MULTIPARAPRAMETER PERSISTENCE MODULES IN THE LARGE SCALE

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ABSTRACT. A persistence module with \( m \) discrete parameters is a diagram of vector spaces indexed by the poset \( \mathbb{N}^m \). If we are only interested in the large scale behavior of such a diagram, then we can consider two diagrams equivalent if they agree outside of a “negligeable” region. In the 2-dimensional case, we classify the indecomposable diagrams up to finitely supported diagrams. In higher dimension, we partially classify the indecomposable diagrams up to suitably finite diagrams, and show that the full classification problem is wild.

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1. INTRODUCTION

In topological data analysis, it is common to study a space endowed with a filtration, which corresponds to a functor \( \mathbb{N} \to \text{Top} \) in the case of a discrete parameter, or \( \mathbb{R}_{\geq 0} \to \text{Top} \) in the case of a continuous real parameter. A multiparameter filtration using \( m \geq 1 \) discrete parameters corresponds to a functor \( \mathbb{N}^m \to \text{Top} \) from the poset \( \mathbb{N}^m \). Taking the \( i \)-th homology group with coefficients in a field \( k \) then yields a functor \( \mathbb{N}^m \to \text{Vect}_k \), called an \( m \)-parameter persistence module. These correspond to multigraded modules over the \( \mathbb{N}^m \)-graded ring \( R = k[t_1, \ldots, t_m] \) with grading \( |t_i| = e_i = (0, \ldots, 1, \ldots, 0) \). For background on multiparameter persistence modules and their applications to topological data analysis, see [CZ09], [Oud15], [Les15], as well as the comprehensive survey [BL23]. This paper concerns the representation theory of finitely generated \( R \)-modules.

Each finitely generated \( R \)-module decomposes (uniquely) into a direct sum of indecomposable modules; see [BCB20] for more general decomposition theorems. For one-parameter persistence modules (the case \( m = 1 \), by the classification of finitely generated modules over a principal ideal
domain, each module decomposes into a direct sum of interval modules, which are the indecomposables. The intervals appearing in the decomposition can then be summarized into a barcode. For multiparameter persistence modules (the case \( m \geq 2 \)), such a classification of indecomposables is unavailable, since the graded ring \( \mathbb{k}[t_1, t_2] \) has wild representation type; see for instance [BL23, §8].

One approach around that problem is to extract from a module certain invariants, notably the rank invariant (introduced by Carlsson and Zomorodian [CZ09]) and its various refinements. The goal is to find invariants that are both computable and significant in applications. That approach is developed for instance in [Vip20], [GC21], [CCS21], [BOO22], [OS22], [KM21], [BBH22], and [CDMW23]. Another approach is to focus on certain families of modules admitting a nice decomposition, such as rectangle-decomposable modules. One then tries to characterize the modules that are of the special form, and to approximate an arbitrary module by such a module. See for instance [BLO20a], [BLO20b], and [ABE+22].

In this paper, we take a different approach. We localize the category of \( R \)-modules until the resulting category admits a classification of indecomposables, or at least a partial classification.

**Organization and results.** Given a simplicial complex \( K \) on \( m \) vertices, we define an abelian category \( \mathcal{L}(K) \) of \( K \)-localized persistence modules (Definition 3.4). Those encode the data of various localizations of an \( R \)-module, where \( K \) parametrizes which variables among \( t_1, \ldots, t_m \) have been inverted. We exhibit \( \mathcal{L}(K) \) as a Serre quotient of \( R \)-modules (Proposition 4.8).

Sections 5–7 then focus on the simplicial complex \( K_m := \text{sk}_{m-3} \Delta^{m-1} \). One reason for that choice is that for any smaller simplicial complex \( K' \subset K_m \), the category \( \mathcal{L}(K') \) is guaranteed to have wild representation type (Proposition 9.7). We construct a torsion pair on \( \mathcal{L}(K_m) \) (Proposition 5.5) and show that the torsion pair splits, so that every object is a direct sum of its torsion part and its torsion-free part (Proposition 5.6). We further decompose the torsion objects into direct sums of indecomposable objects of a specific form (Proposition 6.3). In the case \( m = 2 \), we decompose the torsion-free objects into direct sums of indecomposable objects of a specific form (Proposition 7.11). For \( m \geq 3 \), we show that the analogous decomposition of torsion-free objects does not hold (Proposition 8.4). The results for \( \mathcal{L}(K_2) \) can be summarized as follows.

**Theorem A.** In the category of finitely generated \( \mathbb{k}[s, t] \)-modules up to finite modules, every object decomposes in a unique way as a direct sum of:

- “vertical strips” \([a, b)_1 := s^a \mathbb{k}[s, t]/s^b \mathbb{k}[s, t]\)
- “horizontal strips” \([a, b)_2 := t^a \mathbb{k}[s, t]/t^b \mathbb{k}[s, t]\)
- “quadrants” \([a_1, a_2, \infty) := s^{a_1} t^{a_2} \mathbb{k}[s, t]\).

See Figure 1.

In particular, \( \mathcal{L}(K_2) \) does not have wild representation type. Combining this fact with Proposition 9.7 and Theorem 9.16, we obtain:

**Theorem B.** The category \( \mathcal{L}(K) \) has wild representation type if and only if the simplicial complex \( K \) has more than 3 missing faces. In other words: a missing face of codimension 2 or at least 3 missing faces of codimension 1.

In Section 10, we show that the rank invariant of an \( R \)-module determines the torsion part of \( L_{K_m}(M) \) in \( \mathcal{L}(K_m) \), but not the torsion-free part, even in the case \( m = 2 \). In Section 11, we turn our attention to the Serre subcategory of \( R \)-modules being quotiented out in the construction of \( \mathcal{L}(K) \). We classify all tensor-closed Serre subcategories of \( R \)-mod and show that they are in bijection with simplicial complexes \( K \) on \( m \) vertices (Theorem 11.15). The tensor-closed Serre subcategory corresponding to \( K \) is generated by the Stanley–Reisner ring \( \mathbb{k}[K] \). In Section 12, we revisit \( \mathcal{L}(K_m) \) and show that it is obtained from \( R \)-mod by iteratively quotienting out the simple objects \( m - 1 \) times (Corollary 12.8).
Remark 1.1. The categories we consider arise in algebraic geometry. The category of coherent sheaves on the projective line \( \mathbb{P}^1 \) over the base field \( k \) is described by an equivalence

\[
\text{Coh}(\mathbb{P}^1) \cong \mathbb{Z}\text{-graded } k[s,t]\text{-mod}/\{\text{finite modules}\},
\]

where \( k[s,t] \) is given an \( \mathbb{N} \)-grading with \( |s| = |t| = 1 \), and the right-hand side denotes the Serre quotient by the subcategory of graded modules \( M \) with graded pieces \( M(d) \neq 0 \) in finitely many degrees \( d \in \mathbb{Z} \). See [Har77, Exercise II.5.9] and [TSPA18, Tag 0BXD], as well as [Cox95, §3] and [CLS11, §5.3]. The \( \mathbb{Z}^2 \)-graded variant of our category \( \mathcal{L}(K_2) \) is the bigraded analogue of the right-hand side:

\[
\mathcal{L}(K_2)_{\mathbb{Z}^2} \cong \mathbb{Z}^2\text{-graded } k[s,t]\text{-mod}/\{\text{finite modules}\}.
\]

It was kindly pointed out to us by Colin Ingalls that the latter category is equivalent to torus-equivariant coherent sheaves on \( \mathbb{P}^1 \); see for instance [Per04, §5] and [Per13]. Hence Theorem A provides a decomposition result for such sheaves. It would be interesting to pursue the connections between persistence modules and toric geometry.

Related work. This paper is close in spirit to [HOST19] and uses some similar tools. In that paper, the authors extract computable invariants from \( R \)-modules that are related to the localizations we consider, whereas our focus is the decomposition of objects in the localized categories.

The paper [BBOS20] is also closely related. The authors use torsion pairs and consider decompositions of certain families of 2-parameter persistence modules up to certain subcategories.

Acknowledgments. We thank Colin Ingalls, Thomas Brüstle, and Markus Perling for helpful discussions. We are very grateful to Steffen Oppermann for providing the argument in the proof of Theorem 9.16. We acknowledge the support of the Natural Sciences and Engineering Research Council of Canada (NSERC). Cette recherche a été financée par le Conseil de recherches en sciences naturelles et en génie du Canada (CRSNG). RGPIN-05466-2020 (Stanley), RGPIN-2019-06082 (Frankland).

2. Serre quotients

In this section, we collect a few facts about Serre quotients that are found in [TSPA18, Tag 02MN], [GM03, §II.5 Exercise 9], [Pop73, §4.3, 4.4], or [Gab62, §III].

Definition 2.1. Let \( \mathcal{A} \) be an abelian category. A Serre subcategory of \( \mathcal{A} \) is a full subcategory \( \mathcal{S} \subseteq \mathcal{A} \) that contains 0 and such that for every short exact sequence

\[
0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0
\]

in \( \mathcal{A} \), the object \( B \) lies in \( \mathcal{S} \) if and only if \( A \) and \( C \) lie in \( \mathcal{S} \). In other words, \( \mathcal{S} \) is closed under forming subobjects, quotients, and extensions.

Lemma 2.2. Let \( F : \mathcal{A} \rightarrow \mathcal{B} \) be an exact functor between abelian categories. Then the subcategory \( \ker(F) := \{ A \in \mathcal{A} \mid F(A) \cong 0 \} \) is a Serre subcategory of \( \mathcal{A} \), called the kernel of \( F \).

Lemma 2.3. Let \( \mathcal{S} \subseteq \mathcal{A} \) be a Serre subcategory. Then there exists an abelian category \( \mathcal{A}/\mathcal{S} \) and an exact functor \( q : \mathcal{A} \rightarrow \mathcal{A}/\mathcal{S} \) satisfying the following universal property: For every exact functor \( F : \mathcal{A} \rightarrow \mathcal{B} \) satisfying \( \mathcal{S} \subseteq \ker(F) \), there exists a unique exact functor \( \overline{F} : \mathcal{A}/\mathcal{S} \rightarrow \mathcal{B} \) satisfying \( F = \overline{F} \circ q \), as illustrated in the diagram

\[
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{F} & \mathcal{B} \\
q \downarrow & & \downarrow \overline{F} \\
\mathcal{A}/\mathcal{S} & & 
\end{array}
\]
The category $A/S$ is called the **Serre quotient** of $A$ by $S$.

**Lemma 2.5.** In the setup of diagram (2.4), the induced functor $F: A/S \to B$ is faithful if and only if $S = \ker(F)$ holds.

The following statement is found in [Gab62, Proposition III.2.5] and [Pop73, Theorem 4.4.9].

**Lemma 2.6.** Let $F: A \to B$ be an exact functor between abelian categories. If $F$ admits a right adjoint $G: B \to A$ that is fully faithful (equivalently, such that the counit $\epsilon: FG \Rightarrow 1$ is an isomorphism), then the induced functor $F^*: A/\ker(F) \to B$ is an equivalence of categories.

### 3. Localization of persistence modules

Let us fix some notation that will be used throughout the paper.

**Notation 3.1.** Let $m \geq 1$ be an integer and denote the set $[m] := \{1, 2, \ldots, m\}$. Let $k$ be a field and consider the polynomial ring $R := k[t_1, \ldots, t_m]$, which is $\mathbb{N}^m$-graded with multigrading

$$|t_i| = \vec{e}_i = (0, \ldots, 0, 1, \ldots, 0).$$

In other words, $R$ is the monoid algebra $R = k(\mathbb{N}^m)$. Let $R$-$\text{Mod}$ denote the category of (left) $R$-modules and $R$-$\text{mod}$ the full subcategory of finitely generated $R$-modules.

Viewing the poset $\mathbb{N}^m$ as a category, an $\mathbb{N}^m$-shaped diagram of $k$-vector spaces $M: \mathbb{N}^m \to \text{Vect}_k$ corresponds to a (multigraded) $R$-module $M$, called a **persistence module**. This correspondence forms an isomorphism of categories

$$\text{Fun}(\mathbb{N}^m, \text{Vect}_k) \cong R\text{-Mod}$$

[HOST19, Theorem 2.6] [CZ09]. Given an $R$-module $M$ and $\vec{d} = (d_1, \ldots, d_m) \in \mathbb{N}^m$, let $M(\vec{d})$ denote the $k$-vector space which is the part of $M$ in multidegree $\vec{d}$.

**Notation 3.2.** For a subset $\sigma \subseteq [m]$, denote the localization of monoids $\sigma^{-1}\mathbb{N}^m$, where we identify $i \in [m]$ with the standard basis element $\vec{e}_i \in \mathbb{N}^m$, for instance:

$$\{1, 3\}^{-1}\mathbb{N}^3 = \mathbb{Z} \times \mathbb{N} \times \mathbb{Z}.$$

Denote the localization of rings

$$R_\sigma := R[t_i^{-1} \mid i \in \sigma],$$

which is $\sigma^{-1}\mathbb{N}^m$-graded. For instance, $R_{[m]} = k[t_1^\pm, \ldots, t_m^\pm]$ is $\mathbb{Z}^m$-graded. Likewise, $R_\sigma$-modules are $\sigma^{-1}\mathbb{N}^m$-graded. An $R$-module $M$ can be viewed as $\sigma^{-1}\mathbb{N}^m$-graded by setting

$$M(\vec{d}) = 0 \quad \text{if } \vec{d} \in \sigma^{-1}\mathbb{N}^m \setminus \mathbb{N}^m.$$

Similarly, the category $R$-$\text{Mod}_{\sigma^{-1}\mathbb{N}^m}$ of $\sigma^{-1}\mathbb{N}^m$-graded $R$-modules can be viewed as a full subcategory of $\mathbb{Z}^m$-graded $R$-modules, which we denote $R$-$\text{Mod}_{\mathbb{Z}^m}$.

**Lemma 3.3.** The full subcategory $R$-$\text{Mod}_{\sigma^{-1}\mathbb{N}^m} \subseteq R$-$\text{Mod}_{\mathbb{Z}^m}$ is an abelian subcategory.

**Proof.** Limits and colimits in $R$-$\text{Mod}_{\sigma^{-1}\mathbb{N}^m}$ are computed as in $R$-$\text{Mod}_{\mathbb{Z}^m}$, namely, degreewise in $k$-modules for each multidegree $\vec{d} \in \mathbb{Z}^m$. \qed

**Definition 3.4.** Let $K \subseteq \mathcal{P}([m])$ be a simplicial complex on the vertex set $[m]$. (We do not assume that all the elements $i \in [m]$ are 0-simplices of $K$. In other words, $K$ may have ghost vertices.) A **$K$-localized persistence module** $M$ consists of the following data:

1. For each missing face $\sigma \notin K$, a finitely generated $R_\sigma$-module $M_\sigma$. 

(2) For each $\sigma \subseteq \tau$ with $\sigma \not\in K$ (and hence $\tau \not\in K$), a map of $R_\sigma$-modules $\varphi_{\sigma,\tau} : M_\sigma \to M_\tau$ such that the induced map of $R_\tau$-modules

$$R_\tau \otimes_{R_\sigma} M_\sigma \xrightarrow{\cong} M_\tau$$

is an isomorphism.

The transition maps are required to satisfy $\varphi_{\sigma,\sigma} = \text{id}$ and $\varphi_{\tau,\nu} \circ \varphi_{\sigma,\tau} = \varphi_{\sigma,\nu}$ for all missing faces $\sigma \subseteq \tau \subseteq \nu$.

A morphism of $K$-localized persistence modules $f : M \to N$ consists of a map of $R_\sigma$-modules $f_\sigma : M_\sigma \to N_\sigma$ for each $\sigma \not\in K$, compatible with the transition maps:

$$M_\sigma \xrightarrow{f_\sigma} N_\sigma \quad \downarrow \quad M_\tau \xrightarrow{f_\tau} N_\tau.$$ 

Let $\mathcal{L}(K)$ denote the category of $K$-localized persistence modules. Let $\mathcal{L}(K)_{\mathbb{Z}_m}$ denote the variant of $\mathcal{L}(K)$ where all the $R_\sigma$-modules are $\mathbb{Z}_m$-graded rather than $\sigma^{-1}\mathbb{N}_m$-graded.

**Example 3.5.** In the case $K = \emptyset = \text{sk}_{-2} \Delta^{m-1}$, we have $\mathcal{L}(K) \cong R\text{-mod}$. The other extreme case is $K = \partial \Delta^{m-1} = \text{sk}_{m-2} \Delta^{m-1}$, which yields $\mathcal{L}(K) = R[\mathbb{m}]\text{-mod} \cong \text{vect}_k$, cf. Remark 3.8.

**Example 3.6.** Consider the case $m = 2$ and $K = \{\emptyset\} = \text{sk}_{-1} \Delta^1$. A $K$-localized persistence module $M$ consists of modules

$$M(2) \xrightarrow{\varphi_1} M(1,2) \quad \downarrow \quad M(3)$$

where for each $i \in \{1, 2\}$, $M(i)$ is a $R[t_i^{-1}]$-module, $\varphi_{3-i} : M(i) \to M(1,2)$ is a map of $R[t_i^{-1}]$-modules, and the induced map of $R[t_1^{-1}, t_2^{-1}]$-modules

$$R[t_1^{-1}, t_2^{-1}] \otimes_{R[t_i^{-1}]} M(i) \xrightarrow{\cong} M(1,2)$$

is an isomorphism. We used the notation

$\varphi_i := \varphi_{[m] \setminus \{i\}, [m]} : M_{[m] \setminus \{i\}} \to M_{[m]}$,

so that $\varphi_i$ denotes the localization map inverting $t_i$. We will write $\sigma_i := [m] \setminus \{i\}$.

**Example 3.7.** Consider the case $m = 3$ and $K = \text{sk}_0 \Delta^2 = \{\emptyset, \{1\}, \{2\}, \{3\}\}$. A $K$-localized persistence module $M$ can be displayed as a diagram

$$M_{\sigma_1} \xrightarrow{\varphi_1} M[3] \quad \downarrow \quad M_{\sigma_2} \xrightarrow{\varphi_2} M[3] \quad \downarrow \quad M_{\sigma_3}.$$
Remark 3.8. A graded \( k[t^\pm] \)-module \( M \) corresponds to a \( k \)-vector space
\[
M(0) \cong \text{colim} \left( \cdots \to M(d-1) \xrightarrow{1} M(d) \xrightarrow{1} M(d+1) \to \cdots \right)
\]
In fact, the functor taking the degree 0 part
\[
k[t^\pm]-\text{Mod} \to \text{Vect}_k
\]
is an equivalence of categories. This is in stark contrast with an ungraded \( k[t^\pm] \)-module, which consists of a \( k \)-vector space \( V \) equipped with an automorphism \( \mu_t: V \xrightarrow{\cong} V \). Likewise, for any \( m \geq 1 \), a \( \mathbb{Z}^m \)-graded \( R_{[m]} \)-module \( M \) corresponds to the \( k \)-vector space \( M(\vec{0}) \) in multidegree \( \vec{0} \in \mathbb{Z}^m \).

Because of this, we will write
\[
\dim_k M := \dim_k M(\vec{0}).
\]

Definition 3.9. For \( \sigma \subseteq [m] \) and an \( R_\sigma \)-module \( M \), the rank of \( M \) is
\[
\text{rank } M := \dim_k R_{[m]} \otimes_{R_\sigma} M.
\]
For a simplicial complex \( K \) on the vertex set \( [m] \) and \( M \) a \( K \)-localized persistence module, the rank of \( M \) is
\[
\text{rank } M := \dim_k M_{[m]},
\]
which equals \( \text{rank}(M_\sigma) \) for any missing face \( \sigma \notin K \).

See also the discussion of rank in [HOST19, §3].

Lemma 3.10. The category \( \mathcal{L}(K) \) is an abelian category. Kernels and cokernels in \( \mathcal{L}(K) \) are computed pointwise, i.e., for each \( \sigma \notin K \), we have
\[
(\ker f)_\sigma = \ker(f_\sigma)
\]
\[
(\text{coker } f)_\sigma = \text{coker}(f_\sigma).
\]
In particular, a sequence
\[
0 \to M \to N \to P \to 0
\]
in \( \mathcal{L}(K) \) is exact if and only if for each \( \sigma \notin K \), the sequence of \( R_\sigma \)-modules
\[
0 \to M_\sigma \to N_\sigma \to P_\sigma \to 0
\]
is exact.

4. AN EQUIVALENCE OF CATEGORIES

Definition 4.1. Given a simplicial complex \( K \) on the set of vertices \( [m] \), consider the functor
\[
L_K: R\text{-mod} \to \mathcal{L}(K)
\]
that sends an \( R \)-module \( M \) to the \( K \)-localized persistence module \( L_K(M) \) given by the localizations
\[
L_K(M)_\sigma = R_\sigma \otimes_R M
\]
and the canonical localization maps \( R_\sigma \otimes_R M \to R_\tau \otimes_R M \) for missing faces \( \sigma \subseteq \tau \). We call \( L_K \) the \( K \)-localization functor.

Lemma 4.2. The functor \( L_K \) is exact.

Proof. This follows from Lemma 3.10 and the fact that each localization functor \( R_\sigma \otimes_R - \) is exact, in other words, \( R_\sigma \) is flat as an \( R \)-module. \( \square \)
**Notation 4.3.** For a multidegree \( \vec{d} \in \mathbb{Z}^m \), let \([\vec{d}] \in \mathbb{N}^m\) denote the multidegree obtained by replacing the negative entries \(d_i\) with zeroes, that is:

\[
[\vec{d}] := \sup\{\vec{d}, \vec{0}\}.
\]

For a \(\mathbb{Z}^m\)-graded \(R\)-module \(M\) and \(x \in M\), let \([x]\) denote the shift of \(x\) that lives in multidegree \([|x|]\), that is:

\[
[x] := t^\delta x \quad \text{where} \quad \delta = [|x|] - |x| = \sup\{\vec{0}, -|x|\} = -\inf\{\vec{0}, |x|\}.
\]

For example, with \(|x| = (-3, 5)\), we obtain \([x] = t^3 x\), which lies in bidegree \([(-3, 5)] = (0, 5)\).

**Lemma 4.4.** For \(\sigma \subseteq [m]\), let

\[
i: R\text{-Mod} \hookrightarrow R\text{-Mod}_{\sigma^{-1}\mathbb{N}^m}
\]

denote the inclusion of the full subcategory of \(\mathbb{N}^m\)-graded \(R\)-modules.

(1) The truncation functor \(\tau_{\geq 0}: R\text{-Mod}_{\sigma^{-1}\mathbb{N}^m} \to R\text{-Mod}\)

given by

\[
(\tau_{\geq 0}M)(\vec{d}) = M(\vec{d}) \quad \text{for} \quad \vec{d} \in \mathbb{N}^m
\]

is right adjoint to \(i\).

(2) \(\tau_{\geq 0}\) sends free \(R\)-modules to free \(R\)-modules and finitely generated \(R\)-modules to finitely generated \(R\)-modules. Hence the resulting functor

\[
\tau_{\geq 0}: R\text{-mod}_{\sigma^{-1}\mathbb{N}^m} \to R\text{-mod}
\]

is right adjoint to the inclusion functor.

**Proof.** (1) Let \(M\) be an \(\mathbb{N}^m\)-graded \(R\)-module and \(N\) a \(\sigma^{-1}\mathbb{N}^m\)-graded \(R\)-module. A map of \(R\)-modules \(f: iM \to N\) is determined by maps of \(k\)-modules \(M(\vec{d}) \to N(\vec{d})\) for each multidegree \(\vec{d} \in \mathbb{N}^m\), since \(M\) is zero outside of that range. Since \(R\) is \(\mathbb{N}^m\)-graded, \(f\) corresponds to the data of a map of \(R\)-modules \(M \to \tau_{\geq 0}N\).

(2) For a free \(R\)-module \(R\langle x\rangle\) on one generator \(x\), the truncation is

\[
\tau_{\geq 0}(R\langle x\rangle) \cong R\langle [x] \rangle.
\]

Let \(M\) be an \(R\)-module with presentation

\[
M = R\langle x_\alpha, \alpha \in A \mid y_\beta, \beta \in B \rangle.
\]

Then its truncation admits the presentation

\[
\tau_{\geq 0}(M) \cong R\langle [x_\alpha], \alpha \in A \mid [y_\beta], \beta \in B \rangle
\]

since \(\tau_{\geq 0}\) preserves direct sums and cokernels, by Lemma 3.3.

**Lemma 4.5.**

(1) The functor \(L_K: R\text{-mod} \to \mathcal{L}(K)\) admits a right adjoint \(\rho_K: \mathcal{L}(K) \to R\text{-mod}\).

(2) The counit

\[
\epsilon: L_K\rho_K(M) \xrightarrow{\cong} M
\]

is an isomorphism for all \(M\) in \(\mathcal{L}(K)\).

**Proof.** (1) We claim that the right adjoint \(\rho_K\) is given by

\[
\rho_K(M) = \tau_{\geq 0}\left(\lim_{\sigma \in \mathcal{P}(\{m\}) \setminus K} M_\sigma\right),
\]

\[
\epsilon: L_K\rho_K(M) \xrightarrow{\cong} M
\]

is an isomorphism for all \(M\) in \(\mathcal{L}(K)\).
where the limit over the missing faces $\sigma$ is computed in $R\text{-Mod}_{\mathbb{Z}^m}$. Indeed, the hom-set out of an object $P$ in $R\text{-mod}$ is

$$R\text{-mod} (P, \rho_K (M)) = R\text{-mod} \left( P, \tau_{\geq 0} \left( \lim_{\sigma \in \mathcal{P}([m]) \setminus K} M_\sigma \right) \right)$$

$$= R\text{-Mod}_{\mathbb{Z}^m} \left( P, \lim_{\sigma \in \mathcal{P}([m]) \setminus K} M_\sigma \right) \quad \text{by Lemma 4.4}$$

$$\cong R\text{-Mod}_{\mathbb{Z}^m}^{\mathcal{P}([m]) \setminus K} \left( \text{cst}(P), M \right) \quad \text{hom in (P([m]) \setminus K)-shaped diagrams}$$

$$\cong \mathcal{L}^{\text{no loc}} (K) \left( L_K (P), M \right) \quad \text{see below}$$

$$= \mathcal{L} (K) \left( L_K (P), M \right).$$

Here $\mathcal{L}^{\text{no loc}} (K)$ denotes the category of finitely generated modules over the diagram of rings \{ $R_\sigma \}$ \( \sigma \in \mathcal{P}([m]) \setminus K \), of which $\mathcal{L} (K)$ is the full subcategory of objects satisfying the localization condition $R_\sigma \otimes_{R_\sigma} M_\sigma \cong M_\tau$. The next-to-last isomorphism uses the fact that for each missing face $\sigma \notin K$, the $R_\sigma$-module $M_\sigma$ is $\sigma$-local, which yields

$$R\text{-Mod}_{\mathbb{Z}^m} (P, M_\sigma) \cong R_\sigma\text{-Mod}_{\mathbb{Z}^m} (R_\sigma \otimes_R P, M_\sigma).$$

(2) The counit $\epsilon: L_K \rho_K (M) \to M$ is given in position $\sigma \in \mathcal{P}([m]) \setminus K$ by the map of $R_\sigma$-modules

$$\left( L_K \rho_K (M) \right)_\sigma \overset{\epsilon_\sigma}{\longrightarrow} M_\sigma,$$

$$R_\sigma \otimes_R \rho_K (M)$$

which we will show is an isomorphism. First, the $R$-module $\lim_{\sigma \in \mathcal{P}([m]) \setminus K} M_\sigma$ can have elements of multidegree $D$ with some $d_i < 0$ only if $i$ belongs to all the missing faces $\sigma \in \mathcal{P}([m]) \setminus K$, so that $t_i$ already acts invertibly on all the $M_\sigma$. Thus we may assume without loss of generality that the missing faces have empty intersection $\bigcap_{\mathcal{P}([m]) \setminus K} \sigma = \emptyset$, in which case the truncation does nothing:

$$\tau_{\geq 0} \left( \lim_{\sigma \in \mathcal{P}([m]) \setminus K} M_\sigma \right) = \lim_{\sigma \in \mathcal{P}([m]) \setminus K} M_\sigma.$$

Now fix a missing face $\sigma$. Since localization $R_\sigma \otimes_R$ preserves finite limits, localizing the diagram of $R$-modules whose limit was being computed yields:

$$R_\sigma \otimes_R \left( \lim_{\tau \in \mathcal{P}([m]) \setminus K} M_\tau \right) \cong \lim_{\tau \in \mathcal{P}([m]) \setminus K} (R_\sigma \otimes_R M_\tau). \quad (4.6)$$

For any other missing face $\sigma'$, the transition map

$$\varphi_{\sigma', \sigma, \sigma'}: M_{\sigma'} \to M_{\sigma \cup \sigma'}$$

is inverting the $t_i$ for $i \in \sigma \setminus \sigma'$, in particular becomes an isomorphism after $\sigma$-localizing. Hence the limit (4.6) is computed on the subdiagram past the position $\sigma$:

$$\lim_{\tau \in \mathcal{P}([m]) \setminus K} (R_\sigma \otimes_R M_\tau) \cong \lim_{\sigma \leq \tau \in \mathcal{P}([m]) \setminus K} (R_\sigma \otimes_R M_\tau)$$

$$\cong \lim_{\tau \in \mathcal{P}([m]) \setminus K} M_\tau \quad \text{since $M_\tau$ is already $\sigma$-local}$$

$$\cong M_\sigma.$$
since $M_\sigma$ is initial among said subdiagram. The typical case of the argument is illustrated in the diagram

$$
\begin{array}{c}
M_\sigma \\
\cong
\end{array}
\begin{array}{c}
\phi_{\sigma,\sigma_1\cup\sigma'} \\
\cong
\end{array}
\begin{array}{c}
M_{\sigma_1\cup\sigma'} \\
\end{array}
\begin{array}{c}
\cong
\end{array}
\begin{array}{c}
R_\sigma \otimes_R M_{\sigma_1\cup\sigma'} \\
\end{array}
\begin{array}{c}
\cong
\end{array}
\begin{array}{c}
R_\sigma \otimes_R (\lim_{\tau \in P([m]\setminus K)} M_\tau) \\
\end{array}
\begin{array}{c}
\cong
\end{array}
\begin{array}{c}
R_\sigma \otimes_R R_{\sigma_1\cup\sigma'}. \\
\end{array}
$$

\[\square\]

**Remark 4.7.** Being a right adjoint, the functor $\rho_K : \mathcal{L}(K) \to R\text{-mod}$ preserves limits and thus is left exact. However, $\rho_K$ need not be right exact. For example, consider the case $m = 2$ with $K = \{\emptyset\}$, as in Example 3.6. The quotient map $q : R \to R/((t_1 t_2))$ in $R\text{-mod}$ yields the epimorphism in $\mathcal{L}(K)$

$$L_K(q) : L_K(R) \twoheadrightarrow L_K(R/((t_1 t_2))).$$

However, the map in $R\text{-mod}$

$$\rho_KL_K(q) : \rho_KL_K(R) \twoheadrightarrow \rho_KL_K(R/((t_1 t_2)))$$

is not an epimorphism. Its source is $\rho_KL_K(R) \cong R$ and its target is

$$\rho_KL_K(R/((t_1 t_2))) = \tau \geq 0 \lim_{\tau \in P([m]\setminus K)} \mathbb{k}[t_1, t_2] \twoheadrightarrow 0 \quad \tau \geq 0 \left( \mathbb{k}[t_1, t_2]/(t_1) \oplus \mathbb{k}[t_1, t_2]/(t_2) \right) = \mathbb{k}[t_1, t_2]/(t_1) \oplus \mathbb{k}[t_1, t_2]/(t_2) = R/(t_1) \oplus R/(t_2).$$

The map $\rho_KL_K(q)$ is the map

$$R \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \to R/(t_1) \oplus R/(t_2)$$

with coordinates the two quotient maps. In bidegree $(0, 0)$, the map $\rho_KL_K(q)$ is the diagonal

$$\mathbb{k} \begin{bmatrix} 1 \end{bmatrix} \to \mathbb{k} \oplus \mathbb{k},$$

which is not surjective.

**Proposition 4.8.** For any simplicial complex $K$ on the vertex set $[m]$, the induced functor $\overline{L_K}$ out of the Serre quotient is an equivalence of categories:
Proof. This follows from Lemmas 2.6 and 4.5.

We will argue the $\mathbb{Z}^m$-graded analogue of Proposition 4.8. The localization functor

$$L_K: R\text{-mod}_{\mathbb{Z}^m} \to \mathcal{L}(K)_{\mathbb{Z}^m}$$

is defined the same way, though the construction of a right adjoint as in Lemma 4.5 does \textit{not} work in the $\mathbb{Z}^m$-graded case. Nevertheless, we will reduce the problem to the $\mathbb{N}^m$-graded case by shifting the degrees.

**Proposition 4.9.** For any simplicial complex $K$ on the vertex set $[m]$, the induced functor $\overline{L_K}$ out of the Serre quotient is an equivalence of categories:

$$\xymatrix{ R\text{-mod}_{\mathbb{Z}^m} \ar[r]^{L_K} \ar[d]_{q} & \mathcal{L}(K)_{\mathbb{Z}^m} \ar@{-->}[dl]^{\cong} \ar[d]_{\overline{L_K}} \ar[r] & R\text{-mod}_{\mathbb{Z}^m}/\ker(L_K) \ar@{-->}[dl]^{\cong} }$$

**Proof.** By Lemma 2.5, $\overline{L_K}$ is faithful. We now argue that $\overline{L_K}$ is essentially surjective.

Given a multidegree $\vec{e} \in \mathbb{Z}^m$, let $T_{\vec{e}}: R\text{-mod}_{\mathbb{Z}^m} \cong \rightarrow R\text{-mod}_{\mathbb{Z}^m}$ denote the isomorphism of categories that shifts the modules by $\vec{e}$, that is:

$$(T_{\vec{e}}M)(\vec{d}) = M(\vec{d} - \vec{e}).$$

Also denote by $T_{\vec{e}}: \mathcal{L}(K)_{\mathbb{Z}^m} \cong \rightarrow \mathcal{L}(K)_{\mathbb{Z}^m}$ the induced isomorphism of categories. Now let $M$ be an object of $\mathcal{L}(K)_{\mathbb{Z}^m}$. There is a large enough multidegree $\vec{e} \in \mathbb{Z}^m$ such that the shift $T_{\vec{e}}M$ lies in the full subcategory $\mathcal{L}(K)$. By Proposition 4.8, there is an $R$-module $A$ satisfying $\overline{L_K}(A) \cong T_{\vec{e}}M$. Undoing the shift yields

$$\overline{L_K}(T_{-\vec{e}}A) \cong T_{-\vec{e}}\overline{L_K}(A) \cong T_{-\vec{e}}T_{\vec{e}}M = M.$$  

The same argument works for maps $f: M \to N$ in $\mathcal{L}(K)_{\mathbb{Z}^m}$, showing that $\overline{L_K}$ is full. \qed

5. TORSION PAIR FOR LOCALIZED PERSISTENCE MODULES

In this section and the next two we restrict to the simplicial complex

$$K_m := \text{sk}_{m-3}\Delta^{m-1} = \{ \sigma \in [m] \mid \#\sigma \leq m - 2 \},$$

the codimension 2 skeleton of $\Delta^{m-1}$. We construct two subcategories of $\mathcal{L}(K_m)$, a torsion subcategory $\mathcal{T}$ and a torsion-free subcategory $\mathcal{F}$ which together give a torsion pair. We show that objects in $\mathcal{L}(K_m)$ break into a direct sum of their torsion and torsion-free parts. In the following two sections, we break the torsion objects into direct sums of indecomposables, and also the torsion-free objects in the case $m = 2$.

Background on torsion pairs (also called torsion theories) can be found in [Dic66], [Pop73, §4.8], or [Bor94, §1.12].

**Definition 5.1.** Let $\mathcal{A}$ be an abelian category. A \textbf{torsion pair}, $(\mathcal{T}, \mathcal{F})$ of $\mathcal{A}$ is a pair of full subcategories $\mathcal{T}, \mathcal{F} \subseteq \mathcal{A}$ both closed under isomorphisms, such that:

1. for every object $M \in \mathcal{A}$ there is a short exact sequence
   $$0 \longrightarrow T(M) \longrightarrow M \longrightarrow F(M) \longrightarrow 0 \quad(5.2)$$
   where $T(M) \in \mathcal{T}$ and $F(M) \in \mathcal{F}$, and
2. $\text{Hom}(T, F) = 0$ for any $T \in \mathcal{T}$ and $F \in \mathcal{F}$.
The subcategory $\mathcal{T}$ is called the torsion class and its objects torsion objects and $\mathcal{F}$ is the torsion-free class whose objects are also called torsion-free.

The short exact sequence (5.2) is unique up to isomorphism [Bor94, Proposition 1.12.4] and may be chosen to be functorial in $M$ [BBOS20, Remark 2.4].

For $i \in [m]$ let $\sigma_i = [m] \setminus \{i\}$. For $M \in R_{\sigma_i}$-mod, $T_i(M)$ is the submodule of torsion elements in $M$, so $x \in T_i(M)$ if for some $k$, $t_i^k x = 0$. We define two full subcategories $\mathcal{T}_i, \mathcal{F}_i \subset R_{\sigma_i}$-mod by

\[
\mathcal{T}_i = \{ M \mid T_i(M) = M \} = \{ M \mid M \otimes R_{\{m\}} = 0 \}
\]

\[
\mathcal{F}_i = \{ M \mid T_i(M) = 0 \}.
\]

Note that $T_i$ defined an endofunctor of $R_{\sigma_i}$-mod and we define the endofunctor $F_i$ by $F_i(M) = M/T_i(M)$. We record some of this in a proposition.

**Proposition 5.3.** For every $M$ in $R_{\sigma_i}$-mod we have a short exact sequence

\[
0 \to T_i(M) \to M \to F_i(M) \to 0
\]

where $T_i(M) \in \mathcal{T}_i$ and $F_i(M) \in \mathcal{F}_i$.

Also for any $A \in \mathcal{T}_i$ and $B \in \mathcal{F}_i$, Hom($A$, $B$) = 0, hence $(\mathcal{T}_i, \mathcal{F}_i)$ is a torsion pair.

**Proof.** For the first statement we just need to check that $M/T_i(M)$ is torsion-free, which is standard commutative algebra. The second statement is also straightforward to check. \(\square\)

We now recall the decomposition of objects in $R_{\sigma_i}$-mod into indecomposables, namely interval modules. We use interval notation

\[
[a, b)_i = t_i^a R_{\sigma_i}/t_i^b = \text{coker}(t_i^a R_{\sigma_i} \to t_i^b R_{\sigma_i})
\]

\[
[a, \infty)_i = t_i^a R_{\sigma_i}.
\]

**Proposition 5.4.**

1. The $R_{\sigma_i}$-modules $(a, b)_i$ and $(a, \infty)_i$ are indecomposable.

2. For any $M$ the short exact sequence in Proposition 5.3 splits, and thus $M \cong T_i(M) \oplus F_i(M)$.

3. For any torsion module $M \in \mathcal{T}_i$, there is a decomposition

\[
M \cong \bigoplus_j (a_j, b_j)_i.
\]

For any torsion-free module $M \in \mathcal{F}_i$, there is a decomposition

\[
M \cong \bigoplus_j (a_j, \infty)_i.
\]

**Proof.** The inclusion $k[t_i] \to R_{\sigma_i}$ induces the restriction functor $R_{\sigma_i}$-mod $\to k[t_i]$-mod, which is an equivalence of categories, cf. Remark 3.8. The claims then follow from the same results in $k[t_i]$-mod. \(\square\)

We define these two subcategories of $L(K_m)$:

\[
\mathcal{T} = \{ M \mid M_{\sigma_i} \in \mathcal{T}_i \text{ for all } i \in [m] \} = \{ M \mid M_{[m]} = 0 \}
\]

\[
\mathcal{F} = \{ M \mid M_{\sigma_i} \in \mathcal{F}_i \text{ for all } i \in [m] \}.
\]

Corresponding to $\mathcal{T}$ and $\mathcal{F}$ we have endofunctors $T, F: L(K_m) \to L(K_m)$ defined by

\[
T(M)_{\sigma} = \begin{cases} 
T_{\sigma_i}(M_{\sigma_i}) & \text{if } \sigma = \sigma_i \\
0 & \text{if } \sigma = [m] 
\end{cases}
\]

and

\[
F(M)_{\sigma} = M_{\sigma}/T(M)_{\sigma}.
\]
Proposition 5.5. \( T \) and \( F \) are functorial and for every \( M, T(M) \in \mathcal{T}, F(M) \in \mathcal{F} \) and we have a short exact sequence
\[ 0 \to T(M) \to M \to F(M) \to 0. \]

Also for any \( A \in \mathcal{T} \) and \( B \in \mathcal{F} \), \( \text{Hom}(A, B) = 0 \), hence \((\mathcal{T}, \mathcal{F})\) is a torsion pair in \( \mathcal{L}(K_m) \).

Proof. It is clear that \( T \) and \( F \) are functorial and that \( T \) lands in \( \mathcal{T} \). That \( F \) lands in \( \mathcal{F} \) follows from the fact that \( F_i \) lands in \( \mathcal{F}_i \) by Proposition 5.3 and since cokernels are computed pointwise (Lemma 3.10). That the sequence is exact and that \( M/T(M) \) is torsion-free both follow since cokernels are computed pointwise. Finally \( \text{Hom}(A, B) = 0 \) since for every \( \sigma \notin K_m \), \( \text{Hom}(A_\sigma, B_\sigma) = 0 \) by Proposition 5.3.

Proposition 5.6. For any \( M \) in \( \mathcal{L}(K_m) \), the short exact sequence in Proposition 5.5 splits, and thus \( M \cong T(M) \oplus F(M) \).

Proof. We construct a retraction of the inclusion
\[ I : T(M) \to M. \]

For \( i \in [m] \) by Proposition 5.4 \( I_{\sigma_i} : T(M)_{\sigma_i} \to M_{\sigma_i} \) has a retraction \( r_{\sigma_i} : M_{\sigma_i} \to T(M)_{\sigma_i} \). Since \( T(M)_{[m]} = 0 \), letting \( r_{[m]} = 0 \) defines a retraction to \( I \).

6. Decomposition of torsion objects

We want to describe some objects in \( \mathcal{L}(K_m) \) that will be our indecomposables.

For \( a < b \in \mathbb{N} \) and \( i \in [m] \), we view the \( R_\sigma \)-module \( [a, b)_i \) as an object of \( \mathcal{L}(K_m) \) also denoted (by abuse of notation) \( [a, b)_i \in \mathcal{L}(K_m) \), defined by
\[ [a, b)_i(\sigma) = \begin{cases} [a, b)_i & \text{if } \sigma = \sigma_i \\ 0 & \text{if } \sigma = \sigma_j, j \neq i \\ 0 & \text{if } \sigma = [m]. \end{cases} \]

The objects \([a, b)_i\) are illustrated in Figure 1 in the case \( m = 2 \).

Proposition 6.1. There is an isomorphism in \( \mathcal{L}(K_m) \)
\[ [a, b)_i \cong L_{K_m}(i^a R/t_i^b R). \]

Proposition 6.2. The objects \([a, b)_i\) in \( \mathcal{T} \) are indecomposable.

Proposition 6.3. Each torsion object \( M \in \mathcal{T} \) decomposes as a direct sum of objects of the form \([a, b)_i\).

Proof. Let
\[ A(i) = \bigoplus_j [a_j, b_j)_i \phi(j) \to M_{\sigma_i}. \]

The equality is a definition and the map \( \phi(i) \) is any isomorphism which we get from Proposition 5.4. Overusing notation we can extend \( A(i) \) to a object in \( \mathcal{L}(K_m) \) by letting \( A(i)_{\sigma} = 0 \) if \( \sigma \neq \sigma_i \). Note that this extension exists since \( M \in \mathcal{T} \) and therefore \( M_{[m]} = 0 \).

So we get a map \( \phi(i) : A(i) \to M \) such that \( \phi(i)_{\sigma_i} = \phi(i) : A(i)_{\sigma_i} = A(i) \to M_{\sigma_i} \). Putting these together we get an isomorphism \( \bigoplus_i A(i) \to M \), thus proving the proposition.
7. Decomposition of torsion-free objects in dimension $m = 2$

**Notation 7.1.** For a multidegree $\bar{a} = (a_1, \ldots, a_m) \in \mathbb{N}^m$, define $[\bar{a}, \infty) \in \mathcal{L}(K_m)$ by

$$[\bar{a}, \infty)_\sigma = \begin{cases} [a_i, \infty) & \text{if } \sigma = \sigma_i \\ R_{[m]} & \text{if } \sigma = [m]. \end{cases}$$

The transition map $\varphi_i : [a_i, \infty)_i \to R_{[m]}$ is the inclusion of $R_{\sigma_i}$-submodule.

The object $[\bar{a}, \infty)$ is illustrated in Figure 1 in the case $m = 2$.

![Figure 1](image)

**Lemma 7.2.** For any $\bar{a} \in \mathbb{N}^m$, there is an isomorphism in $\mathcal{L}(K_m)$

$$[\bar{a}, \infty) \cong L_{K_m}(t^\bar{a} R),$$

where $t^{\bar{a}} = t_1^{a_1} \cdots t_m^{a_m}$.

**Lemma 7.3.** Let $M$ be a finitely generated $R_{\sigma_i}$-module, for some $1 \leq i \leq m$. Let $x \in M$ be a (non-zero) element of minimal degree in the product partial order, that is:

$$|x|_i \leq |y|_i \quad \text{for all } y \in M,$$

and let $\langle x \rangle \subseteq M$ denote the $R_{\sigma_i}$-submodule generated by $x$. Then the quotient map $M \to M/\langle x \rangle$ admits a section. In particular, if $M$ is torsion-free, then so is $M/\langle x \rangle$.

**Proof.** By the equivalence of categories $R_{\sigma_i}$-mod $\cong \mathbb{k}[t_i]$-mod, we may assume $m = 1$. The $\mathbb{k}[t]$-module $M$ has an interval decomposition

$$M \cong \bigoplus_j [a_j, b_j)$$

with $b_j \in \mathbb{N} \cup \{\infty\}$. Writing $a = \min_j \{a_j\}$, the element $x \in M$ must lie in the submodule

$$M' = \bigoplus_{a_j = a} [a, b_j)$$

because of its degree $|x| = a$. Consider the number

$$b = a + \min \{\ell \mid t^\ell x = 0\} \in \mathbb{N} \cup \{\infty\}$$

and the submodule

$$M'' = \bigoplus_{a_j = a, b_j \leq b} [a, b_j) \subseteq M'.$$
Then \( x \in M'' \) holds and there is an index \( j \) satisfying \( b_j = b \). There is an automorphism of the \( \mathbb{k} \)-vector space \( M''(a) \cong \mathbb{k}^r \) sending \( x \) to a generator of a summand \([a, b]\) which moreover extends to an automorphism of the \( \mathbb{k}[t] \)-module \( M'' \). Hence we may assume that \( x \) is a generator of a summand \([a, b]\), in which case the quotient module \( M/\langle x \rangle \) consists of the other summands. \( \square \)

Recall the following basic fact from algebra, which will be used in the next proof.

**Lemma 7.4.** Let \( R \) be a commutative ring and \( S \subset R \) a multiplicative set. Let \( f: M \to N \) be a map of \( R \)-modules which is an \( S \)-localization, i.e., isomorphic to \( M \to S^{-1}M \cong (S^{-1}R) \otimes_R M \).

1. Given an \( R \)-submodule \( M' \subseteq M \) and taking \( N' := f(M) \subseteq N \), the composite \( \xymatrix{ M' \ar[r]^{f} & N' \ar[r] & S^{-1}N' } \) is an \( S \)-localization.

2. Given an \( S^{-1}R \)-submodule \( N' \subseteq N \) and taking \( M' := f^{-1}(N') \subseteq M \), the map \( f': M' \to N' \) obtained by restricting \( f \) is an \( S \)-localization.

**Lemma 7.5.** Let \( M \) be a non-zero object in \( \mathcal{F} \). Then there exists a monomorphism \( \mu: [\bar{a}, \infty) \hookrightarrow M \) for some \( \bar{a} \in \mathbb{N}^m \) such that the quotient module \( \text{coker}(\mu) = M/[\bar{a}, \infty) \) is also in \( \mathcal{F} \).

**Proof.** Note that \( M \) being non-zero and torsion-free forces \( M_{[m]} \neq 0 \), which in turn forces \( M_{\sigma_i} \neq 0 \) for all \( 1 \leq i \leq m \). We will construct certain subobjects \( M^{(m)} \subseteq M^{(m-1)} \subseteq \cdots \subseteq M^{(1)} \subseteq M^{(0)} = M \) in \( m \) steps. The process is illustrated in Figure 2 in the case \( m = 2 \).

**Step** \( i = 1 \). Since \( M_{\sigma_i} \neq 0 \) is a torsion-free \( R_{\sigma_i} \)-module, by Proposition 5.4, it admits a finite direct sum decomposition

\[
M_{\sigma_i} \cong \bigoplus_{j} [a_{1,j}, \infty)_1. \tag{7.6}
\]

Take \( a_1 := \min_j \{a_{1,j}\} \) and take the submodule \( M_{\sigma_1}^{(1)} \subseteq M_{\sigma_1} \) corresponding to

\[
M_{\sigma_1}^{(1)} \cong \bigoplus_{a_{1,j}=a_1} \bigoplus_{j} [a_{1,j}, \infty)_1
\]

via the decomposition (7.6), i.e., the summands with the minimal starting point. Now apply the transition map \( \varphi_1: M_{\sigma_1} \to M_{[m]} \) and consider the \( R_{[m]} \)-submodule

\[
M_{[m]}^{(1)} := t_1^{-1} \varphi_1(M_{\sigma_1}^{(1)}) \cong R_{[m]} \otimes_{R_{\sigma_1}} \varphi_1(M_{\sigma_1}^{(1)}) \subseteq M_{[m]}
\]

with transition map

\[
\varphi_1^{(1)}: M_{\sigma_1}^{(1)} \to M_{[m]}^{(1)}
\]

induced by \( \varphi_1 \). Via the decomposition (7.6), \( M_{[m]}^{(1)} \subseteq M_{[m]} \) corresponds to a summand inclusion:

\[
\xymatrix{ M_{[m]}^{(1)} \ar[r] & M_{[m]} \ar[d] \ar[r] & M_{[m]}^{(1)} } \cong \xymatrix{ \bigoplus_{a_{1,j}=a_1} R_{[m]} \ar[r] & \bigoplus_{j} R_{[m]} }.
\]

For the remaining positions \( \ell \neq 1 \), take the pullback

\[
M_{\sigma_\ell}^{(1)} := \varphi_\ell^{-1}(M_{[m]}^{(1)}) \subseteq M_{\sigma_\ell}
\]

with transition map

\[
\varphi_\ell^{(1)}: M_{\sigma_\ell}^{(1)} \to M_{[m]}^{(1)}
\]
induced by \( \varphi_t \). The submodules \( M^{(1)}_{\sigma_1} \subseteq M_{\sigma_1} \) assemble into a subobject \( M^{(1)} \subseteq M \). Note that the condition \( M^{(1)} \neq 0 \) ensures \( M^{(1)} \neq 0 \).

Steps \( i > 1 \). Starting from \( M^{(1)} \), repeat the process for the step \( i = 2 \), which yields a subobject \( M^{(2)} \subseteq M^{(1)} \) satisfying \( M^{(2)} \neq 0 \). Repeat the process for the steps \( i = 3, \ldots, m \). Collect the numbers \( a_i \) into a multidegree \( \bar{a} = (a_1, \ldots, a_m) \in \mathbb{N}^m \).

**Constructing a monomorphism** \( \lambda: (\bar{a}, \infty) \to M^{(m)} \). Note that a morphism \( [\bar{a}, \infty) \to N \) in \( L(K) \) is the same data as a family of elements \( x_i \in N_{\sigma_i}(\bar{a}) \) that all map to the same element in the terminal position:

\[
\varphi_i(x_i) = \varphi_j(x_j) \in N_{[m]}(\bar{a}) \quad \text{for all } 1 \leq i, j \leq m.
\]

We will produce such elements for \( N = M^{(m)} \). By construction, the \( R_{\sigma_m} \)-module \( M^{(m)}_{\sigma_m} \) has an interval decomposition

\[
M^{(m)}_{\sigma_m} \cong \bigoplus_j [a_m, \infty)_{m,j}.
\]

Pick one of the summands \([a_m, \infty)_{m} \hookrightarrow M^{(m)}_{\sigma_m} \) and let \( x_m \in M^{(m)}_{\sigma_m}(\bar{a}) \) be the image of the generator \( t \bar{a} \in [a_m, \infty)_{m}(\bar{a}) \cong k \). Take \( x := \varphi^{(m)}(x_m) \in M^{(m)}_{[m]}(\bar{a}) \). To find the next element \( x_{m-1} \), consider the pullback diagram in \( R_{\sigma_{m-1}} \)-modules

\[
\vdots \exists x_{m-1} \in M^{(m)}_{\sigma_{m-1}} \xrightarrow{\varphi^{(m)}_{m-1}} M^{(m)}_{[m]} \ni x
\]

\[
\exists \bar{x}_{m-1} \in M^{(m-1)}_{\sigma_{m-1}} \xrightarrow{\varphi^{(m-1)}} M^{(m-1)}_{[m]} \ni x
\]

where the rightward maps invert \( t_m-1 \). Since the bottom map is an isomorphism of \( k \)-modules in multidegree \( \bar{a} \), there is an element \( \bar{x}_{m-1} \in M^{(m-1)}_{\sigma_{m-1}} \) satisfying \( \varphi^{(m-1)}_{m-1}(\bar{x}_{m-1}) = x \), hence also an element in the pullback \( x_{m-1} \in M^{(m)}_{\sigma_{m-1}} \) satisfying \( \varphi^{(m)}_{m-1}(x_{m-1}) = x \).

To find the next element \( x_{m-2} \), apply the same argument to the pullback diagram in \( R_{\sigma_{m-2}} \)-modules

\[
\vdots \exists x_{m-2} \in M^{(m)}_{\sigma_{m-2}} \xrightarrow{\varphi^{(m)}_{m-2}} M^{(m)}_{[m]} \ni x
\]

\[
\exists \bar{x}_{m-2} \in M^{(m-2)}_{\sigma_{m-2}} \xrightarrow{\varphi^{(m-2)}} M^{(m-2)}_{[m]} \ni x
\]

Repeat the process to find the remaining elements \( x_i \in M^{(m)}_{\sigma_i}(\bar{a}) \), which jointly define a morphism \( \lambda: [\bar{a}, \infty) \to M^{(m)} \) in \( L(K) \). For each \( 1 \leq i \leq m \), the \( R_{\sigma_i} \)-module \( M^{(m)}_{\sigma_i} \) is torsion-free, hence the
map
\[ \lambda_{\sigma_i} : [a_i, \infty)_i \hookrightarrow M^{(m)}_{\sigma_i} \]
is a monomorphism. By Lemma 3.10, the map \( \lambda : [\bar{a}, \infty) \hookrightarrow M^{(m)} \) is also a monomorphism. Define \( \mu : [\bar{a}, \infty) \hookrightarrow M^{(m)} \) as \( \lambda \) followed by the inclusion \( M^{(m)} \subseteq M \).

The cokernel of \( \mu \) is torsion-free. For each \( 1 \leq i \leq m \), let us show that the \( R_{\sigma_i} \)-module \( \text{coker}(\mu)_{\sigma_i} = M_{\sigma_i} / \langle x_i \rangle \) is torsion-free. Let \( z \in M_{\sigma_i} \setminus \langle x_i \rangle \). We want to show that for any non-zero scalar \( \alpha \in R_{\sigma_i} \), the condition \( \alpha z \in M_{\sigma_i} \setminus \langle x_i \rangle \) still holds. We distinguish two cases, depending where \( z \) lies in the decreasing filtration
\[ M_{\sigma_i} = M^{(0)}_{\sigma_i} \supseteq M^{(1)}_{\sigma_i} \supseteq \cdots \supseteq M^{(m)}_{\sigma_i} . \]

Case: \( z \) does not survive all the way. Assume \( z \in M^{(\ell-1)}_{\sigma_i} \setminus M^{(\ell)}_{\sigma_i} \) for some \( 1 \leq \ell \leq m \). Then the condition \( \alpha z \in M^{(\ell-1)}_{\sigma_i} \setminus M^{(\ell)}_{\sigma_i} \) also holds. In the case \( \ell = \ell \), this follows from the pullback square on the left:

\[
\begin{array}{cccc}
\bigoplus R_{[m]} & \xrightarrow{\text{inc}} & \bigoplus [a_{\ell}, \infty)_{\ell} \\
M^{(\ell)}_{\sigma_i} & \xrightarrow{\varphi^{(\ell)}_i} & M^{(\ell)}_{[a]} & \xrightarrow{\varphi^{(\ell)}_{\ell}} \bigoplus_{\ell-1} M_{\sigma_{\ell}}^{(\ell)} \\
\bigoplus_{\ell-1} M^{(\ell-1)}_{\sigma_i} & \xrightarrow{\text{inc}_{\ell-1}} & \bigoplus_{\ell-1} M^{(\ell-1)}_{[a]} & \xrightarrow{\text{inc}_{\ell-1}} \bigoplus_{\ell-1} M_{\sigma_{\ell}}^{(\ell-1)} \\
\bigoplus_j R_{[m]} & \xrightarrow{\text{inc}_j} & \bigoplus_j [a_{\ell,j}, \infty)_{\ell} \\
\end{array}
\]

and the fact that \( R_{[m]} \) has no zero-divisors. In the case \( i = \ell \), it follows from the rightmost column and the fact that each \( R_{\sigma_{\ell}} \)-module \( [a_{\ell,j}, \infty)_{\ell} \) is torsion-free. In either case, \( \alpha z \) cannot lie in the submodule \( \langle x_i \rangle \subseteq M^{(m)}_{\sigma_i} \subseteq M^{(\ell)}_{\sigma_i} \).

Case: \( z \) survives all the way. Assume \( z \in M^{(m)}_{\sigma_i} \). By construction, the \( R_{\sigma_i} \)-module \( M^{(m)}_{\sigma_i} \) is concentrated in degrees \( \bar{d} \in \sigma_i^{-1}N^m \) with \( d_i \geq a_i \). Since \( x_i \) achieves the minimal degree \( |x_i|_i = a_i \), the quotient \( M^{(m)}_{\sigma_i} / \langle x_i \rangle \) is torsion-free, by Lemma 7.3. \( \square \)

Figure 2. The “scanning” process in the proof of Lemma 7.5.

Lemma 7.7. Let \( M \) be an object in \( \mathcal{F} \).
(1) If rank \( M = 0 \) holds, then \( M = 0 \) holds.
(2) If rank \( M = 1 \) holds, then \( M \) is isomorphic to \([\vec{d}, \infty)\) for some \( \vec{d} \).

Proof. (1) We will show the contrapositive. Given \( M \neq 0 \), we have \( M_{\sigma_i} \neq 0 \) for some \( i \in [m] \). Pick a non-zero element \( x \in M_{\sigma_i} \) and consider the map of \( R_{\sigma_i}\)-modules

\[
f_x : t^{|x|}R_{\sigma_i} \rightarrow M_{\sigma_i},
\]

sending \( 1 \in R_{\sigma_i} \) to \( x \). Since the \( R_{\sigma_i}\)-module \( M_{\sigma_i} \) is torsion-free, the map \( f_x \) is a monomorphism. Since \( R_{[m]} \) is flat over \( R_{\sigma_i} \), applying the localization yields a monomorphism of \( R_{[m]}\)-modules

\[
t^{|x|}R_{[m]} \cong R_{[m]} \otimes R_{\sigma_i} (t^{|x|}R_{\sigma_i}) \xrightarrow{R_{[m]} \otimes f_x} R_{[m]} \otimes R_{\sigma_i} M_{\sigma_i} \cong M_{[m]},
\]

which ensures \( M_{[m]} \neq 0 \).

(2) By Lemma 7.5, there exists a monomorphism \( \mu : [\vec{d}, \infty) \hookrightarrow M \) such that the quotient coker(\( \mu \)) is also in \( \mathcal{F} \). Evaluating at the position \([m]\) yields a short exact sequence of \( R_{[m]}\)-modules

\[
0 \longrightarrow R_{[m]} \xrightarrow{\mu_{[m]}} M_{[m]} \longrightarrow \text{coker}(\mu)_{[m]} \longrightarrow 0
\]

from which we obtain the dimension count

\[
\dim_k \text{coker}(\mu)_{[m]} = \dim_k M_{[m]} - \dim_k R_{[m]} = 1 - 1 = 0.
\]

Part (1) then implies coker(\( \mu \)) = 0, so that \( \mu : [\vec{d}, \infty) \xrightarrow{\cong} M \) is an isomorphism. \( \square \)

Lemma 7.8. For any \( \vec{d} \in \mathbb{N}^m \), the object \([\vec{d}, \infty)\) in \( \mathcal{L}(K) \) is indecomposable.

Proof. Consider a direct sum decomposition \([\vec{d}, \infty) \cong P \oplus Q \). Since the subcategory \( \mathcal{F} \) is closed under subobjects, both summands \( P \) and \( Q \) lie in \( \mathcal{F} \). A dimension count as in the proof of Lemma 7.7 (2) then forces \( P = 0 \) or \( Q = 0 \). \( \square \)

Lemma 7.9. Let \( M \in \mathcal{F} \) be a torsion-free module of rank \( r = \dim_k M_{[m]} \geq 2 \). Then applying the "scanning" Lemma 7.5 \( r - 1 \) times produces an epimorphism in \( \mathcal{F} \)

\[
M \twoheadrightarrow [\vec{a}, \infty).
\]

The process is illustrated in Figure 3.

Proof. By Lemma 7.5, there is a monomorphism \( \mu : [\vec{d}, \infty) \hookrightarrow M \) such that \( M^1 := \text{coker}(\mu) \) lies in \( \mathcal{F} \). Repeating the argument with \( M^1 \), there is a short exact sequence in \( \mathcal{F} \)

\[
0 \longrightarrow [\vec{d}^2, \infty) \xrightarrow{\mu_{[m]}} M^1 \longrightarrow M^2 \longrightarrow 0.
\]

Applying the argument \( r - 1 \) times, we obtain a module \( M^{r-1} \) in \( \mathcal{F} \) satisfying

\[
\text{rank } M^{r-1} = \text{rank } M - (r - 1) = r - (r - 1) = 1.
\]

By Lemma 7.7, there is an isomorphism \( M^{(r-1)} \cong [\vec{a}, \infty) \) for some \( \vec{a} \in \mathbb{N}^m \). Composing the epimorphisms along the way

\[
M \longrightarrow M^1 \longrightarrow M^2 \longrightarrow \cdots \longrightarrow M^{r-1} \cong [\vec{a}, \infty)
\]

yields an epimorphism \( M \rightarrow [\vec{a}, \infty) \). \( \square \)
Lemma 7.10. In the case $m = 2$, let $M$ be a torsion-free module in $\mathcal{F}$. Then any epimorphism $q: M \rightarrow [\bar{a}, \infty)$ constructed as in Lemma 7.9 is split.

Proof. Write $y_i := t^{-a_i}t_i^1 \in [a_i, \infty)$ for the canonical generator, shifted to bidegree $\bar{a}$ for convenience. Those agree in the terminal corner:

$$\varphi_1(y_1) = \varphi_2(y_2) = y = t^\bar{a}1 \in k[t_1^\pm, t_2^\pm] = R_{[2]} = [\bar{a}, \infty]_{[2]}.$$ 

A section of $q$ amounts to preimages $x_i \in M_{\sigma_i}(\bar{a})$ of the generators $y_i$ that agree in the terminal corner: $\varphi_1(x_1) = \varphi_2(x_2) \in M_{[2]}(\bar{a})$, as illustrated in the diagram

In position $\sigma_2 = \{1\}$, consider the epimorphism of $R_{\sigma_2}$-modules $q_{\sigma_2}: M_{\sigma_2} \rightarrow [a_2, \infty)_{2, \sigma_2}$ and pick any preimage $x_2$ of $y_2$. Take $x := \varphi_2(x_2) \in M_{[2]}(\bar{a})$, which is a preimage of $y \in [\bar{a}, \infty)_{[2]}(\bar{a})$. Now the $R_{\sigma_1}$-module $M_{\sigma_1}$ admits an interval decomposition $M_{\sigma_1} \cong \bigoplus_j [a^j, \infty)_{1_j}$. By construction of the epimorphism in Lemma 7.9, we have $a_1 \geq a^j_1$ for all $j$. Hence the localization map $\varphi_1: M_{\sigma_1} \hookrightarrow M_{[2]}$ is an isomorphism in bidegree $\bar{a}$. Therefore there exists a unique $x_1 \in M_{\sigma_1}(\bar{a})$ satisfying $\varphi_1(x_1) = x$. One readily checks that $x_1$ satisfies $\varphi_1q_{\sigma_1}(x_1) = \varphi_1(y_1)$.

Since the localization map $\varphi_1: [a_1, \infty)_{1} \hookrightarrow R_{[2]}$ is injective, this forces $q_{\sigma_1}(x_1) = y_1$, so that $x_1$ is a preimage of $y_1$. \hfill \Box

Proposition 7.11. In the case $m = 2$, each torsion-free object $M \in \mathcal{F}$ decomposes as a direct sum of modules of the form $[\bar{a}, \infty)$. 

\begin{figure}
\centering
\includegraphics[width=\textwidth]{battleship_game.png}
\caption{The “battleship game” in the proof of Lemma 7.9.}
\end{figure}
Proof. Write \( r = \text{rank } M \). The case \( r \leq 1 \) of the statement was proved in Lemma 7.7, so we assume \( r > 1 \). Pick any epimorphism \( M \to [\vec{a}, \infty) \) constructed as in Lemma 7.9, which admits a section, by Lemma 7.10. This produces a splitting \( M \cong P \oplus [\vec{a}, \infty) \), so that the complementary summand \( P \) satisfies
\[
\text{rank } P = \text{rank } M - 1 = r - 1.
\]
By induction on \( r \), we obtain a direct sum decomposition of \( M \) into \( r \) summands of the form \([\vec{d}, \infty)\).

\[
\square
\]

8. Torsion-free objects in dimension \( m \geq 3 \)

In this section, we show that the analogue of Proposition 7.11 fails for \( m \geq 3 \). In Section 9, we will show that it fails miserably.

The following lemma says that the algorithm used in Proposition 7.11 has a detection property: If a torsion-free object admits a direct sum decomposition into objects of the form \([\vec{d}, \infty)\), then the algorithm is guaranteed to find such a decomposition.

**Lemma 8.1.** Let \( M \) be a torsion-free object in \( \mathcal{F} \) of the form
\[
M \cong \bigoplus_j [\vec{a}^j, \infty),
\]
and write \( \vec{b} := \min_j \vec{a}^j \), the minimal multidegree in lexicographic order.

1. The subobject \( M^{(m)} \) constructed in Lemma 7.5 is of the form
\[
M^{(m)} \cong \bigoplus_{\vec{a}^j = \vec{b}} [\vec{a}^j, \infty).
\]

2. Any monomorphism \( \mu: [\vec{b}, \infty) \hookrightarrow M \) constructed in Lemma 7.5 is split.

3. Any epimorphism \( M \twoheadrightarrow [\vec{a}, \infty) \) constructed as in Lemma 7.9 is split.

**Proof.** (1) Since isomorphisms in \( \mathcal{L}(K) \) preserve the multigrading, we may assume \( M = \bigoplus_j [\vec{a}^j, \infty) \). The successive subobjects constructed in Lemma 7.5 consist of fewer and fewer of the summands:
\[
M^{(\ell)} = \bigoplus_j [\vec{a}^j, \infty), \quad \vec{a}^j_i = \vec{b} \text{ for } 1 \leq i \leq \ell.
\]

(2) By the analogue of Lemma 7.3 in \( \mathcal{L}(K) \) instead of \( R_{\sigma_i} \)-modules, the monomorphism \( \lambda: [\vec{b}, \infty) \hookrightarrow M^{(m)} \) is isomorphic to the inclusion of one of the summands. The inclusion \( M^{(m)} \subseteq M \) is also an inclusion of summands. Hence the composite monomorphism \( \mu: [\vec{b}, \infty) \hookrightarrow M \) is split, and its cokernel \( \text{coker}(\mu) \) still satisfies the assumption of part (1). The epimorphism constructed in Lemma 7.9 is then a composite of \( r - 1 \) split epimorphisms, which proves (3).

**Example 8.2.** For \( M \) in \( \mathcal{F} \) and \( \vec{a} \in \mathbb{N}^m \), an epimorphism \( f: M \to [\vec{a}, \infty) \) need not admit a section. Here is an example in the case \( m = 2 \). Consider the map of \( R \)-modules
\[
t_1 R \oplus t_2 R \xrightarrow{g=\begin{bmatrix} \text{inc} & \text{inc} \end{bmatrix}} R
\]
and denote the generators respectively by
\[
\begin{align*}
x &= t_1 \cdot 1 \in t_1 R, & |x| &= (1, 0) \\
y &= t_2 \cdot 1 \in t_2 R, & |y| &= (0, 1) \\
z &= 1 \in R, & |z| &= (0, 0).
\end{align*}
\]
Take $f = L_K(g)$, which can be written using Notation 7.1:

$$M = [(1,0),\infty) \oplus [(0,1),\infty) \rightarrow [(0,0),\infty).$$

The map $f$ is an epimorphism in $\mathcal{L}(K_2)$, since in position $\sigma_1$ it is the epimorphism of $R_{\sigma_1}$-modules

$$[1,\infty)_1 \oplus [0,\infty)_1 \rightarrow [0,\infty)_1$$

and similarly for position $\sigma_2$. However, $f$ does not admit a section $s$. Indeed, in position $\sigma_1$, the only section of $R_{\sigma_1}$-modules $s_{\sigma_1}$ is

$$s_{\sigma_1}(z_1) = t_2^{-1}y_1 \in [0,\infty)_1(0,0)$$

where $y_1$ denotes the image of $y$ under the localization map $t_1R \oplus t_2R \rightarrow M_{\sigma_1}$. Similarly in position $\sigma_2$, the only section of $R_{\sigma_2}$-modules is

$$s_{\sigma_2}(z_2) = t_1^{-1}x_2 \in [0,\infty)_2(0,0).$$

The condition

$$\varphi_1(t_2^{-1}y_1) \neq \varphi_2(t_1^{-1}x_2) \in M_2 = R_2 \oplus R_2$$

prevents $s_{\sigma_1}$ and $s_{\sigma_2}$ from assembling into a compatible section $s$ in $\mathcal{L}(K_2)$.

**Lemma 8.3.** For $m = 3$, there exists a torsion-free object in $F$ of rank 2 that is indecomposable.

*Proof.* Consider the $R$-module

$$M = R\langle w, y, z \mid t_1w = t_3y - t_2z, |w| = (0,1,1), |y| = (1,1,0), |z| = (1,0,1) \rangle$$

and take its localization $M := L_{K_3}(M)$. The “scanning” Lemma 7.5 produces a monomorphism

$$\mu: (0,1,1,\infty) \hookrightarrow M$$

picking out the image of $w$. More precisely, consider the short exact sequence of $R$-modules

$$0 \rightarrow t_1|w| R \rightarrow \mathcal{M} \rightarrow M/\langle w \rangle \rightarrow 0.$$

Applying the exact functor $L_{K_3}$ yields a short exact sequence in $\mathcal{L}(K_3)$

$$0 \rightarrow [(0,1,1,\infty)] \rightarrow M \rightarrow L_{K_3}(\mathcal{M}/\langle w \rangle) \rightarrow 0.$$

Restricting to the positions $\sigma \subseteq [3]$ containing 1 and evaluating the multidegree $\vec{d}$ at $d_1 = 0$ defines functors

$$\mathcal{L}(K_3) \xrightarrow{\text{restrict}} \mathcal{L}(K_3)[t_1^\pm] \xrightarrow{\text{set } d_1 = 0} \mathcal{L}(K_2)$$

whose composite sends the epimorphism $q: M \rightarrow \text{coker}(\mu)$ to the non-split epimorphism from Example 8.2. Hence the epimorphism $q$ itself is not split. By Lemma 8.1, $M$ does not decompose as a direct sum of modules of the form $[\vec{d},\infty)$. By Lemma 7.7, any direct sum decomposition of $M$ would be of that form.

**Proposition 8.4.** For any $m \geq 3$, there exists a torsion-free module in $F$ of rank 2 that is indecomposable.

*Proof.* Here we will denote subsets of $[3]$ by the letter $\tau$, and the graded ring $S = \mathbb{k}[t_1, t_2, t_3]$ to avoid ambiguity with $[m]$. Consider the functor $G: \mathcal{L}(K_3) \rightarrow \mathcal{L}(K_m)$ defined by

$$G(M)_{\sigma_i} = \begin{cases} R_{\sigma_i} \otimes s_{\tau_1} M_{\tau_1} & \text{if } 1 \leq i \leq 3 \\ R_{\sigma_i} \otimes s_{[3]} M_{[3]} & \text{if } 4 \leq i \leq m \end{cases}$$
with the canonical localization maps $\varphi_i : G(M)_{\sigma_i} \to G(M)_{[m]}$. The composite of $G$ followed by the functors

$$\mathcal{L}(K_m) \xrightarrow{\text{restrict}} \mathcal{L}(K_m)[t^\pm_1, \cdots, t^\pm_m] \xrightarrow{\text{set } d_4 = 0, \ldots, d_m = 0} \mathcal{L}(K_3)$$

is an equivalence of categories. Now take $M$ to be the rank 2 indecomposable torsion-free object in $\mathcal{L}(K_3)$ from Lemma 8.3. Then $G(M)$ is a rank 2 torsion-free object in $\mathcal{L}(K_m)$ which is indecomposable.

The functors used in the proof allow us to compare the categories $\mathcal{L}(K)$ for different $K$. The next lemma generalizes the functor $G$. Since we will have different vertex sets $V$, we will include the vertex set in the notation $\mathcal{L}(K;V)$.

**Lemma 8.5.** Let $K$ be a simplicial complex on a vertex set $V$ and $L$ a simplicial complex on the vertex set $W$, with $V \subseteq W$. Assume that $L$ extends $K$, that is:

$$K \subseteq L|_V := L \cap \mathcal{P}(V).$$

Then the following construction defines an exact functor

$$\Phi_{K,L} : \mathcal{L}(K; V) \to \mathcal{L}(L; W).$$

Given $M$ in $\mathcal{L}(K; V)$, the module $\Phi_{K,L}(M)$ in $\mathcal{L}(L; W)$ has at position $\sigma \in L^c$

$$\Phi_{K,L}(M)_\sigma := R_\sigma \otimes_{R_{\sigma \cap V}} M_{\sigma \cap V},$$

which is defined since $\sigma \cap V \in K^c$ is a missing face of $K$. For $\sigma \subseteq \tau$, the transition map $\Phi_{K,L}(M)_\sigma \to \Phi_{K,L}(M)_\tau$ is the composite

$$R_\sigma \otimes_{R_{\sigma \cap V}} M_{\sigma \cap V} \xrightarrow{\text{localize}\otimes \varphi_{\sigma \cap V, \tau \cap V}} R_\tau \otimes_{R_{\tau \cap V}} M_{\tau \cap V} \xrightarrow{\text{flat}} R_\tau \otimes_{R_{\tau \cap V}} M_{\tau \cap V}.$$

**Proof.** Exactness of $\Phi_{K,L}$ follows from Lemma 3.10 and the flatness of $R_\sigma$ as a module over $R_{\sigma \cap V}$. □

**Example 8.6.** The functor $G : \mathcal{L}(K_3) \to \mathcal{L}(K_m)$ in the proof of Proposition 8.4 was the functor $\Phi_{K_3, K_m}$, over the vertex sets $[3] \subseteq [m]$. The functor $\Phi_{K_1, K_2} : \mathcal{L}(K_1) \to \mathcal{L}(K_2)$ has the following effect on objects:

$$M_{\sigma_1} \quad \longleftrightarrow \quad k[t_1, t_2] \otimes_k M_{\sigma_1} \quad \longleftrightarrow \quad k[t^\pm_1, t^\pm_2] \otimes_k M_{[1]} \quad \longleftrightarrow \quad k[t^\pm_1, t_2] \otimes_k M_{[1]}.$$

9. **Wild representation type**

We first review some background about wild representation type. Some references assume that the base field $k$ is algebraically closed, which we do not assume here.

We follow the terminology in [SS07b, §XIX.1].

**Notation 9.1.** Let $k[t_1, t_2]$ denote the $k$-algebra of polynomials in two non-commuting variables, i.e., the free (associative unital) $k$-algebra on two generators, i.e., the tensor algebra $T_k(k^2)$.

For a $k$-algebra $A$, let $A$-$\text{fmod}$ denote the full subcategory of $A$-$\text{Mod}$ consisting of modules that are finite-dimensional over $k$. Note that if $A$ is itself finite-dimensional, an $A$-module is finite-dimensional if and only if it is finitely generated over $A$, i.e., $A$-$\text{fmod} = A$-$\text{mod}$.

**Definition 9.2.** Let $\mathcal{A}$ and $\mathcal{B}$ be $k$-linear abelian categories.
(1) A functor $T: \mathcal{A} \to \mathcal{B}$ is a **representation embedding** if it is $\mathbb{k}$-linear, exact, injective on isomorphism classes (i.e., $T(X) \cong T(Y) \implies X \cong Y$), and sends indecomposables.

(2) The category $\mathcal{A}$ has **wild representation type** if there exists a representation embedding $T: \mathbb{k}(t_1, t_2)$-fmod → $\mathcal{A}$.

(3) A (finite-dimensional) $\mathbb{k}$-algebra $A$ has **wild representation type** if its category of finitely generated modules $A$-mod does.

Since we do **not** assume that the base field $\mathbb{k}$ is algebraically closed, we should say $\mathbb{k}$-wild, but we say wild for short. For different notions of wildness, see the discussion after [SS07b, §XIX Corollary 1.15] and the helpful overviews in [Sim05, §2] and [AS05, §2].

**Lemma 9.3.** For $\mathbb{k}$-linear abelian categories $\mathcal{A}$ and $\mathcal{B}$, a fully faithful exact $\mathbb{k}$-linear functor $T: \mathcal{A} \to \mathcal{B}$ is a representation embedding [SS07b, §XIX Lemma 1.2].

**Lemma 9.4.** A $\mathbb{k}$-linear abelian category $\mathcal{A}$ has wild representation type if and only if for each finite-dimensional $\mathbb{k}$-algebra $B$, there exists a representation embedding $T: B$-mod → $\mathcal{A}$ [AS05, Lemma 2.2].

**Lemma 9.5.** The category $\mathbb{k}[t_1, t_2]$-mod of finitely generated bigraded modules over the bigraded polynomial algebra $\mathbb{k}[t_1, t_2]$ has wild representation type.

**Proof.** The poset $\mathbb{N}^2$ contains a rectangular grid $[m] \times [n]$, which has wild representation type for $m, n \geq 5$ [BL23, §8.2] [Naz75]. By Lemma 9.3, the result follows.

**Remark 9.6.** The analogue of Lemma 9.5 for the ungraded polynomial algebra also holds [SS07b, §XIX Theorem 1.11]. The proof therein works over an arbitrary field $\mathbb{k}$.

**Proposition 9.7.** Let $K$ be a simplicial complex on the vertex set $[m]$ satisfying the strict inclusion $K \subset K_m = sk_{m-3} \Delta^{m-1}$. Then the category $\mathcal{L}(K)$ contains $\mathbb{k}[s, t]$-mod as an exact retract (up to equivalence). In particular, $\mathcal{L}(K)$ has wild representation type.

**Proof.** The condition $K \subset K_m = sk_{m-3} \Delta^{m-1}$ is equivalent to: there is a missing face $\sigma \in K^c$ that does not contain two vertices, say, $k, \ell \notin \sigma$. At position $\sigma$, the variables $t_k$ and $t_\ell$ have not been inverted in $R_\sigma$.

Consider the simplicial complex $\emptyset = \{\}$ on the vertex set $\{k, \ell\} \subseteq [m]$. The condition $\emptyset \subseteq K|_{\{k, \ell\}}$ holds (vacuously), so that Lemma 8.5 provides an exact functor

$$\Phi_{\emptyset, K}: \mathcal{L}(\emptyset; \{k, \ell\}) \to \mathcal{L}(K).$$

On the other hand, inverting all the variables $t_i$ for $i \neq k, \ell$ yields a functor

$$\mathcal{L}(K) \xrightarrow{\text{restrict}} \mathcal{L}(K)|_{t_i^\pm \mid i \neq k, \ell} \xrightarrow{\text{set } d_i=0 \text{ for } i \neq k, \ell} \mathcal{L}(K|_{\{k, \ell\}}) = \mathcal{L}(\emptyset; \{k, \ell\}).$$

(9.8)

Here we used the assumption $k, \ell \notin \sigma$ to obtain the missing faces of the restricted complex:

$$\sigma \cap \{k, \ell\} = \emptyset \in (K^c)|_{\{k, \ell\}} = (K_{\{k, \ell\}})^c$$

$$\implies (K|_{\{k, \ell\}})^c = \mathcal{P}(\{k, \ell\}).$$

The composite of $\Phi_{\emptyset, K}$ followed by the functor in Equation (9.8) is naturally isomorphic to the identity, providing the desired retraction. By Lemma 9.3, $\Phi_{\emptyset, K}$ is a representation embedding. The equivalence $\mathcal{L}(\emptyset; \{k, \ell\}) \cong \mathbb{k}[t_k, t_\ell]$-mod from Example 3.5 together with Lemma 9.5 concludes the proof.

**Notation 9.9.**

1. Let $\mathbb{k}[t]$-mod$_{\leq n}$ denote the full subcategory of $\mathbb{k}[t]$-mod consisting of modules $M$ whose transition maps become isomorphisms past degree $n$:

$$t^{e-d}: M(d) \cong M(e) \text{ for } d \geq n.$$
We will obtain the equivalence as a composite of two equivalences.

**Proof.**

1. **Proposition 9.13.** There is an equivalence of categories $\mathcal{L}(K_m)_{\leq n} \cong \operatorname{rep}_k(Q_n)$.

**Proof.** We will obtain the equivalence as a composite of two equivalences

\[ \mathcal{L}(K_3)_{\leq n} \xrightarrow{\Psi} \operatorname{rep}_k(Q_{n+1})_{\leq n} \xrightarrow{\Theta} \operatorname{rep}_k(Q_n), \]

where $\operatorname{rep}_k(Q_{n+1})_{\leq n}$ denotes the full subcategory of $\operatorname{rep}_k(Q_{n+1})$ where the maps past each vertex with index $n$ are isomorphisms, namely $\alpha_{n+1}$, $\beta_{n+1}$, and $\gamma_{n+1}$.

**The equivalence $\Psi$.** Given a module $M$ in $\mathcal{L}(K_3)_{\leq n}$, consider the sequence of $k$-modules

\[ M_{\sigma_1}(0, n, n) \xrightarrow{t_1} M_{\sigma_1}(1, n, n) \xrightarrow{t_1} \cdots \xrightarrow{t_1} M_{\sigma_1}(n, n, n) \xrightarrow{\varphi_1} M_{[3]}(n, n, n). \] (9.14)

The last map is an isomorphism, since the map of $R_{\sigma_1}$-modules $\varphi_1 : M_{\sigma_1} \to M_{[3]}$ inverts $t_1$, and all the transition maps

\[ t_1^{d-n} : M_{\sigma_1}(n, n, n) \cong M_{\sigma_1}(d, n, n) \]

are isomorphisms for $d \geq n$ by assumption on $M$. Define the representation $\Psi(M)$ as having the sequence (9.14) as its first leg. Construct the second and third legs similarly using $M_{\sigma_2}$ and $M_{\sigma_3}$. This construction yields the desired functor $\Psi$, which is moreover an equivalence.
The equivalence Θ. Given a representation $V$ in $\text{rep}_k(Q_{n+1})_{\leq n}$, construct the representation $\Theta(V)$ by absorbing the last isomorphism into the previous map. That is, take the $k$-vector space $\Theta(V)_s = V_s$ in the terminal position and

$$\Theta(V)_{a_n} = V_{a_{n+1}} \circ V_{a_n} : V_{a_{n-1}} \to V_s,$$

and likewise for the second and third legs. This yields a functor $\Theta$ which is an equivalence, with inverse equivalence inserting an identity at the end of each leg. □

**Definition 9.15.** A finite quiver $Q$ has **wild representation type** if its category of pointwise finite-dimensional representations $\text{rep}_k(Q)$ does. Equivalently, the path algebra $kQ$ has wild representation type.

Wildness of a quiver turns out to be independent of the base field $k$. That is, $Q$ has wild representation type over some field $k$ if and only if $Q$ has wild representation type over any other field $k'$ [Naz73]. The same is true for a finite poset with a unique maximal element [Naz75] [AS05, §1].

The argument in the next proof was kindly provided by Steffen Oppermann.

**Theorem 9.16.** For $m \geq 3$, the category $\mathcal{L}(K_m)$ has wild representation type.

**Proof.** In the proof of Proposition 8.4, we showed that the category $\mathcal{L}(K_3)$ is an exact retract of $\mathcal{L}(K_m)$. By Lemma 9.3, it suffices to show that $\mathcal{L}(K_3)$ has wild representation type. Its full subcategory $\mathcal{L}(K_3)_{\leq n}$ is an exact subcategory, by Lemma 9.11. Thus it suffices to show that $\mathcal{L}(K_3)_{\leq n}$ has wild representation type for $n$ large enough. We have an equivalence $\mathcal{L}(K_3)_{\leq n} \cong \text{rep}_k(Q_n)$ by Proposition 9.13. For $n \geq 3$, the quiver $Q_n$ is known to have wild representation type [SS07b, §XVIII Theorem 4.1] [Naz73, Lemma 8]. □

In the remainder of the section, we interpret this result in terms of the torsion pair on $\mathcal{L}(K_m)$ introduced in Section 5.

**Remark 9.17.** The setup of Definition 9.2 can be generalized to categories $\mathcal{A}$ of the following form. Consider $\mathcal{A}'$ a $k$-linear abelian category and $\mathcal{A} \subseteq \mathcal{A}'$ a non-empty full subcategory that is closed under extensions (in particular finite direct sums) and summands. For example, if the abelian category $\mathcal{A}'$ has a torsion pair $(\mathcal{T}, \mathcal{F})$, then both classes $\mathcal{T}$ and $\mathcal{F}$ are subcategories of that form.

The setup can be generalized further to a $k$-linear (Quillen) exact category $\mathcal{A}$ that is idempotent complete (i.e., in which every idempotent $e : X \to X$ splits). Background on exact categories can be found in [Brih10]. For background on idempotents and decompositions, see [Sha23, §3].

**Lemma 9.18.** Let $\mathcal{B}$ be an abelian category with a torsion pair $(\mathcal{T}, \mathcal{F})$, and let $\mathcal{A} \subseteq \mathcal{B}$ be an abelian subcategory. Assume that for every object $A$ in $\mathcal{A}$, the monomorphism $i : T(A) \to A$ in $\mathcal{B}$ lies in $\mathcal{A}$. Then the torsion pair on $\mathcal{B}$ restricts to a torsion pair $(T \cap \mathcal{A}, F \cap \mathcal{A})$ on $\mathcal{A}$.

**Lemma 9.19.** The torsion pair $(\mathcal{T}, \mathcal{F})$ on $\mathcal{L}(K_m)$ induces a torsion pair $(\mathcal{T}_{\leq n}, \mathcal{F}_{\leq n})$ on the full subcategory $\mathcal{L}(K_m)_{\leq n}$, for any $n \geq 0$.

**Proof.** Let $M$ be an object in the subcategory $\mathcal{L}(K_m)_{\leq n}$, i.e., $M_{\sigma_i}$ lies in $R_{\sigma_i} \cdot \text{mod}_{\leq n}$ for all $1 \leq i \leq m$. Its torsion part $T(M)$ is given by

$$T(M)_{\sigma_i} = T(M_{\sigma_i}),$$

which also lies in $R_{\sigma_i} \cdot \text{mod}_{\leq n}$, so that $T(M)$ lies in $\mathcal{L}(K_m)_{\leq n}$. By Lemma 9.18, the claim follows. □

**Lemma 9.20.** Via the equivalence from Proposition 9.13, the torsion pair on $\mathcal{L}(K_3)_{\leq n}$ corresponds to the two classes:

$$\mathcal{T}_{\leq n} \cong \{ V \in \text{rep}_k(Q_n) \mid V_s = 0 \}$$

$$\mathcal{F}_{\leq n} \cong \{ V \in \text{rep}_k(Q_n) \mid \text{the maps in } V \text{ are monomorphisms} \}.$$
Corollary 9.21. The torsion class $T_{\leq n}$ in $\mathcal{L}(K_3)_{\leq n}$ is equivalent to a product of categories

$T_{\leq n} \cong \text{rep}_k(\bar{A}_n)^3,$

where $\bar{A}_n$ denotes the linearly ordered quiver of type $A_n$ (with $n$ vertices), as illustrated here:

\[ a_0 \overset{\alpha_1}{\longrightarrow} a_1 \cdots \overset{\alpha_{n-1}}{\longrightarrow} a_{n-2} \overset{\alpha_{n-1}}{\longrightarrow} a_{n-1} \]

Corollary 9.22. For $n \geq 3$, the torsion-free class $F_{\leq n}$ in $\mathcal{L}(K_3)_{\leq n}$ has wild representation type.

Proof. Quivers of type $A_n$ have finite representation type, by Gabriel’s theorem [ASS06, §VII Theorem 5.10] [EH18, Theorem 11.1]. By Corollary 9.21, the torsion class $T_{\leq n}$ has finite representation type. But the torsion pair $(T_{\leq n}, F_{\leq n})$ splits and $\mathcal{L}(K_3)_{\leq n}$ has wild representation type, which forces $F_{\leq n}$ to have wild representation type. \qed

Remark 9.23. A closer look at the proof that the quiver $Q_3$ has wild representation type shows Corollary 9.22 directly. In [Nazarova 1981, Theorem 2’] [Nazarova 1975], Nazarova shows that $Q_3$ is wild using representations in which all the $k$-linear maps are subspace inclusions.

10. Relationship to the rank invariant

In this section, we work with the simplicial complex $K = K_m$ and investigate the relationship between the rank invariant of an $R$-module $M$ and the decomposition of the localization $L_{K_m}(M)$ in $\mathcal{L}(K_m)$.

Definition 10.1. Let $M$ be an $R$-module. The rank invariant of $M$ is the function assigning to each pair of multidegrees $\vec{a}, \vec{b} \in \mathbb{N}^m$ with $\vec{a} \leq \vec{b}$ the integer

$\text{rk}_M(\vec{a}, \vec{b}) = \text{rank} \left( M(\vec{a}) \overset{\vec{a} - \vec{b}}{\longrightarrow} M(\vec{b}) \right).$

In general, the rank invariant does not determine the decomposition of $L_{K_2}(M)$ in $\mathcal{L}(K_2)$.

Example 10.2. Let $(t_1, t_2) \subset R$ be the graded ideal generated by $t_1$ and $t_2$ and take the $R$-modules

\[ M = (t_1, t_2) \oplus t_1t_2R \]
\[ N = t_1R \oplus t_2R. \]

Figure 4. The modules $M$ and $N$ in Example 10.2.

The modules $M$ and $N$ have the same rank invariant:

$\text{rk}_M(\vec{a}, \vec{b}) = \begin{cases} 0 & \text{if } \vec{a} = (0, 0) \\ 1 & \text{if } \vec{a} = (a, 0) \text{ or } \vec{a} = (0, a) \text{ for some } a > 0 \\ 2 & \text{if } \vec{a} \geq (1, 1). \end{cases}$
However, their $K_2$-localizations are not isomorphic in $\mathcal{L}(K_2)$:

\[
\begin{align*}
L_{K_2}(M) &\cong [(0,0), \infty) \oplus [(1,1), \infty) \\
L_{K_2}(N) &\cong [(1,0), \infty) \oplus [(0,1), \infty).
\end{align*}
\]

**Lemma 10.3.** Let $M$ be a finitely generated $R$-module. There exists $\vec{d} \in \mathbb{N}^m$ such that for every $\vec{c}' \geq \vec{c} \geq \vec{d}$ in $\mathbb{N}^m$, the transition map of $M$

\[
M(\vec{c}) \xrightarrow{\tau^{\vec{c}'-\vec{c}}} M(\vec{c}')
\]

is an isomorphism.

**Proof.** Since $M$ is finitely generated and $R$ is Noetherian, $M$ is finitely presented. Pick a finite presentation $F_1 \to F_0 \to M$ of $M$ and let $\vec{d} \in \mathbb{N}^m$ be an upper bound for the multidegrees of the generators and relations. For $\vec{c}' \geq \vec{c} \geq \vec{d}$, consider the diagram of $k$-vector spaces with exact rows

\[
\begin{array}{ccc}
F_1(\vec{c}) & \to & F_0(\vec{c}) \to M(\vec{c}) \to 0 \\
\cong & & \cong & \cong \\
F_1(\vec{c}') & \to & F_0(\vec{c}') \to M(\vec{c}') \to 0.
\end{array}
\]

The first (resp. second) downward map is an isomorphism since $\vec{c}$ is greater than or equal to the degrees of the relations (resp. generators) of $M$. Hence the induced map on cokernels is an isomorphism. \qed

In the next statement, we will have a subset $\sigma \subseteq [m]$ and multidegrees viewed as functions $\vec{a} \in \mathbb{N}^{[m] \setminus \sigma}$ and $\vec{c} \in \mathbb{N}^\sigma$. Denote by $\vec{a} \ast \vec{c} \in \mathbb{N}^m$ the function extending $\vec{a}$ and $\vec{c}$.

**Lemma 10.4.** Let $M$ be a finitely generated $R$-module and $\sigma \subseteq [m]$. For every $\vec{a},\vec{b} \in \mathbb{N}^{[m] \setminus \sigma}$, the rank invariant of the localization $M_\sigma := R_\sigma \otimes_R M$ is given by

\[
\operatorname{rk}_{M_\sigma}(\vec{a}, \vec{b}) = \lim_{\vec{c} \in \mathbb{N}^\sigma} \operatorname{rk}_M(\vec{a} \ast \vec{c}, \vec{b} \ast \vec{c}). \tag{10.5}
\]

**Proof.** For a fixed $\vec{a} \in \mathbb{N}^{[m] \setminus \sigma}$, apply Lemma 10.3 to the $k[t_i \mid i \in \sigma]$-module $M(\vec{a} \ast -)$. There exists a $\vec{d} = \vec{d}_\sigma \in \mathbb{N}^\sigma$ such that for every $\vec{c}' \geq \vec{c} \geq \vec{d}$ in $\mathbb{N}^\sigma$, the transition map of $M$

\[
M(\vec{a} \ast \vec{c}) \xrightarrow{\tau^{\vec{c}'-\vec{c}}} M(\vec{a} \ast \vec{c}')
\]

is an isomorphism. The commutative square of transition maps

\[
\begin{array}{ccc}
M(\vec{a} \ast \vec{c}) & \xrightarrow{=} & M(\vec{a} \ast \vec{c}') \\
\cong & & \cong \\
M(\vec{b} \ast \vec{c}) & \xrightarrow{=} & M(\vec{b} \ast \vec{c}')
\end{array}
\]

shows that the ranks on the right-hand side of Equation (10.5) are eventually constant, with the limit value being achieved for all $\vec{c} \geq \sup(\vec{d}_\vec{a}, \vec{d}_\vec{b})$. Moreover, the localization map $M \to M_\sigma$ induces an isomorphism

\[
M(\vec{a} \ast \vec{c}) \xrightarrow{=} M_\sigma(\vec{a} \ast \vec{c})
\]

for all $\vec{c} \geq \vec{d}$, showing that the limit value in Equation (10.5) is indeed the rank $\operatorname{rk}_{M_\sigma}(\vec{a}, \vec{b})$. \qed

**Corollary 10.6.** The rank invariant of an object $M$ of $\mathcal{L}(K_m)$ determines the modules $M_{\sigma_i}$ and $M_{[m]}$ up to isomorphism.

**Proof.** A finitely generated $k[t]$-module is determined by its rank invariant. \qed
**Example 10.7.** For $m = 2$, the rank invariant of an $R$-module $M$ determines the interval decompositions of the $\mathbb{k}[t_1, t_2]$-module $M_{\sigma_1}$ and the $\mathbb{k}[t_1^\pm, t_2]$-module $M_{\sigma_2}$. Each finite interval module $[a, b]_1$ in $M_{\sigma_2}$ contributes a “vertical strip” $[a, b]_1$ to the decomposition of $M$ in $\mathcal{L}(K_2)$. Likewise, each finite interval module $[a, b]_2$ in $M_{\sigma_2}$ contributes a “horizontal strip” $[a, b]_2$ to the decomposition of $M$ in $\mathcal{L}(K_2)$. The number of infinite interval modules $[a, \infty)_1$ in $M_{\sigma_1}$ equals the number of infinite interval modules in $M_{\sigma_2}$, which is $\dim_k M_{[m]}$.

**Proposition 10.8.** For $m = 2$, if an $R$-module $M$ lies in the image of the delocalization functor $\rho_{K_2} : \mathcal{L}(K_2) \to R$-$\text{mod}$ (see Lemma 4.5), then the rank invariant of $M$ determines the decomposition of $L_{K_2}(M)$ in $\mathcal{L}(K_2)$.

**Proof.** By Corollary 10.6, the rank invariant of $M$ always determines the torsion part $T(L_{K_2}M)$ in $\mathcal{L}(K_2)$. Since $M$ lies in the image of the right adjoint $\rho_{K_2}$, the unit map

$$M \xrightarrow{\cong} \rho_{K_2} L_{K_2}(M)$$

is an isomorphism. By additivity of the rank invariant under direct sum, we may assume that $L_{K_2}(M)$ is torsion-free. By Proposition 7.11, $L_{K_2}(M)$ is a direct sum of “quadrant modules” $\bigoplus_i [\vec{d}(i), \infty)$, so that $M$ is a free $R$-module

$$M \cong \rho_{K_2} \left( \bigoplus_i [\vec{d}(i), \infty) \right) \cong \bigoplus_i \vec{t}(i) R.$$

Such an $R$-module is determined (up to isomorphism) by its Hilbert function $\text{rk}_M(\vec{a}, \vec{a})$, in particular by its rank invariant. \hfill \square

**Question 10.9.** Which refinements of the rank invariant of a $\mathbb{k}[t_1, t_2]$-module $M$ determine the torsion-free part of $L_{K_2}(M)$ in $\mathcal{L}(K_2)$?

One could investigate a refinement along the following lines. For any three bidegrees $\vec{a}, \vec{b}, \vec{c} \in \mathbb{N}^2$ satisfying $\vec{a} \leq \vec{c}$ and $\vec{b} \leq \vec{c}$, take the dimension of the intersection of images

$$\dim_k \left( \text{im} \left( M(\vec{a}) \xrightarrow{t^{c-a}} M(\vec{c}) \right) \right) \cap \text{im} \left( M(\vec{b}) \xrightarrow{t^{c-b}} M(\vec{c}) \right).$$

Such an invariant was used in [Zha19] to decompose the representations of the quiver with relations consisting of a commutative square.

### 11. Classification of Tensor-closed Serre Subcategories

In this section we work both in $R$-$\text{mod}$ and $R$-$\text{mod}_{\mathbb{Z}^m}$ and a module will mean a module in one of those categories.

**Lemma 11.1.** The homogeneous primes of $R$ are of the form $< t_{i_1}, \cdots, t_{i_k} >$.

**Proof.** See [HOST19, Lemma 4.35]. \hfill \square

**Notation 11.2.** Let $\text{Spec}(R)$ denote the set of homogeneous prime ideals of $R$. A subset $S$ of $\text{Spec}(R)$ is closed if $p \in S$ and $p \subset q$ implies that $q \in S$. We denote the closure of $S$ by $\overline{S}$.

The support of a module $M$ is $\text{Supp}(M) = \{ \sigma \mid R_{\sigma} \otimes M \neq 0 \}$, where $\sigma^c$ is the complement of $\sigma \subset [m]$. We use $\text{Supp}^c(M) = \{ \sigma \mid R_{\sigma} \otimes M \neq 0 \}$ for the set of complements of the support and also call it the support and consider it a subset of $\mathcal{P}([m])$.

For each $\sigma \in \text{Supp}^c(M)$ the corresponding prime ideal is $p =< t_{i} >_{i \in \sigma}$. In other words $p \in \text{Supp}(M)$ if and only if $p^c \in \text{Supp}^c(M)$. We see that $\text{Supp}^c(M)$ is closed under taking submodules and so a simplicial complex since $\text{Supp}(M)$ is a closed subset of $\text{Spec}(R)$.

We define $\text{Ass}(M)$ to be the associated primes of $M$, by letting $p \in \text{Ass}(M)$ if there exists $x \in M$ such that $p$ is the annihilator of $x$. 
Here are some standard results about support and associated primes [AM69, Exercise 3.19].

**Lemma 11.3.** Let $M, N, L$ be modules and $0 \to L \to M \to N \to 0$ be a short exact sequence.

1. $\text{Ass}(M) = \text{Supp}(M)$
2. $\text{Supp}(L) \cup \text{Supp}(N) = \text{Supp}(M)$.

**Notation 11.4.** For a subcategory $\mathcal{C} \subset \text{R-mod}$ (or $\text{R-mod}_{\mathbb{Z}^m}$), we define its support as $\text{Supp}^\mathcal{C}(M) = \bigcup_{M \in \mathcal{C}} \text{Supp}(M)$. Note that $\text{Supp}^\mathcal{C}(M)$ is also a simplicial complex.

Let $\otimes$ be the tensor product on $\text{R-mod}_{\mathbb{Z}^m}$ which turns it into a symmetric monoidal category. A Serre subcategory $\mathcal{C} \subset \text{R-mod}_{\mathbb{Z}^m}$ is a **tensor ideal** if $M \in \mathcal{C}$ and $N \in \text{R-mod}_{\mathbb{Z}^m}$ then $M \otimes N \in \mathcal{C}$. A Serre subcategory $\mathcal{C} \subset \text{R-mod}$ is a **tensor ideal** if $M \in \mathcal{C}$, $N \in \text{R-mod}_{\mathbb{Z}^m}$, and $M \otimes N \in \text{R-mod}$ then $M \otimes N \in \mathcal{C}$. Note that a tensor ideal in $\text{R-mod}$ is the same as $\mathcal{C} \cap \text{R-mod}$ for a tensor ideal $\mathcal{C}$ in $\text{R-mod}_{\mathbb{Z}^m}$.

For an object $M \in \text{R-mod}$, let $S(M)$ be the smallest Serre subcategory of $\text{R-mod}$ containing $M$ and $S^\otimes(M)$ be the smallest tensor ideal of $\text{R-mod}$ containing $M$.

**Lemma 11.5.** For a prime $p$, $p \in \text{Supp}(M)$ implies that for some $\vec{d}$, $t^{\vec{d}}R/p \in S(M)$ and thus $R/p \in S^\otimes(M)$.

**Proof.** By Lemma 11.3 (1), there is $q \subset p$ with $q \in \text{Ass}(M)$. So for some $\vec{d}$ and $\alpha \in M(\vec{d})$ with anihilator $q$ we get that there is an injective map $t^{\vec{d}}R/q \to M$ and so since Serre subcategories are closed under taking subobjects $t^{\vec{d}}R/q \in S(M)$. Then since Serre subcategories are closed under quotients we get that $t^{\vec{d}}R/p \in S(M)$ and so $R/p \in S^\otimes(M)$.

**Lemma 11.6.** $M \in S^\otimes\left(\bigoplus_{p \in \text{Supp}(M)} R/p\right)$.

**Proof.** We construct a sequence of modules $M(i)$. Let $M(0) = M$. Suppose we have already constructed $M(i)$ and that $\text{Supp}(M(i)) \subset \text{Supp}(M)$. If $M(i) \neq 0$ let $p \in \text{Ass}(M(i)) \subset \text{Supp}(M(i)) \subset \text{Supp}(M)$ and define $M(i+1)$ so there is an exact sequence

$$0 \to t^{\vec{d}}R/p \to M(i) \to M(i+1) \to 0 \quad (11.7)$$

We also have $\text{Supp}(M(i+1)) \subset \text{Supp}(M(i)) \subset \text{Supp}(M)$ by Lemma 11.3 and the induction hypothesis.

The increasing sequence of submodules $\ker(M \to M(i)) \subset M$ must stabilize and so $M(n) = 0$ for some $n$. Then clearly $M(n) \in S^\otimes\left(\bigoplus_{p \in \text{Supp}(M)} R/p\right)$.

As each $R/p$ used has $p \in \text{Supp}M$ we get that $R/p \in S\left(\bigoplus_{p \in \text{Supp}(M)} R/p\right)$ and $t^{\vec{d}}R/p \in S^\otimes\left(\bigoplus_{p \in \text{Supp}(M)} R/p\right)$. Using Equation (11.7) and that Serre subcategories are closed under extensions we get that $M(n-1) \in S^\otimes\left(\bigoplus_{p \in \text{Supp}(M)} R/p\right)$. Continuing in this way we see that $M = M(0) \in S^\otimes\left(\bigoplus_{p \in \text{Supp}(M)} R/p\right)$.

**Lemma 11.8.** $\text{Supp}(M) = \text{Supp}(S(M)) = \text{Supp}(S^\otimes(M))$.

**Proof.** That $\text{Supp}(M) \subset \text{Supp}(S(M)) \subset \text{Supp}(S^\otimes(M))$ follow directly from the definitions. That $\text{Supp}(S(M)) \subset \text{Supp}(M)$ follows since by Lemma 11.3 the closure operations of a Serre subcategory, that is taking subobjects, quotients and extensions do not increase support. Also since $\text{Supp}(M \otimes N) \subset \text{Supp}(M)$ we get that $\text{Supp}(S(M)) \subset \text{Supp}(S^\otimes(M))$.

**Notation 11.9.** For $K$ a simplicial complex on $[m]$ recall the face ring (or Stanley–Reisner ring) $k[K]$ of $K$. For a simplex $< i_1, \ldots, i_k > = \sigma \in \mathcal{P}([m])$ we denote the monomial $t^\sigma = t_{i_1} \cdots t_{i_k}$. Then, let $I(K) = (t^\sigma)_{\sigma \notin K}$ and

$$k[K] = R/I(K).$$

In other words it is the (multigraded) polynomial ring modulo the ideal $I(K)$ generated by missing faces. Remembering we work with finitely generated modules, we call a module $M$, $I$-torsion if $I^sM = 0$ for some $s$. We let $I$-tors denote the full subcategory of $I$-torsion modules.
Lemma 11.10. \( \ker L_K = I(K)\)-tors = \( S^\otimes(k[K]) = S^\otimes(\oplus_{\mathfrak{p} \in K} R/p) \).

Proof. First observe that \( M_\sigma = 0 \iff (t_\sigma)^* M = 0 \) for some \( s \). Then

\[
M \in \ker L_K \iff M_\sigma = 0 \text{ for all } \sigma \not\in K
\]

\[
\iff t_\sigma^* M = 0 \text{ for all } \sigma \not\in K \text{ for some } s
\]

\[
\iff (I(K))^s M = 0 \text{ for some } s
\]

\[
\iff M \text{ is } I(K)\text{-tors.}
\]

This proves the first equality. The last equality follows from Lemma 11.12 and the computation \( \text{Supp}^c(\oplus_{\mathfrak{p} \in K} R/p) = K \) since by Theorem 11.15 \( \text{Supp}^c \) determines tensor ideals in \( R\text{-mod}(R\text{-mod}_{\mathbb{Z}^m}) \).

Since for each \( \mathfrak{p} \in K \) and \( \sigma \not\in K \), \( R_\sigma \otimes R/p = 0 \) and so \( R/\mathfrak{p} \subseteq \ker L_K \). It follows that \( S^\otimes(\oplus_{\mathfrak{p} \in K} R/p) \subseteq \ker L_K \). If \( M \in \ker L_K \) then \( M_\sigma = 0 \) for each \( \sigma \not\in K \) so \( \text{Supp}^c M \subseteq K \) so by Lemma 11.6 \( M \in S^\otimes(\oplus_{\mathfrak{p} \in K} R/p) \). This shows the first class is equal to the last and completes the proof.

Remark 11.11. The open subspace complement of the closed subset \( V(K) \) cut out by the ideal \( I(K) \) is written in polyhedral product language as \( (\mathbb{C}, \mathbb{C}^*)^K \). The homotopy quotient of \( (\mathbb{C}, \mathbb{C}^*)^K \) by \( (S^1)^n \) is the Davis–Januszkiewicz space \( DJ(K) \). Also \( H^*(DJ(K)) \), and therefore the equivariant cohomology of \( (\mathbb{C}, \mathbb{C}^*)^K \), is \( k[K] \) \([BP15, \S 4.3, 4.7]\). Coincidentally the category of \( (\mathbb{C}^*)^m \)-equivariant coherent sheaves on \( (\mathbb{C}, \mathbb{C}^*)^K \) is related to \( R\text{-mod}_{\mathbb{Z}^m} \) modulo the tensor closed Serre subcategory generated by \( H^*(DJ(K)) \).

Lemma 11.12. \( \text{Supp}^c(k[K]) = K \).

Proof. We have an exact sequence

\[
\bigoplus_{\sigma \not\in K} t_\sigma R \xrightarrow{\Sigma_{\sigma \not\in K} f_\sigma} R \to k[K] \to 0
\]

where \( f_\sigma \colon t_\sigma R \to R \) is the inclusion. Tensoring with \( R_\delta \) preserves this exact sequence, so \( k[K] \otimes R_\delta = 0 \) if and only if \( \Sigma_{\sigma \not\in K} f_\sigma \otimes R_\delta \) is a surjection.

Let \( \sigma \subseteq \delta \subseteq [m] \). Recall that \( R_\delta \) then inverts all the elements in \( \delta \). So all \( t_i \) that make up \( t_\sigma \) are inverted and so \( f_\sigma \otimes R_\delta \) becomes an isomorphism. So if there is \( \sigma \not\in K \) such that \( \sigma \subseteq \delta \subseteq [m] \) and \( \Sigma_{\sigma \not\in K} f_\sigma \otimes R_\delta \) is a surjection and \( k[K] \otimes R_\delta = 0 \). So if \( \sigma \subseteq \delta \) for some \( \sigma \not\in K \) then \( \delta \not\in \text{Supp}^c(k[K]) \). Also if \( \delta \not\in K \) then there is a \( \sigma \not\in K \) with \( \sigma \subseteq \delta \). Thus \( \text{Supp}^c(k[K]) \subseteq K \).

On the other hand if \( \sigma \not\subseteq \delta \) then not all the \( t_i \) that make up \( t_\sigma \) are inverted and so \( t_\sigma R_\sigma \otimes R_\delta(\bar{0}) = 0 \). This implies that if for every \( \sigma \not\in K \), \( \sigma \not\subseteq \delta \) then \( \Sigma_{\sigma \not\in K} f_\sigma \otimes R_\delta \) is not surjective, \( k[K] \otimes R_\delta \not= 0 \) and so \( \delta \in \text{Supp}^c(k[K]) \). Observed that any \( \delta \in K \) will have this property. Therefore \( K \subseteq \text{Supp}^c(k[K]) \), and the proof is complete.

Lemma 11.13. For any tensor closed Serre subcategory \( \mathcal{C} \) and \( R \)-module, \( M \)

\[
M \in \mathcal{C} \iff \text{Supp}(M) \subseteq \text{Supp}(\mathcal{C})
\]

(11.14)

Proof. If \( M \in \mathcal{C} \) then clearly \( \text{Supp}(M) \subseteq \text{Supp}(\mathcal{C}) \).

Next assume that \( \text{Supp}(M) \subseteq \text{Supp}(\mathcal{C}) \). Then for each \( p \in \text{Supp}(M) \) there is \( N \in \mathcal{C} \) such that

\[
p \in \text{Supp}(N).
\]

Then \( R/p \in \mathcal{C} \). Since \( \mathcal{C} \) is closed under extensions it is closed under finite direct sums. Then since \( \text{Supp}(M) \) is finite we get that \( \bigoplus_{p \in \text{Supp}(M)} R/p \in \mathcal{C} \). So from Lemma 11.6 \( M \in \mathcal{C} \).

Theorem 11.15. \( \text{Supp}^c \) induces a bijection (of sets)

\[
\text{tensor ideals in } R\text{-mod (or } R\text{-mod}_{\mathbb{Z}^m}) \to \text{simplicial complexes on } [m]
\]

with inverse \( K \mapsto S^\otimes(k[K]) \).
Proof. For a simplicial complex $K$ on $[m]$, we have

$$\text{Supp}^\circ(S^\otimes(k[K])) = \text{Supp}^\circ(k[K]) = K,$$

(11.16)

the first equality being Lemma 11.8 and the second being Lemma 11.12. We conclude that $\text{Supp} \circ S$ is the identity.

The equation also implies that for any tensor closed Serre subcategory $\mathcal{C}$,

$$\text{Supp}\mathcal{C} = \text{Supp}S^\otimes(k[\text{Supp}\mathcal{C}]).$$

(11.17)

Lemma 11.13 then shows that $M \in \mathcal{C}$ if and only if $M \in S\text{Supp}\mathcal{C}$, which shows $S \circ \text{Supp}$ is also the identity and the proof of the theorem has been completed. \hfill \Box

12. Simples

For this section $K$ is again any simplicial complex on $[m]$. In this section we will classify the simples in $\mathcal{L}(K)$ and use that classification to show that when $K$ is a skeleton of $\Delta^{m-1}$, $\mathcal{L}(K)$ is obtained from $R\text{-mod}$ by iteratively quotienting out the simples.

A subset $\sigma \subset [m]$ is a minimal missing face of $K$ if $\sigma \not\subset K$ and for any $\tau \subset \sigma$, $\tau \in K$. For $K = \emptyset$, $\emptyset$ is considered a minimal missing face.

For a minimal missing face $\sigma$ of $K$, consider the $R_\sigma$-module, $S(\sigma) = R_\sigma/(t_i)_{i \in \sigma}$. Let $S(\sigma) \in \mathcal{L}(K)$ be given by the formula

$$S(\sigma)_\tau = \begin{cases} R_\sigma/(t_i)_{i \not\in \sigma} & \text{if } \tau = \sigma \\ 0 & \text{otherwise}. \end{cases}$$

(12.1)

The following alternative way of looking at $S(\sigma)$ also shows the formulas define an element of $\mathcal{L}(K)$.

Lemma 12.2. For a minimal missing face $\sigma \in K$, $S(\sigma) \cong L_KR/(t_i)_{i \not\in \sigma}$.

Lemma 12.3. For any minimal missing face $\sigma$ of $K$ and $\vec{d} \in \mathbb{N}^m$, $t^{\vec{d}}S(\sigma) \in \mathcal{L}(K)$ is simple.

Proof. Observe that $S(\sigma)_\sigma = R_\sigma/(t_i)_{i \not\in \sigma}$ is a simple $R_\sigma$-module. Using Lemma 3.10 this implies that $S(\sigma)$ is simple, and similarly $t^{\vec{d}}S(\sigma) \in \mathcal{L}(K)$ is simple. \hfill \Box

Next we show that these are all the simples in $\mathcal{L}(K)$.

Lemma 12.4. Any simple in $\mathcal{L}(K)$ is isomorphic to $t^{\vec{d}}S(\sigma)$ for some minimal missing face $\sigma$ and some $\vec{d} \in \mathbb{N}^m$.

Proof. Suppose $M \in \mathcal{L}(K)$ is simple. For some $\sigma$ we have that $M_\sigma \neq 0$. By taking points away from $\sigma$ if necessary we can get a minimal missing face $\tau$ with $\tau \subset \sigma$. Since $M_\sigma \cong M_\tau \otimes_{R_\sigma} R_\sigma$, we get that $M_\tau \neq 0$.

Next let $x \in M_\tau$ be a non-zero element. For $i \not\in \tau$, $t_iM_\tau$ is a proper submodule of $M_\tau$ and so $t_iM$ is a proper submodule of $M$. Since $M$ is simple this proper submodule must be the 0 module, and thus $t_iM = 0$ and $t_ix = 0$.

Suppose $x \in M_\tau(\vec{d})$ (recall that means $x$ is in multidegree $\vec{d}$). Then consider the map $t^{\vec{d}}R_\sigma/(t_i)_{i \not\in \sigma} \to M_\tau$ that sends 1 to $x$. Since $x \neq 0$, this extends to a non-zero map $t^{\vec{d}}S(\tau) \to M$, which must be an isomorphism since both $t^{\vec{d}}S(\tau)$ and $M$ are simple. \hfill \Box

The following definition coincides with that in [Gab62, §IV.1] for categories of finitely generated modules and the categories $\mathcal{L}(K)$.

Definition 12.5. Given an abelian category $\mathcal{A}$, we let $\mathcal{A} = \mathcal{A}(-1)$ and $\mathcal{A}(i + 1) = \mathcal{A}(i)/S(\mathcal{A}(i))$ where $S(\mathcal{A})$ is the Serre subcategory generated by the simples in $\mathcal{A}$. The Krull dimension of $\mathcal{A}$, $\text{Kdim}(\mathcal{A})$ is the least $n$ such that $\mathcal{A}(n)$ is equivalent to the 0-category (i.e. the abelian category with one object).
Example 12.6. The trivial abelian category 0 has Krull dimension −1. We have Kdim(vect_k) = 0 and Kdim(k[t_1, \ldots, t_m]-mod) = m.

Proposition 12.7. The category L(K) has Krull dimension Kdim(L(K)) = m − s where s is the smallest cardinality of a missing face of K.

In other words, Kdim(L(K)) is the largest number of non-inverted variables t_i appearing in the rings R_σ for missing faces σ ∈ K^e.

Proof. Using Lemma 12.4, the simples in S(L(K)) are t^dS(σ) where σ runs over the minimal missing faces of K. Let K' denote K together with its minimal missing faces. By Lemma 11.10 L(K) ≅ R-mod/S^0(⊕_{σ∈K} R/p), which yields

L(K)/S(L(K)) ≅ R-mod/S^0(⊕_{σ∈K'} R/p) = L(K').

This implies that for n = m − s, L(K)(n) ≅ L(Δ^{m−1}) ≅ 0 and so Kdim(L(K)) ≤ n. Also we have L(K)(n−1) ≅ L(∂Δ^{m−1}) ≅ vect_k ≠ 0,

which shows Kdim(L(K)) ≥ n, hence Kdim(L(K)) = n. □

Corollary 12.8. For the simplicial complex K = sk_i Δ^{m−1} with −2 ≤ i ≤ m − 1, the category L(K) is obtained from R-mod by iteratively quotienting out the simples i + 2 times. In particular for K_m = sk_{m−3} Δ^{m−1}, L(K_m) is obtained from R-mod by quotienting out the simples m − 1 times.

Remark 12.9. In the setup of [BSS22], if we give the simples weight 1, the Serre subcategory generated by the simples consists of the objects with finite distance from the 0 object.

References

[AS05] D. M. Arnold and D. Simson, Endo-wild representation type and generic representations of finite posets, Pacific J. Math. 219 (2005), no. 1, 1–26, DOI 10.2140/pjm.2005.219.1. MR2174218

[ABE+22] H. Asashiba, M. Buchet, E. G. Escolar, K. Nakashima, and M. Yoshiwaki, On interval decomposability of 2D persistence modules, Comput. Geom. 105 (2022), Paper No. 101879, 33, DOI 10.1016/j.comgeo.2022.101879. MR4402576

[ASS06] I. Assem, D. Simson, and A. Skowroński, Elements of the representation theory of associative algebras. Vol. 1, London Mathematical Society Student Texts, vol. 65, Cambridge University Press, Cambridge, 2006. Techniques of representation theory. MR2197389

[AM69] M. F. Atiyah and I. G. Macdonald, Introduction to commutative algebra, Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969. MR0242802

[BBH22] B. Blanchette, T. Brüstle, and E. J. Hanson, Homological approximations in persistence theory (2022), Preprint, available at arXiv:2112.07632.

[Bor94] F. Borceux, Handbook of Categorical Algebra 2: Categories and Structures, Encyclopedia of Mathematics and its Applications, vol. 51, Cambridge University Press, 1994.

[BBCR20] M. B. Botnan and W. Crawley-Boevey, Decomposition of persistence modules, Proc. Amer. Math. Soc. 148 (2020), no. 11, 4581–4596, DOI 10.1090/proc/14790. MR4143378

[BLO20a] M. B. Botnan, V. Lebovici, and S. Oudot, On rectangle-decomposable 2-parameter persistence modules, 36th International Symposium on Computational Geometry, LIPIcs. Leibniz Int. Proc. Inform., vol. 164, Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2020, pp. Art. No. 22, 16. MR4117735

[BLO20b] ________, Local characterizations for decomposability of 2-parameter persistence modules (2020), Preprint, available at arXiv:2008.02345.

[BOO22] M. B. Botnan, S. Oppermann, and S. Oudot, Signed barcodes for multi-parameter persistence via rank decompositions, 38th International Symposium on Computational Geometry, LIPIcs. Leibniz Int. Proc. Inform., vol. 224, Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2022, pp. Paper No. 19, 18, DOI 10.4230/lipics.socg.2022.19. MR4470898

[BL23] M. B. Botnan and M. Lesnick, An Introduction to Multiparameter Persistence (2023), To appear in the proceedings of ICRA 2020, available at arXiv:2203.14289.
Meta-Diagrams for 2-Parameter Persistence

[CDMW23] N. Clause, T. K. Dey, F. Mémoli, and B. Wang, The Shift-Dimension of Multipersistence Modules (2021), Preprint, available at arXiv:2112.06509.

[HOST19] H. A. Harrington, N. Otter, H. Schenck, and U. Tillmann, The Theory of Multiparameter Persistent Homology, Proceedings of the 39th International Symposium on Computational Geometry (SoCG 2023) (2023), available at arXiv:2303.08270.

[Cox95] D. A. Cox, The homogeneous coordinate ring of a toric variety, J. Algebraic Geom. 4 (1995), no. 1, 17–50. MR1299003

[CLS11] D. A. Cox, J. B. Little, and H. K. Schenck, Toric varieties, Graduate Studies in Mathematics, vol. 124, American Mathematical Society, Providence, RI, 2011. MR2810322

[CDW23] O. Gafvert and W. Chachólski, Stable Invariants for Multiparameter Persistence, Preprint, available at arXiv:2112.06509.
[SS07b] Elements of the representation theory of associative algebras. Vol. 3, London Mathematical Society Student Texts, vol. 72, Cambridge University Press, Cambridge, 2007. Representation-infinite tilted algebras. MR2382332

[TSPA18] The Stacks Project Authors, Stacks Project, 2018.

[Vip20] O. Vipond, Multiparameter persistence landscapes, J. Mach. Learn. Res. 21 (2020), Paper No. 61, 38. MR4095340

[Zha19] Y. Zhang, Decomposition of Certain Representations Into A Direct Sum of Indecomposable Representations, Apr. 2019. Thesis (M.Sc.)–University of Regina.