Causality and Peierls Bracket in Classical Mechanics

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Abstract

Relation between the Peierls and the Poisson bracket is derived in classical mechanics of time-dependent systems. Equal-time Peierls brackets are seen to be the same as the Poisson brackets in simple cases but a proof for a general Hamiltonian is lacking.

1 Introduction

Once the coordinates and momenta of an autonomous system are specified at any fixed time, the phase trajectory of the system is determined for all past and future. But for a system in the presence of external agents, the phase trajectories get affected by these external agents only after the time they are switched on. This requires a formulation of the condition of causality in classical mechanics although it is always assumed implicitly. In this note we find that a bracket, defined by Peierls[1] in 1952 as a covariant bracket for relativistic fields, is the natural bracket for implementing causality in general time-dependent classical systems.

In the autonomous case observables are functions of coordinates and momenta. The time dependence of such an observable is given by the values of the observable along a phase trajectory which is determined by Hamiltonian equations of motion. In the case of time-dependent systems the definition of an observable has to be extended to include time as an extra variable on which it can depend. This necessitates the inclusion of time as a dynamical variable alongwith coordinates and momenta.

The Poisson bracket determines how one quantity \( b(t, q, p) \) changes another quantity \( a(t, q, p) \) when it acts as the Hamiltonian or vice-versa. The Peierls bracket, on the other hand, determines how one quantity \( b(t, q, p) \) when added to the system Hamiltonian \( h \) with an infinitesimal coefficient \( \lambda \) affects changes in

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another quantity \( a(t, q, p) \) and vice-versa. Therefore, we expect a close relationship between the two brackets when the system Hamiltonian is zero. This indeed is true as we prove in section 3 below. We find that the Peierls bracket of any two observables for zero Hamiltonian is related to ‘two-point’ Poisson bracket of time dependent quantities and reduces to the Poisson bracket at equal times.

What is even more interesting is that even for non-zero Hamiltonians the canonical Peierls brackets at equal-times coincide with the canonical Poisson brackets. For unequal times the Peierls bracket is different from the two-point Poisson brackets and is characteristic of the governing Hamiltonian. We can see this for simple Hamiltonians like free particle and Harmonic oscillator or quadratic Hamiltonians. But a proof (or a counter example) of the fact that equal-time Peierls bracket is the same as Poisson bracket for a general Hamiltonian seems to be lacking.

2 Phase trajectories and observables

For simplicity of notation we consider just one coordinate and its momentum. The case of \( N \) degrees of freedom is similar.

The phase trajectories are curves \( \sigma : t \to (t, q = F(t), p = G(t)) \) from one dimensional time manifold \( T \) into the extended phase space \( \Gamma \) with coordinates \((t, q, p)\). The possible trajectories \( \sigma \) are solutions to

\[
\frac{dF}{dt} = \frac{\partial h}{\partial p}, \quad \frac{dG}{dt} = -\frac{\partial h}{\partial q},
\]

(1)

where \( h(t, q, p) \) is the Hamiltonian function.

The rate of change of an observable \( a(t, q, p) \) along a phase trajectory \( \sigma \) is determined by

\[
\frac{da}{dt} \bigg|_{\sigma} = \left( \frac{\partial a}{\partial t} + \{a, h\} \right) \bigg|_{q=F(t), p=G(t)}.
\]

The Poisson bracket \( \{a, b\} \) of two observables \( a(t, q, p) \) and \( b(t, q, p) \) is defined as

\[
\{a, b\} = \left( \frac{\partial a}{\partial q} \frac{\partial b}{\partial p} - \frac{\partial a}{\partial p} \frac{\partial b}{\partial q} \right).
\]

The Poisson bracket as a function of \( t, q, p \) refers to a single time \( t \). Its explicit time dependence, if any, comes from that of \( a \) and \( b \).

For time-dependent systems it is useful to define the value of an observable \( a(t, q, p) \) on a phase trajectory as the integral \( \int a(t, q, p) dt \). We assume that an observable is always multiplied by (‘smeared with’) an appropriate switching function of \( t \) for the integral to exist. If \( a(t, q, p) \) has a Dirac delta function \( \delta(t-t_a) \) as a factor, the value of \( a \) of \( a \) will reduce to the value of the observable at a specific time \( t_a \) on the trajectory.
3 Peierls bracket

The Peierls bracket was originally introduced for relativistic fields by Peierls\[1\] in 1952 and has been promoted extensively by B. S. DeWitt\[2\] as the fundamental covariant object in quantum field theory.

While the Poisson bracket between two observables \( a \) and \( b \) is defined on the whole phase space and is not dependent on the existence of a Hamiltonian, the Peierls bracket refers to a specific trajectory determined by a governing Hamiltonian. The Peierls bracket is related to the change in an observable when the trajectory on which it is evaluated gets shifted due to an infinitesimal change in the Hamiltonian of the system by another observable.

Let the Hamiltonian \( h \) be deformed by a term \( \lambda b \) where \( \lambda \) is a small parameter and \( b(t, q, p) \) an observable. Due to this term the phase trajectory gets modified by amounts proportional to \( \lambda \).

We can compare the values \( \int a(t, q, p) dt \) of an observable \( a \) on the two trajectories \( \sigma_0 \) and \( \sigma_{\lambda b} \):

\[
\sigma_0 : t \to q = F_0(t), p = G_0(t), \\
\sigma_{\lambda b} : t \to q = F(t) = F_0(t) + \lambda F_b(t), \\
p = G(t) = G_0(t) + \lambda G_b(t), \\
\lim_{t \to -\infty} F(t) = F_0(t), \quad \lim_{t \to -\infty} G(t) = G_0(t)
\]

which agree in remote past and then evolve with the Hamiltonians \( h \) and \( h + \lambda b \) respectively. The change in \( \int a(t, q, p) dt \) as \( \lambda \to 0 \) is written

\[
D_b a \equiv \lim_{\lambda \to 0} \frac{1}{\lambda} \left[ \int_{\sigma_{\lambda b}} a dt - \int_{\sigma_0} a dt \right]. \quad (2)
\]

Similarly, we can find the change in \( b \) when \( a \) deforms the Hamiltonian. The Peierls bracket is given by

\[
[a, b] \equiv D_b a - D_a b \quad (3)
\]

For the Hamiltonian \( h + \lambda b \) the phase trajectories are

\[
\frac{dF}{dt} = \left. \left( \frac{\partial h}{\partial p} + \lambda \frac{\partial b}{\partial p} \right) \right|_{q=F(t), p=G(t)} \\
\frac{dG}{dt} = -\left. \left( \frac{\partial h}{\partial q} + \lambda \frac{\partial b}{\partial q} \right) \right|_{q=F(t), p=G(t)}
\]

There are similar equations for \( F_0, G_0 \) when \( \lambda = 0 \).

Now, neglecting \( O(\lambda^2) \),

\[
\left. \left( \frac{\partial h}{\partial p} + \lambda \frac{\partial b}{\partial p} \right) \right|_{F,G} = \left. \left( \frac{\partial h}{\partial p} + \lambda \frac{\partial b}{\partial p} \right) \right|_{F_0,G_0} + \lambda \left. \left( \frac{\partial^2 h}{\partial q \partial p} F_b + \frac{\partial^2 h}{\partial p^2} G_b \right) \right|_{F_0,G_0}.
\]
There is a similar expression for \( dG/dt \). Thus,

\[
\begin{align*}
\frac{dF_b}{dt} &= \left( \frac{\partial^2 h}{\partial q \partial p} F_b + \frac{\partial^2 h}{\partial p^2} G_b \right) \frac{\partial b}{\partial p}, \\
\frac{dG_b}{dt} &= -\left( \frac{\partial^2 h}{\partial q^2} F_b + \frac{\partial^2 h}{\partial p \partial q} G_b \right) \frac{\partial b}{\partial q}.
\end{align*}
\]

The Peierls bracket will depend in general on the governing Hamiltonian \( h \). To relate to the Poisson bracket, where \( b \) and \( a \) act as Hamiltonians to each other, consider the case \( h = 0 \). Then the original trajectories are \( q = F_0 = \text{constant}, p = G_0 = \text{constant} \) and we can write

\[
\begin{align*}
F_b(t) &= \int_{-\infty}^{t} \frac{\partial b}{\partial p}(t') dt \\
&= \int \theta(t - t') \frac{\partial b}{\partial p}(t') dt, \\
G_b(t) &= -\int_{-\infty}^{t} \frac{\partial b}{\partial q}(t') dt \\
&= -\int \theta(t - t') \frac{\partial b}{\partial q}(t') dt,
\end{align*}
\]

where causality is automatically taken into account by the step function \( \theta(t - t') = 1 \) for \( t > t' \) and \( \theta(t - t') = 0 \) for \( t < t' \). The values of \( q \) and \( p \) are fixed by the original trajectory \( q = F_0, p = G_0 \). The change in an observable \( a \) is

\[
D_{b,a} = \int \left( \frac{\partial a}{\partial q}(t) F_b(t) + \frac{\partial a}{\partial p}(t) G_b(t) \right)
\]

\[
= \int_{-\infty}^{t} dt \int_{-\infty}^{\infty} dt' \theta(t - t') \times \left[ \frac{\partial a}{\partial q}(t') \frac{\partial b}{\partial p}(t') - \frac{\partial a}{\partial p}(t') \frac{\partial b}{\partial q}(t') \right].
\]

The expression for \( D_{a,b} \) is similar with the roles of \( a \) and \( b \) interchanged. The two step functions add to unity \( \theta(t - t') + \theta(t' - t) = 1 \) on calculating \( D_{b,a} - D_{a,b} \). The Peierls bracket for the value of observables \( a \) and \( b \) is thus seen to be an integrated ‘two-point’ Poisson bracket with the time dependence at the individual times of the observables:

\[
[a, b] = \int \int dt dt' \{a(t), b(t')\}_{\sigma_0}
\]

where

\[
\{a(t), b(t')\}_{\sigma_0} = \frac{\partial a}{\partial q}(t) \frac{\partial b}{\partial p}(t') - \frac{\partial a}{\partial p}(t) \frac{\partial b}{\partial q}(t')
\]

evaluated at the original trajectory \( \sigma_0 \) with constant \( q = F_0, p = G_0 \).
If the observables $a$ and $b$ are localized in time,

$$a(t, q, p) = \delta(t - t_a)\alpha(t, q, p), \quad b(t, q, p) = \delta(t - t_b)\beta(t, q, p),$$

then the expression for the Peierls bracket is the same as Poisson bracket evaluated at $q = F_0, p = G_0$:

$$[a, b] = \{\alpha(t_a, q, p), \beta(t_b, q, p)\}.$$

For equal times it is just the Poisson bracket. In general (when $\hbar$ is not zero) the Peierls bracket will depend on the second derivatives of the Hamiltonian as seen by the equation 4 above for $F_b, G_b$. In particular, the $[q(t), q(t')]$ Peierls bracket will not be zero for different times and would be equal to the Poisson bracket only at equal time. Simple examples in the next section illustrate this.

4 Examples

As one example take $h = (1/2)(q^2 + p^2)$.

Let $b = qj$ where $j : t \to j(t)$ is a function ‘switching on’ the observable $q$ in some interval. If $t \to q = F(t), p = G(t)$ is the deformed trajectory, then from equation (4)

$$\dot{F}_b = G_b, \quad \dot{G}_b = -F_b - j,$$

and the solution for the deformation $F_b$ is

$$F_b(t) = -\int G_R(t - s)j(s)ds$$

where $G_R$ is the retarded Green’s function satisfying

$$\dot{G}_R(t) + G_R(t) = \delta(t).$$

$G_R(t)$ is equal to $\sin t$ for $t > 0$ and is 0 for $t < 0$. Therefore,

$$F_b(t) = -\int G_R(t - s)j(s)ds,$$

$$G_b(t) = -\int \partial_t G_R(t - s)j(s)ds.$$

If $a$ is the other observable $a = qk$ (with the switching function $k$) then

$$D_b a = -\int \int k(t)G_R(t - s)j(s)ds dt.$$

The Peierls bracket of $a$ and $b$ is

$$[a, b] = -\int \int [k(t)G_R(t - s)j(s)$$

$$+ j(t)G_R(t - s)k(s)]ds dt$$

$$= -\int \int k(t)[G_R(t - s) - G_A(t - s)]j(s)$$

$$= -\int \int k(t)\sin(t - s)j(s).$$
where $G_A(t) = G_R(-t)$ is the corresponding advanced Green’s function. If $k$ and $j$ were both Dirac delta functions supported at $t = t_a$ and $s = t_b$ then this Peierls bracket gives

$$[a, b] = -\sin(t_a - t_b),$$

which is the two-point bracket of $q$ with $q$, depending on phase space points of the original trajectory at times $t_a$ and $t_b$. It is zero at equal times. Physically, since both $a$ and $b$ are coordinates, the Peierls bracket measures how much of canonical momentum is generated in $q$ from time $t_a$ to $t_b$ along the oscillator trajectory if $t_b$ is later than $t_a$ or vice versa in the opposite case.

If $c = lp$ is another observable with $l(t)$ as the switching function then

$$G_c(t) = -\int G_R(t - s)l(s)ds,$$

$$F_c(t) = -\dot{G}_c(t) = \int \partial_t G_R(t - s)l(s)ds.$$

Thus the ‘canonical’ Peierls bracket between $a = qk$ and $c = pl$ is

$$[a, c] = \int \int k(t)\cos(t - s)l(s)dsdt$$

which gives $[q, p] = 1$ at equal times when $a$ and $c$ are localized in time. The Peierls bracket in this case measures how much of canonical momentum is retained in $p$ from time $t_a$ to $t_c$ along the oscillator trajectory if $t_c$ is later than $t_a$ or how much ‘coordinatelessness’ is retained in $q$ from time $t_c$ to $t_a$ if opposite is the case.

We can check that the equal time Peierls brackets remain unchanged and equal to Poisson brackets for some other simple Hamiltonians. For a free particle $h = p^2/2$, the trajectories are $G_0 = \text{constant}$, $F_0(t) = G_0 + F_1$ where $F_1$ is another constant. Let $b = qj$ as before. Then $F_b = G_b$ and $\dot{G}_b = -j$. The solution is

$$F_b(t) = -\int (t - t')\theta(t - t')j(t')dt'$$

$$G_b(t) = -\int \theta(t - t')j(t')dt'$$

which gives

$$[a, b] = -\int \int k(t)(t - t')j(t')dt'dt'$$

for $a = qk$, and

$$[c, b] = -\int \int l(t)j(t')dt'dt'$$

for $c = pl$. Again the equal-time canonical Peierls bracket is the same as the Poisson bracket.
It would be interesting to prove that the equal time canonical Peierls bracket is the same as the canonical Poisson bracket for any Hamiltonian. But a proof appears to be lacking.

References

[1] R. E. Peierls, Proc. Roy. Soc.(London), A214, 143(1952)

[2] B. S. DeWitt, in Relativity, Groups and Topology, C. DeWitt and B. DeWitt (eds.), Blackie and Son, London, 1964.
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