A solvable nonlinear autonomous recursion of arbitrary order

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Abstract
The initial-values problem of the following nonlinear autonomous recursion of order \( p \),
\[
z (s + p) = c \prod_{\ell=0}^{p-1} \left\{ [z (s + \ell)]^{a_\ell} \right\} ,
\]
— with \( p \) an arbitrary positive integer, \( z (s) \) the dependent variable (possibly a complex number), \( s \) the independent variable (a nonnegative integer), \( c \) an arbitrarily assigned, possibly complex, number, and the \( p \) exponents \( a_\ell \) arbitrarily assigned integers (positive, negative or vanishing, so that the right-hand side of the recursion be univalent)—is solvable by algebraic operations, involving the solution of a system of linear algebraic equations (generally explicitly solvable) and of a single polynomial equation of degree \( p \) (hence explicitly solvable for \( p = 1, 2, 3, 4 \)).

1 Introduction
In this paper we point out that the initial-values problem of the following nonlinear recursion of order \( p \),
\[
z (s + p) = c \prod_{\ell=0}^{p-1} \left\{ [z (s + \ell)]^{a_\ell} \right\} ,
\]
(1a)
is solvable by algebraic operations, involving the solution of an (generally explicitly solvable) system of linear algebraic equations, and of a single polynomial equation of degree \( p \) (hence explicitly solvable for \( p = 1, 2, 3, 4 \)). And—trivial as this clarification might be—let us emphasize that "solving " the initial-values problem of the recursion (1a) implies the determination of the function \( F (z (0), z (1), \ldots, z (p - 1); c; a_0, a_1, \ldots, a_{p-1}; s) \) such that, for all values of the independent variable \( s \),
\[
z (s) = F (z (0), z (1), \ldots, z (p - 1); c; a_0, a_1, \ldots, a_{p-1}; s) , \quad s = 0, 1, 2, \ldots,
\]
(1b)
as a consequence of (1a): not solving step-by-step (obviously each being an easy algebraic operation) the recursion (1a) starting from the assigned \( p \) initial values \( z (0), z (1), \ldots, z (p - 1) \)—which is of course not the way to find the solution \( z (s) \) for all values of the independent variable \( s \), this procedure becoming indeed unmanageable for large \( s \).

Notation 1-1. Above and hereafter \( z (s) \) is the dependent variable (generally a complex number), \( s \) is the independent variable (a nonnegative integer), \( c \) is an arbitrarily assigned, possibly complex, number, the \( p \) exponents \( a_\ell \) in the right-hand side of (1a) are arbitrarily assigned integers (positive, negative or vanishing, so that the right-hand side of the recursion be univalent; although the relaxation of this constraint might be worthy of future
investigation), and of course the \( p \) (possibly complex) numbers \( z(0), z(1), \ldots, z(p-1) \) are the arbitrarily assigned initial data. Below the Kronecker symbol \( \delta_{\ell s} \) has the standard meaning: \( \delta_{\ell s} = 1 \) for \( \ell = s \), \( \delta_{\ell s} = 0 \) for \( \ell \neq s \).

This paper can be considered a follow-up to the recent papers [2]-[4], [15] identifying solvable Ordinary Differential Equations (ODEs) and Difference Equations (DEs, or equivalently recursions): its main novelty is to identify a simple class of nonlinear recursions of arbitrary order which are solvable—in the sense specified above and below—without requiring any restriction on their parameters nor on their initial data. The issue of course is whether—or to what extent—this finding is indeed new, due to the difficulty to fully explore the relevant literature (see, for instance, [1], [7]-[14]). We did not find it explicitly reported in the excellent open-access website EqWorld managed by A. D. Polyanin, where it certainly belongs. But see below Remark 3-1.

In the following Section 2 our main results are reported; the corresponding proofs—to the extent they are needed—are provided in Appendix A. And a quite terse Section 3 mentions possible future developments.

## 2 Results

In this Section we report the main results of this paper; in the interest of pedagogical clarity, we firstly consider the simplest case with \( p = 1 \), then with \( p = 2 \), and finally we exhibit the result for arbitrary positive integer \( p \), where \( p \) is of course the order of the recursion.

### 2.1 \( p=1 \)

**Proposition 2-1.** The solution of the initial-value problem for the recursion

\[
z(s+1) = c[z(s)]^a
\]

reads as follows:

\[
z(s) = c^{(1-a')/(1-a)} [z(0)]^{a'};
\]

unless

\[
a = 1,
\]

in which case the solution reads as follows:

\[
z(s) = c^s [z(0)]^{s+1}.
\]

This result is simple enough to make the verification that indeed the formulas (4) provide the solution of the initial-value problem of the recursion (2) a quite simple exercise; we nevertheless provide an explanation of how it has been obtained below, see Appendix A.

**Remark 2.1-1.** It is quite obvious that the result described by Proposition 2-1 holds just as well if (only) the dependent variable \( z(s) \), instead of being a scalar entity, were, for instance, a matrix of arbitrary rank; thereby involving in the recursion (2) a much larger number of (scalar) dependent variables, i.e. all the entries of that matrix.

### 2.2 \( p=2 \)

**Proposition 2-2.** The solution of the initial-values problem for the recursion

\[
z(s+2) = c \ [z(s+1)]^{a_1} \ [z(s)]^{a_0}
\]

is given—for generic values of the 2 (integer) parameters \( a_0 \) and \( a_1 \); for some special values see below—by the following formulas:

\[
z(s) = c^{\gamma(s)} \ [z(1)]^{\alpha_1(s)} \ [z(0)]^{\alpha_0(s)},
\]

with

\[
\alpha_1(s) = [(y_+)^s - (y_-)^s] / d,
\]

\[
\alpha_0(s) = -y_+y_- \ [(y_+)^{s-1} - (y_-)^{s-1}] / d,
\]

\[
\gamma(s) = \log(y_+ / y_-) / d.
\]
\[ \gamma(s) = \left[ (1 - a_1 - a_0) d \right]^{-1} \{ d + (y_+ - 1) (y_+)^s - (y_+ - 1) (y_-)^s \} \]
\[ = (1 - a_1 - a_0)^{-1} \left[ 1 - \alpha(s) - a_0(s) \right], \quad (5d) \]

where
\[ y_{\pm} = a_1 \pm d \div 2, \quad d = \sqrt{(a_1)^2 + 4a_0}. \quad (5e) \]

This finding remains true also in the special case with
\[ a_0 = -(a_1 / 2)^2, \quad (6a) \]

when \( d \) vanishes, \( d = 0 \), by taking the appropriate limit as \( d \to 0 \). Then the solution of the recursion (4) is still given by the expression (5a), but now with
\[ \alpha(s) = s y^s - 1, \quad \gamma(s) = \frac{1 - \alpha(s) - a_0(s)}{1 - a_1 - a_0}, \quad (6b) \]

where
\[ y = a_1 / 2. \quad (6c) \]

While, in the special case
\[ a_1 \neq 2, \quad a_1 + a_0 = 1, \quad (7a) \]

the solution of the initial-values problem of the recursion (4) reads as follows:
\[ z(s) = z(0) c^\gamma(s) \left[ z\left(1\right) / z\left(0\right)\right]^{1-(a_1-1)\gamma/(2-a_1)}, \quad (7b) \]

with
\[ \gamma(0) = \gamma(1) = 0; \quad \gamma(s) = \sum_{\ell=0}^{s-2} (s-1-\ell) \left( a_1 - 1 \right)^\ell, \quad s > 2. \quad (7c) \]

Finally, in the even more special case with
\[ a_1 = 2, \quad a_0 = -1, \quad (8a) \]

the solution of the initial-values problem of the recursion (4) reads as follows:
\[ z(s) = c^{s(s-1)/2} \left[ z\left(1\right)\right]^s \left[ z\left(0\right)\right]^{1-s}. \quad (8b) \]

For an indication of how these findings have been arrived at see Appendix A; but to check their validity it is sufficient to verify that in each case the indicated solution does indeed satisfy the relevant recursion with the arbitrarily assigned initial data \( z_1(0), z_2(0) \).

### 2.3 p is an arbitrary positive integer

The results reported below are for generic values of all involved parameters. And their validity shall be evident—after the explanations provided below—to all readers who have previously digested the results reported in the 2 preceding Subsections.

**Proposition 2-3.** The solution of the initial-values problem of the recursion (1a) is provided by the following formula
\[ z(s) = c^\gamma(s) \prod_{\ell=0}^{p-1} \left\{ [z(\ell)]^{\alpha(s)} \right\}, \quad (9) \]

where of course the \( p \) values \( z(\ell) \) with \( \ell = 0, 1, ..., p-1 \) are the arbitrarily assigned initial data, while the function \( \gamma(s) \), as well as the \( p \) functions \( \alpha(s) \) (with \( \ell = 0, 1, ..., p-1 \)), can be calculated as shown below.

But firstly we note that this formula (9) clearly implies that the exponents \( \gamma(s) \) and \( \alpha(s) \) satisfy the following initial conditions:
\[ \gamma(s) = 0, \quad s = 0, 1, ..., p-1. \quad (10a) \]
\[ \alpha_n (s) = \delta_{ns} , \quad n, s = 0, 1, ..., p - 1 . \]  
(10b)

Next, we define the \( p \) quantities \( y_\ell \), with \( \ell = 0, 1, ..., p - 1 \), as the \( p \) roots of the algebraic equation
\[ y^p - \sum_{\ell=0}^{p-1} (a_\ell y^\ell) = 0 ; \]  
(11)
and we hereafter assume that they are all different—consistently with our genericity premise, introduced to avoid the consideration of a plethora of special subcases. Of course for \( p = 1, 2, 3, 4 \) these roots \( y_\ell \) can be computed explicitly in terms of the \( p \) parameters \( a_\ell \) (and, in some special cases, also for \( p = 6 \) and \( p = 8 \) or \( p = 9 \); see \[5\] and \[6\]).

The exponents \( \gamma (s) \), as well as the \( p \) exponents \( \alpha_\ell (s) \), featured by the right-hand side of the solution formula \( [9] \), are then clearly the solutions of the following linear recursions:
\[ \alpha_n (s + p) = \sum_{\ell=0}^{p-1} [a_\ell \alpha_n (s + \ell)] , \quad n = 0, 1, 2, ..., p - 1 , \]  
(12a)
\[ \gamma (s + p) = 1 + \sum_{\ell=0}^{p-1} [a_\ell \gamma (s + \ell)] ; \]  
(12b)

hence they are given by the following explicit formulas:
\[ \alpha_n (s) = \sum_{\ell=0}^{p-1} [A_{n\ell} \ (y_\ell)^\ell] , \quad n = 0, 1, ..., p - 1 , \]  
(13)
\[ \gamma (s) = \left[ 1 - \sum_{\ell=0}^{p-1} (a_\ell) \right]^{-1} + \sum_{\ell=0}^{p-1} [C_\ell \ (y_\ell)^\ell] , \]  
(14)
where the \( p^2 \) \((s\)-independent\) parameters \( A_{n\ell} \) and the \( p \) \((s\)-independent\) parameters \( C_\ell \) are defined by the following system of linear equations implied by the initial values \( [10] \):
\[ \sum_{\ell=0}^{p-1} [A_{n\ell} \ (y_\ell)^\ell] = \delta_{ns} , \quad n, s = 0, 1, 2, ..., p - 1 , \]  
(15a)
\[ \sum_{\ell=0}^{p-1} [C_\ell \ (y_\ell)^\ell] = - \left[ 1 - \sum_{\ell=0}^{p-1} (a_\ell) \right]^{-1} . \]  
(15b)

These linear algebraic equations are of course solvable, providing explicit—if, for large \( p \), quite complicated—expressions of the \( p^2 + p \) parameters \( A_{n\ell} \) and \( C_\ell \). 

Remark 2-1. A rather trivial extension of the findings reported above is via the change of dependent variables
\[ z (s) = \tilde{z} (s) + f (s) , \]  
(16)
where \( \tilde{z} (s) \) is the new dependent variable and \( f (s) \) an arbitrarily chosen function; or just an \( s\)-independent parameter, in order to maintain the autonomous character of the resulting new recursion satisfied by \( \tilde{z} (s) \). 

3 Outlook

The generalization of the findings reported in this paper to nonautonomous recursions (by other means than that mentioned in Remark 2-1), as well as to systems of nonlinear recursions involving several dependent variables, are interesting tasks; and in that context the possibility to also consider the case of noncommuting dependent variables (such as matrices: see, for instance, \[16\] \[17\]) is another interesting possible development (it is already available in the \( p = 1 \) case; see Remark 2.1-1).
Remark 3-1. This important remark has been added after the present paper was completed. The recursion (1a) which we investigate in this paper is evidently nonlinear, but—as pointed out to us by Paolo Santini—it is easily transformed into a linear recursion via the following simple change of dependent variables:

\[ \ln [z(s)] = x(s) \ , \tag{17a} \]

which indeed transforms the nonlinear recursion (1a) satisfied by the dependent variable \( z(s) \) into the following linear recursion satisfied by the new dependent variable \( x(s) \):

\[ x(s + p) = \ln (c) + \sum_{\ell=0}^{p-1} [a_\ell x(s + \ell)] \ . \tag{17b} \]

It is indeed this kind of change of variables that subtends the findings reported in the present paper, see in particular the ansatz (19). Note however that the solutions of these linear recursions (17b) entail themselves the solution of nonlinear algebraic operations, as indeed displayed by the solutions reported above.

Finally let us emphasize—to complement what was stated above (in the next-to-last paragraph of Section 1)—that the transition from (1a) to (17b) via (17a) implies that the solvability of (1a) can be reduced to the solvability of (17b); which is indeed mentioned in EqWorld, see there Exact Solutions > Functional Equations > 1. Linear Difference and Functional Equations with One Independent Variable > 1.2. Other Linear Difference and Functional Equations > see Items 16 and 15; although a few more developments are still needed to go from the general solution described in EqWorld to the solution of the initial-values problem discussed in our paper. ■

4 Appendix A

In this Appendix we explain how the results—Propositions 2-1 and 2-2—reported in the 2 Subsections 2.1 and 2.2 have been arrived at. The following subdivision in Subsections indeed corresponds to the analogous subdivision in Section 2. As for the main result, as reported in Subsection 2.3, we believe that a more detailed explanation than that provided in the formulation of Proposition 2-3 is unnecessary, given the analogy of that treatment to that provided below in the case with \( p = 2 \).

4.1 \( p=1 \)

In this Subsection we consider the simple recursion

\[ z(s+1) = c [z(s)]^a \tag{18} \]

(in the case with \( a \neq 1 \); the case with \( a = 1 \) being too trivial to require an explicit treatment).

We then introduce the following ansatz

\[ z(s) = c^\gamma(s) [z(0)]^{\alpha(s)} \tag{19a} \]

clearly implying

\[ \gamma(0) = 0 \ , \ \alpha(0) = 1 \ . \tag{19b} \]

We then insert this ansatz (19a) in the recursion (18) and thereby easily get the following linear relations:

\[ \gamma(s+1) = 1 + a\gamma(s) \tag{19c} \]

implying, via the position

\[ \gamma(s) = \tilde{\gamma}(s) + 1/(1-a) \ , \tag{19d} \]

with (via (19b))

\[ \tilde{\gamma}(0) = -1/(1-a) \ , \tag{19e} \]

the recursion

\[ \tilde{\gamma}(s+1) = a \tilde{\gamma}(s) ; \tag{19f} \]
as well as
\[ \alpha(s + 1) = a \alpha(s) . \]  
(19g)

The last 2 recursions clearly imply
\[ \alpha(s) = \alpha(0) a^s , \quad \tilde{\gamma}(s) = \tilde{\gamma}(0) a^s \]  
(19h)
hence, via (19d) and (19e),
\[ \gamma(s) = (1 - a^s) / (1 - a) . \]  
(19i)
The results reported in Proposition 2-1 are thus proven.

4.2 \( p=2 \)

In this Subsection we show how the findings reported in Proposition 2-2 were obtained. Our focus is of course on the recursion (4).

We then introduce the following ansatz:
\[ z(s) = c_{\gamma}^{(s)} [z(1)]^{\alpha_1(s)} [z(0)]^{\alpha_0(s)} \]  
(20a)
clearly implying
\[ \gamma(0) = 0 , \quad \alpha_1(0) = 0 , \quad a_0(0) = 1 , \]
\[ \gamma(1) = 0 , \quad \alpha_1(1) = 1 , \quad a_0(1) = 0 . \]  
(20b)
The introduction of this ansatz in the recursion (4) yields the following simple linear decoupled recursions:
\[ \gamma(s + 2) = 1 + a_1 \gamma(s + 1) + a_0 \gamma(s) , \]  
(21a)
\[ \alpha_n(s + 2) = a_1 \alpha_n(s + 1) + a_0 a_n(s) , \quad n = 1,0 . \]  
(21b)
The solutions of the 2 recursions (21b) clearly read as follows:
\[ \alpha_n(s) = A_{n+}(y_+)^s + A_{n-}(y_-)^s , \quad n = 1,0 , \]  
(22a)
with \( y_{\pm} \) the 2 solutions of the quadratic equation
\[ y^2 - a_1 y - a_0 = 0 \]  
(22b)implying the expression (5e) of \( y_{\pm} \) and \( d \). In these formulas the 4 parameters \( A_{n,\pm} \) (with \( n = 1,0 \)) are \textit{a priori} arbitrary, but they are clearly related to the 4 initial values \( \alpha_n(0) \), \( \alpha_n(1) \) (again, with \( n = 1,0 \)) by the following 4 formulas:
\[ A_{n+} + A_{n-} = \alpha_n(0) , \quad A_{n+} y_+ + A_{n-} y_- = \alpha_n(1) , \quad n = 0,1 , \]  
(23a)
implying, via (20b),
\[ A_{1+} + A_{1-} = 0 , \quad A_{1+} y_+ + A_{1-} y_- = 1 , \]
\[ A_{0+} + A_{0-} = 1 , \quad A_{0+} y_+ + A_{0-} y_- = 0 , \]  
(23b)
hence
\[ A_{1\pm} = \pm 1 / d , \quad A_{0\pm} = \mp y_\mp / d , \]  
(23c)
of course with \( d = \sqrt{(a_1)^2 + 4a_0} \) (see (5c)). Which clearly imply the 2 relations (5b) and (13c).

The next step is to determine the function \( \gamma(s) \). The position
\[ \gamma(s) = \tilde{\gamma}(s) + 1 / (1 - a_0 - a_1) , \]  
(24)
when inserted in the recursion (21a) satisfied by \( \gamma(s) \), implies that \( \tilde{\gamma}(s) \) satisfies the same recursion (21b) satisfied by \( \alpha_n(s) \), hence a repetition of the development detailed just above leads to the solution (7c) for \( \gamma(s) \), thereby completing the derivation of the first part of Proposition 2-2.
As for the remaining part of Proposition 2-2, it is easily seen that the formulas (6b) follow from the formulas (5) by taking the limit $a_0 \to -\left(a_1/2\right)^2$ implying $d \to 0$, and likewise the formulas (7b) with (7c) follow from the formulas (5) by taking the limit $a_0 \to 1 - a_1$ implying $d \to a_1 - 2$, $y_+ \to a_1 - 1$, $y_- \to 1$; while a check of the very special case (8) is quite trivial.

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