On the Real Analyticity of the Scattering Operator for the Hartree Equation

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Abstract
In this paper, we study the real analyticity of the scattering operator for the Hartree equation \(i\partial_t u = -\Delta u + u(V \ast |u|^2)\). To this end, we exploit interior and exterior cut-off in time and space, and combining with the compactness argument to overcome difficulties which arise from absence of good properties for the nonlinear Klein-Gordon equation, such as the finite speed of propagation and ideal time decay estimate. Additionally, the method in this paper allows us to simplify the proof of analyticity of the scattering operator for the nonlinear Klein-Gordon equation with cubic nonlinearity in Kumlin\cite{9}.

Key words: Hartree equation, real analyticity, scattering operator; compactness
MSC: 35P25, 35Q55.

1 Introduction
This paper is devoted to the proof of the real analyticity of scattering operator for the Hartree equation

\[
\begin{align*}
&i\partial_t u = -\Delta u + u(V \ast |u|^2), \quad (t, x) \in \mathbb{R} \times \mathbb{R}^3, \\
u(0) = u_0(x) \in H^1(\mathbb{R}^3).
\end{align*}
\] (1.1)

Here \(u(t, x)\) is a complex valued function defined in \(\mathbb{R}^{1+3}\), \(V(x)\), called potential, is a real valued radial function defined in \(\mathbb{R}^3\), and \(\ast\) denotes the convolution in \(\mathbb{R}^3\). Under suitable assumption on \(V\), Ginibre-Velo\cite{7} proved the scattering theory of the equation (1.1) in the energy space \(H^1\). Attempting to study the (complex) analyticity of the
scattering operator is in vain because $\bar{u}$ is not analytic even if $u$ is. However, following
the W. Strauss suggestion (private communication), we can study the real analyticity
which is still a very interesting issue.

Let $u = \varphi(t, x) + i\psi(t, x)$, $u_0 = \varphi_0(x) + i\psi_0(x)$, and $\varphi(t, x)$, $\psi(t, x)$, $\varphi_0(x)$, $\psi_0(x)$
are real valued functions defined in $\mathbb{R} \times \mathbb{R}^3$ or $\mathbb{R}^3$. Then the integral form of equation [1.1]
\[
u(t) = e^{it\Delta}u_0 - i \int_0^t e^{i(t-s)\Delta}((V \ast |u|^2)u(s))ds
\]
can be rewritten as
\[
\begin{pmatrix}
\varphi \\
\psi
\end{pmatrix} = \begin{pmatrix}
\cos t\Delta & -\sin t\Delta \\
\sin t\Delta & \cos t\Delta
\end{pmatrix}
\begin{pmatrix}
\varphi_0 \\
\psi_0
\end{pmatrix} + \int_0^t \begin{pmatrix}
\sin(t-s)\Delta & \cos(t-s)\Delta \\
-\cos(t-s)\Delta & \sin(t-s)\Delta
\end{pmatrix}
\begin{pmatrix}
\varphi \\
\psi
\end{pmatrix}
\begin{pmatrix}
V \ast (\varphi^2 + \psi^2)
\end{pmatrix}(s)ds.
\]
Setting
\[
U(t) = \begin{pmatrix}
\varphi(t) \\
\psi(t)
\end{pmatrix}
\text{ and } U_0 = \begin{pmatrix}
\varphi_0 \\
\psi_0
\end{pmatrix},
\]
then (1.3) can be transformed into
\[
N(t)U_0 := U(t) = G(t)U_0 - \int_0^t \Delta^{-1}G'(t-s)(V \ast |U|^2)U(s)ds,
\]
where
\[
G(t) = \begin{pmatrix}
\cos t\Delta & -\sin t\Delta \\
\sin t\Delta & \cos t\Delta
\end{pmatrix}
\]
is a unitary group associated with the equation [1.1].

First, we recall the decay estimate and Strichartz estimates in the context of
Schrödinger equation (see [6], [10], [11]).

**Definition 1.1** A pair $(q, r)$ is admissible, denoted by $(q, r) \in \Lambda$, if $r \in [2, 6]$ and $q$ satisfies
\[
\frac{2}{q} = \delta(r) := 3\left(\frac{1}{2} - \frac{1}{r}\right).
\]

**Lemma 1.1** Let $S(t) = e^{it\Delta}$, then
(1) the $L^r' - L^r$ decay estimate
\[
\|S(t)\varphi\|_r \leq C|t|^{-\delta(r)}\|\varphi(x)\|_r',
\]
holds for $2 \leq r \leq \infty$;
(2) the Strichartz estimates
\[
\|S(t)u\|_{L^s(I, L^r(\mathbb{R}^3))} \leq C\|u\|_2,
\]
\[
\left\| \int_0^t S(t-s)f(s)ds \right\|_{L^n(I, L^r(\mathbb{R}^3))} \leq C\|f\|_{L^q(I, L^r(\mathbb{R}^3))}
\]
hold true for any interval $I \subset \mathbb{R}$, and for any admissible pairs $(q, r)$, $(q_j, r_j) \in \Lambda$, $j = 1, 2$. 

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Remark 1.1 Lemma 1.1 still holds for the propagators $G(t)$ and $\Delta^{-1}G'(t)$ by Euler formulae

$$\cos t\Delta = \frac{e^{it\Delta} + e^{-it\Delta}}{2}, \quad \sin t\Delta = \frac{e^{it\Delta} - e^{-it\Delta}}{2}.$$ 

Let $B$ be a Banach space, and

$$\begin{pmatrix} u \\ v \end{pmatrix} \in B \iff \left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\|_B = \|u\|_B + \|v\|_B < \infty.$$ 

Throughout this paper, the symbol $C$ denotes a constant which may be different from line to line, and $C(*)$ denotes the constant which only depends on the parameter $*$. Define the wave operator $W_\pm : U_\pm \mapsto U_0$ as follows: for any $U_\pm \in H^1$, there exists $U_0 \in H^1$ such that

$$\left\| G(t)U_\pm - N(t)U_0 \right\|_{H^1} \to 0, \quad \text{as} \ t \to \pm \infty. \quad (1.9)$$

When the wave operator $W_\pm$ are invertible operators, we can define scattering operator as $S = W_{-1}^{-1} \circ W_- : U_- \mapsto U_+$. Set

$$X = C(\mathbb{R}, H^1(\mathbb{R}^3)) \cap \bigcap_{(q,r) \in \Lambda} L^q(\mathbb{R}, H^1_r(\mathbb{R}^3)).$$

Ginibre-Velo established a complete scattering theory in energy space provided that the potential $V$ satisfies the following assumption:

(H1) $V$ is a real function and $V \in L^{p_1} + L^{p_2}$ for some $p_1, p_2$ satisfying

$$1 < p_2 \leq p_1 < \frac{3}{2}.$$ 

(H2) $V$ is radial and nonincreasing, namely $V(x) = v(r)$ where $v$ is nonincreasing in $\mathbb{R}^+$. Furthermore, for some $\alpha \geq 2$, $v$ satisfies the following condition:

$$(A_\alpha): \quad \text{There exists } a > 0 \text{ and } A_\alpha > 0 \text{ such that }$$

$$v(r_1) - v(r_2) \geq \frac{A_\alpha}{\alpha} (r_2^\alpha - r_1^\alpha) \text{ for } 0 < r_1 < r_2 \leq a.$$ 

In particular, the wave operator $W_\pm$ and the scattering operator $S$ are bounded and continuous from $H^1$ to $H^1$.

Our main result is

**Theorem 1.1** Let $V(x)$ satisfy the assumption (H1) and (H2). Then the operators $W_\pm$ and $S$ are analytic from $H^1$ to $H^1$.

The proof of Theorem 1.1 depends on the following theorem:
**Theorem 1.2** Let $U_0 \in H^1$ and $U(t)$ be the unique solution of (1.4) in $X$. Then the map $U : U_0 \mapsto U(U_0)$ is analytic from $H^1$ to $X$.

For the nonlinear Klein-Gordon equation with cubic nonlinearity, using the contraction mapping principle, Baez-Zhou\cite{1} proved the analyticity of scattering operator on a neighborhood of the space of finite-energy Cauchy data, $H^1 \oplus L^2(\mathbb{R}^3)$. Kumlin\cite{9} generalized the result to entire energy space by means of the Fredholm alternative theorem. The proof in [9] depends on the following two good properties of the linear Klein-Gordon equation:

1. $L^p - L^{p'}$ estimates stated in the following proposition.

**Proposition 1.1 (\cite{4} \cite{9})** Let $K(t) = \frac{\sin t(-\Delta + m^2)^\frac{\delta}{2}}{(-\Delta + m^2)^\frac{\delta}{2} + 1}$, $1 < p \leq 2 < p', \frac{1}{p} + \frac{1}{p'} = 1$ and $\sigma := \frac{1}{2} - \frac{1}{p'}$ and $0 \leq \theta \leq 1$. Then if $(n + 1 + \theta)\sigma \leq 1 + s - s'$,

$$
\|K(t)g\|_{W^{s',s'}} \leq k(t)\|g\|_{W^{s,p}}, \quad t \geq 0,
$$

(1.10)

where

$$
k(t) = \begin{cases} 
t^{-(n-1-\theta)\sigma}, & 0 < t < 1, \\
t^{-(n+1+\theta)\sigma}, & 1 \leq t.
\end{cases}
$$

For suitable $p$, $p'$ and $\theta$, $k(t) \in L^1(\mathbb{R})$. In particular,

$$
\|K(t)g\|_{H^1} \leq C\|g\|_{L^2}.
$$

These estimates are crucial in the proofs of Step 1 and Step 3 in [9].

2. The finite speed of propagation.

The finite speed of propagation of the solution of linear wave equation means that for $t \in [-T,T]$, $T < \infty$,

$$
\left\| \int_0^t K(t-s)u(s)ds \right\|_{L^6(|x|>R)} \sim \left\| \int_0^t K(t-s)\eta_R u(s)ds \right\|_{L^6(\mathbb{R}^3)}.
$$

(1.11)

Namely, the cut-off function $\eta_R(x)$ defined below in (1.15) commutes with the group $K(t)$ in some sense, which plays an important role in the proof of analyticity, see Step 2 in [9].

The arguments in this paper still take advantage of the Fredholm alternative theorem together with the analytic version of implicit function theorem (cf \cite{1} \cite{2} \cite{9}). However we have to overcome some difficulties arise from loss of the good properties (1) and (2) for the Schrödinger equation. Our major innovations are as follows : Comparing with $k(t)$ in (1.10), the kernel $|t|^{-\delta(r)}$ in (1.6) is not in any $L^p(\mathbb{R})$, $1 \leq p \leq \infty$, and the Hardy-Litttlewood-Sobolev inequality can not supply any decay yet. A new approach to deal with the singular kernel is the double localization in time

$$
|t-s|^{-\delta(r)}\chi_{\{|s|\leq T/2\}}\chi_{\{|t| T\}}, \quad |t-s|^{-\delta(r)}\chi_{\{|s|>T/2\}}\chi_{\{|t| T\}},
$$

(1.12)
this together with other techniques helps us to get time decay, where \( \chi_A \) denotes the characteristic function on the interval \( A \). On the other hand, replacing (1.11) by introducing double (interior and exterior) cut-off on space

\[
\int_0^t \Delta^{-1} G'(t - s)((V * |U|^2)\xi_R \Psi_j)(s)ds \rightarrow L^4(|t| \lesssim T, H^1_3(|x| > M)) ,
\]

we obtain decay estimates and overcome the above difficulties by means of compactness principle, where

\[
\begin{align*}
\xi_R(x) &\in C_0^\infty(\mathbb{R}^3), \\
\xi_R(x) = 1 &\text{ if } |x| \leq R, \quad \xi_R(x) = 0 &\text{ if } |x| > 2R \quad \text{and} \\
\eta_R(x) &= 1 - \xi_R(x).
\end{align*}
\]

Based on these decay estimates and the arguments in Kumlin [9], we can prove Theorem 1.1 by the approximate theorem of analytic operator sequence (cf. [8]). However, it is worth mentioning that we make use of compactness arguments and the definition of Frechét derivative to avoid repeating the argument of global time space integrability, and give a more concise proof of Theorem 1.1.

The paper is organized as follows: In Section 2 we give the proof of Theorem 1.2. Section 3 is devoted to the proof of key lemma which consists of the main part of this paper. At last, we supply a brief derivation of Theorem 1.1 in Section 4.

## 2 Proof of Theorem 1.2

To prove Theorem 1.2 we need the following analytic version of implicit function theorem.

**Lemma 2.1** [2] Suppose that \( X, Y, Z \) are Banach spaces and \( Q \) is an open neighborhood of the point \((x, y) \in X \times Y\). Suppose that \( f : Q \rightarrow Z \) is analytic, \( f(x, y) = 0 \) and \( D_2 f(x, y) : X \rightarrow Z \) has a left inverse, where \( D_2 \) indicates the Frechét derivative with respect to the second variable. Then for some open set \( P \) containing \( x \), there exists a unique analytic function \( g : P \rightarrow Y \) such that \( g(x) = y \) and \( f(x', g(x')) = 0 \) for all \( x' \in P \).

For \( U_0 \in H^1 \) and \( \Psi \in X \), we consider the following mapping

\[
R(U_0, \Psi) = G(t)U_0 - \int_0^t \Delta^{-1} G'(t - s)((V * |\Psi|^2)\Psi)(s)ds - \Psi(t).
\]

Since it is linear in \( U_0 \) and multilinear in \( \Psi \), we know that \( R : H^1 \times X \rightarrow X \) is analytic by the nonlinear estimate in [7]

\[
\left\| \int_0^t \Delta^{-1} G'(t - s)((V * |\Psi|^2)\Psi)(s)ds \right\|_X < \infty, \quad \forall \Psi \in X.
\]
On the other hand, \( R(U_0, U) = 0 \) by (1.4). Hence it suffices to prove the invertibility of

\[
D_2R(U_0, U) : X \mapsto X
\]

for each \( U_0 \in H^1 \). By the open mapping theorem, we only need to prove that \( D_2R(U_0, U) \) is injective and surjective.

For \( U_0 \in H^1, \Psi \in X \), one has

\[
D_2R(U_0, U)(\Psi)(t) = -2 \int_0^t \Delta^{-1} G'(t-s)(V*(\Psi U))\Psi(s)ds
\]

\[
- \int_0^t \Delta^{-1} G'(t-s)(V*|U|^2)\Psi(s)ds - \Psi(t). \tag{2.2}
\]

(1) The injectivity of \( D_2R(U_0, U) \).

For simplicity, we always assume that \( V(x) \in L^p \). Let \( D_2R(U_0, U)\Psi = 0 \), then

\[
\|\Psi\|_r \leq C \int_0^t |t-s|^{-\delta(r)} \left( \| (V* (U\Psi))U(s) \|_{L^2} + \| (V* |U|^2)\Psi(s) \|_{L^2} \right) ds
\]

\[
\leq 2C \int_0^t |t-s|^{-\delta(r)} \|V\|_p \|U\|_{L^\infty}^2 \|\Psi\|_r ds, \tag{2.3}
\]

where \( 2 = \frac{1}{p} + \frac{2}{r} + \frac{2}{p} \).

For every \( p \in (1, \frac{3}{2}) \), we can take \( l = r \in (3, 4) \) such that

\[
\|\Psi\|_r \leq C \|V\|_p \|U\|_{L^\infty}^2 \|\Psi\|_r ds. \tag{2.4}
\]

For each \( t \in (0, T) \), one easily verifies that by (2.4)

\[
\|\Psi(t)\|_r \leq Ct^{1-\delta(r)} \text{ess sup}_{s \in (0,t)}\|\Psi(s)\|_r.
\]

We choose \( T \) small enough such that

\[
\text{ess sup}_{t \in (0,T)}\|\Psi(t)\|_r \leq \frac{1}{2} \text{ess sup}_{s \in (0,T)}\|\Psi(s)\|_r.
\]

This implies that \( \Psi \equiv 0 \), a.e. \( t \in [0,T] \) for some \( T > 0 \). Repeating this process on \( (nT, nT+T) \), \( n \in \mathbb{Z} \), we have \( \Psi \equiv 0 \), a.e. \((x,t) \in \mathbb{R}^{3+1}\).

(2) The surjectivity of \( D_2R(U_0, U) \).

Setting

\[
T_{U_0}\Psi(t) = -\int_0^t \Delta^{-1} G'(t-s)((V*|U|^2)\Psi)(s)ds
\]

\[
- 2\int_0^t \Delta^{-1} G'(t-s)((V*(U\Psi))U)(s)ds, \tag{2.5}
\]

\[
6
\]
we have $D_2R(U_0, U) = T_{U_0} - I$. By the Fredholm alternative theorem, our first choosing is to show that $T_{U_0}$ is compact operator from $X$ to $X$ since $T_{U_0} - I$ is injective. However, $T_{U_0}$ may be not compact. To our goal, it suffices to show that $T_{U_0}^2$ is compact. In fact, since

$$T_{U_0}^2 - I = (T_{U_0} - I)(T_{U_0} + I) \tag{2.6}$$

and $T_{U_0} + I$ is also injective, the Fredholm theorem still works. Therefore (2.6) implies that the surjectivity of $T_{U_0} - I$.

Concerning the trilinear form

$$B(\Psi_1, \Psi_2, \Psi_3) := -\int_0^t \Delta^{-1} G'(t-s)(V \ast (\Psi_1 \Psi_2)) \Psi_3(s)ds, \tag{2.7}$$

we have the following nonlinear estimate.

**Lemma 2.2** For $\Psi_j \in X$, $j = 1, 2, 3$, one has

$$\|B(\Psi_1, \Psi_2, \Psi_3)\|_X \lesssim \prod_{j=1,2,3} \|\Psi_j\|_{L^4H^{1/3}_3}. \tag{2.8}$$

**Proof** Using the Strichartz estimates together with the Hölder inequality, we obtain

$$\left\| \int_0^t \Delta^{-1} G'(t-s)(V \ast (\Psi_1 \Psi_2)) \Psi_3(s)ds \right\|_X \lesssim \|V \ast (\Psi_1 \Psi_2)\|_{L^{4/3}H^{1/2}_3}$$

$$\leq \sum_{\{i,j,k\} = \{1,2,3\}} \|V\|_p \|\Psi_i\|_{L^4H^{1/3}_3} \|\Psi_j\|_{L^4L^{6\tilde{p}}} \|\Psi_k\|_{L^4L^{\tilde{p}}} \leq \prod_{j=1,2,3} \|\Psi_j\|_{L^4H^{1/3}_3}, \tag{2.9}$$

where $\tilde{p} = \frac{6\rho}{4\rho - 3}$, and we have used the embedding relation $H^{1/3}_3 \hookrightarrow L^{\tilde{p}}$.

As a direct consequence of Lemma 2.2 we have

$$\|T_{U_0}\Psi(t)\|_X \lesssim \|\Psi\|_{L^4H^{1/3}_3}.$$ 

This implies that $T_{U_0} : L^4H^{1/3}_3 \rightarrow X$ is bounded. Since the composition of compact operator and bounded operator is still compact, it is enough to verify following key lemma.

**Lemma 2.3** Let $U_0 \in H^1$, Then

$$T_{U_0} : X \hookrightarrow L^4H^{1/3}_3 \quad \text{is compact.}$$
3 Proof of Lemma 2.3

Let \( \{ \Psi_j \}_{j=0}^\infty \) be uniformly bounded in \( X \), i.e. \( \| \Psi_j \|_X \leq C \) for constant \( C > 0 \), we shall show that \( \{ T_{U_0} \Psi_j \}_{j=0}^\infty \) has a Cauchy subsequence in \( L^4 H^1_3 \). Our main tool is the Arzela-Ascoli theorem, so it is necessary to localize both the time and the space. The proof can be divided into five steps.

Let \( \xi_R, \eta_R \) be defined as those in (1.15).

Step 1. \( \lim_{T \to \infty} \sup_{j \in \mathbb{N}} \| T_{U_0} \Psi_j \|_{L^4(|t|>T,H^1_3(\mathbb{R}^3))} = 0 \);

Step 2. \( \lim_{R \to \infty} \sup_{j \in \mathbb{N}} \| T_{U_0} (\eta_R \Psi_j) \|_{L^4(|t| \leq T,H^1_3(\mathbb{R}^3))} = 0 \) for all \( T > 0 \),

Step 3. \( \lim_{M \to \infty} \sup_{j \in \mathbb{N}} \| T_{U_0} (\xi_R \Psi_j) \|_{L^4(|t| \leq T,H^1_3(|x|>M))} = 0 \) for all \( T > 0, R > 0 \);

Step 4. \( \{ T_{U_0} (\xi_R \Psi_j) \}_{j=0}^\infty \) has a Cauchy subsequence in \( L^4(|t| \leq T,H^1_3(|x| \leq M)) \) for all \( T > 0, R > 0, M > 0 \);

Step 5. A Cantor diagonalized process.

For the sake of convenience, we first give some useful estimates.

Lemma 3.1 Let \( \| U \|_X \leq C, \| \Psi_j \|_X \leq C \) and \( V \in L^p(\mathbb{R}^3) \). For any \( p \in (1, \frac{3}{2}) \), then

\[
\| (V \ast |U|^2) \Psi_j \|_2 \leq C
\]

and

\[
\| (V \ast |U|^2) \Psi_j \|_{2^\delta} \leq C
\]

hold for sufficient small \( \delta > 0 \).

Proof For any \( p \in (1, \frac{3}{2}) \), by Sobolev embedding theorem it is derived that

\[
\| (V \ast |U|^2) \Psi_j \|_2 \leq \| V \|_p \| U \|_r \| \Psi_j \|_r \leq \| V \|_p \| U \|_r \| D^\delta \Psi_j \|_r \| \Psi_j \|_r \leq \| V \|_p \| U \|_r \| D^\delta U \|_r \| \Psi_j \|_r
\]

where \( 1 + \frac{1}{2} = \frac{1}{p} + \frac{3}{r}, \frac{18}{5}, 6 \).

For \( p, r \) as above, taking \( \delta > 0 \) small enough and using fractional Leibniz formula, we have

\[
\| (V \ast |U|^2) \Psi_j \|_{2^\delta} \leq C \| V \|_p \| U \|_r \| D^\delta \Psi_j \|_r + \| V \|_p \| U \|_r \| D^\delta U \|_r \| \Psi_j \|_r \]
\[
\leq \| V \|_p \| U \|_r \| D^\delta U \|_r \| \Psi_j \|_r \leq \| V \|_p \| U \|_r \| D^\delta U \|_r \| \Psi_j \|_r \leq \| V \|_p \| U \|_r \| D^\delta U \|_r \| \Psi_j \|_r
\]

This shows that the sequence \( \{ (V \ast |U|^2) \Psi_j \}_{j=0}^\infty \) is uniformly bounded in \( H^\delta(\mathbb{R}^3) \).
Lemma 3.2 Let $T < \infty$, $t \in [-T, T]$, then
\[
\left\| \int_0^t \Delta^{-1} G'(t - s)U(s)ds \right\|_{H^{2-\epsilon}(\mathbb{R}^3)} \leq C \|U\|_{L^2([-T, T], L^2(\mathbb{R}^3))}
\] (3.1)
holds for all $\epsilon > 0$.

**Proof** (cf. [5,9]) Let $k \in \mathbb{R}$. $F^k$ denotes the operator on $L^2[-T, T]$ defined by $\hat{F^k}h(n) = (in)^k\hat{h}(n)$, $n \in \mathbb{N} - \{0\}$, and $\hat{F^k}h(0) = \hat{h}(0)$, where $h \in L^2[-T, T]$ and $\hat{\cdot}$ denotes the Fourier transform. Making use of the discrete Plancherel identity and the transformation between time and space regularity, it follows that by taking $2k = 2 - \epsilon$ with $k < 1$
\[
\left\| \int_0^t \Delta^{-1} G'(t - s)U(s)ds \right\|_{H^{2-\epsilon}} \\
\leq \left\| \int_{-T}^T (F^k \chi_{[0,t]})(s) \cdot (F^{-k} \Delta^{-1} G'(t - \cdot)U(\cdot))(s)ds \right\|_{H^{2-\epsilon}} \\
\leq \|F^k\chi_{[0,t]}\|_{L^2([-T,T])} \cdot \|F^{-k} \Delta^{-1} G'(t - \cdot)U(\cdot)\|_{L^2([-T,T], H^{2-\epsilon})} \\
\leq C \|U\|_{L^2([-T,T], L^2)},
\]
where we have used the following estimate
\[
\|F^k\chi_{[0,t]}\|_{L^2([-T,T])} \leq \left( \sum_{n \in \mathbb{Z} - \{0\}} (n^k \cdot \frac{1}{n})^2 + 1 \right)^{\frac{1}{2}} < \infty.
\]

Now, we are in position to prove Lemma 2.3. Note that the multi-linear estimates of two terms of $T_{\nu_0}$ are similar, we only need to estimate the first term.

**Step 1** We make use of an interior time cut-off technique to deal with the convolution kernel. It is easy to show that
\[
\left\| \int_0^t \Delta^{-1} G'(t - s)(V \ast |U|^2)\Psi_j(s)ds \right\|_{L^4(|t| > T, H^1_{\infty}(\mathbb{R}^3))} \\
\leq \left\| \int_0^t |t - s|^{-\frac{3}{2}} \|(V \ast |U|^2)\Psi_j(s)\|_{H^{3/2}_1} ds \right\|_{L^4(|t| > T)} \\
\leq \left\| \int_0^t |t - s|^{-\frac{3}{2}} \chi_{\{|s| > \frac{T}{2}\}}(s) \|(V \ast |U|^2)\Psi_j(s)\|_{H^{3/2}_1} ds \right\|_{L^4(|t| > T)} \\
+ \left\| \int_0^t |t - s|^{-\frac{3}{2}} \chi_{\{|s| \leq \frac{T}{2}\}}(s) \|(V \ast |U|^2)\Psi_j(s)\|_{H^{3/2}_1} ds \right\|_{L^4(|t| > T)} \\
= : I_1 + I_2
\] (3.2)
\]
On the one hand,
\[
I_1 = \left\| \int_0^t |t-s|^{-\frac{1}{4}} \left( (V^2 |U|^2 \Psi_j)(s) \chi_{\{|s| > \frac{T}{2}\}}(s) \right) \right\|_{L^4(|t|>T)} ds
\]
\[
\lesssim \| (V^2 |U|^2 \chi_{\{|s| > \frac{T}{2}\}}(s) \chi_{\{|s| > \frac{T}{2}\}}(s) \|_{L^4(|t|>T)}^4
\]
\[
\lesssim \| (V^2 |U|^2 \chi_{\{|s| > \frac{T}{2}\}}(s) \|_{L^2(\mathbb{R}, L^3)} \| \Psi_j \|_{L^4(\mathbb{R}, H^1)}
\]
\[
+ \| (V^2 |U|^2 \chi_{\{|s| > \frac{T}{2}\}}(s) \|_{L^2(\mathbb{R}, H^1)} \| \Psi_j \|_{L^4(\mathbb{R}, L^6)}
\]
\[
\lesssim \| \Psi_j \| L^4 \left( \| (V^2 |U|^2 \chi_{\{|s| > \frac{T}{2}\}}(s) \|_{L^2(\mathbb{R}, L^3)} + \| (V^2 |U|^2 \chi_{\{|s| > \frac{T}{2}\}}(s) \|_{L^2(\mathbb{R}, H^1)} \right).
\]

Since
\[
\| (V^2 |U|^2 \chi_{\{|s| > \frac{T}{2}\}}(s) \|_{L^2(\mathbb{R}, L^3)} + \| (V^2 |U|^2 \chi_{\{|s| > \frac{T}{2}\}}(s) \|_{L^2(\mathbb{R}, H^1)}
\]
\[
\ll \| V \|_p \| U \|_{L^4(\mathbb{R}, L^6)} + \| V \|_p \| U \|_{L^4(\mathbb{R}, L^6)} \| U \|_{L^4(\mathbb{R}, H^1)}
\]
\[
\ll 2 \| V \|_p \| U \|_{L^4(\mathbb{R}, H^1)} < \infty,
\]
we get
\[
\lim_{T \to \infty} I_1 = 0
\]
by
\[
\| (V^2 |U|^2 \chi_{\{|s| > \frac{T}{2}\}}(s) \|_{L^2(\mathbb{R}, L^3)} + \| (V^2 |U|^2 \chi_{\{|s| > \frac{T}{2}\}}(s) \|_{L^2(\mathbb{R}, H^1)}
\]
\[
= \| (V^2 |U|^2 \chi_{\{|s| > \frac{T}{2}\}}(s) \|_{L^2(\mathbb{R}, L^3)} + \| (V^2 |U|^2 \chi_{\{|s| > \frac{T}{2}\}}(s) \|_{L^2(\mathbb{R}, H^1)}
\]
\[
\to 0, \quad \text{as} \quad T \to + \infty,
\]
where we have used the property of absolute continuity.

On the other hand, similar arguments as deriving the estimate of $I_1$ can be used to get that
\[
I_2 \leq \left( \int_{|t|>T} \int_{\mathbb{R}} |t-s|^{-\frac{1}{4}} \chi_{\{|s| \leq \frac{T}{2}\}}(s) \right) \left( (V^2 |U|^2 \Psi_j)(s) \|_{H^1_{3/2}}(s) \right) ds \right)^{\frac{1}{2}}
\]
\[
\lesssim \left( \int_T^{\infty} |t - \frac{T}{2}|^{-\frac{1}{4}} dt \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}} \chi_{\{|s| \leq \frac{T}{2}\}}(s) \right) \left( (V^2 |U|^2 \Psi_j)(s) \|_{H^1_{3/2}}(s) \right) ds \right)^{\frac{1}{2}}
\]
\[
\lesssim T^{-1/4} \left( \int_{\mathbb{R}} \chi_{\{|s| \leq \frac{T}{2}\}}(s) \right) \left( \int_{\mathbb{R}} ((V^2 |U|^2 \Psi_j)(s) \|_{H^1_{3/2}}(s) \right) ds \right)^{\frac{3+q}{4+q}}
\]
\[
\lesssim T^{-1/4} \| (V^2 |U|^2 \Psi_j)(s) \|_{L^4 H^1_{3/2}}
\]
\[
\leq T^{-\frac{q-3}{8+3q}} \left( \| V \|_p \Psi_j \|_{L^4 H^1_{3/2}} \right) \left( \| U \|_{L^2 L^r} + \| V \|_p \| U \|_{L^4 H^1_{3/2}} \| U \|_{L^2 L^r} \| \Psi_j \|_{L^2 L^r} \right),
\]
here $q = 4 - \frac{4p}{8+3q}$, $r = \tilde{p} = \frac{6p}{4p-3}$. 

10
For any $p \in (1, \frac{2}{3})$, $r \in (3, 6)$, we can choose admissible pairs $(q, r) \in \Lambda$ such that $q \in (2, 4)$, provided that $\varepsilon > 0$ sufficient small. Hence

$$I_2 \lesssim T^{-\frac{2r}{r+2}} (\|V\|_p \|\Psi_j\|_X \|U\|_{\frac{2}{3}}^2) \to 0,$$

as $T \to \infty$.

**Step 2** Different with the proof of Step 1, we perform an interior cut-off on space to get

$$\left\| \int_0^t \Delta^{-1} G^\prime(t-s) \left( (V * |U|^2) \eta_R \Psi_j \right)(s) ds \right\|_{L^4(|t| \leq T, H^1_3(\mathbb{R}^3))}$$

$$\lesssim \|V * |U|^2\|_{L^4(|t| \leq T, L^{1/3})} \|\Psi_j\|_{L^4(|t| \leq T, H^1_3)}$$

$$+ \|V * |U|^2\|_{L^2(|t| \leq T, H^1_{2p})} \|\Psi_j\|_{L^4(|t| \leq T, L^p)}$$

$$\lesssim \|\Psi_j\|_X \left( \|V * |U|^2\|_{L^2(|t| \leq T, L^3)} + \|V * |U|^2\|_{L^2(|t| \leq T, H^1_{2p})} \right),$$

while

$$\|V * |U|^2\|_{L^2(|t| \leq T, L^3)} + \|V * |U|^2\|_{L^2(|t| \leq T, H^1_{2p})}$$

$$\lesssim \|V\|_p \|U\|_{L^4(|t| \leq T, L^3)} + \|V\|_p \|U\|_{L^4(|t| \leq T, L^3)} \|U\|_{L^4(|t| \leq T, H^1_3)}$$

$$\lesssim 2 \|V\|_p \|U\|_{L^4(|t| \leq T, H^1_3)} < \infty.$$ 

Hence, we have

$$\|V * |U|^2\|_{L^2(|t| \leq T, L^3)} + \|V * |U|^2\|_{L^2(|t| \leq T, H^1_{2p})}$$

$$= \|V * |U|^2\|_{L^2(|t| \leq T, L^3)} + \|V * |U|^2\|_{L^2(|t| \leq T, H^1_{2p})}$$

$$\to 0, \quad \text{as } R \to + \infty. \quad (3.5)$$

**Step 3** Observe that for each fixed $j \in \mathbb{N}$,

$$\lim_{M \to \infty} \left\| \mathcal{T}_{U_0}(\xi_R \Psi_j) \right\|_{L^4(|t| \leq T, H^1_3(|x| > M))} = 0.$$ 

In order to prove convergence uniformly on $j \in \mathbb{N}$, we take advantage of the finite $\varepsilon$-cover property of compact set.

By Lemma 3.1 one has

$$\|V * |U|^2\|_{L^1(\mathbb{R}^3)} \lesssim \|V * |U|^2\|_{L^2(|x| \leq 2R)} \lesssim C.$$ 

This together with the Rellich-Kondrachov theorem implies that

$$\{(V * |U|^2) \Psi_j\}_{j=0}^\infty$$

is compact in $L^2(|x| \leq 2R)$.
This fact shows that \( \forall \varepsilon > 0, \exists \) finite set \( \mathcal{A} = \{j_1, j_2, \ldots, j_0\} \) such that \( \forall j \in \mathbb{N}, \exists l \in \mathcal{A} \) satisfying

\[
\|(V \ast |U|^2)\xi_R \Psi_j - (V \ast |U|^2)\xi_R \Psi_l\|_{L^2} < \varepsilon.
\]

Hence, as \( M \) is large enough, we obtain that by Lemma 3.2

\[
\left\| \int_0^t \Delta^{-1} G'(t - s) \left( (V \ast |U|^2)\xi_R \Psi_j \right)(s) ds \right\|_{L^4(|t| \leq T, H^1_3(|x| > M))} \\
\leq \left\| \int_0^t \Delta^{-1} G'(t - s) \left( (V \ast |U|^2)\xi_R \Psi_j - (V \ast |U|^2)\xi_R \Psi_l \right) ds \right\|_{L^4(|t| \leq T, H^1_3)} \\
+ \left\| \int_0^t \Delta^{-1} G'(t - s) \left( (V \ast |U|^2)\xi_R \Psi_l \right)(s) ds \right\|_{L^4(|t| \leq T, H^1_3(|x| > M))} \\
\leq C(T) \sup_{t \in [-T, T]} \left\| \int_0^t \Delta^{-1} G'(t - s) \left( (V \ast |U|^2)\xi_R \Psi_j - (V \ast |U|^2)\xi_R \Psi_l \right) ds \right\|_{H^{2-\epsilon}([\mathbb{R}^3])} + \varepsilon \\
\leq C(T) \sup_{t \in [-T, T]} \|(V \ast |U|^2)\xi_R \Psi_j - (V \ast |U|^2)\xi_R \Psi_l\|_{L^2([\mathbb{R}^3])} + \varepsilon \\
\leq \varepsilon
\]

**Step 4** After localizing \( t \) and \( x \) to bounded domain, we can use the following Arzelà-Ascoli compactness argument.

**Lemma 3.3** A sequence \( \{f_j\}_{j=0}^\infty \) in \( C([-T, T], H^1_3(|x| \leq M)) \) has a convergent subsequence iff

(i) for each \( t \in [-T, T] \), the sequence \( \{f_j(t)\}_{j=0}^\infty \) has a convergent subsequence in \( H^1_3(|x| \leq M) \);

(ii) the sequence \( \{f_j\}_{j=0}^\infty \) is equicontinuous on \([-T, T]\).

We now verify that

\[
f_j(t) = -\int_0^t \Delta^{-1} G'(t - s) \left( (V \ast |U|^2)\xi_R \Psi_j \right)(s) ds
\]

satisfies the two conditions of Lemma 3.3.

By Lemma 3.2 we have for all \( t \in [-T, T] \)

\[
\left\| \int_0^t \Delta^{-1} G'(t - s) \left( (V \ast |U|^2)\xi_R \Psi_j \right)(s) ds \right\|_{H^{2-\epsilon}(|x| \leq M))} \\
\leq \|(V \ast |U|^2)\xi_R \Psi_j\|_{L^2([-T, T], L^2)} \leq C.
\]

(3.6)

This, together with the Rellich-Kondrachov theorem, implies that the sequence \( \{f_j\}_{j=0}^\infty \) satisfies (i) of in Lemma 3.3.
Next, we show that the equicontinuity of the sequence \( \{f_j\}_{j=0}^{\infty} \) on \([T, T]\).

\[
\|f_j(t + h) - f_j(t)\|_{H^1_T} = \left\| \int_t^{t+h} \Delta^{-1} G'(t + h - s)(V*|U|^2)\xi_R \Psi_j(s) \, ds \right\|_{H^1_T} \\
\leq \left\| \int_t^{t+h} (\Delta^{-1} G'(t + h - s) - \Delta^{-1} G'(t - s))(V*|U|^2)\xi_R \Psi_j(s) \, ds \right\|_{H^1_T} \\
+ \left\| \int_t^{t+h} \Delta^{-1} G'(t + h - s)(V*|U|^2)\xi_R \Psi_j(s) \, ds \right\|_{H^1_T} \\
=: J_1 + J_2
\]

Let \( U_j := (V*|U|^2)\xi_R \Psi_j \). By Lemma \([3.2]\) and the compactness as same as that in Step 3, it is derived that

\[
J_1 = \left\| \int_0^t \Delta^{-1} G'(t - s)(G(h) - I)U_j(s) \, ds \right\|_{H^1_T} \\
\leq \left\| \int_0^t \Delta^{-1} G'(t - s)(G(h) - I)U_j(s) \, ds \right\|_{H^{2-\epsilon}} \quad \text{(for some } \epsilon > 0) \\
\leq \left\| (G(h) - I)U_j \right\|_{L^2([-T,T], L^2)} \\
\leq \left\| (G(h) - I)(U_j - U_l) \right\|_{L^2([-T,T], L^2)} + \left\| (G(h) - I)U_l \right\|_{L^2([-T,T], L^2)} \\
\leq 2\|U_j - U_l\|_{L^2([-T,T], L^2)} < \varepsilon \quad (3.7)
\]

uniformly on \( j \in \mathbb{N} \) as \(|h|\) is small enough. Combining the \( L^p-L^{p'} \) estimate with the Hölder inequality, we deduce that

\[
J_2 \leq \int_t^{t+h} |t + h - s|^{-\frac{1}{q}} \|V*|U|^2\Psi_j\|_{H^{3/2}_T} \, ds \\
\leq \left( \int_t^{t+h} |t + h - s|^{-\frac{q'}{2q'}} \, ds \right) \|V*|U|^2\Psi_j\|_{L^q H^{3/2}_T}, \quad (3.8)
\]

where \( \frac{1}{q} + \frac{1}{q'} = 1 \).

Let \( 1 + \frac{2}{3} = \frac{1}{p} + \frac{3}{2} \). One easily verify \( 3 < r < \frac{9}{2} \) for any \( 1 < p < \frac{3}{2} \). This allows us to choose admissible pairs \((q, r) \in \Lambda\) such that

\[
\|V*|U|^2\Psi_j\|_{L^q H^{3/2}_T} \\
\leq \left\| V \right\|_p \left( \|U\|_{L^2 L^r} \|\Psi_j\|_{L^q H^{3/2}_T} + \|U\|_{L^2 L^r} \|\Psi_j\|_{L^q L^r} \|U\|_{L^2 H^1_T} \right) \\
\leq c\|V\|_p \|U\|_{L^q H^{3/2}_T} \|\Psi_j\|_X \leq C \quad (3.9)
\]

and

\[
\int_t^{t+h} |t + h - s|^{-\frac{q'}{2q'}} \, ds \to 0, \quad \text{as } h \to 0. \quad (3.10)
\]
Hence, \( J_2 \to 0 \) uniformly on \( j \in \mathbb{N} \) as \( h \to 0 \).

**Step 5** A Cantor diagonalized process.

For each \( N \in \{1, 2, 3, \cdots \} \), we choose a \( T(N) \) in Step 1, then a \( R(N) \) in Step 2 and then a \( M(N) \) in Step 3 such that

\[
\sup_{j \in \mathbb{N}} \| T_{U_0}(\Phi) \|_{L^4(|t| > T(N), H^1_3(\mathbb{R}^3))} < \frac{1}{N^4},
\]

(3.11)

\[
\sup_{j \in \mathbb{N}} \| T_{U_0}(\eta R(N) \Phi) \|_{L^4(|t| \leq T(N), H^1_3(\mathbb{R}^3))} < \frac{1}{N^4},
\]

(3.12)

\[
\sup_{j \in \mathbb{N}} \| T_{U_0}(\xi R(N) \Phi) \|_{L^4(|t| \leq T(N), H^1_3(|x| > M(N)))} < \frac{1}{N^4}.
\]

(3.13)

In this way, we can choose inductively subsequence \( \{ \Psi_{j,N} \} \) of \( \{ \Psi_{j,N-1} \} \) for every \( N = 1, 2, \cdots \) with \( \Psi_{j,0} = \Psi_j \), such that \( \{ T_{U_0}(\xi R(N) \Psi_{j,N}) \} \) converges in \( L^4(|t| \leq T(N), H^1_3(|x| \leq M(N))) \). Thus the subsequence \( \{ T_{U_0}(\xi R(N) \Psi_{N,N}) \}_{N=1}^{\infty} \) converges in \( L^4(\mathbb{R}, H^1_3(\mathbb{R}^3)) \). This completes the proof of Lemma 2.3.

### 4 Proof of Theorem 1.1

In this paper we still take advantage of the approach in Kumlin [9] to prove Theorem 1.1, and by exploring sufficiently compactness condition we give a more concise proof.

**Lemma 4.1** Let \( H \) be a Hilbert space, \( A_k : H \to H \), \( k = 1, 2, \cdots \), be analytic mappings, uniformly bounded on all compact set \( D \subset H \). Also assume that \( A_k u \to Au \) as \( k \to \infty \) for all \( u \in H \). Then the mapping \( A : H \to H \) is also analytic.

According to Theorem 1.2 one has that \( \mathcal{N}(T) : U_0 \to U(T) \) is analytic from \( H^1 \) to \( H^1 \) for every \( T \in \mathbb{R} \). The wave operators \( W_\pm \) and their inverses can be represented as

\[
W_\pm = \lim_{T \to \pm \infty} \mathcal{N}(-T)G(T)
\]

(4.1)

\[
W^{-1}_\pm = \lim_{T \to \pm \infty} G(-T)\mathcal{N}(T)
\]

(4.2)

Note that \( \mathcal{N}(-T)G(T) \) and \( G(-T)\mathcal{N}(T) \) are analytic on \( H^1 \), and \( G(T) \) is an isometric on \( H^1 \), Lemma 4.1 implies that \( W_\pm, W^{-1}_\pm \) and \( S \) are analytic provided that

\[
\sup_{\Phi \in D} \sup_{T \in \mathbb{R}} \| \mathcal{N}(T)\Phi \|_{H^1} < \infty
\]

(4.3)

for all compact set \( D \subset H^1 \). In fact,

\[
\| \mathcal{N}(T)\Phi \|_{H^1} \leq \| G(T)\Phi \|_{H^1} + \left\| \int_0^T \Delta^{-1} G'(t-s)(V*|U(\Phi)|^2)U(\Phi)(s)ds \right\|_{L^\infty H^1} \\
\leq \| \Phi \|_{H^1} + \| V \|_p \| U(\Phi) \|^3_{L^4 H^1_3}.
\]

(4.4)
Hence, it is enough to prove
\[
\sup_{\Phi \in D} \|U(\Phi)\|_{L^4 H^3_3} < \infty. \tag{4.5}
\]
Now we give the proof of (4.5) briefly by using the finite \(\varepsilon\)-cover again and the definition of Fréchet derivative.

In fact, As a direct result of the scattering theory, we have
\[
\|U(\Phi)\|_{L^4 H^3_3} < \infty
\]
for each \(\Phi \in H^1\). Hence, we need to prove it is bounded uniformly on \(\Phi \in D\).

Since \(D\) is a compact subset of \(H^1\), then for fixed \(0 < \varepsilon_0 < 1\), there exists a finite set \(A = \{\Phi_1, \Phi_2, \ldots, \Phi_n\}\) such that for any \(\Phi \in D\), there exists \(\Phi_i \in A\) satisfying
\[
\|\Phi - \Phi_i\|_{H^1} < \varepsilon_0.
\]
Note that \(U: \Phi \rightarrow U(\Phi)\) is analytic from \(H^1\) to \(L^4 H^3_3\), we easily see that the Fréchet derivative \(U'(\Phi_i)\) is a bounded operator from \(H^1\) to \(L^4 H^3_3\). This yields that
\[
\begin{align*}
\|U(\Phi)\|_{L^4 H^3_3} &\leq \|U(\Phi) - U(\Phi_i)\|_{L^4 H^3_3} + \|U(\Phi) - U(\Phi_i)\|_{L^4 H^3_3} \\
&\leq \|U'(\Phi_i)(\Phi - \Phi_i)\|_{L^4 H^3_3} + o(\varepsilon_0) + \|U(\Phi_i)\|_{L^4 H^3_3} \\
&\leq C\|\Phi - \Phi_i\|_{H^1} + o(\varepsilon_0) + \|U(\Phi_i)\|_{L^4 H^3_3} \\
&\leq C\varepsilon_0 + o(\varepsilon_0) + \|U(\Phi_i)\|_{L^4 H^3_3} < C. \tag{4.6}
\end{align*}
\]
This completes the proof of Theorem 1.1

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