Ferromagnetism in quark matter

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Abstract

We investigate the magnetic property of the cold quark matter employing the magnetic moment $\vec{\mu}'$ as the order parameter. Through analysis of the effective potential $V(\vec{\mu}'^2)$, we find that, at relatively high densities $\rho > \rho_c$, the quark matter is in the normal phase. At $\rho = \rho_c$, the magnetic phase transition takes place and, in the low-density region $\rho < \rho_c$, the quark matter is in the ferromagnetic phase. For up, down and strange quarks, $\rho_c = 1.62\rho_0$, $1.47\rho_0$ and $2.20\rho_0$ ($\rho_0$ the nuclear density), respectively. We also find that the leptons ($e$ and $\mu$) inside the quark matter are in the normal phase.
1 Introduction

Several years ago, the possibility of existence of a quark matter in a ferromagnetic phase has been pointed out [1]. Meanwhile, the possible discovery of a quark star [2, 3] has renewed our interest in this issue.

In Ref. 1), through computing the energy density of a polarized quark matter, it is suggested that the Hartree-Fock state shows a spontaneous magnetic instability at low densities. The QCD coupling constant taken in Ref. 1) is $\alpha_s = g^2 / (4\pi) = 2.2$ and the quark mass is $m = 300$ MeV/c$^2$. In this paper, we examine a possibility of the quark matter in a ferromagnetic phase by choosing a magnetic moment $\vec{\mu}' \propto (e/m) \bar{\psi} \vec{\sigma} \psi$ as an order parameter.

The QCD phase diagram is sketched in Fig. 1, where ‘$T$’ is the temperature and ‘$\mu_B$’ is the baryonic chemical potential. ‘QGP’ stands for the quark-gluon plasma phase, ‘Hadron’ stands for the hadronic phase and ‘CSC’ stands for several phases which include the color super conducting phases. We are interested in the QGP phase near the phase boundaries and $0.6 \text{ GeV} \leq \mu_B$. In this region $T << \mu_B$, so that we ignore the effect due to finite $T$. Thus, the object of our concern is the quark matter with finite baryon-number density at zero temperature.

We have to introduce two kinds of cutoff and then the result depends on the ratio of these two. This is not particular to the present study but is a common feature in the study of effective potential for some order parameter (see, e.g., Ref. 4)). We shall simply equate the two cutoff parameters (Sec. 3), for which we take the baryonic chemical potential $\mu_B$ (Sec. 5).

The plan of the paper is as follows. We introduce in Sec. 2 the closed-time-path (CTP) formalism [5, 6, 7, 8] and derive the formula for the expectation value $\langle \vec{\mu}' \rangle$ under the existence of an uniform external magnetic field $\vec{B}$. In Sec. 3, within the leading logarithmic approximation, we compute $\langle \vec{\mu}' \rangle$ up to $O(\vec{B}^2 \vec{B})$ for $\vec{B}$ and to $O(g^2)$ for the QCD coupling constant. Then, performing the resummation of the leading contributions, we obtain the “resummed form” for $\langle \vec{\mu}' \rangle$. In Sec. 4, we deal with the effective potential $V(\langle \vec{\mu}' \rangle)$. In Sec. 5, numerical analysis of $V(\langle \vec{\mu}' \rangle)$ is made in the range $0.6 \text{ GeV} < \mu_B < 4 \text{ GeV}$. We find that, in the high-density region $\mu_B > \mu_{Bc}$, the quark matter is in a normal phase. As the density is lowered, the quark matter undergoes a magnetic phase transition, at the critical density, into a ferromagnetic
phase. The critical or transition chemical potentials are $\mu_{Bc} = 641, 625$ and $693$ MeV for up ($u$), down ($d$) and strange ($s$) quarks, respectively. We finally discuss the case of leptons and find that they are in the normal phase. Sec. 6 is devoted to summary and outlook.

2 Preliminary

Throughout this paper, we adopt the leading logarithmic approximation,

$$1 << \ln(\Lambda/m), \quad 1 << \ln[\mu_B/(3m)]. \quad (1)$$

Here, $\Lambda$ is the cutoff parameter and $\mu_B$ is the chemical potential being conjugate to the baryon number. The terms whose relative order of magnitude are $O(1/\ln(\Lambda/m))$ and/or $O(1/\ln[\mu_B/(3m)])$ are ignored.

As the order parameter, we adopt one-half of the spin magnetic moment,

$$\vec{\mu}' = \vec{\mu}/2 = \frac{1}{2} \left( \frac{e_q}{2m} \right) \bar{\psi} \vec{\sigma} \psi = \frac{e_q}{4m} \bar{\psi} \gamma_5 \gamma_0 \vec{\gamma} \psi,$$

where and in the following the color index is suppressed. An argument for this choice is given in Appendix A.

2.1 Closed-time-path formalism

We use the closed-time-path (CTP) formalism [5, 6, 7, 8], which is formulated by introducing an oriented closed-time path $T_1 \otimes T_2$ in a complex time plane, that goes from $-\infty$ to $+\infty$ ($T_1$) and then returned from $+\infty$ to $-\infty$ ($T_2$). The field $\phi(x)$ with $x_0 \in T_i$ ($i = 1, 2$) is called type $i$, $\phi_i(x) = \phi(x) \big|_{x_0 \in T_i}$. The Lagrangian density in this formalism is $\hat{L} (\equiv \mathcal{L}(\phi_1) - \mathcal{L}(\phi_2))$.

The Lagrangian of a quark with some flavor reads

$$\hat{L} = \hat{L}_{QCD} + \vec{\mu}_1 \cdot \vec{B}_1 - \vec{\mu}_2 \cdot \vec{B}_2 = \hat{L}^{(2)} + \hat{L}^{(int)}_{QCD},$$

$$\hat{L}^{(2)} = \begin{pmatrix} \psi_1, \bar{\psi}_2 \end{pmatrix} \left[ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \left( i \hat{\gamma} \cdot \vec{m} \right) + \frac{e_q}{4m} \gamma_5 \gamma_0 \begin{pmatrix} \vec{\gamma} \cdot \vec{B}_1 & 0 \\ 0 & -\vec{\gamma} \cdot \vec{B}_2 \end{pmatrix} \right] \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix},$$

(2)
where $\hat{L}^{(\text{int})}_{\text{QCD}}$ is the interaction part of $\hat{L}_{\text{QCD}}$. It is convenient to construct a $(2 \times 2)$-matrix “propagator” $\hat{G}$, the inverse of the kernel of Eq. (2), which is diagonal in a color space. For the purpose of later use (cf. Eq. (8)), it is sufficient to obtain $\hat{G}$ with $\vec{B}_1 = \vec{B}_2 (\equiv \vec{B})$. Straightforward manipulation yields for the $(i,j)$-element of $\hat{G}$,

\[ G_{11}(P) = -G_{22}^*(P) = G_F(P) - N(p_0)[G_F(P) - G_F^*(P)], \]

\[ G_{12}(P) = [\theta(-p_0) - N(p_0)] [G_F(P) - G_F^*(P)], \]

\[ G_{21}(P) = [\theta(p_0) - N(p_0)] [G_F(P) - G_F^*(P)], \]

where

\[
G_F(P) = \left\{ P^2 - m^2 + \left( \frac{e_q}{4m} \right)^2 \vec{B}^2 \right\} \left( \vec{p} + m + \frac{e_q}{4m} \gamma_5 \gamma_0 \vec{B} \right) \\
-2 \frac{e_q}{4m} \vec{p} \cdot \vec{B} \left( -\frac{e_q}{4m} \vec{B} \cdot \vec{B} + m \gamma_5 \gamma_0 - \gamma_5 \vec{p} \gamma_0 \right) \\
+2 \frac{e_q}{4m} p_0 \left( -\frac{e_q}{4m} \vec{B}^2 \gamma_0 + m \gamma_5 \vec{B} \cdot \vec{B} - \gamma_5 \vec{p} \gamma_0 \cdot \vec{B} \right) \\
\times \left[ \Delta_F^{-2}(P) - 2 \left( \frac{e_q}{4m} \right)^2 \Delta_F^{-1}(P) + 2E_p^2 \vec{B}^2 \right] \\
+4 \left( \frac{e_q}{4m} \right)^2 (\vec{p} \cdot \vec{B})^2 + \left( \frac{e_q}{4m} \right)^4 (\vec{B}^2)^2 \right]^{-1},
\]

\[ \Delta_F(P) = \frac{1}{P^2 - m^2 + i0^+}. \]

Here we have used a capital letter for denoting a four vector $P = (p_0, \vec{p}) (p \equiv |\vec{p}|)$ and $E_p = \sqrt{p^2 + m^2}$. In the above equations, “∗” does not apply to the Dirac gamma matrices and $N(p_0)$ is the number density function,

\[ N(p_0) = \theta(p_0) \theta(\mu_B/3 - p_0). \]

**Generating functional and the formula for $\langle \vec{\mu}' \rangle$**

The CTP generating functional is defined [7] by

\[ Z(\vec{B}_1, \vec{B}_2) = \int \prod_{j=1}^2 [\mathcal{D}\psi_j \mathcal{D}\bar{\psi}_j \mathcal{D}A_j] \exp \left[ i \int d^4x \left( \hat{L}_{\text{QCD}} + \vec{\mu}'_1 \cdot \vec{B}_1 - \vec{\mu}'_2 \cdot \vec{B}_2 \right) \right] \rho, \]
where $A_j (j = 1, 2)$ is the gluon field and $\rho$ is the density matrix. An expectation value of the magnetic moment $\vec{\mu}'$ is computed through [7]

$$
\langle \vec{\mu}' \rangle \equiv \frac{1}{2} (\langle \vec{\mu}_1' \rangle + \langle \vec{\mu}_2' \rangle) \bigg|_{\vec{B}_1 = \vec{B}_2 = \vec{B}} = -i \frac{\delta}{2} \left( \frac{\delta}{\delta \vec{B}_1} - \frac{\delta}{\delta \vec{B}_2} \right) \ln Z(\vec{B}_1, \vec{B}_2) \bigg|_{\vec{B}_1 = \vec{B}_2 = \vec{B}}. \tag{8}
$$

We compute $\langle \vec{\mu}' \rangle$ up to $O(\vec{B}^2 \vec{B})$.

### 3 Computation of $\langle \vec{\mu}' \rangle$

#### 3.1 0th order contribution

From Eq. (8), we obtain for the 0th-order contribution ($\mathcal{L}_{\text{QCD}}^{\text{int}} = 0$) to $\langle \vec{\mu}' \rangle$,

$$
\langle \vec{\mu}' \rangle = -i \frac{e_q}{2 \cdot 4m \cdot \left( \gamma_5 \gamma_0 \gamma \right)} \int \frac{d^4P}{(2\pi)^4} \left[ \gamma_5 \gamma_0 \gamma \left( G_{11}(P) + G_{22}(P) \right) \right] = \frac{e_q}{4m} \Im \text{Tr} \int \frac{d^4P}{(2\pi)^4} \left[ \gamma_5 \gamma_0 \gamma G_{11}(P) \right]. \tag{9}
$$

Straightforward manipulation using Eq. (6) yields, up to $O(\vec{B}^2 \vec{B})$,

$$
\Im \text{Tr} \int \frac{d^4P}{(2\pi)^4} \left[ \gamma_5 \gamma_0 \gamma G_{11}(P) \right] = -e_q \frac{\vec{B} \cdot \vec{B}}{3 \cdot 4m} \Im \text{Tr} \int \frac{d^4P}{(2\pi)^4} \left[ 3\Delta_F + 2(2p^2 + 3m^2)\Delta^2_F \right] - 4 \left( \frac{e_q}{4m} \right)^3 (\vec{B} \cdot \vec{B}) \Im \text{Tr} \int \frac{d^4P}{(2\pi)^4} \left[ 15\Delta^2_F + 40(2p^2 + 3m^2)\Delta^3_F + 8(8p^4 + 20m^2p^2 + 15m^4)\Delta^4_F \right]. \tag{10}
$$

Substituting Eq. (3) for $G_{11}$ in Eq. (9) and using Eq. (10), one can compute $\langle \vec{\mu}' \rangle$.

We divide $\langle \vec{\mu}' \rangle$ into two pieces,

$$
\langle \vec{\mu}' \rangle = \langle \vec{\mu}' \rangle_{\text{vac}} + \langle \vec{\mu}' \rangle_{\text{mat}},
$$

where $\langle \vec{\mu}' \rangle_{\text{vac}}$ ($\langle \vec{\mu}' \rangle_{\text{mat}}$) stands for the contribution from the vacuum (matter) sector. $\langle \vec{\mu}' \rangle_{\text{vac}}$ is given by Eq. (9) with $G_F$ for $G_{11}$ (cf Eq. (3)). Thanks to the $O(3, 1)$
symmetry, in Eq. (10), the replacements, \( p^2 \rightarrow -3P^2/4 \) and \( p^4 \rightarrow -5(P^2)^2/8 \) may be made. Thus we obtain
\[
\langle \vec{\mu}^2 \rangle_{\text{vac}} = -4 \left( \frac{e_q}{4m} \right)^2 \bar{B} \text{Im} \int \frac{d^4 P}{(2\pi)^4} m^2 \Delta_{\vec{k}}^2(P) + \frac{4}{3} \left( \frac{e_q}{4m} \right)^4 \bar{B}^2 \bar{B} \text{Im} \int \frac{d^4 P}{(2\pi)^4} \Delta_{\vec{k}}^2(P). 
\]
(11)

In the second term on the right-hand side (RHS), we have dropped the terms that are proportional to \( m^2(e_q/4m)^4 \) and \( m^4(e_q/4m)^4 \), the contribution of which is nonleading in our approximation (cf. Eq. (1)). Introducing a momentum cutoff \( \Lambda \), we have
\[
\int \frac{d^4 P}{(2\pi)^4} \Delta_{\vec{k}}^2(P) = \frac{i}{8\pi^2} \ln(\Lambda/m),
\]
with which we obtain the first terms on the RHS’s of Eqs. (28) and (29) below.

For \( \langle \vec{\mu}^2 \rangle_{\text{mat}} \), we obtain
\[
\langle \vec{\mu}^2 \rangle_{\text{mat}} = \frac{8}{3} \left( \frac{e_q}{4m} \right)^2 \bar{B} \text{Im} \int \frac{d^4 P}{(2\pi)^4} \theta(\mu_B/3 - p_0) \\
\times \left[ 3\Delta_F + 4p^2\Delta_F^2 + \frac{1}{5} \left( \frac{e_q\bar{B}}{4m} \right)^2 \left( 15\Delta_F^2 + 80p^2\Delta_F^2 + 64p^4\Delta_F^4 \right) \right].
\]
(13)

Here the terms that are proportional to \( m^2(e_q/4m)^{2n} \) and \( m^4(e_q/4m)^{2n} \) \( (n = 1, 2) \) have been dropped, which yield the subleading contributions.

### 3.2 \( O(g^2) \) QCD correction to \( \langle \vec{\mu} \rangle \)

Here we compute the \( O(g^2) \) QCD correction to \( \langle \vec{\mu} \rangle \) in Eq. (8). Gluon propagator is diagonal in a color space. Choosing a covariant gauge, we have
\[
D^{\nu\mu}_{11}(Q) = - (D^{\nu\mu}_{22}(Q))^* = D^{\nu\mu}_{F}(Q) = - \frac{g^{\nu\mu}}{Q^2 + i0^+} + \eta \frac{Q^\nu Q^\mu}{(Q^2 + i0^+)^2},
\]
(14)
\[
D^{\nu\mu}_{12}(Q) = \theta(-q^0)[D^{\nu\mu}_{F}(Q) - (D^{\nu\mu}_{F}(Q))^*],
\]
(15)
\[
D^{\nu\mu}_{21}(Q) = \theta(q^0)[D^{\nu\mu}_{F}(Q) - (D^{\nu\mu}_{F}(Q))^*].
\]
(16)

To \( O(g^2) \), we obtain from Eq. (8),
\[
\langle \vec{\mu} \rangle = -i \frac{e_q}{24m} \sum_{j,l=1}^2 \text{Tr} \int \frac{d^4 P}{(2\pi)^4} \gamma_5 \gamma_0 \gamma \left[ \{ G_{1j}(P) \bar{\Sigma}_{ij}(P) G_{1l}(P) \} + \{ 1 \rightarrow 2 \} \right],
\]
\[
\bar{\Sigma}_{jl}(P) = \frac{4i}{3} g^2 (-)^{j+l} \int \frac{d^4 Q}{(2\pi)^4} \gamma^\mu G_{jl}(P + Q) \gamma^\nu D^{\nu\mu}_{ij}(Q).
\]


Using Eqs. (3) - (5) and (14) - (16), we get, after some manipulation,

\[
\langle \tilde{\mu}' \rangle = \frac{e_q}{4m} \text{ImTr} \int \frac{d^4 P}{(2\pi)^4} \gamma_5 \gamma_0 \gamma [1 - 2N(p_0)] G_F(P) \tilde{\Sigma}_F(P) G_F(P), \tag{17}
\]

\[
\tilde{\Sigma}_F(P) = \tilde{\Sigma}_{11}(P) + \theta(p_0) \tilde{\Sigma}_{12}(P) + \theta(-p_0) \tilde{\Sigma}_{21}(P)
\]

\[
= \frac{4i}{3} g^2 \int \frac{d^4 Q}{(2\pi)^4} \gamma^\mu [G_{11}(P + Q) D_F^{\nu\mu}(Q) - \theta(p_0) G_{12}(P + Q) D_{21}^{\nu\mu}(Q)] \gamma^\nu
\]

\[
+ \frac{4i}{3} g^2 \int \frac{d^4 Q}{(2\pi)^4} [G_{11}(P + Q) D_F^{\nu\mu}(Q)
\]

\[
+ N(p_0 + q_0) \left\{ G_F(P + Q) - G_F^*(P + Q) \right\}
\]

\[
\times \left\{ \theta(p_0) \left( \theta(-q_0) D_F^{\nu\mu}(Q) + \theta(q_0) (D_F^{\nu\mu}(Q))^* \right) \right\} \gamma^\nu.
\]

The term with \( \theta(-p_0) D_{12}^{\nu\mu}(Q) \) in the last line vanishes, since \( D_{12}^{\nu\mu}(Q) = 0 \) for \( q_0 > 0 \), \( N(p_0 + q_0) = 0 \) for \( p_0 + q_0 < 0 \). Nevertheless we include it for later convenience. Using Eqs. (3), (15), and (16), we obtain

\[
\tilde{\Sigma}_F(P) = \frac{4i}{3} g^2 \int \frac{d^4 Q}{(2\pi)^4} \gamma^\mu [G_F(P + Q) D_{F}^{\nu\mu}(Q)
\]

\[
- N(p_0 + q_0) \left\{ G_F(P + Q) - G_{F}^*(P + Q) \right\}
\]

\[
\times \left\{ \theta(p_0) \left( \theta(-q_0) D_F^{\nu\mu}(Q) + \theta(q_0) (D_F^{\nu\mu}(Q))^* \right) \right\} \gamma^\nu.
\]

Substituting this for \( \tilde{\Sigma}_F(P) \) in Eq. (17), we finally obtain

\[
\langle \tilde{\mu}' \rangle = \langle \tilde{\mu}' \rangle_1 + \langle \tilde{\mu}' \rangle_2, \tag{18}
\]

\[
\langle \tilde{\mu}' \rangle_1 = \frac{4}{3} g^2 e_q \frac{4m}{\text{ReTr}} \int \frac{d^4 P}{(2\pi)^4} \int \frac{d^4 Q}{(2\pi)^4} \gamma_5 \gamma_0 \gamma
\]

\[
\times [1 - 2N(p_0)] G_F(P) \gamma^\mu G_F(P + Q) D_F^{\nu\mu}(Q) \gamma^\nu G_F(P), \tag{19}
\]

\[
\langle \tilde{\mu}' \rangle_2 = \frac{4}{3} g^2 e_q \frac{4m}{\text{ReTr}} \int \frac{d^4 P}{(2\pi)^4} \int \frac{d^4 Q}{(2\pi)^4} \gamma_5 \gamma_0 \gamma
\]

\[
\times [1 - 2N(p_0)] G_F(P) \gamma^\mu N(p_0 + q_0) \left\{ G_F(P + Q) - G_{F}^*(P + Q) \right\}
\]

\[
\times \left\{ \theta(p_0) D_A^{\nu\mu}(Q) + \theta(-p_0) D_R^{\nu\mu}(Q) \right\} \gamma^\nu G_F(P), \tag{20}
\]

where \( D_R^{\nu\mu} (D_R^{\nu\mu}) \) is the retarded (advanced) gluon propagator:

\[
D_R(Q) = [D_A(Q)]^* = \theta(q_0) D_F(Q) + \theta(-q_0) [D_F(Q)]^*.
\]

The vacuum-sector contribution is given by Eq. (19) with \( N(p_0) = 0 \).
It is necessary to compute (up to \( O(\tilde{B}^2\tilde{B}) \)) the quantity,

\[
\tilde{G} \equiv \text{Tr} [\gamma_5 \gamma_0 \gamma_F (P) \gamma^\mu G_F (P') \gamma_\mu G_F (P)] D_T (Q) \quad (P' = P + Q),
\]

where \( T \) stands for \( F, R, \) or \( A. \) Introducing

\[
\mathcal{I}_{Tmn}^{(T)} \equiv [\Delta_F (P)]^i [\Delta_F (P')]^m ([\Delta_T (Q)]_{m=0})^n \quad (T = F, R, A),
\]

we obtain using Eq. (6),

\[
\tilde{G} = \frac{e_q}{4m_B} (\mathcal{G}_{\text{self}}^{(1)} + \mathcal{G}^{(1)'}) + \left( \frac{e_q}{4m_B} \right)^3 \tilde{B}^2 \tilde{B} \left( \mathcal{G}_{\text{self}}^{(2)} + \mathcal{G}^{(2)'} \right),
\]

\[
\mathcal{G}_{\text{self}}^{(1)} = -8 \left[ \mathcal{I}_{111}^{(T)} + \left\{ 3(Q^2 + 2m^2) - 4 \vec{p} \cdot \vec{q} + 4p_0 q_0 \right\} \mathcal{I}_{211}^{(T)} + \frac{8}{3} p^2 (Q^2 + 2m^2) \mathcal{I}_{311}^{(T)} \right],
\]

\[
\mathcal{G}^{(1)'} = -\frac{32}{3} m^2 \left[ 3p_0 q_0 - \vec{p} \cdot \vec{p} \right] \mathcal{I}_{221}^{(T)},
\]

\[
\mathcal{G}_{\text{self}}^{(2)} = -16 \left[ 3 \mathcal{I}_{211}^{(T)} + \left( -11Q^2 + 8p^2 + 8p_0 p_0' - \frac{8}{3} \vec{p} \cdot \vec{p} \right) \mathcal{I}_{311}^{(T)} + 8p^2 \left( -Q^2 + \frac{4}{3} p_0 p_0' - \frac{4}{15} \vec{p} \cdot \vec{p} \right) \mathcal{I}_{411}^{(T)} \right],
\]

\[
\mathcal{G}^{(2)'} = -8 \left[ -9 \left( \mathcal{I}_{211}^{(T)} + \mathcal{I}_{221}^{(T)} - \mathcal{I}_{220}^{(T)} \right) + 8 \left( 2p_0 p_0' - p'^2 - p^2 - \frac{2}{3} \vec{p} \cdot \vec{p} \right) \mathcal{I}_{221}^{(T)} + 8p^2 \mathcal{I}_{230}^{(T)} + \frac{32}{15} \left( 5p_0 p_0' p^2 - 3p^2 p'^2 - p'^2 \vec{p} \cdot \vec{p} - (\vec{p} \cdot \vec{p})^2 \right) \mathcal{I}_{231}^{(T)} - 8p^2 \mathcal{I}_{313}^{(T)} - \frac{32}{15} \left( 5p^2 p'^2 + (\vec{p} \cdot \vec{p})^2 \right) \mathcal{I}_{330}^{(T)} - \frac{8}{3} p^2 \mathcal{I}_{320}^{(T)} + \frac{32}{3} p^2 \mathcal{I}_{320}^{(T)} + \frac{32}{15} \left( -3p^2 p'^2 - p^2 \vec{p} \cdot \vec{p} + 5p_0 p_0' p^2 - (\vec{p} \cdot \vec{p})^2 \right) \mathcal{I}_{321}^{(T)} \right].
\]

Here \( \mathcal{G}_{\text{self}}^{(j=1,2)} \) comes from the diagram that includes a quark self-energy subdiagram. For \( \mathcal{G}_{\text{self}}^{(2)} + \mathcal{G}^{(2)'} \), the contributions that are proportional to \( m^{2n} \) \( (n \geq 1) \) has been dropped, which do not yield leading contributions.

According to Eqs. (21) - (25), we write \( \langle \vec{p} \rangle_j \) \( (j = 1, 2) \) in Eq. (18), with obvious notation, as

\[
\langle \vec{p} \rangle_j = \langle \vec{p} \rangle_j^{(\text{self})} + \langle \vec{p} \rangle_j^{(2)} \quad (j = 1, 2).
\]

The quark self-energy part \( \Sigma (P) \) consists of two pieces, \( \Sigma (P) = \Sigma_{\text{vac}} (P) + \Sigma_{\text{mat}} (P). \) Here \( \Sigma_{\text{vac}} (P) \) stands for the self-energy part in vacuum theory, while \( \Sigma_{\text{mat}} (P) \) is due to the existence of matter. Eqs. (19) and (20) tell us that \( \langle \vec{p} \rangle_1^{(\text{self})} \) includes \( \Sigma_{\text{vac}} (P) \) and \( \langle \vec{p} \rangle_2^{(\text{self})} \) includes \( \Sigma_{\text{mat}} (P). \)
Observation on Eqs. (19) and (20)

It is not difficult to see that, both in Eq. (19) and in (20), an integration over \( P \) (with \( Q \) fixed) is free from ultraviolet divergence. Then, we introduce a Pauli-Villars regulator (with mass \( \Lambda_{PV} \)) for a gluon propagator, which makes the integrals in Eq. (19) and in Eq. (20) converge. This means that the contribution from the gauge-parameter \( (\eta) \) dependent part of the gluon propagator (Eq. (14)) vanishes, thanks to the Ward identity, and the results (28) - (36) below are independent of \( \eta \). Thus, we use the Feynman gauge, \( \eta = 0 \). In the following we equate \( \Lambda_{PV} \) with \( \Lambda \) in Eq. (12).

3.3 Form for \( \langle \vec{\mu}' \rangle \)

Here we display the resultant form for \( \langle \vec{\mu}' \rangle \),

\[
\langle \vec{\mu}' \rangle = \left[ C_1 + \vec{B}^2 C_2 \right] \vec{B}.
\] (27)

Summing over colors, we have for the contributions to \( C_1 \) and to \( C_2 \) from the vacuum sector and those owing to the existence of matter, in respective order,

\[
C_1^{(vac)} = -\frac{3e_q^2}{32\pi^2} \left[ 1 + \frac{4\alpha_s}{3\pi} \ln \frac{\Lambda}{m} \right] \ln \frac{\Lambda}{m},
\] (28)

\[
C_2^{(vac)} = \frac{e_q^4}{512\pi^2 m^4} \left[ 1 - \frac{4\alpha_s}{3\pi} \ln \frac{\Lambda}{m} \right] \ln \frac{\Lambda}{m},
\] (29)

and

\[
C_1^{(mat)} = \frac{3e_q^2 \mu_B^2}{32\pi^2 9m^2} \left[ 1 + \frac{3\Lambda}{\mu_B} \right],
\] (30)

\[
C_2^{(mat)} = -\frac{e_q^4}{512\pi^2 m^4} \left[ 1 + \frac{3\Lambda^2}{m \mu_B} \right] \ln \frac{\mu_B}{3m}.
\] (31)

Note that \( C_1^{(vac)} \big|_{\alpha_s \to 0} \), \( C_2^{(vac)} \big|_{\alpha_s \to 0} \) and \( C_1^{(mat)} \big|_{\alpha_s \to 0} \) are “exact” (cf. Eqs. (11) and (12)). To logarithmic accuracy, \( \ln \mu_B/(3m) \sim \ln \mu_B/m \). Nevertheless we choose \( \ln \mu_B/(3m) \) because the baryon number of the quark is 1/3 (cf. Eq. (7)). Adding both contributions, we obtain

\[
C_j = C_j^{(vac)} + C_j^{(mat)} = \left( C_j^{(0)} + \frac{4\alpha_s}{3\pi} C_j^{(1)} \right) \left( j = 1, 2 \right),
\] (32)

\[
C_1^{(0)} = -\frac{3e_q^2}{32\pi^2} \left( \ln \frac{\Lambda}{m} - \frac{\mu_B^2}{9m^2} \right),
\] (33)
\begin{align*}
C_1^{(1)} &= -\frac{3e_q^2}{32\pi^2} \left[ \left( \ln \frac{\Lambda}{m} \right)^2 + \frac{\mu_B^2}{9m^2} \ln \frac{3\Lambda}{\mu_B} \right], \\
C_2^{(0)} &= \frac{e_q}{512\pi^2} \frac{1}{m^4} \ln \frac{3\Lambda}{\mu_B}, \\
C_2^{(1)} &= -\frac{e_q}{512\pi^2} \frac{1}{m^4} \left( \ln \frac{3\Lambda}{\mu_B} \right)^2.
\end{align*}

Several comments on the $O(g^2)$ contributions are in order. To logarithmic accuracy, the contributions $\langle \vec{\mu}' \rangle_2$ (Eq. (20)) and $\langle \vec{\mu}' \rangle_1'$ (Eq. (26)) are negligible. In other words, the leading contributions displayed above come from $\langle \vec{\mu}' \rangle_1^{(\text{self})}$, to which the responsible diagrams include the vacuum piece $\Sigma_{\text{vac}}$ of the quark self-energy part. We see the point in more detail.

On $C_1^{(\text{vac})} \in \langle \vec{\mu}' \rangle_1'$: As mentioned above, integration over $P$ (with $Q$ fixed) in Eq. (19) converges. If the $P$-integration yields the term that is proportional to $\ln(\sqrt{-Q^2}/m)$, the remaining $Q$-integration yields $\ln^2(\Lambda/m)$ term. This is, however, not the case.

On $C_2^{(\text{vac})} \in \langle \vec{\mu}' \rangle_1'$: $P$-integration (with $Q$ fixed) in Eq. (19) yields the terms being proportional to $\ln(\sqrt{-Q^2}/m)$, which lead to the $\ln^2(\Lambda/m)$ terms. Cancellation occurs among them and no $\ln^2(\Lambda/m)$ term remains.

On $C_2^{(\text{mat})} \in \langle \vec{\mu}' \rangle_1'$: $Q$-integration using the regularized gluon propagator yields $\ln(\Lambda/m)$ terms. Cancellation occurs among those terms and then no $\ln(\Lambda/m)$ term remains. If the $Q$-integration yields the $\ln[(m^2 - P^2)/m^2]$ term, the remaining $P$-integration results in the leading $\ln^2(\mu_B/(3m))$ contribution. Among the $Q$-integrated terms are the $\ln[(m^2 - P^2)/m^2]$ ones. Adding all of those, cancellation occurs again, so that no $\ln^2(\mu_B/(3m))$ term remains.

On $\langle \vec{\mu}' \rangle_2$: Cancellation occurs between the leading contributions, so that there remains no leading contribution in $C_1^{(\text{vac})}$, $C_1^{(\text{mat})}$, $C_2^{(\text{vac})}$ and $C_2^{(\text{mat})}$.

### 3.4 Resummation

As has been mentioned above, the leading contributions (28) - (31) come from $\langle \vec{\mu}' \rangle_1^{(\text{self})}$, for which the diagrams include $\Sigma_{\text{vac}}$. In the region $(4\alpha_s/3\pi) \ln \Lambda/m \sim 1$ and/or
(4\alpha_s/3\pi) \ln \mu_B/(3m) \sim 1$, the expressions (28) - (31) are not reliable and an improvement is necessary through resummation of $\Sigma_{\text{vac}}$. To this end, it is convenient to start from Eqs. (11) and (13) and, on the RHS’s of them, make the substitutions

\[ P \rightarrow P(1 - \Sigma_w) \equiv Z_2^{-1} P(1 - \Sigma'_w), \]
\[ m \rightarrow m + \Sigma_m \equiv Z_2^{-1} Z_m(m + \Sigma'_m). \tag{37} \]

Here $\Sigma_w = \text{Tr}(P \Sigma_{\text{vac}})/(4P^2)$, $\Sigma_m = \text{Tr} \Sigma_{\text{vac}}/4$ and $Z_2 (Z_m)$ is the wave-function (mass) renormalization constant,

\[ Z_2 = 1 - \frac{2\alpha_s}{3\pi} \ln \frac{\Lambda}{m}, \]
\[ Z_m = 1 + \frac{2\alpha_s}{\pi} \ln \frac{\Lambda}{m}. \]

The substitutions (37) are not made for $m$ within $e_q/(4m)$ and for $p_0$ in $\theta(\mu_B/3 - p_0)$ in Eq. (13). It should be noted that we are using the Feynman gauge here.

The leading-logarithmic contributions (28) - (31) emerge from $Z_2^2$, $Z_m^2$ and the piece $\ln(m^2 - P^2)/m^2$ in $\Sigma_w'$ and $\Sigma_m'$. Then, to logarithmic accuracy, it is sufficient to resum the $\ln(m^2 - P^2)/m^2$ pieces in $\Sigma_w'$ and $\Sigma_m'$, which is gauge-parameter ($\eta$) independent. Thus, Eq. (37) may be simplified as

\[ P \rightarrow Z_2^{-1} P \left(1 - \frac{\alpha_s}{3\pi} \ln \frac{m^2 - P^2}{m^2}\right), \tag{38} \]
\[ m \rightarrow Z_2^{-1} Z_m m \left(1 - \frac{4\alpha_s}{3\pi} \ln \frac{m^2 - P^2}{m^2}\right). \tag{39} \]

Straightforward manipulation yields

\[ C_1^{(\text{vac})} = -\frac{3e_q^2}{32\pi^2} Z_2^2 Z_m^2 \left[1 - \frac{5\alpha_s}{3\pi} \ln \frac{\Lambda}{m} + 7 \left(\frac{2\alpha_s}{3\pi} \ln \frac{\Lambda}{m}\right)^2 \right] \ln \frac{\Lambda}{m}, \tag{40} \]
\[ C_2^{(\text{vac})} = -\frac{e_q^4}{1024\pi} Z_2^2 \left[1 - \frac{2\alpha_s}{3\pi} \ln \frac{\Lambda}{m}\right]^{-3}, \tag{41} \]
\[ C_1^{(\text{mat})} = \frac{3e_q^2 \mu_B^2}{32\pi^2} Z_2^2 \left[1 - \frac{2\alpha_s}{3\pi} \ln \frac{\mu_B}{3m}\right]^{-2}, \tag{42} \]
\[ C_2^{(\text{mat})} = \frac{e_q^4}{1024\pi} Z_2^2 \left[1 - \frac{2\alpha_s}{3\pi} \ln \frac{\mu_B}{3m}\right]^{-3}. \tag{43} \]
Adding both contributions, we obtain

\[ C_1 = C_1^{(\text{vac})} + C_1^{(\text{mat})} \]
\[ = -\frac{3e_q^2}{32\pi^2} Z_2^2 \left[ Z_m^2 \Omega^{-1} \left( 7 - 9\Omega^{-1} + 3\Omega^{-2} \right) \ln \frac{\Lambda}{m} - \frac{\mu_B^2}{9m^2\Omega m^2} \right], \]

(44)

\[ C_2 = C_2^{(\text{vac})} + C_2^{(\text{mat})} \]
\[ = \frac{e_q^4}{1024\pi m^4\alpha_s} Z_2^4 \left( \Omega^{-3} - \Omega_m^{-3} \right), \]

(45)

where

\[ \Omega = 1 - \frac{2\alpha_s}{3\pi} \ln \frac{\Lambda}{m} (= Z_2), \quad \Omega_m = 1 - \frac{2\alpha_s}{3\pi} \ln \frac{\mu_B}{3m}. \]

(46)

### 4 Effective potential

In this section, we study the structure of the effective potential \( V(\langle \vec{\mu} \rangle) \). The “generating function” \( W(\vec{B}^2) \) is related to \( \langle \vec{\mu} \rangle \), Eq. (27), through

\[ W(\vec{B}^2) = \int_0^{\vec{B}} \langle \vec{\mu} \rangle d\vec{\mu} = \frac{C_1}{2} \vec{B}^2 + \frac{C_2}{4} (\vec{B}^2)^2 \]

(47)

with \( C_1 \) and \( C_2 \) as in Eqs. (44) and (45), respectively. \( V(\langle \vec{\mu} \rangle) \) is defined by the following Legendre transformation on \( W(\vec{B}^2) \):

\[ V(\langle \vec{\mu} \rangle) = \langle \vec{\mu} \rangle \vec{B} - W(\vec{B}^2), \]

(48)

\[ \frac{dV(\langle \vec{\mu} \rangle)}{d\langle \vec{\mu} \rangle} = \vec{B}, \]

(49)

where \( \vec{B} (\equiv \vec{B}(\langle \vec{\mu} \rangle)) \) is related to \( \langle \vec{\mu} \rangle \) through Eq. (27).

When one is interested in \( V(\langle \vec{\mu} \rangle) \) with small \( \langle \vec{\mu} \rangle \), it is sufficient to obtain \( \vec{B}(\langle \vec{\mu} \rangle) \) in powers of \( \langle \vec{\mu} \rangle \). For obtaining more accurate information, however, we solve Eq. (27) exactly.

As will be seen below, in the region of our interest, \( C_2 > 0 \). For convenience, let us introduce the dimensionless quantities,

\[ \vec{M} \equiv \frac{3}{2} \sqrt{\frac{3C_2}{|C_1|^3}} \langle \vec{\mu} \rangle = M \frac{\vec{M}}{|M|} \quad (M \geq 0), \]

(50)

\[ \vec{B} \equiv \frac{3C_2}{|C_1|} \vec{B} = B \frac{\vec{M}}{|M|}, \]

(51)

\[ V(M) \equiv \frac{C_2}{|C_1|^2} V(\langle \vec{\mu} \rangle). \]

(52)
Eqs. (27), (48) and (49) turn out, in respective order, to

\[ \mathcal{B}^3(M) + 3\epsilon(C_1)\mathcal{B}(M) - 2M = 0, \]  
\[ \mathcal{V}(M) = \frac{2}{9} M \mathcal{B}(M) - \frac{1}{6} \epsilon(C_1)\mathcal{B}^2(M) - \frac{1}{36} \mathcal{B}^4(M), \]  
\[ \frac{d\mathcal{V}(M)}{dM} = \frac{2}{9} \mathcal{B}(M), \]  

where \( \epsilon(C_1) = \frac{C_1}{|C_1|} \).

Solving Eq. (53), we obtain the following expressions for \( \mathcal{B}(M) \).

\( C_1 > 0 \)

\[ \mathcal{B}(M) = \left( \sqrt{1 + M^2} + M \right)^{1/3} - \left( \sqrt{1 + M^2} - M \right)^{1/3}. \]  

From Eqs. (55) and (56), we see that

\[ \frac{d\mathcal{V}(M)}{dM} > 0 \]  

for \( M > 0 \),

\[ \frac{d\mathcal{V}(M)}{dM} = 0 \]  

at \( M = 0 \).

\( C_1 < 0 \)

(1) \( M \geq 1 \)

Solution (s1):

\[ \mathcal{B}(M) = \left( M + \sqrt{M^2 - 1} \right)^{1/3} + \left( M - \sqrt{M^2 - 1} \right)^{1/3}, \]

\[ \frac{d\mathcal{V}(M)}{dM} > 0. \]

(2) \( M < 1 \)

Solution (s2):

\[ \mathcal{B}(M) = 2 \cos \left( \frac{\arccos M}{3} \right), \]

\[ \frac{d\mathcal{V}(M)}{dM} > 0. \]

Solution (s3):

\[ \mathcal{B}(M) = 2 \cos \left( \frac{4\pi + \arccos M}{3} \right), \]

\[ \frac{d\mathcal{V}(M)}{dM} < 0 \]  

for \( 0 < M < 1 \),

\[ \frac{d\mathcal{V}(M)}{dM} = 0 \]  

at \( M = 0 \).
Solution (s4):

\[ B(M) = 2 \cos \left( \frac{2 \pi + \arccos M}{3} \right), \]

\[ \frac{dV(M)}{dM} < 0. \]

In Fig. 2, we display \( V(M) \) in the case of \( C_1 > 0 \). The curve “0” is \( V(M) \) with Eq. (56) for \( B(M) \). We see that, when \( C_1 > 0 \), the quark matter is in the normal phase.

In Fig. 3(a), we depict \( V(M) \) for \( C_1 < 0 \). The curves (s1), (s2), (s3) and (s4) show \( V(M) \) corresponding, in respective order, to the solutions (s1), (s2), (s3) and (s4) for \( B(M) \). The actual effective potential \( V(M) \) is as in Fig. 3(b). From this figure, we see that, when \( C_1 < 0 \), \( V(M) \) is minimum at \( M = 1 \) or \( \langle \vec{\mu}' \rangle \approx \sqrt{4|C_1|^3/(27C_2)} \), which means that the spontaneous magnetization occurs.

Here we discuss the effect of the higher-order (in \( \vec{B} \)) contribution to \( \langle \vec{\mu}' \rangle \) in Eq. (27),

\[ \langle \vec{\mu}' \rangle = [C_1 + C_2 \vec{B}^2 + C_3(\vec{B}^2)^2] \vec{B}. \] (57)

Eqs. (53) and (54) turn out to be

\[ B^3(M) + 3\epsilon(C_1)B(M) + \frac{|C_1|C_3}{3C_2^2}B^5(M) - 2M = 0, \] (58)

\[ V(M) = \frac{2}{9}M B(M) - \frac{1}{6}\epsilon(C_1)B^2(M) - \frac{1}{36}B^4(M) - \frac{|C_1|C_3}{162C_2^2}B^6(M). \] (59)

From these equations, we see first of all that, near the phase transition point, \( C_1 \approx 0 \), the effects of the “\( C_3 \) term” is very small. We will compute \( V(M) \) for the following two choices for \( |C_1|C_3/C_2^2 \):

\[ \frac{|C_1|C_3}{C_2^2} = -2, +2. \]

The curve “+” (“-”) in Fig. 2 depicts \( V(M) \) with \( |C_1|C_3/C_2^2 = +2 (-2) \). We see that the effect of the “\( C_3 \) term” in Eq. (57) on \( V(M) \) is small and the system is in the normal phase. Fig. 3(c) depicts \( V(M) \) in the case of \( C_1 < 0 \). For \( |C_1|C_3/C_2^2 = +2 \), the behavior of \( V(M) \) is qualitatively the same as that for \( C_3 = 0 \) and the spontaneous magnetization takes place with \( \langle \vec{\mu}' \rangle \approx 0.84\sqrt{4|C_1|^3/(27C_2)} \). On the other hand, for \( |C_1|C_3/C_2^2 = -2 \), \( V(M) \) decreases monotonically as \( M \) increases. If we take this result at face value, the system is in the ferromagnetic phase.
Recalling the fact that $C_1$ is a function of $\Lambda$ and $\mu_B$, we summarize the above observations:

- In the region of $\mu_B$ where $C_1(\Lambda, \mu_B) > 0$, the system is in the normal phase.
- In the region where $C_1(\Lambda, \mu_B) < 0$, the system is in the ferromagnetic phase.
- At $\mu_B = \mu_{Bc}$, where $C_1(\Lambda, \mu_{Bc}) = 0$, a magnetic phase transition between the above two phases takes place.

5 Numerical Analysis

In this section, on the basis of the observation in Sec. 4, we study the magnetic property of the quark matter through the numerical analysis in the region $0.6 \text{ GeV} \leq \mu_B \leq 4 \text{ GeV}$. Let start with two observations.

- As the renormalization scale $\mu_R$ (for the running coupling constant $\alpha_s(\mu_R)$ and the running mass $m(\mu_R)$), we choose $\mu_R = \mu_B$. There is an ambiguity on the choice of $\mu_R$. For computing the pressure of the quark-gluon plasma (QGP) at temperature $T$ and $\mu_B = 0$, $\mu_R \approx 2\pi T$ is chosen in the literature (see, e.g., Ref. 9)). The so-called thermal mass of a zero mass quark in QGP with finite $T$ and $\mu_B$ is $m_{\text{th}} \propto T^2 + \mu_B^2/(3\pi^2)$. These observations suggest the choice, $\mu_R = 2\pi[T^2 + \mu_B^2/(3\pi^2)]^{1/2} \frac{T^{-\alpha_s}}{2\pi} \times (\mu_B/\sqrt{3\pi}) \approx 1.15\mu_B$, which is not very different from $\mu_B$.

- Following Ref. 10), we identify the effective potential $V$ computed above with cutoff $\Lambda$ with the one renormalized at the renormalization scale $\mu_R = \Lambda$ in QCD.

On the basis of these observations, as a crude approximation, we choose $\Lambda = \mu_B$. Then, from Eqs. (45) and (46), $C_2(\mu_B) > 0$, provided that $\Omega(\mu_B) > 0$, which is the case in the region of our interest.

For the QCD $\alpha_s (= g^2/(4\pi))$, we take the running one $\alpha_s(\mu_B)$ in the $\overline{\text{MS}}$ scheme with $\alpha_s(\mu_B = 1.2 \text{ GeV}) = 0.39$. In Fig. 4, $\alpha_s(\mu_B)$ is depicted in the relatively small $\mu_B$ region, $600 \text{ MeV} \leq \mu_B \leq 700 \text{ MeV}$. For the quark masses, we take 1-loop running masses

$$m(\mu_R) = m(\mu_R') \left( \frac{\alpha_s(\mu_R)}{\alpha_s(\mu_R')} \right)^{2/(\pi\beta_0)} \quad (\beta_0 = (11 - 2n_f/3)/(2\pi)), $$
with
\[ m_u(\mu_R = 2 \text{ GeV}) = 3 \text{ MeV/c}^2, \quad m_d(\mu_R = 2 \text{ GeV}) = 6.8 \text{ MeV/c}^2, \]
\[ m_s(\mu_R = 2 \text{ GeV}) = 118 \text{ MeV/c}^2, \quad m_c(\mu_R = 1.2 \text{ GeV}) = 1.2 \text{ GeV/c}^2. \]

\( n_f \) is the number of quark flavors and the quark with mass \( m_q \) is counted for \( m_q < \mu_R \) (= \( \mu_B \)). For the QED fine structure constant \( \alpha \), we take the 1-loop running one. For computing it, we have used the above-mentioned quark masses and, for lepton masses, we have used \( m_e = 0.5 \text{ MeV/c}^2, m_\mu = 106 \text{ MeV/c}^2 \) and \( m_\tau = 1.78 \text{ GeV/c}^2 \).

The relation between the baryon-number density \( \rho \) and the baryonic chemical potential \( \mu_B \) is
\[ \rho(\mu_B) = 3 \times 2 \times \frac{1}{6\pi^2} \sum_{i=u,d,s,c} \theta \left( \frac{\mu_B}{3} - m_i(\mu_B) \right) \left( \frac{\mu_B^2}{9} - m_i^2(\mu_B) \right)^{3/2}, \quad (60) \]
where “3” is the number of colors and “2” comes from the spin degree of freedom.

We do not take into account the QCD correction to the relation (60) except that the running masses have been used. The nuclear density at the center of a nuclei is \( \rho_0 \simeq 0.17 \text{ fm}^{-3} \). The density of neutron stars are in the range, \( \rho = 10^{-3} \rho_0 \sim 10 \rho_0 \).

For a guide of eyes, the relation between \( \rho/\rho_0 \) and \( \mu_B \) is shown in Figs. 5(a) and (b), and the ratio \( \rho_i/\rho \ (i = u, d, s, c) \) is shown in Fig. 5(c).

**Quarks**

As mentioned above, \( C_2(\mu_B) > 0 \) in the region of interest. Numerical analysis shows that
\[
\begin{align*}
C_1(\mu_B) < 0 & \quad \text{for } \mu_B < \mu_{Bc}, \\
C_1(\mu_B) = 0 & \quad \text{for } \mu_B = \mu_{Bc}, \\
C_1(\mu_B) > 0 & \quad \text{for } \mu_B > \mu_{Bc},
\end{align*}
\]
with \( \mu_{Bc} \simeq 641, 625 \) and 693 MeV for, in respective order, \( u, d \) and \( s \) quarks. In Fig. 6, \( C_1(\mu_B) \) is depicted against \( \mu_B \) for \( 600 \text{ MeV} \leq \mu_B \leq 700 \text{ MeV} \), where the coefficient \( d_1 = 1.50 \) and \( 2 \times 10^4 \) for \( u, d \) and \( s \) quarks, respectively. Vertical dashed lines indicate the phase transition points. We see that, at high-density region, \( \mu_B > \mu_{Bc} \), \( C_1 > 0 \) and the quarks are in the normal phase. As the density is lowered, the quarks undergo a magnetic phase transition \( (C_1 = 0) \), at the critical density, into a ferromagnetic phase \( (C_1 < 0) \). We see from Fig. 3(b) (also Fig. 3(c)) that, in
the ferromagnetic phase, there arises spontaneous magnetization whose magnitude is

\[ |\langle \mu' \rangle| = \sqrt{4|C_1| \|/(27C_2)} M \simeq \sqrt{4|C_1| \|/(27C_2)} (\equiv \langle \mu' \rangle) \] (cf. Eq. (50)). We depict \( \langle \mu' \rangle \) in Fig. 7, where \( d_2 = 10^{-4}, 2 \times 10^{-3} \) and 0.1 for \( u, d, \) and \( s \) quarks respectively.

\( C_1 \) is the quantity that determines the phase of the system. Eq. (40) with \( \Lambda = \mu_B \) tells us that the vacuum-sector contribution to \( C_1 \) is negative, \( C_1^{(\text{vac})} < 0 \), in the region of our interest. While, from Eq. (42), \( C_1^{(\text{mat})}(\mu_B) \) is positive and an increasing function of \( \mu_B \). The effect of the QCD interaction does not change the signs of \( C_1^{(\text{vac})} \) and \( C_1^{(\text{mat})} \). At low densities \( C_1^{(\text{mat})} \) is so small that \( C_1 = C_1^{(\text{vac})} + C_1^{(\text{mat})} < 0 \). As \( \mu_B \) is increased \( C_1^{(\text{mat})} \) increases and we arrive at \( C_1 = 0 \) at \( \mu_B = \mu_{Bc} \). As \( \mu_B \) further increases, \( C_1 \) turns out to be positive. To understand the behavior of \( C_1^{(\text{mat})}(\mu_B) \), let us recall that the hamiltonian density describing the interaction of the system with the external magnetic field is \( \mathcal{H}_{\text{int}} = -\mu' \cdot \vec{B} \). This means that the configuration \( \mu' \parallel \vec{B} \) is energetically favorable, which reflects on \( C_1^{(\text{mat})} \bigg|_{\alpha_s = 0} > 0 \), Eq. (30) or (42) (cf. Eq. (27)).

Several observations are in order here.

- In spite of the fact that \( m_u < m_d < m_s \), we have obtained the curious result, \( (\mu_{Bc})_d < (\mu_{Bc})_u < (\mu_{Bc})_s \). This means that, as \( \mu_B \) is lowered from the normal-phase region, the heaviest \( s \) quarks are magnetized first, then follows the lightest \( u \) quarks and finally \( d \) quarks start are magnetized.

- The magnitude of the magnetization is larger for the lighter quark. This is a reflection of the fact that \( \mu' \propto 1/m \).

- The phase transition points \( \mu_{Bc} \approx 641, 625 \) and 693 MeV for, in respective order, \( u, d \) and \( s \) quarks correspond to the densities \( \rho \approx 1.62\rho_0, 1.47\rho_0 \) and \( 2.20\rho_0 \) with \( \rho_0 \) the nuclear density. These values for \( \rho \) are near to the nuclear density \( \rho_0 \) and then our results may be relevant to the quark star.

- As seen from Fig. 4, \( \alpha_s(\mu_B) \) is not very small at \( 600 \text{ MeV} \leq \mu_B \leq 700 \text{ MeV} \). Then, before arriving at definite conclusion, higher-order (in \( \alpha_s \)) contributions should be taken into account.\(^2\)

\(^2\)In this relation, we recall the fact that, for a dense electron system, the higher-order contributions substantially changes [11] the magnetic phase diagram when compared to the calculations using the random-phase approximation.
Here we mention the case with the order parameter $\bar{\mu} = 2\bar{\mu}'$ (see Appendix A). It can readily be seen that $C_1 = 4C_1(\text{Eq. (44)})/4$ and $C_2 = 16C_2(\text{Eq. (45)})$. The phase transition point $\mu_{Bc}$ does not change.

**Leptons**

Our primary interest is to study the magnetic property of the quark matter, like quark star. Such a system is electrically neutral, so that there exists leptons. From the charge neutrality, we obtain the relation between the baryonic chemical potential $\mu_B$ and the leptonic chemical potential $\mu_L$ being conjugate to the lepton number. We see that, for $0.6 \text{ GeV} \leq \mu_B \leq 4 \text{ GeV}$, $\mu_L$ covers the range $174 \text{ MeV} \leq \mu_L \leq 969 \text{ MeV}$. In Fig. 8(a), we depict the relation between them in the region $0.6 \text{ GeV} \leq \mu_B \leq 3 \text{ GeV}$, and, in Fig. 8(b), the ratio $\rho_i/\rho$ $(i = e, \mu)$ is shown in the region $174 \text{ MeV} \leq \mu_L \leq 300 \text{ MeV}$. An abrupt increase in $\mu_L$ at $\mu_B \simeq 2.9 \text{ GeV}$ is due to the opening of the c quark channel, $m_c(\mu_B \simeq 2.9 \text{ GeV}) \simeq 2.9/3 \text{ GeV}/c^2$.

For the QED coupling constant, we take $\alpha = 1/136$ and, for the lepton masses, we take the values cited at the beginning of Sec. 5. Numerical analysis shows that, in the region $174 \text{ MeV} \leq \mu_L \leq 969 \text{ MeV}$, the leptons are in the normal phase.

6 **Summary and Outlook**

In this paper, we have studied the magnetic property of the quark matter through evaluating the effective potential $V(\langle \bar{\mu}' \rangle)$ for the magnetic moment $\bar{\mu}'$ of a quark. We have found that, at low densities $\rho < \rho_c$, the quarks are in the ferromagnetic phase. For the $u$, $d$ and $s$ quarks, $\rho_c \simeq 1.62\rho_0$, $1.47\rho_0$ and $2.20\rho_0$ ($\rho_0$ the nuclear density), respectively. The phase transition occurs at $\rho = \rho_c$ and, at high densities $\rho > \rho_c$, the quarks turn out to be in the normal phase.

We have also studied the case of leptons in the region $174 \text{ MeV} \leq \mu_L \leq 969 \text{ MeV}$ ($\mu_L$ the leptonic chemical potential), which corresponds to $0.6 \text{ GeV} \leq \mu_B \leq 4 \text{ GeV}$. We have found that both $e$ and $\mu$ are in the normal phase.

We list the several points which are left for future study.

1. When more than two kinds of particles are in ferromagnetic phases, for determining the relative direction(s) of their magnetizations, interactions between
them should be taken into account.

2. Study of the quark matter at finite temperature and density.

3. Analysis in the region ‘CSC’ in Fig. 1, where several phases coexist.

4. Undoing the assumption made after Eq. (1) in Appendix A.

5. Improvements of the approximation.

   a) Undoing the leading logarithmic approximation (1).

   b) Resummations for the (quark and gluon) propagators and the quark-gluon vertex.

   c) Computation of the higher-order contributions. This is particularly important for studying the ferromagnetic phase, since $\alpha_s(\mu_B)$ is not very small there.

   d) Resummation of, e.g., the ladder diagrams. From thus obtained effective potential and the renormalization-group formula, one can expect to obtain (approximately) $\Lambda$-independent result. (For the case of chiral and diquark condensations, see, e.g., Ref. 10.)

6. Computation of the effective action, which allows us to study the dynamical aspect of the system.

7. Study of nonequilibrium quark matter, which allows one to deal with the space-time evolution of the system under a given initial data.

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Appendix A Order parameter

We start with the action that describes the interaction between a quark (with charge $e_q$) and uniform external magnetic field $\vec{B}$:

$$
\int d^4x \mathcal{L}_{\text{ext}} = e_q \int d^4x \bar{\psi} \gamma \cdot \vec{A} \psi
$$

where $\psi$ and $\bar{\psi}$ are the quark fields in the interaction representation. Here the color index is suppressed. Using the Dirac equation, we have

$$
0 = \bar{\psi}(x) \left( -i \vec{\partial} - m \right) \gamma \cdot \vec{A}(x) \psi(x) + \bar{\psi}(x) \gamma \cdot \vec{A}(x) \left( i \vec{\partial} - m \right) \psi(x),
$$

which yields

$$
2m \bar{\psi} \gamma \cdot \vec{A} \psi = \bar{\psi} \left[ -i \vec{A} \cdot \vec{\nabla} + A^i \partial_\mu \sigma^{i\mu} + i \partial_i \vec{A} - \partial_\mu \sigma^{\mu i} A^i \right] \psi.
$$

Using $\vec{A} = \vec{B} \times \vec{r}/2$, we finally obtain

$$
\int d^4x \mathcal{L}_{\text{ext}} = \frac{e_q}{2m} \int d^4x \bar{\psi} \left[ -2i \vec{B} \cdot (\vec{r} \times \vec{\nabla}) + \sigma^{ki} \epsilon^{ijk} B^j \right] \psi
$$

We take a statistical ensemble of the systems, each of which has vanishing total angular momentum. We further assume that, in each system, the total angular momentum of quarks and that of antiquarks vanish separately. Incidentally, the antiquarks exist as the virtual particles, through repetition of the $q\bar{q}$-pair productions (from the vacuum) and the pair annihilations. Then, Eq. (1) turns out to

$$
\int d^4x \mathcal{L}_{\text{ext}} = \int d^4x \bar{\mu}' \cdot \vec{B},
$$

$$
\bar{\mu}' = \frac{\bar{\mu}}{2} = \frac{1}{2} \left( \frac{e_q}{2m} \right) \bar{\psi} \sigma \psi = \frac{e_q}{4m} \bar{\psi} \gamma_5 \gamma_0 \gamma \psi.
$$

As the order parameter, we adopt $\bar{\mu}'$.

It is to be noted that if we ignore the contribution from the orbital angular momentum in Eq. (1), we obtain $\bar{\mu}' (= 2\bar{\mu})$ for the order parameter.
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Fig. 1. Sketch of the QCD phase diagram. ‘T’ is the temperature and ‘$\mu_B$’ is the baryonic chemical potential. ‘Hadron’ stands for the hadronic phase, ‘QGP’ for the quark-gluon plasma phase and ‘CSC’ for several phases that include the color super conducting phases.

Fig. 2. $\mathcal{V}(M)$ in the case of $C_1 > 0$ against $M$. The curve with “0 ” is the $\mathcal{V}(M)$ with Eq. (56) for $\mathcal{B}(M)$. The curves with “+ ” and “- ” is $\mathcal{V}(M)$ in Eq. (59) with $|C_1|C_3/C_2^2 = +2$ and $-2$, respectively.

Fig. 3. $\mathcal{V}(M)$ in the case of $C_1 < 0$. (a) The curves (s1), (s2), (s3) and (s4) are $\mathcal{V}(M)$ with, in respective order, the solutions (s1), (s2), (s3) and (s4) for $\mathcal{B}(M)$. (b) Actual effective potential. (c) The curve with “0 ” is as in Fig. (b) and the curves with “+ ” and “- ” are $\mathcal{V}(M)$ in Eq. (59) with $|C_1|C_3/C_2^2 = +2$ and $-2$, respectively.

Fig. 4. $\alpha_s(\mu_B)$ against $600 \text{ MeV} \leq \mu_B \leq 700 \text{ MeV}$.

Fig. 5. (a) The ratio $\rho/\rho_0$ against $0.6 \text{ GeV} \leq \mu_B \leq 1.6 \text{ GeV}$. (b) Same as (a) for $0.6 \text{ GeV} \leq \mu_B \leq 4 \text{ GeV}$. (c) The ratio $\rho_i/\rho$ ($i = u, d, s, c$) against $\mu_B$.

Fig. 6. $C_1(\mu_B)$ against $600 \text{ MeV} \leq \mu_B \leq 700 \text{ MeV}$. $d_1 = 1, 50$ and $2 \times 10^4$ for $u, d$ and $s$ quarks.

Fig. 7. $|\langle \mu' \rangle| = \sqrt{4|C_1|^3/(27C_2^2)}$ against $600 \text{ MeV} \leq \mu_B \leq 700 \text{ MeV}$. $d_2 = 10^{-4}, 2 \times 10^{-3}$ and 0.1 for $u, d$ and $s$ quarks.

Fig. 8. (a) Relation between $\mu_L$ and $\mu_B$. (b) The ratio $\rho_i/\rho$ ($i = e, \mu$) against $\mu_L$. 

22
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