The Continuous quaternion Algebra-Valued Wavelet Transform and the Associated Uncertainty Principle

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Abstract

The purpose of this article is to extend the wavelet transform to quaternion algebra using the kernel of the two-sided quaternion Fourier transform (QFT). We study some fundamental properties of this extension such as scaling, translation, rotation, Parseval’s identity, inversion theorem, and a reproducing kernel, then we derive the associated Heisenberg-Pauli-Weyl uncertainty principle UP. Finally, using the quaternion Fourier representation of the CQWT we generalize the logarithmic UP and Hardy’s UP to the CQWT domain.

Key words: quaternion algebra; quaternion Fourier transform; Admissible quaternion wavelet; Uncertainty principle.

1 Introduction

The wavelet transform (WT) is of great importance due to its applications in different disciplines including: signal analysis, image processing and denoising, pattern recognition, quantum mechanics, astronomy, sampling theory and other fields. The WT was introduced in the classical case, at first for one dimension by Grassman and Morlet [11], Later Murenzi [18] generalized the WT to more than one dimension. Thereafter Brackx and Sommen extended the classical wavelets to Clifford algebra [4, 5]. Considering quaternion algebra as a special case of Clifford algebra, the generalization of the WT to the quaternion framework came quite naturally [19], [20]. In Ref. [19], Traversoni proposed a discrete quaternion wavelet transform using the (two-sided) quaternion Fourier transform (QFT).

Our contribution to these developments is that we introduce the two-dimensional continuous quaternion wavelet transform by means of the kernel of the two-sided QFT, and using the similitude group of the plane. We thoroughly study this generalization of the continuous wavelet transform to quaternion algebra which we call the two-sided continuous quaternion wavelet transform CQWT.

To the best of our knowledge, the study of a CQWT from the similitude groupe $SIM(2)$ using the kernel of the two-sided QFT, has not been carried out yet. In this regard, the novelty in the present work can be stated as follows: following the same processus as in Clifford case [14], and CQWT case based on the kernel of the right sided QFT [1], we construct our new transform and investigate its important properties such as linearity, scaling, rotation, inversion formula, reproducing kernel..., we show that these properties of the two-sided CQWT can be established whenever the quaternion wavelet satisfies a particular admissibility condition. Even our generalization does not verify the spectral QFT representation property used on the case of the two-dimensional CQWT based on the kernel of the (right-sided) QFT [1] which was the key to the demonstration of the Heisenberg principle. However, one could establish the Heisenberg-Pauli-Weyl UP related for the two-sided CQWT using the UP and derivative theorem of the two-sided QFT.

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*unless the two-sided QFT of a function commute with the quaternion basis vector $e_2$, see lemma 4.5 in the present paper.
The hope is that such a transform could be useful in signal processing and optics.

The manuscript is structured as follows: The remainder of the section 2 briefly reviews quaternions and the two-sided QFT. In section 3, we discuss the basis ideas for the construction of a CQWT based on the two-sided QFT, and derive some important properties, We then, in section 4, prove the Heisenberg-Pauli-Weyl inequality related to the CQWT, and extend the corresponding results of logarithmic UP and Hardy’s UP to the CQWT domain respectively. Finally, a conclusion is given in section 5.

## 2 Preliminaries

The quaternion algebra $\mathbb{H}$ over $\mathbb{R}$, is a special Clifford algebra $Cl_{0,2}$, it is an associative non-commutative four-dimensional algebra, its basis : $e_0, e_1, e_2, e_3$ satisfies Hamilton’s multiplication rules

$$ e_1^2 = e_2^2 = -1, e_1e_2 = e_3, e_1e_3 = -e_2e_1. $$

Let $q = \sum_{k=0}^{3} q_k e_k, q' = \sum_{k=0}^{3} q'_k e_k \in \mathbb{H}$.

Then the product $qq'$ is given by

$$ qq' = (q_0q'_0 - q_1q'_1 - q_2q'_2 - q_3q'_3)e_0 + (q_1q'_0 + q_0q'_1 - q_3q'_2 + q_2q'_3)e_1 + (q_2q'_0 + q_3q'_1 + q_0q'_2 - q_1q'_3)e_2 + (q_3q'_0 - q_2q'_1 + q_1q'_2 + q_0q'_3)e_3. $$

We define the conjugation of $q \in \mathbb{H}$ by :

$$ \overline{q} = q_0 e_0 - \sum_{k=1}^{3} q_k e_k. $$

The quaternion conjugation is a linear anti-involution

$$ \overline{qp} = \overline{p} \overline{q}, \quad \overline{p \pm q} = \overline{p} \overline{q}, \quad \overline{p} = p. \quad (2.1) $$

The modulus of a quaternion $q$ is defined by:

$$ |q| = \sqrt{q\overline{q}} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}. $$

It is easy to verify that:

$$ |pq| = |p||q|. $$

And $0 \neq q \in \mathbb{H}$ implies :

$$ q^{-1} = \frac{\overline{q}}{|q|^2}. $$

This means that $\mathbb{H}$ is a normed division algebra.

**Definition 2.1.** A quaternion module $L^2(\mathbb{R}^2, \mathbb{H})$ is given by

$$ L^2(\mathbb{R}^2, \mathbb{H}) = \{ f = \sum_{k=0}^{3} f_k e_k : \mathbb{R}^2 \rightarrow \mathbb{H}, f_k \in L^2(\mathbb{R}^2, \mathbb{R}) \quad k = 0, 1, 2, 3 \}, $$

Let the inner product of $f, g \in L^2(\mathbb{R}^2, \mathbb{H})$ be defined by

$$ (f, g) := \int_{\mathbb{R}^2} f(x) \overline{g(x)} dx. \quad (2.2) $$

If $f = g$, we get the associated norm:

$$ \| f \|_{L^2(\mathbb{R}^2, \mathbb{H})} = \sqrt{(f, f)} = \sqrt{\int_{\mathbb{R}^2} |f(x)|^2 dx}, $$

From (2.2), we obtain the quaternion Schwartz’s inequality
\[ \forall f, g \in L^2(\mathbb{R}^2, \mathbb{H}) : \left\| \int_{\mathbb{R}^2} f(x)g(x)dx \right\|^2 \leq \int_{\mathbb{R}^2} |f(x)|^2 dx \int_{\mathbb{R}^2} |g(x)|^2 dx. \]

We denote by \( S(\mathbb{R}^2, \mathbb{H}) \), the quaternion Schwartz space of \( C^\infty \)- functions \( f \), from \( \mathbb{R}^2 \) to \( \mathbb{H} \), that for all \( m, n \in \mathbb{N} \)

\[ \sup_{t \in \mathbb{R}^2, \alpha_1 + \alpha_2 \leq m} (1 + |f|)^n \left| \frac{\partial^{\alpha_1 + \alpha_2}}{\partial t_1^{\alpha_1} \partial t_2^{\alpha_2}} f(t) \right|_Q < \infty \text{, where } (\alpha_1, \alpha_1) \in \mathbb{N}^2. \]

Let \( \text{SIM}(2) \) denote the similitude group, a subgroup of the affine group of \( \mathbb{R}^2 \), which is given by

\[ \text{SIM}(2) = (\mathbb{R}_+ \times \text{SO}(2)) \otimes \mathbb{R}^2, \]

\[ = \{(a, r_\theta, b), a > 0, \theta \in [0, 2\pi[, b \in \mathbb{R}^2\}. \]

Where \( \text{SO}(2) \) is the special orthogonal group of \( \mathbb{R}^2 \).

The group law of \( \text{SIM}(2) \) is given by

\[ \{x, b\} \{x', b'\} = \{xx', xb + b\}, \text{ where } x = ar, r \in \text{SO}(2). \]

The rotation operator \( r_\theta \in \text{SO}(2) \) acts on \( x = (x_1, x_2) \in \mathbb{R}^2 \) as usual,

\[ r_\theta(x) = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad 0 \leq \theta < 2\pi. \tag{2.3} \]

We define \( L^2(\text{SIM}(2), \mathbb{H}) \) as follows:

\[ L^2(\text{SIM}(2), \mathbb{H}) = \{ f(a, \theta, b) : f(a, \theta, b) = \int_{\mathbb{R}^2} |f(a, \theta, b)|^2 d\mu(a, \theta) db < \infty \}, \]

where \( d\mu(a, \theta) db \) is the left Haar measure on \( \text{SIM}(2) \) with \( d\mu(a, \theta) = a^{-3} \sin(\theta) d\theta d\theta \) and \( d\theta d\theta \) is the Haar measure on \( \text{SO}(2) \).

Let \( L^\infty(\text{SIM}(2), \mathbb{H}) \) be the collection of essentially bounded measurable functions \( f \) with the norm

\[ \| f \|_{L^\infty(\text{SIM}(2), \mathbb{H})} = \sup_{(a, r_\theta, b) \in \text{SIM}(2)} |f(a, \theta, b)|. \]

If \( f \in L^\infty(\text{SIM}(2), \mathbb{H}) \) is continuous, then \( \| f \|_{L^\infty(\text{SIM}(2), \mathbb{H})} = \sup_{(a, r_\theta, b) \in \text{SIM}(2)} |f(a, \theta, b)|. \)

For the sake of simplicity, We write \( L^2(\text{SIM}(2), \mathbb{H}) \) as \( L^2(\text{SIM}(2), \mathbb{H}, d\theta d\theta db) \) and \( d\mu(a, \theta) = d\mu \)

and write the element \( f(a, \theta, b) \) of \( L^2(\text{SIM}(2), \mathbb{H}) \) as \( f(a, r_\theta, b) \).

We introduce an inner product for \( f, g : \text{SIM}(2) \to \mathbb{H} \) as follows :

\[ (f, g) \int_{\text{SIM}(2)} f(a, \theta, b) \overline{g(a, \theta, b)} d\mu(db, \hat{b}). \tag{2.4} \]

and we obtain \( L^2(\text{SIM}(2), \mathbb{H}) \) – norm

\[ \| f \|^2_{L^2(\text{SIM}(2), \mathbb{H})} = \int_{\text{SIM}(2)} |f(a, \theta, b)|^2 d\mu(db, \hat{b}). \tag{2.5} \]

**Definition 2.2.** (the two-sided QFT)
The two-sided QFT with respect to \( e_1, e_2 \) \([15]\), is defined by:

For \( f \) in \( L^1(\mathbb{R}^2, \mathbb{H}) \),

\[ \mathcal{F}^{e_1, e_2} \{ f \} (u) = \hat{f}(u) = \int_{\mathbb{R}^2} e^{-2\pi i u_1 t_1} f(t) e^{-2\pi i u_2 t_2} dt, \text{ where } t, u \in \mathbb{R}^2. \tag{2.6} \]

**Lemma 2.3.** Inverse QFT \([8]\), Thm. 2.5

For \( f, \hat{f} \in L^1(\mathbb{R}^2, \mathbb{H}) \), the inverse transform for the QFT is given by \( \mathcal{F}^{-e_1, -e_2} \)

\[ f(t) = \mathcal{F}^{-e_1, -e_2} \{ \hat{f}(\xi) \} (t) = \int_{\mathbb{R}^2} e^{2\pi i e_1 \xi_1 t_1} \hat{f}(\xi) e^{2\pi i e_2 \xi_2 t_2} d\xi. \tag{2.7} \]

**Lemma 2.4.** Derivative theorem (QFT) \([8]\), Thm. 2.10

If \( f, \frac{\partial^{m+n}}{\partial x_1^m \partial x_2^n} f \in L^2(\mathbb{R}^2, \mathbb{H}) \) for \( m, n \in \mathbb{N} \),

Then

\[ \mathcal{F}^{e_1, e_2} \{ \frac{\partial^{m+n}}{\partial x_1^m \partial x_2^n} f(x) \} (\xi) = (2\pi)^{m+n} (e_1 \xi_1)^m \mathcal{F}^{e_1, e_2} \{ f(x) \} (\xi) (e_2 \xi_2)^n. \tag{2.8} \]
Lemma 2.5. **QFT of laplacian**

Let \( f, \frac{\partial^2}{\partial x^2} f \in L^2(\mathbb{R}^2, \mathbb{H}) \), \( l = 1, 2 \)

One has

\[
\mathcal{F}^{e_1, e_2} \{ \triangle f \} (\xi) = -(2\pi)^2 |\xi|^2 \mathcal{F}^{e_1, e_2} \{ f \} (\xi),
\]

where \( \triangle \) stands the Laplace operator \( \triangle = \sum_{l=1}^{2} \frac{\partial^2}{\partial x_l^2} \).

**Proof.** by linearity of \( \mathcal{F}^{e_1, e_2} \), and using (2.8) we get

\[
\begin{align*}
\mathcal{F}^{e_1, e_2} \{ \triangle f (x) \} (\xi) &= \mathcal{F}^{e_1, e_2} \{ \frac{\partial^2}{\partial x_1^2} f (x) \} (\xi) + \mathcal{F}^{e_1, e_2} \{ \frac{\partial^2}{\partial x_2^2} f (x) \} (\xi) \\
&= (2\pi)^2 (e_1 \xi_1)^2 \mathcal{F}^{e_1, e_2} \{ f (x) \} (\xi) + (2\pi)^2 (e_2 \xi_2)^2 \mathcal{F}^{e_1, e_2} \{ f (x) \} (\xi) \\
&= -(2\pi)^2 (\xi_1^2 + \xi_2^2) \mathcal{F}^{e_1, e_2} \{ f (x) \} (\xi) \\
&= -(2\pi)^2 |\xi|^2 \mathcal{F}^{e_1, e_2} \{ f (x) \} (\xi). \quad \square
\end{align*}
\]

The following lemma states the Plancherel’s formula, specific to the two-sided QFT.

**Lemma 2.6. Plancherel theorem for (QFT) ([7], Thm 3.2)**

For \( f, g \in L^2(\mathbb{R}^2, \mathbb{H}) \):

\[
(f, g) = (\mathcal{F}^{e_1, e_2} f, \mathcal{F}^{e_1, e_2} g).
\]

(2.9)

In particular, if \( f = g \), we find Parseval’s formula,

\[
\|f\|_{L^2(\mathbb{R}^2, \mathbb{H})}^2 = \|\mathcal{F}^{e_1, e_2} f\|_{L^2(\mathbb{R}^2, \mathbb{H})}^2.
\]

(2.10)

The next lemma states that the QFT of a Gaussian function, is also a Gaussian function.

**Lemma 2.7. QFT of a Gaussian ([8], Lem. 3.5)**

\[
\mathcal{F}^{e_1, e_2} \left\{ e^{-\pi |x|^2} \right\}(y) = e^{-\pi |y|^2},
\]

(2.11)

where \( x, y \in \mathbb{R}^2 \).

**Lemma 2.8. SO(\mathbb{R}^2) transformation of the QFT**

The QFT of a signal \( f \in L^2(\mathbb{R}^2, \mathbb{H}) \), with a \( \text{SO}(\mathbb{R}^2) \) transformation \( A = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \),

is given by

\[
\mathcal{F}^{e_1, e_2} \{ f(Ax) \} (\xi) = \frac{1}{2} \left[ \hat{f}(A\xi) + \hat{f}(A^{-1}\xi) + e_1 [ \hat{f}(A\xi) - \hat{f}(A^{-1}\xi) ] e_2 \right].
\]

(2.12)

Proof. The proof was first given by Thomas Bülow ([8], Thm. 2.12) for a real 2D signal \( f \in L^2(\mathbb{R}^2, \mathbb{R}) \). After, the result was generalized by Eckhard Hitzer ([13], Thm. 2.6) for \( f \in L^2(\mathbb{R}^2, \mathbb{H}) \).

### 3 Construction of The quaternion Algebra-Valued Wavelet Transform

#### 3.1 Admissible quaternion Wavelet

**Definition(admissibility condition)**

A two-sided admissible quaternion wavelet is a function \( \varphi \in L^2(\mathbb{R}^2, \mathbb{H}) \), not identically zero, satisfying

\[
0 < C_\varphi = \int_{\mathbb{R}^+} \int_{SO(2)} |\mathcal{F}^{e_1, e_2} \{ \varphi(r_\theta(.)\} (\eta\xi)|^2 d\theta a^{-1} da < \infty.
\]

(3.1)
Example 3.4. Let

$$\phi$$

wavelet

Remark 3.2.

Remark 3.1.

and rotation as

Where

Lemma 3.3.

Thus we have

$$< \varphi_1, \varphi_2 >_{AQW} = \int_{\mathbb{R}^2} \overline{\phi_1}(\xi) \phi_2(\xi) |\xi|^2 d\xi.$$ 

As a consequence, AQW is a left $$\mathbb{H}$$-module.

Using the same technique as in the case of the classical wavelets, the two-dimensional quaternion wavelet $$\phi_{(a,\theta,b)}$$ can be obtained from a mother $$\varphi \in L^2(\mathbb{R}^2, \mathbb{H})$$ by the combination of dilation, translation and rotation as

$$\phi_{(a,\theta,b)}(x) = \frac{1}{a} \varphi \left( r_{-\theta} \left( \frac{x - b}{a} \right) \right),$$ (3.2)

Where $$a \in \mathbb{R}^+, b \in \mathbb{R}^2, r_{\theta}$$ is the rotation given by (2.3).

Remark 3.1. We note that if $$\varphi \in L^2(\mathbb{R}^2, \mathbb{H}),$$ then $$\phi_{(a,\theta,b)} \in L^2(\mathbb{R}^2, \mathbb{H}).$$ Indeed

$$\left\| \phi_{(a,\theta,b)} \right\|^2_{L^2(\mathbb{R}^2;\mathbb{H})} = \frac{1}{a^2} \int_{\mathbb{R}^2} \left| \varphi \left( r_{-\theta} \left( \frac{x - b}{a} \right) \right) \right|^2 dx$$

$$= \frac{1}{a^2} \int_{\mathbb{R}^2} \left| \varphi \left( r_{-\theta} \left( \frac{x - b}{a} \right) \right) \right|^2 dy = \left\| \phi \right\|^2_{L^2(\mathbb{R}^2;\mathbb{H})}.$$ 

Remark 3.2. Generally $$< \varphi_1, \varphi_2 >_{AQW}$$ is quaternion-valued, so it cannot be taken out of inner product, but if $$\varphi_1 = \varphi_2,$$ then $$< \varphi_1, \varphi_2 >_{AQW} = C_{\varphi_1}$$ is real-valued.

Lemma 3.3.

$$\phi_{a,\theta,b}(\xi) = a e^{-2\pi \xi_1 b_1 e_1} F_{e_1,e_2} \{ \varphi \left( r_{-\theta}() \right) \} a \xi) e^{-2\pi \xi_2 b_2 e_2}.$$ (3.3)

Proof.

$$\phi_{a,\theta,b}(\xi) = \frac{1}{a} \int_{\mathbb{R}^2} e^{-2\pi \xi_1 x_1 e_1} \varphi \left( r_{-\theta} \left( \frac{x - b}{a} \right) \right) e^{-2\pi \xi_2 x_2 e_2} dx$$

$$= a \int_{\mathbb{R}^2} e^{-2\pi \xi_1 x_1 e_1} \varphi \left( r_{-\theta} \left( y \right) \right) e^{-2\pi \xi_2 x_2 e_2} dx$$

$$= ae^{-2\pi \xi_1 b_1} \int_{\mathbb{R}^2} e^{-2\pi a \xi_1 x_1 e_1} \varphi \left( r_{-\theta} \left( y \right) \right) e^{-2\pi \xi_2 x_2 e_2} dy e^{-2\pi \xi_2 b_2 e_2}$$

$$= ae^{-2\pi \xi_1 b_1} \left\{ F_{e_1,e_2} \{ \varphi \left( r_{-\theta}() \right) \} a \xi \right\} e^{-2\pi \xi_2 b_2 e_2}.$$ 

Example 3.4. Let $$f(t) = e^{-|t|^2},$$ (2.11) yields

$$F_{e_1,e_2} \{- \frac{1}{4\pi^2} \triangle f \} (\xi) = (2\pi)^2 |\xi|^2 e^{-|\xi|^2}.$$ 

Now, we take

$$\varphi(t) = -\frac{1}{4\pi^2} \triangle f (t), \ t \in \mathbb{R}^2.$$ 

Let’s prove that $$\varphi$$ belongs to AQW.

While it is obvious that $$\varphi \in L^2(\mathbb{R}^2, \mathbb{H}),$$ it has yet to be shown that

$$C_\varphi = \int_{\mathbb{R}^+} \int_{SO(2)} F_{e_1,e_2} \{ \varphi \left( r_{-\theta}() \right) \} a \xi)^2 d\theta a^{-1} da < +\infty.$$ 

Applying lemma 2.8 we have that for all $$r_{\theta} \in SO(2), \ a > 0$$

$$F_{e_1,e_2} \{ \varphi \left( r_{-\theta}() \right) \} a \xi) = \frac{1}{2} \{ \varphi(a r_{-\theta}(\xi)) + \varphi(a r_{\theta}(\xi)) \} + e_1(\varphi(a r_{\theta}(\xi)) - \varphi(a r_{-\theta}(\xi))) e_2$$

$$= a^2 |\xi|^2 e^{-\pi a^2 |\xi|^2}.$$ 

Where in the last equality we applied $$|r_{\theta}(\xi)| = |r_{\theta}(\xi)| = |\xi|.$$ Thus we have

$$C_\varphi = \int_{\mathbb{R}^+} \int_{SO(2)} F_{e_1,e_2} \{ \varphi \left( r_{-\theta}() \right) \} (a \xi)^2 d\theta a^{-1} da$$

$$= 2\pi |\xi|^4 \int_0^{+\infty} a^3 e^{-2\pi a^2 |\xi|^2} da < +\infty.$$ 

Remark 3.5. As $\mathcal{F}^{e_1, e_2} \{ \varphi (r_{-\theta} (\cdot )) \}$ in example 3.4 is real-valued function, then it commutes with $e_2$, hence one see that the wavelet in this example satisfy the assumption of the main theorems of this article.

Furthermore $\hat{\varphi}(0) = 0$. Then $\int_{\mathbb{R}^2} \sum_{k=0}^{3} \varphi_k(x) e_k \, dx = 0$, that is $\int_{\mathbb{R}^2} \varphi_k(x) = 0$, $k=0,1,2,3.$, which means, similar to classical wavelets, that the integral of every component $\varphi_k$ is zero.

3.2 Two-Dimensional Continuous Quaternion Wavelet Transform

Definition 3.6. Let $\varphi \in AQW$, the two-sided CQWT $T_\varphi$ is defined by

$$T_\varphi : L^2(\mathbb{R}^2, \mathbb{H}) \to L^2(SIM(2), \mathbb{H})$$

and

$$f \mapsto T_\varphi f : (a, \theta, b) \mapsto T_\varphi f (a, \theta, b) = (f, \varphi_{a,\theta,b}) = \int_{\mathbb{R}^2} f(x) \frac{1}{a} \varphi (r_{-\theta} (x - b) / a) \, dx.$$ (3.4)

We now investigate some basic properties of the CQWT:

3.3 Its Properties

Proposition 3.7. Let $\varphi, \psi$ are quaternion admissible wavelets.

If $f, g \in L^2(\mathbb{R}^2, \mathbb{H})$, then

(i) Left linearity: $T_\varphi (\lambda f + \mu g ) (a, \theta, b) = \lambda T_\varphi f (a, \theta, b) + \mu T_\varphi g (a, \theta, b)$. For arbitrary quaternion constants $\lambda, \mu \in \mathbb{H}$.

(ii) Anti-linearity: $T_{\lambda \varphi + \mu \psi} f (a, \theta, b) = T_\varphi f (a, \theta, b) \overline{\lambda} + T_\varphi f (a, \theta, b) \overline{\mu}$. Where $\lambda$ and $\mu$ are constants in $\mathbb{H}$.

(iii) Scaling: $T_\varphi f (c \cdot) (a, \theta, b) = \frac{1}{c} T_\varphi f (c \cdot) (ca, \theta, cb) : c \in \mathbb{R}^*.$

(iv) Translation: $T_\varphi \tau_c f (a, \theta, b) = T_\varphi f (a, \theta, b - c).$ where $c \in \mathbb{R}^2$, and $\tau_c$ is the translation operator given by $\tau_c f (\cdot) = f (\cdot - c)$.

(v) Rotation: $T_\varphi f (r_{\omega}) (a, \theta, b) = T_\varphi f (a, \theta + \omega, r_{\omega}(b))$, where $r_{\omega}$ is a rotation.

Proof.

(i) For $\lambda, \mu \in \mathbb{H}$, we have $T_\varphi (\lambda f + \mu g ) (a, \theta, b) = (\lambda f + \mu g, \varphi_{a,\theta,b})$

$= \lambda (f, \varphi_{a,\theta,b}) + \mu (g, \varphi_{a,\theta,b})$

$= \lambda T_\varphi f (a, \theta, b) + \mu T_\varphi g (a, \theta, b).$

(ii) For $\lambda, \mu \in \mathbb{H}$, we have $T_{\lambda \varphi + \mu \psi} f (a, \theta, b) = \int_{\mathbb{R}^2} f(t) \frac{1}{a} \varphi (r_{-\theta} (t - b) / a) dt$

$= \int_{\mathbb{R}^2} f(t) \frac{1}{a} \varphi (r_{-\theta} (\frac{t-b}{a})) dt \overline{\lambda} + \int_{\mathbb{R}^2} f(t) \frac{1}{a} \psi (r_{-\theta} (\frac{t-b}{a})) dt \overline{\mu}$

$= T_\varphi f (a, \theta, b) \overline{\lambda} + T_\varphi f (a, \theta, b) \overline{\mu}$.

(iii) For $c \in \mathbb{R}^*$, we have $T_\varphi f (c \cdot) (a, \theta, b) = (f (c \cdot), \varphi_{a,\theta,b})$

$= \int_{\mathbb{R}^2} f(t) \frac{1}{a} \varphi (r_{-\theta} (\frac{t-b}{a})) dt$

$= \frac{1}{c} \int_{\mathbb{R}^2} f(t) \frac{1}{a} \varphi (r_{-\theta} (\frac{t-b}{ac})) dt$

$= \frac{1}{c} T_\varphi f (c \cdot) (ca, \theta, cb)$.

(iv) For $c \in \mathbb{R}^2$, we have $T_\varphi \tau_c f (a, \theta, b) = (\tau_c f, \varphi_{a,\theta,b})$

$= \int_{\mathbb{R}^2} f(t-c) \frac{1}{a} \varphi (r_{-\theta} (\frac{t-c}{a})) dt$

$= \int_{\mathbb{R}^2} f(t) \frac{1}{a} \varphi (r_{-\theta} (\frac{t-b-c}{a})) dt$

$= T_\varphi f (a, \theta, b - c).$ (By change of variable $t - c = u$)
(v) Applying (3.4) and using the fact that $r^{-1} = -r$, and $r_\theta r_\omega = r(\theta + \omega)$, we obtain
\[
T_\varphi f(\varphi_\omega)(a, \theta, b) = \int_{\mathbb{R}^2} f(r_\omega(t) \varphi(\varphi_\theta(\frac{t}{a}))) dt
\]
\[
= \int_{\mathbb{R}^2} f(u) \frac{1}{\alpha} \varphi(\varphi(\varphi_\theta(\frac{u-r_\omega(b)}{a}))) dt^{-1}(r_\omega) du
\]
\[
= \int_{\mathbb{R}^2} f(u) \frac{1}{\alpha} \varphi(\varphi(\varphi_\theta(\frac{u-r_\omega(b)}{a})))) du
\]
\[
= \int_{\mathbb{R}^2} f(u) \frac{1}{\alpha} \varphi(\varphi(\varphi_\theta(\frac{u-r_\omega(b)}{a})))) du
\]
\[
= T_\varphi f \left( \alpha \theta + \omega, r_\omega(b) \right)
\]

\[\square\]

**Theorem 3.8. (Parseval’s identity for the CQWT).**

Suppose that $\varphi \in AQW$ be a quaternion admissible wavelet, and assume that $\mathcal{F}^{e_1,e_2} \{ \varphi(r_\theta(\cdot)) \}$ commute with $e_2$, then for every $f, g \in L^2(\mathbb{R}^2, \mathbb{H})$, we have
\[
< T_\varphi f, T_\varphi g > = C_\varphi(f, g).
\]

**Proof.**
\[
< T_\varphi f, T_\varphi g > = \int_{SIM(2)} T_\varphi f(a, \theta, b) \overline{T_\varphi g(a, \theta, b)} d\mu db
\]
\[
= \int_{SIM(2)} \left[ \int_{\mathbb{R}^2} f(x) \overline{\varphi(a, \theta, b(x))} dx \right] \left[ \int_{\mathbb{R}^2} g(y) \overline{\varphi(a, \theta, b(y))} dy \right] d\mu db
\]
\[
= \int_{SIM(2)} \left[ \int_{\mathbb{R}^2} \hat{f}(\xi) \overline{\varphi(a, \theta, b(\xi))} d\xi \right] \left[ \int_{\mathbb{R}^2} \hat{g}(\eta) \overline{\varphi(a, \theta, b(\eta))} d\eta \right] d\mu db
\]
\[
= \int_{SIM(2)} \left[ \int_{\mathbb{R}^2} \hat{f}(\xi) e^{2\pi i \xi \beta_2} e^{2\pi i \xi \beta_1} \mathcal{F}^{e_1,e_2} \{ \varphi(r_\theta(\cdot)) \} (a\xi) \right]
\]
\[
\times \mathcal{F}^{e_1,e_2} \{ \varphi(r/'_\theta(\cdot)) \} (a\eta) \overline{\hat{g}(\eta)} d\xi d\eta d\theta a^{-1} da
\]
\[
= \int_{0}^{\infty} \int_{\mathbb{R}^2} \hat{f}(\xi) \delta(\xi - \eta) \mathcal{F}^{e_1,e_2} \{ \varphi(r_\theta(\cdot)) \} (a\xi) \mathcal{F}^{e_1,e_2} \{ \varphi(r_\theta(\cdot)) \} (a\eta) d\xi d\eta d\theta a^{-1} da
\]
In the second equality we applied the orthogonality of harmonic exponential functions.

Further, we get
\[
< T_\varphi f, T_\varphi g > = \int_{0}^{\infty} \int_{SIM(2)} \hat{f}(\xi) \mathcal{F}^{e_1,e_2} \{ \varphi(r_\theta(\cdot)) \} (a\xi) \mathcal{F}^{e_1,e_2} \{ \varphi(r_\theta(\cdot)) \} (a\eta) \overline{\hat{g}(\eta)} d\xi d\eta d\theta a^{-1} da
\]
\[
= \int_{\mathbb{R}^2} \hat{f}(\xi) \left[ \int_{\mathbb{R}^2} \mathcal{F}^{e_1,e_2} \{ \varphi(r_\theta(\cdot)) \} (a\xi) \mathcal{F}^{e_1,e_2} \{ \varphi(r_\theta(\cdot)) \} (a\eta) \overline{\hat{g}(\eta)} d\eta d\theta a^{-1} da \right]
\]
\[
= \left( \hat{f} \mathcal{C}_\varphi \hat{g} \right) \quad \text{(by (2.1) and (3.1))}
\]
\[
= C_\varphi \left( \hat{f} \hat{g} \right) \quad \text{C_\varphi \ is a real constant}
\]
\[
= C_\varphi \left( \hat{f} \hat{g} \right) \quad \text{(by using (2.9))}.
\]

Hence the theorem follows. \[\square\]

**Remark 3.9.** Theorem 3.5 could be interpreted as preservation of energy by CQWT.

The following corollary follows directly from Theorem 3.5

**Corollary 3.10. (Plancherel’s Formula for CQWT).**

Suppose that $\varphi$ is quaternion admissible wavelet with $\mathcal{F}^{e_1,e_2} \{ \varphi(r_\theta(\cdot)) \}$ commute with $e_2$, then for every $f \in L^2(\mathbb{R}^2, \mathbb{H})$ we have
\[
\| T_\varphi f \|_{L^2(SIM(2), \mathbb{H})}^2 = C_\varphi \| f \|_{L^2(\mathbb{R}^2, \mathbb{H})}^2.
\]

Thus, except for the factor $C_\varphi$, CQWT is an isometry from $L^2(\mathbb{R}^2, \mathbb{H})$ to $L^2(SIM(2), \mathbb{H})$.

The inversion formula for the CQWT is given by the following theorem.
Theorem 3.11. \textit{(Inversion theorem for CQWT).}
If \( f \in L^2(\mathbb{R}^2, \mathbb{H}) \), then \( f \) can be reconstructed by the formula
\[
f(t) = \frac{1}{C_\varphi} \int_{SIM(2)} T_\varphi f(a, \theta, b) \varphi_{a, \theta, b} d\mu db. \tag{3.7}
\]

Proof. Applying theorem \([3.5]\) we obtain for every \( g \in L^2(\mathbb{R}^2, \mathbb{H}) \),
\[
C_\varphi(f, g) = \int_{SIM(2)} T_\varphi f(a, \theta, b) \overline{T_\varphi g(a, \theta, b)} d\mu db
= \int_{SIM(2)} T_\varphi f(a, \theta, b) g(a, \theta, b) \overline{(\varphi_{a, \theta, b}(t) g(t))} dt\mu db
= \int_{R^2} [\int_{SIM(2)} T_\varphi f(a, \theta, b) \varphi_{a, \theta, b}(t) d\mu db] g(t) dt
= \left( \int_{SIM(2)} T_\varphi f(a, \theta, b) \varphi_{a, \theta, b}(t) d\mu db, g \right).
\tag{3.8}
\]

Where in the third equality we applied Fubini’s theorem to interchange the order of integrations. Since \((3.8)\) holds for every \( g \in L^2(\mathbb{R}^2, \mathbb{H}) \), it follows, therefore
\[
f(t) = \frac{1}{C_\varphi} \int_{SIM(2)} T_\varphi f(a, \theta, b) \varphi_{a, \theta, b}(t) d\mu db.
\]

Next, let’s establish the Reproducing kernel theorem of CQWT

Theorem 3.12. \textit{(Reproducing Kernel).}
Let \( \varphi \) be a quaternion admissible wavelet.
We have
\[
T_{\varphi'} f(a', \theta', b') = \frac{1}{C_\varphi} \int_{SIM(2)} T_\varphi f(a, \theta, b) (\varphi_{a, \theta, b}, \varphi_{a', \theta', b'}) d\mu db. \tag{3.9}
\]

Proof. \(T_{\varphi'} f(a', \theta', b') = \int_{R^2} f(t) \varphi_{a', \theta', b'}^\prime(t) dt\)
(by \((3.4)\))
\[
= \int_{R^2} [\frac{1}{C_\varphi} \int_{SIM(2)} T_\varphi f(a, \theta, b) \varphi_{a, \theta, b}(t) d\mu db] \varphi_{a', \theta', b'}^\prime(t) dt
= \frac{1}{C_\varphi} \int_{SIM(2)} T_\varphi f(a, \theta, b) \varphi_{a, \theta, b}'(t) dt d\mu db
= \frac{1}{C_\varphi} \int_{SIM(2)} T_\varphi f(a, \theta, b) (\varphi_{a, \theta, b}, \varphi_{a', \theta', b'}) d\mu db.
\]

which completes the proof. \(\square\)

Theorem 3.13. \textit{Let \( \varphi \) be a quaternion admissible wavelet.}
For every \( f \in L^2(\mathbb{R}^2, \mathbb{H}) \), we have \( T_\varphi f \in L^p(\mathbb{R}^2, \mathbb{H}) \), \( 2 \leq p \leq \infty \).
And the following inequality holds
\[
\| T_\varphi f \|_{L^p(SIM(2), \mathbb{H})} \leq C_{\varphi} \| \varphi \|^{1-\frac{2}{p}}_{L^2(\mathbb{R}^2, \mathbb{H})} \| f \|_{L^2(\mathbb{R}^2, \mathbb{H})}.
\]

Proof. For \( p = 2 \), by \((17)\) we have
\[
\| T_\varphi f \|_{L^2(SIM(2), \mathbb{H})} = C_{\varphi} \| f \|_{L^2(\mathbb{R}^2, \mathbb{H})}.
\]

For \( p = \infty \), the theorem is obtained by Hölder’s inequality, indeed
\[
| T_\varphi f (a, \theta, b) | = \left| \int_{R^2} f(x) \varphi_{a, \theta, b}(x) dx \right| \leq \| \varphi \|_{L^2(\mathbb{R}^2, \mathbb{H})} \| f \|_{L^2(\mathbb{R}^2, \mathbb{H})}.
\]

Thus \( \| T_\varphi f \|_{L^\infty(SIM(2), \mathbb{H})} \leq \| \varphi \|_{L^2(\mathbb{R}^2, \mathbb{H})} \| f \|_{L^2(\mathbb{R}^2, \mathbb{H})} \).
Let us show the theorem for the case \( 2 < p < \infty \),

From the two previous cases we have \( T_\varphi \) is a bounded linear operator of type \((2, 2)\) with norm \( C_{\varphi} \),
and it is of type \((2, \infty)\) with norm bounded by \( \| \varphi \|_{L^2(\mathbb{R}^2, \mathbb{H})} \).
Therefore Riesz-Thorin interpolation theorem \([17], \text{Thm. 2.1}\) guarantees that
\[
T_\varphi \text{ is bounded from } L^{p_0}(\mathbb{R}^2, \mathbb{H}) \text{ in } L^{q_0}(SIM(2), \mathbb{H}) \text{ with norm } M_\alpha \text{ such that } M_\alpha \leq (C_{\varphi})^{1-\alpha} \| \varphi \|_{L^2(\mathbb{R}^2, \mathbb{H})}^\alpha,
\]
whith
\[
\frac{1}{p_0} = \frac{1-\alpha}{2} + \frac{\alpha}{2} = \frac{1}{2}, \quad \frac{1}{q_0} = \frac{4-\alpha}{2} + \frac{\alpha}{2} = \frac{1}{2}, \quad 0 < \alpha < 1.
\]
Thus \( p_0 = 2 \), and by taking \( p = g_0, \) \( i.e. \alpha = 1 - \frac{2}{p} \)
we get \( T_\varphi \) is of type \((2, p), \) \( 2 < p < \infty, \) with norm \( M_\alpha \) bounded by \( C_{\varphi} \| \varphi \|_{L^2(\mathbb{R}^2, \mathbb{H})}^{(1-\frac{2}{p})} \).
\(\square\)
4 Uncertainty Principles for the Two- Dimensional Continuous quaternion Wavelet Transform

In this section we will prove the famous Heisenberg-Weyl’s UP, and its logarithmic version for the CQWT. Moreover, we establish Hardy’s theorem in the setting of the CQWT.

4.1 Heisenberg-Weyl’s uncertainty principle

It is known that Heisenberg-Weyl’s UP for the two-sided QFT states that a nonzero quaternion algebra-valued function and its QFT cannot both be sharply localized [7]. In what follows, we will extend the validity of the Heisenberg-Weyl’s inequality for the CQWT.

Lemma 4.1. Let \( \varphi \) be a quaternion admissible wavelet, with \( \mathcal{F}^{e_1,e_2} \{ \varphi (r_{-\theta} (\cdot)) \} \in \mathbb{R} + \mathbb{R} e_2 \). If \( f, \frac{\partial^2}{\partial b_1^2} f \in L^2 (\mathbb{R}^2, \mathbb{H}) \)

Then

\[
\int_{SO(2)} \int_{\mathbb{R}^2} \| \xi_1 \mathcal{F}^{e_1,e_2} \{ T_{\varphi} f (a, \theta, b) \} \|^2_{L^2 (\mathbb{R}^2, \mathbb{H})} \, d\mu = C_{\varphi} \| \xi_1 f \|^2_{L^2 (\mathbb{R}^2, \mathbb{H})}, \quad l = 1, 2,
\]

(4.1)

Proof. For \( l = 1 \), we have

\[
\int_{SO(2)} \int_{\mathbb{R}^2} \| \xi_1 \mathcal{F}^{e_1,e_2} \{ T_{\varphi} f (a, \theta, b) \} \|^2_{L^2 (\mathbb{R}^2, \mathbb{H})} \, d\mu = \int_{SO(2)} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \mathcal{F}^{e_1,e_2} \{ T_{\varphi} f (a, \theta, b) \} (\xi) \overline{T_{\varphi} f (a, \theta, b)} (\xi) \, d\mu d\xi d\mu
\]

By using (2.8), we obtain

\[
\mathcal{F}^{e_1,e_2} \{ \frac{\partial^2}{\partial b_1^2} f (a, \theta, b) \} (\xi) = -(2\pi)^2 \xi_2^2 \mathcal{F}^{e_1,e_2} \{ f (a, \theta, b) \} (\xi),
\]

Then (4.2) becomes

\[
\int_{SO(2)} \int_{\mathbb{R}^2} \| \xi_1 \mathcal{F}^{e_1,e_2} \{ T_{\varphi} f (a, \theta, b) \} \|^2_{L^2 (\mathbb{R}^2, \mathbb{H})} \, d\mu = \int_{SO(2)} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \mathcal{F}^{e_1,e_2} \{ T_{\varphi} f (a, \theta, b) \} (\xi) \overline{T_{\varphi} f (a, \theta, b)} (\xi) \, d\mu d\xi d\mu
\]

(Using (2.9))

By (3.4) again

\[
= a^2 \int_{SO(2)} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \mathcal{F}^{e_1,e_2} \{ T_{\varphi} f (a, \theta, b) \} (\xi) \overline{T_{\varphi} f (a, \theta, b)} (\xi) \, d\mu d\xi d\mu
\]

(by applying (3.3))

\[
= a^2 \int_{SO(2)} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \mathcal{F}^{e_1,e_2} \{ T_{\varphi} f (a, \theta, b) \} (\xi) \overline{T_{\varphi} f (a, \theta, b)} (\xi) \, d\mu d\xi d\mu
\]

(4.2)

Where we assume that \( \mathcal{F}^{e_1,e_2} \{ \varphi (r_{-\theta} (\cdot)) \} \in \mathbb{R} + \mathbb{R} e_2 \), i.e \( \mathcal{F}^{e_1,e_2} \{ \varphi (r_{-\theta} (\cdot)) \} \) commute with \( e_2 \).
Moreover, as
\[ \int_{\mathbb{R}} e^{2\pi (\xi - \eta) b \cdot c_i} d\eta = \delta(\xi - \eta), \] for i = 1, 2.
We obtain
\[ \int_{\mathbb{R}^d} \left\| \xi f^{1, e_2} \{ T_{\varphi} f(\alpha, \theta, \cdot) \} \right\|_{L^2(\mathbb{R}^2, \mathbb{H})}^2 d\mu = \delta(\xi) \sum_{i,j} c_i c_j \int_{\mathbb{R}^2} f^{1, e_2} \{ \varphi(r_{-\theta}(\cdot)) \} (\alpha \xi)^2 d\theta d\alpha \, d\tilde{\xi} \]
\[ = C_\varphi \left\| \xi \hat{f}(\xi) \right\|_{L^2(\mathbb{R}^2, \mathbb{H})}^2. \] (C_\varphi \text{ is a real constant})
For l = 2, the argument is similar to the one used for l = 1.

The proof is complete. \( \square \)

**Lemma 4.2.** Under the same conditions as in Lemma 4.1, one has
\[ \int_{\mathbb{R}^d} \left\| \xi f^{1, e_2} \{ T_{\varphi} f(\alpha, \theta, \cdot) \} \right\|_{L^2(\mathbb{R}^2, \mathbb{H})}^2 d\mu = C_\varphi \left\| \xi \hat{f}(\xi) \right\|_{L^2(\mathbb{R}^2, \mathbb{H})}^2. \]

**Proof.** We have
\[ \int_{\mathbb{R}^d} \left\| \xi f^{1, e_2} \{ T_{\varphi} f(\alpha, \theta, \cdot) \} \right\|_{L^2(\mathbb{R}^2, \mathbb{H})}^2 d\mu = \int_{\mathbb{R}^d} \int_{\mathbb{R}^2} \left\| \xi^{2+2} \{ T_{\varphi} f(\alpha, \theta, \cdot) \} \right\|_{L^2(\mathbb{R}^2, \mathbb{H})}^2 d\xi d\mu = C_\varphi \int_{\mathbb{R}^2} \left\| \xi^2 f(\xi) \right\|_{L^2(\mathbb{R}^2, \mathbb{H})}^2 d\xi + C_\varphi \int_{\mathbb{R}^2} \left\| \xi^2 f(\xi) \right\|_{L^2(\mathbb{R}^2, \mathbb{H})}^2 d\xi \]
(by linearity of the integral, and (21))
\[ = C_\varphi \int_{\mathbb{R}^2} \left\| \xi^2 \hat{f}(\xi) \right\|_{L^2(\mathbb{R}^2, \mathbb{H})}^2 d\xi \]
\[ = C_\varphi \left\| \xi \hat{f}(\xi) \right\|_{L^2(\mathbb{R}^2, \mathbb{H})}^2. \] \( \square \)

The following proposition is Heisenberg-Weyl’s inequality related to the two-sided QFT, it is a generalization of the inequality obtained, in the remark on page 12, in \[7\], for f \( \in S(\mathbb{R}^2, \mathbb{H}) \) with \( \| f \|_{L^2(\mathbb{R}^2, \mathbb{H})} = 1 \).
The proof is quite similar to the one of Thm. 4.1 in \[8\] and will be omitted.

**Proposition 4.3.** \([7]\). Heisenberg-Weyl’s UP associated with two-sided QFT
Let \( f \in S(\mathbb{R}^2, \mathbb{H}) \), we have
\[ \left\| \xi \hat{f}(\xi) \right\|_{L^2(\mathbb{R}^2, \mathbb{H})} \geq \frac{1}{16\pi^2} \left\| f \right\|_{L^2(\mathbb{R}^2, \mathbb{H})}^2. \] (4.3)

**Theorem 4.4.** (Heisenberg-Weyl’s UP associated with CQFT)
For \( \varphi \in S(\mathbb{R}^2, \mathbb{H}) \) satisfying the admissibility condition (3.1), with \( F^{1, e_2} \{ \varphi(r_{-\theta}(\cdot)) \} \in \mathbb{R} + \mathbb{R} e_2 \).
For every \( f \in S(\mathbb{R}^2, \mathbb{H}) \), such as \( \frac{d^2}{dx^2} f \in L^2(\mathbb{R}^2, \mathbb{H}) \). We have the following inequality:
\[ \left\| b \right\|_{L^2(\mathbb{R}^2, \mathbb{H})} \left\| \xi \right\|_{L^2(\mathbb{R}^2, \mathbb{H})} \left\| \hat{f} \right\|_{L^2(\mathbb{R}^2, \mathbb{H})} \geq \frac{1}{\sqrt{C_\varphi} 4\pi} \left\| T_{T_f} \right\|_{L^2(\mathbb{R}^2, \mathbb{H})}. \] (4.3)

**Proof.** Firstly we note that \( f, \varphi \in S(\mathbb{R}^2, \mathbb{H}) \) implies that \( T_{\varphi} f(\alpha, \theta, \cdot) \in S(\mathbb{R}^2, \mathbb{H}) \).
Replacing \( f \) by \( T_{T_f} f(\alpha, \theta, \cdot) \) in (4.3), we get
\[ \left\| \int_{\mathbb{R}^2} |b|^2 \left\| T_{\varphi} f(\alpha, \theta, b) \right\|_{L^2(\mathbb{R}^2, \mathbb{H})}^2 \right\|_{L^2(\mathbb{R}^2, \mathbb{H})} \frac{1}{\sqrt{C_\varphi} 4\pi} \left\| T_{T_f} \right\|_{L^2(\mathbb{R}^2, \mathbb{H})}. \]
Integrating both sides of the above inequality with respect to the Haar measure \( d\mu \), we have
\[ \int_{\mathbb{R}^2} |b|^2 \left\| T_{\varphi} f(\alpha, \theta, b) \right\|_{L^2(\mathbb{R}^2, \mathbb{H})}^2 \right\|_{L^2(\mathbb{R}^2, \mathbb{H})} \frac{1}{\sqrt{C_\varphi} 4\pi} \left\| T_{T_f} \right\|_{L^2(\mathbb{R}^2, \mathbb{H})}. \]
Using the quaternion Schwartz’s inequality, we can write,
\[ \left\| \int_{\mathbb{R}^2} |b|^2 \left\| T_{\varphi} f(\alpha, \theta, b) \right\|_{L^2(\mathbb{R}^2, \mathbb{H})}^2 \right\|_{L^2(\mathbb{R}^2, \mathbb{H})} \frac{1}{\sqrt{C_\varphi} 4\pi} \left\| T_{T_f} \right\|_{L^2(\mathbb{R}^2, \mathbb{H})}. \]
\[ = \frac{1}{\sqrt{C_\varphi} 4\pi} \left\| T_{T_f} \right\|_{L^2(\mathbb{R}^2, \mathbb{H})}. \]
As the first term in the above expression is \( L^2(\mathbb{R}^2, \mathbb{H}) \)-norm, we finally obtain
\[ \left\| b \right\|_{L^2(\mathbb{R}^2, \mathbb{H})} \left\| \xi \right\|_{L^2(\mathbb{R}^2, \mathbb{H})} \left\| \hat{f} \right\|_{L^2(\mathbb{R}^2, \mathbb{H})} \geq \frac{1}{\sqrt{C_\varphi} 4\pi} \left\| T_{T_f} \right\|_{L^2(\mathbb{R}^2, \mathbb{H})}. \]

The proof of theorem 4.4 is complete. \( \square \)
4.2 Logarithmic uncertainty principle

Based on the classical Pitt’s inequality, Becker [3] proved the logarithmic version of Heisenberg’s UP. Recently this principle has been carried out for different two-dimensional time-frequency domain transforms [7, 2, 19]. Here, we derive the logarithmic inequality in CQWT domains.

**Lemma 4.5.** *(Quaternion Fourier representation of the CQWT)*

Let \( \varphi \) be a quaternion admissible wavelet, and \( f \in L^2(\mathbb{R}^2, \mathbb{H}) \). If we assume that \( \hat{f} \) is \( \mathbb{R} + \mathbb{R}_e \)-valued, one has

\[
F^{e_1,e_2} \{ \hat{T}_\varphi f(a, \theta, b) \} (\xi) = a F^{e_1,e_2} \{ \varphi(r_{-\theta}(.) \} (a \xi) \hat{f}(\xi). \quad (4.4)
\]

Proof. We have by (3.4), (2.9) and (3.3)

\[
T_\varphi f(a, \theta, b) = \int_{\mathbb{R}^2} f(\xi) \overline{\varphi_{\alpha\theta, b}(\xi)} d\xi = a \int_{\mathbb{R}^2} \hat{f}(\xi) e^{2\pi i \xi b e_2} F^{e_1,e_2} \{ \varphi(r_{-\theta}(.) \} (a \xi) e^{2\pi i \xi b e_1} d\xi.
\]

Hence by (2.7), we have

\[
\hat{T}_\varphi f(a, \theta, b) = a F^{e_{-1},e_2} \{ F^{e_1,e_2} \{ \varphi(r_{-\theta}(.) \} (a \xi) \hat{f}(\xi) \} (b).
\]

**Proposition 4.6.** [7]

For \( f \in S(\mathbb{R}^2, \mathbb{H}) \), we have

\[
\int_{\mathbb{R}^2} \ln(|y|) |F^{e_1,e_2} \{ f \} (y)|^2 dy + \int_{\mathbb{R}^2} \ln(|y|)|f(t)|^2 dt \geq A \int_{\mathbb{R}^2} |f(t)|^2 dt,
\]

with \( A = -\ln(\pi) + \Gamma'(1)/\Gamma(1) \), and \( \Gamma(\cdot) \) is the Gamma function.

**Theorem 4.7.** Let \( \varphi \in S(\mathbb{R}^2, \mathbb{H}) \) satisfying the admissibility condition given by (3.1), and suppose that \( F^{e_1,e_2} \{ \varphi(r_{-\theta}(.) \} \) commute with \( e_2 \), and let \( f \in S(\mathbb{R}^2, \mathbb{H}) \).

If we assume that \( \hat{f} \) is \( \mathbb{R} + \mathbb{R}_e \)-valued, then we have

\[
C_\varphi \int_{\mathbb{R}^2} \ln(|y|) |\hat{f}(y)|^2 dy + \int_{SIM(2)} \ln(|b|)|T_\varphi f(a, \theta, b)|^2 d\mu db \geq A C_\varphi \| f \|^2_{L^2(\mathbb{R}^2, \mathbb{H})}.
\]

Proof. Replacing \( f \) by \( \hat{T}_\varphi f(a, \theta, \cdot) \) in (4.5), we get

\[
\int_{\mathbb{R}^2} \ln(|y|)|F^{e_1,e_2} \{ T_\varphi f(a, \theta, b) \} (y)|^2 dy + \int_{\mathbb{R}^2} \ln(|b|)|T_\varphi f(a, \theta, \cdot)|^2 db \geq A \int_{\mathbb{R}^2} |T_\varphi f(a, \theta, b)|^2 db.
\]

As \( \| \cdot \| = |\cdot| \), for any \( q \in \mathbb{H} \), and by change of variable in the second term on the left-hand side, and the term on the right-hand side of the above inequality, we have

\[
\int_{\mathbb{R}^2} \ln(|y|)|F^{e_1,e_2} \{ T_\varphi f(a, \theta, \cdot) \} (y)|^2 dy + \int_{\mathbb{R}^2} \ln(|b|)|T_\varphi f(a, \theta, b)|^2 db \geq A \int_{\mathbb{R}^2} |T_\varphi f(a, \theta, \cdot)|^2 db.
\]

We now integrate both sides of this inequality with respect to the measure \( d\mu \), we obtain

\[
\int_{SO(2)} \int_{\mathbb{R}^2} \ln(|y|)|F^{e_1,e_2} \{ T_\varphi f(a, \theta, \cdot) \} (y)|^2 dy d\mu + \int_{SIM(2)} \ln(|b|) |T_\varphi f(a, \theta, b)|^2 d\mu db \geq A \int_{SIM(2)} |T_\varphi f(a, \theta, b)|^2 d\mu db.
\]

(4.6)

We will show the following assumption

\[
\int_{SO(2)} \int_{\mathbb{R}^2} \ln(|y|)|F^{e_1,e_2} \{ T_\varphi f(a, \theta, \cdot) \} (y)|^2 dy d\mu = C_\varphi \int_{\mathbb{R}^2} \ln(|y|) \hat{f}(y) |^2 dy.
\]

(4.7)

Using (4.4), and straightforward computations show that

\[
\int_{SO(2)} \int_{\mathbb{R}^2} \ln(|y|)|F^{e_1,e_2} \{ T_\varphi f(a, \theta, \cdot) \} (y)|^2 dy d\mu = \int_{SO(2)} \int_{\mathbb{R}^2} \ln(|y|) a F^{e_1,e_2} \{ \varphi(r_{-\theta}(.) \} (ay) \hat{f}(y) |^2 dy d\mu
\]

\[
= \int_{SO(2)} \int_{\mathbb{R}^2} a^2 |\ln(|y|)| F^{e_1,e_2} \{ \varphi(r_{-\theta}(.) \} (ay) \hat{f}(y) |^2 d\theta a^{-1} da \int_{\mathbb{R}^2} \ln(|y|) \hat{f}(y) |^2 dy
\]

\[
= \int_{SO(2)} \int_{\mathbb{R}^2} F^{e_1,e_2} \{ \varphi(r_{-\theta}(.) \} (ay) |^2 d\theta a^{-1} da \int_{\mathbb{R}^2} \ln(|y|) \hat{f}(y) |^2 dy
\]
In the last equality we used (3.1). The proof of (4.7) is completed.

Inserting (4.7) into the first integral of (4.6), we obtain

\[
C \varphi \int_{\mathbb{R}^2} \ln(|y|) |\hat{f}(y)|^2 dy + \int_{SIM(2)} \ln(|b|)|\hat{T}_\varphi f(a, \theta, b)|^2 d\mu db \geq A \int_{SIM(2)} |\hat{T}_\varphi f(a, \theta, b)|^2 d\mu db,
\]

Using (3.6) gives

\[
C \varphi \int_{\mathbb{R}^2} \ln(|y|) |\hat{f}(y)|^2 dy + \int_{SIM(2)} \ln(|b|)|\hat{T}_\varphi f(a, \theta, b)|^2 d\mu db \geq A C \varphi \|f\|_2^2_{L^2(\mathbb{R}^2; \mathbb{H})}.
\]

The result follows. \qed

### 4.3 Hardy’s uncertainty principle

We remind that Hary’s UP associated with the QFT asserts that is impossible for a non-zero function and its QFT to both decrease very rapidly as it was given in [8]. Actually, we establish Hardy’s UP for two-dimensional continuous quaternion wavelet transform.

Let us review Hardy’s UP in the quaternion Fourier transform domain as follows.

**Proposition 4.8.** ([8], Thm. 5.3)

Let \(\alpha\) and \(\beta\) are positive constants. Suppose \(f \in L^2(\mathbb{R}^2, \mathbb{H})\) with

\[
|f(x)|_Q = O(e^{-\alpha|x|^2}), \quad x \in \mathbb{R}^2.
\]

\[
|\hat{f}(y)|_Q = O(e^{-\beta|y|^2}), \quad y \in \mathbb{R}^2.
\]

Then, three cases can occur:

(i) If \(\alpha \beta > \pi^2\), then \(f = 0\).

(ii) If \(\alpha \beta = \pi^2\), then \(f = Ae^{-\alpha|x|^2}\), where \(A\) is a quaternion constant.

(iii) If \(\alpha \beta < \pi^2\), then there are infinitely many such functions \(f\).

Based on Proposition 4.8, we derive the corresponding Hardys UP for the CQWT.

**Theorem 4.9.** Let \(\alpha\) and \(\beta\) are positive constants, and Let \(\varphi\) be a quaternion admissible wavelet,\n
Suppose \(f \in L^2(\mathbb{R}^2; \mathbb{H})\) such that \(\hat{f}\) is \(\mathbb{R} + \mathbb{R}e_2\)-valued, we assume that

\[
|T_\varphi f(a, \theta, b)| = O(e^{-\alpha|b|^2}), \quad b \in \mathbb{R}^2
\]

and

\[
|\hat{f}(\xi)| = O(e^{-\beta|\xi|^2}), \quad \xi \in \mathbb{R}^2
\]

for \(\alpha, \beta > 0\).

Then,

(i) If \(\alpha \beta > \pi^2\), then \(T_\varphi f(a, \theta, .) = 0\).

(ii) If \(\alpha \beta = \pi^2\), then \(T_\varphi f(a, \theta, b) = Ae^{-\alpha|b|^2}\), where \(A\) is a quaternion constant.

(iii) If \(\alpha \beta < \pi^2\), then there are infinitely many \(T_\varphi f\).

Proof. Since \(f\) and \(\varphi\) are in \(L^2(\mathbb{R}^2; \mathbb{H})\) we have \(T_\varphi f(a, \theta, .) \in L^2(\mathbb{R}^2; \mathbb{H})\) by the use of Theorem 3.13. Then it follows from (4.4) that

\[
|\mathcal{F}_{e_1, e_2} \{T_\varphi f(a, \theta, b)\}(\xi)| = a |\mathcal{F}_{e_1, e_2} \{\varphi(r_\theta(\cdot))\}(\xi)| |\hat{f}(\xi)| = a |\mathcal{F}_{e_1, e_2} \{\varphi(r_\theta(\cdot))\}(\xi)||\hat{f}(\xi)|
\]

As the quaternion Fourier transform is isometry on \(L^2(\mathbb{R}^2; \mathbb{H})\) ([7], Thm. 3.2), we have \(\varphi \in L^2(\mathbb{R}^2; \mathbb{H})\) therefore.
\[
\mathcal{F}_{e_1,e_2}\left\{T_\varphi f(a,\theta,-b)\right\} (\xi) = O(e^{-\beta |\xi|^2}),
\]

On the other hand, by assumption we obtain

\[
|T_\varphi f(a,\theta,-b)| = |T_\varphi f(a,\theta,-b)| = O(e^{-\alpha |b|^2}),
\]

Hence, it follows from Proposition 4.8 that that if \(\alpha = \pi^2\), then \(T_\varphi f(a,\theta,-b) = Be^{-\alpha |b|^2}\), with \(B\) is a quaternion constant. That is \(T_\varphi f(a,\theta,b) = Be^{-\alpha |b|^2}\).

If \(\alpha > \pi^2\), then \(T_\varphi f(a,\theta,b) = 0\) on \(\mathbb{R}^2\).

If \(\alpha < \pi^2\), then there are infinitely many such functions \(T_\varphi f(a,\theta,\cdot)\), that verify (4.8) and (4.9).

This completes the proof. \(\square\)

5 Conclusion

In this paper, we developed the definition of CQWT using the Kernel of the two-sided QFT, and the similitude group. The various important properties of the CQWT such as scaling, translation, rotation, Parsevals identity, inversion theorem, and a reproducing kernel are established. Using the derivative theorem, and Heisenberg-Weyl UP related to the two-sided QFT, we derived the Heisenberg-Weyl UP associated with the CQWT. Finally, due to the spectral QFT representation of CQWT and based on the logarithmic UP and Hardy’s UP for the two-sided QFT, the forms associated with these UPs have been proved in the CQWT Domain.

With the help of this paper, we hope to introduce an extension of the wavelet transform to Clifford algebra by means of the kernel of Two-sided Clifford Fourier transform defined by Hitzer in [16]. The investigation on this topic will be reported in a forthcoming paper.

| Property | Function | CQWT |
|----------|----------|------|
| Left linearity | \((\lambda f + \mu g)(a,\theta,b)\) | \(\lambda T_\varphi f(a,\theta,b) + \mu T_\varphi g(a,\theta,b)\) |
| Translation | \(f(a,\theta,b-c)\) | \(T_\varphi f(a,\theta,b-c)\) |
| Scaling | \(f(\alpha \cdot)(a,\theta,b)\) | \(\frac{1}{\alpha}T_\varphi f(\alpha \cdot)(\alpha a,\theta,\alpha b)\) |
| Rotation | \(f(r_\omega)(a,\theta,b)\) | \(T_\varphi f(a,\theta + \omega, r_\omega(b))\) |
| Plancherel | \(C_\varphi(f,g)_{L^2(\mathbb{R}^2,H)}\) | \(\frac{1}{T_\varphi f, T_\varphi g}_{L^2(\text{SIM}(2), \mathbb{H})}\) |
| Parseval | \(\sqrt{C_\varphi} \|f\|_{L^2(\mathbb{R}^2,H)}\) | \(\|T_\varphi f\|_{L^2(\text{SIM}(2), \mathbb{H})}\) |

Some important properties of the CQWT are summarized in Table 1.

References

[1] M. Bahri, R. Ashino and R. Vaillancourt, Two-dimensional quaternion wavelet transform, Applied Mathematics and Computation, 218 (2011), pp. 10-21.

[2] M. Bahri, R. Ashino, Logarithmic Uncertainty Principle for quaternion linear canonical , , Proceedings of the 2016 International Conference on Wavelet Analysis and Pattern Recognition, Jeju, South Korea, pp. 10-13 July.

[3] W. Beckner, Pitts inequality and the uncertainty principle. Proc. Amer. Math. Soc. 123(6), pp. 1897-1905 (1995).
[4] F. Brackx, F. Sommen, Clifford-Hermite wavelets in Euclidean space, J. Fourier Anal. Appl., 2000, 6(3): pp. 209-310.

[5] F. Brackx, F. Sommen, The continuous wavelet transform in Clifford analysis. Clifford analysis and its applications, Prague, 2000, 9-26, NATO Sci. Ser II Math Phys Chem. Dordrecht: Kluwer Acad Publ, 2001, 25.

[6] T. Bülow, Hypercomplex spectral signal representations for the processing and analysis of images, Ph.D. Thesis, 1999, Institut für Informatik und Praktische Mathematik, University of Kiel, Germany.

[7] L.P. Chen, K.I. Kou, M.S. Liu, Pitt’s inequality and the uncertainty principle associated with the quaternion Fourier transform, J. Math. Anal. Appl. 423 (2015), pp. 681-700.

[8] Y. El Haoui, S. Fahlaoui, The Uncertainty principle for the two-sided quaternion Fourier transform, Mediterr. J. Math. (2017) doi:10.1007/s00009-017-1024-5.

[9] Y. El Haoui, S. Fahlaoui, Generalized Uncertainty Principles associated with the Quaternionic Offset Linear Canonical Transform, https://arxiv.org/abs/1807.04068, 2018.

[10] T.A. Ell, Quaternion-Fourier transformms for analysis of two-dimensional linear time-invariant partial differential systems. In: Proceeding of the 32nd Conference on Decision and Control, San Antonio, Texas, pp. 1830-1841, 1993.

[11] A. Grossman, J. Morlet, Decomposition of Hardy functions into square integrable wavelets of constant shape, SIAM J Math Anal, 1984, 15: pp. 723-736.

[12] W. Heisenberg, Uber den anschaulichen inhalt der quanten theoretischen kinematik und mechanik. Zeitschrift für Physik 43, pp. 172-198, 1927.

[13] E. Hitzer, Quaternion Fourier transform on quaternion fields and generalizations, Advances in Applied Clifford Algebras, 17 (3) (2007), pp. 497-517.

[14] E. Hitzer, Clifford (Geometric) Algebra Wavelet Transform, in V. Skala and D. Hildenbrand (eds.), Proc. of GraVisMa 2009, 02-04 Sep. 2009, Plzen, Czech Republic, pp. 94-101 (2009). Preprint: http://arxiv.org/abs/1306.1620.

[15] E. Hitzer, S. J. Sangwine, The Orthogonal 2D Planes Split of Quaternions and Steerable Quaternion Fourier Transformations, in E. Hitzer, S.J. Sangwine (eds.), "Quaternion and Clifford Fourier transforms and wavelets", Trends in Mathematics 27, Birkhauser, Basel, 2013, pp. 15-39. DOI: 10.1007/978-3-0348-0603-9_2, Preprint: http://arxiv.org/abs/1306.2157.

[16] E. Hitzer, Two-Sided Clifford Fourier Transform with Two Square Roots of 1 in Cl(p, q), Adv. Appl. Clifford Algebras 24, pp. 313-332 (2014). doi: 10.1007/s00006-014-0441-9, preprint: http://arxiv.org/abs/1306.2092.

[17] F. Linares, G. Ponce, Introduction to Nonlinear Dispersive Equations, Publicações Matemáticas, IMPA, Rio de Janeiro, Brazil, 2004.

[18] R. Murenzi, Wavelet transforms associated to the n-dimensional Euclidean group with dilations, In: Combes J, ed. Wavelet, Time-Frequency Methods and Phase Space. Boston-London: Jones and Bartlett Publishers, 1989, pp. 239246.

[19] L. Traversoni, Imaging analysis using quaternion wavelet, in geometric algebra with applications, in: E.B. Corrochano, G. Sobczyk (Eds.), Science and Engineering, Birkhuser, Boston, 2001.

[20] J. Zhao, L. Peng, Quaternion-valued admissible wavelets associated with the 2-dimensional Euclidean group with dilations, J. Nat. Geom. 20 (1), pp. 21-32, 2001.