A scaleable projection-based branch-and-cut algorithm for the 
\( p \)-center problem

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Abstract

The \( p \)-center problem (\( p \)CP) is a fundamental problem in location science, where we are given customer demand points and possible facility locations, and we want to choose \( p \) of these locations to open a facility such that the maximum distance of any customer demand point to its closest open facility is minimized. State-of-the-art solution approaches of \( p \)CP use its connection to the set cover problem to solve \( p \)CP in an iterative fashion by repeatedly solving set cover problems. The classical textbook integer programming (IP) formulation of \( p \)CP is usually dismissed due to its size and bad linear programming (LP)-relaxation bounds.

We present a novel solution approach that works on a new IP formulation that can be obtained by a projection from the classical formulation. The formulation is solved by means of branch-and-cut, where cuts for demand points are iteratively generated. Moreover, the formulation can be strengthened with combinatorial information to obtain a much tighter LP-relaxation. In particular, we present a novel way to use lower bound information to obtain stronger cuts. We show that the LP-relaxation bound of our strengthened formulation has the same strength as the best known bound in literature, which is based on a semi-relaxation.

Finally, we also present a computational study on instances from the literature with up to more than 700000 customers and locations. Our solution algorithm is competitive with highly sophisticated set-cover-based solution algorithms, which depend on various components and parameters.

Keywords: location; \( p \)-center problem; integer programming formulation; lifting; min-max objective

1 Introduction

The (vertex) \( p \)-center problem (\( p \)CP) is a fundamental problem in location science\(^{[\text{Laporte et al., 2019}]}\)\(^{[\text{Snyder and Shen, 2011}]}\), where we are given customer demand points and potential facility locations, and we want to choose \( p \) of these locations to open a facility such that the maximum distance of any customer demand point to its closest open facility is minimized. The \( p \)CP is \( \text{NP-hard} \)\(^{[\text{Kariv and Hakimi, 1979}]}\) and has applications where it is critical that all customer demand points can be reached quickly, i.e., at least one open facility can be reached quickly by each customer demand point. This includes emergency service locations, such as ambulances or fire stations, and also relief actions in humanitarian crisis\(^{[\text{Çalık and Tansel, 2013}]}\)\(^{[\text{Lu and Shen, 2013}]}\)\(^{[\text{Jia et al., 2007}]}\). Next to applications in location science, the \( p \)CP is also used for clustering of large-scale data\(^{[\text{Geraci et al., 2009}]}\)\(^{[\text{Kleindessner et al., 2019}]}\)\(^{[\text{Malkomes et al., 2015}]}\), for feature selection\(^{[\text{Meinl et al., 2011}]}\) and in computer vision\(^{[\text{Friedler and Mount, 2010}]}\).

A formal definition of the \( p \)CP is as follows. Given an integer \( p \), a set of customer demand points \( I \) with cardinality \( |I| = n \), a set of potential facility locations \( J \) of cardinality \( |J| = m \geq p \) and a distance \( d_{ij} \) from

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a customer demand point \(i\) to the potential facility location \(j\) for every \(i \in I\) and \(j \in J\), find a subset \(S \subseteq J\) with cardinality \(|S| = p\) of facilities to open such that the maximum distance between a customer demand point and its closest open facility is minimized, i.e., such that \(\max_{i \in I} \min_{j \in S} \{d_{ij}\}\) is minimized.

There exists various integer programming (IP) formulations for the pCP. In this paper we focus on the classical textbook formulation, see for example Daskin [2013]. It uses binary variables \(y_j\) to indicate whether a facility opens at the potential facility location \(j \in J\), binary variables \(x_{ij}\) to indicate whether a customer demand point \(i \in I\) is served by the the potential facility location \(j \in J\) and a continuous variable \(z\) to measure the distance in the objective function.

\[
\begin{align*}
\text{(PC1)} & \quad \min & & z \\
\text{s.t.} & & \sum_{j \in J} y_j = p & \quad (1a) \\
& & \sum_{j \in J} x_{ij} = 1 & \forall i \in I & \quad (1b) \\
& & x_{ij} \leq y_j & \forall i \in I, \forall j \in J & \quad (1c) \\
& & \sum_{j \in J} d_{ij} x_{ij} \leq z & \forall i \in I & \quad (1d) \\
& & x_{ij} \in \{0, 1\} & \forall i \in I, \forall j \in J & \quad (1f) \\
& & y_j \in \{0, 1\} & \forall j \in J & \quad (1g) \\
& & z \in \mathbb{R} & & \quad (1h)
\end{align*}
\]

In particular, \((1b)\) makes sure that exactly \(p\) facilities are open, \((1d)\) guarantees that every customer is assigned to a facility, \((1e)\) ensures that customers are only assigned to open facilities, \((1f)\) forces \(z\) to be at least the distance of any customer to its assigned facility, the objective function \((1a)\) minimizes this \(z\) and the constraints \((1f)\) and \((1g)\) make certain that \(x\) and \(y\) are binary. However, this formulation suffers from bad linear programming (LP)-relaxation bounds (see, e.g., Snyder and Shen [2011]) and also has scalability issues, as there are \(O(|J| \cdot |J|)\) variables and constraints and large-scale instances from the literature for the pCP usually have \(I = J = V\) with \(|V|\) in the ten-thousands.

Thus, state-of-the-art exact approaches (see, e.g., Chen and Chen [2009] Contardo et al. [2019]) for the pCP use its connection to the set cover problem (SCP). Given a ground set \(U\) and a set \(W\) of subsets of \(U\), such that the union of the sets in \(W\) is \(U\), in the SCP we want to find a subset \(T\) of \(W\) of minimum cardinality, such that the union of the sets in \(T\) is \(U\). The question of whether the optimal objective function value of an instance of the pCP is less than or equal to a given value \(\bar{z}\) can be modeled by an instance of the SCP: we set \(U = I\), define \(W_j = \{i \in I : d_{ij} \leq \bar{z}\}\) for every \(j \in J\), and set \(W = \cup_{j \in J} W_j\). Then the optimal objective function of pCP is less than or equal to \(\bar{z}\) if and only if the corresponding set cover instance has an optimal solution that uses at most \(p\) sets. State-of-the-art exact approaches for the pCP use this connection and solve the pCP by iteratively solving SCPs for different values of \(\bar{z}\). More details on this approach and also other existing IP formulations are given in Section 1.2.

### 1.1 Contribution and outline

In this paper, we present a novel solution approach that works on an IP formulation that can be obtained by a projection from the classical formulation. The formulation is solved by means of branch-and-cut (B&C), where cuts for customer demand points are iteratively generated. This makes the method suitable for large scale instances, as the complete distance matrix does not need to be kept in memory. Moreover, we show how our formulation can be strengthened with combinatorial information to obtain a much tighter LP-relaxation compared to the LP-relaxation of the classical formulation. This strengthening procedure uses a novel way to use lower bound information to obtain stronger cuts. We also show how our formulation is connected to the SCP and present a computational comparison with state-of-the-art solution algorithms on instances from the literature.
In the remainder of this section, we discuss previous and related work to the pCP. Section 2 contains our new formulation, and Section 3 the strengthening procedure for the inequalities included in our formulation and theoretical results on the connection to the SCP and existing lower bounds from the literature. Section 4 describes implementation details of our B&C based solution algorithm. In Section 5 the computational study is presented. Finally, Section 6 concludes the paper.

1.2 Literature review

The pCP was first mentioned in 1965 by Hakimi (1965) and it was proven to be NP-hard by Kariv and Hakimi (1979) for $p \geq 2$, even in the case that $I = J$ is the set of vertices of a planar graph with maximum degree three and all distances are equal to one. However, there are special cases of pCP that can be solved in polynomial time, for example it can be solved in $O(|I|^2 \log |I|)$ time if $I = J$ is the set of vertices of a tree (Kariv and Hakimi, 1979).

Since the introduction of the pCP there has been an tremendous amount on work on both heuristic and exact solution methods. As our work is on the design of an exact solution algorithm, we focus our literature review on existing exact methods and refer to the recent survey (Garcia-Diaz et al., 2019) for approximation algorithms and heuristics for the pCP.

The first exact solution method for the pCP was proposed by Minieka (1970) and used the above described connection with the SCP. Minieka suggested to start with an arbitrary solution of the pCP and then iteratively solve SCPs in order to find out whether there is a better solution. Let $D = \{d_{ij} : i \in I, j \in J\}$ denote the set of all possible distances and let $d_1, \ldots, d_K$ be the values contained in $D$, so $D = \{d_1, \ldots, d_K\}$. Clearly, the optimal objective function value of pCP is in $D$ and there are at most $|I| \cdot |J|$ potential optimal values. Minieka’s approach can be interpreted as searching for the optimal value in $D$ by first removing some values from $D$ and then going through all remaining values of $D$ one by one starting with the largest one and checking their optimality. Later Garfinkel et al. (1977) elaborated this approach and proposed to use a heuristic to obtain a starting solution in order to further reduce potential optimal values in $D$ and to then use a binary search with the remaining values of $D$ instead of a linear search.

In the early 2000s a lot of work was done on the pCP. Ilhan and Pinar (2001) introduced a two-phase algorithm based on the idea of Minieka, where they solve feasibility linear programs (LPs) in order to obtain a good lower bound in the first phase, and iteratively solve feasibility SCPs within performing a linear search on $D$ starting from the lower bound in the second phase. Later Al-Khedhairi and Salhi (2005) proposed some enhancements to this approach in order to reduce the number of iterations of the second phase. Ilhan et al. (2002) perform a binary search to detect a suitable lower bound by solving LPs, and then perform a linear search on $D$. Caruso et al. (2003) use the connection to the SCP by providing two heuristics and two exact algorithms for the pCP based on weak and strong dominance relationships.

In 2004, Elloumi et al. (2004) introduced a new IP formulation for the pCP. This formulation has a binary variable $y_j$ for $j \in J$ indicating whether facility $j$ opens, analogously to (PCL). Furthermore there is a binary variable for each value in $D$ that indicates whether the optimal value of pCP is less or equal than this value. Towards this end let $u_k = 0$ if all customers have an open facility with distance at most $d_{k-1}$,
otherwise \( u_k = 1 \) for all \( k \in \{2, \ldots, K\} \).

\[
\text{(PCE)} \quad \min \quad d_1 + \sum_{k=2}^{K} (d_k - d_{k-1}) u_k \\
\text{s.t.} \quad \sum_{j \in J} y_j \leq p \\
\sum_{j \in J} y_j \geq 1 \\
u_k + \sum_{j : d_j < d_k} y_j \geq 1 \quad \forall i \in I, \forall k \in \{2, \ldots, K\} \\
u_k \in \{0, 1\} \quad \forall k \in \{2, \ldots, K\} \\
y_j \in \{0, 1\} \quad \forall j \in J
\]

Elloumi et al. (2004) show that the lower bound obtained by the LP-relaxation of this IP formulation (PCE) is tighter than the one from the LP-relaxation of the classical IP formulation (PC1) and how the advantages of their formulation can help in the computation of an optimal solution.

Yet another IP formulation was introduced by Çalık and Tansel (2013). They also have the same binary variables \( y \) as in (PC1), and additionally to that there is a binary variable for each value in \( D \) that indicates whether the optimal value of pCP is exactly this value. Thus, in contrast to the IP formulation from Elloumi et al. (2004) here exactly one of this new binary variables is equal to one. Çalık and Tansel (2013) also investigated the connection of their IP formulation to the one of Elloumi et al. (2004) and provided the best LP-relaxation for the pCP so far. They also utilize their IP formulation within an exact solver for the pCP based on successive restrictions of this new formulation.

Chen and Chen (2009) introduced an LP-relaxation-based iterative SCP-based algorithm, in which they only consider some of the customer demand points in each iteration. So, instead of having few iterations with large sub-problems like previous approaches, they considered many iterations with small sub-problems. Recently, Contardo et al. (2019) have successfully enhanced this idea of considering only a subset of customer demand points and included it into an exact SCP-based binary search algorithm for the pCP, that allows to solve large scale instances.

Finally, several different variants of the pCP have also been investigated, among them the capacitated pCP, where each potential facility location has a maximum capacity, the continuous pCP, where the potential facility locations are not restricted to a set, and several versions with uncertain parameters. We refer to the chapter on the pCP (Çalık et al., 2019) in Laporte et al. (2019) for more details.

2 A new IP formulation for the \( p \)-center problem

In this section, we start by presenting our new IP formulation for the \( p \)-center problem in Section 2.1. Next, we show that the feasible region of the LP-relaxation of our new formulation can be seen as a projection of the feasible region of the LP-relaxation of (PC1) in Section 2.2. Furthermore, we provide details about how our new formulation can be obtained from (PC1) as Benders decomposition in Section 2.3.

2.1 New IP formulation

Our new IP formulation (PC2) for the \( p \)-center problem uses \( y \) and \( z \) variables with the same meaning as in (PC1), i.e., the binary variables \( y_j \) indicate whether a facility opens at location \( j \in J \) and the continuous
variable $z$ represents the distance in the objective function.

$$
\begin{align}
\text{(PC2)} & \quad \min \quad z \\
\text{s.t.} \quad & \sum_{j \in J} y_j = p \\
& z \geq d_{ij} - \sum_{j' : d_{ij'} < d_{ij}} (d_{ij} - d_{ij'}) y_{j'} \quad \forall i \in I, \forall j \in J \\
& y_j \in \{0, 1\} \quad \forall j \in J \\
& z \in \mathbb{R} 
\end{align}
$$

Next, we give combinatorial arguments for the correctness of $\text{(PC2)}$. The constraint (3b) makes sure that exactly $p$ facilities open and (3d) forces $y$ to be binary. The constraints (3c) ensure that $z$ is at least the distance to the nearest open facility for each customer $i \in I$ and work as follows: For each $j \in J$, they ensure that $z$ is at least $d_{ij}$ in case no closer facility $j'$ to customer $i$ (i.e., a facility $j'$ with $d_{ij'} < d_{ij}$) is open. Thus, $z$ has to take at least the distance to the nearest open facility to customer $i$. Moreover, if for $j \in J$ at least one closer facility is open, then let $j'$ be the closest facility to customer $i$ that is open. Then the right-hand side of (3c) is at most $d_{ij} - (d_{ij} - d_{ij'}) = d_{ij'}$ (maybe even more is subtracted from $d_{ij}$), thus $z$ is forced to be larger than a value that is at most $d_{ij'}$, which is the distance of customer $i$ to its closest open facility. Thus, constraints (3c) never lead to an overestimation of the distance of a customer to its nearest open facility.

We observe that formulation $\text{(PC2)}$ has $O(|J|)$ variables and $O(|I| \cdot |J|)$ constraints. Thus, the number of variables is $O(|I|)$ times less than for formulation $\text{(PC1)}$. However, similar to $\text{(PC1)}$, the number of constraints can become prohibitive for solving large scale instances. We thus do not solve $\text{(PC2)}$ directly, but use a branch-and-cut approach, where a lifted version of constraints (3c) are separated on-the-fly. Section 3 describes the lifting of (3c) and Section 4 discusses the separation.

### 2.2 Projection-based point of view

Before exploiting $\text{(PC2)}$ computationally, we show the connection between $\text{(PC1)}$ and $\text{(PC2)}$. In particular, we show that the feasible region of the LP-relaxation of $\text{(PC2)}$ is a projection of the feasible region of the LP-relaxation of $\text{(PC1)}$. Thus, the LP-relaxations of $\text{(PC1)}$ and $\text{(PC2)}$ give the same bound for the pCP.

Towards this end, keep in mind that we have already argued that both IPs model the pCP. We start by examining their LP-relaxations. Let $\mathcal{P}(\text{PC1})$ be the feasible region of the LP-relaxation of $\text{(PC1)}$, i.e.,

$$
\mathcal{P}(\text{PC1}) = \{(x, y, z) \in \mathbb{R}^{|I|+|J|+|J|+1} : (1b), (1c), (1d), (1e), x_{ij} \geq 0 \ \forall i \in I \ \forall j \in J, \ 0 \leq y_j \leq 1 \ \forall j \in J\}
$$

and let $\mathcal{P}(\text{PC2})$ be the feasible region of the LP-relaxation of $\text{(PC2)}$, i.e.,

$$
\mathcal{P}(\text{PC2}) = \{(y, z) \in \mathbb{R}^{|J|+1} : (3b), (3c), 0 \leq y_j \leq 1 \ \forall j \in J\}.
$$

**Theorem 1.** The feasible region $\mathcal{P}(\text{PC2})$ is the projection of $\mathcal{P}(\text{PC1})$ to the space of $(y, z)$-variables, so $\mathcal{P}(\text{PC2}) = \{(y, z) \in \mathbb{R}^{|J|+1} : \exists x \in \mathbb{R}^{|I|+|J|} : (x, y, z) \in \mathcal{P}(\text{PC1})\}.$

**Proof:** We start by showing that any point in the projection of $\mathcal{P}(\text{PC1})$ is in $\mathcal{P}(\text{PC2})$. Towards that end let $(x^*, y^*, z^*)$ be in $\mathcal{P}(\text{PC1})$. Clearly $(y^*, z^*)$ fulfills (3b) due to (1b) and $0 \leq y_j \leq 1$ for all $j \in J$ by construction, so we only have to show that it fulfills (3c).

Consider some $i \in I$ and $j \in J$. Because of (1d) for the right-hand side of (3c) clearly

$$
d_{ij} - \sum_{j' : d_{ij'} < d_{ij}} (d_{ij} - d_{ij'}) y_{j'} \leq d_{ij} - \sum_{j' : d_{ij'} < d_{ij}} (d_{ij} - d_{ij'}) x_{ij'}^* = d_{ij} \left(1 - \sum_{j' : d_{ij'} < d_{ij}} x_{ij'}^*\right) + \sum_{j' : d_{ij'} < d_{ij}} d_{ij'} x_{ij'}^*,
$$

where $d_{ij} = \min\{d_{ij} : j \text{ is open}\}$. The proof holds for all $i \in I$ and $j \in J$.
which is at most $z^*$ because of (1c). Thus, the constraints (3c) are fulfilled for any $i$ and $j$ and therefore $(y^*, z^*)$ is in $\mathcal{P}(PC2)$.

What is left to show is that any point in $\mathcal{P}(PC2)$ is also in the projection of $\mathcal{P}(PC1)$. Let $(y^*, z^*) \in \mathbb{R}^{|J|+1}$ be in $\mathcal{P}(PC2)$. We start by constructing $x^* \in \mathbb{R}^{|J|}$ and then show that $(x^*, y^*, z^*) \in \mathcal{P}(PC1)$.

For every $i \in I$, let $j_i \in \arg\min_{j \in J} \{d_{ij} : \sum_{j' : d_{ij'} < d_{ij}} y^*_{ij'} < 1 \text{ and } \sum_{j' : d_{ij'} \leq d_{ij}} y^*_{ij'} \geq 1\}$, i.e., $j_i$ is an index such that all $y^*$ strictly closer than $d_{ij_i}$ to customer $i$ have weight less than 1, and all $y^*$ with distance at most $d_{ij_i}$ to customer $i$ have weight at least 1. In other words $y^*$ reaches weight 1 at distance $d_{ij_i}$ for customer $i$. Then for every $j \in J$ we set

$$x^*_{ij} = \begin{cases} y^*_{ij} & \text{if } d_{ij} < d_{ij_i}, \\ (1 - \sum_{j' : d_{ij'} < d_{ij}} y^*_{ij'}) / \left(\sum_{j' : d_{ij'} \geq d_{ij}} y^*_{ij'}\right) & \text{if } d_{ij} = d_{ij_i}, \\ 0 & \text{if } d_{ij} > d_{ij_i}. \end{cases}$$

Next we prove that $(x^*, y^*, z^*)$ is in $\mathcal{P}(PC1)$. Clearly $0 \leq y^*_i \leq 1$ for all $j \in J$ holds and (1b) is satisfied because of (3b). Furthermore $x^*_{ij} \geq 0$ for all $i \in I$ and $j \in J$ holds by construction. Moreover

$$\sum_{j \in J} x^*_{ij} = \sum_{j : d_{ij} < d_{ij_i}} y^*_{ij} + \left(1 - \sum_{j : d_{ij} < d_{ij_i}} y^*_{ij}\right) / \left(\sum_{j : d_{ij} \geq d_{ij_i}} y^*_{ij}\right) \sum_{j : d_{ij} = d_{ij_i}} y^*_{ij} = 1,$$

so (1c) holds. In addition to that, due to the fact that $\sum_{j' : d_{ij'} \leq d_{ij}} y^*_{ij'} \geq 1$ holds, we have that $x^*_ij \leq y^*_j$, hence (1d) is fulfilled. Eventually we consider the last constraint (1e). For a fixed $i$, due to (1c) we have

$$\sum_{j \in J} d_{ij} x^*_{ij} = d_{ij_i} \left(1 - \sum_{j \in J} x^*_{ij}\right) + \sum_{j \in J} d_{ij} x^*_{ij} = d_{ij_i} - \sum_{j \in J} (d_{ij} - d_{ij_i}) x^*_{ij},$$

As a consequence of $x^*_{ij} = 0$ for $d_{ij} > d_{ij_i}$ and $x^*_{ij} = y^*_j$ for $d_{ij} < d_{ij_i}$, this is the same as

$$d_{ij_i} - \sum_{j : d_{ij} < d_{ij_i}} (d_{ij} - d_{ij_i}) x^*_{ij} = d_{ij_i} - \sum_{j : d_{ij} < d_{ij_i}} (d_{ij} - d_{ij_i}) y^*_j,$$

which is at most $z^*$ due to (3c) for $i$ and $j_i$. Hence we have shown that $\sum_{j \in J} d_{ij} x^*_{ij} \leq z^*$, so (1e) holds and therefore $(x^*, y^*, z^*)$ is in $\mathcal{P}(PC1)$.

Thus, Theorem 1 confirms that the feasible region of the LP-relaxation of (PC2) is in fact a projection of the feasible region of the LP-relaxation of (PC1). The following corollary is a simple consequence.

**Corollary 2.** The optimal objective function values of the LP-relaxations of (PC1) and (PC2) coincide. Furthermore any optimal $(y^*, z^*)$ of the LP-relaxation of (PC2) is optimal for the LP-relaxation of (PC1) and vice versa.

As a consequence of Corollary 2, the LP-relaxations of (PC1) and (PC2) give the same bound for the pCP.
2.3 Reference to Benders decomposition

We now present an alternative approach on how \(\text{PC2}\) can be obtained from \(\text{PC1}\) as Benders decomposition. Towards this end let \((\text{PC1} - \text{Rx})\) be formulation \(\text{PC1}\) with relaxed \(x\) variables, i.e., \(\text{PC1-Rx}\) is \(\text{PC1}\) without \(1f\) and with the constraint \(x_{ij} \geq 0\) for all \(i \in I\) and for all \(j \in J\).

**Observation 3.** For any feasible solution \((x, y, z)\) of \((\text{PC1-Rx})\), a binary \(\tilde{x}\) can be constructed such that \((\tilde{x}, y, z)\) is a feasible solution of \(\text{PC1}\). Note that both solutions have the same objective function value \(z^∗\).

**Proof.** Let \((x, y, z)\) be a feasible solution of \((\text{PC1-Rx})\). We construct \(\tilde{x}\) in the following way. For any \(i \in I\), we choose \(j_i \in \arg \min_{j \in J} \{d_{ij} : y_{ij} = 1\}\) and define \(x_{ij_i} = 1\) and \(\tilde{x}_{ij} = 0\) for all \(j \in J \setminus \{j_i\}\). Clearly this \(\tilde{x}\) is binary.

What is left to show is that \((\tilde{x}, y, z)\) is feasible for \(\text{PC1}\). Clearly \(1b\) and \(1g\) hold because \((x, y, z)\) is feasible for \(\text{PC1-Rx}\). Moreover it is easy to see that \(1c\), \(1d\) and \(1j\) are fulfilled by construction. What is left so show is that \(1e\) is satisfied. For any \(i \in I\) we have by construction \(\sum_{j \in J} d_{ij} \tilde{x}_{ij} = d_{ij} \tilde{x}_{ij'} = d_{ij}\) and, because \((x, y, z)\) is feasible for \(\text{PC1-Rx}\), equals \(d_{ij}, \sum_{j \in J} x_{ij} = \sum_{j \in J} d_{ij} x_{ij}\). Due to the fact that \(x_{ij} = 0\) whenever \(d_{ij} < d_{ij}\), (because in this case \(y_{ij} = 0\)), this is less or equal to \(\sum_{j \in J} d_{ij} x_{ij}\). This is less or equal to \(z\), because \((x, y, z)\) is feasible for \(\text{PC1-Rx}\). Thus \(1e\) is fulfilled and \((\tilde{x}, y, z)\) is feasible for \(\text{PC1}\).

As a consequence of Observation 3, the pCP can be solved by using formulation \(\text{PC1-Rx}\) instead of \(\text{PC1}\). Our new formulation \(\text{PC2}\) can be seen as being obtained by applying Benders decomposition to \(\text{PC1-Rx}\) to project out the \(x\)-variables. Similar to the Benders decomposition reformulation for the uncapacitated facility location problem (UFL) presented in Fischetti et al. (2017), the obtained formulation i) is compact, i.e., it has a polynomial number of variables and constraints, and ii) has a combinatorial interpretation, that we have already presented. The main difference between the Benders decomposition for the UFL and our new formulation for the pCP is that in the UFL, there is a variable \(z_i\) \(\geq 0\) for every customer \(i \in I\) to measure the cost for each customer \(i\), and the variables are then summed in the objective function, while in our case we have just a single \(z\) as the pCP has the objective function to minimize the maximum distance. We use this connection for an efficient separation of inequalities \(3c\) (resp., a lifted variant of them) in our branch-and-cut algorithm.

3 Lifting the new formulation

In this section we first present a method to lift the LP-relaxation of our new IP formulation \(\text{PC2}\) in Section 3.1 which allows us to utilize a known lower bound and obtain a lower bound that is at least as strong as this lower bound by solving an LP. Then we discuss how to compute the best possible lower bound obtainable this way and give a combinatorial interpretation of it in terms of the set cover problem in Section 3.2. In Section 3.3 we show that our best possible lower bound coincides with the currently best lower bound known from the literature. Finally, we show that only some inequalities of the LP are necessary for obtaining the best possible lower bound in Section 3.4. Aspects relevant for computations are discussed in Section 4.
3.1 Exploiting lifted optimality cuts

The key ingredient to our lifting is the following lemma that involves the lifted optimality cuts (L-OPT).

**Lemma 5.** Let \( LB \geq 0 \) be a lower bound on the optimal objective function value of \((PC2)\). Then for any \( i \in I \) and \( j \in J \) the inequality

\[
z \geq \max\{LB, d_{ij}\} - \sum_{j': d_{ij'} < d_{ij}} (\max\{LB, d_{ij'}\} - \max\{LB, d_{ij''}\}) y_{j'}
\]

(L-OPT)

is valid for \((PC2)\), i.e., every feasible solution of \((PC2)\) fulfills (L-OPT).

**Proof.** Let \( i \in I \), \( j \in J \) and let \((y, z)\) be a feasible solution of \((PC2)\). Then \( y \) is a feasible solution to the considered instance of pCP with objective function value \( z \). In particular, every customer demand point \( i \in I \) has at most distance \( z \) to at least one of the locations indicated in \( y \). As \( LB \) is a lower bound on the optimal objective function value of \((PC2)\) we have \( LB \leq z \). As a consequence, \((y, z)\) is also a feasible solution of the slightly modified instance of pCP where we replace every distance \( d_{ij} \) with \( \max\{LB, d_{ij}\} \). Clearly (L-OPT) is nothing else than (3c) for this slightly modified instance, hence \((y, z)\) fulfills (L-OPT) because it is feasible for this slightly modified instance. Therefore (L-OPT) is valid. \( \Box \)

Now we can use the valid inequalities (L-OPT) to find a stronger LP-relaxation of \((PC2)\) by replacing (3c) by (L-OPT) in the following way.

**Theorem 6.** Let \( LB \geq 0 \) be a lower bound on the optimal objective function value of \((PC2)\). Then

\[
(PCLB) \quad \mathcal{L}(LB) = \min \quad z \quad \text{s.t.} \quad \sum_{j \in J} y_j = p \\
\quad z \geq d_{ij} - \sum_{j': d_{ij'} < d_{ij}} (d_{ij} - \max\{LB, d_{ij'}\}) y_{j'} \quad \forall i \in I, \forall j \in J : d_{ij} > LB
\]

(4a)

(4b)

(4c)

(4d)

(4e)

(4f)

is an LP-relaxation of \((PC2)\). In particular, \( \mathcal{L}(LB) \) is a lower bound on the optimal objective function value of \((PC2)\) with \( \mathcal{L}(LB) \geq LB \).

**Proof.** Obviously an LP-relaxation of \((PC2)\) is obtained by relaxing the binary constraints on \( y \). Due to Lemma 5, this LP-relaxation can be strengthened by adding cuts of the form (L-OPT). Let us consider some \( i \in I \) and \( j \in J \). If \( d_{ij} \leq LB \) then (L-OPT) simplifies to (4d), and if \( d_{ij} > LB \) then (L-OPT) is equivalent to (4c). Furthermore we can disregard the inequalities (3c) because they are superseded by (L-OPT). \( \Box \)

In a nutshell, Theorem 6 allows us to start with a lower bound \( LB \) on \((PC2)\) and solve the LP \((PCLB)\) with \(|J| + 1\) variables and a maximum of \(|I| \cdot |J| + 2|J| + 2\) constraints, in order to obtain a new lower bound \( \mathcal{L}(LB) \) that is as least as good as \( LB \). Our aim is to utilize Theorem 6 in the following way: We start with a lower bound \( LB \) on the optimal objective function value of \((PC2)\), for example the minimum of all distance values of \( D \), so \( LB = \min\{d : d \in D\} \), and then iteratively improve this lower bound by solving \((PCLB)\) in order to obtain the new lower bound \( \mathcal{L}(LB) \) and use this bound as \( LB \) in the next iteration. We give a formal definition of this procedure at the end of Section 3.2.
3.2 Our best possible lower bound

The aim of this section is to give a combinatorial interpretation of the best possible lower bound that we can reach with this procedure, i.e., to consider its convergence. Furthermore, we detail how to compute this best possible lower bound. To this end we start with the following lemmata.

**Lemma 7.** Let \( LB \geq 0 \) be a lower bound on the optimal objective function value of (PC2) such that \( \mathcal{L}(LB) = LB \) and let \((y^*, z^* = LB)\) be an optimal solution of (PCLB). Then

\[
\sum_{j : d_{ij} \leq LB} y_j^* \geq 1
\]

holds for all \( i \in I \).

**Proof.** For any \( i \in I \) let \( j_i \in \arg \min_{j \in J} \{d_{ij} : d_{ij} > LB\} \), i.e., \( j_i \) is the closest facility location to the customer demand point \( i \) that has distance greater than \( LB \). Then (4c) for \( i \) and \( j = j_i \) yields that

\[
LB \geq d_{ij_i} - \sum_{j' : d_{ij'} < d_{ij_i}} (d_{ij_i} - \max\{LB, d_{ij'}\}) y_{j_i}^*
\]

holds. By the choice of \( j_i \) we have \( \max\{LB, d_{ij'}\} = LB \) for all \( j' \), so this implies

\[
(d_{ij_i} - LB) \sum_{j' : d_{ij'} < d_{ij_i}} y_{j_i}^* \geq d_{ij_i} - LB
\]

and together with the fact that \( d_{ij'} < d_{ij_i} \) is equivalent to \( d_{ij'} \leq LB \) for all \( j' \) this yields the desired result

\[
\sum_{j' : d_{ij'} \leq LB} y_{j_i}^* \geq 1.
\]

**Lemma 8.** Let \( LB \geq 0 \) be a lower bound on the optimal objective function value of (PC2), \( y \) such that \( 0 \leq y \leq 1 \) and \( z = LB \). If

\[
\sum_{j' : d_{ij'} \leq LB} y_{j_i}^* \geq 1 \tag{5}
\]

holds for some \( i \in I \), then the inequalities (4c) hold for \((y, z)\) for this \( i \) and all \( j \in J \).

**Proof.** Let \( i \in I \) such that (5) holds. Consider an arbitrary but fixed \( j \in J \) with \( d_{ij} > LB \). Then clearly

\[
d_{ij} - \sum_{j' : d_{ij'} < d_{ij}} (d_{ij} - \max\{LB, d_{ij'}\}) y_{j_i} \leq d_{ij} - \sum_{j' : d_{ij'} \leq LB} (d_{ij} - \max\{LB, d_{ij'}\}) y_{j_i'}
\]

\[
= d_{ij} - (d_{ij} - LB) \sum_{j' : d_{ij'} \leq LB} y_{j_i} \leq d_{ij} - (d_{ij} - LB) = LB = z
\]

holds, and hence (4c) is satisfied for \((y, z)\).

Lemma 7 and Lemma 8 are the key ingredients of the following theorem.

**Theorem 9.** Let \( LB \geq 0 \) be a lower bound on the optimal objective function value of (PC2). Then \( \mathcal{L}(LB) = LB \) holds if and only if there is a fractional set cover solution with radius \( LB \) that uses at most \( p \) sets, i.e., if and only if there is a feasible solution \( y^* \) for

\[
\min \sum_{j \in J} y_j \tag{6a}
\]

s.t.

\[
\sum_{j : d_{ij} \leq LB} y_j \geq 1 \quad \forall i \in I \tag{6b}
\]

\[
0 \leq y_j \leq 1 \quad \forall j \in J \tag{6c}
\]
Note that an infinite series of increases without convergence is not possible since in the procedure LB in an iteration anymore, we set

\[ LB = \text{min}\{d_k \in D : d_k \geq \mathcal{L}(LB)\} \]

Then we iteratively improve this lower bound by first solving (PCLB) in order to obtain the new lower bound \( p \) that uses at most \( p \) sets.

By Theorem 9 there is a fractional set cover solution with radius \( LB \), and all other constraints of (PCLB) are still satisfied. Therefore \( \mathcal{L}(LB) \leq z^* = LB \) holds because \( (y^*, z^*) \) is feasible for (PCLB). As a result \( \mathcal{L}(LB) = LB \), because \( \mathcal{L}(LB) \geq LB \) holds by Theorem 9.

Theorem 9 implies that if there is a fractional set cover solution with radius \( LB \) that uses at most \( p \) sets, then the new bound we obtain by solving (PCLB) will not be better than \( LB \). Also the following corollary is an easy consequence of Theorem 9.

**Corollary 10.** Let \( LB \geq 0 \) be a lower bound on the optimal objective function value of (PC2) with \( \mathcal{L}(LB) = LB \). Then \( \mathcal{L}(LB') = LB' \) holds for all \( LB' \geq LB \).

**Proof.** By Theorem 9 there is a fractional set cover solution with radius \( LB \) that uses at most \( p \) sets for (PC2). Then for any \( LB' \geq LB \) it follows that there is also a fractional set cover solution with radius \( LB' \) that uses at most \( p \) sets for (PC2), so by Theorem 9 it holds that \( \mathcal{L}(LB') = LB' \).

Corollary 10 implies that whenever a lower bound \( LB \) does not yield a better bound by solving (PCLB), this is also true for all values larger than \( LB \). This gives rise to the following definition.

**Definition 11.** Let \( LB^# = \text{min}\{LB \in \mathbb{R} : \mathcal{L}(LB) = LB\} \).

Due to Corollary 10, \( LB^# \) is not only the smallest lower bound, for which solving (PCLB) does not improve the lower bound, but \( LB^# \) is also the supremum over all lower bounds, for which solving (PCLB) improves the bound. Next we reduce the possible values for \( LB^# \).

**Lemma 12.** It holds that \( LB^# \in D \).

**Proof.** It is easy to see that for any \( k \in \{1, \ldots, K - 1\} \) the constraint (6b) is the same for all values of \( LB \) with \( d_k \leq LB < d_{k+1} \), because whether a variable \( y_j \) is present in the sum only depends on whether its distance is less or equal to \( LB \), and \( d_k \) and \( d_{k+1} \) are two consecutive distances of all distances. As a consequence, the optimal objective function value of (PCLB) is the same for all \( LB \) with \( d_k \leq LB < d_{k+1} \). This together with Theorem 9 implies that whether \( \mathcal{L}(LB) = LB \) holds is the same for all \( LB \) with \( d_k \leq LB < d_{k+1} \). As \( LB^# \) is defined as the minimum value of \( LB \) such that \( \mathcal{L}(LB) = LB \), clearly \( LB^# \) coincides with \( d_k \) for some \( k \in \{1, \ldots, K - 1\} \) or with \( d_K \), therefore \( LB^# \in D \) holds.

In order to compute \( LB^# \) we can use the following approach: We start with a lower bound \( LB \) on the optimal objective function value of (PC2), for example the minimum of all distance values in \( D \), so \( LB = d_1 \). Then we iteratively improve this lower bound by first solving (PCLB) in order to obtain the new lower bound \( \mathcal{L}(LB) \), and in a second step we set \( LB = \text{min}\{d_k \in D : d_k \geq \mathcal{L}(LB)\} \). As soon as we do not improve \( LB \) in an iteration anymore, we set \( LB^# = LB \).

It is straightforward to see that the output \( LB^# \) of this procedure gives \( LB^# \) as defined in Definition 11. Note that an infinite series of increases without convergence is not possible since in the procedure \( LB \) only takes values of \( D \) and there are at most \(|I| \cdot |J| \) values in \( D \).

### 3.3 Comparison of our best possible lower bound with the best lower bound from the literature

Next we want to compare our best bound \( LB^# \) to the currently best lower bound known from the literature \( LB^* \), which was introduced by Elloumi et al. (2004). They obtain \( LB^* \) by relaxing the integrality...
of the variables \(y_j\) while keeping the integrality of the variables \(u_k\) of their IP formulation (PCE), so \(LB^*\) is the optimal objective function value of the following MIP.

\[
\text{(PCE – R)} \quad LB^* = \min \; (2a)
\]

\[
\text{s.t. } \; (2b) - (2c) \\
0 \leq y_j \leq 1 \quad \forall j \in J
\]

It turns out that our best bound \(LB^\#\) and the best known bound from the literature \(LB^*\) coincide.

**Theorem 13.** \(LB^\# = LB^*\)

**Proof.** First, we show that \(LB^\# \geq LB^*\) holds. Clearly \(L(LB^\#) = LB^\#\) holds, by Theorem 9 there is a fractional set cover solution \(y^*\) with radius \(LB^*\) that uses at most \(p\) sets. Furthermore by Lemma 12 we have that \(LB^\# \in D\), so there is a \(k^*\) such that \(LB^\# = d_{k^*}\). We set \(u_k^* = 1\) for all \(k \leq k^*\) and \(u_k^* = 0\) for all \(k > k^*\). It is easy to see that \((u^*, y^*)\) is a feasible solution of (PCE-R), whose objective function value equals \(d_{k^*} = LB^\#,\) thus \(LB^\# \geq LB^*\) holds.

Next, we show that \(LB^* \geq LB^\#\) holds. Towards this end let \((u^\circ, y^\circ)\) be an optimal solution of (PCE-R). Elloumi et al. [2004] already observed that due to the structure of the problem there is an index \(k^\circ \in \{2, \ldots, K\}\) such that \(u_k^\circ = 1\) for all \(k \leq k^\circ\) and \(u_k^\circ = 0\) for all \(k > k^\circ\). Then constraint (2d) for \(k = k^\circ + 1\) implies that \(\sum_{j: d_{ij} < d_{k^\circ} + 1} y_j^\circ \geq 1\) holds \(\forall i \in I\), and hence also \(\sum_{j: d_{ij} \leq d_{k^\circ}} y_j^\circ \geq 1\) holds \(\forall i \in I\).

As a consequence, \(y^\circ\) is a fractional set cover solution with radius \(d_{k^\circ}\) that uses at most \(p\) sets. Furthermore \(d_{k^\circ}\) coincides with \(LB^*\), because \(LB^*\) is the optimal objective function value of (PCE-R) and the optimal solution \((u^\circ, y^\circ)\) has objective function value \(d_{k^\circ}\). Thus, \(y^\circ\) is a fractional set cover solution with radius \(LB^*\) that uses at most \(p\) sets, so by Theorem 9 it holds that \(L(LB^*) = LB^*\), therefore \(LB^* \geq LB^\#\).

### 3.4 Inequalities needed for obtaining our best possible lower bound

Now we want to draw our attention to the question of whether all inequalities (4e) are necessary for the convergence result. In fact, the following holds.

**Theorem 14.** Let \(LB \geq 0\) be a lower bound on the optimal objective function value of (PC2). For any \(i \in I\) let \(j_i \in \arg\min_{j \in J} \{d_{ij} : d_{ij} > LB\}\), i.e., \(j_i\) is the closest facility location to the customer demand point \(i\) that has distance more than \(LB\). Then

\[
L'(LB) = \min \; (4a)
\]

\[
\text{s.t. } \; (4b), (4c) - (4d) \\
z \geq d_{ij_i} - \sum_{j' : d_{ij'} < d_{ij_i}} (d_{ij'} - \max\{LB, d_{ij'}\}) y_{ij'} \quad \forall i \in I
\]

(7)

is an LP-relaxation of (PC2). In particular, \(L'(LB)\) gives a lower bound on the optimal objective function value of (PC2) with \(L(LB) \geq L'(LB) \geq LB\). Furthermore \(L'(LB) = LB\) holds if and only if there is a fractional set cover solution with radius \(LB\) that uses at most \(p\) sets.

**Proof.** Clearly \(L'(LB)\) is obtained from \(L(LB)\) by removing some inequalities, so \(L(LB) \geq L'(LB)\) holds. Analogous arguments as the ones in the proof of Theorem 6 prove that \(L'(LB)\) is a lower bound on the optimal objective function value of (PC2). The proof of the remaining statements can be done analogously to the proof of Theorem 9 i.e., with the help of analogous results as Lemma 7 and Lemma 8.

In other words Theorem 14 implies that including only one of the inequalities (4e) for every customer demand point \(i\), namely (7) for the closest facility location to \(i\) that has distance more than \(LB\), is enough to ensure convergence to a fractional set cover solution \(LB^\#\). This reduces the number of needed constraints in \(L'(LB)\) to \(|I| + 2|J| + 2\) instead of potentially \(|I| \cdot |J| + 2|J| + 2\) constraints needed in \(L(LB)\). However, it can be beneficial to include (4e) for a customer demand point \(i\) for several locations \(j\) in order to improve the speed of convergence.

After deriving theoretical properties of our best bound \(LB^\#\) and showing that it is as good as the best bound from the literature, we next exploit \(LB^\#\) computationally.
We now present our solution algorithm to solve the pCP based on formulation (PC2), where we replace (3c) with the lifted optimality cuts \((L\text{-OPT})\). In a nutshell, in our branch-and-cut algorithm we initialize the model with only a subset of inequalities \((L\text{-OPT})\) (i.e., only for a subset of customers \(i \in I\) and potential facility locations \(j \in J\)) for a given valid lower bound \(LB\). We then iteratively separate inequalities \((L\text{-OPT})\) while also updating the current lower bound \(LB\) used in defining \((L\text{-OPT})\) by exploiting the objective value of the current LP-relaxation. In the update of \(LB\) we also exploit that in instances from the literature the distances are integral.

We have implemented different strategies for doing the overall separation scheme (e.g., which inequalities to add, ...), these strategies are detailed in Section 4.2. Before we discuss these strategies, we describe how a violated inequality \((L\text{-OPT})\) can be separated efficiently in Section 4.1. Finally, Section 4.3 describes a primal heuristic which is called during the B&C and driven by the optimal solution of the current LP-relaxation.

Our solution algorithm was implemented in C++ and it is available online under https://msinnl.github.io/pages/instancescodes.html IBM ILOG CPLEX 12.10 with default settings was used as B&C framework. We apply the described separation schemes in the UserCutCallback of CPLEX, which gets called for fractional optimal solutions of the LP-relaxations within the B&C tree. In the LazyConstraintCallback, which gets called for optimal integer solutions, we simply add a single violated inequality \((L\text{-OPT})\) if any exits. This is done, as most of the integer solutions encountered are produced by internal CPLEX heuristics (and are not just integral LP-relaxations) and are very different to the current optimal solution of the LP-relaxations. As a consequence, adding multiple inequalities \((L\text{-OPT})\) for such solutions is usually not useful to help to improve the optimal objective value the LP-relaxation.

## 4 Implementation details

### 4.1 Separation of a violated inequality \((L\text{-OPT})\)

Let \((y^*, z^*)\) be a (fractional) optimal solution of the LP-relaxation at a B&C node (the LP-relaxation of (PC2), where we replace (3c) with the lifted optimality cuts \((L\text{-OPT})\) for a subset of all customers \(i \in I\) and potential facility locations \(j \in J\), \(LB\) a given lower bound on the objective value and \(i \in I\) a given customer. In order to obtain an efficient separation procedure, we can leverage that the cuts for a fixed customer are similar to the Benders cuts for the uncapacitated facility location problem as discussed in Section 2.3. Thus we can use an adapted separation procedure of the one presented in Fischetti et al. (2017) to find the most violated inequality, if there is a violated inequality for the customer \(i \in I\) and the lower bound \(LB\).

The procedure works as follows: Let \(d'_{ij} = d_{ij}\) if \(d_{ij} > LB\) and \(d'_{ij} = LB\) otherwise. Sort the facilities \(j \in J\) in ascending order according to \(d'_{ij}\). Note that we only need to consider facilities with \(y_j^* > 0\) for this sorting as the other facilities do not contribute to the potential violation. In the following, we assume that the locations are ordered in this way, i.e., we have \(d'_{i1} \leq \ldots \leq d'_{ij}\). Let the critical location \(j_i\) be the index such that \(\sum_{j=1}^{j_i-1} y_j^* < 1 \leq \sum_{j=1}^{j_i} y_j^*\). The maximum violation (i.e., the largest right-hand side value for the current \(y^*\)) is then given by

\[
\sum_{j: y_j^* = 1 \text{ and } d_{ij} < d_{ij}} (d_{ij} - \max\{LB, d_{ij}\}) y_j^*
\]

and the inequality \((L\text{-OPT})\) with maximum violation is

\[
z \geq \sum_{j: d_{ij} < d_{ij}} (d_{ij} - \max\{LB, d_{ij}\}) y_j.
\]

For more details on why this procedure gives the largest right-hand side value we refer to Fischetti et al. (2017).

We note that this procedure allows us to calculate \(j_i\) and hence the maximal violation for each customer \(i\) in an efficient way without the need of knowing all the distances \(d_{ij}\). As mentioned above, we only need to know the distances \(d_{ij}\) for each \(j \in J\) with \(y_j^* > 0\) in order to determine \(j_i\), and the number of such \(j\) is
usually rather small compared to $|J|$. This is important for large-scale instances, where the distance matrix cannot be stored in memory and the distances need to be computed on-the-fly when needed. Note however, that for adding such an inequality to the model for a customer $i$, all the distances from $i$ to any $j$ need to be known.

### 4.2 Details of the overall separation schemes

We have implemented two different separation schemes, which differ in the way the violated inequalities that are added at a separation round are selected. Note that due to the size of the encountered instances, always adding all the violated inequalities becomes prohibitive. Moreover, previously added violated inequalities (L-OPT) can also become redundant, because during the course of the algorithm, we iteratively obtain larger values for $LB$, and thus new inequalities (L-OPT) which are strictly stronger than previously added ones can be derived. Moreover, as detailed in Section 4.1, the maximal violation of any cut of the form (L-OPT) can be calculated efficiently for each customer, while for adding an inequality, all the distances need to be known, which is computationally much more costly for large-scale instances. Hence, a carefully engineered separation scheme is needed in order to obtain good performance.

Our two schemes are denoted by maxViolated and fixedCustomer. In both schemes, we use the option purgeable of CPLEX when adding a violated inequality in the root-node of the B&C tree. This setting allows CPLEX to automatically remove a previously violated inequality if CPLEX decides (by an internal mechanism) that this inequality is “useless” for the subsequent solution process. To add a violated inequality at any other node of the B&C tree, we use the method addLocal of CPLEX, which adds the inequality not in a global fashion, but only for the current subtree of the B&C tree. This allows us to use the lower bound of the subtree as $LB$ within the inequalities (L-OPT) instead of using the global lower bound.

**Separation scheme maxViolated** In this scheme, we simply add the most violated inequality (L-OPT) for the maxNumCutsRoot (maxNumCutsTree) customers that have the largest maximum violation at each separation round. We use at most maxNumSepRoot (maxNumSepTree) separation rounds. If the lower bound value does not improve more than $\varepsilon = 1e^{-5}$ for maxNoImprovements separation rounds at a node in the B&C tree (including the root-node), we stop the separation at this node. In this scheme, we do not add any inequalities (L-OPT) at initialization, and set $LB$ to zero (the best deducible lower bound without additional computational effort) in the beginning. Algorithm 1 summarizes the separation scheme which is implemented in the UserCutCallback.

**Separation scheme fixedCustomer** In this scheme, we restrict the separation to a subset $\hat{\mathcal{I}} \subseteq \mathcal{I}$ of the customers, which we then iteratively grow during the course of the algorithm. The idea behind this separation scheme is that for solving the problem to optimality, it can be enough to focus on a subset of the customers due to the min-max structure of the objective function. This is also used in the set-cover-based approaches Chen and Chen (2009); Contardo et al. (2019), where the set cover problem is solved (to optimality) on only a subset of the customers, and then it is checked, if any of the not-considered customers would change the objective function value. If yes, the set cover problem is then adapted with new customers and resolved. Furthermore, note that in this scheme, we sometimes use that $\hat{\mathcal{I}} = \mathcal{J}$ as is customary in the instances from the literature. We mention whenever we use that condition and also detail how to proceed if $\hat{\mathcal{I}} \neq \mathcal{J}$.

Our separation scheme tries to follow a similar avenue within our B&C framework. In particular, we add the most violated inequality (L-OPT) for all customers in $\hat{\mathcal{I}} \subseteq \mathcal{I}$ in each separation round and use at most maxNumSepRoot (maxNumSepTree) separation rounds. If the lower bound value does not improve more than $\varepsilon = 1e^{-5}$ for maxNoImprovements separation rounds at a node in the B&C tree (including the root-node), we stop the separation at this node.

The procedure to initialize the set $\hat{\mathcal{I}}$ is given as follows. We select a sample $\hat{\mathcal{I}}$ of $p + 1$ customers using the following greedy algorithm: We randomly choose an initial customer $i \in \hat{\mathcal{I}}$ and add it to the empty set $\hat{\mathcal{I}}$.

\footnote{Unfortunately, CPLEX does not allow to use the purgeable option when using addLocal.}
We then iteratively grow \( \hat{I} \) by adding the customer \( i \in I \setminus \hat{I} \) which has the maximum distance to its closest customer in \( \hat{I} \). This is done until \(|\hat{I}| = p + 1 \).

This procedure to initialize the set \( \hat{I} \) exploits the assumption that \( I = J \) by using the distances between pairs of customers, which are not necessary available if \( I \neq J \). In the case that \( I \neq J \) and these distances are available, this procedure can be used in the same way. If \( I \neq J \) and these distances are not available, then one can iteratively add to \( \hat{I} \) the customer \( i \in I \setminus \hat{I} \) which has the maximum value of \( \min_{i \in I} \min_{j \in J} \{d_{ij} + d_{ij}'\} \), or \(|\hat{I}| \) can be initialized with \( p + 1 \) random customers.

As initial value for \( LB \) we use \( \min_{i \in I, j \neq I} \{d_{ij}\} \). This is a valid lower bound, since initially \(|\hat{I}| = p + 1 \). Thus at least one customer \( i \in \hat{I} \) cannot be chosen as open facility (which is in any case only possible if \( i \in J \)) in a feasible solution, and so its minimum distance must be the one to another location \( j \neq i \) with \( j \in J \). Note that in the case \( I \neq J \) the condition \( j \neq i \) can become redundant. Furthermore observe that as in the scheme \textit{maxViolated}, we initialize \( LB \) with the best deducible lower bound without additional computational effort.

The customers to grow \( \hat{I} \) are selected as follows: At each separation iteration, take the customer \( i \in I \setminus \hat{I} \) which induces the most violated inequality (L-OPT) (if any exist) and add it to \( \hat{I} \). Moreover, if there are
more than \texttt{maxNoImprovementsFixed} consecutive iterations, in which the lower bound did not improve, we increase \( \hat{I} \) more aggressively by adding more customers. The selection of these additional customers is again driven by maximum violation of \( \text{[L-OPT]} \), but in order to find a diverse set of customers does not consider all \( i \in I \setminus \hat{I} \) as candidates, but a subset \( \bar{I} \subseteq I \setminus \hat{I} \). To build \( \bar{I} \), initially we set \( \bar{I} = I \setminus \hat{I} \). Whenever \( d_{ij} \leq LB \) holds for some \( j \in J \) for a customer \( i \) we just added to \( \hat{I} \), we remove \( j \) from \( \bar{I} \). Note that in this case \( j \) occurs in the maximum violated inequality \( \text{[L-OPT]} \) for customer \( i \). The idea behind this is that if \( i \) and \( j \) are close to each other, then the already added inequality for customer \( i \) may be quite similar to the inequality we would add for \( j \), thus we prevent adding the inequality for \( j \).

Note that for constructing \( \bar{I} \) we exploit that \( I = J \), again by using the distances between pairs of customers. If these are available for \( I \neq J \), our approach can still be used, only \( j \) does not appear in the maximum violated inequality \( \text{[L-OPT]} \) for customer \( i \) in this case. If \( I \neq J \) and these distances are not available, then one could either delete all customers \( \hat{i} \) for which there is a location \( \hat{j} \) such that \( d_{\hat{i}\hat{j}} + d_{i\hat{j}} \leq LB \), or omit the deletion of customers of \( \bar{I} \) completely.

Algorithm 2, Part 1 (continued in Algorithm 2, Part 2) summarizes the separation scheme which is implemented in the \texttt{UserCutCallback}.

4.3 Primal heuristic

In order to obtain primal solutions, we have implemented a greedy heuristic in the \texttt{HeuristicCallback} of CPLEX. It uses the optimal solution \((y^*, z^*)\) of the current LP-relaxation at a B&C node. Let \( S^H \) be the indices of the facilities to open in the solution produced by our heuristic. To construct \( S^H \), we sort all the locations with \( y_j^* > 0 \) in descending order and then iterate through the sorted list. Whenever adding a location \( j \) from the list to \( S^H \) would improve the objective function value of the heuristic solution (which initially is set to \( \infty \)), we add \( j \) to \( S^H \). This is done until \( |S^H| = p \). If needed, we make multiple passes through the list.

5 Computational results

The runs were made on a single core of an Intel Xeon E5-2670v2 machine with 2.5 GHz and 32GB of RAM, and all CPLEX settings were left on their default values. The timelimit for the runs was set to 1800 seconds. The following parameter values were used for our algorithm, they were determined in preliminary computations:

- \( \text{maxNumCutsRoot}: 100 \)
- \( \text{maxNumCutsTree}: 50 \)
- \( \text{maxNumSepRoot}: 1000 \)
- \( \text{maxNumSepTree}: 1 \)
- \( \text{maxNoImprovements}: 100 \)
- \( \text{maxNoImprovementsFixed}: 5 \)

5.1 Instances

We used two sets of instances from the literature to evaluate the performance of our algorithm, these sets are denoted by \texttt{pmedian} and \texttt{TSPLIB} and details are given below.

- \texttt{pmedian}: This is an instance set used in Çalık and Tansel \( (2013) \); Chen and Chen \( (2009) \); Contardo et al. \( (2019) \). The set contains 40 instances. All instances have all customer demand points as potential facility locations, so \( I = J = V \) holds. The number of customer demand points and hence also the number of potential facility locations \( |I| = |J| = |V| \) is between 100 and 900, and \( p \) is between 5 and 200. For the concrete values of \( |V| \) and \( p \) for each instance see Table 1.

- \texttt{TSPLIB}: This set contains larger instances and is based on the TSP-library \( (Reinelt\,1991) \). The number of customer demand points and hence also the number of potential facility locations \( |I| = |J| = |V| \) is between 1621 and 744710, for the concrete values of \( |V| \) see the tables in the appendix. The complete set
Algorithm 2: (Part 1 of 2) Separation scheme fixedCustomer

Input: instance \((I = J, d, p)\) of the pCP, LP-relaxation \((y^*, z^*)\) of current B&C node, ID nodeID of current B&C node, ID prevNodeID of previous B&C node, counter nodeIterations, counter boundNotImprovedCount, counter boundNotImprovedCountFixed, LP-relaxation bound \(z^*\) of the previous iteration, current subset of customers \(\hat{I}\)

1. if nodeID = prevNodeID then
2.   if \(z^* - z^*_\text{prev} < \varepsilon\) then
3.     boundNotImprovedCount ← boundNotImprovedCount + 1
4.     boundNotImprovedCountFixed ← boundNotImprovedCountFixed + 1
5.     if boundNotImprovedCount = maxNoImprovements then
6.       return
7. else
8.   boundNotImprovedCount ← 0
9.   boundNotImprovedCountFixed ← 0
10.  nodeIterations ← 0
11.  maxNumSep ← maxNumSepRoot
12.  if nodeID ≠ rootNode then
13.    maxNumSep ← maxNumSepTree
14.  if nodeIterations > maxNumSep then
15.    return
16.  \(z^*_\text{prev} ← z^*\)
17.  \(LB ← \lceil z^* \rceil\)
18.  customerAndViolation ← ∅
19.  for \(i ∈ \hat{I}\) do
20.    calculate maximal violation of inequalities \((L-OPT)\) for customer \(i\) and given lower bound \(LB\) as described in Section 4.1
21.    if violation > 0 then
22.      customerAndViolation ← customerAndViolation ∪ \{(i, violation)\}
23.  sort customerAndViolation in descending order according to violation
24.  for \(\ell ∈ \{1, \ldots, \text{size(customerAndViolation)}\}\) do
25.    \(i ← \text{customer from customerAndViolation[\ell]}\)
26.    if nodeID = rootNode then
27.      add the maximal violated inequality \((L-OPT)\) for customer \(i\) and lower bound \(LB\) with the option purgeable
28.    else
29.      add the maximal violated inequality \((L-OPT)\) for customer \(i\) and lower bound \(LB\) as locally valid cut with using addLocal

continued in Algorithm 2 Part 2 of 2

of 44 instances was used in Contardo et al. (2019). In other works (Elloumi et al., 2004; Chen and Chen, 2009; Çalık and Tansel, 2013) used only two or three (smaller) instances (with up to 3038 vertices) to test the scalability of the respective algorithms\(^2\)\(^\text{2}\). In Contardo et al. (2019) the instances were used with \(p = 2, 3, 5, 10, 15, 20, 25, 30\), while in the other works also larger values of \(p\) were considered. In the instances all customer demand points are given as two-dimensional coordinates, and the Euclidean distance rounded down to the nearest integer is used as distance. This follows previous literature.

5.2 Results

We first give a comparison of our approaches with the lifted inequalities \((L-OPT)\) with a B&C using our new formulation \((PC2)\) without the lifting on instance set \(pmedian\). Then we describe the results obtained

\(^2\)\(^\text{2}\)In Chen and Chen (2009) also other smaller TSPlib instances with less than 1000 vertices were used, but they are not included in the set used in Contardo et al. (2019).
Algorithm 2: (Part 2 of 2) Separation scheme fixedCustomer

continued from Algorithm 2, Part 1 of 2

30 customerAndViolationOutside ← ∅

31 for \( i \in I \setminus \hat{I} \) do

32 calculate maximal violation of inequalities \([L-OPT]\) for customer \( i \) and given lower bound \( LB \) as described in Section 4.1

33 if violation > 0 then

34 customerAndViolationOutside ← customerAndViolationOutside \( \cup \{ (i, \text{violation}) \} \)

35 sort customerAndViolationOutside in descending order according to the violation

36 if boundNotImprovedCountFixed < maxNoImprovementsFixed then

37 if size(customerAndViolationOutside) ≥ 1 then

38 mostViolatedCustomer ← customer from customerAndViolationOutside[1]

39 if nodeID = rootNode then

40 add the maximal violated inequality \([L-OPT]\) for customer mostViolatedCustomer and lower bound \( LB \) with the option purgeable

41 else

42 add the maximal violated inequality \([L-OPT]\) for customer mostViolatedCustomer and lower bound \( LB \) as locally valid cut with using addLocal

43 else

44 boundNotImprovedCountFixed ← 0

45 \( \hat{I} ← I \setminus \hat{I} \)

46 for \( \ell \in \{1, \ldots, \text{size(customerAndViolationOutside)}\} \) do

47 \( i ← \) customer from customerAndViolationOutside[\( \ell \) ]

48 if \( i \in \hat{I} \) then

49 \( \hat{I} ← \hat{I} \cup \{ i \} \)

50 if nodeID = rootNode then

51 add the maximal violated inequality \([L-OPT]\) for customer \( i \) and lower bound \( LB \) with the option purgeable

52 else

53 add the maximal violated inequality \([L-OPT]\) for customer \( i \) and lower bound \( LB \) as locally valid cut with using addLocal

54 for \( j \) with \( d_{ij} \leq LB \) (this implies that \( j \) occurs in the maximal violated inequality \([L-OPT]\) for customer \( i \) and lower bound \( LB \) ) do

55 \( \hat{I} ← \hat{I} \setminus \{ j \} \)

by using our approaches with the lifted inequalities \([L-OPT]\) for the larger instance set TSPLIB.

Results for the instance set pmedian In Table I we compare the following six different configurations:

- mVnoL: separation scheme maxViolated and inequalities \((3c)\)
- fCnoL: separation scheme fixedCustomer and inequalities \((3c)\)
- mVL: separation scheme maxViolated and lifted inequalities \([L-OPT]\)
- fCL: separation scheme fixedCustomer and lifted inequalities \([L-OPT]\)
- mVHL: configuration mVL and the primal heuristic described in Section 4.3
- fCLH: configuration fCL and the primal heuristic described in Section 4.3

In Table I we see the huge effect of the lifted inequalities \([L-OPT]\). We report the runtime in seconds \((t[s])\), the optimality gap \((g[\%])\), which is computed as \( \frac{UB - LB}{UB} \cdot 100 \), where \( LB \) is the obtained lower bound and \( UB \) is the obtained upper bound and the number of B&C nodes \((\#BC)\). Without lifting, only 11 of the 40 instances are solvable within the timelimit of 1800 seconds, while with lifting all the instances can be
solved to optimality within four seconds runtime. Thus, our lifting has an immense positive effect. For these instances, the primal heuristic has no discernible effect, as they already become solvable very fast when using the lifting. Also the choice of the separation scheme (maxViolated or fixedCustomer) does not influence the quality of the results a lot. We also note that most of the instances can be solved within the root-node of the B&C algorithm when using our lifting.

Results for the instance set TSPlib  In Figures 1-2, we show plots of the runtime and optimality gap aggregated by $p$ for configurations mVL, fCL, mVLH and fCLH, i.e., the by far best configurations for the instance set pmedian that contains smaller instances. We used the same values for $p$ as Contardo et al. (2019). Detailed results for the setting fCLH can be found in the appendix in the Tables 2-9 were we also include the results obtained by Contardo et al. (2019). From the figures, we see that the instances are becoming harder to solve when $p$ becomes larger, which is in tune with the computational experiments of Contardo et al. (2019). We note that for the settings mVL and mVLH the code did not terminate for all instances, as the available memory was not enough. In general, the settings fCL and fCLH perform better than mVL and mVLH. An explanation for this could be that by focusing the separation of inequalities (L-OPT) on the subset $\hat{I}$, the separation procedure becomes more stable, which could allow CPLEX to easier decide which inequalities to remove for further iterations. Hence, the LP-relaxations becomes smaller and easier to solve, and also not so memory-intensive.

To further illustrate the different behavior of both separation schemes, Figure 3 shows the obtained lower bound values at the root node of the B&C tree against the runtime for settings mVLH and fCLH for the TSPlib instance rl11849 for $p = 5$. While both settings manage to solve this instance to optimality in the root node, we see that fCLH converges much faster. The behavior of the lower bound in this instance is a typical behavior we observed for both strategies.

Figures 4a and 4b show the number of added inequalities (L-OPT) for all TSPlib instances with $p = 2$ and $p = 5$ which could be solved by all four considered settings. Theses figures further support that with the scheme fixedCustomer much less inequalities need to be added. We also see that for larger $p$ much more inequalities need to be added. For the scheme fixedCustomer the maximum number of added inequalities for $p = 2$ is slightly over 500, while for $p = 5$ it is around 10000. For the scheme maxViolated this number grows from around 4000 for $p = 2$ to around 80000 for $p = 5$. We note that the number of added inequalities is not necessarily the same as the number of inequalities which are in the final LP-relaxations, as the CPLEX cut management may delete added violated inequalities during the course of the B&C.

From the Figures 1-2 we also see that there is some effect of using the primal heuristic, in particular regarding the optimality gap. This is consistent when comparing our obtained lower and upper bound values with the values obtained by Contardo et al. (2019), i.e., from the comparison we see that our approach provides good lower bounds (sometimes even improving on the lower bound of Contardo et al. (2019), which they obtained within a timelimit of 24 hours), while the upper bounds for some instances could still be improved. This again shows the important contribution of our lifting procedure.

From the results in Tables 2-9 we can also see that in some instances for larger $p$ our approach scales worse compared to the approach of Contardo et al. (2019). We believe that this could be due to the combination of the following factors: i) our method is a “classical” B&C approach, thus the strength of the LP-relaxation is of course crucial. For larger $p$ the feasible region of the LP-relaxation is getting larger, and thus the LP-relaxation may also become weaker; ii) state-of-the-art methods such as Contardo et al. (2019) work by iteratively resolving set cover problems. For these problems, the used MIP-solvers may be able to use additional preprocessing routines, general-purpose cuts or other speed-up tricks, which are nowadays built in such solvers; iii) Contardo et al. (2019) also uses a specific preprocessing which is applied before each set cover problem. The procedure is not described, thus it is also clear, if such a procedure could be directly transferable into our B&C approach; iv) in some cases the obtained primal bound is the bottleneck for our algorithm, even though we already implemented a primal heuristic.
| Name | mVLD | nDelta | #BC | tCoL | #BC | mLH | #BC | #BC | #BC | #BC |
|------|------|--------|-----|------|-----|-----|-----|-----|-----|-----|
| pmld1 | 100  | 5     | 57.72 | 0.00 | 1.00 | 11.59 | 0.00 | 6.36 | 0.00 | 36.36 |
| pmld2 | 200  | 5     | 57.72 | 0.00 | 1.00 | 11.59 | 0.00 | 6.36 | 0.00 | 36.36 |
| pmld3 | 300  | 5     | 57.72 | 0.00 | 1.00 | 11.59 | 0.00 | 6.36 | 0.00 | 36.36 |
| pmld4 | 400  | 5     | 57.72 | 0.00 | 1.00 | 11.59 | 0.00 | 6.36 | 0.00 | 36.36 |
| pmld5 | 500  | 5     | 57.72 | 0.00 | 1.00 | 11.59 | 0.00 | 6.36 | 0.00 | 36.36 |
| pmld6 | 600  | 5     | 57.72 | 0.00 | 1.00 | 11.59 | 0.00 | 6.36 | 0.00 | 36.36 |
| pmld7 | 700  | 5     | 57.72 | 0.00 | 1.00 | 11.59 | 0.00 | 6.36 | 0.00 | 36.36 |
| pmld8 | 800  | 5     | 57.72 | 0.00 | 1.00 | 11.59 | 0.00 | 6.36 | 0.00 | 36.36 |
| pmld9 | 900  | 5     | 57.72 | 0.00 | 1.00 | 11.59 | 0.00 | 6.36 | 0.00 | 36.36 |
| pmld10| 1000 | 5     | 57.72 | 0.00 | 1.00 | 11.59 | 0.00 | 6.36 | 0.00 | 36.36 |

Table 1: Detailed results for the pmedian instances
Figure 1: Plots for runtime and optimality gap for $p = 2, 3, 5, 10$ for the TSPLIB instances. If #instances[k%] in the optimality gap plot does not sum up to 100%, this means that some runs did not finish due to memory issues.

6 Conclusions

In this paper, we present a novel solution approach for the the $p$-center problem, which is a fundamental problem in location science. The solution approach works on a new integer programming formulation that can be obtained by a projection from the classical formulation. The formulation is solved by means of branch-and-cut, where cuts for customer demand points are iteratively generated. This makes the method suitable for large scale instances. Moreover, the formulation can be strengthened with combinatorial information to
Figure 2: Plots for runtime and optimality gap for $p = 15, 20, 25, 30$ for the TSPLIB instances. If 
\#instances[%] in the optimality gap plot does not sum up to 100\%, this means that some runs did not 
finish due to memory issues.

obtain a much tighter LP-relaxation. In particular, we present a novel way to use lower bound information 
to obtain stronger cuts. We show that the obtained formulation has the same lower bound as the best 
known relaxations from the literature (Elloumi et al., 2004; Çalış and Tansel, 2013). In contrast to these 
bounds from the literature, which are based on semi-relaxations, i.e., some variables are kept integer in 
the relaxation, our relaxation only contains continuous variables. Moreover, our approach can be easily 
implemented in a single branch-and-cut algorithm, while existing state-of-the-art approaches for the $p$-center 
problem consist of iterative solution algorithms, where set cover problems are solved repeatedly (Chen and
In order to assess the efficiency of our proposed approach, we conducted a computational study on instances from the literature with up to more than 700000 customers and locations. The results show that for many instances, our solution algorithm is competitive with highly sophisticated solution frameworks from the literature. We describe different strategies on how to separate our lifted cuts, and also present a primal heuristic which uses the information provided by the linear programming relaxation. The solution algorithm is made available online.

Figure 3: Behavior of the lower bound at the root node for the instance rl11849 with $p = 5$ and our two separation schemes

Figure 4: Plots for the number of added inequalities (L-OPT) for all TSPLIB instances which were solved to optimality with all considered settings

We describe different strategies on how to separate our lifted cuts, and also present a primal heuristic which uses the information provided by the linear programming relaxation. The solution algorithm is made available online.

In order to assess the efficiency of our proposed approach, we conducted a computational study on instances from the literature with up to more than 700000 customers and locations. The results show that for many instances, our solution algorithm is competitive with highly sophisticated solution frameworks from the literature.

There are various avenues for further work. Further research regarding potential additional valid inequalities to strengthen the LP-relaxation could be interesting. Another direction could be the development of better primal heuristics to be used within the branch-and-cut algorithm, as for many instances our lower bound was quite good, while the primal bound still had some room for improvement. However, the large-scale nature of the considered instances may make this a challenging task, as even applying simple local search operator could become quite time consuming. A hybridization of our approach with existing set cover-based approaches could also be interesting, because the results show that for some instances our approach works better, while for other instances set-cover-based approaches work better. The investigation if preprocessing procedures from set-cover-based approaches can be transferred to our approach could also be a fruitful topic. It could also be worthwhile to investigate, if there are certain instance-characteristics which may favor a particular approach. Moreover, as our solution approach directly models the problem with optimality-type cuts, our approach could be more suitable when considering stochastic or robust versions of the pCP, compared to the set cover-based approaches, which may no be easily adaptable to such settings.
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A Detailed results for the instance set TSPlib

In the following Tables 2-9 we provide a comparison with the results obtained by Contardo et al. (2019) using an Intel Xeon E5462 with 2.8 Ghz and 16GB RAM. In their runs, the timelimit was set to 24 hours (indicated by TL2 in the tables), while we have a timelimit of 30 minutes (indicated by TL in the tables). Whenever at least one of the two versions finished, the faster runtime is bold. If both versions did not finish, the better upper and the better lower bound is bold.

Table 2: Detailed results for the TSPlib instances with \( \text{p} = 2 \)

| name               | \(|V|\) | LB  | UB  | rCLH g[\%] | #BC | t[s] | Contardo et al. (2019) |
|--------------------|-------|-----|-----|------------|-----|------|------------------------|
| rw1621             | 1621  | 714 | 714 | 0.00       | 0   | 0.11 | 714                    |
| u1s817             | 1817  | 1061| 1061| 0.00       | 0   | 0.20 | 1061                   |
| r1l889             | 1889  | 6931| 6931| 0.00       | 0   | 0.16 | 6931                   |
| miu1979            | 1979  | 3284| 3284| 0.00       | 0   | 0.09 | 3284                   |
| px2398             | 2392  | 6960| 6960| 0.00       | 0   | 0.28 | 6605                   |
| d1512-modif-2500   | 2500  | 9234| 9234| 0.00       | 0   | 0.30 | 9234                   |
| pch3038            | 3038  | 1734| 1734| 0.00       | 0   | 0.27 | 1734                   |
| nu3496             | 3496  | 2140| 2140| 0.00       | 0   | 0.21 | 2140                   |
| ca6663             | 6663  | 25781|25781|0.00       | 0   | 0.53 | 25781                  |
| rl5915             | 5915  | 7385| 7385| 0.00       | 0   | 0.52 | 7385                   |
| rl5934             | 5934  | 7004| 7004| 0.00       | 0   | 0.97 | 7004                   |
| tz6117             | 6117  | 4582| 4582| 0.00       | 0   | 0.85 | 4582                   |
| eg7146             | 7146  | 4643| 4643| 0.00       | 0   | 0.49 | 4643                   |
| pla7397            | 7397  | 310664|310664|0.00       | 0   | 0.43 | 310664                |
| ym7663             | 7663  | 3807| 3807| 0.00       | 0   | 0.25 | 3807                   |
| pm8079             | 8079  | 1813| 1813| 0.00       | 0   | 0.57 | 1813                   |
| ei8246             | 8246  | 1813| 1813| 0.00       | 0   | 0.57 | 1813                   |
| ar9152             | 9152  | 9725| 9725| 0.00       | 0   | 0.68 | 9725                   |
| ja9847             | 9847  | 10099|10099|0.00       | 0   | 0.30 | 10099                 |
| gr9882             | 9882  | 3757| 3757| 0.00       | 0   | 0.71 | 3757                   |
| k29976             | 9976  | 11325|11325|0.00       | 0   | 1.91 | 11325              |
| fi6039             | 6039  | 4922| 4922| 0.00       | 0   | 0.79 | 4922                   |
| rl11849            | 11849 | 7298| 7298| 0.00       | 0   | 1.11 | 7298                   |
| usa13509           | 13509 | 175750|175750|0.00       | 0   | 0.86 | 175750             |
| brd44051           | 44051 | 2970| 2970| 0.00       | 0   | 1.38 | 2970                   |
| moli1415           | 1415  | 3698| 3698| 0.00       | 0   | 1.27 | 3698                   |
| ho14473            | 14473 | 1834| 1834| 0.00       | 0   | 0.96 | 1834                   |
| d15112             | 15112 | 9406| 9406| 0.00       | 0   | 2.05 | 9406                  |
| iti16862           | 16862 | 4677| 4677| 0.00       | 0   | 1.01 | 4677                 |
| d18512             | 18512 | 3301| 3301| 0.00       | 0   | 2.51 | 3301                 |
| vm22775            | 22775 | 4266| 4266| 0.00       | 0   | 1.59 | 4266              |
| swz24978           | 24978 | 4960| 4960| 0.00       | 0   | 1.91 | 4960                 |
| fy82534            | 2534  | 448 | 448 | 0.00       | 0   | 3.00 | 448                  |
| bmn33708           | 33708 | 5167| 5167| 0.00       | 0   | 2.02 | 5167               |
| pla33810           | 33810 | 330462|330462|0.00       | 0   | 6.95 | 330462          |
| bby34656           | 34656 | 474 | 474 | 0.00       | 0   | 2.99 | 474                 |
| pba38478           | 38478 | 469 | 469 | 0.00       | 0   | 2.00 | 469                 |
| ch71009            | 71009 | 19988|19988|0.00       | 0   | 5.63 | 19988           |
| pla65900           | 65900 | 436008|436008|0.00       | 0   | 21.56 | 436008        |
| sra104815          | 104815| 6108 | 6108 | 0.00       | 0   | 16.21 | 6108               |
| usa115475          | 115475| 17745|17745|0.00       | 0   | 10.22 | 17745         |
| ara238025          | 238025| 1484 | 1484 | 0.00       | 0   | 39.25 | 1484             |
| ara489378          | 489378| 5888 | 5888 | 0.00       | 0   | 81.50 | 5888               |
| lrb744710          | 744710| 1930 | 1930 | 0.00       | 0   | 127.01 | 1930          |
Table 3: Detailed results for the TSPlib instances with \( p = 3 \)

| name             | \( |V| \) | LB  | UB  | g\% [0.00] | #BC | t [s] | Contardo et al. [2019] |
|------------------|-------|-----|-----|-----------|-----|------|------------------------|
| rw1621           | 1621  | 624 | 624 | 0.00      | 0   | 0.18 | 624 624 10.80          |
| ul1817           | 1817  | 895 | 895 | 0.00      | 0   | 0.23 | 895 895 9.90           |
| rl1889           | 1889  | 6066| 6066| 0.00      | 0   | 0.35 | 6066 6066 12.30        |
| mu1979           | 1979  | 2326| 2326| 0.00      | 0   | 0.14 | 2326 2326 10.10        |
| pr2392           | 2392  | 5413| 5413| 0.00      | 0   | 1.36 | 5413 5413 17.10        |
| d15112-modif-2500| 2500  | 7952| 7952| 0.00      | 0   | 0.38 | 7952 7952 13.10        |
| pch3038          | 3038  | 1519| 1519| 0.00      | 0   | 0.61 | 1519 1519 12.40        |
| mu3496           | 3496  | 1513| 1513| 0.00      | 0   | 0.72 | 1513 1513 12.40        |
| ca4663           | 4663  | 22680|22680|0.00       | 2   | 1.56 | 22680 22680 20.70      |
| rl5915           | 5915  | 6377| 6377| 0.00      | 0   | 1.62 | 6377 6377 18.50        |
| rl5934           | 5934  | 6005| 6005| 0.00      | 0   | 1.29 | 6005 6005 16.80        |
| tze6117          | 6117  | 4171| 4171| 0.00      | 0   | 3.06 | 4171 4171 21.60        |
| eg7146           | 7146  | 3702| 3702| 0.00      | 0   | 0.55 | 3702 3702 15.30        |
| pla7397          | 7397  | 279242|279242|0.00       | 0   | 2.36 | 279242 279242 32.50    |
| ym7663           | 7663  | 3107| 3107| 0.00      | 0   | 0.67 | 3107 3107 17.10        |
| pm8079           | 8079  | 1382| 1382| 0.00      | 0   | 0.62 | 1382 1382 14.50        |
| eib246           | 8246  | 1522| 1522| 0.00      | 0   | 1.34 | 1522 1522 15.90        |
| ar9152           | 9152  | 8196| 8196| 0.00      | 0   | 1.23 | 8196 8196 24.80        |
| ja9847           | 9847  | 7872| 7872| 0.00      | 0   | 0.64 | 7872 7872 13.50        |
| gr9882           | 9882  | 3170| 3170| 0.00      | 0   | 1.11 | 3170 3170 14.90        |
| kzu976           | 976   | 8244| 8244| 0.00      | 0   | 1.02 | 8244 8244 16.10        |
| fil10639         | 10639 | 3742| 3742| 0.00      | 0   | 1.69 | 3742 3742 13.50        |
| rl11849          | 11849 | 6452| 6452| 0.00      | 0   | 5.10 | 6452 6452 24.20        |
| usa13509         | 13509 | 134489|134489|0.00       | 0   | 1.18 | 134489 134489 23.50    |
| brd14051         | 14051 | 2426| 2426| 0.00      | 0   | 2.14 | 2426 2426 17.90        |
| mo14185          | 14185 | 2992| 2992| 0.00      | 0   | 1.71 | 2992 2992 16.70        |
| hol14473         | 14473 | 1534| 1534| 0.00      | 0   | 1.81 | 1534 1534 19.00        |
| d15112           | 15112 | 8154| 8154| 0.00      | 0   | 5.08 | 8154 8154 32.60        |
| it16862          | 16862 | 3724| 3724| 0.00      | 0   | 1.09 | 3724 3724 20.50        |
| d18512           | 18512 | 2914| 2914| 0.00      | 0   | 6.41 | 2914 2914 30.20        |
| vm22775          | 22775 | 3135| 3135| 0.00      | 0   | 2.86 | 3135 3135 21.80        |
| sw24978          | 24978 | 4120| 4120| 0.00      | 0   | 2.99 | 4120 4120 24.20        |
| fyg28544         | 28544 | 400 | 400 | 0.00      | 0   | 10.00| 400 400 34.60          |
| bn33708          | 33708 | 4213| 4213| 0.00      | 0   | 7.25 | 4213 4213 35.10        |
| pla33810         | 33810 | 302160|302160|0.00       | 0   | 31.67| 302160 302160 154.70   |
| bby34656         | 34656 | 420 | 420 | 0.00      | 0   | 11.43| 420 420 43.30          |
| pba38478         | 38478 | 413 | 413 | 0.00      | 0   | 9.88 | 413 413 33.20          |
| ch71009          | 71009 | 13292|13292|0.00       | 0   | 13.96| 13292 13292 59.90      |
| pla85900         | 85900 | 399677|399677|0.00       | 0   | 127.69|399677 399677 577.30   |
| sra104815        | 104815| 688 | 688 | 0.00      | 0   | 35.93| 688 688 97.70          |
| usai15475        | 15475| 13530|13530|0.00       | 0   | 13.33| 13530 13530 64.30      |
| ara238025        | 238025| 1166| 1166| 0.00      | 0   | 53.05| 1166 1166 153.30       |
| lra498378        | 498378| 4995| 4995| 0.00      | 0   | 260.76|4995 4995 689.40       |
| lrb744710        | 744710| 1702| 1702| 0.00      | 0   | 284.90|1702 1702 567.10       |
Table 4: Detailed results for the TSPlib instances with $p = 5$

| name              | $|V|$ | LB  | UB  | $f_{CLH}$ $g[\%]$ | #BC | t[s] | Contardo et al. (2019) | LB  | UB  | t[s] |
|-------------------|-----|-----|-----|-------------------|-----|------|------------------------|-----|-----|------|
| rw1621            | 1621| 414 | 414 | 0.00              | 0   | 0.33 | 414                    | 414 | 12.50 |
| u1817             | 1817| 715 | 715 | 0.00              | 0   | 1.22 | 715                    | 715 | 15.90 |
| rl1889            | 1889| 4792| 4792| 0.00              | 0   | 1.74 | 4792                   | 4792| 16.30 |
| mu1979            | 1979| 1877| 1877| 0.00              | 0   | 0.56 | 1877                   | 1877| 14.90 |
| pr2392            | 2392| 3827| 3827| 0.00              | 0   | 4.51 | 3827                   | 3827| 33.90 |
| d15112-modif-2500 | 2500| 5856| 5856| 0.00              | 0   | 3.44 | 5856                   | 5856| 16.90 |
| pch5938           | 5938| 1064| 1064| 0.00              | 0   | 1.79 | 1064                   | 1064| 21.20 |
| mu3496            | 3496| 1123| 1123| 0.00              | 0   | 1.11 | 1123                   | 1123| 14.60 |
| ca4663            | 4663| 16837|16837|0.00               | 0   | 1.51 | 16837                  | 16837|16.10 |
| rl5915            | 5915| 4554| 4554| 0.00              | 0   | 5.30 | 4554                   | 4554| 27.00 |
| rl5934            | 5934| 4792| 4792| 0.00              | 0   | 4.85 | 4792                   | 4792| 33.40 |
| tz6117            | 6117| 2918| 2918| 0.00              | 0   | 8.45 | 2918                   | 2918| 37.10 |
| eg7146            | 7146| 2590| 2590| 0.00              | 0   | 2.27 | 2590                   | 2590| 21.90 |
| pla7397           | 7397| 174542|174542|0.00            | 0   | 2.22 | 174542                | 174542|32.70 |
| ym7663            | 7663| 1877| 1877| 0.00              | 0   | 0.95 | 1877                   | 1877| 14.60 |
| pm8079            | 8079| 938 | 938 | 0.00              | 0   | 2.04 | 938                    | 938 | 18.40 |
| el8246            | 8246| 1042| 1042| 0.00              | 0   | 2.71 | 1042                   | 1042| 18.00 |
| ar9152            | 9152| 6752| 6752| 0.00              | 0   | 3.29 | 6752                   | 6752| 21.40 |
| ja9847            | 9847| 2590| 2590| 0.00              | 0   | 2.27 | 2590                   | 2590| 21.90 |
| gr9882            | 9882| 2151| 2151| 0.00              | 0   | 6.48 | 2151                   | 2151| 37.00 |
| kzn976            | 976 | 5995| 5995| 0.00              | 0   | 4.70 | 5995                   | 5995| 29.70 |
| fi10639           | 10639|2739 |2739 |0.00            | 0   | 6.88 | 2739                   | 2739| 23.30 |
| rl11849           | 11849|4873 |4873 |0.00            | 0   | 20.29| 4873                   | 4873| 53.70 |
| usa13509          | 13509|103671|103671|0.00        | 0   | 9.52 | 103671                | 103671|53.40 |
| brd114051         | 114051|1822 |1822 |0.00            | 0   | 18.00| 1822                   | 1822| 22.70 |
| moli4185          | 14185|2143 |2143 |0.00            | 0   | 3.03 | 2143                   | 2143| 30.30 |
| hol14473          | 14473|1112 |1112 |0.00            | 0   | 4.39 | 1112                   | 1112| 20.70 |
| d15112            | 15112|5890 |5890 |0.00            | 0   | 42.21| 5890                   | 5890| 77.40 |
| it16862           | 16862|2811 |2811 |0.00            | 0   | 3.13 | 2811                   | 2811| 33.10 |
| d18512            | 18512|2073 |2073 |0.00            | 0   | 65.95| 2073                   | 2073| 68.40 |
| vm22775           | 22775|2269 |2269 |0.00            | 0   | 6.20 | 2269                   | 2269| 41.70 |
| sw24978           | 24978|3022 |3022 |0.00            | 0   | 14.66| 3022                   | 3022| 45.90 |
| fyg28534          | 28534|261  |261  |0.00            | 0   | 64.21| 261                    | 261 | 164.00|
| bnm33708          | 33708|2747 |2747 |0.00            | 0   | 9.95 | 2747                   | 2747| 53.80 |
| pla33810          | 33810|203597|203597|0.00       | 0   | 179.93|203597                | 203597|769.50 |
| bby34656          | 34656|286  |286  |0.00            | 0   | 118.60|286                   | 286 | 256.70|
| pba38478          | 38478|313  |313  |0.00            | 0   | 71.79| 313                    | 313 | 174.10|
| ch71009           | 71009|10944|10944|0.00          | 11  | 200.89|10944                | 10944|269.30 |
| pla85900          | 85900|269544|269544|0.00      | 0   | 1030.73|269544              | 269544|3212.00|
| sra104815         | 104815|508 |508 |0.00           | 0   | 115.84|508                  | 508 | 237.20|
| usa115475         | 115475|10414|10414|0.00        | 0   | 196.31|10414               | 10414|562.20 |
| ara258825         | 258825|855 |855 |0.00           | 5   | 406.24|855                  | 855 | 578.20|
| lra498378         | 498378|3259|3259|0.00           | 0   | 561.74|3259                | 3259| 2250.30|
| lrb744710         | 744710|1167|1176|0.77          | 0   | TL   | 1170               | 1170| 4781.70|
Table 5: Detailed results for the TSPlib instances with \( p = 10 \)

| name            | \(|V|\) | \(fCLH\) | \(q[\%]\) | \#BC | \(t[s]\) | Contardo et al. (2019) |
|-----------------|-------|---------|-----------|------|---------|------------------------|
| rw1621          | 1621  | 285     | 285       | 0    | 1.21    | 285                    |
| u1817           | 1817  | 458     | 458       | 0    | 16.71   | 458                    |
| r11889          | 1889  | 3078    | 3101      | 0.74 | 584     | 3101                   |
| mut1979         | 1979  | 1161    | 1161      | 0    | 0.61    | 1161                   |
| pr2392          | 2392  | 2581    | 2581      | 0.00 | 550     | 2581                   |
| d15112-modif-2500 | 2500 | 3705    | 3705      | 0.00 | 45      | 3705                   |
| pch3908         | 3908  | 729     | 729       | 0.00 | 169     | 729                    |
| mu49878         | 49878 | 7167    | 7157      | 0.28 | 4.59    | 7167                   |
| ca4663          | 4663  | 10499   | 10499     | 0.00 | 0       | 10499                  |
| rl5915          | 5915  | 3104    | 3262      | 4.84 | 52      | 3137                   |
| rl5934          | 5934  | 3078    | 3143      | 2.07 | 105     | 3092                   |
| d15112          | 15112 | 3705    | 3705      | 0.00 | 45      | 3705                   |
| d18512          | 18512 | 1332    | 1371      | 2.84 | 94      | 1340                   |
| vr27770         | 27770 | 1315    | 1315      | 0.00 | 31.50   | 1315                   |
| sw24978         | 24978 | 2061    | 2061      | 0    | 103.94  | 2061                   |
| fvy28534        | 28534 | 175     | 176       | 0.57 | 36      | 176                    |
| bm33708         | 33708 | 1907    | 1907      | 0    | 1063.64 | 1907                   |
| pla33810        | 33810 | 135441  | 168992    | 19.85| 0       | 136517                 |
| bby34656        | 34656 | 192     | 197       | 2.54 | 2       | 192                    |
| pba38478        | 38478 | 206     | 232       | 11.21| 2       | 207                    |
| ch71009         | 71009 | 7162    | 7599      | 5.75 | 0       | 7183                   |
| plab85900       | 85900 | 176333  | 200715    | 12.15| 0       | 180497                 |
| sra104815       | 104815| 349     | 373       | 6.43 | 0       | 354                    |
| usa115475       | 115475| 6700    | 6866      | 2.42 | 0       | 6736                   |
| ara239625       | 239625| 549     | 573       | 4.19 | 0       | 552                    |
| lra498378       | 498378| 2160    | 2468      | 12.48| 0       | 2232                   |
| lrb744710       | 744710| 748     | 942       | 20.59| 0       | 795                    |

**Note:** The table shows the name of each instance, the number of nodes \(|V|\), the lower bound (LB) and upper bound (UB) of the solution quality, the ratio of the actual solution to the best known solution \(q[\%]\), the number of backtracks \(#BC\), the runtime \(t[s]\) in seconds, and the best known solution from Contardo et al. (2019).
| name              | V  | LB  | UB  | g[%] | #BC | t[s] | Contardo et al. (2019) |
|-------------------|----|-----|-----|------|-----|------|--------------------------|
| rw1621            | 1621 | 217 | 217 | 0.00 | 8   | 6.62 | 217 | 217 | 27.50 |
| u1817             | 1817 | 359 | 359 | 0.00 | 737 | 1366.66 | 359 | 359 | 386.20 |
| rl1889            | 1889 | 2384 | 2384 | 0.00 | 264 | 1146.25 | 2384 | 2384 | 181.60 |
| mu1979            | 1979 | 868 | 868 | 0.00 | 0   | 0.92 | 868 | 868 | 17.50 |
| py2392            | 2392 | 2011 | 2110 | 4.69 | 394 | TL   | 2039 | 2039 | 1581.30 |
| d15112-modif-2500 | 2500 | 2972 | 2972 | 0.00 | 30  | 95.50 | 2972 | 2972 | 110.70 |
| pch6308           | 6308 | 567 | 701 | 19.12 | 276 | TL   | 578 | 578 | 3474.90 |
| rl5462            | 5462 | 1604 | 2144 | 0.00 | 12  | 31.38 | 604 | 604 | 54.40 |
| ca4663            | 4663 | 8296 | 8296 | 0.00 | 0   | 3.80 | 8296 | 8296 | 36.20 |
| rl5915            | 5915 | 2403 | 2768 | 13.19 | 186 | TL   | 2441 | 2441 | 6266.20 |
| rl5934            | 5934 | 2392 | 2393 | 0.04 | 83  | TL   | 2393 | 2393 | 821.20 |
| tz6117            | 6117 | 1508 | 1727 | 12.68 | 459 | TL   | 1528 | 1528 | 764.70 |
| eg74146           | 74146 | 1308 | 1308 | 0.00 | 0   | 3.81 | 1308 | 1308 | 44.70 |
| pla7397           | 7397 | 95270 | 95270 | 0.00 | 0   | 186.73 | 95270 | 95270 | 217.50 |
| ym7663            | 7663 | 1054 | 1054 | 0.00 | 0   | 16.28 | 1054 | 1054 | 67.60 |
| pm8079            | 8079 | 517 | 517 | 0.00 | 0   | 9.59 | 517 | 517 | 52.30 |
| eu8246            | 8246 | 655 | 12.21 | 256 | TL   | 579 | 579 | 2268.70 |
| ar9152            | 9152 | 3250 | 3284 | 1.04 | 9   | TL   | 3250 | 3250 | 397.70 |
| ja9847            | 9847 | 1990 | 1990 | 0.00 | 0   | 9.36 | 1990 | 1990 | 65.10 |
| gp9882            | 9882 | 1038 | 1038 | 0.00 | 0   | 12.58 | 1038 | 1038 | 74.40 |
| kx9976            | 9976 | 3261 | 3261 | 0.00 | 0   | 169.08 | 3261 | 3261 | 433.10 |
| fl10639           | 10639 | 1456 | 1484 | 1.89 | 182 | TL   | 1461 | 1461 | 1030.10 |
| rl11849           | 11849 | 2424 | 3129 | 22.53 | 10  | TL   | 2462 | 2462 | 14808.20 |
| usa13509          | 13509 | 51476 | 57780 | 10.91 | 6   | TL   | 52178 | 52178 | 4740.90 |
| bdr14051          | 14051 | 954 | 1058 | 9.83 | 248 | TL   | 966 | 966 | 1832.50 |
| mol14185          | 14185 | 1110 | 1166 | 4.80 | 166 | TL   | 1118 | 1118 | 692.60 |
| ho14473           | 14473 | 577 | 647 | 10.82 | 105 | TL   | 585 | 585 | 826.10 |
| d15112           | 15112 | 3008 | 3239 | 7.13 | 0   | TL   | 3037 | 3037 | 27897.80 |
| it16862           | 16862 | 1237 | 1237 | 0.00 | 20  | 257.02 | 1237 | 1237 | 142.10 |
| rl18512           | 18512 | 1066 | 1119 | 4.74 | 0   | TL   | 1075 | 1075 | 12403.90 |
| vm22775           | 22775 | 1099 | 1099 | 0.00 | 0   | 33.74 | 1099 | 1099 | 265.60 |
| sv24978           | 24978 | 1631 | 1775 | 8.11 | 19  | TL   | 1637 | 1637 | 1751.50 |
| fyg28554          | 28554 | 139 | 168 | 17.26 | 0   | TL   | 142 | 142 | 151 |
| bm33708          | 33708 | 1469 | 1512 | 2.84 | 1   | TL   | 1475 | 1475 | 3952.00 |
| pla33810          | 33810 | 106684 | 137325 | 22.31 | 0   | TL   | 110019 | 126933 | TL2 |
| bby34656          | 34656 | 152 | 189 | 19.58 | 0   | TL   | 155 | 155 | 163 |
| pba38478          | 38478 | 158 | 208 | 24.04 | 0   | TL   | 160 | 160 | 163 |
| ch71009           | 71009 | 5564 | 6459 | 13.86 | 0   | TL   | 5664 | 5664 | 13621.50 |
| pla59900          | 59900 | 136667 | 185853 | 26.47 | 0   | TL   | 142938 | 159484 | TL2 |
| sra104815         | 104815 | 273 | 347 | 21.33 | 0   | TL   | 277 | 277 | 15204.60 |
| usa115475         | 115475 | 5909 | 6259 | 18.68 | 0   | TL   | 5288 | 5419 | TL2 |
| ara238025         | 238025 | 432 | 511 | 15.46 | 0   | TL   | 441 | 441 | TL2 |
| lra49378          | 49378 | 1647 | 2147 | 25.29 | 0   | TL   | 1755 | 1755 | 44562.60 |
| lrb744710         | 744710 | 586 | 694 | 15.56 | 0   | TL   | 626 | 626 | TL2 |
Table 7: Detailed results for the TSPlib instances with $p = 20$

| name               | $|V|$ | LB   | UB   | fCLH $g$ [%] | #BC | t [s] | Contardo et al. [2019] |
|--------------------|-----|------|------|--------------|-----|------|------------------------|
|                   |     |      |      | Contardo et al. [2019] |
|                   |     |      |      | LB | UB | t [s] |
| rw1621             | 1621 | 186  | 186  | 0.00  | 376  | 21.39 | 186 | 186 | 34.80 |
| u1817              | 1817 | 300  | 310  | 1.29  | 6425 | TL    | 309 | 309 | 2171.80 |
| rl1889             | 1889 | 2060 | 2201 | 6.41  | 632  | TL    | 2089| 2089| 405.00 |
| mu1979             | 1979 | 751  | 751  | 0.00  | 0    | 1.15  | 751 | 751 | 26.20 |
| pt2392             | 2392 | 1694 | 1864 | 9.12  | 358  | 454.56| 519 | 519 | 128.30 |
| d15112-modif-2500  | 2500 | 2519 | 2965 | 15.04 | 227  | TL    | 2573| 2573| 76684.10 |
| pcb3638            | 3638 | 480  | 618  | 22.33 | 124  | TL    | 493 | 493 | 54802.10 |
| ru546              | 546  | 519  | 519  | 0.00  | 358  | 454.56| 519 | 519 | 128.30 |
| ca4663             | 4663 | 7024 | 7024 | 0.00  | 11.04| TL    | 7024| 7024| 57.00 |
| rl5915             | 5915 | 2036 | 2509 | 18.85 | 11   | TL    | 2083| 2083| 57664.10 |
| rl5934             | 5934 | 2068 | 2412 | 14.26 | 68   | TL    | 4367| 4367| 4367.20 |
| tz6117             | 6117 | 1266 | 1306 | 3.06  | 155  | TL    | 1988| 1988| 1988.70 |
| eg7146             | 7146 | 972  | 972  | 0.00  | 3.16 | TL    | 972 | 972 | 51.30 |
| pla7397            | 7397 | 78014| 83860| 6.97  | 19   | TL    | 7881| 7881| 608.20 |
| ym7663             | 7663 | 899  | 899  | 0.00  | 5    | 63.54 | 899 | 899 | 152.10 |
| pm8079             | 8079 | 430  | 430  | 0.00  | 5    | 29.19 | 430 | 430 | 91.40 |
| ei8246             | 8246 | 488  | 533  | 8.44  | 295  | TL    | 497 | 497 | 33150.50 |
| ar9152             | 9152 | 2653 | 2818 | 5.86  | 80   | TL    | 2695| 2695| 6067.80 |
| ja9847             | 9847 | 1724 | 1724 | 0.00  | 0    | 8.76  | 1724| 1724| 66.70 |
| gr9882             | 9882 | 877  | 877  | 0.00  | 4    | 106.71| 877 | 877 | 235.60 |
| kz9976             | 9976 | 2762 | 3168 | 12.82 | 45   | TL    | 2790| 2790| 1540.20 |
| fi10639            | 10639| 1228 | 1500 | 18.13 | 16   | TL    | 1245| 1245| 15688.60 |
| ri11849            | 11849| 2080 | 2629 | 20.88 | 0    | TL    | 2119| 2119| 2119.00 |
| usa13509           | 13509| 43592| 56610| 23.00 | 0    | TL    | 44740| 44740| 16721.90 |
| bbd14051           | 14051| 796  | 909  | 12.43 | 70   | TL    | 803 | 803 | 16721.90 |
| mnl4185            | 4185 | 914  | 1022 | 10.57 | 23   | TL    | 919 | 919 | 3597.10 |
| hol4473            | 4473 | 492  | 566  | 13.07 | 82   | TL    | 497 | 497 | 1470.20 |
| d15112             | 15112| 2539 | 3359 | 24.41 | 0    | TL    | 2581| 2581| 2581.00 |
| it16862            | 16862| 1059 | 1059 | 0.00  | 0    | 134.58| 1059| 1059| 563.60 |
| d18512             | 18512| 902  | 1218 | 25.94 | 0    | TL    | 912 | 912 | 912.00 |
| vm2717             | 2717 | 919  | 967  | 4.96  | 42   | TL    | 932 | 932 | 1491.90 |
| sw24978            | 24978| 1403 | 1662 | 15.58 | 0    | TL    | 1421| 1421| 14835.00 |
| fyg2854            | 2854 | 118  | 154  | 23.38 | 0    | TL    | 128 | 128 | 128.00 |
| bmd3708            | 3708 | 1245 | 1455 | 14.43 | 3    | TL    | 1257| 1257| 17808.70 |
| pla33810           | 33810| 89123| 118855| 25.02 | 0    | TL    | 91302| 91302| 91302.00 |
| bby34656           | 34656| 128  | 159  | 19.50 | 0    | TL    | 128 | 128 | 128.00 |
| pba38478           | 38478| 136  | 175  | 22.29 | 0    | TL    | 136 | 136 | 136.00 |
| ch71009            | 71009| 4637 | 5868 | 20.98 | 0    | TL    | 4798| 4798| 4798.00 |
| pla55900           | 55900| 115361| 163300| 29.36 | 0    | TL    | 119643| 119643| 119643.00 |
| sra104815          | 104815| 227 | 294  | 22.79 | 0    | TL    | 232 | 232 | 232.00 |
| usa115475          | 115475| 4198 | 5855 | 28.30 | 0    | TL    | 4453| 4453| 4453.00 |
| ara238025          | 238025| 359 | 466  | 22.96 | 0    | TL    | 372 | 372 | 372.00 |
| lra498378          | 498378| 1342 | 1726 | 22.25 | 0    | TL    | 1442| 1442| 1442.00 |
| lrb744710          | 744710| 474  | 697  | 31.99 | 0    | TL    | 524 | 524 | 524.00 |
Table 8: Detailed results for the TSP1ib instances with $p = 25$

| name              | $|V|$ | LB | UB  | $f_{\text{CLH}}$ $g[\%]$ | #BC | $t[s]$ | Contardo et al. (2019) | LB | UB  | $t[s]$ |
|-------------------|-----|----|-----|---------------------------|-----|--------|------------------------|----|-----|--------|
| rw1621            | 1621| 166 | 166 | 0.00                      | 8   | 12.83  | 166                    | 166| 38.70|
| u1817             | 1817| 265 | 275 | 3.64                      | 1766 | TL     | 272                    | 272| 1875.80|
| rl1889            | 1889| 1833| 2063| 11.15                      | 294 | TL     | 1866                   | 1866| 1154.80|
| m11979            | 1979| 639 | 639 | 0.00                       | 0   | 2.11   | 639                    | 639| 30.40|
| pr2392            | 2392| 1492| 1956| 23.72                      | 135 | TL     | 1520                   | 1520| 23767.70|
| d15112-modif-2500 | 2500| 2205| 2581| 14.57                      | 225 | TL     | 2243                   | 2243| 5266.20|
| pcbl638           | 638 | 441 | 441 | 0.00                       | 11  | 41.79  | 441                    | 441| 116.80|
| ca4663            | 4663| 5066| 5066| 0.00                       | 13  | 121.12 | 5066                   | 5066| 118.20|
| rl5915            | 5915| 1786| 2201| 18.86                      | 20  | TL     | 1823                   | 1823| 118.20|
| rl5934            | 5934| 1825| 2113| 13.63                      | 19  | TL     | 1850                   | 1850| 25300.40|
| tz6117            | 6117| 1121| 1426| 21.39                      | 237 | TL     | 1152                   | 1152| 1258.10|
| eg7146            | 7146| 855 | 855 | 0.00                       | 0   | 9.76   | 855                    | 855| 79.10|
| pla7397           | 7397| 68964| 78142| 11.75                    | 35  | TL     | 69508                  | 69508| 776.80|
| ym7663            | 7663| 785 | 819 | 4.15                       | 20  | TL     | 785                    | 785| 189.40|
| pm8079            | 8079| 367 | 367 | 0.00                       | 0   | 50.52  | 367                    | 367| 184.80|
| eil246            | 246 | 421 | 532 | 20.86                      | 71  | TL     | 429                    | 429| 184.80|
| ar9152            | 9152| 2392| 2717| 15.27                      | 50  | TL     | 2355                   | 2355| 30531.70|
| ja9847            | 9847| 1362| 1362| 0.00                       | 0   | 43.02  | 1362                   | 1362| 154.80|
| gr9882            | 9882| 772 | 772 | 0.00                       | 59  | 236.54 | 772                    | 772| 532.30|
| kz9976            | 9976| 2439| 2800| 12.89                      | 72  | TL     | 2479                   | 2479| 4666.40|
| fi10639           | 10639| 1075| 1400| 23.21                      | 10  | TL     | 1103                   | 1103| 3122.30|
| rl11849           | 11849| 1823| 2506| 27.25                      | 0   | TL     | 1838                   | 1838| 2099.90|
| usa13509          | 13509| 37471| 46954| 20.20                    | 0   | TL     | 38150                  | 38150| 40578.10|
| brd14051          | 14051| 693 | 843 | 17.79                      | 7   | TL     | 703                    | 703| 184.80|
| mol1185           | 1185| 806 | 907 | 11.14                      | 11  | TL     | 818                    | 818| 42123.30|
| hol14473          | 14473| 433 | 547 | 20.84                      | 99  | TL     | 436                    | 436| 11941.10|
| d15112            | 15112| 2201| 2877| 23.50                      | 0   | TL     | 2233                   | 2233| 2447.80|
| it16862           | 16862| 939 | 1140| 17.63                      | 6   | TL     | 951                    | 951| 5284.40|
| vm22773           | 22773| 798 | 815 | 2.09                       | 1   | TL     | 799                    | 799| 3102.50|
| sw24978           | 24978| 1209| 1567| 22.85                      | 0   | TL     | 1233                   | 1233| 1285.80|
| fyg28554          | 28554| 103 | 133 | 22.56                      | 0   | TL     | 102                    | 102| 112.00|
| bmm37908          | 37908| 1069| 1358| 21.28                      | 0   | TL     | 1106                   | 1106| 1140.80|
| pla33810          | 33810| 80661| 108074| 23.57                   | 0   | TL     | 81800                  | 81800| 88861.00|
| bbv34656          | 34656| 113 | 144 | 21.53                      | 0   | TL     | 113                    | 113| 128.00|
| pba38478          | 38478| 118 | 160 | 26.25                      | 0   | TL     | 118                    | 118| 134.00|
| ch71009           | 71009| 4013| 5491| 26.92                      | 0   | TL     | 4145                   | 4145| 4321.00|
| pla85900          | 85900| 103546| 145695| 28.93                   | 0   | TL     | 107527                 | 107527| 124693.00|
| sra14085          | 14085| 203 | 241 | 15.77                      | 0   | TL     | 206                    | 206| 216.00|
| usa115475         | 115475| 3649| 5347| 31.76                      | 0   | TL     | 3808                   | 3808| 4453.00|
| ara238025         | 238025| 307 | 393 | 21.88                      | 0   | TL     | 319                    | 319| 357.00|
| lra49378          | 49378| 1109| 1608| 31.03                      | 0   | TL     | 1228                   | 1228| 1339.00|
| lrb744710         | 744710| 415 | 636 | 34.75                      | 0   | TL     | 454                    | 454| 533.00|
Table 9: Detailed results for the TSP1lib instances with $p = 30$

| name                   | $|V|$ | LB   | UB   | $f_{CLH}$ $g[\%]$ | $\#BC$ | $t[s]$ | Contardo et al. (2019) |
|------------------------|-----|------|------|-------------------|--------|--------|------------------------|
| rw1621                 | 1621| 147  | 147  | 0.00              | 32     | 14.84  | 147 147 64.80          |
| u1187                  | 1187| 240  | 251  | 4.38              | 2438   | 241    | 241 4228.30           |
| rl1889                 | 1889| 1638 | 1885 | 13.10             | 226    | 1657   | 1657 1603.40          |
| mu1979                 | 1979| 552  | 552  | 0.00              | 0      | 552    | 552 34.40             |
| pr2392                 | 2392| 1351 | 1765 | 23.46             | 362    | 2029   | 2029 50627.40         |
| d15112-modif-2500     | 2500| 1980 | 2604 | 23.96             | 414    | 241    | 241 4228.30           |
| plc30838              | 30838| 382  | 508  | 24.80             | 235    | 386    | 412 152.60            |
| mu1889                | 1889| 396  | 1886 | 33.00             | 43     | 396    | 396 205.70            |
| ca4663                | 4663| 5361 | 5361 | 0.00              | 0      | 5361   | 5361 152.60           |
| rl5915                | 5915| 1618 | 2210 | 26.79             | 10     | 1624   | 1853 152.60           |
| rl5934                | 5934| 1631 | 2116 | 22.92             | 60     | 1658   | 1812 152.60           |
| tz6117                | 6117| 1001 | 1304 | 23.24             | 224    | 1025   | 1142 152.60           |
| eg7146                | 7146| 744  | 744  | 0.00              | 0      | 744    | 744 76.80            |
| pla7397               | 7397| 62057| 71915| 13.71             | 0      | 63770  | 63770 20045.10       |
| ym7663                | 7663| 703  | 748  | 6.02              | 0      | 715    | 715 536.60           |
| pm8079                | 8079| 330  | 330  | 0.00              | 0      | 330    | 330 459.80           |
| ei8246                | 8246| 384  | 508  | 24.41             | 42     | 386    | 412 152.60           |
| ar9152                | 9152| 2050 | 2648 | 22.58             | 14     | 2080   | 2080 10145.70        |
| ja9847                | 9847| 1220 | 1220 | 0.00              | 0      | 1220   | 1220 268.30          |
| gr9882                | 9882| 677  | 677  | 0.00              | 9      | 677    | 677 1024.60          |
| kz9976                | 9976| 2191 | 2820 | 22.30             | 44     | 2230   | 2230 11820.20        |
| fil0639               | 10639| 967  | 1251 | 22.70             | 10     | 974    | 1017 152.60          |
| rl11849               | 11849| 1649 | 2255 | 26.87             | 0      | 1641   | 1855 152.60          |
| usa13509              | 13509| 34137| 45931| 25.12             | 0      | 34771  | 37036 152.60         |
| brd18140              | 18140| 618  | 812  | 26.06             | 0      | 620    | 668 152.60           |
| mo14185               | 14185| 719  | 936  | 23.18             | 0      | 732    | 767 152.60           |
| hol14473              | 14473| 389  | 511  | 23.87             | 30     | 396    | 396 29545.80         |
| d15112                | 15112| 1994 | 2662 | 25.09             | 0      | 2009   | 2254 152.60          |
| it16862               | 16862| 849  | 947  | 10.35             | 57     | 854    | 854 16851.80         |
| d18512                | 18512| 712  | 963  | 26.06             | 0      | 712    | 786 152.60           |
| vm22775               | 22775| 690  | 802  | 13.97             | 7      | 696    | 696 13315.20         |
| sw24978               | 24978| 1063 | 1514 | 28.47             | 0      | 1096   | 1215 152.60          |
| fyg28544              | 28544| 93   | 122  | 23.77             | 0      | 92     | 108 152.60           |
| bms33708              | 33708| 930  | 1221 | 23.83             | 0      | 968    | 1065 152.60          |
| plas3810              | 3810 | 70365| 95203| 26.09             | 0      | 71480  | 83187 152.60         |
| bby34566              | 34566| 101  | 137  | 26.28             | 0      | 102    | 133 152.60           |
| pba38478              | 38478| 107  | 149  | 28.19             | 0      | 107    | 125 152.60           |
| ch71009               | 71009| 3581 | 4947 | 27.61             | 0      | 3724   | 4098 152.60          |
| pla85900              | 85900| 90212| 127681| 29.35            | 0      | 93871  | 114244 152.60       |
| sra10481              | 10481| 182  | 216  | 15.74             | 0      | 186    | 204 152.60           |
| usa115475             | 115475| 3258 | 4770 | 31.70             | 0      | 3391   | 3874 152.60          |
| ara238025             | 238025| 277  | 386  | 28.24             | 0      | 288    | 368 152.60           |
| lra49838              | 49838| 1015 | 1486 | 31.70             | 0      | 1073   | 1192 152.60          |
| lrb744710             | 744710| 370  | 580  | 36.21             | 0      | 403    | 475 152.60           |