RELATIONSHIP BETWEEN COLORING, EMBEDDING AND DECYCLING NUMBER OF A GRAPH

Sajid Hussain¹, Ren Han², Nisar Hussain Khoja³

¹,²School of Mathematical Science, East China Normal University
Shanghai 20062, China
³Karakoram International University, Gilgit-Baltistan, Pakistan

¹sajid.khoja100@yahoo.com, ²hrn@math.ecnu.edu.cn,
³khoja1210@gmail.com.

Corresponding Author: Ren Han

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Abstract

A set $S$ of vertices of a graph $G$ is said to be a decycling set if $G - S$ is acyclic. The size of a minimum decycling set of $G$ is called the decycling number of $G$ and it is denoted by $\psi(G)$. In this paper, our chief objectives are to obtain the upper bound of the decycling number of a graph by using graph chromatics number and its order. The relation of the genus of the surface $\Sigma$ and the decycling number of a graph embedded in surface $\Sigma$ is studied. The decycling number of a planar graph with $n$ vertices is conjectured to be $\frac{n}{2}$, which is shown in this paper if the girth of the graph is at least four. The decycling number of a graph with $n$ vertices and maximum degree three is proved to be at most $\left\lceil \frac{n}{3} \right\rceil$. Also, we completely investigate the decycling number of the hypercube $Q_n$.

Keywords : Decycling number, Chromatic number, Maximum degree, Embedding, Girth, hypercube.

I. Introduction

Graphs throughout this paper are finite, undirected and simple. For general terminologies and theoretic notations, we follow [V]. Let $G = (V, E)$ be a graph and $S \subseteq V(G)$ if $G - S$ is forest, then $S$ is said to be a decycling set of $G$. The cardinality of a smallest decycling set of $G$ is called the decycling number and it is

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signified by $\nabla(G)$. The decycling sets are also called Feedback vertex sets, which are heavily in the area of circuit design and deadlock prevention [IX]. Clearly, $\nabla(G) = 0$ if and only if $G$ is acyclic, and $\nabla(G) = 1$ if and if $G$ contains at least one cycle. In 1997 Beineke and Vandell [III] gives an expository result on the decycling number of the complete graph $K_n$ is $n - 2$ and the decycling number of the complete bipartite graph $K_{r,t}$ is $r - 1$ if $r \leq t$ and they developed a relation between the maximum degree, the order, the size and the decycling number of a graph. And they also proved that $\nabla(G) = \frac{m-n+1}{n-1}$ if $G$ is a connected graph with $n$ vertices, $m$ edges, and maximum degree $\Delta$.

In 2006 Punnim [XVII] explored the relationship between the maximum degree, the order and the decycling number of a graph under the definite condition. Now the question is that, what will be the relationship if the imposed conditions are ignored? In the second part of this paper, firstly we develop the relationship between the chromatic number and the decycling number of a graph. Then we determine the upper bound for the decycling number of a graph using its order and maximum degree. Also, we find the lower bound for the decycling number of a graph using its order and minimum degree.

The decycling number of a graph with maximum degree three has attracted much attention. Zheng et. al. [XXIII] shows that for any cubic graph $G$ of order $n$ without triangles, except for two cubic graphs with $n = 8$, $\nabla(G) \leq \left\lceil \frac{n}{3} \right\rceil$. Punnim [XVII] proved that the decycling number of a graph $G$ with maximum degree at most three, under the special conditions, for example, the condition says that $G$ is not a cubic graph without triangles. In 2017 Ren et al [XIX] explored the relationship between the decycling number and maximum genus of a graph with maximum degree three. Whether the decycling number of a graph $G$ with order $n$ and maximum degree three is at most $\left\lceil \frac{n}{3} \right\rceil$? So, in the third part of this paper we give an affirmative solution.

Albertson and Berman [I] and Erdös et al [VIII] introduced a conjecture on the decycling number of a planer graph, which implies the following

**Conjecture 1** ([I] [VIII]) If $G$ is a connected planar graph with $n$ vertices, then $\nabla(G) \leq \frac{n}{2}$.
This conjecture is still open. In the fourth part of this paper, we prove that the conjecture holds if $G$ is a planar graph with girth at least six. Also, we study the relationship between the decycling number of a graph and the genus of a surface $\Sigma$ in which the graph is embedded.

In 1997 Beineke and Vandell [III] gives the lower bound for the decycling number of hypercubes graph $Q_n$ and they determined the decycling number of $Q_n$ when $n$ is a smaller number. Pike [XVI] discovered the relation between the decycling number of $Q_n$ and Hamming code. In the fifth part of this paper, we show that an upper bound for the decycling number of $Q_n$ is the same as the lower bound in [III]. Thus, the decycling number of $Q_n$ is determined.

Before moving on to our main discussion, we go through some basic terminologies and theoretic notations.

Preliminaries about Graph Theory

Let $G$ be a graph. The degree of $v$ in $G$, denoted by $d_G(v)$, is the number of edges incident with $v$ in $G$. The maximum degree and the minimum degree of $G$ are denoted by $\Delta(G)$ and $\delta(G)$, respectively. If $v$ is a vertex in $G$, then any other vertex which is adjacent to $v$ is called a neighbors of $v$. The set of neighbors of $v$ is denoted by $N(v)$. A graph $H$ is called an irreducible graph if $H$ has not any cut edge. A matching $M$ in a graph $G$ is a set of edges in $G$ such that any two edges in $M$ have not a common end. A cycle with $n$ vertices is said to be an $n$-cycles. The girth of a graph is the length of its shortest cycle and if a graph has no cycle, then it girth will be considered as infinity. $\lceil x \rceil$ is the minimum number which is not less than $x$. In the field of topological graph theory, the graph embedding is a special type of drawing of a graph in the surface $\Sigma$ in which embedded graph’s edges do not intersect to each other, except at their endpoints. A surface $\Sigma$ is called orientable if a clockwise rotation can be made around all points consistently and is called non-orientable otherwise. All the orientable surfaces are homeomorphic to the sphere with $k$ handles, denoted by $S_k$. And all the non-orientable surfaces are homeomorphic to the sphere with $h$ crosscaps, denoted by $N_h$. Where $k$ and $h$ are said to be the genus of the orientable surface and non-orientable surface respectively.

II. The Chromatic Number and the Decycling Number of A Graph

A proper k-coloring of a graph $G$ is a function $f$ from $V(G)$ to the set $\{1,2, \ldots ,k\}$ such that any two vertices are received different colors if they are adjacent
to each other. The minimum number of colors needed to color a connected graph $G$ is called its chromatic number and it is denoted by $\chi(G)$. A subset of vertices receives the same color is called color class, each of the forms an independent set.

**Theorem 2.1** Suppose that $G$ is a graph with $n$ vertices. If $\chi(G) = \alpha$, then,

$$\nabla(G) \leq \frac{\alpha - 1}{\alpha} n - 1$$

**Proof.** Since $\chi(G) = \alpha$, $G$ has a proper coloring with $\alpha$ colors. Let $X_1, X_2, ..., X_\alpha$ be $\alpha$ coloring classes. Without loss of generality, suppose that $X_\alpha$ is the largest coloring class. Then $X_\alpha$ contains $\frac{n}{\alpha}$ vertices. For any vertex $v$ in $V(G) \setminus X_1, X_2, ..., X_\alpha \cup \{v\}$ does not contain any cycle. So $V(G) \setminus (X_\alpha \cup \{v\})$ is a decycling set of $G$. Since $X_\alpha \cup \{v\}$ has at least $\frac{n}{\alpha} + 1$ vertices, we have that $\nabla(G) \leq \frac{\alpha - 1}{\alpha} n - 1$.

By a result in [III], the decycling number of the complete graph $\nabla(K_n) = n - 2$. So, the upper bound in Theorem 2.1 is tight.

**Theorem 2.2** Suppose that $G$ is a graph with $n$ vertices, and suppose that $\alpha$ is a positive integer. If $\nabla(G) \leq \frac{\alpha - 1}{\alpha} n - 1$, then $\chi(G) \geq \alpha$.

**Proof.** Assume that $\chi(G) = \beta < \alpha$. Then $\nabla(G) \leq \frac{\beta - 1}{\beta} n - 1$ by Theorem 2.1. Since

$$\frac{\theta - 1}{\theta} < \frac{\alpha - 1}{\alpha} \text{ if } \beta < \alpha,$$

we have that $\nabla(G) \leq \frac{\alpha - 1}{\alpha} n - 1$, a contradiction. So $\chi(G) \geq \alpha$.

It is important to say that $\chi(G)$ must not be $\alpha$ if $\nabla(G) \leq \frac{\alpha - 1}{\alpha} n - 1$. Let us consider the following graphs.

**Example.** Let $H$ be the graph obtained from $K_{3,3}$ by adding an edge to connect two non-adjacent vertices, say $x$ and $y$. Then $\nabla(H) = 2$ and $S = \{x, y\}$ is a decycling set of $H$, but $\chi(H) = 3$.

**Lemma 2.3** ([VII]) let $G$ be a connected graph. If $G$ is either complete graph nor an odd cycle, then $\chi(G) \leq \nabla(G)$.
Corollary 2.4 Suppose that $G$ is a graph which is neither complete graph nor an odd cycle. If $G$ has $n$ vertices with maximum degree $\Delta$. Then

$$\nabla(G) \leq \frac{\Delta - 1}{\Delta} n - 1$$

Punnim [XVII] shows that if $G$ is a connected $r$-regular graph of order $n > 2r + 2$, then $\nabla(G) \leq \frac{r-2}{r} n$ for all $r \geq 4$. We note that the condition $n > 2r + 2$ cannot be ignored in Punnim’s result. Otherwise, it would be wrong if $G$ is the graph $K_n$ when $n \geq 4$. However, in our results, there is no restriction on order of the graph.

A forest $k$-coloring of a graph $G$ is a function $f$ from $V(G)$ to the set $\{1, 2, \ldots, k\}$ such that each color class induces an acyclic subgraph. The vertex-arboricity of $G$, denoted by $\nu a(G)$, is the minimum number of subsets in a partition of the vertex set of $G$ in such a way that each subset induces an acyclic subgraph (or forest $k$-coloring). In fact, $\nu a(G) \geq 1$ for all non-empty graph $G$ and $\nu a(G) = 1$ iff $G$ itself a forest. We now consider the relationship between the vertex arboricity and the decycling number of a graph.

**Theorem 2.5** Suppose that $G$ is a graph with $n$ vertices. If $\nu a(G) = \beta$, then

$$\nabla(G) \leq \frac{\beta - 1}{\beta} n$$

**Proof.** Let us consider that $f$ is a forest $\beta$-coloring of $G$. Then there is a class $X$ with at least $\frac{n}{\beta}$ vertices. So $V(G) \setminus X$ is a decycling set of $G$. Since $V(G) \setminus X$ contains at most $(1 - \frac{1}{\beta})n$ vertices, we have that $\nabla(G) \leq \frac{\beta - 1}{\beta} n$.

**Lemma 2.6** [VII] For any graph $G$, $\nu a(G) \leq \left\lceil \frac{1+\Delta(G)}{2} \right\rceil$

Theorem 2.7 obtained from Theorem 2.5 and Lemma 2.6

**Theorem 2.7** Suppose that $G$ is a graph with $n$ vertices and maximum degree $\Delta$. Then

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\[ \nabla(G) \leq \left\lfloor \frac{1 + \Delta(G)}{2} \right\rfloor - 1 \]

The upper bound in Theorem 2.7 is tight. It is sufficient to consider the complete graph \( k_n \), where \( n \) is a positive even number. Upon the relation between the minimum degree and the decycling number of a graph, we have the following theorem.

**Theorem 2.8** Suppose that \( G \) is a graph with \( n \) vertices. If \( \delta(G) \geq 4 \), then \( \nabla(G) \leq \delta(G) - 1 \).

**Proof.** Suppose on the contrary that \( \nabla(G) \leq \delta(G) - 2 \). Let \( S \) be a decycling set with \( \delta(G) - 2 \) vertices of \( G \). Then the minimum degree of \( G - S \) is at least two. So \( G - S \) has at least one cycle, a contradiction. Hence \( \nabla(G) \leq \delta(G) - 1 \).

The lower bound in Theorem 2.8 is also sharp. It is sufficient to consider for the complete graph \( k_n \), where \( n \geq 5 \). Next, we determine the decycling number of two graphs.

**Theorem 2.9** Suppose that \( n \geq 5 \) is an integer. Let \( G \) be the graph obtained from the complete graph \( k_n \) by deleting a matching. Then \( \nabla(G) = n - 3 \).

**Proof.** Suppose that \( V(k_n) = \{v_1, v_2, ..., v_n\} \). Without loss of generality, suppose that the matching is \( \{v_1v_2, v_3v_4, ..., v_{2k-1}v_{2k}\} \), where \( 2k \leq n \). Then \( \{v_4, v_5, ..., v_n\} \) is a decycling set of \( G \). So \( \nabla(G) \leq n - 3 \). Since \( \delta(G) = n - 2 \) we have that \( \nabla(G) \geq n - 3 \) by Theorem 2.8. Hence \( \nabla(G) = n - 3 \).

**Theorem 2.10** Suppose that \( n \geq 6 \) is an integer. Let \( G \) be the graph obtained from the complete graph \( k_n \) by deletion, all edges in a subgraph which is isomorphic to \( k_r \), where \( 2 \leq r \leq n - 4 \), then \( \nabla(G) = n - r - 1 \).

**Proof.** Clearly, \( \delta(G) = n - r \). So \( \nabla(G) \geq n - r - 1 \) by Theorem 2.8. Let \( H \) be a subgraph of \( k_n \) which is isomorphic to \( k_r \). Let \( v \) be a vertex in \( V(k_n) \setminus V(H) \). Then the graph induced by \( V(H) \cup \{v\} \) does not contain any cycle. Since \( V(H) \cup \{v\} \) has \( r + 1 \) vertices, we have that \( \nabla(G) \leq n - r - 1 \). Hence \( \nabla(G) = n - r - 1 \).

As we know that \( \nabla(k_n) = n - 2 \) if \( n \geq 3 \). At the end of the section, we consider the inverse proposition.
Theorem 2.11. Let $n \geq 3$ be an integer. If $G$ has $n$ vertices and $\nabla(G) = n - 2$, then $G$ is isomorphic to the complete graph $K_n$.

**Proof.** It is not difficult to find that $G$ is isomorphic to $k_n$ if $\nabla(G) = n - 2$ for $n = 3$ or $n = 4$. Now we suppose that $n \geq 5$, $\nabla(G) = n - 2$, and $V(G) = \{v_1, v_2, ..., v_n\}$. If there are two vertices $v_j$ and $v_k$ belongs to graph $G$, in such a way that they are not adjacent to each other, then $G$ is a subgraph of the graph obtained from $K_n$ by removing the edge $v_jv_k$. Theorem 2.9 implies that, $\nabla(G) \leq n - 3$, a contradiction. So, any two vertices in $G$ are adjacent to each other. Hence $G$ is isomorphic to the complete graph $K_n$.

\[ \nabla(G) \leq \left\lceil \frac{n}{3} \right\rceil \]

**Proof.** Suppose that $G$ is a connected irreducible graph. If $G$ is non-connected, then we similarly consider its components. So $\delta(G) \geq 2$. If $\Delta(G) = 2$, then $G$ is the union of several cycles. Since each cycle has at least three vertices, we have that $\nabla(G) \leq \left\lceil \frac{n}{3} \right\rceil$.

We now suppose that $G$ has at least one vertex with degree three. We use the induction on $n$ to show the proposition. Clearly, $n \geq 4$ in this case. If $n = 4$, then $G$ is one of the $G$ looks like Figure 1.
It is easy to check that $\nabla(G_1) = 1$ and $\nabla(G_2) = 2$. So, the proposition holds.

Assume that the proposition is true if $G$ has at most $n - 1$ vertices, where $n \geq 5$. We now consider that $G$ has $n$ vertices. We claim that $G$ has no any cut vertex. Otherwise, $G$ has a cut edge, a contradiction. So $G$ is a $2$-connected graph.

Let us consider $v$ is a vertex of $G$ with degree three, and suppose that $N(v) = \{v_1, v_2, v_3\}$. If there are two vertices in $N(v)$, say $v_1$ and $v_2$, such that $d_G(v_1) = d_G(v_2) = 2$, then we delete $v$. Thus, each of $v_1$ and $v_2$ has degree one in the present graph. Next, they are deleted. Note that this procedure does not affect the decycling number.

Let $G'$ be the obtained graph. Then $G'$ is a graph with $\nabla(G') \leq 3$, and $\nabla(G) \leq \nabla(G') + 1$. By the induction assumption, $\nabla(G') \leq \left\lfloor \frac{n}{3} \right\rfloor - 1$. So $\nabla(G) \leq \left\lfloor \frac{n}{3} \right\rfloor$.

If there is only one neighbor of $v$, say $v_1$, which has degree two, then we consider two cases. If the neighbor of $v_1$ other than $v$ is one of $v_2$ and $v_3$, say $v_2$, then we delete $v$ at first. Thus, the degrees of $d(v_1) = 1$ and $d(v_2) = 2$, in the present graph. Next, we delete $v_1$, then $v_2$. Let $G''$ be the obtained graph. Then $G''$ is a graph with $\nabla(G'') \leq 3$. By the similar argument as that in the above paragraph,$\nabla(G) \leq \left\lfloor \frac{n}{3} \right\rfloor$.

If $v_1$ is not adjacent to anyone of $v_2$ and $v_3$, then there is no any singular $3$-cycle passing through $v_1$. In this case $v_1$ is contracted into $v$. If the neighbor of $v_1$ other than $v$ has degree two, then we proceed the similar argument, and so on. In the end, we have $\nabla(G) \leq \left\lfloor \frac{n}{3} \right\rfloor$ or each neighbor of $v$ has degree three.
We now suppose that \( d_G(v_i) = 3 \) for \( i = 1, 2, 3 \). Let \( H \) be the graph obtained from \( G \) by deleting \( v \). Then \( d_H(v_i) = 2 \) for \( i = 1, 2, 3 \). If \( v_1 \) is not in any 3-cycle in \( H \) for \( i = 1, 2, 3 \), then it is contracted it’s a neighbor. Need to say that if \( v_i \) is adjacent to one of \( \{v_1, v_2, v_3\}\), say \( v_j \), then \( v_i \) is contracted into \( v_j \) firstly, and \( v_j \) contracted into it’s a neighbor, and so on. Let \( H' \) is a graph with \( \nabla(H') \leq 3 \). By the induction hypothesis \( \nabla(H') \leq \left\lfloor \frac{n}{3} \right\rfloor - 1 \). So \( \nabla(G) \leq \nabla(H') + 1 \leq \left\lfloor \frac{n}{3} \right\rfloor \).

If there is a vertex in \( v_1, v_2 \) and \( v_3 \), say \( v_1 \), such that it is in some singular 3-cycle \( C \), suppose that the neighbors of \( v_1 \) are \( x_1 \) and \( x_2 \). Then \( C = x_1 v_1 x_2 x_1 \). We delete \( x_1 \) at first. Then the degree of \( v_1 \) is one and the degree of \( x_2 \) is at most two in the present graph. Next, \( v_1 \) and \( v_2 \) are deleted in this same order. Note that the deletion of \( v_1 \) and \( x_2 \) does not affect the decycling number. Let \( H'' \) be the obtain graph. If \( H'' \) also has a singular 3-cycle, then the cycle is dealt with as \( C \), and so on. Suppose that \( k \) singular 3-cycles are being deleted. Let \( F \) be the obtained irreducible graph, i.e., \( F \) has no any cut edge.

If \( F \) has not any vertex, then \( n = 3k + 1 \). Considering that \( k \) vertices are deleted, we have that \( \nabla(G) \leq k + 1 \), we have that \( \nabla(G) \leq \left\lfloor \frac{n}{3} \right\rfloor \). If \( F \) has at least one vertex and any edge, then \( \nabla(G) \leq \left\lfloor \frac{n}{3} \right\rfloor \) by a similar argument. If \( F \) has at least one edge, then \( \delta(F) \geq 2 \), since \( F \) is an irreducible graph. In this case, we claim that \( F \) is not a 3-regular graph. Otherwise, \( G \) is not connected, which violates the assumption. So \( F \) has at least one vertex with degree two. If \( F \) has only one vertex with degree two, then \( G \) has a cut edge or is not connected, a contradiction. Hence \( F \) has at least two vertices with degree two. Let \( y_1 \) and \( y_2 \) be two vertices with degree two if \( F \). Then each of them is not in a 3-cycle. Otherwise, \( F \) has a singular 3-cycle. Let \( \geq 4 - cycle \) be a cycle whose length no less than four. If both \( y_1 \) and \( y_2 \) are not in the same \( \geq 4 - cycle \), then they are contracted respectively. Let \( F' \) be obtained graph.

By the induction hypothesis, \( \nabla(F') \leq \left\lfloor \frac{n-k-2}{3} \right\rfloor \), that is \( \nabla(F') \leq \left\lfloor \frac{n}{3} \right\rfloor - k - 1 \). Thus \( \nabla(G) \leq \left\lfloor \frac{n}{3} \right\rfloor \).

Otherwise, suppose that \( C' \) is the \( \geq 4 - cycle \) which contains \( y_1 \) and \( y_2 \). If the length of \( C' \) is at least five, then \( y_1 \) and \( y_2 \) are contracted. By a similar argument as the above, we have that \( \nabla(G) \leq \left\lfloor \frac{n}{3} \right\rfloor \).

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If \( C' \) is a 4-cycle, let \( z, w \) be two vertices in \( V(C) \setminus \{y_1, y_2\} \). If \( C' = zy_1y_2z \), then \( d_F(z) = d_F(w) = 2 \) or \( d_F(z) = d_F(w) = 3 \). Otherwise, one of \( z \) and \( w \) is incident with a cut edge in \( F \), a contraction. If \( d_F(z) = d_F(w) = 2 \), there are two cases to consider. If \( F \) is exactly the cycle \( C' \), then \( n \geq 3k + 5 \) and \( \nabla(F) = 1 \). So
\[
\nabla(G) \leq \left\lceil \frac{n}{3} \right\rceil.
\]

If \( d_F(z) = d_F(w) = 3 \), then we delete the vertex \( z \) the degrees of \( y_1, y_2 \) and \( w \) are 1,1, and 2, respectively, in the present graph. Next, \( y_1, y_2 \) and \( w \) are deleted in this order. The most interesting point is that this procedure does not affect the decycling number. Let \( B \) be the obtained irreducible graph. If \( B \) has not any edge, then \( \nabla(G) \leq \left\lceil \frac{n}{3} \right\rceil \). Otherwise, \( B \) has at least two vertices with degree two by a similar argument as \( F \). Next, a vertex with degree two is contracted into its a neighbor. Let \( B' \) be the obtained graph. So \( \nabla(G) \leq \nabla(B') + k + 2 \). Since \( B' \) has at most \( n - 3k - 6 \) vertices, \( \nabla(B') \leq \left\lceil \frac{n}{3} \right\rceil - k - 2 \) by the induction hypothesis. Hence \( \nabla(G) \leq \left\lceil \frac{n}{3} \right\rceil \).

If \( C' = zy_1wz \), then we proceed the similar arguments as the above paragraph. So \( \nabla(G) \leq \left\lceil \frac{n}{3} \right\rceil \).

IV. Graphs on Surfaces

A surface \( \Sigma \) is a compact connected 2-dimensional manifold without boundary. The orientable surface \( S_k \) \((k \geq 0)\) can be obtained from the sphere by attaching \( k \) handles, where \( k \) is called the genus of the orientable surface. The non-orientable surfaces \( N_h \) \((h \geq 1)\) can be obtained from the sphere by attaching \( h \) crosscaps, where \( h \) is said to be the genus of the non-orientable surface. The classical examples of orientable and non-orientable surfaces are torus \( S_1 \) and projective plane \( N_1 \).

A graph embedding is a special type of drawing of a graph \( G \) on the surface in which embedded graphs edges do not intersect to each other, except at their endpoints. The chromatics number \( \chi(\Sigma) \) of a surface \( \Sigma \) is the largest \( \chi(G) \) such that \( G \) can be embedded in \( \Sigma \).

Let \( \Sigma \) be a surface and \( \gamma \) be a simple closed curve in \( \Sigma \). A simple closed curve
\( \gamma \) is said to be non-contractible if it is not null-homotopic. Let \( \Psi \) be an embedding of a graph \( G \) on a non-spherical surface \( \Sigma \). The face-width of \( \Psi \) denoted by \( fw(\Psi) \), is \( \min\{ |\gamma \cap V(G) | : \gamma \) is a non-contactable simple close curve in \( \Sigma \) and \( \gamma \cap G \subset V(G) \} \).

**Theorem 4.1** [X] (Heawood Map Color Theorem)

(i) \( \chi(S_k) = \left\lceil \frac{7 + \sqrt{1 + 48k}}{2} \right\rceil \), for \( k > 0 \),

(ii) \( (N_h) = \left\lceil \frac{7 + \sqrt{1 + 24h}}{2} \right\rceil \), for \( h = 1 \) and \( h \geq 3 \), and \( (N_h) = 6 \).

The coming up theorem follows from Theorem 2.1 and Theorem 4.1

**Theorem 4.2** Let \( G \) be a connected graph with \( n \) vertices.

If \( G \) is embedded in the orientable surface \( S_k \), where \( k > 0 \), then

\[
\nabla(G) \leq \left\lfloor \frac{7 + \sqrt{1 + 48k}}{2} \right\rfloor - 1 - \frac{n - 1}{2}
\]

(1) Suppose that \( G \) is embedded in the non-orientable surface \( N_h \). Then

\[
\nabla(G) \leq \left\lfloor \frac{7 + \sqrt{1 + 24h}}{2} \right\rfloor - 1 - \frac{n - 1}{2}, \text{ if } h = 0 \text{ or } h > 3,
\]

And \( \nabla(G) \leq \frac{5}{6} n - 1, \text{ if } h = 2. \)

Let \( G \) be a graph embedded on a surface \( \Sigma \). A cycle \( C \) of \( G \) in the surface \( \Sigma \) is called non-contractible if it is not null-homotopic.

The length of the shortest non-contractible cycle is said to be the edge-width. Under the condition of the large edge-width, Thomasson [20] showed the following result.

**Theorem 4.3** [XX] Let \( G \) be a graph which is embedded in an orientable surface such that the edge-width is at least \( 2^{4k+6} \). Then \( \chi(G) \leq 5 \).

By combining Theorem 2.1 with Theorem 4.3, we get the following result.

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Theorem 4.4 Let $G$ be a graph with $n$ vertices is embedded in an orientable surface $S_k$ in such a way that the edge-width is at least $2^{14k+6}$. Then $\nabla(G) \leq \frac{4}{5} n - 1$.

An embedding of a planar graph in the plane is said to be a plane graph. Let us give a lemma at first.

Lemma 4.5 If $G$ is a plane graph with girth at least six, then $G$ has a vertex with a degree at most two.

Proof. Suppose on the contrary that $\delta(G) \geq 3$. Let $S_i$ be the number of vertices with degree $i \geq 3$. The $G$ has $n = \sum_{i\geq3} n_i$ vertices, and $m = \sum_{i\geq3} in_i$ edges. Let $f$ be the number of faces of $G$. By Euler’s formula, we have that

$$n - m + f = \sum_{i\geq3} n_i - \frac{1}{2} \sum_{i\geq3} in_i + f = 2.$$

So $f = 2 + \frac{1}{2} \sum_{i\geq3} tn_i - \sum_{i\geq3} n_i$. Thus

$$6f - 2m = 12 + \sum_{i\geq3} (2i - 6) n_i > 0$$

Since the girth of $G$ is at least six, we have that $6f \leq 2m$, a contradiction. ■

We now consider the decycling number of a plane graph with girth at least six.

Theorem 4.6 If $G$ is a plane graph with $n$ vertices and girth at least six, then

$$\nabla(G) \leq \frac{n}{2}.$$ 

Proof. Assume that the theorem does not hold. Let $G$ be a plane graph with the girth at least six and decycling number more than $\frac{|V(G)|}{2}$ which has as fewer vertices as possible. By lemma 4.5, $G$ has a vertex $v$ such that $d(v) \leq 2$. If $d(v) = 2$, then the vertex $v$ will be deleted. This procedure does not affect the decycling number, there is a clear contradiction. So, we can suppose that $d(v) = 2$.

Let $u$ be a neighbor of $v$ and now we remove $u$ at first, then $v$ is removed, in such a way we obtain a new graph $G'$. If $G'$ has not to cycle, then $\nabla(G) \leq 1$. 

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Obviously, $\nabla(G) \leq \frac{n}{2}$. Otherwise $G'$ has at least one cycle, and the girth of $G'$ is at least six. So $\nabla(G') \leq \frac{|V(G')|}{2}$. Since $G'$ has $n - 2$ vertices and $\nabla(G) \leq \nabla(G') + 1$, we have that $\nabla(G) \leq \frac{n}{2}$, a contradiction. 

For a graph with girth at least six which is embedded in the projective plane, we have the following result.

**Theorem 4.7** Let $G$ be a graph with $n$ vertices and girth at least six. If $G$ is embedded into the projective plane with face-width $k$, then $\nabla(G) \leq \frac{n+k-1}{2}$.

**Proof.** Let $\Sigma$ contains a non-contractible closed curve that intersects the graph at $k$ vertices. Then $k - 1$ ones in those $k$ vertices are eliminated. Thus, we obtained an embedding $\Pi$ of a graph $G'$ with $n - k + 1$ vertices. Obviously, the face-width of $\Pi$ is one. So $G'$ is a planar graph with girth at least six ($\Pi$ can be changed into an embedding in the plane by cutting the surface along a non-contractible closed curve passing the rest one of those $k$ vertices and identifying the vertex with its copy). By Theorem 4.6, $\nabla(G') \leq \frac{n-k+1}{2}$. Since $\nabla(G) \leq \nabla(G') + (K - 1)$, thus we have that $\nabla(G) \leq \frac{n+k-1}{2}$.

**Theorem 4.8** Let $G$ be a graph with $n$ vertices and girth at least six. If $G$ is embedded in the torus with face-width $k$, then $\nabla(G) \leq \frac{n+k}{2}$.

**Proof.** Let $\rho$ be a non-contractible closed curve such that it crosses $k$ vertices in $G$. Then those $k$ vertices are deleted. Thus, we obtain an embedding $\Pi$ of a graph $G'$ with $n - k$ vertices. Moreover, $G'$ is a planar graph with girth at least six. Next, we proceed the similar argument as that in the proof of Theorem 4.7.

**V. The Decycling Number of the Hypercube $Q_n$**

The hypercube $Q_n$ is an important regular graph which has numerous definitions. For $n \geq 2$, it can be reclusively defined as the Cartesian product of $k_2$ and $Q_{n-1}$.
that is \( Q_n = k_2 \times Q_{n-1} \). Beinke and Vandell [III] has been determined the decycling number of \( Q_n \) for \( n \leq 8 \), and they also showed that \( \forall Q_n \geq 112.2^{n-8} \) if \( n \geq 8 \). Since \( 112 = 2^7 - 2^4 \), in such a way that we have the following result.

**Lemma 5.1 [III]** \( \forall (Q_n) \geq 2^{n-1} - 2^{n-4} \) for \( n \geq 8 \).

The hypercube \( Q_n \) is an \( n \)-regular graph with \( 2^n \) vertices. For the convenience, the vertices in \( Q_n \) can be treated as binary words of length \( n \). If \( u \) is a binary word, the number of 1’s is said to be the Hamming weight of \( u \). For two binary words \( x \) and \( y \), the hamming distance between \( x \) and \( y \) in \( Q_n \), denoted by \( \tau Q_n (x, y) \), is the number of bits in which \( x \) and \( y \) differ. Two vertices of \( Q_n \) are adjacent to each other if and only if Hamming distance between them is one.

Since \( \chi(K_2) = 2 \), it is easy to show that \( \chi(Q_n) = 2 \) by the induction on \( n \). Since \( Q_n = k_2 \times Q_{n-1} \), \( Q_n \) can obtain from \( Q_{n-1} \) and its copy \( Q'_{n-1} \) by adding \( 2^{n-1} \) edges. A vertex \( (e_1,e_2,...,e_{n-1},e_n) \) in \( Q_n \) is define as follows. If \( (e_1,...,e_{n-1}) \) is in \( Q_{n-1} \), then \( e_n = 0 \). If \( (e_1,...,e_{n-1}) \) is in \( Q'_{n-1} \), then \( e_n = 1 \). So a 2-coloring of \( Q_n \) can be obtained recursively. Any vertex in \( Q_{n-1} \) with even Hamming weight is colored by 1, and any vertex in \( Q_{n-1} \) with odd Hamming weight is colored by 2. However, any vertex in \( Q'_{n-1} \) with odd Hamming weight is colored by 1, and any vertex in \( Q'_{n-1} \) with even Hamming weight is colored by 2. For the convenience let \( A_n \) be the set of all vertices in \( Q_n \) in which each is colored by 1, and let \( B_n \) the set of all vertices in \( Q_n \) in which each is colored by 2. In the following discussion the 2-coloring \( C(Q_n) \) is always the above coloring.

**Lemma 5.2** For any two words \( x \) and \( y \) in the same coloring class in the 2-coloring \( C(Q_n) \), if \( \tau Q_n (x, y) \geq 3 \), then \( N(x) \cap N(y) = \emptyset \).

**Proof:** Assume that \( N(x) \cap N(y) = \emptyset \). Let \( z \) be a word in \( N(x) \cap N(y) \). Then \( \tau Q_n (x, z) = 1 \) and \( \tau Q_n (y, z) = 1 \). So \( \tau Q_n (x, y) \leq 2 \), a contradiction. ■

**Lemma 5.3** Suppose that \( S \) is subset of \( A_n \) (or \( B_n \)) in the 2-coloring \( C(Q_n) \). If \( N(x) \cap N(y) = \emptyset \) for any two words \( x \) and \( y \) in \( S \), then \( A_n - S \) (or \( B_n - S \)) is a decycling set of \( Q_n \).

**Proof:** Let \( H = Q_n - (A_n - S) \). The \( V(H) = S \cup B_n \). Clearly, both \( S \) and \( B_n \) are independent set. If \( N(x) \cap N(y) = \emptyset \) for any two words \( x \) and \( y \) in \( S \). Similarly \( B_n - S \) is a decycling set of \( Q_n \).

We now give a subset \( S_1 \) in \( Q_1 \) and a subset \( S'_1 \) in \( Q'_1 \).
We now recursively define a subset of $V(Q_1)$ (or $V(Q_1')$) based on $S_1$ (or $S_1'$) for $n \geq 9$. Suppose that $S_{n-8}$ and $S_{n-8}'$ have been constructed. For a word $(e_1, \ldots, e_{n-1}, e_n)$ in $S_{n-7}$, if $(e_1, \ldots, e_{n-1})$ in $S_{n-8}$, then $e_n = 0$; if $(e_1, \ldots, e_{n-1})$ in $S_{n-8}'$, then $e_n = 1$. So $|S_{n-7}| = |S_{n-8}| + |S_{n-8}'|$ ($|X|$ is the cardinality of $X$). For a word $(e_1, \ldots, e_{n-1}, e_n)$ in $S_{n-7}'$ if $(e_1, \ldots, e_{n-1})$ in $S_{n-8}$, then $e_n = 1$; if $(e_1, \ldots, e_{n-1})$ in $S_{n-8}'$, then $e_n = 0$. So $|S_{n-7}'| = |S_{n-8}| + |S_{n-8}'|$.

Using the induction on $n$, it is easy to show the following result.
Lemma 5.4 For \( n \geq 8 \), \(|S_{n-7}| = |S'_{n-7}| = 2^{n-4}\).

\( S_{n-7} \) and \( S'_{n-7} \) has the following properties.

Theorem 5.5 Suppose that \( n \geq 8 \).

(1) For any two words \( x \) and \( y \) in \( S_{n-7} \) (or \( S'_{n-7} \)), \( \tau Q_n (x, y) \geq 3 \) and \( \tau Q'_n (x, y) \geq 3 \).

(2) If \( u \) is a word in \( S_{n-7} \), then there is only one-word \( v \) in \( S'_{n-7} \) such that 
\( \tau Q_{n+1} (u, v) = 2. \) For any word \( w \) in \( S'_{n-7} \), \( \tau Q_{n+1} (u, w) \geq 3 \).

(3) If \( u' \) is a word in \( S'_{n-7} \), then there is one word \( v' \) in \( S_{n-7} \) such that 
\( \tau Q_{n+1} (u', v') = 2. \) For any word \( w' \) in \( S_{n-7} \), \( \tau Q_{n+1} (u', w') \geq 3 \).

Proof. We use the induction on \( n \). If \( n = 8 \), it can be checked that the proposition holds. Assume that the proposition is true for \( n = k \). We now consider the case that \( n = k + 1 \).

(1) Suppose that \( x = (e_1, \ldots, e_k, e_{k-1}) \) and \( y = (f_1, \ldots, f_k, f_{k-1}) \) are two words in \( S_{k-6} \). If \( e_{k+1} = 0 \), then \( (e_1, \ldots, e_k) \in S_{k-7} \). If \( f_{k+1} = 0 \), then \( (f_1, \ldots, f_k) \in S_{k-7} \). By the induction assumption, \( \tau Q_k ((e_1, \ldots, e_k), (f_1, \ldots, f_k)) \geq 3 \). So \( \tau Q_{k+1} (x, y) \geq 3 \). If \( f_{k+1} = 1 \), then \( (f_1, \ldots, f_k) \in S'_{n-7} \). By the induction assumption, \( \tau Q_k ((e_1, \ldots, e_k), (f_1, \ldots, f_k)) \geq 2 \). So \( \tau Q_{k+1} (x, y) \geq 3 \). If \( e_{k+1} = 1 \), then \( (e_1, \ldots, e_k) \in S'_{n-7} \). Proceeding the similar argument, we have that \( \tau Q_{k+1} (x, y) \geq 3 \). If \( x \) and \( y \) are two words in \( S'_{k-6} (x,y) \geq 3 \).

By the similar argument.

(2) Suppose that \( u = (S_1, \ldots, S_k, S_{k-1}) \) and \( v = (t_1, \ldots, t_k, t_{k-1}) \) are two words in \( S_{k-6} \) and \( S'_{n-6} \), respectively. If \( S_{k+1} = 0 \), then \( (S_1, \ldots, S_k) \in S_{k-7} \). If \( t_{k+1} = 0 \), then \( (t_1, \ldots, t_k) \in S'_{n-7} \). By the induction assumption, \( \tau Q_k ((S_1, \ldots, S_k), (t_1, \ldots, t_k)) \geq 3 \). So \( \tau Q_{k+1} (u, v) \geq 3 \) if \( t_{k+1} = 1 \), then \( (t_1, \ldots, t_k) \in S_{k-7} \). If \( (S_1, \ldots, S_k) = (t_1, \ldots, t_k) \), then \( \tau Q_{k+1} (u, v) = 2 \). If \( (S_1, \ldots, S_k) \neq (t_1, \ldots, t_k) \), then \( \tau Q_k ((S_1, \ldots, S_k), (t_1, \ldots, t_k)) \geq 3 \). So \( \tau Q_{k+1} (u, v) \geq 3 \). If \( t_{k+1} = 1 \), then \( (S_1, \ldots, S_k) \in S'_{n-7} \). Continue the similar argument, we got that \( \tau Q_{k+1} (u, v) = 2 \) if \( (S_1, \ldots, S_k) = (t_1, \ldots, t_k) \) and \( Q_{k+1} (u, v) \geq 3 \) otherwise.

(3) Suppose that \( u' \) and \( v' \) are two words in \( S'_{n-7} \) and \( S_{n-7} \), respectively. By a similar argument, as we have given in (2), we get the desired result.

Theorem 5.6 \( \forall (Q_n) \leq 2^{n-1} - 2^{n-4} \) for \( n \geq 8 \).

Proof. We use the induction on \( n \). The base case is that \( n = 8 \). Since each word in

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$S_1$ has even Hamming weight, $S_1$ is a subset of $A_8$ in the 2-coloring $C(Q_8)$. By lemma 5., for any two words $x$ and $y$ in $S_1$, $\tau_{Q_{k+1}^k}(x,y) \geq 3$. By lemma 5.2, $N(x) \cap V(y) = \emptyset$. So $A_8 - S_1$ is a decycling set of $Q_8$ by lemma 5.3. Similar, $B_8 - S'_1$ is a decycling set of $Q'_8$. By lemma 5.4, $A_8 - S_1$ has $2^7 - 2^4$ words. So, the proposition holds.

Assume that the proposition is true for $n = k$. We now consider the case that $n = k + 1$.

Since each word in $S_{n-6}$ has even Hamming weight, $S_{n-6}$ is a subset of $A_{k+1}$ in the 2-coloring $C(Q_{k+1})$. By Lemma 5.5, for any two words $u$ and $v$ in $S_{n-6}$, $\tau_{Q_{k+1}}(u,v) \geq 3$. By Lemma 5.2, $N(u) \cap N(v) = \emptyset$. So $A_{Q_{k+1}} - S_{k-6}$ is a decycling set of $Q_{k+1}$ by lemma 5.3. Similarly, $B_{k+1} - S'_{k-6}$ is a decycling set of $Q'_{k+1}$. By Lemma 5.4, $A_{Q_{k+1}} - S_{k-6}$ has $2^k - 2^{k+3}$ words. Hence $\nabla(Q_n) \leq 2^k - 2^{k+3}$. Thus, the proof is completed.

Theorem below follows from Theorem 5.1 and Theorem 5.6.

Theorem 5.7 $\nabla(Q_n) \leq 2^{n-1} - 2^{n-4}$ for $n \geq 8$.

VI. Conclusion

In this contribution, we successfully obtained the upper bound of the decycling number of a graph by using graph chromatics number and its order, and the relation of the genus of the surface and the decycling number of a graph embedded in surface has been studied. The decycling number of a planar graph with $n$ vertices is conjectured to be $n/2$, which is shown in this paper if the girth of the graph is at least four. The decycling number of a graph with $n$ vertices and maximum degree three is proved to be at most $[n/3]$. Also, we have been investigated the decycling number of the hypercube graphs.

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