On the largest two and smallest six distance Pareto eigenvalues of a graph.

Deepak Sarma
Department of Mathematical Sciences,
Tezpur University, Tezpur-784028, India.
Email address: deepaks@tezu.ernet.in

Abstract
In this article, we establish some bounds involving the largest two distance Pareto eigenvalues of a connected graph. Also we characterize all possible values for smallest six distance Pareto eigenvalues of a connected graph.

Keywords: Pareto eigenvalue, Distance matrix, spectral radius.

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1 Introduction
All our graphs are finite, undirected, connected and simple. Let $G$ be a graph on vertices $\{1,2,\ldots,n\}$. At times, we use $V(G)$ and $E(G)$ to denote the set of vertices and the set of edges of $G$, respectively. For $i,j \in V(G)$, the distance between $i$ and $j$, denoted by $d_G(i,j)$ or simply $d_{ij}$, is the length of a shortest path from $i$ to $j$ in $G$. The distance matrix of $G$, denoted by $D(G)$ is the $n \times n$ matrix with $(i,j)$-th entry $d_{ij}$.

For a column vector $x = (x_1,\ldots,x_n)^T \in \mathbb{R}^n$ we have

$$x^T D(G)x = \sum_{1 \leq i < j \leq n} d_{ij}x_ix_j. \quad (1.1)$$

If vertices $i$ and $j$ are adjacent, we write $i \sim j$. Degree of a vertex $v$ in a graph $G$ will be denoted by $d_G(v)$. By pendent vertex of a graph we mean a vertex of degree 1. The transmission, denoted by $Tr(v)$ of a vertex $v$ is the sum of the distances from $v$ to all other vertices in $G$. The diameter of a connected graph $G$ denoted by $diam(G)$ is the maximum distance between any two vertices in $G$, i.e. $diam(G)$ is the largest entry of $D(G)$. A clique of a graph is a maximal complete subgraph and clique number of a graph is the order of a maximal clique. We denote clique number of a graph $G$ by $\omega(G)$. By $K_n$, $P_n$, $C_n$, $S_n$, and $W_n$ we respectively mean the Complete graph, Path graph, Cycle
graph, Star graph and Wheel graph with \( n \) vertices. The complete bipartite graph with bipartition size \( m \) and \( n \) is represented by \( K_{m,n} \). By \( S_n^+ \) we represent the graph obtained by adding an edge between any two independent vertices in \( S_n \). If \( V_1 \subseteq V(G) \) and \( E_1 \subseteq E(G) \), then by \( G - V_1 \) and \( G - E_1 \) we mean the graphs obtained from \( G \) by deleting the vertices in \( V_1 \) and the edges \( E_1 \) respectively. In particular case when \( V_1 = \{ u \} \) or \( E_1 = \{ e \} \), we simply write \( G - V_1 \) by \( G - u \) and \( G - E_1 \) by \( G - e \) respectively. By \( K_n - e \) is the graph obtained from \( K_n \) by removing any one edge of it. The graph obtained from \( G \) and \( H \) by identifying \( u \in G \) and \( v \in H \) is denoted by \( G_u \ast H_v \). When there is no confusion of vertices we write \( G \ast H \) for the coalescence of the graphs \( G \) and \( H \).

By spectral radius of a symmetric matrix \( M \), we mean its largest eigenvalue and denote it by \( \rho(M) \). Note that for a connected graph \( G \), \( D(G) \) is irreducible nonnegative matrix. Thus by the Perron-Frobenius Theorem, \( \rho(D) \) is simple, and there is a positive eigenvector of \( D(G) \) corresponding to \( \rho(D) \). Such eigenvectors corresponding to \( \rho(D) \) is called Perron vector of \( D(G) \). By an eigenvector we mean a unit eigenvector and by \( M_n \), we denote the class of all real matrices of order \( n \). We use the notation \( A \geq 0 \) to indicate that each component of the matrix \( A \) is nonnegative. Furthermore in places we write \( A \geq B \) to mean \( A - B \geq 0 \).

**Definition 1.1.** A real number \( \lambda \) is said to be a Pareto eigenvalue of \( A \in M_n \) if there exists a nonzero vector \( x(\geq 0) \in \mathbb{R}^n \) such that

\[
Ax \geq \lambda x \quad \text{and} \quad \lambda = \frac{x^T Ax}{x^T x},
\]

also we call \( x \) to be a Pareto eigenvector of \( A \) associated with Pareto eigenvalue \( \lambda \).

A Pareto eigenvalue of \( D(G) \) of a graph \( G \) will be called as distance Pareto eigenvalue of \( G \). Fernandes at.el. in [1] and Seeger at [6] studied the Pareto eigenvalues of adjacency matrix of a graph. Pareto eigenvalue of the distance matrix of a connected graph was first studied in [4]. In this article we study something more about distance Pareto eigenvalues.

This article is organized as follows. Some basic results of distance Pareto eigenvalues of a graph are discussed in **Section 2**. We establish some bounds of the difference and ratio of the largest two distance Pareto eigenvalues of a graph in **Section 3**. We characterize all possible smallest five distance Pareto eigenvalues of a connected graph in **Section 4**. Finally in **Section 5**, we find the possible values of sixth smallest distance Pareto eigenvalue of a connected graph with at least 5 vertices.

## 2 Preliminaries and basic results

For a square matrix \( A \), we use the symbol \( A(i) \) for the principal submatrix of \( A \) obtained by deleting \( i - \text{th} \) row and column of \( A \). In particular if \( D(G) \) be the distance matrix of a graph \( G \) then by \( D(i) \) we will denote the principal submatrix of \( D \) obtained by deleting row and column corresponding to vertex \( i \) of \( G \). By \( 1 \) we denote the column vector of all ones and by \( J \) the matrix of all ones of appropriate size.

For a matrix \( A \) we use \( \rho_k(A) \) and \( \mu_k(A) \) to denote the k-th largest and k-th smallest Pareto eigenvalue of \( A \). For a connected graph \( G \) we simply write \( \rho_k(G) \) and \( \mu_k(G) \) to mean \( \rho_k(D(G)) \) and \( \mu_k(D(G)) \) respectively.
Lemma 2.1. [Weyl’s Inequalities][2] Let $\lambda_i(M)$ denote the $i$-th largest eigenvalue of a real symmetric matrix $M$. If $A$ and $B$ are two real symmetric matrices of order $n$, then

$$\lambda_1(A) + \lambda_i(B) \geq \lambda_i(A + B) \geq \lambda_n(A) + \lambda_i(B) \quad \text{for} \quad i = 1, 2, \ldots, n.$$ 

Lemma 2.2. [3] If $A$ is an irreducible matrix and $A \geq B \geq 0$, $A \neq B$, then $\rho(A) > \rho(B)$.

Lemma 2.3. [2] If $A$ is a symmetric $n \times n$ matrix with $\lambda_1$ as the largest eigenvalue then for any normalized vector $x \in \mathbb{R}^n(x \neq 0)$,

$$x^tAx \leq \lambda_1.$$ 

The equality holds if and only if $x$ is an eigenvector of $A$ corresponding to the eigenvalue $\lambda_1$.

Putting $x = \frac{1}{\sqrt{n}} \mathbb{I}$ in Lemma 2.3 we get the following result as a Corollary.

Corollary 2.4. If $A$ is a symmetric $n \times n$ matrix, then

$$\rho(A) \geq \bar{R}, \quad (2.2)$$

where $\bar{R}$ is the average row sum of the matrix $A$. The equality in (2.2) holds if and only if all the row sums of $A$ are equal.

Theorem 2.5. [5] The scalar $\lambda \in \mathbb{R}$ is a Pareto eigenvalue of $A \in \mathbb{M}_n$ if and only if there exist a nonempty set $J \subset \{1, 2, \ldots, n\}$ and a vector $\xi \in \mathbb{R}^{|J|}$ such that

$$A^J\xi = \lambda \xi,$$

$$\xi_j > 0 \quad \forall j \in J.$$

$$\sum_{j \in J} a_{i,j} \xi_j \geq 0 \quad \forall i \notin J.$$ 

Furthermore, a Pareto eigenvector $x$ associated to $\lambda$ is constructed by setting

$$x_j = \begin{cases} 
\xi_j & \text{if } j \in J \\
0 & \text{otherwise}.
\end{cases}$$

From Theorem 2.5, we get the following result similar to that of [6, Theorem 1].

Theorem 2.6. [4] The distance Pareto eigenvalues of a connected graph $G$ are given by $\Pi(G) = \{\rho(A) : A \in M\}$, where $M$ is the class of all principal sub-matrices of $\mathcal{D}(G)$.

Lemma 2.7. [4] For any positive integer $n$, $\Pi(K_n) = \{0, 1, \ldots, n - 1\}$.

Lemma 2.8. [4] If $G$ is a connected graph with at least two vertices, then

$$\rho_2(G) = \max\{\rho(A) : A \in P\},$$

where $P = \{(\mathcal{D}(G))(v) : v \in V(G), d_v > 1\}$.
Definition 2.9. If $A$ and $B$ are two nonnegative matrices then we say that $A$ dominates $B$ if either of the following two cases hold

1. $A$ and $B$ are of same size and upto permutation similarity $A \geq B$, $A \neq B$.

2. $A$ is permutation similar to $\begin{bmatrix} B & C \\ D & E \end{bmatrix}$ and at least one of $C, D$ and $E$ is a nonzero matrix.

Lemma 2.10. [4] If $A$ and $B$ are two symmetric nonnegative irreducible matrices, then $A$ dominates $B$ implies $\rho(A) > \rho(B)$.

Lemma 2.11. [4] There are $2(n-1)$ distance Pareto eigenvalues of $S_n$ and they are

$$\mu_{2k} = 2(k-1), \mu_{2k-1} = k - 1 + \sqrt{k^2 - 3k + 3}$$

where $k = 1, \ldots, n-1$.

Theorem 2.12. $\rho_1(S_n) - \rho_2(S_n)$ is a decreasing function in $n$.

Proof. Let

$$f(n) = \rho_1(S_n) - \rho_2(S_n) = \sqrt{n^2 - 3n + 3} - n + 2.$$ [Using Lemma 2.11]

Then

$$f'(n) = \frac{n - \frac{3}{2} - \sqrt{(n - \frac{3}{2})^2 + \frac{3}{4}}}{\sqrt{n^2 - 3n + 3}} < 0 \quad \forall n \in \mathbb{N}$$

This completes the proof. \hfill \blacksquare

Theorem 2.13. $\rho_2(S_n^+) = \frac{2n-7+\sqrt{(2n-1)^2-16}}{2}$

Proof. Let $A = D(S_n^+)(v)$, where $v$ is the vertex of $(S_n^+)$ of degree $n-1$. Then upto permutation similarity, we get

$$A = \begin{pmatrix} 2(I - I_{n-3}) & 2I_{n-3,2} \\ 2I_{2,n-3} & I_2 \end{pmatrix}$$

Let $\mathbf{x}$ be the perron vector of $A$. Then due to symmetry we can assume that all the components of $\mathbf{x}$ corresponding to pendent vertices equals $a$ and that of the vertex of degree $n-1$ is $b$. If $\rho$ is the largest eigenvalue of $A$, then from eigen equations we have

$$2(n-4)a + 2 \times 2b = \rho a \quad (2.3)$$

$$2(n-3)a + b = \rho b \quad (2.4)$$
Solving (2.3) and (2.4) we get
\[ \rho^2 - (2n - 7)\rho + 16 - 6n = 0 \]
\[ \therefore \rho = \frac{2n - 7 + \sqrt{(2n - 5)(2n + 3)}}{2} \]

Upto permutation similarity there are exactly 3 principal submatrices of \( D(S_n^+) \) of order \( n - 1 \). But \( A \) dominates the other two, hence the result follows from Lemma 2.8.

By direct verification we get the following result regarding the second largest distance pareto eigenvalue among all unicyclic graphs of order at most six.

**Theorem 2.14.** Among all unicyclic graph of order \( n \leq 6 \), \( \rho_2 \) is minimum for \( C_n \) and second minimum for \( S_n^+ \).

**Conjecture 1.** Among all unicyclic graph \( G \) of order \( n \geq 7 \), \( \rho_2 \geq \frac{2n - 7 + \sqrt{(2n - 1)^2 - 16}}{2} \) with equality if and only if \( G \cong S_n^+ \).

**Theorem 2.15.** \( \rho_2(W_n) = 2(n - 3) \).

**Proof.** Observe that upto permutation similarity there are only two distinct submatrix of \( D(W_n) \) of order \( n - 1 \). Let \( A \) be the principal submatrix of \( D(W_n) \) of order \( n - 1 \) obtained by removing row and column corresponding to vertex of degree \( n - 1 \) and \( B \) be any other principal submatrix of \( D(W_n) \) of order \( n - 1 \). Then \( B \) has constant row sum equal to \( 2 + 2(n - 4) = 2(n - 3) \). Therefore \( \rho(B) = 2(n - 3) \). Since \( B \) dominates \( A \), hence the result follows.

**Theorem 2.16.** If \( k \) be the minimum positive component of the distance Pareto eigenvector of a graph \( G \) corresponding to \( \rho_2 \), then for any real \( t > 0 \)
\[ \rho_1 \geq \frac{\rho_2 + 2tk(n - 1)}{1 + t^2} \]
with equality if and only if \( t = \frac{\sqrt{n - 1}}{\rho_1} = \frac{\rho_1 - \rho_2}{\sqrt{n - 1}} \).

**Proof.** Let \( D(G) = \begin{pmatrix} A & y \\ y^T & 0 \end{pmatrix} \) so that \( \rho_2(G) = \rho(A) \) and \( z \) be the normalized vector with \( z^TAz = \rho_2 \). For \( t > 0 \), we set \( x = \begin{pmatrix} z \\ t \end{pmatrix} \). Then we have
\[ \rho_1 \geq \frac{x^TDx}{x^Tx} \]
\[ = \frac{z^TAz + 2tz^Ty}{1 + t^2} \]
\[ = \frac{\rho_2 + 2tz^Ty}{1 + t^2} \]
\[ \geq \frac{\rho_2 + 2tk \text{trace}(v)}{1 + t^2} \]
\[ \geq \frac{\rho_2 + 2tk(n - 1)}{1 + t^2} \]
Thus the first part is done. Now if the equality holds then equality (2.7) gives

$$\text{trace}(v) = n - 1 \Rightarrow y = I.$$  \hspace{1cm} (2.8)

Equality (2.6) gives

$$z = \frac{I}{\sqrt{n - 1}}.$$  \hspace{1cm} (2.9)

Again equality in (2.5) gives

$$\mathcal{D}x = \rho_1 x.$$  \hspace{1cm} (2.10)

Using (2.8)–(2.10) we get

$$\frac{\rho_1 - \rho_2}{\sqrt{n - 1}} = t I \text{ and } \frac{n - 1}{\sqrt{n - 1}} = \rho_1 t.$$  

Therefore

$$t = \frac{\rho_1 - \rho_2}{\rho_1 \sqrt{n - 1}}.$$  

\[\blacksquare\]

\textbf{Definition 2.17.} We define by $\mathcal{G}_n$ the class of all connected graphs $g$ of order $n$ so that if $\rho_2(g) = \rho(A)$ where $A \in \mathcal{M}_{n-1}$ is a principal submatrix of $\mathcal{D}(g)$ then $A$ has all row (column) sums equal.

\textbf{Note:} A graph $g \in \mathcal{G}_n$ if and only if all non zero components of distance Pareto eigenvector of $g$ corresponding to $\rho_2$ are $\frac{1}{\sqrt{n - 1}}$.

\textbf{Theorem 2.18.} If $k$ be the minimum positive component of the distance Pareto eigenvector of a graph $G$ of order $n$ corresponding to $\rho_2(G)$, then

$$\rho_1 \geq \frac{\rho_2 + \sqrt{\rho_2^2 + 4(n - 1)(2k\sqrt{n - 1} - 1)}}{2}$$

equality holds if and only if $G \in \mathcal{G}_n$.

\textbf{Proof.} Taking $t = \frac{\sqrt{n - 1}}{\rho_1}$ in Theorem 2.16 we get

$$\rho_1 \geq \rho_2 + \frac{2\sqrt{\frac{n - 1}{\rho_1}}k(n - 1)}{1 + \frac{n - 1}{\rho_1}}$$

i.e. $$\rho_1 \geq \frac{\rho_2 + \sqrt{\rho_2^2 + 4(n - 1)(2k\sqrt{n - 1} - 1)}}{2}$$

Now from Theorem 2.16 equality holds in the above expression if and only if $k = \frac{1}{\sqrt{n - 1}}$, i.e. if and only if $G \in \mathcal{G}_n$. \[\blacksquare\]

\textbf{Theorem 2.19.} If $k$ is the minimum positive component of the distance Pareto eigenvector of a graph $G$ corresponding to $\rho_2(G)$, then

$$2\rho_1 - \rho_2 \geq 2k(n - 1),$$

equality holds if and only if $G = K_2$. 

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**Proof.** Taking \( t = 1 \) in Theorem 2.16 we get
\[
\rho_1 \geq \frac{\rho_2 + 2k(n-1)}{2}
\]
i.e. \( 2\rho_1 - \rho_2 \geq 2k(n-1) \).

Now by Theorem 2.16 equality holds if and only if
\[
1 = \frac{\sqrt{n-1}}{\rho_1} = \frac{\rho_1 - \rho_2}{\sqrt{n-1}}
\]
i.e. \( \rho_1 = \sqrt{n-1} \) and \( \rho_2 = 0 \)

which is possible only if \( G = K_2 \).  

**Definition 2.20.** A vertex \( v \) of a connected graph \( G \) of order \( n \) is called pyramidal if \( d_v = n-1 \) and \( G-v \) is connected and regular. Besides we call a connected graph \( G \) to be pyramidal if there exist at least one pyramidal vertex in it.

**Theorem 2.21.** If \( G \) be a connected graph of order \( n \) with \( v \in V(G) \) such that \( Tr(v) \) is minimum, then
\[
\rho_1 \geq \frac{Tr(v) - 1 + \sqrt{(Tr(v) - 1)^2 + 4(n-1)}}{2}
\]
with equality if and only if \( G \) is pyramidal.

**Proof.** Let \( x = (x_1, x_2, \ldots, x_n) \) be the Perron-vector with \( x_i = \min x_k, x_j = \min \{x_k > x_i\} \). From eigenequations, we have
\[
\rho_1 x_i \geq T_i x_j \text{ and } \rho_1 x_j \geq x_i + (T_j - 1)x_j
\]
(2.11)

Now (2.11) gives
\[
\rho_1 (\rho_1 - T_j + 1) \geq T_i \Rightarrow \rho_1^2 - (T_j - 1)\rho_1 - T_i \geq 0
\]
\[
\Rightarrow \rho_1 \geq \frac{T_j - 1 + \sqrt{(T_j - 1)^2 + 4T_i}}{2} \geq \frac{T_j - 1 + \sqrt{(T_j - 1)^2 + 4(n-1)}}{2}
\]

Thus the first part is done.

Now if the equalities hold, then considering all the above equalities we get
\[
v_i \sim v_j, \ x_k = x_j \forall k \neq i \text{ and } d_i = n-1.
\]

Therefore
\[
\rho_1 x_j = x_i + (T_j - 1)x_j \Rightarrow T_k = T_j \forall k \neq i
\]
\[
\Rightarrow T_k = d_k + 2(n-d_k-1) \forall k \neq i = 2(n-1) - d_k
\]
\[
\Rightarrow d_k = d_j \forall i \neq k \text{ and } T_i = n-1.
\]

Hence the result follows.  

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3 Difference and ratio of largest two distance Pareto eigenvalues

From [4] we see that the second largest distance Pareto eigenvalues is always greater than the second largest distance eigenvalue, we now study the difference(ratio) of the largest two distance Pareto eigenvalues of a graph.

Theorem 3.1. For any positive integer \( n \), \( \rho_1(C_n) - \rho_2(C_n) < \frac{1}{n-1} \left\lfloor \frac{n^2}{4} \right\rfloor \).

Proof. Since all row(column) sum of \( D(C_n) \) are equal to \( \left\lfloor \frac{n^2}{4} \right\rfloor \), therefore

\[
\rho_1(C_n) = \left\lfloor \frac{n^2}{4} \right\rfloor. \tag{3.12}
\]

Now upto permutation similarity all the principal sub-matrix of \( D(C_n) \) of order \( n - 1 \) are equal and average row sum of any such matrix is

\[
\frac{n\left\lfloor \frac{n^2}{4} \right\rfloor - 2\left\lfloor \frac{n^2}{4} \right\rfloor}{n - 1} = \frac{n - 2}{n - 1} \left\lfloor \frac{n^2}{4} \right\rfloor = \left\lfloor \frac{n^2}{4} \right\rfloor - \frac{1}{n - 1} \left\lfloor \frac{n^2}{4} \right\rfloor.
\]

Besides all the row(column) sum of principal sub-matrix of \( D(C_n) \) of order \( n - 1 \) are not equal. Therefore using Corollary 2.4 we get

\[
\rho_2(C_n) > \left\lfloor \frac{n^2}{4} \right\rfloor - \frac{1}{n - 1} \left\lfloor \frac{n^2}{4} \right\rfloor. \tag{3.13}
\]

From (3.12) and (3.13) we have

\[
\rho_1(C_n) - \rho_2(C_n) < \left\lfloor \frac{n^2}{4} \right\rfloor - \left\lfloor \frac{n^2}{4} \right\rfloor + \frac{1}{n - 1} \left\lfloor \frac{n^2}{4} \right\rfloor = \frac{1}{n - 1} \left\lfloor \frac{n^2}{4} \right\rfloor.
\]

Conjecture 2. If \( G \) is a connected graph of order \( n \), then \( \rho_1(G) - \rho_2(G) \leq \rho_1(C_n) - \rho_2(C_n) \), equality holds if and only if \( G = C_n \).

Theorem 3.2. For any positive integer \( n \), \( \rho_1(S_n) - \rho_2(S_n) = \sqrt{n^2 - 3n + 3} - n + 2 \).

Proof. From Lemma 2.11 we have

\[
\rho_1(S_n) = n - 2 + \sqrt{n^2 - 3n + 3} \quad \text{and} \quad \rho_2(S_n) = 2(n - 2).
\]

Hence the result follows easily.
**Conjecture 3.** If $G$ is a connected graph of order $n$, then

$$\rho_1(G) - \rho_2(G) \geq \sqrt{n^2 - 3n + 3} - n + 2,$$

equality holds if and only if $G = S_n$.

**Conjecture 4.** Among all connected graphs of order $n$ the sum of $k$ largest distance Pareto eigenvalue is minimum for $S_n$ and maximum for $P_n$.

**Conjecture 5.** Among all bipartite graph the sum of $k$ largest distance Pareto eigenvalue is minimum for $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$

**Theorem 3.3.** For any connected graph $G$, $\rho_1(G) - \rho_2(G) < \min_{u \in V(G)} \sqrt{\sum_{v \in V(G)} d_{uv}^2}$.

**Proof.** Let

$$D(G) = \begin{pmatrix} 0 & x^T \\ x & E \end{pmatrix}, \quad M = \begin{pmatrix} 0 & 0 \\ 0 & E \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} 0 & x^T \\ x & 0 \end{pmatrix}.$$

Then clearly $D = M + N$, where $D = D(G)$. Now by Weyl’s inequality (2.1) we have

$$\rho(D) \leq \rho(M) + \rho(N), \quad (3.14)$$

equality holds if and only if there is a vector which is at a time eigenvector corresponding to $\rho(D)$ of $D$, $\rho(M)$ of $M$ and $\rho(N)$ of $N$.

Now we have

$$\rho(D) = \rho_1(G), \quad (3.15)$$
$$\rho(M) \leq \rho_2(G) \quad (3.16)$$
and $$\rho(N) = \sqrt{x^T x} \quad (3.17)$$

Also there is a particular $u \in V(G)$ such that

$$x_i = d_{ui} \quad (3.18)$$

On using (3.15) and (3.18), inequality (3.14) reduces to the form

$$\rho_1(G) \leq \rho_2(G) + \sqrt{\sum_{v \in V(G)} d_{uv}^2}. \quad (3.19)$$

Taking minimum of $\sum_{v \in V(G)} d_{uv}^2$ over all vertices of $G$ in (3.19), we get

$$\rho_1(G) - \rho_2(G) \leq \min_{u \in V(G)} \sqrt{\sum_{v \in V(G)} d_{uv}^2}. \quad (3.20)$$

If possible suppose the equality holds in (3.20). Then $\exists y \in \mathbb{C}^n$ such that
\[ D y = \rho(D) y \]  
\[ M y = \rho(M) y \]  
\[ N y = \rho(N) y \]

From (3.21) and (3.23) we have

\[ \sum_{v \in V(G)} d_{uv} y_v = \rho(D) y_u \]  
\[ \sum_{v \in V(G)} d_{uv} y_v = \rho(N) y_u \]

Equations (3.24) and (3.25) suggests that \( \rho(D) = \rho(N) \), which is a contradiction to the fact that \( D \) dominates \( N \). Therefore the equality in (3.20) can never hold. Hence the result follows. \[\Box\]

**Lemma 3.4.** If \( G \) be a connected graph and \( x \) be the normalized perron vector of \( D(G) \) then for any \( v \in V(G) \),

\[ \frac{\rho_1}{\rho_2} \leq \frac{1 - x_v^2}{1 - 2x_v^2} \]

equality holds if and only if \( x_u = \frac{d_{uv}}{\sqrt{(\rho_1 - \rho_2)(2\rho_1 - \rho_2)}} \) \( \forall u \neq v \)

**Proof.** Upto permutation similarity we can take \( D = \begin{pmatrix} A & y \end{pmatrix} \begin{pmatrix} y^T & 0 \end{pmatrix} \) and \( x = \begin{pmatrix} z \\ x_v \end{pmatrix} \), where \( D = D(G) \) and \( y_u = d_{uv} \).

Now \( Dx = \rho_1 x \) gives

\[ Az + x_v y = \rho_1 z \]  
\[ y^T z = \rho_1 x_v \]

Also as \( x \) is normalized vector so we have

\[ z^T z = 1 - x_v^2 \]

From (3.26)–(3.28), we get

\[ z^T Az = \rho_1 (1 - 2x_v^2) \]

But from Lemma 2.3, we have

\[ \frac{z^T Az}{z^T z} \leq \rho_2 \]

Equations (3.29) and (3.30) together gives

\[ \frac{\rho_1}{\rho_2} \leq \frac{1 - x_v^2}{1 - 2x_v^2} \]
Thus the first part is done.

Now suppose the equality holds in (3.31), then equality must hold in (3.30) as well. Therefore we have

\[ Az = \rho_2 z. \]

Using equation (3.26) we get

\[ z = \frac{x_v}{\rho_1 - \rho_2} y \]

But equality in (3.31) gives

\[ x_v = \sqrt{\frac{\rho_1 - \rho_2}{2\rho_1 - \rho_2}} \]

Therefore equation (3.32) reduces to

\[ z = \frac{y}{\sqrt{(\rho_1 - \rho_2)(2\rho_1 - \rho_2)}} \]

which is again equivalent to the form

\[ x_u = \frac{d_{uv}}{\sqrt{(\rho_1 - \rho_2)(2\rho_1 - \rho_2)}} \quad \forall u \neq v \]

Conversely if we assume \( x_u = \frac{d_{uv}}{\sqrt{(\rho_1 - \rho_2)(2\rho_1 - \rho_2)}} \) \( \forall u \neq v \), then from equations (3.27) and (3.28), we get

\[ \sum_{u \in V(G)} d_{uv}^2 = \rho_1 x_v \] \( (3.33) \)

\[ \sum_{u \in V(G)} d_{uv}^2 = 1 - x_v^2 \] \( (3.34) \)

Simplifying equations (3.33) and (3.34), we get \( \frac{\rho_1}{\rho_2} = \frac{1-x_v^2}{1-2x_v^2} \)

Hence our proof is complete.

\[ \square \]

**Theorem 3.5.** If \( G \) is a connected graph of order \( n \) then \( \frac{\rho_1}{\rho_2} \leq \frac{n-1}{n-2} \), equality holds if and only if \( G = K_n \).

**Proof.** If \( x \) be the normalised perron vector of \( D(G) \) with \( x_v = \min_{i \in V(G)} x_i \), then

\[ x_v \leq \frac{1}{\sqrt{n}} \] \( (3.35) \)
Therefore by Lemma 3.4 we have

\[
\frac{\rho_1}{\rho_2} \leq \frac{1 - x_v^2}{1 - 2x_v^2} \leq \frac{n - 1}{n - 2}
\] (3.36)

Now equality in (3.35) holds if and only if \( x = \frac{1}{\sqrt{n}} \). Also by Lemma 3.4 equality in (3.36) holds if and only if

\[
x_u = \frac{d_{uv}}{\sqrt{(\rho_1 - \rho_2)(2\rho_1 - \rho_2)}} \quad \forall \ u \neq v
\]

Thus equality in (3.37) holds if and only if

\[
x_u = \frac{d_{uv}}{\sqrt{(\rho_1 - \rho_2)(2\rho_1 - \rho_2)}} = \frac{1}{\sqrt{n}} \quad \forall \ u \neq v
\]

i.e.

\[
d_{uv}^2 = \frac{(\rho_1 - \rho_2)(2\rho_1 - \rho_2)}{n} \quad \forall \ u \neq v
\]

But \( G \) is connected, therefore we must have \( d_{uv} = 1 \) \( \forall \ u \neq v \) and thus \( Tr(v) = n - 1 \). Again \( x_u = \frac{1}{\sqrt{n}} \) \( \forall \ u \neq v \) implies \( x_v = \frac{1}{\sqrt{n}} \). Therefore \( x = \frac{1}{\sqrt{n}} \). Thus \( G \) is transmission regular i.e. all the row sums of \( D(G) \) are equal. Hence \( Tr(v) = n - 1 \) implies \( G = K_n \).

Lemma 3.4 can also be expressed in slightly different form as follows.

**Lemma 3.6.** If \( G \) be a connected graph and \( x \) be the normalized perron vector of \( D(G) \) then for any \( v \in V(G) \),

\[
\rho_1 - \rho_2 \leq \frac{\rho_2 x_v^2}{1 - 2x_v^2}
\]

equality holds if and only if

\[
x_u = \frac{d_{uv}}{\sqrt{(\rho_1 - \rho_2)(2\rho_1 - \rho_2)}} \quad \forall \ u \neq v
\]

Using Lemma 3.6 and proceeding as in Theorem 3.5 the following result can easily be established.

**Theorem 3.7.** If \( G \) is a connected graph of order \( n \) then \( \rho_1 - \rho_2 \leq \frac{\rho_2}{n-2} \), with equality if and only if \( G = K_n \).

4 Smallest five distance Pareto eigenvalues

In this section, we provide all possible values of the smallest five distance Pareto eigenvalues of a connected graph.

**Theorem 4.1.** For any connected graph \( G \) with at least 3 vertices, 0, 1 and 2 are the smallest three distance Pareto eigenvalues of \( G \).
Proof. If $G$ is the complete graph, then the result follows from Lemma 2.7. Suppose $G$ is not complete. Then $0$ being the only $1 \times 1$ principal sub-matrix of $D(G)$ is the smallest Pareto eigenvalue of $D(G)$. Again as $G$ is connected, $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is a $2 \times 2$ principal sub-matrix of $D(G)$ and any other $2 \times 2$ sub-matrix of $D(G)$ dominates $A$. Hence $\mu_2(G) = 1$. Now as $G$ has at least $3$ vertices and $G$ is not complete, therefore $B = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$ is a $2 \times 2$ principal sub-matrix of $D(G)$ and any other $2 \times 2$ principal sub-matrix of $D(G)$ other than $A$ and $B$ dominates both of them. Further $\rho(B) = 2$ and any principal sub-matrix of $D(G)$ of order $3$ or higher has minimum row sum $2$ and hence spectral radius at least $2$. Therefore $\mu_3(G) = 2$.

**Theorem 4.2.** The fourth smallest distance Pareto eigenvalue of a connected non complete graph is $1 + \sqrt{3}$.

Proof. If $G$ is a connected non complete graph of order $n$, then $n \geq 3$ and therefore from Theorem 4.1 we see that $0, 1, 2$ are the smallest three distance Pareto eigenvalues of $G$. Now let $A_1 = J_3 - I_3$, $A_2 = D(P_3)$. Then

$$\rho(A_1) = 2 \quad \text{and} \quad \rho(A_2) = 1 + \sqrt{3}.$$

Now it can be observed from $\text{diam}(G) \geq 2$, therefore any principal sub-matrix of $D(G)$ of order $3$ or higher other than $A_1, A_2$ always dominates either $A_1$ or $A_2$. Besides any principal sub-matrix of $D(G)$ of order $4$ or higher dominating $A_1$ has minimum row sum $3$ and therefore has spectral radius at least $3$. Also if $\text{diam}(G) \geq 3$, then $3 \in \Pi(G)$. On the other hand $\rho(A_2) = 1 + \sqrt{3} < 3$.

Again as $\text{diam}(G) \geq 2$, $D(P_3) = A_2$ is always a principal sub-matrix of $D(G)$. Hence $\mu_4(G) = 1 + \sqrt{3}$.

**Note:** From Lemma 2.7 and Theorem 4.2, we observe that a connected graph $G$ is complete if and only if $1 + \sqrt{3} \notin \Pi(G)$. Also among all connected graphs of given order $n$, $K_n$ is the only graph with all integral distance Pareto eigenvalues.

**Theorem 4.3.** If $G$ is a non complete graph with at least $4$ vertices, then $\mu_5(G) \geq 3$. The equality holds if and only if $\omega(G) \geq 4$ or $\text{diam}(G) \geq 3$.

**Proof.** Let $A_0 = 3(J_2 - I_2)$, $A_1 = J_3 - I_3$, $A_2 = D(P_3)$, $A_3 = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}$,

$A_4 = 2(J_3 - I_3)$, $B_1 = J_4 - I_4$, $B_2 = D(K_4 - e)$, and $B_3 = D(C_4)$.

From Theorem 4.2 we have $\mu_4(G) = \rho(A_2)$.

Now as $\text{diam}(G) \geq 2$, any principal sub-matrix of $D(G)$ of order $3$ or higher other than $A_0, A_1, A_2, A_3, A_4, B_1, B_2, B_3$ dominates at least one of $A_i$ or $B_j$ for $i = 1, 2, 3, 4$ and $j = 1, 2, 3$. Besides any principal sub-matrix of $D(G)$ of order $4$ or higher dominating $A_1$ is either $B_1$ or it dominates $B_1$. Similarly any principal sub-matrix of $D(G)$ of order $4$ or higher dominating $A_2$ is either $B_2$ or it dominates $B_2$ and hence dominates $B_1$.

Again $\min\{\rho(A_0), \rho(A_3), \rho(A_4), \rho(B_1), \rho(B_2), \rho(B_3)\} = 3$, and equality occurs for $A_0$ and $B_1$. 

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Hence $\mu_5(G) \geq 3$. The equality holds if and only if $A_0$ or $B_1$ is a principal sub-matrix of $D(G)$ which is the case if and only if either $K_4$ or $P_4$ is a induced connected subgraph of $G$, i.e. if and only if $\omega(G) \geq 4$ or $\text{diam}(G) \geq 3$.

**Theorem 4.4.** If $G$ is a connected graph of order $n \geq 4$, diameter 2 and $\omega(G) \leq 3$, then

$$
\mu_5(G) = \begin{cases} 
\frac{1 + \sqrt{33}}{2} & \text{if } C_5 \text{ or } S_4^+ \text{ is an induced subgraph of } G, \\
\frac{3 + \sqrt{17}}{2} & \text{if neither } C_5 \text{ nor } S_4^+ \text{ is an induced subgraph of } G \text{ but } K_4 - e \text{ is,} \\
4 & \text{otherwise.}
\end{cases}
$$

**Proof.** If we take $A_0, A_1, A_2, A_3, B_1, B_2, B_3$ as in the Theorem 4.3, then from the proof of Theorem 4.3 it is clear that

$$
\mu_5(G) \geq \min\{\rho(A_3), \rho(A_4), \rho(B_2), \rho(B_3)\} \\
= \rho(A_3) \\
= \frac{1 + \sqrt{33}}{2},
$$

where equality holds if and only if $A_3$ is a principal sub-matrix of $D(G)$, i.e. if and only if $C_5$ or $S_4^+$ is an induced subgraph of $G$. Now if $\mu_5(G) < \frac{1 + \sqrt{33}}{2}$, then neither $C_5$ nor $S_4^+$ is an induced subgraph of $G$ and therefore

$$
\mu_5(G) \geq \min\{\rho(A_4), \rho(B_2), \rho(B_3)\} \\
= \rho(B_2) \\
= \frac{3 + \sqrt{17}}{2},
$$

where equality holds if and only if $B_2$ is a principal sub-matrix of $D(G)$, i.e. if and only if $K_4 - e$ is an induced subgraph of $G$.

Finally if $\mu_5(G) < \frac{3 + \sqrt{17}}{2}$, then none of $K_5, P_4, S_4^+, K_4 - e$ is an induced subgraph of $G$ and therefore either $C_4$ or $S_4$ must be an induced subgraph of $G$. Now if $C_4$ is an induced subgraph of $G$, then $B_3$ is a principal sub-matrix of $D(G)$ and in the other case $A_4$ is a principal sub-matrix of $D(G)$. In the either case, we have

$$
\rho(B_3) = \rho(A_4) = 4.
$$

Hence $\mu_5(G) = 4$.

**Corollary 4.5.** If $G$ is a non complete connected graph with at least 4 vertices, then

$$
3 \leq \mu_5(G) \leq 4.
$$

Furthermore, the left hand equality holds if and only if $\text{diam}(G) \geq 3$ or $\omega(G) \leq 4$ and the right hand equality holds if and only if $G$ does not have $K_4, P_4, C_5, K_4 - e$, and $S_4^+$ as induced subgraph.

**Corollary 4.6.** If $T$ is a tree with at least 4 vertices, then $\mu_5(T) = 4$ or $3$ according as $T$ is a star or not.
Proof. For any tree $T$, we have $\omega(T) = 2$. Therefore if $\text{diam}(T) \geq 3$, then by Theorem 4.3, $\mu_5(T) = 3$.

Now if $\text{diam}(T) = 2$, T must be a star and then by Theorem 4.4, $\mu_5(T) = 4$. □

Corollary 4.7. If $n \geq 4$, then

$$
\mu_5(C_n) = \begin{cases} 
4 & \text{if } n = 4, \\
\frac{1+\sqrt{33}}{2} & \text{if } n = 5, \\
3 & \text{otherwise}.
\end{cases}
$$

Corollary 4.8. For any positive integers $m, n$ with $m + n \geq 4$, $\mu_5(K_{m,n}) = 4$.

5 6th smallest distance Pareto eigenvalue

From Theorem 4.3 and Theorem 4.4, we see that for a connected graph $G$ possible values of $\mu_5(G)$ are 3, 4, $\frac{3+\sqrt{17}}{2}$ and $\frac{1+\sqrt{33}}{2}$. In this section we consider all those four cases and find all possible values of the sixth smallest distance Pareto eigenvalue of a connected graph.

Theorem 5.1. If $G$ is a connected graph with at least 5 vertices and $\mu_5(G) = 4$, then

$$
\mu_6(G) = \begin{cases} 
5 & \text{if } G = K_6, \\
2 + \sqrt{7} & \text{otherwise}.
\end{cases}
$$

Proof. First suppose that $G$ is a complete graph, then by Lemma 2.7, $G = K_6$ is the only complete graph with $\mu_5(G) = 4$ and in this case $\mu_6(G) = 5$.

Now if $G$ is not a complete graph then by Theorem 4.3 and Theorem 4.4, $\mu_5(G) = 4$ implies $\text{diam}(G) = 2$ and $\omega(G) \leq 3$.

So any principal sub-matrix of $D(G)$ of order at most 3 will have row (column) sum at most 4.

Therefore

$$
\mu_6(G) = \min_A \rho(A),
$$

where the minimum is over all principal sub-matrix of $D(G)$ of order 4 or higher with $\rho(A) > 4$.

Now as $\omega(G) \leq 3$, $J_k - I_k$ cannot be a principal sub-matrix of $D(G)$ for $k \geq 5$.

Claim: $S_4$ must be an induced subgraph of $G$.

$\mu_5(G) = 4$ implies that $G$ does not have $K_4$, $K_4 - e$, $P_4$ and $S_4^+$ as an induced subgraph. Therefore only possible induced connected subgraphs of order 4 are $C_4$ and $S_4$. If $S_4$ is an induced subgraph of $G$ then we are done.

Otherwise let $H = C_4$ be an induced subgraph of $G$. Since $G$ has at least 5 vertices, we can choose vertex $w \in V(G) - V(H)$. Again as $\text{diam}(G) = 2$, $w$ must be adjacent to at least two vertices of $H$. But as $S_4^+$ is not an induced subgraph of $G$, $w$ cannot be adjacent to
two adjacent vertices in $H$. Again for the same reason $w$ cannot be adjacent to more than 2 vertices in $H$. Hence $w$ must be adjacent to exactly two vertices in $H$ which are not adjacent in $H$. Thus $S_4$ must be an induced subgraph of $G$ and thereby the claim is established.

Now as $\text{diam}(G) = 2$ and $\omega(G) \leq 3$, it is obvious that any principal sub-matrix of $\mathcal{D}(G)$ of order 4 or higher other than $\mathcal{D}(S_4)$ always dominates $\mathcal{D}(S_4)$.

Hence $\mu_6(G) = \rho(\mathcal{D}(S_4)) = 2 + \sqrt{7}$. □

**Theorem 5.2.** If $G$ is a connected graph with at least 5 vertices and $\mu_5(G) = \frac{3+\sqrt{17}}{2}$, then $\mu_6(G) = 4$.

**Proof.** From Theorem 4.4, $\mu_5(G) = \frac{3+\sqrt{17}}{2}$ implies that $K_4, P_4, C_4$ and $S_4^+$ are not an induced subgraph of $G$ but $K_4 - e$ is. As $K_4$ is not induced subgraph of $G$ any principal sub-matrix of $\mathcal{D}(G)$ of order 5 or higher dominates $\mathcal{J}_5 - I_5$ and has spectral radius greater than 4. Besides $\text{diam}(G) = 2$ implies that spectral radius of any principal sub-matrix of $\mathcal{D}(G)$ of order 1 or 2 is at most 2.

Therefore

$$\mu_6(G) = \min_A \rho(A),$$

where the minimum is over all principal sub-matrix $A$ of $\mathcal{D}(G)$ of order 3 or higher with $\rho(A) > \frac{3+\sqrt{17}}{2}$.

Now let $H = K_4 - e$ is an induced subgraph of $G$ then as $G$ has at least 5 vertices, we can choose vertex $w$ of $G$ such that $w \notin V(H)$. Again $\text{diam}(G) = 2$ implies that $w$ must be adjacent to at least one vertex of $V(H)$.

If $w$ is adjacent to two vertices of $H$ both of degree 3, then $S_4$ is an induced subgraph of $G$ and therefore $M = 2(\mathcal{J}_3 - I_3)$ is a principal sub-matrix of $\mathcal{D}(G)$. Again if $w$ is adjacent to three vertices of $H$ of which two are of degree 2 and the third is of degree 3 then $C_4$ is an induced subgraph of $G$ and therefore $N = \mathcal{D}(C_4)$ is a principal sub-matrix of $\mathcal{D}(G)$.

Now as $G$ cannot have any of $S_4^+, K_4, P_4$ as an induced subgraph, therefore it can be easily observed that there cannot be any other possibilities for $w$.

But $\rho(M) = \rho(N) = 4$. Hence $\mu_6(G) = 4$. □

**Theorem 5.3.** If $G$ is a connected graph with at least 5 vertices and $\mu_5(G) = 3$, then

$$\mu_6(G) = \begin{cases} 
\frac{1+\sqrt{33}}{2} & \text{if } C_5 \text{ or } S_4^+ \text{ is an induced subgraph of } G, \\
\frac{3+\sqrt{17}}{2} & \text{if } C_5 \text{ and } S_4^+ \text{ are not induced subgraph of } G \text{ but } K_4 - e \text{ is}, \\
4 & \text{if } C_5, S_4^+, K_4 - e \text{ are not induced subgraph of } G \text{ but at least one of } K_5, C_6, C_4, S_4, P_5 \text{ is}, \\
\rho(\mathcal{D}(P_4)) & \text{otherwise.} 
\end{cases}$$

**Proof.** From Theorem 4.3, $\mu_5(G) = 3$ implies $\text{diam}(G) = 3$ or $\omega(G) \geq 4$. Proceeding as in the Theorem 5.2 we can show that $\mu_6(G) = \frac{1+\sqrt{33}}{2}$ if $C_5$ or $S_4^+$ is an induced subgraph of $G$ and $\mu_6(G) = \frac{3+\sqrt{17}}{2}$ if $C_5$ and $S_4^+$ are not induced subgraph of $G$ but $K_4 - e$ is.

Now suppose $C_5$, $K_4 - e$ and $S_4^+$ are not induced subgraph of $G$. Then as $\text{diam}(G) \geq 3$, or $\omega(G) \geq 4$ or both, therefore $K_4$ or $C_4$ or $S_4$ or $P_4$ must be induced subgraph of $G$. 16
Thus at least one of $\mathcal{D}(K_4), B_3, A_4, \mathcal{D}(P_4)$ must be a principal sub-matrix of $\mathcal{D}(G)$, where $A_4$ and $B_3$ are as defined in Theorem 4.3. But $\rho(\mathcal{D}(K_4)) = 3$ and so we can ignore it.

Also $\rho(A_4) = \rho(B_3) = 4 < \rho(\mathcal{D}(P_4))$. Besides if $\omega(G) \geq 5$ then $\mathcal{D}(K_5)$ is a principal sub-matrix of $\mathcal{D}(G)$ with $\rho(\mathcal{D}(K_5)) = 4$. Again $A_4$ is a principal sub-matrix of $\mathcal{D}(G)$ if $C_6$ is a induced subgraph of $G$. Also as $\rho(\mathcal{D}(K_5 - e)) > \rho(\mathcal{D}(P_4)) > 4$ and $\rho(\mathcal{D}(K_6)) = 5\rho(\mathcal{D}(P_4))$, so for any sub-matrix of $\mathcal{D}(G)$ of order 5 or higher other than $\mathcal{D}(K_5)$ spectral radius is more than $\rho(\mathcal{D}(P_4))$. Hence if at least one of $K_5, C_6, C_4, S_4, P_5$ is an induced subgraph of $G$, then $\mu_6(G) = 4$ and otherwise $\mu_6(G) = \rho(\mathcal{D}(P_4))$. ■

**Corollary 5.4.** If $T$ is a tree with $n \geq 5$ vertices, then

$$
\mu_6(T) = \begin{cases} 
2 + \sqrt{7} & \text{if } T = S_n, \\
4 & \text{otherwise.}
\end{cases}
$$

**Proof.** If $\text{diam}(T) = 2$, then $T$ must be a star, therefore from Corollary 4.6, $\mu_5(T) = 4$ and thus by Theorem 5.1 $\mu_6(T) = 2 + \sqrt{7}$.

Now if $\text{diam}(T) = 3$, then $T$ must have $S_4$ as induced subgraph as $n \geq 5$. Again if $\text{diam}(T) \geq 4$, then $T$ must have $P_4$ as induced subgraph. Thus in either case by Theorem 5.3 we have $\mu_6(T) = 4$. ■

**Corollary 5.5.** Among all trees with at least 5 vertices, 6th smallest distance Pareto eigenvalue is maximum for the star graph.

**Theorem 5.6.** If $G$ is a connected graph with at least 5 vertices and $\mu_5(G) = \frac{1+\sqrt{33}}{2}$, then

$$
\mu_6(G) = \begin{cases} 
\frac{3+\sqrt{37}}{2} & \text{if } G = C_5, \\
\gamma & \text{if } G = C_3 \ast C_3, \\
\frac{3+\sqrt{17}}{2} & \text{if } K_4 - e \text{ is an induced subgraph of } G, \\
4 & \text{otherwise.}
\end{cases}
$$

where $\gamma$ is the largest root of $x^3 - x^2 - 11x - 7 = 0$

**Proof.** As before, it can be easily shown that if $K_4 - e$ is an induced subgraph of $G$ then $\mu_6(G) = \frac{3+\sqrt{37}}{2}$.

Now suppose that $K_4 - e$ is not an induced subgraph of $G$. As $\mu_5(G) = \frac{1+\sqrt{33}}{2}$, from Theorem 4.3 and Theorem 4.4 $G$ does not have $P_4$ or $K_4$ as induced subgraph but has $C_5$ or $S_4^+$ as induced subgraph.

If $H = C_5$ is an induced subgraph of $G$ then for $n = 5$, $G = C_5$ and $\mu_6(C_5)$ is the spectral radius of any $4 \times 4$ sub-matrix of $\mathcal{D}(C_5)$ i.e. $\mu_6(G) = \frac{3+\sqrt{37}}{2}$. Again if $n \geq 6$, then as $\text{diam}(G) = 2$ and $K_4$ is not an induced subgraph of $G$, therefore any vertex $w \in V(G) - V(H)$ of $G$ must be adjacent to at least two non adjacent vertices of $H$. Thus at least one of $C_4$ and $S_4$ must be an induced subgraph of $G$. Hence $\mu_6(G) = 4$.

Again if $H = S_4^+$ is an induced subgraph of $G$, we take $w \in V(G) - V(H)$. Now as $\text{diam}(G) = 2$, $w$ must be adjacent to at least one vertex of $H$. Besides as $K_4 - e$ is not induced subgraph of $G$, $w$ cannot be adjacent to 3 or more vertices of $H$, also for the same reason $w$ cannot be adjacent to two vertices in the triangle in $H$. 17
If $w$ is adjacent to a single vertex $u \in H$, then $\text{diam}(G) = 2$ implies that $d_H(u) = 3$. Thus $S_4$ is an induced subgraph of $G$. Again if $w$ is adjacent to exactly two vertices $u, v \in H$, with \{\(d_H(u), d_H(v)\} = \{1, 2\}$ then $C_4$ is an induced subgraph of $G$. In either case we get $\mu_6(G) = 4$.

Now if $G$ does not have $C_4$ or $S_4$ as induced subgraph then we are left with only one possibility i.e. every $w \in V(G) - V(H)$ is adjacent to exactly two vertices $u, v \in H$, with \{\(d_H(u), d_H(v)\} = \{1, 3\}$. But in this situation we must have $|V(G) - V(H)| = 1$ as $K_4 - e$ is not an induced subgraph of $G$. Which implies that $G = C_3 * C_3$. It can be directly verified that $\mu_6(C_3 * C_3)$ is the spectral radius of the matrix

$$
\begin{pmatrix}
0 & 1 & 2 & 2 \\
1 & 0 & 1 & 1 \\
2 & 1 & 0 & 1 \\
2 & 1 & 1 & 0 \\
\end{pmatrix}
$$

Therefore $\mu_6(G)$ is the largest root of $x^3 - x^2 - 11x - 7 = 0$.

Combining all the above situations, we get our required result.

**Corollary 5.7.** If $n \geq 5$, then

$$
\mu_6(C_n) = \begin{cases}
\frac{3+\sqrt{37}}{2} & \text{if } n = 5, \\
\gamma & \text{if } n = 7, \\
4 & \text{otherwise.}
\end{cases}
$$

where $\gamma$ is the largest root of $x^3 - x^2 - 11x - 7 = 0$

**Corollary 5.8.** For any positive integers $m, n$ with $m + n \geq 5$, $\mu_6(K_{m,n}) = 2 + \sqrt{7}$.

**References**

[1] R.Fernandes, J.Judice, V.Trevisan, *Complementarity eigenvalue of graphs*, Linear Algebra Appl, 527 (2017), 216-231.

[2] R.A.Horn, C.R.Johnson, *Matrix Analysis*, 19th printing, Cambridge University Press, Cambridge, 2005.

[3] H.Mink. *Nonnegative Matrices*. 2nd Edition, A wiley - interscience publication, John wiley & sons, 1998

[4] M.Nath, D.Sarma. *On Pareto eigenvalue of distance matrix of graphs*, arXiv:1809.07707.

[5] A.Seeger, J.Vincente-Pérez, *On cardinality of Pareto spectra*, Electronics Journal pf linear Algebra, 22 (2011) 758-766.

[6] A.Seeger, *Complementarity eigenvalue analysis of connected graphs*, Linear Algebra Appl, 543 (2018) 205-225.