Geometrically nonlinear forced vibrations of fully clamped multi-span beams carrying multiple masses and resting on a finite number of simple supports

H Fakhreddine*1, A Adiri1, M Chajdi2, S Rifai1 and R Benamar3

1 Laboratoire de Mécanique Productique et Génie Industriel, Ecole Supérieure de Technologie, Hassan II University of Casablanca, B.P.8012, Oasis, Casablanca, Maroc
2 Mohammed V University in Rabat, ENSET - Rabat, MSSM, B.P.6207, Rabat Instituts, Rabat, Morocco
3 Mohammed V University in Rabat, EMI-Rabat, LERSIM, B.P.765 Agdal, Rabat, Morocco

*Corresponding author: hatim.fakhreddine@gmail.com

Abstract. Geometrically nonlinear forced vibrations of fully clamped multi-span beams resting on multiple simple supports and carrying multiple masses may be encountered in many mechanical and civil engineering applications. The theoretical model developed here is based on the Euler-Bernoulli beam theory and the Von Karman geometrical non-linearity assumptions. Harmonic motion is assumed and the nonlinear beam transverse displacement function is expanded as a series of the linear modes, determined by solving first the linear problem. The discretised expressions for the beam total strain and kinetic energies are then derived, and by applying Hamilton’s principle, the problem is reduced to a nonlinear algebraic system solved using an approximate method (the so-called second formulation). The basic function contribution coefficients to the structure non-linear response function and the corresponding backbone curves giving the non-linear amplitude-frequency dependence is determined. Numerical results are given in the neighbourhood of the predominant nonlinear mode shape, based on the single mode approach, for a wide range of vibration amplitudes, showing the effect of the added masses and their locations, as well as the applied uniformly distributed harmonic force on its non-linear dynamic response.

Introduction

In the design of structures, in several fields, such as aeronautics, civil or mechanical engineering, the ultimate objective is to obtain the most optimal structure. The latter, while resisting the constraints imposed by the environment, can be obtained from a weak blow of production. Despite its frequent use and the simplicity of its calculation, linear vibration analysis remains valid only at certain limits, beyond which its use can lead to errors or inaccuracies. In fact, the nonlinear analysis remains more significant, meanwhile, its complexity and the scarcity of the means of computation make its use less frequent. The problem of vibration of beams containing masses has since few decades started to attract the attention of researchers, given its different areas of application. Starting with reference [1] where the author establishes a new formulation of the problem of vibration of a beam carrying only one concentrated mass using Dirac functions to describe the mass effect. Several authors in the literature
investigated the vibrations of a cantilever beam carrying a mass at the free end and determined their natural frequencies and mode shapes [2,3]. In [4], a concentrated mass is added to the vibration of a continuous beam resting on intermediate supports, using polynomial coordinate functions. Laplace transform and the Raleigh-Ritz method was used in [5] to obtain the eigenfrequency parameters and the eigenfrequencies of non-uniform beams carrying concentrated masses at arbitrary locations. Additionally, [6] tackled the vibration problematic of a cantilevered beam carrying a finite number of sprung masses and developed a numerical theory to obtain exactly, for any number of spring masses, the beam natural frequencies and mode shapes. In [7], alternative formulas have been considered for the frequency equation of a cantilever Euler-Bernoulli beam carrying spring-masses. The aforementioned formulations are based on the discretization of the beam by its first bending eigenfunctions. Furthermore, a vibration analysis of a cantilevered Euler-Bernoulli beam was considered in [8], the beam supported two masses, one at its free end and the second at its middle, using three different numerical methods, namely the Raleigh-Ritz method, the Galerkin and the Finite element method. In [9], the vibration of a simply supported beam carrying a rigid mass in the middle was examined. Other studies explored the problem of uniform beams with different spring-masses, such as [10], in which the exact solutions for the natural frequencies and associated modes shapes was determined using the numerical assembly technique for the conventional finite element method. The authors of [11] attempted to solve the problem of transverse vibration of beams carrying point masses while considering their rotary inertia in order to finally give the exact solutions by considering general end conditions modelled by translational and rotational springs. The effect of the masses and their rotary inertia have been clearly observed, in several end conditions. Despite the fact that many works dealt with the problem of beam transverse vibrations with several intermediate elements, such as intermediate supports (rigid or flexible) [12,13], added masses (concentric and inertial) [5,9,14,15], only a few have been investigated geometrically nonlinear vibrations. In [16], the effect of geometrical non-linearity was taken into account in the analysis of a Euler-Bernoulli beam free vibrations with several added masses. The effect of non-linearity on the behaviour of the thin beam has been demonstrated and observed for different end conditions. In perspective, the present work aims to establish the formulation of the problem of geometrically non-linear vibrations of an Euler-Bernoulli beam carrying several masses with rotary inertia, and resting on a finite number of intermediate supports. The mathematical model adopted in the current study is based on Hamilton's principle and spectral analysis, leading to the solution of a nonlinear algebraic system by use of numerical and analytical methods. The nonlinear algebraic system has been solved for a beam carrying several masses and resting on different intermediate supports, using the second formulation developed in [17] and applied previously in [12,18]. The effect of the added masses on the nonlinear behaviour of beams resting on a number of supports was demonstrated, as well as the impact of their rotary inertia. By considering the single mode approach [2], and after determination of the predominant mode corresponding to a harmonic distributed physical excitation, the study of the beam non-linear dynamic behaviour has been studied, and the effect of the mass as well as its position illustrated.

**General formulation**

### 2.1 Linear formulation

The present study considers a beam that satisfies the Euler-Bernoulli conditions, and has the following mechanical and geometrical characteristics: \((L, b, h, I, E, \rho)\) for the length, width, thickness, second moment of the area of the cross-section, Young’s modulus and the mass per unit length. The beam supports three inertial masses at \(x_1, x_2,\) and \(x_3,\) and rests on two fixed supports located at \(x_2\) and \(x_4.\)
In this section, the system linear mode shapes are first determined in order to use them as basic functions in the nonlinear theory developed in the second section [19]. The transverse displacement function $w$ of the beam shown in figure 1 can be expressed at each span as:

$$
\begin{align*}
    w_1(\eta) & \to 0, \eta_1[ \\
    w_2(\eta) & \to \eta_1, \eta_2[ \\
    w_3(\eta) & \to \eta_3, \eta_4[ \\
    w_4(\eta) & \to \eta_4, \eta_5[ \\
    w_5(\eta) & \to \eta_5, 1[ 
\end{align*}
$$

The general solution for transverse vibration in the $(j)^{th}$ span can be written as:

$$
\begin{align*}
    w_j(\eta) = a_j \cosh(\beta_j L(\eta - \eta_{j-1})) + b_j \sinh(\beta_j L(\eta - \eta_{j-1})) + c_j \cos(\beta_j L(\eta - \eta_{j-1})) + d_j \sin(\beta_j L(\eta - \eta_{j-1})) \\
    \eta_{j-1} \leq \eta \leq \eta_j ; \text{ for } j = 1, 2, \ldots, n
\end{align*}
$$

(2)

The constants $a_j, b_j, c_j, d_j$ are determined by the beam end and continuity conditions as follows:

The boundary conditions imposed in the case of a fully clamped beam are:

$$
\begin{align*}
    w_j(\eta)|_{\eta=0} = w_{j+1}(\eta)|_{\eta=L} \\
    \frac{dw_j(\eta)}{d\eta}|_{\eta=0} = \frac{dw_{j+1}(\eta)}{d\eta}|_{\eta=L}
\end{align*}
$$

(3)

(4)

The compatibility conditions for the intermediate support are[20]:

$$
\begin{align*}
    w_j(\eta)|_{\eta_j} = w_{j+1}(\eta)|_{\eta_j} = 0 ; \frac{dw_j(\eta)}{d\eta}|_{\eta_j} = \frac{dw_{j+1}(\eta)}{d\eta}|_{\eta_j}
\end{align*}
$$

(5, 6)
\[
\frac{d^2 w_j(\eta)}{d\eta^2} \bigg|_{\eta_j} = \frac{d^2 w_{j+i,\eta}(\eta)}{d\eta^2} \bigg|_{\eta_j} ; \quad \frac{d^3 w_j(\eta)}{d\eta^3} \bigg|_{\eta_j} = \frac{d^3 w_{j+i,\eta}(\eta)}{d\eta^3} \bigg|_{\eta_j} - \frac{R}{EI}
\]

The compatibility conditions for the added mass are:

\[
\frac{d^2 w_j(\eta)}{d\eta^2} \bigg|_{\eta_j} = w_{j+i,\eta}(\eta) \bigg|_{\eta_j} ; \quad \frac{d^2 w_j(\eta)}{d\eta^2} \bigg|_{\eta_j} = \frac{d^2 w_{j+i,\eta}(\eta)}{d\eta^2} \bigg|_{\eta_j}
\]

\[
\frac{d^2 w_j(\eta)}{d\eta^2} \bigg|_{\eta_j} = \frac{d^2 w_{j+i,\eta}(\eta)}{d\eta^2} \bigg|_{\eta_j} + J \omega^2 \frac{dw_j(\eta)}{d\eta} \bigg|_{\eta_j}
\]

\[
\frac{d^2 w_j(\eta)}{d\eta^2} \bigg|_{\eta_j} = \frac{d^2 w_{j+i,\eta}(\eta)}{d\eta^2} \bigg|_{\eta_j} - M \omega^2 \frac{dw_j(\eta)}{d\eta} \bigg|_{\eta_j}
\]

Where \( R \) denotes the support reaction, \( M_j \) and \( J_j \) are respectively the concentrated mass and the moment inertia of the attached mass.

The application of the compatibility and continuity conditions as well as the satisfaction of the end conditions (clamped at both ends) lead to a homogeneous system. To obtain non-trivial solutions, corresponding to the natural frequencies, the determinant of the system must vanish. The system is then iteratively solved by the newton-Raphson method.

2.2 Non-linear formulation

The total strain energy \( V \) of the beam can be written as the sum of the axial strain energy due to the nonlinear stretching forces induced by the large deflections \( V_a \), and the strain energy due to bending \( V_b \).

\[
V_a = \frac{ES}{8L} \int_0^L \left( \frac{\hat{\omega} w(x,t)}{\hat{\omega}} \right)^2 dx \quad \text{and} \quad V_b = \frac{EI}{2} \int_0^L \left( \frac{\hat{\omega}^2 w(x,t)}{\hat{\omega}^2} \right)^2 dx
\]

\[
T = \frac{1}{2} \rho S \int_0^L \left( \frac{\hat{\omega} w(x,t)}{\hat{\omega}t} \right)^2 dx + \frac{1}{2} \sum_{j=1}^n M \left( \frac{\hat{\omega} w(x,t)}{\hat{\omega}t} \right)_{x=x_j}^2 + \frac{1}{2} \sum_{j=1}^n J \left( \frac{\hat{\omega} w(x,t)}{\hat{\omega}t} \right)_{x=x_j}^2
\]

Consider now a beam excited by a uniformly distributed force. According to [19], the physical force \( F(x,t) \) excites the modes of the structure via a set of generalised forces \( F_i \) which depend on the expression for \( F \), the excitation location, and the mode considered.

The generalised force \( F_i \) corresponding to a mode \( w_i \) are given by:

\[
F_i = \int_0^L F(x,t) w_i(x) dx
\]

The beam shown in figure 1 is supposed to be excited by a distributed harmonic force, the expression of which is given by [19]:

\[
F(x,t) = \phi x(t) \sin(\omega t)
\]
where $F_i(t)$ is the corresponding generalised force given by:

$$F_i(t) = F_i^* \sin(\omega t) \int_0^L w_i(x) dx = f_i^* \sin(\omega t)$$

(18)

It is well known that the dynamic behaviour of a conservative system is governed by Hamilton’s principle, which, by taking into account the forcing term, may be written as in [21], by:

$$\delta \int_0^{2\pi/\omega} (V - T + W) dt = 0$$

(19)

Upon assuming harmonic motion and expanding the transverse displacement $w$ in the form of the following finite series:

$$w(x,t) = \sum_{i=1}^n a_i w_i \sin(\omega t)$$

(20)

Substituting $w$ (in the form defined above), in the expressions for $V_a$, $V_b$ and $T$, one may write:

$$V_a = -\frac{1}{2} a^T a \dot{w}^2(\omega t) ; V_b = -\frac{1}{2} a^T a k_y^2 \dot{w}^2(\omega t)$$

(21, 22)

$$T = \frac{1}{2} \dot{\omega}^2 a^T a m^2 \dot{w}^2(\omega t) ; F = a f_i^* \sin(\omega t)$$

(23, 24)

Where $k_y$ denotes the classical rigidity:

$$K_y = \int_0^L \left( \frac{\partial^2 w_i}{\partial x^2} \right) \left( \frac{\partial^2 w_j}{\partial x^2} \right) dx$$

(25)

$b_{0\ell}$ presents the non-linearity tensor:

$$b_{0\ell} = \frac{ES}{4L} \int_0^L \left( \frac{\partial w_i}{\partial x} \right) \left( \frac{\partial w_j}{\partial x} \right) dx \int_0^L \left( \frac{\partial w_i}{\partial x} \right) \left( \frac{\partial w_j}{\partial x} \right) dx$$

(26)

and, $m_y$ the mass tensor:

$$m_y = \rho S \int_0^L w_i(x) w_j(x) dx + \sum_i M_j w_i(x) w_j(x) + \sum_i J_j \psi_i(x) \psi_j(x)$$

(27)

$$J_j = m_j r_j^2 \text{ with } c_j = \frac{r_j}{l} \text{ and } M_j = \frac{m_j}{\rho A l}$$

(28)

Where $r_j$ denotes the radius of gyration of the attached mass $m_j$ and the amplitudes $a_i$ are unknown as well as the frequency $\omega$. After calculations, the following non-linear system is obtained:

$$[K][A] + \frac{3}{2}[B([A])][A] - \omega^2[M][A] = [F]$$

(29)

For obtaining non-dimensional parameters, one puts:

$$w(x) = hw \frac{x}{L} = hw^* (x^*) ; \frac{m_y}{m^*} = \rho Sh^2 L$$

(30)
The dimensionless generalised force corresponding to the uniformly distributed force is given by:

\[ f^d_i = F^d \frac{L^3}{EI} \int_0^L w^* \, dx \]  

(31)

Substituting these expressions into (29), one obtains the following nonlinear algebraic equation:

\[ \left[ \left( k^* \right)^{\omega^{i2}} \left[ M^* \right] \right] \left[ A \right] + \frac{3}{2} \left[ B^* \left( \left[ A \right] \right) \right] \left[ A \right] = \left[ F^* \right] \]  

(32)

Which can be written under the following tensorial notation:

\[ a \mathbf{k}^*_{ij} - \omega^{i2} a \mathbf{m}^*_{ij} + a a \mathbf{a} \mathbf{b}^*_{ij} = \mathbf{F}^*_i \]  

(33)

In the current study, the calculation of the corresponding generalised forces has led to the conclusion that the nonlinear response, in the case of uniformly distributed force, involves predominately one mode denoted in what follows as \( k \). Consequently, an analysis has been performed in the neighbourhood of this mode, based on the single mode approach.

By considering one mode, one may write the equation (33) in the following form:

\[ (k^*_{ij} - \omega^{i2} m^*_{ij}) a_k + \frac{3}{2} a^* b^*_{ik} = F^*_i \]  

(34)

Finally, the following cubic equation is obtained, the solution of which requires use of Cardan’s method:

\[ \frac{\left( \omega \right)^2}{\omega} = 1 + \frac{3}{2} \left( \frac{b^*_{kj}}{k^*_{ii}} \right) a^* \left( \frac{F^*}{k^*_{ii}} \right) a_k \]  

(35)

Results and discussion:

The following table presents the linear frequencies for different values of \( m \) and \( c \). It can be noticed that by increasing the mass or its rotary inertia, the frequencies of the system decrease, which corresponds to a reduction of the rigidity.

### Table 1. First three eigenvalues for fully clamped beam carrying tree masses and resting on two simple supports.

| \( \eta_1 \) = 1/6, \( \eta_2 \) = 3/6, \( \eta_3 \) = 5/6 | \( \beta_i \) | \( c_1=c_2=c_3=0 \) | \( c_1=c_2=c_3=0.01 \) | \( c_1=c_2=c_3=0.05 \) | \( c_1=c_2=c_3=0.1 \) |
|---|---|---|---|---|---|
| \( M_1=M_2=M_3=0 \) | \( \beta_1 \) | 10.66922 | | | |
| | \( \beta_2 \) | 12.89258 | | | |
| | \( \beta_3 \) | 14.19012 | | | |
| \( M_1=M_2=M_3=0.01 \) | \( \beta_1 \) | 10.51234 | 10.51225 | 10.51018 | 10.50352 |
| | \( \beta_2 \) | 12.68756 | 12.68733 | 12.68183 | 12.66347 |
| | \( \beta_3 \) | 13.93311 | 13.93311 | 13.93311 | 13.93311 |
| \( M_1=M_2=M_3=0.1 \) | \( \beta_1 \) | 9.46362 | 9.46315 | 9.45145 | 9.40775 |
| | \( \beta_2 \) | 11.33454 | 11.33352 | 11.30617 | 11.16775 |
| | \( \beta_3 \) | 12.30612 | 12.30612 | 12.30612 | 12.30612 |
| \( M_1=M_2=M_3=0.5 \) | \( \beta_1 \) | 7.49505 | 7.49439 | 7.47710 | 7.39927 |
| | \( \beta_2 \) | 8.88683 | 8.88565 | 8.85071 | 8.54426 |
The figure 1 a, b shows respectively, the normalised first nonlinear mode shape and the corresponding curvature for the case of an Euler-Bernoulli beam, carrying three masses \( m_1 = 1; \ c_1 = 0.05; m_2 = 2; c_2 = 0.1; m_3 = 0.5; c_3 = 0.01 \) located at \( x_1 = 1/6, \ x_3 = 1/2 \) and \( x_5 = 5/6 \), resting on intermediate supports located at \( x_2 = 1/3 \) and \( x_4 = 2/3 \), denoted in what follows as C-MSMSM-C.

Figure 2. The normalised first non-linear mode and the associated curvature distribution of a C-MSMSM-C beam for various values of the vibration amplitudes.

Figure (a) illustrates, for various vibration amplitudes, the hardening type effect of geometrical non-linearity. It can be seen in figure (b) that the curvatures increase in the vicinity of the clamps. It is quite clear that the effect of the mass and its rotary inertia becomes more important with the increase of the amplitude of vibration, which justifies the objective of the present study, which aims to show that the linear analysis may be inaccurate at large vibration amplitudes.

For the parametric study below, three scenarios are considered:

1st scenario: \( m_1 = m_2 = m_3 = 0.5; c_1 = c_2 = c_3 = 0.01 \)

2nd scenario: \( m_1 = m_2 = m_3 = 1; c_1 = c_2 = c_3 = 0.01 \)

3rd scenario: \( m_1 = m_2 = m_3 = 2; c_1 = c_2 = c_3 = 0.01 \)
The masses added to the beam reduce significantly the linear frequencies. It can be noticed on the backbone curves that by increasing the mass, the frequency ratio increases, which may appear unusual. This is due to the fact that the denominator, corresponding to the linear frequencies, decreases with increasing the added masses, as may be seen in figure 4.

Before studying the forced case, and as previously defined, the determination of the predominant mode is necessary in order to perform the analysis in its neighbourhood. The following tables, present the contribution of the first symmetric linear functions.
Table 2. Percentage of generalised forces exciting the first five symmetric modes of a C-MSMSM-C excited by uniformly distributed force along the whole beam. (1st scenario: \( m_1 = 1; m_2 = 0; m_3 = 0 \)).

| Modes       | 1      | 3      | 5      | 7      | 9      |
|-------------|--------|--------|--------|--------|--------|
| \( \int w_i^* (x) \) | 1.02E-01 | 6.53E-01 | 1.17E-02 | 3.64E-01 | 3.10E-01 |
| \( \int w_i^* (x) / \sum_{i=1}^{n} \left| w_i^* (x) \right| \) | 4.715   | 30.284  | 0.54    | 16.892  | 14.362 |

Table 3. Percentage of generalised forces exciting the first five symmetric modes of a C-MSMSM-C excited by uniformly distributed force along the whole beam. (2nd scenario: \( m_1 = 0; m_2 = 1; m_3 = 0 \)).

| Modes       | 1      | 3      | 5      | 7      | 9      |
|-------------|--------|--------|--------|--------|--------|
| \( \int w_i^* (x) \) | 6.91E-02 | 6.63E-01 | 1.93E-01 | 3.64E-01 | 4.08E-01 |
| \( \int w_i^* (x) / \sum_{i=1}^{n} \left| w_i^* (x) \right| \) | 04.07   | 39.08   | 11.37   | 21.46   | 24.02   |

From the tables, it can be concluded that the force excites predominantly the 3rd mode, an analysis is then established in the vicinity of this mode in order to obtain the following response curves:

![Figure 5](image-url)

**Figure 5.** Comparison between resonance curves obtained for two different positions of the same mass of a C-MSMSM-C beam, based on the single mode approach.

For a given excitation (distributed over the length of the beam) corresponding to \( F^d = 500 \), when the added mass is located between the supports (at \( x_3 = 1/2 \)), it is observed in figure 5 that for any frequency ratio, one obtains a higher amplitude than that of the other scenario. Moreover, the opening of its response curve, leaks in a considerable way the case of free vibrations, which shows that the beam is more affected in the second scenario than the first (mass located at \( x_1 = 1/6 \)).
In the parametric study above, the three new scenarios considered are:

1\textsuperscript{st} scenario: \( m_1 = m_3 = 0; m_2 = 0.5; c_1 = c_3 = 0; c_2 = 0.05 \).

2\textsuperscript{nd} scenario: \( m_1 = m_3 = 0; m_2 = 1; c_1 = c_3 = 0; c_2 = 0.05 \).

3\textsuperscript{rd} scenario: \( m_1 = m_3 = 0; m_2 = 2; c_1 = c_3 = 0; c_2 = 0.05 \).

It is noticeable in figure 6. that for small amplitudes, the response curves vary approximately in a linear way and remain confounded for the three scenarios studied. However, for the same excitation, the maximum vibration amplitude for each frequency ratio is obtained for the first scenario. By increasing the amplitude, and consequently the nonlinear frequency, the ratio of frequencies increases and accentuated by the addition of a heavier mass, which makes it possible to conclude the considerable effect of the added mass on the nonlinear dynamic behaviour of a beam.

**Conclusion**

The nonlinear forced vibration of multi-span beams carrying multiple masses was examined, the mathematical formulation was based on Hamilton's principle and spectral analysis. First of all, using the end conditions, compatibility and continuity conditions, a homogeneous system is obtained, whose nontrivial solutions were obtained iteratively using the Newton-Raphson algorithm. The linear modes obtained have been used as a basic function in the study of nonlinear vibrations. Finally, using the single-mode approach and by determination of the predominant mode, the dynamic behaviour of the considered structure subjected to a uniformly distributed force was studied and the effect of mass positions illustrated.

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