Causal propagation of constraints in bimetric relativity in standard 3+1 form

Mikica Kocic

Department of Physics & The Oskar Klein Centre,
Stockholm University, AlbaNova University Centre,
SE-106 91 Stockholm, Sweden

E-mail: mikica.kocic@fysik.su.se

ABSTRACT: The goal of this work was to investigate the propagation of the constraints in the ghost-free bimetric theory where the evolution equations are in standard 3+1 form. It is established that the constraints evolve according to a first-order symmetric hyperbolic system whose characteristic cone consists of the null cones of the two metrics. Consequently, the constraint evolution equations are well-posed, and the constraints stably propagate.

KEYWORDS: Classical Theories of Gravity, Cosmology of Theories beyond the SM

ArXiv ePrint: 1804.03659
1 Introduction

We study the propagation of the constraints in the Hassan-Rosen (HR) ghost-free bimetric theory [1–4] where the evolution equations are expressed in standard 3+1 form [5]. The HR theory is a nonlinear theory of two interacting classical spin-2 fields, which is closely related to de Rham-Gabadadze-Tolley (dRGT) massive gravity [6–8]. Like in general relativity (GR), the unphysical degrees of freedom in the HR theory are necessarily eliminated by constraints. Also as in GR, one of the difficulties encountered when treating the initial value problem (IVP) is the fact that the bimetric theory is a constrained system.

Suppose that we have reduced the bimetric field equations and posed the IVP for the HR theory. Besides the evolution equations, the IVP setup will comprise a set of constraints \( \{C_n\} \) that obey evolution equations of the form \( \partial_t C_n = F_n(C_1, C_2, \ldots) \) where \( \partial_t C_n = 0 \) if all \( C_n \) vanish. Then, for analytic initial data, if the constraints exactly vanish on the initial manifold, they will be vanishing at all times due to the Cauchy-Kovalevskaya theorem.

Nevertheless, analytic functions are fully determined by the values on a small open set, which badly fits with causality requirements and the notion of the domain of dependence. Also, on physical grounds, assuming analytic initial data is too restrictive because of the diffeomorphism invariance of the theory and the smooth structure of the manifold. In fact, the real problem is in the requirement ‘exactly vanish’ which is unobtainable since the physical quantities are a priori given with some uncertainty.

Because the physical initial data cannot be freely specified, even infinitesimal variations that violate the constraints can lead to significantly different values at subsequent times. As a result, the continuous dependence on initial data will be corrupted, destroying the well-posedness [9] of a mathematical problem that is to correspond to physical reality. Therefore, it is important to show that the constraints propagate in a stable manner for the given IVP setup in bimetric theory. This is in particular relevant to the problem of
evolution in numerical relativity, if the constraints are solved only to get the initial data. Otherwise, if the constraint evolution equations are not well-posed, the unphysical modes will not be bounded but uncontrollably amplified during the free evolution.

The causal propagation of the constraints in general relativity where the evolution equations are in standard 3+1 form as formulated by York [10] was established by Frittelli in [11]. A similar procedure is followed in this work. The starting point of the performed analysis comprises the bimetric equations in standard 3+1 form obtained in [5].

This paper is organized as follows. In the rest of the introduction, we review the HR field equations and the bimetric space-plus-time split à la York. In section 2, after revisiting how the propagation of the constraints works in GR, we show that the bimetric constraints evolve according to a first-order symmetric hyperbolic system whose characteristic cone consists of the null cones of the two metrics. The paper ends with a summary and outlook.

1.1 Bimetric field equations

Let $g$ and $f$ be two metric fields on a $d$-dimensional manifold coupled through a ghost-free bimetric potential [6, 7, 12],

$$V(S) := -m^d \sum_n \beta_n e_n(S), \quad S := (g^{-1}f)^{1/2}. \quad (1.1)$$

Here, $(g^{-1}f)^{1/2}$ is the matrix square root of the linear operator $g^\mu\nu f_{\mu\nu}$, and $e_n(S)$ are the elementary symmetric polynomials [13], the scalar invariants of $S$ which can be expressed,

$$e_{n \geq 1}(S) := S^{[\mu_1 \mu_2 \cdots \mu_n]}, \quad e_0(S) := 1. \quad (1.2)$$

In particular, $e_1(S) = \text{Tr} S$, $e_d(S) = \det S$, and $e_{n > d}(S) = 0$. The potential is parametrized by the set of dimensionless real constants $\{\beta_n\}$ and an overall mass scale $m$. The algebraic form of the potential is due to the necessary condition for the absence of ghosts [14] where the dynamics of each metric is given by a separate Einstein-Hilbert term in the action [1]. The principal branch of $S$ ensures an unambiguous definition of the theory [3].

Since we are interested in the partial differential equations (PDE) governing the theory, we start from the locally given bimetric field equations [15, 16],

$$G_g = \kappa_g V_g + \kappa_g T_g, \quad V_g := -m^d \sum_n \beta_n Y_n(S), \quad (1.3a)$$

$$G_f = \kappa_f V_f + \kappa_f T_f, \quad V_f := -m^d \sum_n \beta_n Y_{d-n}(S^{-1}), \quad (1.3b)$$

where $G_g$ and $G_f$ are the Einstein tensors of the two metrics, $\kappa_g$ and $\kappa_f$ are two different gravitational constants, $T_g$ and $T_f$ are the stress-energy tensors of the matter fields each minimally coupled to a different sector, and $V_g$ and $V_f$ are the effective stress-energy contributions of the bimetric potential (1.1). The function $Y_n(S)$ in (1.3) encapsulates the variation of the bimetric potential with respect to the metrics (note that $Y_{n \geq d} = 0$),

$$Y_n(S) := \sum_{k=0}^n (-1)^{n+k} e_k(S) S^{n-k} = \frac{\partial e_{n+1}(S)}{\partial S}. \quad (1.4)$$
Figure 1. The null cones of the metrics $g$, $f$, and the geometric mean metric $h$. The shift vector of $h$ is denoted as $q$ and given relative to a common time coordinate $t$.

Importantly, the effective stress-energy tensors satisfy the following identities [17, 18],

$$\sqrt{-g} V_{\mu}^\nu + \sqrt{-f} V_{\mu}^\nu - \sqrt{-g} V \delta_{\mu}^\nu = 0,$$

(1.5a)

$$\sqrt{-g} \nabla_{\mu} V_{\mu}^\nu + \sqrt{-f} \tilde{\nabla}_{\mu} V_{\mu}^\nu = 0,$$

(1.5b)

where $\nabla_{\mu}$ and $\tilde{\nabla}_{\mu}$ are the covariant derivatives compatible with $g$ and $f$, respectively. Assuming that the matter conservation laws hold $\nabla_{\mu} T_{g}^\mu_\nu = 0$ and $\nabla_{\mu} T_{f}^\mu_\nu = 0$, the field equations (1.3) imply the bimetric conservation law,

$$\nabla_{\mu} V_{g}^\mu_\nu = 0, \quad \tilde{\nabla}_{\mu} V_{f}^\mu_\nu = 0.$$

(1.6)

The two equations in (1.6) are not independent according to the differential identity (1.5b).

1.2 $N+1$ splitting

In GR, the kinematical and dynamical parts of a metric field can be isolated using the $N+1$ formalism [10, 19]. A similar procedure can be applied to the bimetric theory. However, one must also take into account: (i) the simultaneous $N+1$ decomposition in both sectors, (ii) the parametrization of the metric fields which does not corrupt the separation between the kinematical and the dynamical parts, and (iii) the bimetric conservation law.

For the first, the existence of a common spacelike hypersurface with respect to both metrics is related to the existence of the real square root $S$ by the theorem from [3]. For the second, the parametrization can be based on the geometric mean metric $h = gS = fS^{-1}$. As shown in [5], such a parametrization covers all possible metric configurations which can have the real principal square root $S$. The appearance of the mean metric $h$ is illustrated in figure 1. In particular, the timelike direction of $h$ is always in the intersection of the null cones of $g$ and $f$, and the shift vector of $h$ can be used as the gauge variable. Subsequently, the projection of the bimetric conservation law is straightforward [5], giving the correct number of truly dynamical degrees of freedom.

Now, let us consider a particular metric sector, say $g$. A foliation of the spacetime into a family of spacelike hypersurfaces $\{\Sigma\}$ is assumed with the timelike unit normal $\vec{n}$ on the
slices such that $n_\mu n^\mu = -1$ with respect to $g$. The geometry is being projected onto the spacelike slices using the operator,

$$\perp_\nu := \delta_\nu^\mu + n^\mu n_\nu,$$

where in a suitable chart $x^\mu = (t, x^i)$ we have,

$$n_\mu = (-N, 0), \quad n^\mu = (N^{-1}, -N^{-1}N^i).$$

Here we used the latin indices to denote the spatial components. Any symmetric tensor field $X_{\mu\nu}$ can be decomposed into the perpendicular projection $\rho$, the mixed projection $j$, and the full projection $J$ of $X$ onto $\Sigma$,

$$\rho[X] := n^\mu X_{\mu\nu} n_\nu, \quad j[X]_\mu := -\perp_\mu X^\sigma n_\sigma, \quad J[X]_{\mu\nu} := \perp_\mu X_{\rho\sigma} \perp_\nu,$$

such that,

$$X_{\mu\nu} = \rho[X] n_\mu n_\nu + n_\mu j[X]_\nu + j[X]_\mu n_\nu + J[X]_{\mu\nu},$$

with the trace,

$$X^\mu_\mu = g^\mu_\nu X_{\mu\nu} = \gamma^{ij} J[X]_{ij} - \rho[X] = J[X]_i^i - \rho[X].$$

For $X = g$, we have $\rho = -1$, $j = 0$, and the metric induced on the spatial slices reads,

$$\gamma_{\mu\nu} := J[g]_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu.$$

This implies $g_{\mu\nu} = -n_\mu n_\nu + \gamma_{\mu\nu}$, and the metric can be written in the chart $(t, x^i)$,

$$g = -N^2 dt^2 + \gamma_{ij} (dx^i + N^i dt) (dx^j + N^j dt).$$

Here, $N$ and $N^i$ are the standard lapse function and shift vector, respectively. The shift $N^i$ is a purely spatial vector field, and $\gamma_{ij}$ is a spatial Riemannian metric whose inverse is obtained through $\gamma^{ik} \gamma_{kj} = \delta^i_j$.

The bimetric field equations (1.3) can be decomposed employing the projections (1.9) with the help of the Gauss-Codazzi-Mainardi equations. Let $K_{ij}$ be the extrinsic curvature of the slices with respect to $g$,

$$K_{ij} := -\frac{1}{2} \mathcal{L}_{\bar{n}} \gamma_{ij}, \quad K := \gamma^{ij} K_{ij},$$

where $\mathcal{L}_{\bar{n}}$ denotes the Lie derivative along the vector field $\bar{n}$. The perpendicular and the mixed projection of the Einstein tensor $G_g$ reads,

$$\rho[G_g] = N^2 G_g^{00} = \frac{1}{2} (R + K^2 - K_{ij} K^{ij}),$$

$$j[G_g]_i = -NG_g^0 i = D_j K^j_i - D_i K,$$

where $R = \gamma^{ij} R_{ij}$ is the trace of the Ricci tensor $R_{ij}$ which is defined using the spatial covariant derivatives $D_i$ compatible with $\gamma_{ij}$. The expressions (1.15) will constitute the constraint equations in the $g$-sector. Following York [10], the evolution equations in standard $N+1$ form are obtained by projecting the equations of motion,

$$R_g^\mu_\nu = \kappa_g (V_g + T_g)^\mu_\nu - \frac{1}{d-2} \kappa_g (V_g + T_g)^\sigma_\sigma \delta^\mu_\nu,$$
which are based on the \(d\)-dimensional Ricci tensor. The full spatial projection of \(R_g\) gives,

\[
J[R_g]_{ij} = N^{-1}(\partial_t K_{ij} - \mathcal{L}_N K_{ij}) - N^{-1}D_i D_j N + R_{ij} - 2K_{ik}K_{kj} + KK_{ij}. \tag{1.17}
\]

Combining (1.15) and (1.17) with the stress-energy tensor projections,

\[
\rho := \rho[V_g + T_g], \quad j := j[V_g + T_g], \quad J := J[V_g + T_g],
\]

we get the constraint and the evolution equations in the \(g\)-sector, respectively. The similar expressions are established in the \(f\)-sector where the geometry is projected with respect to the timelike unit normal \(n^f = (M^{-1}, -M^{-1}M')\) relative to \(f\) such that,

\[
f = -M^2 dt^2 + \varphi_{ij}(dx^i + M^jdt)(dx^j + M^jdt). \tag{1.19}
\]

The rest of the variables in the \(f\)-sector are denoted by tildes: the extrinsic curvature \(\tilde{K}_{ij}\), the spatial Ricci tensor \(\tilde{R}_{ij}\), and the spatial covariant derivative \(\tilde{D}_{\ell}\). The stress-energy tensor projections in the \(f\)-sector are (obtained using the timelike unit normal \(\tilde{n}_f\)),

\[
\tilde{\rho} := \rho[V_f + T_f], \quad \tilde{j} := j[V_f + T_f], \quad \tilde{J} := J[V_f + T_f]. \tag{1.20}
\]

Note that one can always find a common spacelike hypersurface with respect to both metrics, if a real principal square root \(S\) exists \([3]\). Moreover, the parametrization \([5]\) will give all possible metric configurations that have a real principal square root \(S\). At the end, the space-plus-time split will comprise the following set of \(N+1\) variables,

\[
\{n, \mathcal{D}, B, V, U, Q, \tilde{n}, \tilde{D}, \tilde{B}, \tilde{V}, \tilde{U}, \tilde{Q}, \lambda\}. \tag{1.21}
\]

These variables do not depend on the shift of the mean metric \(h\) and the lapses \(N\) and \(M\); they are quoted in appendix B as their form is not important for further analysis. (As earlier noted, the timelike direction of \(h\) is always in the intersection of the null cones of \(g\) and \(f\), and the shift vector of \(h\) can be used as the gauge variable; see \([5]\) for more details.)

The evolution equations for the dynamical pairs \((\gamma_{ij}, K_{ij})\) and \((\varphi_{ij}, \tilde{K}_{ij})\) reads,

\[
\partial_t \gamma_{ij} = \mathcal{L}_N \gamma_{ij} - 2NK_{ij}, \tag{1.22a}
\]

\[
\partial_t K_{ij} = \mathcal{L}_N K_{ij} - D_i D_j N + N \left[ R_{ij} - 2K_{ik}K_{kj} + KK_{ij} \right]
- N\kappa_g \left\{ \gamma_{ik}J^k_j - \frac{1}{d-2}\gamma_{ij}(J - \rho) \right\}, \tag{1.22b}
\]

\[
\partial_t \varphi_{ij} = \mathcal{L}_N \varphi_{ij} - 2M\tilde{K}_{ij}, \tag{1.22c}
\]

\[
\partial_t \tilde{K}_{ij} = \mathcal{L}_N \tilde{K}_{ij} - D_i D_j M + M \left[ \tilde{R}_{ij} - 2\tilde{K}_{ik}\tilde{K}^k_j + \tilde{K}\tilde{K}_{ij} \right]
- M\kappa_f \left\{ \varphi_{ik}\tilde{J}^k_j - \frac{1}{d-2}\varphi_{ij}(\tilde{J} - \tilde{\rho}) \right\}. \tag{1.22d}
\]

The scalar and vector constraint equations are,

\[
\mathcal{C} := \frac{1}{2}(R + K^2 - K_{ij}K^{ij}) - \kappa_g \rho = 0, \tag{1.23a}
\]

\[
\tilde{\mathcal{C}} := \frac{1}{2}(\tilde{R} + \tilde{K}^2 - \tilde{K}_{ij}\tilde{K}^{ij}) - \kappa_f \tilde{\rho} = 0, \tag{1.23b}
\]

\[
C_i := D_kK^k_i - D_iK = -\kappa_g \dot{j}_i = 0, \tag{1.23c}
\]

\[
\tilde{C}_i := \tilde{D}_k\tilde{K}^k_i - \tilde{D}_i\tilde{K} = -\kappa_f \tilde{\dot{j}}_i = 0. \tag{1.23d}
\]
The full spatial projection of $V_g + T_g$ and $V_f + T_f$ are given by,

\begin{align}
J^k_{\ j} &= \left[ \sqrt{\tilde{g}} - (Q\tilde{g}) + MN^{-1}U \right]^{k}_{\ j} + J[T_g]^k_{\ j}, \\
\tilde{J}^k_{\ j} &= \sqrt{\rho} \left[ \sqrt{\tilde{g}} - (Q\tilde{g}) + NM^{-1}i\tilde{U} \right]^{k}_{\ j} + J[T_f]^k_{\ j},
\end{align}

(1.24a, 1.24b)

where the traces of $V_g + T_g$ and $V_f + T_f$ are respectively $J - \rho$ and $\tilde{J} - \tilde{\rho}$ by using (1.11). The perpendicular and mixed projections reads,

\begin{align}
\rho &= m^d \sum_n \beta_n c_n(B) + \rho[T_g], \\
\rho &= \sqrt{\sqrt{\rho}} m^d \sum_n \beta_n c_n(B) + \rho[T_f], \\
j_i &= j[V_g]_i + j[T_g]_i, \\
j_i &= j[V_f]_i + j[T_f]_i,
\end{align}

(1.25a, 1.25b, 1.25c)

In the $N+1$ decomposition, we also have to assume specifically that (i) the matter conservation laws $\nabla_\mu T^\mu_{\ \nu} = 0$ and $\nabla_\mu T_f^\mu_{\ \nu} = 0$ hold, and (ii) the conservation law for the bimetric potential $\nabla_\mu V^\mu_{\ \nu} = 0$ holds. The projection of $\nabla_\mu V^\mu_{\ \nu} = 0$ gives the $N+1$ form of the conservation law for the ghost-free bimetric potential [5],

$$C_b := U^i_j \left( D_i \nabla^j - K^j_{\ i} \right) + \tilde{U}^i_j \left( \tilde{D}_i \nabla^j + \tilde{K}^j_{\ i} \right) - D_i [U^i_j, m^j] = 0.$$  

(1.26)

The equation (1.26) is the same as the so-called secondary constraint obtained using the Hamiltonian formalism [4]. The existence of (1.26) is essential for removing the unphysical (ghost) modes. The counting of the degrees of freedom for the system (1.22)–(1.26) is given in table 2 in [5]. Importantly, the lapses of $g$ and $f$, and the shift of the geometric mean $h$ are absent from the constraint equations and from the projected conservation laws.

2 Propagation of constraints

We first consider the propagation of the constraints in general relativity. The GR equations are in fact the same as the HR equations if the bimetric interaction is turned off, $V(S) = 0$. Then $V_g = V_f = 0$, and the two metric sectors decouple.

Let us assume that we have prepared GR data that satisfies the constraint equations on the initial spatial hypersurface at $t = 0$. We then evolve the system by solving the evolution equations for $(\gamma_{ij}, K_{ij})$, which have to satisfy the constraint equations on each spatial hypersurface of $t > 0$. The constraint equations are satisfied due the contracted Bianchi identity provided that the conservation laws $\nabla_\mu T_g^\mu_{\ \nu} = 0$ hold, which is automatically fulfilled if one solves the field equations.

In the $N+1$ decomposition, however, we need to assume specifically that $\nabla_\mu T_g^\mu_{\ \nu} = 0$ holds because we do not demand that the complete set of the field equations is satisfied, but only their dynamical part [20–23],

$$\mathcal{E}^i_{\ j} = 0, \quad \mathcal{E}^i_{\ j} := J[R_g]^i_{\ j} - \frac{1}{d-2} (J - \rho) \delta^i_{\ j}.$$

(2.1)
where $\rho = \rho[T_g]$, $j_i = j[T_g]$, $J^i_j = J[T_g]^i_j$, and $J = J^k_k$. In this context, the matter energy $\rho$ and momentum density $j$ must obey the equation $\nabla_{\mu} T^\mu_{\nu} = 0$. A straightforward manipulation [20, 24] yields the following projections of $\nabla_{\mu} T^\mu_{\nu} = 0$,

\begin{align}
\partial_t \rho &= \mathcal{L}_N \rho - ND_i j^i - 2j^i D_i N + NK \rho + NK^i_i J^i_j, \quad \text{(2.2a)}
\partial_t j_1 &= \mathcal{L}_N j_1 - D_j [N J^j_i] - \rho D_i N + NK j_i. \quad \text{(2.2b)}
\end{align}

These are effectively the evolution equations of the matter fields in the $N+1$ decomposition. An important remark is that, if the bimetric interaction is taken into account, the projection of the bimetric conservation law $\nabla_{\mu} V^\mu_{\nu} = 0$ yields the constraint equation (1.26).

Now, thanks to the contracted Bianchi identity $\nabla_{\mu} G^\mu_{\nu} = 0$ and the filled conservation laws, the projection of the divergence of the field equations (1.3) is obtained similarly to (2.2) by a formal replacement (see also chapter 11 in [20]),

$$\rho \rightarrow \mathcal{C}, \quad j_i \rightarrow C_i, \quad \text{and} \quad J^j_j \rightarrow J [G_g - \kappa_g T_g]_{ij} = \mathcal{E}^i_j + (\mathcal{C} - \mathcal{E}) \delta^i_j, \quad \text{(2.3)}$$

Here we used $\mathcal{C}$ and $C_i$ to denote the violations of the constraints rather than the constraints. As a result, we get the evolution equations for the constraint violations,

\begin{align}
\partial_t C &= \mathcal{L}_N C - ND_i C^i - 2C^i D_i N + NK C + NK \left(K^j_i \mathcal{E}^i_j - K \mathcal{E}\right), \quad \text{(2.4a)}
\partial_t C_i &= \mathcal{L}_N C_i - D_j [N (\mathcal{E}^j_i - \mathcal{E} \delta^j_i)] - 2CD_i N - ND_i C + NK C_i. \quad \text{(2.4b)}
\end{align}

Assuming that the evolution equations hold, $\mathcal{E}^i_j = 0$, we have,

\begin{align}
\partial_t C &= \mathcal{L}_N C - ND_i C^i - 2C^i D_i N + NK C, \quad \text{(2.5a)}
\partial_t C_i &= \mathcal{L}_N C_i - 2CD_i N - ND_i C + NK C_i. \quad \text{(2.5b)}
\end{align}

If the constraints are satisfied on the initial spatial hypersurface at $t = 0$, then $C = 0$, $C_i = 0$, and we have $\partial_t C = 0$ and $\partial_t C_i = 0$. Hence, provided that (2.2) holds, the constraints are preserved for all $t > 0$ by the dynamical evolution, at least in the case when the initial data is analytic. Nonetheless, the same conclusion can be deduced even for the smooth initial data since the system (2.5) is in a symmetric hyperbolic form [11].

Note that the $N+1$ evolution equations (1.22) are not dynamically equivalent to the original Arnowitt-Deser-Misner (ADM) equations [19]. The source of the different behavior is that the ADM equations come from the field equations written in terms of the Einstein tensor $G_g$, while the evolution equations (1.22) are due to York [10] derived from the field equations expressed in terms of the Ricci tensor $R_g$. To see the difference, let us relate the ADM canonical momenta $\pi^{ij}$ conjugate to $\gamma_{ij}$ and the extrinsic curvature $K_{ij}$ by,

$$\pi^{ij} = -\kappa_g \sqrt{\gamma} \left(K^{ij} - K \gamma^{ij}\right). \quad \text{(2.6)}$$

The ADM version [25] of the evolution equation for $K_{ij}$ becomes,

$$\partial_t K_{ij} = \mathcal{L}_N K_{ij} - D_i D_j N + N \left[R_{ij} - 2K_{ik} K_{kj} + K K_{ij}\right] - N \kappa_g \left\{J_{ij} - \frac{1}{d-2} (J - \rho) \gamma_{ij}\right\} + \frac{1}{d-2} N \gamma_{ij} \mathcal{C}. \quad \text{(2.7)}$$
Hence, the ADM and the York equations only differ by an additive term proportional to the scalar constraint $C$, which vanishes for a physical solution.\footnote{Note that $C$ is usually removed during simplification (assuming $C = 0$), which is forbidden here since $C$ rather denotes the violation of the scalar constraint.} In other words, both systems are equivalent only in a subset of the full space of solutions, called the constraint hypersurface (which is a hypersurface in the space of solutions to the evolution equations). As a consequence, although the two versions are physically equivalent, they are not equivalent in their mathematical properties as shown by Frittelli \cite{Frittelli:1999}. In particular, the constraint evolution equations are well-posed for the York version (1.22) and not well-posed for the ADM version (2.7). The reason for a different dynamical behavior is that $C$ contains hidden second derivatives of the spatial metric (inside $R$) which alter the hyperbolic structure of the differential equations. Furthermore, Anderson and York \cite{Anderson:1983} have shown that, assuming a small (but nontrivial) change in the ADM action principle where the independent gauge function is not taken to be the lapse $N = N/\sqrt{\gamma}$ (the weight-minus-one lapse, also called the “slicing density”), the resulting evolution equations for the new momenta $\pi^{ij}$ correspond precisely to those of York.\footnote{As Alcubierre \cite{Alcubierre:1995} noted, perhaps the densitized lapse is more fundamental than the lapse itself.}

We now turn to the HR theory and prove the following statement.

**Proposition 1.** If the evolution equations hold, then the constraints (1.23) and (1.26) satisfy a homogeneous first-order symmetric hyperbolic system. If initially satisfied, then the constraints hold at all times. The characteristic cone of the system comprises the null cones of the two metrics in the tangent space.

**Proof.** The proof is divided into two parts. We begin by imposing the bimetric conservation law (1.26) to hold at all times. Assuming that the evolution equations $\mathcal{E}^i_{j} = 0$ and $\bar{\mathcal{E}}_{i} = 0$ hold, the evolution equations for the constraint violations (1.23) reads,

\[
\begin{align*}
\partial_i C &= \mathcal{L}_N C - N D_i C^i - 2C^i D_i N + NK C, \quad (2.8a) \\
\partial_i C^i &= \mathcal{L}_N C^i - C D_i N - ND^i C + 2NK^i_j C^j + NK C^i, \quad (2.8b) \\
\partial_i \bar{C} &= \mathcal{L}_\bar{M} \bar{C} - \bar{M} \bar{D}_i \bar{C} - 2\bar{C} \bar{D}_i \bar{M} + M \bar{K} \bar{C}, \quad (2.8c) \\
\partial_i \bar{C}^i &= \mathcal{L}_\bar{M} \bar{C}^i - \bar{C} \bar{D}_i \bar{M} - \bar{M} \bar{D}^i \bar{C} + 2M \bar{K}^i_j \bar{C}^j + M \bar{K} \bar{C}^i. \quad (2.8d)
\end{align*}
\]

This is a system of 2d first-order partial differential equations (PDE) in 2d unknown scalar functions $(C, C^i, \bar{C}, \bar{C}^i)$. The system is obtained similarly to (2.4), where we raised the indices of the vector constraints $C^i = \gamma^{ij} C_j$ and $\bar{C}^i = \varphi^{ij} \bar{C}_j$ for convenience. Neglecting the homogeneous terms, the principal part of (2.8) has the following form,

\[
(A^\mu)^I_{\ j} \partial_\mu w^J = \begin{pmatrix}
\partial_i C - N^k \partial_k C + N \partial_k C^k \\
\partial_i C^i - N^k \partial_k C^i + N \gamma^{ik} \partial_k C \\
\partial_i \bar{C} - M^k \partial_k \bar{C} + M \partial_k \bar{C}^k \\
\partial_i \bar{C}^i - M^k \partial_k \bar{C}^i + M \varphi^{ik} \partial_k \bar{C}
\end{pmatrix}, \quad w^J := \begin{pmatrix}C \\ C^i \\ \bar{C} \\ \bar{C}^i\end{pmatrix}. \quad (2.9)
\]
where $I, J = 1, \ldots, 2d$. The matrices in $A^I \partial_I u = A^0 \partial_t u + A^k \partial_k u$ are easily identified as,

\[
(A^0)_{I J} \equiv \delta^I_I, \quad \text{and} \quad (A^k)_{I J} = \left( -N^k \begin{array}{cc} N \delta^k_k & N \delta^k_j \\ N \gamma^{ik} \xi_k & (\xi_0 - N^k \xi_k) \delta^i_j \end{array} \right) \oplus \left( -M^k + M \phi^k_k \right) \begin{array}{c} \left( -M^k + M \phi^k_k \right) \ \delta^i_j \end{array},
\]

(2.10)

The principal symbol of (2.8) is defined as the matrix,

\[
A(\xi) := A^\mu \xi_\mu = \left( \begin{array}{cc} \xi_0 - N^k \xi_k & N \xi_j \\ N \gamma^{ik} \xi_k & (\xi_0 - N^k \xi_k) \delta^i_j \end{array} \right) \oplus \left( \begin{array}{cc} \xi_0 - M^k \xi_k & M \xi_j \\ M \phi^k_k \xi_k & (\xi_0 - M^k \xi_k) \delta^i_j \end{array} \right),
\]

(2.11)

where $\xi_\mu := (\xi_0, \xi_i)$ is an arbitrary covector. The characteristic polynomial of (2.8) is defined as the determinant of the principal symbol, $P(\xi) := \det A(\xi)$, evaluated as,

\[
P(\xi) = (\xi_0 - N^k \xi_k)^{d-2} \left[ (\xi_0 - N^k \xi_k)^2 - N^2 \xi_i \gamma^{ik} \xi_k \right] \times (\xi_0 - M^k \xi_k)^{d-2} \left[ (\xi_0 - M^k \xi_k)^2 - M^2 \xi_i \phi^k_k \xi_k \right].
\]

(2.12)

Observe that the factor in the $g$-sector can be rewritten by noting that,

\[
g^{\mu \nu} \xi_\mu \xi_\nu = -N^{-2} \left[ (\xi_0 - N^k \xi_k)^2 - N^2 \xi_i \gamma^{ik} \xi_k \right].
\]

(2.13)

Thus, the characteristic polynomial has the form,

\[
P(\xi) = N^2 M^2 (\xi_0 - N^k \xi_k)^{d-2} (\xi_0 - M^k \xi_k)^{d-2} (g^{\mu \nu} \xi_\mu \xi_\nu)(\phi^{\mu \nu}) \xi_\sigma).
\]

(2.14)

The system (2.8) is clearly symmetric hyperbolic and well-posed. Therefore, the stable propagation of the constraints (1.23) holds whenever the projected bimetric conservation law (1.26) is imposed at all times. This concludes the first part of the proof.

Let us now assume that the additional constraint $C_b$ (1.26) holds only for the initial data, and we add its development to (2.8) extending the system of PDE. The evolution equation for $C_b$ was treated in [4] and more recently in [27]. It takes the form,

\[
\partial_t C_b = \partial_t (q^i \partial_i C_b) + N \Omega_g + M \Omega_f,
\]

(2.15)

where $q^i$ is the shift vector of the mean metric $h$, and the functions $\Omega_g$ and $\Omega_f$ depend on the $N+1$ phase space variables (the spatial metric components and the extrinsic curvatures).

Since $C_b$ is not kept vanishing at all times, its contribution will appear in (2.8) as [4, 27],

\[
\partial_t C \approx M C_b, \quad \partial_t \bar{C} \approx 0, \quad \partial_t \bar{C} \approx -N C_b, \quad \partial_t \tilde{C} \approx 0.
\]

(2.16)

After neglecting all the homogeneous terms, the only addition to the principal part is $\partial_t C_b - q^i \partial_i C_b$, which comes from (2.15). This does not destroy the symmetry of the original principal symbol (2.11). Hence, the combined PDE system for $(C, C^i, \bar{C}, \tilde{C}, C_b)$ is symmetric hyperbolic and well-posed. This can also be verified by examining the characteristics.

---

3 $\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A - BD^{-1}C) \det D$

4 Note that $\partial_t C_b$ includes the term $N \Omega_g + M \Omega_f$ that does not depend on the constraints. This term must vanish at all times, which yields the ratio between the lapses of the two metrics. This ratio is a function of the phase space variables only. It is computed in [27] for the spherically symmetric case.

5 Here, $\approx$ denotes a weak equality when all the other constraints, except $C_b$, vanish in (2.8).
The characteristic polynomial for the combined PDEs (2.8) and (2.15) has the form,
\[ P_2(\xi) = N^2M^2(\xi_0 - N^k\xi_k)d^2(\xi_0 - M^k\xi_k)d^2(\xi_0 - q^k\xi_k)(g^{\mu\nu}\xi_\mu\xi_\nu)(f^{\rho\sigma}\xi_\rho\xi_\sigma). \] (2.17)

The characteristics are obtained from \( P_2(\xi) = 0 \), where the roots \( \xi_0 \) are interpreted as the characteristic speeds [28]. The characteristics of the propagation of the constraints are surfaces with normal covectors \( \xi_\mu \) that satisfy either (i) \( \xi_0 = N^k\xi_k \), (ii) \( \xi_0 = M^k\xi_k \), (iii) \( \xi_0 = q^k\xi_k \), (iv) \( g^{\mu\nu}\xi_\mu\xi_\nu = 0 \), or (v) \( f^{\rho\sigma}\xi_\rho\xi_\sigma = 0 \). The characteristics (i), (ii), and (iii) are timelike with respect to \( g \), \( f \), and \( h \), respectively, and tangent to the corresponding timelike unit normals \( n_g, n_f \), and \( n_h \) of the metrics. The theorem in [3] asserts that the timelike direction of \( h \), which is determined by the shift vector \( q^i \), is always in the intersection of the null cones of \( g \) and \( f \). The characteristics (iv) and (v) are null and propagate on the null cones of the metrics \( g \) and \( f \) in the tangent space. Note that the PDE system is not strictly hyperbolic since there are multiple roots in (2.17). Nevertheless, it is strongly hyperbolic as one can determine a complete set of eigenvectors for the principal symbol.

As shown, the causal propagation of the constraints in the HR theory works almost the same as the one in GR [11], provided that one crucial condition is fulfilled: the spatial metrics \( \gamma \) and \( \varphi \) must be simultaneously positive definite. Such a condition is guaranteed using the parametrization from [5], supported by the theorem from [3].

If we started from the ADM version (2.7), the terms \( ND^iC \) and \( \tilde{M}D^i\tilde{C} \) would be absent from (2.8), as well as the elements \( N\gamma^{ik}\xi_k \) and \( M\varphi^{ik}\xi_k \) from (2.11). Then, the principal symbol \( A(\xi) \) would have two Jordan blocks of size two, which implies that the system is weakly hyperbolic since the principal symbol does not have a complete set of eigenvectors. Consequently, the Cauchy problem is not well-posed for the ADM version, and the free evolution (where the constraints are solved only to get the initial data) is not guaranteed to give a solution to the HR bimetric field equations (similarly to GR [11, 29]).

We now illustrate the characteristics of the PDE system. The attributes timelike, spacelike, null, or causal will be used to state the causal structure defined by the PDE. The causality relative to metrics will be stated explicitly. For comparison, the causal structure of several physical systems with nontrivial principal polynomials is given in appendix A. The normal cone of the system is defined by the characteristic polynomial (2.17) in the cotangent space as shown in figure 2a (where the planes \( \xi_0 = N^k\xi_k \), \( \xi_0 = M^k\xi_k \), and \( \xi_0 = q^k\xi_k \) are suppressed). The ray cone is defined by the characteristic vectors in the tangent space, shown in figure 2b. In the case of (2.17), the causal cone of the combined system is the convex hull of the causal cones of the two subsystems, that is, the set of all sums of the form \( \xi_0^a + \xi_0^b \), with \( \xi_0^a \) in the null cone of \( g_{\mu\nu} \) and \( \xi_0^b \) in the null cone of \( f_{\mu\nu} \) [30–32]. The convex inner mantle of the normal cone is associated with the convex hull of the ray surface (both are indicated by thicker lines in figure 2). The outer shell of the ray cone is not necessarily convex and therefore it does not coincide with its convex hull. The signal propagation for the combined system is in those spacetime directions obtained by taking sums of the signal-propagation directions for the two systems separately. The outer sheet of the causal cone defines the domain of dependence (called the domain of determinacy of the initial manifold in [28, p. 439]), where upon the dynamical evolution of the initial data, the
Figure 2. The normal cone (a) of the evolution equations for the constraint violations in the cotangent space, and the corresponding ray cone (b) in the tangent space. The geometric mean cones (shown in green) are not part of the normal or the ray cone. The causal cone (c) is the convex hull of the causal cones of the two individual systems, which encloses the ray cones (b).

constraints are satisfied at all points inside the domain. If the two individual subsystems (sectors) had not been interacting, \( V(S) = 0 \), there would be no need for concerning the convex hull. Since the two sectors are coupled (the lapses and shifts are not independent of each other), then the ray surface remains unchanged, but the data must be given in the convex hull (see the footnote on p. 652 in [28]). Note that each FOSH system carries within itself its own initial-value formulation with its own causal cones for signal propagation (rooted in the PDE structure itself). By combining several FOSH systems (turning on interactions between them but not corrupting the highest-order derivatives), the causal cones are also combined. Importantly, all the combined systems appear on an equal footing. This formulation manifests what Geroch [30] calls the democracy of causal cones: no subsystem (or a set of causal cones) has priority over any others.

3 Summary and outlook

The goal was to investigate the causal properties of the constraint evolution equations in the ghost-free bimetric theory. This resulted in a proposition that, if the evolution equations are given in standard 3+1 form, the propagation of the constraints satisfies a first-order symmetric hyperbolic system. To ensure the well-posedness of the constraint evolution equations and the stable propagation of the constraints is necessary, otherwise the unphysical modes will not be bounded but amplified during the free evolution. Some other aspects of the causality in bimetric theory have been investigated in [33–38].

The causal structure appearing in the proof of proposition 1 can be related with the analysis of Schuller et al. [39, 40], who studied gravitational dynamics and partial differential equations for arbitrary tensorial spacetimes carrying predictive, interpretable, and quantizable matter. These three requirements can be translated into the corresponding algebraic conditions on the underlying geometry, which state that the geometry must be bi-hyperbolic, time-orientable, and energy-distinguishing. The authors of [39] further inves-
tigated the gravitational closure of two Klein-Gordon scalar fields on a bimetric geometry, pointing out that the principal polynomial of such a theory is the product of the principal polynomials of the two individual scalar fields [ibid. appendix B]. This is the same algebraic structure as found in the principal polynomial (2.17) of the evolution equations for the constraint violations (2.8) and (2.15). Moreover, the bi-hyperbolicity condition is equivalent to the requirement that the null cones of the metrics $g$ and $f$ intersect in such a way that provides the existence of the real principal square root of $g^{-1}f$ [3]. Therefore, the HR bimetric spacetime can carry predictive, interpretable, and quantizable matter.

Note that the HR evolution equations (1.22) are not themselves a FOSH system. Hence, it would be desirable to cast the HR $N+1$ system into a symmetric hyperbolic form ensuring the well-posedness of the full theory. As a guideline, we already encountered one observation that comes out from the proof of proposition 1. The causal propagation of the constraints in the HR theory works almost the same as the one in GR due to the theorem from [3] and the property that the evolution equations of the two sectors are coupled algebraically. This means that some of the $N+1$ hyperbolic reductions in GR are applicable to the HR theory, such as the Frittelli-Reula [41] formulation, or the Einstein-Christoffel formulation of Anderson and York [42] which employs the earlier mentioned densitized lapse function.\footnote{In fact, another more recent approach by Kidder et al. [43] shows that densitization of the lapse is a necessary condition for the hyperbolicity. The Kidder-Scheel-Teukolsky formulation [43] comprises both Anderson-York [42] and Frittelli-Reula [41] formulations as subsets.}

As a motivation, let us address the question about the well-posedness of the reduced Einstein-matter equations, discussed in [44] and [45]. Consider the reduced Einstein equations written as a FOSH system with basic unknowns denoted by $u$. Suppose that the matter equations can be written in a symmetric hyperbolic form with basic unknowns $v$. If the coupling of these systems is only by terms of order zero, then the combined system is symmetric hyperbolic for $(u,v)$. The necessary condition for this is that the matter equations contain at most first derivatives of the metric (e.g., the Christoffel symbols) and that the stress-energy tensor contains no derivatives of $u$. In the bimetric case, the ‘matter’ is another Einstein sector already in a FOSH form, therefore we would have the reduced Einstein-Einstein system. Since the bimetric stress-energy tensors are purely algebraic with respect to the metrics, the above conditions for the Einstein-matter system can be satisfied, so a local existence theorem for the reduced Einstein-Einstein bimetric system is obtainable. However, the importance of such results should not be overestimated when considering numerical applications since the hyperbolization of the evolution equations does not necessarily contribute to numerical accuracy and stability.\footnote{See [46] for a review of the efforts to reformulate the Einstein equations for stable numerical simulations.} Adapting, for instance, the BSSN formulation [24, 47] might be more attractive for numerical bimetric relativity.\footnote{Such a modification could also be implemented as a component of the Einstein Toolkit [48] which is based on the BSSN evolution system.}

Acknowledgments

I am grateful to Fawad Hassan and Francesco Torsello for numerous fruitful discussions and reading of the manuscript. My special thanks go to Anders Lundkvist for the valuable discussions about the canonical analysis of bimetric constraints.
Figure 3. The normal cone (a), the ray cone (b), and the causal cone (c) for a peculiar hyperbolic PDE of third order. The reciprocal sheets of the normal and the ray cone are depicted in the same color. See appendix A for further explanations.

A Hyperbolic polynomials

In this section, we illustrate possible peculiarities in the geometrical structure of hyperbolic polynomials. We first consider a somewhat artificial example studied on page 588 in [28]. The PDE of interest is of third order,
\[
\left(\partial_t - \partial_y\right)\left(\partial_t + \partial_y\right)^2 - \partial_x^2 \left(2\partial_t + \partial_y\right)\right) u(t,x,y) = 0,
\]
with the principal polynomial \((\xi_0 - \xi_2)(\xi_0 + \xi_2)^2 - \xi_1^2 (2\xi_0 + \xi_2)\) which can be written,
\[
P(\xi) = (\xi_0^2 - \xi_1^2 - \xi_2^2) (\xi_0 + \xi_2) - \xi_1^2 \xi_0.
\]
The normal cone is defined by \(P(\xi) = 0\) in the \((\xi_0, \xi_1, \xi_2)\) space, shown in figure 3a. Note that, if the term \(\xi_1^2 \xi_0\) is absent from (A.2), the normal cone degenerates into the plane \(\xi_0 + \xi_2 = 0\) tangent to the null cone \(\xi_0^2 - \xi_1^2 - \xi_2^2 = 0\). The normal surface is obtained at \(\xi_0 = -1\) in the \((\xi_1, \xi_2)\)-plane solving \(P(-1, \xi_1, \xi_2) = 0\). This is the curve of third order (the folium of Descartes) indicated by thicker lines in figure 3a. The ray cone is obtained by solving \(\dot{x}^\mu = \partial P/\partial \xi^\mu\) where \(\xi\) satisfies \(P(\xi) = 0\). The ray cone and ray surface are shown in figure 3b. The causal cone is the convex hull of the ray cone, shown in figure 3c.

This example reveals two possible peculiar features of hyperbolic polynomials:

(i) The normal surface can have double points, in which case a lid must be added to form the convex hull as the ray cone may not be convex.

(ii) The normal surface may have a sheet extending to infinity (or a point of inflection).

In such a case, the ray surface will have a cusp at an isolated point.

Furthermore, chapter VI in [28] investigates several systems of physical significance which posses the above properties, for instance, Maxwell’s equations in crystal optics which exhibit a feature of an anisotropy where the velocity of a propagating wave depends upon the direction of propagation. In that example, the normal surface and the ray surface are surfaces of the fourth-degree. The ray cone for the crystal optics PDE reduced to 2+1 dimensions is shown in figure 4a. The section of the ray surface for the full 3+1-dimensional case of the reduced Maxwell’s equations is shown in figure 4b.
Finally, we show the causal structure of a symmetric hyperbolic system obtained from the 3+1 equations in GR \[45\]. The characteristics of that system propagate on several null cones, shown in figure 4c. In this example, the null cone of the spacetime metric is not necessarily the outer one, that is, the causal cone is wider than the null cone of the metric.

B \textit{N+1 bimetric variables}

The primary variables in the parametrization \[5\] are the lapses \(N\), \(M\), the spatial vielbeins \(e^a_i\), \(m^a_i\), the overall shift vector \(q^i\) of the geometric mean, and the Lorentz vector \(p^a\) that defines the separation between the two metrics. The separation \(p\) is in fact the boost parameter of the Lorentz transformation that comprises the spatial part \(\Lambda\) and the Lorentz factor \(\lambda\) where \(p = \Lambda v = \lambda v\) and,

\[
\lambda := \left(1 + p^T \delta p\right)^{1/2} = \left(1 - v^T \delta v\right)^{-1/2}, \tag{B.1}
\]

\[
\Lambda := \left(\mathbb{I} + pp^T\right)^{1/2} = \left(\mathbb{I} - vv^T\delta\right)^{-1/2}. \tag{B.2}
\]

The rest of the variables (1.21) are derived from \(N\), \(e^a_i\), \(M\), \(m^a_i\), \(p^a\), and \(q^i\) as,

\[
\gamma := e^T \delta e, \quad \varphi := m^T \delta m, \tag{B.3}
\]

\[
n := e^{-1} v, \quad \tilde{n} := m^{-1} v, \tag{B.4}
\]

\[
\tilde{N} := q + Nn, \quad \tilde{M} := q - M\tilde{n}, \tag{B.5}
\]

\[
\tilde{Q} := e^{-1} \Lambda^2 e, \quad \tilde{Q} := m^{-1} \Lambda^2 m, \tag{B.6}
\]

\[
\tilde{D} := m^{-1} \Lambda^{-1} e, \quad \tilde{D} := e^{-1} \Lambda^{-1} m, \tag{B.7}
\]

\[
\tilde{B} := D^{-1} = e^{-1} \Lambda m = \tilde{D} \tilde{Q} = \tilde{Q}, \quad \tilde{B} := \tilde{D}^{-1} = m^{-1} \Lambda e = D\tilde{Q} = \tilde{Q}, \tag{B.8}
\]

\[
\tilde{V} := -m^d \sum_n \beta_n e_n(\tilde{D}), \quad \tilde{V} := -m^d \sum_n \beta_n e_n(\tilde{D}), \tag{B.9}
\]

\[
\tilde{U} := -m^d \sum_n \beta_n \lambda^{-1} Y_{n-1}(\tilde{B}), \quad \tilde{U} := -m^d \sum_n \beta_n \tilde{D} Y_{n-1}(\tilde{D}), \tag{B.10}
\]

\[
(\tilde{Q}\tilde{\alpha}) := -m^d \sum_n \beta_n BY_{n-1}(\tilde{D}), \quad (\tilde{Q}\tilde{\mu}) := -m^d \sum_n \beta_n \tilde{D} Y_{n-1}(\tilde{D}). \tag{B.11}
\]
Open Access. This article is distributed under the terms of the Creative Commons Attribution License (CC-BY 4.0), which permits any use, distribution and reproduction in any medium, provided the original author(s) and source are credited.

References

[1] S.F. Hassan and R.A. Rosen, Bimetric Gravity from Ghost-free Massive Gravity, JHEP 02 (2012) 126 [arXiv:1109.3515] [INSPIRE].
[2] S.F. Hassan and R.A. Rosen, Confirmation of the Secondary Constraint and Absence of Ghost in Massive Gravity and Bimetric Gravity, JHEP 04 (2012) 123 [arXiv:1111.2070] [INSPIRE].
[3] S.F. Hassan and M. Kocic, On the local structure of spacetime in ghost-free bimetric theory and massive gravity, JHEP 05 (2018) 099 [arXiv:1706.07806] [INSPIRE].
[4] S.F. Hassan and A. Lundkvist, Analysis of constraints and their algebra in bimetric theory, JHEP 08 (2018) 182 [arXiv:1802.07267] [INSPIRE].
[5] M. Kocic, Geometric mean of bimetric spacetimes, arXiv:1803.09752 [INSPIRE].
[6] C. de Rham and G. Gabadadze, Generalization of the Fierz-Pauli Action, Phys. Rev. D 82 (2010) 044020 [arXiv:1007.0443] [INSPIRE].
[7] C. de Rham, G. Gabadadze and A.J. Tolley, Resummation of Massive Gravity, Phys. Rev. Lett. 106 (2011) 231101 [arXiv:1011.1232] [INSPIRE].
[8] S.F. Hassan and R.A. Rosen, Resolving the Ghost Problem in non-Linear Massive Gravity, Phys. Rev. Lett. 108 (2012) 041101 [arXiv:1106.3344] [INSPIRE].
[9] J. Hadamard, Sur les Problèmes aux Dérivées Partielles et Leur Signification Physique, Princeton Univ. Bull. 13 (1902) 49.
[10] J.W. York Jr., Kinematics and Dynamics of General Relativity, pp. 83–126 [INSPIRE].
[11] S. Frittelli, Note on the propagation of the constraints in standard (3+1) general relativity, Phys. Rev. D 55 (1997) 5992 [INSPIRE].
[12] S.F. Hassan and R.A. Rosen, On Non-Linear Actions for Massive Gravity, JHEP 07 (2011) 009 [arXiv:1103.6055] [INSPIRE].
[13] I.G. Macdonald, Symmetric functions and orthogonal polynomials, AMS (1998).
[14] P. Creminelli, A. Nicolis, M. Papucci and E. Trincherini, Ghosts in massive gravity, JHEP 09 (2005) 003 [hep-th/0505147] [INSPIRE].
[15] S.F. Hassan, A. Schmidt-May and M. von Strauss, On Consistent Theories of Massive Spin-2 Fields Coupled to Gravity, JHEP 05 (2013) 086 [arXiv:1208.1515] [INSPIRE].
[16] A. Schmidt-May and M. von Strauss, Recent developments in bimetric theory, J. Phys. A 49 (2016) 183001 [arXiv:1512.00024] [INSPIRE].
[17] S.F. Hassan, A. Schmidt-May and M. von Strauss, Particular Solutions in Bimetric Theory and Their Implications, Int. J. Mod. Phys. D 23 (2014) 1443002 [arXiv:1407.2772] [INSPIRE].
[18] T. Damour and I.I. Kogan, Effective Lagrangians and universality classes of nonlinear bigravity, Phys. Rev. D 66 (2002) 104024 [hep-th/0206042] [INSPIRE].
[19] R.L. Arnowitt, S. Deser and C.W. Misner, *The Dynamics of general relativity*, *Gen. Rel. Grav.* 40 (2008) 1997 [gr-qc/0405109] [inSPIRE].

[20] É. Gourgoulhon, *3+1 Formalism in General Relativity: Bases of Numerical Relativity*, Lecture Notes in Physics, Springer Berlin Heidelberg (2012).

[21] M. Alcubierre, *Introduction to 3+1 Numerical Relativity*, International Series of Monographs on Physics, Oxford University Press, Oxford (2012).

[22] M. Shibata, *Numerical Relativity*, 100 years of general relativity, World Scientific Publishing Company Pte. Limited (2015).

[23] C. Bona, C. Palenzuela-Luque and C. Bona-Casas, *Elements of Numerical Relativity and Relativistic Hydrodynamics: From Einstein’s Equations to Astrophysical Simulations*, Lecture Notes in Physics, Springer Berlin Heidelberg (2009).

[24] M. Shibata and T. Nakamura, *Evolution of three-dimensional gravitational waves: Harmonic slicing case*, *Phys. Rev. D* 52 (1995) 5428 [inSPIRE].

[25] R.L. Arnowitt, S. Deser and C.W. Misner, *Dynamical Structure and Definition of Energy in General Relativity*, *Phys. Rev.* 116 (1959) 1322 [inSPIRE].

[26] A. Anderson and J.W. York Jr., *Hamiltonian time evolution for general relativity*, *Phys. Rev. Lett.* 81 (1998) 1154 [gr-qc/9807041] [inSPIRE].

[27] M. Kocić, A. Lundkvist and F. Torsello, *On the ratio of lapses in bimetric relativity*, arXiv:1903.09646 [inSPIRE].

[28] R. Courant and D. Hilbert, *Methods of Mathematical Physics*, Methods of Mathematical Physics, vol. 2, Interscience Publishers (1962).

[29] T. Baumgarte and S. Shapiro, *Numerical Relativity: Solving Einstein’s Equations on the Computer*, Cambridge University Press (2010).

[30] R. Geroch, *Faster Than Light?*, *AMS/IP Stud. Adv. Math.* 49 (2011) 59 [arXiv:1005.1614] [inSPIRE].

[31] P. Lax, *Hyperbolic Partial Differential Equations*, Courant Lecture Notes in Mathematics, American Mathematical Society (2006).

[32] J. Rauch, *Hyperbolic Partial Differential Equations and Geometric Optics*, Graduate Studies in Mathematics, American Mathematical Society (2012).

[33] K. Izumi and Y.C. Ong, *An analysis of characteristics in nonlinear massive gravity*, *Class. Quant. Grav.* 30 (2013) 184008 [arXiv:1304.0211] [inSPIRE].

[34] X.O. Camanho, G. Lucena Gómez and R. Rahman, *Causality Constraints on Massive Gravity*, *Phys. Rev. D* 96 (2017) 084007 [arXiv:1610.02033] [inSPIRE].

[35] B. Bellazzini, F. Riva, J. Serra and F. Sgarlata, *Beyond Positivity Bounds and the Fate of Massive Gravity*, *Phys. Rev. Lett.* 120 (2018) 161101 [arXiv:1710.02539] [inSPIRE].

[36] K. Hinterbichler, A. Joyce and R.A. Rosen, *Eikonal scattering and asymptotic superluminality of massless higher spin fields*, *Phys. Rev. D* 97 (2018) 125019 [arXiv:1712.10021] [inSPIRE].

[37] J. Bonifacio, K. Hinterbichler, A. Joyce and R.A. Rosen, *Massive and Massless Spin-2 Scattering and Asymptotic Superluminality*, *JHEP* 06 (2018) 075 [arXiv:1712.10020] [inSPIRE].
[38] K. Hinterbichler, A. Joyce and R.A. Rosen, Massive Spin-2 Scattering and Asymptotic Superluminality, *JHEP* 03 (2018) 051 [arXiv:1708.05716] [inSPIRE].

[39] M. Düll, F.P. Schuller, N. Stritzelberger and F. Wolz, Gravitational closure of matter field equations, *Phys. Rev.* D 97 (2018) 084036 [arXiv:1611.08878] [inSPIRE].

[40] F.P. Schuller and C. Witte, How quantizable matter gravitates: A practitioner’s guide, *Phys. Rev.* D 89 (2014) 104061 [arXiv:1402.6548] [inSPIRE].

[41] S. Frittelli and O.A. Reula, First order symmetric hyperbolic Einstein equations with arbitrary fixed gauge, *Phys. Rev. Lett.* 76 (1996) 4667 [gr-qc/9605005] [inSPIRE].

[42] A. Anderson and J.W. York Jr., Fixing Einstein’s equations, *Phys. Rev. Lett.* 82 (1999) 4384 [gr-qc/9901021] [inSPIRE].

[43] L.E. Kidder, M.A. Scheel and S.A. Teukolsky, Extending the lifetime of 3-D black hole computations with a new hyperbolic system of evolution equations, *Phys. Rev. D* 64 (2001) 064017 [gr-qc/0105031] [inSPIRE].

[44] A. Rendall, *Partial differential equations in general relativity*, Oxford Graduate Texts in Mathematics, Oxford University Press (2008).

[45] H. Friedrich and A.D. Rendall, The Cauchy problem for the Einstein equations, *Lect. Notes Phys.* 540 (2000) 127 [gr-qc/0002074] [inSPIRE].

[46] H.-a. Shinkai and G. Yoneda, Reformulating the Einstein equations for stable numerical simulations, gr-qc/0209111 [inSPIRE].

[47] T.W. Baumgarte and S.L. Shapiro, On the numerical integration of Einstein’s field equations, *Phys. Rev. D* 59 (1999) 024007 [gr-qc/9810065] [inSPIRE].

[48] F. Löffler et al., The Einstein Toolkit: A Community Computational Infrastructure for Relativistic Astrophysics, *Class. Quant. Grav.* 29 (2012) 115001 [arXiv:1111.3344] [inSPIRE].