Non-Markovian interaction of many fields

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We study the interaction between several fields initially in coherent states. The solution allows us to explain why coherent states remain coherent states when subject to non-Markovian dissipation. We first study the interaction between two fields and show that this is the building block of the total interaction. We give a completely algebraic solution of this system.

I. INTRODUCTION

It is well known that a coherent state subject to dissipation keeps its form during the dynamics. This is, given the master equation for a field in a lossy cavity at zero temperature

\[ \frac{d\rho}{dt} = \gamma (a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a) \] (1)

with \( a (a^\dagger) \) the annihilation (creation) operator for the cavity mode, \( \gamma \) the decay constant and \( \rho \) the density matrix, if the initial state of the cavity field is a coherent state \( |\alpha\rangle \), then the dynamics shows that it will decay in time as \( |\alpha e^{-\gamma t}\rangle \) (see for instance \[1\]). One possible answer about why the coherent states preserve its form during decay is the fact that coherent states are eigenstates of the annihilation operator, however this argument does not hold for a dissipative two-photon process \[2\]

\[ \frac{d\rho}{dt} = \gamma (a^2\rho a^\dagger - a^\dagger a^2\rho - \rho a^\dagger a^2) \] (2)

even though coherent states are also eigenstates of the annihilation operator squared (so do even and odd coherent states \[3\]).

Both equations above are obtained using Born-Markov approximations \[2, 4\]. In the case in which such approximations are not used, i.e. when the interaction between a harmonic oscillator and a set of harmonic oscillators (the environment) is considered, is not clear how a coherent state decays. Here we will try to answer this question. First we will consider the interaction between two fields, to later generalize the result to a field interacting with many.

II. TWO FIELDS INTERACTING

Consider the Hamiltonian of two interacting fields (we set \( \hbar = 1 \))

\[ H = \omega_a a^\dagger a + \omega_b b^\dagger b + \lambda (a^\dagger b + b^\dagger a), \] (3)

a system like this may be produced by the interaction of two quantized fields with a two-level atom \[5\] by transforming to the interaction picture, i.e. getting rid off the free Hamiltonians, we obtain

\[ H_I = \Delta a^\dagger a + \lambda (a^\dagger b + b^\dagger a), \] (4)

with \( \Delta = \omega_a - \omega_b \), the detuning. It is useful to define normal-mode operators by \[6\]

\[ A_1 = \delta a + \gamma b, \quad A_2 = \gamma a - \delta b, \] (5)

with

\[ \delta = \frac{2\lambda}{\sqrt{2\Omega(\Omega - \Delta)}}, \quad \gamma = \sqrt{\frac{\Omega - \Delta}{2\Omega}} \] (6)

with \( \Omega = \sqrt{\Delta^2 + 4\lambda^2} \) the Rabi frequency. \( A_1 \) and \( A_2 \) are annihilation operators just like \( a \) and \( b \) and obey the commutation relations

\[ [A_1, A_1^\dagger] = [A_2, A_2^\dagger] = 1, \] (7)
moreover, the normal-mode operators commute with each other

\[ [A_1, A_2] = [A_1, A_2^\dagger] = 0. \]  

(8)

In terms of these operator the Hamiltonian (4) becomes

\[ H_I = \mu_1 A_1^\dagger A_1 + \mu_2 A_2^\dagger A_2, \]  

(9)

with \( \mu_{1,2} = (\Delta \pm \Omega)/2 \). Up to here, we have translated the problem of solving Hamiltonian (1) into the problem of obtaining the initial states, for the "bare" modes in the initial states for the normal modes. In order to have a more general way of transforming states from one basis to the other, we note that the vacuum states in both systems \(|0\rangle_a|0\rangle_b\) and \(|0\rangle_{A_1}|0\rangle_{A_2}\) differ only for a phase \( e^{i\phi} \). First note that

\[ A_1|0\rangle_a|0\rangle_b = 0, \]  

(10)

and in a similar way it may be seen the other normal-mode annihilation operator, \( A_2 \), has the same effect. Choosing the phase so that

\[ |0\rangle_a|0\rangle_b = |0\rangle_{A_1}|0\rangle_{A_2}. \]  

(11)

If we consider coherent states as initial states for the interaction, we obtain the evolved wavefunction

\[ |\psi(t)\rangle = e^{-i(t(\mu_1 A_1^\dagger + \mu_2 A_2^\dagger) + \mu_2 A_2^\dagger A_2)} D_a(\alpha) D_b(\beta)|0\rangle_a|0\rangle_b, \]  

\[ = e^{-i(t(\mu_1 A_1^\dagger + \mu_2 A_2^\dagger) + \mu_2 A_2^\dagger A_2)} D_a(\alpha) D_b(\beta)|0\rangle_{A_1}|0\rangle_{A_2}, \]  

(12)

where the \( D_c(\epsilon) = \exp(\epsilon a^\dagger - \epsilon^* c) \) is the Glauber displacement operators \( \hat{c} \). From (12) we can write the operators \( a \) and \( b \) in terms of the operator \( A_1 \) and \( A_2 \) [12] as

\[ |\psi(t)\rangle = e^{-i(t(\mu_1 A_1^\dagger + \mu_2 A_2^\dagger) + \mu_2 A_2^\dagger A_2)} D_{A_1}(\alpha \delta + \beta \gamma) D_{A_2}(\alpha \gamma - \beta \delta)|0\rangle_{A_1}|0\rangle_{A_2}. \]  

(13)

Passing the exponential in the above equation to the right and applying it to the vacuum states we obtain

\[ |\psi(t)\rangle = D_{A_1}(\alpha \delta + \beta \gamma e^{-i\mu_1 t}) D_{A_2}(\alpha \gamma - \beta \delta e^{-i\mu_2 t})|0\rangle_{A_1}|0\rangle_{A_2} \]  

\[ = |(\alpha \delta + \beta \gamma e^{-i\mu_1 t}) A_1(\alpha \gamma - \beta \delta e^{-i\mu_2 t}) A_2. \]  

(14)

Equation (14) shows that in the new basis, coherent states remain coherent during evolution. By transforming back to the original basis we obtain

\[ |\psi(t)\rangle = |\delta(\alpha \delta + \beta \gamma e^{-i\mu_1 t} + \gamma(\alpha \gamma - \beta \delta e^{-i\mu_2 t})_a|_1(\alpha \delta + \beta \gamma e^{-i\mu_1 t} - \delta(\alpha \gamma - \beta \delta e^{-i\mu_2 t})_b, \]  

(15)

i.e. coherent states remain coherent during evolution. This will be used next Section as the building block for the interaction of many modes. In obtaining (14) and (15), we have used the following property

\[ D_c(\epsilon_1) D_c(\epsilon_2) = D_c(\epsilon_1 + \epsilon_1) e^{\frac{1}{2}(\gamma \epsilon_2^* - \epsilon_1^* \gamma_2)}. \]  

(16)

### III. Generalization to \( n \) Modes

Consider the Hamiltonian of the interaction of \( k \) fields

\[ \hat{H} = \sum_j \omega_j \hat{n}_j + \sum_{j \neq k} \lambda_{ij} \left( \hat{a}_i^\dagger \hat{a}_j + \hat{a}_i \hat{a}_j^\dagger \right). \]  

(17)

From the Hamiltonian above, we can produce the following matrix

\[ M = \begin{pmatrix} \omega_1 & \lambda_{12} & \ldots & \lambda_{1k} \\ \lambda_{12} & \omega_2 & \ldots & \lambda_{2k} \\ \lambda_{13} & \lambda_{23} & \ldots & \lambda_{3k} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{1k} & \lambda_{2k} & \ldots & \omega_k \end{pmatrix}. \]  

(18)
We can rewrite the Hamiltonian in the form (9)

$$\hat{H} = \sum_{m}^{n} \mu_{m} \hat{A}_{m}^{\dagger} \hat{A}_{m},$$  \hspace{1cm} (19)

such that

$$[\hat{A}_{k}, \hat{A}_{m}^{\dagger}] = 0,$$  \hspace{1cm} (20)

where we have defined the normal-mode operators $\hat{A}_{k}$ as

$$\hat{A}_{k} = \sum_{i=1}^{n} r_{ki} \hat{a}_{i},$$  \hspace{1cm} (21)

with $r_{ki}$ a real number.

Equation (20) implies that

$$[\hat{A}_{k}, \hat{A}_{m}^{\dagger}] = \sum_{i,j=0}^{n} r_{ki} r_{mj} [\hat{a}_{i}, \hat{a}_{j}^{\dagger}] = \sum_{i}^{n} r_{ki} r_{mi} = 0.$$ \hspace{1cm} (22)

By defining the vector

$$\vec{r}_{k} = (r_{k1}, r_{k2}, ..., r_{kn}),$$  \hspace{1cm} (23)

equation (22) takes the form $\vec{r}_{n} \cdot \vec{r}_{m} = 0$, i.e. they are orthogonal, we will consider them also normalized, $\vec{r}_{k} \cdot \vec{r}_{k} = 1$.

With these vectors we can form the matrix

$$R = \begin{pmatrix}
r_{11} & r_{21} & \cdots & r_{n1} \\
r_{12} & r_{22} & \cdots & r_{n2} \\
\vdots & \vdots & \ddots & \vdots \\
r_{1n} & r_{2n} & \cdots & r_{nn}
\end{pmatrix},$$  \hspace{1cm} (24)

If we combine equations (17), (19) and (21) we obtain the system of equations

$$\sum_{m}^{n} \mu_{m} r_{mi}^{2} = \omega_{i},$$  \hspace{1cm} (25)

$$\sum_{m}^{n} \mu_{m} r_{mi} r_{mj} = \lambda_{ij},$$  \hspace{1cm} (26)

that may be re-expressed in the compact form

$$RDR^{\dagger} = M = \begin{pmatrix}
\omega_{1} & \lambda_{21} & \cdots & \lambda_{n1} \\
\lambda_{12} & \omega_{2} & \cdots & \lambda_{n2} \\
\lambda_{13} & \lambda_{23} & \cdots & \lambda_{n3} \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_{1n} & \lambda_{2n} & \cdots & \omega_{n}
\end{pmatrix},$$  \hspace{1cm} (27)

with

$$D = \begin{pmatrix}
\mu_{1} & 0 & \cdots & 0 \\
0 & \mu_{2} & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \mu_{n}
\end{pmatrix},$$  \hspace{1cm} (28)
i.e. \( D \) is a diagonal matrix whose elements are the eigenvalues of the matrix \( M \), defined from the Hamiltonian. The matrix \( R \) is therefore \( M \)'s eigenvectors matrix. The solution to the Schrödinger equation subject to the Hamiltonian (17) with all the modes initially in coherent states, \( |\psi(0)\rangle = |\alpha_1\rangle_1|\alpha_2\rangle_2...|\alpha_n\rangle_n \), is simply the direct product of coherent states

\[
|\psi(t)\rangle = |\vec{r}_1 \cdot \vec{\beta}(t)\rangle_1|\vec{r}_2 \cdot \vec{\beta}(t)\rangle_2...|\vec{r}_n \cdot \vec{\beta}(t)\rangle_n
\]

(29)

with \( \vec{\beta}(t) = (\vec{r}_1 \cdot \vec{\alpha}_e^{-i\mu_1 t}, \vec{r}_2 \cdot \vec{\alpha}_e^{-i\mu_2 t}, ..., \vec{r}_n \cdot \vec{\alpha}_e^{-i\mu_n t}) \) and the vector \( \vec{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_n) \) is composed by the coherent amplitudes of the initial wave function. Up to here we have shown that the interaction of several modes initially in coherent states does not change the form of those states (remain coherent), but modifies their amplitude. If we choose the interaction constants to be \( \lambda_{ij} \neq 0 \) for \( i \neq j \) and the rest as zero, we are dealing with the interaction between one field and \( n-1 \) fields. If \( n \to \infty \) and the amplitudes \( \alpha_j \) are zero for \( j > 1 \), we deal with the interaction of one field with \( n-1 \) one of them in a coherent state with amplitude \( \alpha_1 \) and the rest in the vacuum. Therefore, the most likely situation we have is the coherent state decaying towards the vacuum while keeping its coherent form.

**IV. CONCLUSIONS**

We have shown that a system of \( n \) interacting harmonic oscillators initially in coherent states, remain coherent during the interaction. In particular, if one considers one field (harmonic oscillator) interacting with many fields (harmonic oscillators), i.e. consider only \( \lambda_{1j} \neq 0 \) and \( \lambda_{ij} \neq 0 \) for \( j > 2 \), and all the others to be zero, we can model non-Markovian system-reservoir interaction. If we consider the system to be in a coherent state and all the others fields that form the environment in a vacuum state (this is also in coherent states with zero amplitude), after evolution, the amplitude of the coherent state will diminish, as one photon will go to another mode, keeping its coherent nature. If the number of modes that form the environment is very large, an event of the photon going back to the system is quite unlikely. Therefore the next probable event is precisely the loss of another photon by the system, etc. until it arrives to a state close to the vacuum. In case the number of modes interacting with the system is infinite, then the vacuum would be the final state of the system. In other words, the total system perform the following transition

\[
|\alpha\rangle_1|0\rangle_2...|0\rangle_n \to |\delta_1\rangle_1|\delta_2\rangle_2...|\delta_2\rangle_n,
\]

(30)

where the coherent amplitudes, \( \delta_k \to 0 \), as \( n \to \infty \).

In conclusion we have given a complete algebraic solution to the problem of \( n \) interacting harmonic oscillators, without Born-Markov approximations.

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