HBT Interference, Wigner functions and MC Simulations

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Abstract

A method of MC simulations including quantum interference, proposed recently by A.Krzywicki and the present author, is explained.

1. Introduction.

The aim of this talk is to explain details of a scheme for MC simulations of multiparticle production including HBT interference which we proposed recently together with Andrzej Krzywicki [1]. The problem became suddenly of great practical importance, when it was realized that the HBT effects may seriously affect some precise measurements of the standard model parameters [2]. As we have heard yesterday from Krzysztof Fialkowski, the existing implementations of quantum interference into standard MC codes suffer from many problems, theoretical as well as practical ones [3, 4]. Therefore construction of a viable MC code correctly including the interference effects is badly needed. To do this, however, it seems first necessary to formulate the problem and the goals to be achieved. It should perhaps be emphasized at this point that the effects of HBT interference are by no means "trivial" or "automatic", as it is sometimes believed. On the contrary, they depend in essential way on the physics of the problem. Consequently, there is no single, unique method of implementing the HBT interference into existing codes. One must therefore be careful to spell out the underlying physical assumptions.
We begin in the next section by a brief reminder of the relation between
the distributions in x-space and in momentum space which will allow to intro-
duce the necessary concepts and to formulate the assumptions. In Section
3 the physical meaning of the procedure is explained in terms of Wigner
functions. Some comments and outlook are given in the last section.

2. Density matrix and relation between x-space and p-space.

Let \( \psi(q_1, q_2, \ldots q_N, \alpha) \equiv \psi(q, \alpha) \) be the probability amplitude for produc-
tion of N particles with momenta \( q_1, q_2, \ldots, q_N \equiv q \). \( \alpha \) denotes a collection of
all other quantum numbers which may be relevant to the process in question
(they may be, e.g., the momenta of other particles which we do not wish to
consider explicitly in a ”semi-inclusive” measurement). The density matrix
in momentum space is then

\[
\rho(q, q') = \int d\alpha \psi(q, \alpha) \psi^*(q', \alpha)
\]

This matrix gives all available information about the system in question. The
observed spectrum of particles reads

\[
\Omega(q) = \int d\alpha |\psi(q, \alpha)|^2 = \rho(q, q)
\]

We see from this formula that measurement of the momentum spectrum pro-
vides only a rather limited information about the system: only the diagonal
elements of the density matrix are determined.

Let us now consider the coordinate space. We write

\[
\psi(q, \alpha) = \int dx <q | x > \psi(x, \alpha)
\]

where \( \psi(x, \alpha) \) is the probability amplitude for producing N particles at the
points \( (x_1, x_2, \ldots, x_N \equiv x) \) and \( <q | x > \) is the known transformation ma-
trix between momentum and coordinate space which shall be specified later.
Introducing the density matrix in coordinate space

\[
\rho(x, x') = \int d\alpha \psi(x, \alpha) \psi^*(x', \alpha)
\]

we obtain the relation

\[
\rho(q, q') = \int dx dx' <q | x > \rho(x, x') <x' | q'
\]
which shows that transformation between description of the system in momentum and in coordinate space requires the knowledge of the full density matrix. The measured distributions (which give only the diagonal elements) are not enough.

To continue, we need an explicit form of the transformation matrix $< q | x >$. As it is different for identical and non-identical particles, we shall treat these two cases separately.

(i) non-identical particles

In this case

$$< q | x > = \exp(iqx) \equiv \exp[i(q_1x_1 + q_2x_2 + ... q_Nx_N)]$$ \hspace{1cm} (6)

(all powers of $2\pi$ are included in normalization of $dx$ and $dq$). Substituting this into (5) we have

$$\rho_0(q, q') = \int dx dx' e^{i(qx-q'x')} \rho(x, x') = \int dx^+ dx^- e^{i(q^-x^++q^+x^-)} \rho(x, x')$$ \hspace{1cm} (7)

where

$$q^+ = \frac{1}{2}(q + q'); q^- = q - q'; x^+ = \frac{1}{2}(x + x'); x^- = x - x'$$ \hspace{1cm} (8)

(from now on we denote the quantities referring to non-identical particles by a subscript 0).

From (7) we obtain for the spectrum of non-identical particles

$$\Omega_0(q) = \rho_0(q, q) = \int dx^+ dx^- e^{-iqx^-} \rho(x^+, x^-).$$ \hspace{1cm} (9)

This formula shows explicitly that the measured momentum spectrum of non-identical particles does not give any information on distribution of particles in coordinate space: the $x^+$ dependence is integrated over. Instead, we obtain information on $x^-$ dependence, i.e., to what degree the density matrix in coordinate space is non-diagonal. In more physical terms, the momentum spectrum gives information only on the coherence properties of the system in the coordinate space. Indeed, the off-diagonal part of the density matrix measures how much the result of the integration over the internal quantum numbers of the system (denoted by $\alpha$ in the Eq.(4)) is affected by the cancellations due to ”incoherent” summation of terms with ”randomly”
distributed phases. For illustration, let us consider a simple parametrization of the density matrix in the form

\[ \rho(x^+, x^-) = \rho(x^+) e^{-\frac{(x^-)^2}{l_c^2(x^+)}}, \]  

(10)

where \( \rho(x^+) \) is the distribution of particles in coordinate space and \( l_c(x^+) \) is the "coherence length". In this case we obtain

\[ \Omega_0(q) = \int dx^+ \rho(x^+) l_c^3(x^+) e^{-\frac{4q^2 l_c^2(x^+)}{2}}, \]  

(11)

and we see explicitly that the momentum spectrum measures the average value of a quantity depending on the coherence length \( l_c(x^+) \).

This point is dramatically emphasized for a limiting case of the system which is fully incoherent in coordinate space, i.e. for which the density matrix is purely diagonal \( (l_c \to 0) \). In this case we obtain the spectrum which is entirely independent of \( q! \) (see e.g. [5] for a more detailed discussion of this result and other effects of incoherence).

One final remark about normalization: one sees from (9) that

\[ \int \Omega_0(q) dq = \int dx^+ \rho(x^+, x^- = 0) = \int dx \rho(x, x). \]  

(12)

(ii) identical particles.

In this case we have to symmetrize the transformation matrix over particle momenta and positions and thus we obtain

\[ \langle q | x \rangle = \frac{1}{(N!)^{1/2}} \sum_P e^{iqPx} \]  

(13)

where \( P \) is a permutation of the numbers \( (1,2,...N) \) and \( q_P \) are the momenta \( (q_1, q_2, ..., q_N) \) ordered according to the permutation \( P \). Introducing (13) into (4) we have

\[ \rho(q, q') = \frac{1}{N!} \sum_{P,P'} \int dx dx' e^{i(q-Px-q'_Px')} \rho(x, x'). \]  

(14)

Using (7) this can be rewritten as

\[ \rho(q, q') = \frac{1}{N!} \sum_{P,P'} \rho_0(q_P, q_P'). \]  

(15)
so that we obtain for the momentum distribution of the identical particles

$$\Omega(q) = \rho(q,q) = \frac{1}{N!} \sum_{P,P'} \int dx^+ dx^- e^{i(q_P - q_{P'}) x^+} \rho(x^+,x^-)$$

(16)

where $q_P = \frac{1}{2}(q_{P'} + q_{P''})$ and $q_P = q_{P'} - q_{P''}$. Eq.(16) shows explicitly that momentum distribution of identical particles gives information on both $x^-$ and $x^+$ dependence of the density matrix. For the example (10) one obtains

$$\Omega(q) = \frac{1}{N!} \sum_{P,P'} \int dx^+ e^{i q_{P'} x^+} \rho(x^+) \lambda^3_c(x^+) e^{-4(q_{P'})^2 \lambda^2_c(x^+)}$$

(17)

which clearly shows that the dependence on momentum differences $q_{P,P'}$ is sensitive to $x^+$ dependence of the particle density in coordinate space $\rho(x^+)$ and of the ”coherence length” $\lambda_c(x^+)$. Three remarks are in order.

(a) The momentum spectrum of N particles given by (14) is expressed in terms of the density matrix of N particles in the coordinate space and thus cannot be reduced (without further assumptions) to the expression involving only single particle density. In particular, it depends on all N-particle correlations in the coordinate space. Usually these correlations are neglected (i.e., the density matrix $\rho(x^+,x^-)$ is written as a product of single particle matrices). Although this is a reasonable procedure in the absence of any additional information, it should be kept in mind that future data may require to include these correlations [1, 7].

(b) The normalization of the spectrum (14) is different from that of non-identical particles given by (12). Integration over particle momenta gives

$$\int \Omega(q) dq = \frac{1}{N!} \sum_{P,P'} \int dx_P \rho(x_P,x_{P'}) = \int \Omega_0(q) dq + \sum_{P' \neq P} \int dx_P \rho(x_P,x_{P'}).$$

(18)

This result shows that the quantum interference changes not only the distribution of produced particles but also the production cross-section (i.e., it acts as final-state interaction).

(c) When all particle momenta are equal to each other we obtain from (2) and (13)

$$\Omega(q) = N! \Omega_0(q) \quad \text{if} \quad q_1 = q_2 = \ldots = q_N$$

(19)
consistent with the standard treatment.

3. Wigner functions.

The density matrix has a clear physical meaning, as seen from Eqs. (1) and (4). Its intuitive meaning is, however, more difficult to grasp. Therefore it is useful to consider a Fourier transform

\[ W(q^+, x^+) = \int dx^- e^{iq^+x^-} \rho(x^+, x^-) \]  \hspace{1cm} (20)

which is the generalization of the well known Wigner function (defined usually for single particle spectrum). It is seen from (20) that \( W(q^+, x^+) \) is a quantum-mechanical generalization of the classical particle density in momentum and in coordinate space (Boltzmann phase-space density).

Using (20) and (9), (14) the particle densities for non-identical and identical particles can be respectively written as

\[ \Omega_0(q) = \int dx W(q, x) \] \hspace{1cm} (21)

\[ \Omega(q) = \frac{1}{N!} \sum_{PP'} \int dx W(\frac{q_P + q_{P'}}{2}, x) e^{i(q_P - q_{P'})x} \] \hspace{1cm} (22)

From these relations one sees again explicitly that while the momentum distribution of non-identical particles does not give any information on the particle distribution in \( x \)-space, the measured momentum spectrum of identical particles is sensitive to \( x \)-dependence of the Wigner function, i.e. to \( x \)-dependence of the distribution.

The advantage of using the Wigner functions is that they appeal to one’s intuition (being the analog of the Boltzmann distribution) and thus the resulting formulas are easier to interpret. Of course it should be kept in mind that this analogy is limited by the fact that a Wigner function is in general locally not positive definite. It can oscillate and, as seen from (21), the oscillations cancel out only after integration over \( x \). However, these oscillations can play a significant role in the Eq. (22) for identical particles by conspiring with oscillating terms in the integrand to contribute significantly to the result. This is how quantum mechanics shows up in the problem. Thus regarding Wigner functions as Boltzmann phase-space density is possible only when they are appropriately smoothed out to remove the oscillations. The price to pay is that, in general, the resulting probabilistic description can only be trusted for when the momentum differences in (22) are not too large.
4. The proposal for MC simulation.

Standard Monte Carlo algorithms generate multiparticle events according to an assumed model for the momentum spectrum $\Omega_0(q)$ which does not include quantum interference. The problem is to correct the weights of these Monte Carlo events once they were generated.

It is clear that this cannot be achieved without additional assumptions. Our proposal is to assume that the corrected spectrum is given by $\Omega(q)$ of the Eq. (22) with the same Wigner function as that present in (21).

I would like to emphasize that this assumption is far from obvious, although it is usually accepted without further comments (see, e.g., [5]). It assumes that the identical and non-identical particles are produced in the same way. Clearly, this can only be an approximation (resonance production, for example, influences differently identical and non-identical particles). As discussed by Bo Andersson at this meeting, it is also violated -generally- in the Lund model [6]. Hopefully it is not unreasonable for events with many particles which we are concerned with.

For an effective use of the Eqs. (21, 22) we need an expression for the Wigner function which reproduces the spectrum $\Omega_0(q)$ for non-identical particles. Therefore we write

$$W(q,x) = \Omega_0(q)w(q,x).$$

(23)

It follows from (21) that $w(q,x)$ obeys the normalization condition

$$\int w(q,x)dx = 1.$$  

(24)

We see that $w(q,x)$ is the quantum analog of the conditional probability: given that particles with momenta $q_1, q_2, ..., q_N$ are present in the final state, $w$ is the probability that they were emitted at the points $x_1, x_2, ..., x_N$.

When (23) is inserted into (22) we obtain for the correcting weights

$$S(q) \equiv \frac{\Omega(q)}{\Omega_0(q)} = \frac{1}{N!} \sum_{P,P'} \frac{\Omega(q_{P} + q_{P'})}{\Omega_0(q_{P})} \hat{w}(q_{P} + q_{P'}, q_{P} - q_{P'})$$

(25)

where

$$\hat{w}(q, \Delta) = \int dx w(q,x)e^{i\Delta x}$$

(26)

To avoid this assumption one needs either a specific model of multiparticle amplitudes (see e.g. [6]) or a direct calculation from the first principles.
with
\[
\hat{w}(q, 0) = 1. \tag{27}
\]

Clearly, \( w(q, x) \) is rather arbitrary and must eventually be determined by analysis of the data. In absence of any information, and to exploit fully the intuitive meaning of Wigner functions, we propose -as a first step- to neglect possible oscillations and to take \( w(q, x) \) in a form which is everywhere positive definite, so that it can indeed interpreted as a probability distribution.

The formula (25) cannot be used at it stands for most of the existing MC algorithms because they use an iterative procedure which provides \( \Omega_0(q) \) only for one given set of momenta and not all the sets needed in (25). This difficulty can be dealt with by observing that one does not make a big error by replacing in (25) \( \Omega_0\left(\frac{q_P + q_{P'}}{2}\right) \) by \( \Omega_0(q_P) \). Indeed, those terms in (25) where this approximation is poor are suppressed by the rapidly decreasing factors \( \hat{w} \) and thus need not be calculated with great precision. Eq.(25) now becomes
\[
S(q) = \frac{1}{N!} \sum_{PP'} \hat{w}\left(\frac{q_P + q_{P'}}{2}, q_P - q_{P'}\right) \tag{28}
\]

The same argument can be used to see that the weights given by (28) are positive, as required for MC simulations. To this end we observe that, as seen from (27), they are certainly positive if the difference between particle momenta are small. Thus the positivity is guaranteed in the region where our approximation for Wigner functions is valid. As we have argued before, outside of this region the non-diagonal \( w(\frac{q_P + q_{P'}}{2}, q_P - q_{P'}) \) are small and -whether positive or not- do not play any role in the sum (28).

To proceed, further working assumptions are needed. In [1] we proposed to start with \( w(q, x) \) in a factorized form, each factor being a superposition of exponentials. Such factorization is most likely not exact (some indications of this were shown by Hans Eggers at this meeting [7]) and may be corrected when the actual data are fitted. For more details we refer the reader to [1].

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