Nonstandard caustics for localized solutions of the 2d shallow water equations with applications to wave propagation and run-up on a shallow beach

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Abstract. The usual semiclassical approximation is not directly applicable to linearized two-dimensional shallow water equations with localized initial data describing long waves (for example, tsunami waves) in a bounded basin of variable depth, because the momentum variables on the geometric optics rays become infinite at the boundary of the region. To obtain asymptotic solutions, one needs to extend the phase space and introduce a modified Maslov canonical operator. The paper gives a brief survey of some results obtained in this direction by the authors and their colleagues.

1. Introduction and statement of the problem
The origin, propagation, and run-up on the coast of long ocean waves are the subject of many publications (e.g., see the monographs [1, 2]). One important model is the system of shallow water equations. Consider a basin of variable depth $D(x, x_2) \in \mathbb{R}^2$, filled with water, which occupies the domain $\Omega = \{x \in \mathbb{R}^2: D(x) > 0\}$ when in equilibrium. We will assume that $D(x)$ is $C^\infty$ and $\nabla D(x) \neq 0$ for $x \in \partial \Omega = \{x \in \mathbb{R}^2: D(x) = 0\}$. The nonlinear shallow water equations for the free surface elevation $\eta = \eta(x, t)$ and the horizontal flow velocity $u = u(x, t)$ have the form (assuming the Earth is flat and neglecting Coriolis forces, bottom friction, etc.)

$$u_t + (u, \nabla)u + g\nabla \eta = 0, \quad \eta_t + (\nabla, (\eta + D(x))u) = 0,$$

where $g$ is the acceleration due to gravity, $\nabla = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})^T$, and $(a, b) = a_1 b_1 + a_2 b_2$. These equations are considered in the time-varying domain $\Omega(t) = \{x \in \mathbb{R}^2: D(x) + \eta(x, t) > 0\}$. Let us supplement them with the initial conditions

$$u|_{t=0} = 0, \quad \eta|_{t=0} = AV\left(\frac{x-x_0}{l}\right),$$

which correspond to the piston model of tsunami wave generation by an instantaneous vertical shift of the bottom over a seismic source. Here $x_0$ is the source location (assumed to be an interior point of $\Omega$), $V(y)$ is a rapidly decaying function determining the initial shape of the free
surface elevation, $A$ is the initial wave height, and $l$ is the horizontal dimension of the source. As a first step in the analysis of the nonlinear system (1), consider the linearized system

$$
u_t + g \nabla \eta = 0, \quad \eta_t + \langle \nabla, D(x) \nu \rangle = 0$$

in the “unperturbed” domain $\Omega$. In fact, we have strong reasons to believe that the solution of problem (3), (2) can be used to obtain a good first approximation to the solution of the nonlinear problem (1), (2) by a relatively simple procedure. Let us very briefly explain why. First, the tsunami wave amplitude in the open ocean is rather small in most cases, and the passage to the linearized system is natural as long as we are far from the boundary $\partial \Omega$ (that is, from the coast). Second, the following series of results suggests that linearization combined with some transformation may also work near $\partial \Omega$. Consider the 1d shallow water equations

$$u_t + uu_x + \eta_x = 0, \quad \eta_t + \left[ (\eta + D)u \right]_x = 0$$

with the bottom profile $D(x) = x$. The Carrier–Greenspan transformation [3]

$$x = z - N + \frac{1}{2} U^2, \quad t = s + U, \quad u = U, \quad \eta = N - \frac{1}{2} U^2$$

reduces Eqs. (4) to the linear system

$$U_s + N_z = 0, \quad N_s + zU_z = 0;$$

formally, this transformation amounts to discarding the nonlinear terms (see the discussion in [4]). Further, it was shown in [5] for the one-dimensional case that even if the bottom profile is nonlinear, there exists a transformation of this kind (which is now asymptotic) allowing one to obtain an asymptotic solution of the nonlinear problem from an asymptotic solution of the linear problem (assuming no wave breakup). In the two-dimensional case, we observe that the characteristics always approach the coast at the right angle. Passing locally to the new variables $(\tilde{x}, \tilde{y})$, where $\tilde{x}$ is the coordinate along the normal to the shore and $\tilde{y}$ is the coordinate along the coast, and freezing the variable $\tilde{y}$, one can transform the asymptotic solution of the linear problem into an asymptotic solution of the nonlinear one with the help of a one-dimensional transformation of the variable $\tilde{x}$ similar to that written above. This scheme has so far been implemented at the physical level of rigor.

Now let us consider only potential solutions of system (3), (2). Then the linearized shallow water equations can be reduced to the two-dimensional wave equation for the free surface elevation $\eta(x, t)$. Thus, we consider the following Cauchy problem for the wave equation with velocity $c(x) = \sqrt{gD(x)}$:

$$\eta_{tt} - \langle \nabla, c^2(x) \nabla \rangle \eta = 0, \quad \eta|_{t=0} = \eta_0(x) := V\left(\frac{x - x_0}{l}\right), \quad \eta_t|_{t=0} = 0,$$

where, the problem being linear, we have omitted the factor $A$ from the initial conditions. The linear problem is considered in the domain $\Omega$. Now the question is whether any boundary conditions on $\partial \Omega$ are needed. It follows from the results in [6] that, since the velocity $c(x)$ vanishes on the boundary, one cannot make the problem well-posed by posing any classical boundary conditions, like the Dirichlet or Neumann conditions, on $\partial \Omega$. Instead, one imposes the requirement of finiteness of the energy integral

$$J^2(t) \equiv \frac{1}{2} \langle \nabla \eta, c^2(x) \nabla \eta \rangle + \frac{1}{2} \| \eta \|^2 < \infty,$$

where the inner product and the norm are taken in the space $L^2(\Omega)$. 
In practice, the width \( l \) of the initial perturbation is usually small compared to the characteristic linear dimension \( L \) of the basin itself. For example, typical dimensions for tsunami waves might be \( l \approx 50–100 \text{ km} \) and \( L \approx 1000–2000 \text{ km} \), which leaves us with the small ratio \( h := l/L \approx 0.1–0.025 \). It is therefore natural to seek the solution in the form of asymptotics with respect to the small parameter \( h \).

The model described by problem (6) assumes an instantaneous source. If the action of the source is stretched in time, then the initial conditions become zero, but there arises a right-hand side; that is, the wave equation becomes inhomogeneous:

\[
\eta_{tt} - \langle \nabla, c^2(x) \nabla \rangle \eta = F(x, t), \quad \eta|_{t=0} = 0, \quad \eta|_{t=0} = 0. \tag{7}
\]

We take

\[
F(x, t) = \frac{1}{\tau} g^\prime \left( \frac{t}{\tau} \right) \int \nabla V \left( \frac{x-x_0}{l} \right),
\]

where \( \tau \) is the characteristic time of the source and \( g(\zeta) \) is a smooth function decaying as \( \zeta \to \infty \) and satisfying the condition \( \int_0^\infty g(\zeta) d\zeta = 1 \); this condition guarantees the convergence \( \tau^{-2} g'(t/\tau) \to \delta'(t) \) as \( \tau \to +0 \), which in turn means the transition of a time-stretched source into an instantaneous one. We assume that the ratio \( l/(\tau c(x_0)) \) is of the order of unity. Under these conditions, it was shown in [7] that the solution of problem (7) asymptotically is the sum of two parts, the transient part and the propagating part. The first of them decays rapidly with time and is supported in a neighborhood of \( x_0 \). The second part is the sum \( \eta_1 + \eta_2 t \), where \( \eta_1 \) and \( \eta_2 \) are the solutions of problem (6) in which the function \( V \) in the initial condition is replaced by some functions \( V_1 \) and \( V_2 \), respectively; these functions are called *equivalent instantaneous sources* and are expressed in closed form via \( V \) and \( g \). (The corresponding formulas, along with a formula for the transient part, can be found in [7].)

This, in what follows we deal with the case of an instantaneous source. For convenience, we nondimensionalize the variables, multiply the equation by the small parameter \( h^2 \), and rewrite problem (6) in the “semiclassical” form with small parameter \( h \),

\[
-h^2 \eta_{tt} - \langle -ih \nabla, c^2(x)(-ih \nabla) \rangle \eta = 0, \quad \eta|_{t=0} = \eta_0(x) := V \left( \frac{x-x_0}{h} \right), \quad \eta|_{t=0} = 0. \tag{8}
\]

2. Asymptotic solution away from the coast

Assume for now that the domain \( \Omega \) is the entire plane \( \mathbb{R}^2 \). Then the asymptotic solution of problem (8) can be obtained with the use of the standard canonical operator (e.g., see [8] and also [9] for new, computationally efficient formulas, where all relevant definitions can be found). Let us briefly describe the construction of the asymptotic solution. The Hamiltonian of the equation in (8) is

\[
H(x, p) = c(x)|p|, \quad \text{where} \quad |p| = (p_1^2 + p_2^2)^{1/2}. \tag{9}
\]

It is a function defined on the phase space \( \mathbb{R}^4 \) with coordinates \( (x, p) = (x_1, x_2, p_1, p_2) \), the variables \( p_j \) being the dual momenta to the corresponding \( x_j \). We construct a Lagrangian manifold \( \Lambda \subset \mathbb{R}^4 \) invariant under the Hamiltonian vector field

\[
V_H = \left\langle H_p(x, p), \frac{\partial}{\partial x} \right\rangle - \left\langle H_x(x, p), \frac{\partial}{\partial p} \right\rangle \tag{10}
\]

as follows. Let \( \Lambda_0 \) be the circle

\[
\Lambda_0 = \{(x, p) \in \mathbb{R}^4 : x = x_0, \ |p| = 1 \}. \tag{11}
\]
The set $\Lambda_0$ can be parameterized by the coordinate $\phi \in \mathbb{R}$ mod $2\pi$ as follows:

$$x = x_0, \quad p = n(\phi), \quad \text{where} \quad n(\phi) = \begin{pmatrix} \cos \phi \\ \sin \phi \end{pmatrix}. $$

The field (10) is nowhere tangent to $\Lambda_0$, and so the union $\Lambda$ of all trajectories of the field (10) issued from $\Lambda_0$ is a two-dimensional $C^\infty$-manifold immersed (i.e., locally embedded) in $\mathbb{R}^2$. The functions $\alpha = (t, \phi)$, where $t$ is the time along the trajectories of the vector field (10), serve as natural coordinates on $\Lambda_0$. We denote the functions defining the immersion $\Lambda \to \mathbb{R}^3$

$$x = X(\phi, t), \quad p = P(\phi, t). \quad (12)$$

The manifold $\Lambda$ is Lagrangian, and the function $\tau(\phi, t) = c(x_0)\tau$ is an action on it (i.e., satisfies the Pfaff equation $d\tau(\phi, t) = (P(\phi, t), dX(\phi, t))$). Let us construct the Maslov canonical operator $K_{(\alpha, dp)}^h$ on the Lagrangian manifold $\Lambda$ with the measure $d\mu = d\phi \wedge dt$ and the initial point $\alpha_0 = (\phi_0, t_0)$, where $\phi_0 = 0$ and $t_0 > 0$ is arbitrarily small.

It was proved in [10] as a development of a formula invented in [11] that the initial condition in (8) can be represented modulo lower-order terms via the canonical operator $K_{(\alpha, dp)}^h$ by the formula

$$V\left(\frac{x}{h}\right) = \sqrt{\frac{h}{2\pi}} e^{-i\pi/4} \int_0^\infty [K_{(\alpha, dp)}^{h/p}(\tilde{V}(p\mu(\phi))e(\tau))] \sqrt{p} d\rho,$$

where $e(\tau)$ is a cutoff function equal to unity in a neighborhood of zero and

$$\tilde{V}(p) = \left(\frac{1}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} e^{-i(p, y)} V(y) dy$$

is the Fourier transform of $V(y)$. Accordingly, by asymptotically solving the Cauchy problem with the initial data given by the integrand in (13) and by integrating the result, one obtains the asymptotic solution of the Cauchy problem (8) in the form

$$\eta(x, t) = \sqrt{\frac{h}{2\pi}} \text{Re} e^{-i\pi/4} \int_0^\infty e^{-\frac{i\rho}{h}c_0t}[K_{(\alpha, dp)}^{h/p}(\tilde{V}(p\mu(\psi))e(\tau - c_0t))] \sqrt{p} d\rho,$$

where $c_0 = c(x_0)$.

3. Phase space of the degenerate problem

Now recall that actually the problem is not in entire $\mathbb{R}^n$ but in the domain $\Omega$. Thus, our phase space is actually $T^*\Omega = \{(x, p) : x \in \Omega, p \in \mathbb{R}^2\}$. One can readily see that the trajectories (12) of the Hamiltonian vector field (10) go to infinity in the variables $p$ in finite time while approaching the boundary in the variables $x$. Thus, the boundary $\partial \Omega$ is some special sort of nonstandard caustic for the problem in question. The formulas described so far do not give the solution at the time when the trajectories hit the boundary or after that. The remedy is to define a broader phase space in which the trajectories are infinitely extendable and then define the canonical operator on Lagrangian manifolds of that phase space.

The phase space $\Phi$ of problem (6) was introduced in [12]. It is a smooth manifold equipped with a smooth mapping (projection) $\pi : \Phi \to \overline{\Omega}$ onto the closure $\overline{\Omega} = \Omega \cup \partial \Omega$ of the open domain $\Omega$ such that $\pi : \pi^{-1}(\Omega) \to \Omega$ is (naturally isomorphic to) the usual cotangent bundle $T^*\Omega$ of $\Omega$. Let us describe the structure of $\Phi$ in a neighborhood of the set $\Phi_{\infty} = \pi^{-1}(\partial \Omega)$. Consider an arbitrary point $x^* = (x^*_1, x^*_2) \in \partial \Omega$. The domain $\Omega$ can be defined in a neighborhood of $x^*$ by one of the four inequalities $\pm x_j > f(x_{3-j}), j = 1, 2$, where $f(y)$ is a smooth function. Consider the case
of the inequality $x_1 > f(x_2)$; the other cases are similar. Set $U = \{(x_1, x_2) \in \mathbb{R}^2 : |x_2 - x_2^0| < \delta,$ $f(x_2) < x_1 < f(x_2) + \delta\}$, where $\delta > 0$ is sufficiently small. Then the set $\pi^{-1}(U) \subset T^*\Omega$ inherits the standard coordinates $(x, p) = (x_1, x_2, p_1, p_2)$ from $T^*\Omega$, where $(x_1, x_2)$ are the coordinates on $\Omega$ and $(p_1, p_2)$ are the dual momenta. Let us make the change of variables

$$\theta = x_1 - f(x_2)p_1^2, \quad y = x_2, \quad q = -1/p_1, \quad \eta = p_2 + f'(x_2)p_1$$

on $\pi^{-1}(U) \setminus \{p_1 = 0\}$. The new coordinates vary in the set $V = \{\{(\theta, y, q, \eta) : q \neq 0, |y - x_2^0| < \delta, 0 < \theta < \delta/q^2\}$, and the inverse change of variables is given by the formulas

$$x_1 = f(y) + q^2 \theta, \quad x_2 = y, \quad p_1 = -1/q, \quad p_2 = \eta + f'(y)/q.$$  

Let $\tilde{V} = \{\{(\theta, y, q, \eta) : |y - x_2^0| < \delta, 0 < \theta < \delta/q^2\}$. The projection $\pi : V \to \Omega$ is defined by the first two formulas in (16) and extends by continuity to a projection $\tilde{\pi} : \tilde{V} \to \partial \Omega$. The points of $\tilde{V}$ mapped by $\pi$ into $\partial \Omega$ are exactly the points with $q = 0$. Now the “interior” chart $(T^*\Omega, (x, p))$ and all possible charts $(\tilde{V}, (\theta, y, q, \eta))$ of the type described above with the transition maps (15), (16) form a differentiable atlas on $\Phi$. (One can readily verify that the naturally defined transition maps between charts of the latter type are smooth.) Note that the change of variables (15) preserves the symplectic structure, $dp_1 \wedge dx_1 + dp_2 \wedge dx_2 = dq \wedge d\theta + dq \wedge dy$. Thus, $\Phi$ is naturally a symplectic manifold. Further, the Hamiltonian $H(x, p) = c(x)|p|$, where $c^2(x)$ is a smooth function having a simple zero on $\partial \Omega$, extend by continuity from $T^*\Omega$ to a smooth function on $\Phi \setminus \{(x, p) \in T^*\Omega : p = 0\}$, and the trajectories of the corresponding Hamiltonian vector field are infinitely extendable in both directions, as desired.

4. Asymptotic solution near the coast
We see that the trajectories of the Hamiltonian system issuing from $\Lambda_0$ extend infinitely in the space $\Phi$, and so now we can define the Lagrangian manifold $\Lambda \subset \Phi$ that is the union of these trajectories. It is still an immersed submanifold, and the coordinates on it are $\alpha = (\phi, t), \phi \in \mathbb{R} \mod 2\pi, t \in (-\infty, \infty)$. We denote the functions determining the immersion $\Lambda \to \Phi$ in the coordinates $(\theta, y, q, \eta)$ by

$$\theta = \Theta(\phi, t), \quad y = Y(\phi, t), \quad q = Q(\phi, t), \quad \eta = \Xi(\phi, t).$$

The modified canonical operator on Lagrangian submanifolds in $\Phi$ was defined in [13] based on the above-described construction [12] of the nonstandard phase space corresponding to such equations and on Fock’s quantization of canonical transformations [14]. We do not present this construction here owing to space limitations and refer the reader to [13]. Once the modified canonical operator has been defined, the asymptotic solution of the Cauchy problem is given by the same formula (14), which, however, now has a different meaning: the modified canonical operator rather than the standard one is used in that formula.

Let us write out the asymptotic solution at a generic point of the boundary for the case of a so-called simple source for which the source shape function has the form $V(y) = W(T_\psi y)$, where

$$T_\psi = \begin{pmatrix} \cos \psi & \sin \psi \\ -\sin \psi & \cos \psi \end{pmatrix}$$

is the matrix of rotation by the angle $\psi$ and

$$W(y) = \frac{A}{1 + (y_1/b_1)^2 + (y_2/b_2)^2}^{3/2}, \quad b_1 \geq b_2 > 0.$$
This source shape is very convenient in computations, because the Fourier transform of \( V(y) \) is given by
\[
\tilde{V}(pm(\phi)) = Ab_1b_2e^{-\rho\beta(\phi)}, \quad \beta(\phi) = \sqrt{b_1^2\cos^2(\phi - \psi) + b_2^2\sin^2(\phi - \psi)}.
\]

Let \( x^* \in \partial\Omega \), and let \( \alpha^* = (\phi^*, t^*) \) be a point such that \( X(\phi^*, t^*) = x \). Then \( \alpha^* \) necessarily belongs to the set \( \Lambda = \Lambda \cap \Phi_{\infty} \) of special focal points on \( \Lambda \) (i.e., points that are projected into the nonstandard caustic \( \partial\Omega \)). These points are indeed focal, because \( \det \frac{\partial X}{\partial \alpha}(\alpha^*) = 0 \) and hence the functions \( (X_1(\alpha), X_2(\alpha)) \) do not form a system of coordinates on \( \Lambda \) in a neighborhood of \( \alpha^* \).

There are two possibilities (not mutually excluding): either \( (Q(\alpha), Y(\alpha)) \) or \( (Q(\alpha), \Xi(\alpha)) \) is a system of local coordinates on \( \Lambda \) in a neighborhood of \( \alpha^* \). If the first possibility holds, then the point \( \alpha^* \) is a \textit{regular focal point}. Now we will write out the values of the asymptotic solution on \( \partial\Omega \) near \( x^* \) assuming that \( \alpha^* \) is regular focal point.\(^1\)

Assume that the domain is locally defined by the inequality \( x_1 > f(x_2) \) in a neighborhood of the point \( x^* \). Then the free surface elevation near \( x^* \) has the asymptotics
\[
\eta(f(x_2), x_2, t) \approx \frac{\sqrt{2D_0A}}{\sqrt{\tan \Gamma} \{X_{\phi}\}^{1/2}} \Re \left( e^{-i\pi m/2} \right) \frac{b_1l_2}{(I - ic_0(T(x_2) - t))^2}.
\]

Here

- The \( l_j = b_j\mu \) are the semiaxes of the source.
- \( I = \sqrt{b_1^2\cos^2(\phi - \psi) + b_2^2\sin^2(\phi - \psi)} \).
- \( D_0 = D(x_0) \) is the depth at the source point, and \( c_0 = \sqrt{gD_0} \).
- \( A \) is the source amplitude.
- \( m \) is the Maslov–Morse index of the trajectory whose projection arrives at the point \( (f(x_2), x_2) \), i.e., the number of focal points on this trajectory, including the endpoint itself.
- \( \tan \Gamma \) is the slope of the bottom at the point \( (f(x_2), x_2) \).
- \( T(x_2) \) is the time at which the projection of the corresponding trajectory arrives at the point \( (f(x_2), x_2) \).

Let \( a = l_2/l_1 \) be the eccentricity of the source in question. Then the maximum modulus of the free surface elevation at the point \( (f(x_2), x_2) \) is
\[
\eta_{\text{max}} = \begin{cases} 
\frac{\sqrt{2D_0A}}{\sqrt{\tan \Gamma} \{X_{\phi}\}^{1/2}} \cos^2(\phi - \psi) + a^2\sin^2(\phi - \psi) & \text{for even } m, \\
\frac{9}{8\sqrt{3}} \frac{\sqrt{2D_0A}}{\sqrt{\tan \Gamma} \{X_{\phi}\}^{1/2}} \cos^2(\phi - \psi) + a^2\sin^2(\phi - \psi) & \text{for odd } m. 
\end{cases}
\]

A similar formula is given in [15] for the free surface elevation near the boundary.

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\(^1\) Note that there can be several points of \( \Lambda \) projected into \( x^* \). However, there are only finitely many of them for each \( t \), and the asymptotics of the solution near \( \alpha^* \) is the sum of contributions of neighborhoods of these focal points. Here we write out the contribution of one point.
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