On the two-loop radiative origin of the smallest neutrino mass and the associated Majorana CP phase

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Abstract

Given a massless neutrino at a superhigh energy scale $\Lambda$ (e.g., in the minimal seesaw model with only two heavy Majorana neutrinos), we calculate quantum corrections to its initially vanishing mass $m_1$ (or $m_3$) and the associated Majorana CP phase $\rho$ (or $\varrho$) at the Fermi scale $\Lambda_F$ by means of the two-loop renormalization-group equations (RGEs) in the standard model and with the help of the latest neutrino oscillation data. The numerical results obtained from our analytical approximations are in good agreement with those achieved by numerically solving the two-loop RGEs. In particular, we confirm that a nonzero value of $m_1$ (or $m_3$) of $\mathcal{O}(10^{-13})$ eV at $\Lambda_F$ can be radiatively generated from $m_1 = 0$ (or $m_3 = 0$) at $\Lambda \approx 10^{14}$ GeV in the SM, and find that $\rho$ (or $\varrho$) may accordingly acquire an appreciable physical value. As a nontrivial by-product, the evolution of all the other (initially nonzero) flavor parameters of massive neutrinos is studied both analytically and numerically, by just keeping their leading (i.e., one-loop) RGE-induced effects.

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1 Introduction

One of the most important tasks in neutrino physics and cosmology is to determine the absolute neutrino mass scale or, equivalently, to tell how small the smallest neutrino mass is. From a phenomenological point of view, the lightest neutrino is allowed to be massless because this expectation is not in conflict with current neutrino oscillation data and cosmological observations \cite{1}. On the theoretical side, however, there is no fundamental symmetry or conservation law to protect a massless neutrino to stay massless, and hence it is most likely to become massive after proper quantum corrections are taken into account \cite{2}.

To generate finite but tiny neutrino masses, one may extend the standard model (SM) of electroweak interactions by adding three heavy (right-handed) neutrino fields $N_{\alpha R}$ (for $\alpha = e, \mu, \tau$) and allowing lepton number violation. In this case the charged-lepton and neutrino mass terms that respect the $SU(2)_L \times U(1)_Y$ gauge symmetry can be written as

\begin{equation}
-L_{\text{lepton}} = \bar{\ell}_L Y_L H E_R + \bar{\ell}_L Y_\nu \tilde{H} N_R + \frac{1}{2} N^c_R M_R N_R + \text{h.c.},
\end{equation}

in which the relevant field notations are self-explanatory, and $M_R$ is a symmetric matrix whose mass scale can be far above the Fermi scale $\Lambda_F \sim 10^2$ GeV. Integrating out the heavy degrees of freedom in Eq. (1) \cite{3}, one is left with the unique dimension-five Weinberg operator \cite{4}

\begin{equation}
O_{\text{Weinberg}} = \frac{\kappa_{\alpha\beta}}{2} \left[ \bar{\ell}_{\alpha L} \tilde{H} H^T \ell_{\beta L} \right]
\end{equation}

with the subscripts $\alpha$ and $\beta$ running over $e, \mu$ and $\tau$, and the effective neutrino coupling matrix $\kappa = Y_\nu M_R^{-1} Y_\nu^T$ is suppressed by a sufficiently high cut-off scale $\Lambda$ \cite{5,9}. Once the electroweak gauge symmetry is spontaneously broken at the Fermi scale $\Lambda_F$, we arrive at the effective Majorana neutrino mass matrix for three light (left-handed) neutrinos:

\begin{equation}
M_\nu = -\kappa \langle H \rangle^2 = -M_D M_R^{-1} M_D^T
\end{equation}

with $M_D = Y_\nu \langle H \rangle$ and the charged-lepton mass matrix $M_l = Y_l \langle H \rangle$, where $\langle H \rangle \simeq 174$ GeV is the vacuum expectation value of the Higgs field. The tiny neutrino masses $m_i$ (for $i = 1, 2, 3$), which equal the singular values of $M_\nu$, are therefore ascribed to the huge mass scale of $M_R$ as compared with the value of $\langle H \rangle$.

Eq. (3) tells us that one of the three light neutrinos is naturally massless in the minimal type-I seesaw scenario with only two heavy Majorana neutrinos \cite{10,12}, simply because in this case the rank of $M_\nu$ is exactly equal to two (i.e., the rank of the $2 \times 2$ mass matrix $M_R$). Combining this observation with current neutrino oscillation data \cite{1,13,15}, one may have either $m_1 = 0$ (normal mass ordering) or $m_3 = 0$ (inverted mass ordering). Note that the vanishing of $m_1$ (or $m_3$) allows one of the Majorana CP phases in the $3 \times 3$ Pontecorvo-Maki-Nakagawa-Sakata (PMNS) neutrino mixing matrix $U$ \cite{16,18}, which is used to diagonalize $M_\nu$ in the $Y_l = \text{Diag} \{y_e, y_\mu, y_\tau\}$ basis (i.e., $U^\dagger M_\nu U^* = D_\nu \equiv \text{Diag} \{m_1, m_2, m_3\}$ in this basis), to automatically disappear. Such a simplified seesaw scenario is therefore more predictive \cite{19}. Of course, assuming $m_1 = 0$ (or $m_3 = 0$) and studying its phenomenological consequences are unnecessarily subject to the minimal seesaw model, since such a conjecture empirically satisfies the principle of Occam’s razor \cite{20}. Here
the main concerns are as follows: (1) whether \( m_1 = 0 \) (or \( m_3 = 0 \)) can be stable against quantum corrections between a superhigh cut-off (or seesaw) scale and the electroweak scale; (2) whether the initially undefined Majorana CP phase \( \rho \) (or \( \varrho \)) can be radiatively generated together with \( m_1 \) (or \( m_3 \)); and (3) how those initially nonzero flavor parameters are modified by the relevant quantum effects.

The first question has essentially been answered by Davidson, Isidori and Strumia \[21\]. Given \( m_{\text{min}} = 0 \) with \( m_{\text{min}} \) being either \( m_1 \) or \( m_3 \) at a superhigh energy scale \( \Lambda \simeq 10^{14} \) GeV, they found \( m_{\text{min}} \sim 10^{-13} \) eV at the Fermi scale \( \Lambda_F \) by considering the two-loop renormalization-group equations (RGEs) of \( M_\nu \) and inputting the preliminary neutrino oscillation data obtained in 2007. Although the Majorana CP phase associated with \( m_{\text{min}} \) was also mentioned in their paper, it was not analytically formulated and numerically evaluated. On the other hand, it is certainly enough to calculate the one-loop RGE-induced quantum corrections to those initially nonzero flavor parameters \[22\], but a transparent analytical formulation of their running effects between \( \Lambda_F \) and \( \Lambda \gg \Lambda_F \) has been lacking.

In this paper we are going to answer the above three questions by means of the two-loop RGEs and with the help of the latest neutrino oscillation data in the SM framework. Different from the previous work done by Davidson et al in Ref. \[21\], here both the smallest neutrino mass (\( m_1 \) or \( m_3 \)) and the associated Majorana CP phase \( \rho \) (or \( \varrho \)) at low energies are analytically formulated by keeping the contributions of all the three neutrino mixing angles, and their magnitudes are evaluated both based on our analytical approximations and by numerically solving the two-loop RGEs. The numerical results obtained in these two ways are in good agreement with each other. In particular, we confirm that a nonzero value of \( m_1 \) (or \( m_3 \)) of \( \mathcal{O}(10^{-13}) \) eV at \( \Lambda_F \) can be radiatively generated from \( m_1 = 0 \) (or \( m_3 = 0 \)) at \( \Lambda \simeq 10^{14} \) GeV in the SM, and find that \( \rho \) (or \( \varrho \)) may accordingly acquire an appreciable physical value. As a nontrivial by-product, the running behaviors of all the other (initially nonzero) flavor parameters of massive neutrinos are calculated both analytically and numerically, by keeping their leading (i.e., one-loop) RGE-induced effects.

### 2 Two-loop RGE-induced corrections

Given the SM-like Yukawa interactions in Eq. (1) and the dimension-five Weinberg operator as the origin of tiny neutrino masses in Eq. (2), an exactly massless neutrino running from a superhigh energy scale \( \Lambda \) down to the Fermi scale \( \Lambda_F \) will stay massless provided only the one-loop RGE of the effective Majorana neutrino coupling matrix \( \kappa \) is taken into account. The reason is simply that \( m_1 = 0 \) (or \( m_3 = 0 \)) requires the rank of \( \kappa \) to be two, but the one-loop quantum corrections to \( \kappa \) do not change its rank. When the two-loop radiative corrections to \( \kappa \) are taken into consideration, however, Davidson et al have pointed out that a nontrivial quantum effect described by the Feynman diagram in Fig. 1 can increase the rank of \( \kappa \) from two to three, and the contributions from all the other two-loop Feynman diagrams are qualitatively trivial and thus quantitatively negligible \[21\]. This interesting observation has been confirmed by our recalculation along the same line of thought. As a straightforward consequence, the initially vanishing neutrino mass at \( \Lambda \) will become nonzero at an energy scale below \( \Lambda \) (e.g., at the Fermi scale \( \Lambda_F \)) thanks to the two-loop RGE evolution.
Figure 1: The dominant two-loop Feynman diagram that can increase the rank of the effective Majorana neutrino coupling matrix $\kappa$ from two to three because of the SM-like leptonic Yukawa interactions as described by Eq. (1).

To be explicit, we write out the RGE of $\kappa$ which includes both the one-loop contributions and the nontrivial two-loop effect originating from Fig. 1 [21]:

$$16\pi^2 \frac{d\kappa}{dt} = \alpha_\kappa \kappa - \frac{3}{2} \left[ (Y_l Y_l^\dagger) \kappa + \kappa (Y_l Y_l^\dagger)^T \right] + \frac{1}{8\pi^2} (Y_l Y_l^\dagger) \kappa (Y_l Y_l^\dagger)^T ,$$

(4)

where $t \equiv \ln(\mu/\Lambda_F)$ with $\mu$ being an arbitrary renormalization scale between $\Lambda_F$ and $\Lambda$, and $\alpha_\kappa \approx -3g_2^2 + 6y_t^2 + \lambda$ with $g_2$, $y_t$ and $\lambda$ standing respectively for the SU(2)$_L$ gauge coupling, the top-quark Yukawa coupling and the Higgs self-coupling constant. It is obvious that the first two terms on the right-hand side of Eq. (4) are the one-loop contributions [23–28], and the last term is the nontrivial two-loop contribution induced by Fig. 1. Without loss of generality, we study the evolution of $M_\nu = -\kappa \langle H \rangle^2$ from $\Lambda$ to $\Lambda_F$ in the basis where $Y_l$ is taken to be diagonal (i.e., $Y_l = \text{Diag}\{y_e, y_\mu, y_\tau\}$). Since $Y_l$ keeps diagonal during the RGE evolution [3], we integrate Eq. (4) and arrive at

$$M_\nu(\Lambda_F) = I_0 T_l [M_\nu(\Lambda) \circ \Omega] T_l ,$$

(5)

where $M_\nu(\Lambda)$ and $M_\nu(\Lambda_F)$ stand respectively for the effective Majorana neutrino mass matrices at $\Lambda$ and $\Lambda_F$, the mathematical symbol "$\circ$" denotes the so-called Hadamard product (also known as the Schur product [29]) which produces a new matrix by multiplying the elements in the same position of the two original matrices with the same dimension [i.e., $(M_\nu \circ \Omega)_{\alpha\beta} = (M_\nu)_{\alpha\beta} \Omega_{\alpha\beta}$], $T_l = \text{Diag}\{I_e, I_\mu, I_\tau\}$ is diagonal but flavor-dependent, and the loop functions $I_0$, $I_\alpha$ and $\Omega_{\alpha\beta}$ (for $\alpha, \beta = e, \mu, \tau$) are defined as

$$I_0 = \exp \left[ -\frac{1}{16\pi^2} \int_0^{\ln(\Lambda/\Lambda_F)} \alpha_\kappa (t) \, dt \right] ,$$

$$I_\alpha = \exp \left[ \frac{3}{32\pi^2} \int_0^{\ln(\Lambda/\Lambda_F)} y_\alpha^2 (t) \, dt \right] ,$$

$$\Omega_{\alpha\beta} = \exp \left[ -\frac{1}{128\pi^4} \int_0^{\ln(\Lambda/\Lambda_F)} y_\alpha^2 (t) y_\beta^2 (t) \, dt \right] .$$

(6)
Figure 2: Changes of $I_0$, $\Delta_\tau$ and $\Delta'_\tau$ with the energy scale $\mu$ below $\Lambda \simeq 10^{14}$ GeV in the SM.

It is clear that the one-loop effects described by $I_0$ and $T_l$ cannot change the rank of $M_\nu(\Lambda)$, but the nontrivial two-loop effect hidden in $\Omega$ is able to increase the rank of $M_\nu(\Lambda)$ from two to three because its contribution to $M_\nu(\Lambda)$ is not flavor-diagonal. Given $y_e^2 \ll y_\mu^2 \ll y_\tau^2 \ll 1$ in the SM \[20\], it is very safe to make the $\tau$-dominance approximations as follows:

$$
T_l \simeq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 + \Delta_\tau \end{pmatrix},
$$
$$
\Omega \simeq \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 - \Delta'_\tau \end{pmatrix},
$$

(7)

where

$$
\Delta_\tau = \frac{3}{32\pi^2} \int_0^{\ln(\Lambda/\Lambda_F)} y_e^2(t) \, dt ,
$$
$$
\Delta'_\tau = \frac{1}{128\pi^4} \int_0^{\ln(\Lambda/\Lambda_F)} y_\tau^4(t) \, dt .
$$

(8)

So $\Delta_\tau$ contributes to every element in the third row and the third column of $M_\nu(\Lambda)$, but $\Delta'_\tau$ only affects the $(3,3)$ element of $M_\nu(\Lambda)$. The values of $\Delta_\tau$ and $\Delta'_\tau$ are both positive in the SM, and their dependence on the energy scale $\mu$ is shown in Fig. 2, where the dependence of $I_0$ on $\mu$ is also illustrated. One can immediately see that $\Delta'_\tau$ is roughly $10^6$ times smaller than $\Delta_\tau$; and their magnitudes are of $\mathcal{O}(10^{-11})$ and $\mathcal{O}(10^{-5})$, respectively, when $\Lambda \simeq 10^{14}$ GeV is fixed and $\mu \lesssim 10^{10}$ GeV holds.

In the chosen basis with $Y_l$ being diagonal, the effective Majorana neutrino mass matrix $M_\nu$ can be reconstructed in terms of the PMNS matrix $U$ and the diagonal neutrino mass matrix
\[ D_\nu = \text{Diag}\{m_1, m_2, m_3\} \] at a given energy scale \( \Lambda \). Namely, \( M_\nu = U D_\nu U^T \). Substituting both Eq. (7) and the decompositions of \( M_\nu \) at \( \Lambda \) and \( \Lambda_F \) into Eq. (5), we obtain the relationship

\[
(U D_\nu U^T)_{\Lambda_F} \simeq I_0 + \left[ UD_\nu U^T + \Delta_{\tau} \left( \begin{array}{ccc}
0 & 0 & \sum_i m_i U_{ei} U_{\tau i} \\
0 & 0 & \sum_i m_i U_{\mu i} U_{\tau i} \\
\sum_i m_i U_{ei} U_{\tau i} & \sum_i m_i U_{\mu i} U_{\tau i} & (2 - r_\tau) \sum_i m_i U_{\tau i}^2
\end{array} \right)_\Lambda \right],
\]

where \( r_\tau \equiv \Delta'_\tau / \Delta_\tau \) signifies the tiny two-loop RGE-induced effect. If one of the three neutrinos is exactly massless at \( \Lambda \), Eq. (9) tells us that the determinant of \( M_\nu^{(\Lambda_F)} = (UD_\nu U^T)_{\Lambda_F} \) must be proportional to \( \Delta_\tau \). It is therefore the diagonal part of Eq. (9) that allows us to calculate a nonzero result of \( m_1 \) (or \( m_3 \)) and the corresponding Majorana CP phase at \( \Lambda_F \) from \( m_1 = 0 \) (or \( m_3 = 0 \)) at \( \Lambda \). In the leading-order approximation, we arrive at

\[
m_1 \simeq -\Delta'_\tau \left[ m_2 (U_{\tau 2} U_{\tau 1})^2 + m_3 (U_{\tau 3} U_{\tau 1})^2 \right]
\]

in the normal neutrino mass ordering case with \( m_1(\Lambda) = 0 \); or

\[
m_3 \simeq -\Delta'_\tau \left[ m_1 (U_{\tau 1} U_{\tau 3})^2 + m_2 (U_{\tau 2} U_{\tau 3})^2 \right]
\]

in the inverted neutrino mass ordering case with \( m_3(\Lambda) = 0 \), where all the neutrino masses and flavor mixing parameters are defined at the Fermi scale \( \Lambda_F \). In view of the fact that \( m_i \) (for \( i = 1, 2, 3 \)) must be real and positive, one may determine the Majorana CP phase associated with \( m_1 \) (or \( m_3 \)) at \( \Lambda_F \) by taking the imaginary part of Eq. (10) or Eq. (11) to be vanishing, and then obtain the explicit expression of \( m_1 \) (or \( m_3 \)) from the real part of Eq. (10) or Eq. (11).

Since the Majorana CP phases of the \( 3 \times 3 \) PMNS matrix \( U \) at a given superhigh energy scale \( \Lambda \) depend on its phase convention, let us take the following popular parametrization \([1]\):

\[
U = P_l \left( \begin{array}{ccc}
c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\
-s_{12}c_{23} - c_{12}s_{13}c_{23}e^{i\delta} & c_{12}c_{23} - s_{12}s_{13}c_{23}e^{i\delta} & c_{13}s_{23} \\
s_{12}s_{23} - c_{12}s_{13}c_{23}e^{i\delta} & -c_{12}s_{23} - s_{12}s_{13}c_{23}e^{i\delta} & c_{13}c_{23}
\end{array} \right) P_\nu ,
\]

in which \( c_{ij} \equiv \cos \theta_{ij} \) and \( s_{ij} \equiv \sin \theta_{ij} \) (for \( ij = 12, 13, 23 \)) with \( \theta_{ij} \) lying in the first quadrant, \( \delta \) is the so-called Dirac CP phase, \( P_l \equiv \text{Diag}\{e^{i\phi_\epsilon}, e^{i\phi_\mu}, e^{i\phi_\tau}\} \) with \( \phi_\epsilon, \phi_\mu \) and \( \phi_\tau \) being the unphysical phases associated with the charged-lepton fields, and \( P_\nu \) is a phase matrix containing two independent Majorana CP phases. Here we choose the phase convention of \( P_l \) as

\[
P_\nu \equiv \begin{cases}
\text{Diag}\{e^{i\rho}, e^{i\sigma}, 1\}, & (m_1 < m_2 < m_3) \\
\text{Diag}\{1, e^{i\rho}, e^{i\sigma}\}, & (m_3 < m_1 < m_2)
\end{cases},
\]

(13)

corresponding to the normal and inverted neutrino mass ordering cases, respectively. Since \( \rho \) (or \( \sigma \)) can always be removed in the \( m_1 = 0 \) (or \( m_3 = 0 \)) limit, only a single Majorana CP phase \( \sigma \) survives when \( M_\nu \) is a rank-two mass matrix. At the Fermi scale \( \Lambda_F \), the PMNS matrix \( U(\Lambda_F) \) can be parametrized in the same form as that of \( U(\Lambda) \). It is convenient to define

\[
\Delta \theta_{ij} \equiv \theta_{ij}(\Lambda_F) - \theta_{ij}(\Lambda), \quad \Delta \delta \equiv \delta(\Lambda_F) - \delta(\Lambda), \quad \Delta \sigma \equiv \sigma(\Lambda_F) - \sigma(\Lambda),
\]

\[
\Delta \phi_\alpha \equiv \phi_\alpha(\Lambda_F) - \phi_\alpha(\Lambda), \quad \Delta \rho \equiv \rho(\Lambda_F) - \rho(\Lambda), \quad \Delta \psi \equiv \psi(\Lambda_F) - \psi(\Lambda),
\]

(14)
so as to describe the strengths of the RGE-induced corrections to the relevant flavor mixing angles and phase parameters. The smallness of such quantum corrections, which are expected to be proportional to either $\Delta_\tau$ or $\Delta'_{\tau}$, makes it reasonable to treat them as small perturbations in the leading-order analytical approximations.

(A) The $m_1 = 0$ case at $\Lambda$

We first calculate the finite values of $m_1$ and $\rho$ at $\Lambda_F$, which originate from $m_1 = 0$ at $\Lambda$ via the two-loop RGE-induced effect. Substituting Eqs. (12) and (13) into Eq. (10), we obtain the following results after a lengthy but straightforward calculation:

$$m_1 \simeq \Delta'_\tau \left( \sin^2 \theta_{12} \sin^2 \theta_{23} + \cos^2 \theta_{12} \sin^2 \theta_{13} \cos^2 \theta_{23} - \frac{1}{2} \sin 2\theta_{12} \sin \theta_{13} \sin 2\theta_{23} \cos \delta \right) \sqrt{\mathcal{F}_1},$$

$$2\rho \simeq \arctan \left( \frac{\mathcal{A}_1}{\mathcal{B}_1} \right),$$

where

$$\mathcal{F}_1 = m_2^2 \left( \cos^2 \theta_{12} \sin^2 \theta_{23} + \sin^2 \theta_{12} \sin^2 \theta_{13} \cos^2 \theta_{23} + \frac{1}{2} \sin 2\theta_{12} \sin \theta_{13} \sin 2\theta_{23} \cos \delta \right)^2 + m_3^2 \cos^4 \theta_{13} \cos^4 \theta_{23} + 2m_2m_3 \cos^2 \theta_{13} \cos^2 \theta_{23} \left[ \sin^2 \theta_{12} \sin^2 \theta_{13} \cos^2 \theta_{23} \cos 2(\sigma + \delta) \right. + \cos^2 \theta_{12} \sin^2 \theta_{23} \cos 2\sigma + \frac{1}{2} \sin 2\theta_{12} \sin \theta_{13} \sin 2\theta_{23} \cos (2\sigma + \delta) \right] + \left. \sin^2 \theta_{13} \sin^2 \theta_{23} \left[ \cos^2 \theta_{12} \sin 2(\sigma - \delta) + \sin^4 \theta_{12} \sin 2(\sigma + \delta) \right] \right),$$

and

$$\mathcal{A}_1 = -m_3 \sin 2\theta_{13} \cos^2 \theta_{23} \left( \sin 2\theta_{12} \cos \theta_{13} \sin 2\theta_{23} \sin \delta - \cos^2 \theta_{12} \sin 2\theta_{13} \cos^2 \theta_{23} \sin 2\delta \right) - m_2 \left\{ 2 \sin 2\theta_{12} \sin \theta_{13} \sin 2\theta_{23} \left( \sin^2 \theta_{13} \cos^2 \theta_{23} - \sin^2 \theta_{23} \right) \right\} \left[ \cos^2 \theta_{12} \sin (2\sigma - \delta) - \sin^2 \theta_{12} \sin (2\sigma + \delta) \right] + \sin^2 \theta_{13} \sin^2 \theta_{23} \left[ \cos^2 \theta_{12} \sin 2(\sigma - \delta) + \sin^4 \theta_{12} \sin 2(\sigma + \delta) \right],$$

$$\mathcal{B}_1 = -m_3 \left\{ \sin^2 \theta_{12} \cos^2 \theta_{13} \sin^2 \theta_{23} - \sin 2\theta_{12} \cos \theta_{13} \sin 2\theta_{23} \left( \sin 2\theta_{12} \cos \theta_{13} \sin 2\theta_{23} \cos \delta \right) - \cos^2 \theta_{12} \sin 2\theta_{13} \cos^2 \theta_{23} \cos 2\delta \right\} - m_2 \left\{ 2 \sin 2\theta_{12} \sin \theta_{13} \sin 2\theta_{23} \left( \sin^2 \theta_{13} \cos^2 \theta_{23} \right) - \sin^2 \theta_{23} \right\} \left[ \cos^2 \theta_{12} \cos (2\sigma - \delta) - \sin^2 \theta_{12} \cos (2\sigma + \delta) \right] + \sin^2 \theta_{12} \cos 2\sigma 
\times \left( \sin^4 \theta_{23} + \sin^4 \theta_{13} \cos^4 \theta_{23} - \sin^2 \theta_{13} \sin^2 2\theta_{23} \right) + \sin^2 \theta_{13} \sin^2 2\theta_{23} \right\} \times \left[ \cos^4 \theta_{12} \cos 2(\sigma - \delta) + \sin^4 \theta_{12} \cos 2(\sigma + \delta) \right] .$$

(B) The $m_3 = 0$ case at $\Lambda$

In the inverted neutrino mass ordering case with $m_3 = 0$ at $\Lambda$, the finite results of $m_3$ and $\varrho$ at $\Lambda_F$ are similarly obtained as follows:

$$m_3 \simeq \Delta'_\tau \left( \cos^2 \theta_{13} \cos^2 \theta_{23} \right) \sqrt{\mathcal{F}_3},$$

$$2\varrho \simeq \arctan \left( \frac{\mathcal{A}_3}{\mathcal{B}_3} \right),$$

(18)
where
\[
F_3 = m_1^2 \left( \sin^2 \theta_{12} \sin^2 \theta_{23} + \cos^2 \theta_{12} \sin^2 \theta_{13} \cos^2 \theta_{23} - \frac{1}{2} \sin 2\theta_{12} \sin \theta_{13} \sin 2\theta_{23} \cos \delta \right)^2 \\
+ m_2^2 \left( \cos^2 \theta_{12} \sin^2 \theta_{23} + \sin^2 \theta_{12} \sin^2 \theta_{13} \cos^2 \theta_{23} + \frac{1}{2} \sin 2\theta_{12} \sin \theta_{13} \sin 2\theta_{23} \cos \delta \right)^2 \\
+ \frac{1}{2} m_1 m_2 \left\{ 2 \sin 2\theta_{12} \sin \theta_{13} \sin 2\theta_{23} \left( \sin^2 \theta_{23} - \sin^2 \theta_{13} \cos^2 \theta_{23} \right) \right\} \left[ \sin^2 \theta_{12} \cos (2\sigma + \delta) \\
- \cos^2 \theta_{12} \cos (2\sigma - \delta) \right] + \sin^2 2\theta_{12} \cos 2\sigma \left( \sin^4 \theta_{23} + \sin^4 \theta_{13} \cos^4 \theta_{23} - \sin^2 \theta_{13} \sin^2 2\theta_{23} \right) \\
+ \sin^2 \theta_{13} \sin^2 2\theta_{23} \left[ \sin^4 \theta_{12} \cos 2 (\sigma + \delta) + \cos^4 \theta_{12} \cos 2 (\sigma - \delta) \right] \}
\]
(19)

and
\[
A_3 = -m_1 \left( 2 \cos^2 \theta_{12} \sin^2 \theta_{13} \cos^2 \theta_{23} \sin 2\delta - \sin 2\theta_{12} \sin \theta_{13} \sin 2\theta_{23} \sin \delta \right) \\
- m_2 \left[ 2 \cos^2 \theta_{12} \sin^2 \theta_{23} \sin 2\sigma + 2 \sin^2 \theta_{12} \sin^2 \theta_{13} \cos^2 \theta_{23} \sin 2 (\sigma + \delta) \right] \\
+ \sin 2\theta_{12} \sin \theta_{13} \sin 2\theta_{23} \sin (2\sigma + \delta) \\
B_3 = -m_1 \left( 2 \sin^2 \theta_{12} \sin^2 \theta_{23} + 2 \cos^2 \theta_{12} \sin^2 \theta_{13} \cos^2 \theta_{23} \cos 2\delta - \sin 2\theta_{12} \sin \theta_{13} \sin 2\theta_{23} \cos \delta \right) \\
- m_2 \left[ 2 \cos^2 \theta_{12} \sin^2 \theta_{23} \cos 2\sigma + 2 \sin^2 \theta_{12} \sin^2 \theta_{13} \cos^2 \theta_{23} \cos 2 (\sigma + \delta) \right] \\
+ \sin 2\theta_{12} \sin \theta_{13} \sin 2\theta_{23} \cos (2\sigma + \delta) \\
\]
(20)

We remark that all the neutrino masses and flavor mixing parameters appearing in Eqs. (15)—(20) take their values at the Fermi scale \( \Lambda_F \). Such a treatment is advantageous to our numerical estimates because it allows us to figure out the radiatively generated values of \( m_1 \) and \( \rho \) (or \( m_3 \) and \( \varrho \)) at \( \Lambda_F \) by directly inputting the experimental data at low energies. Different from \( m_1 \) (or \( m_3 \)), whose running effect from \( \Lambda \) to \( \Lambda_F \) is apparently measured by the value of \( \Delta_F \), the Majorana CP phase \( \rho \) (or \( \varrho \)) is essentially insensitive to a change of the energy scale. This phase parameter is not well defined when \( m_1 = 0 \) (or \( m_3 = 0 \)) exactly holds at \( \Lambda \), but it will become physical soon after the vanishing neutrino mass acquires a tiny nonzero value just a bit below \( \Lambda \). Once \( \rho \) (or \( \varrho \)) is radiatively generated together with \( m_1 \) (or \( m_3 \)), it will almost keep unchanged until \( \Lambda_F \).

At this point it is also worth remarking that our analytical results in Eqs. (15)—(20) are essentially new. In comparison, Davidson et al have only presented the considerably simplified expression of \( m_1 e^{i\rho} \) (or \( m_3 e^{i\varrho} \)) by explicitly taking \( \sin \theta_{12} = 1/\sqrt{3} \), \( \sin \theta_{13} \ll 1 \) and \( \sin \theta_{23} = 1/\sqrt{2} \) in Ref. [21] to give the reader a ball park feeling of the two-loop RGE-induced effect. The latest global analysis of currently available neutrino oscillation data [15], in which the T2K collaboration’s 3\( \sigma \) evidence for \( \delta \neq 0 \) (or \( \pi \)) [30] has been included, yields the best-fit values
\[
\sin^2 \theta_{12} = \left\{ \begin{array}{ll} 0.305 \\
0.303 \\
0.0222 \\
0.0223 \\
0.545 \\
0.551 \\end{array} \right. \\
\sin^2 \theta_{13} = \left\{ \begin{array}{ll} 0.0222 \\
0.0223 \\
0.545 \\
0.551 \\end{array} \right. \\
\sin^2 \theta_{23} = \left\{ \begin{array}{ll} 0.545 \\
0.551 \\
1.28\pi \\
1.52\pi \\end{array} \right. \\
\]
(21)
and
\[
\delta m^2 = \left\{ \begin{array}{ll} 7.34 \times 10^{-5} \text{ eV}^2 \\
7.34 \times 10^{-5} \text{ eV}^2 \\
+2.485 \times 10^{-3} \text{ eV}^2 \\
-2.465 \times 10^{-3} \text{ eV}^2 \\end{array} \right. \\
\Delta m^2 = \left\{ \begin{array}{ll} +2.485 \times 10^{-3} \text{ eV}^2 \\
-2.465 \times 10^{-3} \text{ eV}^2 \\end{array} \right. \\
\]
(22)
where both the normal neutrino mass ordering (upper values) and the inverted one (lower values) have been taken into account, and the two neutrino mass-squared differences are defined as \( \delta m^2 = \)
Figure 3: The numerical result of $m_1$ (or $m_3$) at a given energy scale above $\Lambda_F$, which is radiatively generated from $m_1 = 0$ (or $m_3 = 0$) at $\Lambda \simeq 10^{14}$ GeV in the normal (or inverted) neutrino mass ordering case with $\sigma(\Lambda_F) = 0$, $\pi/4$ or $\pi/2$.

$m_2^2 - m_1^2$ and $\Delta m^2 \equiv m_3^2 - (m_1^2 + m_2^2)/2$. These results will be used in our subsequent numerical estimates of $m_1$ and $\rho$ (or $m_3$ and $\varrho$) at $\Lambda_F$, which are generated from $m_1 = 0$ (or $m_3 = 0$) at $\Lambda \gg \Lambda_F$ via the two-loop RGE evolution.

To compute the evolution of $m_1$ and $\rho$ (or $m_3$ and $\varrho$) with the energy scale $\mu$, we incorporate the two-loop RGE of $\kappa$ described by Eq. (4) into those already known two-loop RGEs of the gauge couplings, the quark and charged-lepton Yukawa couplings and the Higgs self-coupling constant in the SM. Then with $m_1 = 0$ (or $m_3 = 0$) being an input at $\Lambda$, one may choose the initial values of all the other neutrino parameters at $\Lambda$ in such a way that the best-fit values of $\theta_{12}$, $\theta_{13}$, $\theta_{23}$, $\delta$, $\Delta m^2$ and $\Delta m^2$ shown in Eqs. (21) and (22) can be achieved at $\Lambda_F$, where the other Majorana CP phase $\sigma$ is required to acquire a special value 0, $\pi/4$ or $\pi/2$. The exact numerical results of $m_1$ and $\rho$ (or $m_3$ and $\varrho$) in the normal (or inverted) neutrino mass ordering case are obtained by numerically solving the full set of two-loop RGEs, and they are explicitly plotted in Figs. 3 and 4. To compare, the approximate numerical results based on our analytical approximations in Eqs. (15)—(20) are also illustrated in the same figures. In addition, we list the results of $m_1$ and $\rho$ (or $m_3$ and $\varrho$) at $\Lambda_F$ in Table 1, where the values given in the parentheses are obtained by numerically solving the two-loop RGEs.

It is clear that our analytical approximations made in Eqs. (15)—(20) are in good agreement with the results obtained by numerically solving the two-loop RGEs, and the relative accuracy is at...
Figure 4: The numerical result of $\rho$ (or $\varrho$) at a given energy scale above $\Lambda_F$, which is radiatively generated from $m_1 = 0$ (or $m_3 = 0$) at $\Lambda \simeq 10^{14}$ GeV in the normal (or inverted) neutrino mass ordering case with $\sigma(\Lambda_F) = 0$, $\pi/4$ or $\pi/2$.

Table 1: The values of $m_1$ and $\rho$ (or $m_3$ and $\varrho$) at $\Lambda_F \simeq 10^2$ GeV, which are radiatively generated from $m_1 = 0$ (or $m_3 = 0$) at $\Lambda \simeq 10^{14}$ GeV in the normal (or inverted) neutrino mass ordering case with $\sigma(\Lambda_F) = 0$, $\pi/4$ or $\pi/2$. The corresponding results given in the parentheses are obtained by numerically solving the two-loop RGEs.

|            | $\sigma(\Lambda_F)/\pi$ | 0     | 1/4     | 1/2     |
|------------|--------------------------|-------|---------|---------|
| NMO        | $m_1(\Lambda_F)/10^{-13}$ eV | 1.382 (1.377) | 1.258 (1.251) | 1.068 (1.061) |
|            | $\rho(\Lambda_F)/\pi$    | 0.453 (0.453) | 0.476 (0.476) | 0.459 (0.459) |
| IMO        | $m_3(\Lambda_F)/10^{-13}$ eV | 2.991 (2.969) | 2.793 (2.777) | 1.489 (1.482) |
|            | $\varrho(\Lambda_F)/\pi$ | 0.499 (0.499) | 1.677 (1.677) | 1.916 (1.916) |

the $\mathcal{O}(1\%)$ level. Fig. 3 and Table 1 tell us that the value of $m_1$ (or $m_3$) at $\Lambda_F$ is about $10^{-13}$ eV, a result which coincides with the previous estimate made in Ref. [21]. From Fig. 4 or Table 1, one can see that $\rho$ (or $\varrho$) has acquired a physical value at an energy scale just a bit below $\Lambda$, and this value is essentially insensitive to the two-loop RGE evolution between $\Lambda$ and $\Lambda_F$ in the SM. This interesting observation is new, both analytically and numerically. It is obvious that the input of the nontrivial Majorana CP phase $\sigma$ in the $m_1 = 0$ (or $m_3 = 0$) limit at $\Lambda$ may quantitatively affect the radiative generation of a nonzero value of $m_1$ (or $m_3$) and a physical value of $\rho$ (or $\varrho$) at lower energies. That is why invoking a proper flavor symmetry (e.g., the $\mu$-\tau reflection symmetry [35]) may help to fix or constrain the value of $\sigma$ at $\Lambda$. 

10
3 Initially nonzero flavor parameters

As a nontrivial by-product, the one-loop relations between those initially nonzero flavor parameters at \( \Lambda \) and their counterparts at \( \Lambda_F \) will be established here in the case of either \( m_1(\Lambda) = 0 \) or \( m_3(\Lambda) = 0 \). It is unnecessary to consider the two-loop RGE-induced effects on those parameters, simply because such effects have no way to compete with the one-loop contributions.

Substituting Eqs. (12) and (13) into Eq. (9), we obtain the neutrino masses

\[
m_2(\Lambda_F) \simeq I_0 \left[1 + \Delta_r \left(2 \cos^2 \theta_{12} \sin^2 \theta_{23} + 2 \sin^2 \theta_{12} \sin^2 \theta_{13} \cos^2 \theta_{23} + \sin 2 \theta_{12} \sin \theta_{13} \sin 2 \theta_{23} \cos \delta \right)\right] m_2(\Lambda),
\]

\[
m_3(\Lambda_F) \simeq I_0 \left(1 + 2 \Delta_r \cos^2 \theta_{13} \cos^2 \theta_{23}\right) m_3(\Lambda) \tag{23}
\]

in the \( m_1(\Lambda) = 0 \) case; or

\[
m_1(\Lambda_F) \simeq I_0 \left[1 + \Delta_r \left(2 \sin^2 \theta_{12} \sin^2 \theta_{23} + 2 \cos^2 \theta_{12} \sin^2 \theta_{13} \cos^2 \theta_{23} - \sin 2 \theta_{12} \sin \theta_{13} \sin 2 \theta_{23} \cos \delta \right)\right] m_1(\Lambda),
\]

\[
m_2(\Lambda_F) \simeq I_0 \left[1 + \Delta_r \left(2 \cos^2 \theta_{12} \sin^2 \theta_{23} + 2 \sin^2 \theta_{12} \sin^2 \theta_{13} \cos^2 \theta_{23} + \sin 2 \theta_{12} \sin \theta_{13} \sin 2 \theta_{23} \cos \delta \right)\right] m_2(\Lambda) \tag{24}
\]

in the \( m_3(\Lambda) = 0 \) case, where the flavor mixing angles \( \theta_{ij} \) (for \( ij = 12, 13, 23 \)) and the CP-violating phase \( \delta \) are all defined at \( \Lambda_F \).

As for the evolution of three lepton flavor mixing angles from \( \Lambda \) down to \( \Lambda_F \), we have defined \( \Delta \theta_{ij} \equiv \theta_{ij}(\Lambda_F) - \theta_{ij}(\Lambda) \) (for \( ij = 12, 13, 23 \)) in Eq. (14) to describe the RGE-induced effects between the two energy scales. Our one-loop analytical results are

\[
\Delta \theta_{12} \simeq \frac{\Delta_r}{2} \left\{ \sin 2 \theta_{12} \sin^2 \theta_{13} \cos^2 \theta_{23} \left[ \zeta_{32} \sin^2 (\delta + \sigma) + \zeta_{32}^{-1} \cos^2 (\delta + \sigma) - 1 \right] - \left[ (\sin^2 \theta_{23} - \sin^2 \theta_{13} \cos^2 \theta_{23}) \sin 2 \theta_{12} \sin \theta_{13} \sin \theta_{13} \sin \theta_{13} \cos \theta_{23} \cos \delta \right] + \sin 2 \theta_{12} \sin \theta_{13} \sin \theta_{13} \sin \theta_{13} \cos \theta_{23} \cos \delta \right\},
\]

\[
\Delta \theta_{13} \simeq -\frac{\Delta_r}{2} \left\{ \frac{1}{2} \sin 2 \theta_{12} \sin 2 \theta_{23} \cos \theta_{13} \left[ \zeta_{32} \sin (\delta + \sigma) \sin \sigma + \zeta_{32}^{-1} \cos (\delta + \sigma) \cos \sigma - \cos \delta \right] + \sin 2 \theta_{13} \cos^2 \theta_{23} \left[ (\zeta_{32} \sin^2 (\delta + \sigma) + \zeta_{32}^{-1} \cos^2 (\delta + \sigma)) \sin^2 \theta_{12} + \cos^2 \theta_{12} \right] \right\},
\]

\[
\Delta \theta_{23} \simeq -\frac{\Delta_r}{2} \left\{ \sin 2 \theta_{12} \sin \theta_{13} \cos^2 \theta_{23} \left[ \zeta_{32} \sin (\delta + \sigma) \sin \sigma + \zeta_{32}^{-1} \cos (\delta + \sigma) \cos \sigma - \cos \delta \right] + \sin 2 \theta_{23} \left[ (\zeta_{32} \sin^2 \sigma + \zeta_{32}^{-1} \cos^2 \sigma) \cos^2 \theta_{12} + \sin^2 \theta_{12} \right] \right\} \tag{25}
\]

in the \( m_1(\Lambda) = 0 \) case; or

\[
\Delta \theta_{12} \simeq -\frac{\Delta_r}{2} \left\{ \sin 2 \theta_{23} \sin \theta_{13} \left[ \cos \delta + \frac{1}{2} (\zeta_{21} - \zeta_{21}^{-1}) \sin 2 \sigma \sin \delta \right] + (\zeta_{21} \sin^2 \sigma + \zeta_{21}^{-1} \cos^2 \sigma) \right\},
\]

\[
\Delta \theta_{13} \simeq -\frac{\Delta_r}{2} \sin 2 \theta_{13} \cos^2 \theta_{23},
\]

\[
\Delta \theta_{23} \simeq -\frac{\Delta_r}{2} \sin 2 \theta_{23} \tag{26}
\]
in the $m_3(\Lambda) = 0$ case, where we have defined $\zeta_{ij} \equiv (m_i - m_j) / (m_i + m_j)$ with $m_i$ and $m_j$ being the neutrino masses at $\Lambda_F$ (for $i \neq j$ and $i, j = 1, 2, 3$).

At the one-loop level it is well known that $m_1 = 0$ (or $m_3 = 0$) will keep unchanged during the RGE running from $\Lambda$ to $\Lambda_F$, and hence the corresponding Majorana CP phase $\rho$ (or $\varphi$) is not well defined. In this case we only pay attention to the evolution of the remaining two CP-violating phases $\delta$ and $\sigma$ by calculating $\Delta \delta \equiv \delta(\Lambda_F) - \delta(\Lambda)$ and $\Delta \sigma \equiv \sigma(\Lambda_F) - \sigma(\Lambda)$. Their approximate analytical expressions turn out to be

$$
\Delta \delta \simeq \frac{\Delta \tau}{2} \left\{ \sin 2\theta_{12} \sin \theta_{13} \cos 2\theta_{23} \cot \theta_{23} \left[ \zeta_{32} \sin (\delta + \sigma) \cos \sigma - \zeta_{32}^{-1} \cos (\delta + \sigma) \sin \sigma - \sin \delta \right] - \frac{2 \sin \theta_{13} \sin 2\theta_{23} \sin \delta}{\sin 2\theta_{12}} - \sin 2\theta_{23} \sin \delta \left( \frac{\sin 2\theta_{12}}{\sin \theta_{13}} - \frac{2 \sin \theta_{13} \sin^4 \theta_{12}}{\sin 2\theta_{12}} \right) - \sin 2\theta_{23} \left( \frac{\sin 2\theta_{12}}{2 \sin \theta_{13}} \right) \right. \\
- \left. \frac{2 \sin \theta_{13} \cos^4 \theta_{12}}{\sin 2\theta_{12}} \left[ \zeta_{32} \cos (\delta + \sigma) \sin \sigma - \zeta_{32}^{-1} \sin (\delta + \sigma) \cos \sigma \right] + \left( \zeta_{32} - \zeta_{32}^{-1} \right) \right. \\
\times \left[ \left( \cos^2 \theta_{12} \sin^2 \theta_{13} - \sin^2 \theta_{12} \right) \cos^2 \theta_{23} \sin 2 (\delta + \sigma) + \cos^2 \theta_{12} \cos 2\theta_{23} \sin 2 \sigma \right] \right\} 
$$

and

$$
\Delta \sigma \simeq \frac{\Delta \tau}{2} \left\{ \sin 2\theta_{12} \sin \theta_{13} \cot \theta_{23} \left[ \sin \delta - \zeta_{32} \sin (\delta + \sigma) \cos \sigma + \zeta_{32}^{-1} \cos (\delta + \sigma) \sin \sigma \right] \\
+ \left( \zeta_{32} - \zeta_{32}^{-1} \right) \left[ \sin 2\theta_{12} \sin 2\theta_{23} \sin \theta_{13} \sin (\delta + 2\sigma) - \cos^2 \theta_{12} \cos 2\theta_{23} \sin 2 \sigma \\
+ 2 \sin^2 \theta_{12} \sin^2 \theta_{13} \cos^2 \theta_{23} \sin 2 (\delta + \sigma) \right] \right\}
$$

in the $m_1(\Lambda) = 0$ case; or

$$
\Delta \delta \simeq -\frac{\Delta \tau}{2} \left[ \left( \zeta_{21} - \zeta_{21}^{-1} \right) \sin 2\sigma \left( \sin^2 \theta_{23} - \sin^2 \theta_{13} \cos^2 \theta_{23} - \sin \theta_{13} \sin 2\theta_{23} \cot 2\theta_{12} \cos \delta \right) \\
+ \frac{2 \sin \theta_{13} \sin 2\theta_{23} \sin \delta}{\sin 2\theta_{12}} \left( \zeta_{21} \cos^2 \sigma + \zeta_{21}^{-1} \sin^2 \sigma \right) - 2 \sin \theta_{13} \sin 2\theta_{23} \cot 2\theta_{12} \sin \delta \right] 
$$

and

$$
\Delta \sigma \simeq -\frac{\Delta \tau}{2} \left\{ 2 \sin \theta_{13} \sin 2\theta_{23} \sin \delta \left[ \left( \zeta_{21} \cos^2 \sigma + \zeta_{21}^{-1} \sin^2 \sigma \right) \cot 2\theta_{12} - \csc 2\theta_{12} \right] \\
+ \left( \zeta_{21} - \zeta_{21}^{-1} \right) \sin 2\sigma \left[ \left( \sin^2 \theta_{23} - \sin^2 \theta_{13} \cos^2 \theta_{23} \right) \cos 2\theta_{12} \\
- \sin \theta_{13} \sin 2\theta_{23} \cos 2\theta_{12} \cot 2\theta_{12} \cos \delta \right] \right\} 
$$

in the $m_3(\Lambda) = 0$ case. These integral-form analytical results are new, and they are certainly more instructive and transparent than the differential RGEs of the relevant flavor parameters for our understanding of their evolution behaviors from $\Lambda$ to $\Lambda_F$ at the one-loop level.

With the same inputs as summarized in section 2, the evolution of those initially nonzero flavor parameters, including $m_2$ and $m_3$ (or $m_1$ and $m_3$) in the normal (or inverted) neutrino mass ordering case, $\Delta \theta_{ij}$ (for $i j = 12, 13, 23$), $\Delta \delta$ and $\Delta \sigma$, is numerically calculated with the help of both

---

1 One should keep in mind that the unphysical phases $\phi_{\alpha}$ (for $\alpha = e, \mu, \tau$) and $\rho$ (or $\varphi$) at the one-loop level will also evolve with the energy scale $\mu$, and hence their evolution cannot be ignored in deriving the one-loop RGEs of those physical flavor parameters [22–28].
the two-loop differential RGEs and the analytical approximations given in Eqs. (23)—(30). Our numerical results are illustrated in Figs. 5—7. In particular, the values of such flavor parameters at $\Lambda_F$ are explicitly listed in Table 2 where the numbers shown in the parentheses are obtained by numerically solving the two-loop RGEs. Some immediate comments are in order.

- From Eqs. (23) and (24), one can see that the running effects of $m_2$ and $m_3$ (or $m_1$ and $m_2$) in the normal (or inverted) neutrino mass ordering case are mainly governed by an overall factor $I_0$ whose values changing with $\mu$ are shown in Fig. 2, and they are independent of the value of the Majorana CP phase $\sigma(\Lambda_F)$ in the leading-order approximation, as also illustrated in Fig. 5 and Table 2.

- In comparison with Eq. (25), Eq. (26) is much simpler and thus makes it much easier to understand the running behaviors of $\Delta \theta_{ij}$ (for $ij = 12, 13, 23$) in the inverted neutrino mass ordering case. With the best-fit values of $\theta_{ij}, \delta, \delta m^2$ and $\Delta m^2$ given in Eqs. (21) and (22), it is obvious that in the inverted neutrino mass ordering case the evolution of $\Delta \theta_{13}$ and $\Delta \theta_{23}$ is dominated by that of $\Delta \tau$ and independent of the value of $\sigma(\Lambda_F)$ in the leading-order approximation, as also shown in Fig. 6 and Table 2.

- Fig. 6 and Table 2 show that the magnitude of $\Delta \theta_{13}$ is strongly suppressed in the normal neutrino mass ordering case with $\sigma(\Lambda_F) = 0$, mainly because a large cancellation appears in the analytical expression of $\Delta \theta_{13}$ when $\sigma(\Lambda_F) = 0$ is taken. The magnitude of $\Delta \theta_{12}$ is also suppressed in the inverted mass ordering case with $\sigma(\Lambda_F) = \pi/2$, simply because of the suppression caused by the smallness of $\zeta_{21}$ and $\theta_{13}$ when $\sigma(\Lambda_F) = \pi/2$ is taken. In either situation the relative accuracy of our analytical approximations at $\Lambda_F$ becomes worse, and

Figure 5: The numerical results of $m_2$ and $m_3$ (or $m_1$ and $m_2$) at a given energy scale above $\Lambda_F$ in the normal (or inverted) neutrino mass ordering case with $m_1 = 0$ (or $m_3 = 0$) at $\Lambda \simeq 10^{14}$ GeV and $\sigma(\Lambda_F) = 0, \pi/4$ or $\pi/2$. 
Figure 6: The numerical results of $\Delta \theta_{ij}$ (for $ij = 12, 13, 23$) at a given energy scale above $\Lambda_F$ in the normal (or inverted) neutrino mass ordering case with $m_1 = 0$ (or $m_3 = 0$) at $\Lambda \simeq 10^{14} \text{ GeV}$ and $\sigma(\Lambda_F) = 0, \pi/4$ or $\pi/2$.

it reduces from the $\mathcal{O}(1\%)$ level to the $\mathcal{O}(1\%)$ level. Of course, the value of $\Delta \theta_{12}$ is largely enhanced in the inverted neutrino mass ordering case with $\sigma(\Lambda_F) = 0$ or $\pi/4$ as a result of the largeness of $\zeta_{21}^{-1}$, which can easily be seen in Eq. (26).

- As can be seen from Fig. 7 and Table 2, the value of $\Delta \sigma$ in the normal neutrino mass ordering case is much smaller than that in the inverted mass ordering case. In the latter case with $\sigma(\Lambda_F) = \pi/4$ or $\pi/2$, the values of $\Delta \delta$ and $\Delta \sigma$ are largely enhanced thanks to the largeness of $\zeta_{21}^{-1}$. Such a feature is easily understandable with the help of Eqs. (29) and (30).

4 Summary

Given two different neutrino mass-squared differences that have been determined in a number of neutrino oscillation experiments, whether the lightest neutrino $\nu_1$ (or $\nu_3$) can be exactly massless turns out to be an interesting question in neutrino phenomenology. From the perspective of model building, it is always possible to obtain $m_1 = 0$ (or $m_3 = 0$) at the tree level if the flavor structure of the model is properly specified (e.g., in the minimal seesaw model with only two right-handed neutrino states). Then the question becomes whether such a massless neutrino can stay massless
Figure 7: The numerical results of $\Delta \delta$ and $\Delta \sigma$ at a given energy scale above $\Lambda_F$ in the normal (or inverted) neutrino mass ordering case with $m_1 = 0$ (or $m_3 = 0$) at $\Lambda \simeq 10^{14}$ GeV and $\sigma(\Lambda_F) = 0$, $\pi/4$ or $\pi/2$.

against quantum corrections when the energy scale evolves from a superhigh scale $\Lambda$, where the seesaw mechanism or flavor symmetry works, down to the Fermi scale $\Lambda_F$. In the SM framework Davidson et al have given a preliminary answer to this question by taking into account the two-loop RGE-induced effects [21]. Here we have carried out a further study of this issue by paying attention to the two-loop radiative corrections to not only the smallest neutrino mass $m_1$ (or $m_3$) but also the associated Majorana CP phase $\rho$ (or $\varrho$).

In the present work both $m_1$ (or $m_3$) and $\rho$ (or $\varrho$) at an arbitrary energy scale between $\Lambda_F$ and $\Lambda$ have been analytically formulated at the two-loop level, and their magnitudes have been evaluated both based on our analytical approximations and by numerically solving the two-loop RGEs. We find that the numerical results obtained in these two ways are in good agreement with each other. In particular, we have confirmed that a nonzero value of $m_1$ (or $m_3$) of $\mathcal{O}(10^{-13})$ eV at $\Lambda_F$ can be generated from $m_1 = 0$ (or $m_3 = 0$) at $\Lambda \simeq 10^{14}$ GeV via the two-loop quantum corrections in the SM, and found that $\rho$ (or $\varrho$) may accordingly acquire an appreciable physical value at the same level. As a nontrivial by-product, the evolution of all those initially nonzero flavor parameters of massive neutrinos has been calculated both analytically and numerically, by simply keeping their leading (i.e., one-loop) RGE-induced effects.

This study can therefore allow one to draw the conclusion that taking $m_1 = 0$ (or $m_3 = 0$) and
Table 2: The values of $m_2$ and $m_3$ (or $m_1$ and $m_2$) at $\Lambda_F \simeq 10^2$ GeV, together with those of $\Delta \theta_{ij}$ (for $ij = 12, 13, 23$), $\Delta \delta$ and $\Delta \sigma$, in the normal (or inverted) neutrino mass ordering case with $m_1 = 0$ (or $m_3 = 0$) at $\Lambda \simeq 10^{14}$ GeV and $\sigma(\Lambda_F) = 0, \pi/4$ or $\pi/2$. The corresponding results given in the parentheses are obtained by numerically solving the two-loop RGEs.

|                  | $\sigma(\Lambda_F)$ | $\pi/4$          | $\pi/2$          |
|------------------|----------------------|------------------|------------------|
| $m_2 (\Lambda_F)/m_2 (\Lambda)$ | 0.762 (0.762)        | 0.762 (0.762)    | 0.762 (0.762)    |
| $m_3 (\Lambda_F)/m_3 (\Lambda)$ | 0.762 (0.762)        | 0.762 (0.762)    | 0.762 (0.762)    |
| $\Delta \theta_{12}(\Lambda_F)$ | $-9.124 \times 10^{-6}$ | $-8.445 \times 10^{-6}$ | $-8.459 \times 10^{-6}$ |
| $\Delta \theta_{13}(\Lambda_F)$ | $-9.136 \times 10^{-6}$ | $-8.460 \times 10^{-6}$ | $-8.474 \times 10^{-6}$ |
| $\Delta \theta_{23}(\Lambda_F)$ | $-2.092 \times 10^{-7}$ | $-3.193 \times 10^{-6}$ | $-3.131 \times 10^{-6}$ |
| $\Delta \delta(\Lambda_F)$ | $-2.227 \times 10^{-7}$ | $-3.195 \times 10^{-6}$ | $-3.135 \times 10^{-6}$ |
| $\Delta \sigma(\Lambda_F)$ | $-1.772 \times 10^{-5}$ | $-1.474 \times 10^{-5}$ | $-1.129 \times 10^{-5}$ |
| $\Delta \delta(\Lambda_F)$ | $-1.765 \times 10^{-5}$ | $-1.469 \times 10^{-5}$ | $-1.125 \times 10^{-5}$ |
| $\Delta \delta(\Lambda_F)$ | $-6.902 \times 10^{-6}$ | $-5.882 \times 10^{-6}$ | $1.281 \times 10^{-5}$ |
| $\Delta \delta(\Lambda_F)$ | $-6.869 \times 10^{-6}$ | $-5.863 \times 10^{-6}$ | $1.275 \times 10^{-5}$ |
| $\Delta \sigma(\Lambda_F)$ | $5.829 \times 10^{-7}$ | $-5.920 \times 10^{-8}$ | $-4.205 \times 10^{-7}$ |
| $\Delta \sigma(\Lambda_F)$ | $5.807 \times 10^{-7}$ | $-5.876 \times 10^{-8}$ | $-4.190 \times 10^{-7}$ |

switching off the associated Majorana CP phase $\rho$ (or $\bar{\rho}$) is absolutely safe at low energies for the minimal type-I seesaw model and some other neutrino mass models of this kind which naturally predict $m_1 = 0$ (or $m_3 = 0$) at the tree level at a superhigh energy scale.

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