SU(3) Yang-Mills Hamiltonian in the flux-tube gauge:
Strong coupling expansion and glueball dynamics

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Abstract

It is shown that the formulation of the SU(3) Yang-Mills quantum Hamiltonian in the "flux-tube gauge"
\[ A_{a1} = 0 \text{ for all } a = 1, 2, 4, 5, 6, 7 \quad \text{and} \quad A_{a2} = 0 \text{ for all } a = 5, 7, \]
allows for a systematic and practical strong coupling expansion of the Hamiltonian in \( \lambda \equiv g^{-2/3} \)
equivalent to an expansion in the number of spatial derivatives. Introducing an infinite spatial lattice with box length \( a \), the "free part" is the sum of Hamiltonians of Yang-Mills quantum mechanics of constant fields for each box, and the "interaction terms" contain higher and higher number of spatial derivatives connecting different boxes. The Faddeev-Popov operator, its determinant and inverse, are rather simple, but show a highly non-trivial periodic structure of six Gribov-horizons separating six Weyl-chambers. The energy eigensystem of the gauge reduced Hamiltonian of SU(3) Yang-Mills mechanics of spatially constant fields can be calculated in principle with arbitrary high precision using the orthonormal basis of all solutions of the corresponding harmonic oscillator problem, which turn out to be made of orthogonal polynomials of the 45 components of eight irreducible symmetric spatial tensors. First results for the low-energy glueball spectrum are obtained which substantially improve those by Weisz and Ziemann using the constrained approach. Thus, the gauge reduced approach using the flux-tube gauge proposed here, is expected to enable one to obtain valuable non-perturbative information about low-energy glueball dynamics, using perturbation theory in \( \lambda \).

1 Introduction

The Hamiltonian approach [1], with the possibility to use the powerful variational method, has turned out to be very suitable for non-perturbative investigations of Yang-Mills theory.

In this work, the Yang-Mills theory for SU(3) gauge fields \( V^a_\mu(x) \) is considered, defined by the action

\[
S[V] := -\frac{1}{4} \int d^4x \, F^{a}_{\mu\nu} F^{a\mu\nu}, \quad F^{a}_{\mu\nu} := \partial_\mu V^a_\nu - \partial_\nu V^a_\mu + g f^{abc} V^b_\mu V^c_\nu, \tag{1}
\]
invariant under Poincaré and scale transformations, and under local SU(3) gauge transformations \( U[\omega(x)] \equiv \exp(\im \omega_a \lambda_a/2) \)

\[
V^a_\mu(x) \lambda_a/2 = U[\omega(x)] \left( V^a_\mu(x) \lambda_a/2 + \frac{i}{g} \partial_\mu \right) U^{-1}[\omega(x)]. \tag{2}
\]

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The corresponding quantum theory is then obtained by exploiting the time dependence of the gauge transformations to put
\[ V_{a0}(x) = 0 , \quad a = 1, \ldots, 8 , \quad \text{(Weyl Gauge)} \]
and imposing canonical commutation relations on the spatial components of the gauge fields and (the negative of) the chromoelectric fields, using the Schrödinger representation
\[ [\Pi_{ai}(x), V_{bj}(y)] = i\delta_{ab}\delta_{ij}\delta(x-y) \quad \rightarrow \quad \Pi_{ai}(x) = -E_{ai}(x) = -i\delta/\delta V_{ai}(x) . \]
The physical states \( \Psi \) have to satisfy the system of equations (see e.g. [1])
\[ (H - E)\Psi = 0 \quad \text{(Schrödinger equation)} , \]
\[ G_a(x)\Psi = 0 \quad \text{(Gauss law constraints)} , \quad \text{(3)} \]
with the Hamiltonian
\[ H = \int d^3x \frac{1}{2} \sum_{a,i} \left[ \Pi_{ai}^2(x) + B_{ai}^2(V(x)) \right] , \quad \text{(4)} \]
in terms of the chromoelectric fields \(-\Pi_{ai}\) and the chromomagnetic \(B_{ai}(V)\) which are the generators of the residual time independent gauge transformations, commute with the Hamiltonian and satisfy angular momentum commutation relations
\[ [G_a(x), H] = 0 , \quad [G_a(x), G_b(y)] = ig\delta(x-y)f_{abc}G_c(x) , \]
The matrix elements are Cartesian
\[ \langle \Psi_1 | O | \Psi_2 \rangle = \int \prod_{ai} dV_{ai} \Psi_1^* O \Psi_2 . \quad \text{(6)} \]

In order to calculate the eigenstates and their energies, it is useful to implement the non-Abelian Gauss law constraints into the Schrödinger equation by further fixing the gauge using the remaining time-independent gauge transformations.

One possibility, well suited for the high energy sector of the theory, is to impose the Coulomb gauge \( \chi_a(A) = \partial_i A_{ai} = 0 \) describing the dynamics in terms of physical colored transverse gluons. For example, integrating out all higher modes in a small box of size \( \alpha \), Lüscher [2] obtained a weak coupling expansion for the energies of the constant Yang-Mills fields, \( E = \frac{1}{2\alpha} \sum_k^{\infty} \epsilon_k \lambda^k \), \( \lambda \equiv [g(\Lambda_{MS0})]^{2/3} \), with the standard running coupling constant in the MS scheme. Already some time ago, Weisz and Ziemann [3], when applying Lüscher’s results to the case of \( SU(3) \), obtained rather accurate values for the eigensystem of \( SU(3) \) Yang-Mills quantum mechanics of spatially constant fields working directly in the unreduced \( V \)-space discussed above.

For the low-energy sector of \( SU(2) \) Yang-Mills theory, the symmetric gauge [4] has been proven to exist [5] and was shown in [6] to be very well suited to describe non-perturbative glueball dynamics by demonstrating how the symmetric gauge allows for a gauge invariant formulation of \( SU(2) \) Yang-Mills theory on a three dimensional spatial lattice by replacing integrals by sums and spatial derivatives by differences. Using the symmetric gauge and constructing the corresponding physical quantum Hamiltonian of \( SU(2) \) Yang-Mills theory according to the general scheme given by Christ and Lee [1], it was proven in [6] that a strong coupling expansion of the \( SU(2) \) Yang-Mills quantum Hamiltonian in \( \lambda = g^{-2/3} \) can be carried out equivalent to an expansion in the number of spatial derivatives. Introducing an infinite lattice with box length \( a \), a systematic strong coupling expansion of the Hamiltonian in \( \lambda \) can be obtained, with the “free part” being the sum of Hamiltonians of Yang-Mills quantum mechanics of constant fields for each box, and “interaction terms” of higher and higher number of spatial derivatives connecting different
The existence of the flux-tube gauge (7) for SU(3) Yang-Mills theories in $D \geq 2$ spatial dimensions, can be proven by construction for the case of spatially constant fields $1$. One first

1This is the generalisation of the SU(2) gauge fixation introduced by Green and Gutperle [18], first rotating the spatial 1-component into the color-3 direction, $A_{a1}^{SU2} = 0$ for all $a = 1, 2$, and then use remaining color-rotations orthogonal to the $a = 3$ direction, generated by $\sigma_3$, which leave the $a = 3$ components of all $A_{a1}^{SU2}$ for all $i = 1, 2, 3$ unchanged, in order to put in the spatial 2-component $A_{a2}^{SU2} = 0$ for $a = 1$. 

In this work a new gauge, the flux-tube gauge, is proposed, which exists for low-energy SU(3) Yang-Mills theory, and allows for a derivative expansion with a rather simple, but non-trivial Faddeev-Popov operator, showing a 6-fold singularity structure with six Gribov horizons separating six Weyl-chambers. Here, the rather accurate recent results for the low-energy eigensystem of SU(3) Yang-Mills Quantum Mechanics [17] are used, substantially improving the results by Weisz and Ziemann [3].

The article is organised as follows: In Sec.2, the flux-tube gauge is defined and the corresponding unconstrained quantum Hamiltonian presented. In Sec.3, the expansion of the Hamiltonian up to second order in the number of spatial derivatives is carried out. Sec.4 revises the coarse graining method and the calculation of the glueball spectrum (e.g.[12],[13]).

In the standard notation of [1] this gauge condition can be written in the form

\[ \chi_a(A) = (\Gamma_i)_{ab} A_{bi} = 0 , \quad a = 1, ..., 8 . \] (8)

The explicit form of the $8 \times 8$-matrices $\Gamma_i$ which satisfy $\Gamma_i^T \Gamma_i = 1$ is

\[
\Gamma_i \equiv \delta_{i1} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} + \delta_{i2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} . \] (9)

The generalisation of the symmetric gauge from SU(2) Yang-Mills theory to SU(3), proposed in [14]-[16], indeed leads to a unconstrained dynamical description in terms of 16 colorless variables with definite spin quantum numbers, spin-0, spin-1, spin-2, and spin-3, and in principle allows for an expansion of the Hamiltonian in spatial derivatives, but due to an algebraically highly intricate FP-operator, is not practically manageable at the moment.

In the Appendix the proof of the existence of the flux-tube gauge for strong coupling is given.

## 2 Physical SU(3) Quantum Hamiltonian in the flux-tube gauge

### 2.1 Definition of the flux-tube gauge

I shall here choose the flux-tube gauge

\[ A_{a1} = 0 \quad \forall a = 1, 2, 4, 5, 6, 7 \quad \land \quad A_{a2} = 0 \quad \forall a = 5, 7 . \] (7)

In the standard notation of [1] this gauge condition can be written in the form

\[ \chi_a(A) = (\Gamma_i)_{ab} A_{bi} = 0 , \quad a = 1, ..., 8 . \] (8)

The explicit form of the $8 \times 8$-matrices $\Gamma_i$ which satisfy $\Gamma_i^T \Gamma_i = 1$ is

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\Gamma_i \equiv \delta_{i1} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} + \delta_{i2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} . \] (9)

The existence of the flux-tube gauge (7) for SU(3) Yang-Mills theories in $D + 1$ dimensions with $D \geq 2$ spatial dimensions, can be proven by construction for the case of spatially constant fields $1$: One first
diagonalises the spatial 1-component of the gauge field in the fundamental representation, \( A_{a1} = 0 \) for all \( a = 1, 2, 4, 5, 6, 7 \). Then, one uses the remaining gauge-freedom, generated by \( \lambda_3 \) and \( \lambda_8 \), which leave the \( a = 3, 8 \) components of all \( A_{ai} \) for all \( i = 1, 2, 3 \) unchanged, to put the spatial 2-components \( A_{a2} = 0 \) for \( a = 5, 7 \). Such a construction is unique for field-configurations, for which the corresponding homogeneous Faddeev-Popov determinant, to be discussed in Sect. 2.3., is non-vanishing. A more detailed discussion of the existence of the flux-tube gauge and the extension of the proof to the case of spatially slightly varying fields in the sense of a strong-coupling expansion, is given in App. A "On the existence of the flux-tube gauge".

The flux-tube gauge corresponds to the point transformation to the new set of adapted coordinates, the 8 \( q_j \) \((j = 1, ..., 8)\) and the 16 elements \( A_{ai} \)

\[
V_{ai}(q, A) = O_{ab}(q) A_{bi} - \frac{1}{2g} f_{abc} (O(q) \partial_i O^T(q))_{bc},
\]

where \( O(q) \) is an orthogonal \( 8 \times 8 \) matrix adjoint to \( U(q) \) parametrized by the 8 \( q_i \) and the physical fields

\[
A = \begin{pmatrix}
0 & A_{12} & A_{13} \\
0 & A_{22} & A_{23} \\
A_{31} & A_{32} & A_{33} \\
0 & A_{42} & A_{43} \\
0 & 0 & A_{53} \\
0 & A_{62} & A_{63} \\
0 & 0 & A_{73} \\
A_{81} & A_{82} & A_{83}
\end{pmatrix} \equiv (X \ Y \ Z).
\]

2.2 Physical \( SU(3) \) Quantum Hamiltonian

After the above coordinate transformation (10), the non-Abelian Gauss-law constraints become the Abelian conditions

\[
G_a \Psi = 0 \iff \frac{\delta}{\delta q_i} \Psi = 0 \quad \text{(Abelianisation)},
\]

that the physical states should depend only on the physical variables \( A_{ik} \), and the system (3) reduces to the unconstrained Schrödinger equation

\[
H(A, P)\Psi(A) = E\Psi(A).
\]

The correctly ordered physical quantum Hamiltonian [1] in the flux-tube gauge in terms of the physical variables \( A_{ik}(x) \) and the corresponding canonically conjugate momenta \( P_{ik}(x) \equiv -i\delta/\delta A_{ik}(x) \) reads

\[
H(A, P) = \frac{1}{2} \mathcal{J}^{-1} \int d^3x \int d^3y \ P_{ai}(x) \ J K_{ai|bk}(x, y) \ P_{bk}(y) + \frac{1}{2} \int d^3x (B_{ai}(A))^2,
\]

with the kernel

\[
K_{ai|bk}(x, y) \equiv \delta_{ab}\delta_{ik}\delta(x - y) - \langle x \ a | D_i(A)^{*}D(A)^{-1} D_j^{*}(A)^{-1} D_k(A) | y \ b \rangle,
\]

the Jacobian

\[
\mathcal{J} \equiv \det |D(A)|.
\]

Here \( D_i(A) \) is the antisymmetric operator whose matrix elements are the covariant derivatives

\[
\langle x \ a | D_i(A) | y \ b \rangle = D_i(A)^{(x)}_{ab} \delta(x - y) \equiv \left( \delta_{ab}\delta_i^{(x)} - g f_{abc} A_{ci}(x) \right) \delta(x - y),
\]

and \( D^{*}(A) \) the Faddeev-Popov (FP) operator whose matrix elements are defined as

\[
\langle x \ a | D^{*}(A) | y \ b \rangle = D^{*}(A)^{(x)}_{ab} \delta(x - y) \equiv (\Gamma_i)_{ac} D_i(A)^{(x)}_{cb} \delta(x - y) = \left( (\Gamma_i)_{ab}\delta_i^{(x)} - g \gamma_{ab}(x) \right) \delta(x - y),
\]
with the homogeneous part of the FP operator

$$\gamma_{ab}(x) \equiv (\Gamma_i)_{ad} f_{dce} A_{ce}(x).$$  (17)

Its explicit expression is

$$\gamma = \begin{pmatrix}
0 & -X_3 & 0 & 0 & 0 & 0 & 0 & 0 \\
X_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-Y_6/2 & 0 & -Y_4/2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & Y_1/2 & -Y_2/2 & -\sqrt{3}Y_4/2 & 0 & 0 \\
0 & 0 & -X_+ & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & X_+ & 0 \\
-Y_4/2 & 0 & Y_6/2 & Y_1/2 & Y_2/2 & Y_- & 0 & -\sqrt{3}Y_6/2 \\
\end{pmatrix},$$  (18)

using the abbreviations $X_\pm := -(X_3 \pm \sqrt{3}X_8)/2$ and $Y_\pm := -(Y_3 \pm \sqrt{3}Y_8)/2$.

The matrix element of a physical operator $O$ is given by

$$\langle \Psi_1 | O | \Psi_2 \rangle \propto \int \prod_x [dA(x)] J \Psi_1^*[A] O \Psi_2[A].$$  (19)

### 2.3 Existence of an expansion in the number of spatial derivatives

The Green functions of the FP operator (16), and hence the non-local terms of the physical Hamiltonian, can -in contrast for example to the Coulomb gauge- be expanded in the number of spatial derivatives.

$$\langle x \ a | D(A)^{-1} | y \ b \rangle = \frac{1}{g} \left( \gamma^{-1}(x) \right)_{ac} \left[ \delta_{cb} \delta(x-y) + \frac{1}{g} \left( \Gamma_i \right)_{cd} \partial_i^{(x)} \left[ \left( \gamma^{-1}(x) \right)_{db} \delta(x-y) \right] \right]$$

$$+ \frac{1}{g^2} \left( \Gamma_i \right)_{cd} \partial_i^{(x)} \left[ \left( \gamma^{-1}(x) \right)_{de} \left( \Gamma_j \right)_{en} \partial_j^{(x)} \left[ \left( \gamma^{-1}(x) \right)_{nb} \delta(x-y) \right] \right] + \ldots \right].$$

The inverse $\gamma^{-1}$ of the homogeneous part of the Faddeev-Popov operator (18) exists in the regions of non-vanishing determinant

$$\det \gamma = X_3^2 \left( X_3^2 - 3 X_8^2 \right)^2 Y_4 Y_6$$  (20)

and the non-vanishing matrix elements of the inverse $\gamma^{-1}$ are rather simple,

$$\left( \gamma^{-1} \right)_{12} = - \left( \gamma^{-1} \right)_{21} = X_3^{-1},$$

$$\left( \gamma^{-1} \right)_{32} = \frac{1}{2} X_3^{-1} (Y_4/Y_6 - Y_6/Y_4) , \quad \left( \gamma^{-1} \right)_{82} = - \frac{1}{2\sqrt{3}} X_3^{-1} (Y_4/Y_6 + Y_6/Y_4) ,$$

$$\left( \gamma^{-1} \right)_{45} = - \left( \gamma^{-1} \right)_{54} = -X_+^{-1}, \quad \left( \gamma^{-1} \right)_{34} = -\sqrt{3} \left( \gamma^{-1} \right)_{84} = - \frac{1}{2} X_+^{-1} (Y_2/Y_6) ,$$

$$\left( \gamma^{-1} \right)_{35} = - \frac{1}{2} X_+^{-1} (Y_1/Y_6 - 2 Y_+ Y_4) , \quad \left( \gamma^{-1} \right)_{85} = - \frac{1}{2\sqrt{3}} X_+^{-1} (Y_1/Y_6 - 2 Y_+ Y_4) ,$$

$$\left( \gamma^{-1} \right)_{67} = - \left( \gamma^{-1} \right)_{76} = X_-^{-1}, \quad \left( \gamma^{-1} \right)_{36} = \sqrt{3} \left( \gamma^{-1} \right)_{86} = \frac{1}{2} X_-^{-1} (Y_2/Y_4) ,$$

$$\left( \gamma^{-1} \right)_{37} = \frac{1}{2} X_-^{-1} (Y_1/Y_4 - 2 Y_- Y_6) , \quad \left( \gamma^{-1} \right)_{87} = \frac{1}{2\sqrt{3}} X_-^{-1} (Y_1/Y_4 + 2 Y_- Y_6) ,$$

$$\left( \gamma^{-1} \right)_{33} = \sqrt{3} \left( \gamma^{-1} \right)_{83} = Y_4^{-1}, \quad \left( \gamma^{-1} \right)_{38} = -\sqrt{3} \left( \gamma^{-1} \right)_{88} = Y_6^{-1},$$  (21)

grouped into those proportional $X_3^{-1}$, $X_+^{-1}$, and $X_-^{-1}$, respectively and those independent of $X$. Such a "Weyl-decomposition" will lead to a considerable simplification of the non-local part of the kernel (14).

From the definition of the gauge-fixing (9) we have

$$(\Gamma_i)_{ad} \partial_i^{(x)} \left[ \gamma_{cb}^{-1}(x) \delta(y) \right] = (\Gamma_1)_{ad} \partial_1^{(x)} \left[ \gamma_{cb}^{-1}(x) \delta(y) \right] + (\Gamma_2)_{ad} \partial_2^{(x)} \left[ \gamma_{cb}^{-1}(x) \delta(y) \right].$$  (22)
Using the values (21), we find
\[
(\Gamma_1)_{ac} \partial_1^{(x)} \left[ \gamma^{-1}_{cb} (x) \delta(x - y) \right] = Q_{ab}^{(0)} \partial_1^{(x)} \left[ \frac{1}{X_3 (x)} \delta(x - y) \right] + Q_{ab}^{(+)} \partial_1^{(x)} \left[ \frac{1}{X_+ (x)} \delta(x - y) \right] + Q_{ab}^{(-)} \partial_1^{(x)} \left[ \frac{1}{X_- (x)} \delta(x - y) \right],
\]
with the \(8 \times 8\) matrices
\[
Q_{ab}^{(0)} = \delta_a [1 \delta_{b2}] \quad Q_{ab}^{(+)} = -\delta_a [4 \delta_{b5}] \quad Q_{ab}^{(-)} = \delta_a [6 \delta_{b7}],
\]
and
\[
(\Gamma_2)_{ac} \partial_2^{(x)} \left[ \gamma^{-1}_{cb} (x) \delta(x - y) \right] = Q_{ab}^{(3)} \partial_2^{(x)} \left[ \frac{1}{X_+ (x)} \delta(x - y) \right] + Q_{ab}^{(8)} \partial_2^{(x)} \left[ \frac{1}{X_- (x)} \delta(x - y) \right],
\]
with the \(8 \times 8\) matrices
\[
Q_{ab}^{(3)} = \delta_a 3 \delta_{b5} \quad Q_{ab}^{(8)} = -\delta_a 8 \delta_{b7}.
\]

The only non-vanishing combinations turn out to be
\[
\langle x \ a | -g^\gamma *D(A)^{-1} | y \ b \rangle = \delta_{ab} \delta(x - y) + \sum_{n=1}^{\infty} \langle x \ a | \left( Q^{(0)} \partial_1 \frac{1}{X_3} \right)^n | y \ b \rangle + \sum_{n=1}^{\infty} \langle x \ a | \left( Q^{(+)} \partial_1 \frac{1}{X_+} \right)^n | y \ b \rangle + \sum_{n=1}^{\infty} \langle x \ a | \left( Q^{(-)} \partial_1 \frac{1}{X_-} \right)^n | y \ b \rangle + \sum_{n=0}^{\infty} \langle x \ a | Q^{(3)} \partial_2 \frac{1}{X_+} \left( Q^{(+)} \partial_1 \frac{1}{X_+} \right)^n | y \ b \rangle + \sum_{n=0}^{\infty} \langle x \ a | Q^{(8)} \partial_2 \frac{1}{X_-} \left( Q^{(-)} \partial_1 \frac{1}{X_-} \right)^n | y \ b \rangle.
\]

This leads to the possibility to write the non-local part of the kernel (14) in much simpler form.

\subsection{Weyl-decomposition" of the non-local potential}

Writing the non-local part of the kernel (14) with two arbitrary functions \(R_a(x)\) and \(O_a(x)\),
\[
R_a(x) \langle x \ a | *D(A)^{-1} *D^\dagger(A)^{-1} | y \ b \rangle \ O_b(y) ,
\]
in the form
\[
\frac{1}{g^2} R_a(x) \gamma^{-1}_{ac} (x) \langle x \ c | \left( -g^\gamma *D(A)^{-1} \right) \left( -g^\gamma *D(A)^{-1} \right)^\dagger | y \ d \rangle \gamma^{-1}_{db} (y) \ O_b(y) ,
\]
and noting that we can write
\[
\begin{pmatrix}
\gamma^{-1}_{16} T O_b \\
\gamma^{-1}_{2b} T O_b \\
\gamma^{-1}_{2b} T O_b \\
\gamma^{-1}_{2b} T O_b \\
\gamma^{-1}_{2b} T O_b \\
\gamma^{-1}_{2b} T O_b \\
\gamma^{-1}_{2b} T O_b \\
\gamma^{-1}_{2b} T O_b
\end{pmatrix} = \frac{1}{X_3} Q^T \begin{pmatrix}
\tilde{O}_1 \\
\tilde{O}_2 \\
\tilde{O}_3 \\
\tilde{O}_4 \\
\tilde{O}_5 \\
\tilde{O}_6 \\
\tilde{O}_7 \\
\tilde{O}_8
\end{pmatrix},
\]
\[
\gamma^{-1}_{3b} T O_b = \frac{1}{Y_4} \left( O_3 + \frac{1}{\sqrt{3}} O_8 \right), \quad \gamma^{-1}_{8b} T O_b = \frac{1}{Y_6} \left( O_3 - \frac{1}{\sqrt{3}} O_8 \right),
\]
with the two-dimensional matrix
\[
Q := \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix} ,
\]
and the shifted

\[ \begin{align*}
\tilde{O}_1 & := O_1 - \frac{Y_6}{2Y_4} \left( O_3 + \frac{1}{\sqrt{3}} O_8 \right) + \frac{Y_4}{2Y_6} \left( O_3 - \frac{1}{\sqrt{3}} O_8 \right), \\
\tilde{O}_2 & := O_2, \\
\tilde{O}_4 & := O_4 - \frac{Y_1}{2Y_6} \left( O_3 - \frac{1}{\sqrt{3}} O_8 \right) - \frac{Y_4}{Y_6} \left( O_3 + \frac{1}{\sqrt{3}} O_8 \right), \\
\tilde{O}_5 & := O_5 - \frac{Y_2}{2Y_6} \left( O_3 - \frac{1}{\sqrt{3}} O_8 \right), \\
\tilde{O}_6 & := O_6 + \frac{Y_1}{2Y_4} \left( O_3 + \frac{1}{\sqrt{3}} O_8 \right) - \frac{Y_3}{Y_6} \left( O_3 - \frac{1}{\sqrt{3}} O_8 \right), \\
\tilde{O}_7 & := O_7 - \frac{Y_2}{2Y_4} \left( O_3 + \frac{1}{\sqrt{3}} O_8 \right),
\end{align*} \]

we obtain the decomposition ("Weyl-decomposition") into the four terms

\[
\sum_{a,b=1}^{8} R_{a}(x) \langle x \mid a \rangle D(A)^{-1} \left( D(A)^{-1} \right)^\dagger \langle y \mid b \rangle O_b(y) =
\]

\[
\begin{align*}
&\left( \tilde{R}_{1}^\dagger(x) \right)^T \langle x \mid D_1(X_3)^{-1} \left( D_1(X_3)^{-1} \right)^\dagger \langle y \mid \tilde{O}_1(y) \rangle \\
&+ \left( \tilde{R}_{2}^\dagger(x) - \tilde{O}_{2}(y) \right) \left( \tilde{R}_{3}(x) + \frac{1}{\sqrt{3}} \tilde{R}_{8}(x) \right) Y_4^{-1}(x) \right)^T \times \\
&\langle x \mid D_1(-X_+)^{-1} \left( D_1(-X_+)^{-1} \right)^\dagger \langle y \mid \tilde{O}_4(y) \rangle \\
&+ \left( \tilde{R}_{6}^\dagger(x) + \tilde{O}_{6}(y) \right) \left( \tilde{R}_{3}(x) - \frac{1}{\sqrt{3}} \tilde{R}_{8}(x) \right) Y_6^{-1}(x) \right)^T \times \\
&\langle x \mid D_1(X_-)^{-1} \left( D_1(X_-)^{-1} \right)^\dagger \langle y \mid \tilde{O}_7(y) \rangle + \tilde{O}_{7}(y) \left( \tilde{R}_{3}(x) - \frac{1}{\sqrt{3}} \tilde{R}_{8}(x) \right) \rangle \right) \\
&+ \left( \tilde{R}_{3}(x) + \frac{1}{\sqrt{3}} \tilde{R}_{8}(x) \right) \left( \tilde{O}_{3}(x) + \frac{1}{\sqrt{3}} \tilde{O}_{8}(x) \right) \\
&+ \left( \tilde{R}_{3}(x) - \frac{1}{\sqrt{3}} \tilde{R}_{8}(x) \right) \left( \tilde{O}_{3}(x) - \frac{1}{\sqrt{3}} \tilde{O}_{8}(x) \right) \right) \delta(x - y),
\end{align*}
\]

in terms of the two-dimensional Faddeev-Popov operator

\[ \langle x \mid D_1(f) \rangle y := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \delta(x - y) + g \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} f(x) \delta(x - y) \]

\[ \equiv 12 \partial_1(x) \delta(x - y) + g Q f(x) \delta(x - y). \tag{34} \]

The inverse of the two-dimensional FP operator (34) has the expansion

\[ \langle x \mid D_1(f)^{-1} \rangle y = - \frac{1}{g} Q f(x) \left[ 12 \delta(x - y) + \frac{1}{g} Q \partial_1(x) \delta(x - y) \right] \]

\[ - \frac{1}{g^2} 12 \partial_1(x) \left[ \frac{1}{f(x)} \partial_1(x) \delta(x - y) \right] + ... . \tag{35} \]

Thus (using \( Q^T = -Q \) and \( Q^2 = -12 \)) we obtain

\[ \langle x \mid D_1(f)^{-1} \left( D_1(f)^{-1} \right)^\dagger \rangle y = - \frac{1}{g^2} \sum_{m,n=0}^{\infty} \left( -1 \right)^n \frac{1}{f(x)} \left( \partial_1(x) \frac{1}{f(x)} \right)^m \delta(x - y) \left( \frac{1}{f(y)} \partial_1(y) \right)^n \frac{1}{f(y)} Q^{m+n}, \tag{36} \]

and hence an explicit derivative expansion of (33).
3 Expansion of the Hamiltonian in spatial derivatives

In order to obtain a consistent expansion of the physical Hamiltonian in spatial derivatives, also the non-locality of the Jacobian $J$ must be taken into account. This can be achieved in the following way.

3.1 Inclusion of the non-locality of the Jacobian into the Hamiltonian

Rewriting the FP operator as the matrix product

$$^*D_{ab}(A) = -g y^{ac}(A) \left[ \delta_{cb} - \frac{1}{g} \gamma_{cd}^{-1}(A) (\Gamma_i)_{db} \partial_i \right] \equiv -g y^{ac}(A) ^*\bar{D}_{cb}(A) ,$$

the Jacobian $J$ factorizes

$$J = J_0 \bar{J} , (37)$$

with the local

$$J_0 \equiv \det |\gamma| = \prod_x \det |\gamma(x)| , \quad \det |\gamma(x)| = X_3^2 \left( X_3^2 - 3 X_8^2 \right)^2 Y_4 Y_6 , (38)$$

and the non-local $\bar{J} \equiv |\bar{D}|$ . Including furthermore the non-local part of the measure into the wave functional

$$\bar{\Psi}(A) := \bar{J}^{1/2} \Psi(A) ,$$

leads to the transformed Hamiltonian $H = \bar{J}^{1/2} H \bar{J}^{-1/2}$ , Hermitian with respect to the local measure $\bar{J}_0$

$$H(A, P) = \frac{1}{2} J_0^{-1} \int d^3x \int d^3y \, P_{ai}(x) \, J_0 \, K_{ai|\ell k}(x, y) \, P_{\ell k}(y) + \frac{1}{2} \int d^3x \, (B_{ai}(A))^2 + V_{\text{meas}}(A) , (39)$$

at the cost of extra terms$^2$ $V_{\text{meas}}$ arising from the non-local factor $\bar{J}$ of the original measure $J$

$$V_{\text{meas}}(A) = \frac{\delta(0)}{4} \bar{J}_0^{-1} \int d^3x \frac{\delta}{\delta A_{ai}(x)} \left[ J_0 \int d^3y \, K_{ai|\ell k}(x, y) \Delta_{\ell k}(y) \right]$$

$$+ \frac{\delta(0)^2}{8} \int d^3x \int d^3y \, \Delta_{ai}(x) \, K_{ai|\ell k}(x, y) \Delta_{\ell k}(y) , (40)$$

(the regularisation of $\delta(0)$ will be given later) with the original kernel $K$ in (14) and the

$$\Delta_{ai}(x) := \delta(0)^{-1} \frac{\delta \ln \bar{J}}{\delta A_{ai}(x)} = -\delta(0)^{-1} (x b | g \, ^*D^{-1} + \gamma^{-1} | x c) \frac{\delta \gamma_{cb}}{\delta A_{ai}} \equiv U_{bc}(x) \frac{\delta \gamma_{cb}}{\delta A_{ai}} . (41)$$

with the

$$U_{ab}(x) = \frac{1}{g} \gamma^{-1}(1) (\Gamma_i)_{cd} \partial_i \gamma^{-1}_{db}(x) - \frac{1}{g} \gamma^{-1}(1) (\Gamma_i)_{cd} \partial_i \left[ \gamma^{-1}(1) (\Gamma_j)_{mn} \partial_j \gamma^{-1}_{nb}(x) \right] + ... , (42)$$

and the c-numbers $\delta \gamma_{cb}/\delta A_{ai}$ obtained from (18). It turns out that the only non-vanishing $\Delta$ are

$$\Delta_{31}(x) = U_{[12]}(x) + \frac{1}{2} \left( U_{[45]}(x) - U_{[67]}(x) \right) , \quad \Delta_{81}(x) = \frac{\sqrt{3}}{2} \left( U_{[45]}(x) + U_{[67]}(x) \right) , (43)$$

being linear combinations of

$$U_{[12]} = 2 u[X_3] , \quad U_{[45]} = -2 u[X_+] , \quad U_{[67]} = 2 u[X_] , (44)$$

in terms of the functional

$$u[f] := -\frac{1}{g^2} \partial_1 \left[ \frac{1}{f} \partial_1 \frac{1}{f} \right] - \frac{1}{g^2} \partial_1 \left[ \frac{1}{f} \partial_1 \left[ \frac{1}{f} \partial_1 \frac{1}{f} \right] \right] - ... . (45)$$

Hence we have the expansion

$$\Delta_{ai}(x) = \Delta_{ai}^{(\partial_1)}[X(x)] + \Delta_{ai}^{(\partial_1 \partial_1 \partial_1)}[X(x)] + ... \quad a = 3, 8 , (46)$$

containing an even number of spatial derivatives $\partial_1$. 

$^2$Although $V_{\text{meas}}$ is, in principle, part of the electric term of the Hamiltonian, I shall treat it separately in this work as the so-called "measure term".
3.3 Weyl-decomposition of the electric potential

The transformed Hamiltonian (39) can be written in the canonical form

\[ \bar{H}(A, P) = \frac{1}{2} \int d^3x \, J_0^{-1}(x) \, P_{ai}(x) \, J_0(x) \, P_{ai}(x) + V_{\text{elec}}(A, P) + V_{\text{magn}}(A) + V_{\text{meas}}(A) . \]  \hspace{1cm} (47)

The electric potential is given by

\[ V_{\text{elec}}(A, P) = \frac{1}{2} J_0^{-1} \int d^3x \int d^3y \, g_a(x) \, J_0(x) \, a^\dagger D(A)^{-1} \left( D(A)^{-1} \right)^\dagger |y \rangle \, \mathcal{G}_b(y) , \]  \hspace{1cm} (48)

where

\[ g_a := D_i(A)_{ab} P_{bi} = \partial_i P_{ai} + g T_a \]  \hspace{1cm} (49)

in terms of the operators

\[ T_a(A, P) := f_{abc} A_{bi} P_{ci} \equiv T_a^Y(Y, P_Y) + T_a^Z(Z, P_Z) . \]  \hspace{1cm} (50)

Note, that the components of the (non-reduced) \( T_a^Z = -i f_{abc} Z_b \partial / \partial Z_c \) satisfy the \( su(3) \) algebra

\[ [T_a^Z, T_b^Z] = i f_{abc} T_c^Z , \]  \hspace{1cm} (51)

whereas the reduced \( T_a^Y = -i f_{abc} Y_b \partial / \partial Y_c \) do not.

The magnetic potential is

\[ V_{\text{magn}}(A) \equiv \frac{1}{2} \int d^3x \, (B_{ai}(A))^2 , \]  \hspace{1cm} (52)

and the measure terms

\[ V_{\text{meas}}(A) \equiv V_{\text{meas,1}}(A) + V_{\text{meas,II}}(A) , \]  \hspace{1cm} (53)

are given as

\[ V_{\text{meas,1}}(A) = \frac{\delta(0)}{4} J_0^{-1} \int d^3x \left[ \frac{\delta}{\delta X_3(x)} (J_0 \Delta_{31}[x]) + \frac{\delta}{\delta X_8(x)} (J_0 \Delta_{81}[x]) \right] \]  
\[ - \frac{\delta(0)}{4} J_0^{-1} \int d^3x \frac{\delta}{\delta X_8(x)} \left[ J_0 \int d^3y \, \partial_i'(x) \langle x a | D(A)^{-1} \left( D(A)^{-1} \right)^\dagger |y \rangle \, \partial_i'\{ \Delta_{b1}(y) \} \right] \]  
\[ - \frac{\delta(0)}{4} J_0^{-1} \int d^3x \frac{\delta}{\delta Y_a(x)} \left[ J_0 \int d^3y \, D_2(Y)^{(ac)}(x) \langle x c | D(A)^{-1} \left( D(A)^{-1} \right)^\dagger |y \rangle \, \partial_i'\{ \Delta_{b1}(y) \} \right] , \]  \hspace{1cm} (54)

\[ V_{\text{meas,II}}(A) = \frac{\delta(0)^2}{8} \int d^3x \left[ (\Delta_{31}(x))^2 + (\Delta_{81}(x))^2 \right] \]  
\[ + \frac{\delta(0)^2}{8} \int d^3x \int d^3y \, \partial_i[\Delta_{a1}(x) \langle x a | D(A)^{-1} \left( D(A)^{-1} \right)^\dagger |y \rangle \, \partial_i[\Delta_{b1}(y) \rangle , \]  \hspace{1cm} (55)

obtained from (40) noting \( D_i(A)_{ab} \Delta_{b1} \delta_{i1} = \partial_i \Delta_{a1} \).

Using the Weyl-decomposition (33) and the expansion (36) of the corresponding two-dimensional non-local potentials, we can further simplify the electric potential (48) and the measure terms (54) and (55).

3.3 Weyl-decomposition of the electric potential

Using (33), the electric potential (48) can be written in the form

\[ V_{\text{elec}}(A, P) = \frac{1}{2} J_0^{-1} \int d^3x \int d^3y \left( \begin{pmatrix} \tilde{g}_1^T(x) \\ \tilde{g}_2^T(x) \end{pmatrix} \right)^T J_0(x) \left( D_1(X_3)^{-1} \left( D_1(X_3)^{-1} \right)^\dagger \right) \left| y \right> \left( \begin{pmatrix} \tilde{g}_1(y) \\ \tilde{g}_2(y) \end{pmatrix} \right) , \]
\begin{align}
+ \left( \tilde{G}_4^\dagger(x) - \partial_2^x \left[ \left( \tilde{G}_3(x) + \frac{1}{\sqrt{3}} G_8(x) \right) Y_4^{-1}(x) \right] \right)^T \mathcal{J}_0 \times \\
(\mathbf{x} | *D_1(-X_+)^{-1} \left( *D_1(-X_+)^{-1} \right)^\dagger | \mathbf{y}) \left( \tilde{G}_5(y) - \partial_2^y \left[ \left[ \tilde{G}_3(y) + \frac{1}{\sqrt{3}} G_8(y) \right] \right] \right) \\
+ \left( \tilde{G}_6^\dagger(x) + \partial_2^x \left[ \left[ \tilde{G}_3(x) - \frac{1}{\sqrt{3}} G_8(x) \right] Y_6^{-1}(x) \right] \right)^T \mathcal{J}_0 \times \\
(\mathbf{x} | *D_1(X_-)^{-1} \left( *D_1(X_-)^{-1} \right)^\dagger | \mathbf{y}) \left( \tilde{G}_7(y) + \partial_2^y \left[ \left[ \tilde{G}_3(y) - \frac{1}{\sqrt{3}} G_8(y) \right] \right] \right) \\
+ \frac{1}{2} \mathcal{J}_0^{-1} \int d^3 \mathbf{x} \left\{ \left( \tilde{G}_3(x) + \frac{1}{\sqrt{3}} G_8(x) \right) \mathcal{J}_0 \left( \frac{1}{X_+} \right) \left( \frac{1}{X_+} \right)^n \left( \tilde{G}_4^\dagger \right)^m \mathcal{J}_0 \left( \frac{1}{X_+} \right) \left( \frac{1}{X_+} \right)^n \left( \tilde{G}_6^\dagger \right)^m \mathcal{J}_0 \left( \frac{1}{Y_6} \right)^n \left( \tilde{G}_7^\dagger \right)^n \mathcal{J}_0 \left( \frac{1}{Y_6} \right)^n \left( \tilde{G}_5^\dagger \right)^m \mathcal{J}_0 \left( \frac{1}{Y_4} \right) \left( \frac{1}{Y_4} \right)^n \left( \tilde{G}_3^\dagger \right)^m \mathcal{J}_0 \left( \frac{1}{Y_4} \right) \right\},
\end{align}

in terms of the two-dimensional Faddeev-Popov operator (34) and the shifted $\tilde{G}_a$ defined according to (32). Using (35) we can expand the electric potential (56) as

\begin{align}
V_{\text{elec}} &= \frac{1}{2} g_2^2 \mathcal{J}_0^{-1} \sum_{m,n=0}^\infty \frac{(-1)^n}{g^{m+n}} \int d^3 \mathbf{x} \left\{ \left( \tilde{G}_4^\dagger \right)^T \mathcal{J}_0 \left( \frac{1}{X_+} \right) \left( \frac{1}{X_+} \right)^n \left( \tilde{G}_4^\dagger \right) \right\} \\
&+ \left( \tilde{G}_5^\dagger - \partial_2 \left[ \left( \tilde{G}_3 + \frac{1}{\sqrt{3}} G_8 \right) Y_4^{-1} \right] \right)^T \times \\
&\frac{1}{X_+} \left( - \tilde{G}_5^\dagger \right)^m \mathcal{J}_0 \left( - \frac{1}{X_+} \right) \left( \frac{1}{X_+} \right)^n \left( \tilde{G}_5^\dagger \right)^m \mathcal{J}_0 \left( - \frac{1}{X_+} \right) \left( \frac{1}{X_+} \right)^n \left( \tilde{G}_6^\dagger \right)^m \mathcal{J}_0 \left( - \frac{1}{X_-} \right) \left( \frac{1}{X_-} \right)^n \left( \tilde{G}_6^\dagger \right)^m \mathcal{J}_0 \left( - \frac{1}{X_-} \right) \left( \frac{1}{X_-} \right)^n \left( \tilde{G}_7^\dagger \right)^n \mathcal{J}_0 \left( - \frac{1}{X_-} \right) \left( \frac{1}{X_-} \right)^n \left( \tilde{G}_3^\dagger \right)^m \mathcal{J}_0 \left( - \frac{1}{X_-} \right) \left( \frac{1}{X_-} \right)^n \\
&+ \frac{1}{2} \int d^3 \mathbf{x} \left\{ \left( \tilde{G}_3 + \frac{1}{\sqrt{3}} G_8 \right) \mathcal{J}_0 \left( \frac{1}{Y_4} \right) \left( \frac{1}{Y_4} \right)^n \left( \tilde{G}_3 + \frac{1}{\sqrt{3}} G_8 \right) \mathcal{J}_0 \left( \frac{1}{Y_4} \right) \left( \frac{1}{Y_4} \right)^n \left( \tilde{G}_3 + \frac{1}{\sqrt{3}} G_8 \right) \mathcal{J}_0 \left( \frac{1}{Y_4} \right) \left( \frac{1}{Y_4} \right)^n \left( \tilde{G}_3 + \frac{1}{\sqrt{3}} G_8 \right) \mathcal{J}_0 \left( \frac{1}{Y_4} \right) \right\}. 
\end{align}

3.4 Weyl-decomposition of the non-local part of the measure term

The corresponding expressions for the measure terms, using the Weyl-decomposition (33), can be obtained in an analogous way. The first summand of $V_{\text{meas},1}(A)$, reads

\begin{align}
V_{\text{meas},1}(A) &= \frac{\delta(0)}{4} \int d^3 \mathbf{x} \left\{ \left( \frac{2}{X_3} - \frac{1}{X_+} - \frac{1}{X_-} + \frac{\delta}{\delta X_3} \right) u[X_3] \right. \\
&\left. + \left( \frac{2}{X_-} - \frac{1}{X_+} - \frac{1}{X_-} + \frac{\delta}{\delta X_-} \right) u[X_-] \right\} + O(\partial^4),
\end{align}

using the $u$-functionals defined in (45) and

\begin{align}
\mathcal{J}_0^{-1} \frac{\delta}{\delta X_+} \mathcal{J}_0 &= \frac{2}{X_3} + \frac{2}{X_+} + \frac{\delta}{\delta X_+}, \\
\mathcal{J}_0^{-1} \frac{\delta}{\delta X_-} \mathcal{J}_0 &= - \frac{2}{X_3} + \frac{2}{X_+} + \frac{\delta}{\delta X_-},
\end{align}

and is at least of order $O(\partial^2)$ and the second summand at least of order $O(\partial^4)$.
Similarly, the first summand of \( V_{\text{meas}, \Pi}(A) \) reads
\[
V_{\text{meas}, \Pi}(A) = \frac{\delta(0)^2}{4} \int d^3x \left[ (u[X_3] - u[X_+])^2 + (u[X_3] - u[X_-])^2 + (u[X_+] - u[X_-])^2 \right] + O(\partial^6), \tag{60}
\]
is at least of order \( O(\partial^4) \) and the second summand at least of order \( O(\partial^6) \).

### 3.5 Derivative expansion of the transformed Hamiltonian and local measure

Altogether, we obtain an expansion of the transformed physical Hamiltonian (47) in the number of spatial derivatives
\[
\hat{H} = H_0 + \sum_\alpha V_\alpha^{(\partial)} + \sum_\beta V_\beta^{(\partial \partial)} + \ldots, \tag{61}
\]
with the free part \( H_0 \) containing no spatial derivatives, the interaction parts \( V_\alpha^{(\partial)} \) containing one spatial derivative, and \( V_\beta^{(\partial \partial)} \) containing two spatial derivatives, and so on.

The matrix element (19) of a physical operator \( O \) becomes the product of local matrix elements
\[
\langle \Psi_1 | O | \Psi_2 \rangle \propto \prod_x \left[ \int dX(x) \ X_3^2 \left( X_3^2 - 3 X_8^2 \right)^2 \int dY(x) \ Y_4 Y_6 \int dZ(x) \right] \Psi_1^*[X, Y, Z] \ O \Psi_2[X, Y, Z]. \tag{62}
\]

### 4 The free part \( H_0 \)

The free part \( H_0 \) containing no spatial derivatives takes the form
\[
H_0 = \int d^3x \frac{1}{2} \left[ J_0^{-1} P_{ai} J_0 P_{ai} + J_0^{-1} T_a J_0 \left( \gamma^{-1} \gamma^{-1T} \right)_{ac} T_c + \left( B_{\text{hom}}^{ai}(A) \right)^2 \right] = \int d^3x H_0(x), \tag{63}
\]
using the homogenous part \( B_{\text{hom}}^{ai} := (1/2)g_{ijk} f_{abc} A_{b\ j} A_{c\ k} \) of the chromomagnetic field and the operators \( T_a(A, P) \) defined in (50).

Changing to the more convenient polar coordinates
\[
X = \begin{pmatrix}
0 \\
r_X \cos \psi_X \\
0 \\
r_X \sin \psi_X
\end{pmatrix}
\quad Y = \begin{pmatrix}
r_{1Y} \cos \theta_Y \\
r_{1Y} \sin \theta_Y \\
r_Y \cos \psi_Y \\
r_Y \sin \psi_Y
\end{pmatrix}
\]
and gathering together the parts of the kinetic part of the Hamiltonian depending only on one of the three space directions respectively, we can write
\[
H_0[A(x), P(x)] = K_X + K_Y + K_Z +
\begin{align*}
&+ \frac{1}{2} \left[ \frac{1}{r_X^2 \cos^2 \psi_X} \sum_{m=1, 2} \left( \frac{1}{r_{2Y} r_{3Y}} \bar{T}_m Y^+ r_{2Y} r_{3Y} \bar{T}_m + \bar{T}_m \right) \right] \left( \bar{T}_m \right) \\
&\quad + \frac{1}{r_X^2 \cos^2 \left( \psi_X + 2\pi/3 \right)} \sum_{m=4, 5} \left( \frac{1}{r_{2Y} r_{3Y}} \bar{T}_m Y^+ r_{2Y} r_{3Y} \bar{T}_m + \bar{T}_m \right) \left( \bar{T}_m \right) \\
&\quad + \frac{1}{2r_{2Y}^2} \left[ \left( T_3^Z + \frac{1}{\sqrt{3}} T_8^Z \right) - 2i \frac{\partial}{\partial \theta_Y} \right] \left( T_3^Z + \frac{1}{\sqrt{3}} T_8^Z \right) \\
&\quad + \frac{1}{2r_{3Y}^2} \left[ \left( T_3^Z - \frac{1}{\sqrt{3}} T_8^Z \right) - 2i \frac{\partial}{\partial \theta_Y} \right] \left( T_3^Z - \frac{1}{\sqrt{3}} T_8^Z \right) + \frac{1}{2} \left( B_{\text{hom}}^{ai}[X, Y, Z] \right)^2, \tag{65}
\end{align*}
\]
and the second order magnetic part

\[ K_X = -\frac{1}{2} \left[ \frac{\partial^2}{\partial r_X^2} + \frac{7}{r_X} \frac{\partial}{\partial r_X} + \frac{1}{r_X^2} \left( -6 \tan[3\psi_X] \frac{\partial}{\partial \psi_X} + \frac{\partial^2}{\partial \psi_X^2} \right) \right], \]

\[ K_Y = -\frac{1}{2} \left[ \frac{\partial^2}{\partial r_Y^2} + \frac{1}{r_Y} \frac{\partial}{\partial r_Y} + \frac{1}{r_Y^2} \frac{\partial^2}{\partial \psi_Y^2} \right] + \sum_{i=1}^{3} \left( \frac{\partial^2}{\partial r_{iY}^2} + \frac{1}{r_{iY}} \frac{\partial}{\partial r_{iY}} + \frac{1}{r_{iY}^2} \frac{\partial^2}{\partial \theta_{iY}^2} \right), \]

\[ K_Z = -\frac{1}{2} \sum_{a=1}^{8} \left( \frac{\partial}{\partial Z_a} \frac{\partial}{\partial Z_a} \right), \]

(66)

and the shifted non-Hermitian \( \tilde{T}^\gamma_a \) and Hermitian \( \tilde{T}^Z_a \), defined according to formulae (32).

The local part of the Faddeev-Popov determinant, \( J_0 \), becomes

\[ J_0 = \prod_x J_0(x), \quad J_0(x) = r_X^6(x) \cos^2[3\psi_X(x)] r_2Y(x) r_3Y(x), \]

(67)

Together with the Jacobians of the coordinate trafo (64) the measures in matrix elements become

\[ \langle \Psi_1 | \mathcal{O} | \Psi_2 \rangle = \prod_x \int d\mu_X(x) \int d\mu_Y(x) \int d\mu_Z(x) \Psi_1^\dagger \mathcal{O} \Psi_2, \]

(68)

with

\[ \int d\mu_X \propto \int_0^{2\pi} dr_X \int_0^\infty d\psi_X \cos^2[3\psi_X], \]

\[ \int d\mu_Y \propto \int_0^{2\pi} dr_Y \int_0^{2\pi} d\psi_Y \int_0^{2\pi} dr_{1Y} r_{1Y} \int_0^{2\pi} dr_{2Y} r_{2Y} \int_0^{2\pi} dr_{3Y} r_{3Y} \int_0^{2\pi} d\theta_Y, \]

\[ \int d\mu_Z \propto \prod_{a=1}^{8} \int_{-\infty}^\infty dZ_a. \]

(69)

In the space-1 direction the measure has a periodic structure with six zeros at \( \psi_X = (2n + 1)\pi/6 \), \( n = 0, 1, ..., 5 \), which are Gribov horizons separating six Weyl-chambers. For recent discussions on Gribov horizons see e.g. [19]. In the space-3 direction the measure is flat.

\section{5 First and second order interaction terms}

The interaction parts of first and second order in the number of spatial derivatives, are the following:

\subsection{5.1 Magnetic terms}

The first order magnetic part reads

\[ V^{(\partial)}_{\text{magn}} = g \int d^3x f_{abc} A_{bj} A_{ck} \partial_j A_{ak}, \]

(70)

and the second order magnetic part

\[ V^{(\partial\partial)}_{\text{magn}} = \frac{1}{2} \int d^3x [\partial_j A_{ak} \partial_j A_{ak} - \partial_j A_{ak} \partial_k A_{aj}] . \]

(71)

More explicitly, the first order magnetic parts read

\[ V^{(\partial_1)}_{\text{magn}} = g \int d^3x \left[ (f_{abc} A_{b1} A_{c2}) \partial_1 A_{a2} + (f_{abc} A_{b1} A_{c3}) \partial_1 A_{a3} \right], \quad V^{(\partial_2)}_{\text{magn}}, V^{(\partial_3)}_{\text{magn}} \text{ cycl. perm.}, \]

and the second order magnetic parts

\[ V^{(\partial_1\partial_1)}_{\text{magn}} = \frac{1}{2} \int d^3x \left[ (\partial_1 A_{a2})^2 + (\partial_1 A_{a3})^2 \right], \quad V^{(\partial_2\partial_2)}_{\text{magn}}, V^{(\partial_3\partial_3)}_{\text{magn}} \text{ cycl. perm.}, \]

\[ V^{(\partial_1\partial_2)}_{\text{magn}} = -\int d^3x (\partial_2 A_{a3}) (\partial_3 A_{a2}), \quad V^{(\partial_1\partial_3)}_{\text{magn}}, V^{(\partial_2\partial_3)}_{\text{magn}} \text{ cycl. perm.}. \]

(72)

(73)

(74)

(75)

(76)

Note, that there are no higher order magnetic terms.
5.2 Electric terms

Up to second order perturbation theory, only the first order electric parts \( V_{\text{elec}}^{(\partial_1)} \), \( V_{\text{elec}}^{(\partial_2)} \), and \( V_{\text{elec}}^{(\partial_3)} \), and the second order electric parts \( V_{\text{elec}}^{(\partial_1 \partial_1)} \), \( V_{\text{elec}}^{(\partial_2 \partial_2)} \), and \( V_{\text{elec}}^{(\partial_3 \partial_3)} \), are needed. They can be easily read off from the general expression (57). The simplest ones are

\[
V_{\text{elec}}^{(\partial_3 \partial_1)} = \frac{1}{2g} \int d^3x \left\{ \frac{1}{r_X^2 \cos^2 \psi_X} \sum_{m=1}^{1,2} \left( \frac{1}{r_{2Y} r_{3Y}} (\overline{T}_m^Y + \overline{Z}_m) r_{3Y} \partial_3 \overline{P}_m^Z + \text{h.c.} \right) \right. \\
+ \frac{1}{r_X^2 \cos^2 [\psi_X + 2\pi/3]} \sum_{m=4}^{5,\overline{Y}} \left( \frac{1}{r_{2Y} r_{3Y}} (\overline{T}_m^Y + \overline{Z}_m) r_{3Y} \partial_3 \overline{P}_m^Z + \text{h.c.} \right) \\
+ \frac{1}{r_X^2 \cos^2 [\psi_X + 4\pi/3]} \sum_{m=6,7} \left( \frac{1}{r_{2Y} r_{3Y}} (\overline{T}_m^Y + \overline{Z}_m) r_{3Y} \partial_3 \overline{P}_m^Z + \text{h.c.} \right) \left. \right\}, (77)
\]

and

\[
V_{\text{elec}}^{(\partial_1 \partial_3)} = \frac{1}{2g^2} \int d^3x \left\{ \frac{1}{r_X^2 \cos^2 \psi_X} \sum_{a=1}^{1,2} (\partial_a \overline{P}_a^Z)^2 + \frac{1}{r_X^2 \cos^2 [\psi_X + 2\pi/3]} \sum_{a=4,5} (\partial_a \overline{P}_a^Z)^2 \right. \\
+ \frac{1}{r_X^2 \cos^2 [\psi_X + 4\pi/3]} \sum_{a=6,7} (\partial_a \overline{P}_a^Z)^2 + \frac{1}{r_{2Y}^2} (\partial_3 \overline{P}_3^Z)^2 + \frac{1}{r_{3Y}^2} (\partial_3 \overline{P}_3^Z)^2 \left. \right\}. (78)
\]

The remaining ones are given by somewhat longer expressions.

5.3 Measure terms

Since up to second order in the number of spatial derivatives only \( \Delta^{(\partial_1 \partial_1)}_{31} \) and \( \Delta^{(\partial_1 \partial_1)}_{81} \) are non-vanishing, the first order measure parts are vanishing

\[
V^{(\partial_1)}_{\text{meas}} = 0, \quad V^{(\partial_2)}_{\text{meas}} = 0, \quad V^{(\partial_3)}_{\text{meas}} = 0, \quad (79)
\]

and the only non-vanishing second order measure part is given by the lowest order of (58) obtained using the lowest order \( u^{(\partial_1 \partial_1)} \) of the \( u \)-functionals defined in (45), such that

\[
V^{(\partial_1 \partial_1)}_{\text{meas}}(A) = -\frac{\delta(0)}{4g^2} \int d^3x \left\{ \left( \frac{2}{X_3} - \frac{1}{X_+} - \frac{1}{X_-} + \frac{\delta}{\delta X_3} \right) \left( \frac{1}{X_3} \partial_1 \left[ \frac{1}{X_3} \right] \right) \right. \\
+ \left( \frac{2}{X_+} - \frac{1}{X_3} - \frac{1}{X_-} + \frac{\delta}{\delta X_+} \right) \left( \frac{1}{X_+} \partial_1 \left[ \frac{1}{X_+} \right] \right) \right. \\
+ \left( \frac{2}{X_-} - \frac{1}{X_+} - \frac{1}{X_3} + \frac{\delta}{\delta X_-} \right) \left( \frac{1}{X_-} \partial_1 \left[ \frac{1}{X_-} \right] \right) \left. \right\}. (80)
\]

More explicitly, we have

\[
V^{(\partial_3 \partial_1)}_{\text{meas}} = -\frac{\delta(0)^2}{2g^2} \int d^3x \left\{ \left[ \frac{1}{r_X^2} \cos^2 [\psi_X] + \cos [\psi_X + 2\pi/3] \cos [\psi_X + 4\pi/3] \right] \left( \partial_1 \frac{1}{r_X \cos [\psi_X]} \right)^2 \right. \\
+ \left( \psi_X \rightarrow \psi_X + 2\pi/3 \right) + \left( \psi_X \rightarrow \psi_X + 4\pi/3 \right) \right. \\
+ 12 \left( \partial_1 \frac{1}{r_X \cos [3\psi_X]} \right) \left( \partial_1 \frac{1}{r_X \cos [3\psi_X]} \right) \left. \right\}. (81)
\]
5.4 General form of the interaction terms

The interaction parts of first order in the number of spatial derivatives can be written in the formal form

\[ V^{(\partial_s)}_\alpha \equiv \int dx \left[ \tilde{Y}_\alpha[A(x)] \partial_s Y_\alpha[A(x)] + \text{h.c.} \right], \quad s, t = 1, 2, 3, \quad (82) \]

and those of second order in the number of spatial derivatives

\[ V^{(\partial_s \partial_t)}_\beta \equiv \int dx \left[ \partial_s X_\beta[A(x)] \right] \tilde{X}_\beta[A(x)] \left( \partial_t X'_\beta[A(x)] \right) + \text{h.c.} \], \quad s, t = 1, 2, 3, \quad (83) \]

containing only first order derivatives of functions \( \partial_s X, \partial_s X', \partial_s Y \).

Interaction parts containing three or four spatial derivatives, contain also \( \partial^2_s X \) and \( \partial_s \partial_t X \). For fifth and sixth order we need also the third derivative \( \partial^3_s X \), and so on.

6 Coarse graining and strong coupling expansion in \( \lambda = g^{-2/3} \)

In order to make the functional approach used above well defined (e.g. what is meant by \( \delta(0) \)), we should introduce an ultraviolet cutoff which should be rather large in order to have only slightly spatially varying physical fields for which an expansion in the number of spatial derivatives is applicable and the flux-tube gauge well-defined. As for the \( SU(2) \)-case using the symmetric gauge in [6], I shall apply the coarse graining approach and set an ultraviolet cutoff by introducing an infinite hypothetical spatial lattice of granules \( G(n, a) \), here large cubes of length \( a \), situated at sites \( x = an \) (\( n = (n_1, n_2, n_3) \in \mathbb{Z}^3 \)), and considering the averaged variables

\[ A(n) := \frac{1}{a^3} \int_{G(n, a)} dx \ A(x) \quad (84) \]

(where in particular \( \delta(0) \to 1/a^3 \)). Furthermore, the discretised spatial derivatives \( \partial_s A(n), \partial_s^2 A(n), \ldots \) at site \( n \) and in the direction \( s = 1, 2, 3 \) are defined as

\[ \partial_s A(n) := \lim_{N \to \infty} \sum_{m=1}^{N} w_N(m) \frac{1}{2ma} \left( A(n + me_s) - A(n - me_s) \right), \]

\[ \partial_s^2 A(n) := \lim_{N \to \infty} \sum_{m=1}^{N} w_N(m) \frac{1}{(ma)^2} \left( A(n + me_s) + A(n - me_s) - 2A(n) \right) \]

\[ \ldots \ldots \ldots \ldots \]

with the lattice unit vectors \( e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1) \) and the distribution

\[ w_N(m) := 2 \frac{(-1)^{m+1}(N!)^2}{(N-m)!(N+m)!}, \quad 1 \leq m \leq N, \quad \sum_{m=1}^{N} w_N(m) = 1. \quad (85) \]

The values of \( \partial_s A(n), \partial_s^2 A(n), \ldots \) in (85) for a given site \( n \) and direction, say \( s = 1 \), are chosen to coincide with the corresponding derivatives of the Lagrange interpolation polynomials \( I_{2N}(x_1)|_{n_2, n_3}, I_{2N}'(x_1)|_{n_2, n_3}, \ldots \)

\[ I_{2N}(x_1)|_{n_2, n_3} \]

of the interpolation polynomial \( I_{2N}(x_1)|_{n_2, n_3} \) in the \( x_1 \) coordinate, uniquely determined by the series of values \( A(n_1 + n_2, n_3) \) (\( n = -N, \ldots, N \)) obtained via the averaging (84), and then, finally, taking the limit \( N \to \infty \). Note that the

\[ \text{Taking the first, second, ... derivatives of the Lagrange interpolation polynomials } I_{2N}(x), \text{ with given values } y_n \text{ at the equidistant points } x_n = x_0 + na, \text{ with } y_n \text{ at the central point } x_0, \text{ yields: } I_{2N}(x_0) = \sum_{n=1}^{N} w_N(n)(y_n - y_{-n})/(2na). \]

\[ I_{2N}'(x_0) = \sum_{n=1}^{N} w_N(n)(y_{n+1} - y_{n-1})/(2n^2a). \]

\[ I_{2N}''(x_0) = \sum_{n=1}^{N} w_N(n)(y_{n+2} - 2y_{n-1} + y_{n-2})/(2n^3a). \]

\[ \text{Note that the above definition of the spatial lattice derivative looks similar to the SLAC-derivative [20], which solves the fermion doubling problem, but has no well-defined continuum limit (see [21]). The SLAC derivative results from our definition (85) with (86), if the limit } N \to \infty \text{ is taken before the sum over } m \text{ is carried out. The expected absence of the fermion-doubling problem is good news for the generalisation of our approach to include fermions. The absence of a well-defined continuum limit, on the other hand, does not present a problem in our large-box case, since } \lambda \text{ as will be discussed below - we expect that with decreasing length } a \text{ a first transition - at some intermediate scale } - \text{ to the small-box scenario by Lüscher [2, 3] using the Coulomb-gauge will be necessary, before a well-defined limit } a \to 0 \text{ can be taken.} \]
\( (N = 1) \) choice, \( \partial_n A(n)|_{N=1} = (A(n + e_s) - A(n - e_s)) / (2a) \), which includes only the nearest neighbors \( n \pm e_s \), would lead to the same results as (85) for the soft components of the original field \( A(x) \), varying only slightly over several lattice sites, but lead to values falling off faster than (85) for higher momentum components close to \( \pi/a \). For example, we have\(^5\)

\[
\lim_{N \to \infty} \sum_{m=1}^{N} \frac{w_N(m)}{m} \sin[m(ak)] = ak . \tag{87}
\]

Applying furthermore the rescaling transformation (again afterwards dropping the primes)

\[
A = g^{-1/3} a A', \quad P = g^{1/3} a^2 P' , \tag{88}
\]

an expansion of the Hamiltonian in \( \lambda = g^{-2/3} \) can be obtained

\[
H = g^{2/3} a \left[ H_0 + \lambda \sum_{\alpha} \lambda^{(0)} + \lambda^2 \sum_{\beta} \lambda^{(0)} + O(\lambda^3) \right] , \tag{89}
\]

with the dimensionless and coupling constant independent terms \( H_0, \lambda^{(0)}, \lambda^{(0)} \) ... .

The "free" part \( H_0 \) is the just the sum of the Hamiltonians of \( SU(3) \)-Yang-Mills quantum mechanics of constant fields in each box,

\[
H_0 = \sum_n H_0^{QM}(n) , \tag{90}
\]

and the interaction parts \( \lambda^{(0)}, \lambda^{(0)} \), ... are relating different boxes. Those contributing up to second order in strong coupling read

\[
\lambda^{(0)} = \lim_{N \to \infty} \sum_{m=1}^{N} \frac{w_N(m)}{2m} \sum_n \tilde{\lambda}_\alpha(n) \left( \lambda^{(0)}(n + m e_s) - \lambda^{(0)}(n - m e_s) \right) + \text{h.c.} , \tag{91}
\]

\[
\lambda^{(0)} = \lim_{N \to \infty} \sum_{m,m'=1}^{N} \frac{w_N(m)w_N(m')}{4mm'} \sum_n \left[ \lambda^{(0)}(n + m e_s) - \lambda^{(0)}(n - m e_s) \right] + \text{h.c.} \tag{92}
\]

in terms of the dimensionless and coupling constant independent terms \( \lambda, \tilde{\lambda}, \lambda', \tilde{\lambda}', \lambda, \lambda \), which are obtained from those in (82) and (83) putting \( a = 1, g = 1, \delta(0) = 1 \).

The expansion of the unconstrained \( SU(3) \) Yang-Mills Hamiltonian in the number of spatial derivatives is therefore equivalent to a strong coupling expansion in \( \lambda = g^{-2/3} \), just as for the \( SU(2) \) case [6]. It is the analogon of the weak coupling expansion in \( g^{2/3} \) for small boxes by Lüscher [2] for \( SU(2) \) Yang-Mills theory and by Weisz and Ziemann[3] for \( SU(3) \) Yang-Mills theory, and supplies an useful alternative to strong coupling expansions based on the Wilson-loop gauge invariant variables, proposed by Kogut, Sinclair, and Susskind [10] for a 3-dimensional spatial lattice in the Hamiltonian approach, yielding an expansion in \( 1/g^4 \), and by Münster [11] for a 4-dimensional space-time lattice.

### 7 The spectrum of \( SU(3) \) Yang-Mills quantum mechanics

The low energy spectrum \( \epsilon^{(S)PC}_i(n) \) and eigenstates \( \mid \Psi^{(S)PC}_{i,M} \rangle_n \) of \( H_0^{QM} \) at each site \( x_n \) appearing in (90), are the solutions of the Schrödinger eigenvalue problem of \( SU(3) \) Yang-Mills quantum mechanics of spatially constant fields

\[
H_0(A, P) \mid \Psi^{(S)PC}_{i,M} \rangle = \epsilon^{(S)PC}_i \left( \frac{g^{2/3}}{a} \right) \mid \Psi^{(S)PC}_{i,M} \rangle , \tag{93}
\]

\(^5\)Noting \( \lim_{N \to \infty} \sum_{m=1}^{N} m^{2k}w_N(m) = \delta_{0k} \).
sym. 2 tensor | $\mathbf{s}^{\pm}_{[2]}[A] := A_{ai}A_{aj}$, $(i \leq j)$ | $0^+; 2^+$

sym. 3 tensor | $\mathbf{s}^{0}_{[3]}[A] := d_{abc}A_{ai}A_{bj}A_{ck}$, $(i \leq j \leq k)$ | $1^-, 3^-$

sym. 2 tensor | $\mathbf{b}^{\pm}_{[ij]}[A] := B_{ai}^{\text{hom}}B_{aj}^{\text{hom}}$, $(i \leq j)$ | $0^+; 2^+$

vector | $\mathbf{b}^{0}_{ijkl}[A] := d_{abc}B_{ai}^{\text{hom}}B_{bj}^{\text{hom}}A_{ci} + \frac{1}{2}(2s_{ijk}s_{123} - s_{ijj}s_{ikk} - s_{kkk}s_{ijj}), (i \neq j \neq k)$ | $1^-$

axial scalar | $\mathbf{a}^{\pm}_{[i]}[A] := f_{abc}A_{ai}A_{b2}A_{c3} = B_{ai}^{\text{hom}}A_{ai} = B_{a2}^{\text{hom}}A_{a2} = B_{a3}^{\text{hom}}A_{a3}$ | $0^-$

axial vector | $\mathbf{a}^{\pm}_{[i]}[A] := d_{abc}B_{ai}^{\text{hom}}B_{bj}^{\text{hom}}A_{ci}$, $(i = 1, 2, 3)$ | $1^+$

sym. axial 2 tens. | $\mathbf{a}^{\pm}_{[ijkl]}[A] := d_{abc}B_{ai}^{\text{hom}}A_{bk}(d_{cde}A_{di}A_{ej})$, $(i \leq j \land k \neq i, j)$ | $0^-, 2^-$

sym. axial 3 tens. | $\mathbf{a}^{0}_{[ijk]}[A] := d_{abc}B_{ai}^{\text{hom}}B_{bj}^{\text{hom}}B_{ck}^{\text{hom}}$, $(i \leq j \leq k)$ | $1^-, 3^-$

Table 1: Definition of the complete set of eight elementary $SU(3)$-invariant spatial tensors on gauge-reduced $A$-space. The indices $a, b, c$ are summed over, but the spatial indices $i, j, k$ are not, in all lines of the table. Note that for the case $i = j$ in the last line one can choose any of the two $k \neq i, j$, both give the same $a_{[5]}$. The second column shows the degree $[n]$ of the tensor (as a polynomial in $A$). The last column shows the spin components into which the tensor can be decomposed.

with the Hamiltonian (65), after having applied the rescaling trasfo (88), and the measure

$$\langle \Psi_1 | O | \Psi_2 \rangle = \int d\mu_X \int d\mu_Y \int d\mu_Z \Psi_1^* O \Psi_2 ,$$

(94)

with the $d\mu_X, d\mu_Y, d\mu_Z$ given by (71). The solutions, characterised by the quantum numbers of spin $S, M$, parity $P$, and charge conjugation $C$, can in principle be obtained with arbitrary high accuracy. This is discussed in detail, in particular also the explicit expression for the spin-operator in reduced $A$-space, in [17]. For completeness, I give a summary of some of the main results here.

7.1 The corresponding harmonic oscillator problem

Replacing in $H_0(A, P)$ of (65) the magnetic potential by the separable harmonic oscillator potential with free parameter $\omega > 0$

$$\frac{1}{2} (B_{ai}^{\text{hom}}(A))^{2} \rightarrow \frac{1}{2} \omega^2 (A_{ai})^{2} = \frac{1}{2} \omega^2 \left( r_X^2 + (r_Y^2 + r_{2Y}^2 + r_{3Y}^2) + \sum_{a=1}^{8} Z_a^2 \right)$$

(95)

the corresponding harmonic oscillator problem (with the same measure (94))

$$H_{h.o.}(A, P)|\Phi_{i,M}^{(S)PC}\rangle = \epsilon_{i,M}^{(S)PC} \left( \frac{g^{2/3}}{a} \right) |\Phi_{i,M}^{(S)PC}\rangle ,$$

(96)

becomes trigonal in the space of the monomial functionals

$$M[\omega s_{[2]}, \omega^3 s_{[3]}, \omega^2 b_{[4]}, \omega^5/2 b_{[5]}, \omega^3/2 a_{[3]}, \omega^2 a_{[4]}, \omega^5/2 a_{[5]}, \omega^3 a_{[6]}] \exp[-\omega (s_{11} + s_{22} + s_{33})/2] ,$$

(97)

where the $M$ are monomials in the 45 components of eight elementary $SU(3)$-invariant spatial tensors in reduced $A$-space shown in Table 1. For example, $M = 1$ or $M = s_{11} + s_{12} + s_{22} + s_{33}$.

My result is in accordance with a theorem proven by Dittner [22], that the primitive $SU(3)$-invariant tensors in original constrained $V$-space have maximal rank 6 and that their number is 35. The eight irreducible symmetric spatial tensors in Table 1 are indeed maximally of rank 6. There is, however a considerable conceptual simplification in reduced $A$-space in comparison to the constrained $V$ space. For example, in constrained $V$-space, the $SU(3)$-invariant

$$d_{abc}C_{12}^b[V]C_{12}^c[V]C_{12}^a[V] \quad \text{with} \quad C_{12}^a[V] := d_{abc} V_i^a V_j^b ,$$

(98)

discussed in [22], is independent of the eight invariants $s_{11}[V], s_{12}[V], s_{22}[V], s_{111}[V], s_{112}[V], s_{122}[V], s_{222}[V], b_{33}[V]$, using the functionals in Table 1 in terms of unredced $V$ instead of reduced $A$\(^6\), in the

\(^6\)According to well known identities, $b_{33}[V] \equiv (f_{abc} V_i^a V_j^b)(f_{abc} V_i^a V_j^b)$ according to Table 1 is related to the invariant $C_{12}^a[V]C_{12}^b[V]$ via $b_{33}[V] \equiv 3 C_{12}^a[V]C_{12}^b[V] - s_{12}[V]s_{12}[V]$
sense, that it cannot be represented as a sum of products of them. It is, however, not primitive because it is related to them via outer products.

In reduced A-space, however, where outer products of invariant tensors are absent, the corresponding polynomial is indeed reducible

\[
d_{abc}C_{12}^a[A]C_{12}^b[A]C_{12}^c[A] = \frac{1}{18} s_{12}^3[A] - \frac{1}{6} s_{12}^2[A] s_{11}[A] s_{22}[A] - \frac{1}{12} s_{11}^2[A] s_{222}[A]
+ \frac{3}{4} s_{11}[A] s_{122}[A] + \frac{1}{6} s_{12}[A] b_{33}[A]. \tag{99}
\]

Organising the monomial functionals (97) according to the degree \( n \) (as a polynomial in the \( A \)) and the conserved quantum numbers \( S, M, P, C \) and applying a Gram-Schmidt orthogonalisation with respect to the measure (70), we obtain all exact solutions

\[
\Phi^{(S)PC}_{[n]}i,M[A] = P^{(S)PC}_{[n]}i,M[\omega s_{[2]}^0, \omega^3 s_{[3]}^0, \omega^2 b_{[4]}^0, \omega^5/2 b_{[6]}^0, \omega^3/2 a_{[3]}, \omega^2 a_{[4]}, \omega^5/2 a_{[5]}, \omega^3 a_{[6]}] \exp[-\omega (A_{ii})^2/2],
\tag{100}
\]
of the corresponding harmonic oscillator problem (96) with energies

\[
\epsilon_{h.o.}^{(S)PC} = (12 + n) \omega, \tag{101}
\]

where \( n \) is the degree of \( P_{[n]} \) as a polynomial in the \( A \).

For the lowest \( 0^{++} \) eigenstates e.g., we find

\[
\begin{align*}
\epsilon_{h.o.}^{(0)++} &= 12 \omega : \quad P_{[0]}^{(0)++} \propto 1, \\
\epsilon_{h.o.}^{(0)++} &= 14 \omega : \quad P_{[2]}^{(0)++} \propto -2\sqrt{3} + \frac{1}{2} \omega s_{[2]}^{(0)++}, \\
\epsilon_{h.o.}^{(0)++} &= 16 \omega : \quad P_{[4]}^{(0)++} \propto \sqrt{78} - \sqrt{13} \omega s_{[2]}^{(0)++} + \frac{1}{2} \sqrt{\frac{3}{26}} \omega^2 \left(s_{[2]}^{(0)++}\right)^2,
\end{align*}
\]

\[
\begin{align*}
P_{[2]}^{(0)++} &\propto -\frac{1}{2\sqrt{273}} \omega^2 \left(s_{[2]}^{(0)++}\right)^2 + \frac{1}{2} \sqrt{\frac{13}{105}} \omega^2 \left(s_{[2]}^{(2)++}\right)^2, \\
P_{[4]}^{(0)++} &\propto \frac{1}{3} \sqrt{\frac{2}{35}} \omega^2 \left(s_{[2]}^{(0)++}\right)^2 + \frac{1}{3} \sqrt{\frac{1}{14}} \omega^2 \left(s_{[2]}^{(2)++}\right)^2 + \frac{1}{9} \sqrt{\frac{14}{5}} \omega^2 b_{[4]}^{(0)++},
\end{align*}
\tag{102}
\]

(overall prop. const. = \( 2\sqrt{2} \omega^6/\pi^{5/2} \)) and so on, in terms of the spin-0 component

\[
s_{[2]}^{(0)++} = \frac{1}{3} (s_{11} + s_{22} + s_{33}) \tag{103}
\]

the five spin-2 components

\[
\begin{align*}
s_{[2]}^{(2)++} &\propto \frac{1}{\sqrt{6}} (s_{11} + s_{22} - 2 s_{33}) & s_{[2]}^{(2)++} &\propto \pm s_{13} + i s_{23} & s_{[2]}^{(2)++} &\propto -\frac{1}{2} (s_{11} - s_{22}) \mp i s_{12},
\end{align*}
\tag{104}
\]
of the 2-tensor \( s_{[2]}^{++} \) and similarly for the 2-tensor \( b_{[4]}^{++} \). Much more details are presented in [17].

### 7.2 Low-energy eigensystem of \( SU(3) \) Yang-Mills quantum mechanics

Consider the basis of energy eigenstates of the corresponding unconstrained harmonic oscillator Schrödinger equation orthonormal with respect to the Yang-Mills measure (94)

\[
H_{h.o.} \Phi_n[A, \omega] \equiv \left[ T_{\text{kin}} + \frac{1}{2} \omega^2 A_{ai}^2 \right] \Phi_n[A, \omega] = \epsilon_{n}^{h.o.} \Phi_n[A, \omega]. \tag{105}
\]

Then the matrix elements of the unconstrained Yang-Mills Hamiltonian are given as

\[
\mathcal{M}_{mn} := \Phi_{m}^{h.o.}[A, \omega] \left( T_{\text{kin}} + \frac{1}{2} B_{ai}^2[A] \right) \Phi_{n}[A, \omega] = \delta_{nm} \epsilon_{n}^{h.o.} + \frac{1}{2} \Phi_{m}^{h.o.}[A, \omega] \left( B_{ai}^2[A] - \omega^2 A_{ai}^2 \right) \Phi_{n}[A, \omega], \tag{106}
\]
Table 2: Results for lowest excitation energies $\mu_i^{(S)PC}$, the "bare" glueball masses, calculated from the results of [17] using orthogonal polynomials up to 10th and 11th order. The numerical errors are of the order of the last digit in the numbers given.

| $i$ | 0++ | 2++ | 4++ | 6++ | 3++ | 0-- | 2-- | 1++ | 1-- | 3-- | 5-- | 2-- | 4-- |
|-----|------|------|------|------|------|------|------|------|------|------|------|------|------|
| 1   | 2.76 | 2.21 | 4.44 | 6.7  | 6.9  | 5.16 | 7.25 | 6.18 | 3.94 | 3.44 | 5.5  | 5.8  | 5.7  |
| 2   | 4.5  | 4.5  | 6.9  | 6.9  | 8.7  | 7.4  | 8.2  | 8.7  | 5.9  | 5.9  | 8    | 7.4  | 8.1  |

since the kinetic terms $T_{\text{kin}}$ are the same for the Yang-Mills and the corresponding harmonic oscillator problem. We treat $\omega$ as a variational parameter, which in each symmetry sector can be choose to minimize the lowest eigenvalue of the matrix $\mathcal{M}$. The results for the energy eigenvalues and eigenstates obtained in the recent work [17] considerably improve those obtained by Weisz and Ziemann [3] using the constrained functional approach working in unreduced $V$-space.

The spectrum obtained is purely discrete in accordance with the proof by Simon [23] and the value

$$\epsilon_{0}^{++} = 12.5868$$

(107)

for the groundstate energy is found. The energies (relative to $\epsilon_0$)

$$\mu_i^{(S)CP} := \epsilon_i^{(S)CP} - \epsilon_{0}^{++},$$

(108)

of the lowest states are summarized in Table 2. In order to obtain from these "bare" masses the corresponding physical glueball spectrum, which can be compared e.g. with the lattice results [12][13], a proper renormalisation has to be carried out, in order to remove the dependence on the box-length $a$. How this could be accomplished, will be discussed in the next Section.

8 Perturbation theory in $\lambda = g^{-2/3}$

8.1 Free many-glueball states

The eigenstates of the free Hamiltonian

$$H_0 = \frac{g^{2/3}}{a} \sum_n \mathcal{H}_0^{QM} (n)$$

are free many-glueball states, i.e. the tensor product of the glueball-eigenstates of completely decoupled granulas. In particular, the free glueball-vacuum is given as

$$|0\rangle \equiv \bigotimes_n |\Phi_0\rangle_n \rightarrow E_{\text{vac}}^{\text{free}} = N \epsilon_0 \frac{g^{2/3}}{a},$$

($N$ total number of granulas) with all granulas in the lowest state of energy $\epsilon_0$. Furthermore, the free one-glueball states, which can be chosen to be e.g. momentum eigenstates, are

$$|S, M, P, C, i, k\rangle \equiv \frac{1}{\sqrt{N}} \sum_n e^{iak.n} \left[ |\Phi_{i_1,M_1}^{(S)PC}\rangle_n \bigotimes_{m \neq n} |\Phi_0\rangle_m \right] \rightarrow E_{i_1}^{\text{free}} (k) = \mu_{i_1}^{(S)PC} \frac{g^{2/3}}{a} + E_{\text{vac}}^{\text{free}},$$

define the free two-glueball states,

$$|(S_1, M_1, P_1, C_1, i_1, n_1), (S_2, M_2, P_2, C_2, i_2, n_2)\rangle \equiv |\Phi_{i_1,M_1}^{(S_1)P_1C_1}\rangle_{n_1} \bigotimes |\Phi_{i_2,M_2}^{(S_2)P_2C_2}\rangle_{n_2} \left[ \bigotimes_{m \neq n_1,n_2} |\Phi_0\rangle_m \right]$$

$$\rightarrow E_{i_1,i_2}^{\text{free}} (P_1 P_2 C_1 C_2) = \left( \mu_{i_1}^{(S_1)P_1C_1} + \mu_{i_2}^{(S_2)P_2C_2} \right) \frac{g^{2/3}}{a} + E_{\text{vac}}^{\text{free}},$$

and so on. Matrix elements between these free many-glueball states are calculated using the measure (70) with the product over space $x$ replaced by the product over $n$. 

18
8.2 Interacting glueball vacuum

The energy of the interacting glueball vacuum up to \( \lambda^2 \)

\[
E_{\text{vac}} = \mathcal{N} \frac{g^{2/3}}{a} \left[ \epsilon_0 + \lambda^2 \sum_{s=1}^{3} \sum_{\alpha} \langle 0 | \bar{V}_\alpha^{(0)} | 0 \rangle - \lambda^2 \sum_{s=1}^{3} \sum_{\alpha, \alpha'} \sum_{i_1, i_2} \frac{\langle 0 | \bar{V}_\alpha^{(0)} | i_1 \rangle \langle i_2 | \bar{V}_{\alpha'}^{(0)} | 0 \rangle}{\mu_{i_1} + \mu_{i_2}} + \mathcal{O}(\lambda^3) \right]
\]

\[
\equiv \mathcal{N} \frac{g^{2/3}}{a} \left[ \epsilon_0 + \lambda^2 \sum_{s=1}^{3} c_{0[s]} + \mathcal{O}(\lambda^3) \right] = \mathcal{N} \frac{g^{2/3}}{a} \left[ \epsilon_0 + \lambda^2 c_0 + \mathcal{O}(\lambda^3) \right],
\]

is obtained using first and second order perturbation theory. To simplify the notation we have used \(|i\rangle \equiv |S, M, P, C, i, n\rangle\) and \(\mu_i \equiv \mu_i^{(S)PC}\) for the intermediate states.

For the first order contribution one obtains

\[
c_{0[1\text{ord.}]} = \frac{\pi^2}{3} \sum_{\beta} \langle 0 | \bar{X}_\beta^{(s)} | 0 \rangle \left( \langle 0 | \bar{X}_\beta^{(s)} Y_\beta^{(s)} | 0 \rangle - \langle 0 | \bar{X}_\beta^{(s)} | 0 \rangle \langle 0 | Y_\beta^{(s)} | 0 \rangle \right),
\]

and for the second order contribution

\[
c_{0[2\text{ord.}]} = -\frac{\pi^2}{3} \sum_{\alpha, \alpha'} \sum_{i_1, i_2} \frac{\langle 0 | \bar{Y}_\alpha^{(s)} | i_1 \rangle \langle i_2 | \bar{Y}_{\alpha'}^{(s)} | 0 \rangle - \langle i_1 | \bar{Y}_{\alpha'}^{(s)} | 0 \rangle \langle i_2 | \bar{Y}_\alpha^{(s)} | 0 \rangle}{\mu_{i_1} + \mu_{i_2}}.
\]

Due to rotational invariance we should find \(c_{0[1\text{ord.}]} = c_{0[2\text{ord.}]} = c_{0[1\text{ord.}]} + c_{0[2\text{ord.}]} = c_{0[s=3]}\) or

\[
c_{0[1\text{ord.}]} + c_{0[2\text{ord.}]} = c_{0[s=1]} + c_{0[s=2]} + c_{0[s=3]} = c_{0[s=1]} + c_{0[s=2]} + c_{0[s=3]}.
\]

8.3 Interacting glueballs

Including interactions \(V^{(0)}\) and \(Y^{(0)}\) using 1st and 2nd order perturbation theory, we can obtain the following energy of the interacting lowest lying \(J^{PC}\) glueballs. Most glueball excitations are unstable at tree-level, except for the lowest \(\mu_1^{(S)PC}\), which are below threshold for decay into two spin-2 glueballs. For example, the energy of the interacting lowest lying \(0^{++}\) glueball up to \(\lambda^2\), (writing \(|1k\rangle \equiv |0, 0, +, +, 1, k\rangle\)

\[
E_{1^{0++}(k)} - E_{\text{vac}} = \frac{g^{2/3}}{a} \left\{ \frac{e_{1^{0++}}}{3} + \lambda^2 \sum_{s=1}^{3} \sum_{\beta} \langle 1k | \bar{Y}_\beta^{(s)} | 1k \rangle \right.
\]

\[
- \lambda^2 \sum_{s=1}^{3} \sum_{\alpha, \alpha'} \sum_{i_1, i_2} \left[ \frac{\langle 1k | \bar{Y}_\alpha^{(s)} | i_1 \rangle \langle i_2 | \bar{Y}_{\alpha'}^{(s)} | 1k \rangle}{\mu_{i_1} + \mu_{i_2}} + \frac{\langle 1k | \bar{Y}_\alpha^{(s)} | 1k, i, i_2 \rangle \langle i_1, i_1, i_1 | \bar{Y}_{\alpha'}^{(s)} | 1k \rangle}{\mu_{i_1} + \mu_{i_2}} \right]
\]

\[
- \lambda^2 \sum_{s=1}^{3} \sum_{\alpha, \alpha'} \sum_{i_1, i_2} \left[ \frac{\langle 1k | \bar{Y}_\alpha^{(s)} | i_1, i_2 \rangle \langle i_1, i_2 | \bar{Y}_{\alpha'}^{(s)} | 1k \rangle}{\mu_{i_1} + \mu_{i_2}} + \mathcal{O}(\lambda^3) \right] - E_{\text{vac}}
\]

\[
\equiv \mathcal{N} \frac{g^{2/3}}{a} \left\{ \frac{\mu_{0^{++}}}{3} + \lambda^2 \sum_{s=1}^{3} \left[ \frac{\pi^2}{3} (e_{1(s)}^{0++} + c_{1(s), \infty}) + \frac{e_{1(s)}^{0++} + c_{1(s), \infty}}{N \to \infty} \right] I[N, (ak_s)^2] (ak_s)^2 \right\} + \mathcal{O}(\lambda^3)
\]

using (87) and the function

\[
I[N, x^2] = \sum_{m=1}^{N} \frac{w_N^2(m)}{4m^2} \frac{1 - \cos[mx]}{x^2} = I_0[N] - I_2[N] x^2 + I_4[N] x^4 - +..., \]

with the infinite series of divergent weighted square averages (n=0,1,2,..)

\[
I_{2n}[N] = \frac{1}{4(2n + 2)!} \sum_{m=1}^{N} m^{2n} w_N^2(m), \quad \lim_{N \to \infty} I_{2n}[N] = \infty.
\]

19
The $N$-independent coefficients $c_{1[s]}^{(0)++}$, $c_{1[s]}^{(0)+}$ and $c_{1[s],\infty}^{(0)++}$ are products/sums of quantum mechanical expectation-values and transition elements obtained in standard time-independent perturbation theory (PT) based on the eigensystem of the free Hamiltonian $H_0$, which can be obtained with high accuracy in Sect. 7.

For the contributions of first order PT, one obtains (denoting the quantum mechanical state $|1\rangle \equiv |\Phi_1(0)++\rangle$)

\[
c_{1[s]}^{(0)++} = \sum_{\beta} \left\{ \langle 0|\bar{\lambda}_{\beta}^{(s)}|0\rangle \left[ \langle 1|\lambda_{\beta}^{(s)}\lambda_{\beta}^{(s)*}|1\rangle - \langle 0|\lambda_{\beta}^{(s)}\lambda_{\beta}^{(s)*}|0\rangle \right] - \left( \langle 1|\lambda_{\beta}^{(s)}|0\rangle \langle 0|\lambda_{\beta}^{(s)*}|1\rangle + (\lambda' \leftrightarrow \lambda) \right) \right. \\
\left. - \left( \langle 1|\lambda_{\beta}^{(s)}|1\rangle - \langle 0|\lambda_{\beta}^{(s)*}|0\rangle \right) \langle 0|\lambda_{\beta}^{(s)*}|0\rangle + (\lambda' \leftrightarrow \lambda') \right] \\
\right. + \left( \langle 1|\lambda_{\beta}^{(s)}|1\rangle - \langle 0|\lambda_{\beta}^{(s)*}|0\rangle \right) \langle 0|\lambda_{\beta}^{(s)*}|0\rangle - \langle 0|\lambda_{\beta}^{(s)*}|0\rangle \langle 0|\lambda_{\beta}^{(s)*}|0\rangle \right\}, \tag{116}
\]

\[
\tilde{c}_{1[s]}^{(0)++} = \sum_{\beta} \langle 0|\bar{\lambda}_{\beta}^{(s)}|0\rangle \left[ \langle 1|\lambda_{\beta}^{(s)}|0\rangle \langle 0|\lambda_{\beta}^{(s)*}|1\rangle + (\lambda' \leftrightarrow \lambda') \right], \tag{117}
\]

\[
c_{1[s],\infty}^{(0)++} = \sum_{\beta} \left\{ \langle 0|\bar{\lambda}_{\beta}^{(s)}|1\rangle \left[ \langle 1|\lambda_{\beta}^{(s)}\lambda_{\beta}^{(s)*}|0\rangle - \langle 1|\lambda_{\beta}^{(s)}|0\rangle \langle 0|\lambda_{\beta}^{(s)*}|0\rangle \right] \\
\right. + \langle 1|\lambda_{\beta}^{(s)}|0\rangle \left[ \langle 0|\lambda_{\beta}^{(s)*}|1\rangle - \langle 0|\lambda_{\beta}^{(s)*}|0\rangle \langle 0|\lambda_{\beta}^{(s)*}|0\rangle \right] \right\}, \tag{118}
\]

and for the contributions of second order PT

\[
c_{1[s]}^{(0)++} = -\sum_{\alpha,\alpha'} \left\{ \sum_{i \neq 1} \frac{1}{\mu_i} \left( \langle 1|\bar{\lambda}_{\alpha}^{(s)}|i\rangle \langle 0|\lambda_{\alpha'}^{(s)}|1\rangle - (\tilde{Y} \leftrightarrow Y) \right) \left( \langle \alpha'|\bar{\lambda}_{\alpha'}^{(s)}|1\rangle \langle 1\rangle \langle \alpha'|0\rangle \langle 0\rangle \right) - (\tilde{Y} \leftrightarrow Y) \right) \\
\right. + \sum_{i \neq 1} \frac{1}{\mu_i + \mu_1^{(0)+}} \left( \langle 0|\bar{\lambda}_{\alpha}^{(s)}|i\rangle \langle 0|\lambda_{\alpha'}^{(s)}|1\rangle - (\tilde{Y} \leftrightarrow Y) \right) \left( \langle 1|\bar{\lambda}_{\alpha'}^{(s)}|0\rangle \langle i\rangle \langle \alpha'|0\rangle \langle 0\rangle \right) - (\tilde{Y} \leftrightarrow Y) \right) \\
\right. + \sum_{i_1,i_2 \neq 1} \frac{1}{\mu_{i_1} + \mu_{i_2} - \mu_1^{(0)+}} \left( \langle 1|\bar{\lambda}_{\alpha}^{(s)}|i_1\rangle \langle 0|\lambda_{\alpha'}^{(s)}|i_2\rangle - (\tilde{Y} \leftrightarrow Y) \right) \times \\
\left( \langle i_1|\bar{\lambda}_{\alpha'}^{(s)}|1\rangle \langle i_2|\lambda_{\alpha'}^{(s)}|0\rangle - (\tilde{Y} \leftrightarrow Y) \right) + (i_1 \leftrightarrow i_2) \right\}, \tag{119}
\]

\[
\tilde{c}_{1[s]}^{(0)++} = \sum_{\alpha,\alpha'} \left\{ \sum_{i \neq 1} \frac{1}{\mu_i - \mu_1^{(0)+}} \left( \langle 0|\bar{\lambda}_{\alpha}^{(s)}|i\rangle \langle 1|\lambda_{\alpha'}^{(s)}|0\rangle - (\tilde{Y} \leftrightarrow Y) \right) \left( \langle 0|\bar{\lambda}_{\alpha'}^{(s)}|1\rangle \langle i\rangle \langle \alpha'|0\rangle \langle 0\rangle \right) - (\tilde{Y} \leftrightarrow Y) \right) \\
\right. + \sum_{i \neq 1} \frac{1}{\mu_i + \mu_1^{(0)+}} \left( \langle 0|\bar{\lambda}_{\alpha}^{(s)}|i\rangle \langle 0|\lambda_{\alpha'}^{(s)}|1\rangle - (\tilde{Y} \leftrightarrow Y) \right) \left( \langle 1|\bar{\lambda}_{\alpha'}^{(s)}|0\rangle \langle i\rangle \langle \alpha'|0\rangle \langle 0\rangle \right) - (\tilde{Y} \leftrightarrow Y) \right\}, \tag{120}
\]

\[
c_{1[s],\infty}^{(0)++} = \sum_{\alpha,\alpha'} \sum_{i_1,i_2 \neq 1} \frac{1}{\mu_{i_1} + \mu_{i_2} - \mu_1^{(0)+}} \left( \langle 1|\bar{\lambda}_{\alpha}^{(s)}|i_1\rangle \langle 0|\lambda_{\alpha'}^{(s)}|i_2\rangle - (\tilde{Y} \leftrightarrow Y) \right) \times \\
\left( \langle i_2|\lambda_{\alpha'}^{(s)}|0\rangle \langle i_1|\lambda_{\alpha'}^{(s)}|0\rangle - (\tilde{Y} \leftrightarrow Y) \right) + (i_1 \leftrightarrow i_2) \right\}. \tag{121}
\]

Note, that, due to rotational invariance, the $s = 1, 2, 3$ contributions to the coefficients $c_{1[s]}^{(0)++}$, $c_{1[s]}^{(0)+}$ and $c_{1[s],\infty}^{(0)++}$ (adding up first and second order perturbation theory) should give equal contributions,

\[
c_{1[s]}^{(0)++} + c_{1[s]}^{(0)+} + c_{1[s],\infty}^{(0)++} = c_{1[s]}^{(0)++} + c_{1[s]}^{(0)+} + c_{1[s],\infty}^{(0)++} \tag{122}
\]

and an analogous expression for $\tilde{c}_{1[s],\infty}^{(0)++}$.
Furthermore, in order for the result (113) to be finite in the limit $N \to \infty$, and hence renormalisable, the above coefficients $\tilde{c}_{1,\infty}^{(0)++}$ should vanish, i.e.
\[ \tilde{c}_{1,\infty}^{(0)++} = c_{1,\infty}^{(0)++} \quad \text{1st ord.} + \tilde{c}_{1,\infty}^{(0)++} \quad \text{2nd ord.} \equiv 0 \quad \text{for all } s = 1, 2, 3. \] (124)

Denoting in this case, when the conditions of rotational invariance (123) and finiteness (124) are indeed fulfilled,
\[ c_1^{(0)++} := \sum_{s=1}^{3} c_1^{(0)[s]} , \quad \tilde{c}_1^{(0)++} := \tilde{c}_1^{(0)++} = (s=1) = c_1^{(0)++} = \tilde{c}_1^{(0)++} = c_1^{(0)++} , \] (125) (126)

the expression (113) can be written as
\[ E_1^{(0)++}(k) - E_{\text{vac}} = \frac{g^{2/3}}{a} \left\{ \mu_1^{(0)++} + \lambda^2 \left[ \frac{\pi^2}{3} c_1^{(0)++} + \tilde{c}_1^{(0)++} (ak)^2 \right] + O(\lambda^4) \right\} . \] (127)

Finally, in order for the result (113) to be Lorentz invariant, i.e. to fulfill the energy momentum relation for a massive spin-0 particle
\[ E = \sqrt{m^2 + (ak)^2} = m + (1/(2m))(ak)^2 + O((ak)^4) \]
we should have the relation
\[ \tilde{c}_1^{(0)++} = c_1^{(0)++} = 1/2 \mu_1^{(0)++} . \] (128)

The check of these conditions, rotational invariance, finiteness and Lorentz invariance, to order $\lambda^2$, the coefficients $c_1^{(0)++}$, $c_1^{(0)++}$ and $c_1^{(0)++}$, and hence the quantum mechanical matrix elements, have to be calculated explicitly, which is expected to be a rather lengthy, but a very important calculation and in my opinion manageable task for future work. Up to now, to the best of my knowledge, nobody can exclude by general arguments the existence of the above quite naturally arising strong coupling scheme.

### 8.4 Glueball-glueball scattering amplitude

Let us consider the elastic scattering of two identical lowest lying $0^{++}$ glueball excitations with equal and opposite momenta $k^{(2)} = -k^{(1)}$ to the new momenta $k^{(2)\prime} = -k^{(1)\prime}$, corresponding to vanishing center-of-mass momentum relative to the lattice,
\[ K \equiv (k^{(1)} + k^{(2)})/2 = 0 , \quad K' \equiv (k^{(1)\prime} + k^{(2)\prime})/2 = 0 . \] (129)

Furthermore, elasticity and hence energy conservation requires
\[ |k^{(1)\prime}| = |k^{(1)}| = k . \] (130)

Due to global isotropy, the scattering amplitude defined by
\[ A(k^{(1)\prime} - k^{(1)}) := \langle 1 1, K' = 0, k^{(1)\prime} | H | 1 1, K = 0, k^{(1)} \rangle = \langle 1 1, K' = 0, k^{(1)} | H | 1 1, K = 0, k^{(1)} \rangle , \] (131)

should depend only on
\[ q := k^{(1)\prime} - k^{(1)} . \] (132)

Up to order $\lambda^2$, using first and second order perturbation theory, it is found to be
\[ A(q) = \frac{1}{N} \frac{g^{2/3}}{a} \lambda^2 \left\{ \frac{1}{2} \sum_{s=1}^{3} a^2 q_s^2 \left( \tilde{a}_1^{[s]} \right) \right\}_{N \to \infty} + O(\lambda^4) \right\} , \] (133)
with the same function $I[N,x^2]$ defined in (114), tending to infinity for $N \to \infty$ and with the first order coefficients

$$\tilde{d}_1^{1\text{st ord.}} = \sum_\beta \langle 0 | \tilde{x}_\beta^{(s)} | 0 \rangle \langle 1 | \tilde{x}_\beta^{(s)} | 1 \rangle \langle 1 | \tilde{x}_\beta^{(s)} | 1 \rangle ,$$

$$\tilde{d}_1^{1\text{st ord.}} = \sum_\beta \langle 1 | \tilde{x}_\beta^{(s)} | 1 \rangle \left[ \langle 1 | \tilde{x}_\beta^{(s)} | \tilde{x}_\beta^{(s)} | 1 \rangle - \langle 1 | \tilde{x}_\beta^{(s)} | 1 \rangle \langle 0 | \tilde{x}_\beta^{(s)} | 0 \rangle - \langle 0 | \tilde{x}_\beta^{(s)} | 0 \rangle \langle 1 | \tilde{x}_\beta^{(s)} | 1 \rangle \right] ,$$

and the second order coefficients

$$\tilde{d}_1^{2\text{nd ord.}} = 0 ,$$

$$\tilde{d}_1^{2\text{nd ord.}} = \sum_{\alpha,\alpha'} \sum_{i_1 > i_2 > 1} \frac{1}{\mu_{i_1} + \mu_{i_2} - 2\mu_1^{(0)} + (\langle 1 | \tilde{x}_\alpha^{(s)} | i_1 \rangle \langle 1 | \tilde{x}_\alpha^{(s)} | i_2 \rangle - (\tilde{Y} \leftrightarrow Y)) + (\langle i_2 | \tilde{x}_\alpha^{(s)} | 1 \rangle \langle 1 | \tilde{x}_\alpha^{(s)} | 1 \rangle - (\tilde{Y} \leftrightarrow Y)) .$$

As for the case of the single interacting glueball, there are constraints from the conditions of isotropy and finiteness Firstly, due to rotational invariance, the $s = 1, 2, 3$ contributions to the coefficients $\tilde{d}_1^{1\text{st ord.}}$ and $\tilde{d}_1^{2\text{nd ord.}}$ (adding up first and second order perturbation theory) should give equal contributions. Secondly, in order for the result (133) to be finite in the limit $N \to \infty$, and hence renormalisable, the above coefficients $\tilde{d}_1^{1\text{st ord.}}$ should vanish, i.e.

$$\tilde{d}_1^{1\text{st ord.}} = \tilde{d}_1^{2\text{nd ord.}} = 0$$

Hence, in this case, defining

$$\tilde{d}_1 := \tilde{d}_1^{1\text{st ord.}} = \tilde{d}_1^{2\text{nd ord.}} = \tilde{d}_1^{3\text{rd ord.}} ,$$

the scattering amplitude becomes

$$\mathcal{A}(q) = \frac{1}{N^2} \frac{g^{2/3}}{a} \left\{ \lambda^2 \left[ \frac{1}{2} \tilde{d}_1 (aq)^2 \right] + \mathcal{O}(\lambda^3) \right\} ,$$

in lowest order depending only on $|q| = 2k \sin(\theta/2)$. Performing the corresponding integrations and using an appropriate overall normalisation, we obtain an expansion of the physical coupling constant $\lambda_R$ in terms of the original bare $\lambda$

$$\lambda_R = \lambda + c\lambda^2 + c'\lambda^3 + ... ,$$

with some constants $c, c', ...$.

### 9 Conclusions

It has been shown in this work, how the gauge invariant formulation of low energy $SU(2)$ Yang-Mills theory on a three dimensional spatial lattice, obtained by replacing integrals by sums and spatial derivatives by differences, proposed in the earlier work [6], can be generalised to $SU(3)$ with a simple, but highly non-trivial, FP-operator, FP-determinant and inverse FP-operator, which are practically manageable.

This has been achieved proposing a new gauge, the "flux-tube gauge", defined in (7), which is shown to exist by construction. In contrast to the $SU(3)$ symmetric gauge, the Faddeev-Popov operator, its determinant and inverse, are rather simple in the flux-tube gauge, but show a highly non-trivial periodic structure of six Gribov-horizons separating six Weyl-chambers. Such a Weyl structure in the context of models of the QCD-vacuum has been discussed recently e.g. in [24].

Furthermore, as for the case of $SU(2)$Yang-Mills theory in the symmetric gauge, the flux-tube gauge allows for a systematic and practical strong coupling expansion of the $SU(3)$ Hamiltonian in $\lambda \equiv g^{-2/3}$, equivalent to an expansion in the number of spatial derivatives.
Constructing the corresponding physical quantum Hamiltonian of $SU(3)$ Yang-Mills theory in the flux-tube gauge according to the general scheme given by Christ and Lee, a systematic and practical expansion of the $SU(3)$ Hamiltonian in the number of spatial derivatives could be obtained here. Introducing an infinite spatial lattice with box length $a$, the "free part" is the sum of Hamiltonians of Yang-Mills quantum mechanics of constant fields for each box with a purely discrete spectrum("free glueballs"), and the "interaction terms" contain higher and higher number of spatial derivatives connecting different boxes. This expansion has been carried out here explicitly and shown to be equivalent to a strong coupling expansion in $\lambda = g^{-2/3}$ for large box sizes $a$. It is the analogon to the weak coupling expansion in $g^{2/3}$ by Lüscher, applicable for small boxes.

The energy eigensystem of the gauge reduced Hamiltonian of $SU(3)$ Yang-Mills mechanics of spatially constant fields can be calculated in principle with arbitrary high precision using the orthonormal basis of all solutions of the corresponding harmonic oscillator problem, which turn out to be made of orthogonal polynomials of the 45 components of eight irreducible symmetric spatial tensors. Rather accurate first results for the lowest bare glueball masses have been obtained in recent work [17] and substantially improve those obtained by Weisz and Ziemann using the unreduced approach.

Thus, the gauge reduced approach using the flux-tube gauge proposed here, is expected to enable one to obtain valuable non-perturbative information about low-energy glueball dynamics, carrying out perturbation theory in $\lambda$.

Finally, I would like to point out, that the flux-tube gauge exists for low energy Hamiltonian formulations of D+1 dimensional $SU(3)$ Yang-Mills theories with $D \geq 2$ and for strong coupling path-integral formulations of $D \geq 2$ dimensional Euclidean $SU(3)$ Yang-Mills theories. In particular, it might be useful for the gauge-invariant investigation of the strong coupling limit of the Hamiltonian of $2+1$ dimensional $SU(3)$ Yang-Mills theories (and compare e.g. with the continuum Hamiltonian formulation in terms of gauge invariant variables [25]).

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Appendix

A On the existence of the flux-tube gauge

In this appendix will be discussed, under which conditions the flux-tube gauge (7)

$$\chi_a(A) = (\Gamma_i)_{ab} A_{bi}(x) = 0 \quad a = 1,\ldots,8,$$

exists, as a generalisation of the proof for the case of the $SU(2)$ symmetric gauge in App.B of [5].

A gauge $\chi_a(A) = 0$ exists, iff

$$\chi_a(V^\omega) = 0 \quad a = 1,\ldots,8,$$  \hfill (141)

for the gauge transformed

$$V_{ai}^\omega(x) \lambda_a/2 = U^{\dagger}[\omega(x)] \left( V_{bi}(x) \lambda_b/2 + \frac{i}{g} \partial_i \right) U[\omega(x)],$$  \hfill (142)

has a unique solution for $\omega(x)$. The flux-tube gauge therefore exists iff an arbitrary gauge potential $V_{ai}(x)$ can be made to fulfill $(\Gamma_i)_{ab} A_{bi}(x) = 0 \quad a = 1,\ldots,8$, by a unique time-independent gauge transformation. We can write the equations which determine the corresponding gauge transformation $\omega(x)$ as

$$\frac{1}{2} (\Gamma_i)_{ab} \text{Tr} \left[ \lambda_b U^{\dagger}(\omega(x)) V_i(x) U(\omega(x)) \right] = \frac{1}{g} (\Gamma_i)_{ab} \Sigma_{bi}(\omega(x)) \quad a = 1,\ldots,8,$$  \hfill (143)
with
\[
\Sigma_{ai}(\omega(\mathbf{x})) := \frac{1}{2i} \text{Tr} \left[ \lambda_a U^\dagger(\omega(\mathbf{x})) \partial_t U(\omega(\mathbf{x})) \right].
\]  

A.1 Solution of the corresponding infinite-coupling problem

Consider first the homogeneous problem obtained in the infinite coupling limit \(1/g \to 0\),
\[
(\Gamma_i)_{ab} \text{Tr} \left[ \lambda_b U^{(0)\dagger} V_i U^{(0)} \right] = 0, \quad \forall a = 1, \ldots, 8.
\]  

This is solved iff we can find a unique unitary transformation \(U^{(0)}\) such that the \(V_i\) can be rotated into new \(A_i\), which satisfy the gauge condition,
\[
U^{(0)\dagger} V_i U^{(0)} = A_i, \quad (\Gamma_i)_{ab} A_{bi} = 0, \quad \forall a = 1, \ldots, 8.
\]  

It is unique, iff the corresponding homogeneous FP operator \(\gamma(A)\) defined in (17) is invertible [1],
\[
\det(\gamma(A)) \neq 0 \quad \gamma_{ab}(A) \equiv (\Gamma_i)_{ad} f_{dabc} A_{ci}.
\]  

For the case of the flux-tube gauge, the homogeneous equation (145) reads more explicitly
\[
\text{Tr} \left[ \lambda_a U^{(0)\dagger} V_1 U^{(0)} \right] = 0 \quad \forall a = 1, 2, 4, 5, 6, 7 \quad \land \quad \text{Tr} \left[ \lambda_a U^{(0)\dagger} V_2 U^{(0)} \right] = 0 \quad \forall a = 5, 7.
\]  

This can always be achieved: In a first step, one diagonalises the spatial 1-component of the gauge field in the fundamental representation, \(A_{a1} = 0\) for all \(a = 1, 2, 4, 5, 6, 7\). In a second step, one uses the remaining gauge-freedom, generated by \(\lambda_3\) and \(\lambda_8\), which leave the \(a = 3, 8\) components of all \(A_{ai}\) for all \(i = 1, 2, 3\) unchanged, in order to put the spatial 2-components \(A_{a2} = 0\) for \(a = 5, 7\). Hence (148) has the solution
\[
U^{(0)}(\omega_1, \ldots, \omega_8) \equiv U_2^{(0)}(\omega_3', \omega_8') U_1^{(0)}(\omega_1', \ldots, \omega_6')
\]  

with
\[
U_1^{(0)}(\omega') := \exp[i \sum_a \omega_a' \lambda_a/2] \quad U_2^{(0)}(\omega'') := \exp[i \omega_3'' \lambda_3/2] \exp[i \omega_8'' \lambda_8/2]
\]  

such that
\[
U_1^{(0)\dagger} V_1 U_1^{(0)} = A_1, \quad U_2^{(0)\dagger} U_1^{(0)\dagger} V_2 U_1^{(0)} U_2^{(0)} = A_2.
\]  

using the invariance of \(A_1\) under \(U_2\) transformations \(U_2^{(0)\dagger} A_1 U_2^{(0)} = A_1\). It is unique, iff the corresponding homogeneous FP determinant (20) is invertible
\[
\det(\gamma(A)) \propto A_{31}^2 \left( A_{31}^2 - 3 A_{81}^2 \right)^2 A_{42} A_{62} \neq 0.
\]  

A.2 Solution of Equ.(143) using a strong-coupling expansion

In order to discuss the solubility of the full Equ.(143) the following Theorem will be proven:

\textbf{Theorem:} Equ.(143) has a unique solution in the form of a \(1/g\) expansion
\[
U(\omega(\mathbf{x})) = U^{(0)}(\mathbf{x}) \left[ 1 + \sum_{n=1}^{\infty} \left( \frac{1}{g} \right)^n X^{(n)}(\mathbf{x}) \right],
\]  

iff the corresponding homogeneous FP operator is invertible at each \(\mathbf{x}\), i.e. \(\det(\gamma(A(\mathbf{x}))) \neq 0\).

\textbf{Proof:} Equating equal powers of \(1/g\) in the unitarity condition \(U U^\dagger = 1\) lead to the condition of unitarity of \(U^{(0)}\)
\[
U^{(0)\dagger} U^{(0)} = U^{(0)} U^{(0)\dagger} = 1,
\]
as well as the conditions
\[
\begin{align*}
X^{(1)} + X^{(1)\dagger} &= 0, \\
X^{(2)} + X^{(2)\dagger} + X^{(1)\dagger}X^{(1)} &= 0, \\
X^{(n)} + X^{(n)\dagger} + \sum_{i+j=n} X^{(i)}X^{(j)\dagger} &= 0, \\
& \quad \ldots \\
\end{align*}
\]
for the unknown \(X^{(n)}\). Furthermore, inserting (153) and equating equal powers of \(1/g\), one finds that the leading order unitary matrix \(U^{(0)}\) should satisfy
\[
(\Gamma_i)_{ab} \text{Tr} \left[ \lambda_b U^{(0)\dagger}(x) V_i(x) U^{(0)}(x) \right] = 0 \quad \forall a = 1,\ldots,8, \tag{156}
\]
and the \(X^{(n)}\) fulfill the infinite set of equations at each \(x\)
\[
\frac{1}{2} (\Gamma_i)_{ab} \text{Tr} \left[ \lambda_b \left( X^{(1)}U^{(0)\dagger}V_i U^{(0)} + U^{(0)\dagger}V_i U^{(0)} X^{(1)} \right) \right] = (\Gamma_i)_{ab} \Sigma_{bi}^{(0)}, \\
\frac{1}{2} (\Gamma_i)_{ab} \text{Tr} \left[ \lambda_b \left( X^{(2)}U^{(0)\dagger}V_i U^{(0)} + U^{(0)\dagger}V_i U^{(0)} X^{(2)} + X^{(1)\dagger}U^{(0)\dagger}V_i U^{(0)} X^{(1)} \right) \right] = (\Gamma_i)_{ab} \Sigma_{bi}^{(1)}, \\
\frac{1}{2} (\Gamma_i)_{ab} \text{Tr} \left[ \lambda_b \left( X^{(n)}U^{(0)\dagger}V_i U^{(0)} + U^{(0)\dagger}V_i U^{(0)} X^{(n)} + \sum_{i+j=n} X^{(i)\dagger}U^{(0)\dagger}V_i U^{(0)} X^{(j)} \right) \right] = (\Gamma_i)_{ab} \Sigma_{bi}^{(n-1)}, \\
& \quad \ldots \\
\tag{157}
\]
using the \(1/g\) expansion
\[
\Sigma_{ai}(\omega(x)) = \sum_{n=0}^{\infty} \left( \frac{1}{g} \right)^n \Sigma_{ai}^{(n)}(x). \tag{158}
\]
Note, that the \(n\)th term in (158) is given in terms of \(U^{(0)}\) and \(X^{(1)},\ldots,X^{(n-1)}\).

Hence we find that using the expansion (153) of \(U\) in \(1/g\) the solution of (143) reduces to the algebraic problem (154)-(157). We now assume that for a given choice of \(V_i(x)\), the first, homogeneous equation (156) has a unique unitary solution \(U^{(0)}\) such that
\[
U^{(0)\dagger}(x) V_i(x) U^{(0)}(x) = A_i(x) \quad \land \quad (\Gamma_i)_{ab} A_{bi}(x) = 0 \quad \land \quad \det(\gamma(A(x))) \neq 0. \tag{159}
\]
Using this solution and the conditions (155) to express \(X^{(n)\dagger}\) in terms of \(X^{(n)}\) plus terms containing \(X^{(1)},\ldots,X^{(n-1)}\) one can then rewrite the remaining equations (157) into the set of equations for \(X^{(n)}\)
\[
(\Gamma_i)_{ab} \text{Tr} \left[ \lambda_b \left( -X^{(1)}(x)A_i(x) + A_i(x) X^{(1)}(x) \right) \right] = C_{a}^{(0)}(x), \\
(\Gamma_i)_{ab} \text{Tr} \left[ \lambda_b \left( -X^{(2)}(x)A_i(x) + A_i(x) X^{(2)}(x) \right) \right] = C_{a}^{(1)}(x), \\
(\Gamma_i)_{ab} \text{Tr} \left[ \lambda_b \left( -X^{(n)}(x)A_i(x) + A_i(x) X^{(n)}(x) \right) \right] = C_{a}^{(n-1)}(x), \\
& \quad \ldots \\
\tag{160}
\]
where the \(n\)th order \(C^{(n)}\) is given in terms of \(U^{(0)}\) and \(X^{(1)},\ldots,X^{(n-1)}\). In terms of \(X^{(n)}_a := (1/2)\text{Tr} \left[ \lambda_a X^{(n)} \right]\), and using the homogeneous FP operator \(\gamma(A)\), Equs. (160) can be written in the final form
\[
\begin{align*}
\frac{i\gamma_{ab}(A(x)) X^{(1)}_b(x)} &= C^{(0)}_a(x), \\
\frac{i\gamma_{ab}(A(x)) X^{(2)}_b(x)} &= C^{(1)}_a(x), \\
\frac{i\gamma_{ab}(A(x)) X^{(n)}_b(x)} &= C^{(n-1)}_a(x), \\
& \quad \ldots \\
\tag{161}
\end{align*}
\]
This system of Equs. (161) has a unique solution, solving subsequently for \(X^{(1)}, X^{(2)}, \) and so on, iff the homogeneous FP operator is invertible, which completes the proof.
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