Abstract

Using anisotropic R-matrices associated with affine Lie algebras \( \hat{\mathfrak{g}} \) (specifically, \( \hat{A}^{(2)}_{2n} \), \( \hat{A}^{(2)}_{2n-1} \), \( \hat{B}^{(1)}_n \), \( \hat{C}^{(1)}_n \), \( \hat{D}^{(1)}_n \)) and suitable corresponding K-matrices, we construct families of integrable open quantum spin chains of finite length, whose transfer matrices are invariant under the quantum group corresponding to removing one node from the Dynkin diagram of \( \hat{\mathfrak{g}} \). We show that these transfer matrices also have a duality symmetry (for the cases \( \hat{C}^{(1)}_n \) and \( \hat{D}^{(1)}_n \)) and additional \( \mathbb{Z}_2 \) symmetries that map complex representations to their conjugates (for the cases \( \hat{A}^{(2)}_{2n-1} \), \( \hat{B}^{(1)}_n \) and \( \hat{D}^{(1)}_n \)). A key simplification is achieved by working in a certain “unitary” gauge, in which only the unbroken symmetry generators appear. The proofs of these symmetries rely on some new properties of the R-matrices. We use these symmetries to explain the degeneracies of the transfer matrices.
1 Introduction and summary

Quantum spin chains have numerous applications. Being interacting many-body systems, their spectra are generally difficult to determine when the number of spins is large. The simplest anisotropic spin chains are arguably those that are integrable and have quantum group (QG) symmetries. Indeed, integrability can help to determine the spectrum, and QG symmetry can help to explain the degeneracies and multiplicities. The first such example was the $U_q(A_1)$-invariant open spin-1/2 chain [1, 2], whose integrability follows from [3, 4]. Various higher-rank generalizations have been investigated, see e.g. [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 20, 21, 22, 23, 24, 25, 26, 27].

We identify here new families of integrable QG-invariant spin chains, which include as special cases many of the previously-studied models. Specifically, we construct integrable open spin chains of finite length using anisotropic R-matrices associated with several families of affine Lie algebras $\hat{g}$ (namely, $A_{2n}^{(2)}$, $A_{2n-1}^{(2)}$, $B_n^{(1)}$, $C_n^{(1)}$, $D_n^{(1)}$) [28, 29, 30, 31, 32], and with corresponding diagonal K-matrices depending on an integer $p \in [0, n]$ [33, 34, 35, 36], whose transfer matrices $t(u, p)$ (2.20) are invariant under the QG corresponding to removing the $p^{th}$ node from the (extended) Dynkin diagram of $\hat{g}$, as summarized in the middle column of Table 1.

| $\hat{g}$ | QG symmetry | Representation at each site |
|-----------|-------------|-----------------------------|
| $A_{2n}^{(2)}$ | $U_q(B_{n-p}) \otimes U_q(C_p)$ | $(2(n-p) + 1, 1) \oplus (1, 2p)$ |
| $A_{2n-1}^{(2)}$ | $U_q(C_{n-p}) \otimes U_q(D_p)$ ($p \neq 1$) | $(2(n-p), 1) \oplus (1, 2p)$ |
| $B_n^{(1)}$ | $U_q(B_{n-p}) \otimes U_q(D_p)$ ($n > 1, p \neq 1$) | $(2(n-p) + 1, 1) \oplus (1, 2p)$ |
| $C_n^{(1)}$ | $U_q(C_{n-p}) \otimes U_q(C_p)$ | $(2(n-p), 1) \oplus (1, 2p)$ |
| $D_n^{(1)}$ | $U_q(D_{n-p}) \otimes U_q(D_p)$ ($n > 1, p \neq 1, n-1$) | $(2(n-p), 1) \oplus (1, 2p)$ |

Table 1: QG symmetries of the open-chain transfer matrix, where $p = 0, 1, \ldots, n$.

For $p = 0$, the “right” (second) factors in Table 1 are absent; these cases were studied long ago [5, 6, 7, 12, 13]. For $p = n$, the “left” (first) factors in Table 1 are absent; these cases were noticed only recently [18, 26, 27]. For intermediate values $0 < p < n$, the QG symmetries are generally given by a tensor product of two factors, as shown in Table 1; these cases had not been considered until now. Moreover, we prove for all $p \in [0, n]$ that the transfer matrices have these QG symmetries.

A key role in our proof of the QG symmetry of the transfer matrix is played by so-called gauge transformations of the R-matrix and K-matrices. By transforming to a certain “unitary” gauge, the asymptotic monodromy matrix becomes expressed in terms of only the unbroken symmetry generators, which then allows us to invoke the powerful machinery of the Quantum Inverse Scattering Method (QISM). The relevance of this gauge transformation

\footnote{All of these spin chains are open, since closed anisotropic integrable spin chains of finite length with periodic boundary conditions generally do not have such QG symmetry.}

\footnote{Such proofs had been known, following [2], only for the cases with $p = 0$ [6]. For the cases with $p = n$, the QG symmetry was conjectured for the transfer matrices, but was proved only for the corresponding Hamiltonians [26, 27].}
can already be seen from the following observation: the R-matrices of Jimbo [28], which are in the so-called homogeneous “picture” or “gradation” (gauge), have the symmetries in Table 1 with \( p = 0 \) (more precisely, \( \hat{R} = PR \) commutes with the coproducts of the generators of the QG); while the gauge-transformed R-matrices, with the gauge transformation corresponding to \( p = n \) (see Eqs. (3.1) and (3.3) below), have instead the symmetries in Table 1 with \( p = n \).

For the cases \( C_n^{(1)} \) and \( D_n^{(1)} \), the two symmetry factors in Table 1 evidently interchange under \( p \leftrightarrow n - p \). In fact, we show that the corresponding transfer matrices are related by “duality” transformations (4.5), implying that their spectra are equal. For the special case with \( n \) even and \( p = \frac{n}{2} \), the transfer matrix is self-dual (4.18), which gives rise to degeneracies in the spectrum beyond those expected from QG symmetry.

For the cases that at least one of the symmetry factors in Table 1 is of type \( D \) (namely, \( A_{2n-1}^{(2)} \), \( B_n^{(1)} \) and \( D_n^{(1)} \)), we show that the transfer matrices have additional \( Z_2 \) symmetries that map complex representations to their conjugates (5.4), (5.15).

All of these symmetries are useful for understanding the degeneracies in the spectrum of the transfer matrix. In proving these symmetries, we use various properties of the R-matrices (4.1), (5.1), (5.12), which to our knowledge are new, and which may be of independent interest.

The outline of this paper is as follows. The transfer matrix is introduced in Sec. 2. The QG symmetry of the transfer matrix is proved in Sec. 3. The duality symmetry of the transfer matrix (for the cases \( C_n^{(1)} \) and \( D_n^{(1)} \)), and the action of duality on the QG generators, are worked out in Sec. 4. The additional \( Z_2 \) symmetries of the transfer matrix (for the cases \( A_{2n-1}^{(2)} \), \( B_n^{(1)} \) and \( D_n^{(1)} \)), and the action of these symmetries on the QG generators, are worked out in Sec. 5. These symmetries are used in Sec. 6 to explain the degeneracies in the spectrum of the transfer matrix for generic values of the anisotropy parameter \( \eta \). Some interesting remaining open problems are listed in Sec. 7. The R-matrices are recalled in Appendix A, details about the QG generators are presented in Appendix B, and the Hamiltonian is noted in Appendix C. Proofs of several lemmas are outlined in Appendix D.

## 2 Basics

We consider an integrable open quantum spin chain with a vector space \( V = \mathbb{C}^d \) at each of its \( N \) sites, where

\[
d = \begin{cases} 
2n + 1 & \text{for } A_{2n}^{(2)}, B_n^{(1)} \\
2n & \text{for } A_{2n-1}^{(2)}, C_n^{(1)}, D_n^{(1)} 
\end{cases}, \quad n = 1, 2, \ldots.
\]

(2.1)

The Hilbert space (“quantum” space) of the spin chain is therefore \( V^\otimes N \).
2.1 R-matrix

The bulk interactions of the spin chain are encoded in the R-matrix $R(u)$, which maps $V \otimes V$ to itself, and satisfies the Yang-Baxter equation (YBE) on $V \otimes V \otimes V$

$$R_{12}(u-v) R_{13}(u) R_{23}(v) = R_{23}(v) R_{13}(u) R_{12}(u-v).$$  \hfill (2.2)

We use the standard notations $R_{12} = R \otimes I, R_{23} = I \otimes R, R_{13} = P_{23} R_{12} P_{23} = P_{12} R_{23} P_{12}$, where $I$ is the identity matrix on $V$, and $P$ is the permutation matrix on $V \otimes V$

$$P = \sum_{i,j=1}^{d} e_{ij} \otimes e_{ji},$$  \hfill (2.3)

where $e_{ij}$ are the $d \times d$ elementary matrices with elements $(e_{ij})_{\alpha\beta} = \delta_{i,\alpha} \delta_{j,\beta}$.

We consider here the anisotropic R-matrices (with anisotropy parameter $\eta$) corresponding to the following affine Lie algebras \(^3\)

$$\hat{g} = \{ A^{(2)}_{2n}, A^{(2)}_{2n-1}, B^{(1)}_n, C^{(1)}_n, D^{(1)}_n \}. \hfill (2.4)$$

These R-matrices, which are given by Jimbo [28] (except for $A^{(2)}_{2n-1}$, in which case we consider instead Kuniba’s R-matrix [30]), are in the homogeneous picture (gauge).\(^4\) These R-matrices, which can be found in Appendix A, all have the following additional properties: $PT$ symmetry

$$R_{21}(u) \equiv P_{12} R_{12}(u) P_{12} = R_{12}^{t_1 t_2}(u),$$  \hfill (2.5)

unitarity

$$R_{12}(u) R_{21}(-u) = \zeta(u) I \otimes I,$$  \hfill (2.6)

where $\zeta(u)$ is given by

$$\zeta(u) = \xi(u) \xi(-u), \quad \xi(u) = -2 \delta_1 \sinh \left( \frac{1}{2}(u + 4\eta) \right) \sinh \left( \frac{1}{2}(u + \rho) \right),$$  \hfill (2.7)

where $\delta_1$ is given by

$$\delta_1 = \begin{cases} i & \text{for } A^{(2)}_{2n}, A^{(2)}_{2n-1} \\ 1 & \text{for } B^{(1)}_n, C^{(1)}_n, D^{(1)}_n \end{cases} \hfill (2.8)$$

and crossing symmetry

$$R_{12}(u) = V_1 R_{12}^{t_2}(-u - \rho) V_1 = V_2^{t_2} R_{12}^{t_1}(-u - \rho) V_2^{t_2},$$  \hfill (2.9)

where the crossing parameter $\rho$ is given by

$$\rho = \begin{cases} -2\kappa \eta - i\pi & \text{for } A^{(2)}_{2n}, A^{(2)}_{2n-1} \\ -2\kappa \eta & \text{for } B^{(1)}_n, C^{(1)}_n, D^{(1)}_n \end{cases},$$  \hfill (2.10)

\(^3\)We do not consider here the case $A^{(1)}_n$, which does not have crossing symmetry; it has been studied in a similar context in [9, 10, 17].

\(^4\)Bazhanov’s R-matrices [29] are equivalent, but are instead in the principal picture.
with $\kappa$ defined in (A.4). The crossing matrix $V$ is an antidiagonal matrix given by

$$V = \delta_2 \sum_{\alpha=1}^{d} \epsilon_{\alpha} e^{(\bar{\alpha} - \bar{\alpha}')\eta} e_{\alpha\alpha'}, \quad V^2 = \mathbb{I}, \quad (2.11)$$

where $\delta_2$ is given by

$$\delta_2 = \begin{cases} 1 & \text{for } A_{2n}^{(2)}, B_n^{(1)}, D_n^{(1)} \\ i & \text{for } A_{2n-1}^{(2)}, C_n^{(1)} \end{cases},$$

and the other notations are defined in (A.5)-(A.7). The corresponding matrix $M$ is defined by

$$M = V^t V, \quad (2.12)$$

and it is given by the diagonal matrix

$$M = \delta_2^2 \sum_{\alpha=1}^{d} e^{4(\frac{d+1}{2} - a)\eta} e_{\alpha\alpha}. \quad (2.13)$$

### 2.2 K-matrices

The boundary interactions are encoded in the right and left K-matrices, denoted here by $K^R(u)$ and $K^L(u)$, respectively, which map $\mathcal{V}$ to itself.\footnote{Following Sklyanin [4], the right and left K-matrices are usually denoted instead by $K^-(u)$ and $K^+(u)$, respectively. However, we adopt a different notation here in order to avoid confusion with the $\pm$ used in subsequent sections to denote the limits $u \to \pm \infty$.} We choose $K^R(u)$ to be the diagonal $d \times d$ matrix

$$K^R(u) = K^R(u, p) = \text{diag} \left( e^{-u}, \ldots, e^{-u}, \right.\left. \gamma e^{u}, \ldots, \gamma e^{u}, e^{u}, \ldots, e^{u} \right), \quad (2.14)$$

where $p = 0, 1, \ldots, n$, and

$$\gamma = \begin{cases} \gamma_0 e^{(4p-2)\eta + \frac{1}{2}\rho} & \text{for } A_{2n-1}^{(2)}, B_n^{(1)}, D_n^{(1)} \\ \gamma_0 e^{(4p+2)\eta + \frac{1}{2}\rho} & \text{for } A_{2n}^{(2)}, C_n^{(1)} \end{cases}, \quad \gamma_0 = \pm 1, \quad (2.15)$$

where $\rho$ is the crossing parameter (2.10). Unless otherwise noted, all the results in this paper hold for both values ($\pm 1$) of the parameter $\gamma_0$. As observed in [36] (see also [33, 34, 35]), the matrices (2.14) are solutions of the boundary Yang-Baxter equation (BYBE) on $\mathcal{V} \otimes \mathcal{V}$ [4, 37, 38]

$$R_{12}(u - v) K^R_1(u) R_{21}(u + v) K^R_2(v) = K^R_2(v) R_{12}(u + v) K^R_1(u) R_{21}(u - v). \quad (2.16)$$

For $p = 0$, we see that $K^R(u, p)$ in (2.14) is proportional to the identity matrix,

$$K^R(u, 0) \propto \mathbb{I}, \quad (2.17)$$
which is the solution noted in [5]. We emphasize that the solution (2.14) depends on the bulk anisotropy parameter \( \eta \) and the discrete boundary parameters \( p \) and \( \gamma_0 \), but does not have any continuous boundary parameters.

For the left K-matrix, we take
\[
K^L(u) = K^L(u, p) = K^R(-u - \rho, p) M, 
\]
where \( M \) is given by (2.12), which is a solution of the corresponding BYBE [4, 39]
\[
R_{12}(-u + v) K^L_{11}(u) M^{-1}_1 R_{21}(-u - v - 2\rho) M_1 K^L_{22}(v)
= K^L_{22}(v) M_1 R_{12}(-u - v - 2\rho) M^{-1}_1 K^L_{11}(u) R_{21}(-u + v). 
\]

### 2.3 Transfer matrix

The open-chain transfer matrix, which maps the quantum space \( \mathcal{V}^\otimes N \) to itself, is given by[4]
\[
t(u, p) = \text{tr}_a K^L_a(u, p) T_a(u) K^R_a(u, p) \hat{T}_a(u), 
\]
where the single-row monodromy matrices are defined by
\[
T_a(u) = R_{aN}(u) R_{aN-1}(u) \cdots R_{a1}(u), \\
\hat{T}_a(u) = R_{1a}(u) \cdots R_{N-1a}(u) R_{Na}(u), 
\]
and the trace in (2.20) is over the “auxiliary” space, which is denoted by \( a \). The transfer matrix is engineered to satisfy the fundamental commutativity property
\[
[t(u, p), t(v, p)] = 0 \text{ for all } u, v, 
\]
which is the hallmark of integrability. The transfer matrix contains the Hamiltonian (\( \sim t'(0, p) \), see Appendix C) and higher local conserved quantities.

### 3 Quantum group symmetry

We now proceed to show that the transfer matrix (2.20) has QG symmetry, in accordance with the second column in Table 1.

A key step of our argument is to use a gauge transformation to bring the right K-matrix “as close as possible” to the identity matrix. By transforming to this “unitary” gauge, the asymptotic (single-row) monodromy matrix becomes expressed in terms of only the unbroken symmetry generators, which then allows us to bring the powerful QISM machinery to bear on the problem. To this end, we set (see e.g. [28])
\[
\tilde{R}_{12}(u, p) = B_1(u, p) R_{12}(u) B_1(-u, p) = B_2(-u, p) R_{12}(u) B_2(u, p), 
\]
and [39]

\[ \tilde{K}^R(u, p) = B(u, p) K^R(u, p) B(u, p), \]
\[ \tilde{K}^L(u, p) = B(-u, p) K^L(u, p) B(-u, p), \]  \hspace{1cm} (3.2)

where \( B(u, p) \) is a diagonal matrix that maps \( V \) to itself, which we choose as follows

\[ B(u, p) = \text{diag} \left( e^{\pm u \frac{p}{d-2p}}, e^{\pm u \frac{1}{d-2p}}, e^{\pm u \frac{d-2p}{p}} \right). \]  \hspace{1cm} (3.3)

Indeed, this gauge transformation brings \( K^R(u, p) \) (2.14) to a form with mostly 1’s on the diagonal

\[ \tilde{K}^R(u, p) = \text{diag} \left( \frac{1}{\gamma + e^u}, \ldots, \frac{1}{\gamma + e^u}, \gamma e^u + 1, \ldots, \gamma e^u + 1 \right). \]  \hspace{1cm} (3.4)

For \( p = n \), we see that \( \tilde{K}^R(u, n) \) is exactly equal to \( I \) if \( d = 2n \) (i.e., for \( A_{2n-1}^{(2)}, C_n^{(1)}, \) and \( D_n^{(1)} \)); and \( \tilde{K}^R(u, n) \) differs from \( I \) only in the middle matrix element if \( d = 2n + 1 \) (i.e., for \( A_{2n}^{(2)} \) and \( B_n^{(1)} \)).

The matrix \( B(u, p) \) satisfies

\[ B(u, p) B(v, p) = B(u + v, p), \quad B(0, p) = I, \]  \hspace{1cm} (3.5)

as well as

\[ [B_1(u, p) B_2(u, p), R_{12}(v)] = 0. \]  \hspace{1cm} (3.6)

With the help of these properties, it can be shown that the gauge-transformed R-matrix and K-matrices continue to satisfy their respective Yang-Baxter equations. The crossing symmetry (2.9) is also maintained, with [39]

\[ \tilde{V}(p) = V B(\rho, p) = B(-\rho, p) V, \]  \hspace{1cm} (3.7)

and

\[ \tilde{M}(p) = \tilde{V}^t(p) \tilde{V}(p) = B(\rho, p) M B(\rho, p). \]  \hspace{1cm} (3.8)

The transfer matrix (2.20) remains invariant under these transformations [39]

\[ t(u, p) = \text{tr}_a \tilde{K}^L_a(u, p) \tilde{T}_a(u, p) \tilde{K}^R_a(u, p) \tilde{T}_a(u, p), \]  \hspace{1cm} (3.9)

where

\[ \tilde{T}_a(u, p) = \tilde{R}_{aN}(u, p) \tilde{R}_{aN-1}(u, p) \cdots \tilde{R}_{a1}(u, p), \]

\[ \tilde{T}_a(u, p) = \tilde{R}_{1a}(u, p) \cdots \tilde{R}_{N-1a}(u, p) \tilde{R}_{Na}(u, p). \]  \hspace{1cm} (3.10)
As already remarked in the Introduction, prior to any gauge transformation, the R-matrix has the property that $\tilde{R}(u) = PR(u)$ commutes with the coproducts of generators of the “left” quantum group $U_q(g^{(l)})$ in Table 1 with $p = 0$, i.e.\(^6\)

\[
p = 0 : \quad [\tilde{R}(u), \Delta(H_j^{(l)}(0))] = 0 = [\tilde{R}(u), \Delta(E_j^{(l)}(0))], \quad j = 1, \ldots, n. \quad \text{(3.11)}
\]

In contrast, the gauge-transformed R-matrix given by (3.1) and (3.3) with $p = n$ has the property that $\tilde{\tilde{R}}(u,n) = P\tilde{R}(u,n)$ commutes with the coproducts of generators of the “right” quantum group $U_q(g^{(r)})$ in Table 1 with $p = n$, i.e.

\[
p = n : \quad [\tilde{\tilde{R}}(u,n), \Delta(H_j^{(r)}(0))] = 0 = [\tilde{\tilde{R}}(u,n), \Delta(E_j^{(r)}(0))], \quad j = 1, \ldots, n. \quad \text{(3.12)}
\]

We now use such gauge transformations to prove the QG invariance of the open-chain transfer matrix $t(u,p)$ for any integer $p \in [0, n]$.

Let us denote by $\tilde{R}^\pm(p)$ the asymptotic limits of the gauge-transformed R-matrix $\tilde{R}(u,p)$ (3.1)

\[
\tilde{R}^\pm(p) = \lim_{u \to \pm \infty} e^{\pm u} \tilde{R}(u,p),
\]

and we similarly denote by $\tilde{T}_a^\pm(p)$ the asymptotic limits of the gauge-transformed monodromy matrix $\tilde{T}_a(u,p)$ (3.10)

\[
\tilde{T}_a^\pm(p) = \tilde{R}_{aN}^\pm(p) \tilde{R}_{aN-1}^\pm(p) \cdots \tilde{R}_{a1}^\pm(p).
\]

Let us further denote by $\tilde{T}_{i,j}^\pm(p)$ ($1 \leq i, j \leq d$) the matrix elements of $\tilde{T}_a^\pm(p)$ in the auxiliary space, which are operators on the quantum space $\mathcal{V}_{\otimes N}$.

We show in Appendix B that the operators $\tilde{T}_{i,j}^\pm(p)$ can be expressed in terms of (the quantum enveloping algebra of) the unbroken $g$ generators, i.e. the generators of the quantum groups in the second column of Table 1. Hence, in order to demonstrate the QG symmetry of the transfer matrix, it suffices to show that

\[
[\tilde{T}_{i,j}^\pm(p), t(u,p)] = 0 \quad i, j = 1, 2, \ldots, d. \quad \text{(3.15)}
\]

To this end, following [40] (see also [2, 6]), we first establish several lemmas.

**Lemma 1.**

\[
[\tilde{R}^\pm_{12}(p), \tilde{K}_2^R(u,p)] = 0. \quad \text{(3.16)}
\]

A proof is outlined in Secs. D.4 and D.5.

**Lemma 2.**

\[
[\tilde{R}^+_1(p), \tilde{M}_1(p) \tilde{K}_2^L(u,p)] = 0. \quad \text{(3.17)}
\]
Proof. We observe that

\[ \tilde{K}^L(u, p) = \tilde{K}^R(-u - \rho, p) \tilde{M}(p) = \tilde{M}(p) \tilde{K}^R(-u - \rho, p), \]

as follows from (2.18), (3.8) and (3.2). Hence,

\[ \tilde{R}_{12}^\pm(p) \tilde{M}_1(p) \tilde{K}_2^L(u, p) = \tilde{R}_{12}^\pm(p) \tilde{M}_1(p) \tilde{M}_2(p) \tilde{K}_2^R(-u - \rho, p) \]
\[ = \tilde{M}_1(p) \tilde{M}_2(p) \tilde{R}_{12}^\pm(p) \tilde{K}_2^R(-u - \rho, p) \]
\[ = \tilde{M}_1(p) \tilde{M}_2(p) \tilde{K}_2^R(-u - \rho, p) \tilde{R}_{12}^\pm(p) \]
\[ = \tilde{M}_1(p) \tilde{K}_2^R(u, p) \tilde{R}_{12}^\pm(p), \]

where the first and last equalities follow from (3.18); the second equality is a consequence of the fact [39]

\[ [R_{12}(u), M_1 M_2] = 0; \] (3.20)

and the third equality follows from Lemma 1 (3.16).

Lemma 3. \[ \left[ \tilde{R}_{12}^\pm(p) \tilde{T}_1^\pm(p), \tilde{T}_2(u, p) \tilde{K}_2^R(u, p) \tilde{T}_2(u, p) \right] = 0. \] (3.21)

Proof. We recall the gauge-transformed fundamental relation

\[ \tilde{R}_{12}(u_1 - u_2, p) \tilde{T}_1(u_1, p) \tilde{T}_2(u_2, p) = \tilde{T}_2(u_2, p) \tilde{T}_1(u_1, p) \tilde{R}_{12}(u_1 - u_2, p). \] (3.22)

Taking asymptotic limits of \( u_1 \) yields

\[ \tilde{R}_{12}^\pm(p) \tilde{T}_1^\pm(p) \tilde{T}_2(u, p) = \tilde{T}_2(u, p) \tilde{T}_1^\pm(p) \tilde{R}_{12}^\pm(p), \] (3.23)

which further implies

\[ \tilde{T}_2^{-1}(u, p) \tilde{R}_{12}^\pm(p) \tilde{T}_1^\pm(p) = \tilde{T}_1^\pm(p) \tilde{R}_{12}^\pm(p) \tilde{T}_2^{-1}(u, p). \] (3.24)

Therefore,

\[ \tilde{R}_{12}^\pm(p) \tilde{T}_1^\pm(p) \tilde{T}_2(u, p) \tilde{K}_2^R(u, p) \tilde{T}_2^{-1}(-u, p) \]
\[ = \tilde{T}_2(u, p) \tilde{T}_1^\pm(p) \tilde{R}_{12}^\pm(p) \tilde{K}_2^R(u, p) \tilde{T}_2^{-1}(-u, p) \]
\[ = \tilde{T}_2(u, p) \tilde{K}_2^R(u, p) \tilde{T}_1^\pm(p) \tilde{R}_{12}^\pm(p) \tilde{T}_2^{-1}(-u, p) \]
\[ = \tilde{T}_2(u, p) \tilde{K}_2^R(u, p) \tilde{T}_2^{-1}(-u, p) \tilde{R}_{12}^\pm(p) \tilde{T}_1^\pm(p), \] (3.25)

where the first equality follows from (3.23), the second equality follows from Lemma 1 (3.16), and the third equality follows from (3.24). We have therefore demonstrated the commutativity property

\[ \left[ \tilde{R}_{12}^\pm(p) \tilde{T}_1^\pm(p), \tilde{T}_2(u, p) \tilde{K}_2^R(u, p) \tilde{T}_2^{-1}(-u, p) \right] = 0. \] (3.26)
Finally, we see from (3.10) that
\[ T_a^{-1}(u, p) = R_a^{-1}(u, p) \cdots R_N^{-1}(u, p) \]
\[ \propto R_a(-u, p) \cdots R_N(-u, p) = \tilde{T}_a(-u, p), \]
where the second line follows from unitarity (2.6). Substituting into (3.26) we obtain the desired result (3.21).

Lemma 4.
\[ \tilde{M}_1^{-1}(p) \left( (\tilde{R}_{12}(p))^{-1} \right)^{t_2} \tilde{M}_1(p) \tilde{R}_{12}^{t_2}(p) = \mathbb{I}^{\otimes 2}. \]  
(3.28)

Proof. We write the gauge-transformed unitarity condition (2.6) as
\[ \tilde{R}_{12}(u, p) \tilde{R}_{12}^{t_2}(-u, p) = \zeta(u) \mathbb{I}^{\otimes 2}, \]  
(3.29)
and then use crossing symmetry (2.9) to obtain
\[ \tilde{V}_1(p) \tilde{R}_{12}^{t_2}(-u - \rho, p) \tilde{V}_1(p) \tilde{R}_{12}^{t_1}(u - \rho, p) \tilde{V}_1(p) = \zeta(u) \mathbb{I}^{\otimes 2}, \]  
(3.30)
where \( \tilde{V}(p) \) is given by (3.7). By taking asymptotic limits of (3.30) and noting that \( \tilde{V}(p)^2 = \mathbb{I} \), we obtain
\[ \tilde{R}_{12}^{t_2}(p) \tilde{M}_1^{-1}(p) \tilde{R}_{12}^{t_1}(p) \tilde{M}_1(p) = \chi \mathbb{I}^{\otimes 2}, \]  
(3.31)
where \( \chi \) is given by
\[ \chi = \lim_{u \to \pm \infty} e^{\mp 2u} \zeta(u) = \frac{1}{4} \delta_1^2. \]  
(3.32)
Moreover, from (3.29) we obtain
\[ \tilde{R}_{12}^{t_2}(p) \tilde{R}_{12}^{t_1 t_2}(p) = \chi \mathbb{I}^{\otimes 2}, \]  
(3.33)
which implies that
\[ \tilde{R}_{12}^{t_2 t_1}(p) = \chi (\tilde{R}_{12}(p))^{-1}, \quad \text{or} \quad \tilde{R}_{12}^{t_1}(p) = \chi \left( (\tilde{R}_{12}(p))^{-1} \right)^{t_2}. \]  
(3.34)
Substituting into (3.31), we obtain
\[ \tilde{R}_{12}^{t_2}(p) \tilde{M}_1^{-1}(p) \left( (\tilde{R}_{12}(p))^{-1} \right)^{t_2} \tilde{M}_1(p) = \mathbb{I}^{\otimes 2}, \]  
(3.35)
which can be rearranged to give the desired result (3.28).

We are finally ready to prove the main result (3.15), which is equivalent to the following

Proposition 1.
\[ [\tilde{T}_1^\pm(p), t(u, p)] = 0. \]  
(3.36)
Lemma 5. The $R$-matrices for both $C_n^{(1)}$ and $D_n^{(1)}$ obey
\begin{align}
U_1 R_{12}(u) U_1 &= W_2^{*}(u) R_{12}(u) W_2(u), \\
U_2 R_{12}(u) U_2 &= W_1(u) R_{12}(u) W_1(u),
\end{align}
(4.1)

Proof. Recalling that the transfer matrix remains invariant under gauge transformations (3.9), we obtain
\begin{align*}
\hat{T}_1^\pm(p) t(u, p) \\
&= \text{tr}_2 \left\{ \hat{T}_1^\pm(p) \hat{K}_2^L(u, p) \hat{T}_2(u, p) \hat{K}_2^R(u, p) \hat{T}_2(u, p) \right\} \\
&= \text{tr}_2 \left\{ \hat{M}_1^{-1}(p) \hat{M}_1(p) \hat{K}_2^L(u, p) (\hat{R}_{12}^\pm(p))^{-1} \hat{R}_{12}^\pm(p) \hat{T}_1^\pm(p) \hat{T}_2(u, p) \hat{K}_2^R(u, p) \hat{T}_2(u, p) \hat{R}_{12}^\pm(p) \right\} \\
&= \text{tr}_2 \left\{ \hat{M}_1^{-1}(p) (\hat{R}_{12}^\pm(p))^{-1} \hat{M}_1(p) \hat{K}_2^L(u, p) \hat{T}_2(u, p) \hat{K}_2^R(u, p) \hat{T}_2(u, p) \hat{R}_{12}^\pm(p) \right\} \hat{T}_1^\pm(p) \\
&= \ldots \quad (3.37)
\end{align*}

In passing to the third equality, we have used Lemma 2 (3.17) and Lemma 3 (3.21). Then
\begin{align*}
\ldots &= \text{tr}_2 \left\{ \hat{M}_1^{-1}(p) (\hat{R}_{12}^\pm(p))^{-1} \hat{M}_1(p) \hat{K}_2^L(u, p) \hat{T}_2(u, p) \hat{K}_2^R(u, p) \hat{T}_2(u, p) \hat{R}_{12}^\pm(p) \right\} \hat{T}_1^\pm(p) \\
&= \text{tr}_2 \left\{ A_{12} Q_2 \hat{R}_{12}^\pm(p) \right\} \hat{T}_1^\pm(p) \\
&= \text{tr}_2 \left\{ A_{12}^L \hat{R}_{12}^\pm t_2(p) Q_2^{L_2} \right\} \hat{T}_1^\pm(p) = \ldots \quad (3.38)
\end{align*}

In passing to the second line we have made the identifications $A_{12} = \hat{M}_1^{-1}(p) (\hat{R}_{12}^\pm(p))^{-1} \hat{M}_1(p)$ and $Q_2 = \hat{K}_2^L(u, p) \hat{T}_2(u, p) \hat{K}_2^R(u, p) \hat{T}_2(u, p)$. Finally, we obtain
\begin{align*}
\ldots &= \text{tr}_2 \left\{ \hat{M}_1^{-1}(p) (\hat{R}_{12}^\pm(p))^{-1} t_2 \hat{M}_1(p) \hat{R}_{12}^\pm t_2(p) Q_2^{L_2} \right\} \hat{T}_1^\pm(p) \\
&= \text{tr}_2 \left\{ Q_2^{L_2} \right\} \hat{T}_1^\pm(p) \\
&= t(u, p) \hat{T}_1^\pm(p). \quad (3.39)
\end{align*}

In passing to the second line we have used Lemma 4 (3.28); and we have used (3.9) again to pass to the third line. \qed

4 Duality symmetry

We now show that the transfer matrix $t(u, p)$ (2.20) for the cases $C_n^{(1)}$ and $D_n^{(1)}$ has a $p \leftrightarrow n-p$ “duality” symmetry. In order to prove the general result (4.5), we need the following lemma:

Lemma 5. The $R$-matrices for both $C_n^{(1)}$ and $D_n^{(1)}$ obey
\begin{align}
U_1 R_{12}(u) U_1 &= W_2^{*}(u) R_{12}(u) W_2(u), \\
U_2 R_{12}(u) U_2 &= W_1(u) R_{12}(u) W_1(u),
\end{align}
(4.1)
where $U$ and $W(u)$ are the following $(2n) \times (2n)$ matrices

\[
U = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}_{2n \times 2n}, \quad U^2 = I,
\]

\[
W(u) = \begin{pmatrix} 0 & e^{-u/2}I_n \\ e^{u/2}I_n & 0 \end{pmatrix}_{2n \times 2n}, \quad W(u)^2 = I. \tag{4.2}
\]

Furthermore, the $K$-matrices (2.14) and (2.18) obey

\[
W(u) K^R(u, p) W^t(u) = f^R(u, p) K^R(u, n - p)
\]

\[
W^t(u) K^L(u, p) W(u) = f^L(u, p) K^L(u, n - p), \tag{4.3}
\]

where $f^R(u, p)$ and $f^L(u, p)$ are scalar functions given by

\[
f^R(u, p) = \frac{\gamma_0 e^u + e^{(2n-4p)\eta}}{\gamma_0 + e^{(2n-4p)\eta}},
\]

\[
f^L(u, p) = \frac{\gamma_0 e^u + e^{(4p+2n+4)\eta}}{\gamma_0 e^{(4n+4i)\eta} + e^{u+(4p-2n)\eta}} \begin{cases} + & \text{for } C_n^{(1)} \\ - & \text{for } D_n^{(1)} \end{cases}, \tag{4.4}
\]

where $\gamma_0 = \pm 1$ is a parameter appearing in the $K$-matrix, see (2.15).

A proof of (4.1) is outlined in Sec. D.3.

The main duality result is given by the following proposition:

**Proposition 2.** For the cases $C_n^{(1)}$ and $D_n^{(1)}$, the transfer matrix has the duality symmetry

\[
\mathcal{U} t(u, p) \mathcal{U} = f(u, p) t(u, n - p), \tag{4.5}
\]

where $\mathcal{U}$ is the quantum-space operator

\[
\mathcal{U} = U_1 \ldots U_N, \quad \mathcal{U}^2 = I^\otimes N, \tag{4.6}
\]

and the scalar factor $f(u, p)$ is given by

\[
f(u, p) = f^L(u, p) f^R(u, p). \tag{4.7}
\]

**Proof.** We see from (4.1) that the monodromy matrices (2.21) transform as follows

\[
\mathcal{U} T_a(u) \mathcal{U} = W_a(u) T_a(u) W_a(u),
\]

\[
\mathcal{U} \hat{T}_a(u) \mathcal{U} = W^t_a(u) \hat{T}_a(u) W^t_a(u). \tag{4.8}
\]

Evaluating $\mathcal{U} t(u, p) \mathcal{U}$ using the definition (2.20) of the transfer matrix together with (4.8) and (4.3), we arrive at the desired result (4.5).

A similar duality symmetry was noted for the case $A_{n-1}^{(1)}$ in [17].

As a consequence of the duality symmetry (4.5), for each eigenvalue $\Lambda(u, p)$ of $t(u, p)$, there is a corresponding eigenvalue $\Lambda(u, n - p)$ of $t(u, n - p)$ such that

\[
\Lambda(u, p) = f(u, p) \Lambda(u, n - p). \tag{4.9}
\]
4.1 Action of duality on the QG generators

For the case $C_n^{(1)}$, the transfer matrix $t(u,p)$ has the symmetry $U_q(C_{n-p}) \otimes U_q(C_p)$ (see again Table 1), while $t(u,n-p)$ (its image under the duality transformation (4.5)) has the symmetry $U_q(C_p) \otimes U_q(C_{n-p})$. Under a duality transformation, the generators of the “left” symmetry factor of $t(u,p)$ (namely, $U_q(C_{n-p})$) are mapped to the generators of the “right” symmetry factor of $t(u,n-p)$ (which is also $U_q(C_{n-p})$). Similarly, the generators of the “right” symmetry factor of $t(u,p)$ (namely, $U_q(C_p)$) are mapped to the generators of the “left” symmetry factor of $t(u,n-p)$ (which is also $U_q(C_p)$). The case $D_n^{(1)}$ is identical, except with $D$’s replacing the $C$’s. In other words,

$$U_H^{(l)}(p) U = H^{(r)}(n-p), \quad U E_i^{\pm (l)}(p) U = E_i^{\pm (r)}(n-p), \quad i = 1, 2, \ldots, n-p,$$

$$U H^{(r)}(p) U = H^{(l)}(n-p), \quad U E_i^{\pm (r)}(p) U = E_i^{\pm (l)}(n-p), \quad i = 1, 2, \ldots, p.$$ (4.10)

and similarly for the coproducts. In order to obtain the general result (4.17), we need a few lemmas.

Lemma 6. \label{lemma6}

$$W^t(u) = B(u,n-p) U (-u,p) ,$$ (4.11)

where $U$ and $W(u)$ are given by (4.2).

Proof. We evaluate the RHS by writing all three matrices in terms of $n \times n$ blocks:

$$RHS = \begin{pmatrix}
\begin{array}{c|c}
\mathbb{I}_{p \times p} & e^2 \mathbb{I}_{(n-p) \times (n-p)} \\
\hline
 e^{-2} \mathbb{I}_{(n-p) \times (n-p)} & \mathbb{I}_{p \times p}
\end{array}
\end{pmatrix} \begin{pmatrix}
\mathbb{I}_{n \times n} & 0 \\
\hline
0 & \mathbb{I}_{n \times n}
\end{pmatrix} B(-u,p) = \begin{pmatrix}
\begin{array}{c|c}
\mathbb{I}_{p \times p} & e^2 \mathbb{I}_{(n-p) \times (n-p)} \\
\hline
 e^{-2} \mathbb{I}_{(n-p) \times (n-p)} & \mathbb{I}_{p \times p}
\end{array}
\end{pmatrix} \begin{pmatrix}
\mathbb{I}_{n \times n} & 0 \\
\hline
0 & \mathbb{I}_{n \times n}
\end{pmatrix} = \text{LHS} .$$ (4.12)

\[\square\]

Lemma 7. The gauge-transformed $R$-matrices for $C_n^{(1)}$ and $D_n^{(1)}$ obey

$$U_1 \tilde{R}_{12}(u,p) U_1 = U_2 \tilde{R}_{12}(u,n-p) U_2 .$$ (4.13)
Proof. Recalling the definition of the gauge-transformed R-matrix (3.1), we see that
\[ U_1 \tilde{R}_{12}(u,p) U_1 = U_1 B_2(-u,p) R_{12}(u) B_2(u,p) U_1 \]
\[ = B_2(-u,p) U_1 R_{12}(u) U_1 B_2(u,p) \]
\[ = B_2(-u,p) W_2^t(u) R_{12}(u) W_2^t(u) B_2(u,p) \]
\[ = U_2 B_2(-u,n-p) R_{12}(u) B_2(u,n-p) U_2 \]
\[ = U_2 \tilde{R}_{12}(u,n-p) U_2. \] (4.14)
In passing to the third line, we have used the result (4.1); in passing to the fourth line, we use
\[ B(-u,p) W^t(u) = U B(-u,n-p), \quad W^t(u) B(u,p) = B(u,n-p) U, \] (4.15)
which follow from (4.11); and we pass to the last line using again the definition of the gauge-transformed R-matrix. \( \square \)

Lemma 8. The gauge-transformed monodromy matrices for \( C_n^{(1)} \) and \( D_n^{(1)} \) transform under duality as
\[ U \tilde{T}_a(u,p) U = U_a \tilde{T}_a(u,n-p) U_a, \] (4.16)
where \( U \) is given by (4.6).

Proof. This result follows immediately from the definition of \( \tilde{T}_a(u,p) \) (3.10) and the result (4.13). \( \square \)

Finally, taking asymptotic limits of the result (4.16), we obtain the sought-after result:

Proposition 3. For the cases \( C_n^{(1)} \) and \( D_n^{(1)} \), the asymptotic gauge-transformed monodromy matrices \( \tilde{T}_a^\pm(p) \) transform under duality as
\[ U \tilde{T}_a^\pm(p) U = U_a \tilde{T}_a^\pm(n-p) U_a. \] (4.17)
From the result (4.17), we can read off the transformation properties of the coproducts of the QG generators under duality, thereby generalizing (4.10).

4.2 Self-duality

For \( p = \frac{n}{2} \) with \( n \) even, we see that the duality relation (4.5) implies that the transfer matrix is self-dual
\[ [U, t(u, \frac{n}{2})] = 0, \] (4.18)
since \( f(u, \frac{n}{2}) = 1 \). This self-duality symmetry maps the “left” and “right” generators into each other
\[ U H_i^{(l)}(\frac{n}{2}) U = H_i^{(r)}(\frac{n}{2}), \quad U E_i^{\pm}(\frac{n}{2}) U = E_i^{\pm}(\frac{n}{2}), \quad i = 1, 2, \ldots, \frac{n}{2}, \] (4.19)
as follows from (4.10). Hence, this symmetry maps the representations \((1, R)\) and \((R, 1)\) (i.e., with “left” and “right” singlets, respectively) into each other; and therefore these states are degenerate (i.e., have the same transfer-matrix eigenvalue). This degeneracy is discussed further in Section 6.
4.2.1 Bonus symmetry for $\gamma_0 = -1$

For the self-dual cases (namely, $C_n^{(1)}$ and $D_n^{(1)}$ with $p = \frac{n}{2}$ and $n$ even) with $\gamma_0 = -1$, there is an additional (“bonus”) symmetry, which leads to even higher degeneracies for the transfer-matrix eigenvalues.

In order to exhibit this symmetry, it is convenient to introduce the matrix $\bar{U}$, which is similar to the duality matrix $U$ (4.2),

$$\bar{U} = \begin{pmatrix}
\frac{iI}{n} & \frac{iI}{n} \\
\frac{-iI}{n} & \frac{-iI}{n}
\end{pmatrix}_{2n \times 2n}, \quad \bar{U}^2 = I, \quad (4.20)
$$

and which satisfies

$$\bar{U} U = -U \bar{U} = iD, \quad (4.21)$$

where $D$ is the diagonal matrix

$$D = \text{diag} \left( \frac{1}{n}, \ldots, 1, -1, \ldots, -1, \frac{1}{n}, \ldots, 1 \right). \quad (4.22)$$

Similarly to (4.13), we find that the gauge-transformed R-matrix obeys

$$\bar{U}_1 \tilde{R}_{12}(u, \frac{n}{2}) \bar{U}_1 = \bar{U}_2 \tilde{R}_{12}(u, \frac{n}{2}) \bar{U}_2, \quad (4.23)$$

as well as

$$D_1 \tilde{R}_{12}(u, \frac{n}{2}) D_1 = D_2 \tilde{R}_{12}(u, \frac{n}{2}) D_2. \quad (4.24)$$

Moreover, the gauge-transformed right K-matrix (3.4) is equal to $D$ \footnote{We emphasize that the result (4.25) holds only for $\gamma_0 = -1$, and we assume that $\gamma_0 = -1$ in the remainder of this subsection.}

$$\tilde{K}^R(u, \frac{n}{2}) = D. \quad (4.25)$$

It follows from the BYBE (2.16) that

$$\tilde{R}_{12}(u - v, \frac{n}{2}) D_1 \tilde{R}_{21}(u + v, \frac{n}{2}) D_2 = D_2 \tilde{R}_{12}(u + v, \frac{n}{2}) D_1 \tilde{R}_{21}(u - v, \frac{n}{2}). \quad (4.26)$$

The key result is given by the following proposition

**Proposition 4.** For the cases $C_n^{(1)}$ and $D_n^{(1)}$ with $p = \frac{n}{2}$ ($n$ even) and $\gamma_0 = -1$, the transfer matrix has the bonus symmetry

$$[\bar{U}, t(u, \frac{n}{2})] = 0, \quad (4.27)$$

where $\bar{U}$ is the quantum-space operator given by \footnote{Note that $\bar{U}$ contains only one factor of $\bar{U}$; all the other factors are $U$.}

$$\bar{U} = \bar{U}_1 U_2 \cdots U_N, \quad \bar{U}^2 = I^\otimes N. \quad (4.28)$$
Proof. We see from (4.23) that the gauge-transformed monodromy matrices (3.10) transform as follows
\[ \tilde{U} \tilde{T}_a(u, \frac{n}{2}) \tilde{U} = -i U_a \tilde{R}_{a,N}(u, \frac{n}{2}) \tilde{R}_{a,N-1}(u, \frac{n}{2}) \cdots \tilde{R}_{a2}(u, \frac{n}{2}) D_a \tilde{R}_{a1}(u, \frac{n}{2}) \tilde{U}_a , \]
\[ \tilde{U} \tilde{T}_a(u, \frac{n}{2}) \tilde{U} = i U_a \tilde{R}_{1a}(u, \frac{n}{2}) D_a \tilde{R}_{2a}(u, \frac{n}{2}) \tilde{R}_{3a}(u, \frac{n}{2}) \cdots \tilde{R}_{N_a}(u, \frac{n}{2}) U_a , \] (4.29)
where we have also used (4.21). Starting from the gauge-transformed expression for the transfer matrix (3.9), and also making use of (4.25), we obtain
\[ \tilde{U} t(u, \frac{n}{2}) \tilde{U} \]
\[ = \text{tr}_a \tilde{K}_a^L(u, \frac{n}{2}) \tilde{R}_{aN}(u, \frac{n}{2}) \cdots \tilde{R}_{a2}(u, \frac{n}{2}) D_a \tilde{R}_{1a}(u, \frac{n}{2}) D_a \tilde{R}_{2a}(u, \frac{n}{2}) \cdots \tilde{R}_{N_a}(u, \frac{n}{2}) \]
\[ = \text{tr}_a \tilde{K}_a^L(u, \frac{n}{2}) \tilde{R}_{aN}(u, \frac{n}{2}) \cdots \tilde{R}_{a1}(u, \frac{n}{2}) D_a \tilde{R}_{1a}(u, \frac{n}{2}) \cdots \tilde{R}_{N_a}(u, \frac{n}{2}) \]
\[ = \text{tr}_a \tilde{K}_a^L(u, \frac{n}{2}) \tilde{T}_a(u, \frac{n}{2}) D_a \tilde{T}_a(u, \frac{n}{2}) \]
\[ = t(u, \frac{n}{2}) , \] (4.30)
which implies the desired result (4.27). In passing to the first equality, we have used the cyclic property of the trace, and the fact \( U_a \tilde{K}_a^L(u, \frac{n}{2}) U_a = -\tilde{K}_a^L(u, \frac{n}{2}) \); and to pass to the second equality, we have used the result
\[ D_a \tilde{R}_{a1}(u, \frac{n}{2}) D_a \tilde{R}_{1a}(u, \frac{n}{2}) D_a = \tilde{R}_{a1}(u, \frac{n}{2}) D_a \tilde{R}_{1a}(u, \frac{n}{2}) , \] (4.31)
which follows from (4.26).

Recalling the definitions of \( \mathcal{U} (4.6) \) and \( \mathcal{U} (4.28) \) as well as the property (4.21), it is easy to see that
\[ [\mathcal{U} , \mathcal{U} ] = -2i \mathcal{D} , \] (4.32)
where \( \mathcal{D} \) is the quantum-space operator defined by
\[ \mathcal{D} = D_1 = D \otimes I^{\otimes (N-1)} , \quad \mathcal{D}^2 = I^{\otimes N} . \] (4.33)
The fact that \( \mathcal{D} \) commutes with the transfer matrix is now a simple corollary of (4.27):

**Corollary.** For the cases \( C_{(1)}^{(1)} \) and \( D_{(1)}^{(1)} \) with \( p = \frac{n}{2} \) (\( n \) even) and \( \gamma_0 = -1 \), the transfer matrix commutes with the operator \( \mathcal{D} (4.33) \)
\[ [\mathcal{D} , t(u, \frac{n}{2})] = 0 . \] (4.34)

Proof. Using (4.32) and the Jacobi identity, we see that
\[ [\mathcal{D} , t(u, \frac{n}{2})] = i \left[ [\mathcal{U} , \mathcal{U} ] , t(u, \frac{n}{2}) \right] \]
\[ = -i \left[ [\mathcal{U} , t(u, \frac{n}{2})] , \mathcal{U} \right] - i \left[ [t(u, \frac{n}{2}) , \mathcal{U} ] , \mathcal{U} \right] \]
\[ = 0 , \] (4.35)
where the final equality follows from the symmetries (4.18) and (4.27). □
The symmetry (4.34) gives rise to additional degeneracies of the transfer-matrix eigenvalues. Indeed, let \( |\Lambda\rangle \) be a simultaneous eigenket of the transfer matrix and of the self-duality operator \( U \),
\[
t(u, \frac{n}{2}) |\Lambda\rangle = \Lambda(u, \frac{n}{2}) |\Lambda\rangle,
\]
\[
U |\Lambda\rangle = \mu |\Lambda\rangle, \quad \mu = \pm 1.
\]
(4.36)
Since \( U \) and \( D \) do not commute\(^9\), \( |\Lambda\rangle \) is not necessarily an eigenket of \( D \), in which case \( D |\Lambda\rangle \) is a linearly independent eigenket with the same transfer-matrix eigenvalue \( \Lambda(u, \frac{n}{2}) \) as \( |\Lambda\rangle \). Note that \( |\Lambda\rangle \) necessarily belongs to a QG representation of the form \( (R, R) \) or \( (R_1, R_2) \oplus (R_2, R_1) \); hence, the bonus symmetry implies the existence of a second set of states of the form \( (R, R) \) or \( (R_1, R_2) \oplus (R_2, R_1) \). In particular, the degeneracy of the corresponding transfer-matrix eigenvalue becomes doubled as a consequence of the bonus symmetry.

5 \( Z_2 \) symmetries

We now show that the transfer matrix \( t(u, p) \) (2.20) has a discrete “right” \( Z_2 \) symmetry that maps complex representations of \( U_q(D_p) \) to their conjugates for the cases \( A^{(2)}_{2n-1}, B_n^{(1)} \) and \( D_n^{(1)} \); and, for the latter case, there is an additional “left” \( Z_2 \) symmetry that maps complex representations of \( U_q(D_{n-p}) \) to their conjugates. We shall see in Section 6 that these discrete symmetries give rise to degeneracies in the spectrum beyond those expected from QG symmetry.\(^10\)

5.1 The “right” \( Z_2 \)

In order to prove the main result (5.4), we need the following lemma:

**Lemma 9.** The R-matrices for \( A^{(2)}_{2n-1}, B_n^{(1)} \) and \( D_n^{(1)} \) obey
\[
Z_1^{(r)} R_{12}(u) Z_1^{(r)} = Y_2^t(u) R_{12}(u) Y_2(u),
\]
\[
Z_2^{(r)} R_{12}(u) Z_2^{(r)} = Y_1(u) R_{12}(u) Y_1(u),
\]
(5.1)
where \( Z^{(r)} \) and \( Y(u) \) are the following \( d \times d \) matrices
\[
Z^{(r)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & I_{(d-2)\times(d-2)} & 0 \\ 1 & 0 & 0 \end{pmatrix}_{d\times d},
\]
\[
Z^{(r)} = I,
\]
\[
Y(u) = \begin{pmatrix} 0 & 0 & e^{-u} \\ 0 & I_{(d-2)\times(d-2)} & 0 \\ e^u & 0 & 0 \end{pmatrix}_{d\times d},
\]
\[
Y(u)^2 = I.
\]
(5.2)

\(^9\)Indeed, \([U, D] = [U, D] \otimes U \otimes \cdots \otimes U = 2i \tilde{U} \), see (4.21).

\(^{10}\)The \( Z_2 \) symmetry for the case \( A^{(2)}_{2n-1} \) with \( p = n \) was conjectured in [27].
Furthermore, for \( p > 0 \), the \( K \)-matrices (2.14) and (2.18) obey
\[
Y(u) K^R(u,p) Y^t(u) = K^R(u,p),
\]
\[
Y^t(u) K^L(u,p) Y(u) = K^L(u,p).
\]

(5.3)

A proof of (5.1) is outlined in Sec. D.2.

The main result concerning the “right” \( Z_2 \) symmetry is contained in the following proposition:

**Proposition 5.** For the cases \( A^{(2)}_{2n-1} \), \( B^{(1)}_n \) and \( D^{(1)}_n \) with \( p > 0 \), the transfer matrix has the “right” \( Z_2 \) symmetry
\[
[Z^{(r)}, t(u,p)] = 0,
\]
where \( Z^{(r)} \) is the quantum-space operator
\[
Z^{(r)} = Z^{(r)}_1 \ldots Z^{(r)}_N, \quad Z^{(r)2} = \mathbb{I}^\otimes N.
\]

(5.5)

**Proof.** We see from (5.1) that the monodromy matrices (2.21) transform as follows
\[
Z^{(r)} T_a(u) Z^{(r)} = Y_a(u) T_a(u) Y_a(u),
\]
\[
Z^{(r)} \hat{T}_a(u) Z^{(r)} = Y_a^t(u) \hat{T}_a(u) Y_a^t(u).
\]

(5.6)

Evaluating \( Z^{(r)} t(u,p) Z^{(r)} \) using the definition (2.20) of the transfer matrix together with (5.6) and (5.3), we arrive at the result (5.4).

**5.1.1 Action of the “right” \( Z_2 \) on the QG generators**

In order to determine the action of the “right” \( Z_2 \) on the QG generators, we use a set of lemmas that are analogous to (4.11), (4.13) and (4.16), and which have similar proofs:

**Lemma 10.**
\[
Y(u) = B(-u,p) Z^{(r)} B(u,p), \quad p > 0,
\]
where \( Z^{(r)} \) and \( Y(u) \) are given by (5.2).

**Lemma 11.** The gauge-transformed R-matrices (3.1) for \( A^{(2)}_{2n-1} \), \( B^{(1)}_n \) and \( D^{(1)}_n \) with \( p > 0 \) obey
\[
Z^{(r)}_1 \hat{R}_{12}(u,p) Z^{(r)}_1 = Z^{(r)}_2 \hat{R}_{12}(u,p) Z^{(r)}_2.
\]

(5.8)

**Lemma 12.** The gauge-transformed monodromy matrices (3.10) for \( A^{(2)}_{2n-1} \), \( B^{(1)}_n \) and \( D^{(1)}_n \) with \( p > 0 \) transform under the “right” \( Z_2 \) as
\[
Z^{(r)} \hat{T}_a(u,p) Z^{(r)} = Z^{(r)}_a \hat{T}_a(u,p) Z^{(r)}_a.
\]

(5.9)

Finally, taking asymptotic limits of the result (5.9), we obtain the sought-after result:
Proposition 6. For the cases $A_{2n-1}^{(2)}$, $B_n^{(1)}$ and $D_n^{(1)}$ with $p > 0$, the asymptotic gauge-transformed monodromy matrices $\tilde{T}_a^\pm(p)$ transform under the “right” $Z_2$ as
\[
Z^{(r)} \tilde{T}_a^\pm(p) Z^{(r)} = Z_a^{(r)} \tilde{T}_a^\pm(p) Z_a^{(r)}. \tag{5.10}
\]

We can read off from this result how the coproducts of the “right” QG generators transform under this $Z_2$ symmetry. In particular, we observe that
\[
Z_a^{(r)} H_j^{(r)} Z_a^{(r)} = \begin{cases} H_j^{(r)} & \text{for } j = 1, \ldots, p-1, \\ -H_p^{(r)} & \text{for } j = p \end{cases},
\]
\[
Z_a^{(r)} E_j^{\pm(r)} Z_a^{(r)} = \begin{cases} E_j^{\pm(r)} & \text{for } j = 1, \ldots, p-2, \\ E_p^{\pm(r)} & \text{for } j = p-1 \end{cases}. \tag{5.11}
\]

Hence, this transformation maps complex representations of $U_q(D_p)$ to their conjugates.

5.2 The “left” $Z_2$

In order to prove the main result (5.15), we need the following lemma:

Lemma 13. The R-matrix for $D_n^{(1)}$ obeys
\[
Z_1^{(l)} R_{12}(u) Z_1^{(l)} = Z_2^{(l)} R_{12}(u) Z_2^{(l)}, \tag{5.12}
\]
where $Z^{(l)}$ is the following $2n \times 2n$ matrix
\[
Z^{(l)} = \begin{pmatrix} \mathbb{I}_{(n-1) \times (n-1)} & 0 & 1 \\ 0 & 1 & 0 \\ \mathbb{I}_{(n-1) \times (n-1)} \end{pmatrix}_{2n \times 2n}, \quad Z^{(l)} 2 = \mathbb{I}. \tag{5.13}
\]

Furthermore, for $p < n$, the K-matrices (2.14) and (2.18) obey
\[
Z^{(l)} K^R(u,p) Z^{(l)} = K^R(u,p), \quad Z^{(l)} K^L(u,p) Z^{(l)} = K^L(u,p). \tag{5.14}
\]

A proof of (5.12) is outlined in Sec. D.1.

The main result concerning the “left” $Z_2$ symmetry is contained in the following proposition:

Proposition 7. For the case $D_n^{(1)}$ with $p < n$, the transfer matrix has the “left” $Z_2$ symmetry
\[
\left[ Z^{(l)}, t(u,p) \right] = 0, \tag{5.15}
\]
where $Z^{(l)}$ is the quantum-space operator
\[
Z^{(l)} = Z_1^{(l)} \ldots Z_N^{(l)}, \quad Z^{(l)} 2 = \mathbb{I}^\otimes N. \tag{5.16}
\]
Proof. We see from (5.12) that the monodromy matrices (2.21) transform as follows

\[
Z^{(l)} T_a(u) Z^{(l)} = Z_a^{(l)} T_a(u) Z_a^{(l)},
\]
\[
\hat{Z}^{(l)} \hat{T}_a(u) Z^{(l)} = \hat{Z}_a^{(l)} \hat{T}_a(u) Z_a^{(l)}.
\]  (5.17)

Evaluating \( Z^{(l)} t(u, p) Z^{(l)} \) using the definition (2.20) of the transfer matrix together with (5.17) and (5.14), we arrive at the result (5.15).

\[ \Box \]

5.2.1 Action of the “left” \( Z_2 \) on the QG generators

The gauge-transformed R-matrix (3.1) for \( D_n^{(1)} \) with \( p < n \) obeys

\[
Z_1^{(l)} \tilde{R}_{12}(u, p) Z_1^{(l)} = Z_2^{(l)} \tilde{R}_{12}(u, p) Z_2^{(l)},
\]  (5.18)
in view of the property (5.12) and the fact that \([Z^{(l)}, B(u, p)] = 0\) for \( p < n \). Hence, the gauge-transformed monodromy matrices (3.10) transform as follows

\[
Z^{(l)} \tilde{T}_a(u, p) Z^{(l)} = Z_a^{(l)} \tilde{T}_a(u, p) Z_a^{(l)},
\]
\[
\hat{Z}^{(l)} \hat{T}_a(u, p) Z^{(l)} = \hat{Z}_a^{(l)} \hat{T}_a(u, p) Z_a^{(l)}.
\]  (5.19)

Taking asymptotic limits of this result gives the following proposition:

**Proposition 8.** For the case \( D_n^{(1)} \) with \( p < n \), the asymptotic gauge-transformed monodromy matrices \( \tilde{T}_a^{\pm}(p) \) transform under the “left” \( Z_2 \) as

\[
Z^{(l)} \tilde{T}_a^{\pm}(p) Z^{(l)} = Z_a^{(l)} \tilde{T}_a^{\pm}(p) Z_a^{(l)}.
\]  (5.20)

We can read off from this result how the coproducts of the “left” QG generators transform under this \( Z_2 \) symmetry. In particular, we observe that

\[
Z_a^{(l)} H_j^{(l)} Z_a^{(l)} = \begin{cases} H_j^{(l)} & \text{for } j = 1, \ldots, n - p - 1, \\ -H_{n-p}^{(l)} & \text{for } j = n - p \end{cases},
\]
\[
Z_a^{(l)} E_j^{\pm(l)} Z_a^{(l)} = \begin{cases} E_j^{\pm(l)} & \text{for } j = 1, \ldots, n - p - 2, \\ E_{n-p}^{\pm(l)} & \text{for } j = n - p - 1 \end{cases}.
\]  (5.21)

Hence, this transformation maps complex representations of \( U_q(D_{n-p}) \) to their conjugates.

6 Degeneracies of the transfer matrix

The symmetries identified above can be used to understand the degeneracies in the spectrum of the transfer matrix. Most importantly, the QG symmetries of the transfer matrix (3.36), summarized in Table 1, are directly manifested in the degeneracies of the spectrum. Indeed,
for generic values of the anisotropy parameter $\eta$, the $N$-site Hilbert space $V^\otimes N$ can be decomposed into a direct sum of irreducible representations of the corresponding classical group, whose dimensions are generally equal to the degeneracies of the eigenvalues.

For the cases $A_{2n-1}^{(2)}$, $B_n^{(1)}$ and $D_n^{(1)}$, the transfer matrix has an additional “right” $Z_2$ symmetry (5.4) that maps complex representations of $U_q(D_p)$ to their conjugates. Moreover, for the case $D_n^{(1)}$, the transfer matrix also has a “left” $Z_2$ symmetry (5.15) that maps complex representations of $U_q(D_{n-p})$ to their conjugates. Consequently, the degeneracies of eigenvalues corresponding to complex representations are larger than expected from the decomposition of the Hilbert space.

For the cases $C_n^{(1)}$ and $D_n^{(1)}$ with $n$ even and $p = \frac{n}{2}$, the transfer matrix has a self-duality symmetry (4.18) that maps the representations $(1, R)$ and $(R, 1)$ into each other, and therefore those states are degenerate. If $\gamma_0 = -1$, then there is a bonus symmetry (4.27), (4.34) that leads to additional degeneracies.

For the cases $C_n^{(1)}$ and $D_n^{(1)}$ with $n$ odd and $p = \frac{n+1}{2}$, we also observe some higher degeneracies, which presumably can also be attributed to some discrete symmetries that remain to be elucidated.

We now consider examples of each of these cases.

6.1 $A_{2n}^{(2)}$

For $A_{2n}^{(2)}$ and generic values of $\eta$, the degeneracies of the transfer matrix exactly match with the predictions from the decomposition of the Hilbert space based on the QG symmetry. That is, in contrast with the other cases considered below, we do not find any higher degeneracies. As an example, let us consider the case $n = 5$ and $N = 2$ (two sites). By direct diagonalization of the transfer matrix $t(u, p)$ for generic numerical values of $u$ and $\eta$, we find that the degeneracies are as follows:

\[
\begin{align*}
    p = 0 : & \quad \{1, 55, 65\} \\
    p = 1 : & \quad \{1, 1, 3, 18, 18, 36, 44\} \\
    p = 2 : & \quad \{1, 1, 5, 10, 21, 27, 28\} \\
    p = 3 : & \quad \{1, 1, 10, 14, 14, 21, 30, 30\} \\
    p = 4 : & \quad \{1, 1, 3, 5, 24, 24, 27, 36\} \\
    p = 5 : & \quad \{1, 1, 10, 10, 44, 55\}. 
\end{align*}
\]  

(6.1)

In other words, for $p = 0$, one eigenvalue is repeated 65 times, another eigenvalue is repeated 55 times, and another eigenvalue appears only once; and similarly for other values of $p$.

On the other hand, according to Table 1, the symmetry for $A_{2n}^{(2)}$ with $n = 5$ is $U_q(B_{5-p}) \otimes U_q(C_p)$, and the representation at each site is $\mathcal{V} = (11 - 2p, 1) \oplus (1, 2p)$. For generic values of $\eta$, the QG representations are the same as for the corresponding classical groups. Performing
the tensor-product decompositions here and below using LieART [41], we obtain

\( p = 0 : B_5 \) \((11)^{\otimes 2} = 1 \oplus 55 \oplus 65\)

\( p = 1 : B_4 \otimes C_1 \) \((9,1) \oplus (1,2))^{\otimes 2} = 2(1,1) \oplus (1,3) \oplus 2(9,2) \oplus (36,1) \oplus (44,1)\)

\( p = 2 : B_3 \otimes C_2 \) \((7,1) \oplus (1,4))^{\otimes 2} = 2(1,1) \oplus (1,5) \oplus 2(7,4) \oplus (1,10) \oplus (21,1) \oplus (27,1)\)

\( p = 3 : B_2 \otimes C_3 \) \((5,1) \oplus (1,6))^{\otimes 2} = 2(1,1) \oplus 2(5,6) \oplus (10,1) \oplus (1,14) \oplus (14,1) \oplus (1,21)\)

\( p = 4 : B_1 \otimes C_4 \) \((3,1) \oplus (1,8))^{\otimes 2} = 2(1,1) \oplus (3,1) \oplus (5,1) \oplus 2(3,8) \oplus (1,27) \oplus (1,36)\)

\( p = 5 : C_5 \) \((1 \oplus 10)^{\otimes 2} = 2(1) \oplus 2(10) \oplus 44 \oplus 55.\) \( (6.2) \)

Comparing the degeneracies (6.1) with the corresponding tensor-product decompositions (6.2), we see that they exactly match. We obtain similar results for other values of \( n \) and \( N \). The special cases \( p = 0 \) and \( p = n \) are discussed further in [26].

### 6.2 \( A_{2n-1}^{(2)} \)

For \( A_{2n-1}^{(2)} \) and generic values of \( \eta \), the degeneracies of the transfer matrix either match with the predictions from QG symmetry, or are larger due to the “right” \( Z_2 \) symmetry (5.4). As an example, let us consider the case \( n = 5 \) and \( N = 2 \) (two sites). By direct diagonalization of the transfer matrix \( t(u,p) \) for generic numerical values of \( u \) and \( \eta \), we find that the degeneracies are as follows:

\[ p = 0 : \{1, 44, 55\} \]
\[ p = 2 : \{1, 1, 6, 9, 14, 21, 24, 24\} \]
\[ p = 3 : \{1, 1, 5, 10, 15, 20, 24, 24\} \]
\[ p = 4 : \{1, 1, 3, 16, 16, 28, 35\} \]
\[ p = 5 : \{1, 45, 54\}. \] \( (6.3) \)

Note that we exclude the case \( p = 1 \).

On the other hand, according to Table 1, the symmetry for \( A_{2n-1}^{(2)} \) with \( n = 5 \) and \( p \neq 1 \) is \( U_q(C_{5-p}) \otimes U_q(D_p) \), and the representation at each site is \( V = (10 - 2p, 1) \oplus (1, 2p) \). The

---

\(^{11}\)We recall that \( A_1 = B_1 = C_1 \), while the \( D_n \) series starts with \( n = 2 \).
tensor-product decompositions are as follows:

\begin{align*}
p = 0 : & \quad C_5 \\
p = 2 : & \quad C_3 \otimes D_2 \quad \left( (6,1) \oplus (1,4) \right) \otimes^2 = 2(1,1) \oplus (1,3) \oplus (1,\bar{3}) \oplus 2(6,4) \oplus (1,9) \\
p = 3 : & \quad C_2 \otimes D_3 \quad \left( (4,1) \oplus (1,6) \right) \otimes^2 = 2(1,1) \oplus (5,1) \oplus 2(4,6) \oplus (10,1) \\
p = 4 : & \quad C_1 \otimes D_4 \quad \left( (2,1) \oplus (1,8_v) \right) \otimes^2 = 2(1,1) \oplus (3,1) \oplus 2(2,8_v) \oplus (1,28) \oplus (1,35_v) \\
p = 5 : & \quad D_5 \\
\end{align*}

\begin{align*}
p = 0 : & \quad (10) \otimes^2 = 1 \oplus 44 \oplus 55 \\
p = 2 : & \quad (6,1) \oplus (1,4) \otimes^2 = 2(1,1) \oplus (1,3) \oplus (1,\bar{3}) \oplus 2(6,4) \oplus (1,9) \\
p = 3 : & \quad (4,1) \oplus (1,6) \otimes^2 = 2(1,1) \oplus (5,1) \oplus 2(4,6) \oplus (10,1) \\
p = 4 : & \quad (2,1) \oplus (1,8_v) \otimes^2 = 2(1,1) \oplus (3,1) \oplus 2(2,8_v) \oplus (1,28) \oplus (1,35_v) \\
p = 5 : & \quad (10) \otimes^2 = 1 \oplus 45 \oplus 54. \quad (6.4)
\end{align*}

Comparing the degeneracies (6.3) with the corresponding tensor-product decompositions (6.4), we see that they match, except for \(p = 2\). For the latter case, the degeneracies are larger, due to the “right” \(Z_2\) symmetry mapping complex representations of \(D_p\) to their conjugates (here, the \(3\) and \(\bar{3}\)). We obtain similar results for other values of \(n\) and \(N\). The special cases \(p = 0\) and \(p = n\) are discussed further in [27].

### 6.3 \(B_n^{(1)}\)

For \(B_n^{(1)}\) and generic values of \(\eta\), the degeneracies of the transfer matrix also either match with the predictions from QG symmetry, or are larger due to the “right” \(Z_2\) symmetry (5.4). As an example, let us consider the case \(n = 5\) and \(N = 2\) (two sites). By direct diagonalization of the transfer matrix \(t(u,p)\) for generic numerical values of \(u\) and \(\eta\), we find that the degeneracies are as follows:

\begin{align*}
p = 0 : & \quad \{1,55,65\} \\
p = 2 : & \quad \{1,1,6,9,21,27,28,28\} \\
p = 3 : & \quad \{1,1,10,14,15,20,30,30\} \\
p = 4 : & \quad \{1,1,3,5,24,24,28,35\} \\
p = 5 : & \quad \{1,1,10,10,45,54\}. \quad (6.5)
\end{align*}

Note that we again exclude the case \(p = 1\).

On the other hand, according to Table 1, the symmetry for \(B_n^{(1)}\) with \(n = 5\) and \(p \neq 1\) is \(U_q(B_{5-p}) \otimes U_q(D_p)\), and the representation at each site is \(V = (11 - 2p,1) \oplus (1,2p)\). The
tensor-product decompositions are as follows:

\[ p = 0 : B_5 \quad (11)^{\otimes 2} = 1 \oplus 55 \oplus 65 \]

\[ p = 2 : B_3 \otimes D_2 \quad ((7, 1) \oplus (1, 4))^{\otimes 2} = 2(1, 1) \oplus (1, 3) \oplus (1, \bar{3}) \oplus (1, 9) \oplus 2(7, 4) \]
\[ \quad \oplus (21, 1) \oplus (27, 1) \]

\[ p = 3 : B_2 \otimes D_3 \quad ((5, 1) \oplus (1, 6))^{\otimes 2} = 2(1, 1) \oplus 2(5, 6) \oplus (10, 1) \oplus (14, 1) \]
\[ \quad \oplus (1, 15) \oplus (1, 20') \]

\[ p = 4 : B_1 \otimes D_4 \quad ((3, 1) \oplus (1, 8_v))^{\otimes 2} = 2(1, 1) \oplus (3, 1) \oplus (5, 1) \oplus 2(3, 8_v) \oplus (1, 28) \oplus (1, 35_v) \]

\[ p = 5 : D_5 \quad (1 \oplus 10)^{\otimes 2} = 2(1) \oplus 2(10) \oplus 45 \oplus 54. \quad (6.6) \]

Comparing the degeneracies (6.5) with the corresponding tensor-product decompositions (6.6), we see that they match, except for \( p = 2 \). For the latter case, the degeneracies are larger, due to the “right” \( Z_2 \) symmetry mapping complex representations of \( D_p \) to their conjugates (here, the \( 3 \) and \( \bar{3} \)). We obtain similar results for other values of \( n \) and \( N \).

6.4 \( C_n^{(1)} \)

For \( C_n^{(1)} \) and generic values of \( \eta \), the degeneracies of the transfer matrix match with the predictions from QG symmetry, except when \( n \) is even and \( p = \frac{n}{2} \) (in which case there is a self-duality symmetry (4.18)) or when \( n \) is odd and \( p = \frac{n+1}{2} \). Moreover, the spectrum exhibits a \( p \rightarrow n - p \) duality symmetry.

6.4.1 Example 1: even \( n \)

As a first example, let us consider the case \( n = 4 \) and \( N = 2 \) (two sites). By direct diagonalization of the transfer matrix \( t(u, p) \) for generic numerical values of \( u \) and \( \eta \), we find that the degeneracies are as follows:

\[ p = 0 : \quad \{1, 27, 36\} \]
\[ p = 1 : \quad \{1, 1, 3, 12, 12, 14, 21\} \]
\[ p = 2 : \quad \begin{cases} \{1, 1, 10, 16, 16, 20\} & \text{for } \gamma_0 = +1 \\ \{2, 10, 20, 32\} & \text{for } \gamma_0 = -1 \end{cases} \]
\[ p = 3 : \quad \{1, 1, 3, 12, 12, 14, 21\} \]
\[ p = 4 : \quad \{1, 27, 36\}. \quad (6.7) \]

The fact that the degeneracies are the same for \( p \) and \( n - p \) is a consequence of the duality symmetry (4.5), (4.9).

On the other hand, according to Table 1, the symmetry for \( C_n^{(1)} \) with \( n = 4 \) is \( U_q(C_{4-p}) \otimes U_q(C_p) \), and the representation at each site is \( V = (8 - 2p, 1) \oplus (1, 2p) \). The tensor-product
decompositions are as follows:

\[ p = 0 : C_4 \quad \quad (8) \otimes 2 = 1 \oplus 27 \oplus 36 \]
\[ p = 1 : C_3 \otimes C_1 \quad ((6,1) \oplus (1,2)) \otimes 2 = 2(1,1) \oplus (1,3) \oplus 2(6,2) \oplus (14,1) \oplus (21,1) \]
\[ p = 2 : C_2 \otimes C_2 \quad ((4,1) \oplus (1,4)) \otimes 2 = 2(1,1) \oplus (5,1) \oplus (1,5) \oplus 2(4,4) \oplus (10,1) \oplus (1,10) . \quad (6.8) \]

There is no need to display the tensor-product decompositions for \( p > 2 \) due to the symmetry \( p \rightarrow n - p \).

Comparing the degeneracies (6.7) with the corresponding tensor-product decompositions (6.8), we see that they match, except for \( p = 2 \). For the latter case, the degeneracies are larger, due to the self-duality symmetry (4.18) for even \( n \) and \( p = \frac{n}{2} \), which here maps \((1,5)\) to \((5,1)\) (resulting in a 10-fold degeneracy), and also maps \((1,10)\) to \((10,1)\) (resulting in a 20-fold degeneracy). If \( \gamma_0 = -1 \), then the bonus symmetry (4.27), (4.34) implies that the two \((4,4)\) are degenerate (giving rise to a 32-fold degeneracy), as well as the two \((1,1)\) (resulting in a 2-fold degeneracy).

### 6.4.2 Example 2: odd \( n \)

As a second example, let us consider the case \( n = 5 \) and \( N = 2 \) (two sites). By direct diagonalization of the transfer matrix \( t(u,p) \) for generic numerical values of \( u \) and \( \eta \), we find that the degeneracies are as follows:

\[ p = 0 : \quad \{1,44,55\} \]
\[ p = 1 : \quad \{1,1,13,16,16,27,36\} \]
\[ p = 2 : \quad \{1,1,5,21,34,38\} \]
\[ p = 3 : \quad \{1,1,5,21,34,38\} \]
\[ p = 4 : \quad \{1,1,3,16,16,27,36\} \]
\[ p = 5 : \quad \{1,44,55\} \]. \quad (6.9) \]

We see again that the degeneracies are the same for \( p \) and \( n - p \), as a consequence of the duality symmetry (4.5), (4.9).

On the other hand, according to Table 1, the symmetry for \( C_n^{(1)} \) with \( n = 5 \) is \( U_q(C_{5-p}) \otimes U_q(C_p) \), and the representation at each site is \( V = (10 - 2p, 1) \oplus (1, 2p) \). The tensor-product decompositions are as follows:

\[ p = 0 : C_5 \quad \quad (10) \otimes 2 = 1 \oplus 44 \oplus 55 \]
\[ p = 1 : C_4 \otimes C_1 \quad ((8,1) \oplus (1,2)) \otimes 2 = 2(1,1) \oplus (1,3) \oplus 2(8,2) \oplus (27,1) \oplus (36,1) \]
\[ p = 2 : C_3 \otimes C_2 \quad ((6,1) \oplus (1,4)) \otimes 2 = 2(1,1) \oplus (1,5) \oplus 2(6,4) \oplus (1,10) \oplus (14,1) \oplus (21,1) . \quad (6.10) \]

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Again, there is no need to display the tensor-product decompositions for \( p > 2 \) due to the symmetry \( p \to n - p \).

Comparing the degeneracies (6.9) with the corresponding tensor-product decompositions (6.10), we see that they match, except for \( p = 2 \). For the latter case, the degeneracies are larger: the \((1, 10)\) and one \((6, 4)\) are degenerate (resulting in a 34-fold degeneracy); and the \((14, 1)\) and the other \((6, 4)\) are degenerate (resulting in a 38-fold degeneracy). We expect that such degeneracies for odd \( n \) and \( p = \frac{n+1}{2} \) can be attributed to some discrete symmetries, which remain to be elucidated.

6.5 \( D_n^{(1)} \)

For \( D_n^{(1)} \) and generic values of \( \eta \), the degeneracies of the transfer matrix match with the predictions from QG symmetry, except for the following exceptions: when \( n \) is even and \( p = \frac{n}{2} \) (in which case there is a self-duality symmetry (4.18)); when \( n \) is odd and \( p = \frac{n+1}{2} \); and when there are additional degeneracies due to the “right” and “left” \( Z_2 \) symmetries (5.4), (5.15). Moreover, the spectrum exhibits a \( p \to n - p \) duality symmetry.

6.5.1 Example 1: even \( n \)

As a first example, let us consider the case \( n = 6 \) and \( N = 2 \) (two sites). By direct diagonalization of the transfer matrix \( t(u, p) \) for generic numerical values of \( u \) and \( \eta \), we find that the degeneracies are as follows:

\[
\begin{align*}
p = 0 : & \quad \{1, 66, 77\} \\
p = 2 : & \quad \{1, 1, 6, 9, 28, 32, 35\} \\
p = 3 : & \quad \begin{cases} 
\{1, 1, 30, 36, 36, 40\} & \text{for } \gamma_0 = +1 \\
\{2, 30, 40, 72\} & \text{for } \gamma_0 = -1 
\end{cases} \\
p = 4 : & \quad \{1, 1, 6, 9, 28, 32, 35\} \\
p = 6 : & \quad \{1, 66, 77\}.
\end{align*}
\]

(6.11)

Note that we exclude the cases \( p = 1 \) and \( p = n - 1 \). The fact that the degeneracies are the same for \( p \) and \( n - p \) is a consequence of the duality symmetry (4.5), (4.9).

On the other hand, according to Table 1, the symmetry for \( D_n^{(1)} \) with \( n = 6 \) and \( p \neq 1, n - 1 \) is \( U_q(D_{6-p}) \otimes U_q(D_p) \), and the representation at each site is \( V = (12 - 2p, 1) \oplus (1, 2p) \).
The tensor-product decompositions are as follows:

\[ p = 0 : D_6 \quad (12)^{\otimes 2} = 1 \oplus 66 \oplus 77 \]

\[ p = 2 : D_4 \otimes D_2 \quad ((8_v, 1) \oplus (1, 4))^{\otimes 2} = 2(1, 1) \oplus (1, 3) \oplus (1, 3) \oplus (1, 9) \oplus 2(8_v, 4) \]

\[ \quad \oplus (28, 1) \oplus (35_v, 1) \]

\[ p = 3 : D_3 \otimes D_3 \quad ((6, 1) \oplus (1, 6))^{\otimes 2} = 2(1, 1) \oplus 2(6, 6) \oplus (15, 1) \oplus (1, 15) \]

\[ \quad \oplus (20', 1) \oplus (1, 20') \]. \quad (6.12) \]

There is no need to display the tensor-product decompositions for \( p > 3 \) due to the symmetry \( p \to n - p \).

Comparing the degeneracies (6.11) with the corresponding tensor-product decompositions (6.12), we see that they match for \( p = 0 \). For \( p = 2 \), the degeneracies are larger due to the the “right” \( Z_2 \) symmetry (5.4), which maps \((1, 3)\) to \((1, \bar{3})\), and results in a 6-fold degeneracy.

For \( p = 3 \), the degeneracies are larger due to the self-duality symmetry (4.18) for even \( n \) and \( p = \frac{n}{2} \), which maps \((1, 15)\) to \((15, 1)\) (resulting in a 30-fold degeneracy), and also maps \((1, 20')\) to \((20', 1)\) (resulting in a 40-fold degeneracy). If \( \gamma_0 = -1 \), then the bonus symmetry (4.27), (4.34) implies that the two \((6, 6)\) are degenerate (giving rise to a 72-fold degeneracy), as well as the two \((1, 1)\) (resulting in a 2-fold degeneracy).

### 6.5.2 Example 2: odd \( n \)

As a second example, let us consider the case \( n = 5 \) and \( N = 2 \) (two sites). By direct diagonalization of the transfer matrix \( t(u, p) \) for generic numerical values of \( u \) and \( \eta \), we find that the degeneracies are as follows:

\[ p = 0 : \quad \{1, 45, 54\} \]

\[ p = 2 : \quad \{1, 1, 6, 20, 33, 39\} \]

\[ p = 3 : \quad \{1, 1, 6, 20, 33, 39\} \]

\[ p = 5 : \quad \{1, 45, 54\}. \quad (6.13) \]

We again exclude the cases \( p = 1, n - 1 \), and observe that the degeneracies are the same for \( p \) and \( n - p \), as a consequence of the duality symmetry (4.5), (4.9).

On the other hand, according to Table 1, the symmetry for \( D_n^{(1)} \) with \( n = 5 \) and \( p \neq 1, n - 1 \) is \( U_q(D_{5-p}) \otimes U_q(D_p) \), and the representation at each site is \( V = (10 - 2p, 1) \oplus (1, 2p) \). The tensor-product decompositions are as follows:

\[ p = 0 : D_5 \quad \quad (10)^{\otimes 2} = 1 \oplus 45 \oplus 54 \]

\[ p = 2 : D_3 \otimes D_2 \quad ((6, 1) \oplus (1, 4))^{\otimes 2} = 2(1, 1) \oplus (1, 3) \oplus (1, 3) \oplus 2(6, 4) \oplus (1, 9) \]

\[ \quad \oplus (15, 1) \oplus (20', 1). \quad (6.14) \]
Again, there is no need to display the tensor-product decompositions for $p > 2$ due to the symmetry $p \rightarrow n - p$.

Comparing the degeneracies (6.13) with the corresponding tensor-product decompositions (6.14), we see that they match for $p = 0$. For $p = 2$, the 6-fold degeneracy is due to “right” $Z_2$ symmetry, which maps $(1, 3)$ to $(1, \bar{3})$. Moreover, the $(1, 9)$ and one $(6, 4)$ are degenerate (resulting in a 33-fold degeneracy); and the $(15, 1)$ and the other $(6, 4)$ are degenerate (resulting in a 39-fold degeneracy). We expect that such degeneracies for odd $n$ and $p = \frac{n+1}{2}$ can be attributed to some discrete symmetries, which remain to be elucidated.

7 Outlook

Several interesting problems remain to be addressed, some of which we list here.

We have noted the existence of a higher degeneracy of the transfer matrix that occurs for the cases $C_n^{(1)}$ and $D_n^{(1)}$ with $n$ odd and $p = \frac{n+1}{2}$; see Sections 6.4.2 and 6.5.2. These degeneracies are unusual, since they result from the “mixing” of representations of unequal dimensions, such as the $(1, 10)$ and the $(6, 4)$ discussed in Section 6.4.2. In contrast, the self-duality and $Z_2$ symmetries that we identified imply degeneracies of representations of equal dimensions, namely, $(1, R) \leftrightarrow (R, 1)$ and $R \leftrightarrow \bar{R}$, respectively. It would be interesting to find some discrete symmetries that could account for these unusual degeneracies.

For the R-matrices that we have considered (2.4), the $K$-matrices (2.14) do not exhaust the possible diagonal $K$-matrices. Indeed, a few additional diagonal solutions depending on one boundary parameter are known [36]. We expect that the corresponding transfer matrices also have some QG symmetry; however, we leave an investigation of those cases to the future.

We have not considered here the case of the $D_{n+1}^{(2)}$ R-matrix [28], because a corresponding set of $K$-matrices depending on an integer $p = 0, 1, \ldots, n$ is not yet known. It would be interesting to find such a set of $K$-matrices, since the corresponding transfer matrices would presumably have the QG symmetry $U_q(B_{n-p}) \otimes U_q(B_p)$, as well as a $p \leftrightarrow n - p$ duality symmetry, and a self-duality symmetry for even $n$ and $p = \frac{n}{2}$. So far, only the special cases $p = 0$ and $p = n$ have been investigated [27], based on the $D_{n+1}^{(2)}$ K-matrices found in [19]. Interestingly, these K-matrices are not diagonal, but only block-diagonal.

Since the transfer matrix $t(u, p)$ is integrable (2.22), its eigenvalues and eigenvectors can be determined by Bethe ansatz. We expect that, for general values of $p$, the $(g^{(1)}, g^{(r)})$ Dynkin labels of the Bethe states can be related to the numbers of Bethe roots of each type, as was done for $p = 0$ and $p = n$ in [26, 27]. It would be interesting to understand the dependence of the Bethe equations on $p$; and, for the cases $C_n^{(1)}$ and $D_n^{(1)}$, to see how the $p \leftrightarrow n - p$ duality symmetry is manifested in these Bethe equations.
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A R-matrices

The R-matrices are given by

\[
R(u) = c(u) \sum_{\alpha \neq \alpha'} e_{\alpha \alpha} \otimes e_{\alpha \alpha} + b(u) \sum_{\alpha \neq \beta, \beta'} e_{\alpha \alpha} \otimes e_{\beta \beta} \\
+ \left( e(u) \sum_{\alpha < \beta, \alpha \neq \beta'} + \bar{e}(u) \sum_{\alpha > \beta, \alpha \neq \beta'} \right) e_{\alpha \alpha} \otimes e_{\beta \beta} + \sum_{\alpha, \beta} a_{\alpha \beta}(u) e_{\alpha \alpha} \otimes e_{\alpha' \beta'},
\]

where \( e_{\alpha \beta} \) are the elementary \( d \times d \) matrices, with \( d \) given by (2.1). Moreover,

\[
c(u) = 2 \sinh(\frac{u}{2} - 2\eta) \\
b(u) = 2 \sinh(\frac{u}{2}) \\
e(u) = -2e^{-\frac{u}{2}} \sinh(2\eta)
\]

\[
\bar{e}(u) = e^u e(u),
\]

\[
a_{\alpha \beta}(u) = \begin{cases} \\
2 \sinh(\frac{u}{2}) \times \begin{cases} \\
\cosh(\frac{u}{2} - (\kappa - 2) \eta) & \text{for } A_{2n}^{(2)}, A_{2n-1}^{(2)} \\
\sinh(\frac{u}{2} - (\kappa - 2) \eta) & \text{for } B_n^{(1)}, C_n^{(1)}, D_n^{(1)}
\end{cases} \\
b(u) + \begin{cases} \\
2 \sinh(2\eta) \sinh((2n - 1)\eta) & \text{for } B_n^{(1)} \\
-2 \sinh(2\eta) \cosh((2n + 1)\eta) & \text{for } A_{2n}^{(2)}
\end{cases}
\end{cases}
\]

\[
-\delta_{\alpha \beta} \cosh(\frac{u}{2} - \kappa \eta) & \text{for } A_{2n}^{(2)}, A_{2n-1}^{(2)} \\
\epsilon_{\alpha \beta} \epsilon_{\epsilon (\pm \kappa + 2(\alpha - \beta)) n} \sinh(\frac{u}{2}) & \text{for } A_{2n}^{(2)}, A_{2n-1}^{(2)}
\end{cases}
\]

\[
\sinh(\frac{u}{2} - \kappa \eta) & \text{for } B_n^{(1)}, C_n^{(1)}, D_n^{(1)}
\end{cases}
\]

\[
\alpha \leq \beta
\]

\[
\alpha = \beta, \alpha = \alpha'
\]

\[
\alpha \beta
\]

\[
(A.1)
\]

\[
(A.2)
\]

\[
(A.3)
\]
where

\[
\kappa = \begin{cases} 
2n & \text{for } A^{(2)}_{2n-1} \\
2n + 1 & \text{for } A^{(2)}_{2n} \\
2n - 1 & \text{for } B^{(1)}_n \\
2n + 2 & \text{for } C^{(1)}_n \\
2n - 2 & \text{for } D^{(1)}_n 
\end{cases} \quad (A.4)
\]

\[
\epsilon_\alpha = \begin{cases} 
1 & \text{for } 1 \leq \alpha \leq n \\
-1 & \text{for } n + 1 \leq \alpha \leq 2n 
\end{cases} \quad \text{for } A^{(2)}_{2n-1}, C^{(1)}_n \quad (A.5)
\]

\[
\epsilon_\alpha = 1 \quad \text{for } A^{(2)}_{2n}, B^{(1)}_n, D^{(1)}_n
\]

\[
\bar{\alpha} = \begin{cases} 
\alpha - \frac{1}{2} & 1 \leq \alpha \leq n \\
\alpha + \frac{1}{2} & n + 1 \leq \alpha \leq 2n 
\end{cases} \quad \text{for } A^{(2)}_{2n-1}, C^{(1)}_n \quad (A.6)
\]

\[
\bar{\alpha} = \begin{cases} 
\alpha - \frac{1}{2} & 1 \leq \alpha < \frac{d+1}{2} \\
\alpha & \alpha = \frac{d+1}{2} \\
\alpha + \frac{1}{2} & \frac{d+1}{2} < \alpha \leq d 
\end{cases} \quad \text{for } A^{(2)}_{2n}, B^{(1)}_n, D^{(1)}_n
\]

\[
\alpha' = d + 1 - \alpha, \quad \alpha, \beta = 1, \ldots, d . \quad (A.7)
\]

All but one of these R-matrices are the same as in [28], up to the change of variables \( x = e^u, k = e^{2\eta} \) and an overall factor. The one exception is the R-matrix for \( A^{(2)}_{2n-1} \), which we obtain from the \( C^{(1)}_n \) R-matrix in [28] by replacing \( \xi = k^{2n+2} \) by \( \xi = -k^{2n} \); i.e. by changing \( \xi \mapsto -\xi k^{-2} \). It is the same as the \( A^{(2)}_{2n-1} \) R-matrix in the appendix of [30] up to some redefinitions of the anisotropy and spectral parameters, and an overall factor. This \( A^{(2)}_{2n-1} \) R-matrix was used in [13, 25, 27].

**B \quad U_q(g^{(l)}) \otimes U_q(g^{(r)}) \text{ and } \tilde{T}^\pm(p)**

We show here that the asymptotic gauge-transformed monodromy matrix \( \tilde{T}^\pm(p) \) (3.14) can be expressed in terms of coproducts of the generators of a QG of the form \( U_q(g^{(l)}) \otimes U_q(g^{(r)}) \), where \( g^{(l)} \) and \( g^{(r)} \) are (non-affine) simple Lie algebras of type \( B \), \( C \) or \( D \), with rank \( n - p \) and \( p \), respectively. Specifically, the pairs of algebras \((g^{(l)}, g^{(r)})\) are given in Table 2, where \( \hat{g} \) is the affine Lie algebra in the list (2.4) that is associated to the R-matrix. The algebras \( g^{(l)} \oplus g^{(r)} \) are in fact the subalgebras of \( \hat{g} \) obtained by removing the \( p^{th} \) node from the (extended) Dynkin diagram of \( \hat{g} \), which has \( n + 1 \) nodes. We emphasize that the possible values of \( p \) are \( 0, 1, \ldots, n \); it is understood that the “right” algebra \( g^{(r)} \) is absent for \( p = 0 \), while the “left” algebra \( g^{(l)} \) is absent for \( p = n \).
Table 2: Pairs of Lie algebras \((g^{(l)}, g^{(r)})\) corresponding to the affine Lie algebras \(\hat{g}\), where \(p = 0, 1, \ldots, n\).

### B.1 Generators

We denote the generators corresponding to the simple roots of \(g^{(l)}\) and \(g^{(r)}\) by

\[
H_i^{(l)}(p), \quad E_i^{\pm (l)}(p), \quad i = 1, \ldots, n - p,
\]

and

\[
H_i^{(r)}(p), \quad E_i^{\pm (r)}(p), \quad i = 1, \ldots, p,
\]

respectively. (To lighten the notation, we shall refrain from displaying the dependence of these generators on \(p\) when there is no ambiguity in so doing.) The “left” generators satisfy the commutation relations

\[
\left[ H_i^{(l)}(p), H_j^{(l)}(p) \right] = 0,
\]

\[
\left[ H_i^{(l)}(p), E_j^{\pm (l)}(p) \right] = \pm \alpha_i^{(j)} E_j^{\pm (l)}(p),
\]

\[
\left[ E_i^{+ (l)}(p), E_j^{- (l)}(p) \right] = \delta_{i,j} \sum_{k=1}^{n-p} \alpha_k^{(j)} H_k^{(l)}(p), \quad (B.1)
\]

and the “right” generators similarly satisfy the commutation relations

\[
\left[ H_i^{(r)}(p), H_j^{(r)}(p) \right] = 0,
\]

\[
\left[ H_i^{(r)}(p), E_j^{\pm (r)}(p) \right] = \pm \alpha_i^{(j)} E_j^{\pm (r)}(p),
\]

\[
\left[ E_i^{+ (r)}(p), E_j^{- (r)}(p) \right] = \delta_{i,j} \sum_{k=1}^{p} \alpha_k^{(j)} H_k^{(r)}(p). \quad (B.2)
\]

Moreover, the “left” and “right” generators commute with each other

\[
\left[ H_i^{(l)}(p), E_j^{\pm (r)}(p) \right] = \left[ E_i^{\pm (l)}(p), H_j^{(r)}(p) \right] = \left[ E_i^{\pm (l)}(p), E_j^{\pm (r)}(p) \right] = \left[ E_i^{\pm (l)}(p), E_j^{\mp (r)}(p) \right] = 0. \quad (B.3)
\]
The simple roots \( \{\alpha^{(1)}, \ldots, \alpha^{(m)}\} \) (where \( m \) is either \( n - p \) or \( p \)) in the orthogonal basis are given by

\[
\alpha^{(j)} = e_j - e_{j+1}, \quad j = 1, \ldots, m - 1,
\]

\[
\alpha^{(m)} = \begin{cases} 
  e_m & \text{for } B_m \\
  2e_m & \text{for } C_m \\
  e_{m-1} + e_m & \text{for } D_m
\end{cases}
\]

(B.4)

where \( e_j \) are the elementary \( m \)-dimensional basis vectors \( (e_j)_i = \delta_{i,j} \) (i.e., \( e_1 = (1, 0, 0, \ldots, 0) \), \( e_2 = (0, 1, 0, \ldots, 0) \), etc.).

In terms of the \( \hat{g} \) generators\(^\dag\)

\[
H_i = e_{i,i} - e_{d+1-i,d+1-i}, \quad i = 1, \ldots, n,
\]

\[
E_i^+ = e_{i,i+1} + e_{d-i,d+1-i}, \quad i = 1, \ldots, n - 1,
\]

\[
E_n^+ = \begin{cases} 
  e_{n,n+1} + e_{d-n,d+1-n} & \text{if } g^{(l)} = B_{n-p} \text{ i.e., for } A^{(2)}_{2n}, B^{(1)}_n \\
  \sqrt{2}e_{n,n+1} & \text{if } g^{(l)} = C_{n-p} \text{ i.e., for } A^{(2)}_{2n-1}, C^{(1)}_n \\
  e_{n-1,n+1} + e_{n,n+2} & \text{if } g^{(l)} = D_{n-p} \text{ i.e., for } D^{(1)}_n
\end{cases}
\]

\[
E_0^+ = \begin{cases} 
  \sqrt{2}e_{d,1} & \text{if } g^{(r)} = C_p \text{ i.e., for } A^{(2)}_{2n}, C^{(1)}_n \\
  e_{d-1,1} + e_{d,2} & \text{if } g^{(r)} = D_p \text{ i.e., for } A^{(2)}_{2n-1}, B^{(1)}_{n-1}, D^{(1)}_n
\end{cases}
\]

\[
E_i^- = (E_i^+)\dagger, \quad i = 0, 1, \ldots, n,
\]

(B.5)

the “left and “right” generators are given by

\[
H_i^{(l)}(p) = H_{p+i},
\]

\[
E_i^{\pm}(p) = E_{p+i}^{\pm}, \quad i = 1, \ldots, n - p,
\]

(B.6)

and

\[
H_i^{(r)}(p) = -H_{p+1-i},
\]

\[
E_i^{\pm}(p) = E_{p-i}^{\pm}, \quad i = 1, \ldots, p,
\]

(B.7)

respectively. Indeed, one can check that the commutation relations (B.1) - (B.3) are satisfied. Note that the “broken” generators \( E_p^{\pm} \) in (B.5) do not belong to either the “left” (B.6) or “right” (B.7) set of generators; indeed, dropping the \( \hat{g} \) generators \( E_p^{\pm} \) corresponds to deleting the \( p \)th node from the (extended) Dynkin diagram of \( \hat{g} \).

**B.2 Coproducts**

We now present the coproducts for the quantum groups \( U_q(g^{(l)}) \) and \( U_q(g^{(r)}) \).

\(^\dag\)Note that \( e_{ij} \) are the elementary \( d \times d \) matrices introduced below (2.3), where \( d \) is defined in (2.1). We see from (B.6) that the generators in (B.5) with \( i = 1, \ldots, n \) are in fact the generators of \( g^{(l)} \) with \( p = 0 \); and we see from (B.7) that \( E_0^{\pm} \) in (B.5) are the \( n \)th generators of \( g^{(r)} \) with \( p = n \).
B.2.1 “Left” generators

The coproducts for the “left” generators are given by

\[
\Delta(H_j^{(l)}) = H_j^{(l)} \otimes \mathbb{I} + \mathbb{I} \otimes H_j^{(l)}, \quad j = 1, \ldots, n - p,
\]

\[
\Delta(E_j^{(l)}) = E_j^{(l)} \otimes e^{(\eta + i\pi)H_j^{(l)} - \eta H_{j+1}^{(l)}} + e^{-(\eta + i\pi)H_j^{(l)} + \eta H_{j+1}^{(l)}} \otimes E_j^{(l)}, \quad j = 1, \ldots, n - p - 1,
\]

\[
\Delta(E_{n-p}^{(l)}) = \begin{cases} 
E_{n-p}^{(l)} \otimes e^{(\eta + i\pi)H_{n-p}^{(l)}} 
+ e^{-(\eta + i\pi)H_{n-p}^{(l)}} \otimes E_{n-p}^{(l)} & \text{if } g^{(l)} = B_{n-p} \text{ i.e., for } A_{2n}^{(2)}, B_n^{(1)} \\
E_{n-p}^{(l)} \otimes e^{2\eta H_{n-p}^{(l)}} + e^{-2\eta H_{n-p}^{(l)}} \otimes E_{n-p}^{(l)} & \text{if } g^{(l)} = C_{n-p} \text{ i.e., for } A_{2n-1}^{(2)}, C_n^{(1)} \\
E_{n-p}^{(l)} \otimes e^{\eta H_{n-p-1}^{(l)} + (\eta + i\pi)H_{n-p}^{(l)}} 
+ e^{-\eta H_{n-p-1}^{(l)} - (\eta + i\pi)H_{n-p}^{(l)}} \otimes E_{n-p}^{(l)} & \text{if } g^{(l)} = D_{n-p} \text{ i.e., for } D_n^{(1)} 
\end{cases}
\]

(B.8)

These coproducts satisfy

\[
\left[ \Delta(H_i^{(l)}), \Delta(E_j^{(l)}) \right] = \pm \alpha_i^{(j)} \Delta(E_j^{(l)}),
\]

and

\[
\Omega_{ij}^{(l)} \Delta(E_{i}^{+(l)}) \Delta(E_{j}^{-(l)}) - \Delta(E_{i}^{-(l)}) \Delta(E_{j}^{+(l)}) \Omega_{ij}^{(l)} = \begin{cases} 
\delta_{i,j} \frac{\sinh[2\eta \sum_{k=1}^{n-p} \alpha_k^{(j)} \Delta(H_k^{(l)})]}{\sinh(2\eta)} & \text{for } A_{2n}^{(2)}, B_n^{(1)}, D_n^{(1)} \\
\delta_{i,j}(1 + \delta_{i,n-p}) \frac{\sinh[2\eta \sum_{k=1}^{n-p} \alpha_k^{(j)} \Delta(H_k^{(l)})]}{\sinh(2(1+\delta_{i,n-p})\eta)} & \text{for } A_{2n-1}^{(2)}, C_n^{(1)} 
\end{cases}
\]

(B.10)

where \( \Omega_{ij}^{(l)} \) is given by

\[
\Omega_{ij}^{(l)} = \begin{cases} 
e^{-\pi H_{\max(i,j)}^{(l)} } \otimes \mathbb{I} & |i - j| = 1 \text{ for } A_{2n}^{(2)}, B_n^{(1)} \\
\mathbb{I} \otimes \mathbb{I} & \text{otherwise} 
\end{cases}
\]

(B.11)

\[
\Omega_{ij}^{(l)} = \begin{cases} 
e^{-\pi H_{\max(i,j)}^{(l)} } \otimes \mathbb{I} & |i - j| = 1 \text{ and } 1 \leq \min(i, j) \leq n - p - 2 \text{ for } A_{2n-1}^{(2)}, C_n^{(1)}, D_n^{(1)} \\
\mathbb{I} \otimes \mathbb{I} & \text{otherwise} 
\end{cases}
\]

(B.12)
B.2.2 “Right” generators

The coproducts for the “right” generators are given by

\[\Delta(H_j^{(r)}) = H_j^{(r)} \otimes 1 + 1 \otimes H_j^{(r)}, \quad j = 1, \ldots, p,\]
\[\Delta(E_j^{\pm(r)}) = E_j^{\pm(r)} \otimes e^{(\eta+\pi)H_j^{(r)} - \eta H_{j+1}^{(r)}} + e^{-(\eta+\pi)H_j^{(r)} + \eta H_{j+1}^{(r)}} \otimes E_j^{\pm(r)}, \quad j = 1, \ldots, p - 1,\]
\[\Delta(E_p^{\pm(r)}) = \begin{cases} 
E_p^{\pm(r)} \otimes e^{2\eta H_p^{(r)}} + e^{-2\eta H_p^{(r)}} \otimes E_p^{\pm(r)} & \text{if } g^{(r)} = C_p \text{ i.e., for } A_{2n}^{(2)}, C_n^{(1)} \\
E_p^{\pm(r)} \otimes e^{(\eta+\pi)H_p^{(r)}} + e^{-(\eta+\pi)H_{p-1}^{(r)} - \eta H_p^{(r)}} \otimes E_p^{\pm(r)} & \text{if } g^{(r)} = D_p \text{ i.e., for } A_{2n-1}^{(2)}, B_n^{(1)}, D_n^{(1)}
\end{cases}.
\]

These coproducts satisfy

\[\left[\Delta(H_i^{(r)}), \Delta(E_j^{\pm(r)})\right] = \pm \alpha_i^{(j)} \Delta(E_j^{\pm(r)}),\]

and

\[\Omega_{ij}^{(r)} \Delta(E_i^{+(r)}) \Delta(E_j^{-(r)}) - \Delta(E_j^{-(r)}) \Delta(E_i^{+(r)}) \Omega_{ij}^{(r)} = \begin{cases} 
\delta_{i,j} \sinh\left[2\eta \sum_{k=1}^{p} \alpha_k^{(j)} \Delta(H_k^{(r)})\right] & \text{for } A_{2n-1}^{(2)}, B_n^{(1)}, D_n^{(1)} \\
\delta_{i,j} (1 + \delta_{i,p}) \sinh\left[2\eta \sum_{k=1}^{p} \alpha_k^{(j)} \Delta(H_k^{(r)})\right] & \text{for } A_{2n}^{(2)}, C_n^{(1)}
\end{cases},\]

where \(\Omega_{ij}^{(r)}\) is given by

\[\Omega_{ij}^{(r)} = \begin{cases} 
\{ e^{i\pi H_{\max(i,j)}^{(r)}} \otimes 1 & |i - j| = 1 \text{ and } 1 \leq \min(i, j) \leq p - 2 \text{ and } i \neq p & \text{for } A_{2n}^{(2)}, C_n^{(1)} \\
1 \otimes 1 & \text{otherwise}
\end{cases},\]

\[\Omega_{ij}^{(r)} = \begin{cases} 
\{ e^{i\pi H_{\max(i,j)}^{(r)}} \otimes 1 & |i - j| = 1 \text{ and } 1 \leq \min(i, j) \leq p - 2, \\
e^{i\pi (H_i^{(r)} + H_j^{(r)})} \otimes 1 & |i - j| = 2 \text{ and } (i = p \text{ or } j = p) & \text{for } A_{2n-1}^{(2)}, B_n^{(1)}, D_n^{(1)} \\
1 \otimes 1 & \text{otherwise}
\end{cases}.
\]

B.3 \(\tilde{T}^{\pm(p)}\)

The matrix elements of the asymptotic gauge-transformed monodromy matrix \(\tilde{T}^{\pm(p)}\) (3.14) can be expressed in terms of the coproducts of the “left” and “right” generators introduced above. We now exhibit a set of matrix elements \(\tilde{T}_{ij}^{(p)}\) that includes all \(\Delta_{(N)}(E_1^{+(l)})\), \ldots, \(\Delta_{(N)}(E_{n-p}^{+(l)})\) and all \(\Delta_{(N)}(E_1^{+(r)})\), \ldots, \(\Delta_{(N)}(E_{n-p}^{+(r)})\).
For $j \neq n$ and for all the considered affine algebras, we find that

$$
\tilde{T}_{j+1,j}^+(p) = \begin{cases} 
-\psi e^{(\eta+i\pi)\Delta_{(N)}(H_{p-j}^{(r)})} e^{\Delta_{(N)}(H_{p-j+1}^{(r)})} \Delta_{(N)}(E_{p-j}^{+}(r)) & j = 1, \ldots, p - 1 \\
0 & j = p \\
\psi e^{(-\eta+i\pi)\Delta_{(N)}(H_{p-j}^{(l)})} e^{\Delta_{(N)}(H_{p-j+1}^{(l)})} \Delta_{(N)}(E_{p-j}^{-}(l)) & j = p + 1, \ldots, n - 1
\end{cases}, \quad (B.18)
$$

where

$$
\psi = e^{-(\kappa N-1)\eta} 2N^{-1} \sinh(2\eta). \quad (B.19)
$$

The set of matrix elements $\{\tilde{T}_{2,1}^+(p), \ldots, \tilde{T}_{n,n-1}^+(p)\}$ evidently contains all the generators except $\Delta_{(N)}(E_{p}^{+}(r))$ and $\Delta_{(N)}(E_{n-p}^{+}(l))$.

For the $p$-th “right” generator $\Delta_{(N)}(E_{p}^{+}(r))$ we have

$$
\tilde{T}_{1,\sigma(n)}^+(p) = \begin{cases} 
0 & p = 0 \\
-\frac{2}{\sqrt{2}} \psi e^{\eta} \cosh(2\eta) \Delta_{(N)}(E_{p}^{+}(r)) & p = 1, \ldots, n \text{ for } g^{(r)} = C_p \\
\psi e^{(-\eta+i\pi)\Delta_{(N)}(H_{p-1}^{(r)})} e^{\Delta_{(N)}(H_{p}^{(r)})} \Delta_{(N)}(E_{p}^{+}(r)) & p = 2, \ldots, n \text{ for } g^{(r)} = D_p
\end{cases}, \quad (B.20)
$$

where

$$
\sigma(n) = \begin{cases} 
2n - 1 & \text{for } A_{2n-1}^{(2)}, D_n^{(1)} \\
2n & \text{for } B_n^{(1)}, C_n^{(1)} \\
2n + 1 & \text{for } A_{2n}^{(2)}
\end{cases}. \quad (B.21)
$$

For the $(n - p)$-th “left” generator $\Delta_{(N)}(E_{n-p}^{+}(l))$ we have (for $p = 0, 1, \ldots, n - 1$)

$$
\tilde{T}_{n+1,\bar{\sigma}(n)}^+(p) = \begin{cases} 
\psi e^{(-\eta+i\pi)\Delta_{(N)}(H_{n-p}^{(l)})} e^{\Delta_{(N)}(H_{n-p+1}^{(l)})} \Delta_{(N)}(E_{n-p}^{+}(l)) & p \neq n, n - 1 \text{ for } g^{(l)} = B_{n-p}^{(2)} \\
-\frac{2}{\sqrt{2}} \psi e^{\eta} \cosh(2\eta) \Delta_{(N)}(E_{n-p}^{+}(l)) & \text{i.e., for } A_{2n}^{(2)}, B_n^{(1)} \\
-\psi e^{-\eta\Delta_{(N)}(H_{n-p-1}^{(l)})} e^{(\eta+i\pi)\Delta_{(N)}(H_{n-p}^{(l)})} \Delta_{(N)}(E_{n-p}^{+}(l)) & p \neq n, n - 1 \text{ for } g^{(l)} = C_{n-p}^{(2)}, \text{i.e., for } A_{2n-1}^{(2)}, C_n^{(1)}, D_n^{(1)}
\end{cases}, \quad (B.22)
$$

where

$$
\bar{\sigma}(n) = \begin{cases} 
n & \text{for } A_{2n}^{(2)}, A_{2n-1}^{(2)}, B_n^{(1)}, C_n^{(1)} \\
n - 1 & \text{for } D_n^{(1)}
\end{cases}. \quad (B.23)
$$

Similar expressions can be found for $\tilde{T}_{ij}^{-}(p)$ in terms of $\Delta_{(N)}(E_{1}^{-}(l)), \ldots, \Delta_{(N)}(E_{n-p}^{-}(l))$ and $\Delta_{(N)}(E_{1}^{-}(r)), \ldots, \Delta_{(N)}(E_{p}^{-}(r))$. 

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C The Hamiltonian

The transfer matrix (2.20) contains the Hamiltonian \( \mathcal{H}(p) \sim t'(0,p) \). More explicitly, using the regularity properties

\[
R(0) = \xi(0) \mathcal{P},
\]

\[
K^R(0, p) = \mathbb{I},
\]

one obtains

\[
\mathcal{H}(p) = \sum_{k=1}^{N-1} h_{k,k+1} + \frac{1}{2} K^R_1(0, p) + \frac{1}{\text{tr} K^L(0, p)} \text{tr}_a K^L_a(0, p) h_{Na},
\]

where the two-site Hamiltonian \( h_{k,k+1} \) is given by

\[
h_{k,k+1} = \frac{1}{\xi(0)} \mathcal{P}_{k,k+1} R'_{k,k+1}(0).
\]

The Hamiltonian is gauge invariant

\[
\mathcal{H}(p) = \sum_{k=1}^{N-1} \tilde{h}_{k,k+1}(p) + \frac{1}{2} \tilde{K}^R_1(0, p) + \frac{1}{\text{tr} K^L(0, p)} \text{tr}_a \tilde{K}^L_a(0, p) \tilde{h}_{Na},
\]

where the gauge-transformed two-site Hamiltonian is given by

\[
\tilde{h}_{k,k+1}(p) = \frac{1}{\xi(0)} \mathcal{P}_{k,k+1} \tilde{R}'_{k,k+1}(0, p)
\]

\[
= h_{k,k+1} + B'_{k+1}(0, p) - B'_k(0, p),
\]

where we have used the definition (3.1) of the gauge-transformed R-matrix to pass to the second line.

C.1 Special cases

For the special case with \( p = 0 \), the K-matrix \( K^R(u, 0) \) is proportional to the identity matrix (2.17). It follows that only the first term in (C.2) contributes

\[
\mathcal{H}(0) = \sum_{k=1}^{N-1} h_{k,k+1}.
\]

Similarly, for the special case with \( p = n \) and \( d = 2n \), the gauge-transformed K-matrix \( \tilde{K}^R(u, n) \) is proportional to the identity matrix, see (3.4). Hence, only the first term in (C.4) contributes

\[
\mathcal{H}(n) = \sum_{k=1}^{N-1} \tilde{h}_{k,k+1}(n) \quad (d = 2n).
\]

\[\text{Indeed, the second term in (C.2) is evidently proportional to the identity matrix; moreover, using an identity from [5, 26, 27], one can show that the third term in (C.2) is also proportional to the identity matrix.}\]
This explains the observation in [27] that the Hamiltonian for this case is given by a sum of two-body terms. A similar result holds for the special case with \( p = n \) and \( d = 2n + 1 \) [26].

## D Proofs of four lemmas

We outline here proofs of Lemmas 1, 5, 9 and 13 for any value of \( n \). For all of these proofs, it is useful to rewrite the R-matrix (A.1) as follows

\[
R(u) = c(u) R^{(1)} + b(u) R^{(2)} + e(u) R^{(3)} + \bar{e}(u) R^{(4)} + R^{(5)}(u),
\]

where

\[
R^{(1)} = \sum_{\alpha \neq \alpha'} e_{\alpha\alpha} \otimes e_{\alpha\alpha} = \sum_{\alpha} e_{\alpha\alpha} \otimes e_{\alpha\alpha} - e_{n+1,n+1} \otimes e_{n+1,n+1} (1 - \delta_{d,2n}),
\]

\[
R^{(2)} = \sum_{\alpha \neq \beta, \beta'} e_{\alpha\alpha} \otimes e_{\beta\beta} = \sum_{\alpha, \beta} e_{\alpha\alpha} \otimes e_{\beta\beta} - \sum_{\beta \neq \beta'} e_{\beta\beta} \otimes e_{\beta\beta} - \sum_{\beta} e_{\beta\beta'} \otimes e_{\beta\beta'},
\]

\[
R^{(3)} = \sum_{\alpha < \beta, \alpha \neq \beta'} e_{\alpha\beta} \otimes e_{\beta\alpha} = \sum_{\alpha < \beta} e_{\alpha\beta} \otimes e_{\beta\alpha} - \sum_{\beta > \frac{d+1}{2}} e_{\beta\beta'} \otimes e_{\beta\beta'},
\]

\[
R^{(4)} = \sum_{\alpha > \beta, \alpha \neq \beta'} e_{\alpha\beta} \otimes e_{\beta\alpha} = \sum_{\alpha > \beta} e_{\alpha\beta} \otimes e_{\beta\alpha} - \sum_{\beta < \frac{d+1}{2}} e_{\beta\beta'} \otimes e_{\beta\beta'},
\]

\[
R^{(5)}(u) = \sum_{\alpha, \beta} q_{\alpha\beta}(u) e_{\alpha\beta} \otimes e_{\alpha'\beta'}.
\]

We follow a similar basic strategy for all the proofs: express all the matrices in terms of the elementary matrices \( e_{ij} \) and the identity matrix \( I \), perform the matrix products using the identity

\[
e_{ij} e_{kl} = \delta_{jk} e_{il},
\]

and then effectuate the resulting Kronecker deltas. Since many terms are generated by this procedure, we use the software \texttt{Mathematica} to perform the necessary algebra. Since the proofs are too long to present all the details, we explain the main steps, and point out some of the subtleties. We start with the simplest proof (Lemma 13), and then work our way to the most difficult one (Lemma 1).

### D.1 Lemma 13

We wish to prove the relation

\[
Z_1^{(l)} R_{12}(u) Z_1^{(l)} = Z_2^{(l)} R_{12}(u) Z_2^{(l)}
\]

for the \( D_n^{(1)} \) R-matrix. We begin by rewriting \( Z^{(l)} \) (5.13) as

\[
Z^{(l)} = I - e_{n,n} - e_{n+1,n+1} + e_{n,n+1} + e_{n+1,n}.
\]

The relation (D.8) is in fact separately satisfied by each of the terms in the expression (D.1) for the R-matrix, which we now discuss in turn.
D.1.1 $R^{(1)}$ and $R^{(2)}$

Since we consider here only the $D_n^{(1)}$ R-matrix, here $d = 2n$; therefore, the second term in (D.2) is absent. For $R^{(1)}$ and $R^{(2)}$, the sums in $\alpha$ and $\beta$ do not have any restriction of the type $\alpha < \beta$ or $\alpha > \beta$; hence, it is straightforward to show using (D.7) that

\[
Z_1^{(l)} R^{(1)} Z_1^{(l)} = Z_2^{(l)} R^{(1)} Z_2^{(l)},
\]

\[
Z_1^{(l)} R^{(2)} Z_1^{(l)} = Z_2^{(l)} R^{(2)} Z_2^{(l)}. \tag{D.10}
\]

D.1.2 $R^{(3)}$ and $R^{(4)}$

These terms require much more effort. Let us start by considering the first term in $R^{(3)}$, and calculating

\[
Z_1^{(l)} \left( \sum_{\alpha < \beta} e_{\alpha \beta} \otimes e_{\beta \alpha} \right) Z_1^{(l)}. \tag{D.11}
\]

Using the relation (D.7) we obtain an expression depending on Kronecker deltas. But we cannot directly effectuate these Kronecker deltas to evaluate the sums because of the condition $\alpha < \beta$. We can put terms such as $\delta_{n,\beta} \cdot \delta_{n+1,\alpha}$, $\delta_{n,\beta} \cdot \delta_{n,\alpha}$ and $\delta_{n+1,\beta} \cdot \delta_{n+1,\alpha}$ to zero, because they do not obey $\alpha < \beta$. After doing this, we remain with expressions such as

\[
\sum_{\alpha < \beta} e_{n,\beta} \otimes e_{\beta,n} \cdot \delta_{n,\alpha}. \tag{D.12}
\]

Notice that we cannot simply set $\alpha = n$ in this expression. In order to satisfy the condition $\alpha < \beta$, if $\alpha = n$, then $\beta \in \{n+1, \ldots, 2n\}$. Hence, we can rewrite (D.12) as

\[
\sum_{\alpha < \beta} e_{n,\beta} \otimes e_{\beta,n} \cdot \delta_{n,\alpha} = e_{n,n+1} \otimes e_{n+1,n} + \sum_{\beta = n+2}^{2n} e_{n,\beta} \otimes e_{\beta,n}, \tag{D.13}
\]

where we separate the term with $\beta = n+1$ from the sum, since this helps to cancel with other terms. For the same reason, we can rewrite

\[
\sum_{\alpha < \beta} e_{n,\beta} \otimes e_{\beta,n} \cdot \delta_{n+1,\alpha} = \sum_{\beta = n+2}^{2n} e_{n,\beta} \otimes e_{\beta,n+1}. \tag{D.14}
\]

Using similar logic with all of the terms, we obtain

\[
Z_1^{(l)} \left( \sum_{\alpha < \beta} e_{\alpha \beta} \otimes e_{\beta \alpha} \right)Z_1^{(l)} - Z_2^{(l)} \left( \sum_{\alpha < \beta} e_{\alpha \beta} \otimes e_{\beta \alpha} \right)Z_2^{(l)} = e_{1+n,n} \otimes e_{1+n,n} - e_{n,1+n} \otimes e_{n,1+n}. \tag{D.15}
\]

We still must consider the contribution of the second term in $R^{(3)}$

\[
Z_1^{(l)} \left( - \sum_{\beta > \frac{n+1}{2}} e_{\beta \beta} \otimes e_{\beta \beta} \right) Z_1^{(l)}. \tag{D.16}
\]
Notice that, since \( d = 2n \), the condition \( \beta > \frac{d+1}{2} \) is equivalent to \( \beta \geq n + 1 \). Due to this condition, all terms with \( \delta_{n,\beta} \) and \( \delta_{n+1,2n+1-\beta} \) must vanish. Taking this into account, we obtain

\[
Z_1^{(l)} \left( - \sum_{\beta > \frac{d+1}{2}} e_{\beta'} \otimes e_{\beta''} \right) Z_1^{(l)} - Z_2^{(l)} \left( - \sum_{\beta > \frac{d+1}{2}} e_{\beta'} \otimes e_{\beta''} \right) Z_2^{(l)} = -e_{1+n,n} \otimes e_{1+n,n} + e_{n,1+n} \otimes e_{n,1+n},
\]

which exactly cancels with (D.15). We conclude that \( R^{(3)} \) satisfies

\[
Z_1^{(l)} R^{(3)} Z_1^{(l)} = Z_2^{(l)} R^{(3)} Z_2^{(l)}.
\]

We prove that \( R^{(4)} \) satisfies

\[
Z_1^{(l)} R^{(4)} Z_1^{(l)} = Z_2^{(l)} R^{(4)} Z_2^{(l)}
\]

using the same arguments presented for \( R^{(3)} \), but considering \( \alpha > \beta \) instead of \( \alpha < \beta \).

**D.1.3 \( R^{(5)}(u) \)**

For \( R^{(5)}(u) \), there are no restrictions on the sums over \( \alpha \) and \( \beta \); hence, we can directly effectuate all the Kronecker deltas. However, doing this is not enough to show that

\[
Z_1^{(l)} \left( \sum_{\alpha,\beta} a_{\alpha,\beta} (u) e_{\alpha} \otimes e_{\alpha'} \right) Z_1^{(l)} - Z_2^{(l)} \left( \sum_{\alpha,\beta} a_{\alpha,\beta} (u) e_{\alpha} \otimes e_{\alpha'} \right) Z_2^{(l)} = 0.
\]

To this end, it is useful to separate all the terms with \( \alpha, \beta \in \{n, n+1\} \) from the sums. For example,

\[
\sum_{\beta} a_{n,\beta} (u) e_{n+1,\beta} \otimes e_{n,\beta'} = a_{n,n} (u) e_{n,n} \otimes e_{n,n} + a_{n,n+1} (u) e_{n+1,n} \otimes e_{n,n+} + \sum_{\beta=1}^{n-1} a_{n,\beta} (u) e_{n+1,\beta} \otimes e_{n,\beta'} + \sum_{\beta=n+2}^{2n} a_{n,\beta} (u) e_{n+1,\beta} \otimes e_{n,\beta'}.
\]

By doing this, we find that all the terms without sums cancel. The remaining terms can also be seen to cancel by using the following properties of the functions \( a_{\alpha,\beta} (u) \) (A.3) for \( D^{(1)}_n \)

\[
a_{n,n} = a_{n+1,n+1},
\]

\[
a_{n,n+1} = a_{n+1,n},
\]

\[
a_{\alpha,\beta} = a_{\alpha+1,\beta} \quad \text{for} \quad 1 \leq \beta \leq n - 1 \quad \text{and} \quad n + 2 \leq \beta \leq 2n,
\]

\[
a_{\beta,n} = a_{\beta,n+1} \quad \text{for} \quad 1 \leq \beta \leq n - 1 \quad \text{and} \quad n + 2 \leq \beta \leq 2n.
\]

We conclude that

\[
Z_1^{(l)} R^{(5)}(u) Z_1^{(l)} = Z_2^{(l)} R^{(5)}(u) Z_2^{(l)},
\]

which concludes the proof of (D.8).
**D.2 Lemma 9**

We now turn to the proof of the relations

\[
Z_1^{(r)} R_{12}(u) Z_1^{(r)} = Y_2^{(t)}(u) R_{12}(u) Y_2^{(t)}(u), \\
Z_2^{(r)} R_{12}(u) Z_2^{(r)} = Y_1^{(t)}(u) R_{12}(u) Y_1^{(t)}(u),
\]

(D.24)

for the \( A_n^{(2)} \), \( B_n^{(1)} \) and \( D_n^{(1)} \) R-matrices. We begin by rewriting \( Z^{(r)} \) and \( Y(u) \) (5.2) as follows

\[
Z^{(r)} = I - e_{1,1} - e_{d,d} + e_{1,d} + e_{d,1},
\]

\[
Y(u) = I - e_{1,1} - e_{d,d} + e^{-u} e_{1,d} + e^{u} e_{d,1}.
\]

The rest of the proof is very similar to the one for Lemma 13 (D.8). However, whereas in the previous case all the terms are written in such a way that \( \alpha, \beta \in \{n, n + 1\} \) appear explicitly and not inside the sums, here we should write all the terms in such a way that \( \alpha, \beta \in \{1, d\} \) appear explicitly. Another difference is that now not all the terms in the expression (D.1) for the R-matrix separately satisfy the relations (D.24). Indeed, the linear combination \( R^{(3)} + e^u R^{(4)} \) satisfies these relations, but not \( R^{(3)} \) and \( R^{(4)} \) separately. Otherwise, all the intermediate strategies are analogous. At the end, we must use the following properties of the functions \( a_{\alpha \beta}(u) \) (A.3) for \( A_n^{(2)} \), \( B_n^{(1)} \) and \( D_n^{(1)} \)

\[
a_{1,1} = a_{d,d},
\]

\[
a_{d,1} = e^{2u} a_{1,d},
\]

\[
a_{d,\beta} = e^{u} a_{1,\beta} \quad \text{for} \quad 2 \leq \beta \leq d - 1,
\]

\[
a_{\beta,d} = e^{-u} a_{\beta,1} \quad \text{for} \quad 2 \leq \beta \leq d - 1.
\]

(D.26)

**D.3 Lemma 5**

We now present some details about our proof of the duality relation

\[
U_2 R_{12}(u) U_2 = W_1(u) R_{12}(u) W_1(u)
\]

(D.27)

for the \( C_n^{(1)} \) and \( D_n^{(1)} \) R-matrices, for which \( d = 2n \). In contrast with the previous proofs (D.9), (D.25), the matrices \( U \) and \( W(u) \) (4.2) cannot be expressed in the form \((I – \text{few terms})\). We rewrite these matrices instead as

\[
U = \sum_{i=1}^{n} (e_{i,n+i} + e_{n+i,i}),
\]

(D.28)

and

\[
W(u) = \sum_{i=1}^{n} (e^{-u} e_{i,n+i} + e^{u} e_{n+i,i}).
\]

(D.29)

We now proceed to analyze separately the contributions of the terms in the expression (D.1) for the R-matrix to the relation (D.27).
D.3.1 \( R^{(1)} \) and \( R^{(2)} \)

After applying the rule (D.7), we must deal with the ranges of the sums. The ranges for the sums in (D.28) and (D.29) (from 1 to \( n \)) are different from the ones in (D.2) and (D.3) (from 1 to 2\( n \)). We cannot effectuate the Kronecker deltas to evaluate the sums in \( R^{(1)} \) unless we split those sums into two ranges: \( 1 \leq \alpha \leq n \) and \( n + 1 \leq \alpha \leq 2n \). In (D.27) we write the \( U_2 \) on the left hand side of \( R^{(1)} \) with a sum in \( i \), and the \( U_2 \) on the right hand side with a sum in \( j \). For the range \( 1 \leq \alpha \leq n \), all the terms with \( \delta_{i+n,\alpha} \) and \( \delta_{j+n,\alpha} \) are zero, because \( \alpha \) is always smaller than \( n + i \). For \( n + 1 \leq \alpha \leq 2n \), all the terms with \( \delta_{i,\alpha} \) and \( \delta_{j,\alpha} \) are zero, because \( \max(i) \) and \( \max(j) \) are \( n \), while \( \alpha \) is always greater or equal to \( n + 1 \). After applying such arguments, we obtain

\[
U_2 R^{(1)} U_2 = \sum_{\alpha=1}^{n} e_{\alpha,\alpha} \otimes e_{\alpha+n,\alpha+n} + \sum_{\alpha=n+1}^{2n} e_{\alpha,\alpha} \otimes e_{\alpha-n,\alpha-n}.
\]

By applying analogous arguments for the terms with \( W_1(u) \), we find

\[
W_1(u) R^{(1)} W_1(u) = \sum_{\alpha=1}^{n} e_{\alpha+n,\alpha+n} \otimes e_{\alpha,\alpha} + \sum_{\alpha=n+1}^{2n} e_{\alpha-n,\alpha-n} \otimes e_{\alpha,\alpha}.
\]

We conclude that

\[
U_2 R^{(1)} U_2 = W_1(u) R^{(1)} W_1(u),
\]

since the right-hand-sides of (D.30) and (D.31) become identical upon redefining the \( \alpha \)'s in the sums. We prove in a similar way that \( R^{(2)} \) satisfies

\[
U_2 R^{(2)} U_2 = W_1(u) R^{(2)} W_1(u).
\]

D.3.2 \( R^{(3)} \) and \( R^{(4)} \)

The duality relation is not satisfied separately by \( R^{(3)} \) and \( R^{(4)} \), but is instead satisfied by the linear combination \( R^{(3)} + e^{u} R^{(4)} \). That is,

\[
U_2 (R^{(3)} + e^{u} R^{(4)}) U_2 = W_1(u) \left( R^{(3)} + e^{u} R^{(4)} \right) W_1(u).
\]

In order to manage the cases with \( \alpha < \beta \) and \( \alpha > \beta \), we split the sums over \( \alpha \) and \( \beta \) into four ranges:

\[
1 \leq \alpha \leq n \quad \text{and} \quad 1 \leq \beta \leq n,
\]
\[
1 \leq \alpha \leq n \quad \text{and} \quad n + 1 \leq \beta \leq 2n,
\]
\[
n + 1 \leq \alpha \leq 2n \quad \text{and} \quad 1 \leq \beta \leq n,
\]
\[
n + 1 \leq \alpha \leq 2n \quad \text{and} \quad n + 1 \leq \beta \leq 2n.
\]

For each of these ranges, we put to zero terms that contain Kronecker deltas where \( \alpha \) and \( \beta \) are outside of the relevant interval. Again, at the end, it is necessary to redefine \( \alpha \) and \( \beta \) on the sums to see that (D.34) is satisfied.
D.3.3 \( R^{(5)}(u) \)

For this term we also split the sums over \( \alpha \) and \( \beta \) into the four ranges (D.35). All the other strategies are similar to the ones presented above, and we obtain

\[
U_2 R^{(5)}(u) U_2 = W_1(u) R^{(5)}(u) W_1(u). \tag{D.36}
\]

D.4 Lemma 1 for \( d = 2n \)

In order to prove

\[
\left[ \tilde{R}^{+}_{12}(p), \tilde{K}^{R}_{2}(u, p) \right] = 0 \tag{D.37}
\]

for any value of \( n \), we proceed in three steps: finding an explicit expression for the gauge-transformed R-matrix \( \tilde{R}_{12}(u, p) \), performing the limit \( u \to \infty \) in \( e^{-u} \tilde{R}_{12}(u, p) \) to obtain \( \tilde{R}^{+}_{12}(p) \), and finally evaluating the commutator. We consider here the case \( d = 2n \), leaving the case \( d = 2n + 1 \) for the following subsection.

D.4.1 Finding \( \tilde{R}_{12}(u, p) \)

In order to obtain an explicit expression for the gauge-transformed R-matrix \( \tilde{R}_{12}(u, p) \) (3.1), it is useful to rewrite \( B(u) \) (3.3) in terms of elementary matrices

\[
B(u) = e^{\frac{u}{2}} \sum_{i=1}^{p} e_{i,i} + \sum_{i=p+1}^{n} e_{i,i} + \sum_{i=n+1}^{2n-p} e_{i,i} + e^{-\frac{u}{2}} \sum_{i=2n-p+1}^{2n} e_{i,i}, \tag{D.38}
\]

for \( 1 \leq p \leq n - 1 \).\(^{14}\)

We now point out some useful simplifications for the contributions from each of the terms in the expression (D.1) for the R-matrix.

Since \( B(u) \) is a diagonal matrix,

\[
B_{1}(u) R^{(1)} B_{1}(-u) = R^{(1)}, \tag{D.39}
\]

\[
B_{1}(u) R^{(2)} B_{1}(-u) = R^{(2)}. \tag{D.40}
\]

Let us now consider the first term in \( B_{1}(u) R^{(3)} B_{1}(-u) \), where \( \alpha < \beta \). After applying the rule (D.7), we obtain terms such as

\[
\sum_{\alpha < \beta} \sum_{i=2n-p+1}^{2n} \sum_{j=n+1}^{2n-p} e_{i,j} \otimes e_{i,i} \delta_{\alpha, i} \delta_{\beta, j}, \tag{D.41}
\]

for example. Several terms like this appear, but they are all equal to zero, because the \( \delta \)'s force \( \alpha = i \) and \( \beta = j \); but \( i > j \) in this sum, which contradicts the condition \( \alpha < \beta \). For

\(^{14}\)For \( p = 0 \) and \( p = n \), \( \tilde{K}^{R}(u, p) \propto I \), so (D.37) is trivially satisfied.
the second term in $B_1(u) R^{(3)} B_1(-u)$, several terms are zero because the Kronecker deltas force $\beta$ to have values that are not greater than $\frac{d+1}{2}$. Similar arguments can be used for $B_1(u) R^{(4)} B_1(-u)$.

For $B_1(u) R^{(5)}(u) B_1(-u)$, after applying the rule (D.7), we can directly use the $\delta$’s to evaluate the sums, because there are no restrictions on the $\alpha$’s and $\beta$’s. The functions $a_{i,j}(u)$ have different expressions depending on whether $i = j$, $i < j$ or $i > j$. For later convenience, we separately calculate the contributions from each of these three cases. For example, consider the term

$$\sum_{i=2n-p+1}^{2n-p+1} \sum_{j=1}^{p} a_{i,j}(u) e_{i,j} \otimes e_{i',j'}.$$  

(D.42)

This term contributes only to $i > j$, due to the ranges in the sums and the fact $2n-p+1 > p$.

We refrain from displaying the final result for $\tilde{R}_{12}(u, p)$, which is quite lengthy even after the simplifications noted above.

### D.4.2 Performing the large-$u$ limit

We now proceed to perform the limit $u \to \infty$ in $e^{-u} \tilde{R}_{12}(u, p)$. To this end, we need the following results

$$\lim_{u \to \infty} e^{-u} e(u) = 0 = \lim_{u \to \infty} e^{-\frac{u}{2}} e(u) = \lim_{u \to \infty} e^{-\frac{3u}{4}} e(u) = \lim_{u \to \infty} e^{-2u} e(u),$$

$$\lim_{u \to \infty} e^{-u} a_{\alpha\beta}(u) = 0 = \lim_{u \to \infty} e^{-\frac{u}{2}} a_{\alpha\beta}(u) = \lim_{u \to \infty} e^{-2u} a_{\alpha\beta}(u) = \lim_{u \to \infty} e^{-\frac{3u}{2}} a_{\alpha\beta}(u),$$

$$b \equiv \lim_{u \to \infty} e^{-u} b(u) = \frac{1}{2} e^{-\kappa \eta},$$

$$c \equiv \lim_{u \to \infty} e^{-u} c(u) = \frac{1}{2} e^{-(\kappa+2) \eta},$$

$$e \equiv \lim_{u \to \infty} e(u) = -e^{-\kappa \eta} \sinh(2\eta) = \lim_{u \to \infty} e^{-u} e(u),$$

$$a^{(1)} \equiv \lim_{u \to \infty} e^{-u} a_{\alpha\beta}^{(1)}(u) = \frac{1}{2} e^{-(\kappa-2) \eta},$$

$$a^{(2)} \equiv \lim_{u \to \infty} e^{-u} a_{\alpha\beta}^{(2)}(u) = \frac{1}{2} e^{-\kappa \eta},$$

$$a^{(3)}_{\alpha,\beta} \equiv \lim_{u \to \infty} a_{\alpha\beta}^{(3)}(u) = e^{-\kappa \eta} \sinh(2\eta) \left( \delta_{1}^{2} e^{2(\kappa+\alpha-\beta) \eta} e_{\alpha} e_{\beta} - \delta_{\alpha,\beta'} \right),$$

$$a^{(4)}_{\alpha,\beta} \equiv \lim_{u \to \infty} e^{-u} a_{\alpha\beta}^{(4)}(u) = e^{-\kappa \eta} \sinh(2\eta) \left( e^{2(\alpha-\beta) \eta} e_{\alpha} e_{\beta} - \delta_{\alpha,\beta'} \right),$$

(D.43)

where

$$a_{\alpha,\beta}(u) = \begin{cases} a_{\alpha,\beta}^{(1)}(u) & \text{for } \alpha = \beta, \alpha \neq \alpha' \\ a_{\alpha,\beta}^{(2)}(u) & \text{for } \alpha = \beta, \alpha = \alpha' \\ a_{\alpha,\beta}^{(3)}(u) & \text{for } \alpha < \beta \\ a_{\alpha,\beta}^{(4)}(u) & \text{for } \alpha > \beta \\ \end{cases}.$$  

(D.44)
and the definition of $a^{(i)}_{\alpha,\beta}(u)$ can be read off directly from (A.3).

With the help of these results, we find that $\hat{K}^+(u,p) (3.13)$ is given, for $d = 2n$ and $1 \leq p \leq n - 1$, by

$$\hat{K}^+(u,p) = c \sum_{\alpha} e_{\alpha,\alpha} \otimes e_{\alpha,\alpha} + b \sum_{\alpha \neq \beta, \beta'} e_{\alpha,\alpha} \otimes e_{\beta,\beta} - c \left( \sum_{\beta=p+1}^{n} + \sum_{\beta=2n-p+1}^{2n} \right) e_{\beta',\beta} \otimes e_{\beta,\beta'}$$

$$+ c \left( \sum_{\alpha,\beta=1}^{p} + \sum_{\alpha,\beta=p+1}^{n} + \sum_{\alpha,\beta=n+1}^{2n-p} + \sum_{\alpha,\beta=2n-p+1}^{2n} + \sum_{\alpha=1}^{p} + \sum_{\alpha=2n-p+1}^{2n} + \sum_{\alpha=n+1}^{n} \sum_{\beta=p+1}^{n} \right) e_{\alpha,\beta} \otimes e_{\beta,\alpha}$$

$$+ a^{(1)} \sum_{\alpha} e_{\alpha,\alpha} \otimes e_{\alpha',\alpha'} + \sum_{\alpha=1}^{p} \sum_{\beta=2n+1-p}^{2n} a^{(3)} e_{\alpha,\beta} \otimes e_{\alpha',\beta'}$$

$$+ \left( \sum_{\alpha,\beta=1}^{p} + \sum_{\alpha,\beta=p+1}^{n} + \sum_{\alpha,\beta=n+1}^{2n-p} + \sum_{\alpha,\beta=2n-p+1}^{2n} + \sum_{\alpha=n+1}^{n} \sum_{\beta=p+1}^{n} \right) a^{(4)} e_{\alpha,\beta} \otimes e_{\alpha',\beta'} \right). \quad (D.45)$$

**D.4.3 Evaluating the commutator**

In order to evaluate the commutator (D.37), we rewrite $\hat{K}^R(u,p) (3.4)$ in terms of elementary matrices, and obtain

$$\hat{K}^R(u,p) = \mathbb{1} \otimes \left[ \sum_{i=1}^{p} + \left( \frac{\gamma e^u + 1}{\gamma + e^u} \right) \sum_{i=p+1}^{n} + \left( \frac{\gamma e^u + 1}{\gamma + e^u} \right) \sum_{i=n+1}^{2n-p} + \sum_{i=2n-p+1}^{2n} \right] e_{i,i}. \quad (D.46)$$

It is then just a matter of applying the same ideas presented above, and putting to zero all the terms that do not belong to the relevant range. In this way, one can see that each of the terms in (D.45) commutes with (D.46).

**D.5 Lemma 1 for $d = 2n + 1$**

The cases where $d = 2n + 1$ can be analogously proved. However, it is more suitable to separate the “middle” terms in $B(u)$ and $\hat{K}_2^R(u,p)$, i.e. we set

$$B(u) = e^u \sum_{i=1}^{p} e_{i,i} + \sum_{i=p+1}^{n} e_{i,i} + e_{n+1,n+1} + \sum_{i=n+2}^{2n-p+1} e_{i,i} + e^{-\frac{u}{2}} \sum_{i=2n-p+2}^{2n+1} e_{i,i} \quad (D.47)$$

and

$$\hat{K}_2^R(u,p) = \mathbb{1} \otimes e_{n+1,n+1} + \mathbb{1} \otimes \left[ \sum_{i=1}^{p} + \left( \frac{\gamma e^u + 1}{\gamma + e^u} \right) \sum_{i=p+1}^{n} + \left( \frac{\gamma e^u + 1}{\gamma + e^u} \right) \sum_{i=n+2}^{2n-p+1} + \sum_{i=2n-p+2}^{2n+1} \right] e_{i,i}, \quad (D.48)$$
for $1 \leq p \leq n - 1$. For this case, $\tilde{R}^{+}_{12}(p)$ is given by

$$
\tilde{R}^{+}_{12}(p) = e \sum_{\alpha \neq \alpha'} e_{\alpha, \alpha} \otimes e_{\alpha, \alpha} + b \sum_{\alpha \neq \beta, \beta'} e_{\alpha, \alpha} \otimes e_{\beta, \beta} - e \left( \sum_{\beta = 2n - p + 2}^{2n + 1} + \sum_{\beta = p + 1}^{n} \right) e_{\gamma, \beta} \otimes e_{\beta, \beta'}
$$

$$
+ e \left( \sum_{\alpha = 1}^{p} \sum_{\beta = 2n - p + 2}^{2n + 1} + \sum_{\alpha = 1}^{p} \sum_{\alpha = \beta + 1}^{n} + \sum_{\alpha = \beta + 1}^{2n - p + 1} + \sum_{\alpha = \beta + 1}^{n} \right) e_{\alpha, \beta} \otimes e_{\beta, \alpha}
$$

$$
+ a^{(1)} \left( \sum_{\alpha = 1}^{n} + \sum_{\alpha = n + 2}^{2n + 1} \right) e_{\alpha, \alpha} \otimes e_{\alpha', \alpha'}
$$

$$
+ \sum_{\alpha = 1}^{p} \sum_{\alpha = \beta + 1}^{2n - p + 1} + \sum_{\alpha = \beta + 1}^{n} \right) a^{(3)}_{\alpha, \beta} e_{\alpha, \beta} \otimes e_{\alpha', \beta'}
$$

$$
+ \sum_{\alpha = n + 2}^{n} \sum_{\beta = p + 1}^{n} a^{(4)}_{\alpha, \beta} e_{\alpha, \beta} \otimes e_{\alpha', \beta'}
$$

$$
+ a^{(2)} e_{n + 1, n + 1} \otimes e_{n + 1, n + 1} - e \left( e^{2(n + 1)\eta} \sum_{\beta = p + 1}^{n} e^{-2\beta \eta} e_{n + 1, \beta} \otimes e_{n + 1, \beta'}
$$

$$
- e \left( e^{2(n + 1)\eta} \sum_{\beta = n + 2}^{2n - p + 1} e^{-2\beta \eta} e_{\gamma, n + 1} \otimes e_{\beta', n + 1}
$$

$$
+ e \left( \sum_{\alpha = p + 1}^{n} e_{n + 1, \alpha} \otimes e_{n + 1, \alpha} + \sum_{\alpha = n + 2}^{2n - p + 1} e_{n + 1, \alpha} \otimes e_{n + 1, \alpha} \right)
$$

End}

For $p = n$, it is suitable to write $B(u)$ and $\tilde{K}^{+}_{2}(u, p)$ as

$$
B(u) = e^{\frac{\pi}{n}} \sum_{i = 1}^{n} e_{i, i} + e_{n + 1, n + 1} + \sum_{i = n + 2}^{2n + 1} e_{i, i}, \quad \text{(D.50)}
$$

and

$$
\tilde{K}^{+}_{2}(u, p) = I \otimes I - I \otimes e_{n + 1, n + 1} + \left( \frac{\gamma e^{u} + 1}{\gamma + e^{u}} \right) I \otimes e_{n + 1, n + 1}. \quad \text{(D.51)}
$$

For this case, $\tilde{R}^{+}_{12}(p)$ is given by

$$
\tilde{R}^{+}_{12}(p) = e \sum_{\alpha \neq \alpha'} e_{\alpha, \alpha} \otimes e_{\alpha, \alpha} + b \sum_{\alpha \neq \beta, \beta'} e_{\alpha, \alpha} \otimes e_{\beta, \beta} - e \left( \sum_{\beta = n + 2}^{2n + 1} + \sum_{\beta = n + 2}^{n} \right) e_{\gamma, \beta} \otimes e_{\beta, \gamma}
$$

$$
+ e \left( \sum_{\alpha = 1}^{n} \sum_{\beta = n + 2}^{2n + 1} + \sum_{\alpha = \beta + 1}^{n} + \sum_{\alpha = \beta + 1}^{2n - p + 1} + \sum_{\alpha = \beta + 1}^{n} \right) e_{\alpha, \beta} \otimes e_{\beta, \alpha}
$$

$$
+ \sum_{\alpha = n + 2}^{n} \sum_{\beta = p + 1}^{n} e_{\alpha, \beta} \otimes e_{\beta, \alpha}
$$

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\[
+ a^{(1)} \left( \sum_{\alpha=1}^{n} + \sum_{\alpha=n+2}^{2n+1} \right) e_{\alpha,\alpha} \otimes e_{\alpha',\alpha'} + a^{(2)} e_{n+1,n+1} \otimes e_{n+1,n+1} \\
+ \sum_{\alpha=1}^{n} \sum_{\beta=n+2}^{2n+1} a^{(3)}_{\alpha,\beta} e_{\alpha,\beta} \otimes e_{\alpha',\beta'} + \left( \sum_{\alpha,\beta=1}^{n} + \sum_{\alpha,\beta=n+2}^{2n+1} \right) a^{(4)}_{\alpha,\beta} e_{\alpha,\beta} \otimes e_{\alpha',\beta'}.
\]

\text{(D.52)}

For \( p = 0 \), \( \tilde{K}^R(u,p) \propto \mathbb{I} \), so (D.37) is trivially satisfied.

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