Replacement of differential variables in the expression of differential shapes and partial derivative

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Abstract. Methods of differentiating functions defined parametrically, complex functions, and implicitly defined functions can be used to replace variables in differential expressions to solve differential equations. In this clause, and implicitly given a function of one of the main X ways, we will consider only the computational aspect of the replacement of variables, without affecting the problems of theoretical justification of the operations.

1. Introduction

Methods for differentiating functions defined parametrically, complex functions, and implicitly defined functions can be used to replace variables in differential expressions - one of the main methods for solving differential equations. In this section, we will consider only the computational aspect of changing variables, without affecting the problems of theoretical substantiation of the operations performed.

Let a differential expression be given:

\[ \frac{\text{d}y}{\text{d}x} = \frac{\text{d}^2y}{\text{d}x^2} = \ldots = \frac{\text{d}^n y}{\text{d}x^n} \]

it requires to go to the new variable \( t \) and the new function \( u(s) \) by the formulas:

\[ X = f(s, u) \]
\[ Y = f(s, u) \] (1.1)

Since \( u(s) \) is a function of \( s \) then the system (1.1) under certain conditions determines parametrically the function \( y(x) \):

Therefore, the derivative \( y\prime(x) \) can be expressed as follows:

\[ y\prime(x) = \frac{y_s}{x_t} = \frac{f_u u_t}{f_t} \] (1.2)

Let \( Z = y(x) \) Formula (1.2) shows the dependence of \( Z \) on \( s \) and \( u \) i.e. \( Z = h(t,u) \): Consider the system

\[ \begin{cases} X = f(s, u), \\ Y = h(s, u). \end{cases} \]

Conducting arguments similar to the previous, we find \( z\prime x \), i.e. \( y\prime x^2 \). This derivative will be expressed through \( s, u, u, u, \ldots, u^n \). Thus, we can find the derivative of the function \( y(x) \) of any order. Substituting the values found in, get a new differential expression:

\[ \psi(s, u, u, u, \ldots, u^n) \].

In the particular case when only the independent variable \( x \) changes to the variable \( s \) i.e.

\[ \begin{cases} X = f(s, u), \\ Y = h(s, u). \end{cases} \]

formulas are simpler. Namely, differentiating the true equality \( y(f(S))=u(s) \) will get
\[ y_1 f_1' = u_1', \quad y_2 f_2' = u_2' \]

from where:
\[ y_1 f_1' (u_1')^2 + y_2 f_2' (u_2')^2 = \frac{u_1'^2 f_1'-u_2' f_2'}{(f_2')^2} \]

Let a differential expression be given differential variables containing partial derivatives
\[ \Phi(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial x \partial y}, \frac{\partial^2 z}{\partial y^2}, \ldots) \]  

it is required to pass to the new independent variables \( u, v \) and new feature \( w = w(u, v) \), according to the formulas:
\[ \begin{align*}
  x &= f(u, v, w), \\
  y &= g(u, v, w), \\
  z &= h(u, v, w)
\end{align*} \]

Substituting in the last formula:
\[ Z = z(x, y), \quad X = f(u, v, w), \quad Y = g(u, v, w), \quad W = w(u, v), \]

we get the correct equality:
\[ z(f(u, v, w), g(f(u, v, w))) = h(u, v, w). \]

Find the partial derivatives concerning \( u, v \) From both sides of this equality:
\[ \begin{align*}
  z' x f'_u + z' y g'_u &= w'_u \]
\[ z' x f'_v + z' y g'_y &= w'_y
\]

Solving this system concerning \( u'_u, v'_v \) we find their expressions through partial derivatives of functions \( f, g, h \) and partial derivatives of the new function \( w \).

calculating then the partial derivatives concerning \( u, v \) in the equations of system (1-4), we get a new system of four equations for determining four quotients second-order derivatives of the function \( z(x, y) \).

In the particular case when only independent variables change and the values of the function in the particular case when only independent variables, and function values \( z = z(x, y) \) remain the same \( x = f(u, v), \quad y = g(u, v), \quad z(u, v) = w(u, v) \), the formulas are simpler :
\[ \begin{align*}
  z'_x f'_u + z'_y g'_u &= w'_u \\
  z'_x f'_v + z'_y g'_v &= w'_v
\] \]

Finally, if the replacement formulas do not express the old variables in terms of the new, but vice versa, then in the Formulas(1.2) and (1.4) the derivatives of the old and new functions simply change roles.

**Definition:** The partial derivative of the function \( z = f(x, y) \) with respect to the variable \( x \) at the point \((x_0, y_0)\) indicated by one of the following characters \( \frac{\partial f}{\partial x} (x_0, y_0), \frac{\partial z}{\partial x} (x_0, y_0), \)
\( f'_x(x_0, y_0), z'_x(x_0, y_0) \) and is calculated by the formula
\[ \frac{\partial f}{\partial x} (x_0, y_0) = \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x, y_0)-f(x_0, y_0)}{\Delta x} \]

Similarly, the partial derivative of a function \( z = f(x, y) \) by variable \( y \) at the point \((x_0, y_0)\) calculated by the formula :
\[ \frac{\partial f}{\partial y} (x_0, y_0) = f'_y(x_0, y_0) = \lim_{\Delta y \to 0} \frac{f(x_0, y_0+\Delta y)-f(x_0, y_0)}{\Delta y} \]

**Definition:**

The function \( f(x, y) \) is said to be differentiable at the point \((x_0, y_0)\), if its increment can be represented as the sum of the principal part, linear concerning increments of variables, and infinitely small of a higher order than the norm of the vector \((\Delta x, \Delta y)\), composed of increments of variables:
\[ \Delta f(x_0, y_0) \]

**Example.** Find by definition partial derivatives of a function
\[ Z = x \cos y \text{ at the point } (1, 0). \]

**Solution:** By definition, the partial derivative of the function \( f \) concerning the variable \( x \) is calculated by the formula
\[ \frac{\partial f}{\partial x} (x_0, y_0) = f'_x(x_0, y_0) = \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x, y_0)-f(x_0, y_0)}{\Delta x} \]
\[ \frac{\partial z}{\partial x}[1,0] = \lim_{\Delta x \to 0} \frac{f(1+\Delta x,0)-f[1,0]}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(1+\Delta x,0).\cos 0 - \cos 0}{\Delta x} = 1 \]

By definition, the partial derivative of \( f \) concerning \( y \) is calculated by the formula:

\[ \frac{\partial z}{\partial y}[1,0] = \lim_{\Delta y \to 0} \frac{f(1,0+\Delta y)-f[1,0]}{\Delta y} = \lim_{\Delta y \to 0} \frac{\cos (0+\Delta y) - \cos 0}{\Delta y} \]

\[ \lim_{\Delta y \to 0} \frac{-\sin \Delta y}{\Delta y} = \lim_{\Delta y \to 0} \frac{2 \cos ^2 \frac{\Delta y}{2}}{\Delta y} = \lim_{\Delta y \to 0} \frac{2 \frac{\Delta y}{2}}{\Delta y} = \lim_{\Delta y \to 0} \frac{\Delta y}{\Delta y} = 0. \]

You can calculate the partial derivative in the same way as the derivative of a function of one variable, using the rules and the table of derivatives. When calculating the partial derivative concerning the variable \( x \), we consider the second variable \( y \) to be fixed. So, for the function from the last example:

\[ \frac{\partial z}{\partial x} = (x \cdot \cos y)_x = \cos y \cdot x' = \cos y; \quad \frac{\partial z}{\partial x}(1,0) = \cos 0 = 1. \]

\[ \frac{\partial z}{\partial y} = (x \cdot \cos y)_y = x \cdot (\cos y)' = x \cdot \sin y; \quad \frac{\partial z}{\partial y}(1,0) = \sin(0) = 0. \]

2. **Theorem (sufficient condition for differentiability).**

Let the function \( f(x, y) \) has both partial derivatives in some neighborhood of the point \((x_0, y_0)\), which are continuous at the point itself \((x_0, y_0)\). Then the function is differentiable at the point \((x_0, y_0)\).

*Proof:* We write the increment of the function in the form

\[ \Delta f = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0) \left( f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0 + \Delta y) + f(x_0, y_0 + \Delta y) - f(x_0, y_0) \right) \]

By the Lagrange theorem, the increment of a function in a variable \( X \) standing in the first bracket can be represented as a partial derivative with respect to a variable \( X \) at some intermediate point times the increment of the variable. Similarly, we can write the term in the second bracket. Then

\[ \Delta f = \frac{\partial f}{\partial x}(x_0 + \theta_1 \cdot \Delta x, y_0 + \Delta y) \cdot \Delta x + \frac{\partial f}{\partial y}(x_0, y_0 + \theta_2 \cdot \Delta y) \cdot \Delta y \]

We verify the differentiability condition:

\[ \frac{\left| \Delta f - df \right|}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} \leq \frac{\left| \frac{\partial f}{\partial x}(x_0 + \theta_1 \cdot \Delta x, y_0 + \Delta y) \cdot \Delta x + \frac{\partial f}{\partial y}(x_0, y_0 + \theta_2 \cdot \Delta y) \cdot \Delta y \right|}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} + \frac{\left| \frac{\partial f}{\partial x}(x_0, y_0) \right| \cdot \left| \frac{\partial f}{\partial y}(x_0, y_0) \right|}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} \]

As \( \frac{\left| \Delta x \right|}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} \leq 1, \frac{\left| \Delta y \right|}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} \leq 1 \), by continuity of partial derivatives at the point \((x_0, y_0)\), the differences of partial derivatives under the sign of the module tend to zero, then the differentiability condition

\[ \frac{\left| \Delta f - df \right|}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} \to 0 \quad \text{done. The theorem is proved.} \]

**Example:** In differential expression:

\[ \Phi(x, y, z, \frac{\partial z}{\partial x}, \ldots, \frac{\partial z}{\partial y}) = \frac{\partial^2 z}{\partial x^2} + 2 \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} \]

New variables are required to introduce \( u = x \cdot y \), \( v = x \cdot y \cdot x \) and a new function \( w = x \cdot y \cdot z \).

Substituting in the last equation the values of the functions entering into it from \( x \), \( y \):

\[ w(x + y, x - y) = xy - z(x, y) \]

Find the partial derivatives for \( x \) and \( y \) of the equations of this system:

\[
\begin{align*}
\mathcal{W}_u^2 + \mathcal{W}_{uv} + \mathcal{W}_w^2 + \mathcal{W}_v^2 &= -z_{x^2}^2 \\
\mathcal{W}_u^2 - \mathcal{W}_{uv} + \mathcal{W}_u^2 + \mathcal{W}_v^2 &= 1 - z_{xy}^2 \\
\mathcal{W}_u^2 + \mathcal{W}_{uv} - \mathcal{W}_u^2 - \mathcal{W}_v^2 &= 1 - z_{yx}^2 \\
\mathcal{W}_u^2 - \mathcal{W}_{uv} + \mathcal{W}_u^2 + \mathcal{W}_v^2 &= -z_{y^2}^2
\end{align*}
\]
From here:

\[ z_{x^2} = W_{u^2} - W_{uv} - W_{v^2} . \]
\[ z_{xy} = 1 - W_{u^2} + W_{uv} - W_{v^2} . \]
\[ -z_{y^2} = -W_{u^2} + W_{uv} + W_{v^2} . \]

We deliver the found values in \( \Phi \):

\[ 2-4W_{u^2} + 2W_{uv} - 2W_{v^2} = \psi(W_{u^2}, W_{uv}, W_{v^2}). \]

If the function \( z(x, y) \) of the mixed derivatives coincided:

\[ \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} , \]

The fact that, as follows from the system (1.5), they coincide with the functions \( w(u, v) \). Then the expression will take on a very simple form:

\[ \psi = \psi(W_{u^2}) = 2 - 4W_{u^2} . \]

References

[1] G. A. Sviridyuk, N. A. Manakova, The Barenblatt – Zheltov – Kochina model with additive white noise in quasi-Sobolev spaces, J. Comp. Eng. Math., 2016, Volume 3, Issue 1, 61–67.
[2] Craig A. Tracy, Lectures on Differential equations / Craig A. Tracy, Springer, 2017 – p 58 -60.
[3] Grigorian A. Ordinary Differential Equation / Grigorian A., July 2008 – I 17.
[4] V. I. Berdyshev, V. B. Kostousov. "Navigation of moving objects by geophysical fields", Journal of Mathematical Sciences, 2007.
[5] Gabriel N. Ordinary Differential Equations Gabriel N., April 2020 – P 81-89.
[6] Stark, H. M. "Zeta Functions of Finite Graphs and Coverings, Part II", Advances in Mathematics, 20000901.
[7] Huy Vu, Antonio Palacios, Visarath In, Patrick Longhini, Joseph D. Neff. "Two-time-scale analysis of a ring of coupled vibratory gyroscopes", Physical Review E, 2010 p 2-4.
[8] K. S. Charak, D. Rochon. "On Factorization of Bicomplex Meromorphic Functions", Hypercomplex Analysis, 2008.
[9] V. Zh. Sakbaev. "Cauchy Problem for Degenerating Linear Differential Equations and Averaging of Approximating Regularizations", Journal of Mathematical Sciences, 2016.
[10] Jingyuan Li. "Fear of Loss and Happiness of Win: Properties and Applications", Journal of Risk & Insurance, 05/2010.
[11] V. Zh. Sakbaev. "Cauchy Problem for Degenerating Linear Differential Equations and Averaging of Approximating Regularizations", Journal of Mathematical Sciences, 2016.
[12] G. A. Sviridyuk, Kazak V.O. "Study of phase spaces of one class of semilinear equations"/ G. A. Sviridyuk, 2004 p 38-45.
[13] Benjamin A. Alexey O. "Galois theory of difference equations with periodic parameters" arXiv:1009.1159v2 [math.AC] 10 Feb 2011.
[14] Lucia D. V. Charlotte H. "Galois theories of q-difference equations: comparison theorems" February 21, 2020.
[15] Sviridyuk G.A. Linear Equations of Sobolev type and Strong Continiously Semigroup of Resolving Operator with Kernel. [Sviridyuk G.A. Linejnye uravneniya tipa Soboleva i sil’no nepreryvnye polugruppy razreshayushchih operatorov s yadrami]. Doklady Akademii nauk — Doklady Mathematics, 1994, vol.337, no.5, p 550–574.
[16] Kostin V.A., Pisareva S.V. Evolutional Equations with Singularities in Generalized Stepanov Spaces. [Kostin V.A., Pisareva S.V. E ’volyucionnye uravneniya s osobennostyami v obobshennyx prostranstvax Stepanova]. Izvestiya vysshix uchebnixx zavedenij. Matematika — Russian Mathematics, 2007, no.6, p 35–44.