Memory effect, conformal symmetry and gravitational plane waves

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Abstract

We discuss in some detail the interaction of classical particles, including the scattering and memory effect, with a pulse of gravitational plane wave. The key point is the conformal symmetry of gravitational plane waves. In particular, we obtain, in the limit of short pulse, some results for impulsive gravitational waves. Furthermore, in the general case, we give certain conditions which allow us to completely describe the interaction in terms of the singular Baldwin-Jeffery-Rosen coordinates.

1 Introduction

Gravitational waves have been for many years a matter of intensive study and controversy [1]-[6]. Recently they gained a new interest both for experimental and theoretical reasons. Although indirect evidence for their existence had been obtained from the observation of binary pulsar system PSR 1913+16 [7] only recently the direct observation of gravitational waves from a pair of merging black holes [8] and binary neutron star inspiral [9] have been possible. From the theoretical point of view a new interesting set of ideas relating to asymptotic symmetries soft theorems and gravitational memory effect has recently emerged [10]-[15]. The gravitational memory effect consists, roughly speaking, in the change in separation of freely falling particles after the passage of short burst of gravitational wave [5]-[20]. On the other hand, soft graviton theorems are related to gauge transformations (diffeomorphisms) of the asymptotically flat spacetime (which do not tend to identity at the infinity) [21, 22]. Both the issues are related to geodesic (deviation) equations and asymptotically flat metrics. In

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many cases one can restrict the considerations to plane waves. This is motivated by the fact that far from the source one can approximate the gravitational wave in a neighborhood of detector by an exact plane wave (assuming that the back reaction of detector is negligible). Especially interesting is the case of the so called impulsive gravitational waves \[23\]-\[31\].

In view of the above remarks it would be interesting to prove analytically solvable examples of gravitational plane waves. Taking into account that higher symmetry of the system is the more likely it is analytically solvable we consider the wave profile carrying the maximal conformal symmetry, allowed for non-flat plane waves. Such a choice allows us to provide the explicit description of the behaviour of test particles and study of memory effect as well as classical cross section. Moreover, by taking an appropriate limit we give explicit description, including the form of conformal symmetry generators, of Dirac delta profile.

Moreover, we study also in some detail the problem of complete description of gravitational plane waves in Baldwin-Jeffery-Rosen (BJR) coordinates. It is well known that the BJR map does not cover the whole spacetime manifold corresponding to the gravitational plane wave. On the other hand, BJR coordinates seem to be some importance in understanding inequivalent ground states (vacua) \[12\], \[13\]. Therefore, a deeper insight into the structure of BJR coordinates would be profitable.

## 2 Conformal symmetry and geodesics

### 2.1 Preliminaries

In relativity, a Ricci flat plane wave is called a gravitational plane wave, or an exact gravitational wave. In the so called Brinkmann (B) coordinates \[1\] it is described by the metric tensor

$$ g = \sum_{\sigma,\sigma'} K_{\sigma\sigma'}(u) X^\sigma X^{\sigma'} du^2 + 2dudV + \sum_\sigma (dX^\sigma)^2, $$ \hspace{1cm} (2.1)

where indices $\sigma, \sigma' = \pm$ and

$$ \sum_{\sigma,\sigma'} K_{\sigma\sigma'}(u) X^\sigma X^{\sigma'} = \frac{1}{2} A_+(u)((X^+)^2 - (X^-)^2) + A_x(u) X^+ X^- . $$ \hspace{1cm} (2.2)

To begin with, one can consider the linear polarization, i.e., $A_x \equiv 0$. Denoting $A_+(u) \equiv A(u)$, we get the metric

$$ g = \frac{1}{2} A((X^+)^2 - (X^-)^2) du^2 + 2dudV + (dX^+)^2 + (dX^-)^2 , $$ \hspace{1cm} (2.3)

and the Lagrangian of the test particle

$$ L = m \left[ \frac{1}{2} A((X^+)^2 - (X^-)^2) \left( \frac{du}{d\tau} \right)^2 + 2 \frac{du}{d\tau} \frac{dV}{d\tau} + \left( \frac{dX^+}{d\tau} \right)^2 + \left( \frac{dX^-}{d\tau} \right)^2 \right] . $$ \hspace{1cm} (2.4)
Let us define
\[ P^\mu \equiv m \frac{dX^\mu}{d\tau} = \frac{dX^\mu}{d\lambda}, \]  
(2.5)
where \( \lambda = \frac{\tau}{m} \). Then
\[ \text{const} = P_V \equiv g_{\nu \mu} \frac{dX^\mu}{d\lambda} = P^\nu = \frac{d\lambda}{d\lambda}. \]  
(2.6)
Thus
\[ u = \lambda P_V. \]  
(2.7)
Then the geodesic equations are of the form\[ \frac{d^2 X^\sigma}{du^2} = \frac{\sigma A}{2} X^\sigma, \quad \sigma = \pm, \]  
(2.8)
and
\[ \frac{d^2 V}{du^2} + \sum_\sigma \sigma X^\sigma \left( A \frac{dX^\sigma}{du} + \frac{1}{4} \frac{dA}{du} X^\sigma \right) = 0. \]  
(2.9)
The observable quantities are connected with the geodesic deviation equations (see [13] for resume); in the case of the transverse-space coordinates \( X^\sigma \) the latter coincide with eqs. (2.8). Thus the problem of the interaction of the particle with the gravitational waves or the scattering problem can be reduced to the analysis of solutions to eqs. (2.8) and (2.9). Although, in general, this set of equations cannot be explicitly solved, there are some special cases where the solution is possible; they are mainly related to the symmetry of the metric; of course the most interesting cases are the ones with the maximal symmetry.

It is well known that the generic dimension of the isometry group of the gravitational plane waves is five [5, 32, 33]. However, when \( A_1 = \text{const} \) or \( A_2 = \text{const} \) (and a suitable generalization to circular polarization, see e.g., [5]) the dimension of the isometry group of \( g \) is six and then one can explicitly solve the geodesic equations. The first case can be used, for example, to describe the gravitational wave which is a sandwich between two Minkowskian regions; the second one is not geodesically complete but is intensively used in the context of the Penrose limit [34].

The situation becomes more interesting if we take into account the conformal symmetry. First, let us recall that the maximal dimension of the conformal group of, non-flat, metric \( g \), given by (2.1), is seven [35] (see also [36, 37]). On the other hand, the metric \( g \) admits a homothetic vector field. Thus, in the non-flat case, the metric \( g \) with \( A_1 \) or \( A_2 \) (and their suitable generalization to the circular polarization) exhibits the maximal, seven-dimensional, conformal symmetry. Further examples of the gravitational plane waves carrying the maximal conformal symmetry entail a non-homothetic conformal vector field (which is rather rare, even in the case of non-vacuum plane waves). In the case of gravitational plane waves it turns out that, besides two mentioned profiles, there is only one metric family when the dimension of

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1If not otherwise stated, we assume that \( A \) is a continuous function; consequently the geodesics are defined on the whole real line.
the conformal group is seven \[38, 39\] (the dimension of the isometry group is five and there is a proper conformal transformation).

This special family of the gravitational plane waves is given, in the case of the linear polarization, by the metric (2.3) with the profile

\[ A(u) = \frac{c}{(u^2 + \alpha u + \beta)^2}, \tag{2.10} \]

In this case the geodesic equations can be also analytically solved what enables us to explicitly discuss interaction of a classical particle with gravitational plane waves. In order to describe the regular gravitational pulse one should consider (2.10) with nonsingular denominator (then the metric is geodesically complete).

### 2.2 Complete conformal case and its Dirac delta limit

Let us consider the profile (2.10) of the form

\[ A(u) = \frac{2\pi^3}{\varepsilon^3 (u^2 + \varepsilon^2)^2}. \tag{2.11} \]

Such a choice is dictated not only by demanding the completeness of geodesics but also by the fact that the profile (2.11) can be used for the study of the Dirac delta pulse, namely

\[ \lim_{\varepsilon \to 0} A(u) = \delta(u). \tag{2.12} \]

In view of the above we start with \( \varepsilon < \pi \), i.e., sufficiently narrow gravitational pulse. Then the general solution of the geodesic equation takes the form:

\[ X^\sigma(u) = C_1^\sigma \sqrt{u^2 + \varepsilon^2 \sin(a_\sigma \arctan(\frac{u}{\varepsilon})) + C_2^\sigma}, \tag{2.13} \]

or equivalently

\[ X^\sigma(u) = D_1^\sigma \sqrt{u^2 + \varepsilon^2 \sin(a_\sigma \arctan(\frac{u}{\varepsilon})) + D_2^\sigma \sqrt{u^2 + \varepsilon^2 \cos(a_\sigma \arctan(\frac{u}{\varepsilon}))}}, \tag{2.14} \]

and

\[ V(u) = C_4 + C_3 u + \frac{1}{4} \sum_{\sigma} (C_1^\sigma)^2 \left[ u \cos(2a_\sigma \arctan(\frac{u}{\varepsilon})) + 2C_2^\sigma \right] - \varepsilon a_\sigma \sin(2a_\sigma \arctan(\frac{u}{\varepsilon})) \tag{2.15} \]

or equivalently

\[ V(u) = C_4 + C_3 u + \frac{1}{4} \sum_{\sigma} ((D_1^\sigma)^2 - (D_2^\sigma)^2) \left[ u \cos(2a_\sigma \arctan(\frac{u}{\varepsilon})) - \varepsilon a_\sigma \sin(2a_\sigma \arctan(\frac{u}{\varepsilon})) \right] \]

\[ - \frac{1}{2} \sum_{\sigma} D_1^\sigma D_2^\sigma \left[ u \sin(2a_\sigma \arctan(\frac{u}{\varepsilon})) + \varepsilon a_\sigma \cos(2a_\sigma \arctan(\frac{u}{\varepsilon})) \right], \tag{2.16} \]
where
\[ a_\sigma = \sqrt{1 - \frac{\sigma \epsilon}{\pi}}, \quad \sqrt{2} > a_- > 1, \quad 1 > a_+ > 0. \tag{2.17} \]

Of course, one can consider the case \( \epsilon \geq \pi \); then the \( X^- \) solution takes the same form, but in \( X^+ \) (and consequently partially in \( V \)) one should replace \( \sin \) and \( \cos \) by hyperbolic or linear functions; the slowly varying case should coincide with the attractive (repulsive) character of the harmonic oscillator. However, such a choice of \( \epsilon \) can lead not only to the technical modification – see the discussion at the end of Section 3. Thus in what follows we consider the case \( \epsilon < \pi \) which seems more adequate to analyse the burst-like pulse of gravitational waves.

First, we impose the initial conditions. Let us note that the condition \( \dot{X}^\sigma(-\infty) = 0 \) implies \( C_2^\sigma = \frac{\pi}{2} a_\sigma \) and, consequently, we may define
\[ X_0^\sigma = \lim_{u \to -\infty} X^\sigma(u). \tag{2.18} \]

It is worth to notice that \( X_0^\sigma = \frac{a_\sigma}{\sin(a_\sigma \frac{\pi}{2})} X^\sigma(0) \). In consequence, the solution is of the form
\[ X^\sigma(u) = \frac{X_0^\sigma}{\epsilon a_\sigma} \sqrt{u^2 + \epsilon^2 \sin(a_\sigma (\arctan(\frac{u}{\epsilon}) + \frac{\pi}{2}))}. \tag{2.19} \]

The asymptotic behaviour near the future infinity is
\[ X^\sigma(u) \approx X_0^\sigma \left( \frac{u}{\epsilon a_\sigma} \sin(a_\sigma \pi) - \cos(a_\sigma \pi) \right), \quad u \gg 1, \tag{2.20} \]

thus the relative transverse-space distance between two trajectories \( X_1^\sigma, X_2^\sigma \) grows linearly near the future infinity, so they exhibit the so called velocity memory effect \[1, 5, 12, 13, 40\].

\[ \sqrt{\sum_\sigma (\dot{X}_1^\sigma(u) - \dot{X}_2^\sigma(u))^2} \approx \sqrt{\sum_\sigma (X_{in1}^\sigma - X_{in2}^\sigma)^2 \frac{\sin^2(a_\sigma \pi)}{\epsilon^2 a_\sigma^2}}, \quad u \gg 1. \tag{2.21} \]

In general, we define the initial conditions as follows
\[ \dot{X}_0^\sigma \equiv \dot{X}^\sigma(-\infty) = \frac{P_0^\sigma}{P_V}, \quad X_0^\sigma \equiv X^\sigma(0); \tag{2.22} \]

then
\[ D_1^\sigma = C_1^\sigma \cos(C_2^\sigma) = \frac{1}{\sin(\frac{\pi}{2} a_\sigma)} \left( X_0^\sigma \cos\left(\frac{\pi}{2} a_\sigma\right) + \dot{X}_0^\sigma \right), \tag{2.23} \]
\[ D_2^\sigma = C_1^\sigma \sin(C_2^\sigma) = \frac{X_0^\sigma}{\epsilon}. \tag{2.24} \]

\( ^2 \)Dot refers to derivative with respect to \( u \).
Moreover, for a timelike geodesic, i.e. \( P_\mu P^\mu = -m^2 \), one gets
\[
2\dot{V}_{in} \equiv 2\dot{V}(-\infty) = -\frac{1}{P_V}(m^2 + \sum_\sigma (P_{in}^\sigma)^2) < 0. \tag{2.25}
\]

Denoting \( V(0) \equiv V_0 \) one finds
\[
C_3 = \dot{V}_{in} - \frac{1}{4} \sum_\sigma \left[ ((D_1^\sigma)^2 - (D_2^\sigma)^2) \cos(\pi \alpha_\sigma) + 2D_1^\sigma D_2^\sigma \sin(\pi \alpha_\sigma) \right], \tag{2.26}
\]
and
\[
C_4 = V_0 + \frac{1}{2} \sum_\sigma \epsilon \alpha_\sigma D_1^\sigma D_2^\sigma, \tag{2.27}
\]
where \( D_1^\sigma \) and \( D_2^\sigma \) are given by (2.23) and (2.24). Thus we obtain the explicit form of the geodesics, in particular, the final velocities or momenta.

Due to eq. (2.12) and eqs. (2.23)-(2.26) one gets, after some troublesome computations, the geodesic for the metric with the Dirac delta profile
\[
\lim_{\epsilon \to 0} X^\sigma(u) = X_0^\sigma(1 + \frac{\sigma}{2} u \theta(u)) + \dot{X}_{in}^\sigma u, \tag{2.28}
\]
\[
\lim_{\epsilon \to 0} V(u) = C_4 + \dot{V}_{in} u - \frac{1}{4} \theta(u) \sum_\sigma \sigma (X_0^\sigma)^2 - \frac{1}{4} u \theta(u) \sum_\sigma \left( \frac{(X_0^\sigma)^2}{2} + 2\sigma X_0^\sigma \dot{X}_{in}^\sigma \right) - \frac{1}{2} \sum_\sigma X_0^\sigma \dot{X}_{in}^\sigma, \tag{2.29}
\]
where \( \theta(u) \) is the Heaviside step function. Now taking arbitrary \( u_0 < 0 \) one can find the value \( C_4 \) for the Dirac delta function, \( C_4 = V(u_0) - \dot{V}_{in} u_0 + \frac{1}{2} \sum_\sigma X_0^\sigma \dot{X}_{in}^\sigma \) and, consequently, the final form of the geodesics; they agree with the form presented in the literature, see e.g., \([14, 41]\); this confirms the conclusion that the impulsive limit is totally independent of the special form of the original profile \([30]\).

\section{2.3 Conformal symmetry for the Dirac delta profile}

As we have indicated above, the metric \( g \) with the profile (2.11) exhibits the conformal symmetry \([38, 39]\). The generators are strictly related to the decomposition of \( X^\sigma \) in terms of the initial conditions (cf. (2.14), (2.23) and (2.24))
\[
X^\sigma(u) = X_0^\sigma P_1^\sigma(u) + \dot{X}_{in}^\sigma P_2^\sigma(u). \tag{2.30}
\]
The Killing vectors are defined as follows
\[
\hat{V} = \partial_V, \\
\hat{D}_1^\sigma = P_1^\sigma(u) \partial_x - X^\sigma \dot{P}_1^\sigma(u) \partial_V, \tag{2.31} \\
\hat{D}_2^\sigma = P_2^\sigma(u) \partial_x - X^\sigma \dot{P}_2^\sigma(u) \partial_V. 
\]
The nonvanishing commutators are:

$$[\hat{D}_1^\sigma, \hat{D}_2^\sigma] = -\frac{a_\sigma}{\sin(\frac{\pi}{2} a_\sigma)} \hat{V}. \quad (2.32)$$

Next, there is a standard homothetic generator

$$\hat{H} = 2V \partial_V + X^+ \partial_{X^+} + X^- \partial_{X^-}, \quad (2.33)$$

and the proper conformal one

$$\hat{K} = u^2 \partial_u - \frac{1}{2} ((X^+)^2 + (X^-)^2) \partial_v + uX^+ \partial_{X^+} + uX^- \partial_{X^-} + \epsilon^2 \partial_u. \quad (2.34)$$

They satisfy the following commutation rules

$$[\hat{H}, \hat{V}] = -2\hat{V}, \quad [\hat{H}, \hat{D}_1^\sigma] = -D_1^\sigma, \quad [\hat{H}, \hat{D}_2^\sigma] = -\hat{D}_2^\sigma,$$

$$[\hat{K}, \hat{D}_2^\sigma] = \frac{a_\sigma \epsilon^2}{\sin(\frac{\pi}{2} a_\sigma)} \hat{D}_1^\sigma - \frac{\epsilon a_\sigma \cos(\frac{\pi}{2} a_\sigma)}{\sin(\frac{\pi}{2} a_\sigma)} \hat{D}_2^\sigma, \quad (2.35)$$

$$[\hat{K}, \hat{D}_1^\sigma] = \frac{\epsilon a_\sigma \cos(\frac{\pi}{2} a_\sigma)}{\sin(\frac{\pi}{2} a_\sigma)} \hat{D}_1^\sigma - \frac{a_\sigma}{\sin(\frac{\pi}{2} a_\sigma)} \hat{D}_2^\sigma.$$

Now, taking the limit \(\epsilon \rightarrow 0\), one obtains for the Dirac delta profile the following generators of the conformal algebra

$$\hat{V} = \partial_V, \quad \hat{H} = 2V \partial_V + X^+ \partial_{X^+} + X^- \partial_{X^-},$$

$$\hat{D}_2^\sigma = u \partial_\sigma - X^\sigma \partial_V,$$

$$\hat{D}_1^\sigma = \partial_\sigma + \frac{\sigma}{2} \theta(u) \hat{D}_2^\sigma,$$

$$\hat{K} = u^2 \partial_u - \frac{1}{2} ((X^+)^2 + (X^-)^2) \partial_v + uX^+ \partial_{X^+} + uX^- \partial_{X^-}, \quad (2.36)$$

satisfying commutation relations

$$[\hat{H}, \hat{V}] = -2\hat{V}, \quad [\hat{D}_1^\sigma, \hat{D}_2^\sigma] = -\hat{V}, \quad [\hat{H}, \hat{D}_1^\sigma] = -D_1^\sigma, \quad [\hat{H}, \hat{D}_2^\sigma] = -\hat{D}_2^\sigma,$$

$$[\hat{K}, \hat{D}_1^\sigma] = -\hat{D}_2^\sigma, \quad [\hat{K}, \hat{D}_2^\sigma] = -\hat{D}_1^\sigma. \quad (2.37)$$

due to \(u^2 \delta(u) = 0\) or taking \(\epsilon \rightarrow 0\) in (2.35). This yields the conformal algebra of the metric with the Dirac delta profile and generalizes to this case the results for the isometry algebra considered in Refs. [14, 42, 43].
2.4 The cross section and the change of energy

Having explicit form of geodesics one can compute the classical differential scattering cross section associated with the transverse-space scattering map in terms of the outgoing momentum components (see [41] for more details).

\[ d\sigma_{\text{classical}} = dX^+_{\text{in}} dX^-_{\text{in}} = |J| dP^+_{\text{out}} dP^-_{\text{out}}, \]

(2.38)

where \( J \) denotes the Jacobian of the transformation between \( P^\sigma_{\text{out}} \) and \( X^\sigma_{\text{in}} \). Indeed, for the profile (2.11) with \( \epsilon < \pi \), by virtue of (2.7) and (2.19) one gets

\[ P^\sigma_{\text{out}} = P_V X^\sigma_{\text{out}} = \frac{P_V \sin \pi a^\sigma_{\text{in}}}{\epsilon a^\sigma_{\text{in}}} X^\sigma_{\text{in}}, \]

(2.39)

thus

\[ |J| = \frac{-\epsilon^2 a_+ a_-}{P_V^2 \sin(\pi a_-) \sin(\pi a_+)} . \]

(2.40)

Taking the limit \( \epsilon \to 0 \) one obtains

\[ |J| = \frac{4}{P_V^2} , \]

(2.41)

and, consequently, the classical cross section for the Dirac delta profile.

In view of the above it is also tempting to compute the change of energy of a test particle after the pulse has passed. Below we present a superficial approach based on the global B coordinates; however, one should keep in mind that such considerations call for more physical clarification related to the choice of the inertial frame after and before the pulse, measurement problem and etc. To this end let us recall some basic relations between the light-cone and Minkowski approaches to the relativistic particle with constant four-velocity. In our case, i.e., \( A = 0 \), not only \( P_V \) is constant but also \( P^\sigma = P^\sigma = \dot{X}^\sigma P_V \) (cf. (2.5) and (2.7)). Thus, by virtue of (2.8), (2.9), (2.22) and (2.25), in this case the geodesic is of the form

\[ X^\sigma(u) = \frac{P^\sigma}{P_V} u + X^\sigma_{\text{in}} , \]

(2.42)

\[ V(u) = -\frac{m^2 + \sum \sigma P^\sigma P^\sigma}{2P_V^2} u + V_0 . \]

Introducing \( Z, T \) coordinates as follows

\[ u = \frac{Z - T}{\sqrt{2}} , \quad V = \frac{Z + T}{\sqrt{2}} , \]

(2.43)

one obtains the Minkowski metric with the signature \((-\,+,+,+)\) as well as

\[ T(u) = \frac{1}{\sqrt{2}} \left[ -(1 + \frac{m^2 + \sum \sigma P^\sigma P^\sigma}{2P_V^2}) u + V_0 \right] , \]

\[ Z(u) = \frac{1}{\sqrt{2}} \left[ (1 - \frac{m^2 + \sum \sigma P^\sigma P^\sigma}{2P_V^2}) u + V_0 \right] . \]

(2.44)
Let us stress that, due to our convention, eq. (2.43), time runs in the opposite direction to the $u$ coordinate; in consequence, $u = \pm \infty$ corresponds to $T = \mp \infty$. Moreover, we assume $P_V < 0$ in order to obtain the same direction for $\lambda$ and $T$, cf. (2.7). Denoting by

$$\gamma = \frac{-P_V}{\sqrt{2}} \left(1 + \frac{m^2 + \sum_{\sigma} P_{\sigma} P^{\sigma}}{2P_V^2}\right),$$

(2.45)

we have

$$P^{\sigma} = \gamma W^{\sigma}, \quad P_V = \frac{\gamma}{\sqrt{2}} (W^Z - 1),$$

(2.46)

where $W^{\sigma}, W^Z$ are velocities in the Cartesian coordinates. Then the energy of a test particle can be expressed as follows

$$E = \frac{m}{\sqrt{1 - \sum_{\sigma} W_{\sigma} W^{\sigma} - (W^Z)^2}} = \gamma = \frac{P_V}{\sqrt{2}} (\dot{V} - 1).$$

(2.47)

Now, we are in the position to analyse the change of energy after the wave has passed. Namely, due to (2.16)

$$\dot{V}_{out} = \dot{V}(u = \infty) = \dot{V}_{in} - \sum_{\sigma} D_1^{\sigma} D_2^{\sigma} \sin(\pi a_{\sigma}) = \dot{V}_{in} - 2 \sum_{\sigma} \frac{\cos(\frac{\pi}{2} a_{\sigma})}{\epsilon} X_{0}^{\sigma} \left( \frac{\cos(\frac{\pi}{2} a_{\sigma})}{\epsilon} X_{0}^{\sigma} + \dot{X}_{in}^{\sigma} \right).$$

(2.48)

Thus

$$\Delta E = E(u = \infty) - E(u = -\infty) = -\sqrt{2} P_V \sum_{\sigma} \frac{\cos(\frac{\pi}{2} a_{\sigma})}{\epsilon} X_{0}^{\sigma} \left( \frac{\cos(\frac{\pi}{2} a_{\sigma})}{\epsilon} X_{0}^{\sigma} + \dot{X}_{in}^{\sigma} \right),$$

(2.49)

or equivalently

$$\Delta E = -\sqrt{2} P_V \sum_{\sigma} \frac{\cos(\frac{\pi}{2} a_{\sigma})}{\epsilon} X_{0}^{\sigma} \left( \frac{\cos(\frac{\pi}{2} a_{\sigma})}{\epsilon} X_{0}^{\sigma} + \dot{X}_{out}^{\sigma} \right)$$

(2.50)

$$= -\frac{P_V}{\sqrt{2}} \sum_{\sigma} X_{0}^{\sigma} \dot{X}_{0}^{\sigma} \frac{\sin(\pi a_{\sigma})}{\epsilon a_{\sigma}}.$$  

(2.51)

In order to analyse this result let us note that, in our convention, $E(T = \infty) - E(T = -\infty) = -\Delta E$ and $\dot{X}_{out}^{\sigma} = \dot{X}^{\sigma}(u = \infty) = 0$ is equivalent to $\dot{X}^{\sigma}(T = -\infty) = 0$; thus assuming vanishing transverse-space velocities in the past infinity, in particular the particle at rest, we obtain that the energy does not decrease after the wave has passed

$$E(T = \infty) - E(T = -\infty) = -\Delta E = -\sqrt{2} P_V \sum_{\sigma} \frac{\cos^2(\frac{\pi}{2} a_{\sigma})}{\epsilon^2} (X_{0}^{\sigma})^2 \geq 0.$$  

(2.52)

In the general case, the final energy is less, greater or equal to the initial energy, depending on the relations between initial positions and velocities (one can find the suitable conditions), cf. results in Ref. [44]. However, as we mentioned above, further clarifications are necessary.
Finally, taking the limit $\epsilon \to 0$ in eq. (2.49) one obtains the change of energy in the case of the Dirac delta profile:

$$\triangle E = -\frac{P_V}{2\sqrt{2}} \sum_{\sigma} \left( \frac{(X_0^{\sigma})^2}{4} + \sigma X_0^{\sigma} \dot{X}_{in}^{\sigma} \right).$$  (2.53)

3 Plane waves in the Baldwin-Jeffery-Rosen coordinates

3.1 General discussion

In order to analyse the interaction or scattering one should specify the notion of the pulse of gravitational waves. First, it seems that the pulse cannot be defined only by the vanishing of the wavefront in the past (future) infinity since the asymptotic behaviour of geodesic might be, for example, oscillatory. Thus our starting point is the linear behaviour of geodesics near the past (future) infinity. Let us consider the first set of geodesic equations (2.8). It turns out that by means of classical results on differential equations [45], the condition $u^2 A(u) \in L^1(\mathbb{R})$ is sufficient to ensure the linear behaviour of the geodesic, $X^{\sigma}(u) \simeq a^\sigma u + b^\sigma$ for large $|u|$. More intuitively, every integral curve has a unique slant asymptote at $\pm \infty$, distinct integral curves having distinct asymptotes, and every slant straight line is the asymptote of a unique integral curve (moreover, under the assumption that $A$ is of constant sign for large $|u|$ this condition is also necessary). In consequence, we have

$$\lim_{u \to \pm \infty} u^2 A(u) = 0.$$  (3.1)

Now, let us note that the differential equation for $V$ (2.9) can be integrated to the form

$$\dot{V}(u) = -\frac{1}{4} \sum_{\sigma} \sigma (X^{\sigma}(u))^2 A(u) - \frac{1}{2} \sum_{\sigma} (\dot{X}_{\sigma}(u))^2 + \text{const},$$  (3.2)

which, by virtue of eq. (3.1), implies the linear behaviour of $V$ for large $|u|$.

So far we discussed gravitational plane waves in the B coordinates where both the wave and the geodesic are global with no singularity; B coordinates cover the whole plane wave spacetime by a single chart. However, the gravitational plane waves are frequently discussed in the, so called, Baldwin-Jeffery-Rosen (BJR) coordinates for which

$$g = \sum_{\sigma, \sigma'} a_{\sigma\sigma'}(u) dx^{\sigma} dx^{\sigma'} + 2dudv,$$  (3.3)

where $a(u) = (a_{\sigma\sigma'}(u))$ is a positive matrix, see e.g., [2, 3, 12, 13]. The BJR coordinates, in contrast to the B ones, are not harmonic and typically not global, exhibiting $u$ coordinate

\textsuperscript{3}Very often one can come across slightly weaker integrability condition for $uA(u)$, instead of $u^2A(u)$, however, it implies only a finite limit $\frac{X^{\sigma}(u)}{u}$ as $|u|$ tends to the infinity.
singularities. This fact is reflected in transformation between both coordinates. Namely, only a piece of the B manifold can be covered by the BJR coordinates and consequently at least two BJR maps are needed to completely describe the interaction (scattering) of particle by gravitational plane waves. The definition of the BJR coordinates is related to the geodesic equations (2.8) and consequently is not unambiguous. Thus we impose some physical conditions to specify them. Namely, we will require that the B and BJR coordinates coincide in the past and future infinity since there is no gravitational wave.

The next problem is the minimal number of charts to cover the whole B manifold. We will give criteria such that there exist two, say "in" and "out", BJR charts which cover the whole B manifold and we will express all information concerning interaction in terms of them.

To do this let us note that the linear asymptotic behaviour implies that there are solutions $P^\sigma_{\text{out}}(u)$ such that

$$\lim_{u \to -\infty} P^\sigma_{\text{out}}(u) = 1;$$

(3.4)

moreover, due to Theorem 2 in Ref. [45], we have the following estimate

$$|P^\sigma_{\text{out}}(u) - 1| \leq \exp(G(u)) - 1, \quad \text{for } u > 0,$$

(3.5)

where

$$G(u) \equiv \frac{1}{2} \int_{u}^{\infty} \tilde{u} |\sigma A(\tilde{u})| d\tilde{u} = \frac{1}{2} \int_{u}^{\infty} \tilde{u} |A(\tilde{u})| d\tilde{u},$$

(3.6)

which gives $P^\sigma_{\text{out}}(u) > 0$ for $u \geq 0$ provided that

$$G(u) < \ln(2), \quad \text{for } u \geq 0.$$  

(3.7)

Since $G(u) \leq G(0)$ for $u \geq 0$ thus $P^\sigma_{\text{out}}(u) > 0$ for $u \geq 0$ if the inequality

$$\int_{0}^{\infty} u |A(u)| du < 2 \ln(2),$$

(3.8)

holds. Thus (3.8) is a sufficient (but not necessary) condition for $P^\sigma_{\text{out}}(u) > 0$ for $u \geq 0$. On the other hand, applying standard reasoning (see, e.g., [42, 41]) to the solutions $P^+_\text{out}, P^-\text{out}$ one can find $u_0 > 0$ such that $(P^\sigma_{\text{out}}\text{out})(u)$ vanishes at $-u_0 < (4)$.

To simplify our further considerations let us assume that the profile $A$ is an even function, $A(-u) = A(u)$. Then the functions $P^\sigma_{\text{in}}(u) \equiv P^\sigma_{\text{out}}(-u)$ are also solutions of (2.8) and satisfy the following conditions

$$\lim_{u \to -\infty} P^\sigma_{\text{in}}(u) = 1, \quad P^\sigma_{\text{in}}(0) = P^\sigma_{\text{out}}(0), \quad (P^\sigma_{\text{in}}P^-_{\text{in}})(u_0) = 0.$$  

(3.9)

Let us now analyse the linear independence of these solutions. As we showed above, at least one of the functions $P^\sigma_{\text{out}}$ vanishes somewhere, say $P^-_{\text{out}}(-u_0) = 0$; then $P^-_{\text{in}}$ and $P^-_{\text{out}}$ are

\[4\]Some aspects of the behaviour of $P^\sigma$ appear also in the study of caustic and focusing properties of plane waves, see e.g., [11, 66, 17, 48].
linearly independent. Now, if $P^+_{\text{out}}(-u'_0) = 0$ for $u'_0 > 0$ then $P^+_{\text{in}}$ and $P^+_{\text{out}}$ are also linearly independent (and we redefine $u_0$ by $\min(u_0, u'_0)$ in further considerations). If $P^+_{\text{out}} > 0$ on the whole real line and $\lim_{u \to -\infty} P^+_{\text{out}}(u) = 1$ then there exists, in one direction, a globally defined map ($x^+_{\text{in}} = x^+_{\text{out}}$ in further considerations); in the other case $P^+_{\text{out}}$ and $P^+_{\text{in}}$ are linearly independent. Thus without loss of generality we may assume that the Wronskian
\[
W^\sigma(u) \equiv \dot{P}^\sigma_{\text{in}}(u)P^\sigma_{\text{out}}(u) - \dot{P}^\sigma_{\text{out}}(u)P^\sigma_{\text{in}}(u) \equiv W^\sigma = \text{const},
\]
is not zero.

In view of the above the condition (3.8) ensures that two BJR charts are sufficient (when the profile in an even function). Namely, we introduce the BJR coordinate $(u, x^\sigma_{\text{in}}, v^\sigma_{\text{in}})$ for $u < u_0$
\[
X^\sigma = P^\sigma_{\text{in}}(u)x^\sigma_{\text{in}},
\]
\[
V = v^\sigma_{\text{in}} - \frac{1}{4} \sum_{\sigma}(x^\sigma_{\text{in}})\dot{a}^\sigma_{\text{in}}(u),
\]
where $a^\sigma_{\text{in}}(u) = (P^\sigma_{\text{in}}(u))^2$. Then the metric takes the form
\[
g_{\text{in}} = 2dudv^\sigma_{\text{in}} + \sum_{\sigma}a^\sigma_{\text{in}}(dx^\sigma_{\text{in}})^2.
\]
Similarly, by means of the functions $P^\sigma_{\text{out}}$, we introduce the BJR coordinates $(u, x^\sigma_{\text{out}}, v^\sigma_{\text{out}})$ in the region $u > -u_0$ leading to the metric $g_{\text{out}}$ in this region. Let us stress that the metrics (coordinates) coincide with the B ones in the past (future) infinity. In the common domain $(-u_0, u_0)$ we have the transformation rules
\[
x^\sigma_{\text{out}} = x^\sigma_{\text{in}} \frac{P^\sigma_{\text{out}}}{P^\sigma_{\text{in}}},
\]
\[
v^\sigma_{\text{out}} = v^\sigma_{\text{in}} - \sum_{\sigma} \frac{W^\sigma}{2P^\sigma_{\text{out}}}(x^\sigma_{\text{in}})^2,
\]
which transform the metric $g_{\text{out}}$ into $g_{\text{in}}$.

Now let us analyse the behaviour of the geodesics in BJR coordinates in both charts. First, let us recall that the geodesics, for $u < u_0$, are of the form, see e.g. (3.16)
\[
x^\sigma_{\text{in}}(u) = b^\sigma_{\text{in}} H^\sigma_{\text{in}}(u) + c^\sigma_{\text{in}},
\]
\[
v^\sigma_{\text{in}}(u) = -\frac{1}{2} \sum_{\sigma}(b^\sigma_{\text{in}})^2 H^\sigma_{\text{in}}(u) + e^\sigma_{\text{in}}u + d^\sigma_{\text{in}},
\]
where $b^\sigma_{\text{in}}, c^\sigma_{\text{in}}, e_{\text{in}}, d_{\text{in}}$ are some constants and
\[
H^\sigma_{\text{in}}(u) = \int_0^u \frac{1}{(P^\sigma_{\text{in}}(u))^2}d\tilde{u}, \quad u < u_0, \quad H^\sigma_{\text{in}}(0) = 0.
\]
Similarly, in the region $u > -u_0$ we have geodesics $x^\sigma_{\text{out}}(u), v_{\text{out}}(u)$ with some initial parameters $b^\sigma_{\text{out}}, c^\sigma_{\text{out}}, d_{\text{out}}, e_{\text{out}}$.

Let us find the relations between "in" and "out" initial conditions and compare them with the ones obtained in the B coordinates. First, substituting $x^\sigma_{\text{out}}(u), x^\sigma_{\text{in}}(u)$ into (3.14) one obtains after some computations

\[ c^\sigma_{\text{in}} = c^\sigma_{\text{out}}, \quad b^\sigma_{\text{out}} = c^\sigma_{\text{in}} \mathcal{W}^\sigma + b^\sigma_{\text{in}}. \quad (3.18) \]

Next, substituting $v_{\text{out}}(u), v_{\text{in}}(u)$ into (3.15), and using the identity $\mathcal{W}^\sigma H^\sigma_{\text{in}} = 1 - P_{\text{out}}^\sigma P_{\text{in}}^\sigma$ one gets

\[ e_{\text{in}} = e_{\text{out}}, \quad d_{\text{out}} = d_{\text{in}} - \frac{1}{2} \sum_\sigma \mathcal{W}^\sigma (c^\sigma_{\text{in}})^2. \quad (3.19) \]

Thus we expressed all parameters of geodesics after the wave has passed in terms of the initial ones and, consequently, the scattering process in terms of BJR coordinates.

Now we compare these results with the ones obtained in B coordinates. Intuitively, since the B and BJR coordinates coincide for $u \to \pm \infty$ and $c^\sigma_{\text{in}} = x^\sigma_{\text{in}}(0)$ one gets by virtue of eqs. (3.18)

\[ \dot{X}^\sigma_{\text{out}} = \dot{X}^\sigma_{\text{in}} = \dot{X}^\sigma(-\infty) + X^\sigma(0) \frac{\mathcal{W}^\sigma}{P_{\text{in}}^\sigma(0)} = \dot{X}^\sigma_{\text{in}} + X^\sigma(0) \frac{\mathcal{W}^\sigma}{P_{\text{in}}^\sigma(0)}. \quad (3.20) \]

Of course, this formula can be directly confirmed in B coordinates. Indeed, due to (3.1) one obtains

\[ \lim_{u \to \infty} \dot{P}^\sigma_{\text{out}}(u) P_{\text{in}}^\sigma(u) = -\lim_{u \to \infty} \frac{\dot{P}^\sigma_{\text{out}}(u) (P_{\text{in}}^\sigma(u))^2}{P_{\text{in}}^\sigma(u)} = -\frac{\sigma}{2} \lim_{u \to \infty} \mathcal{A}(u) (P_{\text{in}}^\sigma(u))^2 \frac{P_{\text{out}}^\sigma(u)}{P_{\text{in}}^\sigma(u)} = 0, \quad (3.21) \]

(if $P_{\text{in}}^\sigma$ tends to the infinity we use L’Hospital’s rule, otherwise the above limit is immediately zero) thus, by virtue of (3.4), one obtains

\[ \mathcal{W}^\sigma = \lim_{u \to \infty} W^\sigma(u) = \lim_{u \to \infty} \dot{P}^\sigma_{\text{in}}(u) P_{\text{out}}^\sigma(u) = \dot{P}^\sigma_{\text{in}}(\infty). \quad (3.22) \]

On the other hand, standard computations yield

\[ \dot{X}^\sigma(\infty) = \dot{X}^\sigma(-\infty) + X^\sigma(0) \frac{\dot{P}^\sigma_{\text{in}}(\infty)}{P_{\text{in}}^\sigma(0)}. \quad (3.23) \]

In consequence, we get the formula (3.20).

Similarly, the $V$ coordinate and $v_{\text{in}}, v_{\text{out}}$ coincide in past (future) infinity; thus, cf. (3.16) and (3.17), we have

\[ \dot{V}(-\infty) = \dot{v}_{\text{in}}(-\infty) = -\frac{1}{2} \sum_\sigma (b^\sigma_{\text{in}})^2 + e_{\text{in}}, \quad \dot{V}(\infty) = \dot{v}_{\text{out}}(\infty) = -\frac{1}{2} \sum_\sigma (b^\sigma_{\text{out}})^2 + e_{\text{out}}. \quad (3.24) \]
Taking into account $\epsilon_{\text{in}} = \epsilon_{\text{out}}$ and (3.18) we get the jump of the $V$ velocity in terms of $P_{\text{in}}^\sigma$ and $P_{\text{out}}^\sigma$

$$
\dot{V}(\infty) = \dot{V}(\infty) - \sum_{\sigma} \frac{\mathcal{W}\sigma}{P_{\text{in}}^\sigma(0)} X'^\sigma(0) \dot{X}'(\infty) - \frac{1}{2} \sum_{\sigma} (\mathcal{W}\sigma)^2 \frac{(X'^\sigma(0))^2}{(P_{\text{in}}^\sigma(0))^2},
$$

(3.25)

which can be directly confirmed in terms of B coordinates (see (3.22) and (3.20)).

### 3.2 The conformal case and the Dirac delta profile

Let us now apply the above considerations to the profile defined by eq. (2.11) with $\epsilon < \pi$. In this case

$$
\int_0^\infty u |A(u)| du = \epsilon \frac{\pi}{\pi} < 2 \ln 2,
$$

(3.26)

thus the sufficient condition (3.3) is satisfied and we need only two BJR charts. Explicitly, they are defined as follows

$$
P_{\text{in}}^\sigma(u) = \frac{\sqrt{u^2 + \epsilon^2}}{\epsilon a_\sigma} \sin \left( a_\sigma \left( \arctan \left( \frac{u}{\epsilon} \right) + \frac{\pi}{2} \right) \right),
$$

$$
P_{\text{out}}^\sigma(u) = -\frac{\sqrt{u^2 + \epsilon^2}}{\epsilon a_\sigma} \sin \left( a_\sigma \left( \arctan \left( \frac{u}{\epsilon} \right) - \frac{\pi}{2} \right) \right),
$$

(3.27)

and satisfy the desired conditions, i.e., they tend to 1 as $u \to \pm\infty$. Moreover, $P_{\text{in}}^+(u) > 0$ for $u \in \mathbb{R}$ and $P_{\text{in}}^-$ vanishes only at one point $u_0$

$$
u_0 = -\epsilon \cot \left( \frac{\pi}{a_{-}} \right) > 0;
$$

(3.28)

also

$$
\mathcal{W}\sigma = \frac{\sin(\pi a_\sigma)}{\epsilon a_\sigma} \neq 0.
$$

(3.29)

In consequence, we can define two maps, which overlap for $u \in (-u_0, u_0)$, and find the explicit form of geodesics in terms of the BJR coordinates; in fact,

$$
H_{\text{in}}^\sigma(u) = \epsilon a_\sigma \left[ - \cot \left( a_\sigma \left( \arctan \left( \frac{u}{\epsilon} \right) + \frac{\pi}{2} \right) \right) + \cot \left( a_\sigma \frac{\pi}{2} \right) \right],
$$

(3.30)

and $H_{\text{out}}^\sigma(u) = -H_{\text{in}}^\sigma(-u)$ (see eq. (3.16)). Now, using the relations (3.20) and (3.25) one can confirm the results obtained in Section 2.

In this context the case of the Dirac delta profile seems to be especially interesting. Using the above results, in the limit $\epsilon \to 0$, one obtains $u_0 = 2$ and $\mathcal{W}\sigma = \frac{\sigma^2}{2}$ together with

$$
H_{\text{out}}^\sigma(u) = \frac{u}{1 - u \theta(-u) \frac{\sigma}{2}}, \quad u > -2; \quad H_{\text{in}}^\sigma(u) = \frac{u}{1 + \frac{\sigma}{2} u \theta(u)}, \quad u < 2,
$$

(3.31)

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and, consequently, the geodesic for the Dirac delta profile (see also [14, 41]). For example, if \( \dot{x}_\text{in}(\infty) = 0 \) then \( x_\text{in}^\sigma(u) = c_\text{in}^\sigma = x_\text{in}^\sigma(0) = x_\text{out}^\sigma(0) \) for \( u < 2 \). On the other hand, due to (3.18), for \( u > -2 \)

\[
x_\text{out}^\sigma(u) = x_\text{out}^\sigma(0) + \frac{\sigma x_\text{out}^\sigma(0)}{2} H_\text{out}^\sigma(u) = \frac{1 + \frac{\sigma}{2} u \theta(u)}{1 - \frac{\sigma}{2} u \theta(-u)} x_\text{out}^\sigma(0),
\]

and thus

\[
\dot{x}_\text{in}(0) = 0, \quad x_\text{out}(0) = \frac{\sigma x_\text{out}^\sigma(0)}{2} = \frac{\sigma X^\sigma(0)}{2};
\]

so the jump at zero the velocity \( \dot{X}^\sigma(u) \) in the B coordinates, see (2.28), is encoded in two maps in BJR coordinates; the same situation holds for the whole trajectory.

Finally, let us briefly consider the case of large \( \epsilon \). As we mentioned in Section 2, in this case the form of the solution \( X^-\)(u) is also valid. However, for sufficiently large \( \epsilon \) (i.e., \( \epsilon \geq 3\pi \)), by virtue of (3.28), \( u_0 \leq 0 \). In consequence, we cannot cover the whole \( B \) manifold by two, “in” and “out”, BJR maps. Moreover, for \( \epsilon = (n^2 - 1)\pi \), \( n = 2, 3, \ldots \) one obtains \( a_- = n; \) consequently, \( X^- \), defined by (2.19), is a quotient of the polynomial by a function and possesses \( n - 1 \) zeros, complicating in this way the problem of BJR charts. What is more, by virtue of (2.39), \( X^-_\text{in} = 0 \). Thus, there is no velocity memory effect in the \( X^- \) direction; however, there is also no permanent displacement. Namely taking \( \epsilon = (n^2 - 1)\pi \) in (2.19) one checks that

\[
X^-_\text{in} = \lim_{u \to \infty} X^-\(u\) = (-1)^{n-1} X^-_\text{in},
\]

thus \( |X^-_\text{out} - X^-_\text{out2}| = |X^-_\text{in1} - X^-_\text{in2}| \). For example, in the simplest case \( \epsilon = 3\pi \), i.e., \( n = 2 \) one gets

\[
X^-\(u\) = \frac{-u}{u^2 + 9\pi^2} X^-_\text{in},
\]

and \( X^-_\text{out} = -X^-_\text{in} \).

4 Summary and outlook

We discussed analytically some elements of the interaction of classical particles with a pulse of gravitational plane waves. The key point is the conformal symmetry of a certain class of plane waves metrics. In particular, we confirm, directly by taking an appropriate limit, some results for impulsive gravitational waves. Furthermore, we gave certain conditions in order to describe complete interaction (scattering) in terms of BJR coordinates and presented an explicit illustration of such situation.

These results may provide a starting point for further considerations. Let us point out a few of them.

i) In the context of some optical effects in nonlinear gravitational plane waves [10, 49] one can consider massless particles following null geodesics of the metric (2.3) with the profile (2.10).
ii) The generalization to case of polarized gravitational plane waves is also possible. To this
dend let us note that the metric \((2.1)\) with non-zero \(A_x\) also exhibits the conformal symmetry
\([38, 39]\). Thus it would be interesting to analyse some of the recently obtained results \([15, 50]\)
in this case.

iii) It seems that the problem of interaction of a quantum particle with exact gravitational
plane waves (see \([11]\)), especially the quantum cross sections, should be directly computable
in the case of the profile \((2.10)\).

iv) The isometry group of gravitational plane waves can be identified with the so called the
Carrol group (an ultrarelativistic group in 2+1 dimensions) without rotations \([51]\). On the
other hand, the conformal extensions of the Carrol group were classified \([52]\) (some of them
can be identified \([53]\) with the asymptotic symmetries in general relativity). Thus there is a
question concerning the relations between conformal algebra discussed in this paper and these
conformal extensions.

v) There are a number of papers concerning the collision of gravitational plane waves (see
\([54]\) and references therein) in particular the ones with the Dirac delta profile \([55]\). It would
be instructive to describe such a situation by means of the discussed metric, especially in the
case of Dirac delta profile.

vi) One of the main motivations of this paper is an attempt to give a better insight into some
problems occurring in the infrared structure of gravity \([56]\).

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