Cover pebbling number of Comb, Friendship and Helm graphs

R. Prabha¹, B. Sandhiya²*
¹,² Department of Mathematics, Ethiraj College for Women, Chennai – 600008, India.
*sandhiyabasskar14@gmail.com

Abstract. For a connected graph $G$ with a distribution of $p$ pebbles on it, a pebbling move involves the removal of two pebbles from a vertex and an addition of one pebble to an adjacent vertex. The cover pebbling number $\gamma(G)$ is the minimum number of pebbles required to place one pebble on each vertex of the graph $G$ after a sequence of pebbling moves, regardless of the initial configuration of the pebbles. It is known that the problem of computing the cover pebbling number is NP-complete [7]. In this paper we compute the cover pebbling number of Comb, Friendship and Helm graphs.

1. Introduction

Graph pebbling is one of the topics of current interest in graph theory and it has seen a greater progress in recent days. The concept of graph pebbling was initially proposed by Lagarias and Saks and was introduced into literature by Chung [3]. The Cop and the robber’s problem, the problem of transporting consumable resources like petrol, transmitting information in ad hoc network are some of the real time applications of the concept of graph pebbling.

Consider a simple, connected and undirected graph $G$ with a distribution of $p$ pebbles on its vertices. The objective is to be able to place at least one pebble at the target vertex after a sequence of pebbling moves. The minimum number of pebbles that are required to reach any target vertex irrespective of the initial configuration is known as the pebbling number $\pi(G)$. The pebbling game on directed graphs which consists of replacing a pebble on one node by new pebbles on the adjacent nodes, along the directed edges seems to be solved completely by Eriksson [4]. There are various categories of pebbling numbers like Optimal pebbling number [2], Cover pebbling number, generalized pebbling number [1], etc. and in this paper, we consider only the cover pebbling number of a graph.

Let us consider a scenario where information must be communicated simultaneously to various nodes of a network or if army soldiers need to be deployed at the same time. The cover pebbling concept serves as a solution to this type of situation. Given any configuration of $p$ pebbles on a graph $G$, the cover pebbling number $\gamma(G)$ is the least number of pebbles that are required such that after a sequence of pebbling moves every vertex of $G$ has a pebble on it. We say a vertex is covered if it has a pebble on it after a sequence of pebbling moves, else it is said to be uncovered. If all the vertices are covered in a graph, then the graph is said to be solvable.

In the following section we explain the terminologies and the notations used in this paper and in the sections 3, 4 and 5, we study the cover pebbling problem of Comb, Friendship and Helm graphs and compute their exact cover pebbling number.
2. Preliminaries

All graphs considered in this article are simple, connected and undirected. We refer to Bondy and Murty [5] for basic definitions. Let \( V(G) \) and \( E(G) \) denote the vertex set and edge set of a given connected graph \( G \) respectively. The distance \( d(u, v) \) is the length of the shortest path between the vertices \( u \) and \( v \) of the graph \( G \). We find the cover pebbling number via a key vertex of the graph since it is much simpler and convenient [8]. The distance \( d(v) \) of a vertex \( v \) is the sum of its distance from each vertex of \( V(G) \), that is

\[
d(v) = \sum_{u \in V(G)} d(u, v)
\]

for all \( u \in V(G), u \neq v \). A vertex \( v \in V(G) \) is said to be a key vertex if \( d(v) \) is maximum [8]. For a given graph \( G \) with \( n \) vertices, let \( P(v_i) \) denote the number of pebbles on the vertex \( v_i, i = 1, 2 \ldots n \). We say that a vertex \( v \) is a

- D – vertex if \( P(v_i) = 0 \)
- N – vertex if \( P(v_i) = 1 \) or 2
- S – vertex if \( P(v_i) > 2 \)

where D, N and S stands for Demand, Neutral and Supply respectively.

Consider \( w \) to be a weight function that maps an integer \( w(v) > 0 \) to every vertex \( v \) of \( G \), the weighted cover pebbling number is the minimum number of pebbles that ensures whatever may be the initial configuration, there is a sequence of pebbling moves after which \( w(v) \) pebbles will be placed on all the vertices simultaneously. The cover pebbling number is a special case of weighted cover pebbling number \( \gamma_w(G) \) where \( w(v) \) takes the value one.

We say an initial distribution is simple if all the pebbles are placed on one single vertex of the given graph. While computing the cover pebbling number of a graph, we initially consider only simple distribution of pebbles because of the following cover pebbling theorem:

**Theorem 1**[6]: Let \( w \) be a positive goal distribution. To determine the \( w \)-cover pebbling number of a (directed or undirected) connected graph, it is sufficient to consider simple initial distributions. In fact, for any initial distribution that admits no cover pebbling, all pebbles may be concentrated to one of the fat nodes with cover pebbling still not possible.

The cover pebbling numbers of certain fundamental graphs computed in [8] are given below:

- For a path \( P_n, \gamma(P_n) = 2^n - 1, n \geq 1. \)
- For a complete graph \( K_n, \gamma(K_n) = 2n - 1, n \geq 1. \)
- For a wheel graph \( W_n, \gamma(W_n) = 4n - 5, n \geq 3. \)
- For a fan graph \( F_n, \gamma(F_n) = 4n - 3, n \geq 3. \)

3. Cover pebbling number of Comb graphs

3.1. Definition

Consider a Path \( P_n \) on \( n \) vertices. The comb graph is obtained by attaching a pendant edge to every vertex of \( P_n \). It has \( 2n \) vertices and \( 2n - 1 \) edges and is denoted as \( CG_n \).

![Figure 1: Comb graph CG_4.](image-url)
Theorem 1: If \( CG_n \) denotes a comb graph on \( 2n \) vertices then, \( \gamma(CG_n) = 7 + 2\left(\sum_{d=3}^{n} 2^d\right) + 2^{n+1}, n \geq 3 \).

Proof: Let \( CG_n \) be a comb graph. We denote the vertices on the stem of the comb as \( u_1, u_2, u_3, \ldots, u_n \) and the pendant vertices as \( v_1, v_2, v_3, \ldots, v_n \). By the cover pebbling theorem, it is enough to consider a simple initial distribution. Without loss of generality we place the \( 7 + 2\left(\sum_{d=3}^{n} 2^d\right) + 2^{n+1} - 1 \) pebbles at the key vertex \( v_1 \). Then \( 2^d \) pebbles will be used to cover each of the vertices of \( CG_n \) which is at a distance \( d \) from \( v_1 \). Thus, no pebble will remain to cover \( v_i \). Hence, \( \gamma(CG_n) \geq 7 + 2\left(\sum_{d=3}^{n} 2^d\right) + 2^{n+1} \).

Consider an initial distribution of \( 7 + 2\left(\sum_{d=3}^{n} 2^d\right) + 2^{n+1} \) pebbles that admits no cover pebbling. We leave the pebbles of the \( N \)-vertex untouched as it forms the covering. Choose a vertex say \( v_l \) with no pebbles on it and an immediate nearby \( S \)-vertex say \( S_1 \). Start the pebbling move from \( S_1 \) to \( v_l \). It consumes \( 2^d \) pebbles to cover \( v_l \) where \( d = d(v_l, S_1) \). We now have two cases:

Case (i): \( P(S_1) > 2^d \)

We take \( 2^d \) pebbles and cover \( v_l \) such that the remaining pebbles still cover \( S_1 \).

Case (ii): \( P(S_1) \leq 2^d \)

In this case we make pebbling move till \( S_1 \) becomes a \( N \)-vertex and stop. Then look for the next immediate nearby \( S \)-vertex say \( S_2 \) and make pebbling moves. We repeat the process until the vertex \( v_l \) is covered. Definitely, the number of pebbles used to cover \( v_l \) from the \( S \)-vertices will be less than the number of pebbles used to cover from \( v_l \). Proceeding in this manner, we make the graph coverable. Thus, \( \gamma(CG_n) \leq 7 + 2\left(\sum_{d=3}^{n} 2^d\right) + 2^{n+1} \). Hence the proof.

4. Cover pebbling number of Friendship graphs

4.1. Definition

Friendship graph \( F_n \) is a planar undirected graph with \( 2n + 1 \) vertices and \( 3n \) edges. It is constructed by joining \( n \) copies of cycle graph \( C_3 \) with a common vertex.

![Figure 2: Friendship graph \( F_4 \).](image)

Theorem 2: If \( F_n \) denotes a friendship graph, then \( \gamma(F_n) = 5 + 8(n - 1), n \geq 1 \).

Proof: Let \( F_n \) be a friendship graph with a common vertex \( v_0 \) and the vertices of degree 2 be \( v_1, v_2, v_3, \ldots, v_n \). By the cover pebbling theorem, it is enough to consider a simple initial distribution. Without loss of generality we place the \( \left[ 5 + 8(n - 1) \right] - 1 \) pebbles at any one of the vertices of degree 2 say \( v_1 \) which seems to be the key vertex. Then, 2 pebbles will be used to cover \( v_0 \) and \( 2^d \) pebbles will be used to cover each of \( v_i, i = 2, 3, \ldots, n \) vertices which are at a distance \( d \) from \( v_i \). Thus, no pebbles will remain to cover \( v_i \). Hence, \( \gamma(F_n) \geq 5 + 8(n - 1) \).
Consider a non-coverable initial configuration of \(5 + 8(n - 1)\) pebbles. Then there exists at least one vertex with no pebble on it say \(x\).

**Case (i):** \(x = v_0\)

We choose a vertex from \(v_i, i = 1, 2, \ldots n\) that has more than 2 pebbles and make pebbling move to cover \(v_0\).

**Case (ii):** \(x = v_i, i = 1, 2, \ldots 2n\)

For every \(x = v_i, i = 1, 2, \ldots 2n\), we have two adjacent vertices namely, \(v_0\) and \(v_j\). If any one of \(v_0\) and \(v_j\) is an \(S\)-vertex then 2 pebbles are required to cover \(x\) else by pigeon-hole principle, there exists at least one vertex say \(v_k\) with more than \(2^2\) pebbles. Then we cover \(x\) by a pebbling move from \(v_k\).

Continuing in this manner we can cover all the uncovered vertices with pebbles less than \(5 + 8(n - 1)\). Thus, \(\gamma(F_n) \leq 5 + 8(n - 1)\). Hence the proof.

5. **Cover pebbling number of Helm graphs**

5.1. **Definition**

A helm graph \(H_n, n \geq 3\) is the graph obtained from the wheel graph by adding a pendant edge at each vertex on the rim of the wheel \(W_n\).

![Figure 3: Helm graph obtained from \(W_4\).](image)

**Theorem 3:** If \(H_n\) denotes a helm graph then \(\gamma(H_n) = 15 + 2^3(n - 1) + 2^4(n - 3)\) for \(n \geq 3\).

Proof: Let \(H_n\) be a helm graph with a common vertex \(v_0\), vertices on the rim be denoted as \(v_1, v_2, v_3, \ldots v_n\) and the pendant vertices as \(u_1, u_2, u_3, \ldots u_n\). By the cover pebbling theorem, it is enough to consider a simple initial distribution. Hence, we take any one of the pendant vertices as the key vertex say \(u_1\). Now let us place \(15 + 2^3(n - 1) + 2^4(n - 3) - 1\) at \(u_1\). Then \(2^d\) pebbles will be used to cover each of the \(n - 1\) other vertices where \(d\) is the distance of the vertex from \(u_1\). Thus, no pebbles will remain to cover \(u_1\). Hence, \(\gamma(H_n) \geq 15 + 2^3(n - 1) + 2^4(n - 3)\).

Consider a non-coverable initial configuration of \(15 + 2^3(n - 1) + 2^4(n - 3)\), which is the number of pebbles required to cover each of the \((n - 1)\) vertices from \(u_1\). The number of pebbles used to cover \(v_0\) from \(u_1\) is \(2^2\). Since the configuration is non-coverable, there will be at least one vertex with no pebbles on it. Choose one such vertex, say \(x\).

**Case (i):** \(x = v_0\)

We choose a \(S\)-vertex from \(V(H_n) - v_0\) and make pebbling move to cover \(v_0\). If the chosen vertex is one among \(u_1, u_2, u_3, \ldots u_n\) it takes \(2^2\) pebbles as in the case of \(u_1\). If the chosen vertex is one among \(v_1, v_2, v_3, \ldots v_n\) then it takes only two pebbles to cover \(v_0\).

**Case (ii):** \(x = v_i, i = 1, 2, \ldots n\)
The number of pebbles used to cover $x$ from $u_1$ will be $2^d + 2$ where $d$ is the shortest distance of $x$ from $v_1$.

Now let us choose an immediate nearby $S$-vertex say $y$.

- If $y = v_0$, then it takes two pebbles to cover $x$.
- If $y = v_j$, $l \neq j$, then it takes $2^d$ pebbles to cover $x$.
- If $y = u_i$, it takes 2 pebbles and if $y = u_{i+1}$ or $u_{i-1}$ it takes $2^2$ pebbles to cover $x$. For covering $x$ from any other $u_i$ $2^d + 2$ pebbles are required.

Case (iii): $x = u_i, i = 1, 2, ... n$

The number of pebbles used to cover $x$ from $u_1$ will be $2^4$ for $u_3, ... u_{n-1}$ and $2^3$ for $u_2$ and $u_n$. Now let us choose a $S$-vertex, say $w$.

- If $w = v_0$, then it takes $2^2$ pebbles since all the $u_i$'s are at a distance two from $v_0$.
- If $w = v_i$, it takes 2 pebbles and if $w = v_{i+1}$ or $v_{i-1}$ it takes $2^2$ pebbles to cover $x$. Any other $v_i$, requires $2^3$ pebbles to cover $x$.
- If $w = u_j, j \neq i$ we can cover $x$ with $2^4$ pebbles and if $w = u_{j+1}$ or $u_{j-1}$, $2^3$ pebbles are required.

Proceeding this way, we cover all the uncovered vertices. Thus, $\gamma(H_n) \leq 15 + 2^2(n - 1) + 2^4(n - 3)$. Hence the proof.

6. Conclusion
The Cover Pebbling problem is an interesting NP-complete problem in Graph theory. In this paper, we have computed the cover pebbling number of Comb, Friendship and Helm graphs.

References
[1] Lourdusamy A, Generalized Pebbling Number, International Mathematical Forum, 5, 2010, no. 27, 1331-1337.
[2] Friedman T and Wyels, Cindy. (2005). Optimal pebbling of paths and cycles
[3] Chung F R K, Pebbling in Hyper cubes, SIAM J. Disc. Math 2 (1989).
[4] Eriksson H, Pebblings, Electron J Combin. 2, R7, 1995.
[5] Bondy J A Graph Theory with Applications, Elsevier, 1976.
[6] Sjöstrand J, The Cover Pebbling Theorem, Electronic Journal of Combinatorics, 1(2005).
[7] Nathaniel G Watson, The Complexity of Pebbling and Cover Pebbling, 2005.
[8] Subido, Michael E, (2014). The cover pebbling number of the join of some graphs. Applied Mathematical Sciences. 4275-4283