Transfer matrix approach to determining the linear response of all-fiber networks of cavity-QED systems

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Abstract: A semiclassical model is presented for characterizing the linear response of elementary quantum optical systems involving cavities, optical fibers, and atoms. Formulating the transmission and reflection spectra using a scattering-wave (transfer matrix) approach, the calculations become easily scalable. To demonstrate how useful this method is, we consider the example of a simple quantum network, i.e., two cavity-QED systems connected via an optical fiber. Differences between our quasi-exact transfer matrix approach and a single-mode, linearized quantum-optical model are demonstrated for parameters relevant to recent experiments with coupled nanofiber-cavity-QED systems.

I. INTRODUCTION

Fiber-optic systems are excellent candidates for the implementation of large-scale quantum networks owing to low propagation losses and to the variety of optical components that can be readily incorporated in a given setup [1]. These include, as a result of recent developments in the field, short lengths of tapered nanofiber with waists on the order of 400 nm, which are carefully tailored using a heat-and-pull method [2–4]. These in turn enable efficient trapping of atoms in the evanescent field of the nanofiber using laser fields at red- and blue-detuned magic wavelengths [5, 6]. Efficient coupling of atomic excitations into guided fiber modes, and vice versa, has introduced new ways of implementing key elements of quantum communication or computation setups. For example, a quantum memory for single-photon states has been demonstrated using a cloud of cold atoms around a nanofiber [7, 8], while chiral coupling of atoms to waveguide modes has been proposed [9], and signatures of super- and subradiant behaviour of atoms coupled to a nanofiber field have been observed [10, 12].

In order to increase the coupling between the trapped atoms and the light field, fiber Bragg grating (FBG) mirrors confine the optical modes longitudinally [13–15]. An important characteristic of fiber quantum-electrodynamic (fiber-QED) systems is that the cooperativity does not depend on the length of the fiber because of its continuous mode structure. Thus, as the effective mode area is greatly reduced in a tapered nanofiber, even a cavity with relatively low finesse and long length can reach the strong coupling regime [15, 16].

Recent experiments have measured the transmission spectra of a network of cavity quantum electrodynamic (cavity-QED) systems, where two ensembles containing some tens of (cesium) atoms trapped in the evanescent fields of nanofiber cavities were connected by a standard single-mode optical fiber. The dressed states of the atoms with the optical normal modes (cavity-fiber-cavity) were identified in the output spectra [17]. In the same experiment a fiber-dark mode was observed, which could serve as a robust and coherent coupling channel between the two cavity-QED systems. Reflection and transmission spectra from a similar setup showed signatures of normal modes consisting of atomic and optical excitations. Of particular interest was a cavity-dark mode, which couples the atoms via the connecting fiber mode, but without any photons in the cavities [18].

Theoretical output spectra of fiber-optic systems can be obtained from the coherent dynamics of single optical modes and atoms, with incoherent losses accounted for in a master equation approach. However, in the case of the above-mentioned experiments, strong coupling was achieved with cavity lengths on the order of a meter and mirror reflectances as low as 60%. Thus, even though one can obtain good qualitative agreement between theory and experiment, some quantitative discrepancies between single-mode models and the experimental results point to an issue with the usual assumption of high mirror reflectances (i.e., low bandwidth) [17, 18].

In this work, we present an alternative description of the output spectra in the weak driving limit using the well-known concept of transfer matrices. In this regime the atoms can be approximated as linearly polarizable systems [19, 22]. This description is analogous to the one obtained using the input-output relations for weak driving [23]. The transfer matrices characterize the propagation and scattering of traveling waves; thus, they are well-suited to the analysis of large-scale networks. While this method is restricted to the linear regime, where the atoms are not excited substantially, it offers an important check of the validity of single-mode quantum-optical models, which can of course also be applied in the stronger driv-
ing, non-linear regime. The present work also offers significant computational advantages over the single-mode quantum models when considering systems with multiple cavities, coupling fibers, and atoms.

The paper is structured as follows. First we introduce the formalism that we apply to describe the various network components using a transfer matrix approach. Then, we proceed to consider the cases of (i) an empty cavity, (ii) a single cavity-QED system, (iii) two fiber-coupled cavities, and (iv) two fiber-coupled cavity-QED systems, using both the transfer matrix model and the simplest quantum optical model, in which the cavities and coupling fiber are each modeled by single modes. We also show that in certain cases where differences emerge between output spectra obtained from the two models, better agreement can be recovered by simultaneously driving counter-propagating fields in the transfer matrix model in order to better “mimic” excitation of a (single) standing-wave mode in the quantum optical model.

II. SCATTERING AND TRANSFER MATRICES FOR A LINEAR FOUR-PORT DEVICE

First we review the scattering and transfer matrices for a linear four-port device (2 inputs and 2 outputs, e.g., a beam splitter), as shown schematically in Fig.1(a). The scattering matrix $S$ describes the relationship between the two output fields $E_{1(2)}^{(\text{out})}$ and the two input fields $E_{1(2)}^{(\text{in})}$,

$$
\begin{pmatrix}
E_{1(2)}^{(\text{out})}
\end{pmatrix} = S 
\begin{pmatrix}
E_{1(2)}^{(\text{in})}
\end{pmatrix}.
$$

The scattering matrix elements $S_{ij}$ are the amplitude transmission/reflection coefficients defined in Fig.1(b),

$$
S = \begin{pmatrix}
S_{11} & S_{12} \\
S_{21} & S_{22}
\end{pmatrix}.
$$

On the other hand, the transfer matrix $T$ relates the fields on the left-hand side, $E_{1(2)}^{(\text{in})}$ and $E_{2(2)}^{(\text{out})}$, to the fields on the right-hand side, $E_{1(2)}^{(\text{out})}$ and $E_{2(2)}^{(\text{in})}$,

$$
\begin{pmatrix}
E_{1(2)}^{(\text{in})} \\
E_{2(2)}^{(\text{out})}
\end{pmatrix} = T \begin{pmatrix}
E_{1(2)}^{(\text{out})} \\
E_{2(2)}^{(\text{in})}
\end{pmatrix}.
$$

The transfer matrix elements $T_{ij}$ can be expressed in terms of $S_{ij}$ via

$$
T = \begin{pmatrix}
T_{11} & T_{12} \\
T_{21} & T_{22}
\end{pmatrix} = \frac{1}{S_{11}} \begin{pmatrix}
1 & -S_{12} \\
S_{21} & S_{22}
\end{pmatrix},
$$

while the scattering matrix elements $S_{ij}$ are expressed in terms of $T_{ij}$ as

$$
S = \frac{1}{T_{11}} \begin{pmatrix}
1 & -T_{12} \\
T_{21} & T_{22} - T_{12}T_{21}
\end{pmatrix}.
$$

Therefore, the transfer matrix for this series of devices is given by the product of the transfer matrices of the individual elements,

$$
T = T_1 T_2 \cdots T_N.
$$

In the following, we introduce the scattering and transfer matrices for specific fiber-network components.

A. Beam splitter/Mirror

The first example is a lossless beam splitter. The scattering matrix of such a device also applies for a partially reflecting mirror and can be expressed by a unitary matrix [24],

$$
S^{(M)} = e^{i\phi_0} \begin{pmatrix}
e^{i\phi_t} \sqrt{T} & e^{i\phi_t} \sqrt{R} \\
e^{-i\phi_t} \sqrt{R} & e^{-i\phi_t} \sqrt{T}
\end{pmatrix},
$$

where $R$ and $T = 1 - R$ are the reflectance and transmittance of the beam splitter, respectively. For the case of a lossy beam splitter, $T + R < 1$ and $S^{(M)}$ is no longer unitary.

In the following, for the sake of simplicity, we take $\phi_0 = -\pi/2$, $\phi_t = \pi$, $\phi_t = \pi/2$, so that the amplitude transmission/reflection coefficients for both input ports.
are the same and the reflection coefficients are real, $S_{11} = S_{22}$, $S_{12} = S_{21} \in \mathbb{R}$, i.e.,

$$S^{(M)} = \begin{pmatrix} i\sqrt{T} & \sqrt{R} \\ \sqrt{R} & i\sqrt{T} \end{pmatrix}. \quad (9)$$

The corresponding transfer matrix is given by

$$T^{(M)} = \frac{i}{\sqrt{T}} \begin{pmatrix} -1 & \sqrt{R} \\ -\sqrt{R} & T + R \end{pmatrix}. \quad (10)$$

### B. Free propagation

![Figure 3: Propagation over a distance $l$.]

The transfer matrix for free propagation in a lossless medium over a distance $l$ (Fig. 3) is given by

$$T^{(l)} = \begin{pmatrix} e^{-ikl} & 0 \\ 0 & e^{ikl} \end{pmatrix} = \begin{pmatrix} e^{-\omega l/c} & 0 \\ 0 & e^{\omega l/c} \end{pmatrix}, \quad (11)$$

where $k$, $\omega$, $c$ are the wave number, the angular frequency, and the velocity of the propagating wave, respectively.

![Figure 4: Linear loss.]

The transfer matrix for a linear loss $\alpha$ (Fig. 4) is given by

$$T^{(\alpha)} = \begin{pmatrix} 1/\sqrt{1-\alpha} & 0 \\ 0 & \sqrt{1-\alpha} \end{pmatrix} = \begin{pmatrix} 1/\sqrt{\eta} & 0 \\ 0 & \sqrt{\eta} \end{pmatrix}, \quad (12)$$

where $\eta = 1 - \alpha$ is the transmission efficiency. One can also incorporate a linear loss by replacing the real wave number $k$ in Eq. (11) with the complex wave number $k_j$,

$$k_j = k + ik_j', \quad k_j' = \text{Im}k_j = -\frac{\ln \eta_j}{2l}, \quad \left( |e^{ik_j}|^2 = e^{-2k_j' l} = \eta_j \right). \quad (13)$$

Here the indices $j$ refer to the distinct spatial regions of the setup (e.g., cavities) from left to right.

### C. Two-level atom in a waveguide

The coupling rate between an atom and an optical mode describes the strength of the interaction between the dipole moment $d$ of the considered atomic transition and the local electric field produced by the optical mode. Therefore, after mode quantization, it takes the following form (see, e.g., [29]),

$$g = d \sqrt{\frac{\omega_c}{2\varepsilon_0 h V}}, \quad (14)$$

where $V = \frac{Al}{2}$ for a Fabry-Pérot cavity of length $l$ and cross-sectional area $A$. Coupling an atom to a waveguide can be described similarly, but a continuous spectrum of optical modes has to be considered. The transition from one regime to the other can be described via [27, 28]

$$g_W = \frac{1}{\sqrt{\Delta k}} \frac{1}{\sqrt{\Delta k}} g = \frac{1}{\sqrt{\Delta k}} \frac{1}{\sqrt{\Delta k}} g = d \sqrt{\frac{\omega_W}{4\pi\varepsilon_0 h V}}, \quad (15)$$

where the mode volume for a waveguide of length $l'$ is $V = \frac{Al'}{2}$. The above expression also highlights the fact that the coupling strength between the atoms and the waveguide is independent of the length. Thus, we have the following relationship between the two coupling strengths:

$$g = g_W \sqrt{\frac{4\pi}{l}}. \quad (16)$$

![Figure 5: Two-level atom coupled to a one-dimensional waveguide.]

Let us assume that an atom couples to forward and back-propagating one-dimensional guided modes (Fig. 5). Then, considering weak coherent driving fields, the atomic dynamics becomes linear with respect to these fields. Using a real-space version of the input-output relations leads to the following transfer matrix [29],

$$T^{(A)} = \begin{pmatrix} 1 - i\xi & -i\xi \\ i\xi & 1 + i\xi \end{pmatrix}, \quad (17)$$

where

$$\xi = -\frac{\Gamma_{1D}/\Gamma'}{i + 2\Delta_A/\Gamma'}. \quad (18)$$

In Eq. (18), $\Gamma_{1D}$ is the radiative intensity decay rate into the guided mode, given by

$$\Gamma_{1D} = \frac{4\pi g_W^2}{\eta_g}, \quad (19)$$
where \( g_W \) and \( v_g \) are the above defined atom-waveguide coupling rate and the group velocity, respectively.

\( \Gamma' \) is the energy decay rate into all the other modes, and we can usually assume that this decay rate is the same as that in free space, \( \Gamma' \approx \Gamma_0 (= 2\gamma) \). Finally, \( \Delta_A = \omega - \omega_A \) is the probe-atom detuning, where \( \omega_A \) is the atomic transition frequency.

The narrow widths of nanofibers support a special instance of atomic decay into guided modes that is commonly cited as chiral coupling [23]. In this case the atom is only coupled to the field propagating in one direction and not the other. This is included in the above formalism by setting \( \xi \) to 0 in the first row of Eq. (17) when the atom is coupled to the left-propagating mode and in the second row when it is coupled to the right-propagating one.

It is worth comparing the Purcell effect for a waveguide with that for a cavity. As mentioned above, a waveguide enhances the spontaneous emission of an atom coupled to the guided mode, and its decay rate is given by \( \Gamma_{1D} \). On the other hand, a cavity with a large field decay rate \( \kappa \gg (g, \gamma) \) gives rise to a cavity-enhanced spontaneous emission rate given by

\[
\Gamma_{\text{cav}} = 2C\Gamma_0 = 4C\gamma,
\]

where \( C = g^2/(\kappa\gamma) \) is the cooperativity parameter. We can see that

\[
\Gamma_{\text{cav}} = \frac{4g^2}{\kappa} = \frac{8\pi g_W^2}{\kappa l} = \frac{8\pi g_W^2}{v_g(T_1 + T_2 + 2\alpha)/4},
\]

where \( T_{1,2} \) are the transmittances of the two cavity mirrors and \( \alpha \) the intrinsic cavity loss. For \( T_1 = T_2 = 1 \) and \( \alpha = 0 \) we find \( \Gamma_{\text{cav}} = 4\Gamma_{1D} \). The factor of 4 is related to the difference between the travelling-wave and standing-wave description of the interaction. More specifically, a factor of 2 comes from the difference between the mode volume for a cavity and a waveguide, as discussed above. The other factor of 2 comes from the fact that in a waveguide two counter-propagating modes are considered, rather than a single cavity mode [23]. Therefore the same amount of intensity decays into two modes instead of one.

In the following we apply this formalism to simple examples of all-fiber networks. The similarities and differences between the results obtained by the transfer matrix approach and the quantum optical model are highlighted. Note that the specific parameter values that we use for the presented numerical results are largely inspired by the experimental platforms of [17][18].

### III. EMPTY FABRY-PÉROT CAVITY

Here we consider a standing-wave, (fiber) Fabry-Pérot cavity bounded by two FBG mirrors at a distance \( l \), as shown in Fig. 6.

![Figure 6: Empty Fabry-Pérot cavity.](image)

#### A. Transfer matrix approach

The transfer matrix for the cavity is given by

\[
\mathcal{T}^{(\text{FP})} = \mathcal{T}^{(M1)} \mathcal{T}^{(l)} \mathcal{T}^{(M2)}
\]

\[
= \frac{i}{\sqrt{T_1}} \begin{pmatrix} -1 & \sqrt{R_1} \\ -\sqrt{R_1} & T_1 + R_1 \end{pmatrix} \begin{pmatrix} e^{-ikl} & 0 \\ 0 & e^{ikl} \end{pmatrix} \times \frac{i}{\sqrt{T_2}} \begin{pmatrix} -1 & \sqrt{R_2} \\ -\sqrt{R_2} & T_2 + R_2 \end{pmatrix}.
\]

(22)

The elements of this matrix are evaluated as

\[
T_{11}^{(\text{FP})} = \frac{C_{FP}(\omega)}{\sqrt{T_1} T_2} \left[ 1 - \sqrt{R_1 R_2} e^{i\Phi(\Delta_C)} \right],
\]

(23)

\[
T_{12}^{(\text{FP})} = \frac{C_{FP}(\omega)}{\sqrt{T_1} T_2} \left[ -\sqrt{R_2} + \sqrt{R_1 (T_2 + R_2)} e^{i\Phi(\Delta_C)} \right],
\]

(24)

\[
T_{21}^{(\text{FP})} = \frac{C_{FP}(\omega)}{\sqrt{T_1} T_2} \left[ \sqrt{R_1} - (T_1 + R_1) \sqrt{R_2} e^{i\Phi(\Delta_C)} \right],
\]

(25)

\[
T_{22}^{(\text{FP})} = \frac{C_{FP}(\omega)}{\sqrt{T_1} T_2} \left[ -\sqrt{R_1 R_2} + (T_1 + R_1)(T_2 + R_2) e^{i\Phi(\Delta_C)} \right],
\]

(26)

with

\[
C_{FP}(\omega) = e^{-i\pi\omega/\omega_{\text{FSR}}}, \quad \Phi(\Delta_C) = 2\pi \frac{\Delta_C}{\omega_{\text{FSR}}},
\]

(27)

where \( R_{1,2} \) and \( T_{1,2} \) are the reflectance and transmittance of mirrors 1 and 2, and \( l \) and \( \eta \) are the cavity length and effective transmission of single-pass propagation inside the cavity. In Eq. (27), we have introduced the free-spectral range (FSR) of the cavity,

\[
\omega_{\text{FSR}} = 2\pi c \frac{2l}{\eta},
\]

(28)

and the probe-cavity detuning

\[
\Delta_C = \omega - \omega_C,
\]

(29)

where \( \omega_C \) is the cavity resonance frequency that satisfies \( e^{i2\pi \omega_C/\omega_{\text{FSR}}} = 1 \), and therefore

\[
e^{i2kl} = e^{i2\pi \omega/\omega_{\text{FSR}}} = e^{i2\pi \Delta_C/\omega_{\text{FSR}}}.\]

(30)
We obtain the transmission and reflection spectra of the cavity as

\[
T^{(FP)}(\omega) = \left| S^{(FP)}_{11} \right|^2 = \left| \frac{1}{T^{(FP)}_{11}} \right|^2 = \frac{\sqrt{\eta T_1 T_2}}{1 - \eta \sqrt{R_1 R_2} e^{i \Phi(\Delta C)}}^2, \tag{31}
\]

\[
R^{(FP)}(\omega) = \left| S^{(FP)}_{21} \right|^2 = \left| \frac{T^{(FP)}_{21}}{T^{(FP)}_{11}} \right|^2 = \frac{\sqrt{R_1 - \eta (T_1 + R_1) \sqrt{R_2} e^{i \Phi(\Delta C)}}}{1 - \eta \sqrt{R_1 R_2} e^{i \Phi(\Delta C)}}^2. \tag{32}
\]

Let us examine these spectra in more detail. The upper panel of Fig. 7 shows that decreasing the length of the cavity increases the effective linewidth and the free spectral range becomes larger (see also the lower panel in Fig. 7 where higher-order resonances are visible). Lowering the reflectances also increases the linewidth and leads to overlapping resonances (Fig. 8).

If we take the limit of high finesse, \(\alpha, 1 - R_1, 1 - R_2 \ll 1\), and small detunings \(|\Delta C| \ll \omega_{FSR}\), i.e., \(\Phi(\Delta C) \ll 1\), Eqs. (31) and (32) reproduce the results obtained from the single-mode, input-output formalism (as we will see in the following subsection),

\[
R^{(FP)}(\omega) = \left| 1 - \frac{2\kappa_1}{\kappa_C - i \Delta C} \right|^2, \tag{33}
\]
Reflection

Transmission

\[
T^{(FP)}(\omega) = \left| \frac{2\sqrt{\kappa_1\kappa_2}}{\kappa_C - i\Delta C} \right|^2, \tag{34}
\]

where \(\kappa_j\) are the cavity field decay rates through the mirrors \(j = 1, 2\), given by

\[
\kappa_j = \frac{1}{2l} \left[ \frac{c}{2} T_j \right] = \frac{1}{2} \frac{\omega_{\text{FSR}}}{2\pi} T_j, \tag{35}
\]

and \(\kappa_C\) is the total cavity field decay given by

\[
\kappa_C = \frac{1}{2l} \left[ (1 - R_1) + (1 - R_2) + 2\alpha \right]
= \frac{1}{2} \frac{\omega_{\text{FSR}}}{2\pi} \left[ (1 - R_1) + (1 - R_2) + 2\alpha \right], \tag{36}
\]

which is related to the linewidth of the resonance, i.e., the full width at half maximum is given by \(2\kappa_C\) (see Fig. 7 and below).

B. Quantum-optical model

The quantum-optical model is built on the assumption that wherever a Fabry-Pérot cavity is considered, standing wave modes are able to build up. Therefore, most frequently a mode operator \(\hat{a}\) is assigned to the dominant (or most relevant) single standing-wave mode.

The standard way of describing the dynamics of an open quantum optical system is to use the master equation for the reduced density operator, \(\hat{\rho}\), of the system. For a cavity driven by a probe laser of frequency \(\omega\) and strength \(E\), this is given by (in a frame rotating at frequency \(\omega\), and setting \(\hbar = 1\))

\[
\frac{d}{dt} \hat{\rho} = -i \left[ \hat{H}^{(\text{cav})}, \hat{\rho} \right] + \mathcal{L}^{(\text{cav})}[\hat{\rho}], \tag{37}
\]

where

\[
\hat{H}^{(\text{cav})} = -\Delta_C \hat{a}^\dagger \hat{a} + E \left( \hat{a} + \hat{a}^\dagger \right), \tag{38}
\]

\[
\mathcal{L}^{(\text{cav})}[\hat{\rho}] = (\kappa_1 + \kappa_2 + \kappa_{C,1}) \mathcal{D}[\hat{a}] \hat{\rho} = \kappa_C \mathcal{D}[\hat{a}] \hat{\rho}, \tag{39}
\]

and \(\mathcal{D}[\hat{O}] = 2\hat{O} \hat{\rho} \hat{O}^\dagger - \hat{O}^\dagger \hat{O} \hat{\rho} - \hat{\rho} \hat{O}^\dagger \hat{O}\).

Here, \(\kappa_{C,1}\) denotes internal (or intrinsic) loss within the cavity, which, if associated with propagation loss, is related to \(\alpha\) by

\[
\kappa_{C,1} = -\frac{1}{2l} \ln(1 - \alpha), \tag{40}
\]
which reduces to $\kappa_{C,i} = \alpha \lambda / (2l)$ for $\alpha \ll 1$.

The time evolution of the expectation value of the cavity field amplitude can be determined from the master equation as

$$\frac{d}{dt} \langle \hat{a} \rangle = \frac{d}{dt} \text{Tr} [\hat{\rho}(t)] = \text{Tr} \left[ \hat{a} \frac{d}{dt} \hat{\rho}(t) \right] = (i\Delta_C - \kappa_C) \langle \hat{a} \rangle - i\xi. \tag{41}$$

The reflection and transmission spectra can be obtained by expressing the steady state of these equations as a function of the driving frequency $\omega$ [29],

$$\langle \hat{a} \rangle_{ss} = -i \frac{\xi}{\kappa_C - i\Delta_C}, \tag{42}$$

which gives a Lorentzian lineshape for the intracavity intensity as a function of $\Delta_C$.

The reflection and transmission spectra can be calculated when driving from the left by using the steady-state intracavity field combined with the coherent input field via the input-output formalism as

$$R = \left| \frac{\langle \hat{a}_{in,1} \rangle + \sqrt{2\kappa_1} \langle \hat{a}_{ss} \rangle^2}{\langle \hat{a}_{in,1} \rangle^2 + \langle \hat{a}_{in,2} \rangle^2} \right|^2, \tag{43}$$

$$T = \left| \frac{\langle \hat{a}_{in,2} \rangle + \sqrt{2\kappa_2} \langle \hat{a}_{ss} \rangle^2}{\langle \hat{a}_{in,1} \rangle^2 + \langle \hat{a}_{in,2} \rangle^2} \right|^2, \tag{44}$$

where

$$\langle \hat{a}_{in,1} \rangle = i \frac{\xi}{\sqrt{2\kappa_1}}, \quad \langle \hat{a}_{in,2} \rangle = 0, \tag{45}$$

which give back the formulae [33,34].

Whenever the assumptions of high finesse and small detuning become marginal, results from the quantum-mechanical model can be expected to start deviating from the more accurate transfer matrix approach. This is illustrated in Figs. 9 and 10 where the two models are compared for relatively low-finesse cavities. We note again that the transfer matrix model describes a scattering problem and therefore incorporates multiple frequency modes for each optical element (Fig. 10). This causes broadening of the central Lorentzian in Fig. 9.

In contrast, the quantum-optical model considers only a single mode for the cavity, and therefore, captures the spectrum only in the close vicinity of the cavity resonance frequency. Fig. 10 also highlights that as the mirror reflectances and the FSR of the cavity decrease, overlaps between neighbouring resonances develop, and this is of course only accounted for in the TM description.

### IV. SINGLE CAVITY-QED SYSTEM

In this section we consider a simple cavity-QED system, where a single atom is trapped in a cavity field, as depicted in Fig. 11.

**A. Transfer matrix approach**

In this case, the transmission and reflection spectra are given by

$$T^{(\text{cQED})}(\omega) = \left| \frac{1}{T_{11}^{(\text{cQED})}} \right|^2, \tag{46}$$

$$R^{(\text{cQED})}(\omega) = \left| \frac{T_{12}^{(\text{cQED})}}{T_{11}^{(\text{cQED})}} \right|^2, \tag{47}$$

where the transfer matrix corresponding to the whole system is

$$T^{(\text{cQED})} = T^{(M1)}T^{(d)}T^{(A)}T^{(l-d)}T^{(a)}T^{(M2)}$$

$$= \frac{i}{\sqrt{T_1}} \left( \begin{array}{cc} -1 - \sqrt{R_1} & e^{-ikd} \\ -\sqrt{R_1} & T_1 + R_1 \end{array} \right) \left( \begin{array}{cc} e^{-ikd} & 0 \\ 0 & e^{ikd} \end{array} \right) \left( \begin{array}{cc} 1 - i\xi & -i\xi \\ i\xi & 1 + i\xi \end{array} \right) \left( \begin{array}{cc} e^{-ik(l-d)} & 0 \\ 0 & e^{ik(l-d)} \end{array} \right) \frac{i}{\sqrt{T_2}} \left( \begin{array}{cc} -1 & \sqrt{R_2} \\ -\sqrt{R_2} & T_2 + R_2 \end{array} \right), \tag{48}$$

with elements

$$T_{11}^{(\text{cQED})} = -\frac{e^{-ikd}}{\sqrt{T_1T_2}} \left\{ \left[ (1 - i\xi) - i\xi \sqrt{R_1} e^{2ikd} \right] - \left[ i\xi + (1 + i\xi) \sqrt{R_1} e^{2ikd} \right] \sqrt{R_2} e^{2ik(l-d)} \right\}, \tag{49}$$

$$T_{12}^{(\text{cQED})} = -\frac{e^{-ikd}}{\sqrt{T_1T_2}} \left\{ \left[ -(1 - i\xi) + i\xi \sqrt{R_1} e^{2ikd} \right] \sqrt{R_2} + \left[ i\xi + (1 + i\xi) \sqrt{R_1} e^{2ikd} \right] (T_2 + R_2) e^{2ik(l-d)} \right\}, \tag{50}$$

**Figure 11: Single-atom cavity QED setup.**
Transmission

Figure 12: Transmission and reflection spectra of a Fabry-Pérot cavity with an atom inside using the TM model near the atomic resonance frequency $\omega_A = \omega_C$ ($\Delta_A = \Delta_C \equiv \Delta$). The setup is weakly driven from the left. Parameters: $R_1 = 0.8$, $R_2 = 0.85$, $l = 2$ m ($\omega_{FSR}/(2\pi) = 51.64$ MHz), $\eta = 0.98$, $\gamma/(2\pi) = 2.65$ MHz.

Reflection

Figure 13: Same as in Fig. 12 but over a wider range of detunings. Parameters: $R_1 = 0.8$, $R_2 = 0.85$, $\eta = 0.98$, $g = 7\cdot2\pi$ MHz, $\gamma/(2\pi) = 2.65$ MHz.

\[
\mathcal{T}_{11}^{(cQED)} = -\frac{e^{-ikl}}{\sqrt{T_1T_2}} \left\{ \left[ (1-i\xi)\sqrt{R_1} - i\xi(T_1 + R_1)e^{2ikd} \right] - \left[ i\xi\sqrt{R_1} + (1+i\xi)(T_1 + R_1)e^{2ikd} \right] \sqrt{R_2}e^{2ik(l-d)} \right\},
\]

\[
\mathcal{T}_{22}^{(cQED)} = -\frac{e^{-ikl}}{\sqrt{T_1T_2}} \left\{ \left[ -(1-i\xi)\sqrt{R_1} + i\xi(T_1 + R_1)e^{2ikd} \right] \sqrt{R_2} \right.
\]

\[
+ \left. \left[ i\xi\sqrt{R_1} + (1+i\xi)(T_1 + R_1)e^{2ikd} \right] (T_2 + R_2)e^{2ik(l-d)} \right\}.
\]

$\mathcal{T}_i^{(cQED)}$ depend on position $d$ of the atom, which reflects the spatial variation of the atom-field coupling rate associated with the standing-wave nature of the cavity mode [$g(d) \sim g\cos(kd)$] [30]. For simplicity, we can take $d = 0$ to obtain simpler expressions:

\[
\mathcal{T}_{11}^{(cQED)} = \frac{CFP(\omega)}{\sqrt{\eta_1T_2}} \left\{ \left[ (1-i\xi)\sqrt{R_1} - i\xi(T_1 + R_1) \right] \sqrt{R_2}\eta e^{\Phi(\Delta_C)} \right\},
\]

\[
\mathcal{T}_{12}^{(cQED)} = \frac{CFP(\omega)}{\sqrt{\eta_1T_2}} \left\{ \left[ -(1+i\xi)\sqrt{R_1} + i\xi(T_1 + R_1) \right] \sqrt{R_2}\eta e^{\Phi(\Delta_C)} \right\},
\]

\[
\mathcal{T}_{21}^{(cQED)} = \frac{CFP(\omega)}{\sqrt{\eta_1T_2}} \left\{ \left[ (1-i\xi)\sqrt{R_1} + i\xi(T_1 + R_1) \right] \sqrt{R_2}\eta e^{\Phi(\Delta_C)} \right\},
\]

\[
\mathcal{T}_{22}^{(cQED)} = \frac{CFP(\omega)}{\sqrt{\eta_1T_2}} \left\{ \left[ -(1+i\xi)\sqrt{R_1} - i\xi(T_1 + R_1) \right] \sqrt{R_2}\eta e^{\Phi(\Delta_C)} \right\}.
\]

Examples of spectra are shown in Figs. 12 and 13 for the case in which $\omega_C = \omega_A$ (so $\Delta_C = \Delta_A \equiv \Delta$). In-

\[
\begin{align*}
\Delta/(2\pi) (MHz) & \quad \text{Transmission} \\
0 & \quad 0.0 \\
10 & \quad 0.2 \\
20 & \quad 0.4 \\
30 & \quad 0.6 \\
40 & \quad 0.8
\end{align*}
\]

\[
\begin{align*}
\Delta C/(2\pi) (MHz) & \quad \text{Reflection} \\
0 & \quad 0.0 \\
10 & \quad 0.2 \\
20 & \quad 0.4 \\
30 & \quad 0.6 \\
40 & \quad 0.8
\end{align*}
\]
increasing the effective coupling between the atom and the cavity gives enhanced Rabi splitting of the central resonance (Fig. 12), while changing the length of the cavity alters the free spectral range and the intensities of the peaks. We note that, even though only the peak closest to atomic resonance is subject to Rabi splitting, it is clear in Fig. 13 that the coupled atom also has a non-trivial effect on higher order resonances, especially when the FSR is decreased (i.e., cavity length increased).

If we assume \( \alpha, 1 - R_1, 1 - R_2 \ll 1, |\Delta C| \ll \omega_{FSR} \), and \( \Gamma' = 2\gamma \), then using the relation

\[
\Gamma_{1D} = \frac{l}{\nu_y} g^2 = \frac{\pi}{\omega_{FSR}} g^2,
\]

Eqs. (46) and (47) give

\[
T^{(\text{QED})}(\omega) = \frac{2\sqrt{\kappa_1 \kappa_2 (\gamma - i\Delta_A)}}{(\kappa_C - i\Delta_C)(\gamma - i\Delta_A) + g^2}^2, \quad (58)
\]

\[
R^{(\text{QED})}(\omega) = \left| 1 - \frac{2\kappa_1 (\gamma - i\Delta_A)}{(\kappa_C - i\Delta_C)(\gamma - i\Delta_A) + g^2} \right|^2, \quad (59)
\]

which reproduce the single-mode, quantum-optical results in the linear regime ([29], and next subsection).

### B. Quantum optical model

The quantum optical description combines the free time-evolution of a single cavity mode and the field of an atom with the evolution due to their interaction. As only two levels of the atom’s energy spectrum are considered, the spin-1/2 or Pauli matrices \( \sigma_z, \sigma^+ \sigma^- \) are used to represent the atomic excitations [25]. Extending the master equation of the previous section with this contribution, we obtain

\[
\frac{d}{dt} \hat{\rho} = -\frac{i}{\hbar} \left[ \hat{H}^{(\text{QED})}, \hat{\rho} \right] + \mathcal{L}^{(\text{QED})}[\hat{\rho}], \quad (60)
\]

with

\[
\hat{H}^{(\text{QED})} = -\Delta C \hat{a}^\dagger \hat{a} - \Delta_A \hat{\sigma}^+ \hat{\sigma}^- + g (\hat{a}^\dagger \hat{\sigma}^- + \hat{\sigma}^+ \hat{a}) + \mathcal{E} (\hat{a} + \hat{a}^\dagger), \quad (61)
\]
The main characteristic of the obtained transmission and reflection spectra is the spontaneous emission rate of the atom into free space. The last equation can also only be described by the TM approach (Fig. 15).

In contrast with the quantum-optical model, the transfer matrix approach enables the atom to couple to multiple modes, which results in a higher rate of transmission away from resonance (Fig. 14). Note that only the central resonance shows Rabi splitting, which is mostly captured, but slightly overestimated, by the quantum model. This is due to the longer length of the cavity, which results in a decreased FSR and means that multiple resonances interact with the atom. As the QO model only captures the contribution of the central resonance, it overestimates the effect of the atom. The intensity variations and shifts of the higher-order resonances, missing from the QO model, can also only be described by the TM approach (Fig. 15).

V. CONNECTED EMPTY CAVITIES

Figure 18: Connected empty cavities.

Having examined the main building blocks in detail,
A. Transfer matrix approach

The transfer matrix of the coupled-cavity system is calculated from

$$T^{(2FP)} = T^{(M1)} T^{(l_1)} T^{(M2)} T^{(l_2)} T^{(M3)} T^{(l_3)} T^{(M4)}$$

where

$$C_{2FP}(\omega) = e^{-i\pi \left( \frac{1}{\omega_{FSR1}} + \frac{1}{\omega_{FSR3}} + \frac{1}{\omega_{FSR2}} \right)}$$

and

$$\Delta C_1 = \omega - \omega C_1, \quad \Delta C_f = \omega - \omega C_f, \quad \Delta C_2 = \omega - \omega C_2$$

are the detunings of the probe for cavity 1, the connecting fiber (which also constitutes a multimode cavity), and cavity 2, respectively.

Transmission and reflection spectra of the coupled-cavity system are shown in Figs. 16-20 for a variety of parameters, with all optical modes on resonance ($\omega_{C1} = \omega_{C2} = \omega_{C1}$). A defining feature of the spectra, for the chosen parameters, is the triplet structure centered around zero detuning. This can be interpreted in terms of three normal modes – two symmetric “bright” modes and one resonant, anti-symmetric “fiber-dark” mode – that result from coupling the resonant cavities via the connecting-fiber “cavity” 31.

Focusing only on the central triplet, a number of additional features illustrated by Figs. 16, 20 should also be noted. In particular, decreasing the reflectivities of mirrors 2 and 3 shifts the side peaks further away from resonance while having no effect on the central peak (see Fig. 19). The same effect can be observed when the length of the connecting fiber is decreased, i.e., when the free spectral range of that section is increased (see Fig. 19). However, when the reflectivity of one mirror is increased and the other is decreased in appropriate proportions, the side peaks stay at approximately the same position (see Fig. 17). A similarly suppressed effect can be seen if the length of one cavity is increased while the
length of the other cavity is decreased (see Fig. 21). In summary, the positions of the side peaks in the central triplet depend essentially on the overall transmission rate through the connecting fiber, which is determined by a number of factors, including the lengths of the cavities.

B. Quantum-optical model

Assuming short enough lengths and, thus, sufficiently large FSR’s for both the cavities and the connecting fiber, it is reasonable to model the fields of the cavities and the coupling fiber using single modes. The master equation for this system can then be written as

\[
\frac{d}{dt} \hat{\rho} = -\frac{i}{\hbar} \left[ \hat{H}^{(\text{cav})} + \hat{H}^{(\text{cav})} + \hat{H}^{(\text{fiber})}, \hat{\rho} \right] + \mathcal{L}_1^{(\text{cav})}[\hat{\rho}] + \mathcal{L}_2^{(\text{cav})}[\hat{\rho}] + \mathcal{L}^{(\text{fiber})}[\hat{\rho}],
\]

where \( \hat{H}^{(\text{cav})} \) and \( \mathcal{L}_i^{(\text{cav})}[\hat{\rho}] \) are defined similarly to the single-empty-cavity case using cavity-mode operators \( \hat{a}_i \). The additional fiber terms incorporate the independent dynamics of the fiber as well as coherent excitation exchange at rates \( v_1 \) and \( v_2 \) between this mode (operator \( \hat{b} \)) and the cavity modes,

\[
\hat{H}^{(\text{fiber})} = -\Delta C_1 \hat{b}^\dagger \hat{b} + v_1 \left( \hat{b}^\dagger \hat{a}_1 + \hat{a}_1^\dagger \hat{b} \right) + v_2 \left( \hat{b}^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{b} \right),
\]

\[
\mathcal{L}^{(\text{fiber})}[\hat{\rho}] = \kappa_{C1,i} \mathcal{D}[\hat{b}]\hat{\rho},
\]

where \( \kappa_{C1,i} \) is the intrinsic loss rate of the coupling fiber mode.

This master equation gives the following equations of motion,

\[
\frac{d}{dt} \langle \hat{a}_1 \rangle = (i \Delta C_1 - \kappa_{C1}) \langle \hat{a}_1 \rangle - iv_1 \langle \hat{b} \rangle - iE_1,
\]

\[
\frac{d}{dt} \langle \hat{b} \rangle = (i \Delta C_1 - \kappa_{C1,i}) \langle \hat{b} \rangle - iv_1 \langle \hat{a}_1 \rangle - iv_2 \langle \hat{a}_2 \rangle,
\]

\[
\frac{d}{dt} \langle \hat{a}_2 \rangle = (i \Delta C_2 - \kappa_{C2}) \langle \hat{a}_2 \rangle - iv_2 \langle \hat{b} \rangle - iE_2,
\]

where \( E_1 \) and \( E_2 \) are the coherent driving strengths of cavities 1 and 2, respectively (i.e., this allows for probe driving through both \( E_1^{(\text{in})} \) and \( E_2^{(\text{in})} \)). The overall decay rate for cavity 1 is \( \kappa_{C1} = \kappa_1 + \kappa_{C1,1} \), where \( \kappa_1 \) characterizes mirror 1 and \( \kappa_{C1,1} \) represents the intrinsic fiber loss due to absorption and scattering in cavity 1 [17] [18]. Similarly, \( \kappa_{C2} = \kappa_4 + \kappa_{C2,1} \). Note that \( \kappa_1 \) and \( \kappa_4 \) are defined as before in [35], but have different cavity lengths, \( l_1 \) and \( l_2 \) respectively, in their expressions.

The obtained reflection and transmission spectra in Figs. 21 and 22 show an interesting systematic shift of the side peaks compared to the transfer matrix approach. This can be understood by the model provided in [31]. In particular, the coherent coupling strengths are proportional to the decay rates into free space via mirrors 2 and 3, respectively. These effective decay rates are broadened more than can be captured by the single-mode model for

Figure 21: Comparison of spectra for the coupled-cavity system using the TM and QO models, when driving from the left (\( E_1 \neq 0, E_2 = 0 \)). The vertical, grey dashed lines indicate the frequencies \( \pm \sqrt{\nu_1^2 + \nu_2^2} / (2\pi) \). The parameters can be found in Tables IV and V of Appendix B (\( g_1 = g_2 = 0 \)).

Figure 22: Same as in Fig. 21 but with driving from the right (\( E_1 = 0, E_2 \neq 0 \)).
the given parameters. This was also shown for a single cavity in Fig. 9. Thus, the emerging multimode nature of the field in the connecting fiber means more modes coupling to the cavities, increasing the effective cavity-fiber coupling strengths. As the position of the side peaks is determined by these coupling strengths, they shift further away from resonance [31].

By extending the model in [31] to the case of unequal \( \kappa_2 \) and \( \kappa_3 \), the splitting between the center peak and the side peaks is given by [32]

\[
\sqrt{v_1^2 + v_2^2} = \sqrt{2 \left( \frac{\kappa_2}{2\pi} \right)^2 \omega_{FSR} + 2 \left( \frac{\kappa_3}{2\pi} \right)^2 \omega_{FSR}} = \sqrt{\frac{\kappa_2 + \kappa_3}{\pi} \omega_{FSR}} = \frac{1}{\sqrt{2 \pi}} \sqrt{ \omega_{FSR_1} T_2 + \omega_{FSR_2} T_3 } \omega_{FSR} = \frac{1}{2\pi} \sqrt{ (\omega_{FSR_1} T_2 + \omega_{FSR_2} T_3 ) \omega_{FSR}}. \tag{73}
\]

This expression explains the difference in the behavior of the side peaks as a result of changing the central mirror reflectances in Figs. 16 and 17. When both of the mirror reflectances are decreased, the result is a net increase of the side peak's distance from resonance. On the other hand, when only one of the reflectances is decreased, and the other is increased, the net change in the side peak's position is negligible. Both figures show that the expected side-peak position of \( (v_1^2 + v_2^2)^{1/2} \) approximately matches the numerically-obtained curves. Note that as mirror 2 has a lower reflectance than mirror 3, the light field effectively couples less strongly to the fiber field when incident from the right. This results in less intense signatures from the bright reflection peaks (which involve the fiber mode) in the reflection spectrum (Fig. 22).

### VI. CONNECTED CAVITY QED SYSTEMS

In the context of quantum networks, an elementary, but important and topical example is a pair of cavity-QED systems connected by an optical fiber, as shown in Fig. 23. Recent experiments have indeed explored transmission and reflection spectra of this system via various ports [17, 18]. In this section we demonstrate how the TM description can differ to various extents from the standard QO approach for the parameters considered in these experiments.

#### A. Transfer matrix approach

The transfer matrix for the whole system can be calculated as follows:

\[
\mathcal{T}^{(2c\text{QED})} = \mathcal{T}^{(M1)} \mathcal{T}^{(d1)} \mathcal{T}^{(A1)} \mathcal{T}^{(l1-d1)} \mathcal{T}^{(M2)} \mathcal{T}^{(l1)} \mathcal{T}^{(M3)} \mathcal{T}^{(d2)} \mathcal{T}^{(A2)} \mathcal{T}^{(l2-d2)} \mathcal{T}^{(M4)}
\]

\[
= \frac{i}{\sqrt{T_1}} \left( \begin{array}{cc} -1 & \sqrt{R_1} \\ -\sqrt{R_1} & T_1 + R_1 \end{array} \right) \left( \begin{array}{cc} e^{-ik_1d_1} & 0 \\ 0 & e^{ik_1d_1} \end{array} \right) \left( \begin{array}{cc} 1 - i \xi_1 & -i \xi_1 \\ i \xi_1 & 1 + i \xi_1 \end{array} \right) \left( \begin{array}{cc} e^{-ik_1(l_1-d_1)} & 0 \\ 0 & e^{ik_1(l_1-d_1)} \end{array} \right)
\]

\[
\times \frac{i}{\sqrt{T_2}} \left( \begin{array}{cc} -1 & \sqrt{R_2} \\ -\sqrt{R_2} & T_2 + R_2 \end{array} \right) \left( \begin{array}{cc} e^{-ik_2l_2} & 0 \\ 0 & e^{ik_2l_2} \end{array} \right) \frac{i}{\sqrt{T_3}} \left( \begin{array}{cc} 0 & -1 \sqrt{R_3} \\ -\sqrt{R_3} & T_3 + R_3 \end{array} \right) \left( \begin{array}{cc} 0 & -1 \sqrt{R_4} \\ -\sqrt{R_4} & T_4 + R_4 \end{array} \right) \right).
\tag{74}
\]

Taking \( d_1 = d_2 = 0 \) for simplicity gives

\[
\mathcal{T}^{(2c\text{QED})} = \mathcal{T}^{(M1)} \mathcal{T}^{(A1)} \mathcal{T}^{(l1)} \mathcal{T}^{(M2)} \mathcal{T}^{(l1)} \mathcal{T}^{(M3)} \mathcal{T}^{(A2)} \mathcal{T}^{(l2)} \mathcal{T}^{(M4)}
\]

\[
= \frac{C_{2FP}(\omega)}{\sqrt{\eta_1 \eta_2 \eta_1 \eta_2 T_1 T_2 T_3 T_4}} \left( \begin{array}{cc} -1 & \sqrt{R_1} \\ -\sqrt{R_1} & T_1 + R_1 \end{array} \right) \left( \begin{array}{cc} 1 - i \xi_1 & -i \xi_1 \\ i \xi_1 & 1 + i \xi_1 \end{array} \right) \left( \begin{array}{cc} 1 & 0 \eta e^{i \frac{\Delta C_1}{FSR_1}} & \sqrt{R_1} \\ 0 & \eta e^{i \frac{\Delta C_1}{FSR_2}} \end{array} \right) \left( \begin{array}{cc} -1 & \sqrt{R_2} \\ -\sqrt{R_2} & T_2 + R_2 \end{array} \right)
\]

\[
\times \left( \begin{array}{cc} 1 & 0 \eta e^{i \frac{\Delta C_2}{FSR_3}} & \sqrt{R_3} \\ 0 & \eta e^{i \frac{\Delta C_2}{FSR_2}} \end{array} \right) \left( \begin{array}{cc} 1 - i \xi_2 & -i \xi_2 \\ i \xi_2 & 1 + i \xi_2 \end{array} \right) \left( \begin{array}{cc} 1 & 0 \eta e^{i \frac{\Delta C_2}{FSR_4}} & \sqrt{R_4} \\ 0 & \eta e^{i \frac{\Delta C_2}{FSR_4}} \end{array} \right) \left( \begin{array}{cc} -1 & \sqrt{R_4} \\ -\sqrt{R_4} & T_4 + R_4 \end{array} \right). \tag{75}
\]
Figure 24: Transmission and reflection spectra of two cavity-QED systems coupled to each other by an optical fiber, computed using the TM model. Weak driving is considered from the left. The various curves are for varying atom-cavity coupling strengths. The parameters can be found in Table IV of Appendix B except $\eta_1 = 0.99$, $\eta_2 = 0.99$.

We see that the central peak observed in the previous section is split into two peaks, similar to the vacuum Rabi splitting for the one-cavity case. These peaks represent the anti-symmetric fiber-dark modes [17, 18]. Meanwhile, the side peaks corresponding to the symmetric bright modes are shifted further away from resonance. As pointed out in [31], the central peak is an anti-symmetric superposition of the modes of cavities 1 and 2 and has no contribution from the connecting fiber. Therefore we can interpret this splitting as the vacuum Rabi splitting for the one-cavity case. These peaks further emphasizes the delocalized nature of the normal modes [17].

B. Quantum-optical model

The master equation for this system can be written as

$$\frac{d}{dt} \hat{\rho} = -i \left[ \hat{H}_1^{(cQED)} + \hat{H}_2^{(cQED)} + \hat{H}^{(fiber)}, \hat{\rho} \right] + \mathcal{L}_1^{(cQED)}[\hat{\rho}] + \mathcal{L}_2^{(cQED)}[\hat{\rho}] + \mathcal{L}^{(fiber)}[\hat{\rho}],$$

(77)

where $\hat{H}_j^{(cQED)}$ and $\mathcal{L}_j^{(cQED)}[\hat{\rho}]$ are defined as in the single cavity case using cavity operators $\hat{a}_j$, and where an atom is included with spin operators $\hat{\sigma}_j^-$. The additional fiber term incorporates the independent dynamics of the fiber as well as the excitation exchange between this mode and the cavities, similarly to the previous section.

This master equation results in a similar set of dynamical equations as in the coupled-cavity case. In the weak driving limit (linearized regime), these are

$$\frac{d}{dt} \langle \hat{\sigma}_1^- \rangle = (i\Delta_A - \gamma) \langle \hat{\sigma}_1^- \rangle - ig_1 \langle \hat{a}_1 \rangle,$$

(78)

$$\frac{d}{dt} \langle \hat{a}_1 \rangle = (i\Delta_{C1} - \kappa_{C1}) \langle \hat{a}_1 \rangle - ig_1 \langle \hat{\sigma}_1^- \rangle.$$
They are positioned as side peaks in the coupled-cavity spectrum in the coupled-cavity case. This is due to the fact that these
shift in the position of the outermost peaks to the
the two approaches in Figs. 26 and 27, we see a simi-
state expectation values of the cavity fields, we can deter-
dependence on the atom-cavity couplings $g$
the mirrors in the setup the asymmetry in the fiber-cavity
there is an asymmetry in the reflectances of the central
when the system is driven from the right (similar to what
the bright-state contributions in the reflection spectrum

Similarly to the previous subsection, using the steady-
 energizes the cavity fields, we can deter-
determine the reflection and transmission spectra. Comparing
the two approaches in Figs. 26 and 27 we see a simi-
this is due to the fact that these
peaks originate from the optical bright modes, showing
as side peaks in the coupled-cavity spectrum in the
the previous section (Figs. 21 and 22). They are positioned
close to $\pm \sqrt{\Gamma_1^2 + \Gamma_2^2}$, but have an additional (small) depen-
dependence on the atom-cavity couplings $g_1$ and $g_2$

This relationship is also justified by the suppression of the bright-state contributions in the reflection spectrum when the system is driven from the right (similar to what can be observed for the side peaks in Figs. 21 and 22). As there is an asymmetry in the reflectances of the central mirrors in the setup the asymmetry in the fiber-cavity couplings show stronger bright-mode contributions when

reflected from the left than from the right (Figs. 26 and
Overall, the QO and TM descriptions agree reasonably well for the parameters of the recent experiments, thus supporting the use of the single-mode description in these works [17, 18].

VII. INCLUDING A BEAM SPLITTER IN THE CONNECTING FIBER

The spectra for the coupled cavity-QED system shown so far included only four peaks. There are, however, five
normal modes, one of which does not have any cavity-
field contribution, it is a cavity-dark mode. Nevertheless,
it still involves the fibre field, thus by introducing an extra input-output channel which can be monitored, its
presence has been detected by a recent experiment [18].

Modelling this experiment requires a modified approach, as the beam splitter is characterized by a total of
four input and four output ports (Fig. 28). In order to facilitate the calculations, we collect the transfer matrices
corresponding to cavity-QED setups 1 and 2 in the following way:

$$
\begin{pmatrix}
E_1^{\text{in}} \\
E_2^{\text{in}}
\end{pmatrix} = T^{(S1)} \begin{pmatrix}
E_1^{(1)} \\
E_2^{(1)}
\end{pmatrix},
\begin{pmatrix}
E_1^{\text{out}} \\
E_2^{\text{out}}
\end{pmatrix} = T^{(S2)} \begin{pmatrix}
E_1^{(2)} \\
E_2^{(2)}
\end{pmatrix},
$$

where $T^{(S1)}$ and $T^{(S2)}$ are the transfer matrices for setups 1 and 2, respectively.
where

\[
\mathcal{T}^{(S1)} = \mathcal{T}^{(M1)} \mathcal{T}^{(l_1/2)} \mathcal{T}^{(A1)} \mathcal{T}^{(l_1/2)} \mathcal{T}^{(M2)} \mathcal{T}^{(l_1/2)} = \frac{i}{\sqrt{T_1}} \begin{pmatrix} -1 & \sqrt{R_1} \\ -\sqrt{R_1} & T_1 + R_1 \end{pmatrix} \begin{pmatrix} e^{-ik_1 l_1/2} & 0 \\ 0 & e^{ik_1 l_1/2} \end{pmatrix} \begin{pmatrix} 1 - i\xi_1 & -i\xi_1 \\ i\xi_1 & 1 + i\xi_1 \end{pmatrix} \begin{pmatrix} e^{-ik_1 l_1/2} & 0 \\ 0 & e^{ik_1 l_1/2} \end{pmatrix} \\
\times \frac{i}{\sqrt{T_2}} \begin{pmatrix} -1 & \sqrt{R_2} \\ -\sqrt{R_2} & T_2 + R_2 \end{pmatrix} \begin{pmatrix} e^{-ik_2 l_2/2} & 0 \\ 0 & e^{ik_2 l_2/2} \end{pmatrix}.
\]

\[
\mathcal{T}^{(S2)} = \mathcal{T}^{(l_2/2)} \mathcal{T}^{(M3)} \mathcal{T}^{(l_2/2)} \mathcal{T}^{(A2)} \mathcal{T}^{(l_2/2)} \mathcal{T}^{(M4)} = \begin{pmatrix} e^{-ik_2 l_2/2} & 0 \\ 0 & e^{ik_2 l_2/2} \end{pmatrix} i \sqrt{R_3} \begin{pmatrix} -1 & \sqrt{R_3} \\ -\sqrt{R_3} & T_3 + R_3 \end{pmatrix} \begin{pmatrix} e^{-ik_2 l_2/2} & 0 \\ 0 & e^{ik_2 l_2/2} \end{pmatrix} \begin{pmatrix} 1 - i\xi_2 & -i\xi_2 \\ i\xi_2 & 1 + i\xi_2 \end{pmatrix} \\
\times \begin{pmatrix} e^{-ik_2 l_2/2} & 0 \\ 0 & e^{ik_2 l_2/2} \end{pmatrix} i \sqrt{R_4} \begin{pmatrix} -1 & \sqrt{R_4} \\ -\sqrt{R_4} & T_4 + R_4 \end{pmatrix}.
\]

The different directions of the introduced fields \(a\) and \(b\) are named according to the original definition of field 1 travelling from left to right and field 2 from right to left.

The fields impinging on the fiber beam splitter can be described using a higher dimensional scattering matrix based on equation (10):

\[
\begin{pmatrix} E^{(2)}_a \\ E^{(out)}_u \\ E^{(1)}_d \\ E^{(in)}_u \end{pmatrix} = S^{(BS)} \begin{pmatrix} E^{(2)}_a \\ E^{(in)}_u \\ E^{(2)}_d \\ E^{(in)}_u \end{pmatrix},
\]

where

\[
S^{(BS)} = \begin{pmatrix} 0 & i\sqrt{R^{(BS)}} & \sqrt{T^{(BS)}} & 0 \\ i\sqrt{R^{(BS)}} & 0 & 0 & \sqrt{T^{(BS)}} \\ \sqrt{T^{(BS)}} & 0 & 0 & i\sqrt{R^{(BS)}} \\ 0 & \sqrt{T^{(BS)}} & i\sqrt{R^{(BS)}} & 0 \end{pmatrix}.
\]

This relationship can also be expressed with two \(2 \times 2\) matrices, which are the same for a symmetric beam splitter in the following way:

\[
\begin{pmatrix} E^{(2)}_a \\ E^{(out)}_u \end{pmatrix} = S^{(M/BS)} \begin{pmatrix} E^{(in)}_u \\ E^{(2)}_d \end{pmatrix}, \quad (85)
\]

\[
\begin{pmatrix} E^{(out)}_d \\ E^{(1)}_d \end{pmatrix} = S^{(M/BS)} \begin{pmatrix} E^{(1)}_u \\ E^{(in)}_u \end{pmatrix}, \quad (86)
\]

where the corresponding scattering matrix is similar to the one in (9):

\[
S^{(M/BS)} = \begin{pmatrix} i\sqrt{R^{(BS)}} & \sqrt{T^{(BS)}} \\ \sqrt{T^{(BS)}} & i\sqrt{R^{(BS)}} \end{pmatrix}.
\]

In the following, we look at various driving conditions to characterize this setup.

A. Driving from the left-hand side

In the case when the system is only driven from the left,

\[
E^{(in)}_2 = E^{(in)}_u = E^{(in)}_d = 0,
\]

Figure 28: Connected cavity QED systems with a 4-port beam splitter in the connecting fibre.
Figure 29: Spectra of two coupled, empty Fabry-Pérot cavities using the TM approach with weak driving from the left ($E_1^{(in)}$). The various curves correspond to the increasing lengths $l_1$ and $l_2$ of the cavities. The rest of the parameters are the same as in Table IV of Appendix B (empty-cavity case $g_1 = g_2 = 0$).

Figure 30: Looking at the same spectra as in Fig. 29 but with atoms. The parameters for these spectra can be found in Table IV of Appendix B.

The above equations simplify as

\[
\begin{pmatrix}
E_1^{(in)} \\
E_2^{(out)}
\end{pmatrix} = \mathcal{T}^{(2cQED,\alpha)} \begin{pmatrix}
E_1^{(out)} \\
0
\end{pmatrix},
\]

\[(89)\]
which, using equation (5) for the output fields on the left and right, translates into
\[
\frac{E^{(\text{out})}}{E^{(\text{in})}} = \frac{T^{(\text{2cQED}, \alpha)}}{T^{(\text{2cQED}, \alpha)}} = \frac{1}{T^{(\text{2cQED}, \alpha)}}, \quad \text{(90)}
\]
\[
\frac{E^{(\text{out})}}{E^{(\text{in})}} = \frac{1}{T^{(\text{2cQED}, \alpha)}}. \quad \text{(91)}
\]

The beam splitter output fields described by equations \(84 \text{ and } 87\) also simplify in the following way:
\[
E^{(2)} = \sqrt{T^{(\text{BS})}} E^{(2)}, \quad \text{(92)}
\]
\[
E^{(\text{out})} = i \sqrt{R^{(\text{BS})}} E^{(1)}, \quad \text{(93)}
\]
\[
E^{(1)} = \sqrt{T^{(\text{BS})}} E^{(1)}, \quad \text{(94)}
\]
\[
E^{(\text{out})} = i \sqrt{R^{(\text{BS})}} E^{(2)} = i \sqrt{\frac{R^{(\text{BS})}}{T^{(\text{BS})}} E^{(2)}}. \quad \text{(95)}
\]

The field \(E^{(1)}\) can be expressed in terms of \(E^{(\text{in})}\) using \(T^{(\text{S1})}\) and equation (90),
\[
\left( \begin{array}{c}
E^{(1)} \\
E^{(2)}
\end{array} \right) = \left( T^{(\text{S1})} \right)^{-1} \left( \begin{array}{c}
E^{(\text{in})} \\
E^{(\text{out})}
\end{array} \right)
\]
\[
= \left( T^{(\text{S1})} \right)^{-1} \left( \begin{array}{c}
1 \\
\frac{T^{(\text{2cQED}, \alpha)}}{T^{(\text{2cQED}, \alpha)}}
\end{array} \right) E^{(\text{in})}. \quad \text{(96)}
\]

Thus, we obtain the following for the output fields from the fiber:
\[
\frac{E^{(\text{out})}}{E^{(\text{in})}} = \frac{i \sqrt{R^{(\text{BS})}}}{\text{det } T^{(\text{S1})}} \left( T^{(\text{S1})} - T^{(\text{S1})} \frac{T^{(\text{2cQED})}}{T^{(\text{2cQED})}} \right), \quad \text{(97)}
\]
and
\[
\frac{E^{(\text{out})}}{E^{(\text{in})}} = i \sqrt{\frac{R^{(\text{BS})}}{T^{(\text{BS})}}} \text{det } T^{(\text{S1})} \times \left( -T^{(\text{S1})} + T^{(\text{S1})} \frac{T^{(\text{2cQED})}}{T^{(\text{2cQED})}} \right). \quad \text{(98)}
\]

Looking at the output channels through the fiber beam splitter in Fig. 29 gives us some insight about the behaviour of the fiber mode. The central peak corresponds to the antisymmetric superposition of the fields in the two cavities, therefore little or no contribution can be detected in the fiber. The down channel \(E^{(\text{out})}\) is a direct continuation of the laser drive on the left, whereas a reflection from mirror 3 is necessary for the light to reach the up channel \(E^{(\text{out})}\). Therefore, the output field through the up channel shows a small peak on resonance confirming the build-up of a central fiber-mode. The modes reaching the down channel travel in the opposite direction, which gives an extra phase that transforms the small bump on resonance into a dip. The small bump and dip are due to the finite transmission rates of the outer mirrors resulting in overlapping normal modes and, thus, interference between them. Increasing the lengths of both cavities shifts the side peaks closer to resonance. Looking at the same spectra but with atoms in Fig. 30 only the outermost peaks corresponding to the bright modes show the same shift. The fiber channels show the signatures of all five normal modes.

### B. Driving through the beam splitter

In the case when the system is driven only via the beam splitter, directly through the fiber, the following applies:
\[
E^{(\text{in})} = E^{(\text{out})} = 0. \quad \text{(99)}
\]

In terms of the fields scattered by the beam splitter this means that
\[
E^{(2)} = \sqrt{T^{(\text{BS})}} E^{(2)}, \quad \text{(100)}
\]
\[
E^{(\text{out})} = i \sqrt{R^{(\text{BS})}} E^{(1)}, \quad \text{(101)}
\]
\[
E^{(1)} = \sqrt{T^{(\text{BS})}} E^{(1)} + i \sqrt{R^{(\text{BS})}} E^{(1)} = i \sqrt{R^{(\text{BS})}} E^{(1)}. \quad \text{(102)}
\]
\[
E^{(\text{out})} = \sqrt{\frac{R^{(\text{BS})}}{T^{(\text{BS})}} E^{(2)}}. \quad \text{(103)}
\]

Considering the fields impinging on the beam splitter from the two cavities, we have the following expressions from \(84 \text{ and } 87\) using the corresponding scattering matrices,
\[
\left( \begin{array}{c}
E^{(1)} \\
E^{(2)}
\end{array} \right) = S^{(\text{S1})} \left( \begin{array}{c}
0 \\
E^{(2)}
\end{array} \right), \quad \text{(104)}
\]
\[
\left( \begin{array}{c}
E^{(1)} \\
E^{(2)}
\end{array} \right) = S^{(\text{S2})} \left( \begin{array}{c}
E^{(1)} \\
0
\end{array} \right), \quad \text{(105)}
\]

which gives extra conditions for fields \(a\) and \(b\),
\[
E^{(1)} = S^{(\text{S1})} E^{(2)}, \quad \text{(106)}
\]
\[
E^{(2)} = S^{(\text{S2})} E^{(1)}. \quad \text{(107)}
\]

Finally, Eq. (100) provides the necessary link between fields \(a\) and \(b\). Putting Eqs. (100) and (106) together thus gives
\[
E^{(1)} = \sqrt{T^{(\text{BS})}} S^{(\text{S1})} E^{(2)} + i \sqrt{R^{(\text{BS})}} E^{(\text{in})} = \frac{i \sqrt{R^{(\text{BS})}}}{1 - T^{(\text{BS})} S^{(\text{S1})} S^{(\text{S2})}} E^{(\text{in})} \xi E^{(\text{in})}. \quad \text{(108)}
\]

The output fields on the left and right have the expression of
\[
E^{(\text{out})} = S^{(\text{S1})} E^{(2)}, \quad \text{(109)}
\]
\[
E^{(\text{out})} = S^{(\text{S2})} E^{(1)}. \quad \text{(110)}
\]
Figure 31: Spectra of two coupled Fabry-Pérot cavities using the TM approach with weak driving from the up channel \( E^{(in)}_u \). Only cavity 1 contains an atom. The parameters are the same as in recent experiments (Table IV, Appendix B, \( g_2 = 0 \)).

Figure 32: Same as Fig. 31 but now only cavity 2 contains an atom. Parameters in Table IV, Appendix B, except for \( g_1 = 0 \).

which means

\[
\frac{E^{(out)}_1}{E^{(in)}_u} = \frac{i \sqrt{R^{(BS)}} \xi \det [T^{(S1)}] T^{(S2)}_{21}}{T^{(S2)}_{11}}, \tag{111}
\]

\[
\frac{E^{(out)}_2}{E^{(in)}_u} = i \sqrt{R^{(BS)}} \xi \frac{\det [T^{(S1)}] T^{(S2)}_{21}}{T^{(S2)}_{11}} \xi, \tag{112}
\]
The output fields through the beam splitters are

\[
\frac{E_u^{(\text{out})}}{E_u^{(\text{in})}} = -R^{(\text{BS})} \frac{T_{21}^{(S2)}}{T_{11}^{(S2)}} \xi, \quad (113)
\]

\[
\frac{E_d^{(\text{out})}}{E_u^{(\text{in})}} = \sqrt{T^{(\text{BS})}} \left[ 1 + R^{(\text{BS})} \frac{T_{12}^{(S1)}}{T_{11}^{(S1)}} \frac{T_{21}^{(S2)}}{T_{11}^{(S2)}} \xi \right]. \quad (114)
\]

In this case, as Figs. 31-33 show, changing the lengths of the cavities not only shifts the outermost peaks but also affects the visibility of the central peaks. If atoms are only present in cavity 1 (Fig. 31), the inner peaks are much closer to resonance and much more pronounced in the fiber contributions than when both cavities are loaded. This is due to the broken spatial symmetry of the setup, which results in a different set of normal modes. In this setup, increasing the cavity lengths shifts the outer peaks closer to resonance, while the inner peaks are left in place but are enhanced.

In the case of cavity 2 being loaded instead of cavity 1 (Fig. 32), similar features can be observed as in Fig. 31. However, the atom is coupled slightly more strongly to its cavity field, which results in an increased splitting of the inner peaks in the spectrum. The spectra on the left (\(E_1^{(\text{out})}\)) and right (\(E_2^{(\text{out})}\)) are effectively exchanged. It is interesting to note that loading both cavities with atoms (Fig. 33), the spectrum measured on the right (\(E_2^{(\text{out})}\)) is more similar to the one corresponding to the case where only the right-hand-side cavity is loaded (Fig. 32).

Having both cavities loaded with atoms, the spectrum in Fig. 33 undergoes a qualitative change. Instead of four well-defined peaks, mostly two peaks corresponding to the bright modes are detected in the left (\(E_1^{(\text{out})}\)) and right (\(E_2^{(\text{out})}\)) channels. Signatures of the fiber-dark modes can be seen as small independent side peaks, as well. The fiber output channels, on the other hand, show an extra peak on resonance that corresponds to the cavity-dark mode.

As the lengths of the cavities are increased, the dark-mode contribution becomes more and more pronounced. Due to the asymmetry in the experimental setup, the fiber-dark modes are also driven via the fiber. They show up as small contributions with a phase that depends on the direction of propagation.

### C. Quantum-optical model

In this case we extend the master equation with a direct driving of the connecting fiber mode, which also introduces an extra loss channel with rate \(\kappa_{\text{BS}}\). The master equation is thus

\[
\frac{d}{dt} \hat{\rho} = -\frac{i}{\hbar} \left[ \hat{H}^{(\text{cQED})} + \hat{H}^{(\text{cQED})} + \hat{H}^{(\text{fiber-dr})}, \hat{\rho} \right]
\]

\[
+ \mathcal{L}^{(\text{fiber-dr})}[\hat{\rho}] + \mathcal{L}^{(\text{fiber-dr})}[\hat{\rho}] + \mathcal{L}^{(\text{fiber-dr})}[\hat{\rho}], \quad (115)
\]

where the terms describing the evolution of the fiber mode are adjusted to include the effective loss produced by the beam splitter and the direct laser driving, i.e.,

\[
\mathcal{L}^{(\text{fiber-dr})}[\hat{\rho}] = (\kappa_{\text{Cl,}i} + \kappa_{\text{BS}}) \mathcal{D}[\hat{b}] \hat{\rho} = \beta^* \mathcal{D}[\hat{b}] \hat{\rho}, \quad (116)
\]

\[
\hat{H}^{(\text{fiber-dr})} = -\Delta_C \hat{b}^\dagger \hat{b} + v_1 \left( \hat{b}^\dagger \hat{a}_1 + \hat{a}_1^\dagger \hat{b} \right)
\]

\[
+ v_2 \left( \hat{b}^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{b} \right) + \mathcal{E}_u \left( \hat{b} + \hat{b}^\dagger \right), \quad (117)
\]
Figure 34: Comparison of the spectra obtained from the TM and QO models when driving through the beam splitter input port $E_a^{\text{(in)}}$ and when only cavity 1 contains an atom. The parameters for these spectra can be found in Tables IV and V of Appendix B (except for $g_2 = 0$).

Figure 35: Same as FIG. 34 but now only cavity 2 contains an atom. The parameters for these spectra can be found in Tables IV and V of Appendix B (except for $g_1 = 0$).

and $\hat{H}_i^{\text{(cQED)}}$ and $\mathcal{L}^{\text{(cQED)}}_i[\hat{\rho}]$ are defined as in the single cavity case with cavity operators $\hat{a}_i$, and spin operators $\hat{\sigma}_i^{-}$.

The master equation results in a similar set of dynamical equations as before. The only difference is in the equation for the fiber mode:

$$\frac{d}{dt} \langle \hat{b} \rangle = (i\Delta_{CF} - \kappa_{CF}) \langle \hat{b} \rangle - iv_1 \langle \hat{a}_1 \rangle$$
Figure 36: Same as Fig. 34 but both cavities are loaded with atoms. The parameters for these spectra can be found in Tables IV and V of Appendix B.

Figure 37: Same as Fig. 34 but considering empty cavities. The parameters for these spectra can be found in Tables IV and V of Appendix B (except for $g_1 = g_2 = 0$).

\[ -iv_2 \langle \hat{a}_2 \rangle - i\mathcal{E}_u, \]  

(118)  

where $\kappa_{Cf} = \kappa_{Cf,i} + \kappa_{BS}$.

Similarly to the previous subsection, using the steady state expectation values of the cavity fields we can determine the reflection and transmission spectra on the left and the right, as well as through the beam splitter.

The QM description assumes well-defined, standing-wave field modes, between which any spectral overlap is negligible. If the mirror reflectances are low and the
cavity lengths are high, such well-defined standing-wave modes are not so “cleanly” formed. In particular, this is highlighted here by the case of driving through the beam splitter input channel $E_0^{(in)}$. A traveling wave incident through this channel first propagates either to the right or downwards, and the appreciable transmittance of mirrors 3 and 4 ($R_3 = 0.80$ and $R_4 = 0.85$) means that a non-negligible fraction of the incident field can be “lost” from the system without contributing to the buildup of the connecting fiber “mode”. This is why Fig. [34] presents significant differences between the QM and TM descriptions in the right and down output channels ($E_1^{(out)}$ and $E_2^{(out)}$). The spectra for the left and up output channels ($E_2^{(out)}$ and $E_4^{(out)}$), however, show better agreement as a result of longer propagation, plus reflection, enabling interference between counter-propagating fields and better establishment of a standing-wave mode. Note that this is also related to the presence of an atom in cavity 1, and its effect on the overall transmission of this portion of the system.

If, instead, there is an atom only in cavity 2, as in Fig. [35], then the particularly high transmission of mirror 2 ($R_2 = 0.65$) plays a more significant role and substantial differences between QM and TM models also arise in the spectrum from the left output channel ($E_2^{(out)}$).

The presence of strongly-coupled atoms in the cavities generally mitigates the differences between spectra obtained from the TM and QO models, and one observes at least qualitative agreement between the two. This can also be observed in Fig. [36] where both cavities are loaded with atoms. However, without atoms at all, as illustrated in Fig. [37], a striking quantitative and qualitative difference can be observed. In particular, where the QO model predicts only weak emission on resonance in the right output channel ($E_1^{(out)}$), as one would also expect from the associated normal-mode analysis (resonant driving of the connecting fiber should not excite the fiber-dark mode), the TM model gives a pronounced peak and strong emission. Note that we have in fact also observed this resonance in the laboratory, confirming the TM prediction.

The reasoning based upon traveling- and standing-wave modes in the system also applies in this case. The high transmission rates of mirrors 2 and 3, as well as the symmetry-breaking directionality of the incident probe field, means that the simple picture of the probe field driving only a resonant, standing-wave fiber mode is no longer valid. Viewed another way, a significant portion of the probe field incident through $E_4^{(in)}$ can be transmitted on resonance directly to the right ($E_2^{(out)}$) in a single pass.

In order to achieve better agreement between the TM and single-mode QO models, we can consider higher reflectances for the mirrors, as in Fig. [38] where $R_{1-4} = 0.95$. This leads to a “better-defined” fiber mode and “cleaner” coupling of the probe to this mode, which clearly reduces the previous difference between the two approaches, but does not completely eliminate it.
Figure 39: Probing the system through both fiber beam splitter input ports (up $E_u^{(in)}$ and down $E_d^{(in)}$) simultaneously. The parameters for these spectra can be found in Tables IV and V of Appendix B (except for $g_1 = g_2 = 0$).

D. Driving through multiple ports

In the previous subsection we argued that the discrepancies between the predictions of the QO and TM approaches are primarily due to the lower reflectances of the central mirrors, which reduce the effectiveness with which standing-wave modes are established or coupled to by an incident traveling-wave probe. To further justify this argument, we can also try to recover the standing-wave-like character of the field modes by driving the system in opposite directions simultaneously, i.e., through both the up ($E_u^{(in)}$) and down ($E_d^{(in)}$) input ports of the beam splitter.

Using equations (84-87) and the conversion between transfer and scattering matrices (5), we can derive the following set of equations for the output spectra:

\[
E_1^{(out)} = \xi \left\{ \sqrt{T^{(BS)}} \frac{1}{T_{11}^{(S1)}} \frac{1}{T_{11}^{(S2)}} E_1^{(in)} - \left[ T^{(BS)} \frac{T_{12}^{(S1)}}{T_{11}^{(S1)}} \frac{T_{22}^{(S2)}}{T_{11}^{(S2)}} + \frac{T_{12}^{(S2)}}{T_{11}^{(S2)}} \right] E_2^{(in)} \right. \\
- \left. i \sqrt{R^{(BS)}} T^{(BS)} \frac{T_{12}^{(S1)}}{T_{11}^{(S1)}} \frac{1}{T_{11}^{(S2)}} E_d^{(in)} + i \sqrt{R^{(BS)}} \frac{1}{T_{11}^{(S2)}} E_u^{(in)} \right\},
\]

\[
E_2^{(out)} = \xi \left\{ \left[ T^{(BS)} \frac{T_{22}^{(S1)}}{T_{11}^{(S1)}} \frac{T_{22}^{(S2)}}{T_{11}^{(S2)}} + \frac{T_{22}^{(S2)}}{T_{11}^{(S2)}} \right] E_1^{(in)} + \sqrt{T^{(BS)}} \frac{\det T^{(S1)}}{T_{11}^{(S1)}} \frac{\det T^{(S2)}}{T_{11}^{(S2)}} E_2^{(in)} \right. \\
+ \left. i \sqrt{R^{(BS)}} \frac{\det T^{(S1)}}{T_{11}^{(S1)}} E_d^{(in)} + i \sqrt{R^{(BS)}} T^{(BS)} \frac{T_{21}^{(S2)}}{T_{11}^{(S2)}} \frac{\det T^{(S1)}}{T_{11}^{(S1)}} - E_u^{(in)} \right\},
\]

\[
E_d^{(out)} = \xi \left\{ i \sqrt{R^{(BS)}} \frac{T_{12}^{(S1)}}{T_{11}^{(S1)}} E_1^{(in)} - i \sqrt{R^{(BS)}} T^{(BS)} \frac{T_{12}^{(S1)}}{T_{11}^{(S1)}} \frac{T_{12}^{(S2)}}{T_{11}^{(S2)}} \frac{\det T^{(S2)}}{T_{11}^{(S2)}} E_2^{(in)} \right\}.
\]
Figure 40: Same as Fig. 39 but both cavities are loaded with atoms. The parameters for these spectra can be found in Tables IV and V of Appendix B.

\[ + R^{(BS)} \frac{T_{12}^{(S1)}}{T_{11}^{(S1)}} E^{(in)}_d + \sqrt{T^{(BS)}} \left[ 1 + \frac{T_{21}^{(S1)}}{T_{11}^{(S1)}} \right] E^{(in)}_u \}, \quad (121) \]

\[ E^{(out)}_u = \xi \left\{ i \sqrt{R^{(BS)}T^{(BS)}} \frac{1}{T_{11}^{(S1)}} \frac{T_{21}^{(S2)}}{T_{11}^{(S2)}} E^{(in)}_1 + i \sqrt{R^{(BS)}\det T^{(S2)}} \frac{T_{21}^{(S2)}}{T_{11}^{(S2)}} E^{(in)}_2 \right\}, \quad (122) \]

Meanwhile, the QO model simply accounts for the effects of a second probe field with an additional driving term in the fiber Hamiltonian \( H^{(fiber-dr)} \),

\[ \mathcal{E}_d \left( \hat{b} + \hat{b}^\dagger \right), \quad (123) \]

where \( \mathcal{E}_d \) is the driving strength corresponding to the other port.

Driving through both input ports of the beam splitter, we obtain very good agreement between the TM and QO approaches, even for the spectrum detected from the right output channel (Fig. 39). This is due to the counter-propagating contributions from the probe fields enhancing their effective “overlap” with the standing-wave modes upon which the QO approach is based. This is further supported by the good agreement in the case where both cavities are loaded with atoms as well (Fig. 40).

VIII. CONCLUSION

In this work, we have presented a transfer matrix formalism that offers a straightforward and intuitive means of calculating the transmission and reflection spectra of weak probe fields incident upon a network of fibers, fiber cavities, beam splitters, and nanofiber-coupled atoms. It is intrinsically multimode in nature, accounting properly for propagation distance and the case of low mirror reflectance, and thus gives a more accurate description of such networks in its (linear response) regime of validity than quantum-optical models based upon single mode fields.

For examples, we have focused on recent, topical configurations involving coupled, nanofiber-cavity-QED systems [17, 18], for which single-mode quantum-optical models still provide good descriptions of experiments, but where differences with the transfer matrix approach,
though typically quite small, are clearly noticeable. In cases of more substantial differences, we are able to trace the differences to the relatively low reflectances of mirrors, appreciable propagation distances (i.e., small free spectral range), and to the precise way in which the probe laser is incident upon the system, i.e., to an explicit directionality of excitation in the system, which is not accounted for in the quantum-optical model.

The simplicity and flexibility of the transfer matrix model make it well suited to examine quite general fiber-based networks, as well as other, topical atom-nanofiber systems, such as (possibly large) ring-cavity QED setups [16, 34] with potential chiral coupling between the atoms and the waveguide. It might also prove useful in examining the behavior of fiber-based, coherent feedback setups with finite time delays, e.g., a version of the emitter-in-front-of-a-mirror problem (see, e.g., [35, 36]), where atoms couple to a portion of nanofiber, which is in turn linked to a long length of regular fiber terminated at the end by a highly reflecting FBG mirror.

Appendix A: Notations

Each mirror in the schematics is numbered $j$ in increasing order from left to right. Similarly, each cavity and corresponding atom in the schematics is numbered $j$ in increasing order from left to right.

Table I: Mirror and atomic parameters

| Transfer Matrix | Reflectivity $R_j$ | Transmission $T_j$ | Atom's distance from the left mirror $d_j$ |
|-----------------|-------------------|-------------------|------------------------------------------|
| Quantum Optical | Atomic resonance frequency $\omega_A$ | Atomic detuning $\Delta_A = \omega - \omega_A$ | Atomic spontaneous emission $\gamma$ |
|                 | Mirror transmission rates $\kappa_j$ |                    |                                          |

Table II: Cavity parameters

| Transfer Matrix | Cavity length $l_j$ | Effective transmission $\eta_j$ | Cavity loss ratio $\alpha_j = 1 - \eta_j$ | Free spectral range $\omega_{FSR(j)}$ |
|-----------------|---------------------|---------------------------------|------------------------------------------|-------------------------------------|
| Quantum Optical | Resonance frequencies $\omega_Cj$ | Detunings $\Delta_Cj = \omega - \omega_Cj$ | Overall decay rates $\kappa_Cj$ | Intrinsic fibre loss $\kappa_{Cj,i}$ |
|                 | Driving rate $\mathcal{E}_j$ | Atom-cavity coupling strength $g_j$ | Cavity-fiber coupling strength $v_j$ | |

Table III: Parameters of the connecting fiber

| Transfer Matrix | Cavity length $l_f$ | Effective transmission $\eta_f$ | Cavity loss ratio $\alpha_f = 1 - \eta_f$ | Free spectral range $\omega_{FSRf}$ |
|-----------------|---------------------|---------------------------------|------------------------------------------|-------------------------------------|
| Quantum Optical | Resonance frequency $\omega_{Cf}$ | Detuning $\Delta_{Cf} = \omega - \omega_{Cf}$ | Overall decay rates $\kappa_{Cf}$ | Intrinsic fiber loss $\kappa_{Cf,i}$ |
|                 | Outcoupling rate $\kappa_{BS}$ | | | |

Appendix B: Parameters for the coupled cavity-QED experiments

In order to demonstrate that the difference between the transfer matrix results and the quantum optical predictions is measurable, we have used the parameters of recent experiments.
Table IV: CCQED parameters in the transfer matrix (TM) formalism:

| Transfer Matrix                   |
|-----------------------------------|
| Detuning between the optical (atomic) and the driving field \( \Delta = \omega_{Cj} - \omega = \omega_{Aj} - \omega \) |
| Mirror reflectances \( R_1 = 0.8 \) |
| \( R_2 = 0.65 \) |
| \( R_3 = 0.8 \) |
| \( R_4 = 0.85 \) |
| Lengths of the cavities and the connecting fiber \( l_1 = 0.92 \) m |
| \( l_2 = 1.38 \) m |
| \( l_l = 1.8 \) m |
| Free spectral range of the cavities and the connecting fiber \( \omega_{FSR1} = 112.25 \) MHz |
| \( \omega_{FSR2} = 74.833 \) MHz |
| \( \omega_{FSRf} = 57.372 \) MHz |
| Intrinsic transmission of the cavities and the connecting fiber \( \eta_i = \eta_f = 0.97 \) |
| Reflectance of the outcoupling BS \( R^{(BS)} = 0.01 \) |
| Energy decay rate into free space \( \Gamma' = 5.2 \times 2 \pi \) MHz |

Table V: CCQED parameters in the quantum-optical single-mode (QO) approach:

| Quantum Optical                  |
|-----------------------------------|
| Cavity and fiber \( \kappa_1/2\pi = 1.787 \) MHz |
| outcoupling rate \( \kappa_2/2\pi = 0.893 \) MHz |
| \( \kappa_j = \frac{1}{2\pi} \omega_{FSR} (1 - R_j) \) |
| Intrinsic loss rates in the cavities and the fiber \( \kappa_{C1}/2\pi = 0.544 \) MHz |
| \( \kappa_{C2}/2\pi = 0.363 \) MHz |
| \( \kappa_{Cj,i} = - \frac{1}{2\pi} \ln \eta_j \) |
| Cavity-fiber coupling strength \( v_1 = 7.556 \) MHz |
| \( v_2 = 4.664 \) MHz |
| Atom-cavity coupling strengths \( g_1 = 6 \times 2\pi \) MHz |
| \( g_2 = 7 \times 2\pi \) MHz |

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Another origin is the propagation efficiency. This can be understood by, for example, considering the reflection by an atom with no mirrors ($T_1 = T_2 = 1$). The incoming field experiences propagation through a distance $d$ (with an efficiency of $\eta = e^{-2k'd}$) before reaching the atom and after reflecting off the atom and returning to the initial position. The amplitude of the reflected field would obviously depend on $d$.

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