On Gauge Invariance and Ward Identities for the Wilsonian Renormalisation Group

Daniel F. Litim and Jan M. Pawlowski

 Departament ECM & IFAE, Facultat de Física, Universitat de Barcelona Diagonal 647, E-08028 Barcelona, Spain.
bDublin Institute for Advanced Studies, 10 Burlington Road, Dublin 4, Ireland.

We investigate non-Abelian gauge theories within a Wilsonian Renormalisation Group approach. The cut-off term inherent in this approach leads to a modified Ward identity (mWI). It is shown that this mWI is compatible with the flow and that the full effective action satisfies the usual Ward identity (WI). The universal 1-loop β-function is derived within this approach and the extension to the 2-loop level is briefly outlined.

1. Introduction

The Wilsonian Renormalisation Group [1] has proven itself as a powerful tool for studying both perturbative and non-perturbative effects in quantum field theory. One may expect that a suitable formulation for non-Abelian gauge theories provides new insight to non-perturbative effects in QCD. However the Wilsonian approach is based on the concept of a step-by-step integrating-out of momentum degrees of freedom and one may wonder whether this concept can be adopted for gauge theories.

In the current contribution we investigate this question in a path integral approach based on ideas of Polchinski [2]. In this approach a momentum cut-off is achieved by adding a cut-off term \( \Delta_k S \) to the action which is quadratic in the field. This results in an effective action \( \Gamma_k \) to the action which is quadratic in the field. The change of \( \Gamma_k \) under an infinitesimal variation of the scale \( k \) is described by a flow equation which can be used to successively integrate-out of momentum degrees of freedom and one may wonder whether this concept can be adopted for gauge theories.

The introduction of \( \Delta_k S \) seems to break gauge invariance. However, \( \Gamma_k \) satisfies a ‘modified’ Ward identity (mWI). This mWI commutes with the flow and approaches the usual Ward identity (WI) as \( k \to 0 \). Consequently the full effective action \( \Gamma \) satisfies the usual Ward identity. In other words, gauge invariance of the full theory is preserved if the effective action \( \Gamma_{k_0} \) satisfies the mWI at the initial scale \( k_0 \).

2. The Flow Equation

To be more explicit let us briefly outline the derivation of the flow equation: We add the following scale-dependent term to the action (e.g. [3], [4] and references therein):

\[
\Delta_k S[\Phi] = \frac{1}{2} \int d^d x \ \Phi^* R_\Phi k^2 \Phi,
\]

where \( P_\Phi^{-1} \) is proportional to the bare propagator of \( \Phi \), \((\Phi_i) = (\Phi_1, \ldots, \Phi_n)\) is a short-hand notation for all fields. \( d \) is the dimension of space-time. The regulator \( R_\Phi^k \) has the properties:

\[
R_\Phi^k[x] \xrightarrow{p^2 \to 0} k^{d-2d_\Phi} \frac{x}{|x|}, \quad R_\Phi^k[x] \xrightarrow{p^2 \to \infty} 0,
\]

where \( d_\Phi \) are the dimensions of the fields \( \Phi \). The cut-off term \( \Delta_k S \) effectively suppresses modes with momenta \( p^2 \ll k^2 \) in the generating functional. For modes with large momenta \( p^2 \gg k^2 \) the cut-off term vanishes and in this regime the theory remains unchanged. In the limit \( k \to 0 \) we approach the full generating functional \( \Gamma \) since the
cut-off term is removed. In the limit \( k \to \infty \) all momenta are suppressed and the effective action approaches the (gauge fixed) classical action \( S_{cl} + S_{gf} \). Hence \( \Gamma_k \) interpolates between the classical action and the full effective action:

\[
S_{cl} + S_{gf} \xrightarrow{k \to \infty} \Gamma_k \xrightarrow{k \to 0} \Gamma.
\]

(3)

An infinitesimal variation of the generating functional with respect to \( k \) is described by the flow equation. For the generating functional of 1PI Green functions, the effective action \( \Gamma_k \), the flow equation can be written in the form (e.g. [3], [4] and references therein)

\[
\partial_k \Gamma_k[\Phi] = \frac{1}{2} \text{STr} \left\{ G_k^{\Phi^* \Phi} [\Phi] \partial_k R_k^\Phi [P\Phi] \right\},
\]

(4)

with

\[
G_k^{\Phi^* \Phi}_j [\Phi] = \left( \frac{\delta^2 \Gamma_k[\Phi]}{\delta \Phi^*_j \delta \Phi^*_j} + R_k^\Phi [P\Phi] \right)^{-1},
\]

(5)

where \( t = \ln k \) and the trace STr denotes a sum over momenta, indices and the different fields \( \Phi \) including a minus sign for fermionic degrees of freedom. Note that \( \partial_k R_k^\Phi \) serves as a smeared-out \( \delta \)-function in momentum space peaked at about \( p^2 \approx k^2 \). Thus by varying the scale \( k \) towards smaller \( k \) according to \( \frac{1}{2} \) one successively integrates-out momentum degrees of freedom.

3. QCD in a Wilsonian approach

The starting point of our considerations is the classical action of QCD with \( N_c \) colours and \( N_f \) flavours including cut-off terms for the gauge field and the fermions:

\[
S_k[A, \psi, \bar{\psi}] = S_{cl}[A, \psi, \bar{\psi}] + S_{gf}[A] + \Delta_k S_A[\psi, \bar{\psi}],
\]

(6)

where \( S_{cl} \) is just the classical Euclidean action of QCD with fermions in the fundamental representation. We allow for a general linear gauge fixing term

\[
S_{gf}[A] = \frac{1}{2 \xi} \int d^4 x \ l_\mu A^\mu_A l_\nu A^\nu_A,
\]

(7)

which includes general Lorentz gauges (\( l_\mu = \partial_\mu \)) and general axial gauges (e.g. [3], [4]). The cut-off terms are given by

\[
\Delta_k S_A[\psi, \bar{\psi}] = \frac{1}{2} \int d^4 x \ A_\mu A_\mu \ R_{k,\mu\nu}^{A,\alpha\beta},
\]

\[
\Delta_k S_\psi[\psi, \bar{\psi}] = \int d^4 x \ \bar{\psi} \ R_{k,\mu}^{\psi,AB} \psi^B,
\]

(8)

where \( A, B \) and \( a, b \) refer to the fundamental and to the adjoint representations respectively. The indices \( s, t \) are summed over all flavours. A convenient choice for the regulators \( R_{A,k}^A, R_{A,k}^\psi \) is

\[
R_{A,k}^{A,ab}[p^2] = \delta_{ab} \delta_{\mu\nu} \frac{p^2}{e^{p^2/k^2} - 1},
\]

\[
R_{k,\mu\nu}^{\psi,AB}[p^2] = \delta_{st} \delta^{AB} \frac{p^2}{e^{p^2/k^2} - 1}.
\]

(9)

It is easy to see that the regulators in (9) have the demanded properties (2).

The Fadeev-Popov determinant arising from the gauge fixing (2) may be regularised in a similar way. However, for the sake of brevity we drop these terms in the following.

The cut-off terms (8) generate additional terms in the Ward identity for \( \Gamma_k \). This modified Ward identity (mWI) is

\[
W_k^\psi(x) = D_{\mu} A^\mu - D_{\mu} l_\mu A^\mu + J^\psi, a
\]

\[-g \int d^4 y \ f^{a\mu\nu} \delta_{\mu} A^{\nu} + R_{k,\mu\nu}^{A,cd} G_{k,\nu\mu}^{A,ab} \]

\[+g \int d^4 y \ (t^a)^{BC} R_{k,\mu}^{\psi,CD} G_{k,\mu ts}^{\psi,AB} = 0 \]

(10)

where \( t^a \) are the gauge group generators in the fundamental representation and \( l_\mu^* \) is the adjoint of \( l_\mu \). We also have used the definition of the full (field dependent) propagators (2) and have introduced the following short-hand notation:

\[
J^\psi, a = \bar{\psi} A^\mu (t^a)^{AB} \frac{\delta \Gamma_k}{\delta \psi^B} + \frac{\delta \Gamma_k}{\delta \psi^A} (t^a)^{AB} \psi^B.
\]

(11)

The cut-off dependent terms in (10) vanish for \( k \to 0 \). However, the mWI (10) is of use only if one can show that the flow of \( \Gamma_k \) is compatible with (10). For this purpose we examine
with \( \partial_t \mathcal{W}_k^a \). The \( t \)-derivative of the right-hand side of (10) yields expressions dependent on \( \partial_t R^a_k \), \( \partial_t R^\psi_k \) and \( \partial_t \Gamma_k \). For the last of these we use the flow equation (8) and after some algebra we arrive at \([\Phi = (A, \psi, \bar{\psi}), \Phi^* = (A, -\bar{\psi}, \psi)]\):

\[
\partial_t \mathcal{W}_k^a = - \frac{1}{2} \text{Str} \left( \frac{\delta^a}{\delta \Phi^*_i} \partial_t R^\psi_k \frac{\delta}{\delta \Phi^*_i} \right) \mathcal{W}_k^a. \tag{12}
\]

Let us assume that the initial effective action \( \Gamma_{k_0} \) satisfies the mWI which can be achieved at least order-by-order in perturbation theory. Then \( \partial_t \mathcal{W}_{k|k_0} \) is zero since it is proportional to \( \mathcal{W}_{k_0} \). Thus it follows \( \mathcal{W}_k^a = 0 \) for all \( k \) and the mWI is satisfied at all scales, in particular for \( k = 0 \). It approaches the usual WI for \( k \to 0 \) since the cut-off dependent terms vanish. As a consequence we only have to ensure that the initial effective action satisfies the mWI in order to ensure gauge invariance of the full theory.

4. Applications

As a first application we want to present some analytic results. Analytic computations would simplify tremendously if we still dealt with a theory satisfying the usual WI instead of (10). In this case the number of possible terms in the effective action is restricted by gauge symmetry. Even though this cannot be achieved one can get very close to such a situation. For this purpose it is quite convenient to introduce the following regulators:

\[
(R_k[p^2], R_k^\psi[p^2]) \rightarrow (R_k[D_T(\bar{A})], R_k^\psi[D^2(\bar{A})]), \tag{13}
\]

with

\[
D_{\mu, \nu}^{ab} = -D_{\rho}^{ac} D_{\rho}^{cb} - 2g f_{\rho c} F_{\mu \nu}. \tag{14}
\]

and \( \mathcal{D} \) is the Dirac operator in the fundamental representation. Here \( \bar{A} \) is an arbitrary gauge field configuration. Note that if one allows for regulators with a non-trivial group structure the set of regulators \( R_k^A[D_T], R_k^\psi[D^2] \) coincides with the set of regulators \( R_k[A], R_k^\psi[p] \). Thus one may interpret \( \bar{A} \) as an index labeling a family of different cutoffs. The effective action now depends on \( \bar{A} \): \( \Gamma_k = \Gamma_k[A, \bar{A}] \).

It is simple to see that the cut-off dependent terms in (10) are just given by an infinitesimal gauge transformation of \( \bar{A} \), hence leading to the identity

\[
D_{\mu}^{ab}(\bar{A}) \frac{\delta \Gamma_k}{\delta A_{\mu}^{ab}} = -g \int d^4 y f^{abc} R_{k, \mu \nu} D_T(\bar{A}) [G^{AA}, db] + g \int d^4 y \left( i^a \right)^{BC} R_{k, \mu \nu} \left( [\mathcal{D}^2(\bar{A})] G_{k, \mu \nu}^{\psi, \psi, DB}. \tag{15}
\]

We conclude from (10) and (13) that \( \hat{\Gamma}[A, \psi, \bar{\psi}] := \Gamma_k[A, \bar{A} = A, \psi, \bar{\psi}] \) satisfies the usual WI without the cut-off dependent terms:

\[
D_{\mu}^{ab} \frac{\delta \hat{\Gamma}_k}{\delta A_{\mu}^{ab}} - D_{\mu}^{ab} l_{\mu}^{ab} A_{\nu}^b + J^{\psi, a} + g \int d^4 y f^{abc} l_{\nu}^{ab} \delta^{cd} G_{k, \nu \mu}^{AA, db} = 0, \tag{16}
\]

where the gauge field derivative involved in (16) hits both the gauge field \( A \) and the auxiliary field \( \bar{A} \). Note however that the propagator \( G_{k, \nu \mu}^{AA} \) is still the one derived from \( \frac{\delta^2}{\delta A_{\mu}^{ab} \delta A_{\nu}^{ab}} \Gamma_k[A, \bar{A}] \) at \( \bar{A} = A \). Moreover the flow equation for \( \hat{\Gamma}_k \) requires the knowledge of \( G_{k, \nu \mu}^{AA} \), thus slightly spoiling the advantage of dealing with an effective action which satisfies the usual WI even for \( k \neq 0 \).

A possible way to proceed from (16) would be to reformulate it in terms of BRST transformations of the fields. Then one can expand the action \( \hat{\Gamma}_k \) in BRST-invariant terms.

Moreover if we restrict ourselves to gauges where neither \( \xi \) nor \( l_{\mu} \) depend on derivatives, the last term in (16) vanishes and we do not have any integral terms in the WI (see also (8)). This is a very attractive case where all the following considerations simplify tremendously from a technical point of view.

As a consequence of (16) we have gained gauge invariance even for \( k \neq 0 \) which simplifies the expansion of the effective action. The problem is now to distinguish between the gauge field \( A \) and the field \( \bar{A} = A \) which only serves as an auxiliary variable. This is necessary since the flow equation still requires the knowledge of \( G_{k, \nu \mu}^{AA} \) as mentioned above. The \( \bar{A} \)-dependence of \( \partial_t \Gamma_k \) is given by the
following equation:

\[
\frac{\delta}{\delta A} \partial_t \Gamma_k = \frac{1}{2} \partial_t \text{Str} \left \{ C_k^{\Phi^* \Phi} \frac{\delta}{\delta A} \frac{\delta}{\delta A} P_k^\Phi \right \} \quad \text{(17)}
\]

With \((14), (10), (13)\) and \((17)\) we can investigate the effective action analytically. It is worth noting that the flow equation is a ‘1-loop’ equation, even though the loops depend on the full field dependent propagator. Thus heat kernel methods can be employed. However we want to emphasise that the heat kernel is not used as a regularisation method since everything is finite from the outset.

Let us now briefly sketch the calculation of the (perturbative) 1-loop and 2-loop \(\beta\)-function. Since these coefficients of the \(\beta\)-function are universal their calculation serves as a consistency check of the formalism. Moreover it provides some additional insight in how perturbation theory is recovered in this approach.

On the right-hand side of the flow equation \((1)\) we have to insert the initial effective action \(\Gamma_{k_0}\). At 1-loop level it is sufficient to insert the classical action \(S_{ct} + S_{gf} (1)\) with multiplicative renormalisation, namely \(A \rightarrow Z_{A,F,k}^{1/2} A\), \(g \rightarrow g_k = Z_{g,k} g\). Moreover at 1-loop level one can show with the help of \((10)\) and \((17)\) that the usual relation between \(Z_{g,k}\) and \(Z_{F,k}\) is valid. This relation depends on the chosen gauge, e.g. for the axial gauge it leads to \(\partial_t Z_{g,k}^2 / Z_{F,k}^2 = -\partial_t Z_{F,k} / Z_{F,k} F^2\) (see also \((3)\)). \(\partial_t Z_{F,k} / Z_{F,k}\) is calculated by projecting out the term \(F^2\) on the right-hand side of the flow equation \((3)\):

\[
\beta g^2 = -\frac{1}{16\pi^2} g_k^4 \left( \frac{22}{3} N_c - \frac{4}{3} N_f \right) + O(g_k^6), \quad \text{(18)}
\]

the well-known universal 1-loop result for a non-Abelian gauge theory coupled to fermions in the fundamental representation. Moreover it can be shown that this holds true for general linear gauges.

In order to derive the 2-loop coefficient for the \(\beta\)-function one has to take into account not only the renormalisation constants \(Z_F, Z_g\) but also terms which can be derived from the mWI \((10)\) when examined at 1-loop level. Additionally one has to examine \((17)\) which in general is non-zero at 2-loop level \((8)\).

We would like to emphasise that the calculations outlined above not only provide the \(\beta\)-functions but also furnish one with correction terms to the effective action at 2-loop level to all orders of the fields.

5. Conclusions

We have investigated non-Abelian gauge theories coupled to fermions within a Wilsonian renormalisation group approach. Gauge invariance of the effective action at a given (infrared) scale \(k\) is controlled by a modified Ward identity which is compatible with the flow equation. The mWI guarantees gauge invariance for the full effective action at \(k = 0\). By introducing an auxiliary gauge field \(A\) gauge invariance can be restored even for \(k \neq 0\). The price to pay is an additional equation for the \(A\)-dependence.

As a consistency check the 1-loop \(\beta\)-function can be calculated for general linear gauges. Even more so, the extension of this calculation to 2-loop effects is straightforward (even though tedious) \((1)\).

The calculations presented here also give a flavour of the main advantage of the formalism, namely its flexibility concerning possible approximations, in particular beyond perturbation theory. This makes it an appropriate tool for studying non-perturbative physics.

References
1. K. G. Wilson and I. G. Kogut, Phys. Rep. 12 (1974) 75; F. Wegner and A. Houghton, Phys. Rev. A 8 (1973) 401.
2. J. Polchinski, Nucl. Phys. B 231 (1984) 269.
3. M. Reuter and C. Wetterich, Nucl. Phys. B 417 (1994) 181; M. Bonini, M. D’Attanasio and G. Marchesini, ibid. B 421 (1994) 429; U. Ellwanger, Phys. Lett. B 335 (1994) 364.
4. D. F. Litim and J. M. Pawlowski, Phys. Lett. B 435 (1998) 181, hep-th/9802064.
5. D. F. Litim and J. M. Pawlowski, On General Axial Gauges for QCD, these proceedings, hep-th/9809023.
6. D. F. Litim and J. M. Pawlowski, under completion.