BOUNDARY C*-ALGEBRAS FOR ACYCLINDRICAL GROUPS

GUYAN ROBERTSON

(Communicated by Marius Junge)

Abstract. Let $\Delta$ be an infinite, locally finite tree with more than two ends. Let $\Gamma < \text{Aut}(\Delta)$ be an acylindrical uniform lattice. Then the boundary algebra $A_{\Gamma} = C(\partial \Delta) \rtimes \Gamma$ is a simple Cuntz-Krieger algebra whose K-theory is determined explicitly.

1. Introduction

Let $\Delta$ be an infinite, locally finite tree with more than two ends and with boundary $\partial \Delta$. Let $k$ be a positive integer and let $\Gamma$ be a group of automorphisms of $\Delta$ without inversion and with no proper invariant subtree. Say that $\Gamma$ is $k$-acylindrical if the stabilizer of any path of length $k$ in $\Delta$ is trivial [BP, Sel]. The group $\Gamma$ is acylindrical if it is $k$-acylindrical for some integer $k \geq 1$. The main result of this article is

Theorem 1.1. Let $\Gamma < \text{Aut}(\Delta)$ be an acylindrical uniform lattice. Then the boundary algebra $A_{\Gamma} = C(\partial \Delta) \rtimes \Gamma$ is a simple Cuntz-Krieger algebra.

The action of $\Gamma$ on $\partial \Delta$ is amenable, so the maximal crossed product $A_{\Gamma} = C(\partial \Delta) \rtimes \Gamma$ coincides with the reduced crossed product and is nuclear by Proposition 4.8 and Théorème 4.5 in [AD]. The algebra $A_{\Gamma}$ is described in Section 4 below. Cuntz-Krieger algebras were introduced in [CK] and are classified up to isomorphism by their K-theory [K]. A special case, where $\Gamma$ is a free uniform tree lattice, was studied in [R2] by different methods. The K-groups of the boundary algebra $A_{\Gamma}$ are isomorphic to the Bowen-Franks invariants of flow equivalence for a certain subshift of finite type associated with the geodesic flow [C3]. This subshift was studied in [BP, 6.3].

The K-groups of the algebra $A_{\Gamma}$ may be computed explicitly. For example, if $\Gamma = \mathbb{Z}_{l+1} * \mathbb{Z}_{m+1}$ acts on its Bass-Serre tree, where $l, m \geq 1$, then $K_0(A_{\Gamma}) = \mathbb{Z}_{m-1}$ via an isomorphism sending the class of the identity idempotent to $l + m$. It follows that $A_{\Gamma} \cong M_{l+m} \otimes \mathcal{O}_{lm}$, where $\mathcal{O}_n$ denotes the Cuntz algebra, which is generated by $n$ isometries on a Hilbert space whose range projections sum to the identity operator [C1]. It follows that if $\Gamma' = \mathbb{Z}_{l'+1} * \mathbb{Z}_{m'+1}$ and $A_{\Gamma} \cong A_{\Gamma'}$, then $\{l, m\} = \{l', m'\}$.

Remark 1.2. The algebra $A_{\Gamma}$ depends only on $\Gamma$. The group $\Gamma$ and the tree $\Delta$ are quasi-isometric; more precisely, for any base vertex in $\Delta$, the natural mapping from $\Gamma$ onto the orbit of this vertex is a quasi-isometry from $\Gamma$ to $\Delta$. This is a special

Received by the editors June 29, 2007, and, in revised form, October 5, 2007.
2000 Mathematics Subject Classification. Primary 20E08, 46L80.
Key words and phrases. Acylindrical group, boundary, Cuntz-Krieger algebra.
case of the “Fundamental Observation of Geometric Group Theory” \[\text{[Ha, Theorem IV.23]}\]. The mapping from \(\Gamma\) to \(\Delta\) induces a \(\Gamma\)-equivariant homeomorphism of the boundaries of \(\Gamma\) and \(\Delta\).

2. Background

The edges of the tree \(\Delta\) are directed, and each geometric edge of \(\Delta\) corresponds to two directed edges. Let \(\Delta^0\) denote the set of vertices and \(\Delta^1\) the set of directed edges of \(\Delta\). There is a distance function \(d\) defined on the geometric realization of \(\Delta\) which assigns unit length to each edge. Choose an orientation on the set of edges which is invariant under \(\Gamma\). This orientation consists of a partition of \(\Delta^1\) and a bijective involution \(e \mapsto \tau: \Delta^1 \to \Delta^1\) which interchanges the two components of \(\Delta^1\). Each directed edge \(e\) has an initial vertex \(o(e)\) and a terminal vertex \(t(e)\) such that \(o(e) = t(e)\).

Let \(\Gamma\) be a group of automorphisms of \(\Delta\) without inversion and with no proper invariant subtree. Say that \(\Gamma\) is \(k\)-acylindrical, where \(k \geq 1\), if the stabilizer of any path of length \(k\) in \(\Delta\) is trivial \[\text{[BP]}\]. (In \[\text{Sel}\] such a group \(\Gamma\) is said to be \((k-1)\)-acylindrical.) To say that \(\Gamma\) is 1-acylindrical is the same as saying that \(\Gamma\) acts freely on \(\Delta^1\). For example, the action of a free product \(\Gamma = \Gamma_1 * \Gamma_2\) on the associated Bass-Serre tree \[\text{[Ser, I.4.1]}\] is 1-acylindrical. If \(\Gamma = \Gamma_1 *_{\Gamma_0} \Gamma_2\) is a free product with amalgamation over \(\Gamma_0\), then the action of \(\Gamma\) on the associated Bass-Serre tree is 2-acylindrical if \(\Gamma_0\) is malnormal in each \(\Gamma_j\), i.e. \(g^{-1} \Gamma_0 g \cap \Gamma_0 = \{1\}\) for all \(g \in \Gamma_j - \Gamma_0, j = 1, 2\). Every small splitting of a torsion free hyperbolic group gives rise to a 3-acylindrical action \[\text{[Sel]}\].

The boundary \(\partial \Delta\) is the set of equivalence classes of infinite semi-geodesics in \(\Delta\), where two semi-geodesics are said to be equivalent if they agree except on finitely many edges. For the rest of this article we make the following assumptions.

Standing Hypotheses.

1. \(\Delta\) is an infinite locally finite tree with more than two boundary points.
2. \(\Gamma < \text{Aut}(\Delta)\) is a uniform tree lattice.
3. \(\Gamma\) acts without inversion and with no proper invariant subtree.
4. \(\Gamma\) is \(k\)-acylindrical, where \(k \geq 1\).

Remark 2.1. The standing hypotheses imply that \(\partial \Delta\) is uncountable.

Remark 2.2. The assumption that \(\Gamma\) has no proper invariant subtree is part of the definition of “acylindrical” in \[\text{Sel}\] (but not in \[\text{BP}\], which is why it is emphasized separately here). It implies that the action of \(\Gamma\) on \(\partial \Delta\) is minimal. The assumption could have been omitted here if \(\partial \Delta\) were replaced throughout by the limit set \(\Lambda\).

Remark 2.3. The assumption that \(\Gamma\) is a uniform lattice is natural and necessary in the context of the theory of \(\text{[RS]}\). It would be interesting to study \(A_{\Gamma}\) for non-uniform tree lattices. For example, \[\text{[BP]}\] provides an example of a 5-acylindrical action of the Nagao lattice \(\text{PGL}_2(\mathbb{F}_q[t])\).

3. Cuntz-Krieger algebras

It is convenient to use the approach to Cuntz-Krieger algebras developed in \[\text{RS}\]. Choose a nonzero matrix \(M\) with entries in \(\{0, 1\}\). For \(m \geq 0\), let \(W_m\) denote the set of all words of length \(m + 1\) based on the alphabet \(A\) and the transition matrix
M. A word $w \in W_m$ is a formal product $w = a_0a_1 \ldots a_m$, where $a_j \in A$ and $M(a_j+1, a_j) = 1$, $0 \leq j \leq m - 1$. Define $o(w) = a_0$ and $t(w) = a_m$.

Fix a nonempty finite or countable set $D$ (whose elements are “decorations”) and a map $\delta : D \to A$. Let $W_m = \{(d, w) \in D \times W_m; \ o(w) = \delta(d)\}$, the set of “decorated words” of length $m + 1$, and identify $D$ with $W_0$ via the map $d \mapsto (d, \delta(d))$. Let $W = \bigcup_m W_m$ and $\overline{W} = \bigcup_m W_m$, the sets of all words and all decorated words respectively. Define $o : W_m \to D$ and $t : W_m \to A$ by $o(d, w) = d$ and $t(d, w) = t(w)$.

Let $u = a_0a_1 \ldots a_n \in W_m$ and $v = b_0b_1 \ldots b_n \in W_n$. If $t(u) = o(v)$, then there exists a unique product $uv \in W_{m+n}$ defined by

$$uw = a_0a_1 \ldots a_m b_1 \ldots b_n.$$ 

If $w = a_0a_1 \ldots a_l \in W_l$ where $l \geq 0$ and if $p \neq 0$, we say that $w$ is $p$-periodic if $a_{j+p} = a_j$ whenever both sides are defined. Assume that the nonzero $(0, 1)$-matrix $M$ has been chosen so that the following conditions from [RS] hold:

(H2) If $a, b \in A$, then there exists $w \in W$ such that $o(w) = a$ and $t(w) = b$.

(H3) For each nonzero integer $p$, there exists some $w \in W$ which is not $p$-periodic.

**Definition 3.1** ([RS]). The $C^*$-algebra $A_D = A(A, D, M)$ is the universal $C^*$-algebra generated by a family of partial isometries $\{s_{u,v}; u, v \in \overline{W} \text{ and } t(u) = t(v)\}$ satisfying the relations

\begin{align*}
(3.1a) & \quad s_{u,v}^* = s_{v,u}, \\
(3.1b) & \quad s_{u,v}s_{v,w} = s_{u,w}, \\
(3.1c) & \quad s_{u,v} = \sum_{w \in W_m} s_{uw,vw}, \\
(3.1d) & \quad s_{u,v}s_{v,u} = 0, \text{ for } u, v \in \overline{W}_0, u \neq v.
\end{align*}

**Remark 3.2.** If $D$ is finite (as is the case in this article), then $A_D$ is isomorphic to a simple Cuntz-Krieger algebra with identity element $1 = \sum_{u \in W_0} s_{u,u}$ [R1].

**Remark 3.3.** Two decorations $\delta_1 : D_1 \to A$ and $\delta_2 : D_2 \to A$ are said to be equivalent [RS] Section 5] if there is a bijection $\eta : D_1 \to D_2$ such that $\delta_1 = \delta_2 \eta$. Equivalent decorations $\delta_1, \delta_2$ give rise to isomorphic algebras $A_{D_1}, A_{D_2}$.

**Remark 3.4.** Denote by $A_A$ the Cuntz-Krieger algebra with decorating set $A$ and with $\delta$ the identity map. The algebra $A_A$ is isomorphic to the algebra $O_M^A$ generated by a set of partial isometries $\{S_a; a \in A\}$ satisfying the relations $S_a^*S_a = \sum_b M(b, a)S_bS_b^*$ [RS] Remark 3.11]. If $A$ contains $n$ elements and $M(b, a) = 1$, for all $a, b \in A$, then $A_A$ is the Cuntz algebra $O_n$ generated by $n$ isometries whose range projections sum to the identity operator $1$. [C1].

4. THE ALGEBRA ASSOCIATED WITH AN ACYCLINDRICAL GROUP

The Standing Hypotheses (1)–(4) are now in force. A geodesic $\gamma$ in $\Delta$ is a sequence $s_j \in -\infty, \infty$ of vertices such that $d(s_i, s_j) = |i - j|$. A directed segment $\sigma$ of length $n$ is a sequence $(s_0, s_1, \ldots, s_n)$ of vertices such that $d(s_i, s_j) = |i - j|$. Denote
such a directed segment by \([s_0, s_n]\) and let \(\mathcal{S}_n\) be the set of directed segments of length \(n\) in \(\Delta\). Since the group \(\Gamma\) is \(k\)-acylindrical, \(\Gamma\) acts freely on the set \(\mathcal{S}_k\).

The alphabet \(A\) is defined to be \(\Gamma \setminus \mathcal{S}_{k+1}\), the set of \(\Gamma\)-orbits of directed segments of length \(k + 1\) in \(\Delta\). Since \(\Gamma\) is a uniform lattice, \(A\) is finite.

Define a matrix \(M\) with entries in \(\{0, 1\}\) as follows. If \(a, b \in A\), we say that \(M(b, a) = 1\) if and only if \(a = \Gamma \sigma\) and \(b = \Gamma \tau\), where \(\sigma = (s_0, s_1, \ldots, s_{k+1})\), \(\tau = (t_0, t_1, \ldots, t_{k+1})\) are directed segments such that \(s_{j+1} = t_j, 0 \leq j \leq k\). The definition is illustrated in Figure 1.

![Figure 1](image)

**Figure 1.** The condition \(M(b, a) = 1\).

As in Section 3, \(W_m\) denotes the set of all words of length \(m + 1\) based on the alphabet \(A\) and transition matrix \(M\). Let \(\mathcal{W}_m = \Gamma \setminus \mathcal{S}_{m+k+1}\) and let \(\mathcal{W} = \bigcup_m \mathcal{W}_m\). There is a map

\[\alpha : \mathcal{W}_m \to W_m\]

defined by

\[\alpha(\Gamma(s_0, s_1, \ldots, s_{m+k+1})) = (\Gamma[s_0, s_{k+1}]) (\Gamma[s_1, s_{k+2}]) \ldots (\Gamma[s_m, s_{m+k+1}]).\]

**Lemma 4.1.** The map \(\alpha\) is a bijection from \(\mathcal{W}_m\) onto \(W_m\).

**Proof.** Suppose that \(\alpha(\Gamma \sigma) = \alpha(\Gamma \tau)\), where \(\sigma = (s_0, s_1, \ldots, s_{m+k+1})\) and \(\tau = (t_0, t_1, \ldots, t_{m+k+1})\). Then \(\Gamma[s_j, s_{j+k+1}] = \Gamma[t_j, t_{j+k+1}], 0 \leq j \leq m\). For each \(j\), there exists \(g_j \in \Gamma\) such that \(g_j[s_j, s_{j+k+1}] = [t_j, t_{j+k+1}]\), and \(g_j\) is uniquely determined, since \(\Gamma\) acts freely on the set of segments of length \(k\). Now if \(0 \leq j \leq m - 1\), then \([s_j, s_{j+k+1}] \cap [s_{j+1}, s_{j+k+2}] = [s_{j+1}, s_{j+k+1}]\) is a segment of length \(k\). Also \(g_j[s_{j+1}, s_{j+k+1}] = [t_{j+1}, t_{j+k+1}] = g_{j+1}[s_{j+1}, s_{j+k+1}]\). Therefore \(g_j = g_{j+1}\). It follows that \(g_0 \sigma = \tau\) and so \(\Gamma \sigma = \Gamma \tau\). This proves injectivity.

To prove surjectivity, suppose that \(w = a_0 a_1 \ldots a_m \in W_m\). Then by definition \(a_j = \Gamma \sigma_j\) for \(0 \leq j \leq m\), where \(\sigma_j \cap \sigma_{j+1}\) is a segment of length \(k\), for \(1 \leq j \leq m - 1\).

It follows that there is a directed segment \(\sigma = (s_0, s_1, \ldots, s_{m+k+1}) \in \mathcal{S}_{m+k+1}\) such that \(\sigma_j = [s_j, s_{j+k+1}], 0 \leq j \leq m\). Thus \(\alpha(\Gamma \sigma) = w\).

Now fix a vertex \(P \in \Delta^0\). Let \(\overline{\mathcal{W}}_m\) denote the set of directed segments of length \(m+k+1\) which begin at \(P\) and let \(\overline{\mathcal{W}} = \bigcup_{m \geq 0} \overline{\mathcal{W}}_m\). The decorating set is \(D = \overline{\mathcal{W}}_0\), and the decorating map \(\delta : D \to A\) is defined by \(\delta(d) = \Gamma d\). Define \(\overline{\alpha} : \overline{\mathcal{W}} \to \mathcal{W}\) by

\[\overline{\alpha}(\sigma) = (o(\sigma), \alpha(\Gamma \sigma)),\]

where \(o(\sigma) = (s_0, s_1, \ldots, s_{k+1})\) is the initial segment of length \(k + 1\) of \(\sigma\). Also define \(t(\sigma)\) to be the final segment of length \(k + 1\) of \(\sigma\).

**Lemma 4.2.** The map \(\overline{\alpha}\) is a bijection from \(\overline{\mathcal{W}}_m\) onto \(\mathcal{W}_m\), for each \(m \geq 0\).

**Proof.** If \(\overline{\alpha}(\sigma_1) = \overline{\alpha}(\sigma_2)\), then \(o(\sigma_1) = o(\sigma_2)\); moreover \(\Gamma \sigma_1 = \Gamma \sigma_2\) by Lemma 4.1. Since \(\Gamma\) acts freely on \(\mathcal{S}_k\), it follows that \(\sigma_1 = \sigma_2\). Therefore \(\overline{\alpha}\) is injective.

To see that \(\overline{\alpha}\) is surjective, let \(w = (d, w) \in W_m\), where \(w \in W_m\) and \(d \in D\). By Lemma 4.1, there exists \(\sigma \in \mathcal{S}_{m+k+1}\) such that \(\alpha(\Gamma \sigma) = w\). Now

\[\Gamma d = \delta(d) = o(w) = o(\alpha(\Gamma \sigma)) = \Gamma o(\sigma).\]
Replacing $\sigma$ by $g\sigma$ for suitable $g \in \Gamma$ ensures that $o(\sigma) = d$. Then $\sigma \in \overline{\Omega}_j$ and $\overline{\sigma}(\sigma) = \overline{\sigma}$.

Recall that each $\omega \in \partial \Delta$ is represented by a unique semi-geodesic $[s_0, \omega)$ with initial vertex $s_0 \in \Delta^0$. If $\sigma$ is a directed segment with initial vertex $t_0$, let

\[ \Omega(\sigma) = \{ \omega \in \partial \Delta : [t_0, \omega) \text{ contains } \sigma \} . \]

The boundary $\partial \Delta$ has a natural compact totally disconnected topology generated by sets of the form $\Omega(\sigma)$ where $\sigma \in \overline{\Omega}$. The group $\Gamma$ acts on $\partial \Delta$, and one can form the crossed product $C^*$-algebra $C(\partial \Delta) \rtimes \Gamma$. This is the universal $C^*$-algebra generated by the commutative $C^*$-algebra $C(\partial \Delta)$ and the image of a unitary representation $\pi$ of $\Gamma$, satisfying the covariance relation

\[ f(g^{-1}\omega) = \pi(g) \cdot f \cdot \pi(g)^{-1}(\omega) \]

for $f \in C(\partial \Delta)$, $g \in \Gamma$ and $\omega \in \partial \Delta$. It is convenient to denote $\pi(g)$ simply by $g$.

Equation (4.1) implies that for each clopen set $E \subset \partial \Delta$,

\[ \chi_{E} = g \cdot \chi_{E} \cdot g^{-1} . \]

The indicator function $\chi_E$ is continuous and is regarded as an element of the crossed product algebra via the embedding $C(\partial \Delta) \subset C(\partial \Delta) \rtimes \Gamma$. The following is a more precise version of Theorem 1.1.

**Theorem 4.3.** Let $A_\Gamma = C(\partial \Delta) \rtimes \Gamma$. Then $A_\Gamma$ is isomorphic to the Cuntz-Krieger algebra $A_D$ associated with the alphabet $A$, the decorating set $D$ and the transition matrix $M$.

**Proof.** The isomorphism $\phi : A_D \to C(\partial \Delta) \rtimes \Gamma$ is defined as follows. Let $\overline{w}_j = \overline{\sigma}(\sigma_j) \in \overline{W}$, $j = 1,2$, with $t(\overline{w}_1) = t(\overline{w}_2)$. By the definition of $\overline{\sigma}$, there is an element $g \in \Gamma$ such that $gt(\sigma_1) = t(\sigma_2)$. Recall that $t(\sigma_1)$ and $t(\sigma_2)$ are segments of length $k + 1$. Therefore $g$ is unique, since $\Gamma$ acts freely on $S_k$. Define the homomorphism $\phi$ by

\[ \phi(s_{\overline{w}_1\overline{w}_2}) = g \chi_{\Omega(\sigma_1)} = \chi_{\Omega(\sigma_2)} g . \]

This equation defines a *-homomorphism of $A_\Gamma$ because the operators of the form $\phi(s_{\overline{w}_1\overline{w}_2})$ are easily seen to satisfy the relations (11). Since the algebra $A_D$ is simple [RS Theorem 5.9], $\phi$ is injective.

Now $\chi_{\Omega(\sigma)} = \phi(s_{\overline{\omega}\overline{\sigma}})$, where $\sigma \in \overline{\Omega}$ and $\overline{\omega} = \overline{\sigma}(\sigma)$. Since the sets $\Omega(\sigma), \sigma \in \overline{\Omega}$, form a basis for the topology of $\Omega$, the linear span of $\{ \chi_{\Omega(\sigma)} : \sigma \in \overline{\Omega} \}$ is dense in $C(\partial \Delta)$. It follows that the range of $\phi$ contains $C(\partial \Delta)$. To show that $\phi$ is surjective, it therefore suffices to show that the range of $\phi$ contains $\Gamma$. 

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
Let $g \in \Gamma$ and choose an integer $m \geq d(P, g^{-1}P)$. Let $\sigma \in \overline{\mathcal{M}}_m$. Then $\sigma$ is a directed segment of length $m + k + 1$ with initial vertex $P$ and final vertex $Q$, say. Let $\sigma''$ be the directed segment with initial vertex $g^{-1}P$ and final vertex $Q$. Since $m \geq d(P, g^{-1}P)$, it follows that $t(\sigma'') = t(\sigma)$.

Let $\sigma' = g\sigma''$. Then $\sigma'$ is a path beginning at $P$ and $t(\sigma') = gt(\sigma)$. Let $\overline{w_1} = \overline{w}(\sigma)$ and $\overline{w}_2 = \overline{w}(\sigma')$. Then $t(\overline{w}_1) = t(\overline{w}_2)$ and $g\chi_{\Omega(\sigma)} = \phi(s_{\overline{w}_2, \overline{w}_1}) \in \phi(\mathcal{A}_D)$. This holds for each $\sigma \in \overline{\mathcal{M}}_m$. Therefore

$$g = \sum_{\sigma \in \overline{\mathcal{M}}_m} g\chi_{\Omega(\sigma)} \in \phi(\mathcal{A}_D).$$

This shows that the range of $\phi$ contains $\Gamma$, as required.

We now prove that conditions (H2) and (H3) are satisfied. To verify condition (H2), it is enough to show that if $a, b \in A$, then there is a directed segment $\sigma$ such that

$$(4.3) \quad a = \Gamma o(\sigma), \quad b = \Gamma t(\sigma).$$

Let $a = \Gamma \sigma_1$, $b = \Gamma \tau_2$, where $\sigma_1, \tau_2 \in \mathfrak{S}_{k+1}$. By \cite{Ch} Proposition 1 (iii), there is a $\Gamma$-periodic geodesic $\gamma$ containing $\tau_2$. By definition, this means that there is a subgroup of $\Gamma$ which leaves the geodesic $\gamma$ invariant and acts upon it by translation. Choose $\omega \in \partial \Delta$ to be the boundary point of $\gamma$ with

$$\tau_2 \subset \{o(\tau_2), \omega\}.$$

Since the action of $\Gamma$ on $\partial \Delta$ is minimal, there exists $g \in \Gamma$ such that $g\omega \in \Omega(\sigma_1)$. The geodesic $g\gamma$ is $\Gamma$-periodic. Therefore the semi-geodesic $[g o(\tau_2), g\omega]$ contains infinitely many directed segments $\sigma_2$ which are $\Gamma$-translates of $\tau_2$. Choose such a segment $\sigma_2$ far enough away from $g\tau_2$ so that $\sigma_2 \in \Omega(\sigma_1)$. Let $\sigma$ be the directed segment with $o(\sigma) = \sigma_1$ and $t(\sigma) = \sigma_2$. Then (4.3) is satisfied.

To prove that condition (H3) holds, let $p > 0$. Since $\Delta$ has more than two ends, there exist vertices of $\Delta$ which have degree greater than 2. Let $\sigma = (s_0, s_1, \ldots, s_{p+k}) \in \mathfrak{S}_{p+k}$ be a directed segment whose final vertex $s_{p+k}$ has degree greater than 2. Extend $\sigma$ to two different segments:

$$(s_0, s_1, \ldots, s_{p+k}, t), \quad (s_0, s_1, \ldots, s_{p+k}, t') \in \mathfrak{S}_{p+k+1}.$$
Let \( w = \alpha(\Gamma(s_0, \ldots, s_{p+k}, t)) \) and \( w' = \alpha(\Gamma(s_0, \ldots, s_{p+k}, t')) \). Then \( w, w' \in W_p \), \( o(w) = o(w') \) and \( t(w) \neq t(w') \), since \( \Gamma \) acts freely on \( \mathcal{G}_k \) and \( \alpha \) is injective. Therefore at least one of the words \( w, w' \) is not \( p \)-periodic.

5. K-theory and examples

The Standing Hypotheses (1)–(4) remain in force. Thus the algebra \( A_{\Gamma} \) is isomorphic to the Cuntz-Krieger algebra \( A_{D} \) associated with the alphabet \( A \), the set of decorations \( D \) and transition matrix \( M \).

The simple Cuntz-Krieger algebras \( A_{D} \) are purely infinite, nuclear and satisfy the Universal Coefficient Theorem [RS, Remark 6.5]. They are therefore classified by their K-theory [K]. It is convenient to consider the related algebra \( A_{A} \) which is stably isomorphic to \( A_{D} \). Recall from Remark 3.4 that \( A_{A} \) is isomorphic to the algebra \( O_{M} \). The groups \( K_0(A_{A}), K_1(A_{A}) \) are the Bowen-Franks invariants of flow equivalence for a certain subshift associated with \( (\Gamma, \Delta) \). More precisely, according to [C3, Proposition 3.1] the group \( K_0(A_{A}) \) is isomorphic to the abelian group

\[
\mathcal{G}_{\Gamma} = \left\{ A \mid a = \sum_{b \in A} M(a, b) b, a \in A \right\}.
\]

Note that, as the notation suggests, \( \mathcal{G}_{\Gamma} \) depends only on \( \Gamma \) by Remark 1.2. Also \( K_1(A_{A}) \) is the torsion free part of \( \mathcal{G}_{\Gamma} \). Therefore \( A_{A} \) is classified up to stable isomorphism by the group \( \mathcal{G}_{\Gamma} \). Since the algebra \( A_{D} \) is stably isomorphic to \( A_{A} \) [RS, Corollary 5.15], we obtain the following result.

**Theorem 5.1.** Under the Standing Hypotheses (1)–(4), \( K_0(A_{\Gamma}) \cong \mathcal{G}_{\Gamma} \).

To completely classify \( A_{A} \) up to isomorphism, we need to identify the class \([1]\) of the identity idempotent in \( K_0(A_{A}) \) [K]. By Remark 3.2 this class corresponds to the element

\[
\varepsilon = \sum_{a \in A} a \in \mathcal{G}_{\Gamma}.
\]

Here are explicit calculations in the case where \( \Gamma \) is a free product of finite cyclic groups.

5.1. Example. The group \( \Gamma = \mathbb{Z}_{t+1} \ast \mathbb{Z}_{m+1} \) acts on its Bass-Serre tree \([S\text{er}, I.4]\) with an edge \( y = [P, Q] \) as its fundamental domain. The stabilizer \( \Gamma_P \) of \( P \) is isomorphic to \( \mathbb{Z}_{t+1} \) and the stabilizer \( \Gamma_Q \) of \( Q \) is isomorphic to \( \mathbb{Z}_{m+1} \).
By construction, \( \Gamma \) acts freely and transitively on the geometric edges of \( \Delta \). In other words, \( \Gamma \) is 1-acylindrical. The theory applies, with \( k = 1 \), and the alphabet \( A \) is the set of \( \Gamma \)-orbits of directed segments of length 2 in \( \Delta \).

Let \( A_1 = \Gamma_P - \{1\} \) and \( A_2 = \Gamma_Q - \{1\} \), so that \( |A_1| = \ell, |A_2| = m \). Each directed segment of length 2 in \( \Delta \) lies in the \( \Gamma \)-orbit of one of the directed segments \([P, a_2P], [Q, a_1Q]\), for some \( a_1 \in A_1, a_2 \in A_2 \). Let \( \hat{a}_2 = \Gamma[P, a_2P], \hat{a}_1 = \Gamma[Q, a_1Q] \) be the corresponding elements of \( \hat{A}_2, \hat{A}_1 \).

\[
P \\ y \\ Q \\ a_2y \\ a_2P \\ Q \\ y \\ P \\ a_1y \\ a_1Q
\]

The map \( a \mapsto \hat{a} \) is a bijection from \( A_1 \cup A_2 \) onto \( A \). The \( \{0,1\} \)-matrix \( M \) is defined by \( M(\hat{a}, b) = 1 \iff \) either \( a \in A_1, b \in A_2 \) or \( b \in A_1, a \in A_2 \).

\[
P \\ Q = a_2Q \\ a_2P \\ a_2a_1Q
\]

The condition \( M(\hat{a}_1, \hat{a}_2) = 1 \).

By Theorem 5.1

\[
(5.3) \quad K_0(\mathcal{A}_\Gamma) = \left\langle \hat{A}_1 \cup \hat{A}_2 \mid \hat{a} = \sum_{b \in A_{3-j}} \hat{b}, \quad a \in A_j, \quad j = 1, 2 \right\rangle.
\]

The relations on the right side of (5.3) show that all the generators of \( \hat{A}_1 \) are equal and all the generators of \( \hat{A}_2 \) are equal. Therefore

\[
K_0(\mathcal{A}_\Gamma) = \langle \hat{a}_1, \hat{a}_2 \mid \hat{a}_2 = \ell \hat{a}_1, \hat{a}_1 = m \hat{a}_2 \rangle = \langle \hat{a}_1 \mid \hat{a}_1 = l \hat{a}_1 = m \hat{a}_1 \rangle = \mathbb{Z}_{lm-1}.
\]

Recall that the classical Cuntz algebra \( \mathcal{O}_n \) is generated by \( n \) isometries whose range projections sum to the identity operator \( [C_1] \). Now \( K_0(\mathcal{O}_n) = \mathbb{Z}_n \). It follows from the classification theorem \([K]\) that \( \mathcal{A}_\Gamma \) is stably isomorphic to \( \mathcal{O}_{lm} \).

In order to classify \( \mathcal{A}_D \) up to isomorphism, the class of the identity idempotent in \( K_0(\mathcal{A}_\Gamma) \) must be identified. This is done by showing that \( \mathcal{A}_\Gamma \cong \mathcal{A}_A \). Choose the decorating set \( D \) to be the set of directed segments of length 2 which begin at \( P \).

The map \( \delta : d \mapsto \Gamma d \) has domain \( D \) and range \( \hat{A}_2 \). Each of the \( \ell + 1 \) edges with origin \( P \) is the initial edge of precisely one directed segment of length 2 from each \( \Gamma \)-orbit in \( \hat{A}_2 \), and so \( \delta^{-1}(\hat{a}_2) \) contains \( \ell + 1 \) elements, for each \( \hat{a}_2 \in \hat{A}_2 \).

Define a new decorating set

\[
D' = \{(\hat{a}, w) \in (\hat{A}_2 \times W_0) \cup (\hat{A}_1 \times W_1) : o(w) = \hat{a}\}
\]

and a decorating map \( \delta' : D' \to A \) by \( \delta'(\hat{a}, w) = t(w) \). The range of \( \delta' \) is \( \hat{A}_2 \) and for each \( \hat{a}_2 \in \hat{A}_2 \), \( \delta'^{-1}(\hat{a}_2) \) contains \( \ell + 1 \) elements: one element from \( \hat{A}_2 \times W_0 \), and \( \ell \) elements from \( \hat{A}_1 \times W_1 \). It follows that \( \delta' \) is equivalent to \( \delta \), in the sense of Remark 3.3 and so the algebras \( \mathcal{A}_D, \mathcal{A}_{D'} \) are isomorphic.

On the other hand, there is an isomorphism \( \phi : \mathcal{A}_{D'} \to \mathcal{A}_A \) defined by

\[
\phi(\delta(\overline{w_1u_1}, \overline{w_2u_2})) = \delta(\overline{w_1u_1}, \overline{w_2u_2})
\]
for \( \varpi_1, \varpi_2 \in D' \), \( u_1, u_2 \in W \), \( o(u_i) = \delta'(\varpi_i) = t(\varpi_i) \), and \( t(u_1) = t(u_2) \). The proof of this fact is contained in [RS, Lemma 5.15]. Therefore

\[ A_\Gamma \cong A_D \cong A_{D'} \cong A_A. \]

Since \( A \) contains \( l + m \) elements, it follows from (5.2) that the class \([1]\) in \( K_0(A_\Gamma) \) corresponds to the element \( l + m \in \Z_{lm-1} \). Now it is known [C2] that

\[ (K_0(M_k \otimes \O_n), [1]) \cong (\Z_{n-1}, k). \]

This proves that \( A_\Gamma \cong M_{l+m} \otimes \O_{lm} \). Note further that \( A_\Gamma \cong \O_{lm} \) if and only if \( l + m \) is a generator of \( \Z_{lm-1} \); that is, \((l + m, lm - 1) = 1 \).

5.2. Example. More generally, a free product of finite groups \( \Gamma = \Gamma_1 * \Gamma_2 * \cdots * \Gamma_n \) acts on its Bass-Serre tree. The fundamental domain is a tree consisting of \( n \) edges \( y_i \) with terminal vertex \( Q_i \), emanating from a common vertex \( P \). The stabilizer of \( P \) and of each of the edges \( y_j \) is trivial, and the stabilizer of \( Q_j \) is isomorphic to \( \Gamma_j \).

The group \( \Gamma \) is 1-acylindrical and acts freely (but not transitively, in contrast to Example 5.1) on the set of edges of \( \Delta \), with finitely many orbits.

Let \( A_i = \Gamma_i - \{1\} \) and \( \gamma_i = |A_i|, 1 \leq i \leq n \). Each directed segment of length 2 in \( \Delta \) lies in the \( \Gamma \)-orbit of a directed segment of the form \((P, Q_i, aP), a \in A_i \), or \((Q_j, P, Q_k), j \neq k \), as illustrated below.

\[
\begin{array}{c}
P & y_i & Q_i \quad a y_i \quad aP \\
& \downarrow & \quad \downarrow & \quad \downarrow \\
& y_2 & Q_2 \quad y_3 & Q_3 \\
\end{array}
\]

Let \( \hat{A}_i = \{ \hat{a} = \Gamma(P, Q_i, aP) : a \in A_i \} \), and let \( \hat{B} = \{ \hat{b}_{jk} = \Gamma(Q_j, P, Q_k) : j \neq k \} \). Then by Theorem 5.1

\[ K_0(A_\Gamma) = \left( \bigcup_{i} \hat{A}_i \cup \hat{B} \right) \left\{ \hat{a} = \sum_{j \neq i} \hat{b}_{ij}, (\hat{a} \in \hat{A}_i), \quad \hat{b}_{jk} = \sum_{\hat{a} \in \hat{A}_k} \hat{a} \right\}. \]

The relations on the right side of (5.4) show that \( \hat{b}_{jk} \) depends only on \( k \). Therefore, for each \( i \), all the generators in \( \hat{A}_i \) are equal. It follows that

\[ K_0(A_\Gamma) = \langle \hat{a}_i | \hat{a}_i = \sum_{j \neq i} \gamma_j \hat{a}_j \rangle. \]

It is easy to see from (5.5) that \( K_0(A_\Gamma) \) is a torsion group and therefore that \( K_1(A_\Gamma) = 0 \). In other words, the unitary group of \( A_\Gamma \) is connected [C2]. If all the
groups $\Gamma_i$ have the same order, say $\gamma + 1$, and $\delta = \gamma(n - 1) - 1$, then (5.3) simplifies to
\[ K_0(\mathcal{A}_\Gamma) = \mathbb{Z}_{(\gamma+1)\delta} \oplus (\mathbb{Z}_{\gamma+1})^{n-2} \]
with canonical generators $\hat{a}_1, \hat{a}_2 - \hat{a}_1, \hat{a}_3 - \hat{a}_1, \ldots, \hat{a}_{n-1} - \hat{a}_1$.

References

[AD] C. Anantharaman-Delaroche, *Systèmes dynamiques non commutatifs et moyennabilité*, Math. Ann. 279 (1987), 297-315. MR0919508 (89f:46127)

[BP] A. Broise-Alamichel and F. Paulin, *Sur le codage du flot géodésique dans un arbre*, Annales de la Faculté des Sciences de Toulouse Math. (6) 16 (2007), 477–527. MR2379050

[C1] J. Cuntz, *Simple $C^*$-algebras generated by isometries*, Comm. Math. Phys. 57 (1977), 173–185. MR0467330 (57:7189)

[C2] J. Cuntz, *K-theory for certain $C^*$-algebras*, Ann. of Math. 113 (1981), 181–197. MR604046

[C3] J. Cuntz, *A class of $C^*$-algebras and topological Markov chains: Reducible chains and the Ext-functor for $C^*$-algebras*, Invent. Math. 63 (1981), 23–50. MR608527 (82f:46073)

[CK] J. Cuntz and W. Krieger, *A class of $C^*$-algebras and topological Markov chains*, Invent. Math. 56 (1980), 251-268. MR561974 (82f:46073a)

[Ch] S-S. Chen, *Limit sets of automorphism groups of a tree*, Proc. Amer. Math. Soc. 83 (1981), 437–441. MR624949 (83a:05065)

[Ha] P. de la Harpe, *Topics in Geometric Group Theory*, University of Chicago Press, Chicago, 2000. MR1786869 (2001i:20081)

[K] E. Kirchberg, *Exact $C^*$-algebras, tensor products, and the classification of purely infinite algebras*, Proceedings of the International Congress of Mathematicians (Zürich, 1994), Vol. 2, 943–954, Birkhäuser, Basel, 1995. MR1403994 (97g:46071)

[R1] G. Robertson, *Boundary actions for affine buildings and higher rank Cuntz-Krieger algebras. $C^*$-algebras: Proceedings of the SFB Workshop on $C^*$-algebras (Münster, March 8–12, 1999)*, 182–202, Springer-Verlag, 2000. MR1798597 (2001j:46082)

[R2] G. Robertson, *Boundary operator algebras for free uniform tree lattices*, Houston J. Math. 31 (2005), 913–935. MR2188033 (2006m:46088)

[RS] G. Robertson and T. Steger, *Affine buildings, tiling systems and higher rank Cuntz-Krieger algebras*, J. Reine Angew. Math. 518 (1999), 115–144. MR1713322 (2000j:46109)

[Se] Z. Sela, *Acylindrical accessibility for groups*, Invent. Math. 129 (1997), 527–565. MR1465334 (98m:20045)

[Ser] J-P. Serre, *Trees*, Springer-Verlag, Berlin, 1980. MR0607504 (82c:20083)

School of Mathematics and Statistics, University of Newcastle, NE1 7RU, United Kingdom

E-mail address: a.g.robertson@newcastle.ac.uk