Sums with the Möbius Function Twisted by Characters with Powerful Moduli

William D. Banks and Igor E. Shparlinski

Abstract. In a recent work, the authors (2016) have combined classical ideas of A. G. Postnikov (1956) and N. M. Korobov (1974) to derive improved bounds on short character sums for certain nonprincipal characters with powerful moduli. In the present paper, these results are used to bound sums with the Möbius function twisted by characters of the same type, which complements and improves some earlier work of B. Green (2012). To achieve this, we obtain a series of results about the size and zero-free region of $L$-functions with the same class of moduli.

1. Introduction

In [1], the authors combine a classical idea of Postnikov [15,16] with the method of Korobov [12] for estimating double Weyl sums, deriving strong bounds on short character sums in the case that the modulus $q$ has a small core

$$q_2 = \prod_{p|q} p,$$

where the product is over all prime divisors of $q$. The results are then used to obtain improvements of certain bounds of Gallagher [4] and Iwaniec [10] for the corresponding $L$-functions, wider zero-free region for the same $L$-functions (which might possibly include Siegel zeros), and a sharper bound on the error term in the asymptotic formula for primes in short arithmetic progressions modulo a prime power, which leads to an improvement of the Linnik constant for such moduli.

In the present paper, we continue this program by extending and generalizing a number of results from [1]. As a starting point, for a Dirichlet character $\chi$ whose modulus $q$ has a small core $q_2$, we bound Dirichlet polynomials

$$\sum_{M < n \leq M+N} \chi(n)n^{it}$$

following the basic strategy employed in our proof of [1, Theorem 2.2], but some new techniques lead to a bound that has fewer restrictions on $N$ and $t$; see Theorem 7 below. As an application, we establish a zero-free region for $L(s, \chi)$ that is even wider than the region described in [1], and from this we derive exceptionally strong...
bounds on the twisted sums
\[ \sum_{n \leq x} \mu(n)\chi(n) \quad \text{and} \quad \sum_{n \leq x} \Lambda(n)\chi(n), \]
where \( \mu \) and \( \Lambda \) are the Möbius and von Mangoldt functions.

It seems unlikely that the bounds we obtain for the sums (1.2) can also be established for Dirichlet characters \( \chi \) to an arbitrary modulus \( q \). For some time it has been in the folklore that characters to a modulus of small core are special in many respects, and the associated \( L \)-functions tend to behave more like the Riemann zeta function than a general Dirichlet \( L \)-function. Many results have been obtained over the years which support this idea; for example, see [7] and the references therein for bounds on \( L \)-functions which appeal to the arithmetic structure of the modulus \( q \). Thus, it may come as no surprise that one can achieve bounds on the sums (1.2) as strong as those in Theorem 2. It is worth reiterating that those bounds are new, and we believe they are close to best achievable in the absence of new information or a fundamentally new approach.

Our work in this paper is motivated, in part, by a program of Sarnak [17] to establish instances of a general pseudo-randomness principle related to a famous conjecture of Chowla [3]. Roughly speaking, the principle asserts that the Möbius function \( \mu \) does not correlate with any function \( F \) of low complexity, so that
\[ \sum_{n \leq x} \mu(n)F(n) = o \left( \sum_{n \leq x} |F(n)| \right) \quad (x \to \infty). \]
Combining a result of Linial, Mansour, and Nisan [13] with techniques of Harman and Kátai [6], Green [5] has shown that if \( F : \{0, \ldots, N - 1\} \to \{\pm 1\} \) has the property that \( F(n) \) can be computed from the binary digits of \( n \) using a bounded depth circuit, then \( F \) is orthogonal to the Möbius function \( \mu \) in the sense that (1.3) holds; see [5, Theorem 1.1]. Among other things, Green’s proof [5] relies on a bound for a sum with the Möbius function twisted by a Dirichlet character \( \chi \) of modulus \( q = 2^\gamma \). To formulate the result of [5] we denote
\[ M(x, \chi) = \sum_{n \leq x} \mu(n)\chi(n), \]
where \( \chi \) is a Dirichlet character modulo \( q \). We also denote
\[ \widehat{M}_q(x) = \max_{\chi \mod q} |M(x, \chi)|, \]
where the maximum is taken over all Dirichlet characters \( \chi \) modulo \( q \). We remark that although the principal character \( \chi_0 \) is included in the definition (1.5), the pure sum \( M(x, \chi_0) \) can be dropped, as it satisfies a stronger bound than any bounds currently known for \( M(x, \chi) \), \( \chi \neq \chi_0 \); see [19, Chapter V, Section 5, equation (12)] (and also [18] for the best known bound under the Riemann Hypothesis).

According to [5, Theorem 4.1], for some absolute constant \( c_1 > 0 \) and all moduli of the form
\[ q = 2^\gamma \leq e^{c_1 \sqrt{\log x}}, \]
the bound
\[ \widehat{M}_q(x) = O \left( xe^{-c_1 \sqrt{\log x}} \right) \]
holds, where the implied constant is absolute. In the present paper, we improve Green’s result in three directions. Namely, we obtain a new bound which is

- stronger than (1.7) and gives a better saving with a higher power of \( \log x \) in the exponent;
- valid for a larger class of moduli \( q \);
- nontrivial in a broader range of the parameters \( q \) and \( x \).

Our bounds also yield improvements of some other results of Green [5].

2. Statement of results

For any functions \( f \) and \( g \), the notation \( f(x) \ll g(x) \) means that \( |f(x)| \leq c|g(x)| \) holds with some constant \( c > 0 \). Throughout the paper, we denote explicitly any parameters on which the implied constants may depend.

Given a natural number \( q \), its core (or kernel) is the product \( q_\# \) over the prime divisors \( p \) of \( q \); that is, \( q_\# \) is defined as in (1.1). In this paper, we are mainly interested in bounding the sums \( M(x, \chi) \) for moduli \( q \) that have a “suitable” core \( q_\# \).

Specifically, we assume that

\[
(2.1) \quad \min_{p \mid q} \{v_p(q)\} \geq 0.7\gamma \quad \text{with} \quad \gamma = \max_{p \mid q} \{v_p(q)\} \geq \gamma_0,
\]

where \( \gamma_0 > 0 \) is a sufficiently large absolute constant and \( v_p \) denotes the standard \( p \)-adic valuation (that is, for \( n \neq 0 \) we have \( v_p(n) = \alpha \), where \( \alpha \) is the largest integer such that \( p^\alpha \mid n \)). The technical condition (2.1) is needed in order to apply the results of our earlier paper [1]; it is likely that this constraint can be modified and relaxed in various ways.

Remark 1. As in our earlier paper [1], our results are formulated only for primitive characters \( \chi \). This condition can be relaxed somewhat but cannot be discarded entirely in general since the shape of the conductor of \( \chi \) plays an important role in our methods. It is worth mentioning, however, that our work in this paper (and in [1]) is primarily motivated by applications in the important special case that \( q = p^\gamma \) is a prime power, where the condition that \( \chi \) be primitive can be (essentially) removed. In this situation, the conductor of \( \chi \) is also a power of \( p \), so the condition (2.1) is met automatically except when \( q \) is bounded by a constant that depends only on \( q_\# = p \), in which case there are superior results already in the literature. This observation is used to derive Corollary [4] below.

In addition to (1.4) we consider sums of the form

\[
(2.2) \quad \psi(x, \chi) = \sum_{n \leq x} \Lambda(n)\chi(n),
\]

where \( \Lambda \) is the von Mangoldt function; such sums are classical objects of study in analytic number theory (see, e.g., [11, Section 5.9] or [14, Section 11.3]). We expect that the techniques of this paper can be applied to bound other sums of number theoretic interest.

We treat the sums \( M(x, \chi) \) and \( \psi(x, \chi) \) in parallel. Building on the results and techniques of [1] we derive the strongest bounds currently known for such sums when the modulus \( q \) of \( \chi \) satisfies (2.1). In particular, we improve and generalize
Green’s bound (1.7). Moreover, we obtain nontrivial bounds assuming only that

\[ q \leq \exp \left( c_1 (\log x)^{3/2} (\log \log x)^{-7/2} \right) \]

holds rather than (1.6) (that is, our results are nontrivial for much shorter sums of
length less than any power of \( q \)).

**Theorem 2.** Let \( q \) be a modulus satisfying (2.1). There is a constant \( c > 0 \) that depends only on \( q \) and has the following property. Let

\[ Q_1 = \exp \left( (\log q)^{7/3} (\log \log q)^{5/3} \right), \]
\[ Q_2 = \exp \left( (\log q)^{7} (\log \log q)^{-1} \right), \]

and for \( j = 1, 2 \) put

\[
E_j = \begin{cases} 
\exp \left( -c \log x \cdot (\log q)^{-2/3} (\log \log q)^{-4/3} \cdot (\log x)^j \right) & \text{if } x \leq Q_1, \\
\exp \left( -c(\log x \cdot \log q)^{1/2} (\log \log q)^{-1/2} \right) & \text{if } Q_1 < x \leq Q_2, \\
\exp \left( -c(\log x)^{4/7} (\log \log x)^{-3/7} \right) & \text{if } x > Q_2.
\end{cases}
\]

For every primitive character \( \chi \) modulo \( q \) we have

\[
\frac{1}{x} M(x, \chi) \ll E_1 \quad \text{and} \quad \frac{1}{x} \psi(x, \chi) \ll E_2.
\]

We remark that Green’s bound (1.7) applies to arbitrary characters, whereas Theorem 2 is formulated only for primitive characters. However, in the special case that \( q = 2^\gamma \), the conductor \( q_0 \) of an arbitrary Dirichlet character \( \chi \) modulo \( q \) is a power of two (since \( q_0 | q \)), and \( \chi \) is a primitive character modulo \( q_0 \). When \( q_0 \geq 2^{7\alpha} \) we still apply Theorem 2 with \( q_0 \) in place of \( q \), and this only increases the range (2.3) in which we have a nontrivial bound.

We now give a simpler formulation of Theorem 2 that applies to some special families of moduli.

**Corollary 3.** Fix \( \alpha > 0 \), and let \( \mathcal{F} \) be a family of moduli \( q \) satisfying (2.1) such that \( \max_{q \in \mathcal{F}} q_0 < \infty \) and

\[ q = \exp \left( (\log x)^{\alpha + o(1)} \right) \quad (x \to \infty, \; q \in \mathcal{F}). \]

Then the bound

\[
\max \{|M(x, \chi)|, |\psi(x, \chi)|\} \leq x \exp \left( -(\log x)^{\beta(\alpha) + o(1)} \right) \quad (x \to \infty, \; q \in \mathcal{F})
\]

holds, where the maximum is taken over all primitive characters \( \chi \) modulo \( q \in \mathcal{F} \), and

\[
\beta(\alpha) = \begin{cases} 
4/7 & \text{if } \alpha \leq 1/7, \\
(1 + \alpha)/2 & \text{if } 1/7 \leq \alpha \leq 3/7, \\
1 - 2\alpha/3 & \text{if } \alpha \geq 3/7.
\end{cases}
\]

Alternatively, we can define the function \( \beta(\alpha) \) of Corollary 3 by

\[
\beta(\alpha) = \min \{ \max \{4/7, (1 + \alpha)/2\}, 1 - 2\alpha/3 \};
\]

see also Figure 1 for its behavior.
At first glance it may be surprising that $\beta(\alpha)$ is increasing for $\alpha \in [1/7, 3/7]$, that is, that the bound of the theorem improves in this range as the modulus $q$ grows. The reason this happens is that our method relies heavily on bounds for Weyl sums with polynomials over $\mathbb{Q}[X]$. It is well known that such bounds improve in certain ranges as the denominators of the coefficients of the polynomials grow.

Our proof of Theorem 2 is based on new bounds on Dirichlet polynomials and on the magnitudes and zero-free regions of certain $L$-functions, which extend and improve those given by Gallagher [4], Iwaniec [10], and the authors [1]; see Section 4 for more details. We believe these bounds are of independent interest and may have other arithmetic applications. In particular, one can apply these results to estimate character sums and exponential sums twisted by the Liouville function, the divisor function, and other functions of number theoretic interest.

The existence of three distinct ranges in the statement of Theorem 2 reflects the fact that diverse methods are required to obtain bounds for all choices of $\alpha > 0$. The underlying method of [1] offers good control over the nonvanishing of $L$-functions $L(\sigma + it, \chi)$ provided that $t$ is not too large compared to the modulus $q$. A result of [14, Lemma 6.3] leads to an application that necessitates a further split of the range. For the large values of $t$ relative to $q$, our underlying method fails, and one can do no better than the result of Iwaniec [10, Lemma 8].

3. Applications

Theorem 2 improves Green’s result [5, Theorem 4.1] in terms of both the range, increasing the right hand side in (1.6), and the strength of the bound, reducing the right hand side in (1.7). Although one can derive a general result with a trade-off between the range and the strength of the bound, we present only two special cases:

(i) improving the range (1.6) as much as possible while preserving (1.7);
(ii) improving the bound (1.7) as much as possible (in this case, we are able to improve the range as well).

More precisely, for (i) we note that the first bound of Theorem 2 implies that

$$\log (x^{-1}|M(x, \chi)|) \ll \sqrt{\log x}$$

Figure 1. The function $\beta(\alpha)$
holds for a character $\chi$ modulo $q = 2^\gamma$ provided that
\[
\log q \ll (\log x)^{3/4}(\log \log x)^{-2}
\]
and the conductor of $\chi$ satisfies $q_0 \geq 2^\gamma_0$ (so Theorem 2 applies). Since the remaining characters $\chi$ modulo $q$ that have $q_0 < 2^\gamma_0$ compose a finite set, the bound (3.1) can be achieved for those characters on a one-to-one basis; see, e.g., the paper of Hinz [9] and references therein. Using (3.1) to bound $\widehat{M}_q(x)$ we derive the following statement.

**Corollary 4.** For some absolute constant $c_1 > 0$ and all moduli of the form
\[
q = 2^\gamma \leq e^{c_1(\log x)^{3/4}(\log \log x)^{-2}} \quad (\gamma \in \mathbb{N})
\]
the bound (1.7) holds.

**Corollary 5.** For some absolute constant $c_1 > 0$ and all moduli of the form
\[
q = 2^\gamma \leq e^{c_1(\log x)^{9/14}(\log \log x)^{-19/14}} \quad (\gamma \in \mathbb{N})
\]
the bound
\[
\widehat{M}_q(x) \ll xe^{-c_1(\log x)^{4/7}(\log \log x)^{-3/7}}
\]
holds.

Notice that the exponents $3/4$ and $9/14$ that occur in Corollaries 4 and 5 are the maximal roots $\alpha$ of the equations $\beta(\alpha) = 1/2$ and $\beta(\alpha) = 4/7$, respectively (in Figure 1 these roots occur at the rightmost intersection points between the graph of $\beta(\alpha)$ and the horizontal lines at height $1/2$ and $4/7$, respectively).

We also remark that Corollaries 4 and 5 hold with $2^\gamma$ replaced by $p^\gamma$ for any fixed prime $p$; in this case, the constant $c_1 > 0$ can depend on $p$ but is otherwise absolute.

Next, we turn our attention to the following exponential sums twisted by the Möbius function:
\[
S_q(x, a) = \sum_{n \leq x} \mu(n) \exp(2\pi ian/q) \quad (a \in \mathbb{Z}).
\]
We denote
\[
\widehat{S}_q(x) = \max_{a \in \mathbb{Z}} |S_q(x, a)|.
\]
Green has shown (see [5, Corollary 4.2]) that for some absolute constant $c_2 > 0$ and all moduli of the form
\[
q = 2^\gamma \leq e^{c_2 \sqrt{\log x}},
\]
the bound
\[
\widehat{S}_q(x) \ll xe^{-c_2 \sqrt{\log x}}
\]
holds. The proof, as in [5], is obtained by relating $\widehat{S}_q(x)$ to roughly $q$ character sums, getting a bound of the shape $q\widehat{M}_q(x)$, and then exploiting the bound (1.7). Thus, in order to get a nontrivial bound on $\widehat{S}_q(x)$ in this way, the savings in the bound on $\widehat{M}_q(x)$ must be a little larger than $q$. For this reason, even though Corollaries 4 and 5 are suitable for such applications (and lead to an improvement of [5, Corollary 4.2] in both the range and strength of the bound), we need to slightly reduce the range of $q$. Furthermore, a full analogue of these results also
holds for sums with the Möbius function in arithmetic progressions; that is, we can bound the quantity
\[ \hat{D}_q(x) = \max_{a \in \mathbb{Z} \atop \gcd(a,q) = 1} |D_q(x,a)|, \]
where
\[ D_q(x,a) = \sum_{n \leq x, n \equiv a \mod q} \mu(n). \]

**Corollary 6.** For some absolute constant \( c_2 > 0 \) and all moduli of the form
\[ q = 2^\gamma \leq e^{c_2(\log x)^{3/5}(\log \log x)^{-4/5}} \quad (\gamma \in \mathbb{N}) \]
we have
\[ \max \{ \hat{S}_q(x), \hat{D}_q(x) \} \ll x e^{-c_2(\log x)^{4/7}(\log \log x)^{-3/7}}. \]

Note that the exponent 3/5 in Corollary 6 is the root \( \alpha \) of the equation \( \beta(\alpha) = \alpha \), and we have
\[ \min \{ \beta(\alpha) : \alpha \in (0,3/5) \} = 4/7. \]

As with the previous results, Corollary 6 also holds with \( 2^\gamma \) replaced by \( p^\gamma \) for any fixed prime \( p \).

We remark that the methods of this paper can also be used to derive strong bounds on Dirichlet polynomials of the form
\[ M_t(x,\chi) = \sum_{n \leq x} \mu(n) \chi(n) n^t, \]
where \( \chi \) is a primitive Dirichlet character whose modulus \( q \) satisfies (2.1). For moduli of the form \( q = p^\gamma \) with a fixed prime \( p \), one can also bound
\[ \hat{M}_{q,t}(x) = \max_{\chi \mod q} |M_t(x,\chi)|. \]

Finally, we mention that the results of the present paper can be used to estimate Fourier-Walsh coefficients \( \hat{\mu}_n(A) \) associated with the Möbius function, interpolating between the result of Green [5, Proposition 1.2] and that of Bourgain [2, Theorem 1]. Recall that for a natural number \( n \) and a set \( A \in \{0,\ldots,n-1\} \), the coefficient \( \hat{\mu}_n(A) \) is defined by
\[ \hat{\mu}_n(A) = \sum_{(x_0,\ldots,x_{n-1}) \in \{0,1\}^n} \prod_{j \in A} (1 - 2x_j) \mu \left( \sum_{j=0}^{n-1} 2^j x_j \right). \]

Green [5, Proposition 1.2] uses (1.7) to give a nontrivial bound on \( \hat{\mu}_n(A) \) for sets \( A \) of cardinality \(|A| \ll n^{1/2} / \log n\), and this bound on the cardinality is essential to the entire approach of [5]. Bourgain [2, Theorem 1] gives a nontrivial bound on \( \hat{\mu}_n(A) \) for arbitrary sets. Using Theorem 2 in the argument of Green [5] one can obtain a result that is stronger than [5, Proposition 1.2] but which also holds for larger sets.
4. Outline of the proof

To prove Theorem 2, we begin by extending the bound of [1, Theorem 2.2] to cover Dirichlet polynomials supported on shorter intervals; see Theorem 7. Adapting various ideas and tools from [1] to exploit this result on Dirichlet polynomials, we give a strong bound on the size of the relevant $L$-functions $L(s, \chi)$ in the case that $\Im(s)$ is not too large; see Theorem 10. For larger values of $\Im(s)$, we apply a well-known bound of Iwaniec [10, Lemma 8].

Having the upper bounds on these $L$-functions at our disposal, we combine them with certain results and ideas of [1] and [10] to obtain a new zero-free region, which is wider than all previously known regions; see Theorem 12.

Next, we introduce and apply an extension of a result of Montgomery and Vaughan [14, Lemma 6.3] which bounds the logarithmic derivative of a complex function in terms of its zeros; see Lemma 14. Similarly to [14], we use Lemma 14 as a device that allows us to consolidate old and new bounds on the size and the zero-free region of our $L$-functions $L(s, \chi)$, and doing so leads us to Theorem 17, which provides strong bounds on these $L$-functions, their logarithmic derivatives, and their reciprocals.

Finally, to conclude the proof of Theorem 2 in Section 9 we relate (via Perron’s formula) the sums $M(x, \chi)$ and $\psi(x, \chi)$ to the bounds given in Theorem 17. This connection involves the use of a parameter $T$, which we optimize to achieve the desired result. It is worth mentioning that the final optimization step is somewhat delicate since our bounds behave very differently in different ranges (e.g., see (8.5)).

5. Bounds on Dirichlet polynomials

In this section, we study Dirichlet polynomials of the form

$$T_\chi(M, N; t) = \sum_{M < n \leq M+N} \chi(n)n^it \quad (t \in \mathbb{R}).$$

As in [1], to bound these polynomials we approximate each $T_\chi(M, N; t)$ with a sum of the form

$$\sum_{M < n \leq M+N} \chi(n)e(G(n)),$$

where $G$ is a polynomial with real coefficients, and $e(t) = e^{2\pi it}$ for all $t \in \mathbb{R}$. Our result is the following statement, which is more flexible than [1, Theorem 2.2].

**Theorem 7.** For every $C > 1$ there are effectively computable constants $\gamma_0, \xi_0 > 0$ depending only on $C$ such that the following property holds. For any modulus $q$ satisfying (2.1) and any primitive character $\chi$ modulo $q$, the bound

$$T_\chi(M, N; t) \ll N^{1-\xi_0/\gamma^2}$$

holds uniformly for all $M, N, t \in \mathbb{R}$ subject to the conditions

$$M \geq N, \quad q^{\gamma_0} \leq N \leq q^C, \quad \text{and} \quad |t| \leq q^{1/2} \log M / \log q,$$

where $\gamma = (\log q) / \log N$ and the implied constant in (5.1) depends only on $C$.

**Proof.** Let $\gamma_0$ be a positive constant exceeding $e^{200}$, and put

$$\gamma_1 = \frac{\log N}{\log q} \quad \text{and} \quad \varepsilon = \frac{5}{4\gamma_1} = \frac{5 \log q}{4 \log N}.$$
Using (2.1) and (5.2) we have
\[ \log q = \sum_{p \mid q} v_p(q) \log p \leq \gamma \sum_{p \mid q} \log p = \gamma \log q \leq \gamma \gamma_1^{-1} \log N; \]
hence \( \varrho \) lies in \([C^{-1}, \gamma/\gamma_1]\). Let \( s = [\varepsilon \gamma/\varrho] \). Since \( \varepsilon \gamma/\varrho \geq \varepsilon \gamma_1 = \frac{5}{4} \) we have
\[ (5.3) \quad \frac{1}{5} \varepsilon \gamma/\varrho \leq \varepsilon \gamma/\varrho - 1 < s \leq \varepsilon \gamma/\varrho, \]
and therefore \( s \approx \gamma/\varrho \).

Now let \( \nu = \lceil \gamma/(3s) \rceil \). For any real number \( x \), we have the estimate
\[ (1 + x)^{it} = e(tG(x))(1 + O(|t||x|^\nu)), \]
where \( G \) is the polynomial given by
\[ G(x) = \frac{1}{2\pi} \sum_{r=1}^{\nu-1} (-1)^{r+1} \frac{x^r}{r}. \]
Hence, uniformly for \( n \in [M+1, M+N] \) and \( y, z \in [1, q_z^s] \), taking into account that \( M \geq N \), we have
\[ (5.4) \quad (n + q_z^s yz)^{it} = n^{it}(1 + q_z^s yz/n)^{it} = n^{it}e(tG(q_z^s yz/n)) + O(M^{-\nu}|t|q_z^{3s\nu}). \]
Let \( N \) be the set of integers coprime to \( q \) in the interval \([M+1, M+N]\). Shifting the interval \([M+1, M+N]\) by the amount \( q_z^s yz \), where \( 1 \leq y, z \leq q_z^s \), we have the uniform estimate
\[ T_\chi(M, N; t) = \sum_{n \in N} \chi(n)n^{it} = \sum_{n \in N} \chi(n + q_z^s yz)(n + q_z^s yz)^{it} + O(q_z^{3s}). \]
Using (5.4) and averaging over all such \( y \) and \( z \), it follows that
\[ T_\chi(M, N; t) = q_z^{2s} V + O(q_z^{3s} + NM^{-\nu}|t|q_z^{3s\nu}), \]
where
\[ V = \sum_{n \in N} \chi(n)n^{it} \sum_{y, z = 1}^{q_z^s} \chi(1 + q_z^s \varpi yz)e(tG(q_z^s yz/n)). \]
In this expression, we have used \( \varpi \) to denote an integer such that \( n\varpi \equiv 1 \mod q \). Since \( \deg G \leq \gamma/(3s) \), we can proceed in a manner that is identical to the proof of [I] Theorem 2.1 to derive the bound
\[ (5.5) \quad T_\chi(M, N; t) \ll N^{1-\varepsilon_0/\varrho^2} + q_z^{3s} + NM^{-\nu}|t|q_z^{3s\nu} \]
in place of [I] equation (5.16) [note that the proof of [I] Theorem 2.1, to which we refer, requires that \( \chi \) be primitive and that \( \gamma_0 > e^{200} \)]. Below, we show that \( q_z^{3s} \leq N^{5s} \); hence the term \( q_z^{3s} \) can be dropped from (5.5) if one makes suitable initial choices of \( \gamma_0 \) and \( \varepsilon_0 \).
To finish the proof, we need to bound the last term in (5.5). Let \( \vartheta \) be such that \( N^\vartheta = |t| + 3 \). Since \( \nu = \lceil \gamma/(3s) \rceil \), it follows that \( 3s\nu \leq \gamma + 3s \), and by (5.3) we have \( \nu \geq \gamma/(3s) \geq \vartheta/(3\varepsilon) \). Thus, setting \( \kappa = (\log M)/\log N \) it follows that
\[ \lvert t \rvert q_z^{3s\nu} \ll N^{1-\kappa\vartheta/(3\varepsilon) + \vartheta} q_z^{\gamma + 3s}. \]
In view of (2.1) the relation \( q = q_2^\mu \) holds for some \( \mu \in [0.7, 1] \), which implies that \( q_2^\gamma \leq N^{2\varepsilon} \) and (using (5.3) again) that \( q_2^\delta \leq N^{5\varepsilon} \) (as claimed above). Hence,

\[
NM^{-\nu}|t|q_2^{3\nu} \ll N^{1-\kappa_0/(3\varepsilon)+\theta+2\theta+5\varepsilon}.
\]

Inserting this bound into (5.5) and recalling that \( \varepsilon = 1.25\gamma_1^{-1} \), we see that (5.1) is a consequence of the inequality

\[
\vartheta \leq \varrho(\kappa\gamma_1/(3.75) - 2) - \xi_0/\varrho^2 - 6.25\gamma_1^{-1}.
\]

Since \( \varrho \geq C^{-1} \) and \( \gamma_1 \geq \gamma_0 \), this condition is met if \( \gamma_0 \) is sufficiently large and \( \xi_0 \) is sufficiently small, since the last inequality in (5.2) implies that \( \vartheta \leq \frac{1}{2}\varrho\kappa\gamma_1 + o(1) \) as \( N \to \infty \). This completes the proof.

**Corollary 8.** Fix \( C > 1 \), and let \( \gamma_0, \xi_0 > 0 \) have the property described in Theorem 7. Let \( q \) be a modulus satisfying (2.1) and let \( \chi \) be a primitive character modulo \( q \). Put

\[
\tau = |t| + 3, \quad \ell = \log(q\tau), \quad \text{and} \quad Q_0 = \max\left\{q_2^{\gamma_0}, q_2^{4\ell/\log q}\right\}.
\]

Then, uniformly for \( s = \sigma + it \in \mathbb{C} \) and \( M \geq Q_0 \) the bound

\[
\sum_{M < n \leq 2M} \chi(n)n^{-s} \ll \begin{cases} M^{1-\sigma-\xi_0/(\log M)^2/(\log q)} + M^{-\sigma-1}Q_0^2 & \text{if } q_2^{\gamma_0} \leq M \leq q^C, \\ M^{1-\sigma}q^{-c_0} + M^{-\sigma-1}Q_0^2 & \text{if } M > q^C \end{cases}
\]

holds with

\[
c_0 = C(C-1)^2\xi_0
\]

and an implied constant which depends only on \( C \).

**Proof.** For simplicity we denote

\[
U_{\chi}(M) = \sum_{M < n \leq 2M} \chi(n)n^{-s}
\]

and

\[
V_{\chi}(u) = T_{\chi}(M, u; -t) = \sum_{M < n \leq M+u} \chi(n)n^{-it} \quad (0 < u \leq M).
\]

By partial summation it follows that

\[
U_{\chi}(M) = (2M)^{-\sigma}V_{\chi}(M) + \sigma \int_0^M (u + M)^{-\sigma-1}V_{\chi}(u) \, du.
\]

Using the trivial bound \( |V_{\chi}(u)| \leq u + 1 \) for \( 0 < u \leq Q_0 \) we see that

\[
U_{\chi}(M) \ll M^{-\sigma}|V_{\chi}(M)| + M^{-\sigma-1}Q_0^2 + M^{-\sigma-1}\int_{Q_0}^M |V_{\chi}(u)| \, du.
\]

We claim that the bound

\[
V_{\chi}(u) \ll f(u) \quad (u \geq Q_0)
\]

holds, where

\[
f(u) = \begin{cases} u^{1-\xi_0/(\log u)^2/(\log q)} & \text{if } q_2^{\gamma_0} \leq u \leq q^C, \\ uq^{-c_0} & \text{if } u > q^C. \end{cases}
\]

Indeed, let \( u > Q_0 \) be fixed. If \( u \in [q_2^{\gamma_0}, q^C] \), then the bound (5.8) follows immediately from Theorem 7 (note that our choice of \( Q_0 \) and the inequality \( u > Q_0 \)
guarantee that the condition (5.2) of Theorem 7 is met). To bound $V_\chi(u)$ when $u > q^C$, we set

$$J = \left\lceil \frac{u}{q^C} \right\rceil \geq 2 \quad \text{and} \quad N = \frac{u}{J} \in \left( \frac{1}{2} q^C, q^C \right].$$

Let $M_j = M + (j - 1)N$ and put

$$I_j = (M_j, M_j + N] \quad (j \leq J).$$

Since $M_{J+1} = M + u$, we see that the interval $(M, M + u]$ is a disjoint union of the intervals $I_j$, and thus

$$V_\chi(u) = \sum_{j \leq J} \sum_{n \in I_j} \chi(n)n^{-it} = \sum_{j \leq J} T_\chi(M_j, N; -t) \ll JN^{1 - \xi_0 (\log N)^2/(\log q)^2},$$

where we have applied Theorem 7 in the last step. Noting that $JN = u$ and $N > \frac{1}{2} q^C \geq q^{C-1}$ and taking into account (5.6), we obtain (5.8) for $u > q^C$.

Assuming as we may that $\xi_0 < (3C^2)^{-1}$ it is easy to verify that $f(u)$ is an increasing function of $u$. Hence, using (5.8) (in the crude form $V_\chi(u) \ll f(M)$ for all $u \in [Q_0, M]$) to bound the right side of (5.7), we derive that

$$U_\chi(M) \ll M^{-\sigma} f(M) + M^{-\sigma - 1} Q_0^2,$$

and the result follows.

6. Bounds on $L$-functions

We continue to use the notation of Section 5. More specifically, let $\gamma_0, \xi_0 > 0$ have the property described in Theorem 7 with $C = 1001$ (say). Let $q$ be a modulus satisfying (2.1), let $\chi$ be a primitive character modulo $q$, and put

$$\tau = |t| + 3, \quad \ell = \log(q\tau), \quad \text{and} \quad Q_0 = \max \left\{ q_2^{\gamma_0}, q_2^{4\ell/\log q} \right\}.$$

Note that we can replace $\gamma_0$ with a larger value or replace $\xi_0$ with a smaller value without changing the validity of Theorem 7. Following Iwaniec [10] we put

$$Y = \exp \left( 60(\ell \log 2\ell)^{3/4} \right).$$

We now present the following extension of [1, Lemma 6.1].

**Lemma 9.** Suppose that the parameters $X$ and $\eta$ satisfy

$$X \geq Q_0, \quad \eta \in \left( 0, \frac{1}{3} \right), \quad \text{and} \quad \eta \leq \frac{\xi_0 (\log X)^2}{(\log q)^2} - \frac{2 \log \ell}{\log X}.$$

Then for any $s = \sigma + it$ with $\sigma > 1 - \eta$ and any primitive character $\chi$ modulo $q$ we have

$$L(s, \chi) \ll \begin{cases} \eta^{-1} X^{\eta} & \text{if } Y \leq q^{1001}, \\ \eta^{-1} X^{\eta} + Y^{\eta} \ell q^{-c_0} & \text{if } Y > q^{1001}, \end{cases}$$

where $c_0 = 10^9 \xi_0$ and the implied constant in (6.4) is absolute.

**Proof.** Arguing as in the proof of [10] Lemma 8, we see that the bounds

$$\left| \sum_{n > Y} \chi(n)n^{-s} \right| \ll \eta^{-1} q_2^{100\eta}$$

This is a free offprint provided to the author by the publisher. Copyright restrictions may apply.
and

\[ (6.6) \quad \left| \sum_{n \leq Y} \chi(n)n^{-s} \right| \ll \eta^{-1} Y^\eta \]

hold for \( \sigma > 1 - \eta \). In the case that \( Y < 2X \), noting that \( q_{100}^1 \leq Q_0 \leq X \), from a combination of bound (6.5) and (6.6), we get that \( L(s, \chi) \ll \eta^{-1} X^\eta \), and so (6.4) holds. From now on we assume \( 2X \leq Y \).

Let \( L \) be the unique integer such that the quantity \( X_* = 2^{-L} Y \) lies in the interval \( [X, 2X) \); note that \( L \geq 1 \) since \( Y \geq 2X \). Let \( S \) be the collection of numbers of the form \( M = 2^{j-1} X_* \) with \( j = 1, \ldots, L \), and notice that the interval \((X_*, Y]\) splits into \( L \) disjoint subintervals of the form \((M, 2M]\) with \( M \in S \), so that

\[ \sum_{X_*,< n \leq Y} \chi(n)n^{-s} = \sum_{M \in S} \sum_{M < n \leq 2M} \chi(n)n^{-s}. \]

We apply Corollary 8 with \( C = 1001 \). Hence, by (5.6) we can even take a slightly larger value of \( c_0 \) than \( c_0 = 10^9 \xi_0 \).

Now, let \( S_1 \) and \( S_2 \) be the (potentially empty) sets of numbers \( M \in S \) for which \( M \leq q_{1001}^1 \) and \( M > q_{1001}^1 \), respectively. Since \( M \geq Q_0 \) for every \( M \in S \) by (6.3), we have by Corollary 8

\[ \sum_{M \in S_1} \sum_{M < n \leq 2M} \chi(n)n^{-s} \ll \sum_{M \in S_1} \left( M^\eta - \xi_0 \log M^2 \right) + M^\eta - 2 Q_0^2, \]

\[ \sum_{M \in S_2} \sum_{M < n \leq 2M} \chi(n)n^{-s} \ll \sum_{M \in S_2} \left( M^\eta q^{-c_0} + M^\eta - 2 Q_0^2 \right) . \]

Using (6.3) and the fact that \( |S_1| \leq |S| = L \ll \ell \), we have

\[ \sum_{M \in S_1} M^\eta - \xi_0 \log M^2 \ll \sum_{M \in S_1} M^2 \log \ell / \log X \ll \sum_{M \in S_1} \ell^{-2} \ll 1. \]

If \( Y \leq q_{1001}^1 \), then \( S_2 = \emptyset \); for larger values of \( Y \) we use the bound

\[ \sum_{M \in S_2} M^\eta q^{-c_0} \ll Y^\eta \ell q^{-c_0} . \]

Putting the preceding bounds together and taking into account that

\[ \sum_{M \in S} M^\eta - 2 Q_0^2 \leq \sum_{j=1}^L (2^{j-1} X_*)^\eta - 2 \ll Q_0^2 X_*^\eta - 2 \ll X^\eta, \]

we derive that

\[ (6.7) \quad \sum_{X_*,< n \leq Y} \chi(n)n^{-s} \ll \begin{cases} X^\eta & \text{if } Y \leq q_{1001}^1, \\ X^\eta + Y^\eta \ell q^{-c_0} & \text{if } Y > q_{1001}^1. \end{cases} \]

To finish the proof, we observe that

\[ (6.8) \quad \left| \sum_{n \leq X_*} \chi(n)n^{-s} \right| \leq \sum_{n \leq X_*} n^{\eta - 1} \leq 1 + \eta^{-1} (X_*^\eta - 1) \ll \eta^{-1} X^\eta. \]

The result follows by combining the bounds (6.5), (6.7), and (6.8). \( \square \)
Theorem 10. There are constants $A, B > 0$ that depend only on $q^♯$ and have the following property. Put

$$
\eta_1 = \frac{A}{(\log q)^{2/3}(\log \log q)^{1/3}}, \quad \eta_2 = \frac{A \log q}{\ell},
$$

and

$$
T = \exp \left( B(\log q)^{5/3}(\log \log q)^{1/3} \right).
$$

Then for any primitive character $\chi$ modulo $q$ and any $s = \sigma + it$ with $\sigma > 1 - \eta_1$ and $|t| \leq T$ we have

$$
L(s, \chi) \ll \eta_1^{-1},
$$

whereas for any $s = \sigma + it$ with $\sigma > 1 - \eta_2$ and $|t| > T$ we have

$$
L(s, \chi) \ll \eta_2^{-1},
$$

where the implied constants depend only on $q^♯$.

Proof. In what follows, we write

$$
M = \exp \left( \left( \frac{1}{4} \xi_0 \right)^{-1/3} (\log q)^{2/3}(\log \ell)^{1/3} \right)
$$

for some constant $\xi_0$, to be chosen later. We also recall the choice of the parameters (6.1). Adjusting the values of $\gamma_0$ and $\xi_0$ if necessary, we can assume that $M \geq q^γ_0$. Consequently, if $\ell_0$ is determined via the relation

$$
\frac{4\ell_0 \log q^2}{\log q} = \left( \frac{1}{4} \xi_0 \right)^{-1/3} (\log q)^{2/3}(\log \ell_0)^{1/3},
$$

then we have

$$
Q_0 \leq M \iff \ell \leq \ell_0,
$$

and the asymptotic relation

$$
\ell_0 \sim c_1 (\log q)^{5/3}(\log \log q)^{1/3}
$$

holds with some absolute constant $c_1 > 0$.

To meet the conditions (6.3) in Lemma 9 we must have $X \geq \max\{M, Q_0\}$. However, we also want $X$ to be reasonably small in order to derive a strong upper bound on $|L(s, \chi)|$. We take

$$
X = \max\{M, Q_0^2\},
$$

where $c_2 \geq 1$ is a constant that is needed for technical reasons.

Given that the quantity $\eta^{-1}X^n$ appears in the bound (6.4) on $L(s, \chi)$, the optimal choice of $\eta$ (at least for small values of $\ell$) is $\eta = (\log X)^{-1}$. This idea leads us to define

$$
\eta = \begin{cases} 
(\frac{1}{4} \xi_0)^{1/3} / (\log q)^{2/3}(\log \ell)^{1/3} & \text{if } \ell \leq \ell_0, \\
\log q / 4c_2 \ell \log q & \text{if } \ell > \ell_0.
\end{cases}
$$

In particular, we see from (6.11) that

$$
X^n \ll 1.
$$
It is also convenient to observe that
\[ \eta \approx \min \left\{ \frac{1}{(\log q)^{2/3}(\log \ell)^{1/3}}, \frac{\log q}{\ell} \right\} \]
in view of (6.12), and since \( \log(qT) \approx \ell_0 \) we see that \( \eta \) is asymptotic to the first (resp., second) term in the above minimum if \( |t| \leq T \) (resp, \( |t| > T \)). Thus, it suffices to establish that
\[ \eta \in (0, 1/3) \]
and adjusting the value of \( \gamma_0 \) if necessary, we also have \( \eta < 1/3 \); therefore, all of the conditions in (6.3) are met.

For \( \ell > \ell_0 \) we see that (6.15) again holds as a consequence of (6.12) provided that \( c_2 \) is sufficiently large. We can also guarantee that \( \eta < 1/3 \) by taking \( c_2 \) large enough; for example, \( c_2 > \frac{3}{4} \log 2 \) suffices. Hence, for a suitable choice of \( c_2 \) the conditions (6.3) are all met when \( \ell > \ell_0 \).

By Lemma 9 we have
\[ L(s, \chi) \ll \eta^{-1} X^n + Y^n q^{-c_0} \quad (\sigma > 1 - \eta), \]
where \( Y \) is given by (6.2).

Recalling (6.13), we see that to establish (6.14) it suffices to show that
\[ Y^n q^{-c_0} \ll \eta^{-1} \]
with the choices we have made. When \( \eta \asymp (\log q)^{-2/3}(\log \ell)^{-1/3} \) it is easy to see that (6.16) holds whenever \( \ell \ll (\log q)^{20/9}(\log \log q)^{5/9} \). Taking into account (6.12), this establishes (6.16) when \( \ell \leq \ell_0 \).

For larger \( \ell \), note that we can ensure that the inequality
\[ \frac{\log q}{4c_2\ell \log q} \cdot 60(\ell \log 2\ell)^{3/4} \leq c_0 \log q - \log \log q \]
Provided that \( c_2 \) is sufficiently large. Exponentiating both sides of this inequality, we obtain (6.16) when \( \ell > \ell_0 \).

\[ \square \]

7. The zero-free region

We continue to use the notation of the first paragraph of Section 6; in particular we use the parameters defined in (6.1) and (6.2).

The following technical result stems from the work of Iwaniec [10]; the present formulation is due to Banks and Shparlinski [11] Lemma 6.2.

**Lemma 11.** Let \( q \) be a fixed modulus. Let \( \eta \in (0, 1/3) \), \( K \geq e \), and \( T \geq 1 \) be given numbers, which may depend on \( q \). Put
\[ \vartheta = \frac{\eta}{400 \log K}, \]
and suppose that
\[ 8 \log(5 \log 3q) + \frac{24}{\eta} \log(2K/5\vartheta) \leq \frac{1}{15 \vartheta}. \]
Suppose that \(|L(s, \chi)| \leq K\) for all primitive characters \(\chi\) modulo \(q\) and all \(s\) in the region \(\{s \in \mathbb{C} : \sigma > 1 - \eta, |t| \leq 3T\}\). Then there is at most one primitive character \(\chi\) modulo \(q\) such that \(L(s, \chi)\) has a zero in \(\{s \in \mathbb{C} : \sigma > 1 - \vartheta, |t| \leq T\}\). If such a character exists, then it is a real character, and the zero is unique, real, and simple.

The main result of this section is the following.

**Theorem 12.** There are constants \(A, B > 0\) that depend only on \(q^\#\) and have the following property. Put

\[
\vartheta_1 = \frac{A}{(\log q)^{2/3}(\log \log q)^{1/3}}, \quad \vartheta_2 = \frac{A \log q}{\ell},
\]

and

\[
T = \exp \left( B(\log q)^{5/3}(\log \log q)^{1/3} \right).
\]

Then for any primitive character \(\chi\) modulo \(q\), the Dirichlet \(L\)-function \(L(s, \chi)\) does not vanish in the region

\[
\{s \in \mathbb{C} : \sigma > 1 - \vartheta_1, |t| \leq T\} \cup \{s \in \mathbb{C} : \sigma > 1 - \vartheta_2, |t| > T\}.
\]

**Proof.** The proof is similar to that of Theorem 10, but the goal here is to produce a large zero-free region rather than a strong upper bound on \(L(s, \chi)\).

In what follows, we can assume that

\[
Q_0 = q^{4\ell/\log q},
\]

for otherwise \(Q_0 = q^\gamma_0\) and the result is already contained in [1, Theorem 3.2]. Let \(Y\) be defined by (6.2), and as in the proof of Theorem 10 let \(M\) and \(\ell_0\) be defined by (6.9) and (6.10), respectively. Put

\[
\lambda_1 = \frac{(2\xi_0)^{1/3}(\log \ell)^{2/3}}{(\log q)^{2/3}} \quad \text{and} \quad \lambda_2 = \frac{c_0 \log q}{\log Y} = \frac{c_0 \log q}{60(\ell \log 2\ell)^{3/4}}.
\]

One verifies that

\[
\lambda_1 = \frac{\xi_0(\log M)^2}{(\log q)^2} - \frac{2 \log \ell}{\log M}.
\]

Let \(\ell_1\) be determined via the relation

\[
\frac{(2\xi_0)^{1/3}(\log \ell_1)^{2/3}}{(\log q)^{2/3}} \cdot 60(\ell_1 \log 2\ell_1)^{3/4} = c_0 \log q.
\]

Then we have

\[
Y^{\lambda_1} \leq q^{c_0} \iff \ell \leq \ell_1
\]

and the asymptotic relation

\[
\ell_1 \sim c_2 (\log q)^{20/9}(\log \log q)^{-17/9}
\]

for some absolute constant \(c_2 > 0\). Clearly, \(\ell_0 < \ell_1\) for all large \(q\).

To prove the result, we study three cases separately.

**Case 1.** \(\ell \leq \ell_0\). We put \(X = M\) and \(\eta = \lambda_1\). Using (7.1) and the fact that \(M \geq Q_0\) in this case, we easily verify the conditions (6.3) if \(q\) is sufficiently large. By Lemma 9 we have

\[
L(s, \chi) \leq \eta^{-1}X^{\eta} + Y^{\eta}q^{-c_0} \quad (\sigma > 1 - \eta).
\]
By our choices of \( X \) and \( \eta \) it follows that \( \eta^{-1}X^\sigma \ll \ell^3 \). Moreover, \( Y^n\ell q^{-c_0} \ll \ell \) by (7.2) since the asymptotic relation (7.3) implies that \( \ell \leq \ell_1 \) for all large \( q \). Consequently,

\[
\log|L(s, \chi)| \leq 3 \log \ell + O(1) \ll \eta \log X \quad (\sigma > 1 - \eta).
\]

Applying Lemma 11 taking into account that \( \log \ell \asymp \log q \) in view of (6.12), we see that there are numbers \( A_1, B > 0 \) with the following property. Put

\[
\vartheta_1 = \frac{A_1}{(\log q)^{2/3}(\log \log q)^{1/3}} \quad \text{and} \quad T = \exp \left( B(\log q)^{5/3}(\log \log q)^{1/3} \right).
\]

Then there exists at most one primitive character \( \chi \) modulo \( q \) such that \( L(s, \chi) \) has a zero in the region

\[
R_1 = \{ s \in \mathbb{C} : \sigma > 1 - \vartheta_1, \ |t| \leq T \}.
\]

If such a character exists, then it is a real character, and the zero is unique, real, and simple. To rule out the possibility of such an exceptional zero, we note that there are at most \( O(1) \) primitive real Dirichlet characters for which the core of the conductor is the number \( q_2 \). Consequently, after replacing \( A_1 \) with a smaller number depending only on \( q_2 \) (more precisely, on the locations of real zeros of these characters) we can guarantee that \( L(s, \chi) \) does not vanish in \( R_1 \) for any primitive character modulo \( q \).

**Case 2.** \( \ell_0 < \ell \leq \ell_1 \). In this case we put \( X = Q_0^{c_3} \), where \( c_3 \) is a large positive constant, and we take \( \eta = \lambda_1 \) as before. Since \( \ell > \ell_0 \) we have by (6.12)

\[
\frac{\xi_0(\log X)^2}{(\log q)^2} - \frac{2\log \ell}{\log X} \geq \frac{16\xi_0c_0^2(\log q_2)^2}{(\log q)^4} - \frac{\log \ell_0 \cdot \log q}{2c_3\ell_0 \log q_2} \sim \frac{c_4(\log \log q)^{2/3}}{(\log q)^{2/3}},
\]

where

\[
c_4 = 16\xi_0c_1^2c_3^2(\log q_2)^2 - \frac{1}{2c_1c_3 \log q_2}.
\]

Thus, if \( c_3 \) is large enough so that \( c_4 > (2\xi_0)^{1/3} \), then the conditions (6.3) are met for all large \( q \). By Lemma 9 we again have

\[
L(s, \chi) \ll \eta^{-1}X^\sigma + Y^n\ell q^{-c_0} \quad (\sigma > 1 - \eta).
\]

Using our choices of \( X \) and \( \eta \) and taking into account that \( Y^n\ell q^{-c_0} \ll \ell \) as in Case 1, it follows that

\[
\log|L(s, \chi)| \ll \frac{\ell(\log \ell)^{2/3}}{(\log q)^{5/3}} \quad (\sigma > 1 - \eta),
\]

where the implied constant depends only on \( q_2 \). Applying Lemma 11 we see that there are numbers \( A_2, B_2 > 0 \) with the following property. Put

\[
\vartheta_2 = \frac{A_2 \log q}{\ell} \quad \text{and} \quad T_2 = \exp \left( B_2(\log q)^{20/9}(\log \log q)^{-17/9} \right).
\]

Then for any primitive character \( \chi \) modulo \( q \), the \( L \)-function \( L(s, \chi) \) does not vanish in the region

\[
R_2 = \{ s \in \mathbb{C} : \sigma > 1 - \vartheta_2, \ |t| \leq T_2 \}.
\]

**Case 3.** \( \ell > \ell_1 \). In this case we put \( X = Q_0^{c_3} \) as before, but now we set \( \eta = \lambda_2 \). As in Case 2 we have

\[
\frac{\xi_0(\log X)^2}{(\log q)^2} - \frac{2\log \ell}{\log X} \geq (c_4 + o(1)) \frac{(\log \log q)^{2/3}}{(\log q)^{2/3}}.
\]
In view of the asymptotic relation in (7.3), the left side of (7.3) exceeds \( \eta \) for all large \( q \) provided that \( c_4 > c_0/(60c_2^4) \), which we can guarantee by choosing \( c_3 \) sufficiently large at the outset. With these choices the conditions (6.3) are met for all large \( q \). By Lemma 9 we again have

\[
L(s, \chi) \ll \eta^{-1}X^{\eta} + Y^n\ell q^{-c_0} \quad (\sigma > 1 - \eta).
\]

Using our choices of \( X \) and \( \eta \) and taking into account that \( Y^n\ell q^{-c_0} = \ell \) holds in view of the definition of \( \lambda_2 \), it follows that

\[
\log |L(s, \chi)| \ll \frac{\ell^{1/4}}{(\log \ell)^{3/4}} \quad (\sigma > 1 - \eta),
\]

where the implied constant depends only on \( q_6 \). Applying Lemma 11 we see that for any primitive character \( \chi \) modulo \( q \), the \( L \)-function \( L(s, \chi) \) does not vanish in the region

\[
R_3 = \{ s \in \mathbb{C} : \sigma > 1 - \vartheta_2, \ |t| > T_2 \}.
\]

Combining the results of the three cases above and selecting \( A = \min\{A_1, A_2\} \), we complete the proof. \( \square \)

**Remark 13.** The result of Iwaniec [10, Theorem 2] is superior to Theorem 12 when \( |t| \gg (\log q)^2(\log \log q)^3 \). Reducing \( A > 0 \) in Theorem 12 so that

\[
\vartheta_3 = \frac{A}{(\ell \log \ell)^{3/4}} \leq \frac{1}{40000(\log q_2 + (\ell \log 2\ell)^{3/4})}
\]

and adjusting the constant \( B \) if necessary, these results can be neatly combined: for any character \( \chi \) modulo \( q \), the \( L \)-function \( L(s, \chi) \) does not vanish in the region \( \{ s \in \mathbb{C} : \sigma > 1 - \max\{\vartheta_1, \vartheta_2, \vartheta_3\} \} \).

8. **Nonvanishing and bounds on \( L \)-functions, their logarithmic derivatives and reciprocals**

**Lemma 14.** Suppose that \( f(z) \) is analytic in a region that contains a disc \( |z| \leq \Delta \) with \( \Delta > 0 \), that \( |f(z)| \leq B \) in this disc, and that \( f(0) \neq 0 \). Let \( r \) and \( R \) be real numbers such that \( 0 < r < R < \Delta \). Then

\[
\left| \frac{f'(z)}{f(z)} - \sum_{j=1}^J \frac{1}{z_j - z} \right| \leq b(\Delta, r, R) \log \frac{B}{|f(0)|} \quad (|z| \leq r),
\]

where the sum runs over all zeros \( z_j \) of \( f \) for which \( |z_j| \leq R \), and

\[
b(\Delta, r, R) = \frac{2R}{(R - r)^2} + \frac{1}{(R - r) \log(\Delta/R)}.
\]

**Proof.** This is Montgomery and Vaughan [14, Lemma 6.3] in the special case that \( \Delta = 1 \); hence we only sketch the proof.

The explicit form of the upper bound given here can be obtained by keeping track of the constants that arise while combining the Jensen inequality (see [14, Lemma 6.1]) with the Borel-Carathéodory lemma (see [14, Lemma 6.2]). The general case \( \Delta > 0 \) is proved by applying the specific case \( \Delta = 1 \) to the function given by \( F(w) = f(\Delta w) \) for all \( w \in \mathbb{C} \). Writing \( z = \Delta w \) with \( |w| \leq 1 \) for \( |z| \leq \Delta \) we have

\[
\frac{f'(z)}{f(z)} = \Delta^{1 - \frac{1}{2}} \frac{F'}{F}(w).
\]
Replacing $r$ by $r\Delta$ and $R$ by $R\Delta$, the general case follows from the case $\Delta = 1$ of the lemma applied to the function $F$; the details are omitted.\hfill $\square$

Let $q$ be a modulus satisfying (2.1), $\chi$ a primitive character modulo $q$, and put $\tau = |t| + 3$ and $\ell = \log(q\tau)$ as usual. For the remainder of this section, all constants (including constants implied by the symbols $\ll$, $\gg$, etc.) may depend on the core $q$ of $q$ but are otherwise absolute.

The next result combines our work in Sections 6 and 7 with the main results of [10]. To formulate the theorem, we introduce some notation. Let $A,B_1,B_2 > 0$ be fixed real numbers, put

$$
\eta_1 = \frac{A}{(\log q)^{2/3}(\log \log q)^{1/3}}, \quad \eta_2 = \frac{A \log q}{\ell}, \quad \eta_3 = \frac{A}{\ell^{1/2}(\log \ell)^{3/4}},
$$

$$
\vartheta_1 = \frac{1}{2} \eta_1, \quad \vartheta_2 = \frac{1}{2} \eta_2, \quad \vartheta_3 = \frac{1}{2} \ell^{-1/4} \eta_3,
$$

and denote

$$
T_1 = \exp \left( B_1 (\log q)^{5/3}(\log \log q)^{1/3} \right),
$$
$$
T_2 = \exp \left( B_2 (\log q)^{4}(\log \log q) \right).
$$

Define $\eta$ and $\vartheta$ by

$$
(\eta, \vartheta) = \begin{cases} 
(\eta_1, \vartheta_1) & \text{if } |t| \leq T_1, \\
(\eta_2, \vartheta_2) & \text{if } T_1 < |t| \leq T_2, \\
(\eta_3, \vartheta_3) & \text{if } |t| > T_2.
\end{cases}
$$

Finally, put

$$
K = \begin{cases} 
(\log q)^{2/3}(\log \log q)^{1/3} & \text{if } |t| \leq T_1, \\
\ell/\log q & \text{if } T_1 < |t| \leq T_2, \\
\exp(100\ell^{1/4}) & \text{if } |t| > T_2.
\end{cases}
$$

Then, combining Theorems 10 and 12 along with [10, Theorems 1 and 2], we see that one can select the constants $A,B_1,B_2 > 0$ so that the following holds.

**Theorem 15.** For any $s = \sigma + it$ with $\sigma \geq 1 - \eta$ and any primitive character $\chi$ modulo $q$ we have $L(s,\chi) \ll K$, and $L(s,\chi)$ does not vanish in the region $\{ s \in \mathbb{C} : \sigma > 1 - \vartheta \}$.

For technical reasons (see Remark 16 below) we now define

$$
T_3 = \exp \left( B_2 (\log q)^{4}(\log \log q)^{-1} \right).
$$

Note that $T_3 < T_2$; hence all of the preceding results hold if $T_2$ is replaced by $T_3$ since the results of Iwaniec [10] hold generally for all $t \in \mathbb{R}$. Although this modification would lead to a slightly weaker form of Theorem 15, it yields a slightly stronger (and more convenient) form of Theorem 17 below.

We apply Lemma 14 with the choices

$$
f(z) = L(z + 1 + 2\eta + it, \chi),
$$

and

$$
\Delta = 3\eta, \quad r = 2\eta + \vartheta, \quad R = 2.9\eta.
$$

Observe that zeros of $f$ are of the form $\rho - (1 + 2\eta + it)$ where $\rho$ runs through the zeros of $L(s,\chi)$.
By Theorem 15 we can take $B = CK$ with a suitable constant $C$, and using the Euler product we have

$$|f(0)| = \prod_p |1 - \chi(p)p^{-1-2\eta - it}|^{-1} \geq \prod_p |1 + p^{-1-2\eta}|^{-1} = \frac{\zeta(2 + 4\eta)}{\zeta(1 + 2\eta)} \gg \eta;$$

consequently,

$$\log \frac{B}{|f(0)|} \ll \begin{cases} \log \log q & \text{if } |t| \leq T_3, \\ \ell^{1/4} & \text{if } |t| > T_3. \end{cases}$$

Since $R - r = 0.9\eta - \vartheta \asymp \eta$ and $\log(\Delta/R) \asymp 1$, it follows that $b(\Delta, r, R) \asymp \eta^{-1}$. Then Lemma 14 shows that for any $s = \sigma + it$ with $\sigma \in [1 - \vartheta, 1 + 4\eta + \vartheta]$ we have

$$L'(s, \chi) = \sum_{\rho \in \mathcal{Z}(t)} \frac{1}{s - \rho} + O(\Theta),$$

where the sum is over the (possibly empty) set $\mathcal{Z}(t)$ comprised of all zeros $\rho$ of $L(s, \chi)$ for which $|\rho - (1 + 2\eta + it)| \leq R = 2.9\eta$, and

$$\Theta = \begin{cases} (\log q)^{2/3}(\log \log q)^{4/3} & \text{if } |t| \leq T_1, \\ (\ell \log \log q)/\log q & \text{if } T_1 < |t| \leq T_3, \\ (\ell \log \ell)^{3/4} & \text{if } |t| > T_3. \end{cases}$$

Note that the implied constant in (8.4) is absolute.

**Remark 16.** Our motivation for introducing the parameter $T_3$ defined by (8.3) is to account for the fact that $\Theta$, regarded as a function of $t$, has an order of magnitude transition at $|t| \asymp T_3$ (and not at $|t| \asymp T_2$).

**Theorem 17.** For any $s = \sigma + it$ with $\sigma > 1 - \frac{1}{2} \Theta^{-1}$ for $L(s, \chi)$ with a primitive character $\chi$ of conductor $q$, we have the following bounds on the logarithmic derivative

$$(8.6) \quad L'(s, \chi) \ll \Theta,$$

its size

$$(8.7) \quad |\log L(s, \chi)| \leq \frac{3}{4} \log \Theta + O(1),$$

and its reciprocal

$$(8.8) \quad \frac{1}{L(s, \chi)} \ll \Theta.$$

**Proof.** We follow the proof of [14, Theorem 11.4] closely, making only minor modifications. Since

$$\left|\frac{L'}{L}(s, \chi)\right| \leq -\frac{\zeta'}{\zeta}(\sigma) \ll \frac{1}{\sigma - 1},$$

the bound (8.6) is immediate when $\sigma \geq 1 + \Theta^{-1}$. Continuing the argument in that proof, here we set $s_*=1+\Theta^{-1}+it$ and in the same manner derive the upper bound (cf. [14, equation (11.11)])

$$\sum_{\rho \in \mathcal{Z}(t)} \frac{1}{s_* - \rho} \ll \Theta \quad (8.9)$$
and the estimate (cf. [14, equation (11.12)])

\[
(8.10) \quad \sum_{\rho \in \mathcal{Z}(t)} \frac{s - \rho}{s - \rho} = \sum_{\rho \in \mathcal{Z}(t)} \left( \frac{1}{s - \rho} - \frac{1}{s* - \rho} \right) + O(\Theta),
\]

where again \( \mathcal{Z}(t) \) is the set of zeros \( \rho \) of \( L(s, \chi) \) for which \( |\rho - (1 + 2\eta + it)| \leq 2.9\eta \).

Now suppose that \( 1 - \frac{1}{2}\Theta^{-1} \leq \sigma \leq 1 + \Theta^{-1} \). If \( \rho = \beta + i\gamma \in \mathcal{Z}(t) \), then by the definition of \( \mathcal{Z}(t) \) we have

\[
|\beta - (1 + 2\eta)| = |\Re \rho - (1 + 2\eta + it)| \leq 2.9\eta;
\]
hence \( \beta \geq 1 - 0.9\eta \). In view of Theorem [15] we see that \( \beta \in [1 - 0.9\eta, 1 - \vartheta] \).

Consequently,

\[
1 \geq \frac{\Re(s - \rho)}{\Re(s* - \rho)} \geq 1 - \frac{1}{2}\Theta^{-1} - \beta \geq 0.9\eta - \frac{1}{2}\Theta^{-1} \gg 1.
\]

Since \( \Im(s - \rho) = \Im(s* - \rho) \), it follows that \( |s - \rho| \gg |s* - \rho| \) for every \( \rho \in \mathcal{Z}(t) \).

Consequently, for all zeros \( \rho \in \mathcal{Z}(t) \) we have for \( \sigma \geq 1 - \frac{1}{2}\Theta^{-1} \):

\[
\frac{1}{s - \rho} - \frac{1}{s* - \rho} = \frac{s* - s}{(s - \rho)(s* - \rho)} = \frac{1 + \Theta^{-1} - \sigma}{(s - \rho)(s* - \rho)} \ll \frac{\Theta^{-1}}{|s* - \rho|^2} \ll \Re \frac{1}{s* - \rho}.
\]

Summing this over \( \rho \in \mathcal{Z}(t) \) and taking into account \( (8.9) \) and \( (8.10) \), we deduce the bound

\[
\sum_{\rho \in \mathcal{Z}(t)} \frac{1}{s - \rho} \ll \Theta.
\]

In view of \( (8.4) \) this completes the proof of \( (8.6) \), which in turn yields the bounds \( (8.7) \) and \( (8.8) \) via the same argument given in the proof of \( [14] \) Theorem 11.4], making use of the bound

\[
\log L(s, \chi) - \log L(s*, \chi) = \int_{s*}^{s} \frac{L'}{L}(u, \chi) du \ll |s* - s|\Theta \ll 1
\]
in the case that \( 1 - \frac{1}{2}\Theta^{-1} \leq \sigma \leq 1 + \Theta^{-1} \). \( \square \)

9. Proof of Theorem [2]

We continue to use the notation of Section [8]. In particular, all constants (including constants implied by the symbols \( \ll, \gg, \) etc.) may depend on \( q \) but are otherwise absolute.

To bound the sums \( M(x, \chi) \) and \( \psi(x, \chi) \) defined by \( (1.1) \) and \( (2.2) \), we use Perron’s formula in conjunction with the bounds provided by Theorem [14]. The techniques we use are standard and well known; we follow an approach outlined in the book [14] of Montgomery and Vaughan.

As in the proof of [14] Theorem 11.16], we start from the estimates

\[
(9.1) \quad \psi(x, \chi) = -\frac{1}{2\pi i} \int_{\sigma_0-iT}^{\sigma_0+iT} \frac{L'}{L}(s, \chi) \frac{x^s}{s} ds + R_A
\]

and

\[
(9.2) \quad M(x, \chi) = \frac{1}{2\pi i} \int_{\sigma_0-iT}^{\sigma_0+iT} \frac{1}{L(s, \chi)} \frac{x^s}{s} ds + R_B,
\]
where \( \sigma_0 = 1 + (\log x)^{-1} \), \( T \) is a parameter in \([2, x]\) to be specified below, and for \( f = \Lambda \) or \( \mu \) the error term \( R_f \) satisfies the bound

\[
R_f \ll \begin{cases} 
   x (\log x)^2 T^{-1} & \text{if } f = \Lambda, \\
   x (\log x) T^{-1} & \text{if } f = \mu.
\end{cases}
\]

The integrals in (9.1) and (9.2) can be handled with the same argument using (8.6) and (8.8), respectively. For this reason, here we consider only \( f = \Lambda \).

Before proceeding, recall that the parameters \( \vartheta \) and \( \Theta \) are functions of the real variable \( t \) (up to now, the dependence on \( t \) has been suppressed in the notation for the sake of simplicity), and thus we write them as \( \vartheta(t) \) and \( \Theta(t) \). In particular, recalling that by (8.1) and (8.3) we have

\[
T_1 = \exp \left( B_1 (\log q)^{5/3} (\log \log q)^{1/3} \right),
\]

\[
T_3 = \exp \left( B_2 (\log q)^4 (\log \log q)^{-1} \right),
\]

we see from (8.5) that

\[
\Theta(t) \asymp \begin{cases} 
   (\log q)^{2/3} (\log \log q)^{4/3} & \text{if } 2 \leq |t| \leq T_1, \\
   (\log |t| \cdot \log \log q) / \log q & \text{if } T_1 < |t| \leq T_3, \\
   (\log |t| \cdot \log \log |t|)^{3/4} & \text{if } |t| > T_3.
\end{cases}
\]

Here, we have used the fact that \( t = \log(qt) \propto \log t \) whenever \( |t| > T_1 \). Similarly, we see from (8.2) that

\[
\vartheta(t) \asymp \begin{cases} 
   (\log q)^{-2/3} (\log \log q)^{-1/3} & \text{if } 2 \leq |t| \leq T_1, \\
   (\log |t|)^{-1} \log q & \text{if } T_1 < |t| \leq T_2, \\
   (\log |t| \cdot \log \log |t|)^{-3/4} & \text{if } |t| > T_2.
\end{cases}
\]

Consequently, there is a constant \( c \in (0, \frac{1}{2}) \) depending only on \( q \) for which

\[
(9.3) \quad c \Theta(T)^{-1} < \vartheta(t) \quad (|t| \leq T).
\]

Finally, notice that

\[
(9.4) \quad \log x \gg (\log q)^{2/3} (\log \log q)^{4/3} \quad \Rightarrow \quad \Theta(T) \ll \log x,
\]

and we see below that the latter bound is needed in order to derive nontrivial bounds on the sums we are considering.

Following the proof of [14, Theorem 6.9] we denote by \( \mathcal{C} \) a closed contour that consists of line segments connecting the points \( \sigma_0 - iT, \sigma_0 + iT, \sigma_1 + iT, \) and \( \sigma_1 - iT, \) where \( \sigma_1 = 1 - \varepsilon \Theta(T)^{-1} \). From (9.3) and Theorem [15] it follows that \( L(s, \chi) \) does not vanish inside the contour \( \mathcal{C} \); therefore, we have

\[
0 = \oint_{\mathcal{C}} \frac{L'}{L} (s, \chi) \frac{x^s}{s} ds = \left( \int_{\sigma_0 - iT}^{\sigma_0 + iT} + \int_{\sigma_0 + iT}^{\sigma_1 - iT} + \int_{\sigma_1 + iT}^{\sigma_1 - iT} \right) \frac{L'}{L} (s, \chi) \frac{x^s}{s} ds.
\]

By (9.1) and the bound on \( R_{\Lambda} \) we have

\[
\int_{\sigma_0 - iT}^{\sigma_0 + iT} \frac{L'}{L} (s, \chi) \frac{x^s}{s} ds = -2\pi i \psi(x, \chi) + O(x (\log x)^2 T^{-1}).
\]
Next, using (8.6) and the bound $\Theta(T) \ll \log x$ we have
\[
\int_{\sigma_1+iT}^{\sigma_0+iT} L'(s, \chi) s^{-s} ds \ll \Theta(T) \frac{x^{\sigma_0}}{T} \log x \ll \frac{x}{T}.
\]
Finally, using (8.6) and $\Theta(T) \ll \log x$ again we see that
\[
\int_{\sigma_1-iT}^{\sigma_1+iT} L'(s, \chi) s^{-s} ds \ll x^{\sigma_1} \int_{-T}^{T} \Theta(t) \frac{dt}{|t|+1} \ll x^{\sigma_1} \Theta(T) \log T \ll x^{\sigma_1} (\log x)^2.
\]
Putting everything together and recalling the definition of $\sigma_1$, we deduce that
\[
\psi(x, \chi) \ll x(\log x)^2 \left( \frac{1}{T} + \exp \left( -c \frac{\log x}{\Theta(T)} \right) \right).
\]
By a similar argument we have
\[
M(x, \chi) \ll x \log x \left( \frac{1}{T} + \exp \left( -c \frac{\log x}{\Theta(T)} \right) \right).
\]
Note that these bounds are trivial unless $\Theta(T) \ll \log x$, which is the reason we assume (2.3) (cf. (3.1)).

It now remains to optimize $T$. Recall the definitions of $T_1$ and $T_3$ in (8.1) and (8.3) again. The constants $B_1, B_2 > 0$ depend only on $q_2$, and it is clear from our methods that these numbers can be chosen (and fixed) with $B_1 \leq B_2$.

To optimize the bounds in (9.5) we select $T$ so that $\Theta(T) \log T \ll \log x$; this requires only a drop of care.

Case 1. $2 \leq T \leq T_1$. In this range $\Theta(T) \asymp (\log q)^{2/3} (\log \log q)^{4/3}$, so we put
\[
T = \exp \left( \frac{B_1 \log x}{(\log q)^{2/3} (\log \log q)^{4/3}} \right).
\]
With this choice, the condition $2 \leq T \leq T_1$ is verified if
\[
B_1^{-1} (\log q)^{2/3} (\log \log q)^{4/3} \leq \log x \leq (\log q)^{7/3} (\log \log q)^{5/3}.
\]
As the results of Theorem 2 are trivial when
\[
\log x < B_1^{-1} (\log q)^{2/3} (\log \log q)^{4/3},
\]
we can dispense with the first inequality in (9.6), which then simplifies as the condition $x \leq Q_1$.

Case 2. $T_1 < T \leq T_3$. In this range $\Theta(T) \asymp (\log T \cdot \log \log q) / \log q$, and to optimize we put
\[
T = \exp \left( \frac{B_1 (\log x \cdot \log q)^{1/2}}{(\log \log q)^{1/2}} \right).
\]
With this choice, the condition $T_1 < T \leq T_3$ is verified if
\[
(\log q)^{7/3} (\log \log q)^{5/3} \ll \log x \leq (B_2/B_1)^2 (\log q)^{7} (\log \log q)^{-1}.
\]
Since $B_1 \leq B_2$ this includes the range $Q_1 < x \leq Q_2$, so we are done in this case.

Case 3. $T > T_3$. In this range $\Theta(T) \asymp (\log T \cdot \log T)^{3/4}$, so we set
\[
T = \exp \left( \frac{16 B_2 (\log x)^{4/7}}{(\log \log x)^{3/7}} \right).
\]
With this choice, the condition $T > T_3$ is verified if

\[ (9.7) \quad \frac{(\log x)^{4/7}}{(\log \log x)^{3/7}} > \frac{(\log q)^{4}}{16 \log \log q}. \]

However, since $q \geq 3$ and $x > Q_2$ (that is, $\log x > (\log q)^7 (\log \log q)^{-1}$) it follows that

\[ \frac{\log x}{(\log \log x)^{3/4}} > \frac{(\log q)^7 (\log \log q)^{-1}}{(7 \log q - \log \log q)^{3/4}} > \frac{(\log q)^7}{128(\log \log q)^{7/4}}, \]

which implies (9.7) and therefore finishes the proof.

References

[1] W. D. Banks and I. E. Shparlinski, Bounds on short character sums and $L$-functions for characters with a smooth modulus, J. d’Analyse Math., to appear (available from http://arxiv.org/abs/1605.07553).

[2] J. Bourgain, Möbius-Walsh correlation bounds and an estimate of Mauduit and Rivat, J. Anal. Math. 119 (2013), 147–163, DOI 10.1007/s11854-013-0005-2. MR3043150

[3] S. Chowla, The Riemann hypothesis and Hilbert’s tenth problem, Mathematics and Its Applications, Vol. 4, Gordon and Breach Science Publishers, New York-London-Paris, 1965. MR0177943

[4] P. X. Gallagher, Primes in progressions to prime-power modulus, Invent. Math. 16 (1972), 191–201, DOI 10.1007/BF01425492. MR0304327

[5] Ben Green, On (not) computing the Mobius function using bounded depth circuits, Combin. Probab. Comput. 21 (2012), no. 6, 942–951, DOI 10.1017/S0963548312000284. MR2981162

[6] Glyn Harman and Imre Kátai, Primes with preassigned digits. II, Acta Arith. 133 (2008), no. 2, 171–184, DOI 10.4064/aa133-2-5. MR2417463

[7] D. R. Heath-Brown, Hybrid bounds for Dirichlet $L$-functions. II, Quart. J. Math. Oxford Ser. (2) 31 (1980), no. 122, 157–167, DOI 10.1093/qmath/31.2.157. MR576334

[8] Ghaith A. Hiary, An explicit hybrid estimate for $L(1/2+it, \chi)$, Acta Arith. 176 (2016), no. 3, 211–239, DOI 10.4064/aa8433-7-2016. MR3580112

[9] Jürgen G. Hinz, Eine Erweiterung des nullstellenfreien Bereiches der Heckeschen Zetafunktion und Primideale in Idealklassen (German), Acta Arith. 38 (1980/81), no. 3, 209–254, DOI 10.4064/aa-38-3-209-254. MR602191

[10] H. Iwaniec, On zeros of Dirichlet’s $L$ series, Invent. Math. 23 (1974), 97–104, DOI 10.1007/BF01405163. MR0344207

[11] Henryk Iwaniec and Emmanuel Kowalski, Analytic number theory, American Mathematical Society Colloquium Publications, vol. 53, American Mathematical Society, Providence, RI, 2004. MR2061214

[12] N. M. Korobov, The distribution of digits in periodic fractions (Russian), Mat. Sb. (N.S.) 89(131) (1972), 654–670, 672. MR0424660

[13] Nathan Linial, Yishay Mansour, and Noam Nisan, Constant depth circuits, Fourier transform, and learnability, J. Assoc. Comput. Mach. 40 (1993), no. 3, 607–620, DOI 10.1145/174130.174138. MR1370363

[14] Hugh L. Montgomery and Robert C. Vaughan, Multiplicative number theory. I. Classical theory, Cambridge Studies in Advanced Mathematics, vol. 97, Cambridge University Press, Cambridge, 2007. MR2378655

[15] A. G. Postnikov, On the sum of characters with respect to a modulus equal to a power of a prime number (Russian), Izv. Akad. Nauk SSSR. Ser. Mat. 19 (1955), 11–16. MR0068575

[16] A. G. Postnikov, On Dirichlet $L$-series with the character modulus equal to the power of a prime number, J. Indian Math. Soc. (N.S.) 20 (1956), 217–226. MR0084010

[17] Peter Sarnak, Möbius randomness and dynamics, Not. S. Afr. Math. Soc. 43 (2012), no. 2, 89–97. MR0241544

[18] K. Soundararajan, Partial sums of the Möbius function, J. Reine Angew. Math. 631 (2009), 141–152, DOI 10.1515/CRELLE.2009.044. MR2542220

[19] Arnold Walfisz, Weylsche Exponentialsummen in der neueren Zahlentheorie (German), Mathematische Forschungsberichte, XV, VEB Deutscher Verlag der Wissenschaften, Berlin, 1963. MR0220685
DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MISSOURI, COLUMBIA, MISSOURI 65211
Email address: banks@missouri.edu

DEPARTMENT OF PURE MATHEMATICS, UNIVERSITY OF NEW SOUTH WALES, SYDNEY, NEW SOUTH WALES 2052, AUSTRALIA
Email address: igor.shparlinski@unsw.edu.au