New Results on $k$-independence of Hypergraphs

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Abstract
Let $H = (V, E)$ be an $s$-uniform hypergraph of order $n$ and $k \geq 0$ be an integer. A $k$-independent set $S \subseteq V$ is a set of vertices such that the maximum degree in the hypergraph induced by $S$ is at most $k$. The maximum cardinality of a $k$-independent set of $H$ is denoted by $\alpha_k(H)$. In this paper, we give a lower bound of $\alpha_k(H)$ for $H$ in terms of its maximum degree. Furthermore, we prove for all $k \geq 0$ that $\alpha_k(H) \geq \frac{s(k+1)n}{2d+s(k+1)}$, where $d$ is the average degree of $H$.

Keywords  $s$-uniform hypergraphs · $k$-independent set

Mathematics Subject Classification 05C65 · 05C69

1 Introduction
Let $G = (V, E)$ be a graph on $n$ vertices and $k \geq 0$ be an integer. A $k$-independent set $S \subseteq V$ is a set of vertices such that the maximum degree in the graph induced by $S$ is at most $k$. The $k$-independence number of a graph $G$ is the maximum cardinality of a $k$-independence set of $G$ and is denoted by $\alpha_k(G)$. In particular, $\alpha_0(G) = \alpha(G)$ is the usual independence number of $G$.

In recent years, as a generalization of the independence number of graphs, the $k$-independence number of graphs has attracted considerable attention of researchers. The first result on bounding the $k$-independence number was given by Caro et al. [6], where it was shown that if the average degree $d \geq k + 1$, then $\alpha_k(G) \geq \frac{k+2}{2(d+1)}n$. Furthermore, Caro et al. [5] showed that $\alpha_k(G) \geq \frac{(k+1)}{\lfloor d \rfloor + k + 1}n$, where $d$ is the average degree.

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degree of $G$. Recently, Kogan [10] proved that $\alpha_k(G) \geq \frac{(k+1)}{d+k+1}n$ for general $k$, thus improving the previous best bound obtained by Caro et al. In this paper we consider the extension of $k$-independence to hypergraphs. We begin with some notation from [1] and [3].

A hypergraph $H$ is a pair $H = (V, E)$ where $V$ is a set of elements, and $E$ is a set of non-empty subsets of $V$. Therefore, $E$ is a subset of $\mathcal{P}(V)\setminus\{\emptyset\}$, where $\mathcal{P}(V)$ is the power set of $V$. The elements of $V$ are referred as vertices, and $n(H) := |V|$ denotes the cardinality of $V$, while the elements of $E$ are called edges, and $e(H) := |E|$. The rank $r$ of a hypergraph $H = (V, E)$ is the maximum size of an edge in $E$. A hypergraph is said to be $s$-uniform if each edge contains precisely $s$ vertices of $V$, where $s$ is a positive integer and $s \geq 2$. A simple graph is a 2-uniform hypergraph. For a vertex $v \in V(H)$, $\text{deg}(v) = \deg_H(v)$ is the degree of $v$ in $H$, that is the number of edges that contain $v$. We denote the maximum degree of $H$ by $\Delta(H)$ and the average degree $\frac{1}{n(H)}\sum_{v \in V(H)} \text{deg}(v)$ by $d(H)$. For a vertex $v \in V(H)$, $H \setminus \{v\}$ represents the hypergraph $H$ with vertex $v$ and all the edges incident to $v$ removed.

Motivated by the study on the $k$-independence number of graphs, researchers have paid much attention to the independence number of hypergraphs. Some relevant results are presented ([2], [4], [8]), below. For each of the results below, we let $d(v)$ denote the degree of the vertex $v$.

**Theorem 1.1** [6] Let $H = (V, E)$ be an $s$-uniform hypergraph with $s \geq 2$. Then

$$\alpha(H) \geq \sum_{v \in V} \prod_{i = 1}^{d(v)} \left(1 - \frac{1}{i(s - 1) + 1}\right).$$

In 1999, Thiele gave a lower bound for the independence number of general hypergraphs.

**Theorem 1.2** [13] Let $H = (V, E)$ be a hypergraph of rank $r$. Then

$$\alpha(H) \geq \sum_{v \in V} \sum_{i \in N_0^r} \left[\prod_{m=1}^{r} \binom{d_m(v)}{i} \right] \frac{(-1)^i \sum_i}{\sum (m-1) \cdot i + 1},$$

where $d_m(v)$ is the number of edges of size $m$ containing $v$ for $1 \leq m \leq r$ and for every vertex $v \in V$, the degree vector $d(v) = (d_1(v), d_2(v), \cdots, d_r(v)) \in N_r^r$.

In 2012, Csaba et al. gave another lower bound of the independence number for an $s$-uniform hypergraph, and thus improving a result of Spencer [12].

**Theorem 1.3** [7] Let $H = (V, E)$ be an $s$-uniform hypergraph with $n$ vertices and $s \geq 3$. Then

$$\alpha(H) \geq e^{-\frac{\gamma}{s-1}} \cdot \sum_{v \in V} \frac{1}{(d(v) + 1)^{\frac{1}{s-1}}},$$

where $\gamma$ is the Euler-Mascheroni constant.
A generalization for the \textit{k-independence number} of \textit{s}-uniform hypergraphs was obtained by Caro et al. [6], which improved earlier results of Favaron [9]. Let \( \alpha_k(H) \) denote the maximum cardinality of a \( k \)-independent set of \( H \) and call it the \( k \)-independence number of \( H \).

\textbf{Theorem 1.4} [6] For every natural number \( k \) and every \( s \)-uniform hypergraph \( H \) with vertex set \( V \),

\[ \alpha_k(H) \geq \sum_{v \in V} f_{k,s}(d(v)), \]

where the function \( f_{k,s}(i) \) is given by

\[ f_{k,s}(i) = \begin{cases} \frac{1 - i}{i^k}, & 0 \leq i \leq k; \\ \prod_{j=1}^{i-k+1} \left( 1 - \frac{1}{j(s-1)+1} \right), & i > k, \end{cases} \]

for integers \( k \geq 1, s \geq 2, \) and \( i \geq 0 \).

In this paper, we study the \( k \)-independent set of \( s \)-uniform hypergraphs. First, some necessary notation and definitions are presented in Sect. 2. Then, in Sect. 3, for an \( s \)-uniform hypergraph \( H \) of order \( n \), we first give a lower bound of the \( k \)-independence number in terms of the maximum degree; see Theorem 3.4. Furthermore, we give a lower bound for the \( k \)-independence number based on the average degree; see Theorem 3.5. The result of Kogan, mentioned previously, immediately follows from Theorem 3.5 when \( s = 2 \).

\section*{2 Preliminary}

We redefine the induced subhypergraph of a hypergraph as follows.

\textbf{Definition 2.1} Let \( H = (V, E) \) be a hypergraph. For any subset \( S \subseteq V \), \( H[S] = (S, E') \) is the subhypergraph induced by the set \( S \), where \( E' \) consists of all those subsets in \( E \) that are completely contained in \( S \).

Based on the above definition, we give the definition of the \( k \)-independent set of a hypergraph.

\textbf{Definition 2.2} Let \( H = (V, E) \) be a hypergraph on \( n \) vertices and \( k \geq 0 \) be an integer. A \( k \)-independent set \( S \subseteq V \) is a set of vertices such that the maximum degree in the hypergraph induced by \( S \) is at most \( k \).

Let \( \alpha_k(H) \) denote the maximum cardinality of a \( k \)-independent set of \( H \) and call it the \( k \)-independence number of \( H \). For \( k = 0 \) we have \( \alpha_0(H) = \alpha(H) \), where \( \alpha(H) \) is the independence number of \( H \).
Definition 2.3 Define the function $f$ for real $x \geq 0$ in the following manner:

$$f(x) = \frac{1}{1 + x} \left(1 + \frac{\{x\}(1 - \{x\})}{([x] + 1)([x] + 2)}\right),$$

and where $[x]$ is the floor function and $\{x\}$ is the fractional part function.

Note that $f(x) \geq \frac{1}{1 + x}$, and equality holds if and only if $x$ is an integer.

Definition 2.4 Define the function $g$ for real $x \geq 0$ in the following manner:

$$g(x) = \begin{cases} 1, & \text{if } x = 0; \\ \frac{2\lceil x \rceil - x}{\lceil x \rceil \lceil 1 + \lceil x \rceil \rceil}, & \text{if } x > 0, \end{cases}$$

where $\lceil \cdot \rceil$ is the ceiling function.

The following two lemmas are due to Kogan [10].

Lemma 2.5 [10] For all real $x > 0$ we have $f(x) = g(x)$.

Lemma 2.6 [10] The function $f(x)$ is continuous, monotonically decreasing and convex on the interval $[0, \infty)$.

3 Main results

In this section, we will give two lower bounds for the $k$-independence number of $s$-uniform hypergraphs based on the maximum degree or average degree; see Theorems 3.4 and 3.5, respectively.

Definition 3.1 For an $s$-uniform hypergraph $H$, we denote by $\chi_k(H)$ the $k$-chromatic number of $H$, i.e. the minimum number $t$ such that there is a partition $V(H) = V_1(H) \cup V_2(H) \cup \cdots \cup V_t(H)$ of the vertex set $H$ such that $\Delta(H[V_i]) \leq k$ for all $1 \leq i \leq t$.

For an $s$-uniform hypergraph $H$ its chromatic number $\chi(H) = \chi_0(H)$ is the smallest integer $k$ for which $H$ admits a $k$-colouring. We note that if the hypergraph $H$ is a graph, the chromatic number of $H$ coincides exactly with the usual chromatic number.

Let $H = (V, E)$ be a hypergraph with $n$ vertices and $m$ edges, where $E = \{e_1, \ldots, e_m\}$. For a set $J \subset \{1, 2, \ldots, m\}$, we call the family $H' = \{e_j : j \in J\}$ the partial hypergraph generated by the set $J$. For $v \in V$, define the star with centre $v$, denoted by $H(v)$, to be the partial hypergraph formed from the edges containing $v$. A $\beta$-star of a vertex $v$ is a family $H^\beta(v)$ of $H(v)$ such that:

(i) $e \in H^\beta(v) \Rightarrow |e| \geq 2$ (where $|e|$ is the number of vertices in $e$).

(ii) $e, e' \in H^\beta(v) \Rightarrow e \cap e' = \{v\}$.

We call the $\beta$-degree of a vertex $v$ the largest number of edges of a $\beta$-star of $v$. The $\beta$-degree of $v$ is denoted by $d^\beta_H(v)$ and the maximum $\beta$-degree is denote by $\Delta^\beta(H) = \max_{v \in V} d^\beta_H(v)$. The following result was obtained by Lovász in 1968.
Theorem 3.2 [11] For every hypergraph $H$, we have $\chi(H) \leq \Delta^\beta(H) + 1$.

Then we can obtain an upper bound for the $k$-chromatic number of $s$-uniform hypergraphs.

Theorem 3.3 If $H = (V, E)$ is an $s$-uniform hypergraph of maximum degree $\Delta$ and $sk$ is an integer that does not divide $2\Delta$, then $\chi_k(H) \leq \left\lceil \frac{2\Delta}{sk} \right\rceil$.

Proof For an $s$-uniform hypergraph $H$ on $V$ of maximum degree $\Delta$, let $H'$ be the hypergraph on $V$ whose edges are exactly the sets $S$ such that $\Delta(H[S]) = k$. Note that the hypergraph $H'$ needed a stronger condition, such as a set $S$ of vertices of $H$ is an edge of $H'$ if $\Delta(H[S]) = k$ and $S$ is maximal in the sense that $\Delta(H[S \cup \{v'\}]) > k$ for any vertex $v' \notin S$. Otherwise it is possible that a maximum $\beta$-star of $H'$ is the same size as a maximum $\beta$-star of $H$. For example it is possible to have a $\beta$-star $C$ of a vertex $v$ in $H'$, such that for each $S \in C$, $v$ has degree 1 in $H[S]$ even though $\Delta(H[S]) = k$ and if the edges incident to $v$ in $H$ form a $\beta$-star of size $\Delta$ it is possible for $C$ to be size $\Delta$. Then by Definition 3.1 and Theorem 3.2, we have

$$\chi_k(H) \leq \chi(H') \leq \Delta^\beta(H') + 1 \leq \left\lceil \frac{2\Delta}{sk} \right\rceil + 1 \leq \left\lceil \frac{2\Delta}{sk} \right\rceil.$$

Next since $\alpha_k(H) \geq \frac{n}{\chi_k(H)}$, we get a lower bound for the $k$-independence number of $s$-uniform hypergraphs in terms of the maximum degree.

Theorem 3.4 Let $H$ be an $s$-uniform hypergraph of order $n$ and maximum degree $\Delta$ and $k$ be an integer that does not divide $2\Delta$. Then

$$\alpha_k(H) \geq \frac{n}{\left\lceil \frac{2\Delta}{sk} \right\rceil}.$$

Next, we consider the average degree of $s$-uniform hypergraph, and we will give another lower bound for the $k$-independence number of $s$-uniform hypergraphs in terms of the average degree.

Theorem 3.5 Let $k \geq 0$ be an integer. Then for any $s$-uniform hypergraph $H$ of order $n$ and average degree $d$,

$$\alpha_k(H) \geq f\left(\frac{2d}{s(k+1)}\right)n.$$

We will present the proof of the theorem towards the end of the paper.

Corollary 3.6 Let $k \geq 0$ be an integer. Then for any $s$-uniform hypergraph $H$ of order $n$ and average degree $d$,

$$\alpha_k(H) \geq \frac{s(k+1)}{s(k+1)+2d}n.$$
Proof This follows from Theorem 3.5 and Definition 2.3. \hfill\Box

Before proving Theorem 3.5 we will need a few more lemmas and definitions.

Lemma 3.7 Let \( k \geq 0 \) and \( r \geq 0 \) be integers. Let \( H \) be an \( s \)-uniform hypergraph of order \( n \) with \( e \) edges and average degree \( d = \frac{se}{n} \). If \( \frac{s}{2} r (k+1) < d \leq \frac{s}{2} (r+1)(k+1) \) holds and

\[
\alpha_k(H) \geq \frac{2}{r+2} \left( n - \frac{e}{(r+1)(k+1)} \right),
\]

then \( \alpha_k(H) \geq f \left( \frac{2d}{s(k+1)} \right) n. \)

Proof Set \( t = \frac{2d}{s(k+1)}. \) Since \( \frac{s}{2} r (k+1) < d \leq \frac{s}{2} (r+1)(k+1) \), we have \( r < t \leq r+1. \) Hence \( \lceil t \rceil = r+1 \). Thus we have

\[
\alpha_k(H) \geq \frac{2}{r+2} \left( n - \frac{e}{(r+1)(k+1)} \right) = \frac{2}{\lceil t \rceil + 1} \left( n - \frac{dn}{s\lceil t \rceil (k+1)} \right) = \frac{2}{\lceil t \rceil + 1} \left( n - \frac{tn}{2\lceil t \rceil} \right) \quad \text{(as } t = \frac{2d}{s(k+1)} \text{)}
\]

\[
= \frac{2n}{\lceil t \rceil + 1} \left( 1 - \frac{t}{2\lceil t \rceil} \right) = n \frac{2\lceil t \rceil - t}{\lceil t \rceil (\lceil t \rceil + 1)} = g(t)n = f(t)n. \quad \text{(by Lemma 2.5)}
\]

\hfill\Box

Lemma 3.8 Let \( k \geq 0 \) be an integer. If \( H \) be an \( s \)-uniform hypergraph of order \( n \) with \( e \) edges, then

\[
\alpha_k(H) \geq n - \frac{e}{k+1}.
\]

Proof Set \( H_0 = H. \) If there is a vertex \( v_0 \in V(H_0) \) such that \( \deg_{H_0}(v_0) \geq k+1 \), then remove it from the hypergraph \( H_0 \) and call the resulting hypergraph \( H_1 \), that is, \( H_1 = H_0 \setminus \{v_0\} \). Now, if there is a vertex \( v_1 \in V(H_1) \) such that \( \deg_{H_1}(v_1) \geq k+1 \), then remove it from the hypergraph \( H_1 \) and call the resulting hypergraph \( H_2 \), that is, \( H_2 = H_1 \setminus \{v_1\} \). We can repeat this operation iteratively until we get a hypergraph \( H_i \) for some \( i > 0 \) such that \( \Delta(H_i) \leq k \). Notice that \( i \leq \left\lfloor \frac{e}{k+1} \right\rfloor \), as there are \( e \) edges in \( H_0 \), and in each iteration the number of edges in the resulting hypergraph is decreased by at least \( k+1 \). According the definition of the \( k \)-independent set, the maximum degree in \( H_i \) must be less than or equal to \( k \). Because in the process of continuously removing vertices, the vertices that are removed are all degrees greater than \( k \). In such a process, the maximum degree of the vertices in the hypergraph we finally obtained must be less...
than or equal to $k$, so it must satisfy the definition of $k$-independent set. Therefore, $H_t$ is a $k$-independent set, so we find a hypergraph $H_t$ equal $H_t$, that is we find the $H_t$ we needed. Since $V(H_t)$ is a $k$-independent set in $H$ and $|V(H_t)| = n - i \geq n - \frac{e}{k+1}$, we are done.

Corollary 3.9 Let $k \geq 0$ be an integer. If $H$ is an $s$-uniform hypergraph of order $n$ with average degree $0 < d \leq \frac{s}{2}(k+1)$, then

$$\alpha_k(H) \geq f\left(\frac{2d}{s(k+1)}\right)n.$$  

Proof This follows from Lemma 3.8 by setting $r = 0$ in Lemma 3.7. 

Finally we are ready to give the proof of our main result of this section.

Proof of Theorem 3.5 Let $k \geq 0$ be an integer. Recall that we need to prove that for any $s$-uniform hypergraph $H$ of order $n$ and average degree $d$, we have

$$\alpha_k(H) \geq f\left(\frac{2d}{s(k+1)}\right)n.$$  

By induction on integer $r \geq 0$, we will prove that $\alpha_k(H) \geq f\left(\frac{2d}{s(k+1)}\right)n$ for any $s$-uniform hypergraph $H$ of order $n$ and average degree $d \leq \frac{s}{2}(r+1)(k+1)$.

When $r = 0$, the result was verified in Corollary 3.9. Assume that the claim holds for $r \geq 0$ and we will prove it for $r+1$.

By Lemma 3.7 and the induction hypothesis, it suffices to prove that if $H$ is an $s$-uniform hypergraph on $n$ vertices, $e$ edges and average degree $d$ satisfying $\frac{s}{2}r(k+1) < d \leq \frac{s}{2}(r+1)(k+1)$, then

$$\alpha_k(H) \geq \frac{2}{r+2}\left(n - \frac{e}{(r+1)(k+1)}\right).$$  

Let $H'$ is a disjoint union of $(r+2)(k+1)$ copies of $H$, where $d(H) = d(H')$, and the number of vertices $n'$ and number of edges $e'$ in $H'$ are both divisible by $(r+2)(k+1)$. Then

$$\frac{\alpha_k(H')}{n(H')} = \frac{(r+2)(k+1)\alpha_k(H)}{(r+2)(k+1)n} = \frac{\alpha_k(H)}{n}.$$

If $\alpha_k(H') \geq n' f\left(\frac{2d}{s(k+1)}\right)$ then the original hypergraph $H$ satisfies

$$\alpha_k(H) \geq \frac{n'}{(r+2)(k+1)} f\left(\frac{2d}{s(k+1)}\right) = n f\left(\frac{2d}{s(k+1)}\right).$$  

Therefore, we can assume that both $n$ and $e$ are divisible by $(r+2)(k+1)$. We define parameter $t$ as follows:

$$t = \frac{se - n \frac{s}{2}r(k+1)}{\frac{s}{2}(r+2)(k+1)}.$$
Because \( n \) and \( e \) are divisible by \((r + 2)(k + 1)\), \( es = nd \), and \( d > \frac{s}{2}r(k + 1) \), we know that \( t \) is an integer and \( t > 0 \).

Set \( H_0 = H \). If there is a vertex \( v_0 \in V(H_0) \) such that \( \deg_{H_0}(v_0) \geq \frac{s}{2}(r + 1)(k + 1) \), then remove it from the hypergraph \( H_0 \), and denote the resulting hypergraph by \( H_1 \), that is, \( H_1 = H_0 \setminus \{v_0\} \). Now if \( t > 1 \) and there is a vertex \( v_1 \in V(H_1) \) such that \( \deg_{H_1}(v_1) \geq \frac{s}{2}(r + 1)(k + 1) \), then remove it from the hypergraph \( H_1 \) and denote the resulting hypergraph by \( H_2 \), that is, \( H_2 = H_1 \setminus \{v_1\} \).

We repeat this operation iteratively, that is on iteration \( i \) we first check if \( i = t \) or \( \Delta(H_i) < \frac{s}{2}(r + 1)(k + 1) \), and if one of these conditions holds we terminate the process. Otherwise, we pick a vertex \( v_i \in V(H_i) \) such that \( \deg_{H_i}(v_i) \geq \frac{s}{2}(r + 1)(k + 1) \) and remove it from the hypergraph \( H_i \). The resulting hypergraph is denoted by \( H_{i+1} \), that is, \( H_{i+1} = H_i \setminus \{v_i\} \).

Suppose that the process above terminated on iteration \( j \leq t \), that is, the last hypergraph created in the process is \( H_j \). If \( j < t \) then \( \Delta(H_j) \leq \frac{s}{2}(r + 1)(k + 1) - 1 \), and thus by Theorem 3.4, we have

\[
\alpha_k(H_j) \geq \frac{n - t}{2\Delta(H_j)} \geq \frac{n - t}{s(k+1)} \geq \frac{n - t}{s(r+1)(k+1)} = \frac{n - t}{r + 1}.
\]

In the above inequality, \( \frac{2\Delta(H_j)}{sk} \geq \frac{2(\Delta(H_j)+1)}{sk} \) is true when \( \Delta(H_j) \geq k \) and \( \left\lfloor \frac{2\Delta(H_j)}{sk} \right\rfloor = 1 = \left\lfloor \frac{2(\Delta(H_j)+1)}{sk} \right\rfloor \) when \( \Delta(H_j) < k \).

It remains to prove that \( \alpha_k(H_t) \geq \frac{n-t}{r+1} \).

First we notice that

\[
n - t = n - \frac{s \cdot e - n \cdot \frac{s}{2} \cdot r(k + 1)}{\frac{s}{2}(r + 2)(k + 1)} = \frac{(r + 2)(k + 1)n + nr(k + 1) - 2e}{(r + 2)(k + 1)} = \frac{2[n(r + 1)(k + 1) - e]}{(r + 2)(k + 1)}.
\]

Now we claim that \( d(H_t) \leq \frac{s}{2}r(k + 1) \). Notice that as in each iteration at least \( \frac{s}{2}(r + 1)(k + 1) \) edges were removed we have that \( e(H_t) \), the number of edges in hypergraph \( H_t \), satisfies

\[
e(H_t) \leq e - t \cdot \frac{s}{2}(r + 1)(k + 1) = e - \frac{2e - nr(k + 1)}{(r + 2)(k + 1)} \cdot \frac{s}{2}(r + 1)(k + 1) = \frac{(r + 2)(k + 1)e - \frac{s}{2}(r + 1)(k + 1)[2e - nr(k + 1)]}{(r + 2)(k + 1)}.
\]
Since $s \geq 2$, $e = \frac{nd}{s}$ and $d > \frac{s}{2}r(k+1)$, then $e \geq \frac{nr(k+1)}{2}$, so we have
\[
2 \frac{e(H_t)}{n-t} \leq \frac{(r+2)(k+1)e - \frac{s}{2}(r+1)(k+1)[2e - nr(k+1)]}{n(r+1)(k+1) - e} \leq r(k+1).
\]
So we have $\frac{e(H_t)}{n-t} \leq \frac{1}{2}r(k+1)$, it follows that
\[
d(H_t) = s \frac{e(H_t)}{n-t} \leq \frac{s}{2}r(k+1).
\]
Now since $d(H_t) \leq \frac{s}{2}r(k+1)$, we can apply the induction hypothesis on $H_t$. By the induction hypothesis and our observation that $f(x) \geq \frac{1}{1+x}$, we have
\[
\alpha_k(H_t) \geq \frac{n-t}{r+1}.
\]
We conclude that
\[
\alpha_k(H) \geq \alpha_k(H_j)
\geq \frac{n-t}{r+1}
= \frac{2}{r+2} \left( n - \frac{e}{(r+1)(k+1)} \right)
\geq f \left( \frac{2d}{s(k+1)} \right) n \quad \text{(by Lemma 3.7)}
\]
and this is exactly what we need to prove. This concludes the proof of the induction. $\blacksquare$

Now we obtain a lower bound for the $k$-independence number of $s$-uniform hypergraphs based on the average degree. From Corollary 3.6, when $s = 2$, the following result is immediate.

**Theorem 3.10** [10] Let $k \geq 0$ be an integer. Then for any graph $G$ of order $n$ and average degree $d$, we have
\[
\alpha_k(G) \geq \frac{k+1}{k+1+d} n.
\]

**Remark 1** Since
\[
s \geq 2, \Delta \geq d,
\]
when $\Delta > k(k+1)$, we have
\[
\left\lceil \frac{2\Delta}{sk} \right\rceil \geq \frac{2\Delta}{sk} \geq \frac{2d}{s(k+1)} + 1.
\]
Hence, 
\[
\frac{n}{\lceil \frac{2\Delta}{sk} \rceil} < \frac{s(k+1)}{s(k+1) + 2dn}.
\]

Therefore, we find that when \( \Delta > k(k + 1) \), the bound \( \frac{s(k+1)}{s(k+1) + 2dn} \) is better than \( \frac{n}{\lceil \frac{2\Delta}{sk} \rceil} \), i.e., Corollary 3.6 is stronger than Theorem 3.4 when \( \Delta > k(k + 1) \).

**Remark 2** It is really difficult to compare how effective Theorems 3.4 and 3.5 are compared to Theorems 1.1–1.4. But if we only look at it formally, our results are more concise and easier to use.

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**Declarations**

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