Upward radiation condition for wave in essentially stratified medium

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Abstract. Waves from a point source in a homogeneous medium will propagate spherically. This can be described by the Sommerfeld radiation condition. However, it is hard to use simple formula to describe radiation condition for wave propagation in a stratified medium. A generalized Sommerfeld radiation was proposed by Xu in 1990, applying contour integral and asymptotic approximation. The study in this area becomes more active recently. Papers in 2008 and 2009 from different aspects presented rigorous analysis for results similar to Xu’s condition. On the other hand, Chandler-Wolde, Bo Zhang and their coworkers introduced the upward propagating radiation condition.

In this paper we introduce an upward propagating radiation condition for waves in essentially stratified medium with unbounded inhomogeneity or unbounded rough boundary.

The upward propagating radiation conditions can be used to formulate direct and inverse scattering problems in nonlocal perturbed medium.

1. Introduction

Waves from a point source in a homogeneous medium will propagate spherically. This can be described by the Sommerfeld radiation condition. However, it is hard to use simple formula to describe radiation conditions for wave propagation in a stratified medium. A generalized Sommerfeld radiation was proposed by Xu in 1990, applying contour integral and asymptotic approximation. In [13], Xu studied the scattering of a harmonic acoustic wave by an obstacle in a stratified medium. In a stratified medium, sound waves may be trapped by acoustic ducts and caused to propagate horizontally. (See [3], [12], [11]) Therefore, the waves scattered by a compact obstacle or local inhomogeneity in a stratified medium may not satisfy the well-known Sommerfeld radiation condition which is the fundamental assumption of scattering theory in a homogeneous medium. In [13][14], a new radiation condition, which is a generalization of the Sommerfeld radiation condition, was found based on the analysis of the behavior of point source acoustic wave in a stratified medium. Using the generalized Sommerfeld condition, we proved the uniqueness and existence of some exterior problems.

The study in this area becomes more active recently. Recently some papers from different aspects presented rigorous analysis for results similar to Xu’s condition. In [7], [8], Ciraolo and Magnanini presented a similar radiation condition based on the same wave splitting using a weaker form of Sommerfeld type radiation conditions for the various components. In [9], Bonnet-Ben Dhia, Dakhia, Hazard, and Chorfi presented a modal radiation condition using a generalized Fourier transform. The generalized Fourier transform appears as the operator of decomposition on the family of eigenfunctions and generalized eigenfunctions, which diagonalizes the transverse
part of the unperturbed Helmholtz operator. Based on their result, a further asymptotic analysis for continuous spectrum and discrete spectrum should lead to a similar generalized Sommerfeld radiation condition presented in [13] [14].

Nonlocal rough surface or rough medium scattering problems are seen in many applications, including acoustic and electromagnetic wave propagation over outdoor ground and sea surfaces, and have been studied in many papers. It is known that the Sommerfeld radiation condition is not appropriate for such problems. A mathematically rigorous, approach is the so-called upward propagating radiation condition (UPRC) introduced by Chandler-Wilde [4], and further studied by Chandler-Wilde and Zhang [5][6][20].

More references on radiation conditions may be found, for examples, in [1] [2] [7] [8] [9].

In this paper, we study the scattering of time-harmonic acoustic waves in an essentially stratified medium with nonlocal rough surface or nonlocal inhomogeneity. (We will clarify the meaning of essentially stratified medium in the later discussion.) We introduce an upward propagating radiation condition for waves in essentially stratified medium with unbounded inhomogeneity or unbounded rough boundary. This condition plays an important role for formulating direct and inverse scattering problems for unbounded object in a stratified medium.

2. Scattering problem in an essentially stratified medium with nonlocal rough surface or nonlocal inhomogeneity

We study the scattering problems in a nonlocally perturbed layered half-space. Let \( R_2^+ = \{ x \in R^2 | x_2 \geq 0 \} \), where \( x = (x_1, x_2) \).

\[
n_0(x_2) = \begin{cases} n_0, & \text{for } 0 < x_2 < h, \\ 1, & \text{for } h < x_2 < \infty, \end{cases}
\]

where \( n_0 > 1 \) is a constant. We assume that the inhomogeneity is contained in a slab domain \( R_d^2 = \{ x \in R^2 | 0 < x_2 < d < \infty \} \).

The nonlocal perturbed medium scattering problem (Problem M) is defined as following: given incident wave \( u^i \) satisfying

\[
\Delta u^i + k^2 n_0^2(x_2)u^i = f(x), \text{ in } R_2^+,
\]

find the total field \( u = u^s + u^i \) such that

\[
\Delta u^s + k^2 n_0^2(x)u^s = 0, \text{ in } R_2^+,
\]

\[
u(x) = u^s(x) + u^i(x) = 0, \text{ when } x_2 = 0,
\]

The nonlocal rough surface scattering problem (Problem S) is defined as following: given incident wave \( u^i \) satisfying

\[
\Delta u^i + k^2 n_0^2(x_2)u^i = f(x), \text{ in } R_2^+,
\]

find the total field \( u = u^s + u^i \) such that

\[
\Delta u^s + k^2 n_0^2(x_2)u^s = 0, \text{ in } R^2_h,
\]

\[
u(x) = u^s(x) + u^i(x) = 0, \text{ on } \Gamma = \{ (x_1, x_2) | x_2 = h(x_1), -\infty < x_1 < \infty \},
\]

where \( h \in C^1(R) \).

Here \( \Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \), \( u^i, u^s \) and \( u \) are the incident, scattered and total field of the time harmonic waves. \( f \) is a function with compact support.
We need to assign a suitable radiation condition. If the inhomogeneity is in a bounded region, a
generalized Sommerfeld radiation condition can be applied [10]. However, if the inhomogeneity
is unbounded, a kind of upward propagating radiation condition may be more appropriate.

First we recall the Green’s function in stratified medium and its far-field behavior. We outline
the construction of the Green’s function for the time-harmonic waves in a layered half-space and
obtain its free-wave and guided-wave far-field behavior. More details can be seen in [10], also in
[13][14]. A function \( G(\cdot; x_1^0, x_2^0) \) is the outgoing Green’s function from the source at \( x_0^0 = (x_1^0, x_2^0) \) for
the time-harmonic wave in a layered half-space, if \( G(x_1, x_2; x_1^0, x_2^0) \) satisfies

\[
\Delta G + k^2 n_0^2(x_2)G = -\delta(|x - x_0^0|), \text{ in } R_+^2
\]

in the generalized function sense;

\[
G(x_1, 0; x_1^0, x_2^0) = 0,
\]

and satisfies the out-going radiation condition.

Using the Fourier Transform, and considering the outgoing property in horizontal direction,
we obtain a representation of \( G(x_1, x_2; x_1^0, x_2^0) \)

\[
G(x_1, x_2; x_1^0, x_2^0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{G}(\xi, x_2; x_1^0, x_2^0) e^{i\xi|x_1-x_1^0|}d\xi,
\]

\( \tilde{G}(\xi, x_2; x_1^0, x_2^0) \) has representation

\[
\tilde{G}(\xi, x_2; x_1^0, x_2^0) = \frac{\phi^+(\xi, x_2, >)\phi^-(\xi, x_2, <)}{W(\phi^+, \phi^-)},
\]

where

\[
\phi^+(\xi, x_2) = \begin{cases} e^{ix_2}, & h < x_2 < \infty \\ B_1 e^{i\tau_0 x_2} + B_2 e^{-i\tau_0 x_2}, & 0 < x_2 < h, \end{cases}
\]

\[
\phi^-(\xi, x_2) = \begin{cases} \sin(\tau_0 x_2), & 0 < x_2 < h \\ C_1 \cos(\tau x_2) + C_2 \sin(\tau x_2), & h < x_2 < \infty, \end{cases}
\]

where

\[
B_1 = \frac{1}{2} \left(1 + \frac{\tau}{\tau_0}\right) e^{i(\tau-\tau_0)h}, \quad B_2 = \frac{1}{2} \left(1 - \frac{\tau}{\tau_0}\right) e^{i(\tau+\tau_0)h},
\]

\[
C_1 = \cos(\tau h)\sin(\tau_0 h) - \frac{\tau_0}{\tau} \sin(\tau h)\cos(\tau_0 h),
\]

\[
C_2 = \sin(\tau h)\sin(\tau_0 h) + \frac{\tau_0}{\tau} \cos(\tau h)\cos(\tau_0 h),
\]

\( \tau = \sqrt{k^2 - \xi^2} \) and \( \tau_0 = \sqrt{k^2 n_0^2 - \xi^2} \). \( W(\phi^+, \phi^-) \) is the Wronskian

\[
W(\phi^+, \phi^-) = e^{i\tau h} [\tau_0 \cos(\tau_0 h) - i\tau \sin(\tau_0 h)].
\]

We obtain from (10)and (11):

\[
G(x_1, x_2; x_1^0, x_2^0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\phi^+(\xi, x_2, >)\phi^-(\xi, x_2, <)}{W(\phi^+, \phi^-)} e^{i\xi|x_1-x_1^0|}d\xi.
\]

Setting the Wronskian be 0, we have \( \tan(\tau_0 h) = \frac{i\tau}{\tau_0} \), which has at most a finite number of
solutions for \( \xi^2 > k^2 \). The out-going radiation condition as \( |x_1| \to \infty \) implies that the positive
zeros should be used. Denote them by \( k < \xi_1 < \xi_2 < \cdots < \xi_N \). They are poles of the integrant.
There are branch points at $\pm k$ and $\pm kn_0$. Choosing branch cuts $\Gamma$ and $\Gamma_0$ [10] for $\tau = \sqrt{k^2 - \xi^2}$ and $\tau_0 = \sqrt{k^2n_0^2 - \xi^2}$, and denoting $\tau_n = \sqrt{k^2 - \xi_n^2}$ and $\tau_{0n} = \sqrt{k^2n_0^2 - \xi_n^2}$, we can represent the Green's function as:

$$G(x_1, x_2; x_1^0, x_2^0) = G_f(x_1, x_2; x_1^0, x_2^0) + G_g(x_1, x_2; x_1^0, x_2^0)$$

$$= \frac{1}{2\pi} \int_{\Gamma} e^{it\tau}[\tau_0 \cos(\tau_0h) - i\tau \sin(\tau_0h)] d\xi + \sum_{n=1}^{N} \frac{\phi^+(\xi_n, x_2^0)\phi^-(\xi_n, x_2^0)}{\xi_n - \xi} W(\phi^+, \phi^-)|_{\xi=\xi_n}.$$  (16)

For $x_2 > h$,

$$G_g(x_1, x_2; x_1^0, x_2^0) = \sum_{n=1}^{N} e^{-|\tau_n|x_2^0} e^{i\xi_n|x_1^0-x_2^0|} = O \left( e^{-|\tau_n|x_2^0} \right).$$  (17)

$G_g(x_1, x_2; x_1^0, x_2^0)$ corresponds to the guided wave that is trapped in the layer $0 < x_2 < h$.

$$G_f(x_1, x_2; x_1^0, x_2^0) = \frac{1}{2\pi} \int_{\Gamma} e^{i\xi|x_1-x_2^0|+irx_2^0} \sin(\tau_0x_2^0) d\xi$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\xi|x_1-x_2^0|+i\sqrt{\tau_0^2+k_2^2(n_0^2-1)x_2^0}} \Phi_f(\xi) d\xi = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\tau_f(\xi)} \Phi_f(\xi) d\xi.$$  (18)

Using the Method of Stationary Phase, we obtain an approximation of $G_f(x_1, x_2; x_1^0, x_2^0)$ for $x_2 > h$ and $r = |x|$ large:

$$G_f(x_1, x_2; x_1^0, x_2^0) = \frac{e^{ikr}}{\sqrt{2\pi i\tau}} \left[ G^\infty(\theta; x_1^0, x_2^0) \right] + O \left( \frac{1}{r^{3/2}} \right)$$  (19)

$G_f(x_1, x_2; x_1^0, x_2^0)$ corresponds to the free-wave that propagates to the half plane. $G^\infty(\theta; x_1^0, x_2^0)$ is the free-wave far-field pattern.

In summary, a point source wave in a layered medium contains two components, the free-wave component decays at the rate of $O \left( \frac{1}{r^{3/2}} \right)$ in all upward angles, while the guided-wave component decays exponentially in the upward direction, but will not decay in the horizontal direction.

For related discussion of free-wave and guided-wave in stratified media, see [15, 16, 14, 17, 18, 19].

3. Upward propagating radiation condition in essentially stratified medium

Now we consider the scattering problems (M) and (S) in stratified medium. Let $\Phi(x, y)$ denote the free-space Green’s function for $\Delta + k_n^2$, that is, $\Phi(x, y) = \frac{i}{4} H_0^{(1)}(k_n|x - y|)$, for $x, y \in \mathbb{R}^2$, $x \neq y$, where $H_0^{(1)}$ is the Hankel function of the first kind of order zero. The upward radiation condition in a medium without guided wave is defined as:[20]

Given a domain $G \subset \mathbb{R}^2$, say that $v$ satisfies the UPRC in $G$ if, for some $d$,

$$v(x) = 2 \int_{\Gamma_d} \frac{\partial \Phi(x, y)}{\partial y_2} \phi(y) ds(y), \quad x \in U_d,$$  (20)

where $U_d = \{x | x_2 > d\}$, and $\Gamma_d = \{x | x_2 = d\}$.

In some of the papers of Chandler-Wilde and Zhang, conditions are given to prevent any guided wave. For example, in [20], the condition (A3) (absorbing medium), or conditions (A4) and (A5) (a counter-waveguide medium) are given explicitly.
Now we consider a perturbed Helmholtz equation that does not satisfy the aforementioned (A3) nor (A4) and (A5):

\[ \Delta G + k(x)G = \delta(x - x^0), \]

where

\[ k(x) = \begin{cases} \kappa_+ & \text{for } x \in E_R = \{-R < x_1 < R, 0 < x_2 < h\}, \\ \kappa_- & \text{otherwise}. \end{cases} \]

Then

\[ \Delta G + k_+ G = (k_+ - k(x))G + \delta(x - x^0) \]

has solution

\[ G(x, x_0) = \int_{E_R} \Phi(\xi, x) m(\xi) G(\xi, x_0) d\xi + \Phi(x^0, x), \]

where \( m(\xi) = k_+ - k(x) \). Recall

\[ \Phi(\xi, x) = O \left( \frac{1}{\sqrt{|\xi|}} \right) \], and \( G(\xi, x_0) = O \left( \frac{1}{\sqrt{|\xi|}} \right) \), as \( |\xi| \to \infty \),

\[ \int_{E_R} \Phi(\xi, x) m(\xi) G(\xi x_0) d\xi = \int_{E_R} \Phi(\xi, x) (k_+ - k_-) G(\xi, x_0) d\xi \]

\[ = (k_+ - k_-) \int_{-R}^R \int_0^h \Phi(\xi, x) G(\xi, x_0) d\xi_2 d\xi_1 = O(\log R). \]

\[ G(x, x_0) = \int_{E_R} \Phi(\xi, x) m(\xi) G(\xi, x_0) d\xi + \Phi(x^0, x) \]

diverges as \( R \to \infty \).

Actually, as discussed above, the Green’s function for layered medium \( G(x, y) \) has representation

\[ G(x, y) = G_f(x, y) + \sum_{l=1}^N G_l(x, y), \]

with

\[ G_l(x, y) = \frac{\sin(\xi_n, y_2)}{\xi_n e^{i \tau_n h} - k^2} e^{-\tau_n |x_2|} e^{i \xi_n |x_1 - y_1|} \]

which does not decay in the \( x_1 \) direction. This shows that the waves in stratified medium may not satisfy the UPRC (20). That is, a different upward propagating radiation condition is needed for stratified medium.

We introduce an Upward Propagating Radiation Condition for Essentially Stratified Medium (UPRCESM): the scattered wave in an essentially stratified medium satisfies the representation formula

\[ u(x) = \int_{\Gamma_d} \frac{\partial G(x, y)}{\partial y_2} \phi(y) ds(y), \quad x \in U_d, \]

where \( U_d = \{ x | x_2 > d > h \} \),

\[ G(x, y) = G_f(x, y) + \sum_{l=1}^N G_l(x, y), \]

with

\[ G_l(x, y) = \frac{\sin(\xi_n, y_2)}{\xi_n e^{i \tau_n h} - k^2} e^{-\tau_n |x_2|} e^{i \xi_n |x_1 - y_1|}. \]
The volume potential \( u \) with density \( \phi \in L_\infty(E_B) \), where \( E_B \) is a box or a slab containing the inhomogeneity, by

\[
u(x) = \int_{E_B} G(x, y) \phi(y) dy = \int_{E_B} G_f(x, y) \phi(y) dy + \sum_{l=1}^{N} \int_{E_B} G_l(x, y) \phi(y) dy
\]

satisfies the perturbed stratified Helmholtz equation, the related boundary equation, and the UPRCESM.

Since

\[
G_f = O \left( R^{-\frac{1}{2}} \right), \quad \frac{\partial G_f}{\partial \nu} - i k n \infty G_f = O \left( R^{-\frac{3}{2}} \right)
\]

uniformly as \( R \to +\infty \) on the set of \( C_R = \{ x | x = R, x_2 > 0 \} \), we have

\[
\int_{\partial \Omega} |G_f|^2 d\sigma = O(1), \quad \int_{\partial \Omega} \left| \frac{\partial G_f}{\partial \nu} - i k n \infty G_f \right|^2 d\sigma = O(R^{-1})
\]

\[
|G_f(x_1, x_2; x_1^0, x_2^0) - \frac{1}{2\pi} \log r| \leq C, \quad r = |(x_1, x_2) - (x_1^0, x_2^0)|.
\]

\[
|G_f(x_1, x_2; x_1^0, x_2^0) - \Phi(x_1, x_2; x_1^0, x_2^0)| \text{ is uniformly bounded.}
\]

\[
\int_{\partial \Omega} |G_l|^2 d\sigma = O(R), \quad l = 1, 2, ..., N, \quad \text{as } R \to +\infty,
\]

\[
\int_{\partial \Omega} \left| \frac{\partial G_l}{\partial \nu} - i k n \infty G_l \right|^2 d\sigma \text{ vanishes exponentially as } R \to +\infty.
\]

Therefore,

\[
u_f(x) = \int_{\Gamma_h} \frac{\partial G_f(x, y)}{\partial y_2} \phi(y) ds(y), \quad x \in U_d,
\]

has similar convergent properties as the UPRC in perturbed homogeneous medium studied by Chandler-Wilde, Zhang, et al. [5] [6] [20].

However,

\[
u_g(x) = \sum_{l=1}^{N} \int_{\Gamma_h} \frac{\partial G_l(x, y)}{\partial y_2} \phi(y) ds(y) = \sum_{l=1}^{N} \int_{\Gamma_h} \Psi_l(x_2, h) \phi(y_1, h) e^{i \xi_n |x_1 - y_1|} dy_1, \quad x \in U_d,
\]

behaves in the same way as a Fourier transform. Therefore, UPRCESM must be understood in the sense of distribution.

Alternatively, we can represent UPRCESM as

\[
u_f(x) = u(x) - \sum_{l=1}^{N} u_l(x) = \int_{\Gamma_h} \frac{\partial G_f(x, y)}{\partial y_2} \phi(y) ds(y), \quad x \in U_d,
\]

or

\[
u(x) = \sum_{l=1}^{N} u_l(x) + \int_{\Gamma_h} \frac{\partial G_f(x, y)}{\partial y_2} \phi(y) ds(y), \quad x \in U_d,
\]
where \( u_l(x) \) may be represented as
\[
  u_l(x) = \int_{U_l} G_l(x,y) \psi_l(y) dy, \quad \text{for } l = 1, 2, ..., N.
\]

Using the Green’s function for stratified medium, the scattered wave for a point source incident wave satisfies the integral equation
\[
  u(x) + \int_{\Omega} G(\xi, x) m(\xi) u(\xi) d\xi = G(x_0, x).
\]

(29)

Since \(|u(x)|\) is uniformly bounded, we have:

(1) \( u(x) \) satisfies radiation condition in stratified medium, if
\[
  \sup_{x \in \mathbb{R}^2_+} \int_{\Omega} |G(\xi, x) m(\xi)| d\xi \leq C < \infty. \tag{30}
\]

Now we can define the **essentially stratified medium**: A perturbed stratified medium is said essentially stratified, if the perturbation \( m(\xi) = k_+ - k(x) \) satisfies the condition (30).

(2) If
\[
  \sup_{x \in \mathbb{R}^2_+} \int_{\Omega} |G(\xi, x) m(\xi)| d\xi < 1, \tag{31}
\]
then the scattering problem has unique solution.

Following are some examples of essentially stratified media:

(a) There is a function \( x_2 = f(x_1) > 0 \) satisfying \( f(x_1) \leq C|x|^{-\delta} \) for \( \delta > 1 \) as \( |x| \to \infty \), such that the support of inhomogeneity \( \text{supp}(m) \subset \{x: 0 \leq x_2 \leq f(x_1)\} \).

(b) \( |m(x)| \leq C|x|^{-\delta} \) for \( \delta > 1 \) as \( |x| \to \infty \), and the support of inhomogeneity \( \text{supp}(m) \subset \{x: 0 \leq x_2 \leq H\} \).

Similarly we can study rough surface in essentially stratified medium:

Let \( \Gamma := \{x: x_2 = h(x_1) > 0, ||h||_1 \leq C\} \) be the sound-soft surface. If there exists \( x_2 = f(x_1) \) satisfying \( f(x_1) \leq C|x|^{-\delta} \) for \( \delta > 1 \) as \( |x| \to \infty \), such that \( 0 \leq h(x_1) \leq f(x_1) \) for \( x_1 \in \mathbb{R} \), then a solution of the Dirichlet problem \( u|\Gamma = g(x) \) has a representation
\[
  u(x) = \int_{\Gamma} G(x, y) \psi(y) dy, \quad x \in \{x \in \mathbb{R}^2_+ | x_2 > h(x_1)\},
\]
where \( |\psi| \leq C_2 \) is bounded and satisfies the boundary integral equation
\[
  \int_{\Gamma} G(x, y) \psi(y) dy = g(x), \quad x \in \Gamma.
\]

Since
\[
  |G(x, y)| \leq C_1 |x_2|, \quad \text{for } x_2 \text{ small}.
\]
\[
  \int_{\Gamma} |G(x, y) \psi(y) dy| \leq C_2 \int_{\Gamma} |G(x, y)| dy
\]
\[
  \leq C \int_{\Gamma} |y_2| dy = C \int_{-\infty}^{\infty} h(y_1) \sqrt{1 + |h'(x_1)|^2} dy_1 < \infty.
\]

We can use the UPRCESM to establish uniqueness of solution for rough surface scattering problems with a Dirichlet or a Robin type boundary condition on the scattering surface provided the medium is essentially stratified.
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