Monodromy Groups associated to Non-Isotrivial Drinfeld Modules in Generic Characteristic

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Abstract

Let $\varphi$ be a non-isotrivial family of Drinfeld $A$-modules of rank $r$ in generic characteristic with a suitable level structure over a connected smooth algebraic variety $X$. Suppose that the endomorphism ring of $\varphi$ is equal to $A$. Then we show that the closure of the analytic fundamental group of $X$ in $\text{SL}_r(\mathbb{A}_f^F)$ is open, where $\mathbb{A}_f^F$ denotes the ring of finite adèles of the quotient field $F$ of $A$.

From this we deduce two further results: (1) If $X$ is defined over a finitely generated field extension of $F$, the image of the arithmetic étale fundamental group of $X$ on the adèlic Tate module of $\varphi$ is open in $\text{GL}_r(\mathbb{A}_f^F)$. (2) Let $\psi$ be a Drinfeld $A$-module of rank $r$ defined over a finitely generated field extension of $F$, and suppose that $\psi$ cannot be defined over a finite extension of $F$. Suppose again that the endomorphism ring of $\psi$ is $A$. Then the image of the Galois representation on the adèlic Tate module of $\psi$ is open in $\text{GL}_r(\mathbb{A}_f^F)$.

Finally, we extend the above results to the case of arbitrary endomorphism rings.

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1 Analytic monodromy groups

Let $\mathbb{F}_p$ be the finite prime field with $p$ elements. Let $F$ be a finitely generated field of transcendence degree 1 over $\mathbb{F}_p$. Let $A$ be the ring of elements of $F$ which are regular outside a fixed place $\infty$ of $F$. Let $M$ be the fine moduli space over $F$ of Drinfeld $A$-modules of rank $r$ with some sufficiently high level structure. This is a smooth affine scheme of dimension $r-1$ over $F$.

Let $F_\infty$ denote the completion of $F$ at $\infty$, and $\mathbb{C}$ the completion of an algebraic closure of $F_\infty$. Then the rigid analytic variety $M_\mathbb{C}^{an}$ is a finite disjoint union of spaces of the form $\Delta \setminus \Omega$, where $\Omega \subset (\mathbb{P}_\mathbb{C}^{r-1})^{an}$ is Drinfeld’s upper half space and $\Delta$ is a congruence subgroup of $\text{SL}_r(F)$ commensurable with $\text{SL}_r(A)$.

Let $X_\mathbb{C}$ be a smooth irreducible locally closed algebraic subvariety of $M_\mathbb{C}$. Then $X_\mathbb{C}^{an}$ lies in one of the components $\Delta \setminus \Omega$ of $M_\mathbb{C}^{an}$. Fix an irreducible component $\Xi \subset \Omega$ of the pre-image of $X_\mathbb{C}^{an}$. Then $\Xi \to X_\mathbb{C}^{an}$ is an unramified Galois covering with Galois group $\Delta := \text{Stab}_{\Delta}(\Xi)$.

Let $\varphi$ denote the family of Drinfeld modules over $X_\mathbb{C}$ determined by the embedding $X_\mathbb{C} \subset M_\mathbb{C}$. We assume that $\dim X_\mathbb{C} \geq 1$. Since $M$ is a fine moduli space, this means that $\varphi$ is non-isotrivial. It also implies that $r \geq 2$. Let $\eta_\mathbb{C}$ be the generic point of $X_\mathbb{C}$ and $\bar{\eta_\mathbb{C}}$ a geometric point above it. Let $\varphi_\mathbb{C}$ denote the pullback of $\varphi$ to $\bar{\eta_\mathbb{C}}$. Let $\mathbb{A}_f^F$ denote the ring of finite adèles of $F$. The main result of this article is the following:

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Theorem 1.1 In the above situation, if \( \text{End}_{\mathbb{C}}(\varphi_{\mathbb{C}}) = A \), then the closure of \( \Delta_{\Xi} \) in \( SL_r(\mathbb{A}_F) \) is an open subgroup of \( SL_r(\mathbb{A}_F) \).

The proof uses known results on the \( p \)-adic Galois representations associated to Drinfeld modules \( \mathbb{C} \) and on strong approximation \( \mathbb{C} \).

Theorem 1.1 leaves open the following natural question:

**Question 1.2** If \( \text{End}_{\mathbb{C}}(\varphi_{\mathbb{C}}) = A \), is \( \Delta_{\Xi} \) an arithmetic subgroup of \( SL_r(F) \)?

Theorem 1.1 has applications to the analogue of the André-Oort conjecture for Drinfeld moduli spaces: see \[3\]. Consequences for étale monodromy groups and for Galois representations are explained in Sections \[2\] and \[3\]. The proof of Theorem 1.1 will be given in Sections \[4\] through \[7\]. Finally, in Section \[8\] we outline the case of arbitrary endomorphism rings.

For any variety \( Y \) over a field \( k \) and any extension field \( L \) of \( k \) we will abbreviate \( Y_L := Y \times_k L \).

2 Étale monodromy groups

We retain the notations from Section \[1\]. Let \( k \subset \mathbb{C} \) be a subfield that is finitely generated over \( F \), such that \( X_\mathbb{C} = X \times_k \mathbb{C} \) for a subvariety \( X \subset M_k \). Let \( K \) denote the function field of \( X \) and \( K^{\text{sep}} \) a separable closure of \( K \). Then \( \eta := \text{Spec} K \) is the generic point of \( X \) and \( \bar{\eta} := \text{Spec} K^{\text{sep}} \) a geometric point above \( \eta \). Let \( k^{\text{sep}} \) be the separable closure of \( k \) in \( K^{\text{sep}} \). Then we have a short exact sequence of étale fundamental groups

\[ 1 \longrightarrow \pi_1(X^{k^{\text{sep}}}, \bar{\eta}) \longrightarrow \pi_1(X, \bar{\eta}) \longrightarrow \text{Gal}(k^{\text{sep}}/k) \longrightarrow 1. \]

Let \( \hat{A} \cong \prod_{p \neq \infty} A_p \) denote the profinite completion of \( A \). Recall that \( \mathbb{A}_F \cong F \otimes_A \hat{A} \) and contains \( A \) as an open subgroup. Let \( \varphi_\eta \) denote the Drinfeld module over \( K \) corresponding to \( \eta \). Its adelic Tate module \( \hat{T}(\varphi_\eta) \) is a free module of rank \( r \) over \( \hat{A} \). Choose a basis and let

\[ \rho : \pi_1(X, \bar{\eta}) \longrightarrow \text{GL}_r(\hat{A}) \subset GL_r(\mathbb{A}_F) \]

denote the associated monodromy representation. Let \( \Gamma_{\text{geom}} \subset \Gamma \subset \text{GL}_r(\hat{A}) \) denote the images of \( \pi_1(X^{k^{\text{sep}}}, \bar{\eta}) \subset \pi_1(X, \bar{\eta}) \) under \( \rho \).

**Lemma 2.1** \( \Gamma_{\text{geom}} \) is the closure of \( g^{-1} \Delta_\Xi g \) in \( SL_r(\hat{A}) \) for some element \( g \in GL_r(\mathbb{A}_F) \).

**Proof.** Choose an embedding \( K^{\text{sep}} \hookrightarrow \mathbb{C} \) and a point \( \xi \in \Xi \) above \( \bar{\eta} \). Let \( \Lambda \subset F^r \) be the lattice corresponding to the Drinfeld module at \( \xi \). This is a finitely generated projective \( A \)-module of rank \( r \). The choice of a basis of \( \hat{T}(\varphi_\eta) \) yields a composite embedding

\[ \hat{A}^r \cong \hat{T}(\varphi_\eta) \cong \Lambda \otimes_A \hat{A} \hookrightarrow F^r \otimes_A \hat{A} \cong (\mathbb{A}_F)^r, \]

which is given by left multiplication with some element \( g \in GL_r(\mathbb{A}_F) \). Since the discrete group \( \Delta \subset SL_r(F) \) preserves \( \Lambda \), we have \( g^{-1} \Delta g \subset SL_r(\hat{A}) \).

For any non-zero ideal \( \mathfrak{a} \subset A \) let \( M(\mathfrak{a}) \) denote the moduli space obtained from \( M \) by adjoining a full level \( \mathfrak{a} \) structure. Then \( \pi_1 : M(\mathfrak{a}) \to M \) is an étale Galois covering with group contained in \( GL_r(A/\mathfrak{a}) \), and one of the connected components of \( M(\mathfrak{a})^{an} \) above the connected component \( \Delta \backslash \Omega \) of \( M^{an} \) has the form \( \Delta(\mathfrak{a}) \backslash \Omega \) for

\[ \Delta(\mathfrak{a}) := \{ \delta \in \Delta \mid g^{-1} \delta g \equiv \text{id mod } \mathfrak{a} \hat{A} \}. \]

Let \( X(\mathfrak{a})_{k^{\text{sep}}} \) be any connected component of the inverse image \( \pi_1^{-1}(X^{k^{\text{sep}}}) \subset M(\mathfrak{a})_{k^{\text{sep}}} \). Since \( k^{\text{sep}} \) is separably closed, the variety \( X(\mathfrak{a}) \) over \( \mathbb{C} \) obtained by base change is again connected. The
associated rigid analytic variety $X(a)_{\mathbb{C}}^{an}$ is then also connected (cf. [2, Kor. 3.5]) and therefore a connected component of $\pi^{-1}_a(X_{\mathbb{C}}^{an})$. But one of these connected components is $\left(\Delta_{\mathbb{E}} \cap \Delta(a)\right) \setminus \Xi$, whose Galois group over $X_{\mathbb{C}}^{an} \cong \Delta_{\mathbb{E}} \setminus \Xi$ is $\Delta_{\mathbb{E}}/\left(\Delta_{\mathbb{E}} \cap \Delta(a)\right)$. This implies that $g^{-1}\Delta_{\mathbb{E}} g$ and $\pi_1(X_{ksep}, \tilde{\eta})$ have the same images in $\text{GL}_r(A/\mathfrak{a}) = \text{GL}_r(A/\mathfrak{a}A)$. By taking the inverse limit over the ideal $\mathfrak{a}$ we deduce that the closure of $g^{-1}\Delta_{\mathbb{E}} g$ in $\text{SL}_r(A)$ is $\Gamma_{\text{geom}}$, as desired. □

Lemma 2.2 $\text{End}_{K_{\text{sep}}} (\varphi_{\eta}) = \text{End}_{\bar{K}_C} (\varphi_{\bar{K}_C})$.

Proof. By construction $\bar{\eta}_C$ is a geometric point above $\eta$, and $\varphi_{\bar{K}_C}$ is the pullback of $\varphi_{\eta}$. Any embedding of $K_{\text{sep}}$ into the residue field of $\bar{K}_C$ induces a morphism $\bar{\eta}_C \to \bar{\eta}$. Thus the assertion follows from the fact that for every Drinfeld module over a field, any endomorphism defined over any field extension is already defined over a finite separable extension. □

Theorem 2.3 In the above situation, suppose that $\text{End}_{K_{\text{sep}}} (\varphi_{\eta}) = A$. Then

(a) $\Gamma_{\text{geom}}$ is an open subgroup of $\text{SL}_r(\mathbb{A}_F^f)$, and

(b) $\Gamma$ is an open subgroup of $\text{GL}_r(\mathbb{A}_F^f)$.

Proof. By Lemma 2.2 the assumption implies that $\text{End}_{\bar{K}_C} (\varphi_{\bar{K}_C}) = A$. Thus part (a) follows at once from Theorem 1.1 and Lemma 2.1. Part (b) follows from (a) and the fact that $\det(\Gamma)$ is open in $\text{GL}_1(\mathbb{A}_F^f)$. This fact is a consequence of work of Drinfeld [11 §8 Thm. 1] and Hayes [6 Thm. 9.2] on the abelian class field theory of $F$, and of Anderson [11] on the determinant Drinfeld module. Note that Anderson’s paper only treats the case $A = \mathbb{F}_q[T]$; the general case has been worked out by van der Heiden [7 Chap. 4]. Compare also [9] Thm. 1.8. □

3 Galois groups

Let $F$ and $A$ be as in Section 1. Let $K$ be a finitely generated extension field of $F$ of arbitrary transcendence degree, and let $\psi : A \to K(\tau)$ be a Drinfeld $A$-module of rank $r$ over $K$. Let $K_{\text{sep}}$ denote a separable closure of $K$ and

$$\sigma : \text{Gal}(K_{\text{sep}}/K) \to \text{GL}_r(\mathbb{A}_F^f)$$

the natural representation on the adèlic Tate module of $\psi$. Let $\Gamma \subset \text{GL}_r(\mathbb{A}_F^f)$ denote its image.

Theorem 3.1 In the above situation, suppose that $\text{End}_{K_{\text{sep}}} (\psi) = A$ and that $\psi$ cannot be defined over a finite extension of $F$ inside $K_{\text{sep}}$. Then $\Gamma$ is an open subgroup of $\text{GL}_r(\mathbb{A}_F^f)$.

Proof. The assertion is invariant under replacing $K$ by a finite extension. We may therefore assume that $\psi$ possesses a sufficiently high level structure over $K$. Then $\psi$ corresponds to a $K$-valued point on the moduli space $M$ from Section 1. Let $\eta$ denote the underlying point on the scheme $M$, and let $L \subset K$ be its residue field. Then $\psi$ is already defined over $L$, and $\sigma$ factors through the natural homomorphism $\text{Gal}(K_{\text{sep}}/K) \to \text{Gal}(L_{\text{sep}}/L)$, where $L_{\text{sep}}$ is the separable closure of $L$ in $K_{\text{sep}}$. Since $K$ is finitely generated over $L$, the intersection $K \cap L_{\text{sep}}$ is finite over $L$; hence the image of this homomorphism is open. To prove the theorem we may thus replace $K$ by $L$, after which $K$ is the residue field of $\eta$.

The assumption on $\psi$ implies that even after this reduction, $K$ is not a finite extension of $F$. Therefore its transcendence degree over $F$ is $\geq 1$. Let $k$ denote the algebraic closure of $F$ in $K$. Then $\eta$ can be viewed as the generic point of a geometrically irreducible and reduced locally closed algebraic subvariety $X \subset M_k$ of dimension $\geq 1$. After shrinking $X$ we may assume that...
X is smooth. We are then precisely in the situation of the preceding section, with \( \psi = \varphi_\eta \). The homomorphism \( \sigma \) above is then the composite
\[
\Gal(K^{\sep}/K) \cong \pi_1(\eta, \bar{\eta}) \twoheadrightarrow \pi_1(X, \bar{\eta}) \xrightarrow{\rho} \GL_r(A_1^f)
\]
with \( \rho \) as in Section 2. It follows that the groups called \( \Gamma \) in this section and the last coincide. The desired openness is now equivalent to Theorem 4.10 (b).

**Note:** The adelic openness for a Drinfeld module defined over a finite extension of \( F \) is still unproved.

### 4 \( p \)-Adic openness

This section and the next three are devoted to proving Theorem 1.1. Throughout we retain the notations from Sections 1 and 2 and the assumptions \( \dim X \geq 1 \) and \( \End_{K^{\sep}}(\varphi_\eta) = A \). In this section we recall a known result on \( p \)-adic openness. For any place \( p \neq \infty \) of \( F \) let \( \Gamma_p \) denote the image of \( \Gamma \) under the projection \( \GL_r(A_1^f) \to \GL_r(F_p) \).

**Theorem 4.1** \( \Gamma_p \) is open in \( \GL_r(F_p) \).

**Proof.** By construction \( \Gamma_p \) is the image of the monodromy representation
\[
\rho_p: \pi_1(X, \bar{\eta}) \longrightarrow \GL_r(F_p)
\]
on the rational \( p \)-adic Tate module of \( \varphi_\eta \). This is the same as the image of the composite homomorphism
\[
\Gal(K^{\sep}/K) \cong \pi_1(\eta, \bar{\eta}) \twoheadrightarrow \pi_1(X, \bar{\eta}) \xrightarrow{\rho_p} \GL_r(F_p).
\]
Since \( K \) is a finitely generated extension of \( F \), and \( \End_{K^{\sep}}(\varphi_\eta) = A \) by the assumption and Lemma 2.2 the desired openness is a special case of [J Thm. 0.1].

Next let \( \Gamma_p^{\text{geom}} \) denote the image of \( \Gamma_p^{\text{geom}} \) under the projection \( \GL_r(A_1^f) \to \GL_r(F_p) \). Note that this is a normal subgroup of \( \Gamma_p \). Lemma 4.1 immediately implies:

**Lemma 4.2** \( \Gamma_p^{\text{geom}} \) is the closure of \( g^{-1} \Delta_{\Xi} g \) in \( \SL_r(F_p) \) for some element \( g \in \GL_r(F_p) \).

### 5 Zariski density

**Lemma 5.1** The Zariski closure \( H \) of \( \Delta_{\Xi} \) in \( \GL_{r,F} \) is a normal subgroup of \( \GL_{r,F} \).

**Proof.** Choose a place \( p \neq \infty \) of \( F \). Then by base extension \( H_{F_p} \) is the Zariski closure of \( \Delta_{\Xi} \) in \( \GL_{r,F_p} \). Thus Lemma 4.2 implies that \( g^{-1} H_{F_p} g \) is the Zariski closure of \( \Gamma_p^{\text{geom}} \) in \( \GL_{r,F_p} \). Since \( \Gamma_p \) normalizes \( \Gamma_p^{\text{geom}} \), it therefore normalizes \( g^{-1} H_{F_p} g \). But \( \Gamma_p \) is open in \( \GL_r(F_p) \) by Theorem 4.1 and therefore Zariski dense in \( \GL_{r,F_p} \). Thus \( \GL_{r,F_p} \) normalizes \( g^{-1} H_{F_p} g \) and hence \( H_{F_p} \), and the result follows.

**Lemma 5.2** \( \Delta_{\Xi} \) is infinite.

**Proof.** Let \( X, K, k \) and \( \varphi_\eta \) be as in Section 2. Then, as \( M_k \) is affine and \( \dim X \geq 1 \), there exists a valuation \( v \) of \( K \), corresponding to a point on the boundary of \( X \) not on \( M_k \), at which \( \varphi_\eta \) does not have potential good reduction. Denote by \( I_v \subset \Gal(K^{\sep}/K^{\sep}) \) the inertia group at \( v \). By the criterion of Néron-Ogg-Shafarevich [5 §4.10], the image of \( I_v \) in \( \Gamma_p^{\text{geom}} \) is infinite for any place \( p \neq \infty \) of \( F \). In particular, \( \Delta_{\Xi} \) is infinite by Lemma 4.2 as desired.
Alternatively, we may argue as follows. Suppose that $\Delta$ is finite. Then after increasing the level structure we may assume that $\Delta = 1$. Then $\Gamma_{p, \text{geom}} = 1$ by Lemma 5.2 which means that $\rho_p$ factors as

$$\pi_1(X, \bar{\eta}) \to \text{Gal}(k^{\text{sep}}/k) \to \GL_r(F_p).$$

After a suitable finite extension of the constant field $k$ we may assume that $X$ possesses a $k$-rational point $x$. Let $\varphi_x$ denote the Drinfeld module over $k$ corresponding to $x$. Via the embedding $k \subset K$ we may consider it as a Drinfeld module over $K$ and compare it with $\varphi_\eta$. The factorization above implies that the Galois representations on the $p$-adic Tate modules of $\varphi_x$ and $\varphi_\eta$ are isomorphic. By the Tate conjecture (see 12 or 13) this implies that there exists an isogeny $\varphi_x \to \varphi_\eta$ over $K$. Its kernel is finite and therefore defined over some finite extension $k'$ of $k$. Thus $\varphi_\eta$, as a quotient of $\varphi_x$ by this kernel, is isomorphic to a Drinfeld module defined over $k'$. But the assumption $\dim X \geq 1$ implies that $\eta$ is not a closed point of $M_k$; hence $\varphi_\eta$ cannot be defined over a finite extension of $k$. This is a contradiction.

**Proposition 5.3** $\Delta$ is Zariski dense in $\SL_{r,F}$.

**Proof.** By construction we have $H \subset \SL_{r,F}$, and Lemma 5.2 implies that $H$ is not contained in the center of $\SL_{r,F}$. From Lemma 5.1 it now follows that $H = \SL_{r,F}$, as desired.

The above results may be viewed as analogues of André’s results [2, Thm. 1, Prop. 2], comparing the monodromy group of a variation of Hodge structures with its generic Mumford-Tate group. Our analogue of the former is $\Delta$, and by [3] the latter corresponds to $\GL_{r,F}$. In our situation, however, we do not need the existence of a special point on $X$.

6 Fields of coefficients

Let $\bar{\Delta}$ denote the image of $\Delta$ in $\PGL_r(F)$. In this section we show that the field of coefficients of $\bar{\Delta}$ cannot be reduced.

**Definition 6.1** Let $L_1$ be a subfield of a field $L$. We say that a subgroup $\bar{\Delta} \subset \PGL_r(L)$ lies in a model of $\PGL_{r,L}$ over $L_1$, if there exist a linear algebraic group $G_1$ over $L_1$ and an isomorphism $\lambda_1 : G_{1,L_1} \sim \to \PGL_{r,L}$, such that $\bar{\Delta} \subset \lambda_1(G_1(L_1))$. $\Delta$ does not lie in a model of $\PGL_{r,F}$ over a proper subfield of $F$.

**Proof.** As before we use an arbitrary auxiliary place $p \neq \infty$ of $F$. Let $\bar{\Gamma}_{p, \text{geom}} \subset \bar{\Gamma}_p$ denote the images of $\Gamma_{p, \text{geom}} \subset \Gamma_p$ in $\PGL_r(F_p)$. Lemma 5.2 implies that $\bar{\Gamma}_p$ is conjugate to the closure of $\bar{\Delta}$ in $\PGL_r(F_p)$. By Proposition 5.3 it is therefore Zariski dense in $\PGL_{r,F_p}$. On the other hand Theorem 3.1 implies that $\bar{\Gamma}_p$ is an open subgroup of $\PGL_r(F_p)$. It therefore does not lie in a model of $\PGL_{r,F_p}$ over a proper subfield of $F_p$. Thus $\bar{\Gamma}_{p, \text{geom}}$ is Zariski dense and normal in a subgroup that does not lie in a model over a proper subfield of $F_p$, which by [10, Cor. 3.8] implies that $\bar{\Gamma}_{p, \text{geom}}$, too, does not lie in a model over a proper subfield of $F_p$.

Suppose now that $\Delta \subset \lambda_1(G_1(F_1))$ for a subfield $F_1 \subset F$, a linear algebraic group $G_1$ over $F_1$, and an isomorphism $\lambda_1 : G_{1,F_1} \sim \to \PGL_{r,F_1}$. Since $\bar{\Delta}$ is Zariski dense in $\PGL_{r,F}$, it is in particular infinite. Therefore $F_1$ must be infinite. As $F$ is finitely generated of transcendence degree $1$ over $F_p$, it follows that $F_1$ contains a transcendental element, and so $F$ is a finite extension of $F_1$. Let $p_1$ denote the place of $F_1$ below $p$. Since $\bar{\Gamma}_{p, \text{geom}}$ is the closure of $\bar{\Delta}$ in $\PGL_r(F_p)$, it is contained in $\lambda_1(G_1(F_{1,p_1}))$. The fact that $\bar{\Gamma}_{p, \text{geom}}$ does not lie in a model over a proper subfield of $F_p$ thus implies that $F_{1,p_1} = F_p$.

But for any proper subfield $F_1 \subset F$, we can choose a place $p \neq \infty$ of $F$ above a place $p_1$ of $F_1$, such that the local field extension $F_{1,p_1} \subset F_p$ is non-trivial. Thus we must have $F_1 = F$, as desired. □
7 Strong approximation

The remaining ingredient is the following general theorem.

**Theorem 7.1** For \( r \geq 2 \) let \( \Delta \subset \text{SL}_r(F) \) be a subgroup that is contained in a congruence subgroup commensurable with \( \text{SL}_r(A) \). Assume that \( \Delta \) is Zariski dense in \( \text{SL}_r,F \) and that its image \( \overline{\Delta} \) in \( \text{PGL}_r(F) \) does not lie in a model of \( \text{PGL}_r,F \) over a proper subfield of \( F \). Then the closure of \( \Delta \) in \( \text{SL}_r(A_f,F) \) is open.

**Proof.** For finitely generated subgroups this is a special case of [11] Thm. 0.2. That result concerns arbitrary finitely generated Zariski dense subgroups of \( G(F) \) for arbitrary semisimple algebraic groups \( G \), but it uses the finite generation only to guarantee that the subgroup is integral at almost all places of \( F \). For \( \Delta \) as above the integrality at all places \( \neq \infty \) is already known in advance, so the proof in [11] covers this case as well.

As an alternative, we will deduce the general case by showing that every sufficiently large finitely generated subgroup \( \Delta \subset \Delta \) satisfies the same assumptions. Then the closure of \( \Delta \) in \( \text{SL}_r(A_f,F) \) is open by [11], and so the same follows for \( \Delta \), as desired.

For the Zariski density of \( \Delta \) note first that the trace of the adjoint representation defines a dominant morphism to the affine line \( \text{SL}_r,F \to A_1,F, \ g \to \text{tr}(\text{Ad}(g)) \). Since \( \Delta \) is Zariski dense, this function takes infinitely many values on \( \Delta \). As the field of constants in the Zariski closure \( H \) and that \( F \) consisting of a subfield dominant morphism to the affine line \( \text{SL}_r,F \) is open by [11], and so the same follows for \( \Delta \), as desired.

Consider another finitely generated subgroup \( \Delta \subset \Delta \) and let \( \overline{\Delta} \) denote its image in \( \text{PGL}_r(F) \). Consider all triples \((F_1,G_1,\lambda_1)\) consisting of a subfield \( F_1 \subset F \), a linear algebraic group \( G_1 \) over \( F_1 \), and an isomorphism \( \lambda_1 : G_1,F \to \text{PGL}_r,F \), such that \( \overline{\Delta} \subset \lambda_1(G_1(F_1)) \). By [11] Thm. 3.6 there exists such a triple with \( F_1 \) minimal, and this \( F_1 \) is unique, and \( G_1 \) and \( \lambda_1 \) are determined up to unique isomorphism. Consider another finitely generated subgroup \( \Delta \subset \Delta \) and let \( (F_2,H_2,\lambda_2) \) be the minimal triple associated to it. Then the uniqueness of \((F_1,G_1,\lambda_1)\) implies that \( F_1 \subset F_2 \), that \( G_2 \cong G_1,F_2 \), and that \( \lambda_2 \) coincides with the isomorphism \( G_2,F \cong G_1,F \to \text{PGL}_r,F \) obtained from \( \lambda_1 \). In other words, the minimal model \((F_1,G_1,\lambda_1)\) is monotone in \( \Delta \).

For any increasing sequence of Zariski dense finitely generated subgroups of \( \Delta \) we thus obtain an increasing sequence of subfields of \( F \). This sequence must become constant, say equal to \( F_1 \subset F \), and the associated model of \( \text{PGL}_r,F \) over \( F_1 \) is the same up to isomorphism from that point onwards. Thus we have a triple \((F_1,G_1,\lambda_1)\) with \( \overline{\Delta} \subset \lambda_1(G_1(F_1)) \) for every sufficiently large finitely generated subgroup \( \Delta \subset \Delta \). But then we also have \( \overline{\Delta} \subset \lambda_1(G_1(F_1)) \), which by assumption implies that \( F_1 = F \). Thus every sufficiently large finitely generated subgroup of \( \Delta \) satisfies the same assumptions as \( \Delta \), as desired.

**Proof of Theorem 7.1.** In the situation of Theorem 7.1 we automatically have \( r \geq 2 \), so the assertion follows by combining Propositions 5.3 and 6.2 with Theorem 7.1 for \( \Delta \).

8 Arbitrary endomorphism rings

Set \( E := \text{End}_{k_r}(\varphi_{r|k}) \), which is a finite integral ring extension of \( A \). Write \( r = r' \cdot [E:A] \); then the centralizer of \( E \) in \( \text{GL}_r(A_f) \) is isomorphic to \( \text{GL}_{r'}(E \otimes_A A_f) \). Lemma 2.2 implies that all elements of \( E \) are defined over some fixed finite extension of \( K \). This means that an open subgroup of \( \rho(\pi_X,\eta) \) is contained in \( \text{GL}_{r'}(E \otimes_A A_f) \). Thus by Lemma 2.3 the same holds for a subgroup of finite index of \( \Delta \). The following results can be deduced easily from Theorems 7.1 and 8.2 using the same arguments as in [9] end of §2.
Theorem 8.1 In the situation of before Theorem 1.1, for $E := \text{End}_{\overline{\mathbb{Q}}}(\varphi_{\overline{\eta}})$ arbitrary, the closure in $\text{GL}_r(\mathbb{A}_F^f)$ of some subgroup of finite index of $\Delta_{\overline{\eta}}$ is an open subgroup of $\text{SL}_r'(E \otimes_A \mathbb{A}_F^f)$.

Theorem 8.2 In the situation of before Theorem 2.3, for $E := \text{End}_{K_{\text{sep}}}(\varphi_{\eta})$ arbitrary,

(a) some open subgroup of $\Gamma_{\text{geom}} := \rho(\pi_1(X_{K_{\text{sep}}}, \overline{\eta}))$ is an open subgroup of $\text{SL}_r'(E \otimes_A \mathbb{A}_F^f)$, and

(b) some open subgroup of $\Gamma := \rho(\pi_1(X, \overline{\eta}))$ is an open subgroup of $\text{GL}_r'(E \otimes_A \mathbb{A}_F^f)$.

Theorem 8.3 In the situation of before Theorem 3.1, for $E := \text{End}_{K_{\text{sep}}}((\psi)$ arbitrary, suppose that $\psi$ cannot be defined over a finite extension of $F$ inside $K_{\text{sep}}$. Then some open subgroup of $\Gamma := \sigma(\text{Gal}(K_{\text{sep}}/K))$ is an open subgroup of $\text{GL}_r'(E \otimes_A \mathbb{A}_F^f)$.

References

[1] Anderson, G.: $t$-Motives. Duke Math. J. 53, 2 (1986), 457–502.

[2] André, Y.: Mumford-Tate groups of mixed Hodge structures and the theorem of the fixed part. Compositio Math. 82 (1992), 1–24.

[3] Breuer, F.: Special subvarieties of Drinfeld modular varieties. In preparation.

[4] Drinfeld, V. G.: Elliptic modules (Russian). Math. Sbornik 94 (1974), 594–627, = Math. USSR-Sb. 23 (1974), 561–592.

[5] Goss, D.: Basic Structures of Function Field Arithmetic. Ergebnisse 35, Berlin etc.: Springer (1996).

[6] Hayes, D. R.: Explicit Class Field Theory in Global Function Fields. In: Studies in Algebra and Number Theory, Adv. Math., Suppl. Stud. 6, Academic Press (1979), 173–217.

[7] van der Heiden, G.-J.: Weil pairing and the Drinfeld modular curve. Ph.D. thesis, Rijksuniversiteit Groningen, 2003.

[8] Lütkebohmert, W.: Der Satz von Remmert-Stein in der nichtarchimedischen Funktionentheorie. Math. Z. 139 (1974), 69–84.

[9] Pink, R.: The Mumford-Tate conjecture for Drinfeld modules. Publ. RIMS, Kyoto University 33 (1997), 393–425.

[10] Pink, R.: Compact subgroups of linear algebraic groups. J. Algebra 206 (1998) 438–504.

[11] Pink, R.: Strong approximation for Zariski dense subgroups over arbitrary global fields. Comm. Math. Helv. 75 vol. 4 (2000) 608–643.

[12] Taguchi, Y.: The Tate conjecture for $t$-motives. Proc. Am. Math. Soc. 123 (1995), 3285–3287.

[13] Tamagawa, A.: The Tate conjecture and the semisimplicity conjecture for $t$-modules. RIMS Kokyuroku (Proc. RIMS) 925 (1995), 89–94.