Hierarchical Bayesian Control Charts
Based on the Spatial Autoregressive Model

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Abstract:
This paper proposes hierarchical Bayesian control charts based on the spatial autoregressive model for trendy datasets in high-mix low-volume production. The control chart, which is a representative method of statistical process control, is plagued by sample shortages for each product type in high-mix low-volume production. In addition, during manufacturing processing equipment can deteriorate, which, in turn, can result in changing process averages, which, in turn, results in an increase in type I errors. Moreover, in high-mix, low-volume production, spatial dependence exists between different product types due to universal equipment being used. To address these problems, we design control charts that consider the spatial relationships among product types using hierarchical Bayesian modeling based on the spatial autoregressive model. To clarify the properties of the proposed method, we evaluate two production orders: completely random production orders and nonrandom production orders. The results suggest the proposed method is effective, especially in the case of a random production sequence and when there are large differences among product types.

Keywords
High-mix low-volume production, Type I error, Spatial dependence, Tool wear, Trendy datasets

1. Introduction

The control chart, as proposed by Shewhart (1931), is a powerful tool for detecting abnormalities in manufacturing processes. However, control charts assume that large historical datasets can be obtained independently from said processes (Montgomery, 2005), and this premise does not satisfy trendy datasets in high-mix low-volume production (Gu et al., 2014). This is because said control charts do not function as expected with small sample sizes. When manufacturing different product types using the same equipment, various processing conditions, such as temperature, pressure, and target values, will be changed. Therefore, control charts must be designed for each product type, which increases the number of required process parameters. Alternatively, if observations from the process are separated for each product type, the sample size is minimized, which, in turn, negatively affects parameter estimation accuracy. To address this, Hawkins (1987) proposed a self-starting CUSUM control chart, which converts the obtained samples into statistics that follow a normal distribution. Quesenberry (1991) proposed a Q control chart based on the self-starting CUSUM control chart, which converts the obtained samples into Q statistics that follow the standard normal distribution and then manages said statistics. The Q control chart is advantageous insofar that a plurality of product types can be managed with one control chart by converting the observations. A similar approach of converting observations into statistics has been taken by Bothe (1988). However, according to Kawamura et al. (2013), the ability to detect sudden abnormalities in the Q control chart is poor when verifying the actual data obtained from the semiconductor manufacturing process.

In addition, some processes do not satisfy the premise that observations are independent, which causes an increase in type I errors (Snoussi et al., 2007). This can be best illustrated using the cutting process, wherein the process average decreases as manufacturing progresses; here, a traditional control chart would determine the process to be abnormal in spite of the in-control state being present (Laura et al., 1991). Other examples include
chemical reaction processes, such as semiconductor manufacturing, (Kawamura et al., 2013) as well as printing process (Cai et al., 2002); it is necessary to address these appropriately.

In the case of trendy data, it is common to identify appropriate models and manage their residuals (Snoussi et al., 2007, Mandel, 1969). However, Jensen et al. (2006) noted that design of a control chart requires large numbers of trendy observations compared with observations obtained independently. Hence, it is difficult to identify appropriate models from the small sample sizes in high-mix low-volume production, which, in turn, makes it difficult to properly manage processes.

The purpose of this paper is to propose a control chart that functions effectively with respect to trendy datasets in high-mix low-volume production. The problem of small samples is solved by hierarchical Bayesian modeling. In hierarchical Bayesian modeling, samples can be used comprehensively among product types by assuming a prior distribution for the model parameters. Kadoishi et al. (2020) proposed a control chart using a hierarchical Bayesian model that assumed the same normal distribution for multiple parameters and compared it with a control prior distribution for the model parameters. As a result, it was shown that the performance of the control chart improved as the number of product types increased, which is effective for high-mix low-volume production. Alternatively, it is necessary to examine the production order to establish a versatile method. In addition, when the difference in parameters among product types is small, the estimation accuracy drastically improves; in other words, when the difference is large, estimation accuracy is negatively affected. Therefore, to further improve estimation accuracy, spatial dependency is modeled by a spatial autoregressive model. In doing so, we evaluate two production orders: completely random production order and nonrandom production order. Considering trendy datasets for high-mix low-volume production, the tendency of each product type often has spatial dependence due to the manufacturing equipment as well as the environmental conditions, such as temperature and processing method. Finally, Thuthumi et al. (2012) states that assumptions in statistical models are often not satisfied and that spatial effects need to be considered. In light of this information, it is evident that modeling the dependency relationship between parameters of different product types is expected to improve estimation accuracy.

This paper is organized as follows: Section 2 describes the dataset generation for the simulation and design of hierarchical Bayesian control charts based on the spatial autoregressive model; Section 3 outlines the simulation mechanism and comparison method; Section 4 analyzes the performance and properties of the proposed method with several evaluation indexes; Section 5 discusses the results of the analysis for further study and conclusions are given in Section 6.

2. Control Chart Design Based on the Spatial-Autoregressive Model

2.1. Dataset generation

In this paper, we generate datasets \( \{y_i\}_{i=1}^{N} = \{y_1, y_2, y_3, \ldots, y_N\} \) according to the regression model. Note that \( i \) is the order of production. The regression model is a linear and simple model, but it is possible to generate complicated and realistic models by using the variable, \( d_i \), which represents the machine deterioration due to production before making the \( i \)-th product. Datasets are generated by the formula below, where \( \alpha_j \) is the target value and \( \beta_j \) is the degree of decrease due to the deterioration of product type \( j \). Note that \( \beta_j \) is a negative value. The subscript, \( T_i \), denotes the product type of the \( i \)-th product. For example, if the order of production is product type 5, product type 6, and product type 7, then \( T_1 = 5 \), \( T_2 = 6 \), and \( T_3 = 7 \).

\[
\begin{align*}
y_i &= \alpha_{T_i} + \beta_{T_i} + d_i + \epsilon_i \quad (i = 1, 2, \ldots, N), \\
d_i &= d_{i-1} + \beta_{T_{i-1}}, \\
\epsilon_i &\sim N(0, \sigma^2).
\end{align*}
\]

Datasets generated by this model tend to decrease as production progresses.

To generate datasets, the design of parameters and production order must be considered. In parameter design, \( \alpha_j \) and \( \beta_j \) need to be determined; the two scenarios outlined below are considered in this paper.

1. The values of \( \beta_j \) are constant for all product types, while \( \alpha_j \) changes according to product type.
2. The values of \( \alpha_j \) and \( \beta_j \) both change according to product type.

Scenario 1 represents the case in which similar observations are obtained; that is, strong similarity exists among product types. Scenario 2 represents the case in which similar observations are not obtained.

In addition, the two scenarios outlined below are considered for the order of production.

A. All types are produced randomly.
B. Nonrandom production; that is, a certain quantity is produced in the order of product types.

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2.1.1. Parameter design

In Scenario 1, the parameters are designed as follows:

\[ \alpha_j = -\frac{R}{V-1} (j-1), \]  
\[ \beta_j = -0.1, \]  
\[ \sigma_j = 0.1, \]

where \( j = 1, 2, \ldots, V \) represents the product type, \( V \) is the number of product types, and \( R \) is the range of the parameters. For example, when \( R = 0.2 \) and \( V = 3 \), \( \alpha_j = \{0, -0.1, -0.2\} \). In this case, the difference among product types is the target value and product deterioration is the same.

In Scenario 2, the parameters are designed as follows:

\[ \alpha_j = -\frac{R}{V-1} (j-1), \]  
\[ \beta_j = -0.1 - \frac{R}{V-1} (V-j), \]  
\[ \sigma_j = 0.1, \]

where \( V \) is the number of product types and \( R \) is the range of the parameters. For example, when \( R = 0.2 \) and \( V = 3 \), \( \alpha_j = \{0, -0.1, -0.2\} \) and \( \beta_j = \{-0.3, -0.2, -0.1\} \). When the target value, \( \alpha_j \), is large, the deterioration, \( \beta_j \), is small. For example, during the cutting process, deterioration increases with an increase in cutting.

2.1.2. Production-order design

On the one hand, in Scenario A, product types are produced in a random order. In high-mix low-volume production, this order is often used to manufacture according to consumer demand. On the other hand, Scenario B considers nonrandom production order. In this study, it is assumed that five samples are produced according to the order of product types. In examining these scenarios, the effect of production-order changes on the performance of the proposed method is identified.

2.2. Hierarchical Bayesian modeling based on the spatial autoregressive model

In the hierarchical Bayesian model, the same distribution can be assumed for parameters of multiple product types. Therefore, when a sample is obtained, the estimation accuracy of another product type is improved. However, assuming the same distribution for all product types is inefficient because it is treated equally regardless of the strength among product types. Thus, efficient sample use can be explained by the strength of the relationship between product types, which can be expressed by the spatial-autoregressive model.

We propose a method for structuring a model using hierarchical Bayesian modeling as follows:

Level 1:

\[ y_i \sim N(\alpha_{T_i} + \beta_{T_i} + d_i, \sigma_j^2), \]
\[ d_1 = 0, \]
\[ d_i = d_{i-1} + \beta_{T_i-1}. \]

Level 2:

\[ \alpha_j \sim N(\lambda_a \sum_{k=1}^{V} W_{jk}' \alpha_k, \sigma_a^2), \]
\[ \beta_j \sim N(\lambda_b \sum_{k=1}^{V} W_{jk}' \beta_k, \sigma_b^2), \]
\[ D_{jk} = |\alpha_j - \alpha_k| (j, k = 1, 2, \ldots, V), \]
\[ W_{jk} = 1/D_{jk}, \text{ and} \]
\[ W_{jk}' = \frac{w_{jk}}{\sum_{k=1}^{V} W_{jk}}. \]

Level 3:

\[ \sigma_j \sim N(0, 1000^2), \sigma_j \geq 0, \]
\[ \sigma_a \sim N(0, 1000^2), \sigma_a \geq 0, \text{ and} \]
\[ \sigma_b \sim N(0, 1000^2), \sigma_b \geq 0. \]

Level 1 generates the obtained samples from the regression model.

Level 2 generates a target value and a degradation value using the spatial autoregressive model. \( W_{jk} \) is a spatial...
weight matrix, representing the strength of the relationship among product types. The elements of the matrix are defined by the difference in target values. In other words, the smaller the difference between target values, the stronger the relationship among product types. During manufacturing, target values are very likely to be obtained in practice. Note that the diagonal elements are 0 and each row is normalized by Equation (17).

At level 3, an appropriate distribution is assumed independently for each parameter generated at level 1 or 2 using available information about said parameters. However, in this paper, to avoid setting an arbitrary prior probability distribution, a sufficiently wide normal distribution is set as an uninformative prior probability distribution for each parameter. These settings indicate the minimum performance of the proposed method; further performance improvement can be expected by setting the appropriate prior distribution based on the correct prior information of parameters.

2.3. Estimation of control limits

For parameter estimation, Markov chain Monte Carlo (MCMC) sampling is conducted using the No-U-Turn sampler method proposed by Hoffman et al. (2014). In MCMC, the user must specify three hyperparameters; namely, the number of generated random numbers, \(T\), the burn-in period, \(B\), and number of chains, \(C\). In this paper, we set \(T = 1,500\), \(B = 500\), and \(C = 4\). \(\theta^{(t)}(t = 1, 2, \ldots, (T - B) \times C)\) is defined as a random number sequence excluding the burn-in period obtained by MCMC sampling. Then, the estimated values, \(\hat{\theta}_{\text{est}}\), of the parameters, \(\theta^{(t)}\), can be obtained as follows:

\[
\hat{\theta}_{\text{est}} = \frac{1}{(T-B)\times C} \sum_{t=1}^{T-B} \theta^{(t)}, \quad (21)
\]

Using the estimated values, \(\hat{a}_j, \hat{\beta}_j\), and \(\hat{d}_i\), the predicted value, \(\hat{y}_i\), can be obtained as

\[
\hat{y}_i = \hat{a}_j + \hat{\beta}_j + \hat{d}_i, \quad (22)
\]

\[
\hat{d}_1 = 0, \quad (23)
\]

\[
\hat{d}_i = \sum_{k=1}^{n-1} \hat{\beta}_j, \quad (24)
\]

The residuals, \(r_{ij}\), are calculated as

\[
r_i = y_i - \hat{y}_i. \quad (25)
\]

Then, the control limits of hierarchical Bayesian control charts can be obtained as follows:

\[
\text{UCL} = \bar{r} + 3s, \quad (26)
\]

\[
\text{CL} = \bar{r}, \quad (27)
\]

\[
\text{LCL} = \bar{r} - 3s. \quad (28)
\]

Note that \(\bar{r}\) and \(s\) are the mean and standard deviation of \(r_i\) calculated by the following equations:

\[
\bar{r} = \frac{1}{N} \sum_{k=1}^{N} r_k, \quad \text{and} \quad (29)
\]

\[
s = \sqrt{\frac{1}{N-2} \sum_{k=1}^{N} (r_k - \bar{r})^2}. \quad (30)
\]

3. Evaluation of Hierarchical Bayesian Control Charts Based on the Spatial Autoregressive Model

3.1. Comparison method

In order to evaluate the performance of the proposed method, comparisons are made with the control chart outlined by Kadoishi et al. (2020). There are two reasons for this. First, Kadoishi et al. (2020) obtained better results than the maximum likelihood estimation method, which is a general method for parameter estimation. Second, random production involves the production of different product types, so conventional estimation methods require data complementation. Below, a hierarchical Bayesian model assuming the same prior distribution parameters is constructed.

Level 1:

\[
y_i \sim N(\alpha_{T_i} + \beta_{T_i} + d_i, \sigma^2). \quad (31)
\]

\[
d_1 = 0, \quad (32)
\]

\[
d_i = d_{i-1} + \beta_{T_{i-1}}, \quad (33)
\]
Level 2:

\[ a_j \sim N(\mu_a, \sigma_a^2), \]
\[ \beta_j \sim N(\mu_\beta, \sigma_\beta^2), \]

Level 3:

\[ \sigma_j \sim N(0, 1000^2), \sigma_j \geq 0, \]
\[ \mu_a \sim N(0, 1000^2), \]
\[ \mu_\beta \sim N(0, 1000^2), \]
\[ \sigma_a \sim N(0, 1000^2), \sigma_a \geq 0, \text{ and} \]
\[ \sigma_\beta \sim N(0, 1000^2), \sigma_\beta \geq 0. \]

Level 2 sets the same normal distribution for parameters of different product types and Level 3 sets the uninformative prior probability distribution.

### 3.2. Evaluation index

Average run length (ARL) and root mean squared error (RMSE) are used as evaluation indexes. ARL is an average of the run length (RL) obtained by simulation, the latter of which is the number of points existing on the control chart in one trial prior to exceeding the control limits. In the in-control state, a large ARL means a low probability of type I error. When the RL obtained by \( z = \{1, 2, ..., L\} \) trials is defined as \( RL_z \), ARL is

\[ ARL = \frac{\sum RL_z}{L} \]

To consider the accuracy of parameter estimation, the RMSE of the two parameters can be defined as

\[ RMSE_\alpha = \sqrt{\frac{1}{L} \sum_{k=1}^{L} \sum_{j=1}^{V} (a_j - \hat{a}_{jk})^2}, \]
\[ RMSE_\beta = \sqrt{\frac{1}{L} \sum_{k=1}^{L} \sum_{j=1}^{V} (\beta_j - \hat{\beta}_{jk})^2}, \]

where \( \hat{a}_{jk} \) and \( \hat{\beta}_{jk} \) denote the estimates of \( a_j \) and \( \beta_j \) at the \( k \)-th simulation trial. If the accuracy of parameter estimation is good, these values will be small.

### 3.3. Simulation mechanism

It is assumed that \( n = \{5, 10, 20, 50\} \) datasets are obtained for the process with the number of product types, \( V = \{5, 10\} \). In each scenario, the parameter range is \( R = \{0.09, 0.27\} \). This is determined as roughly one to three times the error variance, \( \sigma_j = 0.1 \). Next, the control limits are calculated based on the estimated parameters. Herein, we can assume that new measured values are obtained successively and that the residuals of the measured values and the predicted values of the \( (n + 1) \)-th (and subsequent values) are plotted on the control charts. When plotted points exceed the control limits, one trial is completed and the RL is counted. In other words, the RL starts counting from the \( (n + 1) \)-th measured value. This trial is repeated 1,000 times \( (L = 1,000) \) under the same conditions. Trials with an RL value of 50,000 or more are removed as outliers. Table 1 and Table 2 show the number of removed trials for the control charts based on the spatial autoregressive model and the control charts based on the model with the same assumed normal distribution, respectively.

| Scenario A | Scenario B |
|------------|------------|
| **Scenario 1** | **Scenario 2** | **Scenario 1** | **Scenario 2** |
| \( R = 0.09 \) | \( R = 0.27 \) | \( R = 0.09 \) | \( R = 0.27 \) | \( R = 0.09 \) | \( R = 0.27 \) |
| \( V = 5 \) | \( V = 10 \) | \( V = 5 \) | \( V = 10 \) | \( V = 5 \) | \( V = 10 \) | \( V = 5 \) | \( V = 10 \) |
| 5 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 10 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 20 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 50 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 2 | 2 |

Table 1. Number of removed trials for control charts based on the spatial autoregressive model.
4. Control Chart Performance and Properties

We indicate the performance of the hierarchical Bayesian control chart when the parameters and scenarios are changed.

4.1. Scenario A: Performance and properties

Scenario A represents the case in which products are produced at random, which is the most likely case for high-mix low-volume production. Table 3 shows the ARL values for the hierarchical Bayesian control charts based on the spatial autoregressive model. Similarly, Table 4 shows the ARL values for the control charts assuming the same normal distribution.

Table 3. ARL for control charts based on the spatial autoregressive model (Scenario A).

| Scenario | $R = 0.09$ | $R = 0.27$ | $R = 0.09$ | $R = 0.27$ | $R = 0.09$ | $R = 0.27$ |
|----------|-----------|-----------|-----------|-----------|-----------|-----------|
| $n$      | $V = 5$   | $V = 10$  | $V = 5$   | $V = 10$  | $V = 5$   | $V = 10$  |
| 5        | 30.4      | 66.3      | 25.1      | 46.2      | 17.2      | 27.0      |
| 10       | 77.3      | 182.1     | 66.1      | 120.1     | 31.9      | 43.8      |
| 20       | 212.4     | 400.5     | 180.7     | 346.1     | 78.2      | 97.8      |
| 50       | 966.9     | 1857.4    | 926.0     | 2086.2    | 381.3     | 357.2     |

Table 4. ARL for control charts assuming the same normal distribution (Scenario A).

| Scenario | $R = 0.09$ | $R = 0.27$ | $R = 0.09$ | $R = 0.27$ | $R = 0.09$ | $R = 0.27$ |
|----------|-----------|-----------|-----------|-----------|-----------|-----------|
| $n$      | $V = 5$   | $V = 10$  | $V = 5$   | $V = 10$  | $V = 5$   | $V = 10$  |
| 5        | 28.3      | 73.0      | 29.5      | 75.2      | 23.0      | 22.5      |
| 10       | 68.5      | 203.9     | 77.2      | 197.1     | 32.3      | 35.6      |
| 20       | 244.7     | 529.7     | 210.7     | 610.5     | 75.2      | 75.0      |
| 50       | 965.3     | 2858.8    | 1073.4    | 2655.5    | 343.0     | 293.8     |

From Table 3, it is evident that the ARL increases as the number of the observations increases in the hierarchical Bayesian control charts based on the spatial autoregressive model. Similar results are indicated in Table 4, which are consistent with the results obtained by Kadoishi et al. (2020). Indeed, this is an effective property for high-mix low-volume production. Next, comparing Table 3 and Table 4, in Scenario 1, control charts with the same assumed normal distribution have less type I errors than control charts based on the spatial autoregressive model. Alternatively, in Scenario 2, where the parameters differ depending on product type, control charts based on the spatial autoregressive model are optimal. Table 5 and Table 6 show the RMSE values for the spatial autoregressive model and the model with the same assumed normal distribution. Similarly, Table 7 and Table 8 show the RMSE values.
Scenario 1, when better or equivalent and, when
Tables 5 and Table 6. For the spatial autoregressive model, the estimation accuracy at
assumed normal distribution, respectively.

### 4.2. Scenario B: Performance and properties

Scenario B represents the case where a certain amount is produced in a regular order. In this study, it is assumed
that five instances of each product type are manufactured. Table 9 and Table 10 show the ARL values for the
control charts based on the spatial autoregressive model and the control charts based on the model with the same
assumed normal distribution, respectively.

Table 5. $RMSE_a$ for control charts based on the spatial autoregressive model (Scenario A).

| Scenario 1 | Scenario 2 |
|------------|------------|
| $R = 0.09$ | $R = 0.27$ | $R = 0.09$ | $R = 0.27$ |
| $n$ | $V = 5$ | $V = 10$ | $V = 5$ | $V = 10$ | $V = 5$ | $V = 10$ | $V = 5$ | $V = 10$ |
| 5 | 0.0625 | 0.0557 | 0.1501 | 0.1464 | 0.0612 | 0.0557 | 0.1299 | 0.1352 |
| 10 | 0.0585 | 0.0534 | 0.1365 | 0.1323 | 0.0587 | 0.0532 | 0.1185 | 0.1135 |
| 20 | 0.0572 | 0.0514 | 0.1247 | 0.1164 | 0.0574 | 0.0502 | 0.1094 | 0.1042 |
| 50 | 0.0551 | 0.0487 | 0.1148 | 0.1000 | 0.0563 | 0.0466 | 0.1060 | 0.1025 |

Table 6. $RMSE_a$ for control charts assuming the same normal distribution (Scenario A).

| Scenario 1 | Scenario 2 |
|------------|------------|
| $R = 0.09$ | $R = 0.27$ | $R = 0.09$ | $R = 0.27$ |
| $n$ | $V = 5$ | $V = 10$ | $V = 5$ | $V = 10$ | $V = 5$ | $V = 10$ | $V = 5$ | $V = 10$ |
| 5 | 0.0808 | 0.0611 | 0.1418 | 0.1089 | 0.0773 | 0.0674 | 0.1293 | 0.1206 |
| 10 | 0.0692 | 0.0503 | 0.1225 | 0.1005 | 0.0721 | 0.0632 | 0.1233 | 0.1119 |
| 20 | 0.0615 | 0.0469 | 0.1145 | 0.0963 | 0.0656 | 0.0580 | 0.1090 | 0.1016 |
| 50 | 0.0583 | 0.0446 | 0.1098 | 0.0936 | 0.0613 | 0.0501 | 0.1095 | 0.0942 |

Table 7. $RMSE_{\beta}$ for control charts based on the spatial autoregressive model (Scenario A).

| Scenario 1 | Scenario 2 |
|------------|------------|
| $R = 0.09$ | $R = 0.27$ | $R = 0.09$ | $R = 0.27$ |
| $n$ | $V = 5$ | $V = 10$ | $V = 5$ | $V = 10$ | $V = 5$ | $V = 10$ | $V = 5$ | $V = 10$ |
| 5 | 0.0337 | 0.0174 | 0.0487 | 0.0277 | 0.0430 | 0.0297 | 0.0879 | 0.0692 |
| 10 | 0.0204 | 0.0119 | 0.0299 | 0.0194 | 0.0293 | 0.0220 | 0.0555 | 0.0421 |
| 20 | 0.0125 | 0.0077 | 0.0181 | 0.0116 | 0.0180 | 0.0146 | 0.0302 | 0.0245 |
| 50 | 0.0055 | 0.0035 | 0.0077 | 0.0044 | 0.0082 | 0.0072 | 0.0119 | 0.0104 |

Table 8. $RMSE_{\beta}$ for control charts assuming the same normal distribution (Scenario A).

| Scenario 1 | Scenario 2 |
|------------|------------|
| $R = 0.09$ | $R = 0.27$ | $R = 0.09$ | $R = 0.27$ |
| $n$ | $V = 5$ | $V = 10$ | $V = 5$ | $V = 10$ | $V = 5$ | $V = 10$ | $V = 5$ | $V = 10$ |
| 5 | 0.0336 | 0.0163 | 0.0435 | 0.0198 | 0.0435 | 0.0321 | 0.0892 | 0.0759 |
| 10 | 0.0202 | 0.0106 | 0.0285 | 0.0133 | 0.0311 | 0.0256 | 0.0573 | 0.0508 |
| 20 | 0.0121 | 0.0064 | 0.0162 | 0.0083 | 0.0198 | 0.0172 | 0.0308 | 0.0275 |
| 50 | 0.0055 | 0.0029 | 0.0074 | 0.0037 | 0.0087 | 0.0081 | 0.0119 | 0.0105 |

The spatial autoregressive model and the model assuming the same normal distribution are compared from
Tables 5 and 6. For the spatial autoregressive model, the estimation accuracy at $R = 0.09$ in Scenario 1 is
better or equivalent and, when $R = 0.27$, it is inferior than the model assuming the same normal distribution. In
Scenario 2, when $R = 0.09$, the estimation accuracy is better and, when $R = 0.27$ and $n = 5$, it is better or
equivalent than the model assuming the same normal distribution. Table 7 and Table 8 suggest that the spatial
autoregressive model in Scenario 1 has poor estimation accuracy than the model assuming the same normal
distribution regardless of $R$. Conversely, in Scenario 2, the estimation accuracy is good regardless of $R$. In
particular, when there are many product types, the performance of the proposed method improves.

### 4.2. Scenario B: Performance and properties

Scenario B represents the case where a certain amount is produced in a regular order. In this study, it is assumed
that five instances of each product type are manufactured. Table 9 and Table 10 show the ARL values for the
control charts based on the spatial autoregressive model and the control charts based on the model with the same
assumed normal distribution, respectively.
In comparing Table 9 and Table 10, it is unclear whether the type I errors are minimized or maximized depending on the number of product types and sample size. For example, in Scenario 1, control charts based on the spatial autoregressive model are superior when comparing $R = 0.09$, $V = 10$, and $n = 10$, and they are inferior when comparing $R = 0.09$, $V = 5$, and $n = 10$. To further examine said superiority and inferiority, the $RMSE_{\alpha}$ and $RMSE_{\beta}$ values are shown in Table 11, 12, 13, and 14.

Table 9. ARL for control charts based on the spatial autoregressive model (Scenario B).

| Scenario 1 | Scenario 2 |
|------------|------------|
| $R = 0.09$ | $R = 0.09$ | $R = 0.27$ | $R = 0.27$ |
| $n$ | $V = 5$ | $V = 10$ | $V = 5$ | $V = 10$ | $V = 5$ | $V = 10$ | $V = 5$ | $V = 10$ |
| 5 | 35.4 | 24.1 | 8.0 | 6.2 | 16.3 | 6.8 | 1.0 | 0.01 |
| 10 | 539.1 | 592.2 | 74.1 | 168.2 | 327.4 | 1057.5 | 81.2 | 87.0 |
| 20 | 911.5 | 2231.5 | 289.2 | 630.4 | 968.5 | 1987.9 | 843.6 | 369.9 |
| 50 | 5047.1 | 8940.0 | 2394.3 | 4093.7 | 5257.2 | 10856.7 | 2274.9 | 3802.1 |

Table 10. ARL for control charts assuming the same normal distribution (Scenario B).

| Scenario 1 | Scenario 2 |
|------------|------------|
| $R = 0.09$ | $R = 0.09$ | $R = 0.27$ | $R = 0.27$ |
| $n$ | $V = 5$ | $V = 10$ | $V = 5$ | $V = 10$ | $V = 5$ | $V = 10$ | $V = 5$ | $V = 10$ |
| 5 | 9.6 | 24.9 | 0.3 | 0.03 | 204.1 | 47.5 | 18.6 | 4.6 |
| 10 | 512.7 | 654.8 | 104.1 | 94.3 | 377.9 | 1237.3 | 100.4 | 134.4 |
| 20 | 1353.5 | 1849.1 | 430.0 | 642.0 | 1791.5 | 2887.0 | 333.3 | 760.7 |
| 50 | 5513.9 | 10770.7 | 3126.1 | 4445.0 | 6456.2 | 10887.2 | 3015.7 | 3603.1 |

Table 11. $RMSE_{\alpha}$ for control charts based on the spatial autoregressive model (Scenario B).

| Scenario 1 | Scenario 2 |
|------------|------------|
| $R = 0.09$ | $R = 0.09$ | $R = 0.27$ | $R = 0.27$ |
| $n$ | $V = 5$ | $V = 10$ | $V = 5$ | $V = 10$ | $V = 5$ | $V = 10$ | $V = 5$ | $V = 10$ |
| 5 | 0.0839 | 0.0881 | 0.1752 | 0.1685 | 0.0783 | 0.0712 | 0.1513 | 0.1613 |
| 10 | 0.0735 | 0.0763 | 0.1031 | 0.1092 | 0.1021 | 0.0976 | 0.2052 | 0.2025 |
| 20 | 0.0926 | 0.0893 | 0.1318 | 0.1342 | 0.1159 | 0.1138 | 0.2447 | 0.1781 |
| 50 | 0.1040 | 0.1032 | 0.1611 | 0.1640 | 0.1242 | 0.1325 | 0.2707 | 0.2675 |

Table 12. $RMSE_{\alpha}$ for control charts assuming the same normal distribution (Scenario B).

| Scenario 1 | Scenario 2 |
|------------|------------|
| $R = 0.09$ | $R = 0.09$ | $R = 0.27$ | $R = 0.27$ |
| $n$ | $V = 5$ | $V = 10$ | $V = 5$ | $V = 10$ | $V = 5$ | $V = 10$ | $V = 5$ | $V = 10$ |
| 5 | 0.1127 | 0.1124 | 0.1928 | 0.1881 | 0.1067 | 0.1120 | 0.1592 | 0.1812 |
| 10 | 0.0830 | 0.0820 | 0.0988 | 0.1058 | 0.1056 | 0.1101 | 0.2160 | 0.2189 |
| 20 | 0.0980 | 0.0932 | 0.1313 | 0.1359 | 0.1194 | 0.1198 | 0.2417 | 0.2470 |
| 50 | 0.1073 | 0.1044 | 0.1593 | 0.1597 | 0.1311 | 0.1297 | 0.2741 | 0.2712 |

Table 13. $RMSE_{\beta}$ for control charts based on the spatial autoregressive model (Scenario B).

| Scenario 1 | Scenario 2 |
|------------|------------|
| $R = 0.09$ | $R = 0.09$ | $R = 0.27$ | $R = 0.27$ |
| $n$ | $V = 5$ | $V = 10$ | $V = 5$ | $V = 10$ | $V = 5$ | $V = 10$ | $V = 5$ | $V = 10$ |
| 5 | 0.0221 | 0.0232 | 0.0242 | 0.0240 | 0.0246 | 0.0255 | 0.0339 | 0.0294 |
| 10 | 0.0238 | 0.0254 | 0.0261 | 0.0261 | 0.0253 | 0.0265 | 0.0287 | 0.0277 |
| 20 | 0.0266 | 0.0271 | 0.0323 | 0.0296 | 0.0271 | 0.0273 | 0.0334 | 0.0221 |
| 50 | 0.0284 | 0.0281 | 0.0353 | 0.0316 | 0.0286 | 0.0285 | 0.0371 | 0.0316 |
Table 14. $RMSE_\beta$ for control charts assuming the same normal distribution (Scenario B).

|       | Scenario 1 |       | Scenario 2 |
|-------|------------|--|-----------|
|       | $R = 0.09$ | $R = 0.27$ | $R = 0.09$ | $R = 0.27$ |
| $n$   | $V = 5$  | $V = 10$ | $V = 5$  | $V = 10$ | $V = 5$  | $V = 10$ |
| 5     | 0.0226   | 0.0244   | 0.0253   | 0.0247   | 0.0261   | 0.0270   | 0.0341   | 0.0311   |
| 10    | 0.0244   | 0.0257   | 0.0268   | 0.0268   | 0.0254   | 0.0269   | 0.0298   | 0.0291   |
| 20    | 0.0264   | 0.0269   | 0.0319   | 0.0296   | 0.0275   | 0.0276   | 0.0335   | 0.0306   |
| 50    | 0.0285   | 0.0279   | 0.0359   | 0.0319   | 0.0283   | 0.0283   | 0.0371   | 0.0322   |

From Table 11 and 12, it is evident that the spatial autoregressive model is excellent under almost all conditions with respect to the estimation accuracy of $\alpha$. Similarly, from Table 13 and 14, the same conclusion can be made for $\beta$. In other words, in the control chart based on the same normal distribution, there is a high possibility that abnormally wide control limits are set due to a low estimation accuracy. Accordingly, it is expected that an abnormality cannot be detected in such a control chart.

5. Discussion

To further improve the performance of the proposed method, detailed analysis is conducted. Figure 1 and Figure 2 show the $RMSE_\alpha$ values for each product type in a random production system (Scenario A). These values are calculated by adding the values for $R$ and $n$ together. In the figures, $S$ and $N$ denote the control charts based on a spatial autoregressive model and the control charts with the same assumed normal distribution, respectively.

Consider a case where product types are rearranged according to their target values. From Figure 1 and Figure 2, it is evident that hierarchical Bayesian modeling improves the estimation accuracy of products located at the center of the curve, which can be attributed to the fact that the product type located at the center has more information available when other types are obtained. Alternatively, it is difficult for the product types located at the ends of the curves to effectively use the information of other products. In other words, there is a need to improve the estimation accuracy of products that are not centered. One possibility for said improvement is associated with the scalar parameter, $\lambda$, which determines the strength of autoregression. By using $\lambda$ as a vector, the strength of autoregression can be changed for each product type. In addition, this study used distance decay as the space weight matrix, which can affect the estimation accuracy, and a good effect can be expected by changing it. However, according to Stakhovych et al. (2008), there are no guidelines for determining spatial weight matrix. Therefore, comparison with the case using other spatial weight matrices is required.
6. Conclusion

In this study, we focused on control charts for trendy datasets in high-mix low-volume production. By constructing a hierarchical Bayesian model based on the spatial autoregression model, useful control charts for the given situations were proposed. Indeed, the proposed method has good properties, especially when differences exist in the parameters depending on product type. In addition, the properties of the proposed method have associated guidelines that can be used for further performance improvement. In a future research, it is necessary to verify the detection capability of the proposed method by assuming abnormalities in the process.

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