Quaternionic Hardy Spaces

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Abstract

The theory of slice regular functions of a quaternionic variable, as presented in [18], extends the notion
of holomorphic function to the quaternionic setting. This fast growing theory is already rich of many
results and has interesting applications. In this setting, the present paper is devoted to introduce and study
the quaternionic counterparts of Hardy spaces of holomorphic functions of one complex variable. The
basic properties of the theory of quaternionic Hardy spaces are investigated, and in particular a Poisson-
type representation formula, the notions of outer function, singular function and inner function are given.
A quaternionic (partial) counterpart of the classical \( H^p \)-factorization theorem is proved. This last result
assumes a particularly interesting formulation for a large subclass of slice regular functions, where it is
obtained in terms of an outer function, a singular function and a quaternionic Blaschke product.

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functions and inner functions.

1 Introduction

The theory of slice regular functions of a quaternionic variable (often simply called regular functions) was
introduced in [19], [20], and represents a natural quaternionic counterpart of the theory of complex hol-
omorphic functions. This recent theory has been growing very fast and was developed in a series of papers,
including in particular [4], [5], [8], [16], [17], [22], [29], [31], [33], [34], [35], [36], where most of the recent
advances are discussed. The detailed up-to-date theory appears in the monograph [18], while an extension of
the theory to the case of real alternative algebras is discussed in [24], [25] and [26]. New contributions to the
geometric theory of regular functions appear in [10], [11], [15]. The theory of regular functions is presently

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expanding in many directions, and it has in particular been applied to the study of a non-commutative functional calculus, (see for example the monograph [7] and the references therein) and to contribute to the problem of the construction and classification of orthogonal complex structures in open subsets of the space of quaternions (see [14]). As it appears in the cited literature, the results one obtains in the theory of slice regular functions are inspired by complex analysis, though they often require essential modifications, due to the different nature of zeroes and singularities of regular functions.

Let $\mathbb{H}$ denote the skew field of quaternions. Set $S = \{q \in \mathbb{H} : q^2 = -1\}$ to be the $2$−sphere of purely imaginary units in $\mathbb{H}$, and for $I \in S$ let $L_I$ be the complex plane $\mathbb{R} + RI$, so that

$$\mathbb{H} = \bigcup_{I \in S} L_I.$$  

To recall the definition of slice regular function we will first describe the natural domains of definition for such functions (see [18]).

**Definition 1.1.** Let $\Omega$ be a domain in $\mathbb{H}$ that intersects the real axis. Then:

1. $\Omega$ is called a slice domain if, for all $I \in S$, the intersection $\Omega_I$ with the complex plane $L_I$ is a domain of $L_I$;

2. $\Omega$ is called a symmetric domain if for all $x, y \in \mathbb{R}$, $x + yI \in \Omega$ implies $x + yS \subset \Omega$.

**Definition 1.2.** Let $\Omega \subseteq \mathbb{H}$ be a slice domain. A function $f : \Omega \to \mathbb{H}$ is said to be (slice) regular if, for all $I \in S$, its restriction $f_I$ to $\Omega_I$ is holomorphic, i.e., it has continuous partial derivatives and satisfies

$$\overline{I}f(x + yI) := \frac{1}{2} \left( \frac{\partial}{\partial x} + I \frac{\partial}{\partial y} \right) f_I(x + yI) = 0$$

for all $x + yI \in \Omega_I$.

As shown in [20], if $\Omega$ is the open unit ball $B$ of $\mathbb{H}$, the class of regular functions coincides with the class of convergent power series of type $\sum_{n \geq 0} q^n a_n$, with all $a_n \in \mathbb{H}$.

One of the most fertile chapters of the theory of complex holomorphic functions consists of the theory of Hardy spaces. This theory contains results of great significance that led to important general achievements and subtle applications. In this paper we define the quaternionic counterpart of complex Hardy spaces, and investigate their basic and fundamental properties. Our definition is the following.

**Definition 1.3.** Let $f : B \to \mathbb{H}$ be a regular function and let $0 < p < +\infty$. Set

$$||f||_p = \sup_{I \in S} \lim_{r \to 1^{-}} \left( \frac{1}{2\pi} \int_{0}^{2\pi} |f(re^{i\theta})|^p \, d\theta \right)^{\frac{1}{p}}$$

and set

$$||f||_\infty = \sup_{q \in B} |f(q)|.$$  

Then, for any $0 < p \leq +\infty$, we define the quaternionic Hardy space $H^p(B)$ as

$$H^p(B) = \{ f : B \to \mathbb{H} \mid f \text{ is regular and } ||f||_p < +\infty \}.$$  

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In Section 3 we study the main properties and features of the quaternionic $H^p$-norms, we motivate the chosen definition and establish the initial properties of the quaternionic $H^p$ spaces. In Section 4 we investigate the boundary behavior of functions $f$ in $H^p(B)$, obtaining that for almost every $\theta \in \mathbb{R}$, the limit
\[
\lim_{r \to 1^-} f(re^{I\theta}) = \tilde{f}(e^{I\theta})
\]
exists for all $I \in \mathbb{S}$ and in this case it belongs to $L^p(\partial B_I)$. We then investigate the boundary values of the $*$-product of two functions, each belonging to some $H^p(B)$ space. In Section 5 we extend the classical $H^p$ kernels, and in particular the Poisson kernel, to the quaternionic setting and deduce Poisson-type representation formulas for all $f \in H^p(B)$. In section 6 possible analogues of outer and inner functions and singular factors on $B$ are given, whose definitions (when compared with those used in the complex case) clearly resent of the peculiarities of the non commutative quaternionic setting. Our results here include factorization properties of $H^p$ functions. We are able to identify the Blaschke factor of a function $f$ in $H^p(B)$, built from the zero set of $f$; then we can exhibit a complete factorization result, in terms of an outer, a singular and a Blaschke factor, for a subclass of regular functions, namely for the one-slice-preserving functions. The cases of $H^2(B)$ (defined already in [1]) and $H^\infty(B)$ are special cases, as in the complex setting.

Hardy and Dirichlet spaces in the more general setting of Clifford valued monogenic functions are studied for example in [3].

2 Preliminaries

We recall in this section some preliminary definitions and results that are essential for the comprehension of the rest of the paper. The related proofs can be found in the literature, and in particular in [18]. Let us begin with a basic result that establishes the connection between the class of regular functions and the class of complex holomorphic functions of one complex variable.

**Lemma 2.1** (Splitting Lemma). If $f$ is a regular function on a slice domain $\Omega$, then for every $I \in \mathbb{S}$ and for every $J \in \mathbb{S}$, $J$ orthogonal to $I$, there exist two holomorphic functions $F,G : \Omega_I \to L_I$, such that for every $z = x + yI \in \Omega_I$, we have

\[
f_I(z) = F(z) + G(z)J.
\]

Regular functions on slice domains satisfy the following version of the Identity Principle and the classical Maximum Modulus Principle.

**Theorem 2.2** (Identity Principle). Let $f$ be a regular function on a slice domain $\Omega$. Denote by $Z_f$ the zero set of $f$,

\[
Z_f = \{ q \in \Omega | f(q) = 0 \}.
\]

If there exists $I \in \mathbb{S}$ such that $\Omega_I \cap Z_f$ has an accumulation point in $\Omega_I$, then $f$ vanishes identically on $\Omega$.

**Theorem 2.3** (Maximum Modulus Principle). Let $f$ be a regular function on a symmetric slice domain $\Omega$. If $f$ has a local maximum in $\Omega$, then $f$ is a constant function on $\Omega$.

We can recover the values of a regular function defined on a symmetric slice domain from its values on a single slice $L_I$ (see [4, 5]).
**Theorem 2.4 (Representation Formula).** Let \( f \) be a regular function on a symmetric slice domain \( \Omega \) and let \( I \in \mathbb{S} \). Then, for all \( x + yJ \in \Omega \) with \( J \in \mathbb{S} \), the following equality holds

\[
f(x + yJ) = \frac{1}{2} [f(x + yI) + f(x - yI)] + \frac{J}{2} [f(x - yI) - f(x + yI)].
\]

In particular the function \( f \) is affine on each sphere of the form \( x + yS \) contained in \( \Omega \), namely there exist \( b, c \in \mathbb{H} \) such that \( f(x + yJ) = b + Jc \) for all \( J \in \mathbb{S} \).

Thanks to the Representation Formula it is possible to prove that we can estimate the maximum modulus of a function with its maximum modulus on each slice, (see [23]). In particular, for the case of \( \mathbb{B} \), we have the following result.

**Proposition 2.5.** Let \( f \) be a regular function on \( \mathbb{B} \). If we set \( M = \sup_{z \in \mathbb{B}} |f(z)| \), then

\[
\sup_{q \in \mathbb{B}} |f(q)| \leq 2M.
\]

In the special case of a regular function \( f \) that maps a slice \( L_I \) to itself, we obtain that the maximum (and minimum) modulus of \( f \) is actually attained on the preserved slice.

**Proposition 2.6.** Let \( f \) be a regular function on a symmetric slice domain \( \Omega \) such that \( f(\Omega_I) \subset L_I \) for some \( I \in \mathbb{S} \). Then, for any \( x + yS \subset \Omega \),

\[
\max_{J \in \mathbb{S}} |f(x + yJ)| = \max\{|f(x + yI)|, |f(x - yI)|\}
\]

and

\[
\min_{J \in \mathbb{S}} |f(x + yJ)| = \min\{|f(x + yI)|, |f(x - yI)|\}.
\]

Holomorphic functions defined on a domain \( \Omega_I \), symmetric with respect to the real axis in the complex plane \( L_I \), extend uniquely to the smallest symmetric slice domain of \( \mathbb{H} \) containing \( \Omega_I \).

**Lemma 2.7 (Extension Lemma).** Let \( \Omega \) be a symmetric slice domain and choose \( I \in \mathbb{S} \). If \( f_I : \Omega_I \rightarrow \mathbb{H} \) is holomorphic, then setting

\[
f(x + yJ) = \frac{1}{2} [f_I(x + yI) + f_I(x - yI)] + \frac{J}{2} [f_I(x - yI) - f_I(x + yI)]
\]

extends \( f_I \) to a regular function \( f : \Omega \rightarrow \mathbb{H} \). Moreover \( f \) is the unique extension and it is denoted by \( \text{ext}(f_I) \).

The product of two regular functions is not, in general, regular. To guarantee the regularity we have to use a different multiplication operation, the \( * \)-product, which extends the one classically defined for polynomials with coefficients in a non-commutative field.

**Definition 2.8.** Let \( f, g \) be regular functions on a symmetric slice domain \( \Omega \). Choose \( I, J \in \mathbb{S} \) with \( I \perp J \) and let \( F, G, H, K \) be holomorphic functions from \( \Omega_I \) to \( L_I \) such that \( f_I = F + GJ, g_I = H + KJ \).

Consider the holomorphic function defined on \( \Omega_I \) by

\[
f_I \ast g_I(z) = \left[ F(z)H(z) - G(z)K(\overline{z}) \right] + \left[ F(z)K(z) + G(z)H(\overline{z}) \right] J.
\]

Its regular extension \( \text{ext}(f_I \ast g_I) \) is called the regular product (or \( * \)-product) of \( f \) and \( g \) and it is denoted by \( f \ast g \).
Notice that the ∗-product is associative but generally is not commutative. Its connection with the usual pointwise product is stated by the following result.

**Proposition 2.9.** Let \( f \) and \( g \) be regular functions on a symmetric slice domain \( \Omega \). Then

\[
f \ast g(q) = \begin{cases} 
  f(q)g(f(q)^{-1}qf(q)) & \text{if } f(q) \neq 0 \\
  0 & \text{if } f(q) = 0
\end{cases}
\]

**Corollary 2.10.** If \( f, g \) are regular functions on a symmetric slice domain \( \Omega \) and \( q \in \Omega \), then \( f \ast g(q) = 0 \) if and only if \( f(q) = 0 \) or \( f(q) \neq 0 \) and \( g(f(q)^{-1}qf(q)) = 0 \).

The regular product coincides with the pointwise product for the special class of regular functions defined as follows.

**Definition 2.11.** A regular function \( f : \Omega \to \mathbb{H} \) such that \( f(\Omega_I) \subseteq L_I \) for all \( I \in \mathbb{S} \) is called a slice preserving regular function.

**Lemma 2.12.** Let \( f, g \) be regular functions on a symmetric slice domain \( \Omega \). If \( f \) is slice preserving, then \( fg \) is a regular function on \( \Omega \) and \( f \ast g = fg = g \ast f \).

The following operations are naturally defined in order to study the zero set of regular functions.

**Definition 2.13.** Let \( f \) be a regular function on a symmetric slice domain \( \Omega \) and suppose that for \( z \in \Omega \), the splitting of \( f \) with respect to \( J \) (\( J \in \mathbb{S} \) orthogonal to \( I \)) is \( f^J(z) = F(z) + G(z)J \). Consider the holomorphic function

\[
f^c_I(z) = F(z) - G(z)J.
\]

The regular conjugate of \( f \) is the function defined by

\[
f^c(q) = \text{ext}(f^c_I)(q).
\]

The symmetrization of \( f \) is the function defined by

\[
f^s(q) = f \ast f^c(q) = f^c \ast f(q).
\]

Both \( f^c \) and \( f^s \) are regular functions on \( \Omega \).

We now recall a result (proven in [30]) that relates the norm of a regular function \( f : \mathbb{B} \to \mathbb{H} \) and that of its regular conjugate.

**Proposition 2.14.** Let \( f \) be a regular function on \( \mathbb{B} \). For any sphere of the form \( x + y\mathbb{S} \) contained in \( \mathbb{B} \), the following equalities hold true

\[
\max_{I \in \mathbb{S}} |f(x + yI)| = \max_{I \in \mathbb{S}} |f^c(x + yI)| \quad \text{and} \quad \min_{I \in \mathbb{S}} |f(x + yI)| = \min_{I \in \mathbb{S}} |f^c(x + yI)|.
\]

Before we give the complete characterization of the zero set of a regular function, let us recall another result in this setting, consequence of the Representation Formula 2.4.

**Proposition 2.15.** Let \( f \) be a regular function on a symmetric slice domain \( \Omega \subset \mathbb{H} \) such that \( f(\Omega_I) \subset L_I \) for some \( I \in \mathbb{S} \). If \( f(x + yJ) = 0 \) for some \( J \in \mathbb{S} \setminus \{\pm I\} \), then \( f(x + yK) = 0 \) for any \( K \in \mathbb{S} \).
Theorem 2.16 (Zero set structure). Let \( f \) be a regular function on a symmetric slice domain \( \Omega \). If \( f \) does not vanish identically, then its zero set consists of the union of isolated points and isolated 2-spheres of the form \( x + yS \) with \( x, y \in \mathbb{R}, y \neq 0 \).

Spheres of zeros of real dimension 2 are a peculiarity of regular functions.

Definition 2.17. Let \( f \) be a regular function on a symmetric slice domain \( \Omega \). A 2-dimensional sphere \( x + yS \subset \Omega \) of zeros of \( f \) is called a spherical zero of \( f \). Any point \( x + yI \) of such a sphere is called a generator of the spherical zero \( x + yS \). Any zero of \( f \) that is not a generator of a spherical zero is called an isolated zero (or a non-spherical zero or simply a zero) of \( f \).

An appropriate notion of multiplicity, introduced in [21] for the case of regular quaternionic polynomials, can be given for zeros of regular functions.

Definition 2.18. Let \( f \) be a regular function on a symmetric slice domain \( \Omega \) and let \( x + yS \subset \Omega \) with \( y \neq 0 \). Let \( m, n \in \mathbb{N} \) and \( p_1, \ldots, p_n \in x + yS \) (with \( p_i \neq p_{i+1} \) for all \( i \in \{1, \ldots, n-1\} \)) be such that
\[
f(q) = \left( (q - x)^2 + y^2 \right)^m (q - p_1) \ast (q - p_2) \ast \cdots \ast (q - p_n) \ast g(q)
\]
for some regular function \( g : \Omega \to \mathbb{H} \) which does not have zeros in \( x + yS \). Then \( 2m \) is called the spherical multiplicity of \( x + yS \) and \( n \) is called the isolated multiplicity of \( p_1 \). On the other hand, if \( x \in \mathbb{R} \), then we call isolated multiplicity of \( f \) at \( x \) the number \( k \in \mathbb{N} \) such that
\[
f(q) = (q - x)^k h(q)
\]
for some regular function \( h : \Omega \to \mathbb{H} \) which does not vanish at \( x \).

In [23] it is proved that is possible to write a complete factorization of the zeros of an entire function (i.e. a function that is regular on \( \mathbb{H} \)) and in order to do it, the following results concerning infinite products of quaternions, and infinite regular products of regular functions, there are proved.

Theorem 2.19. Let \( \{a_n\}_{n \in \mathbb{N}} \subset \mathbb{H} \). A sufficient condition for the convergence of the product
\[
\prod_{n \geq 0} (1 + a_n) = (1 + a_0)(1 + a_1) \cdots
\]
is the convergence of the series
\[
\sum_{n \geq 0} |a_n|.
\]

Theorem 2.20. Let \( \{f_n\}_{n \in \mathbb{N}} \) be a sequence of regular functions defined on a symmetric slice domain \( \Omega \). The infinite \( \ast \)-product
\[
\prod_{n \geq 0} (1 + f_n(q))
\]
converges uniformly on compact sets of \( \Omega \) if and only if the infinite product
\[
\prod_{n \geq 0} (1 + f_n(q))
\]
converges uniformly on compact sets of \( \Omega \).
Moreover, where the infinite regular product does converge, it defines a regular function.

**Theorem 2.21.** Let \( \{f_n\}_{n \in \mathbb{N}} \) be a sequence of regular functions defined on a symmetric slice domain \( \Omega \). If the infinite regular product

\[
\prod_{n \geq 0} (1 + f_n(q))
\]

converges uniformly on compact sets of \( \Omega \) to a function \( g \), then \( g \) is regular on \( \Omega \).

The regular conjugate, and the symmetrization, of a regular function \( f \) on a symmetric slice domain \( \Omega \) allow the definition of its regular reciprocal \( f^{-*} \) defined on \( \Omega \setminus Z_f^* \) as

\[
f^{-*} = (f^*)^{-1} f^c,
\]

where \( Z_f^* \) denotes the zero set of the symmetrization \( f^* \). This naturally leads to the definition of regular quotient of regular functions, for more details see [18].

**Proposition 2.22.** Let \( \Omega \) be a symmetric slice domain in \( \mathbb{H} \) and let \( f \) and \( g \) be regular functions on \( \Omega \). If \( T_f : \Omega \setminus Z_f^* \to \Omega \setminus Z_f^* \) is defined as

\[
T_f(q) = f^c(q)^{-1} q f^c(q),
\]

then

\[
f^{-*} g(q) = f(T_f(q))^{-1} g(T_f(q)) \quad \text{for every} \quad q \in B \setminus Z_f^*.
\]

Furthermore, \( T_f \) and \( T_{f^{-*}} \) are mutual inverses so that \( T_f \) is a diffeomorphism.

Regular Moebius transformations are interesting examples of regular quotients (see [32]).

**Definition 2.23.** A regular Moebius transformation is a regular function \( M : \mathbb{B} \to \mathbb{B} \) of the form

\[
M(q) = (1 - q \alpha)^{-*} (q - \alpha) u
\]

where \( \alpha \in \mathbb{B} \) and \( u \in \partial \mathbb{B} \).

In particular, these transformations are regular bijections of the open unit ball onto itself and, by direct computation, it is possible to prove that they map the boundary of the unit ball to itself.

**Proposition 2.24.** A function \( M \) is a regular bijection from \( \mathbb{B} \) to itself if and only if \( M \) is a regular Moebius transformation.

### 3 The quaternionic spaces \( H^p(\mathbb{B}) \)

In this section we give a definition of quaternionic Hardy spaces. The first natural step is to define an appropriate Hardy-type norm. Let \( p \in (0, +\infty) \), \( r \in [0, 1) \), and \( I \in \mathbb{S} \). For any regular function \( f \) in the unit ball \( \mathbb{B} \), let \((f_I)_r\) be the function defined on the unit circle \( \partial \mathbb{B} \) by

\[
(f_I)_r(e^{i\theta}) = f(re^{i\theta})
\]

and let \( M_p(f_I, r) \) be the integral mean

\[
M_p(f_I, r) = \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} |(f_I)_r(e^{i\theta})|^p d\theta \right)^{\frac{1}{p}} = \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^p d\theta \right)^{\frac{1}{p}}. \tag{2}
\]
If \( p = +\infty \), set \( M_\infty(f, r) \) to be defined as
\[
M_\infty(f, r) = \sup_{|q| < r} |f(q)|
\]
for \( 0 < r < 1 \) (and \( |f(0)| \) for \( r = 0 \)).

**Proposition 3.1.** Let \( f : \mathbb{B} \to \mathbb{H} \) be a regular function. Then for any \( p \in (0, +\infty) \) and for any imaginary unit \( I \in \mathbb{S} \), the function \( r \mapsto M_p(f_I, r) \) is increasing. For \( p = +\infty \), the function \( r \mapsto M_\infty(f, r) \) is increasing as well.

**Proof.** The last part of the statement follows directly from the Maximum Modulus Principle 2.3. To prove the first part, fix \( p \in (0, +\infty) \) and \( I \in \mathbb{S} \). We will prove that the function
\[
g : \mathbb{B}_I \to \mathbb{R}, \quad g : z \mapsto |f_I(z)|^p,
\]
is a subharmonic function, and then the statement will follow from classical results, see for instance Theorem 17.5 in [28]. In order to obtain the subharmonicity of \( g(z) \), we will first show that
\[
h : \mathbb{B}_I \to \mathbb{R}, \quad h : z \mapsto \log(|f_I(z)|)
\]
is a subharmonic function. Then, since \( g(z) = e^{ph(z)} \) is the composition of an increasing convex function with a subharmonic function, we will conclude that also \( g \) is subharmonic (see for instance Theorem 17.2 in [28]).

Suppose \( f \neq 0 \) (otherwise the statement is trivially true). Then \( h(z) \) is an upper semicontinuous function. Moreover in the set \( \{ z \in \mathbb{B}_I \mid f_I(z) \neq 0 \} \) the function \( h \) is subharmonic. In fact we have that
\[
\Delta h(z) = 4 \frac{\partial}{\partial z} \frac{\partial}{\partial \overline{z}} \log(|f_I(z)|) = 4 \frac{\partial}{\partial z} \frac{\partial}{\partial \overline{z}} \frac{1}{2} \log(|f_I(z)|^2)
\]
\[
= 2 \frac{\partial}{\partial z} \frac{\partial}{\partial \overline{z}} \log(|F(z)|^2 + |G(z)|^2)
\]
where \( F, G \) are the splitting (holomorphic) functions of \( f_I \) with respect to \( J \in \mathbb{S} \), \( J \) orthogonal to \( I \). Hence (omitting the variable \( z \))
\[
\Delta h = 2 \left( |F'|^2 + |G'|^2 \right) \left( |F|^2 + |G|^2 \right) - (F'F + G'G) \left( FF' + GG' \right).
\]
If we denote by \( \langle , \rangle \) the standard Hermitian product in \((L_I)^2 \simeq \mathbb{C}^2\), the numerator of the last term in equation (4) can be written in vector notation as
\[
|\langle F', G' \rangle|^2 |(F, G)|^2 - \langle (F', G'), (F, G) \rangle \langle (F, G), (F', G') \rangle
\]
\[
= |\langle F', G' \rangle|^2 |(F, G)|^2 - |\langle (F', G'), (F, G) \rangle|^2.
\]
Schwarz inequality for Hermitian product yields that this quantity is always non-negative, and therefore we have that \( h(z) \) is subharmonic where \( f_I \) is non vanishing. It remains to show that \( h \) is still subharmonic in a neighborhood of each zero of \( f_I \). Recall that the zero set of a regular function intersected with a slice \( L_I \), is a discrete subset of \( L_I \) (see Theorem 2.16). Then for any zero \( z_0 \) of \( f_I \), there exists a neighborhood \( U_I \subset \mathbb{B}_I \) where \( z_0 \) is the only point where \( f_I \) vanishes. Hence we have that for all \( r \) such that \( z_0 + B_I(0, r) \subset U_I \) the submean property is trivially satisfied
\[
-\infty = h(z_0) \leq \frac{1}{2\pi} \int_{\partial B_I(0, r)} h(z_0 + re^{i\theta}) d\theta.
\]
Since this condition implies the subharmonicity of \( h \) near \( z_0 \), we can conclude the proof. \( \Box \)
Remark 3.2. The previous result is the analogue of the first statement of the Hardy convexity Theorem in the complex setting, see Theorem 1.5 in [12].

Remark 3.3. We point out that, despite what happens in the complex case, where the function \( z \mapsto \log(|g(z)|) \) is actually harmonic for any non vanishing holomorphic function \( g \), the function \( h \) defined in equation (3) is only subharmonic in general. In fact \( \Delta h = 0 \) if and only if the last term in equation (5) equals zero, that is, if and only if the vector \( (F(z), G(z)) \in (L_1)^2 \), identified by the restriction of the regular function \( f_1 \), is parallel to the vector \( (F'(z), G'(z)) \) identified by the derivative of \( f_1 \). This happens for example if

\[
\begin{cases}
F'(z) = k F(z) \\
G'(z) = k G(z)
\end{cases}
\]

for some \( k \in \mathbb{L} \). In this case \( F \) and \( G \) are exponential (or constant) functions,

\[
\begin{cases}
F(z) = F(0) e^{k z} \\
G(z) = G(0) e^{k z}.
\end{cases}
\]

Thanks to Proposition 3.1, we can give the following Definition.

Definition 3.4. Let \( f : \mathbb{B} \to \mathbb{H} \) be a regular function. If \( p \in (0, +\infty) \), for any \( I \in \mathbb{S} \), we set

\[
\|f_I\|_p = \lim_{r \to 1^-} |M_p(f_I, r)| = \lim_{r \to 1^-} \left( \frac{1}{2\pi} \int_0^{2\pi} |f(r e^{i\theta})|^p d\theta \right)^{\frac{1}{p}}
\]

and

\[
\|f\|_p = \sup_{I \in \mathbb{S}} \|f_I\|_p = \sup_{I \in \mathbb{S}} \lim_{r \to 1^-} \left( \frac{1}{2\pi} \int_0^{2\pi} |f(r e^{i\theta})|^p d\theta \right)^{\frac{1}{p}}.
\]

If \( p = +\infty \), we set

\[
\|f\|_\infty = \lim_{r \to 1^-} |M_\infty(f, r)| = \lim_{r \to 1^-} \sup_{|q| < r} |f(q)| = \sup_{q \in \mathbb{B}} |f(q)|.
\]

Remark 3.5. Notice that \( \|f\|_\infty \) is the uniform norm of \( f \) on \( \mathbb{B} \). Moreover, if we set for any \( I \in \mathbb{S} \)

\[
\|f_I\|_\infty = \sup_{z \in \mathbb{B}_I} |f(z)|,
\]

then we have

\[
\|f\|_\infty = \sup_{I \in \mathbb{S}} \|f_I\|_\infty.
\]

The set where the uniform norm is taken will be \( \mathbb{B}_I \) when considering the restriction \( f_I \) (or its splitting components), and \( \mathbb{B} \) when considering the function \( f \).

Definition 3.6. Let \( p \in (0, +\infty] \). We define the quaternionic Hardy space \( H^p(\mathbb{B}) \) as

\[
H^p(\mathbb{B}) = \{ f : \mathbb{B} \to \mathbb{H} \mid f \text{ is regular and } \|f\|_p < +\infty \}.
\]

Remark 3.7. As in the complex case, for any \( p \in (0, +\infty] \), the space \( H^p(\mathbb{B}) \) is a real vector space. Furthermore if (and only if) \( p \geq 1 \) the function \( \| \cdot \|_p \) satisfies the triangle inequality, and hence it is a norm on \( H^p(\mathbb{B}) \). Moreover, the same relations of inclusions that hold for complex \( H^p \) spaces, hold in the quaternionic setting. In fact, for any \( p, q \) such that \( 0 < p < q \leq +\infty \), thanks to the classical Jensen inequality we have that

\[
H^q(\mathbb{B}) \subset H^p(\mathbb{B}).
\]

Moreover the inclusion is continuous for \( 1 \leq p < q \leq +\infty \).
In analogy with the complex case, the space $H^2(\mathbb{B})$ is special. Indeed the 2-norm turns out to be induced by an inner product (compare with [1]).

**Proposition 3.8.** Let $f \in H^2(\mathbb{B})$ and let $f(q) = \sum_{n \geq 0} q^n a_n$ be its power series expansion. Then the 2-norm of $f$,

$$\|f\|_2 = \sup_{I \in \mathbb{S}} \lim_{r \to 1^-} \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^2 d\theta \right)^{\frac{1}{2}},$$

coincides with

$$\left( \sum_{n \geq 0} |a_n|^2 \right)^{\frac{1}{2}}.$$

**Proof.** We can express the square of the modulus of $f$ in terms of its power series expansion as follows

$$|f(re^{i\theta})|^2 = f(re^{i\theta})^* f(re^{i\theta}) = \sum_{n \geq 0} r^n \overline{a_n} e^{-in\theta} a_n \sum_{m \geq 0} r^m e^{im\theta} a_n = \sum_{n,m \geq 0} r^{n+m} \overline{a_n} e^{i(m-n)\theta} a_m.$$

Thanks to the uniform convergence on compact sets of the series expansion, when we integrate in $\theta$ we get

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^2 d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{n,m \geq 0} r^{n+m} \overline{a_n} e^{i(m-n)\theta} a_m d\theta = \sum_{n,m \geq 0} r^{n+m} \overline{a_n} \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(m-n)\theta} d\theta \right) a_m.$$

Since

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(m-n)\theta} d\theta = \delta_{m,n},$$

where $\delta_{k,j}$ is the Kronecker delta, then

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^2 d\theta = \sum_{n \geq 0} r^{2n} \overline{a_n} a_n = \sum_{n \geq 0} r^{2n} |a_n|^2.$$

Notice that the last quantity in equation (6) does not depend on $I \in \mathbb{S}$. This yields that

$$\|f\|_2^2 = \sup_{I \in \mathbb{S}} \lim_{r \to 1^-} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^2 d\theta = \sup_{I \in \mathbb{S}} \lim_{r \to 1^-} \sum_{n \geq 0} r^{2n} |a_n|^2 = \sum_{n \geq 0} |a_n|^2.$$

Proposition 3.8 suggests to us a way to define an inner product on the space $H^2(\mathbb{B})$. In fact, if $f, g \in H^2(\mathbb{B})$, let $f(q) = \sum_{n \geq 0} q^n a_n$, $g(q) = \sum_{n \geq 0} q^n b_n$ be their power series expansions. Following the proof of the previous result, we obtain that

$$\lim_{r \to 1^-} \frac{1}{2\pi} \int_{-\pi}^{\pi} g(re^{i\theta}) f(re^{i\theta}) d\theta = \sum_{n \geq 0} b_n a_n \quad \text{for any } I \in \mathbb{S}.$$
Therefore it is natural to define, in $H^2(\mathbb{B})$, the inner product

$$
\langle f, g \rangle = \sum_{n \geq 0} \overline{b_n} a_n
$$

that recalls the classic Hermitian product of the space $H^2(\mathbb{D})$.

Looking at the definition of complex Hardy spaces, one could wonder why, instead of the integral mean $M_p(f_I, r)$ defined in (2), taken on a circle, we did not choose the classical integral mean taken on a 3-dimensional sphere. In fact consider

$$
N_p(f, r) = \left( \frac{1}{2\pi r^3} \int_{r^3} |f(q)|^p d\sigma_3(r^3) \right)^{\frac{1}{p}}, \tag{7}
$$

where $\sigma_3(r^3)$ is the usual hypersurface measure of the 3-dimensional sphere $r^3$ and $2\pi^2 r^3$ is the 3-dimensional volume of the 3-dimensional sphere $r^3$. If we then set

$$
N_p(f) = \sup_{0 < r < 1} N_p(f, r),
$$

it turns out that the class of regular functions $f$ such that $N_p(f)$ is finite does not coincide with the space $H^p(\mathbb{B})$ defined earlier. Indeed, using an appropriate change of variable in the integral appearing in (7), it is possible to prove that $N_p(f) \leq ||f||_p$ for any regular function $f$ and any $p \in (0, +\infty)$, thus showing that if $f \in H^p(f)$ then also $N_p(f)$ is finite. To show that the two classes of functions do not coincide, it is possible to exhibit explicit examples. For instance, the function

$$
(f(q) = (1 - q)^{-s}
$$

is such that $N_p(f) < +\infty$ but it does not belong to $H^p(\mathbb{B})$ for all $2 \leq p \leq +\infty$. This example suggests us that a definition of a quaternionic $H^p$ space that relies upon the $N_p$ norm is not convenient. In fact, the function $(1 - q)^{-1}$ has real coefficients, and therefore, its restriction to $\mathbb{B} = \mathbb{B} \cap L_I$ is the holomorphic function $f_I(z) = (1 - z)^{-1}$. It is well known that $f_I(z)$ does not belong to $H^p(\mathbb{B} I)$ for all $2 \leq p \leq +\infty$, hence it would be weird if its regular extension were in the quaternionic $H^p$ space. Furthermore, our choice (2) of the $p$-integral mean allows the slicewise approach to the theory of $H^p$ spaces. Let us, in fact, begin by studying how $||f||_p$ of a regular function $f$ is related with the value of $||f_I||_p$ on the slice $L_I$.

**Proposition 3.9.** A regular function $f$ is in $H^p(\mathbb{B})$ for some $p \in (0, +\infty)$, if and only if there exists $I \in \mathbb{S}$ such that $||f_I||_p$ is bounded.

**Proof.** Let $p \in (0, +\infty)$ and suppose first that $||f||_p < +\infty$. Then trivially, for any $I \in \mathbb{S}$,

$$
||f_I||_p \leq \sup_{I \in \mathbb{S}} ||f_I||_p = ||f||_p < +\infty.
$$

To see the other implication, let $J \in \mathbb{S}$ be such that $||f_J||_p$ is finite, and use the Representation Formula 2.4 to write

$$
||f||_p^p = \sup_{I \in \mathbb{S}} \lim_{r \to 1^-} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{1}{2} \left( f(re^{i\theta}) + f(re^{-i\theta}) \right) + \frac{I J}{2} \left( f(re^{-i\theta}) - f(re^{i\theta}) \right) \right|^p d\theta
$$

$$
\leq \sup_{I \in \mathbb{S}} \lim_{r \to 1^-} \frac{1}{4\pi} \int_{-\pi}^{\pi} \left( |f(re^{i\theta})| + |f(re^{-i\theta})| + |f(re^{i\theta})| + |f(re^{-i\theta})| \right)^p d\theta
$$

$$
= \lim_{r \to 1^-} \frac{2p}{4\pi} \int_{-\pi}^{\pi} \left( |f(re^{i\theta})| + |f(re^{-i\theta})| \right)^p d\theta
$$

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where the last integral does not depend on \( I \in \mathcal{S} \). If \( p \geq 1 \), taking into account the convexity of the map \( x \mapsto x^p \) on the positive real axis, we get that
\[
\|f\|_p^p \leq \lim_{r \to 1^-} \frac{2^p 2p - 1}{4\pi} \int_{-\pi}^{\pi} \left( |f(re^{i\theta})|^p + |f(re^{-i\theta})|^p \right) d\theta = \lim_{r \to 1^-} \frac{2^p}{4\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^p d\theta = 2^{2p-1} \|f_J\|_p^p < +\infty.
\]

On the other hand, if \( 0 < p < 1 \), taking into account the subadditivity on the positive real axis of the map \( x \mapsto x^p \) (it is concave and maps 0 to 0), we have
\[
\|f\|_p^p \leq \lim_{r \to 1^-} \frac{2^p}{4\pi} \int_{-\pi}^{\pi} \left( |f(re^{i\theta})|^p + |f(re^{-i\theta})|^p \right) d\theta = \lim_{r \to 1^-} \frac{2^p}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^p d\theta = 2^p \|f_J\|_p^p < +\infty.
\]
Let \( p = +\infty \). On one side, if \( f \) is bounded, then for any \( I \in \mathcal{S} \)
\[
\|f_I\|_\infty = \sup_{z \in \mathcal{B}_I} |f_I(z)| \leq \sup_{q \in \mathcal{B}} |f(q)| = \|f\|_\infty < +\infty.
\]
On the other side, thanks to Proposition 2.5, if \( J \in \mathcal{S} \) is such that \( \|f_J\|_\infty < +\infty \) we have that
\[
\|f\|_\infty \leq 2 \|f_I\|_\infty < +\infty.
\]

**Remark 3.10.** In particular, we get that if \( \|f_J\|_p < +\infty \) for some \( J \in \mathcal{S} \), we have the following inequalities
\[
\|f_J\|_p \leq \|f\|_p \leq 2 \|f_J\|_p, \quad \text{if } p \in (0, 1),
\]
\[
\|f_J\|_p \leq \|f\|_p \leq 2^\frac{p-1}{p} \|f_J\|_p, \quad \text{if } p \in [1, +\infty),
\]
\[
\|f_J\|_\infty \leq \|f\|_\infty \leq 2 \|f_J\|_\infty \quad \text{if } p = +\infty.
\]

The key fact that allows us to apply classical results to the splitting components of a function in \( H^p(\mathcal{B}) \) is the following.

**Proposition 3.11.** Let \( f \in H^p(\mathcal{B}) \) for some \( p \in (0, +\infty) \). Then for any \( I \in \mathcal{S} \), if the splitting of \( f \) on \( L_I \) with respect to \( J \in \mathcal{S} \), \( J \perp I \), is \( f_I(z) = F(z) + G(z)J \), then the holomorphic functions \( F \) and \( G \) are both in \( H^p(\mathcal{B}_I) \).

**Proof.** Let \( I \in \mathcal{S} \) and consider, for any \( z \in \mathcal{B}_I \), the splitting \( f_I(z) = F(z) + G(z)J \). Then, for any \( z \in \mathcal{B}_I \),
\[
|f_I(z)| = \sqrt{|F(z)|^2 + |G(z)|^2} \geq \max\{|F(z)|, |G(z)|\}. \tag{9}
\]

Hence, for \( p \in (0, +\infty) \),
\[
+\infty > \|f\|_p^p = \sup_{I \in \mathcal{S}} \|f_I\|_p^p \geq \|f_I\|_p^p = \lim_{r \to 1^-} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f_I(re^{i\theta})|^p d\theta \geq \lim_{r \to 1^-} \frac{1}{2\pi} \int_{-\pi}^{\pi} |F(re^{i\theta})|^p d\theta = \|F\|_p^p
\]
and analogously $+\infty > ||f||_p^p \geq ||f_I||_p^p \geq ||f||_p^p$. For $p = +\infty$, formula (9) directly implies that

$$||F||_\infty \leq ||f_I||_\infty \leq ||f||_\infty < +\infty \quad \text{and} \quad ||G||_\infty \leq ||f_I||_\infty \leq ||f||_\infty < +\infty.$$  

\[ \square \]

Notice that Remark 3.10 and Proposition 3.11 imply that, for all $p \in [1, +\infty]$, the space $H^p(\mathbb{B})$ is a Banach space.

The natural guess that if a function is in $H^p(\mathbb{B})$ then its regular conjugate is in $H^p(\mathbb{B})$ as well, is in fact true.

**Proposition 3.12.** Let $p \in (0, +\infty]$ and let $f \in H^p(\mathbb{B})$. Then also the regular conjugate $f^c$ belongs to $H^p(\mathbb{B})$.

**Proof.** If $p = +\infty$ the proof follows directly by Proposition 2.14. Consider then $p \in (0, +\infty)$. For any $I \in \mathfrak{S}$, if $f$ splits on $\mathbb{B}_I$ as $f(z) = F(z) + G(z)J$, then, recalling Definition 2.13, we get that on the same slice $f^c$ can be written as

$$f_I^c(z) = F(\overline{z}) - G(z)J.$$  

Therefore, for $0 \leq r < 1$,

$$M_p(f_I^c, r)^p = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f_I^c(re^{i\theta})|^p d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} |F(re^{i\theta}) - G(re^{i\theta})J|^p d\theta$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left( |F(re^{i\theta})|^2 + |G(re^{i\theta})|^2 \right)^{\frac{p}{2}} d\theta.$$  

If $0 < p < 2$, thanks to the subadditivity on the positive real axis of the map $x \mapsto x^{p/2}$, we get

$$M_p(f_I^c, r)^p \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \left( |F(re^{i\theta})|^p + |G(re^{i\theta})|^p \right) d\theta$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} |F(re^{i\theta})|^p d\theta + \frac{1}{2\pi} \int_{-\pi}^{\pi} |G(re^{i\theta})|^p d\theta = M_p(F, r)^p + M_p(G, r)^p.$$  

Proposition 3.11 yields that both $F$ and $G$ belong to $H^p(\mathbb{B}_I)$, hence

$$||f_I^c||_p^p = \lim_{r \to 1^-} M_p(f_I^c, r)^p \leq \lim_{r \to 1^-} (M_p(F, r)^p + M_p(G, r)^p) = ||F||_p^p + ||G||_p^p < +\infty$$  

and therefore we obtain that $f^c(q) \in H^p(\mathbb{B})$.

If $2 \leq p < +\infty$, thanks to the convexity of the map $x \mapsto x^{p/2}$, we can bound $M_p(f_I^c, r)$ as follows

$$M_p(f_I^c, r)^p \leq \frac{2^{\frac{p}{2}} - 1}{2^{p}} \int_{-\pi}^{\pi} \left( |F(re^{i\theta})|^p + |G(re^{i\theta})|^p \right) d\theta = 2^{\frac{p}{2}-1} (M_p(F, r)^p + M_p(G, r)^p).$$  

Hence, as before,

$$||f_I^c||_p^p \leq 2^{\frac{p}{2}-1} (||F||_p^p + ||G||_p^p) < +\infty,$$  

which concludes the proof.  

\[ \square \]

For the symmetrization of a function in $H^p(\mathbb{B})$ the following result holds true.
Proposition 3.13. For any $p \in (0, +\infty)$, if $f \in H^p(\mathbb{B})$, then the symmetrization $f^* \in H^\frac{p}{2}(\mathbb{B})$. Moreover if $f \in H^\infty(\mathbb{B})$ then also $f^*$ does.

Proof. Let $f \in H^p(\mathbb{B})$ for some $0 < p < +\infty$. For any $r \in [0, 1), I \in \mathbb{S}$ (such that $f(re^{i\theta}) \neq 0$) we have 

$$|f^*(re^{i\theta})| = |f \ast f^*(re^{i\theta})| = |f(re^{i\theta})f^*(f(re^{i\theta})^{-1}re^{i\theta} f(re^{i\theta}))| = |f(re^{i\theta})f^*(re^{i\theta})|$$

for some $J \in \mathbb{S}$. Hence, recalling Proposition 2.14,

$$|f^*(re^{i\theta})| \leq \sup_{I \in \mathbb{S}} |f(re^{i\theta})| \sup_{J \in \mathbb{S}} |f^*(re^{i\theta})| = \left( \sup_{I \in \mathbb{S}} |f(re^{i\theta})| \right)^2 = |f(re^{K\theta})|^2$$

(10)

for a suitable $K \in \mathbb{S}$ which depends on $r$ but does not depend on $I$ (since $\mathbb{S}$ is compact). For such a $K$ inequality (10) implies

$$M_2(f^*_r,r)^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^2 d\theta \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{K\theta})|^p d\theta \leq ||f_k||_p^p \leq ||f||_p^p < +\infty.$$

Taking the limit for $r \to 1^-$, and the supremum in $I \in \mathbb{S}$, we get that $f^* \in H^\frac{p}{2}(\mathbb{B})$. If instead $f \in H^\infty(\mathbb{B})$, we observe that inequality (10) directly implies that, for any $q \in \mathbb{B}$,

$$|f^*(q)| \leq \sup_{q \in \mathbb{B}} |f(q)|^2 = ||f||_\infty^2,$$

and we are done. \qed

The Hölder inequality leads to the following result concerning the $*$-product of regular functions.

Proposition 3.14. Let $p, q \in [1, +\infty]$ be conjugate exponents $\frac{1}{p} + \frac{1}{q} = 1$. If $f \in H^p(\mathbb{B})$ and $g \in H^q(\mathbb{B})$, then $f \ast g \in H^1(\mathbb{B})$ and $g \ast f \in H^1(\mathbb{B})$.

Proof. If $f \equiv 0$ (or $g \equiv 0$) there is nothing to prove. Otherwise, let us consider the case in which $p, q \in (1, +\infty)$. Fix $I \in \mathbb{S}$. For all $r \in [0, 1)$ and $\theta \in [-\pi, \pi)$ such that $f(re^{i\theta}) \neq 0$, set

$$J(r,\theta) = (f(re^{i\theta}))^{-1}I f(re^{i\theta}) \in \mathbb{S}$$

so that recalling Proposition 2.9 we can write

$$f \ast g(re^{i\theta}) = f(re^{i\theta})g(re^{J(r,\theta)\theta}).$$

The function $g$ can be represented as

$$g(re^{J(r,\theta)\theta}) = \frac{1}{2}(g(re^{J(r,\theta)\theta}) + g(re^{-J(r,\theta)\theta})) + \frac{J(r,\theta)}{2}(g(re^{-J(r,\theta)\theta}) - g(re^{J(r,\theta)\theta}))$$

and hence, using the convexity of the function $x \mapsto x^q$ for $x \geq 0$ (since $q > 1$),

$$|g(re^{J(r,\theta)\theta})|^q \leq 2^{q-1}(|g(re^{J(r,\theta)\theta})|^q + |g(re^{-J(r,\theta)\theta})|^q).$$

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Now, using the Hölder inequality we obtain
\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} |f * g(re^{i\theta})| d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})||g(re^{i\theta})| d\theta \\
\leq \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^p d\theta \right)^\frac{1}{p} \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} |g(re^{i\theta})|^q d\theta \right)^\frac{1}{q}
\]
(11)
which leads to
\[
||f * g||_1 \leq 2||f||_p ||g||_q < +\infty.
\]
For the remaining cases \((p = 1, q = +\infty)\) and \((p = +\infty, q = 1)\) the proofs follow the same lines. \(\square\)

**Corollary 3.15.** Let \(p \in [2, +\infty]\). If \(f\) and \(g\) belong to \(H^p(\mathbb{B})\), then \(f * g \in H^1(\mathbb{B})\) and \(g * f \in H^1(\mathbb{B})\).

**Proof.** If \(\frac{1}{p} + \frac{1}{q} = 1\) and \(p \geq 2\), then \(1 \leq q \leq 2\). Therefore \(H^p(\mathbb{B}) \subset H^q(\mathbb{B})\) and hence \(g\) belongs to \(H^q(\mathbb{B})\).

Theorem 3.14 leads to the conclusion. \(\square\)

### 4 Boundary values of regular functions

A very important result, that in the classic case is quite laborious to reach, states that all functions in \(H^p(\mathbb{B})\) have radial limit along almost any ray.

**Proposition 4.1.** Let \(f \in H^p(\mathbb{B})\) for some \(p \in (0, +\infty]\). Then for any \(I \in \mathbb{S}\), the limit
\[
\lim_{r \to 1^-} f(re^{i\theta}) = \tilde{f}(e^{i\theta})
\]
exists for almost every \(\theta \in [0, 2\pi)\).

**Proof.** Let \(I \in \mathbb{S}\). Write the splitting of \(f\) on \(\mathbb{B}_I\) as \(f_I(z) = F(z) + G(z)J\), then Proposition 3.11 yields that the holomorphic functions \(F\) and \(G\) are both in \(H^p(\mathbb{B}_I)\). Classical results in the theory of \(H^p\) spaces (see e.g. [27]) yield that the radial limits
\[
\lim_{r \to 1^-} F(re^{i\theta}) = \tilde{F}(e^{i\theta}) \quad \text{and} \quad \lim_{r \to 1^-} G(re^{i\phi}) = \tilde{G}(e^{i\phi})
\]
exist respectively for almost every \(\theta\) and for almost every \(\phi\). Therefore the radial limit
\[
\lim_{r \to 1^-} f(re^{i\theta}) = \lim_{r \to 1^-} (F(re^{i\theta}) + G(re^{i\theta})J) = \tilde{F}(e^{i\theta}) + \tilde{G}(e^{i\theta})J = \tilde{f}(e^{i\theta})
\]
exists for almost every \(\theta \in [0, 2\pi)\). \(\square\)

**Remark 4.2.** The previous result is slightly stronger than its complex counterpart: in fact, if \(f \in H^p(\mathbb{B})\) for some \(p \in (0, +\infty]\), then on each slice, the radial limit of \(f\) exists along almost any ray.

From now on, we will denote by \(\tilde{f}\) the radial limit of a function \(f \in H^p(\mathbb{B})\). As it happens in the complex case, the function \(\tilde{f}\) is measurable on \(\partial \mathbb{B}\).
**Proposition 4.3.** Let \( f \in H^p(\mathbb{B}) \) for some \( p \in (0, +\infty) \). Then for almost every \( \theta \in [-\pi, \pi) \), the limit
\[
\lim_{r \to 1^-} f(re^{i\theta}) = \tilde{f}(e^{i\theta})
\]
effects for every \( I \in \mathbb{S} \). Namely, the radial limit of \( f \) exists at all points of the sphere \( \cos \theta + (\sin \theta)S \) (contained in the boundary of \( \mathbb{B} \)) for almost every \( \theta \in [-\pi, \pi) \).

**Proof.** Choose \( I, J \in \mathbb{S} \) and use the Representation Formula for \( f \)
\[
f(re^{i\theta}) = \frac{1}{2} \left[ f(re^{i\theta}) + f(re^{-i\theta}) \right] + \frac{JI}{2} \left[ f(re^{-i\theta}) - f(re^{i\theta}) \right].
\]
Since for almost every \( \theta \in [-\pi, \pi) \) the radial limit \( \tilde{f} \) exists at \( e^{i\theta} \) and at \( e^{-i\theta} \), then formula
\[
\tilde{f}(e^{i\theta}) = \frac{1}{2} \left[ \tilde{f}(e^{i\theta}) + \tilde{f}(e^{-i\theta}) \right] + \frac{JI}{2} \left[ \tilde{f}(e^{-i\theta}) - \tilde{f}(e^{i\theta}) \right].
\]
leads to the conclusion. \( \square \)

Moreover, we will show that radial limits of functions in \( H^p(\mathbb{B}) \) can not vanish on a subset of positive measure of the boundary of the ball.

**Proposition 4.4.** Let \( f \in H^p(\mathbb{B}) \) for some \( p \in (0, +\infty) \), \( f \neq 0 \). Then, for any \( I \in \mathbb{S} \), for almost every \( \theta \in [-\pi, \pi) \),
\[
\lim_{r \to 1^-} f(re^{i\theta}) = \tilde{f}(e^{i\theta}) \neq 0.
\]

**Proof.** If \( f \) splits on \( \mathbb{B}_I \) as \( f_I(z) = F(z) + G(z)J \), then the splitting components \( F \) and \( G \) are in \( H^p(\mathbb{B}_I) \) and (thanks to the Identity Principle 2.2) at least one of them is not identically vanishing. Suppose that \( F \neq 0 \) on \( \mathbb{B}_I \). The classical result stated, e.g., in Theorem 17.18, [28], yields that \( \lim_{r \to 1^-} F(re^{i\theta}) = \tilde{F}(e^{i\theta}) \neq 0 \) for almost every \( \theta \in [-\pi, \pi) \). Thanks to the orthogonality of \( I \) and \( J \), we easily conclude that for almost every \( \theta \in [-\pi, \pi) \),
\[
\tilde{f}(e^{i\theta}) = \tilde{F}(e^{i\theta}) + \tilde{G}(e^{i\theta})J \neq 0.
\]
\( \square \)

This easy consequence of the previous result will be used in the sequel.

**Remark 4.5.** If \( f \in H^p(\mathbb{B}) \) for some \( p \in (0, +\infty) \), \( f \neq 0 \), we have that for any \( I \in \mathbb{S} \) and for almost every \( \theta \in [-\pi, \pi) \), there exists \( r_0 > 0 \) such that \( f(re^{i\theta}) \neq 0 \) for all \( r \in [r_0, 1) \).

Let us define the \(*\)-product and the \(*\)-inverse for radial limits. To this aim we prove the following statement.

**Proposition 4.6.** Let \( f \in H^p(\mathbb{B}) \), \( f \neq 0 \), and \( g \in H^q(\mathbb{B}) \) for some \( p, q \in (0, +\infty) \), and let \( \tilde{f} \) and \( \tilde{g} \) be their (almost everywhere) radial limits. For any \( I \in \mathbb{S} \), for almost every \( \theta \in [-\pi, \pi) \),
\[
\lim_{r \to 1^-} f \ast g(re^{i\theta}) = \tilde{f}(e^{i\theta}) \tilde{g}(\tilde{f}(e^{i\theta})^{-1} e^{i\theta} \tilde{f}(e^{i\theta})).
\]
Moreover, for any \( I \in \mathbb{S} \), we have that \( \lim_{r \to 1^-} f^{-\ast} \ast g(re^{i\theta}) \) exists (possibly infinite) for almost every \( \theta \in [-\pi, \pi) \), and when finite,
\[
\lim_{r \to 1^-} f^{-\ast} \ast g(re^{i\theta}) = \tilde{f}(\tilde{f}(e^{i\theta})^{-1} e^{i\theta} \tilde{f}(e^{i\theta}))^{-1} \tilde{g}(\tilde{f}(e^{i\theta})^{-1} e^{i\theta} \tilde{f}(e^{i\theta})).
\]
**Proof.** Proposition 4.4 yields that if \( f \) is not vanishing identically, then for any \( I \in \mathbb{S} \), \( \check{f}(e^{i\theta}) \neq 0 \) for almost every \( \theta \in [-\pi, \pi) \), and the same holds for \( f^\ast \). Set

\[
T(q) = f(q)^{-1} q f(q).
\]

Thanks to Remark 4.5, \( T(re^{i\theta}) = f(re^{i\theta})^{-1} re^{i\theta} f(re^{i\theta}) \) is well defined for any \( I \in \mathbb{S} \), for almost every \( \theta \in [-\pi, \pi) \), and for \( 1 - r \) sufficiently small. Moreover the radial limit

\[
\lim_{r \to 1^-} T(re^{i\theta}) = \check{f}(e^{i\theta})^{-1} e^{i\theta} \check{f}(e^{i\theta}) =: \check{T}(e^{i\theta})
\]

exists for any \( I \in \mathbb{S} \) and almost every \( \theta \in [-\pi, \pi) \). Given any \( I \in \mathbb{S} \), if \( f(re^{i\theta}) \neq 0 \) set \( J(r, \theta) = f(re^{i\theta})^{-1} I f(re^{i\theta}) \in \mathbb{S} \); then, for almost every \( \theta \) the radial limit

\[
\lim_{r \to 1^-} J(r, \theta) = \check{J}(\theta)
\]

exists and belongs to \( \mathbb{S} \). Hence we can write

\[
\check{T}(e^{i\theta}) = \lim_{r \to 1^-} T(re^{i\theta}) = \lim_{r \to 1^-} (r \cos \theta + r \sin \theta (f(re^{i\theta})^{-1} I f(re^{i\theta})) = \cos \theta + (\sin \theta) \check{I}(\theta).
\]

Using the Representation Formula 2.4 twice, for almost every \( \theta \) we can write

\[
\lim_{r \to 1^-} g(T(re^{i\theta})) = \lim_{r \to 1^-} (g(r \cos \theta + r \sin \theta J(r, \theta)))
\]

\[
= \lim_{r \to 1^-} \left( \frac{1}{2} \left( g(re^{i\theta}) + g(re^{-i\theta}) \right) + \frac{J(r, \theta) I}{2} \left( g(re^{-i\theta}) - g(re^{i\theta}) \right) \right)
\]

\[
= \lim_{r \to 1^-} \frac{1}{2} \left( \check{g}(e^{i\theta}) + \check{g}(e^{-i\theta}) \right) + \frac{\check{J}(\theta) I}{2} \left( \check{g}(e^{-i\theta}) - \check{g}(e^{i\theta}) \right)
\]

\[
= \lim_{r \to 1^-} \frac{1}{2} \left( g(re^{i\theta}) + g(re^{-i\theta}) \right) + \frac{\check{J}(\theta) I}{2} \left( \check{g}(e^{-i\theta}) - \check{g}(e^{i\theta}) \right)
\]

\[
= \lim_{r \to 1^-} g(re^{\check{J}(\theta)\theta}) = \check{g}(e^{\check{J}(\theta)\theta}) = \check{g}(\check{T}(e^{i\theta})).
\]

Hence, recalling Proposition 2.9, we have

\[
\lim_{r \to 1^-} f * g(re^{i\theta}) = \lim_{r \to 1^-} f(re^{i\theta}) g(T(re^{i\theta})) = \check{f}(e^{i\theta}) \check{g}(\check{T}(e^{i\theta})).
\]

The same arguments apply also to the proof for the regular quotient. \( \square \)

We are now ready to give the announced definitions.

**Definition 4.7.** Let \( f \in H^p(\mathbb{B}) \) and \( g \in H^q(\mathbb{B}) \) for some \( p, q \in (0, +\infty] \). For any \( I \in \mathbb{S} \), for almost every \( \theta \in [-\pi, \pi) \), let \( \check{f}(e^{i\theta}) \) and \( \check{g}(e^{i\theta}) \) be the radial limits of \( f \) and \( g \). We define the \(*\)-product of \( \check{f} \) and \( \check{g} \) as

\[
\check{f} * \check{g}(e^{i\theta}) = \lim_{r \to 1^-} f * g(re^{i\theta})
\]

for almost every \( \theta \). If moreover \( \check{f}(e^{i\theta}) \neq 0 \) at all points \( e^{i\theta} \) where it is defined, then we can define the \(*\)-quotient of \( \check{f} \) and \( \check{g} \) as

\[
\check{f}^{-*} * \check{g}(e^{i\theta}) = \lim_{r \to 1^-} f^{-*} * g(re^{i\theta})
\]

for almost every \( \theta \). In particular, if \( g \equiv 1 \), we obtain the definition of the \(*\)-inverse of \( \check{f} \).
Thanks to the existence of the radial limit it is possible to obtain integral representations for functions in $H^p(\mathbb{B})$ for $p \in [1, \infty]$. 

**Theorem 4.8.** If $f \in H^p(\mathbb{B})$ for $p \in [1, \infty]$, then, for any $I \in \mathbb{S}$, $f_I$ is the Poisson integral and the Cauchy integral of its radial limit $\tilde{f}_I$, i.e.,

$$f_I(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - r^2}{1 - 2r \cos(\theta - t) + r^2} \tilde{f}_I(e^{i\tau}) dt$$

and

$$f_I(z) = \frac{1}{2\pi} \int_{\partial \mathbb{B}} \frac{d\zeta}{\zeta - z} \tilde{f}_I(\zeta).$$

**Proof.** As we have seen in the proof of Theorem 4.1, if $f \in H^1(\mathbb{B})$ then, for any $I \in \mathbb{S}$, if $f$ splits on $\mathbb{B}_I$ with respect to $J$, as $f_I(z) = F_I(z) + G_I(z)J$, the splitting components $F_I$ and $G_I$ of $f_I$ are in $H^1(\mathbb{B}_I)$. Clearly their radial limits are related (for almost every $\theta$) by the equation $f_I(e^{i\theta}) = \tilde{F}_I(e^{i\theta}) + \tilde{G}_I(e^{i\theta})J$.

Thanks to classical results in this setting, (see e.g. [28], Corollary to Theorem 17.12) we have that both $F_I$ and $G_I$ are the Poisson and the Cauchy integrals of their radial limits $\tilde{F}_I$ and $\tilde{G}_I$. Hence, for the Cauchy integral,

$$f_I(z) = F_I(z) + G_I(z)J = \frac{1}{2\pi} \int_{\partial \mathbb{B}_I} \frac{\tilde{F}_I(\zeta)}{\zeta - z} d\zeta + \frac{1}{2\pi} \int_{\partial \mathbb{B}_I} \frac{\tilde{G}_I(\zeta)}{\zeta - z} d\zeta J$$

$$= \frac{1}{2\pi} \int_{\partial \mathbb{B}_I} \frac{d\zeta}{\zeta - z} \left( \tilde{F}_I(\zeta) + \tilde{G}_I(\zeta)J \right) = \frac{1}{2\pi} \int_{\partial \mathbb{B}_I} \frac{d\zeta}{\zeta - z} \tilde{f}_I(\zeta).$$

Analogously for the Poisson integral, if $z = re^{i\theta}$ for some $r \in (0, 1)$, $\theta \in [-\pi, \pi)$, and $P_r(\theta)$ is the classical Poisson kernel (see e.g. [27], Chapter 3),

$$P_r(\theta) = \frac{1 - r^2}{1 - 2r \cos \theta + r^2},$$

then

$$f_I(re^{i\theta}) = F_I(re^{i\theta}) + G_I(re^{i\theta})J$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta - t) \left( \tilde{F}_I(e^{i\tau}) + \tilde{G}_I(e^{i\tau})J \right) dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta - t) \tilde{f}_I(e^{i\tau}) dt.$$

Thanks to the relations of inclusion of $H^p$ spaces (see Remark 3.7) the proof is complete. \hfill \square

Our next goal is to show that, for any $p \in (0, \infty]$, the radial limits $\tilde{f}_I$ of the restrictions of a function $f$ in $H^p(\mathbb{B})$, are $L^p$ functions on the circle $\partial \mathbb{B}_I$.

**Definition 4.9.** Let $g$ be a (quaternion valued) function defined (almost everywhere) on $\partial \mathbb{B}_I$ and such that $|g|$ is measurable. If $p \in (0, \infty]$ we set $\|g\|_{L^p}$ to be the integral mean

$$\|g\|_{L^p} = \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} |g(e^{i\theta})|^p d\theta \right)^{\frac{1}{p}},$$

If $p = \infty$, we set

$$\|g\|_{L^\infty} = \sup_{\theta \in (-\pi, \pi)} |g(e^{i\theta})|.$$
For any \( p \in (0, +\infty) \), we denote by \( L^p(\partial B_1) \) the standard \( L^p \) space,

\[
L^p(\partial B_1) = \{ g : \partial B_1 \to \mathbb{H} \mid \text{is measurable and } ||g||_{L^p} < +\infty \}.
\]

Let us now point out that, if \( (f_1)_r(e^{i\theta}) = f(re^{i\theta}) \), then

\[
||| (f_1)_r |||_{L^p} = \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^p d\theta \right)^{\frac{1}{p}} = M_p(f_1, r)
\]

for all \( r \in [0, 1) \) and all \( I \in \mathcal{S} \).

**Proposition 4.10.** Let \( f \in H^p(\mathbb{H}) \) for some \( p \in (0, +\infty) \). Then, for any \( I \in \mathcal{S} \), \( \tilde{f}_I - (f_1)_r \) belongs to \( L^p(\partial B) \) and

\[
\lim_{r \to 1^-} ||\tilde{f}_I - (f_1)_r||_{L^p} = 0.
\]

**Proof.** Thanks to Theorem 2.6 in [12], if \( f_I \) splits as \( f_I(z) = F_I(z) + G_I(z)J \), then for both \( F_I \) and \( G_I \) (holomorphic functions in \( H^p(\mathbb{H}) \))

\[
\lim_{r \to 1^-} ||\tilde{F}_I - (F_1)_r||_{L^p} = \lim_{r \to 1^-} ||\tilde{G}_I - (G_1)_r||_{L^p} = 0.
\]

Hence, if \( 1 \leq p < +\infty \), with obvious notations, we get

\[
\lim_{r \to 1^-} ||\tilde{f}_I - (f_1)_r||_{L^p} = \lim_{r \to 1^-} ||\tilde{F}_I + \tilde{G}_I J - (F_1)_r + (G_1)_r J||_{L^p}
\]

\[
\leq \lim_{r \to 1^-} \left( ||\tilde{F}_I - (F_1)_r||_{L^p} + ||\tilde{G}_I J - (G_1)_r J||_{L^p} \right) = 0.
\]

If instead \( 0 < p < 1 \), consider

\[
\lim_{r \to 1^-} ||\tilde{f}_I - (f_1)_r||_{L^p} = \lim_{r \to 1^-} \left( ||\tilde{F}_I + \tilde{G}_I J - (F_1)_r + (G_1)_r J||_{L^p} \right)
\]

\[
= \lim_{r \to 1^-} 1 \int_{-\pi}^{\pi} \left| \tilde{F}_I - (F_1)_r \right|^p + \left| \tilde{G}_I - (G_1)_r \right|^p d\theta
\]

where the last equality is due to the orthogonality of \( I \) and \( J \). Then, recalling that for any \( 0 < p < 2 \) the map \( x \mapsto x^{p/2} \) is subadditive on the positive real axis, we have

\[
\lim_{r \to 1^-} ||\tilde{f}_I - (f_1)_r||_{L^p} \leq \lim_{r \to 1^-} \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} \left( \left| \tilde{F}_I - (F_1)_r \right|^p + \left| \tilde{G}_I - (G_1)_r \right|^p \right)^{\frac{p}{2}} d\theta \right)
\]

\[
= \lim_{r \to 1^-} \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \tilde{F}_I - (F_1)_r \right|^p d\theta + \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \tilde{G}_I - (G_1)_r \right|^p d\theta \right)
\]

\[
= \lim_{r \to 1^-} \left( ||\tilde{F}_I - (F_1)_r||_{L^p}^p + ||\tilde{G}_I - (G_1)_r||_{L^p}^p \right) = 0.
\]

Now we are able to prove the desired result.
Proposition 4.11. Let $f \in H^p(B)$ for some $p \in (0, \infty)$. Then, for any $I \in S$, the function $\tilde{f}_I : \partial B_I \to \mathbb{H}$ defined for almost every $\theta \in [0, 2\pi)$, by

$$\tilde{f}(e^{i\theta}) = \lim_{r \to 1^-} f(re^{i\theta}),$$

does belong to $L^p(\partial B_I)$ and

$$||\tilde{f}_I||_{L^p} = ||f_I||_p.$$  \(\text{Proof.}\) Thanks to Proposition 4.10, $\lim_{r \to 1^-} ||\tilde{f}_I - (f_I)_r||_{L^p} = 0$ for any $0 < p < +\infty$ and for any $I \in S$. If $1 \leq p < +\infty$, for any $r \in (0, 1)$ we have

$$\left|\left|\left|\tilde{f}_I\right|\right|_{L^p} - \left|\left|\left|(f_I)_r\right|\right|_{L^p}\right| \leq \left|\left|\tilde{f}_I - (f_I)_r\right|\left|_{L^p}\right|\right| = 0.$$  

Therefore

$$||f_I||_p = \lim_{r \to 1^-} M_p(f_I, r) = \lim_{r \to 1^-} \left|\left|\left|(f_I)_r\right|\right|_{L^p}\right| = ||\tilde{f}_I||_{L^p}.$$  

If $0 < p < 1$, since for any $r \in (0, 1)$

$$\left|\left|\left|\tilde{f}_I\right|\right|_{L^p} - \left|\left|\left|(f_I)_r\right|\right|_{L^p}\right| \leq \left|\left|\tilde{f}_I - (f_I)_r\right|\left|_{L^p}\right|\right|,$$

then we can use the same argument to prove the assertion. Finally, if $p = +\infty$, on one side, for almost all $\theta$,

$$\left|\tilde{f}_I(e^{i\theta})\right| = \lim_{r \to 1^-} f_I(re^{i\theta}) = \lim_{r \to 1^-} \sup_{0 \leq r < 1} |f_I(re^{i\theta})|$$

yielding

$$||\tilde{f}_I||_{L^\infty} = \sup_{\theta \in (-\pi, \pi)} \left|\tilde{f}_I(e^{i\theta})\right| \leq \sup_{\theta \in (-\pi, \pi)} \sup_{0 \leq r < 1} \sup_{0 \leq \theta < 1} \sup_{z \in B_I} |f_I(e^{i\theta})| \leq \sup_{z \in B_I} |f(z)| = ||f_I||_\infty.$$  

On the other side, since $f_I$ is the Poisson integral of $\tilde{f}_I$ (see Theorem 4.8), for any $r \in (0, 1)$ and for any $\theta \in (-\pi, \pi)$, we have

$$|f_I(re^{i\theta})| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |P_r(\theta - t)\tilde{f}_I(e^{it})| dt.$$  

Since the classical Poisson kernel is a non-negative function on the unit circle with integral mean equal to 1, then for any $r \in (0, 1)$ and for any $\theta \in (-\pi, \pi)$,

$$|f_I(re^{i\theta})| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta - t)|\tilde{f}_I(e^{it})| dt \leq ||\tilde{f}_I||_{L^\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta - t) dt = ||\tilde{f}_I||_{L^\infty}.$$  

Therefore we obtain

$$||f_I||_\infty \leq \sup_{0 \leq r < 1} \sup_{\theta \in (-\pi, \pi)} |f_I(re^{i\theta})| \leq ||\tilde{f}_I||_{L^\infty},$$  

hence concluding the proof. \(\square\)
5 An integral representation

The integral representation that we present in this section allows us to recover the value of a function \( f \), regular in the unit ball \( B \) and continuous up to the closure \( \overline{B} \), from the boundary values of the real parts of its splitting components. Let \( I \) be an element of \( S \). If \( f_I(z) = F_I(z) + G_I(z)J \), then results well known in the complex setting yield that the splitting components \( F_I \) and \( G_I \) can be represented by means of the holomorphic kernel \( H_\theta \) defined, for all \( z \in B_I \), by

\[
H_\theta(z) = \frac{e^{i\theta} + z}{e^{i\theta} - z},
\]

where \( \theta \in [-\pi, \pi) \). We recall that, if \( z = re^{it} \), then

\[
\Re\left(H_\theta(re^{it})\right) = P_r(t - \theta)
\]

where \( P_r(t) \) is the classical Poisson kernel. The integral representations that we obtain for \( F_I \) and \( G_I \) are the following (see [27], Chapter 3 and [28], Chapter 11).

\[
F_I(z) = \Re F_I(0) - \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \Re F_I(e^{i\theta}) d\theta
\]

and

\[
G_I(z) = \Re G_I(0) - \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \Re G_I(e^{i\theta}) d\theta.
\]

If we set

\[
C_0 = \Re F_I(0) + \Re G_I(0) - \left( F_I(0) + G_I(0)J \right),
\]

we can represent \( f_I \) on \( B_I \) in the following manner

\[
f_I(z) = C_0 + \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \left( \Re F_I(e^{i\theta}) + \Re G_I(e^{i\theta})J \right) d\theta.
\]

In order to study the constant \( C_0 \) and the function \( z \mapsto \Re F_I(z) + \Re G_I(z)J \) appearing in equality (12), consider the splitting of \( f \) on \( L_J \) with respect to \( IJ = K \),

\[
f_J(w) = F_J(w) + G_J(w)K,
\]

where \( w \in \overline{B}_J \). On the real axis we can express the value of \( F_J \) (and of \( G_J \)) in terms of the splitting functions \( F_I, G_I \). In fact since \( \overline{B} \cap R = \overline{B}_I \cap \overline{B}_J \), for any \( x \in \overline{B} \cap R \) we have

\[
f_J(x) = f_I(x) = \Re F_I(x) + \Im F_I(x)I + \Re G_I(x)J + \Im G_I(x)IJ,
\]

and hence

\[
F_J(x) = \Re F_I(x) + \Re G_I(x)J.
\]

(13)

Therefore we get

\[
C_0 = \Re F_I(0) + \Re G_I(0) - \left( F_I(0) + G_I(0)J \right) = F_J(0) - \left( F_I(0) + G_I(0)J \right).
\]

Moreover, equality (13) and the Identity Principle 2.2 imply that, if \( \text{ext}(F_J)(q) \) denotes as usual the unique regular extension of \( F_J \) to \( \overline{B} \),
\[
\text{Re } F_I(z) + \text{Re } G_I(z) J = \text{ext}(F_J)(z) \quad \text{for any } z \in \mathbb{B}_I.
\]

Hence the integral representation of \( f_I \) on \( \mathbb{B}_I \) is
\[
f_I(z) = F_J(0) - (F_I(0) + G_I(0)) J + \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \text{ext}(F_J)(e^{i\theta}) d\theta.
\]

If \( q = x + y L \in \mathbb{B} \), using the Representation Formula 2.4 we can retrieve the value of \( f(q) \) in terms of \( f_I(z) \) and \( f_I(\overline{z}) \) (where \( z = x + y i \)) as
\[
f(q) = F_J(0) - (F_I(0) + G_I(0)) J + \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[ \frac{1}{2} \left( \frac{e^{i\theta} + z}{e^{i\theta} - z} + \frac{e^{i\theta} + \overline{z}}{e^{i\theta} - \overline{z}} \right) + \frac{L I}{2} \left( \frac{e^{i\theta} + \overline{z}}{e^{i\theta} - \overline{z}} - \frac{e^{i\theta} + z}{e^{i\theta} - z} \right) \right] \text{ext}(F_J)(e^{i\theta}) d\theta.
\] (14)

Consider now the unique regular extension of the holomorphic kernel \( H_\theta \) to the entire ball \( \mathbb{B} \), namely the regular kernel \( \text{ext}(H_\theta)(q) = (e^{i\theta} - q)^{-*} \ast (e^{i\theta} + q) \).

Notice that the Representation Formula 2.4 applied to the function \( \text{ext}(H_\theta)(q) \), yields that
\[
\frac{1}{2} \left( \frac{e^{i\theta} + z}{e^{i\theta} - z} + \frac{e^{i\theta} + \overline{z}}{e^{i\theta} - \overline{z}} \right) + \frac{L I}{2} \left( \frac{e^{i\theta} + \overline{z}}{e^{i\theta} - \overline{z}} - \frac{e^{i\theta} + z}{e^{i\theta} - z} \right) = (e^{i\theta} - q)^{-*} \ast (e^{i\theta} + q),
\]

hence, from equation (14), we can conclude
\[
f(q) = F_J(0) - (F_I(0) + G_I(0)) J + \frac{1}{2\pi} \int_{-\pi}^{\pi} ((e^{i\theta} - q)^{-*} \ast (e^{i\theta} + q)) \text{ext}(F_J)(e^{i\theta}) d\theta.
\]

We point out that we can write the previous result also with an intrinsic notation. Indeed, set \( \zeta = e^{i\theta} \), \( \zeta \in \partial \mathbb{B}_I \), and consider the corresponding holomorphic kernel \( H_\zeta \) defined, for \( z \in \mathbb{B}_I \), by
\[
H_\zeta(z) = \frac{1 + z \overline{\zeta}}{1 - z \overline{\zeta}} = \frac{\zeta + z}{\zeta - z}.
\]

As in the parametric version, a repeated use of the Representation Formula 2.4 leads to
\[
f(q) = F_J(0) - (F_I(0) + G_I(0)) J + \frac{1}{2\pi I} \int_{\partial \mathbb{B}_I} ((\zeta - q)^{-*} \ast (\zeta + q)) \frac{d\zeta}{\zeta} \text{ext}(F_J)(\zeta),
\]

where
\[
\text{ext}(H_\zeta)(q) = (\zeta - q)^{-*} \ast (\zeta + q)
\]
is the unique regular extension to the entire ball \( \mathbb{B} \) of the holomorphic kernel \( H_\zeta \).

6 Factorization theorems

In the classical setting it is possible to decompose a holomorphic function in \( H^p(\mathbb{D}) \) into its inner and outer factors, see Chapter 5 of [27]. The quaternionic counterparts are defined as follows.
Definition 6.1. A regular function \( E \in H^1(B) \) is an outer function if for any \( f \in H^1(B) \) such that \( |\tilde{E}(q)| = |\tilde{f}(q)| \) for almost any \( q \in \partial B \), we have
\[
|E(q)| \geq |f(q)| \quad \text{for any } q \in B.
\]

In the complex setting, the definition of outer function can be given equivalently in terms of a never-vanishing holomorphic function, expressed by means of the integral representation introduced in Section 5 (see [27], Chapter 5). This correspondence fails to be true for regular functions, since in general we can not reproduce the same construction.

Definition 6.2. A regular function \( I \in H^\infty(B) \) is an inner function if
\[
|I(q)| \leq 1 \quad \text{for any } q \in B \quad \text{and} \quad |\tilde{I}(q)| = 1 \quad \text{for almost any } q \in \partial B.
\]

We recall that, in the complex setting, each inner function can be factored into a product of two distinct types of inner functions, namely a Blaschke product and a singular function. Let us define the quaternionic analogues of singular functions, and then study the analogues of Blaschke products.

Definition 6.3. An inner function \( f \in H^\infty(B) \) is a singular function if \( f \) is non-vanishing on \( B \).

In the complex setting there are two possible approaches to the factorization of an \( H^p \) function (compare [12, 27]). The first one is to begin by the extraction of the outer factor, thus obtaining the inner one. At this point, extracting the zeros one separates the Blaschke product and the singular part. The other possibility is to start with the extraction of the zeros, thus identifying the Blaschke product, and then separate the outer factor from the singular one. In the quaternionic setting, since we can not reproduce the construction of the outer factor, let us begin with the extraction of the zeros of a function \( f \) in \( H^p(B) \) for \( p \in (0, +\infty) \).

Thanks to the characterization of the zero set of regular functions (see Theorem 2.16), recalling Definitions 2.17 and 2.18, we can build a sequence representing the zeros of \( f \).

Definition 6.4. Let \( f \) be a regular function. The sequence of zeros of \( f \) is a sequence \( \{a_n\}_{n \in \mathbb{N}} \), contained in the zero set of \( f \), composed as follows: the isolated zeros are listed according to their isolated multiplicity; the spherical zeros are represented by any element that generates the 2-sphere of zeros together with its conjugate, listed according to their spherical multiplicity. Namely, if \( a_\ell \) generates a spherical zero (not containing \( a_{\ell-1} \)) with spherical multiplicity \( 2m \), then \( a_{\ell+2k} = a_\ell \) and \( a_{\ell+2k+1} = \overline{a_\ell} \) for all \( k = 0, \ldots, m - 1 \).

In analogy with the complex case, we can give the following two definitions (previously introduced in [2]).

Definition 6.5. Let \( a \) be a point in \( B \). The Blaschke factor associated with \( a \) is the regular Moebius transformation defined as
\[
M_a(q) = (1 - q\overline{a})^{-*} (a - q) \frac{\overline{a}}{|a|}
\]
if \( a \neq 0 \), and
\[
M_a(q) = q
\]
if \( a = 0 \).

Definition 6.6. If \( \{a_n\}_{n \in \mathbb{N}} \) is a sequence of points in \( B \) such that the infinite regular product
\[
B(q) = \prod_{n \geq 0} M_{a_n}(q)
\]
converges uniformly on compact sets of \( B \), then \( B \) is called Blaschke product, and it defines a regular function on \( B \) (see Theorem 2.21).
Blaschke products are examples of inner functions.

**Theorem 6.7.** Let

\[ B(q) = \prod_{n \geq 0} M_{a_n}(q) \]

be a Blaschke product. Then \(|B(q)| \leq 1\) for any \(q \in \mathbb{B}\) and \(|\overline{B}(q)| = 1\) for almost any \(q \in \partial \mathbb{B}\).

**Proof.** For \(k \in \mathbb{N}\) consider the finite (regular) product \(B_k\),

\[ B_k(q) = \prod_{n=0}^{k} M_{a_n}(q) = \prod_{n=0}^{k} M_{a_n}(T_n(q)), \]

where (for any \(\ell = 0, \ldots, k\)) \(T_\ell\) is defined iteratively (outside the zero set of \(B_{\ell-1}\)), in view of Proposition 2.9, by

\[ T_\ell(q) = \left( \prod_{j=0}^{\ell-1} M_{a_j}(q) \right)^{-1} q \prod_{j=0}^{\ell-1} M_{a_j}(T_j(q)) = \left( \prod_{j=0}^{\ell-1} M_{a_j}(T_j(q)) \right)^{-1} \prod_{j=0}^{\ell-1} M_{a_j}(T_j(q)). \]

Since \(|T_n(q)| = |q| < 1\) for any \(q \in \mathbb{B}\) and since each factor \(M_{a_n}\) is bounded in modulus by 1 on \(\mathbb{B}\) (see Proposition 2.24), we get that we can bound each finite product,

\[ |B_k(q)| = \prod_{n=0}^{k} |M_{a_n}(T_n(q))| < 1. \]

Thanks to the uniform convergence on compact sets of the finite products to \(B\) we get that

\[ |B(q)| \leq 1 \quad \text{for any } q \in \mathbb{B}. \]

Hence \(B \in H^\infty(\mathbb{B})\) and therefore for any \(I \in \mathbb{S}\), for almost any \(\theta \in [-\pi, \pi)\) there exists the radial limit

\[ \lim_{r \to 1^-} B(re^{i\theta}) = \overline{B}(e^{i\theta}). \]

The same clearly holds true for any finite product \(B_k \in H^\infty(\mathbb{B})\). For any \(k \in \mathbb{N}\), the following regular function

\[ B_k^{-*} \ast B(q) = \prod_{n \geq k+1} M_{a_n}(q) \]

is a Blaschke product (and hence a bounded regular function) and

\[ \lim_{k \to \infty} B_k^{-*} \ast B(q) = 1 \]

uniformly on compact sets of \(\mathbb{B}\). Observe that, for any \(k \in \mathbb{N}\), the finite regular product \(B_k\) (as well as its regular conjugate \(B_k^c\)) is regular up to the closure of \(\mathbb{B}\). Moreover, both \(B_k\) and \(B_k^c\) only have finitely many zeros in the interior of \(\mathbb{B}\). Hence (see Proposition 2.22) the function

\[ \tau_k(q) = \left( B_k^c(q) \right)^{-1} q B_k^c(q) \]
is a diffeomorphism of a neighborhood (for instance a spherical shell) of \( \partial B \) onto itself, it maps the boundary of \( B \) onto itself and it has inverse given by

\[
\tau_k^{-1}(q) = (B_k(q))^{-1}qB_k(q).
\]

Let \( I \in \mathbb{S} \). Proposition 2.22 yields that we can write

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} |B_k^* B(\tau_k^{-1}(re^{i\theta}))| d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| B_k(re^{i\theta}) \right|^{-1} |B(re^{i\theta})| d\theta 
\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \left( \max_{\theta \in [-\pi, \pi]} \left| B_k(re^{i\theta}) \right|^{-1} \right) |B(re^{i\theta})| d\theta.
\]

Since \( B_k \) is a finite Blaschke product, it maps \( \partial B \) to itself and

\[
\lim_{r \to 1} |B_k(re^{i\theta})| = |\tilde{B}_k(e^{i\theta})| = 1,
\]

for any \( \theta \). Hence, for any \( \varepsilon > 0 \) there exists \( r(\varepsilon) \) such that for any \( r(\varepsilon) \leq r < 1 \)

\[
\max_{\theta \in [-\pi, \pi]} \left| B_k(re^{i\theta}) \right|^{-1} \leq 1 + \varepsilon.
\]

Therefore, for any \( \varepsilon > 0 \) there exists \( r \) sufficiently near to 1 such that

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} |B_k^* B(\tau_k^{-1}(re^{i\theta}))| d\theta \leq \frac{1 + \varepsilon}{2\pi} \int_{-\pi}^{\pi} |B(re^{i\theta})| d\theta.
\]

Thanks to Proposition 3.1 and since \( ||\tilde{B}||_{L^\infty} = ||B||_{L^\infty} \leq 1 \), we have then that

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} |B_k^* B(\tau_k^{-1}(re^{i\theta}))| d\theta \leq \frac{1 + \varepsilon}{2\pi} \int_{-\pi}^{\pi} |\tilde{B}(e^{i\theta})| d\theta \leq 1 + \varepsilon.
\] (15)

Set

\[
J_k(r, \theta) = (B_k(re^{i\theta}))^{-1}IB_k(re^{i\theta}) \in \mathbb{S},
\]

so that

\[
\tau_k^{-1}(re^{i\theta}) = r \cos \theta + (r \sin \theta)J_k(r, \theta).
\]

Using the Representation Formula 2.4 we can then write

\[
|B_k^* B(\tau_k^{-1}(re^{i\theta}))| = |B_k^* B(re^{J_k(r, \theta)\theta})|
= \left| \frac{1}{2} (B_k^* B(re^{i\theta}) + B_k^* B(re^{-i\theta})) + \frac{J_k(r, \theta)}{2} (B_k^* B(re^{-i\theta}) - B_k^* B(re^{i\theta})) \right|.
\]

Since \( B_k \) is a diffeomorphism of (a neighborhood of) \( \partial B \) onto itself, we get that for every \( \theta \) the limit

\[
\lim_{r \to 1^-} J_k(r, \theta) = \tilde{J}_k(\theta)
\]

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exists. Hence, recalling Proposition 4.6 and 4.11, applying twice the Representation Formula 2.4, we get
\[
\lim_{r \to 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| B_k^{-*} * B(\tau_k^{-1}(re^{i\theta})) \right| d\theta \\
= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{1}{2} (\bar{B}_k^{-*} * \bar{B}(e^{i\theta}) + B_k^{-*} * B(e^{-i\theta})) + \frac{\hat{J}_k(\theta)I}{2} \right| d\theta \\
= \lim_{r \to 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{1}{2} (B_k^{-*} * B(re^{i\theta}) + B_k^{-*} * B(re^{-i\theta})) + \frac{\hat{J}_k(\theta)I}{2} \right| d\theta \\
= \lim_{r \to 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{1}{2} (B_k^{-*} * B(re^{i\theta}) - B_k^{-*} * B(re^{-i\theta})) \right| d\theta
\]
where \( \hat{J}_k^{-1} \) is the radial limit of \( J_k^{-1} \) (see Proposition 4.6). Recalling inequality (15), we get then that for any \( \varepsilon > 0 \)
\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} \left| B_k^{-*} * B(\tilde{J}_k(e^{i\theta})) \right| d\theta = \lim_{r \to 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| B_k^{-*} * B(\tau_k^{-1}(re^{i\theta})) \right| d\theta \\
\leq \frac{1 + \varepsilon}{2\pi} \int_{-\pi}^{\pi} \left| \tilde{B}(e^{i\theta}) \right| d\theta \leq 1 + \varepsilon.
\]
Now we want to take the limit for \( k \to +\infty \) of the previous inequality. For almost every \( \theta \), the limit
\[
\lim_{k \to +\infty} \tilde{\tau}_k^{-1}(e^{i\theta}) = \lim_{k \to +\infty} (\hat{B}_k(e^{i\theta}))^{-1} e^{i\theta} \tilde{B}_k(e^{i\theta}) = (\tilde{B}(e^{i\theta}))^{-1} e^{i\theta} \tilde{B}(e^{i\theta}) = \tilde{\tau}^{-1}(e^{i\theta})
\]
does exist and it coincides with
\[
\lim_{k \to +\infty} \tilde{\tau}_k^{-1}(e^{i\theta}) = \lim_{k \to +\infty} (\cos \theta + (\sin \theta) \tilde{J}_k(\theta)) = \cos \theta + (\sin \theta) \tilde{J}(\theta),
\]
which implies that \( \tilde{J}_k(\theta) \) converges for almost every \( \theta \). Using again the Representation Formula 2.4, we have then that
\[
\lim_{k \to +\infty} \left| \tilde{B}_k^{-*} * \tilde{B}(\tilde{\tau}_k^{-1}(e^{i\theta})) \right| \\
= \lim_{k \to +\infty} \left| \frac{1}{2} (\tilde{B}_k^{-*} * \tilde{B}(e^{i\theta}) + B_k^{-*} * \tilde{B}(e^{-i\theta})) + \frac{\hat{J}_k(\theta)I}{2} \right| \left| \tilde{B}_k^{-*} * \tilde{B}(e^{-i\theta}) - B_k^{-*} * \tilde{B}(e^{i\theta}) \right|
\]
Recalling that \( B_k^{-*} * B \) converges uniformly on compact sets to the function constantly equal to 1, and that \( \tilde{J}_k(\theta) \) converges, we obtain
\[
\lim_{k \to +\infty} \left| \tilde{B}_k^{-*} * \tilde{B}(\tilde{\tau}_k^{-1}(e^{i\theta})) \right| = 1.
\]
Therefore we get that, for any \( \varepsilon > 0 \)
\[
1 = \lim_{k \to +\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \tilde{B}_k^{-*} * \tilde{B}(\tilde{\tau}_k^{-1}(e^{i\theta})) \right| d\theta \\
\leq \frac{1 + \varepsilon}{2\pi} \int_{-\pi}^{\pi} \left| \tilde{B}(e^{i\theta}) \right| d\theta \leq 1 + \varepsilon,
\]
that finally implies
\[
\left| \tilde{B}(e^{i\theta}) \right| = 1
\]
for almost every \( \theta \).
\[\square\]
In order to show that the sequence of zeros of a function \( f \) in \( H^p(\mathbb{B}) \) is such that the Blaschke product associated with it is convergent, we will use classical results in the theory of complex \( H^p \) spaces, that apply to the symmetrization of \( f \).

**Remark 6.8.** Recall that the symmetrization \( f^* \) of a regular function \( f \) behaves exactly as a holomorphic function on any slice \( L_I \). Hence, if \( f^* \) is in \( H^p(\mathbb{B}) \) (and therefore in \( H^p(\mathbb{B}_I) \) for some \( p \in (0, +\infty) \), and \( f^* \not\equiv 0 \), classical results (see e.g. Theorem 15.23 in [28]) yield that, if \( \{a_n^I\}_{n \geq 0} \) is the sequence of zeros of \( f^* \) in \( \mathbb{B}_I \), listed according to their multiplicity, then the Blaschke condition

\[
\sum_{n \geq 0} (1 - |a_n^I|) < +\infty
\]

is fulfilled.

Consequently,

**Proposition 6.9.** Let \( p \in (0, +\infty] \), \( f \in H^p(\mathbb{B}) \), \( f \not\equiv 0 \) and let \( \{b_n\}_{n \in \mathbb{N}} \) be its sequence of zeros. Then \( \{b_n\}_{n \in \mathbb{N}} \) satisfies the Blaschke condition

\[
\sum_{n \geq 0} (1 - |b_n|) < +\infty.
\]

**Proof.** Let us consider the symmetrization of \( f \), \( f^* \). Thanks to Remark 6.8, for any \( I \in \mathbb{S} \), if \( \{a_n^I\}_{n \in \mathbb{N}} \) is the sequence of zeros of \( f^* \) on \( \mathbb{B}_I \), then it satisfies the Blaschke condition. If \( b_n = b_{1,n} + b_{2,n}I \), let us set \( b'_n = b_{1,n} + b_{2,n}I \) for all \( n \in \mathbb{N} \). Then \( |b_n| = |b'_n| \) for all \( n \in \mathbb{N} \), and \( \{b'_n\}_{n \in \mathbb{N}} \subseteq \{a_n^I\}_{n \in \mathbb{N}} \). Therefore (recalling that \( |a_n^I| < 1 \))

\[
\sum_{n \geq 0} (1 - |b_n|) = \sum_{n \geq 0} (1 - |b'_n|) \leq \sum_{n \geq 0} (1 - |a_n^I|) < +\infty.
\]

The previous result implies that the Blaschke product built from the zeros of a regular function \( f \in H^p(\mathbb{B}) \) for some \( p \in (0, +\infty) \) does converge uniformly on compact sets (compare with [2]).

**Proposition 6.10.** Let \( f \) be in \( H^p(\mathbb{B}) \) for some \( p \in (0, +\infty] \) and let \( \{a_n\}_{n \geq 0} \) be its sequence of zeros. If \( M_{a_n}(q) \) denotes the Blaschke factor associated with \( a_n \),

\[
M_{a_n}(q) = (1 - qa_n)^{-*} (a_n - q) \frac{p_n}{|a_n|}
\]

(\( M_{a_n}(q) = q \) if \( a_n = 0 \), then the Blaschke product

\[
B(q) = \prod_{n \geq 0} M_{a_n}(q)
\]

converges uniformly on compact sets of \( \mathbb{B} \). Moreover, the function \( B \) is regular on \( \mathbb{B} \).
**Proof.** Thanks to Theorem 2.20, we know that the infinite regular product
\[
\prod_{n \geq 0} M_{a_n}(q)
\] (16)
converges uniformly on compact sets if and only if the infinite product
\[
\prod_{n \geq 0} M_{a_n}(q)
\]
does. Furthermore Theorem 2.19 gives us a sufficient condition for the uniform convergence on compact sets of
\[
\prod_{n \geq 0} M_{a_n}(q),
\]namely the uniform convergence on compact sets of the series
\[
\sum_{n \geq 0} |1 - M_{a_n}(q)|.
\]
To estimate the latter, for any \( n \in \mathbb{N} \) let us set \( T_n(q) = (1 - qa_n)^{-1}q(1 - qa_n) \) (in view of Proposition 2.22), and consider
\[
|1 - M_{a_n}(q)| = |1 - (1 - q\pi_n)^{-*}(a_n - q\frac{\pi_n}{|a_n|})| = |1 - (1 - T_n(q)\pi_n)^{-1}(a_n - T_n(q))\frac{\pi_n}{|a_n|}| = |(1 - T_n(q)\pi_n)^{-1}(1 - T_n(q)\pi_n + T_n(q)(\frac{\pi_n}{|a_n|} - |a_n|))| = |(1 - T_n(q)\pi_n)^{-1}(1 - |a_n|)(1 + T_n(q)\frac{\pi_n}{|a_n|})| = |(1 - T_n(q)\pi_n)^{-1}(1 - |a_n|)(1 + T_n(q)\frac{\pi_n}{|a_n|})|.
\]
Recalling that \(|T_n(q)| = |q| < 1 \) and \(|a_n| < 1\), we obtain
\[
|1 - M_{a_n}(q)| \leq (1 - |q|)^{-1}(1 - |a_n|)^2.
\]
Recalling that the sequence of zeros of a function in \( H^p(\mathbb{B}) \) satisfies the Blaschke condition,
\[
\sum_{n \geq 0} (1 - |a_n|) < +\infty,
\]
we get that
\[
\sum_{n \geq 0} |1 - M_{a_n}(q)| < +\infty
\]
and hence the convergence of the infinite regular product (16). The regularity of \( B(q) \) is guaranteed by Theorem 2.21. \( \square \)
We point out that the convergence of $B(q)$ depends only on the moduli $|a_n|$, $n \in \mathbb{N}$. This means that we can build a Blaschke product $\hat{B}(q)$ having exactly the same sequence of zeros $\{a_n\}_{n \in \mathbb{N}}$ of a given regular function $f \in H^p(\mathbb{B})$. In fact, in order to build such a Blaschke product $\hat{B}(q)$, we have to consider the product of Blaschke factors associated with suitable conjugates of the points $a_n$, lying on the same 2-spheres $x_n + y_nS$ generated by $a_n$, taking into account Proposition 2.9.

**Proposition 6.11.** Let $\{a_n\}_{n \in \mathbb{N}}$ be the sequence of zeros of a regular function $f \in H^p(\mathbb{B})$ for some $p \in (0, +\infty]$. Then there exists a Blaschke product $\hat{B}(q)$ having the same sequence of zeros.

**Proof.** We will give the proof in the case in which all the zeros (both isolated and spherical) have multiplicity 1. In this case, we can assume, without loss of generality, that $a_j \neq a_k$ for all $j, k \in \mathbb{N}$. Our aim is now to build a sequence $\{b_n\}_{n \in \mathbb{N}}$, where each $b_n$ is a conjugate of $a_n$, such that the Blaschke product associated with it

$$\hat{B}(q) = \prod_{n \geq 0} M_{b_n}(q)$$

has $\{a_n\}_{n \in \mathbb{N}}$ as its sequence of zeros. The convergence of the Blaschke product is guaranteed by Proposition 6.10. Since the regular multiplication does not conjugate the zeros of the first function in the $*$-product, the first term of the sequence will be equal to $a_0$,

$$b_0 = a_0.$$

For the second term, we need to find $b_1$ such that

$$M_{b_0}(q) * M_{b_1}(q) = M_{a_0}(q) * (1 - q\overline{b_1})^{-*} * (b_1 - q) \frac{\overline{b_1}}{|b_1|}$$

vanishes at $q = a_1$. Notice that, for any $k \in \mathbb{N}$,

$$(1 - q\overline{b_k})^{-*} * (b_k - q) = (b_k - q) * (1 - q\overline{b_k})^{-*},$$

because

$$(b_k - q) * (1 - q\overline{b_k}) = (1 - q\overline{b_k}) * (b_k - q).$$

Hence

$$M_{b_0}(q) * M_{b_1}(q) = M_{a_0}(q) * (b_1 - q) * (1 - q\overline{b_1})^{-*} \frac{\overline{b_1}}{|b_1|},$$

and, thanks to Proposition 2.9, we can write

$$M_{b_0}(q) * M_{b_1}(q) = (M_{a_0}(q)(b_1 - T_1(q))) * \left((1 - q\overline{b_1})^{-*} \frac{\overline{b_1}}{|b_1|}\right),$$

where

$$T_1(q) = (M_{a_0}(q))^{-1} qM_{a_0}(q).$$

Therefore, if we want that this product vanishes at $a_1$, we can set

$$b_1 = T_1(a_1).$$
well defined since $a_1 \neq a_0$. We can iterate this process, setting, for any $n \geq 1$,

$$T_n(q) = \left( \prod_{k=0}^{n-1} M_{b_k}(q) \right)^{-1} q \left( \prod_{k=0}^{n-1} M_{b_k}(q) \right),$$

so that

$$\left( \prod_{k=0}^{n-1} M_{b_k}(q) \right) * M_{b_n}(q) = \left( \prod_{k=0}^{n-1} M_{b_k}(q) \right) * (b_n - q) * (1 - q \bar{b}_n)^{-1} \frac{\bar{b}_n}{|b_n|}$$

$$= \left( \prod_{k=0}^{n-1} M_{b_k}(q) \right) (b_n - T_n(q)) * (1 - q \bar{b}_n)^{-1} \frac{\bar{b}_n}{|b_n|}$$

Hence, if we want that

$$\left( \prod_{k=0}^{n-1} M_{b_k}(q) \right) * M_{b_n}(q)$$

vanishes at $q = a_n$, we have to set

$$b_n = T_n(a_n),$$

well defined since $a_n \neq a_{n-1}$. In the case in which some of the zeros of $f$ have multiplicities greater than 1, the proof follows the same lines; one has only to take into account that:

1. to each isolated zero $a_j$ of $f$ of multiplicity $p > 1$ there corresponds the (regular) power of a Blaschke factor $M_{b_j}^{p}(q)$ which vanishes at $b_j$ with multiplicity $p$;

2. to each spherical zero of $f$ containing $\alpha_k, \beta_k$ and having multiplicity $p > 1$ there corresponds the slice preserving factor $(M_{b_k} * M_{\beta_k}(q))^{*p} = (M_{b_k}(q))^{*p}$.

Remark 6.12. Since we transform the zeros of $f$ by conjugation, all real and spherical zeros of $f$ are not modified by this process.

We are now ready to prove our first result in the direction of finding a factorization for functions in $H^p(B)$.

**Theorem 6.13.** Let $f \in H^p(B)$ for some $p \in (0, +\infty]$. Then we can factor $f$ as

$$f(q) = h * g(q)$$

where $h$ and $g$ are regular functions on $B$ such that $h(q) \neq 0$ for any $q \in B$ and $g$ is a Blaschke product.

**Proof.** As in the proof of Proposition 6.11, we give the proof for the case in which all zeros of $f$ have multiplicity 1 and the sequence of zeros $\{a_n\}_{n \in \mathbb{N}}$ of $f$ is an injective sequence. Let us subdivide the sequence of zeros as

$$\{a_n\}_{n \in \mathbb{N}} = \{\alpha_n\}_{n \in \mathbb{N}} \cup \{\beta_n\}_{n \in \mathbb{N}}$$

where $\{\alpha_n\}_{n \in \mathbb{N}}$ is the sequence corresponding to spherical zeros, while $\{\beta_n\}_{n \in \mathbb{N}}$ is the sequences of isolated ones. Since $\{\alpha_n\}_{n \in \mathbb{N}}$ is contained in the sequence of zeros of the function $f$, clearly

$$\sum_{n \geq 0} (1 - |\alpha_n|) \leq \sum_{n \geq 0} (1 - |a_n|) < +\infty.$$
Hence the Blaschke product associated with \( \{ \alpha_n \}_{n \in \mathbb{N}} \)

\[
B_\alpha(q) = \prod_{n \geq 0} M_{\alpha_n}(q)
\]

converges uniformly on compact sets thus defining a regular function, vanishing exactly at the spherical zeros of \( f \). The function \( B_\alpha(q) \) is slice preserving, in fact it contains only factors of the type \( M_{\alpha_n} \ast M_{\tau_n} = M_{\alpha_n}^{*} \).

Let \( f_\beta(q) \) be the function defined as

\[
f_\beta(q) = B_\alpha^{-*} \ast f(q) = B_\alpha(q) \ast f(q),
\]

so that we can write

\[
f(q) = B_\alpha(q) f_\beta(q).
\]

Since the poles of \( B_\alpha^{-*} \) are spherical zeros of \( f \), then \( f_\beta \) is regular on \( \mathbb{B} \) and its sequence of zeros coincides with \( \{ \beta_n \}_{n \in \mathbb{N}} \). The idea is now to “make spherical” all the zeros of \( f_\beta \). In order to do it, we want to find a Blaschke product \( B_\gamma \) such that

\[
f_\beta \ast B_\gamma(q)
\]

vanishes at all spheres generated by \( \{ \beta_n \}_{n \in \mathbb{N}} \), namely such that the sequence of zeros of \( f_\beta \ast B_\gamma(q) \) is

\[
\{ \beta_n, \overline{\beta}_n \}_{n \in \mathbb{N}}.
\]

We can build \( B_\gamma \) following the lines of the proof of Proposition 6.11. If

\[
B_\gamma(q) = \prod_{n \geq 0} M_\gamma_n(q),
\]

we can define the sequence \( \{ \gamma_n \}_{n \in \mathbb{N}} \) iteratively as follows. The first quaternion, \( \gamma_0 \), is such that

\[
f_\beta \ast M_{\gamma_0}(q) = f_\beta \ast (\gamma_0 - q) \ast (1 - q^{\gamma_0})^{-*} \frac{\gamma_0}{|\gamma_0|}
\]

vanishes at \( q = \beta_0 \) and at \( q = \overline{\beta}_0 \). Since \( f_\beta \) vanishes at \( q = \beta_0 \) and \( f \) does not vanish at \( q = \overline{\beta}_0 \), if \( T_0 \) is defined as

\[
T_0(q) = (f_\beta(q))^{-1} q f_\beta(q),
\]

then we can write

\[
f_\beta \ast M_{\gamma_0}(q) = (f_\beta(q) (\gamma_0 - T_0(q))) \ast \left( (1 - q^{\gamma_0})^{-*} \frac{\gamma_0}{|\gamma_0|} \right).
\]

Hence, if we set

\[
\gamma_0 = T_0(\overline{\beta}_0),
\]

we have that

\[
f_\beta \ast M_{\gamma_0}(q)
\]

vanishes both at \( q = \beta_0 \) and at \( q = \overline{\beta}_0 \). As we have done in order to prove Proposition 6.11, we can iterate the process, setting, for any \( n \geq 1 \),

\[
T_n(q) = \left( f_\beta \ast \prod_{k=0}^{n-1} M_{\gamma_k}(q) \right)^{-1} q \left( f_\beta \ast \prod_{k=0}^{n-1} M_{\gamma_k}(q) \right)
\]

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so that

$$
\left( f_\beta \star \prod_{k=0}^{n-1} M_{\gamma_k} \right) \star M_{\gamma_n}(q) = \left( f_\beta \star \prod_{k=0}^{n-1} M_{\gamma_k} \right) \star (\gamma_n - q) \star \left( (1 - q^{*n})^{*n} \frac{\gamma_n}{|\gamma_n|} \right)
$$

\[
= \left( f_\beta \star \prod_{k=0}^{n-1} M_{\gamma_k} \right) (\gamma_n - T_n(q)) \star \left( (1 - q^{*n})^{*n} \frac{\gamma_n}{|\gamma_n|} \right).
\]

Since $T_n$ is well defined on $\overline{\mathbb{B}}_n$, if we set

$$
\gamma_n = T_n(\overline{\beta}_n)
$$

we get that

$$
\left( f_\beta \star \prod_{k=0}^{n-1} M_{\gamma_k} \right) \star M_{\gamma_n}(q)
$$

vanishes both at $q = \beta_n$ and at $q = \overline{\beta}_n$. The convergence of the infinite product $B_{\overline{\beta}}(q)$ is guaranteed by the fact that it is the Blaschke product associated with the sequence $\{ T_n(\overline{\beta}_n) \}_{n\in\mathbb{N}}$ where each element $T_n(\overline{\beta}_n)$ has the same modulus of $\beta_n$, and each $\beta_n$ is contained in the sequence of zeros of a function in $H^p(\mathbb{B})$. Hence

$$
f_\beta \star B_{\overline{\beta}}(q)
$$

is a regular function that has only spherical zeros, and its sequence of zeros is $\{ \beta_n, \overline{\beta}_n \}_{n\in\mathbb{N}}$. Therefore, if we set

$$
\overline{B}_\beta(q) = \prod_{n\geq 0} \left( M_{\beta_n} \star M_{\overline{n}} \right) (q) = \prod_{n\geq 0} M_{\beta_n}(q),
$$

then we can write

$$
f_\beta \star B_{\overline{\beta}}(q) = \overline{B}_\beta \star h(q) = \overline{B}_\beta(q) \cdot h(q),
$$

for some function $h$, never vanishing and regular on $\mathbb{B}$. To prove the regularity of $h$, it suffices to observe that since $\overline{B}_\beta$ has real coefficients, it is regular and it has exactly the same zeros of $f_\beta \star B_{\overline{\beta}}(q)$, then the regular quotient

$$
\overline{B}_\beta^{-1} \star (f_\beta \star B_{\overline{\beta}})(q) = (\overline{B}_\beta(q))^{-1}(f_\beta \star B_{\overline{\beta}})(q)
$$

is well defined (and regular) on the entire ball $\mathbb{B}$. Consider the regular conjugate of $B_{\overline{\beta}}$, and multiply on the right by $B_{\overline{\beta}}^c$ all terms of equality (17). We obtain

$$
f_\beta \star B_{\overline{\beta}} \star B_{\overline{\beta}}^c(q) = \overline{B}_\beta(q) \cdot h \star B_{\overline{\beta}}^c(q),
$$

that can also be written as

$$
B_{\overline{\beta}}^c(q) f_\beta(q) = \overline{B}_\beta(q) h \star B_{\overline{\beta}}^c(q).
$$

Now notice that $B_{\overline{\beta}}^c(q) = \overline{B}_\beta(q)$ because they both are Blaschke products associated to the same spherical zeros. Therefore it results

$$
f_\beta(q) = h \star B_{\overline{\beta}}^c(q),
$$

that, for $f$, means

$$
f(q) = B_\alpha(q) h \star B_{\overline{\beta}}^c(q) = h \star B_\alpha \star B_{\overline{\beta}}^c(q).
$$

Setting $g(q) = B_\alpha \star B_{\overline{\beta}}^c(q)$ leads to the conclusion of the proof. \qed
Once “extracted” the zeros of a function in $H^p(\mathbb{B})$, we would like to identify its outer factor and its singular part.

**Proposition 6.14.** Let $f \in H^p(\mathbb{B})$ for some $p \in [1, +\infty]$ be such that $f^{-*} \in H^q(\mathbb{B})$ where $\frac{1}{p} + \frac{1}{q} = 1$. Then $f$ is an outer function.

**Proof.** Let $g \in H^p(\mathbb{B})$ be such that $|\tilde{g}| = |\tilde{f}|$ almost everywhere on $\partial \mathbb{B}$. The regular function $h = f^{-*} * g$ belongs to $H^1(\mathbb{B})$ thanks to Proposition 3.14. Therefore recalling Proposition 4.6, we get that for almost every $\theta \in [-\pi, \pi)$,

$$\lim_{r \to 1^-} |h(re^{i\theta})| = \lim_{r \to 1^-} |f^{-*} * g(re^{i\theta})|$$

$$= |\tilde{f}(\tilde{f}^{-1}(e^{i\theta})^{-1} e^{i\theta} \tilde{f}^{-1}(e^{i\theta}))|^{-1} |\tilde{g}(\tilde{f}^{-1}(e^{i\theta})^{-1} e^{i\theta} \tilde{f}^{-1}(e^{i\theta}))|.$$ 

Since $f$ and $\tilde{g}$ coincide almost everywhere at the boundary, we get that for almost every $\theta \in [-\pi, \pi)$,

$$\lim_{r \to 1^-} |h(re^{i\theta})| = 1.$$

Therefore the Poisson integral representation formula guarantees that

$$|h(q)| \leq 1$$

for any $q \in \mathbb{B}$. With the same notation of Proposition 2.22, we obtain that for all $q \in \mathbb{B}$

$$1 \geq |f^{-*} * g(q)| = |f(T_f(q))|^{-1} |g(T_f(q))|.$$ 

Since $f^{-*} \in H^q(\mathbb{B})$ then $f$ has no zeros and $T_f$ is a diffeomorphism of $\mathbb{B}$, yielding

$$|g(q)| \leq |f(q)|$$

for any $q \in \mathbb{B}$. We thus conclude that $f$ is an outer function. \hfill $\square$

**Remark 6.15.** In the case of a regular function $f : \mathbb{B} \to \mathbb{H}$ that is continuous and non-vanishing up to the boundary of $\mathbb{B}$, the previous result implies that $f$ is an outer function (since $f$ and $f^{-*}$ both belong to $H^\infty(\mathbb{B})$).

For slice regular functions that preserve a slice $L_I$, the factorization can be done in a stronger and more satisfactory fashion, that very much resembles its complex counterpart.

**Theorem 6.16.** Let $f \in H^p(\mathbb{B})$ for some $p \in [0, +\infty]$, be such that $f$ maps $\mathbb{B}_I$ to $L_I$ for some $I \in S$. Then we can factor $f$ as

$$f(q) = E * \mathcal{I}(q),$$

where $E$ is an outer function in $H^p(\mathbb{B})$ such that $|E| = |\tilde{f}|$ almost everywhere on the boundary $\partial \mathbb{B}$ and $\mathcal{I}$ is an inner function.

**Proof.** The restriction of $f$ to $\mathbb{B}_I$ is a (complex) holomorphic function $F_I : \mathbb{B}_I \to L_I$ mapping $z \mapsto f_I(z)$. Let us define the function $E_I : \mathbb{B}_I \to L_I$ to be

$$E_I(z) = \exp \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log |F_I(e^{i\theta})| d\theta \right),$$

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namely the outer function of $F_I$. From factorization results in the complex setting, (see for instance [27]), we know that we can write
\[ F_I(z) = E_I(z)I_I(z) \]
where $I_I(z)$ is the inner function of $F_I$. In particular, since both $E_I$ and $I_I$ map $B_I$ to $L_I$, we can also write
\[ F_I(z) = E_I(z)I_I(z) = E_I(z) * I_I(z). \]
Hence
\[ f(q) = \text{ext}(f_I)(q) = \text{ext}(F_I)(q) = \text{ext}(E_I * I_I)(q) = \text{ext}(E_I) * \text{ext}(I_I)(q) \]
where the last equality is due to the Identity Principle 2.2. Let us set (see Proposition 2.6)
\[ I \]
and its uniform norm equals $1$, therefore we get, thanks to (18)
\[
\max_{J \in \mathbb{S}} |I(x + yJ)| = \max\{|I(x + yI)|, |I(x - yI)|\} = \max\{|I_I(x + yI)|, |I_I(x - yI)|\}
\]
for all $x, y$ such that $x + yI \in B_I$. By classical results, the inner function $I_I$ is bounded in modulus by 1 and its uniform norm equals 1, therefore we get
\[
\min_{J \in \mathbb{S}} |I(x + yJ)| = \min\{|I(x + yI)|, |I(x - yI)|\} = \min\{|I_I(x + yI)|, |I_I(x - yI)|\}
\]
for all $x, y$. Moreover, since $|I_I|$ equals 1 almost everywhere on the boundary $\partial B_I$, we have that, for almost every $x + yI$ such that $x^2 + y^2 = 1$,
\[
\max \left\{|I_I(x + yI)|, |I_I(x - yI)|\right\} = \min \left\{|I_I(x + yI)|, |I_I(x - yI)|\right\} = 1.
\]
Therefore, for almost every $x + yJ$ such that $x^2 + y^2 = 1$,
\[
\max_{J \in \mathbb{S}} |I(x + yJ)| = \min_{J \in \mathbb{S}} |I(x + yJ)| = 1
\]
for almost every $q \in \partial B$, $|I(q)| = 1$. To obtain the wanted properties of the modulus of $E$, let us denote by $T$ the transformation
\[ T(q) = (E^c(q))^{-1}qE^c(q). \]
Since $E^c$ is non-vanishing on $B$, $T$ is a diffeomorphism of $B$ with inverse
\[ T^{-1}(q) = (E(q))^{-1}qE(q). \]
Then, thanks to Proposition 2.22 and to inequality (19), we can write
\[
1 \geq |I(T^{-1}(q))| = |E^{-*} * f(T^{-1}(q))| = |E^{-1}(q)f(q)| = |E(q)|^{-1}|f(q)|
\]
for any \( q \in \mathcal{B} \). Hence

\[
|E(q)| \geq |f(q)| \quad \text{for any } q \in \mathcal{B}.
\]

The study of the behavior of the modulus \( |\tilde{E}| \) at the boundary, requires some more effort. First of all, classical results on outer functions (see e.g. [27]) imply that \( E_I \in H^p(\mathbb{B}_I) \). Then, thanks to Proposition 3.9 we get that \( E \in H^p(\mathbb{B}) \). Therefore, for any \( J \in \mathcal{S} \), the function \( E \) has radial limit \( \tilde{E} \) for almost any \( x + yJ \in \partial\mathbb{B}_J \). Hence, also the transformation \( T^{-1}(q) = (E(q))^{-1}qE(q) \) does. This allows us to prove that \( T^{-1} \) maps almost every 2-sphere \( x + yS \subset \partial\mathbb{B} \) one-to-one onto itself. In fact, for almost every sphere \( x + yS \subset \partial\mathbb{B} \) the function \( \tilde{E} \) is defined at all points of \( x + yS \) (see Proposition 4.3). For such a sphere \( x + yS \), thanks to the Representation Formula 2.4 and to the fact that \( E \) preserves \( L_I \), there exist \( b,c \in L_I \) such that

\[
\tilde{E}(x + yJ) = b + Jc \quad \text{for any } J \in \mathcal{S}.
\]

Take \( x + yK \in x + yS \), with \( K \neq I \). We are going to show that we can find the only \( J \in \mathcal{S} \) such that \( T^{-1}(x + yJ) = x + yK \). This is possible if and only if

\[
\left( \tilde{E}(x + yJ) \right)^{-1}q\tilde{E}(x + yJ) = x + yK
\]

namely, if and only if

\[
(b + Jc)^{-1}(x + yJ)(b + Jc) = x + yK,
\]

that is equivalent to

\[
(x + yJ)(b + Jc) = (b + Jc)(x + yK)
\]

and, since \( x \) and \( y \) are real numbers, to

\[
J(b + Jc) = (b + Jc)K
\]

and to

\[
J(b - cK) = c + bK.
\]

Now, since \( K \neq I \) and since \( b,c \in L_I \), necessarily \( b - cK \neq 0 \). Hence

\[
J = (bK + c)(b - cK)^{-1}
\]

solves our problem. Thanks to this property of the map \( T^{-1} \), we get that \( 1 = |\tilde{I}(T^{-1}(q))| \) for almost every \( q \in \partial\mathbb{B} \). The radial limit version of equation (20), holding for almost every \( q \in \partial\mathbb{B} \),

\[
1 = |\tilde{I}(T^{-1}(q))| = |\tilde{E}(q))|^{-1}|\tilde{f}(q)|
\]

leads to the equality

\[
|\tilde{E}(q)| = |\tilde{f}(q)| \quad \text{(21)}
\]

for almost every \( q \in \partial\mathbb{B} \), which leads also to \( ||E||_p = ||\tilde{E}||_{L^p} = ||\tilde{f}||_{L^p} = ||f||_{L^p} \). We want to show now that \( E \) is an outer function. To this aim, let \( g \in H^p(\mathbb{B}) \) be such that

\[
|\tilde{E}(q)| = |\tilde{g}(q)| \quad \text{(22)}
\]
for almost all \( q \in \partial \mathbb{B} \). If we restrict \( E \) and \( g \) to \( \mathbb{B} \), and recall the definition of \( E_I \), we can write (see Proposition 4.3)

\[
\log |E_I(re^{i\theta})| = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta-t) \log |\tilde{f}_I(e^{it})| dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta-t) \log |\tilde{g}_I(e^{it})| dt \geq \log |g_I(re^{i\theta})| \]

where \( P_r(t) \) is the Poisson kernel and where last inequality is due to the subharmonicity of \( \log |g_I| \) (see the proof of Proposition 3.1). As a consequence we get that

\[
1 \geq |E_I(z)|^{-1} |g_I(z)| = |E_I^{-*} * g_I(z)| = |(E^{-*} * g)_I(z)|
\]

for all \( z \in \mathbb{B} \). Since \( E^{-*} * g \) is regular on \( \mathbb{B} \), using Proposition 3.9 we obtain that \( E^{-*} * g \) belongs to \( H^\infty(\mathbb{B}) \). Then the radial limit of \( E^{-*} * g \) exists at almost every point of \( \partial \mathbb{B} \). Equation (22) and Proposition 4.6 guarantee that for every \( J \in \mathbb{S} \)

\[
\lim_{r \to 1^-} |E^{-*} * g(re^{i\theta})| = |\tilde{E}^{-*} * g(e^{i\theta})| = 1
\]

for almost every \( \theta \in [-\pi, \pi) \). Hence, by Proposition 4.11,

\[
||E^{-*} * g||_\infty = ||\tilde{E}^{-*} * g||_{L^\infty} = 1.
\]

Recalling now that \( T^{-1}(q) = (E(q))^{-1}qE(q) \) is a diffeomorphism of \( \mathbb{B} \), thanks to Proposition 2.22 we obtain

\[
1 \geq |E^{-*} * g(T^{-1}(q))| = |E^{-1}(q)g(q)| = |E(q)|^{-1}|g(q)|
\]

i.e.,

\[
|E(q)| \geq |g(q)|
\]

for all \( q \in \mathbb{B} \). This concludes the proof.

**Corollary 6.17.** Let \( f \in H^p(\mathbb{B}) \) for some \( p \in (0, +\infty) \), be such that \( f \) maps \( \mathbb{B} \) to \( L_1 \) for some \( I \in \mathbb{S} \). Then there exist an outer function \( E \in H^p(\mathbb{B}) \), a singular function \( S \in H^\infty(\mathbb{B}) \) and a Blaschke product \( B \) such that

\[
f(q) = E * S * B(q)
\]

for all \( q \in \mathbb{B} \).

**Proof.** Theorem 6.16 allows us to factor \( f = E * I \) as a \(*\)-product of an outer function \( E \) and an inner function \( I \). Theorem 6.13 guarantees now the existence of a Blaschke product \( B \) and a non-vanishing function \( S \) such that \( I = S * B \). Following the lines of the proof of Theorem 6.16 that led us to show that the outer factor of \( f \) belongs to \( H^p(\mathbb{B}) \), one can prove that \( S \) belongs to \( H^\infty(\mathbb{B}) \) and hence is a singular function.

\[
\square
\]
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