ON SOME 2D ORTHOGONAL $q$-POLYNOMIALS

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Abstract. We introduce two $q$-analogues of the 2D-Hermite polynomials which are functions of two complex variables. We derive explicit formulas, orthogonality relations, raising and lowering operator relations, generating functions, and Rodrigues formulas for both families. We also introduce a $q-2D$ analogue of the disk polynomials (Zernike polynomials) and derive similar formulas for them as well including evaluating certain connection coefficients. Some of the generating functions may be related to Rogers–Ramanujan type identities.

1. Introduction

The 2D-Hermite (or complex Hermite) polynomials\{ $H_{m,n}(z_1, z_2)$\}$_{m,n=0}^{\infty}$

$H_{m,n}(z_1, z_2) = \sum_{k=0}^{m\land n} (-1)^k k! \binom{n}{k} \binom{m}{k} z_1^{m-k} z_2^{n-k}$.

(1.1)

were introduced in [24]. Recently several mathematical physicists studied these polynomials from mathematical and physical points of view, [1], [22], [28], [29], [30], [31], [32], [33], [34], [35]. Their combinatorics were studied in [21], [22], and in [20]. Ismail [20] proved a Kibble-Slepian type multilinear generating function for these polynomials while the present authors gave a new proof together with a proof of the original Kibble-Slepian formula for Hermite polynomials in the forthcoming work [23]. Relevant references on the 2D-Hermite polynomials are [27], [28], [30], [31], and [32].

This work introduces two $q$-analogues of the 2D-Hermite polynomials denoted by \{ $H_{m,n}(z_1, z_2|q)$\}, and \{ $h_{m,n}(z_1, z_2|q)$\} and a $q-2D$ sequence of ultraspherical polynomials. The polynomials $H_{m,n}(z_1, z_2|q)$, and \{ $h_{m,n}(z_1, z_2|q)$\} transform to each other as $q \to 1/q$. We produce one orthogonality measure for the

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first family while we give infinity many orthogonality measures for the second family. An orthogonality measure is given for the $q-2D$ ultraspherical polynomials. We also find the raising and lowering operators for both families of $q-2D$ Hermite polynomials together with Sturm-Liouville equations which they satisfy.

In Section 2 we collect all the preliminary results used throughout the paper. Section 3 treats the polynomials $\{H_m,n(z_1, z_2|q)\}$ while Section 4 treats the second family $\{h_m,n(z_1, z_2|q)\}$. In Section 5 we first give the definition of a set or orthogonal polynomials in two variables, the disk polynomials [8 §2.3]. They are also known as Zernike polynomials [34]. The rest of Section 5 contains the definition and properties of the $q-2D$ ultraspherical polynomials denoted by $\{p_m,n(z;|q)\}$. These are $q$-analogues of the disk polynomials. They constitute a $q$-analog which is different from the one introduced by Floris in [10–11], see also [12]. Section 6 has several applications of the results obtained in the earlier sections including multilinear generating functions. In Section 7 we establish moment type representations for $\{H_m,n(z_1, z_2|q)\}$, and $\{h_m,n(z_1, z_2|q)\}$ and give closed form expressions for the connection coefficients in the expansion of $\{H_m,n(z_1, z_2|q)\}$, (respectively $\{h_m,n(z_1, z_2|q)\}$) in $\{h_m,n(z_1, z_2|q)\}$, (respectively $\{H_m,n(z_1, z_2|q)\}$). In addition we give a two dimensional $q$-analogue of the generating function [19 (4.6.29)].

\[
\sum_{n=0}^{\infty} H_{m+n}(x) t^n / n! = \exp(2xt - t^2)H_m(x - t).
\]

A formula that may have ramifications on the theory of partitions is formula [7,10]. In Section 8 we show that the zero sets of $\{H_{m,n}(z, \bar{z}|q)\}$, $\{h_{m,n}(z, \bar{z}|q)\}$ and $\{p_{m,n}(z, \bar{z}; b|q)\}$ are concentric circles in $\mathbb{C}$ centered at $z = 0$. We also show that the limiting distribution of the zeros of $\{H_{m,n}(z, \bar{z}|q)\}$ and $\{p_{m,n}(z, \bar{z}; b|q)\}$ coincide with the support of their measures of orthogonality. The polynomials $\{h_{m,n}(z, \bar{z}|q)\}$ are orthogonal on an bounded sets with respect to different measures. We describe their zero sets as $m, n \to \infty$. The asymptotics involves the zeros of the Ramanujan function to be defined in (2.14). This is similar to the one variable $q$-polynomials in [18]. In Section 9 we show that certain matrices whose entries are formed by $2D$-polynomials are positive definite.

This is the first part in a series of papers on the subject of $2D$ orthogonal polynomials where we study several new families of orthogonal polynomials.

### 2. Preliminaries

In this section we collect all the formulas used in the later sections and mention some of the notation. We shall follow the notation and terminology for special functions and $q$-series in [5, 14, 19, and 25]. We assume the reader is familiar with the notations of $q$-shifted factorials as well as the unilateral and bilateral basic hypergeometric functions $\phi_1$ and $\psi_1$. Moreover we use the notations

\[
\{x\} = \text{the fractional part of } x, \quad \lfloor x \rfloor = x - \{x\}
\]

\[
m \wedge n = \min \{m, n\}.
\]

The $q$-difference and dilatation operators are

\[
(D_q f)(z) = \frac{f(z) - f(qz)}{z - qz}, z \neq 0, \quad \text{and} \quad (\eta_q f)(x) = f(qx),
\]

respectively. If the dependence on $z$ is important we shall use $D_{q,z}$ and $\eta_{q,z}$ instead of $D_q$ and $\eta_q$, respectively. The Leibniz rule for $D_q$ is

\[
D_q^n(fg)(x) = \sum_{k=0}^{n} \binom{n}{k}_q D_q^k f(x) \eta_q^k D_{q^{-k}} g(x).
\]

The $q$-binomial theorem is [14 (II.3)]

\[
\sum_{n=0}^{\infty} \binom{a}{n}_q \frac{z^n}{(q; q)_n} = \frac{(az; q)_\infty}{(z; q)_\infty}, \quad |z| < 1.
\]
The terminating case is

\[ \sum_{k=0}^{n} \binom{n}{k} q(z)^{n-k} (-z)^{k} = (z; q)_n. \tag{2.6} \]

Two important special and limiting case are the Euler identities \[14, (II.1)-(II.2)]

\[ \sum_{n=0}^{\infty} \frac{z^n}{(q; q)_n} = \frac{1}{(z; q)_{\infty}}. \tag{2.7} \]

\[ \sum_{n=0}^{\infty} \frac{(-z)^n}{(q; q)_n} q(z) = (z; q)_\infty. \tag{2.8} \]

The \( q \)-integral is, \[14, \S 1.11\]

\[ \int_0^\infty f(x) dx = (1 - q) \sum_{n=-\infty}^{\infty} q^n f(q^n). \tag{2.9} \]

The \( q \)-Laguerre polynomials are \[25, 3.21.1]\]

\[ L_n^{(\alpha)}(x; q) = \frac{(q^{\alpha+1}; q)_n}{(q; q)_n} \phi_1 \left( \frac{q^{-n}}{q^{\alpha+1}}, q, q^{n+\alpha+1}, \frac{q^{-n} - q \cdot x}{q^{n+\alpha+1}} \right) \tag{2.10}. \]

Their moment problem is indeterminate, that is there are infinitely many orthogonality measures with respect to which the \( q \)-Laguerre polynomials are orthogonal. For a treatment of the \( q \)-Laguerre polynomials and the corresponding moment problem we refer the interested reader to \[19, \S 21.8\]. The little \( q \)-Laguerre, also known as Wall polynomials are defined by \[25, (3.20.1)\]

\[ p_n(x; a|q) = 2 \phi_1 \left( \frac{q^{-n}}{aq}, q, qx \right) = \frac{1}{(q^{-n}/a; q)_n} \phi_0 \left( \frac{q^{-n} 1/x}{q}, \frac{x}{a} \right). \tag{2.11} \]

See also \S 11 of Chapter VI in \[6\].

The \( q \)-Bessel functions \( J^{(2)} \) and \( I^{(2)} \) are defined by

\[ J^{(2)}_{\nu}(z; q) = \frac{(q^{\nu+1}; q)_\infty}{(q; q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+\nu)}}{(q, q^{\nu+1}; q)_n} \left( \frac{z}{2} \right)^{\nu+2n}, \tag{2.12} \]

\[ I^{(2)}_{\nu}(z; q) = \frac{(q^{\nu+1}; q)_\infty}{(q; q)_\infty} \sum_{n=0}^{\infty} \frac{q^{n(n+\nu)}}{(q, q^{\nu+1}; q)_n} \left( \frac{z}{2} \right)^{\nu+2n}, \tag{2.13} \]

respectively, \[19, 14\]. The Ramanujan function is \[19\]

\[ A_q(z) = \sum_{n=0}^{\infty} \frac{q^n}{(q; q)_n} (-z)^n. \tag{2.14} \]

It had has only positive zeros. It appeared in Ramanujan’s Lost Note Book \[20\] with some statements about the asymptotics of its zeros.

Garrett, Ismail, and Stanton \[13\] generalized the Rogers–Ramanujan identities to

\[ \sum_{n=0}^{\infty} q^{\frac{x^2 + m n}{2}} (q; q)_n = \frac{(-1)^m q^{-\frac{1}{2}} a_m(q)}{(q, q^2; q^5)_\infty} + \frac{(-1)^{m+1} a_m(q)}{(q, q^3; q^5)_\infty} \tag{2.15} \]

where

\[ a_m(q) = \sum_{j} q^{j^2} \left[ \frac{m - j - 2}{j} \right], \quad b_m(q) = \sum_{j} q^{j^2} \left[ \frac{m - j - 1}{j} \right]. \tag{2.16} \]

The polynomials \( a_m(q) \) and \( b_m(q) \) were considered by Schur in conjunction with his proof of the Rogers–Ramanujan identities, see \[12\] and \[13\] for details. We shall refer to \( a_m(q) \) and \( b_m(q) \) as the Schur polynomials.
Let $q = e^{-2k^2}$ and $|q| < 1$, the Ramanujan’s identities are

\begin{equation}
\int_{-\infty}^{\infty} e^{-x^2+2mx} \left( -aqe^{-2kx}, -bqe^{-2kx}; q \right) \infty dx = \frac{\sqrt{\pi} \left( abq; q \right) \infty e^{m^2}}{(ae^{2mk}/q, be^{-2mk}/q)^\infty},
\end{equation}

[13] Ex 6.15(i), and

\begin{equation}
\int_{-\infty}^{\infty} \frac{e^{-x^2+2mx} dx}{(ae^{2ikx}/\sqrt{q}, be^{-2ikx}/\sqrt{q})^\infty} = \frac{\sqrt{\pi} e^{m^2} \left( -aqe^{2ikx}, -bqe^{-2ikx}; q \right) \infty}{(abq; q)^\infty},
\end{equation}

[13] Ex 6.15(ii). For $0 < q < 1$, $\Re(a + c) > 0$ and $\Re(b - c) > 0$, Ramanujan extended the beta integral on $(0, \infty)$ to the following integrals,

\begin{equation}
\int_{0}^{\infty} \frac{(-tq^b, -q^{a+1}/t; q) \infty t^{c-1} dt}{(-t, -q/t; q) \infty (1 - q)} = \frac{(q, -q^c, -q^{1-c}, q^{a+b}; q) \infty}{(-1, -q, q^{a+c}, q^{b-c}; q) \infty},
\end{equation}

[14] Ex 6.17[i] and

\begin{equation}
\int_{0}^{\infty} \frac{(-tq^b, -q^{a+1}/t; q) \infty t^{c-1} dt}{(-t, -q/t; q) \infty} = \frac{\Gamma(c) \Gamma(1-c) \left( q^c, q^{1-c}, q^{a+b}; q \right) \infty}{(q, q^{a+c}, q^{b-c}; q) \infty}.
\end{equation}

[14] Ex 6.17(ii). Then,

\begin{equation}
\int_{0}^{\infty} \frac{t^{c-1} dt}{(-t, -q/t; q) \infty (1 - q)} = \frac{(q, -q^c, -q^{1-c}; q) \infty}{(-1, -q; q) \infty}
\end{equation}

and

\begin{equation}
\int_{0}^{\infty} \frac{t^{c-1} dt}{(-t, -q/t; q) \infty} = \frac{\Gamma(c) \Gamma(1-c) \left( q^c, q^{1-c}; q \right) \infty}{(q; q) \infty}.
\end{equation}

The Askey-Roy integral is [14] (4.11.1)

\begin{equation}
\int_{-\pi}^{\pi} \frac{(ce^{i\theta}/\beta, qa^{i\theta}/\beta, ca^{i\theta}/\alpha, ce^{-i\theta}, qa^{-i\theta}; q) \infty d\theta}{(ae^{i\theta}, be^{i\theta}, ae^{-i\theta}, \beta e^{-i\theta}; q) \infty} 2\pi = \frac{(ab\alpha\beta, q/c, \alpha, \beta, q\beta/\alpha; q) \infty}{(a\alpha, a\beta, b\alpha, b\beta, q; q) \infty}.
\end{equation}

3. First $q$-Analogue

The first $q$-analogue of \{H\}_{m,n}(z_1, z_2) is defined by

\begin{equation}
H_{m,n}(z_1, z_2|q) = \sum_{k=0}^{m+n} \left[ \begin{array}{c} m+n \\ k \end{array} \right] \left[ \begin{array}{c} m+k \\ k \end{array} \right] (-1)^k q^{k^2} (q; q)_k z_1^{m-k} z_2^{n-k}.
\end{equation}

Clearly,

\begin{equation}
H_{m,n}(z_2, z_1|q) = H_{n,m}(z_1, z_2|q).
\end{equation}

Theorem 1. The polynomials \{H_{m,n}(z_1, z_2|q)\} satisfy the relations

\begin{equation}
\sum_{m, n=0}^{\infty} H_{m,n}(z_1, z_2|q) \frac{u^m v^n}{(q; q)_m (q; q)_n} = \frac{(uv; q) \infty}{(u z_1, v z_2; q) \infty}
\end{equation}

\begin{equation}
H_{m,n}(q z_1, z_2|q) = H_{m,n}(z_1, z_2|q) - z_1 (1 - q^m) H_{m-1,n}(z_1, z_2|q),
\end{equation}

\begin{equation}
H_{m,n}(z_1, q z_2|q) = H_{m,n}(z_1, z_2|q) - z_2 (1 - q^n) H_{m,n-1}(z_1, z_2|q),
\end{equation}

\begin{equation}
H_{m,n}(q z_1, q z_2|q) q^{-m} = H_{m,n}(z_1, z_2|q) - q^{-1} (1 - q^m) (1 - q^n) H_{m-1,n-1}(z_1, z_2|q)
\end{equation}

\begin{equation}
H_{m,n}(z_1, q z_2|q) q^{-n} = H_{m,n}(z_1, z_2|q) - q^{-1} (1 - q^m) (1 - q^n) H_{m-1,n-1}(z_1, z_2|q),
\end{equation}

\begin{equation}
H_{m,n}(q z_1, z_2|q) q^{-m} = H_{m,n}(z_1, q z_2|q) q^{-n} = H_{m,n}(q z_1, q z_2|q) q^{-m-n} = H_{m,n}(z_1, q z_2|q) q^{-n-m} = H_{m,n}(z_1, q z_2|q).
\end{equation}
Moreover they have the operational representation

\[ (1 - q)^2 D_{q, z_1} D_{q, z_2}; q)_\infty z_1^m z_2^n \]

Before proving Theorem 1 we consider some of its implications. We note that (3.4) and (3.5) are the lowering operator relations

\[ D_{q, z_1} H_{m, n}(z_1, z_2|q) = \frac{1 - q^m}{1 - q} H_{m - 1, n}(z_1, z_2|q), \]
\[ D_{q, z_2} H_{m, n}(z_1, z_2|q) = \frac{1 - q^n}{1 - q} H_{m, n - 1}(z_1, z_2|q), \]

respectively. Moreover we observe that (3.6)–(3.7) imply the symmetry relation

\[ H_{m, n}(q z_1, z_2|q) q^{-m} = H_{m, n}(z_1, q z_2|q) q^{-n}. \]

Indeed (3.12) can be proved directly from the generating function (3.3). Finally we record a possible connection between the generating function (3.3) and partitions. Let \( M(m, n) \) denotes the number of partitions of a positive integer \( n \) with crank = \( m \). Andrews and Garvan [4] established the generating function

\[ \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} M(m, n) z^m q^n = \frac{(q; q)_\infty}{(q z, q/z; q)_\infty}. \]

It is clear that (3.13) is a special case of our generating function (3.3). This suggests that there may be a more refined statistic defined on partitions which will have the generating function (3.3).

**Proof of Theorem 1** The generating function follows from (3.1) and the Euler sums (2.7)–(2.8), (3.4) and (3.5) follow from

\[ (uv; q)_\infty \frac{(u z_1 q, v z_2; q)_\infty}{(u z_1, v z_2; q)_\infty} = (1 - u z_1) \frac{(uv; q)_\infty}{(u z_1, v z_2; q)_\infty}, \]

and

\[ (uv; q)_\infty \frac{(u z_1, v z_2 q; q)_\infty}{(u z_1, v z_2; q)_\infty} = (1 - u z_2) \frac{(uv; q)_\infty}{(u z_1, v z_2; q)_\infty}, \]

and (3.6) and (3.7) follow from

\[ (u q^{-1} v; q)_\infty \frac{(u z_1 q^{-1} z_2, v z_2; q)_\infty}{(u z_1, v z_2; q)_\infty} = (1 - q^{-1} u v) \frac{(uv; q)_\infty}{(u z_1, v z_2; q)_\infty}, \]

and

\[ (u v q^{-1}; q)_\infty \frac{(u z_1 z_2 q^{-1}, v z_2; q)_\infty}{(u z_1, v z_2; q)_\infty} = (1 - q^{-1} u v) \frac{(uv; q)_\infty}{(u z_1, v z_2; q)_\infty}. \]

The first 3-term recurrence follows from (3.4) and (3.6), similarly, the second one can be obtained from (3.5) and (3.7). It can be proved directly. It is clear that \( z_1 H_{m, n}(z_1, z_2|q) - H_{m+1, n}(z_1, z_2|q) \) is

\[ \sum_{k=0}^{(m+1) \wedge n} \left\{ \binom{m}{k} q - \binom{m + 1}{k} q \right\} \binom{n}{k} (-1)^k q^{(m+1-k)}(q; q)_q z_1^{m+1-k} z_2^n - z_2^{n-k} \]

which gives the first recurrence relation after replacing \( k \) by \( k + 1 \). The proof of the second recurrence relation is similar. The representation of (3.9) follows by expanding \(((1 - q)^2 D_{q, z_1} D_{q, z_2}; q)_\infty\) using (2.8). \( \square \)
Theorem 2. The polynomials satisfy the Rodrigues type formula

\[ H_{m,n}(z_1,z_2|q) = \frac{(1-1/q)^{m+n} q^{mn}}{(qz_1z_2;q)_\infty} D_{q^{-1},z_2}^m D_{q^{-1},z_1}^n ((qz_1z_2;q)_\infty) \]

and the raising relations

\[ H_{m+1,n}(z_1,z_2|q) = q^n \frac{1-1/q}{(qz_1z_2;q)_\infty} D_{q^{-1},z_2} H_{m,n}(z_1,z_2|q), \]

\[ H_{m,n+1}(z_1,z_2|q) = q^m \frac{1-1/q}{(qz_1z_2;q)_\infty} D_{q^{-1},z_1} H_{m,n}(z_1,z_2|q). \]

Moreover the polynomials {\( H_{m+1,n}(z_1,z_2|q) \)} have the multiplication formula

\[ H_{m,n}(a z_1,b z_2|q) = \sum_{j=0}^{m\wedge n} \binom{m}{j} \binom{n}{j} \frac{H_{m-j,n-j}(z_1,z_2|q)}{a^{j-m} b^{j-n}} (q,1/ab;q)_j(q;q)_j. \]

Proof. It is clear \((1-1/q)D_{q^{-1},z_2} z_2^n (qz_1z_2;q)_\infty = z_2 (qz_1z_2;q)_\infty\). Therefore the right-hand side of (3.14) is

\[ q^{mn} (1-1/q)^m D_{q^{-1},z_2}^m [z_2^n (qz_1z_2;q)_\infty] \]

\[ = q^{mn} (1-1/q)^m \sum_{k=0}^{m} \binom{m}{k} z_2^k (1-1/q)^k (q^{-k} z_2)^{n+k} (q^{-n} z_2^{m-k}) (1-1/q)^{m-k} \]

\[ = q^{mn} \sum_{k=0}^{m} \binom{m}{k} z_1^{-m-k} z_2^{-n-k} q^{-(m-k)(n-k)} (q^{-n} z_2^{m-k}) = H_{m,n}(z_1,z_2|q), \]

and we have proved (3.14). Formulas (3.15) and (3.16) follow directly from (3.14). The generating function (3.13) implies

\[ \sum_{n=0}^{\infty} H_{m,n}(z_1,z_2|q) \frac{u^m v^n}{(q;q)_m (q;q)_n} = (uv;q)_\infty \frac{(abuv;q)_\infty}{(abuv;q)^2} \]

and (3.17) follow from the \(q\)-binomial theorem (2.5). \(\Box\)

In the next section we shall introduce the polynomials \{\( h_{m,n}(z_1,z_2|q) \}\}, see (4.11) and (4.16). We also note (4.23) which indicates their relation to the \(q\)-Laguerre polynomials, [25]. We now show a connection between the polynomials \( H_{m,n}(z_1,z_2|q) \) and the little \(q\)-Jacobi polynomials, [26].

\[ H_{m,n}(z_1,z_2|q) = q^{mn} i^{m+n} h_{m,n}(z_1/i,z_2/i|q^{-1}) \]

\[ = q^{mn} (q^{-1};q^{-1})_n z_1^{m-n} F_{m-n}(z_1z_2|q^{-1}) \]

\[ = q^{mn} (q^{-m-1};q^{-1})_n z_1^{m-n} P_n \left( z_1z_2,q^{-m-n}|q \right) \]

\[ = (-1)^n \frac{(q;q)_m q^{(n)}(z_1z_2|q)}{(q;q)_{m-n}} z_1^{m-n} P_n \left( z_1z_2,q^{-m-n}|q \right), \]

or

\[ H_{m,n}(z_1,z_2|q) = (-1)^n \frac{(q;q)_m q^{(n)}(z_1z_2|q)}{(q;q)_{m-n}} z_1^{m-n} P_n \left( z_1z_2,q^{-m-n}|q \right), \]

where \( P_n (x;q^a|q) \) is the little \(q\)-Laguerre or Wall’s polynomials, [25].

\[ p_n (x;a|q) = 2^a \left( q^{-n},0 \atop a \right| q;q^x \right). \]

They satisfy the discrete orthogonality relation

\[ \sum_{k=0}^{\infty} a^k q^k (q^{k+1};q)_\infty p_m (q^k;a|q) p_n (q^k;a|q) = \frac{(q;q)_\infty}{(aq;q)_\infty} \frac{(aq)_n (q;q)_\infty}{(aq;q)_\infty} \delta_{m,n}, \]

where \( m,n \in \mathbb{N}_0 \) and \( 0 < a < q^{-1} \).
Theorem 3. The polynomials \( \{H_{m,n}(z,\overline{z}|q)\} \) satisfy the following orthogonality

\[
(3.20) \quad \int_{C} H_{m,n}(z,\overline{z}|q) \overline{H_{s,t}(z,\overline{z}|q)} d\mu(z,\overline{z}) = \frac{q^{mn} (q; q)_m (q; q)_n}{(q; q)_\infty} \delta_{m,s} \delta_{n,t},
\]

where

\[
d\mu(z,\overline{z}) = \frac{d\theta}{2\pi} \otimes \sum_{k=0}^{\infty} \frac{q^k}{(q; q)_k} \delta(r - q^{k/2}),
\]

and \( z = re^{i\theta}, r \in \mathbb{R}^+, \theta \in [0, 2\pi], \) \( m, n, s, t \in \mathbb{N}_0. \)

Proof. We may assume that \( m \geq n \) because of the symmetry \( \overline{H} \). Then apply \( (3.19) \) and change into polar coordinates to get

\[
\begin{align*}
\int_{0}^{2\pi} H_{m,n}(z,\overline{z}|q) \overline{H_{s,t}(z,\overline{z}|q)} d\mu(z,\overline{z}) \\
= (-1)^{n+t} \frac{(q; q)_m}{(q; q)_m-n} \frac{q^m}{(q; q)_s-t} \int_{0}^{2\pi} e^{i\theta(m-n+t-s)} d\theta \\
x \sum_{k=0}^{\infty} \frac{q^k}{(q; q)_k^2} p_n \left( r^2 q^{m-n} \right) p_t \left( r^2 q^{s-t} \right) r^{m-n+s-t} \delta(r - q^{k/2}) \\
= (-1)^{n+t} \frac{(q; q)_m}{(q; q)_m-n} \frac{q^m}{(q; q)_s-t} \delta_{m-n+t-s,0} \\
x \sum_{k=0}^{\infty} \frac{q^{k(1+m-n)}}{q^k} p_n \left( q^k q^{m-n} \right) p_t \left( q^k q^{s-t} \right) q^{mn} (q; q)_m (q; q)_n \delta_{m,s} \delta_{n,t}.
\end{align*}
\]

This completes the proof of the orthogonality relation. \( \square \)

It is clear that the orthogonality relation \( (3.20) \) and the generating function \( (3.3) \) imply the \( q \)-beta integral

\[
(3.21) \quad \int_{C} \frac{d\mu(z,\overline{z})}{(u_1 z, v_1 \overline{z}, u_2 z, v_2 \overline{z}; q)_\infty} = \frac{(u_1 u_2 v_1 v_2; q)_\infty}{(q, u_1 u_2 v_1 v_2, u_1 v_1, v_1 u_2 v_2; q)_\infty}.
\]

The large degree asymptotics of \( H_{m,n}(z,\overline{z}|q) \) are straightforward. Indeed \( (3.1) \) and Tannery’s theorem show that

\[
(3.22) \quad \lim_{m \to \infty} z_1^{-m} H_{m,n}(z_1, \overline{z}_2|q) = z_2^n \sum_{k=0}^{n} \binom{n}{k}_q q^k (-z_1 z_2)^{-k} = z_2^n (1/z_1 z_2; q)_n,
\]

where we used the \( q \) binomial theorem \( (2.6) \) in the last step. Similarly

\[
(3.23) \quad \lim_{n \to \infty} z_2^{-n} H_{m,n}(z_1, \overline{z}_2|q) = z_1^m (1/z_1 z_2; q)_m.
\]

One can similarly show that

\[
(3.24) \quad \lim_{m, n \to \infty} z_1^{-m} z_2^{-n} H_{m,n}(z_1, \overline{z}_2|q) = (1/z_1 z_2; q)_\infty.
\]

It must be noted that the convergence in \( (3.22) - (3.24) \) is uniform on compact subsets of the \( z_1 \) and \( z_2 \) planes.

Theorem 4. The polynomials \( \{H_{m,n}(z_1, z_2|q)\} \) have the generating function

\[
(3.25) \quad \sum_{m,n=0}^{\infty} H_{m,n}(z_1, \overline{z}_2|q) \frac{u^m(a/u; q)_m v^n(b/v; q)_n}{(q; q)_m (q; q)_n}
= \frac{(a z_1, b \overline{z}_2; q)_\infty}{(u z_1 v \overline{z}_2; q)_\infty} 2 \phi_2 \left( \begin{array}{c}
\frac{a}{u}, \frac{b}{v} \\
\frac{a z_1, b \overline{z}_2}{u z_1 v \overline{z}_2} \\
q; uv
\end{array} \right).
\]
Proof. From the explicit representation it follows that the right-hand side of (3.26) is equal to

\[
\sum_{m,n=0}^{\infty} (-1)^k q^k \frac{u^m(a/u; q)_m v^n(b/v; q)_n}{(q; q)_{m+k}(q; q)_{n-k}} z_1^{m-k} z_2^{-n-k} = \sum_{k=0}^{\infty} (-1)^k q^k z_2^{k} \frac{u^k(a/u; q)_k v^k(b/v; q)_k}{(q; q)_k} \sum_{m=0}^{\infty} (v z_2)^m (b q^k/v; q)_m \frac{(aq^k/u; q)_m}{(q; q)_m},
\]

and the theorem follows.

The polynomials \( \{H_{m,n}(z_1, z_2|q)\} \) have an additional orthogonality relation, which we now record.

**Theorem 5.** We have the orthogonality relation

\[
\sum_{j=0}^{p} \sum_{k=0}^{s} \int_{0}^{\pi} (q, e^{2i\theta}, e^{-2i\theta}; q)_\infty \frac{H_{j,k}(re^{i\theta}, re^{-i\theta}|q) H_{s-k,p-j}(re^{i\theta}, re^{-i\theta}|q)}{(q; q)_j(q; q)_{s-k}(q; q)_{p-j}} d\theta = \frac{\pi}{(q, ab; q)_\infty}. \tag{3.26}
\]

**Proof.** A special case of the Askey–Wilson integral is \([19, 14]\)

\[
\int_{0}^{\pi} (e^{2i\theta}, e^{-2i\theta}; q)_\infty \frac{(uv, uv; q)_\infty}{(uv, uv; q)_\infty} d\theta = \frac{\pi}{(uv, uv; q)_\infty}. \tag{3.27}
\]

Therefore

\[
\frac{(q; q)_\infty}{\pi} \int_{0}^{\pi} (e^{2i\theta}, e^{-2i\theta}; q)_\infty \left[ \sum_{j,k,m,n=0}^{\infty} H_{j,k}(re^{i\theta}, re^{-i\theta}|q) H_{m,n}(re^{i\theta}, re^{-i\theta}|q) \frac{u^j v^k (u; q)_m (v; q)_n}{(q; q)_j(q; q)_{m+k}(q; q)_{n-k}} \right] d\theta = \frac{(q; q)_\infty}{\pi} \int_{0}^{\pi} \frac{(uv, uv; q)_\infty (e^{2i\theta}, e^{-2i\theta}; q)_\infty}{(uv, uv; q)_\infty} d\theta = \sum_{s,t=0}^{\infty} (-1)^s q^s (1/r^2; q)_{s,t} (uv)^{s+t}.
\]

Therefore \( j + n = k + m \), say. The coefficient of \((uv)^p\) in the above expression is

\[
r^{2p} \sum_{s=0}^{p} (-1)^s q^s (1/r^2; q)_{p-s} = \frac{r^{2p}(1/r^2; q)_p}{(q; q)_p} \frac{\phi_1}{q^{s-p}},
\]

and the theorem follows.
The application of this theorem to derive sharp bounds for the zeros of \( p_{m,n} \) will done in Section 8.

4. A second \( q \)-Analogue

Our second \( q \)-analogue is defined by the explicit representation

\[
\begin{align*}
  h_{m,n}(z_1, z_2|q) &:= (-1)^n \left( q^{m-n+1}; q \right)_n \sum_{j=0}^{\infty} \frac{(q^{-m}, q^{-n}; q)_j}{(q; q)_j} \left( \frac{-q}{z_1 z_2} \right)^j.
\end{align*}
\]

Equivalently

\[
\begin{align*}
  h_{m,n}(z_1, z_2|q) &= \sum_{j=0}^{m+n} \binom{m}{j} q^{m-j(n-j)} (-1)^j (q; q)_j z_1^{-j} z_2^{-n-j}.
\end{align*}
\]

Note the \((m, n) - (z_1, z_2)\) symmetry

\[
\begin{align*}
  h_{m,n}(z_1, z_2|q) &= h_{n,m}(z_2, z_1|q)
\end{align*}
\]

It is easy to see that

\[
\begin{align*}
  \binom{m}{k} q^{-k} = q^{k(m-k)} \binom{m}{n}.
\end{align*}
\]

Therefore

\[
\begin{align*}
  h_{m,n}(z_1, z_2|q) &= q^{-mn} i^{-m-n} H_{m,n}(iz_1, iz_2|q)
\end{align*}
\]

Theorem 7. The polynomials \( \{h_{m,n}(z_1, z_2|q)\} \) have the following properties

\[
\begin{align*}
  \sum_{m,n=0}^{\infty} h_{m,n}(z_1, z_2|q) q^{(m-n)^2/2} y^{m+n} &= \frac{(-q^{1/2} u z_1, -q^{1/2} v z_2; q)_\infty}{(-uv; q)_\infty},
\end{align*}
\]

\[
\begin{align*}
  h_{m,n}(z_1^{-1}, z_2|q) &= h_{m,n}(z_1, z_2|q) + z_1 (1 - q^m) q^{-m} h_{m-1,n}(z_1, z_2|q),
\end{align*}
\]

\[
\begin{align*}
  h_{m,n}(z_1, z_2^{-1}|q) &= h_{m,n}(z_1, z_2|q) + z_2 (1 - q^n) q^{-n} h_{m,n-1}(z_1, z_2|q),
\end{align*}
\]

\[
\begin{align*}
  q^m h_{m,n}(z_1, z_2|q) &= h_{m,n}(z_1, z_2|q) + (1 - q^m) (1 - q^n) q^{-m-n} h_{m-1,n-1}(z_1, z_2|q),
\end{align*}
\]

\[
\begin{align*}
  q^n h_{m,n}(z_1, z_2|q) &= h_{m,n}(z_1, z_2|q) + (1 - q^m) (1 - q^n) q^{-m-n} h_{m-1,n-1}(z_1, z_2|q),
\end{align*}
\]

\[
\begin{align*}
  q^n z_1 h_{m,n}(z_1, z_2|q) &= h_{m+1,n}(z_1, z_2|q) + (1 - q^n) h_{m,n-1}(z_1, z_2|q),
\end{align*}
\]

and

\[
\begin{align*}
  q^m z_2 h_{m,n}(z_1, z_2|q) &= h_{m,n+1}(z_1, z_2|q) + (1 - q^m) h_{m-1,n}(z_1, z_2|q).
\end{align*}
\]

Moreover they have the Rodrigues type formula

\[
\begin{align*}
  h_{m,n}(z_1, z_2|q) &= (q - 1)^{m+n} (-z_1 z_2; q)_\infty D_q^{m} D_{q,z_1}^{n} \frac{1}{(-z_1 z_2; q)_\infty}.
\end{align*}
\]

Furthermore we also have the operational formula

\[
\begin{align*}
  h_{m,n}(z_1, z_2|q) &= \frac{q^{mn}}{(-q^{-1} (1 - q) z_1 D_{q^{-1}, z_1} D_{q^{-1}, z_2}; q)_\infty} z_1^m z_2^n.
\end{align*}
\]
Before proving Theorem 7 we explore some of its consequences. First note that (4.7)–(4.8) describe lowering operators. Indeed they can be written as

\begin{equation}
D_{q^{-1}, z^1} h_{m,n}(z_1, z_2|q) = \frac{q^{-m} - 1}{q^{-1} - 1} h_{m-1,n}(z_1, z_2|q),
\end{equation}

and

\begin{equation}
D_{q^{-1}, z^2} h_{m,n}(z_1, z_2|q) = \frac{q^{-n} - 1}{q^{-1} - 1} h_{m,n-1}(z_1, z_2|q),
\end{equation}

respectively. The raising operators come from the Rodrigues type formula (4.18). We have

\begin{equation}
h_{m+1,n}(z_1, z_2|q) = (q - 1)(-z_1z_2; q)_{\infty} D_{q,z_1} \left( \frac{h_{m,n}(z_1, z_2|q)}{(-z_1z_2; q)_{\infty}} \right),
\end{equation}

\begin{equation}
h_{m,n+1}(z_1, z_2|q) = (q - 1)(-z_1z_2; q)_{\infty} D_{q,z_2} \left( \frac{h_{m,n}(z_1, z_2|q)}{(-z_1z_2; q)_{\infty}} \right).
\end{equation}

**Proof of Theorem 7** The generating function (4.10) follows from

\[
\sum_{m,n=0}^{\infty} \frac{h_{m,n}(z_1, z_2|q)}{(q; q)_m^m (q; q)_n^n} q^{(m-n)^2/2} u^m v^n
\]

\[
= \sum_{j=0}^{\infty} \frac{(-uv)^j}{(q; q)_j} \sum_{m,j} \frac{(uz_1)^{m-j} q^{(m-j)^2/2}}{(q; q)_m} \sum_{n=j} \frac{(vz_2)^n q^{n^2/2}}{(q; q)_n}
\]

\[
= \sum_{j=0}^{\infty} \frac{(-uv)^j}{(q; q)_j} \left(-q^{1/2} u z_1, -q^{1/2} v z_2; q\right)_{\infty}
\]

\[
= \left(-q^{1/2} u z_1, -q^{1/2} v z_2; q\right)_{\infty}.
\]

Let \( z_1 \to z_1/q \) in (4.6), then

\[
\frac{(-q^{1/2} u z_1, -q^{1/2} v z_2; q)_{\infty}}{(-quv; q)_{\infty}} = \left(1 + u z_1 q^{-1/2}\right) \frac{(-q^{1/2} u z_1, -q^{1/2} v z_2; q)_{\infty}}{(-quv; q)_{\infty}}
\]

implies (4.7). Let \( z_2 \to z_2/q \) in (4.6) we get (4.8).

Let \( u \to uq, z_1 \to z_1/q \) in (4.6), from

\[
\frac{(-q^{1/2} u z_1, -q^{1/2} v z_2; q)_{\infty}}{(-quv; q)_{\infty}} = \left(1 + quv\right) \frac{(-q^{1/2} u z_1, -q^{1/2} v z_2; q)_{\infty}}{(-quv; q)_{\infty}}
\]

we get (4.9), similarly let \( v \to vq, z_2 \to z_2/q \) in (4.6) to get (4.10). To prove (4.13) we first note that

\[
D_{q,z_1} \frac{1}{(-z_1 z_2; q)_{\infty}} = -\frac{z_2}{1 - q (-z_1 z_2; q)_{\infty}}.
\]

Therefore the right-hand side of (4.13) is

\[
(q - 1)^m (-z_1 z_2; q)_{\infty} D_{q,z_2}^m \left( \frac{1}{(-z_1 z_2; q)_{\infty}} \right) z_2^n
\]

\[
= (-1)^m \sum_{k=0}^{m} \binom{m}{k}_q (-z_1)^k n_{q,z_2} \frac{(q; q)_n}{(q; q)_{n+m+k}} q^n z_2^{n+m+k}
\]

\[
= \sum_{k=0}^{m} \binom{m}{k}_q (-1)^{m-k} \frac{(q; q)_n}{(q; q)_{n+m+k}} q^{n+m+k} z_2^{n+m+k} q^{k(n+m+k)}.
\]

Replace \( k \) by \( m - k \) and we get the left-hand side of (4.13). We now come to (4.14). It is clear that

\[
D_{q^{-1}, z^r} = \frac{q^{-r} - 1}{q^{r-1} - 1} \cdots \frac{q^{k-r-1} - 1}{q^{r-k} - 1} z^{r-k} = \left(\frac{q}{q; q}_{r-k} - 1\right)^{k-r} z^{r-k}
\]
The right-hand side of (4.14) is
\[
q^m \sum_{k=0}^{\infty} \binom{m}{k}_q \binom{n}{j}_q (-1)^k (q; q)_k \left( q^{k^2-km-ka} z_1^{m-k} z_2^{-k} \right).
\]
and the proof is complete.

The next theorem gives multiplication formulas for the polynomials \( \{h_{m,n}(z_1, z_2; q)\} \).

**Theorem 8.** We have
\[
(4.19) \quad h_{m,n}(a z_1, b z_2; q) = \sum_{j=0}^{m \land n} \binom{m}{j}_q \binom{n}{j}_q (q, ab; q)_j a^{m-j} b^{n-j} h_{m-j, n-j}(z_1, z_2; q).
\]

**Proof.** The desired result follows from the generating function (4.6) and the q-binomial theorem (4.15). □

We now identify the q-Sturm–Liouville problems whose eigenvalues are \( \{h_{m,n}(z_1, z_2|q)\} \). Combine (4.14) and (4.17) to obtain
\[
(4.20) \quad -\frac{1}{w(z_1 z_2)} D_{q,z_2} (w(z_1 z_2) D_{q^{-1},z_1} h_{m,n}(z_1, z_2; q)) = \frac{1 - q^m}{(1-q)^2} q^{1-m} h_{m,n}(z_1, z_2|q),
\]
where
\[
(4.21) \quad w(z) := 1/(-z|q)_{\infty}.
\]

Similarly
\[
(4.22) \quad -\frac{1}{w(z_1 z_2)} D_{q,z_2} (w(z_1 z_2) D_{q^{-1},z_1} h_{m,n}(z_1, z_2; q)) = \frac{1 - q^n}{(1-q)^2} q^{1-n} h_{m,n}(z_1, z_2|q).
\]

We note the relation
\[
(4.23) \quad h_{m,n}(z_1, z_2; q) = (-1)^n (q; q)_n \left( z_1^{m-n} L_n^{(m-n)}(z_1 z_2; q)ight)
\]

**Theorem 9.** The polynomials \( \{h_{m,n}(z, \overline{z}|q)\} \) satisfy the following orthogonality
\[
(4.24) \quad \int_{\mathbb{R}^2} h_{m,n}(z, \overline{z}; q) h_{s,t}(z, \overline{z}; q) \frac{dxdy}{(-z\overline{z}; q)_{\infty}} = \frac{\pi \log q^{-1} (q; q)_n (q; q)_n \delta_{m,s} \delta_{n,t}}{q^{(m-n)^2/2+(m+n)/2}}
\]
where \( m, n, s, t \in \mathbb{N}_0 \).

**Proof.** There is no loss of generality in assuming that \( m \geq n \) because of the symmetry property (4.3). Apply (4.23) and change into polar coordinates to get
\[
\int_{\mathbb{R}^2} h_{m,n}(z, \overline{z}; q) h_{s,t}(z, \overline{z}; q) \frac{dxdy}{(-z\overline{z}; q)_{\infty}}
= (-1)^{n+t} (q; q)_n (q; q)_t \int_{0}^{2\pi} e^{i\theta(m-n+t-s)} d\theta
\times \int_{0}^{\infty} L_n^{(m-n)}(r^2; q) L_t^{(m-n-t)}(r^2; q) \frac{r^{m-n+s+t+1}dr}{(-r^2; q)_{\infty}}
= (-1)^{n+t} \pi (q; q)_n (q; q)_t \delta_{m-n+t-s,0}
\times \int_{0}^{\infty} L_n^{(m-n)}(x; q) L_t^{(m-n)}(x; q) \frac{x^{(m-n)}dx}{(-x^2; q)_{\infty}}
= (-1)^{n+t} \pi (q; q)_n (q; q)_t \delta_{m-n+t-s,0}
\times (\log q^{-1}) q^{-(m-n)^2/2-(m+n)/2} (q^{n+1}; q)_m \delta_{m,n}
\times \pi (\log q^{-1}) q^{-(m-n)^2/2+(m+n)/2} (q^{n+1}; q)_m \delta_{m,n}
= \frac{\pi (\log q^{-1}) (q; q)_m (q; q)_n \delta_{m,s} \delta_{n,t}}{q^{(m-n)^2/2+(m+n)/2}}
\]
and the proof is complete. □
Note that the orthogonality relation (4.24) and the generating function (4.10) give rise to the integral

\[
\int_{\mathbb{R}^2} \left( -q^{1/2}u_1 z, -q^{1/2}v_1 z, -q^{1/2}v_2 z, -q^{1/2}u_2 \bar{z}; q \right)_\infty dxdy \\
= \pi \ln q^{-1}(-u_1 v_1, -u_2 v_2, -u_1 v_2, -v_1 v_2; q)_\infty.
\]

(4.25)

It is clear that (4.25) is a q-beta integral.

Since the associated moment problem for \( L_0^{(\alpha)}(x; q) \) is indeterminate, they have infinitely many orthogonal measures. Let \( x^\alpha d\mu(x) \) be such a measure, for example,

\[
d\mu(x) = x^{-\alpha} w_{QL}(x; \alpha, \lambda, \lambda) dx, \quad \alpha, \lambda, \gamma > 0,
\]

etc. it is clear that our proof shows that

\[
d\sigma(re^{i\theta}, re^{-i\theta}) = \frac{1}{2} d\theta d\mu(r^2), \quad r \in \mathbb{R}^+, \theta \in [0, 2\pi]
\]

is also an orthogonal measure for \( h_{m,n}(re^{i\theta}, re^{-i\theta}|q) \) where \( r \in \mathbb{R}^+, \theta \in [0, 2\pi] \).

**Theorem 10.** Assume that \( q = e^{-2k^2} \) and \( |q| < 1 \), then we have

\[
(4.26) \quad (-aqe^{2imk}, -bqe^{-2imk}; q) = \sum_{s,t=0}^{\infty} s(t) (a,b|q)_s (q|q)_t \left( \frac{q^{1/2} e^{2imk}}{q^{1/2}} \right)^s \left( \frac{e^{i\theta}}{q^{1/2}} \right)^t,
\]

\[
(4.27) \quad \frac{(ab; q)_\infty}{(-ab, ae^{2mk}, be^{-2mk}; q)_\infty} = \sum_{s,t=0}^{\infty} s(t) (a,b|q) (q|q)_s (q|q)_t \left( \frac{q^{1/2} e^{imk}}{q^{1/2}} \right)^s \left( \frac{e^{i\theta}}{q^{1/2}} \right)^t
\]

and for any \( 0 < q < 1 \)

\[
(4.28) \quad \frac{(aq^a+b; q)_{\infty}}{(-aq^{a+b}, q^{a+c}, b^{b-c}; q)_{\infty}} = \sum_{m,n=0}^{\infty} h_{m,n}(q^a, q^b|q) (q^{a+c}|q)_m (q^{b-c}|q)_n q^{c(m-n)},
\]

where \( \Re(a+c) > 0 \) and \( \Re(b-c) > 0 \).

**Proof.** Equation (4.28) could be proved either from equations (2.10), (2.21) or equations (2.20), (2.22),

\[
\begin{aligned}
(4.29) \quad \frac{(q, -q^c, -q^{1-c}, q^{a+b}; q)_{\infty}}{(-1, -q, -q^{a+b}, q^{a+c}, q^{b-c}; q)_{\infty}} &= \int_{0}^{\infty} \frac{(q^{a+1/t}, -q^b; q)_{\infty}}{(-q^{a+b} - t, -q/t; q)_{\infty}} t^{1-1} dt \\
&= \sum_{m,n=0}^{\infty} \frac{h_{m,n}(q^{a+1/2}, q^{b-1/2}|q)}{(q|q)_m (q|q)_n} q^{(m-n)^2} \int_{0}^{\infty} \frac{t^{e+n-1} dt}{(-t, -q/t; q)_{\infty}} (1-q)
\end{aligned}
\]

Without loss of generality we may assume that \( m \geq n \)

\[
\left( \frac{aq^{a+b}; q}_{\infty} \right) \left( -aq^{a+b}, q^{a+c}, q^{b-c}; q \right)_{\infty} = \sum_{m,n=0}^{\infty} \frac{h_{m,n}(q^a, q^b|q)}{(q|q)_m (q|q)_n} q^{(m-n)^2} \left( -q^{c+1/2}, -q^{a+c+1/2}; q \right)_{\infty} = \sum_{m,n=0}^{\infty} h_{m,n}(q^a, q^b|q) (q|q)_m (q|q)_n q^{c(m-n)}.
\]
For \( q = e^{-2k^2} \) and \(|q| < 1\), observe that

\[
\sqrt{\pi}e^{m^2} (aqe^{2imk}, -bqe^{-2imk}; q) = \int_{-\infty}^{\infty} \frac{(ab; q)_{\infty} e^{-x^2+2mx}}{(ae^{2ikx}/\sqrt{q}, be^{-2ikx}/\sqrt{q}, q)_{\infty}} e^{-x^2+2mx+2iksx-2ikt} dx
\]

\[
= \sum_{s,t=0}^{\infty} \frac{H_{s,t}(a,b)}{(q; q)_s (q; q)_t} q^{(s+t)/2} \int_{-\infty}^{\infty} e^{-x^2+2mx+2iksx-2ikt} dx
\]

\[
= \sqrt{\pi}e^{m^2} \sum_{s,t=0}^{\infty} \frac{H_{s,t}(a,b)}{(q; q)_s (q; q)_t} q^{(s-t)^2/2} \left(q^{1/2}e^{2imk}\right)^s \left(q^{1/2}e^{-2imk}\right)^t.
\]

Equations (4.27) is proved in a similar fashion.

From (4.32) and (3.19) we get

\[
(ab; q)_{\infty} = \sum_{s,t=0}^{\infty} \frac{L_{i(s-t)}(ab; q)}{(q; q)_s (q; q)_t} \left(q^{1/2}e^{imk}\right)^s \left(-a^{-1}q^{1/2}e^{-imk}\right)^t,
\]

and for any \( 0 < q < 1 \)

\[
\frac{(q^{a+b}; q)_{\infty}}{(-q^{a+b}, q^{a+c}, q^{b-c}; q)_{\infty}} = \sum_{m,n=0}^{\infty} \frac{L_n^{(m-n)}(q^{a+b}; q)}{(q; q)_m} (-1)^n q^{(a+c)(m-n)},
\]

where \( \Re(a+c) > 0 \) and \( \Re(b-c) > 0 \), where \( L_n^{(a)}(x; q) \) and \( p_n(x, a|q) \) are the \( q\)-Laguerre and \( q\)-Laguerre polynomials respectively.

The next theorem gives the Plancherel-Rotach asymptotics of the polynomials \( \{h_{m,n}(z_1, z_2)\} \).

**Theorem 12.** For \( a, b \in \mathbb{C} \) and \( 0 < \epsilon < 1 \) the following asymptotic result holds uniformly when \( w_1, w_2 \) are in compact subsets of the complex plane

\[
\lim_{m,n \to \infty} h_{m,n}(w_1 q^{-am-bn}, w_2 q^{-1-am-(1-b)n}; q) \left(q; q\right)_{\infty}^2 = A_q \left( \frac{1}{w_1 w_2} \right)
\]

**Proof.** From the definition (1.2) it follows that

\[
\frac{h_{m,n}(w_1 q^{-am-bn}, w_2 q^{-1-am-(1-b)n}; q) \left(q; q\right)_{\infty}^2}{w_1^m w_2^n q^{(a-b)m-am+2+(b-1)n^2}}
\]

\[
= \sum_{j=0}^{m \wedge n} \frac{(-w_1 w_2)^{-j} q^{2j} \left(q^{m-j+1}, q^{1-n+1}; q\right)_\infty}{(q; q)_j} \to A_q \left( \frac{1}{w_1 w_2} \right),
\]

as \( m, n \to \infty \). \( \square \)

### 5. 2D \( q\)-ULTRASPERICAL POLYNOMIALS

The 2D-ultraspherical polynomials are

\[
C_{m,n}(z_1, z_2) = \sum_{k=0}^{m \wedge n} \left( \begin{array}{c} m \\ k \end{array} \right) \left( \begin{array}{c} n \\ k \end{array} \right) k! (-1)^k \nu_{m+n-k} z_1^{m-k} z_2^{-k}, \quad \nu > -1.
\]

They are also known as the disk polynomials or the Zernike polynomials, \([8]\).
It is clear that

\[ C_{m,n}^\nu(z_1, z_2) = (\nu)_{m+n} z_1^m z_2^n 2F_1 \left( \begin{array}{c} -m, -n \\ -m - n - \nu + 1 \end{array} \middle| \frac{1}{z_1 z_2} \right) \]  

which a constant multiple of the disk polynomials of §2.3 in [8].

They have the generating function

\[ \sum_{m,n=0}^{\infty} C_{m,n}^\nu(z_1, z_2) \frac{u^m v^n}{m! n!} = (1 - uz_1 - vz_2 + uv)^{-\nu}, \]

whose proof is an exercise in the application of the binomial theorem. Next we solve the connection relation between \( C_{m,n}^\nu(z_1, z_2) \) and \( H_{m,n}(z_1, z_2) \). We claim that

\[ C_{m,n}^\nu(z_1, z_2) \]

\[ = \sum_{p=0}^{m+n} \frac{(\nu)_{m+n} m! n!}{p! (m-p)! (n-p)!} H_{m-p,n-p}(z_1, z_2) \, P^\nu(-p; -\nu - m - n + 1; -1) \]

Write the right-hand side of (5.3) as

\[ \int_0^\infty \frac{\nu - 1}{(\nu)} e^{-t+uz_1+vz_2-tuv} dt = \int_0^\infty \frac{(\nu - 1)}{(\nu)} e^{-t+t(t-1)uv} \sum_{r,s=0}^{\infty} H_{r,s}(z_1, z_2) \frac{w^r v^s}{r! s!} t^{r+s} dt \]

\[ = \sum_{r,s=0}^{\infty} H_{r,s}(z_1, z_2) \frac{w^r v^s}{r! s!} \frac{(-1)^j (uv)^{j+k} \Gamma(\nu + s + j + 2k)}{\Gamma(\nu)}. \]

Equating coefficients of \( u^m v^n \) we see that \( m = r + j + k, n = s + j + k \). Let \( p = j + k \). Now (5.4) follows after some manipulations.

It is clear from the generating function (5.3) that the disk polynomials have the convolution property

\[ C_{m,n}^{\nu+j+k}(z_1, z_2) = \sum_{j=0}^{m} \sum_{k=0}^{n} C_{j,k}^\nu(z_1, z_2) C_{m-j,n-k}^{\nu+j+k}(z_1, z_2) \]

Floris [11], see also [10] and [12], introduced the following \( q \)-analogue of the disk polynomials. For \( \alpha, \beta > -1 \) and \( m \in \mathbb{Z}_+ \), the Floris \( q \)-disk polynomials \( R_{l,m}^{(\alpha)}(z, z*; q^2) \) are defined by

\[ R_{l,m}^{(\alpha)}(z, z*; q^2) = \begin{cases} \frac{1}{l-m} P_{m}^{(\alpha,l-m)}(1 - zz*; q^2) & l \geq m \\ P_{m}^{(\alpha,m-l)}(1 - zz*; q^2) (z*)^{m-l} & l \leq m, \end{cases} \]

where

\[ zz* = q^2 zz* + 1 - q^2, \]

and

\[ P_{m}^{(\alpha,\beta)}(x; q) = P_{m}(x; q^\alpha, q^\beta; q) \]

is the little \( q \)-Jacobi polynomials. The \( q \)-disk polynomials \( R_{l,m}^{(\alpha)}(z, z*; q^2) \) satisfy

\[ R_{l,m}^{(\alpha)}(z, z*; q^2) = R_{l,m}^{(\alpha)}(z, z*; q^2), \]

and the orthogonality relation

\[ \frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} R_{l,m}^{(\alpha)}(e^{i\theta} z, e^{-i\theta} z*; q^2) R_{l',m'}^{(\alpha)}(e^{i\theta} z, e^{-i\theta} z*; q^2) d\theta (1 - zz*)^\alpha d\theta (1 - zz*) = \frac{(1 - q^2) (q^2; q^2)_m (q^2; q^2)_m - q^{2m+1} \delta_{l,l'} \delta_{m,m'}}{(1 - q^2) (q^2; q^2)_m (q^2; q^2)_m}, \]

for \( \alpha > -1, l, l', m, m' \in \mathbb{Z}_+. \)
We now introduce our q-analogue of these polynomials. For \(0 < q < 1\) and \(b < q^{-1}\), let us define

\[
(5.6) \quad p_{m,n}(z_1, z_2; b|q) = \sum_{k=0}^{\infty} \binom{m}{k}_q \binom{n}{k}_q \frac{q^{\binom{k}{2}}(q; q)_k (bq; q)_{m+n-k}}{(-1)^k z_1^{m-k} z_2^{-n}} ,
\]

it is clear that

\[
(5.7) \quad p_{m,n}(z_2, z_1; b|q) = p_{n,m}(z_1, z_2; b|q)
\]

then for \(m \geq n\) we have

\[
(5.8) \quad p_{m,n}(z_1, z_2; b|q) = (-1)^n q^{\binom{n}{2}}(bq; q)_m (q^{m-n+1}; q)_n z_1^{m-n} p_n(z_1 z_2; q^{m-n}, b|q),
\]

where \(p_n(x; a, b|q)\) is the little Jacobi polynomials.

**Theorem 13.** For \(0 < q < 1\) and \(b < q^{-1}\), the polynomials \(\{p_{m,n}(z, z; b|q)\}\) satisfy the following orthogonality

\[
(5.9) \quad \int_C p_{m,n}(z, z; b|q) p_{s,t}(z, z; b|q) d\mu(z, z) = \frac{(bq; q)_\infty (q; q)_\infty (q^{m-n}(q, bq; q)_m (q, bq; q)_n)}{1 - bq^{m+n+1}} \delta_{m,s} \delta_{n,t},
\]

where

\[
(5.10) \quad d\mu(z, z) = \frac{d\theta}{2\pi} \otimes \sum_{k=0}^{\infty} \frac{(bq; q)_k q^k}{(q; q)_k} \delta \left( r - q^{k/2} \right),
\]

\(z = re^{i\theta}, r \in \mathbb{R}^+, \theta \in [0, 2\pi]\) and \(m, n, s, t \in \mathbb{N}_0\).

**Proof.** Because of the symmetry \((5.7)\), we may assume that \(m \geq n\). We first use polar coordinates, then apply \((5.8)\) and the orthogonality relation of the little Jacobi polynomials to get

\[
\int_C p_{m,n}(z, z; b|q) p_{s,t}(z, z; b|q) d\mu(z, z)
\]

\[
= (-1)^{n+t} q^{\binom{n}{2} + \binom{t}{2}} (bq; q)_m (bq; q)_s (q^{m-n+1}; q)_n (q^{s-t+1}; q)_t
\]

\[
\times \int_0^{2\pi} e^{i\theta(m-n-t-s)} \frac{d\theta}{2\pi} \sum_{k=0}^{\infty} \frac{(bq; q)_k q^k}{(q; q)_k} \delta \left( r - q^{k/2} \right) p_n(r^2; q^{m-n}, b|q) p_t(r^2; q^{s-t}, b|q)
\]

\[
= (-1)^{n+t} q^{\binom{n}{2} + \binom{t}{2}} (bq; q)_m (bq; q)_s (q^{m-n+1}; q)_n (q^{m-n+1}; q)_t
\]

\[
\times \sum_{k=0}^{\infty} \frac{(bq; q)_k q^k}{(q; q)_k} \rho_n(q^k; q^{m-n}, b|q) p_t(q^k; q^{m-n}, b|q) \delta_{m-n+t-s,0}
\]

\[
= \frac{(bq; q)_\infty q^{mn}(q; bq; q)_m (q; bq; q)_n}{(q; q)_\infty} \delta_{m,s} \delta_{n,t}.
\]

This completes the proof. \(\square\)

**Theorem 14.** The polynomials \(\{p_{m,n}(z_1, z_2; b|q)\}\) have the generating function

\[
(5.11) \quad \sum_{m,n=0}^{\infty} p_{m,n}(z_1, z_2; b|q) (q; q)_m (q; q)_n q^{mn} = \frac{(bq, u; q)_\infty (u z_1, u z_2; q; u v)_\infty}{(u z_1, z_2; q; u v)_\infty} \phi_1 \left( \begin{array}{c} u z_1, u z_2 \no u v \\ u z_1, z_2 \no u v \end{array} \right) (q; bq),
\]

For \(bq, cq < 1\) and \(b \neq 0\), the connection relation between the \(q\)-2D ultraspherical polynomials is given by

\[
(5.12) \quad \frac{p_{m,n}(z_1, z_2; b|q)}{(q; q)_\infty} = \sum_{j=0}^{\infty} \frac{\left( \frac{\pi}{2}; q \right)_j}{(q; q)_j} \left( b q^{m+n+1} \right)^j \frac{p_{m,n}(z_1 q^2, z_2 q^2; c|q)}{(c q; q)_\infty}.
\]
The connection relation between the \( q - 2D \) ultraspherical and \( q - 2D \) Hermite is given by

\[
(5.13) \quad \frac{p_{m,n}(z_1, z_2; b|q)}{(aq; q)_\infty} = \sum_{k=0}^{\infty} \left( \frac{bq^{m+n+1+j}}{(aq; q)_j} \right) H_{m, n} \left( z_1 q^{j/2} | z_2 q^{j/2} \right).
\]

Moreover we have the inverse relation

\[
(5.14) \quad H_{m, n} (z_1, z_2|q) = \sum_{k=0}^{\infty} \left( -\frac{bq^{m+n+1}}{(aq; q)_k} \right) p_{m,n} (z_1 q^{k/2}, z_2 q^{k/2}; b|q).
\]

Proof. Using the explicit definition \( (5.8) \) we see that

\[
\sum_{m,n=0}^{\infty} \frac{p_{m,n}(z_1, z_2; b|q)}{(aq; q)_m (aq; q)_n} u^m v^n = \sum_{k=0}^{\infty} \frac{bq^{m+n+1+k}}{(aq; q)_k} \frac{(z_1 u q^k; q)_m (z_2 v q^k; q)_n}{(aq; q)_m (aq; q)_n} \sum_{k=0}^{\infty} \frac{q^{k/2} (-uvq^k)^k}{(aq; q)_k}.
\]

We then expand \( 1/(aq^{m+n+k+1}; q)_\infty \) using \( (2.8) \) and write the above expression as

\[
= (aq; q)_\infty \sum_{j=0}^{\infty} \frac{(aq; q)_j}{(aq; q)_j} \sum_{m,n=0}^{\infty} \frac{(z_1 u q^j)^m (z_2 v q^j)^n}{(aq; q)_m (aq; q)_n} \sum_{k=0}^{\infty} \frac{q^{k/2} (-uvq^k)^k}{(aq; q)_k}.
\]

From above calculations we see that the generating function could be also written as

\[
\sum_{m,n=0}^{\infty} \frac{p_{m,n}(z_1, z_2; b|q)}{(aq; q)_m (aq; q)_n} u^m v^n = (aq; q)_\infty \sum_{j=0}^{\infty} \frac{(aq; q)_j}{(aq; q)_j} \frac{(uvq^j; q)_\infty}{(u z_1, z_2 v q^j; q)_\infty} \frac{1}{(u z_1, z_2 v q^j; q)_\infty}.
\]

and we find that

\[
\sum_{m,n=0}^{\infty} \frac{p_{m,n}(z_1, z_2; b|q)}{(aq; q)_m (aq; q)_n} u^m v^n = (aq; q)_\infty \sum_{m,n=0}^{\infty} \frac{u^m v^n}{(aq; q)_m (aq; q)_n} \sum_{j=0}^{\infty} \frac{bq^{m+n+1+j}}{(aq; q)_j} H_{m, n} \left( z_1 q^{j/2} | z_2 q^{j/2} \right).
\]

which gives \( (5.13) \). We now prove \( (5.14) \). From

\[
\sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right] q^k = \delta_{n,0}
\]
to get

\[
\frac{1}{(bq; q)_{\infty}} \sum_{k=0}^{\infty} \frac{(-bq^{(m+n)/2+1})^k}{(q; q)_k} q^{(\frac{k}{2})}_{m,n} \left( z_1 q^{k/2}, z_2 q^{k/2}; b | q \right) = \sum_{j,k=0}^{\infty} \frac{(-bq^{(m+n)/2+1})^{k+j}}{(q; q)_k (q; q)_j} q^{(\frac{k}{2})}_{m,n} \left( z_1 q^{(k+j)/2}, z_2 q^{(k+j)/2}; | q \right) = \sum_{\ell=0}^{\infty} \frac{(-bq^{(m+n)/2+1})^{\ell}}{(q; q)_\ell} H_{m,n} \left( z_1 q^{\ell/2}, z_2 q^{\ell/2}; | q \right) \sum_{k=0}^{\ell} \left[ \frac{\ell}{k} \right] q^{(k)} = H_{m,n} \left( z_1, z_2 | q \right).
\]

The connection formula between \( p_{m,n}(z_1, z_2; b | q) \) polynomials can be proved directly by observing that

\[
p_{m,n} \left( z_1, z_2; b | q \right) = \sum_{k=0}^{\infty} \left[ \frac{m}{k} \right] \left[ \frac{n}{k} \right] q^{(\frac{k}{2})}_{m,n} \left( q; q \right)_k (-1)^k z_1^{m-k} z_2^{n-k} \times (cq; q)_{m+n-k} \left( bq^{m+n-k+1}; q \right)_{m+n-k} \sum_{j=0}^{\infty} \frac{(\tilde{\xi}; q)_j}{(q; q)_j} (bq^{m+n-k+1})^j \times \left( bq^{m+n-k+1} \right)^j \sum_{k=0}^{m+n-k} \left[ \frac{m}{k} \right] \left[ \frac{n}{k} \right] q^{(\frac{k}{2})}_{m,n} \left( z_1 q^{\frac{k}{2}}, z_2 q^{\frac{k}{2}}; c | q \right).
\]

This completes the proof of our theorem.

Let us rewrite (5.11) in the form

\[
(\text{5.15}) \quad _2\phi_1 \left( \begin{array}{c}
u z_1, \nu z_2 \\
u v
\end{array} \right| q; bq) = \frac{(uz_1, vz_2; q)_{\infty}}{(bq, uv; q)_{\infty}} \sum_{m,n=0}^{\infty} \frac{p_{m,n} \left( z_1, z_2; b | q \right)}{(q; q)_m (q; q)_n} u^m v^n.
\]

**Theorem 15.** The polynomials \( \{ p_{m,n}(z_1, z_2; b | q) \} \) satisfy the following properties,

\[
(\text{5.16}) \quad D_{q, z_1} p_{m,n} \left( z_1, z_2; b | q \right) = \frac{(1 - bq)}{1 - q} \left( 1 - q^m \right) p_{m-1,n} \left( z_1, z_2; b | q \right),
\]

\[
(\text{5.17}) \quad D_{q, z_2} p_{m,n} \left( z_1, z_2; b | q \right) = \frac{(1 - bq)}{1 - q} \left( 1 - q^n \right) p_{m,n-1} \left( z_1, z_2; b | q \right),
\]

\[
(\text{5.18}) \quad D_{q^{-1}, z_1} \left( \frac{(qz_1 z_2; q)_{\infty}}{(bq z_1 z_2; q)_{\infty}} \right) p_{m,n} \left( z_1, z_2; b | q \right) = \frac{(qz_1 z_2; q)_{\infty} p_{m,n+1} \left( z_1, z_2; bq^{-1} | q \right)}{q^{m-1} (q - 1) (bq z_1 z_2; q)_{\infty}},
\]

\[
(\text{5.19}) \quad D_{q^{-1}, z_2} \left( \frac{(qz_1 z_2; q)_{\infty}}{(bq z_1 z_2; q)_{\infty}} \right) p_{m,n} \left( z_1, z_2; b | q \right) = \frac{(qz_1 z_2; q)_{\infty} p_{m+1,n} \left( z_1, z_2; bq^{-1} | q \right)}{q^{n-1} (q - 1) (bq z_1 z_2; q)_{\infty}},
\]
ON SOME 2D ORTHOGONAL $q$-POLYNOMIALS

$$p_{m,n}(z_1, z_2; b|q) - bq^{n+m-1} (1 - q^m) (1 - q^n) p_{m-1,n-1} (z_1 q, z_2; b|q)$$

$$= p_{m,n}(z_1, z_2; b|q) q^{m-1} (1 - q^m) (1 - q^n) p_{m-1,n-1}(z_1, z_2; b|q),$$

$$p_{m,n}(z_1 q, z_2; b|q) - bq^{2m-1} (1 - q^m) (1 - q^n) p_{m-1,n-1}(z_1, z_2 q; b|q)$$

$$= p_{m,n}(z_1, z_2; b|q) q^{m-1} (1 - q^m) (1 - q^n) p_{m-1,n-1}(z_1, z_2 b|q),$$

$$p_{m,n}(z_1, z_2 q; b|q) - bq^{2n-1} (1 - q^m) (1 - q^n) p_{m-1,n-1}(z_1 q, z_2; b|q)$$

$$= q^n p_{m,n}(z_1, z_2; b|q) - q^{n-1} (1 - q^m) (1 - q^n) p_{m-1,n-1}(z_1, z_2; b|q),$$

$$p_{m,n}(z_1 q, z_2 q; b|q) - bq^{m+n} (1 - q^m) (1 - q^n) p_{m-1,n-1}(z_1, z_2 q; b|q)$$

$$= q^n p_{m,n}(z_1, z_2; b|q) - q^{n-1} (1 - q^m) (1 - q^n) p_{m-1,n-1}(z_1, z_2; b|q).$$

(5.20)

(5.21)

(5.22)

(5.23)

(5.24)

(5.25)

(5.26)

(5.27)

(5.28)

(5.29)

(5.30)

Proof. Applying the difference operator $D_{q,z}$ to the form (6.10) of the generating function we find that

$$\sum_{m,n=0}^{\infty} \frac{D_{q,z} p_{m,n}(z_1, z_2; b|q)}{(q; q)_m (q; q)_n (bq; q)_\infty} u^m v^n = \sum_{j=0}^{\infty} \frac{(bq)_j^j}{(q; q)_j} \frac{(uvq^j; q)_\infty}{(z_2 q^j; q)_\infty} D_{q,z} \frac{1}{(z_1 u q^j; q)_\infty}$$

$$= \frac{1}{1-q} \sum_{j=0}^{\infty} \frac{(bq^2)_j}{(q; q)_j} \frac{(uvq^j; q)_\infty}{(z_2 q^j; q)_\infty} = \frac{1}{1-q} \sum_{m,n=0}^{\infty} \frac{p_{m,n}(z_1, z_2; b|q)}{(bq^2; q)_\infty (q; q)_m (q; q)_n} u^m v^n.$$
to get (5.16). (5.17) is obtained similarly. On the other hand, from (5.8) and the backward shift operator we get (5.18) and by the symmetry (5.7) we have (5.19). From the Heine’s contiguous relation (17.6.17) in [3] we get
\[
2\phi_1 \left( \frac{uq^{-1}z_1q, vz_2}{uq^{-1}v} \middle| q; bq \right) - 2\phi_1 \left( \frac{uz_1, vz_2}{uv} \middle| q; bq \right) = uzb_1 \frac{1 - vz_2}{1 - uv} 2\phi_1 \left( \frac{uz_1q, vz_2}{uvq} \middle| q; bq \right),
\]
(5.31)
\[
2\phi_1 \left( \frac{uq^{-1}z_1q, vz_2}{uq^{-1}v} \middle| q; bq \right) - 2\phi_1 \left( \frac{uz_1, vz_2}{uv} \middle| q; bq \right) = uzb_1 \frac{1 - vz_2}{1 - uvq} 2\phi_1 \left( \frac{uz_1q, vz_2q}{uvq} \middle| q; bq \right),
\]
(5.32)
\[
2\phi_1 \left( \frac{uz_1, vz_2q}{uvq^{-1}} \middle| q; bq \right) - 2\phi_1 \left( \frac{uz_1, vz_2}{uv} \middle| q; bq \right) = uzb_1 \frac{1 - vz_2}{1 - uvq^{-1}} 2\phi_1 \left( \frac{uz_1q, vz_2q}{uvq} \middle| q; bq \right),
\]
(5.33)
and
\[
2\phi_1 \left( \frac{uz_1, vz_2q}{uvq^{-1}} \middle| q; bq \right) - 2\phi_1 \left( \frac{uz_1, vz_2}{uv} \middle| q; bq \right) = uzb_1 \frac{1 - vz_2}{1 - uvq^{-1}} 2\phi_1 \left( \frac{uz_1q, vz_2q}{uvq} \middle| q; bq \right).
\]
(5.34)

Applying (5.15) we get (5.20) from (5.31), (5.21) from (5.32), (5.22) from (5.33) and (5.23) from (5.34). From Heine’s contiguous relation [3 (17.6.18)]

From the contiguous relation (17.6.18) in [3] we get
\[
2\phi_1 \left( \frac{uz_1q, vz_2}{uv} \middle| q; bq \right) - 2\phi_1 \left( \frac{uz_1, vz_2}{uv} \middle| q; bq \right) = uz_1b_1 2\phi_1 \left( \frac{uz_1q, vz_2}{uvq} \middle| q; bq \right),
\]
(5.35)
and
\[
2\phi_1 \left( \frac{uz_1q, vz_2}{uv} \middle| q; bq \right) - 2\phi_1 \left( \frac{uz_1, vz_2}{uv} \middle| q; bq \right) = uz_1b_1 2\phi_1 \left( \frac{uz_1q, vz_2q}{uvq} \middle| q; bq \right).
\]
(5.36)

Applying (5.15) to (5.27) to get (5.28), applying (5.15) to (5.36) to get (5.29).

From the fourth contiguous relation [3 (17.6.19)] we get
\[
2\phi_1 \left( \frac{uqz_1, vz_2}{uqv} \middle| q; bq \right) - 2\phi_1 \left( \frac{uz_1, vz_2}{uv} \middle| q; bq \right) = bq \frac{(1 - vz_2)(uz_1 - u)}{(1 - uv)(1 - uqv)} 2\phi_1 \left( \frac{uqz_1, vz_2}{uqvq} \middle| q; bq \right),
\]
(5.37)
we apply (5.15) to (5.37) to get (5.26).

From the fourth contiguous relation [3 (17.6.20)] we get
\[
2\phi_1 \left( \frac{uq^{-1}z_1q, vz_2}{uq^{-1}v} \middle| q; bq \right) - 2\phi_1 \left( \frac{uz_1, vz_2}{uv} \middle| q; bq \right) = b \frac{(uz_1q - vz_2)}{(1 - uv)} 2\phi_1 \left( \frac{uqz_1, vz_2}{uqv} \middle| q; bq \right).
\]
(5.38)
and
\[
2\phi_1 \left( \begin{array}{c|c} u z_1 q, v z_2 q^{-1} \\ u v \\ \end{array} \right) | q; b q \right) - 2\phi_1 \left( \begin{array}{c|c} u z_1, v z_2 \\ u v \\ \end{array} \right) | q; b q \right) = b \left( \begin{array}{c|c} u z_1 q - v z_2 \\ 1 - u v \\ \end{array} \right) 2\phi_1 \left( \begin{array}{c|c} u q z_1, v z_2 \\ u q v \\ \end{array} \right) | q; b q \right).
\]

From the contiguous relation (17.6.21) we get
\[
\begin{align*}
&v z_2 (1 - u z_1) 2\phi_1 \left( \begin{array}{c|c} u z_1 q, v z_2 \\ u v \\ \end{array} \right) | q; b q \right), \\
&-u z_1 (1 - v z_2) 2\phi_1 \left( \begin{array}{c|c} u z_1, v z_2 q \\ u v \\ \end{array} \right) | q; b q \right), \\
&= (v z_2 - u z_1) 2\phi_1 \left( \begin{array}{c|c} u z_1, v z_2 \\ u v \\ \end{array} \right) | q; b q \right),
\end{align*}
\]
then apply (5.15) we get (5.29).

\[
\begin{align*}
&v z_2 \sum_{m,n=0}^\infty \frac{p_{m,n} (z_1 q, z_2; b q)}{(q; q)_m (q; q)_n} u^{m} v^{n} \\
&-u z_1 \sum_{m,n=0}^\infty \frac{p_{m,n} (z_1, z_2 q; b q)}{(q; q)_m (q; q)_n} u^{m} v^{n} \\
&= (v z_2 - u z_1) \sum_{m,n=0}^\infty \frac{p_{m,n} (z_1, z_2; b q)}{(q; q)_m (q; q)_n} u^{m} v^{n}
\end{align*}
\]

From 5.38, 5.39 we get 5.27 and 5.28 respectively. From the contiguous relation (17.6.22) in 3 we obtain
\[
\begin{align*}
&(u z_1 - z_1 z_2 q) 2\phi_1 \left( \begin{array}{c|c} u z_1 q, v z_2 \\ u v \\ \end{array} \right) | q; b q \right) \\
&- (v z_2 - z_1 z_2 q) 2\phi_1 \left( \begin{array}{c|c} u z_1, v z_2 q \\ u v \\ \end{array} \right) | q; b q \right)
&= (u z_1 - v z_2) (1 - b z_1 z_2 q) 2\phi_1 \left( \begin{array}{c|c} u z_1 q, v z_2 q \\ u v \\ \end{array} \right) | q; b q \right),
\end{align*}
\]
which gives (5.30). \qed

The inversion transformation of quanta \( q \to q^{-1} \) in (4.4) relates the properties of one family of polynomials for \( q > 1 \) to the properties of another family of polynomials with \( 0 < q < 1 \). The polynomials \( p_{m,n} (z_1, z_2; b q) \) are essentially invariant under the quanta inversion transformation,

\[
p_{m,n} (z_1, z_2; b q^{-1})
= (-b)^{m+n} \sum_{k=0}^\infty \frac{[m]}{[k]} \frac{[n]}{[k]} (-b)^{-k} (q; q)_k \left( \frac{q}{b}; q \right)_{m+n-k} \\
\times q^{k(k-m) + k(k-n) - k^2} (q^{\frac{1}{2}})_m (q^{\frac{1}{2}})_n z_1^{m-k} z_2^{n-k}
= (-b)^{m+n} q^{-\frac{m+n+1}{2}} \sum_{k=0}^\infty \frac{[m]}{[k]} \frac{[n]}{[k]} \left( \frac{q}{b}; q \right)_k \frac{q}{b} (q; q)_k \left( \frac{q}{b}; q \right)_{m+n-k} z_1^{m-k} z_2^{n-k}
= (-1)^{m+n} \left( \frac{b}{q} \right)^{(1-\alpha)m+\alpha} q^{-\frac{m+n}{2}} p_{m,n} \left( \left( \frac{b}{q} \right)^\alpha z_1, \left( \frac{b}{q} \right)^{1-\alpha} z_2; 1 \right). 
\]
Therefore, we have established the symmetry

\[ p_{m,n} (z_1, z_2; b|q^{-1}) = \frac{(bq^{-1})^{(1-\alpha)m+n}}{(-1)^{m+n} q^{\frac{n}{2}}} p_{m,n} \left( (b/q)^{\alpha} z_1, (b/q)^{1-\alpha} z_2; 1/b|q \right), \]

for \( \alpha \in \mathbb{C} \).

We now come the asymptotics of \( p_{m,n} (z_1, z_2; b|q) \).

**Theorem 16.** Let \( z_1, z_2 \in \mathbb{C} \), \( bq < 1 \) and \( z_1 z_2 \neq 0 \), then we have

\[ \lim_{m,n \to \infty} p_{m,n} \left( \frac{1}{z_1}, \frac{1}{z_2}; \frac{b|q}{z_1^2} \right) = \left( \frac{1}{z_1 z_2}; q \right)_\infty, \]

uniformly on compact subsets of the \( z_1 \) and \( z_2 \) planes.

The theorem follows from the definition (5.6) and Tannery’s theorem.

### 6. Applications

**Theorem 17.** Let \( |t_i x_i| < \sqrt{q} \) for \( i = 1, 2, 3, 4 \), then we have

\[
\begin{align*}
&\frac{(t_1 x_1 \sqrt{q}, t_2 x_2 \sqrt{q}, t_3 x_3 \sqrt{q}, t_4 x_4 \sqrt{q}, x_1 x_2, x_3 x_4, t_1 t_2 t_3 t_4 x_1 x_2 x_3 x_4 q^2; q)}{(t_1 t_2 x_1 x_2, t_2 t_3 x_1 x_3, t_3 t_4 x_2 x_4, t_1 t_2 t_3 t_4 x_1 x_2 x_3 x_4 q^2; q)_{\infty}} \\
&= \sum \frac{h_{m_1,n_1} (t_1 t_3, t_2 t_4; q)}{(q; q)_{m_1} (q; q)_{n_1}} q^{((m_1-n_1)^2+m_1+n_1+(m_1+m_2+m_3-n_1-n_2-n_3)/2)/2} \\
&\times H_{m_2,n_2} (t_1, t_2; q) H_{m_3,n_3} (t_3, t_4; q) x_1^{m_1+m_2} x_2^{n_1+n_2} x_3^{m_1+m_3} x_4^{n_1+n_3} \\
&\times (-1)^{m_2+m_3-n_2-n_3} \prod_{j=2}^{3} (q; q)_{m_j} (q; q)_{n_j},
\end{align*}
\]

where the summation is over all the nonnegative integers \( m_i, n_i \quad i = 1, 2, 3 \) such that \( m_1 + m_2 + m_3 = n_1 + n_2 + n_3 \).

**Proof.** Observe that

\[
\begin{align*}
&\frac{(x_1 x_2 q^{-1}; q)_{\infty}}{(t_1 x_1 q^{-1/2} z, t_2 x_2 q^{-1/2} z; q)_{\infty}} \frac{(x_3 x_4 q^{-1}; q)_{\infty}}{(t_3 x_3 q^{-1/2} z, t_4 x_4 q^{-1/2} z; q)_{\infty}} \\
&\times \frac{(q^{1/2} t_1 t_3 x_1 x_3 q^{-1}, q^{1/2} t_2 t_4 x_2 x_4 q^{-1}; q)}{(-x_1 x_2 x_3 x_4 q^{-2}; q)_{\infty}} \\
&= \sum \frac{h_{m_1,n_1} (t_1 t_3, t_2 t_4; q)}{(q; q)_{m_1} (q; q)_{n_1}} H_{m_2,n_2} (t_1, t_2; q) H_{m_3,n_3} (t_3, t_4; q) \\
&\times q^{((m_1-n_1)^2-2m_1-2n_1-m_2-n_2-m_3-n_3)/2} \\
&\times (-1)^{(m_1-n_1)^2} x_1^{m_1+m_2} x_2^{n_1+n_2} x_3^{m_1+m_3} x_4^{n_1+n_3} \\
&\times z^{(m_1+m_2+m_3)-(n_1+n_2+n_3)},
\end{align*}
\]
by the $q$-beta integral we have

$$
\frac{(x_1x_2q^{-1}, x_3x_4q^{-1}, t_1t_2t_3t_4x_1x_2x_3x_4; q)_{\infty}}{(q-x_1x_2x_3x_4q^{2}; q)_{\infty}} \times \frac{(t_1t_2t_3x_1q^{-1}, t_2t_3x_2x_3q^{-1}, x_1x_2x_3x_4; q^{-1})_{\infty}}{(q, x_1x_2x_3x_4q^{2}; q^{-1})_{\infty}}
$$

$$
= \sum h_{m_1,n_1} (t_1t_3, t_2t_4|q) H_{m_2,n_2} (t_1t_2|q) H_{m_3,n_3} (t_3t_4|q)
$$

$$
\times \frac{(q, q^{12}; q)_{\infty} \prod_{j=1}^n (q; q)_{m_j} (q; q)_{n_j}}{\pi^2 (m_1-n_1)^2 - 2m_1 - 2n_1 - m_2 - n_2 - m_3 - n_3 / 2 / (2\pi i)}
$$

$$
\times (-1)^{(m_1-n_1)} x_1^{m_1+m_2+n_1+n_2} x_3^{m_3} x_4^{n_1+n_3}
$$

$$
\times \int_{|z|=1} \frac{\phi \left( (m_1-n_1)^2 - 2m_1 - 2n_1 - m_2 - n_2 - m_3 - n_3 + (m_1+m_2+m_3-n_1-n_2-n_3)^2 / 2 \right)}{(4\pi^2)^{1/2}} dz
$$

$$
= \sum h_{m_1,n_1} (t_1t_3, t_2t_4|q) H_{m_2,n_2} (t_1t_2|q) H_{m_3,n_3} (t_3t_4|q)
$$

$$
\times \frac{(q, q^{12}; q)_{\infty} \prod_{j=1}^n (q; q)_{m_j} (q; q)_{n_j}}{\pi^2 (m_1-n_1)^2 - 2m_1 - 2n_1 - m_2 - n_2 - m_3 - n_3 / 2 / (2\pi i)}
$$

$$
\times (-1)^{(m_1-n_1)} x_1^{m_1+m_2+n_1+n_2} x_3^{m_3} x_4^{n_1+n_3},
$$

where the summation is over all the nonnegative integers $m_i, n_i$ such that $m_1 + m_2 + m_3 = n_1 + n_2 + n_3$.

From (3.19) and (3.19) we obtain the following equivalent representation:

**Corollary 18.** Let $|t,x| < q$ for $j = 1, 2, 3, 4$, then we have

$$
\frac{(t_1t_2t_3t_4q^{-1}, x_1x_2x_3x_4; q)_{\infty}}{(t_1t_2t_3x_1q^{-1}, x_2x_3x_4; q^{-1})_{\infty}}
$$

$$
= \sum \frac{L_{m_1,n_1} (t_1t_2t_3q^{-2}; q)_{m_1} (q; q)_{m_2} (q; q)_{m_3} (q; q)_{m_4}}{p_n (t_1t_2q^{-1}, q^{m_2-n_2}; q) p_n (t_3t_4q^{-1}, q^{m_3-n_3}; q)}
$$

$$
\times \frac{m_2}{m_3} \frac{m_3}{n_3} \frac{q^{(m_1-n_1)^2 + n_1^2 + n_2^2 + (m_1+m_2+m_3-n_1-n_2-n_3)^2 - (m_1+m_2+m_3-n_1-n_2-n_3)^2 / 2}}{\pi^2 (m_1-n_1)^2 - 2m_1 - 2n_1 - m_2 - n_2 - m_3 - n_3 / 2 / (2\pi i)}
$$

$$
\times (-1)^{(m_1-n_1)} t_1^{m_1+m_2+n_1} t_3^{m_2-n_2} t_4^{m_3-n_3} x_1^{m_1} x_2^{m_2} x_3^{m_3} x_4^{n_1+n_3},
$$

where the summation is over all the nonnegative integers $m_i, n_i$ such that $m_1 + m_2 + m_3 = n_1 + n_2 + n_3$.

From (3.19) we find that

$$
\sum_{n=-\infty}^{\infty} t^n J_n^{(2)} (x; q) = \frac{\left(\frac{x^2}{4}; q\right)_{\infty}}{\left(\frac{q^2}{4}; q\right)_{\infty}} \frac{\left(-\frac{x^2}{4}; q\right)_{\infty}}{\left(\frac{q^2}{4}; q\right)_{\infty}} = \frac{\left(-\frac{x^2}{4}; q\right)_{\infty}}{\left(\frac{q^2}{4}; q\right)_{\infty}} \sum_{j,k=0}^{\infty} H_j \left(\frac{x}{4}; \frac{q}{4}; q\right)_{j,k}^{q^{j-k}}
$$

$$
= \frac{-x^2}{4}; q_{\infty} \sum_{j,k=0}^{\infty} p_k \left(\frac{x^2}{4}; q^{j-k}; q\right)_{j,k} \left(\frac{x}{4}\right)^{j-k} t^{j-k}
$$

$$
= \frac{-x^2}{4}; q_{\infty} \sum_{n=-\infty}^{\infty} t^n \left(\frac{x}{4}; q\right)_{n} \sum_{k=0}^{\infty} p_k \left(\frac{x^2}{4}; q^n; q\right)_{k}^{q^{n+1}} q_{n+k},
$$

then

$$
J_n^{(2)} (x; q) = \frac{\left(\frac{x^2}{4}; q\right)_{\infty}}{\left(\frac{q^2}{4}; q\right)_{\infty}} \sum_{k=0}^{\infty} p_k \left(\frac{x^2}{4}; q^n; q\right)_{k}^{q^{n+1}} q_{n+k}.$$
for all $n \in \mathbb{Z}$, and

$$(6.3) \quad \frac{J^{(2)}_n (2x; q)}{x^n} = (q^{n+1}, -x^2; q)_\infty \sum_{k=0}^{\infty} \frac{p_k (x^2, q^2; q)}{(q, q^{n+1}; q)_k}$$

for $\alpha > 0$ by analytic continuation.

We now use the Askey–Roy integral (2.23) to derive

$$(aba\beta, c, q/c, ca/\beta, q\beta/\alpha q; q)_\infty$$
$$= \frac{(a, b, q, -c^2 \alpha/q \beta, -q^2 \beta/c^2 \alpha; q)_\infty}{(a, b, q, -c^2 \alpha/q \beta, -q^2 \beta/c^2 \alpha; q)_\infty}$$
$$\times \frac{H_{m_4, n_4} \left(a, \alpha \mid q \right) H_{m_4, n_4} \left(b, \beta \mid q \right)}{H_{m_4, n_4} \left(a, \alpha \mid q \right) H_{m_4, n_4} \left(b, \beta \mid q \right)}$$
$$\times q^{-m_1 + m_2 + m_3 + m_4 - n_1 - n_2 - n_3 - n_4} \frac{d\theta}{2\pi},$$

that is,

$$(aba\beta, c, q/c, ca/\beta, q\beta/\alpha q; q)_\infty$$
$$= \sum \frac{h_{m_1, n_1} \left(c, \beta \mid q \right) h_{m_2, n_2} \left(1/c, \beta/c \mid q \right)}{h_{m_1, n_1} \left(1/c, \beta/c \mid q \right) h_{m_2, n_2} \left(c, \beta \mid q \right)}$$
$$\times \frac{H_{m_3, n_3} \left(a, \alpha \mid q \right) H_{m_4, n_4} \left(b, \beta \mid q \right)}{H_{m_3, n_3} \left(a, \alpha \mid q \right) H_{m_4, n_4} \left(b, \beta \mid q \right)}$$
$$\times q^{-m_1 + m_2 + m_3 + m_4 - n_1 - n_2 - n_3 - n_4} \frac{d\theta}{2\pi},$$

where $|q|, |\alpha|, |\beta|, |a|, |b| < 1$, $\alpha \beta \neq 0$ and the summation is over all the nonnegative integers such that $m_1 + m_2 + m_3 + m_4 - n_1 - n_2 - n_3 - n_4 = 0$.

From (123) and (6.19) we obtain the following equivalent representation,

$$(aba\beta, c, q/c, ca/\beta, q\beta/\alpha q; q)_\infty$$
$$= \sum \frac{L^{(m_1+n_1)} \left(\alpha^2 \beta/\beta \mid q \right) q^{(m_1+n_1)^2/2}}{L^{(m_1+n_1)} \left(\alpha^2 \beta/\beta \mid q \right) q^{(m_1+n_1)^2/2}}$$
$$\times \frac{L^{(m_2+n_2)} \left(\beta^2 \alpha/\beta \mid q \right) q^{(m_2+n_2)^2/2}}{L^{(m_2+n_2)} \left(\beta^2 \alpha/\beta \mid q \right) q^{(m_2+n_2)^2/2}}$$
$$\times \frac{p_{m_3} \left(\alpha q^{m_3-n_3} \mid q \right) p_{m_4} \left(b \beta q^{m_4-n_4} \mid q \right)}{p_{m_3} \left(\alpha q^{m_3-n_3} \mid q \right) p_{m_4} \left(b \beta q^{m_4-n_4} \mid q \right)}$$
$$\times \frac{q^{-m_3-m_4} b^{m_4-n_4} \left(q, q \mid q \right) m_3 \left(q, q \mid q \right) m_4 \left(q, q \right)}{m_3 \left(q, q \right) m_4 \left(q, q \right)},$$

where $|q|, |\alpha|, |\beta|, |a|, |b| < 1$, $\alpha \beta \neq 0$, the summation is over all the nonnegative integers such that $m_1 + m_2 + m_3 + m_4 - n_1 - n_2 - n_3 - n_4 = 0$ and $L^{(\alpha)}_n \left(x \mid q \right)$ and $p_n \left(x \mid a \mid q \right)$ are the $q$-Laguerre and Little $q$-Laguerre polynomials respectively.
Let $a = u e^{i\phi}$, $b = u e^{-i\phi}$, $\alpha = v e^{i\psi}$, $\beta = v e^{-i\psi}$, $c = q^{1/2}$ in Askey and Roy integral to get
\[\int_{-\pi}^{\pi} \frac{(q^{1/2} e^{i\theta} e^{i\psi}; q, q^{1/2} e^{i\theta} e^{-i\psi}; q, q^{1/2} e^{-i\theta} e^{i\psi}; q, q^{1/2} e^{-i\theta} e^{-i\psi}; q)_\infty}{(e^{i\theta} u v e^{i\phi}, e^{-i\theta} u v e^{-i\phi}, e^{-i\theta} u v e^{i\phi}, e^{i\theta} u v e^{-i\phi}; q)_\infty} d\theta = \frac{(u^2 v^2, q^{1/2}, q^{1/2}, q^{1/2} e^{2i\phi}, q^{1/2} e^{-2i\phi}; q)_\infty}{(u v e^{i(\phi+\psi)}, u v e^{i(-\phi-\psi)}, u v e^{-i(\phi-\psi)}, u v e^{-i(\phi+\psi)}; q)_\infty}\]
and
\[\frac{(u^2 v^2, q^{1/2}, q^{1/2}, q^{1/2} e^{2i\phi}, q^{1/2} e^{-2i\phi}; q)_\infty}{(u v e^{i(\phi+\psi)}, u v e^{i(-\phi-\psi)}, u v e^{-i(\phi-\psi)}, u v e^{-i(\phi+\psi)}; q)_\infty} \times \frac{h_{m_1}(\sin \left(i\psi + \frac{\pi}{2}\right) | q) h_{m_2}(\sin \left(i\psi - \frac{\pi}{2}\right) | q)}{(q; q)_{m_1}(q; q)_{m_2}(q; q)_{m_3}(q; q)_{m_4}(vi)^{m_1+m_2}} \times H_{m_3}(\cos \phi | q) H_{m_4}(\cos \psi | q) u^{m_3} v^{m_4}\]
\[= \sum \frac{h_{m_1}(\sin(i\psi + \frac{\pi}{2}) | q) h_{m_2}(i \sin \left(\psi - \frac{\pi}{2}\right) | q)}{(q; q)_{m_1}(q; q)_{m_2}(q; q)_{m_3}(q; q)_{m_4}(vi)^{m_1+m_2}} \times \frac{q^{(m_1^2+m_2^2)/2} H_{m_3}(\cos \phi | q) H_{m_4}(\cos \psi | q) u^{m_3} v^{m_4}}{q^{(m_1^2+m_2^2)/2} H_{m_3}(\cos \phi | q) H_{m_4}(\cos \psi | q) u^{m_3} v^{m_4}}\]
or
\[\frac{(u^2 v^2, q^{1/2}, q^{1/2}, q^{1/2} e^{2i\phi}, q^{1/2} e^{-2i\phi}; q)_\infty}{(u v e^{i(\phi+\psi)+\pi/2}, u v e^{i(-\phi-\psi)+\pi/2}, u v e^{-i(\phi-\psi)+\pi/2}, u v e^{-i(\phi+\psi)+\pi/2}; q)_\infty} \times \frac{h_{m_1}(i \sin \psi | q) h_{m_2}(i \sin \psi | q) (-1)^{m_2}}{(q; q)_{m_1}(q; q)_{m_2}(q; q)_{m_3}(q; q)_{m_4}(vi)^{m_1+m_2}} \times \frac{q^{(m_1^2+m_2^2)/2} H_{m_3}(\cos \phi | q) H_{m_4}(\cos \psi | q) u^{m_3} v^{m_4}}{q^{(m_1^2+m_2^2)/2} H_{m_3}(\cos \phi | q) H_{m_4}(\cos \psi | q) u^{m_3} v^{m_4}}\]
\[= \sum \frac{(-1)^{m_1} q^{(m_1^2+m_2^2)/2} u^{m_3} v^{m_4}}{(q; q)_{m_1}(q; q)_{m_2}(q; q)_{m_3}(q; q)_{m_4}} \times H_{m_1}(\sin \psi | q^{-1}) H_{m_2}(\sin \psi | q^{-1}) H_{m_3}(\cos \phi | q) H_{m_4}(\cos \psi | q)\]
or
\[\frac{(u^2 v^2, q^{1/2}, q^{1/2}, q^{1/2} e^{2i\phi}, q^{1/2} e^{-2i\phi}; q)_\infty}{(u v e^{i(\phi+\psi)+\pi/2}, u v e^{i(-\phi-\psi)+\pi/2}, u v e^{-i(\phi-\psi)+\pi/2}, u v e^{-i(\phi+\psi)+\pi/2}; q)_\infty} \times \frac{H_{m_1}(\sin \psi | q^{-1}) H_{m_2}(\sin \psi | q^{-1}) H_{m_3}(\cos \phi | q) H_{m_4}(\cos \psi | q)}{(-1)^{m_1} q^{-(m_1^2+m_2^2)/2} (q; q)_{m_1}(q; q)_{m_2}(q; q)_{m_3}(q; q)_{m_4} v^{m_3} u^{m_4}}\]
where the summation is over all the nonnegative integers $m_i$, $i = 1, 2, 3, 4$ such that $m_1 + m_3 = m_2 + m_4$.

7. Additional Results

In this section we first derive moment integral representations for $\{H_{m,n}(\zeta, \zeta | q)\}$ and $\{h_{m,n}(\zeta, \zeta | q)\}$. We then derive additional generating functions and expansions. We shall use the terminating $q$-binomial theorem \[24\] in the form
\[\prod_{j=0}^{n-1} (a + bq^j) = \sum_{j=0}^{n} \binom{n}{j} q^j a^{n-j} b^j.

Theorem 19. Let $\mu(\zeta, \bar{\zeta})$ be a normalized orthogonal measure for $H_{m,n}(z, \bar{z} | q)$ and $\nu(\zeta, \bar{\zeta})$ be a normalized measure for $h_{m,n}(z, \bar{z})$ respectively, then we have the integral representations
\[H_{m,n}(z_1, z_2 | q) = \int_{\mathbb{R}^2} \prod_{j=0}^{m-1} \left(z_1 + i\zeta q^{j+1}\right) \prod_{k=0}^{n-1} \left(z_2 + i\bar{\zeta} q^{k+1}\right) d\nu(\zeta, \bar{\zeta}),\]
and

\[ (7.3) \quad q^{\frac{(m-n)^2}{2}} j^{m+n} h_{m,n}(z_1, z_2 | q) = \int_{\mathbb{R}^2} \prod_{j=0}^{m-1} \left( \zeta + iz_1 q^{\frac{j}{2}} + j \right) \prod_{k=0}^{n-1} \left( \zeta + iz_2 q^{\frac{k}{2}} + k \right) d\mu(\zeta, \bar{\zeta}), \]

where \( z_1, z_2 \in \mathbb{C} \) and \( m, n \in \mathbb{N}_0 \).

**Proof.** Let

\[ a_{m,n}(z_1, z_2 | q) = \int_{\mathbb{R}^2} \prod_{j=0}^{m-1} \left( \zeta + iz_1 q^{\frac{j}{2}} + j \right) \prod_{k=0}^{n-1} \left( \zeta + iz_2 q^{\frac{k}{2}} + k \right) d\mu(\zeta, \bar{\zeta}). \]

The form \((7.3)\) of the \(q\)-binomial theorem implies

\[ \prod_{j=0}^{m-1} \left( \zeta + iz_1 q^{\frac{j}{2}} + j \right) \prod_{k=0}^{n-1} \left( \zeta + iz_2 q^{\frac{k}{2}} + k \right) = \sum_{m,n=0}^{\infty} \sum_{j,k=0}^{m-1} \binom{m}{j} \binom{n}{k} q^{\frac{j(k-1)}{2}} j^{m+n} u^{m} v^{n}, \]

and we find that

\[ \sum_{m,n=0}^{\infty} a_{m,n}(z_1, z_2 | q) u^m v^n \]

\[ = \int_{\mathbb{R}^2} \sum_{j,k=0}^{\infty} q^{\frac{j(k-1)}{2}} \binom{m}{j} \binom{n}{k} q^{\frac{j(k-1)}{2}} j^{m+n} u^j v^n d\mu(\zeta, \bar{\zeta}) \]

\[ = \int_{\mathbb{R}^2} \sum_{j,k=0}^{\infty} q^{\frac{j(k-1)}{2}} \binom{m}{j} \binom{n}{k} q^{\frac{j(k-1)}{2}} j^{m+n} u^j v^n d\mu(\zeta, \bar{\zeta}) \]

\[ = \int_{\mathbb{R}^2} \sum_{j,k=0}^{\infty} b_{m,n}(z_1, z_2 | q) u^{m} v^{n} \]

Similarly, let

\[ b_{m,n}(z_1, z_2 | q) = \int_{\mathbb{R}^2} \prod_{j=0}^{m-1} \left( \zeta + iz_1 q^{\frac{j}{2}} + j \right) \prod_{k=0}^{n-1} \left( \zeta + iz_2 q^{\frac{k}{2}} + k \right) d\nu(\zeta, \bar{\zeta}) \]

\[ = \int_{\mathbb{R}^2} \sum_{m,n=0}^{\infty} \sum_{j,k=0}^{\infty} \binom{m}{j} \binom{n}{k} q^{\frac{j(k-1)}{2}} j^{m+n} u^j v^n d\nu(\zeta, \bar{\zeta}) \]

then,

\[ \sum_{m,n=0}^{\infty} b_{m,n}(z_1, z_2 | q) u^m v^n \]

\[ = \int_{\mathbb{R}^2} \sum_{j,k=0}^{\infty} q^{\frac{j(k-1)}{2}} \binom{m}{j} \binom{n}{k} q^{\frac{j(k-1)}{2}} j^{m+n} u^j v^n d\nu(\zeta, \bar{\zeta}) \]

\[ = \int_{\mathbb{R}^2} \sum_{j,k=0}^{\infty} q^{\frac{j(k-1)}{2}} \binom{m}{j} \binom{n}{k} q^{\frac{j(k-1)}{2}} j^{m+n} u^j v^n d\nu(\zeta, \bar{\zeta}) \]

This completes the proof. \( \Box \)

**Remark 20.** Observe that for any fixed \( z_1, z_2 \neq 0 \), \((7.2)\) and \((7.3)\) can be re-casted into
\begin{align}
H_{m,n}(z_1, z_2|q) &= z_1^m z_2^n \int_{\mathbb{R}^2} \left( -\frac{i cz_1^{\frac{i}{2}}}{z_1}; q \right)_m \left( -\frac{i cz_2^{\frac{i}{2}}}{z_2}; q \right)_n d\nu(\zeta, \zeta^*) \\
\text{and} \\
q^\frac{(m-n)^2}{4} t^{m+n} h_{m,n}(z_1, z_2|q) &= \int_{\mathbb{R}^2} \zeta^m \zeta^n \left( -\frac{i z_1^{\frac{i}{2}}}{\zeta}; q \right)_m \left( -\frac{i z_2^{\frac{i}{2}}}{\zeta}; q \right)_n d\mu(\zeta, \zeta^*) .
\end{align}

Using the relation \((a; q)_n = (-qa^{-1})^n q^n(z)/ (qa^{-1}; q)_n\) for \(n = 0, 1, \ldots\) and above equations, we can extend the definitions of \(H_{m,n}(z_1, z_2|q)\) and \(h_{m,n}(z_1, z_2|q)\) to all \(m, n \in \mathbb{Z}\). Of course, we can use these equations and \((a; q)_\infty = (aq^n; q)_\infty\) to extend the definitions of \(H_{m,n}(z_1, z_2|q)\) and \(h_{m,n}(z_1, z_2|q)\) to all \(m, n \in \mathbb{C}\) where the integrals are convergent.

**Corollary 21.** Let \(a, b, z_1, z_2 \neq 0\) such that \(\left| \frac{cdq}{abz_1z_2} \right| < 1\) and \(cq^m, dq^m \neq 1, m \in \mathbb{N}\), then

\[
\sum_{m, n=0}^{\infty} \frac{(a; q)_m (b; q)_n q^{\frac{(m-n)^2}{4}} h_{m,n}(z_1, z_2|q)}{(q, c; q)_m (q, d; q)_n} \left( -\frac{c}{\sqrt{q} a z_1} \right)_m \left( -\frac{d}{\sqrt{q} b z_2} \right)_n \\
= \frac{(c/a, d/b; q)_{\infty}}{(c, d; q)_{\infty}} 2\phi_1 \left( \frac{a, b}{q a b z_1 z_2} \right) \\
= \frac{(c/a, d/b, cdq^{-1}/b z_1 z_2; q)_{\infty}}{(c, d, -cdq^{-1}/ab z_1 z_2; q)_{\infty}} 1\phi_1 \left( \frac{a}{cdq^{-1}/b z_1 z_2}, \frac{cd}{q a b z_1 z_2} \right) .
\]

The equality between the left-hand side and the extreme right-hand side holds when \(z_1 z_2 \neq 0\) without the assumption \(\left| \frac{cdq}{abz_1z_2} \right| < 1\). On the other hand if \(z_1 z_2 \neq 0\) and \(cq^m, dq^m \neq 1, m \in \mathbb{N}\) then

\[
\sum_{m, n=0}^{\infty} q^{m^2 - mn + n^2} h_{m,n}(z_1, z_2|q) c^m d^n = \frac{A_q \left( \frac{cd}{z_1 z_2} \right)}{(cq, dq; q)_\infty},
\]

where \(A_q\) is the Ramanujan function defined in \([2, 14]\). Alternately the generating function \((7.1)\) may be written as

\[
\sum_{m, n=0}^{\infty} q^{m^2 - mn + n^2} h_{m,n}(z_1, z_2|q) c^m d^n = \frac{A_q (cd)}{(cz_1 q^m, dz_2 q^m; q)_{\infty}} .
\]

which holds for any \(c, d, z_1, z_2 \in \mathbb{C}\) such that \(cz_1 q^m, dz_2 q^m \neq 1, m \in \mathbb{N}\).

**Proof.** The integral representation \((7.3)\) and Fubini’s theorem imply

\[
\sum_{m, n=0}^{\infty} \frac{(a; q)_m (b; q)_n q^{\frac{(m-n)^2}{4}} h_{m,n}(z_1, z_2|q)}{(q, c; q)_m (q, d; q)_n} \left( \frac{ic}{ za_1 \sqrt{q}} \right)_m \left( \frac{id}{ b z_2 \sqrt{q}} \right)_n \\
= \int_{\mathbb{R}^2} \sum_{m=0}^{\infty} \frac{(a, -iz_1^{\frac{i}{2}}/q; q)_m}{(q, c; q)_m} \left( \frac{ic \zeta}{ az_1 \sqrt{q}} \right)_m \sum_{n=0}^{\infty} \frac{(b, -iz_2^{\frac{i}{2}}/q; q)_n}{(q, d; q)_n} \left( \frac{id \zeta^*}{ b z_2 \sqrt{q}} \right)_n d\mu(\zeta, \zeta^*) ,
\]

where we used the \(q\)-Gauss sum in the last line, \([14 (II.8)]\). Thus the above equation equals

\[
\frac{(c/a, d/b; q)_{\infty}}{(c, d; q)_{\infty}} \int_{\mathbb{R}^2} \left( \frac{ic \zeta}{ az_1 \sqrt{q}} \frac{id \zeta^*}{ b z_2 \sqrt{q}} \right)_{\infty} d\mu(\zeta, \zeta^*) \\
= \frac{(c/a, d/b; q)_{\infty}}{(c, d; q)_{\infty}} \sum_{m, n=0}^{\infty} \frac{(a; q)_m (b; q)_n}{(q, c; q)_m (q, d; q)_n} \left( \frac{ic}{ az_1 \sqrt{q}} \right)_m \left( \frac{id}{ b z_2 \sqrt{q}} \right)_n \int_{\mathbb{R}^2} \zeta^m \zeta^n d\mu(\zeta, \zeta^*) .
\]
It is straightforward to find that
\[ \int_{\mathbb{R}^2} \zeta^m \zeta^n d\mu(\zeta, \zeta) = (q; q)_n \delta_{m,n}, \]
whence,
\[ \sum_{m,n=0}^{\infty} \frac{(a; q)_m (b; q)_n q^{(m-n)^2} h_{m,n}(z_1, z_2|q)}{(q; c; q)_m (q; d; q)_n} \left( \frac{ic}{a z_1 \sqrt{q}} \right)^m \left( \frac{id}{b z_2 \sqrt{q}} \right)^n = \]
\[ \frac{(c/a, d/b; q)_\infty}{(c, d; q)_\infty} \sum_{n=0}^{\infty} \frac{(a, b; q)_n}{(q; q)_n} \left( -\frac{cd}{ab z_1 z_2 q} \right)^n, \]
which is the first equality in (7.6). The equality between the \( \varphi_1 \) and the \( \varphi_2 \) functions is a special case of Corollary 21. The following theorem is a very interesting special case of Corollary 21.

\[ \varphi_1 \]

**Proof.** Replace \( c \) and \( d \) by \( cz_1 \) and \( dz_2 \), respectively, in (7.6) then let \( a \to \infty \). \( \square \)

A very interesting special case of Corollary 21 is the following theorem.

**Theorem 23.** Let \( cd = -q^s, s = 0, 1, \cdots \). Then the generating function
\[ \sum_{m,n=0}^{\infty} \frac{q^{m^2-n^2} h_{m,n}(z_1, z_2|q)}{(q,cz_1; q)_m (q,dz_2; q)_n} q^m q^n \]
(7.10)
holds when neither \( cz_1 \) nor \( dz_2 \) is \( -q^{-r} \) for \( r = 0, 1, \cdots \).

**Proof.** Apply (7.8) and the Garret–Ismail–Stanton result (2.15). \( \square \)

**Theorem 24.** For any \( m, n \in \mathbb{N}_0 \) and \( z_1, z_2 \in \mathbb{C} \) we have
\[ z_1^m z_2^n = \sum_{k=0}^{\infty} \frac{m!}{k!} \frac{n!}{q^k} (q; q)_k H_{m-k, n-k}(z_1, z_2|q), \]
(7.11)
and
\[ z_1^m z_2^n = q^{-mn} \sum_{k=0}^{\infty} \frac{m!}{k!} \frac{n!}{q^k} (q; q)_k q^{(2)_k} h_{m-k, n-k}(z_1, z_2|q). \]
(7.12)
Proof. From the generating function (4.6) we see that
\[
\frac{1}{(u z_1, v z_2; q)_\infty} = \frac{1}{(u v; q)_\infty} \sum_{m,n=0}^\infty H_{m,n} \left( z_1, z_2 \right| q \right) u^m v^n (q; q)_m (q; q)_n = \sum_{k=0}^\infty \sum_{m,n=0}^\infty H_{m,n} \left( z_1, z_2 \right| q \right) u^{m+k} v^{n+k} (q; q)_m (q; q)_n (q; q)_k.
\]
The expansion (7.11) follows from equating like powers of \( u \) and \( v \). Similarly we apply the generating function (4.6) and find that
\[
\left( -q^{1/2} u z_1, -q^{1/2} z_2 v; q \right)_\infty = (-u v; q)_\infty \sum_{m,n=0}^\infty h_{m,n} \left( z_1, z_2 \right| q \right) q^{(m-n)^2/2} u^m v^n (q; q)_m (q; q)_n
\]
and (7.12) follows.

Theorem 25. The connection relations between \( H_{m,n}(z_1, z_2|q) \) and \( h_{m,n}(z_1, z_2|q) \) are
\[
\begin{align*}
H_{m,n}(z_1, z_2|q) &= q^{-m n} \sum_{s=0}^{m \wedge n} \left[ m \atop s \right]_q \left[ n \atop s \right]_q (q; q)_s q^{(s)} h_{m-s, n-s}(z_1, z_2|q) \sum_{k=0}^s \left[ s \atop k \right]_q (-1)^k q^{(m+n-k)}, \\
h_{m,n}(z_1, z_2|q) &= \sum_{s=0}^{m \wedge n} \left[ m \atop s \right]_q \left[ n \atop s \right]_q (q; q)_s H_{m-s, n-s}(z_1, z_2|q) \sum_{k=0}^s \left[ s \atop k \right]_q (-1)^k q^{(m-k)(n-k)}.
\end{align*}
\]

Proof. The theorem follows from the explicit formulas (7.11), (4.6) and the connection relations (7.11) – (7.12).

Theorem 26. We have the generating functions
\[
\sum_{m,n=0}^\infty H_{m+n,k}(z_1, z_2|q) = \frac{u^m v^n}{(q; q)_m (q; q)_n}
\]
\[
\times z_1^{j-\ell} z_2^{k-\ell} \left( \frac{v q^\ell}{z_1}; q \right)_j \left( \frac{u q^\ell}{z_2}; q \right)_k \left( z_2 v; q \right)_\ell
\]
\[
= \left( \frac{u v q^{j+k}}{u z_1, v z_2; q} \right) \sum_{\ell=0}^{j \wedge k} \left[ j \atop \ell \right]_q \left[ k \atop \ell \right]_q q^{(\ell)} \left( -q; q \right)_\ell
\]
\[
\times z_1^{j-\ell} z_2^{k-\ell} \left( \frac{u q^\ell}{z_1}; q \right)_j \left( \frac{v q^\ell}{z_2}; q \right)_k \left( z_1 u; q \right)_\ell
\]
and
Observe that for

\[ \sum_{m,n=0}^{\infty} h_{m+j,n+k}(z_1, z_2; q) q^{(m-n)^2/2} \frac{u^m}{(q; q)_m} \frac{v^n}{(q; q)_n} \]

\( (-1)^{j+k} q^{\frac{u-v}{2}} \left( -uz q^{k+1} - v z q^{j+1} \right) \sum_{\ell=0}^{j+k} \left[ j \right] \left[ k \right] q^{j-k} \]

(7.16)

\( \times (-1)^\ell (q; q)_\ell u^{k-\ell} v^{j-\ell} \left( \frac{z q^{\ell}}{u} \right)_{k-\ell} \left( \frac{z z q^{\ell}}{u} \right)_{j-\ell} \left( -uv; q \right)_\ell. \)

More generally our q-disk polynomials have the generating function,

\[ \sum_{m,n=0}^{\infty} \frac{p_{m+j,n+k}(z_1, z_2; b q)}{(q; q)_m (q; q)_n} u^{m} v^{n} \]

\( = \left( bq, u v q^{i+j+k}; q \right)_{\infty} \sum_{\ell=0}^{\infty} \left( bq^{1+j+k}; u z_1, v z_2; q, u v q^{i+j+k}; q \right)_{\ell} \sum_{i=0}^{j+k} \left[ j \right] \left[ k \right] q^{j-i} q^{k-i} \]

(7.17)

\( \times (-q^{-i})^{j-i} z_1^{j-i} z_2^{k-i} q^{(j-i)}(q; q)_{j-i} \left( \frac{v}{z_1} q^{i}; q \right)_{j-i-k} \left( z z q^{i}; q \right)_{j-i} \)

\( = \left( bq, u v q^{i+j+k}; q \right)_{\infty} \sum_{\ell=0}^{\infty} \left( u z_1, v z_2; q \right)_{\ell} \left( bq^{1+j+k}; q \right)_{\ell} \sum_{i=0}^{j+k} \left[ j \right] \left[ k \right] q^{j-i} q^{k-i} \]

\( \times (-q^{-i})^{j-i} z_1^{j-i} z_2^{k-i} q^{(j-i)}(q; q)_{j-i} \left( u z_1 q^{i}; q \right)_{j-i} \left( u z_1 q^{i}; q \right)_{j-i}. \)

**Proof.** Observe that for \( k \in \mathbb{N} \) we have

\[ D_{q,u}^k \left( \frac{u^m}{(q; q)_m} \right) = \begin{cases} 0 & m < k \\ \frac{1}{(1-q)^m (q; q)_m} & m \geq k \end{cases}. \]

\[ D_{q,u}^k \left( \frac{a^m}{u} q \right) = \begin{cases} 0 & m < k \\ \frac{a^m}{(a; q)_m} q^{m-k} (\frac{a}{u}; q)_m-k & m \geq k \end{cases}. \]

\[ D_{q,u}^k ((a u; q)_m) = \begin{cases} 0 & m < k \\ \frac{m}{(k; q)_m} q^{m-k} (a u; q)_m-k & m \geq k \end{cases}. \]

and

\[ D_{q,u}^k \left( \frac{a u q^k}{(b u; q)_\infty} \right) = \left( \frac{b}{1-q} \right)^k \left( \frac{a}{b}; q \right)_k \left( a u q^k; q \right)_{\infty}. \]
Then we apply the operator \((1 - q)^{j+k} D_{q,v}^j D_{q,u}^k\) to the generating function for \(h_{m,n}(z_1, z_2|q)\) to get

\[
\sum_{m \geq j, n \geq k} h_{m,n}(z_1, z_2|q) q^{(m-n)^2/2} \frac{v^{m-j} u^{n-k}}{(q; q)_{m-j} (q; q)_{n-k}}
\]

\[
= (1 - q)^{j+k} D_{q,v}^j D_{q,u}^k \left( \frac{(-u z_1 q^{1/2}, -v z_2 q^{1/2}; q)_\infty}{(-uv; q)_\infty} \right)
\]

\[
= (1 - q)^{k} (-1)^j \left( -uv z_1 q^{j+1/2}; q \right)_\infty D_{q,v}^k \left( \frac{z_1}{v} q^{1/2}; q \right)_j \left( -uv z_2 q^{1/2}; q \right)_\infty
\]

\[
= \left( -uv z_1 q^{j+\frac{1}{2}} - v z_2 q^{k+\frac{1}{2}}; q \right)_\infty \sum_{\ell=0}^{k} \binom{j}{\ell} \binom{k}{\ell} \left( -1 \right)^\ell (q; q)_\ell \times v^{j-\ell} \left( \frac{z_1 q^{j+\frac{1}{2}}}{v}; q \right)_{j-\ell} u^{k-\ell} \left( \frac{z_2 q^{k+\frac{1}{2}}}{u}; q \right)_{k-\ell},
\]

and \((7.16)\) follows. Similarly we find,

\[
\sum_{m \geq j, n \geq k} H_{m,n}(z_1, z_2|q) q^{(m-n)^2/2} \frac{v^{m-j} u^{n-k}}{(q; q)_{m-j} (q; q)_{n-k}} = (1 - q)^{j+k} D_{q,v}^j D_{q,u}^k \left( \frac{uv; q)_\infty}{(u z_1, v z_2; q)_\infty} \right)
\]

\[
= \left( uv q^{j+k}; q \right)_\infty \sum_{\ell=0}^{j+k} \binom{j}{\ell} \binom{k}{\ell} q^{(j)} (q; q)_\ell \left( -1 \right)^j (q; q)_\ell z_1^{j-\ell} z_2^{k-\ell} \left( \frac{v q^\ell}{z_1}; q \right)_{j-\ell} \left( \frac{u q^\ell}{z_2}; q \right)_{k-\ell}
\]

and

\[
\sum_{m \geq j, n \geq k} H_{m,n}(z_1, z_2|q) q^{(m-n)^2/2} \frac{v^{m-j} u^{n-k}}{(q; q)_{m-j} (q; q)_{n-k}} = (1 - q)^{j+k} D_{q,v}^j D_{q,u}^k \left( \frac{uv; q)_\infty}{(u z_1, v z_2; q)_\infty} \right)
\]

\[
= \left( uv q^{j+k}; q \right)_\infty \sum_{\ell=0}^{j+k} \binom{j}{\ell} \binom{k}{\ell} q^{(j)} (q; q)_\ell \left( -1 \right)^j (q; q)_\ell z_1^{j-\ell} z_2^{k-\ell} \left( \frac{u q^\ell}{z_2}; q \right)_{k-\ell},
\]

which gives \((7.15)\).

More generally, we have

\[
\sum_{m \geq j, n \geq k} \frac{p_{m,n}(z_1, z_2; b|q)}{(q; q)_{m-j} (q; q)_{n-k}} q^{m-j} u^{n-k} = (1 - q)^{j+k} D_{q,v}^j D_{q,u}^k \left( \frac{uv; q)_\infty}{(u z_1, v z_2; q)_\infty} \right)
\]

\[
= (1 - q)^{j+k} (b q; q)_\infty \sum_{\ell=0}^{j+k} \binom{b q^{1+j+k}}{\ell} (q; q)_\ell \left( u z_1 q^{j+\ell}; q \right)_\infty D_{q,v}^\ell \left( \frac{v}{z_1}; q \right)_j \left( \frac{uv q^{j+k}; q)_\infty}{(z_2 v q^{j+k}; q)_\infty} \right)
\]

\[
= (b q; q)_\infty \sum_{\ell=0}^{j+k} \binom{b q^{1+j+k}}{\ell} (q; q)_\ell \left( u z_1 q^{j+\ell}; q \right)_\infty \sum_{i=0}^{\infty} \binom{j}{i} \binom{k}{i} \left( -q^{-i} \right)^i \left( z_2 q^{-k-i} q^{(i)} \right) (q; q)_i \left( \frac{v}{z_1}; q \right)_j \left( \frac{u q^{j+k}; q)_k}{(z_2 v q^{j+k}; q)_k} \right)
\]

\[
\times \sum_{i=0}^{\infty} \binom{j}{i} \binom{k}{i} \left( -q^{-i} \right)^i \left( z_2 q^{-k-i} q^{(i)} \right) (q; q)_i \left( \frac{v}{z_1}; q \right)_j \left( \frac{u q^{j+k}; q)_k}{(z_2 v q^{j+k}; q)_k} \right)\]
which gives
\[
\sum_{m,n=0}^{\infty} \frac{p_{m+j,n+k}(z_1, z_2; b|q)}{(q; q)_m (q; q)_n} u^m v^n \\
= \frac{(b q, u v q^{j+k}; q)_\infty}{(u z_1, v z_2; q)_\infty} \sum_{\ell=0}^{\infty} \frac{(b q^{1+j+k}; q)_\ell}{(q, u v q^{j+k}; q)_\ell} \sum_{i=0}^{\infty} \left[ \begin{array}{c} j \\ i \end{array} \right]_q \left[ \begin{array}{c} k \\ i \end{array} \right]_q \\
\times (-q^{-\ell}) i z_1^{-i} z_2^{-k-i} q^{j+i}_q(q; q)_i \left( \frac{v}{z_1} q^i; q \right)_{j-i} \left( \frac{u}{z_2} q^j; q \right)_{k-i} (z_2 v q^k; q)_i.
\]
Similarly we have,
\[
\sum_{m \geq j, n \geq k}^{\infty} \frac{p_{m,n}(z_1, z_2; b|q)}{(q; q)_{m-j} (q; q)_{n-k}} u^{m-j} v^{n-k} \\
= (1 - q)^{j+k} (b q; q)_\infty \sum_{\ell=0}^{\infty} \frac{(b q^{j+k+1}; q)_\ell}{(q, u v q^{j+k}; q)_\ell} D_{q,u}^j D_{q,v}^k \left( \frac{u}{z_2} q; k \right) (z_2 v q^{j+k}; q)_\infty \\
= (1 - q)^{j+k} z_2^{-j} (b q; q)_\infty \sum_{\ell=0}^{\infty} \frac{(b q^{j+k+1}; q)_\ell}{(q, u v q^{j+k}; q)_\ell} D_{q,u}^j D_{q,v}^k \left( \frac{u}{z_2} q; k \right) (z_2 v q^{j+k}; q)_\infty \\
\times \sum_{i=0}^{\infty} \left[ \begin{array}{c} j \\ i \end{array} \right]_q \left[ \begin{array}{c} k \\ i \end{array} \right]_q (-q^{-\ell}) i (q; q)_i \\
\times z_1^{-i} z_2^{-k-i} \left( \frac{u q^i}{z_2} q; q \right)_{k-i} \left( \frac{v q^k}{z_1} q; q \right)_{j-i} (u z_1 q^j; q)_i
\]
which gives
\[
\sum_{m,n=0}^{\infty} \frac{p_{m+j,n+k}(z_1, z_2; b|q)}{(q; q)_m (q; q)_n} u^m v^n \\
= \frac{(b q, u v q^{j+k}; q)_\infty}{(u z_1, v z_2; q)_\infty} \sum_{\ell=0}^{\infty} \frac{(u z_1, v z_2; q)_\ell}{(q, u v q^{j+k}; q)_\ell} (b q^{j+k+1})_\ell \\
\times \sum_{i=0}^{\infty} \left[ \begin{array}{c} j \\ i \end{array} \right]_q \left[ \begin{array}{c} k \\ i \end{array} \right]_q (-q^{-\ell}) i z_1^{-i} z_2^{-k-i} q^{j+i}_q(q; q)_i \left( \frac{u q^i}{z_2} q; q \right)_{k-i} \left( \frac{v q^k}{z_1} q; q \right)_{j-i} (u z_1 q^j; q)_i,
\]
which establishes \( \text{Theorem 27} \). \( \square \)

8. Zeros

In this section we study the zeros of the two \( 2D-q \)-Hermite polynomials and the \( q \)-analogue of the Zernike polynomials introduced in this paper. Because all polynomials factor as a function of \( \theta \) times a radial function it is clear that with \( z_1 = z, z_2 = \bar{z} \) the zeros of the polynomials investigated here as functions of \( z \) lie on circles.

Let
\[(8.1) \quad 0 < i_1(q) < i_2(q) < \cdots ,\]
be the zeros of \( A_q(z) \).

**Theorem 27.** Assume that the zeros of \( H_{m,n}(z, \bar{z}|q) \) and of \( h_{m,n}(z, \bar{z}|q) \) lie on the circles with radii
\[(8.2) \quad r_1(H, m, n) > r_2(H, m, n) > \cdots , \quad \text{and} \quad r_1(h, m, n) > r_2(h, m, n) > \cdots \]
respectively. Moreover let the zeros of \( p_{m,n}(z,\bar{z};b|q) \) lie on the circle \( |z| = r_j(p, m, n), j = 1, 2, \ldots \), ordered as
\[
(8.3) \quad r_1(p, m, n) > r_2(p, m, n) > \cdots.
\]

Then
\[
(8.4) \quad \lim_{m, n \to \infty} r_j(H, m, n) = q^{j/2}, \quad j = 1, 2, \ldots,
\]
\[
(8.5) \quad \lim_{m, n \to \infty} q^{(m+n)/2} r_j(h, m, n) = 1/\sqrt{i_j(q)}, \quad j = 1, 2, \ldots,
\]
\[
(8.6) \quad \lim_{m, n \to \infty} r_j(p, m, n) = q^{j/2}, \quad j = 1, 2, \ldots.
\]

**Proof.** The first part, (8.4), follows from (3.24) since its left-hand side converges to its right-hand side on compact subsets of \( \mathbb{C} \). Similarly (8.6) follows from (5.43). Formula (8.5) follows from Theorem 12 since the limit in Theorem 12 is uniform on compact subsets of \( \mathbb{C} \). \( \square \)

It is important to note that the support of the orthogonality measure of \( \{H_{m,n}(z,\bar{z}|q)\} \) and \( \{p_{m,n}(z,\bar{z};b|q)\} \) coincides with the closure of the union of the limiting circles on which the zeros lie. This is similar to the single variable case. It is not surprising that the zeros of the Ramanujan function appear in the leading terms of the asymptotics of zeros of the polynomials \( \{h_{m,n}(z,\bar{z}|q)\} \). This is again similar to the single variable case.

### 9. Positivity Results

**Lemma 28.** For \( N \in \mathbb{N}_0, q \in (0, 1) \) and \( z \in \mathbb{C} \setminus \{0\} \), the following matrices are positive definite
\[
(9.1) \quad \left( \frac{H_{m,n}(iz,iz|q)}{i^{m+n}} \right)_{m,n=0}^N, \quad \left( q^{mn} h_{m,n}(z,\bar{z}|q^{-1}) \right)_{m,n=0}^N,
\]
\[
(9.2) \quad \left( \frac{q^{(m+n)^2} h_{m,n}(iz,iz|q)}{i^{m+n}} \right)_{m,n=0}^N, \quad \left( H_{m,n}(z,\bar{z}|q^{-1}) \right)_{m,n=0}^N.
\]

**Proof.** Observe that
\[
H_{m,n}(iz,iz|q) = q^{mn} h_{m,n}(z,\bar{z}|q^{-1})
\]
\[
= \sum_{k=0}^\infty q^{(\ell)k} q^{-\ell} \left\{ \begin{array}{c} m \\ k \end{array} \right\}_{q} z^m \left\{ \begin{array}{c} n \\ k \end{array} \right\}_{q} z^n
\]
and
\[
q^{(m+n)^2} h_{m,n}(iz,iz|q) = q^{m^2} q^{n^2} H_{m,n}(z,\bar{z}|q^{-1})
\]
\[
= \sum_{j=0}^\infty q^{(\ell)j} q^{-\ell} \left\{ \begin{array}{c} m \\ j \end{array} \right\}_{q} q^{n^2 - mj} z^m \left\{ \begin{array}{c} n \\ j \end{array} \right\}_{q} q^{m^2 - nj} z^n.
\]
\( \square \)

**Theorem 29.** For \( z \neq 0 \) and \( 0 < q < 1 \), there exist sequences \( e_n = \{e_m^{(n)}\}_{m=0}^\infty \) and \( f_n = \{f_m^{(n)}\}_{m=0}^\infty \) with \( e_m^{(n)} = f_m^{(n)} = 0 \) for \( m > n \) such that
\[
(9.3) \quad \int_{\mathbb{R}^2} \sum_{m=0}^\infty \frac{e^{(j)}_m z^m}{z^{2\ell}} \left( -\zeta q^{\frac{j}{z}} \right)_q m \left\{ \sum_{n=0}^k e^{(k)}_n z^n \left( -\zeta q^{\frac{k}{z}} : q \right)_n \right\} d\nu(\zeta, \bar{\zeta})
\]
\[
= \sum_{\ell=0}^\infty \frac{q^{(\ell)}(q; q)_\ell}{|z|^{2\ell}} \sum_{m=\ell}^\infty \left\{ \begin{array}{c} m \\ \ell \end{array} \right\}_{q} e^{(j)}_m z^m \left\{ \sum_{n=\ell}^k \left\{ \begin{array}{c} n \\ \ell \end{array} \right\}_{q} e^{(k)}_n z^n \right\} = \delta_{j,k}.
\]
and

$$\int_{\mathbb{R}^2} \left\{ \sum_{m=0}^{j} f_m^{(j)} (-\zeta)^m \left( \frac{z q^\frac{1}{2}}{z} q \right)_m \right\} \cdot \left\{ \sum_{n=0}^{k} f_n^{(k)} (-\zeta)^n \left( \frac{z q^\frac{1}{2}}{z} q \right)_n \right\} d\mu (\zeta, \bar{\zeta})$$

\[(9.4)\]

$$= \sum_{\ell=0}^{\infty} q^{2 \ell} \left( q ; q \right)_\ell \left\{ \sum_{m=\ell}^{j} \left[ m \atop \ell \right] q^{\frac{m}{2} - m \ell} z m f_m^{(j)} \right\} \left\{ \sum_{n=\ell}^{k} \left[ n \atop \ell \right] q^{\frac{n}{2} - n \ell} z n f_n^{(k)} \right\} = \delta_{j,k}$$

for \(j, k = 0, 1, 2, \ldots\).

**Proof.** Let us define the following inner vector spaces

$$H(N_0; z, q) = \left\{ \{c_n\}_{n=0}^{\infty} \big| c_n \in \mathbb{C}, n \in N_0, \sum_{m,n=0}^{\infty} H_{m,n} (iz, i\bar{z}) \frac{c_m \bar{c}_n}{i^{m+n}} < \infty \right\}$$

with

$$\langle \{c_n\}_{n=0}^{\infty}, \{d_n\}_{n=0}^{\infty} \rangle_H = \sum_{m,n=0}^{\infty} H_{m,n} (iz, i\bar{z}) \frac{c_m \bar{d}_n}{i^{m+n}},$$

where \(\{c_n\}_{n=0}^{\infty}, \{d_n\}_{n=0}^{\infty} \in H(N_0; z, q)\) and

$$h(N_0; z, q) = \left\{ \{c_n\}_{n=0}^{\infty} \big| c_n \in \mathbb{C}, n \in N_0, \sum_{m,n=0}^{\infty} \frac{(m-n)^2}{2} H_{m,n} (iz, i\bar{z}) q \frac{c_m \bar{c}_n}{i^{m+n}} < \infty \right\}$$

with

$$\langle \{c_n\}_{n=0}^{\infty}, \{d_n\}_{n=0}^{\infty} \rangle_h = \sum_{m,n=0}^{\infty} \frac{q^{(m-n)^2}}{2} H_{m,n} (iz, i\bar{z}) q \frac{c_m \bar{d}_n}{i^{m+n}},$$

where \(\{c_n\}_{n=0}^{\infty}, \{d_n\}_{n=0}^{\infty} \in h(N_0; z, q)\). Then, the vectors \(\{\delta_{m,n}\}_{m=0}^{\infty}, \ n = 0, 1, \ldots\) are linearly independent in these spaces. For \(n \in N_0\), let \(e_n = \{e_m(n)\}_{m=0}^{\infty}\) and \(f_n = \{f_m(n)\}_{m=0}^{\infty}\) be the obtained orthonormal bases from the orthogonalization process in \(H(N_0; z, q)\) and \(h(N_0; z, q)\) respectively, then it is clear that \(e_m(n) = f_m(n) = 0\) for \(m > n\). Observe that

$$\frac{H_{m,n} (iz, i\bar{z})}{i^{m+n}} = z^{m-n} \int_{\mathbb{R}^2} \left( -\frac{\zeta q^\frac{1}{2}}{z} q \right)_m \left( -\frac{\bar{\zeta} q^\frac{1}{2}}{z} q \right)_n d\nu (\zeta, \bar{\zeta}),$$

then,

$$\left\{ \{e(n)\}_{m=0}^{\infty}, \{e(m)\}_{m=0}^{\infty} \right\}_H = \sum_{m,n=0}^{\infty} \frac{H_{m,n} (iz, i\bar{z}) e_m(n) \bar{e}_n(n)}{i^{m+n}} = \delta_{j,k},$$

and

$$\sum_{\ell=0}^{\infty} q^{2\ell} \left( q ; q \right)_\ell \left\{ \sum_{m=\ell}^{j} \left[ m \atop \ell \right] q^{\frac{m}{2} - m \ell} z m e_m(n) \right\} \left\{ \sum_{n=\ell}^{k} \left[ n \atop \ell \right] q^{\frac{n}{2} - n \ell} z n e_n(n) \right\} \int_{\mathbb{R}^2} \left( -\zeta q^\frac{1}{2} \right)_m \left( -\bar{\zeta} q^\frac{1}{2} \right)_n d\nu (\zeta, \bar{\zeta}).$$

Similarly, from

$$\frac{q^{(m-n)^2}}{2} H_{m,n} (iz, i\bar{z}) = \int_{\mathbb{R}^2} \left( -\zeta \right)^m \left( -\bar{\zeta} \right)^n \left( \frac{z q^\frac{1}{2}}{z} q \right)_m \left( \frac{\bar{z} q^\frac{1}{2}}{z} q \right)_n d\mu (\zeta, \bar{\zeta}),$$
we get
\[
\left(\left\{ f_m^{(j)} \right\}_{m=0}^{\infty}, \left\{ f_m^{(k)} \right\}_{m=0}^{\infty}\right)_{\mathbb{H}} = \sum_{m,n=0}^{\infty} q^{\frac{(m-n)^2}{2}} h_{m,n}(iz,izf) f_m^{(j)} f_n^{(k)} = \delta_{j,k}
\]
\[
= \sum_{\ell=0}^{\infty} \frac{q^{\ell^2}}{|z|^{2\ell}} \left\{ \sum_{m=\ell}^{j} \left[ \begin{array}{cc} m \\ \ell \end{array} \right] q^{n-m}\ell z^m f_m^{(j)} \right\} \left\{ \sum_{n=\ell}^{k} \left[ \begin{array}{cc} n \\ \ell \end{array} \right] q^{n-\ell}z_n f_n^{(k)} \right\}
\]
\[
= \int_{\mathbb{R}^2} \left\{ \sum_{m=0}^{j} f_m^{(j)} (-\zeta)^m \left( \frac{zq^\ell}{\zeta} ; q \right)_m \right\} \cdot \left\{ \sum_{n=0}^{k} f_n^{(k)} (-\zeta)^n \left( \frac{zq^\ell}{\zeta} ; q \right)_n \right\} d\mu(\zeta, \bar{\zeta}).
\]

Remark 30. If we could inverse the matrices \((9.1)\) and \((9.2)\), then we can determine \(e_n, f_n, \ n = 0, 1, \ldots \) explicitly.

Lemma 31. For \(z \cdot \zeta \neq 0\) and \(m = 0, 1, \ldots \) we have
\[
\zeta^m = \sum_{j=0}^{m} \left[ \begin{array}{cc} m \\ j \end{array} \right] q^{(m-j)} \left( -\zeta q^{1/2} / z ; q \right)_j z^j
\]
and
\[
\zeta^m = \sum_{j=0}^{m} \left[ \begin{array}{cc} m \\ j \end{array} \right] q^{(m-j)} \left( \frac{zq^{1/2}}{\zeta} ; q \right)_j \zeta^j.
\]

Proof. From the \(q\)-binomial theorem we have
\[
\sum_{m=0}^{\infty} \frac{(-\zeta q^{1/2} / z ; q)_m (zt)^m}{(q ; q)_m} = \frac{(-\zeta tq^{1/2} ; q)_\infty}{(zt ; q)_\infty},
\]
to get
\[
\left( -\zeta tq^{1/2} ; q \right)_\infty = \sum_{m=0}^{\infty} \frac{t^m}{(q ; q)_m} q^{m^2/2} \zeta^m
\]
\[
= \sum_{m=0}^{\infty} \frac{t^m}{(q ; q)_m} \sum_{j=0}^{m} \left[ \begin{array}{cc} m \\ j \end{array} \right] q^{(m-j)} \left( -z \right)^{m-j} \left( -\zeta q^{1/2} / z ; q \right)_j z^j,
\]
and \((9.5)\) is obtained by matching the coefficients of \(t^m\).

Similarly, from
\[
1 \left( (zt ; q)_\infty \right) = \frac{1}{(ztq^{1/2} ; q)} \sum_{k=0}^{\infty} \frac{(\zeta t)^k}{(q ; q)_k}
\]
to get \((9.6)\).

Corollary 32. Let \(z \neq 0\) and \(0 < q < 1\), then for \(j, k = 0, 1, \ldots \) we have
\[
\sum_{\ell=0}^{\infty} \frac{q^{\ell^2}}{|z|^{2\ell}} e_j (\ell;q) e_k (\ell;q) = \frac{(q ; q)_j \log q^{-1}}{q^{(j+1)/2}} |z|^{2j} \delta_{j,k},
\]
and
\[
\sum_{\ell=0}^{\infty} \frac{q^{\ell^2}}{|z|^{2\ell}} f_j (\ell;q) f_k (\ell;q) = \frac{\delta_{j,k}}{(q^{1/2}; q)_\infty |z|^{2j}},
\]
where
\[
e_j (\ell;q) = \sum_{m=\ell}^{j} \left[ \begin{array}{cc} m \\ \ell \end{array} \right] q^{(j-m)} (-1)^{j-m}
\]

and

\[ f_j (\ell | q) = \sum_{m=\ell}^{j} \binom{m}{\ell}_q \binom{j}{m}_q q^{\frac{m^2}{2} - m\ell} \left( -q^{1/2} \right)^{j-m}. \]

**Proof.** Observe that

\[
\delta_{j,k} \left( \frac{q}{q^2} \right)_j \log q^{-1} = \int_{\mathbb{R}^2} \zeta^k \overline{\zeta} \, d\nu (\zeta, \overline{\zeta})
\]

\[
= \int_{\mathbb{R}^2} \left\{ \sum_{m=0}^{j} \left[ \begin{array}{c} j \\ m \end{array} \right]_q q^{(j-m)/2} \left( -z \right)^{j-m} \left( -\zeta q^{1/2} / z ; q \right)_m z^m \right\}
\times \left\{ \sum_{n=0}^{k} \left[ \begin{array}{c} k \\ n \end{array} \right]_q q^{(k-n)/2} \left( -z \right)^{k-n} \left( -\zeta q^{1/2} / z ; q \right)_n z^n \right\} d\nu (\zeta, \overline{\zeta}).
\]

Let us take

\[ e^{(j)}_m = \frac{q^{(j+1)} \left( q^2 \right)_j \log q^{-1}}{q^{(j+1)} \left( q^2 \right)_j \log q^{-1}} \left[ \begin{array}{c} j \\ m \end{array} \right]_q q^{(j-m)/2} \left( -z \right)^{j-m}
\]

to get \((9.7)\) and \((9.8)\). Similarly, from

\[
\delta_{j,k} \left( \frac{q^2}{q^2} \right)_j \log q^{-1} = \int_{\mathbb{R}^2} \zeta^k \overline{\zeta} \, d\mu (\zeta, \overline{\zeta})
\]

\[
= \int_{\mathbb{R}^2} \left\{ \sum_{m=0}^{j} \left[ \begin{array}{c} j \\ m \end{array} \right]_q q^{(j-m)/2} \left( zq^{1/2} \right)^{j-m} \left( \frac{zq^{1/2} / \zeta ; q}{m} \right) \right\}
\times \left\{ \sum_{n=0}^{k} \left[ \begin{array}{c} k \\ n \end{array} \right]_q q^{(k-n)/2} \left( zq^{1/2} \right)^{k-n} \left( \frac{zq^{1/2} / \zeta ; q}{n} \right) \right\} d\mu (\zeta, \overline{\zeta}),
\]

we take

\[ f^{(j)}_m = \sqrt{q^{(j+1)} \left( q^2 \right)_j} \left[ \begin{array}{c} j \\ m \end{array} \right]_q \left( -zq^{1/2} \right)^{j-m}
\]

to obtain \((9.9)\) and \((9.10)\). \(\square\)

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