Stochastic integration with respect to canonical \( \alpha \)-stable cylindrical Lévy processes

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Abstract

In this work, we introduce a theory of stochastic integration with respect to symmetric \( \alpha \)-stable cylindrical Lévy processes. Since \( \alpha \)-stable cylindrical Lévy processes do not enjoy a semi-martingale decomposition, our approach is based on a decoupling inequality for the tangent sequence of the Radonified increments. This approach enables us to characterise the largest space of predictable Hilbert-Schmidt operator-valued processes which are integrable with respect to an \( \alpha \)-stable cylindrical Lévy process as the collection of all predictable processes with paths in the Bochner space \( L^\alpha \). We demonstrate the power and robustness of the developed theory by establishing a dominated convergence result allowing the interchange of the stochastic integral and limit.

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1 Introduction

Symmetric \( \alpha \)-stable distributions are popular for modelling random perturbations in the Euclidean space; see e.g. Samorodnitsky and Taqqu [32]. This is because they are

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natural, discontinuous and non-Gaussian generalisations of Brownian motions meeting various empirical requests, such as heavy tails, self-similarity and infinite variance. As perturbations of infinite dimensional systems, such as partial differential equations, models with stable distributions can usually only be found realised as random fields but not as stochastic processes in a Hilbert or Banach space. This fact is not surprising since a random noise with a symmetric $\alpha$-stable distribution does not exist as an ordinary Hilbert space-valued process but only in the generalised sense of Gel’fand and Vilenkin [8] or Segal [34] as cylindrical processes. This is analogous to the standard Brownian motion in an infinite dimensional Hilbert space which only exists as a cylindrical process.

The purpose of this work is to provide a comprehensive and robust theory of stochastic integration for predictable processes with respect to symmetric $\alpha$-stable cylindrical processes, which will lay the foundation for the application of these processes as a model of random perturbations of infinite dimensional systems. We do not only introduce the stochastic integral but also characterise the largest set of predictable Hilbert-Schmidt-valued processes which are integrable with respect to a symmetric $\alpha$-stable cylindrical process. Including predictable integrands in the theory of stochastic integration is important, as solutions of stochastic partial differential equation driven by cylindrical Lévy processes do not necessarily have càdlàg trajectories; see e.g. Brzeźniak et al. [3]. We demonstrate the power of the developed integration theory by establishing a dominated convergence result allowing the interchange of the stochastic integral and limit.

The classical approach to stochastic integration in finite-dimensional spaces is based on the semi-martingale decomposition; see Dellacherie and Meyer [6]. The reverse approach, starting with good integrators, is introduced in Protter [28] or in Kurtz and Protter [15] for an infinite dimensional setting; however, extending the space from adapted, càdlàg to arbitrary predictable integrands in Protter [28] also depends on the semi-martingale decomposition. A completely different approach is introduced in Bichteler [2], where the construction of the stochastic integral for real-valued integrands and integrators mimics that of the Daniell integral in calculus. Another approach defines the stochastic integral as a vector-valued random measure; see e.g. Métivier and Pellaumail [22] or Rao [29]. A decoupling inequality for tangent sequences is the foundation for stochastic integration developed by Kwapień and Woyczyński in [16] and [17]; their approach also allows for the characterisation of the largest space of predictable integrands as a randomised Musielak-Orlicz space. This approach is extended to Hilbert space-valued semi-martingales in Nowak [25].

Turning to stochastic integration with respect to cylindrical processes, an extensive theory has been developed for stochastic integrals with respect to cylindrical Brownian motions, which has recently been extended even to UMD Banach spaces; see van Neerven [37]. Surprisingly, stochastic integration with respect to other cylindrical pro-
cesses than cylindrical Brownian motion is much less considered. In fact, only with respect to cylindrical martingales a stochastic integration theory is developed either by following a Doléans measure approach by Métivier and Pellaumail in [21] and [22], or by constructing a family of reproducing kernel Hilbert spaces in Mikulevičius and Rozovskii in [23] and [24]. For the special case of a cylindrical Lévy process with finite weak second moments, an Itô approach to stochastic integration is developed in Riedle [30]. The reason for the restriction to cylindrical martingales is due to the fact that cylindrical semi-martingales do not enjoy a semi-martingale decomposition, and thus the classical approach to stochastic integration, mentioned above, is not applicable. This obstacle is overcome in Jakubowski and Riedle [12], in which a stochastic integral for adapted, càglàd integrands and arbitrary cylindrical Lévy processes is introduced. The approach is based on a decoupling inequality for tangent sequences, but seems neither to allow an extension to include predictable integrands nor to establish some powerful limit theorems.

The approach in the current work also relies on a decoupling inequality for tangent sequences, but we characterise convergence of the stochastic integral in the semi-martingale topology to be equivalent to convergence in probability in the space of random variables with values in a Bochner space. This approach originates from Kwapień and Woyczyński [19], in which the equivalent topology is defined on the space of random variables with values in a certain Musielak-Orlicz space. This equivalent description of a topology for the convergence of the stochastic integral allows the characterisation of the largest space of predictable integrands.

The paper is organised as follows: we summarise some preliminaries on genuine and cylindrical Lévy processes in Section 2. The theory of stochastic integration for deterministic integrands is introduced in Section 3. The following Section 4 is devoted to providing an approximation result for the predictable integrands under consideration. In Section 5, we explicitly construct the tangent sequence of the Radonified increments, which is fundamental for our approach. In the final Section 6, we state our main result on the equivalent description of the largest space of predictable integrands as a Bochner space $L^p$, and provide its proof. We finish the section with a result on dominated convergence.

2 Preliminaries

Let $G$ and $H$ be separable Hilbert spaces with inner products $\langle \cdot, \cdot \rangle$ and corresponding norms $\| \cdot \|$. Let $(a_k)_{k \in \mathbb{N}}$ and $(b_k)_{k \in \mathbb{N}}$ be orthonormal bases of $G$ and $H$, respectively. We identify the dual of a Hilbert space by the space itself. The Borel $\sigma$-algebra of $H$ is denoted by $\mathcal{B}(H)$ and the open unit ball by $B_H := \{ h \in H : \| h \| < 1 \}$.

The Banach space of bounded linear operators from $G$ to $H$ will be denoted by
\( L(G,H) \) with the operator norm \( \| F \|_{G \to H} \). Its subspace \( L_2(G,H) \) of Hilbert-Schmidt operators is endowed with the norm \( \| F \|_{HS}^2 := \sum_{k=1}^{\infty} \| F a_k \|_2^2 \) for \( F \in L_2(G,H) \). We denote by \( L^0_{\text{Leb}}([0,T],L_2(G,H)) \) the space of Borel measurable functions \( f : [0,T] \to L_2(G,H) \) that satisfy the condition \( \int_0^T \| f(t) \|_{HS}^2 \, dt < \infty \). For \( \alpha \geq 1 \), this is a Banach space if equipped with the usual norm \( \| f \|_{L^\alpha}^2 := \int_0^T \| f(t) \|_{HS}^\alpha \, dt \), and for \( 0 < \alpha < 1 \) it is a metric space under the translation invariant metric \( d(f,g) = \int_0^T \| f(t) - g(t) \|_{HS}^\alpha \, dt \).

For ease of notation, we also use the notation \( \| f \|_{L^\alpha} \) to denote the metric \( d(0,f) \) for \( 0 < \alpha < 1 \).

Let \( (\Omega,\Sigma,P) \) be a complete probability space. We will denote by \( L^0_p(\Omega,H) \) the space of equivalence classes of measurable functions \( X : \Omega \to H \), equipped with the topology of convergence in probability.

Let \( S \) be a subset of \( G \). For each \( n \in \mathbb{N} \), elements \( g_1,\ldots,g_n \in S \) and Borel set \( A \in \mathcal{B}(\mathbb{R}^n) \), we define

\[
C(g_1,\ldots,g_n;A) := \{ g \in G : (g,g_1),\ldots,(g,g_n) \in A \}.
\]

Such sets are called cylindrical sets with respect to \( A \). The set of all these cylindrical sets is denoted by \( \mathcal{Z}(G,S) \), and it is a \( \sigma \)-algebra if \( S \) is finite and otherwise an algebra. We write \( \mathcal{Z}(G) \) for \( \mathcal{Z}(G,G) \).

A set function \( \mu : \mathcal{Z}(G) \to [0,\infty] \) is called a cylindrical measure on \( \mathcal{Z}(G) \) if for each finite dimensional subset \( S \subseteq G \), the restriction of \( \mu \) to the \( \sigma \)-algebra \( \mathcal{Z}(G,S) \) is a \( \sigma \)-additive measure. A cylindrical measure is said to be a cylindrical probability measure if \( \mu(G) = 1 \).

A cylindrical random variable \( X \) in \( G \) is a linear and continuous mapping \( X : G \to L^0_p(\Omega,\mathbb{R}) \). It defines a cylindrical probability measure \( \mu_X \) by

\[
\mu_X : \mathcal{Z}(G) \to [0,1], \quad \mu_X(Z) = P((Xg_1,\ldots,Xg_n) \in A)
\]

for cylindrical sets \( Z = C(g_1,\ldots,g_n;A) \). The cylindrical probability measure \( \mu_X \) is called the **cylindrical distribution** of \( X \). We define the characteristic function of the cylindrical random variable \( X \) by

\[
\varphi_X : G \to \mathbb{C}, \quad \varphi_X(g) = E[e^{\imath g}].
\]

Let \( T : G \to H \) be a linear and continuous operator. By defining

\[
TX : H \to L^0_p(\Omega,\mathbb{R}), \quad (TX)h = X(T^*h)
\]

we obtain a cylindrical random variable on \( H \). In the special case when \( T \) is a Hilbert-Schmidt operator and hence 0-Radonifying by [36, Th. VI.5.2], it follows from [36, Pr. VI.5.3] that the cylindrical random variable \( TX \) is induced by a genuine random
variable \( Y : \Omega \to H \), that is \((TX)h = \langle Y, h \rangle\) for all \( h \in H \). The following result shows that the inducing random variable \( Y \) continuously depends on the Hilbert-Schmidt operator.

**Lemma 2.1.** Let \( X \) be a cylindrical random variable and \((F_n)_{n \in \mathbb{N}}\) a sequence in \( L_2(G, H) \) converging to \( F \) in \( \| \cdot \|_{\text{HS}} \). Then \((F_n X)_{n \in \mathbb{N}}\) converges to \( FX \) in \( L^0_P(\Omega, H) \).

**Proof.** Let \( \mu_X \) denote the cylindrical distribution of \( X \). As the sequence \((F_n)_{n \in \mathbb{N}}\) is compact in \( L_2(G, H) \), the collection of measures \( \{\mu_X \circ F_n^{-1} : n \in \mathbb{N}\} \) is relatively compact in the space of probability measures on \( \mathcal{B}(H) \); see [12, Pr. 5.3]. Continuity of \( X \) implies for all \( h \in H \) that

\[
\lim_{n \to \infty} \langle F_n X, h \rangle = \lim_{n \to \infty} X(F_n^* h) = X(F^* h) = \langle FX, h \rangle \quad \text{in } L^0_P(\Omega, \mathbb{R}).
\]

Together with relative compactness, this implies that \((F_n X)_{n \in \mathbb{N}}\) converges to \( FX \) in \( L^0_P(\Omega, H) \); see e.g. [10, Le. 2.4].

A family \((L(t) : t \geq 0)\) of cylindrical random variables \( L(t) : G \to L^0_P(\Omega, \mathbb{R}) \) is called a cylindrical Lévy process if for each \( n \in \mathbb{N} \) and \( g_1, \ldots, g_n \in G \), the stochastic process

\[
\left( (L(t)g_1, \ldots, L(t)g_n) : t \geq 0 \right)
\]

is a Lévy process in \( \mathbb{R}^n \). The filtration generated by \((L(t) : t \geq 0)\) is defined by

\[
\mathcal{F}_t := \sigma(\{L(s)g : g \in G, s \in [0, t]\}) \quad \text{for all } t \geq 0.
\]

Denote by \( \mathcal{Z}_s(G) \) the collection

\[
\{ \{g \in G : (\langle g, g_1 \rangle, \ldots, \langle g, g_n \rangle) \in B\} : n \in \mathbb{N}, g_1, \ldots, g_n \in G, B \in \mathfrak{B}(\mathbb{B}(\mathbb{R}^n) \setminus \{0\}) \}
\]

of cylindrical sets, which forms an algebra of subsets of \( G \). For fixed \( g_1, \ldots, g_n \in G \), let \( \lambda_{g_1,\ldots, g_n} \) be the Lévy measure of \( (\langle L(t)g_1, \ldots, L(t)g_n \rangle \) : \( t \geq 0 \)). Define a function \( \lambda : \mathcal{Z}_s(G) \to [0, \infty] \) by

\[
\lambda(C) := \lambda_{g_1,\ldots, g_n}(B) \quad \text{for } C = \{g \in G : (\langle g, g_1 \rangle, \ldots, \langle g, g_n \rangle) \in B\},
\]

for \( B \in \mathfrak{B}(\mathbb{B}(\mathbb{R}^n)) \). It is shown in [1] that \( \lambda \) is well defined. The set function \( \lambda \) is called the cylindrical Lévy measure of \( L \).

In this paper, we restrict our attention to canonical \( \alpha \)-stable cylindrical Lévy processes. These are cylindrical Lévy processes with characteristic function \( \varphi_{L(t)}(g) = \exp(-t \|g\|^{\alpha}) \) for each \( t \geq 0 \) and \( g \in G \). By [31, Le. 2.4], the cylindrical Lévy measure \( \lambda \) of the canonical \( \alpha \)-stable cylindrical Lévy processes satisfies the spectral representation

\[
\lambda \circ \pi_{a_1, \ldots, a_n}^{-1}(B) = \frac{\alpha}{c_\alpha} \int_{S_{a_1, \ldots, a_n}} \nu_a(dx) \int_0^\infty 1_B(rx) \frac{1}{r^{1+\alpha}} dr \quad \text{for } B \in \mathfrak{B}(\mathbb{B}(\mathbb{R}^n)), \quad (2.1)
\]

where \( c_\alpha \) is a constant.
where $S_{\mathbb{R}^n} := \{\beta \in \mathbb{R}^n : |\beta| = 1\}$, $c_\alpha > 0$ is a constant dependent only on $\alpha$ and $\nu_n$ denotes a uniform distribution on the sphere $S_{\mathbb{R}^n}$.

Infinitely divisible measures on a Hilbert space $H$ can be defined as in the Euclidean space; see [26]. As in finite dimensions, infinitely divisible distributions are completely characterised by a triplet $(a, \rho, \lambda)$, where $a \in H$, the mapping $\rho : H \to H$ is nuclear and non-negative, and the Lévy measure $\lambda$ is a $\sigma$-finite measure on $\mathfrak{B}(H)$ satisfying $\int_H (\|h\|^2 \wedge 1) \lambda(dh) < \infty$. Given any $\delta > 0$ and Lévy measure $\lambda$ on $\mathfrak{B}(H)$, we say that $\delta \in C(\lambda)$ if $\lambda(\{h \in H : \|h\| = \delta\}) = 0$. A sequence of infinitely divisible measures $\mu_n = (a_n, \rho_n, \lambda_n)$ with associated sequence $(T_n)_{n \in \mathbb{N}}$ of $S$-operators $T_n : H \to H$, which are defined by

$$\langle T_n h_1 , h_2 \rangle = \langle \rho_n h_1 , h_2 \rangle + \int_{\|h\| < 1} \langle h_1, u \rangle \langle h_2, u \rangle \lambda_n(du) \quad \text{for all } h_1, h_2 \in H,$$

converges weakly to an infinitely divisible measure $\mu = (a, \rho, \lambda)$ if and only if the following conditions hold:

1. $a = \lim_{\delta \downarrow 0} \lim_{n \to \infty} \langle a_n \rangle + \int_{1 < \|h\| < \delta} h \lambda_n(dh)$;

2. $\lim_{\delta \downarrow 0} \limsup_{n \to \infty} \int_{\|h\| \leq \delta} \langle h, u \rangle^2 \lambda_n(du) + \langle \rho_n h, h \rangle = \langle \rho h, h \rangle$ for all $h \in H$; (2.3)

3. $\lambda_n \to \lambda$ weakly outside of every closed neighbourhood of the origin; (2.4)

4. $(T_n)_{n \in \mathbb{N}}$ is compact in the space of nuclear operators. (2.5)

The necessity of these conditions can be found in [20, Pr. 5.7.4] and their sufficiency is an adaption of [26, Th. 5.5] to the case of a discontinuous truncation function.

It is well known, even in finite dimensions, that the small jumps of the converging sequence $\mu_n$ may contribute to the Gaussian part of the limit distribution $\mu$; see (2.3). However, if we rule out this situation, we obtain the following result.

**Lemma 2.2.** Let $\mu_n = (0, 0, \lambda_n)$ be a sequence of infinitely divisible measures on $\mathfrak{B}(H)$ converging weakly to $\mu = (0, 0, \lambda)$. Then, it holds that

$$\lim_{n \to \infty} \int_H (\|h\|^2 \wedge 1) \lambda_n(dh) = \int_H (\|h\|^2 \wedge 1) \lambda(dh).$$

**Proof.** It is enough to prove the statement under the assumption that $\lambda_n$ is symmetric for all $n \in \mathbb{N}$, since the general case can be reduced back to this setting by a symmetrisation argument. Let $\delta \in C(\lambda)$ be such that $\delta \in (0, 1)$. Then, it follows directly from
Condition (2.4) that
\[
\lim_{n \to \infty} \int_{\|h\| > \delta} \left( \|h\|^2 \wedge 1 \right) \lambda_n (dh) = \int_{\|h\| > \delta} \left( \|h\|^2 \wedge 1 \right) \lambda (dh).
\]
To establish the result for the integrals over the closed ball of radius \(\delta\), note that the infinitely divisible measures \(\tilde{\mu}_n = (0, 0, \lambda_n|_{B_H(\delta)})\), where \(\lambda_n|_{B_H(\delta)} = \lambda_n(\bar{B}_H(\delta) \cap \cdot)\), converge weakly to the infinitely divisible measure \(\tilde{\mu} = (0, 0, \lambda|_{B_H(\delta)})\). Therefore, the set of measures \(\{\tilde{\mu}_n\}_{n \in \mathbb{N}}\) is weakly compact and it follows from [26, Le. VI.5.3] that
\[
\sup_{n \in \mathbb{N}} \int_H \|h\|^4 \tilde{\mu}_n (dh) < \infty.
\]
By the Skorokhod representation theorem and the Vallée-Poussin theorem [5, Le. Th. II.22], it follows as \(n \to \infty\) that
\[
\int_{B_H(\delta)} \|h\|^2 \lambda_n (dh) \to \int_H \|h\|^2 \tilde{\mu} (dh) = \int_{B_H(\delta)} \|h\|^2 \lambda (dh),
\]
which completes the proof.

\[\square\]

3 Deterministic integrands

The definition of the stochastic integral for deterministic integrands with respect to a canonical \(\alpha\)-stable cylindrical Lévy process \(L\) depends on two classes of step functions. We give in the following a precise definition of what is meant by a step function.

**Definition 3.1.**

1. An \(L_2(G,H)\)-valued step function is of the form
   \[
   \psi: [0,T] \to L_2(G,H), \quad \psi(t) = F_0 \mathbb{1}_{\{0\}}(t) + \sum_{i=1}^{n-1} F_i \mathbb{1}_{(t_i,t_{i+1})}(t),
   \]
   where \(0 = t_1 < \cdots < t_n = T\), \(F_i \in L_2(G,H)\) for each \(i \in \{0,\ldots,n-1\}\). The space of \(L_2(G,H)\)-valued step functions is denoted by \(S_{\text{det}}^{\text{HS}} := S_{\text{det}}^{\text{HS}}(G,H)\).

2. An \(L(H,H)\)-valued step function is of the form
   \[
   \gamma: [0,T] \to L(H,H), \quad \gamma(t) = F_0 \mathbb{1}_{\{0\}}(t) + \sum_{i=1}^{n-1} F_i \mathbb{1}_{(t_i,t_{i+1})}(t),
   \]
   where \(0 = t_1 < \cdots < t_n = T\) and \(F_i \in L(H,H)\) for each \(i \in \{0,\ldots,n-1\}\). The space of \(L(H,H)\)-valued step functions with \(\sup_{t \in [0,T]} \|\gamma(t)\|_{H \to H} \leq 1\) is denoted by \(S_{\text{det}}^{\text{op}} := S_{\text{det}}^{\text{op}}(H,H)\).
Consider an increment $L_{t+1} - L_t$ of the cylindrical Lévy process $L$ and let $F_i \in L^2(G,H)$ for each $i \in \{1,\ldots,n-1\}$. Since Hilbert-Schmidt operators are 0-Radonifying by [36, Th. VI.5.2], it follows from [36, Pr. VI.5.3] that there exist a genuine random variables $F_i(L(t_{i+1}) - L(t_i)) : \Omega \to H$ for each $i \in \{1,\ldots,n-1\}$ satisfying

$$(L(t_{i+1}) - L(t_i))(F_i^* h) = \langle F_i(L(t_{i+1}) - L(t_i)), h \rangle \quad P\text{-a.s. for all } h \in H.$$ 

We call the random variables $F_i(L(t_{i+1}) - L(t_i))$ for each $i \in \{1,\ldots,n-1\}$ Radonified increments. The stochastic integral is defined for any $\psi \in S_{\text{HS}}^\text{det}$ with representation (3.1) as the sum of the Radonified increments

$$I(\psi) := \int_0^T \psi \, dL = \sum_{i=1}^{n-1} F_i(L(t_{i+1}) - L(t_i)).$$

Thus, the integral $I(\psi) : \Omega \to H$ is a genuine $H$-valued random variable.

The following definition of the stochastic integral originates from the theory of vector measures, and was adapted to the probabilistic setting in [35] by Urbanik and Woycicielński.

**Definition 3.2.** A function $\psi : [0,T] \to L^2(G,H)$ is integrable if there exists a sequence $(\psi_n)_{n \in \mathbb{N}}$ of elements of $S_{\text{det}}^\text{HS}$ satisfying

1. $(\psi_n)_{n \in \mathbb{N}}$ converges to $\psi$ Lebesgue a.e.;
2. $\lim_{m,n \to \infty} \sup_{\gamma \in S_{\text{det}}^\text{HS}^{\text{op}}} E\left[\left\| \int_0^T \gamma(\psi_m - \psi_n) \, dL \right\| \wedge 1 \right] = 0.$

In this case, the stochastic integral of the deterministic function $\psi$ is defined by

$$I(\psi) := \int_0^T \psi \, dL := \lim_{n \to \infty} \int_0^T \psi_n \, dL \quad \text{in } L^0_p(\Omega,H).$$

The class of all deterministic integrable Hilbert-Schmidt operator-valued functions is denoted by $I^\text{HS}_\text{det} := I^\text{HS}_\text{det}(G,H)$.

**Remark 3.3.** If Conditions (1) and (2) in Definition 3.2 are satisfied, then completeness of $L^0_p(\Omega,H)$ implies the existence of the limit. Furthermore, it follows that the integral process $(\int_0^t \psi \, dL)_{t \geq 0}$, defined by $\int_0^t \psi \, dL := \int_0^T \mathbb{1}_{[0,t]}(\psi) \, dL$ has càdlàg paths. To see this, note that for each $m,n \in \mathbb{N}$ the process $(\int_0^t (\psi_m - \psi_n) \, dL)_{t \geq 0}$ has càdlàg paths. By an extension of [18, Pr. 8.2.1] to $H$-valued processes and Condition (2) above, we
obtain

\[
\lim_{m,n \to \infty} P \left( \sup_{0 \leq t \leq T} \left\| \int_0^t (\psi_m - \psi_n) \, dL \right\| > \epsilon \right)
\leq 3 \lim_{m,n \to \infty} \sup_{0 \leq t \leq T} P \left( \left\| \int_0^t (\psi_m - \psi_n) \, dL \right\| > \frac{\epsilon}{3} \right) = 0.
\]

By passing on to a suitable subsequence if necessary, we obtain that there exists a subsequence \((\int_0^t \psi_{n_k} \, dL)_{k \in \mathbb{N}}\) that converges uniformly almost surely, which guarantees that the limiting process has càdlàg paths.

The following is the main result of this section, which characterises the space \(I_{HS}^{det}\) of deterministic integrands, i.e. the set of functions \(\psi: [0, T] \to L_2(G, H)\) that satisfy Definition 3.2 for the canonical \(\alpha\)-stable cylindrical Lévy process \(L\).

**Theorem 3.4.** The space \(I_{HS}^{det}\) of deterministic functions integrable with respect to the canonical \(\alpha\)-stable cylindrical Lévy process in \(G\) for \(\alpha \in (0, 2)\) coincides with \(L^\alpha_{Leb}([0, T], L_2(G, H))\).

The rest of this chapter is devoted to proving the above theorem which is divided into a few lemmas in the following.

**Lemma 3.5.** Let \(L\) be the canonical \(\alpha\)-stable cylindrical Lévy process in \(G\) with cylindrical Lévy measure \(\lambda\). Then there exists a constant \(c_\alpha > 0\) such that

\[
\int_0^T \|\psi(t)\|_{HS}^\alpha \, dt \leq c_\alpha \int_0^T \int_H (\|h\|^2 \wedge 1) (\lambda \circ \psi(t)^{-1})(dh) \, dt
\]

for all measurable functions \(\psi: [0, T] \to L_2(G, H)\).

**Proof.** Let \(F\) be an operator in \(L_2(G, H)\). The spectral theorem for compact operators, see e.g. [7, Th. 4.1], guarantees that \(F\) has a decomposition of the form

\[
F = \sum_{j=1}^{\infty} \gamma_j \langle a_j, \cdot \rangle b_j,
\]

where \((a_j)_{j \in \mathbb{N}}\) and \((b_j)_{j \in \mathbb{N}}\) are orthonormal bases of \(G\) and \(H\), respectively, and \((\gamma_j)_{j \in \mathbb{N}}\) is a sequence in \(\mathbb{R}\). Let \(\pi_n: H \to H\) be the projection onto \(\text{Span}\{b_1, \ldots, b_n\}\). We conclude from the spectral representation (2.1) of the stable measure \(\lambda \circ \pi_{a_1, \ldots, a_n}^{-1}\) for each \(n \in \mathbb{N}\)
that

\[ (\lambda \circ F^{-1} \circ \pi_n^{-1})(B_H) = \lambda\left( \left\{ g \in G : \sum_{j=1}^{n} \gamma_j^2 (a_j, g)^2 > 1 \right\} \right) \]

\[ = \lambda \circ \pi_n^{-1}\left( \left\{ x \in \mathbb{R}^n : \sum_{j=1}^{n} \gamma_j^2 x_j^2 > 1 \right\} \right) \]

\[ = \frac{\alpha}{c_\alpha} \int_{S^{n-1}} \int_{0}^{\infty} \mathbb{1}_{\left\{ y \in \mathbb{R}^n : \sum_{j=1}^{n} \gamma_j^2 y_j^2 > 1 \right\}}(r x) \frac{1}{r^{1+\alpha}} \, dr \, \nu_n(dx) \]

\[ = \frac{1}{c_\alpha} \int_{S^{n-1}} \left( \sum_{j=1}^{n} \gamma_j^2 x_j^2 \right)^{\alpha/2} \nu_n(dx), \]

where \( \nu_n \) is a uniform distribution on the sphere \( S^{n-1} \) not necessarily of unit mass. By defining \( c_n := \sum_{j=1}^{n} \gamma_j^2 \) and applying Jensen’s inequality to the concave function \( \beta \mapsto \beta^{\alpha/2} \) with respect to the discrete probability measure \( \{ c_n^{-1} \gamma_1^2, \ldots, c_n^{-1} \gamma_n^2 \} \), we obtain

\[ \frac{1}{c_\alpha} \int_{S^{n-1}} \left( \sum_{j=1}^{n} \gamma_j^2 x_j^2 \right)^{\alpha/2} \nu_n(dx) \geq \frac{c_n^{\alpha/2}}{c_\alpha} \int_{S^{n-1}} \sum_{j=1}^{n} \frac{\gamma_j^2}{c_n} |x_j|^\alpha \nu_n(dx). \]

Letting \( \nu_n^1 = \frac{1}{\nu_n(S^{n-1})} \nu_n \), Lemma 2.4 and A2 in [31] imply

\[ \frac{c_n^{\alpha/2}}{c_\alpha} \int_{S^{n-1}} \sum_{j=1}^{n} \frac{\gamma_j^2}{c_n} |x_j|^\alpha \nu_n(dx) = \frac{c_n^{\alpha/2}}{c_\alpha} \nu_n(S^{n-1}) \sum_{j=1}^{n} \frac{\gamma_j^2}{c_n} \int_{S^{n-1}} |x_j|^\alpha \nu_n(dx) \]

\[ = \frac{c_n^{\alpha/2}}{c_\alpha} \nu_n(S^{n-1}) \frac{\Gamma(\frac{n}{2}) \Gamma(1 + \frac{\alpha}{2})}{\Gamma(\frac{\alpha}{2}) \Gamma(\frac{n + \alpha}{2})} \sum_{j=1}^{n} \frac{\gamma_j^2}{c_n} \]

\[ = \frac{1}{c_\alpha} \left( \sum_{j=1}^{n} \gamma_j^2 \right)^{\alpha/2} \]

\[ = \frac{1}{c_\alpha} \left( \sum_{j=1}^{n} \|F a_j\|^2 \right)^{\alpha/2}, \]

where the last step follows from the spectral representation (3.3). Thus, for all \( n \in \mathbb{N} \) it holds that

\[ \frac{1}{c_\alpha} \left( \sum_{j=1}^{n} \|F a_j\|^2 \right)^{\alpha/2} \leq (\lambda \circ F^{-1} \circ \pi_n^{-1})(B_H). \tag{3.4} \]
Since $\pi_n F \to F$ in $L_2(G,H)$, Lemma 2.1 implies that $((\pi_n \circ F)(L(1)))_{n \in \mathbb{N}}$ converges in probability to the random variable $F(L(1))$. Condition (2.4) yields

$$
\lim_{n \to \infty} (\lambda \circ F^{-1} \circ \pi_n^{-1})(\mathcal{B}_H^c) = (\lambda \circ F^{-1})(\mathcal{B}_H^c).
$$

By taking limits as $n \to \infty$ on both sides in (3.4), we obtain

$$
\frac{1}{c_\alpha} \|F\|_{\alpha HS}^\alpha \leq (\lambda \circ F^{-1})(\mathcal{B}_H^c).
$$

It follows for any measurable function $\psi : [0, T] \to L_2(G,H)$ that

$$
\int_0^T \|\psi(t)\|_{\alpha HS}^\alpha \, dt \leq c_\alpha \int_0^T \int_{\mathcal{B}_H^c} (\lambda \circ \psi(t)^{-1}) (dh) \, dt
$$

$$
\leq c_\alpha \int_0^T \int_H (\|h\|^2 \wedge 1)(\lambda \circ \psi(t)^{-1}) (dh) \, dt,
$$

which completes the proof.

The product measure of two cylindrical measures is defined analogously to the case of Radon measures; see [33, Ch. II.2.2]. The following lemma provides an alternative representation of an integral with respect to the product measure of the cylindrical Lévy measure of $L$ and the Lebesgue measure on a finite interval. To make sense of this, the Lebesgue measure is considered as a cylindrical measure on $\mathcal{B}(\mathbb{R})$.

**Lemma 3.6.** Let $L$ be the canonical $\alpha$-stable cylindrical Lévy process in $G$ with cylindrical Lévy measure $\lambda$. Then we have for each $\psi \in S_{det}^{HS}$ with $\psi(0) = 0$ that

$$
\int_0^T \int_H \left( \|h\|^2 \wedge 1 \right)(\lambda \circ \psi(t)^{-1}) (dh) \, dt = \int_H \left( \|h\|^2 \wedge 1 \right) \left( (\lambda \otimes \text{Leb}) \circ \kappa_{\psi}^{-1} \right) (dh),
$$

where $\kappa_{\psi} : G \times [0, T] \to H$ is defined by $\kappa_{\psi}(g, t) = \psi(t)g$.

**Proof.** First, we show that the result holds for $\psi = F1_{(t_i, t_{i+1})}$, where $F \in L_2(G,H)$ and $0 \leq t_i < t_{i+1} \leq T$. In this case, we see that for all $C \in \mathcal{Z}(H)$

$$
(\lambda \otimes \text{Leb}) \circ \kappa_{\psi}^{-1}(C) = (\lambda \otimes \text{Leb})(F^{-1}(C) \times (t_i, t_{i+1}) = (t_i - t_{i+1})(\lambda \circ F^{-1})(C). \quad (3.5)
$$

Since the cylindrical measure on the right hand side of Equation (3.5) is the cylindrical Lévy measure of the Radonified increment $F(L(t_{i+1}) - L(t_i))$, it extends to a genuine Lévy measure on $\mathcal{B}(\mathbb{R})$ for which we keep the notation $\lambda \circ F^{-1}$. Consequently, the
cylindrical Lévy measure on the left hand side of Equation (3.5) extends to a genuine Lévy measure on \( \mathfrak{B}(H) \), and the two extensions agree on \( \mathfrak{B}(H) \). It follows that

\[
\int_H \left( \|h\|^2 \wedge 1 \right) \left( (\lambda \otimes \text{Leb}) \circ \kappa_{\psi}^{-1} \right) (dh)
= \int_{t_i}^{t_{i+1}} \int_H \left( \|h\|^2 \wedge 1 \right) (\lambda \circ F^{-1}) (dh) \, dt
= \int_0^T \int_H \left( \|h\|^2 \wedge 1 \right) (\lambda \circ \psi^{-1}(t)) (dh) \, dt.
\]

Let \( \psi \in \mathcal{S}^{\text{HS}}_{\text{det}} \) be of the form as in (3.1) with \( \psi(0) = 0 \). For each \( C \in \mathcal{Z}_\ast(H) \) we obtain

\[
\kappa_{\psi}^{-1}(C) = \bigcup_{i=1}^{n-1} \left\{ (g,t) \in G \times [0,T] : F_i g \mathbb{1}_{(t_i,t_{i+1}]} \in C \right\}.
\]

Since the above is a finite union of disjoint cylindrical sets, it follows

\[
\left( (\lambda \otimes \text{Leb}) \circ \kappa_{\psi}^{-1} \right)(C) = \sum_{i=1}^{n-1} \left( (\lambda \otimes \text{Leb}) \circ \kappa_{F_i,1(t_i,t_{i+1})}^{-1} \right)(C).
\]

As the measure on the right side of Equation (3.7) extends to a genuine Lévy measure on \( \mathfrak{B}(H) \) according to the first part of this proof, the measure on the left extends to a genuine Lévy measure on \( \mathfrak{B}(H) \). It follows from Equation (3.6) that

\[
\int_H \left( \|h\|^2 \wedge 1 \right) \left( (\lambda \otimes \text{Leb}) \circ \kappa_{\psi}^{-1} \right) (dh)
= \sum_{i=1}^{n-1} \int_H \left( \|h\|^2 \wedge 1 \right) \left( (\lambda \otimes \text{Leb}) \circ \kappa_{F_i,1(t_i,t_{i+1})}^{-1} \right) (dh)
= \sum_{i=1}^{n-1} \int_0^T \int_H \left( \|h\|^2 \wedge 1 \right) \left( \lambda \circ (F_i,1(t_i,t_{i+1})(t))^{-1} \right) (dh) \, dt
= \int_0^T \int_H \left( \|h\|^2 \wedge 1 \right) \left( \lambda \circ \psi(t)^{-1} \right) (dh) \, dt,
\]

which completes the proof. \( \square \)

**Lemma 3.7.** Let \( L \) be the canonical \( \alpha \)-stable cylindrical Lévy process in \( G \) and \( (\psi_n)_{n \in \mathbb{N}} \) a sequence in \( \mathcal{S}^{\text{HS}}_{\text{det}} \). Then the following are equivalent:

(a) \( \lim_{n \to \infty} \|\psi_n\|_{L^\alpha} = 0; \)

(b) \( \lim_{n \to \infty} \sup_{\gamma \in \mathcal{S}^{\text{op}}_{\text{det}}} E \left[ \left\| \int_0^T \gamma \psi_n \, dL \right\| \wedge 1 \right] = 0. \)
Proof. Let \((\psi_n)_{n \in \mathbb{N}}\) be a sequence of elements of \(\mathcal{S}_{\text{det}}^{\text{HS}}\) with \(\lim_{n \to \infty} \|\psi_n\|_{L^\alpha} = 0\). Corollary 3 in \([14]\) and Markov’s inequality show for all \(\varepsilon > 0\) and \(p < \alpha\) that

\[
\sup_{\gamma \in \mathcal{S}_{\text{det}}^{\text{op}}} P\left( \left\| \int_0^T \gamma \psi_n \, dL \right\| > \varepsilon \right) \leq \frac{1}{\varepsilon^p} \sup_{\gamma \in \mathcal{S}_{\text{det}}^{\text{op}}} E\left[ \left( \left\| \int_0^T \gamma \psi_n \, dL \right\|^p \right]\right]
\leq \sup_{\gamma \in \mathcal{S}_{\text{det}}^{\text{op}}} \frac{1}{\varepsilon^p c_{\alpha,p}} \int_0^T \|\gamma \psi_n\|_{\text{HS}}^\alpha \, dt \leq \frac{1}{\varepsilon^p c_{\alpha,p}} \|\psi_n\|^p_{L^\alpha},
\]

where the last inequality follows from \(\|\gamma \psi_n\|_{\text{HS}} \leq \|\psi_n\|_{L^\alpha}\) for \(\gamma \in \mathcal{S}_{\text{det}}^{\text{op}}\). This proves (b) because of the equivalent characterisation of the topology in \(L_{p}^0(\Omega, H)\).

To prove the converse implication, let \((\psi_n)_{n \in \mathbb{N}}\) be a sequence of elements of \(\mathcal{S}_{\text{det}}^{\text{HS}}\) satisfying Condition (b). Since for each \(n \in \mathbb{N}\), \(\psi_n\) has a representation of the form

\[
\psi_n(t) = F_0^n \mathbb{1}_\{0\}(t) + \sum_{i=1}^{N(n)-1} F_i^n \mathbb{1}_{(t^n_i, t^n_{i+1})}(t),
\]

where \(0 = t^n_0 < \ldots < t^n_{N(n)} = T\), and \(F_i^n \in L_2(G, H)\) for each \(i \in \{0, \ldots, N(n) - 1\}\), the integral \(I(\psi_n)\) satisfies

\[
I(\psi_n) = \sum_{i=1}^{N(n)-1} F_i^n (L(t^n_{i+1}) - L(t^n_i)).
\]

The cylindrical Lévy measure of \(F_i^n(L(t^n_{i+1}) - L(t^n_i))\) is given by \((t^n_{i+1} - t^n_i)(\lambda \circ (F^n_i)^{-1})\) which extends to a genuine Lévy measure on \(\mathcal{B}(H)\) for which we keep the same notation. Independent increments of \(L\) together with Equation (3.7) show that the infinitely divisible random variable \(I(\psi_n)\) has characteristics

\[
\begin{pmatrix}
0, 0, & \sum_{i=1}^{N(n)-1} (t^n_{i+1} - t^n_i)(\lambda \circ (F^n_i)^{-1}) \\
0, & 0, & (\lambda \otimes \text{Leb}) \circ \kappa_{\psi_n}^{-1}
\end{pmatrix}
\]

where \(\kappa_{\psi_n}^{-1}\) is the Lévy measure of \(I(\psi_n)\). We conclude from Lemmata 3.5, 3.6 and 2.2 that

\[
\lim_{n \to \infty} \int_0^T \|\psi_n(t)\|_{\text{HS}}^\alpha \, dt \leq \lim_{n \to \infty} c_\alpha \int_0^T \int_H (\|h\|^2 \wedge 1) (\lambda \circ \psi_n(t)^{-1})(dh) \, dt
\]

\[
= \lim_{n \to \infty} c_\alpha \int_H (\|h\|^2 \wedge 1) (\lambda \otimes \text{Leb}) \circ \kappa_{\psi_n}^{-1}(dh) = 0,
\]

which completes the proof.  

\[\square\]
Proof of Theorem 3.4. If \( \psi \in I_{\text{det}}^{\text{HS}} \) then there exists a sequence \((\psi_n)_{n \in \mathbb{N}} \subseteq I_{\text{det}}^{\text{HS}}\) such that \( \psi_n \to \psi \) Lebesgue a.e. and \( \sup_{\gamma \in I_{\text{det}}^{1,\text{op}}} E[\|I(\gamma(\psi_m - \psi_n))\| \wedge 1] \to 0 \) as \( m,n \to \infty \). This implies \( \|\psi_m - \psi_n\|_{L^\alpha} \to 0 \) by Lemma 3.7. Completeness of \( L^\alpha \) and the fact that \( \psi_n \to \psi \) Lebesgue a.e. allows us to conclude \( \psi \in L^\alpha \). Conversely, if \( \psi \in L^\alpha \) then there exists a sequence \((\psi_n)_{n \in \mathbb{N}}\) of elements in \( I_{\text{det}}^{\text{HS}} \) such that \( \psi_n \to \psi \) Lebesgue a.e. and \( \|\psi_n - \psi\|_{L^\alpha} \to 0 \) by an extension of [9, Re. 1.2.20] to all \( \alpha \in (0,1) \). It follows that \( \|\psi_m - \psi_n\|_{L^\alpha} \to 0 \), which implies \( \sup_{\gamma \in I_{\text{det}}^{1,\text{op}}} E[\|I(\gamma(\psi_m - \psi_n))\| \wedge 1] \to 0 \) by Lemma 3.7 and establishes \( \psi \in I_{\text{det}}^{\text{HS}} \).

4 Predictable Integrands

As in the case of deterministic integrands, we begin by introducing two classes of functions on which our definition of the stochastic integral depend.

Definition 4.1.

(1) An \( L_2(G,H) \)-valued adapted step process \( \Psi: \Omega \times [0,T] \to L_2(G,H) \) is of the form

\[
\Psi(\omega,t) = \left( \sum_{k=1}^{N(0)} F_{0,k} \mathbb{1}_{A_{0,k}}(\omega) \right) \mathbb{1}_{[0)}(t) + \sum_{i=1}^{n-1} \left( \sum_{k=1}^{N(i)} F_{i,k} \mathbb{1}_{A_{i,k}}(\omega) \right) \mathbb{1}_{(t_i,t_{i+1})}(t),
\]

(4.1)

where \( 0 = t_1 < \cdots < t_n = T \), \( A_{0,k} \in F_0 \) and \( F_{0,k} \in L_2(G,H) \) for all \( k = 1,...,N(0) \), \( A_{i,k} \in F_{t_i} \) and \( F_{i,k} \in L_2(G,H) \) for all \( i = 1,...,n-1 \) and \( k = 1,...,N(i) \). The space of all \( L_2(G,H) \)-valued adapted step processes is denoted by \( S_{\text{adp}}^{\text{HS}} := S_{\text{adp}}^{\text{HS}}(G,H) \).

(2) An \( L(H,H) \)-valued adapted step process \( \Gamma: \Omega \times [0,T] \to L(H,H) \) is of the form

\[
\Gamma(\omega,t) = \left( \sum_{k=1}^{N(0)} F_{0,k} \mathbb{1}_{A_{0,k}}(\omega) \right) \mathbb{1}_{[0)}(t) + \sum_{i=1}^{n-1} \left( \sum_{k=1}^{N(i)} F_{i,k} \mathbb{1}_{A_{i,k}}(\omega) \right) \mathbb{1}_{(t_i,t_{i+1})}(t),
\]

(4.2)

where \( 0 = t_1 < \cdots < t_n = T \), \( A_{0,k} \in F_0 \) and \( F_{0,k} \in L(H,H) \) for all \( k = 1,...,N(0) \), \( A_{i,k} \in F_{t_i} \) and \( F_{i,k} \in L(H,H) \) for all \( i = 1,...,n-1 \) and \( k = 1,...,N(i) \). The space of all \( L(H,H) \)-valued adapted step processes with

\[
\sup_{(\omega,t) \in \Omega \times [0,T]} \|\Gamma(\omega,t)\|_{H \to H} \leq 1
\]

is denoted by \( S_{\text{adp}}^{1,\text{op}} := S_{\text{adp}}^{1,\text{op}}(H,H) \).
Let $\Psi \in \mathcal{S}_{\text{adp}}^{\text{HS}}$ be of the form (4.1). Since Hilbert-Schmidt operators are 0-Radonifying by [36, Th. VI.5.2], it follows from [36, Pr. VI.5.3] that there exists an $H$-valued random variable $F_{i,k}^*(L(t_{i+1}) - L(t_i)): \Omega \to H$ for each $i = 1, \ldots, n - 1$ and $k = 1, \ldots, N(i)$, satisfying

$$(L(t_{i+1}) - L(t_i)) (F_{i,k}^* h) = (F_{i,k} (L(t_{i+1}) - L(t_i)), h) \quad P\text{-a.s. for all } h \in H.$$ 

In this case, the stochastic integral of $\Psi$ is defined by

$$I(\Psi) := \int_0^T \Psi(t) \, dL(t) := \sum_{i=1}^{n-1} \sum_{k=1}^{N(i)} \mathbb{1}_{A_{i,k}} F_{i,k} (L(t_{i+1}) - L(t_i)).$$

Thus, the integral $I(\Psi) : \Omega \to H$ is a genuine $H$-valued random variable.

For the purposes of this section, it is convenient to introduce the measure space $(\Omega \times [0, T], \mathcal{P}, P_T)$, where $\mathcal{P}$ denotes the predictable $\sigma$-algebra and the measure $P_T$ is defined by $P_T := P \otimes \text{Leb}|_{[0, T]}$.

**Definition 4.2.** We say that a predictable process $\Psi$ is integrable if there exists a sequence $(\Psi_n)_{n \in \mathbb{N}}$ of processes in $\mathcal{S}_{\text{adp}}^{\text{HS}}$ such that

1. $(\Psi_n)_{n \in \mathbb{N}}$ converges $P_T$-a.e. to $\Psi$,

2. $\lim_{m,n \to \infty} \sup_{\Gamma \in \mathcal{S}_{\text{adp}}} E \left[ \left\| \int_0^T \Gamma (\Psi_m - \Psi_n) \, dL \right\| \wedge 1 \right] = 0$.

In this case, the stochastic integral of $\Psi$ is defined by

$$I(\Psi) := \int_0^T \Psi \, dL = \lim_{n \to \infty} \int_0^T \Psi_n \, dL \quad \text{in } L_P^2(\Omega, H).$$

The class of all integrable $L_2(G, H)$-valued predictable processes is denoted by $\mathcal{T}_{\text{pr}}^{\text{HS}}(G,H)$.

The space $L_P^0(\Omega, L_{\text{Leb}}^0([0, T], L_2(G, H)))$ for $\alpha \in (0, 2)$ denotes the set of all random variables $\Psi : \Omega \to L_{\text{Leb}}^0([0, T], L_2(G, H))$ endowed with the translation invariant metric $\| \cdot \|_{L_\alpha}$ defined by

$$\| \Psi_1 - \Psi_2 \|_{L_\alpha} := E \left[ \| \Psi_1 - \Psi_2 \|_{L_\alpha} \wedge 1 \right].$$

The space $L_{\text{Leb}}^\alpha([0, T], L_2(G, H))$ is separable, see [9, Pr. 1.2.29] for the proof when $\alpha \geq 1$, which can be generalised for all $\alpha > 0$. If $(\Psi(t))_{t \geq 0}$ is an $L_2(G, H)$-valued predictable stochastic process that almost surely has paths in $L_{\text{Leb}}^\alpha([0, T], L_2(G, H))$, then [4, Pr. 3.19] guarantees that the mapping $\Psi : \Omega \to L_{\text{Leb}}^\alpha([0, T], L_2(G, H))$ defines a random variable. Therefore, it is reasonable to write $\Psi \in L_P^0(\Omega, L_{\text{Leb}}^\alpha([0, T], L_2(G, H)))$ to denote this setting.
Lemma 4.3. Let $\Psi$ be a predictable stochastic process in $L^0_P(\Omega, L^0_{\text{Leb}}([0, T], L_2(G, H)))$. Then there exists a sequence $(\Psi_k)_{k \in \mathbb{N}}$ of elements of $\mathcal{S}^\text{HS}_{\text{adp}}$ converging to $\Psi$ both in $\|\cdot\|_{L^0}$ and $P_T$-a.e.

Proof. If $\Psi$ is bounded, then $\Psi \in L^\infty_P(\Omega \times [0, T], L_2(G, H))$. Since the algebra of sets $\mathcal{A}' = \{(s, t) \times B : s < t, B \in \mathcal{F}_s\} \cup \{(0) \times B : B \in \mathcal{F}_0\}$ generates $\mathcal{P}$, we conclude from [9, Le. 1.2.19] and [9, Re. 1.2.20] that there exists a sequence $(\Psi_k)_{k \in \mathbb{N}}$ of uniformly bounded processes in $\mathcal{S}^\text{HS}_{\text{adp}}$ such that $\Psi_k \to \Psi$ $P_T$-a.e.

Let $\mathcal{N}_\omega := \left\{ t \in [0, T] : (\Psi_k(\omega, t) - \Psi(\omega, t))_{m \in \mathbb{N}} \text{ does not converge to } 0 \right\}$. The above implies that $\text{Leb}_{[0, T]}(\mathcal{N}_\omega) = 0$ for almost all $\omega \in \Omega$, that is, there exists an $\Omega_0 \subseteq \Omega$ with $\text{Leb}(\Omega_0) = 1$ such that for all $\omega \in \Omega_0$ we have

$$\text{Leb}_{[0, T]}\left( t \in [0, T] : (\Psi_k(\omega, t) - \Psi(\omega, t))_{m \in \mathbb{N}} \text{ does not converge to } 0 \right) = 0.$$ 

Because $(\Psi_k)_{k \in \mathbb{N}}$ is uniformly bounded and $\Psi$ is bounded, we can conclude from Lebesgue's dominated convergence theorem that $\|\Psi_k(\omega, \cdot) - \Psi(\omega, \cdot)\|_{L^0} \to 0$ as $k \to \infty$ for each $\omega \in \Omega_0$. Another application of Lebesgue's dominated convergence theorem yields

$$\lim_{k \to \infty} \|\Psi - \Psi_k\|_{L^0} = \lim_{k \to \infty} \int_\Omega \left( \|\Psi - \Psi_k\|_{L^0} \land 1 \right) \, dP = 0,$$

which shows the claim if $\Psi$ is bounded. In the case of a general $\Psi$, we define

$$\Psi_n : \Omega \times [0, T] \to L_2(G, H), \quad \Psi_n(\omega, t) = \begin{cases} \Psi(\omega, t) & \text{if } \|\Psi(\omega, t)\|_{HS} \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $\lim_{n \to \infty} \|\Psi - \Psi_n\|_{L^0} = 0$. The first part of the proof shows that for each $n \in \mathbb{N}$ there exists a sequence $(\Psi_{n,k})_{k \in \mathbb{N}} \subseteq \mathcal{S}^\text{HS}_{\text{adp}}$ converging to $\Psi_n$ as $k \to \infty$ in $\|\cdot\|_{L^0}$ and $P_T$-a.e. For each $n \in \mathbb{N}$ choose $k_n \in \mathbb{N}$ such that $\|(\Psi_{n,k_n} - \Psi_{n,k_{n+1}})\|_{L^0} < \frac{1}{n}$. It follows that

$$\lim_{n \to \infty} \|\Psi - \Psi_{n,k_n}\|_{L^0} \leq \lim_{n \to \infty} \left( \|\Psi - \Psi_n\|_{L^0} + \|(\Psi_{n,k_n} - \Psi_{n,k_{n+1}})\|_{L^0} \right) = 0,$$

which completes the proof. \qed
5  Construction of the decoupled tangent sequence

The technique of constructing decoupled tangent sequences is a powerful tool to obtain strong results on a sequence of possibly dependent random variables. In this section, we briefly recall the fundamental definition, see e.g. Kwapień and Woyczyński [18] or de la Peña and Giné [27], and construct the decoupled tangent sequence in our setting which will enable us to identify the largest space of predictable integrands in the next section.

Remark 5.1. We repeatedly use the fact in the following that given a random variable $X$ on $(\Omega, \mathcal{F}, P)$ and another filtered probability space $(\Omega', \mathcal{F}', P')$, the random variable $X$ can always be considered as a random variable on the product space $(\Omega \times \Omega', \mathcal{F} \otimes \mathcal{F}', P \otimes P')$ by defining

$$X(\omega, \omega') = X(\omega) \quad \text{for all } (\omega, \omega') \in \Omega \times \Omega'.$$

In this case, if $X$ is real-valued and $P$-integrable we have $E_P[X] = E_{P \otimes P'}[X]$.

In the next definition, we follow closely Chapter 4.3 of [18].

Definition 5.2. Let $(\Omega, \mathcal{F}, P, (\mathcal{F}_n)_{n \in \mathbb{N}})$ be a filtered probability space and $(X_n)_{n \in \mathbb{N}}$ an $(\mathcal{F}_n)$-adapted sequence of $H$-valued random variables. If $(\Omega', \mathcal{F}', P', (\mathcal{F}'_n)_{n \in \mathbb{N}})$ is another filtered probability space, then a sequence $(Y_n)_{n \in \mathbb{N}}$ of $H$-valued random variables defined on $(\Omega \times \Omega', \mathcal{F} \otimes \mathcal{F}', P \otimes P', (\mathcal{F}_n \otimes \mathcal{F}'_n)_{n \in \mathbb{N}})$ is said to be a decoupled tangent sequence to $(X_n)_{n \in \mathbb{N}}$ if

1. for each $\omega \in \Omega$, we have that $(Y_n(\omega, \cdot))_{n \in \mathbb{N}}$ is a sequence of independent random variables on $(\Omega', \mathcal{F}', P')$;

2. the sequences $(X_n)_{n \in \mathbb{N}}$ and $(Y_n)_{n \in \mathbb{N}}$ satisfy for each $n \in \mathbb{N}$ that

$$\mathcal{L}(X_n | \mathcal{F}_{n-1} \otimes \mathcal{F}'_{n-1}) = \mathcal{L}(Y_n | \mathcal{F}_{n-1} \otimes \mathcal{F}'_{n-1}) \quad P \otimes P' - a.s.$$

The main tool for establishing the stochastic integral in the next section is a cylindrical Lévy process $\tilde{L}$ on an enlarged probability space, whose Radonified increments are decoupled to the Radonified increments of the original canonical $\alpha$-stable cylindrical Lévy process. This cylindrical Lévy process $\tilde{L}$ is explicitly constructed in the following result.

Proposition 5.3. Let $L$ be a cylindrical Lévy process on $G$, $0 = t_0 \leq \ldots \leq t_N = T$ be a partition of $[0, T]$ and for each $n = 1, \ldots, N$ we define $\Theta_n := \sum_{k=1}^{M(n)} F_{n,k} \mathbb{1}_{A_{n,k}}$, where $F_{n,k} \in L_2(G, H)$, $A_{n,k} \in \mathcal{F}_{n-1}$ for all $k = 1, \ldots, M(n)$. By defining cylindrical random variables

$$\tilde{L}(t) : G \to L^0_{P \otimes \mathbb{P}}(\Omega \times \Omega; \mathbb{R}), \quad \left(\tilde{L}(t)g\right)(\omega, \omega') = \left(L(t)g\right)(\omega'),$$

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it follows that \((\tilde{L}(t) : t \geq 0)\) is a cylindrical Lévy process on \(G\) and the sequence of its Radonified increments 
\[
\left( \Theta_n(\tilde{L}(t_n) - \tilde{L}(t_{n-1})) \right)_{n \in \{1, \ldots, N\}}
\]
defined on \((\Omega \times \Omega, \mathcal{F} \otimes \mathcal{F}, P \otimes P, (\mathcal{F}_{t_n} \otimes \mathcal{F}_{t_n})_{n \in \{0, \ldots, N\}})\) is a decoupled tangent sequence to the sequence of Radonified increments 
\[
\left( \Theta_n(L(t_n) - L(t_{n-1})) \right)_{n \in \{1, \ldots, N\}}
\]
defined on \((\Omega, \mathcal{F}, P, (\mathcal{F}_{t_n})_{n \in \{0, \ldots, N\}})\).

Proof. In order to make it easier to follow this proof, we define \(\Omega' = \Omega, \mathcal{F}' = \mathcal{F}, P' = P\) and \(\mathcal{F}'_{t_n} = \mathcal{F}_{t_n}\) for all \(n \in \{0, \ldots, N\}\) and instead of denoting the filtered product space by 
\[
\left( \Omega \times \Omega, \mathcal{F} \otimes \mathcal{F}, P \otimes P, (\mathcal{F}_{t_n} \otimes \mathcal{F}_{t_n})_{n \in \{0, \ldots, N\}} \right),
\]
we write 
\[
\left( \Omega \times \Omega', \mathcal{F} \otimes \mathcal{F}', P \otimes P', (\mathcal{F}_{t_n} \otimes \mathcal{F}'_{t_n})_{n \in \{0, \ldots, N\}} \right).
\]

The fact that for each \(t \geq 0\) the mapping \(\tilde{L}(t) : G \rightarrow L^0_{P \otimes P'}(\Omega \times \Omega', \mathbb{R})\) is continuous follows directly from the definition of \(\tilde{L}\) and Remark 5.1. Thus \(\tilde{L}\) is a cylindrical stochastic process. To prove that it is in fact a cylindrical Lévy process, let us fix \(n \in \mathbb{N}\) and \(g_1, \ldots, g_n \in G\) and consider the \(n\)-dimensional processes \(Y\) and \(Z\) defined by \(Y(t) = (\tilde{L}(t)g_1, \ldots, \tilde{L}(t)g_n)\) and \(Z(t) = (L(t)g_1, \ldots, L(t)g_n)\). It is enough to show that for any \(m \in \mathbb{N}\) and times \(0 \leq t_0 < \cdots < t_m \leq T\) the random variables \(Y(t_m) - Y(t_{m-1}), \ldots, Y(t_1) - Y(t_0)\) and \(Z(t_m) - Z(t_{m-1}), \ldots, Z(t_1) - Z(t_0)\) have the same distribution. Here we only prove that for any \(0 \leq s < t \leq T\) the random variables \(Y(t) - Y(s)\) and \(Z(t) - Z(s)\) have the same distribution. The general case follows analogously. To see this, let \(A \in \mathcal{B}(\mathbb{R}^n)\) be arbitrary. The very definition of \(\tilde{L}\) shows 
\[
(P \otimes P')(Y(t) - Y(s) \in A)
\]
\[
= (P \otimes P') \left( (\tilde{L}(t)g_1 - \tilde{L}(s)g_1, \ldots, \tilde{L}(t)g_n - \tilde{L}(s)g_n) \in A \right)
\]
\[
= (P \otimes P') (\Omega \times \{(L(t)g_1 - L(s)g_1, \ldots, L(t)g_n - L(s)g_n) \in A\})
\]
\[
= P'(((L(t)g_1 - L(s)g_1, \ldots, L(t)g_n - L(s)g_n) \in A)
\]
\[
= P(\Omega \times ((Z(t) - Z(s) \in A).
\]

To show that the Radonified increments of \(\tilde{L}\) satisfy Condition (1) of Definition 5.2, fix some \(\omega \in \Omega\). Then \(\Theta_n(\omega)\) is a (deterministic) Hilbert-Schmidt operator and \((\tilde{L}(t)(\omega, \cdot) : t \geq 0)\) is a cylindrical Lévy process in \(G\). Thus, for a fixed \(\omega \in \Omega\) and \(n \in \{1, \ldots, N\}\), the
Radonified increment $\Theta_n(\omega)(\bar{L}(t_n)(\omega, \cdot) - \bar{L}(t_{n-1})(\omega, \cdot))$ is an $\mathcal{F}'_{t_n}$-measurable $H$-valued random variable on $(\Omega', \mathcal{F}', P')$ independent of $\mathcal{F}'_{t_{n-1}}$. It follows for each $\omega \in \Omega$ that
\[
\left(\Theta_n(\omega)(\bar{L}(t_n)(\omega, \cdot) - \bar{L}(t_{n-1})(\omega, \cdot))\right)_{n \in \{1, \ldots, N\}}
\]
is a sequence of independent random variables.

For establishing Condition (2) of Definition 5.2, we define for each $n \in \{1, \ldots, N\}$ the $H$-valued random variables
\[
X_n := \Theta_n(L(t_n) - L(t_{n-1})) := \sum_{k=1}^{M(n)} \mathbb{1}_{A_{n,k}} F_{n,k}(L(t_n) - L(t_{n-1}))\]
and
\[
Y_n := \Theta_n(\bar{L}(t_n) - \bar{L}(t_{n-1})) := \sum_{k=1}^{M(n)} \mathbb{1}_{A_{n,k}} F_{n,k}(\bar{L}(t_n) - \bar{L}(t_{n-1}))\]
where $F_{n,k}(L(t_n) - L(t_{n-1}))$ and $F_{n,k}(\bar{L}(t_n) - \bar{L}(t_{n-1}))$ refer to the Radonified increments. Choose regular versions of the conditional distributions
\[
(P \otimes P')_{X_n} : \mathcal{B}(H) \times (\Omega \times \Omega') \to [0, 1],
(P \otimes P')_{X_n}(B, (\omega, \omega')) = (P \otimes P')(X_n \in B | \mathcal{F}_{t_{n-1}} \otimes \mathcal{F}'_{t_{n-1}})(\omega, \omega'),
\]
\[
(P \otimes P')_{Y_n} : \mathcal{B}(H) \times (\Omega \times \Omega') \to [0, 1],
(P \otimes P')_{Y_n}(B, (\omega, \omega')) = (P \otimes P')(Y_n \in B | \mathcal{F}_{t_{n-1}} \otimes \mathcal{F}'_{t_{n-1}})(\omega, \omega').
\]
Since $\tilde{L}(t)$ is a cylindrical Lévy process, we obtain for all $h \in H$ and $n \in \mathbb{N}$ that
\[
E_{P \otimes P'} \left[ e^{i(X_n,h)} \big| \mathcal{F}_{t_{n-1}} \otimes \mathcal{F}'_{t_{n-1}} \right] = e^{-(t_n-t_{n-1})\|\Theta_n h\|^\alpha}.
\] (5.1)

In the same way we obtain
\[
E_{P \otimes P'} \left[ e^{i(X_n,h)} \big| \mathcal{F}_{t_{n-1}} \otimes \mathcal{F}'_{t_{n-1}} \right] = e^{-(t_n-t_{n-1})\|\Theta_n h\|^\alpha}.
\] (5.2)

It follows from (5.1) and (5.2) by calculating the conditional expectation from the conditional probability, see e.g. [13, Th. 6.4], that for $P \otimes P'$ a.a. $(\omega, \omega') \in \Omega \times \Omega'$ and for all $u \in H$ we have
\[
\varphi(P \otimes P')(X_n(\cdot,(\omega,\omega')))(u) = \int_H e^{i(h,u)} (P \otimes P') X_n (dh, \omega, \omega') = e^{-(t_n-t_{n-1})\|\Theta_n h\|^\alpha}.
\]

Since characteristic functions uniquely determine distributions on $\mathcal{B}(H)$, we obtain
\[
(P \otimes P')(X_n(\cdot, (\omega,\omega'))) = (P \otimes P') Y_n(\cdot, (\omega,\omega')) \quad P \otimes P' \text{ a.s.},
\]
evidently proving Condition (2) of Definition 5.2.

\[\square\]
6 Characterisation of random integrable processes

The following is the main result of our work characterising the largest space of predictable integrands which are stochastically integrable with respect to a canonical $\alpha$-stable cylindrical Lévy process.

**Theorem 6.1.** The space $\mathcal{S}_{\text{prd}}^{\text{HS}}$ of predictable Hilbert-Schmidt operator-valued processes integrable with respect to a canonical $\alpha$-stable cylindrical Lévy process in $G$ for $\alpha \in (0, 2)$ coincides with predictable processes in $L^0_p\left(\Omega, L^\alpha_{\text{Leb}}([0, T], L^2(G, H))\right)$.

As in the case of deterministic integrands, the above characterisation of the space of integrable predictable processes strongly relies on the equivalent notion of convergences in the two spaces.

**Lemma 6.2.** Let $L$ be the canonical $\alpha$-stable cylindrical Lévy process in $G$ and $(\Psi_n)_{n \in \mathbb{N}}$ a sequence in $\mathcal{S}_{\text{adp}}^{\text{HS}}$. Then the following are equivalent:

(a) $\lim_{n \to \infty} \|\Psi_n\|_{L^\alpha} = 0$;

(b) $\lim_{n \to \infty} \sup_{\Gamma \in \mathcal{S}_{\text{adp}}^{1,\text{op}}} E\left[\left\| \int_0^T \Gamma \Psi_n \, dL \right\| \wedge 1 \right] = 0$.

**Proof.** To prove (a) $\Rightarrow$ (b), let $\epsilon > 0$ be fixed. Lemma 3.7 enables us to choose $\delta > 0$ such that for every $\psi \in \mathcal{S}_{\text{det}}^{\text{HS}}$ we have the implication:

$$\|\psi\|_{L^\alpha} \leq \delta \implies \sup_{\gamma \in \mathcal{S}_{\text{det}}^{1,\text{op}}} E\left[\left\| \int_0^T \gamma \psi \, dL \right\| \wedge 1 \right] \leq \epsilon. \quad (6.1)$$

Since $\lim_{n \to \infty} \|\Psi_n\|_{L^\alpha} = 0$, there exists $n_0 \in \mathbb{N}$ such that the set $A_n := \{\|\Psi_n\|_{L^\alpha} \leq \delta\}$ satisfies $P(A_n) \geq 1 - \epsilon$ for all $n \geq n_0$. Implication (6.1) implies for all $\omega \in A_n$ and $n \geq n_0$ that

$$\sup_{\Gamma \in \mathcal{S}_{\text{adp}}^{1,\text{op}}} P'\left(\omega' \in \Omega': \left\| \int_0^T \Gamma(\omega)\Psi_n(\omega) \, d\overline{L}(\omega, \cdot) \right\| > \epsilon \right) \leq \epsilon.$$

Since $P(A_n) \geq 1 - \epsilon$ for all $n \geq n_0$, we obtain

$$\sup_{\Gamma \in \mathcal{S}_{\text{adp}}^{1,\text{op}}} P\left(\omega \in \Omega: \left\| \int_0^T \Gamma(\omega)\Psi_n(\omega) \, d\overline{L}(\omega, \cdot) \right\| > \epsilon \right) \leq \epsilon$$

$$\geq P(A_n) \geq 1 - \epsilon.$$
Fubini’s theorem implies for all \( n \geq n_0 \) and \( \Gamma \in S^{1,\text{op}}_{\text{adp}} \) that

\[
(P \otimes P') \left( (\omega, \omega') \in \Omega \times \Omega' : \left\| \left( \int_0^T \Gamma \Psi_n \, d\tilde{L} \right) (\omega, \omega') \right\| > \epsilon \right) = \int_\Omega P' \left( \omega' \in \Omega' : \left\| \left( \int_0^T \Gamma(\omega) \Psi_n(\omega) \, d\tilde{L}(\omega, \cdot) \right) (\omega') \right\| > \epsilon \right) P(d\omega) \leq \epsilon + \epsilon(1 - \epsilon).
\]

As \( \epsilon > 0 \) is arbitrary, we obtain

\[
\lim_{n \to \infty} \sup_{\Gamma \in S^{1,\text{op}}_{\text{adp}}} E_{P \otimes P'} \left[ \left\| \int_0^T \Gamma \Psi_n \, d\tilde{L} \right\| \wedge 1 \right] = 0. \tag{6.2}
\]

By the ideal property of \( L_2(G, H) \), for each \( n \in \mathbb{N} \) the integrand \( \Gamma \Psi_n \) lies in \( S^{\text{HS}}_{\text{adp}} \) and has a representation of the form

\[
\Gamma \Psi_n = \Gamma^n_0 F^n_0 1_{(0)} + \sum_{i=1}^{N(n)-1} \Gamma^n_i F^n_i 1_{(t^n_i, t^n_{i+1}]}, \tag{6.3}
\]

where \( 0 = t^n_1 \leq \cdots \leq t^n_{N(n)} = T \), and \( \Gamma^n_i F^n_i \) is an \( F^n_{t^n_i} \)-measurable \( L_2(G, H) \)-valued random variable taking only finitely many values for each \( i = 0, \ldots, N(n) - 1 \). Proposition 5.3 guarantees for each \( n \in \mathbb{N} \) that the sequence of Radonified increments

\[
\left( \Gamma^n_i F^n_i (L(t^n_{i+1}) - L(t^n_{i})) \right)_{i=1,\ldots,N(n)-1}
\]

has the decoupled tangent sequence

\[
\left( \Gamma^n_i F^n_i (\tilde{L}(t^n_{i+1}) - \tilde{L}(t^n_{i})) \right)_{i=1,\ldots,N(n)-1}.
\]

We conclude from the decoupling inequality [18, Pr. 5.7.1.(ii)] that there exists a constant \( c > 0 \) such that, for all \( n \in \mathbb{N} \) and \( \Gamma \in S^{1,\text{op}}_{\text{adp}} \), we have

\[
E_{P \otimes P'} \left[ \left\| \int_0^T \Gamma \Psi_n \, dL \right\| \wedge 1 \right] = E_{P \otimes P'} \left[ \left\| \sum_{i=1}^{N(n)-1} \Gamma^n_i F^n_i (L(t^n_{i+1}) - L(t^n_{i})) \right\| \wedge 1 \right] \leq c E_{P \otimes P'} \left[ \left\| \sum_{i=1}^{N(n)-1} \Gamma^n_i F^n_i (\tilde{L}(t^n_{i+1}) - \tilde{L}(t^n_{i})) \right\| \wedge 1 \right] = c E_{P \otimes P'} \left[ \left\| \int_0^T \Gamma \Psi_n \, d\tilde{L} \right\| \wedge 1 \right]. \tag{6.4}
\]
We conclude from Remark 5.1 and (6.2) that
\[
\lim_{n \to \infty} \sup_{\Gamma \in \mathcal{S}_{\text{adp}}^{1, \text{op}}} E_P \left[ \left\| \int_0^T \Gamma \Psi_n \, dL \right\| \wedge 1 \right] = \lim_{n \to \infty} \sup_{\Gamma \in \mathcal{S}_{\text{adp}}^{1, \text{op}}} E_{P \otimes P'} \left[ \left\| \int_0^T \Gamma \Psi_n \, dL \right\| \wedge 1 \right] = 0,
\]
which shows (b).

For establishing (b) \(\Rightarrow\) (a), we assume the representation (6.3) of \(\Gamma \Psi_n\). We conclude from [18, Pr. 5.7.2] that there exists a constant \(c > 0\) such that
\[
E_{P \otimes P'} \left[ \left\| \int_0^T \Gamma \Psi_n \, d\tilde{L} \right\| \wedge 1 \right] = E_{P \otimes P'} \left[ \left\| \sum_{i=1}^{N(n)-1} \Gamma_i^n F_i^n (\tilde{L}(t_{i+1}^n) - \tilde{L}(t_i^n)) \right\| \wedge 1 \right]
\leq c \max_{\epsilon_i \in \{-1,1\}} E_{P \otimes P'} \left[ \left\| \sum_{i=1}^{N(n)-1} \epsilon_i \Gamma_i^n F_i^n (L(t_{i+1}^n) - L(t_i^n)) \right\| \wedge 1 \right]
= c \max_{\epsilon_i \in \{-1,1\}} E_P \left[ \left\| \sum_{i=1}^{N(n)-1} \epsilon_i \Gamma_i^n F_i^n (L(t_{i+1}^n) - L(t_i^n)) \right\| \wedge 1 \right]
\leq c \sup_{\Gamma \in \mathcal{S}_{\text{adp}}^{1, \text{op}}} E_P \left[ \left\| \sum_{i=1}^{N(n)-1} \Gamma_i^n F_i^n (L(t_{i+1}^n) - L(t_i^n)) \right\| \wedge 1 \right]
= c \sup_{\Gamma \in \mathcal{S}_{\text{adp}}^{1, \text{op}}} E_P \left[ \left\| \int_0^T \Gamma \Psi_n \, dL \right\| \wedge 1 \right]. \tag{6.5}
\]

By choosing \(\Gamma = \text{Id}_{\Omega \times [0,T]}\), the hypothesis on \((\Psi_n)_{n \in \mathbb{N}}\) implies
\[
\lim_{n \to \infty} E_{P \otimes P'} \left[ \left\| \int_0^T \Psi_n \, d\tilde{L} \right\| \wedge 1 \right] = 0.
\]

It follows that there exists a subsequence \((\Psi_{n_k})_{k \in \mathbb{N}}\) of \((\Psi_n)_{n \in \mathbb{N}}\) and a set \(N \subseteq \Omega \times \Omega'\) with \((P \otimes P')(N) = 0\) satisfying
\[
\lim_{k \to \infty} \left( \int_0^T \Psi_{n_k} \, d\tilde{L} \right)(\omega, \omega') = 0 \quad \text{for each} \ (\omega, \omega') \in N^c.
\]

Define the section of the set \(N\) for each \(\omega \in \Omega\) by
\[
N_\omega = \left\{ \omega' \in \Omega' : \lim_{k \to \infty} \left( \int_0^T \Psi_{n_k}(\omega) \, d\tilde{L}(\omega, \cdot) \right)(\omega') \neq 0 \right\},
\]

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where we note that since $\Psi_{n_k}$ are adapted step processes it holds that

$$
\left( \int_0^T \Psi_{n_k} \, d\tilde{L} \right)(\omega, \cdot) = \int_0^T \Psi_{n_k}(\omega) \, d\tilde{L}(\omega, \cdot) \quad \text{for all } \omega \in \Omega.
$$

Fubini’s theorem implies $0 = (P \otimes P')(N) = \int_\Omega P'(N_\omega) \, dP(\omega)$, from which it follows that there exists $\Omega_1 \subseteq \Omega$ with $P(\Omega_1) = 1$ such that $P'(N_\omega) = 0$ for all $\omega \in \Omega_1$. In other words, for each fixed $\omega \in \Omega_1$, the sequence of random variables

$$
\left( \int_0^T \Psi_{n_k}(\omega) \, d\tilde{L}(\omega, \cdot) \right)_{k \in \mathbb{N}}
$$

converges $P'$-a.s. to 0 as $H$-valued random variables on $(\Omega', \mathcal{F}', P')$. Lemma 3.7 implies $\lim_{n \to \infty} \|\Psi_{n_k}(\omega)\|_{L^\alpha} = 0$ for each fixed $\omega \in \Omega_1$. As $P(\Omega_1) = 1$, Lebesgue’s dominated convergence theorem implies

$$
\lim_{n \to \infty} \|\Psi_{n_k}\|_{L^\alpha} = \lim_{n \to \infty} \int_\Omega \left( \|\Psi_{n_k}(\omega)\|_{L^\alpha} \wedge 1 \right) P(\, \mathrm{d} \omega) = 0,
$$

which completes the proof.

**Proof of Theorem 6.1.** If $\Psi \in \mathcal{T}_{\text{prd}}^{\text{HS}}$ then Definition 4.2 guarantees the existence of a sequence $(\Psi_{n})_{n \in \mathbb{N}}$ of elements of $\mathcal{S}_{\text{adp}}^{\text{HS}}$ converging $P_T$-a.e. to $\Psi$ and satisfying

$$
\lim_{m,n \to \infty} \sup_{\Gamma \in \mathcal{S}_{\text{op}}^{1, \text{adp}}} E \left[ \left\| \int_0^T \Gamma(\Psi_m - \Psi_n) \, dL \right\| \wedge 1 \right] = 0.
$$

Lemma 6.2 implies that $\lim_{m,n \to \infty} \|\Psi_m - \Psi_n\|_{L^\alpha} = 0$. Completeness of the space $L^\alpha_P(\Omega, L^\alpha_{\text{Leb}}([0, T], L^2(G, H)))$ and the fact that $(\Psi_{n})_{n \in \mathbb{N}}$ converges $P_T$-a.e. to $\Psi$ yields that the sequence $(\Psi_{n})_{n \in \mathbb{N}}$ has a limit in $L^\alpha_P(\Omega, L^\alpha_{\text{Leb}}([0, T], L^2(G, H)))$ and that this limit necessarily coincides with $\Psi$. Thus $\Psi \in L^\alpha_P(\Omega, L^\alpha_{\text{Leb}}([0, T], L^2(G, H)))$.

To establish the converse inclusion, let $\Psi$ be a predictable process in the space $L^\alpha_P(\Omega, L^\alpha_{\text{Leb}}([0, T], L^2(G, H)))$. Lemma 4.3 guarantees that there exists a sequence $(\Psi_{n})_{n \in \mathbb{N}}$ of elements of $\mathcal{S}_{\text{adp}}^{\text{HS}}$ converging to $\Psi$ in $\|\cdot\|_{L^\alpha}$ and $P_T$-a.e. Consequently, $\lim_{m,n \to \infty} \|\Psi_m - \Psi_n\|_{L^\alpha} = 0$, which implies by Lemma 6.2 that

$$
\lim_{m,n \to \infty} \sup_{\Gamma \in \mathcal{S}_{\text{op}}^{1, \text{adp}}} E \left[ \left\| \int_0^T \Gamma(\Psi_m - \Psi_n) \, dL \right\| \wedge 1 \right] = 0.
$$

Thus $\Psi$ satisfies the conditions of Definition 4.2, which means that $\Psi \in \mathcal{T}_{\text{prd}}^{\text{HS}}$. \qed
Lemma 6.2 is crucial to characterise the space of integrable adapted processes in Theorem 6.1, as it describes convergence of adapted step processes in the space of integrands in terms of convergence in the explicitly given space $L^\alpha$. Having identified the space of integrable adapted processes, we can extend Lemma 6.2 to the general class of integrable processes.

**Corollary 6.3.** Let $L$ be the canonical $\alpha$-stable cylindrical Lévy process in $G$ and $(\Psi_n)_{n \in \mathbb{N}}$ a sequence in $I_{\text{prd}}$. Then the following are equivalent:

**(a)** $\lim_{n \to \infty} \|\Psi_n\|_{L^\alpha} = 0$;

**(b)** $\lim_{n \to \infty} \sup_{\Gamma \in \mathcal{S}_{\text{adp}}^{1,\text{op}}} E \left[ \left\| \int_0^T \Gamma \Psi_n \, dL \right\| \wedge 1 \right] = 0$.

**Proof.** To establish the implication (a) $\Rightarrow$ (b) let $\epsilon > 0$ be fixed. Lemma 6.2 implies that there exists a $\delta(\epsilon) > 0$ such that we have for all $\Psi \in \mathcal{S}_{\text{adp}}$ the implication:

$$
\|\Psi\|_{L^\alpha} < \delta(\epsilon) \quad \Rightarrow \quad \sup_{\Gamma \in \mathcal{S}_{\text{adp}}^{1,\text{op}}} E \left[ \left\| \int_0^T \Gamma \Psi \, dL \right\| \wedge 1 \right] < \epsilon. \quad (6.6)
$$

Since $\lim_{n \to \infty} \|\Psi_n\|_{L^\alpha} = 0$, there exists an $n_0 \in \mathbb{N}$ such that $\|\Psi_n\|_{L^\alpha} < \frac{\delta(\epsilon)}{2}$ for all $n \geq n_0$. As $(\Psi_n)_{n \in \mathbb{N}} \subseteq L^\alpha_0(\Omega, L^\alpha_{\text{Leb}}([0, T], L_2(G, H)))$, Lemma 4.3 guarantees for each $n \in \mathbb{N}$ the existence of a sequence $(\Psi^m_n)_{m \in \mathbb{N}} \subseteq \mathcal{S}_{\text{adp}}$ converging to $\Psi_n$ in $\|\cdot\|_{L^\alpha}$ and $P_T$-a.e. Consequently, we can find $m_0(n, \epsilon) \in \mathbb{N}$ for each $n \in \mathbb{N}$ such that for all $m \geq m_0(n, \epsilon)$ we have $\|\Psi^m_n - \Psi_n\|_{L^\alpha} < \frac{\delta(\epsilon)}{2}$. We obtain for each $n \geq n_0$ and $m \geq m_0(n, \epsilon)$ that

$$
\|\Psi^m_n\|_{L^\alpha} \leq \|\Psi^m_n - \Psi_n\|_{L^\alpha} + \|\Psi_n\|_{L^\alpha} < \delta(\epsilon),
$$

which implies by (6.6) that

$$
\sup_{\Gamma \in \mathcal{S}_{\text{adp}}^{1,\text{op}}} E \left[ \left\| \int_0^T \Gamma \Psi^m_n \, dL \right\| \wedge 1 \right] < \epsilon. \quad (6.7)
$$

Thus, if we fix an $n \geq n_0$ and recall that the integral of $\Psi_n$ is defined to be the limit in probability of the integrals of $\Psi^m_n$ as $m \to \infty$, we obtain from Equation (6.7) that

$$
\sup_{\Gamma \in \mathcal{S}_{\text{adp}}^{1,\text{op}}} E \left[ \left\| \int_0^T \Gamma \Psi_n \, dL \right\| \wedge 1 \right] = \sup_{\Gamma \in \mathcal{S}_{\text{adp}}^{1,\text{op}}} \lim_{m \to \infty} E \left[ \left\| \int_0^T \Gamma \Psi^m_n \, dL \right\| \wedge 1 \right] \leq \lim_{m \to \infty} \sup_{\Gamma \in \mathcal{S}_{\text{adp}}^{1,\text{op}}} E \left[ \left\| \int_0^T \Gamma \Psi^m_n \, dL \right\| \wedge 1 \right] < \epsilon.
$$
To establish the reverse implication \((b) \Rightarrow (a)\), let \(\varepsilon > 0\) be fixed. Lemma 6.2 implies that there exists a \(\delta(\varepsilon) > 0\) such that we have for all \(\Psi \in S^{HS}_{adp}\) the implication:

\[
\sup_{\Gamma \in S^{1,op}_{adp}} E\left[ \left\| \int_0^T \Gamma \Psi \, dL \right\| \wedge 1 \right] < \delta(\varepsilon) \quad \Rightarrow \quad \|\Psi\|_{L^\alpha} < \frac{\varepsilon}{2}.
\] (6.8)

By assumption, there exists an \(n_0 \in \mathbb{N}\) such that for all \(n \geq n_0\) we have

\[
\sup_{\Gamma \in S^{1,op}_{adp}} E\left[ \left\| \int_0^T \Gamma \Psi_n \, dL \right\| \wedge 1 \right] < \frac{\delta(\varepsilon)}{2}.
\] (6.9)

As \((\Psi_n)_{n \in \mathbb{N}} \subseteq T^{HS}_{prd}\), it follows from Theorem 6.1 and Lemma 4.3 that for each \(n \in \mathbb{N}\) there exists a sequence \((\Psi^m_n)_{m \in \mathbb{N}}\) of elements of \(S^{HS}_{adp}\) converging to \(\Psi_n\) in \(\|\cdot\|_{L^\alpha}\) and \(P_T\)-a.e. Consequently, we can find \(m_0(n, \varepsilon) \in \mathbb{N}\) for each \(n \in \mathbb{N}\), such that for all \(m \geq m_0(n, \varepsilon)\) we have

\[
\|\Psi^m_n - \Psi_n\|_{L^\alpha} < \varepsilon/2
\] (6.10)

Definition 4.2 shows that for each \(n \in \mathbb{N}\) there exists an \(m_1(n, \varepsilon) \in \mathbb{N}\) such that for all \(m \geq m_1(n, \varepsilon)\) we have by the reverse triangle inequality that

\[
\left| \sup_{\Gamma \in S^{1,op}_{adp}} E\left[ \left\| \int_0^T \Gamma \Psi_n \, dL \right\| \wedge 1 \right] - \sup_{\Gamma \in S^{1,op}_{adp}} E\left[ \left\| \int_0^T \Gamma \Psi^m_n \, dL \right\| \wedge 1 \right] \right| < \frac{\delta(\varepsilon)}{2}.
\] (6.11)

Combining (6.9) and (6.11) shows for \(n \geq n_0\) and \(m \geq \max\{m_0(n, \varepsilon), m_1(n, \varepsilon)\}\) that

\[
\sup_{\Gamma \in S^{1,op}_{adp}} E\left[ \left\| \int_0^T \Gamma \Psi^m_n \, dL \right\| \wedge 1 \right] < \delta(\varepsilon),
\]

which implies by (6.8) and (6.10) that

\[
\|\Psi_n\|_{L^\alpha} \leq \|\Psi_n - \Psi^m_n\|_{L^\alpha} + \|\Psi^m_n\|_{L^\alpha} < \varepsilon.
\]

As \(\varepsilon > 0\) was arbitrary, this concludes the proof. \(\square\)

**Theorem 6.4.** If \(\Psi \in T^{HS}_{prd}\), then the integral process \((I(\Psi)(t) : t \in [0, T])\) defined by

\[
I(\Psi)(t) := \int_0^T 1_{[0,t]}(s)\Psi(s) \, L(ds) \quad \text{for } t \in [0, T],
\]

is a semi-martingale.
Proof. By [11, Th. 2.1], it suffices to show that the set
\[
\left\{ \int_0^T \Gamma dI(\Psi) : \Gamma \in S_{\text{adp}}^{1,\text{op}} \right\}
\]
is bounded in probability. Suppose, aiming for a contradiction, that it is not the case. Then there exists an \( \epsilon > 0 \) and a sequence \((\Gamma_n)_{n \in \mathbb{N}} \subseteq S_{\text{adp}}^{1,\text{op}} \) satisfying for all \( n \in \mathbb{N} \) that
\[
P\left( \left\| \int_0^T \Gamma_n dI(\Psi) \right\| > n \right) \geq \epsilon. \tag{6.12}
\]
For each \( \Psi \in S_{\text{adp}}^{\text{HS}} \) and \( \Gamma \in S_{\text{adp}}^{1,\text{op}} \), the very definitions of stochastic integrals show
\[
\int_0^T \Gamma dI(\Psi) = \int_0^T \Gamma \Psi dL.
\]
This equality can be generalised to arbitrary \( \Psi \in T_{\text{prd}}^{\text{HS}} \) and \( \Gamma \in S_{\text{adp}}^{1,\text{op}} \) by a standard approximation argument. Using this to rewrite Equation (6.12), we obtain for all \( n \in \mathbb{N} \) that
\[
\epsilon \leq P\left( \left\| \int_0^T \frac{1}{n} \Gamma_n \Psi \right\| > 1 \right) = P\left( \left\| \int_0^T \Gamma_n \Psi dL \right\| > 1 \right). \tag{6.13}
\]
On the other hand, since \( \left\| \frac{1}{n} \Gamma_n \Psi \right\|_{L^2} \to 0 \) as \( n \to \infty \), Corollary 6.3 implies
\[
\lim_{n \to \infty} E \left[ \left\| \int_0^T \frac{1}{n} \Gamma_n \Psi dL \right\|_1 \right] = 0,
\]
which contradicts (6.13) because of the equivalent characterisation of the topology in \( L^0_P(\Omega,H) \).

We finish this section with a stochastic dominated convergence theorem.

**Theorem 6.5.** Let \((\Psi_n)_{n \in \mathbb{N}}\) be a sequence of processes in \( T_{\text{prd}}^{\text{HS}} \) such that

1. \((\Psi_n)_{n \in \mathbb{N}}\) converges \( P_T\)-a.e. to an \( L_2(G,H) \)-valued predictable process \( \Psi \);
2. there exists a process \( \Upsilon \in T_{\text{prd}}^{\text{HS}} \) satisfying for all \( n \in \mathbb{N} \) that
   \[
   \|\Psi_n(\omega,t)\|_{\text{HS}} \leq \|\Upsilon(\omega,t)\|_{\text{HS}} \quad \text{for } P_T\text{-a.a. } (\omega,t) \in \Omega \times [0,T].
   \]

Then it follows that \( \Psi \in T_{\text{prd}}^{\text{HS}} \) and
\[
\lim_{n \to \infty} P\left( \sup_{t \in [0,T]} \left| \int_0^t \Psi_n dL - \int_0^t \Psi dL \right| > \epsilon \right) = 0 \quad \text{for all } \epsilon > 0.
\]
Proof. By assumption, there exists a set $N \subseteq \Omega \times [0, T]$ with $P_T(N) = 0$ such that $\lim_{n \to \infty} \Psi_n(\omega, t) = \Psi(\omega, t)$ and $\|\Psi_n(\omega, t)\|_{HS} \leq \|\Psi(\omega, t)\|_{HS}$ for all $(\omega, t) \in N^c$ and $n \in \mathbb{N}$. Fubini’s theorem yields that

$$0 = P_T(N) = \int_{\Omega} \text{Leb}_{[0,T]}(N^c) P(d\omega),$$

where

$$N^c_\omega := \left\{ t \in [0, T] : \lim_{n \to \infty} \Psi_n(\omega, t) \neq \Psi(\omega, t) \right\} \text{ or } \|\Psi_n(\omega, t)\|_{HS} > \|\Psi(\omega, t)\|_{HS}.$$  

It follows that there exists an $\Omega_1 \subseteq \Omega$ with $P(\Omega_1) = 1$ such that $\text{Leb}_{[0,T]}(N^c_\omega) = 0$ for all $\omega \in \Omega_1$. Consequently, for each $\omega \in \Omega_1$ we have $\|\Psi_n(\omega, t)\|_{HS} \leq \|\Psi(\omega, t)\|_{HS}$ and $\lim_{n \to \infty} \Psi_n(\omega, t) = \Psi(\omega, t)$ for Lebesgue almost every $t \in [0, T]$. Theorem 6.1 guarantees that there exists $\Omega_2 \subseteq \Omega$ with $P(\Omega_2) = 1$ such that $\|\gamma(\omega, \cdot)\|_{L^\alpha} < \infty$ for all $\omega \in \Omega_2$. The classical Lebesgue’s dominated convergence theorem implies that $(\Psi_n(\omega))_{n \in \mathbb{N}}$ converges in $\|\cdot\|_{L^\alpha}$ and its limit $\Psi(\omega, \cdot)$ is in $L^\alpha$ for all $\omega \in \Omega_1 \cap \Omega_2$. Since $P(\Omega_1 \cap \Omega_2) = 1$, Theorem 6.1 shows $\Psi \in \mathcal{I}_{HS}$.

Another application of Lebesgue’s dominated convergence theorem establishes that $\lim_{n \to \infty} \|\Psi_n - \Psi\|_{L^\alpha} = 0$. Corollary 6.3 implies

$$\lim_{n \to \infty} \sup_{\Gamma \in \mathcal{S}_{\text{adp}}^{1, \text{op}}} E\left[\left\| \int_0^T \Gamma(\Psi_n - \Psi) \, dL \right\| \wedge 1 \right] = 0,$$  

which means that the sequence $(I(\Psi_n))_{n \in \mathbb{N}}$ of processes converges in the semi-martingale topology to the process $I(\Psi)$. Since convergence in the semi-martingale topology implies convergence in probability on compact time intervals, we have

$$\lim_{n \to \infty} P\left( \sup_{t \in [0, T]} \left\| \int_0^t \Psi_n \, dL - \int_0^t \Psi \, dL \right\| > \epsilon \right) = 0 \quad \text{for all } \epsilon > 0,$$

which completes the proof.

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