THE SYMPLECTOMORPHISM GROUPS OF $T^2 \times S^2$ ARE JORDAN

IGNASI MUNDET I RIERA

ABSTRACT. A group $G$ is Jordan if there exists a constant $C$ such that any finite subgroup $\Gamma$ of $G$ contains an abelian subgroup whose index in $\Gamma$ is at most $C$. We prove that for any symplectic form $\omega$ on $T^2 \times S^2$ the group of symplectomorphisms $\text{Symp}(T^2 \times S^2, \omega)$ is Jordan. As a corollary we deduce that all ruled symplectic 4-manifolds, and all symplectic manifolds diffeomorphic to the product of two compact Riemann surface, have Jordan symplectomorphism group. We also prove that for any $\epsilon > 0$ there is a symplectic structure $\omega$ on $T^2 \times S^2$ and a finite subgroup $\Gamma$ of the symplectomorphism group $\text{Symp}(T^2 \times S^2, \omega)$ such that any abelian subgroup $A \subseteq \Gamma$ satisfies $|A| \leq \epsilon |\Gamma|$. This refines a recent result of Csikós, Pyber and Szabó, according to which $\text{Diff}(T^2 \times S^2)$ is not Jordan.

1. Introduction

A group $G$ is said to be Jordan [18] if there is some constant $C$ such that any finite subgroup $\Gamma$ of $G$ contains an abelian subgroup whose index in $\Gamma$ is at most $C$. The terminology comes from a classic theorem of Camille Jordan, which implies that $\text{GL}(n, \mathbb{C})$ is Jordan for every $n$ (see [9] and [11] for modern presentations). A number of papers have appeared in the last few years studying whether the automorphism groups of different geometric structures are Jordan or not: these include diffeomorphism groups, groups of birational transformations of algebraic varieties, or automorphism groups of algebraic varieties (see [19] for a survey).

Around twenty years ago, Étienne Ghys conjectured that the diffeomorphism group of any smooth compact manifold is Jordan (see Question 13.1 in [7], and footnote 1 in [15]). This conjecture has been partially confirmed in a number of cases (see the introduction and references in [17]). However, recently Csikós, Pyber and Szabó [3] came up with a counterexample, proving that the diffeomorphism group of $T^2 \times S^2$ is not Jordan (see [17] for more examples). In contrast, in this paper we prove the following.

Theorem 1.1. For any symplectic form $\omega$ on $T^2 \times S^2$ the symplectomorphism group $\text{Symp}(T^2 \times S^2, \omega)$ is Jordan.

We also prove the following refinement of Csikós, Pyber and Szabó’s result.

Theorem 1.2. For any $\epsilon > 0$ there exists a symplectic structure $\omega$ on $T^2 \times S^2$ and a finite subgroup $\Gamma \subset \text{Symp}(T^2 \times S^2, \omega)$ such that any abelian subgroup $A \subseteq \Gamma$ satisfies $|A| \leq \epsilon |\Gamma|$.

To prove Theorem 1.2 we simply observe that a slight modification of the construction in [3] can be made symplectic. More concretely, we prove that for any odd prime $p$ there exists some symplectic form $\omega$ on $T^2 \times S^2$ and a finite subgroup of $\text{Symp}(T^2 \times S^2, \omega)$
isomorphic to the Heisenberg group $\Gamma_p$ over $\mathbb{Z}_p$ (see Section 4). Since $\Gamma_p$ is not abelian, any abelian subgroup $A \subseteq \Gamma_p$ satisfies $|A| \leq p^{-1}|\Gamma_p|$, so taking $p$ big enough Theorem 1.2 follows immediately. Note, on the other hand, that Theorem 1.1 implies that for any symplectic form $\omega$ on $T^2 \times S^2$ the set of primes $p$ such that there exists a monomorphism $\Gamma_p \hookrightarrow \text{Symp}(T^2 \times S^2, \omega)$ is finite.

Combining Theorem 1.1 with the main result in [16] we obtain the following.

**Corollary 1.3.** Let $(M, \omega)$ be a symplectic 4-manifold diffeomorphic to the total space of a $S^2$-fibration over a compact Riemann surface or to the product of two compact Riemann surfaces. Then $\text{Symp}(M, \omega)$ is Jordan.

The proof of Theorem 1.2 can be easily combined with the results in [17] to yield the following.

**Theorem 1.4.** For any $\epsilon > 0$ there exists natural numbers $r, n$ with the following property. Suppose that $(X, \omega)$ is a symplectic manifold with an effective and symplectic action of $\text{SU}(n)$. For any $m$ there exists a symplectic structure $\omega$ on $T^2 \times X$ and a finite group $\Gamma$ acting symplectically on $(T^2 \times X, \omega)$ satisfying $|\Gamma| \geq m$, and furthermore any abelian subgroup $A \subseteq \Gamma$ satisfies $|A| \leq |\Gamma|^\epsilon$.

All manifolds and group actions in this paper will be implicitly assumed to be smooth.

Theorem 1.1 is proved in Section 2, Corollary 1.3 is proved in Section 3, and Section 4 contains the proofs of Theorems 1.2 and 1.4.

### 2. Proof of Theorem 1.1

We prove Theorem 1.1 modulo some results whose proofs are postponed to later paragraphs of this section. Denote throughout this section

$$X = T^2 \times S^2$$

and let

$$\pi : X \to T^2$$

be the projection to the first factor. Fix orientations on $T^2$ and $S^2$ and symplectic forms $\omega_{T^2}$ and $\omega_{S^2}$ compatible with the orientations (we will assume this choices to be fixed throughout this section). Assume that the total volumes of both $\omega_{T^2}$ and $\omega_{S^2}$ are 1. Take the product orientation on $T^2 \times S^2$, so that $\omega_{T^2} + \omega_{S^2}$ is compatible with the orientation (pullbacks are implicit in the notation here and in the rest of the paper).

Suppose that $\omega$ is a symplectic form on $X$ and that $\Gamma \subset \text{Symp}(X, \omega)$ is a finite group. Since both $S^2$ and $T^2$ admit orientation reversing diffeomorphisms we may assume, replacing $\omega$ by $\theta^*\omega$ for a suitable diffeomorphism $\theta$ of $X$, that $[\omega] = \alpha[\omega_{T^2}] + \beta[\omega_{S^2}]$ in $H^2(X; \mathbb{R})$, with $\alpha, \beta$ both strictly positive real numbers (we then conjugate the original action of $\Gamma$ by $\theta$, so that $\Gamma$ acts by symplectomorphisms with respect to the new symplectic form).

We are going to prove that there exists an abelian subgroup $A \subseteq \Gamma$ satisfying

$$|\Gamma : A| \leq 2C_1C_4 \max \left\{ 1, \frac{4\alpha^2}{\beta^2} \right\},$$

(1)
where $C_1$ is the constant in Proposition $2.7$ and $C_4$ is the constant in Proposition $2.9$.

Both $C_1$ and $C_4$ are independent of any choices (in particular, they do not depend on the symplectic form $\omega$).

By a theorem of Lalonde and McDuff [10, Theorem 1.1] there is a diffeomorphism $\xi$ of $X$ so that $\xi^*\omega = \alpha\omega_{T^2} + \beta\omega_{S^2}$. Conjugating the action of $\Gamma$ on $X$ by the diffeomorphism $\xi$, we may assume that $\Gamma \subset \text{Symp}(X, \alpha\omega_{T^2} + \beta\omega_{S^2})$.

Before continuing the proof, we introduce some useful terminology. Suppose that $\Pi : E \to B$ is a fibration of manifolds (by that we mean a locally trivial fibration in the category of smooth manifolds, so in particular $\Pi$ is a submersion). An action of a group $\Gamma$ on $E$ is said to be compatible with $\Pi$ if it sends fibers of $\Pi$ to fibers of $\Pi$. In other words, for any $x, y \in E$ we have $\Pi(x) = \Pi(y)$ if and only if $\Pi(\gamma \cdot x) = \Pi(\gamma \cdot y)$ for any $\gamma \in \Gamma$. This implies that there is an action of $\Gamma$ on $B$ so that if $x \in \Pi^{-1}(b)$ then $\gamma \cdot x \in \Pi^{-1}(\gamma \cdot b)$ for any $\gamma \in \Gamma$.

Let $\kappa_{S^2} \in H_2(X;\mathbb{Z})$ be the homology class represented by $\{t\} \times S^2$ for any $t \in T^2$, and let $\kappa_{T^2} \in H_2(X;\mathbb{Z})$ be the homology class represented by $T^2 \times \{s\}$ for any $s \in S^2$ (we use the chosen orientations of $S^2$ and $T^2$). By Proposition $2.1$ there is an orientation preserving diffeomorphism $\phi : X \to X$ such that the action of $\Gamma$ on $X$ is compatible with the fibration $\pi \circ \phi$, and such that $\phi_* \kappa_{S^2} = \kappa_{S^2}$, where $\phi_*$ is the map induced in homology by $\phi$. Furthermore, there is a $\Gamma$-invariant almost complex structure $J$ on $X$ which is compatible with $\omega$ and with respect to which the fibers of $\pi \circ \phi$ are (almost) complex.

Since $\phi$ is orientation preserving, it leaves invariant the intersection pairing in $H_2(X;\mathbb{Z})$, which is symmetric and bilinear. Using the equalities $\kappa_{S^2} \cdot \kappa_{S^2} = \kappa_{T^2} \cdot \kappa_{T^2} = 0$ and $\kappa_{T^2} \cdot \kappa_{S^2} = 1$, and the fact that $\phi_* \kappa_{S^2} = \kappa_{S^2}$, it follows easily that $\phi_* \kappa_{T^2} = \kappa_{T^2}$. Hence, $\phi$ acts trivially on $H_2(X;\mathbb{Z})$. By duality, the action of $\phi$ on $H^2(X;\mathbb{Z})$ is also trivial, so in particular $\phi^*[\omega] = \alpha[\omega_{T^2}] + \beta[\omega_{S^2}]$.

Consequently, replacing $\omega$ by $\phi^*\omega$, and conjugating both $J$ and the action of $\Gamma$ by $\phi$ we put ourselves in the situation where Proposition $2.7$ applies, and furthermore the new symplectic form $\omega$, which is compatible with the new almost complex structure $J$, satisfies

$$[\omega] = \alpha[\omega_{T^2}] + \beta[\omega_{S^2}].$$

For any subset $H \subseteq \Gamma$ we denote by $X^H$ the set of points fixed by all elements in $H$. By Proposition $2.7$ there is a subgroup $\Gamma_0 \subseteq \Gamma$ satisfying $[\Gamma : \Gamma_0] \leq C_1$, where $C_1$ is a constant which is independent of any choice of symplectic or almost complex structure on $X$, and satisfying one of these properties: (1) $\Gamma_0$ is abelian, or (2) there is some nontrivial $\gamma \in \Gamma$ whose action on $X$ fixes the fibers of $\pi$ and such that the action of $\Gamma_0$ on $X$ fixes $X^\gamma$. The two possibilities are not mutually exclusive.

If $\Gamma_0$ is abelian, then we set $A := \Gamma_0$ and we are done, since such $A$ clearly satisfies (I). Otherwise we know, by Proposition $2.7$ that the restriction of $\pi$ to $F := X^\gamma$ satisfies the hypothesis of Proposition $2.8$. This means that $F$ is a compact orientable surface which is either connected or has two connected components, and that the normal bundle $N \to F$ has a structure of complex line bundle which satisfies $\deg N = 0$ if $F$ is connected and $\deg N|_{F_1} + \deg N|_{F_2} = 0$ if $F$ has two connected components $F_1$ and $F_2$. The degrees are defined using an orientation on $F$ with respect to which the projection
Suppose that $F$ is connected. Since $\deg N = 0$, by Proposition 2.9 there is an abelian subgroup $\Gamma_{ab}\subseteq \Gamma_0$ satisfying $[\Gamma_0 : \Gamma_{ab}] \leq C_4$. So $A := \Gamma_0$ satisfies (1) and we are done.

Now suppose that $F$ has two connected components $F_1$ and $F_2$. We are going to bound the absolute value of the degrees of $\deg N|_{F_j}$ in terms of the numbers $\alpha, \beta$. Let $[F_j] \in H_2(X; \mathbb{Z})$ be the homology class represented by $F_j$, using the orientation on $F_j$ which is compatible with the projection $p$. Since $p$ restricts to a diffeomorphism $F_j \to T^2$ for $j = 1, 2$, we have

$$[F_j] = \kappa_{T^2} + \lambda_j \kappa_{S^2}$$

for some real number $\lambda_j$. Let $T^\text{ver} = \ker d\pi \subset TX$ denote the vertical tangent bundle of the fibration $\pi$. We have $T^\text{ver} = T^2 \times TS^2$, so $c_1(T^\text{ver}) = 2[\omega_{S^2}]$ (the factor of 2 is the Euler characteristic $\chi(S^2)$; recall that $\omega_{S^2}$ has total volume 1). By Proposition 2.7, $F$ intersects each fiber of $\pi$ transversely in two points. This implies that $N$ can be identified with the restriction of $T^\text{ver}$ to $F$, so we have

$$\deg N|_{F_j} = \langle c_1(T^\text{ver}), [F_j] \rangle = \langle 2[\omega_{S^2}], \kappa_{T^2} + \lambda_j \kappa_{S^2} \rangle = 2\lambda_j.$$

Hence,

$$\lambda_j = \frac{\deg N|_{F_j}}{2}.$$

Since by Proposition 2.7 both $F_1$ and $F_2$ are $J$-complex submanifolds and $J$ is compatible with $\omega$, we have, using (2) and the fact that the total volumes of $\omega_{T^2}$ and $\omega_{S^2}$ are 1,

$$0 < \langle [\omega], [F_j] \rangle = \langle \alpha[\omega_{T^2}] + \beta[\omega_{S^2}], \kappa_{T^2} + \lambda_j \kappa_{S^2} \rangle = \alpha + \beta \lambda_j = \alpha + \beta \frac{\deg N|_{F_j}}{2}.$$

Consequently

$$\deg N|_{F_j} > -\frac{2\alpha}{\beta}$$

for $j = 1, 2$. Since $\deg N|_{F_1} + \deg N|_{F_2} = 0$, this implies that

$$| \deg N|_{F_j} | < \frac{2\alpha}{\beta}.$$

Replacing if necessary $\Gamma_0$ by a subgroup of index 2, we may assume that the action of $\Gamma_0$ preserves $F_1$. Using again Lemma 2.5 we deduce that the action of $\Gamma_0$ on $N|_{F_1}$ is effective. By Proposition 2.9 there is an abelian subgroup $\Gamma_{ab}\subseteq \Gamma_0$ such that

$$[\Gamma_0 : \Gamma_{ab}] \leq C_4 \max\{1, (\deg N|_{F_1})^2 \} \leq C_4 \max\left\{1, \frac{4\alpha^2}{\beta^2} \right\}.$$

It follows that $A := \Gamma_0$ satisfies (1), so the proof of the theorem is complete.

2.1. Construction of a $\Gamma$-invariant $S^2$-bundle structure. Recall that $\kappa_{S^2} \in H_2(X; \mathbb{Z})$ denotes the homology class represented by $\{t\} \times S^2$ for any $t \in T^2$.

**Proposition 2.1.** Suppose that $\omega = \alpha \omega_{T^2} + \beta \omega_{S^2}$ is a symplectic form on $X$, with $\alpha, \beta > 0$. Suppose that a finite group $\Gamma$ acts symplectically on $(X, \omega)$. There exists an orientation preserving diffeomorphism $\phi : X \to X$ so that the action of $\Gamma$ is compatible
with the fibration \( \pi \circ \phi \), and a \( \Gamma \)-invariant almost complex structure \( J \) on \( X \) such that the fibers of \( \pi \circ \phi \) are \( J \)-complex. Finally we have \( \phi_* \kappa_{S^2} = \kappa_{S^2} \).

Proof. The proof uses pseudoholomorphic curves and is a slight generalisation of [12, Proposition 4.1] and the note afterwards. We sketch the main ideas for completeness, giving precise references when necessary (the reader not familiar with pseudoholomorphic curve theory may look at the beautiful survey [11] for an introduction targeted to results on 4-dimensional ruled symplectic manifolds).

Let \( J \) denote the Frechet space of \( C^\infty \) almost complex structures on \( X \) which are compatible with \( \omega \). By [13, Proposition 2.50], \( J \) is a contractible space (hence nonempty). Denote for convenience \( A = \kappa_{S^2} \in H_2(X; \mathbb{Z}) \). Choose a complex structure \( J_{S^2} \) on \( S^2 \) compatible with the orientation. Take any \( J \in J \) and define the set

\[
\mathcal{M}(A, J) = \{ u : S^2 \rightarrow X \mid \overline{\partial}_J u = 0, u_*[S^2] = A \}.
\]

Here \( \overline{\partial}_J u = \frac{1}{2}(du \circ J_{S^2} - J \circ du) \) and \([S^2] \in H_2(S^2; \mathbb{Z})\) denotes the fundamental class defined by the orientation. The group \( G \) of complex automorphisms of \( S^2 \) acts on \( \mathcal{M}(A, J) \) by precomposition (by Riemann’s uniformization theorem we have \( G \simeq \text{PSL}(2, \mathbb{C}) \)). The compact open topology on the set of maps from \( S^2 \) to \( X \) induces a topology on \( \mathcal{M}(A, J) \) with respect to which the action of \( G \) is continuous and proper. Gromov compactness theorem implies that \( \mathcal{M}(A, J)/G \) is compact because one cannot write \( A = A_1 + A_2 \) in such a way that both \( A_1 \) and \( A_2 \) belong to the image of the Hurewicz homomorphism \( \pi_2(X) \rightarrow H_2(X; \mathbb{Z}) \), and also \( \langle \omega, A_j \rangle > 0 \) for \( j = 1, 2 \) (hence, no bubbling can occur).

Since \( \langle c_1(TX), A \rangle = 2 > 1 \), the main result in [3] (see also [11, §3.3.2]) implies that \( \mathcal{M}(A, J) \) has a natural structure of smooth oriented manifold of dimension \( 2(\langle c_1(TX), A \rangle + 1) = 6 \), and the action of \( G \) on \( \mathcal{M}(A, J) \) is smooth. By the adjunction formula (see [11, Exercise 3.5]) each \( u \in \mathcal{M}(A, J) \) is an embedding. In particular, the action of \( G \) on \( \mathcal{M}(A, J) \) is free and \( \mathcal{M}(A, J)/G \) has a natural structure of smooth oriented compact surface.

The natural evaluation map \( \psi_J : \mathcal{M}(A, J) \times_G S^2 \rightarrow X \) that sends the class of \( (u, s) \in \mathcal{M}(A, J) \times S^2 \) to \( u(s) \) is an orientation preserving diffeomorphism (see [12, Proposition 4.1] and the note afterwards, and also [11, §4.3] — the latter refers only to fibrations over \( S^2 \), but everything works identically for fibrations over general Riemann surfaces). The fact that the evaluation map is orientation preserving is not explicitly mentioned neither in [12, Proposition 4.1] nor in [11, §4.3], but it is an immediate consequence of the fact that the evaluation map has degree 1. Using the multiplicativity of Euler characteristics in fibrations, it follows that \( \chi(\mathcal{M}(A, J)/G) = 0 \), so that \( \mathcal{M}(A, J)/G \) is diffeomorphic to \( T^2 \). Hence the projection \( f : \mathcal{M}(A, J) \times_G S^2 \rightarrow \mathcal{M}(A, J)/G \) is a fibration over \( T^2 \) with fibers diffeomorphic to \( S^2 \), and its total space is orientable.

Up to isomorphism, there are two fibrations over \( T^2 \) with fiber \( S^2 \) and orientable total space, the trivial one and a twisted one (see e.g. [13, Lemma 6.25]). Their total spaces are not diffeomorphic. Indeed, the twisted fibration can be identified with \( \mathbb{P}(L(1) \oplus L(0)) \), where \( L(d) \rightarrow T^2 \) is a complex line bundle of degree \( d \). A simple computation proves that a generator of \( H^4(\mathbb{P}(L(1) \oplus L(0)); \mathbb{Z}) \) can be represented as the square of an element in \( H^2(\mathbb{P}(L(1) \oplus L(0)); \mathbb{Z}) \). In contrast, the square of any element in \( H^2(T^2 \times S^2; \mathbb{Z}) \) is an even multiple of a generator of \( H^4(T^2 \times S^2; \mathbb{Z}) \). Hence the total spaces of the trivial fibration
and the twisted fibration are not homotopy equivalent. Since $\mathcal{M}(A, J) \times_G S^2 \to X$ is diffeomorphic to $T^2 \times S^2$, the fibration $f$ is the trivial one.

We emphasize that the preceding results hold true for every $J \in \mathcal{J}$. Now let $\mathcal{J}_\Gamma \subset \mathcal{J}$ be the subset of $\Gamma$-invariant almost complex structures. Using again [13, Proposition 2.50], we deduce that $\mathcal{J}_\Gamma$ is contractible and hence nonempty (because it is homeomorphic to the space of $\Gamma$-invariant Riemannian metrics on $X$, and the latter is contractible by the standard trick of averaging arbitrary metrics over the action of $\Gamma$). Now for any $J \in \mathcal{J}_\Gamma$, the diffeomorphism $\phi := \psi_J^{-1} : X \to \mathcal{M}(A, J) \times_G S^2$ and the almost complex structure $J$ satisfy the properties of the theorem. □

2.2. Lemmas on finite groups acting on surfaces.

Lemma 2.2. If $g$ is a nontrivial orientation preserving diffeomorphism of $S^2$ of finite order then $g$ fixes exactly two points.

For the next lemma, recall that a subgroup $G'$ of a group $G$ is said to be characteristic if $G'$ is invariant under all automorphisms of $G$.

Lemma 2.3. There exists some $C_2 \geq 1$ such that any finite group $H$ acting effectively on $S^2$ has a cyclic and characteristic subgroup $H' \subseteq H$ satisfying $[H : H'] \leq C_2$ and such that the fixed point set $(S^2)^{H'}$ consists of two points.

Lemma 2.4. There exists some $C_3 \geq 1$ such that any finite group $H$ acting effectively on $T^2$ has an abelian subgroup $H' \subseteq H$ satisfying $[H : H'] \leq C_3$, and the action of $H'$ on $T^2$ is free.

We now prove the three preceding lemmas. A basic trick in the three proofs is the following. If a finite group $H$ acts by orientation preserving diffeomorphisms on a surface $\Sigma$, then one may take an invariant Riemannian metric on $\Sigma$ and consider the induced conformal structure. The surface $\Sigma$ then becomes a Riemann surface, and the action of $\Sigma$, then one may take an invariant Riemannian metric on $\Sigma$ and consider the induced conformal structure. The surface $\Sigma$ then becomes a Riemann surface, and the action of $\Sigma$ is by Riemann surface automorphisms. At this point we may use results on automorphisms of Riemann surfaces to understand the action of $H$.

Suppose that $\Sigma = S^2$. By Riemann’s uniformization theorem we can identify $\Sigma$ with its Riemann surface structure with $\mathbb{C}P^1$. The automorphism group of $\mathbb{C}P^1$ is $\text{PSL}(2, \mathbb{C})$, acting through the fundamental representation of $\text{SL}(2, \mathbb{C})$ in $\mathbb{C}^2$. So if $H$ is cyclic, say generated by some nontrivial $g \in H$, then $g$ has two fixed points on $\mathbb{C}P^1$, corresponding to the two eigenspaces of $g$ seen as an element of $\text{PSL}(2, \mathbb{C})$. This proves Lemma 2.2.

To prove Lemma 2.3 we argue as follows. Let $q : \text{SL}(2, \mathbb{C}) \to \text{PSL}(2, \mathbb{C})$ be the projection and let $\tilde{H} = q^{-1}(H)$. By Jordan’s theorem applied to $\text{SL}(2, \mathbb{C})$ there is an abelian subgroup $\tilde{H}_0 \subseteq \tilde{H}$ satisfying $[\tilde{H} : \tilde{H}_0] \leq C$ for some integer $C \geq 1$ independent of $H$. Since $\tilde{H}_0 \subseteq \text{SL}(2, \mathbb{C})$, the fact that $\tilde{H}_0$ is abelian implies that $H_0$ is cyclic. So $H_0 := q(\tilde{H}_0)$ is cyclic and satisfies $[H : H_0] \leq C$. Let $H' := \bigcap_{\phi \in \text{Aut}(H)} \phi(H_0)$. We claim that $[H_0 : H'] \leq C!$. To prove the claim, suppose that $H_0$ is generated by $h$. For any $\phi \in \text{Aut}(H)$, $H_0 \cap \phi(H_0)$ has index at most $C$ in $H_0$, because $H_0$ is a subgroup of $H$ and $[H : \phi(H_0)] = [H : H_0] \leq C$. This implies that $h^{C!} \in \phi(H_0)$ (look at the image of $h$ in the quotient $H_0/(H_0 \cap \phi(H_0))$ and use the fact that the latter has at most $C$ elements). Since this is true for all $\phi \in \text{Aut}(H)$, we deduce that $h^{C!} \in H'$, which proves the claim. Consequently, setting $C_2 := C!C$ we have $[H : H'] \leq C_2$. Finally, $H'$ is clearly
a characteristic subgroup of $H$, and $H'$ is cyclic because it is a subgroup of $H_0$, which is cyclic. So the proof of Lemma 2.3 is now complete.

Lemma 2.4 follows from the results in [6 V.4.7].

2.3. Lemmas on finite group actions and invariant submanifolds.

**Lemma 2.5.** Let $E$ be a compact and connected manifold. Suppose that a finite group $H$ acts effectively on $E$ and that $F \subset E$ is a $H$-invariant submanifold. Let $N \to F$ be the normal bundle. The action of $\Gamma$ on $E$ induces, linearising in the normal directions of $F$, an effective action of $\Gamma$ on $N$ by bundle automorphisms.

**Lemma 2.6.** Let $\pi : E \to B$ be a fibration of compact manifolds. Suppose that a finite group $H$ acts on $E$ compatibly with $\pi$, preserving an almost complex structure on $E$, and fixing all fibers of $\pi$. Then the fixed point $E^H$ is an almost complex submanifold and the restriction of $\pi$ to $E^H$ is a fibration of manifolds.

To prove the lemmas we use the following well known trick. Suppose that a finite group $H$ acts on a compact manifold $E$. Let $g$ be a $H$-invariant Riemannian metric on $E$. Let $x \in E$ be any point, and let $H_x \subset H$ be its isotropy group. The action of $H_x$ on $E$ induces a linear action on $T_x E$, and the exponential map $\exp^g_x : T_x E \to E$ is $H_x$-equivariant. This implies that, near $x$, $E^{H_x}$ is a submanifold whose tangent space at $x$ can be identified with the linear subspace $(T_x E)^{H_x} \subset T_x E$. Repeating the same argument at each point of $E^{H_x}$ it follows that $E^{H_x}$ is a closed submanifold of $E$. The same argument implies that for each subgroup $H' \subset H$ the fixed point set $E^{H'}$ is a closed submanifold of $E$.

We now prove Lemma 2.5. Suppose that the action of $H$ on $E$ is effective and fixes a submanifold $F \subset E$. Suppose also that for some nontrivial $h \in H$ and any $x \in F$ we have $h \cdot x = x$. Denoting by $\langle h \rangle$ the group generated by $h$, we deduce that $F \subset E^{\langle h \rangle}$. Since the action of $H$ on $E$ is effective, $E^{\langle h \rangle} \neq E$, and since $E$ is connected this implies that for any $x \in F$ the tangent space $T_x E^{\langle h \rangle}$ is a proper subspace of $T_x E$. Hence there is some tangent vector in $T_x E$ that is not fixed by $h$. Since the action of $h$ fixes each element in $T_x F$, we deduce that the induced action of $h$ on $T_x E / T_x F$ is not trivial. But $T_x E / T_x F$ can be identified with the fiber at $x$ of the normal bundle $N \to F$, so Lemma 2.5 is proved.

Next we prove Lemma 2.6. Suppose that $J$ is an almost complex structure on $E$ which is fixed by the action of $H$. This implies that for any $x \in E^H$ the subspace $(T_x E)^H \subset T_x E$ is $J$-invariant, so $E^H$ is an almost complex submanifold. This proves the first statement of Lemma 2.6. To prove the second statement, suppose that $\pi : E \to B$ is a fibration and that the action of $H$ on $E$ is compatible with $\pi$. To prove that $\pi |_{E^H} : E^H \to B$ is a fibration it suffices to prove, by Ehresmann’s theorem [5], that $\pi |_{E^H} : E^H \to B$ is a submersion (since $E$ is compact and $E^H$ is closed, $E^H$ is compact and hence $\pi$ is proper, so the hypothesis of Ehresmann’s theorem are satisfied). Let $x \in E^H$. The fact that the action of $H$ on $E$ is compatible with $\pi$ implies that $H$ acts on $B$ fixing $\pi(x)$ and the differential of the projection, $d\pi : T_x E \to T_{\pi(x)}B$, is $H$-equivariant. Of course $d\pi$ is surjective, since $\pi$ is a fibration. Since $H$ preserves the fibers of $\pi$, the action of $H$ on the whole $B$ is trivial, hence so is the action of $H$ on $T_{\pi(x)}B$. Since $H$ is finite, its action on $T_x E$ is reductive, and this implies, in view of the preceding observations, that the restriction of $d\pi$ to $(T_x E)^H$ is a surjection. But $(T_x E)^H$ can be identified with $T_x (E^H)$,
so we have proved that $\pi|_{E^H} : E^H \to B$ is a submersion and the proof of Lemma 2.6 is now complete.

2.4. Finite groups of automorphisms of spherical fibrations over $T^2$. Let $J$ be an almost complex structure on $X$ with respect to which the fibers of $\pi$ are complex.

**Proposition 2.7.** There exists a constant $C_1$ with the following property. Suppose that a finite group $\Gamma$ acts effectively on $X$ respecting $J$, and suppose that the action is compatible with the fibration $\pi$. Then there is a subgroup $\Gamma_0 \subseteq \Gamma$ such that $[\Gamma : \Gamma_0] \leq C_1$ and such that at least one of these conditions holds true: (1) $\Gamma_0$ is abelian; (2) there exists a nontrivial element $\gamma \in \Gamma$ whose action on $X$ fixes all fibers of $\pi$, such that the action of $\Gamma_0$ fixes $X^\gamma$; furthermore, $X^\gamma \subset X$ is a closed (almost) complex submanifold intersecting transversely each fiber of $\pi$ and the restriction of $\pi$ to $X^\gamma$ is a 2-sheeted unramified covering.

We emphasize that the constant $C_1$ in the proposition is independent of any choice of symplectic or almost complex structure on $X$.

**Proof.** Let $C_2$ (resp. $C_3$) be the constant given by Lemma 2.3 (resp. Lemma 2.4). We are going to prove that $C_1 := C_2C_3$ has the desired property.

Let $\Gamma$ be a finite group acting on $X$, and assume that the hypothesis in the statement of Proposition 2.7 are satisfied. Since $\Gamma$ preserves both $J$ and the fibers of $\pi$, and the fibers of $\pi$ are $J$-complex, the induced action of $\Gamma$ on each fiber of $\pi$ is orientation preserving. Let $\Gamma_S \subseteq \Gamma$ be the normal subgroup consisting of those $\gamma \in \Gamma$ that fix all fibers of $\pi$, i.e., $\pi(x) = \pi(\gamma \cdot x)$ for all $x \in X$. Let $\Gamma_T = \Gamma/\Gamma_S$ and denote by $q : \Gamma \to \Gamma_T$ the quotient map. The action of $\Gamma$ on $T^2$ factors through an effective action of $\Gamma_T$. By Lemma 2.4 there is an abelian subgroup $\Gamma'_B \subseteq \Gamma_B$ satisfying $[\Gamma_B : \Gamma'_B] \leq C_3$. Let $\Gamma_1 = q^{-1}(\Gamma'_B)$.

If $\Gamma_S = \{1\}$ then $q$ is an isomorphism. Hence $\Gamma_1 \simeq \Gamma'_B$, so $\Gamma_1$ is abelian, and $[\Gamma : \Gamma_1] = [\Gamma_B : \Gamma'_B] \leq C_3 \leq C_1$. So setting $\Gamma_0 := \Gamma_1$ we are done in this case.

Assume for the rest of the proof that $\Gamma_S \neq \{1\}$. Before continuing our arguments, let us pause to prove an auxiliary result. Let $S \subset X$ be any of the fibers of $\pi$. We claim that the action of $\Gamma_S$ on $S$ is effective. Indeed, if for some element $\eta \in \Gamma_S$ generating a subgroup $\langle \eta \rangle \subseteq \Gamma_S$ we had $S^{(\eta)} = S$ then, since by Lemma 2.6 the projection $\pi : S^{(\eta)} \to T^2$ is a fibration, we would deduce that the fibers of $\pi : X^{(\eta)} \to T^2$ are two dimensional closed submanifolds of the fibers of $\pi : X \to T^2$, hence $X^{(\eta)} = X$, contradicting the assumption that $\Gamma$ acts effectively on $X$.

To find the element $\gamma \in \Gamma_S$ and the subgroup $\Gamma_0 \subseteq \Gamma$ satisfying the desired properties we distinguish two cases. Recall that $C_2$ is the constant in Lemma 2.3.

Suppose first that $|\Gamma_S| \leq C_2$. Let $\gamma \in \Gamma_S$ be any nontrivial element and define $\Gamma_0$ to be the centralizer of $\gamma$ in $\Gamma_1$. Since $\Gamma_S$ is normal in $\Gamma_1$ and $|\Gamma_S| \leq C_2$, we necessarily have $[\Gamma_1 : \Gamma_0] \leq C_2$. Furthermore, since $\Gamma_0$ centralizes $\gamma$, the action of $\Gamma_0$ on $X$ fixes $X^\gamma$.

Now suppose that $|\Gamma_S| > C_2$. Let $S \subset X$ be as before any fiber of $\pi$. Since the action of $\Gamma_S$ on $S$ is effective, we may apply Lemma 2.3 and deduce that there is a cyclic and characteristic subgroup $\Gamma'_S \subseteq \Gamma_S$ satisfying $[\Gamma_S : \Gamma'_S] \leq C_2$. In particular, $\Gamma'_S \neq \{1\}$. Since $\Gamma'_S$ is normal in $\Gamma_1$ and $\Gamma'_S$ is characteristic in $\Gamma_S$, we deduce that $\Gamma'_S$ is normal in $\Gamma_1$, so the action of $\Gamma_1$ on $X$ preserves $X^{\Gamma'_S}$. So we can take $\gamma$ to be any generator of $\Gamma'_S$ (so that $X^\gamma = X^{\Gamma'_S}$), and $\Gamma_0 = \Gamma_1$. 
We have $[\Gamma : \Gamma_0] = [\Gamma : \Gamma_1][\Gamma_1 : \Gamma_0] \leq C_3C_2 = C_1$.

Let $F := X^\gamma$. By Lemma 2.6, $p := \pi|_F : F \to T^2$ is a fibration, so $F$ intersects transversely each fiber of $\pi$, and by Lemma 2.2 the intersection of $F$ with any fiber of $\pi$ consists of two points. Hence $p$ is a 2-sheeted unramified covering, so the proof of the proposition is complete. \qed

**Proposition 2.8.** Suppose that $F \subset X$ is an almost complex closed submanifold intersecting transversely each fiber of $\pi$ and such that the restriction of $\pi$ to $F$ is a 2-sheeted (unramified) covering $F \to T^2$. Let $N \to F$ be the normal bundle of the inclusion $F \hookrightarrow X$. Then either $F$ is connected or it has two connected components $F_1$, $F_2$. In the first case, $F$ is diffeomorphic to $T^2$ and $\deg N = 0$; in the second case, $F_j$ is diffeomorphic to $T^2$ for $j = 1, 2$ and $\deg N|_{F_1} + \deg N|_{F_2} = 0$.

Some comments on the statement of the proposition are in order. Note first that the normal bundle $N$ has a structure of complex line bundle inherited by $J$, because $F$ is an almost complex submanifold. On the other hand, the hypothesis of the proposition imply that $F$ is a compact orientable surface. To give a sense to the degree of $N$, we orient $F$ in such a way that $p$ is orientation preserving.

**Proof.** Clearly, either $F$ is connected or has two connected components. A computation with the Euler characteristic shows that in the first case $F$ is a torus. In the second case the restriction of $p$ to each connected component of $F$ is a diffeomorphism, so $F$ is the disjoint union of two tori.

The fibers of $\pi$ are almost complex submanifolds, so the vertical tangent bundle of $\pi$, $T^\text{ver}X = T^2 \times TS^2 \to X$, inherits from $J$ by restriction a complex structure. Since $F$ intersects transversely each fiber of $\pi$ in two points, we can identify $N$, as a complex line bundle, with the restriction of $T^\text{ver}X$ to $F$.

The complex structure on $T^\text{ver}X$ induced by restricting $J$ can be continuously deformed to the complex structure induced by restricting a product complex structure $J_{T^2} \oplus J_{S^2}$ on $X = T^2 \times S^2$, where $J_{S^2}$ is compatible with the chosen orientation of $S^2$: indeed, up to homotopy, a complex structure on a real vector bundle of rank 2 is the same thing as an orientation of the fibers. This deformation does not change the degrees, so it suffices to prove the formulas on the degree of $N$ identifying $N$ with $T^\text{ver}X|_F$ endowed with the complex structure $J_{S^2}$.

Let $\iota : F \to F$ be the involution that exchanges the two points in each fiber of $p : F \to T^2$. We are going to prove that the bundles $\iota^*N$ and $N^*$ are isomorphic. Since $\iota$ is orientation preserving and exchanges the two connected components of $F$ when $F$ is disconnected, the isomorphism $\iota^*N \simeq N^*$ implies the desired properties on the degree of $N$ both when $F$ is connected and when it is not.

Identifying $(S^2, J_{S^2})$ with $\mathbb{C}P^1$, we may think of $X$ as the projectivisation $\mathbb{P}(V)$, where $V \to T^2$ is the trivial rank 2 complex vector bundle. If $t \in T^2$ is any point and $p^{-1}(t) = \{a, b\}$, we can identify $a, b$ with points in $\mathbb{P}(V)_t = \mathbb{C}P^1$, hence with two different lines $L_a, L_b \subseteq \mathbb{C}^2$. Composing the inclusion $L_b \hookrightarrow \mathbb{C}^2$ with the projection $\mathbb{C}^2 \to \mathbb{C}^2/L_a$ we obtain an isomorphism $L_b \simeq \mathbb{C}^2/L_a$, and similarly $L_a \simeq \mathbb{C}^2/L_b$. The vertical tangent bundle $T^\text{ver}X$ at $a$ (resp. $b$) can be naturally identified with $T_a\mathbb{C}P^1 = \text{Hom}(L_a, \mathbb{C}^2/L_a) \simeq \text{Hom}(L_a, L_b)$ (resp. with $T_b\mathbb{C}P^1 = \text{Hom}(L_b, \mathbb{C}^2/L_b) \simeq \text{Hom}(L_b, L_a)$). The canonical
isomorphism $\text{Hom}(L_a, L_b) \simeq \text{Hom}(L_b, L_a)^*$ induces an isomorphism $\iota^* N \simeq N^*$. So the proof of the proposition is complete. \hfill \Box

2.5. Finite groups of automorphisms of a complex line bundle over $T^2$.

**Proposition 2.9.** There is a constant $C_4 \geq 1$ with the following property. Let $L \to T^2$ be a complex line bundle. Assume that a finite group $\Gamma$ acts effectively on $L$ by vector bundle automorphisms. There is an abelian subgroup $\Gamma_{ab} \subseteq \Gamma$ satisfying $[\Gamma : \Gamma_{ab}] \leq C_4 \max\{1, (\deg L)^2\}$.

**Proof.** We are going to prove that $C_4 = C_3$, where $C_3$ is given by Lemma 2.4 has the desired property. Assume that a finite group $\Gamma$ acts effectively on a line bundle $L \to T^2$ by vector bundle automorphisms. Let $\Gamma_0 \subseteq \Gamma$ be the subgroup whose elements act trivially on the base $T^2$. Then $\Gamma_B = \Gamma / \Gamma_0$ acts effectively on $T^2$. Let $p : \Gamma \to \Gamma_B$ be the quotient morphism. By Lemma 2.4 there is an abelian subgroup $\Gamma'_B \subseteq \Gamma_B$ whose induced action on $T^2$ is free and such that $[\Gamma_B : \Gamma'_B] \leq C_3$.

To simplify the notation at this point we replace $\Gamma_B$ by $\Gamma'_B$ and $\Gamma_B$ by $p^{-1}(\Gamma'_B)$. Hence we put ourselves in the situation where a finite group $\Gamma$ acts effectively on $L$ by vector bundle automorphisms in such a way that the induced action on the base $T^2$ is through an abelian quotient $\Gamma_B$ which acts freely on $T^2$. Our aim is to prove that there is an abelian subgroup $\Gamma_{ab} \subseteq \Gamma$ such that

$$[\Gamma : \Gamma_{ab}] \leq \max\{1, (\deg L)^2\}.$$ 

If $\Gamma$ is abelian then we set $\Gamma_{ab} = \Gamma$ and we are done. So we may assume for the rest of the proof that $\Gamma$ is not abelian (although everything that follows makes sense also when $\Gamma$ is abelian).

We have an exact sequence

$$1 \to \Gamma_0 \to \Gamma \xrightarrow{\eta} \Gamma_B \to 1$$

where $\Gamma_B$ is abelian. The subgroup $\Gamma_0 \subseteq \Gamma$ is central because its elements act by homotheties on the fibers of $L$ and the action of $\Gamma$ on $L$ is linear. Furthermore, the action of $\Gamma$ on $L$ defines a monomorphism $\Gamma_0 \hookrightarrow S^1$, since the elements of $\Gamma_0$ act on $L$ as multiplication by a complex number of modulus one. This implies that $\Gamma_0$ is cyclic.

Define a map

$$Q : \Gamma_B \times \Gamma_B \to \Gamma_0$$

as follows. Given elements $a, b \in \Gamma_B$ take lifts $\alpha, \beta \in \Gamma$ and set

$$Q(a, b) := [\alpha, \beta] = \alpha \beta \alpha^{-1} \beta^{-1}.$$ 

The term $\alpha \beta \alpha^{-1} \beta^{-1}$ belongs to $\Gamma_0$ because $\Gamma_B$ is abelian, so $\eta(\alpha \beta \alpha^{-1} \beta^{-1}) = 1$, and $\Gamma_0$ is the kernel of $\eta$. It is straightforward to check that $Q$ is well defined.

**Lemma 2.10.** The map $Q$ has the following properties.

1. For all $a, b, c \in \Gamma_B$ we have $Q(ab, c) = Q(a, c)Q(b, c)$, $Q(a, bc) = Q(a, b)Q(a, c)$ and $Q(a, a) = Q(1, a) = Q(a, 1) = 1$;
2. for any $a, b \in \Gamma_B$ the order of $Q(a, b) \in \Gamma$ divides $\text{GCD}($ord$_B(a), \text{ord}_B(b))$, where ord$_B$ refers to the order of elements in $\Gamma_B$;
3. if $p, q$ are different primes, $a \in \Gamma_B$ is a $p$-element and $b \in \Gamma_B$ is a $q$-element, then $Q(a, b) = 1$. 
(4) if $a, b$ are both $p$-elements, the order of $Q(a, b)$ is at most $\max\{\text{ord}_B(a), \text{ord}_B(b)\}$.

Proof. Suppose that $\alpha, \beta, \gamma \in \Gamma$ satisfy $\eta(\alpha) = a$, $\eta(\beta) = b$ and $\eta(\gamma) = c$. We have
\[
Q(ab, c) = (\alpha \beta \gamma)(\alpha \beta)^{-1} - 1 = \alpha \beta \gamma^{-1} \alpha^{-1} \gamma^{-1} = \alpha \beta \gamma^{-1} \alpha^{-1} \gamma^{-1} = \alpha \gamma^{-1} (\beta \gamma^{-1} \alpha^{-1} \gamma^{-1}) = Q(a, b)Q(b, c).
\]
The proof of $Q(a, bc) = Q(a, b)Q(a, c)$ is identical, and $Q(1, a) = Q(a, 1) = 1$ is immediate. So (1) is proved. To prove (2) we use (1) to deduce $Q(a, b)^{\text{ord}_B(a)} = Q(a^{\text{ord}_B(a)}, b) = Q(1, b) = 1$ and similarly $Q(a, b)^{\text{ord}_B(b)} = 1$. Finally, (3) and (4) follow from (2).

Let $\Gamma_c \subseteq \Gamma_0$ be the subgroup generated by the elements $Q(a, b) \in \Gamma_0$ as $a, b$ run through all elements of $\Gamma_B$. It follows immediately from the definition that $\Gamma_c$ is equal to $[\Gamma,\Gamma]$, the subgroup of $\Gamma$ generated by all commutators $[\alpha, \beta]$ for $\alpha, \beta \in \Gamma$. Hence $\Gamma_c = \{1\}$ if and only if $\Gamma$ is abelian.

Before concluding the proof of Proposition 2.12 we prove three lemmas.

Let $d_c = |\Gamma_c|$.

**Lemma 2.11.** $|\Gamma_B|$ divides the product $d_c \deg L$.

**Proof.** Consider the line bundle $\Lambda = L^{\otimes d_c}$. The action of $\Gamma$ on $L$ induces an action on $\Lambda$ defined by $\gamma \cdot (v_1 \otimes \cdots \otimes v_{d_c}) = (\gamma \cdot v_1) \otimes \cdots \otimes (\gamma \cdot v_{d_c})$, and the subgroup of $\Gamma$ defined as $\Gamma_\Lambda = \{\gamma \in \Gamma \mid \gamma \text{ acts trivially on } \Lambda\}$ coincides with the set elements of $\Gamma_0$ whose order divides $d_c$. Since $\Gamma_0$ is cyclic and $|\Gamma_c| = d_c$, we have $\Gamma_\Lambda = \Gamma_c$. The quotient $\Gamma_\Lambda : = \Gamma/\Gamma_\Lambda = \Gamma/\Gamma_c = \Gamma/[\Gamma,\Gamma]$ acts effectively on $\Lambda$ and defining $\Gamma_{\Lambda,0} : = \Gamma_0/\Gamma_c$ there is an exact sequence
\[
1 \to \Gamma_{\Lambda,0} \to \Gamma_\Lambda \to \Gamma_B \to 1.
\]
The action of $\Gamma_\Lambda$ on $\Lambda$ gives a monomorphism $i : \Gamma_{\Lambda,0} \hookrightarrow S^1$. Since $\Gamma_\Lambda$ is finite and abelian, there is a homomorphism $\rho : \Gamma_\Lambda \to S^1$ which extends $i$. Denote by
\[
\phi : \Gamma_\Lambda \times \Lambda \to \Lambda
\]
the map corresponding to the action of $\Gamma$ on $\Lambda$, so that $\phi(\gamma, \lambda) = \gamma \cdot \lambda$. Define a map
\[
\psi : \Gamma_\Lambda \times \Lambda \to \Lambda
\]
by $\psi(\gamma, \lambda) = \rho(\gamma)^{-1} \phi(\gamma, \lambda)$. The map $\psi$ defines a new action of $\Gamma$ on $\Lambda$, with respect to which $\Gamma_{\Lambda,0}$ acts trivially. Hence this new action factors through an action of $\Gamma_B$ on $\Lambda$ lifting the action on $T^2$. Since the action of $\Gamma_B$ on $T^2$ is free, so is the action of $\Gamma_B$ on $\Lambda$. Consequently, the bundle $\Lambda$ descends to a line bundle on the quotient $T^2/\Gamma_B$. Equivalently, there is a line bundle $\Lambda' \to T^2/\Gamma_B$ satisfying $\Lambda \simeq q^* \Lambda'$, where $q : T^2 \to T^2/\Gamma_B$ is the quotient map. Since $q$ has degree $|\Gamma_B|$, it follows that $\deg \Lambda$ is divisible by $|\Gamma_B|$. Finally, $\deg \Lambda = d_c \deg L$, so the proof is complete.

**Lemma 2.12.** If $\Gamma_c \neq \{1\}$ then $\deg L \neq 0$.

**Proof.** Let us suppose that $\deg L = 0$ and that $\Gamma_c \neq \{1\}$. Then there is an isomorphism $L \simeq q^*_C$, where $q$ is as before the quotient map $T^2 \to T^2/\Gamma_B$ and $C \to T^2/\Gamma_B$ is the trivial line bundle. So the action of $\Gamma_B$ on $T^2$ lifts to an action of $\Gamma_B$ on $L$. Composing with the projection $\Gamma \to \Gamma_B$ we get an action of $\Gamma$ on $L$, which is not effective (because
Γ_c is contained in the kernel of the projection Γ → Γ_B, and hence does not coincide with the original action. Suppose that φ, ψ : Γ × L → L are the maps corresponding to the two actions of Γ on L: say φ corresponds to the original (effective) action and ψ corresponds to the new (non-effective) one. Since both actions lift the same action on T^2, there is a map

ζ : Γ → S^1

such that ψ(γ, λ) = ζ(γ)φ(γ, λ). The map ζ is easily seen to be a morphism of groups (here it is crucial that we are dealing with line bundles and not higher rank vector bundles). Furthermore, since all elements in Γ act nontrivially (resp. trivially) through φ (resp. ψ) we must have ζ(γ) ≠ 1 for any γ ∈ Γ \ {1}. Since S^1 is abelian, any morphism Γ → S^1 factors through \[ Γ/\{ Γ, Γ \} \]. Applying this to ζ, and taking into account that \[ [Γ, Γ] \subset Γ_0 \], it follows that Γ_c = [Γ, Γ] = {1}, a contradiction.

**Lemma 2.13.** We have \( d_c^2 \leq |Γ_B| \).

**Proof.** We first prove that Γ_c can be generated by an element of the form \( Q(a, b) \) for some \( a, b \in Γ_B \). Take to begin with a generator of Γ_c of the form

\[ h = Q(a_1, b_1) \cdots Q(a_r, b_r). \]

Since Γ_B is abelian we can write \( a_i = \prod_p a_{ip}, b_i = \prod_p b_{ip} \), where each product is over the set of primes, and \( a_{ip}, b_{ip} \) are p-elements of Γ_B. In the next arguments we use repeatedly Lemma 2.12. We have

\[ Q(a_i, b_i) = \prod_{p, q} Q(a_{ip}, b_{iq}) = \prod_p Q(a_{ip}, b_{ip}), \]

and hence, if we denote by \( \text{ord} \gamma \) the order of any \( γ ∈ Γ \),

\[ \text{ord} h = \prod_i \text{ord} \prod_p Q(a_{ip}, b_{ip}) = \prod_p \prod_i Q(a_{ip}, b_{ip}) \leq \prod_p \max_i \text{ord} Q(a_{ip}, b_{ip}). \]

Choose for any \( p \) an index \( i(p) \) so that \( Q(a_{i(p)p}, b_{i(p)p}) = \max_i \text{ord} Q(a_{ip}, b_{ip}) \). Let \( a = \prod_p a_{i(p)p} \) and \( b = \prod_p b_{i(p)p} \). We have

\[ d_c = \text{ord} h \leq \prod_p \max_i \text{ord} Q(a_{ip}, b_{ip}) = \text{ord} Q(a, b). \]

This implies that \( Q(a, b) \) is a generator of Γ_c. We claim that the set

\[ S = \{ a^l b^j ∈ Γ_B \mid 0 ≤ i < d_c, 0 ≤ j < d_c \} \]

contains \( d_c^2 \) elements. Otherwise there would exist \( 0 ≤ k < d_c \) and \( 0 ≤ l < d_c \) so that \( a^k b^l = 1 \), hence \( b^{-l} = a^k \). This would imply \( Q(a, b)^k = Q(a^k, b) = Q(b^{-l}, b) = Q(b, b)^{-l} = 1 \). Hence \( \text{ord} Q(a, b) < d_c \), a contradiction with our previous computation. It follows that Γ_B contains at least \( d_c^2 \) elements, so the lemma is proved.

We are now ready to finish the proof of Proposition 2.9. It suffices to consider the case Γ_c ≠ {1}. By Lemma 2.12 we have \( \text{deg} L ≠ 0 \). By Lemma 2.11 the nonvanishing of \( \text{deg} L \) implies that \( |Γ_B| ≤ |d_c \text{deg} L| \). Using this inequality and Lemma 2.13 we have

\[ |Γ_B|^2 ≤ d_c^2 (\text{deg} L)^2 ≤ |Γ_B| (\text{deg} L)^2. \]

Dividing both sides by \( |Γ_B| \) we get \( |Γ_B| ≤ (\text{deg} L)^2 \), which implies \( [Γ : Γ_0] = |Γ_B| ≤ (\text{deg} L)^2 \). Since Γ_0 is abelian, the proof is complete.
3. Proof of Corollary 1.3

Let \((M, \omega)\) be a symplectic manifold diffeomorphic to a \(S^2\)-fibration over a compact Riemann surface \(\Sigma\). If \(\chi(\Sigma) \neq 0\) then \(\chi(M) \neq 0\), so by the main result in [16] the diffeomorphism group of \(M\) is Jordan. A fortiori, so is \(\text{Symp}(M, \omega)\). The only case not covered by [16] is precisely when \(\Sigma = T^2\). In this case, \(M\) is either the trivial fibration \(T^2 \times S^2\) or a twisted fibration. In the first case Theorem 1.1 applies. In the second case, we can consider a degree 2 unramified covering \(\mu : T^2 \to T^2\) and take the pullback \(\mu^*M \to T^2\) of the fibration \(M \to T^2\). There is a degree 2 unramified covering \(\nu : \mu^*M \to M\). Then \(\mu^*M \simeq T^2 \times S^2\), so \(\text{Symp}(\mu^*M, \nu^*\omega)\) is Jordan by Theorem 1.1, and the arguments in [14, §2.3] imply, using \(\nu\), that \(\text{Symp}(M, \omega)\) is also Jordan.

Suppose now that \((M, \omega)\) is a symplectic manifold with \(M\) diffeomorphic to the product of two Riemann surfaces of genuses \(g\) and \(h\). If \(\chi(M) \neq 0\) then [16] implies as before that \(\text{Symp}(M, \omega)\) is Jordan. Now suppose that \(\chi(M) = 0\). Then \(1 \in \{g, h\}\), so suppose that \(g = 1\). If \(h = 0\) then \(M \simeq T^2 \times S^2\), so by Theorem 1.1 \(\text{Symp}(M, \omega)\) is Jordan. Finally, if \(h \geq 1\) then one may find cohomology classes \(\alpha_1, \ldots, \alpha_4 \in H^1(M; \mathbb{Z})\) such that \(\alpha_1 \cup \cdots \cup \alpha_4 \neq 0\), so by [14] the diffeomorphism group of \(M\) is Jordan. Consequently, \(\text{Symp}(M, \omega)\) is Jordan in this case as well.

4. Proof of Theorems 1.2 and 1.4

We begin by explaining a slight modification of the construction in [3] (we modify the presentation in [3] to accommodate the construction in [17]; but note that the construction in [3] can be made symplectic as well). Let \(I\) be an ideal inside a commutative ring \(R\) with unit. Consider the group

\[
T(R, I) = \left\{ A(x, y, z) := \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \in \text{Mat}_{3 \times 3}(R) \mid x, y, z \in I \right\}
\]

with the group structure given by matrix multiplication. For any prime \(p\), \(T(\mathbb{Z}, p\mathbb{Z})\) is a normal subgroup of \(T(\mathbb{Z}, \mathbb{Z})\), so we may define the quotient group \(\Gamma_p := T(\mathbb{Z}, \mathbb{Z})/T(\mathbb{Z}, p\mathbb{Z})\). This group has \(p^3\) elements and it is nonabelian, so the index of any of its abelian subgroups is at least \(p\) (\(\Gamma_p\) is usually called a finite Heisenberg group). Define also

\[
M_p = T(\mathbb{Z}, p\mathbb{Z}) \setminus T(\mathbb{R}, \mathbb{R}).
\]

Then \(\Gamma_p\) acts effectively on \(M_p\) on the left via product of matrices. On the other hand, the projection \(T(\mathbb{R}, \mathbb{R}) \ni A(x, y, z) \mapsto (x, y) \in \mathbb{R}^2\) descends to a projection

\[
M_p \to \mathbb{R}^2/\mathbb{Z}^2 = T^2
\]

which is a principal circle bundle (the structure of principal bundle is induced by right multiplication on \(T(\mathbb{R}, \mathbb{R})\) by elements of the form \(A(0, 0, z) \in T(\mathbb{R}, \mathbb{R})\)). The action of \(\Gamma_p\) on \(M_p\) is by principal bundle automorphisms, lifting an action of \(\Gamma_p\) on \(T^2\) which factors through an effective and free action of \(\mathbb{Z}_p \times \mathbb{Z}_p\).

Let \(L_p = M_p \times \mathbb{C}\). Then \(L_p\) is a complex line bundle over \(T^2\), and \(\Gamma_p\) acts effectively on \(L_p\) by vector bundle automorphisms. The direct sum \(V := L_p \oplus L_p^*\) is a rank two vector bundle of degree 0 on \(T^2\), and hence it is smoothly trivial. This implies that the projectivisation \(\mathbb{P}(V)\) is diffeomorphic to \(T^2 \times \mathbb{C}P^1 = T^2 \times S^2\). On the other hand, since \(p\) is odd the action of \(\Gamma_p\) on \(V\) induces an effective action on \(\mathbb{P}(V)\).
To prove Theorem 1.2 it suffices to check that $\mathbb{P}(V)$ supports a $\Gamma_p$-invariant symplectic form. This follows from a standard construction of symplectic structures on fiber bundles, which we now recall. Let $\omega_{T^2}$ be a $\mathbb{Z}_p \times \mathbb{Z}_p$-invariant symplectic structure on $T^2$. Take a $\Gamma_p$-invariant Hermitian structure $h$ on $V$, and let $P \rightarrow T^2$ be the principal $\text{SU}(2)$-bundle of $h$-unitary frames of $V$. The action of $\Gamma_p$ on $V$ induces an action on $P$. Let $A_0$ be a connection on $P$. Averaging over the action of $\Gamma_p$ we obtain a $\Gamma_p$-invariant connection $A$ on $P$. We can identify $\mathbb{P}(V)$ with $P \times_{\text{SU}(2)} \mathbb{C}P^1$, where the action of $\text{SU}(2)$ on $\mathbb{C}P^1$ is induced by the fundamental representation of $\text{SU}(2)$ on $\mathbb{C}^2$. This action is Hamiltonian with respect to the Fubini–Study symplectic form $\omega_{\mathbb{C}P^1}$ on $\mathbb{C}P^1$. So by Weinstein's theorem (see [13, Theorem 6.10]) there is a closed 2-form $\tilde{\omega}_{S^2}$ on $P \times_{\text{SU}(2)} \mathbb{C}P^1$ whose restriction to each fiber is equal to $\omega_{\mathbb{C}P^1}$. The form $\tilde{\omega}_{S^2}$ constructed in [13, Theorem 6.10] out of the connection $A$ is $\Gamma_p$-invariant. Adding to $\tilde{\omega}_{S^2}$ a big multiple of the pullback of $\omega_{T^2}$ we obtain a symplectic form on $P \times_{\text{SU}(2)} \mathbb{C}P^1$ which by construction is $\Gamma_p$ invariant.

To prove Theorem 1.4 we use exactly the same construction. In [17] it is proved that for any $\epsilon > 0$ there exists natural numbers $r$ and $m$ so that the trivial vector bundle $T^{2r} \times \mathbb{C}^m$ supports effective actions by vector bundle automorphisms of arbitrarily large finite groups $\Gamma$ all of whose abelian subgroups $A$ satisfy $|A| \leq |\Gamma|^\epsilon$. These actions lift an action of $\Gamma$ on $T^{2r}$ by translations, so any constant symplectic structure on $T^{2r}$ is $\Gamma$-invariant. Finally, since $\text{SU}(n)$ is a compact simple Lie group, any symplectic action of $\text{SU}(n)$ is Hamiltonian (see e.g. [2, Corollary 26.4]). With all these ingredients, the previous construction applies to the more general situation of Theorem 1.4 as well.

References

[1] E. Breuillard, An exposition of Jordan’s original proof of his theorem on finite subgroups of $\text{GL}_n(\mathbb{C})$, preprint available at \url{http://www.math.u-psud.fr/~breuilla/Jordan.pdf}.
[2] A. Cannas da Silva, \textit{Lectures on Symplectic Geometry}, Lecture Notes in Mathematics 1764, Springer-Verlag, 2008.
[3] B. Csikós, L. Pyber, E. Szabó, Diffeomorphism groups of compact 4-manifolds are not always Jordan, preprint \url{arXiv:1411.7524}.
[4] C.W. Curtis, I. Reiner, Representation Theory of Finite Groups and Associative Algebras, reprint of the 1962 original, AMS Chelsea Publishing, Providence, RI (2006).
[5] C. Ehresmann, Les connexions infinitésimales dans un espace fibré différentiable. \textit{Colloque de Topologie}, Bruxelles, 1950, pp. 29–55.
[6] H.M. Farkas, I. Kra, \textit{Riemann Surfaces}, 2nd edition, Graduate Texts in Mathematics 71, Springer (1992).
[7] D. Fisher, Groups acting on manifolds: around the Zimmer program, \textit{Geometry, rigidity, and group actions}, 72157, Chicago Lectures in Math., Univ. Chicago Press, Chicago, IL (2011), \url{arXiv:0809.4849}.
[8] H. Hoefer, V. Lizan, J.-C. Sikorav, On genericity for holomorphic curves in 4-dimensional almost-complex manifolds, \textit{J. Geom. Anal.} 7 (1997) 149–159.
[9] C. Jordan, Mémoire sur les équations différentielles linéaires à intégrale algébrique, \textit{J. Reine Angew. Math.} 84 (1878) 89–215.
[10] F. Lalonde, D. McDuff, The classification of ruled symplectic 4-manifolds, \textit{Math. Res. Lett.} 3 (1996), no. 6, 769778.
[11] F. Lalonde, D. McDuff, $J$-curves in symplectic 4-manifolds and the classification of rational and ruled manifolds, Proc. 1994 Symplectic topology program, Newton Institute, ed S. Donaldson and C. Thomas, Cambridge University Press, 1996.
[12] D. McDuff, The structure of rational and ruled symplectic 4-manifolds, \textit{J. Amer. Math. Soc.} 3 (1990), 679–712.
[13] D. McDuff and D. Salamon, *Introduction to Symplectic Topology*, Oxford University Press, 1995, 2nd edition 1998.

[14] I. Mundet i Riera, Jordan’s theorem for the diffeomorphism group of some manifolds, Proc. AMS **138** (2010) 2253–2262.

[15] I. Mundet i Riera, Finite groups actions on manifolds without odd cohomology, preprint [arXiv:1310.6565](http://arxiv.org/abs/1310.6565).

[16] I. Mundet i Riera, Finite group actions on 4-manifolds with nonzero Euler characteristic, preprint [arXiv:1312.3149](http://arxiv.org/abs/1312.3149).

[17] I. Mundet i Riera, Non Jordan groups of diffeomorphisms and actions of compact Lie groups on manifolds preprint [arXiv:1412.6964](http://arxiv.org/abs/1412.6964).

[18] V.L. Popov, On the Makar-Limanov, Derksen invariants, and finite automorphism groups of algebraic varieties. In *Peter Russells Festschrift, Proceedings of the conference on Affine Algebraic Geometry held in Professor Russells honour*, 15 June 2009, McGill Univ., Montreal., volume 54 of Centre de Recherches Mathématiques CRM Proc. and Lect. Notes, pages 289311, 2011.

[19] V.L. Popov, Jordan groups and automorphism groups of algebraic varieties, in Cheltsov et al. (ed.), *Automorphisms in Birational and Affine Geometry*, Springer Proceedings in Mathematics and Statistics **79**, Springer (2014), 185–213.

Departament d’Àlgebra i Geometria, Facultat de Matemàtiques, Universitat de Barcelona, Gran Via de les Corts Catalanes 585, 08007 Barcelona, Spain

E-mail address: ignasi.mundet@ub.edu