TRIVIAL ENDOMORPHISMS OF THE CALKIN ALGEBRA

ANDREA VACCARO

Abstract. We prove that it is consistent with ZFC that every (unital) endomorphism of the Calkin algebra \( \mathcal{Q}(H) \) is unitarily equivalent to an endomorphism of \( \mathcal{Q}(H) \) which is liftable to a (unital) endomorphism of \( \mathcal{B}(H) \). We use this result to classify all unital endomorphisms of \( \mathcal{Q}(H) \) up to unitary equivalence by the Fredholm index of the image of the unilateral shift. Finally, we show that it is consistent with ZFC that the class of C\(^{*}\)-algebras that embed into \( \mathcal{Q}(H) \) is not closed under countable inductive limit nor tensor product.

1. Introduction

Let \( H \) be a separable, infinite-dimensional, complex Hilbert space. The Calkin algebra \( \mathcal{Q}(H) \) is the quotient of \( \mathcal{B}(H) \), the algebra of all linear, bounded operators on \( H \), over the ideal of compact operators \( \mathcal{K}(H) \). Let \( q : \mathcal{B}(H) \to \mathcal{Q}(H) \) be the quotient map.

Over the last 15 years, the study of the automorphisms of the Calkin algebra has been the setting for some of the most significant applications of set theory to C\(^{*}\)-algebras. The original motivation for these investigations is of C\(^{*}\)-algebraic nature, and it dates back to the seminal paper [BDF77]. One of the most prominent questions asked in [BDF77] is whether there exists a K-theory reverting automorphism of \( \mathcal{Q}(H) \) or, more concretely, an automorphism of \( \mathcal{Q}(H) \) sending the unilateral shift to its adjoint. Since all inner automorphisms act trivially on the K-theory of a C\(^{*}\)-algebra, a preliminary question posed in [BDF77] is whether the Calkin algebra has outer automorphisms. The answer turned out to be depending on set theoretic axioms. Phillips and Weaver show in [PW07] that outer automorphisms of \( \mathcal{Q}(H) \) exist if the Continuum Hypothesis CH is assumed, while in [Far11b] Farah proves that the Open Coloring Axiom OCA (see definition 2.1) implies that all the automorphisms of \( \mathcal{Q}(H) \) are inner. It is still unknown whether it is consistent with ZFC that there exists an automorphism of \( \mathcal{Q}(H) \) sending the unilateral shift to its adjoint, since the automorphisms built in [PW07] act like inner automorphisms on every separable subalgebra of \( \mathcal{Q}(H) \).

In this note we investigate the effects of OCA on the endomorphisms of the Calkin algebra. The main consequence of OCA that we show is a complete classification of the endomorphisms of \( \mathcal{Q}(H) \), up to unitary equivalence, by essentially the Fredholm index of the image of the unilateral shift. Two endomorphisms \( \varphi_1, \varphi_2 : \mathcal{Q}(H) \to \mathcal{Q}(H) \) are unitarily equivalent if there is a unitary \( v \in \mathcal{Q}(H) \) such that \( \text{Ad}(v) \circ \varphi_1 = \varphi_2 \). Let \( \text{End}(\mathcal{Q}(H)) \) be the set of all endomorphisms of \( \mathcal{Q}(H) \) modulo unitary equivalence. Let

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End_u(Q(H)) be the set of all the classes in \( \text{End}(Q(H)) \) corresponding to unital endomorphisms. The operation of direct sum \( \oplus \) naturally induces a structure of semigroup on both \( \text{End}(Q(H)) \) and \( \text{End}_u(Q(H)) \), as well as the operation of composition \( \circ \). We fix an orthonormal basis \( \{\xi_k\}_{k \in \mathbb{N}} \) of \( H \) and we let \( S \) be the unilateral shift sending \( \xi_k \) to \( \xi_k + 1 \).

**Theorem 1.1.** Assume OCA. Two endomorphisms \( \varphi_1, \varphi_2 : Q(H) \to Q(H) \) are unitarily equivalent if and only if the following two conditions are satisfied:

1. There is a unitary \( w \in Q(H) \) such that \( w\varphi_1(1)w^* = \varphi_2(1) \).
2. The (finite) Fredholm indices of \( \varphi_1(q(S)) + (1 - \varphi_1(1)) \) and \( \varphi_2(q(S)) + (1 - \varphi_2(1)) \) are equal.

Moreover, the map sending \( \varphi \in \text{End}_u(Q(H)) \) to \( -\text{ind}(\varphi(q(S))) \) is a semigroup isomorphism between \( (\text{End}_u(Q(H)), \oplus) \) and \( (\mathbb{N} \setminus \{0\}, +) \), as well as between \( (\text{End}_u(Q(H)), \circ) \) and \( (\mathbb{N} \setminus \{0\}, \cdot) \).

An explicit description of \( (\text{End}(Q(H)), \oplus) \) and \( (\text{End}(Q(H)), \circ) \) under OCA is given in remark 5.1.

The second consequence of OCA we prove in this paper is related to the recent works [FHV19], [FKV19] and [Vac19, Chapter 2], where methods from set theory are employed in the study of nonseparable subalgebras of \( Q(H) \). Let \( \mathcal{E} \) be the class of all \( C^* \)-algebras that embed into \( Q(H) \). Under CH the Calkin algebra has density character \( \aleph_1 \) (the first uncountable cardinal), hence a \( C^* \)-algebra belongs to \( \mathcal{E} \) if and only if its density character is at most \( \aleph_1 \) (see [FHV19]). Therefore, when CH holds, the class \( \mathcal{E} \) is closed under all operations whose output, when starting from \( C^* \)-algebras of density character at most \( \aleph_1 \), is a \( C^* \)-algebra whose density character is at most \( \aleph_1 \), such as minimal/maximal tensor product and countable inductive limit. It is not clear whether CH is necessary to prove these closure properties, but we prove that they might fail if CH is not assumed, answering [FKV19, Question 5.3].

**Theorem 1.2.** Assume OCA.

1. The class \( \mathcal{E} \) is not closed under minimal/maximal tensor product. Moreover, there exists \( \mathcal{A} \in \mathcal{E} \) such that \( \mathcal{A} \otimes \gamma \mathcal{B} \notin \mathcal{E} \) for every infinite-dimensional, unital \( B \in \mathcal{E} \) and for every tensor norm \( \gamma \).
2. The class \( \mathcal{E} \) is not closed under countable inductive limits.

In particular, both (1) and (2) are independent from ZFC.

Theorems 1.1 and 1.2 are proved in §5 using theorem 1.3 for which we need to introduce a definition.

We say that an endomorphism \( \varphi : Q(H) \to Q(H) \) is trivial if there is a unitary \( v \in Q(H) \) and a strongly continuous (i.e. strong-strong continuous) endomorphism \( \Phi : B(H) \to B(H) \) such that the following diagram commutes.

\[
\begin{array}{ccc}
B(H) & \xrightarrow{\Phi} & B(H) \\
q \downarrow & & \downarrow q \\
Q(H) & \xrightarrow{\text{Ad}(v) \circ \varphi} & Q(H)
\end{array}
\]
With this terminology, the main theorem of [Far11b] says that under OCA all automorphisms of $Q(H)$ are trivial (indeed, up to unitary equivalence, they lift to the identity). We extend this result to all endomorphisms of $Q(H)$.

**Theorem 1.3.** Assume OCA. All endomorphisms of the Calkin algebra are trivial.

The proof of theorem 1.3 occupies both §3 and §4. Similarly to theorem 1.2 theorems 1.1 and 1.3 cannot be proved in ZFC alone, in fact they fail under CH. In [§5] give two examples of non-trivial endomorphisms of $Q(H)$ existing when CH is assumed (examples 5.3 and 5.4), which highlight different levels of failure of the classification of $\text{End}(Q(H))$ in theorem 1.1. In particular, it is possible to find uncountably many inequivalent automorphisms of $Q(H)$ which send the unilateral shift to a unitary of index $-1$ (see example 5.3). Examples 5.3 and 5.4 along with theorem 1.3 entail the following corollary.

**Corollary 1.4.** The existence non-trivial endomorphisms of $Q(H)$ is independent from ZFC.

We remark that, unlike the results concerning automorphisms of $Q(H)$ in [Far11b], the commutative analogue of theorem 1.3 for $\mathcal{P}(\mathbb{N})/\text{Fin}$ does not hold. In [Dow14] it is proved that there are non-trivial endomorphisms of $\mathcal{P}(\mathbb{N})/\text{Fin}$ in ZFC. In this scenario, the best one can hope for is the so called weak Extension Principle, a consequence of OCA + MA introduced and discussed in [Far00, Chapter 4]. A crucial difference in the noncommutative context is the presence of partial isometries allowing to compress/decompress operators into/from infinite-dimensional subspaces of $H$. These objects play a crucial role in the proof of theorem 1.3.

The observations and results exposed in this note are in continuity with the numerous studies investigating the strong rigidity properties induced by the Proper Forcing Axiom (of which OCA is a consequence) on the Calkin algebra and on other nonseparable quotient algebras ([Far11b], [MV18], [Vig18], BFV18: see also [Far00], [Vel93]). The Continuum Hypothesis, on the other hand, grants the opposite effect, allowing to prove the existence of too many maps on these quotients for all of them to be ‘trivial’ ([PW07], [CF14], [FMS13]). Woodin’s $\Sigma^1_2$-absoluteness theorem gives a deeper meta-mathematical motivation for the efficacy of CH in solving these problems (see [Woo02]).

The paper is structured as follows. Section 2 contains preliminaries and definitions. Sections 3 and 4 are devoted to the proof of theorem 1.3. In section 3 we show that locally trivial (see definition 2.2) endomorphisms of $Q(H)$ are, up to unitary transformation, locally liftable with ‘nice’ unitaries in $B(H)$. In section 4 adapting the main arguments from [Far11b] (see also [Far, §18]), we prove that under OCA all endomorphisms of $Q(H)$ are locally trivial, and that all locally trivial endomorphisms are trivial. We remark that OCA is only needed in §4, while the results in §3 require no additional set-theoretic axiom. Finally, §5 contains the proof of theorem 1.2 and of theorem 1.1, some observations about what happens to $\text{End}(Q(H))$ under CH and some open questions.
2. Notation and Preliminaries

The only extra set-theoretic assumption required for our proofs is the Open Coloring Axiom OCA, which is defined as follows.

**Definition 2.1.** Given a set $X$, let $[X]^2$ be the set of all unordered pairs of elements of $X$. For a topological space $X$, a *coloring* $[X]^2 = K_0 \sqcup K_1$ is open if the set $K_0$, when naturally identified with a symmetric subset of $X \times X$, is open in the product topology. For a $K \subseteq [X]^2$, a subset $Y$ of $X$ is $K$-homogeneous if $[Y]^2 \subseteq K$.

**OCA.** Let $X$ be a separable, metric space, and let $[X]^2 = K_0 \sqcup K_1$ be an open coloring. Then either $X$ has an uncountable $K_0$-homogeneous set, or it can be covered by countably many $K_1$-homogeneous sets.

This statement, which contradicts CH, is independent from ZFC and it was introduced by Todorcevic in [Tod89].

For the rest of the paper, fix $\{\xi_k\}_{k \in \mathbb{N}}$ an orthonormal basis of $H$ and identify $\ell_\infty$, the C*-algebra of all uniformly bounded sequences of complex numbers, with the atomic masa of all operators in $\mathcal{B}(H)$ diagonalized by such basis. With this identification, the algebra $c_0 = \ell_\infty \cap \mathcal{K}(H)$ is the set of all sequences converging to zero. Given a set $M \subseteq \mathbb{N}$, $P_M$ denotes the orthogonal projection onto the closure of $\text{span}\{\xi_k : k \in M\}$. If $M = \{k\}$, we simply write $P_k$. Throughout this paper, all the partitions $E$ of $\mathbb{N}$ are always implicitly assumed to be composed by consecutive, finite intervals. Given such a partition $E = \{E_n\}_{n \in \mathbb{N}}$, $\mathcal{D}[E]$ is the von Neumann algebra of all operators in $\mathcal{B}(H)$ for which each span$\{\xi_k : k \in E_n\}$ is invariant.

Equivalently, $\mathcal{D}[E]$ is the set of all operators which commute with $P_{E_n}$ for every $n \in \mathbb{N}$. Given a subset $X \subseteq \mathbb{N}$, $\mathcal{D}_X[E]$ denotes the C*-algebra of all the operators in $\mathcal{D}[E]$ which act as zero on $\text{span}\{\xi_k : k \in E_n\}$ for every $n \in X$.

Given a unital $*$-homomorphism $\Phi : \mathcal{D}[E] \to \mathcal{B}(H)$ such that $\Phi[\mathcal{D}[E] \cap \mathcal{K}(H)] \subseteq \mathcal{K}(H)$, $\tilde{n}_\Phi$ denotes the sequence $\{\text{rk}(\Phi(P_k))\}_{k \in \mathbb{N}}$ of the (finite) ranks of the projections $\Phi(P_k)$. Notice that $\tilde{n}_\Phi$ only depends on how $\Phi$ acts on $\ell_\infty$ and that $n_i = n_j$ whenever $i, j \in E_n$ for some $n \in \mathbb{N}$. For a partition $E = \{E_n\}_{n \in \mathbb{N}}$ of $\mathbb{N}$, $E^{\text{even}}$ is the partition $\{E_{2n} \cup E_{2n+1}\}_{n \in \mathbb{N}}$ and $E^{\text{odd}}$ is the partition $\{E_{2n}\} \cup \{E_{2n+1} \cup E_{2n+2}\}_{n \in \mathbb{N}}$.

The strong topology of $\mathcal{B}(H)$ (and on any subalgebra of $\mathcal{B}(H)$) is the one induced by the pointwise norm convergence on $H$, hence a sequence $\{T_n\}_{n \in \mathbb{N}}$ of operators in $\mathcal{B}(H)$ strongly converges to $T$ iff $T_n\xi \to T\xi$ in norm for every $\xi \in H$.

For every partial isometry $v$ in a C*-algebra $\mathcal{A}$, $\text{Ad}(v)$ is the endomorphism sending $a$ to $va^*v$ for every $a \in \mathcal{A}$.

For every subalgebra $\mathcal{A}$ of $\mathcal{B}(H)$, let $\mathcal{A}/(\mathcal{K}(H) \cap \mathcal{A})$. Given a map $\varphi : \mathcal{A}_Q \to \mathcal{Q}(H)$, the function $\Phi : \mathcal{A} \to \mathcal{B}(H)$ lifts (or is a lift of) $\varphi$ if the following diagram commutes:

\[
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{\Phi} & \mathcal{B}(H) \\
\downarrow \varphi & & \downarrow q \\
\mathcal{A}_Q & \xrightarrow{\varphi} & \mathcal{Q}(H)
\end{array}
\]
Definition 2.2. Given a C*-algebra $A \subseteq B(H)$, we say that an embedding (i.e. an injective $*$-homomorphism) $\varphi : A \rightarrow Q(H)$ is trivial if there exists a unitary $v \in Q(H)$ and a strongly continuous (i.e. strong-strong continuous), $*$-homomorphism $\Phi : A \rightarrow B(H)$ such that $\Phi$ lifts $\text{Ad}(v) \circ \varphi$. An endomorphism $\varphi : Q(H) \rightarrow Q(H)$ is locally trivial if, for every partition $\tilde{E}$, the restriction $\varphi \mid D[\tilde{E}]_{Q}$ is trivial.

Given two operators $T,S \in B(H)$, we use the notation $T \sim_{K(H)} S$ to abbreviate $T - S \in K(H)$. Analogously, for a C*-algebra $A$ and two functions $\Phi_{1}, \Phi_{2} : A \rightarrow B(H)$, $\Phi_{1} \sim_{K(H)} \Phi_{2}$ abbreviates $\Phi_{1}(a) \sim_{K(H)} \Phi_{2}(a)$ for all $a \in A$.

Given a C*-algebra $A \subseteq B(H)$ ($A \subseteq Q(H)$), the commutant $A' \cap B(H)$ ($A' \cap Q(H)$) is the set of all the operators in $B(H)$ ($Q(H)$) commuting with all elements of $A$.

The (Fredholm) index of an operator $T \in B(H)$ is the integer $\dim(\ker(T)) - \text{codim}(T[H])$. An element $a \in Q(H)$ is invertible if and only if it can be lifted to an operator of finite index ([Mur90, Theorem 1.4.16]), which can be assumed to be a partial isometry if $a$ is a unitary.

Remark 2.3. Strongly continuous, unital endomorphisms of $B(H)$ have an extremely rigid structure. Indeed, strong continuity implies that such maps are uniquely determined by how they behave on the projections whose range is 1-dimensional. Since all these projections are Murray-von Neumann equivalent, the same is true for their images, which therefore all have the same rank. For every $m \in \mathbb{N}$, let $\Phi_{m} : B(H) \rightarrow B(H \otimes \mathbb{C}^{m})$ be the map sending $T$ to $T \otimes 1_{m}$. As $B(H)$ and $B(H \otimes \mathbb{C}^{m})$ are isomorphic, with an abuse of notation we consider $\Phi_{m}$ as a map from $B(H)$ into $B(H)$. If $\Phi : B(H) \rightarrow B(H)$ is a unital, strongly continuous endomorphism sending compact operators into compact operators, by the previous observation it is possible to find an $m \in \mathbb{N} \setminus \{0\}$ and a unitary $U \in B(H)$ such that $\Phi = \text{Ad}(U) \circ \Phi_{m}$. Our classification of $\text{End}(Q(H))$ and $\text{End}_{u}(Q(H))$ in theorem 1.1 will be based on this simple observation. Since the commutant of the image of $\ell_{\infty}$ via $\Phi_{m}$ is isomorphic to $\ell_{\infty}(M_{m}^{\infty}C)$ (the C*-algebra of all norm-bounded sequences of $m \times m$ matrices with complex entries), the same is true for the commutant of the image of $\ell_{\infty}$ via $\Phi$.

Every unital endomorphism $\varphi : Q(H) \rightarrow Q(H)$ is uniquely determined by its restrictions $\{\varphi \mid D[\tilde{E}]_{Q}\}$ as $\tilde{E}$ varies among all partitions of $\mathbb{N}$. This is a consequence of the following standard fact (see [Far11], Lemma 1.2, or the proof of [Ell77], Theorem 3.1]).

Proposition 2.4. For every countable set $\{T_{n}\}_{n \in \mathbb{N}}$ in $B(H)$ there exists a partition $\tilde{E}$ of $\mathbb{N}$ such that for every $n \in \mathbb{N}$ there are $T_{n}^{0} \in D[\tilde{E}^{\text{even}}]$ and $T_{n}^{1} \in D[\tilde{E}^{\text{odd}}]$ such that $T_{n} \sim_{K(H)} T_{n}^{0} + T_{n}^{1}$.

3. LOCALLY TRIVIAL ENDOmorphisms

Given a unital, locally trivial endomorphism $\varphi : Q(H) \rightarrow Q(H)$, throughout this section we fix, for every partition $\tilde{E}$ of $\mathbb{N}$, a partial isometry of finite index $v_{\tilde{E}}$ and a strongly continuous, $*$-homomorphism $\Phi_{\tilde{E}} : D[\tilde{E}] \rightarrow B(H)$ such that $\text{Ad}(v_{\tilde{E}}) \circ \Phi_{\tilde{E}}$ lifts the restriction of $\varphi$ to $D[\tilde{E}]_{Q}$. In this part we
show that, up to considering $\text{Ad}(v) \circ \varphi$ for some unitary $v \in \mathcal{Q}(H)$, we can assume that $\Phi_E$ is $\Phi_m$ (as defined in remark 2.2) and $v_E$ is a unitary in the commutant of $\Phi_m[\ell_\infty]$, for every partition $E$. We remark that no extra set-theoretic axiom is required in the present section.

**Remark 3.1.** Notice that for a unital, locally trivial endomorphism $\varphi : \mathcal{Q}(H) \to \mathcal{Q}(H)$, for each partition $E$ the projection $\Phi_E(1)$ is a compact perturbation of the identity, hence its range has finite codimension $r$. Therefore, by multiplying $v_E$ by a suitable partial isometry of index $-r$, we can always assume that $\Phi_E$ is unital, and we will always implicitly do so.

**Lemma 3.2.** Let $\Phi_1 : \ell_\infty \to \mathcal{B}(H)$ and $\Phi_2 : \ell_\infty \to \mathcal{B}(H)$ be two strongly continuous, unital $*$-homomorphisms such that $\Phi_1[\mathcal{C}] \subseteq \mathcal{K}(H)$. Suppose there exist two partial isometries of finite index $v_1, v_2$ such that $\text{Ad}(v_1) \circ \Phi_1 \sim_{\mathcal{K}(H)} \text{Ad}(v_2) \circ \Phi_2$. Then the sequences $\bar{n}_{\Phi_1} = \{\text{rk}(\Phi_1(P_k))\}_{k \in \mathbb{N}}$ and $\bar{n}_{\Phi_2} = \{\text{rk}(\Phi_2(P_k))\}_{k \in \mathbb{N}}$ are eventually equal.

**Proof.** Since $q(v_1)$ and $q(v_2)$ are unitaries in $\mathcal{Q}(H)$, we can assume that $v_1$ is the identity, and we denote $v_2$ by $v$. Let $\bar{n}_{\Phi_1} = \{n_k\}_{k \in \mathbb{N}}$, $\bar{n}_{\Phi_2} = \{m_k\}_{k \in \mathbb{N}}$ and suppose there is an infinite $X \subseteq \mathbb{N}$ such that $n_k > m_k$ for every $k \in X$.

We inductively define a set $Y \subseteq X$ as follows.

Let $y_0$ be the minimum of $X$ and let $Y_0 = \{y_0\}$. Since $\text{rk}(\Phi_1(P_{y_0})) > \text{rk}(\Phi_2(P_{y_0})) \geq \text{rk}(\text{Ad}(v)(\Phi_2(P_{y_0})))$, there is a norm-one vector $\xi_0$ in the image of $\Phi_1(P_{y_0})$ which also belongs to the kernel of $\text{Ad}(v)(\Phi_2(P_{y_0}))$. This is the case since the codimension of $\ker(\text{Ad}(v)(\Phi_2(P_{y_0})))$ is strictly smaller than $\text{rk}(\Phi_1(P_{y_0}))$.

Suppose $Y_k = \{y_0 < \cdots < y_k\} \subseteq X$ and that, for every $h \leq k$, there is a norm-one vector $\xi_h$ such that $\Phi_1(P_{y_h})\xi_h = \xi_h$ and $\|\text{Ad}(v)(\Phi_2(P_{y_h}))\xi_h\| < 1/2$. Let $y_{k+1}$ be the smallest element in $X$ greater than $y_k$ such that

\begin{align*}
(1) \quad & \|\text{Ad}(v)(\Phi_2(P_{y_k}))\Phi_1(P_{y_{k+1}})\xi_h\| < 1/2, \\
(2) \quad & \|\text{Ad}(v)(\Phi_2(P_{y_k \cup \{y_{k+1}\}}))\xi_h\| < 1/2 \text{ for every } h \leq k.
\end{align*}

Such number $y_{k+1}$ exists since all $\text{Ad}(v)(\Phi_2(P_{y_h}))$ and the projection onto span$\{v^*\xi_h : h \leq k\}$ have finite rank and, by strong continuity, the sequences $\{\Phi_1(P_k)\}_{k \in \mathbb{N}}$ and $\{\Phi_2(P_k)\}_{k \in \mathbb{N}}$ strongly converge to zero. Define $Y_{k+1} = Y_k \cup \{y_{k+1}\}$. We have to verify that $Y_{k+1}$ satisfies the inductive hypothesis, namely that for every $h \leq k + 1$ there is a norm-one vector $\xi_h$ such that $\Phi_1(P_{y_h})\xi_h = \xi_h$ and $\|\text{Ad}(v)(\Phi_2(P_{y_{k+1}}))\xi_h\| < 1/2$. For $h \leq k$, pick the $\xi_h$ given by the inductive hypothesis, and the inequality follows by item (2). Since $y_{k+1} \in X$, it follows that $\text{rk}(\Phi_1(P_{y_{k+1}})) > \text{rk}(\Phi_2(P_{y_{k+1}})) \geq \text{rk}(\text{Ad}(v)(\Phi_2(P_{y_{k+1}})))$. There exists thus a norm-one vector $\xi_{k+1}$ in the image of $\Phi_1(P_{y_{k+1}})$ which also belongs to the kernel of $\text{Ad}(v)(\Phi_2(P_{y_{k+1}}))$. This is the case since the codimension of $\ker(\text{Ad}(v)(\Phi_2(P_{y_{k+1}})))$ is strictly smaller than $\text{rk}(\Phi_1(P_{y_{k+1}}))$. Because of this and item (1):

$$
\|\text{Ad}(v)(\Phi_2(P_{y_{k+1}}))\xi_{k+1}\| = \|\text{Ad}(v)(\Phi_2(P_{y_k}))\xi_{k+1}\| \leq \|\text{Ad}(v)(\Phi_2(P_{y_k}))\Phi_1(P_{y_{k+1}})\| < 1/2.
$$

Let $Y = \cup_{k \in \mathbb{N}} Y_k$. We show that, for every $k \in \mathbb{N}$, the following holds

$$
\|(\Phi_1(P_Y) - \text{Ad}(v)(\Phi_2(P_Y)))\xi_k\| \geq 1/2,
$$

\footnote{We denote the restriction of $\Phi_m$ to $\mathcal{D}(E)$ by $\Phi_m$.}
which contradicts $\Phi_1(P_Y) \sim_{K(H)} \text{Ad}(v)(\Phi_2(P_Y))$. The previous inequality follows since, for every $k \in \mathbb{N}$, by strong continuity of $\Phi_1$ we have that $\Phi_1(P_Y)\xi_k = \xi_k$, and by strong continuity of $\Phi_2$ we have that
\[ \|\text{Ad}(v)(\Phi_2(P_Y))\xi_k\| \leq 1/2. \]

\[ \square \]

\textbf{Lemma 3.3.} Let $\varphi : Q(H) \to Q(H)$ be a unital, locally trivial endomorphism. There exists $m \in \mathbb{N}$ such that, for every $E$, the sequence $\tilde{n}_{\Phi_E} = \{rk(\Phi_E(P_k))\}_{k \in \mathbb{N}}$ is, up to a finite number of entries, constantly equal to $m$.

\textit{Proof.} By lemma 3.2 it is enough to show that there exists a partition $E$ such that $\tilde{n}_{\Phi_E}$ is eventually constant. Let $E_1$ be the partition composed by the intervals $\{2k, 2k+1\}$ and $E_2$ be the partition composed by $\{2k+1, 2k+2\}$, as $k$ varies in $\mathbb{N}$. Let $\tilde{n}_{\Phi_{E_1}} = \{n_k\}_{k \in \mathbb{N}}$ and $\tilde{n}_{\Phi_{E_2}} = \{m_k\}_{k \in \mathbb{N}}$. On the one hand we have that $n_{2k} = n_{2k+1}$ and $m_{2k+1} = m_{2k+2}$ for all $k \in \mathbb{N}$, since these couple of numbers belong to the same intervals in $E_1$ and $E_2$ respectively. On the other hand, by lemma 3.2 there is $j \in \mathbb{N}$ such that $n_i = m_i$ for all $i \geq j$. It follows that there is $m \in \mathbb{N}$ such that $n_i = m_i = m$ for all $i \geq j$. \[ \square \]

Let $\Phi : D[E] \to B(H)$ be a strongly continuous, unital $*$-homomorphism such that $\tilde{n}_\Phi$ is eventually constant with value $m$. This is not enough to infer that, even up to a unitary transformation of $H$, $\Phi$ is a compact perturbation of $\Phi_m$. For instance, the map $\Psi : \ell_\infty \to B(H)$ sending $(a_0, a_1, a_2, \ldots) \mapsto (a_0, a_0, a_1, a_2, \ldots)$ is not a compact perturbation of the identity, since it is a compact perturbation of $\text{Ad}(S)$, being $S$ the unilateral shift. Nevertheless, by suitably ‘shifting’ $\Phi$ it is possible to obtain a compact perturbation of $\Phi_m$, as shown in the following lemma.

\textbf{Lemma 3.4.} Let $E$ be a partition of $\mathbb{N}$ and let $\Phi : D[E] \to B(H)$ be a strongly continuous, unital $*$-homomorphism such that $\Phi[D[E] \cap K(H)] \subseteq K(H)$ and such that $\tilde{n}_\Phi$ is eventually constant with value $m \in \mathbb{N} \setminus \{0\}$. There exists a partial isometry $w$ of finite index such that $\text{Ad}(w) \circ \Phi \sim_{K(H)} \Phi_m$.

\textit{Proof.} If two strongly continuous, unital embeddings $\Phi_1, \Phi_2 : D[E] \to B(H)$ are such that $\tilde{n}_{\Phi_1} = \tilde{n}_{\Phi_2}$, then there is a unitary $u \in B(H)$ sending $\Phi_2(P_k)H$ to $\Phi_1(P_k)H$ for every $k \in \mathbb{N}$ such that $\text{Ad}(u) \circ \Phi_1 = \Phi_2$. Thus, it is enough to show that there is a partial isometry $w$ of finite index and a strongly continuous, unital $*$-homomorphism $\Phi : D[E] \to B(H)$ such that $\tilde{n}_\Phi$ is constantly equal to $m$. Let $n_k = m$ for all $k \in E_j$ and all $j \geq h$, which exists by lemma 3.3. Let $k$ be the minimum of $E_k$, let $n = \sum_{k \in E_k} n_k$, and let $r = \text{Tr}m - n$. Denote the projection $\sum_{k \in E_k} P_k$ by $Q$ and fix $\{\zeta_i\}_{i \in \mathbb{N}}$ an orthonormal basis of $H$ such that $\{\zeta_i\}_{i < n}$ is an orthonormal basis of $\Phi(Q)H$. We remark that $Q$ commutes with every element in $D[E]$. Let $S$ be the unilateral shift sending $\zeta_i$ to $\zeta_{i+1}$. The range of the projection $P = \text{Ad}(S^r)(\Phi(Q)) + (1 - S^rS^{-r})$ has dimension $\text{Tr}m$, since it is the space spanned by $\{\zeta_i\}_{i < n}$, and $S^{-r}S^r \geq \Phi(1 - Q)$, since the former is either the identity (if $r \geq 0$) or the orthogonal projection onto $\{\zeta_i\}_{i \geq -r}$ (if $r \leq 0$) and $-r \leq n$. Let $\Psi := (\Phi \circ \text{Ad}(Q)) \oplus (\text{Ad}(S^r\Phi(1 - Q)) \circ \Phi)$, where $\Phi$ is a unital
embedding between $QB(H)Q(\cong M_{\mathcal{E}}(\mathbb{C}))$ and $PB(H)P(\cong M_{\mathcal{K}}(\mathbb{C}))$. The map $\Phi$ is a strongly continuous, unital $*$-homomorphism (multiplicativity follows since $S^*S^* \geq \Phi(1 - Q)$ and $Q$ commutes with every element in $\mathcal{D}[\mathcal{E}]$) with $n_{\Phi}$ constantly equal to $m$ and such that $\Phi \sim_{\mathcal{K}} \text{Ad}(S^*) \circ \Phi$. □

The following lemma, an analogue of [Far11b, Lemma 1.4], shows that unital, locally trivial embeddings which lift to $\Phi_m$ on $\ell_\infty/c_0$ have nice and regular lifts also on the other $\mathcal{D}[\mathcal{E}]Q$'s.

**Lemma 3.5.** Let $\varphi : Q(H) \to Q(H)$ be a unital, locally trivial endomorphism such that $\Phi_m$ lifts $\varphi$ on $\ell_\infty/c_0$. Then, for every partition $\mathcal{E}$, there exists a unitary $u_{\mathcal{E}}$ in $\Phi_m[\ell_\infty'] \cap B(H) \cong \ell_\infty(M_{\mathcal{M}}(\mathbb{C}))$ such that $\text{Ad}(u_{\mathcal{E}}) \circ \Phi_m$ lifts $\varphi_{\mathcal{E}}$ on $\mathcal{D}[\mathcal{E}]Q$.

**Proof.** Fix a partition $\mathcal{E}$, let $v_{\mathcal{E}}$ be a partial isometry of finite index and let $\Phi_{\mathcal{E}} : \mathcal{D}[\mathcal{E}] \to B(H)$ be a strongly continuous, unital $*$-homomorphism such that $\text{Ad}(v_{\mathcal{E}}) \circ \Phi_{\mathcal{E}}$ lifts $\varphi$ on $\mathcal{D}[\mathcal{E}]$. By assumption we have that $\text{Ad}(v_{\mathcal{E}}) \circ \Phi_{\mathcal{E}} \sim_{\mathcal{K}} \Phi_m$ on $\ell_\infty$. By lemmas 3.3 and 3.4 there is a finite index isometry $w$ such that $\text{Ad}(v_{\mathcal{E}}) \circ \Phi_{\mathcal{E}} \sim_{\mathcal{K}} \text{Ad}(w) \circ \Phi_m$ on $\mathcal{D}[\mathcal{E}]$, hence the latter also lifts $\varphi$ on $\mathcal{D}[\mathcal{E}]Q$. We have therefore that $\text{Ad}(w) \circ \Phi_m \sim_{\mathcal{K}} \Phi_m$ on $\ell_\infty$, which entails that $w$ commutes, up to compact operators, with the elements in $\Phi_m[\ell_\infty]$. The commutant of $\Phi_m[\ell_\infty]$ is (isomorphic to) $\ell_\infty(M_{\mathcal{M}}(\mathbb{C}))$ and by [JP72] Theorem 2.1 we have that $w$ is a compact perturbation of an element $u$ in $\ell_\infty(M_{\mathcal{M}}(\mathbb{C}))$. Being an element of $\ell_\infty(M_{\mathcal{M}}(\mathbb{C}))$, the operator $u$ has Fredholm index zero, moreover $u \sim_{\mathcal{K}} w$ entails that its class $q(u)$ in $Q(H)$ is a unitary. Therefore, the polar decomposition of $u$ in $\ell_\infty(M_{\mathcal{M}}(\mathbb{C}))$ provides a unitary $u_{\mathcal{E}}$ in the commutant of $\Phi_m[\ell_\infty]$ such that $u_{\mathcal{E}} \sim_{\mathcal{K}} w$. □

4. **All Endomorphisms are Trivial**

We split the proof of theorem 1.3 in two steps. We first prove that all unital, locally trivial endomorphisms of $Q(H)$ are trivial, then we show that all unital endomorphisms of $Q(H)$ are locally trivial. We use OCA in both proofs. The non-unital case follows from the unital one, since every endomorphism $\varphi : Q(H) \to Q(H)$ can be thought as a unital endomorphism with codomain $Q(\varphi(1)H)$.

4.1. **Locally trivial endomorphisms are trivial.**

**Theorem 4.1.** Assume OCA. Every unital, locally trivial endomorphism $\varphi : Q(H) \to Q(H)$ is trivial.

Fix a unital endomorphism $\varphi : Q(H) \to Q(H)$. The endomorphism $\varphi$ is trivial if and only if $\text{Ad}(v) \circ \varphi$ is trivial for some unitary $v \in Q(H)$. Thus, by lemmas 3.3 and 3.4 we can assume that there is $m \in \mathbb{N}$ such that $\varphi$ lifts to $\Phi_m$ when restricted to $\ell_\infty/c_0$, and that for every partition $\mathcal{E}$ there is a unitary $u_{\mathcal{E}}$ in $\ell_\infty(M_{\mathcal{M}}(\mathbb{C}))$ such that $\text{Ad}(u_{\mathcal{E}}) \circ \Phi_m$ lifts $\varphi$ on $\mathcal{D}[\mathcal{E}]Q$. Given a unitary $u \in \ell_\infty(M_{\mathcal{M}}(\mathbb{C}))$ we denote $\text{Ad}(u) \circ \Phi_m$ by $\Phi^u$ (the endomorphism $\varphi$, and therefore the integer $m$, will be always fixed through this section, hence we omit $m$ in this notation).
The proof of theorem 4.1 is inspired to [Far11b, Section 3], where theorem 4.1 is proved for an automorphism, hence in case \( m = 1 \). The idea is to glue together the various \( u_n \) in a coherent way in order to define a unitary \( u \in \ell_\infty(M_n(\mathbb{C})) \) such that \( \Phi^n \) lifts \( \varphi \) globally. We identify the unitaries in \( \ell_\infty(M_n(\mathbb{C})) \) with elements in \( \mathcal{U}(M_n(\mathbb{C}))^\mathbb{N} \), being \( \mathcal{U}(M_n(\mathbb{C})) \) the unitary group of \( M_n(\mathbb{C}) \).

For \( u = (u(i))_{i \in \mathbb{N}}, v = (v(i))_{i \in \mathbb{N}} \in \mathcal{U}(\ell_\infty(M_n(\mathbb{C}))) \) and \( I \subseteq \mathbb{N} \), define
\[
\Delta_I(u, v) := \sup_{i,j \in I} \|u(i)u^*(j) - v(i)v^*(j)\|.
\]

**Lemma 4.2.** For all \( I \subseteq \mathbb{N} \) and \( u, v \in \mathcal{U}(\ell_\infty(M_n(\mathbb{C}))) \):

1. \( \Delta_I(u, v) \leq 2 \sup_{i \in I} \|u(i) - v(i)\| \).
2. \( \Delta_I(u, v) \geq \sup_{j \in I} \|u(j) - v(j)\| - \inf_{j \in I} \|u(i) - v(i)\| \). In particular, if \( u(k) = v(k) \) for some \( k \in I \), then \( \Delta_I(u, v) \geq \sup_{j \in I} \|u(j) - v(j)\| \).
3. If \( w \in \mathcal{U}(M_n(\mathbb{C})) \) then \( \Delta_I(u, vw) = \Delta_I(u, v) \).
4. \( \Delta_I(u, v) \leq \Delta_I(u, v) + \Delta_J(u, v) \).
5. \( \inf_{w \in \mathcal{U}(M_n(\mathbb{C}))} \sup_{i \in I} \|u(i) - v(i)w\| \leq \Delta_I(u, v) \leq 2 \sup_{i \in I} \|u(i) - v(i)\| \).

**Proof.** The proof of this lemma can be easily inferred from the proof of [Far11b, Lemma 1.5]. We have that
\[
\|u(i)u^*(j) - v(i)v^*(j)\| = \|v^*(i)(u(i)u^*(j) - v(i)v^*(j))u(j)\|
= \|v^*(i)u(i) - 1 + v^*(j)u(j)\|
= \|v^*(i)(u(i) - v(i)) - v^*(j)(u(j) - v(j))\|.
\]

This entails
\[
\|u(i) - v(i)\| - \|u(j) - v(j)\| \leq \|u(i)u^*(j) - v(i)v^*(j)\| \leq \|u(i) - v(i)\| + \|u(j) - v(j)\|,
\]
from which both item (1) and (2) follow. Item (3) is straightforward to check since \( v(i)uwv^*(j) = v(i)v^*(j) \). Notice that, unlike the 1-dimensional case, it is important to consider \( vw \) rather than \( vw \). Item (4) follows by the triangular inequality, since \( \|u(i)u^*(j) - v(i)v^*(j)\| \) is equal to \( \|v^*(i)u(i) - v^*(j)u(j)\| \). Item (5) follows by item (3) plus items (1) and (2). \( \square \)

**Lemma 4.3.** Let \( u = (u(i))_{i \in \mathbb{N}}, v = (v(i))_{i \in \mathbb{N}} \in \mathcal{U}(\ell_\infty(M_n(\mathbb{C}))) \).

1. If \( \lim_{n \to \infty} \|u(i) - v(i)\| = 0 \) then \( \Phi^u \sim_{K(H)} \Phi^v \).
2. \( \Phi^u \sim_{K(H)} \Phi^v \) on \( \mathcal{D}[\overline{E}] \) if and only if \( \limsup_n \Delta_{E_n}(u, v) = 0 \).

**Proof.** This lemma (and its proof) is an adapted version of [Far11b, Lemma 1.6] for endomorphisms. If \( \lim_{n \to \infty} \|u(i) - v(i)\| = 0 \), it means that \( u \sim_{K(H)} v \), hence \( \Phi^u \sim_{K(H)} \Phi^v \).

In order to prove item (2), suppose first that \( \limsup_n \Delta_{E_n}(u, v) = 0 \). For every \( n \in \mathbb{N} \), let \( k_n = \min(E_n) \). Let \( w = (w(i))_{i \in \mathbb{N}} \in \ell_\infty(M_n(\mathbb{C})) \) be the unitary defined, for \( i \in E_n \), as
\[
w(i) := v(i)v^*(k_n)u(k_n).
\]
The unitary \( \sum P_{E_n} \otimes v^*(k_n)u(k_n) \) belongs to the commutant of \( \Phi_m[\mathcal{D}[\overline{E}]] \), hence \( \Phi^w = \Phi^v \) on \( \mathcal{D}[\overline{E}] \). On the other hand, by items (2)-(3) of lemma 1.2
we have that, for \( i \in E_n \), \( \|w(i) - u(i)\| \leq \Delta_{E_n}(u, w) = \Delta_{E_n}(u, v) \). Thus \( \lim_{n \to \infty} \|w(i) - u(i)\| = 0 \) and, by item [11] of this lemma, \( \Phi^u \sim_{K(H)} \Phi^w = \Phi^v \).

To prove the other direction, suppose there is \( \epsilon > 0 \) and a subsequence \( \{n_k\}_{k \in \mathbb{N}} \) such that \( \Delta_{E_{n_k}}(u, v) > \epsilon \). Fix two sequences \( i_k, j_k \in E_{n_k} \) such that

\[
\|u(j_k)u^*(i_k) - v(j_k)v^*(i_k)\| > \epsilon
\]

for every \( k \in \mathbb{N} \). Let \( \eta_k \in \mathbb{C}^m \) be a norm-one vector witnessing the previous inequality. Let \( V \) be the partial isometry in \( D[\vec{E}] \) moving \( \xi_{i_k} \) to \( \xi_{j_k} \) (from the orthonormal basis of \( H \) we fixed at the beginning of [2] for every \( k \in \mathbb{N} \) and sending all other vectors in \( \{\xi_n\}_{n \in \mathbb{N}} \) to zero. We have that, if \( \zeta \in \Phi_m(P_{i_k}) \),

\[
\Phi^u(V)(\zeta) = u\Phi_m(V)u^*(i_k)(\zeta) = u(j_k)u^*(i_k)(\zeta),
\]

\[
\Phi^v(V)(\zeta) = v\Phi_m(V)v^*(i_k)(\zeta) = v(j_k)v^*(i_k)(\zeta).
\]

Thus, for the vector \( \eta_k \) we fixed before (or rather for \( 0 \oplus \cdots \oplus 0 \oplus \eta_k \oplus \cdots \), where the non-zero coordinate appears in the \( i_k \)-th position), we have

\[
\|\Phi^u(V) - \Phi^v(V)\| = \| (u(j_k)u^*(i_k) - v(j_k)v^*(i_k))\| \| \eta_k \| > \epsilon.
\]

Since this holds for every \( k \in \mathbb{N} \), it follows that the difference \( \Phi^u(V) - \Phi^v(V) \) is not compact. \( \square \)

Given a function \( f \in \mathbb{N}^\mathbb{N} \) we define

\[
E^f_n := [f(n), f(n + 1)),
\]

\[
F^f_n := [f^+(n), f^+(n + 1)),
\]

\[
E^{f, even}_n := [2f(n), 2f(n + 2)),
\]

\[
E^{f, odd}_n := [2f(n + 1), 2f(n + 3)).
\]

The corresponding partitions are \( E^f, F^f, E^{f, even} \) and \( E^{f, odd} \) respectively. We shall denote \( E^{f, even} \) and \( E^{f, odd} \) by \( E^{f, even} \) and \( E^{f, odd} \).

**Lemma 4.4.** Let \( \varphi : \mathbb{Q}(H) \to \mathbb{Q}(H) \) be a unital, locally trivial endomorphism which can be lifted to \( \Phi_m \) on \( \ell_\infty/c_0 \) for some \( m \in \mathbb{N} \). For every \( f \in \mathbb{N}^\mathbb{N} \) there is a unitary \( w \in \ell_\infty(M_m(\mathbb{C})) \) such that \( \Phi^w \) lifts \( \varphi \) on both \( D[E^{f, even}] \) and \( D[E^{f, odd}] \).

**Proof.** This proof follows the one of [Far11b, Lemma 3.5]. By assumption there are two unitaries \( u, v \in \ell_\infty(M_m(\mathbb{C})) \) such that \( \Phi^v \) and \( \Phi^v \) lift \( \varphi \) on \( D[E^{f, even}] \) and \( D[E^{f, odd}] \) respectively. We define inductively two unitaries \( u', v' \in \ell_\infty(M_m(\mathbb{C})) \) as follows. For \( i \in [f(0), f(2)) \), let \( u'(i) = u(i) \). If \( u'(i) \) has been defined for \( i < f(2n) \), for \( i \in [f(2n - 1), f(2n + 1)) \) let

\[
v'(i) = v(i)v^*(f(2n - 1))u'(f(2n - 1)).
\]

If \( v'(i) \) has been defined for \( i < f(2n + 1) \), for \( i \in [f(2n), f(2n + 2)) \) let

\[
u'(i) = u(i)u^*(f(2n))v'(f(2n)).
\]

We have that \( v'(f(n)) = u'(f(n)) \), that \( \Phi^u = \Phi^{u'} \) on \( D[E^{f, even}] \) and that \( \Phi^v = \Phi^{v'} \) on \( D[E^{f, odd}] \). This implies that, by item [2] of lemma 4.2

\[
\sup_{i \in E^f_n} \|u'(i) - v'(i)\| \leq \Delta_{E^f_n}(u', v').
\]

On the other hand we have that \( \Phi^{u'} = \Phi^{u} \sim_{K(H)} \Phi^v = \Psi^{v'} \) on hypothesis (remember that \( D[E] \subseteq D[E^{f, even}] \cap D[E^{f, odd}] \)), therefore by
Using the pigeonhole principle, we can assume that there is

\[ \text{Theorems 3.4 and 8.5} \]

\[ \text{there is} \]

\[ f \]

\[ \text{Proof.} \]

\[ \text{Assume} \]

\[ \text{Fix} \]

\[ X. \]

\[ \text{Let} \]

\[ (U \times V) \subseteq X \times Y \] with the product topology. In this setting, it is straightforward to check that \( \mathcal{E} \) is a complete separable metric. We consider

\[ \text{Lemma 4.5 (Far11b Lemma 3.3)}. \]

Assume \( \mathcal{F} \subseteq \mathbb{N}^\mathbb{N} \) is \( \leq^* \)-cofinal.

\( \text{(1) If} \mathcal{F} \text{ is partitioned into countably many pieces, then at least one is} \leq^* \)-cofinal.

\( \text{(2) } (\exists n)(\exists i)(\forall k \geq n)(\exists f \in \mathcal{F})(f(i) \leq n \text{ and } f(i+1) \geq k). \)

\( \text{(3) } \{ f^+ : f \in \mathcal{F} \} \text{ is } \leq^* \)-cofinal.

\[ \text{Lemma 4.6 (Far11b Lemma 3.4)}. \]

Let \( f, g \in \mathbb{N}^\mathbb{N} \) be such that \( g \leq^* f \). For all but finitely many \( n \in \mathbb{N} \) there is an \( i \) such that \( f^+(i) \leq g(n) < g(n+1) \leq f^+(i+2) \). If \( f(m) \geq g(m) \) for all \( m \in \mathbb{N} \), then the previous statement holds for every \( n \in \mathbb{N} \).

Lemma 4.6 entails that if \( g \leq^* f \) then for all but finitely many \( n \in \mathbb{N} \) there is an \( i_n \in \mathbb{N} \) such that \( E^g_{i_n} \subseteq F^f_{i_n} \cup F^f_{i_n+1} \). In particular, if \( f(m) \geq g(m) \) for all \( m \in \mathbb{N} \), then \( D[\mathcal{E}^g] \) is contained in the algebra generated by \( D[\mathcal{E}^f, \text{even}] \cup D[\mathcal{E}^f, \text{odd}] \).

**Proof of theorem 4.1.**

We can assume that \( \varphi : \mathcal{Q}(H) \to \mathcal{Q}(H) \) is locally represented on \( D[\mathcal{E}] \) by \( \Phi^u_\mathcal{E} \), where \( u_\mathcal{E} \) is a unitary in \( \ell_\infty(M_m(\mathbb{C})) \) (see the paragraph after the statement of theorem 4.1). Let \( \mathcal{X} \subseteq \mathbb{N}^\mathbb{N} \times U(M_m(\mathbb{C}))^\mathbb{N} \) be the set of all pairs \((f, u)\) such that \( \Phi^u \) lifts \( \varphi \) on both \( D[\mathcal{E}^f, \text{even}] \) and \( D[\mathcal{E}^f, \text{odd}] \). By lemma 4.3 for every \( f \in \mathbb{N}^\mathbb{N} \) there is \( u \) such that \((f, u) \in \mathcal{X} \).

Fix \( \epsilon > 0 \) and consider the coloring of \([X]^2 = K_0^c \cup K_1^c\), where the pair \((f, u)\), \((g, v)\) has color \( K_0^c\) if there are \( m, n \in \mathbb{N} \) such that \( \Delta_{F^f \cup F^g} (u, v) > \epsilon \).

We consider \( \mathbb{N}^\mathbb{N} \) with the Baire space topology, induced by the metric

\[ d(f, g) = 2^{-\min\{n : f(n) \neq g(n)\}}. \]

This is a complete separable metric. We consider \( U(M_m(\mathbb{C}))^\mathbb{N} \) with the product of the strong operator topology on \( U(M_m(\mathbb{C})) \) and \( \mathcal{X} \) with the product topology. In this setting, it is straightforward to check that \( K_0^c \) is open.

**Claim 4.6.1.** Assume OCA. For every \( \epsilon > 0 \) there are no uncountable \( K_0^c \)-homogeneous subsets of \( \mathcal{X} \).

**Proof.** Fix \( \epsilon > 0 \) and let \( \mathcal{H} \) be an uncountable \( K_0^c \)-homogeneous subset of \( \mathcal{X} \). Let

\[ \mathcal{F} = \{ g^+ : \exists u(g, u) \in \mathcal{H} \}. \]

We can assume that \( \mathcal{H} \), and thus \( \mathcal{F} \), has size \( \aleph_1 \). By OCA and [Tod89, Theorems 3.4 and 8.5] there is \( f \in \mathbb{N}^\mathbb{N} \) which is an upper bound for \( \mathcal{F} \).

Using the pigeonhole principle, we can assume that there is \( \pi \in \mathbb{N} \) such that \( f(m) \geq g^+(m) \) for all \( g^+ \in \mathcal{F} \) and all \( m \geq \pi \), and moreover that \( g^+(i) = h^+(i) \) for all \( g^+, h^+ \in \mathcal{F} \) and all \( i \leq \pi \). By increasing \( f \) by \( f(\pi) \) we can also assume that \( f(m) \geq g^+(m) \) for all \( g^+ \in \mathcal{F} \) and all \( m \in \mathbb{N} \). By lemma 4.6 this
entails that for every \( n \in \mathbb{N} \) there is \( i \in \mathbb{N} \) such that \( E_n^+ = F_n^0 \subseteq F_i^f \cup F_{i+1}^f \).

Let \( u \) be a unitary in \( \ell_\infty(M_m(\mathbb{C})) \) such that \( \Phi^u \) lifts \( \varphi \) on \( D[\tilde{F}_{f,\text{even}}] \) and \( D[\tilde{F}_{f,\text{odd}}] \). Since, by the previous observations, for every \( g^+ \in \mathcal{F} \) we have that \( D[\tilde{F}^g] \) is contained in the algebra generated by \( D[\tilde{F}_{f,\text{even}}] \cup D[\tilde{F}_{f,\text{odd}}] \), it follows that \( \Phi^u \) lifts \( \varphi \) also on \( D[\tilde{F}^g] \) and therefore, by item (2) of lemma 4.3 we have that \( \lim_{n \to \infty} \Delta_{F_n^g}(u,v) = 0 \) for every \( (g,v) \in \mathcal{H} \). By taking an uncountable subset of \( \mathcal{F} \) if necessary, we can assume that there is \( \mathcal{K} \) such that \( \Delta_{F_n^g}(u,v) < \epsilon/2 \) for all \( m \geq \mathcal{K} \) and all \( (g,v) \in \mathcal{H} \). By separability of \( \mathcal{U}(M_m(\mathbb{C}))^\mathbb{N} \) there are \( (g,v), (h,w) \in \mathcal{H} \) such that \( g^+(i) = h^+(i) \) for all \( i \leq \mathcal{K} \) and \( \| u_i - v_i \| < \epsilon/2 \) for all \( i \leq g^+(\mathcal{K}) \). This entails that if \( n, m \in \mathbb{N} \) are such that \( F_n^g \cap F_m^h \neq \emptyset \), then either both \( m, n \leq \mathcal{K} \) or \( m, n \geq \mathcal{K} \). In the former case it follows that \( \Delta_{F_n^g \cap F_m^h}(v,w) < \epsilon \) by item (1) of lemma 4.2 If \( m, n \geq \mathcal{K} \) then we have

\[
\Delta_{F_n^g \cap F_m^h}(w,v) \leq \Delta_{F_n^g \cap F_m^h}(u,v) + \Delta_{F_n^g \cap F_m^h}(u,w) < \epsilon.
\]

This is a contradiction since \( (g,v), (h,w) \in \mathcal{H} \). \( \square \)

By OCA, for every \( \epsilon > 0 \) there is a partition of \( \mathcal{X} \) into countably many \( K_1^\alpha \)-homogeneous sets. Let \( \epsilon_n = 2^{-n} \). Repeatedly using item (1) of lemma 4.3 find sequences \( \mathcal{X} \supseteq \mathcal{X}_0 \supseteq \cdots \supseteq \mathcal{X}_n \supseteq \cdots \) and \( 0 = m(0) < m(1) < \cdots < m(n) < \cdots \) such that \( \mathcal{X}_n \) is \( K_1^e \)-homogeneous and such that the set \( \{ f : (\exists u)(f,u) \in \mathcal{X}_n \} \) is \( \preceq^* \)-cofinal. Let \( m(n) \) be the natural number given by item (2) of lemma 4.3 for \( \mathcal{X}_n \). For each \( n \in \mathbb{N} \) fix a sequence \( \{(f_{n,i},u_{n,i})\}_{i \in \mathbb{N}} \) in \( \mathcal{X}_n \) such that, for some \( j_i \in \mathbb{N} \)

\[
(4.6.1) \quad f_{n,i}(j_i) \leq m(n) < m(n + i) \leq f_{n,i}(j_i + 1).
\]

By compactness of \( \mathcal{U}(M_m(\mathbb{C}))^\mathbb{N} \), we can assume that each sequence \( \{u_{n,i}\}_{i \in \mathbb{N}} \) converges to some \( u_n \).

**Claim 4.6.2.** There is a subsequence \( \{u_{n_k}\}_{k \in \mathbb{N}} \) such that

\[
\sup_{i \in [m(n_k),\infty)} \| u_{n_k}(i) - u_{n_h}(i) \| \leq \epsilon_k
\]

for all \( h \geq k \).

**Proof.** We start by showing that \( \Delta_{(m(n),\infty)}(u_h,u_n) \leq \epsilon_h \) for all \( h < n \). Suppose this is not the case and let \( m(n) \leq i_1 < i_2 \) be such that \( \| u_{h}(i_1)u_n^*(i_2) - u_{n}(i_1)u_n^*(i_2) \| > \epsilon_h \). There is \( j \in \mathbb{N} \) such that

\[
\| u_{h,j}(i_1)u_{h,j}^*(i_2) - u_{n,j}(i_1)u_{n,j}^*(i_2) \| > \epsilon_h
\]

and by (4.6.1) there are \( k_1, k_2 \in \mathbb{N} \) such that

\[
f_{h,j}(k_1) \leq m(n) < i_2 < f_{h,j}(k_1 + 1),
\]

\[
f_{h,j}(k_2) \leq m(h) < i_2 < f_{h,j}(k_2 + 1).
\]

In particular, this entails that \( \Delta_{F_{k_1,j} \cap F_{k_2,j}}(u_{h,j},u_{n,j}) > \epsilon_h \), which is a contradiction since \( (f_h,j,u_{h,j}) \) and \( (f_{n,j},u_{n,j}) \) both belong to \( \mathcal{X}_h \), which is \( K_1^e \)-homogeneous.
By item 5 of lemma 4.2, for every \( h < n \) there is \( w_{h,n} \in \mathcal{U}(M_m(\mathbb{C})) \) such that

\[
\sup_{i \geq m(n)} \| u_n(i) - u_h(i) w_{h,n} \| \leq \epsilon_h.
\]

The unitary \( w_{h,n} \) exists by compactness of \( \mathcal{U}(M_m(\mathbb{C})) \). Given \( h < n < k \in \mathbb{N} \) we have that, for \( i \geq m(k) \)

\[
u_h(i) w_{h,n} \approx_{\epsilon_h} u_n(i) \approx_{\epsilon_h} u_k(i) w_{n,k}^* \approx_{\epsilon_h} u_h(i) w_{h,k} w_{n,k}^*.
\]

hence

\[(4.6.2) \quad \| w_{h,n} - w_{h,k} w_{n,k}^* \| \leq 3\epsilon_h.\]

This can be used to show that there is an infinite \( Y = \{ n_k \}_{k \in \mathbb{N}} \subseteq \mathbb{N} \) such that, for \( i < j \in \mathbb{N} \), then \( \| 1 - w_{n_i,n_j} \| \leq 4\epsilon_{n_i} \). To see this, define a coloring \( M_0 \cup M_1 \) on the triples of elements in \( \mathbb{N} \), by saying that the triple \( i < j < k \) is in \( M_0 \) if and only if

\[
\| 1 - w_{j,k} \| \leq 4\epsilon_j.
\]

Suppose there is an infinite \( M_1 \)-homogeneous set \( Y \). Let \( h \) be the minimum of \( Y \). By compactness of the unit ball of \( M_m(\mathbb{C}) \) there is \( n \in Y \) big enough so that, for some \( j < k < n \) all in \( Y \) we have that \( \| w_{j,n} - w_{k,n} \| < \epsilon_h \). It follows that \( \| w_{j,k} - 1 \| \leq \| w_{j,k} - w_{j,n} w_{n,k}^* \| + \| 1 - w_{j,n} w_{n,k}^* \| \leq 4\epsilon_k \),

which is a contradiction, since the triple \( (h, j, k) \) is supposed to be in \( M_1 \). By Ramsey’s theorem there is an infinite \( M_0 \)-homogeneous set \( Y = \{ n_k \}_{k \in \mathbb{N}} \).

We have therefore, for \( j > i \geq 1 \)

\[
\| 1 - w_{n_i,n_j} \| \leq 4\epsilon_{n_{i-1}}.
\]

Without loss of generality we can assume that \( n_0 \geq 4 \), hence that, for every \( i \geq 1 \) the following holds

\[
4\epsilon_{n_{i-1}} \leq \epsilon_{i-1}/4 = \epsilon_{i+1}.
\]

Summarizing, we have that for every \( k < h \in \mathbb{N} \)

\[
\sup_{i \geq m(n_h)} \| u_{n_k}(i) - u_{n_h}(i) \| \leq \sup_{i \geq m(n_h)} \| u_{n_k}(i) w_{n_k,n_h} - u_{n_k}(i) \| + \| 1 - w_{n_k,n_h} \|
\]

\[
\leq \epsilon_{n_k} + \epsilon_{k+1} \leq \epsilon_k.
\]

Let \( \{ u_{n_k} \}_{k \in \mathbb{N}} \) be the subsequence given by the previous claim and let \( v \in \mathcal{U}(M_m(\mathbb{C}))^\mathbb{N} \) be defined as \( v(i) = u_{n_k}(i) \) for all \( i \in [m(n_k), m(n_k+1)] \) and \( v(i) = u_{n_0}(i) \) for all \( i \leq m(n_0) \). It follows then that \( \| v(i) - u_{n_k}(i) \| < \epsilon_k \) for all \( i \geq m(n_k) \). Given \( j \in \mathbb{N} \) and \( (g, w) \in \mathcal{X}_{n_j} \), we claim that for all \( i \in \mathbb{N} \) we have \( \Delta F^g_{n_j \setminus n_j}(v, w) \leq 3\epsilon_j \). This is the case since for every \( i \in \mathbb{N} \) there is \( h \in \mathbb{N} \) such that

\[
[f^+_{n_j,h}(j_h), f^+_{n_j,h}(j_h + 1)] \supseteq F^g_i \setminus m_{n_j}.
\]
Hence, since both $g$ and $f_{n_j,h}$ belong to $X_n$, we have that $\Delta_{F^q_{m_{n_j}}}(w, u_{n_j,h}) < \epsilon_{n_j}$. By continuity, we also have $\Delta_{F^q_{m_{n_j}}}(w, u_{n_j}) \leq \epsilon_{n_j}$. Thus, in conclusion

$$\Delta_{F^q_{m_{n_j}}}(v, w) \leq \Delta_{F^q_{m_{n_j}}}(v, u_{n_j}) + \Delta_{F^q_{m_{n_j}}}(u_{n_j}, w)$$

$$\leq 2 \sup_{h \in F^q_{m_{n_j}}} \|v(h) - u_{n_k}(h)\| + \epsilon_{n_j}$$

$$\leq 3\epsilon_j.$$  (4.6.3)

We conclude by showing that $\Phi^v$ lifts $\varphi_E$ for every partition $E$. Let $g \in \mathbb{N}^\mathbb{N}$ be such that $E^g = \tilde{E}^g$ and find $u \in \mathcal{U}(M_n(\mathbb{C}))/\mathcal{U}$ such that $\Phi^u$ lifts $\varphi$ on $\mathcal{D}[\tilde{E}^g]$. By item (2) of lemma 4.3, it is enough to show that $\lim_{n \to \infty} \Delta_{E^g_n}(w, u) = 0$. Fix $k \in \mathbb{N}$, and let $(f, w) \in X_{n_k}$ be such that $f \geq^* g$. For all but finitely many $n \in \mathbb{N}$ there is $i \in \mathbb{N}$ such that $E^g_n \subseteq F^{f_i}_{n_i} \cup F^{f_{i+1}}_{n_{i+1}}$. This implies that $\lim_{n \to \infty} \Delta_{E^g_n}(w, u) = 0$, which, by item (3) of lemma 4.2 in turn entails

$$\lim_{n \to \infty} \Delta_{E^g_n}(u, v) \leq \lim_{n \to \infty} \Delta_{E^g_n}(u, w) + \Delta_{E^g_n}(w, v) = \lim_{n \to \infty} \Delta_{E^g_n}(w, v)$$

$$\leq \lim_{n \to \infty} \Delta_{F^{f_i}_{n_i} \cup F^{f_{i+1}}_{n_{i+1}}}(w, v)$$

$$\leq \lim_{n \to \infty} \Delta_{F^{f_i}_{n_i}}(w, v) + \Delta_{F^{f_{i+1}}_{n_{i+1}}}(w, v)$$

$$\leq 6\epsilon_k.$$  (4.6.4)

The inequality above holds for every $k \in \mathbb{N}$, thus $\lim_{n \to \infty} \Delta_{E^g_n}(u, v)$ is zero. □

4.2. All endomorphisms are locally trivial.

**Theorem 4.7.** Assume OCA. Every unital endomorphism $\varphi : \mathcal{Q}(H) \to \mathcal{Q}(H)$ is locally trivial.

**Proof.** Given a partition $E$, we want to find a finite index isometry $v_E$ and a strongly continuous, unital $*$-homomorphism $\Phi_E : \mathcal{D}[\tilde{E}] \to \mathcal{B}(H)$ such that $\text{Ad}(v_E) \circ \Phi_E$ lifts $\varphi$ on $\mathcal{D}[\tilde{E}]$. Without loss of generality, we take a partition $E$ which is composed of intervals whose length is strictly increasing. First we need a fact following from OCA which is proved in [Far11b, §6, §7] (see also [Far §18]). In that paper it is shown that there is a strongly continuous $*$-homomorphism $\Psi : \mathcal{D}[\tilde{E}] \to \mathcal{B}(H)$ which lifts $\varphi$ on $\mathcal{D}_X[\tilde{E}]$, for some infinite $X \subseteq \mathbb{N}$. The paper [Far11b] focuses on automorphisms $\mathcal{Q}(H)$, but these proofs also work for unital endomorphisms. More specifically, [Far11b] Lemma 7.2 and [Far11b] Proposition 7.7 can be used, as shown in the proof of [Far11b] Proposition 7.1, to find an infinite $Y \subseteq \mathbb{N}$ such that $\varphi$ has a $C$-measurable (see [Far11b] §2.1) for a definition) lift $\Psi'$ on $\mathcal{D}_Y[\tilde{E}]$. The proof of [Far11b] Theorem 6.3 (which does not require OCA) shows how to find an infinite $X \subseteq Y$ and refine $\Psi'$ to a strongly continuous $*$-homomorphism $\Psi : \mathcal{D}[\tilde{E}] \to \mathcal{B}(H)$ such that $\Psi$ lifts $\varphi$ on $\mathcal{D}_X[\tilde{E}]$. Alternatively, it is possible to use [MV18 Theorem 8.4] to directly obtain $X$ and $\Psi$. This result however, being a more general statement about corona algebras rather than only about the Calkin algebra, requires the stronger assumption $\text{OCA}_\infty + \text{MA}_\mathbb{R}_1$ (see [MV18 §2.2]). Given such lift
Ψ and \( X \subseteq \mathbb{N} \), the idea now is to exploit the abundance of partial isometries in \( \mathcal{Q}(H) \) to obtain a global unital lift in \( \mathcal{D}[\hat{E}] \). Concretely, we ‘compress’ elements of \( \mathcal{D}[\hat{E}] \) into \( \mathcal{D}_X[\hat{E}] \), apply \( \Psi \) and finally we ‘decompress’ their image in \( \mathcal{B}(H) \) (see also [Far11b, Lemma 4.1]).

Let \( v \in \mathcal{B}(H) \) be a partial isometry such that \( v^*v = 1 \), \( P := vv^* \leq P_X \) belongs to \( \mathcal{D}_X[\hat{E}] \) and such that \( vD[\hat{E}]v^* \subseteq \mathcal{D}_X[\hat{E}] \). Such partial isometry exists since, by assumption, the length of the intervals in \( \hat{E} \) is strictly increasing. Let \( Q \) be the image of \( P \) via \( \Psi \). Since \( P \in \mathcal{D}_X[\hat{E}] \) we have that

\[
q(Q) = \varphi(q(P)).
\]

Let \( w \) be a partial isometry lifting \( q(w) \), hence we have

\[
q(w) = \varphi(q(v)), \quad 1 \sim_{\mathcal{K}(H)} w^*w \text{ and } Q \sim_{\mathcal{K}(H)} ww^*.
\]

**Claim 4.7.1.** Up to a compact perturbation, we can assume that \( w \) satisfies the following properties.

1. \( ww^* \leq Q \).
2. \( w^*w \geq Q \).

**Proof.** We start by proving item (1). Let \( w' \) be a lift of \( \varphi(q(v^*)) \). Then \( w'Q \) is also a lift of \( \varphi(q(v^*)) \), since

\[
q(w'Q) = \varphi(q(v^*)) \varphi(q(P)) = \varphi(q(v^*P)) = \varphi(q(v^*)).
\]

Let \( w' = u|w'Q| \) be the polar decomposition of \( w' \). Since \( |w'Q| \) is a compact perturbation of the identity, we have that \( u \), whose kernel is equal to \( \ker(w'Q) \subseteq \ker(Q) \), is also a lift of \( \varphi(q(v^*)) \) such that \( u^*u \leq Q \). Let \( w \) be \( u^* \). Summarizing, we can assume that both \( 1 - w^*w \) and \( Q - ww^* \) are finite rank projections.

In order to prove item (2), first notice that the space \( K := Q^\perp H \cap w^*wH \) is infinite dimensional, as the former is infinite dimensional and the latter has finite codimension. Let \( n \) be the rank of \( 1 - w^*w \) and fix a set of linearly independent vectors \( \{\zeta_k\}_{k<n} \) in \( K \). Let \( \{\eta_k\}_{k<n} \) be a basis of \( (1 - w^*w)H \) and modify \( w \) to be the operator sending all vectors in \( \{\zeta_k\}_{k<n} \) to zero, sending \( \eta_k \) to \( w(\zeta_k) \) for every \( k < n \), and acting as \( w \) everywhere else. With these (compact) modifications, we have that \( w^*w \geq Q \). \( \square \)

Let \( \{\zeta_k\}_{k<n} \) be an orthonormal basis of \( (1 - w^*w)H \), and let \( \{\zeta_k\}_{k\in\mathbb{N}} \) be an orthonormal basis of \( H \) extending it. Denote the shift operator sending \( \zeta_k \) to \( \zeta_{k+1} \) for all \( k \in \mathbb{N} \) by \( S \) and let \( r \in \mathbb{Z} \) be the difference

\[
r := \text{rk}(Q - ww^*) - \text{rk}(1 - w^*w).
\]

The operator \( S^{-r}S^r \) is either the identity (if \( r \) is zero or positive) or a projection greater than \( w^*w \), therefore, by claim 4.7.1

\[
S^{-r}S^r \geq w^*w \geq Q \geq ww^*.
\]

Consequently, \( \bar{w} := S^r w S^{-r} \) is a partial isometry such that \( \bar{w}^* \bar{w} = S^r w^* w S^{-r} \) and \( \bar{w} \bar{w}^* = S^r \bar{w}^* \bar{w} S^{-r} \). Moreover \( \text{rk}(S^r(Q - ww^*)S^{-r}) = \text{rk}(Q - ww^*) \), since \( S^r \) acts as an isometry on \( (Q - ww^*)H \) and \( S^{-r} \) acts as an isometry on \( S^r(Q - ww^*)H \), which is the case since \( S^{-r}S^r \geq Q \geq Q - ww^* \). The operator \( 1 - S^r w^* w S^{-r} \) is the orthogonal projection onto the span of \( \{\zeta_k\}_{k<r+n} \).
therefore \( \text{rk}(1 - S^* w^* w S^-r) = \text{rk}(1 - w^* w) + r \), thus
\[
\text{rk}(S^*(Q - w w^*) S^-r) = \text{rk}(1 - S^* w^* w S^-r).
\]
Because of this, there is a partial isometry \( w \) such that
\[
(4.7.4) \quad w^* w = 1, \quad w w^* = S^* Q S^-r
\]
and
\[
(4.7.5) \quad w \sim_{K(H)} S^* w S^-r.
\]
We claim that the map
\[
\Phi \bar{\varepsilon} : D[\bar{E}] \to B(H)
\]
\[
a \mapsto w^* w \Psi(v a v^*) S^-r w
\]
is a strongly continuous, unital \(*\)-homomorphism lifting \( \text{Ad}(q(S^*)) \circ \varphi \) on \( D[\bar{E}] \). Strong continuity, linearity and preservation of the adjoint operation follow since \( \Psi \) has these properties. Unitality is a consequence of the definition of \( w \):
\[
\text{Eq. (4.7.4)} \quad w^* w \Psi(v a v^*) S^-r w = \text{Eq. (4.7.4)} \quad w^* w \Psi(v a v^*) S^-r w = 1.
\]
Given \( a, b \in D[\bar{E}] \) we have that
\[
\text{Eq. (4.7.4)} \quad w^* w \Psi(v a v^*) S^-r w = \text{Eq. (4.7.4)} \quad w^* w \Psi(v a v^*) S^-r w = 1.
\]
Finally, for \( a \in D[\bar{E}] \), the following holds
\[
\text{Eq. (4.7.1)} \quad q(w^* w \Psi(v a v^*) S^-r w) = q(S^* w^* S^-r S^* \varphi(q(v a v^*)) q(S^-r S^* w S^-r))
\]
\[
\text{Eq. (4.7.1)} \quad q(S^* w^* S^-r S^* \varphi(q(v a v^*)) q(S^-r S^* w S^-r)) = q(S^* w^* q(Q) \varphi(q(v a v^*)) q(Q) q(S^-r))
\]
\[
\text{Eq. (4.7.1)} \quad q(S^* w^* q(Q) \varphi(q(v a v^*)) q(Q) q(S^-r)) = q(S^* \varphi(q(a)) q(S^-r).
\]

5. Classification and Closure Properties

We are finally ready to prove theorems 1.1 and 1.2.

Proof of theorem 1.1 We start by showing the following claim, which does not require OCA.

Claim 5.0.1. The index of the image of the unilateral shift \( S \) via a trivial, unital endomorphism \( \varphi \) is finite and negative.
Proof. By definition of trivial endomorphism, there is a unitary \( u \in Q(H) \) such that \( \text{Ad}(u) \circ \varphi \) is induced by a strongly continuous endomorphism \( \Phi \) of \( B(H) \), which can assumed to be unital (see remark 5.1). This means that there exists \( m \in \mathbb{N} \setminus \{0\} \) such that, up to unitary transformation, \( \varphi \) lifts to the map \( \Phi_m : B(H) \to B(H \otimes \mathbb{C}^m) \) sending \( T \) to \( T \otimes 1_m \) (see remark 2.3). In particular, the index of \( \varphi(q(S)) \) is \(-m\). \( \square \)

The forward direction of the equivalence is straightforward. Suppose thus that \( \varphi_1, \varphi_2 \) are two endomorphisms of \( Q(H) \) that satisfy conditions (1) and (2) of the statement. By condition (1) we can assume that \( \varphi_1(1) = \varphi_2(1) = p \). If there is a unitary \( v \in Q(H) \) such that \( \text{Ad}(v) \circ \varphi_1 = \varphi_2 \), then \( vpu^* = p \), hence \( v \) and \( p \) commute, which means that \( v \) is a direct sum of a unitary in \( pQ(H)p \) and a unitary in \((1 - p)Q(H)(1 - p)\). The only part of \( v \) acting non-trivially on \( \varphi_1[q(H)] \) is the one in \( pQ(H)p \). Hence, without loss of generality, we can assume that \( p = \varphi_1(1) = \varphi_2(1) = 1 \). By theorem 1.3 both \( \varphi_1 \) and \( \varphi_2 \) are trivial. Therefore, by condition (2) and claim 5.0.1 we have that \( \varphi_1 \) and \( \varphi_2 \), modulo unitary equivalence, lift to the same endomorphism on \( B(H) \), hence they are equal.

The final sentence of the theorem follows from claim 5.0.1 since \( \Phi_m \oplus \Phi_n = \Phi_{m+n} \) and \( \Phi_n \circ \Phi_m = \Phi_{mn} \) for every \( m, n \in \mathbb{N} \setminus \{0\} \). \( \square \)

Remark 5.1. Every non-unital \( \varphi \in \text{End}(Q(H)) \) can be written as a direct sum \( \varphi_1 \oplus 0 \) where \( \varphi_1 \) is unital and 0 is the zero endomorphism of \( Q(H) \). Therefore, by theorem 1.3 every non-unital endomorphism \( \varphi \) is, up to unitary equivalence, equal to \( \Phi_m \oplus 0 \) for some \( m \in \mathbb{N} \). Consider the set \( \mathcal{N} := (\mathbb{N} \times \{0, 1\}) \setminus \{(0, 1)\} \) and, for \( \varphi \in \text{End}(Q(H)) \), define \( \text{ind}(\varphi) := -\text{ind}(\varphi(q(S)) + (1 - \varphi(1))) \), where \( S \) is the unilateral shift. Consider the map:

\[
\Theta : \text{End}(Q(H)) \to \mathcal{N}
\]

\[
\varphi \mapsto \begin{cases} 
(0, 0) & \text{if } \varphi(1) = 0 \\
(\text{ind}(\varphi), 1) & \text{if } \varphi(1) = 1 \\
(\text{ind}(\varphi), 0) & 0 < \varphi(1) < 1 
\end{cases}
\]

By theorem 1.3 and the previous observation, the map \( \Theta \) is a bijection, since all projections \( p \in Q(H) \) such that \( 0 < p < 1 \) are unitarily equivalent in \( Q(H) \). For \( \varphi_1, \varphi_2 \in \text{End}(Q(H)) \) we have that \( \varphi_1 \oplus \varphi_2 \) (and \( \varphi_1 \circ \varphi_2 \)) is non-unital if and only if at least one between \( \varphi_1 \) and \( \varphi_2 \) is non-unital. Therefore, the map \( \Theta \) is a semigroup isomorphism between \( \text{End}(Q(H)), \oplus \) and \( (\mathcal{N}, +) \), where the addition on \( \mathcal{N} \) is defined as \( (n, i) + (m, j) = (n + m, i \cdot j) \). Analogously, \( \Theta \) is an isomorphism between \( \text{End}(Q(H)), \circ \) and \( (\mathcal{N}, \cdot) \), where \( (n, i) \cdot (m, j) = (n \cdot m, i \cdot j) \).

Proof of theorem 2.2.1 Let \( \varphi : Q(H) \to Q(H) \) be a trivial, unital endomorphism. Up to unitary transformation there is \( m \in \mathbb{N} \) such that \( \varphi \) is induced by the map \( \Phi_m : B(H) \to B(H \otimes \mathbb{C}^m) \) sending \( T \) to \( T \otimes 1_m \) (see remark 2.3). The commutant of \( \Phi_m[B(H)] \) in \( B(H \otimes \mathbb{C}^m) \) is isomorphic to \( M_m(\mathbb{C}) \). By [JP72, Lemma 3.2] also the commutant of \( \varphi[Q(H)] \) in the codomain \( Q(H) \) is isomorphic to \( M_m(\mathbb{C}) \). Thus, the commutant of the image of a trivial, unital endomorphism of \( Q(H) \) is always finite dimensional.
Let $B \in \mathbb{E}$ be unital and infinite-dimensional. Consider the algebraic tensor product $Q(H) \otimes_{\text{alg}} B$ and suppose $\psi : Q(H) \otimes_{\text{alg}} B \to Q(H)$ is an embedding. The element $\psi(1)$ is a non-zero projection. Since $\psi(1)Q(H)\psi(1) \cong Q(H)$, we can assume that $\psi$ is unital. On the one hand, by theorem 1.3 the restriction of $\psi$ to $Q(H)$ is unital and trivial, on the other hand $\psi$ sends $B$ into the commutant of $\psi[Q(H)]$, which is a contradiction.

(2) Let $\{A_n\}_{n \in \mathbb{N}}$ be an increasing sequence of finite-dimensional, unital C*-algebra such that $A := \bigcup_{n \in \mathbb{N}} A_n$ is an infinite-dimensional, unital AF-algebra. We have that $Q(H) \otimes A_n \in \mathbb{E}$ for all $n \in \mathbb{N}$, but, by the proof of item (1), $Q(H) \otimes A \notin \mathbb{E}$. □

By the results in [Far11b] we know that it is consistent with ZFC that there is no automorphism of $Q(H)$ sending the unilateral shift $S$ to its adjoint. Combining claim 5.0.1 with theorem 1.3 and with the equality $\text{ind}(q(S^*)) = 1$, we can generalize this statement to endomorphisms of $Q(H)$.

**Corollary 5.2.** Assume OCA. There is no endomorphism $\varphi : Q(H) \to Q(H)$ sending the unilateral shift to its adjoint or to any unitary of index zero.

The following two examples witness the failure of theorems 1.1 and 1.3 when CH holds, suggesting a rather complicated picture of $\text{End}(Q(H))$ in that case. In particular, while $\text{End}(Q(H))$ is countable under OCA (theorem 1.1), CH implies that $\text{End}(Q(H))$ has size $2^{\aleph_1}$, as shown in the following example.

**Example 5.3.** An automorphism is trivial if and only if it is inner. Hence all the inequivalent $2^{\aleph_1}$ outer automorphisms produced in [PW07] are examples of non-trivial endomorphisms of $Q(H)$ (outer automorphisms of $Q(H)$ can also be built using the weak Continuum Hypothesis, see [FSL14] and [Far §18.1]). All the automorphisms built in [PW07] locally behave like an inner automorphism, in particular they behave like an inner automorphism on the unilateral shift. Therefore, under CH it is possible to find $2^{\aleph_1}$ inequivalent automorphisms which all send the unilateral shift to an element of index -1. Hence this invariant is not enough to classify $\text{End}_u(Q(H))$, and theorem 1.1 fails under CH. It would be interesting to investigate whether CH implies that for every endomorphism $\varphi$ of $Q(H)$ there are $2^{\aleph_1}$ inequivalent endomorphisms of $Q(H)$ with the same invariant as $\varphi$.

**Example 5.4.** In [FHV19] it is proved that all C*-algebras of density character $\mathfrak{r}_1$ can be embedded into $Q(H)$ with a map whose restriction to the separable subalgebras is a trivial extension.\footnote{Given a C*-algebra $A$ and an embedding $\theta : A \to Q(H)$, the map $\theta$ is a trivial extension of $A$ if and only if $\theta$ is a $\ast$-homomorphism $\varphi : A \to B(H)$ such that $\varphi = q \circ \theta$. This notion of trivial maps, albeit more common, is different from the one we used throughout this paper in in definition 2.2.} Under CH the density character of $Q(H)$ is $\mathfrak{r}_1$, thus there is a unital endomorphism $\varphi : Q(H) \to Q(H)$ that sends the unilateral shift $S$ to a unitary in $Q(H)$ which lifts to a unitary in $\mathcal{B}(H)$, which has therefore index zero. By claim 5.0.1 this map cannot be a trivial endomorphism. With this example we see that without OCA the range of values assumed by $\text{ind}(\varphi(S))$, as $\varphi$ varies in $\text{End}_u(Q(H))$, can
be strictly larger than the negative numbers. Notice that the existence of an automorphism sending $S$ to $S^*$ would give an example where the index of the image of the shift is positive.

In [Far11a] the author extends his results in [Far11b] to all Calkin algebras on nonseparable Hilbert spaces, showing that the Proper Forcing Axiom implies that all automorphisms of the Calkin algebra on a nonseparable Hilbert space are inner. It would be interesting to known whether those techniques can be generalized to study the semigroup of endomorphisms of the Calkin algebra on a nonseparable Hilbert space.

In conclusion, we remark that an interesting consequence of the simple structure of $\text{End}_u(\mathcal{Q}(H))$ under OCA is that the monoid of $(\text{End}_u(\mathcal{Q}(H)), \circ)$ is commutative (theorem 1.1). To our knowledge, surprisingly, it is not known whether commutativity fails when OCA is not assumed.

Question 5.5. Is it relatively consistent with ZFC that $(\text{End}_u(\mathcal{Q}(H)), \circ)$ is noncommutative?

Notice that, by Woodin’s $\Sigma^2_1$-absoluteness theorem ([Woo02]), this is essentially asking whether commutativity holds under CH.

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(A. Vaccaro) DEPARTMENT OF MATHEMATICS, BEN-GURION UNIVERSITY OF THE NEGEV, P.O.B. 653, BE’ER SHEVA 84105, ISRAEL

E-mail address: vaccaro@post.bgu.ac.il

URL: http://people.dm.unipi.it/vaccaro/index.html