Grassmann extensions of Yang–Baxter maps

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Received 27 October 2015, revised 30 December 2015
Accepted for publication 19 January 2016
Published 22 February 2016

Abstract

In this paper we show that there are explicit Yang–Baxter (YB) maps with Darboux–Lax representation between Grassman extensions of algebraic varieties. Motivated by some recent results on noncommutative extensions of Darboux transformations, we first derive a Darboux matrix associated with the Grassmann-extended derivative nonlinear Schrödinger (DNLS) equation, and then we deduce novel endomorphisms of Grassmann varieties, which possess the YB property. In particular, we present ten-dimensional maps which can be restricted to eight-dimensional YB maps on invariant leaves, related to the Grassmann-extended NLS and DNLS equations. We consider their vector generalisations.

Keywords: Yang–Baxter maps, Grassmann algebraic varieties, Grassmann extensions of Yang–Baxter maps, Grassmann extensions of Darboux transformations, noncommutative extensions of Yang–Baxter maps

1. Introduction

The Yang–Baxter (YB) equation has a fundamental role in the theory of quantum and classical integrable systems. In particular, the set-theoretical solutions of the YB equation, have been of great interest for several researchers in the area of Mathematical Physics. The consideration of such solutions was formally proposed by Drinfeld in [12], although the first examples of such solutions appeared in [23]. Moreover, the study of the set-theoretical...
solutions gained a more algebraic flavour in [6]. We refer to these solutions using the shorter term ‘YB maps’ which was proposed by Veselov in [25]. YB maps are related to several concepts of integrability as, for instance, the multidimensionally consistent equations [1, 2, 5, 21]. Of particular interest are those YB maps which admit Lax representation [24]. They are connected with integrable mappings [25, 26] and they are also related to integrable partial differential equations via Darboux transformations [15].

Moreover, noncommutative extensions of integrable equations have been of great interest over the last decades [9, 10]. Darboux transformations for noncommutative-extended integrable equations were recently constructed; in the case of Grassman-extended NLS equation in [13] and for the supersymmetric KdV equation [27, 28, 31] and the AKNS system [29]. At the same time, the derivation of noncommutative versions of YB maps has gained its interest [11].

In this paper, we make the first attempt to extend the theory of YB maps in the case of Grassmann algebras; in particular, we study the Grassmann extensions of the YB maps related to the nonlinear Schrödinger equation and the derivative nonlinear Schrödinger (DNLS) equation which have recently appeared in [15].

The paper is organised as follows. The second section deals with parametric YB maps and their Lax representations. Moreover, we present some basic properties of Grassmann algebras in order to make this text self-contained, as well as some properties of YB maps which admit Lax representation. In section 3, following [13], we consider a noncommutative (Grassmann) extension of the Darboux transformation for the DNLS equation. In section 4, we employ all the Darboux matrices presented in section 3 and, from their associated refactorisation problems, we construct ten-dimensional YB maps. The entries of the considered Darboux matrices satisfy particular systems of differential-difference equations which possess first integrals. These integrals indicate that the associated YB maps can be restricted to eight-dimensional YB maps on invariant leaves. Moreover, we consider their vector generalisations. Finally, in section 5 we summarise our results and we present some ideas for future work.

2. Preliminaries

Let $A$ be an algebraic variety in $K^N$, where $K$ is any field of zero characteristic (such as $\mathbb{C}$, $\mathbb{R}$ or $\mathbb{Q}$), and let $Y \in \text{End}(A \times A)$ be a map $(x, y) \mapsto (u(x, y), v(x, y))$. The map $Y$ is called a YB map if it satisfies the following YB equation

$$Y^{12} \circ Y^{13} \circ Y^{23} = Y^{23} \circ Y^{13} \circ Y^{12},$$

where $Y^i \in \text{End}(A \times A)$, $i, j = 1, 2, 3, i \neq j$, are defined by the following relations

$$Y^{12}(x, y, z) = (u(x, y), v(x, y), z),$$

$$Y^{13}(x, y, z) = (u(x, z), v(x, z), y),$$

$$Y^{23}(x, y, z) = (x, u(y, z), v(y, z)),$$

where $x, y, z \in A$.

A YB map $Y$ is called reversible if the composition of $\tilde{Y} = \pi Y \pi$ (where $\pi \in \text{End}(A \times A)$ is the permutation map $\pi(x, y) = (y, x)$) with $Y$ is the identity map, namely

$$\tilde{Y} \circ Y = \text{Id}.$$ (3)

Furthermore, we use the term parametric YB map if two parameters $a, b \in K$ are involved in the definition of the YB map, namely we have a map of the following form
satisfying the parametric YB equation
\[
Y_{a,b}^{12} \circ Y_{a,c}^{13} \circ Y_{b,c}^{23} = Y_{b,c}^{23} \circ Y_{a,c}^{13} \circ Y_{a,b}^{12}.
\] (5)

2.1. Grassmann extended varieties

Here, we briefly present the basic properties of Grassmann algebras. For further details one could consult [4]. Let \( G \) be a \( \mathbb{Z}_2 \)-graded algebra over \( \mathbb{C} \) or, in general, over a field \( K \) of characteristic zero. Thus, as a linear space \( G \) is a direct sum \( G = G_0 \oplus G_1 \) (mod 2), such that \( G_i G_j \subseteq G_{i+j} \). Those elements of \( G \) that belong either to \( G_0 \) or to \( G_1 \) are called homogeneous, the ones from \( G_0 \) are called even (bosonic), while those in \( G_1 \) are called odd (fermionic).

By definition, the parity \([a]\) of an even homogeneous element \( a \) is 0, and it is 1 for odd homogeneous elements. The parity of the product \([ab]\) of two homogeneous elements is a sum of their parities: \([ab] = [a] + [b] \). Grassmann commutativity means that \( ba = (-1)^{|a||b|}ab \) for any homogeneous elements \( a \) and \( b \). In particular, \( \alpha^2 = 0 \), for all odd elements \( \alpha \in G_1 \), and even elements commute with all the elements of \( G \).

**Remark 2.1.1.** In the rest of this paper we shall be using Latin letters for even variables, and Greek letters when referring to the odd ones; yet, we shall be using the Greek letter \( \lambda \) when referring to the spectral parameter, despite the fact that \( \lambda \) is an even variable.

A Grassmann extension of an algebraic variety, \( V_G(p_1, \ldots, p_k) \), can be defined similarly to the commutative case:
\[
V_G(p_1, \ldots, p_k) = \{ a_1, \ldots, a_n \in G_0, \alpha_1, \ldots, \alpha_m \in G_1 | p_1 = \ldots p_k = 0, p_i \in \mathbb{C}[a_1, \ldots, a_n, \alpha_1, \ldots, \alpha_m] \}. 
\] (6)

2.1.1. Supertrace and superdeterminant. Let \( M \) be a square matrix of the following form
\[
M = \begin{pmatrix} P & \Pi \\ \Lambda & L \end{pmatrix},
\] (7)
where \( P \) and \( L \) are square matrices of even entries, whereas \( \Pi \) and \( \Lambda \) are matrices with odd entries, not necessarily square.

We define the **supertrace** of \( M \)—and we shall denote it by \( \text{str}(M) \)—to be the following quantity
\[
\text{str}(M) = \text{tr}(P) - \text{tr}(L),
\] (8)
where \( \text{tr}(.) \) is the usual trace of a matrix.

Moreover, we define the **superdeterminant** of \( M \)—and we shall denote it by \( \text{sdet}(M) \)—to be
\[
\text{sdet}(M) = \det(P - \Pi L^{-1}\Lambda)\det(L^{-1}) = \det(P^{-1})\det(L - \Lambda P^{-1}\Pi),
\] (9)
where \( \det(.) \) is the usual determinant of a matrix.
2.2. Lax representations of YB maps

Following Suris and Veselov in [24], we call a Lax matrix for a parametric YB map a square matrix, $L = L(x, \chi; a, \lambda)$, depending on an even variable $x$, an odd variable $\chi$, a parameter $a$ and a spectral parameter $\lambda$, such that the Lax-equation

$$L(u, \zeta; a)L(v, \eta; b) = L(y, \psi; b)L(x, \chi; a)$$  \hspace{1cm} (10)

is satisfied whenever $(u, \xi, v, \eta) = Y_{a,b}(x, \chi, y, \psi)$. Equation (10) is also called a refactorisation problem.

If the Lax-equation (10) has a unique solution, namely it is equivalent to a map

$$(u, v, \xi, \eta) = Y_{a,b}(x, \chi, y, \psi),$$  \hspace{1cm} (11)

then the Lax matrix $L$ is said to be strong [19]. In this case (11) is a YB map and it is reversible [26].

Now, since the Lax equation (10) has the obvious symmetry

$$(u, v, \xi, \eta, a, b) \leftrightarrow (y, \psi, x, \chi, b, a)$$  \hspace{1cm} (12)

we have the following

**Proposition 2.2.1.** If a matrix refactorisation problem (10) yields a rational map $(x, \chi, y, \psi) = Y_{a,b}(u, \xi, v, \eta)$, then this map is birational.

**Proof.** Let $Y: (x, \chi, y, \psi) \mapsto (u, \xi, v, \eta)$ be a rational map corresponding to a refactorisation problem (10), i.e.

$$x \mapsto u = \frac{n_1(x, \chi, y, \psi; a, b)}{d_1(x, \chi, y, \psi; a, b)}, \hspace{1cm} y \mapsto v = \frac{n_2(x, \chi, y, \psi; a, b)}{d_2(x, \chi, y, \psi; a, b)},$$  \hspace{1cm} (13a)

$$\chi \mapsto \xi = \frac{n_3(x, \chi, y, \psi; a, b)}{d_3(x, \chi, y, \psi; a, b)}, \hspace{1cm} \psi \mapsto \eta = \frac{n_4(x, \chi, y, \psi; a, b)}{d_4(x, \chi, y, \psi; a, b)},$$  \hspace{1cm} (13b)

where $n_i, d_i, i = 1, 2, 3, 4$, are polynomial functions of their variables.

Due to the symmetry (12) of the refactorisation problem (10), the inverse map of $Y$, $Y^{-1}: (u, \xi, v, \eta) \mapsto (x, \chi, y, \psi)$, is also rational and it is given by

$$u \mapsto x = \frac{n_1(v, \eta, u, \xi; b, a)}{d_1(v, \eta, u, \xi; b, a)}, \hspace{1cm} v \mapsto y = \frac{n_2(v, \eta, u, \xi; b, a)}{d_2(v, \eta, u, \xi; b, a)},$$  \hspace{1cm} (14a)

$$\xi \mapsto \chi = \frac{n_3(v, \eta, u, \xi; b, a)}{d_3(v, \eta, u, \xi; b, a)}, \hspace{1cm} \eta \mapsto \psi = \frac{n_4(v, \eta, u, \xi; b, a)}{d_4(v, \eta, u, \xi; b, a)},$$  \hspace{1cm} (14b)

Therefore, $Y$ is a birational map. \hfill \square

**Remark 2.2.2.** Functions $d_i(x, \chi, y, \psi; a, b), i = 1, 2, 3, 4$, must be nonnilpotent even-valued.

**Proposition 2.2.3.** If $L = L(x, \chi, a; \lambda)$ is a Lax matrix with corresponding YB map $Y: (x, \chi, y, \psi) \mapsto (u, \xi, v, \eta)$, then $\text{str}(L(y, \psi; b; \lambda)L(x, \chi; a; \lambda))$ is a generating function of invariants of the YB map.
Proof. Since
\[ \text{str}(L(u, \xi, a; \lambda)L(v, \eta, b; \lambda)) = \text{str}(L(y, \psi, b; \lambda)L(x, \chi, a; \lambda)) \]
and function \( \text{str}(L(x, \chi, a; \lambda)L(y, \psi, b; \lambda)) \) can be written as
\[ \text{str}(L(x, \chi, a; \lambda)L(y, \psi, b; \lambda)) = \sum \lambda^k l_k(x, \chi, y, \psi; a, b), \]
from (15) follows that
\[ I(u, \xi, v; a, b) = I(x, \chi, y, \psi; a, b), \]
which are invariants for \( Y \).

Remark 2.2.4. The invariants of a YB map, \( I(x, \chi, y, \psi; a, b) \), may not be functionally independent.

3. Grassmann extensions of Darboux transformations

Let \( L \) be a Lax operator of the following form
\[ L(p, q, \theta, \phi; \lambda) = D_x + U(p, q, \theta; \lambda), \]
where \( U \) is a matrix depending on two even potentials, \( p = p(x) \) and \( q = q(x) \), two odd potentials, \( \theta = \theta(x) \) and \( \phi = \phi(x) \), a spectral parameter \( \lambda \) and a variable \( x \) implicitly through the potentials. In all our cases the dependence on the spectral parameter is polynomial.

By Darboux transformation we understand a map of the following form
\[ L \rightarrow \tilde{L} = MLM^{-1}, \]
where \( \tilde{L} \) is \( L \) updated with potentials \( p_{10} = p_{10}(x) \), \( q_{10} = q_{10}(x) \), \( \theta_{10} = \theta_{10}(x) \) and \( \phi_{10} = \phi_{10}(x) \), namely \( \tilde{L} = L(p_{10}, q_{10}, \theta_{10}, \phi_{10}; \lambda). \) The matrix \( M \) in (18) is called Darboux matrix. Here, we shall be assuming that the matrix \( M \) has the same \( \lambda \)-dependence with \( U \). Moreover, we define the rank of a Darboux transformation to be the rank of the matrix which appears as coefficient of the highest power of the spectral parameter.

In this section we consider the Grassmann extensions of the Darboux matrices corresponding to the NLS equation (see [13]) and the DNLS equation, which we shall use to construct YB maps.

3.1. Nonlinear Schrödinger equation

The Grassmann extension of the Darboux matrix for the NLS equation was constructed in [13]. In particular, the following noncommutative extension of the NLS operator
\[ \mathcal{L} := D_x + U(p, q, \psi, \phi, \xi; \kappa; \lambda) = D_x + \lambda U^1 + U^0, \]

If one sets the odd variables equal to zero, the obtained operator corresponds to the spatial part of the Lax pair for the NLS equation.
was considered, where $U^1$ and $U^0$ are given by

$$\begin{align*}
U^1 &= \text{diag}(1, -1, 0), \quad U^0 = \begin{pmatrix} 0 & 2p & \theta \\ 2q & 0 & \zeta \\ \phi & \kappa & 0 \end{pmatrix},
\end{align*}$$

(19b)

where $p, q \in G_0$ and $\psi, \phi, \zeta, \kappa \in G_1$.

It was shown that all the Darboux transformations of rank 1 associated to this operator are described by the following matrix

$$M(p, q, \theta, \phi; c_1, c_2) = \begin{pmatrix} F + \lambda & p & \theta \\ q_{10} & c_1 & 0 \\ \phi_{10} & 0 & c_2 \end{pmatrix},$$

(20)

where $c_1$ and $c_2$ can be either 1 or 0. In the case where $c_1 = c_2 = 1$, the entries of $M(p, q, \theta, \phi; 1, 1)$ satisfy the following system of differential-difference equations

\begin{align*}
F_x &= 2(pq - p_{10}q_{10}) + \theta \phi - \theta_{10}\phi_{10}, \\
p_x &= 2(Fp - p_{10}) + \theta\kappa, \\
q_{10,x} &= 2(q - q_{10}F) - \kappa_{10}\phi_{10}, \\
\theta_x &= F\theta - \theta_{10} + p\kappa, \\
\phi_{10,x} &= \phi - \phi_{10}F - \zeta_{10}q_{10},
\end{align*}

(21)

and the algebraic equations

\begin{align*}
\theta q_{10} &= (S - 1)\kappa, \\
\phi_{10}p &= (S - 1)\zeta.
\end{align*}

(22)

Moreover, system (21) admits the following first integral

$$\partial_x(F - pq_{10} - \phi_{10}\theta) = 0,$$

(23)

which implies that $\partial_x(s\text{det}(M)) = 0$, since $s\text{det}(M) = \lambda + F - pq_{10} - \phi_{10}\theta$.

### 3.2. DNLS equation

Let us now consider the Lax operator given by

$$\mathcal{L} = D_x + \lambda^2 U^2 + \lambda U^1,$$

(24)

where $U_1$ and $U_2$ are given by

$$\begin{align*}
U^2 &= \mathcal{S}_3, \\
U^1 &= \begin{pmatrix} 0 & 2p & 2\theta \\ 2q & 0 & 0 \\ 2\phi & 0 & 0 \end{pmatrix},
\end{align*}$$

(24b)

and $\mathcal{S}_3 = \text{diag}(1, -1, -1)$. The potentials $p$ and $q$ and the spectral parameter $\lambda$ are even, whereas the potentials $\phi$ and $\theta$ are odd. Moreover, the operator (24) is invariant under the transformation

$$s_1(\lambda): \mathcal{L}(\lambda) \to \mathcal{L}(-\lambda) = \mathcal{S}_3\mathcal{L}(\lambda)\mathcal{S}_3.$$

(25)
We seek a rank 1 Darboux matrix of the following form
\[ M = \lambda^2 M_2 + \lambda M_1 + M_0, \] (26)
where \( M_i, i = 0, 1, 2 \), is a 3 \times 3 matrix, and we assume that \( M \) possesses the same symmetry, (25), as the Lax operator (24), namely
\[ M(-\lambda) = s_3 M(\lambda) s_3, \] (27)
as in the commutative case in [16]. Therefore, for the entries of matrices \( M_i, i = 0, 1, 2 \), we have
\[ M_{i,12} = M_{i,13} = M_{i,21} = M_{i,31} = 0, \quad i = 0, 2, \quad \text{and} \quad M_{i,11} = M_{i,22} = M_{i,33} = M_{i,23} = M_{i,32} = 0. \] (28a
(28b)
Now, the definition (18) implies a second order algebraic equation in \( \lambda \). Equating the coefficients of different powers of \( \lambda \) to zero, we obtain the following system of equations
\[ [U^2, M_2] = 0, \] (29a)
\[ [U^2, M_1] + U_1^1 M_2 - M_2 U_1^j = 0, \] (29b)
\[ M_{2,x} + [U^2, M_0] + U_1^1 M_1 - M_1 U_1^j = 0, \] (29c)
\[ M_{x,1} + U_1^1 M_0 - M_0 U_1^1 = 0, \] (29d)
\[ M_0 = 0. \] (29e)
Equation (29a) is satisfied identically, whereas (29e) implies that matrix \( M_0 \) must be constant. Moreover, since rank \( M_2 = 1 \), we can choose \( M_2 = \text{diag}(f, 0, 0) \). In this case, from equation (29b) we have that the entries of \( M_1 \) are given by
\[ M_{1,12} = f p, \quad M_{1,13} = f \theta, \quad M_{1,21} = q_{10} f \quad \text{and} \quad M_{1,31} = \phi_{10} f. \] (30)
Moreover, from equation (29c) we deduce equation
\[ f_x = 2f(pq - p_{10} q_{10} + \theta\phi - \theta_{10} \phi_{10}). \] (31)
Therefore, matrix \( M \) is of the form:
\[ M(f, p, q_{10}, \theta, \phi_{10}; c_1, c_2) = \lambda^2 \left( \begin{array}{ccc} f & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) + \lambda \left( \begin{array}{ccc} 0 & f p & f \theta \\ q_{10} f & 0 & 0 \\ \phi_{10} f & 0 & 0 \end{array} \right) + \left( \begin{array}{ccc} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & 1 \end{array} \right), \] (32)
where we have chosen the constant matrix \( M_0 \) to be diagonal, namely of the form \( M_0 = \text{diag}(c_1, c_2, 1) \) (one of the parameters along its diagonal can be rescaled to 1).

Finally, due to (29d), the entries of the Darboux matrix must satisfy the following system of equations
\[ p_x = 2p(p_{10} q_{10} - pq + \theta_{10} \phi_{10} - \theta\phi) - \frac{2c_2 p_{10} - c_1 p}{f}, \] (33a)
\[ q_{10,x} = 2q_{10}(p_{10} q_{10} - pq + \theta_{10} \phi_{10} - \theta\phi) - \frac{2c_1 q_{10} - c_2 q}{f}, \] (33b)
\[ \theta_x = 2\phi(p_{10} q_{10} - pq + \theta_{10} \phi_{10} - \theta\phi) + \frac{2c_1 \theta - \theta_{10}}{f}, \] (33c)
where we have made use of (31).

Thus, matrix $M$ given by (32) constitutes a Darboux matrix for the Lax operator (24), if its entries satisfy the system of equations system (31), (33). We can readily show that the latter system admits the following first integral

$$\partial_s(c_2 f - f^2 (pq_{10} + c_2 \theta \phi_{10})) = 0,$$

by straightforward calculation. Using the above first integral we can show that $\partial_s(\text{sdet } M) = 0$.

**Remark 3.2.1.** The Darboux matrix (31) in [16] constitutes the bosonic limit of (32).

## 4. Derivation YB maps

In [15] we considered the case of Darboux matrices associated with Lax operators of NLS type, which correspond to a recent classification of automorphic Lie algebras [7, 8, 20]. We used these Darboux matrices to construct six-dimensional YB maps together with their four-dimensional restrictions on invariant leaves.

In this paper, we are interested in the Grassmann extensions of these YB maps. In particular, we shall discuss the cases of the YB maps associated with the NLS equation [30] and the DNLS equation [14].

### 4.1. NLS case

According to (20) we define the following matrix

$$M(\mathbf{x}; \lambda) = \begin{pmatrix} X + \lambda & x_1 & \chi_1 \\ x_2 & 1 & 0 \\ \lambda_2 & 0 & 1 \end{pmatrix}, \quad \mathbf{x} = (\zeta_1, x_2, \chi_1, \chi_2, X),$$

Then, we substitute $M$ to the Lax equation (10) which leads to a system of polynomial equations. The corresponding algebraic variety is a union of two ten-dimensional components. The first one is obvious from the refactorisation problem, and it corresponds to the trivial YB map, while the second one corresponds to a non-trivial ten-dimensional YB map. In particular, we have the following.

**Proposition 4.1.1.** The matrix refactorisation problem

$$M(\mathbf{u}; \lambda)M(\mathbf{y}; \lambda) = M(\mathbf{y}; \lambda)M(\mathbf{x}; \lambda),$$

where $M = M(\mathbf{x}; \lambda)$ is given by (35), yields the permutation map

$$\mathbf{x} \mapsto \mathbf{u} = \mathbf{y}, \quad \mathbf{y} \mapsto \mathbf{v} = \mathbf{x},$$

and the following ten-dimensional YB map

$$\begin{align*}
x_1 &\mapsto u_1 = y_1 - \frac{X - x_1 x_2 - \chi_1 \chi_2 - Y + y_1 y_2 + \psi_1 \psi_2}{1 + x_1 y_2 + \chi_1 \psi_2}, \\
x_2 &\mapsto u_2 = y_2.
\end{align*}$$


which is non-involutive and birational.

**Proof.** Equation (36) implies that \( v_1 = x_1, \eta_1 = \chi_1, u_2 = y_2, \xi_2 = \psi_2 \), and the following system of equations

\[
U + V = X + Y, \tag{38a}
\]
\[
UV + u_1v_2 + \xi_1\eta_2 = YX + y_1x_2 + \psi_1\chi_2, \tag{38b}
\]
\[
Ux_1 + u_1 = Yx_1 + y_1, \quad U\chi_1 + \xi_1 = Y\chi_1 + \psi_1, \tag{38c}
\]
\[
y_2V + v_2 = y_2X + x_2, \quad \psi_2V + \eta_2 = \psi_2X + \chi_2, \tag{38d}
\]

for \( u_1, \xi_1, U, v_2, \eta_2 \) and \( V \). From (38c), (38d) using (38a), we can express all variables \( u_1, \xi_1, v_2, \) and \( \eta_2 \) in terms of \( Y - U \), as:

\[
u_1 = (Y - U)x_1 + y_1, \quad \xi_1 = (Y - U)\chi_1 + \psi_1 \tag{39a}
\]
\[
v_2 = x_2 - y_2(Y - U), \quad \eta_2 = \chi_2 - \psi_2(Y - U). \tag{39b}
\]

Now, substituting (39) to (38b), using (38a), we obtain

\[
(Y - U)[U(1 + x_1y_2 + \chi_1\psi_2) - X + x_1x_2 + \chi_1\chi_2 - (y_1y_2 + \chi_1\psi_2)Y - y_1x_2 - \psi_1\psi_2] = 0. \tag{40}
\]

From the above follows that either \( U = Y \), which in view of (39) and (38a) implies the permutation map (37), or

\[
U = \frac{X - x_1x_2 - \chi_1x_2 + (x_1y_2 + \chi_1\psi_2)y_2 + \psi_1\psi_2}{1 + x_1y_2 + \chi_1\psi_2}. \tag{41}
\]

Substitution of the latter to (39), implies that \( u_1 \) and \( v_2 \) are given by (37a) and (37g), respectively, while \( \xi_1 \) and \( \eta_2 \) are given by
\[ \xi_i = \psi_i = \frac{X - x_i x_2 - \chi_1 \chi_2 - Y + y_i y_2 + \psi_1 \psi_2}{1 + x_i y_2 + \chi_1 \psi_2} \chi_1 \]
\[ \eta_2 = \chi_2 + \frac{X - x_1 x_2 - \chi_1 \chi_2 - Y + y_1 y_2 + \psi_1 \psi_2}{1 + x_1 y_2 + \chi_1 \psi_2} \psi_1 \psi_2. \]

Now, in the above expressions we multiply both the nominator and the denominator with the conjugate expression of the latter, namely \( 1 + x_1 y_2 - \chi_1 \psi_2 \), and we use the fact that \( \chi_1^2 = \psi_2^2 = 0 \). Then, \( \xi_i \) and \( \eta_2 \) can be written in the form (37c) and (37f).

Finally, it can be readily verified by straightforward calculation that map (37) is non-
is involutive, and its birationality is due to proposition 2.2.1.

**Remark 4.1.2.** The bosonic limit of the above map (namely if we set the odd variables \( x = x_1 = \psi_1 = \psi_2 = 0 \)) is map (4.7) in [15].

4.1.1. Restriction on invariant leaves: extension of Adler–Yamilov map. In this section, we derive an eight-dimensional YB map from map (37), which is the Grassmann extension of the Adler–Yamilov map [3, 17, 22]. Our proof is motivated by the existence of the first integral (23) for system (21).

In particular, we have the following.

**Proposition 4.1.3.**

(1) The quantities \( \Phi = X - x_1 x_2 - \chi_1 \chi_2 \) and \( \Psi = Y - y_1 y_2 - \psi_1 \psi_2 \) are invariants (first integrals) of the map (37).

(2) The ten-dimensional map (37) can be restricted to an eight-dimensional map, \( Y_{a,b} \in \text{End} \{ A_a \times A_b \} \), given by

\[
x \mapsto u = \left( y_1 + \frac{(b - a)(1 + x_1 y_2 - \chi_1 \psi_2)}{(1 + x_1 y_2)^2} x_1, \ y_2, \ \psi_1 + \frac{b - a}{1 + x_1 y_2} \chi_1, \ \psi_2 \right), \tag{42a} \]

\[
y \mapsto v = \left( x_1, \ x_2 + \frac{(a - b)(1 + x_1 y_2 - \chi_1 \psi_2)}{(1 + x_1 y_2)^2} y_2, \ \chi_1, \ \chi_2 + \frac{a - b}{1 + x_1 y_2} \psi_2 \right), \tag{42b} \]

where \( a, b \in G_0 \) and \( A_a, A_b \) are level sets of the first integrals \( \Phi \) and \( \Psi \), namely

\[ A_a = \{ (x_1, x_2, \chi_1, \chi_2, X) \in A^5; \Phi = a \}, \quad A_b = \{ (y_1, y_2, \psi_1, \psi_2, Y) \in A^5; \Psi = b \}. \tag{43} \]

(3) The bosonic limit of map \( Y_{a,b} \) is the Adler–Yamilov map.

**Proof.**

(1) It can be readily verified that (37) implies \( U - u_1 u_2 - \xi \xi_2 = X - x_1 x_2 - \chi_1 \chi_2 \) and \( V - v_1 v_2 - \eta \eta_2 = Y - y_1 y_2 - \psi_1 \psi_2 \). Thus, \( \Phi \) and \( \Psi \) are invariants, i.e. first integrals of the map.

(2) The existence of the restriction is obvious. Using the conditions \( X = x_1 x_2 + \chi_1 \chi_2 + a \) and \( Y = y_1 y_2 + \psi_1 \psi_2 + b \), one can eliminate \( X \) and \( Y \) from (37). The resulting map, \( x \mapsto u(x, y), \ y \mapsto v(x, y) \), is given by (42).
If one sets the odd variables of the above map equal to zero, namely $\chi_1 = \chi_2 = 0$ and $\psi_1 = \psi_2 = 0$, then the map (42) coincides with the Adler–Yamilov map.

Now, one can use the condition $X = x_1 x_2 + \chi_1 \chi_2 + a$ to eliminate $X$ from the Lax matrix (35), i.e.

$$M(x; a, \lambda) = \begin{pmatrix} a + x_1 x_2 + \chi_1 \chi_2 + \lambda & x_1 & \chi_1 \\ x_2 & 1 & 0 \\ \chi_2 & 0 & 1 \end{pmatrix},$$

which corresponds to the Darboux matrix derived in [13]. Now, the Adler–Yamilov map’s extension follows from the strong Lax representation

$$M(u; a, \lambda)M(v; b, \lambda) = M(y; b, \lambda)M(x; a, \lambda).$$

Therefore, the extension of the Adler–Yamilov’s map (42) is a reversible parametric YB map. Moreover, it is easy to verify that it is not involutive. Birationality of map (42) is due to proposition 2.2.1.

To generate invariants of map (42) we use $\text{str}(M(y; b, \lambda)M(x; a, \lambda))$, and we obtain the following

$$T_1 = x_1 x_2 + y_1 y_2 + x_1 \chi_2 + \psi_1 \psi_2,$$

$$T_2 = (a + x_1 x_2 + \chi_1 \chi_2)(b + y_1 y_2 + \psi_1 \psi_2) + x_1 y_2 + x_2 y_1 + \chi_1 \psi_2 - \chi_2 \psi_1,$$

where we have omitted the additive constants. However, $T_1$ and $T_2$ are linear combinations of the following invariants

$$I_1 = x_1 x_2 + y_1 y_2,$$  

$$I_2 = \chi_1 \chi_2 + \psi_1 \psi_2,$$  

$$I_3 = \chi_1 \chi_2 \psi_1 \psi_2,$$  

$$I_4 = b(x_1 x_2 + \chi_1 \chi_2) + a(y_1 y_2 + \psi_1 \psi_2) + y_1 y_2(x_1 x_2 + \chi_1 \chi_2) + x_1 x_2 \psi_1 \psi_2 + y_1 x_2 + y_2 x_1 + \chi_1 \psi_2 - \chi_2 \psi_1.$$  

4.2. DNLS case

According to matrix $M(p, q_{10}, \theta, \phi_{10}; 1, 1)$ in (32) we consider the following matrix

$$M(x; \lambda) = X^2 \begin{pmatrix} X & 0 & 0 \\ 0 & X & 0 \\ 0 & 0 & X \end{pmatrix} + \lambda \begin{pmatrix} 0 & x_1 & \chi_1 \\ x_2 & 0 & 0 \\ \chi_2 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $x = (x_1, x_2, \chi_1, \chi_2, X)$ and, in particular, we have set

$$X := f, \quad x_1 := fp, \quad x_2 := f q_{10}, \quad \chi_1 := f \theta \quad \text{and} \quad \chi_2 := \psi_{10} f.$$  

In this case, the Lax equation implies the following equations

$$U + V + \xi_1 \eta_2 = Y + X + y_1 x_2 + \psi_1 \chi_2,$$  

$$u_2 v_1 = y_2 \eta_1, \quad u_2 \eta_1 = y_2 x_1, \quad \xi_2 v_1 = \psi_2 x_1, \quad \xi_2 \eta_1 = \psi_2 \chi_1,$$  

$$U v_1 = Y \eta_1, \quad u_2 V = y_2 X, \quad U \eta_1 = Y \chi_1, \quad \eta_2 X = \psi_2 X,$$  

4.2. DNLS case
As in the previous section, the algebraic variety consists of two components. The first ten-dimensional component corresponds to the permutation map

\[ \mathbf{x} \mapsto \mathbf{u} = \mathbf{y}, \quad \mathbf{y} \mapsto \mathbf{v} = \mathbf{x}, \]

and the second corresponds to the following ten-dimensional YB map

\[
\mathbf{x} \mapsto \mathbf{u} = \left( \frac{y_1 + f(x, y)}{g(x, y)} x_1, \frac{g(x, y)}{h(x, y)} y_2, \psi_1 + \frac{f(x, y)}{g(x, y)} \chi_1, \frac{g(x, y)}{h(x, y)} \psi_2, \frac{g(x, y)}{h(x, y)} \right),
\]

\[
\mathbf{y} \mapsto \mathbf{v} = \left( \frac{h(x, y)}{g(x, y)} x_1, x_2 + \frac{f(y, x)}{h(x, y)} y_2, \frac{h(x, y)}{g(x, y)} \chi_1, x_2 + \frac{f(y, x)}{h(x, y)} \psi_2, \frac{h(x, y)}{g(x, y)} \right),
\]

where \( f, g \) and \( h \) are given by the following expressions

\[
f(x, y) = X - x_1 x_2 - \chi_1 \chi_2 - Y + y_1 y_2 + \psi_1 \psi_2, \]

\[
g(x, y) = X - x_1 (x_2 + y_2) - \chi_1 (\chi_2 + \psi_2), \]

\[
h(x, y) = Y - (x_1 + \chi_1) y_2 - (\chi_1 + \psi_1) \psi_2. \]

### 4.3. Restriction on invariant leaves

In this section, we show that the map given by (51), (52) can be restricted to a completely integrable eight-dimensional YB map on invariant leaves. As in the previous section, the idea of this restriction is motivated by the existence of the first integral (34).

Particularly, we have the following.

**Proposition 4.3.1.**

1. \( \Phi = X - x_1 x_2 - \chi_1 \chi_2 \) and \( \Psi = Y - y_1 y_2 - \psi_1 \psi_2 \) are invariants of the map (51), (52).
2. The ten-dimensional map (51), (52) can be restricted to an eight-dimensional map, \( \mathcal{F}_{a,b} \in \text{End} \{ A_a \times A_b \} \), given by

\[
x_1 \mapsto u_1 = y_1 + \frac{(a - b)(a - x_1 y_2 + \chi_1 \psi_2)}{(a - x_1 y_2)^2} x_1, \]

\[
x_2 \mapsto u_2 = \frac{(a - x_1 y_2 - \chi_1 \psi_2)(b - x_1 y_2 + \chi_1 \psi_2)}{(b - x_1 y_2)^2} y_2, \]

\[
\chi_1 \mapsto \xi_1 = \psi_1 + \frac{a - b}{a - x_1 y_2} \chi_1, \]

\[
\chi_2 \mapsto \xi_2 = \frac{a - x_1 y_2}{b - x_1 y_2} \psi_2, \]

\[
y_1 \mapsto v_1 = \frac{(b - x_1 y_2 - \chi_1 \psi_2)(a - x_1 y_2 + \chi_1 \psi_2)}{(a - x_1 y_2)^2} x_1, \]
\[ \begin{align*}
y_2 \mapsto y_2 &= x_2 + \frac{(b - a)(b - x_1 y_2 + \chi_1 \psi_2)}{(b - x_1 y_2)^2} y_2, \\
\psi_1 \mapsto \eta_1 &= \frac{b - x_1 y_2}{a - x_1 y_2}, \\
\psi_2 \mapsto \eta_2 &= \chi_2 + \frac{b - a}{b - x_1 y_2} \psi_2,
\end{align*} \tag{53f, 53g, 53h} \]

where \(a, b \in G_0\) and \(A_y, A_i\) are given by \((43)\).

(3) The bosonic limit of the above map is the four-dimensional YB map associated to the DNLS equation.

Proof.

(1) Map \((51), (52)\) implies \(U - u_1 u_2 - \xi \xi = X - x_1 x_2 - \chi_1 \chi_2\) and \(V = v_1 v_2 - \eta \eta = Y - y_1 y_2 - \psi_1 \psi_2\). Therefore, \(\Phi\) and \(\Psi\) are first integrals of the map.

(2) The conditions \(X = x_1 x_2 + \chi_1 \chi_2 + a\) and \(Y = y_1 y_2 + \psi_1 \psi_2 + b\) define the level sets, \(A_y\) and \(A_i\) of \(\Phi\) and \(\Psi\), respectively. Using these conditions, we can eliminate \(X\) and \(Y\) from map \((51), (52)\). The resulting map, \(I_{a,b} : A_y \times A_i \rightarrow A_y \times A_i\), is given by \((53)\).

(3) Setting the odd variables in \((53)\) equal to zero, we obtain map \((4.37)\) in \([15]\).

Now, using condition \(X = x_1 x_2 + \chi_1 \chi_2 + a\), matrix \((47)\) takes the following form

\[
M = \lambda^2 \begin{pmatrix} k + x_1 x_2 + \chi_1 \chi_2 & 0 & 0 \\
0 & 0 & \chi_2 \end{pmatrix} + \lambda \begin{pmatrix} 0 & x_1 & \chi_1 \\
x_2 & 0 & 0 \\
\chi_2 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \end{pmatrix} \tag{54} \]

Map \((53)\) has the following strong Lax representation

\[
M (u; a, \lambda) M (v; b, \lambda) = M (y; b, \lambda) M (x; a, \lambda). \tag{55} \]

Therefore, it is reversible parametric YB map which is birational due to proposition 2.2.1. It can also be verified that it is not involutive.

Regarding the invariants of map \((53)\), the ones which we retrieve from \(\text{str}(M (y; b, \lambda) M (x; a, \lambda))\) are

\[
\begin{align*}
K_1 &= (a + x_1 x_2 + \chi_1 \chi_2) (b + y_1 y_2 + \psi_1 \psi_2) \\
K_2 &= x_1 y_2 + y_1 x_2 + x_2 y_1 + \chi_1 x_2 + \psi_1 \psi_2 + \chi_1 \psi_2 - \chi_2 \psi_1,
\end{align*} \]

where we have omitted the additive constants. However, \(K_1\) is sum of the following quantities

\[
\begin{align*}
I_1 &= (a + x_1 x_2) (y_1 y_2 + \psi_1 \psi_2), \\
I_2 &= \chi_1 x_2 \psi_1 \psi_2, \tag{56a, 56b} \end{align*} \]

which are invariants themselves. Moreover, \(K_2\) is sum of the following invariants

\[
\begin{align*}
I_3 &= (x_1 + y_1) (x_2 + y_2), \quad \text{and} \quad I_4 = (\chi_1 + \psi_1) (\chi_2 + \psi_2). \tag{57} \end{align*}
\]

In fact, the quantities \(C_i = x_i + \chi_i\) and \(\Omega_i = \chi_i + \psi_i, i = 1, 2\), are invariants themselves.
4.4. Vector generalisations: \(4N \times 4N\) maps

In what follows we use the following notation for a vector \(\mathbf{w} = (w_1, \ldots, w_{4N})\)

\[ \mathbf{w} = (w_1, w_2, \omega_1, \omega_2), \quad \text{where} \quad w_1 = (w_1, \ldots, w_N), \quad w_2 = (w_{N+1}, \ldots, w_{2N}) \]

and \(\omega_1 = (w_{2N+1}, \ldots, w_{3N}), \quad \omega_2 = (w_{3N+1}, \ldots, w_{4N})\).

where \(w_1\) and \(w_2\) are even and \(\omega_1\) and \(\omega_2\) are odds. Also

\[ \langle u_1 \rangle = u_1, \quad |u_i\rangle = w_i^T \quad \text{and their dot product with} \quad \langle u_i, w_i \rangle. \]  

4.5. NLS case

Now, we replace the variables in map (42) with \(N\)-vectors, namely we consider the following \(4N \times 4N\) map

\[
\begin{align*}
\langle u_1 \rangle &= \langle y_1 \rangle + f(z; a, b)\langle x_1 \rangle(1 + \langle x_1, y_2 \rangle - \langle x_1, \psi_2 \rangle), \\
\langle u_2 \rangle &= \langle y_2 \rangle, \\
\langle x_1 \rangle &= \langle \psi_1 \rangle + f(z; a, b)\langle x_1 \rangle(1 + \langle x_1, y_2 \rangle - \langle x_1, \psi_2 \rangle), \\
\langle x_2 \rangle &= \langle \psi_2 \rangle,
\end{align*}
\]

where \(f\) is given by

\[
f(z; b, a) = \frac{b - a}{1 + z^2}, \quad z := \langle x_1, y_2 \rangle.
\]

Map (59), (60) is a reversible parametric YB map, for it has the following strong Lax-representation

\[ M(\mathbf{u}; a)M(\mathbf{v}; b) = M(\mathbf{y}; b)M(\mathbf{x}; a), \]

where

\[
M(\mathbf{w}; a) = \begin{pmatrix}
\lambda + a + \langle w_1, w_2 \rangle + \langle \omega_1, \omega_2 \rangle & \langle w_1 \rangle & \langle \omega_1 \rangle \\
\langle w_2 \rangle & \langle \omega_2 \rangle & I_{2N-1}
\end{pmatrix}.
\]

Moreover, map (59), (60) is birational and not involutive.

The invariants of this map are given by

\[
\begin{align*}
K_1 &= a + b + \langle x_1, x_2 \rangle + \langle y_1, y_2 \rangle + \langle x_1, \chi_2 \rangle + \langle \psi_1, \psi_2 \rangle, \\
K_2 &= b(\langle x_1, x_2 \rangle + \langle x_1, \chi_2 \rangle) + a(\langle y_1, y_2 \rangle + \langle \psi_1, \psi_2 \rangle) + (\langle x_1, y_2 \rangle + \langle \psi_1, \psi_2 \rangle)(\langle x_1, x_2 \rangle + \langle \chi_1, \chi_2 \rangle).
\end{align*}
\]

The quantities

\[ I_1 = \langle x_1, x_2 \rangle + \langle \chi_1, \chi_2 \rangle, \quad \text{and} \quad I_2 = \langle y_1, y_2 \rangle + \langle \psi_1, \psi_2 \rangle, \]

\[
\mu_{a,b} = -\frac{1}{2}((x_1, y_2) + \langle \psi_1, \psi_2 \rangle)(\langle x_1, x_2 \rangle + \langle \chi_1, \chi_2 \rangle).
\]

\[ \frac{d \mu_{a,b}}{d t} = -2\mu_{a,b} + 2\langle x_1, x_2 \rangle + 2\langle \chi_1, \chi_2 \rangle - 2\langle y_1, y_2 \rangle - 2\langle \psi_1, \psi_2 \rangle + \lambda\mu_{a,b}.
\]

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are invariant themselves, while $K_2$ can be written as sum of the following

$$I_3 = \langle \chi_1, \chi_2 \rangle \langle \psi_1, \psi_2 \rangle,$$

$$I_4 = b \langle \xi_1, \xi_2 \rangle + a \langle \chi_1, \xi_2 \rangle + \langle \chi_1, \psi_2 \rangle + \langle \chi_1, \psi_2 \rangle \langle \chi_1, \chi_2 \rangle. \quad (65a)$$

$$K_2 = \langle \chi_1, \xi_2 \rangle + \langle \chi_1, \psi_2 \rangle + \langle \chi_1, \chi_2 \rangle + \langle \chi_1, \psi_2 \rangle + \langle \chi_1, \chi_2 \rangle. \quad (65b)$$

### 4.6. DNLS case

Now, replacing the variables in (53) with $N$-vectors we obtain the following $4N \times 4N$-dimensional map

$$\begin{align*}
\langle u_1 \rangle &= \langle y_1 \rangle + h(\xi; \alpha, \beta) \langle \xi_1 \rangle (a - \langle x_1, \xi_2 \rangle + \langle x_1, \psi_2 \rangle), \\
\langle u_2 \rangle &= g(\xi; \alpha, \beta) \langle y_2 \rangle - h(\xi; \alpha, \beta) \langle \xi_1 \rangle, \\
\langle \xi_1 \rangle &= \langle \psi_1 \rangle + f(\xi; \alpha, \beta) \langle \chi_1 \rangle, \\
\langle \xi_2 \rangle &= g(\xi; \alpha, \beta) \langle \psi_2 \rangle,
\end{align*} \quad (66a)$$

and

$$\begin{align*}
\langle v_1 \rangle &= g(\xi; \alpha, \beta) \langle \xi_1 \rangle - h(\xi; \alpha, \beta) \langle x_1 \rangle, \\
\langle v_2 \rangle &= \langle x_2 \rangle + h(\xi; \alpha, \beta) \langle y_2 \rangle (b - \langle x_1, \xi_2 \rangle + \langle x_1, \psi_2 \rangle), \\
\langle \eta_1 \rangle &= g(\xi; \alpha, \beta) \langle \chi_1 \rangle, \\
\langle \eta_2 \rangle &= \langle \chi_2 \rangle + f(\xi; \alpha, \beta) \langle \psi_2 \rangle,
\end{align*}$$

where $f, g$ and $h$ are given by

$$f(\xi; \alpha, \beta) = \frac{a - b}{a - z}, \quad g(\xi; \alpha, \beta) = \frac{a - z}{b - z},$$

$$h(\xi; \alpha, \beta) = \frac{a - b}{(a - z)^2}, \quad z := \langle x_1, \xi_2 \rangle. \quad (67)$$

Map (66), (67) is reversible parametric YB map, as it has the strong Lax-representation (61) where

$$M = \begin{pmatrix}
\lambda (k + \langle x_1, x_2 \rangle + \langle \chi_1, \chi_2 \rangle) & \lambda (\xi_1) & \lambda (\chi_1) \\
\lambda (\xi_2) & \lambda (\chi_2) & \lambda (\psi_2) \\
\lambda (\chi_2) & \lambda (\psi_2) & I_{2N}
\end{pmatrix} \quad (68)$$

Moreover, it is a noninvolutive map and birational.

The invariants we retrieve from the supertrace of the monodromy matrix are given by

$$K_1 = (a + \langle x_1, x_2 \rangle + \langle \chi_1, \chi_2 \rangle) (b + \langle y_1, y_2 \rangle + \langle \psi_1, \psi_2 \rangle)$$

$$K_2 = \langle x_1, x_2 \rangle + \langle y_1, y_2 \rangle + \langle x_1, y_2 \rangle + \langle x_2, y_1 \rangle + \langle \chi_1, \chi_2 \rangle + \langle \psi_1, \psi_2 \rangle +$$

$$\langle \chi_1, \psi_2 \rangle - \langle \chi_2, \psi_1 \rangle,$$

where we have omitted the additive constants. In fact, $K_2$ is a sum of the following invariants

$$I_1 = \langle x_1 + y_1 \rangle, \quad I_2 = \langle x_1 + \psi_1 \rangle, \quad I_3 = \langle x_2 + y_2 \rangle, \quad I_4 = \langle x_1 + \psi_1 \rangle, \quad \langle x_2 + \psi_2 \rangle, \quad (69)$$

and the vectors in the above dot products are invariant themselves, namely

$$\langle x_1 + y_1 \rangle, \quad \langle x_2 + y_2 \rangle, \quad \langle x_1 + \psi_1 \rangle, \quad \langle x_2 + \psi_2 \rangle, \quad (70)$$

are invariants.
5. Conclusions

We showed that there are explicit examples of birational endomorphisms of Grassmann algebraic varieties which possess the YB property. These YB maps are related to non-commutative versions of integrable PDEs via their Lax representations which consist of Darboux matrices for these PDEs. Specifically, we considered the cases of the Grassmann extensions of Darboux transformations corresponding to

(1) the NLS equation;
(2) the DNLS equation.

In the former case a Darboux transformation appeared in [13] and, here, we constructed a Darboux transformation for the latter case. Employing the associated Darboux matrices we derived ten-dimensional maps, which we restricted on invariant leaves to eight-dimensional birational parametric YB maps. The motivation for these restrictions was the fact that the entries of the associated Darboux matrices satisfy particular systems of differential-difference equations which possess first integrals. The latter indicated the invariant leaves. In the case of the NLS equation the derived eight-dimensional YB map, namely map (42), is the Grassmann extension of the Adler–Yamilov map, while in the case of the DNLS equation the result is a novel eight-dimensional YB map, map (53), which, at the bosonic limit, is equivalent to a four-dimensional YB map which appeared recently in [15]. Moreover, we considered the vector generalisations of these eight-dimensional maps.

Our results could be extended in several ways.

(i) find the Poisson structure of the eight-dimensional YB maps;
(ii) study the case of the Lax operator with $\mathbb{D}_2$ symmetries;
(iii) study the corresponding noncommutative entwining systems;
(iv) study the transfer dynamics of all the Grassmann extended YB maps and the entwining systems associated to the Grassmann extended Darboux matrices.

Regarding (i) one needs to find the even–odd Poisson brackets [4] for the maps (42) and (53). In the case of map (53), the invariants $C_i$ and $\Omega_i$ will be Casimirs for the associated Poisson bracket. For (ii), in [16] we studied the Darboux transformations in the case of Lax operators which are invariant under the action of the $\mathbb{D}_2$ reduction group, whereas in [15] we studied the associated YB maps. The Grassmann extension of these Darboux transformations and their associated YB maps in this case is an open problem. With regards to (iii), one can consider Lax triples of Darboux matrices with even and odd entries. Finally, concerning (iv), one can consider the transfer maps for the $n$-periodic problem as defined in [18].

Acknowledgments

The authors would like to thank Vassilis Papageorgiou for the private discussion, and Hovhannes Khudaverdian for the discussion and his useful comments. GGG and AVM acknowledge support from the Leverhulme Trust. The work of AVM is partially supported by the EPSRC (Grant EP/I038675/1 is acknowledged). SKR acknowledges University of Leeds’ William Wright Smith scholarship and would like to thank JE Crowther for the scholarship-contribution to fees. The work of SKR was finalised at the ‘Laboratory of Applied Mathematics & Computer Technology’ of the Faculty of Mathematics & Computer Technology, Chechen State University, Russia.
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