Hecke and Galois Properties of Special Cycles on Unitary Shimura Varieties

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Abstract
We define and study a collection of special cycles on certain non-PEL Shimura varieties for $U(2,1) \times U(1,1)$ that appear naturally in the context of the recent conjectures of Gan, Gross and Prasad on restrictions of automorphic forms for unitary groups and conjectural generalizations of the Gross–Zagier formula. We prove a combinatorial formula expressing the Galois action in terms of the distance function on the Bruhat–Tits buildings for these groups. In addition, we calculate explicitly the Hecke polynomial appearing in the congruence relation conjectured by Blasius and Rogawski. Using methods and recent results of Koskivirta, we prove the congruence relation in this case. Finally, using the action of the local Hecke algebra on the Bruhat–Tits building, we establish explicit relations (distribution relations) between the Hecke action and the Galois action on the special cycles. These relations are key for a work in progress by the author on constructing a new Euler system from these cycles and using it to prove instances of the Bloch–Kato–Beilinson conjecture for certain Galois representations associated to automorphic forms on unitary groups.

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1. Introduction

1.1 Motivation

In [Gro04], Gross outlines a program to link automorphic $L$-functions on one side with special cycles on Shimura varieties on the other side via the Gross–Prasad restriction problems for automorphic representations. A basic case is the case of classical Heegner points on modular curves and restrictions of automorphic representations on $GL_2$ to a non-split torus associated to an imaginary quadratic field. Subsequently, the work of Gan, Gross and Prasad provides Gross–Zagier type conjectures for classical groups [GGP09, §26–27] relating two major conjectures in number theory: the Birch and Swinnerton-Dyer conjecture (and its generalizations to higher dimensional rational cycles and $p$-adic Galois representations in the context of the Bloch–Kato–Beilinson conjectures) and the Langlands reciprocity conjectures.

It is thus of interest to study whether the Gross–Zagier type conjectures from [GGP09] imply new results towards the Bloch–Kato–Beilinson conjectures. Such a program will aim at generalizing Kolyvagin’s proof of the Birch and Swinnerton-Dyer conjecture for the case when the analytic rank of the elliptic curve is at most one [Kol90, Gro91] and provide a more conceptual representation-theoretic understanding of the latter. In order to achieve that, one needs an Euler system similar to the Euler system of Heegner points (special 0-cycles on modular curves) originally considered by Kolyvagin. Since the Heegner point analogue of [GGP09, Conj.27.1] is a higher-dimensional cycle on a Shimura variety, one could hope for Euler system constructed from similar special cycles, but defined over increasing abelian extensions of the reflex field. Using $p$-adic Abel–Jacobi maps, one can obtain cohomology classes in the appropriate Selmer groups of geometric $p$-adic Galois representations appearing in the cohomology of Shimura varieties associated to unitary groups and subsequently, apply Kolyvagin’s method to these classes.

1.2 Main results

This article carries out the first part of such a program for higher rank unitary groups. More precisely, we define a collection of special cycles and study their Hecke and Galois properties. Comparing the two actions (via distribution relations) is the main idea behind the construction of an Euler system from these cycles. We work with $n$-dimensional and $(n-1)$-dimensional unitary groups of isometries where precise Gross–Zagier conjectures relating the non-vanishing of the derivative of an automorphic $L$-function to the height of a cycle have already been stated [GGP09, Conj.27.1]. Lots of progress has been made towards this conjecture starting with the work of W. Zhang using an approach based on the relative trace formula and a conjectural arithmetic fundamental lemma [Zha12]. The latter has been proved in [Zha12] in the case $n = 3$. Partial results have been proved by W. Zhang, Rapoport and Terstiege for general $n$ [RTZ13]. Although the main results of this paper are for $n = 3$, we will do some of the computations for arbitrary $n$ as those will be used in...
1.2.1 Hermitian spaces, unitary groups and Shimura varieties. Let $F$ be a totally real number field with $[F : \mathbb{Q}] = d$ and let $E/F$ be a totally imaginary quadratic extension. Let $\rho_1, \ldots, \rho_d$ be the real places of $F$. Choose an embedding $\bar{\rho}_1 : E \to \mathbb{C}$ that extends the place $\rho_1 : F \to \mathbb{R}$. Moreover, fix embeddings $\iota_\tau : \overline{E}_\tau \to \overline{F}_\tau$ for every finite place $\tau$ of $E$.

Let $n \geq 3$ be an integer and let $(V, \langle ., . \rangle)$ be a non-degenerate Hermitian space of dimension $n$ over $E$. Suppose that $V$ has signature $(n - 1, 1)$ at $\rho_1$ and signatures $(n, 0)$ at each of the places $\rho_2, \ldots, \rho_d$. Suppose that $W \subset V$ is a Hermitian subspace of dimension $n - 1$ that has signature $(n - 2, 1)$ at $\rho_1$ and signatures $(n - 1, 0)$ at $\rho_2, \ldots, \rho_d$. Let $D \subset V$ be the $E$-line that is the orthogonal complement of $W$ with respect to the Hermitian form, i.e., for which

$$V = W \perp D. \quad (1)$$

Associated to $V$ and $W$ are the groups of unitary isometries $U(V)$ and $U(W)$, respectively, defined over $F$. We can then view $H = \text{Res}_{E/F} U(W)$ as an algebraic subgroup of $G = \text{Res}_{E/F} (U(V) \times U(W))$ via the diagonal embedding (i.e., the natural embedding $U(W) \hookrightarrow U(V)$ on the first factor) and the identity map on the second factor. Associated to the $F$-algebraic groups $H$ and $G$ are Shimura data $(H, Y)$ and $(G, X)$ introduced in Section 2.2. There are Shimura varieties $\text{Sh}_{K_H}(H, Y)$ and $\text{Sh}_K(G, X)$ associated to these data and a natural diagonal cycle $\text{Sh}_{K_H}(H, Y) \to \text{Sh}_K(G, X)$ for some compact open subgroups $K_H \subset H(\mathbb{A}_f)$ and $K \subset G(\mathbb{A}_f)$ where $\mathbb{A}_f$ denotes the finite ad"{e}les of $\mathbb{Q}$ (see Section 2.2 for the precise definitions of the open compact subgroups). Considering $G(\mathbb{A}_f)$-translates of a connected component of the small Shimura variety $\text{Sh}_{K_H}(H, Y)$ yields a collection of special cycles $Z_K(g) \subset \text{Sh}_K(G, X)$ for $g \in G(\mathbb{A}_f)$ defined (by Shimura reciprocity laws) over abelian extensions of $E$ (see Section 2.3). These should be thought of as higher-dimensional analogues of higher Heegner points (see [Gro84] and [Gro91]). Our main goal is to compare the Hecke and Galois properties of these cycle that will allow for the construction of an Euler system.

1.2.2 Galois properties of CM cycles. Our first contribution is a calculation of the field of definition of each cycle in terms of the distance function on the corresponding Bruhat–Tits buildings for $U(V)$ and $U(W)$. This is achieved by describing the set $Z_K(G, H)$ of special cycles as well as the set of Galois orbits adelically. The latter can be done using reciprocity laws for the Galois action on the connected components for Shimura varieties associated to the smaller group $H$. It turns out that the orbits of the cycles under the decomposition group at a finite place $\tau$ of $F$ are related to the local double quotients $H_{\tau} \backslash G_{\tau} / K_{\tau}$. Here, $G_{V,\tau} = U(V)(F_\tau)$, $G_{W,\tau} = U(W)(F_\tau)$, $G_\tau = G_{V,\tau} \times G_{W,\tau}$ and $H_\tau$ is the diagonal image of $G_{W,\tau}$ in $G_\tau$. In the case when $\tau$ is inert in $E$ and $K_\tau = K_{V,\tau} \times K_{W,\tau}$ where $K_{V,\tau} \subset G_{V,\tau}$ and $K_{W,\tau} \subset G_{W,\tau}$ are hyperspecial maximal compact subgroups, the double quotient $H_{\tau} \backslash G_{\tau} / K_{\tau}$ is in bijection with the set of $H_{\tau}$-orbits $[L_{V,\tau}, L_{W,\tau}]$ of pairs $(L_{V,\tau}, L_{W,\tau})$ of self-dual local Hermitian $O_{E_\tau}$-lattices $L_{V,\tau} \subset V_\tau$ and $L_{W,\tau} \subset W_\tau$ where $V_\tau = V \otimes_{E_\tau} E_\tau$ and $W_\tau = W \otimes_{E_\tau} E_\tau$. In order to compute the completion at $\tau$ of the field of definition $E(\xi)$ of the cycle $\xi = Z_K(g)$, it suffices to compute the stabilizer of the corresponding pair in $H_\tau$. Indeed, by reciprocity laws for the Galois action on the group of connected components for the Shimura varieties for $H$, this will give us the corresponding norm subgroup of $E_\tau^\times$ which, by local class field theory, will determine the corresponding abelian extension of $E_\tau$. Note that the norm subgroup is of the form $O_n^\times \subset E_\tau^\times$ where $O_n = O_{E_\tau} + \varpi^n O_{E_\tau}$, $O_{E_\tau}$ is a uniformizer (we call $c_\tau([L_{V,\tau}, L_{W,\tau}]) = \varpi^n$ the local conductor at $\tau$). The computation of the local conductor is then given by the following:

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1 Here, each unitary isometry for $W$ is extended to a unitary isometry for $V$ via the identity on $D$.
2 This case is the most important one from the point of view of applying Kolyvagin’s arguments.
Theorem 1.1 (Local Conductor Formula). Let \( n = 3 \) and suppose that \( K_\tau = K_{V, \tau} \times K_{W, \tau} \) where \( K_{V, \tau} \subset G_{V, \tau} \) and \( K_{W, \tau} \subset G_{W, \tau} \) are hyperspecial maximal compact subgroups. Given a \( H_\tau \)-orbit \([L_{V, \tau}, L_{W, \tau}]\) of pairs of lattices \((L_{V, \tau}, L_{W, \tau}) \in L_\tau\), the local conductor is

\[
e_{\tau}\left([L_{V, \tau}, L_{W, \tau}]\right) = \infty \min \left\{ \text{dist}(pr_{W_\tau}(L_{V, \tau})), 2 \text{dist}(pr_{W_\tau}(L_{V, \tau}), L_{W, \tau}) \right\}.
\]

Here, \( pr_{W_\tau}(L_{V, \tau}) \) denotes the convex projection of the hyperspecial point corresponding to \( L_{V, \tau} \) to the Bruhat–Tits building and \text{dist} indicates the distance function on the building \( B(V_\tau) \) of \( G_{V, \tau} \).

Remark 1. The interpretation of the Galois action on the special cycles in \( Z_K(G, H) \) in terms of Bruhat–Tits theory in the above theorem and the well-known fact that Hecke operators can be viewed as adjacency operations on the Bruhat–Tits building already indicates that one should expect a precise formula relating the Hecke and the Galois actions on the spaces of cycles (the main objective of the paper).

1.2.3 Blasius–Rogawski congruence relation. In a fundamental study of the zeta function of a Shimura variety [BR94], Blasius and Rogawski formulated a conjecture known as the congruence relation generalizing the classical Eichler–Shimura relation for modular curves and providing an explicit polynomial, the Hecke polynomial, annihilating the geometric Frobenius \( \Phi_\tau \) acting on the \( \ell \)-adic étale cohomology (we normalize so that geometric Frobenii correspond to uniformizers under the Artin map). More precisely, let \( \tau \) be a finite place of \( F \) that is unramified in \( E \). Since \( G_{\tau} \) is quasi-split over \( E_{\tau} \) but is split over the unramified extension \( E_{\tau} \), \( G_{\tau} \) is unramified in the sense of [BR94, §1.11]. The polynomial \( H_{\tau}(z) \) is defined purely representation theoretically out of the Shimura datum (see Section 4.2 and also [BR94] for the definition) and has coefficients that are elements of the Hecke algebra \( \mathcal{H}(G, K) \) (that is, the algebra (under convolution) of \( K \)-bi-invariant locally constant functions on \( G(A_f) \)). Let \( \ell \) be a prime such that \( \tau \mid \ell \). Let \( \overline{SH_K(G, X)} \) denote the Baily–Borel compactification. Blasius and Rogawski [BR94, p.33] stated the following conjecture generalizing the classical Eichler–Shimura relation:

Theorem 1.2 (Congruence Relation). Let \( \pi_f \) be a cohomological automorphic representation of \( G(A_f) \) occurring in the \( \ell \)-adic étale cohomology \( H^*(\overline{SH_K(G, X)}, \mathcal{Q}_\ell) \) and \( \tau \mid \ell \). Assume that the local representation \( \pi_{\tau} \) is unramified (i.e., \( G_{V, \tau} \) is split over an unramified extension of \( E_{\tau} \)) and let \( H_{\tau}(z, \pi_{\tau}) \) be the specialization of the Hecke polynomial \( H_{\tau}(z) \) at \( \pi_{\tau} \). Then \( H_{\tau}(\Phi_{\tau}, \pi_{\tau}) \) vanishes on \( H^*(\overline{SH_K(G, X)}, \mathcal{Q}_\ell)[\pi_f] \).

Remark 2. Note that the geometric \( \Phi_{\tau} \) acts on the \( \pi_f \)-equivariant part \( H^*(\overline{SH_K(G, X)}, \mathcal{Q}_\ell)[\pi_f] \) as \( \pi_{\tau} \) is unramified.

Remark 3. Koskivirta [Kos13a] verifies the conjecture in the case of PEL-type unitary Shimura varieties that are closely related to ours and in this case, he works on a certain moduli space for \( p \)-isogenies. \textit{A priori}, it is not automatic how one can pass to \( \ell \)-adic étale cohomology (although the latter is known to experts). A desirable version of the congruence relation from the point of view of Euler systems will be a statement on the level of Chow groups.

\[\text{It is a tree in this case and we see from Fig. 1 that the projection of any hyperspecial (black) point is hyperspecial (black) point as well.}\]

\[\text{In this case, } B(V_\tau) \text{ is the tree on Fig. 1 and the distance is the usual distance in the sense of a tree where we normalize so that the distance between a black (hyperspecial) and a white (special, but not hyperspecial) vertices is 1/2.}\]

\[\text{For } GL_2, \text{ if } \tau = p \neq \ell, \text{ the Hecke polynomial is simply the polynomial } H_p(z) = X^2 - T_pX + p \text{ and the classical Eichler–Shimura relation } T_p = Fr_p + Ver_p \text{ is equivalent to the statement that } H_p \text{ vanishes on } Fr_p \text{ acting on the } \ell \text{-adic Tate module } T_{\ell}\mathcal{E} \text{ of an elliptic curve } \mathcal{E}.\]
1.2.4 **Distribution relations.** Kolyvagin’s method of Euler systems developed in [Ko90] and extended in [Rub00] is a way of modeling local $L$-factors algebraically via geometric data (special points or special cycles). The core idea of the method is the existence of a relation (distribution relation) between the action of the Galois group and the action of the Hecke algebra. For the case of an elliptic curve (or more generally, for automorphic forms on $\text{GL}_2$), these relations are very simple to state. It is well-known that the classical Heegner points $\{x_c\}$ on the modular curve $X_0(N)$ for the imaginary quadratic field $E$ satisfy the property that if $p \nmid c$ is inert in $E$ then

$$T_p(x_c) = \text{Tr}_{E[cp]/E[c]}(x_{cp}),$$

the latter considered as an equality in $\mathbb{Z}[\text{CM}]$ where CM indicates the set of points of $X_0(N)$ having CM by $E$. This equality is proved in [Gro91, Prop.3.7(1)] and is known as a distribution relation for Heegner points. Together with the Eichler–Shimura relation [Gro91 Prop.3.7(ii)], one gets an Euler system and derived cohomology classes that, via general global duality arguments, yield upper bounds on Selmer groups. Although rather simple, (3) is not very convenient when generalizing to Shimura varieties for higher-rank groups. An alternative way of restating the above equation is as follows:

$$(\text{Fr}_p^2 - T_p \text{Fr}_p + p) \text{Fr}_p(x_c) = \text{Tr}_{E[cp]/E[c]}(\text{Fr}_p^{-1}(x_c) - (x_{cp})).$$

This is more convenient as the left-hand side is simply the operator that is the value of the Hecke polynomial $H_p(z) = z^2 - T_p z + p$ on $\text{Fr}_p$. In fact, the analogue of the pair $(G, H)$ of algebraic groups in this case is $(\text{GL}_2, E^\times)$ (see [Gro04] for the precise analogy from the point of view of the Gross–Prasad restriction problems).

In the case of unitary groups, both the Hecke algebra $\mathcal{H}(G, K)$ and the Galois group $\text{Gal}(E^{ab}/E)$ act on $\mathbb{Z}[\mathbb{Z}_K(G, H)]$ as explained in Section 6. Given an element $\xi \in \mathbb{Z}[\mathbb{Z}_K(G, H)]$, let $E(\xi)$ be the smallest abelian extension of $E$ such that all special cycles in $\text{Supp}(\xi)$ are defined over $E(\xi)$. Moreover, let $c_\tau(\xi)$ denote the local conductor of $E(\xi)$. Using Theorem 1.2 expressing the Galois action in terms of the distance function on the building, we prove the following relation between the two actions:

**Theorem 1.3.** (Horizontal Distribution Relations) Let $\xi \in \mathbb{Z}[\mathbb{Z}_K(G, H)]$ and let $\tau$ be a place of $F$ that is inert in $E$. Assume that $E(\xi)_{\tau} = E_{\tau}$ and assume that $K = K_{\tau} \times K(\tau)$ for $K_{\tau} = K_{V_{\tau}} \times K_{W_{\tau}} \subset G_{\tau}$ where both $K_{V_{\tau}} \subset G_{V_{\tau}}$ and $K_{W_{\tau}} \subset G_{W_{\tau}}$ are hyperspecial maximal compact subgroups. There exists an element $\xi(\tau) \in \mathbb{Z}[\mathbb{Z}_K(G, H)]$ such that $\tau$ is totally ramified in $E(\xi(\tau))$, the the local conductor $c_\tau(\xi(\tau)) = \omega^2$ and

$$H_\tau(\text{Fr}_\tau)\xi = \text{Tr}_{E(\xi(\tau))/E(\xi),c}(\xi(\tau)).$$

Here, $E(\xi(\tau))_{\tau}$ denotes the completion of $E(\xi(\tau))$ at the unique place of $E(\xi)$ above $\tau$ (note that $E(\xi(\tau))_{\tau}$ is a totally ramified extension of $E(\xi)_{\tau}$ of degree $q+1$ where $q$ is the residue characteristic of $F$ at the place $\tau$).

**Remark 4.** The left-hand side of (5) is the same as $H_\tau(1)\xi$ since $\text{Fr}_\tau$ acts trivially on the cycle $\xi$ as the finite place $\tau$ splits completely in $E(\xi)$. Yet, we write $H_\tau(\text{Fr}_\tau)$ to illustrate the analogy with (4) except that the above theorem increases the local conductor by 2 as opposed to 1 in the $\text{GL}_2$-case. This is not a problem for the arithmetic applications as one can always define norm-compatible cycle over an extension of local conductor one by taking traces of $\xi(\tau)$.

**Remark 5.** The essence of Theorem 1.3 is that it provides a link between Hecke eigenvalues (or the arithmetic information encoded in automorphic $L$-functions) to special cycles and hence (via the $p$-adic Abel–Jacobi maps) to Selmer groups of Galois representations associated to automorphic forms on unitary groups. In the language of Kolyvagin, we say that the elements $\mathbb{Z}[\mathbb{Z}_K(G, H)]$ form an Euler system [Ko90] (see also [Rub00]). Knowing the non-triviality of certain cohomology classes
derived from the Euler system yields upper bounds on the Selmer groups in accordance to the rank part of the Bloch–Kato–Beilinson conjecture. Studying the properties of this Euler system as well as the derived Kolyvagin system is the subject of a work in progress. From that point of view, the above theorem is analogous to the connection between the Galois and Hecke actions on Heegner points (see, e.g., [Gro91, Prop.3.7]).

Remark 6. The above relations are known as horizontal distribution relations (needed for developing the Iwasawa theory as well as for the congruence relation for Euler systems). For the application to Iwasawa theory, one needs vertical distribution relations where the conductors should vary $p$-adically. It is possible to derive vertical distribution relation using similar methods which is the subject of a work in progress joint with Boumasmoud and Brooks. We expect that our Euler system will be useful for establishing new results towards the main conjecture of Iwasawa theory for the relevant Galois representations (an ongoing joint project with Brooks).

Remark 7. The setting $U(1, 1) \hookrightarrow U(2,1) \times U(1,1)$ is relevant since there is a conjectural Gross–Zagier type formula originally stated in [GGP09, Conj.27.1]. As already mentioned, this conjecture is a work in progress initiated by W. Zhang via an approach based on an arithmetic fundamental lemma proved in [Zha12]. Using this conjecture together with the conjectural injectivity of the $p$-adic Abel–Jacobi map, one can prove the rank part of the Bloch–Kato conjecture for a large class of Galois representations that appear in the cohomology of the above Shimura varieties for which the associated automorphic $L$-function vanishes up to order one. In fact, removing the last condition, one can still establish some structure results for the associated Selmer groups conjectural on the non-vanishing of at least one derived class from the Euler system similarly to [Kol91] (see also [JLS09]).

Remark 8. Note that the construction of Euler systems for higher rank groups has been initiated by Cornel [Cor09, Cor10] for the case of $U(n) \subset SO(2n + 1)$. Our setting matches the setting of Gan, Gross and Prasad [GGP09, §27] where there is already an explicit Gross-Zagier type conjecture stated.

1.3 Outline of the article

We introduce the setting for unitary groups, Hermitian lattices, the relevant Shimura data, Shimura varieties and special cycles in detail in Section 2. We prove Theorem 1.1 in Section 3 by studying the action of the small group $H_{\tau}$ on the product $B(V_{\tau}) \times B(W_{\tau})$ of the buildings for the groups $G_{V,\tau}$ and $G_{W,\tau}$. In Section 4 we compute the Hecke polynomial for our unitary groups using a method of Cornel and Koskivirta [Kos13a] reducing the computation to local combinatorics on the Bruhat–Tits buildings via canonical retraction maps on buildings. In Section 5 we discuss a recent proof of the Blasius–Rogawski congruence relation due to J.-S. Koskivirta and adapt it to our case, thus proving Theorem 1.2. Finally, we prove Theorem 1.3 in Section 6 via a local combinatorial argument using Theorem 1.1.

2. Shimura Varieties and Special Cycles

2.1 Unitary Groups

2.1.1 Unitary groups of isometries and similitudes. Using the setting and notation from the introduction, let $G_{V} = \text{Res}_{F/Q} U(V)$ and $G_{W} = \text{Res}_{F/Q} U(W)$ be the groups of unitary isometries of $V$ and $W$, respectively. Then $G = G_{V} \times G_{W}$ and $H = G_{W}$, the latter viewed as an algebraic subgroup of $G$ via the diagonal embedding. We also consider the groups of unitary isometries $GU(V)$ and $GU(W)$. If $R$ is an $F$-algebra then

$$GU(V)(R) = \{ g \in GL(V)(R) : \exists \nu(g) \in R^\times, \forall x, y \in V \otimes R, \langle gx, gy \rangle = \nu(g) \langle x, y \rangle \}.$$
and similarly for $W$. Clearly, $\nu$ (meaning the similitude factor) is a homomorphism and $U(V)(R) = \ker(\nu)$. Throughout, we will denote by $\tilde{G}_V = \text{Res}_{F/Q}GU(V)$ and $\tilde{G}_W = \text{Res}_{F/Q}GU(W)$. Let $\tilde{G} = \tilde{G}_V \times \tilde{G}_W$.

2.1.2 Hermitian lattices and self-dual lattices. By a Hermitian lattice $L \subset V$, we mean a $O_E$-submodule of $V$ of full rank. Similarly, for any finite place $\tau$ of $E$, a local $O_{E_\tau}$-Hermitian lattice will be an $O_{E_\tau}$-submodule of $V_\tau = V \otimes E_\tau$ of full rank. If $L \subset V$ is a global Hermitian lattice, we obtain local Hermitian lattices $L_\tau = L \otimes_{E_\tau} O_{E_\tau} \subset V_\tau$ for any finite place $\tau$ of $E$ and an adelic lattice $\tilde{L} = L \otimes_{E} O_{E} \subset \tilde{V}$ where $\tilde{O}_{E} = O_{E} \otimes \mathbb{Z}$ and $\tilde{V} = V \otimes \tilde{Q}$. Given an $O_{E_\tau}$-lattice $L_\tau \subset V_\tau$, we define its dual lattice $L_\tau^\vee \subset V_\tau$ by

$$L_\tau^\vee = \{ v \in V_\tau : \langle v, L_\tau \rangle \subset O_{E_\tau} \}.$$ 

A lattice $L_\tau$ is self-dual if $L_\tau^\vee = L_\tau$. For each finite place $\tau$ of $E$, the group $G_{V,\tau}$ acts transitively on the set $L(V_\tau)$ of self-dual local Hermitian $O_{E_\tau}$-lattices in $V_\tau$. Similarly, the group $G_{W,\tau}$ acts transitively on the set $L(W_\tau)$ of $O_{E_\tau}$-lattices in $W_\tau$. As we will explain later, $L(V_\tau)$ and $L(W_\tau)$ correspond to the sets of hyperspecial vertices for the Bruhat–Tits buildings for the local unitary groups $G_{V,\tau}$ and $G_{W,\tau}$, respectively.

2.1.3 Self-dual lattices adapted to a decomposition. Consider a decomposition

$$V_\tau = V_\tau' \oplus V_\tau'',$$

of a local Hermitian $E_\tau$-space $V_\tau$, where $V_\tau'$ and $V_\tau''$ are $E_\tau$-vector subspaces equipped with the restriction of the Hermitian form. If $L_\tau \subset V_\tau$ is a self-dual local Hermitian $O_{E_\tau}$-lattice then we say that the decomposition (3) is adapted to $L_\tau$ (or that $L_\tau$ is adapted to (3)) if $L_\tau \cap V_\tau'$ is a self-dual $O_{E_\tau}$-lattice of $V_\tau'$. Note that $L_\tau \cap V_\tau'$ is a self-dual $O_{E_\tau}$-lattice of $V_\tau' \iff L_\tau \cap V_\tau''$ is a self-dual $O_{E_\tau}$-lattice of $V_\tau''$.

2.1.4 Integral structures. Let $\tau$ be a place of $E$ such that the groups $U(V)$ and $U(W)$ are quasi-split at $\tau$. Fix a Witt basis corresponding to the decomposition $V_\tau = W_\tau \perp D_\tau$. If $n = 2m + 1$, this is a decomposition

$$W_\tau = (E_\tau e_1 \oplus E_\tau e_{-1}) \perp \cdots \perp (E_\tau e_m \oplus E_\tau e_{-m}), \quad D = E_\tau e_D,$$

where $H_i = E_\tau e_i \oplus E_\tau e_{-i}$ is a hyperbolic plane with $E_\tau e_i$ and $E_\tau e_{-i}$ being isotropic lines satisfying $\langle e_i, e_{-i} \rangle = 1$ for $i = 1, \ldots, m$ and $e_D$ satisfying $\langle e_D, e_D \rangle = 1$. If $n = 2m$, this is a decomposition

$$W_\tau = E_\tau e_0 \perp (E_\tau e_1 \oplus E_\tau e_{-1}) \perp \cdots \perp (E_\tau e_{m-1} \oplus E_\tau e_{-m+1}), \quad D = E_\tau e_D,$$

where $H_i$ for $i = 1, \ldots, m-1$ and $e_D$ are as above and $e_0$ satisfies $\langle e_0, e_0 \rangle = 1$. In the first case, we define a self-dual lattice $L_{0,\tau} \subset V_\tau$ by

$$L_{0,V_\tau} = O_{E_\tau} e_1 \oplus O_{E_\tau} e_{-1} \oplus \cdots \oplus O_{E_\tau} e_m \oplus O_{E_\tau} e_{-m} \oplus O_{E_\tau} e_D,$$

and in the second case,

$$L_{0,W_\tau} = O_{E_\tau} e_0 \oplus O_{E_\tau} e_1 \oplus O_{E_\tau} e_{-1} \oplus \cdots \oplus O_{E_\tau} e_m \oplus O_{E_\tau} e_{-m} \oplus O_{E_\tau} e_D.$$ 

Let $L_{0,V_\tau} \cap W_\tau$, let $K_{V_\tau} = \text{Stab}_{U(V)(F_\tau)}(L_{0,V_\tau})$ and let $K_{W_\tau} = \text{Stab}_{U(W)(F_\tau)}(L_{0,W_\tau})$. The compact open subgroups $K_{V_\tau}$ and $K_{W_\tau}$ are hyperspecial maximal compact subgroups of $U(V)(F_\tau)$ and $U(W)(F_\tau)$, respectively. Depending on the application (for a fixed finite place $\tau$ of $E$), we will be considering compact open subgroups of the form $K = K_\tau K^{(\tau)}$ where $K_\tau = K_{V_\tau} \times K_{W_\tau} \subset$
\( U(V)(F_\tau) \times U(W)(F_\tau) \) and \( K^{(\tau)} = K^{(\tau)}_V \times K^{(\tau)}_W \subset U(V)(A^{(\tau)}_{F,f}) \times U(W)(A^{(\tau)}_{F,f}) \) being a product of open compact subgroups (note that \( K \) can be viewed as a subgroup of \( G(A_f) \)).

Similarly, for a distinguished finite place \( \tau \) and \( K = K_\tau K^{(\tau)} \) as above, we will be considering compact open subgroups \( \widetilde{K}_V \subset \tilde{G}_V(A_f) \) and \( \widetilde{K}_W \subset \tilde{G}_W(A_f) \) for \( \widetilde{K}_V = \widetilde{K}_V,\tau \times \widetilde{K}_V^{(\tau)} \) and \( \widetilde{K}_W = \widetilde{K}_W,\tau \times \widetilde{K}_W^{(\tau)} \) where \( \widetilde{K}_V,\tau \) and \( \widetilde{K}_W,\tau \) are the hyperspecial maximal subgroups of \( \tilde{G}_V(F_\tau) \) and \( \tilde{G}_W(F_\tau) \), respectively corresponding to the self-dual lattices \( L_{0,V} \) and \( L_{0,W,\tau} = L_{0,V,\tau} \cap W \) and \( \tilde{K}_V^{(\tau)} \) (resp., \( \tilde{K}_W^{(\tau)} \)) is such that \( K^{(\tau)}_V = \tilde{K}_V^{(\tau)} \cap U(V)(A^{(\tau)}_{F,f}) \) (resp., \( K^{(\tau)}_W = \tilde{K}_W^{(\tau)} \cap U(W)(A^{(\tau)}_{F,f}) \)).

### 2.2 Shimura Varieties

Here, we describe the Shimura varieties associated to the unitary groups \( H \) and \( G \).

#### 2.2.5 The groups \( U(V)_E \) and \( GU(V)_E \)

Let \( V = Ev_1 \oplus \cdots \oplus Ev_n \) for some \( E \)-basis \( \{v_1, \ldots, v_n\} \) and let \( J' \) be the matrix \( (v_i, v_j)_{i,j=1}^n \) of the Hermitian form. The unitary group \( U(V) \) over \( E \) can then be described as follows: for any \( F \)-algebra \( R \),

\[
U(V)(R) = \{ M \in GL_n(E \otimes_F R) : TMJ'M = J' \}.
\]

Let \( S \) be an \( E \)-algebra. We have an isomorphism of \( E \)-algebras

\[
E \otimes_F S \cong S \times S, \quad \alpha \otimes x \mapsto (\overline{\alpha}x, \alpha x).
\]

The Galois group \( \text{Gal}(E/F) \) acts on \( E \otimes_F S \) via \( \alpha \otimes x \mapsto \alpha \otimes \overline{x} \) for \( \alpha \in E \) and \( x \in S \). This action corresponds to the action \( (x, y) \mapsto (\overline{y}, \overline{\alpha}) \) on the right-hand side of the above isomorphism. The \( E \)-algebra isomorphism \( \otimes \) yields a group isomorphism

\[
GL_n(E \otimes_F S) \cong GL_n(S \times S) \cong GL_n(S) \times GL_n(S),
\]

and hence,

\[
U(V)_E(S) = \{ (M_1, M_2) \in GL_n(S) \times GL_n(S) : M_1^T J'M_2 = J' \}.
\]

This means that the map \( (M_1, M_2) \mapsto M_1 \) gives an isomorphism of algebraic groups \( U(V)_E \cong_E GL_{n,E} \). Similarly, the group \( GU(V)_E \) is described by

\[
GU(V)_E(S) = \{ (M_1, M_2, \lambda) \in GL_n(S) \times GL_n(S) \times G_m(S) : M_1^T J'M_2 = \lambda J' \},
\]

and \( (M_1, M_2, \lambda) \mapsto (M_1, \lambda) \) yields an isomorphism \( GU(V)_E \cong_E GL_{n,E} \times G_{m,E} \).

#### 2.2.6 Center, derived subgroup and adjoint quotient

The above description of the unitary group \( U(V)_E \) allows us to describe the center \( Z(U(V)) \), the derived subgroup \( U(V)^{\text{der}} \) and the adjoint quotient \( U(V)^{\text{ad}} \). The derived subgroup \( U(V)^{\text{der}} \) is isomorphic to \( SL_n \) over \( F \), i.e., \( U(V)^{\text{der}} \cong \mathcal{T} \cdot SL_n \). The latter is simply connected, so \( U(V)^{\text{der}} \) is simply connected over the fixed algebraic closure \( \overline{F} \).

Similarly, since \( Z(U(V)) \cong \mathcal{T} \cdot G_m \), i.e., \( U(V)^{\text{ad}} \cong \mathcal{T} \cdot PGL_n \).

#### 2.2.7 Maximal torus, characters and co-characters

The algebraic subgroup of diagonal matrices of \( GL_{n,E} \) gives rise to a maximal torus \( T_{V,E} \) of \( GL_{n,E} \). For any \( F \)-algebra \( R \), define a maximal torus \( T_V \subset U(V) \) by \( T_V(R) = T_{V,E}(E \otimes_F R) \cap U(V)(R) \). Denote by \( \chi_1, \ldots, \chi_n : T_{V,E} \to G_{m,E} \) be the standard characters for \( GL_{n,E} \). The corresponding co-characters (i.e., \( \mu_i : GL_{n,E} \to G_{m,E} \) is given by \( \mu_i(x) = \text{diag}(1, \ldots, 1, x, 1, \ldots, 1) \)).

#### 2.2.8 Galois action

Since \( U(V) \) is split over \( E \), the action of the Galois group \( \text{Gal}(\overline{F}/F) \) on both \( X_*(T_V) \) and \( X^*(T_V) \) factors through the action of \( \text{Gal}(E/F) \). Using \( [10] \) to describe explicitly the unitary group together with the fact that \( \text{Gal}(E/F) \) acts on \( E \otimes_F S \) by \( (x, y) \mapsto (\overline{y}, \overline{x}) \) for the \( E \)-algebra isomorphism \( \otimes \), the the action of \( \text{Gal}(E/F) \) on \( U(V)_E \cong_E GL_{n,E} \) is given by
$A \mapsto J'(A^{-1})J'$. In particular, if $\sigma \in \text{Gal}(E/F)$ is the non-trivial element then $\sigma \mu_i = -\mu_i$ for $i = 1, \ldots, n$.

2.2.9 Shimura datum $(G, X)$. Let $S = \text{Res}_{C/R} G_{m,C}$ be the Deligne circle group and let $X_V$ be the $G_V(R)$-conjugacy class of homomorphisms of real algebraic groups

$$h: S_{R} \to G_{V,R}, \quad z \mapsto \text{diag}(1, \ldots, 1, z/z) \times 1_2 \cdots \times 1_d, \quad (12)$$

where we have implicitly used the identification $G_V(R) \cong U(V)(F_{\rho_1}) \times U(V)(F_{\rho_2}) \times \cdots \times U(V)(F_{\rho_d})$ and $1_2 \in U(V)(F_{\rho_1})$ is the identity matrix and all matrices are considered with respect to a basis that diagonalizes the Hermitian form (such a basis always exists as the unitary groups are considered over $R$). Similarly, one defines $X_W$ for the subspace $W \subset V$ whose signature at the place $\tau_1$ of $F$ is $(n - 2, 1)$. Next, consider the identification $S_C \cong G_{m,C} \times G_{m,C}$. Under this identification, $S(R)$ embeds into $S(C)$ via

$$S(R) \cong C^\times \subset C^\times \times C^\times, \quad z \mapsto (z, \overline{z}). \quad (13)$$

The homomorphism $h_C: S_C \to G_{V,C}$, $(z_1, z_2) \mapsto \text{diag}(1, \ldots, 1, z_2/z_1) \times 1_2 \cdots \times 1_d$.

2.2.10 Co-character and reflex field. To calculate the associated co-character $\mu_{h,V}$, consider the homomorphism $G_{m,C} \to S_C \cong G_{m,C} \times G_{m,C}$ given by $z \mapsto (z, 1)$. Then

$$\mu_{h,V}: G_{m}(C) \to G_{V}(C), \quad \lambda \mapsto \text{diag}(1, \ldots, 1, \lambda^{-1}) \times 1_2 \cdots \times 1_d,$$

i.e., $\mu_{h,V} = -\mu_n$ which, according to $2.2.8$, is defined over $E$ with respect to the embedding $\iota: E \to C$, i.e., the reflex field of the Shimura datum $(G_{V}, X_V)$ is $E(G_V, X_V) = E$. It is now clear that the unique irreducible representation of $\tilde{G}_V = \text{GL}_n(C)$ with anti-dominant weight $\mu_{h,V}$ (the corresponding character to $\mu_{h,V}$) is precisely the representation $r_n: \tilde{G}_V \to G_{V}(C) \times G_{V}(C)$ given by $A \mapsto t_A^{-1}$. We define $\mu_{h,W}, \mu_{h,W}$ and $r_{n-1}$ in a similar way. Alternatively [Gro09], it follows from Witt’s theorem that $X_V$ is the space of negative lines in $V \otimes E$ (i.e., $X_V$ is a complex ball of dimension $n - 1$) and $X_W$ is the space of negative lines in $W \otimes C$ (a complex ball of dimension $n - 2$).

Finally, let $\Delta: W \to V \times W$ be the diagonal embedding (that is, the natural inclusion on the first factor and the identity on the second factor). There is an induced diagonal embedding $\Delta: X_W \to X_V \times X_W = X$. Let $H = \Delta(G_V)$. Consider the symmetric space $X = X_V \times X_W$ for the product group $G = G_V \times G_W$ and let $Y = \Delta(X_W) \subset X$. Finally, let $K_H = K \cap H(A_f)$. Note that the representation of $\tilde{G} = \tilde{G}_V \times \tilde{G}_W$ corresponding to $X$ is the product $r_n \boxtimes r_{n-1}$ of the standard representations for $\text{GL}_n(C)$ and $\text{GL}_{n-1}(C)$ (the dimension of this representation is $n(n - 1)$).

2.2.11 Shimura data $(\tilde{G}, X')$ and $(\tilde{G}, \tilde{X})$. Besides the Shimura datum $(G, X)$, we consider two other data for the groups of unitary similitudes $\tilde{G}_V$ (resp., $\tilde{G}_W$). Let $X'_V$ be the $\tilde{G}_V(R)$-conjugacy class of

$$h: S_{R} \to \tilde{G}_{V,R}, \quad z \mapsto \text{diag}(1, \ldots, 1, z/z) \times 1_2 \cdots \times 1_d,$$

where we have identified $\tilde{G}_V(R) \cong GU(F_{\rho_1}) \times \cdots \times GU(F_{\rho_d})$.

Remark 9. The domains $X'_V$ and $X_V$ for the groups $\tilde{G}_{V,R}$ and $G_{V,R}$ are closely related. Yet, as we will see below, $X'_V$ might be a disjoint union of two conjugates of $X_V$ (e.g., in the case when the dimension of the space $V$ is even; for instance, for $GU(1, 1)$).

The other hermitian symmetric domain $\tilde{X}_V$ is defined as the $\tilde{G}_V(R)$-conjugacy class of

$$\tilde{h}: S_{R} \to \tilde{G}_{V,R}, \quad z \mapsto \text{diag}(z, \ldots, z, z, \overline{z}) \times 1_2 \cdots \times 1_d.$$
Similarly, we define Shimura data \((\tilde{G}_W, X'_W), (\tilde{G}_W, \tilde{X}_W), (\tilde{G}, X')\) and \((\tilde{G}, \tilde{X})\). Unlike \((G_V, X_V)\) and \((G_V, X'_V)\), the Shimura datum \((\tilde{G}_V, \tilde{X})\) gives rise to a PEL-type Shimura variety (if \(n = 3\) this is precisely the Picard modular surface). Yet, the Galois action on the connected components on that variety is more complicated.

2.2.12 Shimura varieties. Consider Shimura varieties \(Sh_K(V, X_V)\) and \(Sh_W(G_W, X_W)\) whose complex points are given by

\[
Sh_K(V, X_V)(C) := G(V)(\mathbb{Q})/(G(V)(A_f) \times X_V)/K_V,
\]

\[
Sh_W(G_W, X_W)(C) = G(W)(\mathbb{Q})/(G(W)(A_f) \times X_W)/K_W.
\]

Let \(Sh_K(G, X) = Sh_K(V, X_V) \times Sh_K(W, X_W)\). The complex points of \(Sh_K(G, X)\) are given by

\[
Sh_K(G, X)(C) := G(\mathbb{Q})/(G(A_f) \times X)/K.
\]

We also consider the Shimura varieties \(Sh_{\tilde{K}_V}(\tilde{G}_V, X'_V)\) (resp., \(Sh_{\tilde{K}_W}(\tilde{G}_W, X'_W)\)), \(Sh_{K_V}(\tilde{G}_V, \tilde{X}_V)\) (resp., \(Sh_{\tilde{K}_W}(\tilde{G}_W, \tilde{X}_W)\)) as well as \(Sh_{\tilde{K}}(G, X')\) and \(Sh_{\tilde{K}}(\tilde{G}, \tilde{X})\). Here, \(\tilde{K}_V\) (resp., \(\tilde{K}_W\)) are as in 2.1.4.

2.2.13 Connected components. By [Mil05, Lem.5.13], the connected components of \(Sh_K(G, X)\) (resp., \(Sh_{\tilde{K}}(G, X')\)) are indexed by the double cosets \(G(\mathbb{Q}) \backslash G(A_f)/K\) (resp., \(G(\mathbb{Q}) \backslash \tilde{G}(A_f)/\tilde{K}\)). More precisely, if \(g_1, \ldots, g_r\) (resp., \(\tilde{g}_1, \ldots, \tilde{g}_s\)) are double coset representatives for \(G(\mathbb{Q}) \backslash G(A_f)/K\) (resp., \(G(\mathbb{Q}) \backslash \tilde{G}(A_f)/\tilde{K}\)) then

\[
Sh_K(G, X) = \bigcup_{i=1}^r \Gamma_i \backslash X, \quad \Gamma_i = G(\mathbb{Q}) \cap g_iKg_i^{-1}.
\]

and

\[
Sh_{\tilde{K}}(\tilde{G}, X') = \bigcup_{i=1}^s \tilde{\Gamma}_i \backslash X', \quad \tilde{\Gamma}_i = \tilde{G}(\mathbb{Q}) \cap g_i\tilde{K}_i\tilde{g}_i^{-1}.
\]

2.2.14 Relation between \((G, X)\) and \((\tilde{G}, X')\). Note that the inclusion \(G_V(A_f) \hookrightarrow \tilde{G}_V(A_f)\) induces a natural map \(e_V : Sh_K(V, X_V) \to Sh_{\tilde{K}_V}(\tilde{G}_V, X'_V)\). Similarly, we have a natural map \(e_W : Sh_{K_W}(G_W, X_W) \to Sh_{\tilde{K}_W}(\tilde{G}_W, X'_W)\). We now show that the maps \(e_V\) and \(e_W\) are closed embeddings.

**Lemma 2.1.** The map \(e_V\) is injective and identifies \(Sh_{K_V}(G_V, X_V)\) with an open and closed subset of \(Sh_{\tilde{K}_V}(\tilde{G}_V, X'_V)\).

**Proof.** The condition that \(\dim V = n\) is odd implies that the similitude \(\nu : G_V(\mathbb{R}) \to \mathbb{R}_0^+\) takes values in \(\mathbb{R}^+_\times\). Indeed, if \(g_{R} \in G(V)(\mathbb{R})\) then taking determinants yields \(\nu(g_{R})^n > 0\) and hence, \(\nu(g_{R}) > 0\). The inclusion \(G_V(A_f) \hookrightarrow \tilde{G}_V(A_f)\) induces a map

\[
G_V(\mathbb{Q}) \backslash G_V(A_f)/K_V \hookrightarrow \tilde{G}_V(\mathbb{Q}) \backslash \tilde{G}_V(A_f)/\tilde{K}_V.
\]

We check that this map is injective. Indeed, it suffices to prove that for any \(g', g'' \in G_V(A_f)\) for which \(\tilde{G}_V(\mathbb{Q}) g' \tilde{K}_V = \tilde{G}_V(\mathbb{Q}) g'' \tilde{K}_V\), we have \(G_V(\mathbb{Q}) g' K_V = G_V(\mathbb{Q}) g'' K_V\). Suppose that \(g'' = g_Q g' h\) where \(g_Q \in \tilde{G}_V(\mathbb{Q})\) and \(k \in \tilde{K}_V\). It follows that \(\nu(g_Q)\nu(k) = 1\). Since \(\nu(g_Q) \in \mathbb{Q}_\times\) and \(\nu(k) \in \mathbb{Z}_\times\) and since \(\mathbb{Q}_\times \cap \mathbb{Z}_\times = \{\pm 1\}\), we get that \(\nu(g_Q) = 1\) (since \(\nu(g_Q) > 0\)).

\(\square\)
The above argument uses in an essential way that for matrices in \( \tilde{G}_V(\mathbb{R}) \), \( \nu \) takes values in \( \mathbb{R}^* \) which is no longer true when \( \dim V \) is even. Yet, the statement of the lemma still holds even when \( \dim V \) and we explain that here. Suppose that \( n = 2 \). The matrix \( g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) is then a unitary similitude for the hermitian form \( J' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \) with similitude factor \( \nu(g) = -1 \) since \( \overline{\gamma J' g} = -J' \).

It then follows that \( \tilde{G}_V(\mathbb{R})/\tilde{G}_V(\mathbb{R})^+ \cong \{1, g\} \) and hence
\[
X'_V = X_V \sqcup gX_Vg^{-1}.
\] (14)

Here, \( X_V \) can also be described as the set of negative-definite lines in \( V \) and \( gX_Vg^{-1} \) is the set of positive-definite lines. In addition \( \text{Stab}_{\tilde{G}_V(\mathbb{R})}(X_V) = \tilde{G}_V(\mathbb{R})^+ \). Suppose now that two points \( \begin{pmatrix} g_1, x_1 \end{pmatrix}, \begin{pmatrix} g_2, x_2 \end{pmatrix} \in G_V(\mathbb{Q}) \setminus G_V(A_f) \times X_V/K_V \) give rise to the same image in \( \tilde{G}_V(\mathbb{Q}) \setminus \tilde{G}_V(A_f) \times X'_V \times \tilde{K}_V \). This means that there exists \( g_Q \in \tilde{G}_V(\mathbb{Q}) \) and \( h \in \tilde{K}_V \) such that
\[
g_Qg_1k = g_2 \quad \text{and} \quad g_Qx_1 = x_2.
\]

Taking similitude factors on the first yields that \( \nu(g_Q) = \nu(k) \in \mathbb{Q}^* \cap \mathbb{Z}^* = \{ \pm 1 \} \). If \( \nu(g_Q) = -1 \) then the \( g_Q \) in \( \tilde{G}_V(\mathbb{R})/\tilde{G}_V(\mathbb{R})^+ \) is \( g \) and hence, \( g_Q \) will map the component \( X_V \) to \( gX_Vg^{-1} \). Yet, both \( x_1 \) and \( x_2 \) are points on \( X_V \) which will be a contradiction.

Similarly, we get embeddings \( e_W : \text{Sh}_{\tilde{K}_W}(G_W, X_W) \hookrightarrow \text{Sh}_{\tilde{K}_W}(\tilde{G}_W, X'_W) \) and \( e : \text{Sh}_K(G, X) \hookrightarrow \text{Sh}_K(\tilde{G}, X') \).

2.2.15 Relation between \((\tilde{G}, X')\) and \((\tilde{G}, X)\). The two Shimura data are related by the following homomorphism of algebraic groups
\[
\omega_{\tilde{G},V} : S_R \rightarrow \tilde{G}_V(\mathbb{R}), \quad z \mapsto \text{diag}(z, \ldots, z) \times 1_2 \times \cdots \times 1_d.
\]

Indeed, there is a bijective mapping \( f : X_V \rightarrow \tilde{X}_V \) given by multiplication by \( \omega_{\tilde{G},V} \) (and similarly for \( X_W \)). This yields an isomorphism of Shimura varieties
\[
f : \text{Sh}_K(\tilde{G}, X') \rightarrow \text{Sh}_K(\tilde{G}, X), \quad \tilde{G}(\mathbb{Q})(g, x) \tilde{K} \rightarrow \tilde{G}(\mathbb{Q})(g, \omega_R x) \tilde{K},
\]
where \( \omega_R = (\omega_{\tilde{G},R}, \omega_{W,R}) : S_R \rightarrow \tilde{G}_R \). The morphism \( f \) is defined over \( \mathbb{C} \) and not necessarily defined over the reflex field \( E \). It is Hecke equivariant, but not Galois equivariant. We will compute precisely how it transforms under the Galois group \( \text{Gal}(E^{ab}/E) \) and hence, obtain explicitly its field of definition using the reciprocity law on special points. As we will see, its field of definition will depend on the level structure \( \tilde{K} \). First, associated to the above isomorphism \( f \), one has a 1-cocycle
\[
u : \text{Gal}(\overline{\mathbb{Q}})/E \rightarrow \text{Aut}(\text{Sh}_K(\tilde{G}, X'))_{\overline{\mathbb{Q}}}, \quad \nu(\sigma) := f^{-1}(\sigma \cdot f).
\]

Clearly, \( f \) is defined over \( E \) if and only if \( u \) is trivial and more generally, \( f \) is defined over an extension \( L/E \) if \( u|_{\text{Gal}(\overline{\mathbb{Q}})/L)} \) is trivial. The main idea behind computing the field of definition of \( f \) is that \( \nu(\sigma) \) can be computed using reciprocity laws for special points on the Shimura variety. Before that, we note that \( \omega_{\tilde{G},R} \) arises from the character of algebraic groups over \( F \)
\[
\omega_V : U_F^1 \rightarrow \tilde{G}_V(\mathbb{R}) \cong F \times \cdots \times U(\mathbb{V}), \quad z \mapsto \text{diag}(z, \ldots, z) \times 1_2 \times \cdots \times 1_d.
\]

2.2.16 Reciprocity law on special points. Let \((T, x)\) be a special pair in the sense of [Mil03, p.103] where \( T \subset \tilde{G}_V \) is a torus. Then \( x \) corresponds to a homomorphism \( h_x : S_R \rightarrow \tilde{G}_V(\mathbb{R}) \) that factors through \( T_R \).

Composing \( h_{x,C} \) with the map \( G_{m,C} \rightarrow S_C \cong G_{m,C} \times G_{m,C} \) given by \( z \mapsto (z, 1) \) yields a co-character \( \mu_x : G_{m,C} \rightarrow T_C \). This is the co-character giving the reciprocity law for the
action of $\sigma \in \text{Gal}(E_{\text{ab}}/E)$ on $[g, x] \in \text{Sh}_{\mathbb{R}}(\tilde{G}, X')$. More precisely, $\mu_x$ gives rise to a homomorphism $r(\mu_x, T): \text{Res}_{E/Q} G_{m,E} \to T$ of algebraic groups over $Q$ defined by

$$r(\mu_x, T)(t) := \prod_{\rho: E \to \overline{Q}} \rho(\mu_x(t)).$$

Note that the sum on the right-hand side is defined over $Q$. This gives us a homomorphism $r_x: A_E^\times \to T(A_f)$. Now, if $s \in A_E^\times$ is an idèle whose image in $\text{Gal}(E_{\text{ab}}/E)$ under the Artin map is exactly $\sigma$ then $\sigma[g, x] = [r_x(s)g, x]$.

2.2.17 Cocharacter $\mu_{\omega_V}$ associated to $\omega_V$. Let

$$\mu_{\omega_V}: G_{m,C} \xrightarrow{\omega_{V,R}} G_{m,C} \times G_{m,C} \cong S_R \xrightarrow{\omega_{V,R}} T_{V,C}$$

be the associated cocharacter. The map $f$ then sends $[g, x] \mapsto [g, y]$ where $h_y = h_x \mu_{\omega_V}$. If $\sigma \in \text{Gal}(E_{\text{ab}}/E)$ and $s = (s_f, s_\infty)$ are as above, then

$$\sigma \cdot f([g, x]) = \sigma \cdot [g, y] = [r_y(s)g, y] = [\omega_V(s)\omega_V(s)r_x(s)g, y] = \langle s \rangle f(\sigma \cdot [g, x]),$$

where $\langle s \rangle: \text{Sh}_{\mathbb{R}}(\tilde{G}_V, \tilde{X}_V) \to \text{Sh}_{\mathbb{R}}(\tilde{G}_V, \tilde{X}_V)$ is given by $\langle s \rangle[g', x'] \mapsto [s_f g', x']$ (note that here, we view $s_f$ as an element of $\tilde{G}_V(A_f)$ via the diagonal embedding). We summarize the commutativity of $f$ and $\text{Gal}(\overline{Q}/E)$ in the following:

**Lemma 2.2.** For $\sigma \in \text{Gal}(E_{\text{ab}}/E)$ and $s_f$ as above, we have

$$\sigma \circ f = \langle s \rangle \circ f \circ \sigma.$$  

(16)

2.2.18 Fields of definition of $f$. Consider the reciprocity map

$$\text{rec}_E: A_E^\times / E^\times \to \text{Gal}(E_{\text{ab}}/E).$$

(17)

Given an open subgroup of finite index. We will often make use of the ring class field $E[c]$ of conductor $c \subset O_F$, that is, the abelian extension of $E$ whose corresponding norm subgroup is equal to $E^\times \tilde{O}_c^\times$ where $O_c \subset O_E = O_F + \gamma O_F$ is the order of conductor $c$, i.e., $O_c = O_F + \gamma O_F$. Equivalently, $\text{rec}_E$ induces an isomorphism

$$\text{rec}_E: E^\times / E^\times \tilde{O_c^\times} \cong \text{Gal}(E[c]/E).$$

(18)

As mentioned in [Mil05, p.98], one often replaces the map $\text{rec}_E$ with its reciprocal, the Artin map, $\text{Art}_E: A_E^\times / E^\times \to \text{Gal}(E_{\text{ab}}/E)$ defined by $\text{Art}_E(s) = \text{rec}_E(s)^{-1}$.

Calculating the field of definition of $f$ is now easy: using that $s_f \in Z(\tilde{G}_V(A_f))$ and (15), the corresponding cocycle $u(\sigma)$ from the previous section is trivial on $\sigma \in \text{Gal}(\overline{Q}/E)$ if and only if $s_f \in K$, i.e., $f$ is defined over the abelian extension $E(f)$ whose norm subgroup is precisely $A_f^\times \cap K$ (or $A_f^\times \cap Z(K)$).

**Proposition 2.3.** The morphism $f: \text{Sh}_{\mathbb{R}}(\tilde{G}_V, X') \to \text{Sh}_{\mathbb{R}}(\tilde{G}_V, \tilde{X})$ is defined over the abelian extension $E(f)/E$ given by the norm subgroup $E^\times(\tilde{E}^\times \cap Z(K)) \subset \tilde{E}^\times$.

2.2.19 Galois action on connected components. The derived subgroup $H_{\text{der}}$ of $H$ is simply connected since it is isomorphic (over $\overline{Q}$) to $\text{SL}_{n-1}^d$. Let $T^1 = H / H_{\text{der}}$ (also isomorphic to $\text{Res}_{F/Q} U_F^1$) and let $\nu: H \to T^1$ be the natural quotient map. Let $X(H)$ be the Hermitian subdomain

$$X(H) = \{ x \in X : h_x : S_R \to G_R \text{ factors through } H_R \}.$$ 

Note that $X(H)$ is a connected Hermitian symmetric domain for the group $H(R)$. One can apply [Mil05, Thm.5.17] (a simplified version of Deligne’s results on the structure of the set of connected
2.3 Special cycles on $\text{Sh}_K(G, X)$

2.3.20 The cycles $Z_K(g)$. Given $g \in G(A_f)$, consider the cycle $Z_K(g)$ that is the image of $gK \times Y$ in $\text{Sh}_K(G, X)(C) = G(Q) \setminus (G(A_f)/K \times X)$. Let $Z_K(G, H) = \{ Z_K(g) : g \in G(A_f) \}$ be the space of all such cycles. We have a map

$$Z_K(\bullet) : \text{H}(Q) \setminus G(A_f)/K \to Z_K(G, H).$$

The map is certainly surjective by definition of $Z_K(G, H)$, so we compute its left and right kernels.

Lemma 2.4. (i) The map $Z_K(\bullet) : \text{H}(Q) \setminus G(A_f)/K \to Z_K(G, H)$ is surjective and induces a bijection

$$Z_K(\bullet) : N_{G(Q)}(H(Q)) \setminus G(A_f)/K \to Z_K(G, H).$$

(ii) We have $N_{G(Q)}(H(Q)) = \Delta(H(Q)) \setminus \bigcup \Delta(H(Q)) \setminus Z_H(Q) \subset G(Q)$, where $\Delta : H \to G \times H$ is the diagonal map.

Proof. (i) The condition $Z_K(g') = Z_K(g'')$ is equivalent to $G(Q)(g'K, Y) = G(Q)(g''K, Y)$. This is equivalent to the following statement:

$$\forall y \in Y, \exists gQ \in g'Kg''^{-1} \cap G(Q) \text{ such that } y \in gQY.$$  

This is equivalent to the following statement:

$$\forall y \in Y, \exists gQ \in g'Kg''^{-1} \cap G(Q) \text{ such that } y \in gQY.$$  

The latter means that $Y = \bigcup_{gQ \in G(Q) \cap g'Kg''^{-1}} (Y \cap gQY)$, i.e., $Y$ is a countable union of sets $Y \cap gQY$.

We know claim the following:

Claim: There exists $gQ \in g'Kg''^{-1}$ such that $gQY = Y$.

We will prove this in two different ways: 1) using the fact that the Riemann manifolds $Y$ and $gQY$ are totally geodesic and 2) via a more general argument about real manifolds.

First argument: Baire’s category theorem implies that there exists $gQ \in G(Q) \cap g'Kg''^{-1}$ such that $Y \cap gQY$ contains an open set for $Y$. We claim that $Y = gQY$. Observe that $Y \cap gQY \subset Y$ is a totally geodesic submanifold. Let $U \subset Y \cap gQY$ be the open set for $Y$. Take any point $y \in Y \cap gQY$ and consider any geodesic $\gamma$ through $y$ in $Y$. Since the germ $[\gamma]$ of that geodesics is contained in $U$, by the extension property of geodesics, the entire geodesics is contained in $Y \cap gQY$. Now, using that $Y$ is connected, it follows that $Y$ is contained in $Y \cap gQY$ (indeed, any point $x \in Y$ can be connected by a geodesic to $y$ which, by the above argument, is necessarily in $Y \cap gQY$, i.e., $x \in Y \cap gQ$). Thus, there exists $gQ \in g'Kg''^{-1} \cap G(Q)$ such that $gQY = Y$. We will next compute $\text{Stab}_{G(Q)}(Y)$ by first computing $\text{Stab}_{G(Q)}(\Delta(W))$.  

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Second argument: Given \(y \in Y\), define \(S(y) = \{g \in G(\mathbb{Q}) : gy \in Y\}\) and let \(S(Y) = \{g \in G(\mathbb{Q}) : gY \subseteq Y\}\). Clearly, \(S(y) \supseteq S(Y)\). Using \([21]\), it suffices to show that there exists \(y \in Y\) for which \(S(y) = S(Y)\). For that, call \(g_{\mathbb{Q}} \in G(\mathbb{Q})\) proper if \(g_{\mathbb{Q}}Y \cap Y \subseteq Y\). It suffices to take \(y \in Y - \bigcup_{g_{\mathbb{Q}}gY \subseteq Y} g_{\mathbb{Q}}\).

Let \(K \subseteq G(\mathbb{Q})\) be a subgroup. Then \(K\) is proper if and only if \(g_{\mathbb{Q}}Y \cap Y\) has an empty interior (as a set, with respect to the topology of \(Y\)). Baire’s theorem when implies the non-emptiness.

It thus remains to compute the stabilizer \(S(Y) = \text{Stab}_{G(\mathbb{Q})}(Y)\).

Computing \(\text{Stab}_{G(\mathbb{Q})}(\Delta(W))\): The condition \((g_{\mathbb{Q}}, g_{\mathbb{Q}}W, Q) \in \text{Stab}_{G(\mathbb{Q})}(\Delta(W))\) means that for any \(w \in W\), \((g_{\mathbb{Q}}, g_{\mathbb{Q}}W, Q)\) fixes \(\Delta_V(w, w')\) for some \(w' \in W\), i.e., \(g_{\mathbb{Q}}\Delta_V(g_{\mathbb{Q}}w, Q)\) fixes \(\Delta_V(w)\) for every \(w \in W\). This is equivalent to \(u \in U(W) \times U(D) \subseteq G(V)\) for some \(u \in U(D)\), i.e., \(\text{Stab}_{G(\mathbb{Q})}(\Delta(V)) = \Delta(H(V)) = (1 \times U(D) \times \mathbb{Z}) \subseteq G(\mathbb{Q})\).

Computing \(\text{Stab}_{G(\mathbb{Q})}(\Delta(X_W))\): Let \((g_{\mathbb{Q}}, g_{\mathbb{Q}}W, Q) \in \text{Stab}_{G(\mathbb{Q})}(\Delta(X_W))\). Equivalently, for any negative-definite line \(\ell \in X_W\), there exists a negative definite line \(\ell' \in X_W\) such that \((g_{\mathbb{Q}}, g_{\mathbb{Q}}W, Q)\) fixes \(\Delta_V(\ell, \ell') = \Delta_V(\ell', \ell)\), and \(\Delta_V(\ell') = g_{\mathbb{Q}}\Delta_V(\ell)\).

The latter is equivalent to \(g_{\mathbb{Q}}\Delta_V(g_{\mathbb{Q}}W, Q) = (g_{\mathbb{Q}}W, Q)\Delta_V(g_{\mathbb{Q}}W, Q)\). This means that \(\text{Stab}_{G(\mathbb{Q})}(\Delta(X_W)) = \Delta(H(V)) = (1 \times U(D) \times \mathbb{Z}) \subseteq G(\mathbb{Q})\).

For \((ii)\), \(g \in G(V) \times G_W(Q)\) be an element of the normalizer. Then for any \(h \in H(V)\), \(g_{\mathbb{Q}}h_{\mathbb{Q}}\) fixes \(D\) point wise, i.e., \(H(V)\) fixes \(g_{\mathbb{Q}}^{-1} D\) pointwise. But the only line in \(V\) fixed pointwise by \(H(V)\) is \(D\) itself. Hence, \(g_{\mathbb{Q}}^{-1} D = D\) and so, \(g = ((h, u), h')\). Now, for any \(\tilde{h} \in H(V)\), \(g((h, u), h)g_{\mathbb{Q}}^{-1} \in H(V)\), i.e., \(h' \in H(V)\) which completes the proof.

2.3.21 Description of \(Z_K(g)\) in terms of connected components and Galois action. Alternatively, given \(g \in G(A_f)\), the cycle \(Z_K(g)\) can be described as follows:

**Lemma 2.5.** If \(K_{g, H} = gKg^{-1} \cap H(A_f)\) then \(Z_K(g)\) is the image of the connected component \(H(\mathbb{Q})(K_{g, H} \times Y)\) of \(\text{Sh}_{K_{g, H}}(H, Y)\) under the map\footnote{Recall that all of these maps are defined over \(E\).} \(\text{Sh}_{K_{g, H}}(H, Y) \to \text{Sh}_{gKg^{-1}}(G, X) \xrightarrow{[g]} \text{Sh}_K(G, X)\).

**Proof.** This follows immediately by chasing through the definitions of the maps: indeed, for any element \((h_{\mathbb{Q}}gkg^{-1}, hgy) \in H(\mathbb{Q})(K_{g, H} \times Y)\) (here, \(h_{\mathbb{Q}} \in H(\mathbb{Q}), g \in G(A_f), k \in K\) and \(y \in Y\)) maps to the element \([gk, y]_K \in \text{Sh}_K(G, X)\). Conversely, any \([gk, y]_K \in [gKg^{-1}] \times Y\) is the image of \([gk, y]_K\) for any \(k' \in K_H\).

One can thus describe the Galois action on \(Z_K(G, H)\) using Deligne’s reciprocity law for the connected components of the Shimura varieties \(\text{Sh}_{K_{g, H}}(H, Y)\) for \(g \in G(A_f)\). More precisely, for \(\sigma \in \text{Gal}(\mathbb{Q}_{\text{ab}}/\mathbb{Q})\) and \(s \in A_{\mathbb{Q}}^\times\), such that \(\text{Art}_{\mathbb{Q}}(s) = \sigma\), consider \(r_f(s) \in T^1(A_f)\). It follows from \([\text{Mil}05\text{, Lem.5.21}]\) that \(\nu: H(A_f) \to T^1(A_f)\) is surjective, so there exists \(h_s\) such that \(\nu(h_s) = r_f(s)\) and hence,

\[
Z_K(g)^\sigma = Z_K(h_sg).
\]

Equivalently, \(Z_K(g)^\sigma\) is the image of the connected component \(H(\mathbb{Q})[K_{h_sg, H} \times Y]\) under \([22]\), but for the compact open subgroup \(K_{h_sg, H}\) instead of \(K_{g, H}\).
3. Galois Action on Special Cycles

We now compute the Galois action on the set $Z_K(G,H)$ of cycles on our 3-fold $Sh_K(G,X)$ and prove Theorem 1.1. The statement and the proof are for $n = 3$, although we do expect that a similar local conductor formula should holds for any $n$. Using Lemma 2.4 one can provide an adelic description of the Galois orbits of special cycles, i.e., there is a surjective map

$$\text{Gal}(E^{ab}/E)\backslash Z_K(G,H) \twoheadrightarrow H(A_f)\backslash G(A_f)/K \cong H_r\backslash G_r/K_r \times H^r\backslash G^r/K^r.$$  \hspace{1cm} (23)

The above map has the advantage that one can reduce the problem of computing the Galois action to a local question at each place $\tau$ of $F$ by lifting a Galois orbit to $H_r\backslash G_r/K_r$. For the purpose of our particular application, we will use the fact that $K_r = K_{V,\tau} \times K_{W,\tau}$ where $K_{V,\tau}$ and $K_{W,\tau}$ are hyperspecial maximal compact subgroups of $G_{V,\tau}$ and $G_{W,\tau}$, respectively.

3.1 Bruhat–Tits buildings for unitary groups

The adelic description of Galois orbits has the advantage that one can relate the local products on the right-hand side of (23) to $H_r$-orbits of hyperspecial points on the product of the Bruhat–Tits buildings for $U(V)$ and $U(W)$ for each finite inert place $\tau$ of $F$ for which $K_{V,\tau}$ and $K_{W,\tau}$ are both hyperspecial (these are all, but finitely many places inert places $\tau$). In this case, the quotient $G_r/K_r$ is in bijection with the pairs $(L_{V,\tau}, L_{W,\tau})$ of self-dual Hermitian lattices in $V_{\tau}$ and $W_{\tau}$, respectively.

3.1.1 Buildings, apartments, hyperspecial and special vertices. One way to describe the Bruhat–Tits buildings for unitary groups is via the theory of $p$-adic self-dual norms. This interpretation was initiated by Goldman and Iwahori [GI69] and was then reinterpreted and developed further by Bruhat and Tits [BT87] (see also [Cor09]). It seems to be a suitable approach if one wants to approach the problem for general $n$. The case of groups of unitary similitudes in three variables is treated in detail in [Kos13b, §4.1]. More precisely, let $B(V_{\tau})$ (resp., $B(W_{\tau})$) be the set of self-dual ultrametric norms in $V_{\tau}$ (resp., $W_{\tau}$) in the sense of [Kos13b, p.28]. Given a Witt basis $B$ for $V_{\tau}$, one defines the apartment $A_B$ corresponding to $B$ as the set of all $\alpha \in B(V_{\tau})$ adapted to $B$ in the sense of [Kos13b, Defn.47]. If $R(V_{\tau})$ is the set of all $O_{E_r}$ lattices in $V_{\tau}$ then the norms in a given apartment corresponding to a Witt basis $\{e_+, e_0, e_-\}$ are described (see p.29 of loc. cit.) by the norm functions $f_\lambda: \mathbb{R} \to R(V_{\tau})$ given by

$$f_\lambda(\theta) = \omega^{-[\frac{\lambda + 1}{2}]}O_{E_r} e_+ \oplus \omega^{-[\theta]}O_{E_r} e_0 \oplus \omega^{-[\frac{\lambda - 1}{2}]}O_{E_r} e_- \in R(V_{\tau}),$$

where $[r]$ is denotes the integer part of $r$. Associated to a self-dual norm $\alpha \in B(V_{\tau})$ is the chain of open balls $B^*(\alpha) = \{B(\alpha, \theta): \theta \in \mathbb{R}\}$ where $B(\alpha, \theta) = \{v \in V_{\tau}: \alpha(v) \leq q^\theta\}$. We say that two norms $\alpha'$ and $\alpha''$ are equivalent (and denote it by $\alpha' \sim \alpha''$) if $B^*(\alpha') = B^*(\alpha'')$. A self-dual norm $\alpha \in B(V_{\tau})$ is a vertex of the building $B(V_{\tau})$ if it is the only self-dual norm in its equivalence class $\text{cl}($,$\alpha$), i.e., if $\text{cl}($,$\alpha$) $= \{\alpha\}$. The other equivalence classes of self-dual norms are called facets. Given a facet $X$ with a chain of lattices $B^*(X)$, we say that a self-dual norm $\alpha$ belongs to $X$ if $B^*(\alpha) \in B^*(X)$. Two vertices $\alpha'$ and $\alpha''$ are called neighbors if they are vertices of the same facet. Moreover [Kos13b, p.30], for every $\alpha \in B(V_{\tau})$ the chain $B^*(\alpha)$ is a union of homothety classes. A vertex $\alpha$ is hyperspecial if $B^*(\alpha)$ is a single homothety class. The other vertices are called special. Hyperspecial vertices correspond to $\lambda \in \mathbb{Z}$ whereas special, but not hyperspecial vertices correspond to $\lambda \in \frac{1}{2} + \mathbb{Z}$. Vertices are connected by edges (facets) $X_\lambda = \left[\frac{\lambda + 1}{2}, \lambda, \frac{\lambda + 1}{2}\right]$ for $\lambda \in \frac{1}{2} \mathbb{Z}$.

There is a rather explicit description of the hyperspecial and special vertices of the buildings $B(V_{\tau})$ and $B(W_{\tau})$ of $V_{\tau}$ and $W_{\tau}$, respectively, for the case $n = 3$ in terms of lattices, thus, explaining the graphs in Fig. 1. Recall that an $O_{E_r}$-lattice $L_{V,\tau} \subset V_{\tau}$ (resp., $L_{W,\tau} \subset W_{\tau}$) is self-dual if $L_{V,\tau}^\vee = L_{V,\tau}$. Each self dual lattice in $V_{\tau}$ (resp., $W_{\tau}$) yields a maximal compact subgroup of $G_{V,\tau}$
Let $\text{Hyp}_{V,\tau}$ (resp., $\text{Sp}_{W,\tau}$) denote the set of hyperspecial vertices of $B(V,\tau)$ (resp., $B(W,\tau)$). To understand the incidences, we need to consider lattices that are not self-dual, but almost self-dual. A lattice $L$ in either $V,\tau$ or $W,\tau$ is almost self-dual if $\varpi^\tau L \varpi^\tau \subseteq L \subseteq \varpi^\tau \tau$. Almost self-dual lattices will correspond to special, but not hyperspecial points (the white points in Fig. 1). Let $A$ lattice $\langle x \rangle$ for example, recall that a choice of a Witt basis of the form $\langle e_+^0, e_-^0 \rangle$ fixes the maximal split torus and we can count the number of neighbors of each edge have different colors. The resulting graph $B(V,\tau)$ (Bruhat–Tits building) is a tree (the dimension is equal to the rank of the maximal split torus) and we can count the number of neighbors of each black and white vertex by counting the number of isotropic lines in Hermitian spaces over finite fields. Each black vertex has $q^3 + 1$ white vertices and each white vertex has $q - 1$ black vertices.

3.2 Local conductor and the distance function - proof of Theorem 1.1

We are ready to prove Theorem 1.1 by showing how to express the local conductor in terms of the distance function on the Bruhat–Tits building for the group $U(V,\tau)$. It is through such a combinatorial formula that we can link the Galois action with the Hecke action and thus, obtain distribution relations.
3.2.2 Fields of definition and local conductors. Given a special cycle \( \xi \in Z_K(G, H) \), the field of definition \( E(\xi) \) is an abelian extension of \( E \). The reciprocity law described in 2.3.21 of Section 2.2 allows us to compute the completion \( E(\xi)_\tau \) at a finite place \( \tau \) of \( E \) as follows: let \( (x_\tau)_\tau \)-finite be the element of the right-hand side of (23) corresponding to the Galois orbit \( \text{Gal}(E^{ab}/E)\xi \). Here, \( x_\tau \in H_\tau \backslash G_\tau / K_\tau \) is the trivial coset for all but finitely many finite places \( \tau \) of \( F \). If \( \tau \) is a finite place of \( F \) such that both \( K_{V,\tau} \) and \( K_{W,\tau} \) are hyperspecial maximal compact subgroups, \( G_\tau / K_\tau \) is in bijection with the pairs \( (L_{V,\tau}, L_{W,\tau}) \) of self-dual hermitian lattices in \( V_\tau \) and \( W_\tau \), respectively. For such a \( \tau \), if \( L_\tau \) is the set of these lattices, then \( H_\tau \) acts on \( L_\tau \) and the element \( x_\tau \) corresponds to an \( H_\tau \)-orbit denoted by \( [L_{V,\tau}, L_{W,\tau}] \). In this case, the local conductor \( c_\tau(\xi) \) can be calculated by considering the image of \( \text{Stab}_{H_\tau}(L_{V,\tau}, L_{W,\tau}) \) under the determinant map \( \det: H_\tau \to U^1(F_\tau) \). If \( \varpi \in O_{E_\tau} \) is a uniformizer then for \( n \geq 0 \), the local order of conductor \( \varpi^n \) is \( O_{n,\tau} = O_{F_\tau} + \varpi^nO_{E_\tau} \).

There is a filtration

\[
U(0) = O_0^\times \supset U(1) = O_1^\times \supset \cdots \supset U(n) = O_n^\times \supset \ldots.
\]

The image of this filtration under the map \( r_\tau: E_\tau^\times \to U^1(F_\tau) \) given by \( \nu(s_\tau) = \varpi/s_\tau \) for \( s_\tau \in E_\tau^\times \) yield a filtration on \( U^1(F_\tau) \):

\[
U^1(0) := r_\tau(U(0)) \supset U^1(1) := r_\tau(U(1)) \supset \cdots \supset U^1(n) := r_\tau(U(n)) \supset \ldots.
\]

Calculating the local conductor amounts to detecting the position of

\[
\det(\text{Stab}_{H_\tau}(L_{V,\tau}, L_{W,\tau})) \subset U^1(F_\tau)
\]

with respect to the above filtration. We will show that \( \det(\text{Stab}_{H_\tau}(L_{V,\tau}, L_{W,\tau})) = U^1(n) \) for a unique \( n \) that is calculated purely in terms of the distance function on the building \( B(V_\tau) \) and will thus get that \( c_\tau([L_{V,\tau}, L_{W,\tau}]) = \varpi^n \).

3.2.3 Lines in \( B(V_\tau) \) are apartments. The following lemma is a basic property of buildings over complete local fields proved in great generality in [BT72 Cor.2.8.4]:

**Lemma 3.1.** Any geodesic on the building \( B(V_\tau) \) is contained in an apartment.

The statement is sometimes known as an "extension of geodesics" property (also discussed in [Par00] and [KL97]).

3.2.4 Two relevant apartments in \( B(V_\tau) \). There is a unique self-dual lattice \( L_{D,\tau} = O_{E_\tau} e_0 \) in the line \( D_\tau \). Let

\[
dist(L_{V,\tau}, L_{W,\tau}) = n \quad \text{and} \quad dist(L_{V,\tau}, \text{pr}_{W_\tau}(L_{V,\tau})) = d.
\]

Since \( B(V_\tau) \) is a tree, \( dist(\text{pr}_{W_\tau}(L_{V,\tau}), L_{W,\tau}) = n - d \). Here, \( d \) is the distance from \( L_{V,\tau} \) to the sub-building \( B(W_\tau) \) and \( n \) is the distance between the two hyperspecial vertices corresponding to \( L_{V,\tau} \) and \( L_{W,\tau} \oplus L_{D,\tau} \). Consider two apartments \( A \) and \( \bar{A} \) that will be used in the computation:

- \( A \): an apartment containing the two black vertices \( \text{pr}_{W_\tau}(L_{V,\tau}) \oplus L_{D,\tau} \) and \( L_{W,\tau} \oplus L_{D,\tau} \) and contained entirely in \( B(W_\tau) \) (such an apartment exists due to, e.g., [G163 Prop.1.3]) and is illustrated on Fig. 2. In addition, we choose a Witt basis \( \{e_+, e_0, e_-\} \) for \( A \) in such a way that

\[
\text{pr}_{W_\tau}(L_{V,\tau}) \oplus L_{D,\tau} = \langle e_+, e_0, e_- \rangle \quad \text{and} \quad L_{W,\tau} \oplus L_{D,\tau} = \langle \varpi^{-(n-d)} e_+, e_0, \varpi^{-d} e_- \rangle.
\]

- \( \bar{A} \): an apartment containing the three black vertices of \( B(V_\tau) \) corresponding to the self-dual lattices \( \{L_{V,\tau}, \text{pr}_{W_\tau}(L_{V,\tau}), L_{W,\tau}\} \) and intersecting \( A \) in a half-line contained in \( B(W_\tau) \) whose end point is \( \text{pr}_{W_\tau}(L_{V,\tau}) \). Such an apartment exists: take any line in the building containing the three vertices and use Lemma 3.1 to conclude that the line is an apartment. In our case, \( \bar{A} \) can be visualized as in Fig. 2.
Note that the intersection $\mathcal{A} \cap \tilde{\mathcal{A}} = B(W_\tau) \cap \tilde{\mathcal{A}}$ is a half-line contained in $B(W_\tau)$. Choose a Witt basis $\tilde{\mathcal{B}} = \{\tilde{e}_+, \tilde{e}_0, \tilde{e}_-\}$ such that

$$\text{pr}_{W_\tau}(L_{V,\tau}) \oplus L_{D,\tau} = \langle \tilde{e}_+, \tilde{e}_0, \tilde{e}_- \rangle \quad \text{and} \quad L_{V,\tau} = \langle \omega^d \tilde{e}_+, \tilde{e}_0, \omega^{-d} \tilde{e}_- \rangle. \quad (25)$$

Here, the common half-apartment $\mathcal{A} \cap \tilde{\mathcal{A}}$ is determined precisely by the isotropic vector $e_+$.

### 3.2.5 The change of basis matrix $S$.

Let $S$ be the change of basis matrix from $\mathcal{B}$ to $\tilde{\mathcal{B}}$.

**Lemma 3.2.** The matrix $S$ is of the form

$$S = \begin{bmatrix} 1 & \beta & \gamma \\ 0 & 1 & -\beta \\ 0 & 0 & 1 \end{bmatrix}, \quad \beta, \gamma \in \mathcal{O}_{E_r}, \quad \beta \overline{\beta} + \gamma + \overline{\gamma} = 0. $$

**Proof.** Since $S$ is the change-of-basis matrix for two bases of the same lattice $\text{pr}_{W_\tau}(L_{V,\tau}) \oplus L_{D,\tau}$, $S \in \text{GL}_3(\mathcal{O}_{E_r})$. Let $\delta_V = \text{diag}(\omega, 1, \omega^{-1})$. Using the condition that for each $m \geq 0$,

$$\langle \omega^{-m} \tilde{e}_+, e_0, \omega^m \tilde{e}_- \rangle = \langle \omega^{-m} e_+, e_0, \omega^m e_- \rangle,$$

but $\langle \omega \tilde{e}_+, e_0, \omega^{-1} \tilde{e}_- \rangle \neq \langle \omega e_+, e_0, \omega^{-1} e_- \rangle$, we get $\delta_V^m S \delta_V^{-m} \in \text{GL}_3(\mathcal{O}_{E_r})$ for all $m \geq 0$, but $\delta_V^{-1} S \delta_V \notin \text{GL}_3(\mathcal{O}_{E_r})$ (since $\delta_V^m S \delta_V^{-m}$ is the matrix that transforms the basis $\{\omega^{-m} e_+, e_0, \omega^m e_-\}$ to $\{\omega^{-m} \tilde{e}_+, \tilde{e}_0, \omega^m \tilde{e}_-\}$). The first condition implies that $S$ is upper-triangular. By changing $\{e_+, e_0, e_-\}$ by units in $\mathcal{O}_{E_r}$ if necessary, we can assume it is upper-triangular, unipotent and unitary (the latter is automatic as $S$ is the change-of-basis matrix from one Witt basis to another), and hence,

$$S = \begin{bmatrix} 1 & \beta & \gamma \\ 0 & 1 & -\beta \\ 0 & 0 & 1 \end{bmatrix}, \quad \beta, \gamma \in \mathcal{O}_{E_r}, \quad \beta \overline{\beta} + \gamma + \overline{\gamma} = 0. $$

The fact that $\langle \omega \tilde{e}_+, e_0, \omega^{-1} \tilde{e}_- \rangle \neq \langle \omega e_+, e_0, \omega^{-1} e_- \rangle$ yields

$$\delta_V^{-1} S \delta_V = \begin{bmatrix} 1 & \omega^{-1} \beta & \omega^{-2} \gamma \\ 0 & 1 & -\omega^{-1} \beta \\ 0 & 0 & 1 \end{bmatrix} \notin \text{GL}_3(\mathcal{O}_{E_r}).$$
which implies that $\gamma \in O_{E_r}^\times$ (we are using that $\beta, \gamma \in O_{E_r}$ and $\overline{\beta} + \gamma + \overline{\gamma} = 0$). It remains to show that $\beta \in O_{E_r}^\times$. To prove this, we not only use the fact that the above matrix is not in $GL_3(O_{E_r})$, but we also need to use that the self-dual lattice $\widetilde{L}_1 = \langle \varpi e_+, \overline{e}_0, \varpi^{-1} \overline{e}_- \rangle \notin B(W_r)$ (since the intersection $B(W_r) \cap \widetilde{A}$ is a half-line whose end-point is $(e_+, e_0, e_-) = (e_+, e_0, e_-)$). Note that if $\varpi^{-1} \beta \in O_{E_r}$, then $\widetilde{L}_1 \cap E_{e_0} = O_{E_r} e_0$ which is equivalent to $\widetilde{L}_1 \subseteq B(W_r)$ (we have used that $\widetilde{L}_1 \cap E_{e_0} \subseteq (\widetilde{L}_1 \cap E_{e_0})^\vee$ since $\widetilde{L}_1$ is self-dual). Hence, $\varpi \nmid \beta$ which proves the lemma. 

3.2.6 Computing stabilizers. To compute $Stab_{H_\tau}(L_{V,\tau}, L_{W,\tau})$, we compute $Stab_{H_\tau}(L_{W,\tau}, \text{pr}_{W_r}(L_{V,\tau}))$ with respect to the basis $B$ and then intersect it with $Stab_{G_{V,\tau}}(L_{V,\tau})$ (computed with respect to the basis $\widetilde{B}$ and converted via the change of basis matrix $S$).

3.2.7 Computing $Stab_{H_\tau}(L_{W,\tau}, \text{pr}_{W_r}(L_{V,\tau}))$. Since $\text{dist}(L_{W,\tau}, \text{pr}_{W_r}(L_{V,\tau})) = n - d$, the stabilizer $Stab_{H_\tau}(L_{W,\tau}, \text{pr}_{W_r}(L_{V,\tau}))$ computed with respect to $B$ and viewed as a subgroup of $G_{V,\tau}$ is

$$Stab_{H_\tau}(\text{pr}_{W_r}(L_{V,\tau}), L_{W,\tau}) = G_{V,\tau} \cap \begin{bmatrix} O_{E_r} & O_{E_r} \\ \varpi^{2(n-d)}O_{E_r} & 1 \end{bmatrix} \subset G_{V,\tau}. \quad (26)$$

Here, $G_{V,\tau}$ is viewed as a subgroup of $GL_3(O_{E_r})$ with respect to the Witt basis $B$ and we have used the fact that the stabilizer belongs to $\delta_V^{-(n-d)}GL_3(O_{E_r})\delta_V^{-d} \cap GL_3(O_{E_r})$.

3.2.8 Computing $Stab_{G_{V,\tau}}(L_{V,\tau}, L_{W,\tau})$. With respect to the basis $\widetilde{B}$, $L_{V,\tau} = \langle \varpi^d e_+, \overline{e}_0, \varpi^{-d} \overline{e}_- \rangle$ and $L_{W,\tau} = \langle \varpi^{-(n-d)} e_+, \overline{e}_0, \varpi^{n-d} \overline{e}_- \rangle$. This implies that

$$Stab_{G_{V,\tau}}(L_{V,\tau}, L_{W,\tau}) = G_{V,\tau} \cap \begin{bmatrix} O_{E_r} & \varpi^d O_{E_r} & \varpi^{2d} O_{E_r} \\ \varpi^{-(n-d)}O_{E_r} & O_{E_r} & \varpi^d O_{E_r} \\ \varpi^{2(n-d)}O_{E_r} & \varpi^{n-d}O_{E_r} & O_{E_r} \end{bmatrix}. \quad (27)$$

Here, we have used that the stabilizer lies in $GL_3(O_{E_r})\cap \delta_V^dGL_3(O_{E_r})\delta_V^{-d} \cap \delta_V^{-(n-d)}GL_3(O_{E_r})\delta_V^{-d}$.

3.2.9 Computing $\det(Stab_{H_\tau}(L_{V,\tau}, L_{W,\tau}))$. We are thus left with computing the image under the determinant map of the intersection

$$Stab_{H_\tau}(L_{V,\tau}, L_{W,\tau}) = G_{V,\tau} \cap S^{-1} \begin{bmatrix} O_{E_r} & \varpi^d O_{E_r} & \varpi^{2d} O_{E_r} \\ \varpi^{-(n-d)}O_{E_r} & O_{E_r} & \varpi^d O_{E_r} \\ \varpi^{2(n-d)}O_{E_r} & \varpi^{n-d}O_{E_r} & O_{E_r} \end{bmatrix} S \cap \begin{bmatrix} O_{E_r} & 1 \\ \varpi^{2(n-d)}O_{E_r} & O_{E_r} \end{bmatrix}. \quad (28)$$

Here, we have changed the basis back to $B$. Let $A = \begin{bmatrix} x_{11} & x_{13} \\ x_{31} & x_{33} \end{bmatrix}$ in the above intersection (here, $v(x_{11}), v(x_{13}), v(x_{31}) \geq 0$ and $v(x_{31}) \geq 2(n - d)$. The intersection condition then means that there exists a matrix $B = \begin{bmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{bmatrix}$ with $v(y_{12}), v(y_{23}) \geq n - d, v(y_{13}) \geq 2d, v(y_{21}), v(y_{32}) \geq n - d, v(y_{31}) \geq 2(n - d)$ such that $SA = BS$, i.e.,

$$\begin{bmatrix} 1 & \beta & \gamma \\ 0 & 1 & -\overline{\beta} \end{bmatrix} \cdot \begin{bmatrix} x_{11} & x_{13} \\ x_{31} & x_{33} \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{bmatrix} \cdot \begin{bmatrix} 1 & \beta & \gamma \\ 0 & 1 & -\overline{\beta} \end{bmatrix}. \quad (28)$$

LEMA 3.3. If $c = \min(d, 2(n - d))$ then $\det(Stab_{H_\tau}(L_{V,\tau}, L_{W,\tau})) = U^1(c)$. 

Proof. By comparing the entries on the left and the right-hand sides of (28), we obtain:
We now calculate if \( \det(A) \) (i.e., \( \det(A) - 1 \)) \( \geq c \), i.e., \( \det(A) \in U^1(c) \). Conversely, take any element \( a + \eta b \in O_{E, \tau}^c \) (here, \( a \in O_{E, \tau}^c \), \( b \in p_{\tau}^c \) where \( p_{\tau} \subset O_{E, \tau}^c \) is the maximal ideal) and let \( \lambda = 1 + \eta a^{-1}b \in O_{E, \tau}^c \). Consider the element \( \lambda/\lambda \in U^1(c) \). It remains to show that there exists \( A \in \text{Stab}_{H, \tau}(L_{V, \tau}, L_{W, \tau}) \) such that \( \det(A) = \lambda/\lambda \). There are two cases we will consider:

**Case 1:** \( d \leq 2(n-d) \). In this case \( c = d \) and we will write a upper-triangular matrix with respect to the basis \( B \) stabilizing both \( L_{V, \tau} \) and \( L_{W, \tau} \oplus L_{D, \tau} \) whose determinant is \( \lambda/\lambda \) and whose conjugate determinant is \( \lambda/\lambda \). We will be looking for a matrix of the form

\[
B = \begin{bmatrix}
1 & x & y \\
1 & -x & 0 \\
0 & 0 & 1
\end{bmatrix} \cdot \begin{bmatrix}
\lambda & 1 & 0 \\
1 & -\lambda & 0 \\
0 & 0 & 1
\end{bmatrix} = \begin{bmatrix}
\lambda x & x & \lambda^{-1}y \\
1 & -\lambda^{-1} & 0 \\
0 & 0 & 1
\end{bmatrix},
\]

where \( x\bar{x} + y + \bar{y} = 0 \) (i.e., a unitary matrix). Moreover, the matrix leaves \( L_{V, \tau} \) stable if and only if \( v(x) \geq c \) and \( v(y) \geq 2c \) (note that as it is lower-triangular, it always leaves \( L_{W, \tau} \oplus L_{D, \tau} \) stable). We now calculate

\[
S^{-1}BS = \begin{bmatrix}
1 & -\beta & \bar{x} \\
1 & \bar{y} \\
0 & 1
\end{bmatrix} \cdot \begin{bmatrix}
\lambda x & x & \lambda^{-1}y \\
\bar{x} & 0 & 0 \\
\bar{y} & 0 & 1
\end{bmatrix} = \begin{bmatrix}
\lambda x + \bar{y}(\lambda - 1) & C \\
\lambda x + \lambda^{-1} & \lambda \gamma \gamma x - (x - \bar{y})\bar{b}
\end{bmatrix},
\]

where

\[
C = -\lambda^{-1}y + \beta\lambda^{-1}\bar{x} + \lambda\gamma - (x - \bar{y})\bar{b}.
\]

We thus want to make \( x + (\lambda - 1)\beta = 0 \), i.e., \( x = (1 - \lambda)\beta \). For this particular \( x \), we check immediately that the entry \(-\lambda^{-1}x + \beta(\lambda^{-1} - 1) = 0\) as well. In addition, since \( v(1 - \lambda) \geq d \) then \( v(x) \geq c \). We only need to choose \( y \) so that \( v(y) \geq 2c \). But the only constraint on \( y \) is that \( x\bar{x} + y + \bar{y} = 0 \) and hence, we can choose \( y = s + \eta t \) where \( s = \bar{x}/2 \) and \( t \in p_{\tau}^{2c} \) is arbitrary (the latter will guarantee that \( v(y) \geq 2c \)).

**Case 2:** \( d > 2(n-d) \). In this case \( c = 2(n-d) \). Consider the following matrix (in \( H_{\tau} \) with respect to the basis \( B \)):

\[
A = \begin{bmatrix}
1 - \gamma x & \gamma x \\
x & 1 - \bar{x}x
\end{bmatrix}, \quad x = 1 - \lambda/\lambda.
\]

Note that \( x \in O_{E, \tau}^c \) as \( \gamma + \bar{x} = -\beta\bar{x} \in O_{E, \tau}^c \). We check that \( \det(A) = \lambda/\lambda \). Moreover, using \( x + \bar{x} = (\gamma + \bar{x})x\bar{x} \), we obtain that \( \bar{x}J_2A = J_2 \) where \( J_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \). Moreover, \( v(x) \geq c = 2(n-d) \) as \( \lambda/\lambda \in U^1(c) \). We only need to check that \( B = SAS^{-1} \) is of the form given by (27). But one
computes (using $\beta\beta = -\gamma - \gamma$ that

$$SAS^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -\beta x & 1 + \beta\beta x & 0 \\ x & -\beta x & 1 \end{bmatrix},$$

which proves that $A$ stabilizes the pair $(L_{V,\tau}, L_{W,\tau})$. This proves the lemma.

### 3.3 Invariants of Galois orbits

Given $x = (x_{V}, x_{W}) \in \text{Hyp}_\tau = \text{Hyp}_{V} \times \text{Hyp}_{W}$, define $\text{inv}(x)$ to be the pair $(a, b)$ where $a = \text{dist}(x_{V}, \text{pr}_{W}(x_{V}))$ and $b = \text{dist}(\text{pr}_{W}(x_{V}), x_{W})$. The following result that classifies $H_\tau$-orbits of elements in $\text{Hyp}_\tau$:

**Proposition 3.4.** Two points $x, y \in \text{Hyp}_\tau$ lie on the same $H_\tau$-orbit if and only if $\text{inv}(x) = \text{inv}(y)$.

To prove the proposition, we introduce the notion of a *special apartment* for the building $B(V_\tau)$. An apartment $A$ determined by a Witt basis $\{\overline{\gamma}_+, \overline{\gamma}_-, \overline{\gamma}_0\}$ is called *special* if the intersection $A \cap B(W_\tau)$ is a half-line. Let $S_\tau$ be the set of special apartments.

**Lemma 3.5.** The group $H_\tau$ acts transitively on $S_\tau$.

**Proof.** Suppose that $A'$ and $A''$ are two special apartments. Since the half-apartments $A' \cap B(W_\tau)$ and $A'' \cap B(W_\tau)$ yield two distinct points on the boundary of the building $B(W_\tau)$ and since $H_\tau$ acts transitively on the boundary, conjugating one apartment by an element of $H_\tau$, we can assume that $A' \cap B(W_\tau) = A'' \cap B(W_\tau)$ is a half-line. Similarly to the argument in Section 3.2, consider an apartment $A$ of $B(W_\tau)$ with the property that $A$ contains the half-line that is the intersection. Lemma 3.2 then implies that one can choose Witt bases $B' = \{e'_+, e'_0, e'_-\}$ and $B = \{e_+, e_0, e_-\}$ for $A'$ and $A$, respectively, such that the change-of-basis matrix $S'$ from $B$ to $B'$ is of the form

$$S' = \begin{bmatrix} 1 & \beta' & \gamma' \\ 0 & 1 & -\beta' \\ 0 & 0 & 1 \end{bmatrix}, \quad \beta', \gamma' \in \mathcal{O}_{E_\tau}^+, \beta'\beta' + \gamma' + \overline{\gamma'} = 0.$$

Similarly, we get a matrix $S''$ for $A''$, i.e.,

$$S'' = \begin{bmatrix} 1 & \beta'' & \gamma'' \\ 0 & 1 & -\beta'' \\ 0 & 0 & 1 \end{bmatrix}, \quad \beta'', \gamma'' \in \mathcal{O}_{E_\tau}^+, \beta''\beta'' + \gamma'' + \overline{\gamma''} = 0.$$

The chosen Witt bases $B'$ and $B''$ corresponding to the apartments $A'$ to $A''$ do not transform to each other under a matrix in $H_\tau$ as $S'(S'')^{-1} \notin H_\tau$. Yet, since $\beta', \beta'' \in \mathcal{O}_{E_\tau}^+$, we can modify $B'$ to the basis $\overline{B'} = \{ue'_+, e'_0, e'_-\}$ where $u = \beta'(\beta'')^{-1}$. Clearly, the new basis is still a Witt basis for the same apartment $A'$. The change of basis matrix from $B$ to $\overline{B'}$ is

$$\overline{S}' = \begin{bmatrix} u & \beta' & \gamma' u^{-1} \\ 0 & 1 & -\beta' u^{-1} \\ 0 & 0 & u^{-1} \end{bmatrix},$$

and

$$\overline{S}'(S'')^{-1} = \begin{bmatrix} u & \beta' & \gamma' u^{-1} \\ 0 & 1 & -\beta' u^{-1} \\ 0 & 0 & u^{-1} \end{bmatrix} \cdot \begin{bmatrix} 1 & -\beta'' & \gamma'' \\ 0 & 1 & \beta'' \\ 0 & 0 & u^{-1} \end{bmatrix} = \begin{bmatrix} u & 0 & u\gamma'' \\ 0 & 1 & 0 \\ 0 & 0 & u^{-1} \end{bmatrix} \in H_\tau,$$

which proves the lemma. □
Figure 3: The transitive action of $H_{\tau}$ reduces to the case where $A'$ and $A''$ share the same intersection with $B(W_{\tau})$. We extend this half-line to an apartment $A$ of the sub-building $B(W_{\tau})$.

Proof of Proposition 3.4. Choose a special apartment $A_x$ containing $x_{V_{\tau}}, \text{pr}_{W_{\tau}}(x_{V_{\tau}})$ and $x_{W_{\tau}}$. Such an apartment exists thanks to Lemma 3.1 and the relative position of the buildings $B(V_{\tau})$ and $B(W_{\tau})$ (just choose a line that goes through the three points and intersects $B(W_{\tau})$ in a half-line). Similarly, choose $A_y$ containing $y_{V_{\tau}}, \text{pr}_{W_{\tau}}(y_{V_{\tau}})$ and $y_{W_{\tau}}$. By Lemma 3.5 there exists an element $h \in H_{\tau}$ such that $hA_x = A_y$. Since $h$ preserves distances, it follows that $h$ transforms $x \in \text{Inv}_{\tau}$ to $y \in \text{Inv}_{\tau}$. \hfill \Box

4. Computing the Hecke Polynomial

As before, let $\tau$ be a finite place of $F$ that is inert in $E$ and such that both open compact subgroups $K_{V,\tau}$ and $K_{W,\tau}$ are hyperspecial maximal compact subgroups. By abuse of notation, the unique place of $E$ above $\tau$ is also denoted by $\tau$. Let $k_0 = F_{\tau}$, let $k = E_{\tau}$ and let $\varpi_0$ and $\varpi$ be uniformizers for $k_0$ and $k$, respectively. Let $q$ be the size of the residue field of $k_0$. Some of the computations in this section follow the approach of [Kos13a]. For this section only (as we are working purely locally), we simplify the notation by letting $G_V = G_V(k_0)$, $G_W = G_W(k_0)$, $G = G(k_0)$, $T_V = T_V(k_0)$, $T_W = T_W(k_0)$ and $T = T(k_0)$. In addition, the hyperspecial maximal compact subgroups $K_{V,\tau}$ and $K_{W,\tau}$ are denoted by $K_V \subset G_V$ and $K_W \subset G_W$. We also set $K = K_V \times K_W$. Let $\mathcal{H} = \mathcal{H}(G, K)$ be the local Hecke algebra. Let $\delta_V = \text{diag}(\varpi, 1, \varpi^{-1})$ and $\delta_W = \text{diag}(\varpi, \varpi^{-1})$. Given a co-character $\mu$ of $\hat{G}$, Blasius and Rogawski [BR94, §6] define a polynomial $H_{\tau}(z)$ with coefficients in $\mathcal{H}$ and conjecture that it vanishes on the geometric Frobenius acting on the $\ell$-adic étale cohomology of the corresponding Shimura variety (providing an analogue of the classical Eichler–Shimura relation).

We now compute the polynomial for the Shimura variety $\text{Sh}_K(G, X)$. Given an element $g \in G$, let $1_{KgK}$ be the characteristic function of the double coset $KgK$ viewed as an element of the local Hecke algebra $\mathcal{H}$.

Theorem 4.1. The Hecke polynomial $H_{\tau}(z) \in \mathcal{H}[z]$ at the place $\tau$ for the Shimura datum $(G, X)$ defined in Section 2.2 is given by

$$H_{\tau}(z) = H^{(2)}(z)H^{(4)}(z),$$

(29)
where
\[ H^{(2)}(z) = z^2 - q^2(1_K(1, \delta_w)K - (q - 1)1_K)z + q^6 \in \mathcal{H}[z], \]
and
\[ H^{(4)}(z) = z^4 + d_1 z^3 + d_2 z^2 + d_3 z + d_4 \in \mathcal{H}[z]. \]

Here,
\[ d_1 = -1_{K(\delta_v, \delta_w)K} + (q - 1)(1_{K(1, \delta_v)K} + (q - 1)1_{K(1, \delta_w)K}) - (q - 1)^2, \]
\[ d_2 = q^2 1_{K(\delta_v^2, 1)K} + q^4 1_{K(1, \delta_v)K} - 2q^2(q - 1)1_{K(\delta_v, 1)K} - 2q^4(q - 1)1_{K(1, \delta_w)K} - q^2(q + 1)(q - 1)^2, \]
\[ d_3 = q^6(-1_{K(\delta_v, \delta_w)K} + (q - 1)(1_{K(1, \delta_v)K} + 1_{K(1, \delta_w)K}) - (q - 1)^2), \]
\[ d_4 = q^{12}. \]

4.1 Unramified Local Langlands Correspondence

We state the conjecture for \( G_V \) (it is similar for \( G_W \)).

4.1.1 Unramified local parameters. The action of the Weil group \( W_{k_0} \) on \( \hat{G}_V \) is explained in [BR94 §1.6] and in our case, factors through the projection \( W_{k_0} \to \text{Gal}(k/k_0) \). Let \( L^{G_V} = \hat{G}_V \rtimes W_{k_0} \) be the \( L \)-group of \( G_V \). Let \( \Phi \in W_{k_0} \) be the Frobenius automorphism and let \( \nu: W_{k_0} \to \mathbb{Z} \) be the map that sends an element \( w \in W_{k_0} \) to the unique exponent \( n \) such that \( w \) induces the automorphism \( \Phi^n \) when restricted to the residue field of \( k_0 \). We then have an exact sequence
\[ 0 \to I \to W_{k_0} \overset{\nu}{\to} \mathbb{Z} \to 0, \]
where \( I \subset W_{k_0} \) is the inertia group. Recall [BR94 §1.10] that a local parameter is a homomorphism
\[ \phi: W_{k_0} \times \text{SU}_2(\mathbb{R}) \to L^{G_V}, \tag{30} \]
such that the composition of \( \phi \) with the projection to \( L^{G} \to W_{k_0} \) is the identity and \( \phi(w) \) is semisimple for all \( w \in W_{k_0} \). Two parameters \( \phi_1 \) and \( \phi_2 \) are equivalent if they are conjugated by an element \( g \in \hat{G} \).

To introduce unramified local parameters, note that since \( G_V \) is unramified (i.e., \( G_V \) is quasi-split over \( k_0 \) and splits over the unramified extension \( k \)), the action of \( W_{k_0} \) on \( \hat{G}_V \) factors through the map \( W_{k_0} \overset{\nu}{\to} \mathbb{Z} \) (equivalently, the inertia group acts trivially), i.e., \( \hat{G}_V \rtimes \mathbb{Z} \) is defined and we have a map \( \hat{G}_V \times W_{k_0} \to \hat{G}_V \rtimes \mathbb{Z} \). A local parameter \( \phi \) is unramified if the following two properties are satisfied:

i) \( \phi \) is trivial on \( \text{SU}_2(\mathbb{R}) \),

ii) The composition \( W_{k_0} \overset{\phi}{\to} \hat{G}_V \rtimes W_{k_0} \to \hat{G}_V \rtimes \mathbb{Z} \) factors through \( \nu: W_{k_0} \to \mathbb{Z} \) (i.e., the inertia group is in the kernel of the composition).

Let \( \Phi_{ur}(G_V) \) be the set of equivalence classes of unramified local \( L \)-parameters. Since an unramified local parameter \( \phi \) is uniquely determined by the semi-simple element \( \phi(\Phi) = g \rtimes \Phi \) then the set \( \Phi_{ur}(G_V) \) of equivalence classes of unramified local parameters is in bijection with \( \hat{G}_V \)-orbits of semisimple elements \( g \rtimes \Phi \in L^{G_V} \). As we will see, the latter are easier to describe for the maximal torus \( T_V \).

4.1.2 Unramified representations and unramified local parameters. Let \( K_V \subset G_V \) be a fixed hyperspecial maximal compact subgroup. An irreducible and admissible representation \( \pi \) of \( G_V \) is called unramified if \( \pi^{K_V} \neq 0 \). Let \( \Pi_{ur}(G_V) \) be the set of isomorphism classes of unramified representations of \( G_V \). Following [BR94 Prop.1.12.1], there is a natural bijection between \( \Phi_{ur}(G_V) \) and \( \Pi_{ur}(G_V) \).
that we now explain. First, it follows from [BR94, p.535] that there are canonical isomorphisms $\Phi_{ur}(G_V) \cong \Phi_{ur}(T_V)/\Omega(T_V)$ and $\Pi_{ur}(G_V) \cong \Pi_{ur}(T_V)/\Omega(T_V)$ where $\Omega(T_V) = N_{G_V}(T_V)/T_V$ is the Weyl group. This reduces the problem of relating unramified representations to unramified local parameters from $G_V$ to the maximal torus $T_V$. Let $S_V \subset T_V$ be the the maximal split (over $k_0$) subtorus of $T_V$. It is proved in [BR94] p.534] that

$$\Pi_{ur}(T_V) \simeq \hat{S}_V \simeq \Phi_{ur}(T_V). \tag{31}$$

4.1.3 Satake parameters. In the case of $G_V \times G_W$, the maximal split torus $S = S_V \times S_W$ has dimension 2 since the maximal split tori $S_V$ and $S_W$ of $G_V$ and $G_W$, respectively, are both 1-dimensional. If $\{\alpha, \beta\}$ is the basis for $X_*(S)$ consisting of the cocharacters $\alpha(\varpi_0) \to \text{diag}(\varpi_0, 1, \varpi_0^{-1})$ and $\beta(\varpi_0) = \text{diag}(\varpi_0, \varpi_0^{-1})$ then we can identify $\hat{S} \cong \text{Hom}(X_*(S), C^\times) \cong (C^\times)^2$. Indeed, let

$$t_{a,b} = (\text{diag}(\varpi_0^a, 1, \varpi_0^{-a}), \text{diag}(\varpi_0^b, \varpi_0^{-b})).$$

Let $s : X_*(S) \to C^\times$ be a homomorphism and let $(u, v)$ be the images of $(\alpha, \beta)$ in $(C^\times)^2$. If $\pi(s)$ is the unramified representation corresponding to $s$ under [31] (we apply for both $V$ and $W$ and write it on the product group) then $\pi(s)(t_{a,b}) = u^a v^b$ determines completely the representation $\pi(s)$. Here, the complex numbers $(u, v) \in (C^\times)^2$ are known as the Satake parameters of $\pi(s)$.

4.2 Computing the Hecke Polynomial

We now recall the definition of the polynomial $H_r(z)$ that appears in the Blasius–Rogawski congruence relation (Theorem 12] and compute it in our setting. More precisely, we show the first part of the computation in the more general case when $\dim V = n$ and $\dim W = n - 1$ and then specialize to the case $n = 3$ in the final part.

4.2.4 Hecke polynomials and the congruence relation. Let $r : \hat{G} \to \text{GL}(V)$ be the complex representation of $\hat{G}$ of highest weight the cocharacter $\mu$ of the Shimura datum $(G, X)$. Associated to the fixed finite place $\tau$ is a polynomial (Hecke polynomial) defined by Blasius and Rogawski [BR94, §6] as follows:

$$H_r(z) = \det \left( z - q^{\dim X_r} \gamma(g^\sigma g) \right) \in \mathcal{H}[z]. \tag{32}$$

Here, the Hecke algebra $\mathcal{H} = \mathcal{H}(G, K)$ is identified with the functions on $\hat{G}$ invariant under $\sigma$-conjugation, i.e., the automorphism of $\hat{G}$ given by $y \mapsto g^\sigma y g^{-1}$. When restricted to the maximal torus $\hat{T}$ of $\hat{G}$, the Satake isomorphism identifies the Hecke algebra with the space of functions on $\hat{T}$ that are invariant under both $\sigma$-conjugation and the Weyl group $\Omega(T)$. The strategy to compute the polynomial is then to restrict to the maximal torus $\hat{T}$ where the above determinant can easily be evaluated and then to invert the Satake isomorphism.

4.2.5 The representation $r : \hat{G} \to \text{GL}_{n(n-1)}(C)$. Here, we make explicit the computation of the Hecke polynomial $H_r(z)$. The Hermitian symmetric domain $X$ for the Shimura datum $(G, X)$ has dimension $\dim X = 2n - 3$. The associated co-character $\widehat{\mu}_h$ of $\hat{G}$ can be determined as follows: the Hermitian symmetric domain $X_V$ is the conjugacy class of the embedding $h_V : S \to G_{V,R}$ given by $(z, \bar{z}) \mapsto \text{diag}(1, \ldots, 1, \bar{z}/z)$. The complexification $h_{V,C} : S_C \to G_{V,C}$ is given by $(z_1, z_2) \mapsto \text{diag}(1, \ldots, 1, z_2/z_1)$, i.e., the associated co-character $\mu_V$ of the Shimura datum $(G_V, X_V)$ is $\lambda \mapsto (1, \ldots, 1, \lambda^{-1})$ which corresponds to the character $-\chi_n$ of the dual group $\text{GL}_n(C)$. The representation of $r_{V} : \text{GL}_n(C) \to \text{GL}_n(C)$ of highest weight $-\chi_n$ is precisely the dual of the standard representation, namely, $A_V \mapsto {}^tA_V^{-1}$ for $A_V \in \text{GL}_n(C)$. Similarly, the associated co-character $\mu_W$ of $(G_W, X_W)$ is the character $-\lambda_{n-1}$ of $\text{GL}_{n-1}(C)$, so the representation $r_W$ is the representation $A_W \mapsto {}^tA_W^{-1}$ for $A_W \in \text{GL}_{n-1}(C)$. The representation $r : \hat{G} \to \text{GL}_{n(n-1)}(C)$ associated to $(G, X)$
is then an \( n(n-1) \)-dimensional representation that is the tensor product of the two representations \( r_V \) and \( r_W \) of \( \hat{G}_V \) and \( \hat{G}_W \), respectively, i.e., it is the representation \( r : \hat{G} \to \text{GL}(V \otimes W) \) given by \( (A_V, A_W) \mapsto {}^t A_V^{-1} \otimes {}^t A_W^{-1} \).

4.2.6 Galois action on \( \hat{G} \). The action of \( \text{Gal}(k/k_0) \) on \( \hat{G} \) can be calculated following [BR94] §1.6. Indeed, let \( (B, T) \) be the Borel pair and consider the standard splitting for \( G \), namely:

- \( \tilde{B} = \tilde{B}_V \times \tilde{B}_W \) is the product of the upper-triangular Borel subgroups,
- \( \tilde{T} = \tilde{T}_V \times \tilde{T}_W \) is the product of the diagonal tori,
- \( \{X_\alpha\} \) is the set of matrices \( (a_{ij})_{i,j=1}^n \) where \( a_{ij} = \delta_{ik}\delta_{k+1,j} \) for \( k = 1, \ldots, n-1 \),
- \( \{Y_\beta\} \) is the set of matrices \( (b_{ij})_{i,j=1}^{n-1} \) where \( b_{ij} = \delta_{ik}\delta_{k+1,j} \) for \( k = 1, \ldots, n-2 \).

According to [BR94] §1.8(b), if

\[
J_n = \begin{pmatrix}
-1 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & (-1)^{n-1}
\end{pmatrix}
\]

then the automorphism \( A_V \mapsto J_n {}^t A_V^{-1} J_n \) is the unique non-inner automorphism of \( G_V = \text{GL}_3(C) \) that fixes the standard splitting \( (B_V, T_V, \{X_\alpha\}) \). Similarly, \( A_W \mapsto J_{n-1} {}^t A_W^{-1} J_{n-1} \) is the unique non-inner automorphism of \( G_W = \text{GL}_2(C) \) that fixes the standard splitting \( (B_W, T_W, \{Y_\beta\}) \). One calls the data \( (B, T, \{(X_\alpha, Y_\beta)\}) \) a splitting because it splits the exact sequence

\[1 \to \text{G}^{ad} \to \text{Aut}(G) \to \text{Aut}(\hat{G}) \to 1.\]

We can thus write the Galois action on \( \hat{G} \) as \( \sigma(A_V, A_W) = (J_n {}^t A_V^{-1} J_n, J_{n-1} {}^t A_W^{-1} J_{n-1}) \). The Hecke polynomial for the Shimura datum \( (G, X) \) is

\[
H_r(z) = \det(z - q^{2n-3}r(g^\sigma g)) = \det(z - q^{2n-3}r((A_V, A_W) \cdot \sigma(A_V, A_W))),
\]

where \( g = (A_V, A_W) \in \hat{G} \) with \( A_V \in \text{GL}_n(C) \) and \( A_W \in \text{GL}_{n-1}(C) \). Let

\[
B := r((A_V, A_W) \cdot \sigma(A_V, A_W)) \in \text{GL}_{n(n-1)}(C).
\]

Then (see [Gro98] p.12)

\[
H_r(z) = \sum_{i=0}^{n(n-1)} (-1)^i \text{Tr}
\left( \begin{pmatrix} i \\ B \end{pmatrix} \right) z^{n(n-1)-i}.
\]

(34)

where \( A = A_V \otimes A_W \) for \( (A_V, A_W) \in \hat{G}(C) = \hat{G}_V(C) \times \hat{G}_W(C) \). Here, the coefficients of the polynomial are viewed as functions on \( \hat{G} \). Restricted to the dual torus \( \hat{T} \), let

\[
A_V = \text{diag}(x_1, \ldots, x_n) \quad \text{and} \quad A_W = \text{diag}(y_1, \ldots, y_{n-1}).
\]

Then

\[
(A_V, A_W) \cdot \sigma(A_V, A_W) = \left( \text{diag} \left( \begin{pmatrix} x_1 \\ x_n \end{pmatrix}, \ldots, \begin{pmatrix} x_n \\ x_1 \end{pmatrix} \right), \text{diag} \left( \begin{pmatrix} y_1 \\ y_{n-1} \end{pmatrix}, \ldots, \begin{pmatrix} y_{n-1} \\ y_1 \end{pmatrix} \right) \right).
\]

In this case, (33) turns into

\[
H_r(z) = \prod_{i=1}^n \prod_{j=1}^{n-1} \left( z - q^{2n-3} \frac{x_{n+1-i} y_{n-j}}{x_i y_j} \right).
\]

(35)
This polynomial is invariant under \(\sigma\)-conjugation as well as under the Weyl group \(\Omega(T)\). When \(n = 3\), we rewrite (35) as

\[
H_\tau(z) = \left(z - q^3\frac{y_2}{y_1}\right) \left(z^2 - q^3\left(\frac{x_3}{x_1} + \frac{x_1}{x_3}\right)\frac{y_2}{y_1}z + q^6\left(\frac{y_2}{y_1}\right)^2\right) \times \left(z - q^3\frac{y_1}{y_2}\right) \left(z^2 - q^3\left(\frac{x_3}{x_1} + \frac{x_1}{x_3}\right)\frac{y_1}{y_2}z + q^6\left(\frac{y_1}{y_2}\right)^2\right).
\]

4.2.7 The coefficients of \(H_\tau(z)\) as elements of \(\mathcal{H}(T,T_c)^{\Omega(T)}\). The above representation yields a factorization into two polynomials (one of degree 2 and one of degree 4) whose coefficients are functions that are invariant under \(\sigma\)-conjugation and under the Weyl group \(\Omega(T)\). Indeed, let \(\delta_V = \text{diag}(\varpi_0, 1, \varpi_0^{-1})\), \(\delta_W = \text{diag}(\varpi_0, \varpi_0^{-1})\) and let

\[
s_{0,1} = 1_{\mathcal{H}(T,T_c)}, \quad s_{1,0} = 1_{\mathcal{H}(T,T_c)} + 1_{\mathcal{H}(T,T_c)} \in \mathcal{H}(T,T_c).
\]

The Hecke polynomial can then be written as follows:

\[
H_\tau(z) = (z^2 - q^3s_{0,1}z + q^6)(z^4 - q^3s_{1,0}z^3 + q^6(s_{0,1}^2 + s_{1,0}^2 - 2)z^2 - q^9s_{1,0}z + q^{12}),
\]

viewed as a polynomial in \(\mathcal{H}(T,T_c)[z]\). We now need to obtain the polynomial with coefficients in the original Hecke algebra \(\mathcal{H}(G,K)\) by inverting the Satake transform.

4.3 The Satake Isomorphism
Satake [Sat63] showed that there is an isomorphism \(\mathcal{H}_Q \cong \mathcal{H}_Q(T,T_c)^{\Omega(T)}\) where \(\mathcal{H}_Q = \mathcal{H} \otimes \mathbb{Z} Q\) and \(\mathcal{H}_Q(T,T_c) = \mathcal{H}(T,T_c) \otimes \mathbb{Z} Q\). \(T_c = T \cap K\) and \(\Omega(T) = N_G(T)/T\) is the Weyl group. The isomorphism is defined via the following commutative diagram:

\[
\begin{array}{ccc}
\mathcal{H}(G,K) & \xrightarrow{\sim} & \mathcal{H}(T,T_c) \\
\mathcal{H}(B,L) & \xrightarrow{\delta^{1/2}} & \mathcal{H}(T,T_c).
\end{array}
\]

Here, \(L = B \cap K\) and \(|B| : \mathcal{H}(G,K) \to \mathcal{H}(B,L)\) is the restriction of functions, the map \(S : \mathcal{H}(B,L) \to \mathcal{H}(T,T_c)\) is defined by \(S(1_{gL}) = [L \cap gLg^{-1}]1_{gT}\) for \(g \in T\), i.e., it is obtained by taking quotients by the unipotent radical, \(\delta\) is the sum of the simple positive roots (in other words, the character \(\delta\) of \(T\) is obtained by looking at the action of \(T\) on the Lie algebra of the unipotent radical \(U(B)\), and \(|\cdot|\) is normalized so that \(|\pi| = q^{-2}\). In fact, the above diagram gives an algebra homomorphism \(S \circ |B| : \mathcal{H}(G,K) \to \mathcal{H}(T,T_c)\) (the one inducing the usual Satake isomorphism), and a twisted version \(|\delta|^{1/2} \circ S \circ |B| : \mathcal{H}(G,K) \to \mathcal{H}(T,T_c)\) that we also denote by \(\sim\). It is explained in Wedhorn [Wed00 Prop.1.9] that one has the following commutative diagram:

\[
\begin{array}{ccc}
\mathbb{C}[\Pi_{ur}(G)] & \xrightarrow{\sim} & \mathbb{C}[\Pi_{ur}(T)]^{\Omega(T)} \\
\mathbb{C}[\Pi_{ur}(B)] & \xrightarrow{\sim} & \mathbb{C}[\Pi_{ur}(T)]^{\Omega(T)}.
\end{array}
\]

4.4 Inverting the Satake transform using buildings.
Let \(\text{Hyp}_V\) (resp., \(\text{Hyp}_W\)) denote the set of hyperspecial vertices on the building \(B(G_V)\) (resp., \(B(G_W)\)) and let \(\text{Hyp} = \text{Hyp}_V \times \text{Hyp}_W\). Fix a Witt basis for \(V\) and let \(\mathcal{A}_V\) be the correspond-
ing apartment. A choice of a fundamental chamber \( C \) of \( \mathcal{A}_V \) gives a canonical retraction map 
\( \rho_{\mathcal{A}_V, C} : B(G_V) \to \mathcal{A}_V \) [Gar97, p.53]. Let \( x_0 = (x_{0,V}, x_{0,W}) \) be a pair of hyperspecial vertices whose stabilizer in \( G \) is \( K \). If \( t = (t_V, t_W) \in T \) then the Hecke operator \( 1_{K(t_V,A_W)K} \) acts on \( x_0 \in \text{Hyp} \) as an adjacency operator:

\[
1_{K(t_V,A_W)K}(x_0) = \sum_{(x_V,x_W) \in B(G_V) \times B(G_W) \atop \text{dist}(x_V,x_{0,V}) = d_V \atop \text{dist}(x_W,x_{0,W}) = d_W} (x_V, x_W),
\]

i.e., it is the formal sum of pairs of points on the \( K \)-orbit of \((t_V x_{0,V}, t_W x_{0,W})\). Here, \( d_V \) and \( d_W \) are defined by 
\( q^{d_V} = [K_V : t_V K_V t_V^{-1} \cap K_V] \), 
\( q^{d_W} = [K_W : t_W K_W t_W^{-1} \cap K_W] \), and \( \delta_V \) and \( \delta_W \) are the distance functions on \( B(G_V) \) and \( B(G_W) \), respectively. We will then prove the following result using an idea of Cornut and Koskivirta [Kos13a]:

**Proposition 4.2.** The map \( S1_{K(t_V,A_W)K} \) can be re-written using the canonical retraction map as follows

\[
S1_{K(t_V,A_W)K}(x) = \sum_{(x_V',x_W') \in B(G_V) \times B(G_W) \atop \text{dist}(x_V',x_{0,V}) = d_V \atop \text{dist}(x_W',x_{0,W}) = d_W} (\rho_{\mathcal{A}_V,C_V}(x_V'), \rho_{\mathcal{A}_W,C_W}(x_W')).
\]

The proof uses an auxiliary lemma:

**Lemma 4.3.** Let \( U_V \) be the unipotent radical of \( B_V \). Each \( U_V \)-orbit of \( \text{Hyp}_V \) intersects the apartment \( \mathcal{A}_V \) at a unique point. Similarly, if \( U_W \) is the unipotent radical of \( B_W \) then each \( U_W \)-orbit of \( \text{Hyp}_W \) intersects the apartment \( \mathcal{A}_W \) at a unique point.

**Proof.** It suffices to prove the statement for \( V \) as the case for \( W \) is identical. We have \( \mathcal{A}_V \cap \text{Hyp}_V = T_V x_0 \). Suppose that there are two points \( t_1 x_0 \) and \( t_2 x_0 \) that are in the same \( U_V \)-orbit. Then there is \( u \in U_V \) such that \( t_2 x_0 = ut_1 x_0 \), i.e., \( t_2^{-1}u t_1 \in K \). The latter means (looking only at the diagonal entries for the matrix representation with respect to the Witt basis for \( V_r = W_r \perp D_r \)) that \( t_2^{-1}t_1 \) has entries in \( \mathcal{O}_{E_r} \), i.e., \( t_2^{-1}t_1 \in K \) and hence, \( t_1 x_0 = t_2 x_0 \), since \( K_r = \text{Stab}_{G_r}(x_0) \).

**Remark 10.** There is a more general argument showing the above statement, namely, take a Weyl chamber \( C_V \) contained in the apartment \( \mathcal{A}_V \) and consider its stabilizer (point-wise) \( U_V(0) \subset U_V \) that is a compact open subgroup. Given \( t \in T_V \), the translated chamber \( t C_V \) has a stabilizer \( U_V(t) = t U_V(0) t^{-1} \). Moreover, the union over all \( t \) of \( U_V(t) \) is \( U_V \). Assuming \( t_2 x_{0,V} = ut_1 x_{0,V} \), choose \( t \) such that \( u \in U(t) \). Take any point \( a \in t C_V \) and write

\[
\text{dist}(a, t_2 x_{0,V}) = \text{dist}(ua, ut_1 x_{0,V}) = \text{dist}(a, t_1 x_{0,V}).
\]

Thus, any point \( a \in t C_V \) is equidistant from \( t_1 x_{0,V} \) and \( t_2 x_{0,V} \). But the set of points in \( \mathcal{A}_V \) that are equidistant from \( t_1 x_{0,V} \) and \( t_2 x_{0,V} \) is a hyperplane, hence, the only possibility is if \( t_1 x_{0,V} = t_2 x_{0,V} \).

**Proof.** (Proof of Proposition 4.2) The Hecke algebra \( \mathcal{H}(G,K) \) is isomorphic to \( \mathbb{Q}[K,G/K] \) and the latter is isomorphic to \( \text{End}_{\mathbb{Q}[G]}(G/K) \) (to give a \( \mathbb{Q}[G] \)-equivariant endomorphism \( \varphi \) of \( G/K \), it suffices to specify \( \varphi(K) \) that is \( K \)-invariant). We thus have

\[
\mathcal{H}_q(G,K) \cong \text{End}_{\mathbb{Q}[G]}(G/K) \cong \text{End}_{\mathbb{Q}[G]}(\text{Hyp}).
\]

Moreover, the restriction map \( |_B \) is simply

\[
|_B : \text{End}_{\mathbb{Q}[G]}(\text{Hyp}) \hookrightarrow \text{End}_{\mathbb{Q}[B]}(\text{Hyp}).
\]

On the level of endomorphisms, the Satake transform is the composition

\[
\text{End}_{\mathbb{Q}[G]}(G/K) \hookrightarrow \text{End}_{\mathbb{Q}[B]}(B/L) \hookrightarrow \text{End}_{\mathbb{Q}[T]}(T/T_c).
\]
Since $\text{Hyp} = G \cdot x_0 = BK \cdot x_0 = B \cdot x_0$ and since $B = UT$ where $U$ is the unipotent radical, the last map is induced from the maps $b_y \cdot x_{V,0} \mapsto t_y \cdot x_{V,0}$ (here, $b_y \in B_V$, $b_y = u_y t_y$, $u_y \in U_V$, $t_y \in T_V$) and $b_W \cdot x_{W,0} \mapsto t_W \cdot x_{W,0}$ where $b_W \in B_W$, $b_W = u_W t_W$, $u_W \in U_W$, $t_W \in T_W$. To describe the last map via Bruhat–Tits theory, we now that $U_V$ (resp., $U_W$) fixes pointwise a neighborhood of the point at infinite for a half apartment of $A_V$ (resp., $A_W$). Equivalently, we say that $U_V$ is the stabilizer of a boundary point on the tree (corresponding to a half-line of $A_V$). This means that if $y_V \in A_V$ (resp., $y_W \in A_W$) is a point that is very far and that it is fixed by $U_V$ (resp., $U_W$), we have
\[ \text{dist}(y_V, b_y x_{V,0}) = \text{dist}(u_y y_V, u_y t_y x_{V,0}) = \text{dist}(y_V, t_y x_{V,0}). \]
That means that the image of $b_y x_{V,0}$ under the last map is precisely $\rho_{A_V,C_V}(b_y x_{V,0})$ and the same for $x_{W,0}$. This proves the proposition.

The latter can now be computed explicitly by counting how many points $(x'_V, x'_W) \in B(G_V) \times B(G_W)$ retract to a given point $(y_V, y_W) \in A_V \times A_W$. Below we have shown the building $B(G_V)$ together with the apartment $A_V$:

For brevity, if $a, b \in \mathbb{Z}$, set $t_{a,b} = 1_K(\text{diag}(\omega_0^a, 1, \omega_0^{-a}), \text{diag}(\omega_0^b, \omega_0^{-b}))K$.

Computing $\tilde{t}_{1,0}$. In this case, (38) shows that $S(t_{1,0})(x_0)$ is a sum of points $x_{-1}, x_0$ and $x_1$. To figure out the multiplicities, we need to figure out the number of points on the sphere $S_2(x_{0,V}) = \{x \in B(G_V) : \text{dist}(x, x_{0,V}) = 2\}$ that retract to $x_{-1}, x_0$ and $x_1$, respectively.

i) $x_1$ occurs with multiplicity 1,
ii) $x_0$ occurs with multiplicity $q + 1 - 2$ as the vertices that retract to $x_0$ are precisely the neighbors of $x_{1/2}$ that are different from $x_0$ and $x_1$,
iii) $x_{-1}$ occurs with multiplicity $1 + (q - 1) + (q^3 - 1)q = q^4$.

Thus,
\[ |_B \circ S(t_{1,0}) = 1_{(\delta_{V,1})T_c} + q^4 1_{(\delta_{V,1})T_c} + (q - 1)1_{T_c}. \]

The twisted Satake transform is then
\[ \tilde{t}_{1,0} = q^2(1_{(\delta_{V,1})T_c} + 1_{(\delta_{V,1})T_c}) + (q - 1) = q^2 s_{1,0} + (q - 1). \]

Computing $\tilde{t}_{0,1}$. Similarly, we have that $S(t_{0,1})(x_0)$ is a sum of points $x_{-1}, x_0, x_1$ with multiplicities given by the number of points on the sphere $S_2(x_{0,W}) = \{y \in B(G_W) : \text{dist}(y, x_{0,W}) = 2\}$ retracting to $x_{-1}, x_0$ and $x_1$, respectively. We have

i) $x_1$ with multiplicity 1,
ii) $x_0$ with multiplicity $q + 1 - 2$ (all the neighbors of $x_{1/2}$ lie on $B(G_W)$),
iii) $x_{-1}$ with multiplicity $1 + (q - 1) + (q - 1)q = q^2$. 

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Thus,
\[ |B \circ S(t_{0,1}) = 1_{(1,\delta_W)T_c} + q^2 1_{(1,\delta_W)T_{c}} + (q - 1)1_{T_{c}}. \]
and hence,
\[ \tilde{t}_{0,1} = q \left( 1_{(1,\delta_W)T_c} + 1_{(1,\delta_W^{-1})T_{c}} \right) + (q - 1) = q s_{0,1} + (q - 1). \] (40)

4.4.8 Final form of the Hecke polynomial. Finally, we compute the inverse image of the polynomial \( \tilde{t}_{0,1} \) under the Satake transform using only the identities (40) and (39) to obtain
\[ H^{(2)}(z) = z^2 - q^2(t_{0,1} - (q - 1))z + q^6. \] (41)
and
\[ H^{(4)}(z) = z^4 + (-t_{1,0}t_{0,1} + (q - 1)t_{1,0} + (q - 1)t_{0,1} - (q - 1)^2)z^3 + q^2(t_{1,0}^2 + q^2 t_{0,1}^2 - 2(q - 1)t_{1,0} + 2q^2(q - 1)t_{0,1} - q^4 - 2q^3 + 2q^2 - 2q + 1)z^2 + q^6(t_{1,0}t_{0,1} + (q - 1)t_{1,0} + (q - 1)t_{0,1} - (q - 1)^2)z + q^{12}. \] (42)
which completes the proof of Theorem 4.1.

5. The Congruence Relation of Blasius–Rogawski

We deduce Theorem 1.2 from recent results of J.-S. Koskivirta [Kos13b] building on the work of Büttel and Wedhorn [BW06] establishing the conjecture for unitary Shimura varieties of PEL type. For that, we use the auxiliary PEL Shimura data \((\tilde{G}_V, \tilde{X}_V)\) and \((\tilde{G}_V, X'_V)\).

5.1 Relation between the two Hecke polynomials

We first relate the Hecke polynomials \( H_{V,\tau}(z) \) to \( \tilde{H}_{V,\tau}(z) \) for the Shimura data \((\tilde{G}_V, X'_V)\) and \((\tilde{G}, \tilde{X}_V)\), respectively. We then use that the conjecture is already known for the PEL data \((\tilde{G}_V, \tilde{X}_V)\) and \((\tilde{G}_W, \tilde{X}_W)\) to deduce it for \((\tilde{G}_V, X'_V)\) and \((\tilde{G}_W, X'_W)\), and hence, for \((G_V, \tilde{X}_V)\) and \((G_W, X_W)\).

5.1.1 The action of \( \text{Gal}(k/k_0) \). Given any element \((A, x)\) of the dual group \( \text{GL}_n(\mathbb{C}) \times \mathbb{C}^\times \) of \( \tilde{G}_V \), the action of the non-trivial automorphism \( \sigma \in \text{Gal}(k/k_0) \) is given by
\[ \sigma(A, x) = (J_n^t A^{-1} J_n, \det(A)x). \] (45)
Restricted to the maximal torus \( \tilde{T} \), the action of \( \sigma \) is
\[ \sigma(\text{diag}(x_1, \ldots, x_n), x) = (\text{diag}(x_n^{-1}, \ldots, x_1^{-1}), x_1 \ldots x_nx) \in \text{GL}_n(\mathbb{C}) \times \mathbb{G}_m. \] (46)

5.1.2 Computing \( H_{V,\tau}(z) \) and \( \tilde{H}_{V,\tau}(z) \). Recall that the \( \tilde{G}_V(\mathbb{R}) \)-conjugacy class of embeddings defining \( X'_V \) is
\[ z \mapsto (\text{diag}(1, \ldots, 1, \overline{z}/z), 1), \]
with an associated cocharacter
\[ \mu_V : S_{\mathbb{R}} \rightarrow \tilde{G}_V(\mathbb{R}), \quad \mu_V(\lambda) = (\text{diag}(1, \ldots, 1, \lambda^{-1}), 1). \]
The \( \tilde{G}_V(\mathbb{R}) \)-conjugacy class of embeddings \( \tilde{h}_V : S_{\mathbb{R}} \rightarrow \tilde{G}_V(\mathbb{R}) \) defining \( \tilde{X}_V \) is
\[ z \mapsto (\text{diag}(z, z, \ldots, z, \overline{z}), z), \]
with an associated cocharacter
\[ \tilde{\mu}_V : S_{\mathbb{R}} \rightarrow \tilde{G}_V(\mathbb{R}), \quad \tilde{\mu}_V(\lambda) = \text{diag}(\lambda, \lambda, \ldots, \lambda, 1), \lambda). \]
As before, the representation of the dual group of $\tilde{G}_V$ corresponding to $\mu_V$ is given by $(A_V, x_0) \mapsto t A_V^{-1}$ and we calculate the Hecke polynomial for the Shimura datum $(\tilde{G}_V, X'_V)$ as

$$H_{V, \tau}(z) = \prod_{i=1}^{n} \left( z - q^n \frac{x_{n+1-i}}{x_i} \right).$$

(47)

The representation of the dual group of $\tilde{G}_V$ of highest weight $\bar{\mu}_V$ is given by $^{7}$

$$(A_V, x) \mapsto x \det(A_V)^t A_V^{-1}. $$

Now, let $A_V = \text{diag}(x_1, x_2, \ldots, x_n)$. We calculate

$$(A_V, x)^\sigma(A_V, x) = \left( \text{diag} \left( \frac{x_1}{x_n}, \ldots, \frac{x_n}{x_1} \right), x^2 \prod_{i=1}^{n} x_i \right).$$

Similarly to the computation of $H_{V, \tau}(z)$, when restricting the coefficients to the maximal torus, the polynomial is

$$\tilde{H}_{V, \tau}(z) = \prod_{i=1}^{n} \left( z - q^n x^2 \det(A_V) \frac{x_{n+1-i}}{x_i} \right).$$

(48)

5.1.3 Relating $H_{V, \tau}(z)$ and $\tilde{H}_{V, \tau}(z)$. We now relate the two Hecke polynomials after inverting the Satake transform following the approach in the previous section (based on Bruhat–Tits theory).

**Lemma 5.1.** The Hecke polynomials are related as follows:

$$\tilde{H}_{V, \tau} \left( \frac{1}{\varpi K_{V, \tau} z} \right) = 1_{\varpi^n \hat{K}_{V, \tau}} H_{V, \tau}(z).$$

(49)

**Proof.** The two product terms in (47) and (48) differ by multiplication by the function $(A_V, x) \mapsto x^2 \det(A_V)$ for $A_V \in \text{GL}_n(\mathbb{C})$ and $x \in \mathbb{C}^\times$. By [Kos13b, Lem.12], this function (viewed as a function on the maximal torus) is a function that is stable under $\sigma$-conjugation and under the Weyl group and hence, it corresponds to an element of $\mathcal{H}_C(G_{V, \tau}, K_{V, \tau})$ that is precisely $1_{\varpi^n \hat{K}_{V, \tau}}$ (again, by [Kos13b, Lem.12]).

**5.2 Proof of the conjecture for $(G_V, X_V)$**

The following argument is due to J.-S. Koskivirta: assume that $\tilde{H}_{V, \tau}(\Phi) = 0$. We first want to show that $H_{V, \tau}(\Phi) = 0$, i.e., to deduce the conjecture for the datum $(\tilde{G}_V, X'_V)$. For that, we apply Lemma 2.2 to $\sigma = \Phi$. Recall that $\Phi$ is the geometric Frobenius and hence $\Phi = \text{Art}_E(\iota_\tau(\varpi))$ where $\iota_\tau: E_\tau \to A_E$ is the natural inclusion. We thus apply Lemma 2.2 for $\sigma = \Phi$ and $s = \iota_\tau(\varpi)$ to get

$$\Phi = 1_{\varpi \hat{K}_{V, \tau}} \circ f \circ \Phi \circ f^{-1} = f \circ 1_{\varpi \hat{K}_{V, \tau}} \circ \Phi \circ f^{-1}.$$  

Here, we have used that the operators $1_{\varpi \hat{K}_{V, \tau}}$ and $\langle \varpi \rangle$ coincide on the level of the corresponding Shimura varieties and that $f$ and $1_{\varpi \hat{K}_{V, \tau}}$ commute. We can thus write

$$H_{V, \tau}(\Phi) = H_{V, \tau}(f \circ 1_{\varpi \hat{K}_{V, \tau}} \circ \Phi \circ f^{-1}) = f \circ H_{V, \tau}(1_{\varpi \hat{K}_{V, \tau}} \circ \Phi) \circ f^{-1} = f \circ 1_{\varpi^n \hat{K}_{V, \tau}} \circ \tilde{H}_{V, \tau}(\Phi) \circ f^{-1} = 0,$$

---

7If $(\chi_1, \ldots, \chi_n)$ are the standard characters of $\text{GL}_n(\mathbb{C})$ and if $\chi_0$ is the character of $\text{G}_m$, then the above character is $\chi_1 + \cdots + \chi_{n-1} + \chi_0 = \det -\chi_n + \chi_0$. The dual of the standard representation of $\text{GL}_n(\mathbb{C}) \times \text{G}_m$, i.e., $(A, x) \mapsto t A^{-1}$ has highest weight $-\chi_n$, so the representation corresponding to the above character is $(A, x) \mapsto x \det(A)^t A^{-1}$.

8We follow the usual normalization where arithmetic Frobenii correspond to inverses of uniformizers under the Artin map.
where we have used Lemma 5.1 and the assumption that $\tilde{H}_{V,r}(\Phi) = 0$. Finally, to deduce the conjecture for $\text{Sh}_{K_V}(G_V, X_V)$ from that for $\text{Sh}_{\overline{K}_V}(\overline{G}_V, X'_V)$, we observe that the Shimura variety $\text{Sh}_{K_V}(G_V, X_V)$ embeds $E$-rationally into $\text{Sh}_{\overline{K}_V}(\overline{G}_V, X'_V)$ and hence, the cohomology is a subspace stable by both the Galois and Hecke actions.

5.3 Deducing the conjecture for $(G, X)$

We deduce the congruence relation in the PEL case for the product of

$$\text{Sh}_{K_V}(G_V, X_V) \times \text{Sh}_{K_W}(G_W, X_W)$$

from the conjecture for the individual factors. This can be done as follows: let $\alpha_1, \alpha_2, \alpha_3$ be the roots of $H_{V,r}(z)$ and let $\beta_1, \beta_2$ be the roots of $H_{W,r}(z)$. Then

$$H_{r}(z) = \prod_{i=1}^{3} \prod_{j=1}^{2} (z - \alpha_i\beta_j).$$

The congruence relation for $(G, X)$ is then a trivial consequence of the following:

**Lemma 5.2.** Let $G$ be a group and let $V_1$ and $V_2$ be finite-dimensional representations of $G$ over a field $k$. Let $\overline{k}$ be a fixed algebraic closure of $k$. Let $\varphi_1 : V_1 \to V_1$ and $\varphi_2 : V_2 \to V_2$ be two endomorphisms. Let $H_1$ and $H_2$ be two polynomials such that the endomorphism $H_i(\varphi_i)$ acts as 0 on $V_i$ for $i = 1, 2$. Suppose that $H_1(X) = \prod_{i=1}^{d_1}(X - \alpha_i)$ and $H_2(X) = \prod_{j=1}^{d_2}(X - \beta_j)$ where $\alpha_i, \beta_j \in \overline{k}$ are the roots. Then $H(\varphi_1 \otimes \varphi_2)$ acts as 0 on $V_1 \otimes V_2$ where

$$H(X) = \prod_{i=1}^{d_1} \prod_{j=1}^{d_2} (X - \alpha_i\beta_j).$$

**Proof.** The polynomial $H_{r}(z) = \prod_{i=1}^{d_1} \prod_{j=1}^{d_2} (z - \alpha_i\beta_j)$, so, before we evaluate it on $\varphi_1 \otimes \varphi_2$, we write

$$\varphi_1 \otimes \varphi_2 - \alpha_i\beta_j = (\varphi_1 - \alpha_i) \otimes \varphi_2 + \alpha_i \otimes (\varphi_2 - \beta_j).$$

We can thus write $H(\varphi_1 \otimes \varphi_2)$ (viewed as an element of $\mathbb{Z}[\varphi_1 \otimes \varphi_2]$)

$$H(\varphi_1 \otimes \varphi_2) = ((\varphi_1 - \alpha_1) \otimes \varphi_2 + \alpha_1 \otimes (\varphi_2 - \beta_1)) \cdot$$

$$
\cdot ((\varphi_1 - \alpha_2) \otimes \varphi_2 + \alpha_2 \otimes (\varphi_2 - \beta_2)) \cdot$$

$$\vdots$$

$$
\cdot ((\varphi_1 - \alpha_{d_1}) \otimes \varphi_2 + \alpha_{d_1} \otimes (\varphi_2 - \beta_{d_2})) \cdot$$

Expanding the product, we notice that each term is of the form

$$\gamma(\varphi_1 - \alpha_1)^{e_1} \ldots (\varphi_1 - \alpha_{d_1})^{e_{d_1}} \otimes (\varphi_2 - \beta_1)^{f_1} \ldots (\varphi_2 - \beta_{d_2})^{f_{d_2}}.$$

To prove the lemma, observe that if $e_i \geq 1$ for all $1 \leq i \leq d_1$ then $(\varphi_1 - \alpha_1)^{e_1} \ldots (\varphi_1 - \alpha_{d_1})^{e_{d_1}} = H_1(\varphi_1)\psi$ for some $\psi \in \mathbb{Z}[\varphi_1]$, so $H(\varphi_1 \otimes \varphi_2)(x \otimes y) = \psi H_1(\varphi_1)x \otimes y' = 0$ since $H_1(\varphi_1)x = 0$. If there is $i$ such that $e_i = 0$ then we see that $f_j \geq 1$ for all $1 \leq j \leq d_2$ and using $H_2(\varphi_2)y = 0$, we see that $H(\varphi_1 \otimes \varphi_2)(x \otimes y)$. 

$\square$
Lemma 5.3. Suppose that the congruence relation holds for $\text{Sh}_{K'}(G_V, \mathcal{X}_V)$ and $\text{Sh}_{K''}(G_W, X_W)$. Then it holds for $\text{Sh}_K(G, X)$.

Proof. This is a straightforward application of Lemma 5.2 together with the Künneth formula that decomposes

$$H^p_{\text{ét}}(\text{Sh}_K(G, X); Q_\ell) = \bigoplus_{p+q=n} H^p_{\text{ét}}(\text{Sh}_{K'}(G_V, X_V); Q_\ell) \otimes H^q_{\text{ét}}(\text{Sh}_{K''}(G_W, X_W); Q_\ell).$$

Here, we have to observe that when specialized to $\pi_{V, \tau}$ and $\pi_{W, \tau}$ and $\pi_\tau = \pi_{V, \tau} \boxtimes \pi_{W, \tau}$, respectively, the roots of the Hecke polynomials $H_{\tau}(\pi_\tau; z)$ are the pairwise $\alpha_i \beta_j$ where $\alpha_i$ and $\beta_j$ are the roots of $H_{\tau}(\pi_{V, \tau}; z)$ and $H_{\tau}(\pi_{W, \tau}; z)$, respectively.

6. Distribution Relations and Proof of Theorem 1.3

6.1 The action of the Hecke algebra and the Galois action

The Galois group $\text{Gal}(E^{ab}/E)$ acts on $\mathbb{Z}[\mathbb{Z}_K(G, H)]$ via the Shimura reciprocity law described in Section 2.2. The Hecke algebra $\mathcal{H}(G, K)$ of $K$-bi-invariant locally constant functions on $G(A_f)$ acts on both $\mathbb{Z}[\mathbb{Z}_K(G, H)]$ and $\mathbb{Z}[G(A_f)/K]$ and the action on the former is Galois equivariant. Recall that if $g \in G(A_f)$ then the function $1_{KgK} \in \mathcal{H}(G, K)$ acts as follows: if $KgK = \bigsqcup g_iK$ then for $h \in G(A_f)$, the corresponding endomorphism is $[h] \mapsto \sum [hg_i]$. Here, $[h]$ denotes the class of $h$ in either $G(A_f)/K$ or $\mathbb{Z}_K(G, H)$. We will prove Theorem 1.3 by relating the two actions locally at the fixed place $\tau$, i.e., by relating the action of the local Hecke algebra $\mathcal{H}(G_\tau, K_\tau)$ to the action of the decomposition group at $\tau$.

First, recall that the set $\text{Inv}_\tau = \{(a, b) : a, b \in \mathbb{Z}_{\geq 0}\}$ is in bijection with the set of $H_\tau$-orbits of elements of $\text{Hyp}_\tau$ (Proposition 3.4). Since $K_\tau = K_{V, \tau} \times K_{W, \tau}$ where both $K_{V, \tau}$ and $K_{W, \tau}$ are hyperspecial then the latter is also in bijection with the double quotient $H_\tau \backslash G_\tau / K_\tau$. In addition, the action of $\mathcal{H}(G_\tau, K_\tau)$ on the set of cycles is related to the action of $\mathcal{H}(G_\tau, K_\tau)$ on the set $\text{Hyp}_\tau$. On the other hand, recall from Section 3 that the set of Galois orbits of special cycles is related to

$$H_\tau \backslash G_\tau / K_\tau \times H^{(\tau)} \backslash G^{(\tau)} / K^{(\tau)},$$

and the local conductor $c_\tau(\xi)$ can be determined by looking at the projection to the $\tau$-component and using Theorem 1.1 expressing the local conductor at $\tau$ in terms of the distance function. In order to prove Theorem 1.3 we will use the fact that the action of $\mathcal{H}(G_\tau, K_\tau)$ on $\text{Hyp}_\tau$ is given by adjacency operators and hence, is closely related to the distance function. The idea is to take a cycle whose $\tau$-component has image $(0, 0)$ in $\text{Inv}_\tau$, apply the Hecke polynomial $H_{\tau}(F_\tau)$ to this cycle (yielding a combination of “adjacent” cycles as the coefficients of $H_{\tau}(F_\tau)$ are Hecke operators) and verify that one gets an exact trace (down to $E_\tau$) of an element of $\mathbb{Z}[\mathbb{Z}_K(G, H)]$ whose local conductor is $\varpi^2$. For the latter, we will use the fact that the Galois and the Hecke actions commute.

6.2 Proof of Theorem 1.3

The main idea is to apply the Hecke polynomial on the level of the building (i.e., to $\mathbb{Z}[\text{Hyp}_\tau]$) to a hyperspecial vertex of type $(0, 0)$ and use the fact that each Hecke operator that appears as a coefficient is an adjacency operator, thus, giving the action expressing the resulting element as a sum of certain neighbors. We would then like to descend this expression on the level of cycles (i.e., in $\mathbb{Z}[\mathbb{Z}_K(G, H)]$). Here, some of the points on the building will get identified. Although it is hard to say which neighbors yield the same Galois conjugates, the property that the Galois action commutes with the Hecke action will give us that each Galois conjugate appears an equal number of times among the neighbors of a certain type and thus, using Theorem 1.1 as well as Proposition 3.4 we
can recognize the left-hand side as an exact trace and thus, complete the proof.

6.2.4 The action of the decomposition group. Let \( \xi \in Z_K(G, H) \) be a cycle of local conductor \( c_\tau(\xi) = p_\tau^n \). Since \( \mathcal{H}(G, K) \cong \mathcal{H}(G_\tau, K_\tau) \otimes \mathcal{H}(G^{(r)}, K^{(r)}) \) and since

\[
G(A_f)/K \cong G_\tau/K_\tau \times G^{(r)}/K^{(r)} \cong \text{Hyp}_\tau \times G^{(r)}/K^{(r)},
\]

it suffices to understand the distribution relations locally at \( \tau \). Here, \( \text{Hyp}_\tau \) means the set of hyperspecial vertices for the building of \( G_\tau = G_{V,\tau} \times G_{W,\tau} \). Given integers \( 0 \leq m < n \), the trace of this cycle is defined as

\[
\text{Tr}_{n,m}(\xi) := \sum_{x \in \mathcal{O}_0^\times / \mathcal{O}_1^\times} \xi^{\mathcal{A}rt_{E,\tau}(x)},
\]

where \( \mathcal{A}rt_{E,\tau} : E_1^\times \to \text{Gal}(E_\tau^{ab}/E_\tau) \) is the local Artin map and we view \( \text{Gal}(E_\tau^{ab}/E_\tau) \) as a subgroup of \( \text{Gal}(E^{ab}/E) \) via the chosen embedding \( \iota_\tau : E \to E_\tau \), the above trace is the exact analogue of the trace \( \text{Tr}_{E[p^n]/E[p^n]} \) where \( E[p^n] \) is the ring class field of conductor \( p_\tau^n \). Note that these traces are used in the theory of Heegner points over the anticyclotomic towers.

6.2.5 Galois equivariance. The argument above involving the computation with the building and the Hecke operator yields the following identity:

\[
H_\tau(1)\xi_{0,0} = q(q + 1)C_{0,0}\xi_{0,0} + q(q + 1)C_{0,1}\xi_{0,1} + q(q + 1)C_{1,0}\xi_{1,0} + q(q + 1)C_{2,0}\xi_{2,0} + q(q + 1)C_{3,0}\xi_{3,0} + \sum_{x \in \mathcal{O}_0^\times / \mathcal{O}_1^\times} m_{1,1}(x)\xi_{1,1} + \sum_{x \in \mathcal{O}_0^\times / \mathcal{O}_1^\times} m_{1,2}(x)\xi_{1,2} + \sum_{x \in \mathcal{O}_0^\times / \mathcal{O}_2^\times} m_{2,1}(x)\xi_{2,1},
\]

where we have used (see Theorem [11]) that \( c_\tau(\xi_{1,1}) = c_\tau(\xi_{1,2}) = \omega \) and \( c_\tau(\xi_{2,1}) = \omega^2 \). Moreover, \( m_{i,j}(x) \in \mathbb{Z} \) is the multiplicity of the cycle \( \xi_{i,j}^{\mathcal{A}rt_{E,\tau}(x)} \),

\[
\begin{align*}
C_{0,0} &= q^{16} - q^{15} + q^{14} - 2q^{12} + 4q^{11} - 3q^{10} + 5q^9 - 6q^7 + 5q^6 + 2q^5 - 5q^4 + 4q^3 - 2q + 2, \\
C_{0,1} &= -q^{14} + q^{13} - q^{12} - q^{11} + q^{10} - q^9 - 2q^8 + 2q^7 - 4q^6 - q^3 - q^2 - q + 1, \\
C_{1,0} &= (q + 1)(q^2 - q + 1)(q^2 - 2q^11 + 2q^{10} - 2q^8 + 3q^7 - q^6 - 2q^5 + 3q^4 - q^3 - q^2 + 2q - 1), \\
C_{0,2} &= q^4(q^8 - q^7 + q^6 + q^5 + q^4 + q^3 - q + 1), \\
C_{2,0} &= q^6(q + 1)(q^2 - q + 1)(q^5 - q^4 + q^3 - q + 1), \\
C_{3,0} &= -q^{10},
\end{align*}
\]

and

\[
\begin{align*}
\sum_{x \in \mathcal{O}_0^\times / \mathcal{O}_1^\times} m_{1,1}(x) &= -q^2(q + 1)^2(q^2 - q + 1)(q^{12} + q^9 + 2q^6 - q^5 + q^3 + 1) = q(q + 1)C_{1,1}, \\
\sum_{x \in \mathcal{O}_0^\times / \mathcal{O}_1^\times} m_{1,2}(x) &= (q + 1)^2q^6(q^2 - q + 1)(q^2 + 1)(q^4 - q^2 + 1) = q(q + 1)C_{1,2}, \\
\sum_{x \in \mathcal{O}_0^\times / \mathcal{O}_2^\times} m_{2,1}(x) &= -(q + 1)^2q^{10}(q^2 - q + 1) = q(q + 1)C_{2,1}.
\end{align*}
\]
Here,

\[ C_{1,1} = -q(q+1)(q^2 - q + 1)(q^{12} + q^9 + 2q^6 - q^5 + q^3 + 1) \]
\[ C_{1,2} = (q+1)q^2(q^2 - q + 1)(q^2 + 1)(q^4 - q^2 + 1) \]
\[ C_{2,1} = -(q+1)q^2(q^2 - q + 1). \]

To complete the proof, we use the fact that the Hecke algebra commutes with the action of Gal(\(\mathcal{E}/E\)). The latter implies that \(m_{1,1}(x)\) are all the same for \(x \in O_0^\times/O_1^\times\) and the same for the \(m_{1,2}(x)\). In addition, all \(m_{2,1}(x)\) for \(x \in O_0^\times/O_2^\times\) are the same. This yields the distribution relation

\[ H_\tau(\text{Fr}_\lambda)\xi_{0,0} = \text{Tr}_{2,0}(\xi), \]

where

\[ \xi = C_{0,0}\xi_{0,0} + C_{0,1}\xi_{0,1} + C_{1,0}\xi_{1,0} + C_{0,2}\xi_{0,2} + C_{2,0}\xi_{2,0} + C_{0,3}\xi_{0,3} + C_{1,1}\xi_{1,1} + C_{1,2}\xi_{1,2} + C_{2,1}\xi_{2,1}, \]

and thus, proves Theorem 1.3.

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