MINIMAL SURFACE ENTROPY AND AVERAGE AREA RATIO

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Abstract. On any closed hyperbolizable 3-manifold, we find a sharp relation between the minimal surface entropy (introduced by Calegari-Marques-Neves) and the average area ratio (introduced by Gromov), and we show that, among metrics $g$ with scalar curvature greater than or equal to $-6$, the former is maximized by the hyperbolic metric. One corollary is to solve a conjecture of Gromov regarding the average area ratio.

Our proofs use Ricci flow with surgery and laminar measures invariant under a $\text{PSL}(2, \mathbb{R})$-action.

1. Introduction

The interplay between scalar curvature, area, and topology is a beautiful chapter in mathematics. For an extended overview of the subject containing the most recent developments, the reader can consult Gromov’s Four Lectures on Scalar Curvature [16].

We are interested in studying the area functional on closed manifolds admitting a hyperbolic metric. Variational methods, which were pioneered by Schoen and Yau, are less effective in this setting because the restrictions imposed by the second variation of area are not sharp in the hyperbolic case. On the other hand, the study of length, curvature, and topology has a very rich literature in the negative curvature case, partly because the fact that the geodesic flow is Anosov brings an extra structure to the problem. In the same vein, there is an extra structure for minimal surfaces coming from a “natural” $\text{PSL}(2, \mathbb{R})$-action on the space of minimal immersions (formalized very clearly by Labourie in [28]). The general principle we follow, initiated by Calegari–Marques–Neves in [9], is to combine the rigidity of that action (due to Ratner and Shah) with geometric methods to obtain sharp relations between area, scalar curvature, and minimal surfaces. One consequence is to answer a conjecture of Gromov regarding the least possible value for the average area ratio.

Using spin methods, Min-Oo [32] proved a rigidity theorem for asymptotically hyperbolic manifolds (see also [21, 42]) and Ono [33], Davaux [11] proved sharp spectral inequalities for hyperbolizable manifolds. Andersson, The second author is partly supported by NSF DMS-2005468 and a Simons Investigator Grant.
Cai, and Galloway [2] proved a positive mass theorem for asymptotically hyperbolic manifolds using variational methods.

Let \((M, g)\) be a closed Riemannian orientable 3-manifold admitting a hyperbolic metric \(g_0\). We refer the reader to Section 2 for all the definitions.

1.1. Minimal Surface Entropy. Let \(S_\varepsilon(M)\) denote the set of all homotopy class \(\Pi\) of essential surfaces whose limit set is a \((1 + \varepsilon)\)-quasicircle. An important result of Kahn–Markovic [23, 24] establishes that \(S_\varepsilon(M) \neq \emptyset\) (assuming \(\varepsilon > 0\)) and provides an estimate for the cardinality of \(S_\varepsilon(M)\). We define

\[
\text{area}_g(\Pi) := \inf \{ \text{area}_g(S) : S \in \Pi \}.
\]

Inspired by the following expression for the volume entropy \(E_{\text{vol}}(g)\) on negatively curved manifolds

\[
E_{\text{vol}}(g) = \lim_{L \to \infty} \frac{\ln \# \{ \text{length}_g(\gamma) \leq L : \gamma \text{ closed geodesic in } (M, g) \}}{L}
\]

Calegari, Marques, and Neves defined in [9] the minimal surface entropy

(1) \[ E(g) := \lim_{\varepsilon \to 0} \limsup_{L \to \infty} \frac{\ln \# \{ \text{area}_g(\Pi) \leq 4\pi(L - 1) : \Pi \in S_\varepsilon(M) \}}{L \ln L}. \]

The authors showed in [9] that \(E(g_0) = 2\) and that, among metrics \(g\) with sectional curvature less than or equal to \(-1\), \(E(g) \geq E(g_0) = 2\) and equality implies \(g = g_0\) (up to isometry). The inequality follows almost directly from Gauss equation. On the other hand, to show that the hyperbolic metric is the unique minimizer the authors had to use the rigidity of \(\text{PSL}(2, \mathbb{R})\)-orbits (due to Ratner [35] and Shah [38]).

1.2. Average Area Ratio. Consider another closed hyperbolic 3-manifold \(N\) with an hyperbolic metric \(h_0\). The Grassmanian bundle over \(N\) (or \(M\)) of unoriented tangent 2-planes to \(N\) (or \(M\)) is denoted by \(\text{Gr}_2(N)\) (or \(\text{Gr}_2(M)\)). The metric \(h_0\) induces a natural metric on \(\text{Gr}_2(N)\).

Given \(T : M \to N\) a smooth map, the following function is defined for almost all \((x, P) \in \text{Gr}_2(N)\):

\[
(x, P) \in \text{Gr}_2(N) \mapsto | \wedge^2 T^{-1} |(x, P) := \lim_{\delta \to 0} \frac{\text{area}_g(T^{-1}(P_\delta))}{\delta},
\]

where \(P_\delta \subset N\) is a disc transversal to \(T\), tangent to \(P\) at \(x \in N\), and area \(\delta\).

Gromov [14, page 73] defined \(\text{Area}_T(g/h_0)\) the average area ratio of \(T\) to be

\[
\text{Area}_T(g/h_0) := \frac{1}{\text{vol}_{h_0}(\text{Gr}_2(N))} \int_{\text{Gr}_2(N)} | \wedge^2 T^{-1} |(x, P) dV_{h_0}(x, P).
\]

A more detailed definition is in (2).

In the same paper, Gromov used a second variation argument to show that \(\text{Area}_T(g/h_0) \geq \text{degree } T/3\) if the scalar curvature of \(g\) satisfies \(R(g) \geq -6\).
The reason he got the factor $1/3$ is because the second variation of area is not sharp on hyperbolic manifolds. He conjectured in Remark 2.4.C’ of [14] that assuming $R(g) \geq -6$

$$\text{Area}_{T}(g/h_0) \geq \text{degree} T \text{ holds true and it is sharp.}$$

This question was also addressed in Section 3.H of [15] and in the more recent [16] at the end of Section 5.9.

For surfaces, the correspondent problem was solved by Katok. On a closed surface with two negatively curved metrics $\sigma, \sigma_1$, Katok [26] defined the average length ratio $L(\sigma/\sigma_1) := \int_{T^1S} ||v||_{\sigma}dV_{\sigma_1}(v)$ (he used a different notation). With $h(\sigma)$ denoting the topological entropy of $\sigma$, he showed [26, Theorem A] that

$$h(\sigma) L(\sigma/\sigma_1) \geq h(\sigma_1).$$

Thus when $\sigma_1$ is hyperbolic and $\sigma$ has Gaussian curvature $K(\sigma) \geq -1$ we have $h(\sigma) \leq h(\sigma_1)$ and so $L(\sigma/\sigma_1) \geq 1$.

1.3. **Motivation.** We follow the spirit of [14, 15] and explain, informally, one of the motivations behind $\text{Area}_{Id}(g/g_0)$. An important object in the study of negatively curved manifolds is the 1-dimensional foliation of the unit tangent bundle where each leaf is an orbit for the geodesic flow.

Gromov proposed the following surface analogue: A 2-dimensional foliation $\mathcal{L}$ of $Gr_2(M)$ is a family $\{S_\lambda\}_{\lambda \in \Lambda}$ of complete surfaces of $M$ so that through every point in $(p, \tau) \in Gr_2(M)$ there is a unique $\lambda$ so that $S_\lambda$ passes through $p$ with tangent plane $\tau$. Laminations where each leaf is a stable minimal surface are analogous to the geodesic flow foliation of the unit tangent bundle. There is a canonical foliation $\mathcal{L}_0$ of $Gr_2(M)$ whose leaves are totally geodesic planes.

Given a metric $g$ on $M$ we obtain an area form on each leaf of $\mathcal{L}$. Assuming $\mathcal{L}$ has a transversal measure we can integrate the area of each leaf with respect to this measure and obtain $vol_g(\mathcal{L})$.

The general question Gromov asks is how low can one make $vol_g(\mathcal{L})$ subject to the constraint that $R(g) \geq -6$. There is a “natural” transversal measure on $\mathcal{L}_0$ such that $\text{Area}_{Id}(g/g_0)$ coincides with $vol_g(\mathcal{L}_0)/vol_{g_0}(\mathcal{L}_0)$. Therefore, Gromov’s conjecture when $T = \text{Id}$ can be rephrased as

$$vol_g(\mathcal{L}_0) \geq vol_{g_0}(\mathcal{L}_0) \text{ if } R(g) \geq -6.$$  

The following comparison is also interesting: if we consider a 3-dimensional foliation of $Gr_3(M) \simeq M$, there is only one leaf, the (unit) transverse measure is trivial, and given a metric on $g$, the volume of this foliation is $vol_g(M)$. Schoen conjectured that $vol_g(M) \geq vol_{g_0}(M)$ among all metrics with $R(g) \geq -6$ and this was proven by Perelman using Ricci flow (see [3] for the argument).

1.4. **Main Theorems.** The next theorem uses Ricci flow in the spirit of [8, 31] and PSL($2, \mathbb{R}$)-invariant laminar measures.

**Theorem 1.1.** Suppose $(M, g)$ has $R(g) \geq -6$. Then

$$E(g) \leq 2$$
and equality occurs if and only if $g$ is isometric to the hyperbolic metric.

This result was shown by the first author in [30] under the condition that the (normalized) Ricci flow starting at $g$ converges to the hyperbolic metric faster than $e^{-ct}$ for some $c > 1$. As it is pointed out in [30], there are closed manifolds $\mathbb{H}^3/\Gamma$ for which that will happen for all metrics and closed manifolds $\mathbb{H}^3/\Gamma$ with metrics for which that will not happen. Manifolds containing closed embedded totally geodesic surfaces fall into the second category. Later the second author shared his notes containing an approach to the general case relying on $\text{PSL}(2,\mathbb{R})$-invariant laminar measures. Both authors decided to join their efforts and write a single paper containing that and other related results.

We note that Agol, Storm, and Thurston [4] conjectured that $E_{\text{vol}}(g)$ is maximized by the hyperbolic metric among all metrics with $R(g) \geq -6$.

The next theorem relates the average area ratio and the minimal surface entropy in a sharp way.

**Theorem 1.2.** For every Riemannian metric $(M, g)$ we have

$$\text{Area}_{\text{id}}(g/g_0)E(g) \geq 2$$

and equality holds if and only if $g = cg_0$ for some constant $c > 0$.

Appealing to the previous interpretation of $\text{Area}_{\text{id}}(g/g_0)$, we can restate Theorem 1.2 as saying that for every metric $g$

$$\text{vol}_g(L_0)E(g) \geq 2\text{vol}_{g_0}(L_0)$$

with equality if and only if $g = cg_0$ for some $c > 0$. This is reminiscent of Besson, Courtois, and Gallot [7] which says that

$$\text{vol}_g(M)^{1/3}E_{\text{vol}}(g) \geq 2\text{vol}_{g_0}(M)^{1/3}$$

for every metric $g$ with equality if and only if $g$ is isometric to $g_0$. It is a conjecture of Calegari-Marques-Neves [10] that

$$\text{vol}_g(M)^{2/3}E(g) \geq 2\text{vol}_{g_0}(M)^{2/3}$$

for every metric $g$ with equality occurring only for $g_0$ (up to isometry).

We now deduce two corollaries. The first one follows from combining Theorem 1.1 and Theorem 1.2.

**Corollary 1.3.** Consider $T : M \to N$ a local diffeomorphism with degree $d$. If $(M, g)$ has scalar curvature $R(g) \geq -6$ then

$$\text{Area}_T(g/h_0) \geq d$$

and equality holds if and only if $T$ is a local isometry between $g$ and $h_0$.

This result confirms Gromov’s conjecture for local diffeomorphisms. If we just assume that $T$ has degree $d$ there may exist no local isometry between $M$ and $N$ and thus no obvious optimal map.
Proof of Corollary 1.3. We have
\[ \text{Area}_T(g/h_0) = d \text{Area}_{id}(g/T^*(h_0)). \]
Hence, with \( g_0 := T^*(h_0) \), we have from Theorem 1.2 and Theorem 1.1 that
\[ 2 \text{Area}_T(g/h_0) \geq E(g) \text{Area}_T(g/h_0) = dE(g) \text{Area}_{id}(g/g_0) \geq 2d \]
and equality holds if and only if \( g = g_0 = T^*(h_0) \). \( \square \)

The second corollary is below.

Corollary 1.4. Suppose \( \Sigma \subset M \) is a totally geodesic closed immersion with respect to the hyperbolic metric \( g_0 \). If \( (M, g) \) has scalar curvature \( R(g) \geq -6 \) then, with \( \Pi \) denoting the homotopy class of \( \Sigma \),
\[ \text{area}_g(\Pi) \geq -2\pi \chi(\Sigma) \]
with equality if and only if \( g \) is an hyperbolic metric.

Proof. This follows from the proof of Theorem 1.1, namely Theorem 5.1 applied with \( \Pi_m = \Pi \) for all \( m \in \mathbb{N} \). \( \square \)

Without the condition that \( \Pi \) contains a totally geodesic immersion in the hyperbolic metric the result fails already for \( g = g_0 \).

1.5. Sketch of proofs. We describe succinctly the main ideas behind the proofs of Theorem 1.1 and Theorem 1.2.

Assume that \( (M, g) \) has \( R(g) \geq -6 \). To prove Theorem 1.1 we need to show that for every sequence of homotopy classes \( \Pi_m \in S_{1/m}(M) \) we have
\[ \lim_{m \to \infty} \text{area}_g(\Pi_m)/\text{area}_{g_0}(\Pi_m) \geq 1. \]
Let \( (g_t)_{t \geq 0} \) be a solution to normalized Ricci flow, which we first assume exists for all time and converges to \( g_0 \). We show that if the inequality above fails then for all \( m \) sufficiently large we have for some \( \delta > 0 \) and all \( t \geq 0 \)
\[ \text{area}_{g_t}(\Pi_m)/\text{area}_{g_0}(\Pi_m) \leq 1 - \delta e^{-t}. \]
Stability analysis shows that \( g_t \simeq g_0 + e^{-t}\bar{h} \) as \( t \to \infty \), where \( \bar{h} \) is an eigentensor for the linearization of the trace-free Ricci tensor at the hyperbolic metric. Letting \( m \to \infty \) we obtain measures \( \mu_t \) on the frame bundle of \( M \) so that
\[ \mu_t(1) = \lim_{m \to \infty} \text{area}_{g_t}(\Pi_m)/\text{area}_{g_0}(\Pi_m). \]
Necessarily \( \mu_t(1) \leq 1 - \delta e^{-t} \). Using a form of Gauss equation (see (12)) for these measures we have an identity of the type
\[ 1 = \mu_t(1) + \mu_t(\text{curvature terms}). \]
Combining with the asymptotics \( g_t \simeq g_0 + e^{-t}\bar{h} \) we show
\[ 1 = \mu_t(1) + \mu_t(\text{curvature terms}) = \mu_t(1) + e^{-t}\mu_{+\infty}(\text{terms with } \bar{h}) + o(e^{-t}). \]
The measure $\mu_{+\infty}$ is $\text{PSL}(2, \mathbb{R})$-invariant. Hence, using Ratner’s classification theorem [35] and the fact that $\bar{h}$ is an eigentensor, we show that

$$\mu_{+\infty}(\text{terms with } \bar{h}) = 0.$$  

Thus $1 = \mu_t(1) + o(e^{-t})$, which contradicts $\mu_t(1) \leq 1 - \delta e^{-t}$.

For the general case, we proceed as above but use Perelman’s Ricci flow with surgery [34]. Because $M$ has a hyperbolic metric all surgeries correspond to removing capped horns and replacing them by standard caps. The key point to check is that essential surfaces minimizing area stay away from the capped horns. We achieve this via area comparison.

We now sketch the argument behind Theorem 1.2. We start by improving a construction of Labourie in [28] and find a sequence $\Sigma_m \subset M$ of connected immersed minimal surfaces with respect to $g_0$ which becomes equidistributed in the frame bundle. We show in Proposition 6.3 that

$$\text{Area}_{\text{Id}}(g/g_0) = \lim_{m \to \infty} \frac{\text{area}_g(\Sigma_m)}{\text{area}_{g_0}(\Sigma_m)}.$$  

A counting argument implies that

$$E(g) \geq E(g_0) \lim_{m \to \infty} \frac{\text{area}_{g_0}(\Sigma_m)}{\text{area}_g(\Sigma_m)}$$  

and these two expressions give that $E(g)\text{Area}_{\text{Id}}(g/g_0) \geq 2$. If equality holds we show first that the metric $g$ is Zoll, i.e., every totally geodesic disc with respect to $g_0$ is minimal with respect to $g$, and then we show that $g$ must be a multiple of $g_0$. This proves Theorem 1.2.

1.6. **Acknowledgments.** The first author thanks his advisor Fernando Codá Marques for helpful conversations related to this paper and for his support.

The second author thanks Danny Calegari. We are also thankful to the referee for the suggestions which helped improve the presentation.

2. **Preliminaries**

Assume $(M, g)$ is a closed Riemannian 3-manifold admitting a hyperbolic metric $g_0$ and $N$ another closed hyperbolic 3-manifold with an hyperbolic metric $h_0$.

Suppose $T: M \to N$ is smooth map. Given $(x, P) \in \text{Gr}_2(M)$ we denote by $|\wedge^2 T|_g(x, P)$ the Jacobian of

$$dT_x: P \to T_{T(x)}N,$$

meaning that if $e_1, e_2$ is an orthonormal basis of $P$ and $u_i := dT_x(e_i), i = 1, 2$,

$$|\wedge^2 T|_g(x, P) := \sqrt{h_0(u_1, u_1)h_0(u_2, u_2) - h_0^2(u_1, u_2)}.$$  

Given $y \in N$ a regular value and $\tau = (y, V) \in \text{Gr}_2(N)$ set

$$|\wedge^2 T^{-1}|_g(y, V) = \sum_{x \in T^{-1}(y)} \frac{1}{|\wedge^2 T|_g(x, (dT_x)^{-1}(V))}.$$  

The function $\tau \mapsto |\wedge^2 T^{-1}|_g(\tau)$ is defined almost everywhere on $\text{Gr}_2(N)$. 
For context, if $T$ is transverse to a closed surface $S \subset N$, then $\Sigma := T^{-1}(S)$ is a surface of $M$ with area
\[
\text{area}_g(\Sigma) = \int_S |\lambda^2 T^{-1}|_g(y, T_y S) dA_{h_0}(y).
\]

Gromov [14, page 73] defined the average area ratio of $T$ as
\[
\text{Area}_T(g/h_0) := \int_{\text{Gr}_2(N)} |\lambda^2 T^{-1}|_g(\tau) d\mu_{h_0}(\tau),
\]
where integration is with respect to the unit volume measure $\mu_{h_0}$ on $\text{Gr}_2(N)$ induced by $h_0$. In particular, $\text{Area}_M(h_0/h_0) = 1$. In the definition above it is implicitly assumed that $|\lambda^2 T^{-1}|_g$ is in $L^1$. If that is not the case we define $\text{Area}_T(g/h_0) = \infty$.

We use $\pi_1(M)$ to denote as well its representation into $\text{PSL}(2, \mathbb{C})$. A closed immersed surface $\Sigma \subset M$ is essential if the immersion $\iota : \Sigma \to M$ is $\pi_1$-injective. Essential surfaces lift to discs in $\mathbb{H}^3$. Using the representation of $\pi_1(M)$ into $\text{PSL}(2, \mathbb{C})$ we have $\iota_\ast(\pi_1(\Sigma)) < \text{PSL}(2, \mathbb{C})$ and so we can associate its limit set $\Lambda(\Sigma) \subset \partial_\infty \mathbb{H}^3 \simeq S^2$ (for the definition see for instance [9, Section 2.2]).

Set $C_\varepsilon$ to be the space of $(1 + \varepsilon)$-quasicircles in $\partial_\infty \mathbb{H}^3$ (see [9, Definition 2.3] for precise definition). The group $\pi_1(M)$ acts on $C_\varepsilon$ and preserves $C_0$ (which is the space of all geodesic circles in $S^2 \simeq \partial_\infty \mathbb{H}^3$).

Let $S_\varepsilon(M)$ denote the set of all homotopy class $\Pi$ of essential surfaces with limit set in $C_\varepsilon$. Recall that we defined
\[
\text{area}_g(\Pi) := \inf \{ \text{area}_g(S) : S \in \Pi \}.
\]

From Schoen-Yau [36] there is an immersed minimal surface (with respect to $g$) $\Sigma_g(\Pi) \in \Pi$ which realizes $\text{area}_g(\Pi)$.

From [37], assuming $\varepsilon$ is sufficiently small, for all $\gamma \in C_\varepsilon$ there is a unique embedded area-minimizing disc $D(\gamma) \subset \mathbb{H}^3$ (with respect to $g_0$) with $\partial_\infty D(\gamma) = \gamma \subset \partial_\infty \mathbb{H}^3$ and principal curvatures that can be made arbitrarily small by choosing $\varepsilon$ small enough.

The same argument as in Theorem 4.3 of [29] (adapted to the case where the minimal discs are not necessarily preserved by some surface group), shows that we can find a neighborhood $U$ of $g_0$ and $\bar{\varepsilon}$ small so that for each $\gamma \in C_\varepsilon$ there is a unique non-degenerate area-minimizing disc $\Sigma_g(\gamma)$ with respect to the metric $g$ so that $\partial_\infty \Sigma_g(\gamma) = \gamma$. The discs $\Sigma_g(\gamma)$ and $D(\gamma)$ are at a bounded Hausdorff distance from each other (independent of $g$) and if $g \to g_0$ then $\Sigma_g(\gamma)$ converges to $D(\gamma)$ uniformly in $C^{2,\alpha}$. Therefore we can decrease $U$ and $\bar{\varepsilon}$ so that for all $\gamma \in C_\varepsilon$ there is $f_\gamma \in C^{\infty}(D(\gamma))$ (depending on $g$) such that its graph over $D(\gamma)$ is identical to $\Sigma_g(\gamma)$ and $|f_\gamma|_{2,\alpha} < 1$. There is an ambiguity on the sign of $f_\gamma$ but the quantities we consider, like $|f_\gamma|$, will be sign independent.

\footnote{Using the Poincaré ball model, the orientation preserving isometries of $\mathbb{H}^3$ are identified with the automorphisms of $S^2$ which is the group $\text{PSL}(2, \mathbb{C})$.}
If $\Pi \in S_\varepsilon(M)$ and $g \in \mathcal{U}$ then, with $\gamma := \Lambda(\Sigma_g(\Pi))$, uniqueness implies that $\Sigma_g(\gamma)$ covers a minimal surface $\Sigma_g(\gamma)$ in $\Pi$ which must coincide with $\Sigma_g(\Pi)$ and thus satisfy

$$\text{area}_g(\Pi) = \text{area}_g(\Sigma_g(\gamma)).$$

Given $\gamma \in C_\varepsilon$, let $n(\gamma)$ denote a continuous unit normal vector field along $D(\gamma)$ with respect to $g_0$. Consider the diffeomorphism (using the hyperboloid model)

$$F_\gamma : D(\gamma) \to \Sigma_g(\gamma), \quad x \mapsto \cosh(f_\gamma(x))x + \sinh(f_\gamma(x))n(\gamma)(x).$$

We omit the dependence of $g$ on $F_\gamma$ to avoid too much notation. When $g = g_0$, $f_\gamma = 0$ and thus $F_\gamma$ is the identity.

3. Laminations and Laminar measures

We follow the presentation of Labourie in [28] and add some auxiliary results.

3.1. Laminations. Consider the space of stable minimal conformal immersions $\mathcal{F}(\mathbb{H}^3, \varepsilon)$ (with $\varepsilon \leq \bar{\varepsilon}$) defined in Definition 5.1 of [28], i.e., the space of conformal minimal immersions $\phi : \mathbb{H}^2 \to (\mathbb{H}^3, g_0)$ with $\partial \phi := \phi(\partial_\infty \mathbb{H}^2) \in C_\varepsilon$.

Because $\varepsilon \leq \bar{\varepsilon}$, $D(\partial \phi) = \phi(\mathbb{H}^2)$, $\phi$ is an embedding and stable for the second variation of area. The topology we choose is the same as the one considered in Definition 5.1 of [28] and it makes the map $\phi \mapsto \partial \phi$ continuous (Theorem 5.2 of [28]). Thus

$$\cap_{k \in \mathbb{N}} \mathcal{F}(\mathbb{H}^3, 1/k) = \mathcal{F}(\mathbb{H}^3, 0).$$

Similar to [28] we also consider $\mathcal{F}(M, \varepsilon) := \mathcal{F}(\mathbb{H}^3, \varepsilon)/\pi_1(M)$ with the quotient topology. The group PSL(2, $\mathbb{R}$) acts on $\mathbb{H}^2$ and thus it acts on $\mathcal{F}(M, \varepsilon)$ in the following way:

$$\tau \in \text{PSL}(2, \mathbb{R}), \quad R_\tau : \mathcal{F}(M, \varepsilon) \to \mathcal{F}(M, \varepsilon) \quad R_\tau(\phi) := \phi \circ \tau^{-1}.$$ 

The space $\mathcal{F}(M, \varepsilon)$ together with the PSL(2, $\mathbb{R}$)-action is called the conformal minimal laminations of $M$.

Fix a fundamental domain $\Delta \subset \mathbb{H}^3$ of $M$. Given $\phi \in \mathcal{F}(M, \varepsilon)$, there is a unique lift to an element of $\mathcal{F}(\mathbb{H}^3, \varepsilon)$, denoted by $\tilde{\phi}$ as well, that is uniquely determined by the requirement that $\phi(i) \in \Delta$. Thus for each $\phi \in \mathcal{F}(M, \varepsilon)$ we obtain $\partial \phi \in C_\varepsilon$ but this map is not necessarily continuous. Theorem 5.2 (i) of [28] says that the evaluation map which sends $\phi \in \mathcal{F}(\mathbb{H}^3, \varepsilon)$ to $\phi(i) \in \mathbb{H}^3$ is proper. As a result we deduce at once the lemma below.

Lemma 3.1. The space $\mathcal{F}(M, \varepsilon)$ is sequentially compact.

Given $\phi \in \mathcal{F}(M, \varepsilon)$, let $C(\phi) > 0$ be the conformal factor of $\phi^*(g_0)$. Denote the Gaussian curvature of $D(\partial \phi)$ by $K(\phi)$. From Gauss equation and [37] we have, after making $\varepsilon$ smaller if necessary, $-2 \leq K(\phi) \leq -1$. 

The maximum principle applied to the equation satisfied by $C(\phi)$ implies that

\begin{equation}
\frac{1}{2} \leq \frac{1}{\sup_D(\partial\phi)|K(\phi)|} \leq C(\phi) \leq 1.
\end{equation}

Let $F(M)$ denote the frame bundle of $M$, i.e., $F(M) = \text{PSL}(2,\mathbb{C})/\pi_1(M)$. Fix $\{e_1, e_2\}$ an orthonormal basis of $T_p\mathbb{H}^2$ and, given $\phi \in F(M, \epsilon)$, set

$$e_1(\phi) := d\phi_1(e_1)C(\phi)^{-1/2} \quad \text{and} \quad e_2(\phi) := d\phi_1(e_2)C(\phi)^{-1/2}.$$ 

There is a unique $n(\phi) \in T_{\phi(i)}M$ so that $\{e_1(\phi), e_2(\phi), n(\phi)\}$ is a positive frame. Consider the continuous map

$$\Omega : F(M, \epsilon) \to F(M), \quad \Omega(\phi) = (\phi(i), \{e_1(\phi), e_2(\phi), n(\phi)\}).$$

Given $\phi \in F(M, \epsilon)$, set $\gamma := \partial\phi$ and denote the Jacobian of (see (3))

$$F_{\gamma} \circ \phi : \mathbb{H}^2 \to (\Sigma_\gamma(\gamma), g)$$

by $|\text{Jac}_g(F_{\gamma} \circ \phi)|$. Consider the function

$$\Lambda_g : F(M, \epsilon) \to \mathbb{R}, \quad \Lambda_g(\phi) := |\text{Jac}_g(F_{\gamma} \circ \phi)|(i).$$

If $dA_g$ is the area element of $\Sigma_\gamma(\gamma)$, then $(F_{\gamma} \circ \phi)^*(dA_g) = \Lambda_g(\phi)dx \wedge dy$, in isothermal coordinates. This function is continuous because it is independent of the particular lift of $\phi$ that was chosen.

With $\gamma \in C_\epsilon$, let $v_\gamma(\gamma)$ and $|A^2_g(\Sigma_\gamma(\gamma))$ denote respectively a continuous unit normal vector field along $\Sigma_\gamma(\gamma)$ with respect to $g$ and the norm square of the second fundamental form of $\Sigma_\gamma(\gamma)$ with respect to the metric $g$. Consider the following functions

\begin{equation}
|A|^2_g : F(M, \epsilon) \to \mathbb{R}, \quad \phi \mapsto |A|^2_g(\Sigma_g(\partial\phi))(F_{\partial\phi} \circ \phi(i)),
\end{equation}

\begin{equation}
\text{Ric}(g)(\nu, \nu) : F(M, \epsilon) \to \mathbb{R}, \quad \phi \mapsto \text{Ric}(g)(F_{\partial\phi \circ \phi(i)}(\nu_g(\partial\phi), \nu_g(\partial\phi))),
\end{equation}

\begin{equation}
R(g) : F(M, \epsilon) \to \mathbb{R}, \quad \phi \mapsto R(g)(F_{\partial\phi \circ \phi(i)}).
\end{equation}

The definition of all these functions is independent of the particular lift of $\phi \in F(M, \epsilon)$ that was chosen and thus they are continuous.

### 3.2. Laminar measures.

A laminar measure $\mu$ on $F(M, \epsilon)$ is a probability measure that is invariant under the PSL$(2,\mathbb{R})$-action given by (4).

A laminar measure $\mu$ and the map $\Omega$ defined in (6) induces a probability measure $\Omega_*\mu$ on $F(M)$. That measure is invariant under a PSL$(2,\mathbb{R})$-action which will not coincide in general with the homogeneous action of PSL$(2,\mathbb{R})$ as a subgroup of PSL$(2,\mathbb{C})$. Another issue that needs to be addressed is the fact that the space of laminar measures is not necessarily weakly compact (a related problem is put as an open question in [27]).

**Lemma 3.2.** Let $\mu_k$ be a sequence of laminar measures on $F(M, 1/k)$ so that $\Omega_*\mu_k$ converges weakly to a probability measure $\bar{\mu}$ on $F(M)$. Then $\bar{\mu}$ is invariant under the homogeneous action of PSL$(2,\mathbb{R})$. 
Proof. Let \( \Omega_0 : \mathcal{F}(M, 0) \to F(M) \) be the restriction of \( \Omega \) to \( \mathcal{F}(M, 0) \). Every \( \phi \in \mathcal{F}(M, 0) \) has the property that \( D(\partial \phi) \) is a totally geodesic disc and so \( \phi : \mathbb{H}^2 \to M \) is an isometric immersion. Thus \( \phi \) is uniquely determined by \( \phi(i), e_1(\phi), \) and \( e_2(\phi) \). Hence \( \Omega_0 \) is bijective and a homeomorphism.

Recall the \( \text{PSL}(2, \mathbb{R}) \)-action on \( \mathcal{F}(M, \varepsilon) \) defined in (4). \( \Omega_0 \) induces a \( \text{PSL}(2, \mathbb{R}) \)-action on \( F(M) \) in the following way:

\[
\tau \in \text{PSL}(2, \mathbb{R}), \quad \tilde{R}_\tau : F(M) \to F(M), \quad \tilde{R}_\tau(x) = \Omega_0(R_\tau \circ \Omega_0^{-1}(x)).
\]

This action corresponds to the homogeneous action of \( \text{PSL}(2, \mathbb{R}) \). Thus, given \( f \in C^0(F(M)) \), we need to check that

\[
\bar{\mu}(f \circ \tilde{R}_\tau) = \bar{\mu}(f) \quad \text{for all } \tau \in \text{PSL}(2, \mathbb{R}).
\]

Consider the “projection” of \( \mathcal{F}(M, \varepsilon) \) onto \( \mathcal{F}(M, 0) \) given by

\[
P := \Omega_0^{-1} \circ \Omega : \mathcal{F}(M, \varepsilon) \to \mathcal{F}(M, 0).
\]

Set \( \eta := f \circ \Omega \circ R_\tau \). We have \( f \circ \tilde{R}_\tau \circ \Omega = \eta \circ P \) and thus

\[
\bar{\mu}(f \circ \tilde{R}_\tau) = \lim_{k \to \infty} \Omega_* \mu_k(f \circ \tilde{R}_\tau) = \lim_{k \to \infty} \mu_k(\eta \circ P).
\]

We also have \( \Omega_* \mu_k(f) = \mu_k(\eta) \) for all \( k \in \mathbb{N} \) by \( \text{PSL}(2, \mathbb{R}) \)-invariance and thus

\[
\bar{\mu}(f) = \lim_{k \to \infty} \Omega_* \mu_k(f) = \lim_{k \to \infty} \mu_k(\eta).
\]

In light of these last two identities it suffices to check that

\[
\lim_{k \to \infty} |\mu_k(\eta \circ P) - \mu_k(\eta)| = 0
\]

and this follows at once if we show that

\[
\lim_{k \to \infty} \sup_{\phi \in \mathcal{F}(M, 1/k)} |\eta \circ P(\phi) - \eta(\phi)| = 0.
\]

If this identity does not hold we find \( \delta > 0 \) and \( \phi_k \in \mathcal{F}(M, 1/k) \) so that \( |\eta \circ P(\phi_k) - \eta(\phi_k)| \geq \delta \) for all \( k \in \mathbb{N} \). From Lemma 3.1 we know that, after passing to a subsequence, \( \phi_k \) converges to some \( \phi \in \mathcal{F}(M, 0) \) which must satisfy \( |\eta \circ P(\phi) - \eta(\phi)| \geq \delta \). This is impossible because \( P(\phi) = \phi \).

Let \( \Gamma \) be a Fuchsian subgroup of \( \text{PSL}(2, \mathbb{R}) \) so that \( \mathbb{H}^2/\Gamma \) is a closed hyperbolic surface with genus \( l \). All Fuchsian groups we consider will have this property with no need for further mentioning.

\( \text{PSL}(2, \mathbb{R})/\Gamma \) is the frame bundle of \( \mathbb{H}^2/\Gamma \). With respect to the invariant metric on \( \text{PSL}(2, \mathbb{R})/\Gamma \) we have \( \text{vol}(\text{PSL}(2, \mathbb{R})/\Gamma) = \alpha_0 4\pi(l - 1) \) for some universal constant \( \alpha_0 \).

Suppose \( \phi \in \mathcal{F}(M, \varepsilon) \) is equivariant with respect to a representation of \( \Gamma < \text{PSL}(2, \mathbb{R}) \) in \( \pi_1(M) < \text{PSL}(2, \mathbb{C}) \). Consider \( U \subset \text{PSL}(2, \mathbb{R}) \) a fundamental domain of \( \text{PSL}(2, \mathbb{R})/\Gamma \). Following Proposition 5.5 of [28] we define \( \delta_\phi \) a laminar measure on \( \mathcal{F}(M, \varepsilon) \)

\[
\delta_\phi(f) := \frac{1}{\text{vol}(U)} \int_U f(\phi \circ \tau)d\bar{\nu}(\tau), \quad f \in C^0(\mathcal{F}(M, \varepsilon))
\]
where \( \nu \) is the bi-invariant measure on \( \text{PSL}(2, \mathbb{R}) \).

Consider the covering map \( \pi : \mathbb{H}^3 \rightarrow M \). Equivariance implies that \( D(\partial \phi) \) projects to a closed surface \( D(\partial \phi) \) in \( M \) and that \( \phi \) descends to an immersion from \( \mathbb{H}^2 / \Gamma \) to \( D(\partial \phi) \) that we still denote by \( \phi \). The uniqueness property implies that \( \Sigma_g(\partial \phi) \) projects to a closed surface \( \Sigma_g(\partial \phi) \) on \( M \) that is homotopic to \( D(\partial \phi) \). The map \( F_{\partial \phi} \circ \phi \) is also equivariant and so descends to a map from \( \mathbb{H}^2 / \Gamma \) to \( \Sigma_g(\partial \phi) \) that we denote by \( F_{\partial \phi} \circ \phi \). The unit normal vector field \( \nu_g(\gamma) \) induces a unit normal vector field along \( \Sigma_g(\gamma) \) that we also denote by \( \nu_g(\gamma) \).

For context, suppose \( f \) is a continuous function in \( Gr_2(M) \) and set

\[
\tilde{f} : \mathcal{F}(M, \mathbb{R}) \rightarrow \mathbb{R}, \quad \phi \mapsto f(\pi \circ F_{\partial \phi} \circ \phi(i), d(\pi \circ F_{\partial \phi})(\phi(i))(d\phi(z)(T_z \mathbb{H}^2))).
\]

The function \( \tilde{f} \) is continuous. With \( dA_{hyp} \) denoting the hyperbolic volume form on \( \mathbb{H}^2 \) we have

\[
\int_{\Sigma_g(\partial \phi)} f(x, T_x \Sigma_g(\partial \phi))dA_g(x) = \int_{\mathbb{H}^2 / \Gamma} \rho(\mathcal{F}_{\partial \phi} \circ \phi(z), d\mathcal{F}_{\partial \phi}(\phi(z))(d\phi(T_z \mathbb{H}^2)))|Jac_g(\mathcal{F}_{\partial \phi} \circ \phi)(\phi(z))|dA_{hyp}(z) = \frac{1}{2\kappa_0} \int_{U} \tilde{f}(\phi \circ \tau) \Lambda_g(\phi \circ \tau)\nu(\tau) = 4\pi(l - 1)\delta_\phi(\tilde{f} \Lambda_g).
\]

**Gauss identity for laminar measures.** From Gauss equation we have

\[
4\pi(l - 1) = \text{area}_g(\Sigma_g(\partial \phi)) + \int_{\Sigma_g(\partial \phi)} \text{Ric}(\nu_g(\partial \phi), \nu_g(\partial \phi)) - \frac{1}{3} R(g) + \frac{|A|^2}{2} - \frac{R(g)}{6} + \frac{6}{6} dA_g.
\]

When interpreted in terms of laminar measures this identity becomes

\[
1 = \frac{\text{area}_g(\Sigma_g(\partial \phi))}{4\pi(l - 1)} + \delta_\phi \left( \left[ \text{Ric}(\nu, \nu) - \frac{R(g)}{3} + \frac{|A|^2}{2} \right] \Lambda_g \right) - \delta_\phi \left( \frac{R(g) + 6}{6} \Lambda_g \right),
\]

where the functions \( |A|^2_g \), \( \text{Ric}(\nu, \nu) \), and \( R(g) \) are as defined in (8), (9), and (10).

**4. Proof of Theorem 1.1: Part I**

Recall that \( M \) is a closed manifold with an hyperbolic metric \( g_0 \). Throughout this paper we refer to normalized Ricci flow as a one-parameter family of metrics \( (\bar{g}_t)_{t \in I} \) which solve

\[
\frac{d\bar{g}_t}{dt} = -2\text{Ric}(\bar{g}_t) - 4\bar{g}_t.
\]

\(^3\)The maps \( F_{\partial \phi} \circ \phi \) and \( \pi \circ F_{\partial \phi} \circ \phi \) have the same image but different domains.
Theorem 4.1. There is a neighborhood $\mathcal{V}$ of $g_0$ so that for all $g \in \mathcal{V}$ with $R(g) \geq -6$ the following holds: For any sequence $\Pi_m \in S_{1/m}(M)$ we have

$$\liminf_{m \to \infty} \frac{\text{area}_g(\Pi_m)}{4\pi(l_m - 1)} \geq 1,$$

where $l_m$ is the genus of an essential surface in $\Pi_m$.

If equality holds then $g$ is isometric to $g_0$.

Proof. Consider the neighborhood $\mathcal{U}$ of $g_0$ described in Section 2. From [25, Appendix A] we see that we can find a neighborhood of $g_0$ in the $C^{2,\alpha}$-topology so that for every initial condition in that neighborhood, the normalized Ricci flow exists for all time and converges exponentially fast in the $C^{2,\alpha}$-topology to an hyperbolic metric in $\mathcal{U}$. Reasoning like in [17, Section 17] we can upgrade the convergence and find a smaller open neighborhood $V \subset \mathcal{U}$ of $g_0$ so that for every $g \in V$ the normalized Ricci flow $(\bar{g}_t)_{t \geq 0}$ starting at $g$ exists for all time, does not leave $\mathcal{U}$, and converges exponentially fast to an Einstein metric in $\mathcal{U}$, which must be isometric to $g_0$ from Mostow rigidity. Furthermore, in [25] it is also constructed a family of diffeomorphisms $\{\Phi_t\}_{t \geq 0}$ converging strongly to some diffeomorphism so that $g_t := \Phi_t^* \bar{g}_t$ solves the DeTurck-modified Ricci flow (which is strictly parabolic) and $g_t$ converges to $g_0$ as $t \to \infty$.

The maximum principle implies that the condition $R(g) \geq -6$ is preserved by the normalized Ricci flow because

$$\frac{d}{dt} R(\bar{g}_t) \geq \Delta_{\bar{g}_t} R(\bar{g}_t) + \frac{2}{3} \bar{R}(\bar{g}_t)(R(\bar{g}_t) + 6).$$

4.1. Proof of inequality. Suppose for contradiction that the inequality fails for some metric $g \in \mathcal{V}$. Thus we can find $\Pi_m \in S_{1/m}(M) \cap S_{\varepsilon}(M)$ and $\delta > 0$ so that for all $m \in \mathbb{N}$

$$\frac{\text{area}_g(\Pi_m)}{4\pi(l_m - 1)} \leq 1 - \delta.$$

Because $\bar{g}_t \in \mathcal{U}$ for all $t$, there is $\phi_m \in \mathcal{F}(M, 1/m)$ equivariant with respect to a representation of a Fuchsian subgroup of $\text{PSL}(2, \mathbb{R})$ in $\pi_1(M)$ so that $\Sigma_{g_t}(\partial \phi_m) \in \Pi_m$, $\Sigma_{g_t}(\partial \phi_m)$ depends smoothly on $t$, and $\text{area}_{g_t}(\Pi_m) = \text{area}_{g_t}(\Sigma_{g_t}(\partial \phi_m))$ for all $t \geq 0$. Fix $m \in \mathbb{N}$ and denote by $\nu_t$ the normal vector to $\Sigma_{g_t}(\partial \phi_m)$ with respect to $g_t$. Using $R(g_t) \geq -6$ and Gauss equation

\begin{align*}
\frac{d}{dt} \text{area}_{g_t}(\Pi_m) &= - \int_{\Sigma_{g_t}(\partial \phi_m)} R(g_t) - \text{Ric}(g_t)(\nu_t, \nu_t) + 4 \, dA_{g_t} \\
&= 4\pi(l_m - 1) - \text{area}_{g_t}(\Pi_m) - \int_{\Sigma_{g_t}(\partial \phi_m)} |A|^2 + \frac{R(g_t) + 6}{2} \, dA_{g_t} \\
&\leq 4\pi(l_m - 1) - \text{area}_{g_t}(\Pi_m).
\end{align*}
Solving the ODE obtained by replacing the inequality sign above by an equality sign we get that for all $t \geq 0$

$$\frac{\text{area}_{g_t}(\Pi_m)}{4\pi (l_m - 1)} \leq 1 - e^{-t} \left( 1 - \frac{\text{area}_{g}(\Pi_m)}{4\pi (l_m - 1)} \right) \leq 1 - \delta e^{-t}. \tag{15}$$

Combining (12) with (15) and using the fact that $R(g_t) + 6 \geq 0$ we get

$$\delta e^{-t} \leq \delta \phi_m \left( [\text{Ric}(g_t)(\nu,\nu) - R(g_t)/3 + |A|_{g_t}^2/2] \Lambda(g_t) \right). \tag{16}$$

After passing to a subsequence, $\Omega_n \delta \phi_m$ converges weakly to a unit measure $\bar{\mu}$ on $F(M)$. From Lemma 3.2 we have that $\bar{\mu}$ is invariant under the homogeneous action of $\text{PSL}(2, \mathbb{R})$.

Let $L$ denote the linearization of the traceless Ricci tensor at $g_0$. For every 2-tensor $h$ on $M$ set $\theta(h) : F(M) \to \mathbb{R}$ to be the continuous function given by

$$\theta(h)(x, \{u_1, u_2, n\}) := L(h)x(n, n). \tag{17}$$

Set $h_t := g_t - g_0$. We use $O(\beta)$ to denote a term bounded by $C\beta$, where $C$ does not depend on $m$ or $t$.

**Lemma 4.2.** $\delta \leq \bar{\mu}(e^t \theta(h_t)) + O(e^{-1/3t})$.

**Proof.** Write

- $E_1 := \delta \phi_m(\|A\|_{g_t}^2/2 \Lambda(g_t))$;
- $E_2 := \delta \phi_m([\text{Ric}(g_t)(\nu,\nu) - R(g_t)/3](\Lambda(g_t) - 1))$;
- $E_3 := \delta \phi_m([\text{Ric}(g_t)(\nu,\nu) - R(g_t)/3] - \theta(h_t) \circ \Omega)$.

We now estimate these terms. We know from Proposition A.2 that

$$|h_t|_{C^1} = O(e^{-2/3t}).$$

Fix $m \in \mathbb{N}$ and set $\gamma_m := \partial \phi_m$. The surface $\Sigma_{g_t}(\gamma_m)$ has zero mean curvature with respect to $g_t$. Thus the mean curvature $H_{g_0}(\Sigma_{g_t}(\gamma_m))$ of $\Sigma_{g_t}(\gamma_m)$ with respect to the hyperbolic metric has $C^{0,\alpha}$-norm bounded by $O(e^{-2/3t})$. (This follows from the fact that $h_t$ and $\nabla h_t$ have both that order).

Using the mean convex foliation of $\mathbb{H}^3$ coming from the discs equidistant to $D(\gamma_m)$ that was described by Uhlenbeck in [40, Theorem 3.3], the maximum principle implies that for some constant $c_0$

$$\sup_{m \in \mathbb{N}} |f_{\gamma_m}|_{L^\infty} \leq c_0 \sup_{m \in \mathbb{N}} |H_{g_0}(\Sigma_{g_t}(\gamma_m))|_{L^\infty} = O(e^{-2/3t}).$$

Elliptic regularity implies the existence of a constant $c_1 > 0$ such that

$$\sup_{m \in \mathbb{N}} |f_{\gamma_m}|_{C^{2,\alpha}} \leq c_1 \sup_{m \in \mathbb{N}} (|f_{\gamma_m}|_{L^\infty} + |H_{g_0}(\Sigma_{g_t}(\gamma_m))|_{C^{0,\alpha}}) = O(e^{-2/3t}).$$

Thus $|A|_{g_0}(\Sigma_{g_t}(\gamma_m)) = O(e^{-4/3t} + A_m^2)$, where $A_m := ||A_{g_0}(D(\gamma_m))||_{\infty}$. If $\lambda_j(\gamma_t)$ and $\lambda_j(\gamma_0)$, $j = 1, 2$ denote, respectively, the principal curvatures of $\Sigma_{g_t}(\gamma_m)$ with respect to $g_t$ and $g_0$, we have

$$\lambda_j(\gamma_t) = \lambda_j(\gamma_0) + O(|h_t|_{C^1}), \quad j = 1, 2.$$
Hence \(|A_{g_t}^2(\Sigma_{g_t}(\gamma_m)) = O(e^{-4/3t} + A_m^2)\) and thus
\[
2E_1 = \frac{1}{4\pi(l_k-1)} \int_{\Sigma_{g_t}(\gamma_m)} |A_{g_t}^2 dA_{g_t} = O(e^{-4/3t} + A_m^2).
\]

From Gauss equation, (5), and the definition of \(A_m\), we obtain that
\[|C(\phi_m) - 1| = O(A_m^2).\]

Moreover, from Proposition A.1 we obtain
\[|\text{Jac}_{g_t}(F_{\gamma_m})| = 1 + O(|h_t|_{C^1}) + O(|f_{\gamma_m}|_{C^1}) = 1 + O(e^{-2/3t}).\]

Thus \(|\text{Jac}_{g_t}(F_{\gamma_m} \circ \phi_m)| = |\text{Jac}_{g_t}(F_{\gamma_m})|C(\phi_m) = 1 + O(e^{-2/3t} + A_m^2)\) and this means (see (7)) that
\[
\Lambda_{g_t}(\phi_m) = 1 + O(e^{-2/3t} + A_m^2).
\]

Recall the definitions in (6) and (17). From Proposition A.1 and the fact that \(|h_t|_{C^1}\) and \(\sup_{m \in \mathbb{N}} |f_{\gamma_m}|_{C^2}\) have order \(O(e^{-2/3t})\), we see that
\[
|\text{Ric}(g_t)(\nu, \nu) - R(g_t)/3(\phi_m) = L(h_t)_{\phi_m(i)}(n(\gamma_m), n(\gamma_m)) + O(e^{-4/3t})
\[= \theta(h_t) \circ \Omega(\phi_m) + O(e^{-4/3t}).
\]

Note that \(\theta(h_t) = O(|h_t|_{C^2}) = O(e^{-2/3t})\) and so we obtain from (20) and (19) that both
\[E_2 = \delta_{\phi_m}([\theta(h_t) \circ \Omega + O(e^{-4/3t})](\Lambda_{g_t} - 1)) = O(e^{-4/3t} + A_m^2)
\] and \(E_3 = O(e^{-4/3t})\). As a result,
\[
\delta_{\phi_m} \left( |\text{Ric}(g_t)(\nu, \nu) - R(g_t)/3 + |A_{g_t}^2/2 \right) \Lambda_{g_t}
\]
\[= \delta_{\phi_m} (\theta(h_t) \circ \Omega) + E_1 + E_2 + E_3 = \Omega^* \delta_{\phi_m}(\theta(h_t)) + O(e^{-4/3t} + A_m^2)
\]
and so we have from (16)
\[
\delta e^{-t} \leq \Omega^* \delta_{\phi_m}(\theta(h_t)) + O(e^{-4/3t} + A_m^2).
\]

Making \(m \to \infty\) the result follows because \(A_m \to 0\). \hfill \Box

Consider the operator on symmetric 2-tensors
\[
A(h) = \Delta_{g_0} h - 2(tr_{g_0} h)g_0 + 2h.
\]

We know from Proposition A.2 that \(e^t h_t\) converges in \(W^{k,2}\) to \(\bar{h}\), where \(\bar{h}\) satisfies \(A(\bar{h}) = -\bar{h}\). Thus from Sobolev embedding theorem we have that \(e^t h_t\) converges to \(\bar{h}\) in \(C^2\) and so \(e^t \theta(h_t) \to \theta(\bar{h})\). The previous lemma implies
\[
\delta \leq \bar{\mu}(\theta(\bar{h})).
\]
We now show that \( \overline{\mu}(\theta(\bar{h})) = 0 \), which gives us a contradiction and thus implies that for any sequence \( \Pi_m \in S_{1/m}(M) \) we have

\[
\liminf_{m \to \infty} \frac{\text{area}_g(\Pi_m)}{\text{area}_{g_0}(\Pi_m)} \geq 1.
\]

**Proposition 4.3.** Let \( \eta \) be an ergodic \( \text{PSL}(2, \mathbb{R}) \)-invariant probability measure on \( F(M) \). Then \( \eta(\theta(\bar{h})) = 0 \).

**Proof.** Because \( \bar{h} \) is the lowest eigenfunction for \( A \), it follows from Koiso Bochner formula (see [25]) that \( \bar{h} \) is trace free and divergence free. Hence from the formula for the linearization of the traceless Ricci tensor [6, Theorem 1.174] we have

\[
L(\bar{h}) = -\frac{1}{2}A(\bar{h}) = \frac{\bar{h}}{2}.
\]

Thus for all \( (x, \{u_1, u_2, n\}) \in F(M) \) we have

(22) \( \theta(\bar{h})(x, \{u_1, u_2, n\}) = \frac{1}{2}\bar{h}_x(n, n) \).

From Ratner’s classification theorem [35] we have that \( \eta \) is either the homogeneous Lebesgue measure \( \mu_{\text{Leb}} \) on \( F(M) \) or there is a totally geodesic surface \( S \subset M \) so that \( \eta \) is the homogeneous Lebesgue measure on \( F(S) \), the frame bundle of \( S \).

Suppose the first case occurs. We have \( \text{tr}_{g_0}\bar{h} = 0 \) and thus

\[
\eta(\theta(\bar{h})) = \mu_{\text{Leb}}(\theta(\bar{h})) = \frac{1}{\text{vol}(M)} \int_M \frac{1}{6}\text{tr}_{g_0}\bar{h} dV_{g_0} = 0.
\]

Suppose the second case occurs and choose \( n \) a normal vector field along \( S \). Consider the vector field \( X \) tangent to \( S \) such that

\[
g_0(X, Y) = \nabla_n \bar{h}(Y, n) \quad \text{for every vector field} \ Y \ \text{tangent to} \ S.
\]

Set \( f : S \to \mathbb{R} \), where \( f(x) := \bar{h}_x(n(x), n(x)) \).

**Lemma 4.4.** With \( \Delta_S \) and \( \text{div}_S \) denoting, respectively, the Laplacian and divergence on \( S \), we have \( \Delta_S f + 6f = \text{div}_S X \).

**Proof.** The computation is local and so we can assume that \( S \) is embedded near a given \( x \in S \). Consider an orthonormal basis \( \{e_1, e_2\} \) for the tangent planes of \( S \) near \( x \) and extend the frame \( \{e_1, e_2, n\} \) via parallel transport.
By commuting derivatives and using the fact that $tr_{g_0} \tilde{h} = 0$ we deduce
\[
\sum_{i=1}^{2} (\nabla_n \nabla_{e_i} \tilde{h})(e_i, n) = \sum_{i=1}^{2} (\nabla_{e_i} \nabla_n \tilde{h})(e_i, n) + 2 \sum_{i=1}^{2} \tilde{h}(R(e_i, n)e_i, n) \\
+ 2 \sum_{i=1}^{2} \tilde{h}(e_i, R(e_i, n)n) \\
= \sum_{i=1}^{2} (\nabla_{e_i} \nabla_n \tilde{h})(e_i, n) - 2 \tilde{h}(n, n) + 2 \sum_{i=1}^{2} \tilde{h}(e_i, e_i) \\
= \text{div}_S X(x) - 3f(x).
\]

Differentiating $\text{div} \; \tilde{h} = 0$ in the direction of $n$ we obtain that at $x \in S$
\[
(\nabla_n \nabla_{e_1} \tilde{h})(e_1, n) + (\nabla_n \nabla_{e_2} \tilde{h})(e_2, n) + (\nabla_n \nabla_n \tilde{h})(n, n) = 0.
\]

From (21) and $\mathcal{A}(\tilde{h}) = -\tilde{h}$ we have that $\Delta_{g_0} \tilde{h} = -3\tilde{h}$. Thus the last two identities we derived and the fact that $S$ is totally geodesic imply
\[
\Delta_S f(x) = (\Delta_{g_0} \tilde{h})(n, n) - (\nabla_n \nabla_n \tilde{h})(n, n) = -3f(x) - (\nabla_n \nabla_n \tilde{h})(n, n) \\
= -3f(x) + \sum_{i=1}^{2} (\nabla_n \nabla_{e_i} \tilde{h})(e_i, n) = -6f(x) + \text{div}_S X(x).
\]

\hfill \Box

From (22) we see that $\theta(\tilde{h}) = f/2$ when restricted to $F(S) \subset F(M)$. Because $\eta$ is the unit Lebesgue measure on $F(S)$ we have from Lemma 4.4 that
\[
\eta(\theta(\tilde{h})) = \frac{1}{\text{area}_{g_0}(S)} \int_S \frac{f}{2} \text{d}A_{g_0} = \frac{1}{12 \text{area}_{g_0}(S)} \int_S \text{div}_S X - \Delta_S f \text{d}A_{g_0} = 0.
\]

\hfill \Box

The ergodic decomposition theorem for $\text{PSL}(2, \mathbb{R})$-actions (Theorem 4.2.6 in [26]) gives the existence of a space $B$ with a probability measure $\nu$ on $B$ so that for all $b \in B$ there is an ergodic $\text{PSL}(2, \mathbb{R})$-invariant probability measure $\mu_b$ on $F(M)$ so that $\bar{\mu}(f) = \int_B \mu_b(f) \text{d}\nu(b)$ for all $f \in C^0(F(M))$. Proposition 4.3 implies that $\bar{\mu}(\theta(\tilde{h})) = 0$.

4.2. **Proof of rigidity.** Suppose that for some metric $g \in \mathcal{V}$ with $R(g) \geq -6$ and some sequence $\Pi_m \in S_{1/m}(M)$ we have
\[
\liminf_{m \to \infty} \frac{\text{area}_{g}(\Pi_m)}{4\pi(l_m - 1)} = 1,
\]
where $l_m$ is the genus of an essential surface in $\Pi_m$. Run normalized Ricci flow $(g_t)_{0 \leq t \leq \tau}$ starting at $g$ for a short time interval and set
\[
a(t) := \liminf_{m \to \infty} \frac{\text{area}_{g_t}(\Pi_m)}{4\pi(l_m - 1)}.
\]
From (15) we see that $a(0) = 1$ implies that $a(t) \leq 1$ for all $0 \leq t \leq \bar{t}$ and thus $a(t) = 1$ for all $0 \leq t \leq \bar{t}$.

Suppose that $g$ is not Einstein. From the strong maximum principle applied to the evolution equation of $R(g_t)$ we obtain the existence of $\delta$ so that $R(g_t) \geq -6 + 2\delta$ for all $\bar{t}/2 \leq t \leq \bar{t}$. Thus we see from (14) that for all $\bar{t}/2 \leq t \leq \bar{t}$ and all $m \in \mathbb{N}$

$$\frac{d}{dt}area_{g_t}(\Pi_m) \leq 4\pi(l_m - 1) - (1 + \delta)area_{g_t}(\Pi_m).$$

ODE comparison gives us a contradiction because

$$a(\bar{t}) \leq a(\bar{t}/2)e^{-(1+\delta)\bar{t}/2} + \frac{1 - e^{-(1+\delta)\bar{t}/2}}{1 + \delta} = e^{-(1+\delta)\bar{t}/2} + \frac{1 - e^{-(1+\delta)\bar{t}/2}}{1 + \delta} < 1.$$ 

\[\square\]

5. PROOF OF THEOREM 1.1: PART II

We use Perelman’s Ricci flow with surgery [34] to remove the local condition on Theorem 4.1.

**Theorem 5.1.** Assume $g$ is a metric on $M$ such that $R(g) \geq -6$. For any sequence $\Pi_m \in S_{1/m}(M)$ we have

$$\liminf_{m \to \infty} \frac{\text{area}_{g_t}(\Pi_m)}{4\pi(l_m - 1)} \geq 1,$$

where $l_m$ is the genus of an essential surface in $\Pi_m$.

If equality holds then $g$ is isometric to $g_0$.

**Proof.** We will use the notation and results of [34]. For a more detailed treatment the reader can consult [5].

Note that Ricci flow and normalized Ricci flow (13) differ only by scaling. With $\varepsilon_0$ small, Perelman finds in [34, Section 4] a sequence of manifolds $M_k$ with $M_0 = M$, a discrete set of times $\{t_k\}_{k \in \mathbb{N}_0}$ with $t_0 = 0$, and a sequence of smooth solutions to normalized Ricci flow $(\bar{g}_t)_{t_k \leq t < t_{k+1}}$ on $M_k$ with $\bar{g}_0 = g$ so that the following holds: each $M_{k+1}$ is obtained from $M_k$ by surgery as described in [34, Section 4.4] but, because $M$ is irreducible and contains no embedded projective planes, the only possible surgeries remove $\varepsilon_0$-caps and glue in (perturbed) standard caps. Hence $M_k = M$ for all $k \in \mathbb{N}_0$ (after discarding 3-spheres) and, still following [34, Section 4], there is a sequence of compact sets $\Lambda_k \subset M$ such that $M \setminus \Lambda_k$ is diffeomorphic to a union of open balls, $\bar{g}_t$ converges smoothly to $\bar{g}_{tk+1}$ on $\Lambda_k$ as $t \to t_{k+1}$, and for all $t < t_{k+1}$ close to $t_{k+1}$, the metric $\bar{g}_t$ is such that every $p$ in the boundary of $M \setminus \Lambda_k$ is the center slice of some $\varepsilon_0$-neck.

In Section 7.1 of [34] Perelman argues that $R(\bar{g}_t) \geq -6$ for all $t \geq 0$ (for the normalized Ricci flow with surgery). In Section 7.3 and Section 7.4 of [34], Perelman shows that $(M, \bar{g}_t)$ admits, for all $t$ sufficiently large, a thick-thin decomposition, where the thick part meets the thin part along incompressible tori. Because $M$ is closed and admits an hyperbolic metric,
it has no incompressible tori and thus, for all \( t \) sufficiently large, \((M, \tilde{g}_t)\) coincides with the thick part. Therefore, we obtain from Lemma 7.2 of [34] that \(|\text{Ric}(\tilde{g}_t) + 2\tilde{g}_t|_{C^0}\) is small for all \( t \) sufficiently large. Thus we see from [41] that for all \( t \) sufficiently large \((\tilde{g}_t)_{t \geq 0}\) is a smooth solution to normalized Ricci flow and converges to an hyperbolic metric on \( M \). In particular there are only finitely many surgery times \( t_1 < t_2 < \ldots < t_K \). Using Mostow rigidity we can assume that, after applying a diffeomorphism to \( g_0 \), \((\tilde{g}_t)_{t \geq 0}\) converges smoothly to \( g_0 \) as \( t \to \infty \).

Given a homotopy class \( \Pi \) of essential surfaces, there is \( \Sigma_t \in \Pi \), a minimal surface with respect to \( \bar{g}_t \), such that \( \text{area}_{\bar{g}_t}(\Pi) = \text{area}_{\bar{g}_t}(\Sigma_t) \).

**Lemma 5.2.** We can find \( \varepsilon_0 \) small so that for every homotopy class \( \Pi \) of essential surfaces and every \( t < t_{k+1} \) close to \( t_{k+1} \), \( \Sigma_t \subset \Lambda_k \).

**Proof.** For all \( t < t_{k+1} \) close to \( t_{k+1} \), the metric \( \tilde{g}_t \) is such that every \( p \) in the boundary of \( M \setminus \Lambda_k \) is the center slice of some \( \varepsilon_0 \)-neck. This means that there exists a neighborhood \( N \subset M \) of \( p \) and a diffeomorphism (depending on \( p \) and \( t \))

\[
\Phi : S^2 \times [-\varepsilon_0^{-1}, \varepsilon_0^{-1}] \to N
\]

such that \( p \in \Phi(S^2 \times \{0\}) \) and for some \( \lambda > 0 \) (depending on \( p \) and \( t \))

\[
|\lambda^{-2} \Phi^* \tilde{g}_t - g_{S^2 \times [-\varepsilon_0^{-1}, \varepsilon_0^{-1}]}|_{C^{1/\varepsilon_0}} < \varepsilon_0.
\]

Moreover, \( \Phi(S^2 \times \{0\}) \) is homotopic to a boundary component of \( \Lambda_k \) because \( M \setminus \Lambda_k \) consists of \( \varepsilon_0 \)-caps.

Suppose for contradiction that \( \Sigma_t \) passes through \( p \). We first give the argument in the case that \( \Sigma_t \) is embedded. The region \( M \setminus \Lambda_k \) is diffeomorphic to a disjoint union of balls and so there is a ball \( B_k \) containing \( N \) so that \( \partial B_k = \Phi(S^2 \times \{-1/\varepsilon_0\}) \) and \( \Phi(S^2 \times [-\varepsilon_0^{-1}, \varepsilon_0^{-1}]) \subset B_k \). It cannot be the case that \( \Sigma_t \) is contained in \( B_k \) and thus \( \Sigma_t \) must intersect \( \partial B_k \). We can perturb \( B_k \) slightly so that \( \partial B_k \) intersects \( \Sigma_t \) transversely in a union of circles. Let \( D_p \) be the connected component of \( \Sigma_t \cap B_k \) that contains \( p \). Then \( D_p \) is homeomorphic to an \( m \)-holed sphere for some \( m \), all of the connected components of whose boundary \( \gamma \) correspond to null-homotopic embedded loops in \( \Sigma_t \).

On the one hand, \( \gamma \) can be filled in by a region \( D_\gamma \) in \( \partial B_k \) with area at most roughly \( 4\pi \lambda^2 \). On the other hand, \( D_p \) must intersect every cross-section \( \Phi(S^2 \times \{y\}) \) for \(-1/\varepsilon_0 < y < 0 \). By the monotonicity formula, there is a universal constant \( c \) such that for \( y \in (-1/\varepsilon_0 + 1/2, -1/2) \),

\[
\text{area}_{\bar{g}_t}(D_p \cap \Phi(S^2 \times (y - 1/2, y + 1/2)) > c\lambda^2.
\]

It follows by choosing disjoint unit intervals in \((-1/\varepsilon_0 + 1/2, -1/2)\) that if we chose \( \varepsilon_0 \) such that \( 1/\varepsilon_0 \) is greater than \( 5\pi/c \), then the area of \( D_p \) will be larger than \((4\pi + 1)\lambda^2 \). By cutting out \( D_p \) and gluing in \( D_\gamma \subset \partial B_k \), we could then produce a surface homotopic to \( \Sigma_t \) but with smaller area, which is a contradiction.
In the case that $\Sigma_t$ is only immersed, we lift $\Sigma_t$ to a surface $\Sigma_t$ in the cover $M_{\Sigma_t}$ of $M$ homeomorphic to $\Sigma_t \times \mathbb{R}$ corresponding to the inclusion of $\pi_1(\Sigma_t) \to \pi_1(M)$ for some choice of basepoint on $\Sigma_t$. Since $\Sigma_t$ minimizes area in its homotopy class in $M$, $\Sigma_t$ minimizes area in its homotopy class in $M_{\Sigma_t}$. It then follows from [13][Theorem 2.1] that $\Sigma_t$ is embedded in $M_{\Sigma_t}$.

Assume now that $\Sigma_t$ passes through a point $p$ contained in some ball $B_k$ as above. Since $B_k$ is contractible, it lifts to $M_{\Sigma_t}$, and we can find a lift $\overline{B}_k$ of $B_k$ containing a lift of $p$ lying on $\Sigma_t$. Since $\Sigma_t$ is embedded, we can apply the above arguments to get a contradiction as before.

□

This lemma implies that we can use [8, Lemma 9] and conclude that $\bar{A}(t) := \text{area}_{\bar{g}_t}(\Pi)$ is a Lipschitz function and thus differentiable almost everywhere. Let $\bar{t}$ be a point of differentiability of $\bar{A}$ and consider the function $a(t) := \text{area}_{\bar{g}_t}(\Sigma)$. We have $\bar{A}(t) \leq a(t)$ for all $t$ and $\bar{A}(\bar{t}) = a(\bar{t})$. Hence, arguing like in (14) we deduce

$$\frac{\text{area}_{\bar{g}_t}(\Sigma)}{4\pi (L - 1)} \leq 1 - \delta e^{-t}.$$  

We have that $\bar{g}_t$ converges smoothly to $g_0$ and so this contradicts Theorem 4.1. Thus the inequality in Theorem 5.1 must hold.

If equality holds in Theorem 5.1, the very same argument used as in Theorem 4.1 shows that $g_0$ must be Einstein. □

We can now derive Theorem 1.1 from Theorem 5.1.

**Theorem 5.3.** Suppose $(M, g)$ has $R(g) \geq -6$. Then $E(g) \leq 2$ and equality occurs if and only if $g$ is hyperbolic.

**Proof.** In what follows we will use the fact (from [37]) that for any $\Pi_m \in S_{1/m}(M)$, if $L$ is the genus of an essential surface in $\Pi_m$, then for all $t \geq 0$

$$\frac{\text{area}_{\bar{g}_t}(\Pi_m)}{4\pi (l_m - 1)} \leq 1 - \delta e^{-t}.$$  

We have that $\bar{g}_t$ converges smoothly to $g_0$ and so this contradicts Theorem 4.1. Thus the inequality in Theorem 5.1 must hold.

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**Proof.** In what follows we will use the fact (from [37]) that for any $\Pi_m \in S_{1/m}(M)$, if $L$ is the genus of an essential surface in $\Pi_m$, then we have

$$\frac{\text{area}_{\bar{g}_t}(\Pi_m)}{4\pi (l_m - 1)} \to 1$$  

as $m \to \infty$.

Choose $\delta > 0$. Theorem 5.1 implies the existence of $\varepsilon_0$ so that for all $\Pi \in S_{\varepsilon_0}(M)$ we have

$$\text{area}_{\bar{g}_t}(\Pi) \leq (1 + \delta)\text{area}_g(\Pi).$$  

Hence for all $\varepsilon \leq \varepsilon_0$ we have

$$\#\{\text{area}_g(\Pi) \leq 4\pi (L - 1) : \Pi \in S_{\varepsilon}(M)\} \leq \#\{\text{area}_{\bar{g}_t}(\Pi) \leq (1 + \delta)4\pi (L - 1) : \Pi \in S_{\varepsilon}(M)\}.$$  

□
This inequality and the expression (1) for the minimal surface entropy imply
\[ E(g) \leq E(g_0)(1 + \delta) = 2(1 + \delta). \]
The inequality follows from making \( \delta \to 0 \).

Suppose now that \( E(g) = 2 \) for some metric with \( R(g) \geq -6 \). Reasoning as above we see that if we could find \( \delta > 0 \) and \( \varepsilon_0 \) so that for all \( \Pi \in S_{\varepsilon_0}(M) \)
\[ \text{area}_{g_0}(\Pi) \leq (1 - \delta)\text{area}_g(\Pi), \]
then \( E(g) \leq 2(1 - \delta) \). Hence there is \( \Pi_m \in S_{1/m}(M) \) so that
\[ \liminf_{m \to \infty} \frac{\text{area}_g(\Pi_m)}{\text{area}_{g_0}(\Pi_m)} \leq 1 \]
and Theorem 5.1 implies that \( g \) is hyperbolic. \( \square \)

6. Proof of Theorem 1.2

For each \( m \in \mathbb{N} \), Labourie [28, Theorem 5.7] found \( N_m \in \mathbb{N} \), \( \{\phi^i_m\}_{i=1}^{N_m} \) in \( F(M,1/m) \), and \( 0 < a^1_m, \ldots, a^{N_m}_m < 1 \) with \( a^1_m + \ldots + a^{N_m}_m = 1 \), so that: \( \phi^i_m \) is equivariant with respect to a representation of a Fuchsian subgroup of \( \text{PSL}(2,\mathbb{R}) \) in \( \pi_1(M) < \text{PSL}(2,\mathbb{C}) \) and the laminar measure
\[
\delta_m := \sum_{i=1}^{N_m} a^i_m \delta_{\phi^i_m}
\]
is such that \( \Omega_s \delta_m \) converges to \( \mu_{\text{Leb}} \), the unit Lebesgue homogeneous measure on \( F(M) \).

Proposition 6.1. There is \( \phi_m \) in \( F(M,1/m) \) equivariant with respect to a representation of a Fuchsian group \( \Gamma_m \) of \( \text{PSL}(2,\mathbb{R}) \) in \( \pi_1(M) < \text{PSL}(2,\mathbb{C}) \) such that \( \Omega_s \delta_m \) converges to \( \mu_{\text{Leb}} \) as \( m \to \infty \).

Proof. The space of all closed totally geodesic immersions in \( (M,g_0) \) is countable and so
\[ \mathcal{T} := \{ F(S) \subset F(M) : S \text{ is a closed totally geodesic immersion in } (M,g_0) \} \]
is also countable, where \( F(S) \) denotes the frame bundle of \( S \) which injects naturally in \( F(M) \).

Considering tubular neighborhoods, we can find a decreasing sequence of open sets \( \{B_k\}_{k \in \mathbb{N}} \subset F(M) \) so that for all \( k \in \mathbb{N} \)
\[ \cup_{T \in \mathcal{T}} B_k, \quad \mu_{\text{Leb}}(\partial B_k) = 0, \quad \text{and} \quad \mu_{\text{Leb}}(B_k) \leq 2^{-2k-3}. \]

Lemma 6.2. For each \( j \in \mathbb{N} \), there is \( j \leq m_j \in \mathbb{N} \) and \( \phi_j \in \{\phi^1_{m_j}, \ldots, \phi^{N_{m_j}}_{m_j}\} \)
so that
\[ \Omega_s \delta_{\phi_j}(B_k) \leq 2^{-(k+1)} \quad \text{for all } k \leq j. \]
Proof. For all $k \in \mathbb{N}$, $\Omega_{s}\delta_{m}(B_{k}) \to \mu_{Leb}(B_{k})$ as $m \to \infty$. Thus we can find a strictly increasing sequence of integers $\{m_{j}\}_{j \in \mathbb{N}}$ so that

$$\Omega_{s}\delta_{m_{j}}(B_{k}) \leq 2\mu_{Leb}(B_{k}) \leq 2^{-2(k+1)}$$

for all $k \leq j$.

Relabel $m_{j}$ to be $j$ so that $\delta_{m_{j}}, \phi_{m_{j}}, N_{m_{j}}$, and $a_{m_{j}}$ become $\delta_{j}, \phi_{j}, N_{j},$ and $a_{j}$, respectively.

Consider $\mu_{j}$ a unit measure on $\{1, \ldots, N_{j}\}$ so that $\mu_{j}(i) = a_{j}^{i}$ and set

$$J_{j,k} := \{i \in \{1, \ldots, N_{j}\} : \Omega_{s}\delta_{\phi_{j}^{i}}(B_{k}) \geq 2^{-(k+1)}\}.$$  

From (25) and the definition of $J_{j,k}$ we have that for all $j \geq k$

$$\mu_{j}(J_{j,k}) = \sum_{i \in J_{j,k}} a_{j}^{i} \leq 2^{k+1} \sum_{i \in J_{j,k}} a_{j}^{i} \Omega_{s}\delta_{\phi_{j}^{i}}(B_{k}) \leq 2^{k+1} \Omega_{s}\delta_{j}(B_{k}) \leq 2^{-(k+1)}.$$  

Thus, for all $j \in \mathbb{N}$ and $k \leq j$, $\mu_{j}(\cup_{k=1}^{j} J_{j,k}) \leq \sum_{k=1}^{j} 2^{-(k+1)} \leq 1/2$. Hence

$$A_{j} := \{1, \ldots, N_{j}\} \setminus \cup_{k=1}^{j} J_{j,k} \neq \emptyset$$

and we pick $l_{j} \in A_{j}$. The maps $\phi_{j} := \phi_{l_{j}}$ satisfy the desired conditions. \(\square\)

After passing to a subsequence, $\Omega_{s}\delta_{\phi_{j}}$ converges weakly to a unit measure $\nu$ on $F(M)$. From Lemma 3.2 we have that $\nu$ is invariant under the homogeneous action of $\text{PSL}(2, \mathbb{R})$. The proof will be completed if we show that $\nu = \mu_{Leb}$.

Ratner’s classification theorem [35] implies that every ergodic probability measures on $F(M)$ invariant under the homogeneous action of $\text{PSL}(2, \mathbb{R})$ is either $\mu_{Leb}$ or is supported in some $T \in \mathcal{T}$. Thus, from the ergodic decomposition theorem for $\text{PSL}(2, \mathbb{R})$-actions (Theorem 4.2.6 in [26]), there is $0 \leq \theta \leq 1$ so that the measure $\nu$ decomposes as $\nu = \theta \mu_{Leb} + (1 - \theta) \mu_{T}$, where $\mu_{T}$ is some probability measure on $F(M)$ with support in $\cup_{T \in \mathcal{T}} T$. Moreover we have from Lemma 6.2 that $\nu(B_{k}) \leq 2^{-(k+1)}$ for all $k \in \mathbb{N}$.

Recall that $\cup_{T \in \mathcal{T}} T \subset B_{k}$ for all $k \in \mathbb{N}$ and so

$$1 - \theta = (1 - \theta) \mu_{T}(B_{k}) \leq \nu(B_{k}) \leq 2^{-(k+1)}.$$  

Hence $\theta = 1$, which means that $\nu = \mu_{Leb}$. \(\square\)

**Proposition 6.3.** $\text{Area}_{\text{Id}}(g_{0}) = \lim_{m \to \infty} \frac{\text{area}_{\text{a}}(D(\partial_{0}m))}{2\pi \chi(D(\partial_{0}m))}$.

**Proof.** We have for all $(x, P) \in Gr_{2}(M)$

$$|\wedge^{2} \text{Id}^{-1}|_{g}(x, P) = |\wedge^{2} \text{Id}|_{g}^{-1}(x, P).$$

We abuse notation and also denote by $|\wedge^{2} \text{Id}|_{g}$ the following smooth positive function on the frame bundle $F(M)$

$$|\wedge^{2} \text{Id}|_{g} : F(M) \to \mathbb{R}, \; (y, \{e_{1}, e_{2}, n\}) \mapsto |\wedge^{2} \text{Id}|_{g}(y, \text{span}\{e_{1}, e_{2}\}).$$

We have

$$\text{Area}_{\text{Id}}(g_{0}) = \mu_{Leb}(|\wedge^{2} \text{Id}|_{g}^{-1})$$
and thus the fact that $\Omega_\star \delta_{\phi_m} \to \mu_{Leb}$ implies
\[(26) \quad \text{Area}_{Id}(g/g_0) = \mu_{Leb}(| \wedge^2 \text{Id}|_g^{-1}) = \lim_{m \to \infty} \Omega_\star \delta_{\phi_m}(| \wedge^2 \text{Id}|_g^{-1}).\]

Consider $U_m \subset \text{PSL}(2, \mathbb{R})$ a fundamental domain of $\text{PSL}(2, \mathbb{R})/\Gamma_m$. Then
\[
\Omega_\star \delta_{\phi_m}(| \wedge^2 \text{Id}|_g^{-1}) = \frac{1}{\text{vol}(U_m)} \int_{U_m} | \wedge^2 \text{Id}|_g^{-1} \circ \Omega(\phi_m \circ \tau) d\nu(\tau).
\]

Note that for all $\tau \in \text{PSL}(2, \mathbb{R})$ we have, with $z = \tau(i)$,
\[
| \wedge^2 \text{Id}|_g \circ \Omega(\phi_m \circ \tau) = | \wedge^2 \text{Id}|_g(\phi_m(z), \langle d\phi_m \rangle_z (T_z \mathbb{H}^2)).
\]

Thus, denoting the hyperbolic volume form on $\mathbb{H}^2$ by $dA_{hyp}$,
\[(27) \quad \Omega_\star \delta_{\phi_m}(| \wedge^2 \text{Id}|_g^{-1}) = \frac{1}{2\pi |\chi(D(\partial \phi_m))|} \int_{\mathbb{H}^2/\Gamma_m} | \wedge^2 \text{Id}|_g^{-1}(\phi_m(z), \langle d\phi_m \rangle_z (T_z \mathbb{H}^2)) dA_{hyp}(z)
\]
\[
= \frac{1}{2\pi |\chi(D(\partial \phi_m))|} \int_{D(\partial \phi_m)} \frac{| \wedge^2 \text{Id}|_g^{-1}(y, T_y D(\partial \phi_m))}{C(\phi_m \circ \phi_m^{-1}(y))} dA_{g_0}(y).
\]

Set $A_m := ||A||_{L^\infty(D(\partial \phi_m))}$. From Gauss equation and (5) we have
\[(28) \quad 1 \leq \frac{1}{C(\phi_m)} \leq 1 + A_m^2.
\]

The co-area formula says that
\[
\int_{D(\partial \phi_m)} | \wedge^2 \text{Id}|_g^{-1}(y, T_y D(\partial \phi_m)) dA_{g_0} = \text{area}_g(D(\partial \phi_m))
\]
and thus, combining with (27) and (28), we obtain
\[
\frac{\text{area}_g(D(\partial \phi_m))}{2\pi |\chi(D(\partial \phi_m))|} \leq \Omega_\star \delta_{\phi_m}(| \wedge^2 \text{Id}|_g^{-1}) \leq (1 + A_m^2) \frac{\text{area}_g(D(\partial \phi_m))}{2\pi |\chi(D(\partial \phi_m))|}.
\]

We have that $A_m \to 0$ as $m \to \infty$ and hence we deduce from this inequality and (26) the desired result. \hfill \Box

6.1 **Proof of inequality.** The closed surfaces $D(\partial \phi_m)$ define a homotopy class $\Pi_m \in S_{1/m}(M)$ for all $m \in \mathbb{N}$. We denote the genus of $D(\partial \phi_m)$ by $l_m$.

**Proposition 6.4.** Area$_{Id}(g/g_0)E(g) \geq 2$.

**Proof.** After passing to a subsequence, set
\[
\alpha := \lim_{m \to \infty} \frac{\text{area}_g(\Pi_m)}{4\pi(l_m - 1)}.
\]

Given $\delta$ we have that for all $m$ sufficiently large
\[
\text{area}_g(\Pi_m) \leq (\alpha + \delta)4\pi(l_m - 1).
\]
Let $D^k_m, l^k_m,$ and $\Pi^k_m \in S_{1/m}(M)$ denote, respectively, a $k$-cover of $D(\partial \phi_m)$, its genus, and its homotopy class. From the inequality above we have for all $k \in \mathbb{N}$ and all $m \in \mathbb{N}$ sufficiently large

\[
\frac{\text{area}_g(\Pi^k_m)}{4\pi(l^k_m - 1)} \leq \frac{k}{4\pi(l^k_m - 1)} = \frac{\text{area}_g(\Pi_m)}{4\pi(l_m - 1)} \leq \alpha + \delta.
\]

From the Müller-Puchta’s formula (see [9, Section 4]) there is $c(m) > 0$ so that $D(\partial \phi_m)$ has at least $(c(m))^{2^k_m}$ distinct covers of degree less than or equal to $k$. Thus, if we choose $L^k_m$ so that $4\pi(L^k_m - 1) = (\alpha + \delta)4\pi(l^k_m - 1)$, we have from (29) that for all $k \in \mathbb{N}$ and all $m \in \mathbb{N}$ sufficiently large

\[
\# \{ \text{area}_g(\Pi) \leq 4\pi(L^k_m - 1) : \Pi \in S_{1/m}(M) \} \geq (c(m))^{2^k_m}.
\]

Hence for all $m$ sufficiently large

\[
\lim_{k \to \infty} \frac{\ln \# \{ \text{area}_g(\Pi) \leq 4\pi(L^k_m - 1) : \Pi \in S_{1/m}(M) \}}{L^k_m \ln L^k_m} \geq \lim_{k \to \infty} \frac{2^k_m \ln(c(m))^{2^k_m}}{L^k_m \ln L^k_m} = \frac{2}{\alpha + \delta}.
\]

From the expression for $E(g)$ in (1) we obtain that $(\alpha + \delta)E(g) \geq 2$. Making $\delta \to 0$ we deduce $\alpha E(g) \geq 2$. From Proposition 6.3 we have

\[
\text{Area}_{\text{id}}(g/g_0) = \lim_{m \to \infty} \frac{\text{area}_g(D(\partial \phi_m))}{2\pi |\chi(D(\partial \phi_m))|} \geq \lim_{m \to \infty} \frac{\text{area}_g(\Pi_m)}{4\pi(l_m - 1)} = \alpha
\]

and thus $\text{Area}_{\text{id}}(g/g_0)E(g) \geq 2$. \hfill \Box

6.2. Proof of rigidity. Suppose that $\text{Area}_{\text{id}}(g/g_0)E(g) = 2$. In this case, the proof of the preceding proposition implies that

\[
\lim_{m \to \infty} \frac{\text{area}_g(D(\partial \phi_m))}{\text{area}_g(\Pi_m)} = 1.
\]

The mean curvature of a surface in $(M, g)$ is denoted by $H_g$.

**Lemma 6.5.** $\lim_{m \to \infty} \frac{1}{\text{area}_g(D(\partial \phi_m))} \int_{D(\partial \phi_m)} |H_g|^2 dA_g = 0$.

**Proof.** Suppose that, after passing to a subsequence, there is $\delta > 0$ so that

\[
\int_{D(\partial \phi_m)} |H_g|^2 dA_g \geq 2\delta \text{area}_g(D(\partial \phi_m))
\]

for all $m \in \mathbb{N}$.

The second fundamental form of $D(\partial \phi_m)$ (with respect to $g$) and any of its derivatives are uniformly bounded independently of $m$. Arguing like in [12, Corollary 4.4] we find $t_0 > 0, C_0 > 0$, and $\{|D_m(t)|\}_{0 \leq t \leq t_0}$ a solution to mean curvature flow with initial condition $D_m(0) := D(\partial \phi_m)$ so that both the mean curvature of $D_m(t)$ and its derivative is bounded uniformly by $C_0$. 


With $H_g(t)$, $A_g(t)$, and $\nu_t$ denoting, respectively, the mean curvature, second fundamental form, and normal vector of $D_m(t)$, we have the evolution equation
\[
\frac{d}{dt}H_g(t) = \Delta H_g(t) + (|A_g(t)|^2 + Rc(g)(\nu_t, \nu_t))H_g(t).
\]
Thus there is a constant $C_1 > 0$ (depending only on $C_0$ and $g$) so that for all $m \in \mathbb{N}$ and $0 \leq t \leq t_0$
\[
\frac{d}{dt} \int_{D_m(t)} |H_g(t)|^2 dA_g \geq -C_1 \operatorname{area}_g(D_m(t)) \geq -C_1 \operatorname{area}_g(D(\partial \phi_m)).
\]
Choose $t_1 < \min\{\delta/C_1, t_0\}$. From (31) and the inequality above we have that for all $m \in \mathbb{N}$ and $0 \leq t \leq t_1$
\[
\int_{D_m(t)} |H_g|^2 dA_g \geq \delta \operatorname{area}_g(D(\partial \phi_m)) \geq \delta \operatorname{area}_g(D_m(t)).
\]
Therefore
\[
\frac{d}{dt} \operatorname{area}_g(D_m(t)) = -\int_{D_m(t)} |H_g|^2 dA_g \leq -\delta \operatorname{area}_g(D_m(t))
\]
and thus for all $m \in \mathbb{N}$ we have
\[
\operatorname{area}_g(\Pi_m) \leq \operatorname{area}_g(D_m(t_1)) \leq e^{-\delta t_1} \operatorname{area}_g(D(\partial \phi_m)).
\]
This contradicts (30). \(\square\)

Recall that $\Omega^* \delta \phi_m$ converges to $\mu_{Leb}$ as $m \to \infty$. The previous lemma allows us to apply Theorem A.3 to conclude the following:

For every $\gamma \in C_0$, there is a lift $D_m \subset \mathbb{H}^3$ of $D(\partial \phi_m) \subset M$ so that, after passing to a subsequence, $D_m$ converges on compact sets to the totally geodesic disc $D(\gamma) \subset \mathbb{H}^3$ and
\[
\lim_{m \to \infty} \int_{D_m \cap B_R(0)} |H_g|^2 dA_{g_0} = 0 \quad \text{for all } R > 0.
\]
Thus we obtain that $D(\gamma)$ is a minimal disc for the metric $g$. The arbitrariness of $\gamma$ implies that every totally geodesic disc of $\mathbb{H}^3$ is minimal with respect to $g$.

The next theorem follows from adapting some of the arguments in [1].

**Theorem 6.6.** Let $g$ be a metric on $M$ with property that every totally geodesic surface with respect to $g_0$ is minimal with respect to $g$. There is a constant $c > 0$ so that $g = cg_0$.

**Proof.** We denote the space of Killing symmetric 2-tensors on a Riemannian manifold $(X, h)$ by $K_2(X, h)$. They are characterized by the property (see [39, Section 1]) that a symmetric 2-tensor $k$ lies in $K_2(X, h)$ if and only if for every geodesic $\gamma \subset X$ the function below is constant
\[
t \mapsto k(\gamma'(t), \gamma'(t)).
\]
Given a symmetric 2-tensor $k$ on $X$, we denote the absolute value of its determinant with respect to $h$ by $|k|_h$.

Let $\delta$ denote the Euclidean metric in $\mathbb{R}^3$ and $B \subset \mathbb{R}^3$ denote the open unit ball. Consider the Beltrami-Klein model $(B, g_{hyp})$ for $(\mathbb{H}^3, g_0)$. The important property of this model is that the image of geodesics and totally geodesic discs in $(B, g_{hyp})$ is the same as affine lines and affine planes in $(B, \delta)$.

In [1, Theorem 9.8] the authors classified all metric $h$ on $S^n$ for which every totally geodesic hypersphere is minimal with respect to $h$. They are those for which

$$|h|_{\text{round}}^{-2/(n+1)}h \in K_2(S^n, g_{\text{round}}).$$

A similar reasoning, which we repeat for the sake of completeness, gives

**Proposition 6.7.** A positive definite symmetric 2-tensor $h$ on $B$ has the property that every affine plane is minimal with respect to $h$ if and only if

$$|h|_{g_{hyp}}^{-1/2}h \in K_2(B, g_{hyp}).$$

**Proof.** We will use the following theorem of Hangan [18]: A metric $h$ on $B$ is such that all affine planes have zero mean curvature with respect to $h$ if and only if

$$|h|_{\delta}^{-1/2}h \in K_2(B, \delta).$$

**Lemma 6.8.** With $k$ a 2-tensor on $B$ then

$$k \in K_2(B, g_{hyp}) \iff |g_{hyp}|_{\delta}^{-1/2}k \in K_2(B, \delta).$$

**Proof.** From Hangan’s theorem we have that $|g_{hyp}|_{\delta}^{-1/2}g_{hyp} \in K_2(B, \delta)$ and hence, for every Euclidean geodesic $\gamma \subset B$, we have some constant $c_0 = c_0(\gamma)$ such that for all $t$

$$g_{hyp}(\gamma'(t), \gamma'(t)) = |g_{hyp}|_{\delta}^{1/2}(\gamma(t))c_0.$$

Note that geodesics with respect to $g_{hyp}$ are also straight lines. Hence if we perform a change of variable according to

$$\frac{dt}{ds}(s) = |g_{hyp}|_{\delta}^{-1/4}(\gamma(t(s)))$$

we have that $s \mapsto \sigma(s) := \gamma(t(s))$ is a geodesic for $g_{hyp}$. Therefore

$$\frac{d}{ds}k(\sigma'(s), \sigma'(s)) = \frac{dt}{ds}(s)\frac{d}{dt=t(s)}(k(\gamma'(t), \gamma'(t))|g_{hyp}|_{\delta}^{1/2}(\gamma(t)))$$

and so

$$\frac{d}{ds}k(\sigma'(s), \sigma'(s)) = 0 \iff \frac{d}{dt=t(s)}(k(\gamma'(t), \gamma'(t))|g_{hyp}|_{\delta}^{1/2}(\gamma(t))) = 0.$$

Thus $k$ is constant along hyperbolic geodesics if and only if $|g_{hyp}|_{\delta}^{-1/2}k$ is constant along Euclidean geodesics, which implies the result. □
Using this lemma and the identity \(|h|_\delta = |h|_{g_{\text{hyp}}}|h_{\text{hyp}}|_\delta\) we have

\[ |h|_\delta^{-1/2} h \in K_2(B, \delta) \iff |h|_{g_{\text{hyp}}}^{-1/2} h \in K_2(B, g_{\text{hyp}}). \]

Hangan’s theorem [18] implies the desired result. \[\square\]

Denote the lift of the metric \(g\) to \(\mathbb{H}^3\) by \(\tilde{g}\) as well. We are assuming that every totally geodesic disc in \(\mathbb{H}^3\) is minimal with respect to \(g\) and so the previous proposition implies that \(G := |g|_{g_0}^{-1/2} g \in K_2(M, g_0)\).

The geodesic flow in \((M, g_0)\) is ergodic and so we can choose a geodesic \(\sigma \subset M\) which is dense in the unit tangent bundle. From the fact that \(t \mapsto G(\sigma'(t), \sigma'(t))\) is constant we deduce the existence of a constant \(\alpha\) so that \(G(Y, Y) = \alpha g_0(Y, Y)\) for every vector field \(Y\). This implies that \(G = \alpha g_0\). Using the fact that \(G = |g|_{g_0}^{-1/2} g\) we deduce that \(g = \alpha^{-2} g_0\). \[\square\]

**Appendix A. Auxiliary Results**

A.1. **Asymptotic Expansion.** With \(g \in \mathcal{U}\), set \(h := g - g_0\). Without loss of generality we assume that \(|h|_{C^4} \leq 1\).

Given \(\gamma \in \mathcal{C}_\varepsilon\), Recall that \(\nu_g(\gamma)\) denotes a continuous unit normal vector field along \(\Sigma_g(\gamma)\) with respect to \(g\) and \(n(\gamma)\) denotes a continuous unit normal vector field along \(D(\gamma)\) with respect to \(g_0\).

Recall the diffeomorphism (using the hyperboloid model)

\[ F_\gamma : D(\gamma) \to \Sigma_g(\gamma), \quad x \mapsto \cosh(f_\gamma(x))x + \sinh(f_\gamma(x))n(\gamma)(x). \]

Given \(p \in M\) and \(k \in \mathbb{N}_0\), \(l \in \mathbb{N}\), we denote by \(O_p^k(|h|^l)\) any quantity for which there is a constant \(\alpha_{k,l}\) (independent of \(p\) and \(g \in \mathcal{U}\)) so that its absolute value at \(p\) is bounded by \(\alpha_{k,l} \sum_{j=0}^k |\nabla^j h|^l(p)\). Likewise, given \(\gamma \in \mathcal{C}_\varepsilon\) and \(x \in D(\gamma)\), we denote by \(O_x^k(|f_{\gamma}|^l)\) any quantity for which there is a constant \(\beta_{k,l}\) (independent of \(x, \gamma\), and \(g \in \mathcal{U}\)) so that its absolute value at \(x\) is bounded by \(\beta_{k,l} \sum_{j=0}^k |\nabla^j f_{\gamma}|^l(x)\).

Let \(L\) denote the linearization at \(g_0\) of

\[ h \mapsto \tilde{\text{Ric}}(g_0 + h) := \text{Ric}(g_0 + h) - \frac{R(g_0 + h) + \frac{1}{3}(g_0 + h)}{3}(g_0 + h). \]

**Proposition A.1.** With \(\gamma \in \mathcal{C}_\varepsilon\), \(x \in D(\gamma)\) and \(y = F_\gamma(x)\) we have

\[
\text{Ric}(g)(\nu_g(\gamma), \nu_g(\gamma)) - \frac{1}{3} R(g)(y) = L(h)x(n(\gamma), n(\gamma)) + O_2^1(|h|^2) + O_2^1(|f_{\gamma}|). 
\]

With \(|\text{Jac}_{\gamma} F_\gamma|\) the Jacobian of \(F_\gamma : (D(\gamma), g_0) \to (\Sigma_g(\gamma), g)\) we have

\[
|\text{Jac}_{\gamma} F_\gamma|(x) = 1 + O_2^1(|h|x) + O_2^1(|f_{\gamma}|). 
\]

**Proof.** Set

\[ X := \{ (\gamma, f) : \gamma \in \mathcal{C}_\varepsilon, f \in C^2(D(\gamma)), |f|_{C^2} \leq 1 \}. \]
Given \((\gamma, f) \in X\) set
\[
F(\gamma, f) : D(\gamma) \to \Sigma_g(\gamma), \quad x \mapsto \cosh(f(x))x + \sinh(f(x))n(\gamma)(x)
\]
and \(\Sigma(\gamma, f) := F(\gamma, f)(D(\gamma))\). Let \(n(\gamma, f)\) be the unit normal vector field with respect to \(g_0\) along \(\Sigma(\gamma, f)\), defined so that \(n(\gamma, f)\) depends smoothly on \((\gamma, f)\) and \(n(\gamma, 0) = n(\gamma)\). We have for all \(0 \leq t \leq 1\)
\[
\left| \frac{d^k}{dt^k}(n(\gamma, tf))(F(\gamma, tf)(x)) \right| = O^1_x(|f|^k), \quad k = 1, 2.
\]
Hence, if \(G\) is a 2-tensor on \(M\) with \(|G|_{C^2} \leq 1\), then
\[
\alpha(t) := G(n(\gamma, tf)(F(\gamma, tf)(x)), n(\gamma, f)(F(\gamma, tf)(x)))
\]
satisfies
\[
|\alpha'(0)| \leq (|\nabla G|(x) + |G|)(O^1_x(|f|)) \quad \text{and} \quad \sup_{0 \leq t \leq 1} |\alpha''(t)| = O^1_x(|f|^2).
\]
Therefore, with \(y = F(\gamma, f)(x)\), we obtain from Taylor’s expansion
\[
|G(n(\gamma, f)(y), n(\gamma, f)(y)) - G(n(\gamma)(x), n(\gamma)(x))| \leq |\nabla G|^2(x) + |G(x)|^2 + O^1_x(|f|^2).
\]
Setting \(G = |\nabla^k h|^2 g_0\) in this inequality we deduce that for \(k = 0, 1, 2\).
\[
|\nabla^k h|(y) = O^{k+1}_x(|h|) + O^1_x(|f|).
\]
Setting \(G = L(h)\) in (32) and using the fact that \(L\) is a second order differential operator we deduce
\[
|L(h)x(n(\gamma)(x), n(\gamma)(x)) - L(h)y(n(\gamma, f)(y), n(\gamma, f)(y))| = O^2_x(|h|^2) + O^1_x(|f|^2).
\]
Let \(\nu_g(\gamma, f)\) denote the unit normal vector field along \(\Sigma(\gamma, f)\) with respect to \(g = g_0 + h\) so that \(\nu_{g_0}(\gamma, f) = n(\gamma, f)\) and \(\nu_g\) depends smoothly on its parameters. Then
\[
|\nu_g(\gamma, f)(y) - n(\gamma, f)(y)| = O^0_y(|h|) = O^1_x(|h|) + O^1_x(|f|),
\]
where in the last identity we used (33). Using this identity and (33) we have
\[
L(h)y(\nu_g(\gamma, f)(y), \nu_g(\gamma, f)(y)) = L(h)y(n(\gamma, f)(y), n(\gamma, f)(y)) + O^3_x(|h|^2) + O^1_x(|f|^2).
\]
Combining with (34) we deduce
\[
L(h)y(\nu_g(\gamma, f)(y), \nu_g(\gamma, f)(y)) = L(h)x(n(\gamma)(x), n(\gamma)(x)) + O^3_x(|h|^2) + O^1_x(|f|^2).
\]
Using Taylor’s expansion we have that for every \(y \in M\), every unit vector field \(Y \in T_y M\) and \(g_0 + h \in \mathcal{U}\)
\[
|\text{Ric}(g_0 + h)(Y, Y) - L(h)y(Y, Y)| = O^2_y(|h|^2) = O^3_x(|h|^2) + O^1_x(|f|^2).
\]
With A.2. Stability of Ricci flow. Ricci flow (\(\bar{g}\)) is a non-linear term depending on \(t\), and thus converging strongly to some diffeomorphism \(\Phi\).

\[ \lambda \leq 2 \]

It has discrete spectrum \(1 \leq \lambda_1 \leq \lambda_2 \leq \ldots \)

**Proposition A.2.** Set \(h_t := \Phi_t^* \bar{g}_t - g_0, t \geq 0\). Then

- \(|h_t|_{C^4} \leq O(e^{-2t/3})\);
- For all \(k \in \mathbb{N}\) the tensor \(e^t h_t\) converges in \(W^{k,2}\) as \(t \to \infty\) to \(\bar{h}\), where \(\bar{h}\) is a smooth 2-tensor with \(A(\bar{h}) = -\bar{h}\).

**Proof.** In \([25]\) it is shown that \(|h_t|_{C^2} \leq O(e^{-2t/3})\). Standard estimates (similar to \([19, \text{Lemma 5.3}]\)) show that for all \(k \in \mathbb{N}\), \(|\text{Ric}(\bar{g}_t) + 2\bar{g}_t|_{C^k} \leq O^k(e^{-2t/3})\) and thus \(|\bar{g}_t - g_{+\infty}|_{C^k} \leq O^k(e^{-2t/3})\) as well. The diffeomorphisms \(\{\Phi_t\}_{t \geq 0}\) depend only on \(g_0\) and \(\{\bar{g}_t\}_{t \geq 0}\), and one can check that for all \(k \in \mathbb{N}\), \(|\Phi_t - \Phi_{+\infty}|_{C^k} \leq O^k(e^{-2t/3})\). Thus \(|h_t|_{C^k} \leq O^k(e^{-2t/3})\).

In \([25]\) it is also show that the tensors \(h_t\) satisfy an equation of the form

\[ \frac{dh_t}{dt} = A(h_t) + Q_t, \]

where \(Q_t\) is a non-linear term depending on \(g_0, h_t, \nabla h_t, \nabla^2 h_t\). The important property we need is that \(|Q_t|_{C^k} \leq O^k(|h_t|_{C^{k+2}}^2)\). From here on we consider \(Q_t\) as being a fixed non-homogeneous term where \(|Q_t|_{C^k} \leq O^k(e^{-4t/3})\).

Consider an \(L^2\)-orthonormal basis \(\{u_j\}_{j \in \mathbb{N}}\) for the space of symmetric 2-tensors on \(M\) made of eigentensors for \(A\). Necessarily
\[ h_t = \sum_{j=0}^{\infty} \left( e^{-\lambda_j t} \langle h_0, u_j \rangle + \int_0^t e^{\lambda_j (\tau - t)} \langle Q_\tau, u_j \rangle d\tau \right) u_j. \]

Set \( \eta_t := h_t - f(t) \), where
\[ f(t) := \sum_{\lambda_1=1} \left( e^{-t \lambda_1} \langle h_0, u_1 \rangle + \int_0^t e^{(\tau - t)} \langle Q_\tau, u_1 \rangle d\tau \right) u_1. \]

With \( \eta_t^0 := \eta_t \) and \( k \in \mathbb{N} \), set \( \eta_t^k = A\eta_t^{k-1} \). Because \( \eta_t^k \) is orthogonal to the 1-eigentensors there is some \( \delta \) so that
\[ \int_M A(\eta_t^k) \eta_t^k dV_{g_0} \leq -(1 + 2\delta) \int_M (\eta_t^k)^2 dV_{g_0}. \]

Moreover we have \( \partial_t \eta_t^k = A(\eta_t^k) + O_k(e^{-4t/3}) \), which when combined with the inequality above implies that
\[
\frac{d}{dt} \int_M (\eta_t^k)^2 dV_{g_0} \leq -2(1 + 2\delta) \int_M (\eta_t^k)^2 dV_{g_0} + O_k(e^{-4t/3}) \int_M \eta_t^k dV_{g_0} \\
\leq -2(1 + 2\delta) \int_M (\eta_t^k)^2 dV_{g_0} + O_k(e^{-4t/3}) \left( \int_M (\eta_t^k)^2 dV_{g_0} \right)^{1/2} \\
\leq -2(1 + \delta) \int_M (\eta_t^k)^2 dV_{g_0} + O_k(e^{-8t/3}).
\]

Hence \( \int_M (e^t \eta_t^k)^2 dV_{g_0} \rightarrow 0 \) as \( t \rightarrow \infty \). From the fact that \( \eta_t^{k+1} - \Delta_{g_0} \eta_t^k \) is linear in \( \eta_t^k \) we obtain that for all \( k \in \mathbb{N}_0 \) we have
\[ \int_M (e^t \eta_t^k)^2 + (e^t \Delta_{g_0} \eta_t^k)^2 dV_{g_0} \rightarrow 0 \text{ as } t \rightarrow \infty. \]

This implies that, for all \( k \in \mathbb{N} \), \( e^t \eta_t \) tends to zero in \( W^{k,2} \) as \( t \rightarrow \infty \). The result follows because the 1-eigentensor \( e^t f(t) \) converges smoothly as \( t \rightarrow \infty \) to some smooth tensor \( \bar{h} \). \( \square \)

**A.3. A theorem of Calegari-Marques-Neves.** Consider a sequence \( \phi_i \in \mathcal{F}(M, 1/i) \) equivariant with respect to a representation of a Fuchsian subgroup of \( \text{PSL}(2, \mathbb{R}) \) in \( \pi_1(M) < \text{PSL}(2, \mathbb{C}) \). Let \( G_i < \pi_1(M) \) be the image of that representation. The group \( G_i \) preserves \( D(\partial \phi_i) \subset \mathbb{H}^3 \) and recall that \( D(\partial \phi_i) = D(\partial \phi_i)/G_i \). We assume

- \( \Omega_{t_{\phi_i}} \) converges weakly to a measure \( \nu \) on \( F(M) \) as \( i \rightarrow \infty \), where the measure \( \nu \) is such that so that \( \nu(O) > 0 \) for every open set \( O \);
- there is a sequence of immersed surfaces \( \Sigma_i \subset M \) homotopic to \( D(\partial \phi_i) \) and \( f_i \in C^0(\Sigma_i) \) so that
\[
\lim_{i \rightarrow \infty} \frac{1}{\text{area}_{g_0}(\Sigma_i)} \int_{\Sigma_i} |f_i| dA_{g_0} = 0.
\]
The Hausdorff distance between two sets in $\mathbb{H}^3$ is denoted by $d_H$.

The following theorem corresponds to Theorem 6.1 of [9], where it is assumed that $\Sigma_i$ is area-minimizing with respect to some metric. An inspection of the proof shows that one only needs the areas of $\Sigma_i$ and $D(\partial \phi_i)$ to be comparable and their universal covers to be at a uniform Hausdorff distance from each other.

**Theorem A.3.** Assume the existence of $C > 0$ and, for all $i \in \mathbb{N}$, a covering $\Omega_i \subset \mathbb{H}^3$ of $\Sigma_i$ so that
\[
\text{area}_{g_0}(\Sigma_i) \leq C \text{area}_{g_0}(D(\partial \phi_i)) \quad \text{and} \quad C^{-1} \leq d_H(\Omega_i, D(\partial \phi_i)) \leq C.
\]

For every $\gamma \in C_0$ there is $\eta_i \in \pi_1(M) < \text{Isom}(\mathbb{H}^3)$ such that, after passing to a subsequence, $\eta_i(D(\partial \phi_i))$ converges on compact sets to $D(\gamma)$ and
\[
\lim_{i \to \infty} \int_{\eta_i(\Omega_i) \cap B_R(0)} |f_i \circ \eta_i^{-1}|dA_{g_0} = 0 \quad \text{for all } R > 0.
\]

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