A Derivation of the Catalan Numbers from a Bijection between Permutations and Labeled Trees

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1 Overview

Let $C_n$ denote the $n$th Catalan number, which represents (among other things) the number of distinct binary trees that have $n$ undistinguished nodes and $n+1$ undistinguished leaves. (Here the term node shall refer to nonterminal nodes only.) Now imagine that in such a tree, one assigns $n$ distinct labels to the nodes, and $n+1$ distinct labels to the leaves; let $D_n$ denote the number of possible trees with the nodes and leaves so labeled. Because there are $n!$ ways to label the nodes, and, independently, $(n+1)!$ ways to label the leaves, we see that $D_n = C_n \cdot n!(n+1)!$, and hence that $C_n = D_n / (n!(n+1)!)$.

This note presents a bijection between permutations of length $2n$ and binary trees having $n$ labeled nodes and $n+1$ labeled leaves. From the existence of this bijection one may infer that $D_n = (2n)!$. The familiar formulas for the Catalan numbers follow directly: $C_n = (2n)!/(n!(n+1)!)$ = $\binom{2n}{n}/(n+1)$. More generally, the presented bijection maps permutations of length $dn$ to labeled $n$-node $d$-ary trees, and vice versa. By an argument analogous to that for binary case, one may conclude that the number of distinct unlabeled $d$-ary trees with $n$ nodes and $m = (d-1)n + 1$ leaves is $(dn)!/(n!m!) = \binom{dn}{n}/m$.

Section 2 below illustrates the bijection for the case of binary trees. Section 3 gives an algorithm that maps permutations to labeled $d$-ary trees, and Section 4 gives the inverse algorithm. Section 5 gives a more abstract characterization of the bijection. Section 6 concludes.
2 Tree Construction using Permutations

Let us begin by considering how one might construct a labeled binary tree from tree fragments, using a permutation to guide the construction.

![Figure 1: Tree fragments](image)

Figure 1a depicts a leaf labeled 2 and a node labeled 1. Each of these tree fragments has an upward sprout, marked “+”, directed toward an unresolved parent; the node additionally has left and right downward sprouts, marked “−”, directed toward unresolved children. We associate with each sprout a unit charge of the indicated sign. The depicted leaf thus carries a net charge of +1, while the node carries a net charge of −1 (i.e., one positive and two negative unit charges). In the text, let $\ell_i$ denote a leaf with label $i$, and let $(u \odot_i v)$ denote a node with label $i$ and with left and right subtrees $u$ and $v$, respectively; let “−” denote an unresolved subtree. Then Figure 1a depicts $\ell_2$ and $(− \odot_1 −)$.

Figure 1b shows the same leaf and node after two of the sprouts have been connected by an edge. The resultant structure is the incomplete tree $(\ell_2 \odot_1 −)$. When two sprouts of opposite sign are connected together, their charges annihilate; total charge is thus conserved. In the present instance, the leaf and node each acquire a net charge of zero when they are connected.

In Figure 1c we see the atomic fragments needed to construct an arbitrary labeled 2-node binary tree. (We assume without loss of generality that a labeled 2-node binary tree has node labels 0 and 1, and leaf labels 2, 3, and 4.) The combined charge of all the fragments comes to +1; moreover, since charge is conserved when fragments are connected, any structure built from these fragments must also carry a total charge of +1. A consequence that we rely on below is that any such structure must contain at least one
individual node or leaf whose net charge is positive, and hence +1 (since no other net positive charge can arise at a single node or leaf). As illustrated in the figure, to identify the left and right downward sprouts of node \( i \) we use the numbers \( 2i \) and \( 2i+1 \), respectively. (We do not number upward sprouts.) Unlike node and leaf labels, sprout numbers may not be freely reassigned: they stand in a fixed relationship to the corresponding node labels.

Now suppose we wish to use a permutation—say, \( \langle 3, 2, 0, 1 \rangle \)—to guide the construction of a labeled 2-node binary tree. Figure 2 illustrates the step-by-step construction of a tree from the fragments in Figure 1c. Each panel of Figure 2 augments the construction with a new edge shown as a dashed line. The downward sprouts participating in the new edges are numbered 3, 2, 0, and 1, respectively; in other words, we have used the permutation \( \langle 3, 2, 0, 1 \rangle \) to decide the order in which the downward sprouts acquire children. The child assigned to each new edge is determined as follows: it is a node or leaf whose net charge is +1, and among all such nodes and leaves, it is the one whose label is smallest. (As noted above, at least one such node or leaf must always exist.) Thus, in Figure 2a the new child is \( \ell_2 \), and in Figure 2b it is \( \ell_3 \). The node labels 0 and 1 are smaller than the leaf labels 2 and 3, but it is only after acquiring the children \( \ell_2 \) and \( \ell_3 \) that one of the nodes—node 1—attains a net charge of +1. Node 1 then becomes the new child in Figure 2c; \( \ell_4 \) finally plays the role of new child in Figure 2d. At the end of this construction we have the tree \( (((\ell_3 \circ_1 \ell_2) \circ_0 \ell_3) \circ_1 \ell_4) \). The completed tree can be characterized as a finite function from downward sprout numbers to child labels, namely \( \{0 \mapsto 1, 1 \mapsto 4, 2 \mapsto 3, 3 \mapsto 2\} \), or, equivalently, as a vector of child pointers \( \langle 1, 4, 3, 2 \rangle \). This vector contains every node and leaf
label except that of the tree’s root (node 0 in this example).

Distinct permutations yield distinct labeled trees under this construction, as we shall establish through the algorithms presented below. Here we merely show that the illustrated construction must yield a tree, as opposed to a cyclic or disconnected graph of some kind.

Call a tree or subtree complete if it has no unresolved descendants; call a node complete if it is the root of a complete tree or subtree. Leaves are trivially complete. Now imagine that the acquisition of a child $c$ by a parent $p$ results in a cycle among the tree fragments; this outcome is possible only if $c$ was previously an ancestor of $p$. But since $p$ was previously incomplete (else it could not have acquired the child $c$), its ancestor $c$ was also incomplete. Thus, a cycle can arise only when a node acquires an incomplete child.

At each step of the construction above we required that the new child be a node or leaf with net charge +1. That requirement implied inductively that each such node or leaf was complete: it had no unconnected downward sprouts (which would have contributed negative charge), and its children, if any, must have themselves been complete when it acquired them. By ensuring completeness of new children, the requirement of net charge +1 thus precluded cycles. In addition, the presence of positive charge indicated that a prospective child did not already have a parent, so no separate bookkeeping was needed to make that determination.

Each time an edge was added to the construction, two previously disconnected tree fragments became connected, and the total number of disconnected fragments decreased by one. (The two fragments joined by the edge could not have been connected previously, because if they had been, then connecting them anew would have created a cycle—a possibility we have ruled out.) After the addition of four edges, therefore, the five initial fragments of Figure 1c necessarily coalesced into a connected acyclic graph. The unique node carrying net charge +1 at the end of the construction had to be the root of a complete tree which, by connectivity, could be counted on to contain every node and leaf in the graph.

### 3 A General Algorithm

The construction from Section 2 above extends straightforwardly to $d$-ary trees. Consider a complete $d$-ary tree with $n$ nodes and $m$ leaves. Each node is parent to $d$ edges, giving $dn$ edges altogether. In a construction that adds one edge connecting two fragments on each step, the number of construction steps must equal the number of edges $dn$, and the initial
for $i = 0$ to $dn$

\[
\begin{align*}
\text{if } i < n & \text{ then let } \text{charge}[i] = +1 + d \cdot (-1) \\
\text{if } i \geq n & \text{ then let } \text{charge}[i] = +1
\end{align*}
\]

next $i$

for $i = 0$ to $dn - 1$

\[
\begin{align*}
\text{for } j = 0 \text{ to } dn & \\
\text{if } \text{charge}[j] = +1 & \text{ then exit for next } j
\end{align*}
\]

\[
\begin{align*}
\text{let } k &= \text{perm}[i] \\
\text{let } \text{kid}[k] &= j
\end{align*}
\]

\[
\begin{align*}
\text{let } \text{charge}[j] &= 0 \\
\text{let } \text{charge}[\text{int}(k/d)] &= \text{charge}[\text{int}(k/d)] + 1
\end{align*}
\]

next $i$

Figure 3: Code for mapping a permutation to a labeled tree

number of fragments $n + m$ must exceed this value by 1; that is, $n + m = dn + 1$. This constraint on $d$, $n$, and $m$ can be seen in terms of charge as well: Each of the $dn$ construction steps consumes a positive charge at one of the $n + m$ initial upward sprouts, and a negative charge at one of the $dn$ initial downward sprouts; when the tree is fully connected, only a single positive charge survives at the root. We thus confirm the constraint $(n + m - dn) - (dn - dn) = +1$, or $n + m - dn = +1$, with the corollary $m = (d - 1)n + 1$.

Figure 3 gives a True BASIC program that maps permutations of length $dn$ to labeled $d$-ary trees. Assumed given are $d$, $n$, and a permutation $\text{perm}[0], \ldots, \text{perm}[dn - 1]$ of the integers $0, \ldots, dn - 1$. From these inputs the program computes a representation of a labeled $n$-node $d$-ary tree as a vector of child pointers $\text{kid}[0], \ldots, \text{kid}[dn - 1]$, where $\text{kid}[dq + r]$ gives the label of the $r$th child of node $q$, for $0 \leq q < n$ and $0 \leq r < d$. The integers $0, \ldots, n - 1$ serve as node labels, and $n, \ldots, dn$ as leaf labels.

The code segment marked $a$ in Figure 3 initializes the array $\text{charge}[0], \ldots, \text{charge}[dn]$, which gives, for each $i$, the net charge of the node or leaf labeled $i$. Each node starts out with net charge $1 - d$ (reflecting one upward and $d$ downward sprouts), and each leaf, with $+1$ (reflecting just an upward sprout). As before, we have the invariant $\sum_i \text{charge}[i] \equiv +1$, which again
Figure 4: A correspondence between permutations and labeled binary trees

assures us that at every step of our construction there will be some node or leaf whose net charge is positive.

The remainder of Figure 3 consists of a loop on $i$ that iteratively performs one construction step, as follows: First (in code segment $b$), it finds the smallest label $j$ such that $\text{charge}[j] = +1$; second (in code segment $c$), it makes node or leaf $j$ the child of downward sprout number $k$, where $k = \text{perm}[i]$; and third (in code segment $d$), it updates the charges associated with node or leaf $j$ and with its new parent. The expression $\text{int}(k/d)$ in code segment $d$ denotes $\lfloor k/d \rfloor$, which yields the label of the parent node that owns downward sprout number $k$. When the code in Figure 3 completes after $dn$ loop iterations, the resultant tree’s root label can be ascertained by once again computing $\lfloor k/d \rfloor$; in general, this expression identifies a node that has just acquired a child, and the node that acquires a child last must be the root.

Figure 4 shows a correspondence between permutations of length 4 and labeled 2-node binary trees. This correspondence was obtained by running the algorithm of Figure 3 on each possible permutation of $(0, 1, 2, 3)$ with $d = 2$ and $n = 2$. For readability, the nodes in Figure 4 are designated + and $\times$, and the leaves, $a$, $b$, and $c$. Observe that $a$ is the left-hand operand of + in the first six expressions, which correspond to permutations that start with 0; in the next six, $a$ is the right-hand operand of +, reflecting permutations that start with 1; and so on. Similar patterns within each
group of six expressions reflect the second, third, and fourth components of the
permutations.

4 Inverting the Algorithm

The algorithm of Figure 3 can easily be run “in reverse” to perform the
inverse mapping from labeled \( d \)-ary trees to permutations. That is, given
\( d \), \( n \), and a representation of a tree in the \( \text{kid}[] \) array, the algorithm can be
made to retrace the construction of that tree, and to compute in \( \text{perm}[] \) the
permutation that would have caused that particular tree to be constructed.

To achieve this reversal, one need only replace code segment \( c \) in Figure 3
with code segment \( c' \) from Figure 5. The difference between \( c \) and \( c' \) is that
where the former consults \( \text{perm}[] \) and assigns to \( \text{kid}[] \), the latter does the
opposite: it consults \( \text{kid}[] \) and assigns to \( \text{perm}[] \). Neither array is referenced
anywhere else in the algorithm, so it is immaterial which array is the source
of information, and which the recipient. The only aspect of code segment
\( c \) or \( c' \) that matters to the rest of the algorithm is that (aside from not
disturbing \( i \), \( j \), or \( \text{charge}[] \)) this code segment must furnish in \( k \) a succession
of distinct values from 0, \ldots , \( dn - 1 \). In the case of code segment \( c \), this
requirement is met in that the values \( k \) come from the permutation \( \text{perm}[] \).
In the case of \( c' \), the values \( k \) are the indices of the child pointers that point
to the labels \( j \) of all the non-root nodes and leaves of the given tree; since
each such node or leaf is the target of a unique child pointer, each value
\( k \) in 0, \ldots , \( dn - 1 \) will be furnished exactly once. In either case, we have
\( \text{perm}[i] = k \) and \( \text{kid}[k] = j \) after execution of code segment \( c \) or \( c' \). Thus,
the relationship between \( \text{perm}[] \) and \( \text{kid}[] \) will be the same whether the
algorithm is run forward or “in reverse.”

Code segment \( b \) in Figure 3 chose, from among the nodes and leaves

\[
\begin{align*}
\text{for } k = 0 \text{ to } dn - 1 \\
\text{if } \text{kid}[k] = j \text{ then exit for} \\
\text{next } k \\
\text{let } \text{perm}[i] = k
\end{align*}
\]

Figure 5: Code revision for mapping a labeled tree to a permutation
5 An Abstract Characterization

The mappings presented above also admit a more abstract characterization. Assume $d$ and $n$ fixed, and let $Q = \{0, \ldots, n-1\}$ and $R = \{0, \ldots, d-1\}$. For $q$ in $Q$, let $\phi_q(\ldots, u_r, \ldots)$ denote a node labeled $q$ with immediate subtrees $u_r$ for $r$ in $R$. If $d = 2$, the notation $\phi_q(u_0, u_1)$ is equivalent to $(u_0 \circ q \ u_1)$.

We define the $\text{maxleaf}$ and $\text{height}$ of a tree in the obvious way:

$$\begin{align*}
\text{maxleaf } \ell_p &= p \\
\text{maxleaf } (\phi_q(\ldots, u_r, \ldots)) &= \text{maxleaf } u_r, \\
\text{height } \ell_p &= 0 \\
\text{height } (\phi_q(\ldots, u_r, \ldots)) &= 1 + \text{max} \{\ldots, \text{height } u_r, \ldots\}
\end{align*}$$

We then define the relation $u \prec v$ to hold on trees $u$ and $v$ just if $\text{maxleaf } u < \text{maxleaf } v$, or if $\text{maxleaf } u = \text{maxleaf } v$ and $\text{height } u < \text{height } v$.

Now let $t$ be a $d$-ary tree with node labels $0, \ldots, n-1$, and leaf labels $n, \ldots, dn$. We define the permutation $P(t)$ as follows. First, for $k$ in $0, \ldots, dn-1$, we define $\sigma_k(t)$, or $\sigma_k$ for short, to be the proper subtree $v$ of $t$ such that $v$ is the $r$th child of the node labeled $q$, where $q = \lfloor k/d \rfloor$ and $r = k - dq$. We then take $P(t)$ to be the permutation $\pi$ on $0, \ldots, dn-1$ such that $\sigma_{\pi(0)} \prec \ldots \prec \sigma_{\pi(dn-1)}$. This permutation is well-defined because the proper subtrees of $t$ must be totally ordered under $\prec$: if two such subtrees share the same maximum leaf label (or indeed if they share any node or leaf at all), they must be of different heights, or else they are the same subtree.

Conversely, if $\pi$ is a permutation on $0, \ldots, dn-1$, we define the tree $T(\pi)$ as follows. First, for $q$ in $Q$, we define $\iota(q) = 1 + \text{max} \{\pi^{-1}(dq + r) \mid r \in R\}$. Since $\iota$ is injective, we may unambiguously define, for $i$ in $0, \ldots, dn$,

$$\tau_i = \begin{cases} 
\phi_q(\ldots, \tau_{\pi^{-1}(dq+r)}, \ldots) & \text{if } i = \iota(q) \text{ for some } q \in Q, \\
\ell_{n+i-\# \{q \in Q \mid \iota(q) < i\}} & \text{otherwise}.
\end{cases}$$

We then take $T(\pi)$ to be $\tau_{dn}$.

**Example** Let $d = 2$ and $n = 2$, and suppose the tree $t = \phi_0(\phi_1(\ell_3, \ell_2), \ell_4)$ is given. Its proper subtrees are $\sigma_0 = \phi_1(\ell_3, \ell_2)$, $\sigma_1 = \ell_4$, $\sigma_2 = \ell_3$, and $\sigma_3 = \ell_2$. These subtrees fall in the order $\sigma_3 \prec \sigma_2 \prec \sigma_0 \prec \sigma_1$, which
induces the permutation \( \pi = \langle 3, 2, 0, 1 \rangle \); thus we have \( P(t) = \langle 3, 2, 0, 1 \rangle \). Conversely, suppose the permutation \( \pi = \langle 3, 2, 0, 1 \rangle \) is given. The indices \( \iota(q) \) are then \( \iota(0) = 1 + \max \{\pi^{-1}(0), \pi^{-1}(1)\} = 1 + \max \{2, 3\} = 4 \), and \( \iota(1) = 1 + \max \{\pi^{-1}(2), \pi^{-1}(3)\} = 1 + \max \{1, 0\} = 2 \). We then have \( \tau_0 = \ell_2; \tau_1 = \ell_3; \tau_2 = \phi_1(\tau_{\pi^{-1}(2)}, \tau_{\pi^{-1}(3)}); \tau_3 = \ell_4; \) and \( \tau_4 = \phi_0(\tau_{\pi^{-1}(0)}, \tau_{\pi^{-1}(1)}); \phi_0(\tau_2, \tau_3) = \phi_0(\phi_1(\ell_3, \ell_2), \ell_4) \). \( \square \)

We now sketch a proof that \( P \) and \( T \) are inverses. To begin, suppose we apply \( T \) to a given \( \pi \). In the definition of \( T \), each \( \tau_i \) acquires a distinct root label, hence all labels in \( 0, \ldots, dn \) are represented. Moreover, for \( i \) in \( 0, \ldots, dn - 1 \), the parent of \( \tau_i \) is some \( \tau_{i'} \) with \( i < i' \leq dn \), so by transitivity, each \( \tau_i \) is a subtree of \( \tau_{dn} = T(\pi) \). By construction, we also have \( \tau_i \prec \tau_{i+1} \) for each \( i \), hence \( \tau_0 \prec \cdots \prec \tau_{dn} \). Now suppose we apply \( P \) to \( T(\pi) \).

The \( r \)th child of node \( q \) in \( T(\pi) \) was defined to be \( \tau_{\pi^{-1}(dq+r)} \); so from the definition of \( \sigma_k(T(\pi)) \) it follows that \( \sigma_{dq+r}(T(\pi)) = \tau_{\pi^{-1}(dq+r)} \); equivalently, we have \( \sigma_k(T(\pi)) = \tau_{\pi^{-1}(k)} \) for \( k = 0, \ldots, dn - 1 \). Letting \( i = \pi^{-1}(k) \), so that \( \pi(i) = k \), we obtain \( \sigma_{\pi(i)}(T(\pi)) = \tau_i \) for \( i = 0, \ldots, dn - 1 \), which permits us to rewrite \( \tau_0 \prec \cdots \prec \tau_{dn-1} \) as \( \sigma_{\pi(0)}(T(\pi)) \prec \cdots \prec \sigma_{\pi(dn-1)}(T(\pi)) \). Then by the definition of \( P \), we have \( P(T(\pi)) = \pi \).

Next consider \( \hat{t} = T(P(t)) \). Let \( \pi = P(t) \); then \( P(\hat{t}) = P(T(P(t))) = P(T(\pi)) = \pi \) also. Let us extend \( \pi \) with \( \pi(dn) = dn \), and for all \( t \) let \( \sigma_{dn}(t) \) denote \( t \); we then have \( \sigma_{\pi(0)}(t) \prec \cdots \prec \sigma_{\pi(dn)}(t) = t \), and similarly for \( \hat{t} \). We shall show by induction on \( i \) that \( \sigma_{\pi(i)}(\hat{t}) = \sigma_{\pi(i)}(t) \), and hence that \( \hat{t} = t \). If \( \sigma_{\pi(i)}(\hat{t}) \) is a node, it must have the form \( \phi_q(\ldots, \sigma_{dq+r}(\hat{t}), \ldots) \), with \( \sigma_{dq+r}(\hat{t}) \prec \sigma_{\pi(i)}(\hat{t}) \) for each \( r \) in \( R \). By the inductive hypothesis, \( \sigma_{dq+r}(\hat{t}) = \sigma_{dq+r}(t) \) for each \( r \), hence \( \phi_q(\ldots, \sigma_{dq+r}(\hat{t}), \ldots) \) is also a subtree of \( t \). Alternatively, if \( \sigma_{\pi(i)}(\hat{t}) \) is just a leaf \( \ell_j \), this leaf must also occur as a subtree of \( t \). In either case, we deduce that \( \sigma_{\pi(i)}(\hat{t}) = \sigma_{\pi(i')}(t) \) for some \( i' \geq i \). Symmetrically, we have \( \sigma_{\pi(i)}(t) = \sigma_{\pi(i'')}(\hat{t}) \) for some \( i'' \geq i \). It follows that \( \sigma_{\pi(i)}(\hat{t}) \leq \sigma_{\pi(i'')}(\hat{t}) = \sigma_{\pi(i)}(t) \leq \sigma_{\pi(i')}(t) = \sigma_{\pi(i)}(\hat{t}) \), and hence that \( \sigma_{\pi(i)}(\hat{t}) = \sigma_{\pi(i)}(t) \).

We remark without proof that \( T \) and \( P \) are exactly the mappings of Sections 3 and 4.

6 Conclusion

We have presented an algorithm for mapping permutations to labeled trees, as well as a variant of that algorithm that performs the inverse mapping. By establishing that these mappings are bijective, we have shown that each
of the factorials in the formulas for the Catalan numbers and their $d$-ary analogues can be understood as a count of permutations.

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