A Note on Six-Dimensional Gauge Theories

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Abstract

We study the new “gauge” theories in 5+1 dimensions, and their non-commutative generalizations. We argue that the $\theta$-term and the non-commutative torus parameters appear on an equal footing in the non-critical string theories which define the gauge theories. The use of these theories as a Matrix description of M-theory on $T^5$, as well as a closely related realization as 5-branes in type II\text{B} string theory, proves useful in studying some of their properties.

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1 Introduction

Matrix theory is an attempt at a non-perturbative formulation of M-theory in the lightcone frame. Compactifying it on a torus $T^d$ has been shown to be related to the large $N$ limit of Super-Yang-Mills (SYM) theory in $d + 1$ dimensions. This description is necessarily only partial for $d > 3$, since the SYM theory is then not renormalizable. The SYM theory is defined, for $d = 4$ by the $(2,0)$ superconformal field theory in $5+1$ dimensions, and for $d = 5$ by a non-critical string theory in $5+1$ dimensions. The situation of compactifications to lower dimensions is unclear for now (for recent reviews see [7, 8]).

Recent work on the discrete light-cone quantization (DLCQ) of M-theory sheds light on its Matrix formulation, which is the above mentioned theories for a finite $N$. In particular it clarifies the problems encountered in finding a Matrix description for compactifications of M-theory to low dimensions. It is natural, then, to consider more general backgrounds for Matrix theory, in the cases when it is relatively well-understood. One hopes that this might uncover phenomena relevant to the low dimensional compactifications, but in a better controlled context.

In recent papers, it was shown that longitudinal constant backgrounds of $A^{(3)}$ may be incorporated into Matrix theory, and lead to SYM theories on non-commutative tori. Indeed, the introduction of non-commutative geometry is not surprising -- it has long been suspected to appear in this context, possibly serving as a cutoff. However, for higher dimensional tori, as is the case for conventional SYM theories, the resulting theories are expected to be non-renormalizable and need a UV definition.

For a large 5-torus, the compactified SYM theory can be defined as a particular limit of the parameter space of the non-critical string theory. In this limit there emerges a geometrical description of the base space, and the SYM is a low energy description in this geometrical setting. We suggest to define the SYM theory on a large non-commutative torus in a similar way, as a particular limit of the non-critical string theory, where there emerges a base space which is a non-commutative 5-torus. We refer to this scenario loosely as “compactifying the non-critical string theory on the non-commutative torus”. We present a precise definition of this theory in terms of a decoupled 5-brane theory in what follows.

A different deformation of 6 dimensional gauge theories was considered recently in Refs. [14, 15]. In [14] the theory on the $(p, q)$ fivebranes of a weakly coupled type IIB was studied. The low energy theory is a gauge theory with a rational $\theta$-angle. An extension to irrational values of $\theta$ was suggested in [15]. An essential part of the analysis in [15] was the fact that the bulk theory is not weakly coupled in any duality frame.

In this paper we take the approach advocated in [16]. The finite $N$ theories above are related to DLCQ of M-theory, and have therefore a spacetime interpretation. The relation to spacetime can be turned around to deduce statements about the

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1 Other works relating to longitudinal moduli and extended U-duality have appeared recently.

2 The Hilbert space of the non-critical string theory includes states that live in the near-horizon “throat” but decouples from the asymptotic spacetime physics.
“gauge” theories in 5+1 dimensions. In order to approach these theories as Matrix theories, we are led to study compactifications of M-theory on $T^5$ with longitudinal moduli turned on.

In this approach we reproduce the theories on the non-commutative torus $T^2$ [12, 13], and the “gauge” theories with any $\theta$-angle [14, 15]. Since there is no apparent quantization condition for $\theta$ in spacetime, we are led to believe that the “gauge” theories exist for any value of the $\theta$-angle. Moreover, once they are defined via the non-critical string theory [5], these “gauge” theories are simply T-dual to the theories considered in [12, 13]. This intrinsic T-duality is the U-duality of $M(T^5)$ in the spacetime picture.

The paper is laid out as follows. We begin with a short review of non-commutative geometry as it has appears in recent M-theory literature. We then consider longitudinal backgrounds for $M(T^5)$, and discuss their relation to the above mentioned gauge theories. We describe their implications for the BPS spectrum in the low energy SYM theory as well as in the corresponding non-critical string theory.

2 Non-Commutative Geometry on $T^2$

In this section, we review some key points in the recent literature [12, 13] on the appearance of non-commutative geometry in M-theory compactifications.

Consider the usual process of deriving a Matrix description of M-theory on a 2-torus, with no backgrounds turned on. Following Ref. [10], we fix M-theory parameters, and boost the lightcone. Following this, we must rescale all length parameters. In the resulting IIA theory, the length scales of the torus are small—the physics in this limit is given by a T-dual theory on a large torus. Note that at large radius, the light states are momentum modes, and these dominate any stringy effects. As the radii get small, it is the winding modes that get light, and thus we get a description in terms of these variables.

Now consider what happens when a longitudinal background $A_{12}^{(3)} = \theta$ is turned on in M-theory. In the Matrix description, we arrive at IIA theory compactified on a small torus of size $L_i$, with a $B$-field turned on. The BPS mass spectrum becomes

$$m^2 = \sum_{i=1,2} \left( \frac{1}{L_i^2} (k_i - \theta \epsilon_{ij} w_j)^2 + L_i^2 w_i^2 \right) + \frac{T_{str}}{g_{str}^2} \sum \left( (Q_0 - \theta Q_2)^2 + (T_{str} L_1 L_2 Q_2)^2 \right)$$

(1)

where $Q_{0,2}$ are, respectively, D0- and D2-brane charges and $k_i, w_i$ are momentum and winding modes. In the presence of $\theta$, the 2-brane picks up 0-brane charge and the winding mode picks up momentum. In the relevant limit $L_i \rightarrow 0$, the winding modes are the lightest modes. In a field theory description, one wishes to identify these modes with momentum modes on a dual torus.

For rational $\theta$, we can describe the system by a particular sector in a $2+1$ SYM theory; however, this description is ergodic in $\theta$. Furthermore, for irrational $\theta$ there is no conventional Yang-Mills description.

In the limit $L_i \rightarrow 0$, the Kähler form has a vanishing imaginary part, but retains a non-zero real part. This particular degeneration of the torus has been shown to be
best described by non-commutative geometry. The winding modes are still described as momenta on the dual torus, but this torus has to be taken to be non-commutative in order to reproduce the BPS formula above.

In what follows, we consider compactification on $T^5$. Clearly, if $A^{(3)}$ is of minimal rank, say $A^{(3)}_{12} \neq 0$, then the discussion will follow the above—we will get SYM theories on $T^g \times T^3$. There are ten such configurations possible (later, we will label these by $\theta_{ij}$)—the generic configuration will give a general non-commutative $T^5$. However, clearly this is only a low-energy description and we will suggest an ultra-violet definition of this theory.

### 3 Backgrounds for $M(T^5)$

In the following we discuss gauge theories in 5+1 dimensions, compactified on a 5-torus. As non-renormalizable field theories, they need to be defined in the ultraviolet. We use their description as the low energy limit of the six-dimensional non-critical string theories[5]. We concentrate mainly on two deformations of these theories. One of them is visible in the low energy SYM as the operator $\theta \int Tr(F^3)$, recently discussed in Refs. [14, 15]. The other is a formulation of the gauge theory on a non-commutative 5-torus[12, 13]. These deformations appear very differently in the low energy SYM theory, but are in fact closely related in the non-critical string description.

The main tool we use to investigate the 6-dimensional string theories, compactified on a torus, is their interpretation as Matrix theories for $M(T^5)$. The Matrix description can be derived directly[9, 10], giving the worldvolume theory of $N$ coincident NS5-branes of type IIB string theory. The bulk string theory has a finite string tension, and a vanishing (asymptotic) string coupling, exactly the limit used before to define the gauge theory[5].

The resulting theory is a non-gravitational string theory in 6 dimensions. Since it lacks an intrinsic definition, we have to resort to indirect descriptions of it. One such description is the above embedding in type IIB string theory. Though it is decoupled from the bulk string theory, the intrinsic 5-brane theory can inherit some properties of the bulk theory, provided they are protected from quantum corrections. One such property, the T-duality group $SO(5,5)$, serves as the U-duality group of $M(T^5)$ [4, 5]. Similarly, masses of BPS saturated excitations can be trusted [5], though their bulk interpretation (as bound states of the NS 5-brane with various D-branes) might change in the intrinsic theory.

We turn now to describe the Matrix definition of general points of the moduli space of M-theory on $T^5$. Moduli associated with purely transverse directions have been described before [3, 6], and in the following we concentrate on “longitudinal” moduli. There are 16 such moduli (transforming in the spinor representation of the U-duality group): these are the off-diagonal metric elements (angles) $g_{ij}$ and Wilson lines (constant background values) for the 3-form $A^{(3)}_{ij}$ and for the 6-form of M-theory,

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3 A Matrix definition of the non-critical string theories, and several field theory limits of them, are discussed in Refs. [17].
One can systematically derive the Matrix description along the lines of [9, 10]. Matrix theory on $T^5$ is defined by the system of $N$ 0-branes in type IIA theory with the following parameters:

\[ T_{str} = \frac{R\gamma}{\ell_p^3} \]  \hspace{1cm} (2)
\[ g_{str} = \left( \frac{R}{\gamma\ell_p} \right)^{3/2} \]  \hspace{1cm} (3)
\[ L_i = \frac{R_i}{\gamma} \]  \hspace{1cm} (4)

where $\ell_p, R, R_i$ are, respectively, the 11-dimensional Planck length, the lightcone periodicity, and the transverse $T^5$ lengths, all in the IMF. They are kept finite as the boost parameter $\gamma$ is taken to infinity. The $T_{str}, g_{str}, L_i$ are parameters of the type IIA string theory in which the 0-brane system is embedded. In this frame, the longitudinal moduli are respectively Wilson lines for the RR 1-form, $\theta_i$, background NS 2-forms $\theta_{ij}$ and a Wilson line for the RR 5-form, $\theta$.

Since the 5-torus is small, one needs to perform T-duality to transform it to a 5-torus of a finite volume. The parameters $\theta_{ij}$, appearing in the type IIA theory as 2-form backgrounds, would prevent, in a conventional treatment, getting a finite volume torus using T-duality. As explained in [12, 13] and reviewed above, interpreting the parameters $\theta_{ij}$ in the framework of non-commutative geometry allows us to describe the system as a SYM theory on a finite size, albeit non-commutative, torus.

The resulting type IIB theory, then, has the following parameters:

\[ l_i = \frac{\ell_p^3}{RR_i} \]  \hspace{1cm} (5)
\[ T_{str} = \frac{R\gamma}{\ell_p^3} \]  \hspace{1cm} (6)
\[ \tau \equiv \frac{i}{g_{str}} + \frac{\theta_B}{2\pi} = \frac{iRR_1R_2R_3R_4R_5}{\ell_p^6\gamma} + \theta \]  \hspace{1cm} (7)
\[ A_{ijkl}^{(4)} = \epsilon_{ijklm}\theta_m. \]  \hspace{1cm} (8)

The resulting Matrix description is the system of $N$ Dirichlet 5-branes of type IIB theory with infinite string coupling and tension. This is an alternative definition of the non-critical string theory and its low energy SYM limit. In the SYM the new moduli appear as follows:

- $\theta$ is the coefficient of the operator $\int Tr(F^3)$. In fact, we have reached exactly the system used by Kol [15] to define the generalization of the $(p, q)$ 5-brane theories of [14].

- $\theta_{ij}$ are the parameters of the non-commutative torus on which the SYM is formulated [12].
\[ \theta_i \text{ are coefficients of operators } \int Tr(F_{0i}), \text{ the integral being over 1-cycles of the 5-torus (and time).} \]

This relation between the SYM couplings and the longitudinal moduli is deduced from the well-known coupling of D-branes to RR backgrounds. As is shown below, these parameters are related to masses of BPS saturated states, and are therefore protected from quantum corrections.

We note that in the presence of a non-vanishing \( \theta \) one cannot, in general, use the \( SL(2, \mathbb{Z}) \) duality of type IIB string theory to make the bulk modes weakly interacting. For a rational \( \theta \), this is possible, yielding the \((p, q)\) theories of \([14]\). However, in this case the description becomes discontinuous in \( \theta \), as was noted in \([14]\).

Thus, for a generic non-zero \( \theta \), the SYM cannot be realized as the world-volume theory of any 5-brane in a weakly interacting string theory. We are led then to define these theories, as in \([14]\), as the world-volume theories of D5-branes in a strongly interacting type IIB theory. It would be interesting to understand the detailed mechanism of decoupling from a strongly interacting bulk theory. This could be of interest for studying low-dimensional compactifications of Matrix theory.

The situation is identical to the case considered in section 2. Indeed, describing the configuration in M-theory, we get \( N \) M5-branes wrapping the long cycle of a 2-torus. In the limit relevant here, the torus complex structure has a vanishing imaginary part and a non-zero real part. This is exactly the degeneration of the 2-torus which led naturally to the introduction of non-commutative geometry on the base space \([12, 13]\). In our case, lacking a clear intrinsic description of the non-critical string theory, it is difficult to make an analysis similar to \([12]\).

We end this section by commenting about the relation between the SYM on the non-commutative torus \([12, 13]\), and the SYM theories with \( \theta \) angle. As in any string theory, the base space geometry (commutative or not) is only a low energy artifact. In the non-critical string theory the parameters \( \theta \) and \( \theta_{ij} \) are T-dual to each other. It is only when going to a particular limit of parameter space of the string theory that a particular parameter manifests itself as a \( \theta \) angle or as a geometrical parameter. It is unclear to us if the non-commutativity introduced by these parameters is a low energy statement, or has a deeper connection to the formulation of the non-critical string theory itself.

### 4 Shifts in BPS charges

In this section we discuss the BPS spectrum of the non-critical string theories defined above, in the presence of the parameters \( \theta, \theta_i, \theta_{ij} \). This demonstrates in detail the T-duality transformations relating these parameters. The effect of the parameters is to shift the charges of BPS saturated states, similar to the well-known Witten effect \([19]\).

For the sake of simplicity of notation we consider only the case of one modulus turned on at a time. Iterating the shifts described below gives the more general situation.

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4 Changing in this definition the D5-brane to a NS5-brane, or to any \((p,q)\) 5-brane, only redefines the parameter \( \theta \). The bulk theory is still strongly interacting. We fix the relation of the gauge theory parameters to the spacetime parameters by using D5-branes in the above definition.
The central charges of the system are all visible in the embedding of the system in type IIB string theory. Some of them have a description in the SYM theory, or even as non-critical string excitations. The central charges are the following:

- $m_i$: In the bulk this is the winding number of the fundamental string; it appears in the SYM theory as an electric flux.

- $m_{ij}$: In the bulk this is a 3-brane charge in the 3-plane transverse to $(ij)$. In the SYM theory this is a magnetic flux in the $(ij)$ 2-plane.

- $m_{12345}$: In the bulk this is the NS5-brane charge. It has a finite energy density in 5+1 dimensions, therefore it is invisible to the SYM theory [5].

- $N$: the number of D5-branes in the bulk. This is the rank of the gauge group, or its non-commutative generalization (dimension) in the SYM theory.

All these central charges are expected to be part of the spectrum of the non-critical string theory. The only charges that are transparent in the non-critical string frame are:

- $k_i$: This is the momentum conjugate to translations on the 5-torus, from each point of view (SYM, non-critical string, type IIB).

- $w_i$: This is the D1-brane number in the bulk. In the 5-brane theory it is a winding number of the non-critical string, and appears in the SYM as the instanton number, $I_{ijkl}$, in the 4-plane transverse to $i$.

Turning on a single modulus corresponds to moving in an $SL(2, \mathbb{R})$ subgroup of the moduli space of $M(T^5 \times S^1_{LC})$. One then arranges these central charges into representations of the particular $SL(2, \mathbb{R})$ involved, and borrows the results from [19]. The “magnetic” charge is left unchanged, whereas the “electric” charge is shifted.

One needs to decide which is the magnetic, and which is the electric charge. The former is the charge which, in the limit involved, appears as a solitonic, topologically quantized, charge. In order to make this identification we use the spacetime interpretation of the charges involved, via Matrix theory.

Using this assumption we get the following results:

- When turning on the backgrounds $\theta_{ij}$, we get the following shifts in the charges which appear in the BPS mass formula (we use $i, j, k, l, m$ as a set of cyclic indices of the 5-torus coordinates):
  
  - $N \rightarrow N - \theta_{ij} m_{ij}$.
  
  - $m_i \rightarrow m_i - \theta_{ij} k_j$.

These effects were observed in [12], and are explained by the non-commutative geometry of the base space. We note that these shifts are not expected to be seen in a conventional gauge theory. Turning on $\theta$-like terms indeed cause mixing between charges in the gauge theory. However, the charges that are
“fundamental” shift, while the “solitonic” charges stay fixed. From the above formulas we see that the fixed charges are $k_i, m_{ij}$, while the charges that shift are $N, m_i$. In that sense we are working in a “dual” formulation of the theory where the “fundamental” charges are $N, m_i$, which are solitonic in the usual semi-classical formulation of the gauge theory.

The relation to spacetime, as well as the intrinsic T-duality predicts one more shift:

\[- m_{kl} \rightarrow m_{kl} - \theta_{ij} I_{ijkl} = m_{kl} - \epsilon_{ijklm} \theta_{ij} w_m \]

**•** When turning on the backgrounds $\theta_i$ one gets:

\[- N \rightarrow N - \theta_i m_i. \]
\[- m_{ij} \rightarrow m_{ij} - \theta_i k_j. \]
\[- m_{12345} \rightarrow m_{12345} - \theta_i w_i. \]

As above, these are not the usual shifts in the gauge theory. For example the usual treatment of the term $\theta_i \int Tr(F_{0i})$ would shift the electric fluxes $m_i$, while keeping $N$ fixed.

**•** Finally, when turning on $\theta$, one gets:

\[- N \rightarrow N - \theta m_{12345}. \]
\[- m_i \rightarrow m_i - \theta w_i. \]

The first shift is invisible to the SYM theory. The second is an electric flux carried by instantons in the presence of the $\theta \int Tr(F^3)$ term.

We may summarize all of these shifts by arranging them into representations of the T-duality group, $SO(5, 5)$. We have the singlet $N$, a spinor $M = (m_i, m_{ij}, m_{12345})$, a vector $\Phi = (w_i, k_i)$, and an (anti)spinor $\bar{\Theta} = (\theta_i, \theta_{ij}, \theta)$. The shifts are then the $SO(5, 5)$ covariant expressions

\[
M \rightarrow M - \Phi \cdot \bar{\Theta} \quad (9)
\]
\[
N \rightarrow N - \bar{\Theta} M \quad (10)
\]

It should be possible to realize $\theta, \theta_i, \theta_{ij}$ as parameters of the intrinsic 5-brane theory. A hint, and a constraint on any such formulation are the shifts above. We note that the BPS masses vary continuously with $\Theta$.

To summarize, we have discussed an ultra-violet definition of gauge theories in $5 + 1$-dimensions. These include gauge theories on non-commutative tori, as well as those with an arbitrary $\theta$-angle. These theories can be defined in terms of the non-critical string theories in six dimensions. An intrinsic definition of these theories is lacking, but we are able to define them formally in terms of their embedding in a particular background of type IIB string theory. This embedding enables us to identify the BPS spectra of these theories.

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