Binary Darboux-Bäcklund Transformations for the Manin-Radul Super KdV Hierarchy

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Abstract

We construct the supersymmetric extensions of the Darboux-Bäcklund transformations (DBTs) for the Manin-Radul super KdV hierarchy using the super-pseudo-differential operators. The elementary DBTs are triggered by the gauge operators constructed from the wave functions and adjoint wave functions of the hierarchy. Iterating these elementary DBTs, we obtain not only Wronskian type but also binary type superdeterminant representations of the solutions.
I. INTRODUCTION

In the past few decades, the Darboux-Bäcklund transformations (DBTs) have been shown to be an efficient method to obtain the soliton solutions of the nonlinear integrable systems including Korteweg-de Vries (KdV), Kadomtsev-Petviashili (KP), sine-Gordon and nonlinear Schrödinger equations (see, for example Ref. [1]) etc. In contrast to these results, the application of the DBTs to super integrable systems is quite limited. The reasons are partly due to the complexity of the supersymmetric formulation of super integrable systems. However, as far as the super soliton solutions are concerned, the supersymmetric generalization of the DBTs is urgently needed.

Recently, Liu [2] proposed a DBT for the Manin-Radul super KdV (MR sKdV) equation [3]. The DBT is triggered by a gauge operator which is a first order super-differential operator (SDO) parametrized by a wave function of the associated linear system. By iterating such DBT, Liu and Mañas [4] obtained the supersymmetric version of the Crum transformation for the MR sKdV equation and showed that the solutions can be expressed in terms of Wronskian superdeterminants.

In this paper, we will investigate the DBTs for the MR sKdV hierarchy from super-pseudo-differential operator (SΨDO) point of view. Motivated by the non-supersymmetric cases [1,5–12] and the works described above, we introduce the adjoint DBT which is triggered by an adjoint wave function of the system. This adjoint DBT in combination with the previous one form a binary DBT which is constructed from a wave function and an adjoint wave function. Iterating these elementary DBTs, we obtain not only Wronskian type but also binary type superdeterminant representations for the solutions of the MR sKdV hierarchy which have not been obtained previously.

Our paper is organized as follows: In Sec. II, some basic facts about SΨDO are recalled and the MR sKdV hierarchy is defined. In Sec. III, we introduce the elementary DBTs which are triggered by the gauge operators constructed from the wave functions and the adjoint wave functions of the MR sKdV system. In Sec. IV, we iterate these DBTs to obtain the solutions of the MR sKdV hierarchy. Concluding remarks are presented in Sec. V.

II. PRELIMINARIES

Before considering a specific super integrable system, let us recall some basic facts of the SΨDO which is defined by

$$\Lambda = D^N + \sum_{i=-\infty}^{N-1} U_i D^i \quad N \in \mathbb{Z}$$  \hspace{1cm} (2.1)

where the supercovariant derivative $D \equiv \partial_\theta + \theta \partial$ satisfies $D^2 = \partial$, $\theta$ is the Grassmann variable ($\theta^2 = 0$) which, together with the even variable $x \equiv t_1$, defines the (1|1) superspace with coordinate $(x, \theta)$. The formal inverse of $D$ is introduced by $D^{-1} = \theta + \partial_\theta \partial^{-1}$ which satisfies $DD^{-1} = D^{-1}D = 1$. The coefficients $U_i$ are superfields that depend on the variables $x, \theta$, and $t_i$ and can be represented by $U_i = u_i(t) + \theta v_i(t)$. Since the SΨDO is assumed to be homogeneous under $Z_2$-grading, the grading of the superfield $U_i$ is $|U_i| = N + i \pmod{2}$. 
Here we refer the parity of $U_i$ to be even if $|U_i| = 0$ and odd if $|U_i| = 1$. The supercovariant derivative $D$ satisfies the supersymmetric version of the Leibniz rule [3]:

$$D^i U = \sum_{k=0}^{\infty} (-1)^{|U|^{i-k}} \binom{i}{k} D^k U D^{i-k}$$

(2.2)

where the super-binomial coefficients $\binom{i}{k}$ are defined by

$$\binom{i}{k} = \begin{cases} \binom{i/2}{k/2} & \text{for } 0 \leq k \leq i \text{ and } (i, k) \neq (0, 1) \text{ mod } 2 \\ (-1)^{i+k/2} \binom{i+k-1}{k} & \text{for } i < 0 \\ 0 & \text{otherwise} \end{cases}$$

(2.3)

It is convenient to separate $\Lambda$ into the positive and negative parts as follows:

$$\Lambda_+ = \sum_{i \geq 0} U_i D^i, \quad \Lambda_- = \sum_{i \leq -1} U_i D^i.$$  

(2.4)

The super-residue (sres) of $\Lambda$ is defined by

$$\text{sres}(\Lambda) = U_{-1}$$

(2.5)

and the conjugate operation “*” for SΨDOs can be defined as follows

$$(PQ)^* = (-1)^{|P||Q|} Q^* P^*$$

(2.6)

which implies that $U^* = U$ for arbitrary superfield $U$ and

$$(\partial^i)^* = (-1)^i \partial^i; \quad (D^i)^* = (-1)^{(i+1)/2} D^i.$$  

(2.7)

A simple calculation shows that

$$\Lambda^* = \sum_i (-1)^N (-1)^{(i-1)/2} D^i U_i$$

(2.8)

With these definitions in hand, let us provide some useful identities which will simplify the computations involving compositions of SΨDOs.

**Lemma 1 :**

$$\begin{align*}
  (\Lambda^*)_+ &= (\Lambda_+)^* \quad (\Lambda^*)_- = (\Lambda_-)^* \\
  (D^{-1} \Lambda)_- &= D^{-1}(\Lambda^*)_0 + D^{-1}(\Lambda)_- \\
  (\Lambda D^{-1})_- &= (\Lambda_0 D^{-1} + \Lambda_- D^{-1} \\
  \text{sres}(\Lambda) &= \text{sres}(\Lambda^*) \quad (D \text{sres}(\Lambda)) = \text{sres}(DA - (-1)^{|A|} A D) \\
  \text{sres}(\Lambda D^{-1}) &= (\Lambda_0^* D^{-1} = (\Lambda_0 D^{-1})_0 \\
  \text{sres}(D^{-1} \Lambda_1 \Lambda_2 D^{-1}) &= \text{sres}(D^{-1} \Lambda_1 D^{-1} \Lambda_2 D^{-1}) + \text{sres}(D^{-1} \Lambda_1 D^{-1} \Lambda_2 D^{-1})
\end{align*}$$

(2.9) - (2.14)

where $\Lambda_1 = (\Lambda_1)_+$ and $\Lambda_2 = (\Lambda_2)_+$.

**Proof.** The proofs for these identities are straightforward. Here we only give the proof for the second identity. From the left-hand side, we have
\[(D^{-1} \Lambda)_- = (D^{-1} \Lambda_+)_- + D^{-1} \Lambda_- \tag{2.15}\]

Then using (2.2), we obtain
\[
(D^{-1} \Lambda_+)_- = \left( \sum_{k=0}^{N} \sum_{i=0}^{N} (-1)^{(N+i)(1+k)} \left[ \begin{array}{c} -1 \\ k \end{array} \right] (D^k U_i) D^{-1-k+i} \right)_-
\]
\[
= \sum_{k=0}^{N} \sum_{i=0}^{N} (-1)^{(N+i)(1+i+l)} (-1)^{(i+l)/2} (D^{i+l} U_i) D^{-1-l}
\]
\[
= \sum_{l=0}^{N} (-1)^{(N+l)} (-1)^{(l/2)} (D^l (\Lambda^*)_0) D^{-1-l}
\]
\[
= D^{-1} (\Lambda^*)_0 \tag{2.16}\]

here the relation \((-1)^{(l/2)} = (-1)^{(i-1)/2}\) has been used to reach the third line of (2.16).\square

The MR sKdV hierarchy is defined, in Lax form, as
\[
\partial_t n L = [P_n, L] \quad n = 1, 3, 5 \ldots \tag{2.17}\]
with
\[
L = \partial^2 + v_1 D + v_0 \tag{2.18}\]
\[
P_n = L_n^{n/2} \tag{2.19}\]
where the coefficients \(v_1\) and \(v_0\) are superfields depending on the variables \((\theta, x, t_3, t_5, \ldots)\) with grading \(|v_i| = i \pmod{2}\). We can rewrite the hierarchy equations (2.17) as follows:
\[
\partial_t m P_n - \partial_t n P_m + [P_n, P_m] = 0 \tag{2.20}\]
which is called the zero-curvature condition and is equivalent to the whole set of equations of (2.17). If we can find a set of superfields \(\{v_1, v_0\}\) and hence a corresponding set of super-differential operator (SDO) \(\{P_n\}\) satisfying (2.20), then we have a solution to the MR sKdV hierarchy. In fact, the MR sKdV hierarchy was constructed originally from the MR sKP hierarchy \[\text{[4]}\] associated with the odd SΨDO: \(\Lambda_{MR} = D + \sum_{i=0}^{\infty} U_i D^{-i}\) by the reduction \(L = (\Lambda_{MR}^n)_+\) and has been shown to be bi-Hamiltonian \[\text{[13,14]}\]. Substituting (2.18) and (2.19) into the Lax equation (2.17) for \(n = 3\), one obtains
\[
\partial_{t_3} v_0 = \frac{1}{4} \partial_x (v_{0xx} + 3 v_0^2 + 3 v_1 (D v_0)) \tag{2.21}\]
\[
\partial_{t_3} v_1 = \frac{1}{4} \partial_x (v_{1xx} + 3 v_1 (D v_1) + 6 v_1 v_0) \tag{2.22}\]
which is the MR sKdV equation. By setting \(v_1 = 0\), Eqs. (2.21)-(2.22) reduce to the KdV equation. The Lax equation (2.17) can be viewed as the compatibility condition of the linear system
\[
L \phi = \lambda \phi, \quad \partial_t n \phi = (P_n \phi)_0 \tag{2.23}\]
where \(\phi\) and \(\lambda\) are called wave function and spectral parameter of the hierarchy, respectively. On the other hand, we can also introduce adjoint wave function \(\psi\) which satisfies the linear system
\[ L^* \psi = \eta \psi, \quad \partial_{t_n} \psi = -((P_n)^* \psi)_0 \]  

(2.24)

For convenience, throughout this paper, \( \phi \) and \( \psi \) will stand for wave function and adjoint wave function, respectively. Of course, it should be realized that both \( \phi \) and \( \psi \) are superfields in this formalism. In next section, we will use these (adjoint) wave functions to construct the DBTs for the MR sKdV hierarchy.

### III. THE ELEMENTARY DB TRANSFORMATIONS

We consider the following transformation:

\[ L \to \hat{L} = TLT^{-1} \]  

(3.1)

where \( T = T(\theta, x, t_3, t_5, \cdots) \) is any reasonable SΨDO. To guarantee that such transformation can generate new solutions of the MR sKdV hierarchy, the transformed Lax operator \( \hat{L} \) should preserve the form of (2.18) and satisfy the Lax equation (2.17). It is easy to show that under the transformation (3.1), the operator \( P_n \) then is transformed as

\[ P_n \to \hat{P}_n = TP_nT^{-1} + \partial_{t_n} TT^{-1}. \]  

(3.2)

which preserves the zero curvature condition (2.20). Note that although \( P_n \) is a SDO, the right hand side of (3.2) will in general not be a purely SDO. However if we suitable choose the gauge operator \( T \) such that \( \hat{P}_n \), as defined by (3.2), is also a purely SDO, then \( \{ \hat{L}, \hat{P}_n \} \) represents a valid new solution to the MR sKdV hierarchy. To formulate the DBTs of the MR sKdV hierarchy, let us introduce a superfield \( \Omega \) called bi-linear potential [6] (or squared eigenfunction potential) which is constructed from a wave function and an adjoint wave function and will be useful later on.

**Lemma 2:** For any pair of \( \phi \) and \( \psi \), there exists a bi-linear potential \( \Omega(\psi, \phi) \) satisfying

\[ (D \Omega(\psi, \phi)) = \psi \phi, \]  

(3.3)

\[ \partial_{t_n} \Omega(\psi, \phi) = \text{sres}(D^{-1}_n P_n \phi D^{-1}_n), \]  

(3.4)

**Proof.** To prove the existence of the bi-linear potential \( \Omega \), we have to show that Eqs. (3.3) and (3.4) are compatible. Firstly, we note that (3.3) is consistent with (3.4) for \( n = 1 \). Since

\[ \Omega_x = \text{sres}(D^{-1}_1 \psi P_1 \phi D^{-1}_1) = \text{sres}(D^{-1}_n \psi D^2 \phi D^{-1}_n) = (D \psi) \phi + (-1)^{|\phi|} \psi (D \phi) = (D (D \Omega)). \]  

(3.5)

This implies that

\[ (D \Omega)_x = (D(D(D \Omega))) = (D \Omega)_x. \]  

(3.6)

In general, from (3.3) and (3.4) we have
\[(D\partial_{tn}\Omega) = (Ds_{res}(D^{-1}\psi P_n\phi D^{-1})) = s_{res}(\psi P_n\phi D^{-1} - (-1)^{\psi||\phi} D^{-1}\psi P_n\phi) = \psi(P_n\phi) - (-1)^{\psi||\phi}\phi(P_n^*\psi) = \psi\phi_{tn} + \psi_{tn}\phi = \partial_{tn}(D\Omega). \quad (3.7)\]

Secondly, from (3.4) and (2.14) we have
\[
\partial_{tm}(\partial_{tn}\Omega) - \partial_{tn}(\partial_{tm}\Omega) = s_{res}\partial_{tm}(D^{-1}\psi P_n\phi D^{-1}) - s_{res}\partial_{tn}(D^{-1}\psi P_m\phi D^{-1}) - s_{res}(D^{-1}\psi(\partial_{tm}P_m - \partial_{tn}P_m)\phi D^{-1}) - s_{res}(D^{-1}\psi(\partial_{tn}P_m - \partial_{tm}P_m + [P_n, P_m])\phi D^{-1}) = 0 \quad (3.8)
\]

here the zero-curvature condition of the MR sKdV hierarchy has been used in the last line.$\square$

**Remarks.** $\Omega(\psi, \phi)$ is a superfield with parity $(-1)^{\psi||\phi} + 1$ and can be represented by $\Omega(\psi, \phi) = f^x\psi_2\phi_1 + (-1)^{\psi} f^x\psi_1\phi_2 + \theta\psi_1\phi_1$ where $\phi = \phi_1 + \theta\phi_2$ and $\psi = \psi_1 + \theta\psi_2$.

**Proposition 1:** Let $\chi$ be an even wave function of the linear system, then the gauge operator
\[
T = \chi D\chi^{-1} = D + \alpha \quad \alpha \equiv -\frac{(D\chi)}{\chi} \quad (3.9)
\]

triggers the following DBT:
\[
\hat{L} = TLT^{-1} = \partial^2 + \hat{\nu}_1D + \hat{\nu}_0, \quad (3.10)
\]
\[
\hat{\phi} = (T\phi) = \chi(D\chi^{-1}\phi), \quad (3.11)
\]
\[
\hat{\psi} = ((T^{-1})^{\phi}) = \chi^{-1}\Omega(\psi, \chi), \quad (3.12)
\]

where the transformed coefficients are given by
\[
\hat{\nu}_1 = -\nu_1 - 2\alpha x, \quad (3.13)
\]
\[
\hat{\nu}_0 = \nu_0 + (D\nu_1) + 2\alpha(\nu_1 + \alpha x). \quad (3.14)
\]

**Proof.** We first show that $\hat{L}$ has the same form as $L$. From (3.10) we have
\[
\hat{L} = (\chi D\chi^{-1}L\chi D^{-1}\chi^{-1}) = \chi(D\chi^{-1}(L\chi))D^{-1}\chi^{-1} = 0. \quad (3.15)
\]

which implies that $\hat{L}$ is a SDO. A simple calculation shows that the transformed coefficients $\hat{\nu}_1$ and $\hat{\nu}_0$ are given by (3.13) and (3.14), respectively. Furthermore, the hierarchy equation for $\hat{L}$ is given by
\[
\partial_{tn}\hat{L} = [TP_nT^{-1} + \partial_{tn}TT^{-1}, \hat{L}]. \quad (3.16)
\]

Using the identity
\[
\chi D\chi^{-1}A_+\chi D^{-1}\chi^{-1} = (\chi D\chi^{-1}A_+\chi D^{-1}\chi^{-1})_+ + \chi(D\chi^{-1}(A_+\chi_0)D^{-1}\chi^{-1} \quad (3.17)
\]
and then substituting $\Lambda = L^{n/2}$ ($n$ odd), we obtain

$$\hat{P}_n = T P_n T^{-1} + \partial_n TT^{-1} = (T L^{n/2} T^{-1})_+ + \chi(D\chi^{-1} \partial_n \chi) D^{-1} \chi^{-1} + \partial_n TT^{-1} = \hat{L}_+^{n/2}$$

Hence

$$\partial_n \hat{L} = [\hat{L}_+^{n/2}, \hat{L}]$$  \hspace{1cm} (3.19)

The evolution equations for $\hat{\phi}$ and $\hat{\psi}$ can be verified in a similar way. From (3.11), we have

$$\partial_n \hat{\phi} = \partial_n T \phi + T \partial_n \phi = \partial_n TT^{-1} \hat{\phi} + TP_n T^{-1} \hat{\phi} = (\hat{P}_n \hat{\phi})_0.$$  \hspace{1cm} (3.20)

Similarly,

$$\partial_n \hat{\psi} = \partial_n (T^{-1})^* \psi + (T^{-1})^* \partial_n \psi = ((T^{-1})^* (\partial_n T)^* + (T^{-1})^* P_n T^*) \hat{\psi} = -(\hat{P}_n^* \hat{\psi})_0$$  \hspace{1cm} (3.21)

Having described the first construction of the DBT, we now turn to another construction using adjoint wave function.

**Proposition 2:** Let $\mu$ be an even adjoint wave function of the linear system (2.24), then the gauge operator

$$S = \mu^{-1} D^{-1} \mu = (D - \beta)^{-1}, \quad \beta \equiv -\frac{(D \mu)}{\mu}$$

triggers the following adjoint DB transformation:

$$\hat{L} = SLS^{-1} = \partial^2 + \hat{v}_1 D + \hat{v}_0,$$

$$\hat{\phi} = (S \phi) = \mu^{-1} \Omega(\mu, \phi),$$

$$\hat{\psi} = ((S^{-1})^* \psi) = -\mu (D \mu^{-1} \psi),$$

where

$$\hat{v}_1 = -v_1 + 2\beta_x,$$

$$\hat{v}_0 = v_0 + (Dv_1) - 2(D\beta)_x - 2\beta(v_1 - \beta_x).$$

**Proof.** The proof is similar to the Proposition 1. Here we only mention that Eq.(3.19) can be verified easily by using the identity

$$(\mu^{-1} D^{-1} \mu \Lambda \mu^{-1} D \mu)_- = \mu^{-1} D^{-1} \mu \Lambda_- \mu^{-1} D \mu - (-1)^{|\Lambda|} \mu^{-1} D^{-1} (D \mu^{-1} (\Lambda^*_+ \mu)_0 ) \mu$$

instead of (3.17). \hspace{1cm} □

**Proposition 3:** For any pair of even wave function $\chi$ and odd adjoint wave function $\mu$, the gauge operator

$$R = 1 - \chi \Omega(\mu, \chi)^{-1} D^{-1} \mu$$

triggers the following binary DB transformation:
\[ \hat{L} = RLR^{-1} = \partial^2 + \hat{v}_1 D + \hat{v}_0, \quad (3.30) \]
\[ \hat{\phi} = (R\phi) = \phi - \chi \Omega(\mu, \chi)^{-1}\Omega(\mu, \phi), \quad (3.31) \]
\[ \hat{\psi} = ((R^{-1})^* \psi) = \psi - \mu \Omega(\mu, \chi)^{-1}\Omega(\psi, \chi), \quad (3.32) \]

where
\[ \hat{v}_1 = v_1 - 2\Gamma_x \quad \Gamma \equiv \frac{(D\Omega)\Omega}{\Omega} \quad (3.33) \]
\[ \hat{v}_0 = v_0 + 2(D\Gamma)_x + 2v_1\Gamma - 2\Gamma \alpha_x + 2\alpha \Gamma_x + 2\Gamma \Gamma_x. \quad (3.34) \]

Proof. The gauge operator \( R \) is just a composition of the DB transformation \( T \) and the adjoint DB transformation \( S \). To see this, let us first perform a DB transformation triggered by the wave function \( \chi \). The odd adjoint wave function \( \mu \) is thus transformed to \( \hat{\mu} = \chi^{-1}\Omega(\mu, \chi) \), which is an even one. Then a subsequent adjoint DB transformation \( S \) triggered by \( \hat{\mu} \) is performed and the composition of these two transformations is given by
\[ R = \hat{\mu}^{-1}D^{-1}\hat{\mu}\chi D\chi^{-1} = \chi^{-1}D^{-1}\Omega D\chi^{-1} = 1 - \chi^{-1}D^{-1}\mu. \]

Therefore, the remaining part of the proof is just a corollary of the Propositions 1 and 2.

Remarks. The binary DB transformation can also be constructed from a pair of \( \chi \) and \( \mu \) with \( |\chi| = 1 \) and \( |\mu| = 0 \). A direct calculation shows that
\[ R = \hat{\chi} D \hat{\chi}^{-1} \mu^{-1} D^{-1} \mu = 1 - \chi^{-1}D^{-1}\mu \]
which has the same form as (3.29). We also note that the transformed coefficients (3.33) and (3.34) are just the ones in Ref. [2] where a different approach was presented.

IV. ITERATION OF THE DB TRANSFORMATIONS

We have introduced the elementary DBTs which contain the DBT, the adjoint DBT, and the binary DBT triggered by the gauge operators \( T \), \( S \), and \( R \), respectively. Using these elementary transformations as the building blocks, the more complicated transformations can be constructed from the compositions of these gauge operators. In Ref. [4], by iterating the DB transformation \( T \), the so-called Crum transformation for the MR sKdV equation was constructed and the Wronskian superdeterminant representations for the solutions were obtained. This construction starts with \( n \) wave functions \( \chi_i, i = 0, \cdots, n - 1 \) of the linear system (2.23) with parity \((-1)^i\). Using \( \chi_0 \), say, to perform the first DBT of Proposition 1, then \( \chi_i \) are transformed to \( \hat{\chi}_i \). It is obvious that \( \hat{\chi}_0 = 0 \). The next step is to perform a subsequent DBT triggered by \( \hat{\chi}_1 \), say, which leads to the new wave functions \( \hat{\hat{\chi}}_i \) with \( \hat{\hat{\chi}}_1 = 0 \). Iterating this process such that all the wave functions are used up, then an \( n \)-step DBT with gauge operator \( T_n \) is obtained.

Proposition 4: [4] Let \( \chi_i, i = 0, \cdots, n - 1 \) be wave functions of the linear system (2.23) with parity \((-1)^i\), then after \( n \) iterations of the DBT of Proposition 1, the transformed Lax operator becomes
\[ \hat{L} = T_n L T_n^{-1} = D^n + \sum_{i=0}^{n-1} a_i D^i \quad (4.1) \]
\[ = \partial^2 + \hat{v}_1 D + \hat{v}_0 \quad (4.2) \]
where the coefficients \( a_i \) are defined by
\[ T_n \chi_j = 0 \quad j = 0, \cdots, n - 1 \quad (4.3) \]

and
\[
\begin{align*}
\hat{\nu}_1 &= (-1)^n v_1 - 2(a_{n-1})_x \\
\hat{\nu}_0 &= v_0 - 2(a_{n-2})_x - a_{n-1}((-1)^n v_1 + \hat{\nu}_1) + \frac{1 - (-1)^n}{2}(Dv_1) 
\end{align*} \quad (4.4) \]

\[
\begin{align*}
\hat{\nu}_0 &= v_0 - 2(a_{n-2})_x - a_{n-1}((-1)^n v_1 + \hat{\nu}_1) + \frac{1 - (-1)^n}{2}(Dv_1) 
\end{align*} \quad (4.5) \]

\textbf{Remarks.} Due to the fact that the parity of the gauge operator \( T_n \) depends on \( n \), the cases for \( n \) being even or odd are rather different. In Ref. [4], the coefficients \( a_i \) for both cases have been obtained separately by solving the linear equation (4.3) and the transformed fields \( \hat{\nu}_1 \) and \( \hat{\nu}_0 \) can be expressed in a compact form in terms of superdeterminant. We will present such calculation in a more general case involving the binary DBTs.

Now let us turn to a similar construction using the adjoint wave functions.

\textbf{Proposition 5 :} Let \( \mu_i, i = 0, \cdots, n - 1 \) be adjoint wave functions of the linear system (2.24) with parity \((-1)^i\), then after \( n \) iterations of the adjoint DBT of Proposition 2, the transformed Lax operator becomes
\[
\hat{L} = R_n LS_n^{-1} \quad (S_n^{-1})^* = (-1)^{[n/2]}(D^n + \sum_{i=0}^{n-1} b_i D^i) 
\quad (4.6) \]
\[
\hat{L} = \hat{\nu}_1 D + \hat{\nu}_0 \quad (4.7) \]

where the coefficients \( b_i \) are defined by
\[
(S_n^{-1})^* \mu_j = 0 \quad j = 0, \cdots, n - 1 \quad (4.8) \]

and
\[
\hat{\nu}_1 = (-1)^n v_1 + 2(b_{n-1})_x \quad (4.9) \]
\[
\hat{\nu}_0 = v_0 - 2(b_{n-2})_x + b_{n-1}((-1)^n v_1 + \hat{\nu}_1) + \frac{1 + (-1)^n}{2}(Dv_1) - (D\hat{\nu}_1) \quad (4.10) \]

\textbf{Proof.} Essentially, the proof is exactly the same as Proposition 4. \( \square \)

\textbf{Proposition 6 :} Let \( \chi_0, \cdots, \chi_{n-1} \) be even wave functions of the linear system (2.23) and \( \mu_0, \cdots, \mu_{n-1} \) be odd adjoint wave functions of the linear system (2.24). Then after \( n \) iterations of the binary DBT of Proposition 3, the transformed Lax operator becomes
\[
\hat{L} = R_n LR_n^{-1} \quad R_n = 1 - \sum_{i=0}^{n-1} c_i D^{-1} \mu_i 
\quad (4.11) \]
\[
\hat{L} = \hat{\nu}_1 D + \hat{\nu}_0 \quad (4.12) \]

where the coefficient \( c_i \) are defined by
\[
R_n \chi_j = 0 \quad j = 0, \cdots, n - 1 \quad (4.13) \]

and
\[
\dot{v}_1 = v_1 - 2 \sum_{i=0}^{n-1} (c_i \mu_i)_x 
\tag{4.14}
\]
\[
\dot{v}_0 = v_0 + (v_1 + \dot{v}_1) \sum_{i=0}^{n-1} c_i \mu_i + 2 \sum_{i=0}^{n-1} (c_i (D \mu_i))_x 
\tag{4.15}
\]

Proof. A straightforward calculation. We only remark that the form of the gauge operator \( R_n \) can be verified by induction. \( \square \)

Proposition 7: The transformed coefficients, \( \dot{v}_1 \) and \( \dot{v}_0 \), in Eqs. (4.14)-(4.15) can be expressed as

\[
\dot{v}_1 = v_1 - 2(D^3 \ln \det \Omega) 
\tag{4.16}
\]
\[
\dot{v}_0 = v_0 + (v_1 + \dot{v}_1)(D \ln \det \Omega) + 2 \sum_{i=0}^{n-1} (D^2 \left( \frac{\det \Omega(i)}{\det \Omega} (D \mu_i) \right)) 
\tag{4.17}
\]

where \( \Omega_{ij} \equiv \Omega(\mu_i, \chi_j) \) and \( \Omega(i) \) is constructed from \( \Omega \) with its \( i \)-th row replaced by \( (\chi_0, \ldots, \chi_{n-1}) \).

Proof. The proof follows easily from the Cramer’s formula. \( \square \)

Remarks. The expressions of the transformed fields \( \dot{v}_1 \) and \( \dot{v}_0 \) show that they are unchanged under the interchange of any two wave functions \( \chi_i \) and \( \chi_j \) with \( i \neq j \). Hence the permutability of DBTs is still maintained in this supersymmetric formalism. We also note that the case for \( |\chi_i| = 1 \) and \( |\mu_i| = 0 \) gives the same result.

Finally let us discuss a more general DBT in which the numbers of wave functions and adjoint wave functions are unequal.

Proposition 8: Suppose there are \( n \) adjoint wave functions \( \mu_0, \ldots, \mu_{n-1} \) and \( n + m \) wave functions \( \chi_0, \ldots, \chi_{n+m-1} \) of the MR sKdV system. Assume the parities of \( \mu_i \) are all odd, whereas among \( \chi_i, \chi_0, \ldots, \chi_{n+\lfloor m+1/2 \rfloor} \) are even and \( \chi_{n+\lfloor m+1/2 \rfloor}, \ldots, \chi_{n+m-1} \) are odd. Then after performing the DB transformations triggered by these (adjoint) wave functions, the transformed Lax operator reads

\[
\dot{L} = Q_{(n,m)} L Q^{-1}_{(n,m)} \quad Q_{(n,m)} = D^m + \sum_{i=0}^{m-1} d_i D^i + \sum_{i=0}^{n-1} e_i D^{-1} \mu_i 
\tag{4.18}
\]
\[
\dot{L} = \partial^2 + \dot{v}_1 D + \dot{v}_0 
\tag{4.19}
\]

where the coefficients \( d_i \) and \( e_i \) are defined by

\[
Q_{(n,m)} \chi_i = 0 \quad i = 0, \ldots, n + m - 1 
\tag{4.20}
\]

and

\[
\dot{v}_1 = (-1)^m v_1 - 2(d_{m-1})_x 
\tag{4.21}
\]
\[
\dot{v}_0 = v_0 - 2(d_{m-2})_x - d_{m-1}((-1)^m v_1 + \dot{v}_1) + \frac{1}{2} (-1)^m (Dv_1) 
\tag{4.22}
\]

Proof. The gauge operator \( Q_{(n,m)} \) can be realized easily as follows: since the number of wave functions is larger than that of the adjoint wave functions thus part of even wave functions, say, \( \chi_0, \ldots, \chi_{n-1} \) would be paired with odd adjoint wave functions \( \mu_0, \ldots, \mu_{n-1} \)
to form the binary DB transformation $R_n$ of Proposition 7. However, among the residual $m$ wave functions $\chi_n, \cdots, \chi_{n+m-1}, [(m + 1)/2]$ are even and $[m/2]$ are odd which will form the DB transformation $T_m$ of Proposition 3. Hence $Q_{(n,m)} = T_m R_n$ and its expression in (4.18) can be verified by induction. \(\Box\)

Notice that the parity of the gauge operator $Q_{(n,m)}$ depends on $m$, therefore the case for $m$ being even or odd should be discussed separately.

For $m = 2k$, it is convenient to define the following row vectors

$$
\begin{align*}
\chi^{[0]} &= (\chi_0, \chi_1, \cdots, \chi_{n+k-1}) \\
\chi^{[1]} &= (\chi_{n+k}, \cdots, \chi_{n+2k-1}) \\
p^{[0]} &= (e_0, e_1, \cdots, e_{n-1}, d_0, d_2, \cdots, d_{2k-2}) \\
p^{[1]} &= (d_1, d_3, \cdots, d_{2k-1}) \\
q^{[i]} &= \partial_x^i \chi^{[i]} \quad i = 0, 1
\end{align*}
$$

and matrices

$$
\begin{align*}
\Omega^{[0]} &= \Omega(\mu_i, \chi_j) \quad i = 0, \cdots, n - 1; j = 0, \cdots, n + k - 1 \\
\Omega^{[1]} &= \Omega(\mu_i, \chi_{n+k+j}) \quad i = 0, \cdots, n - 1; j = 0, \cdots, k - 1 \\
\Xi^{[i]} &= \begin{pmatrix} \chi^{[i]} \\
\partial_x \chi^{[i]} \\
\vdots \\
\partial_x^{k-1} \chi^{[i]} \end{pmatrix} \quad i = 0, 1
\end{align*}
$$

where the superscript in bracket stands for the parity of the corresponding vector or matrix.

**Proposition 9:** After solving the linear equation (4.20) for $m = 2k$, the transformed coefficients, $\hat{v}_1$ and $\hat{v}_0$, in Eqs.(4.21)-(4.22) can be expressed as

$$
\begin{align*}
\hat{v}_1 &= v_1 + 2(\frac{\det(\hat{D} - \hat{C} A^{-1} B)}{\det(D - C A^{-1} B)}) x \\
\hat{v}_0 &= v_0 + 2(\frac{s\det M}{\det D}) x - (v_1 + \hat{v}_1)(\frac{\det(\hat{D} - \hat{C} A^{-1} B)}{\det(D - C A^{-1} B)})
\end{align*}
$$

where the supermatrix $M = \begin{pmatrix} A & B \\
C & D \end{pmatrix}$ with

$$
\begin{align*}
A &= \begin{pmatrix} \Omega^{[0]} \\
\Xi^{[0]} \end{pmatrix} \\
B &= \begin{pmatrix} \Omega^{[1]} \\
\Xi^{[1]} \end{pmatrix} \\
C &= (D \Xi^{[0]}) \\
D &= (D \Xi^{[1]})
\end{align*}
$$

and its superdeterminant is defined by [15]

$$
\begin{align*}
s\det M &= \frac{\det(A - BD^{-1}C)}{\det D} = \frac{\det A}{\det(D - C A^{-1} B)}.
\end{align*}
$$

The matrix $\hat{M}$ is constructed from $M$ with its $(n + k)$-th row replaced by its $\partial_x$ derivation, whereas $\hat{D}$ and $\hat{C}$ are constructed from $D$ and $C$ with their last rows replaced by their $D$ derivation.
Proof. For $m = 2k$, the transformed coefficients $\hat{v}_1$ and $\hat{v}_0$ are expressed as

\[
\hat{v}_1 = v_1 - 2(d_{2k-1})_x
\]
\[
\hat{v}_0 = v_0 - 2(d_{2k-2})_x + (v_1 + \hat{v}_1)d_{2k-1}
\]

Since the parity of the operator $Q_{(n,2k)}$ is even, we can rewrite the linear equation (4.21) as follows

\[
(p^{[0]}, p^{[1]})M = -(q^{[0]}, q^{[1]})
\]

(4.35)

Now multiplying $M^{-1}$ to the right of (4.35), we have

\[
(p^{[0]}, p^{[1]}) = -(q^{[0]}, q^{[1]}) \begin{pmatrix}
(A - BD^{-1}C)^{-1} & -AB^{-1}(D - CA^{-1}B) \\
-D^{-1}C(A - BD^{-1}C)^{-1} & (D - CA^{-1}B)^{-1}
\end{pmatrix}
\]

(4.36)

which implies

\[
p^{[0]}(A - BD^{-1}C) = -(q^{[0]} - q^{[1]}D^{-1}C)
\]

(4.37)

\[
p^{[1]}(D - CA^{-1}B) = -(q^{[1]} - q^{[0]}A^{-1}B)
\]

(4.38)

Using the Cramer’s rule, we obtain

\[
d_{2k-2} = p^{[0]}_{n+k} = -\frac{s\text{det}\hat{M}}{\text{det}M}
\]

(4.39)

\[
d_{2k-1} = p^{[1]}_k = -\frac{\text{det}(\hat{D} - \hat{C}A^{-1}B)}{\text{det}(D - CA^{-1}B)}
\]

(4.40)

which lead to the result (4.29)-(4.30).

For $m = 2k + 1$, some notations in Eqs.(4.23)-(4.28) should be modified to

\[
\chi^{[0]} = (\chi_0, \chi_1, \cdots, \chi_{n+k})
\]
\[
\chi^{[1]} = (\chi_{n+k+1}, \cdots, \chi_{n+2k})
\]

(4.41)

\[
p^{[0]} = (d_1, d_3, \cdots, d_{2k-1})
\]
\[
p^{[1]} = (e_0, e_1, \cdots, e_{n-1}, d_0, d_2, \cdots, d_{2k})
\]

(4.42)

\[
q^{[i]} = (D^{2k+1}x^{[i]})
\]

(4.43)

\[
\Omega_{ij}^{[0]} = \Omega(\mu_i, \chi_j)
\]

(4.44)

\[
\Omega_{ij}^{[1]} = \Omega(\mu_i, \chi_{n+k+j+1})
\]

(4.45)

\[
\Xi^{[i]} = \begin{pmatrix}
\frac{\partial}{\partial x^{[0]}}
\frac{\partial}{\partial x^{[0]}}
\vdots
\frac{\partial}{\partial x^{[0]}}
\end{pmatrix}
\]

(4.46)

Then we can define another supermatrix $N = \begin{pmatrix} \hat{A} & \hat{B} \\ \hat{C} & D \end{pmatrix}$ where the matrices $\hat{A}$ and $\hat{B}$ are constructed from the matrices $A$ and $B$ by taking into account the replacements (4.41)-(4.46), whereas $\hat{C}$ does the same thing in addition to removing the last row.

Proposition 10: Solving the linear equation (4.21) for $m = 2k + 1$, the transformed coefficients, $\hat{v}_1$ and $\hat{v}_0$, in Eqs.(4.21)-(4.22) can be expressed as
\[ \hat{v}_1 = -v_1 + 2\frac{\det(\tilde{A} - \tilde{B}D^{-1}\tilde{C})}{\det(\tilde{A} - \tilde{B}D^{-1}\tilde{C})} \]  
(4.47)

\[ \hat{v}_0 = v_0 + (Dv_1) + 2\left( \frac{s\det \tilde{N}}{s\det N} \right) + (v_1 - \hat{v}_1)(\frac{\det(\tilde{A} - \tilde{B}D^{-1}\tilde{C})}{\det(\tilde{A} - \tilde{B}D^{-1}\tilde{C})}) \]  
(4.48)

where \( \tilde{N} \) is constructed from \( N \) with its last row replaced by its \( \partial_x \) derivation, whereas \( \tilde{A} \) and \( \tilde{B} \) are constructed from \( \tilde{A} \) and \( \tilde{B} \) with their last rows replaced by their \( D \) derivation.

**Proof.** For \( m = 2k + 1 \), the linear equation (4.20) now becomes

\[ (p^{[1]}, p^{[0]})\tilde{N} = - (q^{[1]}, q^{[0]}) \]  
(4.49)

Following the procedures for the case of \( m = 2k \), it is easy to show that

\[ d_{2k} = p^{[1]}_{n+k+1} = \frac{\det(\tilde{A} - \tilde{B}D^{-1}\tilde{C})}{\det(\tilde{A} - \tilde{B}D^{-1}\tilde{C})} \]  
(4.50)

\[ d_{2k-1} = p^{[0]}_k = - \frac{s\det \tilde{N}}{s\det N} \]  
(4.51)

which imply that the transformed coefficients \( \hat{v}_1 \) and \( \hat{v}_0 \) are given by (4.47) and (4.48), respectively.

**Remarks.** (1) In both cases \( (m = 2k \) or \( 2k + 1 \)), the expressions for \( \hat{v}_1 \) and \( \hat{v}_0 \) are indeed, independent of the order of the even wave functions as well as the odd ones. (2) For the case that the number of the adjoint wave functions is larger than that of the wave functions in a DBT, we can get a similar result just by exchanging the roles played by them. (3) When we set \( n = 0 \), the binary type expressions for \( \hat{v}_1 \) and \( \hat{v}_0 \) then reduce to the Wronskian type ones obtained in Ref. [4]. Especially, in the \( m = 2k \) case, it is possible to write \( d_{2k-1} \) in terms of superdeterminant.

**V. CONCLUDING REMARKS**

We have studied the solutions of the MR sKdV hierarchy by using DBTs. In addition to the previously known DBT [2,4], we provide the adjoint DBT which can be combined with the former one to form the binary DBT. Using these elementary DBTs, we obtain not only Wronskian type but also binary type solutions for the MR sKdV hierarchy. The super soliton solutions then can be constructed from the trivial one which corresponds to \( v_1 = v_0 = 0 \) by using the formulae derived in Sec. IV. Finally, we would like to remark that our approach involves only the algebra of S\( \Psi \)DOs, hence the formulation is general enough to extend to the other cases such as super KP hierarchies and their reductions. We will leave these discussions to another publication.

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