THERMODYNAMICS AND UNIVERSALITY FOR MEAN FIELD QUANTUM SPIN GLASSES

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ABSTRACT. We study aspects of the thermodynamics of quantum versions of spin glasses. By means of the Lie-Trotter formula for exponential sums of operators, we adapt methods used to analyze classical spin glass models to answer analogous questions about quantum models.

1. INTRODUCTION

Classical spin glass models have seen a flurry of activity over the last few years, see [2, 4, 9, 10, 11, 13, 14, 15, 20, 21, 22] to name a few; in particular [2, 9, 10, 22] all consider aspects of the (generalized) Sherrington-Kirkpatrick model, a mean-field model in which the interactions between spins are mediated by an independent collection of Gaussian random variables. In contrast, though physicists have considered both short and long range quantum spin glass models for quite sometime, see [3, 8, 18, 19], there are few rigorous mathematical results; we mention [5], which provides a proof that the quenched free energy of certain short ranged quantum spin glasses exists.

Here we extend classical spin glass results to quantum models in two directions. First, using the ideas of [11], we demonstrate the existence of the quenched free energy of the Sherrington-Kirkpatrick spin glass with a transverse external field. Next, under conditions made precise below, we give a complementary result which shows that a large class of quantum spin glasses, including the transverse S-K model, satisfies universality. By this we mean that the asymptotics of the free energy of the system do not depend on the type of disorder used to define model. This latter result is based on the work of [4].

One may view this paper as an attempt to adapt the methods of classical spin systems to the analysis of various quantum models. Guerra’s interpolation scheme [9, 10, 11] and the Gaussian integration-by-parts formula are ubiquitous tools in the classical setting. A major theme of the present work is that through the systematic use of the Lie-Trotter product formula (i.e. that

\[ e^{A+B} = \lim_{k \to \infty} \prod_{j=1}^{k} e^{\frac{A}{k}} e^{\frac{B}{k}} \]  

(1.1)

for any pair of operators \(A, B\) one may extend the interpolation scheme and integration-by-parts formula to quantum systems in useful ways.

For concreteness, the remainder of the introduction and the following section use the language of the spin-\(1/2\) representation of \(su(2)\) to describe quantum spin systems, though a number of
our methods apply in larger generality. In this setting, one describes each particle using a two dimensional Hilbert space $\mathbb{C}^2$ along with a representation of $su(2)$ generated by the triplet of Pauli operators $\vec{S} = (S^{(x)}, S^{(y)}, S^{(z)})$.

To represent an $N$-particle system, we introduce the tensor product $\mathbb{C}^{2^N}$ along with a sequence $(\vec{S}_j)_{j=1}^N$ of $N$ copies of the Pauli vector $\vec{S}$, where $\vec{S}_j$ acts on the $j$'th factor. The particles interact by means of the Gibbs-Boltzmann operator $e^{-\beta \mathcal{H}_N}$ associated to the Hamiltonian $\mathcal{H}_N$. Here, $\mathcal{H}_N$ is a self adjoint operator acting on $\mathbb{C}^{2^N}$, typically a polynomial in the $N$-tuple of spin operators $(\vec{S}_j)_{j=1}^N$. For example, the Hamiltonian of the simplest non-trivial quantum spin system, the transverse Ising model, is described by

$$-\mathcal{H}_N = \frac{1}{N} \sum_{i,j=1}^N S_i^{(z)} S_j^{(z)} + \lambda \sum_{j=1}^N S_j^{(x)}$$

(1.2)

where $\lambda > 0$.

Once we specify the Hamiltonian, statistical quantities of the system may be defined. The partition function and free energy of a quantum spin system are defined via the trace of the Gibbs-Boltzmann operator as

$$Z_N(\beta) = \text{Tr} \left( e^{-\beta \mathcal{H}_N} \right)$$

$$f_N(\beta) = -\frac{1}{N\beta} \log Z_N.$$

Self adjoint operators on $\mathbb{C}^{2^N}$ replace functions as observables of the system and the thermal average of an observable $A$ is defined as

$$\langle A \rangle = \frac{\text{Tr} \left( A e^{-\beta \mathcal{H}_N} \right)}{\text{Tr} \left( e^{-\beta \mathcal{H}_N} \right)}.$$

With this formalism we present a few examples of spin glasses of particular interest. Traditionally the modeling of any spin glass necessitates the introduction of disordered interactions between spin. In general, and will be the case here, the interactions are i.i.d. The basic example, the transverse S-K model, has a Hamiltonian defined by

$$-\mathcal{H}_N = \frac{1}{2\sqrt{N}} \sum_{i,j=1}^N J_{i,j} S_i^{(z)} S_j^{(z)} + \lambda \sum_{j=1}^N S_j^{(x)}.$$

A more complicated class of models, the quantum Heisenberg spin glasses, are described by one of the Hamiltonians

$$-\mathcal{H}_N = \frac{1}{2\sqrt{N}} \sum_{i,j=1}^N J_{i,j} \left[ S_i^{(z)} S_j^{(z)} + \alpha S_i^{(x)} S_j^{(x)} + \gamma S_i^{(y)} S_j^{(y)} \right]$$

$$-\tilde{\mathcal{H}}_N = \frac{1}{2\sqrt{N}} \sum_{\nu \in \{x,y,z\}} \sum_{i,j=1}^N J_{i,j}^{(\nu)} S_i^{(\nu)} S_j^{(\nu)},$$

where $\alpha, \gamma \in \mathbb{R}$.
There is a sharp and interesting contrast in the scope of application of our results. Unlike the quantum Heisenberg spin glasses, applying the Lie-Trotter expansion to the transverse S-K model has the added benefit of allowing a natural path integral representation, known as the Feynman-Kac representation, of statistical quantities like the free energy. More precisely, the Feynman-Kac representation allows the expression of the partition function as a measure on càdlàg paths with state space the classical Ising spin configurations on $N$ sites, $\{-1, +1\}^N$. For this and other technical reasons we can only resolve the existence of the free energy for the one quantum model.

The remainder of the paper is structured as follows. Section 2 introduces notation and states the main theorems of the paper. Section 3 presents quantum generalizations of [4] which allow us to prove universality and control the fluctuations of the free energy for a class of quantum spin glasses which include the spin-$1/2$ quantum Heisenberg spin glasses. In addition, we adapt the techniques of [11] to prove exponential decay of the fluctuations of models with Gaussian disorder. Finally, Section 4 details the existence of the pressure of the transverse S-K model.

2. Results

We begin by giving a bit of notation to be used below. We consider a collection $\xi_I$ of random variables, where $I$ is in some finite indexed set $\mathcal{I}$ and denote $\mathbb{E} [\cdot]$ and $\mathbb{P} (\cdot)$ integration with respect to this collection. Examples of index sets that we have in mind include the collection of all $r$-tuples of sites in a system of $N$ particles. In the simplest case the index set $\mathcal{I}$ consists of all pairs $(i,j)$ (or more generally some subset of pairs). Unless otherwise specified, the variables $\xi_I$ are assumed to be i.i.d. according to some fixed random variable $\xi$ satisfying the conditions

$$\mathbb{E} [\xi] = 0, \quad \mathbb{E} [\xi^2] = 1, \quad \mathbb{E} [|\xi|^3] < \infty.$$  

When explicitly considering Gaussian environments we denote the random variables by $g_I$.

Let $\mathcal{S}$ represent the $N$-tuple of Pauli vectors and consider the Hamiltonian

$$\mathcal{H}_N(\xi) = \sum_{I \in \mathcal{I}} \xi_I X_I(\mathcal{S})$$

where each $X_I(\mathcal{S})$ is a self-adjoint polynomial in the spin operators, i.e. $X_I^*(\mathcal{S}) = X_I(\mathcal{S})$. We define the associated partition function and quenched ‘pressure’ by

$$Z_N(\beta, \xi) = \text{Tr} \left( e^{\beta \mathcal{H}_N} \right); \quad \alpha_N(\beta, \xi) = \mathbb{E} [\log Z_N(\beta, \xi)].$$

Note that for convenience we have omitted the minus sign from the expression for the Gibbs-Boltzmann operator. For any operator $A$ we denote its operator norm by $\|A\| := \sup_{v \in V} \left[ \frac{(Av, Av)}{(v,v)} \right]^{1/2}$. With this notation we have the following result:

**Theorem 2.1** Let $\xi$ be random variable with mean 0, variance 1, and $\mathbb{E} [\xi^3] < \infty$ and let $g$ be a standard normal random variable. Let $\mathcal{I}_N$ index the interactions between particles at the system size $N$. Suppose

$$\sum_{I \in \mathcal{I}_N} \|X_I(\mathcal{S})\|^3 = o(N).$$
Then for any $\beta \in \mathbb{R}$, 
\[ \left| \frac{1}{N} \alpha_N(\beta, \xi) - \frac{1}{N} \alpha_N(\beta, g) \right| = o(1) \]
as $N$ tends to infinity. Moreover 
\[ \mathbb{E} \left[ \left| \frac{1}{N} \log Z(\beta, \xi) - \frac{1}{N} \alpha_N(\beta, \xi) \right|^3 \right] \leq \frac{\sqrt{3N} o(N)}{N^3}. \]

Remark 2.2 In the quantum Heisenberg Hamiltonians, the norm of each summand is bounded by $\frac{C}{\sqrt{N}}$ and the number of pairs is of the order $N^2$. As a result, 
\[ \sum_{I \in \mathcal{J}_N} \| X_I(S) \|^3 = O(N^{\frac{3}{2}}) \]
and the theorem is immediately applicable.

The error bounds get better if one assumes the random variable $\xi$ has higher moments. At the extreme end, we consider the case of fluctuations for Gaussian environments:

**Proposition 2.3** Let $g$ be a standard normal random variable. Then 
\[ \mathbb{P} \left( | \log Z_N(\beta, g) - \alpha_N(\beta, g) | \geq u \right) \leq 2e^{-\frac{\sum_{I \in \mathcal{J}_N} \| X_I \|^2}{N^2u^2}}. \]

Remark 2.4 Note the generality with which these results are stated. In the introduction we advertised their application to mean field models. However, choosing the operators $X_I$ and index sets $\mathcal{J}_N$ appropriately allows application to a wide variety of systems.

Next we take up the more subtle question of the existence of the free energy for mean field models. As mentioned above, we specialize to the transverse S-K model:

\[ -H_N(\lambda) = \frac{1}{2\sqrt{N}} \sum_{i,j=1}^{N} g_{i,j} S_i^{(z)} S_j^{(z)} + \lambda \sum_{j=1}^{N} S_j^{(z)}. \]

In light of Theorem 2.1, we have assumed the interactions are Gaussian.

**Theorem 2.5** Let $\beta, \lambda > 0$ be fixed. Then 
\[ \lim_{N \to \infty} \frac{-1}{\beta N} \mathbb{E} \left[ \log \text{Tr} \left( e^{-\beta H_N(\lambda)} \right) \right] \]
exists and is finite. Moreover, there exists a $K > 0$ so that the following concentration property holds: 
\[ \mathbb{P} \left( \left| -\frac{1}{N} \log \text{Tr} \left( e^{-\beta H_N(\lambda)} \right) + \frac{1}{N} \mathbb{E} \left[ \log \text{Tr} \left( e^{-\beta H_N(\lambda)} \right) \right] \right| \geq u \right) \leq 2e^{-\frac{N K u^2}{\beta^2}}. \]

Remark 2.6 For readability, we assume here that the disordered portion of the Hamiltonian in (??) is a two body interaction. However analogous arguments allow one to treat p-spin models.
where we replace the disordered portion by

$$-\mathcal{H}_N^{\text{dis}}(S^{(z)}) = N \sum_{r=1}^{\infty} \frac{a_r}{N^2} \sum_{i_1, \ldots, i_r} g_{i_1 \ldots i_r} \prod_{k=1}^{r} S^{(z)}_{i_k}$$

Here \(\{g_{i_1 \ldots i_r}\}\) is a collection of independent standard Gaussian random variables and \(\sum_{1}^{\infty} a_r^2 q^r\) is even, convex and continuous as a function of \(q\) on \([-1, 1]\). See [2] and [11].

3. Universality and Fluctuations

Before proving Theorem 2.1, we adapt the methods of [4] so as to apply them to a wide variety of quantum spin systems. It turns out that the line of argument given there is robust enough to be followed in the quantum case, though the calculations must be adjusted to accommodate the non-commutative setting.

In the general setup, we consider a collection of self-adjoint operators \(\{X_i\}_{i=1}^{d}\) and \(\mathcal{H}_0\) defined on some finite dimensional Hilbert space. Let

$$\mathcal{H}(\xi) = \sum_{i=1}^{d} \xi_i X_i + \frac{1}{\beta} \mathcal{H}_0$$

$$Z(\beta, \xi) = \text{Tr} \left( e^{\beta \mathcal{H}(\xi)} \right)$$

$$\alpha(\beta, \xi) = \mathbb{E} [\log Z(\beta, \xi)]$$

denote the Hamiltonian, partition function, and quenched ‘pressure’ of a system. We define the thermal average, Duhammel two point function, and the three point function for the operators \(A, B,\) and \(C\) as follows:

$$\langle A \rangle = \frac{\text{Tr} \left( A e^{\beta \mathcal{H}(\xi)} \right)}{Z(\beta, \xi)}$$

$$\langle A, B \rangle = \frac{\int_{0}^{1} \text{Tr} \left( A e^{s \beta \mathcal{H}(\xi)} B e^{(1-u) \beta \mathcal{H}(\xi)} \right) du}{Z(\beta, \xi)}$$

$$\langle A, B, C \rangle = \frac{\int_{0}^{1} \int_{0}^{1} u \text{Tr} \left( A e^{s u \beta \mathcal{H}(\xi)} B e^{(1-s) u \beta \mathcal{H}(\xi)} C e^{(1-u) \beta \mathcal{H}(\xi)} \right) ds du}{Z(\beta, \xi)}.$$
Indeed, we may assume by continuity that 

\[
0
\]

so that

\[
\text{we find that the first and second derivatives of}
\]

\[
Trotter\ expansion,\ i.e.\ that
\]

\[
\text{To expand Equation (3), we calculate the first and second derivatives of}
\]

\[
\text{and define the function}
\]

\[
\text{Lemma 3.2}
\]

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Remark 3.3 There is no correction in this formula when \(\xi\) is Gaussian since the integration-by-parts formula is exact in this case.

Proof. To begin the derivation, we have

\[
\frac{\partial \alpha(\beta, \xi)}{\partial \beta} = \sum_{i=1}^{d} \mathbb{E} \left[ \xi_i (X_i) \right].
\]

Let us define \(\mathcal{H}_i(z) = \sum_{j \neq i} \xi_j X_j + z X_i + \frac{1}{\beta} \mathcal{H}_0\), denote the corresponding Gibbs state by \(\langle \cdot \rangle_i^{(z)}\) and define the function \(F_i(z) = \langle X_i \rangle_i^{(z)}\). Then

\[
\mathbb{E} [\xi_i (X_i)] = \mathbb{E} [\xi_i F_i (\xi_i)].
\]

To expand Equation (3), we calculate the first and second derivatives of \(F_i\). Applying the Lie-Trotter expansion, i.e. that \(e^{A+B} = \lim_{k} \prod_{j=1}^{k} e^{A_{j} e^{B_{j}/k}}\), with \(A = \beta z X_i\) and \(B = \beta \mathcal{H}_i(z) - \beta z X_i\), we find that the first and second derivatives of \(F_i\) take the form:

\[
F_i'(z) = \beta \left[ (X_i, X_i)^{(z)}_i - \left( \langle X_i \rangle_i^{(z)} \right)^2 \right]
\]

\[
F_i''(z) = \beta^2 \left[ 2 (X_i, X_i, X_i)^{(z)}_i - 3 \langle X_i, X_i \rangle_i^{(z)} \langle X_i \rangle_i^{(z)} + 2 \left( \langle X_i \rangle_i^{(z)} \right)^3 \right].
\]

In order to prove the lemma we must bound \(F_i''\). To this end we claim that for any \(a_1, \ldots, a_n > 0\) so that \(a_1 + \cdots + a_n = 1\) and any self-adjoint operators \(X, \mathcal{H}\)

\[
\left| \text{Tr} \left( X e^{a_1 \mathcal{H}} \cdots X e^{a_n \mathcal{H}} \right) \right| \leq \|X\|^n \text{Tr} \left( e^{\mathcal{H}} \right)
\]

(3.2)

Indeed, we may assume by continuity that \(a_j = \frac{k_j}{2m}\) for some \(m > 0\) and some sequence of positive integers \((k_j)\) summing to \(2^m\).

For any sequence of operators \((B_j)_{j=1}^{2n}\),

\[
\left| \text{Tr} \left( \prod_{j=1}^{2k} B_j \right) \right| \leq \prod_{j=1}^{2k} \text{Tr} \left( [B_j B_j^*]^{k} \right)^{\frac{1}{2k}}.
\]

(3.3)

This can be seen as a specific case of the general method of chessboard estimates, Theorem 4.1 of [7]. Applying (3.3) to the left hand side of (3.2) with \(B_j \in \{ X e^{\frac{k_j}{2m}}, e^{\frac{k_j}{2m}} \}\) we have

\[
\left| \text{Tr} \left( X e^{a_1 \mathcal{H}} \cdots X e^{a_n \mathcal{H}} \right) \right| \leq \text{Tr} \left( \left[ X e^{\frac{k_1}{2m}} X \right]^{2^{m-1}} \right)^{\frac{m}{2m}} \text{Tr} \left( e^{\mathcal{H}} \right)^{\frac{m}{2m}}
\]

(3.4)
Another application of (3.3) implies
\[
\text{Tr} \left( \left[ X \cdot e^{2m} X \right]^{2m-1} \right) \leq \text{Tr} \left( X^{2m+1} \right)^{\frac{1}{2}} \text{Tr} \left( e^{2H} \right)^{\frac{1}{2}}.
\]

Using this bound on the right hand side of (3.4) and letting \( m \) pass to infinity proves the bound (3.2).

It follows that \( \| F'' \|_\infty \leq 6 \beta^2 \| X \|_3^3 \). Recalling that \( \mathbb{E} [ \xi^2 ] = 1 \), Lemma 3.1 in conjunction with the previous calculations imply
\[
\left| \frac{\partial \alpha(\beta, \xi)}{\partial \beta} - \beta \mathbb{E} \left[ \sum_{i=1}^d (X_i, X_i) - \langle X_i \rangle^2 \right] \right| \leq 9 \beta^2 \mathbb{E} [ |\xi| ] \left( \sum_{i=1}^d \| X_i \|_3^3 \right).
\]

Next we use this expansion to compare the quenched ‘pressure’ for \( \xi \) to that of a Gaussian environment \( g \) where \( g \) is a standard normal.

**Lemma 3.4** Let \( \{ \xi_i \} \) and \( \{ g_i \} \) be collections of i.i.d random variables distributed according to \( \xi \) and \( g \) respectively. For any \( \beta \in \mathbb{R} \),
\[
| \alpha(\beta, \xi) - \alpha(\beta, g) | \leq 9 |\beta|^3 \mathbb{E} [ |\xi| ] \left( \sum_{i=1}^d \| X_i \|_3^3 \right)
\]

**Proof.** The similarity between the proof of this lemma and that of Proposition 7 [4] means that we will be extremely brief. Consider the interpolating partition function and corresponding quenched ‘pressure’ defined by
\[
Z(s, t - s) = e^{\sqrt{t} \left( \sum_{i=1}^d \xi_i X_i \right) + \sqrt{s} \left( \sum_{i=1}^d g_i X_i \right) + H_0}
\]
\[
\alpha^{(t)}(s) = \mathbb{E} \log Z(s, t - s)
\]
respectively. We have
\[
\alpha^{(t)}(t) = \alpha(\sqrt{t}, \xi); \quad \alpha^{(t)}(0) = \alpha(\sqrt{t}, g).
\]

Lemma 3.2 along with independence between the two environments implies that for all \( s \in [0, t] \) we have
\[
\left| \frac{\partial \alpha^{(t)}(s)}{\partial s} \right| \leq 9 \sqrt{t} \mathbb{E} [ |\xi| ] \left( \sum_{i=1}^d \| X_i \|_3^3 \right)
\]
For \( \beta \geq 0 \), if we let \( t = \beta^2 \) and integrate this inequality the result follows. For \( \beta < 0 \) we instead consider the environments \( -\xi, -g \).

Next we attend to the fluctuations of the ‘pressure’ determined by the random environment \( \{ \xi_i \} \):

**Lemma 3.5** There exists some universal constant \( c > 0 \) so that
\[
\mathbb{E} \left[ \log Z(\beta, \xi) - \alpha(\beta, \xi) \right]^3 \leq c \mathbb{E} [ |\xi| ]^3 \beta^3 \sqrt{d} \left( \sum_{i=1}^d \| X_i \|_3^3 \right).
\]
Proof. Consider the filtration $\mathcal{F}_k = \sigma\{\xi_1 \ldots \xi_k\}, k \geq 1$ determined by the sequence of independent random variables $(\xi_k)$. Let

$$\Delta_i := E\left[\log Z(\beta, \xi) | \mathcal{F}_i\right] - E\left[\log Z(\beta, \xi) | \mathcal{F}_{i-1}\right]$$

We have

$$\log Z(\beta, \xi) - \alpha(\beta, \xi) = \sum_{i=1}^{d} \Delta_i.$$  

Burkholder’s martingale inequality implies the existence of a universal constant $c'$ so that

$$E \left| \sum_{i=1}^{d} \Delta_i \right|^3 \leq c' E \left( \sum_{i=1}^{d} \Delta_i^2 \right)^{\frac{3}{2}}.$$  

To bound the increment $\Delta_i$, consider the partition function

$$Z_i(\beta, \xi) = \text{Tr}\left(e^{\beta \psi_i}\right)$$

where $\psi_i = \psi_i(\xi) := H(\xi) - \xi_i X_i$. Since $Z_i(\beta, \xi)$ is independent of $\xi_i$,

$$\Delta_i = E\left[\log \frac{Z(\beta, \xi)}{Z_i(\beta, \xi)} | \mathcal{F}_i\right] - E\left[\log \frac{Z(\beta, \xi)}{Z_i(\beta, \xi)} | \mathcal{F}_{i-1}\right].$$

We use this identity to estimate $\Delta_i$.

We claim that

$$\frac{Z(\beta, \xi)}{Z_i(\beta, \xi)} \leq e^{\beta|\xi_i|\|X_i\|}.$$  

Indeed, the Lie-Trotter formula implies

$$Z(\beta, \xi) = \lim_{k \to \infty} \text{Tr}\left(\prod_{j=1}^{2^k} e^{\frac{\beta \xi_i}{2^k}} e^{\frac{\beta \xi_j X_j}{2^k}}\right) = \lim_{k \to \infty} \text{Tr}\left(\prod_{j=1}^{2^k} e^{\frac{\beta \xi_i}{2^k}} e^{\frac{\beta \xi_j X_j}{2^k}} e^{\frac{\beta \xi_j X_j}{2^k}}\right).$$

Since

$$\left\| e^{\frac{\beta \xi_j X_j}{2^k}} e^{\frac{\beta \xi_j X_j}{2^k}} \right\| \leq e^{\frac{\beta|\xi_i|\|X_i\|}{2^k}},$$

Inequality (??) applied to the righthand side for $k$ finite gives

$$\text{Tr}\left(\prod_{j=1}^{2^k} e^{\frac{\beta \xi_j X_j}{2^k}} e^{\frac{\beta \xi_j X_j}{2^k}}\right) \leq e^{\beta|\xi_i|\|X_i\|} \text{Tr}\left(e^{\beta \psi_i}\right)$$

from which our claim follows.

From this we estimate:

$$|\Delta_i| \leq \beta\|X_i\|(|\xi_i| + E|\xi_i|).$$
Therefore,

\[
\mathbb{E} \left| \log Z(\beta, \xi) - \alpha(\beta, \xi) \right|^3 \leq c' \mathbb{E} \left( \sum_{i=1}^{d} \Delta_i^2 \right)^{\frac{3}{2}} \leq c' \beta^3 \left( \sum_{i=1}^{d} \|X_i\|^2 \mathbb{E}(|\xi_i| + \mathbb{E}[|\xi_i|]^2) \right)^{\frac{3}{2}} \\
\leq c \beta^3 \mathbb{E} |\xi|^3 \sqrt{d} \left( \sum_{i=1}^{d} \|X_i\|^3 \right)
\]

\(\square\)

**Proof of Theorem 2.1.** Our first theorem now follows as an application of the above machinery: the first statement is an application of Lemma 3.4, while the second follows from Lemma 3.5. \(\square\)

Finally, we consider fluctuations in Gaussian environments:

**Proof of Proposition 2.3.** Let \(Z_\beta(t)\) be defined as the auxiliary partition function given by two independent collections of Gaussian disorder \(g^{(1)}\) and \(g^{(2)}\),

\[
Z_\beta(t) = \text{Tr} \left( e^{\beta \sqrt{T} \sum_{i=1}^{d} g_i^{(1)} X_i + \beta \sqrt{1-t} \sum_{i=1}^{d} g_i^{(2)} X_i + H_0} \right)
\]

with \(t\) an interpolation parameter varying between 0 and 1. Let \(\mathbb{E}_j\) denote the average with respect the random variables \(g^{(j)}\) for \(j = 1, 2\). Given any \(s \in \mathbb{R}\), let

\[
Y(t) = \exp(s \mathbb{E}_2 \log Z_\beta(t)); \quad \phi(t) = \log \mathbb{E}_1 [Y(t)].
\]

We note that

\[
\phi_N(1) - \phi_N(0) = \log \mathbb{E} [\exp(s(\log Z(\beta, g) - \alpha(\beta, g)))]. \quad (3.5)
\]

In order to estimate this difference, consider

\[
\phi'(t) = \frac{s^2 \beta^2}{2 \mathbb{E}_1 [Y(t)]} \mathbb{E}_1 \left[ Y(t) \mathbb{E}_2 \left[ \frac{1}{\sqrt{t}} \left( \sum_{i=1}^{d} g_i^{(1)} X_i \right) - \frac{1}{\sqrt{1-t}} \left( \sum_{i=1}^{d} g_i^{(2)} X_i \right) \right] \right]
\]

where the notation \(\langle \cdot \rangle_t\) represents the Gibbs state induced by the interpolating Hamiltonian. A calculation involving the Lie-Trotter expansion and the Gaussian version of the integration-by-parts formula implies

\[
\phi'(t) = \frac{s^2 \beta^2}{2 \mathbb{E}_1 [Y(t)]} \mathbb{E}_1 \left[ Y(t) \sum_{i=1}^{d} \langle X_i \rangle_t^2 \right]
\]

so that

\[
|\phi'(t)| \leq \frac{s^2 \beta^2 \sum_{i=1}^{d} \|X_i\|^2}{2}. \quad (3.6)
\]

Using Equation (3.6) and the fact that \(e^{|x|} \leq e^x + e^{-x}\) we have

\[
\exp(|s| \log Z(\beta, g) - \alpha(\beta, g)) \leq 2e^{s^2 \beta^2 \sum_{i=1}^{d} \|X_i\|^2}. \quad (3.6)
\]

Finally, applying Markov’s inequality we have

\[
\mathbb{P} \left( |\log Z(\beta, g) - \alpha(\beta, g)| \geq u \right) \leq 2e^{-s^2 \beta^2 \sum_{i=1}^{d} \|X_i\|^2 - su}
\]
for any \( s \in \mathbb{R} \). Optimizing over \( s \) concludes the lemma. \( \square \)

4. EXISTENCE OF THE PRESSURE

Recalling the \( su(2) \) formalism, we choose a preferred basis for \( V_N \) consisting of tensor products of eigenvectors for the operators \( \{ S_j^{(z)} \} \). Denoting the eigenvector for \( S_j^{(z)} \) which corresponds to the eigenvalue \(+1\) by \( |+\rangle \) and the eigenvector for which corresponds to the eigenvalue \(-1\) by \(|-\rangle \), we may identify this preferred basis with classical Ising spin configurations \( \sigma \in \{-1,+1\}^N \). For each \( \sigma \), we denote the corresponding basis vector by \( |\sigma\rangle \).

The proof of Theorem 2.5 proceeds in two steps. The first step consists of a concentration estimate following essentially the same argument as that of Proposition 2.3. Widening the scope beyond the transverse Ising spin glass, for this first step we consider quantum Hamiltonians of the form

\[
H_N := H_N^{\text{dis}}(S(z)) + H_N^{\text{det}}
\]

We assume that only \( H_N^{\text{dis}} \) involves Gaussian disorder and that the deterministic operator \( H_N^{\text{det}} \) takes a sufficiently nice form so as to admit a Feynman-Kac representation in terms of the basis of eigenvectors for the \( S(z) \) operators. By this we mean that

\[
\langle \sigma | \exp(-u H_N^{\text{det}}) | \tilde{\sigma} \rangle \geq 0
\]

for all \( u \geq 0 \) and all spin configurations \( \sigma, \tilde{\sigma} \). Assuming that all off diagonal matrix elements of \( H_N^{\text{det}} \) are negative gives a necessary and sufficient condition which guarantees the existence of a Feynman-Kac representation. As mentioned in Section 2, for our treatment \( H_N^{\text{dis}} \) takes the form

\[
-H_N^{\text{dis}}(S(z)) = \frac{1}{2\sqrt{N}} \sum_{i,j=1}^{N} g_{ij} S_i^{(z)} S_j^{(z)}.
\]

where the collection \( \{ g_{i,j} \} \) is assumed to be i.i.d. standard normal, though more general interactions involving the \( \{ S_j^{(z)} \} \) may be considered.

To illustrate the significance of the Feynman-Kac representation, recall that by adding a suitably chosen diagonal matrix to \( H_N^{\text{det}} \) we can force the rows of the matrix representation of \( H_N^{\text{det}} \) to sum to \( 0 \), which implies that the (modified) Hamiltonian \( H_N^{\text{det}} \) is the generator of a continuous time Markov chain with state space \( \{-1,+1\}^N \).

Consider any matrix element \( \langle \sigma | e^{-u H_N} | \tilde{\sigma} \rangle \). Expanding the exponential for finite \( k \) using the Lie-Trotter formula with \( A = -u H_N^{\text{dis}}(S(z)) \) and \( B = -u H_N^{\text{det}} \) and inserting the complete orthogonal set \( \{ |\sigma'\rangle \} \) between each factor \( e^A e^B \) and then passing to the limit in \( k \), it is not difficult to see that

\[
\langle \sigma | e^{-u H_N} | \tilde{\sigma} \rangle = \int_{\sigma(0)=\sigma} e^{\int_0^u \langle \sigma(u) | -H_N^{\text{dis}}(S(z)) | \sigma(u) \rangle du} \delta_{\sigma(\sigma(0))} \nu(d\sigma)
\]

where \( \nu \) is the induced Markov chain measure and

\[
H_N^{\text{dis}}(\sigma) = \langle \sigma | H_N^{\text{dis}}(S(z)) | \sigma \rangle
\]
Moreover, we can view the augmentation by the diagonal matrix as introducing a weighting to each spin configuration which corresponds to the amount of time the process spends in each state along its trajectory. Thus, the original Hamiltonian yields an un-normalized measure on paths taking values in $\{-1, +1\}^N$. Let us also note, as is the case in the transverse S-K model, that the diagonal matrix is constant.

Returning to our original Hamiltonian (4.1), the upshot is that via the Feynman-Kac transformation we may represent the partition function $Z_N$ associated to the Hamiltonian $H_N$ by

$$Z_N = \text{Tr} \left( e^{-\beta H_N} \right) = \int_{\Omega} e^{\int_0^\beta \frac{1}{2\sqrt{N}} \sum_{i,j=1}^N g_{ij} \sigma_i(u) \sigma_j(u) du} d\nu(\sigma)$$

where $\nu$ is a measure on spin configuration paths determined by $H^\text{det}_N$ and $\Omega$ is the space of càdlàg paths taking values in $[-1,1]$. We use this representation implicitly throughout Section 4. In the case of the transverse Ising model, one may check that the induced measure is in fact (proportional to) the Markov chain measure defined by starting from an initial configuration and evolving in time via a spin flip process determined by flipping the spin value at each site according to the arrivals of independent Poisson processes of rate $\lambda$. In order to prove convergence of the quenched pressure we shall need to consider restricted partition functions $Z_A^A$ defined via the Feynman-Kac transformation by

$$\frac{1}{N} \log Z_A^A = \frac{1}{N} \log \int_A e^{\int_0^\beta \frac{1}{2\sqrt{N}} \sum_{i,j=1}^N g_{ij} \sigma_i(u) \sigma_j(u) du} d\nu(\sigma)$$

where $A = A_N$ is some deterministic (i.e. not depending on Gaussian disorder) subset of $\Omega$.

For any pair of spin configurations $\sigma$ and $\tilde{\sigma}$, let

$$R(\sigma, \tilde{\sigma}) = \frac{1}{N} \sum_{i=1}^N \sigma_i \tilde{\sigma}_i,$$

denote the classical overlap for Ising spin configurations. An easy calculation shows that $\mathbb{E} [\zeta_N(\sigma) \zeta_N(\tilde{\sigma})] = \frac{N}{4} \int_0^\beta \int_0^\beta R^2(\sigma(u), \tilde{\sigma}(s)) du ds$

where the function $R$ is the classical overlap for Ising spin configurations.

**Lemma 4.1** Let $\beta > 0$ be given. Fix any Feynman-Kac representable deterministic Hamiltonian $H^\text{det}_N$. Then

$$\mathbb{P} \left( \frac{1}{N} \log Z_N^A - \frac{1}{N} \mathbb{E} [\log Z_N^A] \geq u \right) \leq 2 \exp \left( -\frac{2u^2}{\beta^2 N} \right).$$
Proof. With the appropriate modifications, the method of proof of Proposition 2.3 may be followed here: Let \( Z_A^A(t) \) be defined as the auxiliary partition function given by two independent collections of Gaussian disorder \( g^{(1)} \) and \( g^{(2)} \),

\[
Z_A^A(t) = \int_A e^{\sqrt{\beta} \zeta^{(1)}_N(\sigma) + \sqrt{1-\beta} \zeta^{(2)}_N(\sigma)} \nu(\sigma)
\]

with \( t \) an interpolation parameter varying between 0 and 1 and \( \zeta^{(i)} \) the Gaussian corresponding to \( g^{(i)} \). Let \( Y_N(t) \) and \( \phi_N(t) \) be defined in terms of \( Z_A^A(t) \) as in (??) and (??).

Replacing the corresponding quantities appearing in the proof of Proposition 2.3 we have, using the Feynman-Kac representation,

\[
\phi'_N(t) = \frac{s^2}{2E_1[Y_N(t)]} E_1 \left[ Y_N(t) \int_{A_N \times A_N} \nu(\sigma) \nu(\tilde{\sigma}) \left( \int_0^\beta \int_0^\beta R^2(\sigma(u), \tilde{\sigma}(s)) du ds \right) w_N(t, \sigma) w_N(t, \tilde{\sigma}) \right]
\]

where \( w_N(t, \sigma) \) is the truncated `Gibbs weight’ corresponding to the event \( A_N \). Thus

\[
|\phi'_N(t)| \leq \frac{s^2 \beta^2 N}{8}.
\]

The bound now follows as in the proof of Proposition 2.3.

\[ \square \]

Remark 4.2 More generally, the p-spin models may also be treated via the method employed here, though the bound stated in the lemma must be modified slightly.

Unfortunately the use of the Feynman-Kac transformation alone does not allow our method to go through. In particular, we were unable to treat the quantum system with deterministic quadratic couplings in the \( x \) and \( y \) directions: the ferromagnetic version does permit a Feynman-Kac representation but the interaction is convex, which turns out to have exactly the wrong sign in the expression for the derivative of the interpolating ‘pressure’.

The only natural example that we found amenable to our method is the transverse field Ising model. Notice that the deterministic portion of this particular Hamiltonian is linear, which simplifies the interpolation scheme that we employ to analyze the thermodynamics at different system sizes. For inverse temperature \( \beta \) and transverse field strength \( \lambda > 0 \), we refer to the partition function of this model by \( Z_N(\beta, \lambda) \) and denote \( p_N(\beta, \lambda) = -\frac{1}{N} \log Z_N(\beta, \lambda) \).

Lemma 4.3 Let \( \beta, \lambda > 0 \) be fixed. Then

\[
\lim_{N \to \infty} \mathbb{E}[p_N(\beta, \lambda)] = p(\beta, \lambda)
\]

exists.

Proof. This proof, like that of the previous lemma, relies on the proof of an analogous statement in [11]. The main idea is to partition the space of paths into subsets on which we may control the time correlated self overlap \( R(\sigma(u), \sigma(s)) \).
To this end, let \( \epsilon, \delta > 0 \) be fixed, where for convenience we assume \( \frac{\beta}{\delta} \in \mathbb{N} \). For any function \( g : [0, \beta] \times [0, \beta] \to [-1, 1] \) we define the event \( A_g(\delta, \epsilon) \) by

\[
A_g(\delta, \epsilon) = \{ \sigma : g(i\delta, j\delta) \leq R(\sigma(i\delta), \sigma(j\delta)) < g(i\delta, j\delta) + \epsilon \ \forall \ i, j \leq \frac{\beta}{\delta} - 1, \\
|g(u, s) - R(\sigma(u), \sigma(s))| \leq 2\epsilon \ \forall \ (u, s) \in [0, \beta] \times [0, \beta] \}.
\]

Let \( S_\delta(\epsilon) \) be the set of functions which are constant on \( [j\delta, (j+1)\delta) \times [k\delta, (k+1)\delta) \) for \( j, k \in \{0, \ldots, \frac{\beta}{\delta} - 1\} \) and take values in \( \{i \epsilon : i \in [-\frac{1}{\epsilon}, \frac{1}{\epsilon}] \cap \mathbb{N}\} \). We define the event

\[
A = A(\delta, \epsilon) = \bigcup_{g \in S_\delta(\epsilon)} A_g(\delta, \epsilon).
\]

Observe that though \( A \) definitely does not cover the full sample space \( \Omega \), it is enough to prove convergence of the truncated pressure

\[
p_N^A(\beta, \lambda) = -\frac{1}{N} \log \int_A e^{\xi_N(\sigma)} d\nu(\sigma).
\]

More precisely suppose \( \epsilon = \epsilon_N, \delta = \delta_N. \) We claim that if \(-\epsilon_N \log \delta_N \) is sufficiently large as \( N \to \infty \)

\[
\lim_{N \to \infty} \mathbb{E} [p_N(\beta, \lambda) - p_N^A(\beta, \lambda)] = 0.
\]

Let \( \Delta N_{i\delta}(\sigma) \) denote the total number of jumps made by the spin path \( \sigma \) in the time interval \([i\delta, (i+1)\delta]\). To determine how large to take \(-\epsilon \log \delta\) with \( N \), let \( A_\epsilon = \{ \sigma : \max_{i \leq \frac{\beta}{\delta} - 1} \Delta N_{i\delta}(\sigma) \geq \frac{\epsilon_N}{\epsilon}\}. \) Then \( A_c \subset A_\epsilon \) so that

\[
\frac{\int_{A_c} \exp \left( \xi_N(\sigma) \right) d\nu(\sigma)}{\int_A \exp \left( \xi_N(\sigma) \right) d\nu(\sigma)} \leq \frac{\int_{A_\epsilon^c} \exp \left( \xi_N(\sigma) \right) d\nu(\sigma)}{\int_{A_c} \exp \left( \xi_N(\sigma) \right) d\nu(\sigma)}
\]

By Jensen’s inequality,

\[
\mathbb{E} \left[ \log \left( 1 + \frac{\int_{A_\epsilon^c} \exp \left( \xi_N(\sigma) \right) d\nu(\sigma)}{\int_{A_c} \exp \left( \xi_N(\sigma) \right) d\nu(\sigma)} \right) \right] \leq \\
\log \left( 1 + \int_{A_\epsilon} \mathbb{E} \left[ \frac{\exp \left( \xi_N(\sigma) \right) d\nu(\sigma)}{\int_{A_c^c} \exp \left( \xi_N(\sigma) \right) d\nu(\sigma)} \right] d\nu(\sigma) \right).
\]

By the Cauchy-Schwarz inequality,

\[
\mathbb{E} \left[ \frac{\exp \left( \xi_N(\sigma) \right)}{\int_{A_c^c} \exp \left( \xi_N(\sigma) \right) d\nu(\sigma)} \right] \leq \mathbb{E} [\exp (2\xi_N(\sigma))]^{\frac{1}{2}} \mathbb{E} \left[ \frac{1}{\int_{A_c^c} \exp \left( \xi_N(\sigma) \right) d\nu(\sigma)}^{\frac{1}{2}} \right]^{\frac{1}{2}}
\]

\[
\leq \mathbb{E} [\exp (2\xi_N(\sigma))]^{\frac{1}{2}} \mathbb{E} \left[ \exp \left( \frac{2}{\nu(A_c^c)} \int_{A_c^c} \xi_N(\sigma) d\nu(\sigma) \right) \right]^{\frac{1}{2}} \nu(A_c^c)^{-1}.
\]
The last line follows from Jensen’s inequality applied with respect to the path measure $\frac{1}{\nu(\mathcal{A}_c^t)}$. Since $2\zeta_N(\sigma)$ is a Gaussian random variable with variance $N \int_0^\beta \int_0^\beta R^2(\sigma(u), \sigma(s))du ds$,

$$
\mathbb{E}[\exp(2\zeta_N(\sigma))] = \exp\left(\frac{N}{2} \int_0^\beta \int_0^\beta R^2(\sigma(u), \sigma(s))du ds\right).
$$

Similarly, after a short calculation we have

$$
\mathbb{E}\left[\exp\left(\frac{2}{\nu(\mathcal{A}_c^t)} \int_{\mathcal{A}_c^t} \zeta_N(\sigma) d\nu(\sigma)\right)\right] = 
\exp\left(\frac{N}{2} \int_{\mathcal{A}_c^t \times \mathcal{A}_c^t} \int_0^\beta \int_0^\beta R^2(\sigma(u), \tilde{\sigma}(s))du ds \frac{d\nu(\sigma)d\nu(\tilde{\sigma})}{\nu(\mathcal{A}_c^t)^2}\right).
$$

As a result of the preceding estimates

$$
\int_{\mathcal{A}_c^t} \mathbb{E}\left[\exp\left(\frac{\zeta_N(\sigma)}{\nu(\mathcal{A}_c^t)}\right)\right] d\nu(\sigma) \leq e^{\frac{N\beta^2}{2} \nu(\mathcal{A}_c^t)}
$$

Standard calculations imply that for all $\epsilon$ small enough and $\delta < \epsilon^2$,

$$
\frac{\nu(\mathcal{A}_c^t)}{\nu(\mathcal{A}_c^t)} \leq \frac{1}{\delta} C N \epsilon \log \delta
$$

for some universal constant $C > 0$. Requiring $-C\epsilon \log \delta \geq \beta^2 + -\frac{1}{N} \log \delta$, putting estimates together and taking the appropriate limits proves our claim. Note that these conditions can be arranged, for example, by letting $\epsilon_N = N^{-1/4}$ and $\delta_N = e^{-N^{1/2}}$ and taking $N$ large.

Thus we are reduced to showing that the mean of the truncated pressure $\mathbb{E}[p_N^*(\beta, \lambda)]$ converges. For convenience of exposition we first consider subsequences of the form $N_k = N_0 n^k$ for some $N_0, n \in \mathbb{N}$. For any $k$, we may view $\sigma \in \Omega_{N_k}$ as an $n$-tuple of spin paths $(\sigma^{(1)}, \ldots, \sigma^{(n)})$ so that $\sigma^{(l)} \in \Omega_{N_k-1}$. In order to compare thermodynamics at consecutive system sizes, let us define the interpolating Hamiltonian

$$
\zeta_{N_k}(t, \sigma) = \sqrt{t} \zeta_{N_k}(\sigma) + \sqrt{1-t} \sum_{l=1}^n \zeta_{N_{k-1}}^{(l)}(\sigma^{(l)}),
$$

where $\zeta_{N_{k-1}}^{(l)}(\sigma^{(l)})$ involve disorder couplings which are mutually independent and independent from the couplings in $\zeta_{N_k}(\sigma)$. In addition, we introduce the partition function

$$
Z_{N_k}^A(t) = \int_{\mathcal{A}} \exp(\zeta_N(t, \sigma))
$$

Let

$$
\mathcal{A}_g = \mathcal{A}_{N_k}(\delta, \epsilon, g) = \{\sigma = (\sigma^{(1)}, \ldots, \sigma^{(n)}) : \sigma^{(l)} \in A_{N_{k-1}}(\delta, \epsilon, g)\}
$$

Obviously $\mathcal{A}_g \subset A_g$. Therefore

$$
- \log Z_{N_k}^{\mathcal{A}_g}(t) \leq - \log Z_{N_k}^{\mathcal{A}_g}(t)
$$
For any \( g \in S_\delta(\epsilon) \) consider
\[
\phi_{N_k}^{(g)}(t) = - \frac{1}{N_k} \mathbb{E} \log Z_{N_k}^{A_g}(t).
\]

After bit of work employing the integration-by-parts formula we arrive at
\[
\frac{d}{dt} \phi_{N_k}^{(g)}(t) = - \frac{1}{8} \mathbb{E} \left[ \left\langle \int_0^\beta \int_0^\beta \mathcal{R}^2(\sigma(u), \sigma(s))du ds - \frac{1}{n} \sum_{i=1}^n \int_0^\beta \int_0^\beta \mathcal{R}^2(\sigma^{(i)}(u), \sigma^{(i)}(s))du ds \right\rangle_t \right]
+ \frac{1}{8} \mathbb{E} \left[ \left\langle \int_0^\beta \int_0^\beta \mathcal{R}^2(\sigma(u), \bar{\sigma}(s))du ds - \frac{1}{n} \sum_{i=1}^n \int_0^\beta \int_0^\beta \mathcal{R}^2(\sigma^{(i)}(u), \bar{\sigma}^{(i)}(s))du ds \right\rangle_t \right]
\]

where \( \langle \cdot \rangle_t^{A_g,(1)} \) corresponds to the truncated Gibbs weight determined by the Hamiltonian \( \zeta_{N_k}(t, \varphi) \) and \( \langle \cdot \rangle_t^{A_g,(2)} \) corresponds to the product Gibbs weight determined by an independent pair of spin paths \( \sigma, \bar{\sigma} \) and Hamiltonian \( \zeta_{N_k}(t, \varphi) + \zeta_{N_k}(t, \bar{\varphi}) \). We stress that corresponding terms in this pair Hamiltonian involve the same realizations of disorder. From the definition of \( A_g \) the first term can be bounded by \( \beta^2 \epsilon \) in absolute value. As \( f \) is convex, the latter term is less than or equal to zero. Thus, by evaluating \( \phi_{N_k}^{(g)} \) at zero and one and applying (4.4) we have
\[
- \frac{1}{N_k} \mathbb{E} \left[ \log Z_{N_k}^{A_g} \right] \leq - \frac{1}{N_{k-1}} \mathbb{E} \left[ \log Z_{N_{k-1}}^{A_g} \right] + \beta^2 \epsilon
\]
for any \( g \in S_\delta(\epsilon) \).

Next, by Lemma 4.1, we have
\[
\mathbb{P} \left( \left| \frac{1}{N_k} \log Z_{N_k}^{A_g}(\beta, \lambda) - \frac{1}{N_k} \mathbb{E} \left[ \log Z_{N_k}^{A_g}(\beta, \lambda) \right] \right| \geq u \right) \leq 2 \exp \left( - \frac{2u^2}{\beta^2 N_k} \right).
\]

Setting \( u = \epsilon \), there exists a set \( S \) and a universal constant \( D > 0 \) so that
\[
\mathbb{P}(S) \geq 1 - \frac{4}{\delta^2 \epsilon} \exp \left( - \frac{2\epsilon^2}{\beta^2 N_k} \right)
\]
and so that on \( S \)
\[
- \frac{1}{N_k} \log Z_{N_k}^{A_g}(\beta, \lambda) \leq - \frac{1}{N_{k-1}} \log Z_{N_{k-1}}^{A_g}(\beta, \lambda) + D \beta^2 \epsilon
\]
for all \( g \in S_\delta(\epsilon) \).

Now let us choose \( \delta = \exp \left( - \frac{1}{2(\beta_0)^2} \right) \) and \( \epsilon = N_k^{-1/4} \). Then a standard application of the Borel-Cantelli lemma implies that there exists \( D > 0 \) so that
\[
\mathbb{P} \left( - \frac{1}{N_k} \log Z_{N_k}^{A_g}(\beta, \lambda) \geq - \frac{1}{N_{k-1}} \log Z_{N_{k-1}}^{A_g}(\beta, \lambda) + D \beta^2 N_k^{-1/4} \right. \text{ for some } g \in S_{\delta(N_k)}(N_k^{-1/4}) \text{ i.o.} = 0.
\]
Therefore, for $N_k$ large enough (with the choice of $\epsilon$ and $\delta$ depending on $N_k$ as above) we have

$$Z_{N_k}^{A N_k} (\beta, \lambda) = \sum_{g \in S_\delta(\epsilon)} Z_{N_k}^{A(\delta, \epsilon, g)} (\beta, \lambda) \geq \sum_{g \in S_\delta(\epsilon)} e^{-\beta^2 N_k^3/4} \left[Z_{N_k}^{A(\delta, \epsilon, g)}(\beta, \lambda)\right]^n \geq e^{-D_\beta^2 N_k^3/4} e^{-(n-1)\beta^2 N_k^1/4} N_k^{1-n} \left[Z_{N_k}^{A N_k} (\beta, \lambda)\right]^n$$

where we have used the inequality $\sum_i x_i^n \geq k^{1-n} (\sum_i x_i)^n$ for $n \geq 1$ and $x_i \geq 0$.

By (4) we have

$$\frac{1}{N_k-1} \mathbb{E} \left[ \log Z_{N_k}^{A N_k} (\beta, \lambda) - \log Z_{N_k}^{A N_k-1} (\beta, \lambda) \right] \to 0 \quad k \to \infty$$

Therefore the truncated sequence of ‘pressures’ $\frac{1}{N_k} \log Z_{N_k}^{A N_k} (\beta, \lambda)$ is nearly decreasing a.s. It follows that the truncated thermodynamic limits exist almost surely.

Let $X_k = -\frac{1}{N_k} \log Z_{N_k}^{A N_k} (\beta, \lambda)$. Since $\{\mathbb{E}[X_k]\}$ is uniformly bounded, the limit is finite a.s. Another application of the concentration inequality (4.1) along with the Borel-Cantelli lemma implies that this limit is a non-random constant $f(\beta, \lambda)$. Further, it is now a small matter to prove convergence of the truncated quenched averages along these subsequences. The bound (4.1) implies that $X_k$ are uniformly integrable, i.e.

$$\lim_{k \to \infty} \sup_{x} \mathbb{E} \left[ |X_k| \mid X_k > x \right] = 0,$$

so we may conclude $\mathbb{E}[X_k] \to f(\beta, \lambda)$ as well. Finally, as $-\epsilon N_k \log \delta N_k = N_k^{3/4}/2\beta^2$, the arguments given above imply that $\lim_k \mathbb{E} \left[p_{N_k} (\beta, \lambda) \right] = f(\beta, \lambda)$. Another application of the concentration inequality then implies that $p_{N_k} (\beta, \lambda) \to f(\beta, \lambda)$ a.s.

However these statements follow from fairly standard arguments based on the above results. □

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