Production Inventory Model for Deteriorating Items with Different Deterioration Rates under Stock and Price Dependent Demand and Shortages

S.R. Sheikh  
Department of Statistics  
V.B. Shah Institute of Commerce and Management  
Amroli, Surat, India  
E-mail: shehnazrsheikh@yahoo.com

Raman Patel  
Department of Statistics,  
Veer Narmad South Gujarat University  
Surat, India.  
E-mail: patelramanb@yahoo.co.in

Abstract

A deteriorating items production inventory model with stock and price dependent demand is developed. Different deterioration rates are considered in a cycle. Shortages are allowed. Numerical example is provided to illustrate the model and sensitivity analysis is also carried out for parameters.

Keywords: Production, Inventory model, Varying deterioration, Price dependent demand, Stock dependent demand, Time varying holding cost, Shortages

1. INTRODUCTION

Most of the items lose their characteristics overtime and this characteristic is defined as deterioration. Ghare and Achrader [2] was the first to describe optimum ordering policies for deteriorating items. Covert and Philip [1] derived an EOQ model for items with weibull distribution deterioration. Shah [11] further extended the model by
considering shortages. Mandal and Phaujdar [5] presented an inventory model for stock dependent consumption rate. Haiping and Wang [4] studied an economic policy model for deteriorating items with time proportional demand. Other research work related to deteriorating items can be found in, for instance (Raafat [9], Goyal and Giri [3], Ruxian et al. [10]).

Tripathy and Mishra [12] dealt with development of an inventory model when the deterioration rate follows Weibull two parameter distribution, demand rate is a function of selling price and holding cost is time dependent. The model was developed by taking care of with and without shortage both cases. Mathew [6] developed an inventory model for deteriorating items with mixture of Weibull rate of decay and demand as function of both selling price and time. A deteriorating items production inventory model with demand dependent production rate was developed by Patel and Patel [8]. Patel and Parekh [7] developed an inventory model with stock dependent demand under shortages and variable selling price.

Generally the products are such that there is no deterioration initially. After certain time deterioration starts and again after certain time the rate of deterioration increases with time. Here we have used such a concept and developed the deteriorating items inventory models.

In this paper we have developed a production inventory model with different deterioration rates. Demand of the product is price and stock dependent for the cycle time under time varying holding cost. Shortages are allowed. To illustrate the model, numerical example is provided and sensitivity analysis of the optimal solutions for major parameters is also carried out.

2. ASSUMPTIONS AND NOTATIONS:

NOTATIONS:
The following notations are used for the development of the model:

\[ P(t) : \text{Production rate is a function of demand (} P(t) = \eta D(t), \eta > 0) \]

\[ D(t) : \text{Demand rate is a linear function of price and inventory level (} a + bI(t) - \rho p, a > 0, 0 < b < 1, \rho > 0) \]

\[ B : \text{Setup cost per order} \]

\[ \text{SeC : Setup cost} \]

\[ c : \text{Purchasing cost per unit} \]

\[ p : \text{Selling price per unit} \]

\[ c_2 : \text{Shortage cost per unit item per unit time} \]
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\( T \): Length of inventory cycle
\( I(t) \): Inventory level at any instant of time \( t \), \( 0 \leq t \leq T \)
\( Q_1 \): Inventory level at \( t_1 \)
\( Q_2 \): Shortages of quantity
\( Q \): Order quantity
\( \theta \): Deterioration rate during \( \mu_1 \leq t \leq t_1 \), \( 0 < \theta < 1 \)
\( \theta t \): Deterioration rate during \( t_1 \leq t \leq t_0 \), \( 0 < \theta < 1 \)
\( \pi \): Total relevant profit per unit time.

**ASSUMPTIONS:**

The following assumptions are considered for the development of the model.

- The demand of the product is declining as a function of price and inventory level.
- Rate of production is a function of demand
- Replenishment rate is infinite and instantaneous.
- Lead time is zero.
- Shortages are allowed and completely backlogged.
- Deteriorated units neither be repaired nor replaced during the cycle time.

**3. THE MATHEMATICAL MODEL AND ANALYSIS:**

Let \( I(t) \) be the inventory at time \( t \) \( (0 \leq t \leq T) \) as shown in figure.
The differential equations which describes the instantaneous states of $I(t)$ over the period $(0, T)$ is given by

$$\frac{dI(t)}{dt} = (\eta - 1)(a + bI(t) - \rho p), \quad 0 \leq t \leq \mu_1$$  \hspace{1cm} (1)

$$\frac{dI(t)}{dt} + \theta I(t) = (\eta - 1)(a + bI(t) - \rho p), \quad \mu_1 \leq t \leq t_1$$  \hspace{1cm} (2)

$$\frac{dI(t)}{dt} + \theta tI(t) = -(a + bI(t) - \rho p), \quad t_1 \leq t \leq t_2$$  \hspace{1cm} (3)

$$\frac{dI(t)}{dt} = -(a + bI(t) - \rho p), \quad t_2 \leq t \leq T$$  \hspace{1cm} (4)

with initial conditions $I(0) = 0$, $I(\mu_1) = S_1$, $I(t_1) = Q_1$, $I(t_0) = 0$, $I(t_2) = -Q_2$ and $I(T) = 0$.

Solutions of these equations are given by

$$I(t) = (\eta - 1)(a - \rho p)\left(t - \frac{1}{2}(\eta - 1)t^2\right).$$  \hspace{1cm} (6)

$$I(t) = S_1\left[1 + (\theta + b)(\mu_1 - t) - \eta b(\mu_1 - t)\right] - (\eta - 1)(a - \rho p)\left[\left(\mu_1 - t\right) + \frac{1}{2}(\theta + b)(\mu_1^2 - t^2) - \frac{1}{2}\eta b(\mu_1^2 - t^2)\right] - (\theta + b)t(\mu_1 - t) + \eta bt(\mu_1 - t)$$  \hspace{1cm} (7)

$$I(t) = \left(a - \rho p\right)\left[\left(t_0 - t\right) + \frac{1}{2}b\left(t_0^2 - t^2\right) + \frac{1}{6}\theta\left(t_0^3 - t^3\right) - bt(t_0 - t)\right] - \frac{1}{6}b\theta t\left(t_0^3 - t^3\right) - \frac{1}{2}\theta t^2(t_0 - t) - \frac{1}{4}b\theta t^2\left(t_0^3 - t^3\right).$$  \hspace{1cm} (8)

$$I(t) = \left(a - \rho p\right)\left[\left(t_0 - t\right) + \frac{1}{2}b\left(t_0^2 - t^2\right) - bt(t_0 - t)\right].$$  \hspace{1cm} (9)

$$I(t) = (\eta - 1)(a - \rho p)\left[t - T - \frac{1}{2}(\eta - 1)b\left(t^2 - T^2\right) + (\eta - 1)bt(t - T)\right].$$  \hspace{1cm} (10)

(by neglecting higher powers of $\theta$)
Putting $t = \mu_1$ in equation (6), we get

$$S_i = (\eta - 1)(a - \rho p)\left(\mu_1 - \frac{1}{2}(\eta - 1)\mu_1^2\right). \quad (11)$$

Putting $t = t_1$ in equations (7) and (8), we have

$$I(t_1) = S_i \left[1 + (\theta + b)\left(\mu_1 - t_1\right) - \eta b\left(\mu_1 - t_1\right)\right]$$

$$- (\eta - 1)(a - \rho p)\left[(\mu_1 - t_1) + \frac{1}{2}(\theta + b)\left(\mu_1^2 - t_1^2\right) - \frac{1}{2}\eta b\left(\mu_1^2 - t_1^2\right)\right]$$

$$- (\theta + b)t_1\left(\mu_1 - t_1\right) + \eta bt_1\left(\mu_1 - t_1\right) \quad (12)$$

$$I(t_1) = (a - \rho p)\left[(t_0 - t_1) + \frac{1}{2}b\left(t_0^2 - t_1^2\right) + \frac{1}{6}\theta\left(t_0^3 - t_1^3\right) - bt_1\left(t_0 - t_1\right)\right]$$

$$- \frac{1}{6}b\theta t_1\left(t_0^3 - t_1^3\right) - \frac{1}{2}\theta t_1^2\left(t_0 - t_1\right) - \frac{1}{4}b\theta t_1^2\left(t_0^2 - t_1^2\right) \quad (13)$$

So, from equations (8) and (9), we have

$$t_1 = \frac{S_i\left(1 + (\theta + b)\mu_1 - \eta b\mu_1\right) - (a - \rho p)((\eta - 1)\mu_1 + t_0)}{S_i\left(\theta + b - \eta b\right) - (a - \rho p)(1 + bt_0) - (\eta - 1)(a - \rho p)(\mu_1 + (\theta + b)\mu_1 - \eta b\mu_1)} \quad (14)$$

From equation (14), we see that $t_1$ is a function of $\mu_1$, $t_0$ and $S_i$, so $t_1$ is not a decision variable.

Similarly putting $t = t_2$ in equations (9) and (10), we have

$$I(t_2) = (a - \rho p)\left[(t_0 - t_2) + \frac{1}{2}b\left(t_0^2 - t_2^2\right) - bt_1\left(t_0 - t_2\right)\right]. \quad (15)$$

$$I(t_2) = (\eta - 1)(a - \rho p)\left[t_2^2 - T^2\right] - \frac{1}{2}(\eta - 1)b\left(t_2^3 - T^3\right) + (\eta - 1)bt_1\left(t_2 - T\right) \quad (16)$$

So from equations (15) and (16), we have

$$t_2 = \frac{t_0 + T\eta - T}{bt_0 + \eta} \quad (17)$$

From equation (17), we see that $t_2$ is a function of $t_0$, $T$ and $\eta$, so $t_2$ is not a decision variable.
Based on the assumptions and descriptions of the model, the total annual relevant profit ($\pi$), include the following elements:

(i) Set-up cost ($SeC$) = $B$

(ii) $HC = \int_0^{t_i} (x+yt)I(t)dt$

\[
= \int_0^{\mu_i} (x+yt)I(t)dt + \int_{\mu_i}^{t_i} (x+yt)I(t)dt + \int_{t_i}^{\tau} (x+yt)I(t)dt
\]

\[
= \frac{1}{8} y(\eta-1)(a-pp)(\eta-1)\mu_i^4 + \frac{1}{2} x(\eta-1)(a-pp)\mu_i^2 \\
+ \frac{1}{6} x\left(S_{1}(-\theta+b+\eta b)-(\eta-1)(a-pp)(1-(\theta+b)\mu_i+\eta b\mu_i)\right)\left(t_{1}^2-\mu_i^2\right) \\
+ x\left(S_{1}(1+(\theta+b)\mu_i-\eta b\mu_i) - (\eta-1)(a-pp)\left(\mu_i + \frac{1}{2}(\theta+b)\mu_i^2 + \frac{1}{2}\eta b\mu_i^2\right)\right)\left(t_{1}^2-\mu_i^2\right)
\]

\[
- \frac{1}{8} y(\eta-1)(a-pp)(\theta+b-\eta b)\left(t_{1}^4-\mu_i^4\right) \\
+ \frac{1}{3} y\left(S_{1}(-\theta+b+\eta b)-(\eta-1)(a-pp)(1-(\theta+b)\mu_i+\eta b\mu_i)\right)\left(t_{1}^3-\mu_i^3\right) \\
+ \frac{1}{2} y\left(S_{1}(1+(\theta+b)\mu_i-\eta b\mu_i) - (\eta-1)(a-pp)\left(\mu_i + \frac{1}{2}(\theta+b)\mu_i^2 + \frac{1}{2}\eta b\mu_i^2\right)\right)\left(t_{1}^3-\mu_i^3\right)
\]

\[
\frac{1}{12} x(a-pp)b\left(t_0^5-t_i^5\right) + \frac{1}{12} x(a-pp)\theta\left(t_0^4-t_i^4\right) \\
+ \frac{1}{6} x(a-pp)\left(b-\theta t_0-\frac{1}{2}b\theta t_0^2\right)\left(t_0^3-t_i^3\right) \\
+ \frac{1}{2} x(a-pp)\left(-1-b t_0-\frac{1}{6}b\theta t_0^2\right)\left(t_0^2-t_i^2\right) + x(a-pp)\left(t_0+\frac{1}{2}b t_0^2 + \frac{1}{6}t_0^3\right)\left(t_0-t_i\right)
\]
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\[
\left[ \frac{5}{72} y(a-\rho p) b0(t_0^6-t_1^6) + \frac{1}{15} y(a-\rho p) \theta(t_0^5-t_1^5) \right] \\
+ \frac{1}{8} y(a-\rho p) \left( b-\theta t_0 - \frac{1}{2} b0t_0^2 \right) \left( t_0^4-t_1^4 \right) \\
+ \frac{1}{3} y(a-\rho p) \left( -1-bt_0 - \frac{1}{6} b0t_0^3 \right) \left( t_0^3-t_1^3 \right) \\
+ \frac{1}{2} y(a-\rho p) \left( t_0 + \frac{1}{2} bt_0^2 + \frac{1}{6} t_0^3 \right) \left( t_0^2-t_1^2 \right)
\]

(by neglecting higher powers of \( \theta \))

(iii) \[ DC = c \left( \int_{t_1}^{t_2} \theta I(t) dt + \int_{t_1}^{t_2} 0 I(t) dt \right) \]

\[ = c \theta \left[ \begin{array}{c}
- \frac{1}{6} (\eta-1)(a-\rho p)(\theta+b-\eta b) \left( t_1^3 - \mu_1^3 \right) \\
+ \frac{1}{2} (S_1 - (\theta-b+\eta b)(\eta-1)(a-\rho p)(-1-(\theta+b)\mu_1+\eta b\mu_1)) \left( t_1^2 - \mu_1^2 \right) \\
+S_1(1+(\theta+b)\mu_1-\eta b\mu_1)(t_1-\mu_1) \\
- (\eta-1)(a-\rho p) \left( \mu_1 + \frac{1}{2} (\theta+b)\mu_1^2 - \frac{1}{2} \eta b\mu_1^2 \right) \left( t_1 - \mu_1 \right) 
\end{array} \right] \]

(iv) \[ SC = - c_2 \left( \int_{t_0}^{t_1} I(t) dt \right) = - c_2 \left( \int_{t_0}^{t_1} I(t) dt + \int_{t_1}^{t_2} I(t) dt \right) \]
\[
\begin{align*}
\rho_p b -1-bt - bθ -t \\
\eta -1 a-ρp -u + θ+b μ - ηbμ \\
\mu -ρp -μ \\
\frac{1}{2}(a-ρp)\left((t_2^2-t_0^2)\right) \\
+ \frac{1}{6}(a-ρp)b\left(t_2^2-t_0^2\right) \\
+ \frac{1}{6}(a-ρp)(-1-bt_o)(t_2^2-t_0^2) \\
+ (a-ρp)\left(t_o + \frac{1}{2}bt_o\right)(t_2-t_0) \\
+ (a-ρp)(t_2-t_0) + (η-1)(a-ρp)\left(-T+\frac{1}{2}(η-1)bT^2\right)(T-t_2) \\
\end{align*}
\]

(v) SR = \[ p\left(\int_0^T (a+bI(t) - ρp)dt\right) \]

\[
\begin{align*}
\left(\int_0^T (a+bI(t) - ρp)dt + \int_0^{t_i} (a+bI(t) - ρp)dt + \int_{t_i}^T (a+bI(t) - ρp)dt \right) \\
\left[ p\left(\frac{1}{6}(η-1) b(a-ρp)(η-1)μ_i^3 + \frac{1}{2}(η-1)b(a-ρp)μ_i^2 + aμ_i -ρpμ_i\right)\right] \\
+ \left[ -(\eta-1) + \left((a-ρp)\left(1+θ+b-ηb\right)\right)\left(μ_i^3 -μ_i^2\right)\right] \\
+ \left(-\left((a-ρp)\left(1+θ+b-ηb\right)\right)\left(μ_i^3 -μ_i^2\right)\right) \\
+ \left[p\left(\frac{1}{12}(a-ρp)b^30\left(t_0^5-t_i^5\right) + \frac{1}{12}(a-ρp)b0\left(t_0^4-t_i^4\right)\right)\right] \\
+ p\left(\frac{1}{6}(a-ρp)b\left(t_0^3-t_i^3\right)\right) \\
+ p\left(\frac{1}{2}(a-ρp)b\left(t_0^2-t_i^2\right)\right) \\
+ p\left(a(t_0-t_i) + (a-ρp)b\left(t_0 \frac{1}{2}bt_o^2 + \frac{1}{6}θt^3\right)\left(t_0-t_i\right) -ρp\left(t_0-t_i\right)\right) \\
\end{align*}
\]
The total profit during a cycle, \( \pi(t_0, T, p) \) consisted of the following:

\[
\pi(t_0, T, p) = \frac{1}{T} [SR - SeC - HC - DC - SC]
\]

(23)

Substituting values from equations (17) to (22) in equation (23), we get total profit per unit. Putting \( \mu_1 = v_1 t_0 \) in equation (21), we get profit in terms of \( t_0, T \) and \( p \). Differentiating equation (23) with respect to \( t_0, T \) and \( p \) and equate it to zero, we have

\[
\frac{\partial \pi(t_0, T, p)}{\partial t_0} = 0, \quad \frac{\partial \pi(t_0, T, p)}{\partial T} = 0, \quad \frac{\partial \pi(t_0, T, p)}{\partial p} = 0
\]

(24)

provided it satisfies the condition

\[
\begin{vmatrix}
\frac{\partial \pi^2(t_0, T, p)}{\partial t_0^2} & \frac{\partial \pi^2(t_0, T, p)}{\partial t_0 \partial T} & \frac{\partial \pi^2(t_0, T, p)}{\partial t_0 \partial p} \\
\frac{\partial \pi^2(t_0, T, p)}{\partial T \partial t_0} & \frac{\partial \pi^2(t_0, T, p)}{\partial T^2} & \frac{\partial \pi^2(t_0, T, p)}{\partial T \partial p} \\
\frac{\partial \pi^2(t_0, T, p)}{\partial p \partial t_0} & \frac{\partial \pi^2(t_0, T, p)}{\partial p \partial T} & \frac{\partial \pi^2(t_0, T, p)}{\partial p^2}
\end{vmatrix} > 0
\]

(25)

4. NUMERICAL EXAMPLE:

Considering \( A = \text{Rs.100}, a = 500, b = 0.05, \eta = 2, c = \text{Rs. 25}, p = \text{Rs. 40}, 0 = 0.05, x = \text{Rs. 5}, y = 0.05, v_1 = 0.30, \eta = 2, \rho = 5, c_2 = 6, \) in appropriate units. The optimal value of \( t_0^* = 0.5793, T^* = 0.8066, p^* = 50.3604, \) and Profit\(^*\) = Rs. 12259.6249.
The second order conditions given in equation (25) are also satisfied. The graphical representation of the concavity of the profit function is also given.

5. SENSITIVITY ANALYSIS:
On the basis of the data given in example above we have studied the sensitivity analysis by changing the following parameters one at a time and keeping the rest fixed.
**Table 1**  
**Sensitivity Analysis**

| Parameter | %  | t₀   | T    | P    | Profit     |
|-----------|----|------|------|------|------------|
| a         | +20% | 0.5890 | 0.7737 | 60.3734 | 17751.1796 |
|           | +10% | 0.5825 | 0.7874 | 55.3661 | 14879.4570 |
|           | -10% | 0.5812 | 0.8339 | 45.3642 | 9891.6576  |
|           | -20% | 0.5869 | 0.8692 | 40.3690 | 7775.5710  |
| θ         | +20% | 0.5639 | 0.7941 | 50.3605 | 12256.4332 |
|           | +10% | 0.5714 | 0.8002 | 50.3604 | 12258.0099 |
|           | -10% | 0.5876 | 0.8133 | 50.3605 | 12261.2801 |
|           | -20% | 0.5963 | 0.8203 | 50.3607 | 12262.9779 |
| x         | +20% | 0.4944 | 0.7438 | 50.3500 | 12236.3265 |
|           | +10% | 0.5333 | 0.7722 | 50.3543 | 12247.2834 |
|           | -10% | 0.6349 | 0.8487 | 50.3697 | 12273.6501 |
|           | -20% | 0.7026 | 0.9009 | 50.3815 | 12289.7562 |
| B         | +20% | 0.6297 | 0.8794 | 50.3949 | 12235.9064 |
|           | +10% | 0.6052 | 0.8440 | 50.3781 | 12247.5116 |
|           | -10% | 0.5520 | 0.7672 | 50.3425 | 12272.3242 |
|           | -20% | 0.5232 | 0.7256 | 50.3249 | 12285.7101 |
| ρ         | +20% | 0.5376 | 0.7851 | 42.0039 | 10167.9029 |
|           | +10% | 0.5556 | 0.7939 | 45.8011 | 11118.4770 |
|           | -10% | 0.6111 | 0.8251 | 55.9394 | 13655.0701 |
|           | -20% | 0.6551 | 0.8523 | 62.9169 | 15400.5432 |
| c₂        | +20% | 0.5881 | 0.7908 | 50.3743 | 12255.3076 |
|           | +10% | 0.5839 | 0.7982 | 50.3678 | 12257.3545 |
|           | -10% | 0.5741 | 0.8161 | 50.3521 | 12262.1585 |
|           | -20% | 0.5682 | 0.8270 | 50.3426 | 12265.0052 |
From the table we observe that as parameter \( a \) increases/ decreases average total profit also increases/ decreases.

Also, we observe that with increase and decrease in the value of \( 0 \) and \( x \), there is corresponding decrease/ increase in total profit.

From the table we observe that as parameter \( B \) increases/ decreases average total profit decreases/ increases.

6. CONCLUSION

In this paper, we have developed a production inventory model for deteriorating items with price and inventory level dependent demand with different deterioration rates. Sensitivity with respect to parameters have been carried out. The results show that with the increase/ decrease in the parameter values there is corresponding increase/ decrease in the value of profit.

REFERENCES

[1] Covert, R.P. and Philip, G.C. (1973): An EOQ model for items with Weibull distribution deterioration; AIIE Transactions, Vol. 5, pp. 323-326.

[2] Ghare, P.M. and Schrader, G.F. (1963): A model for exponentially decaying inventories; J. Indus. Engg., Vol. 14, pp. 238-243.

[3] Goyal, S.K. and Giri, B. (2001): Recent trends in modeling of deteriorating inventory; Euro. J. Oper. Res., Vol. 134, pp. 1-16.

[4] Haiping, U. and Wang, H. (1990): An economic ordering policy model for deteriorating items with time proportional demand; Euro. J. Oper.Res., Vol. 46, pp. 21-27.

[5] Mandal, B.N. and Phujdar, S. (1989): A note on inventory model with stock dependent consumption rate; Opsearch, Vol. 26, pp. 43-46.

[6] Mathew, R.J. (2013): Perisable inventory model having mixture of Weibull lifetime and demand as function of both selling price and time; International J. of Scientific and Research Publication, Vol. 3(7), pp. 1-8.

[7] Patel, R. and Parekh, R. (2014): Deteriorating items inventory model with stock dependent demand under shortages and variable selling price, International J. Latest Technology in Engg. Mgt. Applied Sci., Vol. 3, No. 9, pp. 6-20.

[8] Patel, R. and Patel, S. (2013): Deteriorating items production inventory model with demand dependent production rate and varying holding cost; Indian J. Mathematics Research, Vol. 1, No.1, pp. 61-70.

[9] Raafat, F. (1991): Survey of literature on continuous deteriorating inventory
model, J. of O.R. Soc., Vol. 42, pp. 27-37.

[10] Ruxian, L., Hongjie, L. and Mawhinney, J.R. (2010): A review on deteriorating inventory study; J. Service Sci. and management; Vol. 3, pp. 117-129.

[11] Shah, Y.K. (1977): An order level lot size inventory for deteriorating items; American Institute of Industrial Engineering Transactions, Vol. 9, pp. 108-112.

[12] Tripathy, C.K. and Mishra, U. (2010): An inventory model for Weibull deteriorating items with price dependent demand and time varying holding cost; Applied Mathematical Sciences, Vol. 4, pp. 2171-2179.
