DISCRETE SIGNATURE
AND ITS APPLICATION TO FINANCE

TAKANORI ADACHI AND YUSUKE NARITOMI

Abstract. Signatures, one of the key concepts of rough path theory, have recently gained prominence as a means to find appropriate feature sets in machine learning systems.

In this paper, in order to compute signatures directly from discrete data without going through the transformation to continuous data, we introduced a discretized version of signatures, called “flat discrete signatures”. We showed that the flat discrete signatures can represent the quadratic variation that has a high relevance in financial applications. We also introduced the concept of “discrete signatures” that is a generalization of “flat discrete signatures”. This concept is defined to reflect the fact that data closer to the current time is more important than older data, and is expected to be applied to time series analysis.

As an application of discrete signatures, we took up a stock market related problem and succeeded in performing a good estimation with fewer data points than before.

1. Introduction

The signature, one of the key concepts of rough path theory, is recently considered as a means to find an appropriate feature set in machine learning systems [Chevyrev and Kormilitzin, 2016]. It may become a powerful tool when combining with traditional machine learning techniques such as deep learning. In this paper, we introduce a new concept called discrete signatures, and apply it to some financial problems.

In Section 2, we introduce a concept of flat discrete signatures that is a simple discretization of the traditional signatures defined in [Lyons et al., 2007], but with the head-tail transformation that is an enlargement method of the underlying alphabet set. We show that the head-tail transformation, just like the lead-lag transformation of streams, provides the quadratic variation of any component of the original process. This is important since the quadratic variation has a high relevance in financial applications. When applying flat discrete signatures to time-series analysis, we often encounter the necessity of treating data closer to the present time as more important than older data. In order to address this problem, we generalize flat discrete signatures to reflect the fact. The resulting version is called discrete signatures.

In Section 3, we will make a brief explanation about how we implement the signatures. Actually, an implementation of signatures was made by Patrick Kidger and Terry Lyons as a Python-usable library called Signatory workable with PyTorch, which is written in C++ [Kidger and Lyons, 2021]. We will present yet another, but a very simple implementation using Python by adopting discrete signatures.

In Section 4, as an example of applications of discrete signature to finance, we consider the problem of judging whether a given price-shares process is of the morning or of the afternoon session in Tokyo Stock Exchange. We make a logistic regression with components of discrete signatures as features or explanatory variables. Then we will see that our result is as good as the regression with the whole raw data set with much fewer data points.

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Throughout this paper, we fix the discrete time domain
\[(2.1)\]
\[T := \{t_0, t_1, t_2, \ldots \}\]
with
\[0 = t_0 < t_1 < \cdots < t_n < t_{n+1} < \cdots\]
and a discrete path \(X\) in \(\mathbb{R}^d\) for some fixed positive integer \(d\), which can be written like
\[(2.2)\]
\[T \xrightarrow{X} \mathbb{R}^d\]
\[\forall \quad t \xrightarrow{X_t} X_t = (X^1_t, \ldots, X^d_t).\]

**Definition 2.1.** [Word]
\[(2.3)\]
\[I := \{1, 2, \cdots, d\},\]
\[(2.4)\]
\[I^* := \bigcup_{k=0}^{\infty} I^k.\]

We call an element of \(I\) an **alphabet** and an element of \(I^*\) a **word** or a **multi-index**.

The unique element of \(I^0\) is denoted by \(\lambda\), which is the word with length 0, or the **empty word**.

The **concatenation** of two words \(u \in I^j\) and \(v \in I^k\), denoted by \(u \circ v\), is the word \(w \in I^{j+k}\) defined by for \(i \in \{1, 2, \cdots, j+k\}\),
\[(2.5)\]
\[w_i := \begin{cases} u_i & \text{if } 1 \leq i \leq j, \\ v_{i-j} & \text{if } j+1 \leq i \leq j+k, \end{cases}\]
where \(w_i\) stands for \(w(i)\).

We usually focus a finite subset of \(I^*\) such as
\[(2.6)\]
\[I^{\leq k} := \bigcup_{\ell=0}^{k} I^\ell.\]

First, we will see the traditional definition of (continuous) signatures.

**Definition 2.2.** [Signature [Lyons et al., 2007]]
Let \(\mathbb{R}_+\) be the continuous time domain starting from 0, \(I\) be an alphabet set and \(X : \mathbb{R}_+ \rightarrow \mathbb{R}^I\) be a path. Let \(a, b \in \mathbb{R}_+\) with \(a < b\).

(1) For \(w \in I^*\), \(S(\tilde{X})_{a,b}^w \in \mathbb{R}\) is defined inductively by
\[(2.7)\]
\[S(\tilde{X})_{a,b}^\lambda := 1,\]
\[(2.8)\]
\[S(\tilde{X})_{a,b}^{u \circ v} := \int_a^b S(\tilde{X})_{a,t}^u d\tilde{X}_{t}^i \quad \text{(for } i \in I),\]
where
\[(2.9)\]
\[d\tilde{X}_{t}^i := \dot{\tilde{X}}_{t}^i dt.\]

(2) The (traditional) **signature** of \(\tilde{X}\) over \([a, b]\) is a function \(S(\tilde{X})_{a,b} : I^* \rightarrow \mathbb{R}\) defined by
\[(2.10)\]
\[S(\tilde{X})_{a,b}(w) := S(\tilde{X})_{a,b}^w\]
for \(w \in I^*\).
Because we have a discrete path \( X : T \rightarrow \mathbb{R}^d \), we have to convert \( X \) to an appropriate continuous time path \( \tilde{X} \) before computing its signature.

One of the natural ways to accomplish this is an interpolation. If we adopt the linear interpolation to fill the values between \( t_m \) and \( t_n \), we have for \( t_n \leq t < t_{n+1} \)

\[
\tilde{X}_t^i := \frac{X_{t_m}^i(t_{n+1} - t) + X_{t_{n+1}}^i(t - t_n)}{t_{n+1} - t_n}.
\]

Then for \( t_n \leq t < t_{n+1} \),

\[
\dot{\tilde{X}}_t^i = \frac{X_{t_{n+1}}^i - X_{t_m}^i}{t_{n+1} - t_n}.
\]

Therefore, for \( m, n \in \mathbb{N} \) with \( m < n \), and \( i \in I \),

\[
S(\tilde{X})_{t_m,t_n}^{w \otimes i} = \int_{t_m}^{t_n} S(\tilde{X})_{t_m,t}^w \dot{X}_t^i dt = \sum_{\ell = m}^{n-1} \int_{t_{\ell+1}}^{t_{\ell+1}} S(\tilde{X})_{t_m,t}^w \dot{X}_t^i dt = \sum_{\ell = m}^{n-1} \frac{X_{t_{\ell+1}}^i - X_{t_{\ell}}^i}{t_{\ell+1} - t_{\ell}} S(\tilde{X})_{t_{\ell+1},t_{\ell}}^w.
\]

for some value \( \tilde{S}_\ell \) that satisfies

\[
\tilde{S}_\ell \in \{ S(\tilde{X})_{t_m,t}^w \mid t_\ell \leq s \leq t_{\ell+1} \}
\]

by the mean-value theorem.

Note that some of candidates of \( \tilde{S}_\ell \) are

\[
S(\tilde{X})_{t_m,t_{\ell+1}}^w, \quad \frac{1}{2} \left( S(\tilde{X})_{t_m,t_\ell}^w + S(\tilde{X})_{t_m,t_{\ell+1}}^w \right), \quad S(\tilde{X})_{t_{\ell+1},t_{\ell}}^w.
\]

**Definition 2.3.** For the alphabet set \( I \), we define the extended alphabet set \( \bar{I} \) by

\[
\bar{I} := I \times \{-, +\}.
\]

For an alphabet \( i \in I \), we call the extended alphabets \( i^- := (i, -) \in \bar{I} \) and \( i^+ := (i, +) \in \bar{I} \) the head and the tail of \( i \), respectively.

In the following definition, we will assign the first and the last candidates in \((2.14)\) to heads and tails.

**Definition 2.4.** [Flat Discrete Signature] Let \( I \) be an alphabet set, \( X : T \rightarrow \mathbb{R}^d \) be a discrete path, and \( m, n \in \mathbb{N} \) with \( m < n \).

1. For \( w \in \bar{I}^* \) and \( i \in I \), \( S(X)_{t_m,t_n}^{w \otimes i} \in \mathbb{R} \) is defined inductively by

\[
S(X)_{t_m,t_n}^{\emptyset} := 1,
\]

\[
S(X)_{t_m,t_n}^{w \otimes i^-} := \sum_{\ell = m}^{n-1} S(X)_{t_m,t_\ell}^{w}(X_{t_{\ell+1}}^i - X_{t_\ell}^i),
\]

\[
S(X)_{t_m,t_n}^{w \otimes i^+} := \sum_{\ell = m}^{n-1} S(X)_{t_m,t_{\ell+1}}^{w}(X_{t_{\ell+1}}^i - X_{t_\ell}^i).
\]

2. The flat discrete signature of \( X \) over \([t_m, t_n] \) is a function \( S(X)_{t_m,t_n} : \bar{I}^* \rightarrow \mathbb{R} \) defined by for \( w \in \bar{I}^* \),

\[
S(X)_{t_m,t_n}(w) := S(X)_{t_m,t_n}^w.
\]
Proposition 2.5. For $i, i_1, i_2, i_3 \in I$, $\ast \in \{-, +\}$ and $m, n \in \mathbb{N}$ with $m < n$,
\begin{align}
S(X)_{i,m,n}^* &= \sum_{m \leq t < n} (X_{i, t+1}^i - X_{i, t}^i) = X_{i, m}^i - X_{i, n}^i, \\
S(X)_{i,m,n}^{i_1 \otimes i_2} &= \sum_{m \leq t_1 < t_2 < n} (X_{i_{1}, t_{1}+1}^{i_{1}} - X_{i_{1}, t_{1}}^{i_{1}}) (X_{i_{2}, t_{2}+1}^{i_{2}} - X_{i_{2}, t_{2}}^{i_{2}}), \\
S(X)_{i,m,n}^{i_1 \otimes i_2^+} &= \sum_{m \leq t_1 < t_2 < n} (X_{i_{1}, t_{1}+1}^{i_{1}} - X_{i_{1}, t_{1}}^{i_{1}}) (X_{i_{2}^+, t_{2}+1}^{i_{2}^+} - X_{i_{2}^+, t_{2}}^{i_{2}^+}), \\
S(X)_{i,m,n}^{i_1 \otimes i_2 \otimes i_3} &= \sum_{m \leq t_1 < t_2 < t_3 < n} (X_{i_{1}, t_{1}+1}^{i_{1}} - X_{i_{1}, t_{1}}^{i_{1}}) (X_{i_{2}^+, t_{2}+1}^{i_{2}^+} - X_{i_{2}^+, t_{2}}^{i_{2}^+}) (X_{i_{3}, t_{3}+1}^{i_{3}} - X_{i_{3}, t_{3}}^{i_{3}}), \\
S(X)_{i,m,n}^{i_1 \otimes i_2 \otimes i_3^+} &= \sum_{m \leq t_1 < t_2 < t_3 < n} (X_{i_{1}, t_{1}+1}^{i_{1}} - X_{i_{1}, t_{1}}^{i_{1}}) (X_{i_{2}, t_{2}+1}^{i_{2}} - X_{i_{2}, t_{2}}^{i_{2}}) (X_{i_{3}^+, t_{3}+1}^{i_{3}^+} - X_{i_{3}^+, t_{3}}^{i_{3}^+}), \\
S(X)_{i,m,n}^{i_1 \otimes i_2 \otimes i_3^+} &= \sum_{m \leq t_1 < t_2 < t_3 < n} (X_{i_{1}, t_{1}+1}^{i_{1}} - X_{i_{1}, t_{1}}^{i_{1}}) (X_{i_{2}, t_{2}+1}^{i_{2}} - X_{i_{2}, t_{2}}^{i_{2}}) (X_{i_{3}, t_{3}+1}^{i_{3}} - X_{i_{3}, t_{3}}^{i_{3}}).
\end{align}

Proof. Straightforward. \hfill \blacksquare

You may notice the correspondence between $\{-, +\}$ and $\{<, \leq\}$ in the ranges of summations in Proposition 2.5.

Example 2.6. Suppose that we observed 2 dimensional data in Table 2.1 with $I = \{1, 2\}$.

| Table 2.1. Input data stream |
|-----------------------------|
| $t$ | 0 | 1 | 1.5 | 2.5 | 3 |
| $X^1$ | 1 | 3 | 2 | 5 | 8 |
| $X^2$ | 1 | 4 | 2 | 6 | |

We will fill the missing data in Table 2.1 with their latest values like the data in Table 2.2

| Table 2.2. Filled data stream |
|-----------------------------|
| $t$ | 0 | 1 | 1.5 | 2.5 | 3 |
| $X^1$ | 1 | 3 | 3 | 5 | 8 |
| $X^2$ | 1 | 4 | 2 | 2 | 6 |

Then, the initial segment of the signature $S(X)_{0,3}$ whose words length is less than or equal to 2, has the following values, where $\ast \in \{-, +\}$.
\begin{align}
S(X)_{0,3}^{\lambda} &= 1, \\
S(X)_{0,3}^{i_1} &= 7, \\
S(X)_{0,3}^{i_1 \otimes i_2} &= 16, \\
S(X)_{0,3}^{i_1 \otimes i_2^+} &= 5, \\
S(X)_{0,3}^{i_1 \otimes i_2 \otimes i_3} &= 33, \\
S(X)_{0,3}^{i_1 \otimes i_2 \otimes i_3^+} &= 12, \\
S(X)_{0,3}^{i_1 \otimes i_2 \otimes i_3^+} &= 30, \\
S(X)_{0,3}^{i_1 \otimes i_2 \otimes i_3^+} &= 27.
\end{align}

In [Gyurkó et al., 2014], the quadratic variation of any component of the original process $X$ is provided by introducing the lead-lag transformation of streams. Since the quadratic variation has a high relevance in financial applications, this result was crucial.
The following theorem shows that our **head-tail** transformation also provides a similar functionality.

**Theorem 2.7.** For \( i \in I, \ast \in \{-, +\} \) and \( m, n \in \mathbb{N} \) with \( m < n \),

\[
S(X)_{\ell_1, t_n}^{\ast \otimes i^+} - S(X)_{\ell_1, t_n}^{\ast \otimes i^-} = \sum_{m \leq \ell < n} (X_{\ell+1}^i - X_{\ell}^i)^2.
\]

**Proof.** By (2.21) and (2.22), we have

\[
S(X)_{\ell_1, t_n}^{\ast \otimes i^+} - S(X)_{\ell_1, t_n}^{\ast \otimes i^-} = \sum_{m \leq \ell_1 \leq \ell < n} (X_{t_{\ell+1}}^i - X_{t_{\ell}}^i) (X_{t_{\ell+1}}^i - X_{t_{\ell}}^i) - \sum_{m \leq \ell_1 \leq \ell < n} (X_{t_{\ell+1}}^i - X_{t_{\ell}}^i) (X_{t_{\ell+1}}^i - X_{t_{\ell}}^i) = \sum_{m \leq \ell_1 \leq \ell < n} (X_{t_{\ell+1}}^i - X_{t_{\ell}}^i)^2.
\]

When applying signatures to time-series analysis, we often encounter the necessity of treating data closer to the present time as more important than older data. Let us think to generalize flat discrete signatures to reflect the fact.

Now for \( m < n \), we can rewrite (2.18) as follows.

\[
S(X)_{t_m, t_n}^{w \otimes i^+} = \sum_{\ell = m}^{n-1} (X_{t_{\ell+1}}^i - X_{t_{\ell}}^i) S(X)_{t_m, t_{\ell+1}}^w = S(X)_{t_m, t_{n-1}}^{w \otimes i^+} + (X_{t_n}^i - X_{t_{n-1}}^i) S(X)_{t_m, t_n}^w.
\]

We can read (2.28) as “First \( S(X)_{t_m, t_{n-1}}^{w \otimes i^+} \) is computed at time \( t_{n-1} \), and then \( (t_n - t_{n-1}) \) later, \( S(X)_{t_m, t_n}^w \) and \( S(X)_{t_m, t_n}^{w \otimes i^+} \) are calculated using the (slightly outdated) \( S(X)_{t_m, t_{n-1}}^{w \otimes i^+} \).”

Similarly, we can rewrite (2.17) as follows.

\[
S(X)_{t_m, t_n}^{w \otimes i^-} = \sum_{\ell = m}^{n-1} (X_{t_{\ell+1}}^i - X_{t_{\ell}}^i) S(X)_{t_m, t_{\ell+1}}^w = S(X)_{t_m, t_{n-1}}^{w \otimes i^-} + (X_{t_n}^i - X_{t_{n-1}}^i) S(X)_{t_m, t_n}^w.
\]

This time, we can read (2.29) as “First \( S(X)_{t_m, t_{n-1}}^{w \otimes i^-} \) and \( S(X)_{t_m, t_{n-1}}^w \) are computed at time \( t_{n-1} \), and then \( (t_n - t_{n-1}) \) later, \( S(X)_{t_m, t_n}^{w \otimes i^-} \) is calculated using the (slightly outdated) \( S(X)_{t_m, t_{n-1}}^{w \otimes i^-} \) and \( S(X)_{t_m, t_{n-1}}^w \).”

In the following definition, a generalized version of flat discrete signatures is defined by calculating the outdated terms with a weight of 1 or less, taking into account the elapsed time.

**Definition 2.8.** [discrete Signature] Let \( I \) be an alphabet set, \( X : T \to \mathbb{R}^I \) be a discrete path, \( m, n \in \mathbb{N} \) with \( m < n \), and \( \mu \geq 0 \).

(1) For \( w \in \bar{I}^* \) and \( i \in I \), \( S^\mu(X)_{t_m, t_n}^w \in \mathbb{R} \) is defined inductively by

\[
S^\mu(X)_{t_m, t_n}^\lambda := 1,
\]

\[
S^\mu(X)_{t_m, t_n}^w := \begin{cases} 1 & \text{if } w = \lambda, \\ 0 & \text{otherwise}, \end{cases}
\]

\[
S^\mu(X)_{t_m, t_n}^{w \otimes i^-} := e^{-\mu(t_n - t_{n-1})} \left( S^\mu(X)_{t_m, t_{n-1}}^{w \otimes i^-} + (X_{t_n}^i - X_{t_{n-1}}^i) S^\mu(X)_{t_m, t_{n-1}}^w \right),
\]

\[
S^\mu(X)_{t_m, t_n}^{w \otimes i^+} := e^{-\mu(t_n - t_{n-1})} S^\mu(X)_{t_m, t_{n-1}}^{w \otimes i^+} + (X_{t_n}^i - X_{t_{n-1}}^i) S^\mu(X)_{t_m, t_{n-1}}^w.
\]

(2) The **discrete signature** of \( X \) with the decay rate \( \mu \) over \([t_m, t_n] \) is a function \( S^\mu(X)_{t_m, t_n} : \bar{I}^* \to \mathbb{R} \) defined by for \( w \in \bar{I}^* \),

\[
S^\mu(X)_{t_m, t_n}^w := S^\mu(X)_{t_m, t_n}^w.
\]
Note that $S^0(X)_{t_m,t_n} = S(X)_{t_m,t_n}$.

**Proposition 2.9.** For $i, i_1, i_2 \in I$, $m, n \in \mathbb{N}$ with $m < n$, and $\mu > 0$,

\[
S^\mu(X)_{t_m,t_n}^{-} = \sum_{m \leq t < n} e^{-\mu(t_{n-t})} (X^i_{t_{n-t}} - X^i_t),
\]

\[
S^\mu(X)_{t_m,t_n}^{+} = \sum_{m \leq t < n} e^{-\mu(t_{n-t}+1)} (X^i_{t_{n-t}+1} - X^i_t),
\]

\[
S^\mu(X)_{t_m,t_n}^{-\otimes i_1^{-}} = \sum_{m \leq t_1 < t_2 < n} e^{-\mu(t_{n-t_1})} (X^i_{t_{n-t_1}} - X^i_{t_{t_1}}) (X^{i_1}_{t_{n-t_1}+1} - X^{i_1}_{t_{t_1}+1}),
\]

\[
S^\mu(X)_{t_m,t_n}^{+\otimes i_1^{+}} = \sum_{m \leq t_1 < t_2 < n} e^{-\mu(t_{n-t_1}+1)} (X^i_{t_{n-t_1}+1} - X^i_{t_{t_1}}) (X^{i_1}_{t_{n-t_1}+1} - X^{i_1}_{t_{t_1}+1}),
\]

\[
S^\mu(X)_{t_m,t_n}^{+\otimes i_2^{+}} = \sum_{m \leq t_1 < t_2 < n} e^{-\mu(t_{n-t_1}+1)} (X^i_{t_{n-t_1}+1} - X^i_{t_{t_1}+1}) (X^{i_2}_{t_{n-t_1}+1} - X^{i_2}_{t_{t_1}+1}).
\]

**Proof.** By induction on $n$. \hfill \blacksquare

We have a similar result as Theorem 2.7 for discrete signatures, which tells that discrete signatures can represent “weighted” quadratic variations. Actually, the result is a generalization of Theorem 2.7.

**Theorem 2.10.** For $i \in I$, and $m, n \in \mathbb{N}$ with $m < n$,

\[
S^\mu(X)_{t_m,t_n}^{-\otimes i^{-}} - S^\mu(X)_{t_m,t_n}^{+\otimes i^{+}} = \sum_{m \leq t < n} e^{-\mu(t_{n-t})} (X^i_{t_{n-t}} - X^i_t)^2,
\]

\[
S^\mu(X)_{t_m,t_n}^{+\otimes i^{+}} - S^\mu(X)_{t_m,t_n}^{-\otimes i^{-}} = \sum_{m \leq t < n} e^{-\mu(t_{n-t+1})} (X^i_{t_{n-t+1}} - X^i_t)^2.
\]

**Proof.** The proof is exactly same as that of Theorem 2.7 by using Proposition 2.9. \hfill \blacksquare

**Example 2.11.** Using the same data in Example 2.6, the initial segment of the discrete signature $S^\mu(X)_{0,3}$ with the decay rate $\mu = \log 2 \approx 0.693$ (half-life = 1) whose words length is less than or equal to 2, has the following values.

\[
S^\mu(X)_{0,3}^0 = 1, \quad S^\mu(X)_{0,3}^1 = 3.08, \quad S^\mu(X)_{0,3}^2 = 4.91, \quad S^\mu(X)_{0,3}^3 = 2.70, \quad S^\mu(X)_{0,3}^4 = 4.04,
\]

\[
S^\mu(X)_{0,3}^5 = 3.37, \quad S^\mu(X)_{0,3}^6 = 11.65, \quad S^\mu(X)_{0,3}^7 = 3.33, \quad S^\mu(X)_{0,3}^8 = 12.56,
\]

\[
S^\mu(X)_{0,3}^{9} = 6.74, \quad S^\mu(X)_{0,3}^{10} = 19.57, \quad S^\mu(X)_{0,3}^{11} = 6.66, \quad S^\mu(X)_{0,3}^{12} = 20.16,
\]

\[
S^\mu(X)_{0,3}^{13} = -0.63, \quad S^\mu(X)_{0,3}^{14} = 8.61, \quad S^\mu(X)_{0,3}^{15} = -1.25, \quad S^\mu(X)_{0,3}^{16} = 12.19,
\]

\[
S^\mu(X)_{0,3}^{17} = 0.21, \quad S^\mu(X)_{0,3}^{18} = 13.71, \quad S^\mu(X)_{0,3}^{19} = -1.33, \quad S^\mu(X)_{0,3}^{20} = 18.34.
\]

### 3. An implementation of discrete signature

In this section, we will make a brief description about an implementation of discrete signature with Python [Beazley, 2022]. You can see the whole code `sig.py` and the data `sample1.dat` in Table 2.1 at https://github.com/takanori-adachi/discrete-signature.

Let us explain the functionality of classes in `sig.py` in the following subsections.
3.1. **The class Data.** Suppose we have a data stream like the following tab-separated records, which is corresponding to the data in Table 2.1:

| time | event type | value |
|------|------------|-------|
| 0.0  | 1          | 1.0   |
| 0.0  | 2          | 1.0   |
| 1.0  | 1          | 3.0   |
| 1.0  | 2          | 4.0   |
| 1.5  | 2          | 2.0   |
| 2.5  | 1          | 5.0   |
| 3.0  | 1          | 8.0   |
| 3.0  | 2          | 6.0   |

The class **Data** will perform the conversion from the above data stream to the filled data specified in Table 2.2. It reads the input stream (raw data) from a file and stores it into a list `self.raw_data`. Then, collects the elements of $I$ (the set of event types, `self.I`), $\bar{I}$ (the set of extended event types, `self.barI`) and $\mathcal{T}(\text{timedomain}, \text{self.T})$, converting them into the internal integer values, and preparing dictionaries for the conversions. It finally creates $I$-dimensional discrete path $X$, or `self.X`.

The method `w2mi` converts a word to a list of integers representing alphabets, or elements of $I$ containing in the word. Conversely, `mi2w` converts a list of integers to the corresponding word. The data member `t2i` is the dictionary converting from an actual time to its corresponding index.

3.2. **The class Words.** The class **Words** generates the set $I^\leq k$ as a list of its elements (words). The resulting list of elements of the set $I^\leq k$ is stored in the data member `self.Istar`.

In the flat case, i.e. when $\mu = 0$, we have

$$S(X)^{i-\otimes w} = S(X)^{i+\otimes w}$$

for $i \in I$ and $w \in \bar{I}$ by Proposition 2.5. Therefore, we can identify $i^-$ and $i^+$ for the first alphabet $i \in I$. So, we prepare a separate universe of words `self.IstarHalf` for the case $\mu = 0$.

3.3. **The class Signature.** A signature is initialized with a **Data** object `data` and the maximum length of words $k$. The class **Signature** encapsulate the heart of the computation of discrete signatures.

```python
class Signature(object): # discrete signature
def __init__(self, data, k):
    self.data = data
    self.k = k # maximum length of words

def sig(self, t1, t2, w):
    return (self.sig0(data.t2i[t1], data.t2i[t2], data.w2mi(w)))

def sig0(self, m, n, iss):
    v = 1.0
    if len(iss) > 0:
        w = iss[\-1]
        i = iss[len(iss)\-1]
        j, s = self.i2js(i)
        if s == 0:  # HEAD
            v = self.mu_delta_t[n\-1] * (self.sig0(m, n\-1, iss)
```

\[ T. \text{ADACHI, Y. \text{NARITOMI}} \]

\[
+ \text{data} \cdot \text{delta}_X[n-1,j] \ast \text{self} \cdot \text{sig0}(m, n-1, w) \]

\[
\text{else: \hspace{1em} \# \hspace{1em} TAIL} \\
\quad v = \text{self} \cdot \text{mu} \cdot \text{delta}_t[n-1] \ast \text{self} \cdot \text{sig0}(m, n-1, \text{iss}) \\
\quad + \text{data} \cdot \text{delta}_X[n-1,j] \ast \text{self} \cdot \text{sig0}(m, n, w) \\
\text{return}(v) \\
\]

where \( \text{mu} \cdot \text{delta}_t[n] \) is \( e^{-\mu(t_n+1-t_n)} \), and \( \text{delta}_X[n-1,j] \) is a data member defined in the class \( \text{Data} \) as \( X^i_{t_{n+1}} - X^i_{t_n} \). The function \( \text{i2js} \) converts a given index specifying an element of \( \bar{I} = I \times \{+, -\} \) to a pair \( (j, s) \) where \( j \in I \) and \( s = 0 \) if \( i = j^- \), and \( s = 1 \) if \( i = j^+ \).

The function \( \text{sig} \) simply calls another function \( \text{sig0} \) after converting its arguments to corresponding internal representations. The function \( \text{sig0} \) is a straightforward implementation of equations (2.30), (2.31), (2.32) and (2.33). Note that it uses the recursive call technique.

This simple implementation, however, is not so efficient. In fact, in the recursive call of the function \( \text{sig0} \), it repeats computations many times for the same arguments, which is simply a waste of time.

In order to avoid this extra computation, we will introduce a container object \( \text{Signature.v} \) for holding results of computation so far.

First, we introduce the container object \( \text{Words.v} \). It consists of binary and multinary tree structures. For each word \( w \in \bar{I}^\leq k \), we have a pair

\[
(3.2) \hspace{1em} c_w := (b_w, r_w),
\]

where \( r_w \) is the value of the signature at \( w \), and \( b_w \) is a boolean value that indicates whether \( r_w \) has been calculated or not. The intermediate container \( v_w \) is defined by the following recursive definition.

\[
\begin{align*}
\quad v_w & := (c_w, (v_{w \otimes i_1}, \ldots, v_{w \otimes i_{\bar{d}}})), \quad \text{(for } w \in \bar{I}^\leq(k-1)) \\
\quad v_w & := (c_w, ()), \quad \text{(for } w \in \bar{I}^k)
\end{align*}
\]

where \( \bar{d} \) is the cardinality of \( \bar{I} \) and \( \{i_1, \ldots, i_{\bar{d}}\} = \bar{I} \). Then, the container \( \text{Words.v} \) is defined by \( v_\lambda \).

Next, we construct a container \( \text{Signature.v} \) which is a double list of \( \text{Words.v} \). For each pair of time \( (t_m, t_n) \in T \) with \( m < n \). The function \( \text{Signature.get.c} \) retrieves \( c_w \) from \( v_{m,n} \) for the word \( w \) whose index is \( \text{iss} \). Using \( \text{Signature.v} \), the function \( \text{Signature.sig0} \) can be rewritten as:

\[
\begin{align*}
\text{def} \hspace{1em} \text{sig0}(\text{self}, m, n, \text{iss}) & : \# \text{\hspace{1em} faster \hspace{1em} algorithm \hspace{1em} using \hspace{1em} container \hspace{1em} self.v} \\
\hspace{2em} c & = \text{self} \cdot \text{get.c}(m, n, \text{iss}) \\
\hspace{2em} \text{if} \hspace{1em} c[0] & : \# \text{\hspace{1em} if \hspace{1em} already \hspace{1em} computed} \\
\hspace{4em} \text{return} \hspace{1em} c[1] & \# \text{\hspace{1em} return \hspace{1em} its \hspace{1em} value} \\
\hspace{4em} \# \text{\hspace{1em} otherwise, \hspace{1em} compute \hspace{1em} from \hspace{1em} scratch} \\
\hspace{4em} v & = 1.0 \\
\hspace{2em} \text{if} \hspace{1em} \text{len} (\text{iss}) & > 0: \\
\hspace{4em} w & = \text{iss}[:-1] \\
\hspace{4em} i & = \text{iss}[\text{len}(\text{iss})-1] \\
\hspace{4em} j, s & = \text{self} \cdot \text{i2js}(i) \\
\hspace{4em} \text{if} \hspace{1em} s == 0 & : \# \hspace{1em} \text{HEAD} \\
\hspace{6em} v & = \text{self} \cdot \text{mu} \cdot \text{delta}_t[n-1] \ast (\text{self} \cdot \text{sig0}(m, n-1, \text{iss}) \\
\hspace{7em} + \text{data} \cdot \text{delta}_X[n-1,j] \ast \text{self} \cdot \text{sig0}(m, n-1, w)) \\
\hspace{4em} \text{else: \hspace{1em} \# \hspace{1em} TAIL} \\
\hspace{6em} v & = \text{self} \cdot \text{mu} \cdot \text{delta}_t[n-1] \ast \text{self} \cdot \text{sig0}(m, n-1, \text{iss}) \\
\hspace{7em} + \text{data} \cdot \text{delta}_X[n-1,j] \ast \text{self} \cdot \text{sig0}(m, n, w) \\
\hspace{4em} c[0] & = \text{True} \# \text{\hspace{1em} it \hspace{1em} is \hspace{1em} computed}
\end{align*}
\]
We will use this faster version in Section [4].

4. AN APPLICATION OF DISCRETE SIGNATURE TO FINANCE

As an example of applications of discrete signature to finance, in this section, we consider the problem of judging whether a given price-shares process is of the morning or of the afternoon session in Tokyo Stock Exchange (TSE). TSE has morning (9:00-11:30) and afternoon (12:30-15:00) sessions each trading day. Therefore, each session has 2 hours and 30 minutes.

We use FLEX Full historical data bought from TSE as the raw data. FLEX Full data consists of high frequency tick data from which we can extract several micro dynamic data such as ita data or limit order book data. The time resolution of FLEX Full data is currently 1 microsecond, or $10^{-6}$ second. In the following, time is displayed in minutes. For example, "09:12:34.567890" is represented by the value $9 \times 60 + 12 + 34.567890/60 = 552.5761315$.

4.1. Make a one-minute interval data stream. We extract data stream

\[ \mathcal{D} = \{ D_t \} \]

from FLEX Full data, where $t$ is an observed time in minutes, and, each $D_t$ consists of the following five components:

- $D_t.P_a$ – best ask price,
- $D_t.P_b$ – best bid price,
- $D_t.S_a$ – the total of ask side shares,
- $D_t.S_b$ – the total of bid side shares,
- $D_t.V$ – accumulated execution volume.

We will generate a substream of $\{ D_t \}$ at one-minute interval for each trading session.

First, let us define index sets of one minute interval blocks from the original data by for $n \in \mathbb{N} := \{0, 1, 2, \cdots \}$,

\[
J_n := \{ t \mid n \leq t < n + 1 \text{ and } D_t \in \mathcal{D} \}, \\
\bar{J}_n := \{ t \mid n \leq t \leq n + 1 \text{ and } D_t \in \mathcal{D} \}. 
\]

Next, define pairs of times denoting open and close times of the session.

\[
(N_0, N_1) \in \{(9 \times 60, 11.5 \times 60), (12.5 \times 60, 15 \times 60)\}, \\
N := N_1 - N_0 = 150. 
\]

If $D_{t_{\max,J_{N_0}}} \cdot V = 0$, i.e. the security had not been open in the first minute of the session, we do not use the session as data and throw it away. By assuming $D_{t_{\max,J_{N_0}}} \cdot V > 0$, we pick $D_t$ for each $n = N_0, N_0 + 1, \cdots, N_1$, which is called $\bar{D}_n$, by the following procedure:

\[
\bar{D}_{N_0} := D_{\min,J_{N_0}} \\
\text{for } n \text{ in range}(N_0 + 1, N_1) : \\
\text{if } J_{n-1} = \emptyset : \bar{D}_n := \bar{D}_{n-1} \\
\text{else} : \bar{D}_n := D_{\max,J_{n-1}} \\
\text{if } \bar{J}_{N_{1-1}} = \emptyset : \bar{D}_{N_1} := \bar{D}_{N_1-1} \\
\text{else} : \bar{D}_{N_1} := D_{\max,J_{N_{1-1}}}
\]
Then, we got a one-minute interval data stream
\[(4.6) \quad \{\tilde{D}_n\}_{n=N_0, \ldots, N_1}\]
for each session.

4.2. **Time normalization.** Since our problem is to detect time-related information of the given data stream, we will eliminate clues by normalizing the time. The followings are normalized time and its corresponding components. For \(n = 0, 1, \ldots, N\),

\[(4.7) \quad t_n := \frac{n}{N},\]
\[(4.8) \quad P_{tn}^a := \tilde{D}_{N_0+n}P^a,\]
\[(4.9) \quad P_{tn}^b := \tilde{D}_{N_0+n}P^b,\]
\[(4.10) \quad S_{tn}^a := \tilde{D}_{N_0+n}S^a,\]
\[(4.11) \quad S_{tn}^b := \tilde{D}_{N_0+n}S^b,\]
\[(4.12) \quad V_{tn} := \tilde{D}_{N_0+n}V.\]

Then, our time domain is
\[(4.13) \quad \mathcal{T} := \{t_0, t_1, \ldots, t_N\}.\]

4.3. **Make a discrete path for each session.** We introduce some other statistics. For \(t \in \mathcal{T}\),

\[(4.14) \quad p_t := \ln \frac{P_{tn}^a + P_{tn}^b}{2}, \quad \text{(logarithm of mid-price)}\]
\[(4.15) \quad s_t := P_{tn}^a - P_{tn}^b. \quad \text{(spread)}\]

Next, we construct a discrete path
\[(4.16) \quad X := (X^1, X^2, X^3, X^4) : \mathcal{T} \to \mathbb{R}^I\]
with
\[(4.17) \quad I := \{1, 2, 3, 4\}\]
from which we will compute its discrete signature. For \(t \in \mathcal{T}\),

\[(4.18) \quad X^1_t := \frac{p_t - \langle p \rangle}{\sqrt{\langle p^2 \rangle - \langle p \rangle^2}}, \quad \text{(normalized logarithm of mid-price)}\]
\[(4.19) \quad X^2_t := \frac{s_t - \langle s \rangle}{\sqrt{\langle s^2 \rangle - \langle s \rangle^2}}, \quad \text{(normalized spread)}\]
\[(4.20) \quad X^3_t := \frac{S_{tn}^a - S_{tn}^b}{S_{tn}^a + S_{tn}^b}, \quad \text{(normalized imbalance)}\]
\[(4.21) \quad X^4_t := \frac{V_t}{V_1}, \quad \text{(normalized accumulated volume)}\]

where \(\langle x \rangle := \frac{1}{N} \sum_{t \in \mathcal{T}} x_t\) for any sequence \(\{x_t\}_{t \in \mathcal{T}}\).
4.4. **Experiment and Result**. In the experiment, we used data from January 2020 to July 2021 for 30 names in TOPIX CORE 30. After shuffling date, we use 80% of the whole data for training, and use 20% for test.

The calculated signature is used to determine the morning and afternoon sessions using logistic regression by which a binary decision was made, with 0 for the morning and 1 for the afternoon.

The set of event types or statistics is \( I \) defined in (4.17). We pick the seven sorts of feature sets as subsets of \( \bar{I} \leq k \), \( \{1\} \leq k \), \( \{2\} \leq k \), \( \{3\} \leq k \), \( \{4\} \leq k \), \( \bar{I} \leq k \) itself, \( \{2, 4\} \leq k \), and \( \{w \in \bar{I} \leq k | w \sim /[4^- 4^+]/\} \), where “\( w \sim /[4^- 4^+]/ \)” means “\( w \) matches the pattern \([4^- 4^+]\)”. In other words, it means “\( w \) contains the (extended) alphabets \( 4^- \) or \( 4^+ \)”. We check these patterns for \( k = 1, 2, 3 \).

Table 4.1 shows the accuracy of logistic regression adopting members of the feature set as its explanatory variables.

| Feature set | \( k = 1 \) | \( k = 2 \) | \( k = 3 \) | Number of features |
|-------------|-------------|-------------|-------------|-------------------|
| \( \{1\} \leq k \) | 50.72% | 55.46% | 55.91% | 1 3 7 |
| \( \{2\} \leq k \) | 72.51% | 75.54% | 83.08% | 1 3 7 |
| \( \{3\} \leq k \) | 55.04% | 58.96% | 59.14% | 1 3 7 |
| \( \{4\} \leq k \) | 90.18% | 93.01% | 97.46% | 1 3 7 |
| \( \bar{I} \leq k \) | 89.63% | 98.86% | 99.51% | 4 36 292 |
| \( \{2, 4\} \leq k \) | 89.58% | 98.84% | 99.82% | 2 10 42 |
| \( \{w \in \bar{I} \leq k | w \sim /[4^- 4^+]/\} \) | 90.18% | 97.84% | 99.55% | 1 15 163 |

The statistics “4” (normalized cumulative volume) apparently made the best performance, and the statistics “2” (normalized spread) is next. That is why we tried the \( \{2, 4\} \) case and the last case that treats only words containing “4”. You may see that the values of the sixth and the last cases are better than that of the whole set \( \bar{I} \leq 3 \) case at \( k = 3 \), while the number of features of the \( \{2, 4\} \) case and the last case are much less than the whole set case.

Let us mention the computation speed of obtaining the signature in Table 4.1. The workstation we used for the computation has 2 CPUs. Each CPU has 48 cores, and each core can handle 2 threads. So, the total number of threads is 192, which is the number of affordable distributed parallel processing. We used 150 threads out of 192 for our computation in order to avoid overwhelming the tasks of other users. The computation of all components of \( \bar{I} \leq 4 \) of the signature took 68 minutes and 33.266 seconds.

In order to evaluate the result in Table 4.1 fairly, we also performed logistic regression using the raw data as it is without using signature as a comparison. The result is shown in Table 4.2.

One of the most important points in the comparison is the number of features required to achieve good accuracy. For example, in \( k = 3 \) cases, the logistic regression using all raw 604 data points performs 99.64% accuracy while the logistic regression using 42 components of the discrete signature specified by \( \{2, 4\} \leq 3 \) performs 99.82% accuracy which is slightly better than the former case. In other words, the regression with the feature set specified by the signature can achieve almost the same level of good results as the regression with the whole raw data set with much fewer data points.
Table 4.2. Computation without Signature

| Statistics                             | Accuracy | Number of features |
|----------------------------------------|----------|--------------------|
| Normalized logarithm of mid-price      | 60.14%   | 151                |
| Normalized spread                      | 88.18%   | 151                |
| Normalized imbalance                   | 66.90%   | 151                |
| Normalized cumulative volume           | 99.73%   | 151                |
| All                                    | 99.64%   | 604                |

5. Concluding Remarks

We would like to leave a few remarks before finishing this paper.

The lead-lag transformation needs to double the cardinality $n$ of the time domain $\mathcal{T}$ while our head-tail transformation needs to double the cardinality $d$ of the alphabet set $I$. Then, the ratio of computation times of these two methods will be $\frac{d^2 n}{(2d) n} = \left(\frac{d}{2}\right)^n$. Therefore, the lead-lag transformation will take more time than ours when $d > 2$.

We used the pattern “[4−4+]” in Table 4.1 for specifying the subset of $\bar{I}^{\leq k}$. In general, a subset of $I^*$ is called a language in Mathematical Language Theory [Sipser, 2013]. There are some popular languages in this sense including regular languages and context-free languages. By modifying the class Words with the Python built-in library re, we can easily extend it to handle regular languages, i.e. languages generated by regular expressions. This gives us a more possibility to specify smaller and more appropriate feature sets instead of using whole $\bar{I}^{\leq k}$ whose cardinality is 292 when $k = 3$ in Section 4.

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Graduate School of Management, Tokyo Metropolitan University
Email address: Takanori Adachi <taka.adachi@tmu.ac.jp>

Graduate School of Management, Tokyo Metropolitan University
Email address: Yusuke Naritomi <naritomi-yusuke@ed.tmu.ac.jp>