The Optimal Assignment Problem for a Countable State Space

Marianne Akian, Stéphane Gaubert, and Vassili Kolokoltsov

Abstract. Given a $n \times n$ matrix $B = (b_{ij})$ with real entries, the optimal assignment problem is to find a permutation $\sigma$ of $\{1, \ldots, n\}$ maximising the sum $\sum_{i=1}^{n} b_{i\sigma(i)}$. In discrete optimal control and in the theory of discrete event systems, one often encounters the problem of solving the equation $Bf = g$ for a given vector $g$, where the same symbol $B$ denotes the corresponding max-plus linear operator, $(Bf)_i := \max_{1 \leq j \leq n} b_{ij} + f_j$. The matrix $B$ is said to be strongly regular when there exists a vector $g$ such that the equation $Bf = g$ has a unique solution $f$. A result of Butkovič and Hevery shows that $B$ is strongly regular if and only if the associated optimal assignment problem has a unique solution. We establish here an extension of this result which applies to max-plus linear operators over a countable state space. The proofs use the theory developed in a previous work in which we characterised the unique solvability of equations involving Moreau conjugacies over an infinite state space, in terms of the minimality of certain coverings of the state space by generalised subdifferentials.

1. Introduction

Let $B = (b_{ij})$ be a $n \times n$ matrix with real entries. The optimal assignment problem is to find a permutation $\sigma$ of $\{1, \ldots, n\}$ maximising the sum $\sum_{i=1}^{n} b_{i\sigma(i)}$.

This problem can be interpreted algebraically by introducing the max-plus or tropical semiring, $\mathbb{R}_{\max}$, which is the set $\mathbb{R} \cup \{-\infty\}$, where $\mathbb{R}$ is the set of real numbers, equipped with the addition $(a,b) \mapsto \max(a,b)$ and the multiplication $(a,b) \mapsto a + b$. With these operations, one can define the notions of vectors, matrices, linear operators. In particular, the value of the optimal assignment is nothing but the permanent of the matrix $B$, evaluated in the semiring $\mathbb{R}_{\max}$.

We also associate to the matrix $B$ a linear operator over the max-plus semiring, which sends the vector $f \in \mathbb{R}^n_{\max}$, to the vector $Bf \in \mathbb{R}^n_{\max}$ given by $(Bf)_i := \max_{1 \leq j \leq n} b_{ij} + f_j$ (here we keep the usual notations max and + for scalars, but use the linear operator notation $Bf$ instead of a non linear one like $B(f)$). The map $f \mapsto B(-f)$ is a special case of Moreau conjugacy, see [RW98, Chapter 11, Section E], [Sin97], [AGK02, AGK05].

Butkovič and Hevery [BH85] found a remarkable relation between the equation $Bf = g$ and the optimal assignment problem. They defined a matrix $B$ with

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finite real entries to be strongly regular when there exists a vector \( g \in \mathbb{R}^n \) such that the equation \( Bf = g \) has a unique solution \( f \in \mathbb{R}^n \). They showed that \( B \) is strongly regular if and only if the associated optimal assignment problem has a unique solution. Further properties of strongly regular matrices appeared in [But94, But00]. In particular, the matrix \( B \) is strongly regular if and only if the space generated by its columns is of nonempty interior.

The same notion arose later on in the work of Richter-Gebert, Sturmfels, and Theobald [RGST05], who defined a matrix to be tropically singular if its columns are not in “generic position” in the tropical sense, meaning that they are included in the tropical analogue of a hyperplane. They showed that a (square) matrix is tropically nonsingular if and only if the associated optimal assignment problem has a unique solution. So tropical nonsingularity and strong regularity coincide.

The infinite dimensional version of the optimal assignment problem is nothing but the celebrated Monge-Kantorovich mass transportation problem. The equation \( Bf = g \) is a well known tool in the study of this problem via the infinite dimensional linear programming formulation introduced by Kantorovich. Indeed, a feasible solution of the dual problem of this linear programming problem consists precisely (up to a change of sign) of a pair of functions \( f, g \) such that \( Bf \leq g \), and when \( f \) and \( g \) are optimal, a complementary slackness property shows, at least formally, that \( Bf = g \). This motivates the search of infinite dimensional analogues of the theorem of Butković and Hevery.

The cases in which the state space is non compact can be regarded as degenerate. In this paper, we consider the simplest among these cases: we study the optimal assignment problem over a denumerable state space.

Loosely speaking, this problem aims at finding the optimal marriages in a society with a denumerable number of boys and girls. The interest in these questions goes back to the very origin of matching theory, since infinite graphs were already considered in König’s book [Kön50]. The theory of matching in infinite graphs has been considerably developed after König, we refer the reader to the survey of Aharoni [Aha91], in which generalisations of fundamental results in matching theory, like König’s theorem, Hall’s marriage theorem, or Birkhoff’s theorem on bistochastic matrices, can be found.

In this paper, we extend the theorem of Butković and Hevery to the denumerable setting, under some critical technical assumptions.

Our approach relies on the characterisation of the existence and of the uniqueness of the solution of the equation \( Bf = g \) in terms of covering by generalised subdifferentials given in our previous work [AGK02, AGK05].

This characterisation originates from a result of Vorobyev [Vor67, Theorem 2.6], who dealt with a finite state space and introduced a notion of “minimal resolvent coverings” of \( X \). Vorobyev’s approach was systematically developed by Zimmermann [Zim76, Chapter 3], who considered several algebraic structures and allowed in particular the matrix \( B \) to have \(-\infty\) entries. The sets arising in Vorobyev’s covering were shown to be special cases of subdifferentials in [AGK02, AGK05], leading to an extension of Vorobyev’s theorem to Moreau conjugacies and even to the more general case of “functional Galois connections”. The existence and uniqueness results proved there contain as special cases Vorobyev’s combinatorial result, and some properties of convex analysis (for instance, that an essentially
smooth lower semicontinuous proper convex function on \( \mathbb{R}^n \) has a unique preimage by the Fenchel transform).

In the characterisation of the existence and uniqueness of the solution of \( Bf = g \) in [AGK02, AGK05], some mild compactness assumptions are needed. These assumptions lead us here to require a tightness condition on the kernel, see Assumption (TC) below. The latter is of the same nature as the tightness condition used by Akian, Gaubert and Walsh [AGW05] in denumerable max-plus spectral theory.

We also note that in the denumerable case, the value of the permanent may be ill defined, because the weight of a permutation is the sum of a possibly divergent series. However, the optimality of a permutation can be expressed in full generality, because the difference of weights of permutations make sense under general circumstances, see Definition 2.1. This definition is somehow reminiscent of the treatment of “infinite extremals” in dynamic programming, see [KM97] for more background on this topic.

After a preliminary section introducing the notations and motivating the main assumptions, we formulate our main results in Section 3 as Theorems 3.1, 3.4 and 3.5, and prove them in Sections 4 and 5.

Let us conclude this introduction by listing further references. Motivations to consider Moreau conjugacies or max-plus linear operators with kernels can be found in [Vor67, CG79, Mas87, GM08, BCOQ92, CGQ99, AQV98, KM97, Gun98, LMS01, LM05, McE06]. Recent developments are highly influenced by tropical geometry via the so-called dequantisation procedure [LM05, IMS07]. The Moreau conjugacies, or equivalently, the max-plus linear operators with kernel considered here, are the most natural \((\max, +)\)-linear operators, though they do not exhaust all of them (see e.g. [Aki99, Kol92, LMS01, LS02] and the references therein for classical and recent results on “kernel type” representations). More insight on the notion of tropical singularity is given in the survey [RGST05] and in the monograph [IMS07].

2. Assumptions and preliminary results

Consider a countable set \( X \) (that is a finite or denumerable set), endowed with a distance \( d \), such that bounded sets are finite. For instance one can consider the set of natural numbers \( \mathbb{N} \) or of integer numbers \( \mathbb{Z} \), with the distance \( d(x, y) = |x - y| \), or the set \( \mathbb{Z}^k \) for some \( k \), with the distance \( d(x, y) = \|x - y\| \) where \( \|\cdot\| \) is any norm on \( \mathbb{R}^k \). The previous property of the distance \( d \) implies that it defines the discrete topology on \( X \), that is all subsets of \( X \) are open. In particular, the sets of finite, compact, and bounded subsets of \( X \) coincide. We shall denote them by \( K \).

If \((s_K)_{K \in K}\) is a net with values in the set \( \overline{\mathbb{R}} \) of extended real numbers, indexed by the compact sets of \( X \), we use the notation:

\[
\lim \inf_{K \in K} s_K := \sup_{K \in K} \inf_{K' \subseteq K, K' \succ K} s_{K'}.
\]

We define similarly \( \lim \sup_{K \in K} s_K \) and if both quantities coincide we denote them by \( \lim_{K \in K} s_K \), which we call the limit of \( s_K \) as \( K \) tends to \( X \).

Given a kernel on \( X \), \( b : X \times X \to \mathbb{R}_{\max} \), \( (x, y) \mapsto b_{xy} \), which may be thought of as the square countable matrix \( B = (b_{xy})_{x, y \in X} \in \mathbb{R}_{\max}^{X \times X} \), a possible generalisation of the optimal assignment problem from the finite to the countable state space case
would be to consider the problem

\[(2.1) \quad \text{find a bijection } F : X \to X \text{ maximising } \limsup_{K \in K} \sum_{x \in K} b_{xF(x)}, \]

or the similar problem obtained by replacing the limsup in (2.1) by a liminf. As the limsup in (2.1) may well be infinite, we shall rather use the following stronger definition:

**Definition 2.1.** A bijection $F : X \to X$ is a (global) **solution**, resp. a **strong solution**, of the assignment problem associated to the kernel $b : X \times X \to \mathbb{R}_{\text{max}}$ if

\[(2.2) \quad \liminf_{K \in K} \sum_{x \in K} \left( b_{xF(x)} - b_{xG(x)} \right) \geq 0, \]

resp. if

\[(2.3) \quad \liminf_{K \in K} \sum_{x \in K} \left( b_{xF(x)} - b_{xG(x)} \right) > 0, \]

for any other bijection $G : X \to X$.

If a strong solution exists, then it is obviously a unique solution to the assignment problem.

Given a kernel $b$, we define the Moreau conjugacy $B : \overline{\mathbb{R}}^X \to \overline{\mathbb{R}}^X$ which maps any function $f = (f_x)_{x \in X}$ to the function $Bf = (Bf)_x = \sup_y \left( b_{xy} - f_y \right)$ with the convention that $-\infty$ is absorbing for addition, i.e., $-\infty + \lambda = \lambda + (-\infty) = -\infty$, for all $\lambda \in \overline{\mathbb{R}}$. Here, and in the sequel, the supremum is understood over all the elements of $X$. Like in [AGK05] and mainly for the sake of symmetry, we work here with Moreau conjugacies (2.4) rather than with the max-plus linear maps discussed in the introduction.

We shall need the following assumptions on the kernel $b$:

- **(ZC)** For any $x \in X$, there exist $y, z \in X$ such that $b_{xy} \neq -\infty$ and $b_{xz} \neq -\infty$.
- **(TC)** $\sup\{b_{xy} \mid d(x, y) \geq n\}$ tends to $-\infty$ when $n$ goes to infinity.

Condition (ZC), which means that all the rows and columns of the matrix $B$ are non zero (in the max-plus sense), was already used in [AGK05]. Condition (TC) is a tightness condition. It implies in particular that all the rows and columns of $B$ are tight vectors or measures (related notions were defined and used in [AQV98] for a general topological space $X$, and in [AGW05] for a countable space $X$). Under Condition (ZC,TC), the Moreau conjugacy $B$ sends the set $\mathcal{B}(X)$ of real valued functions on $X$ that are bounded from below, to the set $\mathbb{R}^X$ of all real valued functions on $X$.

By $B^T$ and $b^T$, we shall denote the transpose matrix of $B$ and its kernel, $B^T = (b^T_{xy})_{x,y \in X}, b^T_{xy} = b_{yx}$. The corresponding Moreau conjugacy is then:

\[(B^T g)_y = \sup_x (b_{xy} - g_x). \]

The pair $(B, B^T)$ defines a Galois connection on $\overline{\mathbb{R}}^X$, which means in particular (see [AGK05]) that $B^T$ is a pseudo-inverse of $B$ in the sense that $B \circ B^T \circ B = B$ and $B^T \circ B \circ B^T = B^T$, hence if the equation $Bf = g$ with a given $g \in \overline{\mathbb{R}}^X$ has a solution $f \in \overline{\mathbb{R}}^X$, then necessarily $B^T g$ is also a solution of this equation.
The infinite dimensional theory depends crucially on the class of functions in which the solutions to the equation $Bf = g$ are sought and on the class of bijections for the solutions to the assignment problem. We first introduce some classes of bijections.

**Definition 2.2.** We define the *distance* between two bijections $F,G : X \to X$ as

$$\rho(F,G) = \sup_x d(F(x), G(x)) \in \mathbb{R} \cup \{+\infty\}.$$ 

A bijection $F : X \to X$ is *locally bounded* if it is at a finite distance from the identity map, $I : X \to X$, $x \mapsto x$.

The map $\rho$ satisfies all the properties of a distance except that $\rho(F,G)$ may be infinite. The binary relation defined as $F \sim G$ if $F$ and $G$ are at a finite distance ($\rho(F,G) < \infty$) is clearly an equivalence relation on the set of bijections of $X$, defining a partition of this set into classes. The set of locally bounded bijections is the class of the identity map.

**Property 2.3.** The set of locally bounded bijections $X \to X$ is a subgroup of the group of bijections of $X$.

**Proof.** This follows from $\rho(F^{-1}, I) = \rho(I, F)$ and $\rho(F \circ G, I) \leq \rho(F \circ G, G) + \rho(G, I) = \rho(F, I) + \rho(G, I)$.

**Definition 2.4.** We say that a bijection $F : X \to X$ is a *local solution* (resp. a *local strong solution*) of the assignment problem associated to $b$ if Condition (2.3) (resp. (2.3)) of Definition 2.1 holds for all $G$ within a finite distance from $F$.

Now we define some classes of functions. Recall that $\mathcal{B}(X)$ is the set of real valued functions on $X$, $s = (s_x)_{x \in X}$ that are bounded from below, that is inf$_x s_x > -\infty$. By $\ell_\infty = \ell_\infty(X)$, $\ell_1 = \ell_1(X)$, $\ell_0 = \ell_0(X)$, we shall denote the linear spaces (in the usual sense) of real valued functions on $X$, $s = (s_x)_{x \in X}$, such that respectively $\|s\|_\infty := \sup_x |s_x| < \infty$, $\|s\|_1 := \sum_x |s_x| < \infty$, or the limit $\lim_{x \to \infty} s_x$ exists and is finite. Here, the expression $x \to \infty$ refers to the filter of complements of finite sets of $X$. Equivalently, we may choose arbitrarily a basepoint $\bar{x} \in X$, and set $d(x) := d(x, \bar{x})$. Then, $\lim_{x \to \infty} s_x = a$ if and only if $s_x$ tends to $a$ as $d(x)$ tends to infinity. We shall also denote by $\ell_{0,1} = \ell_{0,1}(X)$ the linear space of functions $s = (s_x)_{x \in X} \in \ell_0$ such that for all $M > 0$, $\|s\|_{0,1,M} := \sup\{\sum_x |s_F(x) - s_x| \mid F : X \to X$, bijection s.t. $\rho(F, I) \leq M\} < +\infty$. This space can be thought of as the space of functions with $\ell_1$ “partial derivatives” and a limit at infinity. In particular when $X = \mathbb{Z}^k$, all semi-norms $\|s\|_{0,1,M}$ are equivalent to $\|s\|_{0,1} = \sum_{e \in E} \sum_x |s_{x+e} - s_x|$, where $E$ is the canonical basis of $\mathbb{R}^k$. For a general set $X$, in particular when $X$ is not included in a finite dimensional normed space (with the distance being defined from the norm), and when the cardinality of the balls of radius $M$ in $X$ is not uniformly bounded, one cannot find a finite set $E$ satisfying the above property, and one cannot replace the semi-norms $\|s\|_{0,1,M}$ by the following simpler ones $\|s\|_{0,1,M} = \sum_x \max_{\|y\| \leq M} |s_y - s_x|$. Indeed, with these semi-norms, it may happen that $\ell_1 \not\subset \ell_{0,1}$, whereas the inclusion holds with our definition of $\ell_{0,1}$, as stated below.

**Property 2.5.** We have $\ell_1(X) \subset \ell_{0,1}(X) \subset \ell_0(X) \subset \ell_\infty(X) \subset \mathcal{B}(X)$. 

PROOF. All these inclusions are clear, except perhaps the inclusion of $\ell_1$ in $\ell_{0,1}$ which follows from well known properties of series with positive terms: if $s \in \ell_1$ then $s$ tends to 0 at infinity and $\sum_x |s_{F(x)} - s_x| \leq \|s \circ F\|_1 + \|s\|_1 = 2\|s\|_1$.  

By $\ell_* = \ell_*(X)$ we shall denote any of the former spaces. They have the following good properties.

**Property 2.6.** For $\ell_*$ being either $\ell_1$, $\ell_0$ or $\ell_\infty$, the space $\ell_*(X)$ is invariant by any bijection $F : X \to X$, meaning that $\phi \circ F \in \ell_*(X)$ when $\phi \in \ell_*(X)$. The space $\ell_{0,1}(X)$ is invariant by any locally bounded bijection $X \to X$.

**Proof.** This is clear for $\ell_\infty$. For $\ell_1$, this follows from properties of series with positive terms. For $\ell_0$, this follows from the fact that, since the image by a bijection $F$ of any finite (compact) set of $X$ is finite, the set $K$ of finite sets is invariant by $F$: $F(K) = K$. For $\ell_{0,1}$, let $\phi = (\phi_x)_{x \in X} \in \ell_{0,1}(X)$ and $F : X \to X$ be a locally bounded bijection, and let us denote by $R = \rho(F,I)$. Since $\ell_0$ is invariant by any bijection, then $\phi \circ F \in \ell_0$. Now, for any $M > 0$, and any bijection $G : X \to X$ such that $\rho(G, I) \leq M$, we have $\rho(F \circ G, I) \leq M + R$, thus

$$
\sum_x |(\phi \circ F)_{G(x)} - (\phi \circ F)_x| \leq \sum_x |\phi_{F \circ G(x)} - \phi_x| + \sum_x |\phi_{F(x)} - \phi_x| 
\leq \|\phi\|_{0,1,M+R} + \|\phi\|_{0,1,R}
$$

hence $\|\phi \circ F\|_{0,1,M} \leq \|\phi\|_{0,1,M+R} + \|\phi\|_{0,1,R} < +\infty$, which shows that $\phi \circ F \in \ell_{0,1}$.

We shall consider the following classes of solutions to the assignment problem.

**Definition 2.7.** A bijection $F : X \to X$ is said to be a $\ell_*$-bijection, with respect to the kernel $b : X \times X \to \mathbb{R}$, if the sequence $(b_{x,F(x)})_{x \in X}$ belongs to $\ell_*(X)$. When in addition $F$ is a solution (in any sense) of the optimal assignment problem, we shall speak of $\ell_*$-solution.

**Remark 2.8.** In general, a solution of the optimal assignment problem associated to the kernel $b$ is necessarily a solution of Problem (2.1), but the converse implication may not be true, because the supremum of the expressions in (2.1) may be infinite. However, if $F$ is a $\ell_1$-bijection, then it is a (strong) solution of the optimal assignment problem associated to the kernel $b$ if and only if it is a (unique) solution of Problem (2.1).

**Definition 2.9.** A kernel $b$ or its corresponding Moreau conjugacy $B$ is said to be $\ell_*$-strongly regular if there exists $g \in \ell_*$ such that (i) $f := B^Tg \in \ell_*$, (ii) $f$ is the unique solution $h$ in $\ell_*$ of the equation $Bh = g$ and (iii) $g$ is the unique solution $h$ in $\ell_*$ of the equation $B^T h = f$. In this case, $g$ (resp. $f$) is said to belong to the $\ell_*$-simple image of $B$ (resp. $B^T$).

Of course it follows from this definition that $B$ is $\ell_*$-strongly regular if and only if $B^T$ is $\ell_*$-strongly regular.

**Remark 2.10.** One can show, see Remark 4.9, that in the case of a finite set $X$, our definition coincides with the standard definition of strong regularity given in the introduction and in [BH85]. In fact we added Condition (iii) in our definition, which turns out to be automatically fulfilled for finite sets $X$. 

DEFINITION 2.11. A matrix \( B = (b_{xy}) \in \mathbb{R}_{\max}^{X \times X} \) (or its kernel \( b \)) is \textit{normal} (resp. \textit{strongly normal}) if all its non-diagonal entries, \( b_{xy} \) with \( x, y \in X \) and \( x \neq y \), are non-positive (resp. negative), and if all its diagonal entries, \( b_{xx} \) for \( x \in X \), are equal to 0.

This definition is literally the same as the usual finite-dimensional one (see But00). The normal (resp. strongly normal) matrices present a class of examples, where the identity map is an obvious locally bounded \( \ell_\ast \)-solution (resp. strong solution) to the assignment problem. As our first result will show, this class of matrices present natural “normal forms” for strongly regular matrices.

DEFINITION 2.12. The kernels \( b, c : X \times X \to \mathbb{R}_{\max} \) are \( \ell_\ast \)-\textit{similar} if there exist two locally bounded bijections \( H, K : X \to X \) and two functions \( \phi \) and \( \psi \) from \( \ell_\ast(X) \) such that

\[
(2.5) \quad c_{xy} = b_{H(x)K(y)} - \phi_x - \psi_y .
\]

When \( H \) (resp. \( K \)) is the identity map, we say that \( b \) and \( c \) are \textit{right} (resp. \textit{left}) \( \ell_\ast \)-\textit{similar}.

When \( X \) is finite, we recover the standard definition (see e.g. But00). Indeed, matrices over the max-plus semiring are invertible if and only if they are the product of a permutation matrix and of a diagonal matrix with real diagonal entries. So, similarity coincides with the usual notion that \( C = PBP' \) for some invertible matrices \( P \) and \( P' \).

PROPERTY 2.13. The relations of (right, left) \( \ell_\ast \)-similarity are equivalence relations.

PROOF. We first consider the relation of \( \ell_\ast \)-similarity. This relation is reflexive since the identity map is locally bounded and the function 0 (identically equal to 0) is in \( \ell_\ast \).

To see that it is symmetric, let \( b \) and \( c \) be \( \ell_\ast \)-similar, that is satisfying (2.5) with locally bounded bijections \( H \) and \( K \), and \( \phi, \psi \in \ell_\ast(X) \). Then

\[
(2.6) \quad b_{xy} = c_{H^{-1}(x)K^{-1}(y)} + (\phi \circ H^{-1})_x + (\psi \circ K^{-1})_y ,
\]

and by Properties 2.0 and 2.3 \( H^{-1} \) and \( K^{-1} \) are locally bounded, and \( \phi \circ H^{-1} \) and \( \psi \circ K^{-1} \) are in \( \ell_\ast(X) \), which shows that \( c \) and \( b \) are \( \ell_\ast \)-similar.

Let us show that \( \ell_\ast \)-similarity is transitive. Assume that \( b \) and \( c \) are \( \ell_\ast \)-similar and that \( c \) and \( c' \) are also \( \ell_\ast \)-similar. This means that there exist locally bounded bijections \( H, K, H', K' \) and functions \( \phi, \psi, \phi', \psi' \in \ell_\ast(X) \) satisfying (2.5) and \( c'_{xy} = c_{H'(x)K'(y)} - \phi'_x - \psi'_y \). Hence \( c'_{xy} = b_{H \circ H'(x)K \circ K'(y)} - (\phi \circ H' + \phi')_x - (\psi \circ K' + \psi')_y \), and by Properties 2.0 and 2.3 and the linearity of \( \ell_\ast(X) \), we get that \( b \) and \( c' \) are \( \ell_\ast \)-similar.

The relations of right and left \( \ell_\ast \)-similarity are treated by requiring \( H, H' \) or \( K, K' \) to be the identity maps in the previous arguments. \qed

REMARK 2.14. In the finite dimensional case, linear programming (or network flow algorithms) yields an effective method to reduce a matrix to a normal matrix by similarity. Indeed, the optimal assignment problem over a finite state space can be formulated as a linear program, the dual of which can be written as

\[
\min_{\phi, \psi} \sum_x \phi_x + \psi_x, \quad \phi, \psi \in \mathbb{R}^X, \quad \phi_x + \psi_y \geq b_{xy}, \forall x, y .
\]
The dual program has an optimal solution \((\phi^*, \psi^*)\), except in the degenerate case in which the primal is not feasible (meaning that there is no permutation \(F\) such that \(b_{xF(x)} > -\infty\) for all \(x\)). By complementary slackness, a permutation \(F\) is optimal if and only if the equality \(\phi^*_x + \psi^*_y = b_{xy}\) holds whenever \(y = F(x)\). It follows that the matrix \(b_{xF(y)} - \phi_x - \psi_{F(y)}\), which is similar to \(b\), is normal.

The importance of the notion of \(\ell_1\)-similarity is basically due to the following results, which are countable analogues to Propositions 3 and 4 in [BH85].

**Proposition 2.15.** Conditions (ZC), (TC) and \(\ell_1\)-strong regularity are each invariant under \(\ell_\infty\)-similarity.

**Proof.** Let \(b\) and \(c\) be \(\ell_1\)-similar kernels on \(X\), thus satisfying (2.5) with locally bounded bijections \(H, K : X \to X\) and \(\phi, \psi \in \ell_1(X)\).

Since \(\ell_1(X) \subset \mathbb{R}^X\), \(b\) satisfies (ZC) if and only if \(c\) does. Moreover, since \(H\) and \(K\) are locally bounded, we get for all \(x, y \in X\):

\[
d(x, y) - \rho(H, I) - \rho(K, I) \leq d(H(x), K(y)) \leq d(x, y) + \rho(H, I) + \rho(K, I),
\]

hence \(d(x, y) \to \infty\) if and only if \(d(H(x), K(y)) \to \infty\), and since \(\phi, \psi \in \ell_\infty \subset \ell_1\), we deduce that \(b\) satisfies (TC) if and only if \(c\) does.

The invariance of \(\ell_1\)-strong regularity follows from the observation that

\[
g = Cf \iff (g + \phi) \circ H^{-1} = B((f + \psi) \circ K^{-1}),
\]

and so \(g = Bf \iff g \circ H - \phi = C(f \circ K - \psi)\). Indeed, let \(b, f, g\) satisfy the properties of Definition 2.9. Then, \(f' = g \circ H - \phi \in \ell_1\), and since \(f' = C f' \circ H^{-1}\), and the last term in the previous equation is equal to \(B^T g = f\), we get that \(f' = f \circ K - \psi \in \ell_\infty\), which shows Property (i) of Definition 2.9 for \(c, f', g'\) instead of \(b, f, g\). Moreover, we have \(C h' = g'\) if and only if \(B h = g\) for \(h = (h' + \psi) \circ K^{-1}\), and since \(h \in \ell_1\), if and only if \(h' \in \ell_1\), we get that Property (ii) of Definition 2.9 for \(c, f', g'\) is equivalent to the same property for \(b, f, g\). By symmetry, the same occurs for Property (iii) of Definition 2.9. \(\square\)

**Proposition 2.16.** The property for a kernel to have a solution or a local solution to the assignment problem is invariant under \(\ell_1\)-similarity. The same is true if the solution is required in addition to be locally bounded, strong, or either a \(\ell_1\), \(\ell_0\) or \(\ell_\infty\)-bijection, or a locally bounded \(\ell_{0,1}\)-bijection, with respect to the kernel.

**Proof.** Let \(b\) and \(c\) be \(\ell_1\)-similar kernels on \(X\), thus satisfying (2.5) with locally bounded bijections \(H, K : X \to X\) and \(\phi, \psi \in \ell_1(X)\). Let \(F, G : X \to X\) be two bijections. We have for any \(K \in \mathcal{K}\),

\[
(2.7) \quad \sum_{x \in K} (c_{xF(x)} - c_{xG(x)}) = \sum_{y \in H(K)} \left( b_{yK \circ F \circ H^{-1}(y)} - b_{yK \circ G \circ H^{-1}(y)} \right) + \sum_{x \in K} \left( (\psi \circ G)_x - (\psi \circ F)_x \right).
\]

Since \(\psi \in \ell_1\), the limit \(\lim_{K \in \mathcal{K}} \sum_{x \in K} \psi_x\) exists. Moreover, since \(F(\mathcal{K}) = \mathcal{K}\), we get

\[
\lim_{K \in \mathcal{K}} \sum_{x \in K} (\psi \circ F)_x = \lim_{K \in \mathcal{K}} \sum_{x \in F(K)} \psi_x = \lim_{K \in \mathcal{K}} \sum_{x \in K} \psi_x,
\]

which implies that \(\lim_{K \in \mathcal{K}} \sum_{x \in K} ((\psi \circ G)_x - (\psi \circ F)_x) = 0\). Using this and \(H(\mathcal{K}) = \mathcal{K}\) in (2.7), we deduce

\[
\liminf_{K \in \mathcal{K}} \sum_{x \in K} (c_{xF(x)} - c_{xG(x)}) = \liminf_{K \in \mathcal{K}} \sum_{y \in K} \left( b_{yK \circ F \circ H^{-1}(y)} - b_{yK \circ G \circ H^{-1}(y)} \right).
\]
Since the map $T(G) := K \circ G \circ H^{-1}$ is a bijective transformation from the set of bijections $X \to X$ to itself, we deduce from the latter relation that $F$ is a solution (resp. a strong solution) of the assignment problem associated to the kernel $c$ if and only if $K \circ F \circ H^{-1}$ is a solution (resp. a strong solution) of the assignment problem associated to the kernel $b$. Since $K$ is locally bounded, the map $T$ is such that $G \sim G' \implies T(G) \sim T(G')$ (recall that $G \sim G'$ iff $\rho(G, G') < \infty$). Since $H$ is also locally bounded, $T$ is a bijective transformation from the set of locally bounded bijections to itself. Hence, a solution (or strong solution, etc.) $F$ for $c$ is locally bounded if and only if the corresponding solution $K \circ F \circ H^{-1}$ for $b$ is locally bounded. Moreover, we also deduce that $F$ is a local solution (resp. a local strong solution) of the assignment problem associated to the kernel $c$ if and only if $K \circ F \circ H^{-1}$ is a local solution (resp. a local strong solution) of the assignment problem associated to the kernel $b$.

Finally, assume that $F$ is a $\ell_\ast$-solution for some space $\ell_\ast$ (which may be different from $\ell_1$), that is $(\varepsilon_{x,F(x)})_{x \in X} \in \ell_\ast(X)$. Composing this sequence with $H^{-1}$, we get that $(\varepsilon_{\ell^{-1}(x),F \circ H^{-1}(x)})_{x \in X} \in \ell_\ast(X)$. Now by (2.5), we get that $b_{K \circ F \circ H^{-1}(x)} = c_{H^{-1}(x)} \cdot (\varepsilon_{\ell^{-1}(x),F \circ H^{-1}(x)} + (\phi \circ H^{-1})_x + (\psi \circ F \circ H^{-1})_x)$ and since $\phi, \psi \in \ell_1(X) \subset \ell_\ast(X)$, we deduce that $(b_{K \circ F \circ H^{-1}(x)})_{x \in X} \in \ell_\ast(X)$ if $\ell_\ast$ is either $\ell_0$, $\ell_\infty$ or $\ell_1$. By symmetry, we have shown, in this case, that $F$ is an $\ell_\ast$-solution of the assignment problem associated to the kernel $c$ if and only if $K \circ F \circ H^{-1}$ is an $\ell_\ast$-solution of the assignment problem associated to the kernel $b$. When $\ell_\ast = \ell_{0,1}$, we need to restrict solutions to be locally bounded.

**Proposition 2.17.** The property for a kernel to have a local solution to the assignment problem is invariant under $\ell_{0,1}$-similarity. The same is true if the solution is required in addition to be locally bounded, strong, or either a $\ell_0$ or $\ell_\infty$-bijection, or a locally bounded $\ell_{0,1}$-bijection, with respect to the kernel.

**Proof.** In view of the arguments of the proof of Proposition 2.16 it is enough to show that

\begin{equation}
(2.8) \quad s_K = \sum_{x \in K} ((\psi \circ G)_x - (\psi \circ F)_x)
\end{equation}

has a zero limit, $\lim_{K \in \mathcal{K}} s_K = 0$, whenever $F$ and $G$ are bijections $X \to X$ that are at a finite distance from each other, and $\psi \in \ell_{0,1}$. Since $\ell_{0,1} \subset \ell_0$, any constant function is in $\ell_{0,1}$, and $s_K$ is invariant when adding a constant to $\psi$, it suffices to consider the case of functions $\psi \in \ell_{0,1}$ such that $\lim_{x \to \infty} \psi_x = 0$. Moreover, since $s_K = \sum_{x \in F(K)} ((\psi \circ G \circ F^{-1})_x - \psi_x)$, $F(K) = \mathcal{K}$ and $\rho(G \circ F^{-1}, I) \leq \rho(G, F) < +\infty$, we may assume that $F = I$ and that $G$ is locally bounded.

Let $M = \rho(G, I) < +\infty$, we get that

\begin{equation}
(2.9) \quad \sum_{x \in X} |(\psi \circ G)_x - \psi_x| \leq \|\psi\|_{0,1,M} < +\infty
\end{equation}

since $\psi \in \ell_{0,1}$. Hence, the sequence $((\psi \circ G)_x - \psi_x)_{x \in X}$ is in $\ell_1$ which implies that $s_K$ is bounded, and, by properties of series with positive terms, we get that

\begin{equation}
(2.10) \quad \limsup_{K \in \mathcal{K}} \sum_{x \notin K} |(\psi \circ G)_x - \psi_x| = \inf_{K \in \mathcal{K}} \sum_{x \notin K} |(\psi \circ G)_x - \psi_x| = 0.
\end{equation}
Hence $s_K$ has a limit. Indeed, for any finite subsets $K_1$ and $K_2$ of $X$, we have
\[
|s_{K_1} - s_{K_2}| \leq |s_{K_1 \cap K_2}| + |s_{K_2 \cap K_1}| \leq 2 \sum_{x \notin (K_1 \cap K_2)} |(\psi \circ G)_x - \psi_x| ,
\]
which implies that
\[
0 \leq \limsup_{K \in \mathcal{K}} s_K - \liminf_{K \in \mathcal{K}} s_K = \inf_{K_1, K_2 \in \mathcal{K}, K_1 \supset K_2} \sup_{K_2 \supseteq K} s_{K_1} - s_{K_2} \leq 2 \sup_{K_1, K_2 \in \mathcal{K}, x \notin (K_1 \cap K_2)} |(\psi \circ G)_x - \psi_x| = 0 .
\]

To show that $s_K$ has a zero limit, it is thus sufficient to prove that $\liminf_{K \in \mathcal{K}} |s_K| = 0$. Since this property means that for all finite sets $K$, $\inf_{K' \supseteq K} |s_{K'}| = 0$, it will hold as soon as for any finite set $K$, there exists a sequence of finite sets $(K_n)_{n \geq 0}$ containing $K$ such that $\lim_{n \to \infty} s_{K_n} = 0$.

Let us show this last property. Consider the sequence $K_n$ such that $K_0 = K$ and $K_{n+1} = K_n \cup G(K_n) \cup G^{-1}(K_n)$ for $n \geq 0$. Then $K_n$ is nondecreasing, and it satisfies $K_n \supseteq K$, $G(K_n) \subset K_{n+1}$ and $G^{-1}(K_n) \subset K_{n+1}$. We have
\[
s_{K_n} = \sum_{x \in G(K_n)} \psi_x - \sum_{x \in K_n} \psi_x = \sum_{x \in G(K_n) \setminus K_n} \psi_x - \sum_{x \in K_n \setminus G(K_n)} \psi_x = \sum_{x \in G(K_n) \setminus K_n} \psi_x - \sum_{x \in G^{-1}(K_n) \setminus K_n} \psi(x) .
\]

Since $G(K_n) \setminus K_n \subset K_{n+1} \setminus K_n$, $G(K_n) \setminus K_n \subset G^{n+1}(K)$, hence its cardinality is less or equal to the cardinality $\# K$ of $K$, and the same is true for $G^{-1}(K_n) \setminus K_n$, we obtain
\[
|s_{K_n}| \leq \# K \left( \max_{x \in K_{n+1} \setminus K_n} |\psi_x| + \max_{x \in K_{n+1} \setminus K_n} |(\psi \circ G)_x| \right) .
\]

Now the sets $K_{n+1} \setminus K_n$ are disjoint. If $K_{n+1} \setminus K_n = \emptyset$ for some $n \geq 0$, then $K_{n+1} = K_n$, and by construction $K_m = K_n$, hence $K_{m+1} \setminus K_m = \emptyset$ for all $m \geq n$. This implies that $|s_{K_n}| = 0$ for all $n \geq m$, hence the sequence $(s_{K_n})_{n \geq 0}$ converges trivially to 0. Otherwise, if all the sets $K_{n+1} \setminus K_n$ are nonempty, one can show, using the fact that they are all disjoint, that for all finite sets $K'$, $K_{n+1} \setminus K_n \subset X \setminus K'$ for $n$ large enough. Since $\lim_{x \to \infty} \psi_x = 0$, we deduce that $\max_{x \in K_{n+1} \setminus K_n} |\psi_x|$ tends to 0. Since the same is true for $\psi \circ G$ instead of $\psi$, Inequality (2.11) implies that the sequence $(s_{K_n})_{n \geq 0}$ converges to 0. This concludes the proof.

From the previous proof, it seems that with Definition 2.1 of a solution to the assignment problem, the invariance by similarities fails under weaker assumptions on similarities, in particular for $\ell_0$ and $\ell_\infty$-similarities. This may hold however if we weaken also the definition of a solution to the assignment problem as follows. In the sequel, we fix a base point $x$ and denote by $B_n$ the ball of centre $x$ and radius $n$ in $X$.

**Definition 2.18.** A bijection $F : X \to X$ is a (global) restricted solution, resp. a strong restricted solution, of the assignment problem associated to the kernel $b : X \times X \to \mathbb{R}_{\max}$ if
\[
\liminf_{n \to \infty} \sum_{x \in B_n} (b_{xF(x)} - b_{xG(x)}) \geq 0 ,
\]
resp. if
\begin{equation}
\liminf_{n \to \infty} \sum_{x \in B_n} (b_x F(x) - b_x G(x)) > 0,
\end{equation}
for any other bijection $G : X \to X$. We say that $F$ is a local (resp. local strong) restricted solution, if (2.12) (resp. (2.13)) holds for all $G$ within a finite distance from $F$.

With this definition, we cannot change the "order" of rows of a matrix, that is we need to consider right-similarities only. From the same arguments as in the proofs of Propositions 2.16 and 2.17 we get that

**Proposition 2.19.** The conclusions of Propositions 2.16 and 2.17 hold true if we replace “solutions” by “restricted solutions” and “similarities” by “right-similarities” in their statements.

Moreover, we can consider $\ell_0$-right-similarities.

**Proposition 2.20.** Assume that $\#B_n - \#B_{n-1}$ is bounded. Then, the property for a kernel to have a locally bounded local restricted solution to the assignment problem is invariant under $\ell_0$-right-similarity. The same is true if the solution is required in addition to be strong, or either a $\ell_0$ or $\ell_\infty$-bijection, with respect to the kernel.

**Proof.** In view of the arguments of the proof of Propositions 2.16 and 2.17 it is enough to show that $s_{B_n}$, defined by (2.18), converges to 0 when $n$ goes to infinity, whenever $F$ and $G$ are locally bounded bijections $X \to X$, and $\psi \in \ell_0$ has a zero limit. Moreover, taking the difference of $\psi_G(x)$ and $\psi_F(x)$ with $\psi_x$ in the expression of $s_{B_n}$, we may assume that $F = I$. Then by the same arguments as in the proof of Proposition 2.17 we get that $s_{B_n} = \sum_{x \in G(B_n) \setminus B_n} \psi_x - \sum_{x \in G^{-1}(B_n) \setminus B_n} \psi_G(x)$. Since $R := \rho(G, I) < +\infty$, we get that $G(B_n) \subset B_{n+R}$ and $G^{-1}(B_n) \subset B_{n+R}$, and by the assumption on the cardinality of $B_n$, we get that the cardinality of $G(B_n) \setminus B_n$ is bounded by some constant $M$. Hence
\begin{equation}
|s_{B_n}| \leq M \left( \max_{x \in B_{n+R} \setminus B_n} |\psi_x| + \max_{x \in B_{n+R} \setminus B_n} |(\psi \circ G)_x| \right).
\end{equation}
Since $\psi_x$ and $\psi_G(x)$ tend to 0 when $x \to \infty$, the r.h.s. of (2.14) tends to 0, which implies that the sequence $(s_{B_n})_{n \geq 0}$ converges to 0. This concludes the proof. \qed

**3. Main results**

In this section, we state the main results, which we prove in Sections 4 and 5.

**Theorem 3.1.** A kernel satisfying (ZC,TC) is $\ell_\times$-strongly regular if and only if it is $\ell_\times$-similar to a strongly normal kernel or if and only if it is $\ell_\times$-right (resp. left)-similar to a strongly normal kernel.

The following counter-example shows that the tightness condition (TC) is useful in the previous result.

**Example 3.2.** Consider $X = \mathbb{N}$ and $b_{xy} = -1/|x - y|$ for $x \neq y$ and $b_{xx} = 0$. The kernel $b$ is clearly strongly normal. It satisfies Condition (ZC), but not Condition (TC). Let $f, g \in \ell_1(X)$ be such that $Bf = g$ and $B^T g = f$. We get that $g_x \geq \lim_{y \to \infty} b_{xy} - f_y = 0$ and symmetrically $f_y \geq 0$. This implies that
The assignment problem for general measurable, (uncountable) state space analysis of the Monge-Kantorovich mass transfer problem, a natural analog of the is

Theorem 3.1 shows in particular that a \( \ell_* \)-strongly regular kernel satisfying (ZC, TC) is \( \ell_* \)-similar to a kernel having a strong solution to the assignment problem. But it is of course interesting to know what can be said about the assignment problem for the regular kernel itself. In the analysis of this question (as well as the inverse one), an important role is played by the following construction.

If \( c : (x, y) \in X \times X \mapsto c_{xy} \in \mathbb{R}_{\max} \) is a kernel, we define the kernel \( c^+ : (x, y) \in X \times X \mapsto c^+_{xy} \in \mathbb{R} \),

\[
(3.1) \quad c^+_{xy} = \sup_{x_0, x_1, \ldots, x_n} c_{x_0x_1} + \cdots + c_{x_{n-1}x_n},
\]

where the sup is taken over \( n \geq 1 \) and over all the sequences \( x_0, x_1, \ldots, x_n \) of elements of \( X \) such that \( x_0 = x \) and \( x_n = y \). The sum \( c_{x_0x_1} + \cdots + c_{x_{n-1}x_n} \) is the weight of the sequence \( x_0, \ldots, x_n \), so that \( c^+_{xy} \) represents the maximal weight of a path of positive length from \( x \) to \( y \).

The sequence \( x_0, \ldots, x_n \) is said to be a circuit if \( x_0 = x_n \). If every circuit has a nonpositive weight, the supremum in (3.1) does not change if one restricts it to those sequences such that the elements \( x_1, \ldots, x_{n-1} \) are pairwise distinct and are distinct from \( x_0 \) and \( x_n \). Note however that unlike in the case in which \( X \) is finite, the fact that every circuit has a nonpositive weight does not imply that \( c^+_{xy} < \infty \) for all \( x, y \in X \), although this turns out to be automatically the case when \( c \) is irreducible, meaning that \( c^+_{xy} > -\infty \) for all \( x, y \in X \), see [AGW05] for more details.

It follows readily from the definition that \( c^+_{xy} \geq c^+_x + c^+_y \). Let us now consider the vector \( f_x = c^+_x \), for some arbitrary \( y \in X \). We deduce from the previous inequality that \( f_x \geq \sup_z (c_z + f_x) \). Moreover, when \( c^+_{yy} \geq 0 \), and a fortiori when \( c^+_{yy} \geq 0 \), it can be checked that the equality holds, for all \( x \in X \) (see for instance [AGW05]).

We shall now apply this construction to the kernel \( c = \tilde{b} \) where

\[
(3.2) \quad \tilde{b}_{xy} = b_{xF(y)} - b_{yF(y)},
\]

and \( F \) is a (possibly local) solution of the assignment problem associated to a kernel \( b \). The kernel \( \tilde{b}^+ \) is obtained by taking \( c = \tilde{b} \) in Equation (3.1). Observe that \( \tilde{b}_{xx} = 0 \) and that the weight of any circuit, with respect to \( \tilde{b} \), is non positive.

As was observed in [Rüis96], the functions \( \tilde{b}^+_{xy} \) turn out to be useful also in the analysis of the Monge-Kantorovich mass transfer problem, a natural analog of the assignment problem for general measurable, (uncountable) state space \( X \).

Define the potential and the inverse potential as the functions on \( X \) given respectively by

\[
(3.3) \quad \tilde{\phi}_x = \sup_y \tilde{b}^+_{xy} \in \mathbb{R} \cup \{+\infty\}, \quad \tilde{\psi}_y = \sup_x \tilde{b}^+_{xy} \in \mathbb{R} \cup \{+\infty\}.
\]

The following simple properties of these functions are crucial:

(i) \( \tilde{b}^+_{xx}, \tilde{\phi}_x \) and \( \tilde{\psi}_y \) are nonnegative for all \( x \) and \( y \) (in fact, take \( n = 1 \) in (3.1)).
(ii) the function \( f = \tilde{\phi} \) satisfies the equation
\[
(3.4) \quad f_x = \sup_y \left( \tilde{b}_{xy} + f_y \right), \quad \forall x \in X.
\]

Similarly, the function \( g = \tilde{\psi} \) satisfies the equation
\[
(3.5) \quad g_y = \sup_x \left( \tilde{b}_{xy} + g_x \right).
\]

Moreover, if \( \tilde{\psi} \) (resp. \( \tilde{\phi} \)) is finite, the function \( -\tilde{\psi} \) (resp. \(-\tilde{\phi}\)) also satisfies (3.4) (resp. (3.5)). Observe that Equation (3.4) can be equivalently written as
\[
(3.6) \quad f = B\psi, \quad \psi_y = b_{F^{-1}(y)\cup F^{-1}(y)} \quad \forall y \in X.
\]

(iii) The function \( f = \tilde{\phi} \) and, if \( \tilde{\psi} \) is finite, the function \( f = -\tilde{\psi} \) satisfy the equation
\[
(3.7) \quad f_x = \sup_y \left( \tilde{b}_{xy}^+ + f_y \right) \quad \forall x \in X.
\]

**Remark 3.3.** When \( b \) is a normal kernel, taking \( F \) to be the identity in the definition of the kernel \( \tilde{b} \), we get \( \tilde{b} = b \), \( \tilde{b}^+ \leq 0 \) and \( \tilde{\phi} = 0 \) for all \( x, y \).

**Theorem 3.4.** (i) If a kernel \( b \) satisfying (ZC,TC) is \( \ell_{0,1} \)-strongly regular, then it has a locally bounded strong local \( \ell_{0,1} \)-solution to its assignment problem. Moreover, if \( F \) denotes this (necessarily unique) solution, and if \( b \) is defined from \( F \) by (3.2), the kernel \( \tilde{b}^+ \) satisfies:
\[
(3.8) \quad \limsup_{x,y\to\infty} \tilde{b}_{xy}^+ \leq 0
\]
and the potentials \( \tilde{\phi} \) and \( \tilde{\psi} \) (defined in (3.3)) are bounded functions.

(ii) If \( b \) is \( \ell_1 \)-strongly regular, then \( F \) is also a global strong \( \ell_1 \)-solution to the assignment problem associated to \( b \).

(iii) Under the assumption that \( \#B_n - \#B_{n-1} \) is bounded, if \( b \) is \( \ell_0 \)-strongly regular, then it has a locally bounded strong local \( \ell_0 \)-solution \( F \) to the assignment problem associated to \( b \), and the kernel \( \tilde{b}^+ \) and potentials \( \tilde{\phi} \) and \( \tilde{\psi} \) satisfy the properties of Point (i).

In order to prove a converse to Theorem 3.4 we shall need the following additional technical assumption on a solution to the assignment problem:

(PC-\( \ell_\ast \)) Either the potential \( \tilde{\phi} \) or the inverse potential \( \tilde{\psi} \) associated to \( b \) and \( F \) belongs to \( \ell_\ast(X) \).

**Theorem 3.5.** Let \( b : X \times X \to \mathbb{R}_{\text{max}} \) be a kernel satisfying (ZC,TC). If \( \ell_\ast \) is either \( \ell_{0,1} \) or \( \ell_1 \), and if the assignment problem associated to \( b \) has a (possibly local) locally bounded strong \( \ell_\ast \)-solution \( F \) satisfying Condition (PC-\( \ell_\ast \)), then \( b \) is \( \ell_\ast \)-strongly regular. If \( \ell_\ast \) is \( \ell_0 \), \( \#B_n - \#B_{n-1} \) is bounded, and if the assignment problem associated to \( b \) has a local locally bounded strong restricted \( \ell_\ast \)-solution \( F \) satisfying Condition (PC-\( \ell_\ast \)), then \( b \) is \( \ell_\ast \)-strongly regular.

**Remark 3.6.** We have to stress an unpleasant small gap between necessary and sufficient conditions: from strong \( \ell_{0,1} \)-regularity it follows that the potential \( \tilde{\phi} \) belongs to \( \ell_\infty \), but in Theorem 3.5 we assume that \( \tilde{\phi} \in \ell_{0,1} \) (which implies (3.8)). However, when considering classes of similar kernels this discrepancy vanishes, as shown by the following direct corollary of Theorem 3.4 and Remark 3.3.
Corollary 3.7. A kernel $b$, satisfying $(ZC, TC)$, is $\ell_*$-strongly regular if and only if it is $\ell_*$-similar to a kernel having a strong solution to the assignment problem satisfying condition $(PC, \ell_*)$.

In the case of a finite set $X$, the technical assumptions in Theorems 3.1 and 3.4 vanish, and we recover the result of Butkovic and Hevery showing that strong regularity is equivalent to the uniqueness of the optimal assignment problem. This result was established in BH85. Theorems 1 and 3] in which the authors considered more generally matrices with entries in a dense commutative idempotent semiring.

4. Coverings and subdifferentials. Proofs of Theorems 3.1 and 3.4

For the analysis of the equation $Bf = g$ (also in a more general setting of uncountable $X$) an important role belongs to the notion of generalised subdifferentials (see for instance ML88, MLS95, AGK05).

Definition 4.1. Let $b : X \times X \to \mathbb{R}_{\text{max}}$ be a kernel and $B$ its associated Moreau conjugacy. Given $f \in \mathbb{R}^X$ and $y \in X$, the subdifferential of $f$ at $y$ with respect $b$ or $B$, denoted $\partial_b f(y)$ or $\partial f(y)$ for brevity is defined as

$$\partial f(y) = \{ x \in X \mid b_{xy} \neq -\infty, \ (Bf)_x = \sup_z (b_{xz} - f_z) = b_{xy} - f_y \}.$$ 

The subdifferential $\partial_b T g(x)$ of $g \in \mathbb{R}^X$ at $x \in X$ with respect to $b^T$ will be denoted by $\partial^T g(x)$ for brevity:

$$\partial^T g(x) = \{ y \in X \mid b_{xy} \neq -\infty, \ (B^T g)_y = \sup_z (b_{zy} - g_z) = b_{xy} - g_x \}.$$ 

Remark 4.2. In the finite dimensional case, if $f, g$ are obtained from optimal dual solutions of the optimal assignment problem (Remark 2.1), every optimal permutation is obtained by selecting precisely one element $F(x)$ in each $\partial^T g(x)$ (in such a way that the same element is never selected twice). A symmetrical interpretation holds with $\partial f(y)$ and the inverse optimal permutation $F^{-1}$.

For a given $f$ the subdifferential is a mapping from $X$ to the set $\mathcal{P}(X)$ of subsets of $X$. For any such mapping $G$, the inverse mapping $G^{-1} : X \mapsto \mathcal{P}(X)$ is defined as $G^{-1}(y) := \{ x \mid y \in G(x) \}$ for $y \in X$. If $Y, Z \subset X$, we say that the family of subsets $\{ G(y) \}_{y \in Y}$ is a covering of $Z$ if $Z \subset \bigcup_{y \in Y} G(y)$.

We shall start with the following well known basic property of subdifferentials that we prove here for the sake of completeness.

Proposition 4.3. If $g = B^T T g$, then $(\partial^T g)^{-1} = \partial B^T g$.

Proof. We have $(\partial^T g)^{-1}(y) = \{ x \mid b_{x,y} \neq -\infty, \ (B^T g)_y = \sup_z (b_{zy} - g_z) = b_{xy} - g_x \}$. The latter relation can be rewritten as $g_x = b_{xy} - (B^T g)_y$, or equivalently $B(B^T g)_x = b_{xy} - (B^T g)_y$, which means that $x \in \partial (B^T g)(y)$. \qed

When $X$ is finite, the following result is due to Vorobyev [Vor67], see also Zimmermann [Zim76, Chapter 3]. In [AGK05, Theorem 3.5], we proved a more general result which applies to the case of a general topological space $X$.

Proposition 4.4. Suppose that $b$ satisfies Conditions $(ZC, TC)$ and that $g \in \mathbb{R}^X$ is such that $B^T g \in \mathcal{B}(X)$. Then $B^T g$ is a solution to the equation $Bf = g$ if and only if $\partial^T g(x) \neq \emptyset$ for all $x$ or equivalently if the family of the subsets $\{(\partial^T g)^{-1}(y)\}_{y \in X}$ is a covering of $X$. 

PROOF. This follows readily from Theorem 3.5 from [AGK05]. We only have to observe that the assumption that \( f = B^T g \in \mathcal{B}(X) \) together with Condition (TC) ensure that the set \( \{ y : b_{xy} - f_y \geq \beta \} \) is finite for any \( x \in X \) and \( \beta \in \mathbb{R} \), which is the crucial condition for the applicability of this theorem. \( \square \)

**Definition 4.5.** Let \( G \) be a mapping from \( X \) to the set of its subsets \( \mathcal{P}(X) \) and let the family of subsets \( \{ G(y) \}_{y \in Y} \) be a covering of \( Z \) with \( Y, Z \subset X \). An element \( y \in Y \) is called **essential** (with respect to this covering) if \( \cup_{z \in Y \setminus y} G(z) \supsetneq Z \). The covering is called **minimal** if all elements of \( Y \) are essential.

When \( X \) is finite, the following result reduces to Vorobiev [Vor67] Theorem 2.6], see also Zimmermann [Zim76, Chapter 3]. In [AGK05] Theorem 4.7], we proved a more general result which applies to the case of a general topological space \( X \), but when \( \mathcal{E} = \mathbb{R}^X \) only.

**Proposition 4.6.** Assume that \( b \) satisfies Conditions (ZC,TC) and that \( g \in \mathbb{R}^X \) is such that \( B^T g \in \mathcal{E} \), where \( \mathcal{E} \) is a linear subspace of \( \mathcal{B}(X) \) containing all the maps \( \delta_y : X \to \mathbb{R} \) such that \( \delta_y(x) = 1 \) if \( x = y \) and \( \delta_y(x) = 0 \) otherwise.

Then \( B^T g \) is the unique solution \( f \in \mathcal{E} \) of the equation \( Bf = g \) if and only if \( \{(\partial^T g)^{-1}(y)\}_{y \in X} \) is a minimal covering of \( X \).

**Proof.** If \( \mathcal{E} \) were replaced by \( \mathbb{R}^X \) in the statement of the proposition while keeping the condition that \( B^T g \in \mathcal{B}(X) \), this would be a consequence of Theorem 4.7 from [AGK05]. This shows in particular the “if” part of the proposition for all subspaces \( \mathcal{E} \).

Let us prove the “only if” by adapting the proof of [AGK05] Theorem 4.7]. Assume that \( g \in \mathbb{R}^X \) is such that \( B^T g \in \mathcal{E} \) is the unique solution \( f \in \mathcal{E} \) of the equation \( Bf = g \). By Proposition 4.4, the family of subsets \( \{(\partial^T g)^{-1}(y)\}_{y \in X} \) is a covering of \( X \). Assume by contradiction that this covering is not minimal, i.e., that there exists \( y_0 \in X \) such that for all \( x \in X \), there exists \( y \in X \setminus y_0 \) such that \( x \in (\partial^T g)^{-1}(y) \). This implies that \( g_x = b_{xy} - (B^T g)_y \), and since \( g \geq BB^T g \), we get:

\[
(4.1) \quad g_x = \sup_{y \in X \setminus y_0} b_{xy} - (B^T g)_y \quad \forall x \in X.
\]

Consider \( f = B^T g + \delta_{y_0} \). Since \( B^T g \in \mathcal{E} \subset \mathbb{R}^X \) and \( \delta_{y_0} \in \mathcal{E} \) and \( c_{y_0} \neq 0 \), we obtain that \( f \in \mathcal{E} \) and that \( f \neq B^T g \). Since \( f \geq B^T g \), we get that \( Bf \leq BB^T g = g \). Moreover, from (4.1), we deduce the reverse inequality \( Bf \geq g \), hence \( f \) is a solution of \( Bf = g \), and we get a contradiction. This concludes the proof. \( \square \)

Proposition 4.6 can be applied in particular to \( \mathcal{E} = \mathcal{B}(X) \) or to \( \mathcal{E} = \ell_* (X) \).

The key point in proving Theorems 3.4 and 3.5 is contained in the following statement.

**Proposition 4.7.** Let \( \mathcal{E} \) be as in Proposition 4.6. Suppose \( g \) and \( B^T g \) belong to \( \mathcal{E} \) and are such that \( f = B^T g \) is the unique solution \( h \in \mathcal{E} \) to the equation \( Bh = g \) and \( g \) is the unique solution \( h \in \mathcal{E} \) to the equation \( B^T h = f \). Then there exists a locally bounded bijection \( F : X \to X \) such that

\[
(4.2) \quad y = F(x) \iff \partial f(y) = \{x\} \iff \partial^T g(x) = \{y\}.
\]
In particular

\begin{align}
(4.3a) & \quad b_{xF(x)} = g_x + f_{F(x)}. \\
(4.3b) & \quad \forall z \neq F(x) \quad g_x > b_{xz} - f_x, \quad \forall z \neq x \quad f_{F(x)} > b_{zF(x)} - g_z.
\end{align}

Remark 4.8. As one easily checks, the inverse statement holds as well: if a locally bounded bijection \( F \) and if the functions \( f, g \in \mathcal{E} \) satisfy \((4.2)\), then \( f = B^T g \) is the unique solution \( h \in \mathcal{E} \) to equation \( Bh = g \) and \( g \) is the unique solution \( h \in \mathcal{E} \) to the equation \( B^T h = f \).

Proof of Proposition 4.7. Applying Proposition 4.6 to the equation \( B^T h = f \) one concludes that for all \( x \) there exists \( y \) such that \( y \in (\partial f)^{-1}(x) \), but \( y \notin (\partial f)^{-1}(z) \) for any \( z \neq x \). In other words \( (\partial f)(y) = \{ x \} \), which by Proposition 4.3 means that \( (\partial^T g)^{-1}(y) = \{ x \} \). Hence, defining the mapping \( F : X \to \mathcal{P}(X) \) by the formula

\[ F(x) = \{ y \mid (\partial f)(y) = \{ x \} \} = \{ y \mid (\partial^T g)^{-1}(y) = \{ x \} \} \subset \partial^T g(x), \]

we deduce that \( F(x) \neq \emptyset \) for all \( x \) and \( F \) is injective in the sense that \( F(x) \cap F(z) = \emptyset \) whenever \( x \neq z \). Applying now Proposition 4.6 to the equation \( Bf = g \) one finds that for all \( y \) there exists \( x \) such that \( (\partial^T g)(x) = \{ y \} \). From this one easily concludes that each set \( F(x) \) contains precisely one point and that \( F \) is surjective, which finally implies that \( F \) is a bijection \( X \to X \) such that \((4.2)\) holds.

From the definition of \( \partial f \) and \( \partial^T g \), and \( Bf = g, B^T g = f \), we deduce from \((4.2)\) that \( g_x = b_{xF(x)} - f_{F(x)} \) and that \( g_z > b_{zF(x)} - f_{F(x)} \) for \( z \neq x \), and \( f_z > b_{xz} - g_x \) for \( z \neq F(x) \), from which \((4.3)\) follows.

Let us show that \( F \) is locally bounded. Indeed, since \( f \) and \( g \) are bounded from below, we get that \( b_{xF(x)} = f_{F(x)} + g_x \) is bounded from below, but since \( b \) satisfies (TC), this implies that \( d(x, F(x)) \) is bounded.

Remark 4.9. When \( X \) is a finite set, the injectivity of the map \( F \) defined in \((4.3)\) implies automatically that \( F(x) \) contains exactly one point and that it is a bijection. Hence, in that case, the proof of Proposition 4.7 only needs the assumption that \( B^T h = f \) has a unique solution \( h \), and the proof is thus much shorter. By symmetry, in that case, one can also prove Proposition 4.7 using the only assumption that \( Bh = g \) has a unique solution \( h \), which is the definition of strong regularity given in BH85. From Proposition 4.6 (or from Vor67 Theorem 2.6), one can also deduce that, when \( X \) is finite, the two assumptions are equivalent, and thus our definition of strong regularity is equivalent to that of BH85, when the set \( X \) is finite.

Proof of Theorem 3.1. Let \( b \) satisfies Condition (ZC, TC). If \( b \) is \( \ell_\ast \)-similar to a strongly normal kernel \( c \), then by Proposition 2.15 \( c \) also satisfies (ZC, TC). Now taking for \( g \) the zero function, we get that \( g \in \ell_* \) and \( f = 0 \in \ell_* \). Moreover, \( \partial_x f(y) = \{ y \} \) and \( \partial_x g(x) = \{ x \} \), thus the covering of \( X \) by \( \{(\partial_x g)^{-1}(y)\}_{y \in X} \) is minimal, and by Proposition 4.6 the equation \( Bh = g \) has a unique solution \( h \in \ell_* \). Similarly, the equation \( B^T h = f \) has a unique solution \( h \in \ell_* \). This shows that \( c \) is \( \ell_\ast \)-strongly regular. Hence, by Proposition 2.15 \( b \) is also \( \ell_\ast \)-strongly regular. This shows the “if” part of the assertion of Theorem 3.1.

Let us show the “only if” part. Assume now that \( b \) is \( \ell_\ast \)-strongly regular, that is there exists \( f, g \in \ell_* \) such that \( f = B^T g \), and the equations \( Bh = g \) and \( B^T h = f \) have both a unique solution in \( \ell_* \). By Proposition 4.7 there exists
a locally bounded bijection $F : X \to X$ satisfying \((4.3)\). From these equations, we deduce that the kernel $c : X \times X \to \mathbb{R}$ such that $c_{xy} = b_{xF(y)} - f_{F(y)} - g_x$ is strongly normal. Since $f \in \ell_\ast(X)$ and $F$ is locally bounded, $f \circ F \in \ell_\ast(X)$, and since $g \in \ell_\ast(X)$, we deduce that $c$ is $\ell_\ast$-right-similar to $b$. Similarly the kernel $c_{F^{-1}(x)F^{-1}(y)}$ is strongly normal and $\ell_\ast$-left-similar to $b$. This finishes the proof of the theorem. \hfill \Box

**Proof of Theorem 3.4.** Let $b$ be a kernel satisfying (ZC,TC). Assume that $b$ is $\ell_\ast$-strongly regular, and let $f$ and $g$ be as in Definition 2.9. By Proposition 4.7 there exists a locally bounded bijection $F : X \to X$ satisfying \((4.3)\). From these equations, we deduce that if $G : X \to X$ is another bijection, $b_{xF(z)} - f_{F(z)} \geq b_{xG(z)} - f_{G(z)}$ for all $x \in X$, and that the inequality is strict when $G(x) \neq F(x)$. Hence if $G \neq F$, we get that

$$
\liminf_{K \in K} \sum_{x \in K} (b_{xF(x)} - b_{xG(x)}) > \liminf_{K \in K} \sum_{x \in K} (f_{F(x)} - f_{G(x)})
$$

as soon as the r.h.s. of this inequality is finite. But, the same arguments as in Propositions 2.16 and 2.17 show that the r.h.s. of \((4.5)\) is a limit and is equal to 0 when either $f \in \ell_1$, or $f \in \ell_{0,1}$ while $F$ and $G$ are at a finite distance. This shows that, when $\ell_\ast = \ell_1$, $F$ is a strong (global) solution to the assignment problem associated to $b$, and that, when $\ell_\ast = \ell_{0,1}$, $F$ is a strong local solution. Similarly,

$$
\liminf_{n \to \infty} \sum_{x \in B_n} (b_{xF(x)} - b_{xG(x)}) > \liminf_{n \to \infty} \sum_{x \in B_n} (f_{F(x)} - f_{G(x)})
$$

as soon as the r.h.s. of this inequality is finite. But, the same arguments as in Proposition 2.20 show the r.h.s. of \((4.6)\) is 0 when $f \in \ell_0$, $F$ and $G$ are locally bounded and $\#B_n - \#B_{n-1}$ is bounded. This shows that when $\#B_n - \#B_{n-1}$ is bounded, and $\ell_\ast = \ell_0$, $F$ is a strong local restricted solution to the assignment problem associated to $b$.

Since $b_{xF(x)} = g_x + f_{F(x)}$, and $\ell_\ast$ is invariant by any locally bounded bijection, $c_{x} \in X$ is in $\ell_\ast$ and thus $F$ is a $\ell_\ast$-bijection. Moreover, by the uniqueness of a strong local solution or of a strong restricted local solution, the solutions $F$ obtained for the $\ell_1$ and $\ell_{0,1}$ cases are the same under the assumptions of Point (ii), and the solutions for the $\ell_{0,1}$ and $\ell_0$ cases are the same under the assumptions of Point (i) and the assumption that $\#B_n - \#B_{n-1}$ is bounded.

It remains to show the properties of $\hat{b}$ defined from $F$ by \((4.2)\), and of the potentials $\hat{\phi}$ and $\hat{\psi}$ defined by \((3.3)\). From \((4.3)\), we deduce that $b_{xy} \leq g_x - g_y$ for all $x, y \in X$, hence

$$
\hat{b}_{xy}^+ \leq g_x - g_y.
$$

Since $g \in \ell_\ast \subset \ell_0$ for all cases of $\ell_\ast$ considered in Theorem 3.4, the r.h.s. of the above inequality \((4.7)\) tends to 0 as $x, y \to \infty$, which shows \((3.8)\). Moreover, by definition of $\hat{\phi}$ and $\hat{\psi}$, we get that $\sup_x \hat{\phi}_x = \sup_y \hat{\psi}_y = \sup_x \hat{b}_{xy}^+$ and using \((4.7)\) and the boundedness of $g$, we get that the functions $\hat{\phi}$ and $\hat{\psi}$ are bounded from above. Since they are also nonnegative functions, they are necessarily bounded. \hfill \Box

5. ”Perestroika” algorithm: proof of Theorem 3.5

Suppose the assumptions of Theorem 3.5 hold true for one of the sets $\ell_\ast$ considered in the statement. Let $F$ be a locally bounded strong local $\ell_\ast$-solution with
\( \ell \) being either \( \ell_{0,1} \) or \( \ell_1 \), or a locally bounded strong local restricted \( \ell_0 \)-solution to the assignment problem associated to the kernel \( b \), satisfying condition (PC-\( \ell \)). We shall consider the case where the potential \( \bar{\phi} \) defined in (3.3) belongs to \( \ell_1(X) \) (the case with the inverse potential is dealt with similarly). Since \( \ell_{*} \subset \ell_0 \), this assumption together with Equation (3.7) implies Condition (3.8).

By Propositions 2.13, 2.16, 2.17 and 2.20 replacing the kernel \( b \) with the \( \ell_{*} \)-right-similar kernel \( c \) such that \( c_{xy} = b_{xy} + \phi_y - \phi_x \), with \( b \) as in (3.2), changes neither Condition (ZC,TC), nor the property of \( \ell_{*} \)-strong regularity, nor the above property of having a locally bounded strong local \( \ell_{*} \)-solution (resp. restricted \( \ell_0 \)-solution) to the assignment problem when \( \ell_{*} \) is \( \ell_1 \) or \( \ell_{0,1} \) (resp. \( \ell_0 \)). Moreover, by the proof of Proposition 2.16, we see that the solution of the assignment problem associated to the kernel \( c \) is the identity map. Since the diagonal entries of \( c \) vanish, we get that \( \tilde{c} = c \), and by (3.2) for \( \tilde{\phi} \) we get that all the entries of \( c \) are nonpositive, hence \( c \) is a normal kernel.

Therefore, denoting the new kernel again by \( b \), we are reduced to the case where \( b \) is a normal kernel and \( F \) is the identity map. From now on, we shall suppose (without loss of generality) that these additional simplifying conditions hold true. Hence \( b \) satisfies the following conditions:

(1) \( b \) is a normal kernel, satisfying Conditions (ZC,TC), and the identity map of \( X \) is a strong local solution or a strong restricted local solution of its associated assignment problem.

This implies in particular that the potential function \( \tilde{\phi} \) associated to \( b \) is identically equal to 0. In order to prove Theorem 3.5 we need to show that \( b \) is necessarily \( \ell_{*} \)-strongly regular. By Theorem 3.1 and the fact that \( b \) satisfies Conditions (ZC,TC), it is enough to show that \( b \) is \( \ell_{*} \)-right-similar to a strongly normal kernel. To this end, we shall construct a function \( \phi \in \ell_1(X) \) such that

\[
  b_{xy} + \phi_y < \phi_x
\]

for all \( x \neq y \in X \), since then the kernel \( c \) with entries \( c_{xy} = b_{xy} + \phi_y - \phi_x \) would be strongly normal and \( \ell_{*} \)-right-similar to \( b \) (\( \ell_{1} \subset \ell_{*} \)). Note that since \( b \) satisfies Condition (TC) and \( \phi \) is bounded, then (5.1) is equivalent to the condition:

\[
  (A(-\phi))_x < \phi_x, \quad \text{for all } x \in X, \quad \text{or to the condition:} \quad (A^T \phi)_y < -\phi_y, \quad \text{for all } y \in X,
\]

where \( A \) is the Moreau conjugacy associated to the kernel \( a \) which coincides with \( b \) except on the diagonal where it is equal to \(-\infty \) (\( a_{xy} = b_{xy} \) if \( x \neq y \) and \( a_{xx} = -\infty \)).

Given a function \( \phi \in \mathcal{B}(X) \) and a kernel \( b : X \times X \rightarrow \mathbb{R}_{\text{max}} \) satisfying

\[
  b_{xy} + \phi_y \leq \phi_x \quad \forall x \neq y,
\]

we define the saturation graph associated to \( \phi \) and \( b \), denoted by \( \text{Sat}(b,\phi) \), or simply \( \text{Sat} \) or \( \text{Sat}(\phi) \), as the (infinite) oriented graph whose edges consist of the pairs \((x,y) \in X \times X \) such that \( x \neq y \) and

\[
  b_{xy} + \phi_y = \phi_x
\]

and whose set of vertices \( V = V(b,\phi) \) is the subset of elements of \( X \) that are adjacent to an edge. As usual by a path of length \( n \geq 1 \) in an oriented graph \( G \) we mean a finite sequence \((x_1, \ldots, x_{n+1})\) of vertices such that \((x_k, x_{k+1})\) is an edge for all \( k = 1, \ldots, n \). and by a circuit (of length \( n \)) we mean a path \((x_1, \ldots, x_{n+1})\) such that \( x_{n+1} = x_1 \). An infinite path leaving (resp. entering) the vertex \( x \) of \( G \) is a sequence \((x_n)_{n \geq 0} \) (resp. \((x_n)_{n \leq 0}\)) such that \( x_0 = x \) and \((x_k, x_{k+1})\) is an edge for all \( k \geq 0 \) (resp. \( k < 0 \)). A string of \( G \) is a sequence \((x_n)_{n \in \mathbb{Z}}\) such that \((x_n, x_{n+1})\) is
an edge for all $n \in \mathbb{Z}$. The length of an infinite path or of a string is infinity. The main properties of the saturation graph associated to the kernel $b$ are collected in the following statement.

**Proposition 5.1.** Let $b : X \times X \to \mathbb{R}_{\text{max}}$ be a kernel satisfying Condition (NC), and $\phi \in \ell_1(X)$ satisfy \((5.2)\), and denote by $\text{Sat}$ their saturation graph and by $V$ the set of its vertices. Then (i) $\text{Sat}$ contains no circuits nor strings. (ii) For all $x \in V$, the set of edges entering or leaving $x$ is finite. (iii) For all $x \in V$, denote by $\text{lp}(x)$ (resp. $\text{ep}(x)$) the supremum of the lengths of all the paths leaving $x$ (resp. entering $x$). Then either $\text{lp}(x)$ or $\text{ep}(x)$ is finite. (iv) If $V$ is nonempty, then the set of its end points is nonempty, where by an end point we mean either an initial point (no edge is entering it) or a final point (no edge is leaving it).

**Proof.** Let us first note that since $b$ satisfies (TC), and $\phi$ is bounded, there exists $M > 0$ such that \((5.1)\) holds for all $x, y$ such that $d(x, y) > M$. This implies that all edges $(x, y)$ of $\text{Sat}$ satisfy $d(x, y) \leq M$.

(i) Suppose now that $\text{Sat}$ has a circuit $(x_1, \ldots, x_{n+1} = x_1)$. We can assume without loss of generality that this circuit is elementary, that is all vertices $x_k$ with $k = 1, \ldots, n$ are distinct. Hence, one can construct a bijection $G : X \to X$ which coincides with the identity map $F$ outside the elements of the circuit, and which acts as $x_k \mapsto x_{k+1}$ on the vertices of the circuit. It is clear that $G$ is locally bounded and different from $F$, and since

$$b_{x_1x_2} + \cdots + b_{x_{n-1}x_n} + b_{x_nx_1} = b_{x_1x_1} + \cdots + b_{x_nx_n},$$

does not hold, which contradicts the assumption that the identity map is a strong or a strong restricted local solution.

Assume next that $\text{Sat}$ contains a string $(x_n)_{n \in \mathbb{Z}}$. Since $\text{Sat}$ contains no circuit, all elements $x_n$ of this sequence are distinct. Hence one can construct a bijection $G : X \to X$ which coincides with the identity map $F$ outside the elements of the string, and which acts as the shift $x_k \mapsto x_{k+1}$ on the string. This bijection is necessarily different from $F$. Moreover, since the distance between the vertices of an edge is bounded by $M$, the bijection $G$ is locally bounded: $\rho(G, I) \leq M$. Finally, by \((5.3)\), we have

$$\sum_{x \in K} (b_{xF(x)} - b_{xG(x)}) = \sum_{x \in K} (\phi_G(x) - \phi_x),$$

and by the same arguments as in the proof of Proposition 2.16 this sum has a zero limit. This contradicts \((2.3)\) or \((2.13)\), and thus the assumption that the identity map is a strong or a strong restricted local solution.

(ii) Since all edges $(x, y)$ of $\text{Sat}$ satisfy $d(x, y) \leq M$, we see that, for all $x \in V$, the set of edges entering or leaving $x$ is included in the ball of centre $x$ and radius $M$ which is finite.

(iii) Choose $x \in V$. As there are no strings in $\text{Sat}$, either all paths leaving $x$ or all paths entering $x$ are finite. Consider, say, the first case. Suppose by contradiction that $\text{lp}(x) = \infty$, that is the lengths of the paths leaving $x$ are not bounded. Hence,

$$\infty = \text{lp}(x) = \sup_y \text{lp}(y),$$

where the supremum is taken over the vertices $y$ such that $(x, y)$ is an edge of $\text{Sat}$. By Point (ii), this set is finite, from which we deduce that at least one of its elements
y is such that \( \ell p(y) = \infty \). Hence by induction one can construct an infinite path leaving \( x \), which contradicts our assumption.

(iv) Again the absence of strings implies that each point belongs to a path that either ends in a final point or starts at an initial point. \( \square \)

**Proof of Theorem 5.2.** Since \( b \) is a normal kernel, the function \( \phi \equiv 0 \) satisfies (5.2), where the equality holds only on the edges \( (x, y) \) of the graph \( \text{Sat}(0) \). Our goal is to change \( \phi \) (by a successive “perestroika”) in such a way that no equality is left, which would yield to (5.1) for all \( x \neq y \). We shall do this by successive elimination of the end points of \( \text{Sat}(0) \).

Namely, let \( \phi \in \ell _1(X) \) satisfy (5.2), and let us denote respectively by \( I_0 = I_0(\phi) \) and \( F_0 = F_0(\phi) \) the sets of the initial points and final points of the saturation graph \( \text{Sat}(\phi) \). By Point (iv) of Proposition 5.1, we know that either \( I_0 \) or \( F_0 \) is nonempty. Assume for instance that \( F_0 \) is nonempty and let \( x \in F_0 \). Then \( b_{xx} + \phi_x < \phi_x \) for all \( z \neq x \), and \( b_{yz} + \phi_z = \phi_y \) for at least one vertex and at most a finite number of vertices \( y \neq x \) of \( \text{Sat}(\phi) \). The first inequality implies that \((A(-\phi))_x < \phi_x \) (by Condition (TC)), hence it is possible to decrease the value of \( \phi \) in all final vertices \( y \) such that \( \phi_y = \phi_y' \). Now, starting from any function \( \psi \in \ell _1(X) \) with positive values, we can choose \( \psi' \) in such a way that \( |\phi_x' - \phi_x| \leq \psi_x \) for all \( x \in X \), which will imply in particular that \( \phi' \in \ell _1(X) \). Indeed, let us take \( \phi'_x = \phi_x - \min(\psi_x, (\phi_x - (A(-\phi))_x)/2) \) for all \( x \in F_0 \) and \( \phi'_x = \phi_x \) elsewhere. Since \( \phi' \leq \phi \), we get that \( b_{yz} + \phi'_z \leq b_{yz} + \phi_z \) for all \( y, z \in X \) such that \( z \neq y \), with equality if and only if \( z \not\in F_0 \). Hence, \( b_{yz} + \phi'_z \leq \phi_y' \) for all \( y \in X \setminus F_0 \) and \( z \neq y \), with equality if and only if \( (y, z) \) is an edge of \( \text{Sat}(\phi) \) and \( z \not\in F_0 \). Moreover, for \( y \in F_0 \) and \( z \neq y \), we have \( b_{yz} + \phi_z \leq (A(-\phi))_y < \phi_y' \), hence \( b_{yz} + \phi'_z < \phi_y' \), and \( (y, z) \) is not an edge of \( \text{Sat}(\phi') \).

Let us now fix a function \( \psi \in \ell _1(X) \), and denote by \( P_F(\phi) \) the function \( \phi' \) obtained from \( \phi \) by the previous construction on the final points of \( \text{Sat}(\phi) \). We denote also by \( P_I(\phi) \) the function \( \phi' \) obtained from \( \phi \) by a similar construction where final points are replaced by initial points (or equivalently the kernel \( b \) is replaced by \( b^T \) and the functions by their opposite). This is one step of our “perestroika” algorithm.

Now, starting from any function \( \phi^0 \in \ell _1(X) \) satisfying (5.2), in particular the function \( \phi^0 = 0 \), one can construct a sequence of functions \( \phi^n \in \ell _1(X) \) by \( \phi^{n+1} = P_F(\phi^n) \). At each step we have \( V(\phi^{n+1}) = V(\phi^n) \setminus F_0(\phi^n) \). Hence, since \( \phi^{n+1} - \phi^n \) has zero entries outside \( F_0(\phi^n) \) and all these sets are disjoint, we get that for all \( x \in X \), \( \phi^n_x \) converges in finite time towards some real \( \phi_x \), and since \( |\phi^n - \phi^0| \leq \psi \) for all \( n \geq 0 \), the function \( \phi = (\phi_x)_{x \in X} \) is in \( \ell _1(X) \). Note that the sequence \( \phi^n \) may stop at step \( n \) if \( F_0(\phi^n) = \emptyset \), in which case, \( \phi \) will be simply \( \phi^n \). Now, since \( \phi^n_x \) converges in finite time for all \( x \in X \), we get easily that \( \phi \) satisfies (5.2), and that \( \text{Sat}(\phi) = \cap _{n \geq 0} \text{Sat}(\phi^n) \). We can then start from \( \psi^0 = \phi \), and construct similarly a sequence \( \psi^n \) using the algorithm \( P_I \) for initial sets. The limit \( \psi \) is again in \( \ell _1(X) \), satisfies (5.2) and \( \text{Sat}(\psi) = \cap _{n \geq 0} \text{Sat}(\psi^n) \).

Let us prove that \( \text{Sat}(\psi) \) is empty or equivalently that \( V(\psi) = \emptyset \), in which case we would have shown that \( \psi \) satisfies (5.1) for all \( x \neq y \). For all \( n \in \mathbb{N} \cup \{ \infty \} \), we shall consider the following subsets of the set of vertices of the saturation graph...
associated to $\phi$:

$$F_n(\phi) := \{x \in V(\phi) \mid \text{lp}(x) = n\}, \quad I_n(\phi) := \{x \in X \mid \text{ep}(x) = n\}. $$

By Point (iii) of Proposition 5.1, we know that for any $\phi \in \ell_1(X)$, and $x \in V(\phi)$, either $\text{lp}(x)$ or $\text{ep}(x)$ is finite, hence

$$V(\phi) = \bigcup_{n \in \mathbb{N} \cup \{\infty\}} F_n(\phi) = \bigcup_{n \in \mathbb{N} \cup \{\infty\}} I_n(\phi) \quad \text{and} \quad F_\infty(\phi) \subset \bigcup_{n \in \mathbb{N}} I_n(\phi)$$

where the unions are disjoint. But the “perestroika” algorithm for final points is such that $\text{Sat}(P(\phi))$ is equal to the subgraph of $\text{Sat}(\phi)$ where all final vertices and all edges entering them are removed. Hence all remaining vertices $y$ in $\text{Sat}(P(\phi))$ are such that $\text{lp}(y)$ is decreased exactly by 1 (and $\text{ep}(y)$ is unchanged), and $V(P(\phi)) = V(\phi) \setminus F_0(\phi)$. We deduce that $F_n(P(\phi)) = F_{n+1}(\phi)$. Similarly $I_n(P(\psi)) = I_{n+1}(\psi)$. Hence, the above sequence $\phi^n$ satisfies $F_0(\phi^n) = F_n(\phi^0)$, thus

$$V(\phi^n) = V(\phi^{n-1}) \setminus F_0(\phi^{n-1}) = V(\phi^0) \setminus (F_0(\phi^0) \cup \cdots \cup F_{n-1}(\phi^0))$$

and $V(\phi) = \bigcap_{n \in \mathbb{N}} V(\phi^n) = F_\infty(\phi^0)$. By a similar argument, we get that $V(\psi) = I_\infty(\phi) = F_\infty(\phi^0) \cap I_\infty(\phi^0) = \emptyset$, which completes the proof of the theorem. \hfill $\square$

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Marianne Akian, INRIA, Saclay—Île-de-France, and CMAP, Ecole Polytechnique, Route de Saclay, 91128 Palaiseau Cedex, France

E-mail address: Marianne.Akian@inria.fr

Stéphane Gaubert, INRIA, Saclay—Île-de-France, and CMAP, Ecole Polytechnique, Route de Saclay, 91128 Palaiseau Cedex, France

E-mail address: Stephane.Gaubert@inria.fr

Vassili Kolokoltsov, Department of Statistics, University of Warwick, Coventry CV4 7AL, UK

E-mail address: v.Kolokoltsov@warwick.ac.uk