SMOOTH STATIONARY WATER WAVES WITH EXPONENTIALLY LOCALIZED VORTICITY

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ABSTRACT. We study stationary capillary-gravity waves in a two-dimensional body of water that rests above a flat ocean bed and below vacuum. This system is described by the Euler equations with a free surface. Our main result states that there exist large families of such waves that carry finite energy and exhibit an exponentially localized distribution of (nontrivial) vorticity. This is accomplished by combining ideas drawn from the theory of spike-layer solutions to singularly perturbed elliptic equations, with techniques from the study of steady solutions of the water wave problem.

1. INTRODUCTION

We consider waves in a two-dimensional body of water that has finite depth. Mathematically, they are modeled as solutions to the incompressible Euler equation
\[ \partial_t v + (v \cdot \nabla) v + \nabla p + ge_2 = 0, \]
on the evolving fluid domain
\[ \Omega(t) = \{(x_1, x_2) \in \mathbb{R}^2: -1 < x_2 < 1 + \eta(t, x_1)\}. \]
Here, \( v = v(t, \cdot): \Omega(t) \to \mathbb{R}^2 \) is the velocity, \( p = p(t, \cdot): \Omega(t) \to \mathbb{R} \) is the pressure, \( g > 0 \) is the constant gravitational acceleration, and \( e_2 = (0, 1) \). Notice that the water is bounded below by a rigid and perfectly flat bed at \( \{x_2 = -1\} \). The upper boundary, given by the graph of \( 1 + \eta \), represents the interface between the water and a region of air which is treated as vacuum. An important feature of this problem is that \( \eta \) is one of the unknowns in the system. For solitary waves, \( \eta \) vanishes as \( |x_1| \to \infty \), and hence the asymptotic depth is normalized to be 2.

The kinematic boundary conditions state that the velocity field does not penetrate the bed:
\[ v_2 = 0 \quad \text{on} \quad x_2 = -1, \]and, along the free surface, we have
\[ \partial_t \eta = -v_1 \partial_{x_1} \eta + v_2 \quad \text{on} \quad x_2 = 1 + \eta(t, x_1). \]
Lastly, on the surface we impose the dynamic condition that
\[ p = \alpha^2 \kappa \quad \text{on} \quad x_2 = 1 + \eta(t, x_1), \tag{1.1e} \]
where \( \alpha > 0 \) is a constant measuring the surface tension and
\[ \kappa = -\frac{\partial_x^2 \eta}{(1 + (\partial_x \eta)^2)^{\frac{3}{2}}} \tag{1.2} \]
is the signed curvature. This is equivalent to the continuity of the stress tensor and the Young–Laplace law.

Because \( g, \alpha > 0 \), we always presume that surface tension is present on the interface and that gravity acts in the bulk. Solutions of (1.1) are therefore called capillary-gravity waves. It is well-known that this system has a conserved total energy \( E \) defined by
\[ E = \frac{1}{2} \int_\Omega |v|^2 \, dx + \int_{\mathbb{R}} \frac{1}{2} g \eta^2 + \alpha^2 \left( \sqrt{1 + (\partial_x \eta)^2} - 1 \right) \, dx_1. \tag{1.3} \]
The first term on the right-hand side represents the kinetic energy, while the second is gravitational potential energy, and the third is the surface potential energy.

In two dimensions, divergence free vector fields can be represented through a stream function, namely,
\[ v = \nabla \bot \Psi := (-\Psi_{x_2}, \Psi_{x_1}). \]
A central object of interest for this paper is the vorticity, which can be written using the stream function as
\[ \omega := \nabla \times v = \Delta \Psi. \]
From the momentum equation (1.1a), we see \( \omega \) satisfies
\[ \partial_t \omega + v \cdot \nabla \omega = 0 \quad \text{on} \quad \Omega(t), \]
and hence the vorticity is transported by the Lagrangian flow.

We are interested in smooth finite energy stationary waves with spatially highly localized vorticity. In this case, the above vorticity equation is equivalent to
\[ \nabla \bot \Psi \cdot \nabla \Delta \Psi = v \cdot \nabla \omega = 0 \quad \text{in} \quad \Omega. \tag{1.4} \]
The kinematic boundary conditions (1.1c)–(1.1d) imply that \( \Psi \) is a constant along each component of \( \partial \Omega \). Without loss of generality, we take
\[ \Psi |_{\partial \Omega} = 0; \tag{1.5} \]
see Section 1.3 for more discussion about this. At the same time, the dynamic condition (1.1e) can be expressed in terms of \( \Psi \) as the well-known Bernoulli equation
\[ \frac{1}{2} |\nabla \Psi|^2 + gx_2 + \alpha^2 \kappa = g \quad \text{on} \quad x_2 = 1 + \eta(x_1). \tag{1.6} \]
A large body of recent work has investigated the existence and qualitative properties of rotational water waves. With a few exceptions, these results pertain to waves without interior stagnation, meaning that the streamlines (level sets of \( \Psi \)) are never closed, and hence the vorticity does not vanish at infinity. In practice, however, many of the effects that generate vorticity are local. It is therefore physically interesting to seek waves having
\[\omega\] concentrated in the near field. As this requires a quite different analytical approach, there are comparatively few rigorous results for waves with localized vorticity, and those that do exist concern either periodic waves or waves with compactly supported vorticity, see the overview below.

1.1. Main theorem. We shall construct large families of solitary stationary water waves with a smooth and highly localized vorticity and a finite energy \( E < \infty \): in a perturbed disk around the origin the vorticity is large and negative, and outside it is positive and exponentially decaying. We call this a vortex spike.

The stream function and the vorticity will have the leading order forms

\[\Psi(x) = U \left( \frac{x - x_*}{\delta} \right) + \ldots \in H^k(\Omega) \cap H^1_0(\Omega), \quad \omega(x) = \frac{1}{\delta^2} \Delta U \left( \frac{x - x_*}{\delta} \right) + \ldots, \quad (1.7)\]

where \(0 < \delta \ll 1, x_* \) is roughly the location of the vorticity to be determined in the proof which will turn out to be very close to the origin in our coordinate system, and \( U \) is a smooth solution to (1.4) on the whole of \( \mathbb{R}^2 \), exponentially decaying as \(|x| \to \infty\). It is well known that (1.4) is satisfied provided that \( \omega = \gamma(\Psi) \), for some vorticity function \( \gamma \). We therefore construct \( \Psi \) as the solution to

\[\Delta \Psi = \frac{1}{\delta^2} \gamma(\Psi) \quad \text{on} \quad \Omega, \quad (1.8)\]

with \( U \) a solution to

\[\Delta U = \gamma(U) \quad \text{in} \quad \mathbb{R}^2. \quad (1.9)\]

We will assume that \( \gamma \) satisfies the following.

(A) \( \gamma \in C^{k_0}(\mathbb{R}, \mathbb{R}), \quad k_0 \geq 2, \quad \gamma(0) = 0, \quad \gamma'(0) = 1, \) and (1.9) has a radial solution \( U \in C^{k_0+2}(\mathbb{R}^2) \cap L^2(\mathbb{R}^2), \) and

(B) the kernel of \(-\Delta + \gamma'(U): H^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)\) is equal to span\{\(\partial_{x_1} U, \partial_{x_2} U\}\).

Prototypical functions fulfilling assumptions (A) and (B) are \( \gamma(t) = t - t^p \), for integers \( p \geq 2 \) and all \( t \geq 0 \), but many others will do as well. Classical results for dimension \( n = 2 \) may be found in for example [2, 3, 17], and a modern summary including the non-degeneracy results in [1]. Under the above assumptions, our main theorem is as follows.

**Theorem 1.1.** For any \( \gamma \) as in Assumptions (A) and (B), there exists \( \delta_0 > 0 \) such that, for each \( \delta \in (0, \delta_0) \), there is a finite energy solution

\[(\Psi, \eta) \in \left( H^1_0(\Omega) \cap H^{k_0}(\Omega) \right) \times H^{k_0}(\mathbb{R})\]

to the stationary water wave problem (1.5), (1.6), and (1.8). Both \( \Psi \) and \( \eta \) are even in \( x_1 \). Moreover, there exists a constant \( C > 0 \), independent of \( \delta \) but depending on \( \gamma \), such that for each \( \delta \in (0, \delta_0) \) there exists \( \tau \) with \(|\tau| \leq C \delta^{-\frac{1}{2}} e^{-\frac{\delta}{\tau}}\) satisfying

\[|\Psi - \Psi_0|_{H^{k_0}(\Omega)} \leq C \delta^{1-2k_0} e^{-\frac{2}{\tau}}, \quad (1.10)\]
Figure 1. Schematic representation of the streamline pattern and free surface. Blue lines indicate positive vorticity, red is negative, and orange is zero. Note that there is a critical layer and all streamlines are closed except the boundary components.

where

$$\Psi_0(x) = U \left( \frac{x_1}{\delta}, \frac{x_2 - \tau}{\delta} \right) - U \left( \frac{x_1}{\delta}, \frac{2 - x_2 - \tau}{\delta} \right) - U \left( \frac{x_1}{\delta}, \frac{-2 - x_2 - \tau}{\delta} \right),$$

and

$$|\eta|_{H^{k_0}(\mathbb{R})} \leq C_0 \delta^{-k_0} e^{-\frac{3}{\delta}}, \quad |\eta - \eta_0|_{H^{k_0}(\mathbb{R})} \leq C_0 \delta^{-k_0} e^{-\frac{3}{\delta}},$$

(1.11)

with

$$\eta_0 = -2\delta^{-2}(g - \alpha^2 D^2)^{-1} \left( (\partial x_2 U(z, \frac{1}{\delta}))^2 \right) - \frac{1}{\alpha \sqrt{g\delta^2}} e^{-\frac{3\sqrt{\gamma}}{\alpha \delta^2}} \ast \left( (\partial x_2 U(z, \frac{1}{\delta}))^2 \right).$$

We first comment on the vorticity and the surface profile given in the above theorem. On the one hand, from Proposition 3.1, Corollary 3.4 and (1.10), we see that the kinetic energy is of $O(1)$. Roughly,

$$|v|_{L^2(\Omega)} = |\nabla \Psi|_{L^2(\Omega)} = |\nabla U|_{L^2(\mathbb{R}^2)} + o(e^{-\frac{1}{\delta^2}}),$$

while the corresponding vorticity is spiked in the sense that

$$\omega = \frac{1}{\delta^2} \gamma \left( U \left( \frac{1}{\delta} \right) \right) + o(e^{-\frac{3}{\delta^2}}), \quad |\omega|_{L^\infty(\Omega)} = O \left( \frac{1}{\delta^2} \right), \quad |\omega|_{L^1(\Omega)} = |\Delta U|_{L^1(\mathbb{R}^2)} + o(e^{-\frac{1}{\delta^2}}).$$

On the other hand, the total vorticity is exponentially small in $0 < \delta \ll 1$:

$$\int_{\Omega} \omega \, dx = \int_{\Omega} \Delta \Psi \, dx = \int_{\partial \Omega} N \cdot \nabla \Psi \, dx = o(e^{-\frac{1}{\delta^2}}).$$

Therefore, as a measure, $\omega \, dx$ converges weakly to 0 as $\delta \searrow 0$. However, the vorticity has a rich spatial structure in a domain on the scale of $O(\delta)$ where its point-wise value is $O(\frac{1}{\delta^2})$. Moreover, as $\omega$ is $O(1)$ in $L^1(\Omega)$, these waves exhibit a highly localized but strong rotational vector field with kinetic energy of order $O(1)$.

Since $\omega$ concentrates far away from $\partial \Omega$ and the total vorticity is exponentially small, $\partial \Omega$ is only weakly impacted by the spike. This fact is reflected in the exponential smallness of $\eta$ in (1.11). According to Proposition 3.1, the leading term $\eta_0$ given in (1.11) satisfies

$$\eta_0(x) \geq \frac{1}{C_0} \delta^{3/2} e^{-2/\delta} \text{ for } |x_1| < C^{-1} \text{ for some } C > 0 \text{ independent of } \delta,$$

while its tail is much smaller. Therefore the concentrated vorticity $\omega$ creates a surface depression in the near field with rapid decay as $|x_1| \to \infty$; see Figure 1.
1.2. **History and relation to our construction.** Rotational steady water waves have been a very active area of research for nearly two decades, beginning with the construction of large-amplitude periodic gravity waves by Constantin and Strauss [8]. These authors used bifurcation theory starting from a fixed shear flow, and their methodology has since been adapted and expanded upon in many ways, see [7]. Most of these works, however, require both that the vorticity is non-localized and that there are no interior stagnation points. In particular, smooth perturbation of a shear flow could never yield decaying vorticity. Interior stagnation and critical layers (regions of closed streamlines), however, can be constructed using variants of this approach. The first paper in this direction was [11] by Ehrnström. Based on it, Wahlén [28] constructed periodic waves with one critical layer, and similar waves were subsequently constructed using a harmonic-functions approach, globally, in [9]. These works all treat constant vorticity and the situation where the linearized problem at the shear flow has a one-dimensional kernel. One can also find steady waves with critical layers bifurcating from two-dimensional [10], three-dimensional [12], and even arbitrarily high dimensional kernels [16] of affine or near-affine vorticity functions, as well as from one-dimensional kernels of constant vorticity with one discontinuity [20]. Very recently, a global theory for analytic vorticity functions allowing for several critical layers has been presented in [27] (one might note that even affine vorticity can yield arbitrarily many vertically aligned stagnation points.) The waves built in this paper have vorticity functions of the next order in this development, as Assumption (A) implies that $\gamma$ is nonlinear with leading-order linear term, although the method of proof is very different.

The first rigorous construction of traveling capillary-gravity waves with localized vorticity in infinite depth is due to Shatah, Walsh, and Zeng [24]. In that paper, two classes of compactly supported vorticity were studied: solitary and periodic waves with a submerged point vortex, and solitary waves with a vortex patch. In the former case, $\omega$ is a Dirac measure supported in the interior of $\Omega$. This can be viewed as a solution to a suitably weakened version of the Euler equations. The proof in [24] was based on a splitting of the velocity field into a rotational and irrotational component, followed by a bifurcation argument beginning at the trivial solution $(\Psi, \eta) = (0, 0)$ with the total vorticity $\int \omega \, dx$ serving as the parameter. While the vortex patch solutions were small amplitude, the authors obtained a global curve of periodic traveling waves with a point vortex. The vortex patches have finite energy and the corresponding vorticity is $C^{0,1}(\Omega)$ and smooth on its support. Later, Varholm [26] extended the ideas in [24] to the finite-depth case with arbitrarily many point vortices, and Le [18] studied the existence and orbital stability of finite dipoles inside an infinite-depth capillary gravity wave. Earlier work in the 50s and 60s that treated point vortices carried by gravity waves in finite depth include [25, 14, 13]. A vortex patch situated near a shoreline and such that the velocity vanishes completely outside a ball has also been constructed in [6], using dynamical systems tools.

The capillary-gravity waves in the current work can be said to live between the above-mentioned types. They can be viewed, for $0 < \delta \ll 1$, as smoothed vortex patches or as the limit, as the period tends to infinity, of steady periodic waves with critical-layers. We note that in [24], (i) $\omega$ is single-signed and either a Dirac measure or in $C^{0,1}(\Omega)$; and (ii) the measure $\omega \, dx$ vanishes absolutely as one approaches the point of bifurcation. By contrast,
in the present paper, the vorticity changes sign and is smooth *throughout* \( \Omega \). Moreover, \( \omega \, dz \) converges to 0 weakly as \( \delta \searrow 0 \), while the \( L^1 \) norm of \( \omega \) and the kinetic energy both remain order \( O(1) \). This surprising feature results from the fact that we do not perturb from a shear flow, but singularly from the ground state \( U \) which has fixed, positive energy. In all these respects, the families constructed in Theorem 1.1 represent a completely new species of water wave.

When \( \omega \) is not compactly supported, it is of little help to decompose the velocity field into rotational and irrotational parts. We are also barred from using shear flows as a model for the stream function. The main new idea is to instead look to the theory of spike and spike-layer solutions to singular perturbations of semi-linear elliptic PDE. These equations typically have the form

\[
\delta^2 \Delta u = u - u^p \quad \text{in } D,
\]

where \( D \subset \mathbb{R}^n \) is a smooth bounded domain, \( p > 1 \), and Dirichlet or Neumann conditions are prescribed on \( \partial D \). Beginning in the late 80s, versions of (1.12) were investigated intensively by the elliptic PDE community resulting in a vast literature; see, for example, [21, 19, 23, 22].

Drawing inspiration from these works, we model our stream function as a rescaled and translated ground state \( U(\cdot - \delta x) \) on the unknown fluid domain represented by a conformal mapping \( \Gamma \). The translation invariance of the problem leads to a degeneracy — as can be seen in Assumption (B) — which is resolved through a Lyapunov–Schmidt reduction. We outline heuristically how to solve the resulting highly degenerate bifurcation equation for \( x_* \) in the next subsection.

To the best of our knowledge, ours is the first work exploring singularly perturbed elliptic equations in the hydrodynamical context. The method bears certain similarities to Li and Nirenberg’s treatment of (1.12) in [19], in particular, the use of a Lyapunov–Schmidt reduction, bundle coordinates in a tubular neighborhood of a family of translates, and boundary correction projections. However, we stress that the steady water wave problem presents substantial new difficulties: the upper boundary is free, the Bernoulli condition (1.6) imposed there is completely nonlinear, and the domain \( \Omega \) is horizontally unbounded.

### 1.3. Heuristic discussions.

In this subsection, we discuss several issues related to the *finite energy/spatial decaying* assumptions on smooth steady (stationary or traveling) solutions on fluid domains extending to horizontal infinity. We first observe that the support of the vorticity of such solutions should be the whole of \( \Omega \). Otherwise, one expects that the vorticity will not be smooth over the boundary of its support, as a consequence of the Hopf lemma for the elliptic equation (1.8).

*Traveling waves.* While we focus on stationary capillary-gravity waves in the current paper, by shifting to a moving reference frame, Theorem 1.1 immediately furnishes families of *traveling* capillary-gravity waves with exponentially localized vorticity. The velocity field for these waves will be an \( H^{k_0-1} \) perturbation of a fixed uniform background current \( ce_1 \neq 0 \), and the vorticity will be spiked in the same sense as before.
On the other hand, smooth finite-energy waves with a non-zero wavespeed are unlikely to exist. In fact, the vorticity level curves for such waves would be closed loops \( C_a = \{ \omega = a \} \), which are transported by the velocity field \( v = (v_1, v_2) \). Therefore

\[ v \cdot \nu = ce_1 \cdot \nu = cv_1 \text{ along } C_a, \]

where \( \nu = (\nu_1, \nu_2) \) is the unit outward normal vector of \( C_a \). This implies that \( |v| \geq \frac{1}{2}|c| \) if \( |\nu_1| > \frac{1}{2} \), which usually happens on an \( O(1) \) proportion of most level curves. Consequently, \( |v| \) is likely to be bounded from below on a set with infinite measure, which is prevented by the finite energy assumption.

**Fluid depth and the boundary condition of the stream function.** In [24], traveling capillary-gravity waves with compact vortex patches were constructed in fluids of infinite depth. Slightly modifying the formula of the rotational part of the velocity fields, actually the same construction should also work with finite depth. However, we do not expect smooth spatially localized stationary waves to exist in infinite depth unless the free surface is overturned.

In fact, let us temporarily not preclude the possibility of \( \Omega \) with infinite depth. Let a solution \( \Psi \) of (1.8) be given satisfying \( v = \nabla_\perp \Psi \in H^1(\Omega) \) and \( (v \cdot N)|_{\partial \Omega} = 0 \) with \( N = (N_1, N_2) \) the outward unit normal to \( \Omega \). The latter condition implies that \( \Psi \) is locally constant on \( \partial \Omega \). Fix \( \Psi = 0 \) on \( S = \text{graph}(1 + \eta) \).

Much as in the proof of Proposition 3.1, \( \Psi \) and its derivatives decay exponentially as \( |x| \to \infty \). Let \( \Gamma \) be the antiderivative of \( \gamma \) with \( \Gamma(0) = 0 \). We multiply (1.8) by \( \Psi x_2 \) and integrate to find

\[
\frac{1}{\delta^2} \int_{\partial \Omega} \Gamma(\Psi) N_2 \, dS = \frac{1}{\delta^2} \int_{\Omega} \partial_{x_2} \Gamma(\Psi) \, dx = \int_{\Omega} \Psi x_2 \Delta \Psi \, dx
\]

\[
= -\int_{\Omega} \nabla \Psi x_2 \cdot \nabla \Psi \, dx + \int_{\partial \Omega} \Psi x_2 \nabla \Psi \cdot N \, dS
\]

\[
= -\frac{1}{2} \int_{\partial \Omega} |\nabla \Psi|^2 N_2 \, dS + \int_{\partial \Omega} \Psi x_2 N \cdot \nabla \Psi \, dS
\]

\[
= \frac{1}{2} \int_{\partial \Omega} |\nabla \Psi|^2 N_2 \, dS, \tag{1.13}
\]

where in the last step above we used that \( \Psi \) is locally constant on \( \partial \Omega \) and thus \( \nabla \Psi = (N \cdot \nabla \Psi) N \) holds there.

The first implication of this equality is that if \( \Psi \) is nontrivial, then \( S \subset \partial \Omega \). Otherwise we would have \( \int_S |\nabla \Psi|^2 N_2 \, dS = 0 \) with \( N_2 > 0 \), which is impossible. This argument does not rely on anything but the regularity of \( \gamma \), in particular, we do not need the full strength of Assumption (A) or (B). Non-existence of deep water solitary waves in the presence of algebraically localized vorticity has been more thoroughly investigated in the recent paper [4].

Now suppose instead that the domain is finite depth, and set \( \partial \Omega = S \cup B \), with \( B = \{x_2 = -1\} \) denoting the flat rigid lower boundary. Suppose also that \( S \cap B = \emptyset \). The properties (i) \( |\eta(x_1)| \to 0 \) as \( |x_1| \to \infty \); (ii) \( \nabla \Psi \in L^2(\Omega) \); and (iii) \( \Psi \) is locally constant
on \( \partial \Omega \), together imply that \( \Psi|_B = \Psi|_S = 0 \) based on a simple Hölder estimate on \( \Psi \) along vertical lines. Therefore, from (1.13), we infer that
\[
\frac{1}{2} \int_{\partial \Omega} |\nabla \Psi|^2 N_2 \, dS = 0. \tag{1.14}
\]

The reduced (degenerate) equation from the Lyapunov–Schmidt reduction. Equation (1.14) is the key to the proof of our main theorem. As mentioned above, we first carry out a Lyapunov–Schmidt reduction argument to reduce the problem to a highly degenerate one-dimensional “bifurcation” equation with the parameter \( \tau \) as in Theorem (1.1). One of the usual techniques to handle those somewhat degenerate bifurcation equations is to first use a blow-up argument to search for a non-degenerate direction of the linearized problem, then employ the implicit function theorem. Even though Proposition 4.5 does imply such linear invertibility of the bifurcation equation, the non-degeneracy we find is far too weak for an (obvious) application of the implicit function theorem to be effective.

Instead, in Section 5 we show that the bifurcation equation is equivalent to the above (1.14). Now, on the free surface, \( N_2 = (1 + (\eta')^2)^{-1/2} > 0 \), while \( N_2 = -1 \) on the flat bed \( B \). If \( \nabla \Psi \) is highly localized close to the surface, then the integral there should dominate so that the left-hand side of (1.14) would be positive, and conversely for \( \nabla \Psi \) concentrated near the bed. This mandates a balancing between the contributions on the surface and bed. That observation is at the heart of the analysis in the last part of Section 5. It also reveals the importance of the translation parameter \( \tau \).

Non-flat bottom and more. With some modifications, the approach of the current paper should also apply when the bed has nontrivial topography\(^1\). Indeed, suppose \( \partial \Omega = S \cup B \), where \( B \) is now a horizontally asymptotically flat rigid bottom, for simplicity taken even in \( x_1 \). Thus we expect the vorticity to be localized at \( \tau e_2 = (0, \tau) \in \Omega \) for some \( \tau \). We can parametrize the unknown \( \Omega \) by a conformal mapping defined on a fixed domain above \( B \) and below \( \{x_2 = 1\} \). Based on Proposition 3.1, one may adjust the basic estimates in Sections 3 and 4 accordingly to carry out the Lyapunov–Schmidt reduction and arrive at a highly degenerate one-dimensional reduced bifurcation equation that would still turn out to be equivalent to (1.14). As in the current paper, the distance from \( \tau e_2 \) would again play a crucial role. Let
\[
d(\tau) = \text{dist}(\tau e_2, \partial \Omega).
\]

Much like [19], stationary solutions are expected to exist with a localized vorticity concentrated near strict local maximums of \( d(\tau) \). However, when multiple localized vorticity locations are considered or when \( B \) is not necessarily even in \( x_1 \), a sphere packing problem arises. See, for example, [15].

\(^1\)This question was also raised by Shuangjie Peng and Shusen Yan during a talk given by the third author.
1.4. Plan. We begin, in Section 2, by rewriting the stationary water wave problem into
an analytically more tractable form. Using a conformal mapping $\Gamma$, the fluid domain is
pulled back to a fixed slab $S_\delta$ of width $2/\delta$; this mapping $\Gamma$ becomes one of the unknowns,
taking the place of $\eta$. We impose the desired ansatz (1.7) on the stream function, thereby
reformulating the problem in terms of the deviation of $\Psi$ from a translated and rescaled
solution $U$ to (1.9).

In Section 3, we obtain leading-order approximations of $U$ and a boundary correction
operator as well as rather precise exponentially small bounds on the remainders.

Section 4 is devoted to the study of the linearized problem at an approximate solution.
Specifically, we prove that there is a small simple eigenvalue $l = l(\delta) = O(e^{2|\tau|/\delta})$
related to the direction of $\partial x_2 U$. The linearized problem is uniformly non-degenerate in
the complementary codimension-1 directions.

All of these tools are used in Section 5 to prove Theorem 1.1. Adopting bundle-type
coordinates over $\tau \in (-\frac{1}{3}, \frac{1}{3})$, we carry out a Lyapunov–Schmidt reduction in the non-
degenerate codimension-1 directions to reduce the problem to a one-dimensional bifurcation
equation. As mentioned above, this bifurcation equation is equivalent to (1.14) and the
proof is completed by an intermediate value theorem argument.

Notation. Throughout the paper, $\preceq$, $\succeq$ and $\eqsim$ indicate relations that are valid up to a
positive factor which can be chosen uniformly in $\delta$ small enough and $\tau \in [-\frac{1}{3}, \frac{1}{3}]$. When
the inequality depends on additional parameters, that will be indicated as in $\gtrsim_\lambda$.

2. Reformulation

As the first step toward proving Theorem 1.1, the stationary water wave problem (1.8),
(1.5), and (1.6) will be reformulated on a fixed domain, and we will build in the spike
ansatz for the stream function mentioned in (1.7). The final product of these efforts is an
equivalent transformed problem (2.24) that is posed on an infinite strip.

2.1. Rescaling and parametrization. We start by introducing new coordinates that
eliminate the free boundary, which comes at the usual cost of increased complexity of the
equations. Given that the highest-order operator in the semi-linear equation (1.8) is the
Laplacian, it is natural to work with conformal mappings. With that in mind, define the
reference domain to be $\mathbb{R} \times (-1, 1)$, which we identify with the complex strip
$$C_{|z_2|<1} = \{z = z_1 + iz_2 \in \mathbb{C} : |z_2| < 1\}.$$ 
We will look for fluid domains $\Omega$ that are expressed as the image of the reference domain
under a near-identity holomorphic mapping. With that in mind, define the reference domain
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Specifically, let $\Gamma = \Gamma_1 + i\Gamma_2 : C_{|z_2|<1} \to \mathbb{C}$ be holomorphic and satisfy
$$|\Gamma|_{H^2}\ll 1, \quad \Gamma(-\bar{\tau}) = -\Gamma(z), \quad \Gamma_2(z_2=-1) = 0.$$
Note this implies that $z_1 \mapsto \Gamma_1$ is odd and $z_1 \mapsto \Gamma_2$ is even. Such a conformal mapping is
uniquely determined by $\Gamma_2|_{z_2=1}$. In fact, since $\Gamma_2$ is harmonic,
$$(\partial_{z_2}^2 - \xi_1^2) F_{x_1}, \Gamma_2 = 0,$$
where $\mathcal{F}_{x_1}$ denotes the Fourier transform in the $x_1$ variable. By construction, $\Gamma_2$ vanishes on the bottom of the domain, and hence it depends analytically upon its trace on the upper boundary, $\Gamma_2(\cdot, 1)$. Explicitly,

$$
(\mathcal{F}_{x_1} \Gamma_2)(\xi_1, x_2) = \frac{\sinh (|\xi_1|(x_2 + 1))}{\sinh (2|\xi_1|)} \mathcal{F}_{x_1} \Gamma_2(\xi_1, 1)
$$

for all $\xi_1 \in \mathbb{R}$, $|x_2| < 1$, so we have

$$
\Gamma_2 = \frac{\sinh (|\partial x_1|(x_2 + 1))}{\sinh (2|\partial x_1|)} \Gamma_2(\cdot, 1) \quad \text{in} \quad C_{|z_2|<1},
$$

(2.2)

and

$$
\partial x_2 \Gamma_2 = |\partial x_1| \coth (2|\partial x_1|) \Gamma_2 \quad \text{on} \quad \{x_2 = 1\}.
$$

(2.3)

Observe that $|\partial x_1| \coth (2|\partial x_1|)$ above is the Dirichlet–Neumann operator on the strip $\{|x_2| < 1\}$ with a homogeneous Dirichlet condition imposed on the lower boundary. The real part $\Gamma_1$ is a harmonic conjugate of $\Gamma_2$ whose one degree of freedom is fixed by the symmetry in $x_1$.

The corresponding fluid domain is taken to be

$$
\Omega := (id + \Gamma)(\{|x_2| < 1\}) = \{(x_1 + \Gamma_1(x), x_2 + \Gamma_2(x)) : |x_2| < 1\}.
$$

It follows that the free surface is parameterized by $x_1 \mapsto (x_1, 1) + \Gamma(x_1, 1)$, for $x_1$ ranging over $\mathbb{R}$. This curve can also be written as the graph

$$
x_2 = 1 + \eta(x_1), \quad \eta = \Gamma_2 \circ (id + \Gamma_1(\cdot, 1))^{-1}.
$$

(2.4)

The stream function can be pulled back,

$$
\Phi = \Psi \circ (id + \Gamma) : \{|x_2| < 1\} \to \mathbb{R},
$$

(2.5)

yielding a new unknown defined on the fixed reference domain. Then the water wave problem (1.8), (1.5), and (1.6) become

$$
\begin{cases}
\delta^2 \Delta \Phi = |1 + \Gamma'|^2 \gamma(\Phi) & \text{in} \quad \{|x_2| < 1\}, \\
\Phi = 0 & \text{on} \quad \{|x_2| = 1\},
\end{cases}
$$

with the transformed Bernoulli condition

$$
\frac{1}{2} \frac{(\partial x_2 \Phi)^2}{|1 + \Gamma|^2} - \alpha^2 \frac{\Im(\Gamma''(1 + \Gamma))}{|1 + \Gamma|^3} + g \Gamma_2 = 0 \quad \text{on} \quad \{x_2 = 1\},
$$

(2.6)

where we used that $|\nabla \Phi| = |\partial x_2 \Phi|$ along $x_2 = 1$ due to the boundary condition on $\Phi$. Here $\Gamma' = \partial_x \Gamma = \partial_{x_1} \Gamma$ in view of the fact that $\Gamma$ is holomorphic, and

$$
\kappa = -\frac{\Im(\Gamma''(1 + \Gamma))}{|1 + \Gamma|^3}
$$

is the signed curvature of the interface.

Recalling the scaling in (1.7), we define

$$
\varphi = \Phi(\delta \cdot),
$$

(2.7)
which then solves a non-dimensionalized version of the problem for $\Phi$ set on the slab
\[ S_\delta = \{ x \in \mathbb{R}^2 : |x_2| < \frac{1}{\delta} \}. \quad (2.8) \]

It is important to realize that this domain is decreasing in $\delta$, so that in particular $S_{2\delta} \subset S_\delta$.

Now it is easy to compute that $\varphi$ satisfies the elliptic equation
\[ \left\{ \begin{array}{l} \Delta \varphi = |1 + \Gamma'(\delta \cdot, 1)|^2 \gamma(\varphi) \quad \text{in } S_\delta \\ \varphi = 0 \quad \text{on } \partial S_\delta, \end{array} \right. \quad (2.9) \]

and the Bernoulli condition translates to
\[ 1 \left( \frac{\partial x_2 \varphi(\cdot, \frac{1}{\delta})}{1 + \Gamma'(\delta \cdot, 1)} \right)^2 - \frac{\alpha^2 \Im(\Gamma''(\delta \cdot, 1)(1 + \Gamma'(\delta \cdot, 1)))}{|1 + \Gamma'(\delta \cdot, 1)|^3} + g \Gamma_2(\delta \cdot, 1) = 0. \quad (2.10) \]

### 2.2. Boundary correction.

Our overarching strategy is to model $\Psi$, and by extension $\varphi$, on a rescaled ground state solution $U$ of (1.9). However, while $U$ is exponentially localized, it does not satisfy the homogeneous boundary conditions in (2.9). We therefore perform a boundary correction, modeled on Assumption (A), subtracting a function from $U$ that shares its trace but is exceedingly smaller in the interior.

For any real-valued continuous function $f$ defined on $\partial S_\delta$, we introduce the extension operator
\[ \text{bc: } f \mapsto f_{bc}, \]

defined uniquely by
\[ (\mathcal{F}_{x_1} f_{bc})(\xi_1, x_2) = \sum_{\pm} \pm \frac{\sinh \left( (\xi_1)(x_2 \pm 1/\delta) \right)(\mathcal{F}_{x_1} f_{\pm})(\xi_1)}{\sinh \left( \frac{2(\xi_1)}{\delta} \right)}, \quad (2.11) \]

where $f_{\pm}$ is the restriction of $f$ on $\{x_2 = \pm \frac{1}{\delta} \}$, and we are using the Japanese bracket notation $\langle \xi_1 \rangle = (1 + |\xi_1|^2)^{1/2}$. Provided that $f_{\pm} \in H^s(\mathbb{R})$, $s > 0$, the function $f_{bc}$ is an $H^{s+\frac{1}{2}}(S_\delta)$-solution\(^2\) of
\[ \left\{ \begin{array}{l} (1 - \Delta)f_{bc} = 0 \quad \text{in } S_\delta, \\ f_{bc} = f \quad \text{on } \partial S_\delta. \end{array} \right. \quad (2.12) \]

Observe that, due to the localization of $U$ and the assumption $\gamma'(0) = 1$, the above problem closely resembles the linearized operator $\gamma'(U) - \Delta$ away from the origin. It is worth noting that, for any $s > 0$,\(^2\)
\[ |f_{bc}|_{H^{s+\frac{1}{2}}(S_\delta)} \lesssim |f|_{H^s(\partial S_\delta)}. \]

\(^2\)In general the solution of (2.12) need not be unique, as $S_\delta$ is an infinite slab.
2.3. The perturbed problem. Let $U$ be the ground state solution given by Assumption (A). As discussed in Section 1, it will be important to consider vertical translates of this function. For each $\tau \in [-\frac{1}{3}, \frac{1}{3}]$, let

$$U(\cdot, \tau) = U(\cdot - \frac{\tau}{3}e_2).$$

(2.13)

With a slight abuse of notation we shall still write $U$ to denote the function $U(\cdot, \tau)$, except when the precise value of $\tau$ becomes important. At other times, it will be more convenient to use the notation $U(\tau)(\cdot)$ rather than $U(\cdot, \tau)$. The value $\frac{1}{3}$ is unimportant; we will find waves for $|\tau|$ exceedingly much smaller. What is important is that the center of vorticity remains closer to the origin than to the boundary of the reference domain, but $1/3$ has no special significance.

We proceed with the ansatz

$$\varphi = u + U - U_{bc},$$

(2.14)

where $bc$ denotes the boundary correction from (2.12). Thus $u$ measures the deviation of $\varphi$ from the rescaled, translated, and boundary corrected ground state. Inserting this into (2.9), we see that it solves the following elliptic PDE set on $S_\delta$:

$$\Delta u = |1 + \Gamma'(\delta)|^2 \gamma(u + U - U_{bc}) - \gamma(U) + U_{bc}$$

$$= \gamma'(U)u + |1 + \Gamma'(\delta)|^2 \gamma(u + U - U_{bc}) - \gamma(U) - \gamma'(U)u + U_{bc}.$$  

(2.15)

Here, we have made use of the facts that $\Delta U = \gamma(U)$ and $\Delta U_{bc} = U_{bc}$. Similarly, the kinematic condition in (2.9) takes the form

$$u = 0 \quad \text{on} \; \partial S_\delta,$$

(2.16)

since $U = U_{bc}$ there.

Consider next the Bernoulli condition (2.10). Direct substitution yields

$$0 = \frac{1}{2\delta^2} \frac{\left(\partial_{x_2}(u + U - U_{bc})(\cdot, \frac{1}{3})\right)^2}{|1 + \Gamma'(\delta)(\cdot, 1)|^2} - \alpha \frac{\text{Im}(\Gamma''(\delta, 1)(1 + \Gamma'(\delta, 1)))}{|1 + \Gamma'(\delta, 1)|^3} + g\Gamma_2(\delta, 1).$$

(2.17)

From the Cauchy–Riemann equations and

$$\Gamma' = \partial_{x_1} \Gamma = \partial_{x_2} \Gamma_2 + i \partial_{x_1} \Gamma_2,$$

any derivatives involving $\Gamma$ can be expressed in terms of derivatives of $\Gamma_2$. Making this replacement in (2.17) yields

$$0 = \frac{1}{2\delta^2} \frac{\left(\partial_{x_2}(u + U - U_{bc})(\cdot, \frac{1}{3})\right)^2}{(1 + \partial_{x_2} \Gamma_2)^2 + (\partial_{x_1} \Gamma_2)^2} - \alpha^2 \frac{(1 + \partial_{x_2} \Gamma_2)(\partial_{x_1} \Gamma_2)^2 - \partial_{x_1} \Gamma_2 \partial_{x_1 x_2} \Gamma_2)}{(1 + \partial_{x_2} \Gamma_2)^2 + (\partial_{x_1} \Gamma_2)^2} + g\Gamma_2.$$  

(2.18)

Here, all terms involving $\Gamma_2$ are evaluated at $(x_1, 1)$. The idea is that, to the leading order in terms of $\Gamma_2$, the right-hand side of (2.18) is determined by the operator $g - \alpha^2 \partial_{x_1}^2$, acting on $\Gamma_2$, which is invertible $H^s(\mathbb{R}) \to H^{s-2}(\mathbb{R})$ for all $s \in \mathbb{R}$. To make this rigorous, let

$$\Gamma_s = \Gamma_2(\cdot, 1).$$

(2.19)
be the trace of $\Gamma_2$ on the top of the reference domain $S_\delta$. We have from (2.18) and (2.3)
\[
\frac{1}{2\delta^2} \left( \partial_{x_2} (u + U - U_{bc}) \left( \frac{\tau}{\pi}, \frac{1}{\delta} \right) \right)^2 - \alpha^2 \frac{(1 + m(D)\Gamma_s)(\Gamma'_s - \Gamma'_m(D)\Gamma'_s)}{((1 + m(D)\Gamma_s)^2 + \Gamma'_s)^{3/2}} + g\Gamma_s = 0, \tag{2.20}
\]
where $D = \partial_{x_1}$, $m(D) = |D| \coth(2|D|)$, and $\Gamma'_s = \partial_{x_1} \Gamma_s$. Let $A(\Gamma_s)$ be a linear operator depending on $\Gamma_s$ acting on $v: \mathbb{R} \to \mathbb{R}$ as
\[
A(\Gamma_s) := \left( g - \alpha^2 \frac{(1 + m(D)\Gamma_s)^2 + \Gamma'_s}{((1 + m(D)\Gamma_s)^2 + \Gamma'_s)^{3/2}} \right) \left( \frac{g}{1 + m(D)\Gamma_s} \right)^2 \frac{1}{2\delta^2} A(\Gamma_s)^{-1} \left( \frac{\partial_{x_2} (u + U - U_{bc}) \left( \frac{\tau}{\pi}, \frac{1}{\delta} \right)}{\left( 1 + m(D)\Gamma_s \right)^{3/2}} \right).
\]
Notice also that $|D|$ preserves the even-odd parity. For a given smooth $\Gamma_s$, $A$ is a zero-order operator on any Sobolev space $H^s_e(\mathbb{R})$, $s \in \mathbb{R}$, where here and elsewhere the subscript ‘$e$’ indicates that the functions are even in $x_1$. More precisely, if $\Gamma_s \in H^s_e(\mathbb{R})$ for $s > 3/2$, then the map
\[
H^s_e(\mathbb{R}) \ni \Gamma_s \mapsto A(\Gamma_s) \in \mathcal{L}(H^{s'}(\mathbb{R})) \text{ is analytic, } s' \in [1 - s, s - 1]. \tag{2.21}
\]
Recall here that $H^{-s}(\mathbb{R})$ is the continuous dual of $H^s(\mathbb{R})$, whence the lower bound $1 - s$ is needed to ensure that products can be made sense of when applying $A(\Gamma_s)$ to $H^s(\mathbb{R})$. Now $A(0) = \text{id}$ and we have the bound
\[
|A(\Gamma_s) - \text{id}|_{\mathcal{L}(H^{s'})} \lesssim |\Gamma_s|_{H^s}.
\]
Thus $A(\Gamma_s) \in \mathcal{L}(H^{s'})$ is invertible for $|\Gamma_s|_{H^s} \ll 1$. We can now isolate the leading-order terms in (2.20) by applying $A(\Gamma_s)^{-1}$ to it:
\[
0 = (g - \alpha^2 D^2)\Gamma_s + \frac{1}{2\delta^2} A(\Gamma_s)^{-1} \left( \frac{\partial_{x_2} (u + U - U_{bc}) \left( \frac{\tau}{\pi}, \frac{1}{\delta} \right)}{\left( 1 + m(D)\Gamma_s \right)^{3/2}} \right), \tag{2.22}
\]
which is valid as long as $|\Gamma_s|_{H^s} \ll 1$.

Now we are roughly in a position to make a rigorous statement of our problem. For any $\delta > 0$, we define
\[
X^k_\delta = H^k_e(S_\delta) \cap H^1_0(S_\delta), \quad k \geq 1, \quad X^0_\delta = L^2_e(S_\delta). \tag{2.23}
\]
Summarizing the analysis of this section, we see that if $u$, $\Gamma_s$, and $\tau$ satisfy
\[
(-\Delta + \gamma'(U))u + F(\tau, u, \Gamma_s) = 0 \quad \text{in } S_\delta \tag{2.24a}
\]
\[
(g - \alpha^2 D^2)\Gamma_s + G(\tau, u, \Gamma_s) = 0 \quad \text{on } \mathbb{R}, \tag{2.24b}
\]
where $F: [-\frac{1}{3}, \frac{1}{3}] \times X^k_\delta \times H^k_e(\mathbb{R}) \to X^{k-2}_\delta$ and $G: [-\frac{1}{3}, \frac{1}{3}] \times X^k_\delta \times H^k_e(\mathbb{R}) \to H^{k-2}(\mathbb{R})$ are the mappings
\[
F(\tau, \cdot): (u, \Gamma_s) \mapsto |1 + \Gamma'(\delta \cdot)|^2 \gamma(u + U - U_{bc}) - \gamma(U) - \gamma'(U)u + U_{bc}, \tag{2.25}
\]
\[\text{Note here that the lower right } s \text{ used in } \Gamma_s \text{ stands for ‘surface’, while the slanted } s \text{ is a (general) regularity index.}\]
and

\[ G(\tau, \cdot) : (u, \Gamma_s) \mapsto \frac{1}{2\delta^2} A(\Gamma_s)^{-1} \left[ \frac{(\partial_{\tau x} (u + U - U_{bc}))}{(1 + |D| \coth (2|D|\Gamma_s)^2 + \Gamma_s^2)} \right], \quad (2.26) \]

then \((\Psi, \eta))\), reconstructed via \((2.5), (2.7), (2.14), (2.1), (2.19)\), and the Cauchy-Riemann equations, will solve the stationary water wave problem \((1.8), (1.5)\), and \((1.6)\). Recall here that \(U\) is a shorthand for \(U(\cdot - \delta \tau e_2)\). For the class of \(\gamma\) satisfying Assumption \((A)\) and Assumption \((B)\), the mappings \(F\) and \(G\) are well defined and continuously differentiable given some basic estimates on \(U\) and \(U_{bc}\) that are derived in the next section. For that reason, we postpone making a precise statement, or offering a proof, until Lemma 3.7.

3. Estimates of the ground state and its boundary corrections

This section is devoted to the estimates of \(U(\cdot, \tau)\), its derivatives, and boundary corrections of the same functions, assuming that Assumptions \((A)\) and \((B)\) from Section 1 hold. Finally, we give some estimates of the nonlinearities \(F\) and \(G\), defined in \((2.25)\) and \((2.26)\), in the elliptic system \((2.24)\), which is equivalent to the original problem \((1.8), (1.5)\), and \((1.6)\) of finding stationary water waves.

**Estimates for** \(U\), \((\partial_{\tau x} U)_{bc}\), and \(\partial_{\tau x} (U_{bc})\).** We start with some basic estimates on the ground state. Recall from Assumption \((A)\) that \(\gamma \in C^{k_0}_0\), for a fixed integer \(k_0 \geq 2\).

**Proposition 3.1.** Under Assumptions \((A)\) and \((B)\), there exists \(\lambda > 0\) such that

\[ \lim_{r \to \infty} (-1)^k r^k e^r \partial_r^k U(r) = \lambda, \quad \text{for all } 0 \leq k \leq k_0 + 2. \quad (3.1) \]

**Remark 3.2.** As

\[ \nabla = \begin{pmatrix} \cos \theta & -\frac{1}{r} \sin \theta \\ \sin \theta & \frac{1}{r} \cos \theta \end{pmatrix} \begin{pmatrix} \partial_r \\ \partial_\theta \end{pmatrix}, \]

and

\[ \partial_{\tau x}^2 = \sin^2 \theta \partial_r^2 + \frac{2 \cos \theta \sin \theta}{r} \partial_{\theta} + \frac{\cos^2 \theta}{r^2} \partial_{\theta}^2 + \frac{\cos^2 \theta}{r} \partial_r - \frac{\cos \theta \sin \theta}{r^2} \partial_\theta, \]

when applied to radial functions, we have

\[ \partial_{\tau x} U = \sin \theta U_r, \quad \partial_{\tau x}^2 U = \sin^2 \theta U_{rr} + \frac{\cos^2 \theta}{r} U_r. \quad (3.2) \]

Proposition 3.1 readily induces signs on the Cartesian derivatives of these functions. In particular, \(\text{sgn} \partial_{\tau x} U = -\text{sgn} x_2 \) globally with

\[ \partial_{\tau x} U \approx -x_2 r^{-\frac{3}{2}} e^{-r}, \quad \partial_{\tau x}^2 U \approx x_2^2 r^{-\frac{5}{2}} e^{-r} \]

when \(r \gg 1\). Note also that

\[ |1 - \gamma'(U)| \lesssim r^{-\frac{3}{2}} e^{-r}, \quad |\gamma(U) - U| \lesssim r^{-1} e^{-2r}, \quad r \gg 1. \]

**Remark 3.3.** For \(|\tau| < \frac{1}{3}\), the function \(U(\cdot, \tau)\) from \((2.13)\) is just a translation of the center and global maximum of the radial function \(U\) from the origin to \((0, \frac{1}{3})\). It follows that Proposition 3.1 applies to \(U(\cdot, \tau)\) with \(r\) changed accordingly.
Proof of Proposition 3.1. The decay rate (3.1) is stated by Li and Nirenberg [19] for the case \( \gamma(t) = t^{-p} \), with a reference to an earlier paper of Berestycki and P. L. Lions [3]. However, while that work could be extended to our setting, as written it does not contain the same sharp result and it is restricted to three or higher dimensions. Here we provide a sketch of a proof that does cover the case of interest; it is based on invariant manifold methods rather than variational techniques. For a reference, see for example [5].

In polar coordinates the semi-linear problem for \( U \) is

\[
\partial_r^2 U = -\frac{1}{r} \partial_r U + \gamma(U). \tag{3.3}
\]

and thus it suffices to obtain the estimate for \( k = 0, 1 \), due to the fact \( \gamma'(0) = 1 \). Letting

\[ w_1 = \frac{1}{2} (U + \partial_r U), \quad w_2 = \frac{1}{2} (U - \partial_r U), \quad s = \frac{1}{r}, \quad \gamma_1(U) = \gamma(U) - U = O(U^2), \]

we rewrite (3.3) as

\[
\begin{cases}
\partial_r w_1 = (1 - \frac{s}{2}) w_1 + \frac{s}{2} w_2 + \frac{1}{2} \gamma_1(w_1 + w_2), \\
\partial_r w_2 = \frac{s}{2} w_1 - (1 + \frac{s}{2}) w_2 - \frac{1}{2} \gamma_1(w_1 + w_2), \\
\partial_r s = -s^2.
\end{cases} \tag{3.4}
\]

Clearly \((0, 0, 0)\) is an unstable equilibrium of the ODE system with \( w_1, w_2, \) and \( s \) being in the unstable, stable, and the center directions, respectively. Therefore there exists a \( C^{k_0} \) center-stable manifold \( W^{cs} \) in a neighborhood of \((0, 0, 0)\) given by a graph

\[ w_1 = \phi(w_2, s), \quad \text{with} \quad \phi \in C^{k_0} \quad \text{and} \quad \phi(0, 0) = 0, \quad \nabla \phi(0, 0) = 0. \]

Even though the center-stable manifold \( W^{cs} \) is usually not unique, the subset \( W^{cs} \cap \{s \geq 0\} \) is indeed unique due to its positive invariance under the ODE flow. We know that \( W^{cs} \) consists of all orbits of (3.4) that grow slower than \( O(e^{\frac{s}{2}}) \) as \( r \to +\infty \), which in particular includes the orbit corresponding to the ground state \( U(r) \) as well as the trivial state \((0, 0, s = \frac{1}{r})\). The latter implies

\[ \phi(0, s) = 0, \quad \text{and thus} \quad w_1 = \phi(w_2, s) = O(|w_2|(|s| + |w_2|)), \quad |w_2|, |s| \ll 1. \]

On \( W^{cs} \), the \( w_2 \) equation in (3.4) and the above properties of \( \phi \) yield

\[
\left| \partial_r w_2 + \left(1 + \frac{s}{2} - \frac{s}{2} \phi_{w_2}(0, s) \right) w_2 \right| = O(w_2^2), \quad |w_2|, |s| \ll 1.
\]

This first implies that \( e^{\frac{s}{2}} w_2 \) is decreasing in \( r \) and so \( \int_{r_0}^{\infty} w_2 \, dr \) converges absolutely. We therefore have

\[
w_2(r) = \left( \frac{r_0}{r} \right)^{\frac{1}{2}} e^{r_0 - r} e^{\frac{s}{2} r} w_2(r_0), \quad \tilde{w}(r) = \int_{r_0}^{r} \left[ \frac{1}{2r'} \phi_{w_2}(0, \frac{1}{r'}) + O(|w_2(r')|) \right] \, dr'.
\]

Since \( \phi_{w_2}(0, s) = O(s) \), the above estimate implies that \( \lim_{r \to +\infty} r^{1/2} e^{r} w_2(r) \) exists and belongs to \((0, \infty)\), which along with the fact that \( w_1 = O(|w_2|(|\frac{1}{r} + |w_2||) \) yields (3.1) for \( k = 0, 1 \).  \( \square \)
Figure 2. Graphs of \( U \) and \( U_{bc} \) along the line \( x_1 = 0 \). On the left, the ground state is centered at the origin; on the right, it is shifted closer to the free surface. See also Corollary 3.6.

The following corollary will be used to analyze the boundary correction operator. For this and the coming results, especially Corollary 3.6, it can be good to consult Figure 2. Note, in particular, that the estimate below essentially concerns the behavior of \( U \) outside of the slab \( S_\delta \) (on the slab reflected over its own boundaries, modulo the translation \( \tau \)).

**Corollary 3.4.** For any \( \tau \in [-\frac{1}{3}, \frac{1}{3}] \), \( 0 \leq k \leq k_0 + 1 \), and \( 0 \leq k' \leq k_0 + 2 \), \( U(\tau) \) and \( \partial_{x_2} U(\tau) \) satisfy
\[
\left| U(\cdot, \pm \frac{2}{3} - \cdot, \tau) \right|_{H^{k'}(S_\delta)}, \left| \partial_{x_2} U(\cdot, \pm \frac{2}{3} - \cdot, \tau) \right|_{H^k(S_\delta)} \asymp \delta^{\frac{1}{4}} e^{-\frac{1+\tau}{\delta}}.
\]

**Proof.** We shall focus on \( \partial_{x_2} U(\cdot, \frac{2}{3} - \cdot, \tau) \) as the others can be handled similarly. Consider the following subset \( S \) of \( S_\delta \)
\[
S = \{ x \in S_\delta : x_1^2 + (\frac{2-\tau}{3} - x_2)^2 < \frac{3-\tau}{2} \}.
\]

Let \((\rho, \beta)\) be the polar coordinates of \((x_1, \frac{2-\tau}{3} - x_2)\) so that
\[
S = \{ (\rho, \beta) : \rho \in (\frac{1+\tau}{3}, \frac{2+\tau}{3}), \beta \in (\beta_0(\rho), \pi - \beta_0(\rho)) \}, \quad \beta_0(\rho) = \sin^{-1} \left( \frac{1-\tau}{\delta \rho} \right).
\]

Since \( |\tau| \leq \frac{1}{3} \), we have \( \sin \beta \approx 1 \) in \( S \), and
\[
\frac{\pi}{2} - \beta_0(\rho) = \sin^{-1} \left( 1 - \left( \frac{1-\tau}{\delta \rho} \right)^2 \right) \approx (\delta \rho - 1 + \tau)^{\frac{1}{2}} \quad \text{for all } \rho \in \left( \frac{1-\tau}{3}, \frac{3-\tau}{3} \right). \tag{3.6}
\]

Along with Proposition 3.1, this implies
\[
I + II := \left( \int_{S} + \int_{S_\delta \setminus S} \right) \left( x_1^2 + (\frac{2-\tau}{3} - x_2)^2 \right)^{-\frac{1}{2}} e^{-2(x_1^2 + (\frac{2-\tau}{3} - x_2)^2)} \frac{1}{2} dx 
\geq \left| \partial_{x_2} U(\cdot, \frac{2}{3} - \cdot, \tau) \right|_{H^{k_0+1}(S_\delta)}^2 \gtrsim \left| \partial_{x_2} U(\cdot, \frac{2}{3} - \cdot, \tau) \right|_{L^2(S_\delta)}^2 \gtrsim I.
\]
Again, it follows from Proposition 3.1 and (3.6) that

$$ I \approx \int_{\frac{3-\tau}{2}}^{3-\tau} \int_{\beta_0(\rho)}^{\pi-\beta_0(\rho)} e^{-2\rho} \, d\beta \, d\rho \approx \int_0^{\frac{3}{2}} (\delta \rho')^{\frac{1}{2}} e^{-\frac{2(1-\tau)}{3} - 2\rho'} \, d\rho' \approx \delta \frac{1}{2} e^{-\frac{2(1-\tau)}{3}}, $$

while

$$ II \lesssim \int_{|x| \geq \frac{3-\tau}{2}} |x|^{-1} e^{-2|x|} \, dx \lesssim \int_3^{\infty} e^{-2\rho} \, d\rho \approx e^{-\frac{2(3-\tau)}{3}}. $$

This completes the proof of the corollary.

In order to estimate the boundary correction operator defined in (2.12), we will need the following auxiliary lemma.

**Lemma 3.5.** Suppose $k \geq 2$ is an integer, $|\tau| \leq \frac{1}{3}$, and $h \in C^k(\mathbb{R}^2, \mathbb{R})$ satisfies

$$ |Dj h(x)| \lesssim (1 + |x - \frac{\tau}{3} e_2|)^{-\frac{1}{2}} e^{-|x - \frac{\tau}{3} e_2|} \quad \text{for } 0 \leq j \leq k, $$

and

$$ |Dj (1 - \Delta) h(x)| \lesssim (1 + |x - \frac{\tau}{3} e_2|)^{-1} e^{-2|x - \frac{\tau}{3} e_2|} \quad \text{for } 0 \leq j \leq k - 2. $$

Then

$$ v(x_1, x_2) := (h|_{\partial S_\delta})_{bc} (x_1, x_2) - h \left( x_1, \frac{2}{3} - x_2 \right) - h \left( x_1, -\frac{2}{3} - x_2 \right) $$

satisfies

$$ |v|_{H^k(S_\delta)} \lesssim \delta \frac{3}{2} e^{-\frac{2(1-|\tau|)}{3}}. $$

Intuitively, this says that the boundary correction of $h$ is, to leading order, found by subtracting the reflections of $h$ over the top and bottom boundaries of the slab.

**Proof.** From the definition of bc in (2.12), we see that $v$ satisfies

$$ \begin{cases} 
(1 - \Delta)v(x_1, x_2) = -(1 - \Delta)h \left( x_1, \frac{2}{3} - x_2 \right) - (1 - \Delta)h \left( x_1, -\frac{2}{3} - x_2 \right) & \text{in } S_\delta, \\
|v|_{x_2=\pm\frac{3}{8}} = -h(x_1, \mp\frac{3}{8}). 
\end{cases} $$

One can immediately deduce the energy estimate

$$ |v|_{H^k(S_\delta)} \lesssim \sum \pm \left| (1 - \Delta)h(\cdot, \pm\frac{2}{3} - \cdot) \right|_{H^{k-2}(S_\delta)} + |h|_{H^{k-\frac{1}{2}}(|x_2|=\frac{3}{8})}. $$

An upper bound of the first term on the right-hand side above can be obtained much as in the proof of Corollary 3.4, and so we only provide a sketch and focus on the “+” case. Let $S$ be given as in (3.5), and split the slab $S_\delta = S \cup (S_\delta \setminus S)$. From the properties assumed...
on \( h \), we see that
\[
| (1 - \Delta) h(\cdot, \frac{2}{3} - \cdot) |_{H^{k-2}(S_3)}^2 \lesssim \left( \int_S + \int_{S_3 \setminus S} \right) \left( x_1^2 + \left( \frac{2 - \tau}{\delta} - x_2 \right)^2 \right)^{-1} e^{-4(x_1^2 + (\frac{2 - \tau}{\delta} - x_2)^2)} \, dx \\
\lesssim \int_{\frac{3 - \tau}{S}}^{\frac{\pi - \beta_0(\rho)}{\beta_0(\rho)}} \int_{\beta_0(\rho)}^2 \rho^{-1} e^{-4\rho} \, d\beta \, d\rho + \int_{|x| \geq \frac{3 - \tau}{\delta}} |x|^{-2} e^{-4|x|} \, dx \\
\lesssim \delta^\frac{3}{2} e^{-\frac{4(1-\tau)}{S}} \int_0^{\frac{2}{3}} (\rho')^\frac{1}{2} \left( \frac{1 - \tau}{\delta} + \rho' \right)^{-\frac{1}{2}} e^{-4\rho'} \, d\rho' + \int_{\frac{2}{\delta}}^\infty \rho^{-1} e^{-4\rho} \, d\rho \lesssim \delta^\frac{3}{2} e^{-\frac{4(1-\tau)}{S}}.
\]

The \( H^{k-\frac{1}{2}}(\partial S_{3/\delta}) \) norm can be estimated by interpolating it between \( H^k \) and \( H^{k-1} \) and then appealing to the assumptions on \( h \):
\[
|h|_{H^{k-\frac{1}{2}}(\{|x| = \frac{\delta}{4}\})}^2 \lesssim \left( \int_0^{\delta^{-\frac{1}{2}}} + \int_{\delta^{-\frac{1}{2}}}^\infty \right) \left( x_1^2 + \frac{(3 - |\tau|)^2}{\delta^2} \right)^{-\frac{1}{2}} e^{-\frac{2}{\delta^2} \left( x_1^2 + \frac{(3 - |\tau|)^2}{\delta^2} \right)^{\frac{1}{2}}} \, dx_1 \\
\lesssim \delta^\frac{3}{4} e^{-\frac{2(3 - |\tau|)^2}{\delta^2}} \int_0^{\infty} e^{-2s} \, ds \lesssim \delta^\frac{3}{2} e^{-\frac{2(3 - |\tau|)^2}{\delta^2}},
\]
where the substitution \( x_1(s) = (s^2 - (3 - |\tau|)^2)^{\frac{1}{2}} \) was used to evaluate the integral on \( \left[ \delta^{-\frac{1}{2}}, \infty \right) \).

Combining the above inequalities concludes the proof of the lemma. \( \square \)

Lemma 3.5 is mainly applied to \( U_{bc} \) and \( (\partial_2 U)_{bc} \) for \( |\tau| \leq \frac{1}{3} \). In fact, (1.9) yields
\[
(1 - \Delta) \partial_{x_2} U = \left( 1 - \gamma'(U) \right) \partial_{x_2} U,
\]
and so the assumption that \( \gamma'(0) = 1 \) together with Proposition 3.1 ensures that \( U \) and \( \partial_{x_2} U \) satisfy the hypotheses of Lemma 3.5. Therefore, in addition to Corollary 3.4 we easily obtain the following estimates, which will be essential to us later.

**Corollary 3.6.** For any \( \tau \in [-\frac{1}{3}, \frac{1}{3}] \), \( U(\tau)_{bc} \) and \( (\partial_{x_2} U)(\tau)_{bc} \) satisfy
\[
|U(\tau)_{bc} - U(\cdot, \frac{2}{3} - \cdot, \tau) - U(\cdot, \frac{2}{3} - \cdot, \tau)|_{H^{k_0+2}(S_3)} \lesssim \delta^\frac{3}{4} e^{-\frac{2(1-|\tau|)}{\delta}} \\
|\partial_{x_2} U(\tau)_{bc} - (\partial_{x_2} U)(\cdot, \frac{2}{3} - \cdot, \tau) - (\partial_{x_2} U)(\cdot, \frac{2}{3} - \cdot, \tau)|_{H^{k_0+1}(S_3)} \lesssim \delta^\frac{3}{4} e^{-\frac{2(1-|\tau|)}{\delta}} \\
|U(\tau)_{bc}|_{H^{k_0+2}(S_3)}, \, |(\partial_{x_2} U(\tau))_{bc}|_{H^{k_0+1}(S_3)} \approx \delta^\frac{1}{4} e^{-\frac{1-|\tau|}{\delta}}.
\]

**Estimating the nonlinearity.** Finally, we give some estimates of the nonlinearities \( F \) and \( G \) occurring in the reformulated water wave problem (2.24).

**Lemma 3.7.** For \( \gamma \) as in Assumptions (A) and (B) and any integer \( 2 \leq k \leq k_0 \), there exists \( \sigma \in (0, 1) \) depending only on \( g \) and \( \alpha \), such that the operators \( F \) and \( G \) given in
\((2.25)\) and \((2.26)\) satisfy

\[
F \in C^{k_0-k+1} \left( (-\frac{1}{3}, \frac{1}{3}) \times H^k_e(S_\delta) \times H^{k-1}_e(S_\delta), H^k_e(R), H^{k-1}_e(S_\delta) \right),
\]

\[
G \in C^\infty \left( (-\frac{1}{3}, \frac{1}{3}) \times H^k_e(S_\delta) \times B_\sigma \left( H^k_e(R) \right), H^{k'}_e(R) \right),
\]

where \(B_\sigma \left( H^k_e(R) \right)\) is the ball in \(H^k_e(R)\) centered at 0 with radius \(\sigma\) and \(k' = k - \frac{3}{2}\) if \(k > 2\) and \(k'\) can be any number smaller than \(k - \frac{3}{2}\) if \(k = 2\). Moreover, for any \(\sigma \in (0, 1)\), \(\sigma_\Gamma \in (0, \sigma)\), \(\tau \in (-\frac{1}{3}, \frac{1}{3})\), \(u \in B_{\sigma_\Gamma} \left( H^k_e(S_\delta) \right)\) and \(\Gamma_S \in B_\sigma \left( H^k_e(R) \right)\), we have

\[
|D_u F|_{\mathcal{L}(H^k_e(S_\delta), H^{k-2}_e(S_\delta))} \lesssim \sigma_u + \delta^{-1} \sigma_\Gamma + \delta^\frac{1}{2} e^{-\frac{1}{\delta} |\tau|}.
\]

\[
|D_{\Gamma_S} F|_{\mathcal{L}(H^k_e(R), H^{k-2}_e(S_\delta))} \lesssim \delta^{-1},
\]

\[
|F(\tau, 0, 0)|_{H^{k-2}_e(S_\delta)} \lesssim \delta^\frac{1}{2} |\log \delta| \delta^\frac{1}{2} e^{-\frac{2(1-|\tau|)}{\delta}},
\]

and

\[
|D_u G|_{\mathcal{L}(H^k_e(S_\delta), H^{k-2}_e(R))} \lesssim \delta^\frac{1}{2} - k \sigma_u + \delta^\frac{3}{2} - k e^{-\frac{1+|\tau|}{\delta}},
\]

\[
|D_{\Gamma_S} G|_{\mathcal{L}(H^k_e(R), H^{k-2}_e(R))} \lesssim \frac{1}{\delta^2} - k (\sigma_u + \delta^\frac{1}{2} e^{-\frac{2(1-|\tau|)}{\delta}}),
\]

\[
|G(\tau, 0, 0)|_{H^{k-2}_e(R)} \lesssim \delta^\frac{1}{2} e^{-\frac{2(1-|\tau|)}{\delta}}.
\]

**Proof.** Verifying the smoothness of \(F\) and \(G\) is tedious but straightforward. The argument is based on (i) standard regularity results on products in Sobolev spaces, properties of the harmonic extension, the trace theorem, and (ii) the \(C^{k_0-k'}\) smoothness of the mapping \(u \in H^1 \mapsto \gamma \circ u \in H^l\) for a given \(\gamma \in C^{k_0}\), which holds for \(l' \leq l\) and \(l > \frac{3}{2} + 1\) in \(n\) dimensions. The small \(\sigma > 0\) is chosen such that the denominator in the definition of \(G\) is bounded away from zero and \(A(\Gamma_S)\) has a bounded inverse, which can be done independent of \(|\tau| \leq \frac{1}{3}\) and small \(\delta > 0\). We omit the details and focus on the quantitative estimates related to \(F\) and \(G\). In what follows, let \(\Gamma = \Gamma_1 + i \Gamma_2 \in H^{k+1/2}_e(S_\delta)\) be the conformal mapping determined by \(\Gamma_S\) through \((2.1)\) and \((2.19)\). Note that this involves just the harmonic extension \((2.2)\) and harmonic conjugate operators.

From the definition of \(F\),

\[
F(\tau, 0, 0) = \gamma(U - U_{bc}) - \gamma(U) + U_{bc}
\]

\[
= \int_0^1 \int_0^1 \gamma''(s_2 U - s_1 s_2 U_{bc})(s_1 U_{bc} - U) U_{bc} \, ds_2 \, ds_1,
\]

which, along with Corollary 3.6, implies that

\[
|F(\tau, 0, 0)|_{H^{k-2}_e(S_\delta)} \lesssim |U U_{bc}|_{H^{k-2}_e(S_\delta)} + |U_{bc}^2|_{H^{k-2}_e(S_\delta)}
\]

\[
\lesssim \sum_{\pm} U(\cdot, \pm \frac{2}{\delta} - \cdot) |U|_{H^{k-2}_e(S_\delta)} + \delta^\frac{1}{2} e^{-\frac{2(1-|\tau|)}{\delta}}.
\]

\[\text{(3.8)}\]
Without loss of generality, we only need to consider the "+" term in the summation. According to Assumption (B) and Proposition 3.1, for any $0 \leq j \leq k - 2$ and $x \in S_{\delta}$,

$$|\partial^j (U(x)U(x_1, \frac{2}{3} - x_2))| \lesssim (1 + |x - \frac{\tau}{3}e_2|)^{-1} \frac{2 - \tau}{3} e_2 - x - \frac{1}{2} e^{-\frac{|x - \tau e_2|}{2} + |\frac{2 - \tau}{3} e_2 - x|}.$$  

Due to the convexity of $t \mapsto (t)$, there exists $\sigma > 0$ independent of $|\tau| \leq \frac{1}{3}$ and small $\delta > 0$ such that, for any $|x_1| \leq \frac{4}{\delta}$,

$$|x - \frac{\tau}{3} e_2| + \frac{2 - \tau}{3} e_2 - x \geq \frac{2(1 - |\tau|)}{3} + \sigma \delta x_1.$$  

It is also clear that

$$|\frac{2 - \tau}{3} e_2 - x| \approx \delta^{-1}, \quad 1 + |x - \frac{\tau}{3} e_2| \gtrsim 1 + |x_2 - \frac{\tau}{3}|, \quad \text{for all } x \in S_{\delta} \text{ with } |x_1| \leq \frac{4}{\delta}.$$  

Therefore

$$\int_{S_{\delta}} (1 + |x - \frac{\tau}{3} e_2|)^{-1} |\frac{2 - \tau}{3} e_2 - x|^{-1} e^{-2(|x - \frac{\tau}{3} e_2| + |\frac{2 - \tau}{3} e_2 - x|)} \, dx$$

$$\lesssim \left( \int_{|x_1| \leq \frac{4}{\delta}} + \int_{|x_1| \geq \frac{4}{\delta}} \right) \int_{\frac{1}{\delta}}^{\frac{1}{3}} (1 + |x - \frac{\tau}{3} e_2|)^{-1} |\frac{2 - \tau}{3} e_2 - x|^{-1} e^{-2(|x - \frac{\tau}{3} e_2| + |\frac{2 - \tau}{3} e_2 - x|)} \, dx \, dx_1$$

$$\lesssim \int_{|x_1| \leq \frac{4}{\delta}} \int_{0}^{\frac{1}{s}} \delta (1 + s)^{-1} e^{-2(2(1 - |\tau|) + \sigma \delta x_1^2)} \, ds \, dx_1 + \int_{|x_1| \geq \frac{4}{\delta}} \delta^2 e^{-2|x_1|} \, dx,$$

and so we have that

$$\int_{S_{\delta}} (1 + |x - \frac{\tau}{3} e_2|)^{-1} |\frac{2 - \tau}{3} e_2 - x|^{-1} e^{-2(|x - \frac{\tau}{3} e_2| + |\frac{2 - \tau}{3} e_2 - x|)} \, dx \lesssim \delta^\frac{1}{2} \log \delta e^{-\frac{4(1 - |\tau|)}{s}}. \quad (3.9)$$

This further implies

$$|U(\cdot, \frac{2}{3} - \cdot)U|^2_{H^{k-2}(S_{\delta})} \lesssim \delta^\frac{1}{2} \log \delta e^{-\frac{4(1 - |\tau|)}{s}}$$

which, with (3.8), furnishes the desired estimate of $F(\tau, 0, 0)$.

Next, observe that, for any $\tilde{u} \in H_{\delta}^k(S_{\delta})$ with $k \geq 2$,

$$D_u F(\tau, u, \Gamma_\delta) \tilde{u} = ((1 + \Gamma'(\delta \cdot))^2 \gamma'(u + U - U_{bc}) - \gamma'(U)) \tilde{u}$$

$$= \left( 2 \Gamma_1(\delta \cdot) + |\Gamma'(\delta \cdot)|^2 \right) \gamma'(u + U - U_{bc}) + (u - U_{bc}) \int_{0}^{1} \gamma''(U + s(u - U_{bc})) \, ds \right) \tilde{u}.$$  

(3.10)

Now, for any $s$ we have the the scaling identity

$$|f(\delta \cdot)|_{H^s(S_{\delta})} = \delta^{s-1} |f|_{H^s(S_{\delta})},$$

and, for $k - \frac{1}{2} \geq \frac{3}{2}$ and $|\Gamma_\delta|_{H^k} < 1$, it holds that

$$|2 \Gamma_1'(\delta \cdot) + |\Gamma'(\delta \cdot)|^2|_{H^{k-\frac{1}{2}}(S_{\delta})} \lesssim \delta^{-1} |\Gamma_\delta|_{H^k(S_{\delta})}.$$
Thus, the $H^{k-2}(S_\delta)$ norm of the last line of (3.10) has the upper bound
\[
O\left(\delta^{-1}|\Gamma s|_{H^k(\mathbb{R})} + |u|_{H^k(S_\delta)} + |U_{bc}|_{H^k(S_\delta)}\right).
\]

Corollary 3.6 therefore gives the claimed estimate of $D_u D_{\Gamma_s} F$.

Regarding $D_{\Gamma_s} F$, we have for any $\tilde{\Gamma}_s \in H^k(\mathbb{R})$ that
\[
D_{\Gamma_s} F(\tau, u, \Gamma_s)\tilde{\Gamma}_s = 2\left(\tilde{\Gamma}_1(\delta \cdot) + \Gamma'(\delta \cdot) \cdot \tilde{\Gamma}(\delta \cdot)\right)\gamma(u + U - U_{bc}),
\]
where $\tilde{\Gamma} = \tilde{\Gamma}_1 + i\tilde{\Gamma}_2$ is the conformal mapping determined by $\tilde{\Gamma}_s$. The estimate on $D_{\Gamma_s} F$ follows from this expression and the scaling of the Sobolev norms explained above.

From the definition of $G$, one can directly compute
\[
G(\tau, 0, 0) = \frac{1}{2\delta^2}(\partial_{x_2}(U - U_{bc}))\left(\frac{x}{2\delta}, \frac{1}{\delta}\right)^2.
\]
Recall the one-dimensional scaling property,
\[
|f(\delta^{-1} \cdot)|_{H^s(\mathbb{R})} \leq \delta^{-s+\frac{1}{2}}|f|_{H^s(\mathbb{R})},
\]
which holds for all $s \in \mathbb{R}$. From Corollary 3.6 and the trace theorem, we then have, for any $m \geq 0$,
\[
|\partial_{x_2}(U - U_{bc})(\cdot, \frac{1}{\delta}) - 2(\partial_{x_2}U)(\cdot, \frac{1}{\delta})|_{H^m(\mathbb{R})} \lesssim |(\partial_{x_2}U)(\cdot, -\frac{3}{4}\delta)|_{H^m(\mathbb{R})} + \delta^{\frac{1}{2}} e^{-\frac{2\delta (|\rho|)}{\delta}}.
\]
Using Proposition 3.1 and the change of variables $x_1(\rho) = (\rho^2 - (\frac{1}{2} \tau)^2)^{\frac{1}{2}}$ we can estimate $(\partial_{x_2}U)(\cdot, \frac{1}{\delta})$ while the terms on the right hand-side are obviously much smaller,
\[
|(\partial_{x_2}U)(\cdot, \frac{1}{\delta})|^2_{H^m(\mathbb{R})} \lesssim \int_0^{\delta^{-2}} + \int_{\delta^{-2}}^{\infty} \left(x_1^2 + (\frac{1}{\delta} \tau)^2\right)^{-\frac{1}{2}} e^{-2(x_1^2 + (\frac{1}{\delta} \tau)^2)} \frac{dx_1}{\delta^2 e^{2\delta}} \lesssim \delta^{\frac{1}{2}} e^{-\frac{2\delta (|\rho|)}{\delta}}.
\]
This implies that
\[
|(\partial_{x_2}(U - U_{bc}))(\cdot, \frac{1}{\delta})|_{H^m(\mathbb{R})} \lesssim \delta^{\frac{1}{2}} e^{-\frac{1}{\delta} |\rho|}.
\]
We therefore obtain the estimate on $G(\tau, 0, 0)$ from the scaling property as
\[
|G(\tau, 0, 0)|_{H^{k-2}(\mathbb{R})} \lesssim \delta^{\frac{1}{2} - (k-2) - 2} \left|\partial_{x_2}(U - U_{bc})(\cdot, \frac{1}{\delta})\right|^2_{H^{k-2}(\mathbb{R})} \lesssim \delta^{1-k} e^{-\frac{2\delta (|\rho|)}{\delta}}.
\]

Consider next the bound on $D_u G$. It is easy to see from the definition of $G$ that,
\[
D_u G(\tau, u, \Gamma_s)\tilde{u} = \frac{1}{\delta^2} A(\Gamma_s)^{-1} \left[\frac{\partial_{x_2}(u + U - U_{bc})}{\partial_{x_2}u}(\cdot, \frac{1}{\delta})\delta^2 \coth (2|D||\Gamma_s)|^2 + \Gamma_s^2\right].
\]
By the trace theorem and (3.14), we have
\[ |(\partial_{x^2}(u + U - U_{bc})\partial_{x^2}\tilde{u})(\cdot, \frac{1}{\delta})|_{H^{k-2}(\mathbb{R})} \]
\[ \lesssim |\partial_{x^2}(u + U - U_{bc})(\cdot, \frac{1}{\delta})|_{H^{k-\frac{3}{2}}(\mathbb{R})} |\partial_{x^2}\tilde{u}(\cdot, \frac{1}{\delta})|_{H^{k-\frac{3}{2}}(\mathbb{R})} \lesssim \left(\sigma_u + \delta^2 e^{-\frac{1}{2\delta^2}}\right) |\tilde{u}|_{H^k(S_{\delta})}. \]
The desired bound on \(D_{\Gamma}G\) then follows from the scaling property.

Finally, for any \(\Gamma_s \in H^k(\mathbb{R})\),
\[ 2\delta^2 D_{\Gamma_s}G((\tau, u, \Gamma_s)\tilde{\Gamma}_s = (D_{\Gamma_s}(A(\Gamma_s)^{-1})\tilde{\Gamma}_s) \frac{(\partial_{x^2}(u + U - U_{bc}))((\cdot, \frac{1}{\delta})^2}{(1 + |D| \coth (2|D|)|\Gamma_s|^2 + \Gamma_s^2)} \]
\[ - 2A(\Gamma_s)^{-1}[\frac{(\partial_{x^2}(u + U - U_{bc}))((\cdot, \frac{1}{\delta})^2}{(1 + |D| \coth (2|D|)|\Gamma_s|^2 + \Gamma_s^2)} \]
\[ \times \left((1 + |D| \coth (2|D|)\Gamma_s)|D| \coth (2|D|)\tilde{\Gamma}_s + \Gamma_s^2 \right). \]
Since \(|u|_{H^k(S_{\delta})} < \sigma_u\) and \(|\Gamma_s|_{H^k(\mathbb{R})} < \sigma < 1\) with \(k \geq 2\), straightforwardly we obtain
\[ |D_{\Gamma_s}G((\tau, u, \Gamma_s)\tilde{\Gamma}_s|_{H^{k-2}(\mathbb{R})} \lesssim \delta^{-2} |(\partial_{x^2}(u + U - U_{bc}))(\cdot, \frac{1}{\delta})|_{H^{k-\frac{3}{2}}(\mathbb{R})} |\tilde{\Gamma}_s|_{H^k(\mathbb{R})} \]
\[ \lesssim \delta^{\frac{1}{2} - k} \left(|u|^2_{H^k(S_{\delta})} + |(\partial_{x^2}(U - U_{bc}))(\cdot, \frac{1}{\delta})|^2_{H^{k-1}(\mathbb{R})}\right) |\tilde{\Gamma}_s|_{H^k(\mathbb{R})}, \]
where the scaling property and the trace theorem have been used. The estimate on \(D_{\Gamma_s}G\) now follows immediately from (3.14).

One notices that \(|D_{\Gamma,F}((\tau, u, \Gamma_s)|\) is not small no matter how small \(u\) and \(\Gamma_s\) are. Fortunately this is an “off diagonal term” in the linearization, which in will be handled by a simple rescaling argument Section 5.

4. Spectral properties

Having the necessary estimates on \(U\) and the boundary correction operator bc now in hand, we next consider the linear operator
\[ L_\tau = -\Delta + \gamma'(U(\tau)): X^2_\delta \to X^0_\delta, \]
in the elliptic equation (2.24a). Recall that the Dirichlet boundary conditions on \(\partial S_\delta\) are encoded in the definition of the space \(X^2_\delta\), and that we usually suppress the dependence on the translation parameter \(\tau\) in the notation for \(U = U(\cdot, \tau) = U(\tau)(\cdot)\).

The inherent difficulty here is that the ground state equation (1.9) implies
\[ \Delta \partial_{x^2}U = \gamma'(U)\partial_{x^2}U, \]
so that \(\partial_{x^2}U\) is in the kernel of \(-\Delta + \gamma'(U)\) viewed as an operator with domain \(H^2(\mathbb{R}^2)\). Working in the strip \(S_\delta\) breaks the vertical translation symmetry and eliminates this kernel direction. It is therefore expected that \(L_\tau\) will be invertible from \(X^2_\delta \to X^0_\delta\), and, indeed, this is proved in Lemma 4.6. However, as \(\delta \searrow 0\), heuristically the strip approximates \(\mathbb{R}^2\),
and so we cannot hope to obtain bounds for \( L_\tau^{-1} : X_\delta^0 \to X_\delta^2 \) that are uniform in \( \delta \). Another way to see this is to note that
\[
L_\tau \partial_{x_2} U = 0 \quad \text{in } S_\delta, \quad \partial_{x_2} U \approx \delta^{\frac{3}{4}} e^{-\frac{1-|\tau|}{8}} \quad \text{on } \partial S_\delta,
\]
as \( U \) and its derivatives decay exponentially. Thus, \( L_\tau \) is nearly degenerate in the direction close to \( \partial_{x_2} U \).

In order to proceed, it is therefore necessary to have detailed information about the behavior of \( L_\tau \) as \( \delta \searrow 0 \). We will prove that there is a positive (simple) eigenvalue \( l = l(\delta, \tau) \) that is exponentially small in \( \delta \) for fixed \( \tau \), and whose eigenfunction \( U_0 \) approaches \( \partial_{x_2} U \) as \( \delta \searrow 0 \). In the orthogonal complement of \( U_0 \), the inverse of \( L_\tau \) is bounded uniformly in \( \delta \). In the next section, we will make use of this fact to perform a Lyapunov–Schmidt type reduction to (2.24a), first solving the problem on a codimension 1 subspace where \( L_\tau \) is well-behaved, and then studying the reduced equation on the near-degenerate direction.

4.1. An approximate eigenfunction. As a preparation for proving the existence of \( l \) and \( U_0 \), we first study the function
\[
U_2(\cdot, \tau) = \partial_{x_2} U(\cdot, \tau) - (\partial_{x_2} U)(\cdot, \tau)_{bc} \in X_\delta^2, \quad (4.1)
\]
which results from taking \( \partial_{x_2} U \) and perturbing it slightly so that the homogeneous boundary condition on \( \partial S_\delta \) is satisfied, see Figure 2. In what follows, the dependence of \( U_2 \) on \( \tau \) will be suppressed when there is no risk of confusion. While \( U_2 \) is not likely to be an eigenfunction itself, we will see that it does help in identifying the asymptotically degenerate direction. Observe that it solves the elliptic problem
\[
\begin{aligned}
(-\Delta + \gamma'(U)) U_2 &= (1 - \gamma'(U)) (\partial_{x_2} U)_{bc} \quad \text{in } S_\delta, \\
U_2 &= 0 \quad \text{on } \partial S_\delta,
\end{aligned}
\]
as \((-\Delta + \gamma'(U)) \partial_{x_2} U = 0 \) and \( \Delta (\partial_{x_2} U)_{bc} = (\partial_{x_2} U)_{bc} \) by the definition of the boundary correction operator. Recall also that \( \gamma \in C^{k_0} \) according to Assumption (A).

Lemma 4.1. For \( |\tau| \leq \frac{1}{3} \), we have \( |U_2|_{H^{k_0+1}(S_\delta)} \approx 1 \) and
\[
|L_\tau U_2|_{H^{k_0-1}(S_\delta)} \lesssim \delta^{\frac{3}{4}} \log \delta \left\| e^{-\frac{2(1-|\tau|)}{8}} \right\| \quad \text{with } 0 < \langle U_2, L_\tau U_2 \rangle_{L^2(S_\delta)} \approx \delta^\frac{3}{4} e^{-\frac{2(1-|\tau|)}{8}}.
\]

Proof. Simply from its definition, it is clear that \( U_2 = O(1) \) in \( H^{k_0+1} \) for \( |\tau| \leq \frac{1}{3} \). On the other hand, from (4.2) and Corollary 3.6, we obtain the estimate
\[
|L_\tau U_2 - (1 - \gamma'(U)) \sum_{\pm} \partial_{x_2} U(\cdot, \pm \frac{2}{9} - \cdot) \|_{H^{k_0-1}(S_\delta)} \lesssim \delta^\frac{3}{4} e^{-\frac{2(1-|\tau|)}{8}}. \quad (4.3)
\]
Without loss of generality, we only need to consider the “+” case. According to Assumption (A) and Proposition 3.1, for any \( 0 \leq k \leq k_0 - 1 \) and \( (x_1, x_2) \in S_\delta \),
\[
|\partial^k ((1 - \gamma'(U(x))) \partial_{x_2} U(x_1, \frac{2}{9} - x_2)) | \lesssim e^{-\frac{2-\tau e_2 - x}{\delta}} \left( 1 + \frac{|x - \frac{\tau}{8} e_2|}{\frac{1}{2}} \right)^\frac{1}{2} \frac{|x - \frac{\tau}{8} e_2 - x|}{\delta} \frac{3}{2}
\]
and thus the claimed bound on \( L_\tau U_2 \) follows from (3.9).
Likewise, using the above estimate on \( L_r U_2 \) in conjunction with Corollary 3.6, (4.1), and (4.2), one can estimate
\[
(U_2, L_r U_2)_{L^2(S_\delta)} = (\partial x_2 U - (\partial x_2 U)_{bc}, L_r U_2)_{L^2(S_\delta)}
\]
\[
= (1 - \gamma'(U)) \partial x_2 U, (\partial x_2 U)_{bc})_{L^2(S_\delta)} + O(e^{-\frac{3(1 - |r|)}{\delta}}).
\]
We concentrate on the first term on the right-hand side, as it will clearly dominate as \( \delta \searrow 0 \). Using the identities
\[
(1 - \gamma'(U)) \partial x_2 U = (1 - \Delta) \partial x_2 U, \quad (1 - \Delta)(\partial x_2 U)_{bc} = 0,
\]
and integrating by parts twice yields:
\[
\int_{\partial S_\delta} [(1 - \gamma'(U)) \partial x_2 U](\partial x_2 U)_{bc} \, dx = \int_{\partial S_\delta} [(1 - \Delta) \partial x_2 U](\partial x_2 U)_{bc} \, dx
\]
\[
= \int_{\partial S_\delta} (-[\partial^2 x_2 U](\partial x_2 U)_{bc} + \partial x_2 U \partial x_2 (\partial x_2 U)_{bc}) N_2 \, dx_1
\]
\[
= \int_{\partial S_\delta} ((\partial x_2 U)_{bc} \partial x_2 (\partial x_2 U)_{bc} - [\partial^2 x_2 U](\partial x_2 U)_{bc}) N_2 \, dx_1.
\]
The first of the boundary integrals we treat by integrating back to the interior domain \( S_\delta \) and using the definition of the boundary correction operator:
\[
\int_{\partial S_\delta} (\partial x_2 U)_{bc} \partial x_2 (\partial x_2 U)_{bc} N_2 \, dx_1 = \int_{S_\delta} (|\nabla (\partial x_2 U)_{bc}|^2 + (\partial x_2 U)_{bc}^2) \, dx \approx \delta^\frac{1}{2} e^{-\frac{2(1 - |r|)}{\delta}}.
\]
The last integral we instead estimate by integrating into the outer domain \( S^c_\delta \), where \( U \) and its derivatives are well defined and exponentially decaying in all radial directions. In analogy to above, we use the elliptic equation that \( \partial x_2 U \) satisfies to find
\[
- \int_{\partial S_\delta} (\partial^2 x_2 U)(\partial x_2 U) N_2 \, dx_1 = \int_{S_\delta} (|\nabla (\partial x_2 U)|^2 + (\partial x_2 U) \Delta (\partial x_2 U)) \, dx
\]
\[
= \int_{S_\delta} (|\nabla (\partial x_2 U)|^2 + \gamma'(U)(\partial x_2 U)^2) \, dx
\]
\[
= \int_{|x_2| \leq \frac{3}{2}} (|\nabla (\partial x_2 U)|^2 + (\partial x_2 U)^2) \, dx + O(e^{-\frac{3(1 - |r|)}{\delta}})
\]
\[
\approx \delta^\frac{1}{2} e^{-\frac{2(1 - |r|)}{\delta}},
\]
where the last bound is from Corollary 3.4. Observe also that \( \text{bth boundary integrals are positive} \). This implies the positivity of \((U_2, L_r U_2)_{L^2(S_\delta)}\) for \( \delta \) small enough, and so the proof is complete. \( \square \)

We wish to show that \( L_r : X^2_\delta \rightarrow X^0_\delta \) is well behaved as \( \delta \searrow 0 \) except in a one-dimensional near-degenerate direction that anticipates the kernel of \(-\Delta + \gamma'(U)\) on \( \mathbb{R}^2 \). To be more precise, define the function spaces
\[
X = H^2_\delta(\mathbb{R}^2), \quad Y = L^2_\delta(\mathbb{R}^2).
\]
Then, under Assumption (B) we see that \(-\Delta + \gamma'(U)\): \(X \to Y\) has a one-dimensional kernel spanned by \(\partial_{x_2}U\). Note that here the even symmetry restriction eliminates the kernel direction generated by \(\partial_{x_2}U\). Let \(P = P(\tau)\) denote the orthogonal projection of \(Y\) onto \(\text{span}\{\partial_{x_2}U\}\); abusing notation somewhat, we use the same symbol for the induced projection \(X \to \text{span}\{\partial_{x_2}U\}\).

The following non-degeneracy result is a direct consequence of Assumption (B).

**Lemma 4.2** (Non-degeneracy in \(\mathbb{R}^2\)). The operator \(-\Delta + \gamma'(U)\): \((I - P)X \to (I - P)Y\) is an isomorphism with bounds uniform in \(|\tau| < \frac{1}{3}\).

Next, we establish an estimate for \(L_\tau: X^2_\delta \to X^0_\delta\). Let \(\varrho \in C^\infty(\mathbb{R}, [0, 1])\) be a smooth cut-off function with

\[
\varrho(r) = \begin{cases} 
1 & \text{for } r < 1/3 \\
0 & \text{for } r > 1/2.
\end{cases}
\]

Given a function \(h: S_\delta \to \mathbb{R}\) we define its (odd) extension \(Eh: \mathbb{R}^2 \to \mathbb{R}\) by

\[
(Eh)(x) := \begin{cases} 
h(x) & \text{for } x \in S_\delta \\
-h(x_1, \pm \frac{3}{8} - x_2)\varrho(|x_2| - \frac{1}{2})\delta & \text{for } \pm x_2 \geq 1/\delta.
\end{cases}
\]

Notice that

\[
|h|_{L^2(S_\delta)} \leq |Eh|_{L^2(\mathbb{R}^2)} \leq 3|h|_{L^2(S_\delta)},
\]

and hence \(h \in L^2(S_\delta)\) if and only if \(Eh \in L^2(\mathbb{R}^2)\). By a standard property of odd extensions, we have in fact that \(h \in X^2_\delta\) if and only if \(Eh \in X\).

Now, let

\[
U_{2}^\perp := \left\{ u \in X^0_\delta : (u, U_2)_{L^2(S_\delta)} = 0 \right\}.
\]

For any \(|\tau| \leq \frac{1}{3}\) and \(u \in U_{2}^\perp\), we see that \(Eu \in Y\) and

\[
\int_{\mathbb{R}^2} (Eu)\partial_{x_2}U \, dx = \int_{S_\delta} (Eu)\partial_{x_2}U \, dx + \int_{S_\delta} u(U_2 + (\partial_{x_2}U)_{bc}) \, dx
\]

\[
= -\sum_{\pm} \int_{\{\frac{1}{8} < x_2 < \frac{3}{8}\}} u(x_1, \pm \frac{3}{8} - x_2)\varrho(\delta|x_2| - 1)\partial_{x_2}U \, dx
\]

\[
+ \int_{S_\delta} u(\partial_{x_2}U)_{bc} \, dx.
\]

Using the \(L^2\) bounds of \(\partial_{x_2}U\) and \((\partial_{x_2}U)_{bc}\) given in Corollary 3.4 and Corollary 3.6, we estimate that

\[
\int_{\mathbb{R}^2} (Eu)\partial_{x_2}U \, dx \lesssim \delta^{\frac{1}{2}} e^{-\frac{1-|\tau|}{\delta}} |u|_{L^2(S_\delta)} \quad \text{for all } u \in U_{2}^\perp,
\]

uniformly for \(|\tau| \leq \frac{1}{3}\) and all small values of \(\delta\). In other words, the extension \(Eu\) is nearly orthogonal to \(\partial_{x_2}U\) in \(L^2_{\text{e}}(\mathbb{R}^2)\). Combining these observation leads to the bound

\[
|PEu|_{L^2(\mathbb{R}^2)} \leq \frac{\left|\int_{\mathbb{R}^2} (Eu)\partial_{x_2}U \, dx\right|}{|\partial_{x_2}U|_{L^2(\mathbb{R}^2)}} \lesssim \delta^{\frac{1}{2}} e^{-\frac{1-|\tau|}{\delta}} |u|_{L^2(S_\delta)},
\]

(4.7)
We compute that, for \( \pm \) as above and let \( u \) can assume without loss of generality that \( \delta \) satisfy (4.9) for all \( \in \mathbb{X} \).

\[
|L_\tau u|_{L^2(S_\delta)} \geq \lambda_0 |u|_{L^2(S_\delta)}, \quad \text{for all } u \in X^2_\delta \cap U^2_\delta. \tag{4.8}
\]

(a) There exists \( \delta_0 > 0 \) and \( \lambda_0 > 0 \) such that, for all \( \delta \in (0, \delta_0) \) and \( |\tau| \leq \frac{1}{\delta} \),

\[
|L_\tau u|_{L^2(S_\delta)} \geq \lambda_0 |u|_{L^2(S_\delta)}, \quad \text{for all } u \in X^2_\delta \cap U^2_\delta. \tag{4.9}
\]

(b) For every \( \theta \in (0, 1) \), there exists \( \delta_0 = \delta_0(\theta) > 0 \) and \( \mu_0 = \mu_0(\theta) > 0 \) such that, for all \( \delta \in (0, \delta_0) \), \( |\tau| \leq \frac{1}{\delta} \), and \( u \in X^2_\delta \) satisfying

\[
|PEu|_{L^2(\mathbb{R}^2)} \leq \theta |u|_{L^2(S_\delta)}, \tag{4.10}
\]

we have

\[
|L_\tau u|_{L^2(S_\delta)} \geq \mu_0 |u|_{L^2(S_\delta)}. \tag{4.11}
\]

Proof. First observe that in light of (4.7), for any fixed \( \theta \in (0, 1) \), any element \( u \in U^2_\delta \) will satisfy (4.9) for \( \delta \) sufficiently small. It therefore suffices to prove part (b). Fix \( \theta \in (0, 1) \) as above and let \( u \) satisfy the near orthogonality condition (4.9) be given. By linearity, we can assume without loss of generality that \( |u|_{L^2(S_\delta)} \leq 1 \).

From the definition (4.4) of the extension \( E \),

\[
[-\Delta + \gamma(U), E] \ u = 0 \quad \text{on } S_\delta \cup S_\delta^c/2.
\]

We compute that, for \( \pm x_2 > 1/\delta \), one has

\[
(( -\Delta + \gamma(U) ) E) \ u ) \ x
\]

\[
= \Delta u(x_1, \pm \frac{2}{\delta} - x_2) q(\pm \delta x_2 - 1) \pm 2 \delta q'((\pm \delta x_2 - 1) \partial_x u(x_1, \pm \frac{2}{\delta} - x_2)
\]

\[
+ \delta^2 u(x_1, \pm \frac{2}{\delta} - x_2) q''((\pm \delta x_2 - 1) - \gamma(U) u(x_1, \pm \frac{2}{\delta} - x_2) q((\pm \delta x_2 - 1).
\]

This leads directly to the following expression for the commutator on the set \( \{ \pm x_2 > 1/\delta \} \)

\[
( [-\Delta + \gamma(U), E] \ u ) \ x
\]

\[
= (( -\Delta + \gamma(U) ) E) \ u ) \ x - ( (E(-\Delta + \gamma(U)) \ u ) \ x)
\]

\[
= \pm 2 \delta q'((\pm \delta x_2 - 1) \partial_x u(x_1, \pm \frac{2}{\delta} - x_2)
\]

\[
+ \delta^2 u(x_1, \pm \frac{2}{\delta} - x_2) q''((\pm \delta x_2 - 1) - \gamma(U) - \gamma(U (x_1, \pm \frac{2}{\delta} - x_2) u(x_1, \pm \frac{2}{\delta} - x_2) q((\pm \delta x_2 - 1).
\]

Now, measuring the left- and right-hand sides of (4.11) in \( L^2(\mathbb{R}^2) \), taking into account the estimates of \( \gamma(U) = 1 + O(U) \) for small \( U \) and Proposition 3.1, we find that

\[
|[-\Delta + \gamma(U), E] u|_{L^2(\mathbb{R}^2)} \lesssim \delta |\partial_x u|_{L^2(S_\delta)} + \delta^2 |u|_{L^2(S_\delta)}.
\]

The \( \partial_x u \) term above can eliminated via interpolation:

\[
|\partial_x u|^2_{L^2(S_\delta)} \lesssim |\Delta u|_{L^2(S_\delta)} |u|_{L^2(S_\delta)} \lesssim |L_\tau u|^2_{L^2(S_\delta)} + |u|^2_{L^2(S_\delta)}
\]

\[
\lesssim |L_\tau u|^2_{L^2(S_\delta)} + |u|^2_{L^2(S_\delta)}.
\]

Inserting this inequality into (4.11), we arrive at the commutator bound

\[
|[-\Delta + \gamma(U), E] u|_{L^2(\mathbb{R}^2)} \lesssim \delta ( |L_\tau u|^2_{L^2(S_\delta)} + |u|^2_{L^2(S_\delta)} ), \tag{4.12}
\]
independent of $\delta$, $\tau$, and $u$.

We are now prepared to prove the estimate (4.10). From Lemma 4.2 we have that

$$
|(-\Delta + \gamma'(U))Eu_{\|L^2(\mathbb{R}^2)}| \geq |(1 - P)Eu_{\|L^2(\mathbb{R}^2)}| \geq (1 - \theta^2)|u|^2_{L^2(S_\delta)},
$$

where the last inequality follows from hypothesis (4.9) and (4.5). On the other hand, together (4.5) and the commutator estimate (4.12) reveal that

$$
|(-\Delta + \gamma'(U))Eu_{\|L^2(\mathbb{R}^2)}| \lesssim |Elu_{\|L^2(\mathbb{R}^2)}| + \delta |u|_{L^2(S_\delta)}.
$$

Combined, (4.13) and (4.14) imply that (4.10) holds when $\delta$ is taken sufficiently small, which completes the proof. \hfill \Box

### 4.2. Construction of a near-degenerate eigenfunction.

In Section 4.1, it was shown that the function $U_2$ roughly aligns with the near-degenerate direction of $L_\tau$ in the sense that the restriction $L_\tau: X^2_\delta \cap U_2 \to X^0_\delta$ is uniformly positive according to (4.8). We now refine our analysis to find a (very small) eigenvalue $\lambda$ and corresponding eigenfunction $U_0$ near $U_2$ that limits to $\partial_{x_2}U$ in some sense as $\delta \searrow 0$. Similar as $P$ above, denote by $P_2$ the $L^2(S_\delta)$ orthogonal projection $X^0_\delta \to \text{span}\{U_2\}$ and also the projection it induces from $X^2_\delta \to \text{span}\{U_2\}$.

#### Lemma 4.4.

Consider the operator

$$
\tilde{L}_\tau: D(\tilde{L}_\tau) = U^\perp_2 \cap X^2_\delta = (1 - P_2)X^2_\delta \to U^\perp_2 = (1 - P_2)X^0_\delta
$$

defined by

$$
\tilde{L}_\tau u = (1 - P_2)L_\tau u, \quad \text{for all } u \in (1 - P_2)X^2_\delta.
$$

There exists $\delta_0 > 0$ such that, for all $|\tau| \leq \frac{1}{3}$ and $\delta \in (0, \delta_0)$, we have that $\tilde{L}_\tau$ is an isomorphism and is self-adjoint as an unbounded and densely defined operator on $U^\perp_2$ with $|\tilde{L}^{-1}_\tau| \approx 1$. 

Proof. Throughout the proof, all norms and inner products are evaluated on the domain \( S_{\delta} \). By definition, \( u \in D(\tilde{L}_\tau^*) \subset U^+_{2} \) and \( \tilde{u} = \tilde{L}_\tau^* u \in U^+_{2} \) if and only if

\[
(\tilde{u}, v)_{L^2} - (u, L_\tau v)_{L^2} = 0 \quad \text{for all } v \in U^+_{2} \cap X^2_{\delta},
\]

which holds if and only if

\[
\left( \tilde{u} + \frac{(u, L_\tau U_2)_{L^2}}{|U_2|^2_{L^2}} U_2, aU_2 + v \right)_{L^2} - (u, L_\tau (aU_2 + v))_{L^2} = 0,
\]

for all \( a \in \mathbb{R} \) and \( v \in U^+_{2} \cap X^2_{\delta} \). Since \( L^*_\tau = L_\tau \) on \( X^0_{\delta} \), we obtain that \( u \in D(\tilde{L}_\tau^*) \) and \( \tilde{u} = \tilde{L}_\tau^* u \) if and only if

\[
u \in U^+_{2} \cap X^2_{\delta} \quad \text{and} \quad u + \frac{(u, L_\tau U_2)_{L^2}}{|U_2|^2_{L^2}} U_2 = L_\tau u.
\]

Thus \( D(\tilde{L}_\tau^*) = U^+_{2} \cap X^2_{\delta} \), and \( \tilde{u} = (I - P_2) L_\tau u = \tilde{L}_\tau u \). This implies that \( \tilde{L}_\tau \) is indeed self-adjoint on its domain in \( U^+_{2} \).

Next, we improve slightly the bound of \( \tilde{L}_\tau \) in (4.8): observe that, for all \( u \in (1 - P_2) X_{\delta}^0 \),

\[
|u|_{H^2(S_{\delta})} \lesssim |u|_{L^2} + |\Delta u|_{L^2} \lesssim |u|_{L^2} + |L_\tau u|_{L^2} \lesssim |L_\tau u|_{L^2} + |P_2 L_\tau u|_{L^2}.
\]

But, due to equation (4.2) satisfied by \( U_2 \) and Lemma 4.1, we know that

\[
|P_2 L_\tau u|_{L^2(S_{\delta})} \simeq |(L_\tau u, U_2)_{L^2(S_{\delta})}| = |(u, L_\tau U_2)_{L^2(S_{\delta})}| \lesssim e^{\frac{-2(1-|\tau|)}{\delta}} |u|_{L^2(S_{\delta})},
\]

and thus

\[
|u|_{H^2(S_{\delta})} \lesssim |\tilde{L}_\tau u|_{L^2(S_{\delta})}. \tag{4.15}
\]

This implies that \( \tilde{L}_\tau \) is an isomorphism from \( U^+_{2} \cap X^2_{\delta} \) to its range — a closed subspace of \( U^+_{2} \). It follows from the self-adjointness of \( \tilde{L}_\tau \) on \( U^+_{2} \) that it is an isomorphism from \( U^+_{2} \cap X^2_{\delta} \) to \( U^+_{2} \). \( \square \)

**Proposition 4.5 (Existence of \( U_0 \)).** For each \( |\tau| < \frac{1}{3} \) and \( \delta > 0 \) sufficiently small, there exists an eigenfunction

\[
U_0 = a_0 (w + U_2) \in X^{k_0+1}_{\delta}, \quad w \in (1 - P_2) X^{k_0+1}_{\delta}, \quad |U_0|_{L^2(S_{\delta})} = 1 \tag{4.16}
\]

of \( L_\tau \) with a real eigenvalue \( \lambda = l(\delta, \tau) \):

\[
L_\tau U_0 = \lambda U_0 \quad \text{in } S_{\delta}.
\]

They obey the estimates

\[
|w|_{H^{k_0+1}(S_{\delta})} \lesssim \delta^{\frac{1}{2}} |\log \delta|^{\frac{1}{2}} e^{-\frac{2(1-|\tau|)}{\delta}}, \quad 0 < \ell(\delta, \tau) \simeq \delta^{\frac{1}{2}} e^{-\frac{2(1-|\tau|)}{\delta}}, \quad 0 < a_0 \simeq 1.
\]

Moreover, for fixed \( \delta, \lambda \) is \( C^{k_0-1} \) in \( |\tau| \) and \( U_0 \in X^{k_0+2}_{\delta} \) is \( C^{k_0-k-1} \) in \( \tau \) for \( 0 \leq k \leq k_0 - 1 \), respectively.
Proof. From Lemma 4.4, we know that there exists $\delta_0 > 0$ such that, for any $\delta \in (0, \delta_0)$, $\tilde{L}_\tau$ is an isomorphism from $(1 - P_2)X_\delta^2$ to $(1 - P_2)X_\delta^0$. By (4.15), its inverse satisfies

$$|\tilde{L}_\tau^{-1}|_{L((1-P_2)X_\delta^0,(1-P_2)X_\delta^2)} \leq \lambda_0^{-1}$$

for all $|\tau| \leq \frac{1}{3}$, (4.17)

for some $\lambda_0 > 0$ independent of $|\tau| < \frac{1}{3}$ and small $\delta > 0$. A function $U_2 + w$, with $w \in (I - P_2)X_\delta^2$, is an eigenfunction corresponding to $l$ if

$$L_\tau(U_2 + w) = l(U_2 + w).$$

Taking the inner product of the above equation with $U_2$ yields

$$l = \frac{(w + U_2, L_\tau U_2)_{L^2(S_\delta)}}{|U_2|_{L^2(S_\delta)}^2}.$$ 

On the other hand, applying $I - P_2$ to the eigenfunction equation and recalling that $P_2 w = 0$, we see that

$$\tilde{L}_\tau w = lw - (I - P_2)L_\tau U_2.$$  

(4.18)

This motivates us to consider the mapping $\Lambda: U_\delta^+ \rightarrow U_\delta^+$ defined by

$$\Lambda(w) = \ell(w)\tilde{L}_\tau^{-1}w - \tilde{L}_\tau^{-1}(1 - P_2)L_\tau U_2,$$

where

$$\ell(w) = \frac{(w + U_2, L_\tau U_2)_{L^2(S_\delta)}}{|U_2|_{L^2(S_\delta)}^2}.$$  

(4.20)

is the presumptive eigenvalue. Clearly a small fixed point $w \in U_\delta^+$ of $\Lambda$ yields an eigenfunction $w + U_2$ of $L_\tau$ close to $U_2$ associated to the eigenvalue $\ell(w)$.

It is straightforward to estimate

$$|\ell(w)| \lesssim |L_\tau U_2|_{L^2(S_\delta)}|w|_{L^2(S_\delta)} + (U_2, L_\tau U_2)_{L^2(S_\delta)},$$  

(4.21)

and

$$|\ell(w_1) - \ell(w_2)| \lesssim |U_2|_{L^2(S_\delta)}^2|L_\tau U_2|_{L^2(S_\delta)}|w_1 - w_2|_{L^2(S_\delta)}.$$  

Likewise, we have

$$|\Lambda(w_1) - \Lambda(w_2)|_{L^2(S_\delta)} \leq |\ell(w_1)||\tilde{L}_\tau^{-1}(w_1 - w_2)|_{L^2(S_\delta)} + |\ell(w_1) - \ell(w_2)||\tilde{L}_\tau^{-1}w_2|_{L^2(S_\delta)}$$

and

$$|\Lambda(0)|_{L^2(S_\delta)} \approx |L_\tau U_2|_{L^2(S_\delta)},$$

where all above inequalities are uniform in $|\tau| \leq \frac{1}{3}$ and small $\delta$. Consequently, Lemma 4.1 and (4.17) imply $\Lambda$ is a contraction map that sends $B_1$, the closed unit ball centered at the origin in $X^0_\delta$, to itself. It therefore has a unique fixed point $w^* = w^*(\delta, \tau) \in (1 - P_2)X^2_\delta \cap B_1$. This yields the eigenvalue $l = \ell(w^*)$ defined by (4.20) and the corresponding eigenfunction $w + U_2$ whose higher Sobolev regularity is due to the ellipticity in (4.18). The normalizing constant $a_0 > 0$ is chosen such that $|U_0|_{L^2(S_\delta)} = 1$. Since $\gamma \in C^{k_0}$ and $U \in C^{k_0+2}$ with exponential decay, it is easy to see that $L_\tau: X^k_\delta \rightarrow X^k_\delta$ is $C^{k_0-k-1}$ in $\tau$ for $k \geq 0$. From
standard spectral theory, the simple eigenvalue $\ell$ is $C^{k_0-1}$ in $\tau$ and the unit eigenfunction $U_0 \in X^{k+2}_\delta$ of $L_\tau$ is $C^{k_0-k-1}$ in $\tau$ for $0 \leq k \leq k_0 - 1$.

As $w$ is a fixed point of the contraction $\Lambda$, its definition (4.19) and Lemma 4.1 imply

$$|w|_{L^2(S_0)} \lesssim |\Lambda(0)|_{L^2(S_0)} \lesssim \delta^{\frac{1}{2}} |\log \delta|^{\frac{1}{2}} e^{-\frac{2(1-|\tau|)}{3}}.$$  

The higher Sobolev norms satisfy similar estimates due to the elliptic regularity given in (4.18). Finally, we conclude from (4.20), the above inequality, and Lemma 4.1, that

$$\|\ell(w) - \frac{(U_2, L_\tau U_2)_{L^2(S_0)}}{|U_2|_{L^2(S_0)}^2}\|_{L^2(S_0)} \leq e^{-\frac{4(1-|\tau|)}{3}}.$$  

Along with Lemma 4.1, this yield the desired estimate on $\ell$. The positivity of $\ell$ is a consequence of the sign of $(U_2, L_\tau U_2)_{L^2(S_0)}$; proved in Lemma 4.1.

Using the estimates just obtained, we can now confirm that $L_\tau$ is invertible (with near-degeneracy in the $U_0$ direction) and, more important, that the inverse of its restriction to the orthogonal complement of $U_0$ is bounded independently of $\delta$. That said, let $U_0$ be given as in Proposition 4.5 and denote its orthogonal complement in $X^0_\delta$ by $U^\perp_0$.

**Lemma 4.6** (Invertibility of $L_\tau$). There exists $\delta_0 > 0$ such that, for all $\delta \in (0, \delta_0)$ and $|\tau| \leq \frac{1}{3}$,

$$L_\tau : X^2_\delta \to X^0_\delta \quad \text{is invertible.} \quad (4.22)$$

Moreover, there exists $\mu_0 = \mu_0(\delta_0) > 0$ such that

$$|L_\tau u|_{L^2(S_0)} \geq \mu_0 |u|_{L^2(S_0)} \quad \text{for all } u \in U^\perp_0 \cap X^2_\delta. \quad (4.23)$$

**Proof.** Since $L_\tau$ is self-adjoint and $U_0$ is an eigenfunction, it is standard that $U^\perp_0$ is invariant under $L_\tau$ in the sense that

$$L_\tau(U^\perp_0 \cap X^2_\delta) \subset U^\perp_0,$$

see also the proof of Lemma 4.4. Because $\ell > 0$, it suffices to prove $L_\tau |U^\perp_0 \cap X^2_\delta$ is an isomorphism to $U^\perp_0$ with an upper bound independent of $|\tau| \leq \frac{1}{3}$ and small $\delta > 0$. In fact, since

$$U_0 = a_0(U_2 + w), \quad \text{with } w \in U^\perp_2, \quad |w|_{L^2(S_0)} \lesssim e^{-\frac{2(1-|\tau|)}{3}},$$

the subspace $U^\perp_0$ is isomorphic to $U^\perp_2$ through $I - P_2$, the projection associated to the orthogonal decomposition $X^0_\delta = \text{span}\{U_2\} \oplus U^\perp_2$. To see this, note that any $v \in U^\perp_0$ can be written as

$$v = v_1 + bU_2, \quad \text{where } v_1 = (I - P_2)v \in U^\perp_2,$$

and

$$b = \frac{(v, U_2)_{L^2(S_0)}}{|U_2|_{L^2(S_0)}^2} = \frac{(v_1, U_0)_{L^2(S_0)}}{|U_2, U_0|_{L^2(S_0)}^2} = \frac{(v_1, w)_{L^2(S_0)}}{|U_2|_{L^2(S_0)}^2}.$$  

The smallness of $w$ implies $b \ll |v|_{L^2(S_0)}$ and thus $L_\tau |U^\perp_0 \cap X^2_\delta$ is an isomorphism to $U^\perp_0$ with a uniform bound due to Lemma 4.4. \qed

The invertibility of $L_\tau$ also holds in higher Sobolev spaces due to elliptic theory.
Corollary 4.7. There exists $\delta_0 > 0$ such that, for all $\delta \in (0, \delta_0)$, $|\tau| \leq \frac{1}{3}$, and $0 \leq k \leq k_0 - 1$,

$$L_\tau : X^{k+2}_\delta \to X^k_\delta$$

is invertible. \hfill (4.24)

Moreover, there exists $\mu_0 = \mu_0(\delta_0) > 0$ such that

$$|L_\tau u|_{H^k(S_\delta)} \geq \mu_0|u|_{H^{k+2}(S_\delta)}$$

for all $u \in U^1_0 \cap X^{k+2}_\delta$. \hfill (4.25)

5. Proof of the main result

In this section we complete the argument leading to the proof of Theorem 1.1.

5.1. Normal bundle coordinates. Recall from Section 2 that the waves we study are represented by two quantities: the boundary value $\Gamma_s$ of a conformal mapping, that determines the fluid domain, and a (rescaled) stationary stream function $\phi$ that gives the velocity field. Our basic approach is to construct waves for which $|\Gamma_s|_{H^k_0} \ll 1$ and $\phi$ is a perturbation of $U(\tau) - U(\tau)_{bc}$, where the parameter $\tau \in (-\frac{1}{3}, \frac{1}{3})$ selects the approximate altitude of the center of vorticity.

At this stage, we have obtained detailed information regarding the spectrum of the linearized operator $L_\tau = -\Delta + \gamma'(U(\tau)) : X^{k+2}_\delta \to X^k_\delta$ and its dependence on $\tau$ and $\delta$. In particular, we proved in Proposition 4.5 that there exists a unique simple eigenvalue $l = l(\delta, \tau)$, associated to an eigenfunction $U_0(\tau)$, that converges to 0 exponentially fast as $\delta \searrow 0$. This presents an obvious obstruction to a na"ive fixed point scheme. We will see that $\tau$ is the key to ameliorating the issue.

To see the connection, observe that the family

$$C := \{ U(\tau) - U(\tau)_{bc} : \tau \in (-\frac{1}{3}, \frac{1}{3}) \}$$

can be viewed as a $C^{k_0+2-k}$ curve in the ambient space $X^k_\delta$. At a fixed $\tau$, the tangent to $C$ is

$$T_\tau C = \partial_\tau (U(\tau) - U(\tau)_{bc}) = \delta^{-1}(\partial_{x_2}U(\tau) - (\partial_{x_2}U(\tau))_{bc}) = \delta^{-1}U_2(\tau) \sim \delta^{-1}U_0(\tau),$$

where the second equality follows from the linearity of the boundary correction operator. The above calculation shows that the tangent direction along the curve $C$ is almost parallel to the near-degenerate subspace.

Therefore, our strategy is to seek a (rescaled) stationary stream function of the form

$$\phi = U(\tau) - U(\tau)_{bc} + v,$$}

with the unknowns

$$(\tau, v) \in \mathcal{X}_{\delta,k} := \left\{ (\tau, v) \mid \tau \in (-\frac{1}{3}, \frac{1}{3}), v \in X^k_\delta \cap U_0(\tau)\right\}, \quad k \geq 2.$$ \hfill (5.2)

This ensures that $v$ avoids the near-degenerate direction of $L_\tau$. While the linear part $g - \alpha^2 D^2$ of the Bernoulli boundary condition (2.24b) is already invertible. We may then perform a Lyapunov–Schmidt reduction: for each fixed $\tau$, we solve for $v$ and $\Gamma_s$, leaving a one-dimensional problem of the form $b(\tau) = 0$, for a certain bifurcation function $b$. Finally,
Figure 4. Schematic of the tubular neighborhood \( N_\delta \) of the curve \( C_\delta \). At each \( \tau \in (-\frac{1}{3}, \frac{1}{3}) \), we locally decompose the space \( X_\delta^k \) into a component \( v \) in the non-degenerate direction \( U_0(\tau)^\perp \), and a component in the near-degenerate direction \( U_0(\tau) \).

we will appeal to an intermediate value theorem argument to infer the existence of solutions to this reduced problem, as anticipated by the model calculation carried out in Section 1.

It is therefore imperative that the Lyapunov–Schmidt reduction be performed in such a way that \( b(\tau) \) is continuous (or even smooth). Because the near-degenerate and non-degenerate subspaces vary as we change \( \tau \), it is natural to view \( X_\delta,k \) as a smooth vector bundle over the base \((-\frac{1}{3}, \frac{1}{3})\), with the fibers being the non-degenerate subspaces

\[
\mathcal{V}_{\delta,k}^\tau := X_\delta^k \cap U_0(\tau)^\perp,
\]

see also Figure 4. According to Proposition 4.5, the \( C^{k_0} \)-regularity of \( \gamma \) ensures that \( \tau \mapsto U_0(\gamma) \in X_{\delta}^{k+2} \) is \( C^{k_0-k-1} \) for \( 0 \leq k \leq k_0 - 1 \), and hence the orthogonal projection \( P_0(\tau) \) onto \( \text{span} U_0(\tau) \) enjoys the same regularity with respect to \( \tau \). It then follows that each \( \tau_0 \in (-\frac{1}{3}, \frac{1}{3}) \) is contained in a neighborhood \( I_0 \) such that the mapping

\[
(\tau, v) \in I_0 \times \mathcal{V}_{\delta,k}^\tau_0 \mapsto (\tau, (1 - P_0(\tau))v) \in \mathcal{V}_{\delta,k}, \quad \text{for} \ 2 \leq k \leq k_0 + 1
\]

is a \( C^{k_0-k+1} \) local trivialization of \( \mathcal{V}_{\delta,k}^\tau \). Note that here and in the sequel, we reserve cursive script for bundles. In the Lyapunov–Schmidt reduction, we fix \( \tau \), while tracking the continuous dependence on it.

Remark 5.1. While continuity in \( \tau \) is sufficient for our purpose, in differential geometry, there are standard notions of smoothness of mappings related to vector bundles based on the smoothness of the trivializations, which allow implicit function theorem type arguments to be carried out as on flat spaces or manifolds. Moreover, it is standard to prove that

\[
\chi: (\tau, v) \in \mathcal{V}_{\delta,k} \mapsto v + U(\tau) - U(\tau)_{bc} \in X_{\delta}^k, \quad \text{for} \ 2 \leq k \leq k_0,
\]

defines a \( C^{k_0-k+1} \) local coordinate map (usually referred to as the normal bundle coordinates) near \( C \).
To simplify notation, we introduce the set
\[ \mathcal{W}_{\delta,k} := \left\{ (\tau, v, \Gamma_s) : \mathcal{X}_{\delta,k} \times H^k_0(\mathbb{R}) \right\}, \]
and endow it with the structure of a vector bundle over \((-\frac{1}{3}, \frac{1}{3})\) having fibers
\[ \mathcal{W}_{\delta,k}^\tau := \mathcal{X}_{\delta,k}^\tau \times H^k_0(\mathbb{R}), \]
and locally trivialized in the obvious way.

5.2. Lyapunov–Schmidt reduction. Let us now reconsider the elliptic system (2.24),
\[
\begin{aligned}
&(-\Delta + \gamma'(U))v + F(\tau, v, \Gamma_s) = 0 \quad \text{in} \ S_{\delta} \\
&(g - \alpha^2D^2)\Gamma_s + G(\tau, v, \Gamma_s) = 0 \quad \text{on} \ \mathbb{R},
\end{aligned}
\]
from this geometrical standpoint. In the previous subsection, we argued that this system is equivalent to finding \(\Gamma_s\) together with a scaled stream function having the ansatz
\[ \varphi = v + U(\tau) - U(\tau)_{bc}, \quad (\tau, v) \in \mathcal{X}_{\delta,k}. \]
As before, we suppress the dependence of \(U\) and \(U_{bc}\) on \(\tau\) whenever there is no risk of confusion. With a slight abuse of notation we also as above view
\[
F(\tau, v, \Gamma_s) = |1 + \Gamma'(\delta \cdot)|^2\gamma(v + U - U_{bc}) - \gamma(U) - \gamma'(U)v + U_{bc} \tag{5.3}
\]
\[
G(\tau, v, \Gamma_s) = \frac{1}{2\delta^2} A(\Gamma_s)^{-1} \left[ \frac{\partial_{x_2} (v + U - U_{bc})(\cdot, \frac{1}{\delta})^2}{(1 + |D| \coth (2|D|\Gamma_s))^2 + \Gamma_s^2} \right], \tag{5.4}
\]
from (2.25) and (2.26) to be the bundle map from a subset (with \(\Gamma_s\) small) of \(\mathcal{W}_{\delta,k}\) to \(\mathcal{W}_{\delta,k-2}\). It is easily seen that the slightly redefined \((F, G)\) enjoy the same regularity as in Lemma 3.7.

Projecting the semilinear elliptic problem into the near-degenerate and non-degenerate subspaces (which are invariant under \(L_\tau\)), we can reconfigure the governing equations as the following system:
\[
P_0 F(\tau, v, \Gamma_s) = 0 \quad \text{in} \ S_{\delta}, \tag{5.5a}
\]
\[
(-\Delta + \gamma'(U))v + (1 - P_0)F(\tau, v, \Gamma_s) = 0 \quad \text{in} \ S_{\delta}, \tag{5.5b}
\]
\[
(g - \alpha^2D^2)\Gamma_s + G(\tau, v, \Gamma_s, \varepsilon) = 0 \quad \text{on} \ \mathbb{R}. \tag{5.5c}
\]

Notice that for a fixed \(\tau\), (5.5b)–(5.5c) are solved on the fiber \(\mathcal{W}_{\delta,k}^\tau\). In the next lemma, we prove that one can always do this, and the solution depends smoothly on \(\tau\). We therefore reduce the system to the one-dimensional equation (5.5a) related to the near-degenerate subspace.

**Lemma 5.2** (Lyapunov–Schmidt reduction). There exists \(C, \delta_0 > 0\) such that, for all \(\delta \in (0, \delta_0)\) and \(\tau \in (-\frac{1}{3}, \frac{1}{3})\), there exists a solution \((\tilde{v}(\tau), \tilde{\Gamma}_s(\tau)) \in \mathcal{W}_{\delta,k_0}^\tau\) to (5.5b)–(5.5c) which is unique in the set
\[
\left\{ (v, \Gamma_s) \in \mathcal{W}_{\delta,k_0}^\tau : |v|_{H^{k_0}(S_{\delta})} + C\delta^{-1}|\Gamma_s|_{H^{k_0}(\mathbb{R})} \leq \delta^{k_0+1} \right\},
\]
and satisfies
\[ |\tilde{v}|_{H^{k_0}(S_\delta)} + C_0^2 \delta^{-1}|\tilde{\Gamma}_s|_{H^{k_0}(\mathbb{R})} \lesssim C_0 \delta^{-k_0} e^{-\frac{2|1-|r|}|S|}, \quad |\tilde{\Gamma}_s - \eta_0|_{H^{k_0}(\mathbb{R})} \lesssim C_0^3 \delta^{-2k_0} e^{-\frac{3|1-|r|}|S|}, \]
where
\[ \eta_0 = -2\delta^{-2}(g - \alpha^2 D^2)^{-1} \left( (\partial_\xi U(\frac{\tau}{\delta}, \frac{1}{\delta}) \right)^2. \]
Moreover, \((v, \Gamma_s) \in H^{k_0}(S_\delta) \times H^{k_0}(\mathbb{R})\) depends continuously on \(\tau\).

Remark 5.3. As a consequence, the system (5.5) is locally equivalent to the one-dimensional problem
\[ 0 = b(\tau) := \left( U_0(\tau), F(\tau, \tilde{\nu}(\tau), \tilde{\Gamma}_s(\tau)) \right)_{L^2(S_\delta)} = \left( U_0, L_\tau \tilde{\nu} + F(\tau, \tilde{\nu}, \tilde{\Gamma}_s) \right)_{L^2(S_\delta)} \quad \text{(5.6)} \]
Also, it is worth noting that, since \(\gamma \in C^k\) for any \(2 \leq k \leq k_0\), the above lemma holds for all such \(k\). The uniqueness property of \((\tilde{\nu}(\tau), \tilde{\Gamma}_s(\tau))\) implies that it is independent of \(k\).

Proof. Let \(\delta \in (0, \delta_0)\) be given, where \(\delta_0\) will determined over the course of the proof, which is largely based on the estimates given in Lemma 3.7. To tame the singular bound \(\delta^{-1}\) of \(D_{\Gamma_s} F\), we introduce the rescaled variable
\[ \tilde{\Gamma}_s := \frac{C}{\delta} \Gamma_s, \]
where \(C > 0\) will be determined independent of \(\tau\) and \(\delta\), and the corresponding scaling of the nonlinearities
\[ \tilde{F}(\tau, v, \tilde{\Gamma}_s) := F(\tau, v, \frac{\delta}{C} \tilde{\Gamma}_s), \quad \tilde{G}(\tau, v, \tilde{\Gamma}_s) := \frac{C}{\delta} G(\tau, v, \frac{\delta}{C} \tilde{\Gamma}_s). \]
Denote by
\[ L_1^{-1}(\tau) = \left( \left( -\Delta + \gamma'(U(\tau)) \right)|_{\mathcal{F}^{k_0}_{\delta,0}} \right)^{-1} : \mathcal{F}^{k_0}_{\delta,0} \rightarrow \mathcal{F}^{k_0}_{\delta,0}, \]
\[ L_2^{-1} = (g - \alpha^2 D^2)^{-1} : H^{k_0-2}(\mathbb{R}) \rightarrow H^{k_0}(\mathbb{R}), \]
where we recall that the existence and boundedness of \(L_1(\tau)^{-1}\) were established in Corollary 4.7. In particular, notice that, because \(-\Delta + \gamma'(U) : \mathcal{F}^{k_0}_{\delta,0} \rightarrow \mathcal{F}^{k_0}_{\delta,0} - 2\) is symmetric with respect to the \(L^2(S_\delta)\) inner product and \(U_0\) is an eigenfunction, the range of \(L_1(\tau)^{-1}\) is contained in \(U_0(\tau)^{-1}\). Then we see that \((\tau, v, \Gamma_s)\) solve (5.5b) and (5.5c) if and only if \((\tau, v, \tilde{\Gamma}_s)\) is a fixed point of the mapping
\[ \Lambda^\tau = (\Lambda_1^\tau(v, \tilde{\Gamma}_s), \Lambda_2^\tau(v, \tilde{\Gamma}_s)) : B \rightarrow \mathcal{W}^{\tau}_{\delta, k_0} \]
given by
\[ \Lambda_1^\tau(v, \tilde{\Gamma}_s) = -L_1^{-1}(\tau) (1 - P_0(\tau)) \tilde{F}(\tau, v, \tilde{\Gamma}_s) \]
\[ \Lambda_2^\tau(v, \tilde{\Gamma}_s) = -L_2^{-1} \tilde{G}(\tau, v, \tilde{\Gamma}_s) \quad \text{(5.7)} \]
on the set
\[ B = \{(v, \tilde{\Gamma}_s) \in \mathcal{W}^{\tau}_{\delta, k_0} : |v|_{H^{k_0}(S_\delta)} + |\tilde{\Gamma}_s|_{H^{k_0}(\mathbb{R})} \leq \delta^{k_0+1}\}. \]
From Lemma 3.7, we have
\[ |\Lambda^\tau(0,0)|_{H^k_0(S_\delta) \times H^k_0(\mathbb{R})} \lesssim C\delta^{-k_0} e^{-\frac{2(1-|\tau|)}{\delta}}, \quad |D\Lambda^\tau|_{C^0(B,\mathcal{L}(\mathbb{R}^n))} \lesssim C^{-1}. \]
Therefore, for a sufficiently large \( C > 0 \), which can be chosen independently of \( \delta \) and \( \tau \), \( \Lambda^\tau \) is a contraction on \( B \), and so it possesses a unique fixed point
\[ (\tilde{v}(\tau), \hat{\Gamma}_s(\tau)) = (\tilde{v}(\tau), C\delta^{-1}\hat{\Gamma}_s(\tau)) \in B. \]
Moreover, we have the estimate
\[ |\tilde{v}(\tau)|_{H^k_0(S_\delta)} + |\hat{\Gamma}_s(\tau)|_{H^k_0(\mathbb{R})} \lesssim |\Lambda^\tau(0,0)|_{H^k_0(S_\delta) \times H^k_0(\mathbb{R})} \lesssim C\delta^{-k_0} e^{-\frac{2(1-|\tau|)}{\delta}}. \]
The continuity of \( \tilde{v}(\tau) \) and \( \hat{\Gamma}_s(\tau) \) follows from the continuity of \( \Lambda^\tau \) in \( \tau \), where we can view it as a mapping defined on a smooth bundle.
Finally, we identify the leading order term of \( \hat{\Gamma}_s(\tau) \). Due to the fixed point property, we have
\[ (g - \alpha^2 D^2)\hat{\Gamma}_s(\tau) = -G(\tau, \tilde{v}, \hat{\Gamma}_s). \]
Lemma 3.7 and the above upper bounds of \( (\tilde{v}, \hat{\Gamma}_s) \) imply
\[ |G(\tau, \tilde{v}, \hat{\Gamma}_s) - G(\tau, 0, 0)|_{H^k_0 - 2(S_\delta)} \lesssim C\delta^{3 - 2k_0} e^{-\frac{3(1-|\tau|)}{\delta}}. \]
From (3.11), (3.12), (3.13), and the scaling property, we have
\[ \left| G(\tau, 0, 0) - 2\delta^{-2} \left( \partial_{x_2} U(\frac{\cdot}{\delta}, \frac{1}{\delta}) \right)^2 \right|_{H^k_0 - 2(S_\delta)} \lesssim \delta^{-\frac{1}{2} + k_0} e^{-\frac{3(1-|\tau|)}{\delta}}, \]
which along with the above inequality yields the desired estimate on \( \hat{\Gamma}_s(\tau) \). \( \square \)

5.3. Proof of the main result.

Proof of Theorem 1.1. The Lyapunov–Schmidt reduction carried out in Lemma 5.2 shows that it suffices to find \( \tau \in (-\frac{1}{3}, \frac{1}{3}) \) with \( b(\tau) = 0 \), where \( b(\tau) \) is defined in (5.6). Our strategy will be to relate the bifurcation equation to the model calculation (1.14).

With that in mind, fix \( \tau \in (-\frac{1}{3}, \frac{1}{3}) \) and recall
\[ (\tilde{v}, \hat{\Gamma}_s) = (\tilde{v}(\tau), \hat{\Gamma}_s(\tau)), \quad U = U(\tau), \quad U_0(\tau) = a_0(U_2 + w), \quad \text{and} \quad \varphi = U - U_{bc} + \tilde{v}, \]
recalling that \( U_0, U_2, a_0, \) and \( w \) were obtained in Section 4. In particular, \( 1 \approx a_0 = a_0(\tau) > 0 \) is a normalizing constant introduced to ensure that \( |U_0|_{L^2} = 1 \).
Since \( (\tilde{v}, \hat{\Gamma}_s) \) solves (5.5b) and (5.5c), we have
\[ L_{\tau}\tilde{v} + F(\tau, \tilde{v}, \hat{\Gamma}_s) = b(\tau)U_0(\tau). \]
Now, let
\[ \psi(\tau) = \varphi(\tau) \circ \left( \text{id} + \delta^{-1}\tilde{\Gamma}(\delta \cdot) \right)^{-1}, \]
where \( \tilde{\Gamma} = \hat{\Gamma}_1 + i\tilde{\Gamma}_2 \) is the holomorphic function constructed from \( \hat{\Gamma}_s \) through (2.19). According to Lemma 5.2, the domain of \( \psi \) is the (slightly) perturbed strip
\[ \tilde{\Omega}(\tau) = \left( \text{id} + \delta^{-1}\tilde{\Gamma}(\delta \cdot) \right)(S_{\delta}) \sim S_{\delta}. \]
For clarity, we use $y = (y_1, y_2)$ as the coordinate variable in $\tilde{\Omega}(\tau)$. It is easy to compute

$$\partial_{y_2} \psi = \left( \frac{i}{1 + \tilde{\Gamma}'(\delta \cdot)} \cdot \nabla \varphi \right) \circ \left( \text{id} + \delta^{-1} \tilde{\Gamma}(\delta \cdot) \right)^{-1},$$

where the complex number $i(1 + \tilde{\Gamma}'(\delta \cdot))^{-1}$ is understood as a two-dimensional vector. Since Corollary 3.6, Proposition 4.5, and Lemma 5.2 together imply that

$$\left| U_0(\tau) - a_0(\tau) \frac{i}{1 + \tilde{\Gamma}'(\delta \cdot)} \cdot \nabla \varphi \right|_{L^2(S_\delta)} \ll 1 = |U_0|_{L^2(S_\delta)},$$

we have that (5.6) holds for $(\tau, \tilde{v}, \tilde{\Gamma}_s)$ if and only if

$$\tilde{b}(\tau) := \left( \frac{i}{1 + \tilde{\Gamma}'(\delta \cdot)} \cdot \nabla \varphi, L_\tau \tilde{v} + F(\tau, \tilde{v}, \tilde{\Gamma}_s) \right)_{L^2(S_\delta)} = 0.$$

By the definitions of $F$ and the boundary correction operator,

$$L_\tau \tilde{v} + F(\tau, \tilde{v}, \tilde{\Gamma}_s) = -\Delta \varphi(\tau) + |1 + \tilde{\Gamma}'(\delta \cdot)|^2 \gamma(\varphi),$$

which, along with the coordinate change $y = x + \tilde{\Gamma}(\delta x)$, gives

$$\tilde{b}(\tau) = \int_{\tilde{\Omega}(\tau)} (-\Delta \psi + \gamma(\psi)) \partial_{y_2} \psi \, dy.$$

Following the same calculation leading to (1.14), we then find that

$$\tilde{b}(\tau) = \frac{1}{2} \int_{\tilde{\Omega}(\tau)} |\nabla \psi|^2 N_2 \, dS_y$$

where

$$N = (N_1, N_2) = \pm \left( \frac{i + i \tilde{\Gamma}'(\delta \cdot)}{|1 + \tilde{\Gamma}'(\delta \cdot)|} \circ \left( \text{id} + \delta^{-1} \tilde{\Gamma}(\delta \cdot) \right)^{-1} \right)$$

is the outward unit normal vector on the upper/lower component of $\partial \tilde{\Omega}(\tau)$, and

$$dS_y = |1 + \tilde{\Gamma}'(\delta \cdot)| \circ \left( \text{id} + \delta^{-1} \tilde{\Gamma}(\delta \cdot) \right)^{-1} \, dx_1$$

the length element along $\partial \tilde{\Omega}(\tau)$. We can rewrite $\tilde{b}(\tau)$ as an integral on $S_\delta$ by reversing the coordinate change:

$$\tilde{b}(\tau) = \frac{1}{2} \int_{\mathbb{R}} \frac{1 + \partial_{x_1} \tilde{\Gamma}_1(\delta x_1, \frac{1}{\delta})}{|1 + \tilde{\Gamma}'(\delta x_1, \frac{1}{\delta})|^2} \left| \partial_{x_2} \varphi(x_1, \frac{1}{\delta}) \right|^2 \, dx_1$$

$$- \frac{1}{2} \int_{\mathbb{R}} \left| 1 + \partial_{x_1} \tilde{\Gamma}_1(\delta x_1, -\frac{1}{\delta}) \right| \left| \partial_{x_2} \varphi(x_1, -\frac{1}{\delta}) \right|^2 \, dx_1.$$  \hspace{1cm} (5.8)

Notice that tangential derivatives do not appear because $\varphi|_{\partial S_\delta} = 0$. Without loss of generality, we just consider the first term. From the definition of $\varphi$, (3.12), (3.13), Lemma
5.2 (taking \( k_0 = 2 \)), and the trace theorem, we obtain
\[
\left| \partial_{x_2} \varphi \left( \cdot, \frac{\tau}{\delta} \right) - 2 \partial_{x_2} U \left( \cdot, \frac{\tau - \tau_1}{\delta} \right) \right|_{L^2(\mathbb{R})} \lesssim |\tilde{v}|_{H^2(S_\delta)} + \delta^{\frac{3}{2}} e^{-\frac{2(1-|\delta|)}{\delta}} \lesssim \delta^{-\frac{2}{3}} e^{-\frac{2(1-|\delta|)}{\delta}},
\]
and
\[
\left| \partial_{x_2} \varphi \left( \cdot, \frac{\tau}{\delta} \right) \right|_{L^2(\mathbb{R})} \lesssim \delta^{\frac{1}{2}} e^{-\frac{1-|\delta|}{\delta}}.
\]
Therefore, again Lemma 5.2 implies
\[
|\tilde{b}(\tau) - \tilde{b}_1(\tau)| \lesssim \delta^{-\frac{7}{4}} e^{-\frac{3(1-|\tau|)}{\delta}},
\]
where
\[
\tilde{b}_1(\tau) := 2 \int_{\mathbb{R}} \left( \partial_{x_2} U(x_1, \frac{\tau - \tau_1}{\delta}) \right)^2 - \left( \partial_{x_2} U(x_1, \frac{\tau}{\delta}) \right)^2 \, dx_1.
\]
Here we have used the radial symmetry of \( U \) to simplify the expression. Clearly, it also implies that \( \tilde{b}_1 \) is odd.

Due to the exponential localization, \( \tilde{b}_1 \) can be effectively determined by integrating only over a \( \delta \)-dependent but compact interval. Indeed, from Proposition 3.1, it is easy to see
\[
|\tilde{b}(\tau) - \tilde{b}_2(\tau)| \lesssim \delta^{-\frac{7}{4}} e^{-\frac{3(1-|\tau|)}{\delta}},
\]
where
\[
\tilde{b}_2(\tau) := 2 \int_{\mathbb{R}} \left( \partial_{x_2} U(x_1, \frac{\tau - \tau_1}{\delta}) \right)^2 - \left( \partial_{x_2} U(x_1, \frac{\tau}{\delta}) \right)^2 \, dx_1.
\]
Since \( \tilde{b}_2 \) is also odd, we consider \( \tau \in (0, \frac{1}{3}) \). Using Proposition 3.1 once more, along with (3.2), we compute that
\[
\tilde{b}_2(\tau) = -4 \int_{-\frac{5}{3}}^{\frac{5}{3}} \int_{-\frac{1}{3}}^{\frac{1}{3}} \partial_{x_2} U \partial_{x_2} U \, dx_2 \, dx_1
\]
\[
= -4 \int_{-\frac{5}{3}}^{\frac{5}{3}} \int_{-\frac{1}{3}}^{\frac{1}{3}} \sin(\theta)U_r \left( \sin^2(\theta)U_{rr} + \frac{\cos^2(\theta)}{r}U_r \right) \, dx_2 \, dx_1
\]
\[
\gtrsim \int_{-\frac{5}{3}}^{\frac{5}{3}} \int_{-\frac{1}{3}}^{\frac{1}{3}} r^{-1} e^{-2r} \, dx_2 \, dx_1,
\]
where we use the fact that \( 0 < \sin \theta \approx 1 \) in this integral region. Let
\[
S = \left\{ x : |x_1| < \frac{5}{3}, \ |x_2 - \frac{1}{3}| \leq \frac{\tau}{\delta}, \ |x| < \frac{1 + \tau}{\delta} \right\},
\]
which has the polar coordinates representation
\[
S = \left\{ (r, \theta) : r \in (\frac{1 + \tau}{\delta}, \frac{1 + \tau}{\delta}), \ \theta \in (\beta(r), \pi - \beta(r)) \right\},
\]
where, because we are restricting to \( \tau \in (0, \frac{1}{3}) \),
\[
\beta(r) = \arcsin \left( \frac{1 + \tau}{\delta r} \right), \quad \frac{\pi}{2} - \beta(r) \approx (\delta r - 1 + \tau)^{\frac{1}{2}}.
\]
Therefore we have
\[ \tilde{b}_2(\tau) \gtrsim \int_S r^{-1} e^{-2r} \, dx_2 \, dx_1 = \int_{\frac{1}{1 - \tau}}^{1 + \tau} \int_{\frac{1}{1 - \tau}}^{\pi - \beta(r)} e^{-2r} \, d\theta \, dr = \int_{\frac{1}{1 - \tau}}^{1 + \tau} (\pi - 2\beta(r)) \, e^{-2r} \, dr \]
\[ \approx \int_{\frac{1}{1 - \tau}}^{1 + \tau} (\delta r - (1 - \tau)) \frac{1}{2} e^{-2r} \, dr = \delta \frac{1}{2} e^{-\frac{2(1 - \tau)}{\tau}} \int_0^{\frac{2\tau}{\delta}} (r') \frac{1}{2} e^{-2r'} \, dr'. \]
This implies that, for \( \frac{\eta}{\tilde{\tau}} \ll 1, \)
\[ \tilde{b}_2(\tau) \gtrsim \tau^\frac{3}{2} \delta^{-1} e^{-\frac{2}{\tau}}, \]
and thus we obtain from (5.9) that there exists \( C > 0 \) independent of \( \delta > 0 \) such that
\[ \tilde{b}(\tau_0) > 0, \quad \tau_0 = C\delta^{-\frac{1}{2}} e^{-\frac{2}{\tilde{\tau}}}. \]

From the oddness of \( \tilde{b}_2 \), we can then conclude that there exists \( \tilde{\tau} \) with \( |\tilde{\tau}| \lesssim \delta^{-\frac{1}{2}} e^{-\frac{2}{\tilde{\tau}}} \) and such that \((\tilde{\tau}, \tilde{v}(\tilde{\tau}), \tilde{\Gamma}_s(\tilde{\tau}))\) is a solution to (5.5), and thus corresponds to a solution to the stationary capillary-gravity wave problem. The stream function is given by
\[ \Psi(x) = (U(\cdot, -\tilde{\tau}e_2) - U(\cdot, -\tilde{\tau}e_2)_{bc} + \tilde{v}(\tilde{\tau})) \circ \left( \frac{1}{\delta} \left( \text{id} + \tilde{\Gamma}(\tilde{\tau}) \right)^{-1} \right), \] (5.10)
defined on
\[ \Omega = \left( \text{id} + \tilde{\Gamma}(\tilde{\tau}) \right) (\{|x_2| < 1\}). \]

From the estimate \( |\tilde{\tau}| \lesssim \delta^{-\frac{1}{2}} e^{-\frac{2}{\tilde{\tau}}} \), Corollary 3.6, and Lemma 5.2, we have
\[ \left| \tilde{v}(\tilde{\tau}) \circ \left( \frac{1}{\delta} \left( \text{id} + \tilde{\Gamma}(\tilde{\tau}) \right)^{-1} \right) \right|_{H^{k_0}(\Omega)} \lesssim \delta^{1-k_0} |\tilde{v}(\tilde{\tau})|_{H^{k_0}(S_\delta)} \lesssim \delta^{1-2k_0} e^{-\frac{2}{\tilde{\tau}}}, \]
and
\[ \left| (U(\cdot, -\tilde{\tau}e_2) - U(\cdot, -\tilde{\tau}e_2)_{bc}) \circ \left( \frac{1}{\delta} \left( \text{id} + \tilde{\Gamma}(\tilde{\tau}) \right)^{-1} \right) - \tilde{\Psi}_0(\tilde{\tau}) \right|_{H^{k_0}(\Omega)} \]
\[ \lesssim \left| (U(\cdot, -\tilde{\tau}e_2) - U(\cdot, -\tilde{\tau}e_2)_{bc} - \tilde{\Psi}_0) \circ \left( \frac{1}{\delta} \left( \text{id} + \tilde{\Gamma}(\tilde{\tau}) \right)^{-1} \right) \right|_{H^{k_0}(\Omega)} \]
\[ + \left| \tilde{\Psi}_0 \circ \left( \frac{1}{\delta} \left( \text{id} + \tilde{\Gamma}(\tilde{\tau}) \right)^{-1} \right) - \tilde{\Psi}_0(\tilde{\tau}) \right|_{H^{k_0}(\Omega)} \]
\[ \lesssim \delta^{1-k_0} e^{-\frac{2}{\tilde{\tau}}} + |\tilde{\Psi}_0(\tilde{\tau})|_{H^{k_0+1}(\mathbb{R}^2)} |\tilde{\Gamma}_s(\tilde{\tau})|_{H^{k_0-\frac{3}{2}}(\mathbb{R})} \lesssim \delta^{1-2k_0} e^{-\frac{2}{\tilde{\tau}}}, \]
where
\[ \tilde{\Psi}_0(x) = U(x, -\tilde{\tau}e_2) - U(x_1, 2 - \tilde{\tau} - x_2) + U(x_1, -2 - \tilde{\tau} - x_2). \]
The desired estimate on \( \tilde{\Psi} \) in Theorem 1.1 follows immediately.

Finally, the corresponding free surface profile \( \eta \) is given by
\[ \eta = \tilde{\Gamma}_s(\tilde{\tau}) \circ \left( \text{id} + \tilde{\Gamma}_1(\tilde{\tau}, \cdot, 1) \right)^{-1}, \]
which clearly satisfies
\[ |\eta|_{H^{k_0}(\mathbb{R})} \lesssim \delta^{1-k_0} e^{-\frac{2}{\delta}}. \]

Using (3.13) and Lemma 5.2, it is straightforward to identify the leading order term of \( \eta \) coinciding with that of \( \tilde{\Gamma}_s(\tilde{\tau}) \) and to obtain the same remainder estimate much as in the above procedure for \( \Psi \). This completes the proof of the main theorem. \( \square \)

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