Optimal Top-k Document Retrieval

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Abstract

Let \( D \) be a collection of \( D \) documents, which are strings over an alphabet of size \( \sigma \), of total length \( n \). We describe a data structure that uses linear space and reports \( k \) most relevant documents that contain a query pattern \( P \), which is a string of length \( p \), in time \( O(p/\log \sigma n + k) \), which is optimal in the RAM model in the general case where \( \lg D = \Theta(\log n) \), and involves a novel RAM-optimal suffix tree search. Our construction supports an ample set of important relevance measures, such as the number of times \( P \) appears in a document (called term frequency), a fixed document importance, and the minimal distance between two occurrences of \( P \) in a document.

When \( \lg D = o(\log n) \), we show how to reduce the space of the data structure from \( O(n \log n) \) to \( O(n(\log \sigma + \log D + \log \log n)) \) bits, and to \( O(n(\log \sigma + \log D)) \) bits in the case of the popular term frequency measure of relevance, at the price of an additive term \( O(\log \varepsilon n \log \sigma) \) in the query time, for any constant \( \varepsilon > 0 \).

We also consider the dynamic scenario, where documents can be inserted and deleted from the collection. We obtain linear space and query time \( O(p(\log \log n)^2 / \log \sigma \log n + k \log \log k) \), whereas insertions and deletions require \( O(\log^{1+\varepsilon} n) \) time per symbol, for any constant \( \varepsilon > 0 \).

Finally, we consider an extended static scenario where an extra parameter \( \text{par}(P, d) \) is defined, and the query must retrieve only documents \( d \) such that \( \text{par}(P, d) \in [\tau_1, \tau_2] \), where this range is specified at query time. We solve these queries using linear space and \( O(p/\log \sigma n + \log^{1+\varepsilon} n + k \log^{1+\varepsilon} n) \) time, for any constant \( \varepsilon > 0 \).

Our technique is to translate these top-k problems into multidimensional geometric search problems. As an additional bonus, we describe some improvements to those problems.

1 Introduction

The design of efficient data structures for document (i.e., string) collections that can report those containing a query pattern \( P \) is an important problem studied in the information retrieval and pattern matching communities (see, e.g., a recent survey \cite{survey}). Due to the steadily increasing volumes of data, it is often necessary to generate a list \( L(P) \) of the documents containing a string pattern \( P \) in decreasing order of relevance. Since the list \( L(P) \) can be very large, in most cases we are interested in answering top-k queries, that is, reporting only the first \( k \) documents from \( L(P) \) for a parameter \( k \) given at query time.

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Inverted files \[12, 39, 6\] that store lists of documents containing certain keywords are frequently used in practical implementations of information retrieval methods. However, inverted files only work when query patterns belong to a fixed pre-defined set of strings (keywords). The suffix tree \[54\], a handbook data structure known since 1973, uses linear space (i.e., \(O(n)\) words, where \(n\) is the total length of all the documents) and finds all the \(occ\) occurrences of a pattern \(P\) in \(O(p + occ)\) time, where \(p = |P|\). Surprisingly, the general document listing problem, that is, the problem of reporting all the documents that contain an arbitrary query pattern \(P\), was not studied until the end of the 90s. Suffix trees and other data structures for standard pattern matching queries do not provide a satisfactory solution for the document listing problem because the same document may contain many occurrences of \(P\). Matias et al. \[42\] described the first data structure for document listing queries; their structure uses \(O(n)\) words of space and reports all \(docc\) documents that contain \(P\) in \(O(p \log D + docc)\) time, where \(D\) is the total number of documents in the collection. Muthukrishnan \[45\] presented a data structure that uses \(O(n)\) words of space and answers document listing queries in \(O(p + docc)\) time. Muthukrishnan \[45\] also initiated the study of more sophisticated problems in which only documents that contain \(P\) and satisfy some further criteria are reported. In the \(K\)-mining problem, we must report documents in which \(P\) occurs at least \(K\) times; in the \(K\)-repeats problem, we must report documents in which at least two occurrences of \(P\) are within a distance \(K\). He described \(O(n)\)- and an \(O(n \log n)\)-word data structures that answer \(K\)-mine and \(K\)-repeats queries, respectively, both in \(O(p + occ)\) time, where \(occ\) is the number of reported documents.

A problem not addressed by Muthukrishnan, and arguably the most important one for information retrieval, is the top-\(k\) document retrieval problem: report \(k\) most highly ranked documents for a query pattern \(P\) in decreasing order of their ranks. The ranking is measured with respect to the so-called relevance of a string \(P\) for a document \(d\). A basic relevance measure is \(tf(P, d)\), the number of times \(P\) occurs in \(d\). Two other important examples are \(mindist(P, d)\), the minimum distance between two occurrences of \(P\) in \(d\), and \(docrank(d)\), an arbitrary static rank assigned to a document \(d\). Some more complex measures have also been proposed. Hon et al. \[37\] presented a solution for the top-\(k\) document retrieval problem for the case when the relevance measure is \(tf(P, d)\). Their data structure uses \(O(n \log n)\) words of space and answers queries in \(O(p + k + \log n \log \log n)\) time. Later, Hon, Shah and Vitter \[37\] presented a general solution for a wide class of relevance measures. Their data structure uses linear space and needs \(O(p + k \log k)\) time to answer a top-\(k\) query. A recent \(O(n)\) space data structure \[38\] enables us to answer top-\(k\) queries in \(O(p + k)\) time when the relevance measure is \(docrank(d)\). However, that result cannot be extended to other more important relevance measures.

Our Results. Hon et al.’s results \[37\] are an important achievement, but their time is not yet optimal. In this paper we describe a linear space data structure that answers top-\(k\) document queries in \(O(p/\log_\sigma n + k)\) time, where \(\sigma\) is the alphabet size of the collection. This is optimal in the \(\Theta(\log n)\)-word RAM model we use, unless the collection has very few documents, \(\lg D = o(\log n)\) (i.e., \(D = o(n^\varepsilon)\) for any constant \(\varepsilon > 0\)). We support the same relevance measures as Hon et al. \[37\].

**Theorem 1** Let \(D\) be a collection of strings (called documents) of total length \(n\) over an integer alphabet \([1, \sigma]\), and let \(w(S, d)\) be a function that assigns a numeric weight to string \(S\) in document \(d\), so that \(w(S, d)\) depends only on the set of starting positions of occurrences of \(S\) in \(d\). Then there exists an \(O(n)\)-word space data structure that, given a string \(P\) of length \(p\) and an integer \(k\),
reports $k$ documents $d$ containing $P$ with highest $w(P,d)$ values, in decreasing order of $w(P,d)$, in $O(p / \log_\sigma n + k)$ time. The time is online on $k$.

Note that the weighting function is general enough to encompass measures $tf(P,d)$, $mindist(P,d)$ and $docrank(d)$. As stated, our solution is online on $k$: It is not necessary to specify $k$ beforehand; our data structure can simply report documents in decreasing relevance order until all the documents are reported or the query processing is terminated by a user.

An online top-$k$ solution using the $tf$ measure solves the $K$-mining problem in optimal time and linear space. An online top-$k$ solution using $mindist$ measure solves the $K$-repeats problem in optimal time and linear space. We remind that Muthukrishnan [45] had solved the $K$-repeats problem using $O(n \log n)$-word space; later Hon et al. [37] reduced the space to linear. Now all these results appear as a natural corollary of our optimal top-$k$ retrieval solution. Our results also subsume those on more recent variants of the problem [38], for example when the rank $docrank(d)$ depends only on $d$ (we just use $w(P,d) = docrank(d)$), or where in addition we exclude those $d$ where $P$ appears less than $K$ times for a fixed pre-defined $K$ (we just use $w(P,d) = docrank(d)$ if $tf(P,d) \geq K$, else 0).

Moreover, we can also answer queries for some relevance metrics not included in Theorem 1. For instance, we might be interested in reporting all the documents $d$ with $tf(P,d) \times idf(P) \geq \tau$, where $idf(P) = \log(N / df(P))$ and $df(P)$ is the number of documents where $P$ appears [6]. Using the $O(n)$-bit structure of Sadakane [55], we can compute $idf(P)$ in $O(1)$ time from the suffix tree locus of $P$. To answer the query, we use our data structure of Theorem 1 in online mode on measure $tf$: For every reported document $d$ we find $tf(P,d)$ and compute $tf(P,d) \times idf(P)$; the procedure is terminated when a document $d_l$ with $tf(P,d_l) \times idf(P) < \tau$ is encountered. Thus we need $O(p / \log_\sigma n + occ)$ time to report all occ documents with $tf \times idf$ scores above a threshold.

When $\log D = o(\log n)$, it is not clear that our time is RAM-optimal. Instead, we show that in this case the space of our data structures can be reduced from $O(n \log n)$ bits to $O(n (\log \sigma + \log D + \log \log n))$. This is $o(n \log n)$ bits unless $\log \sigma = \Theta(\log n)$ (in which case the linear-space data structure is already asymptotically optimal). For the most important $tf$ relevance measure, where we report documents in which $P$ occurs most frequently, we obtain a data structure that uses $O(n (\log \sigma + \log D))$ bits of space. The price of the space reduction is an additive term $O(\log^\epsilon n \log \sigma)$ in the query time, for any constant $\epsilon > 0$.

We also consider the dynamic framework, where collection $\mathcal{D}$ admits insertions of new documents and deletions of existing documents. Those updates are supported in slightly superlogarithmic time per character, whereas the query times are only slightly slowed down. We note that measure $C_w$ is just $O(1)$ for the typical relevance measures $tf$ and $docrank$, and $O(\log n)$ for $mindist$.

**Theorem 2** Let $\mathcal{D}$ be a collection of documents of total length $n$ over an integer alphabet $[1, \sigma]$, and let $w(S,d)$ be a function that assigns a numeric weight to string $S$ in document $d$, so that $w(S,d)$ depends only on the set of starting positions of occurrences of $S$ in $d$, and can be computed in $O(C_w[d])$ time for all the nodes of the suffix tree of document $d$. Then there exists an $O(n)$-word space data structure that, given a string $P$ of length $p$ and an integer $k$, reports $k$ documents $d$ containing $P$ with highest $w(P,d)$ values, in decreasing order of $w(P,d)$, in $O(p (\log n)^2 / \log_\sigma n + \log n + k \log \log k)$ time, online in $k$. The structure can insert new documents and delete existing documents in $O(C_w + \log^{1+\epsilon} n)$ time per inserted character and $O(\log^{1+\epsilon} n)$ per deleted character, for any constant $\epsilon > 0$. 

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We note that a direct dynamic implementation of the solution of Hon et al. [37] would require at least performing \( p + k \) dynamic RMQs, which cost \( \Omega(\log n / \log \log n) \) time [2]. Thus modeling the original problem as a geometric one pays off in the dynamic scenario as well.

Furthermore, we can extend the top-k ranked retrieval problem by allowing a further parameter \( \text{par}(P, d) \) to be associated to any pattern \( P \) and document \( d \), so that only documents with \( \text{par}(P, d) \in [\tau_1, \tau_2] \) are considered. Some applications are selecting a range of creation dates, lengths, or PageRank values for the documents (these do not depend on \( P \)), bounding the allowed number of occurrences of \( P \) in \( d \), or the minimum distance between two occurrences of \( P \) in \( d \), etc.

**Theorem 3** Let \( D \) be a collection of documents of total length \( n \) over an integer alphabet \([1, \sigma]\), let \( w(S, d) \) be a function that assigns a numeric weight to string \( S \) in document \( d \), and let \( \text{par}(S, d) \) be another parameter, so that \( w \) and \( \text{par} \) depend only on the set of starting positions of occurrences of \( S \) in \( d \). Then there exists an \( O(n) \)-word space data structure that, given a string \( P \) of length \( p \), an integer \( k \), and a range \([\tau_1, \tau_2]\), reports \( k \) documents \( d \) containing \( P \) and with \( \text{par}(P, d) \in [\tau_1, \tau_2] \), with highest \( w(P, d) \) values, in decreasing order of \( w(P, d) \), in \( O(p / \log \sigma n + \log \varepsilon n + k \log n) \) time, online in \( k \), for any constant \( \varepsilon > 0 \).

Our solutions map these document retrieval problems into range search problems on multi-dimensional spaces, where points in the grids have associated weights. We improve some of the existing solutions for those problems.

An early partial version of this article appeared in Proc. SODA 2012 [47]. This extended version includes, apart from more precise explanations and fixes, the improvement of the static results to achieve RAM-optimality on the suffix tree traversal, and the new results on the dynamic scenario. The paper is organized as follows. In Section 2 we review the top-k framework of Hon et al. [37] and reinterpret it as the combination of a suffix tree search plus a geometric search problem. We introduce a RAM-optimal suffix tree traversal technique that is of independent interest, and state our results on geometric grids, each of which is related to the results we achieve on document retrieval. Those can also be of independent interest. Sections 3 and 4 describe our basic static solution. Section 5 describes our dynamic solution. In Section 6 we show how the static solution can be modified to reduce its space requirements, and in Section 7 we show how it can be extended to support an additional restriction on the documents sought. Finally, Section 8 concludes and gives future work directions.

## 2 Top-k Framework

In this section we overview the framework of Hon, Shah, and Vitter [37]. Then, we describe a geometric interpretation of their structure and show how top-k queries can be reduced to a special case of range reporting queries on a grid.

Let \( T \) be the generalized suffix tree [64, 43, 62] for a collection of documents \( d_1, \ldots, d_D \), each ending with the special terminator symbol "\$." \( T \) is a compact trie, such that all suffixes of all documents are stored in the leaves of \( T \). We denote by \( \text{path}(v) \) the string obtained by concatenating the labels of all the edges on the path from the root to \( v \). The locus of a string \( P \) is the highest node \( v \) such that \( P \) is a prefix of \( \text{path}(v) \). Every occurrence of \( P \) corresponds to a unique leaf that descends from its locus. We refer the reader to classical books and surveys [5, 32, 18] for an extensive description of this data structure.
We say that a leaf \( l \) is marked with document \( d \) if the suffix stored in \( l \) belongs to \( d \). An internal node \( v \) is marked with \( d \) if at least two children of \( v \) contain leaves marked with \( d \). While a leaf is marked with only one value \( d \) (equal suffixes of distinct documents are distinguished by ordering the string terminators arbitrarily), an internal node can be marked with many values \( d \).

In every node \( v \) of \( T \) marked with \( d \), we store a pointer \( \text{ptr}(v, d) \) to its lowest ancestor \( u \) such that \( u \) is also marked with \( d \). If no ancestor \( u \) of \( v \) is marked with \( d \), then \( \text{ptr}(v, d) \) points to a dummy node \( \nu \) such that \( \nu \) is the parent of the root of \( T \). We also assign a weight to every pointer \( \text{ptr}(v, d) \). This weight is the relevance score of the document \( d \) with respect to the string \( \text{path}(v) \). The following statements hold; we reprove them for completeness.

**Lemma 1** ([37], Lemma 4) The total number of pointers \( \text{ptr}(\cdot, \cdot) \) in \( T \) is bounded by \( O(n) \).

**Proof**: The total number of pointers \( \text{ptr}(v, d), v \in T \), does not exceed the number of nodes marked with \( d \). The total number of internal nodes marked with \( d \) is smaller than the number of leaves marked with \( d \). Since there are \( O(|d|) \) leaves marked with \( d \), the total number of pointers \( \text{ptr}(v, d) \) for a fixed document \( d \) is bounded by \( O(|d|) \), and those \(|d|\) add up to \( n \). \( \square \)

**Lemma 2** ([37], Lemma 2) Assume that document \( d \) contains a pattern \( P \) and \( v \) is the locus of \( P \). Then there exists a unique pointer \( \text{ptr}(u, d) \), such that \( u \) is in the subtree of \( v \) (which includes \( v \)) and \( \text{ptr}(u, d) \) points to an ancestor of \( v \).

**Proof**: If \( d \) contains \( P \) then there is at least one leaf \( u \) marked \( d \) below the locus of \( P \), with a pointer \( \text{ptr}(u, d) \). If there are two maximal (in the sense of ancestorship) nodes \( u \) and \( u' \) below \( v \) with pointers \( \text{ptr}(u, d) \) and \( \text{ptr}(u', d) \), then their lowest common ancestor \( v' \) is also marked. Since \( v \) is an ancestor of \( u \) and \( u' \), \( v \) is \( v' \) or an ancestor of \( v' \) and then \( \text{ptr}(u, d) \) and \( \text{ptr}(u', d) \) must point to \( v' \), not to an ancestor of \( v \). Finally, if \( u \) is a unique maximal node with \( \text{ptr}(u, d) \) (not \( u \) might be \( v \)), then it must point an ancestor of \( v \). \( \square \)

Moreover, in terms of Lemma 2, it turns out that \( \text{path}(u) \) occurs in \( d \) at the same positions as \( \text{path}(v) \). Note that the starting positions of \( P \) and of \( \text{path}(v) \) in \( d \) are the same, since \( v \) is the locus of \( P \), and those are the same as the starting positions of \( \text{path}(u) \) in \( d \). Thus \( w(P, d) = w(\text{path}(u), d) \) for any measure \( w(\cdot, \cdot) \) considered in Theorem 1.

**RAM-optimal suffix tree traversal.** To achieve time \( O(p) \) for the locus search in the suffix tree while retaining linear space, one needs to organize the children of each node in a perfect hash function (phf) [27]. In order to reduce this time to the RAM-optimal \( O(p/ \log_\sigma n) \), we proceed as follows. Let \( l(u, v) \) be the concatenation of string labels from node \( u \) to its descendant \( v \). We collect in a phf \( H(u) \), for the suffix tree root \( u \), all the highest descendants \( v \) such that \( |l(u, v)| \geq \ell = \log_\sigma n \). Those nodes \( v \) are indexed with a key built from the first \( \ell \) symbols of \( l(u, v) \) interpreted as a number of \( \log n \) bits. We build recursively phfs for all the descendants \( u \) identified. Since each suffix tree node is included in at most one hash table, the total size is \( O(n) \) and the total deterministic construction time is \( O(n \log \log n) \) [55]. Now \( P \) is searched for as follows. We take its first chunk of \( \ell \) characters, interpret it as a number, and query the phf of the root. If no node \( u \) is found for that prefix of \( P \), then \( P \) is not in the collection. Otherwise, the string depth of \( u \) is \( \geq \ell \). We check explicitly the extra \( |l(u, v)| - \ell \) symbols in the text, by comparing chunks of \( \ell \) symbols. If there is a match, we continue with the next \( \ell \) unread symbols of \( P \), and so on, until there are less than \( \ell \) symbols to match in the
remaining suffix $P'$ of $P$. At this point we switch to using a weak prefix search (wps) data structure [7]: For each node $u$ holding a phf, we also store this wps data structure with all the nodes $v$ that descend from $u$ where $|l(u,v)| < \ell$. The wps data structure will return the lexicographic range of the nodes of which $P'$ is a prefix (here we identify nodes $v$ with the strings $l(u,v)$). The first in that range is the locus of $P$. One detail is that, if there is no such node (i.e., $P$ has no locus), the wps structure returns an arbitrary value, but this can be easily checked in optimal time in the text. The wps structure requires, for our length $|P'| < \ell$ and in the RAM model, $O(1)$ query time, and can use, for the strings we store, of length $< \ell$, $O(\sqrt{\log n \log \log n})$ bits, or $o(1)$ words, per stored node [7 Thm. 6]. Once again, each node is stored only in one wps structure, so the overall extra space is linear at worst. The wps construction is $O(n \log^c n)$ randomized time. It can be made deterministic $O(n \polylog(n))$ time by using a phf inside the construction [1]. By replacing the wps structure by layered phfs for $(\log \alpha n) / 2^i$ symbols, we would have an additive term $O(\log \log \alpha n)$ in the query time.

**Geometric interpretation.** We index the nodes of $T$ in the following way: All nodes of $T$ are visited in pre-order; we also initialize an index $i \leftarrow 0$. When a node $v$ is visited, if $v$ is marked with values $d_{v_1}, \ldots, d_{v_j}$, we assign indexes $i+1, \ldots, i+j$ to $v$ and set $i \leftarrow i+j$. We will denote by $[l_v, r_v]$ the integer interval bounded by the minimal and maximal indexes assigned to $v$ or its descendants. Values $l_v$ and $r_v$ are stored in node $v$ of $T$. Furthermore, for every $d_{v_t}$, $1 \leq t \leq j$, there is a pointer $\text{ptr}(v, d_{v_t})$ that points to some ancestor $u_t$ of $v$. We encode $\text{ptr}(v, d_{v_t})$ as a point $(i+t, \text{depth}(u_t))$, where depth denotes the depth of a node; depth($v$) = 0. Thus every pointer in $T$ is encoded as a two-dimensional point on an integer $O(n) \times O(n)$ grid. The weight of a point $p$ is that of the pointer $p$ encodes. We observe that all the points have different $x$-coordinates. Thus we obtain a set $S$ of weighted points with different $x$-coordinates, and each point corresponds to a unique pointer.

For the final answers we will need to convert the $x$-coordinates of points found on this grid into document numbers. We store a global array of size $O(n)$ to do this mapping.

**Answering queries.** Assume that top-$k$ documents containing a pattern $P$ must be reported. We find the locus $v$ of $P$ in $O(p / \log \alpha n)$ time. By Lemma [2] there is a unique pointer $\text{ptr}(u, d)$, such that $u$ is a descendant of $v$ (or $v$ itself) and $\text{ptr}(u, d)$ points to an ancestor of $v$, for every document $d$ that contains $P$. Moreover the weight of that point is $w(P, d)$. Hence, there is a unique point $(x, y)$ with $x \in [l_v, r_v]$ and $y \in [0, \text{depth}(v) - 1]$ for every document $d$ that contains $P$. Therefore, reporting top-$k$ documents is equivalent to the following query: among all the points in the three-sided range $[l_v, r_v] \times [0, \text{depth}(v))$, report $k$ points with highest weights. We will call such queries three-sided top-$k$ queries. In Sections [3] and [4] we prove the following result. Theorem [1] is an immediate corollary of it, as $h = \text{depth}(v) - 1$ and $\text{depth}(v) \leq p$, and we can choose $c \geq 1$.

**Theorem 4** A set of $n$ weighted points on an $n \times n$ grid can be stored in $O(n)$ words of space and built in $O(n \log n)$ time, so that for any $1 \leq k, h \leq n$ and $1 \leq a \leq b \leq n$, $k$ most highly weighted points in the range $[a, b] \times [0, h]$ can be reported in decreasing order of their weights in $O(h / \log^c n + k)$ time, for any constant $c$.

**Top-$k$ queries on dynamic collections.** The static suffix tree is replaced by a dynamic one, with search time $O(p(\log \log n)^2 / \log \alpha n + \log n)$ and update time $O(\log n)$ per symbol. We must
also update the grid, for which we must carry out lowest common ancestor queries on the dynamic suffix tree and also insert/delete points (and columns) in the grid. We split the grid of Theorem 4 into horizontal stripes of height \( m = \text{polylog } n \) to obtain improved performance, and query the highest \([p/m]\) of those grids. In the most general case (i.e., the last grid) we carry out a three-sided top-\( k \) query. We address this in Section 5 where in particular we prove the following result on dynamic grids. Theorem 2 is then obtained by combining those results.

**Theorem 5** A set of \( n \) points, one per column on an \( n \times n \) grid, with weights in \([1, O(n)]\), can be stored in \( O(n) \) words of space, so that for any \( 1 \leq k \leq n \), \( 1 \leq h \leq n \) and \( 1 \leq a \leq b \leq n \), \( k \) most highly weighted points in the range \([a, b] \times [0, h]\) can be reported in decreasing order of their weights in \( O(h/\log^c n + \log n + k \log \log k) \) time, online in \( k \), for any constant \( c \). Points (and their columns) can be inserted and deleted in \( O(\log^{1+\varepsilon} n) \) time, for any constant \( \varepsilon > 0 \).

**Parameterized top-\( k \) queries.** We use the same geometric interpretation as described above, but now each pointer \( \text{ptr}(v, d) \) is also associated with the parameter value \( \text{par}(\text{path}(v), d) \). We encode a pointer \( \text{ptr}(v, d) \) as a three-dimensional point \((i + t, \text{depth}(u_t), \text{par}(\text{path}(v), d))\), where \( i, t, \text{and} u_t \) are defined as in the case of nonparameterized top-\( k \) queries. All the documents that contain a pattern \( P \) (with locus \( v \)) and satisfy \( \tau_1 \leq \text{par}(P, d) \leq \tau_2 \) correspond to unique points in the range \([i, r_v] \times [0, \text{depth}(v)] \times [\tau_1, \tau_2]\). Hence, reporting top-\( k \) documents with \( \text{par}(P, d) \in [\tau_1, \tau_2] \) is equivalent to reporting top-\( k \) points in a three-dimensional range. The following result is proved in Section 7 and Theorem 3 is an immediate corollary of it, choosing any \( c \geq 1 \).

**Theorem 6** A set of \( n \) weighted points on an \( n \times n \times n \) grid can be stored in \( O(n) \) words of space, so that for any \( 1 \leq k, h \leq n \), \( 1 \leq a \leq b \leq n \), and \( 1 \leq \tau_1 \leq \tau_2 \leq n \), \( k \) most highly weighted points in the range \([a, b] \times [0, h] \times [\tau_1, \tau_2]\) can be reported in decreasing order of their weights in \( O(h/\log^c n + \log^{1+\varepsilon} n + k \log \log n) \) time, for any constants \( c \) and \( \varepsilon > 0 \).

3 **An \( O(m^f + k) \) Time Data Structure**

In this section we give a data structure that does not yet achieve the desired \( O(h/\log n + k) \) time, but its time depends on the width \( m \) of the grid. This will be used in Section 4 to handle vertical stripes of the global grid, in order to achieve the final result.

We assume that a global array gives access to the points of a set \( S \) in constant time: if we know the \( x \)-coordinate \( p.x \) of a point \( p \in S \), we can obtain the \( y \)-coordinate \( p.y \) of \( p \) in \( O(1) \) time. Both \( p.x \) and \( p.y \) are in \([1, O(n)]\), thus the global array requires \( O(n) \) words of space. We consider the question of how much additional space our data structure uses if this global array is available. The result of this section is summed up in the following lemma, where we consider tall grids of \( m \) columns and \( n \) rows.

**Lemma 3** Assume that \( m \leq n \) and let \( 0 < f < 1 \) be a constant. There exists a data structure that uses \( O(m \log m) \) additional bits of space and construction time. It answers three-sided top-\( k \) queries for a set of \( m \) points on an \( m \times n \) grid in \( O(m^f + k) \) time.

**Proof:** The idea is to partition the points of \( S \) by weights, where the weights are disregarded inside each partition. Those partitions are also refined into a tree. Then we solve the problem by
traversing the appropriate partitions and collecting all the points using classical range queries on unweighted points. A tree of arity \(m^{\Theta(f)}\) yields constant height and thus constant space per point.

More precisely, we partition \(S\) into classes \(S_1, \ldots, S_r\), where \(r = m^{f'}\) for a constant \(0 < f' < f\). For any \(1 \leq i < j \leq r\), the weight of any point \(p_i \in S_i\) is larger than the weight of any point \(p_j \in S_j\). For \(1 \leq i < r\), \(S_i\) contains \(m^{1-f'}\) points. Each class \(S_i\) that contains more than one element is recursively divided into \(\min(|S_i|, r)\) subclasses in the same manner. This subdivision can be represented as a tree: If \(S_i\) is divided into subclasses \(S_{i1}, \ldots, S_{ik}\), we will say that \(S_i\) is the parent of \(S_{i1}, \ldots, S_{ik}\). This tree has constant height \(O(1/f')\).

For every class \(S_j\) we store data structures that support three-sided range counting queries and three-sided range reporting queries, in \(O(|S| \log m)\) and \(O(|S| \log m + occ)\) time respectively. These structures will be described in Section 3.1 and require \(O(m^{1-f'} \log m)\) construction time; note they do not involve weights. This adds up to \(O((1/f')m \log m)\) construction time.

We will report \(k\) most highly weighted points in a three-sided query range \(Q = [a,b] \times [0,h]\) using a two-stage procedure. During the first stage we produce an unsorted list \(L\) of \(k\) most highly weighted points. During the second stage the list \(L\) is sorted by weight.

Let \(Q^k\) denote the set of \(k\) most highly weighted points in \(S \cap Q\). Then \(Q^k\) can be formed as the union of the result of the three-sided query over certain classes, at most \(O(m^f)\) classes per level over a constant number of levels. More precisely, there are \(O(m^{f'})\) classes \(S_c\), such that \(p \in Q^k\) if and only if \(p \in S_c \cap Q\) for some \(S_c\). During the first stage, we identify the classes \(S_c\) and report all the points in \(S_c \cap Q\) using the following procedure. Initially, we set our current tree node to \(\tilde{S} = S\) and its child number to \(i = 1\). We count the number of points inside \(Q\) in the \(i\)-th child \(S_i\) of \(\tilde{S}\). If \(k_i = |S_i \cap Q| \leq k\), we report all the points from \(S_i \cap Q\) and set \(k = k - k_i\). If \(k = 0\), the procedure is completed; otherwise we set \(i = i + 1\) and proceed to the next child \(S_i\) of \(\tilde{S}\). If \(k_i > k\), instead, we set \(\tilde{S} = S_i\), \(i = 1\), and report \(k\) most highly weighted points in the children of \(\tilde{S}\) using the same procedure. During the first stage we examine \(O(m^{f'})\) classes \(S_i\) and spend \(O(m^{f'} \log m + k) = O(m^f + k)\) time in total.

When the list \(L\) is completed, we can sort it in \(O(m^f + k)\) time. If \(L\) contains \(k < m^f\) points, \(L\) can be sorted in \(O(k \log k) = O(m^f)\) time. If \(L\) contains \(k \geq m^f\) points, then we can sort it in \(O(k)\) time using radix sort: As the set \(S\) contains at most \(m\) distinct weights, we store their ranks in an array ordered by \(x\)-coordinate, and thus can sort the result using the ranks instead of the original values. By sorting \(f' \log_2 m\) bits per pass the radix sort runs in time \(O(k)\).

As for the space, the structures in Lemmas 4 and 5 require \(O(\log m)\) bits per point. Each point of \(S\) belongs to \(O(1/f')\) classes \(S_i\). Hence, the total number of points in all classes is \(O(m)\), giving \(O(m \log m)\) bits of total space. The local array of weight ranks also uses \(O(m \log m)\) bits.

\[\square\]

### 3.1 Counting and Reporting Points

It remains to describe the data structures that answer three-sided counting and reporting queries, with no weights involved.

**Lemma 4** Let \(v \leq m \leq n\). There exists a data structure that uses \(O(v \log m)\) additional bits and answers three-sided range counting queries for a set of \(v\) points on an \(m \times n\) grid in \(O(\log v)\) time. It can be built in \(O(v \log v)\) time.

**Proof:** Using the rank space technique [28], we reduce the problem of counting on an \(m \times n\) grid to the problem of counting on a \(v \times n\) grid: let \(\tau(p.x, p.y) = (\text{rank}(p.x, S_x), p.y)\), where the
set $S_x$ consists of the $x$-coordinates of all the points in $S$ and $\text{rank}(a, S) = |\{ e \in S | e \leq a \}|$. Then the mapped set is $S' = \{ \tau(v.x, v.y), (p.x, p.y) \in S \}$. Let $\text{pred}(a, S) = \max\{ e \in S | e \leq a \}$ and $\text{succ}(a, S) = \min\{ e \in S | e \geq a \}$. Then a query $[a, b] \times [0, h]$ on $S$ is equivalent to a query $[a', b'] \times [0, h]$ on $S'$, where $a' = \text{rank}(\text{succ}(a, S_x), S_x)$ and $b' = \text{rank}(\text{pred}(b, S_x), S_x)$. Using standard binary search on an array, we can find $a'$ and $b'$ in $O(\log v)$ time and $v \log m$ bits of space.

We store the points on a variant of the wavelet tree data structure \cite{Bayer1972}. Each node of this tree $W$ covers all the points within a range of $y$-coordinates. The root covers all the nodes, and the two children of each internal node cover half of the points covered by their parent. The leaves cover one point. The $y$-coordinate limits of the nodes are not stored explicitly, to save space. Instead, we store the $x$-coordinate of the point holding the maximum $y$-coordinate in the node. With the global array we can recover the $y$-coordinate in constant time. Each internal node $v$ covering $r$ points stores a bitmap $B_v[1..r]$, so that $B_v[i] = 0$ iff the $i$-th point, in $x$-coordinate order, belongs to the left child (otherwise it belongs to the right child). Those bitmaps are equipped with data structures answering operation $\text{rank}_b(B_v, i)$ in constant time and $r + o(r)$ bits of space \cite{Bender2001}, where $\text{rank}_b(B_v, i)$ is the number of occurrences of bit $b$ in $B_v[1..i]$. Since $W$ has $O(v)$ nodes and height $O(\log v)$, its bitmaps require $O(v \log v)$ bits and its pointers and $x$-coordinates need $O(v \log m)$ bits. The construction time is $O(v \log v)$.

We can easily answer range counting queries $[a', b'] \times [0, h]$ on $W$ \cite{Bender2001}. The procedure starts at the root node of $W$, with the range $[a', b']$ on its bitmap $B$. This range will become $[a_1, b_1] = [\text{rank}_0(B, a' - 1) + 1, \text{rank}_0(B, b')]$ on the left child of the root, and $[a_r, b_r] = [\text{rank}_1(B, a' - 1) + 1, \text{rank}_1(B, b')]$ on the right child. If the maximal $y$-coordinate of the left child is smaller than or equal to $h$, we count the number of points $p$ with $p.x \in [a, b]$ stored in the left child, which is simply $b_1 - a_1 + 1$, and then visit the right child. Otherwise, the maximal $y$-coordinate in the left child is larger than $h$, and we just visit the left child. The time is $O(1)$ per tree level.

**Lemma 5** Let $v \leq m \leq n$. There exists a data structure that uses $O(v \log m)$ additional bits and answers three-sided range reporting queries for a set $S$ of $v$ points on an $m \times n$ grid in $O(\log v + \log m)$ time, to report the occ results. It can be built in $O(v \log v)$ time.

**Proof:** We can reduce the problem of reporting on an $m \times n$ grid to the problem of reporting on a $v \times n$ grid, as described above. The query time is increased by an additive $O(\log v)$ factor, and the space usage increases by $O(v \log m)$ bits of space. Now we sort the $v$ points in $x$-coordinate order, build the sequence $Y[1..v]$ of their $y$-coordinates, and build a Range Minimum Query (RMQ) data structure on $Y$ \cite{Bender2001}. This structure requires only $O(v)$ bits of space, does not need to access $Y$ after construction (so we do not store $Y$), and answers in constant time the query $\text{rmq}(c, d) = \arg \min_{c \leq i \leq d} Y[i]$ for any $c, d$. It is well known that with such queries one can recursively retrieve all the points in the three-sided range in $O(\log m)$ time; see, for example, Muthukrishnan \cite{Muthukrishnan2005}. The construction time is dominated by the sorting of points. $\square$

## 4 An Optimal Time Data Structure

The data structure of Lemma 3 gives us an $O(m^f + k)$ time solution, for any constant $f$, where $m$ is the grid width. In this section we use it to obtain $O(h/\log n + k)$ time.

The idea is to partition the space into vertical stripes, for different stripe widths, and index each stripe with Lemma 3. Then the query is run on the partition of width $m$ so that the $O(m^f)$ time complexity is dominated by $O(h/\log n + k)$. The many partitions take total linear space because the size per point in Lemma 3
is $O(\log m)$, and our widths decrease doubly exponentially. As a query may span several stripes, a structure similar to the one used in the classical RMQ solution \cite{ref} is used. This gives linear space for stripes of width up to $\Omega(\log^2 n)$. Smaller ones are solved with universal tables.

In addition to the global array storing $p.y$ for each $p.x$, we use another array storing the weight corresponding to each $p.x$. As there are overall $O(n)$ different weights, those can be mapped to the interval $[1, O(n)]$ and still solve correctly any top-$k$ reporting problem. Thus the new global array also requires $O(n)$ words of space.

4.1 Structure

Let $g_j = 1/2^j$ for $j = 0, 1, \ldots, r$. We choose $r$ so that $n^{g_r} = O(1)$, thus $r = O(\log \log n)$. The $x$-axis is split into intervals of size $\Delta_j = n^{g_j} \log^2 n$ and $j = 1, \ldots, r$. For convenience, we also define $\Delta_0 = n$ and $\Delta'_j = \Delta_j / \log^2 n = n^{g_j}$. For every $1 \leq j < r$ and for every interval $I_{j,t} = [(t-1)\Delta_j, t\Delta_j-1]$, we store all the points $p$ with $p.x \in I_{j,t}$ in a data structure $E_{j,t}$ implemented as described in Lemma \ref{lemma:S}. $E_{j,t}$ supports three-sided top-$k$ queries in $O((\Delta_j)^f + k)$ time for any constant $0 < f < 1/4$. We also construct a data structure $E_0$ that contains all the points of $S$ and supports three-sided top-$k$ queries in $O(n^{1/4} + k)$ time. To simplify the description, we also assume that $I_{-1} = I_0 = [0, n-1]$ and $E_{-1} = E_0$.

The data structures $E_{j,t}$ for a fixed $j$ contain $O(n)$ points overall, hence by Lemma \ref{lemma:S} all $E_{j,t}$ use $O(n \log \Delta_j) = O(n \log(n^{g_j} \log^2 n)) = (1/2^j)O(n \log n) + O(n \log \log n)$ additional bits of space. Thus all $E_{j,t}$ for $0 \leq j < r$ use $\sum_{j=0}^{r-1}[(1/2^j)O(n \log n) + O(n \log \log n)] = O(n \log n)$ bits, or $O(n)$ words. They also require $O(n \log n)$ total construction time. Since $f < 1/4$, a data structure $E_{j,t}$ supports top-$k$ queries in time $O((\Delta_j)^f + k) = O((n^{g_j} \log^2 n)^f + k) = O(n^{g_j+2}/\log n + k) = O(\Delta'_{j+2}/\log n + k)$ time. For each of the smallest intervals $I_{r,t}$ we store data structures $T_{r,t}$ that use $o(\log^2 n)$ words of space (adding up to $o(n)$) and support three-sided top-$k$ queries in $O(h/\log n + k)$ time. This structure will be described in Section 4.3.

Note that our choice of writing $(n^{g_j} \log^2 n)^f = O(n^{g_j+2}/\log n)$ was arbitrary, because $f$ is strictly less than $1/4$. We could have written $(n^{g_j} \log^2 n)^f = O(n^{g_j+2}/\log^c n)$ for any constant $c$, and this would yield $O(h/\log^c n + k)$ query time. We have chosen to favor simplicity in the exposition, but will return to this point at the end of Section 4.3.

4.2 Queries

We can carry out the query using a range of intervals $I_{j,t}$ of any width $\Delta_j$. The key idea is to use a $j$ value according to the height of the three-sided query, so that the search time in $I_{j,t}$ gives the desired $O(h/\log n)$ time. More precisely, assume we want to report $k$ points with highest weights in the range $[a,b] \times [0,h]$. First, we find the index $j$ such that $\Delta_{j+1} > \max(h,k) \geq \Delta'_{j+2}$. The index $j$ can be found in $O(\log \log(h + k))$ time by linear search if $[a,b]$ is contained in some interval $I_{j,t}$, then we can answer a query in $O(\Delta'_{j+2}/\log n + k) = O(h/\log n + k)$ time using $E_{j,t}$. If $[a,b]$ is contained in two adjacent intervals $I_{j,t}$ and $I_{j,t+1}$, we generate the lists of top-$k$ points in $(a,b) \cap I_{j,t} \times [0,h]$ and $(a,b) \cap I_{j,t+1} \times [0,h]$ in $O(h/\log n + k)$ time, and merge them in $O(k)$ time. To deal with the case when $[a,b]$ spans one or more intervals $I_{j,t}$, we store pre-computed solutions for some intervals.

\footnote{This is $O(h/\log n + k)$ for sure if $\max(h,k) = \Omega(\log n \log \log n)$; otherwise a small table can be used to perform the search in constant time.}
For $1 \leq j \leq r$, we consider the endpoints of intervals $I_{j,t}$. Let $\text{top}_j(a, b, c)$ denote the list of top-$k$ points in the range $[a \cdot \Delta_j, b \cdot \Delta_j - 1] \times [0, c]$ in descending weight order. We store the values of $\text{top}_j(t, t + 2^v, c, \Delta_{j+1})$ for any $t \in [0, n/\Delta_j]$, any $0 \leq v \leq \log_2(n/\Delta_j)$, and any $0 \leq c \leq \Delta_{j+1}$. All the lists $\text{top}_j(\cdot, \cdot, \cdot)$ use space $O((n/\Delta_j)(\Delta_{j+1})^2 \log n) = O(n/\log n)$ words. Hence the total word space usage of all lists $\text{top}_j(\cdot, \cdot, \cdot)$ for $2 \leq j \leq r$ is $O(n \log \log n/\log n) = o(n)$. It can also be built in $o(n)$ time using dynamic programming.

Assume that $[a, b]$ spans intervals $I_{j,t_1+1}, \ldots, I_{j,t_2-1}$; $[a, b]$ also intersects with intervals $I_{j,t_1}$ and $I_{j,t_2}$. Let $a' \Delta_j$ and $b' \Delta_j$ denote the left endpoints of $I_{j,t_1+1}$ and $I_{j,t_2}$, respectively. The list $L_m$ of top-$k$ points in $[a' \Delta_j, b' \Delta_j - 1] \times [0, h]$ can be generated as follows. Intervals $[a' \Delta_j, (a' + 2^v) \Delta_j - 1]$ and $[(b' - 2^v) \Delta_j, b' \Delta_j - 1]$ for $v = \lfloor \log_2((b' - a')/\Delta_j) \rfloor$ cover $[a' \Delta_j, b' \Delta_j - 1]$. Let $L'_m$ and $L''_m$ denote the lists of the first $k$ points in $\text{top}_j(a', a' + 2^v, h, \Delta_{j+1})$ and $\text{top}_j(b' - 2^v, b', \Delta_{j+1})$ (we have $k$ results because $k < \Delta_{j+1}$); similarly we have the results for $c = h$ because $h < \Delta_{j+1}$). We merge both lists (possibly removing duplicates) according to the weights of the points, and store in $L_m$ the set of the first $k$ points from the merged list. Let $L_{t_1}$ and $L_{t_2}$ denote the sets of top-$k$ points in $[a, a' \Delta_j - 1] \times [0, h]$ and $[b' \Delta_j, b] \times [0, h]$. We can obtain $L_{t_1}$ and $L_{t_2}$ in $O(h/\log n + k)$ time using data structures $E_{j,t_1}$ and $E_{j,t_2}$ as explained above. Finally, we can merge $L_m$, $L_{t_1}$, and $L_{t_2}$ in $O(k)$ time; the first $k$ points in the resulting list are the top-$k$ points in $[a, b] \times [0, h]$.

### 4.3 A Data Structure for an $O(\log^2 n) \times n$ Grid.

The data structures $E_{r,t}$ for an interval $I_{r,t}$ use the same general approach as the data structures $E_{j,t}$, at a smaller scale. Note that these structures will be consulted only when $\max(h, k) < C = \Delta_{r+1} = O(1)$. Each interval $I_{r,t}$ is subdivided into $\log^{7/4}/n$ intervals $\tilde{I}_{1}, \tilde{I}_{2}, \ldots$ of width $\log^{1/4}/n$. Let $\tilde{S}$ denote the set that contains the endpoints of $\tilde{I}_{1}, \tilde{I}_{2},\ldots$. For every $x \in \tilde{S}$, each $1 \leq v \leq 2 \log \log n$ and each $h \leq C$, we store the lists $\text{top}(x, x + 2^v, h, C)$. All such lists use $O((n/\log^2 n) C^2 \log^{7/4}/n \log \log n) = o(n)$ space in total.

A query on $I_{r,t}$ is processed as follows. Suppose that $[a, b]$ intersects with intervals $\tilde{I}_{g_1}, \ldots, \tilde{I}_{g_2}$ for some $g_1 \leq g_2$. We find the top-$k$ points from $\text{top}(\tilde{I}_{g_1} \cup \cdots \cup \tilde{I}_{g_2-1}) \times [0, h]$ in $O(k)$ time using lists $\text{top}(\cdot, \cdot, \cdot, \cdot)$, as before. We also find top-$k$ points from $\tilde{I}_{g_1} \cap [a, b]) \times [0, h]$ and $\tilde{I}_{g_2} \cap [a, b]) \times [0, h]$ in $O(k)$ time using data structures for $\tilde{I}_{g_1}$ and $\tilde{I}_{g_2}$, respectively, to be described next. We thus obtain three lists of points sorted by their weights, and merge them in $O(k)$ time as before.

Finally, we describe how to answer queries in $O(k)$ time in the grids $\tilde{I}_g$, of width $\log^{1/4}/n$. We replace the $y$-coordinates of points by their ranks; likewise, the weights of points are also replaced by their ranks. The resulting sequence $X_g$ contains all mapped points in $\tilde{I}_g$ and consists of $O(\log^{1/4}/n \log \log n)$ bits, so all the descriptions of all sequences $X_g$ require $O(n \log \log n)$ bits, or $o(n)$ words. There are $O(\log^{1/2}/n)$ queries that can be asked (considering all the sensible values of $[a, b]$, $h$ and $k$), and the answers require $O(k \log(\log^{1/4}/n)) = O(\log \log n)$ bits. Thus we can store a universal look-up table of size $2^{O(\log^{3/4}/n \log \log n)}O(\log \log n) = o(n)$ words common to all subintervals $\tilde{I}_g$. This table contains pre-computed answers for all possible queries and all possible sequences $X_g$. Hence, we can answer a top-$k$ query on $X_g$ in $O(k)$ time.

A query on $\tilde{I}_g$ can be transformed into a query on $X_g$ by reduction to rank space in the $y$ coordinates. Consider a query range $Q = [a, b] \times [0, h]$ on $\tilde{I}_g$. We can find the rank $h'$ of $h$ among the $y$-coordinates of points from $\tilde{I}_g$ in $O(h) = O(1)$ time by linear search (remember that we store only the reordering of the local $x$-coordinates, and the actual $y$-coordinates are found in the global
Thus our data structure uses $O(n)$ words of space and answers queries in $O(h/\log n + k)$ time. It can be built in $O(n \log n)$ time. As mentioned at the end of Section 4.1, we can obtain any query time of the form $O(h/\log^c n + k)$, for any constant $c$. This completes the proof of Theorem 4, which is given in this general form.

### 4.4 Online Queries

An interesting extension of the above result is that we can deliver the top-$k$ documents in online fashion. That is, after the $O(p/\log \sigma n)$ time initialization, we can deliver the highest weighted result, then the next highest one, and so on. It is possible to interrupt the process at any point and spend overall time $O(p/\log \sigma n + k)$ after having delivered $k$ results. That is, we obtain the same result without the need of knowing $k$ in advance. This is achieved via an online version of Theorem 4 and is based on the idea used, for example, in [13, 37]. We describe it for completeness.

Consider an arbitrary data structure that answers top-$k$ queries in $O(f(n) + kg(n))$ time in the case when $k$ must be specified in advance. Let $k_1 = [f(n)/g(n)]$, $k_i = 2k_{i-1}$, and $s_i = \sum_{j=1}^{i-1} k_j$ for $i \geq 2$. Let $S$ be the set of points stored in the data structure and suppose that we must report top points from the range $Q$ in the online mode. At the beginning, we identify top-$k_1$ points in $O(f(n) + g(n))$ time and store them in a list $L_1$. Reporting is divided into stages. During the $i$-th stage, we report points from a list $L_i$. $L_i$ contains min($k_i, |Q \cap S| - s_i$) top points that were not reported during the previous stages. Simultaneously we compute the list $L_{i+1}$ that contains min($2k_i + s_i, |Q \cap S|) < 4k_i$ top points. We identify at most $2k_i + s_i$ top points in $O(f(n) + 4k_i \cdot g(n)) = O(k_i \cdot g(n))$ time. We also remove the first $s_i$ points from $L_{i+1}$ in $O(k_i)$ time. The resulting list $L_{i+1}$ contains $2k_i = k_{i+1}$ points that must be reported during the next $(i + 1)$-th stage. The task of creating and cutting the list $L_{i+1}$ is executed in such a way that we spend $O(1)$ time when each point of $L_i$ is reported. Thus when all the points from $L_i$ are output, the list $L_{i+1}$ that contains the next $k_{i+1}$ top points is ready and we can proceed with the $(i + 1)$-th stage.

The reporting procedure described above outputs the first $k$ most highly weighted points in $O(f(n) + kg(n))$ time. It can be interrupted at any time.

### 5 A Dynamic Structure

We consider now a scenario where insertions and deletions of whole documents are interspersed with top-$k$ queries. This has two important components: how to maintain the suffix tree and how to maintain the grid. Then we consider how queries are performed. We first consider just the measure $tf$ and then generalize to others at the end.

#### 5.1 Dynamic Suffix Trees

We use a dynamic suffix tree maintenance algorithm where leaves and unary nodes can be inserted and deleted, and lowest common ancestors can be computed, all in constant worst-case time [17].

The updates on leaves and unary nodes are the operations we need to insert and delete all the suffixes of a document in the suffix tree, in time proportional to the length of the document inserted or deleted, whereas the lowest common ancestor queries are necessary to compute the new $\text{ptr}(\cdot, \cdot)$
pointers to insert. In any node we will maintain the up to \( \sigma \) children using a linear-space dynamic predecessor data structure that supports queries and updates in worst-case time \( o((\log \log \sigma)^2) \).

Upon insertion of a new document \( d \) of length \(|d|\), we follow McCreight’s procedure to insert a new string in a generalized suffix tree [8]. First we search for the new document string in the suffix tree, add the corresponding leaf and, if necessary, insert its parent splitting an edge. Then we compute the suffix link of the last node that belonged to the path found, follow the suffix link, descend again as much as possible, create a new leaf and possibly an internal node, create suffix links from the last to the current created nodes, and continue until exhausting the suffix. This requires \( O(|d|) \) suffix tree operations, and total time \( O(|d|(\log \log \sigma)^2) \).

The deletion of a document \( d \) is symmetric to insertion. We find its corresponding string, delete its leaf and possibly its parent if it becomes unary, follow the suffix link, and repeat the process until removing all the leaves and possibly their parents. This takes \( O(|d|(\log \log \sigma)^2) \) time.

In order to accelerate searches we will use a technique analogous to the one used with the static suffix tree. We define \( \ell = \log_\sigma n \), and the level of a node \( v \) as \( \text{lev}(v) = \lceil |l(root, v)|/\ell \rceil \). Note that the level of a node depends on its string depth, and thus it does not change upon updates. Each suffix tree node \( v \) with parent \( u \) such that \( \text{lev}(v) > \text{lev}(u) \) will maintain a predecessor data structure called an accelerator, storing all its highest descendant nodes \( z \) such that \( \text{lev}(z) > \text{lev}(v) \). The key used for the predecessor data structure are the \( \ell \) characters (\( \lg n \) bits) formed by \( l(root, z)[\text{lev}(v) \cdot \ell + 1, (\text{lev}(v) + 1) \cdot \ell] \). Note these keys do not depend precisely on \( v \) being the node holding the accelerator; any other ancestor of \( z \) of the same level of \( v \) yields the same key. Note also that the nodes \( z \) stored in the accelerator of \( v \) are owners of subsequent accelerators.

The predecessor structures hold \( O(n) \) nodes, and thus they require \( o((\log \log n)^2) \) time and linear space [3]. The total extra space is linear because each suffix tree node belongs to at most one predecessor structure.

Upon searches, we start at the root and use the accelerators of successive nodes, using consecutive chunks of \( \ell \) symbols in \( P \). In some cases we may arrive at nodes whose string distance to the previously visited node is more than \( \ell \); in those cases we check the missing symbols directly in the text, also in chunks of \( \ell \) characters. When, finally, the accelerator does not give a node matching the next \( \ell \) symbols of \( P \), we switch to the character-based search. Thus the total search time is \( O(p(\log \log n)^2/\log_\sigma n + (\log_\sigma n)(\log \log \sigma)^2) = O(p(\log \log n)^2/\log_\sigma n + \log n) \).

Those accelerators must be updated upon insertions and deletions of suffix tree nodes. Note that we always know \( l(root, v) \) when we insert or delete a node \( v \). Upon insertion of a leaf \( v \) as a child of a node \( u \), it may turn out that the leaf must be inserted into an accelerator (because \( \text{lev}(v) > \text{lev}(u) \)). We can simply find the nearest ancestor holding an accelerator via at most \( \ell \) parent operations from \( v \). We must also initialize an empty accelerator for \( v \). Symmetrically, when a leaf \( v \) is removed, we may have to remove it from its ancestor’s accelerator. When an edge from \( u \) to \( v \) is split with a new node \( z \), it may be that \( \text{lev}(u) < \text{lev}(z) = \text{lev}(v) \). In this case, \( z \) takes the role of \( v \), “stealing” the accelerator from \( v \) (which needs no change, as explained). We must also replace \( v \) by \( z \) in the accelerator stored at the proper ancestor of \( z \). Another case that requires care is when \( \text{lev}(u) < \text{lev}(z) < \text{lev}(v) \). In this case \( v \) is replaced by \( z \) in the proper ancestor of \( z \), but \( v \) retains its accelerator and \( z \) creates a new accelerator holding only \( v \). Other cases require no action. Upon deletions, the obvious reverse actions are necessary. The total update time can be bounded by \( O(|d| \log n) \) for both insertions and deletions.
5.2 Relating the Suffix Tree and the Grid

Since grid columns will appear and disappear upon document insertions and deletions, we will not associate integers to columns, but just abstract labels. The mapping between the suffix tree and the grid columns will be carried out via a dynamic technique to maintain order in a list $X$ of such abstract labels \[21\] \cite{10}. The data structure supports the operations of creating a new label $y$ as the immediate successor of a given label $x \in X$, deleting a label $y \in X$, and determining which of two labels comes first in $X$, all in constant time. In addition, each suffix tree node $v$ will hold a (classical) doubly-linked list $\text{list}(v)$ storing consecutive labels of $X$, each label corresponding to a grid column where this node induces points, and will maintain pointers to the first and the last node in $\text{list}(v)$. Finally, $v$ will maintain special labels $\text{first}(v)$, $\text{last}(v) \in X$ that do not represent any column, but are the predecessor (resp. successor) in $X$ of the first (resp. last) label in its subtree.

As we insert a new leaf $v$ as the child of $u$, we must create a predecessor of $\text{last}(v')$ to assign to $\text{last}(v)$, where $v'$ is the next sibling of $v$, and then create a predecessor of $\text{last}(v)$ to assign to $\text{first}(v)$. If $v$ is the last child of $u$, then we create a predecessor of $\text{last}(u)$ to assign to $\text{last}(v)$. When, instead, we create a new node $v$ that splits an edge from $u$ to $v'$, then $\text{first}(v)$ will be a new predecessor of $\text{first}(v')$ and $\text{last}(v)$ will be a new successor of $\text{last}(v')$. When a node $v$ is removed, its labels $\text{first}(v), \text{last}(v)$ are also removed from $X$.

As we insert a new document $d$, we must associate new grid columns to the new and existing suffix tree nodes traversed. Each newly created pointer $\text{ptr}(v,d)$ will require creating a new label $t(v,d) \in X$ as the successor of the last one currently held in $\text{list}(v)$ (it will also be stored at the end of $\text{list}(v)$). When $\text{list}(v)$ is empty, the new label $t(v,d)$ must be created as the successor of $\text{first}(v)$.

As we insert new leaves in the suffix tree, we collect them in an array $L[1,|d|]$. We also create the first label $t(v,d)$ of such leaves $v$. Now we sort $L$ by the labels $t(v,d)$, and as a result the new leaves become sorted by their suffix tree preorder. All the internal suffix tree nodes that must be labeled with $d$ are obtained as $u = \text{lca}(v,v')$ for consecutive leaves $v = L[i]$ and $v' = L[i+1]$. We create pointers $\text{ptr}(v,d)$ and $\text{ptr}(v',d)$ towards node $u$, associated to the labels $t(v,d)$ and $t(v',d)$, respectively, and with weights $w(v,d) = w(v',d) = 1$. For each new internal suffix tree node $u = \text{lca}(v,v')$ obtained, we create a new label $t(u,d)$ for the new grid column that $u$ will originate, and associate weight $w(u,d) = 2$ to it (at the end, $w(u,d)$ will be the number of leaves labeled $d$ in the subtree of $u$). Each time $u$ is obtained again (which we know because the last element of $\text{list}(u)$ is already $t(u,d)$), we increase $w(u,d)$ by 1.

Now we have to propagate weights and pointers from internal nodes labeled with $d$ to their nearest ancestors labeled with $d$ (i.e., the nodes that would be their parent in the suffix tree of document $d$). For this sake, the internal nodes $u = \text{lca}(v,v')$ obtained are collected in a new array $I$, of size up to $|d| - 1$, and $I$ is sorted by the labels $t(u,d)$, so that the nodes become sorted by preorder. We traverse $I$ left to right, simulating a recursive preorder traversal of the suffix tree of document $d$, although the nodes are in the generalized suffix tree. Let $u = I[i]$ and $v = I[j]$, initially for $i = 1$ and $j = 2$. If $\text{lca}(u,v) = u$, then $u$ is the parent of $v$ in the suffix tree of $d$. Thus we recursively traverse the subtree that starts in $v = I[j]$, which finishes at a node $v' = I[j']$ that is not anymore a descendant of $v$. Now we check whether $\text{lca}(u,v') = u$ (i.e., $v'$ is the second child of $u$ in the suffix tree of $d$), and so on. At some point, it will hold that $I[j']$ does not descend from $u$, and we have finished the traversal of the subtree of $u$. Along this recursive traversal we will identify the nearest ancestor $u$ labeled $d$ of each node $v$ labeled $d$. For each such pair, and after having processed $v$ and computed $w(v,d)$, we increase $w(u,d) = w(u,d) + w(v,d)$ and generate the pointer $\text{ptr}(v,d)$ pointing to $u$, associated to label $t(v,d)$ and weight $w(v,d)$. 

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All the labels created when inserting a document \( d \) are additionally chained in a (classical) list \( \text{list}(d) \), to facilitate deletion of the document. The overall time of this step is \( O(|d| \log |d|) \), dominated by the sorting via comparisons of labels in \( X \).

Finally, we will create new columns and points in the grid associated to all the pointers \( \text{ptr}(v, d) = u \) created. The label \( t(v, d) \) will be an identifier for the \( x \)-coordinate of the point (we remark that these are not integers, but just labels that can be compared). The \( y \)-coordinate will be the string depth of the target node, \( |l(root, u)| \). This value is stored in the suffix tree node when the node is created and, unlike the tree depth, does not change upon suffix tree updates. The document associated to the new point is the new one, \( d \), and the weight is the value \( w(v, d) \) associated to the source node of the pointer.

Handling the deletion of a document \( d \) is simple. After deleting all the corresponding suffix tree nodes, we follow the chain of labels \( t(v, d) \) in \( \text{list}(d) \), delete them from \( X \) and remove their nodes from the doubly-linked list \( \text{list}(v) \). This takes \( O(|d|) \) additional time. We also remove the columns in the grid corresponding to the labels deleted, and the associated points. Both insertion and deletion times are superseded by those of Section 5.3.

### 5.3 Slim Grids

To achieve faster searches, the grid will be divided into horizontal slices of small height \( r \). For every slice, we maintain a structure that reports \( k \) most highly weighted points from a horizontal range of labels \( [a, b] \) intersected with a vertical range of integers \( [0, y] \). We describe here how those slices are updated and queried for a sublogarithmic value of \( r \), and in Section 5.4 we extend the solution to grids of polylogarithmic height.

Each slice is represented with a B-tree ordered by the labels (i.e., \( x \)-coordinates) of the points, of arity \( \rho \) to \( 2\rho - 1 \), for some sublogarithmic \( \rho \) to be defined later (as usual, the root can have arity as low as 2). Thus the B-tree has height \( O(\log \rho n) \). At each internal node \( u \) with \( a(u) \) children \( v_1, \ldots, v_{a(u)} \), we will store \( a(u) \) arrays \( W_{v_1[0..r-1]}, \ldots, W_{v_{a(u)}[0..r-1]} \). In these arrays \( W_v, W_v[y] \) is the point \( p \) with maximum weight among all points \( (1) \) whose \( x \)-coordinates belong to the subtree of \( v \), \( (2) \) with \( y \)-coordinate equal to \( y \), and \( (3) \) not stored in \( W_u[y] \) for ancestors \( u \) of \( v \) (some \( W_v[y] \) cells can be empty, if no point with \( y \)-coordinate \( y \) exists below \( v \)). We will also store a structure \( W_{root} \) for the root node \( root \). Thus \( W_{root}[y] \) contains the point \( p_r \) of maximum weight among all points with \( y \)-coordinate \( y \); for a child \( v \) of \( root \), \( W_v[y] \) contains the point \( p_v \) of maximum weight among all points \( p \neq p_r \) in the subtree of \( v \) with \( y \)-coordinate \( y \). In general, all points already stored in ancestors are excluded from consideration. We store \( W_v[y] = (x, w, d) \), where \( x \) is the \( x \)-coordinate, \( w \) is the weight, and \( d \) is the document of the point. Each point is also stored in the corresponding leaf node. Those points in \( W_v \) are not used to separate the \( x \)-coordinates of the points in the tree. Instead, new labels \( x(v_1) \ldots x(v_{a(u)}-1) \in X \) will be created and stored at node \( u \), to split the points between its \( a(u) \) consecutive children. That is, the \( x \)-coordinate of any point stored below \( v_i \) will be between \( x(v_{i-1}) \) and \( x(v_i) \). The size of the list \( X \) stays \( O(n) \).

The leaves of the B-tree will store \( r \) to \( 2r - 1 \) points. Leaves store the actual points, even if they are also mentioned in some previous \( W_v \) structure. The points in leaves \( l \) are arranged in an array \( W'_l \), which is similar to the arrays \( W_v \) and list the points in increasing \( y \)-coordinate order, except that \( W'_l \) has no empty cells and some \( y \)-coordinates can be repeated in the points. Therefore the \( W'_l \) cells store the full point data, \( W'_l[j] = (x, y, w, d) \).

To each internal node \( u \) with children \( v_1, \ldots, v_{a(u)} \) we will also associate structures \( Yx_u[0..a(u)r-1] \) and \( Yw_u[0..a(u)r-1] \), where the child numbers and the \( y \)-coordinates of the (up to) \( r \) points of
the $a(u)$ arrays $W_v$, are sorted by their $x$-coordinate label (in $Yx_u$) and by their weight (in $Yw_u$). That is, in $Yx_u$ and $Yw_u$ we store the pair $(i, y)$ for each entry $W_v[y]$, ordered by $W_v[y].x$ (in $Yx_u$) or by $W_v[y].w$ (in $Yw_u$). Each value stored in $Yx_u$ and $Yw_u$ requires $\lg(2\rho r)$ bits, thus all the values in these two structures add up to at most $4\rho r \lg(2\rho r)$ bits. Each time we modify a value in a $W_v$ array, we rebuild from scratch the $Yx_u$ and $Yw_u$ structures of the parent $u$ of $v$.

We will also maintain structures $Yx'_v$ and $Yw'_v$ on the (up to) $2r - 1$ points of leaves $l$, analogous to the $Yx_u$ and $Yw_u$ structures of internal nodes. Instead of the pairs $(i, y)$, structures $Yx'_v$ and $Yw'_v$ will just store positions $j$ of the array $W'_v$ (those positions would coincide with $y$-coordinates in internal nodes). Leaves will also store an array $Yl[0..r - 1]$ where $Yl[y] = j$ if $j$ is the last position where $W'_v[j] < y$. Finally, leaves will store bitmaps $Q_l$ marking in $Ql[j]$ whether the point in $W'_v[j]$ also appears in the $W_v$ array of an ancestor $v$ of $l$.

We will end up choosing $\rho r = o(\log n)$, and thus universal tables of $2^{4\rho r \log(2\rho r)} \cdot O(\log(\rho r)) = o(n)$ bits will be used to query and update the arrays $Yx_u$ and $Yw_u$, in constant time. Similarly, leaves will use even smaller universal tables of $2^{4r \log(2r)} \cdot O(\log r) = o(n)$ bits.

The whole data structure requires linear space, because the leaves contain $\Theta(n)$ points. If the whole grid contains less than $r$ points, we just store the space for them in a leaf.

### 5.3.1 Insertions

Consider the insertion of a new point $(x, y, w, d)$, with label $x \in X$, $y$-coordinate $y \in [0, r)$, weight $w$ and document $d$. While following the normal insertion procedure on the B-tree (where we compare the labels $x(v_i)$ of the nodes with $x$ to decide the insertion path), we look for the highest node $v$ with $W_v[y].w < w$ or with $W_v[y]$ empty. For the first (i.e., highest) such $v$ we find, we set $W_v[y] \leftarrow (x, w, d)$, and then we continue the classical insertion procedure (not looking at $W_v[y]$ entries anymore) until adding the point $(x, y, w, d)$ in a leaf $l$. In the leaf we mark in the corresponding $Q_l$ entry whether we had updated an entry $W_v[y]$ in some ancestor $v$.

If we updated some $W_v[y]$, and it already had a previous value $W_v[y] = (x', w', d')$, we perform a process we call reinsertion of $(x', w', d')$. We restart the process of inserting the point $(x', y, w', d')$ from node $v$ (note that this point already exists in a leaf; reinsertion will not alter the structure of the tree, but just rewrite some $W$ and $Q$ values). In the reinsertion path, if we arrive at a node $v'$ where $W_{v'}[y].w < w$, we set $W_{v'}[y] \leftarrow (x', w', d')$. If there was a previous value $W_{v'}[y] = (x'', w'', d'')$, we continue the reinsertion process for point $(x', w', d')$ from node $v'$, and so on until either we find an empty space in some $W_u[y]$ or we reach the leaf $l$ where the point being reinserted is actually stored. In this latter case, we clear the corresponding bit in $Q_l$, indicating that this point is not stored anymore in an ancestor structure.

Thus we traverse two paths, one for inserting the point, and another for reinserting the point(s) possibly displaced from some $W_v[y]$ structure. This part of the operation requires, in the worst case, $O(\log \log n)$ updates to the structures $Yx_u$ and $Yw_u$ of the parents $u$ of nodes $v$ where $W_v$ is modified, plus an insertion in a leaf.

**Rebuilding structures $Yx_u$ and $Yw_u$.** Upon an assignment $W_v[y] \leftarrow (x, w, d)$, we must rebuild the structures $Yx_u$ and $Yw_u$ of the parent $u$ of $v$. We binary search $Yx_u$ for $x$, and binary search $Yw_u$ for $w$, both in $O(\log(\rho r))$ time. In these binary searches we obtain the actual label and weight of each element of $Yx_u$ and $Yw_u$, respectively, using its $(i, y)$ pair, as $W_v[y].x$ and $W_v[y].w$. These
binary searches give the insertion positions $0 \leq e < 2pr$ and $0 \leq g < 2pr$, respectively, of the pair $(i, y)$ in $Yx_u$ and $Yw_u$. Note that the new contents of $Yx_u$ and $Yw_u$ depend only on their current contents, on the values $e$ and $g$, and on the incoming pair $(i, y)$ (the existing occurrence of $(i, y)$, if any, must be removed). Thus, the new content of $Yx_u$ and $Yw_u$ for each $(e, g, i, y)$ can be precomputed in a universal table of $o(n)$ bits, as explained, so that they are updated in constant time. Therefore the time to update the structures is $O(\log(pr))$, and the cost of a full reinsertion process is $O(\log_n n \log(pr))$.

Removing the value of a cell $W_v[y]$ is analogous. We can insert all the cells of a whole new $W_v$ array, or remove all the cells of a whole $W_v$ array, one by one in time $O(r \log(pr))$. When a whole array is inserted or removed, we have to rename all the labels $i$ in the cells $(i, y)$, but those updates can also be precomputed in universal tables of sublinear size.

**Insertion in leaves.** In leaves $l$, we must actually insert the point, possibly displacing all the entries in $W'_l$ and recalculating $Y_l$, $Yx'_l$, $Yw'_l$ and $Q_l$, all in $O(r)$ time. When a leaf overflows to $2r$ points, we must split it into two leaves $l'$ and $l''$ of $r$ points each. We first remove the array $W'_l$ from the parent $u$ of $l$, clearing the corresponding bits in $Q_l$. Now we distribute the points of $W'_l$ into the new arrays $W'_v$ and $W''_v$, and make $l'$ and $l''$ children of $u$, replacing the old $l$. We create a new label $x(l') \in X$ right after the largest $x$-coordinate in $l'$, and add it to $u$ separating $l'$ and $l''$.

Now we build new arrays $W'_v$ and $W''_v$. Those arrays, as well as the $Y$, $Yx'_v$, $Yw'_v$ and $Q$ structures of $l'$ and $l''$, are built in $O(r)$ time from $W'_l$, $Yx'_l$, $Yw'_l$ and $Q_l$. We mark in $Q'_v$ and $Q''_v$ the points that have been included in $W'_v$ and $W''_v$ (we cannot choose any point for $W'_v$ and $W''_v$ that is already marked in $Q_l$). Finally, we insert $W'_v$ and $W''_v$ in $u$.

The overall time is $O(r)$, but this is dominated by the $O(r \log(pr))$ time needed to update the $Yx_u$ and $Yw_u$ arrays upon the $O(r)$ changes induced by substituting $W_l$ by $W'_v$ and $W''_v$.

**Overflows in internal nodes.** Further, the insertion in the parent $u$ can trigger an overflow in this internal node, if its arity reaches $2r$. We must split $u$, with children $v_1, \ldots, v_{2r}$, into two nodes, $u'$ with children $v_1, \ldots, v_r$ and $u''$ with children $v_{r+1}, \ldots, v_{2r}$. The process is analogous to the case of leaves, but slightly more complicated. We create a new $x$-coordinate $x(u') \in X$ following $x(v_r)$, to separate the points of $u'$ and $u''$. We create the two nodes with their corresponding arrays $W_{v_1}$, and build the tables $Yz$ and $Yw$ of $u'$ and $u''$, in $O(pr)$ time from $Yz_u$ and $Yw_u$.

Now we must create new arrays $W'_{v_1}$ and $W''_{v_1}$ to replace $W_v$ in the parent of $u$. First, we move the points in $W_u[y]$ into $W'_{v_1}[y]$ and $W''_{v_1}[y]$, according to their $x$-coordinate. Now we can get rid of $W_v$, but we still have several empty cells in $W'_{v_1}[y]$ and $W''_{v_1}[y]$. Those are filled with a process we call *uninsertion*: To fill some cell $W'_{v_1}[y]$ (analogously for $u''$), we take the maximum weight in cells $W_{v_1}[y], \ldots, W_{v_r}[y]$. The maximum $W_{v_1}[y], w$ is found in constant time using a universal table on $Yw_{u'}. Then we copy $W'_{v_1}[y] \leftarrow W_{v_1}[y]$, and continue the uninsertion process for $W_{v_1}[y]$. When we finally arrive at uninserting a point from a leaf $l$, all we have to do is to mark the corresponding entry in $Q_l$. Note that uninsertion does not alter the structure of the tree; it just rewrites some $W$ and $Q$ values. The cost of one uninsertion is $O(\log_n n \log(pr))$, to rebuild the affected structures $Yx$ and $Yw$. Thus the $O(r)$ uninsertions in $u'$ and $u''$ add up to $O(r \log_n n \log(pr))$ time, which subsumes the $O(r \log(pr))$ cost to replace $W_u$ by $W'_{v_1}$ and $W''_{v_1}$ in the parent of $u$.

Note that the insertion of a single point could produce one split per level of the B-tree. To avoid this, we use a deamortization technique by Fleischer [26]. This maintains an $(\alpha, 2\beta)$-tree (for $\alpha \leq 2\beta$) storing $n$ keys in the leaves, and each leaf is a bucket storing at most $2\log_\alpha n$ keys.
It supports constant-time insertion and deletion of a key once its location in a leaf is known, guaranteeing at most one split per insertion or deletion. The premises are consistent with our setting, with $\alpha = \beta = \rho$, and storing $O(r) = o(\log n)$ keys per leaf.

5.3.2 Deletions

Deletion of a point $(x, y)$ starts by searching the B-tree for the $x$-coordinate $x$. The point will be found in its leaf, and also possibly in some cell $W_v[y]$ of some internal node $v$. The search takes $O(\log n)$ time because, for internal nodes $u$, we binary search the coordinates $x(v_i)$ stored in $u$ for the correct child $v$, in $O(\log \rho)$ time, and then only have to check if $W_v[y].x = x$. In leaves $l$, we binary search for $x$ in $Ya'_l$ in $O(\log r)$ time.

If the point has to be deleted from some $W_v[y]$, we carry out the uninsertion process already described, in $O(\log \rho n \log(\rho r))$ time. We also remove the point $(x, y)$ itself from leaf $l$. When a leaf $l$ underflows, we merge it with a neighbor leaf and, if necessary, split it again. The merging process is analogous to the splitting and can be easily carried out in $O(\log \rho)$ time, plus $O(r \log(\rho r))$ to update the structures $Yx$ and $Yw$ in the parent.

If an internal node underflows, we also merge it with its neighbor and re-split it if necessary. The merging of two sibling nodes $v$ and $v'$ is carried out in $O(\rho)$ time, including the construction of the $Yx$ and $Yw$ structures for the merged node, $u$. The difficult part is, again, to get rid of the arrays $W_v$ and $W_{v'}$ at the parent node, replacing them by a new $W_{v^*}$ table for the merged node $v^*$. For this sake, we choose the maximum weight between each $W_v[y]$ and $W_{v'}[y]$ and assign it to $W_{v^*}[y]$. The point that was not chosen among $W_v[y]$ and $W_{v'}[y]$ must be reinserted, as before. Finally, we must rebuild the $Yx$ and $Yw$ structures of the parent of $v^*$. The total cost is $O(r \log \rho n \log(\rho r))$, just as for insertions.

Note that, upon leaf or internal node merges, a separating label $x(v)$ becomes unused, and it is removed from $X$. Again, Fleischer’s technique [26] ensures at most one underflow per update.

5.3.3 Queries

Identifying the relevant nodes. To solve a top-$k$ query with label restriction $[a, b]$ and $y$-coordinate restriction $[0, y]$ on the slice, we first identify the $O(\log n)$ ranges of siblings of the B-tree that exactly cover the interval of labels $[a, b]$; plus up to 2 leaf nodes that partially overlap the interval. For each node $u$ that is the parent of a range of children $v_s, \ldots, v_e$ included in the cover, we find the maximum weight in $W_{v_s}[0, y-1]$, $\ldots$, $W_{v_e}[0, y-1]$ and insert the result in a max-priority queue $Q$ sorted by the weights of the points. Such maximum across $W_{v_s}[0, y-1]$ arrays is obtained in constant time using universal tables on $Yw_u$. If $l$ is a leaf partially covering $[a, b)$, then the interval is some $W_l'[0, y']$, where $y' = Y_l'[y]$. In addition, we must binary search $Ya'_l$ for the range $[x_a, x_b]$ corresponding to the interval $[a, b)$. Furthermore, we can only return points whose $Q_l$ bit is not set, to avoid repeated answers. Knowing the range in $Ya'_l[x_a, x_b]$ and the range $W_l'[0, y']$, the maximum weight can be obtained from $Ya'_l$, $Yw_l'$ and $Q_l$ with a universal table, in constant time. Identifying the cover nodes and finding their $O(\log n)$ maxima takes $O(\log n)$ time, and leaves add only $O(\log r)$ time.

Each element inserted in $Q$ coming from a range of siblings will be a tuple $(u, s, e, i, z, k)$, where $u$ is the parent node of the range of children $v_s, \ldots, v_e$ in the cover, $(i, z)$ means that the maximum was found at $W_{v_i}[z]$ ($s \leq i \leq e$), and $k$ indicates that the point $W_{v_i}[z]$ is the $k$th in the range of interest for $u$. All the nodes initially inserted have $k = 1$. 18
The elements inserted in $Q$ coming from leaves $l$ are of the form $[l, j, x_a, x_b, k]$, meaning that the maximum was found in $W'_l[j]$, that the range of interest is $W'_l[0, Y_l[y]]$ and $Yx'[x_a, x_b]$, and that the point is the $k$th in the range of interest. The first insertions use $k = 1$.

We also insert in $Q$ a third kind of tuples, namely, the maximum-weight point in $W_v[0, y - 1]$ with $x$-coordinate in $[a, b]$, for each of the $O(\log_\rho n)$ ancestors $v$ of the cover nodes, as they may also hold relevant points. To find those maxima we consider the parent $u$ of $v$ and binary search $Yx_u$ for $a$ and $b$, to find a mapped interval $Yx_u[x_a, x_b]$, in $O(\log(pr))$ time. Note that this area of $Yx_u$ corresponds to nodes in $W_v$. Then we use universal tables on $Yx_u$ and $Yw_u$ to find the maximum weight of $y$-coordinate below $y$ and in the range $Yx_u[x_a, x_b]$. For these nodes we insert tuples of the form $⟨u, v, z, x_a, x_b, k⟩$ in $Q$, meaning that the maximum was obtained from $W_v[z]$, the range of interest is $Yx_u[x_a, x_b]$, and the point is the $k$th in its range of interest. Therefore the initial computation on these nodes requires $O(\log_\rho n \log(pr))$ time. Recall that the root node of the B-tree will also have a $W$ structure computed (this is easily treated as a special case).

We implement $Q$ as a Thorup’s priority queue [60] on the universe of weights $[1, O(n)]$. Note that we do not need to insert the whole initial set of $O(\log_\rho n)$ tuples in $Q$ if this number exceeds $k$: if a tuple is not among the first $k$, it cannot contribute to the answer. Then we use linear-time selection to find the $k$th largest weight in the tuples and then insert only the first $k$ tuples in $Q$. This structure supports insertions in constant time, thus the initialization of $Q$ takes time $O(\log_\rho n)$.

**Extracting the top-$k$ points.** The first answer to the top-$k$ query is among the $O(k)$ tuples we have inserted in $Q$. Therefore, to obtain the first result, we extract the tuple with maximum weight from $Q$. If it is of the form $[l, j, x_a, x_b, k]$, that is, it comes from a leaf $l$, we report the point $W'_l[j]$, compute the $(k + 1)$th highest-weight point $W'_l[j']$ within $W'_l[0, Y_l[y]]$ and $Yx'_l[x_a, x_b]$ using universal tables, and reinsert tuple $[l, j', x_a, x_b, k + 1]$ in $Q$. If, instead, the maximum tuple extracted from $Q$ is of the form $⟨u, v, z, x_a, x_b, k⟩$, that is, it becomes from an ancestor of a cover node, we report the point $W_v[z]$, compute the $(k + 1)$th highest-weight point $W_v[z']$ with $y$-coordinate below $y$ and within $Yx_u[x_a, x_b]$ using universal tables, and reinsert tuple $⟨u, v, z', x_a, x_b, k + 1⟩$. Finally, if the maximum tuple extracted from $Q$ is of the form $(u, s, e, i, z, k)$, we report the point $W_{vi}[z]$, where $vi$ is the $i$th child of $u$, compute the $(k + 1)$th highest-weight point $W_{v_i}[z']$ in $W_v[0, y - 1], . . . , W_{v_i}[0, y - 1]$ using universal tables, and reinsert tuple $⟨u, s, e, i', z', k + 1⟩$. If the extracted point had $k = 1$, however, it is possible that the next highest-weight element comes from the child $v_i$. Therefore, if $v_i$ is an internal node, we compute the highest-weight point in $YW_{vi}$ with $y$-coordinate below $y$. Let it be the pair $(i''', z''')$, then we insert a new tuple $(v_i, 1, a(v_i), i''', z''', 1)$ in $Q$. If, instead, $v_i$ is a leaf $l = vi$ with $r(l)$ elements, then we find the maximum-weight point $W'_l[j''']$ in $W'_l[0, Y_l[y]]$ using $YW'_l$, and insert the tuple $[l, j''', 1, r(l), 1]$ in $Q$. In all cases the cost to compute and insert the new tuples is constant.

If we carry out $k$ extractions from $Q$, we will also carry out up to $2k$ insertions, thus the size of $Q$ will be $O(k)$ and minima extractions will cost $O(\log \log k)$ [60]. The cost of this part is then $O(k \log \log k)$, and the total query time is $O(\log n + \log r + \log_\rho n \log(pr) + \log_\rho n + k \log \log k)$. Given a constant $0 < \varepsilon < 1/2$, we will choose $r = \rho = \log n$, fulfilling the promise that $pr = o(\log n)$. Therefore, the update cost becomes $O(r \log_\rho n \log(pr)) = O(\log^{1+\varepsilon} n)$, and the query time becomes $O(\log n + k \log \log k)$. Note that the process is not online: We must know $k$ in advance so as to initially limit the size of $Q$ to $k$. We use the technique of Section 4.4 to make the process online in $k$. That is, $k$ is not specified in advance and the process can be interrupted after having produced any number $k$ of results, and the total cost paid will be $O(\log n + \log \log k)$.
5.4 Multiresolution Grids

We extend the result of Section 5.3 to grids of polylogarithmic height $r^c$, for some constant $c$. We will represent the grid at various resolutions and split it into slim grids for each resolution. Consider a virtual perfect tree of arity $r$ and $n$ leaves, so that the $i$th left-ro-right node of height $j$ covers the rows $(i-1) \cdot r^j + 1$ to $i \cdot r^j$. The tree is of height $c$.

For each node $v$ of this tree we store a slim grid of $r$ rows, one per child. All the points whose row belongs to the area covered by the $i$th child of $v$ will be represented as having $y$-coordinate $i$ in the slim grid of $v$.

When a new point $(x, y, w, d)$ is inserted in the grid, we insert it into the $c$ slim grids that cover it, giving it the appropriate row value in each slim grid, and similarly when a point is deleted. The $x$-coordinate labels are shared among all the grids. This arrangement multiplies space and insertion and deletion times by the constant $c$.

Now consider a 3-sided top-$k$ query with the restriction $[a, b]$ on the $x$-coordinates and $[0, y)$ on the $y$-coordinates. The range $[0, y)$ is covered with the union of one range in one slim grid per level of the tree. Let $y_c, \ldots, y_1$ be the child numbers of the path from the root to the $y$-th row of the grid. Then we take the 3-sided query $[a, b] \times [0, y_j)$ at the node of height $j$ in the path.

We start the searches in the $c$ slim grids, and extract the first result from each grid. We insert those local maxima into a new global queue $Q$. Now we repeat $k$ times the process of extracting the next result from $Q$, reporting it, requesting the next result from the grid where the result came from, and inserting it in $Q$. Note that $Q$ can be implemented naively because it contains at most $c$ elements and $c$ is a constant.

Initializing the searches will then require $O(c \log n)$ time, and extracting $k$ results from the slim grids will require $O(k \log \log k)$ time. Managing $Q$ will require $O(ck)$ time even if done naively. Therefore the total time is still $O(\log n + k \log \log k)$. The update time per element stays $O(c \log^{1+\varepsilon} n)$. The process is also online. Then we obtain the following lemma.

**Lemma 6** A set of $n$ points, one per column on an $n \times r^c$ grid, for $r = \log^\varepsilon n$ and any constant $0 < \varepsilon < 1$ and $c \geq 1$, with weights in $[1, O(n)]$, can be stored in $O(n)$ words of space, so that for any $1 \leq k \leq n$, $1 \leq h \leq r^c$ and $1 \leq a \leq b \leq n$, $k$ most highly weighted points in the range $[a, b] \times [0, h]$ can be reported in decreasing order of their weights in $O(\log n + k \log \log k)$ time, online in $k$. Points (and their columns) can be inserted and deleted in $O(\log^{1+\varepsilon} n)$ time.

5.5 The Final Result

We find the locus $v$ of $P$ in the suffix tree in time $O(p(\log \log n)^2 / \log x n + \log n)$. Then the $x$-coordinate range of labels to search for in the grid is $[a, b)$, where $a$ is the first label in $\text{list}(v)$ and $b = \text{last}(v)$. Since we store string depths in the grid, the $y$-coordinate range of the query is $[0, p)$.

Our dynamic grid is horizontally split into bands of $r^c$ rows, for a constant $c$, which are handled as explained in Section 5.3. Therefore, our 3-sided query is translated into 3-sided queries on the first $[p/r^c]$ bands. All but the last will query for the whole row interval $[0, r^c)$, whereas the latter will query for the row interval $[0, (p - 1) \mod r^c]$.

We start the searches in all the bands, and extract the first result from each. If there are more than $k$ bands, we use linear-time selection to keep only the $k$ highest weights. Then we insert the local maxima into a new global queue $Q$. Now we repeat $k$ times the process of extracting the first result from $Q$, and if it came from the $i$th band, then we request the next result from that band.
and insert it in \( Q \) (unless it has no more results, in which case we continue with the remaining bands). Again, \( Q \) will be implemented with Thorup’s priority queue \([60]\).

Initializing the searches will then require \( O(\lceil p/r^c \rceil \log n) \) time, and extracting \( k \) (and inserting other \( k \)) results in \( Q \) will take time \( O(k \log \log k) \). We choose \( r = \lg^\varepsilon n \) for some \( 0 < \varepsilon < 1/2 \), as explained, and rename \( c \) as \((c + 1)/\varepsilon\). Therefore, we obtain a query time of \( O(p/\log^c n + \log n + k \log \log k) \) for the grid. Once again, the scheme can be made online with the technique of Section 4.3. Updating grid points, including extending the grid downwards, requires \( O(\log^{1+\varepsilon} n) \) time. This yields Theorem 5.

We developed our result for \( tf \) as the relevance measure. It is very easy to support others like \( docrank \), but if the weights are not integer numbers, then Thorup’s priority queues \([60]\) cannot be used. In this case we insert all the new weights that appear in a data structure for monotonic list labeling, which assigns them integers in a polynomial universe \([1, n^{O(1)}]\). This adds at most \( O(\log n) \) time per symbol inserted \([20]\) (deletions can be handled by deamortized periodic rebuildings). In general we can support any measure that can be computed in time \( O(|d|C_w) \) over the suffix tree of the document to insert: We explicitly build such suffix tree, compute the relevance measure for all the nodes, and then use them to assign the weights as we insert the nodes in our suffix tree. At the end we delete the suffix tree of \( d \). This suffix tree can be built (and deleted) in \( O(|d|) \) time on integer alphabets \([22]\). Therefore we simply charge \( O(C_w) \) time per character inserted in our text collection. Note that \( C_w \) is \( O(1) \) for measures \( tf \) and \( docrank \). Hon et al. \([37]\) show how to compute \( mindist \) from the suffix tree of the document in \( O(|d| \log |d|) \) time, so \( C_w = O(\log n) \) in this case.

Our scheme works as long as \( \lg n \) has a fixed value (plus \( O(1) \)). We use standard techniques to incrementally rebuild the structure for larger or smaller \( \lg n \) values as more insertions or deletions are processed.

## 6 A Space-Efficient Data Structure

We show now how the space of our static structure can be reduced to \( O(n(\log \sigma + \log D + \log \log n)) \) bits, where \( \sigma \) is the alphabet size and \( D \) is the number of documents, and retain almost the same query time. Our approach is to partition the tree into minitrees, which are represented using narrower grids.

### Partitioning the tree

We define \( z = \Theta(\sigma D \log n) \). We say that a node \( v \in T \) is heavy if the subtree rooted at \( v \) has at least \( z \) leaves, otherwise it is light. A heavy node is fat if it has at least two heavy children, otherwise it is thin.

All the non-fat nodes of \( T \) are grouped into minitrees as follows. We traverse \( T \) in depth-first order. If a visited node \( v \) has two heavy children, we mark \( v \) as fat and proceed. If \( v \) has no heavy children, we mark \( v \) as thin or light, and make \( v \) the root of a minitree \( T_v \) that contains all the descendants of \( v \) (which need not be traversed). Finally, if \( v \) has one heavy child \( v_1 \), we mark \( v \) as thin and make it the root of a minitree \( T_v \). The extent of this minitree is computed as follows. If \( v_i, i \geq 1 \), is a thin node with one heavy child \( v_{i+1} \), we visit nodes \( v_1, v_2, \ldots, v_{j-1} \) and include \( v_i \) and all the descendants of its other children, until either \( v_{j-1} \) has no heavy children or is fat, or \( T_v \) contains more than \( \sigma z \) nodes after considering \( v_j \). Then we continue our tree traversal from \( v_j \). Note that \( T_v \) contains at the very least the descendants of \( v \) by children other than \( v_1 \).

With the procedure for grouping nodes described above, the leaves of minitrees can be parents of nodes not in the minitree. Those child nodes can be either fat nodes or roots of other minitrees.
However, at most one leaf of a minitree can have children in $T$.

Note that the size of a minitree is at most $O(\sigma z)$. On the other hand, as two heavy children have disjoint leaves, there are $O(n/z)$ fat nodes in $T$. Finally, minitrees can contain as little as one node (e.g., for leaves that are children of fat nodes). However, note that a minitree root is either a child of a fat node (and thus there are $O(\sigma n/z)$ minitrees of this kind), or a child of a leaf of another minitree such that the sum of both minitree sizes exceeds $\sigma z$ (otherwise we would have included the root $v_j$ of the child minitree as part of the parent minitree). Moreover, as said, at most one of the leaves of a minitree can be the parent of another minitree, so these minitrees that are “children” of others form chains where two consecutive minitrees cover at least $\sigma z$ nodes of $T$. Thus there are $O(n/(\sigma z))$ minitrees of this second kind. Adding up both cases, there are $O(\sigma n/z) = O(n/(D \log n))$ minitrees in $T$.

**Contracted tree and minitrees.** The pointers in a tree $T$ are defined in the same way as in Section 2. Since we cannot store $T$ without violating the desired space bound, we store a contracted tree $T^c$ and the minitrees $T_v$.

The contracted tree $T^c$ contains all fat nodes of $T$, plus one node $v^c$ for each minitree $T_v$. Each pointer $\text{ptr}(u, d) = u'$ of $T$ is mapped to a pointer $\text{ptr}^c(u^c, d) = (u')^c$ of $T^c$ as follows. If $u$ is a fat node, then $u^c = u$. Else, if $u$ belongs to minitree $T_v$, then $u^c = v^c$. Similarly, if $u'$ is a fat node then $(u')^c = u'$; else if $u'$ belongs to minitree $T_v'$ then $(u')^c = (v')^c$. In other words, nodes of a minitree are mapped to the single node that represents that minitree in $T^c$ and pointers are changed accordingly.

For each minitree $T_v$, we store one additional dummy node $\nu$ that is the parent of $v$. If a leaf $u_h$ of $T_v$ has a heavy child $u' \notin T_v$, we store an additional dummy node $\nu' \in T_v$ that is the only child of $u_h$. Pointers of $T_v$ are modified as follows. Each pointer $\text{ptr}(u, d)$, $u \in T_v$, that points to an ancestor of $v$ is transformed into a pointer $\text{ptr}(u, d)$ that points to $\nu$. Every pointer $\text{ptr}(u', d)$ that starts in a descendant $u''$ of $u_h$ and points to a node $u \in T_v$, $u \neq u_h$, (respectively to an ancestor of $v$) is transformed into $\text{ptr}(v', d)$ that starts in $\nu'$ and points to $u$ (respectively to $\nu$). By Lemma 2, there are at most $D$ such pointers $\text{ptr}(u', d)$. We observe that there is no need to store pointers to the node $u_h$ in the minitree $T_v$ because such pointers are only relevant for the descendants of $u_h$ that do not belong to $T_v$.

**Suffix trees.** The contracted tree $T^c$ consists of $O(n/(D \log n))$ nodes, and thus it would require just $O(n/D)$ bits. The minitrees contain $O(\sigma z)$ nodes, but still an edge of a minitree can be labeled with a string of length $\Theta(n)$. Instead of representing the contracted tree and the minitrees separately, we use Sadakane’s compressed suffix tree (CST) to represent the topology of the whole $T$ in $O(n)$ bits, and a compressed representation [31] Thm. 5.3] of the global suffix array (SA) of the string collection, which takes $O(n \log \sigma)$ bits. This SA representation finds the suffix array interval $[l, r]$ of $P$ in time $O(p/\log_n n + \log^c n \log \sigma)$ for any constant $c > 0$, and a lowest-common-ancestor query for the $l$-th and $r$-th leaves of $T$ finds the locus $u$ of $P$ in $O(1)$ additional time. A bitmap $M[1, n]$ marks which nodes are minitree roots, and another bitmap $C[1, n]$ marks which nodes are fat or minitree roots. Both are indexed with preorder numbers of $T$, which are computed in constant time on the CST. With a simple $O(n)$-bit structure for constant-time marked ancestor queries that is compatible with our CST representation [54] Sec. 4.1], we can find the lowest ancestor $u$ of $u$ marked in $M$ or in $C$. With bitmap $M$ we can identify whether $u$ belongs to a minitree rooted at $v$ (with local preorder $\text{preorder}_{T_v}(u) = \text{preorder}_T(u) - \text{preorder}_T(v)$.
and depth \( depth_{T_v}(u) = depth_T(u) - depth_T(v) \); depths are also computed in constant time). Similarly, with \( C \) and \( M \) we can identify whether \( u \) is a fat node, and find out its preorder in \( T^c \) as \( \text{preorder}_{T^c}(u) = \text{rank}_1(C, \text{preorder}_T(u)) \), in constant time. Its depth in \( T^c \) can be stored in an array indexed by \( \text{preorder}_{T^c} \) in \( O(n/D) \) bits.

**Contracted grid.** We define the grid of the contracted tree \( T^c \) as in Section~\ref{sec:enhanced-cst}, considering all pointers \( ptr^c \). Those are either \( ptr \) pointers leaving from fat nodes, or leaving from inside some minitree \( T_v \) and pointing above \( v \). For every fat node and for every minitree \( T_v \), and for each document \( d \), there is at most one such pointer by Lemma~\ref{lem:fat-tree}. Thus each node of \( T^c \) contributes at most \( D \) pointers \( ptr^c \). As there are \( O(n/(D \log n)) \) nodes, there are \( O(n/\log n) \) pointers \( ptr^c \) in \( T^c \).

Therefore, the grid associated to \( T^c \) is of width \( O(n/\log n) \) and height \( O(n/(D \log n)) \). As there are \( O(n/\log n) \) distinct weights among the \( ptr^c \) pointers, we only store their ranks. This change does not alter the result of any top-\( k \) query. Therefore the data structure of Theorem~\ref{thm:enhanced-cst} on \( T^c \) occupies \( O(n/\log n) \) words, or \( O(n) \) bits.

**Local grids.** The local grid for a minitree \( T_v \) collects the pointers \( ptr \) local to \( T_v \). It also includes at most \( D \) pointers towards its dummy root \( \nu \), and at most \( D \) pointers coming from its node \( \nu' \), if it has one. Overall, \( T_v \) contains \( O(\sigma z) \) pointers and \( O(\sigma z) \) nodes, so its grid is of size \( O(\sigma z) \times O(\sigma z) \). The weights are also replaced by their ranks, so they are also in the range \([1, O(\sigma z)]\). Using Theorem~\ref{thm:enhanced-cst} the minitree requires \( O(\log(\sigma z)) \) bits per node. Added over all the nodes of \( T \) that can be inside minitrees, the total space is \( O(n \log(\sigma z)) = O(n(\log \sigma + \log D + \log \log n)) \). Note that the tree topology is already stored in the CST, so information associated to nodes \( u \in T_v \) such as the intervals \([l_u, r_u]\) can be stored in arrays indexed by preorder numbers.

**Queries.** Given a query pattern \( P \), we find the locus \( u \) of \( P \) and determine whether \( u \) is a fat node or it belongs to a minitree in \( O(p/\log \sigma n + \log^\epsilon n \log \sigma) \) time, as explained. If \( u \) is a fat node, we solve the query on the contracted grid of \( T^c \). Note that this grid does not distinguish among different nodes in the same minitree. But since \( u \) is an ancestor either of all nodes in a minitree or of none of them, such distinction is not necessary.

If \( u \) belongs to a minitree \( T_v \), we answer the query using the corresponding local grid. This grid does not distinguish where exactly the pointers pointing to \( \nu \) lead, nor where exactly the pointers that originate in \( \nu' \) come from. Once again, however, this information is not important in the case where the locus \( u \) of \( P \) belongs to \( T_v \).

Note that we still need to maintain the global array mapping \( x \)-coordinates to document identifiers. This requires \( O(n \log D) \) bits.

**Theorem 7** Let \( D \) be a collection of \( D \) documents over an integer alphabet \([1, \sigma]\) with total length \( n \), and let \( w(S, d) \) be a function that assigns a numeric weight to string \( S \) in document \( d \), that depends only on the set of starting positions of occurrences of \( S \) in \( d \). Then there exists an \( O(n(\log D + \log \sigma + \log \log n)) \)-bit data structure that, given a string \( P \) and an integer \( k \), reports \( k \) documents \( d \) containing \( P \) with highest \( w(P, d) \) values, in decreasing order of \( w(P, d) \), in \( O(p/\log \sigma n + \log^\epsilon n \log \sigma + k) \) time, for any constant \( \epsilon > 0 \).

In case \( p < \log^{1+\epsilon} n \), we can use a different compressed suffix array \([8]\), which gives \( O(p) \) search time, and the overall time becomes \( O(p + k) \).
6.1 A Smaller Structure when using Term Frequencies

In this section we show that the space usage can be further improved if \( w(P, d) = tf \), i.e., when the data structure must report \( k \) documents in which \( P \) occurs most frequently.

Our improvement is based on applying the approach of Theorem 7 to each minitree. The nodes of a minitree are grouped into microtrees; if the structure for a microtree still needs too much space, we store them in a compact form that will be described below.

Let \( z' = \sigma D \log m \), where \( m \) is the number of nodes in a minitree \( T \). Using the same method as in Theorem 7, we divide the nodes of \( T \) into \( O(m/z') \) minutefat nodes and \( O(m/(D \log m)) \) microtrees, so that each microtree contains \( O(\sigma z') \) nodes. We construct the contracted minitree and the contracted grid for \( T \) as in Theorem 7. Both the contracted minitree and the structure for the contracted grid use \( O(m) \) bits. We can traverse a path in the microtree using the implementation of the global suffix tree described in the previous section, as well as compute local preorders and depths, and attach satellite information to microtree nodes.

For every microtree \( T_v \), we define the dummy nodes \( \nu \) and \( \nu' \). Pointers in \( T_v \) are transformed as in the proof of Theorem 7 with regard to \( \nu \) and \( \nu' \).

Let \( m' \) denote the number of nodes in a microtree. If \( \log m' = O(\log \sigma + \log D) \), we implement the local grid data structure described in Theorem 7 for a microtree. In this case we can store a data structure for a microgrid in \( O((\log (m'/D) + O(\log \sigma + \log D)) \) bits per node.

If, instead, \( \log m' = \omega(\log \sigma + \log D) \), since \( \log m' = O(\log(\sigma z')) = O(\log \sigma + \log D + \log \log m) \), it follows that \( \log m' = O(\log(\sigma z')) = O(\log \sigma + \log D + \log \log n)^{O(1)} \). Hence, the size of the microtree is \( m' = (\log \sigma + \log D + \log \log n)^{O(1)} \). The total number of pointers in the microtree is also \( m'' = m' + O(D) = (\log \log n)^{O(1)} \) (since \( \log D = o(\log m') \)). Since all the grids in \( m'' \times m' \), with one point per \( x \)-coordinate, and weights in \([1, m'']\), can be expressed in \( m''(\log m' + \log m'') = O(\log n) \) bits, we can store pre-computed answers for all possible queries on all possible small microtrees.

The only technical difficulty is that weights of some pointers in a microtree can be arbitrarily large. However, as explained below, it is not necessary to know the exact weights of pointers to answer a query on a small microtree.

All pointers \( \text{ptr}(u_t, d) \) where \( u_t \) is a leaf node and \( u_t \neq \nu' \) have weight 1. The weights of \( \text{ptr}(\nu', d) \) can be arbitrarily large. The weight of a pointer \( \text{ptr}(u, d) \) for an internal node \( u \) equals to the sum of weights of all pointers \( \text{ptr}(u', d) \) for the same document \( d \) that lead to \( u \). Thus the weight of \( \text{ptr}(u, d) \) can also be large. We note that there is at most one pointer \( \text{ptr}(\nu', d) \) for each \( d \). Therefore the weight of each pointer \( \text{ptr}(u, d) \) can be expressed as the sum \( w_1(u) + w_2(u) \), where \( w_1(u) \) is the weight of \( \text{ptr}(\nu', d) \) or 0 and \( w_2(u) \leq m' \). In other words, the weight of \( \text{ptr}(u, d) \) differs from the weight of \( \text{ptr}(\nu', d) \) by at most \( m' \).

Let the set \( \mathcal{N} \) contain the weights of all pointers \( \text{ptr}(u_t, d) \) and \( \text{ptr}(\nu', d) \). Let \( \mathcal{N}' = \{ |w/m'|, \lfloor w/m' \rfloor + 1 \mid w \in \mathcal{N} \} \). To compare the weights of any two pointers it is sufficient to know (i) the tree topology (ii) for every leaf \( u_t \), the document \( d \) whose suffix is stored in \( u_t \) (iii) for every \( \text{ptr}(\nu', d) \), the pair \((\text{rank}([w/m'],\mathcal{N}'),w \mod m')\) where \( w \) is the weight of \( \text{ptr}(\nu', d) \). There are \( o(n/\log n) \) possible combinations of tree topologies and possible pairs \((\text{rank}([w/m'],\mathcal{N}'),w \mod m')\). Hence, we can store answers to all possible queries for all microtrees in a global look-up table of size \( o(n) \) bits.

The topology of a microtree can be stored in \( O(m') \) bits. We can specify the index of the document \( d \) stored in a leaf \( u_t \) with \( \log D \) bits. We can specify each pair \((\text{rank}([w/m'],\mathcal{N}'),w \mod m')\) with \( O(\log m') \) bits. Since \( D = O(m'/\log m') \), information from item (iii) can be stored in \( O(m') \) bits. Thus each microtree can be stored in \( O(m' \log D) \) bits if \( \log m' = \omega(\log \sigma + \log D) \).
Summing up, our data for a minitree uses \(O(m(\log \sigma + \log D))\) bits. Therefore the total space usage is \(O(n(\log \sigma + \log D))\) bits.

A query for a pattern \(P\) is answered by locating the locus \(u\) of \(P\). If \(u\) is a fat node in \(T\), the query is answered by a data structure for the contracted grid. If \(u\) belongs to a minitree \(T\) and \(u\) is a minifat node, we answer the query by employing the data structure for the contracted grid of \(T\). If \(u\) belongs to a microtree \(T_v\), the query is answered either by a microgrid data structure or by a table look-up.

**Theorem 8** Let \(D\) be a collection of strings over an integer alphabet \([1, \sigma]\) with total length \(n\), and let \(tf(P,d)\) denote the number of occurrences of \(P\) in \(d\). Then there exists an \(O(n(\log D + \log \sigma))\) bit data structure that, given a string \(P\) and an integer \(k\), reports \(k\) documents \(d\) containing \(P\) with highest \(tf(P,d)\) values, in decreasing order of \(tf(P,d)\), in \(O(p/\log \sigma n + \log^\varepsilon n \log \sigma + k)\) time, for any constant \(\varepsilon > 0\).

### 7 Parameterized Top-\(k\) Queries

In this section we improve a recent data structure that supports two-dimensional top-\(k\) queries [48, Sec. 5]. The structure is similar to our wavelet tree \(W\) described in the proof of Lemma 4. In addition, for the points stored at any node of \(W\), it stores an RMQ data structure that gives in constant time the position of the point with maximum weight within any interval. As explained, this structure [25] uses \(O(t)\) bits if the node of \(W\) handles \(t\) points, and thus the total space of this extended wavelet tree \(W\) is \(O(n)\) words for an \(O(n) \times O(n)\) grid.

They [48] show how to support top-\(k\) queries in a general interval \([a, b] \times [c, d]\) by first identifying the \(O(n)\) nodes \(v \in W\) that cover \([c, d]\), mapping the interval \([a, b]\) to \([a_v, b_v]\) in all those nodes \(v\), and setting up a priority queue with the maximum-weight point of each such interval. Now, they repeat \(k\) times the following steps: (i) extract the maximum weight from the queue and report it; (ii) replace the extracted point, say \(x \in [a_v, b_v]\), by two points corresponding to \([a_v, x - 1]\) and \([x + 1, b_v]\), prioritized by the maximum weight in those ranges.

Their total time is \(O((k + \log n) \log n)\) if using linear space. The \(O(\log n)\) extra factor is due to the need to traverse \(W\) in order to find out the real weights, so as to compare weights from different nodes. However, those weights can be computed in time \(O(\log^\varepsilon n)\) and using \(O(n \log n)\) extra bits [15, 52, 14]. The operations on the priority queue can be carried out in \(O(\log log n)\) time [60]. Thus we have the following result.

**Lemma 7** Given a grid of \(n \times n\) points, there exists a data structure that uses \(O(n)\) words of space and reports \(k\) most highly weighted points in a range \(Q = [a, b] \times [c, d]\) in \(O((k + \log n) \log^\varepsilon n)\) time, for any constant \(\varepsilon > 0\). The structure is built in \(O(n \log n)\) time.

Note this technique automatically admits being used in online mode (i.e., without knowing \(k\) in advance), since we have not made use of \(k\) to speed up the priority queue as in previous sections. We can easily stop the computation at some \(k\) and resume it later.

#### 7.1 Limited Three-Dimensional Queries

In this section we slightly extend the scenario considered above. We assume that each point has an additional coordinate, denoted \(z\), and that \(p.z \leq \log^\alpha n\) for a constant \(\alpha > 0\). Top-\(k\) points in a
three-dimensional range \([a, b] \times [c, d] \times [\beta, \gamma]\) must be reported sorted by their weights. Such queries will be further called \textit{limited three-dimensional top-}\(k\) \textit{queries}. We can obtain the same result as in Lemma 7 for these queries.

Instead of a binary wavelet tree, we use a multiary one \cite{23}, with node degree \(\log^\epsilon n\) and height \(O(\log n / \log \log n)\). Now each node \(v \in W\) has associated a vector \(B_v\) so that \(B_v[i]\) contains the index of the child in which the \(i\)-th point of \(v\) is stored. Using \(B_v\) and some auxiliary data structures, we can obtain the weight of any point at any node in \(O(\log^\epsilon n)\) time \cite{52}. All vectors \(B_v\) and the extra data structures use \(O(n)\) words.

We regard the \(t\) points of each node \(v\) as lying in a two-dimensional grid of \(x\)- and \(z\)-coordinates. Instead of one-dimensional RMQs on the \(x\)\(-coordinates \([a_v, b_v]\), we issue two-dimensional RMQs on \([a_v, b_v] \times [\beta, \gamma]\). The wavelet tree of the basic two-dimensional RMQ data structure \cite{48} handles \(n \times m\) grids in \(O(n \log m)\) bits of space and answers RMQs in time \(O(\log^2 m)\). In our case \(m < \log^\alpha n\) and thus the space is \(O(n \log \log n)\) bits and the query time is \(O((\log \log n)^2)\). Thus the space of the two-dimensional data structures is of the same order of that used for vectors \(B_v\), adding up to \(O(n)\) words. As RMQs are built in linear time, the construction time is \(O(n \log n)\).

Now we carry out a procedure similar to that of the two-dimensional version. The range \([a, b]\) is covered by \(O(\log^{1+\epsilon} n / \log \log n)\) nodes. We obtain all their (two-dimensional) range maxima, insert them in a priority queue, and repeat \(k\) times the process of extracting the highest weight and replacing the extracted point \(x \in [a_v, b_v]\) by the next highest weighted point in \([a_v, b_v]\) (thus we are running these range maxima queries in online mode).

The two-dimensional RMQ structures at nodes \(v\) cannot store the absolute weights within overall linear space. Instead, when they obtain the \(x\)-coordinate of their local grid, this coordinate \(x_v\) is mapped to the global \(x\)-coordinate in \(O(\log^\epsilon n)\) time, using the same technique as above. Then the global array of weights is used. Hence these structures find a two-dimensional maximum weight in time \(O(\log^2 n (\log \log n)^2)\). This is repeated over \(O(\log^{1+\epsilon} n / \log \log n)\) nodes, and then iterated \(k\) times. The overall time is \(O((k + \log^{1+\epsilon} n / \log \log n) \log^\epsilon n (\log \log n)^2)\), which is of the form \(O((k + \log n) \log^\epsilon n)\) by adjusting \(\epsilon\). The times to handle the priority queue are negligible \cite{60}.

\textbf{Lemma 8} \textit{Given a grid of} \(n \times n \times \log^\alpha n\textit{ points, for a constant }\alpha > 0\textit{, there exists a data structure that uses }O(n)\textit{ words of space and reports }k\textit{ most highly weighted points in a range }Q = [a, b] \times [c, d] \times [\beta, \gamma]\textit{ in }O((k + \log n) \log^\epsilon n)\textit{ time, for any constant }\epsilon > 0\textit{. It is built in }O(n \log n)\textit{ time.}

Again, this result holds verbatim in online mode.

\subsection{The Final Result}

We divide the grid into horizontal stripes of height \(r = \lfloor \log^{c+1+\epsilon} n \rfloor\) for any constant \(c\), much as in Section \ref{sec:slim}. We store a data structure for limited three-dimensional top-\(k\) queries for each slim grid, taking \(y\) as the limited coordinate. A query \([a, b] \times [0, b] \times [\tau_1, \tau_2]\) is processed just as in Section \ref{sec:slim} with the only difference that the queries \([a, b] \times [\tau_1, \tau_2] \times [0, y']\) to the local grids now require \(O(\log^{1+\epsilon} n)\) initialization time and then \(O(\log^\epsilon n)\) time per element retrieved, according to Lemma 8. Then, we initialize our global query \(Q\) in time \(O(\lfloor h/r \rfloor \log^{1+\epsilon} n) = O(h/\log^{c+1+\epsilon} n)\), and then extract each new result in time \(O(\log^\epsilon n)\). The time of the priority queue is blurred by adjusting \(\epsilon\). Hence, the total query time is \(O(h/\log^\epsilon n + (k + \log n) \log^\epsilon n)\), and Theorem 3 is proved.
8 Conclusions

We have presented an optimal-time and linear-space solution to top-$k$ document retrieval, which can be used on a wide class of relevance measures and subsumes in an elegant and uniform way various previous solutions to other ranked retrieval problems. We have also presented dynamic variants, space-reduced indexes, and structures that solve extensions of the basic problem. The solutions reduce the problem to ranked retrieval on multidimensional grids, where we also present improved results, some tailored to this particular application, some of more general interest.

After the publication of the conference version of this article \[47\], Shah et al. \[58\] showed how to achieve the optimal $O(k)$ time once the locus of $P$ is known. This is in contrast to our original result, where we used time $O(p + k)$ after having spent time $O(p)$ to find the locus. Their improvement allows one to use these techniques in other scenarios where the locus is obtained in some other way, without the need to search for it directly using $P$. They also extend the results to the important case of the external memory scenario. The new results we obtain in this article about how to search the suffix tree in RAM-optimal time $O(p/\log \sigma n)$, and how to handle the dynamic scenario, nicely complement those results and add up to a rather complete solution to the problem.

There are several relevant research directions, on which we comment next.

**RAM optimality.** In our previous conference version we had achieved time $O(p + k)$, which was optimal only in the comparison model (although we used RAM-based techniques). Now we have improved this result to $O(p/\log \sigma n + k)$, which is optimal in general in the RAM model (considering the case $\lg D = \Theta(\log n)$), because it is the size in words of the input plus the output. Achieving $O(p/\log \sigma n)$ time on the suffix tree, without any polylogarithmic additive penalty, is an interesting result by itself, and we have obtained it without altering the topology of the suffix tree (which is crucial for the invariants of Hon et al. \[37\] to work). However, we do not know if our solution is optimal when there are very few distinct documents, $\lg D = o(\log n)$. The question of whether $O(p/\log \sigma n + k/\log D n)$ time can be achieved is still open.

**Construction time.** Without considering the cost to compute weights $w(path(u), d)$ for all pointers $ptr$ in the suffix tree, the construction time of Hon et al. \[37\] (which achieves suboptimal query time) is $O(n)$. The time to build our grid structure is $O(n \log n)$, to which we must add $O(n \log^\varepsilon n)$ randomized time to achieve RAM-optimal search time in the suffix tree traversal (or $O(n \text{polylog } n)$ deterministic time for a weaker version of it). Is it possible to achieve linear, or at least $O(n \log n)$, deterministic construction time for our data structures?

**Dynamic optimality.** In our dynamic variant, the static RAM-optimal search time in the suffix tree becomes $O(p(\log \log n)^2/\log \sigma n + \log n)$. There are schemes that do better for large $\sigma$, for example $O(p + (\log \log \sigma)^2)$ time \[24\]. Although they do not support deletions yet, this seems to be possible. On the other hand, we obtained $O(\log^{1+\varepsilon} n)$ update time per symbol. A general question is, which is the best search time we can obtain in the dynamic scenario?

**Practical results.** Our solutions are not complex to implement and do not make use of impractical data structures. A common pitfall to practicality, however, is space usage. Even achieving linear space (i.e., $O(n \log n)$ bits) can be insufficient. We have shown that our structure can use,
instead, \( O(n(\log \sigma + \log D)) \) bits for the \( tf \) measure (and slightly more for others), but the constants are still large. There is a whole trend of reduced-space representations for general document retrieval problems with the \( tf \) measure \[57\] \[63\] \[37\] \[19\] \[31\] \[9\] \[29\] \[61\] \[36\] \[49\]. The current situation is as follows \[46\]: One trend aims at the least space usage. It has managed to use just \( D \log(n/D) + O(D) + o(n) \) bits on top of a compressed suffix array of the collection, and the best time complexity it has achieved is \( O(p + k \log^2 k \log^{1+\varepsilon} n) \) for any constant \( \varepsilon > 0 \) \[49\]. Another trend adds to the space the so-called document array \[45\], which uses \( n \log D + o(n \log D) \) bits and enables faster solutions. Currently the fastest one achieves \( O(p + k(\log \sigma \log \log \log n))^{1+\varepsilon} \) time \[34\]: a recent unpublished result \[50\] obtains time \( O(p + \log^\ast k) \). This is very close to optimal, but not yet our \( O(p/\log \sigma n + k) \) time.

In practice, the most compact implementation in this trend \[51\] reaches about 1–2 times the text size (including a representation of the text) and retrieves each of the top-\( k \) results within milliseconds. An implementation of the ideas we propose in this article \[40\] makes use of the fact that, under very general probabilistic models, the average height of the suffix tree (and hence of our grids) is \( O(\log n) \) \[59\]. This enables a simple implementation of our grid-based index that uses 2–3 times the text size and, although its average query time, \( O(p + (k + \log \log n) \log \log n) \), is not optimal, it returns each answer within microseconds \[40\].

**More complex queries.** In the long term, the most interesting open questions are related to extending the one-pattern results to the bag-of-words paradigm of information retrieval. Our model easily handles single-word searches, and also phrases (which is quite complicated with inverted indexes \[65\] \[6\], particularly if their weights have to be computed). Handling a set of words or phrases, whose weights within any document \( d \) must be combined in some form (for example using the \( tf \times idf \) model) is more challenging. We are only aware of some very preliminary results for this case \[16\] \[30\] \[35\], which suggest that it is unlikely that strong worst-case results can be obtained. Instead, one can aim at complexities related to the results achieved with inverted lists on the simpler natural language model. It is interesting to note that our online result allows simulating the left-to-right traversal, in decreasing weight order, of the virtual list of occurrences of any string pattern \( P \). Therefore, for a bag-of-word queries, we can emulate any algorithm designed for inverted indexes which stores those lists in explicit form \[53\] \[3\], therefore extending any such technique to the general model of string documents.

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