Some Applications of Berezin $\theta$–sequences and Berezin symbols

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Abstract. We consider the so-called Berezin $\theta$– sequence, where $\theta$ is an operator valued inner function, for operators on the vector valued Hardy space $H^2_E (\mathbb{D})$, and study the invertibility of some operators on the model space $K_\theta = H^2_E \ominus \theta H^2_E$ via Berezin $\theta$– sequence. By applying of Berezin symbols technique the Toeplitz corona problem in the Bergman space $L^2_\mathcal{A} (\mathbb{D})$ is studied. Moreover, $C$– invertibility and $C$–unitarity of operators are also defined and studied.

Notation

$\mathbb{D}$ Open unit disc in the complex plane $\mathbb{C}$, $\mathbb{D} := \{ z \in \mathbb{C} : |z| < 1 \}$;

$\mathbb{T}$ unit circle, $\mathbb{T} := \partial \mathbb{D} = \{ z \in \mathbb{C} : |z| = 1 \}$;

Hardy classes of analytic functions,

$H^p$ Hardy classes can be identified with spaces of analytic functions on the unit disc $\mathbb{D}$: in particular, $H^\infty$ is the space of all bounded analytic functions on $\mathbb{D}$;

$E, E_*$ separable Hilbert spaces;

$H^2_E$ vector-valued Hardy space $H^2$ with values in $E$;

$L^\infty_{E \rightarrow E_*}$ class of bounded functions on the unit circle $\mathbb{T}$ whose values are bounded operators from $E$ to $E_*$;

$H^\infty_{E \rightarrow E_*}$ operator Hardy class of bounded analytic functions whose values are bounded operators from $E$ to $E_*$;

$K_\theta$ the model space, $K_\theta := H^2_E \ominus \theta H^2_E$ for the inner function $\theta \in H^\infty_{E \rightarrow E_*}$;

$T_\phi$ Toeplitz operator with symbol $\phi \in L^\infty_{E \rightarrow E_*}, T_\phi f := P_+ (\phi f), f \in H^2_E \subseteq E$,

where $P_+$ is an orthogonal projection onto $H^2_E \subseteq E$.

Throughout the paper all Hilbert spaces are assumed to be separable, and also we always assume that in any Hilbert space an orthonormal basis is fixed, so any operator $A : E \rightarrow E_*$ can be identified with its matrix.

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1. Introduction and Preliminaries

Let $E$ and $E$, be two separable Hilbert spaces, and $H^2_E$ be the vector-valued Hardy space $H^2$ with values in $E$. The function $θ, θ ∈ H^∞_{H^2_E}$ is called inner if its angular limiting values $θ(ζ)$ are unitary operators in $E$ for almost all $ζ$ in $T$. We consider the unilateral shift operator $S$ in the space $H^2_E$

$$Sf(z) = zf(z), \quad S^∗f(z) = \frac{f(z) - f(0)}{z},$$

the backward shift operator $S^∗$ (the adjoint of $S$) and its restriction $T$ to $S^∗$–invariant subspace $K_θ$,

$$T = S^∗|K_θ. \quad (1)$$

By the well-known Sz.-Nagy and Foias theorem [21], any contraction $A$ in a Hilbert space $H$ such that

$$s - \lim_n A^n = s - \lim A^∞ = 0,$$

is unitarily equivalent to an operator $T$ of the form (1), and the function $θ$ turns out to be the characteristic operator function of the contraction $A^∗$.

Following [11], and its references, we shall assume the following:

(i) That portion of the spectrum $σ(T)$ of the operator $T$ which lies within the unit disc $D$ consist of characteristic values which are simple poles of the resolvent.

(ii) The system of characteristic functions of operator (1), which correspond to characteristic values lying inside the unit disc $D$, is complete in $K_θ$.

In terms of the characteristic functions $θ$, condition (i) may be written as follows: if $λ_k ∈ σ(T) ∩ D$, then

$$θ(z) = θ_k(z)B_k(z), \quad |z| < 1, \quad (2)$$

where $θ_k, B_k$ are inner functions,

$$B_k(z) := \frac{λ_k - z}{1 - λ_kz}π_k + (I - π_k), \quad |z| < 1,$$

$π_k$ is the orthogonal projector of $E$ onto the subspace ker $θ(λ_k)$ of zeros of $θ(λ_k)$, and $B_k(λ_k)$ is a bounded invertible operator in $E$. Condition (ii) may also be expressed in terms of the characteristic function of the operator $T^*$.

Let $[λ_k]^∞_{k=1} = σ(T) ∩ D$ be a sequence of eigenvalues of operator $T$ of the form (1). It can be proved that the system of eigenvectors of operator $T$ is formed by the functions

$$Φ_k(z) := (1 - |λ_k|^2)^{1/2} \frac{θ̄^{-1}_k(λ_k)e}{1 - λ_kz}, \quad e ∈ π_kE, ||e|| = 1, \quad (3)$$

where $π_k$ is the orthogonal projector onto ker $θ(λ_k)$. The biorthogonal system consisting of eigenvectors of the operator $T^*$ has the form

$$Ψ_k(z) := (1 - |λ_k|^2)^{1/2} θ_k(z) \frac{e}{1 - λ_kz}, \quad e ∈ π_kE, ||e|| = 1. \quad (4)$$

(Note that when speaking of system (3) and (4), we mean any of the systems $\{Φ_k\}$ and $\{Ψ_k\}$ obtained from (3) and (4), respectively, by a choice of an orthonormal basis $\{e_i\}$, in each $π_kE$.) The following lemma has been originally proved by Katsnelson [16], see also Nikolski and Pavlov [18].
Lemma 1.1. If the Carleson condition

$$\inf_k \left| \prod_{n \neq k} \frac{\lambda_k - \lambda_n}{1 - \lambda_k \lambda_n} \frac{\lambda_n}{|\lambda_n|} \right| > 0$$  \hspace{1cm} (C)

is satisfied, then the system \( \{\Psi_k(z)\}_{k=1}^\infty \) is a Riesz basis in \( K_\theta \).

Let \( \mathcal{B}(H) \) denote the Banach algebra of all bounded linear operators on \( H \). The following definition is introduced in \([11]\).

**Definition 1.2.** For any operator \( A \in \mathcal{B}(K_\theta) \), its Berezin \( \theta \)–sequence is defined as

$$A^0(\lambda_k) := (A\Psi_k(z), \Psi_k(z)), k \geq 1.$$  \hspace{1cm} \text{By considering that} \( \theta_k \text{ is an inner function and} \|e\| = 1, \text{it is easy to verify that} \|\Psi_k\|_{H^2} = 1 \text{ for all} k \geq 1, \text{and hence} \|A^0(\lambda_k)\| \leq \|A\| \text{ for all} k \geq 1, \text{that is} \{A^0(\lambda_k)\}_{k=1}^\infty \text{ is a bounded sequence.}$$

For any function \( F \in H^\infty_{E^{-\infty}}, \) the model operator \( F(M_\theta) \) on \( K_\theta \) is defined by \( F(M_\theta)f = P_\theta(Ff), \ f \in K_\theta, \) where \( P_\theta : H^2_{E^\theta} \to K_\theta, P_\theta = I - T_0T_0^* \), is the orthogonal projector onto \( K_\theta \).

In this paper, we will study invertibility of operators \( F(M_\theta), F \in H^\infty_{E^{-\infty}}, \) via Berezin \( \theta \)–sequences.

Note that this paper was mainly motivated with a question of Treil from his paper \([22]\) concerning to Operator Corona Problem, which is stated as follows: does there exist \( \delta > 0 \) (close to 1) such that for any \( F \in H^\infty_{E^{-\infty}}, \) the inequality \( I \geq F(z)F(z) \geq \delta^2 I \) implies that there exists \( G \in H^\infty_{E^{-\infty}}, \) such that \( GF = I \)?

(Remark that the counterexample constructed in \([22]\) works only \( \delta < \frac{1}{2} \), the method from \([23]\) gives counterexample \( \delta < \frac{1}{2} \).)

In the present paper, we investigate the similar question for the model operators \( F(M_\theta), \ f \in H^\infty_{E^{-\infty}} \) (see Section 2). We also study in Section 3 the so-called Toeplitz corona problem in the Bergman-Hilbert space \( L^2_\theta = L^2_\theta(\mathbb{D}). \) Namely, we prove a necessary condition for its solvability. In Section 4, we introduce the notions \( \text{C} \)–in invertible and \( \text{C} \)–unitary operator which are extensions of invertible, unitary and essentially unitary operators, and prove some necessary and sufficient conditions for \( \text{C} \)–unitarity.

Before giving our results, note that the sequence \( X = \{x_n\}_{n \geq 1}, \) where \( x_n \in H, n \geq 1 \) is called a Riesz basis in \( H \) if there exists an isomorphism \( V \) mapping \( X \) onto an orthonormal basis \( \{x_n : n \geq 1\}; \) the operator \( V \) will be called the orthogonalizer of \( X \). It is well known that \( X \) is a Riesz basis in \( H \) if there are positive constants \( c \) and \( C \) such that

$$c \left( \sum_{n \geq 1} |a_n|^2 \right)^{1/2} \leq \left\| \sum_{n \geq 1} a_n x_n \right\| \leq C \left( \sum_{n \geq 1} |a_n|^2 \right)^{1/2} \hspace{1cm} (5)$$

for all finite complex sequence \( \{a_n\}_{n \geq 1} \). Note that if \( V \) is orthogonalizer of the family \( X \), then the product \( r(X) := \|V\|\|V^{-1}\| \) characterizes the deviation of the basis \( X \) from an orthonormal one and \( \|V\| \) and \( \|V^{-1}\| \) are the best constants in the inequality \((5);

\( r(X) \) will be referred to as the Riesz constant of the family \( X \). Obviously, \( r(X) \geq 1 \).

2. Invertibility of model operators via Berezin \( \theta \)–sequences

The main result of this section is the following theorem in which the invertibility of some operators on \( K_\theta \) via Berezin \( \theta \)–sequences, is proved.

**Theorem 2.1.** Let \( \Psi := \{\Psi_k(z)\} = \left\{ \theta_k(z) \frac{1 - |\lambda_k|^2}{1 - \lambda_k z} \right\} \) be a biorthogonal system (of system \((4)) \) consisting of eigenfunctions of the operator \( T^* \), where \( \Lambda := \{\lambda_k\}_{k=1}^\infty \) satisfies Carleson condition \((C)\), and let \( r(\Psi) := \|V\|\|V^{-1}\| \)
be a corresponding Riesz constant of the Riesz basis \( \Psi \) (see Lemma 1.1), where \( V \) is an orthogonalizer of the system \( \Psi \). Let \( F(M_\theta) \) be a model operator acting in \( K_\theta = H_\mathcal{E}^2_\theta \ominus H_\mathcal{E}^2_\theta \) and satisfying the following conditions:

(i) \[
\sum_{k=1}^{\infty} \left( \| F(M_\theta) \|_2^2 - \| \tilde{F}(M_\theta) \|_2^2 \right) < +\infty;
\]

(ii) \[
\inf_{k \geq 1} \left| F(M_\theta)^{\theta}(\lambda_k) \right| =: \delta > r(\Psi) \| V \| b_{F(M_\theta)},
\]

where

\[
b_{F(M_\theta)} := \left( \sum_{k=1}^{\infty} \left( \| F(M_\theta) \|_2^2 - \| \tilde{F}(M_\theta) \|_2^2 \right) \right)^{1/2}.
\]

Then, \( F(M_\theta) \) is invertible in \( K_\theta \) and

\[
\| F(M_\theta)^{-1} \| \leq \left( \frac{\delta}{r(\Psi)} - \| V \| b_{F(M_\theta)} \right)^{-1}.
\]

Proof. By considering that \( \Lambda \in \mathbb{C} \), it follows from Lemma 1.1 that \( \Psi \) is a Riesz basis in the model space \( K_\theta \). Therefore we have that

\[
\| V \|^{-1} \left( \sum_{k=1}^{N} |a_k|^2 \right)^{1/2} \leq \left\| \sum_{k=1}^{N} a_k \Psi_k \right\| \leq \| V \|^{-1} \left( \sum_{k=1}^{N} |a_k|^2 \right)^{1/2} \tag{6}
\]

for all finite sequences \( \{a_k\} \) of complex numbers. By considering (6) and condition (i) of the theorem, we obtain:

\[
\left\| \sum_{k=1}^{N} a_k F(M_\theta)^{\theta}(\lambda_k) \Psi_k(z) \right\| \geq \| V \|^{-1} \left( \sum_{k=1}^{N} |a_k F(M_\theta)^{\theta}(\lambda_k)|^2 \right)^{1/2} \geq \delta \| V \|^{-1} \left( \sum_{k=1}^{N} |a_k|^2 \right)^{1/2} \geq \frac{\delta}{\| V \| \| V^{-1} \|} \left\| \sum_{k=1}^{N} a_k \Psi_k(z) \right\| = \frac{\delta}{r(\Psi)} \left\| \sum_{k=1}^{N} a_k \Psi_k(z) \right\| \tag{7}
\]

for all finite sequences \( \{a_k\}_{k=1}^{N} \). Using condition (ii) of the theorem and inequalities (6), (7), we have for every sequence \( \{a_k\}_{k=1}^{N} \) that
\[
\left\| F(M_0) \sum_{k=1}^{N} a_k \Psi_k(z) \right\| = \left\| \sum_{k=1}^{N} a_k F(M_0) \Psi_k(z) \right\|
\]
\[
= \left\| \sum_{k=1}^{N} a_k \left( F(M_0) \Psi_k - F(M_0)^\theta (\lambda_k) \Psi_k + F(M_0)^\theta (\lambda_k) \Psi_k \right) \right\|
\]
\[
\geq \left\| \sum_{k=1}^{N} a_k F(M_0)^\theta (\lambda_k) \Psi_k(z) \right\| - \left\| \sum_{k=1}^{N} a_k \left( F(M_0) \Psi_k - F(M_0)^\theta (\lambda_k) \Psi_k \right) \right\|
\]
\[
\geq \frac{\delta}{r(\Psi)} \left\| \sum_{k=1}^{N} a_k \Psi_k(z) \right\| - \left\| V \right\| \left\| \sum_{k=1}^{N} a_k \Psi_k(z) \right\| \left\| \sum_{k=1}^{N} \left( F(M_0) \Psi_k - F(M_0)^\theta (\lambda_k) \Psi_k + F(M_0)^\theta (\lambda_k) \Psi_k \right) \right\|^{1/2}
\]
\[
\geq \frac{\delta}{r(\Psi)} \left\| \sum_{k=1}^{N} a_k \Psi_k(z) \right\| - \left\| V \right\| \left\| \sum_{k=1}^{N} a_k \Psi_k(z) \right\| \left\| \sum_{k=1}^{N} \left( F(M_0) \Psi_k - F(M_0)^\theta (\lambda_k) \Psi_k + F(M_0)^\theta (\lambda_k) \Psi_k \right) \right\|^{1/2}
\]
\[
= \frac{\delta}{r(\Psi)} \left\| \sum_{k=1}^{N} a_k \Psi_k(z) \right\| - \left\| V \right\| \left\| \sum_{k=1}^{N} \left( \left\| F(M_0) \right\|^2 - \left| F(M_0)^\theta (\lambda_k) \right|^2 \right) \right\|^{1/2} \left\| \sum_{k=1}^{N} a_k \Psi_k \right\|
\]
\[
= \left( \frac{\delta}{r(\Psi)} - \left\| V \right\| b_{F(M_0)} \right) \left\| \sum_{k=1}^{N} a_k \Psi_k(z) \right\|.
\]
Thus
\[ \left\lVert F(M_\theta) \sum_{k=1}^{N} a_k \Psi_k (z) \right\rVert \geq \left( \frac{\delta}{r(\Psi)} - \| V \| b_{F(M_\theta)} \right) \left\lVert \sum_{k=1}^{N} a_k \Psi_k \right\rVert . \]

Now, since \( \Psi \) is a complete system in \( K_\theta \), we have from the latter that
\[ \left\lVert F(M_\theta) f \right\rVert \geq \left( \frac{\delta}{r(\Psi)} - \| V \| b_{F(M_\theta)} \right) \left\lVert f \right\rVert \]
for all \( f \in K_\theta \).

Similarly, it is easy to show that
\[ \sum_{k=1}^{\infty} \left\lVert F(M_\theta)^* \Psi_k - F(M_\theta)^* \Psi_k \right\rVert^2 \leq \sum_{k=1}^{\infty} \left( 1 - \left| F(M_\theta)^* \lambda_k \right|^2 \right). \]

Then, by similar arguments it can be proved that
\[ \left\lVert F(M_\theta)^* f \right\rVert \geq \left( \frac{\delta}{r(\Psi)} - \| V \| b_{F(M_\theta)} \right) \left\lVert f \right\rVert \]
for all \( f \in K_\theta \).

Hence, we deduce from (8), (9) and condition (ii) of the theorem that \( F(M_\theta) \) is an invertible operator on \( K_\theta \) and
\[ \left\lVert F(M_\theta)^{-1} \right\rVert \leq \left( \frac{\delta}{r(\Psi)} - \| V \| b_{F(M_\theta)} \right)^{-1}, \]
which proves the theorem.

We remark that it easy to see from the proof of Theorem 2.1 that by the same method it can be proved invertibility of more general operators on \( K_\theta \) including, in particular, truncated Toeplitz operators \( T_\varphi \) with symbols \( \varphi \) in \( L^\infty_E \rightarrow E \) (for the related results, see [11, 12]).

3. On the Toeplitz Corona Problem

Let \( dA \) denote Lebesgue area measure on \( \mathbb{D} \), normalized so that the measure of the disc \( \mathbb{D} \) is 1. The Bergman space \( L^2_\mathbb{A} := L^2 (\mathbb{D}) \) is the Hilbert space consisting of the analytic functions on \( \mathbb{D} \) that are square integrable with respect to the measure \( dA \). For \( \varphi \in L^\infty (\mathbb{D}) \), the Toeplitz operator \( T_\varphi \) with symbol \( \varphi \) is defined on \( L^2_\mathbb{A} \) by \( T_\varphi f = P(\varphi f) \), where \( P : L^2 (\mathbb{D}, dA) \rightarrow L^2_\mathbb{A} \) is the orthogonal projector. Using the concrete form of the reproducing kernel \( k_{z, z'} (w) \), \( z, w \in \mathbb{D} \), we can express the Toeplitz operator to be the integral operator:
\[
T_\varphi f = \int_{\mathbb{D}} \varphi (w) f (w) k_{z, z'} (w) dA (w) = \int_{\mathbb{D}} \frac{\varphi (w) f (w)}{1 - w^2} dA (w)
\]
for \( f \) in \( L^2_\mathbb{A} \).

In this section, we will consider the solvability of the operator equation in the set of Toeplitz operators on the Bergman space \( L^2_\mathbb{A} \) (i.e., the Toeplitz Corona Problem in the Bergman space):
\[
X_1 T_\varphi_1 + X_2 T_\varphi_2 + \ldots + X_n T_\varphi_n = I,
\]
where $T_{\varphi_i}, i = 1, 2, \ldots, n,$ are given Toeplitz operators on the Bergman space $L^2_a$.

Note that for $\varphi \in L^\infty(\mathbb{D})$, the Berezin symbol $\check{T}_\varphi$ of the Toeplitz operators $T_\varphi$ on $L^2_a$ is defined as (see Englis [6] and Zhu [24])

\[
\check{T}_\varphi(z) := \left\langle T_\varphi \hat{k}_{\lambda, l_z^2}, \hat{k}_{\lambda, l_z^2} \right\rangle \equiv \left\langle T_\varphi \frac{1 - |z|^2}{(1 - \overline{w}z)^2}, \frac{1 - |z|^2}{(1 - \overline{w}z)^2} \right\rangle = \int_{\mathbb{D}} \varphi(w) |\hat{k}_{\lambda, l_z^2}(w)|^2 dA(w),
\]

where $\hat{k}_{\lambda, l_z^2}$ is the normalized Bergman reproducing kernel of $L^2_a(\mathbb{D})$ given by

\[
\hat{k}_{\lambda, l_z^2}(w) = \frac{1 - |z|^2}{(1 - \overline{w}z)^2} \quad (z \in \mathbb{D}).
\]

For simplicity, we will denote $\varphi := \check{T}_\varphi$. It is well known that $\varphi = \varphi$ for any bounded harmonic function $\varphi$ (see Ahern, Flores and Rudin [1] and Englis [5]). For other results related with Berezin symbols and Toeplitz operators the reader can be found in [2, 5–7, 10, 17, 24, 25]. The following lemma belongs to Axler and Zheng [2].

**Lemma 3.1.** If $\varphi$ is bounded harmonic function, then the function $\lambda \rightarrow \left\| T_{\varphi - \varphi_1(\lambda)} \hat{k}_{\lambda, l_z^2} \right\|_{L^2_a}$ has nontangential limit 0 at almost every point of $\partial \mathbb{D}$.

Now we are ready to formulate and prove our next result.

**Theorem 3.2.** Let $\varphi_1, \varphi_2, \ldots, \varphi_n$ be bounded harmonic functions on $\mathbb{D}$. If there exist the bounded harmonic functions $\psi_1, \psi_2, \ldots, \psi_n$ such that

\[
T_{\psi_1}T_{\psi_2} + T_{\psi_2}T_{\psi_3} + \ldots + T_{\psi_n}T_{\psi_1} = I,
\]

where $T_{\psi_i}, T_{\psi_j}$ ($i = 1, 2, \ldots, n$) are Toeplitz operators on $L^2_a$, then

\[
\text{ess inf}_T \left( |\varphi_1(\zeta)| + \ldots + |\varphi_n(\zeta)| \right) > 0.
\]

**Proof.** Let $T_{\psi_1}T_{\psi_1} + T_{\psi_2}T_{\psi_2} + \ldots + T_{\psi_n}T_{\psi_n} = I$. Then, passing to the Berezin transform, we have:

\[
1 = \left\langle T_{\check{T}_{\psi_1}}, \check{T}_{\psi_1}(\lambda) \right\rangle + \left\langle T_{\check{T}_{\psi_2}}, \check{T}_{\psi_2}(\lambda) \right\rangle + \ldots + \left\langle T_{\check{T}_{\psi_n}}, \check{T}_{\psi_n}(\lambda) \right\rangle = \left\langle T_{\check{T}_{\psi_1}}, \check{k}_{\lambda, l_z^2} \right\rangle + \ldots + \left\langle T_{\check{T}_{\psi_n}}, \check{k}_{\lambda, l_z^2} \right\rangle + \left\langle T_{\check{T}_{\psi_1}}, \check{T}_{\psi_1}(\lambda) \right\rangle + \ldots + \left\langle T_{\check{T}_{\psi_n}}, \check{T}_{\psi_n}(\lambda) \right\rangle
\]

\[
= \left\langle \check{T}_{\psi_1}(\lambda), \check{T}_{\psi_1} \right\rangle + \ldots + \left\langle \check{T}_{\psi_n}(\lambda), \check{T}_{\psi_n} \right\rangle + \left\langle T_{\check{T}_{\psi_1}}, \check{T}_{\psi_1}(\lambda) \right\rangle + \ldots + \left\langle T_{\check{T}_{\psi_n}}, \check{T}_{\psi_n}(\lambda) \right\rangle
\]

\[
= \left\langle \check{T}_{\psi_1}(\lambda), \check{T}_{\psi_1} \right\rangle + \ldots + \left\langle \check{T}_{\psi_n}(\lambda), \check{T}_{\psi_n} \right\rangle + \left\langle T_{\check{T}_{\psi_1}}, \check{T}_{\psi_1}(\lambda) \right\rangle + \ldots + \left\langle T_{\check{T}_{\psi_n}}, \check{T}_{\psi_n}(\lambda) \right\rangle
\]

\[
= \left\langle \check{T}_{\psi_1}(\lambda), \check{T}_{\psi_1} \right\rangle + \ldots + \left\langle \check{T}_{\psi_n}(\lambda), \check{T}_{\psi_n} \right\rangle + \left\langle T_{\check{T}_{\psi_1}}, \check{T}_{\psi_1}(\lambda) \right\rangle + \ldots + \left\langle T_{\check{T}_{\psi_n}}, \check{T}_{\psi_n}(\lambda) \right\rangle
\]
and hence

\[ 1 = \psi_1(\lambda) \varphi_1(\lambda) + \ldots + \psi_n(\lambda) \varphi_n(\lambda) + \left\langle T_{\psi_1} \bar{k}_{A,\lambda}, T_{\psi_1-\varphi_1(\lambda)} \bar{k}_{A,\lambda} \right\rangle + \ldots + \left\langle T_{\psi_n} \bar{k}_{A,\lambda}, T_{\psi_n-\varphi_n(\lambda)} \bar{k}_{A,\lambda} \right\rangle \]

for all \( \lambda \in \mathbb{D} \). Therefore we obtain:

\[ 1 \leq \left| \psi_1(\lambda) \right| \left| \varphi_1(\lambda) \right| + \ldots + \left| \psi_n(\lambda) \right| \left| \varphi_n(\lambda) \right| + \left\| T_{\psi_1} \bar{k}_{A,\lambda} \right\| \left\| T_{\psi_1-\varphi_1(\lambda)} \bar{k}_{A,\lambda} \right\| + \ldots + \left\| T_{\psi_n} \bar{k}_{A,\lambda} \right\| \left\| T_{\psi_n-\varphi_n(\lambda)} \bar{k}_{A,\lambda} \right\|. \]

Note that every bounded harmonic function has nontangential limits at almost every point of \( \mathbb{T} \), and by applying Lemma 3.1 we have that the functions \( \lambda \rightarrow \left\| T_{\psi_i-\varphi_i(\lambda)} \bar{k}_{A,\lambda} \right\| \) (\( i = 1, 2, \ldots, n \)) have nontangential limits 0 at almost every point of \( \mathbb{T} \). Then we have from the latter inequality that

\[ 1 \leq \left| \psi_1(\zeta) \right| \left| \varphi_1(\zeta) \right| + \ldots + \left| \psi_n(\zeta) \right| \left| \varphi_n(\zeta) \right| \]

for almost all \( \zeta \in \mathbb{T} \). This implies that

\[ \text{ess inf}_{\mathbb{T}} \left( \left| \psi_1(\zeta) \right| + \ldots + \left| \psi_n(\zeta) \right| \right) \geq \frac{1}{\max_{1 \leq i \leq n} \left\| \psi_i \right\|_{L^\infty(\mathbb{T})}} > 0, \]

which proves the theorem. \( \Box \)

We do not know: is (10) a sufficient condition in Theorem 3.2?

### 4. On the C– unitarity of C– invertible operators

Recall that the Berezin symbol \( \bar{A} \) of an operator on the reproducing kernel Hilbert space \( \mathcal{H} = \mathcal{H}(\Omega) \) with reproducing kernel \( k_{H,\lambda} \) is defined by

\[ \bar{A}(\lambda) := \left\langle A k_{H,\lambda}, k_{H,\lambda} \right\rangle, \lambda \in \Omega, \]

where \( k_{H,\lambda} := \frac{k_{H,\lambda}}{\left\| k_{H,\lambda} \right\|} \) is the normalized reproducing kernel for the space \( \mathcal{H} \). It is clear that:

(i) \( \bar{A} \) is a bounded function on \( \Omega \) and \( \sup_{\lambda \in \Omega} \left| \bar{A}(\lambda) \right| =: \text{ber}(A) \leq \|A\| ; \text{ber}(A) \) is called the Berezin number of operator \( A \);

(ii) the Berezin set \( \text{Ber}(A) := \text{Range}(\bar{A}) \) is contained in the numerical range \( \text{W}(A) := \{ \langle Ax, x \rangle : \|x\| = 1 \} \) of operator \( A \), and hence \( \text{ber}(A) \leq \omega(A) \), where \( \omega(A) := \sup \{ \langle Ax, x \rangle : x \in H \text{ and } \|x\| = 1 \} \) is the numerical radius of operator \( A \in \mathcal{B}(H) \).

(iii) \( \bar{A} = \overline{A} \), and hence \( \left| \bar{A} \right| = \left| A \right| \).

For more detail about Berezin symbols, and their applications, see for instance Berezin [3, 4], Kac [11, 13, 14], Zhu [24] and Zorboska [25].

**Definition 4.1.** (i) Let \( A, C \in \mathcal{B}(\mathcal{H}) \). We say that \( A \) is a C– invertible operator if there exists an operator \( B \in \mathcal{B}(\mathcal{H}) \) (which is called a C– inverse of \( A \)) such that

\[ BA = AB = C. \]  

(ii) We say that a C– invertible operator \( A \) is C– unitary if \( B = A^* \).
Note that $C$-unitary operator is a generalization of unitary ($C = I$) and essentially unitary ($C = I + K$, where $K$ is compact) operators on a Hilbert space. So, the main result of this section (Theorem 4.1) improves some results of the works [9, 11, 14, 15].

It is easy to see that if $A \in \mathcal{B}(\mathcal{H}(\Omega))$ is a $C$-unitary operator, then $\|A\bar{k}_{H,\lambda}\|^2 = \overline{C}(\lambda)$ and $\|A^\ast \bar{k}_{H,\lambda}\|^2 = \overline{C}(\lambda)$ for all $\lambda \in \Omega$ (which in particular shows that $C \geq 0$, and hence $Ber(C) \subset \{0, +\infty\}$). Indeed, it follows from (11) that

$$\overline{C}(\lambda) = \overline{BA}(\lambda) = \langle BA\bar{k}_{H,\lambda}, \bar{k}_{H,\lambda} \rangle$$

$$= \langle A^\ast A\bar{k}_{H,\lambda}, \bar{k}_{H,\lambda} \rangle = \langle \bar{k}_{H,\lambda}, \bar{k}_{H,\lambda} \rangle$$

and similarly we obtain from $C = AB = AA^\ast$ that $\overline{C}(\lambda) = \|A^\ast \bar{k}_{H,\lambda}\|^2$ for all $\lambda \in \Omega$.

In this section, we give sufficient conditions for $C$-unitarity which are very close to the above mentioned necessary conditions. We denote by $A^{-1C}$ the $C$-inverse of $C$-invertible operator, i.e., $A^{-1C} := B$ in (11).

Note that unitary operators can be characterized as invertible contractions with contractive inverses, i.e., as operators $A$ with $\|A\| \leq 1$ and $\|A^{-1}\| \leq 1$ (see, for instance, Furuta [8]). Sano and Uchiyama [19] improved this result by proving that if $A$ is an invertible operator on the Hilbert space $H$ such that $w(A) \leq 1$ and $w(A^{-1}) \leq 1$, then $A$ is unitary (see also Stampfli [20, Corollary 1]). Now the following question is natural: is it true that if $A$ is invertible and $ber(A) \leq 1$ and $ber(A^{-1}) \leq 1$, then $A$ is unitary?

In the following theorem we partially solve this question. Our proof is based mainly on a Furuta’s argument contained in [8].

**Theorem 4.2.** Let $C \in \mathcal{B}(\mathcal{H}(\Omega))$ be fixed such that $Re(C) \geq 0$. If $A \in \mathcal{B}(\mathcal{H}(\Omega))$ is a $C$-invertible operator such that $\|A^\ast \bar{k}_{H,\lambda}\|^2 \leq Re(C(\lambda))$ and $\|A^{-1C} \bar{k}_{H,\lambda}\|^2 \leq Re(C(\lambda))$ for all $\lambda \in \Omega$, then $A$ is a $C$-unitary operator.

**Proof.** Indeed, we have for every $\lambda \in \Omega$ that

$$\|A^\ast - A^{-1C}\|_{\bar{k}_{H,\lambda}}\|^2 = \langle (A^\ast - A^{-1C}) \bar{k}_{H,\lambda}, (A^\ast - A^{-1C}) \bar{k}_{H,\lambda} \rangle$$

$$= \|A^\ast \bar{k}_{H,\lambda}\|^2 + \|A^{-1C} \bar{k}_{H,\lambda}\|^2 - \langle A^\ast \bar{k}_{H,\lambda}, A^{-1C} \bar{k}_{H,\lambda} \rangle$$

$$- \langle A^{-1C} \bar{k}_{H,\lambda}, A^\ast \bar{k}_{H,\lambda} \rangle$$

$$= \|A^\ast \bar{k}_{H,\lambda}\|^2 + \|A^{-1C} \bar{k}_{H,\lambda}\|^2$$

$$- \langle \bar{k}_{H,\lambda}, AA^{-1C} \bar{k}_{H,\lambda} \rangle - \langle AA^{-1C} \bar{k}_{H,\lambda}, \bar{k}_{H,\lambda} \rangle$$

$$= \|A^\ast \bar{k}_{H,\lambda}\|^2 + \|A^{-1C} \bar{k}_{H,\lambda}\|^2$$

$$- \langle \bar{k}_{H,\lambda}, C \bar{k}_{H,\lambda} \rangle - \langle C \bar{k}_{H,\lambda}, \bar{k}_{H,\lambda} \rangle$$

$$= \|A^\ast \bar{k}_{H,\lambda}\|^2 + \|A^{-1C} \bar{k}_{H,\lambda}\|^2$$

$$- \overline{C}(\lambda) + C(\lambda)\|^2$$

$$= \|A^\ast \bar{k}_{H,\lambda}\|^2 + \|A^{-1C} \bar{k}_{H,\lambda}\|^2 - 2Re(C(\lambda)) \leq 0$$

(by conditions of the theorem). This shows that $(A^\ast - A^{-1C})k_{H,\lambda} = 0$ for all $\lambda \in \Omega$ which implies that $A^{-1C} = A^\ast$ because the system $\{k_{H,\lambda} : \lambda \in \Omega\}$ is complete on $\mathcal{H}$. The theorem is proved. \(\square\)
It is easy to see that the conditions of this theorem implies the inequalities $\text{ber}(AA^*) \leq \sup_{\lambda \in \Omega} \Re(\tilde{C}(\lambda))$ and $\text{ber}(A^{-1c}A^{-1c}) \leq \sup_{\lambda \in \Omega} \Re(\tilde{C}(\lambda))$.

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