Phases fluctuations, self-similarity breaking and anomalous scalings in driven nonequilibrium critical phenomena

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Abstract
We study in detail the dynamic scaling of the three-dimensional (3D) Ising model driven through its critical point on finite-sized lattices. We show explicitly that while finite-size scaling (FSS) at fixed driving rates and finite-time scaling (FTS) on fixed lattice sizes are satisfied for one set of all four observables we measure in their respective scaling regimes, they can be violated for the other set of the observables even in the same regimes. The different behaviors of the two sets of the observables indicate that the usual critical fluctuations can be divided into the so-called phases fluctuations and magnitude fluctuations. The self-similarity of criticality can also be divided into intrinsic and extrinsic self-similarities. The numerical results show that the phases fluctuations lead to the different behaviors while breaking the extrinsic self-similarity gives rise to the violations of the scalings. The set of the observables that violate the scalings is further divided into a primary observable and a secondary observable. Their different leading behaviors enable us to identify four breaking-of-extrinsic-self-similarity exponents for rectifying the violations of either FSS or FTS in either heating or cooling. Crossovers from the extrinsic-self-similarity-breaking-controlled regimes to the usual FSS or FTS regimes are also discussed. In addition, qualitatively different behaviors of the magnitude fluctuations in cooling and in heating and their origin are revealed. Moreover, both FTS and FSS are good down to quite low temperatures in cooling with the extrinsic self-similarity. This indicates that phase ordering can only have an effect at even lower temperatures. Besides, from the quality of curve collapses, we find that the 3D dynamic critical exponent \(z\) in heating and in cooling appears identical but the two-dimensional ones different. Our results demonstrate that new exponents are generally required for scaling in the whole driven process near criticality once the lattice size is taken into account. This opens a new door in critical phenomena and suggest that much is yet to be explored in driven nonequilibrium critical phenomena.

Keywords: self-similarity breaking, phases fluctuations, anomalous scaling, finite-time scaling, finite-size scaling

(Some figures may appear in colour only in the online journal)
1. Introduction

Experimental advances in manipulating real-time evolution of ultracold atoms [1–4] have stimulated renewed interest in driven nonequilibrium critical phenomena [5]. The well-known Kibble–Zurek (KZ) mechanism studies nonequilibrium topological defect formations during a continuous phase transition. First proposed in cosmology [6, 7] and then applied in condensed-matter physics [8, 9], the KZ mechanism divides the cooling process from a disordered phase to a symmetry-broken order phase into adiabatic, impulse, and another adiabatic stages. In the nonequilibrium impulse stage, the evolution of the system is assumed to be frozen. The correlation length of the system then ceases to grow exactly at the boundary between the first adiabatic stage and the impulse stage. This frozen correlation length thus determines the density of the topological defects formed during the transition. Using just equilibrium relations of the correlation length and correlation time in critical phenomena, one can find the dependence of the frozen correlation length and hence the defect density on the cooling rate. This relationship is known as KZ scaling. Although it seems to agree with numerical simulations [10–19], most experimental results require additional assumptions for interpretation of their consistency with the theory [20]. This leads to a reconsideration of the process from a statistical context.

One way is to consider phase-ordering kinetics [21]. This occurs when a system is quenched instantaneously from a disordered phase at high temperatures into a two-phase coexistence region at low temperatures. It has been found that at long times when the characteristic size of the ordered phases is large enough, this size scales with time with a dynamic ordering exponent that is different from the dynamic critical exponent. Upon connecting the two kinds of dynamics, it was concluded that phase ordering was important below the critical temperature [22].

In statistical physics, the divergent correlation time in critical phenomena results in the notorious critical slowing down, which is a stringent situation for accurate estimates of critical properties. Upon noticing the spatial analogue of the divergent correlation length and the well-known finite-size scaling (FSS) [23–26] to circumvent it, finite-time scaling (FTS) was independently proposed [27, 28] on the basis of a renormalization-group theory for driving [29]. In this theory, the rate of the linear driving, which drives the system through its critical point, introduces a finite timescale that serves as the temporal analogue of the system size in FSS. Moreover, the system itself can be driven out of equilibrium, because the finite driving timescale becomes inevitably shorter than the diverging correlation time once the system is close enough to its critical point. The renormalization-group theory of such driven nonequilibrium critical phenomena can be generalized to a weak external driving of an arbitrary form and a series of nonequilibrium phenomena, such as negative susceptibility and competition of various equilibrium and nonequilibrium regimes and their crossovers, as well as the violation of fluctuation-dissipation theorem and hysteresis, arise from the competition of the various timescales stemming from the parameters of the driving [5]. FTS has also been successfully applied to many systems both theoretically [5, 28, 30–42] and experimentally [43, 44]. Equilibrium and nonequilibrium initial conditions have also been considered [5, 45]. Applying to the KZ mechanism, one finds that the impulse stage is just the FTS regime, in which the driven timescale is shorter than the correlation time, and that the KZ scaling results from a special value of the scaling function which describes the entire process [32].

FTS works well in all the three stages of the driving, although the adiabatic stages obey the usual (quasi-) equilibrium critical scaling governed by the equilibrium correlation length and time while the diabatic impulse stage satisfies the nonequilibrium scaling controlled by the driven length and time. Accordingly, a first question is whether the phase ordering or the FTS dominates the cooling process through a critical point. If the phase ordering matters, it seems that one has to introduce a new exponent, the dynamic ordering exponent, to the cooling process [22]. This then leads to another more fundamental question.

So far, in the KZ scaling in particular and the driven nonequilibrium critical phenomena in general, the equilibrium critical exponents including the dynamic critical exponent have been found to characterize the scaling well, apart from the possible effect of the phase ordering. No new exponents surprisingly appear to be needed to describe the apparent driven nonequilibrium process which induce a lot of nonequilibrium phenomena [5] including the KZ mechanism for nonequilibrium topological defect formation. Therefore, a fundamental question is that whether this is true or not.

A seemingly possible case appeared when one extended the KZ scaling to a finite-sized system [33]. In contrast with the previous theoretical results of FTS in a finite-sized system [28], a special scaling of a magnetization squared with a complicated exponent was suggested and verified for the two-dimensional (2D) Ising model exactly at its critical point [33]. However, this exponent was shown to be that of the susceptibility and a revised FTS form was proposed and confirmed within the FTS theory [32]. This revised FTS leads to distinct scalings in heating and in cooling for the order parameter and its squared at the critical point, though not for the susceptibility [32]. Therefore, no new exponents are needed again. Still, there exist some peculiar features [32]. The dynamic critical exponent $z$ estimated from data collapses right at the critical point in heating and in cooling was also found to be different.

In a recent letter [46], through studying the whole driving dynamics and hence the whole scaling functions rather than just at the critical point of the 2D Ising model, it was discovered that the FTS and the FSS of some observable quantities are violated either in heating or in cooling even in their respective FTS and FSS regimes. Such violations of scaling were found to originate from a novel source, the self-similarity breaking of the so-called phases fluctuations. Note the plural form of phase used both to emphasize that at least two phases are involved owing to the symmetry breaking and to distinguish it from the usual phase of a complex field. New breaking-off-self-similarity exponents are then needed. Moreover, these exponents lead to different leading critical exponents for the disordered and the ordered phases rather than identical leading
critical exponents but different amplitudes in the usual critical phenomena.

Symmetry breaking is well known and plays a pivotal role in modern physics. Self-similarity is a kind of symmetry and appears ubiquitously in nature [47, 48]. In contrast to rigorous mathematical objects such as fractals that are self-similar on all scales [47], in nature, self-similarity holds inevitably only within a certain range of scales [48]. This might be regarded as a certain kind of self-similarity breaking with the only consequence of a self-similarity limited to a certain range of scales. Our results thus show that self-similarity can indeed be broken with significant consequences [46].

The self-similarity and its breaking concern with the phases fluctuations. These are the fluctuations of the disymmetric phases within a system near its critical point. For the Ising model, the disymmetric phases are the spin up phase and the spin down phase. These phases can either coexist in or exist alone as the low-temperature ordered phase. Moreover, they can appear as fluctuating clusters even above the critical point. The main point is that, in the proximity of the critical point, the correlated clusters are large enough that can be regarded as the disymmetric phases (with an appropriate order parameter) and the flippings among these phases are then the phases fluctuations. Their spatial self-similarity is embodied in the fact that different-sized lattices contain on average identical number of the large clusters by adjusting the cluster sizes through varying the driving rates, while their temporal self-similarity is reflected in the fact that different driven times, originating from different driving rates, comprise identical times of cluster flippings due to different lattice sizes.

Here, using the three-dimensional (3D) Ising model, we show explicitly how one set of four measured observables satisfy and how the other set of the observables violate FSS (FTS) at fixed driving rates (on fixed lattice sizes) in the same FSS (FTS) regime, and how all the observables satisfy both FSS and FTS once the extrinsic self-similarity holds. We also provide detailed procedures in estimating the four breaking-of-extrinsic-self-similarity, abbreviated as Bressy, exponents for FTS and FSS in heating and in cooling from the different behavior of the observables that violate the scalings. These results, with much more details for the 3D model than those for the 2D model in [46], are new outcomes that confirm the 2D results even though the 3D Bressy exponents themselves have already been listed in [46]. More importantly, a series of further new results are presented. We argue that the different behaviors of the two sets of the observables indicate that the usual critical fluctuations can be divided into the phases fluctuations and magnitude fluctuations. The self-similarity of criticality can also be divided into intrinsic and extrinsic self-similarities, which are determined by the criticality and external conditions such as system sizes and external drivings, respectively. The phases fluctuations resulting in the different behaviors are the necessary condition for the violations of the scalings. Only when the extrinsic self-similarity is broken, can the violations come into reality. The set of the observables that violate the scalings is further divided into a primary observable whose scaling functions is a direct power-law of the scaled variable with the power related to the Bressy exponent and a secondary observable whose scaling function possesses a regular term and may thus appear not to violate the scaling. Crossovers from the extrinsic-self-similarity-breaking-controlled regimes to the usual FSS or FTS regimes are also discussed. In addition, qualitatively different behaviors of the magnitude fluctuations in cooling and in heating are revealed. However, once the extrinsic self-similarity is in place, the peculiar cooling behavior disappears, reflecting the relation between the two kinds of fluctuations. Moreover, both FTS and FSS are good down to quite low temperatures in cooling with the extrinsic self-similarity. This indicates that phase ordering can only have an effect at even lower temperatures. Besides, from the quality of the entire curve collapses for different dynamic critical exponent in heating, we find that the 3D z values in heating and in cooling appear identical but the 2D ones different.

The layout of the remaining part of the paper is the following. First in section 2, we present the comprehensive theory for both FTS and FSS and their combined effects owing to the phases fluctuations. Then in section 3, we clarify the phases fluctuations and their self-similarity. After defining the model and the two sets of observables we measure and providing simulation details in section 4, we report in sections 5 and 6 numerical results to examine the theory, to estimate the Bressy exponents, and to differentiate the possible z values in heating and cooling. First, in section 5, as an appreciation of the richness of the driven nonequilibrium critical phenomena, we first display the evolution of one set of the observables in the FTS regime in heating and cooling, which exhibits qualitatively different behaviors. Then, to test the theory, we study sequentially FSS at fixed rates (section 5.1), FTS on fixed lattice sizes (section 5.2), and the full scaling forms (section 5.4) by fixing a scaled variable both in heating and in cooling. The cases and their details that violate either FSS or FTS in either heating or cooling are summarized in section 5.3 and the exponents originate from the Bressy are extracted in section 5.5 through first identifying the primary observables and then curve collapsing. Second, in section 6, we study the scaling collapses in the whole heating process instead of just at the critical point to investigate the effect stemming from the different dynamic critical exponents in heating and cooling. Finally, in section 7, we provide a detailed summary along with a lot of problems for further study and possible experimental observations of the results.

2. Theory

In this section, we first review the comprehensive theory for both FTS and FSS and their combination, the revised FTS, together with the crossovers between them. Then, it is generalized to account for the new exponents. We specify the spatial and temporal self-similarities of the phases fluctuations in the theory by extending the picture of self-similarity [41] in space to real time. The Bressy exponents are then introduced to rectify the violated scalings when the extrinsic self-similarity is broken. In this regard, the observables that violate scalings are classified into two catalogs, the primary and the secondary observables, whose behavior is derived.
Consider a system with a lateral size \( L \) that is driven from one phase through a critical point to another phase by changing the temperature \( T \) with a rate \( R \geq 0 \) such that

\[
T - T_c \equiv \tau = \pm R t,
\]

where \( T_c \) is the critical point and the plus (minus) sign corresponds to heating (cooling). We have set the zero time at the critical point for simplicity [27]. Accordingly, the time \( t \) can be both negative and positive. To derive the scaling behavior, it is helpful to begin with the scaling hypothesis for the susceptibility \( \chi \)

\[
\chi(\tau, R, L^{-1}, H) = b^{\gamma/\nu} \chi(b^{1/\nu}, R b^\nu, L^{-1} b),
\]

where \( b \) is a scaling factor, \( \gamma, \beta, \nu, \) and \( r \) are the critical exponents for the susceptibility, the order parameter \( M \), the correlation length, and the rate, respectively. In the absence of either \( R \) or \( L \), equation (2) can be obtained from the renormalization-group theory [27–29, 49, 50]. Here, we simply combine them as an ansatz as usual [28]. We have also replaced the time \( t \) with \( R \) because they are related by equation (1). Moreover, from the same equation, we find [51]

\[
r = z + 1/\nu
\]

because \( t \) transforms as \( b^{-z} \) with \( z \) being the dynamic critical exponent [52].

From equation (2), various scaling forms can be readily derived by choosing proper scaling factors [27–29, 32]. On the one hand, setting \( b = R^{-1/r} \) leads to the FTS form

\[
\chi = R^{-\gamma/\nu} F_T(\tau R^{-1/r}, L^{-1} R^{-1/r}),
\]

with \( F_T \) being a universal scaling function. On the other hand, assuming \( b = L \) in equation (2) results in

\[
\chi = L^{\gamma/\nu} F_S(\tau L^{1/r}, RL'),
\]

which is the FSS form under driving, where \( F_S \) is another scaling function. Note that we have neglected constants in the scaled variables of the scaling functions for simplicity.

With the scaling forms equations (4) and (5), various regimes controlled by their dominated length scales can be defined. The FTS scaling form equation (4) dominates if the scaling function \( F_T \) is analytic when its scaling variables are vanishingly small [27–29, 32]. This implies \( R^{-1/r} \ll |\tau|^{-\nu} \) and \( R^{-1/r} \ll L \), all sufficiently large so that the system lies in the scaling regime. These relations dictate that the driving length scale \( \xi_R \sim R^{-1/r} \), be the shortest among the correlation length \( \xi \sim |\tau|^{-\nu} \) and the system size \( L \) in the FTS regime. Because the correlation time \( \zeta \) is asymptotically proportional to \( \xi^\beta \), viz., \( \zeta \sim \xi^\beta \), associated with a finite rate \( R \) is then a driven timescale \( \zeta_R \sim R^{-\beta/\nu} \) beyond which the driving changes appreciably and thus comes the name of FTS. Accordingly, the above conditions can be rephrased as the driven timescale is the shortest among the other long timescales. Similarly, the FSS regime requires \( L \) is the shortest among the other long length scales. One can readily envision another regime governed by the equilibrium correlation length, which is the quasi-equilibrium regime.

Moreover, properties of the regimes can be deduced. Because the scaling functions are analytic at vanishingly small scaled variables, the leading behavior of each regime is thus determined by the factor in front of the scaling function. Therefore, in the FTS regime, the leading behavior of \( \chi \) is proportional to \( R^{-\gamma/\nu} \). This means that as \( R \) decreases, \( \chi \) must increase. When \( R \to 0 \), equilibrium is recovered and \( \chi \) diverges for \( L = \infty \). Moreover, \( \chi R^{\beta/\nu} \) versus \( L^{-1} R^{-1/r} \) at \( \tau = 0 \) must be a horizontal line in the FTS regime, because \( L^{-1} R^{-1/r} \) can be neglected in the regime [32]. Similarly, in the FSS regime, the leading behavior of \( \chi \) is \( L^{\gamma/\nu} \) and so \( \chi L^{-\gamma/\nu} \) versus \( RL' \) must be a horizontal line, in conformity with the negligibility of \( RL' \) in the regime. In the thermodynamic limit \( L \to \infty \), \( \chi \) again diverges for \( R = 0 \) as expected.

In addition, crossovers between different regimes and their characteristics can also be identified [32]. As all the regimes are governed by the same fixed point, every scaling form can also describe the other regimes besides its own one, though, in that case, the scaling function is singular [32]. As a result, all the scaling functions are related. For example, if \( R \) and \( L \) are changed such that \( \xi_R \) becomes longer than \( L \), the system crosses over from the FTS regime to the FSS regime and thus

\[
\mathcal{F}_T = \left( L^{-1} R^{-1/r} \right)^{-\gamma/\nu} \mathcal{F}_S
\]

with proper changes in the scaled variables. This gives rise to a slope change from the FTS regime to the FSS regime [32]. Examples will be given below shortly.

All observables must show similar scaling forms to the susceptibility with their own exponents in the critical region. For example, the order parameter \( M \) must behave

\[
M = R^{\beta/\nu} F_T(\tau R^{-1/r}, L^{-1} R^{-1/r})
\]

in the FTS regime, while in the FSS regime, it becomes

\[
M = L^{-\beta/\nu} F_S(\tau L^{1/r}, RL'),
\]

where \( F_T \) and \( F_S \) are again scaling functions. They are related by \( F_T = (L^{-1} R^{-1/r})^{\nu/\beta} F_S \). Accordingly, exactly at \( \tau = 0 \), \( MR^{-\beta/\nu} \) versus \( L^{-1} R^{-1/r} \) ought to be a horizontal line for such \( R \) and \( L \) that \( L^{-1} R^{-1/r} \ll 1 \), viz, in the FTS regime; whereas it changes to an inclined line with a slope of \( \beta/\nu \) in the FSS regime in which \( L^{-1} R^{-1/r} \) is large [32].

The above scalings are all standard forms of FTS and FSS. However, it was found that, exactly at \( \tau = 0 \), equations (7) and (8) both give rise to different leading characteristics in heating and cooling in the FTS regime although they describe the scalings in both cases well [32]. In particular, the above slope and its change in the frame of \( MR^{-\beta/\nu} \) versus \( L^{-1} R^{-1/r} \) is only valid in heating. In cooling, however, the slope in the FTS regime is not zero (horizontal) but \( d/2 \) [32], where \( d \) is the spatial dimensionality. These different leading characteristics indicate different leading exponents for heating and cooling. To account for this behavior, an additional ingredient is needed.

This ingredient is the phases fluctuations. In the FTS regime, the system divides into regions of a size \( \xi_R \) on average. During cooling from a disordered phase to a symmetry-broken...
ordered phase, the ordered direction of each such region fluctuates freely among all possible directions because of the absence of a symmetry breaking direction. Each direction is just one possible phase of the ordered phase and thus the fluctuations between different directions are just the phases fluctuations. The average of the magnetization of these regions ought to be vanished in the thermodynamic limit owing to the central limit theorem [32]. Therefore, in equation (7), L cannot be neglected no matter how small the scaled variable $L^{-1}R^{-1/\nu}$ is. To satisfy the central limit theorem, $F_T$ must behave as

$$F_T = (L^{-1}R^{-1/\nu})^{d/2}F_{TS}$$  \hspace{1cm} (9)

for small $L^{-1}R^{-1/\nu}$ [32], where $F_{TS}$ is another scaling function. As a result,

$$M = L^{-d/2}R^{-1/2\nu}F_{TS}(rR^{-1/\nu}, L^{-1}R^{-1/\nu}),$$  \hspace{1cm} (10)

upon application of the scaling laws [53–55]

$$\alpha = 2 - d\nu, \quad \alpha + 2\beta + \gamma = 2.$$  \hspace{1cm} (11)

Comparing equation (10) with (7), one sees that the leading FTS behavior of $M$ in cooling is now qualitatively different from heating. Because of equation (9), the slope of $MR^{-1/\nu}$ versus $L^{-1}R^{-1/\nu}$ is now $d/2$ instead of 0 in the FTS regime. In order to have a 0 slope, one has to use $ML^{d/3}R^{-2\nu}$ at $\tau = 0$ from the revised FTS form, equation (10). In this way, the slope of the FSS regime then becomes $-d/2 + \beta/\nu = -\gamma/2\nu$ in cooling [32]. Note that in the FSS regime $L$ is shorter than the correlation length in principle. Accordingly, the system itself is just one phase on average and thus no anomaly equivalent to equation (10) is needed for FSS.

In practical verification of the scaling forms, when there exist more than one scaled variable in the scaling functions, one needs to fix the other variables for exactness. If they are not fixed, for an approximation, good scaling collapses can also be obtained if the leading behavior has been extracted and the other scaled variables are sufficiently small, provided that corrections to scaling [56] that have not been considered are negligible. However, there exist cases in which such an approximative approach does not work no matter how small the other scaled variables are. The revised FTS introduced above is just such a case. Moreover, as revealed in reference [46], the standard FTS, equation (7), and FSS, equation (8), and even the revised FTS, equation (10) itself, all can surprisingly fall into this class in either heating or cooling. In these cases in which the phases fluctuations are strong, the other scaled variables have to be fixed. Note, however, that this is different to the case in which the scaled variables are not small and thus competition among the variables originating from several length scales has to be considered and high-dimensional parameter spaces have to be invoked [39].

Fixing a scaled variable fixes, in fact, the ratio of the two scales involved in the variable and therefore ensures self-similarity. If one fixes $L^{-1}R^{-1/\nu}$ for instance, one fixes the ratio between $L$ and $\xi_R$. In the FTS regime, on the one hand, this means that a series of systems of different sizes $L$ always contain the same number of $\xi_R$ regions because of their different $R$ values used, as was schematically illustrated with checkerboards in figure 1 of reference [46]. This spatial extrinsic self-similarity of the phases fluctuations is indispensable for some observables to have good scaling collapses. In the FSS regime, on the other hand, $\xi_R$ might appear to be irrelevant especially for large values of $L^{-1}R^{-1/\nu}$, because the whole finite-size system is just a phase on average. However, fixing $L^{-1}R^{-1/\nu}$ also implies fixes $L^{-1}R^{-\nu/\gamma}$, the ratio of a finite-size relaxation time over the driven time. This dictates that the fluctuating phases on different lattice sizes survive for the same ratio of duration under driving and thus ensures the extrinsic self-similarity of the phases fluctuations in time. In such a way, good scaling collapses ensure provided that corrections to scaling can be ignored. An example of the temporal self-similarity is given in figures 1(b) and (c), where the lattices of sizes $L = 10$ and $L = 20$ share identical $R^{-\nu/\gamma}/L^\nu \approx 2/1.5$ due to their different heating rates $R$.

If the spatial or temporal extrinsic self-similarity is broken, it is found that some observables violate the standard FSS or FTS and even the revised FTS either in the ordered phase or the disordered phase [46]. The violated scaling of an observable can be rectified by a breaking-of-extrinsic-self-similarity, or Bressy in short, exponent $\sigma$ that is defined as [46]

$$f(L^{-1}R^{-1/\nu}) \propto \left(L^{-1}R^{-1/\nu}\right)^\sigma, \text{ for FTS},$$

$$f(RL') \propto (RL')^{\nu/\beta}, \text{ for FSS},$$  \hspace{1cm} (12)

where $f$ is the scaling function of the observable. The definitions factor out the rate exponent $r$. For a non-integer $\sigma$, the scaling function then depends on the scaled variable singularly in the phase. With such an exponent $\sigma$, the leading behavior of the observable in the phase in which its scaling is violated is then qualitatively different from its usual one in the other phase [46]. This is in stark contrast to the usual equilibrium critical phenomena. There, one can also define different critical exponents above and below $T_c$ [53–55]. However, they ought to be equal because of the absence of singularities across the critical isotherm [54]. Only the amplitudes of the leading singularities and thus the scaling functions above and below $T_c$ differ.

In equation (12), we have not written a scaling function on the right-hand side similar to equation (9). This is because, in the case of $R = 0$ and $L = \infty$, or $RL' = 0$ and $L^{-1}R^{-1/\nu} = 0$, respectively, the usual equilibrium FSS and FTS, respectively, must recover. This means that there is a crossover to the usual behavior under the same condition. Moreover, for the large values of the scaled variables, equation (12) is poor owing to the crossover of the FSS and FTS regimes. As a consequence, equation (12) is only valid in some ranges of the variables. Except for $RL' = 0$ and $L^{-1}R^{-1/\nu} = 0$, the Bressy-exponent dominated regime is believed to be valid for arbitrarily small $RL'$ and $L^{-1}R^{-1/\nu}$ for the FSS in both heating and cooling and the FTS in cooling. Accordingly, the crossovers from this regime to the usual FSS and FTS occur abruptly at the two special points. However, for FTS in heating, we will see in section 5.5 that the crossover happens abruptly at a finite $L^{-1}R^{-1/\nu}$.

The violations of scaling appears always in one set of the observables, either $\chi$ and $\langle m \rangle$ or $\chi'$ and $\langle |m| \rangle$, to be defined in
numbers over the lines are the results of the corresponding timescales divided by the secondary observable.

ular term because the first two terms on the right-hand side directly by equation (12), while the secondary one can also be accounted for by the same \( \sigma \) but needs a constant regular term.

Table 1. Summary of the observables and the phases that violate FSS and FTS in heating and in cooling. The breaking-of-extrinsic-self-similarity exponent \( \sigma \) that rectifies the violated scaling of the primary (pri) observable (obs) in each entry is also given, including the 2D results [46]. The secondary (sec) observables in the parentheses exhibit apparently good scaling because of their regular terms. The figures in the parentheses mark the ranges within which the scaling collapses are of similar quality.

| Pri obs | \( \chi \) | \( \chi' \) | \( \langle |m| \rangle \) | \( \langle |m| \rangle \) |
|---------|---------|---------|----------------|----------------|
| Sec obs | \( \langle m \rangle \) | \( \langle |m| \rangle \) | Disordered | Ordered |
| Phase   | Ordered | Ordered | Disordered | Ordered |
| \( \sigma \) (2D) | -2.75(15) | \( \beta/2\nu \) | -0.625(25) | \( \frac{3}{2} \pm \frac{1}{2} \) |
| \( \sigma \) (3D) | -0.15 ± 0.15 | \( \beta/2\nu \) | -0.584(50)/\( \nu \) | \( \frac{3}{2} \pm \frac{1}{2} \pm \frac{1}{2} \) |

of equation (14) associated with the pair \( \chi' \) and \( \langle |m| \rangle \) are analytic. Note that an observable obeying the standard FSS or FTS under driving needs only a single scaling function no matter whether under heating or cooling. This is in stark contrast to the usual critical phenomena in which two scaling functions are needed for the two phases. Accordingly, we have not written \( F_S \chi' \) and \( F_S \langle |m| \rangle \) for the corresponding parts in \( F_S \langle m \rangle \) for \( 2D \). To verify equation (14), one has to move the two regular terms to the right. The result is obviously \(- F_S \chi - \) from the first equality of the same equation. Therefore, if \( \chi \) collapses well, \( \langle m \rangle \) will naturally collapse well too. This same reasoning applies to the following cases as well.
For the case FSS in cooling, it is $\chi'$ that is the primary observable. A similar equation to equation (14) with the two pairs of observables interchanged follows.

In the case of FTS in heating, it is again $\chi'$ that is the primary observable. The secondary observable is then

$$F_{T[m]}^{2} = F_{T} + \left( L^{-1}R^{-1/\nu} \right)^{d} F_{T[m]}^{2} - F_{T} \chi' + \text{constant} - \left( L^{-1}R^{-1/\nu} \right)^{\sigma},$$

from equations (4), (7), (11), and (12). The scaling variable $L^{-1}R^{-1/\nu}$ in front of $F_{T[m]}^{2}$ does not affect the analyticity of the latter. In cooling, the secondary observable is $\chi'$, whose scaling function becomes

$$F_{T} \chi' = F_{T} \chi - \left( L^{-1}R^{-1/\nu} \right)^{-d} F_{T[m]}^{2} \chi' + \text{constant} - \left( L^{-1}R^{-1/\nu} \right)^{2\sigma - d},$$

since $\langle m \rangle = 0$. The exponent $d$ in the second line comes from equation (9) for the revised scaling since we define $\sigma$ using $F_{T[m]}^{2}$ instead of $F_{T}[m]^{2}$. This is because we will see in section 5 that the usual FTS of $\langle |m| \rangle$ becomes valid if the extrinsic self-similarity of the phases fluctuations is kept and thus the revised scaling is also a result of the extrinsic self-similarity breaking.

If the extrinsic self-similarity breaking is not taken into account, the above theory yields a single scaling form for $\chi$ either in FTS or in FSS but one more combined form, the revised FTS form, for $M$. However, we will see from figure 3 below that $\chi$ appears to have two different appearances but $M$ seems to be similar in heating and cooling. Nevertheless, we will show in the following that the above theory describes the phenomena well and phase ordering can only play a role at rather low temperatures in cooling provided that the phases fluctuations are properly taken into account.

3. Phases fluctuations and their self-similarity

In this section, we elaborate the idea of the phases fluctuations and their self-similarity, although we have introduced them in the last section and will see more numerical evidences for them in section 5 below. We will argue that critical fluctuations consist of two parts: one is the phases fluctuations and the other is magnitude fluctuations. The phases fluctuations can further possess two kinds of self-similarity: one is intrinsic and the other extrinsic. The former one gives rise to the usual critical scaling while the other is related to external conditions such as the system size and can thus be broken with significant consequences.

When a system is subjected to a continuous phase transition from a homogeneous disordered state to an ordered phase, its symmetry is broken spontaneously, resulting in a number of disymmetric phases. For a finite group, this number is given by the index of the group of the new phase in the parent group. Accordingly, the ordered phase can be either a mixture of various such phases forming domains or just a single phase depending on the boundary conditions, defects, external fields and so on. In particular, for the Ising model, spin down and spin up are the two possible phases at low temperatures because of the broken up-down symmetry. Both can either coexist as domains or appear alone. Of course, at a finite temperature sufficiently lower than the critical point, a stable spin-up phase, for example, contains a number of down spins that are fluctuating randomly both spatially and temporally. These fluctuating down spins give rise to the correct equilibrium magnetization of the spin-up phase at the temperature. However, it is apparent that these fluctuating spins forming clusters of various sizes (typified by the correlation length of several lattice sizes) cannot be referred to as the spin-down phase, though they have the same origin. Instead, they are called thermal or spin fluctuations.

Near the critical point, it is well known that the divergent correlation length is responsible for the singular behavior of all measurable quantities and serves as the only relevant length scale. A physical picture of this correlation length scaling hypothesis is droplets of overturned spins in the sea of up spins [57]. However, a droplet of the large size $\xi$ cannot contain all down spins. Accordingly, a better and frequently-used picture is that there are fluctuations on all length scales up to $\xi$ [57] such that the system is statistically self-similar within $\xi$. In equilibrium, in the thermodynamic limit, and in the absence of an externally applied field, at $T > T_{c}$, both spin directions are equally probable and the above picture of self-similarity may be suitable at low temperatures. However, for $T < T_{c}$ but close to $T_{c}$, this picture cannot produce a finite magnetization since each cluster has nearly zero net magnetization. A more realistic picture is then that these large clusters contain predominantly up or down spins such that they have a finite net magnetization, a picture which is an extension of the above picture of the stable phase. Each such large droplet can be regarded as a disymmetric phase that contains fluctuating smaller droplets of the other disymmetric phase. The smaller droplets, in turn, may also contain even smaller fluctuating droplets. However, the averaged magnetization of each large droplet ought to be roughly the equilibrium magnetization of the symmetry-broken phase itself at the temperature with a mean fluctuation of $\sqrt{\xi} L^{1/2}$ or so. These clusters are not frozen in but are fluctuating unless at low temperatures, at which equilibrium domains of only up or down spins may form after coarsening and coagulation of the clusters. Since the magnetization of the symmetry-broken phase decreases to zero when the temperature is raised to the critical point, at which each droplet has equal number of up and down spins on average, a situation which persists up to $T > T_{c}$, such a picture of clusters matches the self-similar one and applies to those temperatures as well. Of course, at too high temperatures, the correlation length is short and the fluctuations become the usual thermal ones. Accordingly, we refer to such large clusters as phases and their fluctuations as ‘phases’ fluctuations. In nonequilibrium situations, these large ‘phases’ clusters may not overturn and thus no phases fluctuations exist. However, the picture of large phases clusters is still valid. The Ising model is a generic example of a system with a discrete symmetry. For a system with a continuous symmetry [53–55], the Goldstone modes [58] are also the phases fluctuations, because
they just change the directions of the broken symmetry and thus the dissymmetric phases.

The most direct evidence of the above picture of the phases fluctuations comes from the time domain rather than the space domain as has been described above. As pointed out in section 2, the system size is the governing scale in FSS. The system is itself a large cluster and hence a phase on average. Therefore, it must acquire a finite magnetization and turn from up to down and vice versa in a time of $L^2$ on average. This is clearly seen in figure 1, in particular, figure 1(a). The survival time $L^2$ is, of course, fluctuating widely, as can also be observed from the figure. However, for the same lattice size $L$, it is the same on average, as is manifest upon comparing figure 1(a) with 1(b) and figure 1(c) with 1(d) for two lattices of identical sizes but different values of $R$. In addition, figure 1 shows that the magnitude of $m$ decreases as $t$ elapses and hence $T$ increases in heating and finally reaches zero with relatively smaller fluctuations.

In figure 2 we show four spatial configurations in cooling as an evidence of the phases fluctuations in space domain. Although the picture is not simply that of a checkerboard schematically illustrated in reference [46] because of the large fluctuations in the cluster sizes, their values of the magnetization, their boundaries, their neighbors and so on, and thus not as convincing as the picture in time domain, large clusters of roughly the size $\xi_R \sim R^{1/\nu} \approx 20$ containing predominantly up or down spins and turning over from up to down and vice versa are evident as the temperature is lowered.

Having confirmed the existence of the phases fluctuations, we can thus separate the critical fluctuations into two parts. One is to form large clusters that are the dissymmetric phases with roughly the order parameter at the temperature at which the system sits and the other is the flipping of these large clusters that are accordingly referred to as the phases fluctuations. One might argue that such a picture had already implied in the previous picture of self-similarity and that it were not necessary to emphasize the phase nature of the large clusters and to coin a new word. However, as will be seen in section 5 below, this picture is indispensable for accounting for the distinctive behavior of two sets of the observables, $\langle m \rangle$ and $\chi$ versus $\langle |m| \rangle$ and $\chi'$ to be defined in section 4 below. In particular, the phases fluctuations manifest themselves as the difference between the two sets of the observables while the magnitude fluctuations are probed by the set of the observables with absolute values that remove the plus and minus signs. This difference can be transparently envisioned from figure 1(a): a huge number of samples having different time series similar to figure 1(a) but fluctuating at random instants will be averaged to a vanishing $\langle m \rangle$ but a finite $\langle |m| \rangle$.

We note that the difference between $\langle m \rangle$ and $\langle |m| \rangle$ stems from the fluctuations of $m$, the magnetization itself, instead of its spatial distribution, of a single sample at a particular instant or temperature. Accordingly, a fixed spatial checkerboard of clusters containing predominantly up and down spins gives rise to a fixed $m$ independent on time or the temperature, in contrast to a fluctuating $m$ shown in figure 1. Different samples may still have their fixed $m$ with different magnitudes and signs. As a result, the difference in the two sets of the observables is still visible. However, figure 2 shows clearly that the clusters are fluctuating rather than fixed in space near the critical point. Only at temperatures sufficiently lower than the critical point can a spatially fixed domain structure exist. Upon taking into account the unambiguously fluctuating phases in time exhibited in figure 1, it is therefore justified to emphasize the fluctuating nature of the phases and to single the phases fluctuations out from the critical fluctuations.

The above division of the critical fluctuations looks like those in systems with a continuous symmetry breaking. In these systems, the critical fluctuations can be separated into longitudinal fluctuations around the symmetry-broken direction and transverse fluctuations, which are Goldstone modes that change the direction of the symmetry-broken phase and are thus the phases fluctuations as mentioned above. Accordingly, the separation of the critical fluctuations into magnitude and phases fluctuations appears as a generalization of the separation into longitudinal and transverse fluctuations in systems with continuous symmetry breaking. However, the longitudinal behavior itself is naively identical with a scalar theory that describes the critical fluctuations of the Ising model [53], if contribution from the transverse directions is ignored [59, 60]. In other words, the longitudinal fluctuations can themselves be divided into the magnitude fluctuations and
the phases fluctuations. Therefore, the similarity of the two divisions is only superficial. Here, the magnitude fluctuations are the critical fluctuations exclusive of the phases fluctuations. They are exhibited by the observables with absolute values that just remove the flipping of the large phases clusters. However, the phases fluctuations also contribute to the magnitude fluctuations via changing the magnitude of $m$, as will be discussed toward the end of section 5.5 below. We will also see in section 5 below that the magnitude fluctuations are generally far weaker than the phases fluctuations.

Phases fluctuations emphasize directly the phase nature of the fluctuations and thus emphasize the origin of the critical fluctuations, the symmetry-broken disymmetric phases. They also provide a somehow vivid picture of the critical fluctuations. More importantly, the phases clusters and their fluctuations are crucial in understanding the dynamics of a driven transition from a disordered phase through a critical point to an ordered phase. A simple example is the KZ mechanism of topological defect formation. As the system is cooled toward its critical point, its correlation length increases and thus the size of the fluctuating phases increases. It is the spatial boundaries of the different disymmetric phases that form the KZ topological defects. Accordingly, this KZ mechanism for the defect formation is transparent from the point of view of phases fluctuations. Upon assuming a frozen correlation length, it thus gives rise directly to the density of the topological defects. However, on the one hand, the phases in neighboring regions have a considerable probability to be identical and thus merge into a larger droplet as evident in figure 2, different from the schematic picture of a regular checkerboard shown in figure 1 of reference [46]. This substantially reduces the number of the topological defects. On the other hand, each phase cluster also contains fluctuations of droplets of the other phases of various sizes smaller than the cluster size and the smaller droplets may contain even smaller droplets of their other phases, thus increasing significantly the number of the topological defects. Consequently, the real density of topological defects can be far different from those reckoned directly from the correlation length. This inevitably leads to disagreement of the KZ scaling with experiments. In addition, phase ordering may occur within the adiabatic region in the ordered phase [22]. Therefore, the density of topological defects may not be a good observable to characterize the dynamics.

A less well-known example is the revised FTS form in cooling. Its origin is also the phases fluctuations. Because each fluctuating phase droplet is independent on the other droplets, all these fluctuating droplets must thus obey the central limit theorem as they are identically distributed. This leads to the special volume factor for equation (10) in the FTS regime for a finite-sized system [32].

Moreover, we will see that the phases fluctuations are crucial not only to cooling but also to heating [46]. This is related to the self-similarity of the fluctuations. Self-similarity of a critical system is described by the renormalization-group theory whose consequence is the scale transformation, equation (2). This equation dictates that the so-called self-similarity be the similarity of different scales at different scaled variables such as temperatures and external fields. Self-similarity may then be reckoned on this similarity [61]. This similarity is evidently satisfied by the phases and their fluctuations. Indeed, different distances to the critical point, for instance, mean different values of $\xi$ and thus different sizes of the large droplets with different ordered parameters. All have a statistically similar picture of the phases and their fluctuations. Only the sizes of the fluctuating droplets are apparently different. In section 5 below, we will find that scalings are poor for the set of the observables with absolute values but good for the other set of the observables for the cases of FTS either in heating or cooling and of FSS in cooling. This might indicate that the magnitude fluctuations alone did not have self-similarity. However, in the case of FSS in heating, it is the set of the observables without taking absolute values that scales well whereas the other set scales poorly. Moreover, we will see that these violations of scaling appear only either in the ordered phase or in the disordered phase, except for the case of FTS in cooling due to the revised FTS. Therefore, both the magnitude and the phases fluctuations must satisfy the scalings and thus possess self-similarity. In fact, we will argue that the violations of scaling originate from another kind of self-similarity, which we have referred to as extrinsic self-similarity.

As pointed out in the last sections, in order to fully describe the scaling behavior, it is essential to maintain an additional self-similarity, the extrinsic self-similarity. This additional self-similarity is to ensure that different-sized lattices are subject to different rates of driving in such a way that all systems contain identical number of either the fluctuating phases clusters [41] or their survival time intervals illustrated numerically in figure 1. Since both the magnitude fluctuations and the phases fluctuations concern the same clusters, the extrinsic self-similarity of one implies the same to the other, except for the case of FSS in which no phases fluctuations mean no flipping of the clusters and hence no temporal extrinsic self-similarity at all. Moreover, we will see that when there exist phases fluctuations, violations of scaling do not necessarily occur; however, if there exist no phases fluctuations and hence both sets of the observables are identical, there are no violations of scaling at all. Accordingly, the phases fluctuations are the necessary condition of the violations. We therefore associate the additional self-similarity with the phases fluctuations rather than the magnitude fluctuations, though breaking of the extrinsic self-similarity of the former also implies breaking that of the latter in the case of FTS. Further, violations of scaling in theobservables with absolute values may not imply that they stem solely from the extrinsic self-similarity breaking of the magnitude fluctuations. In addition, we also note that even if this additional self-similarity is broken, the original intrinsic self-similarity of the phases fluctuations themselves remains. As a consequence, the scalings are still exhibited in one phase even for those observables that violate the scalings in the other phase.

Therefore, we have two kinds of self-similarity of the phases fluctuations. The first one is the intrinsic self-similarity of the phases fluctuations themselves. This intrinsic self-similarity is only limited by the criticality. The second kind of self-similarity is additionally limited by external conditions such as the system size and the external driving and thus is
an extrinsic self-similarity. Both kinds of self-similarity can of course be broken by tuning the system far away from the critical point. However, the second kind can be easily broken by changing the external conditions.

We will see in the following that the differences between heating and cooling stems also from the phases fluctuations. In cooling, the dissymetrical phases can freely fluctuate and thus the phases fluctuations are not restricted. In heating, symmetry is broken from the beginning of the driving due to our initial conditions of a chosen ordered state and thus the phases fluctuations are substantially reduced. This difference in the phases fluctuations play also an important role in the anomalous result in cooling exhibited in figure 3 below. The different phases fluctuations in heating and cooling give rise to different magnitude fluctuations. Therefore, the phases fluctuations are not only a realistic picture for critical fluctuations, but also play a pivotal role in accounting for the dynamic scaling of critical behavior.

4. Model and method

We consider the 3D Ising model defined by the Hamiltonian

\[ H = -J \sum_{(i,j)} \sigma_i \sigma_j, \]

where \( J > 0 \) is a nearest-neighbor coupling constant and will be set to 1 hereafter as an energy unit, \( \sigma_i = \pm 1 \) is a spin on site \( i \) of a simple cubic lattice, and the summation is over all nearest neighbor pairs. Periodic boundary conditions are applied throughout. The critical temperatures [62] and the critical exponents [62–64] of the 3D simple cubic lattice are \( T_c = 1/0.2216595(26) = 4.51142(6) \), \( \nu = 0.6301(4) \), \( \beta = 0.3265(3) \), and \( \gamma = 1.2372(5) \). Most of estimation of the dynamical critical exponent \( z \) is near 2.0. We choose \( z = 2.055 \) here if not mentioned explicitly, which is estimated in the cooling process [32] and close to the previous results [65, 66].

The observables we measure are the order parameter \( M \) and the susceptibility \( \chi \) defined as

\[
M = \left\{ \begin{array}{l}
\langle m \rangle = \left\langle \frac{1}{N} \sum_{i=1}^{N} \sigma_i \right\rangle, \\
\langle |m| \rangle = \left\langle \frac{1}{N} \sum_{i=1}^{N} |\sigma_i| \right\rangle,
\end{array} \right.
\]

\[ \chi = L^d \left( \langle m^2 \rangle - \langle |m| \rangle^2 \right), \]

\[ \chi' = L^d \left( \langle m^2 \rangle - \langle m \rangle^2 \right), \]

where the angle brackets represent ensemble averages and \( N \) is the total number of spins. The first set of definitions, consisting of the first lines in equations (18) and (19), is the usual definitions of the order parameter and its susceptibility, while the second set including the remaining two equations is usually employed when \( \langle m \rangle = 0 \) in the absence of symmetry breaking and thus absolute values are needed. The absolute values disregard the sign change of \( m \) and hence remove the phases fluctuations and ought to probe the magnitude fluctuations. However, as mentioned in the last section, we will see that the phases fluctuations also contribute to the magnitude fluctuations for FTS in cooling. Therefore, probing both sets of observables is invaluable to uncover the secrets of phases transitions and their effects both in cooling and in heating. This is true even when an external field is applied to the system and symmetry is apparently broken. For a strong external field, phases fluctuations may be suppressed. Consequently, both sets of the observables become identical and the usual scaling is applicable [16]. However, if the field is applied only below \( T_c \), phases fluctuations still play an important role [67]. For a weak applied field, richer phenomena emerge [67].

We note that the real susceptibility defined as the change of magnetization due to a unit change of an externally applied field on the left-hand side of equation (19) is not equal to the fluctuations on the right-hand side in a nonequilibrium situation. However, their scaling behaviors are identical [5]. For simplicity, we thus simply use the susceptibility defined in equation (19) to represent the fluctuations. We will generally refer to the order parameter \( M \) and the susceptibility \( \chi \) for both definitions, which accounts for the somewhat odd expression in the first line of equation (19), and stipulate to a particular one when so indicated, except for section 6, where only the first set of the observables is employed.

The single-spin Metropolis algorithm [68] is employed and interpreted as dynamics [69, 70]. The time unit is the standard Monte Carlo step per site, which contains \( N \) randomly attempts to update the spins. For both the heating and cooling processes, we prepare the system far away from \( T_c \) in a definite ordered or a disordered initial configuration at a negative initial time and then heat and cool it, respectively, through \( T_c \) at \( t = 0 \) according to a given \( R \). We check that the initial states create no observable effects once they are sufficiently far away from \( T_c \), because the system can then equilibrate quickly due to the short relaxation time there. Various sizes \( L \) and rates \( R \) are used. All the results are averaged over 10–30 thousand samples.

The reason for chosen one definite initial ordered state for heating is to accelerate the equilibration of the state far away from the critical point. The aim of the equilibration is to eliminate the possible dependence of the results on the initial conditions. This concerns with the initial slip [71] which we do not study here. If we start with a disordered state, the system needs to order at first because \( T < T_c \) and then equilibrate. This necessitates starting at a far lower temperature. The ordering ends most probably either in an up state or a down state for sufficiently long time of equilibration. Choosing a definite state then removes other possibilities. Possibly if the initial state for heating were set as the disordered state, heating might appear to behave similarly to cooling. However, the present heating protocol enriches phenomena.

5. FSS and FTS in heating and cooling

In this section, we will study the FSS and FTS in heating and cooling. However, as an appreciation of how surprising driven nonequilibrium critical phenomena can be, we first
Figure 3. (a) and (b) Susceptibility χ′ and (c) and (d) order parameter ⟨|m|⟩ upon heating, (a) and (c), and cooling, (b) and (d), at the various rates given in the legend on 20 × 20 × 20 simple cubic lattices. The two panels in each column share the same axis of temperature T and its scales. All panels share the same legend.

of all compare behavior in heating with cooling in the FTS regime.

In figure 3, we display the rate dependence of the order parameter and the susceptibility on a fixed lattice size. The most prominent feature is that the susceptibility exhibits a sharp qualitative difference in heating and cooling: The peaks increase in heating whereas decrease in cooling with decreasing $R$. Note that, as pointed out in section 2, exactly at the critical point, the scaling of the susceptibility in heating and cooling was found to be similar, only the order parameter and its squared are different [32]. In contrast, in figure 3, χ appears to have two different forms while $M$ seems to be similar in heating and cooling. In fact, one can see from figures 3(a) and (b) that the susceptibility exactly at $T_c \approx 4.511$ indeed increases with decreasing $R$ both in heating and in cooling, in accordance with equation (4). Only the process below $T_c$ exhibits difference. Also, the order parameter exactly at $T_c$ exhibits different trends with $R$ in heating and cooling in accordance with equations (7) and (10), respectively, although the two sets of curves appear not so different. One might think that the difference come as no surprise. However, the usual differences of critical behavior above and below the critical point are not the critical exponents but only the amplitudes [53, 55]. Yet, the difference as exhibited in figure 3 is the different dependence of the peaks on the rate, a difference which clearly cannot be accounted for by the amplitude itself. In addition, the susceptibility in cooling is much larger than that in heating. It does not vanish and still has large fluctuations at low temperatures for large rates. Accordingly, it seems that new ingredients like phase ordering might be needed to account for the cooling beyond the critical point.

Besides the sharp differences in the susceptibility, figure 3 shows that the positions of the peak maxima behave similarly in heating and cooling. As $R$ is lowered, $\xi_D$ becomes longer and hysteresis weaker. As a consequence, the peak positions get closer to the equilibrium transition temperature $T_c$. The same reason leads to the higher peaks for lower $R$, as the result of heating demonstrates. Accordingly, the result of cooling is strange. We will come back to it toward the end of section 5.4 below.

5.1. FSS at fixed rates

We now study the FSS of both heating and cooling at a fixed rate $R$. This helps to reveal the origin of the differences. We will see that phases fluctuations affect FSS too. In fact, clear evidences exist for the phases fluctuations.

Figures 4(a) and (c) show $M$ and χ in heating for a fixed rate $R = 0.0001$. This rate corresponds to a driving length scale of $\xi_R \sim R^{-1/\nu} \approx 12.5$ or timescale of $\zeta_R \sim R^{-z/\nu} \approx 180.7$. Accordingly, for $L \lesssim \xi_R$ or $L^z \lesssim \zeta_R$, FSS shows and thus observables must depend appreciably on $L$ according to the leading behavior of equations (8) and (5); whereas for $L$ sufficiently large, FTS is exhibited and the results from different lattice sizes vary only slightly due to the negligibility of $L^{-1}R^{-z/\nu}$. One sees from figure 4(a) that ⟨|m|⟩ (solid curves) of different lattice sizes separate at different temperatures near $T_c$ from that of the largest lattice size, which is, of course, closest to equilibrium among the curves. The smaller $L$ is, the lower the separating temperature, indicating the earlier transition to the disordered phase because of its shorter correlation time $L^z$, which is about 27.3, 54.5, 113.5, and 261.1 for $L = 5, 7, 10, 15$, respectively, in the FSS regime. For $L > 15$, the separated curves are closer to each other than those of the smaller sizes, reflecting the crossover to FTS regime. The values of ⟨|m|⟩ are rather large above $T_c$, in agreement with similar results without driving [70, 72]. However, they decrease as $L$ increases and hence ought to be a finite-size
Figure 4. (a) $M$ and (b) its FSS and (c) $\chi$ and (d) its FSS in heating at a fixed rate $R = 0.0001$ on different lattice sizes given in the legend. (e) and (f) Are (b) and (d) in the absence of the two smallest lattice sizes $L = 5$ and $L = 7$. The dashed lines represent results of $\langle m \rangle$ and $\chi$ and the solid lines those of $\langle |m| \rangle$ and $\chi'$. The inset in (d) zooms in on the scale for $\chi'$. All panels share the same legend.

Figure 4. (a) $M$ and (b) its FSS and (c) $\chi$ and (d) its FSS in heating at a fixed rate $R = 0.0001$ on different lattice sizes given in the legend. (e) and (f) Are (b) and (d) in the absence of the two smallest lattice sizes $L = 5$ and $L = 7$. The dashed lines represent results of $\langle m \rangle$ and $\chi$ and the solid lines those of $\langle |m| \rangle$ and $\chi'$. The inset in (d) zooms in on the scale for $\chi'$. All panels share the same legend.

Figure 4. (a) $M$ and (b) its FSS and (c) $\chi$ and (d) its FSS in heating at a fixed rate $R = 0.0001$ on different lattice sizes given in the legend. (e) and (f) Are (b) and (d) in the absence of the two smallest lattice sizes $L = 5$ and $L = 7$. The dashed lines represent results of $\langle m \rangle$ and $\chi$ and the solid lines those of $\langle |m| \rangle$ and $\chi'$. The inset in (d) zooms in on the scale for $\chi'$. All panels share the same legend.

On the other hand, $\langle m \rangle$ (dashed curves) can deviate from the main curve at a rather low temperature. These different separating temperatures of $\langle |m| \rangle$ and $\langle m \rangle$ give rise to different transition temperatures, which are characterized by the peak temperatures of $\chi'$ and $\chi$, respectively. Moreover, both transition temperatures are lower than $T_c$ for those $L \lesssim 15$, though their distances to $T_c$ are much different in magnitude. Furthermore, the peak heights exhibit a different size dependence. While the peak heights of $\chi'$ (solid curves) increase with $L$ rapidly and monotonically, those of $\chi$ (dashed curves) rise somehow mildly for large $L$ and even slightly decrease at $L = 50$. These trends indicate that $L = 20$ lies possibly in the crossover between the FSS and FTS regimes and $L = 50$ in the latter regime. In the FSS regime on the one hand, $\chi$ is proportional to $L^{\nu/\nu}$ according to equation (5). In the FTS regime on the other hand, the dependence on $L$ is slight as mentioned. In addition, in the FSS regime, the cluster size of the disymmetric phases is about $L$ itself, whereas in the FTS regime, it is only $\xi_R$. This may contribute to the peak height difference in $L = 20$ and $L = 50$. Besides, in the FTS regime, fluctuations are suppressed to $\xi_R$ which protects the existing phase and thus the transition can only be delayed rather than advanced.

A prominent feature observed from figures 4(a) and (c) is that the two sets of the observables, $\langle m \rangle$ and $\chi$ versus $\langle |m| \rangle$ and $\chi'$, are markedly different. This, as pointed out in section 3, is a direct exhibition and evidence of the phases fluctuations. As seen in figure 4(a), $\langle m \rangle$ collapses onto $\langle |m| \rangle$ at low temperatures. As $T$ increases, except for the largest lattice which belongs to the FTS regime, $\langle m \rangle$ first deviates from $\langle |m| \rangle$ at intermediate temperatures and then falls to zero or so at higher temperatures. At the low temperatures, the correlation time is short and the system equilibrates in the up phase (because of the chosen initial state) with the correct averaged magnetization of the temperatures, owing to the spin fluctuations of spin-down clusters of the size of the correlation length. The magnetization $m$ of each sample is always positive and thus $\langle m \rangle$ and $\langle |m| \rangle$ coincides. As $T$ increases toward $T_c$, $\xi$ is raised to the size of $L$ and the whole system is one cluster on average. $m$ of the cluster can then overturn in an average time of $L^2$ and fluctuate between up and down as a whole, as exemplified in figure 1. This implies that both directions ought to possess
Figure 5. FSS in heating at a fixed rate $R = 0.00003$ of (a) $M$ and (b) $\chi'$ on different lattice sizes given in the legend. Solid lines are results for $\langle|m|\rangle$ and dashed lines for $\langle m \rangle$. Both panels share the same legend.

similar magnitudes of $m$. In fact, $\langle|m|\rangle$ of different lattice sizes converges almost to a single envelop for $T \lesssim 4.3$ in figure 4(a) means all $m$—no matter whether they are positive or negative—ought to be the same on average and independent on $L$. Because the fluctuations between the two directions with finite values of $m$ happen at random instants, averaging different samples thus gives rise to different $\langle m \rangle$ and $\langle|m|\rangle$. Moreover, the frequency of the fluctuations may depend on the temperature. As a result, in the nonequilibrium heating, there can be no time enough for the cluster to flip freely so that $\langle m \rangle$ can also be finite at some temperatures. This picture thus qualifies us to call the fluctuating clusters as phases, and the deviation of $\langle m \rangle$ from $\langle|m|\rangle$ is thus a direct evidence of the phases fluctuations. Note, however, that although smaller lattice sizes exhibit more pronounced difference between $\langle m \rangle$ and $\langle|m|\rangle$ in that the averaged magnetization is zero at lower $T$ below $T_c$, this is due to their shorter correlation times or survival times and does not mean that their phases fluctuations are stronger. Rather, lattices of larger $L$ values have larger phases cluster sizes and hence stronger fluctuations, as is evident from the larger $\chi$ peaks in figure 4(c). However, the phases fluctuations are suppressed when the absolute values are employed to average. Only the magnitude fluctuations remain. Consequently, $\chi'$ is much smaller than $\chi$ in the FSS regime and its peak appears near $T_c$ as can be seen from figures 4(a) and (c). We note that for the small lattice sizes, the usual random thermal fluctuations—those that produce the temperature dependence of the averaged magnetization—may also contribute to overturn the phases at temperatures quite lower than $T_c$. For a small lattice, with several existing opposite spins, other spins can likely overturn without appreciable energy penalties.

Figures 4(b) and (d) show that the FSS of $\langle m \rangle$ and $\chi$ for the small lattice sizes is severely violated, though that of $\chi$ in the disordered phase beyond the peak appears good. However, the FSS of $\langle|m|\rangle$ and $\chi'$ on small lattice sizes is reasonably good noting that corrections to scaling [56] are likely not small for the small lattice sizes. Indeed, in the absence of the two smallest lattice sizes, the scaling appears quite good even for $L = 15$ bigger than $\xi_R \approx 12.5$, or $RL' \approx 1.92$, as shown in figures 4(e) and (f). This means that $RL'$ can be ignored even up to a value of about 2 in the scaling functions and so $L^c$ of $L = 15$ can be longer than $\xi_R$. This is not unreasonable because we have omitted possible multiplying constants for the scaled variables. In figure 5, we confirm the above conclusions using results at a smaller fixed rate of $R = 0.00003$ whose $\xi_R \approx 17.4$. Again, $\langle|m|\rangle$ and $\chi'$ on $L = 20$ corresponding to $RL' \approx 1.64$ follows well FSS, since the previous value of 1.92 leads back to an $L \approx 21$ for $R = 0.00003$, while the bad FSS of $\langle m \rangle$ remains. Of course, for sufficiently larger lattice sizes, the system must show FTS instead of FSS.

Therefore, upon heating at a fixed rate, FSS is exhibited for the observables with absolute values such as $\langle|m|\rangle$ and $\chi'$. In fact, they were utilized in the early verification of FSS [70, 72]. The FSS of $\chi$ in the disordered phase is also good. This is reasonable because $\chi$ is just $\langle m^2 \rangle$ when $\langle m \rangle = 0$ but $\langle|m|\rangle$ and $\langle|m|\rangle$ which compose $\chi'$ scale well. However, the FSS of $\langle m \rangle$ and $\chi$ in the ordered phase is violated. A possible reason might be the seemingly large deviation from $T_c$ of the $\langle m \rangle$ and $\chi$ curves. One might also imagine that, possibly for very large lattice sizes on which $\langle m \rangle$ and $\chi$ differ negligibly from $\langle|m|\rangle$ and $\chi'$, respectively, their FSS might show. However, comparing figure 4(e) to 5(a), we observe that the curve of $L = 15$ moves to low temperatures and the associated transition is further advanced as $R$ is lowered. Noting that $L$ must not be too larger than $\xi_R$ for FSS to be fulfilled, it is thus unlikely that the FSS of $\langle m \rangle$ at a fixed $R$ could appear good. In fact, as will be shown in sections 5.4 and 5.5 below, the violation does not result from the difference between the two sets of observables but rather from the extrinsic self-similarity breaking.

Under cooling and in the absence of an externally applied field, $\langle m \rangle$ is vanishingly small. $\langle m \rangle$ and $\langle|m|\rangle$ are thus completely different; each large cluster can assume any direction and fluctuate freely, resulting in the strong phases fluctuations. In fact, it is the large phases fluctuations between the two phases that result in the vanishing $\langle m \rangle$ for not too low temperatures at which $\langle m \rangle$ and $\langle|m|\rangle$ in heating are different in figure 4. For lower temperatures, the surviving phases can rarely overturn. However, these phases are selected randomly out of the phases fluctuations and thus the average of many samples again leads to the vanishing $\langle m \rangle$. We thus show $\langle|m|\rangle$ only. Moreover, again due to the phases fluctuations, $\chi$ is far much larger than $\chi'$ and exhibits no peak at all, as seen in figure 6. This is different from the case in heating in which $\chi$ shows
a peak similar to $\chi'$. This in turn reflects an important difference between heating and cooling. In heating, as pointed out in section 3, we start the driving from an ordered initial state and the symmetry is broken from the beginning. The phases fluctuations can only start from the symmetry-broken phase rather than freely fluctuate between the two disymmetric phases and hence are substantially reduced. Furthermore, in contrast with figure 3(b) in the FTS regime, in the FSS regime, $\chi'$ is vanishingly small at low temperatures, a point to which we will come back later on in section 5.2 below. In addition, similar to figure 4 in heating, the curves of $L = 50$ which does not fall in the FSS regime exhibit qualitatively distinct behavior to the other curves.

As demonstrated in figures 6(b) and (d), the FSS of $\langle |m| \rangle$ and $\chi$ appear quite good similar to the corresponding figures in figure 4 for heating. However, in stark contrast with heating, the FSS of $\chi'$ appears reasonable only for $T > T_c$; the low-temperature side including $T_c$ is violated. This cannot stem from the sample size because the separation of different lattice sizes depends on $L$ systematically. In figure 7, we show the results for $R = 0.00003$ for which $L = 20$ exhibits good FSS in heating. The violation of FSS for $\chi'$ at the low-temperature side remains though the rescaled curves appears closer compared with those for $R = 0.00001$ in figure 6. Corrections to scaling [56] cannot be attributed to either, although they are responsible for the poor curve collapses of the two smallest lattice sizes as can be seen from figures 6(d) and (f), and 7. Indeed, the collapse of $\langle |m| \rangle$ becomes better if the two are removed as shown in the inset in figure 7(a). However, the violation of FSS for $\chi'$ remains even without the two curves.

Summarizing, at a fixed $R$, FSS is valid except $\langle m \rangle$ and $\chi$ in heating and $\chi'$ in cooling, all in the ordered phase. On the one hand, in heating, as the phases fluctuations are reduced when the absolute values such as $\langle |m| \rangle$ and $\chi'$ are employed, FSS is remedied. On the other hand, in cooling, the phases fluctuations are so strong that $\langle m \rangle$ is not a valid order parameter and $\chi$ has no peak. Again, when the absolute value is used, the FSS of $\langle |m| \rangle$ appears good. However, that of $\chi'$ is only good in the disordered phase. This goodness in the disordered phase rules out a global revised scaling form similar to equation (10), because it can then be destroyed. Although this might indicate some effects of phase ordering, the good scaling of $\chi$ and $\langle |m| \rangle$ seems to rule them out. As pointed out above, for $R = 0.00003$, $RL_0 \approx 1.64$ falling within the FSS regime even for the largest $L = 20$ and thus corrections to scaling can be safely neglected, as the scaling of $\langle |m| \rangle$ and also $\chi$ show, though the two smallest $L$ values may need. Therefore, the violation of FSS in cooling appears surprising.

Figure 6. (a) $M$ and (b) its FSS, (c) $\chi$ and (d) its FSS, and (e) $\chi'$ and (f) its FSS in cooling at a fixed rate $R = 0.0001$ on different lattice sizes given in the legend, which is shared by all panels.
Figure 7. FSS in cooling at a fixed rate $R = 0.00003$ of (a) $M$ and (b) $\chi'$ on different lattice sizes given in the legend. The inset in (a) enlarges the curves in the absence of the two smallest lattice sizes.

Figure 8. (a) $M$ and (c) and (e) its FTS and (b) $\chi$ and (d) and (f) its FTS in heating on fixed lattice size of $L = 20$, (c) and (d), and $L = 50$, (e) and (f). In contrast with figures 4–7 and all following ones in this subsection, here dashed lines stand for $\langle |m| \rangle$ and $\chi'$ while solid lines for $\langle m \rangle$ and $\chi$ for clarity of scaling collapses. The inset in (e) magnifies the curves in the disordered phase. In (a) and (b), the curves of both lattice sizes are displayed together. $\langle m \rangle$ and $\chi$ of the different lattice sizes overlap except for the smallest rates. The dashed curves of $\langle |m| \rangle$ and $\chi'$, deviating from $\langle m \rangle$ and $\chi$, respectively, at higher temperatures have a larger $L$. In (b), for the two smallest rates, the right $\chi$ peaks have a larger $L$. All panels share the same legend. Three curves in both (a) and (b) have already displayed in figure 3.

5.2. FTS on fixed lattice sizes

In this section, we study the FTS of both heating and cooling on fixed lattice sizes.

Figure 8 shows $M$ and $\chi$ and their FTS in heating on two fixed lattice sizes. In contrast with figure 4 in the FSS regime, the transition is now delayed as mentioned there; the larger $R$ is, the stronger the hysteresis. In the FTS regime, the
evolution is controlled by $\xi_R$ rather than $R$. This is clearly reflected in figures 8(a) and (b) in that the large $R(m)$ and $\chi$ curves for the two different lattice sizes overlap, except for the smallest $R = 0.0001$ and almost the second smallest one in figure 8(b). This means that the smallest rate for $L = 20$ is not in the FTS regime, though $\xi_R \approx 12.5 < L = 20$ for $R = 0.0001$, or $L^{-1}R^{-1/3} \approx 0.627 < 1$. In the last section, we also noted that this same parameter violates FSS. It must thus lie in the crossover regime between FTS and FSS, as has already been pointed out there.

From figures 8(a) and (b), it can be seen that $\langle m \rangle$ and $\chi$ are again different from $\langle |m| \rangle$ and $\chi'$, respectively. Although the differences for large $R$ values appear at $T > T_c$, opposite to FSS in figure 4, those for small $R$ values can occur before the $\chi$ peaks where the transition is happening. Accordingly, the differences reveal again the phases fluctuations rather than spin fluctuations. Figures 8(a) and (b) show that the deviations of $\langle m \rangle$ and $\chi$ to $\langle |m| \rangle$ and $\chi'$, respectively, for a fixed $L$ are reduced as $R$ increases or $\xi_R$ decreases. They also diminish for a fixed $R$ in the FTS regime on the larger $L$. Both trends are related to more regions of the size $\xi_R$. However, in FSS, the deviations occur at lower temperatures for the smaller $L$ and $R$, see figures 4(a) and 4(c), and 5(a). Here, they are completely opposite to the case of FSS in heating but in accordance with the opposite trend of hysteresis versus advance. This may be understood as the driving suppresses fluctuations to the scale set by $\xi_R$ or $\xi_R$, similar to the origin of the hysteresis mentioned above, and thus the phases fluctuations can only occur at high temperatures. Indeed, the fluctuations, as indicated by the $\chi$ peaks, are reduced as $R$ increases for a fixed $L$ due to the smaller $\xi_R$. However, they are hardly affected by $L$ in the FTS regime as the $\langle m \rangle$ and $\chi$ curves for the two different lattice sizes overlap for the large $R$ values. It seems that the size dependence of the deviations of the two sets of the observables is related to the size dependence of $\langle |m| \rangle$. As seen in figures 4(a) and 6(a), $\langle |m| \rangle$ depends on $L$ significantly for $T > T_c$. It takes on almost the same value for the same $L$ no matter whether in heating or in cooling. This is also true for FTS, as can be seen in figures 3(c) and (d) as well as 8(a), because far away from $T_c$, a system is controlled by its short correlation length. Accordingly, all curves of a same lattice size collapse independently on their rates. This is not, of course, due to the choice of the initial configurations, peculiarities of the algorithm used, etc. In fact, such a long tail of finite $\langle |m| \rangle$ values at high temperatures was found in the early Monte Carlo simulations [72] and agrees with its FSS $\langle |m| \rangle \sim L^{-3/4}$, equation (8), and thus, as pointed out in the last section, a finite-size effect.

One sees from figures 8(c)–(f) that while $\langle m \rangle$ and $\chi$ show FTS well and $\langle |m| \rangle$ and $\chi'$ also display better FTS as the lattice sizes get larger, the FTS of $\chi'$ is still poor on $L = 50$ lattices at high temperatures. In fact, as seen in the inset in figure 8(e), the FTS of $\langle |m| \rangle$ is also poor at those temperatures. This is different from the FSS in heating in which it is $\chi$ and $\langle m \rangle$ that display bad scaling but similar to the FSS in cooling in which $\chi$ shows poor scaling though $\langle |m| \rangle$ appears good. Note that the poor scaling here does not arise from the sample size. We checked that more samples only affect slightly the top of the peak.

To study FTS in cooling, we first note that, as seen in figure 3, in cooling $\chi'$ does not vanish and still exhibits large fluctuations at low temperatures different from the FSS case, as mentioned in the last section. This is because the magnetization $m$ of each sample can assume any value between the plus and the minus saturated magnetization in the FTS regime due to the phases fluctuations, whereas it takes on its equilibrium value (with thermal fluctuations around it) at sufficiently low temperatures in the FTS regime. We can thus remove those samples whose magnetization is smaller than a threshold below the peak temperature. As a consequence, the shape of $\chi'$ becomes normal, though the inverse rate dependence as compared to heating remains, as is illustrated in figures 9(a) and (b), which also show the reduction of the fluctuations (the peak size) with the removal. However, the method is rather rough in that both the threshold and the temperature chosen are rather ad hoc, though it confirms the origin of the fluctuations.

In figure 9, we also display the FTS of $\chi'$ and $\langle |m| \rangle$ averaged over all samples and samples with a saturated magnetization only. We further depict the scaling of $\langle |m| \rangle$ using equation (10) with the $L^{-d/2}$ factor omitted because of the fixed $L$. One sees generally and in particular from the insets that removal of the unsaturated samples does not improve but may even worsen the scaling in figure 9(h). As pointed out above, the removal is rough, we therefore do not consider it further in the following. One sees also that the scaling of $\chi'$ is acceptable only for rates bigger than 0.003, corresponding to $\xi_R \approx 4.9$, or $L^{-1}R^{-1/3} \approx 0.246$, consistent with the crossover value of 0.627 found above. Moreover, the scalings only extend to a small range below $T_c$. However, the scaling of $\chi'$ is quite remarkable upon noticing its inverse rate dependence in comparison with heating. In addition, the revised scaling for $\langle |m| \rangle$ is quite good, as seen in figure 9(g). To the contrast, without the revised, the original scaling equation (7) cannot describe the scaling behavior of $M$ both above and below $T_c$ at all.

These results are confirmed in figure 10, in which a larger lattice size is used. One sees again that without taking the phases fluctuations into account, the naive FTS, equation (7), does not work at all even for the large lattice size. As $L$ increases, the range that obeys the scaling extends to further lower temperatures, including that of $\chi'$ despite its peculiar feature. In addition, the scaling of $\chi$ is somehow better than that of $\chi'$.

Therefore, the standard FTS of the order parameter, equation (7), cannot at all describe the scaling behavior of $\langle |m| \rangle$ both above and below $T_c$ in cooling on fixed lattice sizes. The FTS of $\chi'$ along with $\langle |m| \rangle$ in heating is also violated in the disordered phase. Nevertheless, the FTS of all other observables studied in both heating and cooling, including the peculiar $\chi'$ and the revised scaling for $\langle |m| \rangle$ in cooling, is quite good and extends to a larger range for bigger lattice sizes. This seems to indicate that phase ordering cannot enter at least to this range provided that a proper scaling form is employed.

5.3. Summary of the violations of FSS and FTS

In table 1, we summarize the measured quantities that are violated in either FSS or FTS and either in heating or in cooling.
Figure 9. (a) and (b) $\chi'$, (c) and (d) its FTS, (e) and (f) FTS of $\langle |m| \rangle$, and (g) and (h) revised FTS, equation (10), of $\langle |m| \rangle$ averaged over all samples (left column) and samples with an unsaturated magnetization excluded (right column) upon cooling at the rates given in the legend on $20 \times 20 \times 20$ simple cubic lattices. The inset in each panel zooms in on the curves in the high-temperature regimes. Panels in each row share identical ordinates. All panels share the same legend.

The phase that the violation occurs is also given. One sees the violations occur in a set except for the cases of FSS and FTS in cooling whose secondary observables are parentheses. We will see in section 5.5 below that this is because the primary observables $\chi'$ and $\langle |m| \rangle (L^{-1/\nu} R^{-1/r} - d/2)$, the latter having been reduced by the revised FTS, of these two cases exhibit relatively weak violations in the sense that their corresponding exponents are relatively small. Consequently, the regular terms similar to equation (14) and in equation (16) for the two cases, respectively, appear dominant. We point out also that although the phase that the violation occurs for FTS in cooling is designated as ordered in table 1, the revised scaling is needed in fact in both the ordered and the disordered phases, as will be discussed in section 5.5. Only a further rectification occurs in the ordered phase. Including this special case, one sees that in all but one cases the violations occur in the ordered phase. This seems reasonable because the effects of the phases fluctuations appear prominent in the ordered phase. We will see in
section 5.5 that the ordered or the disordered phase of the violations corresponds to the low- or the high-temperature slope of the $\chi$ or $\chi'$ peak, respectively, after the violations are rectified, except for the special case of FTS in cooling, whose primary observable is $\langle |m| \rangle$ instead of $\chi$ or $\chi'$. We also note that the violated scalings are always exhibited in the set with absolute values in all cases but FSS in heating. It seems that the general rule here is that all rules have exceptions and why this is so is yet to be studied.

From table 1, it is obvious that the violations only display in one set of the observables. Consequently, they can only occur when both sets of the observables behave differently. This happens when the phases fluctuations appear. Consequently, no phases fluctuations, no violation of scaling. However, we will see in the following section that the phases fluctuations themselves are not sufficient for the violations. Once the extrinsic self-similarity is kept, no violation occurs at all even though the phases fluctuations are still there, provided that other sub-leading terms and corrections to scaling that have not been considered in the theory are negligible. Therefore, phases fluctuations are the necessary condition for the violations of scaling.

5.4. Full scaling forms

We now study the full scaling forms equations (4) to (10). To this end, we fix $L^{-1}R^{-1/r}$ to a constant and thus a given different $L$ means a different $R$. This is different from the above studies in which either $L$ or $R$ is fixed.

Figure 11 shows both the FSS and FTS of $M$ and $\chi$ in heating according to equations (8), (5), (7), and (4). One sees that as $L$ gets larger and correspondingly $R$ smaller, both FSS and FTS become better and extend to a larger range. This is more transparent if one disregards the three smallest lattice sizes. Moreover, owing to the phases fluctuations, $\langle m \rangle$ and $\langle |m| \rangle$ as well as $\chi$ and $\chi'$ are again different in the disordered phase. However, they all satisfy the scalings and the violated scaling collapses for $\langle |m| \rangle$ and $\chi'$ in figure 8 disappear. Therefore, the phases fluctuations themselves are not sufficient for the violations of the scaling. In addition, the scalings in the high-temperature part appear better than those in the low-temperature parts. The deviation of the small lattice sizes must arise from other sub-leading or correction terms [56] that are not included in the scaling forms. Note that the fixed value of $L^{-1}R^{-1/r} \approx 0.3541$ lies on the verge of FSS regime and corresponds to $RL$ of about 43.86, a number much bigger than the previous ones of 1.92 or even 5.48 for $R = 0.0001$ and $L = 20$ and thus the system cannot fall in the FSS regime. For $R = 0.0001$, for example, the fixed value leads to $L \approx 35.4$, which cannot show FSS as clearly seen in figures 4(e) and (f). However, once $L^{-1}R^{-1/r}$ is fixed, figures 11(c) and (d) show clearly that FSS can still describe the FTS regime, because one can freely insert a certain power of the constant factor to change the scaling form from one type to the other, as equation (6) demonstrated.

The good FSS of both $\langle |m| \rangle$ and $\chi'$ might seem commonplace. For the fixed $L^{-1}R^{-1/r} \approx 0.3541$, the rates on $L = 5$ and 7 are about 0.1248 and 0.0367, respectively. As mentioned above, these relatively large rates substantially reduce the deviations of $\langle |m| \rangle$ from $\langle m \rangle$, which are clearly exhibited in figure 11(a). The driven transitions on these small lattices are thus not advanced compared to figure 4. In figure 12, we depict the FSS and FTS of heating at $L^{-1}R^{-1/r} \approx 1.152$ or $RL' \approx 0.5982$, which puts the system firmly in the FSS regime. Figure 12(a) shows clearly that the transitions are advanced before $T_c = 4.51142(6)$ since all $\langle m \rangle$ curves become almost zero at $4.5 < T_c$. Nevertheless, both FSS and FTS of both sets of the observables are all remarkably well once the small
lattices and thus large rates are removed, even though the difference between the two sets of the observables is even more evident. This confirms that the phases fluctuations alone are not sufficient for the violations of the scaling.

We now turn to the qualitative difference exhibited in figure 3 and the systematic deviations in figures 6(f) and 7 in cooling. Figure 13 displays the FSS and FTS of $\langle|m|\rangle$ and $\chi$ and $\chi'$ in cooling for fixed $L^{-1}R^{-1/r}$ again according to equations (8), (5), (7), and (4), as well as equation (10). One first notices upon comparing figures 13(a) to 11(a) that different $\langle|m|\rangle$ curves of different $L$ cross each other below $T_c$ in cooling. The large fluctuations of $\chi'$ at low temperatures for small rates can also be observed in figure 13(b) and its inset. These again remind us of the uniqueness of cooling. However, with $L^{-1}R^{-1/r}$ being fixed, the rate dependence of the peak heights now resembles that in heating, in sharp contrast with figure 3, although the $\langle|m|\rangle$ curves now behave distinctively as mentioned. These therefore show the dramatic effects of $L^{-1}R^{-1/r}$ in cooling.

Moreover, the FSS and FTS now describe the data well in cooling for sufficiently large $L$ and correspondingly sufficiently small $R$ (so that corrections to scaling can be ignored) for a large region in both sides of $T_c$ similar to heating, as is clearly seen in figure 13. Also, both equations (7) and (10) are equally well because the latter is a special form of the former. So are FSS and FTS regardless of the fact that $RL'$ is now almost 120. These results, together with those in heating in this section, should already confirm the validity of the full scaling forms in describing the processes no matter whether in the FTS or the FSS regime. Indeed, a similar plot has been shown in figure 3(b) of reference [46] for a fixed $RL' \approx 0.548$ in the FSS regime of the 3D Ising model. It is no doubt that the poor FSS in figures 6(f) and 7 and the FTS of $\langle|m|\rangle$ in figures 9(e) and 10(a) all disappear in such a plot. Although the tops of the $\chi'$ peaks scale not so well even for the large $L$, this must result from the sample size, since fluctuations are violent there. Therefore, FTS itself well describes the cooling transition across the critical point for large $L$ and small $R$ and no phase ordering is needed down to at least $(T - T_c)R^{-1/r} < -5$ as seen in figure 13. Note that the frozen

Figure 11. (a) $M$ and (b) $\chi$ and (c) and (d) their respective FSS and (e) and (f) their respective FTS in heating with fixed $L^{-1}R^{-1/r} \approx 0.3541$, which is obtained from $L = 10$ and $R = 0.01$. Solid lines denote results of $\langle|m|\rangle$ and $\chi$' while dashed lines of $\langle|m\rangle$ and $\chi$. All panels share the same legend.
temperature $\hat{T}$ at which the KZ scaling is reckoned is defined just at $(\hat{T} - T_c) R^{-1/\nu} = \pm 1$ [32].

We have confirmed the validity of the full forms equations (4) and (5) in describing the FTS and FSS of both sets of the observables in both heating and cooling, including the seemingly large deviation away from $T_c$ of FSS in heating, provided that other sub-leading and corrections to scaling are negligible. The scaled variables $L^{-1}R^{-1/\nu}$ in FTS or $RL^\nu$ in FSS cannot be neglected in the cases in which scalings are violated even though they are small. Looking back on the inverse rate dependence in figure 3 and the systematic dependence on $L$ in the ordered phase in figures 6(f) and 7(b), one thus convinces oneself that they cannot stem from phase ordering. Rather, it is the scaled variable $RL^\nu$, albeit small, that gives rise to the qualitative difference and the poor scalings at low temperatures in cooling at both fixed $R$ and fixed $L$.

By definition, $\chi'$ removes the effect of the phases fluctuations and ought to probe the magnitude fluctuations. Fixing $L^{-1}R^{-1/\nu}$ renders it in cooling normal indicates that the normal trend of the dependence of $\chi'$ on $R$, viz, the increasing $\chi'$ peak with decreasing $R$, indeed reflects the larger cluster sizes and hence the larger magnitude fluctuations. This is in conformity with the same trend in the FSS regime as exhibited in figure 4(c) for heating and 6(e) for cooling. As phase ordering is ruled out, the qualitatively different dependence on $R$ of $\chi'$ in heating and cooling must thus stem from the extrinsic self-similarity breaking of the phases fluctuations, since the most important difference between heating and cooling is their phases fluctuations arisen from their different initial conditions, as mentioned in section 3. Accordingly, the phases fluctuations ought to contribute to the special behavior of $\chi'$ in cooling. Indeed, overturning a single large cluster changes $m$ of a sample at an instant or a temperature and hence $\langle |m| \rangle$ and $\chi'$. If the extrinsic self-similarity is present by fixing $L^{-1}R^{-1/\nu}$, the change for different lattices is the same on average and thus does not alter the dependence of $\chi'$ on $R$. Otherwise, the bigger the number of the large phases clusters, the smaller the
Figure 13. (a) $\langle |m| \rangle$, (b) $\chi'$, and (c) $\chi$ and (d)–(f) their respective FSS and (g)–(i) their respective FTS in cooling with fixed $L^{-1}R^{-1/r} \approx 0.2687$, which is derived from $L = 30$ and $R = 0.0005$. (j) The revised FTS, equation (10), of (a). The insets in (b) and (g) zoom in on the low- and high-temperature parts, respectively. The inset in (g) exhibits the broadening of the rescaled curve by the small $L$ curves. All panels share the same legend.
contribution of overturning a single cluster. This suppresses the magnitude of \( \langle m \rangle \) and \( \chi' \) at a temperature but expands the distribution of \( m \) and hence increases the entire \( \chi' \) peak, a feature which figure 3(b) exhibits.

The results of this section solve a superficial conflict in our previous descriptions. We pointed out in section 5.2 that the usual FTS form, equation (7), cannot describe the scaling behavior of \( \langle m \rangle \) at all; only the revised form, equation (10) can do. However, in section 1, we also pointed out that both these two scaling forms can describe well the scaling behavior exactly at \( T_c \), though their leading singularities differ, as shown in reference [32]. We emphasize that these two statements do not conflict. In section 5.2, we did not fix \( L^{-1} R^{-1/r} \). Therefore, equation (7) failed, as seen in figures 9(e) and 10(a). When we fix it, it works well, as demonstrated in figure 13(g).

In reference [32], we worked exactly at \( T_c \) only and plotted \( \langle |m| \rangle R^{-3/\nu r} \) with respect to \( L^{-1} R^{-1/r} \). In this way, the different values of the ordinate at \( T_c \) in figures 9(e) and 10(a) appear in different places of the abscissa in the \( \langle |m| \rangle R^{-3/\nu r} \) versus \( L^{-1} R^{-1/r} \) plot, because they just have different \( L^{-1} R^{-1/r} \) values. As a result, the scaling is satisfied. In other words, the full scaling function has been considered in this case and thus the results are no doubt good.

### 5.5. Bressy exponents

In the last section, we have verified unambiguously that the full scaling forms of both FSS and FTS can well describe both sets of the observables in both heating and cooling provided that other sub-leading terms and corrections to scaling are negligible. All the violated scalings summarized in section 5.3 arise thus solely from the scaled variables \( L^{-1} R^{-1/r} \) or \( RL' \) even though they are small. In this section, we study the properties of the violations. We find that new breaking-of-extrinsic-self-similarity, or Bressy, exponents \( \sigma \) can rectify the violations. These exponents lead to different leading singularities of the ordered and disordered phases for the primary observables, in stark contrast to the other observables and in usual equilibrium and dynamic critical phenomena.

To recapitulate, fixing \( L^{-1} R^{-1/r} \) in FTS (or \( RL' \) in FSS) fixes the ratio of \( L \) and \( R^{-1/r} \) in FTS (or \( L' \) and \( R'^{-1/r} \) in FSS) and therefore ensures the spatial (or temporal) self-similarity of the phases fluctuations. This kind of self-similarity is related to the system size or the external driving and is thus the extrinsic self-similarity. Accordingly, all the mentioned violated of scalings result from the breaking of the extrinsic self-similarity. Although the violations appear just in one phase and a global revised form like equation (10) destroys the good scalings in the other phase, we can find exponents to remedy the violated scalings in just one phase according to the theory in section 2. This, however, leads to different leading behavior in the two phases for the observables.

We first consider FSS in heating. In figure 14(a), we show the dependence on \( RL' \) of \( \chi L^{-\gamma/r} \) with \( \chi L^{-\gamma/r} \) cut vertically at fixed values of \( \tau L^{1/r} = (T - T_c) L^{1/r} \) from plots like figure 4(d). The apparent linearity of the three lines indicates the power-law relation in consistence with equation (12). By contrast, similar cuts on \( \langle |m| \rangle L^{3/\nu r} \) shown in figure 14(b) exhibit a relation compatible with equation (14). These indicate that \( \chi \) is the primary observable which obeys equation (12) whereas \( \langle m \rangle \) is the secondary one. Indeed, it is seen from figure 14(c) that a Bressy exponent of \( \sigma = -3 \pm 0.15 \) collapses the low-temperature slope of the \( \chi \) curves rather well, in comparison with the completely separation in the ordered phase shown in figure 4(d) for just one \( R \). Moreover, \( \sigma / r \approx 0.82 \) agrees with the slopes in figure 14(a).
The range of $RL'$ displayed in figure 14 is from 0.12 to 1.92 or so, more than sixteen times. The big value sits on the boundary of the FSS regime, as pointed out in previous sections, while the small value can be further lowered, we believe, by using smaller $R$. However, exactly at $R = 0$, the static case, the usual FSS ought to be recovered. Accordingly, the crossover seems discontinuously. How the two behaviors connect exactly with each other is yet to be studied. In addition, as pointed out in previous sections, while $\tau$ collapses the small value can be further lowered, we believe, by using smaller $R$.

This shows that $\chi'$ is the primary observable. Indeed, as exhibited in figure 15(b), $\sigma = \beta/2\nu$ collapses all the curves in the ordered phase rather well, as compared to the original scaling collapse (the left curves), which resembles the curves in figures 6(f) and 7(b) with fewer rates. Moreover, $\sigma/r \approx 0.071$ agrees with the slopes in figure 15(a), though the latter have relatively large errors due to the small rates as mentioned. Since this exponent is rather small, it can be expected that the secondary observable, $\langle |m| \rangle$ exhibits a good scaling collapse in figures 6(b) and 7(a) and hence has appeared with parentheses in table 1. Conversely, this implies that $\langle |m| \rangle$ can only be secondary.

Next, we study FTS in heating. In figure 16(a), we show the dependence on $L^{-1}R^{-1/\nu}$ of $\chi R^{-1/\nu}$ at fixed values of $\tau R^{-1/\nu}$. Different from figures 14(a) and 15(a), the data appeared not to fit a simple power law. However, as can be seen from figure 16(b) for $L = 70$ and figure 8(f) for $L = 50$, the $\chi$ and $\chi'$ curves do not yet separated from each other and hence the necessary condition of the phases fluctuations is not yet met for the two small fixed values of $\tau R^{-1/\nu}$ at large rates, corresponding to the data on the left. Accordingly, these data must follow the usual FTS and hence hardly depend on $L^{-1}R^{-1/\nu}$. Indeed, the left four data of $\tau R^{-1/\nu} = 4.5$ and three data of $\tau R^{-1/\nu} = 5$ are almost horizontal. Upon excluding these data, linearity emerges, though the errors in the slopes are relatively large. One reason comes from other sub-leading and/or corrections terms. One sees from figure 16(b), together with figures 8(d) and (f), that the large rate curves somehow deviate off the rescaled curves. For the same reason, we omit the rightmost data of $\tau R^{-1/\nu} = 5.5$ in the fit. On the other hand, from the inset of figure 8(e), at least for $R = 0.001$ downwards on the $L = 50$ lattice, $\langle |m| \rangle$ and $\langle m \rangle$ have already separated. Consequently, the behavior on the left in figure 16(c) arises from equation (15) rather than the independence on $L^{-1}R^{-1/\nu}$ of the usual FTS as the case of $\chi'$ just noted. Therefore, $\langle |m| \rangle$ is the secondary observable and $\chi'$ is the primary one. Indeed, figure 16(d) shows that for $\sigma = -(\gamma - 2\beta)/\nu \pm 0.05$ the curves collapse quite well on the high-temperature slope of the peaks, which we have roughly termed as the disordered phase. This $\sigma \approx 0.93$ is compatible with the slopes within the errors in
Table collapse is somehow large. (section 2, this indicates that the bad scaling originates from the rates slope as mentioned. Comparing with other Bressy exponents shown in figures 9(e) and 10(a) recovers. As pointed out in error bars are estimated to be about the sizes of the symbols. The slopes of the three fitted lines in (a) are once ⟨|m|⟩∕⟨R⟩1∕ν for the rates R listed on L = 20 (dashed lines), L = 50 (solid lines), and L = 70 (chain lines) lattices. In (a) and (c), dashed lines connecting symbols are only a guide to the eyes. Solid lines in (a) are linear fits to the data points within the ranges covered by the lines. In (b), chain lines denoting χ′ are those displayed in (d) while solid lines stand for χ. In (b) and (d), curves of identical colors have identical rates. The error bars are estimated to be about the sizes of the symbols. The slopes of the three fitted lines in (a) are −0.79(7), −0.72(12), and −0.68(16) from up to down.

figure 16(a) albeit a little bigger, noticing the large errors in the slope as mentioned. Comparing with other Bressy exponents listed in table 1, we find the range of ν that produces an acceptable collapse is somehow large. (γ = 2β)∕ν turns the leading exponent of χ′ in the disordered phase to 2β∕ν, the exponent of ⟨m2⟩. However, in the 2D model, we have found a different expression that invalidates this conclusion [46]. Therefore, like the case of FSS in cooling, χ′ here is again likely a new exponent.

In the present case, the crossover from the usual FTS regime to the Bressy regime in which the extrinsic self-similarity is important occurs in finite values of L−1R1∕ν and τR−1∕ν. As just seen in figure 16(a), for small enough values of L−1R1∕ν and τR−1∕ν, the two sets of curves do not separate and hence the phases fluctuations are absent. As a result, the usual FTS works well and the Bressy exponent is not needed.

We emphasize again that it is the breaking of the extrinsic self-similarity of the phases fluctuations that leads to the violation of the scaling, rather than the separation of the two sets of curves, though we have invoked the latter to exclude some data above. The separation of the two sets of curves is a transparent exhibition of the phases fluctuations, which is only a necessary condition, because both sets of curves exhibit good scaling collapses as seen in figure 11 once L−1R1∕ν is fixed.

Finally, we turn to FTS in cooling. As seen in figure 13, once L−1R1∕ν is fixed, all FTS of all the four observables considered are good provided that corrections to scaling can be ignored. Even the completely bad scaling of equation (7) shown in figures 9(e) and 10(a) recovers. As pointed out in section 2, this indicates that the bad scaling originates from L−1R1∕ν, which is to ensure the vanishing fluctuations in the thermodynamic limit, or the central limit theorem. Indeed, the revised scaling form, equation (10), describes well the behavior, as demonstrated in figures 9(g) and 10(c). Since keeping the extrinsic self-similarity of the phases fluctuations rectifies the scaling, we regard the exponent d∕2 in equation (9) as the Bressy exponent ν in the case of FTS in cooling, as equation (9) is similar to equation (12) for FTS. The meaning of this similarity is twofold. One is the similarity of the two relations and the other is that the curves collapse well with such extra scalings. However, we note one difference. The extra scaling for the revised scaling (10) makes the scaling collapse well in both phases except for yet another exponent to be considered shortly, whereas that for equation (12) works only for one phase. In other words, the former does not change the leading behaviors of the two phases whereas the latter does. So, we may separate the d∕2 off the Bressy exponent as well.

Apart from the d∕2, one sees from figures 9(c) and (g), and 10(b)–(d) that the rescaled curves generally do not collapse as well as those in figures 13(g)–(j) in the ordered state. As displayed in figure 17(a), ⟨|m|⟩R−3∕ν(L−1R−1∕ν)1−d∕2 at fixed values of (T − Tc)R−1∕ν increases first with L−1R1∕ν and then decreases for large values of L−1R1∕ν. After excluding the five largest values of L−1R1∕ν on the right, which come close to the crossover from FTS to FSS, the linearity is quite good, though the slope increases with |T − Tc|R−1∕ν. This indicates that here ⟨|m|⟩R−3∕ν(L−1R−1∕ν)1−d∕2 is the primary observable. Consequently, χ′Rν∕ν′ at fixed values of (T − Tc)R−1∕ν ought to behave according to equation (16). This is consistent with the behavior shown in figure 17(b); for small values of |T − Tc|R−1∕ν, the slope ν′ = ν − d∕2 is small and χ′Rν∕ν′ is almost a constant for small values of L−1R−1∕ν. As |T −
Figure 17. (a) $\langle |m| \rangle R^{-\beta/\nu}(L^{-1}R^{1/\nu})^{d-2/\nu}$ and (b) $x' R^{\nu/\nu}$ on $L = 20$ (circles), $L = 50$ (squares), and $L = 70$ (triangles) simple cubic lattices at fixed values of $(T - T_c) R^{-1/\nu} = -3$, $-4$, and $-5$ (from down to up) versus $L^{-1} R^{1/\nu}$ in double logarithmic scales for FTS in cooling.

$\langle |m| \rangle R^{-\beta/\nu}(L^{-1}R^{1/\nu})^{d-2/\nu}$ versus $(T - T_c) R^{-1/\nu}$ with various $\sigma' = \sigma - d/2$ values marked near the curves for the rates $R$ listed on (c) $3D L = 50$ (solid lines) and $L = 70$ (dashed lines) lattices and (d) $2D L = 50$ (dashed double dotted lines), $L = 70$ (dashed dotted lines), $L = 100$ (dashed lines), and $L = 150$ (solid lines). Solid lines in (a) are linear fits to the left nine data points. Their slopes are $0.037(16)$, $0.061(23)$, and $0.092(31)$ from down to up. Dashed lines in (b) connecting symbols are only a guide to the eyes. In (a) and (b), the error bars are estimated to be about the sizes of the symbols. In (c) and (d), each curve with a finite $\sigma'$ is displaced by 1.5 to the right with respect to its preceding one.

Moreover, the apparent values of $\sigma$ for the 2D and 3D Ising models appear to be different, though they are small with large errors and there exists no reference for a reliable estimate. In addition, these values appear to have no simple explanation similar to the heating cases and hence do not rule out the possibility of a new single exponent that yields the two values in the two spatial dimensions. Interestingly, the two numerical values may be both about $0.064$ within the estimated large errors, a situation which appears also in FTS and FSS, both in heating. One may then wonder whether the only left case of FSS in cooling should also have similar numerical values or not. However, the numerical value of the 3D $\beta/2\nu$ appears to have no peers due to the small 2D $\beta$ even if $\nu$ is separated similar to the case of FTS in heating.
We have seen that the violations of scaling during heating both for FSS and FTS are prominent and their Bressy exponents are quite big. Yet, that during cooling is less pronounced and the corresponding Bressy exponents are small if the revised scaling and its exponent of \( d/2 \) are excluded. Although the violation of the 3D FSS in cooling shown in figures 6(f) and 7(b) are also evident and appear not to stem from corrections to scaling, its 2D counterpart appears good [46]. This difference may arise from the small 2D value of \( \beta/2\nu = 1/16 = 0.0625 \) in comparison with the same 3D value of about 0.259 [46]. However, one might suspect that larger lattice sizes might remove the violations. For the FTS in cooling, the reduced Bressy exponent \( \sigma' = \sigma - d/2 \) is even difficult to be determined, as exhibited in figure 17, because of no proper reference. Nevertheless, we emphasize that at least the heating results are definite.
Figure 19. FTS of (a) and (b) $M$ and (c) and (d) $\chi$ with fixed $L^{-1} R^{-1/\gamma} \approx 0.3541$ for $z = 2.055$, (a) and (c), and $z = 2.211$, (b) and (d), for the 3D Ising model. Panels in both rows and columns share the same scales and labels. The vertical dashed lines mark $T = T_c$. The legend specifies the lattice sizes used for all the panels.

Figure 20. FTS of (a) and (b) $M$ and (c) and (d) $\chi$ with fixed $L^{-1} R^{-1/\gamma} \approx 0.0758$ for $z = 2.188$, (a) and (c), and $z = 2.122$, (b) and (d), for the 2D Ising model. Panels in both rows and columns share the same scales and labels. The vertical dashed lines mark $T = T_c$. The legend specifies the lattice sizes used for all the panels.
6. FTS in heating with different dynamic critical exponents

In this section, we study the issue of different dynamic critical exponent \( z \) estimated in heating and in cooling. Only heating is considered since the estimated cooling \( z \) agree well with other sources. We employ the usual definitions of \( M \) and \( \chi \) without using absolute values, viz, the first equations in equations (18) and (19), in this section.

In reference [32], the dynamical critical exponent \( z \) was estimated through fitting the expansions of the scaling functions in different regimes right at \( T_c \). The best fitted \( z \) was found to be \( 2.211(2) \) and \( 2.055(5) \) in heating and cooling, respectively, for the 3D Ising model using \( \langle |m| \rangle \) and \( \langle m^2 \rangle \). However, the scaling collapse of \( \chi' \) at \( T_c \) versus \( L^{-1}R^{-1/r} \) in heating for the 3D model requires a \( z \) which is, to the contrast, smaller than that in cooling, a situation which is different from the 2D model (see below). Corrections to scaling and the fact that \( T_c \) lies in the ordered phase in heating whereas it sits in the disordered phase in cooling as seen from figures 11 and 13, as well as a small sample size of no more than 10 thousands, were suggested as possible reasons for the different \( z \) values in heating and cooling. Indeed, figures 11 and 13 show that the scaling exactly at \( T_c \) in heating is poorer than that in cooling. This may give rise to the difference values of the estimated \( z \). Here, we compare the scaling collapses of the two exponents in the whole critical regime rather than just at \( T_c \) itself.

We show the scaling collapses of \( M \) and \( \chi \) on fixed lattice sizes in figure 18 and those for fixed \( L^{-1}R^{-1/r} \) in figure 19. Note that the rescaled \( \chi \) peaks rise with \( z \) because of the bigger \( r \) from equation (3). One sees that the collapses for both values of \( z \) become better as \( L \) increases similar to those in figure 8, although they are both worse than those for fixed \( L^{-1}R^{-1/r} \) in figure 19. Since the curves from the small lattice sizes are absent, the scaling collapses in figures 19(a) and (c) appear far better than figures 11(e) and (f), all have identical fixed \( L^{-1}R^{-1/r} \). It is clear from figures 18 and 19 that the collapses of both \( M \) and \( \chi \) with \( z = 2.055 \) are better than those with \( z = 2.211 \). Therefore, \( z \) appears to be around 2.055 for both heating and cooling from the scaling collapses in the 3D Ising model.

For comparison, we show in figure 20 the same collapses for the 2D Ising model. For the 2D model, the \( z \) values estimated from the collapses of \( \chi' \) at \( T_c \) agree with results obtained from \( \langle |m| \rangle \) and \( \langle m^2 \rangle \), which give rise to \( z \) = 2.188(2) for cooling and \( z \) = 2.1223(9) for heating. Note the different trends in heating and in cooling for the 2D and 3D models. One sees from figure 20 that the difference between the two \( z \) values is only slight, possibly because the two \( z \) values differ less than 0.1. Therefore, from the curve collapses, the 3D \( z \) values in heating and in cooling appear identical whereas the 2D ones different.

7. Summary

We have studied in detail the dynamic scaling of the 3D Ising model driven by heating or cooling through its critical point on finite-size lattices. Two sets of observable quantities, \( \langle |m| \rangle \) and \( \chi \) versus \( \langle m^2 \rangle \) and \( \chi' \), have been investigated. We have studied FTS on fixed lattice sizes and FSS at fixed driving rates as well as their full scaling forms. For the 3D Ising model, it has been found that the FTS is good for \( L^{-1}R^{-1/r} < 0.3 \) or so while the FSS behaves well for about \( RL' < 1.6 \). Fluctuations in heating and cooling in the FTS regime has also been observed to exhibit qualitatively different behavior. More importantly, we have shown that the two sets of the observables exhibit distinctive scaling behavior similar to the 2D model [46]. Only one set of the observables displays good FTS on fixed lattice sizes or FSS at fixed driving rates in their respective scaling regimes according to the standard theory, provided that other sub-leading terms and corrections to scaling are ignored. The other set violates FTS or FSS even in their respective scaling regimes. However, when the scaled variables \( L^{-1}R^{-1/r} \) or \( RL' \) is fixed, the violated scalings including the crossover regime are completely restored in a large temperature range around the critical point. This last result indicates that phase ordering can only have an effect at temperatures beyond this range and far lower than the lower-temperature boundary of the impulse region. As a consequence, it cannot be the origin of the violation of the KZ scaling reckoned exactly on the boundary.

In order to explain these observations, we need to endow large clusters with physics and separate critical fluctuations into magnitude fluctuations that associated with the forming of the large clusters of the spin-up and the spin-down phases and the phases fluctuations, which are the flipping of the large clusters. In equilibrium and in the thermodynamics limit, the large clusters are of the size of the correlation length on average and have roughly the equilibrium magnetization. Thus, at \( T \geq T_c \), their magnetization is roughly zero and the picture is the usual picture of fluctuations on all scales as depicted by Kogut [57]. However, for \( T < T_c \) but near \( T_c \), they have predominantly up or down spins and hence a finite magnetization, thus acting as the up or the down phases. Yet, they are not frozen in but are fluctuating unless \( T \) is low. Under driving or on finite-sized lattices, the size of the phases clusters may be controlled by these additional length scales and hysteresis or advance occur. A most direct evidence of the phases fluctuations comes from the FSS regime in which the whole finite-sized system is just one large cluster on average because the correlation length is longer than the system size at least deep inside the regime. In this case the system is no doubt in one dissymmetric phase and changes to the other with time back and forth, which is a transparent picture of the phases fluctuations. Accordingly, \( \langle m \rangle \) can be vanishingly small and \( \chi \) be large because of the phases fluctuations, whereas \( \langle |m| \rangle \) can be finite and \( \chi' \) be relatively small because the flipping of the clusters are removed and thus both probe the magnitude fluctuations. Also, \( \langle m \rangle \) may be finite at some range of temperatures depending on initial conditions. Therefore, phases fluctuations are the origin of the difference between the two sets of the observables. They also lead to the KZ mechanism since the boundary between the clusters of different dissymmetric phases are just the topological defects. Moreover, because only one set of the observables violates the scalings, if there are no phases fluctuations, no violations of the scalings appear.
The scaled variables \( L^{-1} R^{-1/r} \) and \( RL' \) are the origin of the violated scalings, because once they are fixed, the violations are completely restored, even though the two sets of the observables still behave differently, provided that other subleading terms and corrections to scaling are negligible. This implies that the phases fluctuations are only the necessary condition of the violations. On the one hand, fixing \( L^{-1} R^{-1/r} \) in FTS means that lattices of different sizes are driven by different rates \( R \) so that every lattice contains the same number of phases clusters of the driven length \( R^{-1/r} \) on average. This is the spatial self-similarity. On the other hand, fixing \( RL' \) in FSS indicates that the survival time \( L' \) of the fluctuating phases on every lattice is a fixed fraction of the driven time \( R^{-2/r} \) on average. This is the temporal self-similarity. Therefore, fixing the scaled variables is just to maintain the spatial or temporal self-similarity of the phases clusters. This is an extrinsic self-similarity that is guaranteed by external conditions such as the system size and the driving rate in contrast to the intrinsic self-similarity controlled by the critical point. Since the phases fluctuations are the necessary condition of the violations, it is therefore appropriate to conclude that the breaking of the extrinsic self-similarity of the phases fluctuations results in the violations of FTS and FSS. In addition, it also gives rise to the qualitatively different behaviors of the magnitude fluctuations in heating and in cooling shown in figure 3 since they behave similarly if \( L^{-1} R^{-1/r} \) is fixed. The reason is that the phases fluctuations change the magnetization \( m \) of a sample at an instant or a temperature and hence also change \( \langle |m| \rangle \) and \( \chi' \), the magnitude fluctuations. If their extrinsic self-similarity is present, the change for different lattices is the same on average and thus does not alter the dependence of \( \chi' \) on \( R \). Otherwise, the bigger the number of the large phases clusters, the smaller the contribution of overturning a single cluster. This suppresses the magnitude of \( \langle |m| \rangle \) and \( \chi' \) at a particular temperature but expands the distribution of \( m \) and hence increases the entire \( \chi' \) peak.

Our numerical results have revealed that the violated scalings can also be rectified even the extrinsic self-similarity of the phases fluctuations is broken. This is achieved by introducing breaking-of-extrinsic-self-similarity, abbreviated as Bressy, exponents to collapse the separated rescaled curves onto each other. In this regard, the two observables that violate the scaling exhibit different behaviors. One is a primary observable whose scaling function is a power-law of the scaled variable \( L^{-1} R^{-1/r} \) or \( RL' \) with the power related to the Bressy exponents. The other is referred to as the secondary observable whose scaling function possesses a regular term and may thus appear not to violate the scaling. This is why two secondary observables appear parenthesized in table 1. Because the scalings are violated only in either the ordered phase or the disordered phase, the Bressy exponents then lead to different leading behavior in the two phases, even though the scaling collapses occur only on one slope of the susceptibility peaks. This is in stark contrast with the equilibrium critical phenomena in which the leading behaviors of the two phases are identical, only their amplitudes differ, though with universal amplitude ratios. The Bressy exponents, whose values are summarized in table 1 along with the phases and the observables that violate the scaling, are found to be different for heating and cooling and in FTS and FSS, except for FSS in cooling. They have also different expressions in the 2D and the 3D Ising models, except again for FSS in cooling. This implies that they are most likely new exponents that produce the 2D and 3D values found here. At present, the Bressy exponents of heating for both FSS and FTS are manifest and their absolute values are relatively large, while those of cooling are not so definite and their values are relatively small, especially for the reduced exponent \( \sigma' \) of FTS in cooling in which \( \langle |m| \rangle \) has on peak for reference.

The regime in which Bressy exponents are needed is limited and there exists crossover from this regime to the usual scaling regime. The collapses in finding the Bressy exponents are limited to a certain range of the scaled variable \( L^{-1} R^{-1/r} \) or \( RL' \). First, if \( RL' (L^{-1} R^{-1/r}) \) is too large, crossover from the FSS (FTS) regime to the FTS (FSS) regime occurs and the FSS (FTS) fails. The collapses cannot therefore be good. Then, if \( R = 0 \) or \( L = \infty \), the usual FSS or FTS, respectively, must recover. Except these special points, the Bressy-exponent dominated regime is believed to be valid for arbitrarily small \( RL' (L^{-1} R^{-1/r}) \) for the FSS in both heating and cooling (the FTS in cooling). Accordingly, the crossovers from this regime to the usual FSS and FTS are believed to occur at \( RL' = 0 \) and \( L^{-1} R^{-1/r} = 0 \) abruptly. For FTS in heating, the crossover appears to happen abruptly at a finite \( L^{-1} R^{-1/r} \) as seen in figure 16(a).

In addition, in order to investigate whether heating and cooling have a different dynamic critical exponent \( z \), we have shown FTS collapses in heating using different values of \( z \). The quality of curve collapses shows that the 3D \( z \) values in heating and in cooling appear identical but the 2D ones different, possibly because of the difference in 2D is too small to be resolved by the method.

Half a century have passed since the renormalization-group theory in which critical phenomena culminate was set up. Still, we have seen that, through a simple driving, a series of phenomena emerge centering on the violations of the usual FSS and FTS with novel ideas of the phases fluctuations and self-similarity breaking together with its associated critical exponents, besides the usual driven nonequilibrium critical phenomena [5]. Yet, these are just the tip of an iceberg and a lot of problems are yet to be resolved such as how the extrinsic self-similarity breaking results in the Bressy exponents, what are the Bressy exponents, why in some cases it is one set while in other cases it is the other set of the observables that violate the scaling and the similar questions about the primary and secondary observables and the phase where scaling is violated, how exactly the crossover between the usual scaling and the extrinsic self-similarity-breaking-controlled regimes occurs and so on.

In retrospect, that FTS in general and the KZ scaling in particular and even the revised FTS involve only the equilibrium critical exponents is specific and rather fortunate, given that critical phenomena are generically nonlinear owing to the generically non-integer critical exponents and that nonequilibrium occurs under driving [29]. We have demonstrated that
when the lattice size is considered, new exponents are generally required for the scaling in the whole driving process within the scaling regime. Although these exponents and many other problems are yet to be understood, our findings open a door in critical phenomena and suggest that much is yet to be explored in driven nonequilibrium critical phenomena.

The linear or even nonlinear driving in FTS is readily realized experimentally [5, 27, 43, 44] while experimental verification of FSS has been progressing in the superfluid transition of confined $^4$He [73, 74]. Recent experimental advances in manipulating real-time evolution of ultracold atoms [1–4, 15] provide another platform for studying finite-sized systems. These offer an opportunity to observe the phases fluctuations and their self-similarity experimentally. A suitable case for this is the FTS in finite-sized systems. This is because the system size $L$ needs not be particularly small. Rather, it can comprise a number of the driven length $\xi_R$. Moreover, applying an external field in different protocols in cooling can serve, among others, a consistent check since the field-cooled and zero-field heating processes are described by the same Bressy exponent [67].

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Data availability statement

The data that support the findings of this study are available upon reasonable request from the authors.

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References

[1] Greiner M, Mandel O, Esslinger T, Hänsch T W and Bloch I 2002 Nature 415 39
[2] Kinoshita T, Wenger T and Weiss D S 2006 Nature 440 900
[3] Hofferberth S, Lesanovsky I, Fischer B, Schumm T and Schmiedmayer J 2007 Nature 449 324
[4] Zhang X, Hung C-L, Tung S-K and Chin C 2012 Science 335 1070
[5] Feng B, Yin S and Zhong F 2016 Phys. Rev. B 94 144403
[6] Kibble T W B 1976 J. Phys. A: Math. Gen. 9 1387
[7] Kibble T 2007 Phys. Today 60 47
[8] Zurek W H 1985 Nature 317 505
[9] Zurek W H 1996 Phys. Rep. 276 177
[10] Laguna P and Zurek W H 1997 Phys. Rev. Lett. 78 2519
[11] Yates A and Zurek W H 1998 Phys. Rev. Lett. 80 5477
[12] Antunes N D, Bettencourt L M A and Zurek W H 1999 Phys. Rev. Lett. 82 2824
[13] Stephens G J, Calzetta E A, Hu B L and Ramsey S A 1999 Phys. Rev. D 59 045009
[14] Suzuki S 2009 J. Stat. Mech. P03032
[15] Yukalov V I, Novikov A N and Bagnato V S 2015 Phys. Lett. A 379 1366
[16] Hamp J, Chandran A, Moessner R and Castelnovo C 2015 Phys. Rev. B 92 075142
[17] del Campo A 2018 Phys. Rev. Lett. 121 200601
[18] Gómez-Ruiz F J, Mayo J J and del Campo A 2019 arXiv:1912.04679
[19] Cui J-M, Gómez-Ruiz F J, Huang Y-F, Li C-F, Guo G-C and del Campo A 2020 Commun. Phys. 3 44
[20] del Campo A and Zurek W H 2014 Int. J. Mod. Phys. A 29 1430018
[21] Bray A J 1994 Adv. Phys. 43 357
[22] Biroli G, Cugliandolo L F and Sicilìa A 2010 Phys. Rev. E 81 050101(R)
[23] Fisher M E and Barber M N 1972 Phys. Rev. Lett. 28 1516
[24] Barber M N 1985 Finite-size scaling Phase Transitions and Critical Phenomena vol 8 ed C Domb and J Lebowitz (New York: Academic)
[25] Cardy J (ed) 1988 Finite Size Scaling (Amsterdan: North-Holland)
[26] Privman V (ed) 1990 Finite Size Scaling and Numerical Simulations of Statistical Systems (Singapore: World Scientific)
[27] Gong S, Zhong F, Huang X and Fan S 2010 New J. Phys. 12 043036
[28] Zhong F 2011 Applications of Monte Carlo Method in Science and Engineering ed S Mordechai (Rijeka: Intech) p 469 http://dx.doi.org/10.5772/99pe2
[29] Zhong F 2006 Phys. Rev. E 73 047102
[30] Yin S, Qin X, Lee C and Zhong F 2012 arXiv:1207.1602
[31] Yin S, Mai P and Zhong F 2014 Phys. Rev. B 89 144455
[32] Huang Y, Yin S, Feng B and Zhong F 2014 Phys. Rev. B 90 134108
[33] Liu C-W, Polkovnikov A and Sandvik A W 2014 Phys. Rev. B 89 054407
[34] Liu C-W, Polkovnikov A, Sandvik A W and Young A P 2015 Phys. Rev. E 92 022128
[35] Liu C-W, Polkovnikov A and Sandvik A W 2015 Phys. Rev. Lett. 114 147203
[36] Pelissetto A and Vicari E 2016 Phys. Rev. E 93 032141
[37] Xu N, Castelnovo C, Melko R G, Chamon C and Sandvik A W 2018 Phys. Rev. B 97 024432
[38] Xue M, Yin S and You L 2018 Phys. Rev. A 98 013619
[39] Cao X, Hu Q and Zhong F 2018 Phys. Rev. B 98 245124
[40] Gerster M, Haggenmiller B, Tschirsich G, Silvi P and Montangero S 2019 Phys. Rev. B 100 024311
[41] Li Y, Zeng Z and Zhong F 2019 Phys. Rev. E 100 020105(R)
[42] Mathey S and Diehl S 2020 Phys. Rev. Res. 2 013150
[43] Clark L W, Feng L and Chin C 2016 Science 354 606
[44] Keesling A et al 2019 Nature 568 207
[45] Huang Y, Yin S, Hu Q and Zhong F 2016 Phys. Rev. B 93 024103
[46] Yuan W, Yin S and Zhong F 2021 Chin. Phys. Lett. 38 026401
[47] Mandelbrot B B 1983 The Fractal Geometry of Nature (San Francisco, CA: Freeman)
[48] Meakin P 1998 Fractal, Scaling and Growth Far from Equilibrium (Cambridge: Cambridge University Press)
[49] Brézin E 1982 J. Physique 43 15
[50] Bréniz E and Zinn-Justin J 1985 Nucl. Phys. B 257 867
[51] Zhong F and Chen Q 2005 Phys. Rev. Lett. 95 175701
[52] Hohenberg P C and Halperin B I 1977 Rev. Mod. Phys. 49 435
[53] Ma K S 1976 Modern Theory of Critical Phenomena (New York: Benjamin)
[54] Fisher M E 1982 Scaling, Universality and Renormalization Group Theory ( Lecture Notes Presented at the Advanced Course on Critical Phenomena) (South Africa: The Meren-sky Institute of Physics, University of Stellenbosch)
[55] Cardy J 1996 Scaling and Renormalization in Statistical Physics (Cambridge: Cambridge University Press)
[56] Wegner F J 1972 Phys. Rev. B 5 4529

W Yuan and F Zhong
[57] Kogut J B 1979 Rev. Mod. Phys. 51 659
[58] Goldstone J 1961 Il Nuovo Cimento 19 155
[59] Patashinski A Z and Pokrovski V L 1974 Sov. Phys.-JETP 37 733
[60] Zwerger W 2004 Phys. Rev. Lett. 92 027203
[61] Suzuki M 1983 Prog. Theor. Phys. 69 65
[62] Ferrenberg A M and Landau D P 1991 Phys. Rev. B 44 5081
[63] Pelissetto A and Vicari E 2002 Phys. Rep. 368 549
[64] Kleinert H 1999 Phys. Rev. D 60 085001
[65] Grassberger P 1995 Physica A 214 547
[66] Kikuchi M and Ito N 1993 J. Phys. Soc. Japan 62 3052
[67] Yuan W and Zhong F 2021 J. Phys. Condens. Matter 33 375401
[68] Metropolis N, Rosenbluth A W, Rosenbluth M N, Teller A H and Teller E 1953 J. Chem. Phys. 21 1087
[69] Glauber R J 1963 J. Math. Phys. 4 294
[70] Landau D P and Binder K 2005 A Guide to Monte Carlo Simulations in Statistical Physics 2nd edn (Cambridge: Cambridge University Press)
[71] Janssen H K, Schaub B and Schmittmann B 1989 Z. Phys. B 73 539
[72] Landau D P 1976 Phys. Rev. B 13 2997
[73] Gasparini F M, Kimball M O, Mooney K P and Diaz-Avila M 2008 Rev. Mod. Phys. 80 1009
[74] Perron J K, Kimball M O and Gasparini F M 2019 Rep. Prog. Phys. 82 114501