JOYAL’S SUSPENSION FUNCTOR ON $\Theta$ AND KAN’S
COMBINATORIAL SPECTRA

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Abstract. In [Joyal] where the category $\Theta$ is first defined it is noted that the dimensional shift on $\Theta$ suggests an elegant presentation of the unreduced suspension on cellular sets. In this note we prove that the reduced suspension associated to that presentation is left Quillen with respect to the Cisinski model category structure presenting the $(\infty, 1)$—category of pointed spaces and enjoys the correct universal property. More, we go on to describe how, in forthcoming work, inspired by the combinatorial spectra described in [Kan], this suspension functor entails a description of spectra which echoes the weaker form of the homotopy hypothesis; we describe the development of a presentation of spectra as locally finite weak $\mathbb{Z}$-groupoids.

Introduction

Since 1983, Grothendieck’s suggestion that:

“...the study of homotopical $n$-types should be essentially equivalent to the study of so-called $n$-groupoids...”

has gone from suggestion in [Grothendieck], to conjecture, to theorem in [KapranovVoevodsky], to counter-example in [Simpson], and finally to abiding definition. Through a remarkable instance of Lakatos’, “method of proofs and refutations,” weak $\omega$-groupoid, is now taken as synonymous with spaces by many.

As for analytic models of $\omega$-groupoids perhaps the most intuitive is made possible by the category $\Theta$. If $\Delta$ is the category of composition data for compositions of morphisms in a 1-category, then $\Theta$ is the category of composition data for compositions of morphisms in $\omega$-categories. It is then that there is a Cisinski model category structure on $\hat{\Theta}$ which is Quillen equivalent to the Kan model structure on simplicial sets. The fibrant objects in $\hat{\Theta}$ with respect to that model category structure are then weak $\omega$-groupoids.

In [Joyal] it is observed that the unreduced suspension on unpointed spaces should be presentable by a dimensional shift functor, an endomorphism we call $J$ of $\Theta$. In this note we prove that a related functor, $\Sigma J$, presents the reduced suspension of pointed cellular sets.

Theorem. The functor $\Sigma J : \hat{\Theta} \longrightarrow \hat{\Theta}$ is left Quillen with respect to the Cisinski model structure on $\hat{\Theta}$, equivalent to pointed spaces and the functor $\Sigma J$ enjoys the universal property of the suspension.

In a final section, we then ask: How does this combinatorial presentations of weak $\omega$-groupoids relate to spectra? and describe a forthcoming presentation of spectra as locally finite $\mathbb{Z}$-groupoids.

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1This is the notion of $\Theta$ defining a cellular nerve for higher categories put forth in [Berger1].

2It has also been shown in [Ara1] that the groupoidal analogue to $\Theta$ enjoys this property, that of being a strict test category. It is expected that in future work, the results presented in this note will extend to that category as well.
The proof of the theorem is comprised of three smaller results which depend non-trivially upon each other. The proof that $\Sigma_J$ preserves monomorphisms, hence preserves the cofibrations of a Cisinski model structure, depends on a generalization of a weakening of lemma 2.1.10 of [Cisinski].

**Proposition.** Let $A$ be an incremental skeletal category for which $A^+ \hookrightarrow \text{Mono} (A)$ and let $F: \hat{A} \to \hat{B}$ be a colimit preserving functor. Then $F$ preserves monomorphisms if:

1. $F (A^+) \hookrightarrow \text{Mono} (\hat{B})$;
2. $F$ preserves the monomorphisms $\partial A^a \to A^a$ as such for all objects $a$ of $A$ with $\lambda_A (a) \leq 1$; and
3. for all $a \in \text{Ob} (A)$ with $\lambda_A (a) \geq 2$ and any pair of $A_+$ cells $f : b \to a$ and $g : c \to a$ with $\lambda_A (f) = \lambda_A (g) = 1$, the induced map
   $$F (A^b \times_{A^a} A^c) \to F (A^b) \times_{F (A^a)} F (A^c)$$
   is an epimorphism.

The proof of that proposition hinges on the fact that in any skeletal category $A$, where a skeletal category is a generalization of a Reedy category, the boundaries of cells comprise what Cisinski calls a cellular model for $\hat{A}$.

A cellular model for $\hat{A}$ is a set of monomorphisms of $\hat{A}$ which generate all other monomorphisms under transfinite pushouts of coproducts of those morphisms and in any skeletal category we have that the set $\{ \partial A^a \to a^a \}_{a \in \text{Ob} (A)}$ is a cellular model for $\hat{A}$. In an incremental skeletal category, a skeletal category in which the positive morphisms factor through degree one positive morphisms, an elegant presentation of the boundary of a cell is available.

**Lemma.** Suppose $A$ to be an incremental skeletal category. Then for every $a$ of $A$ with $\lambda_A (a) = n \geq 2$, we’ve an isomorphism

$$\lim \to \left\{ \coprod_{(f,b \to a, g;c \to a) \in X^2} A^b \times_{A^a} A^c \to \coprod_{(f,b \to a) \in X} A^b \right\} \sim \partial A^a$$

where $X = \{ f : b \to a | f \in \text{Mor} (A_+) , \lambda_A (f) = 1 \}$.

The proof that $\Sigma_J$ preserves weak equivalences is got by simple observation, Reedy theory, and the monomorphism preservation of $\Sigma_J$. The last part of the theorem is got by constructing a natural weak equivalence $\Sigma_J \to (\_ \\wedge S^1)$. The construction of the components of that natural weak equivalence on $\Theta$ bears resemblance to the Eckmann-Hilton argument and we end up defining those maps by recursion to the canonical map $\overline{T} \to \overline{0}$. The extension of those weak equivalences on cells to weak equivalences of cellular sets is then had by another application of Reedy theory.

**ON NOTATION**

Given any small category $A$ we denote by $\hat{A}$ the category of presheaves on $A$ and by $\hat{A}_+$ the category of presheaves of pointed sets on $A$. The disjoint base-point functor $\hat{A} \to \hat{A}_+$ we denote by $(\_)_+$. When we make it explicit, the Yoneda embedding will be denoted by $\text{Yon}$. The presheaf on $A$ represented by some object $a$ therein will be denoted $A^a$. In the category $\hat{A}$ the empty presheaf will be denoted $\emptyset_A$ and the single point presheaf will be denoted by $\bullet_A$ in both $\hat{A}$ and $\hat{A}_+$. 
In this section we provide a development of Joyal’s $\Theta$. The presentation here is adapted from [CisinskiMaltsiniotis].

1. Segal’s category $\Gamma$.

**Definition 1.1.** Let $\Gamma$, **Segal’s gamma category**, be the category specified thus:

- $\text{Ob}(\Gamma) = \{\langle k \rangle = \{1, \ldots, k\} | k \geq 1\} \cup \{\langle 0 \rangle = \emptyset\}$,
- $\text{Hom}_\Gamma(\langle n \rangle, \langle m \rangle) = \{\varphi : \langle n \rangle \to \text{Sub}_\Gamma(\langle m \rangle) | \forall (1 \leq i < j \leq m) \varphi(i) \cap \varphi(j) = \emptyset\}$

where for any category $A$ and object $a$ thereof, $\text{Sub}_A(a)$ is the category of sub-objects of $a$ and define the composition of morphisms by setting

\[ \langle \ell \rangle \xrightarrow{\varphi} \langle m \rangle \xrightarrow{\sigma} \langle n \rangle \]

Lemma 1.2. **The functors**

\[ H : \text{FinSet}_\bullet^{\text{op}} \cong \Gamma : G \]

where

\[ H : \text{FinSet}_\bullet^{\text{op}} \xrightarrow{\langle m \rangle_+} \Gamma \]

\[ (f : \langle m \rangle_+ \to \langle n \rangle_+) \xrightarrow{\varphi : \langle n \rangle \to \langle m \rangle} : i \mapsto f^{-1}(i) \]

and

\[ G : \Gamma^{\text{op}} \xrightarrow{\langle m \rangle} \text{FinSet}_\bullet \]

\[ (f : \langle m \rangle \to \langle n \rangle) \xrightarrow{\varphi : \langle n \rangle_+ \to \langle m \rangle_+} : i \mapsto k \]

where by the instance of the symbol $k$ in the definition we mean the unique $k \in \langle m \rangle_+$ such that $i \in f(k)$, comprise the inverse functor of a contravariant equivalence of categories.

Lemma 1.3. **The category** $\text{FinSet}_\bullet$ **has all finite limits and all finite colimits.**

We also record a simple corollary which will be central to the proof that the functor which is the subject of this short work is left Quillen.

**Corollary 1.4.** **The category** $\Gamma$ **has all fibered products and the pullback of the diagram**

\[ \langle m \rangle \]

\[ \langle n \rangle \]

\[ \langle \ell \rangle \]

\[ g \]

\[ f \]
is the set \( \langle n \rangle \times \langle \ell \rangle \langle m \rangle \) whose elements are pairs 
\[
(I, J) \in \text{Sub}_{\text{Set}}(\langle n \rangle) \times \text{Sub}_{\text{Set}}(\langle m \rangle)
\]
such that 
\[
\bigcup_{i \in I} f_i = \bigcup_{j \in J} g_j
\]
which are minimal with respect to the containment partial order.

**Proof.** See appendix. \( \square \)

1.2. The Categorical wreath product.

**Definition 1.5.** Let \( A \) and \( B \) be small categories. Given a functor \( G : B \to \Gamma \), we define 
\[
B \int_G A = \text{Sub}_{\text{Set}}(\langle m \rangle)
\]
with the second notation suppressing the functor \( G \) when the meaning is clear, be the category whose objects are pairs 
\[
[b; (a_i)_{i \in G(b)}] \to [d; (c_i)_{i \in G(d)}]
\]
are comprised of a morphism 
\[
g : b \to d
\]
of \( B \) and a morphism of \( \hat{A} \), 
\[
f = \left( (f_{ji} : a_i \to a_j)_{j \in G(g(i))} \right)_{i \in G(b)} : A^{a_i} \to \prod_{i \in G(b)} A^{a_j}.
\]
The composition 
\[
[b; (a_i)_{i \in G(b)}] \xrightarrow{[g; f]} [d; (c_i)_{i \in G(d)}] \xrightarrow{[r; q]} [\ell; (k_i)_{i \in G(\ell)}]
\]
is denoted \([r \circ g; q \circ f] \) where the meaning of \( r \circ g \) is clear and 
\[
q \circ f = \left( (q_{jk} \circ f_{ki})_{j \in G(r(g)(i))} \right)_{i \in G(b)}
\]
with the values for \( k \in G(d) \) being those unique \( k \) in \( G(g)(i) \) such that \( j \in G(r)(k) \).

**Example 1.6.** We define a functor \( F : \Delta \to \Gamma \) by setting 
\[
F(\langle n \rangle) = \langle n \rangle
\]
and setting for each \( \varphi : [m] \to [n] \), 
\[
F(\varphi) : \langle m \rangle \to \langle n \rangle
\]
to be the function 
\[
F(\varphi) : \langle m \rangle \to \text{Sub}_{\text{Set}}(\langle n \rangle)
\]
given thus: 
\[
F(\varphi)(i) = \{ j | \varphi(i - 1) < j \leq \varphi(i) \}.
\]
Consider then the category \( \Delta \int_F \Delta = \Delta \int \Delta \) and observe that we may sketch the object \([1]; [0]) as
\[
\bullet \to \bullet
\]
and the object $[[1];([1];[0])]$ as

\[
\begin{array}{c}
\bullet \\
\downarrow
\end{array}
\]

with the morphisms between them those we expect from any definition of higher categories. It is also worth noting that

\[
\text{Hom}_{\Delta \int \Delta} \left( [[1];[0]] , [[1],[1];[0]]) \cong \text{Hom}_{\mathcal{G}_r} \left( G^T, G^T \right).
\]

**Lemma 1.7.** The functor $\Delta \int \text{Yon} : \Delta \int C \to \Delta \int \hat{C}$ is a fully faithful limit preserving embedding.

*Proof.* It suffices to observe that by definition

\[
\text{Hom}_{\Delta \int \Delta} \left( [[n];a_1,\ldots,a_n],[[m];b_1,\ldots,b_m] \right)
\]

is isomorphic to

\[
\prod_{f \in \text{Hom}(n,m)} \prod_{i \in (n)} \text{Hom} \left( C^{a_i}, \prod_{k \in F(f)(i)} C^{b_k} \right)
\]

which is isomorphic to $\text{Hom}_{\Delta \int \Delta} \left( [[n];C^{a_1},\ldots,C^{a_n}],[[m];C^{b_1},\ldots,C^{b_m}] \right).$ \hfill $\square$

1.3. The Definition of $\Theta$.

**Definition 1.8.** Let $\gamma : \Delta \to \Delta \int \Delta$ be the functor extending the assignment $\gamma([n]) = [[n];[0]\cdots[0]].$ Note that this functor is an embedding. We may then define the category $\Theta$ to be the colimit

\[
\lim_{\longrightarrow} \left\{ \Delta \xrightarrow{\gamma} \Delta \int \Delta \xrightarrow{\gamma} \cdots \right\}.
\]

This colimit presentation also filters $\Theta$:

\[
\Delta = \Theta_1 \hookrightarrow \Delta \int \Delta = \Theta_2 \hookrightarrow \cdots.
\]

It should also be noted that

\[
\Theta \xrightarrow{\sim} \Delta \int \Theta \xrightarrow{\sim} \Delta \int \Delta \int \Theta \xrightarrow{\sim} \cdots
\]

so we may denote cells, the objects of $\Theta$, in many compatible ways. For example for any $T$ a cell of $\Theta$ we may also write $T = [[n];T_1,\ldots,T_n]$ for some unique $n \in \mathbb{N}$ and unique $T_1,\ldots,T_n$ cells of $\Theta$.

**Definition 1.9.** Let $\overline{\pi}$ be the object of $\Theta$

\[
\overline{[[1];[[1];\cdots[[1];[0]]\cdots]]}
\]

let $s : \overline{\pi} \to \overline{n+1}$ be the morphism $[\text{id};[\text{id};\cdots[\text{id}]]]$, let $t : \overline{\pi} \to \overline{n+1}$ be the morphism $[\text{id};[\text{id};\cdots[\text{id}]]]$, and let $i : \overline{n+1} \to \overline{\pi}$ be the morphism $[\text{id};[\text{id};\cdots[\text{id}]]]$.

**Remark 1.10.** The above specifies a fully faithful embedding of the reflexive globe category $\mathcal{G}_r$ into $\Theta$. 

With notation for the globes in hand, the recursive decomposition suggested by the observation that for any \( T \) a cell of \( \Theta_\ell \) we may also write \( T = [n]; T_1, \ldots, T_n \) for some unique \( n \in \mathbb{N} \) and unique \( T_1, \ldots, T_n \) cells of \( \Theta_{\ell-1} \), can be carried out to provide a useful canonical representation of a cell \( T \) in terms of colimits computed in \( \Theta \).

**Lemma 1.11.** Given any object \( T \) of \( \Theta \), there exists a unique list of non-negative integers, 

\[
(n_0, m_1, n_1, \ldots, n_{\ell-1}, m_{\ell-1}, n_\ell)
\]

with each 

\[
m_i \leq n_{i-1}, n_i,
\]

such that 

\[
\lim_{\to} \left\{ \begin{array}{c}
  n_0 \\
  s^{n_1-m_1} t^{n_0-m_1} n_1 \\
  m_1 \\
  \cdots \\
  s^{n_{\ell-1}-m_{\ell-1}} t^{n_{\ell-1}-m_{\ell-1}} n_{\ell-1} \\
  m_{\ell-1} \\
  n_{\ell}
  \end{array} \right\} \to T.
\]

**Remark 1.12.** It is important to note that the colimit in this lemma is taken in \( \Theta \) and not in \( \hat{\Theta} \). In work of Ara this description of a cell is known as the **globular sum** presentation. It is from careful consideration of this presentation of the cells of \( \Theta \) that Ara proves the universality of \( \Theta \) among categories wherein we may compose globes.

**Proof.** First, we provide a function \( A \) from the set 

\[
\{(n_0, m_1, \ldots, m_{\ell-1}, n_\ell)| n_i, m_j \in \mathbb{N}, \forall 1 \leq i \leq \ell, m_i \leq n_{i-1}, n_i\}
\]

to the set of objects of \( \Theta \) and then prove that the cells \( A(n_0, \ldots, n_\ell) \) together with natural inclusions enjoy the universal property of the requisite colimit.

The function \( A \) is defined recursively. Let 

\[
Z = Z(n_0, m_1, \ldots, m_{\ell-1}, n_\ell)
\]

be the ordered set of indices 

\[
[i_1 < i_2 < \cdots < i_k] = [1 \leq i \leq \ell - 1 | m_i = 0].
\]

Then \( A(n_0, m_1, \ldots, m_{\ell-1}, n_\ell) \) is the tree 

\[
\begin{bmatrix}
A(n_0 - 1, m_1 - 1, \ldots, m_{i_1-1} - 1, n_{i_1} - 1) \\
[1 + |Z|] \\
\vdots \\
A(n_{i_k+1} - 1, m_{i_k+1} - 1, \ldots, m_{\ell-1} - 1, n_\ell - 1)
\end{bmatrix}
\]

where the right hand side is interpreted in \( \triangle \int \Theta \to \Theta \).

To verify that \( A \) indeed computes the requisite colimits observe that 

\[
A(n_0, 0, n_1) = [(1+1); \overline{n_0-1} \overline{n_1}]
\]

which enjoys the universal property of the colimit 

\[
\lim_{\to} \left\{ \begin{array}{c}
  n_0 \\
  \sigma^{n_0} \overline{0} \\
  n_1 \\
  \tau^{n_1}
  \end{array} \right\}.
\]

It then follows by recursion that \( A \) computes the colimit correctly. \( \square \)
1.4. The Category \( \Theta \) is a strict test category.

Proposition 1.13. (Cisinski-Maltsiniotis) The adjunction

\[
i^*_\Theta : \widehat{\Theta} \rightleftarrows \widehat{\Delta} : i^{-1}_\Theta
\]

induced by the co-cellular object \( i_\Theta : \Theta \to \widehat{\Delta} \) which associates to any cell \( T \) the simplicial nerve of the category \( \Theta \downarrow T \), defines a Quillen between the Kan model structure on simplicial sets and a Cisinski model structure on \( \widehat{\Theta} \) whose class of weak equivalences is the class \( \mathcal{W}_\Theta = (i^*_\Theta)^{-1}(\mathcal{W}_\infty) \). What’s more, \( i^*_\Theta \) preserves cartesian products.

Proof. See [CisinskiMaltsiniotis]. \( \square \)

2. Reedy, multi-Reedy, and skeletal structures

In this section we recall three progressively less familiar sorts of structures on (multi)-categories.

2.0.1. Reedy categories.

Definition 2.1. A category \( C \) together with two wide subcategories \( C^- \) and \( C^+ \) along with a degree function \( \lambda_C : \text{Ob}(C) \to \mathbb{N} \) is said to be a Reedy category, if these data satisfy the following hypotheses:

- **factorization:** every morphism \( f : a \to b \) in \( C \) factors uniquely as \( a \xrightarrow{f^-} c \xrightarrow{f^+} b \) with \( f^- \) a morphism of \( C^- \) and \( f^+ \) a morphism of \( C^+ \); and
- **degree:** every morphism \( f : a \to b \) of \( C^- \) has \( \lambda_C(f) = \lambda_C(b) - \lambda_C(a) \leq 0 \) and every morphism \( f : a \to b \) of \( C^+ \) has \( \lambda_C(f) = \lambda_C(b) - \lambda_C(a) \geq 0 \), moreover if in either subcategory we have \( \lambda_C(f) = 0 \), then \( f \) is an identity morphism.

Example 2.2. The most familiar example is that of \( \Delta \) for which the degree function acts as \( \lambda([n]) = n \) and \( \Delta^+ \) consists of the monomorphisms and \( \Delta^- \) consists of the epimorphisms. It’s worth noting that \( \Delta^{\text{op}} \) is also Reedy with the same degree function, \( (\Delta^{\text{op}})^+ = (\Delta^-)^{\text{op}} \), and \( (\Delta^{\text{op}})^- = (\Delta^+)^{\text{op}} \).

2.0.2. Skeletal categories. A generalization of the concept of a Reedy category which behaves better in the presence of non-identity automorphisms is found in [Cisinski].

Definition 2.3. A skeletal category \((A, A^+, A^-, \lambda_A)\) is given by:

1. a small category \( A \);
2. sub-categories \( A^+ \) and \( A^- \); and
3. \( \lambda : \text{Ob}(A) \to \mathbb{N} \)

subject to the constraints:

- **Sk0:** All isomorphisms of \( A \) lie in both \( A^+ \) and \( A^- \) and for any isomorphism \( \varphi : a \to a' : A \), \( \lambda_A(a) = \lambda_A(a') \);
- **Sk1:** If \( \varphi : a \to a' : A^+ \) (resp. \( A^- \)) and \( \varphi \) is not an isomorphism, then \( \lambda_A(a) < \lambda_A(a') \) (resp. \( \lambda_A(a) > \lambda_A(a') \));
- **Sk2:** Each morphism \( \alpha : a \to a' \) admits an essentially unique factorization \( \delta \circ \pi \) where \( \pi : A_- \) and \( \delta : A_+ \); and
- **Sk3:** All morphisms of \( A^- \) admits sections and morphisms \( \pi, \pi' : a \to a' : A^- \) are equal if and only if they admit the same sections.
Definition 2.5. A skeletal category $A$ is said to be incremental if $A^+$ is generated by morphisms $d : s \to t$ such that $\lambda_A(s) + 1 = \lambda_A(t)$. A skeletal category is said to be collapsible if all $A^+$ morphisms admit retractions.

Given a skeletal category $A$ and an object $a$, for each $n \in \mathbb{N}$ let

$$\operatorname{Sk}^n A^a = \lim_{\to \{ f : b \to a \mid \lambda(b) < \lambda(a) \} } A^b.$$  

Given an object $a$ of $A$ with $\lambda(a) = n$, let $\partial A^a = \operatorname{Sk}^{n-1} A^a$.

2.0.3. Multi-Reedy.

Definition 2.6. Given a category $C$, for each $c \in \operatorname{Ob}(C)$, $m \in \mathbb{N}$ and $d_1, \ldots, d_m \in \operatorname{Ob}(C)$, define the sets

$$\operatorname{MHom}(c, (d_1, \ldots, d_m)) = \prod_{i = 1, \ldots, m} \operatorname{Hom}(c, d_i).$$

Let $C(\ast)$ denote the multi-category whose objects are those of $C$ with multi-morphisms $\operatorname{MHom}(c, (d_1, \ldots, d_m))$.

A multi-Reedy structure on $C$ is comprised of:

- a wide subcategory $C^-$ of $C$;
- a wide submulti-category of $C(\ast), C(\ast)^+$; and
- a degree function $\lambda_{C(\ast)} : \operatorname{Ob}(C) \to \mathbb{N}$;

satisfying the two axioms:

**factorization:** Every multi-morphism

$$\alpha_s s_{1, \ldots, m} \in \operatorname{MHom}(c, (d_1, \ldots, d_m))$$

admits a unique factorization $\alpha^+ \circ \alpha^-$ where $\alpha^- : c \to x$ is of $C^-$ and

$$\alpha^+ \in \operatorname{MHom}(x, (d_1, \ldots, d_m))$$

is a morphism of $C(\ast)^+$.

**degree:** For every multi-morphism

$$\alpha_s s_{1, \ldots, m} \in \operatorname{MHom}(c, (d_1, \ldots, d_m))$$

in $C(\ast)^+$ we have that $\lambda_C((\alpha_s)) = \sum_{s = 1, \ldots, m} \lambda_C(d_s) - \lambda_C(c) \geq 0$. For every multi-morphism $(\alpha_s)$ which lies in the embedding $C \hookrightarrow C(\ast)$, $\lambda_c(\alpha) = 0$ if and only if $\alpha$ is an identity. For every $f : a \to b$ of $C^-$ we have $\lambda_C(f) \leq 0$ and equality is attained if and only if $f$ is an identity morphism.

Example 2.7. There is a multi-reedy structure on $\Delta$. Let $\Delta^-$ be the same $\Delta^-$ as in the Reedy structure on $\Delta$ and let $\Delta(\ast)^+$ be comprised of all joint monomorphisms, that is families of maps $f_1, \ldots, f_n$ such that $g = h$ if and only if

$$f_1 \circ g = f_1 \circ h, \ldots, f_n \circ g = f_n \circ h.$$  

See that $\Delta \cap \Delta(\ast)^+ = \Delta^+$ from the usual Reedy structure on $\Delta$.

Proposition 2.8. (Bergner-Rezk) If $C$ admits a multi-Reedy structure $(C, C^-, C(\ast)^+, \lambda_C)$, then

$$(C, C^-, C(\ast)^+ \cap C, \lambda_C)$$

is a Reedy structure on $C$. 
Theorem 2.9. (Bergner-Rezk) If $C$ is equipped with the structure of a multi-reedy category and functor $H : C \to \Gamma$, then the following declarations comprise a multi-reedy structure on $\Delta \int C$:

- let $(\Delta \int C)^-$ be the wide subcategory of $\Delta \int C$ having as morphisms all those morphisms of $\Delta \int C$,
  
  $$[x; y] : [m] ; c_1, \ldots, c_m \to [n] ; e_1, \ldots, e_n],$$

  for which $x : [m] \to [n]$ is of $\Delta^-$, and each $y_{h,j}$ appearing in some $(y_{k,l})_{k \in F(x)(i)} c_i \to \prod_{j \in F(x)(i)} e_j^3$ is of $C^-$. 

- let $\Delta \int C (\ast)^+$ be the wide sub-multi-category of $\Delta \int C (\ast)$ whose morphisms are those multi-morphisms of $\Delta \int C (\ast)$

  $$([x^s; y^s]) : [m] ; c_1, \ldots, c_m \to \prod_{s \in \{1, \ldots, u\}} ([n^s] ; e_1^s, \ldots, e_n^s],$$

  such that the implicit multi-morphism $(x^s) : [m] \to \prod_{s \in \{1, \ldots, u\}} [n^s]$ is of $\Delta (\ast)^+$ where each $x^s : [m] \to [n^s]$ is of $\Delta (\ast)^+$ and that for each $i \in \{1, \ldots, m\}$ the multi-morphism

  $$(y^s_{i,j}) : c_i \to \prod_{s \in \{1, \ldots, u\}} \prod_{j \in F(x^s)(i)} e_j^s$$

  is a $C (\ast)^+$ multi-morphism.

Remark 2.10. $\Theta$ is multi-Reedy, therefore Reedy, and therefore skeletal.

Corollary 2.11. The category $\Theta$ is incremental skeletal.

Proof. Recall that an incremental skeletal category $A$ is one for which any positive degree $A^+$ map admits a factorization through an $A^+$ morphism with co-dimension 1. As with many arguments on $\Theta$, we will prove the claim by recursion on the height of the target.

Suppose $T \to S$ to be a $\Theta^+$ morphism and let $S$ lie in $\Delta = \Theta^+ \hookrightarrow \Theta$. Then $T$ also lies in $\Theta^+$ and $T \to S$ is some $\Delta^+$ map $[n] \to [m]$ which indeed factors as claimed since $\Delta^+$ is generated by the morphisms $d^i$. Now suppose $T \to S$ to be a $\Theta^+$ morphism located in some $\Theta_n \hookrightarrow \Theta$. Such a map is of the form

$$[f : g] : [[n]; T_1, \ldots, T_n] \to [[m]; S_1, \ldots, S_m]$$

with $f : [n] \to [m]$ of $\Delta^+$ whence $f = d^{i_1} \circ d^{i_2} \circ \cdots \circ d^{i_l}$ where $m \geq i_1 \geq \cdots \geq i_l \geq 0$. We note that $[f; f]$ factors as follows into two $\Theta^+$ morphisms: first

$$[d^{i_1} \circ d^{i_2} \circ \cdots \circ d^{i_l}; g^{<i_2-1}; id, \ldots, id]$$

a map

$$[[n]; T_1, \ldots, T_n] \to [[m-1]; S_1, \ldots, S_{i_1-1}, T_{i_1-1(j-1)}, S_{i_1+1}, \ldots, S_m]$$

followed by

$$[d^{i_2}; id, \ldots, id, g_{i_2-1(j-1)}; id, \ldots, id]$$

a map

$$[[m-1]; S_1, \ldots, S_{i_2-1}, T_{i_2-1(j-1)}, S_{i_2+1}, \ldots, S_m] \to [[m]; S_1, \ldots, S_m].$$

\[3\text{recall that } F : \Delta \to \Gamma \text{ is the functor defining wreath products and thus } F(x) \text{ here is a function } [m] \to P([n]) \text{ indicating the target indices of the maps of } C \text{ indexed over some simplex.}\]
We may in fact assume $f = d^i$ and commensurate therewith that
\[ [f; g] = [d^i; \text{id}, \ldots, \text{id}, g^i, \text{id}, \ldots, \text{id}], \]
a map $[[m - 1]; S_1, \ldots, S_{i-1}, T_i, S_{i+2}, \ldots, S_m] \rightarrow [[m]; S_1, \ldots, S_m]$, where the multi-morphism $g^i = g_i^i \times g_{i+1}^i : T_i \rightarrow S_i \times S_{i+1}$.

Now, if $i = m$ then there exists a co-face $S'_m \rightarrow S_m$ by recursion; indeed see that the height of $S_m$ is strictly less than that of $S$ and that $S_m$ is non-empty, so $[f; g]$ may be factored through $[[m]; S_1, \ldots, S'_m] \rightarrow [[m]; S_1, \ldots, S_m]$. A similar argument exists if $i = 0$ so it remains only to be seen that a co-face factorization exists for $0 < i < m$.

If $0 < i < m$ then the multi-morphism $g^i = g_i^i \times g_{i+1}^i : T_i \rightarrow S_i \times S_{i+1}$ is of degree 0, and we are done, or it is of strictly positive degree. In the case where it is of positive degree then suppose $g_i^i : T_i \rightarrow S_i$ contributes to that positivity in the sense that $\lambda(g_i^i) > 0$. A factoring of $T_i \rightarrow S_i$ exists into a $\Theta^-$ map $\alpha^-$ then a $\Theta^+$ map $\alpha^+$. That $\Theta^+$ map $\alpha^+$ has a target of height strictly lower than $S$ so it factors through some co-face of $S_i$, $\alpha^+ = \gamma \circ \beta$ where $\gamma$ is a co-face and $\beta$ is of $\Theta^+$.

In light of this we’ve a multi-morphism factorization

\[
\begin{array}{ccc}
T_i & \xrightarrow{\alpha^-} & T'_i \\
\alpha^- & & \beta \\
& & \gamma \\
\downarrow{g_i^i} & & \downarrow{g_{i+1}^i} \\
S_{i+1} & \longrightarrow & S_{i+1}
\end{array}
\]

which then defines a factoring of $[f; g]$ through the co-face $[\text{id}; \text{id}, \ldots, \text{id}, \gamma, \text{id}, \ldots, \text{id}]$. □

3. CELLULAR MODELS AND MONOMORPHISM PRESERVATION

Definition 3.1. A **cellular model** for a category $A$ is a set of morphisms of $\hat{A}$ which generate the class of monomorphism of $\hat{A}$ under transfinite pushouts of coproducts of those morphisms.

Our focus on skeleta is not merely morbid curiosity but instead stems from the fact that for many skeletal categories $A$, the set $\mathcal{M}_A = \{ \partial A^a \rightarrow A^a | a \in A \}$ comprises a cellular model.

Proposition 3.2. (Cisinski) Let $A$ be a skeletal category and let $\mathcal{M}_A$ denote the set of monomorphisms $\{ \partial A^a \rightarrow A^a \}_{a \in A}$. Then $\mathcal{M}_A$ is a cellular model for $A$ if and only if $A$ is such that no object admits non-trivial automorphisms.

The preservation of monomorphisms by left adjoint functors between presheaf categories may always be reduced then to the preservation of cellular models as sets of monomorphisms. This is one of the incredible strengths of the theory of Cisinski model categories. The model category theoretic statement of cofibration preservation is reduced to a category theoretic one, monomorphism preservation, and more the categories we use to model spaces, $(\infty, 1)$-categories, or $(\infty, n)$-categories often permit further reductions.
Lemma 3.3. Suppose $A$ to be an incremental skeletal category. Then for every $a$ of $A$ with $\lambda_A(a) = n \geq 2$, we have an isomorphism

$$\lim_{\to} \left\{ \coprod_{(f:b \to a, g:c \to a) \in X^2} A^b \times_{A^a} A^c \Rightleftharpoons \coprod_{(f:b \to a) \in X} A^b \right\} \iso \partial A^a$$

where $X = \{ f : b \to a | f \in \text{Mor}(A_+) \land \lambda_A(f) = 1 \}$.

Proof. The factorization axiom and incrementality provide that every cell of $a$, $g : d \to a$, with $\lambda_A(d) \leq n - 1$ factors through some cell of $a$, $g' : d' \to a$ with $\lambda_A(d') = n - 1$. From that observation it follows that the canonical map

$$\prod_{(f:b \to a) \in X} A^b \longrightarrow \partial A^a$$

is an epimorphism whence the induced morphism

$$\lim_{\to} \left\{ \coprod_{(f:b \to a, g:c \to a) \in X^2} A^b \times_{A^a} A^c \Rightleftharpoons \coprod_{(f:b \to a) \in X} A^b \right\} \iso \partial A^a$$

is an isomorphism as indicated. □

From this explicit form we may derive at least one mechanism for proofs that left adjoint functors between presheaf topoi preserve monomorphisms.

Proposition 3.4. Let $A$ be an incremental skeletal category for which $A^+ \hookrightarrow \text{Mono}(A)$ and let $F: \hat{A} \longrightarrow \hat{B}$ be a colimit preserving functor. Then $F$ preserves monomorphisms if:

1. $F(A^+) \hookrightarrow \text{Mono}(\hat{B})$
2. $F$ preserves the monomorphisms $\partial A^a \longrightarrow A^a$ as such for all objects $a$ of $A$ with $\lambda_A(a) \leq 1$.
3. for all $a \in \text{Ob}(A)$ with $\lambda_A(a) \geq 2$ and any pair of $A_+$ cells $f : b \to a$ and $g : c \to a$ with $\lambda_A(f) = \lambda_A(g) = 1$, the induced map

$$F(A^b \times_{A^a} A^c) \longrightarrow F(A^b) \times_{F(A^a)} F(A^c)$$

is an epimorphism.

Proof. Suppose that $F$ satisfies the the first two conditions. Since $\{ \partial A^a \longrightarrow A^a \}_{a \in A}$ comprises a cellular model for $\hat{A}$ and $F$ preserves colimits, it suffices to prove that for all $n \geq 2$ and all $a \in \text{Ob}(A)$ with $\lambda_A(a) \geq 2$, the morphisms

$$F\partial A^a \longrightarrow FA^a$$

are monomorphisms. We will prove this by identifying $F\partial A^a \longrightarrow FA^a$ with a sub-object of $FA^a$.

Let $a$ be given with $\lambda_A(a) \geq 2$ be given, let $X = \{ f : b \to a | f \in \text{Mor}(A) \land \lambda_A(f) = 1 \}$, and define the sub-object $\partial FA^a$ of $FA^a$ by the formula below.

$$\partial FA^a = \bigcup_{f:X} \text{im} \left( Ff : F(A^b) \longrightarrow F(A^a) \right)$$

Let us denote the forgetful functor $\text{Sub}_{\hat{A}}(A^a) \longrightarrow \hat{A}$ by $U$. We note that

$$\lim_{\to} \left\{ \coprod_{(f,g) \in X^2} U(\text{im}(Ff) \cap \text{im}(Fg)) \Rightleftharpoons \coprod_{h:X} U(\text{im}(Fh)) \right\} \iso U\partial FA^a$$
computed in $\widehat{A}$. Since all $f \in X$ are of $A^+$ then by hypothesis we have that $Ff$ is a monomorphism for all $f \in X$ and as a consequence we have that for each $(h : c \rightarrow a) \in X$,
\[ U(\text{im}(Fh)) \xrightarrow{\sim} FA^c, \]
and thus too have we for each $(f : b \rightarrow a, g : c \rightarrow a) \in X^2$,
\[ U(\text{im}(Ff) \cap \text{im}(Fg)) \xrightarrow{\sim} F(A^b) \times_{F(A^e)} F(A^c). \]
Thus
\[ \lim\longrightarrow \left\{ \bigsqcup_{(f,g) \in X^2} \xrightarrow{\sim} F(A^b) \times_{F(A^e)} F(A^c) \xrightarrow{\text{coface maps}} \bigsqcup_{h \in X} FA^c \right\} \xrightarrow{\sim} U\partial FA^a. \]
Our hypothesis however is precisely that $F(A^b \times_{A^e} A^c) \xrightarrow{\sim} F(A^b) \times_{F(A^e)} F(A^c)$ so we have gotten an isomorphism
\[ \lim\longrightarrow \left\{ \bigsqcup_{(f,g) \in X^2} F(A^b) \times_{A^e} F(A^c) \xrightarrow{\text{face maps}} \bigsqcup_{h \in X} FA^c \right\} \xrightarrow{\sim} U\partial FA^a \]
over $FA^a$. Since $F$ preserves colimits, the colimit above is $F\partial A^a$, and thus $F$ preserves $\partial A^a \longrightarrow A^a$ as a monomorphism. □

Definition 3.5. A square

\[
\begin{array}{ccc}
A & \xrightarrow{i} & B \\
\downarrow j & & \downarrow l \\
C & \xrightarrow{k} & D
\end{array}
\]
in a category ‘$\mathcal{C}$’ is said to be absolutely cartesian if, for any functor $F : \mathcal{C} \longrightarrow \mathcal{D}$, the image of that square is again cartesian.

In light of proposition 3.4 any proof of monomorphism preservation by some suitable functor is reduced to the question preserving the relations on co-face maps. What’s more, if we know the co-face pair pullback squares which are absolute for a specific incremental collapsible skeletal category then we may further reduce the question to only those pullbacks which are not absolute. Lest one think this reduction both tautological and useless\(^4\), consider the following corollary, which as an independent lemma in [Cisinski] (lemme 2.1.10) inspired the proposition.

Corollary 3.6. (Cisinski) Let $A$ be a small category and let $F : \widehat{\triangle} \longrightarrow \widehat{A}$ preserve small colimits. Then $F$ preserves monomorphisms if and only if the morphism
\[ F\left(\triangle^d \bigsqcup \triangle^d\right) : F\triangle^0 \bigsqcup F\triangle^0 \longrightarrow F\triangle^1 \]
is a monomorphism of $\widehat{A}$.

Proof. Any monomorphism preserving functor preserves the specified monomorphism, so the content is the converse. Since $\triangle$ is a collapsible incremental skeletal category proposition we have that $G(\triangle^+) \hookrightarrow \text{Mono}(\widehat{A})$ for any functor $G : \widehat{\triangle} \longrightarrow \widehat{A}$ since each morphism of $\triangle^+$ admits a retraction, thus $F(\triangle^+) \hookrightarrow \text{Mono}(\widehat{A})$. Thus monomorphism preservation may be proved by verifying the second and third conditions of prop. 3.4.

\(^4\)The reduction is properly speaking tautological while specifically not useless.
Since $\partial \Delta^0 = \emptyset$, any colimit preserving functor into $\hat{\Delta}$ will preserve $\partial \Delta^0 \to \Delta^0$ as a monomorphism, so the second condition is implied by the hypothesis that

$$F \left( \Delta^d \coprod \Delta^d \right) : F \Delta^0 \coprod F \Delta^0 \to F \Delta^1$$

is a monomorphism. Then, since all the fibered products in the scope of the third condition of the proposition are absolute (see [Cisinski] 2.1.9) the third condition holds. □

### 3.1. Co-faces and wreath products.

**Definition 3.7.** Let a co-face map in a skeletal category $A$ be an $A^+$ morphism $f : a \to b$ with $\lambda_A(f) = 1$.

In this subsection we will study the intersections of co-faces in wreath products for the purpose of applying prop. 3.4.

Suppose $C$ to be a multi-reedy category. Then co-face maps of $\Delta \int C$ are made of co-face maps of $\Delta$ and co-face maps of $C$ in the following sense.

Suppose $[f; g] : [[m] ; c_1 , \ldots , c_m] \to [[n] ; e_1 , \ldots , e_n]$ is of the sub category $(\Delta \int C)^+ \cap (\Delta \int C)$ and that $\lambda_{\Delta \int C}([f; g]) = 1$. Then it is the case that exactly one of the following components is of degree one and all others are of degree zero:

- $f : [m] \to [n]$;
- $(g^1_k) : c_1 \to \prod_{s \in F(f)(1)} e_s$;
- $\ldots$;
- $(g^m_k) : c_m \to \prod_{s \in F(f)(m)} e_s$.

It then follows that either $f = d^k$ for some $k$ and the listed multi-morphisms are degree zero, or $f = \text{id}$ and some multi-morphism is degree 1 while all others are degree 0.

More however is true. If $f = d^k : [n-1] \to [n]$ and $0 < k < n$ then only one of the multi-morphism is strictly a multi-morphism

$$g^k_k \times g^k_{k+1} : c_k \to e_k \times e_{k+1}$$

whereas all others come from the inclusion $C \hookrightarrow C(*)$ and as such must actually be the identity by the degree axiom. Similarly, if $f = d^k : [n-1] \to [n]$ and $k = 0, n$, then all of the multi-morphisms involved are actually morphisms of $\Theta$ and are thus identities. Conversely, if $f = \text{id}$ then all the multi-morphism are actually morphisms and so those of them which are degree zero are in fact identities.

There are thus three sorts of co-face maps in $\Delta \int C$, which we organize into two broader classes:

- $\Delta$ non-trivial
  - purely simplicial where $f = d^0$ or $d^n$ and all other data the identity;
  - one involving both $\Delta$ and $C$, with $f = d^k$ with $k \neq 0, n$, and
    $$g^k_k \times g^k_{k+1} : c_k \to e_k \times e_{k+1}$$
    some degree zero multi-morphism and all other data the identity; and
- $\Delta$ trivial
  - one purely involving $C$ with $g^i_i : c_i \to e_i$ some co-face map of $C$ with all other data the identity.
From these observations we may extract a typology of co-face pairs and then compute fibered products in those distinct cases.

3.2. Intersections of Co-face pairs. In order to describe the intersection of co-face pairs of objects in a category wreathed over $\Delta$ we first need a lemma about fiber products in $\Gamma$.

**Corollary 3.8.** The category $\Gamma$ has all fibered products and the pullback of the diagram

$$
\begin{array}{c}
\langle m \rangle \\
\downarrow g \\
\langle n \rangle \\
\downarrow f \\
\langle \ell \rangle
\end{array}
$$

is the set $\langle n \rangle \times_{\langle \ell \rangle} \langle m \rangle$ whose elements are pairs

$$(I, J) \in \text{Sub}_{\text{Set}}(\langle n \rangle) \times \text{Sub}_{\text{Set}}(\langle m \rangle)$$

such that

$$\bigcup_{i \in I} f_i = \bigcup_{j \in J} g_j = Y \subset \langle \ell \rangle$$

which are minimal with respect to the containment partial order and subsets of $Y$ which admit such presentations.

**Proof.** See appendix. \qed

**Lemma 3.9.** Suppose $C$ to be multi-reedy. Then, given any pair of $\Delta \int C^+$ maps

$$[f; f] : [[n]; a_1, \ldots, a_n] \longrightarrow [[\ell]; c_1, \ldots, c_\ell]$$

and

$$[g; g] : [[m]; b_1, \ldots, b_m] \longrightarrow [[\ell]; c_1, \ldots, c_\ell]$$

with $\lambda_{\Delta \int C}([f; f]), \lambda_{\Delta \int C}([g; g]) \leq 1$, if the fiber product of $f$ and $g$ exists in $\Delta$, the fiber product of $[f; f]$ and $[g; g]$ in $\Delta \int \widehat{\Gamma}$ is

$$[[m] \times_{[\ell]} [n]; x_1, \ldots, x_{m \times n}]$$

where for each $z \in \{1, \ldots, m \times \ell n\}$, $x_z$ is the limit of the diagram $X_z$ of presheaves on $C$ described as follows:

the objects of $X_z$ are the elements of the set

$$\{a_i\}_{i \in I_z} \cup \{c_k\}_{k \in Y_z} \cup \{b_j\}_{j \in J_z}$$

where

$$\bigcup_{i \in I_z} F(f)(i) = \bigcup_{j \in J_z} F(g)(i) = Y_z \subset \langle \ell \rangle$$

is the $z^\text{th}$ element of $F([[n] \times_{[\ell]} [m]]) = \langle n \rangle \times_{\langle \ell \rangle} \langle m \rangle$ which is or, and the set of morphisms is the set of components of the multi-morphisms $f$ or $g$ which connect objects in that set of objects.
Proof. If $\lambda_{\Delta f C} ([[\ell], c_1, \ldots, c_\ell]) = 0$ then $[[\ell], c_1, \ldots, c_\ell] = [0]$ so there are no co-faces to consider.

If $\lambda_{\Delta f C} ([[\ell], c_1, \ldots, c_\ell]) = 1$ then $[[\ell], c_1, \ldots, c_\ell] = [1]$ and then either the fiber product of $f$ and $g$ does not exist in $\Delta$ or $f = g$ whence the claim holds trivially.

We may thus assume that $\lambda_{\Delta f C} ([[\ell], c_1, \ldots, c_\ell]) \geq 2$. Now, since $f$ and $g$ are either $d^i$ for some possibly different choice of $i \in [n]$, or $id_{[n]}$ then $[m] \times_{[\ell]} [n]$ exists in $\Delta$ and is furthermore by an absolute fiber product. What’s more, this must be the simplicial aspect of the fibered product of $[f; f]$ and $[g; g]$ if it is to exist; consider maps into the fiber product from $[0]$.

Consider then that the cartesian square

$$
\begin{array}{ccc}
[n] \times_{[\ell]} [m] & \xrightarrow{\text{\textbullet}} & [m] \\
\downarrow f & & \downarrow g \\
[n] & \xrightarrow{\text{\textbullet}} & [\ell]
\end{array}
$$

is absolutely cartesian by Cisinski’s lemma as both $f$ and $g$ must be co-face maps of $\Delta$ or the identity. Thus $F ([n] \times_{[\ell]} [m]) = \langle n \rangle \times_{\langle \ell \rangle} \langle m \rangle$ which is the set

$$
\left\{ Y \subset \langle \ell \rangle \mid \exists I_Y \subset \langle n \rangle, J_Y \subset \langle m \rangle, \ s.t. \ Y = \bigcup_{i \in I} f_i = \bigcup_{j \in J} g_j \right\}.
$$

What’s more, this set is naturally ordered by being $F ([n] \times_{[\ell]} [m])$. Indeed for each $0 < i \leq n \times_{[\ell]} m$ let

$$
\overline{i} : [1] \rightarrow [n \times_{[\ell]} m]
$$

be the map for which $0 \mapsto i - 1$ and $1 \mapsto i$ and let $F (\overline{i}) (1) = \{ Y_i \} \subset \langle n \rangle \times_{\langle \ell \rangle} \langle m \rangle$.

Let $X$ be the diagram on objects $\{a_1, \ldots, a_n, b_1, \ldots, b_m, c_1, \ldots, c_\ell\}$ with morphisms being all those drawn from $f$ or $g$ and let $X_i$ be the full subcategory of that diagram on the object set $\{a_i\}_{i \in I_Y} \cup \{C_k\}_{k \in I_Y} \cup \{b_j\}_{j \in J_Y}$. For each $i$, let $x_i$ be the limit of the diagram $X_i$.

We’ve then a commutative square

$$
\begin{array}{ccc}
[[m] \times_{[\ell]} [n]; x_1, \ldots, x_{m \times n}] & \xrightarrow{[\overline{f}; \overline{g}]} & [[m]; b_1, \ldots, b_m] \\
\downarrow [\overline{f}; \overline{g}] & & \downarrow [\overline{g}; \overline{g}] \\
[[n]; a_1, \ldots, a_n] & \xrightarrow{[f; f]} & [[\ell]; c_1, \ldots, c_\ell]
\end{array}
$$

where $\overline{f}$ and $\overline{g}$ are got from the cartesian square in $\Delta$ above and for each $k \in \langle n \rangle \times_{\langle \ell \rangle} \langle m \rangle$ the component multi-morphism

$$
\overline{f}^k : x_k \rightarrow \prod_{i \in I_Y} a_i
$$

is that got from the canonical projections and the same is true for the components

$$
\overline{g}^k : x_k \rightarrow \prod_{j \in J_Y} b_j.
$$
We claim that this square is cartesian and more, since \( \Delta \int C \) is generated under colimits by \([0]\) and objects of the form \([(1); e]\) to prove such it suffices to demonstrate that
\[
\text{Hom} \left( [0], \left[ [m] \times_{[e]} [n]; x_1, \ldots, x_{m \times n} \right] \right) \xrightarrow{\sim} [m] \times_{[e]} [n]
\]
and
\[
\text{Hom} \left( \left[ [1]; e \right], \left[ [m] \times_{[e]} [n]; x_1, \ldots, x_{m \times n} \right] \right)
\]
is naturally isomorphic to
\[
\text{Hom} \left( \left[ [1]; e \right], \left[ [n]; a_1, \ldots, a_n \right] \right) \times_{\text{Hom}([1]; [e]; [c], \ldots, [e])} \text{Hom} \left( \left[ [1]; e \right], \left[ [m], b_1, \ldots, b_m \right] \right).
\]
While the former is clear, the latter merits exposition.

A map \( q : [1] \to [m] \times_{[e]} [n] \) defines \( F(q)(1) \subset \langle m \rangle \times \langle e \rangle \langle n \rangle \) and thus for any morphism
\[
[q; q] : \left[ [1]; e \right] \to [m] \times_{[e]} [n]; x_1, \ldots, x_{m \times n}
\]
indexed thereby, \( q \) is a multi-morphism
\[
q : e \to \prod_{k \in F(q)(1)} x_k.
\]
Post-composition with \( [f; f] \times [g; g] \) yields the multi-morphism \( [f \circ q; f \circ q] \times [g \times q; g] \) where \( f \circ q \) is the multi-morphism got from the composition of multi-morphisms
\[
e \to \prod_{k \in F(q)(1)} x_k \to \prod_{k \in F(q)(1)} \prod_{i \in I_k} a_i
\]
and likewise for \( g \circ q \).

Since we’ve already proven \( [n] \times_{[e]} [m] \) correct, to prove that post-composition from the preceding paragraph a monomorphism of sets it suffices to observe that for each \( k \), post-composition with \( \left( x_k \to \prod_{i \in I_k} a_{i}, x_k \to \prod_{j \in J_k} b_{j} \right) \) is a monomorphism since \( x_k \) is a limit. Surjectivity likewise follows from the fact that each \( x_k \) is a particular limit. \( \Box \)

4. Joyal’s shift and the suspension functor \( \Sigma J \)

4.1. the \( p \)-Collapse functor \( K_p \).

**Definition 4.1.** For each \( T = A(n_0, m_1, \ldots, m_\ell, n_\ell) = [(q); T_1, \ldots, T_q] \) of \( \Theta \) and \( p > 0 \) define the functor \( K_p : \Theta \to \Theta \) by setting
\[
K_p(T) = A \left( \min \{ n_0, p \}, \min \{ m_1, p \}, \ldots, \min \{ m_\ell, p \}, \min \{ n_\ell, p \} \right).
\]
Define the functor \( K_0 : \Theta \to \hat{\Theta} \) by setting
\[
K_0(T) = \prod_{i \in [q]} \Theta^{[0]}.
\]
Lemma 4.2. The diagrams

\[ \begin{array}{ccc}
  n_0 & \xrightarrow{i_{\max(0,n_0-p)}} & \min \{n_0,p\} \\
  m_1 & \xrightarrow{i_{\max(0,m_1-p)}} & \min \{m_1,p\} \\
  \vdots & \vdots & \vdots \\
  m_\ell & \xrightarrow{i_{\max(0,m_\ell-p)}} & \min \{m_\ell,p\} \\
  n_\ell & \xrightarrow{i_{\max(0,n_\ell-p)}} & \min \{n_\ell,p\}
\end{array} \]

define morphisms \( C_T : T \to K_p(T) \).

Proof. It suffices to prove that the diagram commutes and reducing our concern to the constituent squares it suffices to show that in the cases \( p \geq n \geq m \), \( n \geq p \geq m \), and \( p \geq n \geq m \) that the squares

\[ \begin{array}{ccc}
  n & \xrightarrow{id} & n \\
  m & \xrightarrow{s^{n-m}} & m
\end{array} \]

commute. In the first case, the square is but

\[ \begin{array}{ccc}
  n & \xrightarrow{i_n-p} & p \\
  m & \xrightarrow{s^{n-m}} & m
\end{array} \]

which commutes as \( n - p \leq n - m \) so \( i_n-p s^{n-m} = s^{p-m} \). In the final case, the square is

\[ \begin{array}{ccc}
  n & \xrightarrow{i_n-p} & p \\
  m & \xrightarrow{s^{n-m}} & m
\end{array} \]

which commute as under these hypotheses \( n - p \geq n - m \) so \( i_n-p s^{n-m} = i^{m-p} \). \( \square \)
These maps do not merely exist but enjoy an important universal property.

**Lemma 4.3.** Suppose $p \geq 1$ is given, suppose $S$ is a cell of $\Theta$ such that $K_p(S) = S$, and let $T$ be any cell of $\Theta$. Then any morphism $T \rightarrow S$ factors through $K_p(T)$.

*Proof.* The proof follows from the globular sum presentation. \qed

**Corollary 4.4.** The assignment $T \mapsto K_p(T)$ defines a functor left adjoint to the inclusion $\Theta_p \hookrightarrow \Theta$.

**Lemma 4.5.** For each non-negative integer $p$ and cell $T = A(n_0, m_1, \ldots, m_\ell, n_\ell)$, the diagram

\[
\begin{array}{c}
\min \{n_0, p\} \\
\min \{m_1, p\} \\
\vdots \\
\min \{m_\ell, p\} \\
\end{array} 
\xrightarrow{s_{n_0 - \min \{n_0, p\}}} \n_0 
\xrightarrow{s_{n_1 - \min \{m_1, p\}}} m_1 
\xrightarrow{s_{n_\ell - \min \{m_\ell, p\}}} m_\ell
\]

induces a natural morphism $D_p : K_p(T) \rightarrow T$. Similarly, the diagram

\[
\begin{array}{c}
\min \{n_0, p\} \\
\min \{m_1, p\} \\
\vdots \\
\min \{m_\ell, p\} \\
\end{array} 
\xrightarrow{s_{n_0 - \min \{n_0, p\}}} \n_0 
\xrightarrow{s_{n_1 - \min \{m_1, p\}}} m_1 
\xrightarrow{s_{n_\ell - \min \{m_\ell, p\}}} m_\ell
\]

induces a natural morphism $F_p : K_p(T) \rightarrow T$.

The reader is encouraged only to read the following proof only so far as required to believe the result.
Proof. It suffices to check that the squares

\[
\begin{array}{ccc}
\min \{n, p\} & \xrightarrow{f_{n-\min \{n, p\}}} & \overline{n} \\
\downarrow & & \downarrow \\
g_{\min \{n, p\}-\min \{m, p\}} & \xrightarrow{g_{n-m}} & \overline{m} \\
\min \{m, p\} & \xrightarrow{f_{m-\min \{m, p\}}} & \overline{m}
\end{array}
\]

commute in the following cases for \(f, g, m, n,\) and \(p\). Either \(f = s\) or \(f = t, g = s\) or \(g = t\) and either (a) \(p \geq n \geq m\), (b) \(n \geq p \geq m\), or (c) \(n \geq m \geq p\). □

\(f = s, g = s, p \geq n \geq m\): in this case the square evaluates to

\[
\begin{array}{ccc}
\overline{n} & \xrightarrow{id} & \overline{n} \\
\uparrow & & \uparrow \\
g_{n-m} & \xrightarrow{id} & g_{n-m} \\
m & \xrightarrow{id} & m
\end{array}
\]

which commutes.

\(f = s, g = s, n \geq p \geq m\): in this case the square evaluates to

\[
\begin{array}{ccc}
\overline{p} & \xrightarrow{s^{n-p}} & \overline{n} \\
\downarrow & & \downarrow \\
s_{p-m} & \xrightarrow{id} & s_{n-m} \\
m & \xrightarrow{id} & m
\end{array}
\]

which commutes.

\(f = s, g = s, n \geq m \geq p\): in this case the square evaluates to

\[
\begin{array}{ccc}
\overline{p} & \xrightarrow{s^{n-p}} & \overline{n} \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
s_{p-m} & \xrightarrow{id} & s_{n-m} \\
m & \xrightarrow{id} & m
\end{array}
\]

\(f = s, g = t, p \geq n \geq m\): in this case the square evaluates to

\[
\begin{array}{ccc}
\overline{n} & \xrightarrow{id} & \overline{n} \\
\uparrow & & \uparrow \\
l_{n-m} & \xrightarrow{id} & l_{n-m} \\
m & \xrightarrow{id} & m
\end{array}
\]

which commutes.
$f = s$, $g = t$, $n \geq p \geq m$: in this case the square evaluates to
\[
\begin{array}{ccc}
p & \xrightarrow{s^{n-p}} & n \\
p & \downarrow & \downarrow \\
p & \xrightarrow{t^{n-m}} & n \\
p & \downarrow & \downarrow \\
m & \xrightarrow{id} & m \\
m & \downarrow & \downarrow \\
m & \xrightarrow{id} & m \\
m & \downarrow & \downarrow \\
m & \xrightarrow{id} & m \\
\end{array}
\]
which commutes by co-globular identity $s \circ t = t \circ t$.

$f = s$, $g = t$, $n \geq m \geq p$: in this case the square evaluates to
\[
\begin{array}{ccc}
p & \xrightarrow{s^{n-p}} & n \\
p & \downarrow & \downarrow \\
p & \xrightarrow{id} & m \\
p & \downarrow & \downarrow \\
p & \xrightarrow{id} & m \\
p & \downarrow & \downarrow \\
p & \xrightarrow{id} & m \\
p & \downarrow & \downarrow \\
p & \xrightarrow{id} & m \\
\end{array}
\]
which commutes by the co-globular identity $t \circ s = s \circ s$.

The analysis for $f = t$ follows mutatis mutandis.

**Corollary 4.6.** Given $p \geq 0$, the morphisms $D_p : K_p(T) \to T$ and $F_p : K_p(T) \to T$ varying over $T$ in $\Theta$ comprise object-wise splittings of $C : \text{id} \to K_p$.

### 4.2. The Shift $J : \Theta \to \Theta$.

**Definition 4.7.** Define the functor $J = (\_)+1 : \Theta \to \Theta$ to be the natural extension of the assignment on objects $T \mapsto [[1]; T]$.

**Remark 4.8.** The purpose of the two notations is clarity, as depending on context one or the other is simpler.

**Lemma 4.9.** The functor $J = (\_)+1$ preserves globular sum decomposition of cells of $\Theta$.

**Proof.** Observe that $A(n_0, m_1, \ldots, m_\ell, n_\ell) = A(n_0 + 1, m_1 + 1, \ldots, m_\ell + 1, n_\ell + 1)$. □

The non degenerate cells of a cell of the form $J(T) = T + 1$ are *almost* in bijection with those of $T$.

**Lemma 4.10.** Given a tree $T$,

\[
\text{Hom}_{\Theta^+}(S, T + 1) = \begin{cases} 
\{d^i, d^0\} & S = [0] \\
\text{Hom}_{\Theta^+}(S', T) & S = S' + 1 \\
\emptyset & S \neq [0], S' + 1
\end{cases}.
\]

**Proof.** Indeed we observe that
\[
\text{Hom}_{\Theta^+}([0], [[1]; T]) \sim \text{Hom}_\Delta([0], [1]) = \{d^i, d^0\},
\]
and if $S = [[n]; S_1, \ldots, S_n]$ with $n \geq 2$ then $\text{Hom}_{\Theta^+}([n], [1]) = \emptyset$ so $\text{Hom}_{\Theta^+}(S, T + 1) = \emptyset$ for $S \neq [0], S' + 1$. □
Corollary 4.11. The functor $J : \Theta \to \Theta$ preserves fibered products of co-face maps.

Proof. Suppose

$$[[m] ; B_1, \ldots, B_m]$$

$$\downarrow [f, g]$$

$$[[n] ; A_1, \ldots, A_n] \xrightarrow{[g, g]} [[\ell] ; C_1, \ldots, C_\ell]$$

to be a pair of co-faces of $[[\ell] ; C_1, \ldots, C_\ell]$. We note that

$$[[m] ; B_1, \ldots, B_m]$$

$$\downarrow [id, [g, g]]$$

$$[[1] ; [[n] ; A_1, \ldots, A_n]] \xrightarrow{[id, [f, g]]} [[\ell] ; C_1, \ldots, C_\ell]$$

is again a diagram consisting of $\Theta^+$ morphisms of degree less than or equal to one, so lemma 3.9 applies and the fiber product formula holds for both diagrams. In particular we see that if

$$[[n] \times_\Theta [m] ; X_1, \ldots, X_{n \times m}]$$

is the original fiber product, then $[[1] ; [[n] \times_\Theta [m] ; X_1, \ldots, X_{n \times m}]]$ is the second fiber product and this is exactly $([[n] \times_\Theta [m] ; X_1, \ldots, X_{n \times m}]) + 1$. □

4.3. The Eckmann-Hilton degeneracies. Given a cell $T$ there is a degeneracy of particular interest $T + 1 \to T$.

Definition 4.12. Let $S$ and $T$ be a pair of cells of $\Theta$. Note that

$K_1(S + 1) = [[1] ; [0]]$

and that

$K_1(T) = [[[t] ; [0] \cdots [0]]$

for some $t \geq 0$. Then let

$F_{1, S, T} : S + 1 \to T$

be defined as the composition

$$S + 1 \xrightarrow{C_{1, S + 1}} [[1] ; [0]] \xrightarrow{(d^1)^{t-1}} [[[t] ; [0] \cdots [0]] \xrightarrow{F_{1, T}} T.$$  

Similarly let

$D_{1, S, T} : S + 1 \to T$

be the composition

$$S + 1 \xrightarrow{C_{1, S + 1}} [[1] ; [0]] \xrightarrow{(d^1)^{t-1}} [[[t] ; [0] \cdots [0]] \xrightarrow{D_{1, T}} T.$$  

Then, given a cell $T$ of $\Theta$ we define the Eckmann-Hilton degeneracy

$E_T : T + 1 \to T$
by recursion. Given a cell $T$ of $\Theta$ we have that

$$T = [[k]; T_1 \cdots T_k]$$

for some $k \geq 0$ and $T_1, \ldots, T_k$ of $\Theta$ and we may set

$$E_T : T + 1 \to T$$

to be the morphism

$$T + 1 = [[1]; [[k]; T_1 \cdots T_k]] \to [[k]; T_1 \cdots T_k] = T$$

induced by the diagram

$$\begin{array}{ccc}
T_1 + 2 & \xrightarrow{\phi_1} & T \\
\downarrow & & \downarrow \\
T & & T \\
\downarrow & & \downarrow \\
T_k + 2 & \xrightarrow{\phi_k} & T
\end{array}$$

where for each $1 \leq i \leq k$ the morphism

$$\phi_i : T_i + 2 = [[1]; T_i + 1] \to [[k]; T_1 \cdots T_k]$$

is given explicitly

$$\left( (d^1)^{k-1} : (F_{1,T_i,T_1}, \ldots, F_{1,T_i,T_{i-1}}, E_{T_i}, D_{1,T_i,T_{i+1}}, \ldots, D_{1,T_i,T_k}) \right),$$

and setting $E_T : \overline{T} \to \overline{T}$ to be the canonical map.

**Lemma 4.13.** The maps $E_T$ comprise the components of a natural transformation

$$E : J \to \text{id}_\Theta.$$

### 4.4. The Suspension functor $\Sigma_J$. For any cell $T$ there is a canonical monomorphism $d^1 \coprod d^0 : \partial T \to \Theta^{T+1}$ and taking the quotient of that target by that monomorphism gives us a suspension functor on the image of $\Theta \hookrightarrow \hat{\Theta}$.

**Definition 4.14.** Let $P : \partial T \to J$ be the natural transformation which is component-wise the monomorphism $d^1 \coprod d^0 : \partial T \to \Theta^{T+1}$. Define the functor

$$\Sigma_J : \hat{\Theta} \to \hat{\Theta}$$

be the left Kan extension along the composition of $\Theta \xrightarrow{\text{yon}} \hat{\Theta} \xrightarrow{\text{inj}} \hat{\Theta}$ of the functor $\Theta \to \hat{\Theta} : T \mapsto \Theta^J(T)/P_T$ where the base point is the unique $0$-cell.

**Lemma 4.15.** There are natural isomorphisms

$$\Sigma_J \Theta^T (S) \sim \Theta^{T+1}(S)^/ \sim \text{Hom}_{\hat{\Theta}} (\Theta_S^+, \Theta_T^{T+1}) \sim \text{Hom}_{\Theta} (S, J (T)) \cup \{\bullet\}$$

**Remark 4.16.** It’s important to note that this natural isomorphism does not respect fibrancy. Indeed while $\Theta^{T+1}$ is fibrant, $\Sigma_J \Theta^T$ is not. The missing cells are precisely those which correspond to the the composition of loops. This seemingly innocuous lemma will feature prominently in the proof that $\Sigma_J$ preserves certain fibered products.
The subsections that follow are devoted to proving that $\Sigma_J$ preserves monomorphisms and that $\Sigma_J$ is naturally weakly equivalent to $(\_)$ $\wedge S^1$. It is then as a corollary that we find $\Sigma_J$ to be a left Quillen suspension functor.

4.4.1. the Functor $\Sigma_J$ preserves monomorphisms.

**Lemma 4.17.** The functor $\Sigma_J : \hat{\Theta}_\bullet \rightarrow \hat{\Theta}_\bullet$ preserves fiber products of co-face maps into cells of the form $[[1];T]$.

**Proof.** The simplicial morphisms into $[1]$ over which a pair of co-face maps in $T + 1$ are organized may appear in only two flavors. The first is the purely simplicial case where $T = [0]$ and the diagram in question is

\[
\begin{array}{c}
[0] \\
\downarrow \quad d^0 \\
[0] \\
\downarrow \quad d^1 \\
[1]
\end{array}
\]

and the second the diagram is cospan comprised of identities\(^5\). We will deal with these two case separately.

In that first purely simplicial case the limit taken in $\hat{\Theta}$ is $\emptyset_\Theta$ and in $\hat{\Theta}_\bullet$ it is $\bullet_\Theta$. Since $\Sigma_J (\bullet_\Theta) = \bullet_\Theta$ we have then only to observe that if a square of the form

\[
\begin{array}{c}
X \\
\downarrow \\
\Sigma_J \Theta_\bullet^{[0]} \\
\downarrow \quad \Sigma_\Theta d^0 \\
\Sigma_J \Theta_\bullet^{[1]}
\end{array}
\]

commutes then $X = \bullet_\Theta$.

In the second case, $T \neq [0]$, so we are concerned with the co-face pairs defined be co-faces $S \rightarrow T$ and $R \rightarrow T$ and we consider the cospans

\[
\begin{array}{c}
[[1];R] \\
\downarrow \\
[[1];S] \\
\downarrow \\
[[1];T].
\end{array}
\]

These cospans are governed by lemma 3.9 so the fiber product in $\triangle \int \hat{\Theta}$ is $[[1];S \times_T R]$. We note then that it suffices to check the universal property against cells as opposed to arbitrary

\(^5\)The configuration

\[
\begin{array}{c}
[1] \\
\downarrow \quad \text{id} \\
[0] \\
\downarrow \quad d^i \\
[1]
\end{array}
\]

is eliminated as for $d^i : [0] \rightarrow [1]$ a co-face it follows that $T = [0]$ whence $\text{id} : [1] \rightarrow [1]$ cannot index a co-face map of $\Theta$. 
presheaves. What’s more, the presheaf

\[ \text{Hom}(\_, \Theta^{S+2}/\sim) \times_{\text{Hom}(\_, \Theta^{R+2}/\sim)} \text{Hom}(\_, \Theta^{R+2}/\sim) \]

is isomorphic to \( \text{Hom}(\_, \Theta^{S+2}) \times_{\text{Hom}(\_, \Theta^{T+2}/\sim)} \text{Hom}(\_, \Theta^{T+2}/\sim) \) as a consequence of lemma 4.15, so our fiber product formula, lemma 3.9, computes the limit of

\[ \Sigma_J \Theta^{[1];R} \]

\[ \Sigma_J \Theta^{[1];S} \to \Sigma_J \Theta^{[1];T} \]

to be \( \Theta^{[1];[1];S \times T;R}/\sim \) or \( \Sigma_J \Theta^{[1];S \times T;R} \) and thus the functor \( \Sigma_J \) preserves fiber products of these co-face maps.

\[ \square \]

**Lemma 4.18.** The functor \( \Sigma_J : \hat{\Theta}_* \to \hat{\Theta}_* \) preserves fiber products of co-face maps into cells of the form

\[ [[\ell]; c_1, \ldots, c_\ell] \]

where \( \ell \geq 2 \).

**Proof.** Since \( \ell \geq 2 \) co-face maps satisfy the hypotheses of lemma 3.9 so we have that the square

\[ [[n] \times [\ell] [m]; X_1, \ldots, X_{n \times m}] \to [[m]; b_1, \ldots, b_m] \]

\[ [[n]; a_1, \ldots, a_n] \to [[\ell]; c_1, \ldots, c_\ell] \]

is cartesian as indicated and the argument for the second case in the previous lemma provides the result m.m. \( \square \)

**Corollary 4.19.** The functor \( \Sigma_J : \hat{\Theta}_* \to \hat{\Theta}_* \) preserves fiber products of co-face maps into cells of dimension 2 or higher.

**Proposition 4.20.** The functor \( \Sigma_J \hat{\Theta}_* \to \hat{\Theta}_* \) preserves monomorphisms.

**Proof.** Since \( \Sigma_J \) preserves colimits and \( \hat{\Theta}_* \) is incremental skeletal proposition 3.4 permits us to reduce the proof to the demonstration that:

1. \( F(A^+) \hookrightarrow \text{Mono}(\hat{B}) \);
2. \( F \) preserves the monomorphisms \( \partial A^a \to A^a \) as such for all objects \( a \) of \( A \) with \( \lambda_A(a) \leq 1 \); and
3. for all \( a \in \text{Ob}(A) \) with \( \lambda_A(a) \geq 2 \) and any pair of \( A_+ \) cells \( f : b \to a \) and \( g : c \to a \) with \( \lambda_A(f) = \lambda_A(g) = 1 \), the canonical map

\[ F(A^b \times_{A^c} A^c) \to F(A^b) \times_{F(A^c)} F(A^c) \]

is surjective.

We have already proven (2) and (3) in the lemmata above, so it suffices to observe that indeed (1) is satisfied. \( \square \)
4.4.2. the Functor $\Sigma_J$ is weakly equivalent to $(\_)^{\land} S^1$. Recall the Eckmann Hilton degeneracies $E_T : T + 1 \to T$ from sub-section 4.3. In this section we will use these degeneracies to describe a natural weak equivalence $\Sigma_J \Theta^T \cong \Theta^{T + 1}/\partial \Theta^{[1]} \to \Theta^T \land S^1$, where by $S^1$ we mean the cellular set $\Theta^{[1]}/\partial \Theta^{[1]}$.

Corollary 4.21. (to lemma 4.10) Given a tree $T$, the non-degenerate cells of $\Theta^{T + 1}/\partial \Theta^T$ are almost in bijection with the non-degenerate cells of $\Theta^T$. More precisely,

$$
\text{Hom}_{\Theta^T}(S, \Theta^{T + 1}/\partial \Theta^T) = \begin{cases} 
\{\bullet\} & S = [0] \\
\text{Hom}_{\Theta^T}(S', T) & S = S' + 1 \\
\emptyset & S \neq [0], S' + 1
\end{cases}
$$

In the definition of the components of the Eckmann-Hilton Degeneracy $E : J \to \text{id}_{\Theta}$, we made use of the maps $D_{1,S,T}$ and $F_{1,S,T}$. These were in turn described as a composition of maps made possible by way of the truncation functor $K_1$.

Lemma 4.22. The assignment $T \mapsto K_0 T$ is functorial and more, there is a natural transformation $M : K_0 \to \text{Yon}$.

Definition 4.23. Let $\text{in}_- : [0] \to K_0(T)$ be the inclusion of the initial $[0]$-cell in the coproduct and let $\text{in}_+ : [0] \to K_0(T)$ be the inclusion of the final $[0]$-cell. In an abuse of notation we may also denote the composites

$$
S \overset{\bullet}{\to} [0] \overset{\text{in}_-}{\longrightarrow} K_0(T) \overset{M}{\longrightarrow} T
$$

by $\text{in}_-$ or $\text{in}_+$ whenever it does not introduce confusion or the economy of notation is worth the sacrifice.

Example 4.24. Suppose $T : \Theta$ is such that

$$
T = [[2]; L - 1, R - 1] = \lim \{L \leftarrow \overline{0} \to R\}
$$

where $L$ and $R$ are objects of $T$ of the form $[[1]; \cdots]$ so that the expressions $L - 1$ and $R - 1$ are clearly defined. Then

$$
T + 1 = [[1]; T] = \lim \{L + 1 \leftarrow \overline{1} \to R + 1\}
$$
In such case we note that the shuffle decomposition of products for cellular sets\footnote{See [Berger1]} puts

\[
\lim_{\longrightarrow} \begin{cases}
\Theta^{[3];[0]}(L-1)(R-1) \\
\Theta^{[2];(L-1)(R-1)} \\
\Theta^{[2];(L-1)(R-1)} \\
\Theta^{[3];(L-1)(R-1)[0]} \\
\Theta^{[3];[0]}(L-1)(R-1) \\
\Theta^{[3];[0]}(L-1)(R-1) \\
\Theta^{[3];[0]}(L-1)(R-1) \\
\Theta^{[3];[0]}(L-1)(R-1)
\end{cases} \sim \Theta^T_+ \land \Theta^T_+
\]

Then, to define a map \( \Sigma \Theta^T_+ \longrightarrow \Theta^T_+ \land S^1 \) it will suffice to define maps from \( L + 1 \) and \( R + 1 \) into that colimit which:

- agree on their common \( T \)-cell;
- and descend to the quotient \( \Theta^T_+ / \partial \Theta^{[1]} \) after the quotient morphism \( \Theta^T_+ \land \Theta^{[1]} \longrightarrow \Theta^T_+ \land S^1 \) is applied.

For the requisite map from \( L + 1 \) into that colimit we use

\[
\left( (d^1)^2; (E_{L-1}, \bullet, \text{in}_-) \right) : \begin{array}{c}
[[1]; L] \\
[1]
\end{array} \longrightarrow \begin{array}{c}
[[3]; L - 1, [0], R - 1] \\
[[2]; L - 1, R - 1]
\end{array}
\]

and similarly for the map \( R + 1 \) into that colimit, we use

\[
\left( (d^1)^2; (\text{in}_+, E_{R-1}, \bullet) \right) : \begin{array}{c}
[[1]; R] \\
[1]
\end{array} \longrightarrow \begin{array}{c}
[[3]; L - 1, R - 1, [0]] \\
[[3]; L - 1, R - 1, [0]]
\end{array}
\]

We then observe that the diagram

\[
\begin{array}{c}
[[1]; L] \quad \left( (d^1)^2; (E_{L-1}, \bullet, \text{in}_-) \right) \quad [[3]; L - 1, [0], R - 1] \\
\begin{array}{c}
[1] \\
[1]
\end{array} \quad \left( (d^1)^2; (E_{L-1}, \bullet, \text{in}_-) \right) \quad [[3]; L - 1, [0], R - 1] \\
\begin{array}{c}
[[1]; R] \quad \left( (d^1)^2; (\text{in}_+, E_{R-1}, \bullet) \right) \quad [[3]; L - 1, R - 1, [0]] \\
\begin{array}{c}
[1] \\
[1]
\end{array} \quad \left( (d^1)^2; (\text{in}_+, E_{R-1}, \bullet) \right) \quad [[3]; L - 1, R - 1, [0]]
\end{array}
\]

commutes whence we’ve defined a morphism from \( T + 1 \) into the target. The map thus specified descends to the quotient \( \Theta^T_+ / \partial \Theta^{[1]} \longrightarrow \Theta^T_+ \land S^1_{\Theta} \).
In general, we may suppose
\[ T = \mathcal{L} ; A_1 \ldots A_\ell = \lim_{\rightarrow} \{ A_1 + 1 \leftarrow \mathcal{L} \rightarrow A_\ell + 1 \} \]
so
\[ T + 1 = \mathcal{L} ; A_1 \ldots A_\ell = \lim_{\rightarrow} \{ A_1 + 2 \rightarrow \mathcal{L} \rightarrow A_\ell + 2 \} \]
where those colimits are taken in \( \Theta \) and not \( \hat{\Theta} \). It is then that the shuffle decomposition of products in cellular sets put
\[
\begin{align*}
\lim_{\rightarrow} & \begin{cases}
\Theta^T & X_0 \\
\Theta^T & X_1 \\
& \vdots \\
\Theta^T & X_{\ell-1} \\
\Theta^T & X_\ell
\end{cases} \\
\sim & \Theta^T \wedge \Theta^{[1]}_+
\end{align*}
\]
where for \( j \in [\ell] \), \( X_j = \Theta^{[\ell+1]; A_1, \ldots, A_j, [0], A_{j+1}, \ldots, A_\ell} \).

We may then describe the map \( T + 1 \rightarrow \Theta^T_+ \wedge \Theta^T_+ \) by way of the maps \( A_i + 2 \rightarrow X_i \) presented as in \( \Theta \) by
\[
\left[ (d_1^\ell; (in_+, \ldots, E_{A_i}, \bullet, in_-, \ldots, in_-)) : [1]; A_i + 1 \rightarrow [\ell + 1]; A_1 \ldots A_i [0] A_{i+1} \ldots A_\ell \right].
\]
As in the example, these are compatible along the gluing faces of the prism, so in passage to the quotient we find a map
\[
\Theta^{T+1}/\Theta^{[0]}_+ \vee \Theta^{[0]}_+ \rightarrow \Theta^T_+ \wedge S^1.
\]
What’s more, it is clear that this map is a weak equivalence.

**Theorem 4.25.** The functor \( \Sigma_J \) is left Quillen with respect to the Cisinski model structure on \( \hat{\Theta}_+ \) equivalent to spaces and the functor \( \Sigma_J \) enjoys the universal property of the suspension.

We’ve already seen the essence of the proof of this theorem in development of the natural weak equivalence over \( \Theta \),
\[
\Theta^{T+1}/\Theta^{[0]}_+ \vee \Theta^{[0]}_+ \rightarrow \Theta^T_+ \wedge S^1.
\]
Since weak equivalence is a homotopical notion as opposed to a purely categorial one, the natural equivalence above need not extend along the Yoneda embedding; the content of the theorem beyond the construction of the map is then that in fact it does. The proof depends on a familiar result in the theory of Reedy categories: Reedy colimits preserve object-wise weak equivalences between Reedy cofibrant objects.

**Proof.** On the image of the functor \( \Theta \hookrightarrow \hat{\Theta}_+ : T \mapsto \Theta^T_+ \), \( \Sigma_J \) enjoys the universal property of the suspension. What remains be seen is that for any pointed cellular set \( X \), the colimit
over the category of elements \( \Theta \downarrow X \), preserves those weak equivalences. It is thus that we must show the induced morphisms

\[
\lim_{T: \Theta \downarrow X} \Theta_T^{+1}/\partial \Theta_T \longrightarrow \lim_{T: \Theta \downarrow X} \Theta_T^+ \land S^1
\]

to be weak equivalences. For this, we invoke the theory of Reedy categories and find our problem is reduced by previous corollary to proving that that the diagrams

\[
S : \Theta \downarrow X \longrightarrow \widehat{\Theta}.
\]

\[
(\Theta^T \longrightarrow X) \longrightarrow \Theta^{T+1}/\partial \Theta^T
\]

and

\[
D : \Theta \downarrow X \longrightarrow \widehat{\Theta}.
\]

\[
(\Theta^T \longrightarrow X) \longrightarrow \Theta_T^+ \land S^1
\]

are Reedy cofibrant, which is here the condition that for all objects of \( \Theta \downarrow X \), the associated latching maps are monomorphisms.

Consider then that for any object \( \Theta^T \longrightarrow X \) of \( \Theta \downarrow X \), the associated latching object of a diagram \( F : \Theta \downarrow X \longrightarrow \widehat{\Theta} \) is the colimit

\[
L_{\Theta^T \rightarrow X} F = \lim_{\Theta^T \longrightarrow X} F(S)
\]

\[
= \lim_{\Theta^T \longrightarrow X} F(S)
\]

\[
\Theta^T \stackrel{\text{Sk}^d(T)-1}{\longrightarrow} \Theta^T \longrightarrow (\Theta^T \rightarrow X),
\]

so to prove that any such diagram \( F \) is Reedy cofibrant, it suffices to show that the functor \( F \) preserves the monomorphisms

\[
\text{Sk}^d(T)-1 (\Theta^T \rightarrow X) \longrightarrow (\Theta^T \rightarrow X).
\]

A fortiori it suffices to show that \( F \) preserves all monomorphisms.

The functor \((\_ \land S^1) : \widehat{\Theta} \longrightarrow \widehat{\Theta} \) is known to be left Quillen hence monomorphism preserving. Proposition 4.20 has it that \( \Sigma_J \) does preserve monomorphisms whence the level-wise weak equivalence \( S \longrightarrow D \), passes to a weak equivalence in \( \widehat{\Theta} \).

What’s more, since \( \Sigma_J \) preserves colimits and \( \widehat{\Theta} \) is locally presentable, there exists a functor which is right adjoint to \( \Sigma_J \).

**Lemma 4.26.** The functor

\[
\Omega : \widehat{\Theta} \longrightarrow \widehat{\Theta}
\]

\[
X \longmapsto \Omega X : T \mapsto \{ x \in X (T+1) | \text{in}_- (x) = \text{in}_+ (x) = \bullet \}
\]

is right adjoint to \( \Sigma_J \).
5. **Kan’s combinatorial spectra and \( \Sigma_J \): Spectra are locally finite weak \( \mathbb{Z} \)-groupoids.**

Kan presents a model of spectra which turns on the observation that on the set of cells of a CW-spectrum the suspension introduces an equivalence relation; we may identify an \( m \)-cell of the \( n \)th space in a CW-spectrum \( \psi : D^m \to X_n \) with an \( m + 1 \)-cell \( \psi' : D^{m+1} \to X_{n+1} \) of the \( n + 1 \)st if \( \psi' \) factors through \( \varphi_n \circ \Sigma \psi \). In this way spectra can be seen to be made up of so called stable cells. Kan realized that if a suspension functor for simplicial sets such that a simplex suspends to another simplex could be had we could model spectra much as we model spaces by simplicial sets.

**Definition.** Let \( K : \Delta \to \Delta \) be the functor which assigns \([n] \mapsto [n + 1], d^i \mapsto d^i, \) and \( s^j \mapsto s^j \). Let \( \Sigma_K \) be the left Kan extension along the composition \( \Delta \xrightarrow{\text{Yon}} \hat{\Delta} \xrightarrow{\cup} \hat{\Delta}_\bullet \) of the functor \( \Delta \to \hat{\Delta}_\bullet \), which assigns

\[
[n] \mapsto \Delta^K([n]) \cup \Delta^n_+ \cup \Delta^0_+,
\]

where the inclusion \( \Delta^n \to \Delta^K([n]) \) is the map \( d^{n+1} \) and the inclusion of the point is opposite that face.

Note then that \( \Sigma_K \Delta^n \) has exactly \( d^0, \ldots, d^n : \Delta^n \to \Sigma_K \Delta^n \) as non-trivial faces and \( d^{n+1} = \bullet \). For any \( \ell \in \mathbb{N} \) then, \( \Sigma^\ell_K \Delta^n_+ \) is an \( \ell \)-sphere with \( n \)-many non-degenerate sides and in the same configuration as those of an \( n \)-simplex. The non-trivial aspect of the combinatorics is dimension invariant.

In order to construct spectra then we can either then stabilize simplicial sets at \( \Sigma_K \) by taking sequential spectra or we can first stabilize \( \Delta \) at \( K \).

**Definition.** Let \( \Delta_{\text{st}} \) be the strict colimit in \( \text{Cat} \) of the diagram

\[
\Delta \xrightarrow{K} \Delta \xrightarrow{K} \cdots.
\]

This category is isomorphic to the category whose set of objects is \( \mathbb{Z} \) with morphisms generated by co-face maps \( d^i : z \to z + 1 \) for each \( i \in \mathbb{N} \) and co-degeneracy maps \( s^j : z + 1 \to z \) for each \( j \in \mathbb{N} \) subject to the co-simplicial relations.

**Definition.** Let \( K^-\text{Sp} \) be the full subcategory of the category of presheaves of pointed sets, \( \widehat{\Delta}_{\text{st}} \), subtended by those presheaves \( X \) such that for all \( z \in \mathbb{Z} \), and \( x \in X(z) \), there exists some \( m \in \mathbb{N} \) such that \( d^{m+i}(x) = \bullet \) for all \( i \in \mathbb{N} \).

The presentation above of Kan’s model, found in [ChenKrizPultr] then culminates in the following proposition.

**Proposition.** (Kan) The category \( K^-\text{Sp} \) is equivalent to the sub-category

\[
\Omega\text{Sp} \left( \widehat{\Delta}_\bullet, \Sigma_K \right) \to \text{Sp} \left( \widehat{\Delta}_\bullet, \Sigma_K \right)
\]

of sequential spectra for which the adjoints

\[
X_n \to \Omega_K X_{n+1}
\]

to the structure maps

\[
\Sigma_K X_n \to X_{n+1}
\]

are isomorphisms.
A replication of that construction with $\hat{\Theta}_\bullet$ in place of $\hat{\Lambda}_\bullet$ and $\Sigma_J$ in place of $\Sigma_K$ subsumes the definition $\Theta_{\text{st}} = \lim \{ \Theta \to \Theta \to \cdots \}$ and a presentation of spectra as a subcategory of $\hat{\Theta}_{\text{st}}\bullet$.

Importantly, as $\Theta$ is the category of composition data for $\omega$-categories, $\Theta_{\text{st}}$ is the category of composition data for $\mathbb{Z}$-categories. It is then that in forthcoming work we will present a model category structure on a subcategory of $J - \text{Sp} \hookrightarrow \hat{\Theta}_{\text{st}}\bullet$ analogous to $K - \text{Sp} \hookrightarrow \hat{\Lambda}_{\text{st}}\bullet$. The fibrant objects will comprise a presentation of locally finite weak $\mathbb{Z}$-groupoids.

Using certain universality observations we also expect that we will be able construct a category $\beta$, such that $\hat{\beta}\bullet$ is equivalent to $J - \text{Sp}$. More, one shortcoming noted by [ChenKrizPultr] of Kan's model is that the tensoring of a combinatorial spectra with a pointed simplicial set does not commute with the natural suspension operation on $\hat{\Delta}_{\text{st}}\bullet$. In the presentation we construct, since suspension and tensoring with spaces are in a sense orthogonal, it is expected that suspension and tensoring with pointed spaces will commute. We should then arrive at a presentation of spectra which enjoys the nice parameterization properties of $K - \text{Sp}$ and also allows for the treatment of the generalized homology of spaces.

**Appendix A. $\Gamma$ category lemmata**

**Corollary A.1.** The category $\Gamma$ has all fibered products and the pullback of the diagram

\[
\begin{array}{ccc}
\langle m \rangle & \xrightarrow{g} & \langle \ell \rangle \\
\langle n \rangle & \xrightarrow{f} & \langle \ell \rangle
\end{array}
\]

is the set $\langle n \rangle \times_{\langle \ell \rangle} \langle m \rangle$ whose elements are pairs $(I, J) \in \text{Sub}_{\text{Set}}(\langle n \rangle) \times \text{Sub}_{\text{Set}}(\langle m \rangle)$ such that

$$
\bigcup_{i \in I} f_i = \bigcup_{j \in J} g_j
$$

which are minimal with respect to the containment partial order.

**Proof.** We prove explicitly that for any $X$ an object of $\Gamma$, we’ve an isomorphism

$$
\text{Hom}_\Gamma(X, \langle n \rangle \times_{\langle \ell \rangle} \langle m \rangle) \xrightarrow{\sim} \text{Hom}_\Gamma(X, \langle n \rangle) \times_{\text{Hom}_\Gamma(X, \langle \ell \rangle)} \text{Hom}_\Gamma(X, \langle m \rangle)
$$

natural in $\langle X \rangle$. A $\Gamma$ morphism $q : \langle X \rangle \to \langle n \rangle \times_{\langle \ell \rangle} \langle m \rangle$ is comprised of assignments

$$
(x \mapsto Q_x)_{x \in X}
$$

with $Q_x \subset \langle n \rangle \times_{\langle \ell \rangle} \langle m \rangle$ being disjoint subsets. The associated $\Gamma$ morphism pair in

$$
\text{Hom}(X, \langle n \rangle) \times \text{Hom}(X, \langle m \rangle)
$$

is then given by

$$
\left( \left( x \mapsto \bigcup_{Y \in Q_x} I_Y \right)_{x \in X} , \left( x \mapsto \bigcup_{Y \in Q_x} J_Y \right)_{x \in X} \right).
$$
These data comprise a pair of functions $X \to P(\langle n \rangle)$ and $X \to P(\langle m \rangle)$ but it is not immediate that disjointness of $Q_x$ from $Q_{x'}$ for $x \neq x'$ implies the disjointness of
$$\bigcup_{Y \in Q_x} I_Y$$
and
$$\bigcup_{Z \in Q_{x'}} I_Z.$$
This disjointness however does follow from the intersection minimality condition on $\langle n \rangle \times \langle \ell \rangle \langle m \rangle \subset C$. It is by hypothesis that for any $Y \in C$ that
$$Y = \bigcup_{k \in I_Y} f_k = \bigcup_{k \in J_Y} g_k,$$
therefore, this function
$$\text{Hom}(X, \langle n \rangle \times \langle \ell \rangle \langle m \rangle) \to \text{Hom}(X, \langle n \rangle) \times \text{Hom}(X, \langle m \rangle)$$
lands in the subset
$$\bigcup_{Y \in Q_x} \bigcup_{i \in I_Y} f_i = \bigcup_{Y \in Q_x} Y = \bigcup_{Y \in Q_x} \bigcup_{j \in J_Y} g_j$$
which implies the commutation of the square.

An inverse function
$$\text{Hom}_\Gamma(\langle X \rangle, \langle n \rangle) \times \text{Hom}_\Gamma(\langle X \rangle, \langle m \rangle) \to \text{Hom}_\Gamma(\langle X \rangle, \langle n \rangle \times \langle \ell \rangle \langle m \rangle)$$
is manifest in the association to an $(a, b)$ on the left the $\Gamma$ morphism which associates to $x$ the unique subset $Q_x \subset \langle n \rangle \times \langle \ell \rangle \langle m \rangle$ such that
$$\bigcup_{Y \in Q_x} Y = \bigcup_{z \in a_x} f_z = \bigcup_{z \in b_x} g_z.$$

□

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