ENVIRONMENTAL GAME MODELING WITH UNCERTAINTIES

YING JI AND SHAOJIAN QU*

Business School, University of Shanghai for Science and Technology
Shanghai 200093, China

FU XING CHEN

School of Science, Shandong University of Technology
Shandong 255049, China

Abstract. We model environmental games with stochastic data based on an imprecise distribution which is assumed to be attached to an a-priori known set. Our model is different from previous games where the probability distribution of the uncertain data is precisely given. Our model is also different from the robust games which presents a robust optimization approach to game models with the uncertain data in a compact convex set without probabilistic information which can lead to overly conservative solutions. A distributionally robust approach is used to cope with our setting in the games by combining the stochastic optimization and robust optimization approaches which can be termed as the distributionally robust environmental games. We show that the existence of an equilibrium for the distributionally robust environmental games under mild assumptions. The computation method for equilibrium, with the first- and second-information about the probability of uncertain data, can be reformulated as a semidefinite programming problem which can be tractably realized. Numerical tests are given to show the efficiency of the proposed methods.

1. Introduction. The climate is changing. It is widely believed that it has caused very serious global risks to human welfare. Extreme weather such as tropical cyclones, heat waves and droughts has already been observed. Many are agreed that climate change may be one of the greatest threats facing the planet. Recent years show increasing temperatures in various regions, and/or increasing extremities in weather patterns. There is now overwhelming scientific consensus that the phenomenon is the result of a greenhouse effect that is caused by increases in atmospheric gases such as water vapor, methane, ozone, and CO₂. Facing such dramatic climate risk, 37 countries met in Kyoto on December 11, 1997 where they ratify the Kyoto Protocol. Under this agreement, each country commits to limit overall Greenhouse Gases (GHG) emission. To meet their respective targets, countries may adopt three market-based mechanisms, namely, Emissions Trading (ET), Clean Development Mechanism (CDM) and Joint Implementation (JI).

Based on the idea behind of these three mechanisms, much research has been done and many environmental friendly technologies have also been developed. This
paper relies on the idea behind the JI mechanism, which is defined in Article 6 of the Kyoto Protocol and admits to countries to earn emission-reduction units (ERUs), that is countries with high abatement costs can meet their targets by investing in countries where the abatement costs are low. The reason for this choice is that, comparing the other two mechanisms, JI is more effective, efficient and politically accepted (Woerdman, 2000). It is worth noting that much research has been done reckoned on the idea behind the JI mechanism [3, 19, 27].

Environmental game based on the JI mechanism has also been extensively done [3, 17, 18]. In their work, the probability distribution of the uncertain parameter is assumed to be sure. However, like climate change itself, environmental game modeling also contains profound uncertainties since its intrinsic randomness and the deviation of our understanding for the complex environment systems. In our model, we assume that the data are contaminated by uncertainty and are described by their imprecise distributions which are assumed to be attached to an a-priori known set. We propose a distributionally robust approach to the environmental game to cope with this data uncertainty.

Our work is related to a few models in the literature. Environmental game is first proposed by [3] to study the joint implementation mechanism of the Kyoto Protocol. Since then, several models have been presented to describe the environmental game [17, 18, 27]. In these models, data are often assumed to be known exactly or random with precise probabilistic distributions. In reality, this is virtually impossible as a large number of ambiguity exists in confirming the distribution. In this paper, we extend the existing models by assuming that the data are stochastic and are described by imprecise distributions which are supposed to be attached to an a-priori known set.

Our model is also related to the uncertain one-shot game in the literature. Harsanyi discusses a one-shot game with incomplete information where a prior probability distribution of the unknown data in the game is assumed to be known by each player [13]. Different from that method, Aghassi and Bertsimas propose a robust one-shot game theory without a probability distribution where the uncertain data in the game is supposed to belong to some known compact and convex set and the players utilize the robust optimization approach to cope with the uncertainty in the game [1]. In that paper, the existence of an equilibrium is proved by considering the boundedness of the uncertainty set and the computation method for an equilibrium point is also provided by assuming the uncertain data attach to a polytope. Our work extends the above two models by assuming that each player has a-priori probabilistic information on the parameter uncertainty.

In the robust approach, the formulation treats the uncertainty to belong to a compact set without the probabilistic information. But in many applications, the probability information is often available. So the robust formulation, as the robust game dose, may lead to overly conservative solutions as such a-priori information can be obtained [6]. Recently, distributionally robust formulation has been widely used to cope with the optimization problems with a-priori probabilistic information of the uncertainty [9, 15, 26, 28]. In this framework, the probabilities of the uncertain parameters are ambiguous and a distributionally robust optimization approach is assumed to deal with the ambiguity. Here the terminology “ambiguity” is in accordance with the meaning in decision science which has been extensively used and is regarded as stochastic with unknown probabilities but, yet is supposed to belong to an a-priori set.
In this paper, we adapt the distributionally robust optimization approach to environmental games under parameter uncertainty including distributional ambiguity. Such games are labelled as “distributionally robust environmental games”, in which each operator has no idea about the true probabilities of data in the game and each operator makes use of a distributionally robust approach to cope with this ambiguity. Such distributionally robust formulation can efficiently incorporate a-priori probabilistic information of the parameter uncertainty. The primary contributions of this paper are as follows. We present a worst-case game model to environmental noncooperative games under parameter uncertainty including distributional ambiguity by utilizing a distributionally robust optimization approach which can be seen as an extension to the robust game model. We prove the existence of an equilibrium under some standard assumptions. We then propose a computation method for an equilibrium with the first- and second information about the probability of the uncertain data. We conduct numerical tests to the distributionally robust environmental game.

The rest of this paper is organized as follows. Section 2 discusses the noncooperative, simultaneous-move, one-shot, \( m \)-person environmental games. We begin with giving a definition of “distributionally robust environmental games” for such games under a parameter’s distributional ambiguity. Some standard assumptions and preliminaries are given to prove the existence of equilibrium for such distributionally robust environmental games. Section 3 shows that the computation for obtaining an equilibrium under distributional ambiguity can be equivalently reformulated as solving a semidefinite programme which can be tractably realized. Numerical results are presented in Section 4. Section 5 concludes with some suggestions for future extension.

**Notations.** Throughout this paper, given a vector \( x \), we use \( x' \) and \( x^i \) to represent its transpose and \( i \)th component respectively. Also, for \( x \), we define \((x^{-i}, u^i)\) as the vector with all components the same as that in \( x \) except the \( i \)th component being \( u^i \), that is \((x^{-i}, u^i) = (x^1, \cdots, x^{i-1}, u^i, x^{i+1}, \cdots, x^m)'\).

### 2. Environmental game and existence of equilibrium.

#### 2.1. Model.** Suppose there are \( m \) operators in the environmental game and each operator \( i \) aims at maximizing her/his welfare function. The welfare function is defined as the difference between the revenue and the damage due to pollution. As we consider the environmental game model, there are typically uncertainties [14]. Usually the distribution of the uncertain parameters is required to be clearly specified. However, this is virtually impossible as much ambiguity exist in confirming the distribution. In this paper, we do not presume any knowledge of the actual distribution of the uncertain parameters and we assume the true distribution information is contained in some family of distributions. Then, we propose a distributionally robust game model by combining the stochastic optimization and robust optimization approaches which can be labelled the minimax stochastic programming approach.

Assume the revenue function \( f_i \) of operator \( i \) is defined as its polluting emission \( x_i \), since the amount of pollution is usually proportional to its supply. Then the revenue function of an operator \( i \) can be given as \( f_i(\tilde{a}, x_i) \), where \( \tilde{a} : \Omega \rightarrow \mathcal{R}^n \) is an uncertain vector represented by a state-space \( \Omega \) and a set (sigma-algebra) \( \mathcal{F} \) of events. Because it is difficult to specify a probability distribution on \((\Omega, \mathcal{F})\), we set \( \mathcal{F} \) be a set of probability measures on \((\Omega, \mathcal{F})\) to cope with the distributional ambiguity.
For a given distribution $\mathbb{P} \in \mathcal{F}$, $\mathbb{E}_{\mathbb{P}}[\cdot]$ is the expectation over the distribution $\mathbb{P}$. We note that although each operator faces the same uncertain parameters, the structural results in this paper remain valid even when the uncertain parameters of each operator are different. As the cost of the environmental damage relies on all emissions $(x^1, \cdots, x^m)$, the cost function of the operator $i$ can be defined as $c_i(\tilde{a}, x^1, \cdots, x^m)$. Thus, the welfare function of operator $i$ is defined as,

$$
\pi_i(\tilde{a}; x^1, \cdots, x^m) := f_i(\tilde{a}, x_i) - c_i(\tilde{a}, x^1, \cdots, x^m).
$$

(1)

In this paper, we propose a distributionally robust game to deal with the distributional ambiguity by the minimax stochastic programming approach. Then, in our distributional robust model, the worst case expected welfare function within a family of possible distributions can be obtained and the best response of operator $i$'s to the other operators’ strategies $x^{-i} \in S_{-i}$ must be subject to

$$
\arg \max_{u^i \in S_i} \left\{ \inf_{\mathbb{P} \in \mathcal{F}} \mathbb{E}_{\mathbb{P}} \left[ \pi_i(\tilde{a}; x^{-i}, u^i) \right] \right\},
$$

where $S_i$ is the environmental constraint of operator $i$ and $S_{-i}$ is the aggregate environmental constraint of operators except operator $i$, i.e., $S_{-i} := (S_1, \cdots, S_{i-1}, S_{i+1}, \cdots, S_m)$. We assume that the emission $x^i$ is bounded from above by the maximum allowable emission $\bar{x}^i$ and from below by the minimum allowable emission $0$, i.e., $S_i := \{x^i | 0 \leq x^i \leq \bar{x}^i\}$, for $i = 1, \cdots, m$.

We now introduce the concept of equilibrium to the distributionally robust environmental game.

**Definition 2.1.** An equilibrium is defined as a tuple of strategies if each operator’s strategy is a best response to the other operator’s strategies.

According to Definition 2.1, $(x^1, \cdots, x^m) \in S$ is an equilibrium of our distributionally robust environmental game with incomplete information iff $\forall i \in \{1, \cdots, m\}$,

$$
x^i \in \arg \max_{u^i \in S_i} \left\{ \inf_{\mathbb{P} \in \mathcal{F}} \mathbb{E}_{\mathbb{P}} \left[ \pi_i(\tilde{a}; x^{-i}, u^i) \right] \right\},
$$

(2)

### 2.2. Assumptions and preliminary results

Next, we first state a number of standard assumptions from the literature, and then present some preliminary results for proving the existence of an equilibrium.

**Assumption 2.1.** The revenue function of each risk-averse operator $(f_i(\tilde{a}, x^i), i = 1, \cdots, m)$ can be defined as $f_i(\tilde{a}, x^i) := \min_{j \in \{1, \cdots, n\}} \tilde{a}_j f_{ij}(x^i)$, where $\tilde{a}_j \geq 0$, $f_{ij}(x^i)$ is twice continuously differentiable in $x^i$, nonnegative and its derivatives satisfy $f_{ij, x^i}(x^i) > 0$, $f_{ij, xx^i}(x^i) \leq 0$, for $x^i \geq 0$, $j \in \{1, \cdots, n\}$, $i = 1, \cdots, m$.

When the stochastic variable $\tilde{a}$ has a specific probability and $n_i = 1$, this definition has been used in the numerical test by [18]. In their paper, a two-player game is considered and the revenue function is defined as $f_i(\tilde{a}, x^i) := \tilde{a}_i f_i(x^i)$, where $\tilde{a}_i$ is a stochastic variable with known probability and function $f_i(x^i)$ satisfies the properties in Assumption 2.1. The functional definition of $f_i$ is also extensively used in distributionally robust optimization [9]. This assumption implies that $f_{ij}(x^i)$ is increasing, concave, and sufficiently smooth.

**Assumption 2.2.** The cost function of the environmental damage of each risk-averse operator $c_i(\tilde{a}, x^1, \cdots, x^m)$ can be defined as $c_i(x^1, \cdots, x^m) := \max_{j \in \{1, \cdots, n\}} \tilde{a}_j$
Both the revenue function and the cost function of each operator, and sufficiently smooth. Note that Assumption 2.2 holds, for example, for an uncertain line cost function, \( c_i(\bar{a}, x^1, \ldots, x^m) := \bar{a}(x^1 + \cdots + x^m) \), where \( \bar{a} > 0 \), \( i = 1, \ldots, m \) and \( x_j \geq 0 \), \( j = 1, \ldots, m \). When \( n_i = 1 \) and the distribution of \( \bar{a} \) is specified, this definition for the cost function is similar with the cost function in the numerical examples given by [18]. In short, \( c_i(x^1, \ldots, x^m) \) is increasing, convex about the emission of each operator, and sufficiently smooth.

**Assumption 2.3.** Both the revenue function and the cost function of each operator \( (f_i(\bar{a}, x^i), \text{ and } c_i(\bar{a}, x^1, \ldots, x^m), \ i = 1, \ldots, m) \) are measurable with respect to the probability measure \( P \) on \( (\Omega, \mathcal{F}) \). Further, we assume that \( \pi_i(\bar{a}; x^1, \ldots, x^m) \) is nonnegative for any \( \bar{a} \) and \( (x^1, \ldots, x^m) \).

The first part of Assumption 2.3 states that the set \( \mathcal{F} \) of probability measures on \( (\Omega, \mathcal{F}) \) must guarantee these two functions measurable. The second part is a common assumption in environmental games since each operator focuses on maximizing his/her welfare. If we define \( g'_i(x^1, \ldots, x^m) := f_i(x^i) - c_i(x^1, \ldots, x^m) \), then we can rewrite the welfare function \( \pi_i \) as follows

\[
\pi_i(x^1, \ldots, x^m) = \min_{j \in \{1, \ldots, n\}} \bar{a}_j g'_i(x^1, \ldots, x^m), \quad i = 1, \ldots, m. \tag{3}
\]

It is obvious that this welfare function is no less than \( \hat{\pi}_i(x^1, \ldots, x^m) := f_i(\bar{a}, x^i) - c_i(\bar{a}, x^1, \ldots, x^m) \). In the rest of this paper, we assume that \( \pi_i \) is defined as in (3). The reasons for this choice are as follows. First, we focus on the maximization of welfare in this paper. Therefore, \( \pi \) is a conservative approximation to \( \hat{\pi} \). Second, \( \pi_i \) keeps better properties than \( \hat{\pi}_i \), as shown in the proof of the existence of an equilibrium later. Finally, this definition makes the computational development effort in the following section easier. We note that in some special cases, \( \pi_i \) is actually equivalent to \( \hat{\pi}_i \). A particular case for \( \pi_i \) can be defined as

\[
\pi_i(\bar{a}; x^1, \ldots, x^m) := \min_{j \in \{1, \ldots, n\}} \bar{a}_j \left[ (1 + x^i) - \prod_{i=1}^m x^i \right], \tag{4}
\]

where operator \( i \)'s revenue and cost functions are

\[
f_i(\bar{a}, x^i) = \min_{j \in \{1, \ldots, n\}} \bar{a}_j (1 + x^i) \quad \text{and} \quad c_i(\bar{a}, x^1, \ldots, x^m) = \max_{j \in \{1, \ldots, n\}} \bar{a}_j \prod_{i=1}^m x^i.
\]

Another example is

\[
\pi_i(\bar{a}; x^1, \ldots, x^m) := \min_{j \in \{1, \ldots, n\}} \bar{a}_j \left[ (1 + x^i) - \sum_{i=1}^m x^i \right], \tag{5}
\]

where operator \( i \)'s cost function \( c_i(\bar{a}, x^1, \ldots, x^m) = \max_{j \in \{1, \ldots, n\}} \bar{a}_j \sum_{i=1}^m x^i \).

In this paper, we discuss the existence of equilibria in distributionally robust environmental games by using the fixed point theorem proposed by [20] which has been extensively used to prove the existence of the equilibria [1, 7, 21]. Before continuing to the existence theorem, we first give Kakutani’s fixed point theorem and a relevant definition that of–upper-semi continuity.
Definition 2.2. A point-to-set mapping $\psi : S \to 2^S$ is said to be upper semi-continuous if $y^n \in \psi(x^n)$, $n = 1, 2, 3, \cdots$, $\lim_{n \to \infty} x^n = x$, $\lim_{n \to \infty} y^n = y$ imply that $y \in \psi(x)$.

Theorem 2.3. (Kakutani’s fixed point theorem). Suppose $S$ is a closed, bounded, and convex set in Euclidean space, and $\psi$ is an upper semi-continuous point-to-set mapping from $S$ to the family of closed, convex subsets of $S$, then $\exists x \in S$ such that $x \in \psi(x)$.

To utilize the above fixed point theorem, we first need to construct a suitable correspondence and then prove that its fixed point is an equilibrium point. As such, we first need the following lemma about the equicontinuity of the set of functions $\{E_p[\pi_i(\hat{a}; x^{-i}, x^i)]\}$, $\mathbb{P} \in \mathcal{F}$, which forms the basis for utilizing Kakutani’s fixed point theorem. For any $(y^1, \cdots, y^m)$ and $(x^1, \cdots, x^m)$, we define the following metric induced by the infinity norm:

$$\| (y^1, \cdots, y^m) - (x^1, \cdots, x^m) \|_\infty = \max_{i \in \{1, \cdots, m\}} |x^i - y^i|.$$ 

Lemma 2.4. Suppose there is a constant $M > 0$ such that for any $\mathbb{P} \in \mathcal{F}$, $E_p[\max_{j \in \{1, \cdots, n\}} \hat{a}_j] \leq M$. Suppose Assumptions 2.1-2.3 hold, then $\forall \mathbb{P} \in \mathcal{F}$, $i \in \{1, \cdots, m\}$, function $E_p[\pi_i(\hat{a}; x^{-i}, x^i)]$ is continuous on $(x^1, \cdots, x^m)$.

Proof. To show continuity, we only need to prove that for any given $\epsilon > 0$, $\exists \delta(\epsilon)$ such that for any $(y^1, \cdots, y^m)$ and $(x^1, \cdots, x^m)$,

$$\| (y^1, \cdots, y^m) - (x^1, \cdots, x^m) \|_\infty \leq \delta(\epsilon)$$

then, $\forall \mathbb{P} \in \mathcal{F}$, $i \in \{1, \cdots, N\}$, the following inequality holds

$$|E_p[\pi_i(\hat{a}; x^{-i}, x^i)] - E_p[\pi_i(\hat{a}; y^{-i}, y^i)]| \leq \epsilon.$$ 

(7)

It follows from Assumptions 2.1-2.3 that for any $\mathbb{P} \in \mathcal{F}$, $i \in \{1, \cdots, m\}$,

$$|E_p[\pi_i(\hat{a}; x^{-i}, x^i)] - E_p[\pi_i(\hat{a}; y^{-i}, y^i)]| \leq E_p[|\pi_i(\hat{a}; y^1, \cdots, y^m) - \pi_i(\hat{a}; x^1, \cdots, x^m)|]
\leq E_p\left[\min_{j \in \{1, \cdots, n\}} \hat{a}_j |g^j_1(x^1, \cdots, x^m) - g^j_1(y^1, \cdots, y^m)|\right]
\leq E_p[\max_{j \in \{1, \cdots, n\}} \hat{a}_j \max_{j \in \{1, \cdots, n\}} |g^j_1(x^1, \cdots, x^m) - g^j_1(y^1, \cdots, y^m)|]
\leq M \max_{j \in \{1, \cdots, n\}} |g^j_1(x^1, \cdots, x^m) - g^j_1(y^1, \cdots, y^m)|.$$ 

(8)

Therefore, it follows from the continuity of $g^j_1$ on $(x^1, \cdots, x^m)$ that for any any given $\epsilon/M > 0$, $\exists \delta(\epsilon)$, if $(y^1, \cdots, y^m)$ and $(x^1, \cdots, x^m)$ satisfies (6), then

$$|g^j_1(x^1, \cdots, x^m) - g^j_1(y^1, \cdots, y^m)| \leq \epsilon/M, \quad j = 1, \cdots, n.$$ 

This together with (8) implies that (7) holds.

The above lemma is important for proving the continuity of the function $\rho_i$ defined later which is a key result for our existence proof. To this end, define

$$\rho_i^\mathcal{F}(x^1, \cdots, x^m) := \inf_{\mathbb{P} \in \mathcal{F}} E_p[\pi_i(\hat{a}; x^{-i}, x^i)]$$

(9)

$i \in \{1, \cdots, m\}$. From the definition of (9), we have the following conclusions.
Lemma 2.5. Suppose there is a constant $M > 0$ such that for any $\mathbb{P}_i \in \mathbb{F}$, $\mathbb{E}_i[\max_{j \in \{1, \ldots, n\}} a_{ij}] \leq M$. Suppose Assumptions 2.1-2.3 apply, then the following conclusions hold: \( \forall i \in \{1, \ldots, m\} \)

(i) given $x^{-i}$ fixed, $\rho_{ij}^F$ is concave in $x^i$;

(ii) $\rho_{ij}^F$ is continuous on $\mathbb{R}^m$.

Proof. From Assumptions 2.1 and 2.2, function $g_i^j(x^{-i}, x^i) = f_{ij}(x^i) - c_{ij}(x^{-i}, x^i)$ is concave about $x^i$. Therefore, for any given $x^{-i}$ fixed that for any $\lambda \in [0, 1]$, $x^i, y^i \geq 0$,

$$\inf_{\mathbb{P}_i \in \mathbb{F}} \mathbb{E}_i \left[ \pi_i \left( \tilde{a}; x^{-i}, \lambda x^i + (1 - \lambda)y^i \right) \right] = \inf_{\mathbb{P}_i \in \mathbb{F}} \mathbb{E}_i \left[ \min_{j \in \{1, \ldots, n\}} a_{ij}^x \left( x^{-i}, \lambda x^i + (1 - \lambda)y^i \right) \right] \geq \inf_{\mathbb{P}_i \in \mathbb{F}} \mathbb{E}_i \left[ \min_{j \in \{1, \ldots, n\}} \left( \lambda a_{ij}^x \left( x^{-i}, x^i \right) + (1 - \lambda) a_{ij}^y \left( x^{-i}, y^i \right) \right) \right] \geq \inf_{\mathbb{P}_i \in \mathbb{F}} \mathbb{E}_i \left[ \lambda \min_{j \in \{1, \ldots, n\}} a_{ij}^x \left( x^{-i}, x^i \right) + \min_{\mathbb{P}_i \in \mathbb{F}} \mathbb{E}_i \left[ \left( 1 - \lambda \right) \min_{j \in \{1, \ldots, n\}} a_{ij}^y \left( x^{-i}, y^i \right) \right] \right] = \lambda \inf_{\mathbb{P}_i \in \mathbb{F}} \mathbb{E}_i \left[ \pi_i \left( \tilde{a}; x^{-i}, x^i \right) \right] + (1 - \lambda) \inf_{\mathbb{P}_i \in \mathbb{F}} \mathbb{E}_i \left[ \pi_i \left( \tilde{a}; x^{-i}, y^i \right) \right].$$

Therefore $\rho_{ij}^F$ is concave in $x^i$ and (i) holds.

From the piecewise linearity of $\pi_i$ over $\tilde{a}$, the linearity of the expectation operator, and Lemma 2.4, (ii) holds. \( \square \)

2.3. Existence of equilibrium. On the basis of the above lemma, we now propose an existence theorem of an equilibrium point in environmental games with distributions ambiguous.

Theorem 2.6. (Existence of equilibrium for distributionally robust environmental games) Suppose for any $\mathbb{P}_i \in \mathbb{F}$, $\mathbb{E}_i [\tilde{a}]$ is bounded. Then, there is at least an equilibrium point for the distributionally robust environmental game.

Proof. Theorem 2.3 will be applied to complete the proof for the existence of an equilibrium point satisfying (2). For this purpose, we define a point-to-set mapping

$$\psi(x) = \left\{ y \in S \mid y^i \in \arg \max_{u_i \in S_i} \left\{ \inf_{\mathbb{P}_i \in \mathbb{F}} \mathbb{E}_i \left[ \pi_i \left( \tilde{a}; x^{-i}, u^i \right) \right] \right\}, \ i = 1, \ldots, N \right\}.$$ 

According to the definition, for any $x \in S$, $\psi(x) \subset S$. With the help of $\psi$, the proof for the existence of an equilibrium point satisfying (2) is equivalent to apply the fixed point Theorem 2.3 to show that there is an fixed point for $\psi$ over $S$. To apply Theorem 2.3, the convexity and upper semi-continuity of $\psi$ are needed.

We first show that $\psi$ is convex for $x \in S$. Given any $(y^1, \ldots, y^m), (z^1, \ldots, z^m) \in \psi(x^1, \ldots, x^m)$. Then by the definition of $\psi$, for any $u^i \in S_i$, $\rho_{ij}^F(\tilde{x}^{-i}, u^i) = \rho_{ij}^F(\tilde{x}^{-i}, z^i)$ $\leq \rho_{ij}^F(\tilde{x}^{-i}, u^i), \ i = 1, \ldots, N$, which means that for any $\lambda \in [0, 1]$, the following inequality holds

$$\lambda \rho_{ij}^F(\tilde{x}^{-i}, y^i) + (1 - \lambda) \rho_{ij}^F(\tilde{x}^{-i}, z^i) \leq \rho_{ij}^F(\tilde{x}^{-i}, u^i), \ i = 1, \ldots, N.$$ 

It follows from the above inequality and the convexity of $\rho_{ij}^F$ with $x$ that $\rho_{ij}^F(\tilde{x}^{-i}, y^i) + (1 - \lambda) z^i) \leq \lambda \rho_{ij}^F(\tilde{x}^{-i}, y^i) + (1 - \lambda) \rho_{ij}^F(\tilde{x}^{-i}, z^i) \leq \rho_{ij}^F(\tilde{x}^{-i}, u^i), \ i = 1, \ldots, N, \ \frac{1}{\lambda} \rho_{ij}^F(\tilde{x}^{-i}, y^i) + (1 - \lambda) z^i) \leq \lambda \rho_{ij}^F(\tilde{x}^{-i}, y^i) + (1 - \lambda) \rho_{ij}^F(\tilde{x}^{-i}, z^i) \leq \rho_{ij}^F(\tilde{x}^{-i}, u^i), \ i = 1, \ldots, N,$

which together with the closedness and convexity of $S$ implies that $\lambda(y^1, \ldots, y^m) + (1 - \lambda) (z^1, \ldots, z^m) \in \psi(x^1, \ldots, x^m)$. Hence $\psi$ is convex.
Next, we show that $\psi$ is upper semi-continuous with $x \in S$. For this purpose, suppose there are two sequences $\{x_n\} \subset S$, $\{y_n\}$ with $x_n \to x$, $y_n \to y$, $n \to \infty$, where $y_n \in \psi(x_n)$, $n = 1, 2, 3, \cdots$. Then according to the definition of $\psi$, for any $u^i \in S_i$, the following inequality holds, $\rho^i(x^{-i,n}, y^{i,n}) \geq \rho^i(x^{-i,n}, u^i)$, $\forall i \in \{1, \cdots, m\}$, $n = 1, 2, \cdots$. Taking the limit on the both sides of the above inequality and considering the continuity of $\rho^i$, the following inequality can be obtained, $\rho^i(x^{-i}, y^i) \geq \rho^i(x^{-i}, u^i)$, $\forall i \in \{1, \cdots, N\}$, $u^i \in S_i$. Hence $y \in \psi(x)$, that is $\psi$ is upper semi-continuous about $x \in S$.

Therefore, $\psi$ satisfies the conditions of Theorem 2.3, which implies that there is an equilibrium point satisfying (2) for the distributionally robust environmental game.

\[ \square \]

3. Computation for equilibrium. To obtain an equilibrium point, we need to estimate the worst-case expectation over a family of distributions and then to optimize it on the parameters $(x^1, \cdots, x^m)$. However, even estimating expectation is computationally intractable since the expectation is a multidimensional integration problem as shown by \cite{24}. Nevertheless, optimizing the decision variables $u^i$ in (2) can easily be cast as a standard stochastic optimization problem which can be approximately solved by sample average approximation (SAA). However under distributional ambiguity, the computation for (2) calls for the ability to evaluate the worst-case expectation over a family of distributions and then optimizing it over the decision variables, $u^i$. In this section, we provide a computation method to obtain the equilibrium under distributional ambiguity which is based on the reformulation of (2) as a linear semi-definite optimization problem. To this end, we need the following assumption.

**Assumption 3.1.** Assume that function $g^i(x^{-i}, x^i)$ is linear about $x^i$ for any given $x^{-i}$, $\forall j \in \{1, \cdots, n\}$, $i \in \{1, \cdots, m\}$.

This assumption implies that the welfare function $\pi_i$ defined by (3) is piecewise linear about $x^i$ for any given $x^{-i}$. Although piecewise linear welfare functions are not commonly used in the context of expected utility theory, they present enough approximation to any desired function with a high level of accuracy by using the first-order Taylor expansion. Examples to this assumption can be seen as Definitions (4) and (5). The linearity of $g_i(x^{-i}, x^i)$ about $x^i$ will be used in developing the computation method for obtaining an equilibrium.

Denote $\tilde{z} = (\tilde{z}_1, \cdots, \tilde{z}_\tau)$ as a vector of $\tau$ random variables defined over $\Omega$ and we assume that the uncertain payoffs for $i$th player are affinely dependent on $\tau$ given random variables or factors $\tilde{z}_1, \cdots, \tilde{z}_\tau$, that is

\[ U := \left\{ \tilde{a} : \exists b^0_j, b_j^t \in R, j \in N, t \in T \mid \tilde{a}_j(\omega) = b^0_j + \sum_{t=1}^{\tau} b^t_j \tilde{z}_t(\omega), \forall \omega \in \Omega, j \in N \right\} \tag{10} \]

where $N := \{1, \cdots, n\}$, $T := \{1, \cdots, \tau\}$. This factor model for uncertain parameters has been comprehensively utilized in the robust optimization literature where the descriptive statistics of these factors are specified and lay the basis for describing the family of distributions $\mathcal{F}$, see for instance, \cite{2, 5, 9, 10}.

For the computation of the equilibrium point, we assume that $\tilde{z}$ is described by its mean $\mu$ and covariance $Q$, then we define the set $\mathcal{F}$ of the family of distributions that have the same first- and second-order moments as follows:

\[ \mathcal{F}(\mu, Q) := \left\{ \mathbb{P} : \mathbb{E}_{\mathbb{P}}[\tilde{z}] = \mu, \mathbb{E}_{\mathbb{P}}[\tilde{z}\tilde{z}'] = Q + \mu\mu' \right\}. \tag{11} \]
Without any loss of generality, we suppose the second-moment information $Q$ is positive definite. When the uncertain parameter $\tilde{a}$ are affinely dependent on $\tilde{z}$ as defined in (10), then for any given $\tilde{a} \in U$ and $(x^1, \cdots, x^m)$, we explicitly define

$$\rho_i^F(\tilde{a}; x^1, \cdots, x^m) := \inf_{\mathcal{F}} \mathbb{E}_\mathcal{F} \left[ \pi_i \left( \tilde{a}; x^{-i}, x^i \right) \right].$$

(12)

With the distributions set $\mathcal{F}$ of $\tilde{z}$ defined as (11), $\rho_i^F(\tilde{a}; x^1, \cdots, x^m)$ is equivalent to the following semi-infinite programming:

$$\inf_{\mathcal{F} \in \mathbb{F}(\mu, Q)} \mathbb{E}_\mathcal{F} \left[ \pi_i \left( \tilde{a}; x^{-i}, x^i \right) \right]$$

s.t. $\mathbb{E}_\mathcal{F}(\tilde{z}) = \mu$

$$\mathbb{E}_\mathcal{F}(\tilde{z}\tilde{z}') = Q + \mu(\mu)'.$$

The above problem can be transformed into an conic programming problem as shown in the section 3.1 which can be tractably realized. Then we show that the computation for equilibrium point is equivalent to solving a nonlinear semidefinite programming.

3.1. Tractable formulation for equilibrium point. We first show that with $U$ as (10) for any $i \in \{1, \cdots, m\}$, $\rho_i^F$ defined in (12) can be equivalently transformed into an semidefinite programme.

**Theorem 3.1.** For any given $\tilde{a} \in U$ with $U$ defined as (10), for any $i \in \{1, \cdots, m\}$, we have $\rho_i^F(\tilde{a}; x^1, \cdots, x^m)$ with the distributions set $\mathcal{F}$ of $\tilde{z}$ defined as (11) equivalent to the optimal values of the following tractable problem,

$$\sup_{\eta_i, \kappa_i, \mu} \eta_i + \kappa_i \mu + \text{tr}(H^i(Q + \mu(\mu)'))$$

s.t. $\left( \begin{array}{c} (b^0)' \cdot g_i(x^{-i}, x^i) - \eta_i \frac{1}{2}(B' g_i(x^{-i}, x^i) - \kappa_i)' \\ \frac{1}{2}(B' g_i(x^{-i}, x^i) - \kappa_i)' - H^i \end{array} \right) \in \mathbb{S}^{r+1}_+$(13)

where $g_i(x^{-i}, x^i) := (g'_1(x^{-i}, x^i), \cdots, g'_m(x^{-i}, x^i))^t$, $b^0 := (b^0_1, \cdots, b^0_m)^t$, $B = (b')_j \in \mathcal{N} \in \mathbb{R}^{n \times r}$, $\text{tr}(\cdot)$ is the corresponding matrix trace and $\mathbb{S}^{r}_+$ (respectively, $\mathbb{S}^r$) is the set of symmetric positive semidefinite matrices (respectively, symmetric matrices) in $\mathbb{R}^{r \times r}$.

**Proof.** By the strong duality results of [16] (see also the proof of Proposition 4 in [22,]), (12) can be equivalently transformed into

$$\sup_{\eta_i, \kappa_i, \mu} \eta_i + \kappa_i \mu + \text{tr}(H^i(Q + \mu(\mu)'))$$

s.t. $\eta_i + \kappa_i y + y' H^i y \leq \pi_i (b^0 + B y; x^{-i}, x^i), \forall y \in \mathbb{R}^r$

$$\eta_i \in \mathbb{R}, \kappa_i \in \mathbb{R}^r, H^i \in \mathbb{S}^r.$$

(14)

By the definition of $\pi_i$, then the first constraint in the above problem is equivalent to

$$\left( \begin{array}{c} 1 \\ y \end{array} \right)' \left( \begin{array}{c} \eta_i - (b^0)' \cdot g_i(x^{-i}, x^i) \frac{1}{2}(\kappa_i - B' g_i(x^{-i}, x^i))' \\ \frac{1}{2}(\kappa_i - B' g_i(x^{-i}, x^i))' H^i \end{array} \right) \left( \begin{array}{c} 1 \\ y \end{array} \right) \leq 0, \forall y \in \mathbb{R}^r,$$

which can be easily transformed into a positive semi-definite conic constraint. Then, the conclusion holds. \qed
It follows from (2) that for any given $\tilde{a} \in U$ with $U$ defined as (10), $(x^1, \cdots, x^m)$ as an equilibrium point of distributionally robust environmental game is equivalent to finding $(x^1, \cdots, x^m)$ such that
\[
\rho_i^\tilde{a}(\tilde{a}; x^i, x^i) - \max_{u^i \in S_i} \rho_i^u(\tilde{a}; x^i, u^i) \geq 0, \quad \forall x^i \in S_i, \ i = 1, \cdots, m. \tag{15}
\]
The above relationships (15) together with Theorem 3.1 implies that the computation for equilibrium point $(x^1, \cdots, x^m)$ can be realized, $\forall i \in \{1, \cdots, N\}$, $\exists \eta_i \in \mathcal{R}, \ k_i \in \mathcal{R}^r$, $H^i \in \mathcal{S}^r$, such that $(x^i, \eta_i, k_i, H^i)$ is a maximizer of the following conic programming problem,
\[
\max_{x^i, \eta_i, k_i, H^i} \eta_i + k_i \mu + tr(H^i (Q^i + \mu \mu'))
\]
\[
s.t. \left( \begin{array}{c}
(b^i)' g_i(x^{-i}, x^i) - \eta_i \frac{1}{2}(B^i g_i(x^{-i}, x^i) - k_i)^' \\
\frac{1}{2}(B^i g_i(x^{-i}, x^i) - k_i) - H^i
\end{array} \right) \in \mathcal{S}^{r+1}
\]
\[x^i \in S_i.
\]
We now describe the method for computing the equilibrium point. Suppose $H^i = (h^i_{jk}) \in \mathcal{S}^r$ and define $\hat{h}^i := (h_{i1}, \cdots, h_{i+r}, \cdots, h_{i1}, \cdots, h_{i+r})' \in \mathcal{R}^{r^2}$,
\[
\mathcal{A}_i(x^i, \eta_i, k_i, \hat{h}^i) := \left( \begin{array}{c}
(b^i)' g_i(x^{-i}, x^i) - \eta_i \frac{1}{2}(B^i g_i(x^{-i}, x^i) - k_i)^' \\
\frac{1}{2}(B^i g_i(x^{-i}, x^i) - k_i) - H^i
\end{array} \right).
\]
Let
\[
A^i := \left( \begin{array}{cc}
(b^i)' g_i(x^{-i}, 1) & \frac{1}{2}B^i g_i(x^{-i}, 1) \\
\frac{1}{2}B^i g_i(x^{-i}, 1) & 0
\end{array} \right) \in \mathcal{S}^{r+1}, \tag{18}
\]
\[
\hat{B}^i := \left( \begin{array}{ccc}
0 & \cdots & 1^t, \cdots, 0 \\
\cdots & \cdots & \cdots \\
-\frac{1}{2} (0, \cdots, 1^t, \cdots, 0) & 0
\end{array} \right) \in \mathcal{S}^{r+1}, \tag{19}
\]
\[
C^i = \left( \begin{array}{ccc}
-1 & 0 & 0 \\
0 & 0 & 0
\end{array} \right) \in \mathcal{S}^{r+1}, \tag{20}
\]
\[
D^i_{jk} := \left( \begin{array}{ccc}
0 & 0 & 0 \\
0 & 1^t_{jk}
\end{array} \right) \in \mathcal{S}^{r+1}, \tag{21}
\]
where $b^i$ is the $k$-th column of $B$, $(0, \cdots, 1^t, \cdots, 0) \in \mathcal{R}^r$ and $I^t_{jk} \in \mathcal{R}^{r \times r}$ denotes that the $j \times k$-th entry is 1, zero elsewhere. With the above definitions, we can rewrite $\mathcal{A}_i(x^i, \eta_i, k_i, H^i)$ as follows,
\[
\mathcal{A}_i(x^i, \eta_i, k_i, \hat{h}^i) = A^i x^i + \sum_{\ell=1}^r k_{i\ell} \hat{B}^i_{\ell} + \sum_{j=1}^r \sum_{k=1}^r h^i_{jk} D^i_{jk} + \eta_i C^i.
\]
\[
\mathcal{A}_i \text{ defined as above is a matrix-valued function from } \mathcal{R}^{1+r+r^2+1} \text{ to } \mathcal{S}^{r+1} \text{ and it is linear about } (x^i, \eta_i, k_i, H^i). \]
Therefore from (22) and (23), the equilibrium point \((x^1, \cdots, x^m)\) can be calculated, \(\forall i \in \{1, \cdots, N\}\), \(\exists \eta_i \in \mathcal{R}, \kappa_i \in \mathcal{R}^+, H^i \in \mathcal{S}^+\), such that \((x^i, \eta_i, \kappa_i, H^i)\) is a maximizer of the following linear semidefinite program,

\[
\begin{align*}
\max_{x^i, \eta_i, \kappa_i, H^i} & \quad f_i(x^i, \eta_i, \kappa_i, H^i) \\
\text{s.t.} & \quad \mathcal{S}_i(x^i, \eta_i, \kappa_i, H^i) \in \mathcal{S}^+_{i+1}, \\
& \quad x^i \in S_i,
\end{align*}
\]

whose dual with \(S_i := \{x^i | 0 \leq x^i \leq \bar{x}^i\} (i = 1, \cdots, m)\) is

\[
\begin{align*}
\min_{r_i \in \mathcal{R}_+, \tau \in \mathcal{S}^+_{i+1}} & \quad -r_i \\
\text{s.t.} & \quad tr(A^i Y_i) + r_i \leq 0, \\
& \quad tr(C^i Y_i) + 1 = 0 \\
& \quad tr(D_{jk}^i Y_i) + \mu_k = 0, t = 1, \cdots, \tau, \\
& \quad \mu_j \mu_k + q_{jk}^i = 0, j, k = 1, \cdots, \tau.
\end{align*}
\]

According to the above discussions, the computational method for equilibrium point can be presented by the following theorem.

**Theorem 3.2.** Given the distributional set \(\mathcal{F}(\mu, Q)\) of \(\bar{z}\), the computation for a distributionally robust equilibrium point \((x^1, \cdots, x^N)\) is equivalent to for all \(i \in \{1, \cdots, N\}\), there exists \(\eta_i \in \mathcal{R}, \kappa_i \in \mathcal{R}^+, H^i \in \mathcal{S}^+, Y_i \in \mathcal{S}^+_{i+1}\) and \(r_i \geq 0\) such that \((x^i, \eta_i, \kappa_i, H^i, Y_i, r_i)\) satisfies

\[
\begin{align*}
f_i(x^i, \eta_i, \kappa_i, H^i) &= -r_i, \\
Y_i, \mathcal{S}_i(x^i, \eta_i, \kappa_i, H^i) &\in \mathcal{S}^+_{i+1}, \\
tr(A^i Y_i) + r_i &\leq 0, \\
tr(C^i Y_i) + 1 = 0 \\
tr(D_{jk}^i Y_i) + \mu_k &\leq 0, t = 1, \cdots, \tau, \\
\mu_j \mu_k + q_{jk}^i &\leq 0, j, k = 1, \cdots, \tau \\
r_i &\geq 0, \bar{x}^i \leq x^i 
\end{align*}
\]

**Proof.** It follows from (24), (25) and the LP strong duality that the necessary condition (26) holds. For the sufficiency condition, that is if (26) holds, then for all \(i \in \{1, \cdots, N\}\) and for \(x^{-i}, (x^i, \eta_i, \kappa_i, H^i)\) is feasible to (24) and \((Y_i, r_i)\) is feasible to (25). The LP weak duality and \(f_i(x^i, \eta_i, \kappa_i, H^i) = -r_i\) implies that \((x^i, \eta_i, \kappa_i, H^i)\) is an optimal solution to (24) and \((Y_i, r_i)\) is an optimal solution to (25). \(\square\)

### 3.2. Computational method

**Theorem 3.2** shows that the computation for distributionally robust equilibrium is equivalent to solving a special system including linear semidefinite matrices inequalities, multilinear equalities, and inequalities. In what follows, we show that such system can actually be transformed into a convex semidefinite program of the form

\[
\begin{align*}
\min_{v \in \mathcal{R}^L} & \quad \phi(v) \\
\text{s.t.} & \quad \mathcal{B}(v) + B \in \mathcal{S}^+_{L}
\end{align*}
\]

where \(\phi(v) : \mathcal{R}^L \to \mathcal{R}\) is a continuously differentiable convex function, \(\mathcal{B} : \mathcal{R}^L \to \mathcal{S}^L\) is a linear function defined as (22), \(B \in \mathcal{S}^L\). Let \(Y_i := (y_{jk}^i) \in \mathcal{S}^+_{i+1}, y^i := \)
A condition is needed, that is, \( \exists l \) such that a saddle point to the Lagrange function \( Y \) is a minimizer to (27), then there exists \( v \) and sufficient optimality condition since the convexity of (27). Further, if \( l \rightarrow (27) \) and the corresponding optimal value \( \phi \), \( v \) and \( \phi \) is a solution to system (29) iff \( \bar{v} \) is a solution to (27). Since for any \( l \rightarrow (27) \) and the corresponding optimal value \( \phi \), \( v \) and \( \phi \) is a second order differentiable but not second order continuously differentiable.

Evidently, (27) is a convex semidefinite program with nonlinear objective function \( \phi \). We present the first-order optimality condition for (27) which is a necessary condition for problem (27) iff there are matrices \( Y, S \) such that (26) can be transformed into the solution for the system (27). There is a variety of methods to solve problem (27), for example, interior point algorithms [4]; SSP algorithms based on SQP methods for standard nonlinear programs [8]; gradient based algorithms [23]. In this paper, to solve problem (27) we use CVX, a package for specifying and solving convex programs [11, 12].

\[(y_{11}, y_{1, r+1}, y_{21}, \ldots, y_{1, r+1}, y_{1, r+1, r+1}) \in \mathcal{R}^{(r+1)^2}, \quad I_{jk}^r \in \mathbb{S}^{r+1} \text{ be matrices with the } j \times k \text{ entry } 1 \text{ and zero elsewhere. Then } Y \text{ can be reexpressed as a linear function with } \mathcal{C} : \mathcal{R}^{(r+1)^2} \rightarrow \mathbb{S}^{r+1}\]

\[\mathcal{C}(y) := Y = \sum_{j=1}^{n+1} \sum_{k=1}^{n+1} y_{jk}^i I_{jk}, \quad (28)\]

where \( Y \) is defined as in (26). This implies that system (26) can be generalized as the following system

\[\varphi_m(v) = 0, \quad \forall m \in E, \quad \varphi_m(v) \leq 0, \quad \forall m \in I, \quad \mathcal{B}(v) + B \in \mathbb{S}_+^L, \quad v \in \mathcal{R}^l, \quad (29)\]

where \( \forall m, \varphi_m : \mathcal{R}^l \rightarrow \mathbb{R} \) is linear about \( v \), \( E \) and \( I \) denote the index set for equality and inequality respectively.

Define

\[\phi(v) := \frac{1}{2} \sum_{m \in E} |\varphi_m(v)|^2 + \frac{1}{2} \sum_{m \in I} \max \{\varphi_m(v), 0\}^2. \quad (30)\]

Then the solution for system (29) is equivalent to the solution for (27). Since for any \( v \in \mathcal{R}^l, \phi(v) \geq 0 \), so \( \bar{v} \) is a solution to system (29) iff \( \bar{v} \) is an optimal solution to (27) and the corresponding optimal value \( \phi(\bar{v}) = 0 \). We note that the function \( \phi(\cdot) \) is second order differentiable but not second order continuously differentiable.

Evidently, (27) is a convex semidefinite program with nonlinear objective function \( \phi \) defined as (30). We first define the Lagrangian function \( \mathcal{L} : \mathcal{R}^l \times \mathbb{S}^L \rightarrow \mathbb{R} \) to (27) as follows,

\[\mathcal{L}(v, Y) := \phi(v) - \text{tr}(\mathcal{B}(v) + B)Y. \quad (31)\]

Since \( \phi(\cdot) \) is continuously differentiable about \( v \) and \( \mathcal{B}(\cdot) \) is linear about \( v \), so the gradient for function \( \mathcal{L}(\cdot, \cdot) \) about \( v \) can be presented by

\[\nabla_v \mathcal{L}(v, Y) = \nabla \phi(v) - \nabla_v \text{tr}(\mathcal{B}(v) + B)Y. \quad (32)\]

We present the first-order optimality condition for (27) which is a necessary and sufficient optimality condition since the convexity of (27). Further, if \( v^* \) is a minimizer to (27), then there exists \( Y^* \in \mathbb{S}_+^L \) such that the pair \( (v^*, Y^*) \) is a saddle point to the Lagrange function \( \mathcal{L} \) defined as (31). To this end, the Slater’s condition is needed, that is \( \exists v \in \mathcal{R}^l \) such that \( \mathcal{B}(v) + B \in \mathbb{S}_+^L \).

**Theorem 3.3.** Suppose Slater’s condition holds at \( v^* \). Then \( v^* \) is a minimizer to problem (27) iff there are matrices \( Y, S \in \mathbb{S}^L \) such that

\[\nabla_v \mathcal{L}(v^*, Y) = 0, \quad \mathcal{B}(v^*) + B - S = 0, \quad SY = 0, \quad S, Y \in \mathbb{S}_+^L. \quad (33)\]

**Proof.** The proof follows directly from the first order analysis for nonlinear semidefinite programs by [25].

The above theorem implies that to obtain the optimal solution to problem (33) can be transformed into the solution for the system (27). There is a variety of methods to solve problem (33), for example, interior point algorithms [4]; SSP algorithms based on SQP methods for standard nonlinear programs [8]; gradient based algorithms [23]. In this paper, to solve problem (27) we use CVX, a package for specifying and solving convex programs [11, 12].
4. Numerical tests. So far, we have discussed the noncooperative environmental game. In this section, we consider the numerical tests for both cooperative and umbrella environmental games. In the cooperative environmental game, all the operators aim to optimize the sum of their individual welfare under a joint environmental constraint, i.e.,

$$\max_{x \in S} \left\{ \inf_{P \in \mathcal{F}} \left[ \sum_{i=1}^{m} \pi_i (\tilde{a}; x) \right] \right\},$$

where \( S := \{ x \in \mathbb{R}^m | x^i \geq 0, \sum_{i=1}^{m} x^i \leq \sum_{i=1}^{m} \bar{x}^i \} \). While in the umbrella environmental game, for every choice of the rival’s strategies, each operator acts in a selfish manner and focuses on optimizing his/her individual welfare under a common environmental constraint, i.e., the \( i \)-th operator solves the following optimization problem, \( i = 1, \ldots, m \)

$$\max_{u^i \in S_i (x^{-i})} \left\{ \inf_{P \in \mathcal{F}} \left[ \pi_i (\tilde{a}; x^{-i}, u^i) \right] \right\},$$

where \( S_i (x^{-i}) := \{ u^i | u^i \geq 0, u^i + \sum_{j \neq i} \sum_{j=1}^{m} x^j \leq \sum_{i=1}^{m} \bar{x}^i \}, i = 1, \ldots, m \).

We consider the environmental game with two players, under incomplete information where the parameter is stochastic with an imprecise distribution which is assumed to be attached to an a-priori known set. The welfare functions are given as

$$\pi_i (\tilde{a}; x^1, x^2) := \tilde{a}_i (1 + x^i) - \max_{k=1,2} \tilde{a}_k \prod_{j=1}^{2} x^j, i = 1, 2,$$

where the stochastic parameter \( \tilde{a} \) is defined as (10) and the distributional set of \( \tilde{z} \) is given by \( \mathcal{F} (\mu, Q) \). In the numerical tests, we assume that \( \tau = 2, \mu = (1, 1)^t, Q \in \mathbb{R}^{2 \times 2} \) is the unit matrix, and the upper bound of \( x \) is \( \bar{x}^1 = \bar{x}^2 = 2 \).

We compute the efficiency as the sum of each operator’s welfare under the corresponding equilibrium obtained by solving the noncooperative, cooperative and umbrella environmental games. For comparison, we consider both symmetric and asymmetric networks with the parameters for defining \( \tilde{a} \) as the following two cases, respectively: the symmetric case (TSC)

$$b^t_i = \frac{1}{2}, i = 1, 2, t = 0, 1, 2;$$

and the asymmetric case (TASC)

$$b^0_1 = b^2_1 = 1, b^1_1 = b^2_2 = 1, b^1_2 = b^2_2 = \frac{1}{2}.$$  

The results are reported in Table 1 with \( \delta = 0 \), in which “NSCW”, “CSCW” and “USCW” represent the corresponding sum of each operator’s welfare under the equilibria obtained by solving noncooperative, cooperative and umbrella environmental games. To show the stability of the distributionally robust approach, we perturb the distributional set \( \mathcal{F} (\mu, Q) \) as follows,

$$\mathcal{F}_\delta (\mu, Q) := \{ P | \mathbb{E} [\tilde{z}] = \mu + \delta e, \mathbb{E} (\tilde{z}^2) = Q + \delta I \},$$

where \( e := (1, \cdots, 1)^t \in \mathbb{R}^\tau \) and \( I \in \mathbb{R}^{\tau \times \tau} \) is the identity matrix. Under different values of \( \delta \), we compute the corresponding “NSCW”, “CSCW” and “USCW”. The results are given in Table 1. It is easy to see that the maximal welfare for
the system can be obtained in the cooperative setting and there is larger welfare improvement in the symmetric than in the asymmetric case. The equilibria in both noncooperative and umbrella settings are same. This is because that the equilibrium in noncooperative game is also the corresponding equilibrium in umbrella setting. With \( \delta \in [0.01, 0.1] \), the results in Table 1 show that the remarkable stability of the distributional robust approach in both the symmetric and asymmetric environmental games.

| \( \delta \) | NSCW  | CSCW  | USCW  |
|------------|-------|-------|-------|
| TSC        | 0     | 2.0001| 2.446 | 1.2269|
| TASC       | 0     | 1.8   | 2.1333| 1.8   |
| TSC        | 0.01  | 2     | 2.4456| 2     |
| TASC       | 0.01  | 1.801 | 2.1565| 1.801 |
| TSC        | 0.03  | 1.9981| 2.4448| 1.9981|
| TASC       | 0.03  | 1.803 | 2.1602| 1.803 |
| TSC        | 0.05  | 1.9906| 2.444 | 1.9906|
| TASC       | 0.05  | 1.805 | 2.1625| 1.805 |
| TSC        | 0.1   | 1.9811| 2.4421| 1.9811|
| TASC       | 0.1   | 1.8155| 2.1681| 1.8155|

5. Conclusion. In this paper we have introduced a class of distributionally robust environmental games. The existence theorem of an equilibrium is proved under standard assumptions. The computation of equilibrium, with the family of distributions that have the same first- and second-order moments, can be equivalently transformed into solving an semidefinite programming problem which is tractable. The results of the numerical tests show the efficiency of the model in describing environmental game with the probabilistic ambiguity of data as an a-priori known set. In the computation, we suppose the support set is \( \mathbb{R}^n \). However, in environmental games, the support set is often compact and convex. Therefore, it is interesting to present the computation method for the support set being compact and convex. Further extending the method proposed in this paper to other problems is also our future studies.

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