Singular solutions for vibration control problems

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Abstract. Optimal control problem for the system of partial differential equations of hyperbolic type is considered. By using the Fourier method this problem is reduced to the optimal control problem for the corresponding Fourier coefficients. For some special initial data we prove the existence of optimal solutions with a countable number of switchings on a finite time interval and optimal spiral-like solutions which attain the origin in a finite time making a countable number of rotations. The problem of controlling the vibrations of the Timoshenko beam is considered as an example of the optimal control problem for linear system of PDE.

1. Optimal control problem for linear system of PDE

We consider a control system
\begin{equation}
\frac{\partial^2}{\partial t^2} p(x,t) + A_x p(x,t) = g(x,t), \quad x \in G, \quad t \in \mathbb{R}_+
\end{equation}
subject to initial
\begin{equation}
p(x,0) = v_0(x), \quad p_t(x,0) = v_1(x)
\end{equation}
and boundary conditions
\begin{equation}
B_x p|_{\partial G} = 0
\end{equation}
Here, $p(x,t)$ and $g(x,t)$ are $m$-dimensional vector-valued functions defined on $G \times [0, \infty)$, $G$ is a compact subset of $\mathbb{R}^n$ with smooth boundary $\partial G$, $A_x$ is a linear differential operator of order 2 in $G$, $B_x$ is a linear differential operator of order 1 in a neighborhood of $\partial G$. Suppose that $p(x,t) \in L_2(G \times [0, \infty), \mathbb{R}^m)$, $g$ is a measurable function in $t$, $g(\cdot, t) \in L_2(G, \mathbb{R}^m)$, $p(\cdot, t), \frac{\partial^2}{\partial t^2} p(\cdot, t), v_0(x), v_1(x) \in H^2(G, \mathbb{R}^m)$. The external force $g(x,t)$ is considered as a control function. We assume that $g(\cdot, t)$ is bounded in the following sense
\begin{equation}
\|g(\cdot, t)\|_{L_2(G, \mathbb{R}^m)}^2 \leq 1
\end{equation}
We study the following control problem: to find the admissible control such that the corresponding solution of (1)-(3) minimize the functional
\begin{equation}
\int_0^\infty \|p(\cdot, t)\|_{L_2(G, \mathbb{R}^m)}^2 \, dt \rightarrow \inf
\end{equation}
Assume that $A_x$ with domain $D_A = \{ y \in H^2(G, \mathbb{R}^m) : B_x y = 0 \}$ is a self-adjoint operator, has the discrete spectrum $\{ \lambda_j \}_{j=1}^\infty$ and the orthonormal complete system of eigenfunctions $\{ h_j (x) \}_{j=1}^\infty$ that are smooth in $G$ and satisfying (3). We seek a solution of (1)-(5) in the form

$$p(x,t) = \sum_{j=1}^\infty q_j(t)h_j(x)$$

where $q_j(t) = (p, h_j)_{L^2(G, \mathbb{R}^m)}$, $j = 1, 2, \ldots$, are Fourier coefficients. Expand the functions $g, v_0, v_1$ in the basis $\{ h_j (x) \}_{j=1}^\infty$

$$g(t,x) = \sum_{j=1}^\infty u_j(t)h_j(x), \quad v_0(x) = \sum_{j=1}^\infty s_jh_j(x), \quad v_1(x) = \sum_{j=1}^\infty r_jh_j(x)$$

Substituting (6) and (7) into (1)-(5) we get an optimal control problem for the Fourier coefficients in the space $l^2$:

$$\int_0^\infty \sum_{j=1}^\infty q_j^2(t)dt \to \min$$

$$(9) \quad \dot{q}_j(t) + \lambda_j q_j(t) = u_j(t), \quad \sum_{j=1}^\infty u_j^2(t) \leq 1$$

$$(10) \quad q_j(0) = s_j, \quad \dot{q}_j(0) = r_j, \quad j = 1, 2, \ldots$$

Problem (1)-(5) with the external force in the form $g(x,t) = u(t)f(x)$ was considered in [1]. In this case the corresponding control problem for the Fourier coefficients has the same form but (9) is replaced by

$$(11) \quad \dot{q}_j(t) + \lambda_j q_j(t) = C_j u(t), \quad -1 \leq u(t) \leq 1$$

where $C_j \in \mathbb{R}$. In [1] for the problem (8), (10), (11) it was constructed the optimal synthesis containing singular extremals and extremals with accumulation of switchings.

In the present paper for some initial conditions (2) we prove that the behavior exhibited by the solutions of (8)-(10) is similar to that of the problem with scalar control.

Note that the system (1)-(3) governs vibrations of beams, strings and other mechanical models. At the end of this paper we will consider, as an example, the problem of controlling the vibrations of the Timoshenko beam.

2. Problem with $k$-dimensional control

Let $h_1(x), h_2(x), \ldots, h_k(x)$ be the orthonormal eigenfunctions of the operator $A_x$, and $\lambda_1, \lambda_2, \ldots, \lambda_k$ be the corresponding eigenvalues. Consider the $k$-dimensional subspace $\mathcal{X}$ in $L^2(G, \mathbb{R}^m)$ spanned on $h_1, h_2, \ldots, h_k$. Assume $v_0(x), v_1(x) \in \mathcal{X}$. Then one can prove that the subspace $\mathcal{X}$ is an integral manifold of the flow of optimal trajectories of the problem (1)-(5).

It means that for any $t > 0$ the corresponding optimal solution $p(x,t)$ and the optimal control $g(x,t)$ can be written in the form

$$p(x,t) = \sum_{j=1}^k q_j(t)h_j(x), \quad g(x,t) = \sum_{j=1}^k u_j(t)h_j(x)$$

where $q_j(t)$ and $u_j(t)$ satisfy (9)-(10). Thus we can reduce (8)-(10) to the following problem with $k$-dimensional control:

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\[ \int_0^\infty \|q(t)\|^2 \, dt \to \min \]  
(12)

\[ \dot{q} = z, \quad \dot{z} = \Lambda q + u, \quad \|u(t)\| \leq 1 \]  
(13)

\[ q(0) = s, \quad z(0) = r. \]  
(14)

Here \( q, z, u \in \mathbb{R}^k \), \|\cdot\| means the Euclidean norm of \( \mathbb{R}^k \), \( \Lambda = \text{diag}\{-\lambda_1, -\lambda_2, \ldots, -\lambda_k\} \).

To construct optimal solutions for the problem (12)-(14) we use the Pontryagin Maximum Principle. Define the Hamiltonian

\[ H(q, z, \phi, \psi) = -\frac{1}{2} \langle q, q \rangle + \langle z, \phi \rangle + \langle \Lambda q, \psi \rangle + \langle u, \psi \rangle \]
where \( \phi, \psi \) are adjoint variables. We have the system of equations of the Pontryagin maximum principle

\[ \dot{q} = z, \quad \dot{z} = \Lambda q + \dot{u}, \quad \dot{\phi} = \frac{1}{2} q - \Lambda \psi, \quad \dot{\psi} = -\phi. \]  
(15)

The optimal control \( \hat{u}(t) \) satisfies the maximum condition:

\[ H(q(t), z(t), \phi(t), \psi(t), \hat{u}(t)) = -\frac{1}{2} \langle q, q \rangle + \langle z, \phi \rangle + \langle \Lambda q, \psi \rangle + \max_{\|u(t)\| \leq 1} \langle u, \psi \rangle \]

The maximum condition gives: if \( \psi \neq 0 \), then \( \hat{u}(t) = \psi/\|\psi\| \); if \( \psi = 0 \), then any control \( \|u(t)\| \leq 1 \) meets the maximum condition.

A solution \( \xi(t) = (u(t), z(t), q(t), \phi(t), \psi(t)) \) of the system (15) is called a singular one on the interval \((t_1, t_2)\) if \( \psi(t) = 0 \) for all \( t \in (t_1, t_2) \). It was proved [2] that in the problem (12)-(14) the origin \((z, q, \phi, \psi) = 0\) with \( \hat{u}(t) = 0 \) is the only singular solution. For sufficiently small initial data the optimal solutions attain the origin in a finite time \( T \) and the corresponding optimal control function \( \hat{u}(t) \) does not have a limit as \( t \to T \). In the case \( k = 1 \) it was shown that the behaviour of the optimal solutions of (12)-(14) in the vicinity of the origin is similar to the optimal synthesis in the two-dimensional Fuller problem with a scalar control. Namely, the optimal control \( u(t) \) has an infinite number of switchings in a finite time interval (chattering mode). In this paper we study the case \( k = 2 \).

3. Model problem

Consider the following optimal control problem with two-dimensional control which is called a model problem.

\[ \int_0^\infty \|q(t)\|^2 \, dt \to \inf \]  

\[ \dot{q} = u, \quad \|u(t)\| \leq 1, \quad q(0) = s^0, \quad \dot{q}(0) = r^0, \quad q, u \in \mathbb{R}^2 \]

Note that the only difference between the model problem and (12)–(14) is the summand \( \Lambda q \) in (13).

The Hamiltonian system (15) for the model problem has a group of symmetries \( G = O(2) \times \mathbb{R}_+ \) that acts on \( \xi(t) = (u(t), z(t), q(t), \phi(t), \psi(t)) \) as follows.

Let \( \zeta \in O(2) \) then \( \zeta \circ \xi \overset{\text{def}}{=} (\zeta u, \zeta z, \zeta q, \zeta \phi, \zeta \psi) \). Let \( \lambda \in \mathbb{R}_+ \) then \( g_\lambda(\xi(t)) \overset{\text{def}}{=} (u(t/\lambda), \lambda z(t/\lambda), \lambda^2 q(t/\lambda), \lambda^3 \phi(t/\lambda), \lambda^4 \psi(t/\lambda)) \). The action of the group \( G \) respects the set of solutions of the model problem. This allows to find some explicit solutions in the model problem.
Proposition 1 ([3]). There exists a one-parameter family of optimal solutions of the model problem that contains logarithmic spirals

\[ q^*(t) = A_1 t^2 e^{i \alpha \ln|T-t|}, \quad z^*(t) = A_2 t e^{i \alpha \ln|T-t|}, \quad u^*(t) = -e^{i \alpha \ln|T-t|}, \]

(16)

\[ \alpha = \pm \sqrt{5}, \quad A_1 = \frac{1}{126} \cdot (4 + i\alpha)(3 + i\alpha), \quad A_2 = -A_1(2 + i\alpha) \]

and all its possible rotations and reflections. They approach the origin in a finite time \( T \) with countable number of rotations.

Unlike trajectories of a focus for linear differential equations, these optimal logarithmic spirals approach the origin in a finite time, while making a countable number of rotations.

4. Blowing up the singularity

Unfortunately, the problem (12)-(14) does not have a symmetry group. However, using the blowing up the singularity at the origin we prove that in the neighborhood of the origin the behavior of optimal solutions for (12)-(14) is determined by optimal solutions of the model problem. Consider the blowing up [4] the singularity at the origin by the map \( B : (q, z, \phi, \psi) \mapsto (\mu, \tilde{q}, \tilde{z}, \tilde{\phi}, \tilde{\psi}) \):

\[ \tilde{z} = \frac{z}{\mu}, \quad \tilde{q} = \frac{q}{\mu^2}, \quad \tilde{\phi} = \frac{\phi}{\mu^3}, \quad \tilde{\psi} = \frac{\psi}{\mu^4}, \quad \mu = \left( |z|^{24} + |q|^{12} + |\phi|^8 + |\psi|^6 \right)^{1/24} \]

where \( \mu \in \mathbb{R}_+ \), and \( (\tilde{q}, \tilde{z}, \tilde{\phi}, \tilde{\psi}) \in \mathbb{R}^8 \) lies on the manifold \( C_0 = \{ |z|^{24} + |q|^{12} + |\phi|^8 + |\psi|^6 = 1 \} \).

Compare the blowup for the model problem with the blowup for system (15).

\[
\begin{align*}
\text{Problem (12)-(14)} & \quad \quad & \text{Model problem} \\
\mu' &= \mu \mathcal{M}(\tilde{q}, \tilde{z}, \tilde{\phi}, \tilde{\psi}) & \mu' &= \mu \mathcal{M}_0(\tilde{q}, \tilde{z}, \tilde{\phi}, \tilde{\psi}) \\
\psi' &= -\tilde{\phi} - 4\tilde{\psi} \mathcal{M} & \psi' &= -\tilde{\phi} - 4\tilde{\psi} \mathcal{M}_0 \\
\phi' &= \tilde{q} - 3\tilde{\phi} \mathcal{M} & \phi' &= \tilde{q} - 3\tilde{\phi} \mathcal{M}_0 \\
\tilde{q}' &= \tilde{z} - 2\tilde{q} \mathcal{M} & \tilde{q}' &= \tilde{z} - 2\tilde{q} \mathcal{M}_0 \\
\tilde{z}' &= \Lambda \mu^2 \tilde{q} + u - \tilde{z} \mathcal{M} & \tilde{z}' &= u - \tilde{z} \mathcal{M}_0 \\
u &= \tilde{\psi}/|\tilde{\psi}| & u &= \tilde{\psi}/|\tilde{\psi}| \\
\mathcal{M} &= \frac{1}{24} \left( 24 |\tilde{z}|^{22} \langle \tilde{z}, u \rangle + 12 |\tilde{q}|^{10} \langle \tilde{q}, \tilde{z} \rangle + 8 |\tilde{\phi}|^6 \langle \phi, \tilde{q} \rangle - 6 |\tilde{\psi}|^4 \langle \psi, \phi \rangle + 24 \mu^2 |\tilde{z}|^{22} \langle \tilde{z}, \tilde{q} \rangle \right) & \mathcal{M}_0 &= \frac{1}{24} \left( 24 |\tilde{z}|^{22} \langle \tilde{z}, u \rangle + 12 |\tilde{q}|^{10} \langle \tilde{q}, \tilde{z} \rangle + 8 |\tilde{\phi}|^6 \langle \phi, \tilde{q} \rangle - 6 |\tilde{\psi}|^4 \langle \psi, \phi \rangle \right)
\end{align*}
\]

Note that if we put \( \mu = 0 \) into the system (17) then we get the system (18).

We get that the blowup (18) for the model problem coincides with the blowup (17) for the
system (17) as $\mu = 0$. One can prove that there is the contraction in the $\mu$-direction and the expansion in the other directions (see figure 1).

Summarizing the above, we have that the spirals (16) represent two circles in the quotient space $(\tilde{q}, \tilde{z}, \tilde{\phi}, \tilde{\psi})$ depending on the direction of rotation. Both of the circles are hyperbolic periodic trajectories and their stable manifolds are woven in logarithmic spirals. Moreover these circles exist in the Hamiltonian system (15) and the eigenvalues for the model problem circles and for the general Hamiltonian system circles coincide. So we can reconstruct the stable manifold in the Hamiltonian system (15) and it is also woven of the analogues of the logarithmic spirals. Thus the following proposition is proved.

**Proposition 2.** There exists the optimal spiral-like solution of (12)–(14) which attains the singular point in finite time making a countable number of rotations.

5. Timoshenko beam

Consider a linear model of a rotating Timoshenko beam [5–7]

$$w_{tt}(x,t) - \frac{1}{\gamma} w_{xx}(x,t) + \frac{1}{\gamma} \xi_x(x,t) = g_1(x,t)$$

$$\zeta_{tt}(x,t) - \zeta_{xx}(x,t) + \frac{1}{\gamma} \zeta(x,t) - \frac{1}{\gamma} w_x(x,t) = g_2(x,t)$$

Here, the $X$-axis coincides with the beam when it’s at rest, $w(x,t)$ is the displacement of the center line of the beam in the direction that is perpendicular to the $X$-axis and $\zeta$ is the rotation angle of the cross-section area at the location $x \in [0,1]$ and time $t$. The controls are introduced as forcing terms $g_1(x,t)$ and $g_2(x,t)$ in the right sides of both equations. Assume that $\int_0^1 (g_1^2(x,t) + g_2^2(x,t)) \, dx \leq 1$ for all $t \geq 0$. We impose the following boundary conditions:

$$w(0,t) = \zeta(0,t) = 0, \quad w_x(1,t) - \zeta(1,t) = 0, \quad \zeta_x(1,t) = 0$$

These boundary conditions mean that the beam is clamped at the left end and free at the right end. The initial state of the beam is

$$w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x), \quad \zeta(x,0) = \zeta_0(x), \quad \zeta_t(x,0) = \zeta_1(x)$$

We study the minimization problem of the deviation of the beam from the equilibrium state in the sense of the functional

$$\int_0^\infty \int_0^1 (w^2(x,t) + \zeta^2(x,t)) \, dx \, dt$$

The control problem for the Timoshenko beam refers to the class of problems (1)-(5). Indeed, define a linear operator $A_x : D \to H$ by

$$A_x \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\gamma} \eta''_1 + \frac{1}{\gamma} \eta'_2 \\ -\frac{1}{\gamma} \eta'_1 - \eta''_2 + \frac{1}{\gamma} \eta_2 \end{pmatrix}$$

where $H = L^2([0,1], \mathbb{C}^2)$ and

$$D = \left\{ \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \in H \mid \eta_1(0) = \eta_2(0) = 0, \eta'_1(1) - \eta'_2(1) = 0, \eta''_2(1) = 0 \right\}$$

It was proved [5,6] that $A_x$ is self-adjoint in $H$ and positive. Hence all the above results on the singularities of optimal solutions are applied to the problem of controlling the vibration of the Timoshenko beam.
6. Conclusion
We have considered a control system of partial differential equations that governs vibrations in many mechanical systems. Vibration control problems are widely used in many areas of engineering (robots with flexible links, spacecrafts with flexible antennas, etc). We have studied the problem of minimizing the deviation from the rest position. This problem was reduced to the optimal control problem for the corresponding Fourier coefficients. For some initial data we have proved the existence of optimal solutions with a countable number of switchings on a finite time interval and optimal spiral-like solutions which attain the origin in a finite time making a countable number rotations. We have shown that optimal solutions contained the singular arc and the chattering mode. Thus, optimal controls have a complicated structure and are difficult to implement in control devices. We believe that knowledge of the behavior of exact optimal solutions can be useful in numerical modeling of mechanical processes.

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