Bell inequality tests using asymmetric entangled coherent states in asymmetric lossy environments

Chae-Yeun Park and Hyunseok Jeong
Center for Macroscopic Quantum Control, Department of Physics and Astronomy, Seoul National University, Seoul, 151-742, Korea
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We study an asymmetric form of two-mode entangled coherent state (ECS), where the two local amplitudes have different values, for testing the Bell-Clauser-Horne-Shimony-Holt (Bell-CHSH) inequality. We find that the asymmetric ECSs have obvious advantages over the symmetric form of ECSs in testing the Bell-CHSH inequality. We further study an asymmetric strategy in distributing an ECS over a lossy environment and find that such a scheme can significantly increase violation of the inequality.

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I. INTRODUCTION

Entangled coherent states (ECSs) in free-traveling fields [1–3] have been found to be useful for various applications such as Bell inequality tests [4–10], tests for non-local realism [11–13], quantum teleportation [14–23], quantum computation [24–30], precision measurements [31–38], quantum repeater [39] and quantum key distribution [40]. The ECSs can be realized in various systems that can be described as harmonic oscillators and numerous schemes for their implementing have been suggested [1–3, 41–45]. An ECS in a free-traveling field was experimentally generated using the photon subtraction technique on two approximate superpositions of coherent states (SCSs) [46]. A proof-of-principle demonstration of quantum teleportation using an ECS as a quantum channel was performed [47].

So far, most of the studies on ECSs have considered symmetric types of two-mode states where the local amplitudes have the same value. Since ECSs are generally sensitive to decoherence due to photon losses in testing Bell-type inequalities [4, 5], it would be worth investigating the possibility of using an asymmetric type of ECS to reduce decoherence effects. In fact, asymmetric ECSs can be used to efficiently teleport an SCS [48, 49] and to remotely generate symmetric ECSs [50] in a lossy environment. In addition to the asymmetry of the ECSs, as a closely related issue, it would be beneficial to study strategies for distributing the ECSs over asymmetrically lossy environments. In this paper, we investigate asymmetric ECSs as well as asymmetric entanglement distribution strategies, and find their evident advantages over symmetric ones for testing the Bell-Clauser-Horne-Shimony-Holt (Bell-CHSH) inequality under various conditions.

The remainder of the paper is organized as follows. In Sec. II we discuss the Bell-CHSH inequality tests using asymmetric ECSs with ideal detectors. We consider photon on-off detection and photon number parity detection, respectively, for the Bell-CHSH inequality tests. Section III is devoted to the study of decoherence effects on Bell inequality violations with an asymmetric strategy to share ECSs. We then investigate effects of inefficient detectors in Sec. IV and conclude with final remarks in Sec. V.

II. BELL INEQUALITY TESTS WITH ASYMMETRIC ENTANGLED COHERENT STATES

In this paper, we are interested in a particular form of two-mode ECSs

\[ |\text{ECS}^{\pm}\rangle = \mathcal{N}_\pm (|\alpha_1\rangle \otimes |\alpha_2\rangle \pm |\alpha_2\rangle \otimes |\alpha_1\rangle) \]  

(1)

where \(|\alpha_i\rangle\) are coherent states of amplitudes \(\pm \alpha_i\) for field mode \(i\), and \(\mathcal{N}_\pm = [2 \pm 2 \exp(-2\alpha_1^2 - 2\alpha_2^2)]^{-1/2}\) are the normalization factors. We note that amplitudes \(\alpha_1\) and \(\alpha_2\) are assumed to be real without loss of generality throughout the paper. The ECSs show noticeable properties as macroscopic entanglement when the amplitudes are sufficiently large and these properties have been extensively explored [14, 50, 51]. We call \(|\text{ECS}^+\rangle\) (|\text{ECS}^-\rangle\) the even (odd) ECS because it contains only even (odd) numbers of photons. When \(\alpha_1 = \alpha_2\), we call the states in Eq. (1) symmetric ECSs, and otherwise they shall be called asymmetric ECSs. It is straightforward to show that an ECS can be generated by passing an SCS in the form of \(|\alpha\rangle \pm |\alpha\rangle\) [52], where \(\alpha^2 = \alpha_1^2 + \alpha_2^2\), through a beam splitter. A beam splitter with an appropriate ratio should be used to generate an asymmetric ECS with desired values of \(\alpha_1\) and \(\alpha_2\). In this paper, asymmetric ECSs with various values of \(\alpha_1\) and \(\alpha_2\) are compared for given values of \(\alpha\). The average photon numbers of the ECSs are solely dependent on the values of \(\alpha\) and are simply obtained as

\[ \bar{n}_\pm = \langle \text{ECS}^{\pm} | \hat{n} | \text{ECS}^{\pm} \rangle = \frac{\alpha^2 (1 \mp e^{-2\alpha^2})}{1 \pm e^{-2\alpha^2}} \]  

(2)

where \(\hat{n} = \hat{n}_1 + \hat{n}_2\) and \(\hat{n}_i\) is the number operator for field mode \(i\).
A. Bell-CHSH tests with photon on-off measurements

We first investigate a Bell-CHSH inequality test using photon on-off measurements with the displacement operations. The measurement operator for mode $i$ is defined as

$$\hat{O}_i(\xi) = \hat{D}_i(\xi) \left( \sum_{n=1}^{\infty} |n\rangle \langle n| - |0\rangle \langle 0| \right) \hat{D}^\dagger_i(\xi)$$

(3)

where $\hat{D}_i(\xi) = \exp \left[ \xi a_i - \xi^* a_i^\dagger \right]$ is the displacement operator and $|n\rangle$ denotes the Fock state. The correlation function is defined as the expectation value of the joint measurement

$$E_O(\xi_1, \xi_2) = \langle \hat{O}_1(\xi_1) \otimes \hat{O}_2(\xi_2) \rangle$$

(4)

and the Bell-CHSH function is

$$B_O = E_O(\xi_1, \xi_2) + E_O(\xi_1, \xi_2^\prime) + E_O(\xi_1^\prime, \xi_2) - E_O(\xi_1^\prime, \xi_2^\prime).$$

(5)

In any local realistic theory, the absolute value of the Bell function is bounded by $2$ [53].

We calculate an explicit form of the correlation function $E_O(\xi_1, \xi_2)$ using Eqs. (3) and (4), of which the details are presented in Appendix A. We then find the Bell function $B_O$ using Eq. (5) and its absolute maximum values $|B_O|_{\text{max}}$ over the displacement variables $\xi_1$, $\xi_1^\prime$, $\xi_2$, and $\xi_2^\prime$ together with $\bar{n}_\pm$, $\alpha_1$, and $\alpha_2$. It requires a numerical multivariable maximization method, and we use the Broyden-Fletcher-Goldfarb-Shanno (BFGS) algorithm throughout this paper.

The optimized Bell functions against the average photon numbers $\bar{n}_\pm$ for $|\text{ECS}^\pm\rangle$ are presented in Fig. 1. The solid curves show the results for symmetric ECSs while the dotted curves correspond to general cases (asymmetric ones). The Bell functions are numerically optimized over all displacement variables and amplitudes under the condition of Eq. (2). The Bell violations occur both for the even and odd ECSs regardless of the values of $\bar{n}_\pm$ that are consistent with the results in Ref. [3]. The odd ECS, $|\text{ECS}^-\rangle$, violates the inequality more than the other one, $|\text{ECS}^+\rangle$ for a given average photon number. The Bell function for state $|\text{ECS}^-\rangle$ reaches up to $|B_O|_{\text{max}} \approx 2.743$ while the maximum Bell function for $|\text{ECS}^+\rangle$ is $|B_O|_{\text{max}} \approx 2.131$. This is due to the facts that the odd ECS is maximally entangled regardless of the value of $\bar{n}_-$ and the on-off measurement can effectively reveal nonlocality of the ECSs when the amplitudes are small.

It is obvious from Fig. 1 that one can increase the amount of violations by using asymmetric ECSs for certain regions of $\bar{n}_-$. In the case of state $|\text{ECS}^-\rangle$, this difference appears for $\bar{n}_- \gtrsim 1.43$. This difference in the optimized Bell function reaches its maximum value $\approx 0.053$ for $\bar{n}_- \approx 2.24$. The even ECS also shows a small increase of the violation when an asymmetric ECS is used in place of the symmetric ECS for $\bar{n}_+ \gtrsim 2.83$. The maximum difference is $\approx 0.007$ when $\bar{n}_+ \approx 2.93$.

In order to further investigate the advantages of the asymmetric ECSs, we plot the optimized Bell function $|B_O|_{\text{max}}$ as a function of $\alpha_1$ and $\alpha_2$ for $|\text{ECS}^-\rangle$ in Fig. 2. The blue line indicates the point for the maximum Bell function for each value of $\bar{n}_-$. The unsmooth change in the blue line at $\alpha_1 = \alpha_2 \approx 0.77$ ($\bar{n}_- \approx 1.43$) results from the numerical optimization process where local maximum values are compared with changes of related parameters, i.e., the displacement variables and amplitudes. We note that such comparisons among local maxima at a number of different parameter regions lead to unsmooth changes in several plots throughout this paper [53]. In fact, the blue line splits to two symmetric curves from the point of $\bar{n}_- \approx 1.43$, according to our numerical calculation (which is obvious because $\alpha_1$ and $\alpha_2$ are simply interchangeable), while we plot only one of the curves for convenience.

The results in Figs. 1 and 2 shows that when the average photon number is relatively large, the asymmetric ECS outperforms the symmetric one in testing the Bell-CHSH inequality with photon on-off measurements and displacement operations. On the other hand, when the average photon number is small, the symmetric ECS gives larger Bell violations.
FIG. 2. (Color online) Optimized Bell-CHSH function $|B_0|_{\text{max}}$ using photon on-off measurements as a function of amplitudes $\alpha_1$ and $\alpha_2$ for $|\text{ECS}\rangle$. For each point of $\alpha_1$ and $\alpha_2$, the Bell function is optimized over the displacement variables. The blue line indicates the states that show the maximum Bell violations among the states for the same the average photon numbers.

B. Bell-CHSH tests with photon number parity measurements

We now consider the photon number parity measurements with the displacement operators for both modes. The displaced parity measurement is given by

$$\hat{\Pi}_i(\xi) = \hat{D}_i(\xi) \sum_{n=0}^{\infty} \left( |2n\rangle \langle 2n| - |2n+1\rangle \langle 2n+1| \right) \hat{D}_i^\dagger(\xi)$$

(6)

where $i \in \{1, 2\}$ denotes each mode and the correlation function is

$$E_{\Pi}(\xi_1, \xi_2) = \langle \hat{\Pi}_1(\xi_1) \otimes \hat{\Pi}_2(\xi_2) \rangle$$

(7)

with which the Bell function $B_{\Pi}$ can be constructed using Eq. (5). We have obtained and presented an explicit form of the correlation function in Appendix A.

The optimized Bell-CHSH functions are plotted for varying $\bar{n}_\pm$ using parity measurement in Fig. 3. As implied by the apparent overlaps between the cases of symmetric ECSs (solid curve) and those of the asymmetric ECSs (dotted), our numerical investigations show that unlike the case with on-off measurements, the asymmetric ECSs do not show any larger Bell violations. Figure [4] shows the optimized Bell function of $|\text{ECS}\rangle$ for amplitudes $\alpha_1$ and $\alpha_2$. The blue line in the figure indicates the states with which the maximum violations are obtained for the same average photon numbers. The figure shows that the asymmetry of the amplitudes does not improve the amount of violations in the case of the parity measurements.

III. BELL-CHSH INEQUALITY TESTS UNDER DECOHERENCE EFFECTS

In this section, we consider decoherence effects on Bell nonlocality tests using an ECS. We will study the Bell-CHSH inequality violation using a general form of ECS.
where $\tau$ operators $\hat{\gamma}$ becomes $2\tau$ photon losses occur only one of the field modes but the ECS is sent to Bob at a distance. In the latter case, the ECS is first generated in the location of Alice and one mode of the field is sent to Bob. In the presence of photon loss, the time evolution of the density operator $\rho$ is described by the master equation

$$\frac{\partial \rho}{\partial \tau} = \hat{J} \rho + \hat{L} \rho,$$  

where $\tau$ is the interaction time, and the Lindblad super-operators $\hat{J}$ and $\hat{L}$ are defined as $\hat{J} \rho = \gamma \sum_i \hat{a}_i \rho \hat{a}_i^\dagger$ and $\hat{L} \rho = -\gamma/2 \sum_i (\hat{a}_i^\dagger \hat{a}_i \rho + \rho \hat{a}_i^\dagger \hat{a}_i)$ with a decay rate $\gamma$.

We consider two different strategies, i.e. symmetric (A) and asymmetric (B) ones, when distributing an ECS over a distance to Alice and Bob as illustrated in Fig. 5. In the case of strategy A, photon loss occurs symmetrically for both modes of the ECS during the decoherence time $\tau$. On the other hand, in strategy B, an ECS is first generated in the location of Alice and one mode of the ECS is sent to Bob at a distance. In the latter case, photon losses occur only one of the field modes but the decoherence time becomes $2\tau$. Assuming a zero temperature bath and a decay rate $\gamma$ for both cases, a direct calculation of the master equation leads to the states

$$\rho^+_A(\alpha_1, \alpha_2, t) = \mathcal{N}_\pm^2 \left\{ |\sqrt{t} \alpha_1 \rangle \langle \sqrt{t} \alpha_1| \otimes |\sqrt{t} \alpha_2 \rangle \langle \sqrt{t} \alpha_2| \right\},$$

$$\rho^-_A(\alpha_1, \alpha_2, t) = \mathcal{N}_\pm^2 \left\{ |\sqrt{t} \alpha_1 \rangle \langle -\sqrt{t} \alpha_1| \right\} \otimes \left\{ |\sqrt{t} \alpha_2 \rangle \langle \sqrt{t} \alpha_2| \right\},$$

for strategies A and B, respectively, where $t = e^{-\gamma \tau}$ and $\tau = 1 - t$ is the normalized time. If the cross terms in Eqs. (9) and (10) vanish, the states become classical mixtures of two distinct states and quantum effects generally disappear. We observe that the cross term in $\rho^+_A$ is proportional to $e^{-2(1-t)(\alpha_1^2 + \alpha_2^2)}$ and it is proportional to $e^{-2(1-t)\alpha_1 \alpha_2}$ in $\rho^-_B$. This implies that one may reduce decoherence effects using strategy B and by making the amplitude smaller for the mode in which loss occurs (i.e., by making the field mode sent to Bob in Fig. 5 to have the smaller amplitude). Such a strategy was also applied to the tele-amplification protocol [47] and the distributed generation scheme for ECSs [48].

### A. Symmetric and asymmetric strategies for entanglement distribution

In the presence of photon loss, the time evolution of a density operator $\rho$ is described by the master equation as

$$\frac{\partial \rho}{\partial \tau} = \hat{J} \rho + \hat{L} \rho,$$

where $\tau$ is the interaction time, and the Lindblad super-operators $\hat{J}$ and $\hat{L}$ are defined as $\hat{J} \rho = \gamma \sum_i \hat{a}_i \rho \hat{a}_i^\dagger$ and $\hat{L} \rho = -\gamma/2 \sum_i (\hat{a}_i^\dagger \hat{a}_i \rho + \rho \hat{a}_i^\dagger \hat{a}_i)$ with a decay rate $\gamma$.

We consider two different strategies, i.e. symmetric (A) and asymmetric (B) ones, when distributing an ECS over a distance to Alice and Bob as illustrated in Fig. 5.

In the case of strategy A, photon loss occurs symmetrically for both modes of the ECS during the decoherence time $\tau$. On the other hand, in strategy B, an ECS is first generated in the location of Alice and one mode of the ECS is sent to Bob at a distance. In the latter case, photon losses occur only one of the field modes but the decoherence time becomes $2\tau$. Assuming a zero temperature bath and a decay rate $\gamma$ for both cases, a direct calculation of the master equation leads to the states

$$\rho^+_A(\alpha_1, \alpha_2, t) = \mathcal{N}_\pm^2 \left\{ |\sqrt{t} \alpha_1 \rangle \langle \sqrt{t} \alpha_1| \otimes |\sqrt{t} \alpha_2 \rangle \langle \sqrt{t} \alpha_2| \right\},$$

$$\rho^-_A(\alpha_1, \alpha_2, t) = \mathcal{N}_\pm^2 \left\{ |\sqrt{t} \alpha_1 \rangle \langle -\sqrt{t} \alpha_1| \right\} \otimes \left\{ |\sqrt{t} \alpha_2 \rangle \langle \sqrt{t} \alpha_2| \right\},$$

for strategies A and B, respectively, where $t = e^{-\gamma \tau}$ and $\tau = 1 - t$ is the normalized time. If the cross terms in Eqs. (9) and (10) vanish, the states become classical mixtures of two distinct states and quantum effects generally disappear. We observe that the cross term in $\rho^+_A$ is proportional to $e^{-2(1-t)(\alpha_1^2 + \alpha_2^2)}$ and it is proportional to $e^{-2(1-t)\alpha_1 \alpha_2}$ in $\rho^-_B$. This implies that one may reduce decoherence effects using strategy B and by making the amplitude smaller for the mode in which loss occurs (i.e., by making the field mode sent to Bob in Fig. 5 to have the smaller amplitude). Such a strategy was also applied to the tele-amplification protocol [47] and the distributed generation scheme for ECSs [48].

### B. Bell-CHSH tests with on-off measurements under photon loss effects

We first consider Bell-CHSH tests with on-off measurements in comparing strategies A and B regarding robustness to the decoherence effects. An explicit form of the correlation functions calculated using Eqs. (9) and (10) can be found Appendix A. We numerically optimize the corresponding Bell-CHSH function, $|B_{O\max}|$, over all displacement variables and amplitudes $\alpha_1$ and $\alpha_2$ for given $r$. The numerically optimized Bell functions against the normalized time are presented for both $\rho_A^+$ and $\rho_B^+$ in Fig. 6 where the decrease of violations due to the decoherence effects is apparent. The optimizing values of amplitudes $\alpha_1$ and $\alpha_2$ are found between 0.4 and 1.4, where relatively smaller values correspond to large values of $r$. We observe that strategy B leads to larger violations than strategy A for $r \gtrsim 0.07$ when we use the even ECS $\rho^+$. For the smaller value of $r$, strategy A gives slightly larger violations where the difference is up to $\lesssim 0.001$. On the other hand, the odd ECS $\rho^-$ shows larger violations for strategy B than strategy A regardless of the value of $r$. As explained in Sec. IIA, the discontinuities of the first derivative at $r \approx 0.09$ for $\rho_A^+$ and $r \approx 0.07$ for $\rho_B^-$
FIG. 6. (Color online) Optimized Bell-CHSH function $|B|_{\text{max}}$ against the normalized time $r$ for (a) even ECSs and (b) odd ECSs using on-off measurements. The dotted curves indicate the optimized results for strategy A for entanglement distribution and the solid curves correspond to the results for strategy B explained in the main text. Amplitudes $\alpha_1$, $\alpha_2$ together with displacement variables are all numerically optimized to find the maximum values of the Bell function for given $r$. The optimizing values of $\alpha_1$ and $\alpha_2$ are found between 0.4 and 1.4, where relatively smaller values correspond to large values of $r$. The dot-dashed horizontal line indicates the classical limit, 2.

in Fig. 6(a) emerge from the numerical optimization process where local maxima for different parameter regions are compared.

The Bell violations of the odd ECSs for varying $\alpha_1$ and $\alpha_2$ for $r = 0.2$, numerically optimized for the displacement variables, are shown in Fig. 7. Figure 7(a) clearly shows the asymmetric behavior of the optimized Bell function under strategy A. Figure 7(b) is for the case of asymmetric decoherence (strategy B). For a given average photon number, the optimizing value of $\alpha_2$ in strategy B is smaller than that of strategy A. It shows that an asymmetric ECS can reduce the asymmetric decoherence effects by lessening the amplitude of the mode which suffers decoherence and adjusting the other mode which is free from decoherence. These results are consistent with previous studies for Bell inequality tests using hybrid entanglement [56, 57] and quantum teleportation [45, 48], where the amplitude of the mode that suffers decoherence should be kept small in order to optimize Bell violations or teleportation fidelities.

It is important to note from Fig. 7 that strategy B shows significantly larger violation than that of strategy A. In strategy A (Fig. 7(a)), the maximum value of the Bell function is $\approx 2.054$ for $\alpha_1 \approx 0.81$ and $\alpha_2 \approx 0.74$. On the other hands, the maximum value for strategy B (Fig. 7(b)) is $\approx 2.145$ for $\alpha_1 \approx 0.76$ and $\alpha_2 \approx 0.50$. The absolute value of violation in strategy B is almost three times larger than that of strategy A. It is a remarkable advantage of using asymmetric ECSs with the asymmetric distribution scheme in testing the Bell-CHSH inequality.

C. Bell-CHSH tests with photon number parity measurement under photon loss effects

Figure 8 presents the numerically optimized Bell function $|B|_{\Pi}$ with parity measurements against the normalized time $r$ (see Appendix A for details). For both the even and odd ECSs, strategy B give significantly larger violations over strategy A. The optimizing values of $\alpha_1$ and $\alpha_2$ approach infinity for $r \to 0$. However, we are interested in values that can be obtained within a realistic
that of photon loss in the entangled state is the order of

difference between the effect of detection inefficiency and
by photon loss (Eqs. (9) and (10)) in the entangled state
normalized decoherence time $r$.

FIG. 8. (Color online) Optimized Bell-CHSH function against
normalized decoherence time $r$ for photon parity measure-
ment (a) for even ECS and (b) for odd ECS. The dotted curves
show the optimized results for strategy A and the solid curves
show the results for strategy B. The dot-dashed line indicates
the classical limit $2$.

range (e.g. $\bar{n}_+ < 5$). We thus plot in Fig. 9 the opti-
mization results for the even ECSs under a normalized
decoherence time ($r = 0.1$) for varying $\alpha_1$ and $\alpha_2$. In
contrast to the case of on-off measurements, as shown in
Fig. 9(a), the asymmetry of the amplitudes of the ECS
does not improve the amount of Bell violations for strat-
agy A where decoherence occurs symmetrically. Like the
results of the on-off measurements, the asymmetry of the amplitudes in ECSs increases the amount of Bell viola-
tions for strategy B (Fig. 9(b)). In this case, the maximum
value of optimized Bell function is $|\mathcal{B}_{II}|_{\text{max}} \approx 2.146$ for
$\alpha_1 \to \infty$ and $\alpha_2 \approx 0.46$. A similar value $|\mathcal{B}_{II}|_{\text{max}} \approx 2.131$
can be obtained for $\alpha_1 = 2$ and $\alpha_2 \approx 0.44$ where the
average photon number is given by $n_+ \approx 4.19$. This value
is much larger than that of strategy A, which only shows
$|\mathcal{B}_{II}|_{\text{max}} \approx 2.014$ for $\alpha_1 = \alpha_2 \approx 0.44$.

IV. EFFECTS OF DETECTION INEFFICIENCY

Here we investigate the effects of detection inefficiency
in the Bell-CHSH inequality tests. An imperfect pho-
todetector with efficiency $\eta$ can be modeled using a per-
fact photodetector together with a beam splitter of trans-
missivity $\sqrt{\eta}$ in front of it [44]. Meanwhile, decoherence
by photon loss (Eqs. (9) and (10)) in the entangled state
can also be modeled using a beam splitter. The only dif-
ference between the effect of detection inefficiency and
that of photon loss in the entangled state is the order of

FIG. 9. (Color online) Using parity measurement for even
ECSs, we optimize the Bell-CHSH function against $\alpha_1$ and
$\alpha_2$ for $r = 0.1$ (a) in strategy A and (b) in strategy B. All
displacement variables are optimized for given $\alpha_1$ and $\alpha_2$.

We compare these two cases in Appendix B where the re-
results show that the two cases lead the same Bell-CHSH
function except for the coefficients of the displacement
variables which pass away during the optimization pro-
cess.

We have numerically investigated both cases of on-off
and parity measurements with limited efficiencies, $\eta_1$ for
mode 1 and $\eta_2$ for mode 2, for both even and odd ECSs.
As we find that the even ECS is better for on-off mea-
surements and the odd ECS is better for parity measure-
ments, as already implied in Figs. 6 and 8, we present the
two corresponding cases in Figure 11. It shows that the
on-off measurement scheme generally gives much larger
violations compared to the parity measurement scheme.
This is consistent with the results of the case with pho-
ton loss effects studied in the previous section (see Figs.
6(b) and 8(a)).

In detail, for the case of the displaced on-off measure-
ments with the odd ECS (Figs. 11(a) and 11(b)), optimi-
izing values of $\alpha_1$ and $\alpha_2$ lie between 0.4 to 1.4, which is
experimentally feasible [46, 52]. We find that the asym-
metric ECS (Fig. 11(b)) shows larger violations compared
to the symmetric ECS (Fig. 11(a)). When $\eta = \eta_1 = \eta_2$, a
The detection efficiency of $\eta \geq 0.771$ is required for violation of $|B_{0}| \geq 2.001$ with the symmetric ECS while a smaller efficiency $\eta \geq 0.745$ is sufficient for the same amount of violation with the asymmetric ECS. When the efficiencies for modes 1 and 2 are $\eta_{1} = 0.75$ and $\eta_{2} = 1$, the asymmetric ECS shows the optimized Bell quantity of 2.305 which is larger than 2.269 for the symmetric ECS.

In the case of the parity measurements using the even ECSs, the improvement by the asymmetric ECS is even larger. For example, comparing $\eta_{1} = 0.98$ line in Fig. 11(c) and (d), we find that there is a violation $|B_{0}| \geq 2.01$ for $\eta_{2} \geq 0.760$ if we use the asymmetric form of ECS. However, much larger efficiency $\eta_{2} \geq 0.805$ is needed when we use the symmetric ECS. The optimizing values of $\alpha_{1}$ and $\alpha_{2}$ lie between 0.02 to 1.64 if $\eta_{1} < 0.99$ and $\eta_{2} < 0.99$. However, we note that if one of the detectors is perfect, the optimizing value of the amplitude of that mode goes to infinity.

V. REMARKS

In this paper, we have studied asymmetric ECSs as well as asymmetric lossy environments for Bell-CHSH inequality tests. We first investigated the violations of Bell-CHSH inequality using perfect on-off detectors and ideal ECSs. We have shown that the asymmetric form of ECS could give larger violations of the Bell-CHSH inequality in some region of the averaged photon numbers of the ECSs. On the other hand, in the case of photon number parity measurements, we could not find apparent improvement of the Bell violations using the asymmetric form of ECS with perfect detectors.

We then studied Bell-CHSH violations under the effects of decoherence on the ECSs. We considered two different schemes for distributing entanglement under photon loss, i.e., symmetric and asymmetric schemes. In the symmetric scheme, the photon losses occur in both modes of the ECSs. On the other hands, photon loss occurs only in one of the two modes in the case of the asymmetric scheme. We showed that the asymmetric form of ECS can increase the amount of violations significantly under the asymmetric scheme compared to the case of the symmetric scheme. For example, when the normalized time is $r = 0.2$, the Bell-CHSH function using photon on-off detectors for the asymmetric loss scheme shows the maximum value of 2.145, which is much larger than that for the symmetric case, 2.054. A similar improvement can be made in the case of photon number parity measurements; when $r = 0.1$ with the ECS of average photon number $\approx 4.19$, the optimized Bell function under the symmetric loss scheme is about 2.014 but it is 2.131 under the asymmetric scheme.

We have also investigated effects of inefficient detectors. We show that the asymmetric form of ECSs lowers the detection efficiency required for violation of the Bell-CHSH inequality. For example, a detection efficiency of $\eta \geq 0.771$ is required for a Bell-CHSH violation of $|B_{0}| \geq 2.001$ with the symmetric ECS, but a smaller efficiency $\eta \geq 0.745$ is sufficient for the same amount of violation with the asymmetric ECS. This improvement is even more significant for the case of parity measurements particularly when the detection efficiencies of two detectors differ much. When the detection efficiency for mode 1 is $\eta_{1} = 0.98$, the required detection efficiency for...
mode 2 is \( \eta_2 \gtrsim 0.81 \) to show the violation of \( |B_{11}| \gtrsim 2.01 \) using the symmetric ECS is used, but it is \( \eta_2 \gtrsim 0.76 \) when the asymmetric ECS is used.

In summary, our extensive study reveals that the asymmetric form of ECSs and the asymmetric scheme for distributing entanglement enable one to effectively test the Bell-CHSH inequality with the same resources.

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**Appendix A: Correlation functions for on-off and parity measurements**

Here, we present the explicit forms of the correlation functions defined in Eqs. (14) and (17) under lossy effects. Instead of computing the correlation function for every case, we calculate the results for an ECS under the beam splitters with transmissivities \( \eta_1 \) and \( \eta_2 \) for mode 1 and mode 2, respectively, and show that these results are applicable to all the cases discussed in this paper.

\[
E_O(\xi, \chi) = \text{Tr}[\rho^+ \hat{O}_1(\xi) \otimes \hat{O}_2(\chi)]
\]

\[
= \frac{2}{2 \pm 2e^{-2\eta_1-2\eta_2}} \left[ 1 \pm 2e^{-2\eta_1-2\eta_2} - 2(\xi^2 + \chi^2 - \xi^2 - \chi^2) e^{2\eta_1} \cos(2\eta_1\xi) 
+ e^{2\eta_2} - 2(\xi^2 + \chi^2 - \xi^2 - \chi^2) e^{2\eta_2} \cos(2\eta_2\chi) 
\right]
\]

\[
E_\Pi(\xi, \chi) = \text{Tr}[\rho^+ \hat{\Pi}_1(\xi) \otimes \hat{\Pi}_2(\chi)]
\]

\[
= \frac{1}{2 \pm 2e^{-2\eta_1-2\eta_2}} \left[ 1 \pm 2e^{-2\eta_1-2\eta_2} \exp \left( -2(\xi^2(\eta_1 + 1) + \alpha_2^2(\eta_2 + 1) + \chi^2 + \chi^2 + \xi^2 + \xi^2) \right) 
\right]
\]

where \( \xi = \xi_\tau + i\xi_\chi \) and \( \chi = \chi_\tau + i\chi_\chi \), and \( \hat{O}_1(\xi) \) and \( \hat{\Pi}_1(\xi) \) were defined in Eqs. (34) and (36). By substituting \( \eta_1 = \eta_2 = 1 \), we can get a correlation functions for the case of perfect detectors. Likewise, \( \eta_1 = \eta_2 = t \) gives the correlation functions for distribution strategy A and \( \eta_1 = 1, \eta_2 = t^2 \) gives the correlation function for strategy B. As we will discuss below, we also can use these results for the cases of imperfect detectors even though they differ in the order of the displacement operators and the beam splitters.

The beam splitter operator with transmissivity \( \sqrt{\eta} \) between modes \( a \) and \( b \) can be represented by \( \hat{B}_{ab} = \exp[(\cos^{-1}(\eta) \hat{a}^\dagger_1 \hat{a}_b - \hat{a}_a \hat{a}^\dagger_2)] \) [61]. If a coherent-state \( \langle \beta \rangle \) for mode \( C \) is mixed with the vacuum for mode \( v \), the initial coherent-state dyadic becomes

\[
\text{Tr}_v([\hat{B}_{Cu} \langle \alpha \rangle \langle \beta \rangle]_C \otimes \langle 0 \rangle \langle 0 \rangle_v)_{\hat{B}_{Cu}^\dagger}
\]

\[
= \exp[-\frac{1}{2} \chi(2|\alpha|^2 + |\beta|^2 - 2\alpha^2)|\sqrt{\eta}\rangle \langle \sqrt{\eta}| \beta \rangle] .
\]

(A1)

We now apply this result to a pure ECS \( |\text{ECS}\rangle \) mixed with the vacuum modes through two beam splitters with transmissivities \( \eta_1 \) and \( \eta_2 \), and the result is

\[
\rho^+ = [\langle \alpha_1\rangle \langle \alpha_2 \rangle \eta_1 \eta_2]
\]

\[
= N_{\pm}^2 \{ [\sqrt{\eta_1} \alpha_1 \rangle \langle \sqrt{\eta_1} \alpha_1 \} \otimes [\sqrt{\eta_2} \alpha_2 \rangle \langle \sqrt{\eta_2} \alpha_2 \} \]
\]

\[
\pm e^{-2[1-(1-\eta_1)\alpha_1^2+(1-\eta_2)\alpha_2^2]}
\]

\[
\{ [\sqrt{\eta_1} \alpha_1 \rangle \langle \sqrt{\eta_1} \alpha_1 \} \otimes [\sqrt{\eta_2} \alpha_2 \rangle \langle \sqrt{\eta_2} \alpha_2 \} \}
\]

\[
+ \{ [\sqrt{\eta_1} \alpha_1 \rangle \langle -\sqrt{\eta_1} \alpha_1 \} \otimes [\sqrt{\eta_2} \alpha_2 \rangle \langle -\sqrt{\eta_2} \alpha_2 \} \}
\]

\[
+ \{ [\sqrt{\eta_1} \alpha_1 \rangle \langle \sqrt{\eta_1} \alpha_1 \} \otimes [\sqrt{\eta_2} \alpha_2 \rangle \langle -\sqrt{\eta_2} \alpha_2 \} \}
\]

\[
+ \{ [\sqrt{\eta_1} \alpha_1 \rangle \langle -\sqrt{\eta_1} \alpha_1 \} \otimes [-\sqrt{\eta_2} \alpha_2 \rangle \langle -\sqrt{\eta_2} \alpha_2 \} \}
\]

It is straightforward to check that if we let \( \eta_1 = \eta_2 = t \) then the state will be exactly the same to Eq. (9). In addition, \( \eta_1 = 1 \) with \( \eta_2 = t^2 \) will give Eq. (10).

Using the result, the correlation functions for on-off measurement and parity measurement are obtained as...
Appendix B: The order of photon loss and displacement operator in correlation

In this section, we show that ECSs give the same optimized value of the Bell function independent of the order of the displacement and the photon loss by a beam splitter. Note that it is sufficient to only consider two coherent-state dyadics of the form $|\gamma\rangle \langle\gamma|$ and $|\gamma\rangle \langle-\gamma|$. This is because ECSs only contain this kind of dyadics and a photon loss by a beam splitter is superlinear (linear in the space of density matrices) and a displacement operator is linear.

First, suppose that the states undergo a beam splitter first and displacement second. From the above, the dyadics would be

$$\begin{align*}
\langle\gamma\rangle \langle\gamma\rangle &\longrightarrow |\sqrt{\eta}\gamma\rangle \langle\sqrt{\eta}\gamma| \\
\langle\gamma\rangle \langle-\gamma\rangle &\longrightarrow \exp[-2(1-\eta)|\gamma|^2] |\sqrt{\eta}\gamma\rangle \langle-\sqrt{\eta}\gamma|.
\end{align*}$$

Now we apply the displacement operator $\hat{D}(\xi) = \exp[\xi a^* - \xi^* a]$ to photon lost dyadics. Then the states would be

$$\begin{align*}
\langle\gamma\rangle \langle\gamma\rangle &\longrightarrow |\sqrt{\eta}\gamma - \xi\rangle \langle\sqrt{\eta}\gamma - \xi| \\
\langle\gamma\rangle \langle-\gamma\rangle &\longrightarrow \exp[-2(1-\eta)|\gamma|^2 + \sqrt{\eta}(\xi^*\gamma - \xi^*\gamma^*)] |\sqrt{\eta}\gamma - \xi\rangle \langle-\sqrt{\eta}\gamma - \xi|.
\end{align*}$$

Next, let us consider the case we employ the operators to opposite order, displacement first and photon loss second. After the displacement, the dyadics would be

$$\begin{align*}
\langle\gamma\rangle \langle\gamma\rangle &\longrightarrow |\gamma - \xi\rangle \langle\gamma - \xi| \\
\langle\gamma\rangle \langle-\gamma\rangle &\longrightarrow \exp[\xi^*\gamma - \xi^*\gamma^*] |\gamma - \xi\rangle \langle-\gamma - \xi|.
\end{align*}$$

Applying photon loss to these dyadics using Eq. (A1), we finally obtain

$$\begin{align*}
\langle\gamma\rangle \langle\gamma\rangle &\longrightarrow |\sqrt{\eta}(\gamma - \xi)\rangle \langle\sqrt{\eta}(\gamma - \xi)| \\
\langle\gamma\rangle \langle-\gamma\rangle &\longrightarrow \exp[-2(1-\eta)|\gamma|^2 + \eta(\xi^*\gamma - \xi^*\gamma^*)] |\sqrt{\eta}(\gamma - \xi)\rangle \langle\sqrt{\eta}(\gamma - \xi)|.
\end{align*}$$

These results show that if we replace $\xi$ with $\sqrt{\eta}\xi$ in Eq. (B1), it will be exactly the same to Eq. (B2). This means that the optimized values of the Bell functions for all displacement variables do not depend on the order of loss and displacement.

References

[1] B. Yurke and D. Stoler, Phys. Rev. Lett. 57, 13 (1986).
[2] A. Mecozzi and P. Tombesi, Phys. Rev. Lett. 58, 1055 (1987).
[3] B. C. Sanders, Phys. Rev. A 45, 6811 (1992).
[4] A. Mann, B. C. Sanders, and W. J. Munro, Phys. Rev. A 51, 989 (1995).
[5] R. Filip, J. Reháček, and M. Dusek, J. Opt. B: Quantum Semiclassical Opt. 3, 341 (2001).
[6] D. Wilson, H. Jeong, and M. S. Kim, J. Mod. Opt. 49, 851 (2002).
[7] H. Jeong, W. Son, M. S. Kim, D. Ahn, and Č. Brukner, Phys. Rev. A 67, 012106 (2003).
[8] J. Wenger, M. Hafezi, F. Grosshans, R. Tualle-Brouri, and P. Grangier, Phys. Rev. A 67, 012105 (2003).
[9] M. Stobinska, H. Jeong and T. C. Ralph, Phys. Rev. A 75, 052105 (2007).
[10] H. Jeong, Phys. Rev. A 78, 042101 (2008).
[11] C.-W. Lee and H. Jeong, Phys. Rev. A 80, 052105 (2009).
[12] C. C. Gerry, J. Mimih, and A. Benmoussa, Phys. Rev. A 80, 022111 (2009).
[13] H. Jeong, M. Paternostro, and T. C. Ralph, Phys. Rev. Lett. 102, 060403 (2009).
[14] Y. Lim, M. Paternostro, M. Kang, J. Lee, and H. Jeong, Phys. Rev. A 85, 062112 (2012).
[15] B.T. Kirby and J.D. Franson, Phys. Rev. A 87, 053822 (2013).
[16] B.T. Kirby and J.D. Franson, Phys. Rev. A 89, 033861 (2014).
[17] M. Paternostro and H. Jeong, Phys. Rev. A 81, 032115 (2010).
[18] C.-W. Lee, M. Paternostro, and H. Jeong, Phys. Rev. A 83, 022102 (2011).
[19] X. Wang, Phys. Rev. A 64, 022302 (2001).
[20] S. J. van Enk and O. Hirota, Phys. Rev. A 64, 022313 (2001).
[21] H. Jeong, M. S. Kim, and J. Lee, Phys. Rev. A 64, 052308 (2001).
[22] H. Jeong and M. S. Kim, Quantum Info. Comp. 2, 208 (2002).
[23] N. B. An, Phys. Rev. A 68, 022321 (2003).
[24] P. T. Cochrane, G. J. Milburn, W. J. Munro, Phys. Rev. A 59, 2631 (1999).
[25] H. Jeong and M. S. Kim, Phys. Rev. A 65, 042305 (2002).
[26] T. C. Ralph, A. Gilchrist, G. J. Milburn, W. J. Munro, and S. Glancy, Phys. Rev. A 68, 042319 (2003).
[27] A. P. Lund, T. C. Ralph, and H. L. Haselgrove, Phys. Rev. Lett. 100, 030503 (2008).
[28] P. Marek and J. Fiurášek, Phys. Rev. A 82, 014304 (2010).
[29] C. R. Myers and T. C. Ralph, New. J. Phys. 13, 115015 (2011).
[30] J. Kim, J. Lee, S.-W. Ji, H. Nha, P. M. Anisimov and J. P. Dowling, Optics Communications 337, 79 (2015).
[31] C. C. Gerry and R. A. Campos, Phys. Rev. A 64, 063814 (2001).
[32] C. C. Gerry, A. Benmoussa, and R. A. Campos, Phys.
One may ask whether this unsmooth change implies that our results around \( \bar{n} \approx \frac{1}{2} \) are not the global maxima. We cannot exclude such a possibility because we do not have an analytical evidence to claim that all the results obtained using the BFGS algorithm [54] throughout the paper are the global maxima. This is a typical problem in an optimization process with multiple parameters as we do in the present paper with 8 to 10 parameters (assuming \( \alpha_1 \) and \( \alpha_2 \) are real without loss of generality).

Additionally, a review of previous studies can provide insights into the topic. For instance, B. Yurke and D. Stoler, Phys. Rev. A 35, 4846 (1987), and P. Tombesi and A. Mecozzi, J. Opt. Soc. Am. B 4, 1700 (1987), have explored similar systems and methodologies. Similarly, C. C. Gerry, Phys. Rev. A 55, 2478 (1997), and B. C. Sanders and D. A. Rice, Opt. Quant. Electronics 31, 525 (1999), have contributed to the theoretical foundations of similar processes.

In addition, the work of A. Ourjoumtsev, F. Ferreyrol, R. Tualle-Brouri, and P. Grangier, Nature Phys. 5, 189(2009), and J. S. Neergaard-Nielsen, Y. Eto, C.-W. Lee, H. Jeong, and M. Sasaki, Nature Photonics 7, 439-443 (2013), have further expanded the understanding of these phenomena. K. Park, S.-W. Lee, and H. Jeong, Phys. Rev. A 86, 062301 (2012), have also contributed valuable insights into related areas.

Moreover, the implications of these findings extend beyond the specific context of the paper, reaching into broader applications. For instance, S. J. D. Phoenix, Phys. Rev. A 41, 5132 (1990), and R. A. Campos, B. E. A. Saleh, and M. C. Teich, Phys. Rev. A 40, 1371 (1989), have considered analogous systems and methodologies, offering further perspectives on the subject.

[54] R. Fletcher, Practical Methods of Optimization (Second Edition) (Wiley, 1987).

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