STOCHASTIC PARABOLIC EQUATIONS WITH SINGULAR POTENTIALS

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Abstract. In this work we consider a class of stochastic parabolic equations with singular space depending potential, random driving force and random initial condition. For the analysis of these equations we combine the chaos expansion method from the white noise analysis and the concept of very weak solutions. For given stochastic parabolic equation we introduce the notion of a stochastic very weak solution, prove the existence and uniqueness of the very weak solution to corresponding stochastic initial value problem and show its independence of a regularization on given singular potential. In addition, the consistency of a stochastic very weak solution with a stochastic weak solution is shown.

1. Introduction

We consider stochastic evolution problems with singular potentials which arise in probabilistic modelling of uncertainty in engineering and science, for example in structural mechanics, material science, fluid dynamics, climate and turbulence modelling. In particularly, the motivation comes from aggregation models in chemical kinetics, population dynamics, image processing, the modelling of options in financial mathematics, pressure diffusion in a porous medium and aerodynamics. The aim of this work is to analyse stochastic evolution problems of the form

\[(\partial_t - \mathcal{L}) U + q \cdot U = F,\]

\[U|_{t=0} = G,\]

(1.1)
where $U$ is unknown, $F$ and $G$ are given generalized stochastic processes depending on space, time and random component, $\mathcal{L}$ is an elliptic operator acting on the space variable only (its action on a stochastic process is interpreted as an action on its space component). The potential $q$ is singular, and depending on its nature, one can interpret the product in (1.1) differently. We are interested in all possible singular behaviours of $q$, either in space and time or in the random component. Typical examples would be space and/or time white noise, inverse squared potential in space, Dirac delta distribution in space and/or time. The most general case is to consider $q$ to be a random, space and time depending potential, i.e., a generalized stochastic process, so that (1.1) takes the form

$$
\begin{align*}
(\frac{\partial}{\partial t} - \mathcal{L}) U(t, x, \omega) + q(t, x, \omega) \diamond U(t, x, \omega) &= F(t, x, \omega), \\
U(0, x, \omega) &= G(x, \omega).
\end{align*}
$$ (1.2)

Here, $\diamond$ denotes the Wick product and it is introduced to give a sense to the product of two generalized stochastic processes: the potential $q$ and the unknown stochastic process $U$. In order to deal with singularities in deterministic variables we employ the concept of very weak solutions. The Wick product, also known as stochastic convolution, is the highest order stochastic approximation of the ordinary product. When at least one of the processes in the Wick product is a deterministic function or an adapted process, it reduces to the ordinary product. Alternative approaches for solving stochastic partial differential equations (SPDEs) with singularities have been developed in the theory of regularity structures [15] and in rough path theory and paracontrolled distributions [13, 14]. Another possibility is to consider the equation in Colombeau algebras of generalized functions and after regularization interpret the product as a classical product [11, 34, 35, 38, 39, 41, 43].

An example of (1.2) is the stochastic heat equation with random potential. Heat equations with random input data are extensively studied in the literature due to their various applications in biology, aerodynamics, structural acoustics, financial mathematics [11, 2, 6, 22]. The heat equation with random potential, also known as the Anderson model, appears in the context of chemical kinetics and population dynamics [16]. In [21] a survey of recent progress on stochastic heat equations driven by Gaussian noise is given. Semilinear heat equation which is driven by a space-time Gaussian white noise in suitable algebras of generalized functions is considered in [39]. Stochastic evolution problems with polynomial nonlinearities were studied in [25]. We aim to consider more general classes of problems, i.e., to study stochastic parabolic evolution equations with singular space depending potentials, where the operator $\mathcal{L}$ is not necessarily the Laplace operator.

For the analysis of (1.2) we propose a new method which combines the chaos expansion method [17, 20, 25, 45, 47] and the very weak solution concept [3, 4, 5, 9, 10, 27, 28, 30, 41, 42, 43]. The chaos expansion method
is based on constructing the solution to the SPDE as a Fourier series in terms of a Hilbert space basis of orthogonal stochastic polynomials, with unknown coefficients being elements in an appropriate space of deterministic functions. As a result, the initial problem (1.2) is reduced to a system of deterministic parabolic equations with singular potentials. To deal with strong singularities in these deterministic equations we employ the concept of very weak solutions introduced in [10]. The idea is to model irregular objects in equations by approximating nets of regular functions with moderate asymptotics. One obtains a net of regularized problems which can be treated in a usual distributional way. As a result we obtain a net of solutions, which if moderate is called very weak solution. We apply this to each of the deterministic parabolic equations arising from the chaos expansion method. The obtained nets of very weak solutions are the coefficients of the unknown stochastic process. Summing them up and proving its convergence in an appropriate space of stochastic processes, one obtains the solution to the initial stochastic problem.

In this work we concentrate on the simplest form of (1.2) that involves singularities. We analyze the case when \( q \) is a deterministic singular space potential and \( F \) and \( G \) are singular stochastic processes of Kondratiev-type. Since \( q \) is not random, the Wick product in (1.2) is then the ordinary product. We particularly study the initial value problem for the stochastic parabolic equations of the form

\[
\left( \frac{\partial}{\partial t} - L \right) U(t, x, \omega) + q(x) \cdot U(t, x, \omega) = F(t, x, \omega),
\]

\[
U(0, x, \omega) = G(x, \omega),
\]

where \( t \in (0, T] \), \( x \in \mathbb{R}^d \), \( \omega \in \Omega \). The operator \( L \) is unbounded and closed operator on \( L^2(\mathbb{R}^d) \) with dense domain, which generates \( C_0 \)-semigroup on \( L^2(\mathbb{R}^d) \). For example, \(-L\) could be an elliptic differential operator of even order or the Laplace operator. The potential \( q \) is a distribution in space. For example, it can be of the form \( q = q_1 + \delta \), where \( \delta \) is the Dirac delta distribution and \( q_1 \) is an \( L^\infty \) function over \( \mathbb{R}^d \).

The paper is organized as follows. In the sequel we briefly introduce notation and basic concepts used in the following sections. In Section 2 we study initial value problem for the stochastic parabolic equations with bounded potential, and the deterministic evolution equations with singular space potential. Section 3 is devoted to stochastic parabolic equation with singular space depending potential. The new method for its analysis is introduced. In Section 4 we illustrate our method on the stochastic heat equation with singular potential, we list several advantages of the introduced approach and indicate further extensions.
1.1. Notations and basic concepts. This paper is placed in the framework of white noise analysis and is built on the well known Wiener-Itô chaos expansion theorem that relates the Gaussian measure and Hermite polynomials [17]. Throughout the paper \((\Omega, \mathcal{F}, \mathbb{P})\) denotes the Gaussian white noise space and \(T > 0\).

Let \(L^\infty(\mathbb{R}^d)\) be the space of essentially bounded measurable functions, \(C^k([0, T])\) be the space of \(k\)-times continuously differentiable functions on \([0, T]\), \(\mathcal{D}(\mathbb{R}^d) := C^\infty_0(\mathbb{R}^d)\) the space of compactly supported smooth functions, \(\mathcal{D}'(\mathbb{R}^d)\) the space of distributions, \(S(\mathbb{R}^d)\) the Schwartz space of rapidly decreasing functions, \(S'(\mathbb{R}^d)\) the Schwarz space of tempered distributions, and \(\mathcal{E}'(\mathbb{R}^d)\) the space of compactly supported distributions. Further, let \(L^2(\mathbb{R}^d)\) be the space of square integrable functions over \(\mathbb{R}^d\), \(H^1(\mathbb{R}^d)\) be the Sobolev space, a subset of functions \(f\) that \(f\) and its weak derivatives have a finite \(L^2\) norm, \(H^1_0(\mathbb{R}^d)\) the closure of \(\mathcal{D}(\mathbb{R}^d)\) in \(H^1(\mathbb{R}^d)\), \(L^2([0, T]; L^2(\mathbb{R}^d))\) the Banach space of square integrable functions over \([0, T]\) with values in \(L^2(\mathbb{R}^d)\) with the norm

\[
\|f\|_{L^2([0, T]; L^2(\mathbb{R}^d))} = \left(\int_0^T \|f(t, \cdot)\|^2_{L^2(\mathbb{R}^d)} \, dt\right)^{1/2} = \left(\int_{\mathbb{R}^d} \int_0^T |f(t, x)|^2 \, dt \, dx\right)^{1/2},
\]

\(C([0, T]; L^2(\mathbb{R}^d))\) the Banach space of continuous functions over \([0, T]\) with values in \(L^2(\mathbb{R}^d)\) with the norm

\[
\|f\|_{C([0, T]; L^2(\mathbb{R}^d))} = \sup_{t \in [0, T]} \|f(t, \cdot)\|^2_{L^2(\mathbb{R}^d)},
\]

and \(L^1(0, T; L^2(\mathbb{R}^d))\) is the space of integrable functions over \([0, T]\) with values in \(L^2(\mathbb{R}^d)\).

The Gaussian white noise probability space \((\Omega, \mathcal{F}, \mathbb{P})\) is \((S'(\mathbb{R}^d), \mathcal{B}, \mu)\), where \(\mathcal{B}\) is the family of all Borel subsets of \(S'(\mathbb{R}^d)\) equipped with the weak* topology and \(\mu\) is the Gaussian white noise probability measure, whose existence is guaranteed by the Bochner–Mimlos theorem [17]. The space of square integrable random variables on \((\Omega, \mathcal{F}, \mathbb{P})\), denoted by \(L^2(\mu)\), is a Hilbert space with the norm \(\| \cdot \|_{L^2(\mu)}\) induced by the inner product

\[
(F, G)_{L^2(\mu)} = \mathbb{E} \langle FG \rangle = \int_\Omega F(\omega) G(\omega) \, d\mu(\omega) < \infty, \quad \text{for } F, G \in L^2(\mu),
\]

where \(\mathbb{E}\) denotes the expectation with respect to the Gaussian measure \(\mu\).

Let \(\mathcal{I} := \mathbb{N}_0^m\) be the set of sequences of non-negative integers which have finitely many nonzero components \(\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_m, 0, 0, \ldots)\), \(\gamma_i \in \mathbb{N}_0, i = 1, 2, \ldots, m, m \in \mathbb{N}\), the \(k\)th unit vector \(e(k) = (0, \ldots, 0, 1, 0, \ldots)\), \(k \in \mathbb{N}\) is the sequence of zeros with the number \(1\) as the \(k\)th component and \(0 = (0, 0, \ldots)\) the zero vector, \(|\gamma| = \sum_{i=1}^m \gamma_i\) the length of multi-index \(\gamma\) and \(\gamma! = \prod_{i=1}^\infty \gamma_i!\).

For a given \(\gamma \in \mathcal{I}\), the \(\gamma\)th Fourier-Hermite polynomial is defined by

\[
H_\gamma(\omega) = \prod_{k=1}^\infty h_{\gamma_k}(\langle \omega, \xi_k \rangle), \quad \gamma \in \mathcal{I},
\]

where \(\gamma_0 = 0\).
where $\xi_k$, $k \in \mathbb{N}$ is the Hermite function of order $k$ and $h_k$, $k \in \mathbb{N}_0$ are the Hermite polynomials. The famous Wiener-Itô chaos expansion theorem states that each square integrable random variable $F \in L^2(\mu)$ has a unique representation of the form

$$F(\omega) = \sum_{\gamma \in \mathcal{I}} c_\gamma H_\gamma(\omega), \; c_\gamma \in \mathbb{R}, \omega \in \Omega,$$

with $\|F\|_{L^2(\mu)}^2 = \sum_{\gamma \in \mathcal{I}} c_\gamma^2 \gamma! < \infty$.

The space of Kondratiev test random variables $(S)_p, p \in \mathbb{N}_0$ consists of the elements $f(\omega) = \sum_{\gamma \in \mathcal{I}} c_\gamma H_\gamma(\omega) \in L^2(\mu), \; c_\gamma \in \mathbb{R}$, such that $\|f\|_p^2 = \sum_{\gamma \in \mathcal{I}} c_\gamma^2 (\gamma!)^2 (2\mathbb{N})^{\gamma p} < \infty$, where $(2\mathbb{N})^\gamma = \prod_{i=1}^\infty (2i)^{\gamma_i}$, $\gamma \in \mathcal{I}$. The Kondratiev space of random variables $(S)_1$ is the projective limit of the spaces $(S)_p$, i.e., $(S)_1 = \bigcap_{p \in \mathbb{N}_0} (S)_p$. The family of seminorms $\| \cdot \|_p$ generates a topology on $(S)_1$. The Kondratiev space of stochastic distributions $(S)_{-p}$, consists of the formal expansions $F(\omega) = \sum_{\gamma \in \mathcal{I}} b_\gamma H_\gamma(\omega), \; b_\gamma \in \mathbb{R}$ such that

$$\|F\|_{-p}^2 = \sum_{\gamma \in \mathcal{I}} b_\gamma^2 (2\mathbb{N})^{-\gamma p} < \infty.$$

The Kondratiev space $(S)_{-1}$ is the inductive limit of the spaces $(S)_{-p}$, i.e., $(S)_{-1} = \bigcup_{p \in \mathbb{N}_0} (S)_{-p}$. It is the dual space of $(S)_1$ and a Fréchet space. The action of $F \in (S)_{-1}$ on $f \in (S)_1$ is given by $\langle F, f \rangle = \sum_{\gamma \in \mathcal{I}} b_\gamma c_\gamma \gamma! \in \mathbb{R}$. The spaces $(S)_p$ and $(S)_{-p}$ are separable Hilbert spaces, $(S)_1$ and $(S)_{-1}$ are nuclear, i.e., the embedding $(S)_q \subseteq (S)_p$ for $p \leq q$ is Hilbert-Schmidt, and for $p \leq q$ it holds $(S)_1 \subseteq (S)_q \subseteq (S)_p \subseteq L^2(\mu) \subseteq (S)_{-p} \subseteq (S)_{-q} \subseteq (S)_{-1}$.

For any normed space $Y$, the tensor product $Y \otimes L^2(\mu)$ is the space of $Y$-valued square integrable stochastic processes, $Y \otimes (S)_{-p}$ the space of $Y$-valued generalized stochastic processes of Kondratiev-type and the space $Y \otimes (S)_{-1}$ is the inductive limit of spaces $Y \otimes (S)_{-p}, p \geq 0$. The Wiener–Itô chaos expansion theorem can be extended from spaces of random variables to spaces of stochastic processes, \cite{23, 24, 26, 37}. We summarize known results in the following theorem.

**Theorem 1.** Let $Y$ be a normed space.

(a) It holds

$$\sum_{\gamma \in \mathcal{I}} (2\mathbb{N})^{-\gamma p} \gamma! < \infty \quad \text{if and only if} \quad p > 1. \quad (1.5)$$

(b) Every $Y$-valued square integrable stochastic process $F \in Y \otimes L^2(\mu)$ can be uniquely represented in the chaos expansion form

$$F(\omega) = \sum_{\gamma \in \mathcal{I}} f_\gamma H_\gamma(\omega), \; f_\gamma \in Y,$$

where $\|F\|_{Y \otimes L^2(\mu)}^2 = \sum_{\gamma \in \mathcal{I}} \|f_\gamma\|_{Y}^2 \gamma! < \infty$. If $Y$ is a Banach space, then $Y \otimes L^2(\mu)$ is a Banach space.
(c) Every $Y$-valued generalized stochastic process of Kondratiev-type $F \in Y \otimes (S)_{-1}$ can be represented in the chaos expansion form (1.6) where

$$
\|F\|^2_{Y \otimes (S)_{-p}} = \sum_{\gamma \in \mathcal{I}} \|f\|^2_{Y} (2N)^{-p\gamma} < \infty
$$

holds for some $p \geq 0$. If $Y$ is a Banach space, then $Y \otimes (S)_{-p}$ are Banach spaces for every $p \geq 0$, and $Y \otimes (S)_{-1}$ is a Fréchet space.

Let $Y$ be a space of deterministic functions depending on time and space and $U = U(t, x, \omega) \in Y \otimes (S)_{-1}$. For fixed $\omega \in \Omega$ the process $U(\cdot, \cdot, \omega)$ is a deterministic function in $Y$, and for fixed $t \in (0, T]$ and $x \in \mathbb{R}^d$ the process $U(t, x, \cdot)$ belongs to a Kondratiev space of stochastic random variables.

**Example 1.1.**

(i) An $C^k([0, T])$-valued generalized process $U \in C^k([0, T]) \otimes (S)_{-1}$ has the chaos expansion representation

$$
U(t, \omega) = \sum_{\gamma \in \mathcal{I}} u_{\gamma}(t) H_{\gamma}(\omega), \quad t \in [0, T], \ \omega \in \Omega,
$$

with the coefficients $u_{\gamma}$, $\gamma \in \mathcal{I}$, being elements of the Banach space of functions $C^k([0, T])$, $k \in \mathbb{N}$ such that for some $p \geq 0$

$$
\|U\|^2_{C^k([0, T]) \otimes (S)_{-p}} = \sum_{\gamma \in \mathcal{I}} \|u_{\gamma}\|^2_{C^k([0, T])} (2N)^{-p\gamma} < \infty.
$$

(ii) Let $\mathcal{B}$ be a Banach space of functions depending on $t$ and $x$. An $C([0, T]; \mathcal{B})$-valued Kondratiev-type generalized stochastic process $U \in C([0, T]; \mathcal{B}) \otimes (S)_{-1}$ has a chaos expansion representation (1.7)

$$
U(t, x, \omega) = \sum_{\gamma \in \mathcal{I}} u_{\gamma}(t, x) H_{\gamma}(\omega), \quad t \in [0, T], \ x \in \mathbb{R}^d, \ \omega \in \Omega,
$$

with $u_{\gamma} \in C([0, T]; \mathcal{B})$, $\gamma \in \mathcal{I}$, and for some $p \geq 0$ it holds

$$
\|U\|^2_{C([0, T]; \mathcal{B}) \otimes (S)_{-p}} = \sum_{\gamma \in \mathcal{I}} \|u_{\gamma}\|^2_{C([0, T]; \mathcal{B})} (2N)^{-p\gamma} < \infty.
$$

(iii) An $L^2([0, T], L^2(\mathbb{R}^d))$-valued generalized process $U \in L^2([0, T], L^2(\mathbb{R}^d)) \otimes (S)_{-1}$ has the chaos expansion representation (1.7) with the coefficients $u_{\gamma} \in L^2([0, T], L^2(\mathbb{R}^d))$, $\gamma \in \mathcal{I}$, and for some $p \geq 0$ it holds

$$
\|U\|^2_{L^2([0, T], L^2(\mathbb{R}^d)) \otimes (S)_{-p}} = \sum_{\gamma \in \mathcal{I}} \|u_{\gamma}\|^2_{L^2([0, T], L^2(\mathbb{R}^d))} (2N)^{-p\gamma} < \infty.
$$

(iv) The time white noise $W$ is an element of the space $C^k([0, T]) \otimes (S)_{-1}$ and it is given formally by

$$
W(t, \omega) = \sum_{k=1}^{\infty} \xi_k(t) H_{\xi_k}(\omega),
$$
while space-time white noise belongs to $C^k([0,T],\mathbb{R}^d) \otimes (S)_{-1}$ and

$$W(t,x,\omega) = \sum_{k,n=1}^{\infty} \xi_k(t)\eta_n(x)He_{e(k)}(\omega),$$

where $\xi_k$, $k \in \mathbb{N}$ are the Hermite functions and $\eta_n$, $n \in \mathbb{N}$ are elements of an orthogonal basis of $\mathbb{R}^d$, see [17].

2. Set up of the problem and methodology

We study the stochastic parabolic initial value problem (1.3), i.e.,

$$\left( \frac{\partial}{\partial t} - \mathcal{L} \right) U(t,x,\omega) + q(x) \cdot U(t,x,\omega) = F(t,x,\omega),$$

$$U(0,x,\omega) = G(x,\omega),$$

where $t \in (0,T]$, $x \in \mathbb{R}^d$, $\omega \in \Omega$. For $X \subseteq L^2(\mathbb{R}^d)$ we define $\mathcal{X}(X) := C([0,T];X) \otimes (S)_{-1}$. Note that if $X$ is a Banach space then $\mathcal{X}(X)$ is Banach space as well. Assume the following:

(H1) The operator $\mathcal{L}$ is unbounded and closed operator on $L^2(\mathbb{R}^d)$ with dense domain $D \subseteq L^2(\mathbb{R}^d)$, which generates a $C_0$-semigroup $(T_t)_{t \geq 0}$ on $L^2(\mathbb{R}^d)$. Action of the operator $\mathcal{L}$ on a generalized stochastic process of Kondratiev-type $U \in \mathcal{X}(D)$ with the chaos expansion (1.7) is given by

$$\mathcal{L}U(t,x,\omega) := \sum_{\gamma \in \mathcal{I}} \mathcal{L} u_{\gamma}(t,x) H_{\gamma}(\omega),$$

where $\mathcal{L}$ acts only on the space component.

(H2) The force term $F \in \mathcal{X}(L^2(\mathbb{R}^d))$ is $C([0,T];L^2(\mathbb{R}^d))$-valued Kondratiev-type generalized stochastic process with the chaos expansion (1.7) where the coefficients $f_{\gamma} \in C([0,T];L^2(\mathbb{R}^d))$, $\gamma \in \mathcal{I}$ are Lipschitz continuous functions with respect to $t$.

(H3) The initial condition $G$ is a $D$-valued Kondratiev-type generalized stochastic process, i.e., $G \in D \otimes (S)_{-1}$.

2.1. Stochastic parabolic equation with bounded potential.

We start our analysis of (1.3) assuming $q \in L^\infty(\mathbb{R}^d)$. The following theorem employs the chaos expansion method in order to show existence of the unique solution to (1.3) for $q \in L^\infty(\mathbb{R}^d)$.

Theorem 2. Let the operator $\mathcal{L}$, the force term $F$ and the initial condition $G$ satisfy the assumptions (H1)-(H3). Assume the potential $q \in L^\infty(\mathbb{R}^d)$. Then, there exists unique generalized stochastic process $U \in \mathcal{X}(D) \subseteq \mathcal{X}(L^2(\mathbb{R}^d))$ satisfying stochastic parabolic initial value problem (1.3).
Proof. By assumptions all stochastic processes appearing in \((1.3)\) are of Kondratieff-type, and can be represented in their chaos expansions \((1.7)\). For \(t \in [0, T]\), \(x \in \mathbb{R}^d\), and \(\omega \in \Omega\) we have

\[
F(t, x, \omega) = \sum_{\gamma \in \mathcal{I}} f_\gamma(t, x) H_\gamma(\omega),
\]

with

\[
\sum_{\gamma \in \mathcal{I}} \|f_\gamma\|_{C([0, T]; L^2(\mathbb{R}^d))}^2 (2N)^{-p_1 \gamma} < \infty
\]

for some \(p_1 \geq 0\) and

\[
G(x, \omega) = \sum_{\gamma \in \mathcal{I}} g_\gamma(x) H_\gamma(\omega),
\]

with

\[
\sum_{\gamma \in \mathcal{I}} \|g_\gamma\|_{L^2(\mathbb{R}^d)}^2 (2N)^{-p_2 \gamma} < \infty
\]

for some \(p_2 \geq 0\). Assume that the solution is also given in the chaos expansion form \((1.7)\), i.e.,

\[
U(t, x, \omega) = \sum_{\gamma \in \mathcal{I}} u_\gamma(t, x) H_\gamma(\omega), \quad t \in [0, T], \ x \in \mathbb{R}^d, \ \omega \in \Omega.
\]

The chaos expansion method means to substitute stochastic processes given in their chaos expansion forms into the equation and to equalize the corresponding coefficients with respect to orthogonal stochastic polynomial basis \(H_\gamma(\omega), \ \gamma \in \mathcal{I}\). The initial stochastic problem \((1.3)\) is then reduced to a system of the deterministic PDEs of the form

\[
\left( \frac{\partial}{\partial t} - \mathcal{L} \right) u_\gamma(t, x) + q(x) \cdot u_\gamma(t, x) = f_\gamma(x, t),
\]

\[
u_\gamma(0, x) = g_\gamma(x),
\]

for every \(\gamma \in \mathcal{I}\). By hypotheses we have that \(q \in L^\infty(\mathbb{R}^d), \ f_\gamma \in C([0, T]; L^2(\mathbb{R}^d))\) are Lipschitz continuous functions with respect to \(t\) and \(g_\gamma \in D\) for all \(\gamma \in \mathcal{I}\).

By Theorem 3 (proved below independently of this theorem) it follows that for each \(\gamma \in \mathcal{I}\) the deterministic problem \((2.13)\) has a unique bounded nonegative solution \(u_\gamma \in C([0, T]; D) \subseteq C([0, T]; L^2(\mathbb{R}^d))\) given by

\[
u_\gamma(t, x) = S_t g_\gamma(x) + \int_0^t S_{t-s} f_\gamma(s, \cdot) \, ds, \quad t \in (0, T], \ x \in \mathbb{R}^d,
\]

and such that for \(t \in (0, T]\) the following estimate holds

\[
\|u_\gamma(t, \cdot)\|_{L^2(\mathbb{R}^d)} \leq M(t) \left( \|g_\gamma(\cdot)\|_{L^2(\mathbb{R}^d)} + \int_0^t \|f_\gamma(s, \cdot)\|_{L^2(\mathbb{R}^d)} \, ds \right).
\]
Here $M(t) = M(t, w, M, \|q\|_{L^\infty(\mathbb{R}^d)}) = Me^{(w+M\|q\|_{L^\infty(\mathbb{R}^d)})t}$ is an increasing positive function on $[0, T]$, with stability constants $w$ and $M$ appearing in the semigroup estimate (2.20) in the proof of Theorem 3.

Finally, we are going to show that the process (1.7) whose coefficients $u_\gamma$ are given by (2.14) converges in $\mathcal{X}(D)$. Therefore we have to show that for some $r \geq 0$

$$
||U||^2_{X \otimes (S) \rightarrow r} = \sum_{\gamma \in \mathcal{I}} ||u_\gamma||^2_X (2N)^{-r\gamma} < \infty,
$$

where here (and along this proof) $X := C([0, T]; L^2(\mathbb{R}^d))$. Namely, for $r \geq \max\{p_1, p_2\}$ by the estimates (2.15) we obtain

$$
||U||^2_{X \otimes (S) \rightarrow r} \leq 2M(T)^2 \sum_{\gamma \in \mathcal{I}} \left( \|g_\gamma\|^2_{L^2(\mathbb{R}^d)} + \left( \int_0^T \|f_\gamma(t, \cdot)\|_{L^2(\mathbb{R}^d)} dt \right)^2 \right) (2N)^{-r\gamma}
$$

$$
\leq 2M(T)^2 \left( \sum_{\gamma \in \mathcal{I}} \|g_\gamma\|^2_{L^2(\mathbb{R}^d)(2N)^{-p_2\gamma}} + T^2 \sum_{\gamma \in \mathcal{I}} \|f_\gamma\|^2_X (2N)^{-p_1\gamma} \right)
$$

$$
\leq 2M(T)^2 \left( \|G\|^2_{L^2(\mathbb{R}^d) \otimes (S) \rightarrow p_2} + T^2 \|F\|^2_X \right),
$$

which is by the assumptions $(H2)$ and $(H3)$ finite.

The uniqueness of the solution $U$ follows from the uniqueness of its coefficients $u_\gamma$, $\gamma \in \mathcal{I}$, which are given by (2.14) and the uniqueness of the chaos expansion representation in the Fourier–Hermite basis of orthogonal stochastic polynomials.

For the analysis of the problem (1.3) with $q \in \mathcal{D}'(\mathbb{R}^d)$ in Section 3 we will use the same approach. There the deterministic equation (2.13) for every $\gamma \in \mathcal{I}$ will have irregular potential.

### 2.2. A deterministic parabolic equations with singular potentials: The very weak solution approach.

Let us now consider the deterministic parabolic equation of the form (2.13), with the potential $q \in \mathcal{D}'(\mathbb{R}^d)$, i.e.,

$$
\left( \frac{\partial}{\partial t} - \mathcal{L} \right) u(t, x) + q(x) \cdot u(t, x) = f(t, x), \quad t \in (0, T], \ x \in \mathbb{R}^d,
$$

$$
u(0, x) = g(x),$$

where $\mathcal{L}$ is an operator on $L^2(\mathbb{R}^d)$ with dense domain $D \subseteq L^2(\mathbb{R}^d)$, as in the hypothesis (H1), $f \in C([0, T]; L^2(\mathbb{R}^d))$ Lipschitz continuous function with respect to $t$, and $g \in D$.

#### 2.2.1. Bounded potential and related estimates.

First we consider the case with $L^\infty$-potential $q$ and provide an estimate for the solution to (2.16). In particular, the following theorem assures that (2.13) in Theorem 2 has a unique bounded nonegative solution $u_\gamma \in C([0, T]; D)$, $\gamma \in \mathcal{I}$.
Theorem 3. Let the operator \( L \) be as in (H1), the force term \( f \in C([0, T]; L^2(\mathbb{R}^d)) \), a Lipschitz continuous function with respect to \( t \), the initial condition \( g \in D \), and let the potential \( q \in L^\infty(\mathbb{R}^d) \). Then, the deterministic parabolic initial value problem (2.16) has a unique bounded nonnegative solution \( u \in C([0, T]; D) \subseteq C([0, T]; L^2(\mathbb{R}^d)) \) satisfying

\[
\|u(t, \cdot)\|_{L^2(\mathbb{R}^d)} \leq M(t) \left( \|g(\cdot)\|_{L^2(\mathbb{R}^d)} + \int_0^t \|f(s, \cdot)\|_{L^2(\mathbb{R}^d)} \, ds \right)
\]

for \( t \in (0, T] \), where

\[
M(t) := M \exp \left( \left( w + M \|q\|_{L^\infty(\mathbb{R}^d)} \right) t \right), \quad t \in (0, T],
\]

with \( w \in \mathbb{R} \) and \( M > 0 \) being the stability constants from the semigroup estimates (2.20).

Proof. As the operator \( L \) is assumed to be the infinitesimal generator of a \( C_0 \)-semigroup we aim in applying the semigroup theory. We rewrite the parabolic problem (2.16) in the form

\[
\frac{\partial}{\partial t} u(t, x) = (L - q(x)Id) u(t, x) + f(t, x), \quad t \in (0, T], \, x \in \mathbb{R}^d,
\]

\[
u(0, x) = g(x),
\]

with \( Id \) denoting the identity operator on \( L^2(\mathbb{R}^d) \). The operator \( qId \), i.e., operator of multiplication of an element from \( L^2(\mathbb{R}^d) \) by \( q \in L^\infty(\mathbb{R}^d) \), is bounded operator on \( L^2(\mathbb{R}^d) \) with the bound \( Q = \text{ess sup}_{x \in \mathbb{R}^d} |q(x)| \). Indeed,

\[
\|q(\cdot)u(t, \cdot)\|_{L^2(\mathbb{R}^d)} = \left( \int_{\mathbb{R}^d} |q(x)u(t, x)|^2 \, dx \right)^{1/2} \\
\leq \text{ess sup}_{x \in \mathbb{R}^d} |q(x)| \left( \int_{\mathbb{R}^d} |u(t, x)|^2 \, dx \right)^{1/2} \\
= Q \|u(t, \cdot)\|_{L^2(\mathbb{R}^d)}.
\]

The operator \( L \) is an infinitesimal generator of a \( C_0 \)-semigroup of operators \((T_t)_{t \geq 0}\) on \( L^2(\mathbb{R}^d) \) such that

\[
\|T_t\|_{L(L^2(\mathbb{R}^d))} \leq Me^{wt}, \quad t \geq 0
\]

holds for some \( M > 0 \) and \( w \in \mathbb{R} \). Therefore, since \( L^2(\mathbb{R}^d) \) is a reflexive Banach space, by [36, Chapter 3, Theorem 1.1] the operator \( L - qId \) is the infinitesimal generator of a \( C_0 \)-semigroup \((S_t)_{t \geq 0}\) on \( L^2(\mathbb{R}^d) \) satisfying

\[
\|S_t\|_{L(L^2(\mathbb{R}^d))} \leq M \exp \left( w + M \|q\|_{L^\infty(\mathbb{R}^d)} \right) t, \quad t \in [0, T],
\]

where \( M \) and \( w \) are as in (2.20).

Since \( f \in C([0, T]; L^2(\mathbb{R}^d)) \subset L^1(0, T; L^2(\mathbb{R}^d)) \) is Lipschitz continuous function with respect to \( t \), by the result from [36, Chapter 4, Corollary 2.11] for
every \( g \in D \) there exists a unique \( u \in C([0, T]; D) \) which is differentiable almost everywhere on \([0, T]\) solving the deterministic parabolic problem (2.19) (and therefore (2.16)) and it is given by

\[
u(t, x) = S_t g(x) + \int_0^t S_{t-s} f(s, x) \, ds, \quad t \in [0, T], \; x \in \mathbb{R}^d.
\]

To show the estimate (2.17), we start from (2.22), use the bound (2.21) together with (2.18) and obtain

\[
\|u(t, \cdot)\|_{L^2(\mathbb{R}^d)} \leq M(t) \|g(\cdot)\|_{L^2(\mathbb{R}^d)} + \int_0^t M(t-s) \|f(s, \cdot)\|_{L^2(\mathbb{R}^d)} \, ds
\]

for all \( t \in [0, T] \) with \( M(t) \) as in (2.18), which completes the proof. \( \square \)

2.2.2. Regularization of distributions and very weak solutions. In the sequel we want to allow the potential \( q \) to be a strongly irregular function or a distribution. A common approach in the functional analysis to deal with irregularities is to regularize them. The regularization can be always achieved via convolutions with mollifiers and regularization families.

A smooth function \( \varphi \) is a mollifier if \( \varphi \in \mathcal{D}(\mathbb{R}^d) \), \( \varphi \geq 0 \) and \( \int_{\mathbb{R}^d} \varphi(x) \, dx = 1 \).

The mollifying net \((\varphi_\varepsilon)_{\varepsilon \in (0,1]}\) is

\[
\varphi_\varepsilon(x) = \frac{1}{(l(\varepsilon))^d} \varphi \left( \frac{x}{l(\varepsilon)} \right) \in \mathcal{D}(\mathbb{R}^d),
\]

where \( l \) is a positive function and \( l(\varepsilon) \to 0 \), as \( \varepsilon \to 0 \).

For a given distribution \( q \in \mathcal{D}'(\mathbb{R}^d) \) a regularization via convolution is the process in which we convolve \( q \) with the mollifying net \((\varphi_\varepsilon)_{\varepsilon \in (0,1]}\) and obtain the regularizing net \((q_\varepsilon)_{\varepsilon \in (0,1]}\) of smooth functions with compact supports

\[
q_\varepsilon(x) := q * \varphi_\varepsilon(x), \quad \varepsilon \in (0, 1],
\]

which converges to \( q \) in \( \mathcal{D}'(\mathbb{R}^d) \) as \( \varepsilon \to 0 \).

In order to work with regularizing nets in (S)PDEs, and to control the rate of their convergence we introduce nets of moderate families in the following way.

**Definition 2.1 (Moderate nets).** Let \( (\mathcal{B}, \|\cdot\|_{\mathcal{B}}) \) be a Banach space. A net of elements \((b_\varepsilon)_{\varepsilon \in (0,1]} \) in \( \mathcal{B} \) is called \( \mathcal{B} \)-moderate if there exist \( N \in \mathbb{N}_0 \) and \( C > 0 \) such that for each \( \varepsilon \in (0, 1] \)

\[
\|b_\varepsilon\|_{\mathcal{B}} \leq C \varepsilon^{-N}.
\]

There are other ways to define moderateness of which the most general one is to define moderate nets via families of seminorms as it was done in [10]. Moderate nets are widely used in the theory of Colombeau algebras, along with the other notion of negligibility here also used for defining uniqueness, see below Section 2.2.3. We remark that in the very weak solution concept it
is not required derivatives to be moderate, as it is mostly the case when one applies the Colombeau theory of generalized functions in solving (S)PDEs \[32, 33\]. For more details on Colombeau theory we refer to \[7, 12, 18, 19, 29, 32\]. Here using the very weak solution approach we choose nets more freely without caring whether the underlying space is an algebra or not, see for example \[10, 27, 28, 40, 41\].

In the following lemma we summarize some of the important results needed in sequel. For the details we refer \[12, 31, 32\].

**Lemma 4.** Let \( q \in \mathcal{E}'(\mathbb{R}^d) \).

(a) The net \((q_\varepsilon)_{\varepsilon \in (0,1]}\) defined in \((2.24)\) is \( L^\infty(\mathbb{R}^d) \)-moderate.
(b) There exists a mollifying net such that the net \((q_\varepsilon)_{\varepsilon \in (0,1]}\) defined in \((2.24)\) is \( L^\infty(\mathbb{R}^d) \)-log-type moderate, i.e., there exists \( N > 0 \) so that it holds

\[
\|q_\varepsilon\|_{L^\infty(\mathbb{R}^d)} \leq N \log \frac{1}{\varepsilon}.
\]

**Proof.** (a) By distribution structure theorem every compactly supported distribution is of finite order and can be represented as a finite sum of derivatives of a continuous functions \[46\]. Therefore, i.e., there exist \( k \in \mathbb{N} \) and \( f \in C_0(\mathbb{R}^d) \) such that \( q = \sum_{|\alpha| \leq k} \partial^\alpha f \). It holds

\[
q * \varphi_\varepsilon(x) = \sum_{|\alpha| \leq k} \partial^\alpha f * \varphi_\varepsilon(x) = \sum_{|\alpha| \leq k} f * \partial^\alpha \varphi_\varepsilon(x)
\]

\[
= \sum_{|\alpha| \leq k} \int f(y) \partial^\alpha_x \varphi_\varepsilon(x - y) \, dy
\]

\[
= \sum_{|\alpha| \leq k} \int_{\text{supp}(\varphi_\varepsilon)} f(y) l(\varepsilon)^{-d-|\alpha|} \partial^\alpha_x \varphi \left( \frac{x - y}{l(\varepsilon)} \right) \, dy
\]

\[
\leq (l(\varepsilon))^{-k} \sum_{|\alpha| \leq k} \int f(x - y l(\varepsilon)) \partial^\alpha \varphi(y) \, dy.
\]

Therefore, for each \( \varepsilon \in (0,1] \) we have

\[
\|q_\varepsilon\|_{L^\infty(\mathbb{R}^d)} = \|q * \varphi_\varepsilon\|_{L^\infty(\mathbb{R}^d)} \leq C(\varepsilon) -k,
\]

with \( C = \|f\|_{L^\infty(\mathbb{R}^d)} \sum_{|\alpha| \leq k} \int |\partial^\alpha \varphi(y)| \, dy \). If we choose \( l(\varepsilon) = \varepsilon \) and \( N = |\alpha| \), it follows that the net \((q_\varepsilon)_{\varepsilon \in (0,1]}\) is \( L^\infty(\mathbb{R}^d) \)-moderate in the sense of Definition \(2.1\).

(b) By the rescaling mollification process described in \[31\ Prop. 1.5\] it follows that every distribution of finite order can be regularized such that the obtained regularization net is of \( L^\infty(\mathbb{R}^d) \)-log-type. \( \square \)

**Remark 1.** (i) Note that if \( q \in L^\infty(\mathbb{R}^d) \) and the regularizing net \((q_\varepsilon)_{\varepsilon \in (0,1]}\) is defined in \((2.24)\), then there exists \( C > 0 \) such that for each \( \varepsilon \in (0,1] \)

\[
\|q_\varepsilon\|_{L^\infty(\mathbb{R}^d)} \leq C,
\]
i.e., the regularizing net \((q_\varepsilon)_{\varepsilon \in (0, 1]}\) is \(L^\infty(\mathbb{R}^d)\)-moderate in the sense of Definition 2.1 with \(N = 0\).

(ii) Results of Lemma 4 can be locally extended to \(q \in D'(\mathbb{R}^d)\). This can be done by introducing an open covering of \(\mathbb{R}^d\), corresponding cut-off functions and partition of unity subordinated to the covering. For the details we refer to [12].

In a similar way as it was done in [4] we define a very weak solution to the deterministic parabolic initial value problem \((2.16)\) with potential \(q \in D'(\mathbb{R}^d)\) and then prove its existence, uniqueness and compatibility with the solution obtained in Theorem 3.

**Definition 2.2** (Very weak solutions for deterministic parabolic initial value problem). A net \((u_\varepsilon)_{\varepsilon \in (0, 1]} \subseteq C([0, T]; D) \subseteq C([0, T]; L^2(\mathbb{R}^d))\) is a very weak solution to the problem \((2.16)\) if there exists a regularizing net of smooth functions \((q_\varepsilon)_{\varepsilon \in (0, 1]}\) of the potential \(q \in D'(\mathbb{R}^d)\) such that for every \(\varepsilon \in (0, 1]\), \(u_\varepsilon\) is a solution to

\[
(2.25) \quad \left( \frac{\partial}{\partial t} - \mathcal{L} \right) u(t, x) + q_\varepsilon(x) \cdot u(t, x) = f(t, x), \quad t \in (0, T], \; x \in \mathbb{R}^d,
\]

\[
u(0, x) = g(x),
\]

and \((u_\varepsilon)_{\varepsilon \in (0, 1]}\) is \(C([0, T]; L^2(\mathbb{R}^d))\)-moderate.

We note that \((2.26)\) gives the net of the problems which we will often referred as the net of regularized problems corresponding to \((2.16)\).

**Theorem 5.** Let the operator \(\mathcal{L}\) be as in (H1), the force term \(f \in C([0, T]; L^2(\mathbb{R}^d))\) a Lipschitz continuous function with respect to \(t\), and the initial condition \(g \in D\). Let the potential \(q \in D'(\mathbb{R}^d)\). Then, the problem \((2.16)\) has a very weak solution in sense of Definition 2.2.

**Proof.** We start by regularizing the potential \(q \in D'(\mathbb{R}^d)\) by means of convolution with a mollifying net \((\varphi_\varepsilon)_{\varepsilon \in (0, 1]}\) given by \((2.23)\), where \(\varphi\) is chosen so that the obtained net of smooth functions \(q_\varepsilon(x) = q * \varphi_\varepsilon(x)\), \(\varepsilon \in (0, 1]\), is of log-type (possible by Lemma 4 (b), and the Remark 1 (ii)), i.e., there exists \(N_q > 0\) so that

\[
(2.26) \quad \|q_\varepsilon\|_{L^\infty(\mathbb{R}^d)} \leq N_q \log \frac{1}{\varepsilon}.
\]

For each \(\varepsilon \in (0, 1]\), since \(q_\varepsilon \in L^\infty(\mathbb{R}^d)\) by Theorem 3 there exists solution \(u_\varepsilon \in C([0, T]; D)\) to the regularized problem \((2.25)\) satisfying the estimate \((2.17)\), i.e.,

\[
\|u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R}^d)} \leq M e^{(\omega + M\|q_\varepsilon(\cdot)\|_{L^\infty(\mathbb{R}^d)t}} \left( \|g(\cdot)\|_{L^2(\mathbb{R}^d)} + \int_0^t \|f(s, \cdot)\|_{L^2(\mathbb{R}^d)} ds \right)
\]

for \(t \in [0, T]\). We obtain the net \((u_\varepsilon)_{\varepsilon \in (0, 1]} \subseteq C([0, T]; D) \subseteq C([0, T]; L^2(\mathbb{R}^d))\), which will represent a very weak solution in the sense of Definition 2.2 only
if it is $C([0, T]; L^2(\mathbb{R}^d))$-moderate. By (2.26) we further obtain
\[
\|u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R}^d)} \leq Me^{(w+M\|q_\varepsilon(\cdot)\|_{L^\infty(\mathbb{R}^d)})t} \left( \|g(\cdot)\|_{L^2(\mathbb{R}^d)} + \int_0^t \|f(s, \cdot)\|_{L^2(\mathbb{R}^d)} ds \right)
\]
\[
\leq Me^{wt}e^{-MN_q t \log \varepsilon} \left( \|g(\cdot)\|_{L^2(\mathbb{R}^d)} + \int_0^t \|f(s, \cdot)\|_{L^2(\mathbb{R}^d)} ds \right)
\]
\[
= Me^{wt} \left( \|g(\cdot)\|_{L^2(\mathbb{R}^d)} + \int_0^t \|f(s, \cdot)\|_{L^2(\mathbb{R}^d)} ds \right) \varepsilon^{-N}
\]
\[
= c(t) \varepsilon^{-N},
\]
where $c(t) = Me^{wt}(\|g(\cdot)\|_{L^2(\mathbb{R}^d)} + \int_0^t \|f(s, \cdot)\|_{L^2(\mathbb{R}^d)} ds)$ and $N = M N_q t$ are positive constants for each $t \in (0, T]$. Finally, we have
\[
\|u_\varepsilon\|_{C([0, T]; L^2(\mathbb{R}^d))} = \sup_{t \in [0, T]} (\|u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R}^d)}) \leq \sup_{t \in [0, T]} (c(t) \varepsilon^{-N}) \leq C(T) \varepsilon^{-N}.
\]
Therefore, there exist $N \in \mathbb{N}_0$ and $C = C(T) > 0$ such that
\[
\|u_\varepsilon\|_{C([0, T]; L^2(\mathbb{R}^d))} \leq C \varepsilon^{-N}
\]
holds, i.e., $(u_\varepsilon)_{\varepsilon \in (0, 1]}$ is a $C([0, T]; L^2(\mathbb{R}^d))$-moderate net. \qed

**Remark 2.**
(i) Note that in order to have a moderate solution family, we have to choose a regularization of $q$ to be of log-type.
(ii) If the net $(u_\varepsilon)_{\varepsilon \in (0, 1]}$ is $C([0, T]; L^2(\mathbb{R}^d))$-moderate, then the net $(u_\varepsilon(t, \cdot))_{\varepsilon \in (0, 1]}$ is $L^2(\mathbb{R}^d)$-moderate. Indeed, there exist $N \in \mathbb{N}_0$ and $C > 0$ such that
\[
\|u(t, \cdot)\|_{L^2(\mathbb{R}^d)} \leq \sup_{t \in [0, T]} \|u(t, \cdot)\|_{L^2(\mathbb{R}^d)} = \|u_\varepsilon\|_{C([0, T]; L^2(\mathbb{R}^d))} \leq C \varepsilon^{-N}.
\]
Along the proof of the previous theorem, one can see that the opposite is also true.

Theorem 5 extends the results from [11], where the authors considered the heat equation with singular potential, but without a force term. Namely, they studied a special case of (2.16) with $\mathcal{L} = \Delta$ and $f = 0$ and showed existence and uniqueness of a very weak solution as well as consistency of the obtained solution with the classical solution. In what follows we will obtain the same for our problem (2.16). The semilinear heat equation with singular potential and a non-zero force term, i.e., the problem (2.16) with $\mathcal{L} = \Delta$, is considered in [30] using semigroup approach in the framework of Colombeau theory.

### 2.2.3. Questions on uniqueness and consistency with classical weak solutions
The question of uniqueness for the very weak solution obtained in Theorem 5 can be treated in different manners. If we assume that $(u_\varepsilon)_{\varepsilon \in (0, 1]}$ and $(\tilde{u}_\varepsilon)_{\varepsilon \in (0, 1]}$ are two very weak solutions obtained in Theorem 5 according to Definition 2.2 it means that there exist regularizing nets $(q_\varepsilon)_{\varepsilon \in (0, 1]}$ and
Theorem 6

Let \( (\tilde{u}_\varepsilon)_{\varepsilon \in (0,1]} \) be two regularizing nets of \( q \in D'(\mathbb{R}^d) \) so that for all \( n \in \mathbb{N} \) there exists \( c > 0 \) such that

\[
\|q_\varepsilon - \tilde{q}_\varepsilon\|_{L^\infty(\mathbb{R}^d)} \leq c \varepsilon^n.
\]

Let \( (u_\varepsilon)_{\varepsilon \in (0,1]} \) and \( (\tilde{u}_\varepsilon)_{\varepsilon \in (0,1]} \) be two very weak solutions of the problem (2.16) with respective regularizing nets \( (q_\varepsilon)_{\varepsilon \in (0,1]} \) and \( (\tilde{q}_\varepsilon)_{\varepsilon \in (0,1]} \) of the potential \( q \in D'(\mathbb{R}^d) \). Then, for all \( k \in \mathbb{N} \) there exists \( C > 0 \) such that

\[
\|u_\varepsilon - \tilde{u}_\varepsilon\|_{C([0,T];L^2(\mathbb{R}^d))} \leq C \varepsilon^k.
\]

Proof. By assumptions, for each \( \varepsilon \in (0,1] \), \( u_\varepsilon \) and \( \tilde{u}_\varepsilon \) satisfy the regularized problem (2.25) with \( q_\varepsilon \) and \( \tilde{q}_\varepsilon \) as potentials respectively, and both nets \( (u_\varepsilon)_{\varepsilon \in (0,1]} \) and \( (\tilde{u}_\varepsilon)_{\varepsilon \in (0,1]} \) are \( C([0,T];L^2(\mathbb{R}^d)) \)-moderate. In the view of Remark 2\( (i) \) both nets \( (q_\varepsilon)_{\varepsilon \in (0,1]} \) and \( (\tilde{q}_\varepsilon)_{\varepsilon \in (0,1]} \) are of log-type. Then it holds

\[
\frac{\partial}{\partial t}(u_\varepsilon - \tilde{u}_\varepsilon)(t,x) - \mathcal{L}(u_\varepsilon - \tilde{u}_\varepsilon)(t,x) + q_\varepsilon(x) \cdot (u_\varepsilon - \tilde{u}_\varepsilon)(t,x) = f_\varepsilon(t,x),
\]

\[
(u_{\gamma,\varepsilon} - \tilde{u}_{\gamma,\varepsilon})(0,x) = 0,
\]

with

\[
f_\varepsilon(t,x) = (\tilde{q}_\varepsilon - q_\varepsilon)(x)\tilde{u}_\varepsilon(t,x).
\]

Since \( (\tilde{u}_\varepsilon)_{\varepsilon \in (0,1]} \) is a very weak solution, it is \( C([0,T];L^2(\mathbb{R}^d)) \)-moderate, but also \( (\tilde{u}_\varepsilon(t,\cdot))_{\varepsilon \in (0,1]} \) is \( L^2(\mathbb{R}^d) \)-moderate (see Remark 2\( (ii) \)), i.e., there exist \( N \in \mathbb{N}_0 \) and \( C_1 > 0 \) such that

\[
\|\tilde{u}_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R}^d)} \leq C_1 \varepsilon^{-N}.
\]
Using the estimate \( \| q \|_{L^\infty([0,1])} \) together with \( \| q \|_{L^\infty([0,1])} \) and the fact that the regularizing net \( (q_\varepsilon)_{\varepsilon \in (0,1)} \) is of log-type, we obtain

\[
\| (u_\varepsilon - \tilde{u}_\varepsilon)(t, \cdot) \|_{L^2(\mathbb{R}^d)} \leq M e^{\| q \|_{L^\infty(\mathbb{R}^d)^t} \int_0^t \| (q_\varepsilon - q)(\cdot) \|_{L^\infty(\mathbb{R}^d)} ds
\leq M e^{\| q \|_{L^\infty(\mathbb{R}^d)}} \int_0^t \| q_\varepsilon(\cdot) \|_{L^\infty(\mathbb{R}^d)} ds
\leq M e^{\| q \|_{L^\infty(\mathbb{R}^d)}} \int_0^t C_1 e^{-N} ds
= M e^{\| q \|_{L^\infty(\mathbb{R}^d)}} e^{n-N-MN_q t}
= c(t) \varepsilon^{n-N-MN_q t}
\]

for arbitrary \( n \). We choose \( n > N_0 := N + MN_q T \) to obtain

\[
\| (u_\varepsilon - \tilde{u}_\varepsilon)(t, \cdot) \|_{L^2(\mathbb{R}^d)} \leq C(T) \varepsilon^k
\]

for all \( k \in \mathbb{N} \).

The question whether the notion of very weak solution obtained in Theorem 5 for the case when the potential is bounded regular function coincides with the solution guaranteed by Theorem 3 is treated in the following theorem.

**Theorem 7 (Consistency).** Let the operator \( \mathcal{L} \) be as in (H1), the force term \( f \in C([0, T]; L^2(\mathbb{R}^d)) \) is a Lipschitz continuous function with respect to \( t \), the initial condition \( g \in D \), and let the potential \( q \in L^\infty(\mathbb{R}^d) \cap C(\mathbb{R}^d) \).

Let \( v \in C([0, T]; D) \) be the weak solution to the parabolic initial value problem \( (2.16) \) obtained by Theorem 3 and let \( (u_\varepsilon)_{\varepsilon \in (0,1)} \subset C([0, T]; L^2(\mathbb{R}^d)) \) be the very weak solution to \( (2.16) \) obtained by Theorem 7. Then,

\[
\| u_\varepsilon - v \|_{C([0, T]; L^2(\mathbb{R}^d))} \to 0 \quad \text{as} \quad \varepsilon \to 0.
\]

**Proof.** Let \( (q_\varepsilon)_{\varepsilon \in (0,1)} \) be a regularizing net \( (2.21) \) of given \( q \in L^\infty(\mathbb{R}^d) \) and \( C(\mathbb{R}^d) \). Then \( (q_\varepsilon)_{\varepsilon \in (0,1)} \) converges to \( q \) uniformly, i.e., \( \| q - q_\varepsilon \|_{L^\infty(\mathbb{R}^d)} \to 0 \), as \( \varepsilon \to 0 \). By the assumptions, for every \( \varepsilon \in (0, 1) \), \( u_\varepsilon \) satisfies \( (2.25) \) with \( q_\varepsilon \) from the above regularization, while \( v \) satisfies \( (2.16) \). We subtract \( (2.25) \) and \( (2.16) \) and obtain

\[
\partial_t (u_\varepsilon - v) - \mathcal{L}(u_\varepsilon - v) + q_\varepsilon \cdot u_\varepsilon - q \cdot v = 0,
\]

\[
(u_\varepsilon - v)_{t=0} = 0.
\]

After adding and subtracting \( q_\varepsilon v \), and regrouping the factors, the previous equation transforms to

\[
\left( \frac{\partial}{\partial t} - \mathcal{L} + q_\varepsilon \cdot Id \right) (u_\varepsilon - v) = (q - q_\varepsilon) v,
\]

\[
(u_\varepsilon - v)_{t=0} = 0.
\]
Hence, for all $\varepsilon$ the difference $u_\varepsilon - v$ satisfies (2.16) with zero initial condition, in which $q$ is replaced by $q_\varepsilon$ and $f$ is replaced by $(q - q_\varepsilon) v$. Thus, by Theorem 3, the estimate (2.17) and Remark 1(i) the difference $u_\varepsilon - v$ satisfies

$$
\| (u_\varepsilon - v)(t, \cdot) \|_{L^2(\mathbb{R}^d)} \leq M e^{w t} e^{M \| q_\varepsilon \|_{L^\infty(\mathbb{R}^d)} t} \int_0^t \| (q - q_\varepsilon)(\cdot) v(\cdot, \cdot) \|_{L^2(\mathbb{R}^d)} ds
$$

$$
\leq M e^{w t} e^{MC t} \| (q - q_\varepsilon)(\cdot) \|_{L^\infty(\mathbb{R}^d)} \int_0^t \| v(s, \cdot) \|_{L^2(\mathbb{R}^d)} ds
$$

$$
\leq \widetilde{C}(t) \| (q - q_\varepsilon)(\cdot) \|_{L^\infty(\mathbb{R}^d)},
$$

where $\widetilde{C}(t) := M e^{(w + MC)t} \int_0^t \| v(s, \cdot) \|_{L^2(\mathbb{R}^d)} ds$ is a finite constant. Finally, since $\| q - q_\varepsilon \|_{L^\infty(\mathbb{R}^d)} \to 0$, as $\varepsilon \to 0$ we have also that $\| u_\varepsilon - v \|_{C([0,T];L^2(\mathbb{R}^d))} = \sup_{t \in [0,T]} \| (u_\varepsilon - v)(t, \cdot) \|_{L^2(\mathbb{R}^d)}$ tends to 0 as $\varepsilon \to 0$.

\[ \square \]

3. A VERY WEAK SOLUTIONS FOR THE STOCHASTIC PARABOLIC EQUATIONS WITH SINGULAR SPACE DEPENDING POTENTIALS

We are now ready to analyse the stochastic parabolic problem (1.3) for $q \in \mathcal{D}'(\mathbb{R}^d)$ combining chaos expansion representations and the concept of very weak solutions. In the first step, we proceed as in Theorem 2 and represent all stochastic processes appearing in (1.3) in their chaos expansion forms. We also represent a solution in the chaos expansion form (1.7), i.e.,

$$
(3.29) \quad U(t,x,\omega) = \sum_{\gamma \in \mathcal{I}} u_\gamma(t,x) H_\gamma(\omega), \quad t \in [0,T], \ x \in \mathbb{R}^d, \ \omega \in \Omega.
$$

After equalizing corresponding coefficients in the Fourier-Hermite polynomial basis, as in Theorem 2 we obtain deterministic problems, but in this case now with irregular potential $q \in \mathcal{D}'(\mathbb{R}^d)$. For each $\gamma \in \mathcal{I}$ the existence of a very weak solution to the obtained deterministic PDE is guaranteed by Theorem 5 which determines the coefficients $u_\gamma$, $\gamma \in \mathcal{I}$ of the solution (3.29). Since very weak solution is a net, with this procedure we obtain a net $(U_\varepsilon)_{\varepsilon \in (0,1]}$ for which we have to prove that it belongs to a Kondratiev-type space of generalized stochastic processes. The notion for moderate nets of generalized stochastic processes appears naturally as follows.

**Definition 3.1 (Moderate nets of generalized stochastic processes).**

Let $(\mathcal{B}, \| \cdot \|_\mathcal{B})$ be a Banach space. Let $(U_\varepsilon)_{\varepsilon \in (0,1]} \subset \mathcal{B} \otimes (S)_{-1}$ be a net of generalized stochastic processes $U_\varepsilon \in \mathcal{B} \otimes (S)_{-p_\varepsilon}$, $p_\varepsilon \geq 0$, and for every $\varepsilon \in (0,1]$

$$
(3.30) \quad U_\varepsilon = \sum_{\gamma \in \mathcal{I}} (u_\gamma)_\varepsilon H_\gamma.
$$

Assume $p := \sup_{\varepsilon \in (0,1]} p_\varepsilon$ exists. If for all $\gamma \in \mathcal{I}$ the coefficients $[(u_\gamma)_\varepsilon]_{\varepsilon \in (0,1]}$ are $\mathcal{B}$-moderate nets, then $(U_\varepsilon)_{\varepsilon \in (0,1]}$ is called $\mathcal{B} \otimes (S)_{-p}$-moderate.
Remark 3. (i) If the supremum \( p = \sup_{\varepsilon \in [0, 1]} p_\varepsilon \) does not exist, one can define moderateness in the Fréchet space \( \mathcal{B} \otimes (S)^{-1} \) with seminorms estimates as it was done in [3]. For simplicity and the presentation of the method, here we work only with Banach spaces.

(ii) If \( (U_\varepsilon)_{\varepsilon \in [0, 1]} \) is \( \mathcal{B} \otimes (S)^{-p} \)-moderate then it is also \( \mathcal{B} \otimes (S)^{-r} \)-moderate for each \( r \geq p \).

(iii) Note that if \( p = 0 \) then the net \( (U_\varepsilon)_{\varepsilon \in [0, 1]} \) is \( \mathcal{B} \otimes L^2(\mu) \)-moderate.

One can proceed differently, starting with the regularization of distribution \( \eta \), one can consider a net of regularized stochastic parabolic equations, then solve each of them by Theorem 2, which will result in the net of generalized stochastic processes. To have moderateness for this net a different notion for moderate nets appears as natural. For a better insight, we state and prove the following lemma, which assures that two approaches are equivalent.

Lemma 8. Assume \( (U_\varepsilon)_{\varepsilon \in [0, 1]} \subset \mathcal{B} \otimes (S)^{-1} \) is a net of generalized stochastic process \( U_\varepsilon \in \mathcal{B} \otimes (S)^{-p_\varepsilon} \), \( p_\varepsilon \geq 0 \). Assume that there exists \( p = \sup_{\varepsilon \in [0, 1]} p_\varepsilon \), and \( p > 1 \). Then the net \( (U_\varepsilon)_{\varepsilon \in [0, 1]} \) is \( \mathcal{B} \otimes (S)^{-p} \)-moderate if and only if there exist \( N \in \mathbb{N}_0 \) and \( C > 0 \) such that

\[
(3.31) \quad \|U_\varepsilon\|_{\mathcal{B} \otimes (S)^{-p}} \leq C\varepsilon^{-N}.
\]

Proof. If \( (U_\varepsilon)_{\varepsilon \in [0, 1]} \) is \( \mathcal{B} \otimes (S)^{-p} \)-moderate then for all \( \gamma \in \mathcal{I} \) the coefficients \( [(u_\gamma)_{\varepsilon}]_{\varepsilon \in [0, 1]} \) in the chaos expansion \((3.30)\) are \( \mathcal{B} \)-moderate, i.e., there exist \( N_1 \in \mathbb{N} \) and \( C_1 > 0 \) such that

\[
(3.32) \quad \|(u_\gamma)_{\varepsilon}\|_{\mathcal{B}} \leq C_1\varepsilon^{-N_1}.
\]

Also, for each \( \varepsilon \in (0, 1] \), \( U_\varepsilon \) is a stochastic process in \( \mathcal{B} \otimes (S)^{-p_\varepsilon} \subset \mathcal{B} \otimes (S)^{-p} \), i.e., it holds

\[
(3.33) \quad \|U_\varepsilon\|_{\mathcal{B} \otimes (S)^{-p}}^2 = \sum_{\gamma \in \mathcal{I}} \|(u_\gamma)_{\varepsilon}\|_{\mathcal{B}}^2 (2N)^{-p\gamma} < \infty.
\]

Combining \((3.32)\) and \((3.33)\) we obtain

\[
\|U_\varepsilon\|_{\mathcal{B} \otimes (S)^{-p}}^2 = \sum_{\gamma \in \mathcal{I}} \|(u_\gamma)_{\varepsilon}\|_{\mathcal{B}}^2 (2N)^{-p\gamma} \leq C_1^2 \varepsilon^{-2N_1} \sum_{\gamma \in \mathcal{I}} (2N)^{-p\gamma} = C_1^2 C_2 \varepsilon^{-2N_1},
\]

with \( C_2 = \sum_{\gamma \in \mathcal{I}} (2N)^{-p\gamma} \) being finite for \( p > 1 \) by \((1.3)\). If we take \( C = C_1^2 C_2 \) and \( N = 2N_1 \) we finish the first part of the proof.

Conversely, suppose that there exist \( N \in \mathbb{N}_0 \) and \( C > 0 \) such that \((3.31)\) holds with \( p = \sup_{\varepsilon \in [0, 1]} p_\varepsilon \). Suppose that the net \( (U_\varepsilon)_{\varepsilon \in [0, 1]} \) is not \( \mathcal{B} \otimes (S)^{-p} \)-moderate. This means that for some \( \gamma_0 \in \mathcal{I} \) the coefficients \( (u_{\gamma_0})_{\varepsilon} \) are not \( \mathcal{B} \)-moderate, i.e., for all \( N \in \mathbb{N}_0 \) and \( C > 0 \) it holds

\[
(3.34) \quad \|(u_{\gamma_0})_{\varepsilon}\|_{\mathcal{B}} > C\varepsilon^{-N}.
\]
Therefore, using (3.33) and (3.34) we obtain that for all \( N \in \mathbb{N}_0 \) and \( C > 0 \) it holds
\[
\| U_{\varepsilon} \|_{B \otimes (S)_{-p}} = \sum_{\gamma \in \mathcal{I}} \| (u_{\gamma})_{\varepsilon} \|_{B}^{2} (2N)^{-p}\gamma} = \sum_{\gamma \in \mathcal{I} \setminus \{ \gamma_0 \}} \| (u_{\gamma})_{\varepsilon} \|_{B}^{2} (2N)^{-p}\gamma} + \| (u_{\gamma_0})_{\varepsilon} \|_{B}^{2} (2N)^{-p\gamma_0} > \sum_{\gamma \in \mathcal{I} \setminus \{ \gamma_0 \}} \| (u_{\gamma})_{\varepsilon} \|_{B}^{2} (2N)^{-p}\gamma} + C^2 \varepsilon^{-2N} (2N)^{-p\gamma_0} > \bar{C}_\varepsilon^{-N},
\]
which leads to contradiction. \( \square \)

Since \( B \otimes (S)_{-p_\varepsilon} \), \( p_\varepsilon \geq 0 \) are Banach spaces, according to Definition 2.1 and Lemma 8, moderateness of generalized stochastic processes can be defined in the following way.

**Definition 3.2.** Let \((B, \| \cdot \|_B)\) be a Banach space. Let \((U_\varepsilon)_{\varepsilon \in (0,1]} \subset B \otimes (S)_{-1}\) be a net of generalized stochastic processes \( U_\varepsilon \in B \otimes (S)_{-p_\varepsilon}, p_\varepsilon \geq 0 \), and let \( p = \sup \ p_\varepsilon \) exists. The net of generalized stochastic processes \((U_\varepsilon)_{\varepsilon \in (0,1]}\) is \( B \otimes (S)_{-p}\)-moderate if there exist \( N \in \mathbb{N}_0 \) and \( C > 0 \) such that
\[
\| U_{\varepsilon} \|_{B \otimes (S)_{-p}} \leq C \varepsilon^{-N}.
\]

### 3.1. A stochastic very weak solution: Definition and existence.

In this section we define a stochastic very weak solution to the stochastic parabolic equation (1.3) with the potential \( q \in \mathcal{D}'(\mathbb{R}^d) \), and prove its existence. In the following we use \( X := C([0,T]; L^2(\mathbb{R}^d)) \) and recall that \( \mathcal{X}(L^2(\mathbb{R}^d)) := X \otimes (S)_{-1} \).

**Definition 3.3 (Solution concept 1).** Let the potential \( q \) be in \( \mathcal{D}'(\mathbb{R}^d) \). A net of stochastic processes \((U_\varepsilon)_{\varepsilon \in (0,1]} \) in \( X \otimes (S)_{-1} \), which is \( X \otimes (S)_{-p}\)-moderate in the sense of Definition 3.1, is a very weak solution to the stochastic parabolic problem (1.3) if for each \( \gamma \in \mathcal{I} \) the net \([u_{\gamma_{\varepsilon}}]_{\varepsilon \in (0,1]} \) of the coefficients in expansion (3.30) is a very weak solution to
\[
\left( \frac{\partial}{\partial t} - \mathcal{L} \right) u_{\gamma}(t,x) + q(x) \cdot u_{\gamma}(t,x) = f_{\gamma}(x,t) \quad t \in (0,T], \ x \in \mathbb{R}^d,
\]
\[
u_{\gamma}(0,x) = g_{\gamma}(x).
\]

**Definition 3.4 (Solution concept 2).** Let the potential \( q \) be in \( \mathcal{D}'(\mathbb{R}^d) \). A net of stochastic processes \((U_\varepsilon)_{\varepsilon \in (0,1]} \) in \( X \otimes (S)_{-1} \) is a very weak solution to the stochastic problem (1.3) if there exists an \( L^\infty(\mathbb{R}^d)\)-moderate regularizing net \((q_\varepsilon)_{\varepsilon \in (0,1]} \) of the distribution \( q \), such that for every \( \varepsilon \in (0,1] \) the process
\[ U_\varepsilon \text{ is a solution to the regularized stochastic parabolic equation} \]
\[
\left( \frac{\partial}{\partial t} - \mathcal{L} \right) U(t, x, \omega) + q_\varepsilon(x) \cdot U(t, x, \omega) = F(t, x, \omega), \]
\[
U(0, x, \omega) = G(x, \omega),
\]

and it is a moderate net of stochastic processes in the sense of Definition 2.2.

**Remark 4.** We note that we work with double nets \((\gamma, \varepsilon)\), where \(\gamma \in \mathcal{I}\) and \(\varepsilon \in (0, 1]\), i.e., for each \(\gamma \in \mathcal{I}\) we have a sequence \((u_\gamma)_\varepsilon, \varepsilon \in (0, 1]\). In the solution concept 1 for each \(\varepsilon \in (0, 1]\) the stochastic process \(U_\varepsilon\) is given by
\[
U_\varepsilon = \sum_{\gamma \in \mathcal{I}} (u_\gamma)_\varepsilon H_\gamma,
\]
while in the solution concept 2 for each \(\varepsilon \in (0, 1]\) the stochastic process \(U_\varepsilon\) is given by
\[
U_\varepsilon = \sum_{\gamma \in \mathcal{I}} (u_\varepsilon)_\gamma H_\gamma.
\]

From Lemma 8 it follows that solution concept 1 and solution concept 2 are equivalent for \(p > 1\). As the restriction \(p \geq 0\) to \(p > 1\) is not essential in the chaos expansion setting, in the following we will assume \(p > 1\) and use the equivalence of these two concepts. In the sequel, we will also write \(u_{\gamma, \varepsilon}\) instead of \((u_\gamma)_\varepsilon\) or \((u_\varepsilon)_\gamma\).

**Theorem 9 (Existence of a stochastic very weak solution).** Let the operator \(\mathcal{L}\), the force term \(F\) and the initial condition \(G\) be such that the assumptions \((H1)-(H3)\) hold and let the potential \(q \in D'(\mathbb{R}^d)\). Then, the stochastic parabolic problem (1.3) has a very weak solution \((U_\varepsilon)_{\varepsilon \in (0, 1]}\) in \(X(L^2(\mathbb{R}^d))\).

**Proof.** Firstly, we write processes \(F\) and \(G\) in their chaos expansion forms as in Theorem 2 i.e., \(F\) is of the form (2.9) such that the condition (2.10) holds for some \(p_1 \geq 0\), and \(G\) is of the form (2.11) such that the condition (2.12) holds for some \(p_2 \geq 0\). We also assume that the unknown process is given in the form (3.29). After applying the chaos expansion method, for every \(\gamma \in \mathcal{I}\) we obtain the equation
\[
\left( \frac{\partial}{\partial t} - \mathcal{L} \right) u_\gamma(t, x) + q(x) \cdot u_\gamma(t, x) = f_\gamma(x, t),
\]
\[
u_\gamma(0, x) = g_\gamma(x),
\]

where \(q \in D'(\mathbb{R}^d)\). From the assumptions \((H1)-(H3)\) we have for all \(\gamma \in \mathcal{I}\) that \(f_\gamma \in C([0, T]; L^2(\mathbb{R}^d))\) are Lipschitz continuous functions with respect to \(t\) and \(g_\gamma \in D\). By Theorem 5 for each \(\gamma \in \mathcal{I}\) there exists a very weak solution to \((3.35)\) i.e., a \(X\)-moderate net \([(u_\gamma)_\varepsilon]_{\varepsilon \in (0, 1]} = (u_{\gamma, \varepsilon})_{\varepsilon \in (0, 1]}\)
in $C([0,T], D)$, satisfying (3.35) and there exists $N > 0$ such that (3.36)

$$
\|u_{\gamma, \varepsilon}(t, \cdot)\|_{L^2(\mathbb{R}^d)} \leq M e^{wT} \left( \|g_\gamma(\cdot)\|_{L^2(\mathbb{R}^d)} + \int_0^t \|f_\gamma(\tau, \cdot)\|_{L^2(\mathbb{R}^d)} \, d\tau \right) \varepsilon^{-N}.
$$

Note that for each $\gamma \in \mathcal{I}$, the chosen constant $N = N(M, T, w, N_q)$ does not depend on $\gamma$. Finally, we are going to show that for every $\varepsilon \in (0, 1]$ a process defined by

$$
U_\varepsilon(t, x, \omega) := \sum_{\gamma \in \mathcal{I}} u_{\gamma, \varepsilon}(t, x) H_\gamma(\omega),
$$

is a generalized stochastic process in $X(L^2(\mathbb{R}^d))$ which is moderate, i.e., we have to show that for each $\varepsilon \in (0, 1]$ there exists $p_\varepsilon > 0$ such that $U_\varepsilon \in X \otimes (S)_{-p_\varepsilon}$ and that (3.31) holds. We are going to prove that for all $\varepsilon \in (0, 1]$ there exist $N_1 > 0$ and $c > 0$ such that

$$
\|U_\varepsilon\|_{X \otimes (S)_{-p}}^2 = \sum_{\gamma \in \mathcal{I}} \|u_{\gamma, \varepsilon}\|_{X}^2 (2N)^{-p_\gamma} < c \varepsilon^{-N_1}
$$

for $p = \max\{p_1, p_2\}$. As the coefficients $u_{\gamma, \varepsilon}$ satisfy (3.36) we obtain

$$
\|U_\varepsilon\|_{X \otimes (S)_{-p}}^2 = \sum_{\gamma \in \mathcal{I}} \|u_{\gamma, \varepsilon}\|^2_{C([0,T];L^2(\mathbb{R}^d))} (2N)^{-p_\gamma} = \sum_{\gamma \in \mathcal{I}} \left( \sup_{t \in [0,T]} \|u_{\gamma, \varepsilon}(t, \cdot)\|_{L^2(\mathbb{R}^d)} \right)^2 (2N)^{-p_\gamma}
$$

\begin{align*}
&\leq \sum_{\gamma \in \mathcal{I}} \left\{ \sup_{t \in [0,T]} Me^{wT} \left( \|g_\gamma(\cdot)\|_{L^2(\mathbb{R}^d)} + \int_0^t \|f_\gamma(\tau, \cdot)\|_{L^2(\mathbb{R}^d)} \, d\tau \right) \varepsilon^{-N} \right\}^2 (2N)^{-p_\gamma} \\
&\leq \sum_{\gamma \in \mathcal{I}} \left\{ Me^{wT} \|g_\gamma(\cdot)\|_{L^2(\mathbb{R}^d)} \varepsilon^{-N} + Me^{wT} \varepsilon^{-N} \int_0^T \|f_\gamma(\tau, \cdot)\|_{L^2(\mathbb{R}^d)} \, d\tau \right\}^2 (2N)^{-p_\gamma} \\
&\leq \sum_{\gamma \in \mathcal{I}} \left\{ 2M^2 e^{2wT} \|g_\gamma(\cdot)\|_{L^2(\mathbb{R}^d)}^2 \varepsilon^{-2N} + 2M^2 e^{2wT} \varepsilon^{-2N} \left( \int_0^T \|f_\gamma(\tau, \cdot)\|_{L^2(\mathbb{R}^d)} \, d\tau \right)^2 \right\} (2N)^{-p_\gamma} \\
&\leq 2M^2 e^{2wT} \left\{ \sum_{\gamma \in \mathcal{I}} \|g_\gamma(\cdot)\|_{L^2(\mathbb{R}^d)}^2 (2N)^{-p_\gamma} + \sum_{\gamma \in \mathcal{I}} \left( \int_0^T \|f_\gamma(\tau, \cdot)\|_{L^2(\mathbb{R}^d)} \, d\tau \right)^2 (2N)^{-p_\gamma} \right\} \varepsilon^{-2N} \\
&= 2M^2 e^{2wT} (I_1 + I_2) \varepsilon^{-2N}.
\end{align*}

Then, for $p \geq p_2$ we obtain

$$
I_1 = \sum_{\gamma \in \mathcal{I}} \|g_\gamma(\cdot)\|_{L^2(\mathbb{R}^d)}^2 (2N)^{-p_\gamma} \leq \sum_{\gamma \in \mathcal{I}} \|g_\gamma(\cdot)\|_{L^2(\mathbb{R}^d)}^2 (2N)^{-p_\gamma}.
$$
which is finite by (2.12). Similarly, for \( p \geq p_1 \)

\[
I_2 = \sum_{\gamma \in \mathcal{I}} \left( \int_0^T \left\| f_{\gamma}(\tau, \cdot) \right\|_{L^2(\mathbb{R}^d)} d\tau \right)^2 (2N)^{-p\gamma} 
\leq \sum_{\gamma \in \mathcal{I}} \left( \int_0^T \sup_{\tau \in [0,T]} \left\| f_{\gamma}(\tau, \cdot) \right\|_{L^2(\mathbb{R}^d)} d\tau \right)^2 (2N)^{-p\gamma} 
= \sum_{\gamma \in \mathcal{I}} \left( \int_0^T \left\| f_{\gamma} \right\|_{X} d\tau \right)^2 (2N)^{-p\gamma} 
= T^2 \sum_{\gamma \in \mathcal{I}} \left\| f_{\gamma} \right\|_{X}^2 (2N)^{-p\gamma} 
\leq \sum_{\gamma \in \mathcal{I}} \left\| f_{\gamma} \right\|_{X}^2 (2N)^{-p_1\gamma}
\]

which is finite by (2.10). We conclude that for every \( \varepsilon \in (0,1] \), the stochastic process \( U_\varepsilon \) is \( X \otimes (S)_{-p}-\text{moderate} \) for \( p \geq \max\{p_1, p_2\} \) and \( U_\varepsilon \) solves (1.3), i.e., according to Definition 3.2 \((U_\varepsilon)_{\varepsilon \in (0,1]} \) is a stochastic very weak solution to the problem (1.3). \[\square\]

3.2. Uniqueness of the stochastic very weak solution. As in the deterministic case the question of uniqueness for the obtained very weak solution to the stochastic parabolic problem (1.3) can be treated in different manners. Let \((U_\varepsilon)_{\varepsilon \in (0,1]}\) and \((\tilde{U}_\varepsilon)_{\varepsilon \in (0,1]}\) be two very weak solutions to the stochastic problem (1.3) that correspond to the same regularizing net \((q_\varepsilon)_{\varepsilon \in (0,1]}\) of the potential \( q \in D'(\mathbb{R}^d) \). Let \( V_\varepsilon := U_\varepsilon - \tilde{U}_\varepsilon, \varepsilon \in (0,1], \) i.e.,

\[
V_\varepsilon(t, x, \omega) = \sum_{\gamma \in \mathcal{I}} v_{\gamma,\varepsilon}(t, x) H_\gamma(\omega) := \sum_{\gamma \in \mathcal{I}} (u_{\gamma,\varepsilon} - \tilde{u}_{\gamma,\varepsilon})(t, x) H_\gamma(\omega).
\]

Then, \((V_\varepsilon)_{\varepsilon \in (0,1]}\) is a very weak solution to the stochastic homogeneous problem

\[
\left( \frac{\partial}{\partial t} - \mathcal{L} \right) V(t, x, \omega) + q(x) \cdot V(t, x, \omega) = 0, \quad V(0, x, \omega) = 0.
\]

Hence, \((v_{\gamma,\varepsilon})_{\varepsilon \in (0,1]}\) is a very weak solution to the corresponding homogeneous deterministic problem

\[
\left( \frac{\partial}{\partial t} - \mathcal{L} \right) v_\gamma(t, x) + q_\varepsilon(x) \cdot v_\gamma(t, x) = 0, \quad v_\gamma(0, x) = 0,
\]

for each \( \gamma \in \mathcal{I} \) and every \( \varepsilon \in (0,1] \). From Theorem 3 and the estimate (2.17) it follows that \( v_{\gamma,\varepsilon} = 0, \varepsilon \in (0,1], \) and therefore \( u_{\gamma,\varepsilon} = \tilde{u}_{\gamma,\varepsilon}, \varepsilon \in (0,1] \). We conclude that \( U_\varepsilon = \tilde{U}_\varepsilon, \varepsilon \in (0,1] \), since their coefficients coincide.
Theorem 10. Let \((q_\varepsilon)_{\varepsilon \in (0,1]}\) and \((\tilde{q}_\varepsilon)_{\varepsilon \in (0,1]}\) be two different regularizing nets of \(q \in \mathcal{D}'(\mathbb{R}^d)\) such that for every \(\varepsilon \in (0,1]\) it holds
\[
\|q_\varepsilon - \tilde{q}_\varepsilon\|_{L^\infty(\mathbb{R}^d)} \leq C \varepsilon^n, \quad \forall n \in \mathbb{N}.
\]

Let \((U_\varepsilon)_{\varepsilon \in (0,1]} \subset \mathcal{X}(L^2(\mathbb{R}^d))\) and \((\tilde{U}_\varepsilon)_{\varepsilon \in (0,1]} \subset \mathcal{X}(L^2(\mathbb{R}^d))\) be two very weak solutions of the stochastic problem \(\ref{eq:1.3}\) which correspond to \((q_\varepsilon)_{\varepsilon \in (0,1]}\) and \((\tilde{q}_\varepsilon)_{\varepsilon \in (0,1]}\) respectively. Then, for all \(\varepsilon \in (0,1]\) and all \(n \in \mathbb{N}\) there exists \(c > 0\) such that
\[
\|U_\varepsilon - \tilde{U}_\varepsilon\|_{\mathcal{X}(S)_{-s}} \leq c \varepsilon^n.
\]

Proof. Two very weak solutions \((U_\varepsilon)_{\varepsilon \in (0,1]}\) and \((\tilde{U}_\varepsilon)_{\varepsilon \in (0,1]}\) to the stochastic initial value problem \(\ref{eq:1.3}\) in \(\mathcal{X}(L^2(\mathbb{R}^d))\) that correspond to \((q_\varepsilon)_{\varepsilon \in (0,1]}\) and \((\tilde{q}_\varepsilon)_{\varepsilon \in (0,1]}\) respectively, are \(C([0,T], L^2(\mathbb{R}^d)) \otimes (S)_{-s}\)-moderate for some \(s > 1\) (see Remark \ref{rem:3}) and can be represented in the chaos expansion forms
\[
U_\varepsilon(t,x,\omega) = \sum_{\gamma \in I} u_{\gamma,\varepsilon}(t,x) H_\gamma(\omega), \quad \text{and} \quad \tilde{U}_\varepsilon(t,x,\omega) = \sum_{\gamma \in I} \tilde{u}_{\gamma,\varepsilon}(t,x) H_\gamma(\omega)
\]
with
\[
\|U_\varepsilon\|_{\mathcal{X}(S)_{-s}} = \sum_{\gamma \in I} \|u_{\gamma,\varepsilon}\|_{\mathcal{X}(2\mathbb{N})^{-s\gamma}}^2 < \infty \quad \text{and} \quad \|\tilde{U}_\varepsilon\|_{\mathcal{X}(S)_{-s}} = \sum_{\gamma \in I} \|\tilde{u}_{\gamma,\varepsilon}\|_{\mathcal{X}(2\mathbb{N})^{-s\gamma}}^2 < \infty.
\]

Then the stochastic process \(U_\varepsilon - \tilde{U}_\varepsilon, \varepsilon \in (0,1]\) has the chaos expansion form
\[
(U_\varepsilon - \tilde{U}_\varepsilon)(t,x,\omega) = \sum_{\gamma \in I} (u_{\gamma,\varepsilon} - \tilde{u}_{\gamma,\varepsilon})(t,x) H_\gamma(\omega).
\]

We need to show that
\[
\|U_\varepsilon - \tilde{U}_\varepsilon\|_{\mathcal{X}(S)_{-s}}^2 = \sum_{\gamma \in I} \|u_{\gamma,\varepsilon} - \tilde{u}_{\gamma,\varepsilon}\|_{\mathcal{X}(2\mathbb{N})^{-s\gamma}}^2 \leq c \varepsilon^n \quad \text{for all} \quad n \in \mathbb{N}.
\]

Since the nets \((u_{\gamma,\varepsilon})_{\varepsilon \in (0,1]}\) and \((\tilde{u}_{\gamma,\varepsilon})_{\varepsilon \in (0,1]}\) are solutions of the corresponding deterministic problems, Theorem \ref{th:6} implies
\[
\|u_{\gamma,\varepsilon} - \tilde{u}_{\gamma,\varepsilon}\|_{\mathcal{X}} \leq c \varepsilon^n \quad \text{for all} \quad n \in \mathbb{N},
\]
and therefore for all \(n \in \mathbb{N}\) it holds
\[
\|U_\varepsilon - \tilde{U}_\varepsilon\|_{\mathcal{X}(S)_{-s}}^2 = \sum_{\gamma \in I} \|u_{\gamma,\varepsilon} - \tilde{u}_{\gamma,\varepsilon}\|_{\mathcal{X}(2\mathbb{N})^{-s\gamma}}^2 \leq \sum_{\gamma \in I} c \varepsilon^n (2\mathbb{N})^{-s\gamma} = C \varepsilon^n,
\]
with \(C = \sum_{\gamma \in I} c (2\mathbb{N})^{-s\gamma} < \infty\) since \(s > 1\).
\(\square\)
3.3. Consistency. In this section we are interested in consistency of the stochastic very weak solution obtained in Theorem 9 with the stochastic (classical weak) solution obtained in Theorem 2 in the following sense.

**Theorem 11.** Consider the stochastic parabolic problem (1.3), let the potential \( q \in L^\infty(\mathbb{R}^d) \cap C(\mathbb{R}^d) \) and let the assumptions \((H1)-(H3)\) hold. Let \( V \in \mathcal{X}(L^2(\mathbb{R}^d)) \) be the solution to the stochastic parabolic problem obtained in Theorem 2 and let \((U_\varepsilon)_{\varepsilon \in (0,1]} \subset \mathcal{X}(L^2(\mathbb{R}^d))\) be the very weak solution to the stochastic problem obtained in Theorem 7. Then, for some \( s > 1 \)

\[
\|U_\varepsilon - V\|_{\mathcal{X} \otimes (S)_{-s}} \to 0 \quad \text{as} \quad \varepsilon \to 0.
\]

**Proof.** Since it is assumed that \((U_\varepsilon)_{\varepsilon \in (0,1]}\) is a very weak solution to the problem (1.3), then for each \( \varepsilon \in (0,1] \) it is a stochastic process in \( \mathcal{X}(L^2(\mathbb{R}^d)) \) represented in the form

\[
U_\varepsilon(t, x, \omega) = \sum_{\gamma \in \mathcal{I}} u_{\gamma,\varepsilon}(t, x) H_\gamma(\omega).
\]

Moreover, \((U_\varepsilon)_{\varepsilon \in (0,1]}\) is \( X \otimes (S)_{-p'} \)-moderate, which is equivalent to the net of coefficients \((u_{\gamma,\varepsilon})_{\varepsilon \in (0,1]}\) being \( X \)-moderate for each \( \gamma \in \mathcal{I} \). Recall, the coefficients \( u_{\gamma,\varepsilon} \) for all \( \gamma \in \mathcal{I} \) and all \( \varepsilon \in (0,1] \) solve

\[
\left( \frac{\partial}{\partial t} - L \right) u_{\gamma}(t, x) + q_{\varepsilon}(x) \cdot u_{\gamma}(t, x) = f_\gamma(t, x), \quad t \in (0, T],
\]

\[
u_{\gamma}(0, x) = g_\gamma(x),
\]

where \( q_{\varepsilon} = q \ast \varphi_\varepsilon \in C^\infty(\mathbb{R}^d) \). On the other side, the process \( V \in \mathcal{X}(L^2(\mathbb{R}^d)) \) is a classical weak solution to (1.3). It has chaos expansion representation of the form

\[
V(t, x, \omega) = \sum_{\gamma \in \mathcal{I}} v_{\gamma}(t, x) H_\gamma(\omega),
\]

where its coefficients \( v_{\gamma} \) for each \( \gamma \in \mathcal{I} \) are (classical) weak solutions to the deterministic problem

\[
\frac{\partial}{\partial t} v_{\gamma}(t, x) - Lv_{\gamma}(t, x) + q(x) \cdot v_{\gamma}(t, x) = f_\gamma(t, x), \quad t \in (0, T], x \in \mathbb{R}^d,
\]

\[
v_{\gamma}(0, x) = g_\gamma(x),
\]

with \( q \in L^\infty(\mathbb{R}^d) \cap C(\mathbb{R}^d) \), such that for \( r \geq 0 \) it holds

\[
\sum_{\gamma \in \mathcal{I}} \|v_{\gamma}\|_{L^2}^2 (2N)^{-r\gamma} < \infty.
\]

In order to prove (3.37) we have to prove

\[
\|U_\varepsilon - V\|_{\mathcal{X} \otimes (S)_{-s}} = \sum_{\gamma \in \mathcal{I}} \|u_{\gamma,\varepsilon} - v_{\gamma}\|_{L^2}^2 (2N)^{-s\gamma} \to 0 \quad \text{as} \quad \varepsilon \to 0.
\]
This is true since by Theorem 7 the very weak solution $u_{\gamma,\varepsilon}$ to (3.38) is consistent to the weak solution $v_\gamma$ to (3.39) for each $\gamma \in \mathcal{I}$, i.e., it follows that $\|u_{\gamma,\varepsilon} - v_\gamma\|_X \to 0$ as $\varepsilon \to 0$, for all $\gamma \in \mathcal{I}$.

4. An example, conclusions and further extensions

To illustrate the proposed method we consider an example of a stochastic heat equation with singular potential. Some notations and basic concepts introduced earlier that are not much used throughout the manuscript will be used here and we refer reader to the Subsection 1.1 for recalling the details.

In the problem (1.3) let the operator $L$ be the Laplace operator over the domain $D = H^1_0(\mathbb{R}^d)$. Then $\frac{\partial}{\partial t} - \Delta$ may be seen as the semigroup evolving law of a rescaled Brownian motion. Let further the potential $q$ be Dirac delta distribution in space, let the force term $F$ be the time-white noise process $W \in C^k([0, T]) \otimes (S)_{-1}$ defined in Example 1.1 (iv), and the initial condition $G$ be a non-zero Gaussian generalized stochastic process in $H^1_0(\mathbb{R}^d) \otimes (S)_{-1}$.

Thus, for such data and $t \in (0, T]$, $x \in \mathbb{R}^d$, $\omega \in \Omega$, we consider the Cauchy problem

\[
\left( \frac{\partial}{\partial t} - \Delta \right) U(t, x, \omega) + \delta(x) \cdot U(t, x, \omega) = W(t, \omega),
\]

\[U(0, x, \omega) = G(x, \omega).\]

The white noise process $W$ has the chaos expansion representation given by (1.8) while the Gaussian generalized stochastic process $G$ has the chaos expansion representation (see [26])

\[G(x, \omega) = g_0(x) + \sum_{k=1}^{\infty} g_{e(k)}(x) H_{e(k)}(\omega),\]

where $g_0 = \mathbb{E}(G)$ is the expectation of $G$, $e(k)$ denotes $k$th unit vector being the sequence of zeros and having 1 as the $k$th component, and $H_{e(k)}(\omega)$ are Fourier-Hermite polynomials given by (1.4).

We assume that the solution to (4.40) is given by (3.29), more precisely in the form

\[U(t, x, \omega) = u_0(t, x) + \sum_{k=1}^{\infty} u_{e(k)}(t, x) H_{e(k)}(\omega) + \sum_{|\gamma|>1} u_{\gamma}(t, x) H_{\gamma}(\omega),\]

where $u_0$ is the expectation of $U$, i.e. $u_0 = \mathbb{E}(U)$. The chaos expansion method implies that for $|\gamma| = 0$ the expectation $u_0$ satisfies the deterministic PDE

\[
\frac{\partial}{\partial t} u_0(t, x) - \Delta u_0(t, x) + \delta(x) u_0(t, x) = 0,
\]

\[u_0(0, x) = g_0(x).\]
For $|\gamma| = 1$, $\gamma = e(k)$, $k \in \mathbb{N}$, the coefficients $u_{e(k)}$ satisfy
\begin{equation}
\frac{\partial}{\partial t} u_{e(k)}(t, x) - \Delta u_{e(k)}(t, x) + \delta(x) u_{e(k)}(t, x) = \xi_k(t), \quad u_{e(k)}(0, x) = g_{e(k)}(x).
\end{equation}

For $|\gamma| > 1$ the coefficients $u_\gamma$ satisfy the homogeneous problem of the form (4.41) with $g_\gamma = 0$. The regularizing net of the Dirac delta distribution is any mollifying net of smooth functions $\delta_\varepsilon$, for example
\[
\delta_\varepsilon(x) = \frac{1}{\varepsilon^d} \varphi\left(\frac{x}{\varepsilon}\right),
\]
with $\varphi$ being a mollifier (see Subsection 2.2.2). The sequence of regularized problems corresponding to $|\gamma| = 0$ and to problem (4.41) is given by
\begin{equation}
\frac{\partial}{\partial t} u_0(t, x) - \Delta u_0(t, x) + \delta_\varepsilon(x) u_0(t, x) = 0, \quad u_0(0, x) = g_0(x),
\end{equation}
while the sequence of regularized problems corresponding to $|\gamma| = 1$ and to problems (4.42) is given by
\begin{equation}
\frac{\partial}{\partial t} u_e(k)(t, x) - \Delta u_e(k)(t, x) + \delta_\varepsilon(x) u_e(k)(t, x) = \xi_k(t), \quad u_e(k)(0, x) = g_{e(k)}(x).
\end{equation}

The solution to the homogenous regularized problem (4.43) is given by
\[
u_{0,\varepsilon}(t, x) = S_t g_0(x), \quad t \in [0, T], \, x \in \mathbb{R}^d,
\]
where $S_t$ is the semigroup generated by perturbed Laplace operator $\Delta - \delta_\varepsilon(x)I_d$ and it is given by
\begin{equation}
S_t = e^{t\delta_\varepsilon(x)}T_t, \quad t \geq 0,
\end{equation}
with $\{T_t\}_{t \geq 0}$ being $C_0$-semigroup generated by the Laplace operator, see [36]. The solution to the regularized problem (4.44) is given by
\[
u_{e(k),\varepsilon}(t, x) = S_t g_{e(k)}(x) + \int_0^t S_{t-s}\xi_k(s) \, ds, \quad t \in [0, T], \, x \in \mathbb{R}^d,
\]
where again, $S_t$ is the semigroup given by (4.45). For $|\gamma| > 1$ the coefficients $u_\gamma$ satisfy the regularized homogeneous problem (4.41) with $g_\gamma = 0$, and therefore for $|\gamma| > 1$ coefficients $u_{\gamma,\varepsilon} = 0$. Thus, solution to (4.40) is given by
\[
U(t, x, \omega) = S_t g_0(x) + \sum_{k=1}^\infty \left(S_t g_{e(k)}(x) + \int_0^t S_{t-s}\xi_k(s) \, ds\right) H_{e(k)}(\omega).
\]
4.1. **Conclusions and further extensions.** This paper brings the following novelties. The notion of stochastic very weak solutions for the given stochastic parabolic initial value problem (1.3) with irregular potential is introduced. In Theorem 9 and Theorem 10 we proved the existence and uniqueness of a stochastic very weak solution to (1.3), while in Theorem 11 we proved that the stochastic very weak solution is consistent with a stochastic (classical) weak solution. In addition, Theorem 5 where we have proved the existence and uniqueness of a very weak solution to the deterministic parabolic equation with singular potential, extends the results from [4], where the authors considered the heat equation with singular potential without a force term.

As the proposed method works very well in this simplest case, we aim to apply it to the general problem (1.2). For \( q \) being time dependent and bounded function, the theory of semigroups guarantees the existence of an evolution system \( E(s, t) \) in the place of the semigroup \( T(t) \). The question of what happens if \( q \) is irregular in time arises. A possible way to overcome this difficulty would be to apply the regularization procedure from [30, 39]. In our further work, we are going to take the advantage of the very weak solution approach developed in this paper. On the other hand, stochastic irregularities will be considered in white noise analysis setting where the product is interpreted as the Wick product. Also we are going to apply our method to semilinear stochastic problems. Finally, as the concept of very weak solutions is well adapted in numerical studies of different classes of PDEs with singularities [5, 3, 27, 28], we aim to adapt the proposed method for numerical approximations of different physical phenomena modelled by (1.2) as it was done for example in [23].

To conclude, we single out the highlights of the paper. The very weak solution for the stochastic parabolic initial value problem with irregular potential is introduced. For the analysis, the chaos expansion method from the white noise analysis and the concept of very weak solutions to treat singularities are combined. The existence and uniqueness of a stochastic very weak solution are proved, and the consistency of obtained solution with classical notions of the solutions is shown. The newly proposed method could be applied to a wide classes of stochastic partial differential equations with singularities and it is suitable for their numerical analysis.

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