Exponentially Convergent Receding Horizon Strategy for Constrained Optimal Control

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Abstract
Receding horizon control has been a widespread method in industrial control engineering as well as an extensively studied subject in control theory. In this work, we consider a lag $L$ receding horizon strategy that applies the initial $L$ optimal controls from each quadratic program to each receding horizon. We investigate a discrete-time and time-varying linear-quadratic optimal control problem that includes a nonzero reference trajectory and constraints on both state and control. We prove that, under boundedness and controllability conditions, the solution obtained by the receding horizon strategy converges to the solution of the full problem interval exponentially fast in the length of the receding horizon for some lag $L$. The exponential rate of convergence provides a systematic way of choosing the receding horizon length given a desired accuracy level. We illustrate our theoretical findings using a small, synthetic production cost model with real demand data.

Keywords Constrained optimal control · Receding horizon control · Sensitivity analysis

Mathematics Subject Classification (2010) 49N10 · 49N35 · 49Q12

1 Introduction
Receding horizon control (RHC), also known as model predictive control, has been a widely used feedback strategy in various industrial control applications, e.g., [1, 3, 7, 8, 27]. RHC can be applied to a broad class of optimal control problems, including those with nonlinear dynamics, time-delay systems, and constraints on state and control [18, 19]. The essence of RHC is to obtain the current control action by solving an optimal control problem defined...
on a finite horizon extending from the current time point \( k \). The finite-horizon problem uses the current state of the system as its initial state, yields sequences of optimal controls and states, applies the optimal control at time point \( k \) to the system, and uses the optimal state at \( k + 1 \) as the initial state of the next receding horizon problem. This on-line feature of RHC makes the method adaptive to changing system parameters, since only a finite horizon extending into the future is required for the current control [18], and particularly attractive when off-line computation of the control policy is difficult [22].

Several results prove the stability of RHC for constrained linear and nonlinear systems. For example, reference [21] proves that RHC yields an asymptotically stable closed-loop system for continuous-time nonlinear systems. In addition, reference [15] establishes stability of RHC for discrete-time, time-varying, constrained nonlinear systems. Both references employ the value function as a Lyapunov function for the stability analysis. Stability results for linear systems can be found, for example, in [16, 17, 24, 25]. More recently, studies [9–11, 26] have shown stability conditions and performance estimates without stabilizing terminal constraints or terminal costs under suitable controllability conditions for both discrete and continuous-time systems. In addition, [5] extends the previous studies by taking into account the state and control constraints.

In this work, we consider a slight variation of the standard RHC described in the references above. In particular, on each receding horizon, instead of applying the optimal control at only the current time point \( k \), we apply the optimal controls at the initial \( L \) time points for some lag \( L > 0 \), and the next receding horizon starts at time point \( k + L \). The same receding horizon strategy is considered in other references for continuous time, in order to account for the sampling time; see, for example, [13, 14]. Building on our recent analytical developments in [30], we prove that for an inequality-constrained time-varying linear system with a quadratic cost function and nonzero reference trajectory, the optimal states and controls obtained by this receding horizon strategy converge, for some lag \( L \), to the solutions on the full problem interval exponentially fast in the length of the receding horizon. The appropriate lag \( L \) in the result is determined by the problem parameters and, in particular, by the controllability properties of the constrained system. Our analysis directly investigates the solutions of a related equality-constrained control problem and connects them to the original inequality-constrained problem through a sensitivity analysis. Specifically, we consider the following problem.

\[
\begin{align*}
\min & \quad \frac{1}{2} \sum_{k=n_1}^{n_2-1} u_k^T R_k u_k + (x_k - d_k)^T Q_k (x_k - d_k) + (x_{n_2} - d_{n_2})^T Q_{n_2} (x_{n_2} - d_{n_2}) \quad (1a) \\
\text{s.t.} & \quad x_{k+1} = A_k x_k + B_k u_k, \quad n_1 \leq k \leq n_2 - 1, \quad x_{n_1} = x_{n_1}^0, \quad (1b) \\
& \quad \tilde{P}_{k+1} x_{k+1} + \tilde{C}_k u_k \geq \tilde{q}_k, \quad n_1 \leq k \leq n_2 - 1. \quad (1c)
\end{align*}
\]

In (1), we refer to \( x_k, u_k, \) and \( d_k \) as the state, control, and reference trajectory, respectively. Problem (1) lacks nonlinear dynamics, which is studied in some stability analyses of RHC, for example, [13, 21]. However, we include the inequality path constraint (1c) of state and control as considered in [24, 25]. Furthermore, we allow a nonzero reference trajectory \( d_k \) and prove the exponential convergence of RHC solutions to the solutions of problem (1) instead of a fixed equilibrium point [13, 14].

In our proofs, we use two important results in optimal control theory. One is developed in [31], where the authors prove that for an unconstrained, switched-time, and discrete-time linear-quadratic optimal control problem, the optimal trajectory stabilizes exponentially under some mild conditions. They also give an estimate of the exponential rate, which we
use here. The other one is established in [29], where the authors propose a Riccati-based approach for solving linear-quadratic optimal control problems subject to linear equality path constraints. They derive a solution procedure based on solving the KKT conditions via the Riccati recursion. We borrow similar manipulations and reductions of the KKT conditions here. Moreover, a few results regarding the Riccati recursion, closed-loop matrix, and sensitivity analysis are nearly the same as those in our previous work [30]. We present those proofs in the Appendix. We note that our previous work [30] had more complex algebra since it did not use the KKT-based ideas from [29], had bound constraints on control only, and did not investigate RHC convergence; these features are present in this work.

The rest of the article is organized as follows. In Section 2, we consider an equality constrained subproblem of (1) and investigate the dependences of solutions on the initial state and terminal reference. In Section 3, we define the lag \( L \) receding horizon strategy and prove the exponential convergence of RHC solutions based on results derived in Section 2. In Section 4, we demonstrate our theoretical findings using a synthetic production cost model with real demand data.

## 2 Path-constrained Linear-quadratic Problem

In this section, we mainly consider a subproblem of the constrained linear-quadratic optimal control problem (1). For (1), we have that \( A_k \in \mathbb{R}^{n \times n}, B_k \in \mathbb{R}^{n \times m}, \tilde{P}_k \in \mathbb{R}^{r \times n}, \tilde{C}_k \in \mathbb{R}^{r \times m}, \) and \( Q_k \in \mathbb{R}^{n \times n}, R_k \in \mathbb{R}^{m \times m} \) are positive definite. We make the following uniform boundedness assumption.

**Assumption 1** For any \( n_1, n_2, \) and \( n_1 \leq k \leq n_2, \) we assume that

\begin{align*}
(\text{a}) \quad & \|A_k\|_2 \leq \tilde{C}_A, \quad \|B_k\|_2 \leq \tilde{C}_B, \quad \|Q_k\|_2 \leq \tilde{C}_Q, \quad \|R_k\|_2 \leq C_R, \quad \|\tilde{P}_k\|_2 \leq C_P, \quad \|\tilde{C}_k\|_2 \leq C_C, \\
(\text{b}) \quad & \lambda_{\text{min}}(Q_k) \geq \lambda_Q > 0, \quad \lambda_{\text{min}}(R_k) \geq \lambda_R > 0.
\end{align*}

Note that we use symbols with a tilde for some upper bounds in Assumption 1 (a), since we reserve the corresponding straight symbols for the frequently used quantities defined later in Lemma 1. The subproblem of (1) we investigate is an equality-constrained problem obtained by considering some active subsets of the polyhedral path constraint (1c).

### 2.1 Equality-constrained Subproblem

To define the equality-constrained subproblem, we let \( I_k \subset \{1, \ldots, r\} \) be some index set of the constraint (1c) attaining the bound. Let \( P_k = \tilde{P}_k(I_k, :) \) and \( C_k = \tilde{C}_k(I_k, :) \) be the corresponding submatrices, and denote \( q_k = \tilde{q}_k(I_k) \). Then the equality constraint corresponding to the sequence of index sets \( \mathcal{I} \triangleq \{I_k\} \) is \( P_{k+1}x_{k+1} + C_ku_k = q_k \). The equality-constrained problem we consider is hence the following:

\begin{alignat}{2}
\min \quad & \frac{1}{2} \sum_{k=n_1}^{n_2-1} u_k^T R_k u_k + (x_k - d_k)^T Q_k (x_k - d_k) + (x_{n_2} - d_{n_2})^T Q_{n_2} (x_{n_2} - d_{n_2}) \quad (2a) \\
\text{s.t.} \quad & x_{k+1} = A_k x_k + B_k u_k, \quad n_1 \leq k \leq n_2 - 1, \quad x_{n_1} = x_0, \quad (2b) \\
& E_k x_k + H_k u_k = q_k, \quad n_1 \leq k \leq n_2 - 1. \quad (2c)
\end{alignat}
where we denote
\[ E_k = P_{k+1}A_k, \quad H_k = P_{k+1}B_k + C_k. \quad (3) \]

Note that if \( I_k = \emptyset \), then we have \( P_k = 0 \), \( C_k = 0 \), \( q_k = 0 \), and hence \( E_k = 0 \), \( H_k = 0 \). If \( \mathcal{I} \) is the active set of problem (1) at optimality, then problems (1) and (2) have the same solutions. Note that \( H_k \) in (3) is determined by the sequence of index sets \( \mathcal{I} \) encoding the equality constraints under consideration, and hence we define the following uniform boundedness property of \( H_k \) in terms of the index sets.

**Definition 1** Given a sequence of index sets, let \( H_k \) be as in (3). With some \( \lambda_H > 0 \), the sequence of index sets is uniformly bounded from below with respect to \( \lambda_H \), denoted as \( \text{UDB}(\lambda_H) \), if for any \( n_1 \leq k \leq n_2 \) and \( I_k \neq \emptyset \), \( H_k \) has full row rank and
\[ \lambda_{\text{min}}(H_k H_k^T) \geq \lambda_H > 0. \]

From (3), the number of rows of \( H_k \) equals to the number of equality constraints corresponding to the index set \( I_k \). Therefore, by requiring \( H_k \) to have full row rank and be uniformly bounded from below, Definition 1 restricts the total number of equality path constraints for a sequence of index sets that is \( \text{UDB}(\lambda_H) \). In the rest of this subsection, we restrict our attention to the index sets that are \( \text{UDB}(\lambda_H) \). First, we define some matrices frequently used in the subsection.

**Definition 2** For some sequence of index sets that is \( \text{UDB}(\lambda_H) \), we define the following matrices for \( n_1 \leq k \leq n_2 - 1 \):
\[
\begin{align*}
\hat{H}_k &= \begin{cases} (H_k R_k^{-1} H_k^T)^{-1}, & I_k \neq \emptyset \\ 0, & I_k = \emptyset \end{cases}, \\
\hat{Q}_k &= Q_k + E_k^T \hat{H}_k E_k, \\
\hat{A}_k &= A_k - B_k R_k^{-1} H_k^T \hat{H}_k E_k, \\
\hat{B}_k &= B_k - B_k R_k^{-1} H_k^T \hat{H}_k H_k R_k^{-1}, \\
\hat{X}_k &= R_k^{-1} - R_k^{-1} H_k^T \hat{H}_k H_k R_k^{-1}, \\
\hat{q}_k &= B_k R_k^{-1} H_k^T \hat{H}_k q_k.
\end{align*}
\]

Note that \( \hat{A}_k \) and \( \hat{B}_k \) are modifications, inspired from [29], of \( A_k \) and \( B_k \), respectively, by taking into account the equality constraints (2c) as determined by the index sets. The rationale for Definition 2 will be made clear later in Lemma 4, which investigates the KKT conditions. To prepare for that, we first derive some properties for the matrices defined.

**Lemma 1** Under Assumption 1, for any \( n_1, n_2 \), and \( n_1 \leq k \leq n_2 \), if the sequence of index sets \( \mathcal{I} \) is \( \text{UDB}(\lambda_H) \), then we have
\[
\begin{align*}
\| \hat{A}_k \|_2 &\leq C_A, \\
\| \hat{B}_k \|_2 &\leq C_B, \\
\| \hat{Q}_k \|_2 &\leq C_Q, \\
\| \hat{H}_k \|_2 &\leq C_H, \\
\| \hat{E}_k \|_2 &\leq C_E, \\
\| \hat{H}_k \|_2 &\leq C_{\hat{H}},
\end{align*}
\]
for some \( C_A, C_B, C_Q, C_H, C_E, C_{\hat{H}} > 0 \) independent of \( n_1, n_2 \), and the particular choice of \( \mathcal{I} \). Here \( \lambda_Q \) is the same as that in Assumption 1.
Proof From Assumption 1 and Definition 2, we have
\[
\|H_k\|_2 \leq \|P_{k+1}B_k\|_2 + \|C_k\|_2 \leq CP \tilde{C}_B + C_C \triangleq C_H,
\]
\[
\|E_k\|_2 \leq \|P_{k+1}A_k\|_2 \leq CP \tilde{C}_A \triangleq C_E,
\]
\[
\|\hat{H}_k\|_2 \leq (\lambda_H/C_R)^{-1} \triangleq C_{\hat{H}},
\]
\[
\|\hat{A}_k\|_2 \leq \tilde{C}_A + \tilde{C}_B C_H C_E/\lambda_R \triangleq C_A,
\]
\[
\|\hat{B}_k\|_2 \leq \tilde{C}_B + \tilde{C}_B^2 C_H^2/\lambda_R \triangleq C_B,
\]
\[
\|\hat{Q}_k\|_2 \leq \tilde{C}_Q + C^2_E C_H \triangleq C_Q,
\]
and \(\lambda_{\min}(\hat{Q}_k) \geq \lambda_{\min}(Q) \geq \lambda_Q\).
\]

Note that throughout the article, we use the notations \(A \succeq B\) and \(A \succ B\) to mean \(A - B\) is symmetric positive semidefinite and symmetric positive definite, respectively.

Lemma 2 For \(n_1 \leq k \leq n_2 - 1\), if the sequence of index sets \(\mathcal{I}\) is UDB(\(\lambda_H\)), then we have \(\hat{R}_k = \hat{B}_k R_k^{-1} \hat{B}_k\), and hence \(R_k \succeq 0\).

Proof Since \(X_k = R_k^{-1} - R_k^{-1} H_k^T \hat{H}_k H_k R_k^{-1}\), we have \(X_k^T = X_k\) and
\[
X_k^T R_k X_k = R_k^{-1} - 2R_k^{-1} H_k^T \hat{H}_k H_k R_k^{-1} + R_k^{-1} H_k^T \hat{H}_k (H_k R_k^{-1} H_k^T) \hat{H}_k H_k R_k^{-1} \tag{4}
\]
\[
\overset{\text{Def. 2}}{=} R_k^{-1} - 2R_k^{-1} H_k^T \hat{H}_k H_k R_k^{-1} + R_k^{-1} H_k^T \hat{H}_k \hat{H}_k R_k^{-1} H_k R_k^{-1} = X_k.
\]

Note that \(\hat{B}_k = B_k X_k R_k\) from Definition 2, so we have
\[
\hat{R}_k = B_k X_k \hat{B}_k \overset{(4)}{=} B_k X_k R_k X_k B_k = (B_k X_k R_k) R_k^{-1} (R_k X_k B_k^T) = \hat{B}_k R_k^{-1} \hat{B}_k^T.
\]

Since \(R_k \succ 0\), we have that \(\hat{R}_k \succeq 0\).

To prove the results in this subsection, we employ the approach in [29] by considering the KKT conditions of problem (2). We first define and derive properties for the following matrices, some of which are similar to those in [29].

Definition 3 For some sequence of index sets that is UDB(\(\lambda_H\)), define the following backward recursions for \(n_1 \leq k \leq n_2 - 1\):
\[
K_{n_2} = Q_{n_2}, \quad T_{n_2} = -Q_{n_2} d_{n_2}, \tag{5a}
\]
\[
W_k = R_k + \hat{B}_k^T K_{k+1} \hat{B}_k, \tag{5b}
\]
\[
M_k = (I + \hat{R}_k K_{k+1})^{-1}, \tag{5c}
\]
\[
D_k = M_k \hat{A}_k, \tag{5d}
\]
\[
T_k = D_k^T T_{k+1} + \hat{A}_k^T K_{k+1} M_k \hat{q}_k - E_k^T \hat{H}_k q_k - Q_k d_k, \tag{5e}
\]
\[
K_k = \hat{Q}_k + \hat{A}_k^T K_{k+1} D_k. \tag{5f}
\]
Lemma 3 For \( n_1 \leq k \leq n_2 - 1 \), if the sequence of index sets \( \mathcal{I} \) is UDB(\( \lambda_H \)), then \( M_k \) is well defined, \( K_k \succ 0 \), \( W_k \succ 0 \), \( K_{k+1}M_k \succ 0 \), and
\[
M_k^T = I - K_{k+1}M_k \hat{R}_k. 
\] (6)

Proof We prove the statement by backward induction based on (5a)–(5f). To start, we have \( K_{n_2} = Q_{n_2} \succ 0 \) as the induction basis. Suppose \( K_{k+1} \) is positive definite. Then we have \( W_k \succ R_k \succ 0 \), and
\[
I + \hat{R}_k K_{k+1} = (K_{k+1}^{-1} + \hat{R}_k)K_{k+1}. 
\]
Since \( \hat{R}_k \succeq 0 \) as shown in Lemma 2, we have that \( I + \hat{R}_k K_{k+1} \) is invertible and hence \( M_k \) is well defined. Also we have
\[
K_{k+1}M_k^{(5c)} = K_{k+1}(I + \hat{R}_k K_{k+1})^{-1} = (I + \hat{R}_k K_{k+1})K_{k+1}^{-1} = (K_{k+1}^{-1} + \hat{R}_k)^{-1} \succ 0,
\]
which implies that
\[
K_k^{(5f),(5d)} = \hat{Q}_k + \hat{A}_k^T (K_{k+1}M_k) \hat{A}_k \succeq \hat{Q}_k \succeq Q_k \succ 0,
\]
so that \( K_k \) is positive definite. By induction we have that \( M_k \) is well defined, \( K_k \succ 0 \), \( W_k \succ 0 \), and \( K_{k+1}M_k \succ 0 \) for all \( n_1 \leq k \leq n_2 - 1 \).

Note that since \( K_k \) is symmetric,
\[
M_k^{-T}(I - K_{k+1}M_k \hat{R}_k) \overset{(5e)}{=} (I + K_{k+1} \hat{R}_k)(I - K_{k+1}M_k \hat{R}_k)
\]
\[
= I + K_{k+1}(I - M_k - \hat{R}_k K_{k+1}M_k) \hat{R}_k
\]
\[
\overset{(5f)}{=} I + K_{k+1}(I - M_k^{-1}M_k) \hat{R}_k
\]
\[
= I.
\]
Therefore (6) holds. \( \square \)

Note that (6) is also stated (without proof) in [29]. Now we derive a recursion of the optimal states of problem (2) by investigating the KKT conditions.

Lemma 4 Let \( u_k^* \) and \( x_k^* \) be the optimal controls and states of problem (2), and let \( \lambda_k^* \) and \( \eta_k^* \) be the Lagrange multipliers associated with the dynamical constraint (2b) and the equality constraint (2c), respectively. For \( n_1 \leq k \leq n_2 - 1 \), if the sequence of index sets \( \mathcal{I} \) is UDB(\( \lambda_H \)), then we have
\[
u_k^* = R_k^{-1}(H_k^T \eta_k^* - B_k^T \lambda_k^*),
\]
(7a)
\[
\eta_k^* = \hat{H}_k(-E_k x_k^* + H_k R_k^{-1} B_k^T \lambda_k^* + q_k),
\]
(7b)
\[
\lambda_k^* = K_{k+1}x_k^* + T_{k+1},
\]
(7c)
\[
x_{k+1}^* = D_k x_k^* - M_k \hat{R}_k T_{k+1} + M_k \hat{q}_k,
\]
(7d)
where \( K_k \) and \( T_k \) are given by the backward recursions (5a)–(5f).
Proof The KKT conditions of problem (2) are

\begin{align}
R_k u^*_k + B_k^T \lambda^*_k - H_k^T \eta^*_k &= 0, \quad n_1 \leq k \leq n_2 - 1, \quad (8a) \\
Q_k(x^*_k - d_k) + A_k^T \lambda^*_k - \lambda^*_{k-1} - E_k^T \eta^*_k &= 0, \quad n_1 + 1 \leq k \leq n_2, \quad (8b) \\
Q_{n_2}(x^*_{n_2} - d_{n_2}) - \lambda^*_{n_2-1} &= 0, \quad (8c) \\
x^*_{k+1} = A_k x^*_k + B_k u^*_k, \quad n_1 \leq k \leq n_2 - 1, \quad (8d) \\
E_k x^*_k + H_k u^*_k &= q_k, \quad n_1 \leq k \leq n_2 - 1. \quad (8e)
\end{align}

Condition (8a) directly gives (7a). Substituting (7a) into (8e) gives

\[ E_k x^*_k + H_k R_k^{-1}(H_k^T \eta^*_k - B_k^T \lambda^*_k) = q_k, \]

\[ \Rightarrow \ E_k x^*_k + (H_k R_k^{-1}H_k^T) \eta^*_k - H_k R_k^{-1}B_k^T \lambda^*_k = q_k, \]

and this gives (7b) from Definition 2. Substituting (7a) and (7b) into (8d) gives

\[ x^*_{k+1} = A_k x^*_k + B_k R_k^{-1}(H_k^T \hat{\eta}_k - B_k^T \lambda^*_k) \]

\[ = A_k x^*_k + B_k R_k^{-1}(H_k^T \hat{\eta}_k - E_k x^*_k + H_k R_k^{-1}B_k^T \lambda^*_k + q_k) - B_k^T \lambda^*_k \]

\[ = (A_k - B_k R_k^{-1}H_k^T \hat{\eta}_k) x^*_k - B_k (R_k^{-1} - R_k^{-1} H_k^T \hat{\eta}_k H_k R_k^{-1}) B_k^T \lambda^*_k + B_k R_k^{-1} H_k^T \hat{\eta}_k q_k \]

\[ \text{Def. 2} \Rightarrow \hat{A}_k x^*_k - \hat{R}_k \lambda^*_k + \hat{q}_k. \quad (9) \]

Substituting (7b) into (8b) gives

\[ Q_k(x^*_k - d_k) + A_k^T \lambda^*_k - \lambda^*_{k-1} - E_k^T \hat{\eta}_k(-E_k x^*_k + H_k R_k^{-1}B_k^T \lambda^*_k + q_k) = 0 \]

\[ \Rightarrow (Q_k + E_k^T \hat{\eta}_k E_k)x^*_k - Q_k d_k + (A_k - B_k R_k^{-1}H_k^T \hat{\eta}_k E_k)^T \lambda^*_k - \lambda^*_k - E_k^T \hat{\eta}_k q_k = 0. \]

From Definition 2, we then have, for \( n_1 \leq k \leq n_2 - 1 \),

\[ \lambda^*_k - 1 = \hat{A}_k x^*_k - \hat{R}_k K_{k+1} x^*_{k+1} - \hat{R}_k T_{k+1} + \hat{q}_k. \quad (10) \]

From (8c) we also have

\[ \lambda^*_{n_2-1} = Q_{n_2}(x^*_{n_2} - d_{n_2}). \quad (11) \]

We prove (7c) and (7d) by backward induction. The statement (7c) holds for \( k = n_2 - 1 \) from (11) and (5a). Suppose (7c) holds for \( \lambda^*_k \). Then substituting (7c) into (9) gives

\[ x^*_{k+1} = \hat{A}_k x^*_k - \hat{R}_k K_{k+1} x^*_{k+1} - \hat{R}_k T_{k+1} + \hat{q}_k, \]

which leads to

\[ (I + \hat{R}_k K_{k+1}) x^*_{k+1} = \hat{A}_k x^*_k - \hat{R}_k T_{k+1} + \hat{q}_k. \]

Therefore from (5c) we have

\[ x^*_{k+1} = M_k(\hat{A}_k x^*_k - \hat{R}_k T_{k+1} + \hat{q}_k), \quad (12) \]

and (7d) holds for \( x^*_{k+1} \). Then for \( \lambda^*_k \), using (10) and (7c), we have

\begin{align}
\lambda^*_k &= \hat{Q}_k x^*_k + \hat{A}_k (K_{k+1} x^*_{k+1} + T_{k+1}) - Q_k d_k - E_k^T \hat{\eta}_k q_k \\
&= (\hat{Q}_k + \hat{A}_k K_{k+1} M_k \hat{A}_k)x^*_k + \hat{A}_k (I - K_{k+1} M_k \hat{R}_k) T_{k+1} + \hat{A}_k K_{k+1} M_k \hat{q}_k - Q_k d_k - E_k^T \hat{\eta}_k q_k \\
&= (\hat{Q}_k + \hat{A}_k K_{k+1} D_k)x^*_k + \hat{A}_k M_k^T T_{k+1} + \hat{A}_k K_{k+1} M_k \hat{q}_k - Q_k d_k - E_k^T \hat{\eta}_k q_k \\
&= K_k x^*_k + T_k, \quad (5f),(5d),(5e)
\end{align}
and hence (7c) holds for $\lambda_{k-1}^*$. 

In the following, we investigate the notion of controllability for the system (2b)–(2c). We define the following controllability matrix for the sequence of index sets $I$ in terms of the sequence pair $\{\hat{A}_k, \hat{B}_k\}$ in Definition 2.

**Definition 4** For some $n_1 \leq q \leq n_2$, $t > 0$, and some sequence of index sets $I$ that is UDB($\lambda_H$), define the controllability matrix associated with time steps $[q, q + t - 1]$ as

$$C_{q,t}(I) = \left[ \hat{B}_{q+t-1}, \hat{A}_{q+t-1}\hat{B}_{q+t-2}, \ldots, \left( \prod_{l=1}^{t-1} \hat{A}_l \right) \hat{B}_q \right].$$

To see the relationship between $C_{q,t}(I)$ and the controllability of the equality constrained system (2b)–(2c), we start by defining the notion of controllability for the system (2b)–(2c).

**Definition 5** Given a sequence of index sets, define $E_k$ and $H_k$ as in (3). At time step $q$, the system

$$x_{k+1} = A_kx_k + B_ku_k, \quad q \leq k \leq n_2 - 1, \quad x_q = x_q^0,$$

$$E_kx_k + H_ku_k = q_k, \quad q \leq k \leq n_2 - 1$$

is controllable in $t$ steps if for any $x_q^0$ and $\bar{x}$, there exist admissible controls $\{\bar{u}_k\}_{k=q+1:q+t-1}$ and corresponding states $\{\bar{x}_k\}_{k=q+1:q+t}$ satisfying (13) and $\bar{x}_{q+t} = \bar{x}$.

**Proposition 1** If the sequence of index sets $I$ is UDB($\lambda_H$) and the resulting constrained system (2b)–(2c) of problem (2) is controllable at time point $q$ in $t$ steps, then $C_{q,t}(I)$ has full row rank.

**Proof** The system (2b)–(2c) being controllable in $t$ steps implies that there exist admissible controls $\{\bar{u}_k\}_{k=q+1:q+t-1}$ and corresponding states $\{\bar{x}_k\}_{k=q+1:q+t}$ so that $\bar{x}_{q+t} = \bar{x}$ for any $\bar{x}$. Then we have, for $q \leq k \leq q + t - 1$,

$$\bar{x}_{k+1} = A_k\bar{x}_k + B_k\bar{u}_k \quad \text{(2c)}$$

$$\overset{\text{Def.}}{=} A_k\bar{x}_k + B_k\bar{u}_k - B_kR_k^{-1}H_k^T \hat{H}_k(E_k\bar{x}_k + H_k\bar{u}_k - q_k)$$

$$\overset{\text{Def.}}{=} \hat{A}_k\bar{x}_k + \hat{B}_k\bar{u}_k + \hat{q}_k,$$

which means that the same sequences $\{\bar{u}_k\}$ and $\{\bar{x}_k\}$ also satisfy the linear dynamics

$$x_{k+1} = \hat{A}_kx_k + \hat{B}_ku_k + \hat{q}_k$$

and that $\bar{x}_{q+t} = \bar{x}$. In other words, (14) can be controlled in $t$ steps to $\bar{x}$. Since $\bar{x}$ is arbitrary, it follows that $C_{q,t}(I)$ has full row rank.

Proposition 1 connects the controllability of the equality-constrained system (2b)–(2c) to the full rank of a related controllability matrix $C_{q,t}(I)$. For our purpose, however, we need a uniform boundedness property of the controllability matrix, which is stronger than the standard assumption of merely full rankness.

**Definition 6** For some sequence of index sets $I$ that is UDB($\lambda_H$), let $\hat{A}_k$, $\hat{B}_k$ be as in Definition 2. With some $0 < t < n_2 - n_1$ and $\lambda_C > 0$, the sequence of index sets is uniformly completely controllable with respect to $\lambda_C$, denoted as UCC($\lambda_C$), if the sequence
pair \(\{\hat{A}_k, \hat{B}_k\}\) is uniformly completely controllable [15, Definition 3.1], i.e., for any \(n_1 \leq q \leq n_2\),

\[
\lambda_{\text{min}} \left( C_{q,t}(I)C_{q,t}^T(I) \right) \geq \lambda_C > 0.
\]

The main purpose of this subsection is to investigate the dependencies of the solutions of problem (2) on the initial value \(x_n^0\) and terminal reference \(d_{n_2}\) for some sequence of index sets that is UDB(\(\lambda_H\)) and UCC(\(\lambda_C\)). To start, we derive properties for the quantities defined in Definition 3. The proofs of the results regarding the Riccati matrix \(K_k\) and closed-loop matrix \(D_k\) are structurally the same as those in [30], and hence they are provided in the Appendix.

**Lemma 5** For \(n_1 \leq k \leq n_2 - 1\), if the sequence of index sets \(I\) is UDB(\(\lambda_H\)), we have

\[
\begin{align*}
M_k &= I - \hat{B}_k W_k^{-1} \hat{B}_k^T K_{k+1}, \\
K_k &= \hat{Q}_k + \hat{A}_k^T K_{k+1} \hat{A}_k - \hat{A}_k^T K_{k+1} \hat{B}_k W_k^{-1} \hat{B}_k^T K_{k+1} \hat{A}_k,
\end{align*}
\]

where \(M_k\), \(K_k\), and \(W_k\) are from Definition 3.

**Proof** Definition 3 and Lemma 2 imply that

\[
M_k = (I + \hat{B}_k R_k^{-1} \hat{B}_k^T K_{k+1})^{-1}.
\]

Then we have

\[
M_k^{-1} \left(I - \hat{B}_k W_k^{-1} \hat{B}_k^T K_{k+1}\right)
\]

\[
= \left(I + \hat{B}_k R_k^{-1} \hat{B}_k^T K_{k+1}\right) \left(I - \hat{B}_k W_k^{-1} \hat{B}_k^T K_{k+1}\right)
\]

\[
= I + \hat{B}_k R_k^{-1} \hat{B}_k^T K_{k+1} - \hat{B}_k W_k^{-1} \hat{B}_k^T K_{k+1} - \hat{B}_k R_k^{-1} \hat{B}_k^T K_{k+1} \hat{B}_k W_k^{-1} \hat{B}_k^T K_{k+1}
\]

\[
= I + \hat{B}_k \left(R_k^{-1} W_k - R_k^{-1} \hat{B}_k^T K_{k+1} \hat{B}_k\right) W_k^{-1} \hat{B}_k^T K_{k+1}
\]

\[
= I + \hat{B}_k \left(R_k^{-1} \left(W_k - \hat{B}_k^T K_{k+1} \hat{B}_k\right)\right) - I \right) W_k^{-1} \hat{B}_k^T K_{k+1}
\]

\[
= (5b) \quad I + \hat{B}_k \left(R_k^{-1} R_k - I\right) W_k^{-1} \hat{B}_k^T K_{k+1}
\]

\[
= I.
\]

Hence, \(M_k = I - \hat{B}_k W_k^{-1} \hat{B}_k^T K_{k+1}\). Substituting (15a) into (5d) and (5f), we have

\[
K_k = \hat{Q}_k + \hat{A}_k^T K_{k+1} \left(I - \hat{B}_k W_k^{-1} \hat{B}_k^T K_{k+1}\right) \hat{A}_k,
\]

which proves (15b).

**Proposition 2** Under Assumption 1, if the sequence of index sets \(I\) is UDB(\(\lambda_H\)) and UCC(\(\lambda_C\)), then for any \(n_1 \leq q \leq n_2\), we have \(\|K_q\|_2 \leq \beta\) for some \(\beta > 0\) independent of \(n_1, n_2\), and the particular choice of \(I\).

**Proof** See Appendix A.1; also see [30, Proposition 2.7].
Proposition 3 Under Assumption 1, for any \( q \leq j \leq n_2 - 1 \), if the sequence of index sets \( \mathcal{I} \) is UDB(\( \lambda_H \)) and UCC(\( \lambda_C \)), then we have

\[
\left\| \prod_{l=q}^{j} D_l \right\|_2 \leq C_1 \rho^{j-q+1},
\]

where \( C_1 = \sqrt{\beta/\lambda_Q} \), \( \rho = 1/\sqrt{1 + (\lambda_Q/\beta)} \), and \( C_1, \rho \) are independent of \( n_1, n_2 \), and the particular choice of \( \mathcal{I} \).

Proof See Appendix A.2; also see [30, Proposition 2.8].

In the following, we show the dependencies of the solutions to problem (2) on the initial state and terminal reference decay exponentially. To start, we prove a short lemma about the recursion defined in (5e).

Lemma 6 For \( n_1 + 1 \leq k \leq n_2 \), if the sequence of index sets \( \mathcal{I} \) is UDB(\( \lambda_H \)) and UCC(\( \lambda_C \)), then we have

\[
\| \nabla_{d_{n_2}} T_k \|_2 \leq C_s \rho^{n_2-k}
\]

for some \( C_s > 0 \) independent of \( n_1, n_2 \), and the particular choice of \( \mathcal{I} \).

Proof Recursion (5e) gives

\[
\nabla_{d_{n_2}} T_k = - \left( \prod_{l=k}^{n_2-1} D_l \right)^T Q_{n_2}.
\]

The statement is proved by using Proposition 3 and taking \( C_s = \tilde{C}_Q C_1 \) for \( \tilde{C}_Q \) defined in Assumption 1.

Proposition 4 Let \( x_k^* \) and \( u_k^* \) be the optimal states and controls of problem (2). Under Assumption 1, if the sequence of index sets \( \mathcal{I} \) is UDB(\( \lambda_H \)) and UCC(\( \lambda_C \)), then

\[
\| \nabla_{x_{n_1}} x_k^* \|_2 \leq Z_1 \rho^{k-n_1}, \quad \| \nabla_{d_{n_2}} x_k^* \|_2 \leq Z_2 \rho^{n_2-k}, \quad \| \nabla_{x_{n_1}} u_k^* \|_2 \leq Z_1 \rho^{k-n_1}, \quad \| \nabla_{d_{n_2}} u_k^* \|_2 \leq Z_2 \rho^{n_2-k},
\]

for some \( Z_1, Z_2 > 0 \) independent of \( n_1, n_2 \), and the particular choice of \( \mathcal{I} \).

Proof From Assumption 1 and Lemma 1 we have, for \( X_k \) and \( \hat{R}_k \) defined in Definition 2,

\[
\| X_k \|_2 \leq 1/\lambda_R + C_B^2 C_H/\lambda_R \overset{\Delta}{=} C_X, \quad \| \hat{R}_k \|_2 \leq \tilde{C}_Q^2 C_X \overset{\Delta}{=} \tilde{C}_R.
\]

Lemmas 5 and 1 and Proposition 2 give

\[
\| M_k \|_2 \leq 1 + C_B^2 \beta/\lambda_R \overset{\Delta}{=} C_M, \quad \| D_k \|_2 \leq \| M_k \|_2 \| \hat{A}_k \|_2 \leq C_M C_A \overset{\Delta}{=} C_D,
\]

where the first inequality uses the relation \( W_k \geq R_k \), which is given by (5b) and Lemma 3. From Lemma 4 we have

\[
x_{k+1}^* = D_k x_k^* - M_k \hat{R}_k T_{k+1} + M_k \hat{q}_k,
\]

(16)
so
\[
\|\nabla x_n^* x_k^*\|_2 = \left\| \prod_{l=1}^{k-1} D_l \right\|_2 \leq C_1 \rho^{k-n_1},
\] (17)
which follows from Proposition 3. Also, from (16) we have
\[
\nabla d_{n_2} x_{k+1}^* = D_k \left( \nabla d_{n_2} x_k^* \right) - M_k \hat{R}_k \left( \nabla d_{n_2} T_{k+1} \right)
\] (18)
and \(\nabla d_{n_2} x_{n_1} = 0\). From Recursion (18) we have
\[
\nabla d_{n_2} x_k^* = - \sum_{i=n_1+1}^{k} \left( \prod_{l=i}^{k-1} D_l \right) M_{i-1} \hat{R}_{i-1} \left( \nabla d_{n_2} T_i \right),
\]
from which, using Proposition 3 and Lemma 6, we have, for \(C \triangleq C_1 M C_R C_s\),
\[
\|\nabla d_{n_2} x_k^*\|_2 \leq \sum_{i=n_1+1}^{k} C_1 \rho^{k-i} M C_R C_s \rho^{n_2-i}
\]
\[
= C \rho^{n_2-k} \sum_{i=n_1+1}^{k} \rho^{2(k-i)}
\]
\[
\leq \frac{C}{1 - \rho^2} \rho^{n_2-k}.
\] (19)
Equation (7a) gives
\[
u_k^* = R^{-1}_k (H_k^T \eta_k^* - B_k^T \lambda_k^*)
\]
\[
\overset{(7b)}{=} R^{-1}_k H_k^T \hat{H}_k \left( -E_k x_k^* + H_k R^{-1}_k B_k^T \lambda_k^* + q_k \right) - R^{-1}_k B_k^T \lambda_k^*
\]
\[
\overset{\text{Def. 2}}{=} -R^{-1}_k H_k^T \hat{H}_k E_k x_k^* - X_k B_k^T \lambda_k^* + R^{-1}_k H_k^T \hat{H}_k q_k
\]
\[
\overset{(7c)}{=} -R^{-1}_k H_k^T \hat{H}_k E_k x_k^* + R^{-1}_k H_k^T \hat{H}_k q_k - X_k B_k^T (K_{k+1} x_{k+1}^* + T_{k+1})
\]
\[
\overset{(7d)}{=} -R^{-1}_k H_k^T \hat{H}_k E_k x_k^* + R^{-1}_k H_k^T \hat{H}_k q_k - X_k B_k^T K_{k+1} (D_k x_k^* - M_k \hat{R}_k T_{k+1} + M_k \hat{q}_k) - X_k B_k^T T_{k+1}
\]
\[
= -R^{-1}_k H_k^T \hat{H}_k E_k + X_k B_k^T K_{k+1} (D_k x_k^* - M_k \hat{R}_k) x_k^* + R^{-1}_k H_k^T \hat{H}_k q_k
\]
\[
- X_k B_k^T (I - K_{k+1} M_k \hat{R}_k) T_{k+1} - X_k B_k^T K_{k+1} M_k \hat{q}_k.
\]
In this expression, for the term multiplying \(x_k^*\) and \(T_{k+1}\), we have the following from Assumption 1 and Lemma 1, respectively:
\[
\|R^{-1}_k H_k^T \hat{H}_k E_k + X_k B_k^T K_{k+1} D_k\|_2 \leq C_H \hat{C}_H E / \lambda_R + C_X \tilde{C}_B \beta C_D \overset{\Delta}{=} Y_1,
\]
\[
\|X_k B_k^T (I - K_{k+1} M_k \hat{R}_k)\|_2 \leq C_X \tilde{C}_B (1 + \beta C_M C_R) \overset{\Delta}{=} Y_2.
\]
Note that from (5e), \(T_k\) does not depend on the initial value \(x_{n_1}\). Therefore we have
\[
\|\nabla x_n^* u_k^*\|_2 \leq Y_1 \left\| \nabla x_n^* x_k^* \right\|_2 \overset{(17)}{\leq} Y_1 C_1 \rho^{k-n_1}
\] (20)
and
\[
\|\nabla d_{n_2} u_k^*\|_2 \leq Y_1 \left\| \nabla d_{n_2} x_k^* \right\|_2 + Y_2 \left\| \nabla d_{n_2} T_{k+1} \right\|_2 \overset{(19), \text{Lemma 6}}{\leq} \frac{Y_1 C}{1 - \rho^2} \rho^{n_2-k} + \frac{Y_2 C_s}{\rho} \rho^{n_2-k}.
\] (21)
Considering (17), (19), (20), and (21) and letting
\[ Z_1 = \max (C_1, Y_1 C_1), \quad Z_2 = \max \left( \frac{C}{1 - \rho^2}, \frac{Y_1 C}{1 - \rho^2} + \frac{Y_2 C_s}{\rho} \right), \]
prove the statement.

Proposition 4 is the main result of this subsection. It shows that the effect of the initial state (or terminal reference) on the solutions of problem (2) decays exponentially fast in the time distance between the solution and the initial (or terminal) time point. Moreover, under the uniform boundedness Assumption 1, the decay rate is independent of the problem interval \([n_1, n_2]\), and the particular choice of the index set given it is UDB(\(\lambda_H\)) and UCC(\(\lambda_C\)). This property is essential for proving that a receding horizon strategy approximates the solution on the full horizon in Section 3. We now conclude this subsection with a boundedness result of the solutions and adjoint variables of problem (2).

Assumption 2 For any \(n_1, n_2\) and \(n_1 \leq k \leq n_2\), we have \(\|d_k\|_2 \leq m_0\) and \(\|\lambda_{n_1}^0\|_2 \leq u_0\).

Lemma 7 Let \(x_k^*\) and \(\lambda_k^*\) be the optimal states and adjoint variables of problem (2), respectively. Under Assumptions 1 and 2, if the sequence of index sets \(I\) is UDB(\(\lambda_H\)) and UCC(\(\lambda_C\)), then we have
\[ \|x_k^*\|_2 \leq C_g, \quad n_1 + 1 \leq k \leq n_2; \quad \|\lambda_k^*\|_2 \leq C_\lambda, \quad n_1 \leq k \leq n_2 - 1 \]
for some \(C_g, C_\lambda > 0\) independent of \(n_1, n_2\), and the particular choice of \(I\).

Proof In (5e), denote \(\overline{T}_{n_2} = T_{n_2}\), and for \(k < n_2\),
\[ \overline{T}_k = \hat{A}_k^T K_{k+1} M_k \hat{q}_k - E_k^T \hat{H}_k q_k - Q_k d_k, \]
so that \(T_k = D_k^T T_{k+1} + \overline{T}_k\) and \(T_{n_2} = -Q_{n_2} d_{n_2}\). As a result, we have
\[ T_k = \sum_{i=k}^{n_2} \left( \prod_{l=k}^{i-1} D_l \right) \overline{T}_i. \]
From Assumption 1 and Definition 2 we have
\[ \|q_k\|_2 \leq U, \quad \|\hat{q}_k\|_2 \leq \overline{C}_B C_H C_{\hat{H}} U / \lambda_R \triangleq \overline{C}_\hat{q}. \]
From Lemmas 5 and 1 and Proposition 2 we have \(\|M_k\|_2 \leq 1 + C_\beta^2 / \lambda R \triangleq C_M\). Then using Assumption 1, Lemma 1, and Proposition 2, we have
\[ \|\overline{T}_k\|_2 \leq C_A \beta C_M C_\hat{q} + C_E C_{\hat{H}} U + \overline{C}_Q m_0 \triangleq C_{\overline{T}}. \]
Combining the above with Proposition 3, we have
\[ \|T_k\|_2 \leq \sum_{i=k}^{n_2} C_T C_1 \rho^{i-k} \leq C_T C_1 / (1 - \rho) \triangleq C_T. \quad (22) \]
Denote \(G_k \triangleq -M_k \hat{R}_k T_{k+1} + M_k \hat{q}_k\). Then, from Lemma 4, we have \(x_{k+1}^* = D_k x_k^* + G_k\). Thus,
\[ x_k^* = \sum_{i=n_1}^{k-1} \left( \prod_{l=i+1}^{k-1} D_l \right) G_i + \left( \prod_{l=n_1}^{k-1} D_l \right) x_{n_1}. \]
Note that \( \|G_k\|_2 \leq CM_C R C_T + C_M C_q \Rightarrow C_G \). Thus, from Proposition 3 we have
\[
\|x^*_k\| \leq \sum_{i=n_1}^{k-1} \left( C_G C_1 \rho^{k-i-1} \right) + C_1 u_0 \rho^{k-n_1} \leq C_1 C_G / (1 - \rho) + C_1 u_0 \Rightarrow C_g. \tag{23}
\]

Next, we prove that the bound on \( \lambda^*_k \). Lemma 4 gives
\[
\lambda^*_k = K_{k+1} x^*_{k+1} + T_{k+1}.
\]
Using Proposition 2, (22) and (23), we have
\[
\|\lambda^*_k\|_2 \leq \beta \|x^*_{k+1}\|_2 + C_T \leq \beta C_g + C_T \Rightarrow C_\lambda.
\]
This completes the proof. \( \square \)

### 2.2 Path-constrained Inequality Problem

In this subsection, we return to the inequality-constrained problem (1). Using the results established for (2), we investigate the solutions and adjoint variables of (1). We make the following controllability assumption of the active set of problem (1) at optimality.

**Assumption 3** Let \( \mathcal{A} \) be the active set of problem (1) at optimality. Then
(a) \( \mathcal{A} \) is UDB(\( \lambda_H \)) as defined in Definition 1;  
(b) the equality-constrained system (2b)–(2c) corresponding to \( \mathcal{A} \) is controllable in \( t \) steps as defined in Definition 5 for any \( n_1 \leq q \leq n_2 \); 
(c) under (b), \( C_{q,t}(\mathcal{A}) \) has full row rank by Proposition 1, and we further assume that \( \mathcal{A} \) is UCC(\( \lambda_C \)) as defined in Definition 6.

**Corollary 1** Let \( x^*_k \) and \( \lambda^*_k \) be respectively the optimal states and adjoint variables of problem (1). Under Assumptions 1, 2, and 3, we have
\[
\|x^*_k\|_2 \leq C_g, \quad n_1 + 1 \leq k \leq n_2; \quad \|\lambda^*_k\|_2 \leq C_\lambda, \quad n_1 \leq k \leq n_2 - 1
\]
for \( C_g, C_\lambda > 0 \) defined in Lemma 7.

**Proof** Note that when the sequence of index sets for problem (2) is the active set \( \mathcal{A} \) of problem (1) at optimality, problems (1) and (2) have the same solutions and adjoint variables. Since \( \mathcal{A} \) is UDB(\( \lambda_H \)) and UCC(\( \lambda_C \)) by Assumption 3, applying Lemma 7 gives the result. \( \square \)

Note that the constant upper bounds in Corollary 1 depend on \( \lambda_H \) and \( \lambda_C \) in Assumption 3.

### 3 Lag L Receding Horizon Strategy

In this section, we prove an exponentially decaying approximation error for a lag \( L \) receding horizon strategy. Let \( N > L \) be the length of each but the last receding horizon, and let
\( n_0 = \lfloor (n_2 - n_1 - N + 1) / L \rfloor + 1 \) be the number of receding horizons. Then for \( i = 1, \ldots, n_0 \), define the \( i \)th receding horizon \( \mathcal{R}_i = [n'_1(i), n'_2(i)] \) as
\[
n'_1(i) = n_1 + L(i - 1), \quad n'_2(i) = \begin{cases} n'_1(i) + N - 1, & 1 \leq i \leq n_0 - 1, \\ n_2, & i = n_0. \end{cases}
\] (24)

For simplicity, we denote \( m \triangleq n'_1(n_0) \) to be the starting index of the last receding horizon. Note that with (24), we have \( n'_1(i + 1) = n'_1(i) + L \) and that the length \( N_1 = n_2 - m + 1 \) of the last receding horizon satisfies \( N \leq N_1 < N + L \). On a receding horizon \( \mathcal{R}_i \), we define the following parametrized problem whose parameters are the initial state and terminal reference.

**Definition 7** For \( i = 1, \ldots, n_0 \), define the parametrized problem \( P^i_\theta \) with \( \theta = (\theta(h), \theta(d)) \) as follows:
\[
\min \frac{1}{2} \sum_{k=n'_1(i)}^{n'_2(i)-1} w_k^T R_k w_k + (h_k - d_k)^T Q_k(h_k - d_k) + (h_{n'_2(i)} - d_{n'_2(i)})^T Q_{n'_2(i)}(h_{n'_2(i)} - d_{n'_2(i)}) \tag{25a}
\]
\[
\text{s.t. } h_{k+1} = A_k h_k + B_k w_k, \quad n'_1(i) \leq k \leq n'_2(i) - 1, \tag{25b}
\]
\[
P_{k+1} h_{k+1} + \tilde{C}_k w_k \geq \tilde{q}_k, \quad n'_1(i) \leq k \leq n'_2(i) - 1, \tag{25c}
\]
\[
h_{n'_1(i)} = \theta(h), \quad d_{n'_2(i)} = \theta(d), \tag{25d}
\]

where \( d_{n'_1(i):n'_2(i)-1} \) is the reference trajectory of problem (1).

The parametrized problem \( P^i_\theta \) is essentially a subproblem of (1) restricted on the receding horizon \( \mathcal{R}_i \) with terminal reference parametrized by \( \theta(d) \) and reinitialized with \( \theta(h) \). Denote \( x^*_k(P^i_\theta), u^*_k(P^i_\theta) \) as the optimal state and control, respectively, of problem \( P^i_\theta \) at some time point \( k \in \mathcal{R}_i \). Then the RHC policy (e.g., [13, 18]) is the sequence \( \tilde{u}_k \}_{k=n_1}^{n_2} \) defined as
\[
\tilde{u}_{n'_1(i)+j-1} = u^*_{n'_1(i)+j-1}(P^i_{\theta_0(i)}), \quad 1 \leq j \leq L, \quad 1 \leq i \leq n_0 - 1,
\]
\[
\tilde{u}_k = u^*_k(P^{n_0}_{\theta_0(n_0)}), \quad m \leq k \leq n_2 - 1,
\] (26)

where \( \theta_0(i) = (\tilde{x}_{n'_1(i)}, d_{n'_2(i)}) \), and the state sequence \( \tilde{x}_k \}_{k=n_1}^{n_2} \) is defined as
\[
\tilde{x}_{n'_1(1)} = x_{n_1}^0,
\]
\[
\tilde{x}_{n'_1(i)+j} = x^*_{n'_1(i)+j}(P^i_{\theta_0(i)}), \quad 1 \leq j \leq L, \quad 1 \leq i \leq n_0 - 1,
\]
\[
\tilde{x}_k = x^*_k(P^{n_0}_{\theta_0(n_0)}), \quad m + 1 \leq k \leq n_2.
\] (27)

In other words, the RHC policies \( \tilde{u}_{n'_1(i)+j-1} \) for \( 1 \leq j \leq L \) and \( 1 \leq i \leq n_0 - 1 \) are obtained by solving problem \( P^i_{\theta_0(i)} \) on \( \mathcal{R}_i \) initialized with \( \tilde{x}_{n'_1(i)} = \tilde{x}_{n'_1(i-1)+L}, \) which in turn is obtained by solving \( P^{i-1}_{\theta_0(i-1)} \) on \( \mathcal{R}_{i-1} \). On the last receding horizon, \( \tilde{u}_k \) for \( m \leq k \leq n_2 - 1 \) are defined as the optimal controls of problem \( P^{n_0}_{\theta_0(n_0)} \) on \( \mathcal{R}_{n_0} \). To bound the error of this RHC strategy, we need to relate the solutions of problems \( P^i_{\theta_0(i)} \) to those of problem (1). To start, we consider a different choice of the parameter \( \theta_1(i) \). The following result establishes a connection between the solutions of (1) and those of \( P^i_{\theta_1(i)} \).
Proposition 5 Let \((u_{n_1}^*, x_{n_1+1}^*)\) and \(\lambda_k^*\) be the solutions and optimal adjoint variables of problem (1) with some initial value \(x_{0,n_1}^*\), and let \(\eta_k^*\) be the optimal Lagrange multipliers associated with the path constraints (1c). For each \(i = 1, \ldots, n_0\), define

\[
\hat{h}_{n_i}^* = \begin{cases} 
  x_{n_i}^*, & i = 1 \\
  x_{n_i}^*, & i = 2, \ldots, n_0
\end{cases}
\]

\[
\hat{d}_{n_i}^* = \begin{cases} 
  -Q_{n_i}^{-1}(\lambda_{n_i}^*(i) - \hat{P}P_{n_i}^{T}(i)\eta_{n_i}^*(i)) + x_{n_i}^*, & i = 1, \ldots, n_0 - 1 \\
  0, & i = n_0
\end{cases}
\]

Then \(\left(u_{n_i}^*(i), n_{i}^*(i) - 1, x_{n_i}^*(i) + 1, x_{n_i}^*(i)\right)\) satisfies the KKT conditions and the second-order sufficient conditions of problem \(P_{\theta_i}^i\) with \(\theta_i(i) = \left(\hat{h}_{n_i}^*, \hat{d}_{n_i}^*\right)\).

Proof The KKT conditions for problem (1) are

\[
R_k h_k^* + B_k^T \lambda_k^* - \tilde{C}_k^T \eta_k^* = 0, \quad k \in [n_1, n_2 - 1],
\]

\[
Q_k(x_k^* - d_k) + A_k^T \lambda_k^* - \lambda_{k-1}^* - \tilde{P}_k \eta_{k-1}^* = 0, \quad k \in [n_1 + 1, n_2 - 1],
\]

\[
Q_{n_2}(x_{n_2}^* - d_{n_2}) - \lambda_{n_2-1}^* - \tilde{P}_{n_2} \eta_{n_2-1}^* = 0,
\]

\[
x_{k+1}^* = A_k x_k^* + B_k u_k^*, \quad k \in [n_1, n_2 - 1],
\]

\[
x_{n_1} = x_{n_1}^*,
\]

\[
\tilde{P}_{k+1} x_{k+1}^* + \tilde{C}_k u_k^* \geq \tilde{q}_k, \quad k \in [n_1, n_2 - 1],
\]

\[
\eta_k^* \geq 0, \quad k \in [n_1, n_2 - 1],
\]

\[
\eta_{k}^*(j) (\tilde{P}_{k+1}(j, :)^T x_{k+1}^* + \tilde{C}_k(j, :)^T u_k^* - \tilde{q}_k(j)) = 0, \quad k \in [n_1, n_2 - 1].
\]

Then for problem \(P_{\theta_i}^i\), the KKT conditions are satisfied by \(\left(u_{n_i}^*(i), n_{i}^*(i) - 1, x_{n_i}^*(i) + 1, x_{n_i}^*(i)\right)\) with the same Lagrange multipliers \(\lambda_k^*\) and \(\eta_k^*\) as follows. Note that for problem \(P_{\theta_i}^i\), the initial state and terminal reference are \(\hat{h}_{n_i}^*(i)\) and \(\hat{d}_{n_i}^*(i)\) defined in (28):

\[
R_k h_k^* + B_k^T \lambda_k^* - \tilde{C}_k^T \eta_k^* = 0, \quad k \in [n_1(i), n_2(i) - 1],
\]

\[
Q_k(x_k^* - d_k) + A_k^T \lambda_k^* - \lambda_{k-1}^* - \tilde{P}_k \eta_{k-1}^* = 0, \quad k \in [n_1(i) + 1, n_2(i) - 1],
\]

\[
Q_{n_2}(x_{n_2}^* - d_{n_2}) - \lambda_{n_2-1}^* - \tilde{P}_{n_2} \eta_{n_2-1}^* = 0,
\]

\[
x_{k+1}^* = A_k x_k^* + B_k u_k^*, \quad k \in [n_1(i), n_2(i) - 1],
\]

\[
x_{n_1(i)} = \hat{h}_{n_i}^*(i),
\]

\[
\tilde{P}_{k+1} x_{k+1}^* + \tilde{C}_k u_k^* \geq \tilde{q}_k, \quad k \in [n_1(i), n_2(i) - 1],
\]

\[
\eta_k^* \geq 0, \quad k \in [n_1(i), n_2(i) - 1],
\]

\[
\eta_{k}^*(j) (\tilde{P}_{k+1}(j, :)^T x_{k+1}^* + \tilde{C}_k(j, :)^T u_k^* - \tilde{q}_k(j)) = 0, \quad k \in [n_1(i), n_2(i) - 1].
\]

where (30a)–(30b) and (30f)–(30h) directly follow from (29a)–(29b) and (29f)–(29h), respectively. Equation (30c) follows from the definition of \(\hat{d}_{n_i}^*(i)\) in (28). Equation (30d) follows from (29d), (30e), and (28). The second-order condition is satisfied by virtue of the strong convexity of the problem. \(\Box\)

Lemma 8 Under Assumptions 1, 2, and 3, for \(\hat{h}_{n_i}^*(i)\) and \(\hat{d}_{n_i}^*(i)\) defined in (28), we have, for \(i = 1, \ldots, n_0\),

\[
\|\hat{h}_{n_i}^*(i)\|_2, \quad \|\hat{d}_{n_i}^*(i)\|_2 \leq C_{\theta}
\]
for some $C_\theta > 0$ independent of $i, n_1,$ and $n_2$.

**Proof** For $i = 1, \ldots, n_0 - 1$, KKT conditions of problem (1) give

$$\lambda_{n_2(i)-1}^* + \tilde{F}^T n_2(i) \eta_{n_2(i)-1}^* = Q n_2(i) \left( x_{n_2(i)}^* - d_{n_2(i)}^* \right) + A^T n_2(i) \lambda_{n_2(i)}^*.$$

Then Assumptions 1 and 2 and Corollary 1 give

$$\|\lambda_{n_2(i)-1}^* + \tilde{F}^T n_2(i) \eta_{n_2(i)-1}^*\| \leq \tilde{C}_Q (C_g + m_0) + \tilde{C}_A C_\lambda.$$

Combining this with (28), we have

$$\|\lambda_{n_2(i)-1}^* + \tilde{F}^T n_2(i) \eta_{n_2(i)-1}^*\| \leq \left( \tilde{C}_Q (C_g + m_0) + \tilde{C}_A C_\lambda \right) / \lambda Q + C_g \Delta C_d,$$

for $i = 1, \ldots, n_0 - 1.$ From (28), we have $\|\hat{d}_{n_2(i)}\| \leq \min(C_{d_\theta}, m_0)$ for $i = 1, \ldots, n_0.$ Similarly, from Assumption 2, Corollary 1, and (28) we have $\|\hat{h}_{n_2(i)}\| \leq \min(u_0, C_g)$ for $i = 1, \ldots, n_0.$ Taking $C_\theta = \min(C_{d_\theta}, m_0, u_0, C_g)$ concludes the proof.

Proposition 5 shows that the solution of $P^i_{\theta_1(i)}$ is identical to the solution, restricted to $\mathcal{R}_i$, of problem (1). However, problem $P^i_{\theta_1(i)}$ is formal and cannot be defined without first solving problem (1). Hence, we need to investigate the relationship between solutions of problems $P^i_{\theta_1(i)}$ and $P^i_{\theta_0(i)}$, the latter of which gives the RHC solutions. Since problem $P^i_{\theta_1(i)}$ can be viewed as resulting from a perturbation of the parameters of problem $P^i_{\theta_0(i)}$, we employ the following parametric sensitivity results derived from [6].

**Definition 8** For $\theta \in \mathbb{R}^q$, define the one-sided directional derivative of $y(\theta)$ along a direction $p \in \mathbb{R}^q$ at $\theta_0$ as

$$D_p y(\theta_0) = \lim_{t \downarrow 0} \frac{y(\theta_0 + tp) - y(\theta_0)}{t},$$

given that the limit exists.

**Lemma 9** Consider the following parametrized quadratic programming problem,

$$\min f(y, \theta) \deq y^T G y / 2 + y^T c(\theta) + \theta^T F \theta + y^T c_1 + \theta^T c_2 + C \quad (31)$$

s.t. $Ay - r \leq 0,$

$$By - d(\theta) = 0,$$

where $G, F$ are positive definite, $\theta \in \mathbb{R}^q$, and $A^T = \left[ a_1, \ldots, a_m \right] \in \mathbb{R}^{n \times m}.$ Denote the solution of problem (31) as $y(\theta)$. When $\theta = \theta_0$, let $y_0 = y(\theta_0)$ and the Lagrange multiplier corresponding to $y_0$ be $\lambda$. Denote $I(y_0, \theta_0) = \{ i : a_i^T y_0 = r_i, i = 1, \ldots, m \}$ be the set of active inequality constraints, $I_+(y_0, \theta_0, \lambda) = \{ i \in I(y_0, \theta_0) : \lambda_i > 0 \}$ and $I_0(y_0, \theta_0, \lambda) = \{ i \in I(y_0, \theta_0) : \lambda_i = 0 \}$ be the subsets of $I(y_0, \theta_0)$ with positive and zero Lagrange multipliers, respectively. If the linear independence constraint qualification (LICQ) [23, Definition 12.4] holds at $y(\theta_0)$, then for any direction $p \in \mathbb{R}^q$, we have

$$D_p y(\theta_0) = \left( \frac{dy^T_{I(\theta)}(\theta)}{d\theta} \right)_{\theta=\theta_0} p.$$
where $y_{I'(\theta_0)}^n(\theta)$ is the solution of the problem

$$
\begin{align*}
\min & \quad f(y, \theta) = y^T G y / 2 + y^T c(\theta) + \theta^T F \theta + y^T c_1 + \theta^T c_2 + C \\
\text{s.t.} & \quad A_{I'(\theta_0)} y - r' = 0, \\
& \quad By - d(\theta) = 0
\end{align*}
$$

(32)

and where $I'(\theta_0) = I_+(y_0, \theta_0, \bar{\lambda}) \cup I_1$ for some $I_1 \subset I_0(y_0, \theta_0, \bar{\lambda})$ and $A_{I'(\theta_0)} = [a_i']_{i \in I'(\theta_0)}$, $r' = [r_i]_{i \in I'(\theta_0)}$.

Proof See Appendix A.3; see also [30, Lemma 3.5].

Note that problem $P_{\theta}^i$ has the same structure as that defined by (31), and Lemma 9 connects the dependence on parameters of the solutions for the inequality-constrained problem (31) with that of a related equality-constrained problem (32), whose equality constraints are subsetted from the active constraints of (31) at optimality. The equality-constrained problem has smooth and regular KKT conditions which facilitate the derivation for the dependence of solutions on parameters as shown in Section 2.1. Now we are ready to investigate the effect on solutions of perturbing the parameters of $\Pi$ on parameters as shown in Section 2.1. We let $P_{\theta}^i$ be the equality-constrained problem (32), whose equality constraints are connected to the solutions of $P_{\theta}^i$ for $\theta = \theta_0 + s(\theta_1 - \theta_0)$ and $s \in [0, 1]$. Let $P_{\theta_0} = (\theta_0)$ be the solution of problem $P_{\theta_0}$ at optimality. We let

$$
E_k(s) = P_{k+1}(s) A_k, \quad H_k(s) = P_{k+1}(s) B_k + C_k(s).
$$

Thus, the active constraints of $P_{\theta_0}$ are

$$
\begin{align*}
h_{k+1} &= A_k h_k + B_k w_k, \quad n_1' \leq k \leq n_2' - 1, \quad h_{n_1'} = \theta_0(s), \\
E_k(s) h_k + H_k(s) w_k &= q_k(s), \quad n_1' \leq k \leq n_2' - 1
\end{align*}
$$

(34a)

(34b)

where $\theta_0 = (\theta_0^{(h)}, \theta_0^{(d)})$. As in Assumption 3 (a), we make the following uniform boundedness assumption about the active constraints of $P_{\theta_0}$.

Assumption 4 For $i = 1, \ldots, n_0$ and $s \in [0, 1]$, let $\theta_0(i) = (\overline{x}_{n_1(i)}, \overline{d}_{n_2(i)})$, $\theta_1(i) = (\hat{h}_{n_1'(i)}, \hat{d}_{n_2'(i)})$ as defined in (28), and $\theta_0(i) = \theta_0(i) + s(\theta_1(i) - \theta_0(i))$. Then we assume that the active sets of problems $P_{\theta_0}^i$ at optimality are UDB($\lambda_H$) as in Definition 1.

In particular, Assumption 4 implies that $H_k(s)$ defined in (33) has full row rank, with which we can now apply Lemma 9 to problem $P_{\theta_0}$.

Lemma 10 Denote $\theta_0 = (\overline{x}_{n_1'}, \overline{d}_{n_2'})$ and $\theta_1 = (\hat{h}_{n_1'}, \hat{d}_{n_2'})$ as defined in (28). For $\theta = (\theta^{(h)}, \theta^{(d)})$, let $x(\theta)$ be the solution of problem $P_0$. Under Assumption 4, for $s \in [0, 1]$ and $\theta_s = \theta_0 + s(\theta_1 - \theta_0)$, we have

$$
D_{\theta_1 - \theta_0} x(\theta_s) = \left( \frac{dy_s(\theta)}{d\theta} \right)_{\theta = \theta_0 + s(\theta_1 - \theta_0)} (\theta_1 - \theta_0),
$$
and \( y_s(\theta) \) is the solution of the following equality-constrained problem:

\[
\min_{2} \frac{1}{2} \sum_{k=n'_1}^{n'_2-1} w_k^T R_k w_k + (h_k - d_k)^T Q_k (h_k - d_k) + \left( h_{n'_2} - \theta_s(d) \right)^T Q_{n'_2} \left( h_{n'_2} - \theta_s(d) \right) \quad (35a)
\]

s.t. \( h_{k+1} = A_k h_k + B_k w_k, \quad n'_1 \leq k \leq n'_2 - 1, \quad h_{n'_1} = \theta_s(d), \)

\[
E_k'(s) h_k + H_k'(s) w_k = q_k'(s), \quad n'_1 \leq k \leq n'_2 - 1, \quad (35b)
\]

where rows of \( E_k'(s) \) and \( H_k'(s) \) are respectively subsets of rows of \( E_k(s) \) and \( H_k(s) \) defined by the active constraints of \( P_{\theta_s} \) at optimality as in (33). In other words, \( E_k'(s) h_k + H_k'(s) w_k = q_k'(s) \) is a subset of the equality constraints (34b).

**Proof** Problem \( P_{\theta_s} \) is an instance of problem (31) with the following parameters:

\[
G = \text{diag} \left( [ R_{n'_1} \cdots R_{n'_2-1} Q_{n'_1+1} \cdots Q_{n'_2} ] \right), \quad c(\theta) = \begin{bmatrix} 0_{(n'_2-n'_1)m+(n'_2-n'_1-1)n} \\ -Q_{n'_2} \theta_s(d) \end{bmatrix},
\]

\[
A = \begin{bmatrix} \ldots & -\tilde{C}_{n'_2} & \ldots \\ \ldots & -\tilde{P}_{n'_2} & \ldots \\ -B_{n'_2} & \ldots & I \end{bmatrix}, \quad r = \begin{bmatrix} \ldots & -\tilde{q}_{n'_2} & \ldots \\ -\tilde{q}_{n'_1} & \ldots & \ldots \end{bmatrix}, \quad F = \begin{bmatrix} Q_{n'_1}/2 \\ Q_{n'_2}/2 \end{bmatrix},
\]

\[
B = \begin{bmatrix} -B_{n'_1} & \ldots & \ldots & -A_{n'_1+1} & I \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ -B_{n'_2} & \ldots & \ldots & -A_{n'_2-1} & I \end{bmatrix}, \quad d(\theta) = \begin{bmatrix} A_{n'_1} \theta_s^{(h)} \\ 0_{(n'_2-n'_1-1)n} \end{bmatrix}.
\]

Here \( Ax \leq r \) and \( Bx = d(\theta) \) correspond respectively to the inequality constraints (25c) and the dynamical constraints (25b). Note that \( A \) and \( B \) have the same number of columns. \( G \) and \( F \) are positive definite from Assumption 1. The quantities \( c_1, c_2, \) and \( C \) of problem (31) do not enter in the proof, so their definitions are not shown.

Let \( \overline{A}(s) \) be the matrix whose rows are subsets of rows of \( A \) corresponding to the active constraints \( P_{k+1}(s) h_{k+1} + C_k(s) w_k = q_k(s) \) at optimality for problem \( P_{\theta_s} \). Then we define

\[
\overline{A}(s) \triangleq \begin{bmatrix} -C_{n'_1}(s) & \ldots & -P_{n'_1+1}(s) \\ \ldots & \ldots & \ldots \\ -C_{n'_2-1}(s) & \ldots & -P_{n'_2}(s) \end{bmatrix}.
\]

In the following we show that rows of \( \overline{A}(s) \) and \( B \) are linearly independent. Denote

\[
\alpha^T = \begin{bmatrix} \alpha_{n'_1+1}^T \ldots \alpha_{n'_2}^T \end{bmatrix}, \quad \beta^T = \begin{bmatrix} \beta_{n'_1}^T \ldots \beta_{n'_2-1}^T \end{bmatrix}.
\]

Then

\[
\alpha^T \overline{A}(s) + \beta^T B = \begin{bmatrix} x_{n'_1} \ldots x_{n'_2-1} \end{bmatrix} \begin{bmatrix} y_{n'_1+1} \ldots y_{n'_2} \end{bmatrix},
\]

where

\[
x_k = -\alpha_{k+1}^T C_k(s) - \beta_k^T B_k, \quad y_k = \begin{cases} \beta_{k-1}^T A_k - \alpha_k^T P_k(s), & k < n'_2, \\ \beta_{n'_2-1}^T A_k - \alpha_k^T P_{n'_2}(s), & k = n'_2. \end{cases}
\]
Now we let $\alpha^T \Xi(s) + \beta^T B = 0$. We show that $\alpha, \beta = 0$ by backward induction. We have
\[
\begin{cases}
    x_{n_2}' - 1 = 0, \\
    y_{n_2}' = 0
\end{cases}
\Rightarrow
\begin{cases}
    \alpha^T_{n_2} C_{n_2}' - 1(s) + \beta^T_{n_2} B_{n_2}' - 1 = 0, \\
    \beta^T_{n_2} = \alpha^T_{n_2} P_{n_2}(s)
\end{cases}
\Rightarrow
\begin{cases}
    \alpha^T_{n_2} C_{n_2}' - 1(s) + \alpha^T_{n_2} P_{n_2}(s) B_{n_2}' - 1 = 0
\end{cases}
(33), Asmp. 4
\Rightarrow
\begin{cases}
    \alpha_{n_2}' = 0 \Rightarrow \beta_{n_2}' - 1 = 0.
\end{cases}

Suppose $\alpha_k, \beta_{k-1} = 0$ for some $n_1' + 1 < k \leq n_2'$. Then we have
\[
\begin{cases}
    x_{k-2} = 0, \\
    y_{k-1} = 0
\end{cases}
\Rightarrow
\begin{cases}
    \alpha^T_{k-1} C_{k-2}(s) + \beta^T_{k-2} B_{k-2} = 0, \\
    \beta^T_{k-2} = \beta^T_{k-1} A_{k-1} + \alpha^T_{k-1} P_{k-1}(s)
\end{cases}
\Rightarrow
\begin{cases}
    \alpha^T_{k-1} C_{k-2}(s) + \alpha^T_{k-1} P_{k-1}(s) B_{k-2} = 0
\end{cases}
(33), Asmp. 4
\Rightarrow
\begin{cases}
    \alpha_{k-1} = 0 \Rightarrow \beta_{k-2} = 0.
\end{cases}

So LICQ holds for problem $P_{\theta_s}$ at optimality. Directly applying Lemma 9 concludes the proof. □

Problem (35) is an equality-constrained problem for which the results derived in Section 2, especially the exponential decay property of the dependence of solutions on the initial state and terminal reference, can be applied under certain assumptions. In the following, we investigate the controllability conditions for problem (35). Denote the active set of problem $P_{\theta_s(i)}$ and the index set for the corresponding equality constraints of problem (35) as $A_{s(i)}$ and $I_{s(i)}$, respectively.

**Lemma 11** Under Assumption 4, for $s \in [0, 1]$ and $i = 1, \ldots, n_0$, we have that $I_{s(i)}$ is UDB($\lambda H$) as defined in Definition 1.

*Proof* Since $I_{s(i)} \subset A_{s(i)}$, rows of $H_k'(s)$ are subsets of rows of $H_k(s)$, then the conclusion follows from Assumption 4. □

**Lemma 12** If the equality-constrained system (34) corresponding to the active sets of $P_{\theta_s}$ is controllable at $q$ in $t$ steps, then the controllability matrix $C_{q,t}(I_{s})$ defined by the subsetted system (35b)–(35c) has full row rank.

*Proof* If system (34) can be controlled to an arbitrary state $\bar{x}$ in $t$ steps with some admissible controls and corresponding states, then the subsetted system (35b)–(35c) can also be controlled to $\bar{x}$ in $t$ steps with the same controls and states, because the feasible set defined by (35b)–(35c) contains that defined by (34). As a result, from Proposition 1 $C_{q,t}(I_{s})$ has full row rank. □

**Assumption 5** For $i = 1, \ldots, n_0$ and $s \in [0, 1]$, let $\theta_0(i) = (\hat{x}_{n_1'(i)}, d_{n_2'(i)})$, $\theta_1(i) = (\hat{h}_{n_1'(i)}, \hat{d}_{n_2'(i)})$ as defined in (28), and $\theta_s(i) = \theta_0(i) + s(\theta_1(i) - \theta_0(i))$. We assume that there exists $\lambda_C > 0$ so that for any $n_1'(i) \leq q \leq n_2'(i), 1 \leq i \leq n_0$:

(a) the system (34) defined by $A_s(i)$ is controllable at $q$ for $t$ steps;
Assumption 5(a) assumes controllability of problem $P_{\theta_0}$ only at optimality. Proposition 1 and Lemma 12 imply that $C_{q,t}(A_s(i))$ and $C_{q,t}(I_s(i))$ are bounded below, and Assumption 5(b) in addition assumes that the lower bounds are uniform for all time points and receding horizons. Now we are ready to bound the distance between solutions of $P_{\theta_0}$ and $P_{\theta_1}$, which by Proposition 5, is also the distance between solutions of RHC and problem (1). The assumptions needed in the proof are summarized as follows. Assumptions 1 and 2 give uniform bounds for the problem parameters. Assumption 3 assumes uniform boundedness and controllability conditions needed in the proof are summarized as follows. Assumptions 1 and 2 give uniform boundedness and controllability for the active set of problem (1) at optimality. Lastly, Assumptions 4 and 5 give the uniform boundedness and controllability conditions, respectively, for the active sets of problems $P^i_{\theta_s(i)}$ for all receding horizons $i = 1, \ldots, n_0$ and $s \in [0, 1]$.

**Theorem 6** For $1 \leq i \leq n_0$, let \( u^*_n(i) \) be the solution of problem $P^i_{\theta_s(i)}$ which from Proposition 5 is exactly the solution of problem (1) restricted to $R_i$. Let $\tilde{u}_k$ and $\tilde{x}_k$ be the receding horizon control and state defined in (26) and (27), respectively. Under Assumptions 1–5, we have, for some lag $L$ so that $Z_1 \rho^L < 1$ and receding horizon length $N > L$,

$$\|x_k^* - \tilde{x}_k\|_2, \|u_k^* - \tilde{u}_k\|_2 \leq C_d Z_2 \left(1 + \frac{Z_1}{1 - Z_1 \rho^L}\right) \rho^{N-L-1}$$

for $n_1 \leq k \leq n_2 - 1$. Here $C_d > 0$ is independent of $N, n_1, n_2, Z_1, Z_2$, and $\rho$ are as in Proposition 4.

**Proof** Assumption 1 and Lemma 8 give, for $i = 1, \ldots, n_0$,

$$\left\|\hat{d}_{n_2(i)} - d_{n_2(i)}\right\|_2 \leq C_0 + m_0 \Delta = C_d. \tag{36}$$

Let $\theta_0(i) = (\tilde{x}_{n_1(i)}, d_{n_1(i)}, \theta_1(i) = (\hat{h}_{n_1(i)}, \hat{d}_{n_1(i)})$ as defined in (28), and $\theta_s(i) = \theta_0(i) + s(\theta_1(i) - \theta_0(i))$ for $s \in [0, 1]$. Note that for $1 \leq j \leq L, x^*_{n_1(i)+j}$ is the optimal state of problem (1), which by Proposition 5 is also that of problem $P^i_{\theta_1(i)}$, while $\tilde{x}_{n_1(i)+j}$ is the optimal state of problem $P^i_{\theta_0(i)}$ by (27). Denote $\tilde{x}_{n_1(i)}(\theta_s(i))$ and $\tilde{p}_{n_1(i)}(\theta_s(i))$ for $n_1(i) \leq k \leq n_2(i)$ as the optimal control and state of problem $P^i_{\theta_1(i)}$, and $\hat{s}_{n_1(i)}(\theta_s(i))$ and $\hat{p}_{n_1(i)}(\theta_s(i))$ as those of the corresponding subsetted equality-constrained problem (35). Then for $1 \leq i \leq n_0 - 1$, we have from Proposition 5 and Lemma 10

$$x^*_{n_1(i)+j} - \tilde{x}_{n_1(i)+j} = \int_0^1 D_{\theta_1(i) - \theta_0(i)} \tilde{p}^*_{n_1(i)+j}(\theta_s(i)) ds$$

$$= \int_0^1 \left[ \nabla_{h_{n_1(i)}} P^*_{n_1(i)+j}(\theta_s(i)) - \nabla_{d_{n_1(i)}} P^*_{n_1(i)+j}(\theta_s(i)) \right] ds.$$

Lemma 11 states that the index set of the corresponding problem (35) is UDB($\lambda_H$), and Assumption 5(b) further states it is UCC($\lambda_C$). Note that the exponential bounds obtained in Proposition 4 are independent of the problem interval and the particular choice of
the equality constraint index set, which is UDB(λ_H) and UCC(λ_C). Therefore, applying Proposition 4, we have, for 1 \leq i \leq n_0 - 1 and 1 \leq j \leq L,
\|x^*_n(i)+j - \tilde{x}^*_{n_1}(i)+j\|_2 \leq C_d Z_2 \rho^{n^*_2(i) - n'_1(i) - j} + Z_1 \rho^j \|\hat{h}_{n_1(i)} - \tilde{x}^*_{n_1(i)}\|_2
\tag{36}
\leq C_d Z_2 \rho^{n^*_2(i) - n'_1(i) - j} + Z_1 \rho^j \|x^*_n(i) - \tilde{x}^*_{n_1(i)}\|_2
\tag{28}
= C_d Z_2 \rho^N - j - 1 + Z_1 \rho^j \|x^*_n(i) - \tilde{x}^*_{n_1(i)}\|_2.
\tag{37}

When j = L, note that n'_1(i) + L = n'_1(i + 1), so (37) becomes
\|x^*_n(i+1) - \tilde{x}^*_{n_1(i+1)}\|_2 \leq C_d Z_2 \rho^{N-L-1} + Z_1 \rho^L \|x^*_n(i) - \tilde{x}^*_{n_1(i)}\|_2.

From this recursion, at the starting index of \mathcal{R}_i for 1 \leq i \leq n_0 we have
\|x^*_n(i) - \tilde{x}^*_{n_1(i)}\|_2 \leq \frac{1 - (Z_1 \rho^L)_{i-1}}{1 - Z_1 \rho^L} C_d Z_2 \rho^{N-L-1}
\leq C_d Z_2 \rho^{N-L-1}/(1 - Z_1 \rho^L).
\tag{38}

Note that L is chosen so that Z_1 \rho^L < 1. Substituting (38) into (37), we have, for 1 \leq i \leq n_0 - 1 and 1 \leq j \leq L,
\|x^*_n(i)+j - \tilde{x}^*_{n_1(i)+j}\|_2 \leq C_d Z_2 \rho^{N-j-1} + Z_1 \rho^j \|x^*_n(i) - \tilde{x}^*_{n_1(i)}\|_2
\leq C_d Z_2 \rho^{N-L-1} + Z_1 \|x^*_n(i) - \tilde{x}^*_{n_1(i)}\|_2
\tag{39}
\leq C_d Z_2 \left(1 + \frac{Z_1}{1 - Z_1 \rho^L}\right) \rho^{N-L-1}.

Now we prove the approximation error bound for the RHC policies. For 1 \leq i \leq n_0 - 1 and 1 \leq j \leq L, u^*_n(i)+j-1 is the optimal control of problem (1), which by Proposition 5 is also that of problem P^I_{\theta_1(i)}, whereas \tilde{u}^*_n(i)+j-1 is the optimal control of problem P^I_{\theta_0(i)} by (26). Therefore, by Proposition 5 and Lemma 10 we have
\begin{align*}
u^*_n(i)+j-1 - \tilde{u}^*_n(i)+j-1 \\
= \int_0^1 D\theta_1(i) - \theta_0(i) \tilde{x}^*_n(i)+j-1(\theta_1(i)) \, ds \\
= \int_0^1 \left[\nabla h^*_n(i)+j-1(\theta_1(i)) \nabla d^*_n(i)+j-1(\theta_1(i)) \right] \left[\hat{h}_{n_1(i)} - \tilde{x}^*_{n_1(i)}\right] \, ds.
\end{align*}

Proposition 4 and (36) give
\begin{align*}
\|\nu^*_n(i)+j-1 - \tilde{u}^*_n(i)+j-1\|_2 \leq C_d Z_2 \rho^{n^*_2(i) - n'_1(i) - j + 1} + Z_1 \rho^{j-1} \|\hat{h}_{n_1(i)} - \tilde{x}^*_{n_1(i)}\|_2
\tag{28}
\leq C_d Z_2 \rho^N - j + Z_1 \rho^{j-1} \|x^*_n(i) - \tilde{x}^*_{n_1(i)}\|_2
\leq C_d Z_2 \rho^{N-L-1} + Z_1 \|x^*_n(i) - \tilde{x}^*_{n_1(i)}\|_2
\tag{38}
\leq C_d Z_2 \left(1 + \frac{Z_1}{1 - Z_1 \rho^L}\right) \rho^{N-L-1}.
\tag{40}
\end{align*}
On the last receding horizon $\mathcal{R}_{n_0} = [n'_1(n_0), n_2]$, we have, for $m + 1 \leq k \leq n_2$,

$$x^*_k - \tilde{x}_k = \int_0^1 D_{\theta_1(n_0) - \theta_0(n_0)} \hat{p}^*_k(\theta_s(n_0)) \, ds$$

$$= \int_0^1 \left[ \nabla h_{n'_1(n_0)} p^*_k(\theta_s(n_0)) - \nabla d_n \hat{p}^*_k(\theta_s(n_0)) \right] \left[ \hat{h}_{n'_1(n_0)} - \tilde{x}_{n'_1(n_0)} \right] ds,$$

and for $m \leq k \leq n_2 - 1$,

$$u^*_k - \tilde{u}_k = \int_0^1 D_{\theta_1(n_0) - \theta_0(n_0)} \hat{p}^*_k(\theta_s(n_0)) \, ds$$

$$= \int_0^1 \left[ \nabla h_{n'_1(n_0)} s^*_k(\theta_s(n_0)) - \nabla d_n \hat{p}^*_k(\theta_s(n_0)) \right] \left[ \hat{h}_{n'_1(n_0)} - \tilde{x}_{n'_1(n_0)} \right] ds,$$

from which we have

$$\|x^*_k - \tilde{x}_k\|_2, \|u^*_k - \tilde{u}_k\|_2 \leq Z_1 \rho^{k - n'_1(n_0)} \|\hat{h}_{n'_1(n_0)} - \tilde{x}_{n'_1(n_0)}\|_2$$

$$\leq Z_1 \rho^{k - n'_1(n_0)} \|x^*_1(n_0) - \tilde{x}_{n'_1(n_0)}\|_2$$

$$\leq C_d Z_2 Z_1 \rho^{N - L - 1}/(1 - Z_1 \rho^L). \quad (41)$$

Combining (38), (39), (40) and (41) concludes this proof. \qed

Note that since the quantities $Z_1$ and $\rho$ are independent of the problem interval $[n_1, n_2]$, so is the choice of $L$. Therefore, Theorem 6 proves the approximation error of the RHC solution with an appropriate lag $L$ decays exponentially in the length $N$ of the receding horizon regardless of the full problem interval under the uniform boundedness and controllability conditions. The exponential decay rate provides systematic means to choose the length of the receding horizon given a desired accuracy level. We also note that it is a bit surprising that the choice $L = 1$ may sometimes not satisfy our assumptions; that case occurs when $Z_1 \rho > 1$. In particular, we note that in reference [11] it was shown that, for nonlinear RHC with zero reference, under controllability conditions similar to ours [11, Assumption 3.1] stabilization does occur for any $L$ (though note that the exponential decay of the initial value effect is assumed in that reference [11, (3.2)] as opposed to proved, as we do in Proposition 4, assuming the space-state form definition of controllability; moreover, the problem described in [11] is unconstrained, though nonlinear). That result was shown by analyzing the descent properties of the Lyapunov function induced by the objective of the RHC problem, given that the system had the zero state vector as a reference. Therefore the objective function value acted as a metric of state size, and reducing it meant approaching optimality and the stable solution. In the case with nonzero reference signal that we treat here, however, only the difference between the current value of the objective and the global minimum can at best play that role, and the latter value is not known a priori. While for analytical results, this may be less of an issue, several of the assumptions used in [11] are not obviously extensible to the case of non-zero reference signal, as is the case with, for example, of [11, Assumption 3.1]. Further difficulties are created by the fact that the infinite horizon objective function [11, (2.3)] for $N = \infty$ may be $\infty$ in the nonzero reference case. While it is certainly unsatisfying that, when the reference signal $d_k$ goes to zero, we do not recover the results from [11], we do not see an obvious way to extend our technique or combine it with the one in [11].
4 Numerical Results

In this section, we apply the receding horizon strategy to the following production cost model and verify some of the theoretical results.

\[
\begin{align*}
\min \quad & \sum_{k=1}^{T} c_1 (x_k - d_k)^2 + c_2 x_k^2 + u_k^2 \\
\text{s.t.} \quad & x_{k+1} = x_k + u_k, \\
& 0 \leq x_k \leq G.
\end{align*}
\] (42a)

In this model, \(d_k\) is the hourly electricity demand to be satisfied, for which we employ the estimated hourly demand data in the northern Illinois region for year 2016 provided by PJM Interconnection [12]. The demand can be satisfied by two generators: one with a high quadratic cost \(c_1 = 10\) and the other one with a low quadratic cost \(c_2 = 5\). The low-cost generator has a limited generation level, modeled by the upper bound (42c) on the generation \(x_k\). The generator with a high cost serves the remaining loads \(d_k - x_k\). We initialize problem (42) by setting the initial state \(x_0\) to be the average demand of year 2015 on the same hour as the initial time point. We note that problem (42) has the form of problem (1). Moreover, the active set of problem (42) at optimality is UDB as in Definition 1 with \(\lambda_H = 1\) since \(H_k\) defined in (3) is either 1 or \(-1\) if (42c) is active at \(k\).

We implement the receding horizon strategy described in Section 3 with lag \(L = 1\). Specifically, we solve a short version of problem (42) on a receding horizon \(\mathcal{R}_i = [n_1'(i), n_2'(i)]\) with length \(N \ll T\) and initial value \(\tilde{x}_{n_1'(i)}\), obtain the optimal control \(\tilde{u}_{n_1'(i)}\) and state \(\tilde{x}_{n_1'(i)+1}\), then reinitialize at time point \(n_1'(i) + 1\) with \(\tilde{x}_{n_1'(i)+1}\) to solve the problem on the next receding horizon \(\mathcal{R}_{i+1} = [n_1'(i) + 1, n_2'(i) + 1]\). Problem (42) is solved on the full horizon and each receding horizon using the Ipopt software [4]. The model was defined by using the Julia/JuMP interface [20].

We investigate the solution accuracy of the receding horizon strategy with different choices of the generation upper bound \(G\) to verify our theoretical findings. Denote

![Fig. 1](image_url)  
**Fig. 1** Relative error in states \(\|x^* - \tilde{x}\|_2/\|x^*\|_2\) at each receding horizon length (hour) \(N\) for \(G = 8000, \ldots, 14000\).
Fig. 2 Relative error in controls $\|u^* - \tilde{u}\|_2/\|u^*\|_2$ at each receding horizon length (hour) $N$ for $G = 8000, \ldots, 14000$

$x^* = \{x^*_k\}$ and $u^* = \{u^*_k\}$ as the optimal state and control of problem (42) on the full horizon, and $\tilde{x} = \{\tilde{x}_k\}$ and $\tilde{u} = \{\tilde{u}_k\}$ as those obtained by the receding horizon strategy. Figures 1 and 2 show the relative approximation errors of the optimal states and controls, respectively, for different choices of $G$. We observe exponential decay of the approximation error in the length of the receding horizon for all cases tested.

Assumption 3 requires problem (42) being controllable in $t$ steps at optimality. We thus investigate numerically the longest period $t$ for which problem (42) is not controllable. Since the problem is one-dimensional, controllability holds when no constraint is active. Therefore, we calculate the longest contiguous period for which (42c) is active at optimality. Table 1 shows $t$ in hours for different choices of $G$. The longest periods of uncontrollability are all less than two days, which is well covered by the one-year problem horizon (42). At small $G$, though, the uncontrollability period may be longer than the small horizons used by RHC for which the error decay is displayed in Figs. 1 and 2. In summary, the example problem (42) satisfies Assumptions 1 and 2, the boundedness of the data, by direct inspection, Assumption 3, the UBD and UCC (controllability) properties by the discussion above, and Assumption 4, the UDB property (Definition 1) for the short horizon RHC problems. Assumption 5 is satisfied only for large values of $G$. The numerical experiments satisfy the exponential decay of the approximation errors for RHC as proved in Theorem 6, and while our assumptions do not all appear necessary, they are certainly sufficient as implied by that theorem.

Table 1 The longest period $t$ (hours) for which the optimal solutions of problem (42) satisfy the constraint (42c)

| $G$    | 8000 | 10000 | 12000 | 14000 |
|--------|------|-------|-------|-------|
| $t$    | 44   | 15    | 11    | 2     |
Table 2 The longest period $t$ (hour) for which the optimal solutions of problem (43) satisfy at least one constraint, proportion $p$ of optimal solutions of problem (43) satisfying both constraints for $G = 12000$ (top) and $U = 600$ (bottom)

| $U$ | 200 | 400 | 600 | 800 | 1000 |
|-----|-----|-----|-----|-----|------|
| $t$ | 38  | 46  | 20  | 12  | 12   |
| $p$ | 0.01% | 0.09% | 0.01% | 0.03% | 0.03% |

| $G$ | 11000 | 12000 | 13000 | 14000 | 15000 |
|-----|-------|-------|-------|-------|-------|
| $t$ | 21    | 20    | 12    | 11    | 11    |
| $p$ | 0.05% | 0.01% | 0 | 0 | 0 |

We show in the following example that even with further moderate violation of the conditions, we may still have exponential decay of the approximation errors for the RHC solutions.

\[
\begin{align*}
\min & \quad \sum_{k=1}^{T} c_1(x_k - d_k)^2 + c_2x_k^2 + u_k^2 \\
\text{s.t.} & \quad x_{k+1} = x_k + u_k, \\
& \quad 0 \leq x_k \leq G, \\
& \quad -U \leq u_k \leq U.
\end{align*}
\]

Problem (43) has one more box constraint (43d) on the controls $u_k$ to model a limited capacity for the low-cost generator to change its output. One example of such a situation is the combination of a fast but expensive gas plant and a cheap but slow coal plant. If both constraints (43c) and (43d) are active at optimality for some time point $k$, the corresponding $H_k$ does not have full row rank and hence the UDB condition is not satisfied. In fact, Table 2 shows that a small proportion $p$ of the optimal solutions of problem (43) does have both constraints active for different choices of $G$ and $U$, hence violating Assumption 3(a) and

Fig. 3 Relative error in states $\|x^* - \tilde{x}\|_2/\|x^*\|_2$ at each receding horizon length (hour) $N$ for $U = 200, \ldots, 1000$ and $G = 12,000$
Assumption 4. Nonetheless, the longest periods $t$ of uncontrollability when at least one constraint is active are less than two days even under the tightest bounds of $U = 200$ or $U = 400$. Therefore, the controllability condition, Assumption 3(b) still holds for this example, in addition to the boundedness Assumptions 1 and 2.

Figures 3 and 4 show the relative approximation errors of the optimal states and controls, respectively, for a fixed $G$. We observe exponential decay of the approximation error in the length of the receding horizon for different choices of $U$. Moreover, the rate of decay is faster for a larger bound $U$ on the control. The decay rate $\rho$ in Theorem 6 depends on the
quantity $\beta$ defined in Proposition 2, which in turn depends on the uniform lower bound $\lambda_C$ of the controllability matrix defined in Definitions 4 and 6. A larger bound on the control improves the controllability of the problem and should lead to a faster rate of convergence, as is indeed observed here. Figures 5 and 6 plot the relative errors for different upper bound $G$ on the generation. Similarly, we observe exponential rate of decay for the approximation error in the length of receding horizon. Furthermore, in this case the decay rate is larger for a smaller choice of $G$. Recall that $G$ is the generation upper limit of the slow plant and our model (43) assumes the remaining load will be satisfied fully by a high-cost fast plant. A smaller $G$ indicates that more demand is met by the fast plant, and therefore the system is more controllable, resulting in a faster rate of convergence (for example, if $G = 0$, then we get the optimal solution to be $x_k = u_k = 0$, $\forall k$ and the convergence occurs in one step for any horizon).

5 Conclusions

RHC has made a significant impact on industrial control engineering and received extensive study of its theoretical characteristics. We investigate the convergence of its solution with respect to the length of the receding horizon for a linear-quadratic path-constrained optimal control problem.

The version of RHC considered in this work applies the model predictive control every $L$ steps. Our theoretical result, Theorem 6, shows that, under some boundedness and controllability conditions, the RHC solution converges to the full horizon solution exponentially fast in the length of receding horizon for a certain choice of $L$. The exponential rate of convergence allows a principled way of choosing the length of the receding horizon and the control frequency, both important parameters for applications, to achieve a desired accuracy. Our problem admits a nonzero reference trajectory, which to the best of our knowledge is not assumed in the existing stability analysis of RHC. The inclusion of a reference trajectory
makes the analysis different from previous approaches since now the convergence is with respect to the solution of the full horizon problem instead of a fixed equilibrium point. Therefore our proofs do not rely on the value function, as most RHC stability analyses do, but instead expose the solution properties of an equality-constrained subproblem and then use sensitivity analysis to connect it to the solution of the original problem. We verify numerically the exponential rate of convergence for a small, synthetic production cost model under various parameter settings. In this example, a lag $L = 1$ is sufficient to observe the exponential decay for the approximation error of the RHC solutions.

The class of optimal control problems investigated here is only one instance of the problems to which RHC can be applied. In particular, although we consider state and control constraints that are common in RHC literature, we do not include other intricate but practical features such as nonlinear dynamics or time delay. Moreover, our theory certifies only that an $L$, which is computable in terms of the problem data, exists; but $L = 1$ may not always satisfy our conditions. In future work, we will investigate extending the results to other complicating features as well as investigate whether we can obtain similar results with weaker assumptions, as our numerical results seem to indicate is possible.

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Appendix A: Proofs of Results in Sections 2 and 3

A.1 Proof of Proposition 2

For any $x_q \in \mathbb{R}^n$, consider the standard linear-quadratic problem:

$$\begin{align*}
\min_{x_k} & \quad n_2^{-1} \sum_{k=q}^{n_2-1} u_k^T R_k u_k + x_k^T \hat{Q}_k x_k + x_{n_2}^T \hat{Q}_{n_2} x_{n_2} \\
\text{s.t.} & \quad x_{k+1} = \hat{A}_k x_k + \hat{B}_k u_k, \quad q \leq k \leq n_2 - 1.
\end{align*}$$

(44a)

(44b)

For $k \geq q$, successively applying (44b) gives, for $j \geq 0$,

$$x_{q+j} - \left( \prod_{l=0}^{j-1} \hat{A}_{q+l} \right) x_q = \begin{bmatrix} \hat{B}_{q+j-1} \hat{A}_{q+j-1} \hat{B}_{q+j-2} \cdots \left( \prod_{l=0}^{j-1} \hat{A}_{q+l} \right) \hat{B}_q \end{bmatrix} \begin{bmatrix} u_{q+j-1} \\ \vdots \\ u_q \end{bmatrix},$$

(45)

and for $j = t$, (45) reduces to

$$x_{q+t} - \left( \prod_{l=0}^{t-1} \hat{A}_{q+l} \right) x_q = C_{q,t} \begin{bmatrix} u_{q+t-1} \\ \vdots \\ u_q \end{bmatrix}.$$
The index set being UCC($\lambda_C$) implies that $C_{q,t}$ is uniformly completely controllable and in particular that $C_{q,t}$ has full row rank. Therefore, there exists $\hat{u} = (\hat{u}_q^T, \ldots, \hat{u}_{q+t-1}^T)^T$ so that

$$ - \left( \prod_{l=0}^{t-1} \hat{A}_l \right) x_q = C_{q,t} \begin{bmatrix} \hat{u}_{q+t-1} \\ \vdots \\ \hat{u}_q \end{bmatrix} . \quad (46) $$

Several $\hat{u}$ satisfy this relationship; we consider the one defined by

$$ \hat{u} = - C_{q,t}^T \left( C_{q,t} C_{q,t}^T \right)^{-1} \left( \prod_{l=0}^{t-1} \hat{A}_q + l \right) x_q. $$

Denote the corresponding states generated with $\hat{u}_q$: $\hat{x}_{q}^{q+t-1}$ as $\hat{x}_{q}^{q+t-1}$. Then $\hat{x}_q^{q+t} = 0$ by (46).

Lemma 1 implies that

$$ \max_{1 \leq j \leq t} \| \left[ \hat{B}_{q+j-1} \hat{A}_{q+j-1} \hat{B}_{q+j-2} \ldots \left( \prod_{l=0}^{j-1} \hat{A}_q + l \right) \hat{B}_q \right] \|_2 \leq \max_{1 \leq j \leq t} \left( C_B + C_A C_B + \cdots + C_A^{j-1} C_B \right) \leq \frac{C_B \left( 1 - C_A^t \right)}{1 - C_A} \triangleq M. $$

Then from Definition 6 and Lemma 1, we have

$$ \| \hat{u} \| \leq \frac{M}{\lambda_C} C_A^t \| x_q \|. \quad (47) $$

From (45), we have, for $1 \leq j \leq t - 1$,

$$ \| \hat{x}_q^{q+j} \| \leq C_A^j \| x_q \| + M \| \hat{u} \| \leq \left( C_A^j + \frac{M^2}{\lambda_C} C_A^t \right) \| x_q \|. \quad (48) $$

Now we let $\hat{u}_k = 0$ for $k \geq q + t$. Then it follows that $\hat{x}_k = 0$ for $k \geq q + t$. Also note that since (44) is a standard linear-quadratic regulator problem, the optimal value is given by $x_q^T K_q x_q$ [2]. As a result, we have the following.

$$ x_q^T K_q x_q = \min_{u_k} \sum_{k=q}^{n_2-1} x_k^T \hat{Q}_k x_k + u_k^T R_k u_k + x_{n_2}^T Q_{n_2} x_{n_2} \leq \sum_{k=q}^{n_2-1} \hat{x}_k^T \hat{Q}_k \hat{x}_k + \hat{u}_k^T R_k \hat{u}_k + x_{n_2}^T Q_{n_2} x_{n_2} \leq \sum_{k=q}^{q+t-1} \hat{x}_k^T \hat{Q}_k \hat{x}_k + \hat{u}_k^T R_k \hat{u}_k \leq \sum_{k=q}^{q+t-1} \| \hat{x}_k \|^2 + \sum_{k=q}^{q+t-1} \| \hat{u}_k \|^2 \leq C_Q \sum_{k=q}^{q+t-1} \| \hat{x}_k \|^2 + C_R \sum_{k=q}^{q+t-1} \| \hat{u}_k \|^2 \leq C_Q \left( 1 + \sum_{i=1}^{t-1} \left( C_A^i + \frac{M^2}{\lambda_C} C_A^t \right)^2 \right) \| x_q \|^2 + C_R \frac{M^2 C_A^{2t}}{\lambda_C^2} \| x_q \|^2. \quad (47), (48) $$
Letting
\[ \beta = C_Q \left( 1 + \sum_{i=1}^{t-1} \left( C^i_A + \frac{M^2}{\lambda C} C'^i_A \right)^2 \right) + C_R \frac{M^2 C^2_A}{\lambda^2 C} \]
completes the proof. Note that \( \beta \) depends only on the quantities in Assumption 1, Definitions 1 and 6, and Lemma 1, which are independent of \( n_1, n_2 \), and the particular choice of \( \mathcal{I} \) given it is UDB(\( \lambda_H \)) and UCC(\( \lambda_C \)).

### A.2 Proof of Proposition 3

Define \( L_k = -W_k^{-1} \hat{B}_k^T K_{k+1} \hat{A}_k \). Then from Lemma 5 and (5d) we have \( D_k = \hat{A}_k + \hat{B}_k L_k \). In [2] the recursion (15b) is shown to be equivalent to
\[ K_k = D_k^T K_{k+1} D_k + \hat{Q}_k + L_k^T R_k L_k. \]  
(49)

For \( q \leq j \leq n_2 - 1 \), define \( x_{j+1} = D_j x_j \). Then (49) and Proposition 2 imply that
\[ x_j^T K_j x_j \geq x_{j+1}^T K_{j+1} x_{j+1} + x_j^T \hat{Q}_j x_j \]
Prop. 2 \[ \geq x_{j+1}^T K_{j+1} x_{j+1} + \frac{\lambda Q}{\beta} x_j^T K_j x_j \]
\[ \geq \left( 1 + \frac{\lambda Q}{\beta} \right) x_{j+1}^T K_{j+1} x_{j+1}. \]  
(50)

Here we used the bounds from Lemma 1 and the fact that \( x_j^T K_j x_j \geq x_{j+1}^T K_{j+1} x_{j+1} \), as implied by (49) and the positive definiteness of \( \hat{Q}_k \) and \( R_k \). Also we have
\[ x_j^T K_j x_j \geq x_j^T \hat{Q}_j x_j \geq \lambda Q \| x_j \|^2. \]  
(51)

As a result, for \( n_2 - 1 \geq j \geq q \), we have the following:
\[ \left\| \prod_{l=q}^{j} D_l x_l \right\|^2 \leq \left\| x_{j+1} \right\|^2 \]
\[ \leq \frac{1}{\lambda Q} x_{j+1}^T K_{j+1} x_{j+1} \]
\[ \leq \frac{1}{\lambda Q \left( 1 + \frac{\lambda Q}{\beta} \right)} x_j^T K_j x_j \]
\[ \leq \frac{1}{\lambda Q \left( 1 + \frac{\lambda Q}{\beta} \right)} \left( \frac{\lambda Q}{\beta} \right)^{j-q+1} x_q^T K_q x_q \]
Prop. 2 \[ \leq \frac{\beta}{\lambda Q \left( 1 + \frac{\lambda Q}{\beta} \right)} \left( \frac{\lambda Q}{\beta} \right)^{j-q+1} \| x_q \|^2, \]

where the third inequality is obtained by repeatedly applying (50).

### A.3 Proof of Lemma 9

Let
\[ L(y, \theta) = y^T G y/2 + y^T c(\theta) + \lambda^T (Ay - r) + \phi^T (By - d(\theta)) + \theta^T F \theta + y^T c_1 + \theta^T c_2 + C \]
be the Lagrangian of problem (31). Then we have
\[ \nabla^2_{(y, \theta)} L = \begin{bmatrix} G & \nabla_{\theta} c \\ \nabla_{\theta}^T c & * \end{bmatrix}. \]
Since $G$ and $F$ are positive definite and LICQ holds at $y_0$, then from [6, Theorem 5.53] and [6, Remark 5.55] we have

$$D_p y(\theta_0) = \arg\min_{h \in S} \left[ h^T \quad p^T \right] \left( \nabla^2_{(y, \theta)} I(y_0, \theta_0) \right) \left[ \begin{array}{c} h \\ p \end{array} \right]$$

$$= \arg\min_{h \in S} h^T G h/2 + p^T \left( \nabla^T_\theta c(\theta_0) \right) h,$$

where $S$ is the solution of the following linearized problem,

$$\begin{align*}
\min_s & \quad (G y_0 + c(\theta_0) + c_1)^T s + \left( \nabla^T_\theta c(\theta_0) y_0 + 2 F \theta_0 + c_2 \right)^T p \\
\text{s.t.} & \quad Bs - (\nabla_\theta d(\theta_0)) p = 0, \\
& \quad A_{I_0(y_0, \theta_0)} s \leq 0,
\end{align*}$$

and $S$ is given by

$$S = \left\{ s : \left[ \begin{array}{c} B \\ -\nabla_\theta d(\theta_0) \end{array} \right] \left[ \begin{array}{c} s \\ p \end{array} \right] = 0, \quad \left[ A_{I_+(y_0, \theta_0, \bar{\lambda})} 0 \right] \left[ \begin{array}{c} s \\ p \end{array} \right] = 0, \quad \left[ A_{I_0(y_0, \theta_0, \bar{\lambda})} 0 \right] \left[ \begin{array}{c} s \\ p \end{array} \right] \leq 0 \right\}.$$ 

Thus the directional derivative $D_p y(\theta_0)$ of $y(\theta)$ along direction $p$ at $\theta_0$ is the solution of the problem

$$\begin{align*}
\min_h & \quad h^T G h/2 + p^T \left( \nabla^T_\theta c(\theta_0) \right) h \\
\text{s.t.} & \quad Bh - (\nabla_\theta d(\theta_0)) p = 0, \\
& \quad A_{I_+(y_0, \theta_0, \bar{\lambda})} h = 0, \\
& \quad A_{I_0(y_0, \theta_0, \bar{\lambda})} h \leq 0.
\end{align*}$$

(52)

Let $I_1$ be the set of active inequality constraints of problem (52). Then $I_1 \subset I_0(y_0, \theta_0, \bar{\lambda})$, and let $I'(\theta_0) = I_1 \cup I_+(y_0, \theta_0, \bar{\lambda})$. The KKT condition of problem (52) is hence

$$\tilde{G} \overset{\Delta}{=} \left[ \begin{array}{cccc} G & A^T_1(y_0) & B^T \\
A_{I'(\theta_0)} & 0 & 0 \\
B & 0 & 0 \end{array} \right], \quad \tilde{G} = \left[ \begin{array}{c} h^* \\
h_1^* \\
h_2^* \end{array} \right] = \left[ \begin{array}{c} -\nabla_\theta c(\theta_0) p \\
0 \\
\nabla_\theta d(\theta_0) p \end{array} \right],$$

for some Lagrange multipliers $\phi_1^*$ and $\phi_2^*$. Since LICQ holds at $y_0$, rows of $A_{I'(\theta_0)}$ and $B$ are linearly independent. Together with the fact that $G$ is positive definite, we have that $\tilde{G}$ is invertible. Denote the first row of $\tilde{G}^{-1}$ to be $[p_{11} \quad p_{12} \quad p_{13}]$. Then, we have

$$D_p y(\theta_0) = h^* = (-p_{11} \nabla_\theta c(\theta_0) + p_{13} \nabla_\theta d(\theta_0)) p.$$

On the other hand, for problem (32) with $I'(\theta_0)$ constructed above, the KKT condition is

$$\tilde{G} \left[ \begin{array}{c} y^*_{I'(\theta_0)}(\theta) \\
\psi_1^* \\
\psi_2^* \end{array} \right] = \left[ \begin{array}{c} -c(\theta) \\
r' \\
d(\theta) \end{array} \right],$$

for some Lagrange multipliers $\psi_1^*$ and $\psi_2^*$. Since $\tilde{G}$ is invertible, we have $y^*_{I'(\theta_0)}(\theta) = -p_{11} c(\theta) + p_{12} r' + p_{13} d(\theta)$. It follows that

$$\left. \frac{dy^*_{I'(\theta_0)}(\theta)}{d\theta} \right|_{\theta=\theta_0} = -p_{11} \nabla_\theta c(\theta_0) + p_{13} \nabla_\theta d(\theta_0).$$
As a result, we have

\[ D_p y(\theta_0) = \left( \frac{dy^*_{I(\theta_0)}(\theta)}{d\theta} \right) \bigg|_{\theta = \theta_0} p, \]

which proves the claim.

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