1. Introduction

It is strongly desirable to find new global Torelli theorem for Calabi-Yau manifold, which extends the celebrated global Torelli theorem for polarized K3 surface. The work of B.Gross [6] is closely related with this problem. In fact, at the Hodge theoretical level, B.Gross [6] have constructed certain canonical real polarized variation of Hodge structures over each irreducible tube domain, and then asked for the possible algebraic geometrical realizations of them(cf. §8 [6]).

The very unique property shared by all canonical PVHSs is the Calabi-Yau like property. For a complete family of polarized Calabi-Yau $n$-folds $f : \mathcal{X} \to S$, we consider the $\mathbb{Q}$-PVHS $\mathcal{V}$ formed by the primitive middle rational cohomologies of fibers. Let $(E, \theta)$ be the system of Hodge bundles associated with $\mathcal{V}$. By the very definition of Calabi-Yau manifold, we have the first property of $(E, \theta)$:

$$\dim E^{n,0} = 1.$$  

The Bogomolov-Todorov-Tian unobstructedness theorem for the moduli space of Calabi-Yau manifolds gives us the second property of $(E, \theta)$:

$$\theta : T_S \cong \text{Hom}(E^{n,0}, E^{n-1,1}).$$

On the other hand, by Prop.4.1 and Prop.5.2 [6], we see that the canonical $\mathbb{R}$-PVHSs of B.Gross also have the above two properties. As a definition, we shall say that any PVHS whose associated Higgs bundle possesses the above two properties is Calabi-Yau like.
The purpose of this paper is three-fold. The first one is to extend B.Gross’s construction of PVHS to each irreducible bounded symmetric domain in a natural way. This is a straightforward step. We will then introduce a series of new invariants associated with IVHS (cf. §2 for the introduction of IVHS and other invariants), which are particularly pertinent for Calabi-Yau like IVHS. We call them characteristic subvarieties. Our main result Theorem 3.3 is to identify the characteristic subvarieties of the Calabi-Yau like PVHS over an irreducible bounded symmetric domain with the characteristic bundles of N.Mok (cf. [7]). These infinitesimal invariants present then the nontrivial obstructions to the algebraic geometrical realization problem of B.Gross. This part forms the second purpose. The last aim of this paper is to prove the generating property proposed by B.Gross (cf. §5, [6]). This also gives us a chance to describe the Calabi-Yau like PVHS over each irreducible bounded symmetric domain. Hopefully, the information contained in this part will be used elsewhere.

The present work is motivated by our project, joint with Ralf Gerkmann, of disproof of modularity of the moduli spaces of certain Calabi-Yau manifolds. Recently, we are able to use the results of this work to disprove modularity of the moduli space of Calabi-Yau 3-folds arising from eight planes of $P^3$ in general positions [5].

Acknowledgements: The authors would like to thank Ngaiming Mok for his explanation of the notion of Characteristic bundles, and thank Eckart Viehweg for his interests and helpful discussions on this work.

2. The Calabi-Yau Like PVHS over Bounded Symmetric Domain

Our construction of PVHS can follow that of B.Gross (cf. §3, [6]), but deviates at one point in the beginning. Either, from an explicit point of view, we can also follow §4, [8]. In particular, the notion of $C$-PVHS of weight $n$ used in this paper was defined in Def.4.6, [8]. Our basic observation, though rather trivial, is that we do not need restricting ourselves to the bounded symmetric domains of tube
type. The price for considering all bounded symmetric domains at the same time is the lack of real structure in the resulting PVHS in general, but the Calabi-Yau like property of the PVHS always remains.

Let $D$ be an irreducible bounded symmetric domain, and let $G$ be the identity component of the automorphism group of $D$. We fix an origin $0 \in D$, then the isotropy subgroup of $G$ at $0$ is a maximal compact subgroup $K$. By Prop.1.2.6 [3], $D$ determines a special node $v$ of the Dynkin diagram of the simple complex Lie algebra $\mathfrak{g}^C = \text{Lie}(G) \otimes \mathbb{C}$. By the standard theory on the finite dimensional representations of semi-simple complex Lie algebras(cf. [4]), we know that the special node $v$ also determines a fundamental representation $W$ of $\mathfrak{g}^C$. By the Weyl’s unitary trick, $W$ gives rise to an irreducible complex representation of $G$. We shall remark that only in the case of tube domain $D$ the representation $W$ of $G$ admits an invariant real form. Given a torsion free discrete subgroup $\Gamma$ of $G$, we obtain from the representation $W$ the complex local system

$$\mathbb{W} = W \times_\Gamma D$$

over the locally symmetric variety $\Gamma \backslash D$.

By the construction on the last paragraph of §4, [8], we know that $\mathbb{W}$ is a complex polarized variation of Hodge structures. We denote by $(E, \theta)$ be the associated system of Hodge bundles with $\mathbb{W}$. It is helpful to illustrate the above construction in the simplest case.

**Example 2.1.** Let $D = SU(p,q)/S(U(p) \times U(q))$ be a type A bounded symmetric domain. Then

$$G = SU(p,q), K = S(U(p) \times U(q)), \mathfrak{g}^C = sl(p + q, \mathbb{C}).$$

The special node $v$ of the Dynkin diagram of $sl(p + q, \mathbb{C})$ corresponding to $D$ is the $p$-th node. Let $\mathbb{C}^{p+q}$ be the standard representation of $Sl(p + q, \mathbb{C})$. Then the
fundamental representation denoted by $v$ is

$$W = \bigwedge^p \mathbb{C}^{p+q}.$$  

The group $G$ preserves a hermitian symmetric bilinear form $h$ with signature $(p, q)$ over $\mathbb{C}^{p+q}$. Then $D$ is the parameter space of the $h$-positive $p$-dimensional vector subspace of $\mathbb{C}^{p+q}$. By fixing an origin $0 \in D$, we obtain an $h$-orthogonal decomposition

$$\mathbb{C}^{p+q} = \mathbb{C}_+^p \oplus \mathbb{C}_-^q.$$  

The corresponding Higgs bundle to $W$ is of form

$$E = \bigoplus_{i+j=n} E^{i,j}$$  

where $n = \min(p, q)$ is the rank of $D$. The Hodge bundle $E^{n-i,i}$ is the homogenous vector bundle determined by, at the origin $0$, the irreducible $K$-representation

$$(E^{n-i,i})_0 = \bigwedge^{p-i} \mathbb{C}_+^p \otimes \bigwedge^i \mathbb{C}_-^q.$$  

We conclude this section with the following description and characterization of $W$, which can be proved similarly as in [6].

**Theorem 2.2.** Let $D = G/K$ be an irreducible bounded symmetric domain of rank $n$ and $\Gamma$ be a torsion free discrete subgroup of $G$. Let $W$ be the irreducible PVHS over the locally symmetric variety $X = \Gamma \backslash D$ constructed above. Then $W$ is Calabi-Yau like. Namely the following two properties hold for the associated system of Hodge bundles $(E, \theta)$:

1. $\dim E^{n,0} = 1$
2. $\eta : T_X \rightarrow \text{Hom}(E^{n,0}, E^{n-1,1}).$

Furthermore, among all irreducible representations of $G$, the fundamental representation $W$ determined by the special node $v$ is the unique one which gives rise to a Calabi-Yau like PVHS over $D$.  

3. The Characteristic Subvariety and The Main Result

We start with a general system of Hodge bundles

\[ (E = \bigoplus_{p+q=n} E^{p,q}, \theta = \bigoplus_{p+q=n} \theta^{p,q}) \]

over a complex manifold \( X \) with \( \dim E_{n,0} \neq 0 \). By the integrality of Higgs field \( \theta \), the \( k \)-iterated Higgs field factors as \( E \to E \otimes S^k(\Omega_X) \). It induces in turn the following natural map

\[ \theta^k : S^k(T_X) \to \text{End}(E). \]

By the Griffiths’s horizontal condition, the image of \( \theta^k \) is contained in the subbundle

\[ \bigoplus_{p+q=n} \text{Hom}(E^{p,q}, E^{p-k,q+k}) \subset \text{End}(E). \]

We interest ourselves in the projection of \( \theta^k \) into the first component of the above subbundle. Abusing the notation a little bit, we denote the composition map still by \( \theta^k \). That is, we concern the following map

\[ \theta^k : S^k(T_X) \to \text{Hom}(E^{n,0}, E^{n-k,k}). \]

We have a tautological short exact sequence of analytic coherent sheaves defined by the iterated Higgs field \( \theta^k \):

\[ 0 \to I_k \to S^k(T_X) \xrightarrow{\theta^k} J_k \to 0. \]

We define a sheaf of graded \( \mathcal{O}_X \)-algebras \( J_k \) by putting

\[ J^i_k = \begin{cases} S^i\Omega_X & \text{if } i < k \\ S^i\Omega_X/\text{Im}((J_k)^* \otimes S^{i-k}\Omega_X \xrightarrow{\text{mult.}} S^i\Omega_X) & \text{if } i \geq k. \end{cases} \]

**Definition 3.1.** For \( k \geq 0 \), we call

\[ C_k = \text{Proj}(J_{k+1}) \]

the \( k \)-th characteristic subvariety of \((E, \theta)\) over \( X \).
Because $\theta^{n+1} = 0,
C_k = \mathbb{P}(T_X), \ k \geq n$

where $\mathbb{P}(T_X)$ is the projective tangent bundle of $X$. For $0 \leq k \leq n - 1$, the natural surjective morphism of graded $\mathcal{O}_X$-algebras
\[
\bigoplus_{i=0}^{\infty} S^i \Omega_X \twoheadrightarrow J_k
\]
gives a proper embedding over $X$,

\[
\begin{array}{ccc}
C_k & \hookrightarrow & \mathbb{P}(T_X) \\
p_k & & p \\
\downarrow & & \downarrow \\
X & & \mathbb{P}(T_X) \\
\end{array}
\]

The next lemma gives a simple criterion to characterize when a nonzero tangent vector at the point $x \in X$ has its image in $(C_k)_x = p_k^{-1}(x)$.

**Lemma 3.2.** Let $v \in (T_X)_x$ be a non-zero tangent vector at $x$. Then its image $[v] \in (\mathbb{P}(T_X))_x$ lies in $(C_{k-1})_x$ if and only if $v^k \in (I_k)_x$, the stalk of $I_k$ at $x$.

**Proof:** $(C_{k-1})_x \subset (\mathbb{P}(T_X))_x$ is defined by the homogenous elements contained in $((J_k)^*)_x$. Thus $[v] \in (C_{k-1})_x$ if and only for all $f \in ((J_k)^*)_x$, $f([v]) = 0$. Now we choose a basis $\{e_1, \ldots, e_m\}$ for $(T_X)_x$ and the dual basis $\{e_1^*, \ldots, e_m^*\}$ for $(\Omega_X)_x$.

Claim: $f([v]) = 0$ if and only $f(v^k) = 0$. In the latter, we consider $f$ as a linear form on $(S^k(T_X))_x$.

Proof of Claim: Let $I = (i_1, \ldots, i_m)$ denote the multi-index with $i_j \neq 0$ for all $j$. And
\[
I! = i_1! \cdots i_m!, \ |I| = i_1 + \cdots + i_m.
\]
We write $v = \sum_{i=1}^{m} a_i e_i$ and $f = \sum_{|I|=k} b^I e^I$. Then considering $f$ as a polynomial of degree $k$ on $(T_X)_x$, we have
\[
f(v) = \sum_{|I|=k} b^I a^I.
\]
On the other hand, we have
\[ v^k = k! \sum_{|I|=k} \frac{1}{I^!} a^I e^I \]
where \( a^I = a_{i_1}^{i_1} \cdots a_{i_m}^{i_m} \) etc. And by Ex.B.12,[4], the canonically dual basis of \((S^k(\Omega_X))_x\) to the natural basis \(\{e^I, |I| = k\}\) of \((S^k(T_X))_x\) is \(\{\frac{1}{I^!}(e^*)^I, |I| = k\}\). Hence, evaluating \(f\) as a linear form of \((S^k(T_X))_x\) at \(v^k\), we obtain
\[ f(v^k) = k! \left( \sum_{|I|=k} b^I a^I \right). \]
It is clear now that our claim holds.

Finally, it is easy to see that \(v^k \in (S^k(T_X))_x\) lies in \((I_k)_x\) if and only if for all \(f \in ((J_k)^*)_x\), considered as a linear form of \((S^k(T_X))_x\), \(f(v^k) = 0\). Therefore, the lemma follows.

Our main result identifies the characteristic subvarieties of the Calabi-Yau like PVHS over an irreducible bounded symmetric domain with the characteristic bundles of N.Mok.

**Theorem 3.3.** Let \(D\) be an irreducible bounded symmetric domain of rank \(n\). The notations are as Theorem 2.2. Then, for \(1 \leq k \leq n - 1\), the \(k\)-th characteristic subvariety \(C_k\) of \((E, \theta)\) over \(X\) coincides with \(k\)-th characteristic bundle \(S_k\) over \(X\).

By the second property of being Calabi-Yau like, \(C_0\) is always empty. For the self-containedness of this paper, we would like to describe briefly the notion of characteristic bundles and refer to §1, Ch.6, Append. IV [7] for a fuller account.

The \(k\)-th characteristic bundle \(S_k\) over \(X = \Gamma \backslash D\) is firstly defined over \(D\). It is a projective subvariety of \(\mathbb{P}(T_D)\) and homogenous under the natural action of automorphism group \(G\) on the projective tangent bundle of \(D\). By taking quotient under the left action of \(\Gamma\), one obtains the \(k\)-th characteristic bundle over \(X\). So it suffices to describe the construction of characteristic bundle at one
point of $D$. At the origin $0$ of $D$, the vectors contained in the fiber $(S_k)_0$ are in fact determined by a rank condition. We have the isotropy representation of $K$ on the tangent space $(T_D)_0$. Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and choose a maximal set of strongly orthogonal positive non-compact roots

$$
\Psi = \{\psi_1, \cdots, \psi_n\}.
$$

Let $e_i, 1 \leq i \leq n$ be a root vector corresponding to the root $\psi_i$. Then the set $\Psi$ determines a distinguished disk

$$
\triangle^n \subset D
$$

passing through the origin $0$, and

$$
(T_{\triangle^n})_0 = \sum_{1 \leq i \leq n} C e_i \subset (T_D)_0.
$$

Moreover, for any nonzero element $v \in (T_D)_0$, there exist an element $k \in K^C$ such that

$$
k(v) = \sum_{1 \leq i \leq r(v)} e_i.
$$

Such an expression for the vector $v$ is unique and the natural number $r(v)$ is called the rank of $v$. Then, for $1 \leq k \leq n - 1$, one defines

$$
(S_k)_0 = \{[v] \in (\mathbb{P}(T_D))_0 | 1 \leq r(v) \leq k\}.
$$

By the definition, we have a natural inclusion

$$
S_1 \subset \cdots \subset S_{n-1} \subset \mathbb{P}(T_D),
$$

We can add two trivially defined characteristic bundles by putting

$$
S_0 = \emptyset, \ S_n = \mathbb{P}(T_D).
$$

$(\mathbb{P}(T_D))_0$ is then decomposed into a disjoint union of irreducible $K^C$ orbits

$$
(\mathbb{P}(T_D))_0 = \bigsqcup_{1 \leq k \leq n} \{(S_k)_0 - (S_{k-1})_0\}.
$$
**Example 3.4.** Let $D$ be the type A tube domain of rank $n$. Then

$$D = SU(n, n)/SU(n) \times U(n).$$

One classically represents $D$ as a space of matrices

$$D = \{ Z \in M_{n,n}(\mathbb{C}) | I_n - \bar{Z}^t Z > 0 \}.$$

At the origin $0 \in D$,

$$(T_D)_0 \simeq M_{n,n}(\mathbb{C}).$$

The action of

$$K^C \simeq S(Gl(n, \mathbb{C}) \times Gl(n, \mathbb{C}))$$

defined by

$$M \mapsto AMB^{-1}, \text{ for } M \in M_{n,n}(\mathbb{C}) \text{ and } (A, B) \in Gl(n, \mathbb{C}) \times Gl(n, \mathbb{C})$$

gives the isotropy representation of $K^C$ on $(T_D)_0$. Then the rank of a vector $M \in (T_D)_0$ defined above is just the rank of $M$ as matrix. Let $(\tilde{S}_k)_0$ be the lifting of $(S_k)_0$ in $(T_D)_0$. Therefore, for $1 \leq k \leq n - 1$, we have

$$(\tilde{S}_k)_0 - (\tilde{S}_{k-1})_0 = S(Gl(n, \mathbb{C}) \times Gl(n, \mathbb{C}))/P_k$$

where

$$P_k = \{(A, B) \in S(Gl(n, \mathbb{C}) \times Gl(n, \mathbb{C})) | A \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix} B \}.$$

We claim that

$$(\tilde{S}_k)_0 - (\tilde{S}_{k-1})_0 = (2n - k)k.$$
where $A_{11}, B_{11}$ are $k$ by $k$ and $A_{22}, B_{22}$ are $n-k$ by $n-k$. Then the constraints required by $P_k$ give the linearly independent equations

\[
\begin{align*}
A_{11} &= B_{11} \\
A_{21} &= 0 \\
B_{12} &= 0
\end{align*}
\]

The number of equations is totally

\[k^2 + 2k(n-k) = k(2n-k).\]

But the dimension of $P_k$ is then

\[2n^2 - 1 - k(2n-k),\]

and the dimension of $(\tilde{S}_k)_0 - (\tilde{S}_{k-1})_0$ is

\[2n^2 - 1 - (2n^2 - 1 - k(2n-k)) = k(2n-k).\]

Now let $D$ be an irreducible bounded symmetric domain of rank $n$, and let

\[i : \Delta^n = \Delta_1 \times \cdots \times \Delta_n \hookrightarrow D\]

be a polydisc embedding. We are going to study the decomposition of $i^*\mathbb{W}$ into a direct sum of irreducible PVHSs over the polydisc. The following proposition is a key ingredient in the proof of Theorem 3.3.

**Proposition 3.5.** Let $p_i, 1 \leq i \leq n$ be the projection of the polydisc $\Delta^n$ into the $i$-th direct factor $\Delta_i$. Then each non-unitary irreducible component contained in $i^*\mathbb{W}$ is of form

\[p_i^*(L \otimes k_i) \otimes \cdots \otimes p_n^*(L \otimes k_n)\]

with

\[0 \leq k_i \leq 1, \text{ for } \forall i\]

where $L$ is the weight 1 PVHS coming from the standard representation of $SL(2, \mathbb{R})$. As a consequence, there exists a unique component of form

\[p_i^*L \otimes \cdots \otimes p_n^*L\]

in $i^*\mathbb{W}$ by the Calabi-Yau like property of $\mathbb{W}$.
Proof: It is known that the polydisc embedding
\[ i: \Delta^\times \hookrightarrow D, \]
determined by a maximal set of strongly orthogonal noncompact roots \( \Psi \subset h^* \), lifts to a group homomorphism
\[ \phi: Sl(2, \mathbb{R})^\times \to G. \]
Our problem is to study the decomposition of \( W \) with respect to all \( Sl(2, \mathbb{R}) \) director factors of \( \phi \).

We can in fact reduce to the study of only one direct factor. This is because a permutation of direct factors can be induced from an inner automorphism of \( G \), which implies the restriction to each direct factor is isomorphic to each other. Furthermore, we can assume that the highest root \( \tilde{\alpha} \) appears in our chosen \( \Psi \) without lose of generality. (cf. Prop.1, Ch.5,[7])

Let \( s_{\tilde{\alpha}} \) be the distinguished \( sl_2 \)-triple in the complex simple Lie algebra \( g^C \) corresponding to \( \tilde{\alpha} \). Let
\[ W = \bigoplus_{\beta \in \Phi} W_\beta \]
be the weight decomposition of \( W \) with respect to the Cartan subalgebra \( h \). Then by (14.9),[4], it is clear that all irreducible component in \( W \) with respect to \( s_{\tilde{\alpha}} \) is contained in
\[ W_{[\beta]} = \bigoplus_{n \in \mathbb{Z}} W_{\beta+n\tilde{\alpha}}. \]
And for the \( \beta \) appearing in the boundary of \( \Phi \), we know by (14.10),[4] that the largest component in \( W_{[\beta]} \) has dimension equal to \( \beta(H_{\tilde{\alpha}}) + 1 \). Our proof boils down to show the following claim.

Claim: For all \( \beta \) appearing in the boundary of \( \Phi \), we have
\[ |\beta(H_{\tilde{\alpha}})| \leq 1. \]
Proof of Claim: We first note that
\[ |\beta(H_{\tilde{\alpha}})| = \frac{2(\beta, \tilde{\alpha})}{(\tilde{\alpha}, \tilde{\alpha})} \]
defines a convex function on \( \Phi \). The maximal value will be achieved for the vertices of \( \Phi \), namely, the orbit of highest weight \( \omega \) of \( W \) under the Weyl group \( W(R) \). Since the Weyl group preserves the Killing form, we will show that
\[ |\omega(H_{s(\tilde{\alpha})})| \leq 1, \text{ for } \forall s \in W(R). \]

The above inequality holds obviously for \( s = id \). Actually, We let \( \alpha_0 \) be the simple root which is the special node determined by \( D \) in the last section. Then in the expression of \( \tilde{\alpha} \) as a linear combination of simple roots, the coefficient before \( \alpha_0 \) is one. Therefore,
\[
\omega(H_{\tilde{\alpha}}) = \frac{2(\omega, \tilde{\alpha})}{(\tilde{\alpha}, \tilde{\alpha})} = \frac{2(\omega, \alpha_0)}{(\tilde{\alpha}, \tilde{\alpha})} = \frac{(\alpha_0, \alpha_0)}{(\tilde{\alpha}, \tilde{\alpha})} = 1.
\]

Now we have to separate the exceptional cases from the ongoing proof because of the complicated description of the Weyl group in the exceptional cases. In the following, we use the same notation as the appendix of [1]. Let \( \{\varepsilon_1, \cdots, \varepsilon_l\} \) be the standard basis of the Euclidean space \( \mathbb{R}^l \), and \( \sigma \) denotes a permutation of index.

Type \( A_{l-1} \): The highest root \( \tilde{\alpha} = \varepsilon_1 - \varepsilon_l \). The Weyl group permutes the basis elements. All fundamental weights \( \omega_i, 1 \leq i \leq l-1 \) correspond to a special node.
Then
\[
|\omega_1(H_{s(\tilde{\alpha})})| = |(\omega_1, s(\tilde{\alpha}))| \\
= \left| \left( \sum_{j=1}^{i} \epsilon_j - \frac{i}{l+1} \sum_{j=1}^{l} \epsilon_j, \epsilon_{\sigma(1)} - \epsilon_{\sigma(l)} \right) \right| \\
= \left| \left( \sum_{j=1}^{i} \epsilon_j, \epsilon_{\sigma(1)} - \epsilon_{\sigma(l)} \right) \right| \\
\leq 1
\]

Type $B_l$: The highest root $\tilde{\alpha} = \epsilon_1 + \epsilon_2$. The Weyl group permutes the basis elements, or acts by $\epsilon_i \mapsto \pm \epsilon_i$. Then
\[
|\omega_1(H_{s(\tilde{\alpha})})| = |(\omega_1, s(\tilde{\alpha}))| \\
= \left| (\epsilon_1, \pm \epsilon_{\sigma(1)} \pm \epsilon_{\sigma(2)}) \right| \\
\leq 1
\]

Type $C_l$: The highest root $\tilde{\alpha} = 2\epsilon_1$. The Weyl group permutes the basis elements, or acts by $\epsilon_i \mapsto \pm \epsilon_i$. Then
\[
|\omega_l(H_{s(\tilde{\alpha})})| = \frac{1}{2} |(\omega_l, s(\tilde{\alpha}))| \\
= \frac{1}{2} \left| \sum_{i=1}^{l+1} \epsilon_i, \pm 2 \epsilon_{\sigma(1)} \right| \\
= 1
\]

Type $D_l$: The highest root $\tilde{\alpha} = 2\epsilon_1$. The Weyl group permutes the basis elements, or acts by $\epsilon_i \mapsto (\pm 1)_i \epsilon_i$ with $\prod_i (\pm 1)_i = 1$. We have three special nodes in this case. It suffices to check $\omega_1$ and $\omega_l$. For $\omega_1$, we have
\[
|\omega_1(H_{s(\tilde{\alpha})})| = |(\omega_1, s(\tilde{\alpha}))| \\
= \left| (\epsilon_1, \pm \epsilon_{\sigma(1)} \pm \epsilon_{\sigma(2)}) \right| \\
\leq 1
\]
For $\omega_l$, we have

$$|\omega_l(H_{s(\tilde{\alpha})})| = |(\omega_l, s(\tilde{\alpha}))|$$

$$= \frac{1}{2} \left| \sum_{i=1}^{l+1} \varepsilon_i, \pm \varepsilon_{\sigma(1)} \pm \varepsilon_{\sigma(2)} \right|$$

$$\leq 1$$

Now we treat with the exceptional cases. In the following, we shall compute the largest value of $|\beta(H_{\tilde{\alpha}})|$ among all weights $\beta$ in $\Phi$. The results will particularly imply the claim.

Type $E_6$: Let $\{\alpha_1, \cdots, \alpha_6\}$ be the set of simple roots of simple Lie algebra of type $E_6$. The highest root is then

$$\tilde{\alpha} = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6.$$ 

A 6-tuple $(a_1, \cdots, a_6)$ denotes the weight $\beta = \sum_{i=1}^{6} a_i \omega_i$. There are two special nodes in this case and it suffices to study either of them. The following table lists all elements of $\Phi$ for the fundamental representation $\omega_1$:

| $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ | $a_6$ |
|-------|-------|-------|-------|-------|-------|
| (1, 0, 0, 0, 0, 0) | (−1, 0, 1, 0, 0, 0) | (0, 0, −1, 1, 0, 0) | (0, 1, 0, −1, 1, 0) |
| (0, 1, 0, 0, −1, 1) | (0, −1, 0, 0, 1, 0) | (0, 1, 0, 0, −1) | (0, −1, 0, 1, −1, 1) |
| (0, 0, 1, −1, 0, 1) | (0, −1, 0, 1, 0, −1) | (1, 0, −1, 0, 0, 1) | (0, 0, 1, −1, 1, −1) |
| (1, 0, −1, 0, 1, −1) | (0, 0, 1, 0, −1, 0) | (−1, 0, 0, 0, 0, 1) | (1, 0, −1, 1, −1, 0) |
| (−1, 0, 0, 0, 1, −1) | (1, 1, 0, −1, 0, 0) | (−1, 0, 0, 1, −1, 0) | (1, −1, 0, 0, 0, 0) |
| (−1, 1, 1, −1, 0, 0) | (0, 1, −1, 0, 0, 0) | (−1, −1, 1, 0, 0, 0) | (0, −1, −1, 1, 0, 0) |
| (0, 0, 0, −1, 1, 0) | (0, 0, 0, 0, −1, 1) | (0, 0, 0, 0, 0, −1) |

For an element $(a_1, \cdots, a_6)$ in the above table, we have

$$\beta(H_{\tilde{\alpha}}) = (\beta, \tilde{\alpha})$$

$$= (\sum_{i=1}^{6} a_i \omega_i, \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6)$$

$$= a_1 + 2a_2 + 2a_3 + 3a_4 + 2a_5 + a_6.$$
According to this formula, it is straightforward to compute that the largest value of $|\beta(H_\alpha)|$ is equal to one.

Type $E_7$: Let $\{\alpha_1, \cdots, \alpha_7\}$ be the set of simple roots of simple Lie algebra of type $E_7$. We can choose the maximal set $\Psi$ of the strongly orthogonal noncompact roots to be

$$\{\psi_1 = 2\alpha_1+2\alpha_2+3\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+\alpha_7, \psi_2 = \alpha_2+\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+\alpha_7, \psi_3 = \alpha_7\}.$$  

It is simpler to use $\psi_3$ to verify our statement, instead of $\psi_1$ which is the highest root $\tilde{\alpha}$. As the last case, a 7-tuple $(a_1, \cdots, a_7)$ denotes the weight $\beta = \sum_{i=1}^7 a_i \omega_i$. The following table lists all elements of $\Phi$ for the fundamental representation $\omega_7$:

| Element | Element | Element | Element |
|---------|---------|---------|---------|
| $(0,0,0,0,0,1)$ | $(0,0,0,0,1,-1)$ | $(0,0,0,1,-1,0)$ | $(0,0,1,-1,0,0)$ |
| $(0,1,1,-1,0,0)$ | $(1,1,-1,0,0,0)$ | $(0,-1,1,0,0,0)$ | $(1,-1,-1,1,0,0)$ |
| $(-1,1,0,0,0,0)$ | $(1,0,0,-1,1,0)$ | $(-1,-1,0,1,0,0)$ | $(1,0,0,-1,1,0)$ |
| $(-1,0,1,-1,1,0)$ | $(1,0,0,0,0,-1)$ | $(0,0,-1,0,1,0)$ | $(-1,0,1,0,-1,1)$ |
| $(1,0,0,0,0,0,0)$ | $(0,0,0,0,0,1)$ | $(0,0,1,0,0,0,0)$ | $(0,0,1,0,0,0,0,0)$ |
| $(0,1,0,0,1,-1,1)$ | $(0,0,1,0,1,0,0)$ | $(0,0,1,0,0,0,0,0)$ | $(0,0,1,0,0,0,0,0,0)$ |
| $(0,0,1,0,0,1)$ | $(0,0,1,0,0,0,1)$ | $(0,0,1,0,0,0,0,1)$ | $(0,0,1,0,0,0,0,0,1)$ |
| $(0,0,0,1,0,0)$ | $(0,0,1,0,0,0)$ | $(0,0,1,0,0,0,0)$ | $(0,0,1,0,0,0,0,0)$ |
| $(0,0,0,0,1,0)$ | $(0,0,1,0,0,0,0)$ | $(0,0,1,0,0,0,0,0)$ | $(0,0,1,0,0,0,0,0,0)$ |

Then for an element $(a_1, \cdots, a_7)$ in the above table, we have

$$\beta(H_{\alpha_7}) = (\beta, \alpha_7)$$

$$= \left(\sum_{i=1}^7 a_i \omega_i, \alpha_7\right)$$

$$= a_7.$$
It is straightforward to see the largest value of $|\beta(H_{\alpha_j})|$ is one. This completes the whole proof.

We can now proceed to prove our main result Theorem 3.3.

**Proof:** It suffices to prove the isomorphism over $D$, and we obtain the claimed isomorphism by taking quotient under the left action of $\Gamma$. Since the constructions on both sides are $G$-equivariant, it is enough to show the isomorphism at the origin of $D$. Over the origin 0, we have the adjoint action of $K^C$ on the holomorphic tangent space $(T_D)_0$ and the dual action on $(\Omega_D)_0$. Since the Higgs field of a locally homogenous VHS is $G$-equivariant, for each $k$, $(J_k)_0 \subset (S^k\Omega_D)_0$ is $K^C$-invariant. This implies $C_k$ is $K^C$-invariant. So we can obtain a decomposition of $(\mathbb{P}(T_D))_0$ into disjoint union of $K^C$ orbits as follows:

$$(\mathbb{P}(T_D))_0 = \coprod_{1 \leq k \leq n} \{ (C_k)_0 - (C_{k-1})_0 \}.$$ 

For $1 \leq k \leq n$, we put

$$v_k = e_1 + \cdots + e_k.$$ 

It is clear that

$$v_k \in (S_k)_0 - (S_{k-1})_0$$

and $K^C(v_k) = (S_k)_0 - (S_{k-1})_0$. We claim that

Claim: For $1 \leq k \leq n$,

$$v_k \in (C_k)_0 - (C_{k-1})_0.$$ 

This claim implies the inclusion of $K^C$ orbit for each $k$

$$(S_k)_0 - (S_{k-1})_0 \subset (C_k)_0 - (C_{k-1})_0,$$ 

and hence the equality for each $k$. Therefore, for $1 \leq k \leq n - 1$,

$$(C_k)_0 = (S_k)_0.$$ 

Proof of Claim: We define $\gamma_k : \Delta \to D$ be the composition map

$$\Delta \xrightarrow{\text{diag.}} \Delta_1 \times \cdots \times \Delta_k \hookrightarrow \Delta^n \hookrightarrow D.$$
Obviously, $\gamma_k'(0) = v_k$. By Proposition 3.5, we know that

$$\gamma_k^* W = L^{\otimes k} \oplus V'$$

where $V'$ is a PVHS with weight $\leq k - 1$. Now that

$$\theta_{v_k}(E) = \theta_{v_k}(\gamma_k^* E),$$

we see

$$\theta^k(v_k) \neq 0, \theta^{k+1}(v_k) = 0.$$  

Using the criterion Lemma 3.2, the claim is then proved.

4. Enumeration of Calabi-Yau like PVHS and the Generating Property of Gross

Let $(E, \theta)$ be a system of Hodge bundles over $X$. We use the same notation as previous sections. We note that

$$I = \bigoplus_{k \geq 1} I_k$$

forms a graded ideal of the symmetric algebra

$$\text{Sym}(T_X) = \bigoplus_{k \geq 0} S^k T_X.$$  

It is trivial to see that

$$I_k = S^k(T_X), \text{ for } k \geq n + 1.$$  

In §5, Gross asked if $I$ is generated by $I_2$ for the Calabi-Yau like PVHS over an irreducible tube domain. We can assert this generating property for each Calabi-Yau like PVHS over an irreducible bounded symmetric domain.

**Theorem 4.1.** We use the same notation as Theorem 2.2. Then the graded ideal $I$, formed by the kernel of iterated Higgs field, is generated by the degree 2 graded piece $I_2$. That is, the multiplication map

$$I_2 \otimes S^{k-2}(T_X) \to I_k$$

is surjective for all $k \geq 2$. 
It suffices to prove the surjectivity for $k \leq n + 1$ where $n = \text{rank}(D)$. In fact, for $k \geq n + 2$, we have

$$I_2 \otimes S^{n-1}(T_X) \otimes (T_X)^{\otimes k-n-1} \to I_{n+1} \otimes (T_X)^{\otimes k-n-1}$$
$$= S^{n+1}(T_X) \otimes (T_X)^{\otimes k-n-1}$$
$$\to S^k(T_X) = I_k.$$ 

By the integrality of Higgs field, the above surjective map factors through $I_2 \otimes S^{k-2}(T_X)$. As the proof of Theorem 3.3, we can work on the level of bounded symmetric domain and prove the statement at the origin as $K$-representations. The theorem will be proved case by case. In the classical case, we shall also describe the system of Hodge bundles $(E, \theta)$ associated with the Calabi-Yau like PVHS $\mathcal{W}$ using the Grassmannian description of classical symmetric domain.

Let $D$ be an irreducible bounded symmetric domain. By fixing an origin of $D$, we obtain an equivalence of categories of homogenous vector bundles and finite dimensional complex representations of $K$. Since $K$ has one dimensional center, a finite dimensional complex $K$-representation is written as $\mathbb{C}(l) \otimes V$ where $V$ is a representation of the semisimple part $K'$ of $K$ and is determined by the induced action of the complexified Lie algebra $\mathfrak{k}'\mathbb{C}$. In the following, the same notation for a $K$-representation and the corresponding homogenous vector bundle will be used when the context causes no confusion. All the isomorphisms are isomorphism between homogenous bundles. A highest weight representation of $sl(n, \mathbb{C})$ will be denoted interchangeably by $\Gamma_{a_1, \ldots, a_{n-1}}$ and $S_{\lambda}(\mathbb{C}^n)$ (cf. §15.3, [4]).

4.1. **Type A.** The irreducible bounded symmetric domain of type A is $D_{p,q}^I = G/K$ where

$$G = SU(p, q), K = S(U(p) \times U(q)).$$

Let $V = \mathbb{C}^{p+q}$ be a complex vector space equipped with a Hermitian symmetric bilinear form $h$ of signature $(p, q)$. Then $D_{p,q}^I$ parameterizes the dimension $p$
complex vector subspaces $U \subset V$ such that
\[ h|_U : U \times U \to \mathbb{C} \]
is positive definite. This forms the tautological subbundle $S \subset V \times D$ of rank $p$ and denote by $Q$ the tautological quotient bundle of rank $q$. We have the natural isomorphism of holomorphic vector bundles
\[ T_{D_{p,q}} \cong \text{Hom}(S, Q). \quad (\ast)_1 \]
The standard representation $V$ of $G$ gives rise to a weight 1 PVHS $\mathbb{V}$ over $D_{p,q}$, and its associated Higgs bundle
\[ F = F^{1,0} \oplus F^{0,1}, \eta = \eta^{1,0} \oplus \eta^{0,1} \]
is determined by
\[ F^{1,0} = S, F^{0,1} = Q, \eta^{0,1} = 0, \]
and $\eta^{1,0}$ is defined by the above isomorphism. The Calabi-Yau like PVHS is
\[ \mathbb{W} = \bigwedge^p \mathbb{V} \]
and its associated system of Hodge bundles $(E, \theta)$ is then
\[ (E, \theta) = \bigwedge^p (F, \eta). \]
Since
\[ \mathfrak{p}^C = \mathfrak{sl}(p, \mathbb{C}) \oplus \mathfrak{sl}(q, \mathbb{C}), \]
by Schur’s lemma, a finite dimensional irreducible complex representation of $\mathfrak{p}^C$ is of form
\[ \Gamma_{a_1, \ldots, a_{p-1}} \otimes \Gamma'_{b_1, \ldots, b_{q-1}} \]
We put $V_1 = \mathbb{C}^p$ to be the representation space $\Gamma_{0, \ldots, 0}$ of $\mathfrak{sl}(p, \mathbb{C})$ and $V_2 = \mathbb{C}^q$ the representation space $\Gamma'_{0, \ldots, 0, 1}$ of $\mathfrak{sl}(q, \mathbb{C})$. In the remaining subsection, we shall assume $p \leq q$ in order to simplify the notations in the argument.
Lemma 4.2. We have isomorphism
\[ T_{D_{p,q}} \simeq V_1 \otimes V_2. \]

Then, for \( k \geq 2 \), we have isomorphism
\[ S^k(T_{D_{p,q}}) \simeq \bigoplus_{\lambda} S_\lambda(V_1) \otimes S_\lambda(V_2) \]
where \( \lambda \) runs through all partitions of \( k \) with at most \( p \) rows. Under this isomorphism, the \( k \)-th iterated Higgs field for \( k \leq p \),
\[ \theta^k : S^k(T_{D_{p,q}}) \rightarrow \Hom(E^{p,0}, E^{p-k,k}) \]
is identified with the projection map onto the irreducible component
\[ \bigoplus_{\lambda} S_\lambda(V_1) \otimes S_\lambda(V_2) \rightarrow S_{\lambda^0}(V_1) \otimes S_{\lambda^0}(V_2) \]
where \( \lambda^0 = (1, \ldots, 1) \).

Proof: By \((\ast)_1\), we have isomorphism
\[ T_{D_{p,q}} \simeq V_1 \otimes V_2. \]

The formula in Ex.6.11,\([4]\) gives the decomposition of \( S^k(V_1 \otimes V_2) \) with respect to \( sl(p, \mathbb{C}) \oplus sl(q, \mathbb{C}) \):
\[ S^k(V_1 \otimes V_2) = \bigoplus_{\lambda} S_\lambda(V_1) \otimes S_\lambda(V_2) \]
where \( \lambda \) runs through all partitions of \( k \) with at most \( p \) rows. Since the center of \( K \) acts on \( (T_{D_{p,q}})_0 \) trivially, it acts on \( (S^k(T_{D_{p,q}}))_0 \) trivially too. Hence the second isomorphism of the statement follows. For the last statement, it suffices to show \( \theta^k \) is a non-zero map because \( \Hom(E^{p,0}, E^{p-k,k}) \) is irreducible. But this follows directly from the definition of the Higgs field \( \theta \) as \( p \)-the wedge power of \( \eta \). The lemma is proved.

From the lemma, we know that
\[ \theta^2 \simeq pr : \Gamma_{0,\ldots,2} \otimes \Gamma'_{0,\ldots,2} \oplus \Gamma_{0,\ldots,0,1,0} \otimes \Gamma'_{0,\ldots,0,1,0} \rightarrow \Gamma_{0,\ldots,0,1,0} \otimes \Gamma'_{0,\ldots,0,1,0}. \]
So by definition,
\[ I_2 \cong \Gamma_{0,\ldots,0,2} \otimes \Gamma'_{0,\ldots,0,2}. \]

Now we proceed to prove that \( I_2 \otimes S^{k-2}(T_{D_{p,q}}) \) generates \( I_k \). By the above lemma and the Formula 6.8 [4], we have
\[
I_2 \otimes S^{k-2}(T_{D_{p,q}}) \cong \bigoplus_{\mu} (S^2(V_1) \otimes S_{\mu}(V_1)) \otimes (S^2(V_2) \otimes S_{\mu}(V_2))
\]
\[
= \bigoplus_{\mu} \left[ \bigoplus_{\nu^1_{\mu}} (S_{\nu^1_{\mu}}(V_1) \otimes S_{\nu^2_{\mu}}(V_2)) \right]
\]
\[
= \bigoplus_{\mu} \bigoplus_{\nu^1_{\mu},\nu^2_{\mu}} (S_{\nu^1_{\mu}}(V_1) \otimes S_{\nu^2_{\mu}}(V_2))
\]

where \( \mu \) runs through all partitions of \( k - 2 \) with at most \( p \) rows and for a fixed \( \mu, \nu_i^i, i = 1, 2 \) runs through those Young diagrams by adding two boxes to the Young diagram of \( \mu \) without in the same column. Let \( \lambda \) be a Young diagram corresponding to a direct factor of \( I_k \) under the isomorphism oin the above lemma. Since
\[
\bigoplus_{\mu} \bigoplus_{\nu_{\mu}} (S_{\nu_{\mu}}(V_1) \otimes S_{\nu_{\mu}}(V_2)) \subset \bigoplus_{\mu} \bigoplus_{\nu^1_{\mu},\nu^2_{\mu}} (S_{\nu^1_{\mu}}(V_1) \otimes S_{\nu^2_{\mu}}(V_2))
\]

it is enough to show that \( \lambda \) can be obtained by a Young diagram \( \mu \) by adding two boxes without in the same column. Actually, by the above lemma the partition \( \lambda \) of \( I_k \) has the property that, either for some \( 1 \leq i_0 \leq p - 1 \),
\[
\lambda_{i_0} > \lambda_{i_0 + 1} = 0,
\]
or for some \( 1 \leq i_0 \leq p \),
\[
2 \leq \lambda_1 = \cdots = \lambda_{i_0} > \lambda_{i_0 + 1} = 0.
\]

In the first case, we can choose \( \mu \) as
\[
\mu_i = \begin{cases} 
\lambda_i - 1 & \text{if } i = i_0, i_0 + 1, \\
\lambda_i & \text{otherwise}.
\end{cases}
\]
In the second case, we choose $\mu$ as

$$\mu_i = \begin{cases} 
\lambda_i - 2 & \text{if } i = i_0, \\
\lambda_i & \text{otherwise}.
\end{cases}$$

The proof of Theorem 4.1 in the type A case is therefore completed.

4.2. Type B, Type $D^R$. For $n \geq 3$, we let

$$G = \text{Spin}(2, n), K = \text{Spin}(2) \times_{\mu_2} \text{Spin}(n).$$

Then $D^IV_n = G/K$ is the bounded symmetric domain of type B when $n$ is odd, of type $D^R$ when $n$ even. Let $(V_\mathbb{R}, Q)$ be a real vector space of dimension $n + 2$ equipped with a symmetric bilinear form of signature $(2, n)$. Then $D^IV_n$ is one of connected components parameterizing all $Q$-positive two dimensional subspace of $V_\mathbb{R}$. In order to see clearer the complex structure of $D^IV_n$, we complexify $(V_\mathbb{R}, Q)$, to obtain $(V = V_\mathbb{R} \otimes \mathbb{C}, Q)$. Then it is known that $D^IV_n$ is an open submanifold of the quadratic hypersurface defined by $Q = 0$ in $\mathbb{P}(V) \simeq \mathbb{P}^{n+1}$, which is just the compact dual of $D^IV_n$. For a $Q$-isotropic line $L \subset V$, we define its polarization hyperplane to be

$$P(L) = \{ v \in V | Q(L, v) = 0 \}.$$

So for each point of $D^IV_n$, we obtain a natural filtration of $V$ by

$$L \subset P(L) \subset V.$$

Varying the points on $D^IV_n$, the above filtration yields a filtration of homogenous bundles

$$S \subset Q(S) \subset V \times D^IV_n.$$

On the other hand, we have a commutative diagram

$$\begin{array}{ccc}
T_{D^IV_n} & \xrightarrow{\simeq} & \text{Hom}(L, \frac{Q(L)}{L}) \\
\cap & & \cap \\
T_{\mathbb{P}(V), [L]} & \xrightarrow{\simeq} & \text{Hom}(L, \frac{V}{L})
\end{array}$$

whose top horizontal line gives the isomorphism of tangent bundle

$$T_{D^IV_n} \simeq \text{Hom}(S, \frac{Q(S)}{S}).$$

\[ (*)_2 \]
We also notice that $Q$ descends to a non-degenerate bilinear form on $\frac{Q(L)}{L}$, so that we have a natural isomorphism

$$\left(\frac{Q(S)}{S}\right)^* \simeq \frac{Q(S)^*}{S}. \quad (\ast')_2$$

We put now

$$E^{2,0} = S, E^{1,1} = \frac{Q(S)}{S}, E^{0,2} = \frac{V \times D^!_n}{Q(S)},$$

and

$$\theta^{2,0} \rightarrow E^{1,1} \otimes \Omega_{D^!_n},$$

$$\theta^{1,1} \rightarrow E^{0,2} \otimes \Omega_{D^!_n}$$

are determined by the isomorphism $(\ast)_2$ and $(\ast')_2$, and $\theta^{0,2} = 0$. The Higgs bundle

$$(E = \bigoplus_{p+q=2} E^{p,q}, \theta = \bigoplus_{p+q=2} \theta^{p,q})$$

is the associated system of Hodge bundle with Calabi-Yau like PVHS $\mathcal{W}$.

Let $m = \left[\frac{n}{2}\right]$ be the rank of $\mathfrak{so}(n)$, and $\Gamma_{a_1,\ldots,a_m}$ denotes a highest weight representation of $\mathfrak{so}(n)$. In terms of this notation, we have

$$E^{2,0} \simeq \mathbb{C}(-2) \otimes \Gamma_{0,\ldots,0}, E^{1,1} \simeq \mathbb{C} \otimes \Gamma_{1,0,\ldots,0}, E^{0,2} \simeq \mathbb{C}(2) \otimes \Gamma_{0,\ldots,0}.$$  

The following easy lemma makes Theorem 4.1 in the cases of type $B$ and type $D^R$ clear.

**Lemma 4.3.** We have isomorphisms

$$T_{D^!_n} \simeq \mathbb{C}(2) \otimes \Gamma_{1,0,\ldots,0},$$

$$S^2(T_{D^!_n}) \simeq \mathbb{C}(4) \otimes \Gamma_{2,0,\ldots,0} \oplus \mathbb{C}(4) \otimes \Gamma_{0,\ldots,0}$$

$$I_2 \simeq \mathbb{C}(4) \otimes \Gamma_{2,0,\ldots,0},$$

$$I_2 \otimes T_{D^!_n} \simeq \mathbb{C}(6) \otimes \Gamma_{3,0,\ldots,0} \oplus \mathbb{C}(6) \otimes \Gamma_{1,0,\ldots,0} \oplus \mathbb{C}(6) \otimes \Gamma_{1,1,0,\ldots,0}$$

$$S^3(T_{D^!_n}) \simeq \mathbb{C}(6) \otimes \Gamma_{3,0,\ldots,0} \oplus \mathbb{C}(6) \otimes \Gamma_{1,0,\ldots,0}.$$
4.3. **Type C.** We fix \( n \geq 2 \). Let

\[
G = Sp(2n, \mathbb{R}), K = U(n).
\]

Then \( D_{n}^{III} = G/K \) is the bounded symmetric domain of type C. \( D_{n}^{III} \) is known as the Siegel space of degree \( n \). Let \((V_{\mathbb{R}}, \omega)\) be a real vector space of dimension \( 2n \) equipped with a skew symmetric bilinear form \( \omega \). As before, we denote also by \((V, \omega)\) the complexification. And

\[
h(u, v) := i\omega(u, \bar{v})
\]

defines a hermitian symmetric bilinear form over \( V \). Then \( D_{n}^{III} \) parameterizes the maximal \( \omega \)-isotropic and \( h \)-positive complex subspaces of \( V \). The standard representation \( V \) of \( G \) gives a weight 1 \( \mathbb{R} \)-PVHS \( V \) over \( D_{n}^{III} \). Let \((F, \eta)\) be the associated Higgs bundle with \( V \). Then \( F^{1,0} \) is simply the tautological subbundle over \( D_{n}^{III} \) and \( F^{0,1} \) is the \( h \)-orthogonal complement of \( F^{1,0} \). Clearly, we have a natural embedding of bounded symmetric domains

\[
\iota: D_{n}^{III} \hookrightarrow D_{n,n}^{I}.
\]

It induces a commutative diagram:

\[
\begin{array}{ccc}
T_{D_{n}^{III}} & \xrightarrow{\simeq} & S^2(F^{0,1}) \\
\cap & \downarrow & \cap \\
\iota^*(T_{D_{n,n}^{I}}) & \xrightarrow{\simeq} & (F^{0,1}) \otimes^2
\end{array}
\]

The Higgs field \( \eta^{1,0} \) is defined by the composition of maps

\[
T_{D_{n}^{III}} \simeq S^2(F^{0,1}) \hookrightarrow (F^{0,1}) \otimes^2 \simeq \text{Hom}(F^{1,0}, F^{0,1}).
\]

The Calabi-Yau like PVHS \( \mathbb{W} \) is the unique weight \( n \) sub-PVHS of \( \bigwedge^n(V) \). In fact, we have a decomposition of \( \mathbb{R} \)-PVHS

\[
\bigwedge^n(V) = \mathbb{W} \oplus V'
\]

where \( V' \) is a weight \( n - 2 \) \( \mathbb{R} \)-PVHS. Therefore the corresponding Higgs bundle \((E, \theta)\) to \( \mathbb{W} \) is a sub-Higgs bundle of \( \bigwedge^n(F, \eta) \).
Let $V_1 = (F^{0,1})_0$ be the standard representation of $K$. It is straightforward to obtain the following

**Lemma 4.4.** We have isomorphism

$$T_{D^{III}} \cong S_{(2)}(V_1).$$

Then, for $k \geq 2$, we have isomorphism

$$S^k(T_{D^{III}}) \cong \bigoplus \lambda S_{\lambda}(V_1)$$

where $\lambda = \{\lambda_1, \cdots, \lambda_l\}$ runs through all partitions of $2k$ with each $\lambda_i$ even and $l \leq n$. Under this isomorphism, for $k \leq n$, the $k$-th iterated Higgs field $\theta^k$ is identified with the projection map onto the irreducible component $S_{\lambda^0}(V_1)$ where $\lambda^0 = (2, \cdots, 2)$.

By the lemma, we know that

$$\theta^2 \cong pr : S_{(4)}(V_1) \oplus S_{(2,2)}(V_1) \to S_{(2,2)}(V_1)$$

and then $I_2 \cong S_{(4)}(V_1)$. Applying the Formula 6.8,[4] to decompose $I_2 \otimes S^{k-2}(T_{D^{III}})$, we obtain

$$I_2 \otimes S^{k-2}(T_{D^{III}}) \cong S_{(4)}(V_1) \otimes \bigoplus \mu S_{\mu}(V_1)$$

$$\cong \bigoplus \mu (S_{(4)}(V_1) \otimes S_{\mu}(V_1))$$

$$= \bigoplus \mu [(\bigoplus \nu S_{\nu}(V_1))]$$

where $\mu$ runs through all partitions of $2(k - 2)$ with the property as that in Lemma 4.4, and for a fixed $\mu$, $\nu_\mu$ runs through those Young diagrams by adding four boxes to the Young diagram $\mu$ without in the same column. The partition $\lambda$ of an irreducible component in $I_k$ is of form

$$\lambda_1 \geq \cdots \geq \lambda_s > \lambda_{s+1} = \cdots = \lambda_l \geq 2.$$
We may then take \( \mu \) to be
\[
\mu_i = \begin{cases} 
\lambda_i - 2 & \text{if } i = s, l, \\
\lambda_i & \text{otherwise}
\end{cases}
\]
And we then define \( \nu \) by adding two boxes to \( \lambda_s \) and \( \lambda_l \) in \( \mu \) respectively to obtain the starting \( \lambda \). Therefore Theorem 4.1 in the type C case is proved.

4.4. Type \( D^\mathbb{H} \). For \( n \geq 3 \), we let
\[
G = SO^*(2n), K = U(n).
\]
Then \( D^\mathbb{H}_n = G/K \) is the bounded symmetric domain of type \( D^\mathbb{H} \). We recall that
\[
G \simeq \{ M \in \text{Sl}(2n, \mathbb{C}) | MI_{n,n} M^* = I_{n,n}, MS_n M^* = S_n \}
\]
where \( I_{n,n} \) denotes the matrix \( \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix} \) and \( S_n \) denotes the matrix \( \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} \).

Let \( (V, h, S) \) be a complex vector space of dimension \( 2n \) equipped with a hermitian symmetric form \( h \) and symmetric bilinear form \( S \), where, under the identification \( V \simeq \mathbb{C}^{2n} \), \( h \) is defined by the matrix \( I_{n,n} \) and \( S \) is defined by the matrix \( S_n \). Then \( D^\mathbb{H}_n \) parameterizes all \( n \)-dimensional \( S \)-isotropic and \( h \)-positive complex subspaces of \( V \). The standard representation \( V \) of \( G \) determines a weight 1 PVHS \( V \). Its associated Higgs bundle \( (F, \eta) \) is determined in a similar manner as type C case. Namely, \( F^{1,0} \) is simply the tautological subbundle and \( F^{0,1} \) is its \( h \)-orthogonal complement. The natural embedding
\[
\iota' : D^\mathbb{H}_n \hookrightarrow D_{n,n}^U
\]
induces a commutative diagram:
\[
\begin{array}{ccc}
T_{D^\mathbb{H}_n} & \overset{\sim}{\longrightarrow} & \Lambda^2(F^{0,1}) \\
\cap & \downarrow & \cap \\
\iota'^*(T_{D_{n,n}}) & \overset{\sim}{\longrightarrow} & (F^{0,1})^{\otimes 2}
\end{array}
\]
And the Higgs field \( \eta^{1,0} \) is induced by the composition of maps
\[
T_{D^\mathbb{H}_n} \simeq \Lambda^2(F^{0,1}) \hookrightarrow (F^{0,1})^{\otimes 2} \simeq \text{Hom}(F^{1,0}, F^{0,1}). \quad (\ast)_4
\]
The Calabi-Yau PVHS $\mathbb{W}$ comes from a half spin representation. We write the corresponding Higgs bundle as

$$(E = \bigoplus_{p+q=[n^2]} E^{p,q}, \theta = \bigoplus_{p+q=[n^2]} \theta^{p,q}).$$

Then the Hodge bundle is

$$E^{p,q} = \bigwedge^{n-2q} F^{1,0},$$

and the Higgs field $\theta^{p,q}$ is induced by the natural wedge product map

$$\bigwedge^2 F^{0,1} \otimes \bigwedge^{2q} F^{0,1} \to \bigwedge^{2q+2} F^{0,1}.$$

While type $D^H$ case enjoys many similarity with type C case, there is one difference we would like to point out. That is, the Calabi-Yau PVHS $\mathbb{W}$ is not a sub-PVHS of $\bigwedge^n V$. In fact, the PVHS $\bigwedge^n V$ is the direct sum of two irreducible PVHSs. One of them, say $V'$, has

$$\bigwedge^n (F^{1,0}) \otimes \bigwedge^0 (F^{0,1}) \simeq (\bigwedge^n (F^{1,0}))^\otimes 2$$

as the first Hodge bundle. For this irreducible $V'$, we have an inclusion of PVHS

$$V' \subset Sym^2(\mathbb{W}).$$

Let $V_1 = (F^{0,1})_0$ be the dual of the standard representation of $K$. It is straightforward to obtain the following

**Lemma 4.5.** We have isomorphism

$$T_{D^{H^I}} \simeq S_{(1,1)}(V_1).$$

Then, for $k \geq 2$, we have isomorphism

$$S^k(T_{D^{H^I}}) \simeq \bigoplus_{\lambda} S_{\lambda}(V_1)$$

where $\lambda = \{\lambda_1, \cdots, \lambda_l\}$ runs through all partitions of $2k$ with $l \leq n$ and each entry of the conjugate $\lambda'$ of $\lambda$ even. Under this isomorphism, for $k \leq [\frac{n}{2}]$, the $k$-th iterated Higgs field $\theta^k$ is identified with the projection map onto the irreducible component $S_{\lambda^0}(V_1)$ with $\lambda^0 = (k, k)$. 
By the lemma, we know that
\[ \theta^2 \simeq pr : S_{(1,1,1,1)}(V_1) \oplus S_{(2,2)}(V_1) \to S_{(2,2)}(V_1) \]
and then \( I_2 \simeq S_{(1,1,1,1)}(V_1) \). By Formula 6.9,\[4\], we have
\[ I_2 \otimes S^{k-2}(T_{D_n^H}) \simeq S_{(1,1,1,1)}(V_1) \otimes \bigoplus \mu S_{\mu}(V_1) \]
\[ \simeq \bigoplus \mu (S_{(1,1,1,1)}(V_1) \otimes S_{\mu}(V_1)) \]
\[ = \bigoplus \mu \left( \bigoplus \nu S_{\nu}(V_1) \right) \]
where \( \mu \) runs through all partitions of \( 2(k - 2) \) with the property as that in Lemma 4.5 and for a fixed \( \mu \), \( \nu_\mu \) runs through those Young diagrams by adding four boxes to the Young diagram of \( \mu \) without in the same row. We observe that we are in the conjugate case of that of type C. Theorem 4.1 in type \( D^H \) case follows easily.

4.5. **Type E.** There are two exceptional reducible bounded symmetric domains. We first discuss the \( E_6 \) case. In this case,
\[ G = E_{6,2}, K = U(1) \times_{\mu_4} Spin(10). \]

Then \( D^V = G/K \) is a 16-dimensional bounded symmetric domain of rank 2. There are two special nodes in the Dynkin diagram of \( E_6 \). But they induce isomorphic bounded symmetric domains. We take the first node so that the fundamental representation corresponding to this special node is \( W_{27} \). Let \( (E = \oplus_{p+q=2} E^{p,q}, \theta) \) be the corresponding Higgs bundle to \( W \). Then we have isomorphism
\[ E^{2,0} \simeq \mathbb{C}(-2), E^{1,1} \simeq \mathbb{C} \otimes \Gamma_{0,0,0,1,0}, E^{0,2} \simeq \mathbb{C}(2) \otimes \Gamma_{1,0,0,0,0}. \]

Furthermore, it is straightforward to obtain the following
Lemma 4.6. We have isomorphism

\[ T_X \simeq \mathbb{C}(2) \otimes \Gamma_{0,0,0,1,0} \]

\[ S^2(T_X) \simeq \mathbb{C}(4) \otimes \Gamma_{0,0,0,2,0} \oplus \mathbb{C}(4) \otimes \Gamma_{1,0,0,0,0} \]

\[ I_2 \simeq \mathbb{C}(4) \otimes \Gamma_{0,0,0,2,0} \]

\[ I_2 \otimes T_X \simeq \mathbb{C}(6) \otimes \Gamma_{0,0,0,3,0} \oplus \mathbb{C}(6) \otimes \Gamma_{1,0,0,1,0} \oplus \mathbb{C}(6) \otimes \Gamma_{0,0,1,1,0} \]

\[ S^3(T_X) \simeq \mathbb{C}(6) \otimes \Gamma_{0,0,0,3,0} \oplus \mathbb{C}(6) \otimes \Gamma_{1,0,0,1,0} \]

We continue to discuss the left case, which have already appeared in [6]. Let

\[ G = E_{7,3}, K = U(1) \times_{\mu_3} E_6. \]

Then \( D^{VI} = G/K \) is of dimension 27 and rank 3. We refer the reader to §4,[6] for the description of Hodge bundles. The corresponding lemma to Lemma 4.6 is the following

Lemma 4.7. We have isomorphism

\[ T_X \simeq \mathbb{C}(2) \otimes \Gamma_{1,0,0,0,0,0} \]

\[ S^2(T_X) \simeq \mathbb{C}(4) \otimes \Gamma_{2,0,0,0,0,0} \oplus \mathbb{C}(4) \otimes \Gamma_{0,0,0,0,0,1} \]

\[ I_2 \simeq \mathbb{C}(4) \otimes \Gamma_{2,0,0,0,0,0} \]

\[ I_2 \otimes T_X \simeq \mathbb{C}(6) \otimes \Gamma_{1,0,0,0,0,1} \oplus \mathbb{C}(6) \otimes \Gamma_{3,0,0,0,0,0} \oplus \mathbb{C}(6) \otimes \Gamma_{1,0,1,0,0,0} \]

\[ S^3(T_X) \simeq \mathbb{C}(6) \otimes \Gamma_{1,0,0,0,0,1} \oplus \mathbb{C}(6) \otimes \Gamma_{3,0,0,0,0,0} \oplus \mathbb{C}(6) \otimes \Gamma_{0,0,0,0,0,0} \]

\[ I_3 \simeq \mathbb{C}(6) \otimes \Gamma_{1,0,0,0,0,1} \oplus \mathbb{C}(6) \otimes \Gamma_{3,0,0,0,0,0} \]

\[ I_2 \otimes S^2(T_X) \simeq \mathbb{C}(8) \otimes \Gamma_{4,0,0,0,0,0} \oplus \mathbb{C}(8) \otimes \Gamma_{2,0,0,0,0,1} \oplus \mathbb{C}(8) \otimes \Gamma_{0,0,0,0,0,2} \]

\[ \oplus \mathbb{C}(8) \otimes \Gamma_{2,0,1,0,0,0} \oplus \mathbb{C}(8) \otimes \Gamma_{0,0,2,0,0,0} \oplus \mathbb{C}(8) \otimes \Gamma_{0,0,1,0,0,1} \]

\[ \oplus \mathbb{C}(8) \otimes \Gamma_{1,0,0,0,0,0} \oplus \mathbb{C}(8) \otimes \Gamma_{1,1,0,0,0,0} \oplus \mathbb{C}(8) \otimes \Gamma_{2,0,0,0,0,1} \]

\[ S^4(T_X) \simeq \mathbb{C}(8) \otimes \Gamma_{4,0,0,0,0,0} \oplus \mathbb{C}(8) \otimes \Gamma_{2,0,0,0,0,1} \oplus \mathbb{C}(8) \otimes \Gamma_{0,0,0,0,0,2} \]

\[ \oplus \mathbb{C}(8) \otimes \Gamma_{1,0,0,0,0,0} \]

Lemma 4.6 and Lemma 4.7 make it clear that the generating property of Gross also holds for the exceptional cases. Then the proof of Theorem 4.1 is completed.
References

[1] Bourbaki,N: *Lie groups and Lie algebras*, Chapters 4-6, Springer, 2002.

[2] Carlson,J; Green,M; Griffiths,P; Harris,J; *Infinitesimal variations of Hodge structures* (I). Compositio Mathematica 50(1983), 109-205.

[3] Deligne,P; *Variétés de Shimura: interprétation modulaire, et techniques de construction de modèles canoniques*, Proc. Sympos. Pure Math., XXXIII, 247-289, 1979.

[4] Fulton,W; Harris,J; *Representation theory, A first course*, GTM 129.

[5] Gerkmann,R; Sheng,M; Zuo,K; *Disproof of modularity of moduli space of CY 3-folds coming from eight planes of $\mathbb{P}^3$ in general positions*, in Preparation.

[6] Gross,B; *A remark on tube domains*, Math. Res. Lett., Vol.1, 1-9, 1994.

[7] Mok,N; *Metric rigidity theorems on Hermitian locally symmetric manifolds*, Series in Pure Mathematics, Vol. 6. 1989.

[8] Zucker,S; *Locally homogenous variations of Hodge structure*, L’Enseignement Mathématique, 27, No.3-4, 243-276, 1981.

E-mail address: msheng@math.ecnu.edu.cn

E-mail address: kzuo@mathematik.mainz-uni.de