HFQPOs and discoseismic mode excitation in eccentric, relativistic discs. I. Hydrodynamical simulations

Janosz W. Dewberry,1⋆ Henrik N. Latter,1 Gordon I. Ogilvie,1 Sebastien Fromang2
1 DAMTP, University of Cambridge, CMS, Wilberforce Road, Cambridge, CB3 0WA, UK
2 Laboratoire AIM, CEA/DSM-CNRS-Université Paris 7, Irfu/Departement d’Astrophysique, CEA-Saclay, 91191 Gif-sur-Yvette, France

Accepted XXX. Received YYY; in original form ZZZ

ABSTRACT
High-frequency quasi-periodic oscillations (HFQPOs) observed in the emission of black hole X-ray binary systems promise insight into strongly curved spacetime. ‘Discoseismic’ modes with frequencies set by the intrinsic properties of the central black hole, in particular ‘trapped inertial waves’ (r-modes), offer an attractive explanation for HFQPOs. To produce an observable signature, however, such oscillations must be excited to sufficiently large amplitudes. Turbulence driven by the magnetorotational instability (MRI) does not appear to provide the necessary amplification, but r-modes may still be excited via interaction with accretion disc warps or eccentricities. We present global, hydrodynamic simulations of relativistic accretion discs, which demonstrate for the first time the excitation of trapped inertial waves by an imposed eccentricity in the flow. While the r-modes’ saturated state depends on the vertical boundary conditions used in our unstratified, cylindrical framework, their excitation is unambiguous in all runs with eccentricity \( e \gtrsim 0.005 \) near the ISCO. These simulations provide a proof of concept, demonstrating the robustness of the trapped inertial wave excitation mechanism in a non-magnetized context. We explore the competition between this excitation, and damping by MHD turbulence and radial inflow in a companion paper.

Key words: accretion, accretion discs – black hole physics – hydrodynamics – instabilities – waves – X-rays: binaries

1 INTRODUCTION

On time-scales of weeks to months, X-ray binaries thought to contain a black hole (hereafter BHBs) progress through distinct emission states during their outbursts. These emission states are distinguished in part by the relative dominance of a ‘thermal’ component with a characteristic temperature of \( \sim 1 \)keV, vis-a-vis a harder ‘power-law’ component at higher energies. During ‘superluminous’ or ‘very high’ emission states in which both of these emission components are strong, BHBs occasionally exhibit high-frequency quasi-periodic oscillations (HFQPOs). While subtle (quality factors \( Q \gtrsim 2 \)), infrequent, and poorly understood, HFQPOs excite considerable theoretical interest; their frequencies of \( \sim 50 - 450 \)Hz are comparable with the orbital and epicyclic frequencies expected in the very inner regions of a relativistic accretion disc, and so are insensitive to variations in luminosity, and scale inversely with estimates of black hole mass. These characteristics suggest that a robust model for HFQPOs may provide measurements of, in particular, black hole spin angular momentum (Remillard & McClintock 2006; Done et al. 2007; Belloni et al. 2012; Belloni & Motta 2016; Motta 2016).

Nearly every model for HFQPOs appeals to the effects of general relativity on orbital motion, specifically the horizontal and vertical epicyclic frequencies \( \kappa \) and \( \Omega_z \), both of which deviate from the orbital frequency \( \Omega \) near the black hole. The ‘relativistic precession model’ (RPM; Stella & Vietri 1998; Stella et al. 1999) takes perhaps the simplest approach, associating quasi-periodic oscillations with the orbital frequency and both the nodal and periastron precession frequencies \( \Omega - \Omega_z \) and \( \Omega - \kappa \) (e.g., Motta et al. 2014). The appearance of multiple HFQPOs with frequencies in near-integer ratios in some BHBs (in particular GRO J1655-40) has led to the development of models that appeal to resonances between either fluid ‘blobs’ orbiting at radii where \( \kappa \) and \( \Omega_z \) achieve the same commensurability (Kluźniak & Abramowicz 2001; Abramowicz & Kluźniak 2001), or the global acoustic oscillations of narrow, pressure-supported, non-Keplerian accretion tori (Rezzolla et al. 2003; Blaes et al. 2006; Horák 2008; Fragile et al. 2016). Amongst several problems, it is unclear how such oscillations achieve observable amplitudes (see discussion in Dewberry et al. 2019).

The model considered in this paper posits that HFQ-
POs may be identified with waves excited in relativistic thin, radially extended discs. As first recognized by Kato & Fukue (1980) and Okazaki et al. (1987), general relativistic effects close to a black hole modify the radial profile of $\kappa$ to form a trapping cavity for inertial oscillations (r-modes). These waves possess frequencies close to the maximum epicyclic frequency, and semi-analytic theory shows they are subject to excitation via a non-linear coupling with eccentricities and warps in the accretion flow, such deformations arising naturally from either the tidal influence of the secondary or the misalignment of the disc’s orbit and the black-hole spin angular momentum (Kato 2004, 2008; Ferreira & Ogilvie 2008). In this paper, we demonstrate the excitation of these waves in fully non-linear three-dimensional numerical simulations, and take some steps toward evaluating their ensuing saturated state. We focus here on the unmagnetized problem, and on eccentric (rather than warped) disks. This hydrodynamic work provides a point of comparison for MRI-turbulent simulations considered in a companion paper.

Run the with code RAMSES, our simulations utilize a pseudo-Newtonian Paczynski-Wiita potential and omit vertical gravity for simplicity. Material is permitted to flow through an inner boundary placed within the innermost stable circular orbit (ISCO), and thus our simulations contain a physical disk edge and a transonic plunging region. The plunging region offers less reflection to spiral density waves, suppressing their growth via the corotation instability considered by Lai & Tsang (2009), Fu & Lai (2011) and Fu & Lai (2013). Meanwhile, we impose a non-axisymmetric pressure gradient at the outer boundary that forces the inward propagation of eccentricity, and the formation of a twisted disc.

Even simulations with low values of eccentricity ($\lesssim 0.005$) reveal the excitation of trapped inertial waves. In the presence of reflecting vertical boundaries they take the form of global standing modes, while periodic vertical boundaries lead to a saturation involving spurious elevator flows. The trapped r-modes produce peaks in the power spectral density (PSD) in a narrow range encompassing the predicted frequency and radius. Through a non-linear coupling at larger amplitudes, the trapped inertial waves additionally excite secondary inertial-acoustic modes at nearly double their frequency. Meanwhile, the waves rearrange angular momentum locally and hence dynamically reshape the trapping region in which they are confined; as a consequence, their frequencies vary, possibly explaining the low quality factor of observed HFQPOs.

The paper takes the following structure. In Section 2 we provide theoretical background, while details of the numerical framework are discussed in Section 3. We present our simulations exhibiting r-mode excitation in Section 4, before concluding in Section 5. Finally, test simulations and a resolution study are discussed in the appendices.

## 2 THEORETICAL BACKGROUND

In this section, we provide an introduction to discoseismology and introduce the non-linear excitation mechanism responsible for r-mode amplification. Those already familiar with the subject may skip to Sections 3 and 4.

### 2.1 Discoseismic oscillations

Consider isothermal, hydrodynamic perturbations to a thin, purely rotating, barotropic fluid disc with angular velocity $\Omega = \Omega(r)$. Such linear disturbances are governed by the local dispersion relation (Okazaki et al. 1987):

$$k_r^2 = \left(\frac{\omega^2 - \kappa^2 (\omega^2 - m^2 \Omega^2)}{\hat{c}_s^2 r^2}\right).$$

Here $k_r$ is the radial wavenumber of the perturbation, $c_s$ is the isothermal sound speed, and $\hat{c} = \omega - m \Omega$ for $\omega$ the (complex) mode frequency and $m$ the azimuthal wavenumber. Meanwhile, $\kappa$ and $\Omega$ are once again the horizontal and vertical epicyclic frequencies. Finally, $n$ is a separation constant associated with a decomposition of the perturbations’ vertical structure in terms of Hermite polynomials, and gives the number of nodes in the vertical direction. In a cylindrical model excluding vertical gravity and density stratification, the perturbations’ vertical structure may be described by a vertical wavenumber $k_z$, and $m \Omega^2$ replaced by $k_z^2 r_s^2$.

Equation (1) is derived from a Newtonian analysis, but permits the approximate inclusion of relativistic effects on wave propagation through the characteristic frequencies $\kappa$ and $\Omega_r$. A common practice is to calculate these frequencies from a ‘pseudo-Newtonian’ Paczynski-Wiita potential given by Equation (4). In a centrifugally supported disc, the radial profile for the horizontal epicyclic frequency so calculated shares two important properties with that derived from a fully general relativistic treatment of orbital motion (Okazaki et al. 1987). First of all, both radial profiles for $k^2$ pass through zero, defining the innermost stable circular orbit within which fluid elements or particles will be unstable to horizontal displacements. Secondly, the combination of the asymptotic limit $k \propto r^{-3/2}$ at large $r$ with the existence of an ISCO implies that $k^2$ must achieve a maximum. This non-monotonic behavior, illustrated by the black dashed line in Fig. 1, is the source of a relativistic disc’s capability to support discrete modes of oscillation irrespective of radial boundary conditions.

Oscillatory, non-evanescent solutions require $k_r^2 > 0$. Equation (1) then predicts different families of modes. The simplest, vertically homogeneous $(n = 0)$ ‘inertial-acoustic’ modes (AKA density waves, but hereafter called f-modes) can propagate where $\hat{c}_s^2 > \omega^2$. Kato & Fukue (1980) posit that in a relativistic disc, axisymmetric inertial-acoustic waves with $m = 0$ and a frequency $\omega < \kappa$ could be trapped between the ISCO and the peak in epicyclic frequency, if the oscillations are reflected at the disc edge. This trapping is illustrated by the orange sinusoidal lines in Fig. 1. More generally, non-axisymmetric inertial-acoustic modes with $m \neq 0$ can propagate where $\omega > k + m \Omega$ or $\omega < m \Omega - k$ (see the red sinusoidal lines in Fig. 1).

Trapped f-modes play a role in a few different models for HFQPOs. They may be viscously overstable in some contexts (Chen & Tsam 1995; Afshordi & Paczynski 2003; Chan 2009; Ferreira 2010). Alternatively, non-axisymmetric inertial-acoustic modes may be excited via a transmission of wave energy across their corotation radii (where $\omega/m$ matches $\Omega$). This excitation mechanism appeals to the same principles underlying the Papaloizou-Pringle instability (Papaloizou & Pringle 1984); non-axisymmetric f-modes propagating within corotation can be seen as carrying negative
wave energy, and grow in amplitude if they transmit positive wave energy to the outer disc. In relativistic discs, a positive gradient in fluid ‘vortensity’ $\kappa^2/(2\Omega z)$ at the corotation radius facilitates such transmission. However, the energy exchange excites large-scale normal modes only if the inner boundary reflects the oscillations (e.g., Lai & Tsang 2009; Fu & Lai 2011, 2013). Such reflection might be caused by, e.g., a steep density gradient or a neutron star’s surface, but is less likely in an accretion disc around a black hole delimited by a plunging region.

In contrast, vertically structured oscillations can exist independently of the inner boundary. For non-zero $n$ (or $k_z$) dispersion relation (1) describes higher frequency acoustic ‘p-modes,’ which propagate where $\omega^2 > \max[\kappa^2, n\Omega^2] = n\Omega^2$, and lower frequency inertial ‘r-modes,’ localized to radii where $\omega^2 < \min[\kappa^2, n\Omega^2] = \kappa^2$. Axisymmetric ($m = 0$) r-modes are of the greatest interest for discoseismic theories of HFQPOs, as the maximum in $\kappa^2$ serves as a ‘self-trapping region,’ providing inner and outer turning points at the radii where $\omega^2 = \kappa^2$ (as illustrated by the dark blue sinusoidal line plotted in Fig. 1). In addition to eliminating the need for reflection at the inner boundary, such confinement to a narrow annulus might provide protection from damping by radial inflow. Importantly, the frequencies of these waves track the maximum of the epicyclic frequency and therefore, when matched to observations, might be used as diagnostics of black hole mass and spin.

Finally, the cyan lines in Fig. 1 also indicate the regions of confinement for non-axisymmetric r-modes with azimuthal wavenumber $m = 1$. Such oscillations may be similarly confined away from radial boundaries, but have been disfavoured as an explanation for HFQPOs because they are strongly damped at corotation (Li et al. 2003; Latter & Balbus 2009). Non-axisymmetric r-modes with frequencies large enough that their corotation radius lies outside of their trapping region (as illustrated by the higher frequency cyan line in Fig. 1) might avoid this damping, but would then have frequencies too high to explain HFQPOs in systems containing black holes with any non-negligible spin angular momentum (Wagoner 2012). On the other hand, lower frequency non-axisymmetric r-modes lying within their corotation radius do play an ancillary role in the excitation mechanism for $m = 0$ r-modes considered in this paper (Kato 2004, 2008; Ferreira & Ogilvie 2008; Oktariani et al. 2010).

Figure 1. Schematic wave propagation diagram illustrating the radii of localization for both axisymmetric ($m = 0$) and non-axisymmetric ($m \neq 0$) inertial and inertial-acoustic waves (r-modes and f-modes, resp.). The sinusoidal lines indicate the regions within which the modes of a given character and azimuthal structure are predicted to be oscillatory by dispersion relation (1). Their intersections with the characteristic frequencies $\Omega$, $\kappa$, and $\Omega \pm \kappa$ define resonant radii beyond which the oscillations become evanescent. Here $r_g$ and $\omega_g$ are the gravitational radius and frequency (see Section 3.3), and frequencies $\nu$ given in Hz are calculated for a $10M_\odot$ Schwarzschild black hole.

HFQPOs in eccentric discs

2.2 Excitation of axisymmetric r-modes

The excitation mechanism considered by Kato (2003, 2004, 2008), Ferreira & Ogilvie (2008) and Oktariani et al. (2010) is essentially parametric and involves i) a non-axisymmetric eccentric (warping) disc deformation with $m = 1, n = 0$ ($n = 1$), ii) the fundamental, axisymmetric trapped inertial wave with $m = 0, n = 1$, and iii) an intermediate non-axisymmetric wave with $m = 1, n = 1$ ($n = 1 \pm 1$), propagating within its corotation radius. For a disc warp or eccentricity with nearly zero frequency, wave coupling rules require that both the axisymmetric and non-axisymmetric r-modes have $\omega \approx \max[\kappa]$. The dark blue and (lower) cyan oscillations in Fig. 1 illustrate the possibility for this matching of frequencies to occur within the expected trapping region.

In coupling with the deformation, the fundamental trapped axisymmetric r-mode gives rise to the intermediate non-axisymmetric mode. Since the intermediate $m = 1$ r-mode propagates within corotation, it can be seen as carrying negative wave energy, and so its generation causes the axisymmetric r-mode to grow in amplitude. If the non-axisymmetric wave is damped by dissipative processes in the disc, then the fundamental trapped r-mode continues to dump negative energy into the intermediate mode, achiev-
ing sustained growth as a consequence (Kato 2004, 2008; Ferreira & Ogilvie 2008).

The scenario of r-mode amplification through non-linear coupling with warps and eccentricities provides a differentiating factor which might explain the appearance of HFQPOs only in the very high emission state. Ferreira & Ogilvie (2009) found that high accretion rates, as are seen in this state, may be necessary for disc deformations to propagate to the inner regions of a black hole accretion disc. Such a differentiating factor is sorely missing from other HFQPO models. Nevertheless, the three-wave coupling approach taken by Ferreira & Ogilvie (2008) assumes that the disc deformations have amplitudes small enough that they may be treated as linear normal modes. The excitation mechanism remains to be validated in fully non-linear simulations, and in turn pitted against possible damping by MHD turbulence.

3 SIMULATION SETUP AND TESTS

3.1 Equations

The simulations presented in this paper have been run with the code RAMSES (Teyssier 2006; Fromang et al. 2006), which uses a finite volume, high-order Godunov method. In the absence of magnetic fields and explicit dissipation, the code solves the equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0,$$

$$\frac{\partial (\rho \mathbf{u})}{\partial t} + \nabla \cdot (\rho \mathbf{u}\mathbf{u}) + \nabla P = -\rho \nabla \Phi.$$

Here $\rho$, $u$, $P$, and $\Phi$ are the mass density, fluid velocity, gas pressure and gravitational potential, respectively. For simplicity, we supplement Equations (2)-(3) with an isothermal equation of state $P = c_s^2 \rho$, where $c_s$ is a purely constant sound speed. The potential $\Phi$ is taken to be the Paczynski-Wiita potential:

$$\Phi = \frac{-GM}{r - 2r_g},$$

where $G$ is the gravitational constant, $M$ is the mass of the central black hole and, for $c$ the speed of light, $r_g = GM/c^2$ is the characteristic gravitational radius.

3.2 Numerical framework

We use a version of RAMSES which solves Equations (2)-(3) on a uniform, cylindrical grid (Faure et al. 2014). Under the cylindrical approximation (e.g., Armitage 1998; Hawley 2001; Sorathia et al. 2012), vertical gravity and density stratification are ignored. The simulations are therefore radially global in the sense that curvature and non-local variations in the background equilibrium are taken into account. HLLD Riemann solvers are used in this and our companion paper. LLF and HLL solvers have also been considered, and found to be more diffusive but to provide similar results. Numerical expense precludes the simulation of the entire radial extent of the disc, and so simulation domains are limited to an annular region defined by inner and outer radii $r_0$ and $r_1$. The vertical and azimuthal domains are taken as $z \in [-H, H]$ and $\phi \in [0, 2\pi)$, where $H = c_s/\Omega(r_{ISCO})$ is the isothermal scale height at the ISCO.

3.3 Physical parameters

In cylindrical coordinates ($r, \phi, z$), the most basic equilibrium to consider is that of rotation dictated by the force balance

$$r \Omega^2 = \frac{1}{\rho} \frac{dP}{dr} + \frac{d\Phi}{dr}.$$

For a globally isothermal equation of state, $\Omega = \Omega(r)$ is assumed to be a function only of radius. The gravitational radius $r_g = GM/c^2$ defines the unit of length in the simulations. With $c = G = M = 1$, time units are in $\omega_g^{-1}$, where $\omega_g = c^3/(GM)$ is the gravitational frequency. In these units, the ISCO is located at $r_{ISCO} = 6r_g$, while the orbital period at this radius is given by $T_{orb} \approx 61.56$. The most relevant parameter is the sound speed, $c_s$, which with an isothermal equation of state serves as a direct proxy for temperature and disc thickness. We consider values $c_s = 0.01c, 0.02c$, which correspond to aspect ratios $H/r \sim 0.016, 0.033$ at the ISCO.

3.4 Boundary conditions

In the 3D simulations described in Section 4, we have chosen an inner boundary $r_0 = 4r_g < r_{ISCO}$. Our reasons for placing $r_0$ within the ISCO are threefold: first, allowing material to flow through the ISCO is more physically realistic. Second, this choice produces simulations more directly comparable to those of Reynolds & Miller (2009). Third, setting $r_0 < r_{ISCO}$ significantly reduces the excitation of spiral density waves via the corotational instability considered by Fu & Lai (2009) and Fu & Lai (2013) (see Appendix A2). Note that we do discuss test simulations with the inner boundary placed at the ISCO itself ($r_0 = 6r_g$) in Appendix A1 and Appendix A2.

In our primary simulations, we use a ‘diode’ outflow boundary condition at this inner boundary. In the ghost cells, this boundary condition matches density and vertical mass flux to the value in the closest cell of the active domain. The radial mass flux $\rho u_r$ is set to the value in the last value in the active domain as long as this value is negative (i.e., flowing inward, out of the simulation domain); otherwise, it is set to zero. Finally, a perturbation from the background ‘pseudo-Keplerian’ rotational velocity determined by the Paczynski-Wiita potential is calculated at the innermost active cell, and added to an extrapolation of this background rotational velocity in each ghost cell.

We choose an outer radial boundary condition that allows us to produce an eccentric deformation in the simulation domain. At the outer radius $r_1$, we set $u_r = u_z = 0$ and match $u_\phi$ to pseudo-Keplerian rotation. We then generate disc eccentricity by imposing a non-axisymmetric density profile in the ghost cells at the outer radial boundary. This produces a non-axisymmetric pressure gradient in the outer disc, which in turn forces the inward propagation of eccentricity. Specifically, at each time-step the density in the

---

1 This uniform grid version of RAMSES is freely available at https://sourcesup.renater.fr/projects/dumses/
ghost cells is set to $\rho(r = r_1, \phi, t) = \rho_E(r = r_1, \phi + \omega_p t)$, where $\rho_E$ and $\omega_p < 0$ are the density and the precession frequency calculated for an eccentric eigenmode solution to the non-linear secular theory considered by Barker & Ogilvie (2016). Calculated for a Keplerian background, these non-linear eccentric eigenmode solutions are inappropriate close to the ISCO, but their use at the outer boundary allows for the organic generation of a deformed disc.

The eccentric modes are calculated as described by Barker & Ogilvie (2016), but with length and time units scaled to match the code units of $r_g$ and $\omega_g^{-1}$. The precession frequencies are very small under this scaling, but must be imposed in the boundary conditions to produce a quasi-steady initial condition. We characterize the non-axisymmetric density profiles imposed at $r_1$ by the maximum eccentricity found in the corresponding (Newtonian) eccentric mode, denoted as $A_f$. It should be noted that the maximum eccentricities induced in a given simulation are significantly lower than $A_f$, due to the diode condition imposed at $r_0$.

We impose periodic boundary conditions in azimuth. While periodic vertical boundary conditions are technically the appropriate choice for a cylindrical model, we find that in conjunction with the excitation of inertial waves in eccentric discs, such BCs lead to the formation of mean vertical flows which are unlikely to form outside of the cylindrical approximation, and may be considered nonphysical (see Section 4.3). We therefore focus (unless otherwise stated) on simulations with the vertical velocity set to zero in the ghost cells at $z = \pm H$ (periodic boundary conditions are still imposed upon the other fluid variables).

Aside from halting the growth of the spurious mean vertical flows, this choice of vertical boundary condition primarily affects the phase of vertically structured oscillations; in particular, the $r$-modes in our simulations have vertical structure described essentially by sines for vertical velocity, and cosines for the rest of the fluid variables. The oscillations’ horizontal velocity components therefore possess even symmetry with respect to the mid-plane, and might be identified most closely with $n = 2$ $r$-modes in a fully stratified model (although a distinction between modes based upon this symmetry is less meaningful in a cylindrical, unstratified framework).

3.5 Initial conditions

To generate an initial condition to utilize in three-dimensional simulations, we follow a three-step process. We first initialize 1D simulations with density set to a floor value within $\rho_{ISCO}$, and a constant background $\rho_0$ without. Equation (5) determines the initial rotational velocity. Following a redistribution of angular momentum, these simulations produce a steady state in which the density falls off smoothly toward the ISCO, and mass continues to trickle through the inner boundary ($M/M_{tot} \approx 10^{-8} \omega_g$).

The flow is transonic in that the inward radial velocity surpasses the sound speed in amplitude, but this occurs only in the inner, evacuated regions where the density becomes very small ($\sim 10^{-4} \rho_0$). The plots in Fig. 2 illustrate the radial profiles of the density and horizontal velocities, along with the horizontal epicyclic frequency (calculated as $\kappa^2 = 2\omega[2\Omega + r\Omega, \omega]$) for simulations run with sound speeds $c_s = 0.01c$ and $0.02c$. The figure indicates that although deviations from the initial condition are small, they result in an outward shift of $\kappa$’s maximum. This shift is important to note before attempting to diagnose trapped inertial oscillations.

After integrating these 1D simulations for $1000\Omega_{orb}$, we copy the resulting radial profiles for each fluid variable at each azimuthal grid cell to initialize 2D simulations in $(r, \phi)$ aimed at producing an eccentric, relativistic disc. The outer radial boundary condition described in Section 3.5 then produces discs with non-circular streamlines. Without significant reflection at $\rho_{ISCO}$, the eccentricity continually propagates through the inner boundary as an eccentric ‘traveling wave.’ The resulting quasi-steady state achieved after $1000\Omega_{orb}$, is that of a twisted eccentric disc, with streamlines composed of concentric ellipses with semi-major axes angled with respect to one another. This is illustrated in Fig. 3, which shows a colour-map of radial velocity overlaid by streamlines calculated from a 2D simulation with $c_s = 0.02c$.

Notably, for large enough forcing amplitudes at the outer boundary, shocks form in the evacuated regions within the ISCO (see the left edge of Fig. 3). However, for forcing amplitudes less than or equal to those implemented here, the streamlines remain smooth in the expected trapping region close to the maximum in $\kappa$ (located at $r \geq 8r_g$ for the simulation shown in Fig. 3). While we do not believe these shocks significantly affect $r$-mode excitation, they may play a role in damping small-scale fluctuations which grow in the inner (unstable) regions in our control simulations of circular discs, but which disappear with even a very small eccentricity.

We use the end states of the 2D simulations as the initial condition for our 3D simulations. The two-dimensional, eccentric disc flow is copied at each vertical grid cell, and superimposed with white noise velocity perturbations at amplitudes $\leq 0.01c_s$. 

Figure 2. Radial profiles for the density, horizontal velocity components and horizontal epicyclic frequency calculated at $1000\Omega_{orb}$ in 1D simulations run with $c_s = 0.01c$ (dashed) and $c_s = 0.02c$ (dotted). The solid lines describe the initial conditions.

MNRA S 600, 1–14 (2019)
weak ‘zonal flows’ (vertically homogeneous modifications to
motion 4. They further reveal that the oscillations give rise to
provide a point of comparison for the modes discussed in Sec-
Appendix A2 we also present the results of preliminary test

3.7 Test simulations

In addition to the 3D simulations discussed in Section 4, in
Appendix A we also present the results of preliminary test
simulations. In Appendix A1, we describe the evolution of
linear r-modes inserted by hand into ‘2.5D’ (axysymmetric)
simulations run with resolutions and boundary conditions
matching those of our simulations of eccentric discs. These
simulations confirm that RAMSES is capable of capturing
the pure oscillatory behavior of r-modes in isolation, and
provide a point of comparison for the modes discussed in Sec-

Table 1. Table summarizing 3D, hydrodynamic, pseudo-
Newtonian simulations of relativistic discs, performed on the do-
main [4r_s, 18r_s] × [0, 2\pi] × [-H, H]. From left to right, the table
lists the i) simulation label, ii) amplitude for eccentricity forcing,
iii) approximate eccentricity near the trapping region, iv) sound
speed, v) resolution, and vii) estimated r-mode growth rate. All
simulations have a run-time of T_{\text{max}} = 5000c_\text{s}^{-1}.

| Label  | A_f | e_\text{tr} | c_s/c | N_r × N_\phi × N_z | s/\omega_g |
|--------|-----|------------|------|-------------------|-----------|
| ctrl2  | 0.02c | 0.02 | 512 × 512 × 32 | - |
| f02c2  | 0.025 | 0.003 | 512 × 512 × 32 | - |
| f05c2  | 0.05 | 0.006 | 512 × 512 × 32 | 4.3 × 10^{-3} |
| f07c2  | 0.075 | 0.009 | 512 × 512 × 32 | 4.9 × 10^{-3} |
| f10c2  | 0.10 | 0.013 | 512 × 512 × 32 | 5.8 × 10^{-3} |
| f10c2p| 0.10 | 0.013 | 512 × 512 × 32 | 5.3 × 10^{-3} |
| ctrl11 | 0.02c | 0.02 | 1024 × 1024 × 32 | - |
| f05c1 | 0.05 | 0.002 | 1024 × 1024 × 32 | 1.0 × 10^{-3} |
| f10c1 | 0.10 | 0.006 | 1024 × 1024 × 32 | 3.3 × 10^{-3} |

the background equilibrium flow) through a local redistribu-
tion of angular momentum. These zonal flows can alter the
trapping region, and may be responsible for small changes in
r-mode frequency over time.

In Appendix A2, we validate the performance of our
code in the non-linear regime by reproducing results from
Fu & Lai (2013). In addition to points of comparison and
insight into our 3D runs, both suites of simulations provide
motivation for our choice to allow material to flow through
an inner boundary placed within the ISCO.

4 RESULTS

In this section we describe our primary 3D simulations of
eccentric, relativistic discs. We demonstrate that trapped
inertial oscillations (like those considered in Section A1) can
be excited by eccentric structures imposed as detailed in
Section 3.5, and describe the modes’ non-linear saturation.

Table 1 provides a summary of the runs considered. All
have been initialized with the end states of 2D planar simu-
lations like the one pictured in Fig. 3, superimposed with
white noise velocity perturbations. Each simulation covers a
domain [r_0, r_1] × [\phi_0, \phi_1] × [z_0, z_1] = [4r_s, 18r_s] × [0, 2\pi] × [-H, H],
with H the scale height at the ISCO. A resolution of
512 × 512 × 32 at c_s = 0.02c then yields grid cells with
aspect ratio dr/d\phi /dz ≈ 2:6:1 at the ISCO. The cells are
elongated azimuthally, but both the twisted eccentric modes
and the oscillations with which they couple involve only low
azimuthal wavenumbers, which should be more than ade-
quately resolved. Indeed, doubling resolution in each (and
every) spatial direction reveals growth rate convergence for
rd\phi /dr ≲ 3 (see Appendix B). Simulation f10c2p is analogo-
us to f10c2 but has been run with purely periodic vertical
boundary conditions for comparison.

The table also lists parameters describing r-mode exci-
tation. A_f describes the maximum eccentricity of the eigen-
mode used to generate the non-axysymmetric density profile
imposed at the outer boundary. Due to the continual propa-
gation through r_0, the resulting eccentricity within the disc
is much lower. This is indicated by the values listed for e_\text{tr},
which gives an approximate measure of the disc eccentricity in the expected r-mode trapping region. This is found by calculating $e_{tr} \sim \max |u_r|/(u_\phi)_{h,z}$ at the radius of maximum $\kappa$ indicated by Fig. 2. Note that the values of eccentricity measured for a given forcing amplitude are smaller for lower sound speeds, due to a tighter ‘winding’ of the twisted eccentric mode close to the ISCO. The values of $e_{tr}$ listed in Table 1 are relatively low, but prove sufficient to excite trapped inertial waves: the final column lists estimates of the oscillations’ growth rates, calculated as described in the following section.

4.1 Volume-averaged quantities

Fig. 4 illustrates the growth of the volume-averaged vertical kinetic energy density in the simulations listed in Table 1. This can be used to diagnose vertical oscillations (Reynolds & Miller 2009). Note that the cylindrical framework does not capture the ‘breathing’ which occurs in fully 3D, density stratified models of eccentric discs due to variation in vertical gravity around elliptical streamlines (Ogilvie 2001). The vertical kinetic energy density can therefore be dissociated from the 2D eccentric structures considered in these simulations, and directly associated with the modes they excite.

The left-hand panels show volume averages taken over the entire simulation domain, and normalized by the volume averaged pressure for simulations run with $c_s = 0.02c$ (top) and $c_s = 0.01c$ (bottom). Volume averaged pressure corresponds to the vertical kinetic energy the oscillations would have if they moved at the sound speed, so Fig. 4 (left) implies a saturated rms $|v_z^2|^{1/2} \sim 0.015c_s$ for simulation f10c2. This value is low, but is tied directly to the small disc eccentricities (of only $e_{tr} \lesssim 0.013$ near the trapping region) that excite them.

Meanwhile, the right-hand panels show the fraction of vertical energy contained in the annular domain $r/r_g \in [7, 9]$. Figs. 4 (right) show that the enhancement of vertical oscillations’ amplitudes indicated by Figs. 4 (left) takes place within the trapping region expected for r-modes. The growth seen in simulation f10c2 is mirrored by its counterpart with periodic vertical boundary conditions, f10c2p.

Notably, the circular disc simulations without forced eccentricity (ctrl1 and ctrl2) do show a small growth in vertical kinetic energy. This is due to the activity of high-$m$ fluctuations close to and within the ISCO. Regardless of
Figure 5. Space-time diagrams showing mid-plane, azimuthally averaged, radial profiles of radial mass flux, $\langle \rho u_r \rangle_\phi(r, z = 0, t)$, for the simulations with $c_s = 0.02c$ listed in Table 1.

their origin, these fluctuations contribute an averaged vertical kinetic energy which is dwarfed by that exhibited in the non-circular disc simulations, and disappears with even a very small amplitude eccentricity forcing (see simulation f02c2). As mentioned in Section 3.7, this may be due to a damping of the fluctuations by the shocks which form due to streamline intersection in the inner evacuated regions.

Because they do not exhibit r-mode excitation, we exclude simulations ctrl1, ctrl2 and f02c2 from Figs. 4 (right), and from r-mode growth rate estimations. We calculate the latter from our measurements of vertical kinetic energy density averaged over the domain $D$: assuming $(\rho v_t^2)_D \sim \langle \rho_0 \delta v_z^2 \rangle_D$, we fit exponentials of the form $\exp[2\pi t]$ to find the growth rates listed in Table 1. We do not observe the same quadratic scaling of growth rate with eccentricity found by Ferreira & Ogilvie (2008), but we note that there are significant differences between our simulation set-up and their calculations. For one thing, the twisted, travelling eccentric modes used in our simulations are inherently different from the $m = 1$ standing modes considered by Ferreira & Ogilvie (2008). Eccentricity gradients play a role in r-mode excitation, and will differ between their disc deformations and ours. Additionally, although the stationary shocks mentioned in Section 3.5 lie interior to the trapping region for the forcing amplitudes considered, they may still play some role in inhibiting growth rates.

4.2 Mode characterization

The space-time diagrams shown in Figs. 5 and 6 show mid-plane, azimuthally averaged profiles of radial mass flux over time for simulations run with $c_s = 0.02c$ and $0.01c$, respectively. Along with Figs. 4 (right), these space-time diagrams clearly reveal the emergence of oscillatory activity localised to the expected trapping region. The periodic nature of the growth in kinetic energy shown in Fig. 4 is particularly well exhibited by the space-time diagrams for runs f07c2, f10c2 and f10c1.

By eye, the frequencies match predictions of $\omega/\omega_g \sim \max[k_{\text{P-W}}] \sim 0.035$ (or periods of $\sim 181 \omega_g^{-1}$), where $k_{\text{P-W}}$ is the horizontal epicyclic frequency determined solely by a Paczynski-Wiita potential. We confirm this quantitatively by computing the power spectral density $P(r, \omega)$ using the temporal data shown in the space-time diagrams of az-
im mutually averaged radial mass flux given in Figs. 5 and 6. The heatmap in Fig. 7 (left) shows arbitrarily normalized power for simulation f10c2. The peaks at very low frequency at all radii correspond to a secular change in the background flow due to the precession of the twisted eccentric disc. Meanwhile, the strong peak at frequencies close to the maximum in $\kappa_{PW}$ (plotted with a white dashed line) indicates the growth of a trapped inertial wave. The peak is slightly offset from the analytical maximum for the epicyclic frequency calculated from a Paczynski-Wiita potential because of the modification to $\kappa$ which occurs in the 1D simulations run to establish an initial condition. The profile for $\kappa$ illustrated in Fig. 2 is plotted with a dotted white line, and its peak aligns with the peak in power. The rotation profile calculated analytically from a Paczynski-Wiita potential is also plotted with a solid white line.

The peak in power corresponding to the trapped inertial wave in Fig. 7 (left) is surrounded by broadband noise. This blurring of the signal may be related to an outward shift of the r-modes’ localization during the oscillations’ saturation. This shift is most apparent in the space-time diagrams for the r-modes’ localization during the oscillations’ saturation. blurring of the signal may be related to an outward shift of wave in Fig. 7 (left) is surrounded by broadband noise. This also plotted with a solid white line.

calculated analytically from a Paczynski-Wiita potential is illustrated in Fig. 2 is plotted with a dotted white line, and the entire simulations run to establish an initial condition. The profile for $\kappa$ illustrated in Fig. 2 is plotted with a dotted white line, and its peak aligns with the peak in power. The rotation profile calculated analytically from a Paczynski-Wiita potential is also plotted with a solid white line.

However, after saturation the signal remains somewhat blurred. We believe this dynamical variation issues at least in part from the mode’s back reaction on the underlying orbital motion. As demonstrated in Appendix A1, trapped inertial waves redistribute angular momentum locally, causing deviations from the orbital equilibrium, possibly in the form of zonal flows. If sufficiently strong, this redistribution alters the properties of the epicyclic frequency, in turn influencing the location and shape of the trapping region — and hence the location and frequency of the trapped wave itself. We do not observe quasi-steady zonal flows directly in our 3D simulations, as seen in the local boxes of Wienkers & Ogilvie (2018); it may be that the system is too dynamic for them to appear unambiguously. But the test simulations described in Appendix A1 strongly suggest that trapped inertial waves will reconfigure the background flow as they reach non-linear saturation. The interplay between a trapped wave, its trapping region, and its time-varying redistribution of angular momentum would induce variation in the oscillation frequency, and might provide an explanation for HFQPOs’ observed low quality factors.

Fig. 8 shows a visualization of the trapped inertial wave growing in simulation f10c2. This composite colour-map has been constructed by averaging over snapshots, taken each period during the interval $1500 < t_{\omega g} < 2000$. The plot shows a trapped inertial wave at $r \sim 8 r_g$ with $k_z \sim \pi/\hat{H}$, along with another dynamical feature worth noting: secondary oscillations are faintly visible as vertical columns to the right of the trapped mode. These columns correspond to axisymmetric inertial-acoustic f-modes, and the space-time diagrams for f07c2, f10c2 and f10c1 shown in Figs. 5 and 6. The black and dark blue lines in Fig. 7 (right) show PSDs integrated over the radial domain $[7 r_g, 9 r_g]$ and the entire disc (resp.), illustrating the dominance of the trapped inertial mode over the background noise. The black dashed and dotted lines correspond to the maxima in the analytical and actual profiles for the horizontal epicyclic frequency.

However, after saturation the signal remains somewhat blurred. We believe this dynamical variation issues at least in part from the mode’s back reaction on the underlying orbital motion. As demonstrated in Appendix A1, trapped inertial waves redistribute angular momentum locally, causing deviations from the orbital equilibrium, possibly in the form of zonal flows. If sufficiently strong, this redistribution alters the properties of the epicyclic frequency, in turn influencing the location and shape of the trapping region — and hence the location and frequency of the trapped wave itself. We do not observe quasi-steady zonal flows directly in our 3D simulations, as seen in the local boxes of Wienkers & Ogilvie (2018); it may be that the system is too dynamic for them to appear unambiguously. But the test simulations described in Appendix A1 strongly suggest that trapped inertial waves will reconfigure the background flow as they reach non-linear saturation. The interplay between a trapped wave, its trapping region, and its time-varying redistribution of angular momentum would induce variation in the oscillation frequency, and might provide an explanation for HFQPOs’ observed low quality factors.

Fig. 6. Same as in Fig. 5, but for simulations run with $c_s = 0.01c$. HFQPOs in eccentric discs

\[ \text{inverted over the radial domain} \]

The black and dark blue lines in Fig. 7 (right) show PSDs simulations $f07c2$, $f10c2$ and $f10c1$ shown in Figs. 5 and 6. This shift is most apparent in the space-time diagrams for the r-modes’ localization during the oscillations’ saturation. blurring of the signal may be related to an outward shift of wave in Fig. 7 (left) is surrounded by broadband noise. This also plotted with a solid white line.

calculated analytically from a Paczynski-Wiita potential is illustrated in Fig. 2 is plotted with a dotted white line, and its peak aligns with the peak in power. The rotation profile calculated analytically from a Paczynski-Wiita potential is also plotted with a solid white line.

However, after saturation the signal remains somewhat blurred. We believe this dynamical variation issues at least in part from the mode’s back reaction on the underlying orbital motion. As demonstrated in Appendix A1, trapped inertial waves redistribute angular momentum locally, causing deviations from the orbital equilibrium, possibly in the form of zonal flows. If sufficiently strong, this redistribution alters the properties of the epicyclic frequency, in turn influencing the location and shape of the trapping region — and hence the location and frequency of the trapped wave itself. We do not observe quasi-steady zonal flows directly in our 3D simulations, as seen in the local boxes of Wienkers & Ogilvie (2018); it may be that the system is too dynamic for them to appear unambiguously. But the test simulations described in Appendix A1 strongly suggest that trapped inertial waves will reconfigure the background flow as they reach non-linear saturation. The interplay between a trapped wave, its trapping region, and its time-varying redistribution of angular momentum would induce variation in the oscillation frequency, and might provide an explanation for HFQPOs’ observed low quality factors.

Fig. 8 shows a visualization of the trapped inertial wave growing in simulation f10c2. This composite colour-map has been constructed by averaging over snapshots, taken each period during the interval $1500 < t_{\omega g} < 2000$. The plot shows a trapped inertial wave at $r \sim 8 r_g$ with $k_z \sim \pi/\hat{H}$, along with another dynamical feature worth noting: secondary oscillations are faintly visible as vertical columns to the right of the trapped mode. These columns correspond to axisymmetric inertial-acoustic f-modes, and the space-time diagrams for f07c2, f10c2 and f10c1 shown in Figs. 5 and 6. In Fig. 7 (left), the f-modes correspond to a weak band in power at radii $r > 9 r_g$ with nearly double the trapped inertial wave’s frequency. This is shown by the cyan line in Fig. 7 (right), which gives ten times the power integrated over radii from $r = 10 r_g$ to the outer boundary.

These vertically homogeneous f-modes cannot be attributed to ‘leakage’ or ‘tunneling’ of the trapped inertial waves out of their effective potential well, as they do not share the same vertical wavenumber. Moreover, they only begin to propagate after the growth of the trapped inertial waves, which suggests that they may be the result of an additional non-linear coupling. The most likely possibility is that this follows a self-interaction of the r-mode; the standing mode formed within the trapping region involves a superposition of oscillations with frequency $\omega \approx \max |x|$, vertical wavenumber $k_z \approx \pi/\hat{H}$, and azimuthal wavenumber $m = 0$. Their non-linear interaction might then be expected to produce an f-mode with $\omega_f \approx 2 \max |x|$, $k_f \approx \pi/\hat{H} - \pi/\hat{H} = 0$ and
$m_f = 0$. Examination of the space-time diagrams in Figs. 5 and 6 suggests that axisymmetric $f$-mode excitation provides r-modes with an additional route to non-linear saturation. This non-linear interaction of modes with different physical character is interesting in light of observations of multiple HFQPOs with frequencies in near-integer ratios in some sources, although the inclusion of magnetic fields and density stratification will change the relevant mode couplings.

### 4.3 Periodic vertical boundary conditions

Lastly, we touch on f10c2p, a simulation run using the same setup as f10c2 but with purely periodic vertical boundary conditions imposed at $z = \pm H$ (rather than $u_z = 0$). The lightest lines in Figs. 4 (top) illustrate a similar evolution of vertical kinetic energy between f10c2p and f10c2. Simulation f10c2p also initially exhibits the growth of an oscillation in the expected trapping region, as illustrated by the space-time diagram showing azimuthally averaged radial mass flux in Fig. 5 (bottom).

However, the periodicity at later times ($r \gtrsim 2000\omega_g^{-1}$) is altered, due to the formation of mean vertical flows like the one shown in Fig. 9. These ‘elevator flows’ are named as such because they appear in the azimuthally averaged flow as columns or ‘elevators’ of constant $u_z$. They originate in the interaction between incident waves which travel upward and downward in the absence of vertical gravity; in the simulations with $u_z$ set to zero in the ghost cells, the incident waves self-organize into global modes like the one shown in Fig. 8, but with periodic BCs they simply shear out and form a vertically homogeneous pattern. Such an occurrence is not likely in the presence of vertical gravity, so the elevators may be assumed non-physical, and their appearance highlights the limitations of the cylindrical model. While the vertically local framework used in this paper is sufficient to demonstrate r-mode excitation via non-linear coupling with disc eccentricity, a true characterization of the saturated state may require more expensive simulations fully incorporating vertical density stratification.

### 5 CONCLUSIONS

In this paper, we have explored the excitation of trapped inertial waves, one explanation for the high-frequency, quasi-periodic oscillations which appear in the emission of black
hole X-ray binary systems. Expanding upon previous three-wave coupling analyses of the problem, we have run global, hydrodynamic simulations of eccentric, relativistic discs. These simulations exhibit trapped r-mode excitation in the presence of sufficiently non-circular streamlines. In the cylindrical, unstratified framework employed, the saturated state depends on the vertical boundary condition; global standing modes form if vertical velocity is set to zero in the ghost cells, but with periodic vertical BCs the formation of elevator flows (mean vertical flows which are approximately stationary and vertically homogeneous, but vary radially) precludes this outcome.

The saturation of the trapped modes involves two processes: i) the time-dependent reconfiguration of the trapping region (mode oscillation frequency), and ii) the secondary excitation of axisymmetric inertial-acoustic (density) waves. Trapped inertial waves can transport angular momentum locally, and upon attaining sufficient amplitude can re-organise the background orbital flow and, in particular, the local epicyclic frequency; this re-organisation may, in certain instances, take the form of a quasi-steady zonal flow. As a consequence, the mode will re-shape the trapping region that contains it. We find that this process is dynamic and time-dependent, leading to an oscillation frequency which continually shifts, in turn blurring the signal and possibly accounting for lower quality factors.

The secondary inertial-acoustic waves result from a non-linear interaction between trapped inertial modes with vertical wavenumbers of equal amplitude and opposite sign. The appearance of distinct frequencies due to non-linear mode coupling is intriguing in light of observations of multiple HFQPOs in some sources. As a consequence of wave coupling rules, the density waves exhibited by our simulations are excited with a frequency roughly twice that of the trapped r-modes. This differs from the often mentioned ratio of 3:2 (e.g., Motta et al. 2014), although the peaks in power for the f-modes vary over a broad enough range in frequency so as not to be inconsistent with this commensurability (see Fig. 7, right).

In any case, the analogous coupling of r-modes in a fully global framework including vertical gravity will likely differ in detail, since in a density stratified model standing modes can no longer be considered as the superposition of Fourier components with positive and negative wavenumbers. The non-linear coupling of fully global trapped r-modes nearing saturation may then produce additional oscillations with different vertical structures and frequencies than the f-modes described in Section 4. Fully exploring the possibility that multiple HFQPOs might result from the non-linear interaction of discoseismic modes will therefore likely require more numerically expensive simulations including vertical gravity. While sufficient for capturing the essentials of the r-mode excitation mechanism, the cylindrical model utilized in our simulations also excludes vertical resonances and a potential for r-mode coupling with warping disc deformations. The eventual necessity of fully 3D simulations is further indicated by the growth of non-physical elevator flows with the use of periodic vertical boundary conditions.

Several other aspects of the discoseismic model for HFQPOs remain to be fully explored. Most importantly, the extent to which r-modes are actively damped by turbulent fluctuations must be considered, along with the capability of the excitation mechanism discussed in this paper to overcome such damping. The isothermal model considered here is also oversimplified. More complicated thermodynamics, along with radiative physics, should be included to produce a fully representative model. Lastly, the relationship between oscillations in a disc and periodic higher energy emission from coronal plasma has yet to be considered in great detail. This relationship must be addressed to connect discoseismic models to observations, since HFQPOs appear imprinted on the higher energy power-law (rather than thermal) component of X-ray binaries’ emission.

Nevertheless, the simulations presented in this paper provide a proof of concept, demonstrating a proclivity toward discoseismic mode excitation in relativistic discs which are eccentric. The viability of the discoseismic model for HFQPOs likely depends most heavily on the robustness of such mode excitation in the presence of damping by radial inflow and MHD turbulence. We explore this competition between damping and driving through global, MRI-unstable simulations of eccentric, relativistic discs in a companion paper.

ACKNOWLEDGEMENTS

J. Dewberry thanks Adrian Barker, Roman Rafikov, Pascale Garaud, and Omer Blaes for helpful discussions. This work was funded by the Cambridge International Trust, the Vassar College De Golier Trust, and the Cambridge Philosophical Society.

REFERENCES

Abramowicz M. A., Kluzniak W., 2001, A&A, 374, 19

Figure 9. Meridional snapshot of azimuthally averaged vertical mass flux in the simulation f10c2p (taken at tωr ~ 3700), which was run with periodic vertical boundary conditions. The colour-plot shows an elevator flow, which has overwhelmed the r-mode.
In this Appendix we present axisymmetric \((r,z)\) and purely 2D \((r-\phi)\) simulations run to establish a baseline and test our version of RAMSES. These test simulations utilize quasi-rigid boundary conditions at the radial boundaries; while radial and vertical velocities are set to zero, the density in the ghost cells is matched to the value in the last cell in the active domain. Meanwhile, the azimuthal velocity is matched to the (extrapolated) equilibrium rotation defined by Equation (5). Note that in these test simulations we have also placed the inner boundary \(r_0\) at the ISCO \((r_{\text{ISCO}} = 6g)\), rather than at the radius of \(4g\) used in the runs described in Section 4.

### A1 Linear r-mode evolution

To provide a point of comparison for our simulations following the non-linear excitation of \(r\)-modes, we have run simulations to verify their linear evolution. Table A1 summarizes axisymmetric simulations initialized with a hydrodynamic trapped inertial wave inserted by hand. The simulations are axisymmetric in that the fields depend only on \(r\) and \(z\). Aside from a change in the location of the inner boundary to \(r_0 = r_{\text{ISCO}} = 6g\), we set up these runs to match those described in Section 4; we again choose a vertical domain \(z/r_g \in [-H,H]\), use equivalent resolutions, and implement the same reflecting vertical boundary conditions described in Section 3.4.

We generate linear oscillations to impose in the initial conditions using the method of calculating cylindrical, hydrodynamic modes described in Dewberry et al. (2018). We initialize standing modes with perturbations \(\delta X(r,z)\) to a given variable \(X\) calculated as \(\delta X(r,z) = R e^{i(k_r r + k_z z)} + \delta X(r,-k_z) e^{-i k_z z}/2\). Here the \(\delta X(r, z_k)\) are Fourier components calculated with a vertical wavenumber \(k_z = \pi/|H|\) chosen so that one wavelength of the oscillation spans the vertical domain. These standing modes are superimposed on top of the pseudo-Keplerian flow with a relative amplitude of \(<10^{-4}\) compared to the background.

The space-time diagram in Fig. A1 plots radial profiles of mid-plane radial mass flux over time for the simulation with \(c_s = 0.02c\). The plot may be compared with the space-time diagrams given in Fig. 5, noting that these axisymmetric simulations do not include the inner, evacuated region present in our primary runs. Fig. A1 illustrates oscillatory behavior very similar to that exhibited by the 3D simulations described in Section 4, although in an isolated context the purely oscillatory \(r\)-modes remain better trapped, and maintain more coherent frequencies. These frequencies match the predictions of the linear calculations very closely (see Table A1).

As mentioned in the text, the linear trapped inertial waves produce weak, vertically homogeneous ‘zonal flows’ via a local redistribution of angular momentum. This process may be caused by the intrinsic momentum transport of the oscillation itself through its Reynolds stress. Ideal linear inertial modes possess perturbations in \(u_r\) and \(u_\phi\) which are exactly out of phase and thus no momentum flux is possible.
vertical average is calculated as \( \phi \) and over the full azimuthal range.

To test the performance of our code in the non-linear regime, we considered in this section a potential modification of the background equilibrium for the azimuthal velocity. Since the linear r-modes are periodic in \( z \), this vertical average removes the oscillation and leaves only the growing zonal flow. The amplitudes of the zonal flows produced by the linear r-modes considered in this section are inconsequential. Given sufficient amplitude, however, zonal flows can play a more significant role in altering flow dynamics (see, e.g., Wienkers & Ogilvie 2018). The most relevant effect to the simulations considered in Section 4 is a potential modification of the profile for \( \kappa \).

Figure A1. Space-time diagram showing mid-plane radial mass flux over time for the simulation initialized with a linear r-mode and \( c_s = 0.02c \).

Figure A2. Spacetime diagram showing radial profiles of the vertically averaged fluctuations from equilibrium for azimuthal velocity in the linear eigenmode simulation with \( c_s = 0.02c \). This vertical average is calculated as \( \int (u_\phi - r\Omega)dz/\int dz \).

But damping (in this case by numerical viscosity) will disturb the phase relation between the velocity perturbations, producing a non-zero flux and causing the growth of a zonal flow over time.

Fig. A2 illustrates the growth of a zonal flow in the simulation with \( c_s = 0.02c \). The space-time diagram in the figure shows a vertical average of the fluctuation from the background equilibrium for the azimuthal velocity. Since the linear r-modes are periodic in \( z \), this vertical average removes the oscillation and leaves only the growing zonal flow. The amplitudes of the zonal flows produced by the linear r-modes considered in this section are inconsequential. Given sufficient amplitude, however, zonal flows can play a more significant role in altering flow dynamics (see, e.g., Wienkers & Ogilvie 2018). The most relevant effect to the simulations considered in Section 4 is a potential modification of the profile for \( \kappa \).

Table A2. Table of parameters describing simulations run to follow the growth of inertial-acoustic modes excited by the corotation instability. All three simulations have been run with a density profile \( \Sigma \propto r^{-6} \), a sound speed \( c_s = 0.06c \), a simulation domain \([6r_g, 24r_g] \times [0, 2\pi] \), and a resolution of \( N_r \times N_\phi = 1024 \times 2048 \). The azimuthal wavenumber \( m_p \) describes the azimuthal structure of initial density perturbations which are random in radius (\( m_p = \) None corresponds to a perturbation random in both \( r \) and \( \phi \)). Meanwhile, \( s \) and \( s_{FL13} \) are the growth rates of inertial-acoustic (spiral density) waves measured in our simulations and by Fu & Lai (2013).

| Name | \( m_p \) | \( s/\omega_g \) | \( s_{FL13}/\omega_g \) |
|------|---------|----------------|----------------|
| m0   | None    | \( 6.1 \times 10^{-3} \) | \( 7.5 \times 10^{-3} \) |
| m2   | 2       | \( 5.9 \times 10^{-3} \) | \( 6.5 \times 10^{-3} \) |
| m3   | 3       | \( 5.9 \times 10^{-3} \) | \( 7.5 \times 10^{-3} \) |

Figure A3. Snapshot of radial velocity taken at \( t = 25T_{orb} \), in simulation m3.

A2 Spiral density wave excitation

To test the performance of our code in the non-linear regime and over the full azimuthal range \( \phi \in [0, 2\pi] \), we have simulated the growth of spiral density waves in relativistic discs via the corotation instability (Lai & Tsang 2009; Fu & Lai 2011). The numerical results of these simulations are closely comparable with those of Fu & Lai (2015).

Table A2 summarizes the relevant parameters used in these test simulations, which have been chosen to mirror those of Fu & Lai (2013). To allow continued outward wave propagation, a non-reflective, wave damping outer boundary condition (de Val-Borro et al. 2006) at \( r_I \) supplements the quasi-rigid inner boundary condition used in the previous section. Upon initialization, we seed the corotational instability with density perturbations \( \delta \Sigma \lesssim 10^{-4} \Sigma_0 \) which are random in radius and have either \( m = 2 \), \( m = 3 \) or random structure in azimuth. The growth rates listed in Table A2 have been calculated from time series tracking the maximum amplitude in radial velocity. We find growth rates comparable to those found by Fu & Lai (2013), and similarly observe saturation of the instability when the maximum radial velocity grows sonic values. Fig. A3 shows a snapshot of radial velocity from the simulation initialized with \( m = 3 \) perturbations, illustrating the growth of an \( m = 3 \) f-mode.

The corotational instability significantly disrupts linear eccentric modes (calculated, e.g., as described by Ferreira...
Table B1. Table summarizing simulations run with the same parameters as f10c2 ($A_f = 0.1$, $e_{tr} \sim 0.013$, $c_s = 0.02c$, simulation domain $[4r_g, 18r_g] \times [0, 2\pi] \times [-H, H]$) but different grid resolutions. From left to right, the table lists the i) simulation label, ii) resolution, and iii) estimated r-mode growth rate. All simulations were run for $5000\omega_g^{-1}$ except for f10c2hhh, which was run for $2500\omega_g^{-1}$.

| Label   | $N_t \times N_\phi \times N_z$ | $s/\omega_g$ |
|---------|---------------------------------|-------------|
| f1c2lll | 256 x 256 x 16                  | $5.7 \times 10^{-3}$ |
| f1c2llm | 256 x 256 x 32                  | $6.2 \times 10^{-3}$ |
| f1c2lml | 256 x 512 x 16                  | $5.8 \times 10^{-3}$ |
| f10c2   | 512 x 256 x 16                  | $4.8 \times 10^{-3}$ |
| f1c2mmh | 512 x 512 x 64                  | $5.6 \times 10^{-4}$ |
| f1c2hmh | 1024 x 1024 x 32                | $5.3 \times 10^{-3}$ |
| f1c2hmm | 1024 x 512 x 32                 | $4.7 \times 10^{-3}$ |
| f1c2hhh | 1024 x 1024 x 64                | $5.5 \times 10^{-3}$ |

& Ogilvie 2008) initialized in 2D simulations with a reflective boundary placed at the ISCO, even with the use of a buffer zone which relaxes the flow toward the disc eccentricity on a dynamical timescale. However, we find that growth rates are greatly reduced with the implementation of a zero-gradient inner boundary condition, and disappear almost entirely when the inner boundary $r_0$ is placed within the ISCO.

APPENDIX B: RESOLUTION STUDY

To check that the resolutions used in Section 4 are sufficient to capture r-mode excitation, we have run simulations with the same parameters as f10c2, but with the number of grid points both halved and doubled in each (and every) direction. Table B1 lists the labels, resolutions and estimated r-mode growth rates associated with these simulations. Notably, simulations f1c2llm and f1c2hmm, which have more azimuthally elongated grid cells, do show depressed r-mode growth rates. For aspect ratios $r d\phi/dr \lesssim 3$ at the ISCO, however, these growth rates return to the values measured for f10c2.

Like Fig. 4, Fig. B1 plots the total average of vertical kinetic energy (top) and the fraction contained between $7r_g$ and $9r_g$ (bottom) for simulations f1c2lll, f10c2 and f1c2hhh (the three simulations in which resolutions are altered in all three dimensions simultaneously). The plots in Fig. B1 suggest that increasing resolution produces a small reduction in vertical kinetic energy at later times. However, both indicate a minimal impact on initial r-mode growth.

This paper has been typeset from a \LaTeX file prepared by the author.

Figure B1. Top: volume averaged vertical kinetic energy density, normalized by volume averaged pressure for three of the simulations listed in Table B1. Bottom: fraction of vertical kinetic energy contained within the annulus defined by $r \in [7r_g, 9r_g]$. 

MNRAS 000, 1–14 (2019)