The identification of physical degrees of freedom is sometimes obscured in the path integral formalism, and this makes it difficult to impose some constraints or to do some approximations. I review a number of cases where the difficulty is overcome by deriving the path integral from the operator form of the partition function after such identification has been made.

I. INTRODUCTION

It is a great pleasure for me to contribute to this volume in honor of Prof. Yu. A. Simonov. This is an occasion for me to remember again the origin of our friendship which extended over the years well beyond what it would appear from our joint papers.

The subject of my contribution is focussed on my recent interests in several problems which have a common feature: The identification of degrees of freedom in a path integral. Indeed there are many situations where such identification is helpful or necessary. An old example is the thermodynamics of gauge theories, but recently I met with many others. The first I will discuss here is how to find actions exactly equivalent to the standard ones but closer to the continuum at finite lattice spacing. This includes the definition of the couplings of the chemical potential, an issue of particular importance in QCD. The chemical potential is used to fix the expectation value of some charge operator. Fixing instead the value of the charge, namely selecting a specific charge sector is somewhat more difficult, but interesting pieces of information can be obtained by an expansion in a given charge sector of the fermion determinant in series of the number of fermions. The last problem I will consider is how to describe the low lying excitations of fermionic systems, both relativistic and nonrelativistic, by means of effective bosons, in short how to bosonize them. All these problems require the identification of the relevant degrees of freedom. In
the last case to determine the structure of the composite bosons in terms of the fermionic constituents, in the first case to show the equivalence of different actions.

Now the path integral formalism is widely used because of its flexibility and the possibility of numerical applications, but the identification of degrees of freedom is not always easy, while it can be conveniently achieved in the operator form of the partition function. Therefore in all the above problems I will first identify the relevant degrees of freedom in the Fock space where the partition function is defined, then I will introduce the appropriate constraints or approximations and finally I will derive the constrained or approximated path integral.

As I said all the subjects I mentioned have a common feature in the role played by the identification of degrees of freedom, but are otherwise very different. Therefore the motivations for their investigation are given separately in the relative Sections. In Section 2 I will report the results I will use later about the standard derivation of the path integral from the operator form of the partition function. In Section 3 I will show how to derive an action different from the standard one and closer to the continuum in the nonrelativistic case. In Section 4 I will carry out the corresponding derivation for relativistic field theories, confining myself to the couplings of the chemical potential. In Section 5 I will discuss the case of a given charge sector. In Section 6 I will present a general method of bosonization valid for relativistic and nonrelativistic theories and in Section 7 my conclusions.

II. THE STANDARD DERIVATION OF THE EUCLIDEAN PATH INTEGRAL FROM THE PARTITION FUNCTION

Let me introduce some definitions. I denote by $\tau$ the temporal lattice spacing, by $N_0$ the number of temporal sites, by $x_0$ or $t$ the temporal component of the site position vector $x$, by $T$ the temperature, by $\mu$ the chemical potential, by $\hat{Q}$ the (electric, baryon...) charge operator, and by $\hat{T}(x_0)$ the transfer matrix.

In the nonrelativistic case $\mathcal{T}$ is expressed in terms of the Hamiltonian

$$\mathcal{T} = \exp (-\tau H),$$

and the Hamiltonian is the generator of continuous time translations. In relativistic theories instead only the transfer matrix is known in general, and the above equation can be used to
define a Hamiltonian, but only as the generator of discrete translations by the time spacing \( \tau \). Another important difference is that the nonrelativistic interactions are generally quartic in the fields while the relativistic ones are quadratic. Both features contribute to make the derivation of the euclidean path integral different in the two cases.

\( \hat{T} \) is defined in terms of particle-antiparticle creation-annihilation operators \( \hat{c}^\dagger, \hat{d}^\dagger, \hat{c}, \hat{d} \) acting in a Fock space. It depends on the time coordinate \( x_0 \) only through the dependence on it of other fields (for instance gauge fields). In fact the creation and annihilation operators do not depend on \( x_0 \). They depend on the spatial coordinates \( x \) and on the internal quantum numbers (Dirac, color and flavor indices in the case of QCD), comprehensively represented by \( I, J,... \).

In the transfer matrix formalism often one has to do with quantities at a given (Euclidean) time \( x_0 \). For this reason we adopt a summation convention over spatial coordinates and intrinsic indices at fixed time. So for instance, for an arbitrary matrix \( M \), we will write

\[
\hat{c}^\dagger M(x_0) \hat{c} = \sum_{x,y,I,J} \hat{c}^\dagger_{x,I} M_{x,I,y,J}(x_0) \hat{c}_{y,J}.
\] (2)

In this notation the charge operator \( \hat{Q} \) has the form

\[
\hat{Q} = \hat{c}^\dagger \hat{c} - \hat{d}^\dagger \hat{d}.
\] (3)

The grand canonical partition function can be written as a time-ordered product

\[
Z = \text{Tr} \left\{ \exp \left( \frac{\mu}{T} \hat{Q} \right) \prod_{x_0} \hat{T}(x_0) \right\},
\] (4)

which, using the relation \( T^{-1} = \tau N_0 \), and assuming the conservation of \( \hat{Q} \) is conveniently rewritten as

\[
Z = \text{Tr} \left\{ \prod_{x_0} \left[ \hat{T}(x_0) \exp \left( \mu \tau \hat{Q} \right) \right] \right\}.
\] (5)

The standard way \([6, 7]\) to obtain the path integral form of \( Z \) is to write all the operators in normal order and introduce between the factors in Eq.(5) the identity

\[
\mathcal{I} = \int [dc^+ dc dd^+ dd] \exp(-c^+ c - d^+ d) |cd\rangle \langle cd|,
\] (6)

where the basis vectors are coherent states

\[
|cd\rangle = |\exp(-c^\dagger c - d^\dagger d)\rangle.
\] (7)
The $c^+, c, d^+, d$ are holomorphic/Grassmann variables and satisfy periodic/antiperiodic boundary conditions in time for bosons/fermions respectively [6]. They have the label of the time slice where the identity operator is introduced. For the other indices they are subject to the same convention as the creation and annihilation operators. The main property of coherent states is that they are eigenstates of the annihilation operators

$$\hat{c}|cd\rangle = c|cd\rangle.$$ (8)

To get the functional form of the partition function it is only necessary to evaluate the matrix elements $\langle c_1 d_1 | T | c d \rangle$. This, as anticipated in the Introduction, must be done in different ways for nonrelativistic and relativistic theories.

The first case appears more difficult because of the quartic interactions, but since in general there are no ultraviolet divergencies (a detailed discussion of the departure from the standard form of the path integral in the presence of singular potentials can be found in [8]) we can make without any error the approximation

$$\exp\left(-\tau \hat{H}\right) \sim 1 - \tau \hat{H}.$$ (9)

Let us then consider a many-body system with the Hamiltonian written in normal form

$$\hat{H}(\hat{a}^\dagger, \hat{a}) = \sum_{x,y} \left\{ \hat{a}^\dagger_x h_{x,y} \hat{a}_y + \hat{a}^\dagger_y v_{x,y} \hat{a}_x \hat{a}_x \right\}.$$ (10)

The standard expression of the euclidean path integral associated to this Hamiltonian is

$$Z = \int [da^* da] \exp(-S)$$ (11)

where

$$S = \tau \sum_{t=1}^{N_0} \left\{ a_{t+1}^* \nabla_t a_t + H(a_{t+1}^*, a_t) \right\}$$ (12)

is the action and I denoted by

$$\nabla_t f_t = \frac{1}{\tau} (f_{t+1} - f_t)$$ (13)

the right discrete time derivative. Notice the time splitting between the fields and their conjugates, which implies a departure from the classical expression not only in derivative, but also in potential terms. Needless to say, neglecting this time splitting introduces finite errors [6].
In the relativistic case the transfer matrix can be written

\[ T_{x_0} = T_{x_0}^\dagger T_{x_0} \]  

(14)

where

\[ T = \exp \left( \hat{c}^\dagger M \hat{c} + \hat{d}^\dagger M^T \hat{d} \right) \exp \left( \hat{c} N \hat{d} \right) \]  

(15)

where the upper script \( T \) means transposed and I do not need to specify the matrices \( M, N \). The matrix elements of \( T \) can be exactly evaluated, yielding the standard form of the Euclidean path integral.

**III. ANTINORMAL ORDERING: A DIFFERENT ACTION IN THE NONRELATIVISTIC CASE**

In the standard form of the action the time is split in the fields and their conjugates. This is an artifact which makes the equations unnecessarily different from the continuous, somewhat more complicated and somewhat confusing. In gauge theories, for instance, such time splitting introduces a coupling of the chemical potential to the temporal gauge fields which has been considered of physical significance, while it can be altogether avoided in a different derivation of the path integral. I will first illustrate such a derivation in the nonrelativistic case.

Instead of the normal order I write the Hamiltonian in antinormal order (all the annihilation operators to the left of the creation ones)

\[ \hat{H} = \sum_{x,y} \left\{ h_0 + \hat{a}_x h'_x, y \hat{\hat{a}}^\dagger_y + \hat{a}_x \hat{a}_y v_{x,y} \hat{\hat{a}}^\dagger_y \hat{\hat{a}}^\dagger_x \right\} \]  

(16)

where

\[ h_0 = N^3 (\sigma h_{x,x} + w) \]

\[ h'_x, y = \sigma h_{x,y} - w \delta_{x,y} \]  

(17)

In the above equations

\[ w = \sigma v_{x,x} + \sum_z v_{x,z} \]  

(18)

\( N^3 \) is the number of spatial sites and \( \sigma = -1 \) for fermions and +1 for bosons. I assume that the potential is sufficiently regular for \( w \) to exist, otherwise a regularization is needed or a
more drastic change in our procedure. Then I expand the transfer matrix and in each term I insert the identity between the creation and annihilation operators. For the rightmost factor before taking the trace I move the creation operators to the leftmost position. In this way the term with the time derivative remains unchanged, but in the other terms all the fields appear at the same time

\[ S' = \tau \sum_t \left\{ h_0 + a_{t+1}^* \nabla_t a_t + (h' - h) a_t^* a_t + H(a_t^*, a_t) \right\}. \] (19)

We can easily check on solvable models that this action gives the right results and that the terms arising from the rearrangement in antinormal ordering cannot be neglected.

In the case of fermions the time splitting between the fields and their conjugates in the potential terms can be avoided in the same way as for bosons, but also in a simpler way. In fact the Grassmann fields, unlike the holomorphic variables, are independent from their conjugates, so that the simple transformation

\[ a_{t+1}^* \rightarrow a_t^* \] (20)

eliminates the time splitting everywhere, with the obvious exception of the term with the time derivative, which is changed into the left one.

Because in this case we have two different derivations of the path integral, by their comparison we can get nontrivial identities.

**IV. ANTINORMAL ORDERING: A DIFFERENT COUPLING OF THE CHEMICAL POTENTIAL IN RELATIVISTIC FIELD THEORIES**

In relativistic field theories the artificial time splitting affects only the coupling of the chemical potential. In the Hasenfratz–Karsch–Kogut formulation \cite{9}, which is the standard one, for Wilson fermions such coupling takes the form

\[ \delta S = 2K \sum_x q \left\{ \exp(\mu - 1) P_0^{(+)} U_0 T_0^{(+)} + \exp(-\mu) - 1 \right\} P_0^{(-)} T_0^{(-)} U_0^{(+)} \right\} q \] (21)

where \( q \) is the quark field, \( K \) is the hopping parameter, \( U_0 \) the temporal link variable and

\[
\begin{align*}
P_0^{(\pm)} &= \frac{1}{2} (\mathbb{1} \pm \gamma_0) \\
T_0^{(\pm)} f(x_0) &= f(x_0 \pm 1).
\end{align*}
\] (22)
Because of gauge invariance, in the presence of the time splitting a coupling with the temporal links is needed, and this led to the conclusion that a nonvanishing contribution of the chemical potential must necessarily involve a Polyakov loop. But Creutz showed in a toy model $[10]$ that this is not true, and I will report $[4]$ in the sequel how to avoid such artifact in the full fledged QCD.

I write the exponential of the charge in the following way

$$\exp(\mu a_0 \hat{Q}) = \int \left[ dc^+ dc dd^+ dd \right] \exp(\delta S - c^+ c - d^+ d) |cd\rangle \langle cd|$$  \hspace{1cm} (23)

where

$$\delta S = (1 - \cosh(\mu a_0)) \left( c^+ c + d^+ d \right) + \sinh(\mu a_0) \left( c^+ c - d^+ d \right).$$  \hspace{1cm} (24)

The expression of $\delta S$ is obtained by expanding the exponential of the charge operator, putting all the terms in antinormal form, inserting in each monomial the unity between the set of annihilation and the set of creation operators and replacing them by their Grassmannian eigenvalues. For the rightmost exponential of the charge before taking the trace one has to move the creation operators to the left of all the operators appearing under trace.

After this, the construction of the path integral proceeds in the standard way, and we get the standard action with the exception of the coupling of the chemical potential where all the fields appear at the same time

$$\delta S = \sum_x \mathcal{Q}(1 - \cosh(\mu a_0)) + \sinh(\mu a_0) \gamma_0]Bq.$$  \hspace{1cm} (25)

Here I only need to say that the matrix $B$ does not depend on $U_0$. What is important is to notice that the quark field and its conjugate are at the same time, and then the temporal Wilson variable disappeared.

V. EXPANSION OF THE FERMION DETERMINANT IN THE NUMBER OF FERMIONS IN A GIVEN CHARGE SECTOR

The use of the chemical potential is a way to impose a given expectation value for some conserved charge. The alternative option of selecting a given sector of the charge in the path integral presents additional difficulties, but something can be learned by a series expansion of the fermion determinant $[5]$. To be concrete we will refer to the case of QCD, but the method can be applied to other cases with appropriate modifications.
In the absence of any condition on the baryon number the quark determinant is
\[ \det Q = \int [d\mathbf{q} dq] \exp S_q, \] (26)
where \( S_q \) is the quark action and \( Q \) the quark matrix. My strategy is to write \( \det Q \) as the trace of the transfer matrix acting in the quark Fock space, impose the restriction to a given baryonic sector, and then rewrite the trace as the determinant of a modified quark matrix. The round trip is done by mapping the Grassmann algebra generated by the quark fields into the Fock space following the construction of Lüscher [7]. But while his paper is based on the mere existence of the map, in order to enforce the projection I will make use of a concrete realization by means of coherent states.

The first step is then to write the unconstrained determinant as a trace in the Fock space
\[ \det Q = \text{Tr} \hat{T}. \] (27)
The second step is to impose the restriction to a sector with baryon number \( n_B \) by inserting in the trace the appropriate projection operator \( \hat{P}_{n_B} \)
\[ \det Q|_{n_B} = \text{Tr} \left( \hat{T} \hat{P}_{n_B} \right) \int [dx^+ dx dy^+ dy] \exp \left( -x^+ x - y^+ y \right) \langle x, y| \hat{T} \hat{P}_{n_B} |- x, -y \rangle \] (28)
which will be expressed in terms of the determinant of a modified quark matrix. The kernel \( \langle x, y| \hat{T} \hat{P}_{n_B} |- x, -y \rangle \) has the integral form
\[ \langle x, y| \hat{T} \hat{P}_{n_B} |- x, -y \rangle = \int [dz^+ dz dw^+ dw] \exp \left( -z^+ z - w^+ w \right) \langle x, y| \hat{T} |z, w \rangle \langle z, w| \hat{P}_{n_B} |- x, -y \rangle. \] (29)
The expression of the kernel \( \langle x, y| \hat{T} |z, w \rangle \) will not be reported here, while that of \( \hat{P}_{n_B} \) can easily be derived
\[ \langle z, w| \hat{P}_{n_B} |- x, -y \rangle = \sum_{r=0}^{\infty} (-1)^{n_B} \langle (\hat{y}w^+)^r (\hat{x}z^+)^{(n_B+r)} (\hat{x}z^+) \rangle \frac{1}{((n_B + r)!r!)^2}. \] (30)
Since \( \hat{x}^+, \hat{y}^+ \) are creation operators of quarks and antiquarks respectively, we see that the \( r \)-th term of this series gives the gauge-invariant contribution of \( n_B \) valence quarks plus \( r \) quark-antiquark pairs.

Needless to say, for \( n_B = 0 \), \( \det Q|_{n_B} \) does not reduce to the unconstrained determinant: Indeed also baryonic states are present in the unconstrained determinant, while they are absent in \( \det Q|_{n_B=0} \). In QCD at nonvanishing temperature it makes a difference whether
we impose or we do not the condition \( n_B = 0 \). In view of the relatively low value of the critical temperature with respect to the nucleon mass, however, we do not expect significant effects from the restriction to a given baryon sector unless we go to exceedingly high temperatures.

Let me now proceed to derive our final result. By evaluating the vacuum expectation values appearing in the last equation we express the kernel of the projection operator in terms of Grassmann variables only

\[
\langle z, w | \hat{P}_{n_B} | x, -y \rangle = \sum_{r=0}^{\infty} (-1)^{n_B} \frac{1}{(n_B + r)! r!} (z^+ x)^{n_B + r} (w^+ y)^r.
\] (31)

To evaluate the integral of Eq. (29) I rewrite the above equation in exponential form

\[
\langle z, w | \hat{P}_{n_B} | x, y \rangle = \sum_{r=0}^{\infty} \frac{1}{(n_B + r)! r!} \frac{\partial^{n_B + r}}{\partial j_1^{n_B + r} \partial j_2^{r}} \exp(-j_1 z^+ x - j_2 w^+ y) |_{j_1=j_2=0}.
\] (32)

The integrals of Eqs. (28), (29) are Gaussian and we get the constrained determinant in terms of the determinant of a modified quark matrix

\[
\det Q|_{n_B} = \sum_{r=0}^{\infty} \frac{1}{(n_B + r)! r!} \frac{\partial^{n_B + r}}{\partial j_1^{n_B + r} \partial j_2^{r}} \det(Q + \delta Q_1 + \delta Q_2) |_{j_1=j_2=0}.
\] (33)

The explicit form of the variations \( \delta Q_1, \delta Q_2 \) of the quark matrix is not important here, but we warn the reader that there is an error in their expression in ref. [4].

VI. BOSONIZATION IN MANY-BODY SYSTEMS AND RELATIVISTIC FIELD THEORIES

The low energy collective excitations of many-fermion systems can be described by effective bosons. Well known examples are the Cooper pairs of Superconductivity, the bosons of the Interacting Boson Model of Nuclear Physics, the chiral mesons and the quark pairs of color superconductivity in QCD. In all these cases the effective bosons are generated by attractive interactions, but effective bosons can arise also in the presence of repulsive forces, like in the Hubbard model [11]. Some of the effective bosons are Goldstone bosons, and then there is a general theory which tells that they live in the coset space of the group which is spontaneously broken and dictates how they are related to the original fields [12]. But there is no general procedure to reformulate the fermionic theory in terms of the effective bosonic degrees of freedom, even though there are several recipes for specific cases which
are reviewed in [13]. A more flexible approach is based on the Hubbard–Stratonovich transformation which linearizes the fermionic interaction by introducing bosonic auxiliary fields which are then promoted to physical life. The typical resulting structure is that of chiral theories [14]. But in such an approach an energy scale emerges naturally, and only excitations of lower energy can be described by the auxiliary fields. Moreover in renormalizable relativistic field theories like QCD, the fermion Lagrangian is quadratic to start with, so that the Hubbard–Stratonovich transformation cannot be used. One can add quartic interactions as irrelevant operators, and this can help in numerical simulations, but has not led so far to a formulation of low energy QCD in terms of chiral mesons.

I present a new approach to bosonization which does not suffer from the above limitations and can be applied to theories with quartic and quadratic interactions as well. It is based on the evaluation of the partition function restricted to the bosonic composites of interest. By rewriting the partition function so obtained in functional form we get the euclidean action of the composite bosons from which in the nonrelativistic theories we can derive the Hamiltonian. In this way I derived the Interacting Boson Model from a Nuclear Hamiltonian. In the case of pure pairing, I reproduce the well known results for the excitations corresponding to the addition and removal of pairs of fermions, as well as for the seniority excitations which are inaccessible by the Hubbard–Stratonovich method. Indeed at least in this example this theory does not have the structure of a chiral expansion.

For the relativistic case an investigation is in progress.

Let me start by defining the composites in terms of the fermion operators $\hat{c}$

$$\hat{b}_{J}^{\dagger} = \frac{1}{2} \hat{c}^{\dagger} B_{J}^{\dagger} \hat{c}^{\dagger} = \frac{1}{2} \sum_{m_{1},m_{2}} \hat{c}_{m_{1}}^{\dagger} \left( B_{J}^{\dagger} \right)_{m_{1},m_{2}} \hat{c}_{m_{2}}^{\dagger}. \quad (34)$$

In the above equation $m$ represents all the fermion intrinsic quantum numbers and position coordinates and $J$ the quantum numbers of the composites. I assume all the structure matrices $B_{J}$ to have one and the same dimension which I denote by $2\Omega$. The fermionic operators have canonical anticommutation relations, while for the composites

$$[\hat{b}_{J_{1}}, \hat{b}_{J_{2}}^{\dagger}] = \frac{1}{2} \text{tr}(B_{J_{1}} B_{J_{2}}^{\dagger}) - \hat{c}^{\dagger} B_{J_{2}}^{\dagger} B_{J_{1}} \hat{c}. \quad (35)$$

It is then natural to require the normalization

$$\text{tr}(B_{J_{1}}^{\dagger}, B_{J_{2}}) = 2\delta_{J_{1},J_{2}}. \quad (36)$$
A convenient way to get the euclidean path integral from the trace of the transfer matrix is to use coherent states of composites. Therefore I introduce the operator

\[ P = \int db^* db (\langle b|b \rangle)^{-1}|b\rangle\langle b| \]

where

\[ |b\rangle = |\exp(b \cdot \hat{b}^\dagger)\rangle. \]  

I adopted the convention

\[ b \cdot \hat{b}^\dagger = \sum_J b_J \hat{b}_J^\dagger. \]

If the \( \hat{b} \)'s where operators of elementary bosons \( P \) would be the identity in the boson Fock space. I would like \( P \) to be the identity in the fermion subspace of the composites. To see the action of \( P \) on composite operators let us first consider the case where there is only one composite with structure function satisfying the equation

\[ B^\dagger B = 1. \]

Then we find

\[ \langle b|b \rangle^{-1} = \left( 1 + \frac{1}{\Omega} b^*_i b \right)^{-\Omega} \]

and

\[ \langle b_1|\langle \hat{b}^\dagger \rangle^n \rangle = C_n (b^*)^n, \]

where

\[ C_n = \frac{\Omega!}{(\Omega - n)! \Omega^n} = \left( 1 - \frac{1}{\Omega} \right) \left( 1 - \frac{2}{\Omega} \right) \ldots \left( 1 - \frac{n - 1}{\Omega} \right). \]

Now we can determine the action of \( P \) on the composites

\[ P|\langle \hat{b}^\dagger \rangle^n \rangle = \left( 1 - \frac{n}{\Omega} \right)^{-1} \left( 1 - \frac{n + 1}{\Omega} \right)^{-1} |\langle \hat{b}^\dagger \rangle^n \rangle \]

which shows that \( P \) behaves approximately like the identity with an error of the order of \( n/\Omega \). It is perhaps worth while noticing that in the limit of infinite \( \Omega \) we recover exactly the expressions valid for elementary bosons, in particular

\[ \langle b_1|b \rangle = \left( 1 + \frac{1}{\Omega} b^*_i b \right)^\Omega \to \exp(b^*_i b), \ \Omega \to \infty. \]

It might appear that the treatment of states with \( n \sim \Omega \) is precluded, but this is not true. Indeed if we are interested in states with \( n = \pi + \nu \) for an arbitrary reference state \( \pi \), we redefine \( P \) according to

\[ P_\pi = \frac{(\Omega - \pi)^2}{\Omega^2} P_0. \]
We then have
\[ \mathcal{P}_\pi |(\hat{b}^\dagger)^n\rangle = \left(1 - \frac{\nu}{\Omega - \pi}\right)^{-1} \left(1 - \frac{\nu + 1}{\Omega - \pi}\right)^{-1} |(\hat{b}^\dagger)^n\rangle \] (47)
which shows that \( \mathcal{P}_\pi \) behaves like the identity in the neighborhood of the reference state up to an error of order \( \nu/(\Omega - \pi) \), namely the measure \( \langle b|b\rangle^{-1} \) is essentially uniform.

In the general case of many composites we have
\[ \langle b_1|b\rangle = [\det (\mathbb{I} + \beta_1\beta)]^{\frac{1}{2}} \] (48)
where the matrix \( \beta \) is
\[ \beta = b \cdot B^\dagger \] (49)
and
\[ \langle b_1|\cdots|b_1\rangle = \frac{\partial^{n_0}}{\partial x_{i_0}} \cdots \frac{\partial^{n_i}}{\partial x_{i_i}} \exp\left\{\frac{1}{2} \text{Tr} \ln[\mathbb{I} + (x \cdot B^\dagger)(b^*_1 \cdot B)]\right\}|_{x=0}. \] (50)

We must now make an assumption which replaces Eq. 40, namely that all the eigenvalues of the matrices \( B_j^\dagger B_j \) are much smaller than \( \Omega \). Then we find again that \( \mathcal{P} \) approximates the identity with an error of order \( 1/\Omega \).

Now we are equipped to carry out the program outlined at the beginning. The first step is the evaluation of the partition function \( Z_C \) restricted to fermionic composites. To this end we divide the inverse temperature in \( N_0 \) intervals of spacing \( \tau \)
\[ \frac{1}{T} = N_0 \tau \] (51)
and write
\[ Z_c = \text{tr} (\mathcal{P}\mathcal{T})^{N_0} \] (52)
where \( \mathcal{T} \) is the transfer matrix. In the nonrelativistic case \( \mathcal{T} \) is expressed in terms of the Hamiltonian
\[ \mathcal{T} = \exp \left( -\tau \hat{H} \right), \] (53)
and the Hamiltonian is the generator of continuous time translations. In relativistic field theories instead only the transfer matrix is known in general, and the above equation can be used to define a Hamiltonian, but only as the generator of discrete translations by the time spacing \( \tau \).

At this point we must evaluate the matrix element \( \langle b_1|\mathcal{T}|b\rangle \) and to do this we must distinguish between relativistic field theories and many-body systems. In the first case the
transfer matrix is a product of exponentials of quadratic forms in the fermion operators, and the matrix element can be directly and exactly evaluated. In the second case one must at an intermediate stage expand with respect to the time spacing $\tau$. This does not introduce any error because one can retain all the terms which give finite contributions in the limit $\tau \to 0$. We will report here only the nonrelativistic calculation. The most general Hamiltonian can be written

$$\hat{H} = \hat{c}^{\dagger} h_0 \hat{c} - \sum_K g_K \frac{1}{2} \hat{c}^{\dagger} F_K^{\dagger} \hat{c}^{\dagger} \frac{1}{2} \hat{c} F_K \hat{c},$$

(54)

where $K$ represents all the necessary quantum numbers. The single-particle term includes the single-particle energy with matrix $e$, any single-particle interaction with external fields described by the matrix $M$ and the chemical potential $\mu$

$$h_0 = e + M - \mu.$$  

(55)

Therefore we will be able to solve the problem of fermion-boson mapping by determining the interaction of the composite bosons with external fields. Assuming for the potential form factors the normalization

$$\text{tr}(F_K^{\dagger} F_K) = 2 \Omega$$

(56)

and setting

$$\Gamma_t = (1 + \beta_t^* \beta_{t-1})^{-1}$$

(57)

we get the euclidean action

$$S = \tau \sum_t \left\{ \frac{1}{2\tau} \text{tr}[\ln(1 + \beta_t^* \beta_t) - \ln \Gamma_t] - H_1 + \frac{1}{4} \sum_K g_K \text{tr} \left[ \text{tr}(\Gamma_t \beta_t^* F_K^{\dagger}) \text{tr}(\Gamma_t F_K \beta_{t-1}) \right] 
- 2 \text{tr} \left( \Gamma_t F_K^{\dagger} F_K \right) - \text{tr}[\Gamma_t \beta_t^* F_K^{\dagger} , \Gamma_t F_K \beta_{t-1}]_+ \right\} + \frac{1}{2} \text{tr} \left[ \beta_t^* (\beta_{t-1} h^T + h \beta_{t-1}) \right],$$

(58)

where $[\ldots,\ldots]_+$ is an anticommutator. This action differs from that of elementary bosons because

i) the time derivative terms (contained in the first line) are non canonical

ii) the coupling of the chemical potential (which appears in $h$) is also noncanonical, since it is not quadratic in the boson fields

iii) the function $\Gamma$ becomes singular when the number of bosons is of order $\Omega$, which reflects the Pauli principle.
We remind the reader that the only approximation done concerns the operator $P$. Therefore these are to be regarded as true features of compositeness.

The bosonization of the system we considered has thus been accomplished. In particular the fermionic interactions with external fields can be expressed in terms of the bosonic terms which involve the matrix $\mathcal{M}$ (appearing in $h$) and the dynamical problem of the interacting (composite) bosons can be solved within the path integral formalism. Part of this problem is the determination of the structure matrices $B_I$. This can be done by expressing the energies in terms of them and applying a variational principle which gives rise to an eigenvalue equation.

The Hamiltonian can be derived by standard procedures.

VII. CONCLUSION

I showed that there is a number of problems which can be easily dealt with in the operator form of the partition function. Only afterwards it can be given the functional form of the path integral which is more convenient for many purposes.

There are other examples I left over for different reasons. Among these I would like to mention the problem of the restriction of gauge theories to physical states, which has not yet found a general and satisfactory solution.
[9] P.Hasenfratz and F.Karsch, Phys. Lett. B 125, 308(1983) ; J.Kogut, M.Matsuoka, M.Stone, H.W.Wyld, J.H.Shenker, J.Shighemitsu and D.K.Sinclair, Nucl. Phys. B 225[FS9] 93(1983)

[10] M.Creutz, Foundations of Phys. 30, 487(2000)

[11] M.Cini and G.Stefanucci, cond-mat/0204311 v1

[12] S. Weinberg, The Quantum Theory of fields, Ed. by Cambridge University Press, 1996

[13] A.Klein and E.R.Marshalek, Rev. Mod. Phys. 63, 375(1991)

[14] V.A.Miransky, Dynamical Symmetry Breaking in Quantum Field Theories, Ed. by World Scientific, 1993

[15] S.Caracciolo and F.Palumbo, in progress