, either by an environment simulator [9] or by an actor-critic algorithm [18], can be combined with the theoretical results in this paper. In addition, other variations of MFC and MFG can be considered for future research, including risk-sensitive mean-field setting [47], robust mean-field games [3] and mean-field models with partially observed information [38].

Broader Impact

Theoretical understanding of reinforcement learning is essential in evaluating its potential for more general applications involving real world big systems. Along this line, researchers still have a long way to accomplish a comprehensive understanding for different problem settings such as discrete vs continuous, linear-quadratic vs general, classical LQR vs multi-agent control/game. In this work, we are motivated to extend our understanding of the policy gradient algorithm to the problem of continuous-time linear-quadratic mean-field control and game under a unified framework. Our analysis serves as a step towards filling in some small theoretical gaps in the big picture.

A Simulation for Model-based MFC and MFG

In this section, we give some numerical results to demonstrate the linear convergence of policy gradient algorithm for MFC and MFG, and make some numerical comparison of them as well. We consider the following setting:

$$A = \begin{pmatrix} -1 & 0.1 & -0.05 \\ 0.05 & -1 & -0.05 \\ 0 & 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} -0.5 \\ -0.5 \\ 0.8 \end{pmatrix},$$

and $\bar{A} = -0.5A$, $\bar{B} = -0.5B$, $D = \bar{D} = I_3$, $Q = 0.1I_3, \bar{Q} = 0.05I_3$, $R = 1, \bar{R} = 2$. The continuous-time Riccati equation has the following solution

$$P^* = \begin{pmatrix} 0.04979778 & 0.00336704 & -0.00080209 \\ 0.00336704 & 0.0499634 & -0.00082373 \\ -0.00080209 & -0.00082373 & 0.04927204 \end{pmatrix}.$$
We can also manually check that the conditions in Assumption 7 hold. Actually \( L_1 = 0.030110, L_2 = 0.570206, L_3 = 2.020098 \) and \( L_0 = 0.631032 < 1 \).

For MFC, we start iterations from \( K = 0, L = 0 \), which are indeed stabilizing. We choose \( \eta = 0.01 \) and let the policy gradient method run for \( N = 200 \) updates. The linear convergence can be clearly seen from the left plot of Figure 1, where we plot \( \log(J(K, L) - J(K^*, L^*)) \) against \( n = 1, 2, \ldots, N \). For MFG, we start iterations from \( K = 0, b = 0, \mu_x = 0.5(1, 1, 1)^\top, \mu_u = 0.5 \), and set \( \eta = 0.005 \), the total number of iterations \( S = 10 \) for the outer loop, and for each \( s = 1, \ldots, 10 \) the number of iterations \( N_s = 20 \) for the inner policy gradient updates. The right plot of Figure 1 shows \( \log(J_{\mu_s}(K_{\pi_s}, b_{\pi_s}) - J_{\mu^*}(K^*, b^*)) \) against \( s = 1, 2, \ldots, 10 \). The linear convergence of the algorithm matches well with our theoretical results. Note that here \( J(K, L) \) is the cost of the MFC problem (14), while \( J_{\mu_s}(K_{\pi_s}, b_{\pi_s}) \) is the the cost of the drifted LQR problem (22) corresponding to MFG. It is not hard to calculate that \( J(K^*, L^*) = 0.598563 \) and \( J_{\mu^*}(K^*, b^*) = 0.298066 \), where \( J_{\mu^*} \) is smaller as it ignores the dynamics of the conditional mean \( \mathbb{E}_0[X_t], \mathbb{E}_0[u_t] \).

![Linear convergence of MFC](image1)

![Linear convergence of MFG](image2)

**Figure 1:** Linear Convergence of Policy Gradient for MFG and MFC. The figure on the left for MFC uses initial values \( K = 0, L = 0 \), learning rate \( \eta = 0.01 \) and plots \( \log(J(K, L) - J(K^*, L^*)) \) against the iterations \( n = 1, 2, \ldots, 200 \). The figure on the right for MFG runs Algorithm 1 with the initial values \( K = 0, b = 0, \mu_x = 0.5(1, 1, 1)^\top, \mu_u = 0.5 \), learning rate \( \eta = 0.005 \), the total number of iterations \( S = 10 \) for the outer loop, and for each \( s = 1, \ldots, 10 \), the number of iterations \( N_s = 20 \) for the inner policy gradient updates. It plots \( \log(J_{\mu_s}(K_{\pi_s}, b_{\pi_s}) - J_{\mu^*}(K^*, b^*)) \) against \( s = 1, 2, \ldots, 10 \).

Since MFG and MFC share the same model dynamics and cost function, we can compare the cost they achieve together in Figure 2. As the target of MFC is indeed minimizing the total cost, the effective control of policy gradient guarantees that the cost of MFC (green curve) converges to the optimal level at a linear rate. However, each agent of MFG only cares minimizing the cost with a given estimate of the mean-field state, i.e. solving the drifted LQR problem. Even when the estimate \( \mu_s \) gets very close to the optimal \( \mu^* \) and the Nash equilibrium is approximately obtained, the total cost of MFG (blue curve) is much larger than the optimal level. This is expected since in MFG, obviously agents have no control over the mean-field state and do not access \( \hat{Q}, \hat{R} \) at all.

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Figure 2: Total Cost of MFG and MFC. The cost of MFC (green curve) converges to the optimal level (red line) at a linear rate, while the cost of MFG (blue curve) fails to converge to the optimal level, although Nash equilibrium has been reached. For MFG, we get the cost every $N_s = 20$ inner policy gradient iterations.

B Proofs for Section 2

Proposition 10 (Proposition 7).

$$\nabla_K J(K) = 2(RK - B^\top P_K)\Sigma_K = 2E_K\Sigma_K, \tag{34}$$

where we define $E_K := RK - B^\top P_K$.

Proof Rewrite the Lyapunov equation (9) as $\phi(K, P_K) = 0$, where $\phi$ is a function of two independent arguments, defined as

$$\phi(K, P_K) := (A - BK)^\top P_K + P_K(A - BK) + Q + K^\top RK.$$  

Taking differential on both sides, we have

$$0 = \nabla_K \phi(K, P_K) dK + \nabla_{P_K} \phi(K, P_K) dP_K$$

$$= [(-BdK)^\top P_K + P_K(-BdK) + (dK)^\top RK + K^\top RdK] + [(A - BK)^\top dP_K + dP_K(A - BK)],$$

or equivalently,

$$(A - BK)^\top dP_K + dP_K(A - BK) + (K^\top R - P_K B) dK + (dK)^\top (RK - B^\top P_K) = 0. \tag{35}$$
Note that (3) and (35) have similar structures. We apply the trace operator to (5) left multiplied by $dP_K$ and (35) left multiplied by $\Sigma_K$, and then take the difference to obtain

$$\text{tr}(dP_KDD^\top) = \text{tr}[\Sigma_K(K^\top R - P_KB)dK + \Sigma_K(dK)^\top(RK - B^\top P_K)]$$

$$= \text{tr}[2\Sigma_K(K^\top R - P_KB)dK].$$

From (10), by definition, we have

$$\text{tr}[(\nabla_K J)\nabla_K dK] = dJ(K) = \text{tr}(dP_KDD^\top).$$

Comparing the above two equations, we conclude $\nabla_K J(K) = 2(RK - B^\top P_K)\Sigma_K$.

\[\boxed{\text{Lemma 11 (Solution of continuous Lyapunov equation). Suppose } W \text{ is stable. The solution } Y \text{ of continuous Lyapunov equation}}\]

$$WY + YW^\top + Q = 0$$

\[\text{can be written as}\]

$$Y = \int_0^\infty e^{W\tau}Qe^{W^\top\tau}d\tau. \quad (36)$$

In the following, given $K$ such that $A - BK$ is stable, we define two operators $\mathcal{T}_K, \mathcal{F}_K$ on symmetric matrix $X$ as

$$\mathcal{T}_K(X) := \int_0^\infty e^{(A-BK)\tau}Xe^{(A-BK)^\top\tau}d\tau,$$

$$\mathcal{F}_K(X) := (A - BK)X + X(A - BK)^\top.$$

Then

$$\mathcal{F}_K \circ \mathcal{T}_K + I = 0,$$

or

$$\mathcal{T}_K = -\mathcal{F}_K^{-1}.$$

Additionally, from (5) we have

$$\Sigma_K = \mathcal{T}_K(DD^\top).$$

\[\text{Lemma 12 (Perturbation of } P_K). \text{ Assume } K, K' \text{ are both stable. Then}\]

$$P_{K'} - P_K = \int_0^\infty e^{(A-BK')\tau}[(E_K^\top(K' - K) + (K' - K)^\top E_K + (K' - K)(K' - K)R(K' - K))]e^{(A-BK')\tau}d\tau.$$
In other words, let \( P \) in which \( J \) is the unknown matrix. Recalling Lemma 11, we finish the proof.

Proof Taking the difference between two equations \( (10) \) corresponding to \( K' \) and \( K \), we have

\[
0 = (A - BK')^\top P_{K'} + P_{K'}(A - BK')^\top - (A - BK' + B(K' - K))^\top P_K + P_K(A - BK' + B(K' - K))^\top + (K' - K + K)^\top R(K' - K + K) - K^\top RK
\]

\[
= (A - BK')^\top (P_{K'} - P_K) + (P_{K'} - P_K)(A - BK')^\top - (K' - K)^\top B^\top P_K + P_K B(K' - K) + (K' - K + K)^\top R(K' - K)
\]

\[
= (A - BK')^\top (P_{K'} - P_K) + (P_{K'} - P_K)(A - BK')^\top + E_K^\top (K' - K) + (K' - K)^\top E_K + (K' - K)^\top R(K' - K).
\]

In other words, \( P_{K'} - P_K \) is the solution of the continuous Lyapunov equation

\[
(A - BK')^\top Y + Y(A - BK') + E_K^\top (K' - K) + (K' - K)^\top E_K + (K' - K)^\top R(K' - K) = 0,
\]

in which \( Y \) is the unknown matrix. Recalling Lemma 11 we finish the proof.

Lemma 13 (Lemma 2). The cost function is gradient dominated \( [26] \), that is

\[
J(K) - J(K^*) \leq \frac{\|\Sigma_{K'}\|}{\sigma_{\min}(R)\sigma_{\min}^2(DD^\top)} \text{tr}(\nabla_K J(K)^\top \nabla_K J(K)).
\] (37)

In additional, we have the following lower bound for \( J(K) - J(K^*) \)

\[
J(K) - J(K^*) \geq \frac{\sigma_{\min}(DD^\top)}{\|R\|} \text{tr}(E_K^\top E_K).
\] (38)

Proof Based on \( [10] \) and Lemma 12 we have

\[
J(K') - J(K)
\]

\[
= \text{tr}[(P_{K'} - P_K)DD^\top]
\]

\[
= \text{tr} \left[ \int_0^\infty e^{(A - BK')^\top \tau} [E_K^\top (K' - K) + (K' - K)^\top E_K + (K' - K)^\top R(K' - K)] e^{(A - BK')^\top \tau} DD^\top d\tau \right]
\]

\[
= \text{tr} \left[ \int_0^\infty e^{(A - BK')^\top \tau} DD^\top e^{(A - BK')^\top \tau} d\tau [E_K^\top (K' - K) + (K' - K)^\top E_K + (K' - K)^\top R(K' - K)] \right]
\]

\[
= \text{tr}[\Sigma_K'[E_K^\top (K' - K) + (K' - K)^\top E_K + (K' - K)^\top R(K' - K)]]
\]

\[
= \text{tr}[\Sigma_K'[(K' - K + R^{-1}E_K)^\top R(K' - K + R^{-1}E_K) - E_K^\top R^{-1}E_K]].
\]

On one hand, letting \( K' = K^* \), we have

\[
J(K) - J(K^*) = \text{tr}[\Sigma_K'[E_K^\top R^{-1}E_K - (K^* - K + R^{-1}E_K)^\top R(K^* - K + R^{-1}E_K)]]
\]

\[
\leq \text{tr}[\Sigma_K' E_K^\top R^{-1}E_K]
\]

\[
\leq \frac{\|\Sigma_K'\|}{\sigma_{\min}(R)} \text{tr}(E_K^\top E_K)
\]

\[
\leq \frac{\|\Sigma_K'\|}{\sigma_{\min}(R)\sigma_{\min}^2(\Sigma_K)} \text{tr}(\nabla_K J(K)^\top \nabla_K J(K))
\]

\[
\leq \frac{\|\Sigma_K'\|}{\sigma_{\min}(R)\sigma_{\min}^2(DD^\top)} \text{tr}(\nabla_K J(K)^\top \nabla_K J(K)).
\]
The last inequality follows from the fact that \( \Sigma_K \geq DD^\top \geq \sigma_{\min}(DD^\top) \cdot I_d \).

On the other hand, letting \( K' = K - R^{-1}E_K \), we have

\[
J(K) - J(K') = \text{tr}[\Sigma_{K'}E_K^\top R^{-1}E_K].
\]

Then

\[
J(K) - J(K^*) \geq J(K) - J(K') \\
\geq \text{tr}[\Sigma_{K'}E_K^\top R^{-1}E_K] \\
\geq \frac{\sigma_{\min}(DD^\top)}{\|R\|} \text{tr}(E_K^\top E_K).
\]

\[\blacksquare\]

**Lemma 14** (Perturbation analysis of \( \Sigma_K \)) Suppose \( A - BK \) is stable and

\[
\|K' - K\| \leq \frac{\sigma_{\min}(Q)\sigma_{\min}(DD^\top)}{4J(K)\|B\|},
\]

then \( A - BK' \) is also stable and

\[
\|\Sigma_{K'} - \Sigma_K\| \leq 4 \left( \frac{J(K)}{\sigma_{\min}(Q)} \right)^2 \frac{\|B\|}{\sigma_{\min}(DD^\top)} \|K' - K\|.
\]

**Proof** The first claim is easy to prove with Lemma 10 in [33]. The second claim is similar to Appendix C.4 in [17]. We first claim

\[
\|\Sigma_K\| \leq \frac{J(K)}{\sigma_{\min}(Q)} \text{ and } \|T_K\| \leq \frac{\|\Sigma_K\|}{\sigma_{\min}(DD^\top)},
\]

and it is clear to see that

\[
\|F_{K'} - F_K\| \leq 2\|B\|\|K' - K\|.
\]

Then

\[
\|T_K\|\|F_{K'} - F_K\| \leq \frac{2J(K)\|B\|\|K' - K\|}{\sigma_{\min}(Q)\sigma_{\min}(DD^\top)} \leq \frac{1}{2}.
\]

Then we have

\[
\|\Sigma_{K'} - \Sigma_K\| = \|(T_{K'} - T_K)(DD^\top)\| \leq \|T_K\|\|F_{K'} - F_K\|\|\Sigma_{K'}\| \\
\leq \|T_K\|\|F_{K'} - F_K\|(\|\Sigma_K\| + \|\Sigma_{K'} - \Sigma_K\|)
\]

Therefore,

\[
\|\Sigma_{K'} - \Sigma_K\| \leq 2\|T_K\|\|F_{K'} - F_K\|\|\Sigma_K\| \\
\leq 4 \left( \frac{J(K)}{\sigma_{\min}(Q)} \right)^2 \frac{\|B\|}{\sigma_{\min}(DD^\top)} \|K' - K\|.
\]
So it remains to show the claim in [39]. The first claim can be seen from

\[ J(K) = \text{tr}(\Sigma_K(Q + K^\top R K)) \geq \text{tr}(\Sigma_K)\sigma_{\text{min}}(Q) \geq \sigma_{\text{min}}(Q) \cdot \sigma_{\text{min}}(Q). \]

The second claim can be shown from the following fact. For any unit vector \( v \in \mathbb{R}^d \) and unit spectral norm matrix \( X \),

\[
v^\top \mathcal{T}_K(X)v = \int_{0}^{\infty} \text{tr}(X e^{(A-BK)^\top \tau} e^{(A-BK)^\top})d\tau \leq \int_{0}^{\infty} \text{tr}(DD^\top e^{(A-BK)^\top \tau} e^{(A-BK)^\top})d\tau \cdot ||(DD^\top)^{-1/2}X(DD^\top)^{-1/2}|| = (v^\top \Sigma_K v) \cdot ||(DD^\top)^{-1/2}X(DD^\top)^{-1/2}|| \leq ||\Sigma_K|| \sigma_{\text{min}}^2(DD^\top).
\]

We now complete the proof.

**Lemma 15** (Estimate of one-step GD). Suppose \( K' = K - \eta \nabla_K J(K) \) with

\[
\begin{align*}
\eta &\leq \min \left\{ \frac{3\sigma_{\text{min}}(Q)}{8J(K)\|R\|}, \frac{1}{16} \left( \frac{\sigma_{\text{min}}(Q)\sigma_{\text{min}}(DD^\top)}{J(K)} \right)^2 \frac{1}{\|B\|\|\nabla_K J(K)\|} \right\}, \\
\end{align*}
\]

then

\[
J(K') - J(K^*) \leq \left( 1 - \frac{\eta \sigma_{\text{min}}(R)\sigma_{\text{min}}^2(DD^\top)}{\|\Sigma_K\|} \right) (J(K) - J(K^*)).
\]

**Proof** By the proof of Lemma 2, we have

\[
J(K) - J(K') = 2 \text{tr}[\Sigma_K' (K - K')^\top E_K] - \text{tr}[\Sigma_K' (K - K')^\top R (K - K')]
\]

\[
= 4\eta \text{tr}(\Sigma_K' \Sigma_K E_K^\top E_K - 4\eta^2 \text{tr}(\Sigma_K' \Sigma_K E_K^\top RE_K)
\]

\[
\geq 4\eta \text{tr}(\Sigma_K' E_K^\top E_K \Sigma_K) - 4\eta\|\Sigma_K' - \Sigma_K\| \text{tr}(\Sigma_K E_K^\top E_K) - 4\eta^2\|\Sigma_K'\|\|R\| \text{tr}(\Sigma_K E_K^\top E_K \Sigma_K)
\]

\[
\geq 4\eta \text{tr}(\Sigma_K' E_K^\top E_K \Sigma_K) - 4\eta \frac{||\Sigma_K' - \Sigma_K||}{\sigma_{\text{min}}(\Sigma_K)} \text{tr}(\Sigma_K E_K^\top E_K \Sigma_K) - 4\eta^2 ||\Sigma_K'\|\|R\| \text{tr}(\Sigma_K E_K^\top E_K \Sigma_K)
\]

\[
= 4\eta \left( 1 - \frac{||\Sigma_K' - \Sigma_K||}{\sigma_{\text{min}}(\Sigma_K)} - \eta \|\Sigma_K'\|\|R\| \right) \text{tr}(\nabla_K J(K)^\top \nabla_K J(K))
\]

\[
\geq 4\eta \frac{\sigma_{\text{min}}(R)\sigma_{\text{min}}^2(DD^\top)}{\|\Sigma_K\|} \left( 1 - \frac{||\Sigma_K' - \Sigma_K||}{\sigma_{\text{min}}(DD^\top)} - \eta \|\Sigma_K'\|\|R\| \right) (J(K) - J(K^*)).
\]

The condition on \( \eta \) ensures

\[
\|K' - K\| \leq \frac{\sigma_{\text{min}}(Q)\sigma_{\text{min}}(DD^\top)}{4J(K)\|B\|},
\]

so by Lemma 14

\[
\frac{||\Sigma_K' - \Sigma_K||}{\sigma_{\text{min}}(DD^\top)} \leq 4\eta \left( \frac{J(K)}{\sigma_{\text{min}}(Q)\sigma_{\text{min}}(DD^\top)} \right)^2 \|B\|\|\nabla_K J(K)\| \leq \frac{1}{4}.
\]
with the assumed $\eta$. Then
\[
\|\Sigma_{K'}\| \leq \|\Sigma_K\| + \|\Sigma_{K'} - \Sigma_K\| \leq \frac{J(K)}{\sigma_{\min}(Q)} + \frac{\sigma_{\min}(DD^\top)}{4} \leq \frac{J(K)}{\sigma_{\min}(Q)} + \frac{\|\Sigma_{K'}\|}{4},
\]
which implies $\|\Sigma_{K'}\| \leq \frac{4J(K)}{3\sigma_{\min}(Q)}$. Hence,
\[
1 - \frac{\|\Sigma_{K'} - \Sigma_K\|}{\sigma_{\min}(DD^\top)} - \eta\|\Sigma_{K'}\|\|R\| \geq 1 - \frac{4J(K)}{3\sigma_{\min}(Q)} \geq \frac{1}{4},
\]
with the assumed $\eta$. Now we have
\[
J(K) - J(K') \geq \eta\frac{\sigma_{\min}(R)\sigma_{\min}^2(DD^\top)}{\|\Sigma_{K'}\|}(J(K) - J(K^*)),
\]
which is equivalent to the desired conclusion.

**Theorem 16** (Theorem 3). With an appropriate constant setting of the stepsize $\eta$ in the form of
\[
\eta = \text{poly}\left(\frac{\sigma_{\min}(Q)}{C(K_0)}, \sigma_{\min}(DD^\top), \frac{1}{\|B\|}, \frac{1}{\|R\|}\right),
\]
and number of iterations
\[
N \geq \frac{\|\Sigma_{K'}\|}{\eta\sigma_{\min}^2(DD^\top)\sigma_{\min}(R)} \log \frac{J(K_0) - J(K^*)}{\varepsilon},
\]
the iterates of gradient descent enjoys
\[
J(K_N) - J(K^*) \leq \varepsilon.
\]

**Proof** Iterating the gradient decent for $N$ times, from Lemma 15, we know
\[
J(K_N) - J(K^*) \leq \left(1 - \eta\frac{\sigma_{\min}(R)\sigma_{\min}^2(DD^\top)}{\|\Sigma_{K'}\|}\right)^N (J(K_0) - J(K^*)).
\]
Therefore, if $N$ is chosen as the above, we can make the right hand side smaller than $\varepsilon$.

**C  Proofs for Section 4**

**Proposition 17** (Proposition 4). The optimal intercept $b^K$ to minimize $J_2(K, b)$ for any given $K$ is that
\[
b^K = -(KQ^{-1}A^\top + R^{-1}B^\top)(AQ^{-1}A^\top + BR^{-1}B^\top)^{-1}a \quad \text{(40)}
\]
Furthermore, $J_2(K, b^K)$ takes the form of
\[
J_2(K, b^K) = a^\top(AQ^{-1}A^\top + BR^{-1}B^\top)^{-1}a \quad \text{(41)}
\]
which is independent of $K$. 18
The problem of \( \min_b J_2(K, b) \) is equivalent to the following constrained optimization

\[
\min\left( \mu \begin{pmatrix} b \\ b \end{pmatrix}^\top \begin{pmatrix} Q + K^\top RK & -K^\top R \\ -RK & R \end{pmatrix} \begin{pmatrix} \mu \\ b \end{pmatrix} \right)
\text{ s.t. } (A - BK)\mu + (a + Bb) = 0
\]

Using the Lagrangian multiplier method, we have

\[
2M \begin{pmatrix} \mu \\ b \end{pmatrix} + N\lambda = 0, \quad N^\top \begin{pmatrix} \mu \\ b \end{pmatrix} + a = 0,
\]

where

\[
M = \begin{pmatrix} Q + K^\top RK & -K^\top R \\ -RK & R \end{pmatrix}, \quad N = \begin{pmatrix} (A - BK)^\top \end{pmatrix}.
\]

Therefore, it is not hard to derive the optimal \((\mu^K, b^K)\) is

\[
\begin{pmatrix} \mu^K \\ b^K \end{pmatrix} = -M^{-1}N(N^\top M^{-1}N)^{-1}a.
\]

And the optimal value of \( J_2(K, b) \) is \( J_2(K, b^K) = a^\top (N^\top M^{-1}N)^{-1}a \). By some simple calculation,

\[
M^{-1} = \begin{pmatrix} Q^{-1} & Q^{-1}K^\top \\ KQ^{-1} & KQ^{-1}K^\top + R^{-1} \end{pmatrix},
\]

and \( N^\top M^{-1}N = AQ^{-1}A^\top + BR^{-1}B^\top \). Therefore, the final optimal

\[
\begin{pmatrix} \mu^K \\ b^K \end{pmatrix} = -\begin{pmatrix} Q^{-1}A^\top \\ KQ^{-1}A^\top + R^{-1}B^\top \end{pmatrix} \left( AQ^{-1}A^\top + BR^{-1}B^\top \right)^{-1}a.
\]

**Theorem 18** (Theorem 5). With the stepsize \( \eta \) in the form of

\[
\eta = \text{poly} \left( \frac{\sigma_{\min}(Q)}{C(K_0)}, \sigma_{\min}(DD^\top), \frac{1}{\|B\|}, \frac{1}{\|R\|} \right),
\]

and number of iterations

\[
N \geq \frac{\|\Sigma_{K^*}\|_F}{\eta\sigma_{\min}(DD^\top)\sigma_{\min}(R)} \log \frac{J_1(K_0) - J_1(K^*)}{\varepsilon},
\]

the iterates of gradient descent enjoys \( J_1(K_N) - J_1(K^*) \leq \varepsilon \). If we follow \( b^K = -(KQ^{-1}AA^\top + R^{-1}B^\top)(AQ^{-1}A^\top + BR^{-1}B^\top)^{-1}a \), we have

\[
J(K_N, b^{K_N}) - J(K^*, b^*) \leq \varepsilon.
\]

Furthermore,

\[
\|K_N - K^*\|_F \leq \sigma_{\min}(R)^{-1/2}(DD^\top)\sqrt{\varepsilon}, \quad \|b^{K_N} - b^*\|_2 \leq C_b(a)\sigma_{\min}(R)^{-1/2}(DD^\top)\sqrt{\varepsilon},
\]

where \( C_b(a) = \|Q^{-1}A^\top \left( AQ^{-1}A^\top + BR^{-1}B^\top \right)^{-1}a\|_2 \) is a constant depending on the intercept \( a \).
Proof. We only need to show the bound for $K_N$ and $b^{K_N}$ in [43]. From the proof of Lemma 2, we showed that for any $K, K'$,

$$J_1(K) - J_1(K') = \text{tr}[\Sigma_K[E_{K'}^T(K - K') + (K - K')^T E_{K'} + (K - K')^T R(K - K')].$$

Choosing $K' = K^*$, since $E_{K^*} = 0$, we get

$$J_1(K) - J_1(K^*) = \text{tr}[\Sigma_K (K - K^*)^T R(K - K^*)] \geq \sigma_{\min}(R), \sigma_{\min}(DD^\top)\|K_N - K^*\|^2_F.$$ 

Therefore, if $(K_N, b^{K_N})$ makes $J(K_N, b^{K_N}) - J(K^*, b^*) = J_1(K) - J_1(K^*)$ $\leq \varepsilon$, we surely obtain $\|K_N - K^*\|^2_F \leq \sigma_{\min}(R)\sigma_{\min}(DD^\top)\varepsilon$.

The bound for $b^{K_N}$ is straightforward as

$$\|b^{K_N} - b^*\|_2 \leq \|K_N - K^*\|_2\|Q^{-1}A^T(AQ^{-1}A^T + BR^{-1}B^\top)^{-1}a\|_2 \leq C_b(a)\|K_N - K^*\|_F \leq C_b(a)\sigma_{\min}^{-1/2}(R)\sigma_{\min}^{-1/2}(DD^\top)\sqrt{\varepsilon}.$$ 

\[\blacksquare\]

D Proofs for Section 5

Proposition 19 (Proposition 8). Under Assumption 7, the operator $\Lambda(\cdot) = \Lambda_2(\cdot, \Lambda_1(\cdot))$ is $L_0$-Lipschitz, where $L_0$ is given in Assumption 7. Moreover, there exists a unique Nash equilibrium pair $(\mu^*, \pi^*)$ of the MFG.

Proof. Consider the linear policies $\pi_{K,b}(x) = -Kx + b$. Define the distance metric of the linear policy as follows

$$d(\pi_{K_1,b_1}, \pi_{K_2,b_2}) = \|K_1 - K_2\|_2 + \|b_1 - b_2\|_2.$$  

(44)

Then for the mapping $\Lambda_1(\mu)$, as the optimal $K^*$ does not depend on $\mu$, we have for any $\mu_1, \mu_2 \in \mathbb{R}^{d+k}$,

$$d(\Lambda_1(\mu_1), \Lambda_1(\mu_2)) = \|b^*_\mu_1 - b^*_\mu_2\|_2 \leq \left\| (K^*Q^{-1}A^T + R^{-1}B^\top)(AQ^{-1}A^T + BR^{-1}B^\top)^{-1}A \right\|_2 \|\mu_{1,x} - \mu_{2,x}\|_2 \leq L_1(\|\mu_{1,x} - \mu_{2,x}\|_2 + \|\mu_{1,u} - \mu_{2,u}\|_2)$, 

(45)

For the mapping $\Lambda_2(\mu, \pi)$, with the same optimal policy $\pi \in \Pi$ under some $\mu \in \mathbb{R}^{d+k}$, for any
\(\mu_1, \mu_2 \in \mathbb{R}^{d+k}\), it holds that

\[
\|\Lambda_2(\mu_1, \pi) - \Lambda_2(\mu_2, \pi)\|_2 = \|\mu_{new,x}(\mu_1) - \mu_{new,x}(\mu_2)\|_2 + \|\mu_{new,u}(\mu_1) - \mu_{new,u}(\mu_2)\|_2 \\
\leq \|(A - BK^*)^{-1}\bar{A}\|_2\|\mu_{1,x} - \mu_{2,x}\|_2 \\
+ \|\mu_{1,u} - \mu_{2,u}\|_2 \\
+ \|K^*(A - BK^*)^{-1}\bar{B}\|_2\|\mu_{1,x} - \mu_{2,x}\|_2 \\
+ \|K^*(A - BK^*)^{-1}\bar{B}\|_2\|\mu_{1,u} - \mu_{2,u}\|_2 \\
\leq L_2(\|\mu_{1,x} - \mu_{2,x}\|_2 + \|\mu_{1,u} - \mu_{2,u}\|_2) = L_2\|\mu_1 - \mu_2\|_2. \tag{46}
\]

With the same mean-field variable \(\mu\), since any two optimal policies \(\pi_1\) and \(\pi_2\) share the same \(K^*\), we also have the following bound

\[
\|\Lambda_2(\mu_1) - \Lambda_2(\mu_2)\|_2 \leq \left(\|(A - BK^*)^{-1}B\|_2 + \|I + K^*(A - BK^*)^{-1}B\|_2\right)\|b_{\pi_1} - b_{\pi_2}\|_2 \\
= L_3\|b_{\pi_1} - b_{\pi_2}\|_2. \tag{47}
\]

Therefore, combining (45), (46), (47), we obtain for any \(\mu_1, \mu_2 \in \mathbb{R}^{d+k}\),

\[
\|\Lambda(\mu_1) - \Lambda(\mu_2)\|_2 = \|\Lambda_2(\mu_1, \Lambda_1(\mu_1)) - \Lambda_2(\mu_2, \Lambda_1(\mu_2))\|_2 \\
\leq \|\Lambda_2(\mu_1, \Lambda_1(\mu_1)) - \Lambda_2(\mu_1, \Lambda_1(\mu_2))\|_2 + \|\Lambda_2(\mu_1, \Lambda_1(\mu_2)) - \Lambda_2(\mu_2, \Lambda_1(\mu_2))\|_2 \\
\leq L_3 d(\Lambda_1(\mu_1), \Lambda_1(\mu_2)) + L_2\|\mu_1 - \mu_2\|_2 \\
\leq (L_1 L_3 + L_2) \|\mu_1 - \mu_2\|_2 = L_0\|\mu_1 - \mu_2\|_2. \tag{48}
\]

So given the assumption that \(L_0 < 1\), the operator \(\Lambda(\cdot)\) is a contraction. By Banach fixed-point theorem, we conclude that \(\Lambda(\cdot)\) has a unique fixed point, which gives the unique Nash equilibrium pair. This completes the proof of the proposition.

\[\square\]

**Theorem 20 (Theorem 4).** For a sufficiently small tolerance \(0 < \varepsilon < 1\), we choose the number of iterations \(S\) in Algorithm 1 such that

\[
S \geq \frac{\log(2\|\mu_0 - \mu^*\|_2 \cdot {\varepsilon}^{-1})}{\log(1/L_0)} \tag{49}.
\]

For any \(s = 0, 1, \ldots, S - 1\), define

\[
\varepsilon_s = \min\left\{2^{-2}\|B\|_2^{-2}\|(A - BK^*)^{-1}\|_2^{-2}, \sigma_{\min}(R)\sigma_{\min}(DD^\top)\right\} \tag{50}
\]

\[
= 2^{-2s-4}(L_3C_b(\mu_s) + 2C_K(\mu_s))^{-2}\varepsilon^2 \cdot \sigma_{\min}(R)\sigma_{\min}(DD^\top), \tag{51}
\]

where

\[
C_b(\mu_s) = \|Q^{-1}A^\top (AQ^{-1}A^\top + BR^{-1}B^\top)^{-1}\bar{\alpha}_{\mu_s}\|_2 \tag{52}
\]

\[
C_K(\mu_s) = \left(\|\bar{\alpha}_{\mu_s}\|_2 + (1 + L_1\|\mu_s\|_2)\|B\|_2\right) \cdot \left(\|(A - BK^*)^{-1}\|_2 + (1 + \|K^*\|_2)\|(A - BK^*)^{-1}\|_2\|B\|_2\right). \tag{53}
\]
In the s-th policy update, we choose the stepsize $\eta$ as in Theorem 3 and number of iterations

$$N_s \geq \frac{\|\Sigma K^*\|}{\eta \sigma_2 \min(DD^\top) \sigma_{\min}(R)} \log \frac{J_{\mu_s,1}(K_{\pi_s}) - J_{\mu_s,1}(K^*)}{\epsilon_s},$$

such that $J_{\mu_s}(K_{\pi_{s+1}}, b_{\pi_{s+1}}) - J_{\mu_s}(K^*, b_{\mu_s}^*) \leq \epsilon_s$ where $K^*, b_{\mu_s}^*$ are parameters of the optimal policy $\pi_\mu^* = \Lambda_1(\mu_s)$ generated from mean-field state/action $\mu_s$, $J_{\mu_s}(K, b_s) = J_{\mu_s}(\pi)$ is defined in the drifted MFG problem (22), and $J_{\mu_s,1}(K_{\pi_s})$ is defined in (18) corresponding to $J_{\mu_s}(K_{\pi_s}, b_{\pi_s})$. Then it holds that

$$\|\mu_S - \mu^*\|_2 \leq \epsilon, \quad \|K_{\pi_S} - K^*\|_F \leq \epsilon, \quad \|b_{\pi_S} - b^*\|_2 \leq (1 + L_1)\epsilon. \quad (54)$$

Here $\mu^*$ is the Hash mean-field state/action, $K_{\pi_S}, b_{\pi_S}$ are parameters of the final output policy $\pi_S$, and $K^*, b^*$ are the parameters of the Nash policy $\pi^* = \Lambda_1(\mu^*)$.

**Proof** Define $\mu_{s+1}^* = \Lambda(\mu_s)$ as the mean-field state/action generated by the optimal policy $\pi_{\mu_s}^* = \Lambda_1(\mu_s)$. Then by (26) and (27), we know that $\mu_{s+1}^* = (\mu_{s+1,x}^T, \mu_{s+1,u}^T)^T$, and

$$\mu_{s+1,x} = -(A - BK^*)^{-1}(BB_{\mu_s}^* + \tilde{\alpha}_{\mu_s}), \quad \mu_{s+1,u} = b_{\mu_s}^* + K^*(A - BK^*)^{-1}(BB_{\mu_s}^* + \tilde{\alpha}_{\mu_s}).$$

Therefore, by triangle inequality,

$$\|\mu_{s+1} - \mu^*\|_2 \leq \|\mu_{s+1} - \mu_{s+1}^*\|_2 + \|\mu_{s+1}^* - \mu^*\|_2 = E_1 + E_2. \quad (55)$$

Next we bound $E_1$ and $E_2$ separately.

The bound for $E_2$ is straightforward. From Proposition 8 we have

$$E_2 = \|\mu_{s+1} - \mu^*\|_2 = \|\Lambda(\mu_s) - \Lambda(\mu^*)\|_2 \leq L_0\|\mu_s^* - \mu^*\|_2,$$

where $L_0 = L_1L_3 + L_2$ is defined in Assumption 7.

The bound for $E_1$ is more involved.

$$E_1 = \|\mu_{s+1} - \mu_{s+1}^*\|_2 = \|\mu_{s+1,x} - \mu_{s+1,x}^*\|_2 + \|\mu_{s+1,u} - \mu_{s+1,u}^*\|_2 \leq \left(\|A - BK^*\|^{-1} B\|_2 + \|I + K^*(A - BK^*)^{-1} B\|_2\right)\|b_{\pi_s} - b_{\mu_s}^*\|_2 + \|BB_{\pi_s} + \tilde{\alpha}_{\mu_s}\|_2 \left(\|A - BK_{\pi_s}^*\|^{-1} - (A - BK^*)^{-1}\|_2\right) + \|KK_{\pi_s} - K^*(A - BK_{\pi_s})^{-1}\|_2 = F_1 + F_2.$$

From Theorem 5, we have

$$\|b_{\pi_{s+1}} - b_{\mu_s}^*\|_2 \leq C_b(\mu_s)\sigma_{\min}(R)^{-1/2}(DD^\top)\sigma_{\min}(R)^{-1/2} \sqrt{\epsilon_s}, \quad \text{where } C_b(\mu_s) = \|Q^{-1}A^T(AQ^{-1}A^T + BR^{-1}B^T)^{-1}\|_2.$$  

Recall that $L_3 = \|(A - BK^*)^{-1} B\|_2 + \|I + K^*(A - BK^*)^{-1} B\|_2$ is defined in Assumption 7. Now let us bound $F_2$. 

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Firstly,
\[
\|Bb_{\pi_{s+1}} + \hat{\alpha}_{\mu_s}\|_2 \leq \|Bb_{\mu_s}^* + \hat{\alpha}_{\mu_s}\|_2 + \|B\|_2 \|b_{\pi_{s+1}} - b_{\mu_s}^*\|_2
\]
\[
\leq (\|\hat{\alpha}_{\mu_s}\|_2 + L_1\|B\|_2\|\mu_s\|_2) + \|B\|_2 C_0(\mu_s)\sigma_{\min}^{-1/2}(R)\sigma_{\min}^{-1/2}(DD^\top)\sqrt{\varepsilon_s}
\]
\[
\leq \|\hat{\alpha}_{\mu_s}\|_2 + (L_1\|\mu_s\|_2 + 1)\|B\|_2,
\]
if we choose \(\varepsilon_s\) such that \(C_0(\mu_s)\sigma_{\min}^{-1/2}(R)\sigma_{\min}^{-1/2}(DD^\top)\sqrt{\varepsilon_s} \leq 1\). The second inequality is due to \(L_1\)-Lipschitz of \(\Lambda_1(\cdot)\). Secondly,
\[
\|(A - BK_{\pi_{s+1}})^{-1} - (A - BK^*)^{-1}\|_2 \leq \|(A - BK_{\pi_{s+1}})^{-1}\|_2 \|(A - BK^*)^{-1}\|_2 \|B(K_{\pi_{s+1}} - K^*)\|_2.
\]
Therefore,
\[
\|(A - BK_{\pi_{s+1}})^{-1} - (A - BK^*)^{-1}\|_2 \leq \frac{\|(A - BK^*)^{-1}\|_2^2 \|B\|_2 \|K_{\pi_{s+1}} - K^*\|_2}{1 - \|(A - BK^*)^{-1}\|_2 \|B\|_2 \|K_{\pi_{s+1}} - K^*\|_2}
\]
\[
\leq 2\|(A - BK^*)^{-1}\|_2 \|B\|_2 \|K_{\pi_{s+1}} - K^*\|_2,
\]
if we choose \(\varepsilon_s\) such that \(\|(A - BK^*)^{-1}\|_2 \|B\|_2 \|K_{\pi_{s+1}} - K^*\|_2 \leq \|(A - BK^*)^{-1}\|_2 \|B\|_2 \|K_{\pi_{s+1}} - K^*\|_2 \leq \frac{1}{2}\) where we use the bound \(\|K_{\pi_{s+1}} - K^*\|_2 \leq \sigma_{\min}^{-1/2}(R)\sigma_{\min}^{-1/2}(DD^\top)\sqrt{\varepsilon_s}\) from Theorem 5. Lastly,
\[
\|K_{\pi_{s+1}}(A - BK_{\pi_{s+1}})^{-1} - K^*(A - BK^*)^{-1}\|_2
\]
\[
\leq \|K_{\pi_{s+1}} - K^*\|_2 \|K_{\pi_{s+1}}(A - BK_{\pi_{s+1}})^{-1}\|_2 + \|K^*\|_2 \|(A - BK_{\pi_{s+1}})^{-1} - (A - BK^*)^{-1}\|_2
\]
\[
\leq \|K_{\pi_{s+1}} - K^*\|_2 \|K_{\pi_{s+1}}(A - BK_{\pi_{s+1}})^{-1}\|_2 + 2\|K^*\|_2 \|(A - BK^*)^{-1}\|_2 \|B\|_2 \|K_{\pi_{s+1}} - K^*\|_2
\]
\[
\leq 2\|K_{\pi_{s+1}} - K^*\|_2 \|K_{\pi_{s+1}}(A - BK_{\pi_{s+1}})^{-1}\|_2 + 2\|K^*\|_2 \|(A - BK^*)^{-1}\|_2 \|B\|_2 \|K_{\pi_{s+1}} - K^*\|_2,
\]
where the last inequality assumes \(\|(A - BK^*)^{-1}\|_2 \|B\|_2 \|K_{\pi_{s+1}} - K^*\|_2 \leq 1/2\) again. Combining the above derivations, we reach the following bound for \(F_2\)
\[
F_2 \leq 2C_K(\mu_s)\|K_{\pi_{s+1}} - K^*\|_2 \leq 2C_K(\mu_s)\sigma_{\min}^{-1/2}(R)\sigma_{\min}^{-1/2}(DD^\top)\sqrt{\varepsilon_s},
\] (57)
where
\[
C_K(\mu_s) = \left(\|\hat{\alpha}_{\mu_s}\|_2 + (1 + L_1\|\mu_s\|_2)\|B\|_2\right)\left(\|(A - BK^*)^{-1}\|_2 + (1 + \|K^*\|_2)\|(A - BK^*)^{-1}\|_2 \|B\|_2\right).
\]
Combining the bounds (56) and (57), we have
\[
E_1 \leq (L_3C_b(\mu_s) + 2C_K(\mu_s))\sigma_{\min}^{-1/2}(R)\sigma_{\min}^{-1/2}(DD^\top)\sqrt{\varepsilon_s}.
\]
Finally, we hope to choose \(\varepsilon_s\) such that \(E_1 \leq \varepsilon \cdot 2^{-s-2}\), which will be sufficient to prove the theorem. Therefore, we just need to set \(\varepsilon_s\) as follows
\[
\varepsilon_s = \min \left\{ 2^{-2\|B\|_2^2\|(A - BK^*)^{-1}\|_2^2, \frac{1}{C_b(\mu_s)^{-2}}, 2^{-2s-4}(L_3C_b(\mu_s) + 2C_K(\mu_s))^{-2}\|B\|_2^2 \right\} \cdot \sigma_{\min}(R)\sigma_{\min}(DD^\top).
\]
With the bounds of $E_1$ and $E_2$, we have shown from (55) that
\[ \| \mu_{s+1} - \mu^* \|_2 \leq L_0 \| \mu_s - \mu^* \|_2 + \varepsilon \cdot 2^{-s-2}. \] (58)

Iterating over $s$ and noting that $L_0 < 1$, we have
\[ \| \mu_S - \mu^* \|_2 \leq L_0^S \| \mu_0 - \mu^* \|_2 + \varepsilon / 2. \]

Therefore, if we choose $S > \log(2\| \mu_0 - \mu^* \|_2 \cdot \varepsilon^{-1}) / \log(1/L_0)$, we have $\| \mu_S - \mu^* \|_2 < \varepsilon$.

Finally we show the bounds for $K_{\pi_S}$ and $b_{\pi_S}$. Since $K^*$ does not depend on $\mu_s$, for any iteration $s$ including the last iteration $S$, we directly get
\[ \| K_{\pi_S} - K^* \|_F \leq \sigma^{-1/2}_\min (R) \sigma^{-1/2}_\min (DD^\top) \sqrt{\varepsilon_S} \leq \varepsilon, \] (59)
from Theorem [5]. By the triangle inequality,
\[ \| b_{\pi_S} - b^* \|_2 \leq \| b_{\pi_S} - b^*_{\mu_S} \|_2 + \| b^*_{\mu_S} - b^* \|_2 \]
\[ \leq C_0(\mu_S) \sigma^{-1/2}_\min (R) \sigma^{-1/2}_\min (DD^\top) \sqrt{\varepsilon_S} + L_1 \| \mu_S - \mu^* \|_2 \]
\[ \leq (1 + L_1)\varepsilon, \] (60)
where the second inequality comes from Theorem [5] and the last inequality comes from the choice of $\varepsilon_S$. Thus we now complete the proof of the theorem.
References

[1] Brian D O Anderson and John B Moore. *Optimal control: linear quadratic methods*. Courier Corporation, 2007.

[2] Brandon Araki, John Strang, Sarah Pohorecky, Celine Qiu, Tobias Naegeli, and Daniela Rus. Multi-robot path planning for a swarm of robots that can both fly and drive. In *2017 IEEE International Conference on Robotics and Automation (ICRA)*, pages 5575–5582. IEEE, 2017.

[3] Dario Bauso, Hamidou Tembine, and Tamer Başar. Robust mean field games with application to production of an exhaustible resource. *IFAC Proceedings Volumes*, 45(13):454–459, 2012.

[4] Alain Bensoussan, Jens Frehse, Phillip Yam, et al. *Mean field games and mean field type control theory*, volume 101. Springer, 2013.

[5] Dimitri P Bertsekas. *Dynamic programming and optimal control*, volume 1. Athena scientific Belmont, MA, 1995.

[6] Jingjing Bu, Lillian J Ratliff, and Mehran Mesbahi. Global convergence of policy gradient for sequential zero-sum linear quadratic dynamic games. *arXiv preprint arXiv:1911.04672*, 2019.

[7] Pierre Cardaliaguet and Saeed Hadikhanloo. Learning in mean field games: the fictitious play. *ESAIM: Control, Optimisation and Calculus of Variations*, 23(2):569–591, 2017.

[8] René Carmona, François Delarue, et al. *Probabilistic Theory of Mean Field Games with Applications I-II*. Springer, 2018.

[9] René Carmona, Mathieu Laurière, and Zongjun Tan. Linear-quadratic mean-field reinforcement learning: convergence of policy gradient methods. *arXiv preprint arXiv:1910.04295*, 2019.

[10] Enrique Munoz de Cote, Alessandro Lazaric, and Marcello Restelli. Learning to cooperate in multi-agent social dilemmas. In *Proceedings of the fifth international joint conference on Autonomous agents and multiagent systems*, pages 783–785, 2006.

[11] Dimos V Dimarogonas and Karl H Johansson. Stability analysis for multi-agent systems using the incidence matrix: Quantized communication and formation control. *Automatica*, 46(4):695–700, 2010.

[12] Magnus Egerstedt and Xiaoming Hu. Formation constrained multi-agent control. *IEEE transactions on robotics and automation*, 17(6):947–951, 2001.

[13] Romuald Elie, Julien Perolat, Mathieu Laurière, Matthieu Geist, and Olivier Pietquin. On the convergence of model free learning in mean field games. In *AAAI Conference on Artificial Intelligence (AAAI 2020)*, 2020.

[14] Robert Elliott, Xun Li, and Yuan-Hua Ni. Discrete time mean-field stochastic linear-quadratic optimal control problems. *Automatica*, 49(11):3222–3233, 2013.
[15] Jacob Engwerda. *LQ dynamic optimization and differential games*. John Wiley & Sons, 2005.

[16] Jian Fang. The LQR controller design of two-wheeled self-balancing robot based on the particle swarm optimization algorithm. *Mathematical Problems in Engineering*, 2014, 2014.

[17] Maryam Fazel, Rong Ge, Sham M Kakade, and Mehran Mesbahi. Global convergence of policy gradient methods for the linear quadratic regulator. *arXiv preprint arXiv:1801.05039*, 2018.

[18] Zuyue Fu, Zhuoran Yang, Yongxin Chen, and Zhaoran Wang. Actor-critic provably finds Nash equilibria of linear-quadratic mean-field games. *arXiv preprint arXiv:1910.07498*, 2019.

[19] Xin Guo, Anran Hu, Renyuan Xu, and Junzi Zhang. Learning mean-field games. In *Advances in Neural Information Processing Systems*, pages 4967–4977, 2019.

[20] Johannes Heinrich and David Silver. Deep reinforcement learning from self-play in imperfect-information games. *arXiv preprint arXiv:1603.01121*, 2016.

[21] Jianhui Huang and Na Li. Linear–quadratic mean-field game for stochastic delayed systems. *IEEE Transactions on Automatic Control*, 63(8):2722–2729, 2018.

[22] Minyi Huang, Peter E Caines, and Roland P Malhamé. Individual and mass behaviour in large population stochastic wireless power control problems: centralized and Nash equilibrium solutions. In *42nd IEEE International Conference on Decision and Control (IEEE Cat. No. 03CH37475)*, volume 1, pages 98–103. IEEE, 2003.

[23] Minyi Huang, Roland P Malhamé, Peter E Caines, et al. Large population stochastic dynamic games: closed-loop mckean-vlasov systems and the Nash certainty equivalence principle. *Communications in Information & Systems*, 6(3):221–252, 2006.

[24] Edward Hughes, Joel Z Leibo, Matthew Phillips, Karl Tuyls, Edgar Dueñez-Guzman, Antonio García Castañeda, Iain Dunning, Tina Zhu, Kevin McKee, Raphael Koster, et al. Inequity aversion improves cooperation in intertemporal social dilemmas. In *Advances in neural information processing systems*, pages 3326–3336, 2018.

[25] Sham M Kakade. A natural policy gradient. In *Advances in neural information processing systems*, pages 1531–1538, 2002.

[26] Hamed Karimi, Julie Nutini, and Mark Schmidt. Linear convergence of gradient and proximal-gradient methods under the polyak-lojasiewicz condition. In *Joint European Conference on Machine Learning and Knowledge Discovery in Databases*, pages 795–811. Springer, 2016.

[27] Jean-Michel Lasry and Pierre-Louis Lions. Jeux à champ moyen. i–le cas stationnaire. *Comptes Rendus Mathématique*, 343(9):619–625, 2006.

[28] Jean-Michel Lasry and Pierre-Louis Lions. Jeux à champ moyen. ii–horizon fini et contrôle optimal. *Comptes Rendus Mathématique*, 343(10):679–684, 2006.
[29] Jean-Michel Lasry and Pierre-Louis Lions. Mean field games. *Japanese journal of mathematics*, 2(1):229–260, 2007.

[30] Joel Z Leibo, Vinicius Zambaldi, Marc Lanctot, Janusz Marecki, and Thore Graepel. Multi-agent reinforcement learning in sequential social dilemmas. *arXiv preprint arXiv:1702.03037*, 2017.

[31] Riccardo Minciardi and Roberto Sacile. Optimal control in a cooperative network of smart power grids. *IEEE Systems Journal*, 6(1):126–133, 2011.

[32] Volodymyr Mnih, Koray Kavukcuoglu, David Silver, Alex Graves, Ioannis Antonoglou, Daan Wierstra, and Martin Riedmiller. Playing atari with deep reinforcement learning. *arXiv preprint arXiv:1312.5602*, 2013.

[33] Hesameddin Mohammadi, Armin Zare, Mahdi Soltanolkotabi, and Mihailo R Jovanović. Convergence and sample complexity of gradient methods for the model-free linear quadratic regulator problem. *arXiv preprint arXiv:1912.11899*, 2019.

[34] Matej Moravčík, Martin Schmid, Neil Burch, Viliam Lisý, Dustin Morrill, Nolan Bard, Trevor Davis, Kevin Waugh, Michael Johanson, and Michael Bowling. Deepstack: Expert-level artificial intelligence in heads-up no-limit poker. *Science*, 356(6337):508–513, 2017.

[35] OpenAI. Openai five. [https://blog.openai.com/openai-five/](https://blog.openai.com/openai-five/), 2018.

[36] Simon Parsons and Michael Wooldridge. Game theory and decision theory in multi-agent systems. *Autonomous Agents and Multi-Agent Systems*, 5(3):243–254, 2002.

[37] Naci Saldi, Tamer Başar, and Maxim Raginsky. Markov-Nash equilibria in mean-field games with discounted cost. *SIAM Journal on Control and Optimization*, 56(6):4256–4287, 2018.

[38] Naci Saldi, Tamer Başar, and Maxim Raginsky. Approximate Nash equilibria in partially observed stochastic games with mean-field interactions. *Mathematics of Operations Research*, 44(3):1006–1033, 2019.

[39] Elham Semsar-Kazerooni and Khashayar Khorasani. Multi-agent team cooperation: A game theory approach. *Automatica*, 45(10):2205–2213, 2009.

[40] Shai Shalev-Shwartz, Shaked Shammah, and Amnon Shashua. Safe, multi-agent, reinforcement learning for autonomous driving. *arXiv preprint arXiv:1610.03295*, 2016.

[41] Jeff Shamma. *Cooperative control of distributed multi-agent systems*. John Wiley & Sons, 2008.

[42] David Silver, Aja Huang, Chris J Maddison, Arthur Guez, Laurent Sifre, George Van Den Driessche, Julian Schrittwieser, Ioannis Antonoglou, Veda Panneershelvam, Marc Lanctot, et al. Mastering the game of go with deep neural networks and tree search. *nature*, 529(7587):484, 2016.
[43] David Silver, Guy Lever, Nicolas Heess, Thomas Degris, Daan Wierstra, and Martin Riedmiller. Deterministic policy gradient algorithms. In Eric P. Xing and Tony Jebara, editors, Proceedings of the 31st International Conference on Machine Learning, volume 32 of Proceedings of Machine Learning Research, pages 387–395, Beijing, China, 22–24 Jun 2014. PMLR.

[44] David Silver, Julian Schrittwieser, Karen Simonyan, Ioannis Antonoglou, Aja Huang, Arthur Guez, Thomas Hubert, Lucas Baker, Matthew Lai, Adrian Bolton, et al. Mastering the game of go without human knowledge. Nature, 550(7676):354–359, 2017.

[45] Richard S Sutton and Andrew G Barto. Reinforcement learning: An introduction. MIT press, 2018.

[46] Richard S Sutton, David A McAllester, Satinder P Singh, and Yishay Mansour. Policy gradient methods for reinforcement learning with function approximation. In Advances in neural information processing systems, pages 1057–1063, 2000.

[47] Hamidou Tembine, Quanyan Zhu, and Tamer Başar. Risk-sensitive mean-field games. IEEE Transactions on Automatic Control, 59(4):835–850, 2013.

[48] Oriol Vinyals, Igor Babuschkin, Junyoung Chung, Michael Mathieu, Max Jaderberg, Wojciech M Czarnecki, Andrew Dudzik, Aja Huang, Petko Georgiev, Richard Powell, et al. Alphastar: Mastering the real-time strategy game starcraft ii. DeepMind blog, page 2, 2019.

[49] Jan Willems. Least squares stationary optimal control and the algebraic riccati equation. IEEE Transactions on Automatic Control, 16(6):621–634, 1971.

[50] Erfu Yang and Dongbing Gu. Multiagent reinforcement learning for multi-robot systems: A survey. Technical report, tech. rep, 2004.

[51] Kaiqing Zhang, Zhuoran Yang, and Tamer Basar. Policy optimization provably converges to Nash equilibria in zero-sum linear quadratic games. In Advances in Neural Information Processing Systems, pages 11598–11610, 2019.

[52] Pingjian Zhang. Some results on two-person zero-sum linear quadratic differential games. SIAM journal on control and optimization, 43(6):2157–2165, 2005.

[53] Xun Yu Zhou and Duan Li. Continuous-time mean-variance portfolio selection: A stochastic lq framework. Applied Mathematics and Optimization, 42(1):19–33, 2000.