Pfaffian Systems of A-Hypergeometric Equations
I: Bases of Twisted Cohomology Groups

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Abstract: We consider bases of Pfaffian systems for $A$-hypergeometric systems. These are given by Gröbner deformations, they also provide bases for twisted cohomology groups. For a hypergeometric system associated with a class of order polytopes, these bases have a combinatorial description. The size of the bases associated with a subclass of the order polytopes has a growth rate of polynomial order.

1 Introduction

Let $g(x,t) = \sum_{a \in \mathcal{A}} x_a t^a$, $t^a = t_1^{a_1} \cdots t_d^{a_d}$ be a generic sparse polynomial in $t = (t_1, \ldots, t_d)$ with the support on a finite set of points $\mathcal{A} \subset \mathbb{Z}^d$. The coefficients $x_a, a \in \mathcal{A}$ are denoted by $x_i, i = 1, \ldots, n$. The function defined by the integral

$$\Phi(x) = \int_{C_1} g(x,t)^{\alpha} t^\gamma dt,$$  

over a cycle $C_i$ in the $t$-space is called the $A$-hypergeometric function of $x$ with parameters $\alpha \in \mathbb{C}, \gamma_i \in \mathbb{C}$ [11], [1], [9]. It is known that the $A$-hypergeometric function satisfies a system of linear partial differential equations in $x$, which is called the $A$-hypergeometric system or equation. The $A$-hypergeometric system is a holonomic system, and the operators of the system generate a zero-dimensional ideal in the ring of differential operators with rational function coefficients (see, e.g., [12] Chapter 6). $A$-hypergeometric systems have been studied for the past 25 years (see, e.g., [10], [11], [21]), and they have applications in various fields.

Let $F$ be a vector-valued function in $x_1, \ldots, x_n$. We suppose the length of $F$ is $r$ and that $F$ is a column vector. Let $P_i(x), i = 1, \ldots, n$, be $r \times r$ matrices satisfying

$$\frac{\partial P_i}{\partial x_j} + P_i P_j = \frac{\partial P_j}{\partial x_i} + P_j P_i$$

for all $i \neq j$. The system of linear differential equations

$$\frac{\partial F}{\partial x_i} = P_i(x) F, \ i = 1, \ldots, n$$
is called a Pfaffian system. We also call the system of linear differential operators \( \frac{\partial}{\partial x_i} - P_i \) a Pfaffian system. The number \( r \) is called the size or the rank of the Pfaffian system. For a given zero-dimensional left ideal in the ring of differential operators with rational coefficients, it is well known that an associated Pfaffian system can be obtained by a Gröbner basis method (see, e.g., [18, Appendix]). The matrix \( P_i \) is an analogy of the companion matrix for a zero-dimensional ideal in the ring of polynomials. Some computer algebra systems can perform this translation. However, in general, this computation is difficult, and we wish to provide an efficient method for translating the \( A \)-hypergeometric system into a Pfaffian system.

Twisted cohomology groups can be used as a geometric method for finding a Pfaffian system associated with a given definite integral that contains parameters (see, e.g., the book by Aomoto-Kita [3]). This approach is as follows: (1) obtain a basis for a twisted cohomology group, and (2) calculate the Pfaffian system associated with that basis. We will use this approach to obtain a Pfaffian system, and in this paper, we consider step (1).

Gel’fand, Kapranov, and Zelevinsky expressed \( A \)-hypergeometric functions with regular singularities as pairings of twisted cycles and twisted cocycles [11]. Esterov and Takeuchi expressed confluent \( A \)-hypergeometric functions as pairings of rapid-decay twisted cycles and twisted cocycles [9]. The cohomology groups that come from geometry and are associated with \( A \)-hypergeometric systems were discussed by Adolphson and Sperber [2]. The next step is to obtain explicit bases for these twisted cohomology groups. Orlik and Terao provided the \( \beta nbc \) bases for the twisted cohomology groups associated with hyperplane arrangements [20]. Aomoto, Kita, Orlik, and Terao [4] provided a basis for a class of confluent hypergeometric integrals. In this paper, we will give a computational method for determining the bases of the twisted cohomology groups associated with generic sparse polynomials or any \( A \)-hypergeometric system, and we will also give a combinatorial method for a class of generic sparse polynomials.

Let \( R_n \) be the ring of differential operators with rational function coefficients in \( n \)-variables. The first step in finding a Pfaffian system associated with a zero-dimensional left ideal \( I \) in \( R_n \) is to obtain a basis for \( R_n/I \) as a \( \mathbb{C}(x) \)-vector space. It is well known that a basis can be obtained by computing a Gröbner basis of \( I \) in \( R_n \). When the set \( \{u_1, \ldots, u_r\} \) is the basis, there exists a matrix \( P_i(x) \) such that \( \partial/\partial x_i U \equiv P_i U \mod I \), \( U = (u_1, \ldots, u_r)^T \). We can show that \( \partial/\partial x_i - P_i \) is a Pfaffian system, and we call \( \{u_1, \ldots, u_r\} \) the basis of the Pfaffian system. In Theorem 1 we show that Gröbner deformations give bases and provide an algorithm that is more efficient than computing the Gröbner basis of \( I \) itself. In Theorem 2 we show that this gives a basis of the twisted cohomology group.

Our theorems are not only useful for computations, but they also pose interesting theoretical problems in commutative algebra and combinatorics. We study the hypergeometric system associated with a class of order polytopes (see, e.g., [14]). We prove that the bases of Pfaffian systems or twisted cohomology groups have combinatorial descriptions (Theorems 3 and 6). The size of the
Pfaffian system associated with a subclass of the order polytopes has a growth rate of polynomial order (Theorem 5).

We will close this introduction by explaining our motivation for this study from algebraic statistics (see, e.g., [12]). The function \( g(x, t)^\alpha t^\gamma / \Phi(x) \) or \( \exp(g(x, t)) t^\gamma / \Phi(x) \) can be regarded as a probability distribution function on \( C_i \) with parameters \( x, \alpha, \gamma \) satisfying certain conditions. This distribution, which we will call the \( A\)-distribution, is a generalization of the Beta distribution or the Gamma distribution. In this context, the function \( \Phi(x) \) is called the normalizing constant of the \( A\)-distribution. In [18], some new statistical methods were proposed. These were the holonomic gradient method (HGM) and the holonomic gradient descent (HGD). The HGM is a method for numerically evaluating the normalizing constant, which is a function of the parameters \( x \), for a given unnormalized probability distribution, and the HGD uses the HGM to obtain the maximal likelihood estimate. The key step for both of these methods is to construct a Pfaffian system associated with the normalizing constant. The size of the Pfaffian system determines the complexity of the HGM and the HGD (see, e.g., [16]). The HGM and HGD lead us to the following fundamental goals for applying \( A\)-hypergeometric systems to statistics.

1. Find an efficient method for constructing a Pfaffian system associated with a given \( A\)-hypergeometric system.
2. Find a subclass of \( A\)-hypergeometric systems for which the associated Pfaffian systems are of moderate size.

We expect that our results will yield a new class of exponential probability distributions for which we can efficiently apply the holonomic gradient method (HGM) and the holonomic gradient descent (HGD). Construction algorithms for Pfaffian systems, utilizing the results of this paper and examples of numerical evaluations, will be discussed in the next paper, which is currently in preparation.

2 Bases for the Pfaffian System

We denote by \( A = (a_{ij}) \) a \( d \times n \)-matrix whose elements are integers. We suppose that the set of the column vectors of \( A \) spans \( \mathbb{Z}^d \). Let \( s_1, \ldots, s_d \) be indeterminates. Following the notation in [21], we denote by \( H_A(s) \) a left ideal generated in the Weyl algebra

\[
D(s) = C(s_1, \ldots, s_d) \langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \rangle, \quad \partial_i = \partial / \partial x_i
\]
that

Theorem 1 to a function $I$ in $\mathbb{R}$ of $\mathbb{R}^n$.

operator ideals are nothing but Gröbner deformations of $H$. We are interested in bases of $\mathbb{R}^n$ (see, e.g., [1], [21]). In other words, we have $\dim E = \text{deg in}_w(I_A) = \deg I_A$. Let $u_1, \ldots, u_r$ be a monomial basis of $R_n/(R_n I_A)$, where the left ideal $I$ is generated by $\text{in}_w(I_A)$ and $E_i - s_i$, $i = 1, \ldots, d$ in $R_n$. Then, the set $\{u_1, \ldots, u_r\}$ is a basis of the vector space $R_n/(R_n I_A)$.

Proof. We denote by $r$ the normalized volume of $A$. Since the $s_i$ are indeterminate, the holonomic rank of $I$ and $H_A(s)$ are $r$ by Adolphson’s theorem (see, e.g., [1], [21]). In other words, we have $\dim C_{(s,x)}(R_n/(R_n I_A)) = r$ and $\dim C_{(s,x)} R_n / J = r$.

We may assume that the $u_i$ are expressed as monomials in terms of Euler operators $\partial_j = x_j \partial_j$. When we regard $J$ as a system of linear differential equations, it has $r$ linearly independent solutions of the form $x^\rho$, where $\rho \in C(s)^n$. We denote them by $g_i = x^{\rho(i)}$, $i = 1, \ldots, r$. Since the $g_i$ are linearly independent solutions, the Wronskian determinant $\det(u_i \cdot g_j)$ is not identically

$$
\sum_{j=1}^n a_{ij} x_j \partial_j - s_i, \quad (i = 1, \ldots, d) \tag{1}
$$

$$
\prod_{i=1}^n \partial_i^{\rho_i} - \prod_{j=1}^n \partial_j^{\rho_j} \tag{2}
$$

(with $u, v \in \mathbb{N}_0^n$ running over all $u, v$ such that $Au = Av$).

Here, $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. We call the ideal generated in $\mathbb{C}[\partial_1, \ldots, \partial_n]$ by the elements of the form (2) the affine toric ideal and denote it by $I_A$. We denote by $E_i - s_i$ the operator $\mathbb{I}$. For complex parameters $\beta_i$, the system of linear differential equations $(E_i - \beta_i)f = 0$ $(i = 1, \ldots, d)$, $\sum_{i=1}^n \partial_i^{\rho_i} - \sum_{j=1}^n \partial_j^{\rho_j} = 0$ $(Au = Av)$ is called the $A$-hypergeometric system of differential equations or just the $A$-hypergeometric system. We will sometimes call the ideal $H_A(s)$ the $A$-hypergeometric system (with indefinite parameters).

Let $R_n$ be the ring of differential operators with rational function coefficients

$$
\mathbb{C}(s, x)(\partial_1, \ldots, \partial_n). \tag{3}
$$

We are interested in bases of $R_n/(R_n H_A(s))$ as the vector space over the field $\mathbb{C}(s, x)$. Any basis of the vector space yields an associated Pfaffian system or an integrable connection associated with $H_A(s)$. Let $u_1, \ldots, u_r$ be a basis of $R_n/(R_n H_A(s))$. For $u_j$, there exist rational functions $p_{ij}^k \in \mathbb{C}(s, x)$ such that $\partial_i u_j \equiv \sum k=1^r p_{ij}^k u_k \mod R_n H_A(s)$. The action of a differential operator $u$ to a function $F$ is denoted by $u \cdot F$. The system of differential equations $\partial_i \cdot F = (p_{ij}^k | 1 \leq j, k \leq r)F$, where $F$ is a vector valued function of size $r$, is called a Pfaffian system, and $\{u_i\}$ is called a basis of the Pfaffian system.

Bases can be described by those of simpler quotients, for which the denominator ideals are nothing but Gröbner deformations of $H_A(s)$, as in the following theorem.

Theorem 1 Let $w \in \mathbb{Z}^n$ be a generic weight vector for the affine toric ideal $I_A$ such that $\deg in_w(I_A) = \deg I_A$. Let $u_1, \ldots, u_r$ be a monomial basis of $R_n/(R_n J)$, where the left ideal $J$ is generated by $\text{in}_w(I_A)$ and $E_i - s_i$, $i = 1, \ldots, d$ in $R_n$. Then, the set $\{u_1, \ldots, u_r\}$ is a basis of the vector space $R_n/(R_n H_A(s))$.
equal to 0. The solution \( g_j \) can be extended to a solution \( f_j \) of \( H_A(s) \) such that \( g_j \) is the leading monomial of \( f_j \) with respect to the weight vector \( w \) (see, e.g., [21, Chapters 2 and 3]). The series \( f_j \) is expressed as \( f_j = g_j \sum_{t \in M_j} C_t x^t, \) \( C_0 = 1, \) where \( M_j \) denotes the set of lattice points in a cone and \( C_t \) is a constant belonging to \( C(s) \). The series converges in the space of convergent power series \( g_j \cdot \mathcal{O}(U) \{ M_j \} \), where \( U \) is an open set in the \( s \)-space and \( \mathcal{O}(U) \) is the space of holomorphic functions on \( U \) [19]. We replace \( x_i \) by \( x_i t_i^w \) for all \( i \) in \( f_j \) and denote by \( x t_j^w \) the vector \( (x_1 t_1^w, \ldots, x_n t_n^w) \). From the construction algorithm of \( f_j \), we may assume that \( f_j(x t_j^w) = g_j(x t_j^w)(1+O(t)) \) when \( t \to 0 \) as a function of \( t \) when \( x \) is fixed and \( s \) lies in \( U \).

Let us prove \( W = \det(u_i \cdot f_j) \neq 0 \). We denote by \( u_i(\rho(j)) \) the constant \((u_i \cdot x^{\rho(j)})/x^{\rho(j)}\). Under this notation, we have \( x^{-\rho(j)} u_i \cdot g_j = u_i(\rho(j)) \) and

\[
(x^{-\rho(j)}(u_i \cdot f_j))(x t_j^w) = \sum_{\ell \in M_j} u_i(\rho(j) + \ell) C_t x^t t_j^w. \tag{4}
\]

Note that \( \ell w > 0 \) for \( \ell \neq 0 \) and \( \ell \in M_j \). Therefore, we have

\[
\det(x^{-\rho(j)} u_i \cdot f_j)(x t_j^w) = \det(x^{-\rho(j)} u_i \cdot g_j)(x t_j^w) + O(t) \tag{5}
\]

from [11] when \( x \) is fixed and \( t \to 0 \). This implies that the Wronskian determinant

\[
\det(u_i \cdot f_j) = \left( \prod_i x^{\rho(j)} \right) \det(x^{-\rho(j)} u_i \cdot f_j)
\]

is not identically equal to 0. Therefore the \( u_i \) are linearly independent in \( R_n/(R_n H_A(s)) \). Q.E.D.

Let \( M \) be a monomial ideal in \( C[\theta] \). When \( M \) is generated by \( \partial^a \), the distraction \( \hat{M} \subset C[\theta] \) is generated by \( \prod_{i=1}^n \theta_i(\theta_i - 1) \cdots (\theta_i - \alpha_i + 1) \), where \( \theta_i = x_i \partial_i \) [21, p.68]. Let \( M = \text{in}_w(I_A) \). Then, the ideal \( J \) in Theorem [11] is generated by \( \hat{M} \) and \( \sum_{j=1}^n a_{ij} \theta_j - s_i, i = 1, \ldots, d \) in the polynomial ring \( C(s)[\theta] \), gives a basis of \( R_n/R_n H_A(s) \) by the replacement \( \theta_i = x_i \partial_i \).

Corollary 1 Retain the assumptions of Theorem [11]. The set of the monomial basis of \( C(s)[\theta]/J \), where \( J \) is the ideal generated by \( \hat{M} \) and \( \sum_{j=1}^n a_{ij} \theta_j - s_i, i = 1, \ldots, d \) in the polynomial ring \( C(s)[\theta] \), gives a basis of \( R_n/R_n H_A(s) \) by the replacement \( \theta_i = x_i \partial_i \).

3 Bases of Twisted Cohomology Groups

Let \( A_1 = (a_1, \ldots, a_{n_1}), \ldots, A_k = (a_{n_k-1+1}, \ldots, a_{n_k}), a_i \in \mathbb{Z}^m \). To each matrix \( A_j \), we associate a generic sparse polynomial in \( t \)

\[
f_j(x, t) = \sum_{i=n_j-1+1}^{n_j} x_i t^{\alpha_i}, \tag{6}
\]

where \( t^b = \prod_{i=1}^m t_i^{b_i} \). For parameters \( \alpha_1, \ldots, \alpha_k \) and \( \gamma_1, \ldots, \gamma_m \), we consider the integral

\[
\Phi(\alpha, \gamma; x) = \int_C P(x, t) dt_1 \cdots dt_m, \quad P(x, t) = \prod_{j=1}^k f_j(x, t)^{\alpha_j} t^\gamma \tag{7}
\]
Define the connection \( P \) and \( \beta \) by is maximal [11]. Set

\[
A = \begin{pmatrix}
1 & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 1 & \cdots & 1 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & 0 & 0 & \cdots & 0 & 1 & \cdots & 1 \\
\end{pmatrix}
\tag{8}
\]

and \( \beta = (\alpha_1, \ldots, \alpha_k, -\gamma_1 - 1, \ldots, -\gamma_m - 1)^T \), where we assume that the rank of \( A \) is maximal \([11]\). Set \( P' = P|_{\alpha=\gamma=1} \). Let \( n = \sum_{i=1}^k n_i \), and define a projection \( p \) by

\[
p : \mathbb{C}^{m+n} \setminus V(P') \ni (x, t) \mapsto x \in \mathbb{C}^n.
\]

We regard \( P \) as a function in \( t = (t_1, \ldots, t_m) \) with the parameter vector \( x \). Define the connection \( \nabla \) with rational function coefficients by

\[
\nabla = d + \sum_{j=1}^m \left( \frac{\partial P}{\partial t_j}/P \right) dt_j
\tag{9}
\]

where \( d \) is the exterior derivative with respect to the variables \( t_1, \ldots, t_m \).

**Theorem 2** Assume that the matrix \( A \) is expressed as [6]. Let \( \alpha, \gamma \) be generic parameters, and let \( \{u_1, \ldots, u_r\} \) be a basis as given in Theorem [1]. Then the set of rational expressions \( \{(u_i \cdot P)/P\} \) is a basis of the twisted cohomology group \( H^m(p^{-1}(x), \nabla) \) when \( x \) lies outside of an analytic set.

**Proof.** It follows from the local triviality theorem [21, 5.1 Corollaire] that the projection \( p \) is a locally trivial map on a Zariski open subset \( U \) of \( \mathbb{C}^n \). Take a point \( x_0 \) in \( U \). Then the inverse image \( p^{-1}(U') \) of a small neighborhood \( U' \) of \( x_0 \) is isomorphic, as a smooth manifold, to the direct product of \( p^{-1}(x_0) \times U' \). Therefore, we can form a basis from the twisted homology group \( H_m(p^{-1}(x), \mathcal{P}_x), x \in U' \) of the form \( \sum c_i \Delta_i \otimes P \), where \( c_i \) is a constant that does not depend on \( x \) and \( \Delta_i \) is a smooth simplex that does not depend on \( x \). Here, \( \mathcal{P}_x \) is the local system defined by \( P \) at \( x \). Note that \( \mathcal{P}_x \) and \( \mathcal{P}_{x'} \) are isomorphic for any \( x, x' \in U' \).

Let \( \{u_1, \ldots, u_r\} \) be a basis of \( R_n/(R_n H_A(s)) \) as given in Theorem [1]. For generic parameters \( \alpha \) and \( \gamma \) and a twisted cycle \( C_j = \sum c_{jk} \Delta_{jk} \otimes P \), where \( c_{jk} \) and \( \Delta_{jk} \) does not depend on the parameter \( x \), the integral

\[
u_i \Phi(\alpha, \gamma; x) = \sum c_{jk} \int_{\Delta_{jk}} u_i \cdot P dt = \sum c_{jk} \int_{\Delta_{jk}} \frac{u_j \cdot P}{P} dt
\]

can be regarded as a pairing \( \langle \varphi_i, C_j \rangle \) of the twisted cocycle \( \varphi_i = \frac{u_i \cdot P}{P} dt \in H^m(p^{-1}(x), \nabla) \) and the twisted cycle \( C_j \). Since the matrix-valued function
((φ
i
, C
j
)) is a fundamental set of solutions of the Pfaffian system for the A-hypergeometric system H
A
(β), its determinant does not vanish out of an analytic set. This implies that the pairing of H
m
(p
-1
(x), ∇) × H
m
(p
-1
(x), P
x
) is perfect out of the analytic set in the x space. Thus, the set u
i
• P/P is a basis of the twisted cohomology group. Q.E.D.

**Remark 1** Our Theorem 2 can be generalized to a matrix A for which the toric ideal I
A
 is not necessarily a homogeneous ideal. It follows from Esterov and Takeuchi [9] that we can form a basis from the rapid-decay homology cycles for which the support does not depend locally on the parameter x. Therefore, we can make an analogous argument for the case of twisted cycles, and the perfectness theorem by Hien [15] proves Theorem 2 for any A that defines an A-hypergeometric system.

**Example 1** Consider the matrix

\[
A = \begin{pmatrix}
  x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\
  1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
  0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\
  0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\
  0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 
\end{pmatrix}
\]

(which are the vertices of the order polytope associated to the distributive lattice of Figure 1; see Section 4). The basis given by Theorem 1 is \{1, ∂
5
, ∂
6
, ∂
7
, ∂
8
, ∂
2
\}. It is determined by computing a Gröbner basis for the ideal J or ˜J by a computer. Note that this computation is easier than computing the Gröbner basis of H
A
(β). The corresponding basis of the twisted cohomology group is \{1, t
1
, t
2
, t
3
, t
4
, t
5
, t
6
, t
7
, t
8
\}, where Q = x_1 + x_2t_1 + x_3t_2 + x_4t_3 + x_5t_1t_2 + x_6t_1t_3 + x_7t_2t_3 + x_8t_1t_2t_3 and dt = dt_1dt_2dt_3.

**Example 2** Let

\[
A' = \begin{pmatrix}
x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\
  1 & 0 & 0 & 1 & 1 & 0 & 1 \\
  0 & 1 & 0 & 1 & 0 & 1 & 1 \\
  0 & 0 & 1 & 0 & 1 & 1 & 1 
\end{pmatrix}
\]
The hypergeometric system associated with $A'$ is the confluent system of the previous example. Put $Q = x_3t_1 + x_3t_2 + x_4t_3 + x_5t_1t_2 + x_5t_1t_3 + x_7t_2t_3 + x_8t_1t_2t_3$ and $dt = dt_1dt_2dt_3$. Then the integral $\int_C \exp(Q) t^\gamma dt$, where $C$ is a rapid decay cycle, is a solution. The toric ideal $I_{A'}$ is obtained formally by setting $\partial_1 = 1$ in $I_A$ of Example[1]. A basis given by Theorem 1 is \{1, $\partial_5, \partial_6, \partial_7, \partial_8, \partial_9^2$\}. The corresponding basis of the twisted cohomology group is \{1, $t_1t_2dt, t_1t_3dt, t_2t_3dt, t_1t_2t_3dt, (t_1t_2t_3)^2dt$\}.

These two examples illustrate that the results in Sections 2 and 3 give a general method for determining the bases of twisted cohomology groups by computing a set of standard monomials with a Gröbner basis of $J$ (Theorem 1) or $\tilde{J}$ (Corollary 1).

4 A-hypergeometric Systems for Order Polytopes

For some classes of generic sparse polynomials or $A$, we can calculate by hand the set of standard monomials for $J$ or $\tilde{J}$.

First, recall the order polytope of a finite partially ordered set ([14], p. 115]). Let $P = \{a_1, \ldots, a_d\}$ be a finite partially ordered set with $|P| = n$. A poset ideal of $P$ is a subset $\alpha$ of $P$ such that if $a \in \alpha$, $b \in P$, and $b \leq a$, then $b \in \alpha$. Thus in particular the empty set and $P$ itself are poset ideals of $P$. Let $J(P)$ denote the distributive lattice ([14], p. 118]) consisting of all poset ideals of $P$, ordered by inclusion. For example, if $P$ is the disjoint union of two chains of length 2 and length 3 shown in Figure 2 then $L = J(P)$ is the distributive lattice shown in Figure 3.

Let $e_1, \ldots, e_d$ denote the standard unit coordinate vectors of $\mathbb{R}^d$. If $\beta$ is a subset of $P$, then we write $w_\beta$ for the $(0, 1)$-vector $\sum_{a \in \beta} e_i \in \mathbb{R}^d$. The order polytope $O(P) \subset \mathbb{R}^d$ of $P$ is the convex hull of the finite set $\{w_\alpha : \alpha \in J(P)\}$. Its dimension is $\dim O(P) = d$.

Let $K = \mathbb{C}(\xi_\alpha)_{\alpha \in J(P)}$ denote the rational function field in $|J(P)|$ variables over $\mathbb{C}$. Let $A = K[t_1, \ldots, t_d, s]$ denote the polynomial ring in $n$ variables over $K$. If $\beta$ is a subset of $P$, then we write $u_\beta$ for the square-free monomial $\prod_{a \in \beta} t_\alpha$. Let $K[O(P)]$ denote the subalgebra of $A$ that is generated by those square-free monomials $u_\beta$ with $\beta \in J(P)$. The semigroup ring $K[O(P)]$ was in-
in $K$.

In general, however, this is difficult. When $C < \alpha$ is an ordering

$P$  

$\{a_1, a_2, a_3, b_1, b_2, b_3, b_4\}$

of $P = \{1, 2, 3\}$, a complete answer can be found, as shown below. For example, the

poset $P$ of Figure 4 can be decomposed into the chains $a_1 < a_2 < a_3$ and $b_1 < b_2 < b_3 < b_4$.

Now, suppose that a finite poset $P$ can be decomposed into two chains

$C_p : a_1 < \cdots < a_p$ of length $p - 1$ and $C_q : b_1 < \cdots < b_q$ of length $q - 1$, where

introduced in [13]. We call $K[O(P)]$ the toric ring of $O(P)$. The Krull dimension

Let $K[\{x_\alpha\}_{\alpha \in \mathcal{J}(P)}]$ denote the polynomial ring in $|\mathcal{J}(P)|$ variables over $K$, and define the surjective ring homomorphism $\pi : K[\{x_\alpha\}_{\alpha \in \mathcal{J}(P)}] \to K[O(P)]$

by setting $\pi(x_\alpha) = u_\alpha$. Its kernel $I_O(P)$ is called the toric ideal of $O(P)$. It is known [13] that $I_O(P)$ is generated by quadratic binomials

\[
\alpha \beta - \alpha x_\beta, \quad (10)
\]

such that $\alpha$ and $\beta$ are incomparable in the distributive lattice $\mathcal{J}(P)$. We fix an ordering $<$ of variables of $K[\{x_\alpha\}_{\alpha \in \mathcal{J}(P)}]$ with the property that if $\alpha > \beta$ in $\mathcal{J}(P)$, then $x_\alpha < x_\beta$. Let $<_{rev}$ be the reverse lexicographic order on $K[\{x_\alpha\}_{\alpha \in \mathcal{J}(P)}]$ induced by the ordering $<$. In [13] it was shown that the set of binomials (10) is the reduced Gröbner basis of $I_O(P)$ with respect to $<_{rev}$. Thus $in_{<_{rev}}(I_O(P))$ is generated by those square-free quadratic monomial $x_\alpha x_\beta$ such that $\alpha$ and $\beta$ are incomparable in $\mathcal{J}(P)$.

Let

\[
\theta_i = \sum_{\alpha \in J} \xi_\alpha x_\alpha - \eta_i, \quad 1 \leq i \leq d,
\]

and let

\[
\theta_0 = \sum_{\alpha \in J} \xi_\alpha x_\alpha - \eta_0,
\]

where $\eta_i \in K$. It then follows that the sequence $(\theta_0, \theta_1, \ldots, \theta_d)$ is a system of parameters of both the residue rings $K[\{x_\alpha\}_{\alpha \in \mathcal{J}(P)}]/I_O(P)$ and $K[\{x_\alpha\}_{\alpha \in \mathcal{J}(P)}]/in_{<_{rev}}(I_O(P))$.

The fundamental goal is to find a $K$-basis of the zero-dimensional residue ring

\[
K[\{x_\alpha\}_{\alpha \in \mathcal{J}(P)}]/(in_{<_{rev}}(I_O(P)), \theta_0, \theta_1, \ldots, \theta_d).
\]

(11)

In general, however, this is difficult. When $P$ can be decomposed into two
chains, a complete answer can be found, as shown below. For example, the

poset $P$ of Figure 4 can be decomposed into the chains $a_1 < a_2 < a_3$ and $b_1 < b_2 < b_3 < b_4$.
$p \geq 1$ and $q \geq 1$. Let $\mathcal{A}$ denote the set of those pairs $(i, j)$, where $0 \leq i \leq p$ and $0 \leq j \leq q$, for which $\{a_1, \ldots, a_i, b_1, \ldots, b_j\}$ is a poset ideal of $P$. In particular, $(0, 0), (p, q) \in \mathcal{A}$. When $(i, j) \in \mathcal{A}$, we write $\alpha_{i,j}$ for $\{a_1, \ldots, a_i, b_1, \ldots, b_j\}$. For example, $\alpha_{0,0} = \emptyset$ and $\alpha_{p,q} = P$. We then have $L = \{ \alpha_{i,j} : (i, j) \in \mathcal{A} \}$. When $(i, j) \in \mathcal{A}$, we write $\xi_{i,j}$ for $\alpha_{i,j}$ and $x_{i,j}$ for $\alpha_{i,j}$. Let

$$\theta_{i*} = \sum_{0 \leq j \leq q, (k,j) \in \mathcal{A}} \xi_{k,j}x_{k,j} - \eta_{i*}, \quad 0 \leq i \leq p$$

and

$$\theta_{s,j} = \sum_{0 \leq i \leq p, j \leq q, (i,\ell) \in \mathcal{A}} \xi_{i,\ell}x_{i,\ell} - \eta_{s,j}, \quad 0 \leq j \leq q.$$ 

In particular,

$$\theta_{0*} = \theta_{*0} = \sum_{(i,j) \in \mathcal{A}} \xi_{i,j}x_{i,j} - \eta_0,$$

with $\eta_0 = \eta_{0*} = \eta_{*0}$. Let $K[x] = K[\{x_{i,j} \mid 0 \leq i \leq p, 0 \leq j \leq q, (i,j) \in \mathcal{A}\}]$ and

$$J = \langle \text{in}_{\text{rev}}(I_{\mathcal{O}(P)}), \{\theta_{i*} \mid 0 \leq i \leq p, \{\theta_{s,j} \mid 0 \leq j \leq q \rangle, \quad (12)$$

where

$$\text{in}_{\text{rev}}(I_{\mathcal{O}(P)}) = \{ x_{i,j}x_{k,\ell} : i < k, \ell < j, (i,j) \in \mathcal{A}, (k,\ell) \in \mathcal{A} \}.$$

Then the residue ring $\mathcal{R}[x] / J$ is $K[x]/J$. Let $<_{\text{rev}}$ denote the reverse lexicographic order on $K[x]$ induced by the ordering of the variables, as follows: $x_{i,j} > x_{k,\ell}$ if either $i + j < k + \ell$ or $i + j = k + \ell$ with $i > k$.

**Lemma 1** In $K[x]/\text{in}_{\text{rev}}(J)$,

$$x_{i,j}x_{i',j'} = x_{i,j}x_{i',j} = 0,$$

where $(i, j), (i, j')$ and $(i', j)$ belong to $\mathcal{A}$.

**Proof.** Let $i < i'$. Then

$$\theta_{i*,x_{i,j}} - x_{i,j} \left( \sum_{i' \leq k, j \leq \ell, (k,\ell) \in \mathcal{A}} \xi_{k,\ell}x_{k,\ell} + b_{i*} \right)$$

belongs to $\text{in}_{\text{rev}}(I_{\mathcal{O}(P)})$. Hence

$$x_{i,j} \left( \sum_{i' \leq k, j \leq \ell, (k,\ell) \in \mathcal{A}} \xi_{k,\ell}x_{k,\ell} + b_{i'} \right)$$

belongs to $J$. Thus its initial monomial $x_{i,j}x_{i',j'}$ belongs to $\text{in}_{\text{rev}}(J)$. Let $i = i'$. Let $f$ be the polynomial

$$\theta_{i*,x_{i,j}} - \xi_{i,j}^{-1} \theta_{s,j} \left( \sum_{0 \leq k \leq -1, (i,k) \in \mathcal{A}} \xi_{i,k}x_{i,k} \right),$$

and write $f = f_1 + f_2$, where $f_2 \in \text{in}_{\text{rev}}(I_{\mathcal{O}(P)})$ and where none of the monomials appearing in $f_1$ belongs to $\text{in}_{\text{rev}}(I_{\mathcal{O}(P)})$. Then $f_1 \in J$ and the initial monomial of $f_1$ is $x_{i,j}^2$. Hence $x_{i,j}^2 \in \text{in}_{\text{rev}}(J)$. Similarly, $x_{i,j}x_{i',j'} \in \text{in}_{\text{rev}}(J)$. Q.E.D.
Lemma 2  For each $0 \leq i \leq p$, we write $j_i^*$ for the smallest integer for which $(i, j_i^*) \in A$. For each $0 \leq j \leq q$, we write $i^*_j$ for the smallest integer for which $(i^*_j, j) \in A$. Then $x_{i, j_i^*}$ and $x_{i^*_j, j}$ belong to $\text{in}_{<rev}(J)$.

Proof. Since $\theta_{i, i^*}$ and $\theta_{i^*, j}$ belong to $J$, their initial monomials $x_{i, j_i^*}$ and $x_{i^*_j, j}$ belong to $\text{in}_{<rev}(J)$. Q.E.D.

Let $S$ denote the set of square-free monomials of $K[x]$ of the form

$$x_{i_1, j_1} x_{i_2, j_2} \ldots x_{i_r, j_r},$$  \hspace{1cm} (13)

with each $(i_k, j_k) \in A \setminus \{ (x_{i, j}^* : 0 \leq i \leq p) \cup \{ x_{i^*_j, j} : 0 \leq j \leq q \} )$ such that

$$0 < i_1 < i_2 < \cdots < i_r \leq p, \quad 0 < j_1 < j_2 < \cdots < j_r \leq q, \quad r = 0, 1, 2, \ldots .$$

Theorem 3  The set of standard monomials of $\text{in}_{<rev}(J)$ is equal to $S$.

Proof. In [7], it was proven that the number of standard monomials of degree $r$ coincides with the number of maximal chains of $\mathcal{J}(P)$ with $r$ descents. Recall that the descents of a maximal chain

$$\alpha_{0, 0} = \alpha_{i_0, j_0} < \alpha_{i_1, j_1} < \cdots < \alpha_{i_{p+q}, j_{p+q}} = \alpha_{p, q}$$

of $\mathcal{J}(P)$ are those $\alpha_{i_k, j_k}$ with $1 \leq k < p + q$ such that

$$i_{k-1} = i_k < i_{k+1}, \quad j_{k-1} < j_k = j_{k+1}, \quad j_{k+1} \neq j_{i_{k+1}}.$$
Now, given a square-free monomial (13) of degree \( r \), we can associate a unique maximal chain whose descents are
\[
\alpha_{i_1 - 1,j_1}, \alpha_{i_2 - 1,j_2}, \ldots, \alpha_{i_r - 1,j_r},
\]
in the obvious way (see Figure 4).

Hence the number of square-free monomials (13) of degree \( r \) is less than or equal to that of standard monomials of degree \( r \). On the other hand, since Lemmata 1 and 2 guarantee that each standard monomial must belong to \( S \), it follows that \( S \) is the set of standard monomials of \( \text{in}_{< \text{rev}}(J) \), as desired. Q.E.D.

**Remark 2** When the variables \( \eta_{i*} \) and \( \eta_{*j} \) are ignored in the rational function field, we can work with a system of parameters consisting of homogeneous elements and both Lemma 2 and Lemma 1 are valid without modification. This observation is crucial to our argument of counting the number of standard monomials in the proof of Theorem 3. We also note that \( \xi_{ij} \) may be specialized to any nonzero number for \( K = \mathbb{C} \) without changing the claims of this section.

**Example 3** Let \( P \) be the finite poset of Figure 7 and let \( L = J(P) \) be the distributive lattice shown in Figure 8. Then the standard monomials of \( \text{in}_{< \text{rev}}(J) \) are \( 1; x_{1,1}; x_{1,2}; x_{2,2}; \) and \( x_{1,1}x_{2,2} \).

Let us turn to the discussion of \( A \)-hypergeometric systems. Let \( P_{p,q} \) denote the disjoint union of two chains \( C_p : a_1 < \cdots < a_p \) of length \( p - 1 \) and \( C_q : b_1 < \cdots < b_q \) of length \( q - 1 \). Let \( \alpha_{i,j} \), where \( 0 \leq i \leq p \) and \( 0 \leq j \leq q \), be the poset ideal \( \{ a_1, \ldots, a_i, b_1, \ldots, b_j \} \). In particular \( \alpha_{0,0} = \emptyset \). This is a special and interesting subclass of poset ideals. We regard the vector \( w_\alpha, \alpha \in J(P_{p,q}) \), as a column vector and construct a matrix \( A_{p,q} \) with these column vectors and a row vector \( (1, 1, \ldots, 1) \). For example, \( A_{2,2} \) is
\[
\begin{pmatrix}
00 & 01 & 02 & 10 & 11 & 12 & 20 & 21 & 22 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1
\end{pmatrix}.
\]
By elementary row transformations, we transform the matrix $A_{p,q}$ into the matrix $\bar{A}_{p,q}$ of the form (8) with $k = p + 1$, $n_1 = \cdots = n_k = q + 1$, and $a_i = 0 \in \mathbb{R}^q$ when $i \equiv 1 \mod q + 1$, $a_{i+1} = e_k \in \mathbb{R}^q$ when $i \equiv k \mod q + 1$. For example, $A_{2,2}$ can be transformed into

$$\bar{A}_{2,2} = \begin{pmatrix} 00 & 01 & 02 & 10 & 11 & 12 & 20 & 21 & 22 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}.$$  

We note that $A_{p,q}$ and $\bar{A}_{p,q}$ define the same $A$-hypergeometric system.

The matrix $A$, which represents a poset $P$ that can be decomposed into two chains (as considered in this section), is obtained by removing some columns from $\bar{A}_{p,q}$. For example, the $A$ that represents Figure 8 is obtained by deleting the seventh column of the matrix $\bar{A}_{2,2}$. Therefore, for $A$, the sparse polynomials $f_j$ that can be decomposed into two chains are linear in $t$. In particular, the $f_j$'s for $P_{p,q}$ are in the general linear position. It follows from the integral representation (7) that the $\bar{A}_{p,q}$-hypergeometric system agrees with the Aomoto-Gel’fand system $E(p + 1, (p + 1) + (q + 1))$ [3], because the matrix $\bar{A}_{p,q}$ defines a hyperplane arrangement $V(\prod t_i \prod f_j)$ in a general position. The initial ideal in $\mathfrak{m}_P(I_A)$ is a square-free monomial ideal. In particular, it follows from Corollary 4 that the standard monomials of the ideal $J$ defined in (12) provide a basis of the Pfaffian system for $H_A(s)$ when $x_{ij}$ is replaced by $\partial_{ij}$.

Let $\mathcal{S}$ be the set of standard monomials given in Theorem 3 for the poset ideal $J(P)$. Then, the set $\mathcal{S}|_{x_{ij} \rightarrow \partial_{ij}}$ gives a basis of the Pfaffian system for the $A$-hypergeometric system.

From our Theorems 2 and 3 and the correspondence that we have explained above, we have the following theorem.

**Theorem 4** Let $A$ be the matrix representing a poset $P$ that can be decomposed into two chains, and let $\mathcal{S} = \{u_1, \ldots, u_r\}$ be the set of standard monomials given in Theorem 3 with $x_{ij}$ replaced by $\partial_{ij}$. Set $Q = \prod_{j=1}^k f_j(x, t)^{\alpha_j} t^\gamma$ and $Q' = Q|_{x_{ij} = \gamma_j = 1}$. Then, the set of rational forms

$$\frac{u_i \cdot Q}{Q} dt_1 \cdots dt_m, \quad i = 1, \ldots, r$$  

(14)

is a basis of the twisted cohomology group $H^m(\Omega^*(sQ'), \nabla)$ when $\alpha_i, \gamma_j$ are generic complex numbers.

In the case $P = P_{p,q}$, this theorem is a different presentation of the celebrated work of K. Aomoto, who gave a basis for the twisted cohomology group associated with a hyperplane arrangement in a general position (see, e.g., [3] Theorem 9.6.2). In a more general result, Orlik and Terao gave bases of twisted cohomology groups with hyperplane arrangements in terms of the $\beta nbc$
basis \cite[6.3]{20}. Our theorem gives bases for twisted cohomology groups in a very
different way for a class of hyperplane arrangements obtained by restricting the
arrangements in the general position to the $x_{ij} = 0$’s.

**Example 4** The $A$-hypergeometric system associated with Figure 8 is the re-
striction of $E(3, 6)$ to $x_{20} = 0$. Figure 9 illustrates the arrangement that repre-
sents it.

5 Rank of a Class of Order Polytopes

We now turn to the discussion of the normalized volume of order polytopes. It follows from \cite{22} that the normalized volume of the order polytope $O(P)$ is
equal to $e(P)$, the number of linear extensions of $P$. Recall that an antichain
of $P$ is a subset $B$ of $P$ such that if $a$ and $b$ belong to $B$ with $a \neq b$, then $a$
and $b$ are incomparable in $P$. The width of $P$ is the supremum of cardinalities
of antichains of $P$. The length of a chain $C$ is $|C| - 1$. The rank of $P$ is the
supremum of lengths of chains of $P$.

**Lemma 3** Fix positive integers $d$ and $r$. Let $P$ be the disjoint union of $d$ chains
$C_1, \ldots, C_d$, and assume that the length of each chain $C_i$ with $1 \leq i < d$ is at
most $r - 1$. Then there exists a polynomial $f(n)$ in $n$ of degree $r(d - 1)$ such
that $e(P)$ is at most $f(n)$, where $n = |P|$.

**Proof.** Let $\ell_i$ denote the length of $C_i$. Then the number of linear extensions
of $P$ is

$$e(P) = \binom{n}{\ell_1, \ell_2, \ldots, \ell_d} = \frac{n!}{\ell_1!\ell_2! \cdots \ell_d!}.$$  

Since $\ell_d = n - \sum_{i=1}^{d-1} \ell_i \geq n - r(d - 1)$, it follows that

$$e(P) \leq \frac{n!}{\ell_d!} \leq \frac{n!}{(n - r(d - 1))!}.$$  

Let

$$f(n) = n(n-1)(n-2) \cdots (n-r(d-1)+1),$$  

14
which is a polynomial in $n$ of degree $r(d - 1)$. Then $e(P) \leq f(n)$, as required. Q.E.D.

**Theorem 5** Fix positive integers $d$ and $r$. Let $P$ be a finite partially ordered set, and suppose that there exists a chain $C$ of $P$ such that

(i) the width of $P \setminus C$ is at most $d - 1$;

(ii) the rank of $P \setminus C$ is at most $r - 1$.

Then there exists a polynomial $f(n)$ in $n$ of degree $r(d - 1)$ such that $e(P)$ is at most $f(n)$, where $n = |P|$.

**Proof.** Since the width of $P \setminus C$ is at most $d - 1$, Dilworth’s theorem guarantees the existence of $d - 1$ chains $C_1, \ldots, C_{d-1}$ of $P \setminus C$, where the length of each $C_i$ is at most $r - 1$, such that $P \setminus C = C_1 \cup C_2 \cup \cdots \cup C_{d-1}$ and $C_i \cap C_j = \emptyset$ for $i \neq j$. Hence there exists a partially ordered set $Q$ that is the disjoint union of $d$ chains $C'_1, \ldots, C'_d$, where the length of each $C'_i$ with $1 \leq i < d$ is at most $r - 1$, such that there is an order-preserving bijection $\varphi : Q \to P$. Hence $e(P) \leq e(Q)$. Thus the desired result follows from Lemma. Q.E.D.

Let us turn to the discussion of $A$-hypergeometric systems. By Theorem 3 we can regard the rank $e(P)$ of the hypergeometric system associated with the order polytope $\mathcal{O}(P)$ ($n = |P|$) has a polynomial growth property with respect to $n$. This is good news, since the rank determines the complexity of the holonomic gradient method [15].

## 6 Bouquet

We now wish to introduce a “bouquet” of finite distributive lattices. Let $P_1, \ldots, P_q$ be finite posets, where

$$P_i = \{a_1^{(i)}, \ldots, a_d^{(i)}\}, \quad 1 \leq i \leq q,$$

and let $L_i = \mathcal{J}(P_i)$ be the distributive lattice consisting of all poset ideals of $P_i$. The finite meet-semilattice $(\bigcup_{i=1}^q (\mathcal{J}(P_i)))$ is called the bouquet of $L_1 = \mathcal{J}(P_1), \ldots, L_q = \mathcal{J}(P_q)$. For example, if $q = 3$ and $P_1 = P_2 = P_3$ are the finite poset $P_{1,1}$ shown in Figure 10 then the Hasse diagram of the bouquet of $L_1, L_2, L_3$ is shown in Figure 11.

Let $e_j^{(i)}$, $1 \leq i \leq q, 1 \leq j \leq d_i$, denote the standard unit coordinate vectors of $\mathbb{R}^d$, where $d = d_1 + \cdots + d_q$. If $\beta$ is a subset of $P_i$, then we write $w_\beta$ for the $(0,1)$-vector $\sum_{\alpha \in \beta} \alpha e_j^{(i)} \in \mathbb{R}^d$. In particular, $w_\emptyset$ is the origin of $\mathbb{R}^d$. Let $\mathcal{O}(P_1, \ldots, P_q) \subset \mathbb{R}^d$ denote the convex hull of the finite set

$$\{ w_\alpha : \alpha \in \bigcup_{i=1}^d \mathcal{J}(P_i) \}.$$
Let $K = \mathbb{C}(\{ \xi_\alpha : \alpha \in \bigcup_{i=1}^d \mathcal{J}(P_i) \}, \{ q_j^{(i)} : 1 \leq i \leq q, 1 \leq j \leq d_i \}, \eta_0)$ denote the rational function field in $|\bigcup_{i=1}^d \mathcal{J}(P_i) + (d + 1)|$ variables over $\mathbb{C}$, and let

$$A = K[\{ t_j^{(i)} : 1 \leq i \leq q, 1 \leq j \leq d_i \}, s]$$

be the polynomial ring in $d + 1$ variables over $K$. If $\beta$ is a subset of $P_i$, then we write $u_\beta$ for the square-free monomial $(\prod_{j \in \beta} t_j^{(i)})s$. The toric ring $K[\mathcal{O}(P_1, \ldots, P_q)]$ of $\mathcal{O}(P_1, \ldots, P_q)$ is the subalgebra of $A$ that is generated by those square-free monomials $u_\alpha$ with $\alpha \in \bigcup_{i=1}^d \mathcal{J}(P_i)$. Its Krull dimension is $d + 1$.

Let $K[\{ x_\alpha \}] = K[\{ x_\alpha : \alpha \in \bigcup_{i=1}^d \mathcal{J}(P_i) \}]$ denote the polynomial ring in $|\bigcup_{i=1}^d \mathcal{J}(P_i)|$ variables over $K$, and define the surjective ring homomorphism

$$\pi : K[\{ x_\alpha \}] \to K[\mathcal{O}(P_1, \ldots, P_q)]$$

by setting $\pi(x_\alpha) = u_\alpha$. Its kernel is the toric ideal $I_{\mathcal{O}(P_1, \ldots, P_q)}$ of $\mathcal{O}(P_1, \ldots, P_q)$. It follows that $I_{\mathcal{O}(P_1, \ldots, P_q)}$ is generated by those quadratic binomials

$$x_\alpha x_\beta - x_{\alpha \land \beta} x_{\alpha \lor \beta},$$

(15)

where both $\alpha$ and $\beta$ belong to $\mathcal{J}(P_i)$ for some $1 \leq i \leq q$ and where $\alpha$ and $\beta$ are incomparable in $\mathcal{J}(P_i)$.

We fix an ordering $<$ of the variables of $K[\{ x_\alpha \}]$ with the property that if both $\alpha$ and $\beta$ belong to $\mathcal{J}(P_i)$ for some $1 \leq i \leq q$ and if $\alpha > \beta$ in $\mathcal{J}(P_i)$, then $x_\alpha < x_\beta$. Let $<_{\text{rev}}$ denote the reverse lexicographic order on $K[\{ x_\alpha \}]$ induced by the ordering $<$. It then follows that the set of binomials is the reduced Gröbner basis of $I_{\mathcal{O}(P_1, \ldots, P_q)}$ with respect to $<_{\text{rev}}$. Thus the initial ideal in $<_{\text{rev}}(I_{\mathcal{O}(P_1, \ldots, P_q)})$ of $I_{\mathcal{O}(P_1, \ldots, P_q)}$ with respect to $<_{\text{rev}}$ is generated by those
square-free quadratic monomials \( x_\alpha x_\beta \) such that both \( \alpha \) and \( \beta \) belong to \( \mathcal{J}(P_i) \) for some \( 1 \leq i \leq q \) and that \( \alpha \) and \( \beta \) are incomparable in \( \mathcal{J}(P_i) \).

Let

\[ \theta^{(i)}_j = \sum_{\alpha} \xi_\alpha x_\alpha - \eta_j^{(i)}, \quad 1 \leq i \leq q, \ 1 \leq j \leq d_i, \]

and let

\[ \theta_0 = \sum_{\alpha \in \mathcal{J}(P)} \xi_\alpha x_\alpha - \eta_0. \]

It then follows that the sequence

\[ (\theta_0, \theta^{(1)}_1, \ldots, \theta^{(1)}_{d_1}, \ldots, \theta^{(2)}_2, \ldots, \theta^{(2)}_{d_2}, \ldots, \theta^{(q)}_q) \]

is a system of parameters of both the residue rings \( K[\{x_\alpha\}_\alpha]/I_{\mathcal{O}(P_1,\ldots,P_q)} \) and \( K[\{x_\alpha\}_\alpha]/\text{in}_{<_{\text{rev}}} (I_{\mathcal{O}(P_1,\ldots,P_q)}) \).

Now, suppose that each poset \( P_i \) can be decomposed into two chains, and write \( S_i \) for the set of standard monomials, which is obtained using Theorem 3 for the residue class ring arising from \( \mathcal{O}(P_i) \). By virtue of the fact that no role of \( \theta_0 \) is required in the proof of Lemma 1, it follows that the set of standard monomials of

\[ K[\{x_\alpha\}_\alpha]/\text{in}_{<_{\text{rev}}} (I_{\mathcal{O}(P_1,\ldots,P_q)}), \theta_0, \theta^{(1)}_1, \ldots, \theta^{(q)}_q) \]  

(16)

with respect to \( <_{\text{rev}} \) is a subset of

\[ \left\{ \prod_{i=1}^q u_i : u_i \in S_i, \ 1 \leq i \leq q \right\}. \]  

(17)

Finally, the computation of the number of standard monomials based on the equality (7) of [6, Theorem 1.4] together with the information on the facets of the order polytopes ([22, p. 10]) guarantee the following theorem.

**Theorem 6** The set of standard monomials of the residue class ring (16) with respect to \( <_{\text{rev}} \) coincides with the set of square-free monomials (17).

**Example 5** For the bouquet of Figure 11, the set of standard monomials obtained by Theorem 8 (labeling variables as in Figure 11 and replacing \( x_{ij} \) by \( \partial_{ij} \)) is

\[ \{\partial_{11}^{k_1} \partial_{22}^{k_2} \partial_{33}^{k_3} | k_i \in \{0,1\}\}. \]

We will show that this bouquet represents the Lauricella hypergeometric function \( F_A \) of three variables [5, Chapitre VII]. We note that the twisted cohomology groups associated with the \( F_A \) are studied in [17] in a quite different way. We consider \( A \)-hypergeometric system associated with the lattice shown in Figure 11. The independent variables of the system will be denoted by \( p_{00}, p_{01}, p_{02}, p_{03}, p_{10}, p_{20}, p_{30}, p_{11}, p_{22}, p_{33} \) or simply as \( 00, 01, 02, 03, 10, 20, 30, 11, 22, 33 \) if no confusion arises. The differential operators of the system will be denoted by
By analogous calculations, we have
\[ f(\theta_{00}, \theta_{01}, \theta_{02}, \theta_{03}, \theta_{10}, \theta_{11}, \theta_{20}, \theta_{22}, \theta_{33}) \] or simply as 00, 01, 02, 03, 10, 20, 30, 11, 22, 33 if no confusion arises. The toric ideal associated with the lattice is generated by
\[ 10 \cdot 01 - 00 \cdot 11, 20 \cdot 02 - 00 \cdot 22, 30 \cdot 03 - 00 \cdot 33. \] (18)

The underlined terms are the leading terms for the reverse lexicographic order such that 00 < other variables, and the set is a Gröbner basis with this order. Hence, the A-hypergeometric system has a solution of the form
\[ p^7 f \left( \frac{10 \cdot 01}{00 \cdot 11}, \frac{20 \cdot 02}{00 \cdot 22}, \frac{30 \cdot 03}{00 \cdot 33} \right). \] (19)

Set
\[
\begin{align*}
x &= \frac{10 \cdot 01}{00 \cdot 11}, \\
y &= \frac{20 \cdot 02}{00 \cdot 22}, \\
z &= \frac{30 \cdot 03}{00 \cdot 33}.
\end{align*}
\]

The differential operator \( p_{10} p_{01} p_{00} p_{11} (\partial_{10} \partial_{01} - \partial_{00} \partial_{11}) \) can be written as \( \theta_{10} \theta_{01} - x \theta_{00} \theta_{11}, \) where \( \theta_{ij} = p_{ij} \partial_{ij} \) (the Euler operator). We will derive a differential operator that annihilates the function \( f \) from this operator. Apply \( \theta_{11} \) to \( p^7 f(x, y, z) \). Then, we have \( p^7(\gamma_{11} - \theta_z)f \). Apply \( \theta_{00} \) to this function. Then, we have
\[ \gamma_{00} \gamma_{11} p^7 f + p^7 \gamma_{11} (-\theta_x - \theta_y - \theta_z)f - \gamma_{00} p^7 x f - p^7 (-1) x f_x - p^7 x (-\theta_x - \theta_y - \theta_z)f_x. \]
This can be factored as
\[ p^7(\theta_x + \theta_y + \theta_z - \gamma_{00})(\theta_x - \gamma_{11})f. \]

An analogous calculation leads us to
\[ \theta_{10} \theta_{01} p^7 f(x, y, z) = p^7(\theta_x + \gamma_{01})(\theta_x + \gamma_{10})f. \]

Therefore, the function \( f(x, y, z) \) satisfies
\[ ((\theta_x + \gamma_{01})(\theta_x + \gamma_{10}) - x(\theta_x + \theta_y + \theta_z - \gamma_{00})(\theta_x - \gamma_{11})) f = 0. \] (20)

By analogous calculations, we have
\[
\begin{align*}
((\theta_y + \gamma_{02})(\theta_y + \gamma_{20}) - y(\theta_x + \theta_y + \theta_z - \gamma_{00})(\theta_y - \gamma_{22})) f &= 0, \\
((\theta_z + \gamma_{03})(\theta_z + \gamma_{30}) - z(\theta_x + \theta_y + \theta_z - \gamma_{00})(\theta_z - \gamma_{33})) f &= 0.
\end{align*}
\]

By these equations for the function \( f \), we conclude that the function \( g = x^{-\gamma_{01}} y^{-\gamma_{02}} z^{-\gamma_{03}} f(x, y, z) \) satisfies the differential equations for the Lauricella function \( F_A, n = 3 \).

The bouquet of \( n \) squares stands for the Lauricella \( F_A \) of \( n \) variables.

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