On hyperelliptic $C^\infty$-Lefschetz fibrations of four-manifolds

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Abstract

We show that hyperelliptic symplectic Lefschetz fibrations are symplectically birational to two-fold covers of rational ruled surfaces, branched in a symplectically embedded surface. This reduces the classification of genus 2 fibrations to the classification of certain symplectic submanifolds in rational ruled surfaces.

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Introduction

A differentiable Lefschetz fibration (DLF) of an oriented, differentiable four manifold $M$ is a proper differentiable map $q : M \to S^2$ with finitely many critical points $Q_1, \ldots, Q_\mu \in M$ in disjoint fibers and such that locally near $Q_r$ there exist complex coordinates $z, w$ on $M$ and $t$ on $S^2$ with $q : (z, w) \mapsto zw$. If $(z, w)$ and $t$ can be chosen compatible with the orientations of $M$ and $S^2$, then $q : M \to S^2$ is called a symplectic Lefschetz fibration (SLF). According to an observation of Gompf a DLF is a SLF iff there exists a symplectic form $\omega$ on $M$ compatible with $q$ in the sense that smooth fibers of $q$ are symplectic submanifolds with respect to $\omega$. An isomorphism of two DLF’s or SLF’s $q : M \to S^2$ and $q' : M' \to S^2$ is an oriented diffeomorphism of total spaces $\Psi : M \to M'$ respecting the fibration structures.

Given two SLF’s (or DLF’s) $q : M \to S^2$, $q' : M' \to S^2$ and an (orientation preserving) diffeomorphism of smooth fibers $\Phi : q^{-1}(s_0) \cong q'^{-1}(s'_0)$ the fiber connected sum of $q$ and $q'$ is the SLF (DLF) obtained by identifying the boundaries of $q^{-1}(S^2 \setminus B_\varepsilon(s_0))$ and $q'^{-1}(S^2 \setminus B_\varepsilon(s'_0))$ via $\Phi$. The origin of this paper is a question of one of the authors (G.T.) if every SLF can be decomposed into a fiber sum of algebraic...
Lefschetz fibrations. This is true for elliptic fibrations \((g = 1, \text{MC})\). To the authors knowledge there still exists no example of a SLF that is known to not decompose in this way. Discussions during the 1993/1994 stay of the first named author at the Courant Institute suggested that at least for fibrations by curves of genus 2, one should expect algebraicity when all singular fibers are irreducible. The work took on more concrete form during the academic year 1997/1998, where we looked at this problem from the point of view of symplectic geometry inside \(S^2\)-bundles \(P \to S^2\). This paper contains the first, rather elementary step of reduction to the classification problem of symplectic submanifolds of \(P\) up to isomorphism. The authors have a program to attack the slightly stronger problem of classification up to isotopy with the technique of pseudo-holomorphic curves.

The results of this paper were presented at the Gökova Geometry-Topology conference (May 1998) and at the Geometry conference at Schloss Ringberg (June 1998).

While this paper was under preparation we learnt that Ivan Smith proved similar results in his Oxford thesis. We apologize for any duplication of results.

1 Hyperelliptic fibrations and branched covers

Our main focus in this paper will be on so-called hyperelliptic SLF’s, which are defined in terms of the monodromy as we will now describe. Let \(s_1, \ldots, s_\mu \in S^2\) be the critical values of \(q\). The restriction of \(q\) to \(S^2 \setminus \{s_1, \ldots, s_\mu\}\) is a fiber bundle with fibers a closed oriented surface \(\Sigma\) of some genus \(g\), the genus of the DLF or SLF. The corresponding monodromy representation

\[
\rho: \pi_1(S^2 \setminus \{s_1, \ldots, s_\mu\}) \to \text{MC}_g = \text{Diff}^+(\Sigma)/\text{Diff}^0(\Sigma)
\]

sends closed loops running counterclockwise once around some \(s_r\) (simple loops) to a Dehn-Twist. Since for loops \(\gamma, \gamma'\) the product \(\gamma\gamma'\) means running through \(\gamma\) first, we let the mapping class group act on \(\Sigma\) from the right to make \(\rho\) a homomorphism. Note also that we do admit Dehn-twists along contractible loops. These occur iff an irreducible component of the singular fiber is an embedded sphere of self-intersection \(-1\). The Dehn-twist is right-handed if the fibration is oriented locally near \(s_r\), otherwise left-handed. The monodromy representation determines the fibration in the following precise way.

**Theorem 1.1** [Ka] There is a one-to-one correspondence between the set of isomorphism classes of SLF’s of genus \(g \geq 2\) and with \(\mu\) singular fibers and the set of representations \(\rho: \pi_1(S^2 \setminus \{s_1, \ldots, s_\mu\}) \to \text{MC}_g\) sending simple loops to right-handed Dehn-twists modulo composition with inner automorphisms of \(\text{MC}_g\). The same result holds for \(g = 1\) provided there is at least one irreducible singular fiber.

By the same proof the analogous statement holds for DLF’s and arbitrary Dehn-twists. Note also that \(\pi_1(S^2 \setminus \{s_1, \ldots, s_\mu\})\) can be generated by \(\mu\) simple loops \(\gamma_1, \ldots, \gamma_\mu\) obeying the single relation \(\gamma_1 \cdots \gamma_\mu = 1\), so a DLF (SLF) can be defined simply by a relation between (right-handed) Dehn-twists.

To define the hyperelliptic mapping class group we fix once and for all a two-sheeted branched cover \(\kappa: \Sigma \to S^2\), necessarily having \(2g + 2\) branch points. Then the hyperelliptic mapping class group \(\text{HMC}_g \subset \text{MC}_g\) is the subgroup generated by
diffeomorphisms of $\Sigma$ that are compatible with $\kappa$, that is, which commute with the hyperelliptic involution swapping the sheets of $\kappa$. Finally a DLF or SLF is hyperelliptic if the monodromy representation is equivalent (by an inner automorphism of $\text{MC}_g$) to one taking values in $\text{HMC}_g$.

The hyperelliptic mapping class group is a $\mathbb{Z}/2\mathbb{Z}$-extension of the mapping class group $\text{MC}(S^2, 2g + 2)$ of $S^2$ with $2g + 2$ marked points. The copy of $\mathbb{Z}/2\mathbb{Z}$ is of course generated by the hyperelliptic involution, while the map to $\text{MC}(S^2, 2g + 2)$ is the map to the induced diffeomorphism of the base, mapping the set of $2g + 2$ branch points to itself (well-definedness of this map requires some work [Bi2]). To write down a presentation of $\text{MC}(S^2, 2g + 2)$ let $x_1, \ldots, x_{2g+1}$ be a standard generating set of Dehn-twists of $(S^2 = \mathbb{C} \cup \{\infty\}; P_1, \ldots, P_{2g+2})$, $P_r = \exp(\frac{2\pi ir}{2g+2})$, represented by a diffeomorphism $\Phi_r$ that is identical outside of a small neighbourhood of the straight line connecting $P_r$ and $P_{r+1}$, while these points get exchanged. Then $\text{MC}(S^2; 2g + 2)$ has the presentation [Bi2, p.164]

$$\text{MC}(S^2, 2g + 2) = \left\langle x_1, \ldots, x_{2g+1} \mid x_i x_j = x_j x_i \quad (|i-j| \geq 2); \ x_i x_{i+1} x_i = x_{i+1} x_i x_{i+1} \quad x_1 \ldots x_{2g+1} x_{2g+1} \ldots x_1 = 1 \quad (x_1 \ldots x_{2g+1})^{2g+2} = 1 \right\rangle$$

The $x_i$ lift to right-handed Dehn-twists on $\Sigma$, that we will denote by the same notation. These Dehn-twists still fulfill the braid relations, but $x_1 \ldots x_{2g+1} x_{2g+1} \ldots x_1$ becomes the hyperelliptic involution $I$. In fact, $\text{HMC}_g$ has the presentation [Bi2]

$$\text{HMC}_g = \left\langle x_1, \ldots, x_{2g+1}, I \mid x_i x_j = x_j x_i \quad (|i-j| \geq 2); \ x_i x_{i+1} x_i = x_{i+1} x_i x_{i+1} \quad I x_i = x_i I; \ I^2 = 1 \quad (x_1 \ldots x_{2g+1})^{2g+2} = 1 \right\rangle$$

This should be contrasted to the complexity of $\text{MC}_g$ ($g \geq 3$), which needs one more Dehn-twist for a generating set and a number of more involved relations [We]. Our main suggestion in this paper is to study hyperelliptic SLF’s by means of branch loci in $S^2$-fibrations over $S^2$. This is based on Theorem 1.2 below.

Before stating the theorem we need a few more notions. A map $\sigma : \tilde{M} \to M$ between oriented, differentiable four-manifolds will be called the blow-up of $M$ in $P \in \tilde{M}$ if $\sigma$ is a diffeomorphism away from $P$ and if there is a neighbourhood $U$ of $P$ such that $\sigma^{-1}(U) \to U$ is oriented diffeomorphic to the blow up of a point in an open set in $\mathbb{C}^2$. The contracted set $\sigma^{-1}(P)$ will be called exceptional divisor.

It is a well-known fact that holomorphic fibrations over $\mathbb{P}^1$ with general fiber $\mathbb{P}^1$ and smooth total space are simply blow ups of $\mathbb{P}^1$-bundles over $\mathbb{P}^1$. As an ad-hoc definition we define $S^2$-fibrations over $S^2$ in the differentiable setting as blow ups of $S^2$-bundles over $S^2$. One class of $S^2$-fibrations will be of special interest, namely those where each singular fiber is of type $\Lambda_2$, which by definition means that locally it is obtained from an $S^2$-bundle by blowing up one point and then another general point on the exceptional divisor thus obtained. Such fibers consist of a chain of embedded spheres with self-intersections $-1$, $-2$, $-1$.

**Theorem 1.2** A SLF $q : M \to S^2$ is hyperelliptic iff it fits into a commutative diagram of the form

$$\begin{array}{cccc}
\tilde{q} : & \tilde{M} & \to & \tilde{P} & \to & S^2 \\
\sigma \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
q : & M & \to & P & \to & S^2
\end{array}$$

(1)
1. $\sigma$ is the blow up of $M$ in the nodes of reducible singular fibers of $q$

2. $\tilde{\rho} : \tilde{P} \to S^2$ is an $S^2$-fibration with all singular fibers of type $A_2$

3. $\tilde{\kappa}$ is a two-fold branched cover with branch locus $B + B_E \subset \tilde{P}$ where $\tilde{\kappa}^{-1}(B_E)$ is the exceptional divisor of $\sigma$ (so $B_E$ is a disjoint union of $(-2)$-spheres in the fibers of $\tilde{\rho}$) and $B$ has no components in the fibers of $\tilde{\rho}$

4. $\rho$ contracts $B_E$, leading to an orbifold $P$ with an $A_1$-singularity (locally isomorphic to $C^2$ modulo the diagonal $\mathbb{Z}/2\mathbb{Z}$-action) for each component of $B_E$

Before we can give the proof we need some preparations. We begin with the braid group $B(S^2, 2g + 2)$ of $S^2$ with $2g + 2$ strands [B]. It has the presentation

$$B(S^2, 2g + 2) = \left\{ x_1, \ldots, x_{2g+1} \middle| x_i x_j = x_j x_i \text{ for } |i - j| \geq 2; x_i x_{i+1} x_i = x_{i+1} x_i x_{i+1} \text{ for } 1 \leq i \leq 2g+1 \right\}.$$

Our notations are consistent in that the element $x_i$ represents the movement that exchanges the $i$-th and $(i + 1)$-th strands by a positive half-twist. So the natural homomorphism $B(S^2, 2g + 2) \to \text{MC}(S^2, 2g + 2)$ induces the presentation of $\text{MC}(S^2, 2g + 2)$ above. One can show that $(x_1 \cdot \ldots \cdot x_{2g+1})^{2(2g+2)} = 1$ in this group [B, p154f]. Geometrically this relation can be seen by pulling $2g + 2$ parallel strands across $\infty$: This operation results in two full twists of the bunch of strands (in fact, $(x_1 \cdot \ldots \cdot x_{2g+1})^{2g+2}$ represents one twist). On the other hand, on the side of the mapping class group $\text{MC}(S^2, 2g + 2)$ a full twist induces a rotation of $S^2$ by $2\pi$, and hence is trivial. So $B(S^2, 2g + 2)$ is another (central) extension of $\text{MC}(S^2, 2g + 2)$ by $\mathbb{Z}/2\mathbb{Z}$, non-isomorphic to $\text{HMC}_g$.

The importance of $B(S^2, 2g + 2)$ in our situation comes from the fact that homomorphisms

$$\rho' : \pi_1(C \setminus \{s_1, \ldots, s_n\}) \to B(S^2, n)$$

(modulo inner automorphisms of $B(S^2, n)$) classify isotopy classes of closed submanifolds $B^* \subset S^2 \times (C \setminus \{s_1, \ldots, s_n\}$) mapping to $C \setminus \{s_1, \ldots, s_n\}$ as $n$-sheeted unbranched covers via monodromy. We will use the following lemma to relate the monodromy of the branch locus to the monodromy of the fibration.

**Lemma 1.3** For $\varepsilon < \min\{\text{dist}_{r\neq 0}(s_r, s_{r'})\}/2$ put $S^* = C \setminus B_{\varepsilon}(\{s_1, \ldots, s_n\})$, $P^* = S^* \times S^2$ and let $\kappa : M^* \to P^*$ be a two-to-one cover with branch locus $B^* \subset P^*$. We furthermore assume that the composition $B^* \hookrightarrow P^* \to S^*$ is a $(2g + 2)$-sheeted unbranched cover. Let $p : P^* \to S^*$ and $q : M^* \to S^*$ be the corresponding fiber bundles. Choose $s_0 \in S^*$ and compatible isomorphisms $q^{-1}(s_0) \simeq \Sigma$ and $p^{-1}(s_0) \cap B^* \simeq (S^2; P_1, \ldots, P_{2g+2})$.

Then the monodromy representations describing $M^* \to S^*$ and $B^* \hookrightarrow P^* \to S^*$

$$\rho : \pi_1(S^*) \to \text{HMC}_g$$

$$\rho' : \pi_1(S^*) \to B(S^2, 2g + 2)$$

are compatible: They induce the same homomorphism into $\text{MC}(S^2, 2g + 2)$. 

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[4]
Proof. Let $\gamma : S^1 \to S^*$ be a closed loop, $\gamma(0) = s_0$. Let $\mathcal{F}$ be a horizontal foliation for $p_\gamma : P_\gamma := S^1 \times \gamma P^* \to S^1$ such that $B_\gamma = S^1 \times \gamma B^* \subset P_\gamma$ is a leaf. The monodromy $\rho(^\gamma)$ is represented by the flow of $S^2$ obtained by following the leaves and using the given trivialization $P_\gamma = S^1 \times S^2$. The corresponding element of $MC(S^2, 2g + 2)$ can then be represented by the self-diffeomorphism $\Phi_{\gamma, \mathcal{F}}$ from lifting the identical path $S^1 \to S^1$. Note that $\Phi_{\gamma, \mathcal{F}}$ maps $B \cap \gamma^{-1}(s_0)$ onto itself because $B_\gamma$ is a leaf.

On the other hand, the fact that $B_\gamma$ is a leaf allows us to lift $\mathcal{F}$ to a foliation $\bar{\mathcal{F}}$ of $M^*$. Thus $\rho(^\gamma)$ is represented by a self-diffeomorphism $\Psi_{\gamma, \mathcal{F}}$ of $\Sigma = q^{-1}(s_0)$ commuting with $\Phi_{\gamma, \mathcal{F}}$ via $\kappa$:

\[
\begin{array}{ccc}
\Sigma & \xrightarrow{\Psi_{\gamma, \mathcal{F}}} & \Sigma \\
\kappa \downarrow & & \downarrow \kappa \\
S^2 & \xrightarrow{\Phi_{\gamma, \mathcal{F}}} & S^2
\end{array}
\]

This is what we claimed. \hfill $\diamond$

Another ingredient in the proof of the theorem is a list of local models for (possibly singular) branch loci $B \subset \Delta \times S^2$ producing all occurring Dehn-twists. For Lefschetz fibrations the monodromy can be represented by a Dehn-twist $\Psi$ along an embedded circle $C \simeq S^1 \subset \Sigma$, the vanishing cycle, cf. e.g. [Ka, Thm.2.1]. Up to a diffeomorphism of $\Sigma$ there are $2 + \left[\frac{g}{2}\right]$ different kinds of embedded circles. They can be distinguished by checking if $\Sigma \setminus C$ is connected or disconnected and in the latter case by the genera of the two components. Accordingly there are only $2 + \left[\frac{g}{2}\right]$ essentially different kinds of Dehn-twists. For monodromies of Lefschetz pencils along simple loops the type of Dehn-twist can be read off from the genera of the irreducible components of the enclosed singular fiber. If $\Psi$ commutes with the hyperelliptic involution $I$ (the hyperelliptic case) then $C$ is isotopic to $I(C)$ and we can even assume $C = I(C)$:

**Lemma 1.4** Let $\kappa : \Sigma \to S^2$ be a two-fold branched cover and $I : \Sigma \to \Sigma$ the hyperelliptic involution exchanging the branches of $\kappa$. Assume that $\gamma : C \subset \Sigma$ is a simple closed curve which is homotopic to $I(C)$. Then $C$ is isotopic to a simple closed curve $C' \subset \Sigma$ with $C' = I(C')$.

**Proof.** We endow $\Sigma$ with the complex structure making $\kappa$ a holomorphic covering, so $\Sigma$ becomes a hyperelliptic Riemann surface and $\kappa$ the 1-canonical map. The Poincaré metric $h$ on $\Sigma$ is then invariant under the hyperelliptic involution $I$. Recall the classical fact that in any free homotopy class of closed loops on a closed surface with negative curvature there is a unique closed geodesic loop. We define $C' \subset \Sigma$ as the unique geodesic loop with respect to $h$ freely homotopic to $C$. By $I$-invariance of $h$ the loop $I(C')$ is also a geodesic loop, homotopic to $I(C)$. But by hypothesis, $I(C)$ is homotopic to $C$, and hence $C' = I(C')$. The proof is finished with the theorem of Baer saying that simple closed loops are isotopic if and only if they are homotopic [Br]. \hfill $\diamond$

Let us therefore assume $C = I(C)$. We distinguish two cases: (1) $C \cap \text{Fix}(I) = \emptyset$ (2) $C \cap \text{Fix}(I) \neq \emptyset$. We will see that the former case covers exactly the disconnecting circles.

In the first instance, $\bar{C} = \kappa(C)$ is an embedded circle in $S^2 \setminus \{P_1, \ldots, P_{2g+2}\}$ and $\kappa : C \to \bar{C}$ is a two-fold unbranched cover. The simple Dehn-twist $\Psi$ along $C$ thus pushes down to a double Dehn-twist $\Phi$ along $\bar{C}$; $\kappa \circ \Psi = \Phi \circ \kappa$. Moreover, since the
monodromy ($\in \mathbb{Z}/2\mathbb{Z}$) of $\kappa$ along $\bar{C}$ is non-trivial, $\bar{C}$ divides $\{P_1, \ldots, P_{2g+2}\}$ into two subsets of odd cardinality, say $\{P_1, \ldots, P_{2h+1}\}$, $\{P_{2h+2}, \ldots, P_{2g+2}\}$, $h \geq 0$.

In the second instance, write $\kappa : z \mapsto w = z^2$ in appropriate complex coordinates $z = x + iy$, $w$ centered at some $P \in C \cap \text{Fix}(I)$ and in $\kappa(P)$ respectively, so $I : z \mapsto -z$. Possibly after a rotation we may write $C$ locally as graph of a real function $\varphi$ with $\varphi(0) = 0$, that is $C = \{x + i \varphi(x)\}$. The property $C = I(C)$ then implies $\varphi(-x) = -\varphi(x)$ and, locally,

$$\bar{C} = \{(x^2 - \varphi^2(x)) + 2ix\varphi(x)\}.$$  

This shows that $\bar{C}$ is an embedded, compact, connected curve in $S^2$ with one end for each point $P_i \in \bar{C}$. Since $\bar{C}$ has thus at least one end, the classification of plane, embedded curves implies that $\bar{C}$ is a closed intervall connecting exactly two points $P_i \neq P_j$. It is a well-known, elementary fact that a Dehn-twist $\Phi$ along a circle enclosing $\bar{C}$ (say the boundary of an $\epsilon$-neighbourhood of $\bar{C}$) commutes with a diffeomorphism $\Psi'$ of $\Sigma$ that is isotopic to $\Psi$. We have thus described all Dehn-twists in $\text{HMC}_g$ as lifts of explicit diffeomorphisms $\Phi$ of $(S^2; P_1, \ldots, P_{2g+2})$.

We now write down elements of the braid group $B(S^2, 2g + 2)$ producing these diffeomorphisms $\Phi$ up to isotopy. To be precise we write $S^2 = \mathbb{C} \cup \{\infty\}$ and assume

$$\bar{C} = \{|z| = 2\}; \quad P_r = e^{\frac{2\pi ri}{\kappa - 1}} \quad \text{for } r = 1, \ldots, 2h + 1; \quad |P_r| > 1 \quad \text{for } r > 2h + 1$$

in the first instance and

$$\bar{C} = [-1, 1]; \quad P_1 = -1; \quad P_2 = 1; \quad |P_r| > 1 \quad \text{for } r > 2$$

$(i = 1, j = 2)$ in the second instance. The double Dehn-twist along $\bar{C}$ (case 1) is then induced by the braid $(t \in [0, 1])$

$$P_r(t) = e^{\left(\frac{\pi r}{\kappa + 1} + 2it\right)2\pi i} \quad \text{for } r = 1, \ldots, 2h + 1; \quad P_r(t) = P_r \quad \text{for } r > 2h + 1,$$

while the Dehn-twist along $\bar{C}$ (case 2) is induced by the half-twist

$$P_1(t) = -e^{\pi it}; \quad P_2(t) = e^{\pi it}; \quad P_r(t) = P_r \quad \text{for } r > 2.$$  

These braids occur as monodromies of the following holomorphic curves $\bar{B} \subset (2\Delta) \times \mathbb{P}^1$ along $\gamma(t) = e^{2\pi it} \in 2\Delta$:

\begin{align*}
(B_{h,g-h}) & \quad \left\{(s, z) \in (2\Delta) \times \mathbb{P}^1 \mid \prod_{r=1}^{2h+1} (z - e^{\frac{2\pi ri}{\kappa} s^2}) = 0 \right\} \\
& \quad \cup (2\Delta) \times \{P_{2h+2}, \ldots, P_{2g+2}\} \\
(B_{\text{irr}}) & \quad \left\{(s, z) \in (2\Delta) \times \mathbb{P}^1 \mid z^2 - s = 0 \right\} \\
& \quad \cup (2\Delta) \times \{P_1, \ldots, \hat{P}_i, \ldots, \hat{P}_j, \ldots, P_{2g+2}\}
\end{align*}

where in the second line the entries with a hat are to be omitted. The second curve is smooth, but unless $h = 0$ the first curve has a singularity of multiplicity $2h + 1$ that in algebraic geometry is traditionally called infinitely close multiple point (here: of multiplicity $2h + 1$). After one blow up the strict transform of this curve becomes an ordinary singular point of the same multiplicity $2h + 1$ (locally the union of $2h + 1$ smooth curves with disjoint tangent planes), so another blow up desingularizes (hence infinitely close). The first non-trivial case $g = 2$, $h = 1$ is depicted in Figure 1 below.
As third preparatory remark let $p : P = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(n)) \to \mathbb{P}^1 = S^2$ be a $\mathbb{P}^1$-bundle and $B \subset P$ a pseudomanifold projecting to $S^2$ as unbranched cover (of degree $d$) away from finitely many points. We want to relate the homology class of $B$ to a “virtual number of critical values” of $B \to S^2$ that we are going to define momentarily. In the sequel we will reserve $H, S_\infty, F \subset P$ for (the classes of) a positive section, a negative section and a fiber of $P$ respectively, so $H^2 = n$ and $S_\infty = -n$. Since $H_2(P, \mathbb{Z}) = \langle H, F \rangle$ we may write $B \sim d \cdot H + l \cdot F$ in homology. To determine $l$ we describe $B$ by its developing map

$$\Phi_B : \mathbb{P}^1 \to S^d_{P^1} P,$$

sending $s \in S^2 = \mathbb{P}^1$ to the 0-cycle $p^{-1}(s) \cap B$ (with multiplicities). Here $S^d_{P^1} P = P \times_{\mathbb{P}^1} \ldots \times_{\mathbb{P}^1} P/S_d$ is the $d$-fold fibered symmetric product of $P$ with itself. Since $S^d_{\mathbb{P}^1} \simeq \mathbb{P}^d$ the target of $\Phi_B$ is a $\mathbb{P}^d$-bundle over $\mathbb{P}^1$. Its homology is thus generated by a fiber class $\eta$ and some horizontal class $\xi$, for which we take the reduced divisor of points in $S^d_{\mathbb{P}^1} P$ parametrizing (zero-dimensional) cycles with support meeting $S_\infty \subset P$. Let $D \subset S^d_{\mathbb{P}^1} P$ be the discriminant locus corresponding to cycles with higher multiplicities. Note that $D$ is the push-forward of the generalized diagonal in $P \times_{\mathbb{P}^1} \ldots \times_{\math{P}^1} P$ under the symmetrization map, so $D$ is an irreducible and reduced divisor. We define the virtual number of critical values of $B \to S^2$ by

$$\nu_{\text{virt}}(B) := \deg \Phi_B^* D.$$

**Lemma 1.5** $D \sim 2(d - 1)\xi + nd(d - 1)\eta.$

**Proof.** We start writing $D \sim a\xi + b\eta$ for some $a, b \in \mathbb{Z}$. To determine $a$ and $b$ we consider two special cases. The first one is $B \sim dH$ represented as union of graphs of $d$ pairwise different multiples $\lambda_i s$ of a section $s$ (i.e. viewing $P$ as compactification of $\mathcal{O}_{P^1}(n)$). The discriminant of this $d$-tuple is a non-zero constant (the discriminant of the $\lambda_i$) times $s^{d(d-1)}$, which counted with multiplicities has $nd(d - 1)$ zeroes. This shows $b = nd(d - 1)$. The other case is $B \sim (d - 1)H + S_\infty \sim dH - nF$ represented by only $d - 1$ of the sections and $S_\infty$. We obtain $-an + b = n(d - 1)(d - 2)$, so $a = 2(d - 1)$, which can also be seen as the degree of the discriminant $D \subset S^d_{\mathbb{P}^1} \simeq \mathbb{P}^d$. ⋄

Since $\Phi_B$ is a section of $S^d_{\mathbb{P}^1} P \to \mathbb{P}^1$ we have $\deg \Phi_B^* \eta = 1$. On the other hand, $\deg \Phi^*_B \xi = S_\infty : B = l$ by the definition of $\xi$. We conclude
Proposition 1.6 The number $\mu_{\text{virt}}$ of virtual branch points of the projection $B \subset \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(n)) \to \mathbb{P}^1$ is related to the homology class $B = dH + lF$ in the following way
\[
\mu_{\text{virt}} = 2(d-1)l + nd(d-1).
\]

For later reference let us also record quickly the contribution to $\mu_{\text{virt}}(B)$ of our local standard models: For the case $B_{\text{irr}}$ of smooth critical points one easily checks in local coordinates of $\mathcal{S}^4 \mathbb{P}^1$ that the contribution is $+1$. The case $B_{h,g-h}$ with a point of multiplicity $2h+1$ is not so obvious. We may however reduce to the previous case by a holomorphic deformation of $B$ to a smooth curve. To this end we first translate each of the $2h+1$ smooth irreducible components slightly to make them meet transversally. Since each pair of irreducible components meets in 2 points we obtain
\[
2 \cdot \binom{2h+1}{2} = 2h(2h+1)
\]

nodes (ordinary double points). In turn each node can be smoothed, leading to 2 simple critical points of the projection to the base (that is, of type $B_{\text{irr}}$). The contribution to $\mu_{\text{virt}}(B)$ is therefore
\[
4h(2h+1).
\]

Note that this number coincides with the number of generators in the corresponding braid monodromy $(x_1 \cdot \ldots \cdot x_{2h})^{2(2h+1)}$. In fact, this deformation argument is another way to deduce this monodromy (cf. [CR, p.83ff] for $h=1$).

We are now ready to prove the theorem.

Proof of Theorem 1.2. The case $g = 1$, which would require some extra considerations, follows from the classification of Moishezon and Livné [Mi, II.2], so we assume $g \geq 2$. We arrange the critical values $s_1, \ldots, s_\mu$ of $q : M \to S^2$ to be the $\mu$-th roots of unity ($S^2 = \mathbb{C} \cup \{\infty\}$). For sufficiently small $\varepsilon > 0$ let $S^* = \mathbb{C} \setminus B_\varepsilon(\{s_1, \ldots, s_\mu\})$ and $\rho : \pi_1(S^2 \setminus \{s_1, \ldots, s_\mu\}) \to \text{HMC}_g$ the monodromy representation. Let $\gamma_r$ be the loop that runs from 0 once counterclockwise around $s_r$ and which does not enclose any $s_r'$, $r' \neq r$. Then $\pi_1(S^*)$ is the free group generated by $\gamma_1, \ldots, \gamma_\mu$, while $\pi_1(S^2 \setminus \{s_1, \ldots, s_\mu\})$ is the quotient by the relation $\gamma_1 \cdot \ldots \cdot \gamma_\mu = 1$. The monodromy representation is then given by Dehn-twists $\tau_r = \rho(\gamma_r) \in \text{HMC}_g$ fulfilling
\[
\tau_1 \cdot \ldots \cdot \tau_\mu = 1.
\]

Let $\hat{\tau}_r$ be the image of $\tau_r$ in $\text{MC}(S^2, 2g+2)$. There are distinguished lifts $\hat{\tau}_r \in B(S^2, 2g+2)$ as follows: By the discussion above we know that a non-trivial $\tau_r$ is conjugate either to $x_1$ or to $(x_1 \cdot \ldots \cdot x_{2h})^{4h+2} (1 \leq h \leq [g/2])$, and the same expression holds for $\hat{\tau}_r$. Now simply represent the conjugating element as word in the $x_r$, and interpret the whole word as element of $B(S^2, 2g+2)$. While the lift of the conjugating element is not unique, the product is well-defined because $B(S^2, 2g+2)$ is a central extension of $\text{MC}(S^2, 2g+2)$.

By this process we might loose triviality of the monodromy along the circle $\gamma_1 \cdot \ldots \cdot \gamma_\mu$ at infinity, but at least
\[
\hat{\tau}_1 \cdot \ldots \cdot \hat{\tau}_\mu \in \{1, (x_1 \ldots x_{2g+1})^{2g+2}\}.
\]
The $\hat{\tau}_r$ determine a $(2g + 2)$-fold unbranched cover $B^* \hookrightarrow S^* \times S^2 \rightarrow S^*$ having $\hat{\tau}_r$ as braid monodromy. In fact, let $D = \{(z_1, \ldots, z_{2g+2}) \in (S^2)^{2g+2} | z_i = z_j \text{ for some } i \neq j\}$ be the generalized diagonal in $(S^2)^{2g+2}$, and

$$A_r = (A_r^k)_{k \in [0, 1]} : \ x \cdot (S^2)^{2g+2} \rightarrow D$$

with $A_r(0) = (P_1, \ldots, P_{2g+2})$, $A_r(1) = (P_{\sigma_r(1)}, \ldots, P_{\sigma_r(2g+2)})$ be a movement of the $2g + 2$ branch points of $\kappa : \Sigma \rightarrow S^2$ representing $\hat{\tau}_r$. So $\sigma_r \in S_{2g+2}$ is the permutation of strands induced by $\hat{\tau}_r$. Instead of giving the construction over $S^*$ we construct $B^*$ over the diffeomorphic domain

$$U = \left( B_1(0) \cup \bigcup_{r=1}^\mu B_2(\varepsilon(s_r)) \right) \cup \bigcup_{r=1}^\mu \bigcup_{k=1}^{2g+2} \left\{ (s_r + \rho e^{\pi i (t - \frac{1}{2}) + i \arg s_r}, A_r^k(t)) \bigg| t \in (-\frac{1}{2}, \frac{3}{2}), \rho \in (\varepsilon, 2\varepsilon) \right\} .$$

Now we want to fill in the holes $\overline{B_\varepsilon}(s_r)$. The braid monodromies $A_r$ around the $s_r$ have been discussed above. We saw that they are represented either by half-twists of a pair of strands or by a double twist of an odd number of strands. In any case, any $A_r$ can be deformed to the monodromy $T$ of one of the standard families \([2]\) by a family of braids $(s \in [0, 1])$

$$A_r[s] : [0, 1] \rightarrow (S^2)^{2g+2} \rightarrow D, \ A_r[0] = A_r, A_r[1] = T.$$ 

Using this isotopy of braids one can glue in the standard models \([2]\) to arrive at a possibly singular surface $B \subset \mathbb{C} \times S^2$ with

1. $B \rightarrow \mathbb{C} \setminus \{s_1, \ldots, s_\mu\}$ is an unbranched covering
2. If $q^{-1}(s_r) \subset M$ is irreducible then $B \subset \mathbb{C} \times S^2$ is locally around $s_r$ isomorphic to $B_{irr}$
3. If $q^{-1}(s_r) \subset M$ is a union of closed surfaces of genera $h$ and $g - h (h \leq g - h)$, then $B \subset \mathbb{C} \times S^2$ is locally around $s_r$ isomorphic to $B_{h, g-h}$
4. The global monodromy is either trivial or $(x_1 \cdots x_{2g+1})^{2g+2}$.

Here “locally isomorphic” refers to the existence of a local diffeomorphism of the ambient spaces mapping the branch surfaces to each other.

Next we have to extend $B$ across $\infty$. If the global monodromy is trivial we use an isotopy of the monodromy over $\{|s| = 2\}$ to the trivial braid $[0, 1] \times (P_1, \ldots, P_{2g+2})$ to connect $B \cap (B_2(0) \times S^2)$ to the product surface $(\mathbb{C} \setminus B_4(0)) \times (P_1, \ldots, P_{2g+2})$, which readily extends over $\infty$ in the product bundle $\tilde{P} := S^2 \times S^2 \rightarrow S^2$.

In the other case of non-trivial monodromy $(x_1 \cdots x_{2g+1})$ we may extend over $\infty$ in the non-trivial $S^2$-bundle $\tilde{P} \rightarrow S^2$, which can be realized holomorphically as the first Hirzebruch surface $\mathbb{P}^2(\mathcal{O} \oplus \mathcal{O}(1))$: $\mathbb{P}$ can be constructed by patching trivial bundles over half spheres via

$$S^1 \rightarrow \text{Diff}(S^2 = \mathbb{C} \cup \{\infty\}), \ e^{2\pi i t} \mapsto \left( z \mapsto e^{2\pi i t} \cdot z \right).$$

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The trivial braid along the equator in the trivialization at $\infty$ clearly extends (as graph of $2g + 2$ sections) over the half-sphere at $\infty$. Applying the patching function to this trivial braid we obtain however one full twist on the bunch of all strands. We pointed out above that this braid is nothing but $(x_1 \ldots x_{2g+1})^{2g+2}$. This shows that we can extend $B \cap (B_2(0) \times S^2)$ to a surface in $\tilde{P}$ with no additional branching. We keep the notation $\tilde{B}$ for the resulting extended surface.

From the local description of the singularities of $\tilde{B}$ we can desingularize $\tilde{B}$ by two consecutive blow ups of $\tilde{P}$ for each singular point. We denote this blow up by $\tilde{\rho} : \tilde{P} \to \tilde{P}$. The result is an $S^2$-fibration $\tilde{P} \to S^2$ with as many singular fibers of type $\Lambda_2$ as $q : M \to S^2$ has reducible singular fibers. Let $B \subset \tilde{P}$ be the strict transform of $\tilde{B}$, and $B_E$ the union of vertical $(-2)$-spheres that arise in the blow up process. To produce a two-fold covering $\tilde{M} \to \tilde{P}$ with branch locus $B + B_E$ we only have to show that $B + B_E \in H_2(\tilde{P}, \mathbb{Z})$ is 2-divisible, see Proposition [4,5] in the appendix.

**Lemma 1.7** $B + B_E \in H_2(\tilde{P}, \mathbb{Z})$ is 2-divisible.

**Proof.** The class of $B + B_E$ differs by a 2-divisible class from $\tilde{\rho}^* \tilde{B}$. In fact, if $Q \in \tilde{B}$ is a point of multiplicity $2h + 1$ and $\rho_1 : P_1 \to \tilde{P}$ is the blow up in $Q$ with exceptional divisor $E \subset P_1$ then

$$\rho_1^* \tilde{B} = B_1 + (2h + 1)E,$$

where $B_1$ is the strict transform of $\tilde{B}$, cf. Fig. 1. Another blow up $\rho_2 : P_2 \to P_1$ in the singular point $B \cap E$ with exceptional divisor $\Gamma' \subset P_2$ and strict transform $B_2$ of $B$ and $\Gamma$ of $E$ leads to

$$\rho_2^* \rho_1^* \tilde{B} = \rho_2^* B_1 + (2h + 1)\rho_2^* E = B_2 + (2h + 1)\Gamma' + (2h + 1)(\Gamma + \Gamma').$$

This differs from $B_2 + \Gamma$ by $2((2h + 1)\Gamma' + h\Gamma)$. Doing this successively at all singular points of $\tilde{B}$ and noticing that the union of the $\Gamma$ equals $B_E$, we conclude that it suffices to prove 2-divisibility of $\tilde{B}$.

We now employ the notion of virtual number of critical points introduced above. Let as before $\tilde{P} = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(n))$ $(n \in \{0, 1\})$ and write $\tilde{B} \sim dH + lF$ with $d = 2g + 2$. We have to show that $l \equiv 0 \pmod{2}$. Proposition [1,6] tell us that

$$\mu_{\text{virt}}(\tilde{B}) = 2(d - 1)l + nd(d - 1),$$

so $\mu_{\text{virt}}(\tilde{B}) - nd(d - 1) \equiv 0 \pmod{4}$ if $l \equiv 0 \pmod{2}$. We also noted that $\mu_{\text{virt}}(\tilde{B})$ is a sum of local contributions that coincide with the signed length of the word in $x_1, \ldots, x_{2g+2}$ representing the braid monodromy around critical values. That this length minus $nd(d - 1)$ is divisible by 4 must be related to the fact that our braid monodromy representation comes from a representation into $\text{HMC}_g$, for there are clearly surfaces $\tilde{B} \subset \tilde{P}$ homologous to $dH + lF$ with $l$ odd. We introduce an auxiliary group $\tilde{B}(S^2, 2g + 2)$ as fibered product

$$\tilde{B}(S^2, 2g + 2) \to \tilde{B}(S^2, 2g + 2) $$

$$\text{HMC}_g \to \text{MC}(S^2, 2g + 2).$$

Explicitly we have to drop the relation $(x_1 \ldots x_{2g+1})^{2g+2} = 0$ in $\text{HMC}_g$, or replace the relation $I = 1$ by $Ix_i = x_iI$, $I^2 = 1$ in $\tilde{B}(S^2, 2g + 2)$. In other words,

$$\tilde{B}(S^2, 2g + 2) = \left\langle x_1, \ldots, x_{2g+1}, I \mid x_i x_j = x_j x_i \quad |i - j| \geq 2; \quad x_i x_{i+1} x_i = x_{i+1} x_i x_{i+1} \quad I = x_1 \ldots x_{2g+1} x_{2g+1} \ldots x_1; \quad I x_i = x_i I; \quad I^2 = 1 \right\rangle$$

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As in the case of lifting $\bar{\tau}_r \in MC(S^2, 2g + 2)$ to $\bar{\tau}_r \in B(S^2, 2g + 2)$ there exists a distinguished lift $\bar{\tau}_r$ of $\tau_r$ to $\tilde{B}(S^2, 2g + 2)$. By definition $\bar{\tau}_r$ maps to $\bar{\tau}_r \in B(S^2, 2g + 2)$ and

$$\bar{\tau}_1 \cdot \ldots \cdot \bar{\tau}_\mu = \begin{cases} 1 & n = 0 \\ (x_1 \ldots x_{2g+1})^{2g+2} & n = 1 \end{cases}$$

The relations in $\tilde{B}(S^2, 2g + 2)$ have the virtue of preserving the signed length modulo 4. This together with the length $nd(d - 1)$ of the total monodromy shows that

$$\text{length}(\bar{\tau}_1 \cdot \ldots \cdot \bar{\tau}_\mu) - nd(d - 1) \equiv 0 \quad (4)$$

as claimed. This concludes the proof of 2-divisibility of $\tilde{B}$, hence of $B + B_E$. $\diamond$

The proof of the theorem is now almost finished: By 2-divisibility of $B + B_E$ there exists a two-fold cover $\tilde{k} : \tilde{M} \to \tilde{P}$ branched along $B + B_E$. Every connected component $\Gamma$ of $B_E$ is a sphere of self-intersection $-2$. Hence $\kappa^{-1}(\Gamma)$ is a $(−1)$-sphere and can be contracted. Let $\sigma : \tilde{M} \to M$ be the blow down of all these $(−1)$-spheres. A local study of this process shows that every contracted component results in an ordinary critical point of the induced fibration $q' : M \to S^2$, which is thus a SLF. By Lemma 1.3 and the construction, the monodromy of $q'$ coincides with the one coming from $q : \tilde{M} \to S^2$. The result of Kas implies that $q$ and $q'$ are isomorphic SLF’s.

Finally, it is well known (and can be easily verified) that the contraction of a $(−2)$-sphere $\Gamma$ leads to the orbifold singularity

$$\mathbb{C}^2/(\mathbb{Z}/2\mathbb{Z}) \simeq \{(z, w, y) \in \mathbb{C}^3 \mid z^2 + w^2 + y^2 = 0\}.$$

This shows that $q : M \to S^2$ factorizes over the orbifold $P$ as described in statement 4 of the proposition.

The converse assertion, that a SLF of this form is hyperelliptic, is trivial. $\diamond$

**Remark 1.8** The theorem shows that hyperelliptic SLF’s $M \to S^2$ admit a hyperelliptic involution $\tau : M \to M$ with fix locus the embedded surface $\sigma(\tilde{k}^{-1}(B))$ union the nodes of reducible singular fibers ($= \sigma(\bar{k}^{-1}(B_E))$). This may fail if one admits multiple singular fibers. $\diamond$

A SLF $q : M \to S^2$ may of course admit several hyperelliptic structures, that is, choices of two-fold covers $\Sigma = q^{-1}(s) \to S^2$ commuting with the monodromy up to a diffeomorphism of $S^2$ and up to isotopy. Essentially different choices lead to non-isomorphic hyperelliptic involutions $\tau : M \to M$, hence to non-isomorphic pairs $\tilde{B} \subset \tilde{P}$. An example is provided by elliptic Lefschetz fibrations: Moishezon and Livné have shown that elliptic Lefschetz fibrations are determined by their number of singular fibers $\mu$: It is always $\mu \equiv 0(12)$ and the monodromy can be put into the form $(x_1, x_2)^{\mu/2}$ $[\text{Mo}]$. But the case $\mu = 12$ arises either with $B \sim 4H + 2F$ in $\mathbb{P}^1 \times \mathbb{P}^1$ or with $B \sim 4H$ in $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(1))$ (cf. Theorem 3.1).

The theorem motivates the following definition.

**Definition 1.9** Let $\tilde{p} : \tilde{P} \to S^2$ be an $S^2$-fibration with $t$ singular fibers of type $\Lambda_2$ and $P \to S^2$ the associated $S^2$-bundle, $\tilde{p} : \tilde{P} \to P$. An embedded surface $B \subset \tilde{P}$ will be called branch surface (of type $(g, \mu, t)$) if
• the composition $B \to \bar{P} \to S^2$ is a branched cover of degree $2g + 2$ with $\mu - t$ ordinary critical points (that is, of the form $z \mapsto z^2$ in appropriate complex coordinates)

• let $\bar{p}^{-1}(s) = F \cup \Gamma \cup \Gamma'$ with $F^2 = -1$, $\Gamma^2 = -2$, $\Gamma'^2 = -1$ be a singular fiber. Then $B \cap \bar{p}^{-1}(s), B \cap \Gamma = \emptyset, B \cdot F \equiv 1$ (2)

• $\bar{p}_*[B] \in H_2(\bar{P}, \mathbb{Z})$ is 2-divisible

The theorem can then more concisely be phrased as

**Corollary 1.10** For $g \geq 2$ there is a one-to-one correspondence between isomorphism classes of SLF’s with hyperelliptic structure $(\bar{q} : M \to S^2, \kappa : q^{-1}(s) \to S^2)$, of genus $g$ with $\mu$ singular fibers, $t$ of which are reducible, and isomorphism classes of pairs $(\bar{P}, B)$ with $\bar{P} \to S^2$ an $S^2$-fibration with $t$ fibers of type $\Lambda_2$ and $B \subset \bar{P}$ a branch surface of type $(g, \mu, t)$. Moreover, for $g = 2$ the hyperelliptic structure is determined by the fibration up to isomorphism.

**Proof.** Starting from a branch surface $B \subset \bar{P}$ the two-fold cover branched along $B \cup B_E$ is a hyperelliptic SLF $q : M \to S^2$. The proof of the theorem now produces a possibly singular branch surface $\bar{B}'$ in the $S^2$-bundle $\bar{P}$ associated to $\bar{P}$ (the isomorphism class of which was determined by the monodromy at infinity). There was one non-uniqueness involved in the construction of $\bar{B}'$. To wit, if $g = 2h$ is even then for any reducible singular fiber of type $B_{h,h}$ we can choose either $(x_1 \cdots x_h)^{4h+2}$ or $(x_{h+2} \cdots x_{2h})^{4h+2}$ as word representing the corresponding Dehn-twist. This amounts to select one of the two connected components of $S^2 \setminus \kappa(C)$ if $C \subset \Sigma$ is the $I$-invariant embedded circle supporting the Dehn-twist. However, both choices lead to the same branch surface after blow up, where this non-uniqueness materializes in a choice of one of the two $(-1)$-curves to be contracted first. This shows that the desingularization $\bar{B}' \subset \bar{P}$ of $\bar{B}'$ is isomorphic to $B \subset \bar{P}$.

For $g = 2$ any cover $\kappa : \Sigma \to S^2$ is compatible with the monodromy and hence gives rise to a hyperelliptic structure. But any two such covers $\kappa, \kappa'$ with the same branch set in $S^2$ differ only by a diffeomorphism of $\Sigma$. But the mapping class group $\text{MC}_2 = \text{HMC}_2$ is an extension of the mapping class group of $(S^2, \{P_1, \ldots, P_6\})$ by the hyperelliptic involution. This shows that for $g = 2$ there is indeed only one hyperelliptic structure on $\Sigma$ up to isomorphy.

Note that in contrast to $B \subset \bar{P}$ the possibly singular branch surface $\bar{B}$ in the $S^2$-bundle $\bar{P}$ associated to $\bar{P}$ is not unique.

**Remark 1.11** The construction of the two-fold cover from the possibly singular surface $\bar{B} \subset \bar{P}$ suggests the question why ordinary singular points do not occur in $\bar{B}$. For multiplicity $h$ these have typical braid monodromy $(x_1 \cdots x_{h-1})^h$ (one full twist of the first $h$ strands) and they require only one blow up for desingularization, so they seem to be more basic than infinitely close singular points. But then $\bar{M} \to \bar{P}$ is unbranched over the critical point of $\bar{P} \to S^2$ (the intersection of the exceptional divisor of $\bar{P} \to \bar{P}$ with the strict transform of the fiber of $P \to S^2$). Thus $\bar{M} \to S^2$ acquires a singular fiber with two critical points that connect two smooth components of genera $\frac{h}{2} - 1$ and $g - \frac{h}{2}$ ($h \equiv 0$ is forced by 2-divisibility of $B$). The monodromy of $\bar{M}$ is therefore the product of two commuting Dehn-twists of non-disconnecting type.
It is also interesting to note the result of a slight (locally holomorphic) perturbation of the projection $q : M \to S^2$. It can be shown that for ordinary singular points in $B$ this is equivalent to smoothing $B$ inside $\tilde{P}$. This is an example of Brieskorn’s simultaneous resolution [Br], which suggested the notion of inessential singularities of branch loci in algebraic geometry [Ho][Ps]. In the case of an ordinary singular point of multiplicity $h$ we obtain $h(h - 1)$ critical points of type $B_{1r}$, so the “stabilization” of the fibration belonging to an ordinary singularity of multiplicity $h$ is a fibration with $h^2 - h$ irreducible singular fibers.

2 Study of the branch locus

We now want to collect more information on the (non-vertical components of) branch loci $B \subset \tilde{P}$, or, equivalently $B \subset \tilde{P}$. We begin by showing that $B$ is symplectic. Without more effort this can be proved for SLF’s factorizing (possibly after blow up) over an arbitrary branched covering and the blow-up of a symplectic fiber bundle.

**Theorem 2.1.** We assume that $q : M \to S^2$ is a SLF and that the induced fibration of a blow up $\sigma : \tilde{M} \to M$ factorizes over a branched covering $\tilde{\kappa}$ (any degree, any branch type)

$$\tilde{M} \xrightarrow{\tilde{\kappa}} N \xrightarrow{p} S^2$$

with $p : N \to S^2$ the blow up of a symplectic fiber bundle. Let $B \subset N$ be the union of the non-vertical components of the branch locus and assume $B \cap \text{Crit}(p) = \emptyset$. Let $\omega_N$ be a symplectic form on $N$ such that the general fibers of $p$ are symplectic and denote by $\omega_{S^2}$ a symplectic form on $S^2$.

Then $B$ is symplectic with respect to $\omega_N + k \cdot p^*\omega_{S^2}$ for sufficiently large $k$.

**Proof.** The composition $p|_B : B \hookrightarrow N \to \mathbb{P}^1$ is a branched covering. We endow $B$ with the unique complex structure making this covering holomorphic. Then $p^*\omega_{\mathbb{P}^1}|_B$ is a Kähler form on $B$ away from the critical set, so $\omega_N + k \cdot p^*\omega_{\mathbb{P}^1}|_B$ will be non-degenerate away from arbitrarily small neighbourhoods of the critical set. It remains to show that for each critical point $P \in B$, the restriction $\omega_N$ to $T_PB$ is non-degenerate and agrees with the complex orientation.

By holomorphicity of $p|_B$ there are local holomorphic coordinates $w$ for $B$ near $P$ and $t$ for $\mathbb{P}^1$ with $(p|_B)^*t = w^m$ for some $m \geq 2$. Extend $w$ to a smooth function on a neighbourhood of $P$ in $N$. Since $p_*T_PB = 0$, the tuple $(z := p^*t, w)$ is a complex coordinate system on $N$, and $\omega_N|_{T_PB}$ is non-degenerate. Now if $Q \in \kappa^{-1}(P)$ is a branch point of $\tilde{\kappa}$ of order $b \geq 2$, then $M$ can be described near $Q$ by

$$\{(z, w, u) \in \mathbb{C}^3 \mid u^b = z - w^m\}.$$ 

So $(u, w)$ are complex coordinates on $M$ near $Q$ and $p : (u, w) \mapsto u^b + w^m$. Since by hypothesis $q : M \to \mathbb{P}^1$ is oriented and Lefschetz, we see that necessarily $b = m = 2$ and $(u, w)$ is compatible with the orientation of $M$. In turn $(z, w)$ agrees with the orientation of $N$ (and of $\mathbb{P}^1$). The equality of complex line bundles

$$T_PB = \ker p_* = \det \mathcal{O}_M \otimes \mathcal{O}_{\mathbb{P}^1}$$

together with the fact that $p : N \to \mathbb{P}^1$ is symplectic with respect to $\omega_N$ then shows that the orientations of $T_PB$ given by $\omega_N|_{T_PB}$ on one hand and by $w$ on the other hand coincide. \hfill \Diamond
Remark 2.2 The authors believe that symplecticity of the branch locus is an essential observation, because it reduces the possible isomorphy classes of branch loci drastically. There are in fact good chances that in $S^2$-bundles over $S^2$, and in $\mathbb{P}^2$, symplectic submanifolds are classified even up to isotopy by their homology, just as in the holomorphic situation. In view of Theorem 1.2 this would in particular imply that every hyperelliptic SLF with only irreducible singular fibers is (isomorphic to) a holomorphic (hyperelliptic, Lefschetz) fibration. At least for surfaces inside $S^2$-bundles over $S^2$ of bidegree $(2, d)$ or $(3, d)$ this can indeed be shown by studying the braid monodromy. Without symplecticity there would be no hope for a simple classification, as one can always paste non-trivial 2-knots locally without changing the homology class. Our approach to the isotopy problem of symplectic submanifolds also suggests that holomorphicity of hyperelliptic fibrations is true provided the number $\mu$ of singular fibers is large compared to the number $t$ of reducible singular fibers (a sufficient condition might be $\mu/t > 18$), so reducible singular fibers might be interpreted as obstructions to holomorphicity. In fact, known examples of non-holomorphic, hyperelliptic SLF’s have small ratio $\mu/t$, e.g. $\mu/t = 4$ for the recent example in \cite{OzSt}. \hfill \diamond

We can also prove that unless $M \to S^2$ is trivial, $B$ has at most two connected components, and if $B$ is disconnected one of them can be taken as negative section $S_\infty$ of $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(m))$ with $m$ even.

Proposition 2.3 Let $B \subset \tilde{P}$ be a branch surface (Definition 1.9). Then either $B$ is connected or the reduction $(\tilde{P}, \tilde{B})$ is diffeomorphic (as $S^2$-bundle over $S^2 = \mathbb{P}^1$) to

- $(\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(2n)), S_\infty \cup \tilde{B}')$ with $n > 0$ and $\tilde{B}' \sim (2g + 1)H$ connected
- $(\mathbb{P}^1 \times \mathbb{P}^1, H_1 \cup \ldots \cup H_{2g+2})$ where $H_i$ are disjoint sections of $P$

Proof. By the local study of the braid monodromy it suffices to investigate connectedness of a smoothing of $B \subset \tilde{P}$. In other words, we may assume that $P = \tilde{P}$ and $B = \tilde{B}$ is already smooth. Let us first consider the case $P \cong \mathbb{P}^1 \times \mathbb{P}^1$. If $B$ has more than two components, say $B_i \sim a_iH + b_iF$, $i = 1, 2, 3$ (necessarily $a_i > 0$), then $b_i/a_i = 0$ for the pairwise intersection is zero. Now Lemma 1.6 applied to $B_i$ shows that $B_i \to \mathbb{P}^1$ is unbranched, hence $a_i = 1$ for all $i$ and we get case (2).

If $B$ has two components we may write them as $S' \sim aH + bF, B' \sim cH + dF$ with $a, c > 0$, $a + c = 2g + 2$, not both $b$ and $d$ equal to 0, and ordered in such a way that $(S')^2 = 2ab \leq (B')^2 = 2cd$. Now $0 = S' \cdot B' = ad + bc$, so $b \leq 0$ and $d \geq 0$. Moreover, since $S'$ is symplectic, the adjunction formula shows

$$2g(S') - 2 = (\langle -2H - 2F \rangle(aH + bF) + (S')^2 = -2b - 2a + 2ab$$

hence $g(S') = -b(1 - a) - a + 1$. But $a \geq 1$ and $b \leq 0$ so $g(S') \leq 0$, which implies $g(S') = 0$, $a = 1$, $c = 2g + 1$. Lemma 2.4 below thus shows that we are dealing with case (2) with $n = -b$.

The other possibility is $P$ diffeomorphic (as fiber bundle) to $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(1))$. If $B$ were not connected write $B = B_1 \cup B_2$ with $B_1^2 \leq B_2^2$, $B_1 = aS_\infty + bF$, $B_2 = cH + dF$, $a, c > 0$, $a + c = 2g + 2$. The adjunction formula for $B_1$ yields

$$2g(B_1) - 2 = (\langle -2S_\infty - 3F \rangle(aS_\infty + bF) + (aS_\infty + bF)^2 = -(a - b)^2 - a \leq -a < 0,$$
so \( g(B_1) = 0, a = 1 \) and \( b \in \{0, 1\} \). If \( b = 1 \) the equation \( 0 = B_1 \cdot B_2 = ad + bc \) implies \( d = -c \) and then \( B_2^2 = -c^2, B_1^2 = 1 \) contradicts \( B_1^2 \leq B_2^2 \). The remaining case is \( B_1 = S_\infty, B_2 = (2g+1)H \). But then \( B_1 + B_2 = (H - F) + (2g+1)H = (2g+2)H - F \) is not a 2-divisible class and can thus not occur as branch locus of a two-fold covering.

\[ \diamond \]

We owe one lemma.

**Lemma 2.4** Suppose that \( P \to \mathbb{P}^1 \) is a ruled surface and \( S \subset P \) is a section with \( S^2 = -n < 0 \). Then there exists an isomorphism of fibers spaces \( P \simeq \mathbb{P}(O \oplus O(n)) \) carrying \( S \) to the negative section \( S_\infty \).

**Proof.** Removing \( S \) we obtain a fiber bundle \( P \setminus S \to \mathbb{P}^1 \) with structure group \( \text{Aff}(\mathbb{C}) = \mathbb{C} \times \mathbb{C}^* \), the complex affine transformations of \( \mathbb{C} \). Conversely, fiber bundles with fiber \( (\mathbb{C}, \text{Aff}(\mathbb{C})) \) correspond exactly to pairs \((P, S)\) of \( \mathbb{P}^1 \)-bundles with section. Now write \( P \setminus S \) as two copies of \( \Delta \times \mathbb{C} \) glued over an annulus \( A \subset \Delta \) via a map \( \theta : A \to \text{Aff}(\mathbb{C}) \). But \( \mathbb{C}^* \subset \text{Aff}(\mathbb{C}) \) is a deformation retract and by degree theory any map \( A \to \mathbb{C}^* \) is homotopic to a holomorphic map, in fact to \( \theta_k : z \to z^k \) for some \( k \in \mathbb{Z} \). So after a change of trivialization we may assume \( \theta = \theta_k \). But gluing two copies of \( A \subset \Delta \) via \( \theta_k \) we obtain \( \mathbb{P}(O \oplus O(k)) \setminus S_\infty \) and this extends to an isomorphism \( P \simeq \mathbb{P}(O \oplus O(k)) \) mapping \( S \) to \( S_\infty \). The number \( k \) is determined by \( k = -(S_\infty)^2 = -S^2 = n \). \[ \diamond \]

**Remark 2.5** In two recent papers Fuller claimed that any SLF can be factorized into a simple 3-fold branched cover of an \( S^2 \)-fibration over \( S^2 \) followed by the bundle projection \([\text{Fu}1], [\text{Fu}2]\). Unfortunately, our observation on symplecticity of the branch locus shows that this is impossible: The covering being simple forces the branch locus to decompose into two parts, each of which having covering degree at least two over \( S^2 \). As we saw in the proof of Proposition 2.3 this is impossible for homological reasons unless the branch locus is trivial. The problem in op.cit. is that the branch locus, which is first constructed in \( \Delta \times S^2 \), can not be extended over the fiber over \( \infty \in S^2 \). \[ \diamond \]

### 3 Topological invariants of hyperelliptic SLF’s

We will see in this section that the description as relative minimalization of a branched cover allows the computation of topological invariants of hyperelliptic SLF’s in terms of the genus \( g \) and the types and numbers of singular fibers.

Let as before \( q : M \to S^2 \) be a hyperelliptic SLF of genus \( g \) with \( \mu \) singular fibers, \( t \) of which are reducible of types \( h_1, \ldots, h_t \) (the minimum of the genera of the two irreducible components). Let \( \sigma : \bar{M} \to M \) be the blow up of the \( t \) nodes of reducible singular fibers, \( \bar{k} : \bar{M} \to N := \tilde{P} \) the two-fold cover with branch locus \( B_N = B \cup B_E \), and \( \bar{\rho} : N \to \tilde{P} \) the birational map to an \( S^2 \)-bundle \( \tilde{P} = \mathbb{P}(O \oplus O(n)) \).

Some more notations for later reference: \( E_1, \ldots, E_t \subset \bar{M} \) are the exceptional fibers of \( \sigma \), \( \Gamma_1 \cup \Gamma_1', \ldots, \Gamma_t \cup \Gamma_t' \) are the exceptional fibers of \( \bar{\rho} \) with \( \Gamma_i^2 = -2, \Gamma_i'^2 = -1, \bar{B} = \bar{\rho}(B), B = \kappa^{-1}(B_N) \).

For the following it will be convenient to work in the almost complex category as follows: We know that \( B \cup B_E \subset N \) is symplectic (with respect to \( \omega_N' := \omega_N + k \cdot p^* \omega_{\mathbb{P}^1} \) for \( k \gg 0 \): Theorem 2.1), so there exists an \( \omega_N' \)-tamed almost complex structure \( J \) on
N making \( B \cup B_E \) a \( J \)-holomorphic curve. We may also assume that \( J \) is compatible with the fiber structure, so \( N \to \mathbb{P}^1 \) is a holomorphic map of almost complex manifolds (near the critical values we can choose \( B \) holomorphic with respect to an integrable complex structure). Clearly, \( J \) lifts to an almost complex structure \( \tilde{J} \) on \( \tilde{M} \) such that \( \kappa : \tilde{M} \to N \) is \( (\tilde{J}, J) \)-holomorphic. By the standard construction of a symplectic structure \( \tilde{\omega} \) on branched covers (unique up to deformation equivalence), \( \tilde{J} \) is \( \tilde{\omega} \)-tame for an appropriately chosen \( \tilde{\omega} \) on \( \tilde{M} \).

The advantage of introducing (tamed) almost complex structures is that we may now employ almost complex analogues of well-known formulas from the theory of complex surfaces. To give them the usual shape, we introduce the following notation for topological invariants of an almost complex four-manifold \((M, J)\):

\[
K_M := -c_1(M), \quad \chi(M) := \frac{c_1^2(M) + e(M)}{12},
\]

where \( c_i(M) \) denotes the \( i \)-th Chern class of \((T_M, J)\) and \( e(M) = c_2(M) \) is the Euler class. We identify \( H^4(M, \mathbb{Z}) = \mathbb{Z} \) and \( H_2(M, \mathbb{Z}) \) with \( H^2(M, \mathbb{Z}) \) by Poincaré-duality, as is customary in the theory of complex surfaces. With this understood we have in our situation

\[
\begin{align*}
K_{\tilde{M}} &= \sigma^*K_M + \sum_i E_i, \quad \text{so} \quad K_{\tilde{M}}^2 = K_M^2 + t \\
K_{\tilde{M}} &= \kappa^*K_N + \tilde{B} = \kappa^*(K_N + \frac{1}{2}B_N), \quad \text{so} \quad K_{\tilde{M}}^2 = 2(K_N + \frac{1}{2}B_N)^2 \\
\tilde{\rho}^*\tilde{B} &= B + \sum_i(2h_i + 1)(\Gamma_i + 2\Gamma_i') \\
&= B + B_E + \sum_i(2h_i\Gamma_i + (2h_i + 1)2\Gamma_i') \\
K_N &= \tilde{\rho}^*K_{\tilde{\rho}} + \sum_i(\Gamma_i + 2\Gamma_i')
\end{align*}
\]

We are now ready to compute \( e \) and \( c_1^2 \).

**Theorem 3.1** Let \( q : M \to S^2 \) be the hyperelliptic SLF of genus \( g \) with \( \mu \) singular fibers, \( t \) of which are reducible, of types \( h_1, \ldots, h_t \), associated to a branch locus \( B \subset \tilde{P} = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(n)) \) homologous to \((2g + 2)H + 2kF\). Then

\[
\begin{align*}
\mu &= (2g + 1)(4k + n(2g + 2)) - \sum_i (8h_i^2 + 4h_i - 1) \\
e(M) &= 4 - 4g + \mu \\
c_1^2(M) &= (g - 1)(4k + n(2g + 2) - 8) - \sum_i (2h_i - 1)^2 \\
\tau(M) &= -(g + 1)(4k + n(2g + 2)) + \sum_i (4h_i^2 + 4h_i - 1) \\
\chi(M) &= \frac{g}{4}(4k + n(2g + 2)) - g + 1 - \sum_i h_i^2
\end{align*}
\]

**Proof.** By the discussion following the definition of \( \mu_{\text{virt}} \) we have

\[
\mu_{\text{virt}}(\tilde{B}) = (\mu - t) + \sum_{i=1}^t 4h_i(2h_i + 1),
\]

but also \( \mu_{\text{virt}}(\tilde{B}) = 2(2g + 1)2k + n(2g + 2)(2g + 1) \) by Lemma 1.6, which yields the formula for \( \mu \).
The formula for the Euler number can be proven by taking \( U \subset S^2 \) to be an \( \varepsilon \)-neighbourhood of the critical locus and noting
\[
e(M) = e(q^{-1}(S \setminus U)) + e(q^{-1}(\bar{U})).
\]
Over \( S \setminus U \) the fibration is locally trivial, so the first term gives \( e(S \setminus U) \cdot e(F) \), which for \( S = \mathbb{P}^1 \) equals \((2 - \mu)(2 - 2g)\). On the other hand \( q^{-1}(\bar{U}) \) is a deformation retract of the set of singular fibers, and each of these having only one node (that we think of as contracted circle on a general fiber) contributes \(3 - 2g\). Putting things together we obtain the stated formula.

As for \( c_1^2(M) = K_M^2 \), using (3) the computation is straightforward: The formulas for \( K_N \) and \( B_N \) yield
\[
K_N + \frac{1}{2}B_N = \rho^*(K_{\tilde{p}} + \frac{1}{2}\tilde{B}) - \sum_i \left((2h_i - 1)\Gamma'_i + (h_i - 1)\Gamma_i\right),
\]
so we get
\[
\left(K_N + \frac{1}{2}B_N\right)^2 = \left(-2H + (n - 2)F + (g + 1)H + kF\right)^2 + \sum_i \left((2h_i - 1)\Gamma'_i + (h_i - 1)\Gamma_i\right)^2
\]
\[
= (g - 1)(2k + n(g + 1) - 4) + \sum_i (-2h_i^2 + 2h_i - 1).
\]
But also
\[
K_M^2 = K_M^2 + t = 2(K_N + \frac{1}{2}B_N)^2 + t,
\]
from which the result follows. Index \( \tau \) and “holomorphic Euler characteristic” \( \chi \) can be computed in terms of \( \mu \), \( h_i \) and \( g \).

Observe that \( n \) and \( k \) occur only in the combination \( 4k + n(2g + 2) \) and can therefore be eliminated from the formula for \( c_1^2 \) in terms of \( \mu \), \( h_i \) and \( g \).

The fundamental group of a SLF is a quotient of the fundamental group of a general fiber and can generally be arbitrary \[^{[AmBoKaPa]}\]. However, under the absence of reducible singular fibers one can show simply-connectedness of hyperelliptic SLF’s.

**Proposition 3.2** Non-trivial, hyperelliptic SLF’s without reducible fibers are simply-connected.

**Proof.** The handlebody decomposition of \( M \) associated to a DLF \( q : M \to S^2 \) \[^{[Ka]}\] implies the following well-known description of the fundamental group of \( M \): Letting \( \delta_1, \ldots, \delta_\mu : S^1 \hookrightarrow \Sigma \) be the vanishing cycles, it holds
\[
\pi_1(M) = \pi_1(\Sigma)/\langle \delta_1, \ldots, \delta_\mu \rangle.
\]
A more convenient interpretation is as the fundamental group of the topological space \( \Sigma/\langle \delta_1, \ldots, \delta_\mu \rangle \) obtained from \( \Sigma \) by contracting the vanishing cycles (Seifert-Van Kampen).

Now if \( q : M \to S^2 \) is hyperelliptic and without reducible fibers, let \( M \to P = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(n)) \to \mathbb{P}^1 \) with branch locus \( B \subset P \) be the factorization over a two-fold cover as in Theorem \[^{[1.2]}\]. We can choose the \( \delta_i \) as lifts under \( \kappa : \Sigma \to S^2 \) of segments \( \tilde{\delta}_i :
[0, 1] \to S^2 joining a pair of critical points of \( \kappa \), cf. the discussion following Lemma 1.4.

Then \( \Sigma/(\delta_1, \ldots, \delta_\mu) \) is a two-fold cover of \( S^2/(\bar{\delta}_1, \ldots, \bar{\delta}_\mu) \). The latter space is a wedge of spheres. Moreover, since the branch locus \( B \) has at most two connected components (Proposition 2.3), the map

\[
\kappa' : \Sigma/(\delta_1, \ldots, \delta_\mu) \to S^2/(\bar{\delta}_1, \ldots, \bar{\delta}_\mu)
\]

is branched only in the vertex of the wedge and in at most one more point. But any connected, two-fold cover of \( S^2 \) branched in at most two points is simply connected. Thus \( \pi_1(M) = \pi_1(\Sigma)/(\delta_1, \ldots, \delta_\mu) = 0 \) as claimed.

\[\Box\]

4 Symplectic Noether-Horikawa surfaces

We conjectured in Remark 2.2 that hyperelliptic SLF’s with only irreducible singular fibers are holomorphic. In this chapter we discuss some more properties of this class of symplectic four-manifolds. We mostly restrict ourselves to the case \( g = 2 \), the reason being that in the holomorphic situation this class of complex surfaces (admitting Lefschetz fibrations of genus 2 without reducible fibers up to deformation) are distinguished in the theory of complex surfaces by their topological invariants. Moreover, there is a simple classification up to algebraic deformations: It consists of 3 infinite series, and each single class is swept out by one algebraically irreducible family. As in the topological situation above, they are two-fold covers of rational surfaces. The precise result is:

**Theorem 4.1** ([Ho], [Ch, p.51]) Let \( q : M \to \mathbb{P}^1 \) be a non-trivial genus 2 fibration (holomorphic with smooth total space, but not necessarily Lefschetz) without reducible fibers. Then \( q \) can be (holomorphically) deformed to a Lefschetz fibration

\[
q' : M \xrightarrow{\kappa} \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(n)) \xrightarrow{P} \mathbb{P}^1,
\]

where \( \kappa \) is a two-fold cover branched along a smooth curve \( B \subset \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(n)) \) and such that one of the following holds

(I) \( n = 0, B \sim 6H + 2kF \) is connected, \( k > 0 \)

(II) \( n = 1, B \sim 6H + 2kF \) is connected, \( k \geq 0 \)

(III) \( n = 2k > 0, B = S_\infty \cup B' \) with \( B' \sim 5H \) connected

Moreover, each of these classes comprises exactly one deformation type with irreducible parameter space.

\[\Box\]

Using specialization to reducible branch curves (e.g. 6 sections, each linearly equivalent to \( H \), plus \( 2k \) fibers in case I), it is not hard to compute the braid monodromies. The result in terms of our standard generators \( x_1, \ldots, x_5 \) of \( B(S^2, 6) \) is [Ch] p.115

(I) \( (\mu = 20k) \quad (\tau_1, \ldots, \tau_\mu) = (x_1, x_2, x_3, x_4, x_5, x_5, x_4, x_3, x_2, x_1)^{2k} \)

(II) \( (\mu = 20k + 30) \quad (\tau_1, \ldots, \tau_\mu) = (x_1, x_2, x_3, x_4, x_5, x_5, x_4, x_3, x_2, x_1)^{2k} \)

(III) \( (\mu = 40k) \quad (\tau_1, \ldots, \tau_\mu) = (x_1, x_2, x_3, x_4)^{10k} \)
where $\tau_r$ is the braid monodromy along $\gamma_r$ ($\gamma_1, \ldots, \gamma_\mu$ a standard generating set for $\pi_1(S^2 \setminus \{t_1, \ldots, t_\mu\}$ as above). The explicit form shows that all such surfaces arise as fiber connected sums of the three basic fibrations $I_1$, $\Pi_0$ and $III_2$ (the fiber connected sum is unique except in the last case because the monodromy generates the whole mapping class group).

From the monodromy representation one derives also easily that $M$ is simply connected (cf. below). Since we know signature $\tau$ and Euler characteristic $e$, it remains to investigate the parity of the intersection form to classify these complex surfaces up to homeomorphy. It turns out that $M$ is spin (has even intersection form) only for type $III_k$ with $k \equiv 0 \pmod{2}$ [PsPtXi].

From a complex surface point of view these surfaces are minimal except in cases $II_0$ and $III_2$ and of general type provided $\mu > 40$ (i.e. excluding types $I_1$, $I_2$, $II_0$, $III_2$). Moreover, they fulfill

$$c_1^2(M) \equiv 0 \pmod{2}, \quad \chi(M) = \frac{1}{2} c_1^2(M) + 3,$$

so they lie on the Noether line. It is also worthwhile to discuss the exceptional cases as they are the basic building blocks:

- $I_1$: Projecting onto the second factor inside $\mathbb{P}^1 \times \mathbb{P}^1$ we obtain a $\mathbb{P}^1$-fibration over $\mathbb{P}^1$ with 12 singular fibers, so this surface is rational.

- $I_2$: Similarly we obtain an elliptic fibration over $\mathbb{P}^1$ with 36 singular fibers; this is (hence) a fiber connected sum of three copies of the basic elliptic fibration of Livné/Moishezon mentioned above.

- $II_0$: Since $S_\infty$ is disjoint from the branch locus, $M$ contains two $(-1)$-curves. Contracting these we obtain a two-fold cover of $\mathbb{P}^2$ branched along a sextic, which is a K3-surface.

- $III_2$: After contraction of the $(-1)$-curve over $S_\infty$ one obtains a surface with $c_1^2 > 0$ which is (thus) of general type.

Conversely, any minimal complex surface with these invariants is of this form, or a two-fold cover of $\mathbb{P}^2$ branched along a curve of degree 8 or 10 [Ho]. There is a similar classification for $c_1^2$ odd. Surfaces on the Noether-line are therefore sometimes called Noether-Horikawa surfaces.

Let us now come back to symplectic geometry. We suggest the term symplectic Noether-Horikawa surfaces for minimal symplectic surfaces on the Noether line that are (symplectically) birational to a (symplectic) two-fold cover of a rational surface ($\mathbb{P}^2$ or a blow-up of $\mathbb{P}(O \oplus O(n))$). If $M$ is a cover of $\mathbb{P}^2$ branched along a symplectic surface of degree $2d$, a computation as in the previous chapter shows

$$c_1^2(M) = 2d^2 - 12d + 18, \quad e(M) = 4d^2 - 6d + 6, \quad \chi(M) = \frac{d^2 - 3d + 4}{2},$$

so $M$ is on the Noether line iff $d = 4$ ($c_1^2 = 2, \chi = 4$) or $d = 5$ ($c_1^2 = 8, \chi = 7$). Otherwise one can use the technique of pseudo-holomorphic curves to show that a branched cover of a rational ruled surface is a hyperelliptic SLF without reducible singular fibers. Conversely, it follows immediately from Theorem [3.1] that a hyperelliptic SLF is a symplectic Noether-Horikawa surface iff all singular fibers are irreducible.
Proposition 4.2 Let \( p : P \rightarrow \mathbb{P}^1 \) be a rational ruled surface and \( \kappa : \tilde{M} \rightarrow P \) a two-fold cover branched along a symplectic submanifold \( B \subset P \). Then \( p \) can be deformed to an \( S^2 \)-fibration \( p' : P \rightarrow \mathbb{P}^1 \) such that \( q = p' \circ \kappa : \tilde{M} \rightarrow \mathbb{P}^1 \) is a (hyperelliptic) SLF without reducible singular fibers.

Proof. (sketch) Choose a tamed almost complex structure \( J \) on \( P \) making \( B \) a \( J \)-holomorphic curve. Let \( S, F \subset P \) be a section and a fiber. One can show that for any tamed \( J \), the moduli space of (not necessarily irreducible, but connected) \( J \)-holomorphic curves \( C \) homologous to \( F \) is isomorphic to \( S \), simply by sending \( C \) to the unique point of intersection with \( S \). Since these \( J \)-holomorphic curves are pairwise disjoint for homological reasons, and rational by the adjunction formula, we obtain another \( S^2 \)-fibration \( p' : P \rightarrow S \simeq \mathbb{P}^1 \). It can be deformed to \( p \) by connecting \( J \) to the integrable complex structure within the space of tamed almost complex structures. For all this we refer to [Mc, §4] for details. Moreover, by choosing the path \( \{J_t\} \) generic, one can arrange the family of fibers to be transversal to the family of \( J_t \)-holomorphic curves \( \{B_t\} \). This means that for any \( J_t \) in the path, \( p'\big|_{B_t} \) has only finitely many simple branch points \( (z \mapsto z^2) \). Then \( p' \circ \kappa \) is a DLF, which is oriented because the intersection indices of \( B \) with the fibers are positive. \( \diamond \)

Remark 4.3 1) A deeper question concerns the characterization of symplectic Noether-Horikawa surfaces by invariants. It would be surprising, should \((c_1^2, \chi)\) alone suffice as in the algebraic case.

2) A similar argument as in the proof shows that to prove holomorphicity of a hyperelliptic SLF, it suffices to show that \((B \hookrightarrow P)\) is diffeomorphic to a holomorphic curve inside a rational surface. The isomorphism will automatically respect the fibration structures (connect the pull-back of the integrable complex structure to an almost complex structure \( J \) that is compatible with \( p : P \rightarrow \mathbb{P}^1 \) and making \( B \) a \( J \)-holomorphic curve).

3) This proposition together with the braid monodromy computation for algebraic curves by specialization also shows that the isotopy problem for symplectic submanifolds is equivalent to the following group theoretic problem: The subset of half-twists in \( B(S^2, d) \) is stable under inner automorphisms. On \( \mu \)-tuples \( (\tau_1, \ldots, \tau_\mu) \) of half-twists consider the equivalence relation generated by

\[
(\tau_1, \ldots, \tau_r, \tau_{r+1}, \ldots, \tau_\mu) \sim (\tau_1, \ldots, \tau_{r-1}, \tau_r \tau_{r+1} \tau_r^{-1}, \tau_{r+2}, \ldots, \tau_\mu)
\]

\((Hurwitz-equivalence)\). The claim is that any \( (\tau_1, \ldots, \tau_\mu) \) with \( \prod_r \tau_r \in \{1, (x_1 \cdot \cdots \cdot x_{d-1})^d\} \) is Hurwitz-equivalent to one of

\[
(x_1, \ldots, x_{d-1}, x_{d-1}, \ldots, x_1)^{\mu \frac{d}{d-2}}
(x_1, \ldots, x_{d-1}, x_{d-1}, \ldots, x_1)^{\frac{\mu - d(d-1)}{2d-2}}(x_1, \ldots, x_{d-1})^d
(x_1, \ldots, x_{d-2})^{\frac{d}{d-2}}.
\]

\( \diamond \)
Appendix: Two-fold covers

A degree two map \( \kappa : M \rightarrow N \) of \( n \)-dimensional, oriented, differentiable manifolds whose set of critical values is a codimension two, oriented submanifold \( B \subset N \) is called two-fold cover of \( N \) with branch locus \( B \) if locally along \( B \) there exist oriented local coordinates \((x_1, \ldots, x_n)\) of \( M \) and \((y_1, \ldots, y_n)\) of \( N \) such that

\[
\kappa : (x_1, \ldots, x_n) \mapsto (y_1, \ldots, y_n) = (x_1^2 + x_2^2, x_1 x_2, x_3, \ldots, x_n).
\]

The fibers having cardinality two, the corresponding unbranched covers \( M \setminus \kappa^{-1}(B) \rightarrow N \setminus B \) are Galois with group \( \mathbb{Z}/2\mathbb{Z} \). The generator of this group is the orientation preserving covering transformation \( \iota : M \rightarrow M \) swapping the two elements of a generic fiber (an involution).

To construct \( M \) with branch locus \( B \subset N \) we assume that the homology class of \( B \) is divisible by 2 in \( H_{n-2}(M; \mathbb{Z}) \) (we will soon see that this is always true for branch loci of two-fold covers). Then \( B \) is Poincaré-dual to the first Chern class of the square of a complex line bundle \( L \). Let \( s \) be a transverse section of \( L^\otimes 2 \) with zero locus \( B \). The preimage of the zero section of the quadratic map

\[
L \rightarrow L^\otimes 2, \quad t \mapsto t^2 - s
\]

defines an oriented, connected submanifold \( M \subset L \). The bundle projection \( \kappa : L \rightarrow N \) exhibits \( M \) as two-fold cover simply branched along \( B \). We call \( M = M_{L,s} \rightarrow N \) the two-fold branched cover associated to \( L \) and \( s \).

It turns out that all two-fold covers arise in this way.

**Proposition 4.4** Let \( \kappa : M \rightarrow N \) be a two-fold cover with branch locus \( B \). Then there exists a complex line bundle \( L \), a transverse section \( s \) of \( L^\otimes 2 \) and a diffeomorphism of \( M \) with \( M_{L,s} \) over \( N \).

Moreover, the pair \((L, s)\) is uniquely determined by \( M \) up to canonical isomorphism (of line bundles over \( N \) with section).

**Proof.** Away from the branch locus we define \( L \) at some \( x \in N \) as the space of linear forms \( \lambda \) on \( \text{Map}(\kappa^{-1}(x); \mathbb{C}) \) that change signs under application of the involution \( \iota \). Explicitly, writing \( \kappa^{-1}(x) = \{P, Q\} \) an element \( f \in \text{Map}(\kappa^{-1}(x); \mathbb{C}) \) is given by the pair \((f(P), f(Q)) \in \mathbb{C}^2 \) and the linear forms to be considered have the form \( \lambda(a, b) = c \cdot (a - b) \) for some \( c \in \mathbb{C} \). To extend \( L \) across \( B \) let \( M \rightarrow N \) locally be given by \((z, w) \mapsto (u, v) = (z^2, w) \) where \( z, u \) are complex valued and \( w, v \) are \((n - 2)\)-tuples of coordinates. Since \( \iota^*z = -z \) the linear form \( \lambda_0 \) sending the values \( \pm \sqrt{u} \) of the function \( z \) on the fibers of \( \kappa \) to 1 is a frame for \( L \) away from \( B \). The extension can thus be defined by taking \( \lambda_0 \) as frame at the point of \( B \) under study.

Next we observe that \( L \otimes L \) has a canonical section \( s \) sending \((a,b), (c,d)\) to \((a-b)(c-d)\). This section has a simple zero along \( B \) since in the local coordinates \( z, w, u, v \) we get

\[
s(u, v) = (z - (-z)) \cdot (z - (-z)) = 4z^2 \cdot \lambda_0^2(u, v) = 4u \cdot \lambda_0^2(u, v).
\]

To identify \( \kappa : M \rightarrow N \) with \( \kappa_{L,s} : M_{L,s} \rightarrow N \) let \( \pm \lambda \in L_x \) be the solutions to \( t^2 = s \) over some \( x \in N \). So \( \kappa^{-1}(x) = \{-\lambda, \lambda\} \). Let \( f(P) = 1, f(Q) = 0 \). Then \( s_x(f, f) = 1 \) and \( \lambda(f) = 1 \) or \( \lambda(f) = -1 \). There is thus a canonical bijection between \( \kappa^{-1}(x) \) and

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\(\kappa_{L,s}^{-1}(x)\) by sending \(P\) to \(\lambda(f) \cdot \lambda\) and \(Q\) to \(-\lambda(f) \cdot \lambda\). This identification clearly extends over \(B\) as one easily checks using the local description.

In view of the functorial nature of the construction it follows also that a diffeomorphism of two-fold coverings \(M \simeq M'\) induces an isomorphism of the associated complex line bundles \(L \simeq L'\) whose square carries the canonical sections \(s, s'\) into each other.

It is natural to ask to what extend the information given by \(L\) and \(s\) is already contained in \(B\).

**Proposition 4.5** For any 2-divisible, oriented submanifold \(B \subset N\) there are exactly \(\# H^1(N; \mathbb{Z}/2\mathbb{Z})\) isomorphism classes of two-fold covers of \(N\) with branch locus \(B\).

In particular, if \(H^1(N; \mathbb{Z}/2\mathbb{Z}) = 0\) there is a one-to-one correspondence between isomorphism classes of two-fold covers of \(N\) and 2-divisible, oriented submanifolds of \(N\), given by the branch locus.

**Proof.** Let \(\mathcal{A}\) be the sheaf of germs of complex valued smooth functions on \(N\), and \(\mathcal{A}^*\) those without zeroes (that we view as multiplicative abelian sheaf). We will use the following short exact sequences of sheaves:

\[
0 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow \mathcal{A}^* \xrightarrow{i_2} \mathcal{A}^* \rightarrow 0
\]

\[
0 \rightarrow \mathbb{Z} \xrightarrow{-} \mathcal{A} \xrightarrow{\exp} \mathcal{A}^* \rightarrow 0
\]

From the last sequence we obtain isomorphisms \(H^i(N; \mathcal{A}^*) \simeq H^{i+1}(N; \mathbb{Z})\) (as a soft sheaf \(\mathcal{A}\) has vanishing higher cohomology). A little diagram chase shows that these are compatible with the cohomology sequences of the other two sequences as follows:

\[
\begin{array}{ccc}
H^0(N; \mathcal{A}^*) & \xrightarrow{i_2} & H^0(N; \mathcal{A}^*) \\
\downarrow & & \downarrow \\
H^1(N; \mathcal{A}^*) & \rightarrow & H^1(N; \mathbb{Z}/2\mathbb{Z}) \\
\downarrow & & \downarrow \\
H^1(N; \mathcal{A}^*) & \rightarrow & H^1(N; \mathcal{A}^*) \\
\end{array}
\]

Here all vertical arrows are isomorphisms. Thus \(H^1(N; \mathbb{Z}/2\mathbb{Z})\) is (non-canonically) isomorphic to the direct sum of the 2-torsion in \(H^2(N; \mathbb{Z})\) and the cokernel of the squaring map \(\mathcal{A}^*(N) \rightarrow \mathcal{A}^*(N)\). We claim that these two data act effectively on the set of isomorphism classes of pairs \((L, s)\).

In fact, the 2-torsion of \(H^2(N; \mathbb{Z})\) exactly parametrizes isomorphism classes of complex line bundles \(L\) with \(2c_1(L)\) a given class (here the Poincaré-dual of \(B\)). Once \(L\) is fixed, the space of isomorphisms \(L \rightarrow L\) induces an action on the space of (transverse) sections \(s\) of \(L^{\otimes 2}\). Now for any two sections \(s, s'\) having the same zero locus \(B\) the function \(\lambda = s'/s\) has no zeros (so \(\lambda \in \mathcal{A}^*(N)\)) and multiplication by \(\lambda\) induces the (unique) isomorphism of \(L^{\otimes 2}\) carrying \(s\) to \(s'\). This comes from an isomorphism of \(L\) iff \(\lambda\) can be written as a square. Conversely, multiplication of \(s\) by any non-square \(\lambda\) leads to a pair \((L, \lambda s)\) that is not equivalent to \((L, s)\) in the considered sense. This concludes the proof of the proposition.

**Remark 4.6** In the holomorphic category, given \(B\) and \(L\) there is at most one isomorphism class of sections \(s\) of \(L^{\otimes 2}\) having a holomorphic representative. The reason
is that by the identity theorem any two holomorphic sections of a holomorphic line bundle with the same zero divisor differ only by a constant. Allowing changes of the complex structure of $M$ (leaving $B$ as complex submanifold) might however lead to different classes of sections.

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